On $\ell$-parabolic Hecke algebras of symmetric groups

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Abstract

Let $H = H_q(n)$ be the Hecke algebra of the symmetric group of degree $n$, over a field of arbitrary characteristic, and where $q$ is a primitive $\ell$-th root of unity in $K$. Let $H_\rho$ be an $\ell$-parabolic subalgebra of $H$. We give an elementary explicit construction for the basic algebra of a non-simple block of $H_\rho$. We also discuss homological properties of $H_\rho$-modules, in particular existence of varieties for modules, and some consequences.

1 Introduction

Let $H = H_q(n)$ be the Hecke algebra of a symmetric group over some field $K$, where $q \in K$ is a primitive $\ell$-th root of unity with $\ell > 1$, and where $K$ has characteristic $p \geq 0$. One would like to understand homological properties of $H$, and how they may differ from those of group algebras.

Standard parabolic subalgebras of $H$ are an analog of subgroup algebras for group algebras. In [8] a vertex theory was developed; and it is now known that a vertex of an indecomposable $H$-module is $\ell - p$ parabolic (see [8], [11], and [21] for the general case). For any composition $\rho$ of $n$, there is an associated parabolic subalgebra $H_\rho$ of $H$, which is isomorphic to the Hecke algebra of the standard Young subgroup $W_\rho$ of the symmetric group $W$ of degree $n$. The algebra $H_\rho$ is $\ell$-parabolic if all parts of $\rho$ are $\ell$ or 1, and it is $\ell - p$ parabolic if all parts are $1, \ell, \ell p, \ell p^2, \ldots$. This suggests that $\ell - p$ parabolic subalgebras should be the analogs of group algebras of $p$-groups, and that $\ell$-parabolic subalgebras should play a role similar to that of group algebras of elementary abelian $p$-groups.

Here we study $\ell$-parabolic subalgebras $H_\rho$ of $H$, the smallest one is $H_\rho(\ell)$. We give an elementary and explicit construction for the basic algebra of the principal block of $H_\rho(\ell)$, the unique non-simple block. (The structure is known, over characteristic 0 it may be found in [15] or [20] by a rather indirect approach.) By taking tensor powers, this gives explicit basic algebras for an arbitrary non-simple block of any $\ell$-parabolic
subalgebra $H_\rho$. Recall that, in general, a basic algebra $A$ of an algebra $\Lambda$ is the smallest algebra which is Morita equivalent to $\Lambda$, it is unique up to isomorphism. When the field is large enough, the simple $A$-modules are one-dimensional. For many problems working, with a basic algebra has great advantages.

In the second part of the paper, we study homological properties of $\ell$-parabolic Hecke algebras. Given a module $M$ of some algebra $\Lambda$, it is a basic question whether or not it is projective. More generally, one wants to know the complexity of $M$, that is the rate of growth of a minimal projective resolution. To answer these, and understand other invariants, one can exploit cohomological support varieties of modules, or perhaps rank varieties, when they exist.

We show that $\ell$-parabolic Hecke algebras have a theory of support varieties constructed via Hochschild cohomology; this can be mostly extracted from existing literature. As a consequence, one also can describe possible tree classes of Auslander-Reiten components.

Rank varieties were originally introduced by J. Carlson [7] for group algebras of elementary abelian $p$-groups over fields of characteristic $p$. They were generalized to a class of quantum complete intersections in [4] (see also [19] for a more detailed account). We show that any non-simple block of an $\ell$-parabolic Hecke algebra has a subalgebra $R$ which controls directly projectivity of modules in the block. By [4], this algebra $R$ has a theory of rank varieties; though it does not seem to extend to $H_\rho$ in general.

Let $\mathcal{X}$ be the category of indecomposable $H$-modules which are not projective restricted to $H_\rho$, it is a union of stable Auslander-Reiten components. We show that one can also describe the possible tree classes of components containing modules in $\mathcal{X}$.

Section 2 contains background on Hecke algebras and basic algebras. Section 3 constructs the basic algebra of $H_q(\ell)$, and in Section 4 we discuss homological properties. Analogues for group algebras of symmetric groups also hold, with $p$ instead of $\ell$.

Our approach to Hecke algebras is based on [9], and for background on homological properties we refer to [3]. We work with right modules.

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## 2 Preliminaries

2.1 Throughout, let $n$ be a positive integer, and let $W$ be the symmetric group on $\{1, 2, \ldots, n\}$. The following is based on [9].

Let $K$ be a field, and let $q$ be a primitive $\ell$-th root of unity in $K$ where $\ell \geq 2$. The Iwahori Hecke algebra $H = H_q(n)$ has generators $T_1, \ldots, T_{n-1}$ and relations

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} (1 \leq i < \ell - 1), \quad T_i T_j = T_j T_i (|i-j| \geq 2)$$
\[ T_i^2 = (q - 1)T_i + qI \ (1 \leq i \leq \ell - 1). \]

It has basis \( \{ T_w : w \in W \} \). If \( w = (i \ i + 1) =: s_i \) then \( T_w = T_i \). The algebra \( H_q(n) \) is a deformation of the group algebra \( KW \). It is a symmetric algebra but is not a Hopf algebra.

Let \( B \) be the set of basic transpositions, that is \( B = \{ s_1, \ldots, s_{n-1} \} \). Recall that the length of an element \( w \in W \) is the minimal \( k \geq 0 \) such that \( w \) can be written as \( w = v_1v_2\ldots v_k \) for \( v_i \in B \). If so then \( T_w = T_{v_1}\ldots T_{v_k} \).

An element of permutations of the form \( w = v_1v_2\ldots v_k \), for \( v_i \in B \). If so then \( T_w = T_{v_1}\ldots T_{v_k} \).

We have \( \ell \geq 3 \).

A composition of \( n \) is a tuple of non-negative integers \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_a) \) such that \( \sum_i \lambda_i = n \). For such a composition, let \( W_\lambda \) be the corresponding standard Young subgroup of \( W \), it is the direct product \( W_{[1,\lambda_1]} \times W_{[\lambda_1+1,\lambda_2]} \times \ldots \times W_{[\sum_{i=1}^{a-1} \lambda_i+1,n]} \).

Here \( [u,v] = \{ x \in \mathbb{N} \mid u \leq x \leq v \} \), we refer to such a set as an interval. The parabolic subalgebra \( H_\lambda \) is Hecke algebra of the Young subgroup \( W_\lambda \), it is isomorphic to the tensor product of algebras \( H_q(\lambda_i) \) with disjoint supports.

2.2 (a) In order to work with a parabolic subalgebra \( H_\lambda \) of \( H \), one needs to use distinguished coset representatives. That is, there is a set \( \mathcal{D}_\lambda \) of permutations such that \( W = \bigcup_{d \in \mathcal{D}_\lambda} W_\lambda d \) (the disjoint union), and moreover \( d \) is the unique element of minimal length in its coset. If so, then \( (\mathcal{D}_\lambda)^{-1} = \{ d^{-1} \mid d \in \mathcal{D}_\lambda \} \) is a system of left coset representatives, where each element is the unique element of minimal length in its coset.

To compute the elements in \( \mathcal{D}_\lambda \), let \( t^\lambda \) be the standard \( \lambda \)-tableau in which the numbers \( 1, 2, \ldots, n \) appear in order along successive rows. With this, \( \mathcal{D}_\lambda \) is the set of permutations \( d \) such that the tableau \( t^\lambda d \) is a row standard tableau.

(b) In order to work with pairs of parabolic subalgebras \( H_\lambda \) and \( H_\mu \) for compositions \( \lambda, \mu \), we need double cosets. Let \( \mathcal{D}_{\lambda,\mu} = \mathcal{D}_\lambda \cap \mathcal{D}_\mu^{-1} \), this is a system of distinguished \( W_\lambda - W_\mu \) double coset representatives in \( W \), and if \( d \in \mathcal{D}_{\lambda,\mu} \) then \( d \) is the unique element of minimal length in the double coset \( W_\lambda d W_\mu \). In 1.7 of [2] it is described how to compute \( \mathcal{D}_{\lambda,\mu} \). We will use a modification below.

2.3 Let \( R \) be a non-empty interval contained in \( \{ 1, 2, \ldots, n \} \), suppose \( W_R \) is the group of permutations of \( R \), and \( H_q(R) \) is the Hecke algebra of \( W_R \). In \( H_q(R) \) we consider \[ x_R := \sum_{w \in W_R} T_w, \quad y_R := \sum_{w \in W_R} (-q)^{-l(w)}T_w. \]

Then \( x_R \) and \( y_R \) span the trivial, and the sign module of \( H_q(R) \), that is \[ x_R T_w = q^{l(w)}x_R \quad \text{and} \quad y_R T_w = (-1)^{l(w)}y_R \quad (w \in W_R). \]

The elements \( x_R, y_R \) are central in \( H_q(R) \), and its product is zero if \( R \) has size \( \geq 2 \) and \( \ell \geq 3 \).

We have \[ x_R^2 = (\sum_{w \in W_R} q^{l(w)}x_R \quad \text{and} \quad y_R^2 = ((\sum_{w \in W_R} q^{-1}l(w)y_R. \]

As is well-known, if \( u \) is a variable then \( \sum_{w \in W_R} u^{l(w)} = \prod_{m=1}^\ell [m]_u =: [r]_u! \) where \( [m]_u = 1 + u + \ldots + u^{m-1} = \frac{1-u^m}{1-u} \). Assume now that \( r < \ell \), then if we substitute \( u = q \)
or \(q^{-1}\) we get a non-zero element in \(K\). This means that we have idempotents \(c_R x_R\) and \(\delta_R y_R\) where \(c_R = \frac{1}{|\rho|}\) and \(\delta_R = \frac{1}{|\rho|}\), setting \(\delta = q^{-1}\). Note that the condition \(r < \ell\) is essential; if \(r = \ell\) then \(x_R^2 = 0 = y_R^2\).

More generally, if \(\rho\) is a composition of \(n\), define \(x_{\rho}, y_{\rho}\), to be the product of the \(x_{\rho_i}\), or \(y_{\rho_i}\), where the \(\rho_i\) are the support sets of the standard Young subgroup \(W_{\rho}\). Then we have

\[ x_{\rho}^2 = c_{\rho} x_{\rho}, \quad y_{\rho}^2 = \delta_{\rho} y_{\rho}\]

where \(c_{\rho}\) and \(\delta_{\rho}\) are non-zero scalars. Suppose \(\rho\) is a composition of \(R\) and \(|R| < \ell\), then \(x_{\rho} x_{\rho} = 0\) is a non-zero scalar multiple of \(x_{\rho}\), and \(y_{\rho} y_{\rho} = 0\) is a non-zero scalar multiple of \(y_{\rho}\). If \(\rho_i \geq 2\) for at least one \(i\) then \(y_{\rho_i} x_{\rho} = 0 = x_{\rho} y_{\rho}\) and \(x_{\rho} y_{\rho} = 0 = y_{\rho} x_{\rho}\).

2.4 Let \(\#\) be the automorphism of \(H\) given by \(T_w \mapsto (-q)^{(w)} (T_{w^{-1}})^{-1}\). Twisting by this automorphism interchanges \(x_R\) and \(y_R\) for any interval. There is also an anti-involution \(*\) on \(H\) defined on the basis, by \(T_w \mapsto T_{w^{-1}}\), it fixes \(x_R\) and \(y_R\) for any interval \(R\). Recall from [9] the symmetric bilinear form

\[ (T_u, T_v) = \begin{cases} q^{(w)}, & u = v \\ 0, & \text{else.} \end{cases} \]

This satisfies

\((h_1 h_2, h_3) = (h_1, h_3 \cdot h_2^*)\) and \((h_1, h_2 h_3) = (h_2^* h_1, h_3)\) (\(h_i \in H\)).

If one defines \(f(h_1, h_2) := (h_1, h_2^*)\) then, as one can check, \(f\) is a symmetrizing form. That is, it is symmetric, associative, and non-degenerate, and it shows that \(H\) is a symmetric algebra.

2.5 The combinatorics for representations of Hecke algebras is similar to that for symmetric groups, with \(\ell\) instead of \(p\), as it is proved in [9] and [10], see also [17]. For each partition \(\lambda\) of \(n\), there is a Specht module \(S^\lambda\). When \(\lambda\) is \(\ell\)-regular, this has a unique simple quotient \(D^\lambda\), and these are the simple \(H\)-modules, up to isomorphism. The blocks are labelled by \(\ell\)-cores \(\kappa\) and weight \(w\), for \(n = |\kappa| + w \ell\). The principal block is the block containing the trivial module, that is \(S^\lambda\) for \(\lambda = (n)\). For our construction we only use that when \(n = \ell\), the principal block has precisely \(\ell - 1\) simple modules.

2.6 We recall facts about basic algebras, for details see for example [2]. Assume \(A\) is any finite-dimensional \(K\)-algebra, a basic algebra associated to \(A\) is an algebra \(e Ae\) where \(e\) is an idempotent of the form \(e = e_1 + \ldots + e_m\) with \(e_i\) orthogonal primitive idempotents, such that \(e_i A \) is not isomorphic to \(e_j A\) for \(i \neq j\), and such that every indecomposable projective \(A\)-module is isomorphic to some \(e_i A\). If so, for large enough \(K\), the \(e Ae\) is isomorphic to an algebra \(KQ/I\) where \(Q\) is a unique quiver (directed graph), \(KQ\) is the path algebra of \(Q\), and \(I\) is an admissible ideal of \(KQ\) (that is \(KQ_N \leq I \leq KQ^2\) where \(KQ^r\) is the span of the paths of lengths \(\geq r\). The vertices of \(Q\) correspond to the idempotents \(e_i\), and the arrows \(i \rightarrow j\) are in bijection with a basis of the vector space \(e_i (\text{rad} A/\text{rad}^2 A) e_j\). The connected components of \(Q\) are in bijection with the blocks of the algebra \(A\) (ie the indecomposable direct summands of \(A\) as an algebra). For a connected component of \(Q\) with vertices \(V \subseteq \{1, 2, \ldots, m\}\) set \(e_V := \sum_{i \in V} e_i\), then \(e_V A e_V\) is a basic algebra for a block of \(A\).
3 The principal block of $H_q(\ell)$

We take $H := H_q(\ell)$. The principal block of this algebra is the block containing the trivial module.

Suppose $\ell = 2$, that is $q = -1$. Then we have $(T_1 + 1)^2 = 0$ and the algebra map $K[X] \to H$ sending $X \mapsto T_1 + 1$ induces an isomorphism $K[X]/(X^2) \cong H$. The algebra $H$ is its own basic algebra. We assume from now that $\ell \geq 3$.

It is known (see [15] or [20] for the case $K = \mathbb{C}$) that a basic algebra of this block is a Brauer tree algebra, isomorphic to $kQ/I$ where $Q$ is of the form

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1 \rightarrow 2 \rightarrow \cdots \rightarrow l - 2 \rightarrow l - 1
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and if we label the arrow $i \rightarrow i + 1$ by $\alpha_i$, and the arrow $i + 1 \rightarrow i$ by $\beta_i$, then the ideal $I$ is generated by the relations

(3.1) $\beta_i \alpha_i = \alpha_{i+1} \beta_{i+1}$ (1 \leq i \leq l - 2), $\alpha_1 \beta_1 \alpha_1$, $\beta_{l-2} \alpha_{l-2} \beta_{l-2}$.

This is a Brauer tree algebra. In order to give a direct construction of the basic algebra, as a subalgebra of $H$, we start by finding a set of $l - 1$ orthogonal idempotents. They will be of the form consider $x_{RYS}$ where $R, S$ are non-empty intervals which form a partition of $\{1, 2, \ldots, \ell\}$. As explained in 2.3, we have idempotents which are non-zero scalar multiples of $x_R, y_S$, and since $x_R, y_S$ commute, then also $x_{RYS}$ is a non-zero scalar multiple of an idempotent in $H$.

3.1 Idempotents

Such idempotents have certain factorisations.

Let $R$ be an interval with $1 \leq |R| < \ell$, and let $W = W_R$. Let $\rho$ be a composition of $R$ and

$$W = \bigcup_{d \in D_{\rho}} W_{d}.\rho$$

**Lemma 3.1** With this, we have

(a) $y_R = y_{\rho}\sigma = \sigma^*y_{\rho}$ where $\sigma = \sum_{d \in D_{\rho}}(-q)^{-l(d)}T_d$.

(b) $x_R = x_{\rho}\psi = \psi^*x_{\rho}$ where $\psi = \sum_{d \in D_{\rho}} T_d$.

**Proof** (a) Let $w \in W_R$, then $w = hd_i$ for $h \in W_{\rho}$ and $d_i \in D_{\rho}$, and then $l(w) = l(h) + l(d_i)$ and $T_w = T_hT_{d_i}$. Therefore we can write $y_R$ as

$$\sum_{w}(-q)^{-l(h)-l(d_i)}T_hT_{d_i}$$

and this gives the required factorisation. We apply the anti-involution $^*$ (see 2.4) and get $y_R = y_R^* = \sigma^*y_{\rho}$. Part (b) is similar. □
Corollary 3.2 Let $U$ and $V$ be intervals contained in $\{1, 2, \ldots, \ell\}$.

(a) If $|U \cap V| \geq 2$ then $x_U y_V = 0 = y_U x_V$.

(b) Otherwise $x_U y_V$ and $y_U x_V$ are non-zero.

Proof Let $G = U \cap V$.

(a) Assume $|G| \geq 2$. With the notation as in Lemma 3.1 we have $x_U = \psi x_G$ and $y_V = y_G \sigma$, and since $x_G y_G = 0$ it follows that $e_U f_V = 0$.

(b) Suppose $|G| \leq 1$, then $e_U$ and $f_V$ have disjoint supports, and then the product is non-zero. Similarly $f_U e_V$ is non-zero. □

Definition 3.3 We define for $1 \leq r \leq \ell - 1$ and $r + s = \ell$, 

$$
\varepsilon_r := \begin{cases} 
  c_r x_{[1, r]} y_{[s+1, \ell]} & r \text{ odd} \\
  c_r y_{[1, r]} x_{[r+1, \ell]} & r \text{ even}
\end{cases}
$$

where $c_r$ is the non-zero scalar as described in 2.3 such that $\varepsilon_r$ is an idempotent.

With these, we will show the following:

Proposition 3.4 We have $\varepsilon_r \varepsilon_u = 0$ for $r \neq u$. Moreover

$$
dim \varepsilon_r H \varepsilon_u = \begin{cases} 
  0 & |r - u| \geq 2 \\
  1 & u = r + 1 \\
  2 & r = u.
\end{cases}
$$

To proceed further, let $\lambda = (r, s)$ and $\mu = (u, v)$ be the compositions of $\ell$ with two non-zero parts. Slightly more general, we write $\varepsilon_\lambda$ for one of $y_{[1, r]} x_{[r+1, \ell]}$ or $x_{[1, r]} y_{[r+1, \ell]}$, and we write $\varepsilon_\mu$ for one of $y_{[1, u]} x_{[u+1, \ell]}$ or $x_{[1, u]} y_{[u+1, \ell]}$.

Lemma 3.5 The space $\varepsilon_\lambda H \varepsilon_\mu$ is the span of the sets $\{\varepsilon_\lambda T_d \varepsilon_\mu \mid d \in D_{\lambda, \mu}\}$.

Proof The space $\varepsilon_\lambda H \varepsilon_\mu$ is spanned by all $\varepsilon_\lambda T_w \varepsilon_\mu$ for $w \in W$. Any such $w$ can be written as $w = h_1 d h_2$ where $h_1 \in W_\lambda$ and $h_2 \in W_\mu$, and $d$ is a minimal length representative for the double coset containing $w$. Then $T_w = T_{h_1} T_d T_{h_2}$, and the claim follows. □

Definition 3.6 Assume Proposition 3.4 and Lemma 3.3. Let $1 \leq r \leq \ell - 2$. Define

$$
\alpha_r := \varepsilon_r T_d \varepsilon_{r+1}, \quad \beta_r := \varepsilon_{r+1} T_{d^{-1}} \varepsilon_r
$$

where $d$ is the distinguished double coset representative such that $\varepsilon_r T_d \varepsilon_{r+1} \neq 0$ (see 3.4 and 3.5).

We will write down $\alpha_r, \beta_r$ explicitly below, and we will prove as the main result:

Theorem 3.7 Let $B$ be the subalgebra of $H$ generated by the set

$$
\{\varepsilon_r \mid 1 \leq r \leq \ell - 1\} \cup \{\alpha_r, \beta_r \mid 1 \leq r \leq \ell - 2\}.
$$

Then $B = \varepsilon H \varepsilon$ where $\varepsilon = \sum_{r=1}^{\ell-1} \varepsilon_i$ and $B$ is a basic algebra for the principal block of $H$.

After possibly rescaling some $\beta_r$ the elements $\alpha_r, \beta_r$ satisfy the relations (3.1).
3.2 Towards $\varepsilon_\lambda H \varepsilon_\mu$.

Let $\lambda = (r, s)$ and $\mu = (u, v)$ be compositions of $\ell$ with all parts non-zero. We fix a distinguished representative $d \in D_{\lambda, \mu}$ and analyze the element $\varepsilon_\lambda T_d \varepsilon_\mu$, which lies in $H_\lambda T_d H_\mu = (T_d)(T_d)^{-1} H_\lambda T_d \cap H_\mu$. By [9], Theorem 2.7, we have that

$$(T_d)^{-1} H_\lambda T_d \cap H_\mu = H_\nu,$$

where $W_\nu$ is the Young subgroup generated by the set of basic transpositions contained in $d^{-1} W_\lambda d \cap W_\mu$. We will now compute $W_\nu$, and also the Young subgroup $U$ of $W_\lambda$ such that $d^{-1} Ud = W_\nu$.

(1) The elements $d \in D_{\lambda, \mu}$ are in bijection with the $2 \times 2$ matrices $A$ with entries in $\mathbb{Z}_{\geq 0}$ having row sums $u, v$ and column sums $r, s$. Take such a matrix $A = \begin{pmatrix} t & u-t \\ r-t & x \end{pmatrix}$.

Note that if $t = r$ or $t = u$, then $d$ is the identity. We observe that the matrix $A$ as above corresponds to $d \in D_{\lambda, \mu}$ if and only if its transpose corresponds to $d^{-1} \in D_{\mu, \lambda}$.

Example 3.8 (1) Let $\lambda = (r, s)$ and $\mu = (s-1, r+1)$, where $r$ is even,

$\varepsilon_\lambda = y_{[1,r]} x_{[r+1,\ell]}$ and $\varepsilon_\mu = x_{[1,s-1]} y_{[s,\ell]}$

They are non-zero scalar multiples of $\varepsilon_r$ and $\varepsilon_{r+1}$ respectively. We take the matrix $A$ and the corresponding $d$ as

$$A = \begin{pmatrix} 0 & s-1 \\ r & 1 \end{pmatrix}, \quad d = \begin{pmatrix} 1 & 2 & \cdots & r & r+1 & \cdots & \ell-1 & \ell \\ s & s+1 & \cdots & \ell-1 & 1 & \cdots & s-1 & \ell \end{pmatrix}.$$

We will see later that with this, $\varepsilon_r T_d \varepsilon_{r+1}$ and $\varepsilon_{r+1} T_{d^{-1}} \varepsilon_r$ are non-zero.

(2) Let $\lambda = (r, s)$ and $\mu = (s+1, r-1)$, where $r$ is even, and

$\varepsilon_\lambda = y_{[1,r]} x_{[r+1,\ell]}$ and $\varepsilon_\mu = x_{[1,s+1]} y_{[s+2,\ell]}$.

They are non-zero scalar multiples of $\varepsilon_r$ and $\varepsilon_{r-1}$ respectively. We take the matrix $A$ and the corresponding permutation $d'$ as

$$A = \begin{pmatrix} 1 & s \\ r-1 & 0 \end{pmatrix}, \quad d' = \begin{pmatrix} 1 & 2 & \cdots & r & r+1 & \cdots & \ell \\ 1 & s+2 & \cdots & \ell & 2 & \cdots & s+1 \end{pmatrix}.$$
Proposition 3.11 With the above setting, we have

\[ W_\nu = W_{[1,t]} \times W_{[u+1,u+(r-t)]} \times W_{[r+1,u+(r-t)+1]} \times W_{[u+(r-t)+1,\ell]}, \]

\[ U = W_{[1,t]} \times W_{[t+1,r]} \times W_{[r+1,u+(r-t)]} \times W_{[u+(r-t)+1,\ell]} . \]

The relevant factors are trivial when \( t = 0 \) or \( t = r \) or \( t = u \) or \( u + (r-t) = \ell \).

\[ \text{Proof } \]

To compute \( d^{-1}s_id \), we have an elementary observation: Let \( d \) be the permutation which maps \( j \mapsto a_j \), for \( 1 \leq j \leq n \). Then for \( 1 \leq i < n \) we have that \( d^{-1}s_id \) is the transposition \((a_i, a_{i+1})\). Namely, \( d^{-1}s_id \) is a transposition, and one sees directly that it takes \( a_i \mapsto a_{i+1} \).

We apply this to \( d \) as above, then

\[ d^{-1}s_id = \begin{cases} 
  s_i & 1 \leq i \leq t - 1 \\
  s_{i+(u-r)} & t + 1 \leq i \leq r - 1 \\
  s_{i-(r-t)} & r + 1 \leq i \leq u + (r-t) - 1 \\
  s_i & u + (r-t) + 1 \leq i \leq \ell - 1 
\end{cases} \]

Moreover \( d^{-1}s_id \) is not a basic transposition for \( i = t, r, u + (r-t) \). With this, the Lemma follows. \( \square \)

(2) Consider first \( \varepsilon_\lambda = y_{[1,r]}x_{[r+1,\ell]} \). We factorize \( y_{[1,r]} \) and \( x_{[r+1,\ell]} \) as described in Lemma 3.1 (taking \( W_\lambda = \bigcup_{y \in D} gU \)) and we get

\[ \varepsilon_\lambda = \zeta y_{[1,t]}y_{[t+1,u+(r-t)]}x_{[r+1,u+(r-t)]}x_{[u+(r-t)+1,\ell]} . \]

Here we write \( \zeta \) for the product of the elements called \( \psi^* \), and \( \sigma^* \). Then

Lemma 3.10 (a) If \( \varepsilon_\lambda = y_{[1,r]}x_{[r+1,\ell]} \), then \( \varepsilon_\lambda T_d = (\zeta T_d)\varepsilon_\nu(d) \) where

\[ \varepsilon_\nu(d) := y_{[1,t]}y_{[u+1,u+(r-t)]}x_{[t+1,u]}x_{[u+(r-t)+1,\ell]} \]

(b) For \( \varepsilon_\lambda = x_{[1,r]}y_{[r+1,\ell]} \) we get the same formula with \( x, y \) interchanged.

\[ \text{Proof } (a) \]

With the above,

\[ \varepsilon_\lambda T_d = \zeta T_d(T_d)^{-1}y_{[1,t]}y_{[t+1,r]}x_{[r+1,u+(r-t)]}x_{[u+(r-t)+1,\ell]}T_d = \zeta T_d y_{[1,t]}(y_{[u+1,u+(r-t)]}x_{[t+1,u]}x_{[u+(r-t)+1,\ell]}T_d = (\zeta T_d)\varepsilon_\nu(d) \]

Part (b) is analogous. \( \square \)

Proposition 3.11 With the above setting, we have \( \varepsilon_\lambda T_d\varepsilon_\mu = 0 \) if and only if \( \varepsilon_\nu(d)\varepsilon_\mu = 0 \).
Proof We must show that if $\varepsilon_{\nu(d)} \varepsilon_{\mu}$ is not zero then $(\zeta T_d) \varepsilon_{\nu(d)} \varepsilon_{\mu}$ is non-zero. Suppose $\varepsilon_{\mu(d)} \varepsilon_{\mu} \neq 0$, then it is a linear combination of $T_w$ with $w \in W_{\mu}$. We have $W_{\lambda} = \bigcup_{g \in G} g U$ and then $d^{-1} W_{\lambda} d = \bigcup_{g \in D} d^{-1} g d W_{\nu}$, where $D$ is the set of distinguished coset representatives. Suppose $g d v = g' d v'$ for $g, g' \in D$, and $v, v' \in W_{\mu}$. Then it follows that $d^{-1} g d d^{-1} g d = d^{-1} W_{\lambda} d \cap W_{\mu} = W_{\nu}$. The $d^{-1} g d$ are coset representatives and it follows that $g = g'$, and then also $v = v'$.

For each $v$ in the support of $\varepsilon_{\nu(d)} \varepsilon_{\mu}$, we have $T_d T_d T_v = T_g d v$, using that $d$ is a distinguished double coset representative. We have just proved that there is no repetition amongst these elements $g d v$ as $g$ varies through $D$, and $v$ varies through $W_{\mu}$, and hence we have an expression of $(\zeta T_d) \varepsilon_{\nu(d)} \varepsilon_{\mu}$ in terms of the basis of $H$, with non-zero coefficients. $\square$

**Corollary 3.12** With the same notation, the set $\{T_g d w \mid w \in W_{\mu}, g \in D\}$ is linearly independent.

This is part of the above proof.

### 3.2.1 Proof of Proposition 3.4

Let $\lambda = (r, s)$ and $\mu = (u, v)$ with each of $r, s, u, v$ non-zero.

(I) Assume that $r, u$ have the same parity. We consider first the case when $\varepsilon_{\lambda} = y_{[1, r]} x_{[r + 1, \ell]}$ and $\varepsilon_{\mu} = y_{[1, u]} x_{[u + 1, \ell]}$. Let $d \in D_{\lambda, \mu}$, then by Lemma 3.10 we have $\varepsilon_{\lambda} T_d = (\zeta T_d) \varepsilon_{\nu(d)}$ where $\varepsilon_{\nu(d)}$ is the element (†) of 3.10. Note that any two factors of $\varepsilon_{\nu(d)}$ commute.

(i) Suppose $t + 1 < u$ then $x_{[t + 1, u]} y_{[1, u]} = 0$, and hence $\varepsilon_{\nu(d)} \varepsilon_{\mu} = 0$.

(ii) Suppose $1 < (r - t)$ then $y_{[u + 1, u + (r - t)]} x_{[u + 1, \ell]} = 0$ and again $\varepsilon_{\nu(d)} \varepsilon_{\mu} = 0$.

This leaves $t + 1 = u$ or $t = u$, and $t + 1 = r$, or $t = r$. We assume that $u$ and $r$ have the same parity, therefore $u = r$ and it is equal to $t$ or $t + 1$. In particular $\varepsilon_{\lambda} = \varepsilon_{\mu}$, and consequently we have $\varepsilon_{\lambda} H \varepsilon_{\mu} = 0$ for $\lambda \neq \mu$ in this case.

If $r = t$ then $d$ is the identity and $\varepsilon_{\lambda} T_d \varepsilon_{\lambda}$ is non-zero (by 2.3). Suppose $r = t + 1$, then $\varepsilon_{\nu(d)} \varepsilon_{\lambda} = y_{[1, r - 1]} x_{[r + 1, \ell]} y_{[1, r]} x_{[r + 1, \ell]}$ which by 2.3 is a non-zero scalar multiple of $\varepsilon_{\lambda}$, and hence $\varepsilon_{\lambda} T_d \varepsilon_{\lambda}$ is non-zero. These are two non-zero elements supported on different double cosets, so they are linearly independent. Hence $\varepsilon_{\lambda} H \varepsilon_{\mu}$ is 2-dimensional.

Now assume $\varepsilon_{\lambda} = x_{[1, r]} y_{[r + 1, \ell]}$ and $\varepsilon_{\mu} = x_{[1, u]} y_{[u + 1, \ell]}$. Then the same proof works with $x, y$ interchanged, and we get again that $\varepsilon_{\lambda} H \varepsilon_{\mu}$ is zero for $\lambda \neq \mu$ and is 2-dimensional otherwise. This proves Proposition 3.4 when $r, u$ have the same parity.

(II) Now we deal with idempotents $\varepsilon_r$ for $r$ even and $\varepsilon_v$ and $v$ for $v$ odd. Suppose $\varepsilon_{\lambda} = y_{[1, r]} x_{[r + 1, \ell]}$ and $\varepsilon_{\mu} = x_{[1, u]} y_{[u + 1, \ell]}$. Let $d \in D_{\lambda, \mu}$, then $\varepsilon_{\lambda} T_d = (\zeta T_d) \varepsilon_{\nu(d)}$ with $\varepsilon_{\nu(d)}$ as in (†) above.

(i) Suppose $t \geq 2$, then $y_{[1, r]} x_{[1, u]} = 0$ and hence $\varepsilon_{\nu(d)} \varepsilon_{\mu} = 0$.

(ii) Suppose $u + (r - t) + 1 < \ell$, then $x_{[u + (r - t) + 1, \ell]} y_{[u + 1, \ell]} = 0$. and hence $\varepsilon_{\nu(d)} \varepsilon_{\mu} = 0$.

This leaves $t = 0$ or $t = 1$, and $u + (r - t) \geq \ell - 1$. Recall from the first part of Section 3.2, the bottom left entry of the matrix $A$ is $x = t + (s - u) = t + (v - r) \geq 0$. 9
(a) Assume $t = 0$, then $s \geq u$ and $v \geq r$ and in fact $v > r$ since $v, r$ have different parity (and then $s > u$). In this case we have $u + r = \ell - 1$ since $r \neq v$. It follows that $r + 1 = v$ (hence $\varepsilon_v = \varepsilon_{r+1}$). Moreover

$$(\varepsilon_v = \varepsilon_{r+1}) \quad \varepsilon_{\nu(d)}\varepsilon_\mu = x_{[u+1,\ell-1]}y_{[1,u]}\varepsilon_\mu$$

which is a non-zero scalar multiple of $\varepsilon_\mu$.

(b) Assume $t = 1$ then similarly $u + (r - 1) = \ell$ and $r - 1 = v$ (hence $\varepsilon_v = \varepsilon_{r-1}$).

Moreover

$$(\varepsilon_v = \varepsilon_{r-1}) \quad \varepsilon_{\nu(d)}\varepsilon_\mu = x_{[u+1,\ell]}y_{[2,u]}\varepsilon_\mu$$

which again is a non-zero scalar multiple of $\varepsilon_\mu$. Hence in this case $\varepsilon_r H\varepsilon_v$ is 1-dimensional if $v = r \pm 1$ and is zero otherwise.

(II') Consider now idempotents for $s$ even and $u$ odd; we take $\varepsilon_\lambda = y_{[1,r]}x_{[r+1,\ell]}$ and $\varepsilon_\mu = x_{[1,u]}y_{[u+1,\ell]}$, and let $d \in D_{\lambda,\mu}$. As in (II) we get $\varepsilon_{\nu(d)}\varepsilon_\mu \neq 0$ only for $t = 0$ and $t = 1$.

(a) When $t = 0$, it follows that $u + r = \ell - 1$ and $v - r = 1 = s - u$, so $\varepsilon_u = \varepsilon_{s-1}$. Moreover

$$(\varepsilon_u = \varepsilon_{s-1}) \quad \varepsilon_{\nu(d)}\varepsilon_\mu = y_{[u+1,\ell-1]}x_{[1,u]}\varepsilon_\mu$$

which is a non-zero scalar multiple of $\varepsilon_\mu$.

(b) When $t = 1$, we must have $u + r - 1 = \ell$ and $u - 1 = s$ so that $\varepsilon_u = \varepsilon_{s-1}$. Moreover

$$(\varepsilon_u = \varepsilon_{s-1}) \quad \varepsilon_{\nu(d)}\varepsilon_\mu = y_{[u+1,\ell]}x_{[2,u]}\varepsilon_\mu$$

and this also is a non-zero scalar multiple of $\varepsilon_\mu$. Hence $\varepsilon_s H\varepsilon_u$ is 1-dimensional for $\varepsilon_u = \varepsilon_{s \pm 1}$ and is zero otherwise. This completes the proof of Proposition 3.4. □

**Remark 3.13** We collect information from the above proof.

1. The matrix $A$ in (II)(a) corresponds to $d$ in Example 3.8(1), this defines $\alpha_r$ as in Definition 3.6 for $r$ even.
2. The matrix $A$ in (II)(b) corresponds to $d'$ in Example 3.8(2), this defines $\beta_{r-1}$ as in Definition 3.6 for $r$ even.
3. The permutation associated to the matrix in part (II')(a) (the inverse of the permutation as in (1) with $s - 1$ instead of $r$) defines $\beta_{s-1}$.
4. The permutation associated to the matrix in part (II')(b) (the inverse of the permutation as in (2) with $s = r - 1$) defines $\alpha_s$.

### 3.3 Arrows and relations

is local, isomorphic to $K[T]/(T^2)$.

**Lemma 3.14** The algebra $\varepsilon_1 H \varepsilon_1$ has basis $\varepsilon_1, x_{[1,\ell]}$. It is local, isomorphic to $K[X]/(X^2)$.  

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Proof The element \( x_{[1,\ell]} \) spans the trivial \( H \)-module, and its square is zero. One checks that \( x_{[1,\ell]} = \varepsilon_1 x_{[1,\ell]} \varepsilon_1 \). Clearly \( x_{[1,\ell]} \) and \( \varepsilon_1 \) are linearly independent. We have proved in Proposition 3.4 that \( \varepsilon_1 H \varepsilon_1 \) is 2-dimensional, and it follows that it has basis \( \{ \varepsilon_1, x_{[1,\ell]} \} \). The Lemma follows. \( \square \)

We will now prove Theorem 3.7.

1. Clearly \( \alpha_r \alpha_{r+1} = 0 \) since it is an element in \( \varepsilon_r H \varepsilon_{r+2} \) which is zero by Proposition 3.4. Similarly \( \beta_{r+1} \beta_r = 0 \).
2. The products \( \alpha_r \beta_r \) and \( \beta_r \alpha_r \) are non-zero:

   We fix \( r \) and write \( \alpha = \alpha_r \) and \( \beta = \beta_r \). To prove \( \alpha \beta \neq 0 \), it suffices to show \( f(\alpha \beta, 1) \neq 0 \) where \( f \) is the symmetrizing form as in 2.4. Using the symmetry property, and the fact that the \( \varepsilon \) are idempotent, and also that \( \varepsilon_r T_d \varepsilon_{r+1} = c(\zeta T_d) \varepsilon_{r+1} \) where \( c \) is a non-zero scalar, we get

\[
\begin{align*}
f(\alpha \beta, 1) &= f(\varepsilon_r T_d \varepsilon_{r+1} T_{d-1}, 1) \\
&= f(\varepsilon_r T_d \varepsilon_{r+1}, T_{d-1}) \\
&= (\varepsilon_r T_d \varepsilon_{r+1}, T_d).
\end{align*}
\]

where \((-,-)\) is the bilinear form as in 2.4. We have that \( \varepsilon_r T_d \varepsilon_{r+1} = (\zeta T_d) \varepsilon_{\nu(d)} \varepsilon_{\mu} \) with the notation as in Proposition 3.11 (with \( \varepsilon_{r+1} \) a non-zero scalar multiple of \( \varepsilon_{\mu} \). We know that \( (\zeta T_d) \varepsilon_{\nu(d)} \varepsilon_{\mu} \) is non-zero, therefore by Corollary 3.12 it is a unique linear combination of the set \( \{ T_{gdw} \mid g \in D, w \in W_\mu \} \). Therefore

\[
((\zeta T_d) \varepsilon_{\nu(d)} \varepsilon_{\mu}, T_d) = cq^{l(d)}
\]

where \( c \) is the coefficient of the identity in \( \varepsilon_{\nu(d)} \varepsilon_{\mu} \) and this is non-zero (by 2.2).

2. \( \alpha_r \beta_r \alpha_r = 0 \) and \( \beta_r \alpha_r \beta_r = 0 \). In particular each \( \varepsilon_r H \varepsilon_2 \) contains a non-zero nilpotent element, and hence is a 2-dimensional local algebra:

   We start with \( r = 1 \). It suffices to show that \( \alpha_1 \beta_1 \) is not a unit in \( \varepsilon_1 H \varepsilon_1 \). Consider right multiplication with \( \alpha_1 \), this is a module homomorphism \( H \varepsilon_1 \rightarrow H \varepsilon_2 \). This would be injective if \( \alpha_1 \beta_1 \) were a unit, in particular \( x_{[1,\ell]} \alpha_1 \neq 0 \). However

\[
x_{[1,\ell]} \alpha_1 = x_{[1,\ell]} \varepsilon_1 T_d \varepsilon_2 = x_{[1,\ell]} T_d \varepsilon_2 = q^{l(d)} x_{[1,\ell]} \varepsilon_2 = 0.
\]

So we have a contradiction, and \( \alpha \beta \) must be nilpotent and is therefore a scalar multiple of \( x_{[1,\ell]} \). By the above computation, we see directly that \( \alpha \beta \alpha = 0 \). We can also deduce \( \beta \alpha \beta = 0 \): Namely it is of the form \( a \beta \) for \( a \in K \) and then \( 0 = (\alpha \beta)^2 = a(\alpha \beta) \) and \( a = 0 \). This now shows that \( \beta \alpha \in \varepsilon_2 H \varepsilon_2 \) is nilpotent and then the algebra \( \varepsilon_2 H \varepsilon_2 \) must be local as well. By induction on \( r \) repeating the arguments, the claim follows.

3. Up to rescaling, the commutativity relations hold:

   The non-zero elements \( \beta_r \alpha_r \) and \( \alpha_{r+1} \beta_{r+1} \) are both in the 1-dimensional radical of \( \varepsilon_r H \varepsilon_r \) and hence must be scalar multiples of each other. So we can take \( \beta_2' = a_2 \beta_2 \) for \( 0 \neq a_2 \in K \) so that \( \beta_1 \alpha_1 = a_2 \beta_2' \).
Inductively, suppose we have scaled arrow so that for \( i \leq r \) the commutativity relation \( \beta'_{i-1} \alpha_{i-1} = \beta_i \alpha_i \). If \( r < \ell - 2 \) then \( \beta'_{r+1} = a_{r+1} \beta_{r+1} \) so that \( \alpha_r \beta'_r = \alpha_{r+1} \beta'_{r+1} \). We repeat this, and when we reach \( r = \ell - 2 \), we are done.

(4) There are no further relations: The algebra \( B \) is equal to \( \varepsilon H \varepsilon \) where \( \varepsilon = \sum_{r=1}^{\ell-1} \varepsilon_r \). Our computations show that it has dimension \( 2(\ell - 1) + 2(\ell - 2) = 4(\ell - 1) - 2 \). This is also the dimension of the algebra \( KQ/I \) in (3.1). One defines an algebra map \( KQ \to B = \varepsilon H \varepsilon \) by mapping the elements \( e_r \in KQ \) (corresponding to paths of length zero) to \( \varepsilon_r \), and taking arrows \( \alpha_r, \beta_r \) to the elements in \( \varepsilon H \varepsilon \) we called \( \alpha_r \) and the rescaled \( \beta_r \), and extend to products and linear combinations. It is surjective, and the ideal \( I \) is in the kernel. Then by dimensions, \( I \) is the kernel and we have shown that \( \varepsilon H \varepsilon \) is the algebra as defined above. □

Recall that the socle of a module is the largest semisimple submodule, and the top is the largest semisimple quotient.

**Corollary 3.15** (a) Each \( \varepsilon_r H \) is indecomposable projective. Its radical has basis \( \{ \alpha_r, \alpha_{r-1}, \alpha_r \beta_r \} \). (omitting \( \alpha_{r-1} \) for \( r = 1 \) and taking \( \alpha_{r-2} \beta_{r-1} \) instead of \( \alpha_r \beta_r \) for \( r = \ell - 2 \)).

(b) The socle of \( \varepsilon_r H \) is simple, spanned by \( \alpha_r \beta_r \) or \( \alpha_{r-1} \beta_{r-1} \).

(c) The simple \( \varepsilon H \varepsilon \)-modules are precisely the 1-dimensional quotients \( \varepsilon_r H / \text{rad} \varepsilon_r H \), and the socle of \( \varepsilon H \varepsilon \) (as a left or right module) is spanned by the elements \( \alpha_r \beta_r \).

**Proof** Since \( \varepsilon_r \) is an idempotent, the module \( \varepsilon_r H \) is projective. Its endomorphism algebra is \( \varepsilon_r H \varepsilon_r \), and we have proved that it is local. Hence \( \varepsilon_r H \) is indecomposable. All other statements follow from the proof of Theorem 3.7 □

**Remark 3.16** (1) The module \( \varepsilon_r H \) is the projective cover of \( D^{(s+1,1^{r-1})} \) for \( r \geq 1 \). This follows from the decomposition number approach to such blocks, see for example [15], together with the observation that \( \varepsilon_1 H \) is the projective cover of the trivial module.

**Example 3.17** Assume \( \ell = 3 \). The algebra \( H = H_q(3) \) has dimension 6 and hence is equal to its basic algebra \( B \) by dimensions. The presentation as a Hecke algebra is not compatible with the presentation (3.1). The generators in our presentation are \( \varepsilon_1 = c_1 x_{[1,2]} \) and \( \varepsilon_2 = c_2 y_{[1,2]} \) (with \( c_1, c_2 \) non-zero scalars), and the arrows are \( \beta_1 = \varepsilon_2 T_2 \varepsilon_1 \) and \( \alpha_1 = \varepsilon_1 T_2 \varepsilon_2 \).
4 Homological properties

4.1 Controlling projectivity

Given a module, a basic question is to determine whether or not it is projective. We give an easy criterion. We show (with \( \ell \geq 3 \)) that there is an associated truncated polynomial algebra which directly detects projectivity.

Consider the basic algebra \( A \) for the principal block of \( H_q(\ell) \) as in Theorem 3.7. We choose and fix non-zero elements \( c_r \in K \) for \( 1 \leq i \leq \ell - 2 \) such that \( c_r + c_{r+1} \neq 0 \).

Such elements exist, \( K \) has at least two non-zero elements.)

Lemma 4.1 Let

\[
\tilde{\alpha} := \sum_{r=1}^{\ell-2} \alpha_r \quad \text{and} \quad \tilde{\beta} := \sum_{r=1}^{\ell-2} c_r \beta_r, \quad \text{and} \quad \rho := \tilde{\alpha} + \tilde{\beta}.
\]

Then \( \rho^2 \) spans the socle of \( A \), and \( \rho^3 = 0 \).

Proof First observe that \( \tilde{\alpha}^2 = 0 \) and \( \tilde{\beta}^2 = 0 \). Then

\[
\rho^2 = c_1 \alpha_1 \beta_1 + \sum_{r=2}^{\ell-3} (c_r + c_{r+1}) \alpha_r \beta_r + c_{\ell-2} \beta_{\ell-2} \alpha_{\ell-2}
\]

All scalar coefficients are non-zero by construction. Moreover \( \varepsilon_r \rho^2 \varepsilon_r \) spans the socle of \( \varepsilon_r H \), hence \( \rho^2 \) spans the socle of \( A \) and \( \rho^3 = 0 \). □

Now take a block of an \( \ell \)-parabolic subalgebra which is not simple. This is Morita equivalent to the tensor product \( A^\otimes m \) of copies of \( A \), let \( B \) be this algebra. Let \( \rho_i \in B \) be the tensor where at the \( i \)-th place take \( \rho \), and all other factors are the identity. Then let \( R = R_B \) be the subalgebra of \( B \) generated by \( \rho_1, \ldots, \rho_m \). The \( \rho_i \) commute, and \( \rho_i^2 = 0 \) so that \( R \) is a truncated polynomial algebra.

Proposition 4.2 Assume \( M \) is an indecomposable \( B \)-module. Then \( M \) is projective if and only if the restriction to \( R \) is projective, which is true if and only if \( (\rho_1 \rho_2 \cdots \rho_m)^2 \) does not annihilate \( M \).

Proof The socle of \( B \) is spanned by the socle of \( R \) which is spanned by \( (\rho_1 \rho_2 \cdots \rho_m)^2 \). An indecomposable \( B \)-module \( M \) is projective if and only if the socle of \( B \) does not annihilate \( M \), this holds since the algebra is self-injective. □

We would like to know when the algebra \( B \) is projective as a module over \( R \).

Lemma 4.3 Let \( A \) the basic algebra of the principal block of \( H_q(\ell) \) and \( B = A^\otimes m \). Then \( B \) is projective as an \( R \)-module if and only if \( \ell = 3 \).
Proof (1) We reduce to the case $m = 1$. We have $B = A^\otimes m$ and $R = R_0^\otimes m$ where
$R_0 = K[Z]/(Z^3)$. Clearly, if $A$ is projective over $R_0$ then $B$ is projective over $R$. On the
other hand, if $B$ is not projective over $R$ then it has a direct summand annihilated by $\rho$ and
then $A$ has a direct summand annihilated by some $\rho_i$ and hence is not projective.

(2) Now consider $m = 1$. Assume first that $\ell = 3$, we can see that $A$ is projective as a
(left or right) $R_0$-module, one checks e.g. that $A = R_0 e_1 \oplus R_0 e_2$.

Now assume $A$ is projective, hence free, over $R_0$; then the dimension of $A$ is a multiple of
3. We know that $\dim A = 2(2\ell - 3)$, so this implies $\ell = 3r$ and then $\dim A = (4r - 2) \cdot 3$.
Then have that $A$ is the direct sum of $4r - 2$ copies of $R_0$. So assume $A = \oplus_{i=1}^{4r-2} R_0 z_i$
and each summand is isomorphic to $R_0$. It follows that $\rho^2 z_i$ is non-zero and hence $z_i$
is not in the radical of $A$ but $\rho z_i$ is in the radical of $A$. Therefore $A/\mathrm{rad} A$ has a basis
consisting of the cosets of the $z_i$. Now $A/\mathrm{rad} A$ has dimension $\ell - 1 = 3r - 1$. So we
must have that $3r - 1 = 4r - 2$ that is $r = 1$ and $\ell = 3$. \(\square\)

This suggests that $R$ might only be useful for $H_\rho$ when $\ell = 3$. (For $\ell = 2$, $H_\rho$ itself is
a truncated polynomial algebra).

4.2 Support varieties

From now we assume that $K$ is algebraically closed. To control projectivity of modules, and
understand some large-scale behaviour, one can often exploit cohomological sup-
port varieties. Hecke algebras are not Hopf algebras but one can use use Hochschild
cohomology. We recall relevant definitions and facts, here $\Lambda$ is a finite-dimensional
algebra. We refer to [14] for details. The Hochschild cohomology $HH^*(\Lambda)$ is a graded
commutative algebra and it acts on $\text{Ext}^*_\Lambda(M,M)$ for any $\Lambda$-module $M$, and with condi-
tion (Fg), one may use a sufficiently large commutative subalg ebra to define a variety
for $M$:

**Definition 4.4** The algebra $\Lambda$ satisfies the finite generation hypothesis (Fg) if the
Hochschild cohomology $HH^*(\Lambda)$ is Noetherian and moreover $\text{Ext}^*_\Lambda(\Lambda/\mathrm{rad}\Lambda, \Lambda/\mathrm{rad}\Lambda)$
is a finitely generated $HH^*(\Lambda)$-module.

With this, any $\Lambda$-module $M$ has a support variety $V(M)$ whose properties are similar
to that of support varieties for group representations. The dimension of $V(M)$ is equal
to the complexity of $M$, in particular $V(M)$ is trivial if and only if $M$ is projective.
Furthermore, it gives information on Auslander-Reiten components.

**Definition 4.5** Assume $\mathcal{X}$ is a subcategory of $\text{mod-}\Lambda$, which is the union of stable
Auslander-Reiten components up to projectives. We say that there are enough periodic
modules for $\mathcal{X}$ if for every non-projective indecomposable module $M$ in $\mathcal{X}$ there is some
$\Omega$-periodic module $W$ such that $\text{Hom}_\Lambda(W,M) \neq 0$.

With this one can construct suitable subadditive functions on Auslander-Reiten com-
ponents and deduce that the tree class of any component is one of
(\(T\)) A Dynkin diagram of type ADE, or a Euclidean diagram, or one of the infinite
trees $A_\infty, D_\infty$ or $A_\infty^\infty$.

For details, see for example [3], or [4] 2.11 and 2.12.
We consider the case when $\Lambda = H_\rho$, an $\ell$-parabolic Hecke algebra. Then we have:

**Theorem 4.6** The algebras $H_\rho$ satisfy (Fg). In particular
(a) $H_\rho$-modules have finite complexity.
(b) The category of $H_\rho$-modules has enough periodic modules.
(c) The tree class of any stable AR component belongs to $(T)$.

**Proof** The condition (Fg) is Morita invariant, so it suffices to show that the basic algebra for each block of $H_\rho$ satisfies (Fg). For the basic algebra $A$ of $H_q(\ell)$ this follows from [12] (the algebra has radical cube zero). Then by [6] (Corollary 4.8), every tensor power of $A$ satisfies (Fg). Parts (b) and (c) follow directly, by applying results from [14]. □

**Remark 4.7**
(1) So far this deals with $H_\rho$. If the field has characteristic zero, Linckelmann’s work [16] implies that even $H$ satisfies (Fg). It is open whether it is true for non-zero characteristic and therefore there are many open questions. For example it is not even known whether the trivial module for $H_q(6)$ when $\ell = 3$ and $p = 2$ has finite complexity.
(2) Over characteristic zero, an explicit presentation of the cohomology of $H$, that is, of $\text{Ext}_H^*(K, K)$ with $K$ the trivial module, was determined in [5]. This is used by [18] to develop a support variety theory.

### 4.3 Rank varieties

For some algebras, modules have rank varieties (and often it is known that they are essentially the same as the support varieties). Rank varieties were first introduced for group algebras of elementary abelian $p$-groups over fields of characteristic $p$ by J. Carlson [7]. This was generalized to quantum complete intersections in [4]. A recent result in [1] introduces a different type of rank variety.

The quantum complete intersections of [4] include truncated polynomial algebras, in particular the algebra $R$ we have constructed earlier. Rank varieties can be used directly to construct enough periodic modules, see [4]; see also [19]. Simon Schmider uses this to show:

**Theorem 4.8** Suppose $\ell \geq 3$ and $p$ does not divide $\ell - 1$. Then all components in a block of wild representation type of $H_\rho$ have tree class $A_\infty$.  

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4.4 Beyond \( \ell \)-parabolic subalgebras

Let \( H_\rho \) be some maximal \( \ell \)-parabolic subalgebra. Define \( \mathcal{X} \) to be the full subcategory of \( \text{mod-} H \) with objects the \( H \)-modules \( M \) whose restriction to \( H_\rho \) is not projective.

**Proposition 4.9** (a) The category \( \mathcal{X} \) is the union of stable Auslander-Reiten components.
(b) It has enough periodic modules, hence each tree class belongs to the list \((T)\).

**Proof** (a) Clearly \( \mathcal{X} \) is closed under syzygies, and under Auslander-Reiten translation \( \tau(\cong \Omega^2) \). Let \( 0 \to \tau(M) \to E \to M \to 0 \) be an almost split sequence, and assume \( M \) is in \( \mathcal{X} \), then also \( \tau(M) \) is in \( \mathcal{X} \). Take an indecomposable non-projective summand \( E' \) of \( E \), then by general theory there is an almost split sequence

\[
(*) \quad 0 \to \tau(E') \to U \oplus \tau(M) \to E' \to 0
\]

(for some module \( U \)). Assume for a contradiction \( E'_{H_\rho} \) is projective, then the restriction of \((*)\) to \( H_\rho \) is split. As well \( \tau(E') \) is not in \( \mathcal{X} \) and so \( E' \oplus \tau(E') \) is projective restricted to \( H_\rho \), and we have the contradiction that \( \tau(M) \) is projective as a module for \( H_\rho \).

(b) Let \( M \) be in \( \mathcal{X} \), then there is a periodic \( H_\rho \)-module \( W \) such that \( \text{Hom}_{H_\rho}(W, M) \neq 0 \). Now, \( H \) is projective as a module for \( H_\rho \), so \( \text{Hom}_H(W \otimes_{H_\rho} H, M) \neq 0 \). Furthermore, the induced module \( W \otimes_{H_\rho} H \) is periodic up to projective summands. □

In general, it is not known which tree classes for \( H \)-modules occur, except for blocks of finite type (with tree class \( A_{\ell-1} \)), and for tame type (with one Euclidean component and otherwise tubes, with tree class \( A_\infty \), see [13].)

**Example 4.10** We give an example of a non-projective module in \( \mathcal{X} \). Let \( \ell = 3 \) and \( n = 6 \), and assume \( p = 2 \). Let \( Y = Y^{(3,3)} \) be the q-Young module labelled by partition \((3,3)\), this is the non-projective summand of the q-permutation module \( M^{(3,3)} \). One can show that it has a chain of submodules

\[
0 \subset U_1 \subset U_2 \subset Y
\]

where both \( U_1 \) and \( Y/U_2 \) are 1-dimensional and isomorphic to the trivial module. Moreover \( U_2/U_1 \) is uniserial of length three with composition factors \( D^{(5,1)}, D^{(3,3)}, D^{(5,1)} \). It is not projective, but one can show that its restriction to \( H_\rho \) is projective.

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