Finite groups with the same power graph

M. Mirzargar\textsuperscript{a} and R. Scapellato\textsuperscript{b}

\textsuperscript{a}Faculty of Science, Mahallat Institute of Higher Education, Mahallat, Islamic Republic of Iran; \textsuperscript{b}Dipartimento di Matematica, Politecnico di Milano, Milano, Italy

\textbf{ABSTRACT}

The power graph \(P(G)\) of a group \(G\) is a graph with vertex set \(G\), where two vertices \(u\) and \(v\) are adjacent if and only if \(u \neq v\) and \(u^m = v\) or \(v^m = u\) for some positive integer \(m\). In this paper, we raise and study the following question: For which natural numbers \(n\) every two groups of order \(n\) with isomorphic power graphs are isomorphic? In particular, it is proved that all such odd number \(n\) are cube-free and also they are not multiples of 16 in general. Moreover, we show that if two finite groups have isomorphic power graphs and one of them is nilpotent, the same is true for the other one.

\textbf{1. Introduction}

There are many different ways to associate a graph to the given group, including the commuting graphs [4], prime graphs [12], and of course, Cayley graphs, which have a long history and applications [14]. Graphs associated with groups and other algebraic structures have been investigated since they have some interesting applications [2, 23], in particular related to automata theory [15, 16].

Let \(G\) be a finite group. The undirected power graph \(P(G)\) is the undirected graph with vertex set \(G\), where two vertices \(a, b \in G\) are adjacent if and only if \(a \neq b\) and \(a^m = b\) or \(b^m = a\) for some positive integer \(m\). Likewise, the directed power graph \(\vec{P}(G)\) is the directed graph with vertex set \(G\), where for two vertices \(a, b \in G\) there is an arc from \(a\) to \(b\) if and only if \(a \neq b\) and \(b = a^m\) for some positive integer \(m\). In [3], you can see a survey of results and open questions on power graphs, also it is explained that the definition given in [13] covers all undirected graphs as well. This means that the undirected power graphs were also defined in [13] for the first time (see [3, 7, 19] for more detailed explanations). These papers used only the brief term power graph, even though they covered both directed and undirected power graphs. For a group, \(G\), the digraph \(\vec{P}(G)\) was considered in [20] as the main subject of study. In order to measure how close the power graph is to the commuting graph, Aalipour [1] introduced the enhanced power graph which lies in between. In [9], the metric dimension of the power graphs has been studied. Cameron proved in [6] that, if \(G_1\) and \(G_2\) are finite groups whose undirected power graphs are isomorphic, then their directed power graphs are also isomorphic. Clearly, the converse is also true. As remarked for instance in [8], \(P(G)\) is connected for every \(G\) and \(P(G)\) is complete if and only if \(G\) is a cyclic group order 1 or prime-power. Clearly, \(G \cong H\) implies \(P(G) \cong P(H)\). The
converse is false for finite groups in general. For example, if $p$ is an odd prime and $m > 2$, besides the elementary abelian group $H$ of order $p^m$, there are non-abelian groups $G$ of order $p^m$ and exponent $p$, so $H$ and $G$ are non-isomorphic but have isomorphic power graphs. On the other hand, it is shown in [5, 18] that if both $G$ and $H$ are abelian then $P(G) \cong P(H)$ implies $G \cong H$. Also in [18], it is proved that if $G$ is one of the following finite groups:

1. A simple group,
2. A cyclic group,
3. A symmetric group,
4. A dihedral group,
5. A generalized quaternion group,

and $H$ is a finite group such that $P(G) \cong P(H)$ implies that $G \cong H$.

Following [17, 22], two finite groups $G$ and $H$ are said to be conformal if and only if they have the same number of elements of each order. Such groups need not be isomorphic (see the above example of groups of exponent $p$). The relevance of this concept to power graphs is due to the fact that, as proved by Cameron [6], two finite groups with isomorphic undirected power graphs are conformal. Note that the converse is not true. For example, two groups of order 16 with the same numbers of elements of each order, e.g. $C_4 \times C_4$ and $C_2 \times Q_8$ are SmallGroup(16, 2) and SmallGroup(16, 4) in small group library of GAP, respectively [10]. Their power graphs are not isomorphic. In fact, in the group $C_4 \times C_4$, each element of order 2 has four square roots, but in $C_2 \times Q_8$, the involution in $Q_8$ has twelve square roots and the other two involutions have none. In [17], an algorithm is described to find the number of elements of a given order in abelian groups, so if $G$ and $H$ are finite conformal abelian groups, then $G \cong H$.

In [22], the following question was investigated:

**Question:** For which natural numbers $n$ every two conformal groups of order $n$ are isomorphic?

In [22], the set of all such numbers was denoted by $S$ and odd and square-free elements of $S$ were characterized.

In this paper, we raise another question along the same lines:

**Question:** For which natural numbers $n$, every two groups of order $n$ with isomorphic power graphs are isomorphic?

Let us denote the set of all such numbers by $\tilde{S}$. Since two finite groups with isomorphic power graphs are conformal, it is easy to see that $S \subseteq \tilde{S}$.

In this paper, we follow the terminology and notation of [11] for graphs and [21] for groups. All groups and graphs considered here are finite. In Section 2, we shall give an answer to the aforementioned question according with the prime power decomposition of $n$. Moreover, it will be shown that all odd elements of $\tilde{S}$ are cube-free. In Section 3, we shall prove that if a group is nilpotent, so are all groups having the same power graph. Moreover, we show a similar statement for groups having a normal Hall subgroup.

2. Orders of groups characterized by their power graphs

In this section, we study the set $\tilde{S}$, often exploiting methods and results already used for $S$.

In [22, Lemma 1], it is proved that if $p$ and $q$ are prime and $q|(p − 1)$, then $p^2q \in S$ if and only if $q = 2$. Since $S \subseteq \tilde{S}$, the following result is straightforward.
**Proposition 2.1.** If \( p \) is an odd prime number, then \( 2p^2 \in S \).

Note that \( 8 \in S \), because the two non-abelian groups of order 8 are either the dihedral group \( D_8 \) or the quaternion group \( Q_8 \), and the number of elements of order 4 in these groups is 2 and 6, respectively. There are three abelian groups of order 8, which are pair-wise non-conformal and non-conformal to \( D_8 \) or \( Q_8 \). Therefore \( 8 \in S \) and \( 8 \in \bar{S} \).

The following result shows that \( \bar{S} \) contains natural numbers with an arbitrary number of prime factors.

**Theorem 2.2.** If \( n \notin \bar{S} \) and \((n,k) = 1\), then \( nk \notin \bar{S} \).

**Proof.** Let \( G \) and \( G' \) be non-isomorphic groups of order \( n \) and \( P(G) \cong P(G') \). Without loss of generality, we may assume that \( G \) and \( G' \) have the same elements and for each \( x \in G \), \( x' \) is the same in both \( G \) and \( G' \), so their power graphs coincide. Let \( H \) be a group of order \( k \). Note that if \((a,x) \) and \((b,y) \) are elements of \( G \times H \), then \((b,y) \) is a power of \((a,x) \) if and only if \( b \) and \( y \) are powers of \( a \) and \( x \) respectively. Namely, if \( b = a^r \) and \( y = x^s \), since \((n,k) = 1\) we can apply the Chinese Reminder Theorem to the system and \( t \equiv r \pmod{\text{ord}(a)} \) and \( t \equiv s \pmod{\text{ord}(x)} \). We get \((a,x)^t = (a',x') = (b,y) \). The same argument applies to \( G' \times H \). Therefore, the power graphs \( P(G \times H) \) and \( P(G' \times H) \) coincide. On the other hand, \( P(G \times H) \) and \( P(G' \times H) \) are non-isomorphic, so \( nk \notin \bar{S} \). \( \square \)

**Lemma 2.3.** Let \( G \) be a 2-group and \( A \) be an elementary abelian 2-group. Two vertices \((a,x),(b,y)\) of the graph \( P(G \times A) \) are adjacent if and only if one of the following holds:

1. \( x = y = 1 \) and \( b \) is a power of \( a \);  
2. \( x = y \neq 1 \) and \( b \) is an odd power of \( a \);  
3. \( x \neq 1, y = 1 \) and \( b \) is an even power of \( a \);  
4. \( x = 1, y \neq 1 \) and \( a \) is an even power of \( b \).

**Proof.** Note first if \( x \neq y, x \neq 1, \) and \( y \neq 1 \), then neither \((a,x)\) nor \((b,y)\) can be a power of the other one in the group \( G \times A \). Therefore, we can restrict our attention to (1)–(4) for what \( x \) and \( y \) are concerned. Besides, (1) is obvious. If \( x = y \neq 1 \) and \((a,x)^h = (b,x)\), then \( h \) must be odd, hence the vertices are adjacent whenever \( b \) is an odd power of \( a \); this proves (2). In case (3), adjacency is possible only if \((a,x)^h = (b,1)\) for some \( h \), and \( h \) is necessarily even. Likewise, we get (4). \( \square \)

**Theorem 2.4.** Let \( n = 2^\omega_0 p_1^{\omega_1} \cdots p_r^{\omega_r} \) \((r \geq 0)\). If \( \omega_0 \geq 4 \) or there exists \( i \neq 0 \) such that \( \omega_i \geq 3 \), then \( n \notin \bar{S} \).

**Proof.** We first consider the case \( \omega_0 \geq 4 \). Suppose \( n = 16 \). Let \( G = K_2 \times C_4 \) be the direct product of a cyclic group of order 4 and a Klein four-group and let \( H \) be the central product of the dihedral group of order 8 and cyclic group of order 4. The groups \( G \) and \( H \) have IDs 10 and 13 respectively among the groups of order 16 in GAP’s SmallGroup library. The graphs \( P(G) \) and \( P(H) \) are isomorphic because both of them, except for the edges incident with 1, consists of four triangles attached by the only element of order 2 having square roots. Therefore \( n \notin \bar{S} \).

Assume that \( n = 2^\omega_0 \) but \( \omega_0 > 4 \), and let \( A \) be an elementary abelian 2-group of order \( 2^{\omega_0 - 4} \). By Lemma 2.3, the graphs \( P(G \times A) \) and \( P(H \times A) \) are isomorphic, while \( G \times A \not \cong H \times A \). Therefore \( n \notin \bar{S} \).

If \( n \) is not a power of 2, we have \( n = 2^{\omega_0} k \), with \( k \neq 1 \) odd. Since \( 2^{\omega_0} \notin \bar{S} \) by the above argument and \((2^{\omega_0}, k) = 1\), by Theorem 2.2 we get \( n \notin \bar{S} \).
Let us now consider the case where there exists \( i \neq 0 \) such that \( x_i \geq 3 \). Let \( G_1 \) and \( G_2 \) be non-isomorphic groups of order \( p_i^{3k} \) and exponent \( p_i \). Without loss of generality, we may assume that \( G_1 \) and \( G_2 \) have the same elements and that for each \( x \in G_1 \) the \( k \)-power of \( x \) is the same in both \( G_1 \) and \( G_2 \). Let \( G' \) be a group of order \( p_i^{3k} \). For each \( (x, a) \in G_1 \times G' \), the \( k \)-power of \( (x, a) \) is the same in both \( G_1 \times G' \) and \( G_2 \times G' \); therefore \( (x, a) \) and \( (y, b) \) are adjacent in \( P(G_1 \times G') \) if and only if they are adjacent in \( P(G_2 \times G') \). Hence \( P(G_1 \times G') = P(G_2 \times G') \). Since these two groups of order \( n \) are not isomorphic, we conclude that \( n \not\in \mathcal{S} \).

\[\square\]

**Corollary 2.5.** Every odd element of \( \mathcal{S} \) is cube-free.

As mentioned above, we have \( S \subseteq \mathcal{S} \). On the other hand, when we look closely at computer programming, we notice that many small numbers belong to both \( S \) and \( \mathcal{S} \) or to neither. It is then natural to ask whether this inclusion is indeed strict.

**Theorem 2.6.** The set \( \mathcal{S} \setminus S \) is non-empty. Its smallest element is 72.

**Proof.** With the help of GAP’s SmallGroup library, we found out that all numbers less than 71 do not belong to \( \mathcal{S} \setminus S \). Applying GAP’s SmallGroup library to 72, we come across with two conformal groups of order 72 (whose IDs are 35 and 40, which we name them \( G \) and \( G' \) respectively). Therefore, 72 \( \not\in \mathcal{S} \).

Table 1 displays the number of elements of their groups for each possible order. In view of the information provided therein, both groups must have a normal 3-Sylow subgroup, which is elementary abelian, and nine 2-Sylow subgroups, which are dihedral. The generators and relators provided by GAP lead to the following presentation for \( G \):

\[
G = \langle a, b, x, y \mid a^3 = b^3 = x^2 = y^2 = 1, (xa)^2 = (xb)^2 = 1, ab = ba, ay = ya, by = yb, (xy)^4 = 1 \rangle.
\]

The set \( M = \{1, (xy)^2, y, xyx\} \) is a subgroup of \( G \). Let us prove that each element of \( M \) commutes with each element of the 3-Sylow subgroup \( P = \langle a, b \rangle \). Namely \( y \) commutes with \( a \) and \( b \) because of the relations \( ay = ya \) and \( by = yb \). Moreover, by using \( (xa)^2 = 1 \) and \( ay = ay \), we get: \( (xy)a = xy^{-1}x = x^{-1}yx = a(xy) \). Likewise, \( (xy)b = b(xy) \) from \( (xb)^2 = 1 \) and \( by = yb \). Clearly, \( a \) and \( b \) must also commute with the remaining elements of \( M \). Note that, considering \( G \) as a semidirect product of \( P \) by \( Q \) through a homomorphism \( f : D \to \text{Aut}(P) \), the kernel of \( f \) is \( M \). The group \( C = \langle a, b, M \rangle \) has order 36. The products of the 3 elements of order 2 in \( M \) by the 8 elements of order 3 in \( \langle a, b \rangle \) give exactly 24 elements of order 6, hence \( C \) contains all the elements of order 6. In view of the above remarks and of Table 1, the elements of \( G \) are distributed as follows: in \( C \), besides the unity, there are all the elements of order 3 and of order 6, plus three elements of order 2 (in \( M \)); in \( G \setminus C \) there are all the elements of order 4 plus the remaining eighteen elements of order 2. The only elements whose roots are of interest for the power graph \( P(G) \), are those of order 2 and of order 3. Since \( M \) has index 2 in every 2-Sylow subgroups, it is normal and the intersection of two of them contains it but cannot be larger, hence the nine 2-Sylow subgroups pair-wise intersect in \( M \). Therefore \( (xy)^2 \) has as roots all the elements of order 4, while none of the remaining elements of order 2 has roots. Each element of order 3 has as roots the three elements of order 6 obtained by multiplying it by the nontrivial elements of \( M \).

Now, we come to the group \( G' \) whose ID in SmallGroup library is 40. The generators and relators provided by GAP lead to the following presentation for \( G' \):

\[
G' = \langle a', b', x', y' \mid a'^3 = b'^3 = x'^2 = y'^2 = 1, (xa')^2 = (xb')^2 = 1, a'b' = b'a', ay' = y'a', by' = y'b', (xy')^4 = 1 \rangle.
\]
Let \( G = \langle a, b, x, y, z | a^3 = b^3 = x^2 = y^3 = z^2 = 1, (xy)^2 = z, ab = ba, xax = a^{-1}, xbx = b, yay = b, yby = a, zaz = a^{-1}, zbx = b^{-1} \rangle. \)

About the subgroups of \( G' \) we have the group \( P = \langle a, b \rangle \) which is the unique 3-Sylow subgroup of \( G' \), \( D = \langle x, y \rangle \) is one of the 2-Sylow subgroups of \( G' \) and dihedral group. As in the case of \( G \), this group is a semidirect product of \( P \) by \( D \) through a homomorphism \( f' : D \to \text{Aut}(P) \). This time, \( f' \) is injective. Because the three automorphisms induced by \( x \), \( y \) and \( z \) generate a group of order 8. There are many ways to see that the power graph of \( G' \) cannot be isomorphic with that of \( G \). For instance, unlike what happens in \( G \), there are no elements of order 2 of \( G' \) whose roots are all the elements of order 4. There are no elements of order 2 of \( G' \) with 8 roots of order 3. Each element of order 3 has three roots of order 6, but unlike what happens with \( G \), the cubes of such root depend on the element. For example, for \( a \) we get \( a^{-1}xy, ab^{-1}xy, a^{-1}b^{-1}xy \), whose cubes are equal to \( yxy \); for \( b \) we get \( b^{-1}x, ab^{-1}x, a^{-1}b^{-1}x \), whose cubes are equal to \( x \). Therefore, \( P(G') \) cannot be isomorphic with \( P(G) \), so 72 \( \not\in \mathcal{S} \).

We tried to generalize the example of groups of orders 72. If we could prove for every odd prime \( p \), the number \( 8p^2 \in \mathcal{S} \), then the set \( \mathcal{S} \) is infinite. Define the following two groups:

\[
G = \langle a, b, x, y | a^p = b^p = x^2 = y^3 = 1, (xa)^2 = (xb)^2 = 1, ab = ba, ay = ya, by = yb, (xy)^4 = 1 \rangle, \\
G' = \langle a, b, x, y | a^p = b^p = x^2 = y^3 = z^2 = 1, (xy)^2 = z, ab = ba, xax = a^{-1}, xbx = b, yay = b, yby = a \rangle.
\]

We checked that \( G \) and \( G' \) are conformal and \( P(G') \) cannot be isomorphic with \( P(G) \). This is not enough to prove that for every odd prime \( p \), \( 8p^2 \in \mathcal{S} \), because there might be further pairs of conformal groups of the given order. However, we suspect that such pairs do not exist, which led us to the following conjecture:

**Conjecture 2.7.** The set \( \mathcal{S} \) is infinite.

### 3. Power graphs of nilpotent groups and groups having a normal Hall subgroup

As a further application of the necessary condition of conformality for groups having the same power graph, we are going to show here some situations where a property of a group \( G \) is inherited by all groups with the same power graph. It is known that this is not the case for abelian groups (namely, in the Introduction we recalled that for each elementary abelian groups of non-prime order there is a non-abelian group with the same power graph).

**Theorem 3.1.** If \( G \) and \( H \) are conformal and \( H \) is nilpotent, then also \( G \) is nilpotent.

**Proof.** Since \( G \) and \( H \) are conformal, for each prime \( p \) dividing the common group order, \( G \) has the same number of elements of \( p \)-power order as \( H \). Since \( H \) is nilpotent, the number of elements of \( p \)-power order is equal to the order of the \( p \)-Sylow subgroups of \( G \) and \( H \). Thus \( G \) contains only one \( p \)-Sylow subgroup for each prime dividing the group order hence is nilpotent. \( \square \)

**Corollary 3.2.** If \( P(G) \cong P(H) \) and \( H \) is nilpotent, then also \( G \) is nilpotent.

A subgroup of a finite group is said to be a Hall subgroup if its order and index are relatively prime.

**Theorem 3.3.** Let \( G \) and \( H \) be conformal groups. If \( H \) has a normal Hall subgroup of order \( m \) and \( G \) is solvable, then also \( G \) has a normal Hall subgroup of order \( m \).

**Proof.** Since \( G \) and \( H \) are conformal, the elements of \( H \) whose order divides \( m \) are exactly \( m \), hence the same happens for \( G \). Since \( G \) is solvable and \( m \) is prime with \( |G|/m \), by Hall's
Theorem it must have a Hall subgroup of order \( m \), which is normal because it contains all the elements whose order divides \( m \).

Corollary 3.4. If \( P(G) \cong P(H) \), \( H \) has a normal Hall subgroup of order \( m \), and \( G \) is solvable, then also \( G \) has a normal Hall subgroup of order \( m \).

4. Conclusion

We know that the power graph of a group does not always determine the group up to isomorphism. It is therefore interesting to find out for which pairs of groups \( G \) and \( H \), \( P(G) \cong P(H) \) implies \( G \cong H \). A necessary condition for this to happen is that the two groups are conformal. This is why the concept of conformal groups seems to be relevant in order to investigate this question.

Therefore, in the present paper we studied the integers \( n \) such that any pair of groups of order \( n \) the isomorphism of the power graphs grants the isomorphism of the groups themselves. Further insights, also exploiting the concept of conformal groups, were given by showing that some structural properties of \( G \) are inherited by the groups \( H \) such that \( P(G) \cong P(H) \).

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