NON-EQUILIBRIUM EVOLUTION OF A ‘TSUNAMI’: DYNAMICAL SYMMETRY BREAKING

Daniel Boyanovsky\textsuperscript{(a)}, Hector J. de Vega\textsuperscript{(b)}, Richard Holman\textsuperscript{(c)}, S. Prem Kumar\textsuperscript{(c)}, Robert D. Pisarski\textsuperscript{(d)}

\textsuperscript{(a)} Department of Physics and Astronomy, University of Pittsburgh, PA. 15260, U.S.A
\textsuperscript{(b)} LPTHE, Université Pierre et Marie Curie (Paris VI) et Denis Diderot (Paris VII), Tour 16, 1er. étage, 4, Place Jussieu 75252 Paris, Cedex 05, France
\textsuperscript{(c)} Department of Physics, Carnegie Mellon University, Pittsburgh, P.A. 15213, U.S.A.
\textsuperscript{(d)} Department of Physics, Brookhaven National Laboratory, Upton, NY 11973, U.S.A.

Abstract

We propose to study the non-equilibrium features of heavy-ion collisions by following the evolution of an initial state with a large number of quanta with a distribution around a momentum $|\vec{k}_0|$ corresponding to a thin spherical shell in momentum space, a ‘tsunami’. An $O(N)$ $(\Phi^2)^2$ model field theory in the large $N$ limit is used as a framework to study the non-perturbative aspects of the non-equilibrium dynamics including a resummation of the effects of the medium (the initial particle distribution). In a theory where the symmetry is spontaneously broken in the absence of the medium, when the initial number of particles per correlation volume is chosen to be larger than a critical value the medium effects can restore the symmetry of the initial state. We show that if one begins with such a symmetry-restored, non-thermal, initial state, non-perturbative effects automatically induce spinodal instabilities leading to a dynamical breaking of the symmetry. As a result there is explosive particle production and a redistribution of the particles towards low momentum due to the nonlinearity of the dynamics. The asymptotic behavior displays the onset of Bose condensation of pions and the equation of state at long times is that of an ultrarelativistic gas although the momentum distribution is non-thermal.

11.10.-z, 11.30.Qc, 11.15.Tk
I. INTRODUCTION

The Relativistic Heavy Ion Collider (RHIC) at Brookhaven and the Large Hadron Collider (LHC) at CERN will provide an unprecedented range of energies and luminosities that will hopefully probe the Quark-Gluon Plasma and Chiral Phase transitions. The basic picture of the ion-ion collisions in the energy ranges probed by these accelerators as seen in the center-of-mass frame (c.m.), is that of two highly Lorentz-contracted ‘pancakes’ colliding and leaving a ‘hot’ region at mid-rapidity with a high multiplicity of secondaries [1]. At RHIC for \( \text{Au} + \text{Au} \) central collisions with typical luminosity of \( 10^{26} \text{cm}^2 \text{s} \), and c.m. energy \( \approx 200 \text{GeV/n} \), a multiplicity of 500-1500 particles per unit rapidity in the central rapidity region is expected [2–4]. At LHC for head on \( \text{Pb} + \text{Pb} \) collisions with luminosity \( 10^{27} \text{cm}^2 \text{s} \) at c.m. energy \( \approx 5 \text{TeV/n} \), the multiplicity of charged secondaries will be in the range \( 2000 – 8000 \) per unit rapidity in the central region [4]. At RHIC and LHC typical estimates [1–6] of energy densities and temperatures near the central rapidity region are \( \varepsilon \approx 1 – 10 \text{GeV/fm}^3, T_0 \approx 300 – 900 \text{MeV} \).

Since the lattice estimates [4,6] of the transition temperatures in QCD, both for the QGP and Chiral phase transitions are \( T_c \approx 160 – 200 \text{MeV} \), after the collision the central region will be at a temperature \( T > T_c \). In the usual dynamical scenario that one [1] envisions, the initial state cools off via hydrodynamic expansion through the phase transition down to a freeze-out temperature, estimated to be \( T_F \approx 100 \text{MeV} \), at which the mean free-path of the hadrons is comparable to the size of the expanding system.

The initial state after the collision is strongly out of equilibrium and there are very few quantitative models to study its subsequent evolution. There are perturbative and non-perturbative phenomena that contribute to the processes of thermalization and hadronization. The perturbative (hard and semihard) aspects are studied via parton cascade models which assume that at large energies the nuclei can be resolved into their partonic constituents and the dynamical evolution can therefore be tracked by following the parton distribution functions through the perturbative parton-parton interactions [8–12]. Parton cascade models (including screening corrections to the QCD parton-parton cross sections) predict that thermalization occurs on time scales \( \approx 0.5 \text{fm/c} \). After thermalization, and provided that the mean-free path is much shorter than the typical interparticle separation, further evolution of the plasma can be described with boost-invariant relativistic hydrodynamics [1,14]. The details of the dynamical evolution between the parton cascade through hadronization, and eventual description via hydrodynamics is far from clear but will require a non-perturbative treatment. The non-perturbative aspects of particle production and hadronization typically envisage a flux-tube of strong color-electric fields, in which the field energy leads to production of \( \bar{q}q \) pairs [15,16]. Recently the phenomenon of pair production from strong electric fields in boost-invariant coordinates was studied via non-perturbative methods that address the non-equilibrium aspects and allow a comparison with hydrodynamics [17].

The dynamics near the phase transition is even less understood and involves physics beyond the realm of perturbative methods. For instance, considerable interest has been sparked recently by the possibility that disoriented chiral condensates (DCC’s) could form during the
evolution of the QCD plasma through the chiral phase transition [18]- [23]. Rajagopal and Wilczek [24] have argued that if the chiral phase transition occurs strongly out of equilibrium, spinodal instabilities [25] could lead to the formation and relaxation of large pion domains. This phenomenon could provide a striking experimental signature of the chiral phase transition and could provide an explanation for the Centauro and anti-Centauro (JACEE) cosmic ray events [26]. An experimental program is underway at Fermilab to search for candidate events [27,28]. Most of the theoretical studies of the dynamics of the chiral phase transition and the possibility of formation of DCC’s have focused on initial states that are in local thermodynamic equilibrium (LTE) [29]- [32].

We propose to study the non-equilibrium aspects of the dynamical evolution of highly excited initial states by relaxing the assumption of initial LTE (as would be appropriate for the initial conditions in a heavy-ion collision). Consider, for example, a situation where the relevant quantum field theory is prepared in an initial state with a particle distribution sharply peaked in momentum space around $\vec{k}_0$ and $-\vec{k}_0$ where $\vec{k}_0$ is a particular momentum. This configuration would be envisaged to describe two ‘pancakes’ or ‘walls’ of quanta moving in opposite directions with momentum $|\vec{k}_0|$. In the target frame this field configuration would be seen as a ‘wall’ of quanta moving towards the target and hence the name ‘tsunami’ [33]. Such an initial state is out of equilibrium and under time evolution with the proper interacting Hamiltonian, non-linear effects should result in a redistribution of particles, as well as particle production and relaxation. The evolution of this strongly out of equilibrium initial state would be relevant for understanding phenomena such as formation and relaxation of chiral condensates. Starting from such a state and following the complete evolution of the system thereon, is clearly a formidable problem even within the framework of an effective field theory such as the linear $\sigma$-model.

In this article we consider an even more simplistic initial condition, where the occupation number of particles is sharply localized in a thin spherical shell in momentum space around a momentum $|\vec{k}_0|$, i.e. a spherically symmetric version of the ‘tsunami’ configuration. The reason for the simplification is purely technical since spherical symmetry can be used to reduce the number of equations. Although this is a simplification of the idealized problem, it will be seen below that the features of the dynamics contain the essential ingredients to help us gain some understanding of more realistic situations.

We consider a weakly coupled $\lambda \Phi^4$ theory ($\lambda \sim 10^{-2}$) with the fields in the vector representation of the $O(N)$ group. Anticipating non-perturbative physics, we study the dynamics consistently in the leading order in the $1/N$ expansion which will allow an analytic treatment as well as a numerical analysis of the dynamics.

The pion wall scenario described above is realized by considering an initial state described by a Gaussian wave functional with a large number of particles at $|\vec{k}_0|$ and a high density is achieved by taking the number of particles per correlation volume to be very large. As in finite temperature field theory, a resummation along the lines of the Braaten and Pisarski [34] program must be implemented to take into account the non-perturbative aspects of the physics in the dense medium. As will be explicitly shown below, the large $N$ limit in the case under consideration provides a resummation scheme akin to the hard thermal loop program.
The dynamical evolution of this spherically symmetric “tsunami” configuration described above reveals many remarkable features: i) In a theory where the symmetry is spontaneously broken in the absence of a medium, when the initial state is the $O(N)$ symmetric, high density, “tsunami” configuration, we find that there exists a critical density of particles depending on the effective (HTL-resummed) coupling beyond which spinodal instabilities are induced leading to a dynamical symmetry breaking. ii) When these instabilities occur, there is profuse production of low-momentum pions (Goldstone bosons) accompanied by a dramatic re-arrangement of the particle distribution towards low momenta. This distribution is non-thermal and its asymptotic behavior signals the onset of Bose condensation of pions. iii) The final equation of state of the “pion gas” asymptotically at long times is ultrarelativistic despite the non-equilibrium distributions.

The paper is organized as follows: In Section II we introduce the model under consideration and describe the non-perturbative framework, namely the large $N$ approximation. Section III is devoted to the construction of the wave functional and a detailed description of the initial conditions for the problem. The dynamical aspects are covered in Section IV. We first outline some issues dealing with renormalization and then provide a qualitative understanding of the time evolution using wave functional arguments. We argue that the system could undergo dynamical symmetry breakdown and provide analytic estimates for the onset of instabilities. We present the results of our numerical calculations in Section V which confirm the robust features of the analytic estimates for a range of parameters. In Section VI we analyze the details of symmetry breaking and argue that the long time dynamics can be interpreted as the onset of formation of a Bose condensate even when the order parameter vanishes.

Finally in Section VII we present our conclusions and future avenues of study.

II. THE MODEL

As mentioned in the introduction we consider a $\lambda \Phi^4$ theory with $O(N)$ symmetry in the large-$N$ limit with the Lagrangian,

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \Phi) (\partial^\mu \Phi) - \frac{m_B^2}{2} (\Phi \cdot \Phi) - \frac{\lambda}{8N} (\Phi \cdot \Phi)^2$$

(2.1)

where $\Phi$ is an $O(N)$ vector, $\Phi = (\sigma, \pi)$ and $\pi$ represents the $N-1$ pions, $\pi = (\pi^1, \pi^2, ..., \pi^{N-1})$. We then shift $\sigma$ by its expectation value in the non-equilibrium state

$$\sigma(\vec{x}, t) = \sqrt{N} \phi(t) + \chi(\vec{x}, t) \quad ; \quad \langle \sigma(\vec{x}, t) \rangle = \sqrt{N} \phi(t) .$$

(2.2)

We refer the interested reader to several articles which discuss the implementation of the large $N$ limit (see for e.g. [33, 39, 41, 47]). The $1/N$ series may be generated by introducing an auxiliary field $\alpha(x)$ which is an algebraic function of $\Phi^2(x)$, and then performing the functional integral over $\alpha(x)$ using the saddle point approximation in the large $N$ limit.
It can be shown that the leading order terms in the expansion can be easily obtained by the following Hartree factorisation of the quantum fields \[38,39,44,45,\]

\[
\begin{align*}
\chi^4 &\to 6\langle \chi^2 \rangle \chi^2 + \text{constant} ; \\
\chi^3 &\to 3\langle \chi^2 \rangle \chi \\
(\vec{\pi} \cdot \vec{\pi})^2 &\to 2\langle \vec{\pi}^2 \rangle \vec{\pi}^2 - \langle \vec{\pi}^2 \rangle^2 + O(1/N) \\
\vec{\pi}^2 \chi^2 &\to \langle \vec{\pi}^2 \rangle \chi^2 + \langle \vec{\pi}^2 \rangle \chi^2 ; \\
\vec{\pi}^2 \chi^2 &\to \langle \vec{\pi}^2 \rangle \chi .
\end{align*}
\] (2.3)

All expectation values are to be computed in the non-equilibrium state. In the leading order large \(N\) limit we then obtain,

\[
\begin{align*}
L &= -\frac{1}{2} \vec{\pi} \cdot (\partial^2 + M_\pi^2(t)) \vec{\pi} - \frac{1}{2} \chi (\partial^2 + M_\chi^2(t)) \chi - \chi V'(\phi(t), t) + \frac{N\lambda}{8} \langle \pi^2 \rangle^2, \\
M_\pi^2(t) &= m_B^2 + \frac{\lambda}{2} [\phi^2(t) + \langle \pi^2 \rangle], \\
M_\chi^2(t) &= m_B^2 + \frac{\lambda}{2} [3\phi^2(t) + \langle \pi^2 \rangle], \\
V'(\phi(t), t) &= \sqrt{N} \left( \ddot{\phi} + \frac{\lambda}{2} [\phi^2 + \langle \pi^2 \rangle] \phi + m_B^2 \phi \right), \\
\langle \pi^2 \rangle &= \langle \vec{\pi}^2 \rangle / N.
\end{align*}
\] (2.4-2.8)

This approximation allows us to expand about field configurations that are far from the perturbative vacuum. In particular it is an excellent tool for studying the behaviour of matter in extreme conditions such as high temperature or high density \[17,35–39,44,45,\].

One way to obtain the non-equilibrium equations of motion is through the Schwinger-Keldysh Closed Time Path formalism. This is the usual Feynman path integral defined on a complex time contour which allows the computation of in-in expectation values as opposed to in-out S-matrix elements. For details see the references \[40\]. The Lagrangian density in this formalism is given by

\[ L_{neq} = L[\Phi^+] - L[\Phi^-], \] (2.9)

with the fields \(\Phi^\pm\) defined along the forward (+) and backward (−) time branches. The non-equilibrium equations of motion are then obtained by requiring the expectation value of the quantum fluctuations in the non-equilibrium state to vanish i.e. from the tadpole equations \[39\],

\[ \langle \chi^\pm \rangle = \langle \vec{\pi}^\pm \rangle = 0. \] (2.10)

In the leading order approximation of the large \(N\) limit, the Lagrangian for the \(\chi\) field is quadratic plus linear and the tadpole equation for the \(\chi\) leads to the equation of motion for the order parameter or the zero mode

\[ \ddot{\phi} + \frac{\lambda}{2} [\phi^2(t) + \langle \pi^2 \rangle(t)] \phi(t) + m_B^2 \phi(t) = 0. \] (2.11)
In this leading approximation the non-equilibrium action for the $N - 1$ pions is,
\[
\int d^4x [\mathcal{L}_{\pi^+} - \mathcal{L}_{\pi^-}] = \int d^3x dt \left\{ -\frac{1}{2} \vec{\pi}^+ \cdot \partial^2 \vec{\pi}^+ - \mathcal{M}^2_{\pi^+}(t) \vec{\pi}^+ \cdot \vec{\pi}^+ \right\} - (\rightarrow -) \quad \text{(2.12)}
\]

We have not written the action for the $\chi$ field fluctuations because they decouple from the dynamics of the pions in the leading order in the large $N$ limit [38,39].

Having introduced the model and the non-perturbative approximation scheme the next step is to construct an appropriate non-equilibrium initial state or density matrix.

Although one could continue the analysis of the dynamics using the Schwinger-Keldysh method, we will study the dynamics in the Schrödinger representation in terms of wave-functionals because this will display the nature of the quantum states more clearly. We find it convenient to work with the Fourier-transformed fields defined as,
\[
\vec{\pi}(\vec{x}, t) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} e^{-i\vec{k} \cdot \vec{x}} \vec{\eta}_{\vec{k}}(t),
\quad \text{(2.13)}
\]
where we have chosen to quantize in a box of finite volume $V$ that will be taken to infinity at the end of our calculations. The Hamiltonian for the pions is given by
\[
H_{\pi} = \sum_{\vec{k}} \left( \frac{1}{2} \vec{\Pi}_{\vec{k}} \cdot \vec{\Pi}_{-\vec{k}} + \frac{1}{2} \omega^2_{\vec{k}}(t) \vec{\eta}_{\vec{k}} \cdot \vec{\eta}_{-\vec{k}} \right) - \frac{N\lambda}{8} \left( \sum_{\vec{k}} \langle \vec{\eta}_{\vec{k}} \cdot \vec{\eta}_{-\vec{k}} \rangle \right)^2,
\quad \text{(2.14)}
\]
where
\[
\omega^2_{\vec{k}}(t) = \vec{k}^2 + \mathcal{M}^2_{\pi}(t)
\quad \text{(2.15)}
\]
is the effective time dependent frequency and $\mathcal{M}^2_{\pi}(t)$ is given by Eq.(2.3). To leading order in the large $N$ limit the theory becomes Gaussian and the non-linearities are encoded in a self-consistency condition, since the frequency (2.15) depends on $\langle \vec{\pi}^2 \rangle$ and this expectation value is in the time dependent state, as displayed by the set of equations (2.5-2.8).

### III. THE INITIAL STATE

As stated in the introduction, our ultimate goal is to model and study the non-equilibrium aspects of the evolution of an initial, highly excited state that relaxes following high energy, heavy-ion collisions. An idealized description of the associated physics would be to consider two wave packets made up of very high energy components representing the heavy ions and moving with a highly relativistic momentum toward each other. The goal would be to follow the dynamical evolution of the wavefunctionals corresponding to this situation, thus clearly elucidating the non-equilibrium features involved in the phase transition processes following the interactions of the wave packets. This initial state could be described by a distribution of particles, sharply peaked around some special values $\vec{k}_0$ and $-\vec{k}_0$ in momentum space. The evolution of this state then follows from the functional Schrödinger equation.
Even with the simplification of a scalar field theory such a program is very ambitious and beyond the present numerical capabilities. One of the major difficulties is that selecting one particular momentum breaks rotational invariance and the evolution equations depend on the direction of wave vectors even in the Gaussian approximation. (This statement will become clear below).

In this article however, we choose to study a much simpler description of the initial state which is characterized by a high density particle distribution in a thin spherical ‘shell’ in momentum space. We propose an initial particle distribution that has support concentrated at $|\vec{k}_0|$. This particular state does not provide the necessary geometry for a heavy ion collision, however it does describe a situation in which initially there is a large multiplicity of particles in a small momentum ‘shell’, there is no special beam-axis and the pions are distributed equally in all directions with a sharp spatial momentum. This is a rotation invariant state that describes a highly out of equilibrium situation and that will relax during its time evolution (a spherical “tsunami”).

A. The Wave Functional:

Since in the leading order approximation in the large $N$ expansion the theory has become Gaussian (at the expense of a self-consistency condition), we choose a Gaussian ansatz for the wave-functional at $t = 0$. The reason for this choice is that upon time evolution this wave functional will remain Gaussian and will be identified with a squeezed state functional of pions.

\[ \Psi(t = 0) = \Pi_k N_k(0) \exp \left( -\frac{A_k(0)}{2} \vec{\eta}_k \cdot \vec{\eta}_{-k} \right) . \] (3.1)

This state will then evolve according to the Hamiltonian given in Eq. (2.14) which is essentially a harmonic oscillator Hamiltonian with self-consistent, time-dependent frequencies. The functional Schrödinger equation is given by

\[ i \frac{\partial \Psi}{\partial t} = H \Psi. \] (3.2)

The last term in the Hamiltonian (2.14) which is independent of the fields (a time dependent ‘vacuum energy term’) can be absorbed in an overall time dependent phase of the wave functional. Removing this term by a phase redefinition, the functional Schrödinger equation becomes

\[ i \dot{\Psi}[\eta] = \sum_k \left[ -\frac{\hbar^2}{2} \frac{\delta^2}{\delta \vec{\eta}_k \delta \vec{\eta}_{-k}} + \omega_k^2(t) \vec{\eta}_k \cdot \vec{\eta}_{-k} \right] \Psi[\eta] \] (3.3)

which then leads to a set of differential equations for the covariance $A_k$. The time dependence of the normalization factors $N_k$ is completely determined by that of the $A_k$ as a consequence of unitary time evolution. The state for arbitrary time $t$ takes then the form:
\[ \Psi(t) = \Pi_N N_k(t) \exp \left[ -\frac{A_k(t)}{2} \vec{\eta}_k \cdot \vec{\eta}_{-k} \right]. \quad (3.4) \]

The evolution equations for the covariance are obtained by taking the functional derivatives and comparing powers of \( \eta_k \) on both sides. We obtain the following evolution equations \[ \text{[32,38]} \]

\[ i \dot{A}_k(t) = A_k^2(t) - \omega_k^2(t), \quad (3.5) \]
\[ N_k(t) = N_k(0) \exp \left[ \int_0^t A_Ik(t') dt' \right], \quad (3.6) \]

with \( A_k = A_{Rk}(t) + iA_{Ik}(t) \). The equal time two-point correlation function in the time evolved non-equilibrium state is given by

\[ \langle \vec{\eta}_k \cdot \vec{\eta}_{-k} \rangle = \frac{\langle \Psi | \vec{\eta}_k \cdot \vec{\eta}_{-k} | \Psi \rangle}{\langle \Psi | \Psi \rangle} = \frac{\int \mathcal{D} \vec{\eta}_q \mathcal{D} \vec{\eta}_{-q} \Pi_N \Pi_q \exp \left[ -\frac{A_q(t)}{2} \vec{\eta}_q \cdot \vec{\eta}_{-q} \right]}{\int \mathcal{D} \vec{\eta}_q \Pi_N \Pi_q \exp \left[ -\frac{A_q(t)}{2} \vec{\eta}_q \cdot \vec{\eta}_{-q} \right]} = \frac{N}{2A_{Rk}(t)}, \quad (3.7) \]

leading to the self-consistency condition

\[ \langle \pi^2 \rangle(t) = \sum_k \frac{1}{2A_{Rk}(t)}. \quad (3.8) \]

Formally, one can also represent these two-point equal time correlators in terms of functional integrals over the closed time path contour where the initial state is chosen to be the Gaussian functional described above. However the explicit and rather simple ansatz for the wave functional enables one to obtain the two-point functions directly in a rather straightforward manner. Moreover, the wave functional approach will permit a much clearer understanding of the physics of the problem. The Ricatti equation (3.5) can be cast in a simpler form by writing \( A_k \) in terms of new variables \( \phi^*_k \) as

\[ A_k(t) = -i \frac{\dot{\phi}^*_k(t)}{\phi^*_k(t)}, \quad (3.9) \]

leading to the simple equation for the new variables

\[ \ddot{\phi}^*_k + \omega_k^2(t) \phi^*_k = 0. \quad (3.10) \]

In terms of these mode functions we find that the real and imaginary parts of the covariance \( A_k \) are given by
\[
A_{Rk}(t) = \frac{i}{2} \frac{\dot{\phi}_k^* \phi_k - \dot{\phi}_k \phi_k^*}{|\phi_k|^2}, 
\]
\[
A_{Ik}(t) = -\frac{d}{dt} \ln |\phi_k(t)|^2. 
\]

From the differential equation for the \(\phi_k(t)\) given by Eq. (3.10) it is clear that the combination that appears in the numerator of Eq. (3.11) is the Wronskian \(\Omega_k\) of the differential equations and will consequently be determined from the initial conditions alone. The expression for the quantum fluctuations \(\langle \pi^2 \rangle = \langle \bar{\pi}^2 \rangle / N\) is given by,

\[
\langle \pi^2 \rangle(t) = \int \frac{d^3k}{(2\pi)^3} \langle \eta_k(t) \eta_{-k}(t) \rangle = \int \frac{d^3k}{(2\pi)^3} \frac{|\phi_k(t)|^2}{2 \Omega_k}. 
\]

The mode functions \(\phi_k\) have a very simple interpretation: they obey the Heisenberg equations of motion for the pion fields obtained from the Hamiltonian (2.14). Therefore we can write the Heisenberg field operators as

\[
\bar{\pi}(\vec{x}, t) = \frac{1}{\sqrt{V}} \sum_k \frac{1}{\sqrt{2\Omega_k}} \left[ \bar{a}_k \phi_k(t) e^{i\vec{k} \cdot \vec{x}} + \bar{a}_k^* \phi_k^*(t) e^{-i\vec{k} \cdot \vec{x}} \right] 
\]

where \(\bar{a}_k, \bar{a}_k^*\) are the time independent annihilation and creation operators with the usual Bose commutation relations.

**B. Initial Conditions:**

Within this Gaussian ansatz for the wave functional, the initial conditions are completely determined by the initial conditions on the mode functions \(\phi_k(t)\). In order to physically motivate the initial conditions we now establish the relation between the particle number distribution and these mode functions.

Since, in a time dependent situation there is an ambiguity in the definition of the particle number, we define the particle number with respect to the eigenstates of the instantaneous Hamiltonian (2.14) at the initial time, i.e.

\[
\hat{n}_k = \frac{1}{\omega_k(0)} \left( -\frac{1}{2} \frac{\delta^2}{\delta \eta_k^* \delta \eta_{-k}} + \frac{\omega_k^2(0)}{2} \eta_k \cdot \eta_{-k} \right) - \frac{1}{2} 
= \frac{1}{\omega_k(0)} \left[ \frac{1}{2} \bar{\Pi}_k \cdot \bar{\Pi}_{-k} + \frac{\omega_k^2(0)}{2} \eta_k \cdot \eta_{-k} \right] - \frac{1}{2}. 
\]

Here, \(\omega_k(0)\) is the frequency (2.15) evaluated at \(t = 0\), i.e. the curvature of the potential term in the functional Schrödinger equation (3.3) at \(t = 0\) and provides a definition of the particle number (assuming that \(\omega_k^2(0) > 0\)). The expectation value of the number operator in the time evolved state is then
\[ n_k(t) = \langle \Psi | \hat{n}_k | \Psi \rangle = \left[ A_{Rk}(t) - \omega_k(0) \right]^2 + A_{ik}^2(t) \]
\[ = \frac{\Delta_k(t)^2 + \delta_k(t)^2}{4 \omega_k(0) A_{Rk}(t)} , \]  
(3.16)

where \( \Delta_k \) and \( \delta_k \) are defined through the relations,
\[ A_{Rk}(t) = \omega_k(0) \left[ 1 + \Delta_k(t) \right] \quad ; \quad A_{ik}(t) = \omega_k(0) \delta_k(t) . \]  
(3.18)

In terms of the mode functions \( \phi_k \) and \( \dot{\phi}_k \) the expectation value of the number operator is given by
\[ n_k(t) = \frac{1}{4 \Omega_k \omega_k(0)} \left[ |\dot{\phi}_k(t)|^2 + \omega_k^2(0)|\phi_k(t)|^2 \right] - \frac{1}{2} . \]  
(3.19)

The quantity \( \delta_k(t) \) appears as the phase of the wave function and will be chosen to be zero at \( t = 0 \) for simplicity,
\[ A_{ik}(0) = 0 \quad ; \quad \delta_k(0) = 0 . \]  
(3.20)

Assuming \( \delta_k(0) = 0 \), the initial conditions on the \( \phi_k(0) \) variables can be obtained at once from Eq. (3.9) and are found to be,
\[ \dot{\phi}_k^*(0) = i\Omega_k = i\omega_k(0) \left[ 1 + \Delta_k(0) \right] \quad ; \quad \phi_k^*(0) = 1 , \]  
(3.21)

where \( \Omega_k \) is the Wronskian \( W[\phi_k(t)^*, \phi_k(t)] \). Hence the wave functional of the system at \( t = 0 \) can be specified completely (up to a phase) by the single function \( \Delta_k(0) \).

Using Eq. (3.17) one can easily solve for \( \Delta_k(0) \) in terms of the initial particle spectrum
\[ \Delta_k \equiv \Delta_k(0) = 2 \left[ n_k(0) \pm \sqrt{n_k(0)^2 + n_k(0)} \right] . \]  
(3.22)

Which of the two solutions will give us interesting physics is a more subtle question that we shall address in the next section when we discuss the dynamics of the problem.

Before moving on to the description of the dynamics let us briefly summarize what we have done. We proposed a rather simple description of a large multiplicity, high energy particle collision process by preparing an initial state with an extremely high number density of particles concentrated at momenta given by \(|\vec{k}| = k_0\). Consistent with the leading order in a large \( N \) approximation, we chose a Gaussian ansatz for our wave functional, parametrized by the variables \( \phi_k(t) \), \( \dot{\phi}_k(t) \) (or alternatively \( \Delta_k(t) \) and \( \delta_k(t) \)) and the initial conditions on these variables are determined completely by the choice of the particle distribution function \( n_k(0) \) at \( t = 0 \) (Eq. (3.22)). The next step will be to obtain the renormalized equations of motion and then to study the dynamics analytically as well as numerically.
IV. THE DYNAMICS

From the discussion in the previous sections we see that the following set of equations for the order parameter $\phi(t)$ and the mode functions $\phi_k(t)$ must be solved self-consistently in order to study the dynamics:

$$\frac{d^2 \phi(t)}{dt^2} + \left( m_B^2 + \frac{\lambda}{2} [\phi^2(t) + \langle \pi^2 \rangle_B] \right) \phi(t) = 0 , \quad (4.1)$$

$$\frac{d^2 \phi_k^*(t)}{dt^2} + \left( k^2 + m_B^2 + \frac{\lambda}{2} [\phi^2(t) + \langle \pi^2 \rangle_B] \right) \phi_k^*(t) = 0 , \quad (4.2)$$

$$\phi_k^*(0) = 1 ; \quad \dot{\phi}_k^*(0) = i \Omega_k , \quad (4.3)$$

with the self-consistent condition

$$\langle \pi^2 \rangle_B = \frac{\int d^3 k (2\pi)^\frac{3}{2}}{2\Omega_k} |\phi_k(t)|^2 . \quad (4.4)$$

The quantities in the above equations must be renormalized. This is achieved by first demanding that all the equations of motion be finite and then absorbing the divergent pieces into a redefinition of the mass and coupling constant respectively,

$$m_B^2 + \frac{\lambda}{2} [\langle \pi^2 \rangle_B + \phi^2(t)] = m_R^2 + \frac{\lambda_R}{2} [\langle \pi^2 \rangle_R + \phi^2(t)] = M^2_\pi R \pi(t) . \quad (4.5)$$

A detailed derivation of the renormalization prescriptions requires a WKB analysis of the mode functions $\phi_k(t)$ that reveals their ultraviolet properties. Such an analysis has been performed elsewhere [38,39]. In summary the mass term will absorb quadratic and logarithmic divergences while the coupling constant will acquire a logarithmically divergent renormalization [38,35]. In particular

$$\langle \pi^2 \rangle_R (t) = \int \frac{d^3 k}{(2\pi)^\frac{3}{2}} \left\{ \frac{|\phi_k(t)|^2}{2\Omega_k} - \frac{1}{2k} + \frac{\theta(k - \kappa)}{4k^3} M^2_\pi(t) \right\} \quad (4.6)$$

with $\kappa$ an arbitrary renormalization scale.

Introducing the effective mass of the particles at the initial time as

$$M^2_R = M^2_\pi R \pi(t = 0) , \quad (4.7)$$

we recognize that this effective mass has contributions from the non-equilibrium particle distribution and is the analog of the hard-thermal loop (HTL) resummed effective mass in a scalar field theory. Recall, however, that the initial distribution is not thermal. In a scalar theory, the HTL effective mass is obtained by summing the daisy and superdaisy diagrams [11] which is precisely the resummation implied in the leading order in the large $N$ approximation. To see this more clearly consider the case in which the order parameter
vanishes, i.e. \( \phi(t) \equiv 0 \); then the effective mass at the initial time can be written as a gap equation

\[
M_R^2 = m_B^2 + \frac{\lambda}{4\pi^2} \int \frac{k^2dk}{2\Omega_k(0)}
\]

\[
= m_B^2 + \frac{\lambda}{4\pi^2} \int \frac{k^2dk}{2\omega_k(0)} + \frac{\lambda}{4\pi^2} \int \frac{k^2dk}{2\omega_k(0)} \left[ -\frac{\Delta_k(0)}{1 + \Delta_k(0)} \right]
\]

where we have used the relation \( \Omega_k(0) = \omega_k(0)(1 + \Delta_k) \). The second term in the above expression is the usual contribution obtained at zero temperature (and zero density) for the (self-consistent) renormalized mass parameter \( M_R^2 \) i.e the one loop tadpole (with \( \omega_k(0) = \sqrt{k^2 + M_R^2} \)). The third term contains the non-equilibrium effects associated with the particle distributions and vanishes when \( n_k \to 0 \). This term is finite (since \( n_k(0) \) is assumed to be localized within a small range of momenta). For a given distribution \( n_k(0) \), the solution to the self-consistent gap equation \((4.8)\) gives the effective mass, dressed by the medium effects. This is indeed very similar to the finite temperature case in which the tadpole term provides a contribution \( \propto \lambda T^2 \) in the high temperature limit. We will see later that a term very similar to this can be extracted in the limit in which the distribution \( n_k(0) \) is very large.

Since the relevant scale is the quasiparticle mass \( M_R \), we will choose to take \( M_R^2 > 0 \) to describe an initial situation in which the \( O(N) \) symmetry is unbroken.

It is convenient for numerical purposes to introduce the following dimensionless quantities:

\[
q = \frac{k}{M_R}; \quad \tau = M_R t; \quad \varphi^2(\tau) = \frac{\lambda_R \phi^2(t)}{2M_R^2}; \quad g = \frac{\lambda_R}{8\pi^2}; \quad W_q = \frac{\Omega_k}{M_R}
\]

\[
g(\Sigma(\tau)) = \frac{\lambda}{2M_R^2} \left[ \langle \pi^2 \rangle_R(t) - \langle \pi^2 \rangle_R(0) \right].
\]

In terms of these dimensionless quantities the equations of motion \((4.1, 4.2)\) with the initial conditions \((4.3)\) become

\[
\left[ \frac{d^2}{d\tau^2} + 1 + \varphi^2(\tau) - \varphi^2(0) + g(\Sigma(\tau)) \right] \varphi(\tau) = 0 ,
\]

\[
\left[ \frac{d^2}{d\tau^2} + q^2 + 1 + \varphi^2(\tau) - \varphi^2(0) + g(\Sigma(\tau)) \right] \phi_q(\tau) = 0 ; \quad \phi_q(0) = 1 ; \quad \dot{\phi}_q(0) = -iW_q ,
\]

\[
g(\Sigma(\tau)) = g \int q^2dq \left\{ \frac{[\phi_q(\tau)]^2 - 1}{W_q} + \frac{\theta(q - 1)}{2q^3} \left( \frac{M_{R,\pi}^2(\tau)}{M_R^2} - 1 \right) \right\},
\]

where we have chosen the renormalization scale \( \kappa = M_R \) for simplicity.

In order to make our statements precise in the analysis of the spherically symmetric “tsunami”, we will assume the initial particle distribution to be Gaussian and peaked at some value \( q_0 \) and width \( \xi \) so that
\[ n_q(0) = \frac{N_0}{I} \exp \left( - \frac{(q - q_0)}{\xi} \right)^2, \]  

(4.13)

where \( N_0 \) is the total number of particles in a correlation volume \( M^{-3} \) and \( I \) is a normalization factor. The case \( N_0 \gg 1 \) corresponds to the high density regime with many particles in the effective correlation volume.

**A. Preliminary considerations of the dynamics:**

Before engaging in a full numerical solution of the evolution equations, we can obtain a clear, qualitative understanding of the main features of the evolution by looking at the quantum mechanics of the wave functional.

The dynamics is different for the different solutions for \( \Delta_q \) in Eq. (3.22) and can be understood with simple quantum mechanical arguments:

1. **Case I**

\[ \Delta_q = 2[n_q(0) + \sqrt{n_q^2(0) + n_q(0)}]: \]

In this case

\[ \Delta_q(0) \approx 4 n_q(0) \gg 1 \text{ for } \frac{|q - q_0|}{\xi} \approx 1 \]

\[ \approx 0 \text{ for } \frac{|q - q_0|}{\xi} \gg 1, \]

(4.14)

and the covariance of the wave functional (3.1) is given by

\[ A_q(0) = \Omega_q(0) = \omega_q(1 + \Delta_q) \gg \omega_q \text{ for } \frac{|q - q_0|}{\xi} \approx 1 \]

\[ \approx \omega_q \text{ for } \frac{|q - q_0|}{\xi} \gg 1. \]

(4.15)

Now, for each wave vector \( \vec{q} \) we have a Gaussian wave-function which is the ground state of a harmonic oscillator with frequency \( \Omega_q(0) \), but whose evolution is determined by a Hamiltonian for a harmonic oscillator of frequency \( \omega_q(0) \) at very early times. For the modes \( q \) such that \( \Omega_q \gg \omega_q(0) \), the wave function is very narrow compared to the second derivative of the potential and there is a very small probability for sampling large amplitude field configurations (i.e. large \( n_{\vec{q}} \)). It is a property of these squeezed states that whereas the wave-functional is narrowly localized in field space, it is a wide distribution in the canonical momentum (conjugate to the field) basis. This wave function will spread out under time evolution to obtain a width compatible with the frequency \( \omega_q \), i.e. the covariance \( A_q(\tau) \) will **diminish in time** and the fluctuation...
\[ \langle |\vec{n}_q|^2 \rangle (\tau) \propto \frac{1}{A_Rq(\tau)} \]  

(4.16)

will increase in time. This will in turn cause the time dependent frequency in the Hamiltonian, \( \omega_q(\tau) \) to increase with time since the frequency and the fluctuations are directly related via the self-consistency condition. The resulting dynamics is then expected to approach an oscillatory regime in which the width of the wavefunctional and the frequency of the harmonic oscillator are of the same order. Under these circumstances, there is the possibility that for a particular range of parameters (coupling, central momentum of the distribution and particle density) parametric amplification can occur \cite{39,44} that could result in particle production and redistribution of particles as will be discussed below within an early time analysis. We will see that this case corresponds to a “tsunami” configuration in a theory which is symmetric even in the absence of the medium.

2. Case II

\[ \Delta_q = 2[n_q(0) - \sqrt{n_q^2(0) + n_q(0)}]: \]

In this case

\[ \Delta_q(0) \approx -1 + \frac{1}{4n_{q0}} \quad \text{for} \quad \frac{|q - q_0|}{\xi} \approx 1 \]

\[ \approx 0 \quad \text{for} \quad \frac{|q - q_0|}{\xi} >> 1, \]

(4.17)

and the covariance is

\[ A_q(0) \approx \frac{\omega_q}{4n_q(0)} << \omega_q \quad \text{for} \quad \frac{|q - q_0|}{\xi} \approx 1 \]

\[ \approx \omega_q \quad \text{for} \quad \frac{|q - q_0|}{\xi} >> 1. \]

(4.18)

Therefore, large amplitude field configurations with momenta in the narrow “tsunami” shell, now have a high probability of being realized. As before, the wave function for each \( \vec{k} \) mode corresponds to the ground state of a harmonic oscillator of frequency

\[ \Omega_q(0) = \frac{\omega_q}{4n_{q0}(0)} \]

which evolves with a Hamiltonian for a harmonic oscillator with frequency \( \omega_q(0) \). In this case the wave function for \( q \approx q_0 \) is spread out over field amplitudes much larger than

\[ 1/\sqrt{\omega_{q0}} \]

and it is localized in the canonical momentum basis.
Under time evolution the wave function will tend to be squeezed i.e. it will be forced to diminish its width and to become localized inside the potential well. This implies that the covariance \( A_q(\tau) \) will increase under time evolution, while the fluctuation (4.16) and the time dependent frequency \( \omega_q(\tau) \) will decrease, i.e. the potential ‘flattens out’.

In this case the quantity \( g\Sigma(\tau) \) (the renormalized quantum fluctuations) in the evolution equations (4.11) decreases and as will be seen below, under certain conditions, can become negative.

There is thus a possibility of inducing spinodal instabilities in the quantum fluctuations. To see this, consider the case in which \( \varphi \equiv 0 \) and the effective mass squared \( M^2_{R\pi}(\tau) = 1 + g\Sigma(\tau) \) in the equation (4.11) becomes negative, i.e. when \( g\Sigma(\tau) < -1 \). The modes for which \( q^2 < |M^2_{R\pi}(\tau)| \) will see an inverted harmonic oscillator and they will begin to grow almost exponentially, resulting in copious particle production for these modes as can be seen from the expression for the particle number as a function of time (3.19).

This situation, in which the potential turns into a maximum at the origin dynamically, corresponds to symmetry breaking, since the minimum will be away from the origin. In this case the dynamics will result in a re-arrangement of the particle distribution: spinodal instabilities will arise, long-wavelength modes will begin to get populated at the expense of the initial non-equilibrium distribution. The spinodal instabilities will in turn result in an increase in the fluctuations that will tend to cancel the negative contribution to \( g\Sigma \) from the initial non-equilibrium distribution. Eventually a stationary regime should ensue in which the instabilities are turned-off and the distribution of particles will be peaked at low momenta.

At this point we want to emphasize that the possibility for the onset of spinodal instabilities is purely dynamical. In contrast to previous studies of dynamics in spinodally unstable situations [24, 29, 25, 32] in which an initially symmetric state is evolved with a broken symmetry Hamiltonian, in the present case the initial state and the effective Hamiltonian are symmetric and the instability is a consequence of the non-equilibrium dynamics.

The above analysis of the dynamics, based on the quantum mechanical analogy will be shown to be accurate in the next section where we present the details of the numerical evolution.

**B. Early Time Analysis:**

A more quantitative understanding of these cases can be achieved by studying the early time behaviour of the solutions and setting \( \varphi \equiv 0 \).

1. Case I

In this case with \( \Delta_q \) given by (4.14) and focusing on the very early time during which backreaction effects can be ignored, the solution to the mode equations (Eq. (4.11)) with the
initial conditions given by Eq. (4.3) is simply a superposition of plane waves with frequency $\omega_q(0)$:

$$\phi_q^*(\tau) \approx \cos(\omega_q(0)\tau) - i(1 + \Delta_q) \sin(\omega_q(0)\tau).$$

(4.19)

The renormalized quantum fluctuations which are dominated by the modes within the highly populated momentum shell are given by,

$$g\Sigma(\tau) \approx -g \int_0^{\Lambda} q^2 dq \frac{\sin^2(\omega_q(0)\tau)[1 - (1 + \Delta_q)^2]}{\omega_q(1 + \Delta_q)}.$$  

(4.20)

If the initial distribution of particles is sufficiently sharp, a qualitative understanding of the early time dynamics can be obtained by a saddle point analysis of the contribution from the region of large occupation number.

In this limit $g\Sigma(\tau)$ is approximately given by,

$$g\Sigma(\tau) \approx +\frac{4gN_0}{\omega_{q_0}} \sin^2(\omega_{q_0}\tau).$$

(4.21)

The mode equations now become

$$\left[ \frac{d^2}{d\tau^2} + q^2 + 1 + \frac{2gN_0}{\omega_{q_0}} - \frac{2gN_0}{\omega_{q_0}} \cos(2\omega_{q_0}\tau) \right] \phi_q(\tau) = 0.$$  

(4.22)

This is a Mathieu equation whose solutions are of the Floquet form [42]. The first and broadest instability band is centered at the value of $q$ given by

$$q^2 = q_0^2 - \frac{2gN_0}{\omega_{q_0}}.$$  

(4.23)

The width of the unstable band depends on the parameter

$$Q = \frac{gN_0}{\omega_{q_0}^3}.$$  

(4.24)

and can be read off in reference [42]. There is a rather small window of relevant parameters that could allow appreciable parametric amplification. Whether the backreaction effects allow the unstable band to remain under time evolution resulting in large particle production and redistribution of particles is a detailed dynamical question that will be studied numerically below.

2. Case II

The dynamics in this case can be understood by the heuristic arguments presented below. We work with the solution $\Delta_q = 2[n_q(0) - \sqrt{n_q^2(0) + n_q(0)}]$, which in the limit $N_0 \gg 1$ yields (4.17)
\[ \Delta_q \approx -1 + \frac{1}{4n_{q_0}} \quad \text{for} \quad q \approx q_0 \]
\[ \approx 0 \quad \text{otherwise.} \]

leading now to the following approximate form for the fluctuation at early times:

\[ g \Sigma(\tau) \approx -\frac{4gN_0}{\omega_{q_0}} \sin^2(\omega_{q_0} \tau) \]

when the backreaction effects can be ignored. The first feature to note is that \( g \) and \( N_0 \) appear together in such a way that the effective coupling is now \( gN_0 \) and hence the physics is intrinsically non-perturbative when \( N_0 \approx 1/g \). This situation is very similar to that in high temperature field theory wherein the relevant dimensionless quantity is \( T/m(T) \) (\( m(T) \) is the temperature corrected effective mass) and the effective coupling constant for long-wavelength physics is \( \lambda T/m(T) \). In this situation the non-perturbative hard-thermal-loop resummation is required.

Secondly the expression for \( g \Sigma(\tau) \) is always less than or equal to zero. Notice that unlike Case I, \( g \Sigma(\tau) \) is negative [see Eq. (4.21)]. In particular \( g \Sigma(0) = 0 \) and then \( g \Sigma(\tau) \) becomes negative i.e. it begins to decrease. The fact that the fluctuations decrease was exactly what we had expected from the wave functional analysis presented in the previous section.

Furthermore we see that when \( 4gN_0/\omega_{q_0} > 1 \) at very early times there will be an unstable band of wave-vectors. An estimate of the width of the band can be provided by averaging the time dependence of \( g \Sigma \) over one period of oscillation. This estimate yields the band of wave-vectors

\[ 0 < q < \sqrt{\frac{2gN_0}{\omega_{q_0}} - 1} = q_m \]

which will become spinodally unstable. The mode functions for these wavevectors will grow exponentially at early times and their contribution to the fluctuation \( g \Sigma(\tau) \) will grow – this is a back-reaction mechanism that will tend to shut-off the instabilities.

This means that if we begin with a completely \( O(N) \) symmetric state i.e. \( \varphi(0) = \dot{\varphi}(0) = 0 \) and if we choose \( N_0 \) large enough such that \( 1 + g \Sigma \approx 1 - 4gN_0 \sin^2(\omega_{q_0} \tau)/\omega_{q_0} < 0 \), spinodal instabilities will be triggered and the symmetry will be spontaneously broken. The condition for spinodal instabilities to appear is given by

\[ \frac{4gN_0}{\omega_{q_0}} > 1 \]

which determines the critical value of the particle number in a correlation volume in terms of the coupling and the peak momentum of the distribution.

In the preceding sections we provided an intuitive understanding of the underlying mechanism of symmetry breaking in terms of a quantum mechanical analogy. We now provide an alternative argument to clarify the physical mechanism for the dynamical symmetry breaking. The argument begins with the expression for the ‘dressed’ mass in Eq. (4.8) which we write in terms of dimensionless quantities as
\[ M^2_R = m^2_R + g M^2_R \int \frac{q^2 dq}{\omega_q(0)} \left[ -\frac{\Delta_q}{1 + \Delta_q} \right], \quad (4.30) \]

\[ m^2_R = m^2_B + g \int \frac{q^2 dq}{\omega_q(0)}. \quad (4.31) \]

The second term is dominated by the peak in the initial particle distribution. Using eqns. (4.25,4.26) and using a saddle point approximation assuming a sharp distribution, we obtain the relationship

\[ M^2_R \left[ 1 - \frac{4gN_0}{\omega_{q_0}} \right] = m^2_R. \quad (4.32) \]

Then choosing the effective mass \( M^2_R > 0 \) as we have done throughout, we see that when the condition for spinodal instabilities in Eq. (4.29) is fulfilled then it must be that \( m^2_R < 0 \). Therefore the renormalized mass squared in the absence of the medium is negative and the medium effects, i.e., the non-equilibrium distribution of particles dresses this mass making the effective, medium ‘dressed’ mass squared positive. Thus in the absence of medium the potential was a (spontaneous) symmetry breaking potential. The initial distribution restores the symmetry at \( \tau = 0 \) much in the same way as in finite temperature field theory at temperatures larger than the critical temperature. However the initial state is strongly out of equilibrium and its time evolution re-distributes the particles towards low momentum and the spinodal instabilities result from the squeezing of the quantum state as explained above.

This situation must be contrasted to that in Case I above. The same argument, now applied to Case I leads to the result

\[ M^2_R \left[ 1 + \frac{4gN_0}{\omega_{q_0}} \right] = m^2_R. \quad (4.33) \]

We clearly see that with a positive effective mass, Case I corresponds to the situation in which the theory was symmetric even without the medium effects (i.e. \( m^2_R > 0 \)).

Thus we obtain a physical picture of the different cases: in Case I the symmetry was unbroken without a medium and remains unbroken when the large density of particles is added. By contrast in Case II, the symmetry is spontaneously broken in the absence of a medium, the high density initial state restores the symmetry in a state out of equilibrium. Under time evolution the dynamics then redistributes the particles producing spinodal instabilities and breaking the symmetry.

We reiterate that the second case represents a novel situation which is in a sense, contrary to what happens at high temperature where thermal fluctuations suppress the possibility of long-wavelength instabilities.

The issue of symmetry breaking is a subtle one here. If we begin with symmetric initial conditions, \( \varphi(0) = \dot{\varphi}(0) = 0 \), the wavefunctional will always be symmetric since the evolution will maintain this symmetry. In order to test whether the symmetry is spontaneously
broken or not, one must provide an initial state that is slightly asymmetric, with a very small initial expectation value $\varphi(0) \neq 0$, and follow the subsequent time evolution. If the expectation value oscillates around zero, then the symmetry is not spontaneously broken since the minimum of the ‘dynamical effective potential’ is at the origin in field space. If the expectation value begins rolling away from zero and reaches a stationary value away from zero then one can assert that there is a dynamical minimum away from the origin and the symmetry is spontaneously broken. Thus the test of symmetry breaking requires an initial condition with a small value of the order parameter.

C. The Late Time Regime

The asymptotic value of the order parameter can be obtained by analyzing the full dynamics of the theory and depends on the initial conditions. This reflects the fact that there is no static effective potential description of the physics. However, some information about the asymptotic state (when $\dot{\varphi}(\infty) = \ddot{\varphi}(\infty) = 0$) can be obtained from the equation of motion (4.10) by setting $\ddot{\varphi}(\infty) = 0$ which yields the sum rule

$$1 + \varphi^2(\infty) + g\Sigma(\infty) = 0$$

(4.34)

provided $\varphi(\infty) \neq 0$. This sum rule guarantees that the pions are the asymptotic massless Goldstone bosons since

$$M^2_\pi(\tau) = m^2 R + \lambda R \left[ \phi^2(\infty) + \langle \pi^2 \rangle_R(\infty) \right]$$

(Eq. (2.5)) and the sum rule is a consequence of the Ward identities associated with the global $O(N)$ symmetry.

The non-linear evolution of the mode functions results in a redistribution of particles within the spinodally unstable band. The distribution becomes more peaked at low momentum and the effective potential flattens resulting in a non-perturbatively large distribution of Goldstone bosons at low momentum.

V. NUMERICAL ANALYSIS

1. Case I

We have investigated the possibility of parametric amplification in this case in a wide region of parameters but always in the dense regime $N_0 >> 1$ and varying the center of the distribution. We find numerically that the backreaction effects shut off the parametric instabilities rather soon allowing only small particle production and redistribution of particles. Typically the distribution develops peaks but remains qualitatively unchanged and the dynamics is purely oscillatory.
2. Case II

The numerical analysis of the problem involves the solution of the coupled set of equations (4.10), (4.11) and (4.12) appended with initial conditions. We choose the zero mode initial conditions to be \( \phi(0) = 10^{-3}, \dot{\phi}(0) = 0 \) while the mode functions satisfy \( \phi_q(0) = 1; \dot{\phi}_q(0) = -i\omega_q(0)(1 + \Delta_q) \). Here \( \omega_q(0) = \sqrt{q^2 + 1} \) and

\[
\Delta_q = 2[n_q(0) - \sqrt{n_q^2(0) + n_q(0)}].
\]

We have tested the numerics with a momentum cutoff \( \Lambda = 25 \) in units of the renormalized mass \( M_R \) and found that after renormalization the numerical results are insensitive to the value of the cutoff provided it is chosen to be much larger than the largest wave-vector which becomes spinodally unstable. The initial particle distribution is chosen to be

\[
n_q(0) = \frac{N_0 I e^{-(q-5)^2}}{q},
\]

where the total initial number of particles is taken to be to be \( N_0 = 2000 \), the coupling is fixed at \( g = 10^{-2} \) and the initial value of the order parameter is taken to be \( \phi(0) = 10^{-3} \).

Results:

Fig. (1) shows \( g\Sigma(\tau) \) vs. \( \tau \) and the effective ‘pion’ mass squared \( M_{\pi R}^2(\tau) \). We see clearly that spinodal instabilities are produced and the quantitative features of the dynamics are in agreement with the estimates established for the early time dynamics given by Eq.(4.27). We see that the pions become massless, asymptotically. The distribution function \( n_q(\tau) \) multiplied by the coupling \( g \) is shown in Fig. (2) at different times, clearly demonstrating how the distribution changes in time. As a consequence of the spinodal instabilities the long-wavelength modes grow exponentially and the ensuing particle production for these modes populates the band of unstable modes. In particular the amplitudes of the spinodal instabilities the long-wavelength modes that become spinodally unstable grow to be non-perturbatively large of order \( 1/g \) and dominate the dynamics completely. At earlier times the initial peak in the distribution at \( q \approx 5 \) can still be seen, but at later times it is overwhelmed by the distribution at long wavelengths. Fig. (3) shows a zoom-in of the distribution functions \( (gn_q) \) vs. \( q \) at \( \tau = 30, 80 \) near \( q = 0 \) and also near the peak of the initial distribution, around \( q_0 \approx 5 \). We see a remnant of the original peak, slightly shifted to the right but much broader than the initial distribution and with about half the original amplitude. After \( \tau \approx 10 \) the distributions do not vary much in this region of momenta, but they do vary dramatically at low momenta. Fig. (4) shows the total number of particles as a function of \( \tau \). We see clearly that initially the total number of particles diminishes because the fluctuations decrease at early times. The long-wavelength modes begin to grow because of spinodal instabilities but their contributions are suppressed by phase space. Only when their amplitudes become non-perturbatively large, is the particle production at long-wavelengths an effective contribution to the total particle number. When this happens, there is an explosive burst of particle production following which the total number of particles remains fairly constant throughout the evolution. After the spinodal instabilities are shut-off, which for the values chosen for the numerical evolution correspond
to $\tau \approx 2$, the dynamics becomes non-linear. Whereas during the initial stages the dynamics is in the linear regime, after backreaction effects have shut-off the spinodal instabilities the further evolution of the distribution functions is a consequence of the non-linearities.

Fig. (5) exhibits one of the clear signals of symmetry breaking. The order parameter begins very near the origin, but once the spinodal instabilities kick in, the origin becomes a maximum and the order parameter begins to roll away from it. Notice that the order parameter reaches a very large value, which is the dynamical turning point of the trajectory, before settling towards a non-zero value. We find that the value of the turning point and the final value of the order parameter depend on the initial conditions. To illustrate the non-perturbative growth of modes clearly, we have plotted the quantity $g |\phi_q(\tau = 5)|^2 - g |\phi_q(\tau = 0)|^2$ in Fig. (6) which shows how the amplitude of the long wavelength modes becomes non-perturbatively large and of order $1/g$.

We have also carried out the numerical evolution with $g = 10^{-2}, N_0 = 4000$ and $g = 10^{-3}, N_0 = 40000$ with the same value of $q_0$ and found the same quantitative behavior, proving that the relevant combination is $gN_0$ as revealed by the analytic estimates above. We have also confirmed that for $gN_0 << 1$ there are no spinodal instabilities and the dynamics is purely oscillatory without a redistribution of the particles and with no appreciable particle production. When the peak of the initial distribution function is beyond the spinodally unstable band $q > q_m$ (see Eq. (4.28)), the original distribution is depleted and broadened somewhat with irregularities and wiggles but remains qualitatively unchanged (see fig. 3). However, when the peak of the initial distribution is within the spinodally unstable band there is a complete re-distribution of particles towards low momentum. The original distribution disappears under time evolution and after the spinodal time only the low momentum modes are populated.

VI. SYMMETRY BREAKING, ENERGY, PRESSURE AND EQUATION OF STATE:

A. Onset of Bose Condensation:

We have seen both from the numerical evolution and from the argument based on the sum rule (4.34) which is a result of the Ward identities and Goldstone’s theorem, that the effective mass term vanishes asymptotically. Therefore the asymptotic equation of motion for the mode functions is that of a massless free field. In particular the asymptotic solution for the $q = 0$ mode is given by

$$\phi_0(\tau \to \infty) = A + B\tau$$

(6.1)

where $A$ and $B$ are complex coefficients that can only be obtained from the full time evolution. However because the Wronskian

$$\phi_0(\tau)\dot{\phi}_0(\tau) - \phi_0^*(\tau)\dot{\phi}_0(\tau) = 2iW_0$$

(6.2)
is constant in time, neither $A$ nor $B$ can vanish \[43\]. This situation must be contrasted with that for the $q \neq 0$ modes whose asymptotic behavior is of the form

$$
\phi_q(\tau \to \infty) = \alpha_q e^{iq\tau} + \beta_q e^{-iq\tau}.
$$

(6.3)

This causes the number of particles at zero momentum to grow asymptotically as $\tau^2$ whereas the number saturates for the $q \neq 0$ modes. The three dimensional phase space conspires to cancel the contribution from the $q = 0$ mode to the total number of particles, energy and pressure, which, from the numerical evolution (see Fig.(4)) are seen to remain constant at long times. This situation is very similar to that in Bose-Einstein condensation where the excess number of particles at a fixed temperature goes into the condensate, while the total number of particles outside the condensate is fixed by the temperature and the chemical potential. The $q = 0$ mode will become macroscopically occupied when $\tau \sim \sqrt{V}$ where $V$ is the volume of the system (i.e. the number of particles in the zero momentum mode becomes of the order of the spatial volume). When this happens this mode must be isolated and studied separately from the $q \neq 0$ modes because its contribution to the momentum integral will be cancelled by the small phase space at small momentum. Again the situation is very similar to the case of the usual Bose-Einstein condensation. Notice that this argument is independent of a non-vanishing order parameter $\varphi$ and leads to the identification of the zero momentum mode as a Bose condensate that signals spontaneous symmetry breaking even when the order parameter remains zero. Since the effective mass is zero we identify the condensing quanta as pions and therefore this mechanism is a novel form of pion condensation in the absence of direct scattering.

When scattering is included, beyond the leading order in the large $N$ approximation, the formation of the Bose condensate will require a detailed understanding of the different time scales. The time scale for the collisionless process described above must be compared to the time scale for collisional processes that would tend to deplete the condensate. If spinodal instabilities causing non-perturbative particle production at low momentum occur on much shorter time scales than collisional redistribution then we would expect that there will be a non-perturbatively large population at low momenta that could be interpreted as a coherent condensate.

The spinodal instabilities seen in this article are similar to those which lead to the formation of Disoriented Chiral Condensates \[24\,25\,29\,32\]. However we emphasize that unlike most of the previously studied scenarios for DCC formation in which a ‘quench’ into the spinodal region was introduced \textit{ad-hoc}, in the present situation the spinodal instabilities are of \textit{dynamical origin}. We have studied a situation where the vacuum theory has symmetry breaking minima (with $m_R^2 < 0$ in Eq. \[4.32\]) but the initial state is highly excited with the particle density larger than a critical value leading to a symmetry restored theory in the medium. However this initial state is strongly out of equilibrium and its dynamical evolution automatically induces spinodal instabilities.
B. Energy, Pressure and Equation of State:

As mentioned in the introduction the goal of our study is to understand the dynamical evolution of strongly out of equilibrium states. In the usual investigations of the dynamics of the quark gluon plasma one uses a hydrodynamic description in which the energy density, pressure and all the thermodynamic variables depend only on proper time \[1,14\]. The hydrodynamic equations are then a consequence of the conservation laws which are appended with an equation of state to determine the evolution completely. The hydrodynamic regime corresponds to the case when the collisional mean free path is shorter than the wavelength of the hydrodynamic collective modes, and therefore the concept of local thermodynamic equilibrium is warranted.

A valid question in the situation that we have studied in this article, is whether and when an equation of state is a meaningful concept. In the leading order in the large $N$ expansion there are no collisional processes (these arise at $O(1/N)$) and therefore the concept of a hydrodynamic regime in is not applicable in principle. Furthermore since the state considered is spatially homogeneous the pressure will depend on time rather than on proper time. Since the energy is conserved and the pressure evolves with time, an equation of state will have a meaning only when the evolution has reached the asymptotic regime.

The energy density is given by

$$ E = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} m_B^2 \phi^2 + \frac{\lambda B}{8} \phi^4 + \frac{1}{4 \pi^2} \int k^2 dk \frac{k^2}{2 \Omega_q} \left( |\dot{\phi}_q(\tau)|^2 + \omega_q^2(\tau)|\phi_q(\tau)|^2 \right) - \frac{\lambda B}{8} \delta^2(\pi^2)^2. \quad (6.4) $$

The last term which arises in a consistent large $N$ expansion, is extremely important in that after renormalization it provides a negative contribution which can interpreted as part of the effective potential \[44,39\]. Using the equation of motion for the order parameter and the mode functions it is straightforward to show that the energy is conserved and the last term is necessary to ensure energy conservation. Since the energy is conserved it can be renormalized by a subtraction at $\tau = 0$, and therefore is finite in terms of the renormalized quantities.

The pressure is given by the following expression,

$$ \frac{p + E}{NV} = \dot{\phi}^2 + \frac{1}{2 \pi^2} \int k^2 dk \frac{k^2}{2 \Omega_q} \left[ |\dot{\phi}_q(\tau)|^2 + \frac{k^2}{3} |\phi_q(\tau)|^2 \right]. \quad (6.5) $$

Unlike the energy density, the pressure is not a constant of the motion and needs proper subtractions to render it finite. The detailed expressions for both the renormalized energy and pressure can be found in references \[39,45,46\]. However, rather than computing the total energy density and pressure, we will study the contributions from the modes that are highly populated and whose amplitudes become non-perturbatively large ($\approx 1/g$). Asymptotically, when the effective mass vanishes and the low momentum modes become highly populated with amplitudes of $O(1/g)$ the renormalized energy density is given by (see ref. \[39,47\] for the explicit expression of the renormalized energy density)
\[
\frac{E}{NV} = \frac{1}{4 \pi^2} \int_0^{k_m} \frac{k^2 dk}{2 \Omega_q} \left[ |\dot{\phi}_q(\tau)|^2 + \omega_q^2(\tau) |\phi_q(\tau)|^2 \right] + \mathcal{O}(g)
\] (6.6)

where \(k_m\) is the largest spinodally unstable wave vector at early times and \(\mathcal{O}(g)\) represents terms that are perturbatively small. Using the asymptotic solutions for the mode functions given by Eq. (6.3) and neglecting the strongly oscillatory phases that average out at long times we obtain

\[
\frac{E}{NV} = \frac{M_R^4}{2 \pi^2} \int_0^{q_m} \frac{q^4 dq}{2 W_q} \left[ |\alpha_q|^2 + |\beta_q|^2 \right] + \mathcal{O}(g).
\] (6.7)

Similarly, neglecting the contribution of modes with small amplitudes, we find that the renormalized pressure plus energy density is given by,

\[
\frac{p + E}{NV} = \frac{4}{3} \frac{M_R^4}{2 \pi^2} \int_0^{q_m} \frac{q^2 dq}{2 W_q} \left[ |\alpha_q|^2 + |\beta_q|^2 \right],
\] (6.8)

so that in the asymptotic regime

\[
p = \frac{E}{3},
\] (6.9)

independent of the particle distribution which is non-thermal. This is one of the important results of this work. In Fig. (7) we show the trace of the energy momentum tensor \(E - 3P\) as a function of time, for the same value of parameters as for Figs. (1-5). Clearly, the trace vanishes asymptotically.

During the early stages of the dynamics when spinodal instabilities arise and develop with profuse particle production, an equation of state cannot be defined. The dynamics cannot be described in terms of hydrodynamic evolution. Since the processes under consideration are collisionless, there is no local thermodynamic equilibrium and an equation of state is ill-defined.

\section*{VII. CONCLUSIONS}

We have studied the evolution of an \(O(N)\)-symmetric quantum field theory, prepared in a strongly out-of-equilibrium initial state. The initial state was characterized by a particle distribution localized in a thin spherical shell peaked about a non-zero momentum, a spherical “tsunami”. The formulation of this scenario resulted from a simplification of the idealized ‘colliding-pancake’ description of a heavy-ion collision. For a large density of particles in the initial state, the ensuing dynamics is non-perturbative and consequently we studied the \(O(N)\) theory in the leading order in the large \(N\) limit which is a systematic non-perturbative approximation scheme. When the tree-level theory has vacua that spontaneously break the symmetry and the number of particles within a correlation volume at \(t = 0\) is so high that the symmetry is restored initially, spinodal instabilities are then induced dynamically resulting in profuse particle production for low momenta.
This situation is to be contrasted with the usual studies of DCC’s where the initial state is assumed to satisfy LTE (local thermodynamic equilibrium) and the spinodal instabilities are introduced either via an \textit{ad-hoc} quench or via cooling due to hydrodynamic expansion which is also introduced phenomenologically.

Backreaction of the long wavelength fluctuations eventually shuts off these instabilities and the nonlinearities redistribute particles towards low momenta. We thus find asymptotically in time a novel form of pion condensation at low momentum, out of thermal equilibrium. Furthermore, a macroscopic condensate of the Bose-Einstein type will form at much longer times ($mt \sim \sqrt{V}$).

When the spinodal instabilities shut off we find that the asymptotic ‘quasiparticles’ are massless pions with a non-thermal, non-perturbative distribution function peaked at low momentum but with an ultrarelativistic equation of state.

We believe that these phenomena point out to very novel and non-perturbative mechanisms for particle production and relaxation that are collisionless, strongly out of local thermodynamic equilibrium and cannot be described in the early stages via a coarse-grained hydrodynamic evolution. These are the result of strongly out of equilibrium initial states of high density that could potentially be of importance in the dynamics of heavy ion collisions at high luminosity accelerators.

A more realistic treatment, modelling a collision will require an initial state which breaks the rotational invariance and selects out a beam-axis along which the colliding pions move in opposite directions. However, the analysis of such initial conditions is beyond the present numerical capabilities and will be deferred to a future work.

An upshot of this study of high density, non-equilibrium particle distributions is the following tantalizing theoretical question: can one extract a resummation scheme, or an effective theory akin to the Hard Thermal Loop effective expansion for arbitrary \textit{non-equilibrium}, \textit{non-thermal} distributions such as the “tsunami” configuration for \textit{gauge theories}? Possible answers and consequences of such initial states for gauge theories will be discussed in a forthcoming article\cite{47}.

VIII. ACKNOWLEDGEMENTS:

D. B. thanks the N.S.F for partial support through the grant awards: PHY-9605186 and LPTHE for warm hospitality. R. H. and S. P. K. were supported by DOE grant DE-FG02-91-ER40682. S. P. K. would like to thank BNL for hospitality during the progress of this work. H. J. de V. thanks BNL and U. of Pittsburgh for warm hospitality. The work of R.D.P. is supported by a DOE grant at Brookhaven National Laboratory, DE-AC02-76CH00016. The authors acknowledge partial support by NATO.
REFERENCES

[1] J. D. Bjorken, Phys. Rev. D 27, 140 (1983).
[2] L. P. Csernai, “Introduction to Relativistic Heavy Ion Collisions”, (John Wiley and Sons, England, 1994).
[3] C. Y. Wong, “Introduction to High-Energy Heavy Ion Collisions”, (World Scientific, Singapore, 1994).
[4] J. W. Harris and B. Muller, Annu. Rev. Nucl. Part. Sci. 46, 71 (1996). B. Muller in Particle Production in Highly Excited Matter, Eds. H.H. Gutbrod and J. Rafelski, NATO ASI series B, vol. 303 (1993). B. Muller, The Physics of the Quark Gluon Plasma Lecture Notes in Physics, Vol. 225 (Springer-Verlag, Berlin, Heidelberg, 1985).
[5] J-e. Alam, S. Raha and B. Sinha, Phys. Rep. 273, 243 (1996).
[6] H. Meyer-Ortmanns, Rev. of Mod. Phys. 68, 473 (1996).
[7] H. Satz, in Proceedings of the Large Hadron Collider Workshop ed. G. Jarlskog and D. Rein (CERN, Geneva), Vol. 1, page 188; and in Particle Production in Highly Excited Matter, Eds. H.H. Gutbrod and J. Rafelski, NATO ASI series B, vol. 303 (1993).
[8] X.N. Wang and M. Gyulassy, Phys. Rev. D44, 3501 (1991); Phys. Rev. D 45, 844 (1992).
[9] K. Geiger and B. Muller, Nucl. Phys. B369, 600 (1992).
[10] K. Geiger, Phys. Rep. 258, 237 (1995); Phys. Rev. D46, 4965 (1992); Phys. Rev. D47, 133 (1993); Quark Gluon Plasma 2, Ed. by R. C. Hwa, (World Scientific, Singapore, 1995).
[11] K. J. Eskola and X. N. Wang, Phys. Rev. D49, 1284 (1994).
[12] K. J. Eskola, hep-ph/9708472 (Aug. 1997).
[13] For a recent review see: E. Shuryak in Quark Gluon Plasma 2, Ed. by R. C. Hwa, (World Scientific, Singapore, 1995).
[14] F. Cooper, G. Frye and E. Schonberg, Phys. Rev. D11, 192 (1975).
[15] T. S. Biro, H. B. Nielsen and J. Knoll, Nucl. Phys. B245, 449 (1984).
[16] K. Kajantie and T. Matsui, Phys. Lett. B164 (1985), 373; G. Gatoff, A. K. Kerman and T. Matsui, Phys. Rev. D36, 114 (114).
[17] Y. Kluger, J. M. Eisenberg, B. Svetitsky, F. Cooper and E. Mottola, Phys. Rev. Lett. 67, 2427 (1991); Phys. Rev. D 45, 4659, (1992); Phys. Rev. D48, 190 (1993); F. Cooper, in Particle Production in Highly Excited Matter NATO ASI series B, eds. H. Gutbrod and J. Rafelski, Vol. 303 (1993).
[18] A. Anselm, Phys. Letters B217, 169 (1989); A. Anselm and M. Ryskin, Phys. Letters B226, 482 (1991).
[19] J. P. Blaizot and A. Krzywicki, Phys. Rev. D46, 246 (1992); Acta Physica Polonica B23, 561 (1992). See also J. D. Bjorken’s contribution to the proceedings of the ECT workshop on Disoriented Chiral Condensates, available at http://www.cern.ch/WA98/DCC; G. Amelino Camelia, J. D. Bjorken and S. E. Larsson, hep-ph/9706530.
[21] K. L. Kowalski and C. C. Taylor, “Disoriented Chiral Condensate: A white paper for the Full Acceptance Detector, CWRUTH-92-6.
[22] J. D. Bjorken, K. L. Kowalski and C. C. Taylor: “Observing Disoriented Chiral Condensates”, (SLAC-CASE WESTERN preprint 1993) hep-ph/9309235; “Baked Alaska”, (SLAC-PUB-6109) (1993).
[23] For recent reviews on the subject, see: K. Rajagopal, in Quark Gluon Plasma 2, ed. R. Hwa, World Scientific (1995); S. Gavin, Nucl. Phys. A590 (1995), 163c; J. P. Blaizot and A. Krzywicki, hep-ph/9606263 (1996).
[24] K. Rajagopal and F. Wilczek, Nucl. Phys. B379, 395 (1993), Nucl. Phys. B404, 577 (1993).
[25] D. Boyanovsky, D.-S. Lee and A. Singh, Phys. Rev. D48, 800, (1993).
[26] C. M. G. Lattes, Y. Fujimoto and S. Hasegawa, Phys. Rep. 65, 151 (1980); G. J. Alner et al Phys. Rep. 154, 247 (1987).
[27] J. D. Bjorken, “t864 (Minimax): A search fo Disoriented Chiral Condensate at the Fermilab Collider, hep-ph/9610379 (1996). (See also the homepage at fnmine.fnal.gov).
[28] See the homepage of the WA98 collaboration at CERN:www.cern.ch/WA98/DCC.
[29] S. Gavin, A. Goksch and R. D. Pisarski, Phys. Rev. Lett., 72, 2143 (1994); S. Gavin and B. Muller, Phys. Lett. B329, 486 (1994).
[30] J. Randrup, Phys. Rev. Lett. 77, (1996), LBL report 38125 (1995); 39328 (1996); hep-ph/9611228 (1996); hep-ph/9612453.
[31] F. Cooper, Y. Kluger, E. Mottola and J. P. Paz, Phys. Rev. D51, 2377 (1995); Y. Kluger, F. Cooper, E. Mottola, J. P. Paz and A. Kovner, Nucl. Phys. A590, 581c (1995); M. A. Lampert, J. F. Dawson and F. Cooper, Phys. Rev. D54, 2213-2221 (1996), F. Cooper, Y. Kluger and E. Mottola, Phys. Rev. C 54, 3298 (1996).
[32] D. Boyanovsky, H.J. de Vega and R. Holman, Phys. Rev. D51, 734 (1995).
[33] Robert D. Pisarski, “Nonabelian Debye screening, tsunami waves, and worldline fermions” To appear in the proceedings of the International School of Astrophysics “D. Chalonge”, Erice, Italy, Sept. 4-15, 1997; also based on a talk given at the RIKEN BNL Workshop on “Non-equilibrium many body dynamics”, Upton, N.Y., Sept. 23-25, 1997, hep-ph/9710370.
[34] E. Braaten and R. D. Pisarski, Nucl. Phys. B337, 569 (1990); E. Braaten and R. D. Pisarski, Nucl. Phys. B339, 310 (1990); E. Braaten and R. D. Pisarski, Phys. Rev. Lett. 64, 1338 (1990); R. D. Pisarski, Phys. Rev. Lett. 63, 1129 (1989); E. Braaten and R. D. Pisarski, Phys. Rev. D42, 2156 (1990); J. C. Taylor and S. M. H. Wong, Nucl. Phys. B346, 115 (1990); J. Frenkel and J. C. Taylor, ibid B334, 199, (1990); ibid B374, 156 (1992).
[35] F. Cooper and E. Mottola, Mod. Phys. Lett. A2, 635 (1987).
[36] F. Cooper, S.-Y. Pi and P. N. Stancioff, Phys. Rev. D34, 3831 (1986).
[37] F. Cooper, S. Habib, Y. Kluger, E. Mottola, J. P. Paz, P. R. Anderson, Phys. Rev. D50, 2848 (1994).
[38] D. Boyanovsky, H.J. de Vega, R. Holman, Phys. Rev. D49, 2769, (1994).
[39] D. Boyanovsky, H. J. de Vega, R. Holman and J. F. J. Salgado,
Phys. Rev. D54, 7570 (1996);
D. Boyanovsky, D. Cormier, H. J. de Vega and R. Holman,
Phys. Rev. D55, 3373 (1997).
[40] J. Schwinger, J. Math. Phys. 2, 407 (1961); K. T. Mahanthappa, Phys. Rev. 126, 329 (1962); P. M. Bakshi and K. T. Mahanthappa, J. Math. Phys. 41, 12 (1963); L. V. Keldysh, JETP 20, 1018 (1965); K. Chou, Z. Su, B. Hao And L. Yu, Phys. Rep. 118, 1 (1985); A. Niemi and G. Semenoff, Ann. of Phys. (NY) 152, 105 (1984); N. P. Landsmann and C. G. van Weert, Phys. Rep. 145, 141 (1987); E. Calzetta and B. L. Hu, Phys. Rev. D41, 495 (1990); ibid D37, 2838 (1990); J. P. Paz, Phys. Rev. D41, 1054 (1990); ibid D42, 529(1990).
[41] R. R. Parwani, Phys. Rev. D45, 4695, (1992).
[42] Handbook of Mathematical Functions, M. Abramowitz and I. Stegun, (Dover Publications, N.Y. 1970).
[43] We thank E. Mottola and F. Cooper for conversations held a long time ago that helped to understand this argument.
[44] D. Boyanovsky, H. J. de Vega, R. Holman, D.-S. Lee and A. Singh,
Phys. Rev. D51, 4419 (1995).
[45] D. Boyanovsky, H. J. de Vega and R. Holman, “Erice Lectures on Inflationary Cosmology”, in the Proceedings of the 5th Erice Chalonge School on Astrofundamental Physics, Ed. N. Sanchez and A. Zichichi, (Kluwer) hep-ph-9701304.
[46] J. Baacke, K. Heitmann and C. Päßzold, Phys. Rev. D55, 2320 (1997) and hep-ph/970627.
[47] D. Boyanovsky, H. J. de Vega, R. Holman, S. P. Kumar and R. D. Pisarski, in preparation.
FIG. 1. $g\Sigma(\tau)$ and $M^2(\tau)$ vs. $\tau$ respectively, with the initial distribution Eq. (5.2), $N_0 = 2000$; $g = 10^{-2}$; $\varphi(0) = 10^{-3}$; $q_0 = 5$
FIG. 2. Distribution function $g_{nq}$ vs. $q$ for $\tau = 0, 2, 5, 10$ respectively with the same parameters as in Fig. [1]
FIG. 3. $g_{nq}(\tau = 30, 80)$ vs. $q$. with the same parameters as in Fig. [I], zoomed in the region near $q \approx 0$ and the peak of the initial distribution $q_0 = 5$. 
FIG. 4. Total number of particles $N(\tau)$ vs. $\tau$. with the same parameters as in Fig. [1]
FIG. 5. $\varphi(\tau)$ vs. $\tau$. with the same parameters as in Fig. (1)
FIG. 6. $\Delta_q(\tau = 5) = g|\phi_q(\tau = 5)|^2 - g|\phi_q(\tau = 0)|^2$ vs. $q$ for the same parameters as in Fig. (1).
FIG. 7. $E - 3P$ vs. $\tau$ for the same parameters as in Fig. (1)