1 Introduction: High-energy scattering and the small-\(x\) problem

Understanding the high-energy behaviour of hadronic interactions from first principles represents a major challenge for theoretical particle physics. This problem is intimately related to that of high parton densities: At high energy, the QCD cross-sections are controlled by the “small–\(x\)” gluons in the hadron wave-function — i.e., the gluons with small longitudinal momenta —, whose density grows rapidly with the energy (or with decreasing \(x\)), because of the enhanced radiative processes. (See Ref. [1] for recent reviews and more references).

Even though QCD is asymptotically free, the high-energy behaviour is not necessarily perturbative: It is the transferred momentum \(Q^2\) (and not the center-of-mass energy squared \(s\)) which controls the running of the QCD coupling \(\alpha_s(Q^2)\). In fact, one expects the total cross sections at very large energies to be controlled by the soft, non-perturbative physics. But perturbation theory may still apply to the evolution of these cross-sections with \(s\).

However, ordinary perturbation theory fails to meet this expectation. By resumming the dominant radiative corrections at high energy, the BFKL equation leads to an expression for the gluon density which grows like a power of \(s\), and which, with increasing \(s\), is driven towards softer and softer transverse momenta \(Q^2\), where perturbation theory cannot be trusted any longer ("infrared diffusion"). The power-law increase of the gluon distribution with \(s\) entails a similar law for the total cross-section, which thus violates the Froissart unitarity bound \(\sigma \leq \ln^2 s\).

But BFKL, and also DGLAP, are linear evolution equations, which neglect the interactions among the small-\(x\) gluons. With increasing energy, recombination effects favoured by the high density should become more and more important, and eventually lead to a saturation of the parton densities, i.e., a limitation of their growth with \(s\). In terms of scattering, this would correspond to the unitarization of the scattering amplitudes at fixed impact parameter.

For given \(Q^2\), non-linear effects should become important when the energy is sufficiently high for the gluons to overlap by a factor \(1/\alpha_s\) (to compensate for the smallness of their interactions \(\propto \alpha_s\)). Equivalently, for a given energy, saturation should occur for those gluons having a sufficiently large transverse size \(1/Q^2\), larger than the critical value \(1/Q_s^2(x)\) at which the “packing factor” becomes \(\sim 1/\alpha_s\). This requires sufficiently low transverse momenta \(Q^2 \lesssim Q_s^2(x)\), with:

\[
Q_s^2(x) \simeq \frac{\alpha_s N_c}{N_c^2 - 1} \frac{x G(x, Q_s^2(x))}{\pi R^2},
\]  

(1)

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where \( xG(x, Q^2) = dN/d\tau \) is the gluon distribution, i.e., the number of gluons with longitudinal momentum fraction \( x \) and transverse size \( \Delta x_\perp \sim 1/Q \) per unit rapidity \( \tau \equiv \ln(1/x) \sim \ln s \). Eq. (1), together with the BFKL prediction \( xG(x, Q^2) \sim s^\omega \), with \( \omega = 4\bar{\alpha}_s \ln 2 \) \((\bar{\alpha}_s \equiv \alpha_s N_c/\pi)\), leads to the conclusion that the saturation scale \( Q_s^2(x) \) should increase as a power of the energy, and also as a power of the atomic number \( A \) (for a nucleus):

\[
Q_s^2(x, A) \approx \Lambda_{QCD}^2 x^{-\lambda} A^\delta,
\]

where \( \delta \approx 1/3 \) and \( \lambda \approx 4.84\bar{\alpha}_s \) in the BFKL approximation.\(^6\) Phenomenological fits of the \( F_2 \) data at HERA using saturation\(^6\) lead to a somewhat smaller value \( \lambda \approx 0.3 \).

Eq. (2) has an important consequence: for sufficiently large \( A \) and/or high enough energy, the saturation scale is a hard scale, \( Q_s^2 \gg \Lambda_{QCD}^2 \), so weak coupling methods should be applicable\(^3\). This opens the way towards perturbative studies of the high energy limit.

But although the coupling is small, standard perturbative techniques fail to apply because of the high-density effects, which call for all-order resumptions. A crucial observation, which allowed for significant technical progress and a deeper physical insight, is that the small-\( x \) regime is a semi-classical regime, because of the large occupation numbers\(^7\). This leads one to treat the small-\( x \) gluons as the classical Weizs\"acker–Williams field radiated by a random colour charge distribution: that of the fast partons with larger \( x \).

With increasing energy, or decreasing \( x \), new quantum modes become relatively “fast” and must be included in the colour source seen by the external probe. Thus, the classical description of the small-\( x \) gluons is to be seen as an effective theory valid at a given value of \( x \), and whose “action” is evolving with \( x \)\(^7\). The definitive form of the equation describing this evolution has been given in Ref. [7] (see also Ref. [8]), where the interpretation of the saturated gluons as a Colour Glass Condensate\(^7\) has been also proposed. It is my purpose in this talk to briefly review this physical picture and its mathematical formulation.

## 2 The effective theory for the Colour Glass Condensate

The classical field equations read:

\[
(D_{\nu} F^{\nu\mu})_a(x) = \delta^{\mu+} \rho_a(x^-, x_\perp)
\]

where the colour current in the r.h.s. has just a plus component since the fast partons are moving at nearly the speed of light in the positive \( z \), or \( x^+ \), direction. (I use light-cone vector notations: \( x^\pm = (t \pm z)/\sqrt{2} \), \( x_\perp = (x, y) \), and similarly for the other vectors.)

The colour charge density \( \rho_a(x^-, x_\perp) \) is localized near the light-cone \((x^- \approx 0)\), because of Lorentz contraction, and is frozen, i.e., it is independent of the (light-cone) time \( x^+ \), because the dynamics of the fast partons is slowed down by Lorentz time dilation. In other terms, the changes in the configuration of the fast partons occur over a time scale which is much larger than the duration of a collision at small-\( x \). This allows for a kind of Born-Oppenheimer approximation, in which one studies first the dynamics of the classical fields for a given configuration \( \rho_a(x^-, x_\perp) \) of the colour sources, and then one averages over all the possible configurations. The weight function \( W_\tau[\rho] \) for this averaging is obtained by integrating out the fast partons, so it depends upon the rapidity scale \( \tau \equiv \ln(1/x) \) at which one considers the effective theory.

For instance, the unintegrated gluon distribution is obtained as (with \( k^+ = xP^+ = P^+e^{-\tau} \)):

\[
\varphi_\tau(k_\perp) \equiv \frac{d^3N}{d\tau d^2k}\bigg|_{k^+} = \frac{1}{4\pi^3} \langle F_a^{+i}(k^+, k_\perp) F_a^{-i}(-k^+, -k_\perp) \rangle_\tau,
\]

\(^6\)We assume that the QCD coupling is running with \( Q_s^2(x) \), which is reasonable since the saturated gluons have typical momenta of order \( Q_s \).
where \( F_a^{+i} = \partial^+ A_a^i \) is the “electric field” in the LC-gauge \( A_a^+ = 0 \), and the matrix element in the r.h.s. is computed in the classical theory as (with \( \vec{x} = (x^-, x_\bot) \)):

\[
\langle F_a^{+i}(\vec{x})F_a^{+j}(\vec{y}) \rangle_\tau = \int D[\rho] W_\tau[\rho] F_a^{+i}(\vec{x})F_a^{+j}(\vec{y}),
\]

(5)

where \( \mathcal{F}_a^{+i} \equiv F_a^{+i}[\rho] \) is the solution to the classical Yang-Mills equations in this LC gauge. At small \( x \), the colour fields become strong, \( F_a^{+i} \sim 1/g \), so the classical solution must be computed exactly. Given the special geometry of the source, this exact solution is known indeed:

\[
\mathcal{F}^{+i}(\vec{x}) \simeq \delta(x^-) \frac{i}{g} V(\partial^i V^\dagger)(x_\bot), \quad V^\dagger(x_\bot) \equiv P \exp \left\{ ig \int dz^- \alpha_a(z^-, x_\bot) T^a \right\},
\]

(6)

with \( \alpha^a(\vec{x}) \) the solution to the 2-dimensional Poisson equation \(-\nabla_\bot^2 \alpha^a(\vec{x}) = \rho_a(\vec{x}) \) (this is the “Coulomb” field in the infinite momentum frame). In terms of the ordinary electric \((E_a^i)\) and magnetic \((B_a^i)\) fields, eq. (6) describes a plane-wave–like configuration in which \( E_\bot \) and \( B_\parallel \) are transverse to each other and also to the direction of propagation, and have equal magnitudes.

The average over \( \rho \) in eq. (5) is reminiscent of that performed for systems with a frozen disorder, like \textit{spin glasses}. A spin glass is a collection of magnetic impurities (the “spins” \( S_i \)) randomly distributed in some non-magnetic host, with lattice points \( i, j, ... \) The disorder can be characterized by treating the spin-spin couplings — the “link variables” \( J_{ij} \) — as fixed, but random. In reality, the \( J_{ij} \)'s can change with time, but their changes occur only on time scales much larger than any characteristic time scale for the dynamics of the spins. Thus, when studying, e.g., the thermalization of the spin system, one can assume that thermal equilibrium is reached for each given configuration of the \( J_{ij} \)'s, and then average over the latter. The free energy is obtained as:

\[
F = -T \int D[J] W[J] \ln Z[J], \quad Z[J] = \sum_{\{S\}} e^{-\beta H_J[S]},
\]

(7)

where \( H_J[S] = -\sum_{i,j} J_{ij} S_i S_j \) and \( W[J] \) is the weight function for the link variables. Clearly, there is a formal analogy between eqs. (5) and (7), with \( J_{ij} \leftrightarrow \rho_a(\vec{x}) \) and \( S_i \leftrightarrow F_a^{+i}(\vec{x}) \). In this analogy, spin is replaced by colour, so one can characterize the small-\( x \) gluonic matter described by the effective theory in eqs. (5) and (7) as a “colour glass”.

It is also instructive to contrast this physical situation to what happens in a \textit{plasma}. There, the charged particles are very mobile, so they can rapidly adapt themselves to the changes in the electromagnetic background field \( A^\mu \). It is then appropriate to first compute the induced current \( j^\mu[A] \) by “integrating out” the charged particles in the background of a given field \( A^\mu \), and only then solve the Maxwell equations with source \( j^\mu[A] \). That is, for a plasma, the analogs of the operations in eqs. (5) and (7) above are performed in the \textit{opposite} order.

3 Non-linear evolution and saturation

When the rapidity \( \tau \) is increased by \( d\tau \), i.e., the hadron is further accelerated, the quantum gluons with rapidities \( \tau' \) in the interval \( \tau < \tau' < \tau + d\tau \) get effectively frozen (since they are time dilated w.r.t. a collision at the new rapidity scale \( \tau + d\tau \)), and therefore must be incorporated in the effective theory. They become a part of the colour glass.

This entails a change in the properties of the colour source \( \rho \) — its support and its correlation functions \( \langle \rho(1) \rho(2) \cdots \rangle \) — which can be absorbed into an appropriate “renormalization” of the weight function \( W_\tau \rightarrow W_{\tau + d\tau} \). The result of a lengthy analysis in which quantum modes are integrated out in the background of the colour fields \( \mathcal{F}_a^{+i}(\vec{x}) \) generated at the previous steps
in the evolution, is the following, non-linear evolution equation for $W_{\tau}[\rho]$:

$$\frac{\partial W_{\tau}[\rho]}{\partial \tau} = \frac{1}{2} \int_{x_{\perp}, y_{\perp}} \frac{\delta}{\delta \rho_a^{\perp}(x_{\perp})} \chi^{ab}(x_{\perp}, y_{\perp})[\rho] \frac{\delta}{\delta \rho_b^{\perp}(y_{\perp})} W_{\tau}[\rho].$$

(8)

This is a functional Fokker-Planck equation. It describes quantum evolution towards small $x$ as a random walk on the functional space spanned by $\rho_a(x^-, x_{\perp})$. The kernel $\chi^{ab}(x_{\perp}, y_{\perp})[\rho]$, which plays the role of a “diffusion coefficient”, is the correction to the 2-point function of $\rho$ induced by the quantum fluctuations. It is itself a non-linear functional of the original source $\rho$, upon which it depends via the Wilson lines $V$ and $V^\dagger$, eq. (8).

The functional equation above can be converted into ordinary evolution equations for the correlation functions of the Wilson lines. This generates the same equations as obtained by Balitsky, Kovchegov and Weigert within different approaches. However, progress can be made also via direct investigations of the functional equation (8), for which approximate solutions have been obtained. These are briefly described below:

When the source is weak, corresponding to a low-density system, or high transverse momenta $k_{\perp}^2 \gg Q_s^2(\tau)$, the Wilson lines can be expanded to lowest order in the Coulomb field $\alpha_a$ (e.g., $V^\dagger(x_{\perp}) \approx 1 + i g \int dx^- \alpha_a(x^-, x_{\perp}) T^a$), and eq. (8) reduces to the linear BFKL equation for the unintegrated gluon distribution of eq. (4). This leads to the usual, exponential, increase with the rapidity: $\varphi_{\tau}(k_\perp) \propto e^{\omega_\tau}$.

However, when the density becomes as large as $\varphi_{\tau}(k_\perp) \sim 1/\alpha_s$ — this happens at rapidities $\tau$ such as $k_{\perp}^2 \lesssim Q_s^2(\tau)$ (cf. eqs. (1)-(3)) —, then the colour fields are strong $\alpha_a \sim 1/g$, so the Wilson are rapidly oscillating and average away to zero: $V \approx V^\dagger \approx 0$. Then the r.h.s. of eq. (8) simplifies drastically: The kernel $\chi$ becomes independent of $\rho$: $\chi(k_\perp) = (1/\tau)k_{\perp}^2$, and vanishes as $k_{\perp}^2 \to 0$, which shows that colour neutrality is achieved over an area $\gtrsim 1/Q_s^2(\tau)$.

With this simplified kernel, eq. (8) is readily solved, and the corresponding gluon distribution can be then computed. The final result reads (up to a numerical factor):

$$\varphi_{\tau}(k_\perp) \sim \frac{N_c - 1}{N_c} \alpha_s \ln \frac{Q_s^2(\tau)}{k_{\perp}^2} \propto \tau, \quad \text{for } k_\perp \ll Q_s(\tau),$$

(9)

and exhibits gluon saturation: At low momenta $k_\perp \lesssim Q_s(\tau)$, the gluon density grows only linearly with $\tau$, i.e., logarithmically with the energy. Thus, with increasing $s$, the new partons are predominantly produced at large transverse momenta $\gtrsim Q_s(\tau)$.

This suggest that saturation is a natural mechanism to restore unitarity: For an external probe with given resolution $Q^2$, the scattering amplitude $T$ saturates, $T = 1$, at any impact parameter where the condition $Q^2 < Q_s^2(\tau)$ is satisfied. But to convincingly demonstrate the unitarization of the total cross-section, one must still show that the area of the “black disk” where $T = 1$ is not increasing faster than $\ln^2 s$. This requires extending the previous analyses of eq. (8) by including the dependence upon the impact parameter.

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