CMC DOUBLINGS OF MINIMAL SURFACES VIA MIN-MAX

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ABSTRACT. Let $\Sigma^2 \subset M^3$ be a minimal surface of index 0 or 1. Assume that a neighborhood of $\Sigma$ can be foliated by constant mean curvature (cmc) hypersurfaces. We use min-max theory and the catenoid estimate to construct $\varepsilon$-cmc doublings of $\Sigma$ for small $\varepsilon > 0$. Such cmc doublings were previously constructed for minimal hypersurfaces $\Sigma^n \subset M^{n+1}$ with $n + 1 \geq 4$ by Pacard and Sun [21] using gluing methods.

1. INTRODUCTION

Let $M$ be a Riemannian manifold. A minimal hypersurface $\Sigma \subset M$ is a critical point of the area functional on $M$. A constant mean curvature (cmc) hypersurface is a critical point of the area functional subject to variations that preserve the enclosed volume. A fundamental problem in geometry is to construct minimal and cmc hypersurfaces in a given manifold.

Min-max methods have long proven to be a powerful tool for constructing minimal surfaces. In 1981, Pitts [22], building on work of Almgren [1], used min-max methods to show that every closed manifold $M^{n+1}$ with $3 \leq n + 1 \leq 6$ contains a smooth, embedded minimal hypersurface. Schoen and Simon [23] improved this to $3 \leq n + 1 \leq 7$. In fact, the work of Schoen and Simon shows that every $M^{n+1}$ with $n + 1 \geq 3$ contains a minimal hypersurface which is smooth and embedded up to a set of codimension 7.

In 1982, Yau [27] conjectured that every closed manifold contains infinitely many minimal surfaces. Marques and Neves devised a program to prove Yau’s conjecture by developing a detailed understanding of the Morse theory of the area functional on a manifold. This program has now been carried out to great success. In [8], Irie, Marques, and Neves showed that Yau’s conjecture is true for a generic metric on $M^{n+1}$ with $3 \leq n + 1 \leq 7$. In fact, they proved more: generically the union of all minimal surfaces in $M$ is dense in $M$. A crucial ingredient in the proof was the Weyl law for the volume spectrum proven by Liokumovich, Marques, and Neves [14].
Later Marques, Neves, and Song [19] improved the result in [8] by showing that, for a generic metric on $M$, some sequence of minimal surfaces becomes equidistributed in $M$. Gaspar and Guaraco [6] showed that the Weyl law and equidistribution results also hold in the Allen-Cahn setting. In the non-generic case, Song [25] has shown that a closed manifold of dimension $3 \leq n + 1 \leq 7$ with an arbitrary metric contains infinitely many minimal hypersurfaces. Thus Yau’s conjecture is fully resolved for these dimensions. In the higher dimensional case, Li [13] has shown that for a generic metric on $M^{n+1}$ with $n + 1 \geq 8$ there are infinitely many minimal hypersurfaces of optimal regularity.

Recently, Zhou [28] proved the multiplicity one conjecture of Marques and Neves [17]. Using this, Marques and Neves [17] [18] were able to prove the following: for a generic metric on $M^{n+1}$ with $3 \leq n + 1 \leq 7$ there is a smooth, embedded, two-sided, index $p$ minimal hypersurface for every $p \in \mathbb{N}$. Moreover, the area of these surfaces grows with $p$ according to the Weyl law for the volume spectrum [14].

Min-max methods for constructing constant mean curvature surfaces have only been developed more recently. Fix a number $h > 0$. Define a functional $A^h$ on open sets in $M$ with smooth boundary by setting

$$A^h(\Omega) = \text{Area}(\partial \Omega) - h \text{Vol}(\Omega).$$

It is known that the critical points of $A^h$ are precisely those sets $\Omega$ whose boundary has constant mean curvature $h$ with respect to the inward pointing normal vector. In [30], Zhou and Zhu developed a min-max theory for the $A^h$ functional, and used this theory to show that every closed manifold $M^{n+1}$ with $3 \leq n + 1 \leq 7$ admits a smooth almost-embedded $h$-cmc hypersurface for every $h > 0$. In [29], Zhou and Zhu extended the theory to construct more general prescribed mean curvature hypersurfaces. Zhou [28] used this to give a proof of the multiplicity one conjecture of Marques and Neves [17]. Earlier work of Chodosh and Mantoulidis [3] had shown that the multiplicity one conjecture was true for dimension $n + 1 = 3$ in the Allen-Cahn setting.

Another technique for constructing minimal and constant mean curvature hypersurfaces is the so-called gluing method. Starting from a collection of nearly minimal surfaces, one joins them together in a carefully chosen manner and then shows that the resulting surface can be perturbed to be minimal (or to have constant mean curvature). Kapouleas and Yang [11] used this technique to construct minimal doublings of the Clifford torus in $S^3$. Kapouleas has also used it to construct constant mean curvature surfaces of high genus in $\mathbb{R}^3$ [9], and to construct minimal doublings of the equator in $S^3$ [10]. In [21], Pacard and Sun used gluing methods to construct constant mean curvature
doublings of minimal hypersurfaces. The following theorem is a special case of their results (see Theorem 2.1 and Corollary 2.1 in [21]).

**Theorem 1** (Pacard and Sun). Let \( n + 1 \geq 4 \). Let \( \Sigma^n \subset M^{n+1} \) be an embedded minimal hypersurface. Assume that the Jacobi operator \( J \) for \( \Sigma \) is invertible, and that the unique solution \( \phi \) to \( J\phi = 1 \) does not change sign, and that \( \phi \) has a non-degenerate critical point. Then for every sufficiently small \( \varepsilon > 0 \), there is an embedded \( \varepsilon \)-cmc hypersurface which is a doubling of \( \Sigma \).

It is natural to ask whether surfaces produced by gluing methods can also be produced by variational techniques. In the case of the Clifford torus in \( S^3 \), Ketover, Marques, and Neves [12] proved the catenoid estimate and used it to give a min-max construction of the doublings of Kapouleas and Yang [11]. In this paper we show that, in certain circumstances, cmc doublings like those of Pacard and Sun can be constructed using min-max methods. Our results apply in the case \( 3 \leq n + 1 \leq 7 \). In the remainder of the introduction, we give a heuristic explanation of the min-max construction of cmc doublings.

1.1. The Stable Case. Fix a dimension \( 3 \leq n + 1 \leq 7 \). Suppose that \( \Sigma^n \subset M^{n+1} \) is an embedded, two-sided, stable, minimal hypersurface. Assume that a neighborhood of \( \Sigma \) can be foliated by \( \beta \)-cmc \( \Sigma^\beta \) whose mean curvature vectors point towards \( \Sigma \). Every strictly stable minimal surface admits such a neighborhood by the implicit function theorem and the maximum principle. A degenerate stable minimal surface may or may not admit such a neighborhood.

Let \( \Omega^\varepsilon \) be the open set in between \( \Sigma^\varepsilon \) and \( \Sigma^{-\varepsilon} \). Then \( \Omega^\varepsilon \) is a critical point of \( A^\varepsilon \). Moreover, using the second variation formula for \( A^\varepsilon \), one can check that \( \Omega^\varepsilon \) is strictly stable for \( A^\varepsilon \). Thus \( \Omega^\varepsilon \) is a strict local minimum for \( A^\varepsilon \) in the smooth topology. Now, by the isoperimetric inequality, the empty set is also a local minimum for \( A^\varepsilon \). Thus one can attempt to do min-max for the \( A^\varepsilon \) functional over all 1-parameter families of open sets connecting the empty set to \( \Omega^\varepsilon \).

Theorem 6 and Theorem 7 are the main results of this paper in the stable case. In Theorem 6, we formalize the min-max argument outlined above to construct an \( \varepsilon \)-cmc doubling of \( \Sigma \). The key tool in the proof is the min-max theory for the \( A^\varepsilon \) functional introduced by Zhou and Zhu in [30]. We also borrow ideas from previous mountain pass type arguments for minimal surfaces. See De Lellis and Ramic [11], Marques and Neves [16], and Montezuma [20]. In the case \( n = 2 \), we are further able to show that the \( \varepsilon \)-cmc doubling constructed in Theorem 6 consists of two parallel copies of \( \Sigma \) joined by a small
catenoidal neck. This is the content of Theorem 7. The proof of this theorem is based on work of Chodosh, Ketover, and Maximo [2].

1.2. The Index 1 Case. Fix a dimension $3 \leq n + 1 \leq 7$. Let $\Sigma^n \subset M^{n+1}$ be an embedded, two-sided, index 1, minimal hypersurface. Let $L$ be the Jacobi operator on $\Sigma$ and assume that $L$ is non-degenerate and that the unique solution $\phi$ to $L\phi = 1$ is positive. The assumption that $L$ is non-degenerate together with the fact that $\phi > 0$ implies that a neighborhood of $\Sigma$ is foliated by cmc hypersurfaces. Again let $\Sigma^\beta$ denote the $\beta$-cmc in this foliation and note that the mean curvature vector of $\Sigma^\beta$ points away from $\Sigma$. Moreover, the surface $\Sigma^\beta$ lies at a height on the order of $\beta$ over $\Sigma$.

Now fix a small number $\varepsilon > 0$ and consider an $\varepsilon$-cmc doubling $\Lambda^\varepsilon$ of $\Sigma$. If $\Lambda^\varepsilon$ arises from the construction of Pacard and Sun, there is a decomposition

$$\Lambda^\varepsilon = \Lambda_+ \cup \Lambda_- \cup N$$

where $N$ is a small neck, and $\Lambda_+$ and $\Lambda_-$ are each diffeomorphic to $\Sigma$ with a small ball removed. The sheet $\Lambda_+$ is the graph of a function of small norm over $\Sigma^\varepsilon$, and the sheet $\Lambda_-$ is the graph of a function of small norm over $\Sigma^{-\varepsilon}$. From this structure, we expect that the index of $\Lambda^\varepsilon$ is three, where the three deformations decreasing $A^\varepsilon$ correspond to varying the height of $\Lambda_+$, varying the height of $\Lambda_-$, and pinching the neck. Thus $\Lambda^\varepsilon$ should be the solution to a three parameter min-max problem.

Based on this, we construct a three parameter family of surfaces $\Phi$ parameterized by the cube

$$X = \left\{ (x, y, t) : -\frac{\varepsilon}{2} \leq x \leq \frac{\varepsilon}{2}, -\frac{\varepsilon}{2} \leq y \leq \frac{\varepsilon}{2}, 0 \leq t \leq R \right\},$$

where $R \gg \varepsilon$ is a fixed small number. To define $\Phi$, first let $\Phi(0,0,0) = \Sigma^\varepsilon \cup \Sigma^{-\varepsilon}$. Think of this as a top sheet $\Sigma^\varepsilon$ at height $\varepsilon$ and a bottom sheet $\Sigma^{-\varepsilon}$ at height $-\varepsilon$. Then extend $\Phi$ to the rest of $X$ as follows: changing the $x$-coordinate varies the height of the top sheet by up to $\pm \varepsilon/2$, changing the $y$-coordinate varies the height of the bottom sheet by up to $\pm \varepsilon/2$, and increasing the $t$-coordinate opens up a neck between the two sheets.

This family $\Phi$ has two important properties.

(i) The surface $\Sigma^\varepsilon \cup \Sigma^{-\varepsilon}$ is an index two critical point of $A^\varepsilon$ and the bottom face of the cube $X$ is a two parameter family of deformations that decreases $A^\varepsilon$. 
(ii) The surface $\Sigma^\varepsilon \cup \Sigma^{-\varepsilon}$ maximizes $A^\varepsilon$ over the boundary of $X$.

To see property (ii), first observe that $\Sigma^\varepsilon \cup \Sigma^{-\varepsilon}$ maximizes $A^\varepsilon$ over the bottom face of the cube. Second, note that by opening a neck up to a fixed size $R \gg \varepsilon$, we can ensure that $A^\varepsilon(S) < A^\varepsilon(\Sigma^\varepsilon \cup \Sigma^{-\varepsilon})$ for every surface $S$ in the top face of the cube. Finally, consider a surface $T$ in the boundary of the bottom face of the cube. Since $\Sigma$ is unstable, there is a uniform constant $c$ such that

$$A^\varepsilon(T) < A^\varepsilon(\Sigma^\varepsilon \cup \Sigma^{-\varepsilon}) - c\varepsilon^2.$$ 

On the other hand, by the catenoid estimate of Ketover, Marques, and Neves [12], it is possible to open a neck between the two sheets in $T$ without ever increasing the area by more than $C\varepsilon^2/|\log \varepsilon|$. Therefore, we can ensure that $\Sigma^\varepsilon \cup \Sigma^{-\varepsilon}$ also maximizes $A^\varepsilon$ over the side faces of the cube.

Theorem 25 and Theorem 26 are the main results of this paper in the index 1 case. In Theorem 25, we construct $\varepsilon$-cmc surfaces $\Lambda^\varepsilon$ in $M$ by doing min-max for the $A^\varepsilon$ functional over all families of surfaces $\Psi$ parameterized by the cube $X$ with $\Psi = \Phi$ on $\partial X$. These surfaces $\Lambda^\varepsilon$ have the property that $\text{Area}(\Lambda^\varepsilon) \to 2\text{Area}(\Sigma)$ as $\varepsilon \to 0$. In Theorem 26, we show that for a generic metric on $M$ the surfaces $\Lambda^\varepsilon$ of Theorem 25 are doublings of $\Sigma$.

1.3. Organization. The rest of the paper is organized as follows. Section 2 reviews some concepts from geometric measure theory as well as some definitions and theorems from Zhou’s min-max theory. Section 3 constructs cmc doublings in the stable case. Section 4 constructs cmc doublings in the index 1 case. Appendix A contains a quantitative minimality theorem that is needed to check that the width of certain homotopy classes is non-trivial. Appendix B proves that a certain class of metrics is generic.

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2. Notation and Preliminaries

Let $M^{n+1}$ be a smooth, closed Riemannian manifold. We begin by introducing some tools from geometric measure theory.

- The set $\mathcal{I}_k(M, \mathbb{Z}_2)$ is the space of $k$-dimensional rectifiable flat chains mod 2 in $M$.
• The flat norm on $\mathcal{I}_k(M, \mathbb{Z}_2)$ is denoted by $\mathcal{F}$, and the mass norm on $\mathcal{I}_k(M, \mathbb{Z}_2)$ is denoted by $\mathcal{M}$.

• Given $T \in \mathcal{I}_k(M, \mathbb{Z}_2)$, the notation $|T|$ stands for the varifold induced by $T$.

• The $\mathcal{F}$ metric on $\mathcal{I}_{n+1}(M, \mathbb{Z}_2)$ is defined by

$$\mathcal{F}(\Omega_1, \Omega_2) = \mathcal{F}(\Omega_1, \Omega_2) + \mathcal{F}(|\partial \Omega_1|, |\partial \Omega_2|)$$

where $\mathcal{F}$ on the right hand side is Pitts’ $\mathcal{F}$-metric on varifolds.

• Following Marques and Neves, an embedded minimal cycle in $M$ is defined to be a varifold $V$ of the form

$$V = a_1 \Gamma_1 + \ldots + a_\ell \Gamma_\ell$$

where the $\Gamma_i$ are disjoint, smooth, embedded minimal surfaces in $M$ and the $a_i$ are positive integers.

• Given $\varepsilon > 0$, define $A^\varepsilon : \mathcal{I}_{n+1}(M, \mathbb{Z}_2) \to \mathbb{R}$ by $A^\varepsilon(\Omega) = \text{Area}(\partial \Omega) - \varepsilon \text{Vol}(\Omega)$.

The following definitions are due to Zhou in [28]. Let $X$ be a cubical complex and let $Z$ be a subcomplex of $X$. Fix an $\mathcal{F}$-continuous map $\Phi : X \to \mathcal{I}_{n+1}(M, \mathbb{Z}_2)$.

**Definition 2.** The $(X, Z)$-homotopy class of $\Phi$ consists of all sequences $\{\Psi_i\}_i$ with the following properties. First, each $\Psi_i$ is an $\mathcal{F}$-continuous map $X \to \mathcal{I}_{n+1}(M, \mathbb{Z}_2)$. Second, for each $i$, there is a flat continuous homotopy $H_i : [0, 1] \times X \to \mathcal{I}_{n+1}(M, \mathbb{Z}_2)$ such that

1. $H_i(0, x) = \Psi_i(x)$,
2. $H_i(1, x) = \Phi(x)$,
3. $\limsup_{i \to \infty} \left[ \sup_{z \in Z, t \in [0, 1]} \mathcal{F}(\Phi(z), H_i(t, z)) \right] = 0$.

**Definition 3.** Let $\Pi$ be the $(X, Z)$-homotopy class of $\Phi$. Fix an $\varepsilon > 0$. Given a sequence $\{\Psi_i\}_i$ in $\Pi$ we let

$$L^\varepsilon(\{\Psi_i\}_i) = \limsup_{i \to \infty} \left[ \max_{x \in X} A^\varepsilon(\Psi_i(x)) \right].$$
The width of the homotopy class \( \Pi \) is then defined by

\[
L^\varepsilon(\Pi) = \inf_{\{\Psi_i\}_i \in \Pi} L^\varepsilon(\{\Psi_i\}_i).
\]

**Definition 4.** Let \( \Gamma \) be a smooth, immersed, constant mean curvature hypersurface in \( M \). Then \( \Gamma \) is said to be almost-embedded provided for every point \( p \in M \) either

(i) \( \Gamma \) is embedded in a neighborhood of \( p \), or

(ii) \( \Gamma \) decomposes into the union of two embedded pieces \( \Gamma_1 \) and \( \Gamma_2 \) in a neighborhood of \( p \) with \( \Gamma_1 \) on one side of \( \Gamma_2 \).

The following min-max theorem for the \( A^\varepsilon \) functional is due to Zhou. See Theorem 1.7 and Theorem 3.1 in [28].

**Theorem 5 (Zhou).** Assume that the min-max width \( \Pi \) is non-trivial, i.e., that

\[
L^\varepsilon(\Pi) > \max_{z \in Z} A^\varepsilon(\Phi(z)).
\]

Then there is a smooth, almost-embedded \( \varepsilon \)-cmc hypersurface \( \Lambda^\varepsilon \) in \( M \), and there is an open set \( \Theta^\varepsilon \) in \( M \) with \( \partial \Theta^\varepsilon = \Lambda^\varepsilon \) and \( A^\varepsilon(\Theta^\varepsilon) = L^\varepsilon(\Pi) \). Moreover, the index of \( \Lambda^\varepsilon \) as a critical point of \( A^\varepsilon \) is at most the dimension of \( X \).

3. The Stable Case

3.1. **Statement of Results.** We now formalize the assumptions outlined in the introduction. Fix a dimension \( 3 \leq n + 1 \leq 7 \). Let \( (M^{n+1}, g) \) be a closed Riemannian manifold and let \( \Sigma^n \subset M^{n+1} \) be a closed, connected, two-sided, minimal hypersurface. Also assume the following.

(S-i) There is a neighborhood \( U \) of \( \Sigma \) and a smooth function \( f \) on \( U \) and a number \( \alpha > 0 \) such that \( -\alpha < f < \alpha \) on \( U \).

(S-ii) The level set \( \Sigma^\beta := f^{-1}(\beta) \) is a closed hypersurface diffeomorphic to \( \Sigma \) with constant mean curvature \( |\beta| \) for \( |\beta| < \alpha \). Moreover \( \Sigma^0 = \Sigma \).

(S-iii) The mean curvature vector of \( \Sigma^\beta \) points toward \( \Sigma \) for each \( |\beta| < \alpha \).

(S-iv) The gradient \( \nabla f \) does not vanish anywhere on \( U \setminus \Sigma \).
For future reference, we will refer to this collection of assumptions as (S). Let \( \Omega^\varepsilon \) be the region contained between \( \Sigma^\varepsilon \) and \( \Sigma^{-\varepsilon} \).

Our main theorems in the stable case are the following.

**Theorem 6.** Fix \((M^{n+1}, g)\) and \(\Sigma\) for which the assumptions (S) hold. Then there is a smooth, almost-embedded \(\varepsilon\)-cmc \(\Lambda^\varepsilon\) contained in \(\Omega^\varepsilon\). Moreover, there is an open set \(\Theta^\varepsilon \subset \Omega^\varepsilon\) with \(\Lambda^\varepsilon = \partial \Theta^\varepsilon\), and the index of \(\Lambda^\varepsilon = \partial \Theta^\varepsilon\) as a critical point of \(A^\varepsilon\) is at most 1.

**Theorem 7.** Assume further that \(n = 2\). Then the surface \(\Lambda^\varepsilon\) from the previous theorem admits a decomposition
\[
\Lambda^\varepsilon = \Lambda^\varepsilon_+ \cup \Lambda^\varepsilon_- \cup N
\]
where each \(\Lambda^\varepsilon_\pm\) is the graph of a function of small norm over \(\Sigma\) minus a ball and \(N\) is a catenoidal neck.

### 3.2. Sweepouts.

We would like to use a mountain pass type argument to produce an \(\varepsilon\)-cmc. We now introduce the maps that will serve as sweepouts.

Fix a number \(0 < \varepsilon < \alpha\). For each \(0 < \beta < \alpha\), let \(\Omega^\beta = \{-\beta < f < \beta\}\) denote the open set between \(\Sigma^{-\beta}\) and \(\Sigma^\beta\). Also fix a small number \(\eta > 0\) to be specified later and let \(\Omega^* = \Omega^{\varepsilon + \eta}\).

**Proposition 8.** There is an \(\mathbf{F}\)-continuous map \(\Phi: [0, 1] \to \mathcal{I}_{n+1}(\Omega^*, \mathbb{Z}_2)\) with \(\Phi(0) = \emptyset\) and \(\Phi(1) = \Omega^\varepsilon\).

**Proof.** The map \(\Psi: [0, 1] \to \mathcal{I}_{n+1}(\Omega^*, \mathbb{Z}_2)\) given by \(\Psi(t) = \Omega^{t\varepsilon}\) is continuous in the flat topology. By Lemma A.1 in Zhou and Zhu [30], it is possible to construct a sequence \(\{\phi_i\}_i\) of better and better discrete approximations to \(\Psi\). Applying Zhou’s discrete to continuous interpolation theorem (Theorem 1.12 in [28]) produces the required map \(\Phi\) from the sequence \(\{\phi_i\}_i\). \(\square\)

**Definition 9.** Let \(\Phi\) be the map constructed in the previous proposition. A sweepout is an \(\mathbf{F}\)-continuous map \(\Psi: [0, 1] \to \mathcal{I}_{n+1}(\Omega^*, \mathbb{Z}_2)\) with \(\Psi(0) = \emptyset\) and \(\Psi(1) = \Omega^\varepsilon\) that is flat homotopic to \(\Phi\) relative to \(\partial[0, 1]\). More precisely, this last statement means that there is a flat continuous map \(H: [0, 1] \times [0, 1] \to \mathcal{I}_{n+1}(\Omega^*, \mathbb{Z}_2)\) such that

1. \(H(0, t) = \Phi(t)\),
2. \(H(1, t) = \Psi(t)\),
3. \(H(s, 0) = \emptyset\),
4. \(H(s, 1) = \Omega^\varepsilon\),

for all \(s\) and \(t\).
Remark 10. Let $X = [0, 1]$ and $Z = \{0, 1\}$. Note that a sweepout $\Psi$ is essentially an element of the $(X, Z)$-homotopy class of $\Phi$ as defined in Section 2. However, we require that $\Psi(0)$ exactly equals $\Phi(0)$ and that $\Psi(1)$ exactly equals $\Phi(1)$. Moreover, all sets in a sweepout $\Psi$ are required to be contained in the set $\Omega^*$.

Definition 11. The min-max width $W^\varepsilon$ is defined by

$$W^\varepsilon = \inf_{\text{sweepouts } \Psi} \left[ \sup_{t \in [0,1]} A^\varepsilon(\Psi(t)) \right].$$

Definition 12. A critical sequence is a sequence of sweepouts $\{\Psi_i\}_i$ with the property that

$$\lim_{i \to \infty} \left[ \sup_{t \in [0,1]} A^\varepsilon(\Psi_i(t)) \right] = W^\varepsilon.$$

Definition 13. Let $\{\Psi_i\}_i$ be a critical sequence. The associated critical set $C(\{\Psi_i\}_i)$ is the collection of all varifolds of the form

$$V = \lim_{i \to \infty} |\partial \Psi_i(t_i)|$$

with $t_i \in [0,1]$ and $\lim_{i \to \infty} A^\varepsilon(\Psi_i(t_i)) = W^\varepsilon$. Note that the critical set is always non-empty and compact.

3.3. Non-trivial Width. Fix $(M, g)$ and $\Sigma$ satisfying the assumptions (S) and fix a number $0 < \varepsilon < \alpha$. Recall that the notation $\Omega^\beta$ denotes the open set between $\Sigma^{-\beta}$ and $\Sigma^\beta$. Also $\eta > 0$ is a fixed small number and $\Omega^* = \Omega^{\varepsilon+\eta}$. The number $W^\varepsilon$ is the min-max width of the collection of all paths in $I_{n+1}(M, \mathbb{Z}_2)$ joining $\emptyset$ to $\Omega^\varepsilon$ while staying inside $\Omega^*$.

The goal of this section is to show that $W^\varepsilon > \max\{A^\varepsilon(\Omega^\varepsilon), 0\}$. The fact that $W^\varepsilon > A^\varepsilon(\emptyset) = 0$ is a consequence of a suitable isoperimetric inequality.

Proposition 14 (See Theorem 2.15 in [30]). There are constants $C$ and $V$ such that

$$\text{Area}(\partial \Omega) \geq C \text{Vol}(\Omega)^{n/(n+1)}$$

whenever $\Omega \in I_{n+1}(M, \mathbb{Z}_2)$ satisfies $\text{Vol}(\Omega) < V$.

Corollary 15. The width $W^\varepsilon$ is positive.

Proof. Choose a small number $0 < v < \min\{V, \text{Vol}(\Omega^\varepsilon)\}$. Let $\Psi: [0, 1] \to I_{n+1}(\Omega^*, \mathbb{Z}_2)$ be a sweepout. By continuity, there must be some $\Omega$ in the image of $\Psi$ with $\text{Vol}(\Omega) = v$. It follows that

$$A^\varepsilon(\Omega) = \text{Area}(\partial \Omega) - \varepsilon \text{Vol}(\Omega)$$

$$\geq C \text{Vol}(\Omega)^{n/(n+1)} - \varepsilon \text{Vol}(\Omega) = \text{Vol}(\Omega)^{n/(n+1)} \left(C - \varepsilon \text{Vol}(\Omega)^{1/(n+1)}\right).$$
The number on the right hand side is positive provided \( v \) is taken sufficiently small. \( \square \)

It remains to show that \( W^\varepsilon > A^\varepsilon(\Omega^\varepsilon) \). To begin, we first show that \( \Omega^\varepsilon \) is a strictly stable critical point of \( A^\varepsilon \).

**Proposition 16.** Assume that \((M, g)\) and \(\Sigma\) satisfy the assumptions \((S)\). Then \( \Omega^\varepsilon \) is a strictly stable critical point of \( A^\varepsilon \).

**Proof.** Let \( N \) be the outward pointing normal vector to \( \partial \Omega^\varepsilon \). The second variation formula for \( A^\varepsilon \) says that
\[
\delta^2 A^\varepsilon \bigg|_{\Omega^\varepsilon} (uN) = -\int_{\Sigma^\varepsilon} uL_\varepsilon u - \int_{\Sigma_{-\varepsilon}} uL_{-\varepsilon} u,
\]
where \( L_\beta \) is the Jacobi operator on \( \Sigma^\beta \). Hence to prove the claim it suffices to show that the lowest eigenvalue of \( L_\beta \) is positive for \( \beta = \pm \varepsilon \).

We will prove this for \( L_\varepsilon \), the argument for \( L_{-\varepsilon} \) being essentially identical. Let \( H \) be the mean curvature operator on \( \Sigma^\varepsilon \) (computed with respect to \( N \)). It is known that \( L_\varepsilon \) is the linearization of \( H \). For \( \gamma \) close enough to \( \varepsilon \), we can write \( \Sigma^\gamma \) as a normal graph of a function \( \varphi_\gamma \) over \( \Sigma^\varepsilon \). Define
\[
\psi = \frac{d}{d\gamma} \bigg|_{\gamma=\varepsilon} (\varphi_\gamma)
\]
and note that \( \psi \geq 0 \). Differentiating the equation \( H(\varphi_\gamma) = -\gamma \) and evaluating at \( \gamma = \varepsilon \) shows that \( L_\varepsilon \psi = -1 \).

The existence of a non-negative solution to this equation implies that the lowest eigenvalue of \( L_\varepsilon \) is positive. Indeed, let \( \lambda \) be the lowest eigenvalue of \( L_\varepsilon \) and let \( \zeta > 0 \) be the associated eigenfunction so that \( L_\varepsilon \zeta + \lambda \zeta = 0 \). Since
\[
\int_{\Sigma^\varepsilon} \zeta = -\int_{\Sigma^\varepsilon} \zeta L_\varepsilon \psi = -\int_{\Sigma^\varepsilon} \psi L_\varepsilon \zeta = \lambda \int_{\Sigma^\varepsilon} \psi \zeta
\]
it follows that \( \lambda \) must be positive. \( \square \)

The desired inequality for the width now follows from the quantitative minimality results in Appendix A.

**Proposition 17.** There are positive constants \( \delta \) and \( C \) such that
\[
A^\varepsilon(\Omega) \geq A^\varepsilon(\Omega^\varepsilon) + C \mathcal{F}(\Omega, \Omega^\varepsilon)^2
\]
for all \( \Omega \in \mathcal{I}_{n+1}(\Omega^*, \mathbb{Z}_2) \) with \( \mathcal{F}(\Omega, \Omega^\varepsilon) < \delta \).
Proof. Proposition \[16\] says that $\Omega^\varepsilon$ is strictly stable for $A^\varepsilon$. Hence the desired result follows from Corollary \[40\] in Appendix A. $\square$

**Corollary 18.** The width satisfies $W^\varepsilon > A^\varepsilon(\Omega^\varepsilon)$.

Proof. Let $\delta$ and $C$ be the constants from Proposition \[17\]. Without loss we can assume that $\delta < \text{Vol}(\Omega^\varepsilon)$. Let

$$
\Psi: [0,1] \to \mathcal{I}_{n+1}(\Omega^*, \mathbb{Z}_2)
$$

be a sweepout. By continuity there is some $\Omega$ in the image of $\Psi$ with $\mathcal{F}(\Omega, \Omega^\varepsilon) = \delta/2$. But then Proposition \[17\] implies that

$$
A^\varepsilon(\Omega) \geq A^\varepsilon(\Omega^\varepsilon) + \frac{C\delta^2}{4},
$$

and the corollary follows. $\square$

3.4. **A Deformation Lemma.** The goal of this section is to prove a deformation lemma that will be used to show that the min-max surface lies in the interior of $\Omega^*$. The proof closely follows an argument of Marques and Neves \[17\], and relies on the existence of a deformation that pushes currents away from $\partial \Omega^*$ while simultaneously decreasing $A^\varepsilon$.

**Proposition 19.** It is possible to find an open set $\Omega^{**}$ with

$$
\Omega^\varepsilon \subset \subset \Omega^{**} \subset \subset \Omega^*
$$

together with a Lipschitz vector field $Z$ supported on $\Omega^* \setminus \Omega^\varepsilon$ with flow $\varphi_t$ such that the following properties hold.

(i) $\text{supp}((\varphi_1)_# \Omega) \subset \Omega^{**}$, for all $\Omega \in \mathcal{I}_{n+1}(\Omega^*, \mathbb{Z}_2)$

(ii) $A^\varepsilon((\varphi_1)_# \Omega) \leq A^\varepsilon(\Omega)$, for all $\Omega \in \mathcal{I}_{n+1}(\Omega^*, \mathbb{Z}_2)$

Proof. Recall that the cmc foliation near $\Sigma$ is given by the level sets of a function $f$ and that $\nabla f \neq 0$ on a neighborhood $W$ of $\Sigma^{-\varepsilon} \cup \Sigma^\varepsilon$. By taking $\eta$ small enough, we can assume that $\Omega^* \setminus \Omega^\varepsilon \subset W$. Define a vector field $X = \nabla f / |\nabla f|^2$ on $W$. Then define

$$
Z = \begin{cases} 
-(f - \varepsilon)X, & \text{on } \Omega^* \setminus \Omega^\varepsilon \\
0, & \text{otherwise}
\end{cases}
$$

and note that $Z$ is a Lipschitz vector field on $\Omega^*$. 
Fix some \( \Omega \in \mathcal{I}_{n+1}(\Omega^*,\mathbb{Z}_2) \) and let \( \nu \) be the outward pointing normal vector to \( \partial \Omega \). According to the first variation formula,
\[
\delta A^\varepsilon \bigg|_\Omega (Z) = \int_{\partial \Omega} \text{div}_\sigma Z - \varepsilon \int_{\partial \Omega} \langle Z, \nu \rangle \, d\mathcal{H}^n.
\]
To understand the right hand side, we need to compute \( \text{div}_\sigma Z \).

Let \( \phi_t \) denote the flow of \( X \) and let \( x \) denote a point in \( \Sigma^\varepsilon \). Choose a point \( y = \psi(x,t) \) and let \( \sigma \subset T_y M \) be an \( n \)-plane. Let \( \mathbf{X}_i \) be coordinates on a neighborhood of \( x \) in \( \Sigma^\varepsilon \). Then the map \( \psi(x,t) = \phi_t(x) \) gives coordinates on a neighborhood of \( y \). Define \( e_i = \partial \psi / \partial x_i \) and note that \( \partial \psi / \partial t = X \). Let \( N = \nabla f / |\nabla f| \) be the unit normal vector to the surfaces \( \Sigma^\beta \) and let \( A \) denote the second fundament form of the surfaces \( \Sigma^\beta \). As in Marques and Neves [17], we compute
\[
\langle \nabla e_i Z, e_j \rangle = -\langle Z, A(e_i, e_j) \rangle,
\]
\[
\langle \nabla N Z, N \rangle = \langle \nabla N (-f - \varepsilon)X, N \rangle = -1 - (f - \varepsilon)\langle \nabla N X, N \rangle.
\]

Also we have
\[
\langle \nabla e_i Z, N \rangle = (f - \varepsilon) \left( \frac{\partial \psi}{\partial x_i}, \nabla \phi_0 N \right) = \frac{f - \varepsilon}{|\nabla f|} \langle e_i, \nabla N N \rangle,
\]
\[
\langle e_i, -\nabla N Z \rangle = \left( e_i, N \left( \frac{f - \varepsilon}{|\nabla f|} \right) N + \frac{f - \varepsilon}{|\nabla f|} \nabla N N \right) = \frac{f - \varepsilon}{|\nabla f|} \langle e_i, \nabla N N \rangle,
\]
and so
\[
\langle \nabla e_i Z, N \rangle = -\langle e_i, \nabla N Z \rangle.
\]

Using this one can compute \( \text{div}_\sigma Z \) as follows.

Let \( v_1, \ldots, v_n \) be an orthonormal basis for \( \sigma \). We can arrange that \( v_1, \ldots, v_{n-1} \) are tangent to \( \Sigma^{\varepsilon+t} \) and that \( v_n = (\cos \theta)u + (\sin \theta)N \) for some unit vector \( u \) which is tangent to \( \Sigma^{\varepsilon+t} \) and orthogonal to \( v_1, \ldots, v_{n-1} \). Let \( H \) be the mean curvature vector for \( \Sigma^{\varepsilon+t} \). Then from the above computations one finds
\[
\text{div}_\sigma Z = \left( \langle \nabla_u Z, u \rangle + \sum_{i=1}^{n-1} \langle \nabla_{v_i} Z, v_i \rangle \right) + \langle \nabla_{v_n} Z, v_n \rangle - \langle \nabla_u Z, u \rangle
\]
\[
= -\langle Z, H \rangle + (\cos^2 \theta - 1)\langle \nabla_u Z, u \rangle + \sin^2 \theta \langle \nabla N Z, N \rangle
\]
\[
= -\varepsilon \frac{(f - \varepsilon)}{|\nabla f|} - \sin^2 \theta \left( 1 + (f - \varepsilon)\langle \nabla N X, N \rangle + (f - \varepsilon)\langle X, A(u, u) \rangle \right).
\]
Therefore, provided \( \eta \) is small enough, it follows that
\[
\text{div}_\sigma Z - \varepsilon \langle Z, \nu \rangle \leq -\frac{\varepsilon(f - \varepsilon)}{|\nabla f|} + \varepsilon|Z| = 0.
\]
Hence following the flow of \( Z \) decreases \( A^\varepsilon \).

**Corollary 20.** There exists an open set \( \Omega^{**} \) with \( \Omega^\varepsilon \subset \subset \Omega^{**} \subset \subset \Omega^* \) and a critical sequence \( \{\Psi_i\}_i \) such that
\[
\text{supp}(\Psi_i(x)) \subset \Omega^{**}
\]
for all \( i \) and all \( x \in [0, 1] \).

**Proof.** Let \( \{\Phi_i\}_i \) be a critical sequence. Let \( \varphi_i \) denote the flow of \( Z \). Define \( \Psi_i(x) = (\varphi_1)_# \Phi_i(x) \) for \( x \in [0, 1] \). By the previous proposition, \( \{\Psi_i\}_i \) is as required.

### 3.5. Constructing the Min-Max Surfaces

We can now perform a min-max argument to construct the doublings. The following min-max theorem essentially follows from Theorem 5. The theorem is not an immediate consequence of Theorem 5 because we require that the surfaces in a sweepout are contained in \( \Omega^* \). However, it is straightforward to modify the proof of Theorem 5 to handle our situation.

**Theorem 21.** Assume that \( W^\varepsilon > \max\{0, A^\varepsilon(\Omega^\varepsilon)\} \). Then for any critical sequence \( \{\Psi_i\}_i \) there is a varifold \( V \in C(\{\Psi_i\}_i) \) that is induced by a smooth, almost-embedded \( \varepsilon \)-cmc hypersurface \( \Lambda^\varepsilon \). There is an open set \( \Theta^\varepsilon \subset \subset \Omega^* \) such that \( \partial \Theta^\varepsilon = \Lambda^\varepsilon \) and \( A^\varepsilon(\Theta^\varepsilon) = W^\varepsilon \). Moreover, there is a bound \( \text{ind}(\Lambda^\varepsilon) \leq 1 \).

**Proof.** We outline the necessary changes to the proof of Theorems 1.7 and 3.1 in [28]. Let \( X = [0, 1] \) and \( Z = \{0, 1\} \). Let \( \Phi \) be the map from Proposition 8. Zhou defines the \((X, Z)\)-homotopy class of \( \Phi \) to consist of all sequences \( \{\Psi_i\}_i \) such that each \( \Psi_i \) is flatly homotopic to \( \Phi \) and
\[
\lim_{i \to \infty} \max \{F(\Psi_i(0), \emptyset), F(\Psi_i(1), \Omega^\varepsilon)\} = 0.
\]
However, because the domain \( X \) is one dimensional, the interpolation results of Zhou show that nothing changes if we instead insist that \( \Psi_i(0) = \emptyset \) and \( \Psi_i(1) = \Omega^\varepsilon \) for all \( i \). This leads to the notion of homotopy in Definition 9.

Now let \( \Psi \) be a sweepout. Assume that \( \Psi' \) is obtained from \( \Psi \) by either the pulltight procedure, the combinatorial argument, or the deformations in the index estimates. Note that we can arrange so that the following property is true: if \( W \) is an open set and \( \text{supp}(\Psi(t)) \subset W \) for all \( t \in [0, 1] \) then \( \text{supp}(\Psi'(t)) \subset W' \) for all \( t \in [0, 1] \) where \( W' \) is a slightly larger open set.
containing $W$. Therefore, by Corollary [20] we can perform all the arguments of Zhou on a critical sequence $\{\Psi_i\}_i$ while always staying inside $\Omega^*$. □

We can now prove the first main theorem.

**Proof.** (Theorem 6) Corollary 15 and Corollary 18 show that
\[ W^\varepsilon > \max\{0, A^\varepsilon(\Omega^\varepsilon)\}. \]
Therefore Theorem 21 applies to produce $\Lambda^\varepsilon$ and $\Theta^\varepsilon$ satisfying the conclusion of Theorem 6. □

### 3.6. Topology of the Min-Max Doubling

The goal of this section is to show that the min-max surfaces constructed above consist of two parallel copies of $\Sigma$ joined by a small catenoidal neck. For this section only, we require that $n + 1 = 3$.

Choose a sequence $\varepsilon_j \to 0$. Let $\Lambda_j = \Lambda^\varepsilon_j$ be the $\varepsilon_j$-cmc given by Theorem 6. Note that $\Lambda_j$ converges to $\Sigma$ in the Hausdorff distance. Hence by the compactness theorem for cmcs with bounded area and index (Zhou [28]), there is a point $p \in \Sigma$ such that (up to a subsequence) $\Lambda_j$ converges locally smoothly to $\Sigma$ away from $p$.

**Proposition 22.** The convergence $\Lambda_j \to \Sigma$ occurs with multiplicity two.

**Proof.** First we show that the multiplicity is at most two. To prove this, it suffices to show that
\[ \limsup_{\varepsilon \to 0} W^\varepsilon \leq 2 \text{Area}(\Sigma). \]
Fix some $\varepsilon > 0$. Since the map $\Phi : [0, 1] \to \mathcal{I}_{n+1}(M, \mathbb{Z}_2)$ given by $\Phi(t) = \Omega^\varepsilon_t$ can be interpolated to a sweepout, it follows that
\[ W^\varepsilon \leq \max_{\beta \in [0, \varepsilon]} A^\varepsilon(\Omega^\beta) \leq \max_{\beta \in [0, \varepsilon]} \text{Area}(\partial\Omega^\beta). \]
The quantity on the right hand side converges to $2 \text{Area}(\Sigma)$ as $\varepsilon \to 0$.

It remains to show that the multiplicity is at least 2. To prove this, it suffices to show that
\[ \liminf_{\varepsilon \to 0} W^\varepsilon \geq 2 \text{Area}(\Sigma). \]
To see this, recall that
\[ W^\varepsilon \geq A^\varepsilon(\Omega^\varepsilon) = \text{Area}(\partial\Omega^\varepsilon) - \varepsilon \text{Vol}(\Omega^\varepsilon). \]
Again the quantity on the right hand side converges to $2 \text{Area}(\Sigma)$ as $\varepsilon \to 0$. □
Proposition 23. The surface $\Lambda_j$ is connected.

Proof. Otherwise there would be a component $\Lambda'_j$ of $\Lambda_j$ which is graphical over $\Sigma$. The maximum principle shows that such a surface $\Lambda'_j$ cannot exist. \qed

Corollary 24. The index of $\Lambda_j$ is one.

Proof. Suppose to the contrary that $\text{ind}(\Lambda_j) = 0$. By the curvature estimates for stable cmcs (see Zhou [28]), the convergence $\Lambda_j \to \Sigma$ would consequently occur smoothly everywhere. But, since $\Sigma$ is two-sided, it is impossible for a connected surface $\Lambda_j$ to converge smoothly to $\Sigma$ with multiplicity two. \qed

We can now give the proof of Theorem 7.

Proof. (Theorem 7) The proof is based on results of Chodosh, Ketover, and Maximo [2]. Although the results in [2] are stated for minimal hypersurfaces, one can check that they continue to hold in our setting. For the sake of completeness, we sketch the details of the argument.

Let $A_j$ denote the second fundamental form of $\Lambda_j$. Recall that stable cmcs have curvature estimates (see Zhou [28]). Therefore we must have

$$
\lim_{j \to \infty} \max_{x \in \Lambda_j} |A_j(x)| = \infty
$$

since the convergence $\Lambda_j \to \Sigma$ is not smooth near $p$. By a point picking argument together with the fact that $\text{ind}(\Lambda_j) = 1$, it is possible to find a constant $C > 0$ and a sequence of points $p_j \in \Lambda_j$ with $|A_j(p_j)| \to \infty$ and such that

$$
|A_j(x)| \text{dist}_M(x, p_j) \leq C
$$

for all $x \in \Lambda_j$. Moreover, it is clear that $p_j \to p$.

Fix a small number $\sigma > 0$. Choose a sequence $\eta_j \to 0$ for which $\text{dist}_M(p_j, p) < \eta_j$ and

$$
\lim_{j \to \infty} \eta_j |A_j(p_j)| = \infty.
$$

We claim that for $j$ sufficiently large there is a bound

$$
|A_j(x)| \text{dist}_M(x, p_j) \leq \frac{1}{4}
$$

for all $x \in \Lambda_j \cap (B(p, \sigma) \setminus B(p_j, \eta_j))$. Suppose not. Then there would be points $x_j \in \Lambda_j \cap (B(p, \sigma) \setminus B(p_j, \eta_j))$ with

$$
|A_j(x_j)| \text{dist}_M(x_j, p_j) > \frac{1}{4}.
$$
Let $\Lambda'_j$ be the surface $\Lambda_j$ rescaled by a factor $\text{dist}_M(x_j, p_j)^{-1}$ about the point $p_j$. Let $A'_j$ denote the 2nd fundamental form of $\Lambda'_j$, and given a point $x \in \Lambda_j$ let $x'$ denote the corresponding point in $\Lambda'_j$.

Notice that $|A'_j(x')| = |A_j(x)|\text{dist}_M(x_j, p_j)$, and hence the surfaces $A'_j$ have uniform curvature bounds on compact sets that do not include the origin. Moreover, $|A'_j(0)| \geq |A_j(p_j)|\eta_j \to \infty$ as $j \to \infty$. Therefore, (up to a subsequence) the surfaces $A'_j$ converge locally smoothly away from the origin to a complete, embedded minimal surface $\Lambda'$ with multiplicity two. Since the mean curvature vectors of the two sheets of $\Lambda'_j$ point toward each other, it follows that $\Lambda'$ must be stable. Hence $\Lambda'$ is a plane. But this means that $|A'_j(x'_j)| \to 0$, and this contradicts the way the points $x_j$ were chosen.

Next one combines the preceding curvature estimate with a Morse theory argument (Lemma 3.1 in [2]) to conclude that $\Lambda_j \cap B(p, \sigma)$ and $\Lambda_j \cap B(p_j, \eta_j)$ have the same topology. We are now reduced to showing that $\Lambda_j \cap B(p_j, \eta_j)$ is topologically a catenoid. Let $\Lambda''_j$ be the surface $\Lambda_j$ rescaled by a factor $\eta_j^{-1}$ about the point $p_j$. It is equivalent to check that $\Lambda''_j \cap B(0, 1)$ is a catenoid.

Let $\Lambda'''_j$ be the surface $\Lambda''_j$ rescaled by a factor $|A''_j(0)|$ about the origin. Then $\Lambda'''_j$ has uniform curvature estimates everywhere. Thus (up to a subsequence) the surfaces $\Lambda'''_j$ converge locally smoothly to a complete, embedded, two-sided, non-flat minimal hypersurface $\Lambda''' \subset \mathbb{R}^3$. Moreover, we have $\text{ind}(\Lambda'''_j) \leq 1$. By the results in [3] and [15], it follows that $\Lambda'''$ is a catenoid. Fix a radius $R > 0$ so that $|A''(y)| \text{dist}(y, 0) < 1/4$ for all $y \in \Lambda''' \setminus B(0, R)$.

We claim that for $j$ sufficiently large there is a bound

$$|A''_j(y)| \text{dist}(y, 0) \leq \frac{1}{4}$$

for all $y \in \Lambda''_j \cap (B(0, 2) \setminus B(0, R/|A''_j(0)|))$. Suppose not. Then there would be points $y_j \in \Lambda''_j \cap (B(0, 2) \setminus B(0, R/|A''_j(0)|))$ with

$$|A''_j(y_j)| \text{dist}(y_j, 0) > \frac{1}{4}.$$

Let $\Lambda'''_j$ be the surface obtained by scaling $\Lambda''_j$ by a factor $\text{dist}(y_j, 0)^{-1}$ about the origin.
We claim that $|A_j'''(0)| \to \infty$ as $j \to \infty$. Suppose this were not the case. Then since

$$|A_j'''(0)| = |A_j''(0)| \text{dist}(y_j, 0),$$

it must be that

$$\frac{R}{|A_j''(0)|} \leq \text{dist}(y_j, 0) \leq \frac{B}{|A_j''(0)|}$$

for some constant $B$. But then (up to a subsequence) $A_j'''$ must converge to a surface $A''' = aA''$ where

$$\frac{1}{B} \leq a \leq \frac{1}{R}.$$

Now observe that

$$\frac{1}{4} < |A'''(\text{dist}(y_j, 0)^{-1}y_j) = a^{-1}|A'''(a^{-1} \text{dist}(y_j, 0)^{-1}y_j)|.$$ 

This contradicts the choice of $R$. Therefore it must be that $|A_j'''(0)| \to \infty$ as $j \to \infty$.

The surfaces $A_j'''$ have uniform curvature estimates on compact subsets that do not include the origin. Hence arguing as above, it follows that (up to a subsequence) the surfaces $A_j'''$ converge locally smoothly to a plane away from the origin. This contradicts the way the points $y_j$ were chosen. Finally one repeats the Morse theory argument with this curvature estimate to deduce that $A_j'' \cap B(0, R/|A_j''(0)|)$ has the same topology as $A_j''' \cap B(0, R),$ it follows that $A_j'' \cap B(0, R/|A_j''(0)|)$ is topologically a catenoid, as needed. This completes the proof of Theorem 7.\[\square\]

4. **The Index 1 Case**

4.1. **Statement of Results.** Now consider the index 1 case. Fix a dimension $3 \leq n + 1 \leq 7$. Let $(M^{n+1}, g)$ be a closed Riemannian manifold and let $\Sigma^n \subset M^{n+1}$ be a closed, connected, two-sided, minimal hypersurface. Also assume the following.

(U-i) The hypersurface $\Sigma$ has index 1 and the Jacobi operator $L$ for $\Sigma$ is non-degenerate. Moreover, the unique solution $\phi$ to $L\phi = 1$ is positive.

Note that by assumption (U-i) and the implicit function theorem, there is a neighborhood of $\Sigma$ that is foliated by constant mean curvature hypersurfaces whose mean curvature vectors point away from $\Sigma$. More precisely, we have the following.
(i) There is a neighborhood $U$ of $\Sigma$ and a smooth function $f : U \to (-\beta, \beta)$.

(ii) For each $\varepsilon \in (-\beta, \beta)$, the set $\Sigma^\varepsilon = f^{-1}(\varepsilon)$ is a smooth hypersurface with constant mean curvature $|\varepsilon|$. Moreover, $\Sigma^0 = \Sigma$.

(iii) For each $\varepsilon \in (-\beta, \beta)$, the mean curvature vector of $\Sigma^\varepsilon$ points away from $\Sigma$.

The next theorem is the main result of the paper in the index 1 case.

**Theorem 25.** Fix $(M, g)$ and $\Sigma$ for which the assumption (U-i) holds. Then for each small $\varepsilon > 0$, there is a smooth, almost-embedded hypersurface $\Lambda^\varepsilon$ of constant mean curvature $\varepsilon$ in $M$. The index of $\Lambda^\varepsilon$ is at most 3 and $\text{Area}(\Lambda^\varepsilon) \to 2 \text{Area}(\Sigma)$ as $\varepsilon \to 0$.

To ensure that $\Lambda^\varepsilon$ is a doubling of $\Sigma$, we have to make an additional assumption. Namely, suppose the following additional property holds.

(U-ii) The varifold $2\Sigma$ is the only embedded minimal cycle in $M$ with area $2 \text{Area}(\Sigma)$.

Then we have the following.

**Theorem 26.** Fix $(M, g)$ and $\Sigma$ for which the assumptions (U-i) and (U-ii) hold. Then the surfaces $\Lambda^\varepsilon$ from Theorem 25 converge to $2\Sigma$ as varifolds as $\varepsilon \to 0$.

**Remark 27.** It is natural to ask whether hypothesis (U-ii) significantly restricts the applicability of Theorem 26. In Appendix B we show that (U-ii) holds for a generic set of metrics on $M$.

4.2. **Construction of the three parameter family.** In this section, we formally construct the three parameter family $\Phi$ described in the introduction. Fix $(M, g)$ and $\Sigma$ satisfying the assumption (U-i) and fix a small number $\varepsilon > 0$. For simplicity, we give the construction in the case where $n + 1 = 3$. The cases $4 \leq n + 1 \leq 7$ are similar but easier since one can use cylindrical necks rather than catenoidal ones.

Before constructing the three parameter family, we need to introduce some notation. Write $\Sigma^\beta$ as the normal graph of a function $\psi_\beta$ over $\Sigma$. Recall that $\phi$ is a positive function on $\Sigma$ that solves $L\phi = 1$, and observe that $\psi_\beta/\beta \to \phi$ smoothly as $\beta \to 0$. 

The following notation is taken from [12]. Fix a point \( p \in \Sigma \) and for \( x \in \Sigma \) let \( r(x) \) be the distance from \( x \) to \( p \). Fix a number \( R > 0 \) to be specified later. For each \( 0 \leq t \leq R \) define a function \( \eta_t \) on \( \Sigma \) by
\[
\eta_t(x) = \begin{cases} 
1, & \text{if } r(x) \geq t \\
\frac{1}{\log(t)}(\log t^2 - \log r(x)), & \text{if } t^2 \leq r(x) \leq t \\
0, & \text{if } r(x) \leq t^2.
\end{cases}
\]
This function \( \eta_t \) will be used to construct the necks.

**Definition 28.** Let \( X = [-\varepsilon/2, \varepsilon/2]^2 \times [0, R] \) and define \( \Phi: X \to \mathcal{I}^{n+1}(M, \mathbb{Z}_2) \) as follows. First, for each \((x, y, t) \in X\) let \( S(x, y, t) \) be the union of the graph of \( \eta_t \psi_{x+y} \) with the graph of \( \eta_t \psi_{-x+y} \). This is a piecewise smooth surface. Choose a point \( q \in \Sigma \) with \( r(q) \gg R \). Then let \( \Phi(x, y, t) \) be the open set in \( M \) such that \( \partial \Phi(x, y, t) = S(x, y, t) \) and \( q \notin \Phi(x, y, t) \). The family \( \Phi \) is continuous in the \( \mathbf{F} \) topology.

In the next sequence of propositions, we prove the two key properties of the family \( \Phi \) outlined in the introduction.

**Proposition 29.** The surface \( \Sigma^\varepsilon \cup \Sigma^{-\varepsilon} \) is an index two critical point of \( A^\varepsilon \). Moreover, there is a constant \( c > 0 \) that doesn’t depend on \( \varepsilon \) such that
\[
A^\varepsilon(\Phi(x, y, 0)) \leq A^\varepsilon(\Phi(0, 0, 0)) - c(x^2 + y^2)
\]
for all \((x, y, 0) \in X\).

**Proof.** Since \( \Sigma \) is an index one critical point of \( A^0 \), it follows that \( \Sigma^\varepsilon \) is an index one critical point of \( A^\varepsilon \). Likewise \( \Sigma^{-\varepsilon} \) is an index one critical point of \( A^\varepsilon \) and therefore the union \( \Sigma^\varepsilon \cup \Sigma^{-\varepsilon} \) is an index two critical point of \( A^\varepsilon \). Next we study how \( A^\varepsilon(\Sigma^t) \) depends on \( t \). Let \( L_t \) be the Jacobi operator on \( \Sigma^t \). Since the Jacobi operator on \( \Sigma \) is non-degenerate, \( L_t \) is also non-degenerate for all sufficiently small \( t \). Moreover, the unique solution \( f_t \) to \( L_t f_t = 1 \) is uniformly positive for \( t \) small enough. Since
\[
\frac{d}{dt} A^\varepsilon(\Sigma^t) = \int_{\Sigma^t} (t - \varepsilon) f_t \, dv_{\Sigma^t},
\]
it follows that there is a constant \( c > 0 \) such that
\[
A^\varepsilon(\Sigma^t) \leq A^\varepsilon(\Sigma^0) - c|t - \varepsilon|^2
\]
for all \( 0 \leq t \leq 2\varepsilon \). The same reasoning applies to \( \Sigma^{-\varepsilon} \) and this implies the proposition. \( \square \)
Lemma 30. Let $c$ be the constant from the previous proposition. Then for all $\varepsilon$ sufficiently small and all $(x, y, t) \in X$ there is an inequality
\[
\text{Area}(\partial \Phi(x, y, t)) \leq \text{Area}(\partial \Phi(x, y, 0)) + \frac{c\varepsilon^2}{2}.
\]
Moreover, $\text{Area}(\partial \Phi(x, y, R)) < \text{Area}(\partial \Phi(x, y, 0))$ for all choices of $x$ and $y$.

Proof. This essentially follows from the proof of Theorem 2.4 in [12]. We include the details for the sake of clarity. Let $\gamma = \varepsilon + x$ and let $g_{\gamma,t} = \psi_{\gamma\eta_t}/\gamma$. Note that there is a bound $\|g_{\gamma,t}\|_{L^\infty} \leq C$ where $C$ is a constant that does not depend on $\gamma$ or $t$.

For a function $f$ on $B(p,R) \subset \Sigma$, let $S_f$ be the normal graph of $f$ over $B(p,R)$. Proposition 2.5 in [12] gives the existence of an $h_0 > 0$ so that for $h \leq h_0$ there is an expansion
\[
\text{Area}(S_{h_{\gamma,t}}) \leq \text{Area}(B_t) - \text{Area}(B_{t^2})
+ \frac{h^2}{2} \int_{B_t \setminus B_{t^2}} (|\nabla g_{\gamma,t}|^2 - g_{\gamma,t}^2(|A|^2 + \text{Ric}(N,N)))
+ Ch^3 \int_{B_t \setminus B_{t^2}} (1 + |\nabla g_{\gamma,t}|^2).
\]
Moreover, the constants $h_0$ and $C$ do not depend on $\varepsilon$ or $t$.

In particular, for $\gamma < h_0$ we can set $h = \gamma$ in the above expansion to get
\[
\text{Area}(S_{\psi_{\gamma\eta_t}}) \leq \text{Area}(B_t) - \text{Area}(B_{t^2})
+ \frac{\gamma^2}{2} \int_{B_t \setminus B_{t^2}} (|\nabla g_{\gamma,t}|^2 - g_{\gamma,t}^2(|A|^2 + \text{Ric}(N,N)))
+ C\gamma^3 \int_{B_t \setminus B_{t^2}} (1 + |\nabla g_{\gamma,t}|^2).
\]
Recall that $\psi_{\gamma}/\gamma \to \phi$ smoothly as $\gamma \to 0$. Therefore, taking $R$ small enough and $\varepsilon$ small enough, we get that
\[
\frac{\gamma^2}{2} \left| \int_{B_t \setminus B_{t^2}} g_{\gamma}^2(|A|^2 + \text{Ric}(N,N)) \right| \leq \frac{c\gamma^2}{128}.
\]
Shrinking $\varepsilon$ further to absorb the $\gamma^3$ terms, this implies that
\[
\text{Area}(S_{\psi_{\gamma\eta_t}}) \leq \text{Area}(B_t) + \frac{c\gamma^2}{128} + \gamma^2 \int_{B_t \setminus B_{t^2}} |\nabla g_{\gamma,t}|^2.
\]
Finally, using the logarithmic cutoff trick as in [12] together with the fact that $\psi/\gamma \to \phi$ as $\gamma \to 0$, it follows that
\[
\int_{B_r \setminus B_2} |\nabla g_{\gamma,t}|^2 \leq \frac{c}{128} + \frac{A}{|\log t|}
\]
where $A$ is a constant that does not depend on $\gamma$ or $t$. For $R$ small enough, this implies that
\[
\text{Area}(S_{\psi,\eta t}) \leq \text{Area}(B_R) + \frac{c\gamma^2}{32}
\]
for all $t \in [0, R]$.

Therefore, letting $\Omega^+ = \{ f > 0 \} = \cup_{\beta > 0} \Sigma^\beta$, it follows that
\[
\begin{align*}
\text{Area}(\partial \Phi(x,y,t) \cap \Omega^+) - \text{Area}(\partial \Phi(x,y,0) \cap \Omega^+) &
\leq \text{Area}(S_{\psi,\eta t}) - \text{Area}(B_R)(1 - C\varepsilon^2) \\
&\leq C\varepsilon^2 \text{Area}(B_R) + \frac{c\gamma^2}{32} \\
&\leq \frac{c\varepsilon^2}{4}
\end{align*}
\]
provided $R$ is small enough. A similar argument shows that the above inequality is also true with $\Omega^+$ replaced by $\Omega^- = \{ f < 0 \} = \cup_{\beta < 0} \Sigma^\beta$. This proves the lemma.

**Proposition 31.** For every $(x,y,t) \in \partial X$ it holds that
\[
A^\varepsilon(\Phi(x,y,t)) \leq A^\varepsilon(\Phi(0,0,0))
\]
with equality if and only if $(x,y,t) = (0,0,0)$.

**Proof.** Fix a point $(x,y,t) \in \partial X$. The proposition is clearly true if $t = 0$, and the proposition is true if $t = R$ by the previous lemma. So assume that $0 < t < R$. The previous lemma implies that
\[
\text{Area}(\partial \Phi(x,y,t)) \leq \text{Area}(\partial \Phi(x,y,0)) + \frac{c\varepsilon^2}{2}.
\]
It follows that
\[
A^\varepsilon(\Phi(x,y,t)) = \text{Area}(\partial \Phi(x,y,t)) - \varepsilon \text{Vol}(\Phi(x,y,t))
\leq A^\varepsilon(\Phi(x,y,0)) + \frac{c\varepsilon^2}{2}
\leq A^\varepsilon(\Phi(0,0,0)) - \frac{c\varepsilon^2}{2}.
\]
This proves the proposition.
4.3. Non-trivial Width. Again fix \((M^{n+1}, g)\) and \(\Sigma\) satisfying assumption (U-i) and fix a small number \(\varepsilon > 0\). Let \(\Pi\) be the \((X, \partial X)\)-homotopy class of the map \(\Phi\) constructed in the previous section. Let \(\Omega^\varepsilon = \Phi(0, 0, 0)\) so that \(\partial\Omega^\varepsilon = \Sigma^\varepsilon \cup \Sigma^{-\varepsilon}\). The goal of this section is to prove that the width of \(\Pi\) is non-trivial, i.e., to check that

\[
L^\varepsilon(\Pi) > A^\varepsilon(\Omega^\varepsilon) = \max_{(x,y,t) \in \partial X} A^\varepsilon(\Phi(x, y, t)).
\]

The proof is based on the quantitative minimality results in Appendix A.

**Proposition 32.** There are constants \(\gamma > 0\) and \(\eta > 0\) and \(C > 0\) such that the following property holds. If \(\Psi: X \to I_{n+1}(M, \mathbb{Z}_2)\) is an \(F\)-continuous map with

\[
\sup_{(x,y,t) \in \partial X} F(\Psi(x,y,t), \Phi(x,y,t)) < \eta
\]

then there is a point \((x_0, y_0, t_0) \in X\) such that

\[
A^\varepsilon(\Psi(x_0, y_0, t_0)) \geq A^\varepsilon(\Omega^\varepsilon) + C\gamma^2.
\]

**Proof.** Let \(\delta > 0\) and \(C > 0\) be the constants from Theorem 36 applied to \(\Sigma^\varepsilon \cup \Sigma^{-\varepsilon} = \partial\Omega^\varepsilon\). Fix some \(0 < \gamma < \delta/4\) and then choose a constant \(\eta > 0\) to be specified later. Consider a map \(\Psi\) as in the statement of the proposition. If \(\eta\) is small enough, it is possible to find a piecewise linear surface \(S \subset X\) such that the following properties hold.

- \(\gamma < F(\Psi(p), \Omega^\varepsilon) < 2\gamma\) for all \(p \in S\)

- \(\partial S\) is a connected curve in the bottom face of \(X\) that encloses \((0, 0, 0)\). Moreover, \(\text{dist}(\partial S, (0, 0, 0)) > d\) for some positive constant \(d\) that doesn’t depend on \(\Psi\).

This can be done, for example, by taking a suitable simplicial approximation to the function

\[
(x,y,t) \in X \mapsto F(\Psi(x,y,t), \Omega^\varepsilon).
\]

Note that \(A^\varepsilon(\Phi(p)) \leq A^\varepsilon(\Omega^\varepsilon) - d_1\) for all \(p \in \partial S\). Here \(d_1 > 0\) is a constant that does not depend on \(\Psi\).

Fix a small number \(\alpha > 0\). By Theorem 3.8 in [18], if \(\eta\) is small enough there exists an \(F\)-continuous homotopy

\[
H: \partial S \times [0, 1] \to I_{n+1}(M, \mathbb{Z}_2)
\]

with the properties that

- \(H(p, 0) = \Psi(p)\) for all \(p \in \partial S\), and
\[ H(p, 1) = \Phi(p) \text{ for all } p \in \partial S, \text{ and} \]
\[ F(H(p, s), \Phi(p)) < \alpha \text{ for all } p \in \partial S \text{ and all } s \in [0, 1]. \]

For an appropriate choice of \( \alpha \), this ensures that
\[ A^\varepsilon(H(p, s)) \leq A^\varepsilon(\Phi(p)) + \frac{d_1}{2} < A^\varepsilon(\Omega^\varepsilon) \quad (1) \]
for all \( p \in \partial S \) and all \( s \in [0, 1] \).

Now let \( S_1 = S \cup_{\partial S} (\partial S \times [0, 1]) \) and define a map \( \Psi_1 : S_1 \to \mathcal{L}_{n+1}(M, \mathbb{Z}_2) \) by letting \( \Psi_1 = \Psi \) on \( S \) and letting \( \Psi_1 = H \) on \( \partial S \times [0, 1] \). Note that part (ii) of Theorem \ref{thm:main} applies to the family \( \Psi_1 \) parameterized by \( S_1 \). Therefore, there is some point \( q \in S_1 \) such that
\[ A^\varepsilon(\Psi_1(q)) \geq A^\varepsilon(\Omega^\varepsilon) + C\mathcal{F}(\Psi_1(q), \Omega^\varepsilon)^2. \]
By \( (1) \), the point \( q = (x_0, y_0, t_0) \) must belong to \( S \). Thus we have exhibited a point \( (x_0, y_0, t_0) \in X \) with
\[ A^\varepsilon(\Psi(x_0, y_0, t_0)) \geq A^\varepsilon(\Omega^\varepsilon) + C\gamma^2, \]
and the proposition follows. \( \square \)

**Corollary 33.** The width of \( \Pi \) satisfies \( L^\varepsilon(\Pi) > A^\varepsilon(\Omega^\varepsilon) \).

**Proof.** This is an immediate consequence of Proposition \ref{prop:area-bound} \( \square \)

### 4.4. Construction of the Doublings

Fix \((M^{n+1}, g)\) and \(\Sigma\) satisfying assumption (U-i). In this section \(\varepsilon\) will be allowed to vary, and so we write \(X^\varepsilon\), \(\Phi^\varepsilon\), and \(\Pi^\varepsilon\) to emphasize the dependence of these objects on \(\varepsilon\).

**Proof.** (Theorem \ref{thm:main}) Corollary \ref{cor:area-bound} shows that
\[ L^\varepsilon(\Pi^\varepsilon) > \max_{(x,y,t)\in\partial X^\varepsilon} A^\varepsilon(\Phi^\varepsilon(x,y,t)), \]
and therefore \(\Pi^\varepsilon\) satisfies all the hypotheses of Theorem \ref{thm:main}. Hence min-max produces an almost embedded \(\varepsilon\)-cmc hypersurface \(\Lambda^\varepsilon = \partial \Theta^\varepsilon\) in \(M\) with \(A^\varepsilon(\Theta^\varepsilon) = L^\varepsilon(\Pi^\varepsilon)\) and \(\text{ind}(\Lambda^\varepsilon) \leq 3\).

Observe that
\[ A^\varepsilon(\Phi^\varepsilon(0,0,0)) \leq L^\varepsilon(\Pi^\varepsilon) \leq \max_{(x,y,t)\in X^\varepsilon} A^\varepsilon(\Phi^\varepsilon(x,y,t)), \]
and that both bounds for \(L^\varepsilon(\Pi^\varepsilon)\) converge to \(2\text{Area}(\Sigma)\) as \(\varepsilon \to 0\). Therefore the area of \(\Lambda^\varepsilon\) converges to \(2\text{Area}(\Sigma)\) as \(\varepsilon \to 0\). \(\square\)
Proof. (Theorem 26) Assume additionally that (U-ii) holds. By the compactness theorem for cmc surfaces with bounded area and index, there is an embedded minimal cycle $V$ in $M$ with $\|V\|(M) = 2 \text{Area}(\Sigma)$ such that $\Lambda^\varepsilon \to V$ as $\varepsilon \to 0$ (up to a subsequence). Assumption (U-ii) implies that $V = 2\Sigma$. □

4.5. The Non-foliated Case. We close this section with some remarks on the non-foliated case. Assume that $\Sigma \subset M$ is an index one, non-degenerate minimal hypersurface. Let $L$ be the Jacobi operator on $\Sigma$ and let $\phi$ be the solution to $L\phi = 1$. One can show that $\phi$ has at most two nodal domains. In the case of exactly two nodal domains, $\phi$ changes sign and thus there is no cmc foliation of a neighborhood of $\Sigma$.

Nevertheless, it is still possible to foliate a neighborhood of $\Sigma$ by surfaces whose mean curvature vectors point away from $\Sigma$. Let $H$ be the mean curvature operator on $\Sigma$, and let $\zeta > 0$ be the first eigenfunction of $L$. Then by the implicit function theorem, for every small $\beta > 0$ there is a smooth function $\psi_\beta$ on $\Sigma$ with $H(\psi_\beta) = \beta\zeta$. The surfaces $\Sigma^\beta = \text{graph}(\psi_\beta)$ foliate a neighborhood of $\Sigma$.

Let $x$ be a system of coordinates on $\Sigma$ and let $(x,t)$ be Fermi coordinates on a tubular neighborhood of $\Sigma$. Let $h$ be a smooth, positive function on $M$ such that $h(x,t) = \zeta(x)$ on a tubular neighborhood of $\Sigma$. Fix some $\varepsilon > 0$ and note that $\Sigma^\varepsilon$ is a critical point of the $A^\varepsilon h$ functional defined by

$$A^\varepsilon h(\Omega) = \text{Area}(\partial\Omega) - \varepsilon \int_\Omega h.$$ 

Using the prescribed mean curvature (pmc) min-max theory of Zhou and Zhu [29] and the same arguments as above, one can show that there are $\varepsilon h$-pmc surfaces $\Lambda^\varepsilon$ with $\text{Area}(\Lambda^\varepsilon) \to 2\text{Area}(\Sigma)$. Generically these are doublings of $\Sigma$.

Appendix A. Quantitative Minimality

This appendix contains a quantitative minimality result for the $A^\varepsilon$ functional. This result is needed to check that the widths of the min-max families in the paper are non-trivial. The result is based on the following theorem of Inauen and Marchese [7].
Theorem 34. (\cite{7} Theorem 4.3) Let $F$ be an elliptic parametric functional on $M^{n+1}$. Let $\Sigma_n \subset M^{n+1}$ be a smooth, closed, hypersurface which is a non-degenerate, index $k$ critical point for $F$. Then there are constants $r > 0$, $c > 0$, $\delta > 0$, and $C > 0$ and a smooth $k$-parameter family of surfaces

$$((\Sigma_v)_{v \in B^k_r})$$

such that the following properties hold.

(i) For every $v \in B^k_r$, the surface $\Sigma_v$ is homologous to $\Sigma$ and satisfies $F(\Sigma_v, \Sigma) < \delta$ and $F(\Sigma_v) \leq F(\Sigma) - c|v|^2$.

(ii) Let $S^k$ be an abstract $k$-manifold with $\partial S^k = \partial B^k_r$. Then for any continuous family of integral currents $(\tilde{\Sigma}_v)_{v \in S}$, each homologous to $\Sigma$ with $F(\tilde{\Sigma}_v, \Sigma) < \delta$ for all $v \in S$ and $\tilde{\Sigma}_v = \Sigma_v$ for $v \in \partial S$, it holds that

$$\sup_{v \in S} \left[ F(\tilde{\Sigma}_v) - CF(\tilde{\Sigma}_v, \Sigma) \right]^2 \geq F(\Sigma).$$

Remark 35. Let $u_1, \ldots, u_k$ be the eigenfunctions for the second variation of $F$ on $\Sigma$ with negative eigenvalues. Let

$$(\psi_v)_{v \in B^k_r}$$

be a family of smooth functions on $\Sigma$ for which the map

$$v \in B^k_r \mapsto \left( \int_\Sigma \psi_v u_1, \ldots, \int_\Sigma \psi_v u_k \right) \in \mathbb{R}^k$$

is a diffeomorphism onto a neighborhood of 0. Then by inspecting the proof of Theorem 4.3 in \cite{7} along with the proofs of Theorem 4 and Theorem 5 in \cite{26}, one sees that it is possible to take $\Sigma_v = \text{graph}(\psi_v)$ in the above theorem.

Unfortunately, Theorem 34 does not apply directly in our setting since the $A^\varepsilon$ functional cannot be written globally as an elliptic parametric functional. Nevertheless, we have the following.

Theorem 36. Let $\Sigma = \partial \Omega$ be a smooth, closed, hypersurface in $M$ which is a non-degenerate, index $k$ critical point for $A^\varepsilon$. Then there are constants $r > 0$, $c > 0$, $\delta > 0$, and $C > 0$ and a smooth $k$-parameter family of open sets

$$((\Omega_v)_{v \in B^k_r})$$

such that the following properties hold.
(i) For every \( v \in B^k_r \), the set \( \Omega_v \) satisfies \( F(\Omega_v, \Omega) < \delta \) and \( A^c(\Omega_v) \leq A^c(\Omega) - c|v|^2 \).

(ii) Let \( S^k \) be an abstract \( k \)-manifold with \( \partial S^k = \partial B^k_r \). Then for any \( F \) continuous family \( (\tilde{\Omega}_v)_{v \in S} \)

in \( \mathcal{I}_{n+1}(M, \mathbb{Z}_2) \) with \( F(\tilde{\Omega}_v, \Omega) < \delta \) for all \( v \in S \) and \( \tilde{\Omega}_v = \Omega_v \) for \( v \in \partial S \), there is a point \( v \in S \) such that

\[
\sup_{v \in S} \left[ A^c(\tilde{\Omega}_v) - C_F(\tilde{\Omega}_v, \Omega)^2 \right] \geq A^c(\Omega).
\]

Moreover, the inequality is strict unless \( \tilde{\Omega}_v = \Omega \).

Let \( u_1, \ldots, u_k \) be the eigenfunctions for the Jacobi operator on \( \Sigma \) with negative eigenvalues. Let \( (\psi_v)_{v \in \mathcal{P}} \) be a family of smooth functions on \( \Sigma \) for which the map

\[
v \in B^k_r \mapsto \left( \int_{\Sigma} \psi_v u_1, \ldots, \int_{\Sigma} \psi_v u_k \right) \in \mathbb{R}^k
\]

is a diffeomorphism onto a neighborhood of 0. Then it is possible to choose \( \Omega_v \) above so that \( \partial \Omega_v = \mathrm{graph}(\psi_v) \).

To prove Theorem 5.3, one essentially copies the arguments from [7] and observes that they continue to hold with \( F \) replaced by \( A^c \). We include the details for completeness.

**Proof.** Let \( u_1, \ldots, u_k \) be the eigenfunctions for the Jacobi operator on \( \Sigma \) with negative eigenvalues. Pick a smooth function \( \tilde{f}: M \to \mathbb{R}^k \) such that

\[
\tilde{f}(x) = 0, \quad \text{and} \quad \nabla \tilde{f}(x) = (u_1(x), \ldots, u_k(x))
\]

for all \( x \in \Sigma \). Let \( K \) be a very large constant and define

\[
G(\Theta) = A^c(\Theta) + K \left\| \int \tilde{f} d\Theta \right\|_2
\]

for \( \Theta \in \mathcal{I}_{n+1}(M, \mathbb{Z}_2) \). It follows from [26] that the functional \( G \) is lower-semicontinuous with respect to flat convergence, and \( \Sigma = \partial \Omega \) is a strictly stable critical point of \( G \).

**Lemma 37.** There is some \( \delta > 0 \) such that \( G(\Omega) < G(\Theta) \) for all \( \Theta \neq \Omega \) with \( F(\Theta, \Omega) < \delta \).
Proof. Suppose for contradiction that this is not the case. Then there are sets \( \Omega_i \neq \Omega \) with \( \mathcal{F}(\Omega_i, \Omega) \to 0 \) and \( G(\Omega_i) \leq G(\Omega) \). Define

\[
G_i(\Theta) = G(\Theta) + \lambda |\mathcal{F}(\Theta, \Omega) - \mathcal{F}(\Omega_i, \Omega)|,
\]

where \( \lambda > 0 \) is a constant to be specified later. Let \( \Omega'_i \) be a minimizer of \( G_i \). Passing to a subsequence, \( \Omega'_i \to \Omega' \) in the flat topology. The proof of Lemma 3.3 in [7] applies verbatim to show that \( \Omega' \) minimizes \( G_0(\Theta) = G(\Theta) + \lambda |\mathcal{F}(\Theta, \Omega)| \) over all \( \Theta \in \mathcal{I}_{n+1}(M, \mathbb{Z}_2) \).

Next one verifies the analog of Lemma 3.5 in [7].

**Lemma 38.** There are constants \( \delta > 0 \) and \( C > 0 \) such that

\[
G(\Omega) - G(\Theta) \leq C \mathcal{F}(\Omega, \Theta)
\]

for all \( \Theta \in \mathcal{I}_{n+1}(M, \mathbb{Z}_2) \).

Proof. Note that

\[
G(\Omega) - G(\Theta) = [\text{Area}(\partial \Omega) - \text{Area}(\Theta)] - \varepsilon [\text{Vol}(\Omega) - \text{Vol}(\Theta)] - K \left\| \int f\,d\partial \Theta \right\|^2 \\
\leq [\text{Area}(\partial \Omega) - \text{Area}(\Theta)] - \varepsilon [\text{Vol}(\Omega) - \text{Vol}(\Theta)] \\
\leq [\text{Area}(\partial \Omega) - \text{Area}(\Theta)] + \varepsilon \mathcal{F}(\Omega, \Theta).
\]

By Lemma 3.5 in [7], there is a constant \( C \) such that \( [\text{Area}(\partial \Omega) - \text{Area}(\partial \Theta)] \leq C \mathcal{F}(\Omega, \Theta) \), and the lemma follows. \( \square \)

The proof of Lemma 3.6 in [7] now applies verbatim to show that \( \Omega \) is the only minimizer of \( G_0 \). Thus the minimizers \( \Omega'_i \) converge to \( \Omega \) in the flat topology. We claim that in fact \( \Omega'_i \to \Omega \) in the \( \mathbf{F} \)-topology. Indeed, since \( \Omega'_i \) minimizes \( G_i \), there is an inequality

\[
G(\Omega'_i) + \lambda |\mathcal{F}(\Omega'_i, \Omega) - \mathcal{F}(\Omega_i, \Omega)| \leq G_i(\Omega_i) = G(\Omega_i) \leq G(\Omega).
\]

This implies that

\[
\text{Area}(\partial \Omega'_i) - \varepsilon \text{Vol}(\Omega'_i) \leq \text{Area}(\partial \Omega) - \varepsilon \text{Vol}(\Omega),
\]

and it follows that

\[
\limsup \text{Area}(\partial \Omega'_i) \leq \text{Area}(\partial \Omega)
\]

since \( \text{Vol}(\Omega'_i) \to \text{Vol}(\Omega) \). This proves the \( \mathbf{F} \)-convergence.
Now observe that the varifolds $|\Omega_i'|$ have uniformly bounded first variation. This implies that they satisfy a monotonicity formula with uniform constants. Since $\Omega_i' \to \Omega$ in the $F$-topology, it follows that $\partial \Omega_i'$ is eventually contained in a tubular neighborhood of $\Sigma$. According to White [26], this implies that $G(\Omega_i') > G(\Omega)$, and this is a contradiction. This establishes Lemma 37. □

Lemma 39. There are constants $\delta > 0$ and $C > 0$ such that 
\[ G(\Omega) \leq G(\Theta) + C\mathcal{F}(\Omega, \Theta)^2 \]
for all $\Theta \in \mathcal{I}_{n+1}(M, \mathbb{Z}_2)$ with $\mathcal{F}(\Omega, \Theta) < \delta$.

Proof. Think of $C > 0$ as a fixed constant to be chosen later. Suppose for contradiction that the claim fails. Then there are sets $\Omega_i \neq \Omega$ with $\mathcal{F}(\Omega_i, \Omega) \to 0$ and
\[ G(\Omega_i) + C\mathcal{F}(\Omega_i, \Omega)^2 \leq G(\Omega) \]
Define 
\[ H_i(\Theta) = G(\Theta) + \lambda[\mathcal{F}(\Theta, \Omega) - \mathcal{F}(\Omega_i, \Omega)]^2 \]
where $\lambda > 0$ is a constant to be specified later. Let $\Omega_i'$ be a minimizer of $H_i$. Passing to a subsequence, $\Omega_i' \to \Omega'$ in the flat topology. The proof of Lemma 4.1 in [7] applies verbatim to show that $\Omega'$ minimizes 
\[ H_0(\Theta) = G(\Theta) + \lambda\mathcal{F}(\Theta, \Omega)^2 \]
over all $\Theta \in \mathcal{I}_{n+1}(M, \mathbb{Z}_2)$.

We claim that $\Omega$ is the unique minimizer of $H_0$ provided $\lambda$ is large enough. Suppose for contradiction that there is some $\Omega_1 \neq \Omega$ with $H_0(\Omega_1) \leq H_0(\Omega)$. Then 
\[ G(\Omega_1) + \lambda\mathcal{F}(\Omega_1, \Omega)^2 \leq A^\varepsilon(\Omega) \]
which implies that 
\[ \mathcal{F}(\Omega_1, \Omega)^2 \leq \frac{A^\varepsilon(\Omega) - A^\varepsilon(\Omega_1)}{\lambda} \leq \frac{A^\varepsilon(\Omega) + \varepsilon \text{Vol}(M)}{\lambda}. \]
In particular, if $\lambda$ is large enough then Claim 37 applies to $\Omega_1$ and so $G(\Omega_1) > G(\Omega)$. This is a contradiction.

Since $\Omega$ is the unique minimizer of $H_0$, it follows that $\Omega_i' \to \Omega$ in the flat topology. The same argument as above shows that this convergence is actually in the $F$-topology. Again the varifolds $|\Omega_i'|$ satisfy a monotonicity formula with uniform constants and hence are eventually contained in a tubular neighborhood of $\Sigma$. This contradicts Theorem 1.1 in [7] since the $A^\varepsilon$ functional can locally be written as an elliptic parametric functional. (This is because the
volume form $\omega$ on $M$ is exact in a tubular neighborhood of $\Sigma$.) This establishes Lemma 39.

Finally Theorem 36 follows from Lemma 39 as explained in [26]. □

Note that Theorem 36 has the following corollary.

**Corollary 40.** Let $\Sigma = \partial \Omega$ be a smooth, closed, $\varepsilon$-cmc in $M$ which is strictly stable for $A^\varepsilon$. Then there are constants $\delta > 0$ and $C > 0$ such that every $\bar{\Omega} \in \mathcal{I}_{n+1}(M, \mathbb{Z}_2)$ with $F(\bar{\Omega}, \Omega) < \delta$ satisfies $A^\varepsilon(\bar{\Omega}) \geq A^\varepsilon(\Omega) + CF(\bar{\Omega}, \Omega)^2$.

**Appendix B. Generic Metrics**

It is natural to ask whether assumption (U-ii) poses a significant restriction to the applicability of Theorem 26. The following proposition addresses this question. It shows that assumption (U-ii) holds for a generic set of metrics $g$ on $M$.

**Proposition 41.** Let $M$ be a closed manifold. There is a (Baire) generic set $\mathcal{G}$ of smooth metrics on $M$ with the following property: if $g \in \mathcal{G}$ then for any closed, connected, embedded minimal hypersurface $\Sigma$ in $(M, g)$ the varifold $2\Sigma$ is the only embedded minimal cycle in $(M, g)$ with area $2 \text{Area}(\Sigma)$.

Proposition 41 is a corollary of the following result of Marques and Neves [18]. Given a metric $g$ on $M$ and $C > 0$ and $I \in \mathbb{N}$, let $\mathcal{M}_{C,I}(g)$ denote the collection of all closed, connected, embedded minimal hypersurfaces in $(M, g)$ with area at most $C$ and index at most $I$.

**Proposition 42.** ([18] Proposition 8.6) Let $g$ be a bumpy metric on $M$, and fix $C > 0$ and $I \in \mathbb{N}$. There exist metrics $\tilde{g}$ arbitrarily close to $g$ in the smooth topology such that the following properties hold.

(i) The set $\mathcal{M}_{C,I}(\tilde{g}) = \{\Sigma_1, \ldots, \Sigma_N\}$ is finite and every surface in $\mathcal{M}_{C,I}(\tilde{g})$ is non-degenerate.

(ii) The areas $\text{Area}_{\tilde{g}}(\Sigma_1), \ldots, \text{Area}_{\tilde{g}}(\Sigma_N)$ are linearly independent over $\mathbb{Q}$.

**Remark 43.** Note that property (ii) above immediately implies the following weaker property.

(iii) Let $A = a_1 \text{Area}_{\tilde{g}}(\Sigma_1) + \ldots + a_N \text{Area}_{\tilde{g}}(\Sigma_N)$ for some integers $a_i \geq 0$. If $A = 2 \text{Area}_{\tilde{g}}(\Sigma_i)$ for some $i$ then $a_i = 2$ and all the other $a_j$’s are zero.
Proof. (Proposition \[11\]) Given $C > 0$ and $I \in \mathbb{N}$, let $G_{C,I}$ be the collection of all metrics $g$ on $M$ for which properties (i) and (iii) above hold (with $g$ in place of $\tilde{g}$). We claim that $G_{C,I}$ is open and dense in the set of all smooth metrics on $M$.

First we show that $G_{C,I}$ is open. Fix some $g \in G_{C,I}$ and write $M_{C,I}(g) = \{\Sigma_1, \ldots, \Sigma_N\}$. Since every surface in $M_{C,I}(g)$ is non-degenerate, there is a neighborhood $U$ of $g$ such that for any $\tilde{g} \in U$ and any $i = 1, \ldots, N$ there is a unique minimal surface $\Sigma_i(\tilde{g})$ in $(M, \tilde{g})$ that is smoothly close to $\Sigma_i$. Moreover, these surfaces $\Sigma_i(\tilde{g})$ are all non-degenerate. By Sharp’s compactness theorem \[24\], it follows that there is a potentially smaller neighborhood $U_1$ of $g$ such that $M_{C,I}(\tilde{g}) \subseteq \{\Sigma_1(\tilde{g}), \ldots, \Sigma_N(\tilde{g})\}$ for all $\tilde{g} \in U_1$. Taking an even smaller neighborhood $U_2$ of $g$, it is then possible to ensure that condition (iii) holds for all $\tilde{g} \in U_2$.

Next we show that $G_{C,I}$ is dense. Consider any metric $g$ on $M$. Since bumpy metrics are dense, there is a bumpy metric $g_1$ on $M$ arbitrarily close to $g$. Applying Proposition 3.2 to $g_1$ then yields $g_2 \in G_{C,I}$ that is arbitrarily close to $g_1$. Thus there is a metric $g_2 \in G_{C,I}$ arbitrarily close to $g$ in the smooth topology.

To conclude the proof, take sequences $C_n \to \infty$ and $I_n \to \infty$ and define

$$G = \bigcap_n G_{C_n,I_n}.$$ 

Then $G$ is Baire generic, and every metric $g \in G$ satisfies the conclusion of Proposition \[11\]. □

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