Discrete Laplace-Beltrami Operator Determines Discrete Riemannian Metric

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Abstract

The Laplace-Beltrami operator of a smooth Riemannian manifold is determined by the Riemannian metric. Conversely, the heat kernel constructed from its eigenvalues and eigenfunctions determines the Riemannian metric. This work proves the discrete analogy on Euclidean polyhedral surfaces that the discrete Laplace-Beltrami operator and the discrete Riemannian metric (unique up to a scaling) are mutually determined by each other. Given an Euclidean polyhedral surface, its Riemannian metric is represented as edge lengths, satisfying triangle inequalities on all faces. The Laplace-Beltrami operator is formulated using the cotangent formula, where the edge weight is defined as the sum of the cotangent of angles against the edge. We prove that the edge lengths can be determined by the edge weights unique up to a scaling using the variational approach. The constructive proof leads to a computational algorithm that finds the unique metric on a topological triangle mesh from a discrete Laplace matrix.

I. Introduction

Laplace-Beltrami operator plays a fundamental role in Riemannian geometry [6]. In real applications, a smooth metric surface is usually represented as a triangulated mesh. The manifold heat kernel is estimated from the discrete Laplace-Beltrami operator. Discrete Laplace-Beltrami operators on triangulated surface meshes span the entire spectrum of geometry processing applications, including mesh parameterization, segmentation, reconstruction, compression, remeshing [3], [5], [10]. A lot of research has been done on the discretization of Laplace operator and theoretical convergence analysis [9], [8]. The most well-known and widely-used discrete formulation of Laplace operator over meshing [3], [5], [10]. A lot of research has been done on the discretization of Laplace operator and theoretical convergence analysis [9], [8]. The most well-known and widely-used discrete formulation of Laplace operator over meshing [3], [5], [10].

Conversely, if \( f \) is a surjective map, and Eqn. (1) holds, then \( f \) is an isometry.

\[ \text{Theorem 2.1 ([7]): Let } f : (M_1, g_1) \rightarrow (M_2, g_2) \text{ be a diffeomorphism between two Riemannian manifolds. If } f \text{ is an isometry, then } \]
\[ K_1(x, y, t) = K_2(f(x), f(y), t), \quad \forall x, y \in M, \quad t > 0. \]  

Conversely, if \( f \) is a surjective map, and Eqn. (1) holds, then \( f \) is an isometry.

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B. Discrete Case

Definition 2.1 (Polyhedral Surface): An Euclidean polyhedral surface is a triple \((S,T,d)\), where \(S\) is a closed surface, \(T\) is a triangulation of \(S\) and \(d\) is a metric on \(S\) whose restriction to each triangle is isometric to an Euclidean triangle.

Definition 2.2 (Cotangent Edge Weight [1], [4]): Suppose \([v_i, v_j]\) is a boundary edge of \(M\), \([v_i, v_j]\) \(\in\partial M\), then \([v_i, v_j]\) is associated with one triangle \([v_i, v_j, v_k]\), the angle against \([v_i, v_j]\) at the vertex \(v_k\) is \(\alpha\), then the weight of \([v_i, v_j]\) is given by \(w_{ij} = \frac{1}{2} \cot \alpha\). Otherwise, if \([v_i, v_j]\) is an interior edge, the two angles against it are \(\alpha, \beta\), then the weight is \(w_{ij} = \frac{1}{2} (\cot \alpha + \cot \beta)\).

Definition 2.3 (Discrete Laplace Matrix): The discrete Laplace matrix \(L = (L_{ij})\) for an Euclidean polyhedral surface is given by

\[
L_{ij} = \begin{cases} -w_{ij} & i \neq j \\ \sum\limits_{k} w_{ik} & i = j \end{cases}.
\]

Because \(L\) is symmetric, it can be decomposed as \(L = \Phi \Lambda \Phi^T\) where \(\Lambda = \text{diag}(\lambda_0, \lambda_1, \cdots, \lambda_n)\), \(0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n\), are the eigenvalues of \(L\), and \(\Phi = (\phi_0|\phi_1|\phi_2|\cdots|\phi_n)\), \(L\phi_i = \lambda_i \phi_i\), are the orthonormal eigenvectors, such that \(\phi_i^T \phi_j = \delta_{ij}\).

Definition 2.4 (Discrete Heat Kernel): The discrete heat kernel is defined as \(K(t) = \text{exp}(-\lambda t) \Phi^T\).

The Main Theorem, called Global Rigidity Theorem, in this work is as follows:

Theorem 2.2: Suppose two Euclidean polyhedral surfaces \((S,T,d_1)\) and \((S,T,d_2)\) are given, \(L_1 = L_2\), if and only if \(d_1\) and \(d_2\) differ by a scaling.

Corollary 2.3: Suppose two Euclidean polyhedral surfaces \((S,T,d_1)\) and \((S,T,d_2)\) are given, \(K_1(t) = K_2(t), \forall t > 0\), if and only if \(d_1\) and \(d_2\) differ by a scaling.

Proof: Note that, \(\frac{dK(t)}{dt}|_{t=0} = -L\). Therefore, the discrete Laplace matrix and the discrete heat kernel mutually determine each other.

C. Proof Overview for Main Theorem 2.2

We fix the connectivity of the polyhedral surface \((S,T)\). Suppose the edge set of \((S,T)\) is sorted as \(E = \{e_1, e_2, \cdots, e_m\}\), where \(m = |E|\), the face set as \(F\), and a triangle \([v_i,v_j,v_k]\) \(\in F\) as \(\{i,j,k\} \in F\). We denote an Euclidean polyhedral metric as \(d = (d_1,d_2,\cdots,d_m)\), where \(d:E\to\mathbb{R}^+\) is the edge length function, \(d_i = d(e_i)\) is the length of edge \(e_i\). Let

\[
E_d(2) = \{(d_1,d_2,d_3)|d_i + d_j > d_k\}
\]

be the space of all Euclidean triangles parameterized by the edge lengths, where \(\{i,j,k\}\) is a cyclic permutation of \(\{1,2,3\}\). In this work, for convenience, we use \(u = (u_1,u_2,\cdots,u_m)\) to represent the metric, where \(u_k = \frac{1}{2}d_k^2\).

Definition 2.5 (Admissible Metric Space): Given a triangulated surface \((S,K)\), the admissible metric space is defined as

\[
\Omega_u = \{(u_1,u_2,u_3,\cdots,u_m)\mid \sum\limits_{k=1}^{m} u_k = m, (\sqrt{u_i},\sqrt{u_j},\sqrt{u_k}) \in E_d(2), \forall \{i,j,k\} \in F\}.
\]

Definition 2.6 (Energy): An energy \(E: \Omega_u \to \mathbb{R}\) is defined as:

\[
E(u_1,u_2,\cdots,u_m) = \int_{(1,1,\cdots,1)} \sum\limits_{k=1}^{m} w_k(\mu)d\mu,
\]

(2)

where \(w_k(\mu)\) is the cotangent weight on the edge \(e_k\) determined by the metric \(\mu\).

Lemma 2.4: Suppose \(\Omega \subset \mathbb{R}^n\) is an open convex domain in \(\mathbb{R}^n\), \(E: \Omega \to \mathbb{R}\) is a strictly convex function with positive definite Hessian matrix, then \(\nabla E: \Omega \to \mathbb{R}^n\) is a smooth embedding.

We show that \(\Omega_u\) is a convex domain in \(\mathbb{R}^m\), the energy \(E\) is convex. According to Lemma 2.4, the gradient of the energy \(\nabla E(d): \Omega \to \mathbb{R}^m\), \(\nabla E: (u_1, u_2, \cdots, u_m) \to (w_1, w_2, \cdots, w_m)\) is an embedding. Namely the metric is determined by the edge weight unique up to a scaling.

III. Euclidean Triangle

Given a triangle \(\{i,j,k\}\), three corner angles denoted by \(\{\theta_i, \theta_j, \theta_k\}\), three edge lengths denoted by \(\{d_i, d_j, d_k\}\). In this case, the problem is trivial. Given \((w_i, w_j, w_k) = (\cot \theta_i, \cot \theta_j, \cot \theta_k)\), we can compute \((\theta_i, \theta_j, \theta_k)\) by taking the arctan function. Then the normalized edge lengths are given by

\[
(d_i, d_j, d_k) = \frac{3}{\sin \theta_i + \sin \theta_j + \sin \theta_k}(\sin \theta_i, \sin \theta_j, \sin \theta_k).
\]
Lemma 3.1: Suppose an Euclidean triangle is with angles \(\{\theta_i, \theta_j, \theta_k\}\) and edge lengths \(\{d_i, d_j, d_k\}\), angles are treated as the functions of the edge lengths \(\theta_i(d_i, d_j, d_k)\), then
\[
\frac{\partial \theta_i}{\partial d_i} = \frac{d_i}{2A}, \quad \frac{\partial \theta_j}{\partial d_j} = -\frac{d_j}{2A} \cos \theta_k,
\]
where \(A\) is the area of the triangle.

Lemma 3.2: In an Euclidean triangle, let \(u_i = \frac{1}{2}d_i^2\) and \(u_j = \frac{1}{2}d_j^2\) then
\[
\frac{\partial \cot \theta_i}{\partial u_j} = \frac{\partial \cot \theta_j}{\partial u_i}.
\]

Corollary 3.3: The differential form \(\omega = \cot \theta_i d u_i + \cot \theta_j d u_j + \cot \theta_k d u_k\) is a closed 1-form.

Definition 3.1 (Admissible Metric Space): Let \(u_i = \frac{1}{2}d_i^2\), the admissible metric space is defined as
\[
\Omega_u := \{(u_i, u_j, u_k) | (\sqrt{u_i}, \sqrt{u_j}, \sqrt{u_k}) \in E(2), u_i + u_j + u_k = 3\}.
\]

Lemma 3.4: The admissible metric space \(\Omega_u\) is a convex domain in \(\mathbb{R}^3\).

Proof: Suppose \((u_i, u_j, u_k) \in \Omega_u\) and \((\tilde{u}_i, \tilde{u}_j, \tilde{u}_k) \in \Omega_u\), then from \(\sqrt{u_i} + \sqrt{u_j} > \sqrt{u_k}\), we get \(u_i + u_j + 2\sqrt{u_i u_j} > u_k\).
Define \((u^3_i, u^3_j, u^3_k) = \lambda(u_i, u_j, u_k) + (1 - \lambda)(\tilde{u}_i, \tilde{u}_j, \tilde{u}_k)\), where \(0 < \lambda < 1\). Then
\[
u_i^3 + \nu_j^3 + 2\sqrt{\nu_i^3 \nu_j^3} \geq \lambda(u_i + u_j + 2\sqrt{u_i u_j}) + (1 - \lambda)(\tilde{u}_i + \tilde{u}_j + 2\sqrt{\tilde{u}_i \tilde{u}_j}) > \lambda u_k + (1 - \lambda)\tilde{u}_k = u^3_k.
\]
This shows \((u^3_i, u^3_j, u^3_k) \in \Omega_u\).

Definition 3.2 (Edge Weight Space): The edge weights of an Euclidean triangle form the edge weight space
\[
\Omega_{\theta} = \{\left((\cot \theta_i, \cot \theta_j, \cot \theta_k) | 0 < \theta_i, \theta_j, \theta_k < \pi, \theta_i + \theta_j + \theta_k = \pi\right)\}.
\]

Note that,
\[
\cot \theta_k = -\cot(\theta_i + \theta_j) = \frac{1 - \cot \theta_i \cot \theta_j}{\cot \theta_i + \cot \theta_j}.
\]

Lemma 3.5: The energy \(E : \Omega_u \to \mathbb{R}\)
\[
E(u_i, u_j, u_k) = \int_{(1,1,1)} \cot \theta_i d \tau_i + \cot \theta_j d \tau_j + \cot \theta_k d \tau_k
\]
is well defined on the admissible metric space \(\Omega_u\) and is convex.

Proof: According to Corollary 3.3, the differential form is closed. Furthermore, the admissible metric space \(\Omega_u\) is a simply connected domain. The differential form is exact, therefore, the integration is path independent, and the energy function is well defined.

Then we compute the Hessian matrix of the energy,
\[
H = -\frac{2R^2}{A} \begin{bmatrix}
\frac{1}{\cos \theta_i} & \frac{-\cos \theta_i}{\cos \theta_j} & \frac{-\cos \theta_i}{\cos \theta_k} \\
\frac{-\cos \theta_i}{\cos \theta_i} & \frac{1}{\cos \theta_j} & \frac{-\cos \theta_j}{\cos \theta_k} \\
\frac{-\cos \theta_i}{\cos \theta_i} & \frac{-\cos \theta_j}{\cos \theta_i} & \frac{1}{\cos \theta_k}
\end{bmatrix} = -\frac{2R^2}{A} \begin{bmatrix}
(\eta_1, \eta_1) & (\eta_1, \eta_j) & (\eta_1, \eta_k) \\
(\eta_1, \eta_j) & (\eta_j, \eta_j) & (\eta_j, \eta_k) \\
(\eta_1, \eta_k) & (\eta_j, \eta_k) & (\eta_k, \eta_k)
\end{bmatrix}.
\]

Let \(n_i\) be the perpendicular from the incenter of the triangle and orthogonal to the edge \(e_i\), \(d_i n_i + d_j n_j + d_k n_k = 0\), \(\eta_i = n_i / rd_i, \eta_j = n_j / rd_j, \eta_k = n_k / rd_k\), where \(r\) is the radius of the incircle of the triangle. Suppose \((x_i, x_j, x_k) \in \mathbb{R}^3\) is a vector in \(\mathbb{R}^3\), then
\[
[x_i, x_j, x_k] \begin{bmatrix}
(\eta_1, \eta_1) & (\eta_1, \eta_j) & (\eta_1, \eta_k) \\
(\eta_1, \eta_j) & (\eta_j, \eta_j) & (\eta_j, \eta_k) \\
(\eta_1, \eta_k) & (\eta_j, \eta_k) & (\eta_k, \eta_k)
\end{bmatrix} \begin{bmatrix}
x_i \\
x_j \\
x_k
\end{bmatrix} = ||x_i \eta_i + x_j \eta_j + x_k \eta_k||^2 \geq 0.
\]
If the result is zero, then \((x_i, x_j, x_k) = \lambda(u_i, u_j, u_k), \lambda \in \mathbb{R}\). That is the null space of the Hessian matrix. In the admissible metric space \(\Omega_u, u_i + u_j + u_k = C (C = 3)\), then \(d_{u_i} + d_{u_j} + d_{u_k} = 0\). If \((d_{u_i}, d_{u_j}, d_{u_k})\) belongs to the null space, then \((d_{u_i}, d_{u_j}, d_{u_k}) = \lambda(u_i, u_j, u_k)\), therefore, \(\lambda(u_i + u_j + u_k) = 0\). Because \(u_i, u_j, u_k\) are positive, \(\lambda = 0\). In summary, the energy on \(\Omega_u\) is convex.

Theorem 3.6: The mapping \(\nabla E : \Omega_u \to \Omega_{\theta}, (u_i, u_j, u_k) \to (\cot \theta_i, \cot \theta_j, \cot \theta_k)\) is a diffeomorphism.

Proof: The energy \(E(u_i, u_j, u_k)\) is a convex function defined on the convex domain \(\Omega_u\), according to Lemma 2.4, \(\nabla E : (u_i, u_j, u_k) \to (\cot \theta_i, \cot \theta_j, \cot \theta_k)\) is a diffeomorphism.
IV. EUCLIDEAN POLYHEDRAL SURFACE

A. Closed Surfaces

Suppose a polyhedral surface \((S, T, d)\) with the admissible metric space and the edge weight (see Section II-B).

**Lemma 4.1:** The admissible metric space \(\Omega_a\) is convex.

**Proof:** For a triangle \(\{i, j, k\} \in F\), define
\[
\Omega_u^{ijk} := \{(u_i, u_j, u_k)|(\sqrt{u_i}, \sqrt{u_j}, \sqrt{u_k}) \in E_d(2)\}.
\]

Similar to the proof of Lemma 3.4, \(\Omega_u^{ijk}\) is convex. The admissible metric space for the mesh is
\[
\Omega_u = \bigcap_{\{i, j, k\} \in F} \Omega_u^{ijk} \bigcap \{(u_1, u_2, \cdots, u_m) | \sum_{k=1}^{m} u_k = m\},
\]
the intersection \(\Omega_u\) is still convex. ■

**Lemma 4.2:** The differential form \(\omega\) defined on \(\Omega_u\)
\[
\omega = \sum_{\{i, j, k\} \in F} \omega_{ijk} = \sum_{i=1}^{m} 2w_idu_i,
\]
is a closed 1-form, where \(\omega_{ijk}\) on each face is given in Corollary 3.3, \(w_i\) is the edge weight on \(e_i\).

**Lemma 4.3:** The energy function
\[
E(u_1, u_2, \cdots, u_n) = \sum_{\{i, j, k\} \in F} E_{ijk}(u_1, u_2, \cdots, u_n) = \int_{\{1, 1, \cdots, 1\}} \sum_{i=1}^{n} w_idu_i
\]
is well defined and convex on \(\Omega_u\), where \(E_{ijk}\) is the energy on the face, defined in Eqn. (5).

**Proof:** For each face \(\{i, j, k\} \in F\), the Hessian matrices of \(E_{ijk}\) are semi-positive definite, therefore, the Hessian matrix of the total energy \(E\) is semi-positive definite. Similar to the proof of Lemma 3.5, the null space of the Hessian matrix \(H\) is
\[
kerH = \{\lambda(d_1, d_2, \cdots, d_n), \lambda \in \mathbb{R}\}.
\]
The tangent space of \(\Omega_u\) at \(u = (u_1, u_2, \cdots, u_n)\) is denoted by \(T\Omega_u(u)\). Assume \((du_1, du_2, \cdots, du_n) \in T\Omega_u(u)\), then from \(\sum_{i=1}^{m} u_i = m\), we get \(\sum_{i=1}^{m} du_i = 0\). Therefore,
\[
T\Omega_u(u) \cap KerH = \{0\},
\]
hence \(H\) is positive definite restricted on \(T\Omega_u(u)\). So the total energy \(E\) is convex on \(\Omega_u\). ■

**Theorem 4.4:** The mapping on a closed Euclidean polyhedral surface \(\mathcal{V}E : \Omega_u \rightarrow \mathbb{R}^m, (u_1, u_2, \cdots, u_n) \rightarrow (w_1, w_2, \cdots, w_n)\) is a smooth embedding.

**Proof:** The admissible metric space \(\Omega_u\) is convex as shown in Lemma 4.1, the total energy is convex as shown in Lemma 4.3. According to Lemma 2.4, \(\mathcal{V}E\) is a smooth embedding. ■

B. Open Surfaces

**Corollary 4.5:** The mapping on an Euclidean polyhedral surface with boundaries \(\mathcal{V}E : \Omega_u \rightarrow \mathbb{R}^m, (u_1, u_2, \cdots, u_n) \rightarrow (w_1, w_2, \cdots, w_n)\) is a smooth embedding.

It can be proven by using double covering technique [2].

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