A Proper Extension of Noether’s Symmetry Theorem for Nonsmooth Extremals of the Calculus of Variations*

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Abstract

For nonsmooth Euler-Lagrange extremals, Noether’s conservation laws cease to be valid. We show that Emmy Noether’s theorem of the calculus of variations is still valid in the wider class of Lipschitz functions, as long as one restrict the Euler-Lagrange extremals to those which satisfy the DuBois-Reymond necessary condition. In the smooth case all Euler-Lagrange extremals are DuBois-Reymond extremals, and the result gives a proper extension of the classical Noether’s theorem. This is in contrast with the recent developments of Noether’s symmetry theorems to the optimal control setting, which give rise to non-proper extensions when specified for the problems of the calculus of variations.

Keywords: calculus of variations, Euler-Lagrange extremals, DuBois-Reymond extremals, Noether’s theorem, Lipschitz admissible functions.

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1 Introduction

Let $L(t, x, v)$ be a given $C^1 ([a, b] \times \mathbb{R}^n \times \mathbb{R}^n; \mathbb{R})$ function (the Lagrangian). The fundamental problem of the calculus of variations consists to minimize the integral functional

$$J [x(.)] = \int_a^b L \left( t, x(t), \dot{x}(t) \right) dt \quad (1)$$

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over a certain class $\mathcal{X}$ of functions $x : [a, b] \to \mathbb{R}^n$ satisfying the boundary conditions $x(a) = \alpha, x(b) = \beta$. The problem is usually solved with the help of the famous Euler-Lagrange equation,

$$\frac{d}{dt} \frac{\partial L}{\partial v} (t, x(t), \dot{x}(t)) = \frac{\partial L}{\partial x} (t, x(t), \dot{x}(t)) ,$$

which is a first-order necessary optimality condition. Each solution of (2) is called an Euler-Lagrange extremal. Condition (2) is obtained in most textbooks from the assumption that minimizers are smooth, or assuming they are piecewise smooth functions. In this last situation the Euler-Lagrange equations are interpreted as holding everywhere except possibly at finitely many points.

In 1918 Emmy Noether [8, 9] established a general theorem asserting that the invariance of the integral functional (1) under a group of transformations depending smoothly on a parameter $s$, implies the existence of a conserved quantity along the Euler-Lagrange extremals. As corollaries, all the conservation laws known to classical mechanics are easily obtained. For a survey of Noether’s theorem and its generalizations see [10]. Noether’s theorem, as is found in the many literature of physics, calculus of variations and optimal control, is formulated with $X$ being smooth. A typical example is $x(\cdot) \in \mathcal{X} = C^2$ (cf. e.g. [1, 4, 6]).

Given that the Euler-Lagrange equation (2) makes sense when $x(\cdot)$ has merely essentially bounded derivative – the biggest class $\mathcal{X}$ for which (2) is still valid is the class Lip of Lipschitz functions (cf. e.g. [2, §2.2]) – it is expected that the conclusion of Noether’s theorem can still be defended in the wider class of Lipschitz functions. This is indeed the case, as it follows from the Pontryagin maximum principle and the results in [11, 12]. As far as for the fundamental problem of the calculus of variations the Pontryagin maximum principle reduces to the Euler-Lagrange necessary condition (2) and to the Weierstrass necessary condition

$$L (t, x(t), v) - L (t, x(t), \dot{x}(t)) \geq \frac{\partial L}{\partial v} (t, x(t), \dot{x}(t)) \cdot (v - \dot{x}(t)) \quad \forall v \in \mathbb{R}^n ,$$

which are distinct necessary conditions even in the $C^2$-smooth case, this does not give a proper extension of Noether’s theorem to the class of Lipschitz functions (the generalization does not reduce to the classical formulation when $\mathcal{X} = C^2$, since we are restricting the set of Euler-Lagrange extremals to those which satisfy Weierstrass’s necessary optimality condition (3)). In the present article we show that to formulate Noether’s theorem for admissible Lipschitz functions, one does not need to restrict the set of Euler-Lagrange extremals to Pontryagin extremals, being enough the restriction to those Euler-Lagrange extremals

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1In this situation equations (2) are interpreted in the almost everywhere sense.
satisfying the DuBois-Reymond condition:

\[
\frac{\partial L}{\partial t} (t, x(t), \dot{x}(t)) = \frac{d}{dt} \left\{ L(t, x(t), \dot{x}(t)) - \frac{\partial L}{\partial v} (t, x(t), \dot{x}(t)) \right\} \cdot \dot{x}(t) .
\] (4)

We remark that the DuBois-Reymond first-order necessary optimality condition (4) is valid when \( \mathcal{X} \) is the class of Lipschitz functions, and that (4) is a consequence of the Euler-Lagrange and Weierstrass conditions. For \( \mathcal{X} = C^2 \), (4) follows from the Euler-Lagrange equation (2) alone (every Euler-Lagrange \( C^2 \)-extremal is a DuBois-Reymond \( C^2 \)-extremal), and therefore our generalization of Noether’s theorem to the class of Lipschitz functions (cf. 4) gives a proper extension of the smooth result.

2 Review of Noether’s Symmetry Theorem

The universal principle described by Noether, asserts that the invariance of a problem with respect to a one-parameter group of transformations implies the existence of a conserved quantity along the smooth Euler-Lagrange extremals.

Definition 1 (cf. [6]). If \( C^2 \ni h^s(t, x) = (h^s_t(t), h^s_x(x)) : [a, b] \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n \), \( s \in (-\varepsilon, \varepsilon) \); \( h^0(t, x) = (t, x) \) for all \( (t, x) \in [a, b] \times \mathbb{R}^n \); and

\[
\int_{h^0_t(a)}^{h^s_t(b)} L \left( t^s, h^s_x (x(t^s)) , \frac{d}{dt} h^s_x (x(t^s)) \right) \, dt^s = \int_a^b L (t, x(t), \dot{x}(t)) \, dt ,
\] (5)

for \( t^s = h^s_t(t) \), all \( s \in (-\varepsilon, \varepsilon) \), and all \( x(\cdot) \in C^2 ([a, b]; \mathbb{R}^n) \); then the fundamental problem of the calculus of variations is said to be invariant under \( h^s \).

Theorem 1 (Classical Noether’s Theorem). If the fundamental problem of the calculus of variations is invariant under \( h^s \), in the sense of Definition 1 then

\[
\left. \frac{\partial L}{\partial v} (t, x(t), \dot{x}(t)) \cdot \frac{\partial}{\partial s} h^s_x(x(t)) \right|_{s=0} + \left[ L (t, x(t), \dot{x}(t)) - \frac{\partial L}{\partial v} (t, x(t), \dot{x}(t)) \cdot \dot{x}(t) \right] \frac{\partial}{\partial s} h^s_t(t) \right|_{s=0} \tag{6}
\]

is constant in \( t \in [a, b] \) along every Euler-Lagrange \( C^2 \)-extremal.

The following result for nonsmooth extremals, is a trivial corollary from the optimal control results in [11] [12].

In the autonomous case, when the Lagrangian \( L \) does not depend on the time variable \( t \) \( (L = L(x, v)) \), the DuBois-Reymond condition is known as the second Erdmann equation. For a survey of the classical optimality conditions and related matters, we refer the reader e.g. to [5] Ch. 2].

3The opposite is not in general true. For an example of a DuBois-Reymond \( C^2 \)-extremal which is not an Euler-Lagrange extremal see [11] p. 24].
Theorem 2. If the fundamental problem of the calculus of variations is invariant under \( h_s \), in the sense that Definition 1 holds with (5) satisfied for all \( x(\cdot) \in \text{Lip}([a, b]; \mathbb{R}^n) \), then (6) is constant in \( t \) along every \( x(\cdot) \in \text{Lip}([a, b]; \mathbb{R}^n) \) satisfying simultaneously the Euler-Lagrange (2) and Weierstrass (3) necessary conditions.

Theorem 2 restrict the conclusion of Noether’s theorem to Pontryagin extremals. The following question comes immediately to mind: Is it really necessary to restrict the set of nonsmooth Euler-Lagrange extremals in order to guarantee that (6) is conserved? In Section 3 we show that a restriction is indeed necessary: we provide an example of a Lipschitz Euler-Lagrange extremal which is not a Weierstrass extremal, and which fails to preserve (6).

While Pontryagin extremals are a natural choice in optimal control, in the context of the calculus of variations such restriction seems to be unnatural: Theorem 2 does not simplify to Theorem 1 in the \( C^2 \) smooth case (Euler-Lagrange equation differs from Weierstrass’s necessary condition in the \( C^2 \) smooth case). This means that Theorem 2 does not give a proper extension of the classical Noether’s theorem. In Section 4 we give a proper restriction of the set of nonsmooth Euler-Lagrange extremals for which Noether’s theorem can still be asserted (Theorem 3).

3 Nonsmooth Euler-Lagrange Extremals may fail to satisfy Noether’s Conserved Quantities

Let us consider the fundamental problem of the calculus of variations with the Lagrangian given by \( L(v) = (v^2 - 1)^2 \): to minimize the functional

\[
J [x(\cdot)] = \int_0^1 \left\{ [\dot{x}(t)]^2 - 1 \right\}^2 \, dt
\]

over the class \( \mathcal{X} = \text{Lip} \) of Lipschitz functions \( x(\cdot) \) on the interval \([0, 1]\) satisfying \( x(0) = x(1) = 0 \). From the Euler-Lagrange equation (2) one obtains that any solution of this problem must satisfy

\[
\left\{ [\dot{x}(t)]^2 - 1 \right\} \dot{x}(t) = \text{const} \quad \text{a.e. on } [0, 1].
\]

As far as the problem is time-invariant (one can choose \( h_s(t) = t + s, \ h_s^e(x) = x \) in Definition 1), Noether’s conserved quantity (6) coincides with the DuBois-Reymond condition (4):

\[
\left\{ [\dot{x}(t)]^2 - 1 \right\}^2 - 4 \left\{ [\dot{x}(t)]^2 - 1 \right\} [\dot{x}(t)]^2 = \text{const} \quad \text{a.e. on } [0, 1],
\]

that is,

\[
\left\{ [\dot{x}(t)]^2 - 1 \right\} \left\{ 1 + 3 [\dot{x}(t)]^2 \right\} = \text{const} \quad \text{a.e. on } [0, 1].
\]
As is easily seen, there exist Lipschitz solutions of (7) which are not solutions of (8) (there are Lipschitz Euler-Lagrange extremals which are not DuBois-Reymond extremals, and which not preserve the quantity (6) of Noether’s theorem). In fact, any Lipschitz function \( x(\cdot) \) satisfying \( \dot{x}(t) \in \{-1, 0, 1\}, t \in [0, 1] \), is an Euler-Lagrange extremal. Among them, only \( \dot{x}(t) \equiv 0 \) and those with \( \dot{x}(t) \pm 1 \) satisfy (8).

### 4 Main Result

We formulate our Noether theorem for nonsmooth extremals under a more general notion of invariance than the one in Definition 1. We require the symmetry transformation to leave the problem invariant up to first order terms in the parameter, and up to exact differentials. \(^4\)

**Definition 2.** The integral functional (7) is quasi-invariant under a one-parameter group of \( C^1 \)-transformations \( (t, x) \rightarrow (T(t, x, \dot{x}, s), X(t, x, \dot{x}, s)) \), \( |s| < \varepsilon \), up to the gauge-term \( \Phi(t, x, \dot{x}) \) if, and only if,

\[
\frac{d}{dt} \Phi(t, x(t), \dot{x}(t)) = \frac{d}{ds} \left\{ L \left( T(t, x(t), \dot{x}(t), s), X(t, x(t), \dot{x}(t), s), \frac{dT(t, x(t), \dot{x}(t), s)}{dt}, \frac{dX(t, x(t), \dot{x}(t), s)}{dt} \right) \right\}_{s=0}
\]

for all \( x(\cdot) \in \text{Lip}([a, b]; \mathbb{R}^n) \).

**Remark 1.** As in the classical context, we are assuming that the parameter transformations \( (t, x) \rightarrow (T(t, x, \dot{x}, s), X(t, x, \dot{x}, s)) \) reduce to the identity for \( s = 0 \), that is,

\[
T(t, x, v, 0) = t, \quad X(t, x, v, 0) = x,
\]

for any choice of \( t, x, \) and \( v \).

**Remark 2.** It is obvious that the invariance notion used in connection with Theorem 2 implies the quasi-invariance up to a gauge-term in Definition 2.

**Remark 3.** In the 1918 original paper of Emmy Noether \[8, 9\], Noether explains that the derivatives of the trajectories \( x \) may also occur in the parameter group of transformations. This possibility has been widely forgotten in the literature of the calculus of variations, the only exception seeming to be the textbook of I. M. Gelfand and S. V. Fomin \[4\]. Such possibility is, however, very interesting from the point of view of optimal control (cf. \[11, 12\]) and is included in Definition 2. An example with relevance in Physics, showing that the dependence of the invariance transformations on the derivatives can be crucial in order to obtain a conservation law, can be found in \[7\].

\(^4\)The exact differentials are called gauge-terms in the literature (cf. \[10\]).
Theorem 3 (Noether’s theorem for Lipschitz functions). If $L$ is quasi-invariant under the one-parameter group of time-space transformations

$$(t, x) \longrightarrow (T(t, x, \dot{x}, s), X(t, x, \dot{x}, s))$$

up to the gauge-term $\Phi (t, x, \dot{x})$, then

$$
\left. \left[ L \left( t, x(t), \dot{x}(t) \right) - \frac{\partial L}{\partial v} \left( t, x(t), \dot{x}(t) \right) \cdot \dot{x}(t) \right] \frac{\partial T}{\partial s} \left( t, x(t), \dot{x}(t), s \right) \right|_{s=0} + \left. \frac{\partial L}{\partial v} \left( t, x(t), \dot{x}(t) \right) \cdot \frac{\partial X}{\partial s} \left( t, x(t), \dot{x}(t), s \right) \right|_{s=0} - \Phi(t, x(t), \dot{x}(t))
$$

is constant in $t \in [a, b]$ along any $x(\cdot) \in \text{Lip}([a, b]; \mathbb{R}^n)$ satisfying (2) and (4) (along any Lipschitz Euler-Lagrange extremal which is also a Lipschitz DuBois-Reymond extremal).

Proof. Having in mind that for $s = 0$ we have the identity transformation, $T(t, x, \dot{x}, 0) = t$, $X(t, x, \dot{x}, 0) = x$, condition (3) yields

$$
\frac{d}{dt} \Phi(t, x(t), \dot{x}(t)) = \frac{\partial L}{\partial t} \left( t, x(t), \dot{x}(t) \right) \frac{\partial T}{\partial s} \left( t, x(t), \dot{x}(t), s \right) \bigg|_{s=0} + \left. \frac{\partial L}{\partial x} \left( t, x(t), \dot{x}(t) \right) \cdot \frac{\partial X}{\partial s} \left( t, x(t), \dot{x}(t), s \right) \right|_{s=0} + \left. \frac{\partial L}{\partial v} \left( t, x(t), \dot{x}(t) \right) \cdot \left( \frac{d}{dt} \frac{\partial X}{\partial s} \left( t, x(t), \dot{x}(t), s \right) \right) \right|_{s=0} - \dot{x}(t) \cdot \frac{d}{dt} T \left( t, x(t), \dot{x}(t), s \right) \bigg|_{s=0} + L \left( t, x(t), \dot{x}(t) \right) \frac{d}{dt} \frac{\partial T}{\partial s} \left( t, x(t), \dot{x}(t), s \right) \bigg|_{s=0}. \tag{10}
$$

From (2) one can write

$$
\frac{\partial L}{\partial x} \left( t, x(t), \dot{x}(t) \right) \cdot \frac{\partial X}{\partial s} \left( t, x(t), \dot{x}(t), s \right) \bigg|_{s=0} + \frac{\partial L}{\partial v} \left( t, x(t), \dot{x}(t) \right) \cdot \frac{d}{dt} \frac{\partial X}{\partial s} \left( t, x(t), \dot{x}(t), s \right) \bigg|_{s=0} = \frac{d}{dt} \left( \frac{\partial L}{\partial v} \left( t, x(t), \dot{x}(t) \right) \cdot \frac{\partial X}{\partial s} \left( t, x(t), \dot{x}(t), s \right) \right) \bigg|_{s=0}. \tag{11}
$$
while from (4) one gets

\[ \frac{\partial L}{\partial t}(t, x(t), \dot{x}(t)) \cdot \frac{\partial}{\partial s} T(t, x(t), \dot{x}(t), s) \bigg|_{s=0} + L(t, x(t), \dot{x}(t)) \frac{d}{dt} \frac{\partial}{\partial s} T(t, x(t), \dot{x}(t), s) \bigg|_{s=0} - \frac{\partial L}{\partial v}(t, x(t), \dot{x}(t)) \cdot \dot{x}(t) \frac{d}{dt} \frac{\partial}{\partial s} T(t, x(t), \dot{x}(t), s) \bigg|_{s=0} \]

\[ = \frac{d}{dt} \left\{ \left( L(t, x(t), \dot{x}(t)) - \frac{\partial L}{\partial v}(t, x(t), \dot{x}(t)) \cdot \dot{x}(t) \right) \frac{\partial}{\partial s} T(t, x(t), \dot{x}(t), s) \bigg|_{s=0} \right\} . \] (12)

Substituting (11) and (12) into (10),

\[ \frac{d}{dt} \left\{ \frac{\partial L}{\partial v}(t, x(t), \dot{x}(t)) \cdot \frac{\partial}{\partial s} X(t, x(t), \dot{x}(t), s) \bigg|_{s=0} - \Phi(t, x(t), \dot{x}(t)) + \left( L(t, x(t), \dot{x}(t)) - \frac{\partial L}{\partial v}(t, x(t), \dot{x}(t)) \cdot \dot{x}(t) \right) \frac{\partial}{\partial s} T(t, x(t), \dot{x}(t), s) \bigg|_{s=0} \right\} = 0 , \]

and the pretended conclusion is obtained. \( \square \)

References

[1] V. M. Alekseev, V. M. Tikhomirov, and S. V. Fomin. *Optimal control*. Consultants Bureau, New York, 1987. Zbl 0689.49001 MR 89e:49002

[2] L. Cesari. *Optimization—theory and applications*. Springer-Verlag, New York, 1983. Zbl 0506.49001 MR 85c:49001

[3] F. H. Clarke. *Methods of dynamic and nonsmooth optimization*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1989. Zbl 0696.49003 MR 91j:49001

[4] I. M. Gelfand and S. V. Fomin. *Calculus of variations*. Prentice-Hall Inc., Englewood Cliffs, N.J., 1963. Zbl 0127.05402 MR 28:3353

[5] M. Giaquinta and S. Hildebrandt. *Calculus of variations I. The Lagrangian formalism*. Springer-Verlag, Berlin, 1996. Zbl 0853.49001 MR 98b:49002a

[6] J. Jost and X. Li-Jost. *Calculus of variations*. Cambridge University Press, Cambridge, 1998. Zbl 0913.49001 MR 2000m:49002

[7] S. Moyo and P. G. L. Leach. A note on the construction of the Ernàkov-Lewis invariant. *J. Phys. A: Math. Gen.*, 35:5333–5345, 2002.

[8] E. Noether. Invariante variationsprobleme. *Gött. Nachr.*, pages 235–257, 1918. JFM 46.0770.01
[9] E. Noether. Invariant variation problems. *Transport Theory Statist. Phys.*, 1(3):186–207, 1971. English translation of the original paper [8]. Zbl 0292.49008 MR 53:10538

[10] W. Sarlet and F. Cantrijn. Generalizations of Noether’s theorem in classical mechanics. *SIAM Rev.*, 23(4):467–494, 1981. Zbl 0474.70014 MR 83c:70020

[11] D. F. M. Torres. Conservation laws in optimal control. In *Dynamics, bifurcations, and control (Kloster Irsee, 2001)*, volume 273 of *Lecture Notes in Control and Inform. Sci.*, pages 287–296. Springer, Berlin, 2002. Zbl pre01819752 MR 2003c:49037

[12] D. F. M. Torres. On the Noether theorem for optimal control. *European Journal of Control*, 8(1):56–63, 2002.