THE UNREASONABLE EFFECTIVENESS OF NONSTANDARD ANALYSIS

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ABSTRACT

The aim of my talk is to highlight a hitherto unknown computational aspect of Nonstandard Analysis. In particular, we provide an algorithm which takes as input the proof of a mathematical theorem from ‘pure’ Nonstandard Analysis, i.e. formulated solely with the nonstandard definitions (of continuity, integration, differentiability, convergence, compactness, et cetera), and outputs a proof of the associated effective version of the theorem. Intuitively speaking, the effective version of a mathematical theorem is obtained by replacing all its existential quantifiers by functionals computing (in a specific technical sense) the objects claimed to exist. Our algorithm often produces theorems of Bishop’s Constructive Analysis [2].

The framework for our algorithm is Nelson’s syntactic approach to Nonstandard Analysis, called internal set theory [4], and its fragments based on Gödel’s T as introduced in [1]. Notable results are that applying our algorithm to theorems involving nonstandard compactness, we rediscover, depending on the formulation of the latter, either totally boundedness, the preferred notion of compactness in constructive and computable analysis, or equivalences from the foundational program Reverse Mathematics [5]. Currently, we can treat theorems ‘up to’ the Stone-Weierstraß theorem: Some proofs not involving Nonstandard Analysis can be treated too using the (more complicated) framework from [3].

Finally, we establish that a theorem of Nonstandard Analysis has the same computational content as its ‘highly constructive’ Herbrandisation. Thus, we establish an ‘algorithmic two-way street’ between so-called hard and soft analysis, i.e. between the worlds of numerical and qualitative results.

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