GIBBS MEASURES FOR
SELF-INTERACTING WIENER PATHS

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Abstract. In this note we study a class of specifications over \(d\)-dimensional Wiener measure which are invariant under uniform translation of the paths. This degeneracy is removed by restricting the measure to the \(\sigma\)-algebra generated by the increments of the coordinate process. We address the problem of existence and uniqueness of Gibbs measures and prove a central limit theorem for the rescaled increments. These results apply to the study of the ground state of the Nelson model of a quantum particle interacting with a scalar boson field.

Keywords: Gibbs measures, Nelson model, scaling limits.

MSC (2000): 82B05; 60K35

1. Introduction

The theory of Gibbs measures for lattice spin models or continuous point processes is by now a well established subject of probability theory. References on the subject are the book of Georgii [12], the early monograph by Preston [15], the readable and concise introduction of Föllmer [11] and the pedagogical review by van Enter et al. [17].

Gibbs measures on path spaces are a more unexplored domain and only recently a series of works started a systematic study of a class of Gibbs measures on paths motivated by applications in Quantum Mechanics [4, 2, 3, 1]. However, part of these results are obtained through functional analytic approaches which does not help to fully understand the probabilistic structure of these models.

An interesting class of Gibbs measures is obtained by perturbing the Wiener measure \(W\) on \(C(\mathbb{R}, \mathbb{R}^d)\) by the exponential of a (finite-volume) energy of the form

\[
H_T(x) = \int_{-T}^T V(x_t) dt + \int_{-T}^T dt \int_{-T}^T ds W(x_t, x_s, t-s)
\]

where \(x\) is the path and the functions \(V\) and \(W\) are interpreted as interactions potentials. In this way we obtain finite-volume measures \(\mu_T\) given by

\[
\mu_{\lambda,T}(dx) = \frac{e^{-\lambda H_T(x)}}{Z_T} W(dx)
\]

The study of these measures in the limit \(T \to \infty\) has been addressed in the works cited above and a series of conditions on \(V\) and \(W\) have been found which are sufficient for the existence of the limit (in the topology of local weak convergence) and for its uniqueness.
Here we are interested in a class of models whose energy function enjoys an invariance under shift of the paths \( x \to x + c \) where \( c \) is a fixed vector in \( \mathbb{R}^d \). In particular we take \( V = 0 \) and let \( W(\xi, \xi', t) = W(\xi - \xi', t) \) so that our finite-volume energy reads

\[
H_T(x) = \int_{-T}^{T} dt \int_{-T}^{T} ds W(x_t - x_s, t - s).
\]

A relevant model for which we will prove existence of a unique Gibbs measure for small coupling is the \((d = 3, \text{UV regularized})\) Nelson model corresponding to the interaction energy given by the function

\[
W(\xi, t) = -\frac{1}{1 + |\xi|^2 + |t|^2}.
\]  

Nelson model is a member of a wide class of potentials justified by applications to Quantum Mechanics which are in the form

\[
W(\xi, t) = \int_{\mathbb{R}^d} \frac{dk}{2\omega(k)} |\rho(k)|^2 e^{-ik \cdot \xi - \omega(k)t}.
\]  

where \( \omega(k) : \mathbb{R}^d \to \mathbb{R}_+ \) is the dispersion law of a scalar boson field. Nelson model corresponds to the case \( \omega(k) = |k| \) and \( \rho(k) \) with a fast decay at infinity (ultraviolet cutoff) and \( |\rho(0)| > 0 \) (no infrared cutoff).

In [5] Betz and Spohn prove that if \( \rho \) satisfy the following integrability conditions:

\[
\int_{\mathbb{R}^d} dk |\rho(k)|^2 (\omega(k)^{-1} + \omega(k)^{-2} + \omega(k)^{-3}) < \infty
\]  

then the rescaled coordinate process \( X_t^{(\varepsilon)} = \varepsilon^{-1/2} X_{t/\varepsilon} \) weakly converge as \( \varepsilon \to 0 \) to a \( d \)-dimensional Brownian motion. Note that this result does not cover the \( d = 3 \) Nelson model since eq. (4) is not satisfied. Moreover they proved that if

\[
\int_{\mathbb{R}^d} dk |\rho(k)|^2 |k|^2 (\omega(k)^{-2} + \omega(k)^{-4}) < \infty,
\]

the limiting Brownian motion has a non-trivial diffusion constant (i.e. different from zero). Their approach is based on the use of an auxiliary Gaussian field which allow to “linearize” the interaction and to see the process \( X \) as the projection of a Markov process on a larger state space. The functional central limit theorem then follows by using a technique due to Kipnis and Varadhan.

In [16] it is pointed out that this class of models can be naturally recast as models of spins on \( \mathbb{Z} \) with single spin space \( C([0, 1], \mathbb{R}^d) \). This is obtained by cutting the path into pieces and considering each piece as a spin. In this representation the interaction becomes multi-body and under natural conditions on the time decay of \( W \), this multi-body interaction is long range with a power decay. The main problem with this approach is that the Wiener measure does not factorizes appropriately after such decomposition. Moreover the shift invariance of the Hamiltonian prevents to have tightness of the family of measures \( \{\mu_{\lambda, T}\}_{T > 0} \) (e.g. when pinned at \( \{-T, T\} \)).

The natural solution to both problems is to consider increments of the Wiener path as the basic variables. No relevant information is lost on the (local or global) behavior of the path. Since increments over disjoint intervals are independent, the
reference measure now factorizes over the product of countably many independent degrees of freedom. Techniques form the theory of one dimensional spin systems can be successfully applied. This approach will give also estimates on the decay of correlation and on the strong mixing coefficients of the Gibbs measures. Then the central limit theorem (CLT) can be proved using standard results on strongly mixing processes. However we should point out that our conditions for the CLT does not cover the 3d Nelson model.

**Plan of the paper.** In Sec.

2 we introduce the space of increments and the reference Gaussian measure induced by the Wiener measure on the paths. In Sec.

3 we prove existence of the Gibbs measures under very mild conditions and in Sec.

4 we give some more restrictive conditions under which uniqueness can be proved. Sec.

5 is devoted to the proof of uniform lower bounds for the diffusion constant under the Gibbs measures. Finally in Sec.

6 we prove the CLT for a suitable class of interactions.

### 2. Brownian Increments

Fixed a finite union of intervals $I \subseteq \mathbb{R}$ consider the linear subspace $\mathcal{X}_I \subset C(I^2, \mathbb{R}^d)$ such that $x \in \mathcal{X}_I$ iff $x_{st} = 0$ for any $t \in I$ and satisfy the cocycle condition

$$x_{su} + x_{ut} = x_{st}, \quad s,t,u \in I$$

(5)

Let $\mathcal{X} := \mathcal{X}_\mathbb{R}$ and $\mathcal{X}_T := \mathcal{X}_{[-T,T]}$ for each $T > 0$.

On $\mathcal{X}$ consider the Gaussian measure $\mu$ such that, if $\{X_{st}\}_{s,t \in \mathbb{R}}$ is the coordinate process:

$$\mathbb{E}[X_{st}] = 0, \quad \mathbb{E}[X_{st}^2] = |s-t|$$

and $X_{st}, X_{uv}$ are independent iff $(s, t)$ and $(u, v)$ are disjoint intervals. This process can be easily understood as deriving from a $d$-dimensional Brownian motion $B$ by setting $X_{st} = B_t - B_s$. Take $a \leq b$ and let $\mathcal{F}_{[a,b]} = \sigma(X_{st} : a \leq s \leq t \leq b)$ the $\sigma$-field generated by the increments in the interval $[a, b]$ and let $\mathcal{F}_{[a,b]}^{T,c} = \sigma(X_{st} : [s,t] \subset [-T,T]\setminus[a,b])$. In general, for a finite union of intervals $I \subset \mathbb{R}$ we let $\mathcal{F}_I = \sigma(X_{ts} : [t,s] \subseteq I)$. Given an interval $I$ and two paths $x, y \in \mathcal{X}$ we let $z = (x \otimes_I y)$ the unique path in $\mathcal{X}$ such that $z|_{I \times I} = x|_{I \times I}$ and $z|_{I \times I^c} = y|_{I \times I^c}$. Translations $\{\tau_a : a \in \mathbb{R}\}$ acts on $\mathcal{X}$ in the canonical way: $(\tau_a x)_{st} = x_{s+a,t+a}$.

In the following $C$ will stay for any positive constant, not necessarily the same from line to line and not depending on anything else unless otherwise stated.

### 3. Gibbs Measures

The specification $\Pi$ is the family of proper probability kernels $\pi_I(\cdot|\cdot)$ with $I$ running on the set $\mathcal{I}$ of all intervals of $\mathbb{R}$ such that

$$\pi_I(dx|y) = Z_I^{-1}(y)e^{-\lambda U_I(x)-\lambda V_I(x,y)} \mu_I \times \delta_{I^c,y}(dx)$$

(6)

for any $I \in \mathcal{I}$, where $\delta_{I^c,y}$ is the Dirac measure on $\mathcal{X}_{I^c}$ concentrated on the path $y|_{I^c \times I^c}$, $U_I$ is an $\mathcal{F}_I$ measurable function and $V_I$ is $\mathcal{F}_I \times \mathcal{F}_{I^c}$ measurable given by

$$U_I(x) = \int_{I \times I} dt ds W(x_{st}, t-s)$$

(7)
\[ V_t(x, y) = \int_{J(I)} dt ds \left[ W((x \otimes_I y)_{st}, t - s) - W((0 \otimes_I y)_{st}, t - s) \right] \]  

(8)

where \( W \in C(\mathbb{R}^d \times \mathbb{R}; \mathbb{R}) \) and 

\[ J(I) = (I^+ \times I^-) \cup (I^- \times I^+) \cup (I \times I^c) \cup (I^c \times I) \]  

(9)

with \( I^+ \) and \( I^- \) the positive and negative half lines cut off by \( I \) \( (I^c = I^- \cup I^+) \). Note that \( J(I) \) is the “cone” of influence of the increments in the interval \( I \). That is, increments outside \( J(I) \) are independent from increments inside \( I \) and any increment inside \( J(I) \) can be written as a sum of an increment in \( I \) and increments outside \( J(I) \).

**Remark 1.** The specification \( \Pi \) is translation invariant (cfr. Preston [15], page 48), i.e. for any \( a \in \mathbb{R} \) and any \( I \in \mathcal{I} \) we have \( \pi_I(\tau_a f(x,y)) = \pi_{\tau_a(I)}(f|y) \) where \( \tau_a \) acts on functions as \( \tau_a f(x) = f(\tau_a(x)) \) and on intervals as \( \tau_a([b,c]) = [b + a, c + a] \).

**Remark 2.** The definition (8) of the interaction between the increments in \( I \) and outside \( I \) is justified by the fact that only relative energies matter in the specification \( \Pi \). As we will shortly see, we can allow for models where the irrelevant constant we subtracted can be infinite.

For \( x \in \mathcal{X} \), \( \xi \in \mathbb{R}^d \), \( a \geq 0 \) define

\[ Q(x, \xi, a) := \int_{-\infty}^{0} dt \int_{0}^{\infty} ds \left| W(\xi + x_{st}, a + s - t) - W(x_{st}, a + s - t) \right|. \]

On \( W \) we will make the following assumptions:

**(H1)** \( \sup_{x \in \mathcal{X}} |U_I(x)| \leq C|I|^2 \) and \( \sup_{x \in \mathcal{X}} \int_{-\infty}^{\infty} dt |W(x_{0t}, |t|)| < \infty \);

**(H2)** For \( \xi \in \mathbb{R}^d \), \( \sup_{a \geq 0} \sup_{x \in \mathcal{X}} Q_T(x, \xi, a) \leq C(1 + |\xi|) \).

**Lemma 3.** Assume \((H1)\) and \((H2)\), then the specification \( \Pi \) is well defined and 

\[ |V_{[a,b]}(x,y)| \leq C(1 + |b - a| + |x_{ab}|) \]  

(10)

for any \( a < b \) and all \( x, y \in \mathcal{X} \).

**Proof.** Let \( I = [a, b] \) for some \( a < b \). By hypothesis \((H1)\) \( U_I(x) \) is well under control so we have only to care about \( V_I \). According to eq. (9) the cone \( J(I) \) can be split in two regions \( R_1 = I^+ \times I^- \cup I^- \times I^+ \) and \( R_2 = I \times I^c \cup I^c \times I \). The second requirement of hypothesis \((H1)\) is enough to show that

\[ \int_{R_2} dt ds \left[ W((x \otimes_I y)_{st}, t - s) - W((0 \otimes_I y)_{st}, t - s) \right] \leq C|I|. \]

For the region \( R_1 \) we proceed as follows. Let \( [a, b] = I \) and consider the integral

\[ J = \int_{I^+ \times I^-} dt ds \left[ W((x \otimes_I y)_{st}, t - s) - W((0 \otimes_I y)_{st}, t - s) \right] \]

\[ = \int_{b}^{+\infty} dt \int_{-\infty}^{a} ds \left[ W((x \otimes_I y)_{st}, t - s) - W((0 \otimes_I y)_{st}, t - s) \right] \]

\[ = \int_{b}^{+\infty} dt \int_{-\infty}^{a} ds \left[ W(x_{ab} + y_{tb} + y_{sa}, t - s) - W(y_{tb} + y_{sa}, t - s) \right] \]
where we used the cocycle property to write \((x \otimes_I y)\) as a sum of increments. By a change of variables we obtain

\[
\mathcal{J} = \int_0^{+\infty} dt \int_{-\infty}^{0} ds \left[ W(x_{ab} + y_{b+t,b} + y_{a+s,a}, t - s + b - a) - W(y_{b+t,b} + y_{a+s,a}, t - s + b - a) \right]
\]

and now it is not difficult to check that \(\mathcal{J} = Q(\tilde{y}, x_{ab}, b - a)\) for some \(\tilde{y} \in \mathcal{X}\) so the integral \(\mathcal{J}\) is well defined under hypothesis \((H2)\). Applying the same argument for the set \(I^- \times I^+\) and summarizing what we have found we end up with the bound \(\mathcal{J}\).

**Theorem 4.** Assume \((H1)\) and \((H2)\), then there exists at least one translation invariant Gibbs measure for the specification \(\Pi\).

**Proof.** Existence will follow from Theorem 3.1 of Preston (15) once we have shown that the specification \(\Pi\) satisfy the following two conditions (referred respectively as (3.11) and (3.8) in the given reference):

1. **(uniform control)** for every interval \(I\) there exists a finite measure \(\omega_I\) on \(\mathcal{F}_I\) and another interval \(K\) such that the kernels \(\{\pi_K(\cdot|y)|_{\mathcal{F}_I} : y \in \mathcal{X}\}\) are uniformly absolutely continuous w.r.t \(\omega_I\);
2. **(quasilocality)** given some local function \(f\) (i.e. \(f \in \mathcal{F}_I\) for some interval \(I\)), some interval \(K\) and \(\varepsilon > 0\) then there exists an interval \(K'\) and a \(\mathcal{F}_{K'}\) measurable function \(g_{K'}\) such that \(|\pi_K(f|y) - g_{K'}(y)| < \varepsilon\) for any \(y \in \mathcal{X}\).

Let us check (a). Fix \(y \in \mathcal{X}\) and take an interval \(K\) such that \(I \subset K\). For \(f \in \mathcal{F}_I\), \(f \geq 0\) we have

\[
\pi_K(f|y) = \frac{\int_{X_K} f(x) e^{-\lambda U_K(x) - \lambda V_K(x,y)} \mu_K(dx)}{\int_{X_K} e^{-\lambda U_K(x) - \lambda V_K(x,y)} \mu_K(dx)}
= \frac{\int_{X_{I}} \mu_I(dz) \int_{X_{I}} \mu_I(dx) f(x) e^{-\lambda U_K(x \otimes_I z) - \lambda V_K(x \otimes_I z,y)}}{\int_{X_{I}} \mu_I(dz) \int_{X_{I}} \mu_I(dx) e^{-\lambda U_K(x \otimes_I z) - \lambda V_K(x \otimes_I z,y)}}
\tag{11}
\]

where \(H = K \setminus I\). From Lemma 3 and setting \(K = [a, b]\) we have the following uniform bound

\[
|V_K(x \otimes_I z, y)| \leq C(1 + |K| + |(x \otimes_I z)_{ab}|) \tag{12}
\]

The expression on the r.h.s. can be bounded using the inequality

\[
|(x \otimes_I z)_{ab}| = |z_{ac} + x_{cd} + z_{db}| \leq |z_{ac}| + |z_{db}| + |x_{cd}|
\]

where we set \(I = [c, d]\) so that \(a < c < d < b\) and used the cocycle property. Moreover \(|U_K(x)| \leq C|K|^2\) so, for some constant \(M_K\), we obtain

\[
\pi_K(f|y) \leq M_K \frac{\int_{X_K} \mu_I(dz) \int_{X_I} \mu_I(dx) f(x) e^{\lambda C |x_{cd}| + \lambda C (|z_{ac}| + |z_{db}|)}}{\int_{X_K} \mu_I(dz) \int_{X_I} \mu_I(dx) e^{-\lambda C |x_{cd}| - \lambda C (|z_{ac}| + |z_{db}|)}}
\leq M_K \frac{\int_{X_I} \mu_I(dz) e^{\lambda C (|z_{ac}| + |z_{db}|)} \int_{X_I} \mu_I(dx) f(x) e^{\lambda C |x_{cd}|}}{\int_{X_I} \mu_I(dz) e^{-\lambda C (|z_{ac}| + |z_{db}|)} \int_{X_I} \mu_I(dx) e^{-\lambda C |x_{cd}|}}
\]

where we used the cocycle property to write \((x \otimes_I y)\) as a sum of increments. By a change of variables we obtain
Under $\mu_K(dz)$ the r.v. $z_{ac}$ and $z_{db}$ have Gaussian distribution so their modulus is exponentially integrable. The denominator is strictly greater than zero by Jensen’s inequality and the integrability of the modulus, so setting
\[
\omega_I(dx) := \frac{e^{\lambda C|x_{cd}|}}{\int_X e^{\lambda C|x_{cd}|} \mu_I(dx)}
\]
we have that there exists a constant $M_K'$ such that $\pi_K(f|y) \leq M_K' \omega_I(f)$ for any $f \in F_I$ and any $y \in X$, proving property $(a)$.

Let us now turn to quasilocality. Given $I,K,f$ as in $(b)$, for any interval $K'$ such that $K \subset K'$ define
\[
g_{K'}(y) := \frac{\int_X \mu_K(dx) f(x \otimes_K y) e^{-\lambda U_K(x) - \lambda V_{K'}(x,y)}}{\int_X \mu_K(dx) f(x \otimes_K y) e^{-\lambda U_K(x) - \lambda V_{K'}(x,y)}}
\]
where
\[
V_{K'}(x,y) := \int_{J(K) \cap K'} dt ds [W((x \otimes_I y)_{st}, t - s) - W((0 \otimes_I y)_{st}, t - s)].
\]

Observe that, by definition, $g_{K'}$ is a $F_{K'}$ measurable function and by $(H2)$ and arguments like those used in lemma 3 we see that $V_{K'}(x,y)$ converges uniformly to $V_K(x,y)$ as $K' \uparrow \mathbb{R}$. Then $g_{K'}(y)$ is $F_{K \cup I}$ measurable and $g_{K'}(y) \to \pi_K(f|y)$ uniformly as $K' \uparrow \mathbb{R}$ (cfr. Preston [15], Prop. 5.3). This proves $(b)$. So we can apply Thm 3.7 of Preston and conclude that the set of Gibbs measures for the specification $\Pi$ is non-empty.

Moreover conditions $(a)$ and $(b)$ and the translation invariance of the specification $\Pi$ are also enough to apply Thm. 4.3 of Preston which ensures that there exists at least one Gibbs measure which is invariant under the countable abelian group of rational translations $\{\tau_a : a \in \mathbb{Q}\}$. This is enough to conclude. $\square$

Remark 5. The hypothesis $(H2)$ above is more general than to require the potential to be absolutely summable (see [12]), which would be equivalent to have
\[
\int_{-\infty}^{0} ds \int_{0}^{\infty} dt |W(x_{st}, t - s)| < \infty
\]
uniformly for $x \in X$. The Nelson model given by eq. (2) is an example of non-absolutely summable potential for which is necessary to consider relative energies.

Remark 6. Nelson model satisfies assumptions $(H1)$ and $(H2)$. In particular, for $(H2)$ we have
\[
\nabla_\xi Q(x, \xi, a) = 2 \int_{-\infty}^{0} dt \int_{0}^{\infty} ds \frac{1}{(1 + |\xi + x_{st}|^2 + |a + s - t|^2)^2}(\xi + x_{st})
\]
(15)
so that
\[|\nabla \xi Q(x, \xi, a)| \leq 2 \int_{-\infty}^{0} dt \int_{0}^{\infty} ds \frac{1}{(1 + |\xi + x_{st}|^{2} + |a + t|^{2})^{3/2}} \]
\[\leq 2 \int_{-\infty}^{0} dt \int_{0}^{\infty} ds \frac{1}{(1 + |a + s|^{2})^{3/2}} \]
\[\leq C(1 + |a|)^{-1} \]
(16)

Define the measure
\[\mu_{\lambda,T}(dx) := Z_{\lambda,T}^{-1} e^{-\lambda H_{T}(x)} \mu(dx) \] (17)
on \(X\) with
\[H_{T}(x) := \int_{-T}^{T} dt \int_{-T}^{T} ds W(x_{st}, s - t) \]

Corollary 7. Assume (H1) and (H2). Then the family of measures \(\{\mu_{\lambda,T}\}_{T \geq 1}\) is tight (for the topology of local convergence). Any cluster point is a Gibbs measure for the specification \(\Pi\).

Proof. Consider the approximate kernels \(\pi_{I}^{T}\) given by
\[\pi_{I}^{T}(dx|y) = Z_{I,T}^{-1}(y)e^{-\lambda U_{I}(x) - \lambda V_{I}^{T}(x,y)} \mu_{I} \times \delta_{[-T,T] \setminus I,y}(dx) \] (18)
for any interval \(I \subseteq [-T, T]\), where \(V_{I}^{T} = V_{I}^{[-T,T]}\) (recall eq. (14)). Then for \(f \in F_{I}\) with \(I \subseteq [-T, T]\) we have
\[\mu_{\lambda,T}(f) = \int_{X} \pi_{I}^{T}(f|y) \mu_{\lambda,T}(dy) \] (19)
for any interval \(K \subseteq [-T, T]\). The kernels \(\pi_{I}^{T}\) can be controlled like the kernels \(\pi_{K}\) with the measure \(\omega_{I}\) defined in eq. (13), i.e. \(\pi_{I}^{T}(f|y) \leq M_{K}' \omega_{I}(f)\) for some constant \(M_{K}'\) depending on \(K\). Then same is true for the measures \(\mu_{\lambda,T}\) showing the tightness. The continuity of the function \(W\) and the absolute convergence of the integrals defining the potentials \(U\) and \(V\) ensures that the functions \(y \mapsto \pi_{I}^{T}(f|y)\) and \(y \mapsto \pi_{I}(f|y)\) are continuous for any continuous bounded local \(f\), moreover \(\pi_{I}^{T}(f|y) \rightarrow \pi_{I}(f|y)\) as \(T \rightarrow \infty\) uniformly in \(y\). Then passing to the limit in eq. (19) we get that any accumulation point \(\mu_{\lambda}\) of the family \(\{\mu_{\lambda,T}\}_{T}\) satisfy the equation \(\mu_{\lambda}(f) = \mu_{\lambda}(\pi_{K}(f|\cdot))\) for any interval \(K\) and any continuous bounded local \(f\). This implies that \(\mu_{\lambda}\) is a Gibbs measure for the specification \(\Pi\). \(\square\)

4. Uniqueness

In the rest of this paper we will assume always that conditions (H1) and (H2) holds. A straightforward condition for uniqueness of the Gibbs measure is given by (H3)
\[\int_{0}^{\infty} dt \int_{-\infty}^{0} ds |W(x_{st}, t - s) - W(0, t - s)| \leq C \]
uniformly in \(x \in X\).

Note that (H3) implies (H2).
Proposition 8. Assuming (H1) and (H3) there exists a unique Gibbs measure for the specification \( \Pi \).

Proof. Observe that under (H3) and for any interval \( I \) we have a uniform bound \( |V_I(x, y)| \leq C_V \) independent of \( I \). Indeed the cone \( J(I) \) can be written as the (non-disjoint) union of the sets \((I^-)^c \times I^-\), \((I^-)^c \times (I^-)^c\), \((I^+)^c \times (I^+)^c\), \((I^+)^c \times I^+\) and for each of these sets \( w \) the double integral in the definition of \( V_I \) is bounded uniformly thanks to (H3).

Then for any bounded \( f \in \mathcal{F}_I \) with \( f \geq 0 \)
\[
\pi_I(f|y) = \frac{\int_{X_I} f(x)e^{-\lambda U_I(x) - \lambda V_I(x,y)} \mu_I(dx)}{\int_{X_I} e^{-\lambda U_I(x)} \mu_I(dx)} \leq e^{2\lambda C_V} \frac{\int_{X_I} f(x)e^{-\lambda U_I(x)} \mu_I(dx)}{\int_{X_I} e^{-\lambda U_I(x)} \mu_I(dx)} \tag{20}
\]
so \( \pi_I(f|y) \leq e^{2\lambda C_V} \tilde{\omega}_I(f) \) where
\[
\tilde{\omega}_I(dx) := \frac{e^{-\lambda U_I(x)} \mu_I(dx)}{\int_{X_I} e^{-\lambda U_I(x)} \mu_I(dx)}
\]
Moreover we can also estimate \( \pi_I(f|y) \) from below as \( \pi_I(f|y) \geq e^{-2\lambda C_V} \tilde{\omega}_I(f) \). Together these bounds imply:
\[
\pi_I(f|y) \geq e^{-4\lambda C_V} \pi_I(f|z) \tag{21}
\]
uniformly for any couple \( z, y \in \mathcal{X} \) and any interval \( I \). This, according to a theorem of Georgii [12] (Prop. 8.38) implies uniqueness of the Gibbs measure. \( \square \)

Remark 9. Condition (H3) is satisfied whenever
\[
\sup_{\xi} |W(\xi, t)| \leq C(1 + |t|)^{-(2+\delta)}
\]
for some \( \delta > 0 \) or when
\[
W(\xi, t) = C(1 + |\xi|^2 + |t|^2)^{-(3/2+\delta)}
\]
always for some \( \delta > 0 \).

4.1. Mapping to a discrete model. For any size \( L > 0 \) the space \( \mathcal{X} \) splits into a product space of countably many copies of \( \mathcal{X}_L = C([0, L]^2, \mathbb{R}^d) \) as follows. Introduce the sequence on intervals \( \tau_i = [iL, (i+1)L], \ i \in \mathbb{Z} \) and for \( x \in \mathcal{X} \) let \( x_i = x|_{\tau_i \times \tau_i} \) considered as an element of \( \mathcal{X}_L \). Let \( F : \mathcal{X} \to \mathcal{X}_L^\mathbb{Z} \) be the map \( (Fx)(i) = x_i \) for any \( i \in \mathbb{Z} \) and \( x \in \mathcal{X} \). Note that it is well defined the inverse map \( F^{-1} : \mathcal{X}_L^\mathbb{Z} \to \mathcal{X} \) and that through \( F \) we can identify \( \mathcal{X} \simeq \mathcal{X}_L^\mathbb{Z} \). Moreover using \( F^{-1} \) the measure \( \mu \) factorizes accordingly : \( \mu = F_*^{-1}(\otimes_{n \in \mathbb{Z}} \mu_L^{(n)}) \) where each \( \mu_L^{(n)} \) is the Wiener measure over increments in the interval \( \tau_n \). Using this mapping the path measure can be seen as a one-dimensional (unbounded) lattice spin system with single spin space \( \mathcal{X}_L \).

In particular we can see the total energy \( W_I(x|y) = U_I(x) + V_I(x, y) \) in a bounded interval as originating from potentials \( U_{ij} \) such that
\[
U_{ij}(x) = \int_{jL}^{(j+1)L} dt \int_{jL}^{(j+1)L} ds W(x_{ts}, t - s)
\]
in the following way: $W_I(x|y) = \sum_{i,j: I \cap I_{ij} \neq \emptyset} U_{ij}(x \otimes I y)$ where $I_{ij}$ is the smallest closed interval containing the set $\{iL, (i+1)L, jL, (j+1)L\}$.

Introduce condition (H3b):

(H3b) There exists constants $C < \infty$ and $\gamma > 2$ such that

$$\sup_{\xi \in \mathbb{R}^d} |W(\xi, t)| \leq C(1 + |t|)^{-\gamma}.$$

Let $\pi_{I,[-n,n]}(\cdot|y)$ be the following probability kernels $\pi_{I,[-n,n]}(\cdot|y) = \pi_{[-n,n]}(\cdot|y)|_{\mathcal{F}_I}$. Note that the map $y \mapsto \pi_{I,[-n,n]}(\cdot|y)$ is $\mathcal{F}_{[-n,n]}$ measurable.

**Theorem 10.** Under condition (H3b) there exist a unique Gibbs measure $\mu_\lambda$ for the specification $\Pi$. Moreover for any interval $I = [a, b]$, the marginal distribution $\mu_{\lambda, I}$ on $\mathcal{F}_I$, satisfy

$$\mathbb{E}^\mu \left| \frac{d\mu_{\lambda, I}}{d\mu_1} - \frac{d\pi_{I,[-n,n]}(\cdot|y)}{d\mu_1} \right| \leq \chi(n - \max(|a|, |b|))$$

uniformly in $y \in \mathcal{X}$ where $\chi(n)$ is a decreasing function which can be chosen to be $\chi(n) = A'|\log n|^{3-\gamma}|n|^{2-\gamma}$ for some constant $A' > 0$.

**Proof.** The theorem follows directly from Thm. 1 in [9] (whose proof is contained in [10]). Indeed it is easy to check that under condition (H3b) the discretized model (e.g. with $L = 1$) fulfills all the required hypotheses. In particular, setting $M_U(k) = \sup_{i-j \geq k} \sup_x |U_{ij}(x)|$, it holds that $\sum_{k \geq 0} kM_U(k) < \infty$ and there exists a non-increasing function $\psi(n) = A'|n|^{2-\gamma}$ such that $\sum_{k \geq n} kM_U(k) \leq \psi(n)$. So, according to this result we can choose the function $\chi(n)$ to be $\chi(n) = A'|\log n|^{3-\gamma}|n|^{2-\gamma}$. \square

4.2. The contraction technique. In a more general setup than conditions (H3) or (H3b) we are able to prove the uniqueness of the Gibbs measure in the small coupling regime (i.e. when $\lambda$ is small) using the contraction technique introduced by Dobrushin [7, 8].

Let $\|\cdot\|$ denote the sup norm on $\mathcal{X}_L$ and $\|\cdot\|_1$ the Lipschitz semi-norm on $C(\mathcal{X}_L, \mathbb{R})$:

$$\|f\|_1 := \sup_{x \neq y \in \mathcal{X}_L} \frac{|f(x) - f(y)|}{\|x - y\|}$$

(23)

Given two probability measures $\mu, \nu$ on $\mathcal{X}_L$ define the Kantorovich-Rubinstein-Vaserstein (KRV) distance

$$d(\mu, \nu) := \sup_f \frac{|\mu(f) - \nu(f)|}{\|f\|_1}$$

where the sup is taken for $f \in C(\mathcal{X}_L, \mathbb{R})$.

Let $C(\mathcal{X})$ the space of continuous functions on $\mathcal{X}$ which are uniform limit of bounded local functions (i.e. depending only on finitely many $\mathcal{X}_L$ factors). For $f \in C(\mathcal{X})$ let

$$\|f\|_{1,i} := \sup \left\{ \frac{|f(x) - f(y)|}{\|x_i - y_i\|} : x, y \in \mathcal{X}, x_j = y_j \text{ for } j \neq i \right\}$$

(24)
and let $\text{Lip}(\mathcal{X})$ the class of functions $f \in C(\mathcal{X})$ which satisfy

$$|f(x) - f(y)| \leq \sum_i \|x_i - y_i\|_i f \leq \sum_i \|f\|_i < \infty$$

Let $\pi_i := \pi_{\tau_i}$. Define the Dobrushin interaction matrix $C$ as

$$C_{ik} := \sup \left\{ \frac{d(\pi_i(\cdot|y), \pi_i(\cdot|z))}{\|y_k - z_k\|} \mid y, z \in \mathcal{X} : y_j = z_j \text{ for } j \neq k \right\}$$ (25)

A tempered measure $\nu$ is a measure on $\mathcal{X}$ for which there exists $z \in \mathcal{X}$ such that

$$\sup_i \int_{\mathcal{X}} \|x_i - z_i\| \nu(dx) < \infty$$

Note that, as a by product of the proof of Thm. 4 we have that, under conditions $(H1)$ and $(H2)$ there exists tempered Gibbs measures for the specification $\Pi$.

The following result is originally due to Dobrushin [7, 8]. The formulation in terms of the KRV metric is taken from Föllmer [11] (see also [13]) where interested readers can find the relative proofs.

**Theorem 11** (Dobrushin’s uniqueness theorem). Whenever

$$\sup_k \lim_{n \to \infty} \sum_i (C^n)_{ik} = 0$$ (26)

there exists a unique tempered Gibbs measure $\mu_\lambda$ for the specification $\Pi$.

**Remark 12.** Note that a sufficient condition for eq. (26) is given by $\sup_k \sum_i C_{ik} < 1$.

Next we will apply these general results to our model. Let us introduce another class of potentials which is more general than $(H3)$:

$(H4)$ For some $K(W) < \infty$ and some $\alpha > 3$ we have

$$\sup_{\xi \in \mathbb{R}^d} |\nabla \xi \nabla \xi W(\xi, t)| \leq \frac{K(W)}{(1 + |t|)^\alpha}.$$ (26)

**Theorem 13.** Under hypothesis $(H4)$ we have

$$C_{ij} \leq C \lambda K(W) \sigma^2 (1 + |i - j|)^{2-\alpha}.$$ (27)

for some constant $C < \infty$ and some $\sigma^2 < \infty$. Therefore, for small $\lambda$ there exists a unique tempered Gibbs measure $\mu_\lambda$ for the specification $\Pi$.

**Proof.** We want to check that the specification $\Pi$ satisfy the requirements for the application of Dobrushin’s uniqueness criterion [26].
Take $y, z \in \mathcal{X}$ and let $y^r = y + r(z - y)$ for $r \in [0, 1]$. Then use the fundamental theorem of calculus to write

$$\pi_i(f|z) - \pi_i(f|y) = \int_0^1 dr \partial_r \left[ Z_i(y^r)^{-1} \int_{X_L} f(x \otimes_i y) e^{-\lambda U_i(x) - \lambda V_i(x, y^r)} \mu_i(dx) \right]$$

$$+ \left[ Z_i(z)^{-1} \int_{X_L} [f(x \otimes_i z) - f(x \otimes_i y)] e^{-\lambda U_i(x) - \lambda V_i(x, z)} \mu_i(dx) \right]$$

$$= \int_0^1 dr A(r) + B$$

where we let

$$Z_i(y) := \int_{X_L} e^{-\lambda U_i(x) - \lambda V_i(x, y)} \mu_i(dx).$$

The $B$ term is estimated by $|B| \leq \sum_{k \neq i} \| f \|_1, k \| z_k - y_k \|$ which is finite if $f \in \text{Lip}(\mathcal{X})$.

Let $J_i = J_1(\tau_i)$ and $J_{ij} = J_i \cap J_j$. For simplicity take first $y, z \in \mathcal{X}$ such that $y = z$ outside $\tau_j$ and compute

$$|\partial_r (V_i(x, y^r) - V_i(w, y^r))| \leq \int_{J_{ij}} dt ds \sup_{\xi} |\nabla_\xi \nabla_\xi W(\xi, t - s)| \| x_i - w_i \| \| y_j - z_j \|$$

If $i \neq j$ this integral contains only contributions with time span greater than $L|i - j - 1|$ so if we assume that

$$\sup_{\xi} |\nabla_\xi \nabla_\xi W(\xi, t)| \leq K(W)(1 + |t|)^{-\alpha}$$

we get

$$|\partial_r (V_i(x, y^r) - V_i(w, y^r))| \leq C K(W)(1 + |i - j|)^{2-\alpha} \| x_i - w_i \| \| y_j - z_j \|.$$

For general $y, z \in \mathcal{X}$ we have

$$|\partial_r (V_i(x, y^r) - V_i(w, y^r))| \leq C K(W) \sum_{j \neq i} (1 + |i - j|)^{2-\alpha} \| x_i - w_i \| \| y_j - z_j \|. \quad (28)$$

And with $\pi_i(\cdot|y^r) = \nu^r$

$$A(r) = \partial_r \int f(\cdot \otimes_i y) d\nu^r = \partial_r \left[ Z_i(y^r)^{-1} \int_{X_L} f(x \otimes_i y) e^{-\lambda U_i(x) - \lambda V_i(x, y^r)} \mu_i(dx) \right]$$

$$= \int_{X_L} (f(x \otimes_i y) - f(w \otimes_i y)) [\lambda \partial_r V_i(x \otimes_i y^r) - \partial_r \log Z_i(y^r)] \nu^r(dx) \quad (29)$$

for any $w \in X_L$.

This implies

$$|A(r)|^2 \leq \int_{X_L} (f(x \otimes_i y) - f(w \otimes_i y))^2 \nu^r(dx) \text{Var}_{\nu^r} [\lambda \partial_r V_i(\cdot \otimes_i y^r)]$$

$$\leq \| f \|_{1,i}^2 \int_{X_L} \| x_i - w \|^2 \nu^r(dx) \text{Var}_{\nu^r} [\lambda \partial_r V_i(\cdot \otimes_i y^r)]$$

$$\leq \| f \|_{1,i}^2 \sigma^2 \text{Var}_{\nu^r} [\lambda \partial_r V_i(\cdot \otimes_i y^r)] \quad (30)$$
where $\sigma_t^2 := \sup_{y \in \mathcal{X}} \inf_{w \in \mathcal{X}_t} \int_{\mathcal{X}} \|x_i - w\|^2 \pi_t(dx | y)$. Eq. \ref{eq:28} implies

$$\{\text{Var}_{\nu, r} [\lambda \partial_i V_i (\cdot \otimes_i y')]\}^{1/2} \leq C K(W) \sigma_t \sum_{j} \lambda (1 + |i - j|)^{2-\alpha} \|y_j - z_j\|$$

and using this estimate into eq. \ref{eq:30} we get

$$|A(r)| \leq C K(W) \|f\|_{1, \sigma_t^2} \sum_{j} \lambda (1 + |i - j|)^{2-\alpha} \|y_j - z_j\|.$$ 

Then

$$|\pi_i(f|y) - \pi_i(f|z)| \leq C \lambda K(W) \|f\|_{1, \sigma_t^2} \sum_{j \neq i} (1 + |i - j|)^{2-\alpha} \|y_j - z_j\|$$

$$+ \sum_{j \neq i} \|f\|_{1,j} \|z_j - y_j\|$$

(31)

According to this bound, if $f \in \text{Lip}(\mathcal{X})$ we have also $\pi_i(f|\cdot) \in \text{Lip}(\mathcal{X})$ provided $\alpha > 3$.

For $f(x) = f(x_i)$ the bound (31) reads

$$|\pi_i(f|y) - \pi_i(f|z)| \leq C \lambda K(W) \|f\|_{1, \sigma_t^2} \sum_{j \neq i} (1 + |i - j|)^{2-\alpha} \|y_j - z_j\|$$

which means that

$$d(\pi_i(\cdot|y), \pi_i(\cdot|z)) \leq C \lambda K(W) \sigma_t^2 \sum_{j \neq i} (1 + |i - j|)^{2-\alpha} \|y_j - z_j\|. \quad \text{(32)}$$

Moreover if $y = z$ outside $\tau_j$ (i.e. if $y|_{\tau_k} = z|_{\tau_k}$ for any $k \neq j$) then

$$d(\pi_i(\cdot|y), \pi_i(\cdot|z)) \leq C \lambda K(W) \sigma_t^2 (1 + |i - j|)^{2-\alpha} \|y_j - z_j\|$$

(33)

which in turn implies $C_{ij} \leq C \lambda K(W) \sigma_t^2 (1 + |i - j|)^{2-\alpha}$. So provided $\sigma = \sup_i \sigma_i < \infty$ and $\alpha > 3$ we have that $\sum_i C_{ij} \leq C' \sigma^2 \lambda K(W) L^{-1}$ and for $\lambda$ sufficiently small we obtain $\sum_i C_{ij} < 1$ and by applying Dobrushin criterion we can conclude that there exists a unique tempered Gibbs measure $\mu_\lambda$ associated to the specification $\Pi$. The condition $\sigma^2 < \infty$ follows easily from (H1) and (H2) and the computations of Thm. \ref{thm:4}.

\begin{remark}
In the case of the Nelson model we have

$$|\nabla_\xi \nabla_\zeta W(\xi, t)| \leq C(1 + |t|^2)^{-2}$$

uniformly in $\xi \in \mathbb{R}^d$, so $\alpha = 4$ and we can apply the above result.
\end{remark}

5. Lower bound on the diffusion constant

Here we provide a lower bound for the second moment of the increments under the Gibbs measure. This will be useful below in the proof of the CLT.

\begin{theorem}
Under condition (H4) there exists a positive constant $\sigma^2_- > 0$ such that

$$\mathbb{E}^{\mu_\lambda, t}[X_{ab}^2] \geq \sigma^2_- |a - b|$$

uniformly in $T, a, b$.
\end{theorem}
Proof. Integration by parts on \( \mu \) is given by the formula
\[
\mathbb{E}^\mu[X_{ab}F] = \int_a^b dt \mathbb{E}^\mu[D_tF]
\]
where \( D_t \) is the Malliavin derivative (see e.g. [14]). This, of course, when both sides make sense. By integration by parts we have for disjoint intervals \([a, b], [c, d]\) with \( b < c \):
\[
\mathbb{E}^{\mu,T}[X_{ab}X_{cd}] = -\mathbb{E}^{\mu,T}[A_{ab}X_{cd}] = \mathbb{E}^{\mu,T}[-B_{ab,cd} + A_{ab}A_{cd}]
\]
with
\[
A_{ab} = \int_a^b ds \lambda D_s(H_T(X)) \quad \text{and} \quad B_{ab,cd} = \int_a^b ds \int_c^d dt \lambda D_t D_s(H_T(X)).
\]

While
\[
\mathbb{E}^{\mu,T}[X_{ab}^2] = |a - b| - \mathbb{E}^{\mu,T}[A_{ab}X_{ab}]
\]
Apply Cauchy-Schwartz inequality to \( \mathbb{E}^{\mu,T}[A_{ab}X_{ab}] \) to get the lower bound
\[
\mathbb{E}^{\mu,T}[X_{ab}^2] \geq |a - b| - (\mathbb{E}^{\mu,T}[A_{ab}^2])^{1/2} (\mathbb{E}^{\mu,T}[X_{ab}^2])^{1/2}.
\]
One more integration by parts on the r.h.s of eq. (35) gives
\[
\mathbb{E}^{\mu,T}[X_{ab}^2] = |a - b| - \mathbb{E}^{\mu,T}[B_{ab,ab}] + \mathbb{E}^{\mu,T}[A_{ab}^2]
\]
which gives another inequality
\[
\mathbb{E}^{\mu,T}[A_{ab}^2] \leq \mathbb{E}^{\mu,T}[X_{ab}^2] + \mathbb{E}^{\mu,T}[|B_{ab,ab}|].
\]
Use the fact that \(|x + y|^{1/2} \leq |x|^{1/2} + |y|^{1/2}\) to get
\[
(\mathbb{E}^{\mu,T}[A_{ab}^2])^{1/2} \leq (\mathbb{E}^{\mu,T}[X_{ab}^2])^{1/2} + (\mathbb{E}^{\mu,T}[|B_{ab,ab}|])^{1/2}.
\]
Combining together eq. (35) and eq. (38) we get the lower bound
\[
2 \mathbb{E}^{\mu,T}[X_{ab}^2] \geq |a - b| - (\mathbb{E}^{\mu,T}[|B_{ab,ab}|])^{1/2} (\mathbb{E}^{\mu,T}[X_{ab}^2])^{1/2}.
\]
If we compute the Malliavin derivatives, we get:
\[
D_t(H_T(X)) = 2 \int_{-T}^T du \int_u^T dv D_t W(X_{uv}, u - v) = 2 \int_{-T}^T du \int_u^T dv W_x(X_{uv}, u - v)
\]
and
\[
D_tD_s(H_T(X)) = 2 \int_{-T}^T du \int_u^T dv D_tD_s W(X_{uv}, u - v) = 2 \int_{-T}^S du \int_u^T dv W_{xx}(X_{uv}, u - v)
\]
where \( W_x(\xi, t) = \nabla_\xi W(\xi, t) \) and \( W_{xx}(\xi, t) = \nabla_\xi \nabla_\xi W(\xi, t) \).
Using condition (H4) we get
\[ |D_tD_s(H_T(X))| \leq 2 \int_{-T}^s du \int_t^T dv \frac{C}{(1 + |u - v|)^{\alpha}} \leq C(1 + |s - t|)^{2 - \alpha}. \] (42)

And since \( \alpha > 3 \) we obtain the bound \( \mathbb{E}^{\mu,\lambda,T}[|B_{a,b}|] \leq C|a - b| \) uniformly in \( T \).

Using this last estimate in the inequality (39) above we get
\[ 2 \mathbb{E}^{\mu,\lambda,T}[X_{a,b}^2] \geq |a - b| - (C\lambda)^{1/2}|a - b|^{1/2} (\mathbb{E}^{\mu,\lambda,T}[X_{a,b}^2])^{1/2} \]
Calling \( y = (\mathbb{E}^{\mu,\lambda,T}[X_{a,b}^2])^{1/2} |a - b|^{-1/2} \) we have \( 2y^2 - 1 + (C\lambda)^{1/2}y \geq 0 \) with \( y \geq 0 \). This implies that \( y \geq \sigma_- \) for some \( \sigma_- > 0 \) and we have obtained the lower bound \( \mathbb{E}^{\mu,\lambda,T}[X_{a,b}^2] \geq \sigma_-^2|a - b| \).

\[ \square \]

**Remark 16.** Using \( 2|a| \leq |a|^2 + |b|^2 \) we have
\[ \mathbb{E}^{\mu,\lambda,T}[X_{a,b}^2] \leq |a - b| + \frac{1}{2} \mathbb{E}^{\mu,\lambda,T}[X_{a,b}^2] + \frac{1}{2} \mathbb{E}^{\mu,\lambda,T}[A_{a,b}^2] \]
and then \( \mathbb{E}^{\mu,\lambda,T}[X_{a,b}^2] \leq 2|a - b| + \mathbb{E}^{\mu,\lambda,T}[A_{a,b}^2] \). This equation and the decay of correlations can provide upper bounds for the diffusion constant.

6. Diffusive behavior

According to Thms. 10 and 13 under condition (H3b) or (H4) (the latter in the small coupling regime) we have a unique Gibbs measure which we denote \( \mu_\lambda \) as above and for which we have polynomial decay of correlations. Then we address the problem of the long-time behavior of the increment process under diffusive rescaling. Thm. 15 rules out the possibility of sub-diffusive behavior of the paths whenever the interaction decays fast enough. This holds irrespective of the magnitude of the coupling constant \( \lambda \) and of the character of the potential (attractive or repulsive). The next theorem establishes that (under some more restrictive assumption than those needed to obtain uniqueness of the Gibbs measure) we actually have diffusive behavior of the increment process.

**Theorem 17.** Assume (H4) holds. Then in either of the following two situations:

(a) condition (H3b) with \( \gamma > 3 \), or
(b) small \( \lambda \) and \( \alpha > 3 \);

the r.v. \( X_{\varepsilon,s}^\varepsilon = \varepsilon^{1/2}X_{\varepsilon,1_t,\varepsilon,1_s} \) weakly converges to an isotropic Gaussian vector as \( \varepsilon \to 0 \).

**Proof.** Consider the random variables \( Y_i = \langle v, X_{L,i,L(i+1)} \rangle \in \mathcal{F}_{\tau_i} \) for some fixed vector \( v \in \mathbb{R}^d \) and for \( \Lambda \subset \mathbb{Z} \) let \( \mathcal{A}_\Lambda = \sigma(X_i, i \in \Lambda) \). If \( \Lambda_1, \Lambda_2 \subset \mathbb{Z} \) let \( d(\Lambda_1, \Lambda_2) = \inf\{|n - k|, n \in \Lambda_1, k \in \Lambda_2\} \). Define the following mixing coefficients for the measure \( \mu_\Lambda \):
\[ \alpha_{l,k}(n) = \sup\{|\mu_\lambda(A_1 \cap A_2) - \mu_\lambda(A_1)\mu_\lambda(A_2)| : A_i \in \mathcal{A}_{\Lambda_i}, |\Lambda_1| \leq k, |\Lambda_2| \leq l, d(\Lambda_1, \Lambda_2) \geq n\} \]
and
\[ \rho(n) = \sup\{|\text{Cov}_{\mu_\lambda}(Z_1, Z_2)| : Z_i \in L^2(\mu_\lambda, \mathcal{A}_{\{k_i\}}), \|Y_i\|_2 \leq 1, |k_1 - k_2| \geq n\} \]
where \( \| \cdot \|_p \) denote the \( L^p \) norm with respect to the measure \( \mu_\lambda \).
Then, according to a theorem of Bolthausen [6], if the following two conditions hold
\[ \|Y_i\|_{2+\delta} < \infty, \quad \sum_{m=1}^{\infty} \alpha_{2,\infty}(m) < \infty \]  \hfill (43)
for some \( \delta > 0 \) and if
\[ \sigma^2 = \sum_n \text{Cov}_{\mu_\lambda}(Y_0, Y_n) > 0 \]  \hfill (44)
then the r.v. \( S_n = (\sigma^2 n)^{-1/2} \sum_{0 \leq k \leq n} Y_k \) converges to a standard Gaussian r.v.

So it will be enough to check the two conditions (43) and (44). For eq. (44) note that \( \sigma^2 = \lim_{n \to \infty} n^{-1} E_{\mu_\lambda}(X_{Ln,0})^2 \) and by the bound proved in Thm. 15 we have that this quantity is bounded below by \( \sigma^2 Ln \) with \( \sigma^2 > 0 \), so that we can conclude \( \sigma^2 > 0 \).

As for condition (43) note that each \( Y_i \) has moments of any order due to the fact that the expectation on the measure \( \mu_\lambda \) can be bounded above by expectation on the measure \( \omega|_{[0,L]} \) defined in the proof of Thm. 1 which can be directly estimated and shown to be finite.

Under assumption (a) we are in the conditions to apply Thm. 10 and the inequality (22) is enough to prove that \( \alpha_{2,\infty}(n) \leq C \chi(n) = C'n^{2-\gamma} \). Then for \( \gamma > 3 \) condition (43) can be satisfied by choosing \( \delta \) sufficiently large.

It remains to check that the mixing coefficient \( \alpha(n) \) is sufficiently summable in case (b). A technical difficulty is that the contraction coefficients for the KRV metric are not suitable to estimate the strong mixing coefficients \( \alpha_{l,k}(m) \). This because we cannot uniformly approximate the indicator functions (needed to estimate probabilities) with functions in \( \text{Lip}(\mathcal{X}) \). However this technical difficulty can be easily overcome by using a slightly different norm in the Dobrushin contraction technique. Let \( \|f\|_{*,i} \) the following local “quasi”-Lipschitz semi-norm:
\[ \|f\|_{*,i} := \sup \left\{ \frac{|f(x) - f(y)|}{\theta_{x,y} + \|x_i - y_i\|} : x, y \in \mathcal{X} \quad x_j = y_j \text{ for } j \neq i \right\} \]
where \( \theta_{x,y} = 1 \) if \( x \neq y \) and \( \theta_{x,y} = 0 \) otherwise, for any two elements \( x, y \in \mathcal{X}_L \).

This new semi-norm would replace the semi-norm \( \|f\|_{1,i} \) defined in eq. (24). Then it is easy to see that all the arguments carry over also with this new semi-norm and that under this semi-norm we can approximate uniformly the indicator function of a set \( A \in \mathcal{A}_\Lambda \) by the Lipschitz functions \( \Gamma_{A,\rho}(x) = \exp(-\rho^{-1} \inf_{y \in A} |x - y|) \).

Then, adapting Prop. 2.5 of [13] we are able to prove that, under the condition \( \sup_i \sum_j C_{ij}|i - j|^{1+\epsilon} < \infty \) for some \( \epsilon > 0 \) (i.e. \( \alpha > 4 \)) we have \( \alpha_{2,\infty}(n) \leq C'n^{1-\epsilon} \) and this is enough to conclude the proof.

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References

[1] V. Betz. Existence of Gibbs measures relative to Brownian motion. *Markov Process. Related Fields*, 9(1):85–102, 2003.

[2] V. Betz, F. Hiroshima, J. Lőrinczi, R. A. Minlos, and H. Spohn. Ground state properties of the Nelson Hamiltonian: a Gibbs measure-based approach. *Rev. Math. Phys.*, 14(2):173–198, 2002.

[3] V. Betz and J. Lőrinczi. Uniqueness of Gibbs measures relative to Brownian motion. *Ann. Inst. H. Poincaré Probab. Statist.*, 39(5):877–889, 2003.

[4] V. Betz, J. Lőrinczi, and H. Spohn. Gibbs measures on brownian paths: Theory and applications. 2003.

[5] V. Betz and H. Spohn. A central limit theorem for gibbs measures relative to brownian motion. 2003.

[6] E. Bolthausen. On the central limit theorem for stationary mixing random fields. *Ann. Prob.*, 10:1047–1050, 1982.

[7] R. L. Dobrushin. Description of a random field by means of conditional probabilities and the conditions governing its regularity. *Theory Probab. Appl.*, 13:197–244, 1968.

[8] R. L. Dobrushin. Prescribing a system of random variables by conditional distributions. *Theory Probab. Appl.*, 15:458–486, 1970.

[9] R. L. Dobrushin. Analyticity of correlation functions in one-dimensional classical system with slowly decreasing potentials. *Comm. Math. Phys.*, 32:269–289, 1973.

[10] R. L. Dobrušin. Analyticity of correlation functions in one-dimensional classical systems with polynomially decreasing potential. *Mat. Sb. (N.S.),* 94(136):16–48, 159, 1974.

[11] H. Föllmer. Random fields and diffusion processes. In *École d’Été de Probabilités de Saint-Flour XV–XVII, 1985–87*, volume 1362 of *Lecture Notes in Math.*, pages 101–203. Springer, Berlin, 1988.

[12] H.-O. Georgii. *Gibbs measures and phase transitions*, volume 9 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1988.

[13] H. Künsch. Decay of correlations under Dobrushin’s uniqueness condition and its applications. *Comm. Math. Phys.*, 84:207–222, 1982.

[14] D. Nualart. *The Malliavin calculus and related topics*. Probability and its Applications (New York). Springer-Verlag, New York, 1995.

[15] C. Preston. *Random fields*. Springer-Verlag, Berlin, 1976. Lecture Notes in Mathematics, Vol. 534.

[16] H. Spohn. Effective mass of the polaron: a functional integral approach. *Ann. Physics*, 175(2):278–318, 1987.

[17] A. C. D. van Enter, R. Fernández, and A. D. Sokal. Regularity properties and pathologies of position-space renormalization-group transformations: scope and limitations of Gibbsian theory. *J. Statist. Phys.*, 72(5-6):879–1167, 1993.