LARGE DEVIATIONS OF THE EMPIRICAL FLOW FOR CONTINUOUS TIME MARKOV CHAINS

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Abstract. We consider a continuous time Markov chain on a countable (finite or infinite) state space and prove a joint large deviation principle for the empirical measure and the empirical flow, which accounts for the total number of jumps between pairs of states. We give a direct proof using tilting and an undirect one by contraction from the empirical process.

Keywords: Markov chain, Large deviations principle, Entropy, Empirical flow.

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1. Introduction

One of the most important contributions in the theory of large deviations is the series of papers of Donsker and Varadhan [17]. Here the authors develop a general approach to the study of large deviations for Markov processes both in continuous and discrete time. They establish large deviations principles (LDP) for the empirical measure and for the empirical process associated to a Markov process.

Given a sample path of the process on the finite time window \([0, T]\), the corresponding empirical measure is a probability measure on the state space that associates to any measurable subset the fraction of time spent on it. A LDP for the empirical measure is usually called a level 2 LDP.

Given a sample path of the process in the finite time window \([0, T]\), the corresponding empirical process is a probability measure on paths defined on the infinite time window \((-\infty, +\infty)\). More precisely it is the unique stationary (with respect to time shift) probability measure that gives weight 1 to periodic paths (of period \(T\)) such that there exists a period \([t, t + T]\) where they coincide with the original sample path. A LDP for the empirical process is usually called a level 3 LDP.

Let us restrict our discussion to the case of a Markov chain on a countable (finite or infinite) state space, which is the actual framework of this paper. The case of discrete time Markov chains has been much more investigated with respect to the case of continuous time.

The general picture of the discrete time case is the following (see for example [15, 22]). The rate function for the level 3 LDP coincides with the relative entropy density. The rate function for the level 2 LDP has instead in general only a variational representation, which cannot be solved explicitly even for symmetric jump probabilities. A very natural and much studied object is the \(k\)-symbols empirical measure. This is a probability measure on strings of symbols with length \(k\) obtained from the frequency of appearance in the sample path. With a suitable
periodization procedure the $k$–symbols empirical measures constitute a consistent family of measures that are exactly the $k$ marginals of the empirical process. For each $k > 1$, and in particular for $k = 2$ the rate function for the LDP associated to the $k$ symbols empirical measure has an explicit expression.

In the continuous time setting, quite surprisingly, there are much less results available and the general picture is less clear. The aim of this paper is to partly fill the gap. As already mentioned, level 2 and level 3 LDPs have been proved in [17]. For the empirical process the rate function is the relative entropy density. For the empirical measure the rate function has instead only a variational representation. Only in the case of reversible Markov chains the corresponding variational problem can be solved and the rate function is related to the Dirichlet form.

In the continuous time setting a natural generalization of the 2–symbols empirical measure is the so called empirical flow. Given a sample path of the Markov chain in the finite time window $[0, T]$, the corresponding empirical flow is a measure on the pairs of states, assigning to each pair the number of jumps performed by the path along this pair of states times a factor $1/T$. In this paper we prove a joint LDP for the empirical measure and the empirical flow. The rate function is explicit and is given by a sum of Poisson like terms constrained by a zero divergence condition for the empirical flow. The LDP is proved on the space of summable flows with a suitable topology but also some other topological frameworks are discussed. Despite the discrete time case in which the empirical measure is the marginal of the empirical 2–symbols measure, in the continuous time case empirical measure and flow can be arbitrary and have not to satisfy any compatibility condition.

The joint rate function for the empirical measure and flow first appeared in [24] through an heuristic derivation. Always in [24] it was then used to recover by contraction the Donsker–Varadhan rate function for the empirical measure in the case of a state space with only two elements. Being a LDP intermediate among level 2 and level 3, the authors called it a level 2.5 LDP. Later in [2], motivated by statistical applications, the authors have showed that the contraction on the empirical measure of the rate function proposed by [24] leads to the Donsker-Varadhan rate function in the case of finite state space. In [14] a weak level 2.5 LDP has been proved. Finally in [1] LDPs for flows and currents have been discussed in relation to non equilibrium thermodynamics.

In the present paper we give a rigorous proof of a full LDP for Markov chains on a countable state space. As a condition assuring the exponential tightness we assume a stronger version of the Donsker–Varadhan condition (alternatively of the hypercontractivity condition in [16]) for the exponential tightness of the empirical measure. For a finite state space the exponential tightness is trivially satisfied, and the proof is strongly simplified. We present two different approaches: a direct derivation is obtained using a perturbation of the original Markov measure (under an additional technical assumption), while an indirect derivation is obtained by contraction from the level 3 LDP.

In [7] we will recover the LDP for the empirical measure by contraction from the joint LDP proved here. In a companion paper [8] we will discuss several applications and consequences of our results like LDPs for currents, Gallavotti–Cohen symmetries and computations in specific models.

Finally we mention some recent results about fluctuations of currents and fluxes inspiring and motivating the present work. We already mentioned the paper [1].
In [3, 4, 5, 9, 10, 11] LDPs for currents of interacting particle systems in the hydrodynamic scaling limit were studied. This was a breakthrough in the study of non equilibrium models of interacting particle systems that for example revealed the possibility of a dynamical phase transition based on current fluctuations. LDPs for the currents in diffusion processes on \( \mathbb{R}^n \) and their symmetries were studied in [6]. In [26] and [27] LDPs for the currents of the Brownian motion on a compact Riemann manifold are obtained. We mention also the recent preprint [31] on the joint large deviations for the empirical measure and flow for a renewal process on a finite graph.

In the next section we fix our notation and state our main results. At the end of that section we outline the structure of the paper.

2. Notation and results

We consider a continuous time Markov chain \( \xi_t, t \in \mathbb{R}_+ \) on a countable (finite or infinite) state space \( V \). The Markov chain is defined in terms of the jump rates \( r(x,y), x \neq y \) in \( V \), from which one derives the holding times and the jump chain [36, Section 2.6]. Since the holding time at \( x \in V \) is an exponential random variable of parameter \( r(x) := \sum_{y \in V} r(x,y) \), we need to assume that \( r(x) < +\infty \) for any \( x \in V \).

The basic assumptions on the chain are the following:

(A1) for each \( x \in V \), \( r(x) = \sum_{y \in V} r(x,y) \) is finite;

(A2) for each \( x \in V \) the Markov chain \( \xi_t^x \) starting from \( x \) has no explosion a.s.;

(A3) the Markov chain is irreducible, i.e. for each \( x, y \in V \) and \( t > 0 \) the event \( \{ \xi_t^x = y \} \) has strictly positive probability;

(A4) there exists a unique invariant probability measure, that is denoted by \( \pi \).

As in [36], by invariant probability measure \( \pi \) we mean a probability measure on \( V \) such that

\[
\sum_{y \in V} \pi(y) r(y,x) = \sum_{y \in V} \pi(y) r(y,x) \quad \forall x \in V \tag{2.1}
\]

where we understand \( r(x,x) = 0 \). We recall some basic facts from [36], see in particular Section 3.5 and Theorem 3.8.1 there. Assuming (A1) and irreducibility (A3), assumptions (A2) and (A4) together are equivalent to the fact that all states are positive recurrent. In (A4) one could remove the assumption of uniqueness of the invariant probability measure, since for an irreducible Markov chain there can be at most only one. Under the above assumptions, \( \pi(x) > 0 \) for all \( x \in V \), the Markov chain starting with distribution \( \pi \) is stationary (i.e. its law is left invariant by time-translations), and the ergodic theorem holds, i.e. for any bounded function \( f : V \rightarrow \mathbb{R} \) and any initial distribution

\[
\lim_{T \to +\infty} \frac{1}{T} \int_0^T dt f(\xi_t) = \langle \pi, f \rangle \quad \text{a.s.} \tag{2.2}
\]

where \( \langle \pi, f \rangle \) denotes the expectation of \( f \) with respect to \( \pi \). Finally, we observe that if \( V \) is finite then (A1) and (A2) are automatically satisfied, while (A3) implies (A4).

We consider \( V \) endowed with the discrete topology and the associated Borel \( \sigma \)-algebra given by the collection of all the subsets of \( V \). Given \( x \in V \), the distribution
of the Markov chain $\xi^x_t$ starting from $x$, is a probability measure on the Skorohod space $D(\mathbb{R}_+; V)$ that we denote by $\mathbb{P}_x$. The expectation with respect to $\mathbb{P}_x$ is denoted by $\mathbb{E}_x$. In the sequel we consider $D(\mathbb{R}_+; V)$ equipped with the canonical filtration, the Skorohod topology, and the completion of the associated Borel $\sigma$–algebra with respect to $\mathbb{P}_x$, $x \in V$. The canonical coordinate in $D(\mathbb{R}_+; V)$ is denoted by $X_t$. The set of probability measures on $V$ is denoted by $\mathcal{P}(V)$ and it is considered endowed with the topology of weak convergence and the associated Borel $\sigma$–algebra. Since $V$ has the discrete topology, the weak convergence of $\mu_n$ to $\mu$ in $\mathcal{P}(V)$ is equivalent to the pointwise convergence of $\mu_n(x)$ to $\mu(x)$ for any $x \in V$.

2.1. **Empirical measure and empirical flow.** Given $T > 0$ the empirical measure $\mu_T: D(\mathbb{R}_+; V) \rightarrow \mathcal{P}(V)$ is defined by

$$
\mu_T (X) = \frac{1}{T} \int_0^T dt \, \delta_{X_t},
$$

where $\delta_y$ denotes the pointmass at $y$. Given $x \in V$, the ergodic theorem implies that the empirical measure $\mu_T$ converges $\mathbb{P}_x$ a.s. to $\pi$ as $T \to \infty$. In particular, the sequence of probabilities $\{\mathbb{P}_x \circ \mu_T^{-1}\}_{T>0}$ on $\mathcal{P}(V)$ converges to $\delta_{\pi}$.

We denote by $B$ the countable set of ordered edges without loops in $V$ and by $E$ the subset of $B$ given by ordered edges with strictly positive jump rate:

$$
B := \{(y, z) \in V \times V : y \neq z\},
$$

$$
E := \{(y, z) \in B : r(y, z) > 0\}.
$$

For each $T > 0$ we define the empirical flow as the map $Q_T: D(\mathbb{R}_+; V) \rightarrow [0, +\infty]^B$ given by

$$
Q_T(y, z)(X) := \frac{1}{T} \sum_{0 \leq t \leq T} \delta_y(X_{t^-})\delta_z(X_t) \quad (y, z) \in B. \quad (2.3)
$$

Namely, $TQ_T(y, z)$ is $\mathbb{P}_x$ a.s. the number of jumps from $y$ to $z$ in the time interval $[0, T]$ of the Markov chain $\xi^x$.

**Remark 2.1.** By the graphical construction of the Markov chain, the random field $\{TQ_T(y, z)\}_{(y, z) \in B}$ under $\mathbb{P}_x$ is stochastically dominated by the random field $\{Z_{y, z}\}_{(y, z) \in B}$ given by independent Poisson random variables, $Z_{y, z}$ having mean $Tr(y, z)$. This fact will be frequently used in the rest of the paper.

We denote by $L^1_+(E)$ the collection of absolutely summable functions on $E$ and by $\| \cdot \|$ the associated $L^1$–norm. The set of nonnegative elements of $L^1_+(E)$ is denoted by $L^+_+(E)$. In what follows, given $Q \in \mathbb{R}_+^E$ we will think of $Q$ as element of $[0, +\infty]^B$ setting $Q(y, z) := 0$ for all $(y, z) \in B \setminus E$. In particular, we have the inclusions

$$
L^1_+(E) \subset \mathbb{R}_+^E \subset [0, +\infty]^B.
$$

Due to the above identification, since the chain is not explosive, for each $T > 0$ we also have $\mathbb{P}_x$ a.s. that $Q_T \in L^+_+(E)$.

Given a flow $Q \in L^+_+(E)$ we let its divergence $\text{div } Q: V \rightarrow \mathbb{R}$ be the function defined by

$$
\text{div } Q (y) = \sum_{z : (y, z) \in E} Q(y, z) - \sum_{z : (z, y) \in E} Q(z, y), \quad y \in V. \quad (2.4)
$$
Namely, the divergence of the flow $Q$ at $y$ is given by the difference between the flow exiting from $y$ and the flow entering into $y$. Observe that the divergence maps $L^1_+(E)$ into $L^1(V)$.

Finally, to each probability $\mu \in \mathcal{P}(V)$ we associate the flow $Q^\mu \in \mathbb{R}_+^E$ defined by

$$Q^\mu(y, z) := \mu(y) r(y, z) \quad (y, z) \in E. \quad (2.5)$$

Note that $Q^\mu \in L^1_+(E)$ if and only if $(\mu, r) < +\infty$. Moreover, in this case, by (2.1) $Q^\mu$ has vanishing divergence if only if $\mu$ is invariant for the Markov chain $\xi$, i.e. $\mu = \pi$.

We now discuss the law of large numbers for the empirical flow. As follows from simple computations (see [35, Lemma II.2.3] and [25, App. 1, Lemma 5.1]), which have to be generalized to the case of unbounded $r(\cdot)$ by means of [36, Sec. 2.8] and Remark 2.1 for each $(y, z) \in E$ the process

$$M_T(y, z) = T Q_T(y, z) (X) - \int_0^T ds \delta_y(X_s) r(y, z) \quad (2.6)$$

is a martingale with respect to $\mathbb{P}_x$, $x \in V$. Moreover, the predictable quadratic variation of $M_T(y, z)$, denoted by $\langle M(y, z) \rangle_T$ is given by

$$\langle M(y, z) \rangle_T = \int_0^T ds \delta_y(X_s) r(y, z).$$

In view of the ergodic theorem (2.2), we conclude that for each $x \in V$ and $(y, z) \in E$ the family of real random variables $Q_T(y, z)$ converges, in probability with respect to $\mathbb{P}_x$, as $T \to +\infty$ to $Q^\pi(y, z)$. We refer to Remark 3.3 for an alternative proof.

2.2. Compactness conditions. The classical Donsker-Varadhan theorem [17, 14, 16, 37] describes the LDP associated to the empirical measure. The main purpose of the present paper is to extend this result by considering also the empirical flow.

Below we will state two LDPs (Theorem 2.5 and Theorem 2.8) for the joint process given by the empirical measure and flow. In Theorem 2.5 the flow space is given by $L^1_+(E)$ endowed of the bounded weak* topology and, in order to have some control at infinity in the case of infinite state space $V$, compactness assumptions are required. In Theorem 2.8 the flow space is given by $[0, +\infty]^B$ endowed of the product topology and weaker assumptions are required (the same of [17]). On the other hand, the rate function has not always a computable form.

Let us now state precisely the compactness conditions under which Theorem 2.5 holds (at least one of the following Conditions (2.2) and (2.3) has to be satisfied). To this aim, given $f: V \to \mathbb{R}$ such that $\sum_{y \in V} r(x, y) |f(y)| < +\infty$ for each $x \in V$, we denote by $L f: V \to \mathbb{R}$ the function defined by

$$L f(x) := \sum_{y \in V} r(x, y) [f(y) - f(x)], \quad x \in V. \quad (2.7)$$

**Condition 2.2.** There exists a sequence of functions $u_n: V \to (0, +\infty)$ satisfying the following requirements:

(i) For each $x \in V$ and $n \in \mathbb{N}$ it holds $\sum_{y \in V} r(x, y) u_n(y) < +\infty$. In the sequel $L u_n: V \to \mathbb{R}$ is the function defined by (2.7).

(ii) The sequence $u_n$ is uniformly bounded from below. Namely, there exists $c > 0$ such that $u_n(x) \geq c$ for any $x \in V$ and $n \in \mathbb{N}$. 
(iii) The sequence \( u_n \) is uniformly bounded from above on compacts. Namely, for each \( x \in V \) there exists a constant \( C_x \) such that for any \( n \in \mathbb{N} \) it holds \( u_n(x) \leq C_x \).

(iv) Set \( v_n := -Lu_n/u_n \). The sequence \( v_n : V \rightarrow \mathbb{R} \) converges pointwise to some \( v : V \rightarrow \mathbb{R} \).

(v) The function \( v \) has compact level sets. Namely, for each \( \ell \in \mathbb{R} \) the level set \( \{ x \in V : v(x) \leq \ell \} \) is finite.

(vi) There exist a strictly positive constant \( \sigma \) and a positive constant \( C \) such that \( v \geq \sigma r - C \).

Replacing in Condition 2.2 the strictly positive constant \( \sigma \) with zero one obtains the same assumptions of Donsker and Varadhan for the derivation in [17]–(IV) of the LDP for the empirical measure of the Markov chain satisfying (A1),...,(A4) (shortly, we will say that the Donsker–Varadhan condition is satisfied).

We recall that the Donsker-Varadhan theorem for the empirical measure still holds under a suitable compactness condition concerning the hypercontractivity of the underlying Markov semigroup, see e.g. [16]. Also in this case we need a stronger version that is detailed below.

Recall that \( \pi \) is the unique invariant measure of the chain. The maps \( P_t f(x) := \mathbb{E}(f(\xi^t)) \), \( t \in \mathbb{R}_+ \), define a strongly continuous Markov semigroup on \( L^2(V,\pi) \). We write \( D_\pi \) for the Dirichlet form associated to the symmetric part \( S = (L + L^*)/2 \) of the generator \( L \) in \( L^2(V,\pi) \). Since the time–reversed dynamics is described by a Markov chain on \( V \) with transition rates \( r^*(x,y) := \pi(y)r(y,x)/\pi(x) \), it holds

\[
D_\pi(f) = \frac{1}{4} \sum_{x \in V} \sum_{y \in V} (\pi(x)r(x,y) + \pi(y)r(y,x))(f(y) - f(x))^2, \quad f \in L^2(V,\pi).
\]

One can take the above expression as definition of \( D_\pi \), avoiding all technicalities concerning infinitesimal generators. One says that the Markov chain \( \xi \) satisfies the logarithmic Sobolev inequality if there exists a constant \( c_{LS} \in (0, +\infty) \) such that for any \( \mu \in \mathcal{P}(V) \) it holds

\[
\text{Ent}(\mu|\pi) \leq c_{LS} D_\pi \left( \sqrt{\mu/\pi} \right)
\]

where \( \text{Ent}(\mu|\pi) \) denotes the relative entropy of \( \mu \) with respect to \( \pi \).

**Condition 2.3.**

(i) The Markov chain satisfies a logarithmic Sobolev inequality.

(ii) The exit rate \( r \) has an exponential moment with respect to the invariant measure. Namely, there exists \( \sigma > 0 \) such that \( \langle \pi, \exp \{ \sigma r \} \rangle < +\infty \).

(iii) The graph \((V,E)\) is locally finite, that is for each vertex \( y \in V \) the number of incoming and outgoing edges in \( y \) is finite.

Item (iii) is here assumed for technical convenience and it should be possible to drop it. Item (i) is the hypercontractivity condition assumed in [16] to deduce the Donsker-Varadhan theorem for the empirical measure. Item (ii) is here required to prove the exponential tightness of the empirical flow in \( L^1(V,E) \).

**2.3. LDP with flow space \( L^1(E) \) endowed of the bounded weak* topology.**

We consider the space \( L^1(E) \) equipped with the so-called bounded weak* topology. This is defined as follows. Recall that the (countable) set \( E \) is the collection of
ordered edges in $V$ with positive jump rate. Let $C_0(E)$ be the collection of the functions $F: E \to \mathbb{R}$ vanishing at infinity, that is the closure of the functions with compact support in the uniform topology. The dual of $C_0(E)$ is then identified with $L^1(E)$. The weak* topology on $L^1(E)$ is the smallest topology such that the maps $Q \in L^1(E) \to \langle Q, f \rangle \in \mathbb{R}$ with $f \in C_0(E)$ are continuous. Given $\ell > 0$, let $B_\ell := \{ Q \in L^1(E) : ||Q|| \leq \ell \}$ be the closed ball of radius $\ell$ in $L^1(E)$ ($|| \cdot ||$ being the standard $L^1$-norm). In view of the separability of $C_0(E)$ and the Banach-Alaoglu theorem, the set $B_\ell$ endowed with the weak* topology is a compact Polish space. The bounded weak* topology on $L^1(E)$ is then defined by declaring a set $A \subseteq L^1(E)$ open if and only if $A \cap B_\ell$ is open in the weak* topology of $B_\ell$ for any $\ell > 0$. The bounded weak* topology is stronger than the weak* topology (they coincide only when $|E| < +\infty$) and for each $\ell > 0$ the closed ball $B_\ell$ is compact with respect to the bounded weak* topology. The space $L^1(E)$ endowed with the bounded weak* topology is a locally convex, complete linear topological space and a completely regular space (i.e. for every closed set $C \subseteq L^1(E)$ and every element $Q \in L^1(E) \setminus C$ there exists a continuous function $f : L^1(E) \to [0, 1]$ such that $f(Q) = 1$ and $f(Q') = 0$ for all $Q' \in C$). Moreover, it is metrizable if and only if the set $E$ is finite. We refer to [32, Sec. 2.7] for the proof of the above statements and for further details.

We regard $L^1_+(E)$ as a (closed) subset of $L^1(E)$ and consider it endowed with the relative topology and the associated Borel $\sigma$-algebra. Accordingly, the empirical flow $Q_T$ will be considered as a measurable map from $D(\mathbb{R}_+; V)$ to $L^1_+(E)$, defined $\mathbb{P}_x$ a.s., $x \in V$. Recalling that we consider $\mathcal{P}(V)$, the set of probability measures on $V$, with the topology of weak convergence, we finally consider the product space $\mathcal{P}(V) \times L^1_+(E)$ endowed with the product topology and regard the couple $(\mu_T, Q_T)$ where $\mu_T$ is the empirical measure and $Q_T$ the empirical flow, as a measurable map from $D(\mathbb{R}_+; V)$ to $\mathcal{P}(V) \times L^1_+(E)$ defined $\mathbb{P}_x$ a.s., $x \in V$.

Below we state the LDP for the family of probability measures on $\mathcal{P}(V) \times L^1_+(E)$ given by $\{\mathbb{P}_x \circ (\mu_T, Q_T)^{-1}\}$ as $T \to +\infty$. Before stating precisely the result, we introduce the corresponding rate function. Let $\Phi: \mathbb{R}_+ \times \mathbb{R}_+ \to [0, +\infty]$ be the function defined by

$$\Phi(q, p) := \begin{cases} q \log \frac{q}{p} - (q - p) & \text{if } q, p \in (0, +\infty) \\ p & \text{if } q = 0, p \in [0, +\infty) \\ +\infty & \text{if } p = 0 \text{ and } q \in (0, +\infty). \end{cases} \tag{2.10}$$

We point out that, given $p > 0$ and letting $N_t, t \in \mathbb{R}_+$ be a Poisson process with parameter $p$, the sequence of real random variables $\{N_T/T\}$ satisfies a large deviation principle on $\mathbb{R}$ with rate function $\Phi(\cdot, p)$ as $T \to \infty$. This statement can be easily derived from the Gärtner-Ellis theorem, see e.g [15, Thm. 2.3.6]. Recalling (2.4) and (2.5), we let $I: \mathcal{P}(V) \times L^1_+(E) \to [0, +\infty]$ be the functional defined by

$$I(\mu, Q) := \begin{cases} \sum_{(y, z) \in E} \Phi(Q(y, z), Q^\mu(y, z)) & \text{if } \text{div } Q = 0, \langle \mu, r \rangle < +\infty \\ +\infty & \text{otherwise.} \end{cases} \tag{2.11}$$

**Remark 2.4.** As proved in Appendix [2], if $\langle \mu, r \rangle = +\infty$ the series in (2.11) diverges. Hence the condition $\langle \mu, r \rangle < +\infty$ can be removed from the first line of (2.11).
Theorem 2.5. Assume the Markov chain satisfies (A1)–(A4) and at least one between Conditions 2.2 and 2.3. Then as \( T \to +\infty \) the family of probability measures \( \{\mathbb{P}_x \circ (\mu_T, Q_T)^{-1}\} \) on \( \mathcal{P}(V) \times L^1_+(E) \) satisfies a large deviation principle, uniformly for \( x \) in compact subsets of \( V \), with good and convex rate function \( I \). Namely, for each not empty compact set \( K \subset V \), each closed set \( C \subset \mathcal{P}(V) \times L^1_+(E) \), and each open set \( A \subset \mathcal{P}(V) \times L^1_+(E) \), it holds
\[
\lim_{T \to +\infty} \sup_{x \in K} \frac{1}{T} \log \mathbb{P}_x \left( (\mu_T, Q_T) \in C \right) \leq \inf_{(\mu, Q) \in C} I(\mu, Q), \tag{2.12}
\]
\[
\lim_{T \to +\infty} \inf_{x \in K} \frac{1}{T} \log \mathbb{P}_x \left( (\mu_T, Q_T) \in A \right) \geq \inf_{(\mu, Q) \in A} I(\mu, Q). \tag{2.13}
\]

As discussed in Lemma 3.3, under the above assumptions it holds \( \langle \pi, r \rangle < +\infty \). In particular, \( I(\mu, Q) = 0 \) if and only if \( (\mu, Q) = (\pi, Q^\pi) \). Hence, from the above LDP one derives the LLN for the empirical flow in \( L^1_+(E) \), improving the pointwise version discussed at the end of Section 2.1. In addition, the function \( I(\cdot, \cdot) \) has an affine structure:

Proposition 2.6. Let \( (\mu, Q) \in \mathcal{P}(V) \times L^1_+(E) \) satisfy \( I(\mu, Q) < +\infty \). Then

(i) All edges in the support \( E(Q) \) of \( Q \) must connect vertexes in the support of \( \mu \), i.e., if \( Q(y, z) > 0 \) then \( y, z \in \text{supp}(\mu) \).

(ii) \( I(\mu, Q) \) has the following affine decomposition. Consider the oriented graph (\( \text{supp}(\mu), E(Q) \)) and let \( K_j, j \in J \), be the family of its oriented connected components. Consider the probability measure \( \mu_j \) on \( V \) concentrated on \( K_j \) defined as \( \mu_j := \frac{\mu|_{K_j}}{\mu(K_j)} \). Consider the flow \( Q_j \in L^1_+(E) \) defined as
\[
Q_j(y, z) = \begin{cases} 
\frac{Q(y, z)}{\mu(K_j)} & \text{if } (y, z) \in E, \ y, z \in K_j, \\
0 & \text{otherwise}.
\end{cases}
\]

Then we have \( (\mu, Q) = \sum_{j \in J} \mu(K_j)(\mu_j, Q_j) \) and
\[
I(\mu, Q) = \sum_{j \in J} \mu(K_j)I(\mu_j, Q_j). \tag{2.14}
\]

(iii) The oriented connected components of the oriented graph (\( \text{supp}(\mu), E(Q) \)) coincide with the connected components of the unoriented graph (\( \text{supp}(\mu), E^u(Q) \)), where
\[
E^u(Q) := \left\{ (y, z) : (y, z) \in E(Q) \text{ or } (z, y) \in E(Q) \right\}.
\]

For the unfamiliar reader, the definition of (oriented) connected components is recalled after Remark 4.2. Note that the oriented components of (\( \text{supp}(\mu), E(Q) \)) coincide with the irreducible classes of the Markov chain on \( \text{supp}(\mu) \) with transition rates \( r(y, z) := Q(y, z)/\mu(y) \). Moreover, note that due to Item (i) the graph (\( \text{supp}(\mu), E(Q) \)) is well defined. The proof of the above proposition is given after Lemma 4.3.

2.4. LDP with flow space \([0, +\infty]^B\) endowed of the product topology. When considering the product topology on \([0, +\infty]^B\) we take \([0, +\infty]^B\) endowed of the metric making the map \( x \to \frac{x}{1+x} \in [0, 1] \) an isometry. Namely, on \([0, +\infty]^\mathbb{N}\) we take the metric \( d(\cdot, \cdot) \) defined as \( d(x, y) = |x/(1+x) - y/(1+y)| \). It is standard to define on the space \([0, +\infty]^B\) a metric \( D(\cdot, \cdot) \) inducing the product topology: enumerating the bonds in \( B \) as \( b_1, b_2, \ldots \) we set \( D(Q, Q') := \sum_{n=1}^{[B]} 2^{-n}d(Q(b_n), Q'(b_n)) \).
We write \( \mathcal{M}_S \) for the space of stationary probabilities on \( D(\mathbb{R}; V) \) endowed of the weak topology. Given \( R \in \mathcal{M}_S \) we denote by \( \hat{\mu}(R) \in \mathcal{P}(V) \) the marginal of \( R \) at a given time and by \( \hat{Q}(R) \) the flow in \([0, +\infty]^B\) defined as \( \hat{Q}(R)(y, z) := \mathbb{E}_R \left[ Q_T(y, z) \right] \) for all \((y, z) \in B\), where \( \mathbb{E}_R \) denotes the expectation with respect to \( R \). It is simple to check that the above expectation does not depend on the time \( T > 0 \):

**Lemma 2.7.** Given an oriented bond \((y, z) \in B\) and a stationary process \( R \in \mathcal{M}_S \), the expectation \( \mathbb{E}_R \left[ Q_T(y, z) \right] \in [0, +\infty) \) does not depend on \( T > 0 \).

**Proof.** Since \( R \) is stationary, fixed \( t \in \mathbb{R} \) it holds \( R(X_t \neq X_{t-}) = 0 \). In particular, given \( T > 0 \) and an integer \( n \) for \( R \)-a.a. \( X \in D(\mathbb{R}; V) \) it holds

\[
Q_T(y, z)(X) = \frac{1}{n} \sum_{j=0}^{n-1} Q_{T/n}(y, z)(\theta_{jT/n}X).
\]

Above we have used the notation \((\theta_s, X) : X_{s+t}\). From the above identity and the stationarity of \( R \), taking the expectation w.r.t. \( R \) one gets \( f(T) = f(T/n) \), where \( f(T) := \mathbb{E}_R \left[ Q_T(y, z) \right] \). Then by standard arguments one gets that \( f(T) = f(1) \) as \( T \) varies among the positive rational numbers. Since for \( 0 < t_1 \leq T \leq t_2 \) it holds \( t_1 f(t_1) \leq T f(T) \leq t_2 f(t_2) \) it is trivial to conclude that \( f(T) \) is constant as \( T \) varies among the positive real numbers. \( \square \)

We can now state our second main result:

**Theorem 2.8.** Assume the Markov chain satisfies (A1)–(A4) together with Donsker–Varadhan condition. Consider the space \( \mathcal{P}(V) \times [0, +\infty]^B \), with \( \mathcal{P}(V) \) endowed of the weak topology and \([0, +\infty]^B\) endowed of the product topology. Then the following holds:

(i) As \( T \to +\infty \) the family of probability measures \( \{\mathbb{P}_x \circ (\mu_T, Q_T)^{-1}\} \) on \( \mathcal{P}(V) \times [0, +\infty]^B \) satisfies a large deviation principle with good rate function

\[
\tilde{I}(\mu, Q) := \inf \left\{ H(R) : R \in \mathcal{M}_S, \hat{\mu}(R) = \mu, \hat{Q}(R) = Q \right\}
\]  
(2.15)

Above \( H(R) \) denotes the entropy of \( R \) with respect to the Markov chain \( \xi \) as defined in (7)–(IV) (see Section 4). Moreover we have

\[
\begin{cases}
\tilde{I}(\mu, Q) = I(\mu, Q) & \text{if } Q \in L^1_+(E), \\
\tilde{I}(\mu, Q) = +\infty & \text{if } Q \notin \mathbb{R}^E_+.
\end{cases}
\]  
(2.16)

(ii) If in addition Condition \ref{cond:2.1} is satisfied, then the rate function \( \tilde{I} \) is given by

\[
\tilde{I}(\mu, Q) := \begin{cases}
I(\mu, Q) & \text{if } Q \in L^1_+(E), \\
+\infty & \text{otherwise}.
\end{cases}
\]  
(2.17)

Since Condition \ref{cond:2.1} implies the Donsker–Varadhan condition, the above theorem under Condition \ref{cond:2.2} implies the variational characterization

\[
I(\mu, Q) = \inf \left\{ H(R) : R \in \mathcal{M}_S, \hat{\mu}(R) = \mu, \hat{Q}(R) = Q \right\}, \quad (\mu, Q) \in \mathcal{P}(V) \times L^1_+(E).
\]

In addition, note that \ref{eq:2.16} does not cover the case \( Q \in \mathbb{R}^E_+ \setminus L^1_+(E) \).
2.5. Outline. The rest of the paper is devoted to the proofs of Theorems 2.5 and 2.8. Sections 3 and 4 contain preliminary results. Then in Section 5 we give a direct proof of Theorem 2.5. For this proof it is necessary to add the condition that the graph \((V, E)\) is locally finite.

In Sections 6, 7, and 8 we remove the above condition and prove both Theorems 2.5 and 2.8 by projection from the large deviations principle for the empirical process proven by Donsker and Varadhan in \([17]–(IV)\). We discuss the details only for the Donsker-Varadhan type compactness conditions. For this reason, we added item (iii) as a separate requirement in the hypercontractivity type Condition 2.3. By using similar arguments to the ones here presented, it should be possible to remove it from Theorem 2.8 and prove the first statement in Theorem 2.8 by assuming only items (i) and (ii) in Condition 2.3.

Finally, in Section 9 we discuss some examples from birth and death processes and compare the different compactness conditions.

3. Exponential estimates

In this section we collect some preliminary results that will enter in the proof of Theorems 2.5 and 2.8. Between other, we prove the exponential tightness in \(L^1_s(E)\) of the empirical flow when at least one between Conditions 2.2 and 2.3 holds.

3.1. Exponential local martingales. We start by comparing our Markov chain with a perturbed one. Let \(\xi\) be a continuous time Markov chain on \(V\) with jump rates \(\hat{r}(y, z), y \neq z \in V\). We assume that \(\hat{r}(y) := \sum_{z \in V} \hat{r}(y, z) < +\infty\) for all \(y \in V\), thus implying that the Markov chain \(\hat{\xi}\) is well defined at cost to add a coffin state \(\partial\) to the state space in case of explosion \([35]\) Ch. 2. We write \(\hat{\mathbb{P}}_x\) for the law on \(D([0, T]), \mathcal{D}(\mathbb{R}_+, V \cup \{\partial\})\) of the above Markov chain \(\hat{\xi}\) starting at \(x \in V\). We denote by \(\rho_T\) the map \(\rho_T : D([0, T], \mathcal{D}(\mathbb{R}_+, V \cup \{\partial\}) \to D([0, T], \mathcal{D}(\mathbb{R}_+, V \cup \{\partial\})\) given by restriction of the path to the time interval \([0, T]\). We now assume that \(\hat{r}(y, z) = 0\) if \((y, z) \notin E\). Then, restricting the probability measures \(\mathbb{P}_x \circ \rho_T^{-1}\) and \(\hat{\mathbb{P}}_x \circ \rho_T^{-1}\) to the set \(D([0, T], V)\) (no explosion takes place in the interval \([0, T]\)), we obtain two reciprocally absolutely continuous measures with Radon–Nykodim derivative

\[
\frac{d\mathbb{P}_x \circ \rho_T^{-1}}{d\hat{\mathbb{P}}_x \circ \rho_T^{-1}}\bigg|_{D([0, T], V)} = \exp \{-T(\mu_T, \hat{\mu} - \hat{\mu})\} \prod_{(y, z) \in E} \left[\frac{\hat{r}(y, z)}{r(y, z)}\right]^{TQ_T(y, z)}.
\]

This formula can be checked very easily. Indeed, calling \(\tau_1(X) < \tau_2(X) < \tau_N(X)\) the jump times of the path \(X\) in \([0, T]\) (below \(N(X) < +\infty\) almost surely) we have

\[
\mathbb{P}_x \circ \rho_T^{-1}\left(N(X) = n, X(\tau_i) = x_i \forall i : 1 \leq i \leq n, \tau_i \in (t_i, t_i + dt_i) \forall i : 1 \leq i \leq n\right)
\]

\[
= \left[\prod_{i=0}^{n-1} e^{-r(x_i)(t_{i+1}-t_i)r(x_i, x_{i+1})} e^{-r(x_n)(T-t_n)} dt_1 \cdots dt_n\right],
\]

where \(t_0 := 0, x_0 := x, 0 \leq t_1 < t_2 < \cdots < t_n \leq T, n = 0, 1, 2, \ldots\). Since a similar formula holds also for the law \(\hat{\mathbb{P}}_x \circ \rho_T^{-1}\), one gets \((3.1)\).

As immediate consequence of the Radon–Nykodim derivative \((3.1)\) we get the following result:
Lemma 3.1. Let $F: E \to \mathbb{R}$ be such that $r^F(y) := \sum_z r(y,z)e^{F(y,z)} < +\infty$ for any $y \in V$. For $t \geq 0$ define $M^F_t : D(\mathbb{R}_+, V) \to (0, +\infty)$ as

$$M^F_t := \exp \left\{ t \left[ (Q_t, F) - (\mu_t, r^F) \right] \right\}$$

(3.2)

where $(Q_t, F) = \sum_{(y,z) \in E} Q_t(y,z)F(y,z)$. Then for each $x \in V$ and $t \in \mathbb{R}_+$ it holds $\mathbb{E}_x(M^F_t) \leq 1$.

Proof. By (3.1) $\langle \hat{r}(y,z), x \rangle := r(y,z)e^{F(y,z)}$, $\mathbb{E}_x(M^F_t) = \hat{\mathbb{P}}_x(D([0,t]; V)) \leq 1$.

Remark 3.2. It is simple to check that the process $M^F$ is a positive local martingale and a supermartingale with respect to $\mathbb{P}_x$, $x \in V$.

Remark 3.3. Fixed $(y,z) \in E$, taking in Lemma 3.1 $F := \pm \lambda \delta_{y,z}$ with $\lambda > 0$ and applying Chebyshev inequality, one gets for $\delta > 0$ that the events $\{Q_t(y,z) > \mu_t(y)r(y,z)(e^\delta - 1)/\lambda + \delta\}$ and $\{Q_t(y,z) < \mu_t(y)r(y,z)(1 - e^{-\delta})/\lambda - \delta\}$ have $\mathbb{P}_x$–probability bounded by $e^{-t\delta\lambda}$. Using that $(e^{x+\lambda}-1)/\lambda = \pm 1 + o(1)$ and since $\mu_t(y) \to \pi(y)$ as $t \to +\infty$, $\mathbb{P}_x$–a.s. by the ergodic theorem (2.2), taking the limit $t \to +\infty$ and afterwards taking $\delta, \lambda$ arbitrarily small, one recovers the LLN of $Q_t(y,z)$ towards $\pi(y)r(y,z)$ discussed in Section 2.1.

The next statement is deduced from the previous lemma by choosing there $F(y) = \log(u(y))/u(y))$, $(y,z) \in E$ for some $u: V \to (0, +\infty)$.

Lemma 3.4. Let $u: V \to (0, +\infty)$ be such that $\sum_z r(y,z)u(z) < +\infty$ for any $y \in V$. For $t \geq 0$ define $M^u_t : D(\mathbb{R}_+, V) \to (0, +\infty)$ as

$$M^u_t := \frac{u(X_t)}{u(X_0)} \exp \left\{ t \left[ \frac{\mu_t}{u} - \frac{Lu}{u} \right] \right\}.$$ 

(3.3)

Then for each $x \in V$ and $t \in \mathbb{R}_+$ it holds $\mathbb{E}_x(M^u_t) \leq 1$.

3.2. Exponential tightness. We shall prove separately the exponential tightness of the empirical measure and of the empirical flow. We first discuss the case in which Condition 2.2 holds. Then the proof of the exponential tightness of the empirical measure is essentially a rewriting of the argument in (17) in the present setting. On the other hand, the proof of the exponential tightness of the empirical flow depends on the extra assumption $\sigma > 0$ in item (vi) of Condition 2.2.

Lemma 3.5. Assume Condition 2.2 hold and let the function $v$ and the constants $c, C_x, C, \sigma$ be as in Condition 2.2. Then for each $x \in V$ it holds

$$\mathbb{E}_x \left( e^{T(\mu_T, v)} \right) \leq \frac{C_x}{c}, \quad \mathbb{E}_x \left( e^{T(\mu_T, r)} \right) \leq e^{TC} \frac{C_x}{c}.$$ 

(3.4)

Proof. The second bound in (3.4) follows trivially from the first one and item (vi) in Condition 2.2. To prove the first bound, let $v_n$ be the sequence of functions on $V$ provided by Condition 2.2 and recall that $v_n = -Lu_n/u_n$. In view of the pointwise convergence of $v_n$ to $v$ and Fatou lemma

$$\mathbb{E}_x \left( e^{T(\mu_T, v)} \right) \leq \lim_n \mathbb{E}_x \left( e^{T(\mu_T, v_n)} \right) = \lim_n \mathbb{E}_x \left( \exp \left\{ T \left( \mu_T, -\frac{Lu_n}{u_n} \right) \right\} \right) \leq \frac{C_x}{c},$$

where the last step follows from Lemma 3.4 and items (ii)–(iii) in Condition 2.2.

The following provides the exponential tightness of the empirical measure and the empirical flow.
We consider the exponential local martingale of Lemma 3.1 choosing \( \ell \) here.

\[
\lim_{T \to +\infty} \frac{1}{T} \log \mathbb{P}_x(\mu_T \notin K_\ell) \leq -\ell, \quad (3.5)
\]

\[
\lim_{T \to +\infty} \frac{1}{T} \log \mathbb{P}_x(\|Q_T\| > A_\ell) \leq -\ell. \quad (3.6)
\]

In particular, the empirical measure and flow are exponentially tight.

**Proof.** We first prove (3.5). For a sequence \( a_\ell \) we deduce (3.5) for any \( \mathbb{P} \)-inequality, the empirical measure and flow are exponentially tight.

From item (vi) in Condition 2.2 (for this step we only need it with \( \sigma = 0 \)) we deduce (3.6). By the exponential Chebyshev inequality and Lemma 3.5 we then get

\[
\mathbb{P}_x \left( \mu_T(K^c_\ell) > \frac{1}{T} \right) \leq \mathbb{P}_x \left( \mu_T(\mu_T, r) > \frac{a_\ell}{T} - C \right) 
\]

\[
\leq \exp \left\{ -T \left[ \frac{a_\ell}{T} - C \right] \right\} \mathbb{E}_x \left( e^{T(\mu_T, v)} \right) \leq \frac{C_x}{e} \exp \left\{ -T \left[ \frac{a_\ell}{T} - C \right] \right\}.
\]

By choosing \( a_\ell = \ell^2 + C_\ell \) the proof is now easily concluded.

Let us now prove (3.6). By the second bound in Lemma 3.5 and Chebyshev inequality, \( \mathbb{P}_x \left( \mu_T(\mu_T, r) > \lambda \right) \leq \frac{C_x}{e} e^{-T(\sigma \lambda - C)} \) for any \( \lambda > 0 \). In particular we obtain that

\[
\mathbb{P}_x \left( \mu_T(\mu_T, r) > A'_\ell \right) \leq \frac{C_x}{e} e^{-T \ell}, \quad A'_\ell := \sigma^{-1}(\ell + C).
\]

Hence, it is enough to show that for each \( x \in V \) there exists a sequence \( A_\ell \uparrow +\infty \) such that for any \( T > 0 \) and any \( \ell \in \mathbb{N} \)

\[
\mathbb{P}_x \left( \|Q_T\| > A_\ell, \mu_T(\mu_T, r) \leq A'_\ell \right) \leq e^{-T \ell}. \quad (3.7)
\]

We consider the exponential local martingale of Lemma 3.1 choosing there \( F : E \to \mathbb{R} \) constant, \( F(x, y) = \lambda \in (0, +\infty) \) for any \( (x, y) \in E \). We deduce

\[
\mathbb{P}_x \left( \|Q_T\| > A_\ell, \mu_T(\mu_T, r) \leq A'_\ell \right) 
\]

\[
= \mathbb{E}_x \left( e^{-T \left[ \lambda \|Q_T\| - (e^{\lambda} - 1)(\mu_T, r) \right]} M_T^{F} \mathbb{I}_{\left\{ \|Q_T\| > A_\ell \right\}} \mathbb{I}_{\left\{ \mu_T(\mu_T, r) \leq A'_\ell \right\}} \right) 
\]

\[
\leq \exp \left\{ -T \left[ \lambda A_\ell - (e^{\lambda} - 1)A'_\ell \right] \right\}
\]

where we used Lemma 3.1 in the last step. The proof of (3.7) is now completed by choosing \( A_\ell = \lambda^{-1} \ell + \lambda^{-1}(e^{\lambda} - 1)A'_\ell \).

Recalling that the closed ball in \( L^1(E) \) is compact with respect to the bounded weak* topology, the exponential tightness of the empirical flow is due to (3.6). \( \square \)

We next discuss the exponential tightness when Condition 2.2 is assumed.
Proposition 3.7. Fix $x \in V$. If item (i) in Condition 2.3 holds then the sequence of probabilities $\{P_x \circ \mu_T^{-1}\}$ on $P(V)$ is exponentially tight. If furthermore it holds also item (ii) in Condition 2.3 then the sequence of probabilities $\{P_x \circ Q_T^{-1}\}$ on $L^1_+(E)$ is exponentially tight.

While the first statement is a consequence of the general results in [10], we next give a direct and alternative proof also of this result. We premise an elementary lemma.

Lemma 3.8. Let $\pi \in P(V)$ be such that $\pi(x) > 0$ for any $x \in V$. There exists a decreasing function $\psi_\pi : (0, 1) \to (0, +\infty)$ such that $\lim_{s \downarrow 0} \psi_\pi(s) = +\infty$ and
\[
\sum_{x \in V} \pi(x) \psi_\pi(\pi(x)) < +\infty.
\]

Proof. By choosing a suitable order in $V$, it is enough to prove the lemma when $V = \mathbb{N}$ and $\pi$ is decreasing, i.e. $\pi(k+1) \leq \pi(k)$, $k \in \mathbb{N}$. Let us first observe that there exists a positive increasing sequence $a_\pi(k)$ such that $\lim_k a_\pi(k) = +\infty$ and $\sum_k \pi(k) a_\pi(k) < +\infty$. Indeed, it is simple to check that the explicit choice $a_\pi(k) = \left[\frac{k+b}{k^{1/3}}\right]^{-1}$, where $R_k := \sum_{i=k}^\infty \pi(i)$, meets the above requirements. By setting
\[
\psi_\pi(s) := \sup \{a_\pi(k) : \pi(k) \geq s\}
\]
we then conclude the proof. \qed

Proof of Proposition 3.7. We prove first the exponential tightness of the empirical measure. Let $\pi$ be the invariant measure of the chain, $\psi_\pi$ be the function provided by Lemma 3.8 and $\alpha := \sum_x \pi(x) \psi_\pi(\pi(x)) < +\infty$. We define $v : V \to (0, +\infty)$ as
\[
v(x) := \log \frac{\psi_\pi(\pi(x))}{\alpha}, \quad x \in V.
\]
Then, in view of Lemma 3.8, $v$ has compact level sets and $\langle \pi, e^v \rangle = 1$.

By the proof of Proposition 3.6 it is enough to show the following bound. For each $x \in V$ there exist constants $\lambda, C_x > 0$ such that for any $T > 0$
\[
\mathbb{E}_x \left(e^{\lambda T (\mu_T, v)} \right) \leq C_x. \tag{3.8}
\]

We now proceed using spectral estimates. When $\sup_{x \in V} r(x) < +\infty$ the tools used below are discussed in [25] App. 1, Sec. 7]. We drop this boundedness assumption and proceed formally. In Appendix A we give a rigorous derivation of our bounds, covering the case $\sup_{x \in V} r(x) = +\infty$. By the Feynman-Kac formula,
\[
\mathbb{E}_x \left(e^{\lambda T (\mu_T, v)} \right) \leq \frac{1}{\pi(x)} \mathbb{E}_x \left(e^{\lambda \int_0^T dt v(X_t)} \right) \leq \frac{1}{\pi(x)} \exp \left[T \sup \operatorname{spec} \{S + \lambda v\} \right]
\]
where we recall $S = (L + L^*)/2$ and we used [25] App. 1, Lemma 7.2] in the last step. The operator $S + \lambda v$ is understood as a self-adjoint operator in $L^2(V, \pi)$. By the variational characterization of the maximal eigenvalue of self-adjoint operators,
\[
\sup \operatorname{spec} \{S + \lambda v\} = \sup_{f : \operatorname{supp}(f^2) = 1} \left\{ -D_\pi(f) + \lambda \langle f \pi \rangle \right\} = \sup_{\mu \in P(V)} \left\{ -D_\pi(\sqrt{\mu/\pi}) + \lambda \langle \mu, v \rangle \right\}. \tag{3.9}
\]
The last equality in (3.9) states that we can restrict the supremum to non-negative functions. The validity of the inequality $\geq$ among the second and the third term in (3.9) is immediate. The validity of the converse inequality follows easily by the inequality $D_\pi(f) \geq D_\pi(|f|)$. Recalling the basic entropy inequality $\langle \mu, v \rangle \leq \log \langle \pi, ev \rangle + \text{Ent}(\mu|\pi)$ and choosing $\lambda \in (0, 1/\sqrt{\lambda_4}]$, the logarithmic Sobolev inequality (2.9) now implies $\sup \text{spec} \{ S + \lambda v \} \leq \lambda \log \langle \pi, ev \rangle = 0$ which concludes the proof of (3.8).

To prove the second statement, one can proceed as in the proof of (3.6) in Proposition 3.9 under Condition 2.3. Indeed, that proof is based on the exponential bounds given by (3.8). The first bound corresponds here to (3.8). The same arguments used to derive (3.8) and item (ii) in Condition 2.3 imply the exponential bound (3.8) with $v$ replaced by $r$. \hfill $\square$

We conclude with a simple observation on the stationary flow:

Lemma 3.9. Assume at least one between Conditions 2.2 and 2.3 to hold. Then $\langle \pi, r \rangle < +\infty$, equivalently $Q^T \in L^1_v(E)$.

Proof. The thesis is trivially true under Condition 2.3. Let us assume Condition 2.2. By Lemma 3.5 we have $E_x(e^{T(\mu_T r)}) \leq e^{TC}C_2/c$. We restrict to $V$ infinite, the finite case being obvious. Enumerating the points in $V$ as $\{x_n\}_{n \geq 0}$, by the ergodic theorem (2.2) fixed $N$ there exists a time $T_0 = T_0(N) > 0$ and a Borel set $A \subset D(\mathbb{R}_+; V)$ such that (i) $\mathbb{P}_x(A) \geq 1/2$ and (ii) $\mu_T(x_n) \geq \pi(x_n)/2$ for all $T \geq T_0$ and $n \leq N \mathbb{P}_x$-a.s. on $A$. Hence, for all $T \geq T_0$ it holds $e^{T\sigma} \sum_{n=0}^N \pi(x_n)r(x_n)/2 \leq E_x(e^{T\sigma} \sum_{n=0}^N \mu_T(x_n)r(x_n); A) \leq E_x(e^{T\sigma} \mu_T r) \leq e^{TC}C_2/c$. This implies that $\sum_{n=0}^N \pi(x_n)r(x_n) \leq 2C/\sigma$. To conclude it is enough to take the limit $N \to +\infty$. \hfill $\square$

4. Structure of divergenceless flows in $L^1_v(E)$

In this section we show that any divergenceless flow in $L^1_v(E)$, and more in general any divergenceless flow in $\mathbb{R}_+^d$ with zero flux towards infinity, can be written as superposition of flows along self avoiding finite cycles. See [21] for other problems related to cyclic decompositions of divergenceless flows on graphs and [31] for similar decompositions for divergenceless vector valued measures on $\mathbb{R}^d$.

We first introduce some key graphical structures. A finite cycle $C$ in the oriented graph $(V, E)$ is a sequence $(x_1, \ldots, x_k)$ of elements of $V$ such that $(x_i, x_{i+1}) \in E$ when $i = 1, \ldots, k$ and the sum in the indices is modulo $k$. A finite cycle is self avoiding if for $i \neq j$ it holds $x_i \neq x_j$. Given $(x, y) \in E$, if there exists an index $i = 1, \ldots, k$ such that $(x, y) = (x_i, x_{i+1})$ we write $(x, y) \in C$. Similarly, given $x \in V$, if there exists an index $i = 1, \ldots, k$ such that $x = x_i$ we say that $x \in C$. The collection of all the self avoiding finite cycles in $(V, E)$ is a countable set which we denote by $C$. In the sequel we shall mostly regard elements $C \in C$ as finite subsets of $E$ and denote by $|C|$ the corresponding cardinality. Consider an invading sequence $V_n \nearrow V$ of finite subsets $V_n$. This means a sequence such that $|V_n| < +\infty$, $V_n \subset V_{n+1}$ and moreover $\bigcup_n V_n = V$. For any fixed $n$ we define $E_n := \{(y, z) \in E : y, z \in V_n\}$,

$$E_n := \{(y, z) \in E : y, z \in V_n\}, \tag{4.1}$$
and observe that it is an invading sequence of edges. Given a flow \( Q \in \mathbb{R}_{+}^{E} \), we define

\[
E(Q) := \{(y, z) \in E : Q(y, z) > 0\},
\]

\[
M_n(Q) := \max_{(y, z) \in E_n} Q(y, z),
\]

\[
\phi^+_n(Q) := \sum_{y \in V_n, z \notin V_n} Q(y, z),
\]

\[
\phi^-_n(Q) := \sum_{y \notin V_n, z \in V_n} Q(y, z).
\]

Moreover, we say that \( Q \) has zero flux towards infinity if there exists an invading sequence \( V_n \not

\)

\[
\lim_{n \to +\infty} \phi^+_n(Q) = 0.
\]

We say that \( Q \) admits a cyclic decomposition if there are constants \( \hat{Q}(C) \geq 0 \), \( C \in \mathcal{C} \) such that

\[
Q = \sum_{C \in \mathcal{C}} \hat{Q}(C) \mathbb{1}_C.
\]

Namely, for each \( (y, z) \in E \) it holds \( Q(y, z) = \sum_{C \in \mathcal{C}, C \ni (y, z)} \hat{Q}(C) \). We emphasize that the constants \( \hat{Q}(C) \), \( C \in \mathcal{C} \), are not uniquely determined by the flow \( Q \).

**Lemma 4.1.** Let \( Q \in \mathbb{R}_{+}^{E} \) be a flow having zero flux towards infinity and such that \( \text{div} \ Q = 0 \). Then \( Q \) admits a cyclic decomposition (4.7). In particular, any divergenceless flow \( Q \in L_1^+(E) \) has a cyclic decomposition.

**Proof.** Since (4.6) holds for any invading sequence of vertices if \( Q \in L_1^+(E) \), the second statement follows directly from the former on which we concentrate.

On a finite graph any divergence free flow admits a cyclic decomposition. The proof follows classical arguments (see e.g. [21, 29]). If \( Q \) has finite support, i.e. if \( |E(Q)| < +\infty \), the thesis follows directly by the analogous result on finite graphs. We will then consider only the case of infinite support, using below the result in the finite case. Remember that \( V_n \) is the invading sequence satisfying (4.0).

We assume \( |E(Q)| = +\infty \) and \( \text{div} \ Q = 0 \). Due to the zero divergence condition, a discrete version of the Gauss theorem guarantees that \( \phi^+_n(Q) = \phi^-_n(Q) \). We define by an iterative procedure a sequence of flows \( Q^i \), \( i \geq 0 \), with infinite support and having zero flux towards infinity as follows. We set \( Q^0 := Q \) and explain how to define \( Q^{i+1} \) knowing \( Q^i \). First, we define \( n_i := \inf \{ n \in \mathbb{N} : M_n(Q^i) > \phi^+_n(Q^i) \} \). Since \( Q^i \neq 0 \), it must be \( n_i < +\infty \). Indeed, \( \phi^+_n(Q^i) \) is a sequence in \( n \) converging to zero, while \( M_n(Q^i) \) is a non decreasing sequence not identically zero. Let \( g \) be a ghost site and define the flow \( Q^i_g \) on a finite graph having vertices \( V_{n_i} \cup \{g\} \) as

\[
\begin{cases}
Q^i_g(y, z) := Q^i(y, z), & (y, z) \in E_{n_i}, \\
Q^i_g(y, g) := \sum_{z \notin V_{n_i}} Q^i(y, z), & y \in V_{n_i}, \\
Q^i_g(g, y) := \sum_{z \in V_{n_i}} Q^i(z, y), & y \in V_{n_i}.
\end{cases}
\]

Roughly speaking, the flow \( Q^i_g \) is obtained from \( Q^i \) by collapsing all vertices outside \( V_{n_i} \) into a single vertex, called \( g \). By construction we have \( \text{div} \ Q^i_g = 0 \). Calling \( \mathcal{C}^i_{n_i} \),
the collection of self avoiding cycles of the finite graph and using the validity of the cyclic decomposition in the finite case, we have
\[ Q^i_g = \sum_{C \in \mathcal{C}^g_i} \mathbb{I}_C. \] (4.8)
We claim that in the decomposition (4.8) there exists a self avoiding cycle \( C_i \) not visiting the ghost site \( g \) and such that \( Q^g_g(C_i) > 0 \). Let us suppose by contradiction that our claim is false and let \( (x^*, y^*) \in E_{n_i} \) be such that \( Q^i(x^*, y^*) = M_{n_i}(Q^i) \).
Then we have
\[ M_{n_i}(Q^i) = Q^i(x^*, y^*) = \sum_{C \in \mathcal{C}^g_i} \mathbb{I}_i(C) \mathbb{I}_{(x^*, y^*)} \leq \sum_{C \in \mathcal{C}^g_i} \mathbb{I}_i(C) = \phi_n^+(Q^i). \]
The last equality follows by the fact that any cycle with positive weight in \( \mathcal{C}^g_i \) has to contain necessarily the ghost site \( g \). This contradicts the definition of \( n_i \), thus proving our claim.

At this point, we know that there exists a self avoiding cycle \( C_i := (x_1, \ldots, x_k) \) such that \( x_j \in V_{n_i} \) and \( Q^i(x_j, x_{j+1}) > 0 \) for any \( j \) (the sum in the indices is modulo \( k \)). We fix \( m_i := \min_{j=1, \ldots, k} Q^i(x_j, x_{j+1}) \) and define
\[ Q^{i+1} := Q^i - m_i \mathbb{I}_{C_i} = Q - \sum_{j=0}^i m_j \mathbb{I}_{C_j}. \]
With this definition we have that \( Q^{i+1} \) is still a flow in \( \mathbb{R}^E_+ \), it satisfies \( \text{div} \, Q^{i+1} = 0 \), it has zero flux towards infinity and infinite support. Moreover
\[ |E_{n_i} \cap E(1_{Q^{i+1}})| \leq |E_{n_i} \cap E(Q^i)| - 1. \] (4.9)
Condition (4.9) implies that \( \lim_{i \to +\infty} n_i = +\infty \). Hence, fixed any \((y, z) \in E\), for \( i \) large it holds
\[ Q^i(y, z) \leq M_{n_i-1}(Q^i) \leq \phi_{n_i-1}^+(Q^i) \leq \phi_{n_i-1}^+(Q) \] (4.10)
(for the first inequality note that \( (y, z) \in E_{n_i-1} \) for \( i \) large, for the second one use the definition of \( n_i \), for the third one observe that by construction \( Q^i \leq Q \)).

Since the r.h.s. of (4.10) converges to zero when \( i \) diverges we obtain \( \lim_{i \to +\infty} Q^i(y, z) = 0 \) for any \((y, z) \in E\). Finally we get
\[ \lim_{i \to +\infty} \left( Q(y, z) - \sum_{j=0}^i m_j \mathbb{I}_{C_j}(y, z) \right) = \lim_{i \to +\infty} Q^{i+1}(y, z) = 0. \]
The above limit trivially implies that \( Q = \sum_{j=0}^\infty m_j \mathbb{I}_{C_j} \). \( \Box 

**Remark 4.2.** It is easy to see that Lemma 3.1 remains valid if the condition of zero flux towards infinity is satisfied just by the reduced flow \( q \in \mathbb{R}_+^E \) defined as
\[ q(y, z) := \begin{cases} Q(y, z) & \text{if } (z, y) \notin E, \\ Q(y, z) - \min \{Q(y, z), Q(z, y)\} & \text{if } (z, y) \in E. \end{cases} \]

Given an oriented graph \((V, E)\) with countable \( V, E \) we say that it is connected if for any \( y, z \in V \) there exist \( x_1, \ldots, x_n \) such that \( x_1 = y, x_n = z \) and \( (x_i, x_{i+1}) \in E, i = 1, \ldots, n - 1 \). To every oriented graph we can associate an unoriented graph \((V, E^u)\) for which \((y, z) \in E^u \) if at least one among \((y, z)\) and \((z, y)\) belongs to \( E \). We say that the unoriented graph \((V, E^u)\) is connected if for any \( y, z \in V \) there exist \( x_1, \ldots, x_n \) such that \( x_1 = y, x_n = z \) and \( (x_i, x_{i+1}) \in E^u, i = 1, \ldots, n - 1 \).
The following lemma will be useful.

**Lemma 4.3.** Let \((V, E)\) be an oriented graph with countable \(V, E\) such that there exists a flow \(Q \in L^1_+(E)\) with \(Q(y, z) > 0\) for any \((y, z) \in E\) and \(\text{div} \, Q = 0\). In this case \((V, E)\) is connected if and only if \((V, E^u)\) is connected.

**Proof.** Trivially if the oriented graph is connected then also the unoriented one is connected. We prove the converse implication. Assume that \((V, E^u)\) is connected and suppose by contradiction that \(V^y\) is strictly included in \(V\); for some \(y\), \(V^y\) being the set of vertices that can be reached from \(y\) by oriented paths. Note that \(V^y\) is nonempty since \(y \in V^y\). Given \(A, B \subset V\) we set \(Q(A, B) := \sum_{(a, b) \in E \cap (A \times B)} Q(a, b)\). Since \(Q\) is divergenceless and \(Q \in L^1_+(E)\) (and therefore in the following sums the summation order can be arbitrarily chosen), it holds

\[
0 = \sum_{v \in V^y} \text{div} \, Q(v) = Q(V \setminus V^y, V) - Q(V, V \setminus V^y) = Q(V \setminus V^y, V^y) - Q(V^y, V \setminus V^y).
\]

We point out that \(Q(V^y, V \setminus V^y) = 0\) since the definition of \(V^y\) implies that \(v \in V^y\) if \(u \in V^y\), \(v \in V\) and \((u, v) \in E\). To get a contradiction we can show that \(Q(V \setminus V^y, V^y) > 0\). Since \(Q\) is positive, we only need to show that there exists a directed bond \((u, v) \in E\) such that \(u \not\in V^y\) and \(v \in V^y\). Here we use that the unoriented graph is connected. Indeed, the nonempty set \(V^y\) is connected to its complement \(V \setminus V^y\) in the unoriented graph. Hence there exist \(u \not\in V^y\) and \(v \in V^y\) such that \((u, v) \in E\) or \((v, u) \in E\). The case \((v, u) \in E\) cannot take place by definition of \(V^y\).

We can now give the proof of Proposition 2.6.

**Proof of Proposition 2.6.** From the definition of \(I(\mu, Q)\) and \(\Phi\) we trivially have that \(Q(y, z) > 0\) implies \(\mu(y) r(y, z) > 0\) and therefore \(y \in \text{supp}(\mu)\). Suppose by contradiction that \(z \not\in \text{supp}(\mu)\). Then there would be a nonzero ingoing flow at \(z\) and therefore a nonzero outgoing flow at \(z\) (since \(I(\mu, Q) < +\infty\) implies \(\text{div} \, Q = 0\)). As a consequence there must exist an edge \((z, u) \in E\) such that \(Q(z, u) > 0\). As proven at the beginning, this implies that \(z \in \text{supp}(\mu)\), hence a contradiction. This completes the proof of Item (i). Item (iii) is an immediate consequence of Lemma 4.3. It remains to prove Item (ii). To this aim we first observe that \(\text{div} \, Q_j = 0\). Indeed, the following property (P) holds: given \(y \in V\) we have that \(z\) belongs to the same oriented connected component of \(y\) if \(Q(y, z) > 0\) or \(Q(z, y) > 0\) (apply Item (iii)). This property and the zero divergence of \(Q\) imply that \(\text{div} \, Q_j = 0\) and that, by definition \((4.11)\) and Remark 2.4

\[
I(\mu_j, Q_j) = \sum_{(y, z) \in E \cap (V_j^+ \times V_j^-)} \Phi(Q_j(y, z), Q_j^\mu(y, z)) + \sum_{(y, z) \in E \cap (V_j^0 \times V_j^0)} Q_j^\mu(y, z).
\]

Always property (P) implies that

\[
I(\mu, Q) = \sum_j \left\{ \sum_{(y, z) \in E \cap (V_j^+ \times V_j^-)} \Phi(Q(y, z), Q^\mu(y, z)) + \sum_{(y, z) \in E \cap (V_j^0 \times V_j^0)} Q^\mu(y, z) \right\}.
\]

To conclude compare \((4.11)\) with \((4.12)\) using that \(Q(y, z) = \mu(K_j) Q_j(y, z)\) and \(Q^\mu(y, z) = \mu(K_j)Q_j^\mu(y, z)\) if \((y, z) \in E\) with \(y \in V_j\).
4.1. An approximation result for the function $I(\mu, Q)$. Let $\mathcal{S}$ be the subset of $\mathcal{P}(V) \times L^1_+(E)$ given by the elements $(\mu, Q)$ with $I(\mu, Q) < +\infty$ and such that the graph $(\text{supp}(\mu), E(Q))$ is finite and connected.

Proposition 4.4. Fix $(\mu, Q) \in \mathcal{P}(V) \times L^1_+(E)$. There exists a sequence $\{\langle \mu_n, Q_n \rangle \} \subset \mathcal{S}$ such that $(\mu_n, Q_n) \to (\mu, Q)$ in $\mathcal{P}(V) \times L^1_+(E)$ and

$$\lim_{n \to +\infty} I(\mu_n, Q_n) \leq I(\mu, Q).$$

(4.13)

As proven below, the convergence $(\mu_n, Q_n) \to (\mu, Q)$ in $\mathcal{P}(V) \times L^1_+(E)$ holds also with $L^1_+(E)$ endowed of the $L^1$-norm (strong topology).

Proof. We consider only elements $(\mu, Q)$ such that $I(\mu, Q) < +\infty$, otherwise the thesis is trivially true. In particular, it must be $\text{div} Q = 0$. First we show that $\mathcal{S}$ is $I$–dense in the set $\mathcal{S}^*$ of elements $(\mu, Q) \in \mathcal{P}(V) \times L^1_+(E)$ with finite support (i.e. with finite sup$(\mu)$ and $E(Q)$) and $\text{div} Q = 0$. Then we show that $\mathcal{S}^*$ is $I$–dense in the set of elements $(\mu, Q) \in \mathcal{P}(V) \times L^1_+(E)$ with $\text{div} Q = 0$.

Let $(\mu, Q) \in \mathcal{S}^*$ and denote by $K_1, \ldots, K_n$ the connected components of the graph $(\text{supp}(\mu), Q)$ (recall Proposition 2.4). Since $(V, E)$ is connected, for any pair of components $(K_i, K_j)$ we can fix an oriented path $\gamma_{i,j}$ on $E$ going from $K_i$ to $K_j$. Let respectively $\bar{V}$ and $\bar{E}$ be the vertices and the edges belonging to some path $\gamma_{i,j}$. We consider the finite connected oriented graph $(\text{supp}(\mu) \cup \bar{V}, E(Q) \cup \bar{E})$. On this graph we define an irreducible Markov chain having unitary rate associated to each oriented edge in $E(Q) \cup \bar{E}$, i.e. the rate for a jump from $y$ to $z$ is $I((y, z) \in E(Q) \cup \bar{E})$. We call $\pi^*$ its unique invariant measure. Then $(\pi^*, Q^*) \in \mathcal{S}$, where $Q^*$ is defined as $Q^*(y, z) = \pi^*(y)$ if $(y, z) \in E(Q) \cup \bar{E}(\gamma)$ and zero otherwise. Consequently for any $\varepsilon > 0 \varepsilon (\pi^*, Q^*) + (1 - \varepsilon)(\mu, Q)$ is an element of $\mathcal{S}$ converging to $(\mu, Q)$ when $\varepsilon \to 0$ (even with $L^1_+(E)$ endowed of the strong topology). Since in the case of finite support $I$ can be written as a finite sum it is not difficult to show that

$$\lim_{\varepsilon \to 0} I(\varepsilon (\pi^*, Q^*) + (1 - \varepsilon)(\mu, Q)) = I(\mu, Q).$$

(4.13)

We now show that $\mathcal{S}^*$ is $I$–dense in the set of elements $(\mu, Q) \in \mathcal{P}(V) \times L^1_+(E)$ with $\text{div} Q = 0$. To this aim, we fix $(\mu, Q)$ with $\text{div} Q = 0$ and $I(\mu, Q) < +\infty$. By Lemma 1.1 the cyclic decomposition (4.7) of $Q$ holds. We fix an invading sequence $V_n \searrow V$ of finite subsets and call $E_n$ the edges in $E$ connecting vertices in $V_n$ (recall (4.1)). Finally, we construct the sequence $(\mu_n, Q_n) \in \mathcal{S}^*$ by

$$\mu_n := \frac{\mu|_{V_n}}{\mu(V_n)}, \quad Q_n := \sum_{\{C \in \mathcal{C} : C \subset E_n\}} \hat{Q}(C) \mathbb{1}_C.$$

For $n$ large $\mu(V_n) > 0$ and the definition is well posed. Clearly $(\mu_n, Q_n)$ converges to $(\mu, Q)$ (also considering the strong topology of $L^1_+(E)$). It remains to show (4.13). By construction $\text{div} Q_n = 0$ and $(\mu_n, r) < +\infty$, hence, recalling (4.11),

$$I(\mu_n, Q_n) = \sum_{y \in V_n} \sum_{z \in V : (y, z) \in E} \Phi(Q_n(y, z), Q^\mu_n(y, z)).$$

We claim that $\Phi(Q_n(y, z), Q^\mu_n(y, z)) = 0$ if $(y, z)$ is as in the above sum and $Q^\mu_n(y, z) = 0$. Indeed, since $y \in V_n$ it must be $Q^\mu(y, z) = 0$. Since $I(\mu, Q) < +\infty$ it must then be $Q(y, z) = 0$, and therefore $Q_n(y, z) = 0$, thus leading to our claim.
To treat the general case we proceed as follows. Recall the definition of $\Phi$ given in (2.10). Given $0 \leq q' \leq q$ and $p' \geq p > 0$, let $\alpha, \beta \geq 0$ be respectively defined by $q' = q(1-\alpha)$ and $p' = p(1+\beta)$. Then we have

$$
\Phi(q', p) - \Phi(q, p) = q' \left( \log \frac{q'}{p'} - \log \frac{q}{p} \right) + (q' - q) \log \frac{q}{p} + (q - q') + (p' - p)
$$

$$
\leq (q' - q) \log \frac{q}{p} + (q - q') + (p' - p) = -\alpha \Phi(q, p) + (\alpha + \beta) p \leq (\alpha + \beta) p.
$$

By construction, it holds $\mu_n(y) \geq \mu(y)$ for $y \in V_n$ and $Q_n(y, z) \leq Q(y, z)$ for $(y, z) \in E_n$. Therefore, by letting $\beta_n := [\mu(V_n)]^{-1} - 1$ and $\alpha_n : E_n \to [0, 1]$ be defined by $Q_n(y, z) = Q(y, z) [1 - \alpha_n(y, z)]$ when $(y, z) \in E(Q)$ we obtain

$$
I(\mu_n, Q_n) \leq I(\mu, Q) + \sum_{y \in V_n} \left[ \beta_n + \alpha_n(y, z) \right] \mu(y) r(y, z).
$$

Above we used our previous claim. If $I(\mu, Q) < +\infty$ then it necessarily holds $\langle \mu, r \rangle < +\infty$. We can therefore assume that $\mu(y) r(y, z), (y, z) \in E$, is summable. Since $\beta_n, \alpha_n(y, z) \downarrow 0$ and the maps $\alpha_n(\cdot)$ are uniformly bounded, by dominated convergence we conclude the proof. \hfill \Box

5. Direct proof of Theorem 2.5

In this section we give a direct proof of Theorem 2.5 independent from the LDP for the empirical process. As already mentioned, the proof works only under an additional condition that we assume here: each vertex in $V$ is the extreme of only a finite family of edges in $E$. This assumption implies that, given $\phi \in C_0(V)$, the function $\nabla \phi : E \to \mathbb{R}$ defined as $\nabla \phi(y, z) = \phi(y) - \phi(z)$ belongs to $C_0(E)$. As a consequence, the map

$$
L^1_+(E) \ni Q \to \langle \phi, \text{div } Q \rangle = -\langle \nabla \phi, Q \rangle \in \mathbb{R}
$$

(5.1)
is continuous. Since a linear functional on $L^1_+(E)$ is continuous w.r.t. the bounded weak* topology if and only if it is continuous w.r.t. the weak* topology [22], by definition of weak* topology the map defined in (5.1) is continuous (w.r.t. the bounded weak* topology) if and only if $\nabla \phi \in C_0(E)$. Hence, our additional condition is equivalent to the fact that (5.1) is continuous for any $\phi \in C_0(V)$. An explicit example where (5.1) becomes not continuous for $\phi = 1_{x, x} \in V$ is given in Appendix A.

5.1. Upper bound. Given $\phi \in C_0(V)$ and $F \in C_c(E)$ (i.e. $\phi$ vanishes at infinity and $F$ is nonzero only on a finite set) let $I_{\phi, F} : \mathcal{P}(V) \times L^1_+(E) \to \mathbb{R}$ be the map defined by

$$
I_{\phi, F}(\mu, Q) := \langle \phi, \text{div } Q \rangle + \langle Q, F \rangle - \langle \mu, r_F - r \rangle
$$

(5.2)

where $r_F : V \to (0, +\infty)$ is defined by $r_F(y) = \sum_{z \in V} r(y, z) e^F(y, z)$ and $\langle \phi, \text{div } Q \rangle = \sum_{y \in V} \phi(y) \text{div } Q(y)$.

Lemma 5.1. Fix $x \in V$. For each $\phi \in C_0(V)$, $F \in C_c(E)$, and each measurable $B \subset \mathcal{P}(V) \times L^1_+(E)$, it holds

$$
\limsup_{T \to +\infty} \frac{1}{T} \log \mathbb{P}_x \left( (\mu_T, Q_T) \in B \right) \leq - \inf_{(\mu, Q) \in B} I_{\phi, F}(\mu, Q).
$$

Proof of Lemma 5.1. Fix $x \in V$. For each $\phi \in C_0(V)$, $F \in C_c(E)$, and each measurable $B \subset \mathcal{P}(V) \times L^1_+(E)$, it holds

$$
\limsup_{T \to +\infty} \frac{1}{T} \log \mathbb{P}_x \left( (\mu_T, Q_T) \in B \right) \leq - \inf_{(\mu, Q) \in B} I_{\phi, F}(\mu, Q).
$$

Proof of Lemma 5.1. Fix $x \in V$. For each $\phi \in C_0(V)$, $F \in C_c(E)$, and each measurable $B \subset \mathcal{P}(V) \times L^1_+(E)$, it holds

$$
\limsup_{T \to +\infty} \frac{1}{T} \log \mathbb{P}_x \left( (\mu_T, Q_T) \in B \right) \leq - \inf_{(\mu, Q) \in B} I_{\phi, F}(\mu, Q).
$$

Proof of Lemma 5.1. Fix $x \in V$. For each $\phi \in C_0(V)$, $F \in C_c(E)$, and each measurable $B \subset \mathcal{P}(V) \times L^1_+(E)$, it holds

$$
\limsup_{T \to +\infty} \frac{1}{T} \log \mathbb{P}_x \left( (\mu_T, Q_T) \in B \right) \leq - \inf_{(\mu, Q) \in B} I_{\phi, F}(\mu, Q).
$$
Proof. Fix \( x \in V \) and observe that the following pathwise continuity equation holds \( \mathbb{P}_x \) a.s.

\[
\delta_y(X_T) - \delta_y(X_0) + T \text{div} Q_T(y) = 0 \quad \forall \ y \in V.
\]

(5.3)

Fix \( F \in C_c(E) \) and \( \phi \in C_0(V) \) and recall the semimartingale \( \mathbb{M}^F \) introduced in Lemma 3.1. In view of (5.2) and (5.3), for each \( T > 0 \) and each measurable set \( B \subset \mathcal{P}(V) \times L^1_+(E) \)

\[
\mathbb{P}_x((\mu_T, Q_T) \in B) = \mathbb{E}_x\left( \exp\left\{ -T I_{\phi,F}(\mu_T, Q_T) - [\phi(X_T) - \phi(x)] \right\} \mathbb{M}^F_T \mathbb{1}_B(\mu_T, Q_T) \right)
\]

\[
\leq \sup_{(\mu, Q) \in B} e^{-T I_{\phi,F}(\mu, Q)} \mathbb{E}_x\left( \exp\left\{ -[\phi(X_T) - \phi(x)] \right\} \mathbb{M}^F_T \mathbb{1}_B(\mu_T, Q_T) \right).
\]

Since \( \phi \) is bounded, the proof is now achieved by using Lemma 3.1. \( \square \)

We can conclude the proof of the upper bound in Theorem 2.5. In view of the exponential tightness proven in Subsection 3.2, it is enough to prove (2.12) for compacts. By our additional assumption, we get that the map \( I_{\phi,F} \) is continuous. Fix \( x \in V \). By Lemma 5.1 and the min-max lemma in [25, App. 2, Lemma 3.3] for each compact \( K \subset \mathcal{P}(V) \times L^1_+(E) \) it holds

\[
\lim_{T \to +\infty} \frac{1}{T} \log \mathbb{P}_x((\mu_T, Q_T) \in K) \leq - \inf_{(\mu, Q) \in K} \sup_{\phi \in C_0(V)} I_{\phi,F}(\mu, Q),
\]

where the supremum is carried out over all \( \phi \in C_0(V) \) and \( F \in C_c(E) \). Recalling (2.11), it is now simple to check (see Appendix B) that for each \( (\mu, Q) \in \mathcal{P}(V) \times L^1_+(E) \) it holds

\[
I(\mu, Q) = \sup_{\phi \in C_0(V)} I_{\phi,F}(\mu, Q),
\]

(5.4)

which concludes the proof of the upper bound.

Trivially, the function \( I_{\phi,F} \) can be thought of as the restriction to \( \mathcal{P}(V) \times L^1_+(E) \) of a continuous linear function defined on \( \mathbb{R}^V \times L^1(E) \) where \( \mathbb{R}^V \) has the product topology. This observation and (5.4) imply the convexity and lower semicontinuity of \( I \).

5.2. Lower bound. Recall the following general result concerning the large deviation lower bound.

Lemma 5.2. Let \( \{P_n\} \) be a sequence of probability measures on a completely regular topological space \( X \). Assume that for each \( x \in X \) there exists a sequence of probability measures \( \{\tilde{P}_n^x\} \) weakly convergent to \( \delta_x \) and such that

\[
\lim_{n \to \infty} \frac{1}{n} \text{Ent}(\tilde{P}_n^x | P_n) \leq J(x)
\]

(5.5)

for some \( J : X \to [0, +\infty] \). Then the sequence \( \{P_n\} \) satisfies the large deviation lower bound with rate function given by \( \text{sc}^{-1} J \), the lower semicontinuous envelope of \( J \), i.e.

\[
(\text{sc}^{-1} J)(x) := \sup_{U \in \mathcal{N}_x} \inf_{y \in U} J(y)
\]

where \( \mathcal{N}_x \) denotes the collection of the open neighborhoods of \( x \).
This lemma has been originally proven in [23] Prop. 4.1] in a Polish space setting. The proof given in [30 Prop. 1.2.4] applies also to the present setting of a completely regular topological space.

Recall the definition of the set $\mathcal{S}$ given before Proposition 4.4. $\mathcal{S}$ is given by the elements $(\mu, Q) \in \mathcal{P}(V) \times L^1_+(E)$ with $I(\mu, Q) < +\infty$ and such that the graph $\{(\mu, Q) \in \mathcal{P}(V) \times L^1_+(E) \}$ is finite and connected.

First we prove the entropy bound (5.5) with $J$ given by the restriction of $I$, as defined in (2.11), to $\mathcal{S}$, that is

$$J(\mu, Q) := \begin{cases} I(\mu, Q) & \text{if } (\mu, Q) \in \mathcal{S} \\ +\infty & \text{otherwise.} \end{cases}$$

(5.6)

Then we complete the proof of the lower bound (2.13) by showing that the lower semicontinuous envelope of $J$ coincides with $I$.

**Lemma 5.3.** Fix $x \in V$ and set $P_T := \mathbb{P}_x \circ (\mu_T, Q_T)^{-1}$. For each $(\mu, Q) \in \mathcal{S}$ there exists a sequence $\{P_{T_n}^{(\mu, Q)}\}$ of probability measures on $\mathcal{P}(V) \times L^1_+(E)$ weakly convergent to $\delta_{(\mu, Q)}$ and such that

$$\lim_{T \to +\infty} \frac{1}{T} \mathrm{Ent}(P_{T_n}^{(\mu, Q)}|P_T) \leq I(\mu, Q).$$

*Proof.* First we discuss the case when $x \in K := \text{supp}(\mu)$. We denote by $\tilde{P}_{T}^{(\mu, Q)}$ the distribution of the Markov chain $\tilde{\xi}^x$ on $V$ starting from $x$ and having jump rates

$$\tilde{r}(y, z) := \begin{cases} \frac{Q(y, z)}{\mu(y)} & \text{if } (y, z) \in E(Q) \\ 0 & \text{otherwise.} \end{cases}$$

(5.7)

Observe that this perturbed chain can be thought of as an irreducible chain on the finite state space $K$. Moreover, the condition $\text{div} Q = 0$ implies that $\mu$ is the invariant probability measure.

Set $\tilde{P}_{T}^{(\mu, Q)} := \mathbb{P}_x \circ (\mu_T, Q_T)^{-1}$. The ergodic theorem for finite state Markov chains and the law of large numbers for the empirical flow discussed in Section 2.4 imply that $\{\tilde{P}_{T}^{(\mu, Q)}\}$ converges weakly to $\delta_{(\mu, Q)}$. A straightforward computation of the Radon-Nikodym density (recall (3.1)) yields

$$\frac{1}{T} \mathrm{Ent}(P_{T_n}^{(\mu, Q)}|P_T) \leq \frac{1}{T} \mathrm{Ent}(\tilde{P}_{x}^{(\mu, Q)}|[0, T])$$

$$= \sum_{y \in K, z \in \{y, z\} \in E} \tilde{P}_{x}^{(\mu, Q)}(Q_T(y, z) \log \frac{Q(y, z)}{\mu(y)r(y, z)} - \mu_T(y) \left[ \frac{Q(y, z)}{\mu(y)} - r(y, z) \right])$$

(5.8)

where the subscript $[0, T]$ denotes the restriction to the interval $[0, T]$ (above we used the convention $0 \log 0 := 0$).

Since $T \tilde{P}_{x}^{(\mu, Q)}(Q_T(y, z)) = \tilde{P}_{x}^{(\mu_T, \tilde{r})}(\mu_T(y))$ (adapt (2.10) to the present setting) and since $\mu_T(y) \to \mu(y) \tilde{P}_{x}^{(\mu, Q)}$–a.s. by ergodicity, the r.h.s. of (5.8) converges in the limit $T \to +\infty$ to

$$\sum_{y, z \in K; \{y, z\} \in E} \left( Q(y, z) \log \frac{Q(y, z)}{\mu(y)r(y, z)} + \mu(y)r(y, z) - Q(y, z) \right) + \sum_{y \in K} \mu(y) \sum_{z \in K} r(y, z),$$

that is $I(\mu, Q)$.
When \( x \not\in K \) then there exists an oriented path on \((V, E)\) from \( x \) to \( K \) since \((V, E)\) is connected. In this case the perturbed Markov chain \( \tilde{\xi}^z \) is defined with rates \([\text{5.7}]\) with exception that \( \tilde{r}(y, z) := r(y, z) \) for any \((y, z)\) belonging to the oriented path from \( x \) to \( K \) (fixed once for all). Since after a finite number of jumps that Markov chain reach the component \( K \), it is easy conclude the proof by the same computations as before. \( \square \)

Recall \([\text{2.11}]\) and \([\text{5.6}]\). As already noted, the variational characterization \([\text{5.4}]\) implies that \( I \) is lower semicontinuous and convex on \( \mathcal{P}(V) \times L_+^1(E) \). In particular, the inequality \( sc^{-J} \geq I \) holds. The proof of the equality \( I = sc^{-J} \) is therefore completed by Proposition \([4.3]\)

6. **Projection from the empirical process: proof of Theorems \([2.5]\) \([2.8]\)**

We recall the definition of the empirical process referring to \([17]–(IV); [37]\) for more details. We consider the space \( D(\mathbb{R}; V) \) endowed of the Skorohod topology and write \( X \) for a generic element of \( D(\mathbb{R}; V) \). Given \( X \in D(\mathbb{R}^+; V) \) and \( t > 0 \), the path \( X^t \in D(\mathbb{R}; V) \) is defined by

\[
\begin{cases}
X^t_s := X_s & \text{for } 0 \leq s < t, \\
X_{s+t}^t := X^t_s & \text{for } s \in \mathbb{R}.
\end{cases}
\]

Writing \( \mathcal{M}_S \) for the space of stationary probabilities on \( D(\mathbb{R}; V) \) endowed of the weak topology, given \( X \in D(\mathbb{R}^+; V) \) and \( t > 0 \) we denote by \( \mathcal{R}_{t, X} \) the element in \( \mathcal{M}_S \) such that

\[
\mathcal{R}_{t, X}(A) = \frac{1}{t} \int_0^t \chi_A(\theta_s X^t) \, ds, \quad \forall A \subset D(\mathbb{R}; V) \text{ Borel},
\]

where \((\theta_s X^t)_u := X^t_{s+u}\). Since \( X \rightarrow \mathcal{R}_{t, X} \) is a Borel map from \( D(\mathbb{R}^+; V) \) to \( \mathcal{M}_S \), for each \( x \in V \) it induces a probability measure \( \Gamma_{t,x} \) on \( \mathcal{M}_S \) defined as \( \Gamma_{t,x} := \mathbb{P}_x \circ \mathcal{R}_{t,X}^{-1} \).

The above distribution \( \Gamma_{t,x} \) corresponds to the \( t \)-periodized empirical process.

Let us denote by \( \bar{R} \) the stationary process in \( \mathcal{M}_S \) associated to the Markov chain \( \xi \) and having \( \pi \) as marginal distribution. By the ergodic theorem \([2.2]\), \( \Gamma_{t,x} \) weakly converges to \( \delta_{\bar{R}} \) as \( t \rightarrow +\infty \), for each \( x \in V \). As proven in \([17]–(IV); [37]\), under the Donsker–Varadhan condition, for each \( x \in V \) as \( t \rightarrow +\infty \) the family of probability measures \( \Gamma_{t,x} \) satisfies a LDP with rate \( t \) and rate function the density of relative entropy \( H \) w.r.t. the Markov chain \( \xi \).

We briefly recall the definition of \( H \) and some of its properties, referring to \([17]\)–(IV) for more details. Given \(-\infty \leq s \leq t \leq \infty\), let \( \mathcal{F}_s^t \) be the \( \sigma \)-algebra in \( D(\mathbb{R}; V) \) generated by the functions \((X_t)_{s \leq t \leq t} \). Let \( R \in \mathcal{M}_S \) and \( R_{0, X} \) be the regular conditional probability distribution of \( R \) given \( \mathcal{F}_t^\infty \), evaluated on the path \( X \). Then \( H(R) \in [0, \infty] \) is the only constant such that \( H(t, R) = tH(R) \) for all \( t > 0 \), where

\[
H(t, R) := \mathbb{E}_R \left[ H_{\mathcal{F}_t^0} \left( R_{0, X} \mid \mathbb{P}_{X_0} \right) \right], \quad (6.1)
\]

\( H_{\mathcal{F}_t^0} (R_{0, X} \mid \mathbb{P}_{X_0}) \) being the relative entropy of \( R_{0, X} \) w.r.t. \( \mathbb{P}_{X_0} \) thought of as probability measures on the measure space \( D(\mathbb{R}; V) \) with measurable sets varying in the \( \sigma \)-subalgebra \( \mathcal{F}_t^0 \). The entropy \( H(R) \) can be also characterized as the limit
\[ H(R) = \lim_{t \to \infty} H(t, R)/t, \]

where

\[ \dot{H}(t, R) := \sup_{\varphi \in B(F_T^0)} \left[ E_R(\varphi) - E_R(\log E_{X_0}(e^{\varphi})) \right] \]  

(6.2)

and \( B(F_T^0) \) denotes the family of bounded \( F_T^0 \)-measurable functions on \( D(\mathbb{R}; V) \). Below we will frequently use that

\[ tH(R) = H(t, R) \geq \dot{H}(t, R) = \sup_{\varphi \in Y_1(t)} E_R(\varphi), \]  

(6.3)

where \( Y_1(t) \) is the family of functions \( \varphi \in B(F_T^0) \) such that \( E_x(\varphi) \leq 1 \) for all \( x \in V \) (the last identity is an immediate restatement of (6.2)).

In the following proposition we investigate some key identities concerning the map \( R \to (\hat{\mu}(R), \hat{Q}(R)) \). Recall the definition of \( \hat{Q}(R) \) given before Lemma 2.7.

**Proposition 6.1.** Assume the Markov chain satisfies (A1)–(A4). Then \( \hat{\mu}(\mathcal{R}_T, X) = \mu_T(X) \) and \( \hat{Q}(\mathcal{R}_T, X) = Q_T(X^T) \in L^1(B) \) for \( \mathbb{P}_x \)-a.a. \( X \in D(\mathbb{R}_+; V) \).

**Proof.** The fact that \( \hat{\mu}(\mathcal{R}_T, X) = \mu_T(X) \mathbb{P}_x \)-a.s. has already been observed in (IV). Let us prove that \( \hat{Q}(\mathcal{R}_T, X) = Q_T(X^T) \mathbb{P}_x \)-a.s. It is convenient to introduce the following notation: given \( (y, z) \in B, X \in D(\mathbb{R}_+; V) \) and \( I \subset \mathbb{R}_+ \) we write \( N_I(y, z)(X) \) for the number of jumps along \( (y, z) \) performed by the path \( X \) at some time in \( I \). In addition we write \( N_T(y, z)(X) \) for \( N_{[0, T]}(y, z)(X) \). Equivalently, \( N_T(y, z)(X) = TQ_T(y, z)(X) \). Given \( T > 0 \) we fix a value \( a \in (0, T) \). Then we have

\[ \hat{Q}(y, z)(\mathcal{R}_T, X) = \frac{1}{aT} E_{\mathcal{R}_T, X}(N_a(y, z)) = \frac{1}{aT} \int_0^T N_a(y, z)(\theta_s X^T) \, ds \]

\[ = \frac{1}{aT} \int_0^T N_{[s, s+a]}(y, z)(X^T) \, ds. \]

Let us write \( 0 \leq t_1 < t_2 < \cdots < t_n \leq T \) for the times in \( [0, T] \) at which the path \( X^T \) jumps from \( y \) to \( z \). Note that \( n = N_T(y, z)(X^T) \). We denote by \( \pi_T : \mathbb{R} \to \mathbb{R}/TZ \) the canonical projection of \( \mathbb{R} \) on the circle of length \( T \). It maps bijectively \( [0, T] \) on \( \mathbb{R}/TZ \). Moreover, we define the set \( \Theta_T(y, z)(X^T) := \{ \pi_T(t_1), \pi_T(t_2), \ldots, \pi_T(t_n) \} \).

Since \( T > a \) the number \( N_{[s, s+a]}(y, z)(X^T) \) of jumps from \( y \) to \( z \) made by \( X^T \) in the time interval \([s, s+a]\) coincides with the cardinality of \( \Theta_T(y, z)(X^T) \cap \pi_T([s, s+a]) \).

Hence

\[ \hat{Q}(y, z)(\mathcal{R}_T, X) = \frac{1}{aT} \int_0^T |\Theta_T(y, z)(X^T) \cap \pi_T([s, s+a])| \, ds = \]

\[ \sum_{k=1}^n \frac{1}{aT} \int_0^T 1_{\pi_T(y_k) \in \pi_T([s, s+a])} \, ds = \sum_{k=1}^n \frac{1}{T} = Q_T(y, z)(X^T). \]  

(6.4)

\[ \square \]

Note that, since \( \mathbb{P}_x \)-a.s. time \( T \) is not a jump time, it holds

\[ Q_T(y, z)(X^T) = \begin{cases} Q_T(y, z)(X) + \frac{1}{T} & \text{if } X_{T-} = y, X_0 = z, \quad \mathbb{P}_x \text{-a.s.} \\ Q_T(y, z)(X) & \text{otherwise} \end{cases}. \]  

(6.5)

In what follows, in order to allow a better overview of the proof of Theorems 2.5 and 2.8, we focus on the main steps, postponing some technical details in subsequent
sections. We start with Theorem 2.8 since the product topology on the flow space is simpler.

6.1. Proof of Theorem 2.8 The proof is based on the generalized contraction principle related to the concept of exponential approximation discussed in [15]. To this aim, given \( \varepsilon \in (0,1/2) \), we fix a continuous function \( \varphi_{\varepsilon} : \mathbb{R} \to [0,1] \) such that \( \varphi_{\varepsilon}(x) = 0 \) if \( x \not\in (0,1) \) and \( \varphi_{\varepsilon}(x) = 1 \) if \( x \in [\varepsilon, 1 - \varepsilon] \). For each \((y,z) \in B \) we consider the continuous and bounded function \( F_{y,z}^\varepsilon : D(\mathbb{R}; V) \to \mathbb{R} \) defined as

\[
F_{y,z}^\varepsilon(X) := \left\{ \sum_{s \in [0,1]} \varphi_{\varepsilon}(s) \mathbb{I}(X_{s^-} = y, X_s = z) \right\} \wedge \varepsilon^{-1}.
\]

Then, we define \( \hat{Q}_{\varepsilon} : \mathcal{M}_S \to [0, +\infty]^B \) as \( \hat{Q}_{\varepsilon}(y,z)(R) := E_R(F_{y,z}^\varepsilon) \). Note that \( \hat{Q}_{\varepsilon} \) maps \( \mathcal{M}_S \) into \([0, \varepsilon^{-1}]^B \).

**Proposition 6.2.** Assume the Markov chain satisfies (A1)–(A4). Consider the space \([0, +\infty]^B \) endowed with the product topology and the Borel \( \sigma \)-algebra. Then the following holds:

(i) The map \((\hat{\mu}, \hat{Q}) : \mathcal{M}_S \to \mathcal{P}(V) \times [0, +\infty]^B \) is measurable and the map \( \hat{\mu} : \mathcal{M}_S \to \mathcal{P}(V) \) is continuous.

(ii) The maps \( \hat{Q}_{\varepsilon} : \mathcal{M}_S \to [0, +\infty]^B \), parameterized by \( \varepsilon \in (0,1/2) \), are continuous and satisfy

\[
\lim_{\varepsilon \downarrow 0} \sup_{H(R) \leq \alpha} |\hat{Q}(y,z)(R) - \hat{Q}_{\varepsilon}(y,z)(R)| = 0,
\]

\[
\lim_{\varepsilon \downarrow 0} \lim_{T \uparrow \infty} \frac{1}{T} \log \Gamma_{T,x} \left( |\hat{Q}(y,z) - \hat{Q}_{\varepsilon}(y,z)| > \delta \right) = -\infty,
\]

for any \( x \in V \), \( \alpha > 0 \), \( \delta > 0 \) and any bond \((y,z) \in B \).

As shown below, if \( H(R) < +\infty \) then \( \hat{Q}(R) \in \mathbb{R}_+^E \). In addition \( \hat{Q}_{\varepsilon} \) always assumes finite values. In particular, the quantities appearing in (6.6) and (6.7) are finite and the subtraction is meaningful. We postpone the proof of Proposition 6.2 to Section 7 and conclude the proof of Theorem 2.8.

To prove item (i) up to (2.15) we apply Theorem 4.2.23 in [15]. Identity (6.6) corresponds to formula (4.2.24) there, while identity (6.7) states that the family of probability measures \( \{\Gamma_{T,x} \circ (\hat{\mu}, \hat{Q}_{\varepsilon})^{-1}\} \) is an exponentially good approximation of the family \( \{\Gamma_{T,x} \circ (\mu, \hat{Q})^{-1}\} \). Combining the last observations with the LDP of the empirical process proved in [17]–(IV), one gets the thesis for the family of probability measures \( \{\mathbb{P}_x \circ (\mu_T, \hat{Q}_T)^{-1}\} \) on \( \mathcal{P}(V) \times [0, +\infty]^B \) where \( \hat{Q}_T(X) := Q_T(X^T) \) (use Proposition 6.1). At this point, due to Theorem 4.2.13 in [15], we only need to prove that the families of probability measures \( \{\mathbb{P}_x \circ (\mu_T, Q_T)^{-1}\} \) and \( \{\mathbb{P}_x \circ (\mu_T, \hat{Q}_T)^{-1}\} \) are exponentially equivalent. It is enough to show that for each \( \delta > 0 \) it holds

\[
\lim_{T \to +\infty} \frac{1}{T} \log \mathbb{P}_x(D(\hat{Q}_T, Q_T) > \delta) = -\infty,
\]

where \( D(\cdot, \cdot) \) denotes the metric of \([0, +\infty]^B \) introduced at the beginning of Subsection 2.3. By (6.5) \( Q_T(y,z) = Q_T(y,z) \) with exception of at most one random bond and in this case \( \hat{Q}_T(y,z) = Q_T(y,z) + 1/T \). Since \( |a/(1+a) - (a+\Delta)/(1+a+\Delta)| \leq \Delta \) for \( a, \Delta \geq 0 \), we conclude that \( D(\hat{Q}_T, Q_T) \leq 1/T \), thus allowing to end the proof.
6.2. Proof of (2.10).

6.2.1. Proof of (2.10) for \( Q \not\in \mathbb{R}_+^E \). We distinguish two cases: \( Q(y,z) \in (0, +\infty] \) for some \((y,z) \in B \setminus E\) and \( Q(y,z) = +\infty\) for some \((y,z) \in E\). In the first case we take the function \( \varphi(X) := \lambda \mathbb{I}(Q_T(y,z) > 0) \) for fixed \( \lambda > 0 \). Trivially, \( \varphi \in Y_1(T) \). Hence by (3.3) we get

\[
TH(R) \geq \tilde{H}(T, R) \geq E_R(\varphi) = \lambda R(Q_T(y,z) > 0) .
\]

Since the last probability is positive, the thesis \( H(R) = +\infty \) follows by taking \( \lambda \) arbitrarily large.

Let us now consider the second case. By Remark 2.1 (stochastic domination), it holds \( C := \sup_{x \in V} E_x(e^{Q_T(y,z)}) < +\infty \). Hence, the proof in the second case is similar to the one in the first case taking \( \varphi := Q_T(y,z) \mathbb{I}(Q_T(y,z) \leq \lambda) - \log C \) and \( \lambda > 0 \) arbitrarily large.

6.2.2. Proof that \( I(\mu, Q) \leq \tilde{I}(\mu, Q) \) for \((\mu, Q) \in \mathcal{P}(V) \times L_1(E)\).

**Lemma 6.3.** Given \( R \in \mathcal{M}_S \) with \( \tilde{Q} = \hat{Q}(R) \in \mathbb{R}_+^E \), it holds

\[
\sum_{z : (y,z) \in E} \hat{Q}(y,z) = \sum_{z : (z,y) \in E} \hat{Q}(z,y) .
\]

**Proof.** The thesis follows by considering the \( R \)-expectation of the following identity on \( D([0,T];V) \):

\[
\mathbb{I}(X_T = y) + \sum_{z : (y,z) \in E} TQ_T(y,z) = \mathbb{I}(X_0 = y) + \sum_{z : (z,y) \in E} TQ_T(z,y) . \quad \square
\]

Fix \((\mu, Q) \in \mathcal{P}(V) \times L_1(E)\). By Lemma 6.3 if \( \text{div } Q \not= 0 \) then there is no \( R \in \mathcal{M}_S \) such that \( Q = \hat{Q}(R) \) and therefore \( \tilde{I}(\mu, Q) = +\infty = I(\mu, Q) \). Hence, from now on we can restrict to \( \text{div } Q = 0 \). Fix \( R \in \mathcal{M}_S \) such that \( Q = \hat{Q}(R) \) and \( \mu = \hat{\mu}(R) \) (the absence of such an \( R \) would imply \( \tilde{I}(\mu, Q) = +\infty \) and there would be nothing to prove).

We first consider the case that there is some edge \((y,z) \in E \) with \( Q(y,z) > 0 \) and \( \mu(y) = 0 \). Trivially in this case \( I(\mu, Q) = +\infty \). Let us prove that \( \tilde{I}(\mu, Q) = +\infty \). To this aim, given \( \varepsilon > 0 \), we define the function \( F_\varepsilon : E \to \mathbb{R} \) as \( F_\varepsilon(u,v) = \log \frac{Q(y,z)}{\varepsilon r(y,z)} \mathbb{I}((u,v) = (y,z)) \). Let \( e^{\varphi_\varepsilon} := M^{F_\varepsilon}_T \) be the supermartingale introduced in Lemma 3.1

\[
\varphi_\varepsilon = TQ_T(y,z) \log \frac{Q(y,z)}{\varepsilon r(y,z)} - T\mu_T(y) \left[ \frac{Q(y,z)}{\varepsilon} - r(y,z) \right] . \quad (6.9)
\]

We take \( \varepsilon \) small enough so that \( \log \frac{Q(y,z)}{\varepsilon r(y,z)} > 0 \) and define for \( \ell > 0 \) the new function \( \varphi_{\varepsilon, \ell} \) as in the r.h.s. of (6.9) with \( Q_T(y,z) \) replaced by \( Q_T(y,z) \wedge \ell \). Then \( \varphi_{\varepsilon, \ell} \leq \varphi_\varepsilon \) and by Lemma 3.1 we conclude that \( \varphi_{\varepsilon, \ell} \in Y_1(T) \). Applying (6.3) we conclude that

\[
H(R) \geq E_R(\varphi_{\varepsilon, \ell})/T = E_R(Q_T(y,z) \wedge \ell) \log \frac{Q(y,z)}{\varepsilon r(y,z)} .
\]

Taking first the limit \( \ell \to +\infty \) and afterwards \( \varepsilon \to 0 \), we get that \( H(R) = +\infty \), thus implying \( \tilde{I}(\mu, Q) = +\infty \).
Due to the previous result, we restrict to the case that \( \mu(y) > 0 \) if \( Q(y, z) > 0 \), with \( (y, z) \in E \). Then we fix an invading sequence \( E_n \supset E \) of finite subsets of \( E \) and consider the function \( F_n : E \to \mathbb{R} \) defined as 

\[
  r(y, z) \in F_n(y, z) := \begin{cases} 
  
  \frac{Q(y, z)}{\mu(y)}, & \text{if } (y, z) \in E_n, \\
  r(y, z), & \text{otherwise,}
  
  \end{cases}
\]

with the convention that \( 0/0 = 0 \). Note that the above ratio is well defined since \( \mu(y) > 0 \) if \( Q(y, z) > 0 \). Let \( e^{\varphi_n} := M^n_T \) be the supermartingale introduced in Lemma [3.1]

\[
\varphi_n = T \sum_{(y,z) \in E_n} \left\{ Q_T(y, z) \log \frac{Q(y, z)}{\mu(y)r(y, z)} - \mu_T(y) r(y, z) \left[ \frac{Q(y, z)}{\mu(y)r(y, z)} - 1 \right] \right\}.
\]

Since \( \varphi_n \) is unbounded, for \( \ell > 0 \) we consider the cut-off

\[
\varphi_{n,\ell} := \begin{cases} 
  \varphi_n, & \text{if } |\varphi_n| \leq \ell, \\
  \varphi_n / \ell, & \text{if } |\varphi_n| > \ell.
  
\end{cases}
\]

We stress that the sum in the definition of \( \varphi_n \) is finite. Since \( |\varphi_{n,\ell}| \leq |\varphi_n| \in L^1(R) \) (recall that \( Q = \tilde{Q}(R) \in L^1_+(E) \)), by the Dominated Convergence Theorem it holds \( \lim_{\ell \to +\infty} \mathbb{E}_R(\varphi_{n,\ell}) = \mathbb{E}_R(\varphi_n) \). Moreover, applying Remark [2.1] and using the notation introduced there, by enlarging the probability space we obtain that there exist positive constants \( A_n, B_n \) depending only on \( n \) such that

\[
|\varphi_{n,\ell}(X)| \leq |\varphi_n(X)| \leq A_n \sum_{(y,z) \in E_n} Z_{y,z} + B_n
\]

sampling \( X \) by \( \mathbb{P}_x \). This implies that \( \log \mathbb{E}_x(e^{\varphi_{n,\ell}}) \) is bounded uniformly in \( x \in V \) and therefore, applying twice the Dominated Convergence Theorem we conclude that

\[
\lim_{\ell \to +\infty} \mathbb{E}_R \log \mathbb{E}_x(e^{\varphi_{n,\ell}}) = \lim_{\ell \to +\infty} \sum_{x \in V} \mu(x) \log \mathbb{E}_x(e^{\varphi_{n,\ell}}) = \sum_{x \in V} \mu(x) \log \mathbb{E}_x(e^{\varphi_n}) \leq 0.
\]

Note that the last bound follows from Lemma [3.1]. As a consequence

\[
\lim_{\ell \to +\infty} \{ \mathbb{E}_R(\varphi_{n,\ell}) - \mathbb{E}_R \log \mathbb{E}_x(e^{\varphi_{n,\ell}}) \} \geq \mathbb{E}_R(\varphi_n).
\]

Combining the above estimate, [0.2] and [0.3], we conclude that

\[
H(R) \geq \tilde{H}(T, R)/T \geq \mathbb{E}_R(\varphi_n)/T = \sum_{(y,z) \in E_n} \Phi(Q(y, z), Q'(y, z)). \tag{6.10}
\]

To conclude we take the limit \( n \to +\infty \), obtaining \( H(R) \geq I(\mu, Q) \) for each \( R \in \mathcal{M}_S \) such that \( \tilde{\mu}(R) = \mu, \tilde{Q}(R) = Q \). This implies that \( \tilde{I}(\mu, Q) \geq I(\mu, Q) \).

6.2.3. Proof that \( I(\mu, Q) \geq \tilde{I}(\mu, Q) \) for \( (\mu, Q) \in \mathcal{P}(V) \times L^1_+(E) \). As consequence of the first part of Theorem [2.5] (already proved), the function \( \tilde{I} \) is lower semicontinuous. Consider the sequence \( \{(\mu_n, Q_n)\}_{n \geq 0} \) in \( S \) converging to \( (\mu, Q) \) as stated in Proposition [4.2]. The set \( S \) has been defined in Section [1.3] as the subset of \( \mathcal{P}(V) \times L^1_+(E) \) given by the elements \( (\mu, Q) \) with \( I(\mu, Q) < +\infty \) and such that the graph \( (\supp(\mu), E(Q)) \) is finite and connected. For each \( n \) we consider the continuous time Markov chain \( (\xi^{(n)}_t) \) on \( V \) with jump rates \( r_n(y, z) = Q_n(y, z)/\mu_n(y) \) with the convention \( 0/0 = 0 \). Since \( I(\mu_n, Q_n) < +\infty \) it cannot be \( Q_n(y, z) > 0 \) and \( \mu_n(y) = 0 \), hence the above ratio is well defined. Since \( \mu_n \) and \( Q_n \) have finite
support, the Markov chain \( \xi^{(n)} \) is indeed a Markov chain with finite effective state space. In particular, explosion does not take place. The bound \( I(\mu_n, Q_n) < +\infty \) implies also that \( \text{div} Q_n = 0 \), hence we get that \( \mu_n \) is an invariant measure for \( \xi^{(n)} \). We define \( R_n \) as the stationary Markov chain \( \xi^{(n)} \) with marginal \( \mu_n \), then \( \hat{Q}(R_n) = Q_n \). By the Radon–Nykodim derivative \( \hat{Q}(\cdot) = Q \) and the definition of the entropy \( H(\cdot) \), we get that \( \hat{I}(\mu_n, Q_n) \leq H(R_n) = I(\mu_n, Q_n) \). Invoking the lower semicontinuity of \( \hat{I} \) and Proposition 4.4 we get the thesis.

6.3. Proof of (2.14). Let us take \((\mu, Q)\) with \( \mu \in \mathcal{P}(V) \) and \( Q \in \mathbb{R}_+^E \setminus L^1_+ (E) \). We need to prove that \( \hat{I}(\mu, Q) = +\infty \). Let \( R \in \mathcal{M}_S \) be such that \( \hat{\mu}(R) = \mu \) and \( \hat{Q}(R) = Q \) (we assume \( R \) exists, otherwise the thesis is trivially true). We fix an invading sequence \( V_n \) of finite sets, define \( E_n := \{(y, z) \in E : y, z \in V_n\} \) and \( F_{n}(y, z) := \mathbb{I}((y, z) \in E_n) \) for \( (y, z) \in E \). Then we know that \( \mathbb{E}_x \left( \exp \{ M_{T,\ell}^{F_n} \} \right) \leq 1 \) for all \( x \in V \), using the same notation of Lemma 3.1. Again we need to work with functions in \( \mathcal{B}(\mathcal{F}_T^n) \). To this aim, given \( \ell > 0 \) we define \( M_{T,\ell}^{F_n} \) as the supermartingale \( M_{T,\ell}^{F_n} \) except that the empirical flow \( Q_T(y, z) \) is replaced by \( Q_T(y, z) \wedge \ell \) for all edges \((y, z)\). Then (note that \( rF_n \geq r \)) \( M_{T,\ell}^{F_n} \in \mathcal{B}(\mathcal{F}_T\ell) \) and \( M_{T,\ell}^{F_n} \leq M_{T,\ell}^{F_n} \), thus implying that \( M_{T,\ell}^{F_n} \in \mathcal{F}_T(\ell) \). By (6.3) this implies that

\[
H(R) \geq \hat{H}(T, R) / T \geq \lim_{\ell \to \infty} \mathbb{E}_R \left( \frac{M_{T,\ell}^{F_n}}{T} \right) / T = \sum_{(y, z) \in E_n} Q(y, z) - \mathbb{E}_R (\mu_T (rF_n - r)).
\]

(6.11)

The conclusion then follows from the next result:

Claim 6.4. Assume Condition 2.2 (where the constants \( \sigma, C \) are defined). Then for each \( R \in \mathcal{M}_S \) it holds

\[
\| \hat{Q}(R) \| \leq H(R)(1 + e/\sigma) + C e/\sigma. \tag{6.12}
\]

Proof. Let us first prove (6.12) knowing that \( H(R) \geq \mathbb{E}_R (v(X_0)) \). We come back to (6.11) and take first the limit \( T \to +\infty \) and afterwards the limit \( n \to +\infty \). Since \( 0 \leq rF_n - r \leq e r \) and, by Fubini–Tonelli and stationarity, \( \mathbb{E}_R (\mu_T (r)) = \mathbb{E}_R (r(X_0)) \), we conclude that

\[
\| \hat{Q} \| = \| Q \| = \lim_{n \to +\infty} \sum_{(y, z) \in E_n} Q(y, z) \leq H(R) + e \mathbb{E}_R (r(X_0)).
\]

By Condition 2.2 \( \mathbb{E}_R (r(X_0)) \leq \mathbb{E}_R (v(X_0)) / \sigma + C / \sigma \). Combining with \( H(R) \geq \mathbb{E}_R (v(X_0)) \) we get the thesis.

Let us now prove that \( H(R) \geq \mathbb{E}_R (v(X_0)) \). Since both \( H(R) \) and \( E_R (v(X_0)) \) are affine in \( R \) (see (IV)) and since all stationary processes are convex combinations of ergodic stationary processes, it is enough to prove the claim for an ergodic \( R \in \mathcal{M}_S \). Given \( k, T > 0 \) and \( W \subset V \) we define \( v^{(k)} := v \wedge k \) and \( \varphi(X) := \mathbb{I}(X \in W) \int_0^T v^{(k)}(X_s) ds \). Trivially, \( \varphi \in \mathcal{B}(\mathcal{F}_T) \). Then, by the definition of \( \hat{H}(T, R) \), it holds

\[
\hat{T} H(R) \geq \hat{H}(T, R) \geq \mathbb{E}_R (\varphi) - \mathbb{E}_R (\log \mathbb{E}_{X_0} (e^\varphi)) \geq \mathbb{E}_R \left( \int_0^T v^{(k)}(X_s) ds; X_0 \in W \right) - \max_{x \in W} \log (C_x e). \tag{6.13}
\]
In the last inequality we have used Lemma 3.5 and the inequality $v^{(k)} \leq v$. At this point, we divide \((6.13)\) by $T$. Since $R$ is ergodic, by Birkhoff ergodic theorem (note that $v^{(k)}(X_0) \in L^1(R)$ since $v^{(k)}$ is bounded) we know that

$$
\lim_{T \to \infty} \frac{1}{T} \int_0^T v^{(k)}(X_s) ds = \mathbb{E}_R(v^{(k)}(X_0)), \quad R\text{-a.s.}
$$

Taking the limit $T \to \infty$ and applying the Dominated Convergence Theorem we conclude that

$$
H(R) \geq \mathbb{E}_R(v^{(k)}(X_0)) \mathbb{P}(X_0 \in W).
$$

At this point it is enough to take the limit $k \to \infty$ and afterwards to take $W$ arbitrarily large and invading all $V$. \hfill \Box

### 6.4. Proof of Theorem 2.5

The proof uses the results of [19], where the notion of exponentially good approximation and the contraction principle are extended to the case of completely regular space as image space of the projection. To this aim we recall some further properties of the bounded weak* topology on $L^1_+(E)$.

We define $\mathcal{A}$ as the set of sequences $a = (a_n)_{n \geq 1}$ of functions in $C_0(E)$ such that $\|a_n\|_\infty \to 0$. Given $a \in \mathcal{A}$ we introduce the pseudometric $d_a$ on $L^1_+(E)$ as

$$
d_a(Q, Q') := \sup_{n \geq 1} (Q - Q', a_n).
$$

Writing $B_a(Q, r) := \{Q' \in L^1_+(E) : d_a(Q, Q') < r\}$, the family of sets $\{B_a(Q, r)\}$, with $a \in \mathcal{A}$, $Q \in L^1_+(E)$ and $r > 0$, form a basis for $L^1_+(E)$. This follows from Def. 2.7.1 and Cor. 2.7.4 in [32]. In addition, the family $\mathcal{D}$ of pseudometrics $\{d_a : a \in C_0(E)\}$ is separating, i.e. given $Q \neq Q'$ in $L^1_+(E)$ there exists $a \in \mathcal{A}$ such that $d_a(Q, Q') > 0$. The above two properties (basis and separating family of pseudometrics) make $L^1_+(E)$ a so called gauge space. Indeed, one can prove that the concepts of completely regular space and gauge space are equivalent [18][Ch. IX].

Due to the above observations on the gauge structure of $L^1_+(E)$ we are in the same settings of [19]. In what follows we restrict to the case $|E| = +\infty$, thus implying $|E| = +\infty$ due to the irreducibility of the Markov chain $\xi$ (the finite case is much simpler). Fix an enumeration $(e_n)_{n \geq 1}$ of $E$. Consider the maps $\hat{Q}, \hat{Q}_\varepsilon$ entering in Proposition 6.2 and define the maps $\hat{Q}, \hat{Q}_\varepsilon : \mathcal{M}_S \to L^1_+(E)$ by

$$
\begin{align*}
\hat{Q}(R) &= \begin{cases} 
\hat{Q}(R) & \text{if } \hat{Q}(R) \in L^1_+(E), \\
0 & \text{otherwise},
\end{cases} \\
\hat{Q}_\varepsilon(R)\langle e_n &= \begin{cases} 
\hat{Q}_\varepsilon(R)\langle e_n & \text{if } n \leq \varepsilon^{-1}, \\
0 & \text{otherwise}.
\end{cases}
\end{align*}
$$

### Proposition 6.5

Assume the Markov chain satisfies (A1)–(A4) and Condition [2.2]. Consider the space $L^1_+(E)$ endowed of the bounded weak* topology and the Borel $\sigma$-algebra. Then the following holds:

(i) The map $\hat{Q} : \mathcal{M}_S \to L^1_+(E)$ is measurable while the maps $\hat{Q}_\varepsilon : \mathcal{M}_S \to L^1_+(E)$ are continuous.
(ii) For each $a \in \mathcal{A}$

$$
\lim_{\varepsilon \downarrow 0} \sup_{R \in \mathcal{M}_S : H(R) \leq \alpha} d_\varepsilon (\hat{Q}(R), \hat{Q}_\varepsilon(R)) = 0,
$$

(6.14)

for any $x \in V$, $\alpha > 0$, $\delta > 0$.

The proof is given in Section 8.

As a byproduct of Proposition 6.2, the extended contraction principle in [19], the LDP of the empirical process and Theorem 2.8 (ii) we can conclude the proof of Theorem 2.5. Let us be more precise. We apply Theorem 1.13 in [19]. Formula (6.14) corresponds to formula (1.14) in [19], while formula (6.15) means that the family of probability measures $\{\Gamma_{T,x} \circ (\bar{\mu}, \bar{Q}_\varepsilon)^{-1}\}$ is a $(d_\varepsilon)_{a \in \mathcal{A}}$-exponentially good approximation of the family $\{\Gamma_{T,x} \circ (\bar{\mu}, \bar{Q})^{-1}\}$. On the other hand, we have that $\hat{Q} = \hat{Q} \in L_1^t(E) \Gamma_{T,x}-a.s.$, while by Proposition 6.1 the random variable $\hat{Q}$ sampled according to $\Gamma_{T,x}$ has the same law of $\hat{Q}_T(X) := Q_T(X')$ with $X \in D(\mathbb{R}_+; V)$ sampled according to $\mathbb{P}_x$. Hence, by Corollary 1.10 in [19] we only need to prove that the families of probability measures $\{\mathbb{P}_x \circ (\mu_T, Q_T)^{-1}\}$ and $\{\mathbb{P}_x \circ (\mu_T, \bar{Q}_T)^{-1}\}$ are $(d_\varepsilon)_{a \in \mathcal{A}}$-exponentially equivalent on $\mathcal{P}(V) \times L_1^t(E)$. It is enough to show for each $\varepsilon > 0$ and $a \in \mathcal{A}$ that

$$
\lim_{T \to +\infty} \frac{1}{T} \log \mathbb{P}_x (d_\varepsilon (\hat{Q}_T, Q_T) > \delta) = -\infty,
$$

(6.16)

Since by (6.5) $d_\varepsilon (\hat{Q}_T, Q_T) \leq \|a\|_\infty / T$, we get the thesis.

7. Exponential approximations: Proof of Proposition 6.2

Item (i) is straightforward. We concentrate on item (ii). Since $\mathcal{M}_S$ is endowed with the weak topology and since $F_{y,z}$ is a continuous bounded function on $D(\mathbb{R}; V)$ we conclude that $\hat{Q}_\varepsilon$ is continuous.

7.1. Proof of (6.2). As already proved in the previous section (independently from the content of Proposition 6.2), $\bar{I}(\mu, Q) = +\infty$ if $Q \notin \mathbb{R}_e$. Hence, given $R \in \mathcal{M}_S$ with $H(R) < +\infty$, it must be $\hat{Q}(y, z)(R) < \infty$ for all $(y, z) \in B$ and $\hat{Q}(y, z)(R) = 0$ if $(y, z) \in B \setminus E$. Since $\hat{Q}_\varepsilon(y, z) \leq \hat{Q}(y, z)$, the same claim holds for $\hat{Q}_\varepsilon$ instead of $\hat{Q}$. In particular, equation (6.6) is meaningful and is trivially true for $(y, z) \in B \setminus E$. We then restrict to $(y, z) \in E$. Below $R \in \mathcal{M}_S$ is such that $H(R) \leq \alpha$.

Recall the definition of $N_1(y, z)$ and $N_T(y, z)$ given in the proof of Proposition 6.1. We can estimate

$$
|\hat{Q}(y, z)(R) - \hat{Q}_\varepsilon(y, z)(R)|
\leq \mathbb{E}_R(N_1(y, z); N_1(y, z) \geq \varepsilon^{-1}) + \mathbb{E}_R(N_{[0, \varepsilon]}[1 - \varepsilon, 1](y, z)).
$$

(7.1)

By stationarity (see the proof of Lemma 2.7)

$$
\mathbb{E}_R(N_{[0, \varepsilon]}(y, z)) = \mathbb{E}_R(N_{[1 - \varepsilon, 1]}(y, z)) = \varepsilon \mathbb{E}_R(N_1(y, z)) = \varepsilon \hat{Q}(y, z)(R).
$$

Consider $\ell \in \mathbb{R}_+$ and apply (6.3) with $t = 1$ and $\varphi = N_1(y, z) \wedge \ell - r(y, z)(e - 1)$ (note that $\varphi \in V_1(t)$ by Remark 2.3). We get for $R \in \mathcal{M}_S$ such that $H(R) \leq \alpha$

$$
\alpha + r(y, z)(e - 1) \geq H(R) + r(y, z)(e - 1) \geq \mathbb{E}_R(N_1(y, z) \wedge \ell).
$$

(7.2)
Since by the Monotone Convergence Theorem \( \lim_{\ell \to +\infty} E_R (N_1(y, z) \wedge \ell) = \hat{Q}(y, z)(R) \), taking the limit \( \ell \to +\infty \) on both extreme sides of \( (7.2) \) we deduce
\[
\alpha + r(y, z)(e - 1) \geq \hat{Q}(y, z)(R).
\]
From the above inequality we get that the last term in \( (7.1) \) converges uniformly to zero on \( \{ R \in M_S : H(R) \leq \alpha \} \) as \( \epsilon \downarrow 0 \). To conclude, it remains to prove that \( \lim_{\epsilon \downarrow 0} E_R (N_1(y, z); N_1(y, z) \geq \epsilon^{-1}) = 0 \). To this aim, given \( \gamma, \ell > 0 \) we define on \( D(\mathbb{R}; V) \) the function
\[
\varphi_{\gamma, \ell, \epsilon} := \gamma N_1(y, z) \mathbf{1}(\ell \geq N_1(y, z) \geq \epsilon^{-1}) - C(\gamma, \epsilon)
\]
where \( C(\gamma, \epsilon) := \sup_{x \in V} \log E_x (e^{\gamma N_1(y, z)} \mathbf{1}(N_1(y, z) \geq \epsilon^{-1})) \). Due to Remark 2.1 (stochastic domination), we get \( C(\gamma, \epsilon) < +\infty \) and \( \lim_{\epsilon \downarrow 0} C(\gamma, \epsilon) = 0 \). By construction \( \varphi_{\gamma, \ell, \epsilon} \in Y_1(t) \) for \( t \geq 1 \). Applying \( (6.3) \) we get for \( t \geq 1 \) that
\[
E_R (\varphi_{\gamma, \ell, \epsilon}) \leq \hat{H}(t, R) \leq tH(R) \leq t\alpha.
\]
Taking \( \ell \to \infty \), we conclude that \( E_R (N_1(y, z); N_1(y, z) \geq \epsilon^{-1}) \leq t\alpha / \gamma + C(\gamma, \epsilon) / \gamma \). Taking first the limit \( \epsilon \downarrow 0 \) and afterwards the limit \( \gamma \uparrow \infty \), we conclude that the expectation \( E_R (N_1(y, z); N_1(y, z) \geq \epsilon^{-1}) \) is negligible as \( \epsilon \downarrow 0 \). \( \square \)

7.2. Proof of \( (6.7) \). We restrict to \( T > 1 \) (the generic case could be treated by the same arguments of the proof of Proposition 6.1). Recall the definition of the projection \( \pi_T \) and set \( \Theta_T(y, z)(X^T) \) given there. \( P_x \)-a.s. it holds
\[
\hat{Q}_\epsilon(y, z)(R_{T, X}) = \frac{1}{T} \int_0^T \left\{ \sum_{u \in [s, s + 1]} \varphi_\epsilon(u - s) \right\} \wedge \epsilon^{-1} ds.
\]
For each \((y, z) \in B \) and \( \epsilon > 0 \) we define the functions \( G_\epsilon(y, z) \) and \( H_\epsilon(y, z) \) on \( D(\mathbb{R}; V) \) as
\[
G_\epsilon(y, z)(X) := \frac{1}{T} \int_0^T |\Theta_T(y, z)(X^T) \cap \pi_T([s + \epsilon, s + 1 - \epsilon])| \wedge \epsilon^{-1} ds,
\]
\[
H_\epsilon(y, z)(X) := \frac{1}{T} \int_0^T |\Theta_T(y, z)(X^T) \cap \pi_T([s + \epsilon, s + 1 - \epsilon])| ds.
\]
By the same argument used in identity \( (6.4) \), it holds
\[
H_\epsilon(y, z)(X) = (1 - 2\epsilon)Q_T(y, z)(X^T) = (1 - 2\epsilon)\hat{Q}(y, z)(R_{T, X}).
\]
Trivially, it holds \( \hat{Q}_\epsilon(y, z)(R_{T, X}) \geq \hat{Q}_\epsilon(y, z)(R_{T, X}) \geq G_\epsilon(y, z)(X) \). Using \( (7.4) \) and the last bounds, we can estimate
\[
P_x \left( \hat{Q}_\epsilon(y, z)(R_{T, X}) \right) \geq \hat{Q}_\epsilon(y, z)(R_{T, X}) \geq \delta
\]
\[
\leq P_x \left( \hat{Q}_\epsilon(y, z)(R_{T, X}) - G_\epsilon(y, z) \geq \delta \right)
\]
\[
\leq P_x \left( \hat{Q}_\epsilon(y, z)(R_{T, X}) - H_\epsilon(y, z) \geq \delta / 2 \right) + P_x \left( H_\epsilon(y, z) - G_\epsilon(y, z) \geq \delta / 2 \right)
\]
\[
= P_x \left( 2\epsilon Q_T(y, z)(X^T) \geq \delta / 2 \right) + P_x \left( H_\epsilon(y, z) - G_\epsilon(y, z) \geq \delta / 2 \right).
\]
(7.5)
In order to prove the super-exponential estimate \( (6.7) \) it is enough to prove a super-exponential estimate for both terms in the last line of \( (7.5) \).
Since, by the graphical construction, under $\mathbb{P}_x$ the process $\{TQ_T(y, z)(X)\}_{T \in \mathbb{R}^+}$ is dominated by a Poisson process $\{Z_T\}_{T \in \mathbb{R}^+}$ with parameter $r(y, z)$ we have
\[
\lim_{\varepsilon \downarrow 0} \lim_{T \to +\infty} \frac{1}{T} \log \left( \mathbb{P}_x \left( 2\varepsilon Q_T(y, z)(X^T) \geq \delta/2 \right) \right) \\
\leq \lim_{\varepsilon \downarrow 0} \lim_{T \to +\infty} \frac{1}{T} \log \left( \mathbb{P} \left( 2\varepsilon (Z_T + 1)/T \geq \delta/2 \right) \right) \\
\leq \lim_{\varepsilon \downarrow 0} -\Phi \left( \frac{\delta}{4\varepsilon}, r(y, z) \right) = -\infty.
\]
We used a LDP for the Poisson process (the extra $1/T$ term is irrelevant) and the explicit form of the rate functional.

It remains to bound the last term in (7.6). For simplicity of notation we restrict to $T$ integer (the general case can be treated similarly). We define $\psi_\varepsilon (r) = r \mathbb{1}(r > \varepsilon^{-1})$. Given $j = 0, 1, \ldots, T - 1$ and $s \in [j, j + 1)$ we have
\[
|\Theta_T(y, z)(X^T) \cap \pi_T([s + \varepsilon, s + 1 - \varepsilon])| \\
\quad - |\Theta_T(y, z)(X^T) \cap \pi_T([s + \varepsilon, s + 1 - \varepsilon]\setminus \varepsilon^{-1})| \\
\leq \psi_\varepsilon \left( |\Theta_T(y, z)(X^T) \cap \pi_T([j, j + 2])| \right).
\]
Hence, we can estimate
\[
H_\varepsilon(y, z)(X) - G_\varepsilon(y, z)(X) \leq \frac{1}{T} \sum_{j=0}^{T-1} \psi_\varepsilon \left( |\Theta_T(y, z)(X^T) \cap \pi_T([j, j + 2])| \right). \tag{7.6}
\]
By the graphical construction of Markov chains, under $\mathbb{P}_x$ the set of jump times for a jump from $y$ to $z$ can be identified with a suitable subset of an homogeneous Poisson point process on $\mathbb{R}_+$ with intensity $r(y, z)$. In particular, it is possible to define a probability measure $\mathcal{P}$ on the product space $D(\mathbb{R}_+; V) \times D(\mathbb{R}_+; \mathbb{N})$ such that

(i) the marginal of $\mathcal{P}$ on $D(\mathbb{R}_+; V)$ equals $\mathbb{P}_x$;
(ii) the marginal of $\mathcal{P}$ on $D(\mathbb{R}_+; \mathbb{N})$ is the law of a Poisson process with parameter $r(x, y)$,
(iii) calling $(X_t)_{t \in \mathbb{R}_+}$ and $(Z_t)_{t \in \mathbb{R}_+}$ the generic elements of respectively $D(\mathbb{R}_+; V)$ and $D(\mathbb{R}_+; \mathbb{N})$, it holds $\mathcal{P}$-a.s.
\[
N_{[a, b]}(y, z)(X) \leq Z_b - Z_a, \quad \forall a < b \in \mathbb{R}_+.
\]
Due to the above coupling and since on the interval $[0, T]$ the paths $X$ and $X^T$ can differ at most in $T$, we can estimate $\mathcal{P}$-a.s.
\[
\psi_\varepsilon \left( |\Theta_T(y, z)(X^T) \cap \pi_T([j, j + 2])| \right)
\leq \begin{cases} 
\psi_\varepsilon (Z_{j+2} - Z_j) & \text{if } 0 \leq j \leq T - 2, \\
\psi_\varepsilon ((Z_T - Z_{T-1}) + Z_1 + 1) & \text{if } j = T - 1.
\end{cases} \tag{7.7}
\]
Now we introduce the nondecreasing function $\hat{\psi}_\varepsilon (r) := 2r \mathbb{1}(r > \varepsilon^{-1}/2)$ satisfying the inequality $\hat{\psi}_\varepsilon (a + b) \leq \hat{\psi}_\varepsilon (a) + \hat{\psi}_\varepsilon (b)$. Then (7.6) and (7.7) imply $\mathcal{P}$-a.s. that
\[
H_\varepsilon(y, z)(X) - G_\varepsilon(y, z)(X) \leq \frac{2}{T} \sum_{j=0}^{T-1} \hat{\psi}_\varepsilon (Z_{j+1} - Z_j + 1).
\]
At this point we recall that under $\mathcal{P}$ the random variables $(Z_{j+1} - Z_j)_{0 \leq j \leq T-1}$ are independent Poisson random variables with parameter $r(y, z)$. Hence we can estimate

$$
\lim_{\epsilon \to 0} \lim_{T \to +\infty} \frac{1}{T} \log \left[ \mathbb{P}_x \left( H_\epsilon(y, z) - G_\epsilon(y, z) \geq \delta/2 \right) \right] = \lim_{\epsilon \to 0} \lim_{T \to +\infty} \frac{1}{T} \log \left[ \mathcal{P} \left( H_\epsilon(y, z) - G_\epsilon(y, z) \geq \delta/2 \right) \right] 
$$

$$
\leq \lim_{\epsilon \to 0} \lim_{T \to +\infty} \frac{1}{T} \log \left[ \frac{\mathcal{P}}{2} \left( \frac{1}{T} \sum_{j=0}^{T-1} \hat{\psi}_\epsilon(Z_{j+1} - Z_j + 1) \geq \delta/2 \right) \right] 
$$

$$
\leq \lim_{\epsilon \to 0} -I_\epsilon(\delta/2) = -\infty. \quad (7.8)
$$

In the above chain of inequalities we used Cramer Theorem for the sum of the independent random variables $2\hat{\psi}_\epsilon(Z_{j+1} - Z_j + 1)$ calling $I_\epsilon$ the associated rate function. The divergence in the last line follows by the following argument. Let $\Lambda_\epsilon(\lambda) := \log \mathbb{E} \left( e^{\lambda \hat{\psi}_\epsilon(Z_1 - Z_0)} \right)$. By the Monotone Convergence Theorem $\Lambda_\epsilon(\lambda)$ converges to zero for each $\lambda \in \mathbb{R}$ as $\epsilon$ goes to zero. Since the rate function $I_\epsilon$ is the Legendre transform of $\Lambda_\epsilon$, we get for each fixed $\lambda \in \mathbb{R}$ that

$$
I_\epsilon(\delta/2) \geq \frac{\delta \lambda}{2} - \Lambda_\epsilon(\lambda).
$$

Hence, $\liminf_{\epsilon \to 0} I_\epsilon(\delta/2) \geq \delta \lambda/2$. By the arbitrariness of $\lambda$ we get the thesis.

8. Exponential approximations: Proof of Proposition 6.5

The measurability of $\hat{Q}$ can be checked by straightforward arguments. Let us prove that $\hat{Q}_\epsilon$ is continuous w.r.t. the bounded weak* topology of $L^1_\omega(E)$. As stated in Prop. 6.2 each map $\hat{Q}_\epsilon(y, z) : \mathcal{M}_S \to [0, \epsilon^{-1}]$ is continuous and bounded. In addition it holds $\|\hat{Q}_\epsilon(R)\| \leq \epsilon^{-2}$ for all $R \in \mathcal{M}_S$. The thesis then follows from Corollary 2.7.3 in [32].

8.1. Proof of (6.15). Due to Proposition 6.1 the law of $\hat{Q}$ under $\Gamma_{T, x}$ is the same of the law of $Q_T(X^T)$ under $\mathbb{P}_x$. Moreover, it holds $Q_T(X^T) \in L^1_\omega(E) \mathbb{P}_x$-a.s. In particular, we get that $\hat{Q} = Q_T \Gamma_{T, x}$-a.s. In addition, by Proposition 6.6 we have

$$
\lim_{\ell \uparrow \infty} \lim_{T \uparrow \infty} \frac{1}{T} \log \Gamma_{T, x} \left( \|\hat{Q}\| \geq \ell \right) = -\infty. \quad (8.1)
$$

Due to (8.1) in order to prove (6.15) we only need to show for any $\ell > 0$ that

$$
\lim_{\epsilon \to 0} \lim_{T \uparrow \infty} \frac{1}{T} \log \Gamma_{T, x} \left( d_\alpha(\hat{Q}, \hat{Q}_\epsilon) > \delta, \|\hat{Q}\| \leq \ell \right) = -\infty. \quad (8.2)
$$

Since $\alpha \in \mathcal{A}$, there exits $\tilde{n} \geq 1$ such that $\|a_n\|_\infty \leq \delta/(2\ell)$ for all $n \geq \tilde{n}$. Note that, since $\hat{Q}(y, z)(R) = \hat{Q}_\epsilon(y, z)(R)$, it holds $\|\hat{Q}(R)\| \geq \|\hat{Q}_\epsilon(R)\|$ and $\|\hat{Q}(R)\| \geq \|\hat{Q}(R) - \hat{Q}_\epsilon(R)\|$ for any $R \in \mathcal{M}_S$. Then for any $n \geq \tilde{n}$ we have $|<\hat{Q}(R) - \hat{Q}_\epsilon(R), a_n>| \leq \delta/2$ if $\|\hat{Q}(R)\| \leq \ell$. Therefore, in order to prove (8.2) we only need to show for any $\ell > 0$ that

$$
\lim_{\epsilon \to 0} \lim_{T \uparrow \infty} \frac{1}{T} \log \Gamma_{T, x} \left( \exists n : 1 \leq n \leq \tilde{n} \text{ s.t. } |<\hat{Q} - \hat{Q}_\epsilon, a_n>| > \delta/2, \|\hat{Q}\| \leq \ell \right) = -\infty. \quad (8.3)
$$
Since $a_n \in C_0(E)$ we can find a finite subset $E' \subset E$ such that $|a_n(e)| \leq \delta/4\ell$ for all $n : 1 \leq n \leq \bar{n}$ and $e \in E \setminus E'$. Estimating
\[|<\hat{Q} - \bar{Q}_\varepsilon, a_n>| \leq \sum_{(y, z) \in E'} \left|\left(\hat{Q}(y, z) - \bar{Q}_\varepsilon(y, z)\right) a_n(y, z)\right| + \|\hat{Q} - \bar{Q}_\varepsilon\| \sup_{e \in E \setminus E'} |a_n(e)|,\]
we reduce the proof of (8.3) to the proof of
\[
\lim_{\varepsilon \downarrow 0} \lim_{T \uparrow \infty} \frac{1}{T} \log \Gamma_{T, x}\left(\hat{Q}(y, z) - \bar{Q}_\varepsilon(y, z)\right) > \beta = -\infty, \quad \forall (y, z) \in E, \forall \beta > 0. \tag{8.4}
\]
This follows from (6.7).

8.2. Proof of (6.14). By arguments similar to the ones used in the previous proof, the thesis follows thanks to the bound (6.12) in Claim 6.4 and (6.6).

9. Birth and death processes

Birth and death processes are nearest–neighbor continuous time Markov chains on $\mathbb{Z}_+$ with jump rates $r(k, k + 1) = b_k$ and $r(k + 1, k) = d_{k+1}$, $k \geq 0$. We assume the birth rate $b_k$ and the death rate $d_k$ to be strictly positive. We also assume
\[
Z := \sum_{k=0}^{+\infty} \frac{b_0 b_1 \cdots b_{k-1}}{d_1 d_2 \cdots d_k} < +\infty \tag{9.1}
\]
and
\[
\sum_{k=0}^{+\infty} \frac{d_1 d_2 \cdots d_k}{b_1 b_2 \cdots b_k} = +\infty. \tag{9.2}
\]
Then assumptions (A1)–(A4) holds. Indeed, (A1) and (A3) are trivially satisfied. Due to the presence of a leftmost point (the origin), equation (2.1) reduces to the detailed balance equation and admits normalizable solutions if and only if (9.1) is fulfilled. In particular, one obtains a unique invariant probability given by
\[
\pi(0) = \frac{1}{Z}, \quad \pi(k) = \frac{1}{Z} \frac{b_0 b_1 \cdots b_{k-1}}{d_1 d_2 \cdots d_k} \quad k \geq 1. \tag{9.3}
\]
Having (9.1), condition (9.2) is equivalent to non–explosion (A2) (combine Corollary 3.18 in [13] with (9.2)) and can be rewritten as $\sum_{k=1}^{+\infty} 1/\pi(k) b_k = +\infty$. Note that condition (9.2) is equivalent to recurrence (combine [39] Ex. 1.3.4] with [39] Th. 3.4.1). Under the above assumptions, the logarithmic Sobolev inequality holds if and only if (see Table 1.4 in [12] [Ch. 1])
\[
\sup_{k \geq 1} \frac{\pi(k, +\infty) \log \left(\frac{1}{\pi(k, +\infty)}\right)}{\sum_{j=0}^{k-1} \frac{1}{\pi(j)b_j}} < +\infty. \tag{9.4}
\]

Possible absence of exponential tightness of the empirical measure. We first discuss a case in which the empirical measure fails to be exponentially tight. Consider constant birth and death rates, i.e. $b_k = \beta$ and $d_k = \delta$. Then (9.1) and (9.2) together are equivalent to the condition $\gamma := \beta/\delta \in (0, 1)$. In particular, $\pi$ is geometric with parameter $\gamma$, i.e. $\pi(k) = (1 - \gamma)\gamma^k$. Consider an event in which in the time interval $[0, T]$ there are $O(T)$ jumps (typical behavior) but all the jumps are to the right (atypical behavior). The probability of such an event is “only” exponentially small in $T$ and therefore the empirical measure cannot be exponentially tight. To be more precise, we write $N_T$ for the number of jumps performed in the
time interval \([0, T]\). Since the holding time at site \(k\) is exponential of parameter \(\beta\) if \(k = 0\) and \(\beta + \delta\) if \(k \geq 1\), \(N_T\) stochastically dominates [is stochastically dominated by] a Poisson random variable with mean \(\beta T \left[ (\beta + \delta)T \right]\). Hence, with probability \(1 - o(1)\), \(N_T\) has value in \(I := [\beta T/2, 2(\beta + \delta)T]\). By conditioning on \(N_T\), it is then simple to check that with probability at least \((1 - o(1)) \left[ (\beta/(\beta + \delta)) \right]^{2(\beta + \delta)T - 1}\) the following event \(\mathcal{A}_T\) takes place: the random variable \(N_T\) has value in \(I\) and all the jumps are to the right. Under the above event \(\mathcal{A}_T\), \(\mu_T = \sum_{i=0}^{N_T} \delta_i/T\). Take now a compact set \(K \subset \mathcal{P}(V)\). By Prohorov’s theorem, \(K\) is a tight family of probability measures and therefore, given \(\varepsilon > 0\), there exists a compact (finite) set \(V\) such that \(\mu(K) \leq \varepsilon\) for all \(\mu \in \mathcal{K}\). Taking \(T\) large enough, under the event \(\mathcal{A}_T\) the empirical measure \(\mu_T\) cannot fulfills the above requirement. Hence

\[\mathbb{P}_0(\mu_T \not\in \mathcal{K}) \geq \mathbb{P}_0(\mu_T(K^c) > \varepsilon) \geq \mathbb{P}_0(\mathcal{A}_T) \geq (1 - o(1)) \left[ (\beta/(\beta + \delta)) \right]^{2(\beta + \delta)T - 1}.\]

The above estimate proves that the empirical measure cannot be exponentially tight. In particular neither Condition 2.2 nor 2.3 holds (even with \(\sigma = 0\)).

**Condition 2.3** Assume now

\[
\lim_{k \to \infty} d_k = +\infty, \quad \lim_{k \to \infty} \frac{b_k}{d_k} < 1. \tag{9.5}
\]

Trivially, (9.1) and (9.2) are satisfied. We show that Condition 2.2 holds. As \(u_n\), we pick the constant sequence \(u(k) = A^k, k \in \mathbb{Z}_+\) for some \(A > 1\) to be chosen later. Since \(u_n\) does not depend on \(n\), it is enough to check Condition 2.2. Items (i)–(iv) then hold trivially; moreover setting \(d_0 := 0\) we get

\[v(k) = -\frac{Lu}{u}(k) = d_k \left( 1 - \frac{1}{A} \right) + b_k(1 - A), \quad k \in \mathbb{Z}_+.
\]

Since \(r(k) = b_k + d_k\), for each \(\sigma \in (0, 1)\) we can write \(v(k) = \sigma r(k) + d_k(1 - \sigma - 1/A) - b_k(A - 1 + \sigma)\). By (4.1), choosing \(A\) large items (v) and (vi) hold. Observe that (9.3) is satisfied when \(d_k = k\) and \(b_k = \lambda \in (0, +\infty)\). In this case \(\pi\) is Poisson with parameter \(\lambda\). This implies that \(e^{-\lambda k/k!} \leq \pi(k, +\infty) \leq \lambda^k/k!\) (for the last bound estimate \(\pi(i) \leq e^{-\lambda} \lambda^i/(i - i)!\) for \(i \geq k\)). Using these bounds, by simple computations one can check from (9.3) that the logarithmic Sobolev inequality (4.4) does not hold. This shows there are cases in which Condition 2.2 holds but Condition 2.3 does not.

**Condition 2.3** Let now focus our attention on Condition 2.3. As already mentioned, the validity of the logarithmic Sobolev inequality is equivalent to (4.4) (assuming (4.1) and (9.2)). Similarly for Condition 2.2 the validity of (4.1) implies that the function \(r(\cdot)\) has compact level set. In fact, suppose that there exists an infinite subset \(W \subset \mathbb{N}\) such that \(c := \sup_{k \in W} r(k) < +\infty\). By detailed balance, for each \(k \in W\) we have \(\pi(k - 1)b_{k-1} = d_k \pi(k) \leq c \pi(k)\). Hence, we can bound from below the \(k\)-term in (9.3) by

\[
\pi[k, +\infty) \log \frac{1}{\pi[k, +\infty)} \cdot \frac{1}{\pi(k - 1)b_{k-1}} \geq \frac{1}{c} \log \frac{1}{\pi[k, +\infty)}
\]

which diverges as \(k\) goes to infinity.

We next exhibit a choice in which Condition 2.2 holds. We take \(b_k = (k + 1)\) and \(d_{k+1} = 2b_k\) for \(k \geq 0\). Observe that such rates satisfy (9.4), and therefore (9.1) and (9.2). The invariant probability \(\pi\) is \(\pi(k) = 2^{-k - 1}\). In remains to estimate

\[\sum_{j=0}^{k-1} (\pi(j)b_j)^{-1} = \sum_{j=1}^{k} 2^j/j.\]

Supposing for simplicity \(k\) even, we observe that
bounded from above. We then consider the continuous–time Markov chain
that πξ
ipation w.r.t. the law of the Markov chain
While
\begin{equation}
\text{Condition 2.3 it holds}
\end{equation}
\text{Theorem 2.5 does not hold in the
strong topology of } L^1_\pi(E). \text{ We choose the birth and death rates as } b_k = (k + 1)/2 \text{ and } d_k = k; \text{ in particular } \pi \text{ is geometric with parameter } 1/2. \text{ Since (9.4) holds, Condition 2.3 is satisfied. We shall show that the level sets of } I \text{ in (2.11) are not compact in the strong topology of } L^1_\pi(E). \text{ Set }
\begin{align*}
\mu^n &:= (1 - \frac{1}{n}) \pi + \frac{1}{2n} \left[ \delta_n + \delta_{n+1} \right] \\
Q^n &:= (1 - \frac{1}{n}) Q^\pi + \frac{1}{2} \left[ \delta_{(n,n+1)} + \delta_{(n+1,n)} \right].
\end{align*}
While \{\mu^n\} converges to \pi in \mathcal{P}(\mathbb{Z}_+), observe that \{Q^n\} converges to \pi in the bounded weak* topology of \( L^1_\pi(E) \) but it is not compact in the strong topology of \( L^1_\pi(E) \). Since \( \lim_{n \to \infty} \|Q^n\| = 0 \), it is simple to check that \( \lim_{n \to \infty} I(\mu^n, Q^n) < +\infty \). This implies that the level sets of \( I \) are not compact in the strong topology of \( L^1_\pi(E) \).

Appendix A. Complements to the proof of Proposition 3.7

We explain here how to prove the bound
\begin{equation}
\mathbb{E}_x \left( e^{\lambda T} (\mu_T, v) \right) \leq \lambda \pi(e^v), \quad \forall \lambda \in (0, 1/c_{LS})
\end{equation}
using the theory reported in [25] App. 1, Sec. 7 for countable Markov chains with \( \sup_{x \in V} r(x) < +\infty \). Since the function \( v \) diverges at infinity, it is bounded from below and has finite level sets \( V_n := \{ x \in V : v(x) \leq n \} \). We define \( v_n(x) := v(x) \mathbb{I}_{x \in V_n} \) and set for \( x, y \in V \)
\begin{align*}
r_n(x,y) &:= \begin{cases} 
  r(x,y) & \text{if } x \in V_n \\
  r(x,y)/r(x) & \text{if } x \notin V_n 
\end{cases} \\
\pi_n(x) &:= \begin{cases} 
  \pi(x) & \text{if } x \in V_n \\
  \frac{Z_n r(x)}{2 \pi(x) r(x)} & \text{if } x \notin V_n 
\end{cases}
\end{align*}
where \( Z_n \) is the normalizing constant making \( \pi_n \) a probability measure on \( V \). Due to Condition 2.3 it holds \( (\pi, r) < +\infty \), thus implying that \( Z_n \) is well defined and that \( \lim_{n \to \infty} Z_n = 1 \).

We notice that the function \( r_n : V \to (0, +\infty), \ r_n(x) := \sum_{y \in V} r_n(x,y), \) is bounded from above. We then consider the continuous–time Markov chain \( \xi^{(n)} \) in \( V \) with transition rates \( r_n(\cdot, \cdot) \). Since \( \pi_n(x)r_n(x,y) = \pi(x)r(x,y)/Z_n \) we derive that \( \pi_n \) is the unique invariant distribution of \( \xi^{(n)} \). We denote by \( \mathbb{E}_x^{(n)} \) the expectation w.r.t. the law of the Markov chain \( \xi^{(n)} \) starting at \( x \) and by \( A_n \) the subset of \( D([0,T]; V) \) defined as \( A_n = \{ X : X_t \in V, \forall t \in [0,T] \} \). Then we have
\begin{equation}
\mathbb{E}_x \left( e^{\mu T} (\mu_T, v) \right) = \lim_{n \to \infty} \mathbb{E}_x \left( e^{\mu T} (\mu_T, v_n) \mathbb{I}_{A_n} \right) = \lim_{n \to \infty} \mathbb{E}_x^{(n)} \left( e^{\mu T} (\mu_T, v_n) \mathbb{I}_{A_n} \right) \tag{A.2}
\end{equation}
(the first identity follows from the monotone convergence theorem). Since \( v_n \) and \( r_n \) are bounded function, we can apply [25] App. 1, Lemma 7.2 and get
\begin{equation}
\mathbb{E}_x \left( e^{\mu T} (\mu_T, v_n) \mathbb{I}_{A_n} \right) \leq \mathbb{E}_x^{(n)} \left( e^{\mu T} (\mu_T, v_n) \mathbb{I}_{A_n} \right) \leq \sup_{\mu \in \mathcal{P}(V)} \left\{ -D_{\pi_n}(\sqrt{\mu}) + \lambda \langle \mu, v_n \rangle \right\}. \tag{A.3}
\end{equation}
if $v_n \leq v$ we have $\langle \mu, v_n \rangle \leq \langle \mu, v \rangle$, while from the identity $\pi_n(x)r_n(x, y) = \pi(x)r(x, y)/Z_n$ we get $D_{\pi_n}(\sqrt{\mu}) = D_{\pi}(\sqrt{\mu})/Z_n$. Combining (A.2) and (A.3), we conclude that for any $\varepsilon \in (0, 1)$ it holds

$$\mathbb{E}_x \left( e^{\lambda T (\mu, v)} \right) \leq \sup_{\mu \in \mathcal{P}(V)} \left\{ -D_{\pi}(\sqrt{\mu})(1 - \varepsilon) + \lambda \langle \mu, v_n \rangle \right\}.$$  

The above result, the entropy inequality and the logarithmic Sobolev inequality imply (A.1).

Appendix B. Proof of (5.4)

We call $\tilde{I}(\mu, Q)$ the r.h.s. of (5.4). Trivially it holds $\tilde{I}(\mu, Q) = +\infty = I(\mu, Q)$ if $\operatorname{div} Q \neq 0$. In the sequel we assume $\operatorname{div} Q = 0$. Then, equation (5.4) reads

$$I(\mu, Q) = \sup_{F \in C_c(E)} I_F(\mu, Q)$$

where $I_F(\mu, Q) := \langle Q, F \rangle - \langle \mu, r_F - r \rangle$. If for some $y \in \mathcal{V}$ and $(y, z) \in E$ it holds $\mu(y) = 0$ and $Q(y, z) > 0$, then taking $F = \lambda \delta_{(y,z)}$ with $\lambda \rightarrow +\infty$ we obtain that $I(\mu, Q) = +\infty$. On the other hand

$$I(\mu, Q) \geq \Phi(Q(y, z), Q^\mu(y, z)) = \Phi(Q(y, z), 0) = +\infty.$$  

As a consequence, from now on we can restrict to $(\mu, Q)$ such that $\operatorname{div} Q = 0$ and $Q(y, z) = 0$ for all $(y, z) \in E$ with $\mu(y) = 0$. Calling $E_+ := \{(y, z) \in E : \mu(y) > 0\}$ we get that

$$I_F(\mu, Q) = \sum_{(y, z) \in E_+} \left\{ Q(y, z)F(y, z) - \mu(y)r(y, z)(e^{F(y, z)} - 1) \right\}.$$  

At this point, it is simple to check that, varying $F(y, z)$, the supremum of the above addendum is given by $\Phi(Q(y, z), Q^\mu(y, z))$ and the value of the above addendum for $F(y, z) = 0$ is zero. Hence,

$$\tilde{I}(\mu, Q) = \sum_{(y, z) \in E_+} \Phi(Q(y, z), Q^\mu(y, z)) = \sum_{(y, z) \in E} \Phi(Q(y, z), Q^\mu(y, z)).$$

We now claim that the above expression is $+\infty$ if $\langle \mu, r \rangle = +\infty$, thus concluding the proof. To this aim we observe that for $0 \leq q < p/2$ it holds $\Phi(q, p) \geq p(1 - \log 2)/2$. Indeed, the thesis is trivially true if $q = 0$, while for $q > 0$ we can write $\Phi(q, p) = pf(q/p)$ where $f(x) = x \log x + 1 - x$. Since $f(x)$ is decreasing for $0 < x < 1$, one has $\Phi(q, p) \geq pf(1/2)$ for $0 \leq q < p/2$. Hence, setting $c := 2/(1 - \log 2)$, our claim follows from the bound

$$\langle \mu, r \rangle = \sum_{(y, z) \in E} Q^\mu(y, z) \\
\leq \sum_{(y, z) \in E : \dfrac{Q(y, z)}{Q^\mu(y, z)} < 2 \dfrac{Q^\mu(y, z)}{2}} 2Q^\mu(y, z) + \sum_{(y, z) \in E : \dfrac{Q^\mu(y, z)}{2}} c \Phi(Q(y, z), Q^\mu(y, z)) + 2\|Q\|_1.$$  

Appendix C. An example with discontinuous divergence

Consider the oriented graph $(\mathcal{V}, E)$ where $\mathcal{V} = \mathbb{N} \cup \{v, w\}$ and $E$ is given by the oriented bonds of the form $(v, n)$, $(n, w), (w, v)$ for some $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ we define $Q^{(n)}$ as the flow of unitary flux associated to the cycle $(v, n, w, v)$, i.e. $Q^{(n)} = I_{(v, n)} + I_{(n, w)} + I_{(w, v)}$. We claim that $Q^{(n)}$ converges to $Q := I_{(v, w)}$ in
$L^1_+(E)$ (endowed of the bounded weak* topology). Since $\|Q^{(n)}\| = 3$, the sequence $(Q^{(n)})_{n \in \mathbb{N}}$ is bounded in the strong topology of $L^1_+(E)$. In particular, $Q^{(n)} \to Q$ in the bounded weak* topology if and only if $Q^{(n)} \to Q$ in the weak* topology, and therefore if and only if $\langle \phi, Q^{(n)} \rangle \to \langle \phi, Q \rangle$ for each $\phi \in C_0(E)$. By construction we have

$$\langle \phi, Q^{(n)} \rangle = \phi(v, n) + \phi(n, w) + \phi(w, v) \to \phi(w, v) = \langle \phi, Q \rangle,$$

thus concluding the proof of our claim.

We observe that, despite $\text{div} \, Q^{(n)} = 0$ for all $n \in \mathbb{N}$, it holds $\text{div} \, Q \neq 0$. This example shows that the map $L^1_+(E) \ni Q \mapsto \text{div} \, Q(x) \in \mathbb{R}$, with $x \in V$, is not in general a continuous map.

**Appendix D. Minimizing sequence $R_n$ for $\tilde{I}(\mu, Q)$ with $(\mu, Q) \in \mathcal{P}(V) \times L^1_+(E)$**

In this appendix we come back to the variational problem determining $\tilde{I}(\mu, Q)$ in (2.15) and exhibit a minimizing sequence $R_n \in \mathcal{M}_S$ for it when $Q \in L^1_+(E)$ and $\tilde{I}(\mu, Q) < +\infty$.

It is convenient to fix some notation. Given $(\mu, Q) \in \mathcal{P}(V) \times L^1_+(E)$ with $\text{div} \, Q = 0$ and $Q(y, z) = 0$ if $\mu(y) = 0 \; \forall (y, z) \in E$, we consider the continuous time Markov chain $\xi$ on $V$ with probability rate $\tilde{r}(y, z)$ for a jump from $y$ to $z$ defined as $\tilde{r}(y, z) := Q(y, z)/\mu(y)$, with the convention that $0/0 = 0$. Let us restrict to the case of $Q$ having finite support. Then all communicating classes of $\xi$ are finite and therefore explosion does not appear. Since $\text{div} \, Q = 0$ the probability measure $\mu$ is an invariant distribution for the Markov chain $\xi$ (in the algebraic sense specified by (2.1)). Applying now Theorem 3.5.6 in [30] to each communicating class of $\xi$, we conclude that the Markov chain $\xi$ starting with distribution $\mu$ is stationary. In what follows, we write $\mathbb{P}[\mu, Q]$ for the law on $D(\mathbb{R}; V)$ of the stationary Markov chain $\xi$ with marginal $\mu$. Then $\mathbb{P}[\mu, Q] \in \mathcal{M}_S$, the associated flow $Q$ equals $Q$ while the entropy $H(\mathbb{P}[\mu, Q])$ can be easily computed. Indeed, combining (3.1) with definition (6.1) we conclude that

$$H(\mathbb{P}[\mu, Q]) = \mathbb{E}_{\mathbb{P}[\mu, Q]} \left[ \sum_{(y,z): Q(y,z) > 0} Q(y, z) \log \frac{Q(y, z)}{\mu(y)r(y, z)} - \sum_{y \in V} \mu_T(y)(\tilde{r}(y) - r(y)) \right].$$

Since $Q$ has finite support, the integrand in r.h.s. of the above formula is given by a finite sum and, commuting sum with expectation, we get

$$H(\mathbb{P}[\mu, Q]) = \sum_{(y,z) \in E} \Phi(Q(y, z), Q^n(y, z)),$$

where

$$Q^n(y, z) = \mu(y)r(y, z). \quad (D.1)$$

Since $I(\mu, Q) = \tilde{I}(\mu, Q) < +\infty$, we know that it must be $\langle \mu, r \rangle < +\infty$, div $Q = 0$ and that $Q(y, z) = 0$ if $\mu(y) = 0$. If $Q$ has finite support, due to (D.1) we conclude that $H(\mathbb{P}[\mu, Q]) = I(\mu, Q)$, thus proving our claim. In order to treat the general case we observe that by Lemma 1.3 we can write $Q = \sum_{C \in \mathcal{C}} \tilde{Q}(C) \mathbb{I}_C$. We fix an invading sequence $V_n \nearrow V$ of vertex sets such that, setting $E_n := \{(x, y) \in E : x, y \in V_n\}$ and $Q_n := \sum_{C \in \mathcal{C} : C \subseteq E_n} \tilde{Q}(C) \mathbb{I}_C$, it holds $\|Q - Q_n\| \leq 2^{-n}\|Q\|$. It is trivial to check that there exists a partition $(J^{(k)}_n)_{k \geq 1}$ of the set $\{C \in \mathcal{C} : C \subseteq E_n\}$ such that

(i) $J^{(k)}_n$ is a finite set for all $k \geq 1$;
(ii) Setting \( Q_n^{(k)} := \sum_{C \in J_n^{(k)}} \hat{Q}(C) \mathbb{1}_C \) it holds \( \|Q_n^{(k)}\| \leq 2^{-k+1} \|Q - Q_n\| \).

Note that, due to the above conditions, it holds \( \|Q_n^{(k)}\| \leq 2^{-k-n+1} \|Q\| \).

Setting \( \gamma_n := 1 - 2^{-n} \) and \( \alpha_n^{(k)} := 2^{-n-k} \), the convex combination

\[
\mathbb{R}_n := \gamma_n \mathbb{P}[\mu, Q_n/\gamma_n] + \sum_{k=1}^{\infty} \alpha_n^{(k)} \mathbb{P}[\mu, Q_n^{(k)}/\alpha_n^{(k)}]
\]

is a stationary process in \( \mathcal{M}_S \) such that \( \hat{Q}(\mathbb{R}_n) = Q, \hat{\mu}(\mathbb{R}_n) = \mu \). At this point, it remains to prove the following claim:

**Claim D.1.** It holds \( I(\mu, Q) \geq \lim_{n \to \infty} H(\mathbb{R}_n) \).

**Proof.** By the convexity of \( H \) proved in [17]–(IV) we can bound

\[
H(\mathbb{R}_n) \leq \gamma_n H(\mathbb{P}[\mu, Q_n/\gamma_n]) + \sum_{k=1}^{\infty} \alpha_n^{(k)} H(\mathbb{P}[\mu, Q_n^{(k)}/\alpha_n^{(k)}])
\]

To compute the above entropies we use formula (D.1), getting

\[
H(\mathbb{P}[\mu, Q_n/\gamma_n]) = \sum_{(y,z) \in E} \Phi(Q_n(y,z)/\gamma_n, Q^\mu(y,z)), \quad (D.2)
\]

\[
H(\mathbb{P}[\mu, Q_n^{(k)}/\alpha_n^{(k)}) = \sum_{(y,z) \in E} \Phi(Q_n^{(k)}(y,z)/\alpha_n^{(k)}, Q^\mu(y,z)). \quad (D.3)
\]

In order to study the contribution of (D.2) take \( \gamma \in (0, 1), p > 0 \) and \( 0 \leq q \leq q' \). Then (with the convention that \( 0 \log 0 = 0 \)) we get

\[
0 \leq \gamma \Phi(q, p) \leq q \log \frac{q}{p} + q \log \frac{1}{\gamma} + \gamma p \leq q \log \frac{q'}{p} + q \log \frac{1}{\gamma} + \gamma p
\]

\[
\leq q' \log \frac{q'}{p} \mathbb{1}(q' > p) + q \log \frac{1}{\gamma} + \gamma p
\]

\[
= [\Phi(q', p) + q' - p] \mathbb{1}(q' > p) + q \log \frac{1}{\gamma} + \gamma p \leq \Phi(q', p) + q' + q' \log \frac{1}{\gamma} + \gamma p.
\]

Applying the above bound to the case \( q = Q_n(y,z), q' = Q(y,z), p = Q^\mu(y,z) \) and \( \gamma = \gamma_n \), by the conditions \( I(\mu, Q) < +\infty, \|Q\| < +\infty, \|Q^\mu\| < +\infty \) and the dominated convergence theorem we conclude that the r.h.s. of (D.2) converges to \( I(\mu, Q) = \sum_{(y,z) \in E} \Phi(Q_n(y,z), Q^\mu(y,z)). \)
In order to study the contribution of \( \text{(D.3)} \) we observe that
\[
\sum_{k=1}^{\infty} \alpha_n^{(k)} H \left( P[\mu, Q_n^{(k)}/\alpha_n^{(k)}] \right)
= \sum_{k=1}^{\infty} \sum_{(y,z) \in E} Q_n^{(k)}(y,z) \left[ \log \frac{Q_n^{(k)}(y,z)}{Q^\mu(y,z)} + \log(1/\alpha_n^{(k)}) \right] - ||Q - Q_n|| + (1 - \gamma_n)||Q^\mu||
\leq \sum_{(y,z) \in E} (Q - Q_n)(y,z) \log \frac{Q(y,z)}{Q^\mu(y,z)} + \sum_{k=1}^{\infty} ||Q_n^{(k)}|| \log(1/\alpha_n^{(k)}) + (1 - \gamma_n)||Q^\mu||
\leq \sum_{(y,z) \in E; Q(y,z) > Q^\mu(y,z)} \sum_{k=1}^{\infty} (Q - Q_n)(y,z) \log \frac{Q(y,z)}{Q^\mu(y,z)} + \sum_{k=1}^{\infty} ||Q_n^{(k)}|| \log(1/\alpha_n^{(k)}) + (1 - \gamma_n)||Q^\mu||.
\]
\( \text{(D.4)} \)

Let us analyze the last member. Since \( ||Q_n^{(k)}|| \leq 2^{k-n+1}||Q|| \) and since \( \gamma_n \to 1 \) the last two terms in the last member goes to zero. In remains to control the first sum in the last member. Since for \( (y,z) \in E \) such that \( Q(y,z) > Q^\mu(y,z) \) it holds
\[
0 \leq (Q - Q_n)(y,z) \log \frac{Q(y,z)}{Q^\mu(y,z)} \leq \Phi(Q(y,z), Q^\mu(y,z)) + Q(y,z)
\]
and the above r.h.s. is summable as \( (y,z) \) varies in \( E \) (recall that \( I(\mu, Q) < +\infty \)), by the dominated convergence theorem we conclude that the first sum in the last member of \( \text{(D.4)} \) is zero as \( n \to \infty \). Coming back to \( \text{(D.4)} \) we have
\[
\limsup_{n \to \infty} \sum_{k=1}^{\infty} \alpha_n^{(k)} H \left( P[\mu, Q_n^{(k)}/\alpha_n^{(k)}] \right) \leq 0.
\]
\( \square \)

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