Localization, Isomorphisms and Adjoint Isomorphism in the Category $\text{Comp}(A - Mod)$

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Abstract

A and B are considered to be non necessarily commutative rings and X a complex of $(A-B)$ bimodules. The aim of this paper is to show that:

1. The functors $\overline{\text{EXT}}^n_{\text{Comp}(A-Mod)}(X, -)$ : $\text{Comp}(A - Mod) \rightarrow \text{Comp}(B - Mod)$ and $\text{Tor}_n^{\text{Comp}(B-Mod)}(X, -)$ : $\text{Comp}(B - Mod) \rightarrow \text{Comp}(A - Mod)$ are adjoint functors.

2. The functor $S^{\mathcal{C}}(-)$ commute with the functors $X \otimes -$ , $\text{Hom}^\ast(X, -)$ and their corresponding derived functors $\overline{\text{EXT}}^n_{\text{Comp}(A-Mod)}(X, -)$ and $\text{Tor}_n^{\text{Comp}(B-Mod)}(X, -)$.

Keywords: saturated multiplicative subset, left Ore conditions, localization, category of complexes, functors $S^{-1}(-)$ and $S^{\mathcal{C}}(-)$, $\text{Hom}^\ast$ functor, tensor product functor, derived functors

1. Introduction

The adjunction study between $\text{Hom}$ functor and tensor product functor has been done by several authors in the category $\text{A - Mod}$ of $A$-modules (see Rotman, J. J. (1972), theorem 2.76 for instance). That is the functors $\text{Hom}_A(M, -)$ and $M \otimes -$ , where $M$ is a $(A - B)$ bimodule, are adjoint functors. Its analogue, considered in the category of complexes, has equally been shown in (Beck, V. (2008), corollary 5.16). Otherwise the functors $\text{Hom}^\ast(X, -)$ and $X \otimes -$ are adjoint functors, where $X$ is a complex of $(A - B)$ bimodules.

Now since on the one hand $\text{Hom}^\ast(X, -)$ and $\overline{\text{EXT}}^0_{\text{Comp}(A-Mod)}(X, -)$, where $\overline{\text{EXT}}^n$ is considered to be the n-th functor derived of $\text{Hom}^\ast$, are isomorphic and on the other hand $X \otimes -$ and $\text{Tor}_0^{\text{Comp}(B-Mod)}(X, -)$, where $\text{Tor}_0^{\text{Comp}(B-Mod)}$ is the n-th derived functor of the tensor product functor $X \otimes -$ , are isomorphic then we can conclude that $\overline{\text{EXT}}^n_{\text{Comp}(A-Mod)}(X, -)$ and $\text{Tor}_0^{\text{Comp}(B-Mod)}(X, -)$ are adjoint functors. Besides, in (Dembele, B., Maouia, B., & Sanghare, M. (2020)) we showed that the functor $S^{\mathcal{C}}(-)$ commute with the functors tensor product, $\text{Hom}^\ast$, $\overline{\text{EXT}}^n$ and $\text{Tor}_n$ on the objects. So, the question is of course this: if we can have the generalization of that results. Otherwise if the functors $\overline{\text{EXT}}^n_{\text{Comp}(A-Mod)}(X, -)$ : $\text{Comp}(A - Mod) \rightarrow \text{Comp}(B - Mod)$ and $\text{Tor}_n^{\text{Comp}(B-Mod)}(X, -)$ : $\text{Comp}(B - Mod) \rightarrow \text{Comp}(A - Mod)$ are adjoint functors. Equally, if $S^{\mathcal{C}}(-)$ commute in the general case with the functors tensor product, $\text{Hom}^\ast$, $\overline{\text{EXT}}^n$ and $\text{Tor}_n$. So let $A$ and $B$ be two rings, $X$ a complex of $(A - B)$ bimodules, $C$ a complex of $A$-modules and $n$ an integer, we organize this work as following:

we give some definitions and preliminary results in our first section for reminder.

In our second section we prove the following results:

1. $\overline{\text{EXT}}^{n+1}_{\text{Comp}(A-Mod)}(X, -)$ : $\text{Comp}(A - Mod) \rightarrow \text{Comp}(B - Mod)$ and $\overline{\text{EXT}}^n_{\text{Comp}(A-Mod)}(K_0, -)$ : $\text{Comp}(A - Mod) \rightarrow \text{Comp}(B - Mod)$, where $K_0$ is considered to be the $0-th$ kernel of $X$, are isomorphic;

2. $\text{Tor}_n^{\text{Comp}(B-Mod)}(X, -)$ : $\text{Comp}(B - Mod) \rightarrow \text{Comp}(A - Mod)$ and $\text{Tor}_{n+1}^{\text{Comp}(B-Mod)}(K_0, -)$ : $\text{Comp}(B - Mod) \rightarrow \text{Comp}(A - Mod)$ are isomorphic;
3. $\text{EXT}_{\text{Comp}(A-\text{Mod})}^n(X, -) : \text{Comp}(A-\text{Mod}) \rightarrow \text{Comp}(B-\text{Mod})$ and $\text{Tor}_{\text{Comp}(B-\text{Mod})}^n(X, -) : \text{Comp}(B-\text{Mod}) \rightarrow \text{Comp}(A-\text{Mod})$ are adjoint functors;

4. if $A$ is a subring of $B$, $S$ a saturated multiplicative subset of $A$ and $B$ satisfying the left Ore conditions then:

$\text{EXT}_{\text{Comp}(S^{-1}A-\text{Mod})}^n(S^{-1}C(X), -) : \text{Comp}(S^{-1}A-\text{Mod}) \rightarrow \text{Comp}(S^{-1}B-\text{Mod})$ and $\text{Tor}_{\text{Comp}(S^{-1}B-\text{Mod})}^n(S^{-1}C(X), -) : \text{Comp}(S^{-1}B-\text{Mod}) \rightarrow \text{Comp}(S^{-1}A-\text{Mod})$ are adjoint functors.

And finally, in the last section, we show the following results:

1. $\text{EXT}_{\text{Comp}(S^{-1}A-\text{Mod})}^{n+1}(S^{-1}C(C), S^{-1}C(-))$ and $\text{EXT}_{\text{Comp}(S^{-1}A-\text{Mod})}^n(S^{-1}C(K_0), S^{-1}C(-))$ are isomorphic;

2. $\text{Tor}_{\text{Comp}(S^{-1}A-\text{Mod})}^{n+1}(S^{-1}C(C), S^{-1}C(-))$ and $\text{Tor}_{\text{Comp}(S^{-1}A-\text{Mod})}^n(S^{-1}C(K_0), S^{-1}C(-))$ are isomorphic;

3. $S^{-1}C(X \otimes -) : \text{Comp}(B-\text{Mod}) \rightarrow \text{Comp}(S^{-1}A-\text{Mod})$ and $S^{-1}C(S^{-1}C(-)) : \text{Comp}(B-\text{Mod}) \rightarrow \text{Comp}(S^{-1}A-\text{Mod})$ are isomorphic;

4. If $X$ is of finite type then $S^{-1}C\text{Hom}^*(X, -) : \text{Comp}(A-\text{Mod}) \rightarrow \text{Comp}(S^{-1}B-\text{Mod})$ and $\text{Hom}^*(S^{-1}C(X), S^{-1}C(-)) : \text{Comp}(A-\text{Mod}) \rightarrow \text{Comp}(S^{-1}B-\text{Mod})$ are isomorphic;

5. If $X$ is of type $FP_\infty$ then $S^{-1}C\text{EXT}_{\text{Comp}(A-\text{Mod})}^n(X, -) : \text{Comp}(A-\text{Mod}) \rightarrow \text{Comp}(S^{-1}B-\text{Mod})$ and $\text{EXT}_{\text{Comp}(S^{-1}A-\text{Mod})}^n(S^{-1}C(X), S^{-1}C(-)) : \text{Comp}(A-\text{Mod}) \rightarrow \text{Comp}(S^{-1}B-\text{Mod})$ are isomorphic;

6. $S^{-1}C\text{Tor}_{\text{Comp}(B-\text{Mod})}^n(X, -) : \text{Comp}(B-\text{Mod}) \rightarrow \text{Comp}(S^{-1}A-\text{Mod})$ and $\text{Tor}_{\text{Comp}(S^{-1}B-\text{Mod})}^n(S^{-1}C(X), S^{-1}C(-)) : \text{Comp}(B-\text{Mod}) \rightarrow \text{Comp}(S^{-1}A-\text{Mod})$ are isomorphic.

2. Definitions and Preliminary Results

Definition and Proposition 2.1

The category of complexes of left $A$-modules is the category denoted by $\text{Comp}(A-\text{Mod})$ such that:

1. objects are complexes of left $A$-modules.

A complex of left $A$-modules $C$ is a sequence of homomorphisms of left $A$-modules $(C^n \xrightarrow{d^n} C^{n+1})_{n \in \mathbb{Z}}$ such that $d^{n+1} \circ d^n = 0$, for all $n \in \mathbb{Z}$.

2. Morphisms are maps of complexes of left $A$-modules. Let $C$ and $D$ be two complexes, a map of complexes of left $A$-modules $f : C \rightarrow D$ is a sequence of homomorphisms of left $A$-modules $(f^n : C^n \rightarrow D^n)_{n \in \mathbb{Z}}$ such that $f^{n+1} \circ d^n = d^n \circ f^n$ for $n \in \mathbb{Z}$.

Proposition 2.2

Let $A$ be a ring and $S$ a saturated multiplicative subset of $A$ verifying the left Ore conditions. Then the relation:

$S^{-1}C() : \text{Comp}(A-\text{Mod}) \rightarrow \text{Comp}(S^{-1}A-\text{Mod})$ such that

1. if $C := \ldots \rightarrow C^n \xrightarrow{d^n} C^{n+1} \rightarrow \ldots$ is an objet of $\text{Comp}(A-\text{Mod})$ then :

$$S^{-1}C(C) := \ldots \rightarrow S^{-1}C^n \xrightarrow{S^{-1}d^n} S^{-1}C^{n+1} \rightarrow \ldots$$

is an objet of $\text{Comp}(S^{-1}A-\text{Mod})$

2. if $f : C \rightarrow D$ is a morphism of $\text{Comp}(A-\text{Mod})$ then

$$S^{-1}C(f) : S^{-1}C(C) \rightarrow S^{-1}C(D)$$

is a morphism of $\text{Comp}(S^{-1}A-\text{Mod})$
Then $S^{-1}_C()$ is an exact covariant functor.

**Proof**

see (Dembele, B., Maaouia, B., F., & Sanghare, M. (2020)), proposition 1

**Definition and proposition 2.3:**

Let $X$ be a complex of $(A-B)$-bimodules and let be the following correspondance:

$$X \otimes - : \text{Comp}(B - \text{Mod}) \rightarrow \text{Comp}(A - \text{Mod})$$

such that:

- If $Y \in \text{Ob}(\text{Comp}(B - \text{Mod}))$ then $X \otimes Y$ is a complex of left $A$-modules such that:

  $$(X \otimes Y)^n = \bigoplus_{t \in \mathbb{Z}} X^t \otimes Y^{n-t}$$

  $$\delta^n_{X \otimes Y}(x \otimes y) = d^t_X(x) \otimes y + (-1)^t x \otimes Y^{n-t}(y)$$

- If $f : Y_1 \rightarrow Y_2$ is a map of complexes of $\text{Comp}(B - \text{Mod})$ then

  $$(X \otimes -)(f) : X \otimes Y_1 \rightarrow X \otimes Y_2$$

  such that:

  $$(X \otimes -)(f)^n : (X \otimes Y_1)^n \rightarrow (X \otimes Y_2)^n$$

  $$x \otimes y \mapsto x \otimes f^{n-t}(y)$$

is a map of complexes of $\text{Comp}(A - \text{Mod})$.

Then $X \otimes -$ is a covariant functor that is right exact.

**Proof**

see [Dembele, B., Maaouia, B., F., & Sanghare, M. (2020)], definition and proposition 2

**Definition and proposition 2.4:**

Let $X$ be a complex of $(A-B)$-bimodules. Let be the following correspondence:

$$\text{HOM}^\ast(X, -) : \text{Comp}(A - \text{Mod}) \rightarrow \text{Comp}(B - \text{Mod})$$

such that:

- If $Y$ is a complex of left $A$-modules then $\text{HOM}^\ast(X, -)(Y) = \text{HOM}^\ast(X, Y)$ is a complex of left $B$-modules such that:

  $$(\text{HOM}^\ast(X, Y))^n = \bigcap_{t \in \mathbb{Z}} \text{Hom}^\ast_A(X^t, Y^{n+t})$$

  and $\delta_{\text{HOM}^\ast(X,Y)}$ is defined as following:

  $$\left(\delta^n_{\text{HOM}^\ast(X,Y)}\right)_t : \text{Hom}^\ast_A(X^t, Y^{n+t}) \rightarrow \text{Hom}^\ast_A(X^t, Y^{n+t+1})$$

  $$g' \mapsto d^t_yn'g' + (-1)^{n+1}g'^{t+1}d^t_X$$

- If $f : Y_1 \rightarrow Y_2$ is a morphism of $\text{Comp}(A)$ then:

  $$\text{HOM}^\ast(X, -)(f)^n : \text{HOM}^\ast(X, Y_1)^n \rightarrow \text{HOM}^\ast(X, Y_2)^n$$

  $$(g')_t \mapsto (f^{n+1} \circ g')_t$$

is a morphism of $\text{Comp}(B - \text{Mod})$.  

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Then $\text{Hom}^*(X, -)$ is a covariant functor that is left exact.

**Proof**

see [Dembele, B., Maaouia, B., F., & Sanghare, M. (2020)], definition and proposition 3

**Definition 2.5**

Let $C$ be a complex of left $A$-modules and $C_\bullet$ a projective resolution of $C$ such us:

$C_\bullet := \ldots \to P_{n+1} \xrightarrow{d_{n+1}} P_n \to \ldots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} C \to 0.$

Then we said that $\text{Ker}(d_n)$ is the $n$-th kernel of $C_\bullet$, and we denote it by $K_n$.

**3. Adjoint Isomorphism Between $\text{Ext}$ and $\text{Tor}$ in $\text{Comp}(A - \text{Mod})$**

**Definition 3.1**

Let $C$ and $D$ be two categories, $F : C \to D$ and $G : D \to C$ two functors. It is said that the couple $(F, G)$ is adjoint if for any $A \in \text{Ob}(C)$ and for any $B \in \text{Ob}(D)$, there is an isomorphism:

$r_{A,B} : \text{Hom}_C(A, G(B)) \to \text{Hom}_D(F(A), B)$

so that:

a) For any $f \in \text{Hom}_C(A', A)$, the following diagram is commutative:

\[
\begin{array}{ccc}
\text{Hom}_C(A, G(B)) & \xrightarrow{\text{Hom}(f, G(B))} & \text{Hom}_C(A', G(B)) \\
\downarrow r_{A,B} & & \downarrow r'_{A',B} \\
\text{Hom}_D(F(A), B) & \xrightarrow{\text{Hom}(F(f), B)} & \text{Hom}_D(F(A'), B)
\end{array}
\]

b) For any $g \in \text{Hom}_D(B, B')$, the following diagram is commutative:

\[
\begin{array}{ccc}
\text{Hom}_C(A, G(B)) & \xrightarrow{\text{Hom}(A, G(g))} & \text{Hom}_C(A, G(B')) \\
\downarrow r_{A,B} & & \downarrow r'_{A,B'} \\
\text{Hom}_D(F(A), B) & \xrightarrow{\text{Hom}(F(A), g)} & \text{Hom}_D(F(A), B')
\end{array}
\]

**Lemma 3.2**

Let $C$ be a complex of left $A$-modules and $C_\bullet$ projective resolution of $C$ of $n$-th kernel $\text{Ker}(d_n) = K_n$. Then the functors $\text{Ext}^n_{\text{Comp}(A-\text{Mod})}(C, -)$ and $\text{Ext}^n_{\text{Comp}(A-\text{Mod})}(K_0, -)$ are isomorphic where $\text{Ext}^n_{\text{Comp}(A-\text{Mod})}(X, -)$ is the $n$-th right derived functor of $\text{Hom}^*(X, -)$.

**Proof**

Since $\ldots \to P_{n+1} \xrightarrow{d_{n+1}} P_n \to \ldots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} C \to 0$ is a projective resolution of $C$ then $\ldots \to P_{n+1} \xrightarrow{d_{n+1}} P_n \to \ldots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} K_0 \to 0$ is a projective resolution of $K_0$. So on the one hand:

$\text{Ext}^{n+1}_{\text{Comp}(A-\text{Mod})}(C, D) \cong \text{Ext}^n_{\text{Comp}(A-\text{Mod})}(K_0, D), \forall D \in \text{Ob}(\text{Comp}(A - \text{Mod}))$

On the other hand, by doing the same thing for maps of complexes, we get the result.

**Lemma 3.3**

Let $C$ be a complex of $A$-modules and $C_\bullet$ projective resolution of $C$ of $n$-th kernel $\text{Ker}(d_n) = K_n$. Then the functors $\text{Tor}^n_{\text{Comp}(A-\text{Mod})}(C, -) \cong \text{Tor}^n_{\text{Comp}(A-\text{Mod})}(K_0, -)$

where $\text{Tor}^n_{\text{Comp}(A-\text{Mod})}(X, -)$ is the $n$-th left derived functor of $X \otimes -$.
Proof
The proof is the same as the one of the previous lemma.

Lemma 3.4:
Let $X$ be a complex of $(A-B)$-bimodules. Then the functors

\[ \text{Hom}^*(X, -) : \text{Comp}(A-\text{Mod}) \to \text{Comp}(B-\text{Mod}) \]

and

\[ X \otimes - : \text{Comp}(B-\text{Mod}) \to \text{Comp}(A-\text{Mod}) \]

are adjoint functors.

Proof
see [Beck, V. (2008), p 180 ]

Theorem 3.5
Let $X$ be a complex of $(A-B)$-bimodules. Then the functors

\[ \text{Ext}^n_{\text{Comp}(A-\text{Mod})}(X, -) : \text{Comp}(A-\text{Mod}) \to \text{Comp}(B-\text{Mod}) \]

and

\[ \text{Tor}^n_{\text{Comp}(B-\text{Mod})}(X, -) : \text{Comp}(B-\text{Mod}) \to \text{Comp}(A-\text{Mod}) \]

are adjoint functors.

Proof
For $n = 0$, we have on the one hand $\text{Ext}^0_{\text{Comp}(A-\text{Mod})}(X, -) \cong \text{Hom}^*(X, -)$ and on the other hand

\[ \text{Tor}^0_{\text{Comp}(B-\text{Mod})}(X, -) \cong X \otimes - . \]

And according to lemma 3.4 $\text{Hom}^*(X, -)$ and $X \otimes -$ are adjoint functors. Therefore $\text{Ext}^0_{\text{Comp}(A-\text{Mod})}(X, -)$ and $\text{Tor}^0_{\text{Comp}(B-\text{Mod})}(X, -)$ are actually adjoint functors.

Suppose now by induction that the relation is verified for all $k < n$ and show that it is verified for $k = n$. That is $\text{Ext}^n_{\text{Comp}(A-\text{Mod})}(X, -)$ and $\text{Tor}^n_{\text{Comp}(B-\text{Mod})}(X, -)$ are adjoint functors.

According to lemma 3.2 $\text{Ext}^n_{\text{Comp}(A-\text{Mod})}(C, -) \cong \text{Ext}^{n-1}_{\text{Comp}(A-\text{Mod})}(K_0, -)$ and according to lemma 3.3 $\text{Tor}^n_{\text{Comp}(B-\text{Mod})}(C, -) \cong \text{Tor}^{n-1}_{\text{Comp}(B-\text{Mod})}(K_0, -)$. By hypothesis $\text{Ext}^{n-1}_{\text{Comp}(A-\text{Mod})}(K_0, -)$ and $\text{Tor}^{n-1}_{\text{Comp}(B-\text{Mod})}(K_0, -)$ are adjoint functors then $\text{Ext}^n_{\text{Comp}(A-\text{Mod})}(X, -)$ and $\text{Tor}^n_{\text{Comp}(B-\text{Mod})}(X, -)$ are adjoint functors.

Theorem 3.6
Let $B$ be a ring, $A$ a sub-ring of $B$, $S$ a saturated multiplicative subset of $A$ and $B$ satisfying the left and right Ore conditions and $X$ a complex of $(A-B)$-bimodules. Then the functors $\text{Ext}^n(S^{-1}A^{-1}X, -) : \text{Comp}(A-\text{Mod}) \to \text{Comp}(S^{-1}B-\text{Mod})$ and $\text{Tor}^n(S^{-1}B^{-1}X, -) : \text{Comp}(S^{-1}B-\text{Mod}) \to \text{Comp}(S^{-1}A-\text{Mod})$ are adjoint functors.

Proof
Since $X$ is a complex of $(A-B)$ bimodules then $S^{-1}X$ is a complex of $(S^{-1}A - S^{-1}B)$ bimodules. Then according to theorem 3.5 the functors $\text{Ext}^n(S^{-1}A^{-1}X, -) : \text{Comp}(S^{-1}A^{-1}X, -) : \text{Comp}(S^{-1}A^{-1}X, -) \to \text{Comp}(S^{-1}B^{-1}X, -) \to \text{Comp}(S^{-1}A^{-1}X, -)$ and $\text{Tor}^n(S^{-1}B^{-1}X, -) : \text{Comp}(S^{-1}B^{-1}X, -) \to \text{Comp}(S^{-1}A^{-1}X, -)$ are adjoint functors.

4. Isomorphisms and localization in $\text{Comp}(A-\text{Mod})$

Definition 4.1
Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, $F$ and $G$ two functors with same variance from $\mathcal{C}$ to $\mathcal{D}$. A natural transformation or functorial morphism from $F$ to $G$ is a map $\Phi : F \to G$ so that:

- If $F$ and $G$ are covariant, then

\[
\Phi : \text{Ob}(\mathcal{C}) \to \text{Mor}(\mathcal{D})
\]

\[
M \mapsto \Phi_M
\]

is a map such that $\Phi_M : F(M) \to G(M)$ and for any $f \in \text{Mor}(\mathcal{C})$ so that $f : M \to N$, then the following diagram is commutative:

\[
\begin{array}{ccc}
F(M) & \xrightarrow{F(f)} & F(N) \\
\Phi_M & \downarrow & \Phi_N \\
G(M) & \xrightarrow{G(f)} & G(N)
\end{array}
\]
If $F$ and $G$ are contravariant then the following diagram is commutative:

\[
\begin{array}{ccc}
F(N) & \xrightarrow{F(f)} & F(M) \\
\Phi_N & \Downarrow & \Phi_M \\
G(N) & \xrightarrow{G(f)} & G(M)
\end{array}
\]

If $\Phi_M$ is an isomorphism for all $M$ then $\Phi$ is called functorial isomorphism.

**Definition 4.2**

1. We say that a complex of left $A$-modules $C$ is bounded if for $|n|$ large, $C^n = 0$.
2. We say that a complex of left $A$-modules $C$ is of finite type if $C$ is bounded and for all $n \in \mathbb{Z}$, $C^n$ is of finite type.
3. We say that a complex of left $A$-modules $C$ is of type $FP_\infty$ if it has a projective resolution:

\[
\cdots \to P_n \xrightarrow{d_n} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} C \to 0
\]

with $P_n$ is a finite type complex of left $A$-modules for all $n \geq 0$.

**Lemma 4.3**

Let $C$ be a complex of $A$-modules and $C_\bullet$ a projective resolution of $C$ of $n$th kernel $\text{Ker}(d_n) = K_n$. Then the functors $\text{EXT}_{\text{Comp}^{(S^{-1}A-\text{Mod})}}^n(S_C^1(C),S_C^1(1))$ and $\text{EXT}_{\text{Comp}^{(S^{-1}A-\text{Mod})}}^n(S_C^1(K_0),S_C^1(1))$ are isomorphic where $\text{EXT}_{\text{Comp}^{(S^{-1}A-\text{Mod})}}^n(S_C^1(X),S_C^1(1))$ is the $n$th right derived functor of $\text{HOM}^n(S_C^1(X),S_C^1(1))$.

**Proof**

As the one of **Lemma 3.2**

**Lemma 4.4**

Let $C$ be a complex of $A$-modules and $C_\bullet$ a projective resolution of $C$ of $n$th kernel $\text{Ker}(d_n) = K_n$. Then

\[
\text{Tor}_{n+1}^{\text{Comp}^{(S^{-1}A-\text{Mod})}}(S_C^1(C),S_C^1(1)) \cong \text{Tor}_n^{\text{Comp}^{(S^{-1}A-\text{Mod})}}(S_C^1(K_0),S_C^1(1))
\]

where $\text{Tor}_n^{\text{Comp}^{(S^{-1}A-\text{Mod})}}(S_C^1(X),S_C^1(1))$ is the $n$th left derived functor of $S_C^{-1}(X) \otimes S_C^{-1}(1)$.

**Proof** As the one of **Lemma 3.2**

**Theorem 4.5**

Let $B$ be a ring, $A$ a sub-ring of $B$, $S$ a saturated multiplicative subset of $A$ and $B$ verifying the left Ore conditions and $X$ a complex of $A - B$ bimodules.

Let be the functors $S_C^{-1}(X \otimes -) : \text{Comp}(B - \text{Mod}) \to \text{Comp}(S^{-1}A - \text{Mod})$ and $S_C^{-1}(X) \otimes S_C^{-1}(-) : \text{Comp}(B - \text{Mod}) \to \text{Comp}(S^{-1}A - \text{Mod})$ such that:

1. for all complex of left $B$-modules $Y$ we have:
   (a) $S_C^{-1}(X \otimes Y) = S_C^{-1}(X \otimes -)(Y)$
   (b) $S_C^{-1}(X) \otimes S_C^{-1}(Y) = S_C^{-1}(X) \otimes S_C^{-1}(Y)$

2. for all map of complexes $f : Y_1 \to Y_2$ we have:
   (a) $S_C^{-1}(X \otimes f) : S_C^{-1}(X \otimes Y_1) \to S_C^{-1}(X \otimes Y_2)$
   (b) $S_C^{-1}(X) \otimes S_C^{-1}(f) : S_C^{-1}(X) \otimes S_C^{-1}(Y_1) \to S_C^{-1}(X) \otimes S_C^{-1}(Y_1)$

Then $S_C^{-1}(X \otimes -)$ and $S_C^{-1}(X) \otimes S_C^{-1}(-)$ are isomorphic.
Proof

we know, according to the proof of theorem 6 in [Dembele, B., Maaouia, B., F., & Sanghare, M. (2020)], that for all complex of left \(A\) modules \(Y\) there exist an isomorphism \(\Phi_Y : S^1_c(X \otimes Y) \rightarrow S^1_c(X) \otimes S^1_c(Y)\) such that:

\[
\Phi^m_Y : S^1_c(X \otimes f^m) \rightarrow S^1_c(X) \otimes S^1_c(Y) \qquad \sum_{x \otimes p_{m-t}} S^1_c(X \otimes Y)
\]

Now it remained to prove, for all map of complexes \(f : Y_1 \rightarrow Y_2\), the commutativity of the following diagram:

\[
\begin{array}{ccc}
S^1_c(X \otimes f) & \rightarrow & S^1_c(Y) \\
\Phi_{Y_1} & & \Phi_{Y_2} \\
S^1_c(X) \otimes S^1_c(Y_1) & \rightarrow & S^1_c(Y) \\
\end{array}
\]

That is for all integer \(m\) the following diagram is commutative:

\[
\begin{array}{ccc}
S^1_c(X \otimes f^m) & \rightarrow & S^1_c(Y) \\
\Phi_{Y_1}^m & & \Phi_{Y_2}^m \\
S^1_c(X) \otimes S^1_c(Y_1) & \rightarrow & S^1_c(Y) \\
\end{array}
\]

So let \(\sum_{x \otimes p_{m-t}} \in S^1_c(X \otimes (Y_1)^{m-t})\). We have on one hand:

\[
\Phi_{Y_2}^m \circ S^1_c(X \otimes f^m) \left( \sum_{x \otimes p_{m-t}} \right) = \Phi_{Y_2}^m \left( \sum_{x \otimes p_{m-t}} \right) = \sum_{x \otimes \frac{p_{m-t}}{s}} \frac{f^m(p_{m-t})}{s}
\]

And on the other hand we have:

\[
(S^1_c(X) \otimes S^1_c(f)^m) \circ \Phi_{Y_1}^m \left( \sum_{x \otimes p_{m-t}} \right) = (S^1_c(X) \otimes S^1_c(f)^m) \left( \sum_{x \otimes \frac{p_{m-t}}{s}} \right) = \sum_{x \otimes \frac{f^m(p_{m-t})}{s}}
\]

Theorem 4.6

Let \(B\) be a ring, \(A\) a sub-ring of \(B\), \(S\) a satrured multiplicative subset of \(A\) and \(B\) verifying the left Ore conditions and \(X\) a complex of \((A, B)\) bimodules of finite type.

Let be the functors \(S^1_c Hom^*(X, -) : Cmp(A - Mod) \rightarrow Comp(S^1 B - Mod)\) and \(Hom^*(S^1_c(X), S^1_c(Y)) : Cmp(A - Mod) \rightarrow Comp(S^1 B - Mod)\) such that:

1. for all complex of left \(A\)-modules \(Y\) we have:
   (a) \(S^1_c Hom^*(X, -)(Y) = S^1_c Hom^*(X, Y)\)
   (b) \(Hom^*(S^1_c(X), S^1_c(Y)) = Hom^*(S^1_c(X), S^1_c(Y))\)

2. for all map of complexes \(f : Y_1 \rightarrow Y_2\) we have:
   (a) \(S^1_c Hom^*(X, f) : S^1_c Hom^*(X, Y_1) \rightarrow S^1_c Hom^*(X, Y_2)\)
   (b) \(Hom^*(S^1_c(X), S^1_c(f)) : Hom^*(S^1_c(X), S^1_c(Y_1)) \rightarrow Hom^*(S^1_c(X), S^1_c(Y_2))\)

Then \(S^1_c Hom^*(X, -)\) and \(Hom^*(S^1_c(X), S^1_c(-))\) are isomorphic.

Proof

we know that according to the proof of theorem 7 in [Dembele, B., Maaouia, B., F., & Sanghare, M. (2020)] that for all complex of left \(A\) modules \(Y\) there exist an isomorphism \(\Phi_{X,Y} : S^1_c Hom^*(X, Y) \rightarrow Hom^*(S^1_c(X), S^1_c(Y))\) such that:

\[
\Phi_{X,Y} \left( \frac{g_t}{\sigma} \right) = \frac{1}{s} \cdot \frac{g(p)}{\sigma}
\]
Now let \( f : Y_1 \rightarrow Y_2 \) be a map of complexes, let us show the commutativity of the following diagram:

\[
\begin{array}{ccc}
S_c^{-1}\text{Hom}^*(X, Y_1) & \xrightarrow{\phi_{(X,Y_1)}} & S_c^{-1}\text{Hom}^*(X, Y_2) \\
\downarrow{\phi_{(X,Y_1)}} & & \downarrow{\phi_{(X,Y_2)}} \\
\text{Hom}^*(S_c^{-1}(X), S_c^{-1}(Y_1)) & \xrightarrow{\text{Hom}^*(\phi_{(X,Y_1)})} & \text{Hom}^*(S_c^{-1}(X), S_c^{-1}(Y_2))
\end{array}
\]

That is for all integers \( m \) and \( t \) the following diagram commutative:

\[
\begin{array}{ccc}
S^{-1}\text{Hom}(X', (Y_1)^{m+t}) & \xrightarrow{\phi_{(X',Y_1)^{m+t}}} & S^{-1}\text{Hom}(X', (Y_2)^{m+t}) \\
\downarrow{\phi_{(X',Y_1)^{m+t}}} & & \downarrow{\phi_{(X',Y_2)^{m+t}}} \\
\text{Hom}(S^{-1}X', S^{-1}(Y_1)^{m+t}) & \xrightarrow{\text{Hom}(\phi_{(X',Y_1)^{m+t}})} & \text{Hom}(S^{-1}X', S^{-1}(Y_2)^{m+t})
\end{array}
\]

So let \( \frac{g_t}{s} \in S^{-1}\text{Hom}(X', (Y_1)^{m+t}) \). At first we have

\[
\phi_{(X',Y_1)^{m+t}} \circ S^{-1}\text{Hom}(X', f^{m+t}) \left( \frac{g_t}{s} \right) \left( \frac{P}{S} \right) = \phi_{(X',Y_2)^{m+t}} \left( \frac{f^{m+t} \circ g_t}{s} \right) \left( \frac{P}{S} \right) = \frac{1}{s} \left( \frac{f^{m+t} \circ g_t}{s} \right) \left( \frac{P}{S} \right)
\]

And secondly:

\[
\text{Hom}(S^{-1}X', S^{-1}f^{m+t}) \circ \phi_{(X',Y_2)^{m+t}} \left( \frac{g_t}{s} \right) \left( \frac{P}{S} \right) = S^{-1} \left( f^{m+t} \circ \phi_{(X',Y_1)^{m+t}} \left( \frac{g_t}{s} \right) \right) \left( \frac{P}{S} \right) = \frac{1}{s} \left( \frac{f^{m+t} \circ g_t}{s} \right) \left( \frac{P}{S} \right)
\]

**Theorem 4.7**

Let \( B \) be a ring, \( A \) a sub-ring of \( B \), \( S \) a saturated multiplicative subset of \( A \) and \( B \) verifying the left Ore conditions and \( X \) a complex of \( (A - B) \) bimodules of type \( FP_\infty \).

Then the functors \( S_c^{-1}\text{EXT}^0(X, -) : \text{Comp}(A - \text{Mod}) \rightarrow \text{Comp}(S_c^{-1}B - \text{Mod}) \) and \( \text{EXT}^0_c(S_c^{-1}(X), S_c^{-1}(1)) : \text{Comp}(A - \text{Mod}) \rightarrow \text{Comp}(S_c^{-1}B - \text{Mod}) \) are isomorphic.

**Proof**

Let us show it by induction on \( n \).

On one part we have:

\[
\text{Hom}^*(X, -) \cong \text{EXT}^0(X, -)
\]

and so

\[
S_c^{-1}\text{Hom}^*(X, -) \cong S_c^{-1}\text{EXT}^0(X, -)
\]

and other part we have:

\[
\text{Hom}^*(S_c^{-1}(X), S_c^{-1}(1)) \cong \text{EXT}^0(S_c^{-1}(X), S_c^{-1}(1))
\]

According to **Theorem 4.6** \( S_c^{-1}\text{Hom}^*(X, -) \cong \text{Hom}^*(S_c^{-1}(X), S_c^{-1}(1)) \) and then \( S_c^{-1}\text{EXT}^0(X, -) \cong \text{EXT}^0(S_c^{-1}(X), S_c^{-1}(1)) \).

That show us that the relation is true for \( k = 0 \).

Assume that it is true for all \( k < n \) and show that it is true for \( n \).

According to **Lemma 3.2** we have:

\[
\text{EXT}^n_{\text{Comp}(A - \text{Mod})}(C, -) \cong \text{EXT}^n_{\text{Comp}(A - \text{Mod})}(K_0, -)
\]

and so

\[
S_c^{-1}\text{EXT}^n_{\text{Comp}(A - \text{Mod})}(C, -) \cong S_c^{-1}\text{EXT}^n_{\text{Comp}(A - \text{Mod})}(K_0, -)
\]

And according to **Lemma 4.3** we have:

\[
\text{EXT}^n(S_c^{-1}(X), S_c^{-1}(1)) \cong \text{EXT}^n(S_c^{-1}(K_0), S_c^{-1}(1))
\]

By hypothesis we have:

\[
S_c^{-1}\text{EXT}^n_{\text{Comp}(A - \text{Mod})}(K_0, -) \cong \text{EXT}^n(S_c^{-1}(K_0), S_c^{-1}(1))
\]
Thus $S^{-1}_{C} \mapsto \text{Ext}_{\text{Comp}(A - \text{Mod})}^{n}(C, -) \cong \text{Ext}_{\text{Comp}(A - \text{Mod})}^{n}(S^{-1}_{C}(X), S^{-1}_{C}())$.

**Theorem 4.8**

Let $B$ be a ring, $A$ a sub-ring of $B$, $S$ a saturated multiplicative subset of $A$ and $X$ a complex of $(A - B)$ bimodules. Then the functors $S^{-1}_{C} \text{Tor}_{n}^{\text{Comp}(A - \text{Mod})}(X, -) : \text{Comp}(B - \text{Mod}) \rightarrow \text{Comp}(S^{-1}A - \text{Mod})$ and $\text{Tor}_{n}^{\text{Comp}(S^{-1}A - \text{Mod})}(S^{-1}_{C}(X), S^{-1}_{C}()) : \text{Comp}(B - \text{Mod}) \rightarrow \text{Comp}(S^{-1}A - \text{Mod})$ are isomorphic.

**Proof**

Let us show it by induction on $n$.

On one part: $X \bigotimes S^{-1}_{C}(X, -) \cong \text{Tor}_{0}^{\text{Comp}(A - \text{Mod})}(X, -)$ and so $S^{-1}_{C}(X) \bigotimes S^{-1}_{C}() \cong \text{Tor}_{0}^{\text{Comp}(S^{-1}A - \text{Mod})}(S^{-1}_{C}(X), S^{-1}_{C}())$.

According to **Theorem 4.5** $S^{-1}_{C}(X \bigotimes -) \cong S^{-1}_{C}(X) \bigotimes S^{-1}_{C}()$ and so $S^{-1}_{C} \text{Tor}_{0}^{\text{Comp}(A - \text{Mod})}(X, -) \cong \text{Tor}_{0}^{\text{Comp}(S^{-1}A - \text{Mod})}(S^{-1}_{C}(X), S^{-1}_{C}())$ and the relation is true for $k = 0$.

Suppose that the relation is true for all $k < n$ and prove that it is true for $n$.

According to **Lemma 3.3** we have: $\text{Tor}_{n}^{\text{Comp}(A - \text{Mod})}(X, -) \cong \text{Tor}_{n-1}^{\text{Comp}(A - \text{Mod})}(K_{0}, -)$

then $S^{-1}_{C} \text{Tor}_{n}^{\text{Comp}(A - \text{Mod})}(X, -) \cong S^{-1}_{C} \text{Tor}_{n-1}^{\text{Comp}(S^{-1}A - \text{Mod})}(K_{0}, -)$

We have also according to **Lemma 4.4** $\text{Tor}_{n}^{\text{Comp}(S^{-1}A - \text{Mod})}(S^{-1}_{C}(X), S^{-1}_{C}()) \cong \text{Tor}_{n-1}^{\text{Comp}(S^{-1}A - \text{Mod})}(S^{-1}_{C}(K_{0}), S^{-1}_{C}())$

By hypothesis we have:

$S^{-1}_{C} \text{Tor}_{n-1}^{\text{Comp}(S^{-1}A - \text{Mod})}(K_{0}, -) \cong \text{Tor}_{n-1}^{\text{Comp}(S^{-1}A - \text{Mod})}(S^{-1}_{C}(K_{0}), S^{-1}_{C}())$

Thus $S^{-1}_{C} \text{Tor}_{n}^{\text{Comp}(A - \text{Mod})}(X, -) \cong \text{Tor}_{n}^{\text{Comp}(S^{-1}A - \text{Mod})}(S^{-1}_{C}(X), S^{-1}_{C}())$.

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