MINIMIZING CURVES IN PROX-REGULAR SUBSETS OF RIEMANNIAN MANIFOLDS

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Abstract. We obtain a characterization of the proximal normal cone to a prox-regular subset of a Riemannian manifold. Moreover, some properties of Bouligand tangent cones to prox-regular sets are described. We prove that for a prox-regular subset $S$ of a Riemannian manifold, the metric projection $P_S$ to $S$ is locally Lipschitz on an open neighborhood of $S$ and it is directionally differentiable at boundary points of $S$. Finally, a necessary condition for a curve to be a minimizing curve in a prox-regular set is derived.

1. Introduction

Closed subsets of Hilbert spaces satisfying an external sphere condition with uniform radius have been studied as generalizations of convex sets, mostly in relation to uniqueness of the metric projection and smoothness of the distance function. In the fundamental paper [10] where the finite dimensional case is considered, these sets were called sets with positive reach. Then various equivalent definitions related to this property have been presented independently by several authors; see [7, 19] and the references therein. Among them, one can mention the notions of $\varphi$-convexity (as titled $p$-convexity) and prox-regularity of sets which were introduced in [8] and [19], respectively. It was shown in [5] that certain properties which hold globally for convex sets are still valid locally for $\varphi$-convex sets.

Differentiability properties of the metric projection onto closed convex sets are of interest in sensitivity analysis of variational inequalities and optimal control problems. Moreover, the regularity of the metric projection onto a sufficiently regular submanifold $M$ of $\mathbb{R}^n$ as well as the regularity of the corresponding distance function have significant role in various aspects of analysis; see [18]. A classical example is the Dirichlet problem for quasilinear partial differential equations, where the manifold of interest is the boundary of the underlying domain; see, for instance, [11].

The example presented by J. Kruskal [15] shows that, in general such a projection is not directionally differentiable, even in finite dimensional

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spaces. By directionally differentiable at a point we mean that the directional derivative exists for all directions through that point. This is weaker than the existence of the gradient at that point.

The problem of differentiability of the metric projection for a closed locally convex subset $S$ of a finite dimensional Riemannian manifold $M$ was studied in [23] and it was proved in [12] that for a closed totally convex subset $S \subset M$, there exists an open set $W$ containing $S$ such that the metric projection is locally Lipschitz on $W$.

In [3] the notion of $\varphi$-convex sets was extended to Hadamard manifolds and it was shown that if $S$ is a $\varphi$-convex subset of an infinite-dimensional Hadamard manifold $M$, then there exists a neighborhood $U$ of $S$ in $M$ such that the metric projection $P_S : U \rightarrow S$ is single-valued and locally Lipschitz. Moreover, it was proved that under the same assumptions on $S$ and $M$, there exists a neighborhood $U$ of $S$ in $M$ such that $d^2_S$ is $C^1$ with locally Lipschitz gradient on $U \setminus S$. On the other hand, in [13] the notion of prox-regular sets was introduced on Riemannian manifolds as a subclass of regular sets. In [20] we proved that the two classes of $\varphi$-convex sets and prox-regular sets coincide in the setting of Riemannian manifolds.

The problem of existence and uniqueness of geodesics on a Riemannian manifold without boundary is a classical subject of differential geometry and global nonlinear analysis and is particularly fit to a treatment by variational methods. However, in the case of Riemannian manifolds with boundary or certain subsets of a manifold without boundary, strong irregularities appear in the energy functional and new techniques are needed for dealing with these problems. In [4, 5] $\varphi$-convex subsets of a Real Hilbert space were considered and using an infinitesimal definition of geodesics in the framework of Sobolev spaces the author characterized these geodesics as critical points of an energy functional on a suitable path spaces.

The aim of this paper is to study minimizing curves in a prox-regular subset $S$ of a Riemannian manifold $M$. To this end, we use some powerful tools from nonsmooth analysis and an adapted variational technique. Applying the first variation formula, we give a necessary condition for an admissible curve $\gamma : [a, b] \rightarrow M$ in $S$ to be minimizing. Indeed, this curve has the property that

$$D_t \dot{\gamma}(t) \in N^P_S(\gamma(t)),$$

for every $t \in [a, b]$ except for finitely many points, provided that $S$ has a $C^2$ boundary, where $N^P_S(x)$ is the proximal normal cone at $x \in S$. To prove this result, we address the problem of the directional differentiability of the metric projection $P_S$ at boundary points of $S$. Employing Shapiro’s variational principle [22], we show that for a prox-regular subset $S$ of a Riemannian manifold $M$, the projection map $P_S$ is locally Lipschitz on an open neighborhood of $S$ which generalizes the result of [3] to the Riemannian setting. Moreover, we prove that $P_S$ is directionally differentiable at boundary points.
of $S$. We also obtain a useful characterization of Bouligand tangent cone to a prox-regular set.

The paper is organized as follows. In Section 2 we present some basic constructions and preliminaries in Riemannian geometry and nonsmooth analysis, widely used in the sequel. Section 3 is devoted to the study of Bouligand and proximal normal cones. Then we obtain a characterization of the proximal normal cone to a prox-regular set. We also show that $P_S$ is a locally Lipschitz retraction from a neighborhood of $S$ to $S$. In Section 4 differentiability properties of the metric projection $P_S$ to a prox-regular subset $S$ of a Riemannian manifold are investigated which leads to a characterization of Bouligand tangent cone. Section 5 is concerned with the necessary condition for a curve $\gamma$ to be a minimizing curve in a prox-regular set whose boundary is a $C^2$ submanifold of $M$. Moreover, some relevant examples are presented.

2. Preliminaries and notations

Let us recall some notions of Riemannian manifolds and nonsmooth analysis; see, e.g., [6, 9, 21]. Throughout this paper, $(M, g)$ is a finite-dimensional Riemannian manifold endowed with a Riemannian metric $g_x = \langle \cdot, \cdot \rangle_x$ on each tangent space $T_x M$ and $\nabla$ is the Riemannian connection of $g$. For every $x, y \in M$, the Riemannian distance from $x$ to $y$ is denoted by $d(x, y)$. Moreover, $B(x, r)$ and $\overline{B}(x, r)$ signify the open and closed metric ball centered at $x$ with radius $r$, respectively. For a smooth curve $\gamma: I \to M$ and $t_0, t \in I$, the notation $L_{t_0 t}^\gamma$ is used for the parallel transport along $\gamma$ from $\gamma(t_0)$ to $\gamma(t)$. When $\gamma$ is the unique minimizing geodesic joining $\gamma(t_0)$ to $\gamma(t)$, we use $L_{t_0 t}$ instead of $L_{t_0 t}^\gamma$. Furthermore for a smooth vector field $X$ along $\gamma$, $D_t X$ is the covariant derivative of $X$ along $\gamma$. For a fixed point $z \in M$, the function $\varphi: M \to \mathbb{R}$ defined by $\varphi(x) = d^2(x, y)$ is $C^\infty$ on any convex neighborhood of $z$ and for every $x$ in a convex neighborhood of $z$, $\nabla \varphi(x) = -2 \exp_x^{-1} z$.

Let $S$ be a nonempty closed subset of $M$. The proximal normal cone to $S$ at $x \in S$, is denoted by $N_S^P(x)$ and $\xi \in N_S^P(x)$ if and only if there exists $\sigma > 0$ such that

$$\langle \xi, \exp_x^{-1} y \rangle \leq \sigma d^2(x, y),$$
for every \( y \in U \cap S \), where \( U \) is a convex neighborhood of \( x \). The metric projection to \( S \), denoted by \( P_S \), is defined by

\[
P_S(z) = \left\{ x \in S : d(x, z) = \inf_{y \in S} d(y, z) \right\} \quad \forall z \in M.
\]

Moreover, \( \text{Unp}(S) \) is considered as the set of all points \( z \in M \) with the property that \( P_S(z) \) is single-valued. Then according to [20, Lemma 4.11], the projection map \( P_S : \text{Unp}(S) \to S \) is continuous. For every \( x \in S \) we also define

\[
\text{reach}(S, x) := \sup \{ r \geq 0 : B(x, r) \subseteq \text{Unp}(S) \},
\]

It is worth mentioning that the function \( x \mapsto \text{reach}(S, x) \) is continuous on \( S \); see [2, 14] for more details.

In order to deduce the Lipschitz property and directional differentiability of \( P_S \), we use the following variational principal by A. Shapiro [22]. Let \( f, g : X \to \mathbb{R} \) be two functions on a Hilbert space \( X \) and \( S, T \subset X \). Consider the optimization problems

(2.1) \[
\min_{x \in S} f(x)
\]

and

(2.2) \[
\min_{x \in T} g(x).
\]

Let \( x_0 \) and \( \bar{x} \) be some optimal solutions of (2.1) and (2.2), respectively and suppose that there exist a neighborhood \( W \) of \( x_0 \) and \( \alpha > 0 \) such that for every \( x \in S \cap W \),

(2.3) \[
f(x) \geq f(x_0) + \alpha \| x - x_0 \|^2.
\]

Also, suppose that \( \bar{x} \in W \) and \( f \) and \( g \) are Lipschitz on \( W \) with Lipschitz constants \( k_1 \) and \( k_2 \), respectively. Then

(2.4) \[
\| \bar{x} - x_0 \| \leq \alpha^{-1}\kappa + 2\delta_1 + \alpha^{-1/2}(k_1\delta_1 + k_2\delta_2)^{1/2},
\]

where \( \kappa \) is a Lipschitz constant of \( h(x) = g(x) - f(x) \) on \( W \) and

\[
\delta_1 = \sup_{x \in T \cap W} d(x, S \cap W),
\]

\[
\delta_2 = d(x_0, T \cap W).
\]

3. Local Lipschitzness of metric projection

In this section we first derive some properties of Bouligand tangent cones to prox-regular sets which we need in the sequel. Let us begin by recalling some required definitions; see [13, 20].

The closed subset \( S \) of \( M \) is said to be prox-regular at \( \bar{x} \in S \) if there exist \( \varepsilon > 0 \) and \( \sigma > 0 \) such that \( B(\bar{x}, \varepsilon) \) is convex and for every \( x \in S \cap B(\bar{x}, \varepsilon) \) and \( v \in N_S^P(x) \) with \( \| v \| < \varepsilon \),

\[
\langle v, \exp_x^{-1} y \rangle \leq \sigma d^2(x, y) \quad \text{for every } y \in S \cap B(\bar{x}, \varepsilon).
\]
Moreover, $S$ is called prox-regular if it is prox-regular at each point of $S$.

In [20, Theorem 3.4], we proved that every $\varphi$-convex subset of a Riemannian manifold $M$ is prox-regular and conversely, for every prox-regular subset $S$ of $M$ there exists a continuous function $\varphi : S \to [0, \infty)$ such that $S$ is $\varphi$-convex. Recall that a closed subset $S \subseteq M$ is called $\varphi$-convex if for every $x \in S$ and $v \in N^\varphi_S(x)$

$$\langle v, \exp^{-1}_x y \rangle \leq \varphi(x) \|v\|^2(x, y),$$

for every $y \in U \cap S$, where $U$ is a convex neighborhood of $x$ and $\varphi : S \to [0, \infty)$ is a continuous function. Note that this definition is independent of the choice of any convex neighborhood of $x$.

Let $S \subseteq M$ be a closed subset and $x \in S$. The Bouligand (or contingent) tangent cone to $S$ at $x$ is defined as

$$T^B_S(x) := \left\{ \lim_{i \to \infty} \frac{\exp^{-1}_x z_i}{t_i} : z_i \in U \cap S, z_i \to x \text{ and } t_i \downarrow 0 \right\},$$

where $U$ is a convex neighborhood of $x$ in $M$. It was shown in [13] that when $S$ is prox-regular, $T^B_S(x)$ is a convex cone for every $x \in S$.

**Lemma 3.1.** Let $S \subseteq M$ be a prox-regular set and $x \in S$. Then

(i) $T^B_S(x) = \left( N^P_S(x) \right)^\circ$,

(ii) $(T^B_S(x))^\circ = N^P_S(x)$.

**Proof.** Assertion (i) can be obtained from [13, Lemma 3.7]. Indeed, we have

$$T^C_S(x) \subseteq T^B_S(x) \subseteq \left( N^P_S(x) \right)^\circ = \left( N^C_S(x) \right)^\circ = T^C_S(x),$$

where $T^C_S(x)$ and $N^C_S(x)$ are (Clarke) tangent and normal cone to $S$ at $x$, respectively.

Assertion (ii) follows from the fact that $N^P_S(x)$ is closed and convex. Hence $(\left( N^P_S(x) \right)^\circ)^\circ = N^P_S(x)$. $\square$

According to [20, Proposition 4.2], for every point $x$ in a closed prox-regular subset $S$ of $M$, $\text{reach}(S, x) > 0$. This property of prox-regular sets helps us to prove the following topological property of these sets.

**Lemma 3.2.** If $S$ is a closed set with the property that $\text{reach}(S, x) > 0$ for every $x \in S$, then $S$ is locally connected.

**Proof.** Let $x \in S$ and $U$ be an open neighborhood of $x$ in $M$. We are going to verify that there exists a neighborhood $V$ of $x$ in $M$ such that $V \subseteq U$ and $V \cap S$ is connected.

If this fails to be the case, then for all positive integer $n$ large enough so that $B(x, 1/n)$ is convex and $B(x, 1/n) \subseteq U$, the set $S_n := S \cap B(x, 1/n)$ is not connected. Suppose that $A_n$ is the connected component of $S_n$ contains $x$, the set $B_n$ is another connected component of $S_n$ and $y_n$ is an arbitrary point of $B_n$. Let $\gamma : [0, 1] \to M$ be the unique minimizing geodesic joining $x, y_n$ and hence its image is entirely in $B(x, 1/n)$. 
Note that $P_S(\gamma(t)) \in S \cap B(x, 1/n)$ for every $t \in [0, 1]$, since
\[
d(P_S(\gamma(t)), x) \leq d(P_S(\gamma(t)), \gamma(t)) + d(\gamma(t), x) \\
\leq d(y_n, \gamma(t)) + d(\gamma(t), x) \\
= d(y_n, x) < 1/n.
\]

We now claim that the image of $\gamma$ on $[0, 1]$ is not entirely in $\Unp(S)$. Otherwise, the continuity of $P_S$ on $\Unp(S)$ ([20, Lemma 4.11]) implies that the set $P_S(\gamma([0, 1]))$ is connected. Since $P_S(\gamma([0, 1])) \subseteq S_n$ and contains $x$, we have $P_S(\gamma([0, 1])) \subseteq A_n$. It follows that $y_n \in A_n$ which contradicts our choice of $y_n$. Then there exists a sequence $\{z_n\}$ such that $z_n \notin \Unp(S)$ and $d(x, z_n) < 1/n$. It implies that $\text{reach}(S, x) = 0$ and this contradiction completes the proof of the lemma. 

Lemma 3.2 implies that every closed prox-regular subset of $M$ is locally connected.

Example 3.3. A well known example of a connected set which is not locally connected is the comb space,
\[C = ([0, 1] \times 0) \cup (K \times [0, 1]) \cup (0 \times [0, 1]),\]
in $\mathbb{R}^2$ where $K = \{1/n : n \in \mathbb{N}\}$. Note that this set is not prox-regular, because for every $x \in (0 \times [0, 1])$, $\text{reach}(C, x) = 0$.

In the following theorem, we obtain a characterization of proximal normal cones to prox-regular subsets of $M$.

Theorem 3.4. Suppose that $S$ is a closed subset of $M$ with the property that its boundary, denoted by $\partial S$, is an embedded $k$-dimensional submanifold of $M$ and $x \in \partial S$. Then
(a) If $\partial S$ is $C^1$, then $N^P_S(x) \subseteq T^\perp_x \partial S$ where $T^\perp_x \partial S$ is the normal space to $\partial S$ at $x$.
(b) If in addition $S$ is prox-regular with nonempty interior and $\partial S$ is $C^2$, then there exist a neighborhood $U$ of $x$ in $M$ and a $C^2$ submersion $\psi : U \to \mathbb{R}$ such that $U \cap \partial S = \psi^{-1}(0)$ and proximal normal cone to $S$ at $x$ is one of the following
\[N^P_S(x) = \text{cone} \{\nabla \psi(x)\},\]
or
\[N^P_S(x) = \text{span} \{\nabla \psi(x)\}.
\]

Proof. Since $\partial S$ is an embedded $k$-dimensional submanifold of $M$, there exists a neighborhood $U$ of $x$ in $M$ such that $U \cap \partial S$ is a level set of a submersion $\psi : U \to \mathbb{R}^{n-k}$, $\psi = (\psi_1, \ldots, \psi_{n-k})$. If $\partial S$ is $C^1$, then along the same lines as the proof of [6, Proposition 1.9], we have
\[N^P_S(x) \subseteq N^P_{\partial S}(x) \subseteq \text{span} \{\nabla \psi_i(x) : i = 1, \ldots, n-k\} = T^\perp_x \partial S.
\]

If in addition $S$ is prox-regular with nonempty interior and $\partial S$ is $C^2$, then $\partial S$ is a codimension 1 submanifold of $M$. Moreover, by Lemma 3.2, $S$ is locally connected and hence by shrinking $U$ if necessary, we may assume
that $U$ is convex and $U \cap S$ is connected. If $U \cap S^0 = \emptyset$ where $S^0$ denotes the interior of $S$ (or there exists a neighborhood $V \subseteq U$ of $x$ such that $V \cap S^0 = \emptyset$), then $U \cap S = U \cap \partial S$ and by [6, Proposition 1.9], we have

$$N^P_S(x) = N^P_{\partial S}(x) = \text{span } \{ \nabla \psi(x) \}.$$  

Now let $U \cap S^0$ be nonempty. Since $U \cap S^0$ is connected and $U \cap \partial S = \psi^{-1}(0)$, we have

$$\psi(U \cap S^0) \subseteq (-\infty, 0) \text{ or } \psi(U \cap S^0) \subseteq (0, +\infty).$$

Replacing $\psi$ by $-\psi$ if necessary, we can assume that $\psi(y) \leq 0$ for every $y \in U \cap S$. Let $\xi := \lambda \nabla \psi(x)$ for some $\lambda \geq 0$. For given $\sigma > 0$, we define

$$h(y) := \langle -\xi, \exp x y \rangle + \sigma d^2(x, y) + \lambda \psi(y),$$

for every $y \in U$. Then $\nabla h(x) = 0$ and for $\sigma$ sufficiently large, $\text{Hess } h(x)$ is positive definite because for every $v \in T_xM$ we have

$$\text{Hess } h(x)(v)^2 = \frac{d^2}{dt^2} \bigg|_{t=0} \left(h\left(\exp_x(tv)\right)\right)$$

$$= \frac{d}{dt} \bigg|_{t=0} \left(\langle -\xi, tv \rangle + \sigma t^2 \|v\|^2 + \lambda \psi\left(\exp_x(tv)\right)\right)$$

$$= 2\sigma \|v\|^2 + \lambda \text{Hess } \psi(x)(v)^2.$$  

Therefore $h$ has a local minimum at $x$ and so there exists a neighborhood $V$ of $x$ such that $V \subseteq U$ and for every $y \in V \cap S$ we have

$$\langle \xi, \exp x y \rangle \leq \sigma d^2(x, y) + \lambda \psi(y) \leq \sigma d^2(x, y).$$

It follows that $\xi \in N^P_S(x)$ which completes the proof of the theorem. $\square \quad \square$

It is worth mentioning that in part (b) of Theorem 3.4 if the interior of $S$ is empty, then $S = \partial S$ and by [6, Proposition 1.9] we have

$$N^P_S(x) = \text{span } \{ \nabla \psi_i(x) : i = 1, \ldots, n - k \} = T_x^\perp \partial S.$$  

**Example 3.5.** Let $S$ be the set $((0) \cup [1, +\infty)) \times \mathbb{R}$ in $\mathbb{R}^2$ and consider the points $(0, 0), (1, 0) \in \partial S$. Then $S$ is prox-regular and has a smooth boundary. At the point $(0, 0)$ we have $\psi(x, y) = x$, $N^P_S(0, 0) = \text{span } \{(1, 0)\}$ and at the point $(1, 0),$ 

$$\psi(x, y) = 1 - x \text{ and } N^P_S(1, 0) = \text{cone } \{(-1, 0)\}.$$  

In what follows, the closed set $S^c \cup \partial S$ is denoted by $\hat{S}$. Note that $\partial \hat{S} \subseteq \partial S$ and if the point $x \in \partial S$ is such that $x \notin \partial \hat{S}$, then $x$ is the interior point of $\hat{S}$.

**Theorem 3.6.** Suppose that $S$ is prox-regular and $\partial S$ is a $C^2$ submanifold of $M$. If $x \in \partial S$, then

$$(3.1) \quad T^B_S(x) \cap T^B_{\hat{S}}(x) = T_x \partial S.$$  

**Proof.** In the case when $S^0 = \emptyset$, we have $T^B_S(x) = T_x \partial S$ and $T^B_{\hat{S}}(x) = T_x M$. So we assume that the interior of $S$ is nonempty. Let $U$ and the submersion $\psi : U \to \mathbb{R}$ be the ones applied in the proof of Theorem 3.4. If $U \cap S^0 = \emptyset$
(or there exists a neighborhood \( V \subseteq U \) of \( x \) such that \( V \cap \partial S^o = \emptyset \), then \( U \subseteq \hat{S} \) and \( T^B_{\hat{S}}(x) = T_x M \). Hence the expression (3.1) holds.

We now consider the case in which \( U \cap \partial S^o \) is nonempty and for every neighborhood \( V \) of \( x \) contained in \( U \), \( V \cap \partial S^o \neq \emptyset \). Then \( U \cap \partial \hat{S} = U \cap \partial S \) and we claim that

\[
N^P_{\hat{S}}(x) = \text{cone} \{ -\nabla \psi(x) \}.
\]

Indeed, Since \( \text{Unp}(\partial S) \subseteq \text{Unp}(\hat{S}) \) and \( \partial S \) is a \( C^2 \) submanifold of \( M \), for every \( z \in \partial \hat{S} \subseteq \partial S \) we have

\[
\text{reach}(\hat{\hat{S}}, z) > \text{reach}(\partial S, z) > 0.
\]

Then \( \text{reach}(\hat{S}, z) > 0 \) for every \( z \in \hat{S} \) and by Lemma 3.2, \( \hat{S} \) is locally connected. Without loss of generality, we assume that \( U \cap \hat{S} \) is connected. By the choice of \( \psi \) we have \( \psi(y) \leq 0 \) for every \( y \in U \cap S \). On the other hand, \( \psi \) is a submersion on \( U \) and \( U \cap \partial \hat{S} = U \cap \partial S = \psi^{-1}(0) \). Then \( \psi(y) \geq 0 \) for every \( y \in U \cap \hat{S} \) and the claim is proved by a procedure similar to the proof of Theorem 3.4. So we have

\[
N^P_S(x) \cup N^P_{\hat{S}}(x) = T^N_x \partial S \quad \text{and} \quad N^P_S(x) \cap N^P_{\hat{S}}(x) = \{0\}.
\]

Let us now prove the expression (3.1). Since \( U \cap \partial \hat{S} = U \cap \partial S \), the set \( \hat{S} \) is prox-regular and applying Lemma 3.1, we deduce that \( T_x \partial S \subseteq T^B_{\hat{S}}(x) \cap T^B_S(x) \). Let \( v \in T^B_{\hat{S}}(x) \cap T^B_S(x) \) and \( w \in T^N_x \partial S \) be arbitrary. Without loss of generality, we assume that \( w \in N^P_S(x) \). Thus \( -w \in N^P_{\hat{S}}(x) \) and applying Lemma 3.1, we have \( \langle v, w \rangle \leq 0 \) and \( \langle v, -w \rangle \leq 0 \). It follows that \( v \in T_x \partial S \).

**Theorem 3.7.** Let \( S \) be a \( \varphi \)-convex subset of \( M \), \( x \in S \) and \( U \) be a convex neighborhood of \( x \). Then

\[
d \left( \exp_x^{-1} y, T^B_S(x) \right) \leq \varphi(x)d^2 (x, y),
\]

for every \( y \in U \cap S \).

**Proof.** Since \( T^B_S(x) \) is a closed convex subset of \( T_x M \), for any \( v \in U \cap S \) there exists a unique vector \( v \in T^B_S(x) \) such that \( d \left( \exp_x^{-1} y, T^B_S(x) \right) = \| \exp_x^{-1} y - v \| \). Therefore we have

\[
\exp_x^{-1} y - v \in N^P_{T^B_{\hat{S}}(x)}(v).
\]

Let us now show that \( N^P_{T^B_{\hat{S}}(x)}(v) \subseteq N^P_{T^B_S(x)}(v) \). Clearly, \( \langle \xi, v \rangle = 0 \) for every \( \xi \in N^P_{T^B_{\hat{S}}(x)}(v) \). Let \( \xi \in N^P_{T^B_{\hat{S}}(x)}(v) \), then for every \( w \in T^B_S(x) \),

\[
\langle \xi, w \rangle = \langle \xi, w - v \rangle + \langle \xi, v \rangle \leq 0.
\]

Thus \( \xi \in \left(T^B_S(x)^\circ \right) \) and by Lemma 3.1 it follows that \( \xi \in N^P_{T^B_S(x)}(v) \).
Hence \( \exp_x^{-1} y - v \in N^P_S(x) \) and so we have
\[
\left\langle \frac{\exp_x^{-1} y - v}{\|\exp_x^{-1} y - v\|}, \exp_x^{-1} y \right\rangle \leq \varphi(x) d^2(x, y).
\]
This implies that \( \|\exp_x^{-1} y - v\| \leq \varphi(x) d^2(x, y) \) which completes the proof. 

We are now ready to prove that the projection map \( P_S \) is locally Lipschitz on an open set containing \( S \), where \( S \) is a prox-regular subset of \( M \). In [13], this property of prox-regular sets is verified in the special case in which \( M \) is a Hadamard manifold.

Recall that the Hessian of a \( C^2 \) function \( \psi \) on \( M \) is defined by
\[
\text{Hess} \, \psi(x)(v, w) := \langle \nabla_X \nabla \psi, Y \rangle(x),
\]
for every \( x \in M \) and \( v, w \in T_x M \) where \( X, Y \) are any vector fields such that \( X(x) = v \) and \( Y(x) = w \) and \( \nabla \psi \) denotes the gradient of \( \psi \).

**Lemma 3.8.** Let \( M \) be a Riemannian manifold and \( x \in M \). Assume that \( R > 0 \) and \( k_0 > 0 \) are given such that \( |k| \leq k_0 \) for every sectional curvature \( k \) on \( B(x, R) \). Then the function \( \psi(z) := d^2(x, z) \) is smooth on \( B(x, r) \) for every \( r > 0 \) with \( r < \min \left\{ r(x), R, \frac{\pi}{2 \sqrt{k_0}} \right\} \) and
\[
\text{Hess} \, \psi(z)(w)^2 \geq c(z) \|w\|^2,
\]
for every \( z \in B(x, r) \) and \( w \in T_z M \), where
\[
c(z) = \min \left\{ 2, 2 \sqrt{k_0} d(x, z) \cot \left( \sqrt{k_0} d(x, z) \right) \right\}.
\]

**Proof.** Let \( z \in B(x, r) \) and \( w \in T_z M \). Thus according to the proof of [1, Proposition 2.2], we have
\[
\text{Hess} \, \psi(z)(w)^2 = 2l \langle D_l X(l), X(l) \rangle,
\]
where \( l = d(x, z) \), \( X \) is the unique Jacobi field along \( \gamma \) with the property that \( X(0) = 0 \) and \( X(l) = w \) and \( \gamma \) is the unique minimizing geodesic, parameterized by arc length, such that \( \gamma(0) = x \) and \( \gamma(l) = z \).

Let \( w = w^T + w^\perp \) be the orthogonal decomposition of \( w \) where \( w^T \) is tangent to \( \gamma \) and \( w^\perp \) is orthogonal to \( \gamma \) at \( z \). Using Propositions 2.3 and 2.4 of Chapter IX of [16], the Jacobi field \( X \) can be decomposed into \( X = X^T + X^\perp \) where \( X^T \) and \( X^\perp \) are Jacobi fields along \( \gamma \) with the property that \( X^T \) and \( D_l X^T \) are tangent to \( \gamma \) and \( X^\perp \) and \( D_l X^\perp \) are orthogonal to \( \gamma \). So \( X^T(l) = w^T \) and \( X^\perp(l) = w^\perp \) and using the proof of [1, Proposition
Let the solution of the equation $2.2$.

Hess $\psi(z)(w)^2 = 2l \langle D_t X(l), X(l) \rangle$

\[
= 2l \langle D_t X^\top(l), X^\top(l) \rangle + 2l \langle D_t X^1(l), X^1(l) \rangle \\
\geq 2l \left( \frac{1}{l} \|w^\top\|^2 \right) + 2l \sqrt{k_0} \cot(l \sqrt{k_0}) \|w^\top\|^2 \\
= 2\|w^\top\|^2 + 2l \sqrt{k_0} \cot(l \sqrt{k_0}) \|w^\top\|^2 \\
\geq c(z)\|w\|^2
\]

where $c(z) = \min \left\{ 2, 2\sqrt{k_0} d(x, z) \cot \left( \sqrt{k_0} d(x, z) \right) \right\}$. \hfill $\square$

\begin{theorem}
Suppose that $S$ is a closed prox-regular subset of a Riemannian manifold $M$. Then $P_S$ is locally Lipschitz on an open set $V$ containing $S$.
\end{theorem}

\begin{proof}
Since $S$ is prox-regular, there exists a continuous function $\varphi : S \to [0, \infty)$ such that $S$ is $\varphi$-convex. Let $x \in S$ and let $R > 0$ be such that $R < r(x)$ and $B(x, R)$ has compact closure and $B(x, R) \subseteq \text{up}(S)$. Suppose that $k_0 > 0$ and $\rho > 0$ are two constants such that $|k| \leq k_0$ for every sectional curvature $k$ on $B(x, R)$ and $\varphi(z) \leq \rho$ for every $z \in B(x, r) \cap S$. Consider $\vec{r} > 0$ given by $\vec{r} \leq r(z)$ for every $z \in B(x, R)$.

Let $a \in \mathbb{R}$ be the solution of the equation $2t \cot(t) = 1$ on the interval $(0, \frac{\pi}{2})$. So we have $2t \cot(t) > 1$ for every $t \in (0, a)$. We now choose $r > 0$ such that

\[ r < \min \left\{ \frac{R}{2}, \vec{r}, \frac{1}{4\rho}, \frac{a}{\sqrt{k_0}} \right\}. \]

We now define the following optimization problems

\begin{align*}
\text{(3.2) } & \min_{s \in \text{exp}_x^{-1}(S \cap B(x, R))} d^2 (x_1, s) = \min_{v \in \exp_x^{-1}(S \cap B(x, R))} d^2 (x_1, \exp_x v), \\
\text{(3.3) } & \min_{s \in \text{exp}_x^{-1}(S \cap B(x, R))} d^2 (x_2, s) = \min_{v \in \exp_x^{-1}(S \cap B(x, R))} d^2 (x_2, \exp_x v).
\end{align*}

Let $P_S(x_1) = s_1$ and $P_S(x_2) = s_2$, hence $s_1 \in B(x, R)$ and $s_1$ is the optimal solution of (4.1). Moreover, $s_1 \in B(x_1, r) \subseteq B(x, R)$ because $x \in S$ and $d(x_1, s_1) \leq d(x_1, x) < r$.

We claim that there exists a positive constant $\sigma$ such that

\[ d^2 (x_1, s) \geq d^2 (x_1, s_1) + \sigma d^2 (s, s_1), \]

for every $s \in S \cap B(x_1, r)$.

Let $s \in S \cap B(x_1, r)$ and $\gamma(t) = \exp_{s_1} \left( t \exp_{s_1}^{-1} s \right)$ be the unique geodesic joining $s_1$ and $s$ which is entirely in $B(x_1, r)$. We now define $\psi(z) := d^2 (x_1, z)$ for every $z \in M$. Then using the Taylor expansion, there exists $t_0 \in (0, 1)$ such that

\begin{align*}
\text{(3.4) } & d^2 (x_1, s) = d^2 (x_1, s_1) - 2 \langle \exp_{s_1}^{-1} x_1, \exp_{s_1}^{-1} s \rangle + \frac{1}{2} \text{Hess } \psi(x_0)(v_0)^2,
\end{align*}

where $x_0$ is the solution of the equation $2.2$.

\end{proof}
where \( x_0 = \gamma(t_0) \) and \( v_0 = \dot{\gamma}(t_0) \). By Lemma 3.8,
\[
\operatorname{Hess} \psi(x_0)(v_0)^2 \geq c(x_0)\|v_0\|^2 = c(x_0) \, d^2(s, s_1),
\]
where \( c(x_0) = \min \{ 2, 2\sqrt{k_0} \, d(x_1, x_0) \cot \left( \sqrt{k_0} \, d(x_1, x_0) \right) \} \). Since \( x_0 \in B(x_1, r) \), by the choice of \( r \) we have
\[
2\sqrt{k_0} \, d(x_1, x_0) \cot \left( \sqrt{k_0} \, d(x_1, x_0) \right) > 1,
\]
and so \( c(x_0) > 1 \). Moreover, \( \exp^{-1}_{s_1} x_1 \in N^p_S(s_1) \), hence
\[
\langle \exp^{-1}_{s_1} x_1, \exp^{-1}_{s_1} s \rangle \leq \varphi(s_1) d(x_1, s_1) d^2(s, s_1) \leq \rho r d^2(s, s_1).
\]
Therefore (3.4) turns into
\[
(3.5) \quad d^2(x_1, s) \geq d^2(x_1, s_1) + \left( \frac{1}{2} - 2\rho r \right) d^2(s, s_1).
\]
We put \( \sigma = \left( \frac{1}{2} - 2\rho r \right) \), hence our choice of \( r \) guarantees that \( \sigma > 0 \) and the proof of the claim is complete.

Suppose that \( \exp_x(w_i) = s_i \) for \( i = 1, 2 \), then (3.5) implies that
\[
d^2(x_1, \exp_x(v)) \geq d^2(x_1, \exp_x(w_1)) + \frac{\sigma}{c_1^2} \| v - w_1 \|^2,
\]
for every \( v \in \exp^{-1}_x(S \cap B(x_1, r)) \), where \( c_1 \) is the Lipschitz constant of \( \exp^{-1}_x \) on \( B(x, R) \).

Let \( c_2 \) be a Lipschitz constant of \( \exp_x \) on \( B(0, R) \), then by Shapiro’s variational principle we finally get
\[
d (P_S(x_1), P_S(x_2)) = d (\exp_x w_1, \exp_x w_2) \\
\leq c_2 \| w_1 - w_2 \| \\
\leq 2\kappa c_2 c_1^2 \, d(x_1, x_2),
\]
where \( \kappa \) is a positive constant such that \( 2\kappa \, d(x_1, x_2) \) is a Lipschitz constant of the function \( f(v) = d^2(x_1, \exp_x(v)) - d^2(x_2, \exp_x(v)) \) on the neighborhood \( W := \exp^{-1}_x(B(x_1, r)) \) of \( w_1 \). \( \square \)

We recall that a continuous map \( r : X \to A \) from a topological space \( X \) to a subspace \( A \) of \( X \) is said to be a retraction if the restriction to \( A \) of \( r \) is the identity map. A subset \( S \) of a Riemannian manifold \( M \) is called \( \mathcal{L} \)-retract if there exist a neighborhood \( V \) of \( S \), a retraction \( r : V \to S \) and a positive constant \( L \) such that
\[
d(x, r(x)) \leq L d_S(x), \quad \forall x \in V.
\]

**Proposition 3.10.** If \( S \) is a prox-regular subset of \( M \), then \( S \) is \( \mathcal{L} \)-retract with \( L = 1 \).

**Proof.** According to Theorem 3.9, the projection map \( P_S : V \to S \) is a locally Lipschitz retraction from a neighborhood \( V \) of \( S \) to \( S \). \( \square \) \( \square \)
4. Directional differentiability of the metric projection at a boundary point

In this section by applying Shapiro’s variational principle, we investigate the directional differentiability of the projection map $P_S$ at the boundary points of $S$ where $S$ is a prox-regular subset of a Riemannian manifold $M$. Let us recall the definition of directional differentiability for maps between two Riemannian manifolds.

**Definition 4.1.** Let $f : M \to N$ be a map between two Riemannian manifolds, $x \in M$ and $(V, \phi)$ be a chart of $N$ at the point $f(x)$. We define the directional derivative of $f$ at $x$ in the direction $v \in T_x M$ as

$$f'(x; v) := \lim_{t \to 0^+} \frac{\phi(f(\text{exp}_x(tv))) - \phi(f(x))}{t},$$

when the limit exists.

Moreover, the map $f$ is said to be directionally differentiable at $x$ if the directional derivative $f'(x; v)$ exists for all $v \in T_x M$.

In fact, $f'(x; v)$ is the right-handed derivative of the curve $\gamma(t) := f(\text{exp}_x(tv))$ at $t = 0$.

**Theorem 4.2.** Let $S$ be a prox-regular subset of $M$ and $x \in S$. Then $P_S$ is directionally differentiable at $x$ and for every $v \in T_x M$

$$P'_S(x; v) = P_{T^B_S(x)}(v),$$

where $P_{T^B_S(x)}$ denotes the metric projection to $T^B_S(x)$.

**Proof.** Prox-regularity of $S$ implies the existence of a continuous function $\varphi : S \to [0, \infty)$ such that $S$ is $\varphi$-convex. Let $B(x, r) \subseteq \text{Unp}(S)$ be a convex ball with compact closure and $v \in T_x M$. We are going to show that

$$\lim_{t \to 0^+} \frac{\exp_x^{-1}(P_S(\text{exp}_x(tv)))}{t} = P_{T^B_S(x)}(v).$$

Since $T^B_S(x)$ is a closed convex cone in $T_x M$, we have

$$P_{T^B_S(x)}(tv) = tP_{T^B_S(x)}(v) \quad \forall t \geq 0,$$

and so equivalently we must prove that

$$\lim_{t \to 0^+} \frac{\|\exp_x^{-1}(P_S(\text{exp}_x(tv))) - P_{T^B_S(x)}(tv)\|}{t} = 0.$$

This means that

$$\|\exp_x^{-1}(P_S(\text{exp}_x(tv))) - P_{T^B_S(x)}(tv)\| = o(t).$$

To this end, let $t > 0$ be given such that $t < \frac{r}{2\|v\|}$ and consider the following optimization problems

$$\min_{w \in T^B_S(x)} \|w - tv\|^2$$(4.1)
and

\[ (4.2) \quad \min_{y \in S \cap B(x,r)} d^2(y, \exp_x tv) = \min_{y \in \exp_x (S \cap B(x,r))} d^2(\exp_x w, \exp_x tv). \]

Note that \( \exp_x tv \in B(x,r) \subseteq \text{Unp}(S) \), then we get \( \bar{x} = P_S(\exp_x(tv)) \). Since \( x \in S \), we have

\[ d(\bar{x}, x) \leq d(\bar{x}, \exp_x(tv)) + d(\exp_x(tv), x) \leq 2d(\exp_x(tv), x) = 2t\|v\| < r, \]

so \( \bar{x} \in B(x,r) \). Hence \( \bar{v} = \exp_x^{-1}(P_S(\exp_x(tv))) \) is the optimal solution of (4.2).

Let \( v^* \) be the optimal solution of (4.1), then \( v^* = P_{T^B_S(x)}(tv) \). Furthermore, using the proof of [22, Theorem 3.1],

\[ \|w - tv\|^2 \geq \|v^* - tv\|^2 + \|w - v^*\|^2, \]

and (2.3) is the case for \( \alpha = 1 \). We take \( \bar{r} := 2t\|v\| \) and \( W := B(0, \bar{r}) \subseteq T_x M \), then \( \bar{v}, v^* \in W \) and by Shapiro’s variational principle,

\[ \|\bar{v} - v^*\| \leq \vartheta(t), \]

where

\[ \vartheta(t) = \kappa(t) + 2\delta_1(t) + (k_1(t)\delta_1(t) + k_2(t)\delta_2(t))^{1/2}, \]

and \( \kappa(t) \) is a Lipschitz constant of the function

\[ h_t(w) := d^2(\exp_x w, \exp_x tv) - \|w - tv\|^2, \]

on \( W \). Moreover, \( k_1(t) \) and \( k_2(t) \) are Lipschitz constants of the functions \( f_t(w) := \|w - tv\|^2 \) and \( g_t(w) := d^2(\exp_x w, \exp_x tv) \) on \( W \), respectively and

\[ \delta_1(t) = \sup \{d(\exp_x^{-1} y, T^B_S(x)) : y \in S \cap B(x, \bar{r})\}, \]

\[ \delta_2(t) = d(v^*, \exp_x^{-1}(S \cap B(x, \bar{r}))). \]

We now show that \( \vartheta(t) = o(t) \). Indeed,

\[ k_1(t) \leq 6t\|v\| \]

and

\[ k_2(t) = \max_{w \in W} \|\langle -2 \exp_x^{-1}(w), \exp_x(tv), d\exp_x(w)\rangle\| \leq \max_{w \in W} (2d(\exp_x(w), \exp_x(tv))\|d\exp_x(w)\|) \leq 6t\|v\|, \]

where \( l = \max_{w \in W} \|d\exp_x(w)\| \). Hence \( k_1(t) \to 0 \) and \( k_2(t) \to 0 \) as \( t \to 0^+ \). Moreover, by Theorem 3.7, for every \( y \in S \cap B(x, r) \)

\[ d(\exp_x^{-1} y, T^B_S(x)) \leq \varphi(x)d^2(x, y). \]

So we have

\[ \lim_{y \to x} \frac{d(\exp_x^{-1} y, T^B_S(x))}{d(x, y)} = 0, \]

and this implies that \( \delta_1(t) = o(t) \). Also, \( \delta_2(t) = o(t) \) since

\[ \delta_2(t) = d(tv_0, \exp_x^{-1}(S \cap B(x, \bar{r}))) \leq c_1 d(\exp_x(tv_0), S), \]
where $v_0 = P_{T_B^\varepsilon(x)}(v)$ and $c_1$ is a Lipschitz constant of $\exp_{x}^{-1}$ on $B(x, r)$.

It remains only to verify that $\kappa(t) = o(t)$. Indeed, for every $w \in B(0, r)$ and $z \in T_x M$,

$$\nabla h_t(w)(z) = -2\left\langle \exp_{\exp_{x}(w)}^{-1}(v) \exp_{x}(tv), d\exp_{x}(w)z \right\rangle - 2\langle w - tv, z \rangle.$$ 

For fixed $w, z$ we define

$$F(t) = \left\langle \exp_{\exp_{x}(w)}^{-1}(v) \exp_{x}(tv), d\exp_{x}(w)z \right\rangle,$$

for every $t$ with $|t| < \frac{2\|v\|}{\kappa}$. The Taylor expansion gives

$$F(t) = F(0) + F'(0)t + o(t) \quad \forall t.$$ 

The values $F(0)$ and $F'(0)$ is obtained as follows: according to [16, Lemma 3.5, p. 250] we have

$$F(0) = \left\langle \exp_{\exp_{x}(w)}^{-1}(v), d\exp_{x}(w)z \right\rangle = \left\langle d\exp_{\exp_{x}(w)}(\gamma(1)) \left( \exp_{\exp_{x}(w)}^{-1}(v) \right), z \right\rangle,$$

where $\gamma$ is the geodesic $\gamma(t) = \exp_{x}(tw)$ and hence

$$\dot{\gamma}(1) = -\exp_{\exp_{x}(w)}^{-1}(v).$$

For simplicity, let us write $\bar{w} = \exp_{\exp_{x}(w)}^{-1}(v)$. Thus using [16, Theorem 3.1],

$$d\exp_{\exp_{x}(w)}(\bar{w})(\bar{w}) = J(1),$$

where $J$ is the Jacobi field along the geodesic $\bar{\beta}$ joining $\exp_{x}(w), x$ satisfying the properties $\bar{\beta}(0) = \bar{w}$, $J(0) = 0$ and $D_tJ(0) = \bar{w}$. In fact, $\beta(t) = \gamma(1 - t)$ and $J(t) = t\bar{\beta}(t)$ and so $J(1) = -w$ and $F(0) = -\langle w, z \rangle$.

Also we have

$$F'(0) = \left\langle d\exp_{\exp_{x}(w)}^{-1}(v), d\exp_{x}(w)z \right\rangle = \left\langle d\exp_{\exp_{x}(w)} \left( \exp_{\exp_{x}(w)}^{-1}(v) \right) \left( d\exp_{\exp_{x}(w)}^{-1}(v) \right), z \right\rangle = \left\langle \langle v, z \rangle \right\rangle.$$ 

It follows that $\nabla h_t(w)(z) = o(t)$ for every $w \in B(0, r)$ and $z \in T_x M$. This implies that $\kappa(t) = o(t)$. \hfill \square

Using Theorem 4.2, we obtain the following characterization of Bouligand tangent cone to a prox-regular set.

**Corollary 4.3.** Let $S$ be a closed prox-regular subset of $M$ and $x \in S$. Then $v \in T_x^S(x)$ if and only if there exists a continuous curve $\alpha : [0, \varepsilon) \to S$ such that $\alpha(0) = x$ and $\dot{\alpha}(0^+) = v$, where $\dot{\alpha}(0^+)$ is the right-handed derivative of $\alpha$ at 0.
Proof. Let $v \in T^B_x(S)$. We choose $\varepsilon > 0$ such that $\exp_x(tv) \in \text{Unp}(S)$ for all $t \in [0, \varepsilon)$. We now define

$$\alpha(t) := P_S(\exp_x(tv)) \quad \forall t \in [0, \varepsilon).$$

Then by Theorem 4.2,

$$\dot{\alpha}(0^+) = P'_S(x; v) = P_{T^B_S(x)}(v) = v.$$

The proof of the converse statement is straightforward. \hfill \Box \Box

5. Minimizing curves in prox-regular sets

Our goal in this section is to derive a necessary condition for a curve $\gamma$ to be a minimizing curve between its endpoints in a prox-regular set. To this end, we employ the first variation formula. Let $S \subseteq M$ be a closed prox-regular set whose boundary is a $C^2$ Riemannian submanifold of $M$.

In this situation, a continuous map $\gamma : [a, b] \rightarrow M$ is called a piecewise regular curve if it is a piecewise $C^2$ curve with nonzero derivatives. Moreover, by an admissible curve we mean a piecewise regular curve $\gamma : [a, b] \rightarrow M$ which is entirely in $S$. An admissible curve $\gamma$ in $S$ is said to be minimizing if $\mathcal{L}(\gamma) \leq \mathcal{L}(\tilde{\gamma})$ for all admissible curves $\tilde{\gamma}$ with the same endpoints where $\mathcal{L}(\gamma)$ denotes the length of $\gamma$ in $M$.

An admissible family of curves in $S$ is a continuous map $\Gamma : [0, \varepsilon) \times [a, b] \rightarrow M$ with the property that $\Gamma(s, t) \in S$ for all $(s, t) \in [0, \varepsilon) \times [a, b]$ and there exists a partition $a = a_0 < \cdots < a_k = b$ of $[a, b]$ such that $\Gamma|[0, \varepsilon) \times [a_{i-1}, a_i]$ is $C^2$ for every $i = 1, \ldots, k$. A variation of an admissible curve $\gamma : [a, b] \rightarrow M$ is an admissible family $\Gamma$ in $S$ such that $\Gamma(0, t) = \gamma(t)$ for all $t \in [a, b]$ and if in addition $\Gamma(s, a) = \gamma(a)$ and $\Gamma(s, b) = \gamma(b)$ for all $s \in [0, \varepsilon)$, then it is called a proper variation.

Recall that if $\Gamma$ is a variation of $\gamma$, then the piecewise $C^1$ vector field $V$ along $\gamma$ defined by $V(t) = \frac{d}{ds}|_{s=0^+} \Gamma(s, t)$ is called the variation field of $\Gamma$, where $\frac{d}{ds}|_{s=0^+}$ denotes the right-handed derivative of $\Gamma(., t) : [0, \varepsilon) \rightarrow M$ at $s = 0$. Note that according to Corollary 4.3, if $V$ is the variation field of a variation along $\gamma$, then

$$V(t) \in T^B_S(\gamma(t)) \quad \forall t \in [a, b].$$

In the following, we investigate when a vector field along an admissible curve $\gamma$ is the variation field of a variation of $\gamma$.

Lemma 5.1. Suppose that the closed set $S$ is prox-regular and $\partial S$ is a $C^2$ submanifold of $M$. If $x \in \partial S$ and $v \in (T^B_x(S) \setminus T_x\partial S) \cup \{0\}$, then there exists $\varepsilon > 0$ such that $\exp_x(tv) \in S$ for all $t \in [0, \varepsilon)$.

Proof. Assuming the contrary, there exists a sequence $\{t_n\}$ such that $t_n \downarrow 0$ and $\exp_x(t_nv) \in S \setminus \hat{S}$. Moreover,

$$v = \lim_{n \to \infty} \exp^{-1}_x(\exp_x(t_nv))/t_n.$$
Hence $v \in T^{B}_{S}(x)$ and so by Theorem 3.6, we have $v \in T_{x}\partial S$. This contradiction completes the proof.

**Lemma 5.2.** Suppose that $\gamma : [a, b] \to M$ is an admissible curve and $V$ is a piecewise $C^{2}$ vector field along $\gamma$. If for any $t \in [a, b]$ with $\gamma(t) \in \partial S$ we have

$$V(t) \in \left(T^{B}_{S}(\gamma(t)) \setminus T_{\gamma(t)}\partial S\right) \cup \{0\},$$

then $V$ is the variation field of a variation $\Gamma$ of $\gamma$.

**Proof.** Lemma 5.1 along with the compactness of $[a, b]$ imply that there exists $\varepsilon > 0$ such that the map $\Gamma : [0, \varepsilon] \times [a, b] \to M$ defined by $\Gamma(s, t) := \exp_{\gamma(t)}(sV(t))$ is the desired variation of $\gamma$ in $S$. □ □

The following theorem gives a necessary condition for a curve to be minimizing in $S$.

**Theorem 5.3.** Let $\gamma : [a, b] \to M$ be a unit speed admissible curve. If $\gamma$ is minimizing in $S$, then

$$D_{t}\gamma(t) \in N^{B}_{S}(\gamma(t)),$$

for every $t \in [a, b]$ except for finitely many points.

**Proof.** If the interior of $S$ is empty, then (5.1) evidently holds, since in this case $N^{B}_{S}(x) = T^{1}_{x}\partial S$ for every $x \in S$. Therefore we assume that the interior of $S$ is nonempty.

Let $a = a_{0} < a_{1} < \cdots < a_{k} = b$ be a partition of $[a, b]$ such that $\gamma$ is $C^{2}$ on each subinterval $[a_{i-1}, a_{i}]$ and $t_{0} \in [a, b]$ be such that $t_{0} \neq a_{i}$ for each $i$. We suppose that $t_{0} \in (a_{j-1}, a_{j})$ for some $j$, $1 \leq j \leq k$ and we get $x_{0} := \gamma(t_{0})$. If $x_{0} \in S^{o}$, then there exist an open neighborhood $U$ of $x_{0}$ in $M$ and a positive number $\delta$ such that $\gamma(t) \in U \subset S$ for all $t \in I_{0} := [t_{0} - \delta, t_{0} + \delta]$. Hence $\gamma|I_{0}$ is minimizing in $M$ and this implies that $D_{t}\gamma(t_{0}) = 0$.

Assume that $x_{0} \in \partial S$ and let the open neighborhood $U$ and the submersion $\psi : U \to \mathbb{R}$ be the ones applied in the proof of Theorem 3.4. Clearly $D_{t}\gamma(t_{0}) = 0$ or there is a positive number $\varepsilon$ such that $\gamma(t) \in U \cap \partial S$ for all $t \in I := [t_{0} - \varepsilon, t_{0} + \varepsilon] \subset (a_{j-1}, a_{j})$. So it suffices to check that (5.1) holds in the latter case. Indeed, $\gamma|I$ is minimizing in the Riemannian submanifold $\partial S$ of $M$. Then we have

$$D_{t}\gamma(t) \in T^{1}_{\gamma(t)}\partial S \quad \forall t \in I.$$

If $U \cap S^{o} = \emptyset$ (or there exists a neighborhood $V \subseteq U$ of $x$ such that $V \cap S^{o} = \emptyset$), then $N^{B}_{S}(x) = T^{1}_{x}\partial S$. Otherwise, in order to deduce that $D_{t}\gamma(t_{0}) \in N^{B}_{S}(\gamma(t_{0}))$, by Lemma 3.1 it suffices to show that

$$\langle D_{t}\gamma(t_{0}), v \rangle \leq 0 \quad \forall v \in T^{B}_{S}(\gamma(t_{0})).$$

If this fails to hold, then there is $\eta \in T^{B}_{S}(\gamma(t_{0}))$ such that

$$\langle D_{t}\gamma(t_{0}), \eta \rangle > 0.$$ 

Thus the inclusion (5.2) implies that $\eta \notin T_{\gamma(t_{0})}\partial S$. 

□ □
We now construct a vector field along $\gamma$ such that for any $t \in [a, b]$ with $\gamma(t) \in \partial S$, \[ V(t) \in (T^B_S(\gamma(t))) \setminus T_{\gamma(t)}\partial S) \cup \{0\}. \]

We define $\nabla(t) := L^\gamma_{\xi, \eta}$ and $g(t) := \langle \nabla(t), \nabla \psi(\gamma(t)) \rangle$ for all $t \in I$. According to Theorem 3.4, $N^B_S(\gamma(t)) = \text{cone}\{\nabla \psi(\gamma(t))\}$ for all $t \in I$. Then $g(t_0) < 0$ and the continuity of $g$ implies that $g(t) < 0$ on a possibly smaller neighborhood of $t_0$. It follows that \[ \nabla(t) \in T^B_S(\gamma(t)) \setminus T_{\gamma(t)}\partial S \quad \forall t \in I, \]

without loss of generality. By shrinking $I$ if necessary, we can assume that

\[
\langle \nabla(t), D_t \hat{\gamma}(t) \rangle > 0 \quad \forall t \in I.
\]

We choose a bump function $\phi \in C^\infty(\mathbb{R})$ with support in $I$ such that $\phi(t) \equiv 1$ on $[c_1, c_2]$, where $c_1, c_2$ is such that $t_0 - \varepsilon < c_1 < c_2 < t_0 + \varepsilon$. We now define $V(t) := \phi(t)\nabla(t)$ for all $t \in [a, b]$. Then $V$ is the desired vector field along $\gamma$.

Applying Lemma 5.2 to the vector field $V$ along $\gamma$ gives rise to a variation $\Gamma : [0, \varepsilon] \times [a, b] \to M$ of $\gamma$ in $S$ such that $V$ is its variation field. Since $\gamma$ is minimizing in $S$, for all $s \in [0, \varepsilon]$ we have $\mathcal{L}(\Gamma_s) \geq \mathcal{L}(\Gamma_0)$ where $\Gamma_s$ is an admissible curve on $[a, b]$ defined by $\Gamma_s(t) := \Gamma(s, t)$. This implies that $\frac{d}{ds}|_{s=0^+} \mathcal{L}(\Gamma_s) \geq 0$. Using the first variation formula (see [17, Theorem 6.3]) we conclude that

\[ \int^b_a \langle V(t), D_t \hat{\gamma}(t) \rangle dt + \sum_{i=1}^{k-1} \langle V(a_i), \Delta_i \hat{\gamma} \rangle \leq 0, \]

where $\Delta_i \hat{\gamma} := \hat{\gamma}(a^+_{i+1}) - \hat{\gamma}(a^-_{i})$. Since $V(a_i) = 0$ for each $i = 1, \ldots, k$, we have

\[ \int^b_a \langle V(t), D_t \hat{\gamma}(t) \rangle dt \leq 0. \]

On the other hand,

\[ \int^b_a \langle V(t), D_t \hat{\gamma}(t) \rangle dt \geq \int^{c_2}_{c_1} \langle \nabla(t), D_t \hat{\gamma}(t) \rangle dt > 0, \]

a contradiction which establishes that

\[ D_t \hat{\gamma}(t_0) \in N^B_S(\gamma(t_0)). \]

\[ \blacksquare \]

Example 5.4. Let $S^2$ be the 2-sphere of radius one in $\mathbb{R}^3$ with the round metric $g^o$ which is induced from the Euclidean metric on $\mathbb{R}^3$. Consider spherical coordinates $(\theta, \phi)$ on the subset $S^2 - \{(x, y, z) : x \leq 0, y = 0\}$ of the sphere defined by

\[ (x, y, z) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \quad 0 < \theta < \pi, -\pi < \phi < \pi. \]
It is known that the round metric is $g^o = d\theta^2 + \sin^2 \theta \, d\phi^2$ in spherical coordinates. Also, Christoffel symbols of $g^o$ in spherical coordinates are
\[
\Gamma_{ij}^0 = \begin{pmatrix} 0 & 0 \\ 0 & -\sin \theta \cos \theta \end{pmatrix}, \quad \Gamma_{ij}^\theta = \begin{pmatrix} 0 & \cos \theta / \sin \theta \\ \cos \theta / \sin \theta & 0 \end{pmatrix}.
\]

Let $S$ be the closed subset of $S^2$ which is obtained by removing the sector $\theta_0 < \theta \leq \pi$ from $S^2$ where $\frac{\pi}{2} < \theta_0 < \pi$. According to [20, Theorem 4.18], $S$ is a prox-regular subset of $S^2$.

Note that for every $p \in \partial S$, the map $\psi : U \to \mathbb{R}$ defined by $\psi(\theta, \phi) = \theta - \theta_0$ is the desired submersion which is used in Theorem 3.4. Hence applying Theorem 3.4, we obtain
\[N_S^\theta(p) = \text{cone}\{(1,0)\} = \{\lambda \partial / \partial \theta : \lambda \geq 0\}.
\]

Clearly, the unit speed curve $\gamma$ defined by
\[\gamma(t) := \left(\theta_0, -\frac{t}{\sin \theta_0}\right) \quad \forall t \in I := [-\pi/2 \sin \theta_0, \pi/2 \sin \theta_0],\]
is a minimizing curve in $S$ joining $(\theta_0, -\pi/2)$ and $(\theta_0, \pi/2)$. It can be found that the curve $\gamma$ satisfies the necessary condition (5.1). Indeed, we have
\[D_t \dot{\gamma}(t) = -\frac{\cos \theta_0}{\sin \theta_0} \frac{\partial}{\partial \theta} \quad \forall t \in I.
\]
Then putting $\lambda := -\cos \theta_0 / \sin \theta_0$, we observe that $\lambda > 0$ and (5.1) holds.

On the other hand, consider another admissible curve $\alpha$ in $S$ joining $(\theta_0, -\pi/2)$ and $(\theta_0, \pi/2)$ defined by
\[\alpha(t) := \begin{cases} (\theta_0 - t, -\pi/2) & 0 \leq t \leq \theta_0 - \pi/2 \\ (\pi/2, t - \theta_0) & \theta_0 - \pi/2 \leq t \leq \theta_0 + \pi/2 \\ (t - \theta_0, \pi/2) & \theta_0 + \pi/2 \leq t \leq 2\theta_0 \end{cases}
\]
Note that $D_t \dot{\alpha}(t) = 0$ and $\alpha$ satisfies (5.1), but it is not a minimizing curve in $S$, since $L(\gamma) = \pi \sin \theta_0 < \pi \leq L(\alpha)$.

**Example 5.5.** Let $H^2$ be the hyperbolic plane; that is, the upper half-plane in $\mathbb{R}^2$ with the metric $g_H = (dx^2 + dy^2) / y^2$. The Riemannian distance between two points $z_1 = (x_1, y_1), z_2 = (x_2, y_2)$ of $H^2$ is as
\[d(z_1, z_2) = 2 \ln \frac{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} + \sqrt{(x_2 - x_1)^2 + (y_2 + y_1)^2}}{2 \sqrt{y_1 y_2}}.
\]

We consider the subset $S = \{(x, y) : 1 \leq y \leq 2\}$ of $H^2$. It is evident that $S$ is not convex in the Hadamard manifold $(H^2, g_H)$. The metric projection $P_S$ is obtained as follows:
\[P_S(x, y) = \begin{cases} (x, 1) & \text{if } 0 < y < 1 \\ (x, 2) & \text{if } y > 2. \end{cases}
\]
Since for instance in the case when $y > 2$, for every $s \neq x$ we have
\[d((x, y), (s, 2)) > \ln \frac{y}{2} = d((x, y), (x, 2)).
\]
Then by [20, Corollary 4.20], $S$ is a prox-regular subset of $H^2$.

Note that for every $z = (s, 2) \in \partial S$, the map $\psi : H^2 \to \mathbb{R}$ defined by $\psi(x, y) = y - 2$ is the desired submersion which is needed in Theorem 3.4. Hence

$$N_S^R(z) = \text{cone}\{4\partial/\partial y\} = \{\lambda \partial/\partial y : \lambda \geq 0\}.$$  

We now consider the unit speed curve $\gamma$ in $S$ defined by

$$\gamma(t) := (2t, 2) \quad \forall t \in \mathbb{R}.$$ 

So $D_t\dot{\gamma}(t) = 2\frac{\partial}{\partial y}$ for all $t \in \mathbb{R}$ and it follows that the curve $\gamma$ satisfies the necessary condition (5.1).

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