Approach to the construction of the spaces

\[ SD^p[\mathbb{R}^\infty] \text{ for } 1 \leq p \leq \infty \]

Hemanta Kalita\(^1\), Bipan Hazarika\(^2,\ast\)

\(^1\)Department of Mathematics, Patkai Christian College (Autonomous), Dimapur, Patkai 797103, Nagaland, India

\(^2\)Department of Mathematics, Gauhati University, Guwahati 781014, Assam, India

Email: hemanta30kalita@gmail.com; bh_rgu@yahoo.co.in;

Abstract. The objective of this paper is to construct separable Banach spaces \( SD^p[\mathbb{R}^\infty] \) for \( 1 \leq p \leq \infty \), each of which contains the \( L^p[\mathbb{R}^\infty] \) spaces, as well as finitely additive measures, as compact dense embedding. Also these spaces contains Henstock-Kurzweil integrable functions.

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1. Introduction and Preliminaries

Henstock integral was first developed by R. Henstrock and J. Kurzweil independently during 1957-1958 from Riemann integral with the concept of tagged partitions and gauge functions. Henstock integral (HK-integral) is a kind of non-absolute integral and contain Lebesgue
integral (see [RH, RH1]).

The most important of the finitely additive measures is the one generated by HK-integral, which generalizes the Lebesgue, Bochner and Pettis integrals (see [AL, RAG, RH, CS]). As a major drawback of HK-integrable function space is not naturally Banach space. In [TY], Yeong mentioned if canonical approach can developed then the above drawback can be solved. Gill and Zachary [TW], introduced a new class of Banach spaces $K^p[\Omega]$, $\forall 1 \leq p \leq \infty$ and $\Omega \subset \mathbb{R}^n$ which are canonical spaces. These spaces are separable that contains the corresponding $L^p$ spaces as dense, continuous, compact embedding. There main aim was to find these spaces contains the Denjoy integrable functions as well as additive measures (see [AL]). They found that these spaces are perfect for highly oscillatory functions that occur in quantum theory and non linear analysis. In [TW], Gill and Zachary obtained few interesting work for the canonical $K^p[\mathbb{R}^n]$ spaces with the Jones family of spaces $SD^p[\mathbb{R}^n]$, $1 \leq p \leq \infty$. So, we attempt to construct a class of separable Banach spaces $SD^p[\mathbb{R}\infty]$, $1 \leq p \leq \infty$ which also contain the non-absolutely integrable functions. These space contains the test functions $D[\mathbb{R}\infty]$, as dense continuous embedding.

Throughout our paper, we suppose the notation $\mathbb{R}\infty$ and assume that $I$ is understood. Our study focused on the main class of Banach spaces $SD^p[\mathbb{R}\infty]$, $1 \leq p \leq \infty$. These spaces are natural for the HK-integrable functions and contains the $L^p$ spaces, $1 \leq p \leq \infty$ as continuous dense and compact embedding.

Using the approach of Yeong [TY], Gill and Zachary [TW] we will discussed about Henstock integrable functions for the strong Jones functions space $SD^p[\mathbb{R}\infty]$, $1 \leq p \leq \infty$ where $\mathbb{R}\infty$, which is extension of $\mathbb{R}^n$, so we use same approach for this.
**Definition 1.1.** [CS] A function \( f : [a, b] \to \mathbb{R} \) is Henstock integrable if there exists \( A \in \mathbb{R} \) and for \( \epsilon > 0 \) there exists a gauge function \( \delta : [a, b] \to \mathbb{R} \) such that for each tagged partition \( (P, (c_k)_{k=1}^n) \) that is \( \delta(x) \)-fine, we have

\[
|R(f, P) - A| < \epsilon.
\]

Or a function \( f : [a, b] \to \mathbb{R} \) is Henstock integrable if there exists a function \( F : [a, b] \to \mathbb{R} \) and for every \( \epsilon > 0 \) there is a function \( \delta(t) > 0 \) such that for any \( \delta \)-fine partition \( D = \{(u, v), t\} \) of \([a, b]\), we have

\[
|| \sum [f(t)(v - u) - F(u, v)] || < \epsilon,
\]

where the sum \( \sum \) is run over \( D = \{(u, v), t\} \) and \( F(u, v) = F(v) - F(u) \). We write \( H \int_{I_0} f = F(I_0) \).

**Definition 1.2.** [GM] Let \( A_n = A \times I_n \) and \( B_n = B \times I_n \) (\( n \)th order box sets in \( \mathbb{R}^\infty \)). We define

1. \( A_n \cup B_n = (A \cup B) \times I_n \);
2. \( A_n \cap B_n = (A \cap B) \times I_n \);
3. \( B_n^c = B^c \times I_n \).

**Definition 1.3.** [GM] We define \( \mathbb{R}_I^n = \mathbb{R}^n \times I_n \). If \( T \) is a linear transformation on \( \mathbb{R}^n \) and \( A_n = A \times I_n \), then \( T_I \) on \( \mathbb{R}_I^n \) is defined by \( T_I[A_n] = T[A] \). We define \( B[\mathbb{R}_I^n] \) to be the Borel \( \sigma \)-algebra for \( \mathbb{R}_I^n \), where the topology on \( \mathbb{R}_I^n \) is defined via the class of open sets \( D_n = \{U \times I_n : U \text{ is open in } \mathbb{R}^n \} \). For any \( A \in B[\mathbb{R}_I^n] \), we define \( \lambda_\infty(A_n) \) on \( \mathbb{R}_I^n \) by product measure \( \lambda_\infty(A_n) = \lambda_n(A) \times \Pi_{i=n+1}^{\infty} \lambda_I(I) = \lambda_n(A) \).

**Theorem 1.4.** [GM] \( \lambda_\infty(\cdot) \) is a measure on \( B[\mathbb{R}_I^n] \) is equivalent to \( n \)-dimensional Lebesgue measure on \( \mathbb{R}^n \).
Corollary 1.5. The measure $\lambda_\infty(\cdot)$ is both translationally and rotationally invariant on $(\mathbb{R}_I^n, B[\mathbb{R}_I^n])$ for each $n \in \mathbb{N}$.

We can construct a theory on $\mathbb{R}_I^n$ that completely parallels that on $\mathbb{R}^n$. Since $\mathbb{R}_I^n \subset \mathbb{R}^{n+1}_I$, we have an increasing sequence, so we define $\hat{\mathbb{R}}_I^n =\lim_{n \to \infty} \mathbb{R}_I^n = \bigcup_{k=1}^{\infty} \mathbb{R}_I^k$. In [GM], shown that it can extend the measure $\lambda_\infty(\cdot)$ to $\mathbb{R}_I^\infty$. We take measurable functions as follows.

Let $x = (x_1, x_2, \ldots) \in \mathbb{R}_I^\infty$, $I_n = \prod_{k=n+1}^{\infty} [-\frac{1}{2}, \frac{1}{2}]$ and let $h_n(\hat{x}) = \chi_{I_n}(\hat{x})$, where $\hat{x} = (x_i)_{i=n+1}^{\infty}$.

Recalling $\mathbb{R}_I^\infty$ be the closure of $\hat{\mathbb{R}}_I^n$ in the induced topology from $\mathbb{R}^\infty$. From our construction, it is clear that a set of the form $A = A_n \times (\prod_{k=n+1}^{\infty} \mathbb{R})$ is not in $\hat{\mathbb{R}}_I^n$ for any $n$. So, $\hat{\mathbb{R}}_I^n \neq \mathbb{R}_I^\infty$. The natural topology for $\mathbb{R}_I^\infty$ is that induced as a closed subspace of $\mathbb{R}^\infty$. Thus if $x = (x_n), y = (y_n)$ are sequences in $\mathbb{R}_I^\infty$ define a metric on $\mathbb{R}_I^\infty$ as

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}.$$ 

Then $\mathbb{R}_I^\infty$ is a translation invariant complete metric space with respect to the metric $d$.

Theorem 1.6. $\mathbb{R}_I^\infty = \mathbb{R}^\infty$.

We call $\mathbb{R}_I^\infty$ the essentially bounded version of $\mathbb{R}^\infty$. We may lose some of pathology of $\mathbb{R}^\infty$ by replacing it with $\mathbb{R}_I^\infty$, however, this is not true. Recalling infinite product measure for unbounded measures induces problems because it can fail to make sense for two additional reasons. Let $A_i$ have measure $1 + \epsilon$ for all $i$, it is easy to see that $\lambda_\infty(A) = \prod_{i=1}^{\infty} \lambda(A_i) = \infty$. On the other hand, if each $A_i$ has measure $1 - \epsilon$, then $\lambda_\infty(A) = \prod_{i=1}^{\infty} \lambda(A_i) = 0$. Thus the class of sets $A \in B[\mathbb{R}_I^\infty]$ for which $0 < \lambda_\infty(A) < \infty$ is relatively small. It follows that the sets of measure zero need not be small nor the sets of infinite measure be large.
Definition 1.7. If \( f \in SD^p[\mathbb{R}^\infty] \), we define the integral of \( f \) by

\[
\int_{\mathbb{R}^\infty} f(x) d\lambda_\infty(x) = \lim_{n \to \infty} \int_{\mathbb{R}^n} f_n(x) d\lambda_n(x),
\]

where \( f_n \in SD^p[\mathbb{R}^n] \) for all \( n \) and the family \( \{f_n\} \) is a Cauchy sequence.

Theorem 1.8. If \( f \in SD^p[\mathbb{R}^\infty] \), then the integral of \( f \) defined in Definition 1.7 exists and is unique for every \( f \in SD^p[\mathbb{R}^\infty] \).

Proof. Since the family of functions \( \{f_n\} \) is Cauchy, it is follows that if the integral exists, it is unique. To prove existence, follow the standard argument and first assume that \( f(x) \geq 0 \). In this case, the sequence can always be chosen to be increasing, so that the integral exists. The general case now follows by the standard decomposition. \( \square \)

We denote \( \mathbb{N}_0^\infty \) be the set of all multi-index infinite tuples \( \alpha = (\alpha_1, \alpha_2, \ldots) \), with \( \alpha_i \in \mathbb{N} \) and all but a finite number of entries are zero (also see [TG]).

Definition 1.9. The Schwartz space \( \mathcal{S}[\mathbb{R}^\infty] \) is the topological space of functions \( f: \mathbb{R}^\infty \to \mathbb{C} \) such that \( f \in C^\infty[\mathbb{R}^\infty] \) and \( x^\alpha \partial^\beta f(x) \to 0 \) as \( |x| \to \infty \) for every pair of multi-indices \( \alpha, \beta \in \mathbb{N}_0^\infty \), and \( f \in \mathcal{S}[\mathbb{R}^\infty] \), let

\[
||f||_{\alpha,\beta} = \sup_{\mathbb{R}^\infty} |x^\alpha \partial^\beta f|.
\]

If \( x = (x_1, x_2, \ldots) \in \mathbb{R}^\infty \) and \( \alpha \in \mathbb{N}_0^\infty \), \( \alpha = (\alpha_1, \alpha_2, \ldots) \), we define \( x^\alpha = \prod_{k=1}^{\infty} x_k^{\alpha_k} \) a product of real or complex numbers.

A sequence of functions \( \{f_k : k \in \mathbb{N}\} \) converges to a function \( f \) in \( \mathcal{S}[\mathbb{R}^\infty] \) if

\[
||f_n - f||_{\alpha,\beta} \to 0
\]
as \( k \to \infty \) for every \( \alpha, \beta \in \mathbb{N}_0^\infty \).

That is the Schwartz space consists of smooth functions whose derivatives decay at infinity faster than any power. For details on Schwartz space and distribution functions we refer \([MA, GT, LG, KB, SY]\).

**Theorem 1.10.** \( \mathcal{S}[\mathbb{R}_I^\infty] \) (respectively \( \mathcal{S}'[\mathbb{R}_I^\infty] \)) is a Fréchet space, which is dense in \( \mathbb{C}_0[\mathbb{R}_I^\infty] \).

**Proof.** It is an easy proof similar to the proof of (p 90 Theorem 2.88 of \([TG]\)). \(\square\)

**Definition 1.11.** A tempered distribution \( T \) on \( \mathbb{R}_I^\infty \) is a continuous linear functional \( T : \mathcal{S}[\mathbb{R}_I^\infty] \to \mathbb{C} \). The topological vector space of tempered distributions is denoted by \( \mathcal{S}'[\mathbb{R}_I^\infty] \) or \( \mathcal{S}' \). If \( \langle T, f \rangle \) denotes the value of \( T \in \mathcal{S}' \) acting on \( f \in \mathcal{S} \), then a sequence \( \{T_k\} \) converges to \( T \) in \( \mathcal{S}' \). Written \( T_k \to T \) if \( \lim_{k \to \infty} \langle T_k, f \rangle = \langle T, f \rangle \) for every \( f \in \mathcal{S} \).

**Purpose of the paper:**

The purpose of this paper is to introduce a class of Banach spaces on \( \mathbb{R}_I^\infty \) which contain the non-absolutely integrable functions, but also contains the Schwartz test function space as dense and continuous embedding.

2. **The Jones family of spaces** \( SD^p[\mathbb{R}_I^\infty] \) \( 1 \leq p \leq \infty \)

The theory of distributions is based on the action of linear functional on a space of test function. In \([TG]\), Gill and Zachary introduced another class of Banach spaces which contain the non-absolutely integrable functions, but also contains the Schwartz test function space as a dense and continuous embedding.

We define the space \( SD^p[\mathbb{R}_I^\infty] \) with the help of the space \( SD^p[\mathbb{R}_I^n] \).
We see that $SD^p[\mathbb{R}^n] \subset SD^p[\mathbb{R}^{n+1}]$. Thus we can define $SD^p[\hat{\mathbb{R}}^\infty] = \bigcup_{n=1}^\infty SD^p[\mathbb{R}^n]$. So, we can say for $1 \leq p \leq \infty$, a measurable function $f \in SD^p[\hat{\mathbb{R}}^\infty]$ if there is a Cauchy sequence $\{f_n\} \subset SD^p[\hat{\mathbb{R}}^\infty]$ with $f_n \in SD^p[\mathbb{R}^n]$ and $\lim_{n \to \infty} f_n(x) = f(x)$ $\lambda_\infty$-a.e.

To construct the space $SD^p[\mathbb{R}^\infty]$, for $1 \leq p \leq \infty$ we start with the space $L^1[\mathbb{R}^\infty]$. Let $\{E_n\}_{n=1}^\infty$ be a dense set of functions on the unit ball of $L^\infty[\mathbb{R}^\infty]$ so that $\|E_n\|_{L^\infty} = 1$. If $\{t_n\}_{n=1}^\infty$ is a set of positive numbers that sum to one, define inner product on $L^1[\mathbb{R}^\infty]$ by

$$< f, g > = \sum_{n=1}^\infty t_n \left[ \int_{\mathbb{R}^\infty} E_n(x)f(x)d\lambda_\infty(x) \right] \left[ \int_{\mathbb{R}^\infty} E_n(x)g(x)d\lambda_\infty(x) \right].$$

Easily we can find that this inner product and that

$$\|f\|^2 = < f, f > = \sum_{n=1}^\infty t_n \left| \int_{\mathbb{R}^\infty} E_n(x)f(x)d\lambda_\infty(x) \right|^2.$$

We call the completion of $L^1[\mathbb{R}^\infty]$ with the above inner product is a Hilbert space, which we denote $SD^2[\mathbb{R}^\infty]$.

**Theorem 2.1.** For each $p$, $1 \leq p \leq \infty$, we have

1. The space $L^p[\mathbb{R}^\infty] \subset SD^2[\mathbb{R}^\infty]$ as a continuous, dense and compact embedding.

2. $\mathcal{M}[\mathbb{R}^\infty] \subset SD^2[\mathbb{R}^\infty]$, $\mathcal{M}[\mathbb{R}^\infty]$ is the space of finitely additive measures on $\mathbb{R}^\infty$, as a continuous dense and compact embedding.

**Proof.** (1) As $L^p[\mathbb{R}^n] \subset SD^2[\mathbb{R}^n]$, for each $p$, $1 \leq p \leq \infty$ as a continuous, dense and compact embedding. However $SD^2[\mathbb{R}^\infty]$ is the closure of $\bigcup_{n=1}^\infty SD^2[\mathbb{R}^n]$. It follows $SD^2[\mathbb{R}^\infty]$ contains $\bigcup_{n=1}^\infty L^p[\mathbb{R}^n]$ which is dense in $L^p[\mathbb{R}^\infty]$, as it’s closure.
(2) As $L^1[\mathbb{R}_I^\infty] \subset SD^2[\mathbb{R}_I^\infty]$ and $\mathcal{M}([R_I]) = \{L^1([R_I])\}^{**}$. It gives $\bigcup_{n=1}^{\infty}\{\mathcal{M}([R_I])\} = \bigcup_{n=1}^{\infty}\{L^1([R_I])\}^{**}$. Since $f \in SD^2[\mathbb{R}_I^\infty]$ is the limit of a sequence $\{f_n\} \subset \bigcup_{n=1}^{\infty}SD^2[\mathbb{R}_I^\infty]$. So $\mathcal{M}([R_I^\infty]) = \bigcup_{n=1}^{\infty}\{L_1([R_I^\infty])\}^{**}$ and hence $\mathcal{M}([R_I^\infty]) \subset SD^2[\mathbb{R}_I^\infty]$. □

**Definition 2.2.** We call $SD^2[\mathbb{R}_I^\infty]$ the Jones-strong distribution Hilbert space on $\mathbb{R}_I^\infty$. Let $\alpha$ be a multi-index of non negative integers $\alpha = (\alpha_1, \alpha_2, \ldots)$ with $|\alpha| = \sum_{j=1}^{\infty} \alpha_j$. If $D$ denotes the standard partial differential operator, let $D^\alpha = D^{\alpha_1}D^{\alpha_2}_2 \ldots$.

2.0.1. **Test function and Distribution in $R_I^\infty$.** Here our space is $\mathbb{R}_I^\infty$. We replace $\mathbb{R}_I^\infty$ with its support in $\mathbb{R}^n$ of $\mathbb{T}[G]$. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$ be multi-index of non negative integers, with $|\alpha| = \sum_{k=1}^{\infty} \alpha_k$.

We define the operators $D^\alpha_\infty$ and $D_{\alpha,\infty}$ by $D^\alpha_\infty = \prod_{k=1}^{\infty} \frac{\partial^{\alpha_k}}{\partial x_k}$ and $D_{\alpha,\infty} = \prod_{k=1}^{\infty} \left(\frac{1}{2\pi i} \frac{\partial}{\partial x_k}\right)^{\alpha_k}$.

Let $C_c[\mathbb{R}_I^\infty]$ be the class of infinitely differentiable functions on $\mathbb{R}_I^\infty$ with the compact support and impose the natural locally convex topology $\tau$ on $C_c[\mathbb{R}_I^\infty]$ to obtain $D[\mathbb{R}_I^\infty]$.

**Definition 2.3.** A sequence $\{f_m\}$ converges to $f \in D[\mathbb{R}_I^\infty]$ with respect to the compact sequential limit topology if and only if there exists a compact set $K \subset \mathbb{R}_I^\infty$, which contain the support of $f_m \rightarrow f$ for each $m$ and $D^\alpha f_m \rightarrow D^\alpha f$ uniformly on $K$, for every multi-index $\alpha \in \mathbb{N}_0^\infty$.

Let $u \in C^1[\mathbb{R}_I^\infty]$ and suppose that $\phi \in C_c^\infty[\mathbb{R}_I^\infty]$ has its support in a unit ball $B_r, r > 0$.

Then

$$
\int_{\mathbb{R}_I^\infty} (\phi uy_i) d\lambda_\infty = \int_{\partial B_r} (u\phi)v ds - \int_{\mathbb{R}_I^\infty} (u\phi y_i) d\lambda_\infty,
$$
where \( v \) is the unit outward normal to \( B_r \). Since \( \phi \) vanishes on the \( \partial B_r \), then

\[
\int_{\mathbb{R}_I^\infty} (\phi u_y) d\lambda_\infty = - \int_{\mathbb{R}_I^\infty} (u \phi_y) d\lambda_\infty, \quad 1 \leq i \leq \infty.
\]

So, in general case, for any \( u \in C^m[\mathbb{R}_I^\infty] \) and any multi-index \( \alpha = (\alpha_1, \alpha_2, \ldots) \), with \( |\alpha| = \sum_{i=1}^\infty \alpha_i = m \),

\[
\int_{\mathbb{R}_I^\infty} \phi (D^\alpha u) d\lambda_\infty = (-1)^m \int_{\mathbb{R}_I^\infty} u (D^\alpha \phi) d\lambda_\infty.
\]

(1)

**Lemma 2.4.** A function \( u \in L_1^1[\mathbb{R}_I^\infty] \) if it is Lebesgue integrable on every compact subset of \( \mathbb{R}_I^\infty \).

**Proof.** We know \( u \in L_1^1[\mathbb{R}_I^n] \) if it is Lebesgue integrable on every compact subset of \( \mathbb{R}_I^n \).

So, \( u \in L_1^1[\bigcup_{n=1}^\infty \mathbb{R}_I^n] \) if it is Lebesgue integrable on every compact subset of \( \bigcup_{n=1}^\infty \mathbb{R}_I^n \).

That is a function \( u \in L_1^1[\mathbb{R}_I^\infty] \) if it is Lebesgue integrable on every compact subset of \( \mathbb{R}_I^\infty \). \( \square \)

**Remark 2.5.** With the Lemma [2.4], we can conclude the Equation [1] is fit even if \( D^\alpha u \) does not exist according to our normal definition.

**Definition 2.6.** If \( \alpha \) is a multi-index and \( u, v \in L_1^1[\mathbb{R}_I^\infty] \), we say that \( v \) is the \( \alpha^{th} \)-weak (or distributional) partial derivative of \( u \) and we write \( D^\alpha u = v \) provided

\[
\int_{\mathbb{R}_I^\infty} u (D^\alpha \phi) d\lambda_\infty = (-1)^{|\alpha|} \int_{\mathbb{R}_I^\infty} \phi v d\lambda_\infty
\]

for all functions \( \phi \in C^\infty_c[\mathbb{R}_I^\infty] \). Thus \( v \) is in the dual space \( D'[\mathbb{R}_I^\infty] \) of \( D[\mathbb{R}_I^\infty] \).
Lemma 2.7. If a weak \( \alpha \)-\( \partial \)-partial derivatives exists for \( u \), then it is unique \( \lambda_{\infty} \)-a.e.

Theorem 2.8. \( D[\mathbb{R}^\infty_I] \subset SD^2[\mathbb{R}^\infty_I] \) as continuous embedding.

Proof. Since \( D[\mathbb{R}^n] \subset SD^2[\mathbb{R}^n] \) as a continuous embedding. So, \( D[\mathbb{R}^\infty_I] \subset SD^2[\mathbb{R}^\infty_I] \) as a continuous embedding. Clearly by construction of \( D[\mathbb{R}^\infty_I] \) and \( SD^2[\mathbb{R}^\infty_I] \), so easily we can show \( D[\mathbb{R}^\infty_I] \subset SD^2[\mathbb{R}^\infty_I] \) as a continuous embedding. \( \square \)

More analytical way we can state the above theorem as follows:

Theorem 2.9. Let \( D[\mathbb{R}^\infty_I] \) be \( C^\infty_c[\mathbb{R}^\infty_I] \) equipped with the standard locally convex topology (test functions). If \( \phi_j \rightarrow \phi \) in \( D[\mathbb{R}^\infty_I] \), then \( \phi_j \rightarrow \phi \) in the norm topology of \( SD^2[\mathbb{R}^\infty_I] \), so that \( D[\mathbb{R}^\infty_I] \subset SD^2[\mathbb{R}^\infty_I] \) as continuous embedding.

Proof. Suppose that \( \phi_j \rightarrow \phi \) in \( D[\mathbb{R}^\infty_I] \). By definition, there exists a compact set \( \mathcal{K} \subset \mathbb{R}^\infty_I \), which is the support of \( \phi_j \rightarrow \phi \) and \( D^\alpha \phi_j \) converges to \( D^\alpha \phi \) uniformly on \( \mathcal{K} \) for every multi-index \( \alpha \). Let \( \{\mathcal{E}_{\mathcal{K}_i}\} \) be the set of all \( \mathcal{E}_i \), with support \( \mathcal{K}_i \subset \mathcal{K} \). If \( \alpha \) is a multi-index, we have

\[
\lim_{j \rightarrow \infty} \left\| D^\alpha \phi_j - D^\alpha \phi \right\|_{SD} = \lim_{j \rightarrow \infty} \left[ \sum_{i=1}^{\infty} t_{\mathcal{K}_i} \left| \int_{\mathbb{R}^\infty_I} \mathcal{E}_{\mathcal{K}_i}(x) \left[ D^\alpha \phi_j(x) - D^\alpha \phi(x) \right] d\lambda_{\infty}(x) \right|^2 \right]^{\frac{1}{2}} \\
\leq M \lim_{j \rightarrow \infty} \sup_{x \in \mathcal{K}} \left| D^\alpha \phi_j(x) - D^\alpha \phi(x) \right| = 0.
\]

\( \square \)

Corollary 2.10. Let \( D[\mathbb{R}^\infty_I] \) be \( C^\infty_c[\mathbb{R}^\infty_I] \) equipped with the standard locally convex topology (test functions). If \( \phi_j \rightarrow \phi \) in \( D[\mathbb{R}^\infty_I] \), then
Proof. By the Theorem 2.1, since $\alpha$ is arbitrary, we see that $D[R^\infty_I] \subset SD^2[R^\infty_I]$ as a continuous embedding. Since $C^\infty_c[R^\infty_I]$ is dense in $L^1[R^\infty_I]$, so $D[R^\infty_I]$ is dense in $SD^2[R^\infty_I]$. □

Theorem 2.11. Let $D[R^\infty_I]$ be $C^\infty_c[R^\infty_I]$ equipped with the standard locally convex topology (test functions). If $T \in D'[R^\infty_I]$, then $T \in SD^2[R^\infty_I]'$ so that $D'[R^\infty_I] \subset SD^2[R^\infty_I]'$ as a continuous dense embedding.

Proof. As $D[R^\infty_I]$ is locally dense convex subspace of $SD^2[R^\infty_I]$, then every continuous linear functional, $T$ defined on $D[R^\infty_I]$, can be extended to a continuous linear functional on $SD^2[R^\infty_I]$.

By Riesz representation theorem, every continuous linear functional $T$ defined on $SD^2[R^\infty_I]$ is of the form $T(f) = \langle f, g \rangle_{SD}$, for some $g \in SD^2[R^\infty_I]$. So, $T \in SD^2[R^\infty_I]'$ and $T \leftrightarrow g$ for each $T \in D'[R^\infty_I]$. So, it is possible to map $D'[R^\infty_I]$ into $SD^2[R^\infty_I]$ as a continuous dense embedding. □

Theorem 2.12. For any $f, g \in SD^2[R^\infty_I]$ and any multi-index $\alpha$, we have

$$< D^\alpha f, g >_{SD[R^\infty_I]} = (-i)^\alpha < f, g >_{SD[R^\infty_I]}.$$ 

Proof. Let $\mathcal{E}_k \in C^\infty_c[R^\infty_I]$. Then for $f \in SD^2[R^\infty_I]$, we have

$$\int_{R^\infty_I} \mathcal{E}_k(x) D^\alpha f(x) d\lambda_{\infty}(x) = (-1)^{|\alpha|} \int_{R^\infty_I} D^\alpha \mathcal{E}_k(x) f(x) d\lambda_{\infty}(x)$$

$$= (-i)^\alpha \int_{R^\infty_I} \mathcal{E}_k(x) f(x) d\lambda_{\infty}(x).$$
Now, for any \( g \in SD^2[R^\infty_I] \),
\[
< D^\alpha f, g >_{SD^2[R^\infty_I]} = (-i)^{\alpha} < f, g >_{SD^2[R^\infty_I]}.
\]

\[\square\]

**Theorem 2.13.** The function space \( S[R^\infty_I] \), of rapid decrease at infinity are contained in \( SD^2[R^\infty_I] \) as continuous embedding, so that \( S'[R^\infty_I] \subset SD^2[R^\infty_I]' \).

**Proof.** Since \( S[R^n_I] \subset SD^2[R^n_I] \) continuous embedding, so that \( S'[R^n_I] \subset SD^2[R^n_I]' \). The remaining proof is easy, so we left to the reader. \( \square \)

In general we construct the space \( SD^p[R^\infty_I] \) for \( f \in L^p[R^\infty_I] \), define
\[
||f||_{SD^p[R^\infty_I]} = \left\{ \left( \sum_{k=1}^{\infty} t_k \left| \int_{R^\infty_I} E_k(x) D^\alpha f(x) d\lambda_\infty(x) \right|^p \right)^{\frac{1}{p}} \right\}, \text{ for } 1 \leq p < \infty;
\]
\[
\sup_{k \geq 1} \left| \int_{R^\infty_I} E(x) D^\alpha f(x) d\lambda_\infty(x) \right|, \text{ for } p = \infty.
\]

It is easy to see that \( ||f||_{SD^p[R^\infty_I]} \) defines a norm on \( L^p[R^\infty_I] \). If \( SD^p[R^\infty_I] \) is completion of \( L^p[R^\infty_I] \) then we have the following.

**Theorem 2.14.** For \( 1 \leq p \leq \infty \)

(1) If \( f_n \to f \) weakly in \( L^p[R^\infty_I] \) then \( f_n \to f \) strongly in \( SD^p[R^\infty_I] \).

(2) \( SD^p[R^\infty_I] \) is uniformly convex, for \( 1 < p < \infty \).

(3) If \( 1 < p < \infty \) and \( \frac{1}{p} + \frac{1}{q} = 1 \), then dual space of \( SD^p[R^\infty_I] \) is 
\( SD^q[R^\infty_I] \).

(4) \( SD^\infty[R^\infty_I] \subset SD^p[R^\infty_I] \) for \( 1 < p < \infty \).

**Proof.** (1) If \( \{f_n\} \) is weakly convergence in \( L^p[R^\infty_I] \) with limit \( f \). Then
\[
\int_{R^\infty_I} E_k(x)|f_n(x) - f(x)|d\lambda_\infty(x) \to 0 \text{ for each } k.
\]
For each \( f_n \in SD^p[\mathbb{R}_1^n] \) for all \( n \), then we have
\[
\lim_{n \to \infty} \int_{\mathbb{R}_1^n} \mathcal{E}_k(x)|\mathcal{D}^\alpha(f_n(x) - f(x))|d\lambda_\infty(x) \to 0.
\]

(2) We know \( L^p[\mathbb{R}_1^n] \) is uniformly convex for each \( n \) and that is dense and compactly embedded in \( SD^q[\mathbb{R}_1^n] \) for \( 1 \leq q \leq \infty \). So, \( \bigcup_{n=1}^{\infty} L^p[\mathbb{R}_1^n] \) is uniformly convex for each \( n \) and that is dense and compactly embedded in \( \bigcup_{n=1}^{\infty} SD^p[\mathbb{R}_1^n] \) for \( 1 \leq p \leq \infty \). However \( L^p[\hat{\mathbb{R}}_1^\infty] = \bigcup_{n=1}^{\infty} L^p[\mathbb{R}_1^n] \). That is \( L^p[\hat{\mathbb{R}}_1^\infty] \) is uniformly convex, dense and compactly embedded in \( SD^p[\hat{\mathbb{R}}_1^\infty] \) for \( 1 \leq p \leq \infty \).

As \( SD^p[\hat{\mathbb{R}}_1^\infty] \) is the closure of \( SD^p[\hat{\mathbb{R}}_1^\infty] \). Therefore \( SD^p[\hat{\mathbb{R}}_1^\infty] \) is uniformly convex.

(3) From (2) we have that \( SD^p[\hat{\mathbb{R}}_1^\infty] \) is reflexive, for \( 1 < p < \infty \). Since
\[
\{SD^p[\mathbb{R}_1^k]\}^* = SD^q[\mathbb{R}_1^k], \quad \frac{1}{p} + \frac{1}{q} = 1, \forall k \text{ and }
\]
\[
SD^p[\mathbb{R}_1^k] \subset SD^p[\mathbb{R}_1^{k+1}], \forall k \quad \Rightarrow \quad \bigcup_{k=1}^{\infty} \{SD^p[\mathbb{R}_1^k]\}^* = \bigcup_{k=1}^{\infty} SD^q[\mathbb{R}_1^k], \quad \frac{1}{p} + \frac{1}{q} = 1.
\]

Since each \( f \in SD^p[\hat{\mathbb{R}}_1^\infty] \) is the limit of a sequence \( \{f_n\} \subset \bigcup_{k=1}^{\infty} SD^p[\mathbb{R}_1^k] \), we see that \( \{SD^p[\hat{\mathbb{R}}_1^\infty]\}^* = SD^q[\hat{\mathbb{R}}_1^\infty] \), for \( \frac{1}{p} + \frac{1}{q} = 1 \).

(4) Let \( f \in SD^\infty[\hat{\mathbb{R}}_1^\infty] \). This implies
\[
\left| \int_{\hat{\mathbb{R}}_1^\infty} \mathcal{E}_k(x)|\mathcal{D}^\alpha f(x)|d\lambda_\infty(x) \right| \text{ is uniformly bounded for all } k.
\]
It follows that
\[
\left| \int_{\hat{\mathbb{R}}_1^\infty} \mathcal{E}_k(x)|\mathcal{D}^\alpha f(x)|d\lambda_\infty(x) \right| \leq M||f||_{SD^p[\hat{\mathbb{R}}_1^\infty]} < \infty.
\]

So, \( f \in SD^p[\hat{\mathbb{R}}_1^\infty] \). □
We recall the space

\[ X^m_p[R^n] = \{ B_\alpha * g = (1 - \Delta)^{-\alpha} g : g \in L^p[R^n], \; 0 < \alpha < n, \; 0 < \alpha < m \} \]

is coincides with \( W^m_p[R^n] \) when \( 1 < p < \infty \) and \( m > 0 \), where \( B_\alpha \) is the Bessel potential of order \( \alpha \), \( \Delta \) is the Laplacian and * is the convolution operator.

We define \( W^m_p[R^n] \) is the space of all functions \( u \in L^1_{loc}[R^n] \) whose weak derivative \( \partial^\alpha u \in L^p[R^n] \) for every \( \alpha \in \mathbb{N}_0^\infty \) with \( |\alpha| = m \).

**Theorem 2.15.** \( W^m_p[R^n] \subset SD^2[R^n] \) as a continuous dense embedding, for all \( m \) and all \( p \).

**Proof.** As \( W^m_p[R^n] \subset SD^2[R^n] \) as continuous dense embedding. However \( SD^2[R^n] \) is the closure of \( \bigcup_{k=1}^\infty SD^2[R^k] \).

That is \( SD^2[R^n] \) contains \( \bigcup_{k=1}^\infty SD^2[R^k] \) which is dense in \( W^m_p[R^n] \) as it’s closure.

Hence, \( W^m_p[R^n] \subset SD^2[R^n] \) as continuous dense embedding. \( \Box \)

In the last, we call a function \( f \) such that \( \int_{R^n} |\mathcal{E}_k(x)f(x)\lambda(x)|^p < \infty \) for every compact set \( \mathcal{K} \) in \( R^n \) is said to be in \( L^p_{loc}[R^n] \).

**2.0.2. Functions of Bounded variation.** The objective of this section is to show that every HK-integrable function is in \( SD^2[R^n] \). To do this, we need to discuss a certain class of functions of bounded variation in the sense of Cesari (see [GL]) are well known for working in PDE (partial differential equations) and geometric measure theory. Also we consider the function of bounded variation in Vitali sense (see [TY]) are applied in applied mathematics and engineering for error estimation associated with research in control theory, financial derivatives,
robotics, high speed networks and in calculation of certain integrals. We developed this portion through the Definition 3.38 and 3.39 of [TG].

**Definition 2.16.** A function \( f \in L^1[\mathbb{R}^\infty] \) is said to be bounded variation i.e. \( f \in BV_c[\mathbb{R}^\infty] \) if \( f \in L^1[\mathbb{R}^\infty] \) there exists a signed Radon measure \( \mu_i \) such that

\[
\int_{\mathbb{R}^\infty} f(x) \frac{\partial \phi(x)}{\partial x_i} d\lambda_\infty(x) = - \int_{\mathbb{R}^\infty} \phi(x) d\mu_i(x_i)
\]

for \( i = 1, 2, 3, \ldots, \infty \) for all \( \phi \in C^\infty_0[\mathbb{R}^\infty] \)

**Definition 2.17.** A function \( f \) with continuous partial derivatives is said to be of bounded variation i.e. \( f \in BV_v[\mathbb{R}^\infty] \) if for all \( D_n = \{(a_i, b_i) \times I_n\}, 1 \leq i \leq n \) for all \( (a_i, b_i) \) is an interval in \( \mathbb{R}^n \),

\[
V(f) = \lim_{n \to \infty} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} \left| \frac{\partial^n f(x)}{\partial x_1 \partial x_2 \cdots \partial x_n} \right| d\lambda_\infty(x) < \infty.
\]

**Definition 2.18.** We define \( BV_{v,0}[\mathbb{R}^\infty] \) by

\[
BV_{v,0}[\mathbb{R}^\infty] = \{ f(x) \in BV_v[\mathbb{R}^\infty] : f(x) \to 0 \text{ as } x_i \to \infty \},
\]

where \( x_i \) is any component of \( x \).

**Theorem 2.19.** The space \( HK[\mathbb{R}^\infty] \) of all HK-integrable functions is contained in \( SD^2[\mathbb{R}^\infty] \).
Proof. Since $ \mathcal{E}_m(x) $ is continuous and differentiable, therefore $ \mathcal{E}_m(x) \in BV_{v,0}[R^\infty_I] $ so that for $ f \in HK[R^\infty_I] $, gives

$$
||f||_{SD^2[R^\infty_I]} = \sum_{m=1}^{\infty} t_m \left| \int_{R^\infty_I} \mathcal{E}_m(x) f(x) d\lambda_\infty(x) \right|^2 \\
\leq \sup_m \left| \int_{R^\infty_I} \mathcal{E}_m(x) f(x) d\lambda_\infty(x) \right|^2 \\
\leq ||f||_{HK}^2 [\sup_m V(\mathcal{E}_m)]^2 < \infty.
$$

So, $ f \in SD^2[R^\infty_I]. $ \hfill \Box

3. Conclusion

We have constructed a new class of separable Banach spaces, $ SD^p[R^\infty_I], \, 1 \leq p \leq \infty, $ which contain each $ L^p $-space as a dense continuous and compact embedding. These spaces have the remarkable property that, for any multi-index $ \alpha, \, ||D^\alpha u||_{SD} = ||u||_{SD}. $ We have shown that our spaces contain the non-absolutely integrable functions and the space of test functions $ D[R^\infty_I], $ as a dense continuous embedding. We have discussed their basic properties and their relationship to $ D[R^\infty_I], \, S[R^\infty_I] $ and $ S[R^\infty_I].$

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