Density matrix of radiation of black hole with fluctuating horizon

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Abstract

The density matrix of Hawking radiation is calculated in the model of black hole with fluctuating horizon. Quantum fluctuations smear the classical horizon of black hole and modify the density matrix of radiation producing the off-diagonal elements. The off-diagonal elements may store information of correlations between radiation and black hole. The smeared density matrix was constructed by convolution of the density matrix calculated with the instantaneous horizon with the Gaussian distribution over the instantaneous horizons. The distribution has the extremum at the classical radius of the black hole and the width of order of the Planck length. Calculations were performed in the model of black hole formed by the thin collapsing shell which follows a trajectory which is a solution of the matching equations connecting the interior and exterior geometries.

1 Introduction.

From the time of Hawking’s discovery that black holes radiate with the black-body radiation, the problem of information stored in a black hole [1] attracted much attention. Different ideas were discussed, in particular those of remnants [2, 3, 4], ”fuzziness” of the black hole [5, 6] and refs. therein, quantum hair [7, 8, 9] and refs.therein., and smearing of horizon by quantum fluctuations [10, 11, 12, 13]. The underlying idea of the last approach is that small fluctuations of the background geometry lead to corrections to the form of the density matrix of radiation. These corrections are supposed to account for correlations between the black hole and radiation and contain the imprint of information thrown into the black hole with the collapsing matter.

The idea that horizon of the black hole is not located at the rigid position naturally follows from the observation that a black hole as a quantum object is described by the wave functional over geometries [14, 15, 16]. In particular, the sum over horizon areas yields the black hole entropy.

In papers [12, 13] the density matrix of black hole radiation was calculated in a model with fluctuating horizon. Horizon fluctuations modify the Hawking density matrix producing off-diagonal elements. Horizon fluctuations were taken into account by convolution the density matrix calculated with the instantaneous horizon radius $\bar{R}$ with the black hole wave function which was taken in the Gaussian form $\psi(R) = N^{-1/2} e^{-(R-2MG)^2/2\sigma^2}$. Effectively the wave function introduces the smearing of the classical horizon radius $\bar{R} = 2MG$. The width of the distribution, $\sigma$, was taken of order the Plank lengths $l_p$ [10, 12, 13]. In paper [10] it was stated that the ”horizon fluctuations do not invalidate the semiclassical derivation of the Hawking effect until the black hole mass approaches the Planck mass”.

In this note we reconsider calculation the density matrix of radiation emitted from the black hole formed by the collapsing shell. The shell is supposed to follow the infalling trajectory which is the exact solution to the matching equations connecting the interior (Minkowski) and exterior

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(Schwarzschild) geometries of the space-time [17, 18]. In this setting one can trace propagation of a ray (we consider only s-modes) through the shell from the past to the future infinity. For the rays propagating in the vicinity of the horizon we obtain an exact formula connecting $v_{in}$ at the past infinity and $u_{out}$ at the future infinity.

We obtain the expression for the "smeared" density matrix of Hawking radiation of the black hole with the horizon smeared by fluctuations. In the limit $\sigma/MG \to 0$ the smeared density matrix turns to the Hawking density matrix. The smeared density matrix is not diagonal and can be expressed as a sum of the "classical part" and off-diagonal correction which is roughly of order $O(\sigma/MG)$ of the classical part. As a function of frequencies $\omega_{1,2}$ of emitted quanta the distribution is concentrated around $\omega_1/\omega_2 = 1$ with the width of order $(\sigma/MG)\ln^{1/2}(MG/\sigma)$.

The paper is constituted as follows. In Sect. 2 we review the geometry of the thin collapsing shell which follows a trajectory consisting of two phases. The trajectory is a solution of the matching equations connecting the internal and external geometries of the shell. We trace propagation of a light ray from the past to future infinity. In Sect.3 we introduce the wave function of the shell which saturates the uncertainty relations. In Sect.4, we calculate the density matrix of black hole radiation smeared by horizon fluctuations. Following the approach of paper [20] calculation is performed by two methods: by the " $i\varepsilon$" prescription and by using the normal-ordered two-point function. In Sect.5, using the exact expressions for the smeared radiation density matrix, we study the diagonal "classical" part of the density matrix and the off-diagonal elements.

## 2 Geometry of the thin collapsing shell

In this section we introduce notations and review the geometry of space with collapsing thin spherical shell [17, 18]. Outside of the shell the exterior geometry is Schwarzschild space-time, the interior geometry is Minkowsky space-time. In the Eddington-Finkelstein coordinates the metric of the exterior space-time is

$$ds^2_{(ext)} = -(1 - R/r) dv^2 + 2dvdr + r^2d\Omega^2, \quad r > R$$

where

$$v = t + x(r), \quad u = t - x(r).$$

$$x(r) = r + R \ln \left( \frac{r}{R} - 1 \right).$$

and

$$v - u = 2x(r).$$

The metric of the interior space-time is

$$ds^2_{(int)} = -dV^2 + 2dVdr + r^2d\Omega^2,$$

where

$$V = T + r, \quad U = T - r.$$ The light rays propagate along the cones $v, u = const$ in the exterior and along $V, U = const$ in the interior regions.

Trajectory of the shell is $r = R_s(\tau)$, where $\tau$ is proper time on the shell. The matching conditions of geometries on the shell, at $r = R_s$, are

$$dV - dU = 2dR_s, \quad dv - du = \frac{2dR_s}{1 - R/R_s}, \quad dUdV = (1 - R/R_s)dudv,$$

(3)
Figure 1: Penrose diagram for collapsing shell. For \( u < 0 \) the shell is in the phase I, for \( u > 0 \) in the phase II. \( v_H \) is the point of horizon formation.

where the differentials are taken along the trajectory. From the matching conditions follow the equations

\[
2R_s'(1 - U') = U'^2 - (1 - R/R_s), \tag{4}
\]
\[
2\dot{R}_s(1 - \dot{V}) = -\dot{V}^2 + (1 - R/R_s). \tag{5}
\]

Here prime and dot denote derivatives over \( u \) and \( v \) along the trajectory.

The trajectory of the shell consists of two phases \[18\]

I. \( u < 0 \) : \( R_s(u) = R_0 = \text{const.} \]

II. \( u > 0 \) : \( v = \text{const.}, V = \text{const.} \]

From the equations (4), (5) are obtained the following expressions for the trajectory;

In the phase I

\[
U(u) = L_0 u - 2R_0 + 2R, \quad V(v) = L_0(v - 2x(R_0)) + 2R, \tag{6}
\]

where \( L_0 = (1 - R/R_0)^{1/2} \).
In the phase II

\[ V = 2R, \quad U = 2R - 2R_s, \quad v = 2x(R_0), \quad u = 2x(R_0) - 2x(R_s). \]  

(7)

Horizon is formed at \( U_H = 0, \quad u \to \infty \) and \( V_H = 2R, \quad v_H = 2x(R_0) \).

We consider the modes propagating backwards in time. At \( I^- \) the ray is in phase I, after crossing the shell it reaches \( I^+ \) in the phase II. Let the in-falling ray be at \( I^- \) at \( v_1 < v_0 \), where \( v_0 = 2x(R_0) - 2RL_0^{-1} \) is the point at which \( V(v_0) = 0 \). Between the points 1-2 the ray propagates outside the shell in the phase I with \( \nu_1 < v_0 \). At the point 2 the ray crosses the shell and we have \( v_2 = v_1 \) and \( V_2(v_1) = L_0(v_1 - 2x(R_0)) + 2R = L_0(v_1 - v_0) \). The ray propagates in the interior of the shell, and at the point 3, at \( r = 0 \), we have \( V_3 = V_2 \). Reflection condition at the point 3 is \( V_3 = U_3 \). At the crossing point 4 we have \( U_4 = U_3 \), where

\[ U_4 = -2R_s(4) + 2R, \quad u_4 = -2x(R_s(4)) + 2x(R_0). \]

Here \( R_s(4) \) stands for the radial position of the shell trajectory at the point 4. The equation for \( u_4 \) can be written as

\[ \frac{u_4}{2R} = \frac{(R_0 - R) - (R_s - R)}{R} + \ln \frac{R_0 - R}{R_s - R}. \]

In the region \( R_s(4) \sim R \), where \( U_4 \ll R \), neglecting in the first term \( R_s - R \) as compared with \( R_0 - R \), we obtain the approximate equation for \( u_4 \)

\[ \frac{u_4}{2R} = \frac{R_0 - R}{R} + \ln \frac{-U_4/2}{R_0 - R}. \]

(8)

Thus, we have

\[ v_1 - v_0 = L_0^{-1}V_2 = L_0^{-1}V_3 = L_0^{-1}U_4(u_4) = -L_0^{-1}2(R_0 - R)e^{-(u_4 - 2R_0)/R - 1}. \]  

(9)

Removing the indices, we obtain our final result as

\[ v = v_0 - 2(eL_0)^{-1}(R_0 - R)e^{-(u - 2R_0)/R}. \]  

(10)

The above formulas are purely classical, modifications due to back reaction of Hawking radiation are neglected.

3 Quantum black hole

Quantum nature of horizons of the black holes was discussed in the work of Carlip and Teitelboim [14], where it was shown that the area of horizon \( A \) and the opening angle, \( \Theta \) or, equivalently, the deficit angle \( 2\pi - \Theta \) form the canonical pair. In paper [13] it was shown that canonical pair is formed by the opening angle and the Wald entropy \( S_W \) [19]

\[ \left\{ \Theta, \frac{S_W}{2\pi} \right\} = 1. \]  

(11)

When the black hole is quantized, the Poisson bracket is promoted to the commutation relation

\[ [\hat{\Theta}, \hat{S_W}] = i\hbar. \]  

(12)
The wave function of the black hole satisfies the relation
\[-i \frac{\partial \Psi}{\partial S_W} = 2\pi \hbar \Theta \Psi.\] (13)

The minimal uncertainty $\Delta S_W \Delta \Theta = \hbar/2$ wave function is
\[\Psi(\Theta) \sim e^{C(\Theta-2\pi)^2/\hbar} < S_W > \Theta,\] (14)
where $C \sim < S_W >$. For the spherically symmetric configurations which we consider the wave function written through the instantaneous horizon radius $R$ is
\[|\Psi(R)| = N^{-1} e^{-\frac{(R-\bar{R})^2}{4\sigma^2}}.\] (15)

The scale of horizon fluctuations is $\sigma \sim l_p$ [10], where $l_p^2 = \hbar G$ is the Planck length and $\bar{R} = 2MG$ is the classical horizon radius of the black hole of the mass $M$. The normalization factor $N$ is
\[N^{-2} = \int_0^\infty 4\pi dRR^2 e^{-\frac{(R-\bar{R})^2}{2\sigma^2}} \simeq \sigma \bar{R}^2.\] (16)

4 Hawking radiation from the black hole formed by the shell

Let us turn the calculation of Hawking radiation of the massless real scalar field in the background of the black hole formed by the shell. To perform quantization of the field, we restrict ourselves to the $s$-wave modes. Expanding the scalar field in the orthonormal set of solutions $u_i^-$ of the Klein-Gordon equation which at the past null infinity $I^-$ have only positive frequency modes we have
\[\varphi = \sum_i (a_i u_i^- + a_i^+ u_i^{-(+)}).\] (17)

The scalar product of the fields is
\[\langle \varphi_1, \varphi_2 \rangle = i \int_\Sigma d\Sigma^\mu \varphi_2^* \partial_\mu \varphi_1.\] (18)

Alternatively the field $\varphi$ can be expanded at the hypersurface $\Sigma^+ = I^+ \oplus H^+$ where $I^+$ is the future null infinity and $H^+$ is the event horizon
\[\varphi = \sum_i (b_i u_i^{(+)} + b_i^+ u_i^{(+)*)} + c_i q_i + c_i^+ q_i^*).\] (19)

Here $\{u_i^{(+)}\}$ is the orthonormal set of modes which contain at the $I^+$ only positive frequencies and $\{q_i\}$ is the orthonormal set of solutions of the wave equation which contains no outgoing components [1]. The operators $a_i, a_i^+$ and $b_i, b_i^+$ are quantized with respect to the vacua $|in >$ and $|out >$ correspondingly.

The modes $u_i^{(+)}$ can be expanded in terms of the modes $u_i^{(-)}$
\[u_i^{(+)} = \sum_j (\alpha_{ij} u_j^{(-)} + \beta_{ij} u_j^{(-)*}),\] (20)
where $\alpha_{ij}$ and $\beta_{ij}$ are given by the scalar products
\[\alpha_{ij} = (u_i^{(+)}, u_j^{(-)*}), \quad \beta_{ij} = -(u_i^{(+)}, u_j^{(-)}).\]
For the spherically-symmetric collapse, the basis for the in- and outgoing modes is
\[ u_{\omega lm}^{(-)} \big| I^- \sim \frac{1}{\sqrt{4\pi \omega}} e^{-i\omega u} r Y_{lm}(\theta, \varphi), \quad u_{\omega lm}^{(-)} \big| I^+ \sim \frac{1}{\sqrt{4\pi \omega}} e^{-i\omega u} Y_{lm}(\theta, \varphi). \]
Omitting the angular parts, the modes \( u_\omega^{(-)} \) and \( u_\omega^{(+)} \) are
\[ u_\omega^{(-)}(v) \big| I^- \sim \frac{1}{\sqrt{4\pi \omega}} e^{-i\omega v}, \quad u_\omega^{(+)}(u) \big| I^+ \sim \frac{1}{\sqrt{4\pi \omega}} e^{-i\omega u}. \] (21)
From (10) we find
\[ u(v) = 2R_0 + 2R \left[ -\ln(eL_0) + \ln \frac{R_0 - R}{R} - \ln \frac{v_0 - v}{2R} \right] = F(R) - 2R \ln \frac{R_0 - R}{R}, \]
where \( F(R) = 2R_0 + 2R[C + \ln((R_0 - R)/R)], \ C = -\ln(eL_0). \) To simplify formulas, we consider the case \( R_0 \gg R, \) so \( \ln((R_0 - R)/R) \simeq \ln R_0/R, \) and
\[ F(R) \simeq 2R_0 + 2R[C + \ln(R_0/R)]. \]
Note that both \( v_0 \) and \( u(v) \) have explicit dependence on \( R. \)

The Bogolubov coefficient
\[ \beta_{\omega_1 \omega_2} = i \int dv u_\omega^{(+)}(v) \frac{\partial}{\partial v} u_\omega^{(-)}(v) = i \int \frac{dv}{\sqrt{\omega_1 \omega_2}} e^{-i\omega_1 u(v)} \frac{\partial}{\partial v} e^{-i\omega_2 v} \]
smeread by horizon fluctuaneons is obtained by convoluting it with the function \( |\Psi^2| \) \[ \bar{\beta}_{\omega_1 \omega_2} = \int_0^\infty dR R^2 e^{-(R-R)^2/2\sigma^2} N^2 \beta_{\omega_1 \omega_2} \sim \]
\[ \sim \left( \omega_2/\omega_1 \right)^{1/2} \int_{-\infty}^{v_0} dv e^{-i\omega_1 F(R) - 2R \ln((v_0 - v)/2R) - i\omega_2 v} dR N^2 R^2 e^{-(R-R)^2/2\sigma^2}. \]
Direct evaluation of the smeared Bogolubov coefficient (22) yields (cf.[20])
\[ \bar{\beta}_{\omega_1 \omega_2} \sim \int dR R^2 N^2 e^{-(R-R)^2/2\sigma^2} R \left( \frac{e^{-i\omega_1 F(R)} + i\omega_1 v_0 \Gamma(2Ri\omega_1)}{(-2Ri\omega_1 + \epsilon)^{2Ri\omega_1}} \right). \] (23)

### 4.1 Method 1

Following the paper [20] we consider two ways of calculating the density matrix
\[ \rho_{\omega_1 \omega_2} = i \int_{I^-} dv u_\omega^{(+)}(v) \frac{\partial}{\partial v} u_\omega^{(+)*}(v) = i \int_{I^-} dv \int d\omega' d\omega'' \beta_{\omega_1 \omega'} \bar{\beta}_{\omega_2 \omega''} u_\omega^{(-)}(v) \frac{\partial}{\partial v} u_\omega^{(-)*}(v) \] (24)
where in the last equality we used that the modes \( \{ u_\omega^{(-)} \} \) form the orthonormal set of functions on \( I^- \). The density matrix smeared by horizon fluctuations is
\[ \bar{\rho}_{\omega_1 \omega_2} = \int d\omega' \beta_{\omega_1 \omega'} \bar{\beta}_{\omega_2 \omega'} \int_0^\infty dR_1 R_1^2 e^{-(R_1-R)^2/2\sigma^2} N^2 \int_0^\infty dR_2 R_2^2 e^{-(R_2-R)^2/2\sigma^2} N^2. \] (25)

**\footnote{Hereafter we abandon the numerical and greybody factors.}**
Substituting (23), we have

\[ \tilde{\rho}_{\omega_1 \omega_2} \sim \int_0^\infty d\omega' \frac{\sqrt{\omega_1 \omega_2}}{\omega'} R_1 R_2 (-2iR_1 \omega' + \varepsilon)^{-2iR_1 \omega_1} (2iR_2 \omega' + \varepsilon)^{2iR_2 \omega_2} \Gamma(-2iR_1 \omega_1) \Gamma(2iR_2 \omega_2) \times e^{-i\omega_1 F(R_1) + i\omega_2 F(R_2)} dR_1 dR_2 R_1^2 R_2^2 N^4 e^{-(R_1 - \bar{R})^2/2\sigma^2} e^{-(R_2 - \bar{R})^2/2\sigma^2} \]  

(26)

The terms with \( v_0 \) have cancelled. Integrating over \( \omega' \), we obtain

\[ \tilde{\rho}_{\omega_1 \omega_2} \sim \sqrt{\omega_1 \omega_2} \int dR_1 dR_2 R_1^2 R_2^2 \delta(R_1 \omega_1 - R_2 \omega_2) \left( \frac{R_1}{R_2} \right)^{-2iR_1 \omega_1} \times \frac{1}{R_1 \omega_1 \sinh(2\pi R_1 \omega_1)} e^{-i\omega_1 F(R_1) + i\omega_2 F(R_2)} N^4 e^{-(R_1 - \bar{R})^2/2\sigma^2} e^{-(R_2 - \bar{R})^2/2\sigma^2} \]

\[ = \sqrt{\omega_1 \omega_2} \int dR_1 dR_2 R_1^2 R_2^2 \frac{1}{R_1 \omega_1 \omega_2} \delta \left( R_2 - R_1 \frac{\omega_1}{\omega_2} \right) \times \left( e^{4\pi R_1 \omega_1} - 1 \right)^{-1} e^{-2iR_0(\omega_1 - \omega_2)} N^4 e^{-(R_1 - \bar{R})^2/2\sigma^2} e^{-(R_2 - \bar{R})^2/2\sigma^2}, \]

where, taking into account the \( \delta \)-function, we substituted

\[ (-2iR_1)^{-2iR_1 \omega_1} (2iR_2)^{2iR_2 \omega_2} \Rightarrow (-R_1/R_2)^{-2iR_1 \omega_1} = e^{2\pi R_1 \omega_1} (R_1/R_2)^{-2iR_1 \omega_1} \]

\[ \Gamma(-2iR_1 \omega_1) \Gamma(2iR_2 \omega_2) \Rightarrow \frac{1}{2R_1 \omega_1 \sinh(2\pi R_1 \omega_1)} \]

and

\[ e^{-i\omega_1 F(R_1) + i\omega_2 F(R_2)} \Rightarrow e^{2iR_1 \omega_1} \ln R_1 - 2iR_2 \omega_2 \ln R_2 = (R_1/R_2)^{2iR_1 \omega_1}. \]

Integration over \( R_2 \) yields

\[ \tilde{\rho}_{\omega_1 \omega_2} \sim \frac{1}{\sqrt{\omega_1 \omega_2}} \int dR_1 R_1^5 \left( \frac{\omega_1}{\omega_2} \right)^3 \left( e^{4\pi R_1 \omega_1} - 1 \right)^{-1} e^{-2iR_0(\omega_1 - \omega_2)} \times \frac{1}{R^4 \sigma^2} \exp \left\{ -\frac{1}{2\sigma^2} \left( R_1 \sqrt{1 + \omega_1^2/\omega_2^2} - \bar{R} \frac{1 + \omega_1/\omega_2}{\sqrt{1 + \omega_1^2/\omega_2^2}} \right)^2 \right\} \exp \left\{ -\frac{\bar{R}^2(\omega_1 - \omega_2)^2}{2\sigma^2(\omega_1^2 + \omega_2^2)} \right\} \]

(27)

Because \( \bar{R}/\sigma \gg 1 \) both exponents have sharp extrema. Integrating over \( R_1 \), we arrive to the density matrix of the form

\[ \tilde{\rho}_{\omega_1 \omega_2} \sim \frac{1}{\sqrt{\omega_1 \omega_2}} \left( \frac{\omega_1}{\omega_2} \right)^3 \left( 1 + \omega_1/\omega_2 \right)^5 \bar{R} \frac{\bar{R}}{\sigma(1 + \omega_1^2/\omega_2^2)^{1/2}} \left( e^{4\pi \bar{R}(\omega_1 + \omega_2)\omega_1 \omega_2/(\omega_1^2 + \omega_2^2)} - 1 \right)^{-1} \times e^{-2iR_0(\omega_1 - \omega_2)} \exp \left\{ -\frac{\bar{R}^2(\omega_1 - \omega_2)^2}{2\sigma^2(\omega_1^2 + \omega_2^2)} \right\} \]

(28)

4.2 Method 2

Alternatively, the density matrix can be presented in the following form

\[ \tilde{\rho}_{\omega \omega} = \int d\omega \int_{\Sigma} d\Sigma^{\mu}_{1} u^{(+)*}_{\omega \ell m}(x_1) \partial_{\mu} u_{\omega_1 \ell \ell_1 m}(x_1) \int_{\Sigma} d\Sigma^{\nu}_{2} u_{\omega \ell m}^{(+)}(x_2) \partial_{\nu} u_{\omega_1 \ell \ell_1 m}(x_2) \]

(30)
where for the initial value hypersurface can be taken either \( I^- \) or \( I^+ \). Expanding \( \varphi \) in the basis \( \{ u^{(-)} \} \)

\[
< \text{in}| \varphi(x_1)\varphi(x_2) |\text{in} > = \int d\omega_1 u^{(-)}(x_1) u^{(-)*} (x_2)
\]

where |\text{in} > and |\text{out} > are vacuum states at \( I^- \) and \( I^+ \), and using the relation

\[
< \text{in} | : \varphi(x_1)\varphi(x_2) : |\text{in} > = < \text{in} | \varphi(x_1)\varphi(x_2) |\text{in} > - < \text{out} | \varphi(x_1)\varphi(x_2) |\text{out} > ,
\]

we obtain

\[
\rho_{\omega_1\omega_2} = \int \Sigma_1 d\Sigma_1 \int \Sigma_2 [ u^{(+)*}(x_1) \frac{\partial}{\partial \mu} [ u^{(+)}(x_2) \frac{\partial}{\partial \nu} ] < \text{in} | : \varphi(x_1)\varphi(x_2) : |\text{in} > (31)
\]

To perform calculation of (31) one can use the expansion of the two-point function

\[
< \text{in} | : \varphi(x_1)\varphi(x_2) : |\text{in} > \text{ on } I^+ \text{ to obtain [20]}
\]

\[
\rho_{\omega_1\omega_2} \sim (\omega_1\omega_2)^{-1/2} \int_{I^+} du_1 du_2 e^{-iu_1\omega_1 + cu_2\omega_2} \left( \frac{(dv/du)(u_1)(dv/du)(u_2)}{(v(u_1) - v(u_2) - i\varepsilon)^2} - \frac{1}{(u_1 - u_2 - i\varepsilon)^2} \right) . \quad (32)
\]

where for \( v(u) \) we take the function (10). For the density matrix modified by horizon fluctuations we obtain

\[
\rho_{\omega_1\omega_2} \sim (\omega_1\omega_2)^{-1/2} \int_{I^+} du_1 \int_{I^+} du_2 e^{-iu_1\omega_1 + cu_2\omega_2} \left( \frac{e^{-(u_1-2R_0)/2R_1} e^{-(u_2-2R_0)/2R_2}}{(e^{-(u_1-2R_0)/2R_1} - e^{-(u_2-2R_0)/2R_2} - i\varepsilon)^2} \right) \quad (33)
\]

where for \( v(u) \) is taken the function (10). Extracting in the denominator the factor \( (e^{-(u_1-2R_0)/2R_1} - (u_2-2R_0)/2R_2)^2 \), shifting \( u_1 - 2R_0 \Rightarrow u_i \) and changing variables \( u_i/4R_i \Rightarrow u_i \), we obtain

\[
\tilde{\rho}_{\omega_1\omega_2} \sim (\omega_1\omega_2)^{-1/2} \int_{-\infty}^{\infty} du_1 \int_{-\infty}^{\infty} du_2 e^{-4i\omega_1 u_1 R_1 + 4i\omega_2 u_2 R_2 - 2i\varepsilon R_0 (\omega_1 - \omega_2)} \sinh^{-2}(u_1 - u_2 - i\varepsilon) \times e^{-(R_1-R)^2/2\sigma^2 -(R_2-R)^2/2\sigma^2} N^4 R_1^2 R_2^2 dR_1 dR_2 . \quad (34)
\]

Performing the contour integration over \( u_1 \) around the pole in the upper half plane using the formula

\[
\int_{-\infty}^{\infty} dy \frac{e^{-iy} y}{\sinh^2(y - z - i\varepsilon)} = 2\pi e^{-i\omega z} e^{\pi \omega} - 1 ,
\]

we have

\[
\tilde{\rho}_{\omega_1\omega_2} \sim (\omega_1\omega_2)^{-1/2} \int dR_1 dR_2 R_1^2 R_2^2 N^4 \omega_1 R_1 e^{4\pi\omega_1 R_1} - 1 \int du_2 e^{-4i\omega_1 u_1 u_2 + 4i\omega_2 u_2 u_2} \times e^{-2iR_0 (\omega_1 - \omega_2)} e^{-(R_1-R)^2/2\sigma^2 -(R_2-R)^2/2\sigma^2} . \quad (35)
\]

Integration over \( u_2 \) yields

\[
\tilde{\rho}_{\omega_1\omega_2} \sim (\omega_1\omega_2)^{-1/2} \int dR_1 dR_2 R_1^2 R_2^2 N^4 \omega_1 R_1 e^{4\pi\omega_1 R_1} - 1 \delta(\omega_1 R_1 - \omega_2 R_2) \times e^{-2iR_0 (\omega_1 - \omega_2)} e^{-(R_1-R)^2/2\sigma^2 -(R_2-R)^2/2\sigma^2} . \quad (36)
\]
Integrating over $R_2$ and removing the $\delta$-function, we obtain
\[
\bar{\rho}_{\omega_1 \omega_2} \sim \frac{1}{\sqrt{\omega_1 \omega_2}} \int dR_1 R_1^5 \left( \frac{\omega_1}{\omega_2} \right)^3 \left( e^{4\pi R_1 \omega_1} - 1 \right)^{-1} e^{-2i R_0 (\omega_1 - \omega_2)} \tag{37}
\]
\[
\times \frac{1}{R^2 \sigma^2} \exp \left\{ - \frac{1}{2\sigma^2} \left( R_1 \sqrt{1 + \omega^2_1/\omega_2^2} - \bar{R} \frac{1 + \omega_1/\omega_2}{\sqrt{1 + \omega^2_1/\omega_2^2}} \right)^2 \right\} \exp \left\{ - \frac{\bar{R}^2 (\omega_1 - \omega_2)^2}{2\sigma^2 (\omega_1^2 + \omega_2^2)} \right\}.
\]

Expression (37) is identical to that obtained by method 1 in (28).

5 Diagonal and off-diagonal parts of the density matrix

In the limit $\bar{R}/\sigma \to \infty$ the expression
\[
\frac{\bar{R}}{\sigma \sqrt{1 + \omega^2_1/\omega_2^2}} \exp \left\{ - \frac{\bar{R}^2 (1 - \omega_1/\omega_2)^2}{2\sigma^2 (1 + \omega^2_1/\omega_2^2)} \right\} \tag{38}
\]
becomes the delta function $\delta(1 - \omega_1/\omega_2)$. The density matrix $\bar{\rho}_{\omega_1 \omega_2}$ Eq. (37) turns into the formula for the Hawking spectrum
\[
\rho_{\omega_1 \omega_2} \sim \sqrt{\frac{\omega_1}{\omega_2}} \delta(\omega_1 - \omega_2) \left( e^{2\pi \bar{R}(\omega_1 + \omega_2)} - 1 \right)^{-1}. \tag{39}
\]

The smeared density matrix contains the off-diagonal elements. Because the density matrix has the sharp maximum at $\omega_1/\omega_2 = 1$, it is natural to divide it into the "classical" contribution
\[
\rho_{\omega_1 \omega_2}^{cl} \sim \Theta \left( 2\frac{\sigma}{\bar{R}} \ln^{1/2} \frac{\bar{R}}{\sigma} \left| 1 - \frac{\omega_1}{\omega_2} \right| \right) \frac{\bar{R}}{\sigma} \exp \left\{ - \frac{\bar{R}^2 (1 - \omega_1/\omega_2)^2}{4\sigma^2} \right\} \sqrt{\frac{\omega_1}{\omega_2}} \left( e^{2\pi \bar{R}(\omega_1 + \omega_2)} - 1 \right)^{-1} \tag{40}
\]
and the off-diagonal correction. As mentioned above, in the classical contribution the factor multiplying $(e^{2\pi \bar{R}(\omega_1 + \omega_2)} - 1)^{-1}$ in the limit $\bar{R}/\sigma \to \infty$ turns into the $\delta$-function.

At $\omega_1/\omega_2 = 1$ the expression (38) equals $R/(\sigma \sqrt{2})$. At $\omega_1/\omega_2 = 1 \pm 2(\sigma/\bar{R}) \ln^{1/2}(\bar{R}/\sigma)$ (38) is of order unity. Stated differently, at the distance $2(\sigma/\bar{R}) \ln^{1/2}(\bar{R}/\sigma)$ from the extremum, the off-diagonal part is of order $O(\sigma/\bar{R})$ of the classical expression at the point of extremum. To make this difference explicit we extract the factor $\sigma/\bar{R}$:
\[
\rho_{\omega_1 \omega_2} = \rho_{\omega_1 \omega_2}^{cl} + \frac{\sigma}{\bar{R}} \Delta \rho_{\omega_1 \omega_2}, \tag{41}
\]
where
\[
\frac{\sigma}{\bar{R}} \Delta \rho_{\omega_1 \omega_2} \sim \rho_{\omega_1 \omega_2} \Theta \left( \left| 1 - \frac{\omega_1}{\omega_2} \right| - 2 \frac{\sigma}{\bar{R}} \ln^{1/2} \frac{\bar{R}}{\sigma} \right). \tag{42}
\]

It is of interest to evaluate the contribution of small distances to the smeared density matrix (cf. [20]). It is convenient to use the method 2. Starting from (33) and making the change of variables $u_i \to u_i R_i/\bar{R}$, we have
\[
\bar{\rho}_{\omega_1 \omega_2} \sim (\omega_1 \omega_2)^{-1/2} \int du_1 \int du_2 e^{-4i\omega_1 u_1 R_i/\bar{R} + 4i\omega_2 u_2 R_2/\bar{R} \sinh^{-2} \left( \frac{u_1 - u_2}{\bar{R}} - i\varepsilon \right)} \times N^4 e^{-(R_1 - \bar{R})^2/2\sigma^2 - (R_2 - \bar{R})^2/2\sigma^2} R_1^2 R_2^2 dR_1 dR_2, \tag{43}
\]
where we omitted the irrelevant for the estimate terms.

Introducing \( z = (u_1 - u_2)/2 \), \( y = (u_1 + u_2)/2 \), we integrate over \( y \) in the interval \((-\infty, \infty)\) and over \( z \) in the interval \((-\alpha, \alpha)\):

\[
\bar{\rho}_{\omega_1 \omega_2} \sim (\omega_1 \omega_2)^{-1/2} \int_{-\alpha}^{\alpha} dz \frac{R_1 R_2}{R^2} e^{-iz(R_1 \omega_1 + R_2 \omega_2)/R} R \delta(R_1 \omega_1 - R_2 \omega_2) \frac{1}{\sinh^2(z/R - i\varepsilon)} \times N^4 e^{-(R_1 - R)^2/2\sigma^2 - (R_2 - R)^2/2\sigma^2} R^2 R_1 R_2 dR_1 dR_2.
\]

(44)

Integrating over \( R_2 \), we obtain the expression structurally similar to (28) and (37). Because this expression has sharp extremum at \( R_1 = \bar{R} \) and \( \omega_1/\omega_2 = 1 \), for our estimates we can set in the integrand \( R_1 \) and \( \omega_1/\omega_2 \) equal to the extremal values.

The resulting density matrix is

\[
I \sim (\omega_1 \omega_2)^{-1/2} \int_{-\alpha}^{\alpha} dze^{-i\omega z} \frac{\bar{R}^2}{\sinh^2(2\bar{R}z)} \Theta \left( \frac{\sigma}{\bar{R}} \ln \frac{\bar{R}}{\sigma} - \left| 1 - \frac{\omega_1}{\omega_2} \right| \right).
\]

(45)

The integral in (45) was estimated in [20] for \( \omega R < 1 \) and it was shown that that the ratio of (45) to the Hawking spectrum is

\[
\frac{I(\omega \bar{R}, \alpha/\bar{R})}{(e^{4\pi\alpha R} - 1)^{-1}} \sim \alpha/\bar{R}.
\]

(46)

Taking \( \alpha \sim \sigma \ln \bar{R}/\sigma \) and assuming for an estimate that the mass of the black hole is of order of several solar masses, we obtain that \( \alpha/\bar{R} \sim (\sigma/M) \ln(M/\sigma) \ll 1 \).

6 Discussion

In this paper we discussed modifications of the density matrix of radiation of the black hole formed by the collapsing shell resulting from horizon fluctuations of black hole. Horizon fluctuations are inherent to the black hole considered as a quantum object. In distinction with the original Hawking calculation based on the rigid horizon, horizon fluctuations provide the off-diagonal matrix elements of the density matrix. Qualitatively, the off-diagonal matrix elements account for correlations between the particles in radiation and for information stored in these correlations.

The construction of the density matrix discussed in the present note is parallel to that of papers [12, 13], where the density matrix with the off-diagonal corrections was obtained in the form \( \rho_H(\omega, \bar{\omega}) + C_{BH}^{1/2} \Delta \rho(\omega, \bar{\omega}) \), where \( \rho_H \) is the original Hawking matrix and the off-diagonal correction is of order \( C^{1/2} \), where \( C_{BH} = l_p^2/4\pi M^2 G^2 \), where \( M \) is the mass of the shell. The fact that the expansion parameter in both approaches is the same is rather obvious because \( \sigma/M \) is the only dimensionless parameter connecting the horizon radius and the scale of fluctuations.

The details of calculations and the actual form of the off-diagonal terms in the density matrix obtained in the present paper and in the papers [12] are different.

Because the structural form (but not the explicit form) of the smeared density matrix obtained in the present paper is similar to that in papers [12, 13], we arrive at the same qualitative conclusions concerning the information problem as in these papers. It is possible to construct the \( N \)-particle density matrix \( \rho^{(N)} \) having dimensionality \( N \times N \) and to calculate the entropy of radiation \( S/N = -Tr(\rho^{(N)} \ln \rho^{(N)}) \). Calculating the information contained in radiation, which is defined as the difference between the thermal Bekenstein-Hawking entropy \( S_H \) of radiation, \( I = S_H - S \), one obtains the qualitatively correct Page purification curve [21, 22, 23].
However, the above results pose a question. In [4, 5] it was shown that the Schwarzschild metric admits construction of "nice slices". The nice slices are at $r \simeq \text{const}$ inside the horizon, and one can take $r \sim M/2$. For $M \gg l_p$ horizon fluctuations which are on the scale $l_p$ are insignificant for particle production on the nice slices. If, however, the horizon fluctuations are somehow connected with hair (in spirit of [8, 9] and refs. therein), then the niceness is broken and horizon fluctuations can be connected with the release of information from the black hole.

The expressions for the density matrix discussed in the present paper refer to eternal black holes. Because of the outgoing flux of particles, the mass of collapsing shell is not constant, but decreases with time
\[
\frac{\partial M(u)}{\partial t} = - \langle T_{uu} \rangle \equiv -L_H.
\]
Here $T_{uu}$ is the $uu$ component of the radiation stress tensor. In papers [17, 18] it was found that for the mass of black hole $M(u) \gg m_p$, where $m_p$ is the Planck mass, the backreaction of black hole radiation does not prevent formation of the event horizon. When the outgoing flux is small and slowly varying, the calculation is self-consistent. The metric of the exterior geometry of the shell at large distances $r$ becomes
\[
ds^2 \simeq - \left(1 - \frac{2M(u)}{r} \right) dv^2 + 2dvdu + r^2 d\Omega^2
\]
where $dM(u)/du = -L_H$, and $L_H \sim 1/M^2(u)$.

For the case considered in Sect.2 at the leading order
\[
L_H = \frac{1}{48\pi R_s^3} \left(2 - \frac{3M}{R_s} \right),
\]
where $M$ is the mass of the shell. Substituting $R_s(U) = -(U - 4M)/2$, we have
\[
L_H = \frac{M}{3\pi (U - 4M)^4}.
\]
In the near-horizon region $U \to 0$, and we obtain $L_H \simeq 1/(768\pi M^2)$. This shows that our semiclassical treatment is valid.

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