Relativistic Harmonic Oscillator Revisited

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Abstract

The familiar Fock space commonly used to describe the relativistic harmonic oscillator, for example as part of string theory, is insufficient to describe all the states of the relativistic oscillator. We find that there are three different vacua leading to three disconnected Fock sectors, all constructed with the same creation-annihilation operators. These have different spacetime geometric properties as well as different algebraic symmetry properties or different quantum numbers. Two of these Fock spaces include negative norm ghosts (as in string theory) while the third one is completely free of ghosts. We discuss a gauge symmetry in a worldline theory approach that supplies appropriate constraints to remove all the ghosts from all Fock sectors of the single oscillator. The resulting ghost free quantum spectrum in d+1 dimensions is then classified in unitary representations of the Lorentz group SO(d,1). Moreover all states of the single oscillator put together make up a single infinite dimensional unitary representation of a hidden global symmetry SU(d,1), whose Casimir eigenvalues are computed. Possible applications of these new results in string theory and other areas of physics and mathematics are briefly mentioned.

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I. INTRODUCTION

The relativistic harmonic oscillator in $d$ space and 1 time dimensions that will be discussed in this paper is the straightforward generalization of the non-relativistic case by replacing position and momentum by their relativistic counterparts $x^\mu, p^\mu$ as $SO(d,1)$ vectors. 

There is a long history of studies of the relativistic harmonic oscillator. Some of these were motivated by possible physical applications of the relativistic oscillator as an “imperfect model” to approximate bound states of quarks in a relativistic setting. This involved solving the relativistic oscillator eigenvalue equation in the space of the relative coordinate $x^\mu = x_1^\mu - x_2^\mu$

$$\frac{1}{2} \left(-\partial^\mu \partial_\mu + x^\mu x_\mu\right) \psi_\lambda(x) = \lambda \psi_\lambda(x), \quad (1.1)$$

and associating the eigenvalue $\lambda$ with the mass of the bound state.

Some solutions of this equation appeared in earlier papers, but the Lorentz symmetry properties of these solutions remained obscure to this day. Lorentz covariant solutions based on a vacuum state $\psi_{\text{vac}}(x) \sim \exp(-x^\mu x_\mu/2)$ that is a Lorentz invariant Gaussian have a number of problems, including issues of infinite norm and negative norm states, that were suppressed with ad hoc arguments for the sake of going forward with the physical application. More careful analyses, that paid attention to Lorentz properties by using infinite dimensional unitary representations of $SO(3,1)$ relevant for this problem, suggest that there are solutions of this equation in different spacelike and timelike patches that should be matched across the lightcone $x^\mu x_\mu = 0$. Several examples of this covariant approach using generalized relativistically invariant potentials $V(x^\mu x_\mu)$ that may be different in different patches were also studied. Proposals to confine the solutions to only part of the spacelike region were also discussed.

It is fair to say that there remains open questions regarding the symmetry properties of the solutions of this differential equation. Understanding the symmetry properties of the solutions will be the focus of the present paper.

The same equation arises as a building block in string theory. The phase space $X^\mu(\tau, \sigma)$, $P^\mu(\tau, \sigma)$ of an open relativistic string can be expressed in terms of its normal modes

$$X^\mu = x_0^\mu(\tau) + \sqrt{2} \sum_{n=1}^{\infty} x_n^\mu(\tau) \cos(n\sigma), \quad P^\mu = \frac{1}{\pi} p_0^\mu(\tau) + \frac{\sqrt{2}}{\pi} \sum_{n=1}^{\infty} p_n^\mu(\tau) \cos(n\sigma) \quad (1.2)$$

Feynman called this approach an imperfect model. Indeed, as is now known, the physically correct description of systems such as quark-antiquark bound states is formulated in the context of quantum chromo-dynamics. Approximations to chromo-dynamics for slow moving heavy quarks is handled in terms of a non-relativistic potential $V(\vec{r}) = \alpha |\vec{r}| - \beta/|\vec{r}|$, rather than the relativistic oscillator, while for fast moving light quarks this approach is not an accurate model.

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3 We absorb all dimensionful parameters as well as the frequency of the oscillator by rescaling the $x^\mu, p^\mu$. 
Except for the center of mass mode \((x_0^\mu, p_0^\mu)\) that behaves like a free particle, the normal modes \((x_n^\mu, p_n^\mu)\) are relativistic harmonic oscillator modes with frequency \(\omega_n = n\). The quantum wavefunction of a string in position space depends on all of these modes

\[
\psi (X^\mu) = \psi (x_0^\mu, x_1^\mu, x_2^\mu, \cdots).
\]  

(1.3)

This is the string field that appears in string field theory \([12][13]\). It obeys a differential equation \((L_0 - 1) \psi (X^\mu) = 0\) where \(L_0\) is the zeroth Virasoro operator which is basically a sum of operators

\[
Q_n = \frac{1}{2} (p_n^2 + n^2 x_n^2) \text{ of the type that appears in Eq.}(1.1)^4
\]

\[
L_0 = -\partial_0^\mu \partial_0^\mu + \sum_{n=1}^{\infty} \left( -\partial_n^\mu \partial_n^\mu + n^2 x_n^\mu x_n^\mu \right) - a. 
\]  

(1.4)

If this had been the only equation for the string field \(\psi (X^\mu)\), then the solution would have been a direct product of solutions of Eq.\((1.1)\) with a restriction on the sum of the eigenvalues

\[
\psi (X^\mu) \sim e^{ik \cdot x_0} \prod_{n=1}^{\infty} \psi_{\lambda_n} (x_n), \quad \sum_{n=1}^{\infty} \lambda_n = (1 - k^2).
\]  

(1.5)

Here the center of mass momentum \(k^\mu\) gives the mass-squared of the relativistic string state \(M^2 \equiv -k^2 = k_0^2 - \vec{k}^2\). However, \(\psi (X^\mu)\) must also obey the Virasoro constraints \(L_n \psi (X^\mu) = 0\). Therefore solutions for the free string field \(\psi (X^\mu)\) are linear combinations of \((1.5)\) with different \(\lambda_n\)’s that satisfy the same mass level, taken with coefficients such that the Virasoro constraints are also obeyed. Such solutions were obtained in the covariant quantization approach, which also provided a proof of the absence of negative norm ghosts in string theory \([14]-[16]\).

As will be explained in section \((III)\), upon a closer examination it becomes evident that the relativistic Fock space treatment of string theory \([17]\) inadvertently specializes to only the spacelike sector of every normal mode without any warning, namely

\[
x_n^\mu x_n^\mu \geq 0 \text{ and } p_n^\mu p_n^\mu \geq 0 \text{ for every string mode } n \geq 1.
\]  

(1.6)

This can give only non-negative eigenvalues \(\lambda_n \geq 0\), and hence Eq.\((1.5)\) is solved for \(k^2\) by mostly timelike center of mass momenta \(k^\mu k_\mu < 0\), or positive \(M^2\). The exception is the tachyon state that is forced to have spacelike momentum \(k^\mu\) when all \(\lambda_n = 0\), and hence \(M^2 = -k^2 = -1\) gives a tachyon

\[
\psi (X^\mu) \sim \langle X|0, k \rangle \sim e^{ik \cdot x_0} \exp \left( -\frac{1}{2} \sum_{n=1}^{\infty} nx_n^\mu x_n^\mu \right).
\]  

(1.7)

when all string modes \(x_n^\mu\) are in the spacelike region. For excited levels this expression is multiplied by polynomials in the various \(x_n^\mu\).

\[\text{The constant } a = \frac{1}{2} (d + 1) \sum_n n \text{ subtracts the vacuum energy of all the oscillators. After this renormalization the Virasoro constraint is determined as } L_0 = 1.\]
In view of the fact that the single oscillator equation (1.1) has solutions in different spacetime regions as indicated above, a natural question arises of whether there might be more general solutions to string theory beyond the spacelike region of Eq. (1.6). This is not an easy question to answer, both because there are the Virasoro constraints to deal with, and because there is still obscurity in the previously known solutions of the relativistic oscillator equations (1.1).

This bring us to the main topic of the current paper. We will investigate the single relativistic oscillator without prejudice as to its possible physical applications. Our main interest is to clarify the symmetry and unitarity or lack thereof of its various solutions in various parts of spacetime. At the end we will point out possible applications of our findings.

Our key observations will follow from hidden symmetries not discussed before. First we point out that the symmetries of Eq. (1.1) go beyond the Lorentz symmetry $\text{SO}(d, 1)$. There is a hidden symmetry $\text{SU}(d, 1)$ that includes $\text{SO}(d, 1)$, and therefore all solutions, unitary or non-unitary, must fall into irreducible representations of $\text{SU}(d, 1)$. Apparently this was never explored in previous investigations of Eq. (1.1).

After clarifying the symmetry aspects we will build three different Fock spaces by using the same relativistic harmonic oscillator creation-annihilation operators. This includes a spacelike, timelike and mixed spacetime sectors that are distinct from each other. While the spacelike or timelike sectors have negative norm states, the mixed case is completely free of negative norm ghosts and is covariant under $\text{SO}(d, 1)$ and $\text{SU}(d, 1)$ in infinite dimensional unitary representations. There may be more solutions in more intricate spacetime sectors than those described in this paper, but we will not attempt to investigate them here (see comments following Eq. (A16) and footnote (18)).

For the single harmonic oscillator we will also discuss a worldline gauge symmetry that removes ghosts and thereby introduces a constraint. The covariant quantization of this constrained model is in agreement with the general discussion. On the other hand, a gauge fixed quantization does not capture all the sectors but is in agreement with the sectors describable in that gauge. This simple example illustrates how a gauge fixed theory can fail to capture all the gauge invariant sectors of a gauge invariant theory\(^5\).

The new phenomena uncovered here both in the covariant quantization as well as the gauge fixed quantization of the relativistic oscillator may provide tools and rekindled interest to revisit string theory.

\(^5\) Another example is that the usual treatment of the lightcone gauge in string theory fails to capture the folded string sectors of string theory [18]-[20].
II. RELATIVISTIC HARMONIC OSCILLATOR AND SU(d, 1)

For the sake of clarity, parts of our presentation, including this section, will include some material that may be quite familiar to many readers, but this will be compensated by simple observations that are not that familiar.

The operator $Q = \frac{1}{2} (p \cdot p + x \cdot x)$ which is being diagonalized, $Q\psi_\lambda = \lambda \psi_\lambda$, can be written as usual in terms of Lorentz covariant oscillators

$$a_\mu = \frac{1}{\sqrt{2}} (x_\mu + ip_\mu), \quad \bar{a}_\mu = \frac{1}{\sqrt{2}} (x_\mu - ip_\mu).$$

(2.1)

The covariant quantization rules

$$[x_\mu, p_\nu] = i \eta_{\mu\nu},$$

(2.2)

with the SO($d, 1$) Minkowski metric $\eta_{\mu\nu}$, lead to the relativistic quantum oscillator commutation rules

$$[a_\mu, \bar{a}_\nu] = \eta_{\mu\nu} = diag (-1, 1, 1, \cdots, 1).$$

(2.3)

In a unitary Hilbert space the operators $x_\mu, p_\mu$ are Hermitian; in that case $\bar{a}_\mu$ is the Hermitian conjugate of $a_\mu$, i.e. $\bar{a}_\mu = (a_\mu)\dagger$. A unitary Hilbert space without ghosts (negative norm states) is possible only and only if $x_\mu, p_\mu$ are hermitian or equivalently if $\bar{a}_\mu = (a_\mu)\dagger$.

In what follows we will seek unitary Hilbert spaces, but along the way we also come across non-unitary Fock spaces in which $\bar{a}_\mu \neq (a_\mu)\dagger$. Therefore we prefer the more general notation $\bar{a}_\mu$ in order not to confuse it with the hermitian conjugate of $a_\mu$ when such vector spaces arise.

In terms of $a_\mu, \bar{a}_\mu$ the operator $Q$ takes the form

$$Q = \frac{1}{2} (p \cdot p + x \cdot x) = \bar{a} \cdot a + \frac{d+1}{2} = a \cdot \bar{a} - \frac{d+1}{2}.$$ (2.4)

This operator $Q$ has a larger symmetry than the evident Lorentz symmetry of the dot products $\bar{a} \cdot a = \eta_{\mu\nu} a_\mu a_\nu$. The hidden symmetry is U($d, 1$) whose generators are

$$U (d, 1) \text{ generators: } \bar{a}_\mu a_\nu.$$ (2.5)

All of these $(d+1)^2$ generators commute with $Q$

$$[Q, \bar{a}_\mu a_\nu] = [\bar{a} \cdot a, \bar{a}_\mu a_\nu] = 0,$$

(2.6)

hence $Q$ has U($d, 1$) symmetry, and the spectrum of $Q$, whether unitary or non-unitary, must be classified as irreducible representations of U($d, 1$) =SU($d, 1$) × U(1) unless the symmetry is broken by boundary conditions. The U(1) part is just the number operator $J_0$

$$J_0 \equiv \bar{a} \cdot a = a \cdot \bar{a} - (d+1),$$

(2.7)

6 See the last paragraph of the Appendix for an example of how the SU($d, 1$) symmetry is broken to SO($d, 1$) in the purely spacelike sector.
which is essentially the operator $Q$ up to a shift. Therefore the non-trivial part is $\text{SU}(d, 1)$ with $(d + 1)^2 - 1$ generators that correspond to the traceless tensor

$$J_{\mu\nu} = \left( \bar{a}_\mu a_\nu - \frac{1}{d+1} \eta_{\mu\nu} \bar{a} \cdot a \right) = \left( a_\nu \bar{a}_\mu - \frac{1}{d+1} \eta_{\mu\nu} a \cdot \bar{a} \right)$$

(2.8) that satisfies $\eta^{\mu\nu} J_{\mu\nu} = 0$. The Lorentz generators $L_{\mu\nu}$ for $\text{SO}(d, 1)$ correspond to the antisymmetric part of the tensor $J_{\mu\nu}$

$$L_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu = -i \left( \bar{a}_\mu a_\nu - \bar{a}_\nu a_\mu \right) = -i \left( a_\nu \bar{a}_\mu - a_\mu \bar{a}_\nu \right).$$

(2.9)

The $L_{\mu\nu}$ are hermitian by construction as long as $x_\mu, p_\mu$ are hermitian. So a unitary representation of the Lorentz group will be obtained if and only if $\bar{a}_\mu = (a_\mu)^\dagger$. We know that unitary representations of non-compact groups are infinite dimensional except for the singlet. Hence $\bar{a}_\mu = (a_\mu)^\dagger$ can be satisfied only on singlets or on infinite dimensional representations of the Lorentz or the $\text{SU}(d, 1)$ symmetry.\textsuperscript{7}

In the following we will see that there are different Fock spaces disconnected from each other, all of which contribute to the full unitary spectrum of $Q$. These Fock spaces are built with the same oscillators $\bar{a}_\mu, a_\nu$ but are based on three different vacua with different $\text{SU}(d, 1)$ or $\text{SO}(d, 1)$ symmetry properties as well as different space-time geometric properties. This shows that there are some surprising features of the relativistic harmonic oscillator that are fundamentally different from the non-relativistic one.

Our aim is to identify the physically acceptable unitary sector of the theory that contains no ghosts and find ways in which the physical sectors can be singled out by an appropriate set of constraints.

III. SYMMETRIC VACUUM, NON-UNITARY FOCK SPACE

We will start with the standard approach to the relativistic oscillator Fock space used by most authors, including string theorists\textsuperscript{17}. The corresponding relativistic differential equation $\left( -\frac{1}{2} \partial_\mu \partial_\mu + \frac{1}{2} \eta^{\mu\nu} x_\mu \right) \psi(x) = \lambda \psi(x)$ in position space, in the purely spacelike sector, is solved in Appendix A in 1 + 1 dimensions. Although the Fock space approach in this section and the

\textsuperscript{7} To be more accurate we should distinguish between fundamental and anti-fundamental representations of $\text{SU}(d, 1)$ by using different indices to label them. For example, we can use undotted indices $a_\mu = \frac{1}{\sqrt{2}} (x_\mu + ip_\mu)$ to emphasize that $a_\mu$ is in the the fundamental representation and dotted indices $\bar{a}_\mu = \frac{1}{\sqrt{2}} (x_\mu - ip_\mu)$ to emphasize that $\bar{a}_\mu$ is in the anti-fundamental representation. Indices are raised or lowered with the Minkowski metric $\eta^{\mu\nu}$ that has mixed indices, such as $\bar{a}^\mu = \eta^{\mu\nu} \bar{a}_\nu$, and $a^\mu = \eta^{\mu\nu} a_\nu$. Because we will not have much use for it we will forgo this more accurate notation and use the same type of indices on all creation or annihilation oscillators. The reader should understand that a lower index on the operator $\bar{a}$ is really meant to be a dotted index $\bar{a}_\mu$, while an upper index on $\bar{a}$ is undotted $\bar{a}^\mu$. The opposite is true for the operators $a_\mu, a^\mu$.
position space approach of Appendix A are in full agreement, a great deal of complementary insight about the issues regarding spacetime regions is gained from considering the properties of the probability amplitude $\psi_\lambda(x)$ in position space. So the reader may benefit from studying the Appendix and comparing it to the Fock space approach in this section.

What we want to emphasize is that the familiar Fock space approach yields only part of the quantum states of this relativistic system. After explaining this, we will discuss a much larger Fock space of quantum states in the following section.

The oscillator approach begins by assuming a normalized Lorentz invariant vacuum state that has finite positive norm and is annihilated by the operators $a_\mu$

$$\langle 0|0 \rangle = 1, \quad a_\mu|0 \rangle = 0, \quad L_{\mu\nu}|0 \rangle = 0.$$  \hspace{1cm} (3.1)

The U(1) quantum number or the level number of this state is zero

$$J_0|0 \rangle = \bar{a} \cdot a|0 \rangle = 0.$$  \hspace{1cm} (3.2)

A usually unstated property of this vacuum is that it also requires a spacelike region for $x^\mu$ as well as for $p^\mu$ since, as a probability amplitude in position space or momentum space, it has the form

$$\langle x|0 \rangle \sim e^{-x^2/2} \quad \text{and} \quad \langle p|0 \rangle \sim e^{-p^2/2}, \quad x^\mu, p^\mu \text{ spacelike.}$$  \hspace{1cm} (3.3)

The minus sign in the exponent follows from satisfying $a_\mu|0 \rangle = 0$ in position or momentum spaces, namely

$$a_\mu|0 \rangle = \frac{1}{\sqrt{2}} \left( x_\mu + ip_\mu \right)|0 \rangle = 0 \leftrightarrow \left\{ \begin{array}{l} \frac{1}{\sqrt{2}} \left( x_\mu + \frac{\partial}{\partial x^\mu} \right) e^{-\frac{1}{2}x^2} = 0, \\ \frac{i}{\sqrt{2}} \left( \frac{\partial}{\partial p^\mu} + p_\mu \right) e^{-\frac{1}{2}p^2} = 0, \end{array} \right\}.$$  \hspace{1cm} (3.4)

Spacelike regions $x \cdot x > 0$ and $p \cdot p > 0$ are necessary so that the Gaussian is integrable at infinity

$$\langle 0|0 \rangle \sim \int d^{n+1}x \ e^{-x^2} < \infty, \quad \text{or} \quad \langle 0|0 \rangle \sim \int d^{n+1}p \ e^{-p^2} < \infty,$$  \hspace{1cm} (3.5)

to give a finite norm $\langle 0|0 \rangle = 1$. Actually these integrals are infinite as they stand because, unlike the Euclidean analogs in which both radial and angular integrals are finite, in the present case the “angular” part contains boost parameters with an infinite range (see e.g. parametrization in Eq.(A1) and Fig.1). For a finite norm this infinity must be divided out (see footnote (11)).

It is also possible to restrict to a timelike region by starting from another Lorentz invariant “vacuum” state $|0'\rangle$ to construct a different Fock space. This second alternative is not considered usually. The vacuum $|0'\rangle$ is defined by being annihilated by $\bar{a}_\mu$ rather than by $a_\mu$

$$\langle 0'|0' \rangle = 1, \quad \bar{a}_\mu|0' \rangle = 0, \quad L_{\mu\nu}|0' \rangle = 0,$$

$$\bar{a}_\mu|0' \rangle = \frac{1}{\sqrt{2}} \left( x_\mu - ip_\mu \right)|0' \rangle = 0 \leftrightarrow \left\{ \begin{array}{l} \frac{1}{\sqrt{2}} \left( x_\mu - \frac{\partial}{\partial x^\mu} \right) e^{\frac{1}{2}x^2} = 0, \\ \frac{i}{\sqrt{2}} \left( \frac{\partial}{\partial p^\mu} - p_\mu \right) e^{\frac{1}{2}p^2} = 0. \end{array} \right\}$$
It corresponds to a normalizable vacuum with \( x^\mu \) and \( p^\mu \) in the timelike region, \( x \cdot x < 0 \) and \( p \cdot p < 0 \), to be able to normalize \( \langle 0'|0' \rangle = 1 \)
\[
\langle x|0' \rangle \sim e^{x^2/2} \quad \text{and} \quad \langle p|0' \rangle \sim e^{p^2/2}, \quad x^\mu, p^\mu \text{ timelike.} \tag{3.6}
\]
The U(1) quantum number or the level number of this state is \(- (d + 1)\)
\[
J_0|0' \rangle = \bar{a} \cdot a|0' \rangle = [a \cdot \bar{a} - (d + 1)]|0' \rangle = - (d + 1)|0' \rangle. \tag{3.7}
\]
so it is clearly distinguishable from the spacelike vacuum.

The Fock space based on the vacuum \( |0' \rangle \) is not usually considered because it contains negative norm states for spacelike oscillators, but by contrast it contains positive norms for timelike oscillators. For example the 1-particle excitation \( a_\mu|0' \rangle \) has norm
\[
\langle 0'|\bar{a}_\nu a_\mu|0' \rangle = -\eta_{\mu\nu}, \quad \left\{ \begin{array}{ll}
\text{negative for spacelike } \mu, \nu \\
\text{positive for timelike } \mu, \nu
\end{array} \right. \tag{3.8}
\]
However, we will see that the physical states in this Fock space sector involve always pairs of spacelike and timelike oscillators, such as \( a \cdot \bar{a}|0' \rangle \). Such paired oscillator states have positive norm. In this respect, the spacelike or timelike vacua stand at an equal footing. We will see that while the spacelike vacuum leads to a positive spectrum for \( Q \), the timelike case leads to a negative spectrum. Whether the negative or positive spectra are suitable in physical applications depends on the physical interpretation of the operator \( Q = \frac{1}{2} (p \cdot p + x \cdot x) \) in some physical context.

This begins to show that there are several disconnected sectors of Fock spaces in the spectrum of the relativistic harmonic oscillator. As we will see below both of these Fock spaces lead to non-unitary vector spaces from which we will need to fish out a subset of positive norm states. Furthermore, in the next section, we will discuss a completely different Fock space that is based on a Lorentz non-invariant vacuum \( |\tilde{0} \rangle \) that leads to a completely unitary infinite dimensional Hilbert space.

In the rest of this section we discuss mainly the Fock space based on the spacelike vacuum \( |0 \rangle \) and only give results or make comments about the very similar Fock space based on the timelike vacuum \( |0' \rangle \).

In either spacelike or timelike cases, since the vacuum respects the SO\((d, 1)\) symmetry, one should expect to find that all the states in either Fock space can be classified as irreducible unitary or non-unitary representations of SO\((d, 1)\). Furthermore, the restriction to a spacelike or timelike region is consistent with an SU\((d, 1)\) symmetric vacuum since we can verify that under an infinitesimal SU\((d, 1)\) transformation we obtain
\[
J_{\mu\nu}|0 \rangle = 0, \quad J_{\mu\nu}|0' \rangle = 0, \tag{3.9}
\]
by using the two forms of \( J_{\mu\nu} \) given in Eq.(2.8). Hence the Fock spaces built on these invariant vacua must be classified as complete irreducible unitary or non-unitary representations not just of SO\((d, 1)\) but of SU\((d, 1)\).
The total level operator can be written out in more detail as
\[ J_0 = \bar{a} \cdot a = (\bar{a}_0 a_0) + \bar{a}_i a_i \] (3.10)

Note how the number operator in the timelike direction \((-\bar{a}_0 a_0)\) works to give a positive number for the level in the spacelike Fock space even when the excitation is in the timelike direction:
\[ (-\bar{a}_0 a_0) \langle \bar{a}_0 | 0 \rangle = (\bar{a}_0 | a_0, \bar{a}_0 \rangle | 0 \rangle \langle \bar{a}_0 | 0 \rangle) = (-1)^2 = (+1) \langle \bar{a}_0 | 0 \rangle. \] (3.11)

Therefore the total level operator \(J_0\) on the covariant states \(\bar{a}_\mu|0\rangle\), excited in either the time or space directions \(\mu\), has \(J_0\) eigenvalue +1.

Similarly, the excited states at a general level \(J_0 = n\) in the spacelike Fock space are constructed by applying \(n\) creation operators either in space or time directions
\[ \bar{a}_{\mu_1} \bar{a}_{\mu_2} \cdots \bar{a}_{\mu_n} | 0 \rangle = \text{SU}(d, 1) \text{ tensor} \sim \begin{array}{c} \mu_1 \mu_2 \mu_3 \cdots \mu_n \end{array} \] (3.12)

This is a symmetric SU\((d, 1)\) or U\((d, 1)\) tensor corresponding to a single row Young tableau as indicated. So, this collection of states at level \(J_0 = n\) form a finite dimensional irreducible representation of SU\((d, 1)\).

The above SU\((d, 1)\) representation can be reduced into irreducible representations of SO\((d, 1)\). This is done by decomposing the symmetric tensor above into a sum of traceless tensors (trace is defined by contracting with the Minkowski metric \(\eta^{\mu\nu}\))
\[ \{(\bar{a}_{\mu_1} \bar{a}_{\mu_2} \cdots \bar{a}_{\mu_n} - \text{trace}) | 0 \rangle + \cdots \} = \text{SO}(d, 1) \text{ traceless tensors}. \] (3.13)

For example at level \(J_0 = 2\) we have one SO\((d, 1)\) tensor of rank 2 and one of rank zero as listed below
\[ \left( \bar{a}_{\mu_1} \bar{a}_{\mu_2} - \frac{\eta_{\mu_1\mu_2}}{d+1} \bar{a} \cdot \bar{a} \right) | 0 \rangle, \text{ and } \bar{a} \cdot \bar{a} | 0 \rangle. \] (3.14)

Similarly at level \(n\) there are the following irreducible tensors of rank \(r\)
\[ r = n, (n-2), (n-4), \cdots, (0 \text{ or } 1). \] (3.15)

At level \(J_0 = n\), each traceless tensor of rank \(r\) listed in Eq.(3.15) is the basis for a separate finite dimensional irreducible representation of SO\((d, 1)\).

All finite representations of non-compact groups, except the singlet, are non-unitary. Therefore all SU\((d, 1)\) or SO\((d, 1)\) representations that emerge in this Fock space at all levels \(n\), except the singlets, are non-unitary. Hence at every level \(J_0 = n\) there are many negative norm states that are unphysical. We have to discuss the types of constraints that can eliminate the ghosts to obtain a physical theory.
Let us now identify the *negative norm states* which appear among the SU(d,1) or SO(d,1) states in Eqs.(3.13-3.15). These are all the ones that contain an odd number of timelike oscillators. For example, the state \( \bar{a}_0 |0 \rangle \) has negative norm\(^8\):

\[
\text{norm} = \langle 0 | \bar{a}_0 a_0 | 0 \rangle = \langle 0 | [a_0, \bar{a}_0] | 0 \rangle = (-1) \langle 0 | 0 \rangle = -1.
\]

The states at a fixed level \( n \) that have an even number of \( \bar{a}_0 \)'s and any number of spacelike oscillators, such as \((\bar{a}_0)^m (\bar{a}_{i_1}, \bar{a}_{i_2} \cdots \bar{a}_{i_{n-m}}) | 0 \rangle\), have positive norm for every even \( m = \) 0, 2, 4, \cdots , \((n \text{ or } n-1)\). A constraint that eliminates all negative norm states in the spacelike region is to demand a reflection symmetry from every state under the operation \( \bar{a}_0 \rightarrow -\bar{a}_0 \) and similarly for \( a_0 \rightarrow -a_0 \). This can be achieved through the operator\(^9\) \( T = \exp (i \pi \bar{a}_0 a_0) \) which gives \( T a_0 T^{-1} = -a_0 \) and \( T \bar{a}_0 T^{-1} = -\bar{a}_0 \), and the boost generator changes sign \( TL^0 L^{-1} = -L^0 \). Therefore a ghost free spectrum is obtained by demanding the following constraint

\[ T | \phi \rangle = (+1) | \phi \rangle, \quad \iff \left\{ \begin{array}{l} \text{ghost free, unitary subset of states,} \\ \text{but not SO}(d,1) \text{ covariant.} \end{array} \right. \]

However, such states by themselves break the Lorentz symmetry since they cannot make up complete irreducible representations of SO(d,1) for any non-zero \( n \). In the absence of this constraint, in any *finite* dimensional representation of SO(d,1), other than the singlet, there will always be states with an odd number of timelike oscillators. For example at level 2 the irreducible tensor in Eq.(3.14) contains the negative norm states

\[
\bar{a}_0 a_i | 0 \rangle.
\]

Therefore, to eliminate the negative norm states all finite representations of SO(d,1) must be discarded by some consistent set of constraints. This leaves only the SO(d,1) singlets\(^10\) at each even level \( J_0 = 2k \)

\[
(\bar{a} \cdot \bar{a})^k | 0 \rangle, \quad k = 0, 1, 2, 3, \cdots \text{ positive norm } \leftrightarrow \text{no ghosts.}
\]

The eigenvalue of \( Q \) on these states is \( \lambda = 2k + \frac{d+1}{2} \)

\[
Q \left[ (\bar{a} \cdot \bar{a})^k | 0 \rangle \right] = \left[ (\bar{a} \cdot \bar{a})^k | 0 \rangle \right] \left( 2k + \frac{d+1}{2} \right).
\]

\(^8\) The negative norm also implies that \( \langle 0 | x_0 x_0 | 0 \rangle \) and \( \langle 0 | p_0 p_0 | 0 \rangle \) are negative as seen from \( \langle 0 | x_0 x_0 | 0 \rangle = \frac{1}{2} \langle 0 | (a_0 + \bar{a}_0) (a_0 + \bar{a}_0) | 0 \rangle = \frac{1}{2} \langle 0 | a_0 \bar{a}_0 | 0 \rangle = -\frac{1}{2} \). If \( x_0 \) were hermitian then \( x_0 x_0 \) would have to be a positive operator with positive expectation value. But in this Fock space \( x_0, p_0 \) are not hermitian, equivalently \( \bar{a}_0 \) is not the hermitian conjugate of \( a_0 \), and this is why negative norms arise.

\(^9\) A similar operator for the spacelike region is \( S = \exp (i \pi a_0 \bar{a}_0) \).

\(^10\) This is in the case of a single oscillator, as in the current simplified problem. If there are additional degrees of freedom then one can find constraints that lead to more interesting ghost-free solutions. For example, in string theory, with an infinite number of oscillators, the Virasoro constraints eliminate ghosts while allowing non-singlets of SO(d,1).
These states have positive norms since \( \vec{a} \cdot \vec{a} = -\vec{a}_0\vec{a}_0 + \vec{a}_i\vec{a}_i \) insures that every term in \((-\vec{a}_0\vec{a}_0 + \vec{a}_i\vec{a}_i)^k\) contains only an even number of \(\vec{a}_i\)'s. All the \(SO(d, 1)\) generators \(L^{\mu\nu}\) in Eq. (2.9) annihilate these states since \([L^{\mu\nu}, \vec{a} \cdot \vec{a}] = 0\) gives

\[
L^{\mu\nu} \left( (\vec{a} \cdot \vec{a})^k |0\rangle \right) = (\vec{a} \cdot \vec{a})^k L^{\mu\nu}|0\rangle = 0, \text{ Lorentz singlets.} \tag{3.21}
\]

So, if the Fock space is restricted to the Lorentz invariant subset, then there are no ghosts.

The position space probability amplitude for these states is determined as

\[
\psi_{k}^{(+)}(x) \sim \langle x | (\vec{a} \cdot \vec{a})^k |0\rangle = \left[ \frac{1}{2} (x - \partial) \cdot (x - \partial) \right]^k e^{-\frac{1}{2}x\mu x_\mu}, \text{ spacelike } x^\mu.
\]

where \(\alpha_k, \tilde{\alpha}_k\) are appropriate normalization constants. For example, for \(k = 1\) it becomes

\[
\psi_{1}^{(+)}(x) \sim (2x^2 - (d + 1)) e^{-\frac{1}{2}x^2}.
\]

More generally this gives the generalized Laguerre polynomial \(L_{k}^{d+1}(x^2)\) with argument \(x^2\) multiplying the Gaussian \(e^{-\frac{1}{2}x^2}\mu x^\mu\).

\[
\psi_{k}^{(+)}(x) = \alpha_k e^{-\frac{1}{2}x^2} \sum_{m=0}^{k} (-1)^m \left( \frac{k + \frac{d-1}{2}}{k - m} \right) \frac{(x \cdot x)^m}{m!} = \alpha_k e^{-\frac{1}{2}x^2} L_{k}^{d+1}(x^2),
\]

where \(\alpha_k\) is an overall constant. It can be checked that this \(\psi_{k}^{(+)}(x)\) is indeed a solution of the relativistic differential equation in \(d+1\) dimensions, with the specified eigenvalue for every positive integer \(k\)

\[
\frac{1}{2} \left[ -\partial^\mu \partial_\mu + x^\mu x_\mu \right] \psi_{k}^{(+)}(x) = \left( 2k + \frac{d+1}{2} \right) \psi_{k}^{(+)}(x), \quad k = 0, 1, 2, \ldots \tag{3.23}
\]

Furthermore, these wavefunctions clearly have positive norm \(\int d^{d+1}x |\psi_{k}^{(+)}(x)|^2\) for all \(k\). We see that according to the symmetry criteria, and unitarity, only these states are admissible as quantum states in the spacelike Fock space\(^\text{11}\).

Similarly, there is another set of \(SU(d, 1)\) singlet states \((a \cdot a)^k |0\rangle\) in the timelike Fock space given by substituting \(a_\mu\) instead of \(\vec{a}_\mu\) and using \(|0\rangle\) instead of \(|0\rangle\).

\[
J_{\mu\nu} \left[ (a \cdot a)^k |0\rangle \right] = 0, \tag{3.24}
\]

\(^1\text{Recall the infinite integrals mentioned following Eq. (3.6). These resurface again in the norm above. For example, in the simplified case in Eq. (1.13) the delta function normalization } \delta(m' - m) \text{ blows up for } m' = m. \text{ This will be a common infinite factor for all Lorentz invariant wavefunctions. The infinity can be avoided by redefining norm by simply not integrating over the extra boost parameters, since those parameters do not appear in the Lorentz invariant wavefunctions. If such a redefinition is not adapted, the infinities may be an argument to discard all of the Lorentz invariant states } \psi_k^\pm (x). \text{ By comparison note that the unitary states based on the Lorentz non-invariant vacuum } |\vec{0}\rangle \text{ discussed in section (IV) have no infinities in their norms.} \)
This follows from the form of \( d \), to? If we apply an infinitesimal \( \text{SU}(d,1) \) classification as complete \( \text{SU}(d,1) \) multiplets. Which \( \text{SU}(d,1) \) invariants, but what are their \( \text{SU}(d,1) \) properties? The \( \text{SU}(d,1) \) symmetry of \( Q \) and of the vacuum exhibited in Eqs. (2.8,3.9) require that the spectrum be classified as complete \( \text{SU}(d,1) \) multiplets. Which \( \text{SU}(n,1) \) multiplets do these states correspond to? If we apply an infinitesimal \( \text{SU}(d,1) \) transformation on the \( \text{SO}(d,1) \) singlets, we find

\[
J_{\mu\nu} \left[ (\bar{a} \cdot \bar{a})^k |0\right] = 2k \left[ \bar{a}_\mu \bar{a}_\nu - \frac{\eta_{\mu\nu}}{d+1} (\bar{a} \cdot \bar{a}) \right] (\bar{a} \cdot \bar{a})^{k-1} |0\right].
\]

We see on the right hand side that, except for the case of \( k = 0 \), we generate inadmissible negative norm states. This also shows that the states \((\bar{a} \cdot \bar{a})^k |0\) with \( k \neq 0 \) are not in a singlet of \( \text{SU}(d,1) \) so that they must be part of non-unitary representations of \( \text{SU}(d,1) \). Hence even though the states \((\bar{a} \cdot \bar{a})^k |0\) are unitary with respect to \( \text{SU}(d,1) \), they are not consistent with an \( \text{SU}(d,1) \) symmetry-consistent unitary spectrum, except for \( k = 0 \).

What happened to the \( \text{SU}(d,1) \) symmetry? It got broken by the boundary conditions of restricting the Fock space inadvertently to a purely spacelike region (see last paragraph of Appendix A for more insight). If one wishes to be consistent with \( \text{SU}(d,1) \) covariance, and also restrict to the spacelike region, then only the vacuum state can be kept in the spectrum.

In a broken \( \text{SU}(d,1) \) scenario all Lorentz singlet states \((\bar{a} \cdot \bar{a})^k |0\) are admissible. Similarly, in a broken \( \text{SO}(d,1) \) scenario all states of the form (3.17) with an even number of \( a_0 \)'s can be included in the ghost free Hilbert space. But, in an exact \( \text{SU}(d,1) \) scenario only the vacuum state \( |0\) can be included. A similar statement applies to the purely timelike sector where only the second vacuum state \( |0'\) can be included.

We see that, in a \( \text{SU}(d,1) \) symmetry-consistent spacelike or timelike Fock spaces, all states other than the vacuum states \( |0\), \( |0'\) must be thrown away by some consistent set of constraints since otherwise the theory cannot be both consistent with its \( \text{SU}(d,1) \) symmetry and also be free

\[Q\left[(a \cdot a)^k |0'\right] = -\left(2k + \frac{d+1}{2}\right) \left[(a \cdot a)^k |0'\right],\]  \hspace{1cm} (3.25)

\[\psi_k^{(-)}(x) = \tilde{\beta}_k \left[\frac{1}{2} \left( x + \partial \right) \cdot \left( x + \partial \right) \right]^k e^{i\frac{x^\mu x_\mu}{2}} \sim \langle x | (a \cdot a)^k |0'\rangle, \text{ timelike } x^\mu.\]  \hspace{1cm} (3.26)

The \( \psi_k^{(-)}(x) \) are related to the \( \psi_k^{(+)}(x) \) by an analytic continuation of \( x^2 \rightarrow -x^2 \) from the spacelike to the timelike region, so they can also be expressed in terms of the Laguerre polynomials

\[\psi_k^{(-)}(x) = \gamma_k e^{i\frac{x^2}{2}} \sum_{m=0}^{k} \left( \frac{k + \frac{d-1}{2}}{k - m} \right) \frac{(x \cdot x)^m}{m!} = \gamma_k e^{i\frac{x^2}{2}} L_{\frac{d-1}{2}}^{\frac{d-1}{2}} (-x^2),\]

However, it must be emphasized that, as computed\(^\text{12}\) in Eq. (3.25), the \( \psi_k^{(-)}(x) \) have the opposite signs for the eigenvalues of \( Q \) as compared to the \( \psi_k^{(+)}(x) \).

\(^\text{12}\) This follows from the form of \( Q = a \cdot \bar{a} - \frac{d+1}{2} \) given in Eq. (2.24), and from the fact that \([a \cdot \bar{a}, (a \cdot a)] = -2 (a \cdot a)\).
of ghosts. One possibility is to choose the constraint to be \( J_{\mu\nu} = 0 \) but this is too restrictive because, as we will see, it throws away the big sector of unitary states that we will discuss in the next section. Less restrictive is a constraint of the form

\[
\left[ \frac{1}{2} (p^2 + x^2) - \lambda_0 \right] = 0, \text{ no ghosts only for } \lambda_0 = \pm \frac{d+1}{2}.
\]

When \( \lambda_0 = \frac{d+1}{2} \) the constraint can be satisfied only by \(|0\rangle\) and when \( \lambda_0 = -\frac{d+1}{2} \) it can be satisfied only by \(|0'\rangle\). For other values of \( \lambda_0 \) that appeared in the spectrum above, such as \( \lambda = \pm \left( n + \frac{d+1}{2} \right) \), the constraint allows also negative norm states in non-unitary representations of SU\((d,1)\) with a Young tableau with \( n \) boxes as in Eq.(3.12), so only \( n = 0 \) is admissible. We see that the only possible constraint of this form can only involve \( \lambda_0 = \pm \frac{d+1}{2} \), leading to only one of the possible states: either \(|0\rangle\) or \(|0'\rangle\).

A constraint of the type (3.28) with general \( \lambda_0 \) emerges as a natural outcome in a worldline theory as a consequence of a gauge symmetry on the worldline as we will see in detail in section \( \text{(VI)} \). That kind of local symmetry is reasonable because it can be used to eliminate the ghosts that come from timelike directions, thus guaranteeing a unitary theory.

If \( \lambda_0 \) is in the range \(-\frac{d+1}{2} < \lambda < \frac{d+1}{2}\) no state in the spacelike or timelike sectors can satisfy the constraint (3.28). So, with such a constraint all the states in the purely spacelike or purely timelike sectors, including \(|0\rangle\) and \(|0'\rangle\) would be excluded.

But in the next section we will find that this type of constraint is satisfied by many more states beyond those that appeared in the spacelike or timelike Fock spaces discussed in this section. There is a large sector of positive norm quantum states that cannot be built by starting from the conventional Lorentz invariant vacuum states \(|0\rangle\), \(|0'\rangle\), and those additional states are compatible with the SU\((d,1)\) symmetry, not as singlets, but as infinite dimensional unitary representations whose Casimir eigenvalues are determined by \( \lambda_0 \).

**IV. UNITARY FOCK SPACE, NON SYMMETRIC VACUUM**

We will now take a different approach to solving the eigenvalue problem \( Q\psi_{\lambda} = \lambda\psi_{\lambda} \). Rather starting with a Lorentz invariant vacuum state as is usually done, we will consider solving the differential equation

\[
\frac{1}{2} \left( -\partial^\mu \partial _\mu + x^\mu x_\mu \right) \psi_{\lambda} (x) = \lambda \psi_{\lambda} (x) .
\]

without paying attention at first to its Lorentz covariance properties \([2][3][4]\). We will then clarify the symmetry properties of the solutions by appealing to the hidden symmetry SU\((d,1)\).

We can obtain solutions by separating it in the \( x^0, \vec{x} \) variables,

\[
\frac{1}{2} \left[ \left( -\partial^2 + x^2 \right) - \left( -\partial^2_0 + x^2_0 \right) \right] \psi_{\lambda} (\vec{x}, x_0) = \lambda \psi (\vec{x}, x_0) ,
\]

14
with a wavefunction of the form
\[ \psi_\lambda (\vec{x}, x_0) = A_{\lambda_a} (\vec{x}) B_{\lambda_b} (x_0), \quad \lambda = (\lambda_a - \lambda_b), \] (4.3)
such that
\[ \frac{1}{2} \left( -\partial^2 + x^2 \right) A_{\lambda_a} (\vec{x}) = \lambda_a A_{\lambda_a} (\vec{x}), \quad \frac{1}{2} \left( -\partial_0^2 + x_0^2 \right) B_{\lambda_b} (x_0) = \lambda_b B_{\lambda_b} (x_0). \] (4.4)

In a unitary Hilbert space in which \( x^\mu, p^\mu \) are all hermitian operators, both \( \lambda_a \) and \( \lambda_b \) must be positive since the operators \( \frac{1}{2} (\hat{p}^2 + \hat{x}^2) \) as well as \( \frac{1}{2} (p_0^2 + x_0^2) \) are positive. In fact, from the study of the Euclidean harmonic oscillator in \( d \) dimensions and 1 dimension respectively we already know all possible eigenvalues and eigenstates\(^\text{13}\) for \((\lambda_a, A_{\lambda_a} (\vec{x}))\) and for \((\lambda_b, B_{\lambda_b} (x_0))\), where

\[ \lambda_a = n_a + \frac{d}{2}, \text{ with } n_a = 0, 1, 2, 3, \ldots, \]
\[ \lambda_b = n_b + \frac{1}{2}, \text{ with } n_b = 0, 1, 2, 3, \ldots. \] (4.5)

Furthermore, we know that the wavefunctions take the form\(^\text{13}\)
\[ A_{\lambda_a} (\vec{x}) = e^{-\frac{1}{2} \vec{x}^2} \times \text{(polynomial of degree } n_a \text{ in the variables } x_i), \]
\[ B_{\lambda_b} (x_0) = e^{-\frac{1}{2} x_0^2} \times \text{(polynomial of degree } n_b \text{ in the variable } x_0). \] (4.6)

In this basis there is infinite degeneracy for the same eigenvalue of \( Q \rightarrow \lambda \), since eigenstates with different \( n_a, n_b \) can lead to the same eigenvalue \( \lambda = \lambda_a - \lambda_b = n + \frac{d-1}{2} \). Thus with both \( m, n \) even integers or with both \( m, n \) odd integers we can write

\[ n_a = \frac{m+n}{2}, \quad n_b = \frac{m-n}{2}, \]

at fixed \( n \), all \( m \geq |n| \) gives infinite degeneracy.

All solutions with the same eigenstate \( \lambda \) can be constructed from (infinite) linear combinations of the ones above, but they all must have the form
\[ \psi_\lambda (x^\mu) = e^{-\frac{1}{2} (\vec{x}^2 + x_0^2)} \times \text{(polynomials in the variables } x^\mu), \] (4.8)
\[ \lambda = n + \frac{d-1}{2}, \text{ with } n = 0, \pm 1, \pm 2, \pm 3, \ldots, \]

\(^\text{13}\) The wavefunction of an arbitrary excited state of the \( d \)-dimensional Euclidean harmonic oscillator at eigenvalue \( \lambda = n + d/2 \), and SO\((d)\) orbital angular momentum quantum number \( l \), has the form
\[ A^{n_l}_{i_1 i_2 \ldots i_l} (\vec{x}) = e^{-\frac{1}{2} \vec{x}^2} |\vec{x}|^l L_n^{l-1+d/2} (\vec{x}^2) T_{i_1 i_2 \ldots i_l} (\hat{x}). \]

Here \( T_{i_1 i_2 \ldots i_l} (\hat{x}) \) is the symmetric \textit{traceless} tensor of rank \( l \) constructed from the unit vector \( \hat{x}_i = x_i / |\vec{x}| \) (this is equivalent to the spherical harmonics in \( d = 3 \) space dimensions). \( L_n^\alpha (z) \) is the generalized Laguerre polynomial with argument \( z = \vec{x}^2 \), and indices \( \alpha = n \) and \( \beta = l - 1 + d/2 \). The quantum numbers take the following values: The excitation level \( n \) is any positive integer \( n = 0, 1, 2, 3, \ldots \), while at fixed \( n \) the allowed values of \( l \) are \( l = n, (n-2), (n-4), \ldots, (1 or 0) \).
It is evident that these solutions have positive norm since the integrals converge in all spacetime directions and they are positive

$$\langle \psi_\lambda | \psi_\lambda \rangle = \int d^{d+1}x |\psi_\lambda(x^\mu)|^2 = 1. \quad (4.9)$$

We have at hand definitely a unitary basis, but what are the Lorentz symmetry properties of these solutions?

The striking contrast to the solutions in the previous section is that the exponent \((\vec{x}^2 + x_0^2)\) is not Lorentz invariant, and hence these solutions and the solutions of the previous section are mutually exclusive. They each span different Hilbert spaces and the spacetime geometric properties are very different. The Lorentz symmetry properties of the solutions (4.8) are not yet evident.

On the other hand, the operator \(Q\) is invariant under SU\((d,1)\) and its Lorentz subgroup SO\((d,1)\), so we must be able to organize the solutions at each value \(\lambda\) in terms of the representations of SU\((d,1)\) and any of its subgroups. These representations are automatically unitary since we have already insured that \(x^\mu, p^\mu\), and therefore the Lorentz generators \(L_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu\), are hermitian in this basis. Hence, we must expect infinite dimensional unitary representations of SO\((d,1)\) and of SU\((d,1)\) at each \(\lambda\). In fact, this is in agreement with the infinite degeneracy at each \(\lambda\) noted above. There remains to answer what precisely are these unitary representations, and how to label states with quantum numbers within the representation?

We now answer this question. We will explain below that at each \(\lambda\) there is a single irreducible unitary representation of SU\((d,1)\) whose Casimir eigenvalues are completely determined by \(\lambda\) and \(d\). We will give the detailed content of this representation in the group theoretical basis when SU\((d,1)\) is decomposed into SU\((d) \times U(1)\). In this way we will be able to determine the SU\((d)\), and the angular momentum SO\((d) \subset SU(d)\), quantum numbers of each quantum state.

The starting point is a new vacuum state \(|\tilde{0}\rangle\) which is different than the Lorentz invariant vacuum states \(|0\rangle, |0'\rangle\) of the previous section. The new vacuum state is defined as the state for which the excitation numbers \(n_a, n_b\) are both zero. Hence, it is defined by the following equations

$$\bar{a}_0 |\tilde{0}\rangle = a_i |\tilde{0}\rangle = 0, \text{ so } \bar{a}_0 \text{ rather than } a_0 \text{ acts as annihilator}. \quad (4.10)$$

The position space representation of this state justifies this definition since the oscillators \(a_\mu, \bar{a}_\mu\) defined in Eq. (2.1) have the following form in position space and therefore they act on the state \(|\tilde{0}\rangle\) as creators/annihilators as indicated

$$\langle x|\tilde{0}\rangle \sim \exp \left(-\frac{x_0^2 + \vec{x}^2}{2} \right) \quad (4.11)$$

| annihilators: \(\bar{a}_0 = \frac{1}{\sqrt{2}} \left(x_0 + \frac{\partial}{\partial x_0}\right)\), \(a_i = \frac{1}{\sqrt{2}} \left(x_i + \frac{\partial}{\partial x_i}\right)\) \quad (4.12) |

| creators: \(a_0 = \frac{1}{\sqrt{2}} \left(x_0 - \frac{\partial}{\partial x_0}\right)\), \(\bar{a}_i = \frac{1}{\sqrt{2}} \left(x_i - \frac{\partial}{\partial x_i}\right)\) \quad (4.13) |
The extra sign in front of $\frac{\partial}{\partial x_0}$ in $a_0, \tilde{a}_0$ is due to lowering the timelike index with the Minkowski metric $p_0 = -i \frac{\partial}{\partial x^0} = +i \frac{\partial}{\partial x_0}$. Then it is convenient to define the excitation number operators as

$$\hat{N}_a = \tilde{a}_i a_i, \quad \hat{N}_b = a_0 \tilde{a}_0,$$

where the orders of $a_0 \tilde{a}_0$ are reversed compared to traditional notation. The eigenvalues $(n_a, n_b)$ of these operators vanish on $|\tilde{0}\rangle$

$$\hat{N}_a |\tilde{0}\rangle = 0, \quad \hat{N}_b |\tilde{0}\rangle = 0. \quad (4.15)$$

It should be noted that the Lorentz covariant commutation rule in the timelike direction $[a_0, \tilde{a}_0] = -1$ indicates that an excited state of the form $(a_0)^{n_b} |\tilde{0}\rangle$ is correctly identified as an eigenstate of $\hat{N}_b = a_0 \tilde{a}_0$ with eigenvalue $n_b$

$$\hat{N}_b \{(a_0)^{n_b} |\tilde{0}\rangle\} = [a_0\tilde{a}_0, (a_0)^{n_b}] |\tilde{0}\rangle = a_0 \{\tilde{a}_0, (a_0)^{n_b}\} |\tilde{0}\rangle = n_b \{(a_0)^{n_b} |\tilde{0}\rangle\}. \quad (4.16)$$

The general state of the form (4.5) with $n_a, n_b$ excitations has the Fock space representation

$$|n_a, n_b\rangle = (\tilde{a}_{i_1} \tilde{a}_{i_2} \cdots \tilde{a}_{i_{n_a}}) (a_0)^{n_b} |\tilde{0}\rangle, \quad (4.17)$$

where each index $i_k$ labels a vector of $SO(d)$ as well as the fundamental representation of $SU(d)$.

In term of these, the total level operator $J_0 = \tilde{a}_i a_i - a_0 \tilde{a}_0 - 1$, which we identified in Eq.(2.7) becomes $J_0 = \tilde{a}_i a_i - a_0 \tilde{a}_0 - 1$, or

$$J_0 = \hat{N}_a - \hat{N}_b - 1. \quad (4.18)$$

Therefore the total level of the vacuum state $|\tilde{0}\rangle$ is

$$J_0 |\tilde{0}\rangle = \left(\hat{N}_a - \hat{N}_b - 1\right) |\tilde{0}\rangle = (-1) |\tilde{0}\rangle. \quad (4.19)$$

We contrast this $(-1)$ eigenvalue with the $J_0$ eigenvalues of the vacua $|0\rangle, |0'\rangle$ which were 0 and $(-d - 1)$ respectively, as shown in Eqs.(3.2,3.7). We also see that the $Q \rightarrow \lambda$ eigenvalue of the vacuum is $\lambda = \frac{d-1}{2}$

$$Q |\tilde{0}\rangle = \left(J_0 + \frac{d+1}{2}\right) |\tilde{0}\rangle = \frac{d-1}{2} |\tilde{0}\rangle. \quad (4.20)$$

Similarly, for the general state $|n_a, n_b\rangle$ we have

$$J_0 |n_a, n_b\rangle = (n_a - n_b - 1) |n_a, n_b\rangle, \quad Q |n_a, n_b\rangle = \left(n_a - n_b + \frac{d-1}{2}\right) |n_a, n_b\rangle \quad (4.21)$$

in agreement with Eq.(4.8).

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14 It may be helpful to define a new notation for the timelike oscillators, $\tilde{a}_0 \equiv b$ and $a_0 \equiv \tilde{b}$, so that the operators that have the bar on top, namely $\tilde{b}, \tilde{a}_i$ are creation operators. Indeed the $b, \tilde{b}$ satisfy the usual commutation rules with the +1 on the right hand side, $[b, \tilde{b}] = [\tilde{a}_0, a_0] = +1$ similar to $[a_i, \tilde{a}_j] = \delta_{ij}$. Then $N_b = \tilde{b}b = a_0 \tilde{a}_0$ is the familiar excitation number.
It must now be emphasized that the vacuum state $|\tilde{0}\rangle$ is neither Lorentz nor SU($d$, 1) invariant since the Lorentz boost operators $L_{0i} = i(\bar{a}_0a_i - \bar{a}_ia_0)$ or the SU($d$, 1) generators $J_{0i} = \bar{a}_0a_i$ contain two creation operators. So the vacuum $|\tilde{0}\rangle$ cannot be invariant under the subset of SO($d$, 1) or SU($d$, 1) infinitesimal transformations generated by the operators that contain double creation

$$L_{0i}|\tilde{0}\rangle \neq 0, \quad J_{0i}|\tilde{0}\rangle \neq 0. \quad (4.22)$$

However, this structure of double creators or double annihilators is tailor made for the oscillator approach to representation theory for non-compact groups or supergroups developed in [21]-[24]. Using those techniques we will classify the states as parts of infinite dimensional unitary representations as explained below.

First we note that the oscillators $a_{\mu}$ that are in the fundamental representation of SU($d$, 1) contain both creation and annihilation operators (see footnote (14) for $a_0 \equiv \bar{b}$)

$$a_{\mu} = \begin{pmatrix} a_0 \\ a_i \end{pmatrix} = \begin{pmatrix} \bar{b} \\ a_i \end{pmatrix}. \quad (4.23)$$

Therefore a general SU($d$, 1) transformation mixes creation with annihilation operators. Similarly the anti-fundamental representation given by $\bar{a}_{\mu} = (\bar{a}_0 \bar{a}_i) = (b \bar{a}_j)$ has the same property, and so does the adjoint representation of SU($d$, 1) which classifies the generators as the traceless product of the fundamental and anti-fundamental

$$J_{\mu\nu} = \bar{a}_{\mu}a_{\nu} - \frac{\eta_{\mu\nu}}{d+1} \bar{a} \cdot a = \begin{pmatrix} \bar{a}_0a_0 + \frac{\bar{a}_0a_j}{d+1} & \bar{a}_0a_j - \frac{\delta_{ij} \bar{a} \cdot a}{d+1} \\ \bar{a}_ja_0 & \bar{a}_ja_j - \frac{\delta_{ij} \bar{a} \cdot a}{d+1} \end{pmatrix} \equiv \begin{pmatrix} J_{00} & J_{0j} \\ J_{ij} & J_{ij} \end{pmatrix}. \quad (4.24)$$

All of these $J_{\mu\nu}$ are symmetries of the operator $Q$ as we noted earlier. The double annihilation part of $J_{\mu\nu}$ is the upper right corner $J_{0j} = \bar{a}_0a_j = ba_j$ and the double creation part is the lower left corner $J_{ij} = \bar{a}_i\bar{b}$ of this matrix. Note that the $d \times d$ matrix $J_{ij}$ has a traceless part $q_{ij}$ while its trace is related to the remaining generator $J_{00}$ as follows

$$J_{ij} = q_{ij} + \delta_{ij} \frac{J_{00}}{d}, \quad J_{00} = \frac{q_0}{d+1}. \quad (4.25)$$

The generators of the subgroup SU($d$) $\times$ U$_q$ $(1) \times$ U$_{J_{0}}$ $(1) \subset$ SU($d$, 1) $\times$ U$_{J_{0}}$ $(1)$ are then

$$q_{ij} = \bar{a}_ia_j - \frac{\delta_{ij}}{d} \hat{N}_a, \quad q_0 = \hat{N}_a + d \left( \hat{N}_b + 1 \right), \quad J_{00} = \hat{N}_a - \hat{N}_b - 1. \quad (4.26)$$

The general excited state in Eq.(4.17) $|n_a, n_b\rangle$ can now be identified by its SU($d$) $\times$ U$_q$ $(1) \times$ U$_{J_{0}}$ $(1)$ quantum numbers, by using a Young tableau as follows

$$|n_a, n_b\rangle = \left( \bar{a}_{i_1}\bar{a}_{i_2} \cdots \bar{a}_{i_{n_a}} \right) (a_0)^{n_b} |\tilde{0}\rangle \quad (4.27)$$

$$= \left[ \begin{array}{cccc} i_1 & i_2 & i_3 & \cdots & i_{n_a} \\ q_0 & n_0 \end{array} \right] \quad (4.28)$$
\[ q_0 = n_a + d(n_b + 1), \quad n_0 = n_a - n_b - 1, \] (4.29)

Note that the eigenvalue \( q_0 \) is a positive integer such that \( q_0 - n_a = d(n_b + 1) \) is positive and furthermore it is a non-zero multiple of \( d \). The Young tableau corresponds to a completely symmetric SU\((d)\) tensor of rank \( n_a \) which fully describes the SU\((d)\) content of the state \(|n_a, n_b\rangle\).

This tensor together with the labels \( \tilde{q}_0 \to q_0 \) and \( J_0 \to n_0 \), or equivalently \( Q \to \lambda = n_0 + \frac{d+1}{2} = n_a - n_b + \frac{d-1}{2} \), are a complete set of quantum numbers for any representation of SU\((d, 1) \times U(J_0) \) that appears in this theory.

The orbital angular momentum \( l \) of any state corresponds to its SO\((d)\) representation. The rank \( l \) of a traceless symmetric tensor determines the angular momentum. The completely symmetric tensor of SU\((d)\) in Eq.(4.27) is decomposed into traceless symmetric tensors of rank \( l \) as follows

\[
\text{SO\((d)\) tensors: } \lambda = n_a, (n_a - 2), (n_a - 4), \cdots, (1 \text{ or } 0) .
\] (4.30)

where each state with angular momentum \( l \) at levels \( n_b \) and \( n_a = l + 2r \) is given by

\[
(\bar{a}_{i_1} \bar{a}_{i_2} \cdots \bar{a}_{i_l} - \text{trace}) (\bar{a}_j \bar{a}_j)^{r} (a_0)^{n_\lambda} |\bar{0}\rangle ,
\] (4.31)

Hence the states \(|n_a, n_b\rangle\) contain a direct sum of states of the type (4.31) with the angular momenta \( l \) specified in Eq.(4.30).

Now we are ready to identify all the states in the same infinite dimensional representation of SU\((d, 1) \times U(J_0) \). For a fixed \( J_0 \), or equivalently a fixed \( Q = J_0 + \frac{d+1}{2} \to n + \frac{d-1}{2} \), we must include all the states \(|n_a, n_b\rangle\) that satisfy \( n_a - n_b = n \). These may be presented as a direct sum of states, meaning any linear combination of those states

\[
\lambda = \frac{d-1}{2} + n : \left\{ \sum_{k=0}^{\infty} (\bar{a}_{i_1} \bar{a}_{i_2} \cdots \bar{a}_{i_k+n}) (a_0)^{k} |\bar{0}\rangle, \text{ if } n \geq 0 \right. \\
\left. \sum_{k=0}^{\infty} (\bar{a}_{i_1} \bar{a}_{i_2} \cdots \bar{a}_{i_k}) (a_0)^{k+n} |\bar{0}\rangle, \text{ if } n \leq 0 \right. 
\] (4.32)

More explicitly we give the example of \( n = 0 \) by writing it out

\[
\lambda = \frac{d-1}{2} : |\bar{0}\rangle \oplus [\bar{a}_i a_0 |\bar{0}\rangle] \oplus [\bar{a}_i \bar{a}_j (a_0)^2 |\bar{0}\rangle] \oplus [\bar{a}_i \bar{a}_j \bar{a}_k (a_0)^3 |\bar{0}\rangle] \oplus \cdots,
\] (4.33)

and similarly for \( n = 1, -1 \)

\[
\lambda = \frac{d-1}{2} + 1 : \bar{a}_i |\bar{0}\rangle \oplus [\bar{a}_i \bar{a}_j a_0 |\bar{0}\rangle] \oplus [\bar{a}_i \bar{a}_j \bar{a}_k (a_0)^2 |\bar{0}\rangle] \oplus \cdots,
\] (4.34)

\[
\lambda = \frac{d-1}{2} - 1 : a_0 |\bar{0}\rangle \oplus [\bar{a}_i (a_0)^2 |\bar{0}\rangle] \oplus [\bar{a}_i \bar{a}_j (a_0)^3 |\bar{0}\rangle] \oplus \cdots.
\] (4.35)

Evidently each distinct value of \( \lambda \) completely determines the allowed \(|n_a, n_b\rangle\) and the corresponding SU\((d) \times U(1) \times U(1)\) tensors of each infinite dimensional tower. Note also that for each \( \lambda \) there is a single tower.

It is easy to show that each tower at fixed \( \lambda \) is an irreducible representation of SU\((d, 1)\). Under an SU\((d, 1)\) group transformation \( g = \exp (i \omega^{\mu \nu} J_{\mu \nu}) \) towers with differing eigenvalues
\( \lambda \neq \lambda' \) cannot mix with each other since \( J_{\mu \nu} \) commutes with \( Q \). Hence a single tower with fixed \( \lambda \) is irreducible under the \( \text{SU}(d, 1) \) group transformation. Furthermore, all the states within each tower mix because the double creation operators \( J_{i0} = \bar{a}_i a_0 = \bar{a}_i \bar{b} \) and the double annihilation operators \( J_{0j} = \bar{a}_0 a_j = b a_j \) applied repeatedly mix all the states under the \( \text{SU}(d, 1) \) group transformation \( g = \exp (i \omega^{\mu \nu} J_{\mu \nu}) \).

In fact, all states in a given tower are obtained by applying repeatedly the double creation \( \text{SU}(d, 1) \) group generators \( J_{i0} = \bar{a}_i a_0 = \bar{a}_i \bar{b} \) on the lowest state

\[
|\text{tower}\rangle_\lambda = \left\{ \sum_{k=0}^{\infty} \bigoplus (J_{i_10} J_{i_20} \cdots J_{i_k0}) \right\} |\text{lowest}\rangle_\lambda. \tag{4.36}
\]

Therefore, only the lowest state in the tower is sufficient to label uniquely the \( \text{SU}(d, 1) \) content of the entire tower. These unique labels correspond to the \( \text{SU}(d) \) Young tableau and the \( U_1(1) \) charge

\[
\hat{q}_0 = \hat{N}_a + d \left( \hat{N}_b + 1 \right),
\]

identified in Eq. (4.26). These are the appropriate quantum numbers for the basis \( \text{SU}(d) \times U_1(1) \subset \text{SU}(d, 1) \) at a fixed \( \lambda \)

\[
|\text{lowest}\rangle = \begin{pmatrix}
  n_1 \\
  n_2 \\
  n_3 \\
  \vdots \\
  n_a
\end{pmatrix}, \quad q_0(\lambda) = n_a (d + 1) - d \left( \lambda - \frac{d + 1}{2} \right). \tag{4.37}
\]

We can easily compute the Casimir operators for the irreducible unitary representations identified above. The quadratic Casimir operator of \( \text{SU}(d, 1) \) is given by

\[
C_2 = \frac{1}{2} J_{\mu \nu} \eta^{\nu \lambda} J_{\lambda \sigma} \eta^{\sigma \mu} = \frac{1}{2} \left( J_{ij} J_{ji} + (J_{00})^2 - J_{i0} J_{0i} - J_{0i} J_{i0} \right). \tag{4.38}
\]

After inserting the oscillator form of the \( J_{\mu \nu} \) given in Eq. (4.24), and rearranging the oscillators we find that \( C_2 \) is rewritten as a function of only the \( U_1(1) \) generator

\[
C_2 (\text{SU} (d, 1)) = \frac{d J_0}{2} \left( \frac{J_0}{d + 1} + 1 \right). \tag{4.39}
\]

Hence \( C_2 \) is diagonal on any state \( |n_a, n_b\rangle \)

\[
C_2 |n_a, n_b\rangle = \frac{d (n_a - n_b - 1) (n_a - n_b + d)}{2 (d + 1)} |n_a, n_b\rangle, \tag{4.40}
\]

and it has the same eigenvalue for all the states in the same tower as follows

\[
C_2 |\text{tower}\rangle_\lambda = \frac{1}{2} d \left( \lambda - \frac{d - 1}{2} \right) \left[ \frac{1}{d + 1} \left( \lambda - \frac{d - 1}{2} \right) + 1 \right] |\text{tower}\rangle_\lambda \tag{4.41}
\]

Similarly, all \( \text{SU}(d, 1) \) Casimir operators \( C_n \sim Tr \ (J)^n \) are found to be only a function of \( J_0 \), so all Casimir eigenvalues are functions of only \( \lambda \).

This result on the Casimirs \( C_n \) confirms that the full \( \text{SU}(d, 1) \) properties of each tower are completely determined by the eigenvalue of the operator \( Q \rightarrow \lambda \). Indeed, as seen explicitly in
Eqs. (4.32)-(4.34), all the states in each tower, and their $\text{SU}(d) \times U_q(1) \times U_{J_0}(1)$ quantum numbers, are pre-determined by the fixed value of $\lambda$.

Now that we have determined that each $|\text{tower}\rangle_\lambda$ corresponds to a single unitary representation of $\text{SU}(d,1)$, what can we say about which unitary representations of the Lorentz group $\text{SO}(d,1)$ classify the quantum states? In particular which eigenvalues of the $\text{SO}(d,1)$ Casimir $C_2 = \frac{1}{2} L_{\mu\nu} L^{\mu\nu}$ appear? This is predetermined by the group theoretical branching rules $\text{SU}(d,1) \to \text{SO}(d,1)$ as applied to each representation. From this it is evident that each $|\text{tower}\rangle_\lambda$ of the type (4.32) can be written as an infinite direct sum of unitary representations of $\text{SO}(d,1)$.

$$|\text{tower}\rangle_\lambda = \sum \oplus |\text{SO}(d,1) \text{ irreps}\rangle_\lambda.$$  (4.42)

It is not easy to see directly in the oscillator formalism precisely which eigenvalues of $C_2 (\text{SO}(d,1)) = \frac{1}{2} L_{\mu\nu} L^{\mu\nu}$ appear in this sum. This is because the natural Fock basis $|n_a, n_b\rangle$ we used above is labelled by the eigenvalues of the operators $\hat{N}_a, \hat{N}_b$ which are not simultaneous observables with this Casimir

$$\left[ \frac{1}{2} L_{\mu\nu} L^{\mu\nu}, \hat{N}_a \right] \neq 0, \quad \left[ \frac{1}{2} L_{\mu\nu} L^{\mu\nu}, \hat{N}_b \right] \neq 0,$$  (4.43)

although $\hat{N}_a - \hat{N}_b$ is. So, we do not expect that the operator $\frac{1}{2} L_{\mu\nu} L^{\mu\nu}$ would be diagonal in the basis $|n_a, n_b\rangle$. Indeed if we construct the $\text{SO}(d,1)$ Casimir operator

$$C_2 (\text{SO}(d,1)) = \frac{1}{2} L_{\mu\nu} L^{\mu\nu} = -\frac{1}{2} (J_{\mu\nu} - J_{\nu\mu}) (J^{\mu\nu} - J^{\nu\mu}),$$  (4.44)

$$= -(J_{\mu\nu} J^{\mu\nu}) + J_{\mu\nu} J^{\nu\mu} = -(J_{\mu\nu} J^{\mu\nu}) + 2C_2 (\text{SU}(d,1)),$$  (4.45)

we see that the last part $2C_2 (\text{SU}(d,1))$ is diagonal on each state of the $|\text{tower}\rangle_\lambda$, but the first part $J_{\mu\nu} J^{\mu\nu}$ contains double creation and double annihilation pieces and hence it cannot be diagonal in the basis $|n_a, n_b\rangle$. However, it is guaranteed that this basis can be rearranged to the form (4.42), as a superposition of unitary representations of the Lorentz group $\text{SO}(d,1)$ with diagonal $\frac{1}{2} L_{\mu\nu} L^{\mu\nu}$, simply because at fixed $n$ we have an irreducible representation of $\text{SU}(d,1)$. When each $\text{SO}(d,1)$ representation in (4.42) is branched down to the $\text{SO}(d)$ subgroup of $\text{SO}(d,1)$, then the $\text{SO}(d)$ quantum numbers must agree with those given in Eq. (4.30), namely $l = n_a, (n_a - 2), \ldots , (0 \text{ or } 1)$. So, we can deduce that those $\text{SO}(d,1)$ representations that contain this set of angular momenta must enter in expressing $|n_a, n_b\rangle$ in terms of an $\text{SO}(d,1)$ basis.

V. UNITARITY CONSTRAINTS ON THE FULL THEORY

We have examined above three distinct Fock spaces based on the three vacua $|0\rangle, |0'\rangle, |\bar{0}\rangle$. All the states in these Fock spaces are eigenstates of the same operator $Q$. After including the unitarity condition we found all the physically acceptable positive norm states.

21
In the quantum theory the existence of different sectors is the analog of different boundary conditions on the solutions of a given differential equation. We saw that the unitary sectors based on $|0\rangle$ and $|0\rangle'$ are all Lorentz invariant and they are distinguished from each other by being in the spacelike or timelike regions of spacetime. On the other hand, none of the unitary states $|n_a, n_b\rangle$ or $|\text{towers}\rangle_\lambda$ based on the vacuum $|\tilde{0}\rangle$ are Lorentz singlets, since $C_2$ is non-vanishing on any of them. So, the different sectors may be distinguished on the basis of their Lorentz, SU($d, 1$) and geometric properties.

In the absence of boundary conditions that naturally emerge for a specific physical system all sectors are a priori included. How can we insure that negative norm ghosts will not appear? We saw that although the sector $|\tilde{0}\rangle$ is free of ghosts, the sectors $|0\rangle$, $|0\rangle'$ contained them. It is only by imposing unitarity by “hand”, or equivalently by requiring Lorentz singlets (which may be viewed as a boundary condition), that we could distinguish the positive norm singlets in the sectors $|0\rangle$, $|0\rangle'$. However, requiring Lorentz invariants only as boundary conditions on the solutions of the entire theory eliminates also the $|\tilde{0}\rangle$ sector completely.

A more comprehensive set of constraints is of the form
\[ \frac{1}{2} (p^2 + x^2) - \lambda_0 = 0. \] (5.1)
This allows states from all sectors $|0\rangle$, $|0\rangle'$, $|\tilde{0}\rangle$ as long as $\lambda_0$ is an eigenvalue of $Q = \frac{1}{2} (p^2 + x^2)$. The possible eigenvalues in each sector were

- $|0\rangle : \lambda = \frac{d + 1}{2} + \text{(positive integer)}$ (5.2)
- $|0\rangle' : \lambda = -\frac{d + 1}{2} - \text{(positive integer)}$ (5.3)
- $|\tilde{0}\rangle : \lambda = \frac{d - 1}{2} + \text{(positive or negative integer)}$ (5.4)

We argued in Eq.(3.28) that the only way to avoid ghosts in the spacelike or timelike sectors was to choose $\lambda_0 = \pm \frac{d + 1}{2}$. Such values of $\lambda_0$ include only the vacua $|0\rangle$, $|0\rangle'$ respectively in the spacelike and timelike sectors, and also the infinite number of states in the $|\text{tower}\rangle_\lambda$ in the $|\tilde{0}\rangle$ sector. Moreover, if we choose $\lambda_0$ in the range $\lambda_0 = 0, \pm 1, \pm 2, \cdots, \pm \frac{d - 1}{2}$ we include only the corresponding towers $|\text{tower}\rangle_{\lambda_0}$ in the $|\tilde{0}\rangle$ sector, but no states at all from the spacelike or timelike sectors based on $|0\rangle$, $|0\rangle'$.

Hence, if the theory is restricted to the following range only
\[ -\frac{d + 1}{2} \leq \frac{1}{2} (p^2 + x^2) \leq \frac{d + 1}{2}, \text{ unitary range,} \] (5.5)

15 In a theory with more degrees of freedom more general constraints can also be considered, see footnote (10).

16 We have not discussed at all the possibility of solutions in the spacelike and timelike sectors that are matched across the lightcone $x^2 = 0$ as outlined following Eq.(M). It is possible that those are already accounted for in the $|\tilde{0}\rangle$ sector, but we are not certain if there are additional ones. If those have $\lambda$’s within the range in Eq.(5.5) they will be part of the constrained theory.
then it is guaranteed to be a unitary theory without any negative norm ghosts. If \( \frac{1}{2} (p^2 + x^2) \) is taken outside of this range then there will always be ghosts coming from the sectors \( |0\rangle, |0'\rangle \). For definiteness we list all the quantum states that satisfy this range

\[
\lambda = \frac{d+1}{2} : |0\rangle \oplus \bar{a}_i \sum_{m=0}^{\infty} \oplus (\bar{a}_{i_1} \bar{a}_{i_2} \cdots \bar{a}_{i_m}) (a_0)^m |\tilde{0}\rangle, \tag{5.6}
\]

\[
\lambda = \frac{d-1}{2} : \sum_{m=0}^{\infty} \oplus (\bar{a}_{i_1} \bar{a}_{i_2} \cdots \bar{a}_{i_m}) (a_0)^m |\tilde{0}\rangle, \tag{5.7}
\]

\[
\lambda = \frac{d-3}{2} : \sum_{m=0}^{\infty} \oplus (\bar{a}_{i_1} \bar{a}_{i_2} \cdots \bar{a}_{i_m}) (a_0)^{m+1} |\tilde{0}\rangle, \tag{5.8}
\]

\[
\lambda = \frac{-d-3}{2} : \sum_{m=0}^{\infty} \oplus (\bar{a}_{i_1} \bar{a}_{i_2} \cdots \bar{a}_{i_m}) (a_0)^{m+d-2} |\tilde{0}\rangle, \tag{5.9}
\]

\[
\lambda = \frac{-d-1}{2} : \sum_{m=0}^{\infty} \oplus (\bar{a}_{i_1} \bar{a}_{i_2} \cdots \bar{a}_{i_m}) (a_0)^{m+d-1} |\tilde{0}\rangle, \tag{5.10}
\]

\[
\lambda = \frac{-d+1}{2} : |0'\rangle \oplus \sum_{m=0}^{\infty} \oplus (\bar{a}_{i_1} \bar{a}_{i_2} \cdots \bar{a}_{i_m}) (a_0)^{m+d} |\tilde{0}\rangle, \tag{5.11}
\]

Note that the cases of \( \lambda = \pm \frac{d+1}{2} \) includes the Lorentz singlets \( |0\rangle, |0'\rangle \), but these singlets do not appear for the other listed values of \( \lambda \). Furthermore note that only for \( \lambda = + \frac{d+1}{2} \) there is an additional \( \bar{a}_i \) outside of the sum in Eq.\((5.6)\). This makes \( |0\rangle \) evidently orthogonal to the tower at \( \lambda = + \frac{d+1}{2} \). The lowest state in each case has \( \text{SO}(d) \) angular momentum zero \( l = 0 \). Only the case of \( \lambda = - \frac{d+1}{2} \) has two zero angular momentum states one of which is a \( \text{SU}(d,1) \) singlet while the other is not.

**VI. WORLDLINE THEORY WITH GAUGE SYMMETRY**

A theory with constraints is obtained by constructing a gauge invariant action. Each constraint is the generator of a gauge symmetry. The gauge symmetry can be used to eliminate degrees of freedom and in particular it can remove ghosts and render the theory to be unitary.

A constraint of the type

\[
\phi(x, p) = \frac{1}{2} (p^2 + x^2) - \lambda_0 = 0 \tag{6.1}
\]

is obtained in the following worldline theory

\[
S(\lambda_0) = \int d\tau \left( \dot{x}^\mu p_\mu - e(\tau) \left[ \frac{1}{2} (p^2 + x^2) - \lambda_0 \right] \right) \tag{6.2}
\]
where $e(\tau)$ is the gauge field that plays a role of a Lagrange multiplier locally on the worldline at each instant $\tau$. The gauge transformations with a local parameter $\Lambda(\tau)$ are

$$
\delta_{\Lambda}x^\mu(\tau) = \Lambda(\tau)p^\mu(\tau), \quad \delta_{\Lambda}p^\mu(\tau) = -\Lambda(\tau)x^\mu(\tau), \quad \delta_{\Lambda}e(\tau) = \frac{d}{d\tau}\Lambda(\tau). \quad (6.3)
$$

The Lagrangian transforms to a total derivative

$$
\delta_{\Lambda}S(\lambda_0) = \int d\tau \frac{d}{d\tau} \left( \frac{1}{2} (p^2 - x^2) \Lambda(\tau) - \lambda_0 \Lambda(\tau) \right) \to 0. \quad (6.4)
$$

which can be dropped in the variation of the action (note $p^2 - x^2$, not $p^2 + x^2$). Hence this action has a local gauge symmetry $\delta_{\Lambda}S = 0$.

One consequence of the gauge symmetry is to impose constraint (6.1) as the equation of motion for the gauge field

$$
0 = \frac{\partial S}{\partial e(\tau)} = \phi(x, p) = \frac{1}{2} (p^2 + x^2) - \lambda_0. \quad (6.5)
$$

The generator of the gauge transformations is $\phi(x, p)$. Saying that $\phi(x, p)$ vanishes is equivalent to saying that the generator of gauge transformations vanishes, meaning that the sector that satisfies it must be gauge invariant.

There are various ways to quantize the theory defined by the $S(\lambda_0)$ above. The first approach is covariant quantization in which we work with the quantum rules $[x_\mu, p_\nu] = i\eta_{\mu\nu}$, in an enlarged Hilbert space that includes all the degrees freedom, including the redundant gauge degrees of freedom that are part of $x^\mu, p^\mu$. Then among the quantum states in this enlarged space we pick the gauge invariant physical states by demanding that they satisfy the vanishing of the gauge generator

\begin{equation}
\text{gauge invariants} : \left[ \frac{1}{2} (p^2 + x^2) - \lambda_0 \right] |\text{physical}\rangle = 0. \quad (6.6)
\end{equation}

If we follow this approach we see that the gauge invariant states $\langle x|\text{physical}\rangle = \psi_{\lambda_0}(x)$ are only those that satisfy the differential equation of the relativistic harmonic oscillator with a fixed eigenvalue $\lambda_0$

$$
\left( -\frac{1}{2} \partial^\mu \partial_\mu + \frac{1}{2} x^\mu x_\mu \right) \psi_{\lambda_0}(x) = \lambda_0 \psi_{\lambda_0}(x). \quad (6.7)
$$

There is no mention of boundary conditions and therefore we must include all sectors that solve this constraint. This is the problem we analyzed in the previous sections. From that analysis we conclude that provided $\lambda_0$ is chosen as one of the quantized values in the range (5.5), then the resulting theory $S(\lambda_0)$ is guaranteed to be a ghost free unitary theory.

Outside of this range we expect that ghosts will be present. Therefore $S(\lambda_0)$ with $\lambda_0$ fixed to any one of the values $\lambda_0 = -\frac{d+1}{2}, -\frac{d-1}{2}, \ldots, \frac{d-1}{2}, \frac{d+1}{2}$, leads to a physically acceptable unitary theory.
A second approach is non-covariant quantization in which we first choose a gauge and solve the constraint once and for all. The phase space that solves $\frac{1}{2}(p^2 + x^2) = \lambda_0$ is then automatically a parametrization of the gauge invariant sector. However one must be careful that there may be more than one sector of phase space that can solve this equation at the classical level. If we choose a gauge in which the timelike degree of freedom is eliminated, then the remaining Euclidean degrees of freedom cannot introduce any negative norm ghosts. The quantum states are then automatically unitary, but one must check that non-linear expressions are properly quantum ordered so as to insure that the global symmetries of the theory have not been violated. Only if the global symmetries are treated properly - in the present case SU(d,1) and its subgroup SO(d,1) - can one declare that the theory has been successfully quantized in the gauge fixed version. In what follows we show how this is done in the present theory defined by the action $S(\lambda_0)$, and how the results agree with the SU(d,1) properties of the covariant quantization approach.

VII. GAUGE FIXED QUANTIZATION

We can choose a gauge that reduces the theory to the purely spacelike harmonic oscillator. Let us first consider the following canonical transformation from $(x_0(\tau), p_0(\tau))$ to $(t(\tau), H(\tau))$ at the classical level (i.e. quantum ordering ignored)

$$
x_0(\tau) = \sqrt{2H(\tau) + 2c} \sin(t(\tau)), \quad p_0(\tau) = \sqrt{2H(\tau) + 2c} \cos(t(\tau)),
$$

(7.1)

where $c$ is some constant that will be fixed later. This covers the entire $(x_0, p_0)$ plane if $H(\tau) + c \geq 0$. The new set $(t, H)$ is canonical as can be seen by computing the corresponding term in the Lagrangian

$$
-\dot{x}_0\dot{p}_0 = -iH + \text{total derivatives}.
$$

The total derivatives can be dropped since they are irrelevant in the action. The Lagrangian in Eq. (6.2) takes the form

$$
L = -iH + \dot{x}^i p_i - e \left[ \frac{1}{2} (\vec{p}^2 + \vec{x}^2) - H - c - \lambda_0 \right],
$$

(7.2)

which shows that the constraint $\phi(x, p)$ that vanishes in the physical sector now has taken the form

$$
\phi(x, p) = \frac{1}{2} (\vec{p}^2 + \vec{x}^2) - H - c - \lambda_0 = 0.
$$

(7.3)

Next we choose the gauge

$$
t(\tau) = \tau,
$$

(7.4)

and solve the constraint $\phi(x, p) = 0$ to determine the canonical conjugate of the gauge fixed $t$, namely $H(\tau)$

$$
H = \frac{1}{2} (\vec{p}^2 + \vec{x}^2) - c - \lambda_0.
$$

(7.5)
The gauge fixed form of the action $S(\lambda_0)$ above describes precisely the spacelike non-relativistic harmonic oscillator after using $i=1$

$$S_{\text{fixed}}(\lambda_0) = \int d\tau \left( \partial_\tau \vec{x} \cdot \vec{p} - \left[ \frac{1}{2} (\vec{p}^2 + \vec{x}^2) - c - \lambda_0 \right] \right).$$ \hspace{1cm} (7.6)

It is possible to fix the constant $c$ in terms of $\lambda_0$, but this is not necessary at this stage because $(-c - \lambda_0)$ seems as an irrelevant constant that may be dropped. We will wait till we compute SU($d,1$) Casimir eigenvalues at the quantum level to learn the role of $c$ and its relationship to $\lambda_0$ when we compare the results of covariant quantization to those of the gauge fixed quantization.

The quantum states of this non-relativistic harmonic oscillator in $d$ Euclidean dimensions are well known. They are constructed by defining creation-annihilation operators $a_i, \bar{a}_i$ in the usual way and applying them on a vacuum $|\tilde{0}\rangle$ that diagonalizes this Hamiltonian

$$a_i |\tilde{0}\rangle = 0, \quad \langle \vec{x}|\tilde{0}\rangle \sim \exp \left( -\frac{1}{2} \vec{x}^2 \right).$$ \hspace{1cm} (7.7)

The general quantum state is a superposition of the following states that make up a tower

$$|\text{tower} \rangle_{\lambda_0} = \sum_{n_a=0}^\infty \oplus |n_a\rangle = \sum_{n_a=0}^\infty \oplus (\bar{a}_{i_1} \bar{a}_{i_2} \cdots \bar{a}_{i_{n_a}}) |\tilde{0}\rangle$$

$$\sim \sum_{n_a=0}^\infty \oplus \begin{bmatrix} i_1 \ i_2 \ i_3 \ \cdot \ \cdot \ \cdot \ i_{n_a} \end{bmatrix} .$$ \hspace{1cm} (7.8)

We compare this spectrum to the towers listed in Eqs.(5.7-5.10). From the comparison we see that the gauge fixed version reproduces the spectrum of the covariant quantum theory for the action $S(\lambda_0)$ at fixed values of $\lambda_0$, provided $\lambda_0$ is fixed to one of the values

$$\lambda_0 = \frac{d+1}{2}, \frac{d-1}{2}, \frac{d-3}{2}, \cdots, -\frac{d-3}{2}, -\frac{d-1}{2},$$ \hspace{1cm} (7.10)

but not the value $\lambda_0 = -\frac{d+1}{2}$, since in that last case there is an additional state $|0\rangle'$ in Eq.(5.11) which does not show up in Eq.(7.8).

As we will see below, the gauge fixed version (7.8) reproduces the subtlety that for $\lambda_0 = \frac{d+1}{2}$ there is a Lorentz invariant state $|0\rangle$ as listed in Eq.(5.6). That is, at $\lambda_0 = \frac{d+1}{2}$ the tower in (7.8) is actually split into two representations of SU($d,1$). But the gauge fixed version could not reproduce the other Lorentz invariant state $|0\rangle'$ at $\lambda_0 = -\frac{d+1}{2}$ in Eqs.(5.11). Similarly, the unitary sector $|n_a, n_b\rangle$ for all $n_b < n_a$ that appears in covariant quantization is entirely missed in the fixed gauge. By contrast all the states $|n_a, n_b\rangle$ for $n_b \geq n_a$, are recovered in the gauge fixed version (7.8) even those beyond the list in (7.10).

The discrepancy between covariant quantization and gauge fixed quantization is attributable to an assumption made inadvertently when making the gauge choice. Namely the canonical
transformation (7.1) is valid only when $\sqrt{H + c}$ is real. After using Eq. (7.5), we see that the reality condition requires

$$0 \leq H + c = \frac{1}{2} (p^2 + x^2) - \lambda_0.$$  

(7.11)

Hence, in the present gauge we have evidently limited ourselves to the quantum states that satisfy $\lambda_0 \leq \frac{1}{2} (p^2 + x^2)$. This explains why the gauge fixed version of the theory defined by $S_{\text{fixed}}(\lambda_0)$ can be related to the covariant theory $S(\lambda_0)$ only under this condition, and does not necessarily cover all the gauge invariant sectors of the theory defined by $S(\lambda_0)$ (for a similar example in string theory, see footnote [5]). This is consistent with the fact that the gauge fixed version could not reproduce all the unitary sectors with $\lambda \geq d + 1/2$. In the guaranteed unitary range $-d + 1/2 \leq \lambda_0 \leq d + 1/2$, all the states except the Lorentz invariant state $|0\rangle$ at $\lambda_0 = -d + 1/2$ are recovered. The missing state $|0\rangle$ should be recoverable by exploring other gauge choices, but we will not pursue this more careful gauge fixing in this paper.

VIII. SU($d, 1$) AND SO($d, 1$) SYMMETRY IN GAUGE FIXED THEORY

We now discuss the unitary representations of the global symmetry SU($d, 1$) and SO($d, 1$) in the gauge fixed version, paying attention to quantum ordering of operators. In particular, we want to show that the gauge fixed version agrees with the covariant version when we compute eigenvalues of the Casimir operator $C_2 (\text{SU}(d, 1))$.

In the gauge fixed version, the timelike oscillator $\bar{a}_0 = \frac{1}{\sqrt{2}} (x_0 - ip_0)$ is computed in terms of the spacelike oscillators $a_i, \bar{a}_i$ after inserting the canonical transformation (7.1) and the gauge $t(\tau) = \tau$. At the classical level this takes the form

$$\bar{a}_0 (\tau) = i e^{i\tau} \sqrt{H + c} = i e^{i\tau} \sqrt{\bar{a}_i (\tau) a_i (\tau) + c}.$$  

(8.1)

At the quantum level one must address operator ordering ambiguities. Since $c$ has not been fixed so far, we absorb all such ambiguities into $c$ and define the quantum version of $a_0$ with the orders of $\bar{a}_i a_i$ as given above. We can now compute the generator of $U_{J_0}(1)$ at the quantum level in the gauge fixed version and find the constant value $J_0 = -c$

$$J_0 = \bar{a} \cdot a = -\bar{a}_0 a_0 + \bar{a}_i a_i = -c.$$  

(8.2)

Recall that in the covariant version $Q = J_0 + \frac{d+1}{2}$, so when $Q, J_0$ are fixed to $Q = \lambda_0$ and $J_0 = -c$, we determine $c$ as

$$c = \frac{d + 1}{2} - \lambda_0.$$  

(8.3)

We see that $c$ is positive only if $\lambda_0 \leq \frac{d+1}{2}$. This is necessary since the square root $\sqrt{a_i a_i + c}$ was defined for all eigenvalues of the operator $\bar{a}_i a_i$ only if $c$ is positive $c \geq 0$. 

27
The generators of SU($d, 1$) can now be computed in the gauge fixed version by inserting the gauge fixed form of $a_0$ and $\bar{a}_0$ into the expression of $J_{\mu\nu}$ given in Eq. (2.8)

$$J_{00} = \hat{N}_a + \frac{cd}{d+1}, \quad J_{ij} = \bar{a}_i a_j + \frac{c}{d+1} \delta_{ij}$$

$$J_{0i} = i e^{i\tau} (\bar{N}_a + c)^{\frac{1}{2}} a_i, \quad J_{i0} = -i e^{-i\tau} \bar{a}_i (\hat{N}_a + c)^{\frac{1}{2}}$$

where $\hat{N}_a = \bar{a}_i a_i$ is the number operator. Note that $J_{00} = \delta_{ij} J_{ij}$ is not independent as expected from $\eta^{\mu\nu} J_{\mu\nu} = 0$. The non-linear generators $J_{0i}, J_{i0}$ must satisfy the following commutation rules according to the SU($d, 1$) algebra (the commutator is evaluated with all $\bar{a}_i (\tau)$ and $a_j (\tau)$ at equal $\tau$)

$$[J_{0i}, J_{i0}] = \delta_{ij} J_{00} - \eta_{00} J_{ji}. \quad (8.6)$$

We can check explicitly that this commutator is indeed satisfied for any constant $c$. The critical point in the calculation is to use the property $a_i \hat{N}_a = (\hat{N}_a + 1) a_i$, leading to $a_i f (\hat{N}_a) = f (\hat{N}_a + 1) a_i$ for any function of $\hat{N}_a$, and similarly for the hermitian conjugate, $\bar{a}_i f (\hat{N}_a + 1) = f (\hat{N}_a) \bar{a}_i$. Then we can compute the commutator $[J_{0i}, J_{i0}]$ as follows

$$[J_{0i}, J_{i0}] = \left((\hat{N}_a + c)^{\frac{1}{2}} a_i \right) \left(\bar{a}_j (\hat{N}_a + c)^{\frac{1}{2}} \right) - \left(\bar{a}_j (\hat{N}_a + c)^{\frac{1}{2}} a_i \right) \left((\hat{N}_a + c)^{\frac{1}{2}} \right)$$

$$= a_i (\hat{N}_a - 1 + c)^{\frac{1}{2}} (\hat{N}_a - 1 + c)^{\frac{1}{2}} \bar{a}_j - \bar{a}_j (\hat{N}_a + c) a_i$$

$$= a_i (\hat{N}_a - 1 + c) \bar{a}_j - \bar{a}_j (\hat{N}_a + c) a_i$$

$$= (\hat{N}_a + c) a_i \bar{a}_j - (\hat{N}_a - 1 + c) \bar{a}_j a_i$$

$$= \delta_{ij} (\hat{N}_a + c) + \bar{a}_j a_i$$

$$= \delta_{ij} \left(J_{00} + \frac{c}{d+1} \right) + \left(J_{ji} - \frac{c}{d+1} \delta_{ij} \right)$$

$$= \delta_{ij} J_{00} + J_{ji}, \quad (8.8)$$

in agreement with SU($d, 1$) as in Eq. (8.6). It is easy to check that the rest of the commutation rules for SU($d, 1$) are satisfied

$$[J_{\mu\nu}, J_{\lambda\sigma}] = \eta_{\nu\lambda} J_{\mu\sigma} - \eta_{\mu\sigma} J_{\lambda\nu}. \quad (8.9)$$

Hence we have constructed correctly the SU($d, 1$) algebra. This implies that we have successfully quantized the theory $S (\lambda_0)$ in the gauge fixed version.

We can now learn the properties of the SU($d, 1$) representation by analyzing the transformation properties of the states. The Young tableaux in Eq. (7.8) already inform us about their transformation properties under the subgroup SU($d$). To learn the transformation rules under the coset generators $J_{i0}, J_{0i}$ we apply these non-linear forms on the states. We see that $J_{i0}, J_{0i}$ create or annihilate excitations

$$J_{i0} |n_a\rangle = \bar{a}_i (\hat{N}_a + c)^{\frac{1}{2}} |n_a\rangle \sim |n_a + 1\rangle \sqrt{n_a + c}, \quad (8.10)$$

28
\[ J_{0i}|n_a\rangle = (\hat{N}_a + c)^{\frac{1}{2}}a_i|n_a\rangle \sim |n_a - 1\rangle \sqrt{n_a - 1 + c}, \quad (8.11) \]

so they mix all SU\((d)\) Young tableaux with each other for all values of \(n_a\). So SU\((d,1)\) transformations connect all levels \(n_a\) to each other, thus showing that the SU\((d,1)\) representation is infinite dimensional as long as \(c > 0\).

When \(c = 0\), we see that all operators \(J_{\mu\nu}\) in Eqs. (8.4, 8.5) annihilate the vacuum state

\[ [J_{\mu\nu}]_{c=0} |\hat{0}\rangle = 0, \quad (8.12) \]

Therefore for \(c = 0\) the vacuum state is SU\((d,1)\) and Lorentz invariant and we must identify it with the Lorentz invariant state \(|\hat{0}\rangle\) listed in Eq. (5.6)

\[ [ |\hat{0}\rangle \text{ in gauge fixed version with } c = 0 ] \leftrightarrow [ |0\rangle \text{ in covariant version}]. \quad (8.13) \]

Furthermore, when \(c = 0\), all the states starting with \(n_a = 1\) form an irreducible infinite dimensional representation, so they can be written just like Eq. (5.6)

\[ c = 0, \text{ or } \lambda_0 = \frac{d + 1}{2} : |\hat{0}\rangle \oplus \bar{a}_i \sum_{m=0}^{\infty} \oplus (\bar{a}_{i_1} \bar{a}_{i_2} \cdots \bar{a}_{i_m}) |\hat{0}\rangle. \quad (8.14) \]

Hence at \(\lambda_0 = \frac{d+1}{2}\) we have identified a SU\((d,1)\) or SO\((d,1)\) singlet, together with an infinite dimensional unitary representation of SU\((d,1)\) whose lowest state has angular momentum \(l = 1\). For all the other cases of \(-\frac{d-1}{2} \leq \lambda_0 \leq \frac{d-1}{2}\) the lowest state has angular momentum zero \(l = 0\) but it is not a Lorentz or SU\((d,1)\) singlet. At \(\lambda_0 = -\frac{d+1}{2}\), according to covariant quantization in Eq. (5.11), we should expect a Lorentz singlet together with another zero angular momentum state as part of an infinite dimensional representation, but the Lorentz invariant state \(|0^\prime\rangle\) is missed in the gauge fixed version.

It is interesting to compute the Casimir operator \(C_2\) \((\text{SU}(d,1))\) in the gauge fixed version. To do so we insert the gauge fixed \(J_{\mu\nu}\) of Eqs. (8.4, 8.5) into Eq. (4.38) and manipulate orders of operators as in Eq. (8.7). After rearranging operators we find that \(C_2\) is just a constant determined by \(c\) as follows

\[ C_2 = \frac{1}{2} \left( J_{ij} J_{ji} + (J_{00})^2 - J_{0i} J_{0i} - J_{0i} J_{i0} - J_{ii} J_{00} \right) \]

\[ = \left\{ \begin{array}{l}
\frac{1}{2} (\bar{a}_i a_j + \frac{c}{d+1} \delta_{ij}) (\bar{a}_j a_i + \frac{c}{d+1} \delta_{ij}) + \frac{1}{2} \left( \hat{N}_a + \frac{c a}{d+1} \right)^2 \\
-\frac{1}{2} \bar{a}_i (\hat{N}_a + c)^{\frac{1}{2}} (\hat{N}_a + c)^{\frac{1}{2}} a_i - \frac{1}{2} (\hat{N}_a + c)^{\frac{1}{2}} a_i \bar{a}_i (\hat{N}_a + c)^{\frac{1}{2}} 
\end{array} \right\} \]

\[ = \frac{(-c)^d}{2} \left( 1 + \frac{(-c)}{d+1} \right) \quad (8.15) \]

This is the same result as the covariant approach \((4.39)\) with \(J_0\) fixed in the gauge fixed version to \(J_0 = -c\), consistent with Eq. (8.2).
It may be interesting to discuss also the SO$(d,1)$ content of each tower. The hermitian Lorentz generators are

$$\text{SO}(d,1): \quad L_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu = -i (J_{\mu\nu} - J_{\nu\mu}) \quad (8.18)$$

which take the following explicit forms in terms of oscillators

rotation : \quad L_{ij} = -i (\bar{a}_i a_j - \bar{a}_j a_i), \quad (8.19)

boost : \quad L_{0i} = -i \left( (\hat{N}_a + c)^{\frac{1}{2}} a_i - \bar{a}_i (\hat{N}_a + c)^{\frac{1}{2}} \right) \quad (8.20)

It is emphasized that these operators satisfy the SO$(d,1)$ Lie algebra

$$[L_{\mu\nu}, L_{\lambda\sigma}] = -i (\eta_{\mu\lambda} L_{\nu\sigma} + \eta_{\nu\sigma} L_{\mu\lambda} - \eta_{\mu\sigma} L_{\nu\lambda} - \eta_{\nu\lambda} L_{\mu\sigma}), \quad (8.21)$$

and in particular the commutator of two boosts gives SO$(d)$ rotations at the quantum level

$$[L_{0i}, L_{0j}] = -i L_{ij} = i \eta_{00} L_{ij}. \quad (8.22)$$

This can be checked explicitly for our non-linear $L_{0i}$ by using the same methods as Eq.(8.7).

Since the $L_{\mu\nu}$ are hermitian they act in infinite dimensional unitary representations of the Lorentz group. This implies that each tower of SU$(d,1)$ at fixed $\lambda_0$ splits into an infinite number of irreducible SO$(d,1)$ towers, the precise content of which SO$(d,1)$ representations appear depend on the constant $c$.

In this section we exhibited new interesting non-linear oscillator representations of SU$(d,1)$ which should have generalizations to other non-compact groups. This type of oscillator representation was not previously considered in \[21\]-\[24\]. The new non-linear expressions for the generators given in Eqs.(8.4,8.5) were obtained by starting from previous oscillator methods and then replacing some of those oscillators by non-linear expressions in terms of the other oscillators. The same method was used to find new interesting SU$(2,3)$ symmetry properties based on twistors \[25\] that describe spinning particles in various 1T-physics systems and explain dualities among them. This non-linear approach to constructing generators and representations of non-compact groups could be of interest in many applications in both physics and mathematics.

IX. NON-RELATIVISTIC OSCILLATOR AS A RELATIVISTIC SYSTEM

While the focus in this paper was the relativistic harmonic oscillator, we were led to the non-relativistic case as a consequence of a gauge choice. Looking at this process in reverse, this shows that the non-relativistic oscillator provides a non-linear realization of a relativistic system. So the non-relativistic oscillator must have some hidden relativistic symmetry of its own. This is possibly a surprising proposition, but it is true as explained below.

In $d$ Euclidean dimensions the non-relativistic oscillator has evident SO$(d)$ symmetry and also a well known SU$(d)$ hidden symmetry that leaves the Hamiltonian invariant. However the
discussion above suggests that we should seek an even larger hidden symmetry SU(d, 1) that includes Lorentz symmetry SO(d, 1).

We recall that the generator of the gauge symmetry of the relativistic action \( S(\lambda_0) \) is \( \phi(x, p) = Q(x, p) - \lambda_0 \) as in Eq. (6.5). By using Poisson brackets \( \delta_\lambda A(x, p) = \Lambda \{ A(x, p), \phi(x, p) \} \) the gauge transformation rules for all observables \( A(x, p) \) are obtained. In particular note that the gauge transformations of \( \delta_\lambda x^\mu \) and \( \delta_\lambda p^\mu \) in Eq. (6.3) follow in this way. Since the SU(d, 1) generators \( J_{\mu \nu} \) commute with the SU(d, 1) invariant \( Q \) as shown in Eq. (2.6), it must have vanishing Poisson brackets with the gauge generator \( \phi(x, p) \) when the \( J_{\mu \nu} \) of Eq. (2.8) is written out in terms of phase space

\[
\{ J_{\mu \nu}(x, p), \phi(x, p) \} = 0 \iff \delta_\lambda J_{\mu \nu} = 0. \quad (9.1)
\]

Therefore, the \( J_{\mu \nu} \) are gauge invariant physical observables.

Since both \( S(\lambda_0) \) and its global symmetry generators \( J_{\mu \nu} \) are gauge invariants, it must be that their gauge fixed versions \( S_{\text{fixed}}(\lambda_0), J_{\mu \nu}^{\text{fixed}} \) also maintain the same SU(d, 1) global symmetry properties. That is, when written out in terms of the remaining Euclidean degrees of freedom \( \vec{x}, \vec{p} \) we must find that \( J_{\mu \nu}^{\text{fixed}} \) is the generator of SU(d, 1) symmetry of the non-relativistic harmonic oscillator action

\[
S_{\text{non.rel.}} = \int d\tau \left( \partial_\tau \vec{x} \cdot \vec{p} - \frac{1}{2} (\vec{p}^2 + \vec{x}^2) \right). \quad (9.2)
\]

The explicit form of \( J_{\mu \nu}^{\text{fixed}}(\vec{x}, \vec{p}, \tau) \) is obtained directly from Eqs. (8.4, 8.5). If these \( J_{\mu \nu}^{\text{fixed}} \) are symmetry generators they must be conserved when the equations of motion of the non-relativistic oscillator are used

\[
\frac{d}{d\tau} J_{\mu \nu}^{\text{fixed}}(\vec{x}(\tau), \vec{p}(\tau), \tau) = 0, \quad \text{for} \quad \dot{x}_i = p_i, \quad \dot{p}_i = -x_i \quad \text{or} \quad \dot{a}_i = -i a_i. \quad (9.3)
\]

Note that \( J_{00}^{\text{fixed}}, J_{i0}^{\text{fixed}} \) depend explicitly on \( \tau \) in addition to the implicit dependence on \( \tau \) that comes through \( \vec{x}(\tau), \vec{p}(\tau) \). Indeed this extra dependence on \( \tau \) is essential to show that the \( J_{00}^{\text{fixed}}, J_{i0}^{\text{fixed}} \) are conserved.

Since we have already shown that these \( J_{\mu \nu}^{\text{fixed}}(\vec{x}, \vec{p}, \tau) \) close to form the SU(d, 1) Lie algebra at the quantum level at any \( \tau \), they also satisfy the same property at the classical level under Poisson brackets. Using these generators we can define infinitesimal SU(d, 1) transformation laws by using Poisson brackets at any fixed \( \tau \), namely \( \delta_\omega \vec{x} = \frac{1}{2} \omega_{\mu \nu} \{ \vec{x}, J_{\mu \nu}^{\text{fixed}}(\tau) \} \) and \( \delta_\omega \vec{p} = \frac{1}{2} \omega_{\mu \nu} \{ \vec{p}, J_{\mu \nu}^{\text{fixed}}(\tau) \} \). More explicitly the transformation laws at any \( \tau \) are

\[
\delta_\omega \vec{x}(\tau) = \frac{1}{2} \omega_{\mu \nu} \frac{\partial J_{\mu \nu}^{\text{fixed}}(x, p, \tau)}{\partial p}, \quad \delta_\omega \vec{p}(\tau) = -\frac{1}{2} \omega_{\mu \nu} \frac{\partial J_{\mu \nu}^{\text{fixed}}(x, p, \tau)}{\partial \vec{x}}. \quad (9.4)
\]

The transformations under the SU(d) x U(1) subgroup are familiar hidden symmetry transformations of the non-relativistic harmonic oscillator. However, the transformations generated by the classical

\[
\frac{1}{\sqrt{2}} \left( J_{i0}^{\text{fixed}} + J_{0i}^{\text{fixed}} \right) = \sqrt{\frac{1}{2} (\vec{p}^2 + \vec{x}^2) + c (x_i \cos \tau - p_i \sin \tau)}, \quad (9.5)
\]
are new non-linear symmetry transformations that were not noted before. It can now be verified that the non-relativistic harmonic oscillator action above is indeed invariant under all of the SU(d, 1) transformations. It can be verified that the new transformations give
\[ \delta \omega S_{\text{non.rel.}} = \int dt \left( \text{stuff} \right) \rightarrow 0, \]
where the total derivative can be dropped in the transformation of the action, thus verifying the expected SU(d, 1) global symmetry. Again the explicit \( \tau \) dependence generated by the expressions in (9.5,9.6) is crucial for this result. A consequence of this symmetry via Noether’s theorem is that the \( J_{\text{fixed}}^i 0 \pm J_{\text{fixed}}^i 0 \) given in Eq.(9.5,9.6) are conserved, as already claimed above in Eq.(9.3).

This hidden symmetry of the non-relativistic harmonic oscillator was not known before. These transformations leave the action, not the Hamiltonian, invariant. As a consequence of the symmetry all the states of the non-relativistic harmonic oscillator taken together at all energy levels must fit into irreducible unitary representations of SU(d, 1) and its Lorentz subgroup SO(d, 1).

Note that the parameter \( c \) is used to construct the non-linear generators \( J_{0i}(c) \) and \( J_{0i}(c) \) in Eq.(8.5), so the SU(d, 1) transformations are different for every \( c \). This means different representations of SU(d, 1) can be realized on the same Fock space consisting of all the states in Eq.(7.8). They will transform differently as a representation basis depending on the choice of the parameter \( c \). When \( c \neq 0 \), all the states form a single irreducible representation of SU(d, 1) with Casimir eigenvalue \( C_2(\text{SU}(d, 1)_c) = -\frac{cd^2}{2} \left( 1 - \frac{c}{d+1} \right) \). The lowest state of this infinite tower has zero SO(d) orbital angular momentum \( l = 0 \) since it is the vacuum state \( |\hat{0}\rangle \). The branching of the SU(d, 1) representation into representations of the Lorentz group SO(d, 1) depend on \( c \), so we expect to describe different relativistic content by using the same non-relativistic harmonic oscillator degrees of freedom.

The \( c = 0 \) case is special, because then the vacuum state \( |\hat{0}\rangle \) of the non-relativistic harmonic oscillator is a singlet of SU(d, 1)_0 and of SO(d, 1), so it is a Lorentz invariant as explained in Eq.(8.12). The remaining states at all energy levels given in Eq.(8.14) make up a single irreducible unitary representation of SU(d, 1)_0 with Casimir 0. The lowest energy state of this \( c = 0 \) infinite tower is \( \bar{a}_i|\hat{0}\rangle \) which has SO(d) angular momentum \( l = 1 \). This is clearly different SO(d, 1) content compared to the \( c \neq 0 \) case for which the lowest state of the irreducible tower had angular momentum \( l = 0 \). This different SU(d, 1) or SO(d, 1) rearrangement of the same states for different values of \( c \) seems surprising when viewed from the perspective of the non-relativistic oscillator. However, when compared to the corresponding \( |\text{towers}\rangle_0 \) in Eqs.(5.6-5.11) in covariant quantization, the hidden information in \( c \) about the SO(d, 1) properties become evident. The comparison shows that \( c \) corresponds to the various powers of \( a_0 \) applied on the vacuum \( |\hat{0}\rangle \) to get the lowest state \( (a_0)^{c-1}|\hat{0}\rangle \) in different towers (for \( c \geq 1 \)). The additional information gained from the Lorentz properties of \( a_0 \) in covariant quantization explains why the same non-relativistic Fock space (7.8)
relates to different relativistic SO($d, 1$) or SU($d, 1$) content as the value of $c$ changes.

Note that if the starting point were the non-relativistic oscillator, then there would be no conditions on the value of $c$ for constructing the SU($d, 1$)$_c$ generators in Eq.(5.3). Of course when $c$ is quantized as indicated before, $c = 0, 1, 2, \cdots , (d + 1)$, the non-linear structures $J_{0i}, J_{i0}$ correspond to just a gauge fixed sector of the relativistic oscillator with a unitarity constraint. Other values of $c$ on the real line seem to describe relativistic systems beyond the oscillator.

Note that $c$ is a Lorentz invariant, therefore in physical applications it could be related to certain relativistically invariant observables, such as the mass of a bound state.

Such relativistic properties of the non-relativistic oscillator may lead to further insights.

X. More Revisits?

We have shed new light on the symmetries and the quantum sectors of the relativistic harmonic oscillator. Since much of this was not noted before, it may lead to additional new observations in old or new applications of this commonly used dynamical system.

Of course, for each physical system there may be various sets of new constraints not discussed in this paper that would influence the allowed physical states as noted in footnote (10). In particular the richer structure of the many oscillators in string theory leads to the Virasoro constraints for removing ghosts rather than those in Eq.(3.28). Whatever the ghost killing constraints may be, it would be of interest to reanalyze the relevant systems to find out whether the additional Fock spaces discussed in this paper lead to additional quantum states that may reveal new physical properties.

This paper is not focused on string theory, but rather on the single relativistic harmonic oscillator. Our initial aim was to clarify some facts about the symmetry aspects of the relativistic oscillator that appeared confusing. The clarification provided here leads us to ask what happens in string theory? In what follows we provide some brief preliminary remarks on this topic.

Past work in string theory has been carried out by relying on the Fock space built exclusively from the covariant spacelike vacuum $|0\rangle$ of section (III), while being unaware of the other Fock space sectors with more general geometry discussed in sections (III,IV). As is well known from previous study of string theory, although not made previously explicit, the spacelike sector is completely consistent. Its results have been reproduced in many approaches, leading to the remarkable properties of string scattering amplitudes.

The question that arises now is not whether anything was wrong with that treatment of strings, but whether there might be more physical phenomena in string theory beyond the usual self consistent spacelike sector, and hence beyond the Veneziano amplitudes. The question is natural since the conventional relativistic Fock space used in string theory inadvertently excludes
a huge sector of unitary quantum states for each single mode as discussed in section (IV). As made clear following Eq. (A16), the relativistic oscillator actually likes to cross between spacelike and timelike regions. Such allowed motions of each single mode simply have never entered the discussion, and therefore there is much room for investigation.

In that connection, it is worth noting that from the earliest period of string theory there has been indications that the lightcone gauge fails to capture all of the gauge invariant physics in string theory (see footnote (5)). A similar phenomenon of missing gauge invariant sectors was seen in the gauge fixed relativistic oscillator discussed in this paper. Therefore gauge fixed treatments, while being quite revealing, cannot be trusted as being complete.

These observations provide new motivation to revisit the covariant quantization of string theory to see whether the concepts discussed in this paper play a role. In the standard treatment of string theory each mode is associated with the spacelike vacuum $|0⟩$, so the standard overall string vacuum is $|0, 0, 0, · · ·⟩$, where each 0 corresponds to a mode. Is it possible to have string configurations built on more complicated vacua, such as $|0, 0, 0', · · ·⟩$ etc. where the various modes could be in various spacetime regions? It is not so easy to answer this question because of the Virasoro constraints.

The sector with all the modes in the timelike Fock space based on $|0', 0', 0', · · ·⟩$, abbreviated as $|0'⟩$, is not difficult to decipher because the analysis is parallel to the usual treatment. The only change is that in this sector all creation annihilation operators $α^μ_n, α^−_n$ switch roles relative to the familiar spacelike sector. Then we find that this sector has a lot of serious problems. The eigenvalues of $Q_n = \frac{1}{2} (p^2_n + n^2 x^2_n)$ are strictly negative and $L'_0 = p^2_0 + \sum Q_n + a$, which is normal ordered relative to $|0'⟩$, has only negative eigenvalues. Hence the Virasoro constraint $L'_0 = 1$ gives only tachyons. The Virasoro constraints $L_{−n}|φ^0⟩$ with $n > 0$ (not $L_n$) can be satisfied by using the same arguments as [14]-[16] but switching $α^μ_n$ with $α^−_n$ at every step. However, the solutions still have ghosts at every mass level because the oscillators $α^i_n$ in $d$ space dimensions produce negative norm states (as opposed to only one time component $α^0_n$ in the usual arguments). Evidently this sector is not acceptable on physical grounds and must be eliminated with some consistent set of gauge symmetries or other arguments. The supersymmetric version of string theory may avoid this sector alltogether, but this needs to be investigated more explicitly.

A more interesting case is the ghost free fully unitary sector based on the vacuum of type $|0, 0, 0, · · ·⟩$ which we abbreviate as $|0⟩$. For example the string state $|k, 0⟩$ has a spacetime configuration of the form (note the relative + sign in $(x^2_{n0} + x^2_n)$)

$$
ψ(X) ∼ ⟨X|k, 0⟩ ∼ e^{ik·x_0} \exp \left( -\frac{1}{2} \sum_{n=1}^{∞} n(x^2_{n0} + x^2_n) \right),
$$

(10.1)

where $x^μ_n$ can be in any spacetime region unlike the usual string field in Eq. (1.7) where $x^μ_0$ was strictly spacelike. This is one of the eigenstates of $L_0$. There are now an infinite number of eigenstates for each eigenvalue of $Q_n = \frac{1}{2} (p^2_n + n^2 x^2_n)$, as explained in section (IV), leading to
the same eigenvalue of $L_0$. All of these states are in infinite dimensional unitary representations of SU($d, 1$). After applying the Virasoro constraints the solutions get rearranged into representations of the overall Poincaré symmetry\textsuperscript{17}. The good thing is that there are no ghosts at all in this Fock space. However, it is not straightforward to solve the Virasoro constraints for string states built on $|k, \bar{0}\rangle$ because the creation-annihilation operators in the time direction $\alpha_n^0, \alpha_{-n}^0$ have their roles inverted while those in the space directions $\bar{\alpha}_n, \bar{\alpha}_{-n}$ remain the same. Solutions seem likely to exist but none are known at this stage. If solutions of the Virasoro constraints can be exhibited they would be of great interest in string theory. This seems to be a challenging problem that we leave to future work.

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APPENDIX A: SO($1, 1$) OSCILLATOR IN POSITION SPACE

In this appendix we solve the differential equation $\left(-\frac{1}{2}\partial^\mu \partial_\mu + \frac{1}{2} x^\mu x_\mu\right) \psi_\lambda(x) = \lambda \psi_\lambda(x)$ in the purely spacelike region\textsuperscript{18} and show that we arrive at the same conclusion as the oscillator approach using the Fock space methods of section (III). For simplicity we will concentrate on 1-space and 1-time dimensions. Therefore the Lorentz symmetry is SO($1, 1$) while the larger hidden symmetry is SU($1, 1$).

We will discuss the spacelike region shown in Fig.1, knowing that the timelike region is similar

\textsuperscript{17}The separate SU($d, 1$) of each single oscillator is not expected to survive in string theory because the Virasoro constraints couple all the modes, including the center of mass mode, to each other. Certainly there is at least an overall Poincaré symmetry, and the states get rearranged into representations of Poincaré with its little group (e.g. SO($d$) for massive states). Of course then the infinite dimensional SU($d, 1$) representations dissociate (they already are in the SU($d$) $\times$ U(1) basis in Eq.(4.32)) and rearranged properly according to Poincaré (or a larger hidden symmetry if any such thing remains).

\textsuperscript{18}There are more general Lorentz covariant solutions that have different forms in various spacelike and timelike regions with continuity conditions across the lightcone $x^\mu x_\mu = 0$ in Fig.1. This will become evident in the discussion following Eq.(A16). For this kind of solution the setting in section (IV) is more convenient. In this section we will seek solutions with support only in the spacelike regions, because those are the only ones described by the standard SO($d, 1$) covariant Fock space approach discussed in section (III), to which we compare the solutions in this Appendix.
as indicated in section (III). Accordingly we parametrize \( x^\mu \) as follows to insure spacelike \( x^\mu \)

\[
x^0 = |x| \sinh \theta, \quad x^1 = x \cosh \theta,
\]
both \( x, \theta \) range from \(-\infty \) to \(+\infty \)  \hspace{1cm} (A1)

This parametrization matches the parabolas in Fig.1 for fixed positive or negative values of \( x \), and as \( x \) is varied the entire spacelike region is covered. The differentials

\[
dx^0 = \varepsilon (x) \sinh \theta dx + |x| \cosh \theta d\theta, \quad dx^1 = \cosh \theta dx + x \sinh \theta d\theta \hspace{1cm} \text{(A2)}
\]

\[
dx = -\varepsilon (x) \sinh \theta dx^0 + \cosh \theta dx^1, \quad d\theta = \frac{\cosh \theta \, dx^0}{|x|} - \frac{\sinh \theta \, dx^1}{x} \hspace{1cm} \text{(A3)}
\]

where \( \varepsilon (x) \equiv \text{sign} (x) \), are useful to compute the derivatives by using the chain rule \( \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial x^\nu} \partial_\nu + \frac{\partial x^\nu}{\partial x^\mu} \partial_\nu \), to obtain

\[
\frac{\partial}{\partial x^0} = \varepsilon (x) \left[ \frac{\cosh \theta}{x} \partial_\theta - \sinh \theta \partial_x \right], \quad \frac{\partial}{\partial x^1} = -\frac{\sinh \theta}{x} \partial_\theta + \cosh \theta \partial_x. \hspace{1cm} \text{(A4)}
\]

The SO(1, 1) boost generator becomes (note extra sign due to raising/lowering the timelike index \( p^0 = -i \partial / \partial x^0 = +i \partial / \partial x^0 \))

\[
L^{01} = x^0 p^1 - x^1 p^0 = -ix^0 \frac{\partial}{\partial x^1} - ix^1 \frac{\partial}{\partial x^0} = -i\varepsilon (x) \partial_\theta. \hspace{1cm} \text{(A5)}
\]

\[
\begin{array}{c}
\text{Fig.1- Parabolas in the spacelike region of } (x^0, x^1) \\
\quad \text{at some fixed } x = \pm a \text{ and any } \theta.
\end{array}
\]

The operator \( Q \) in \( x^\mu \) space is then computed as

\[
Q = \frac{1}{2} (p \cdot p + x \cdot x) = \frac{1}{2} \left[ -\partial_\theta^2 - \frac{1}{x} \partial_x + \frac{1}{x^2} \partial_\theta^2 \right] + \frac{1}{2} x^2. \hspace{1cm} \text{(A6)}
\]

The solution of the eigenvalue equation \( Q \psi_{\lambda m} = \lambda \psi_{\lambda m} \) takes the separable form

\[
\psi_{\lambda m} (x, \theta) = x^{-1/2} F_{\lambda m} (x) e^{im\theta}, \hspace{1cm} \text{(A7)}
\]

where the factor of \( x^{-1/2} \) is inserted for convenience. The eigenvalue \( m \) of the operator \( (-i \partial_\theta) \) must be real if \( L^{01} = -i\varepsilon (x) \partial_\theta \) is to be hermitian. This condition on \( m \) imposes unitarity hence...
only positive norms are possible (see footnote 8). The range of \( m \) is the entire continuous real line \(-\infty < m < \infty\). Then \( F_{\lambda m} (x) \) satisfies

\[
\left\{ -\partial_x^2 - \frac{m^2}{x^2} + \frac{1}{4} + x^2 - 2\lambda \right\} F_{\lambda m} (x) = 0. \tag{A8}
\]

This is a one dimensional problem with an effective potential that has an attractive (negative) component

\[
V_{\text{eff}} (x) = -\frac{m^2}{2x^2} + \frac{1}{2} x^2. \tag{A9}
\]

\( V_{\text{eff}} (x) \) is plotted in Fig.2. For this shape of potential we expect that there are normalizable bound states. We also need to define a normalization and include in the spectrum only the normalizable solutions of this equation.

Fig.2 - Dashed line is for \( m = 0 \), solid line is for \( m \neq 0 \).

We can choose the square integrable norm

\[
\langle \psi_{\lambda m} | \psi_{\lambda' m'} \rangle = \int d^2 x \, (\psi_{\lambda m} (x))^* \, \psi_{\lambda' m'} (x) \tag{A10}
\]

\[
= \int_{-\infty}^{\infty} dx F_{\lambda m}^* (x) \, F_{\lambda' m'} (x) \int_{-\infty}^{\infty} d\theta e^{i(m' - m)\theta} \tag{A11}
\]

\[
= \delta (m - m') 2\pi \int_{-\infty}^{\infty} dx F_{\lambda m}^* (x) \, F_{\lambda' m'} (x) \tag{A12}
\]

\[
= \delta (m - m') \delta_{kk'} \tag{A13}
\]

In this case we must require a finite integral in \( x \) space

\[
2\pi \int_{-\infty}^{\infty} dx F_{km}^* (x) \, F_{k' m} (x) = \delta_{kk'}. \tag{A14}
\]

Next we solve for the allowed values of \( k, m \). The Schrödinger equation in Eq. (A8) is related to the confluent hypergeometric equation and the solutions are given by a linear superposition of the confluent hypergeometric functions \( M (a, b, x^2), U (a, b, x^2) \). The solution that is well behaved at \( x^2 \rightarrow \infty \) is given by

\[
\psi_{\lambda m} (x) = \alpha e^{-x^2/2} x^{im} U \left( \left[ \frac{1}{2} - \frac{1}{2} \lambda + \frac{1}{2} im \right], [1 + im], x^2 \right), \tag{A15}
\]
where $\alpha$ is a normalization constant. This expression is even when $m$ is replaced by $-m$ due to the property $U(a, b, z) = z^{1-b} U(1 + a - b, 2 - b, z)$. This is in agreement with the unitarity condition of Eq. (3.17), since the operator $T$ in that equation also reverses the sign of the boost operator $L^0_i \rightarrow -L^0_i$, and hence demands that only states that are even under $m \rightarrow -m$ can appear in the unitary spectrum. Since we have already demanded unitarity of $L^0_i$, it has to be true that only states even under $m \rightarrow -m$ should emerge automatically in the spectrum of $Q$.

The behavior at $x \rightarrow \pm \infty$ is convergent $\psi_{\lambda m}(x) \sim |x|^\lambda e^{-\frac{1}{2} x^2} (1 + O(1/x^2))$. The small $x \rightarrow 0$ behavior is given by (replace $m$ by $m \mp i\varepsilon$ with small real $\varepsilon$)

$$
\psi_{\lambda m}(x) \rightarrow \begin{cases} 
\text{if } m \neq 0: & \frac{\Gamma(1 \pm im)}{\Gamma(\frac{1}{2} \pm \frac{1}{2} im)} |x|^{\pm m} x^{\mp 2\varepsilon}, \varepsilon \rightarrow 0^+, \\
\text{if } m = 0: & \frac{-1}{\Gamma(\frac{1}{2} \mp \frac{1}{2} \lambda)} (\ln x^2 + O(1)).
\end{cases} \tag{A16}
$$

Therefore, the norm $\int dx |x| |\psi_{\lambda m}(x)|^2$ is integrable at $x = 0, \pm \infty$, hence $\psi_{\lambda m}(x)$ is normalizable. This is in line with expectations on the basis of the shape of the effective potential in Fig.2.

The probability density $|\psi_{\lambda m}(x, \theta)|^2$ does not generally vanish at $x = 0$, which is everywhere at the lightcone $x^\mu x_\mu = 0$ in Fig.1. The physical meaning of this result is that the oscillating particle in a spacelike region has generally a non-vanishing probability at the lightcone. A similar computation in the timelike region will also show that the lightcone is an allowed region of spacetime. Therefore it would make sense to match the probability amplitude in the spacetime region to the one in the timelike region at the lightcone. Then we would get solutions in which the oscillating particle moves easily from the spacetime to the timelike regions and vice versa. This kind of general solution will be discussed in a more convenient setting in section (IV).

There are however quantum states in which the leakage from the spacetime to the timelike regions do not occur at all. This is seen by examining Eq. (A16) and noting that for Lorentz singlets ($m = 0$) the probability amplitude vanishes at the lightcone when $\frac{1}{2} - \frac{1}{2} \lambda$ is a negative integer or zero. Hence only for the following quantized values of $m, \lambda$ it is consistent to have a purely spacelike relativistic harmonic oscillator

$$
m = 0, \text{ and } \lambda = 1 + 2k, \text{ with integer } k = 0, 1, 2, 3, \cdots. \tag{A17}
$$

For these values of $\lambda$ the solution $U$ reduces to a polynomial as follows

$$
\psi_k = \tilde{\alpha}_k e^{-x^2/2} U(-k, 1, x^2) = \alpha_k e^{-x^2/2} L^0_k(x^2), \tag{A18}
$$

where $L^0_k(x^2)$ is the Laguerre polynomial with argument $x^2$.

$$
L^0_k(x^2) = \sum_{m=0}^{k} \frac{(-1)^m k!}{(m!)^2 (k-m)!} x^{2m}, \text{ } k = 0, 1, 2, 3, \cdots \tag{A19}
$$

So, the probability density $x |\psi|^2$ vanishes at the lightcone.
This result in $d = 1$ is in full agreement with the oscillator approach of section (III) for general $d$. The oscillator method, which was valid only for the spacelike region, also yielded only Lorentz singlets Eq. (3.20) as the only positive norm states in a unitary representation of the Lorentz group SO$(d, 1)$. Furthermore, the eigenvalues of $Q \rightarrow \lambda = 1 + 2k$ agree when specialized to $d = 1$.

What happened to the finite dimensional Lorentz representations with ghosts that showed up in the Fock space approach in section (III)? Those had emerged in Fock space by applying oscillators $\bar{a}_\mu$ on the vacuum $|0\rangle$. What do we get if we follow the same approach in position space? To investigate this we start with the oscillators in the Cartesian basis

\begin{align}
    a_0 &= \frac{1}{\sqrt{2}} \left( -x^0 + \frac{\partial}{\partial x^0} \right), \quad \bar{a}_0 = \frac{1}{\sqrt{2}} \left( -x^0 - \frac{\partial}{\partial x^0} \right) \tag{A20} \\
    a_1 &= \frac{1}{\sqrt{2}} \left( x_1 + \frac{\partial}{\partial x_1} \right), \quad \bar{a}_1 = \frac{1}{\sqrt{2}} \left( x_1 - \frac{\partial}{\partial x_1} \right) \tag{A21}
\end{align}

and transform them to $(x, \theta)$ basis as

\begin{align}
    a_0 &= \frac{\varepsilon(x)}{\sqrt{2}} \left( -\sinh \theta (x + \partial_x) + \frac{\cosh \theta}{x} \partial_\theta \right), \tag{A22} \\
    \bar{a}_0 &= \frac{\varepsilon(x)}{\sqrt{2}} \left( -\sinh \theta (x - \partial_x) - \frac{\cosh \theta}{x} \partial_\theta \right) \tag{A23} \\
    a_1 &= \frac{1}{\sqrt{2}} \left( \cosh \theta (x + \partial_x) - \frac{\sinh \theta}{x} \partial_\theta \right), \tag{A24} \\
    \bar{a}_1 &= \frac{1}{\sqrt{2}} \left( \cosh \theta (x - \partial_x) + \frac{\sinh \theta}{x} \partial_\theta \right) \tag{A25}
\end{align}

Clearly, $a_0, a_1$ both annihilate the ground state $\psi_{\text{vac}}(x, \theta) = \langle x | 0 \rangle = e^{-x^2/2}$ since it is independent of $\theta$ and satisfies $(x + \partial_x) e^{-x^2/2} = 0$

\begin{equation}
    a_0 |0\rangle \rightarrow a_0 e^{-x^2/2} = 0, \quad a_1 |0\rangle \rightarrow a_1 e^{-x^2/2} = 0. \tag{A26}
\end{equation}

If we try to create states with the $\bar{a}_1, \bar{a}_0$, we automatically obtain solutions to the differential equation, but we see that the $\theta$ dependence is not normalizable as follows

\begin{align}
    \bar{a}_0 |0\rangle &\Rightarrow \frac{\varepsilon(x)}{\sqrt{2}} \left( -\sinh \theta (x - \partial_x) - \frac{\cosh \theta}{x} \partial_\theta \right) e^{-x^2/2} = -\sqrt{2} |x| e^{-x^2/2} \sinh \theta \tag{A27} \\
    \bar{a}_1 |0\rangle &\Rightarrow \frac{1}{\sqrt{2}} \left( \cosh \theta (x - \partial_x) + \frac{\sinh \theta}{x} \partial_\theta \right) e^{-x^2/2} = \sqrt{2} x e^{-x^2/2} \cosh \theta \tag{A28}
\end{align}

These are solutions, but do not have the unitary form $e^{\pm i m \theta}$.

\begin{equation}
    \text{Indeed, the boost } L^{01} = -i \varepsilon(x) \partial_\theta \text{ is hermitian only for the } e^{\pm i m \theta} \text{ basis, it is not hermitian for the } (\sinh \theta, \cosh \theta) \text{ or } e^{\pm \theta} \text{ basis. Therefore, such excited states cannot be included in the spectrum if unitarity is imposed from the beginning as was done in this section.}
\end{equation}
We emphasize that the oscillator states $|0\rangle$, $|1\rangle$ are excluded for two reasons. First, they are not in a unitary representation of the Lorentz group $SO(1,1)$ or of the hidden symmetry group $SU(1,1)$, second they are not normalizable according to the square integrable norm defined above because their norm diverges for the $\theta$ integral $\int_{-\infty}^{\infty} d\theta \sinh^2 \theta = \infty$, etc. It is important to emphasize that the square integrable norm above is different than the Fock space norm. On that issue note that $|0\rangle$, $|1\rangle$ are normalizable if one uses the definition of norm in the non-unitary Fock space of section (3.9), however this admits negative as well as positive norms.

Following the oscillator approach in position space we obtain square integrable normalizable states only for the singlets as follows. We compute $\vec{a} \cdot \vec{a}$ and note that it is independent of $\theta$

$$\vec{a} \cdot \vec{a} = -\vec{a}_0 \vec{a}_0 + \vec{a}_1 \vec{a}_1 = \frac{1}{2} \left( x - \frac{1}{x} - \partial_x \right) (x - \partial_x).$$

Therefore, $(\vec{a} \cdot \vec{a})^k$ creates $\theta$-independent excited states, which are Lorentz singlets. For $k = 1$ we can now compute the oscillator state in Eq.(3.20). This gives

$$\langle \vec{a} \cdot \vec{a} \rangle \langle x|0 \rangle = \frac{1}{2} \left( x - \frac{1}{x} - \partial_x \right) (x - \partial_x) e^{-x^2/2} = 2 (x^2 - 1) e^{-x^2/2},$$

(A30)

which is in agreement with Eq.(A18) for $k = 1$

$$\psi_1(x) = \alpha e^{-x^2/2} L^0_1(x^2) = \alpha e^{-x^2/2} (1 - x^2).$$

(A31)

More generally we can verify that the oscillator states $(\vec{a} \cdot \vec{a})^k \langle x|0 \rangle$ reproduce the Laguerre polynomials

$$\psi_k(x) \sim (\vec{a} \cdot \vec{a})^k \langle x|0 \rangle \sim \alpha_k x e^{-x^2/2} L_k^0(x^2).$$

(A32)

(A33)

These are certainly normalizable in $x$-space, and have positive norm, so they are included in the positive norm spectrum. This is in complete agreement with the results for general $d$ of section (III).

In the present approach the selection of the correct set of states emerged automatically on the basis of normalizability and unitarity of the Lorentz generator $L^0_1$ with the chosen norm of Eqs.(A10,A14). Of course, this amounts to the same criterion of section (III).

However, in the present approach we did not see so far why only the vacuum state $\langle x|0 \rangle$ must be kept. For this, we apply the $SU(1,1)$ generators, such as $\vec{a}_0 a_1$ or $\vec{a}_1 a_0$ on the states $\psi_{\lambda m}(x, \theta)$ and note that this takes us out of the unitary space $e^{im\theta}$ as explained in Eqs.(A27,A29). This means that the restriction to only the spacelike region, plus unitarity, breaks generally the $SU(1,1)$ covariance of the problem. This is like breaking symmetries via boundary conditions.
The covariance can be fully maintained only in the vacuum state. Thus, if one is to seek solutions that are consistent with SU(1, 1) covariance, then only the vacuum state can satisfy this criterion. Again this is in agreement with the Fock space approach of section (3.9).

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