Three-block p-branes in various dimensions

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Abstract

It is shown that a Lagrangian, describing the interaction of the gravitation field with the dilaton and the antisymmetric tensor in arbitrary dimension spacetime, admits an isotropic p-brane solution consisting of three blocks. Relations with known p-brane solutions are discussed. In particular, in ten-dimensional spacetime the three-block p-brane solution is reduced to the known solution, which recently has been used in the D-brane derivation of the black hole entropy.

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1 Introduction

Recently, isotropic p-brane solutions in supergravity and string theory have been extensively studied, see, for example [1]–[23]. Investigation of these solutions is important for the superstring duality [24]–[26] and for recent D-brane derivation of the black hole entropy [27]–[33]. One believes that a fundamental eleven-dimensional M-theory or twelve-dimensional F-theory exist and that five known ten-dimensional superstring theories and eleven-dimensional supergravity are low energy limits of these theories [3], [4], [24]–[26]. The study of p-brane solutions in an arbitrary dimension spacetime might gain better understanding of the structure of superstring theory.

We shall consider the following action [1], [8]:

$$I = \frac{1}{2\kappa^2} \int d^D x \sqrt{-g} \left( R - \frac{1}{2} \nabla^2 \phi \right)^2 - \frac{1}{2(q + 1)!} e^{-\alpha \phi} F_{q+1}^2, \quad (1.1)$$

where $F_{q+1}$ is the $q + 1$ differential form, $F_{q+1} = dB_q$ and $\phi$ is the dilaton. All ten-dimensional supergravity theories contain the terms from (1.1) if $D = 10$, $q = 2$, $\alpha = 1$.

A class of p-brane solutions of (1.1) for arbitrary $D$ with the metric

$$ds^2 = H^k[H^{-N} \eta_{\mu\nu} dy^\mu dy^\nu + dx^\alpha dx^\alpha], \quad (1.2)$$

was found in [1], [2]–[4]. Here parameters $k$ and $N$ are functions of the parameters $\alpha$, $D$ and $q$ in the action (1.1). The solution (1.1) consists of two blocks. The first block consists of variables $y$ and the other of variables $x$, so we shall call it the two-block p-brane. This solution depends on one harmonic function $H(x)$.

The aim of this note is to present a solution of (1.1) with the metric of the form

$$ds^2 = H_1^{\frac{2}{D-2}} H_2^{\frac{2(q+2-D)}{D-2}} [(H_1 H_2)^{-\frac{2}{q}} \eta_{\mu\nu} dy^\mu dy^\nu + H_2^{-\frac{2}{q}} dz^m dz^m + dx^\alpha dx^\alpha], \quad (1.3)$$

here $\mu, \nu = 0, 1, \ldots, q-1$; $\eta_{\mu\nu}$ is a flat Minkowski metric; $m, n = 1, 2, \ldots, D - 2q - 2$ and $\alpha, \beta = 1, \ldots, q + 2$.

The solution is valid for

$$\alpha = \pm q \sqrt{\frac{2}{D-2}}. \quad (1.4)$$
This value corresponds to the supersymmetry \( [1] \) of the action (1.1). It is interesting that we get this value of \( \alpha \) not from supersymmetry, but as a solution of a system of algebraic equations (see Appendix B).

Here \( H_1 \) and \( H_2 \) are two arbitrary harmonic functions of variables \( x^\alpha \)

\[
\Delta H_1 = 0, \quad \Delta H_2 = 0. \tag{1.5}
\]

The dilaton field is

\[
\phi = \sqrt{\frac{2}{D - 2}} \ln \left( \frac{H_2}{H_1} \right). \tag{1.6}
\]

Non-vanishing components of the differential form are given by

\[
A_{\mu_1...\mu_q} = \sqrt{\frac{2}{q}} \epsilon_{\mu_1...\mu_q} H_1^{-1} \tag{1.7}
\]

\[
F^{\alpha_1...\alpha_{q+1}} = \sqrt{\frac{2}{q}} H_1^{-\frac{D-4}{2}} H_2^{-\frac{2(D-q-2)}{q(D-2)}} \epsilon^{\alpha_1...\alpha_{q+1}\beta} \partial_\beta H_1^{-1}. \tag{1.8}
\]

Here \( \epsilon_{123...q} = 1, \epsilon^{123...q+2} = 1 \)

The solution (1.3) consists of three blocks, the first block consists of variables \( y \), another of variables \( z \) and the other of variables \( x \). We shall call it the three-block p-branes solution.

The solution (1.3) depends on two harmonic functions. We shall show that for \( q=2 \) there is a three-block solution depending on three harmonic functions \( H_1(x), H_2(x), K(x) \):

\[
ds^2 = H_1^{-\frac{D-4}{2}} H_2^{-\frac{D-2}{2}} [-dy_0^2 + dy_1^2 + K(dy_0 - dy_1)^2] \\
+ H_1^{-\frac{D-2}{2}} H_2^{-\frac{D-2}{2}} dz^m dz^m + H_1^{-\frac{D-4}{2}} H_2^{-\frac{D-2}{2}} dx^\alpha dx^\alpha \tag{1.9}
\]

This solution in \( D = 10 \) dimension spacetime \([17]\) has been considered in recent derivation of the black hole entropy \([27]-[33]\) in the D-brane approach \([34]-[35]\).

The paper is organized as follows. In Section 2 the three-block solution is derived. In Section 3 particular examples of the solution (1.3) and their relations with known solutions are discussed. In Appendices A,B and C technical details are collected.
2 Solution of the Einstein equations

The Einstein equations for the action (1.1) read

\[ R_{MN} - \frac{1}{2}g_{MN}R = T_{MN}, \]  

(2.1)

where the energy-momentum tensor is

\[ T_{MN} = \frac{1}{2} (\partial_M \phi \partial_N \phi - \frac{1}{2} g_{MN} (\partial \phi)^2) + \frac{1}{2q!} e^{-\alpha \phi} (F_{M_{q+1}...M_q} F^{M_{q+1}...M_q} - \frac{1}{2(q+1)} g_{MN} F^2). \]  

(2.2)

The equation of motion for the antisymmetric field is

\[ \partial_M (\sqrt{-g} e^{-\alpha \phi} F^M_{M_{q+1}...M_q}) = 0, \]  

(2.3)

where \( F \) is a subject of the Bianchi identity

\[ \varepsilon^{M_{q+1}...M_q+2} \partial_{M_{q+1}} F_{M_{2}...M_{q+2}} = 0. \]  

(2.4)

The equation of motion for the dilaton is

\[ \partial_M (\sqrt{-g} g^{MN} \partial_N \phi) + \frac{\alpha}{2(q+1)!} \sqrt{-g} e^{-\alpha \phi} F^2 = 0. \]  

(2.5)

We shall solve equations (2.1)-(2.5) using the following Ansatz for the metric

\[ ds^2 = e^{2A(x)} \eta_{\mu \nu} dx^\mu dx^\nu + e^{2F(x)} \delta_{nm} dz^n dz^m + e^{2B(x)} \delta_{\alpha \beta} dx^\alpha dx^\beta, \]  

(2.6)

where \( \mu, \nu = 0, ..., q+1, \eta_{\mu \nu} \) is a flat Minkowski metric, \( m, n = 1, 2, ..., D-2q-2 \) and \( \alpha, \beta = 1, ..., q+2 \). Here \( A, B \) and \( C \) are functions on \( x \); \( \delta_{nm} \) and \( \delta_{\alpha \beta} \) are Kronecker symbols.

Non-vanishing components of the differential forms are

\[ A_{\mu_1...\mu_q} = h_{\mu_1...\mu_q} C(x), \]  

(2.7)

\[ F^{\alpha_1...\alpha_{q+1}} = \frac{1}{\sqrt{-g}} e^{\alpha \phi} \epsilon^{\alpha_1...\alpha_{q+1} \beta} \partial_\beta C(x), \]  

(2.8)
where $h, \gamma$ are constants.

By applying the above Ansatz and using the components of the Ricci tensor from Appendix A, one can reduce the ($\mu\nu$)-components of (2.1) to the equation

$$(q - 1)\Delta A + (q + 1)\Delta B + r\Delta F$$

$$+ \frac{q(q - 1)}{2} (\partial A)^2 + \frac{r(r + 1)}{2} (\partial F)^2 + \frac{q(q + 1)}{2} (\partial B)^2$$

$$+ q(q - 1)(\partial A\partial B) + r(q - 1)(\partial A\partial F) + rq(\partial B\partial F) =$$

$$-\frac{1}{4}(\partial \phi)^2 - \frac{\gamma^2}{4}(\partial \chi)^2 e^{2qB+\alpha\phi+2\chi} - \frac{h^2}{4}(\partial C)^2 e^{-\alpha\phi-2qA+2C}, \hspace{1cm} (2.9)$$

$(nm)$-components of (2.1) to the following equation:

$q\Delta A + (q + 1)\Delta B + (r - 1)\Delta F$

$$+ \frac{q(q + 1)}{2} (\partial A)^2 + \frac{q(q + 1)}{2} (\partial B)^2 + \frac{r(r - 1)}{2} (\partial F)^2$$

$$+ q^2(\partial A\partial B) + q(r - 1)(\partial A\partial F) + q(r - 1)(\partial B\partial F) =$$

$$-\frac{1}{4}(\partial \phi)^2 + \frac{h^2}{4}(\partial C)^2 e^{-\alpha\phi-2qA+2C} - \frac{\gamma^2}{4}(\partial \chi)^2 e^{2qB+\alpha\phi+2\chi} \hspace{1cm} (2.10)$$

and $(\alpha\beta)$-components to the equation:

$$-q\partial\alpha\partial\beta A - q\partial\alpha\partial\beta B - r\partial\alpha\partial\beta F$$

$$-q\partial\alpha A\partial\beta A + q\partial\alpha B\partial\beta B - r\partial\alpha F\partial\beta F + q(\partial\alpha A\partial\beta B + \partial\alpha B\partial\beta A)$$

$$+ r(\partial\alpha B\partial\beta F + \partial\alpha F\partial\beta B) + \delta_{\alpha\beta}[q\Delta A + q\Delta B + r\Delta F$$

$$+ \frac{q(q + 1)}{2} (\partial A)^2 + \frac{r(r + 1)}{2} (\partial F)^2$$

$$+ \frac{q(q - 1)}{2}(\partial B)^2 + q(q - 1)(\partial A\partial B) + r(q - 1)(\partial F\partial B) + qr(\partial A\partial F)] =$$

$$= \frac{1}{2}\partial\alpha\phi\partial\beta\phi - \frac{1}{4}\delta_{\alpha\beta}(\partial\phi)^2 - \frac{h^2}{2} e^{-\alpha\phi-2qA+2C}[\partial\alpha C\partial\beta C - \frac{\delta_{\alpha\beta}}{2}(\partial C)^2]$$
\[ -\frac{\gamma^2}{2} e^{2qB+\alpha\phi+2\chi}[\partial_\alpha\chi \partial_\beta\chi - \frac{\gamma\delta_{\alpha\beta}}{2}(\partial\chi)^2], \]  

(2.11)

where we use notations \((\partial A\partial B) = \partial_\alpha A \partial_\beta F\) and \(r = D - 2q - 2\).

The equations of motion \((2.3)\) for a part of components of the antisymmetric field are identically satisfied and for the other part they are reduced to a simple equation:

\[ \partial_\alpha (e^{-\alpha\phi-2qA+C} \partial_\alpha C) = 0. \]  

(2.12)

For \(\alpha\)-components of the antisymmetric field we also have the Bianchi identity:

\[ \partial_\alpha (e^{\alpha\phi+2Bq+\chi} \partial_\alpha \chi) = 0. \]  

(2.13)

The equation of motion for the dilaton has the form

\[
\partial_\alpha (e^{qA+qB+F_{r}} \partial_\alpha \phi) + \frac{\alpha\gamma^2}{2} e^{\alpha\phi+3qB+rF+2\chi} (\partial_\alpha \chi)^2
- \frac{\alpha h^2}{2} e^{-\alpha\phi-qA+qB+rF+2C} (\partial_\alpha C)^2 = 0. 
\]  

(2.14)

We have to solve the system of equations \((2.9)-(2.14)\) for unknown functions \(A, B, F, C, \chi, \phi\). We shall express \(A, B, F\) and \(\phi\) in terms of two functions \(C\) and \(\chi\). In order to kill exponents in \((2.9)-(2.14)\) we impose the following relations:

\[ qA + rF + qB = 0, \]  

(2.15)

\[ \alpha\phi + 2\chi + 2qB = 0, \]  

(2.16)

\[ 2C - 2qA - \alpha\phi = 0. \]  

(2.17)

Under these conditions equations \((2.12),(2.13)\) and \((2.14)\) will have the following forms, respectively,

\[ \partial_\alpha (e^{-C} \partial_\alpha C) = 0, \quad \partial_\alpha (e^{-\chi} \partial_\alpha \chi) = 0, \]  

(2.18)

\[ \Delta \phi + \frac{\alpha\gamma^2}{2} (\partial_\alpha \chi)^2 - \frac{\alpha h^2}{2} (\partial_\alpha C)^2 = 0. \]  

(2.19)
One rewrites (2.18) as
\[ \Delta C = (\partial C)^2, \quad \Delta \chi = (\partial \chi)^2. \] (2.20)

Therefore (2.19) will have the form
\[ \Delta \phi + \frac{\alpha \gamma^2}{2} \Delta \chi - \frac{\alpha h^2}{2} \Delta C = 0. \] (2.21)

From (2.21) it is natural to set
\[ \phi = \phi_1 C + \phi_2 \chi, \] (2.22)
where
\[ \phi_1 = \frac{\alpha h^2}{2}, \quad \phi_2 = -\frac{\alpha \gamma^2}{2}. \] (2.23)

From equations (2.15), (2.16) and (2.17) it follows that $A$, $B$ and $F$ can be presented as linear combinations of functions $C$ and $\chi$:
\[ A = a_1 C + a_2 \chi, \] (2.24)
\[ B = b_1 C + b_2 \chi, \] (2.25)
\[ F = f_1 C + f_2 \chi, \] (2.26)
where
\[ a_1 = \frac{4 - \alpha^2 h^2}{4q}, \quad a_2 = \frac{\alpha^2 \gamma^2}{4q}, \] (2.27)
\[ b_1 = -\frac{\alpha^2 h^2}{4q}, \quad b_2 = \frac{\alpha^2 \gamma^2 - 4}{4q}, \] (2.28)
\[ f_1 = \frac{\alpha^2 h^2 - 2}{2r}, \quad f_2 = \frac{2 - \alpha^2 \gamma^2}{2r}. \] (2.29)

Let us substitute expressions (2.22), (2.24)–(2.26) for $\phi$, $A$, $B$, $F$ into (2.9)–(2.11). We get relations containing bilinear forms over derivatives on $C$ and $\chi$. We assume that the coefficients in front of these bilinear forms vanish. Therefore we get the system of twelve quartic (actually quadratic) equations
for three unknown parameters $\alpha, h$ and $\gamma$. These relations are presented and solved in Appendix B. The solution is:

$$\alpha^2 = q^2 \frac{2}{r + 2q}, \quad h^2 = \gamma^2 = \frac{2}{q} \quad (2.30)$$

$$a_1 = \frac{q + r}{q(r + 2q)}, \quad a_2 = \frac{1}{r + 2q}, \quad (2.31)$$

$$b_1 = -\frac{1}{r + 2q}, \quad b_2 = -\frac{q + r}{q(r + 2q)}, \quad (2.32)$$

$$f_1 = -\frac{1}{r + 2q}, \quad f_2 = \frac{1}{r + 2q}, \quad (2.33)$$

Finally, we get 8 different solutions which differ in terms of signs $\alpha, h, \gamma$.

They are

$$h = \pm \sqrt{\frac{2}{q}}, \quad \gamma = \pm \sqrt{\frac{2}{q}}, \quad \alpha = \pm q \sqrt{\frac{2}{r + 2q}}.$$

From (2.23) we get that

$$\phi_1 = \frac{\alpha}{q} \quad \text{and} \quad \phi_2 = -\frac{\alpha}{q},$$

so

$$\phi_1 = \pm \sqrt{\frac{2}{r + 2q}}, \quad \phi_2 = \mp \sqrt{\frac{2}{r + 2q}}. \quad (2.34)$$

Unknown functions in the metric (2.6) were presented in terms of functions $C$ and $\chi$, which satisfy the equations (2.20). Let us express $C$ and $\chi$ in terms of $H_1$ and $H_2$ by using the formulas:

$$e^{-C} = H_1, \quad e^{-\chi} = H_2. \quad (2.35)$$

Then from (2.20) one gets that $H_1$ and $H_2$ are harmonic functions

$$\Delta H_1 = 0, \quad \Delta H_2 = 0. \quad (2.36)$$

By using expressions for $a_1, a_2, b_1, b_2, f_1, f_2, \phi_1, \phi_2$ in terms of $\alpha, h, \gamma$ and $C, \chi$ in terms of $H_1$ and $H_2$ one gets:
$$A = \frac{-D - q - 2}{q(D-2)} \ln H_1 - \frac{1}{D-2} \ln H_2, \quad (2.37)$$

$$B = \frac{1}{D-2} \ln H_1 + \frac{D - q - 2}{q(D-2)} \ln H_2, \quad (2.38)$$

$$F = \frac{1}{D-2} \ln H_1 - \frac{1}{D-2} \ln H_2, \quad (2.39)$$

$$\phi = \pm \sqrt{\frac{2}{D-2}} \ln H_1 \pm \sqrt{\frac{2}{D-2}} \ln H_2. \quad (2.40)$$

Finally, by using (2.37)-(2.39) we get the expression for the metric (2.6)

$$ds^2 = H_1^{-2(D-q-2)/q(D-2)} H_2^{-2} \eta_{\mu\nu} dy^\mu dy^\nu +$$

$$H_1^{-2(D-q-2)/q(D-2)} dz^m dz^m + H_1^{-2(D-q-2)/q(D-2)} dx^\alpha dx^\alpha, \quad (2.41)$$

that is equivalent to (1.3) presented in Introduction.

### 3 Particular cases

In this section different particular cases of the solution (2.41) are discussed. 

#### Boosted solution

For $q = 2$ equations (2.1) admit a solution depending on three harmonic functions. Let us use the following ansatz for the metric

$$ds^2 = e^{2A(x)} [-dy_0^2 + dy_1^2 + K(dy_0 - dy_1)^2]$$

$$+ e^{2F(x)} dz^m dz^m + e^{2B(x)} dx^\alpha dx^\alpha, \quad (3.1)$$

where $K$ is a function of variables $x$. We shall use ansatz (2.8), (2.9) for the antisymmetric field and coordinates

$$v = y_1 + y_0, \quad u = y_0 - y_1.$$
For this ansatz the form of equations of motion (2.13) and (2.14) as well as the form of $uv$, $vv$ and also $mn$ and $\mu\nu$-components of the Einstein equations does not change. The form for $uu$-component changes. For details see Appendix C. For example, for $uv$-components of the Einstein equations we have

$$-\Delta A + \Delta B + (\partial A)^2 + \frac{r(r + 1)}{2} (\partial F)^2 + 3(\partial B)^2 + 2(\partial A\partial B) + r(\partial A\partial F) + 2r(\partial F\partial B) = -\frac{1}{4}(\partial \phi)^2 - \frac{\gamma^2}{4}(\partial C)^2 - \frac{\gamma^2}{4}(\partial \chi)^2. \quad (3.2)$$

Here we use the relation

$$2A + rF + 2B = 0. \quad (3.3)$$

The Einstein equations for $uu$-component has the form

$$-\frac{1}{2}\Delta K + K [ -\Delta A + \Delta B + (\partial A)^2 + \frac{r(r + 1)}{2} (\partial F)^2 + 3(\partial B)^2 + 2(\partial A\partial B) + r(\partial A\partial F) + 2r(\partial F\partial B)] - \partial K [2\partial A + r\partial F + 2\partial B] = -K\frac{1}{4}(\partial \phi)^2 + \frac{\gamma^2}{4}(\partial C)^2 + \frac{\gamma^2}{4}(\partial \chi)^2. \quad (3.4)$$

We see that in equation (3.4) the terms containing $K$ without derivatives are canceled due to equation (3.2). Terms with the first order derivative are canceled due to relation (3.3). Therefore, if we assume that

$$\Delta K = 0, \quad (3.5)$$

we get that the metric

$$ds^2 = H_1^{-\frac{\partial_{\phi}}{2}} H_2^{-\frac{\partial_{\mu}}{2}} [-d\gamma_0^2 + d\gamma_1^2 + K(d\gamma_0 - d\gamma_1)^2 + H_1^{-\frac{\partial_{\phi}}{2}} d\gamma^m d\gamma^m + H_1^{-\frac{\partial_{\phi}}{2}} H_2^{-\frac{\partial_{\gamma}}{2}} dx^a dx^a] \quad (3.6)$$

solves the theory.
Magnetic charge

Let us take $H_1 = 1$, $H_2 = H$. This ansatz corresponds to zero electric field (see (2.7)) and non-zero magnetic charge. In this case the three-block metric reduces to two-block metric

$$ds^2 = H^{-\frac{2}{q(D+q-2)}}[\eta_{\mu\nu}dy^\mu dy^\nu + dz^m dz^m] + H^{\frac{D+q}{2}}dx^\alpha dx^\alpha.$$  \hspace{1cm} (3.7)

More general solution for arbitrary $\alpha$ has been presented in \[1, 7]-[9], \[21\].

The metric is:

$$ds^2 = H^\sigma[H^{-N}\eta_{\mu\nu}dy^\mu dy^\nu + dx^\alpha dx^\alpha].$$ \hspace{1cm} (3.8)

$$N = \frac{4}{\Delta}, \quad \sigma = \frac{4(D-q-2)}{\Delta(D-2)},$$

$$\Delta \equiv a^2 + \frac{2q(D-q-2)}{D-2}.$$ 

Here $\mu, \nu = 0, 1, ..., D-q-3$; $\eta_{\mu\nu}$ is a flat Minkowski metric and $\alpha = 1, 2, ..., q+2$. It depends on one harmonic function $H(x)$.

Electric charge

Let us take $H_1 = H$, $H_2 = 1$. This ansatz corresponds to zero magnetic field and non-zero electric charge. In this case the three-block metric reduces to two-block metric

$$ds^2 = H^{-2\frac{D+q-2}{q(D-q-2)}}\eta_{\mu\nu}dy^\mu dy^\nu + H^{\frac{2}{D-q-2}}[dz^m dz^m + dx^\alpha dx^\alpha].$$ \hspace{1cm} (3.9)

More general solution for arbitrary $\alpha$ has been presented in \[1, 36\].

Non-dilaton solution

Let us take $H_1 = H_2 = H$. This ansatz corresponds to equal electric and magnetic charges. In this case the three-block metric reduces to a two-block metric plus flat Euclidean metric

$$ds^2 = H^{-\frac{2}{q(D+q-2)}}\eta_{\mu\nu}dy^\mu dy^\nu + dz^m dz^m + H^{\frac{2}{D-q-2}}dx^\alpha dx^\alpha.$$ \hspace{1cm} (3.10)
Self-dual solution

In the particular case of $D = 2q + 2$ magnetic and electric fields fill all spacetime, and the three-block metric reduces to the two-block metric

$$ds^2 = (H_1H_2)^{-\frac{1}{2}}\eta_{\mu\nu}dy^\mu dy^\nu + (H_1H_2)^{\frac{1}{2}}dx^\alpha dx^\alpha,$$  \hspace{1cm} (3.11)

For this particular case $\alpha = \pm \sqrt{q}$.

**D=4 and q=1**

If D=4 and q=1 then

$$ds^2 = -(H_1H_2)^{-1}dy_0^2 + H_1H_2(dx_1^2 + dx_2^2 + dx_3^2)$$  \hspace{1cm} (3.12)

This metric was obtained in [6] for an action with two antisymmetric fields.

**D=10, q=2, K≠0**

If we take $D = 10$, $q = 2$, $K \neq 0$ then we get the solution [17, 28, 22]

$$ds^2 = H_1^{-\frac{q}{4}}H_2^{-\frac{q}{4}}(-dy_0^2 + dy_1^2 + K(dy_0 - dy_1)^2)$$
$$+ H_1^{-\frac{1}{4}}H_2^{-\frac{1}{4}}(dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2) + H_1^{-\frac{1}{4}}H_2^{-\frac{1}{4}}(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2).$$  \hspace{1cm} (3.13)

**D=10, q=1**

Let us also mention the following solution for $D = 10$, $q = 1$

$$ds^2 = H_1^\frac{1}{4}H_2^\frac{1}{4}[-(H_1H_2)^{-2}dy_0^2 +$$
$$H_2^{-2}(dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2 + dz_5^2 + dz_6^2) +$$
$$dx_1^2 + dx_2^2 + dx_3^2)],$$  \hspace{1cm} (3.14)
4 Concluding Remarks

To conclude, we have constructed the three-block solution (1.3) of the Lagrangian (1.1) and discussed its reduction to known solutions. It is interesting that the value of the dilaton parameter \( \alpha \) (1.4) was obtained as a solution of the system of algebraic equations which follows from the Einstein equations for the three-block metric (2.6). There is no such fixing for the two-block p-brane (1.2). As it is known, this value of \( \alpha \) leads to a supersymmetric theory [1]. It would be interesting to gain better understanding of a mechanism responsible for this fixing. It would be also interesting to generalize the three-block solution to n-block solutions \( (n \geq 4) \) and to see the connection with the harmonic superposition of M-branes [19, 21, 18].

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Appendix

A Three-block metric

Let us calculate the left hand side of the Einstein equations for the metric

\[ ds^2 = e^{2A(x)} \eta_{\mu\nu} dy^\mu dy^\nu + e^{2F(x)} dz^m dz^m + e^{2B(x)} dx^\alpha dx^\alpha, \]  

(A.1)

\( \mu, \nu = 0, \ldots, q - 1 \), \( \eta_{\mu\nu} \) is a flat Minkowski metric with signature \((-, +, \ldots)\);

\( m, n = 1, \ldots, r \); \( r = d - q \); \( \alpha, \beta = 1, \ldots, D - d \). \( A \), \( B \) and \( C \) are functions on \( x \).

For this metric non-vanishing Christoffel symbols have the form:

\[ \Gamma^\mu_{\nu\alpha} = \delta^\nu_{\alpha} \partial_{\alpha} A, \]  

(A.2)

\[ \Gamma^\alpha_{\mu\nu} = -h_{\mu\nu} e^{2(A-B)} \partial_{\alpha} A, \]  

(A.3)

\[ \Gamma^\alpha_{mn} = -\delta_{mn} e^{2(F-B)} \partial_{\alpha} F, \]  

(A.4)

\[ \Gamma^m_{n\alpha} = \delta^m_n \partial_{\alpha} F, \]  

(A.5)

\[ \Gamma^\alpha_{\beta\gamma} = \delta^\alpha_{\beta} \partial_{\gamma} B + \delta^\alpha_{\gamma} \partial_{\beta} B - \delta^\gamma_{\beta} \partial_{\alpha} B. \]  

(A.6)

The Riemann tensor is defined as

\[ R^M_{NKL} = \partial_K \Gamma^M_{NL} - \partial_L \Gamma^M_{NK} + \Gamma^P_{KL} \Gamma^M_{PN} - \Gamma^M_{PL} \Gamma^P_{NK}. \]  

(A.7)

The Ricci tensor is

\[ R_{MK} = R^N_{MNK}. \]

For the metric (A.1) the components of the Ricci tensor are:

\[ R_{\mu\nu} = -h_{\mu\nu} e^{2(A-B)} [\Delta A + q (\partial A)^2 + \tilde{d}(\partial A \partial B) + r (\partial A \partial F)], \]  

(A.8)

\[ R_{mn} = -\delta_{mn} e^{2(F-B)} [\Delta F + r (\partial F)^2 + q (\partial A \partial F) + \tilde{d}(\partial B \partial F)], \]  

(A.9)

\[ R_{\alpha\beta} = -q \partial_{\alpha} \partial_{\beta} A - r \partial_{\alpha} \partial_{\beta} F - \tilde{d} \partial_{\alpha} \partial_{\beta} B - q \partial_{\alpha} A \partial_{\beta} A - r \partial_{\alpha} F \partial_{\beta} F + \tilde{d} \partial_{\alpha} B \partial_{\beta} B \]

\[ + q (\partial_{\alpha} A \partial_{\beta} B + \partial_{\alpha} B \partial_{\beta} A) + r (\partial_{\alpha} B \partial_{\beta} F + \partial_{\alpha} F \partial_{\beta} B) \]
\[ + \delta_{\alpha\beta} [-\Delta B - \tilde{d}(\partial B)^2 - q(\partial A\partial B) - r(\partial F\partial B)], \tag{A.10} \]

where \( \tilde{d} = D - d - 2, \ r = d - q. \)

Scalar curvature is

\[ R = e^{-2B} [-2q\Delta A - 2(\tilde{d} + 1)\Delta B - 2r\Delta F \]

\[ - q(q + 1)(\partial A)^2 - \tilde{d}(\partial B)^2 - r(r + 1)(\partial F)^2 \]

\[ - 2 qr(\partial A\partial F) - 2q\tilde{d}(\partial A\partial B) - 2r\tilde{d}(\partial B\partial F)], \tag{A.11} \]

The left hand side of the Einstein equations read

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \eta_{\mu\nu} e^{2(A-B)} [(q - 1)\Delta A + (\tilde{d} + 1)\Delta B + r\Delta F \]

\[ + \frac{q(q - 1)}{2} (\partial A)^2 + \frac{r(r + 1)}{2} (\partial F)^2 + \frac{\tilde{d}(\tilde{d} + 1)}{2} (\partial B)^2 \]

\[ + \tilde{d}(q - 1)(\partial A\partial B) + r(q - 1)(\partial A\partial F) + r\tilde{d}(\partial F\partial B), \tag{A.12} \]

\[ R_{mn} - \frac{1}{2} g_{mn} R = \delta_{mn} e^{2(F - B)} [q\Delta A + (\tilde{d} + 1)\Delta B + (r - 1)\Delta F \]

\[ + \frac{q(q + 1)}{2} (\partial A)^2 + \frac{r(r - 1)}{2} (\partial F)^2 + \frac{\tilde{d}(\tilde{d} + 1)}{2} (\partial B)^2 \]

\[ + \tilde{d}q(\partial A\partial B) + q(r - 1)(\partial A\partial F) + \tilde{d}(r - 1)(\partial F\partial B)], \tag{A.13} \]

\[ R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = -q\partial_\alpha\partial_\beta A - \tilde{d}\partial_\alpha\partial_\beta B - r\partial_\alpha\partial_\beta F \]

\[ - q\partial_\alpha A\partial_\beta A + \tilde{d}\partial_\alpha B\partial_\beta B - r\partial_\alpha F\partial_\beta F \]

\[ + q(\partial_\alpha A\partial_\beta B + \partial_\alpha B\partial_\beta A) + r(\partial_\alpha B\partial_\beta F + \partial_\alpha F\partial_\beta B) \]
\[ + \delta_{\alpha\beta} \left[ q \Delta A + \tilde{d} \Delta B + r \Delta F + \frac{q(q + 1)}{2} (\partial A)^2 + \frac{r(r + 1)}{2} (\partial F)^2 + \frac{\tilde{d}(\tilde{d} - 1)}{2} (\partial B)^2 
\]

\[ + q(\tilde{d} - 1)(\partial A \partial B) + qr(\partial A \partial F) + r(\tilde{d} - 1)(\partial F \partial B) \right]. \]  

(A.14)
B System of algebraic equations

Here we present a system of equations which follows from equations (2.9) - (2.11). This system is

\[-a_1 + b_1 + \frac{q(q - 1)}{2}a^2_1 + \frac{r(r + 1)}{2}f^2_1 + \frac{q(q + 1)}{2}b^2_1 + q(q - 1)a_1b_1 + r(q - 1)a_1f_1 + rqf_1b_1 + \frac{\phi^2_1}{4} + \frac{h^2}{4} = 0; \] (B.15)

\[-a_2 + b_2 + \frac{q(q - 1)}{2}a^2_2 + \frac{r(r + 1)}{2}f^2_2 + \frac{q(q + 1)}{2}b^2_2 + q(q - 1)a_2b_2 + r(q - 1)a_2f_2 + rqb_2f_2 + \frac{\phi^2_2}{4} + \frac{\gamma^2}{4} = 0; \] (B.16)

\[q(q - 1)a_1a_2 + r(r + 1)f_1f_2 + q(q + 1)b_1b_2 + q(q - 1)(a_1b_2 + a_2b_1) + r(q - 1)(a_2f_1 + a_1f_2) + rq(f_1b_2 + f_2b_1) + \frac{\phi_1\phi_2}{2} = 0; \] (B.17)

\[-f_1 + b_1 + \frac{q(q + 1)}{2}a^2_1 + \frac{r(r - 1)}{2}f^2_1 + \frac{q(q + 1)}{2}b^2_1 + q^2a_1b_1 + q(r - 1)a_1f_1 + q(r - 1)f_1b_1 + \frac{\phi^2_1}{4} - \frac{h^2}{4} = 0; \] (B.18)

\[-f_2 + b_2 + \frac{q(q + 1)}{2}a^2_2 + \frac{r(r - 1)}{2}f^2_2 + \frac{q(q + 1)}{2}b^2_2 + q^2a_2b_2 + q(r - 1)a_2f_2 + q(r - 1)b_2f_2 + \frac{\phi^2_2}{4} + \frac{\gamma^2}{4} = 0; \] (B.19)
\[ q(q+1)a_1a_2 + r(r-1)f_1f_2 + q(q+1)b_1b_2 \]

\[ + q^2(a_1b_2 + a_2b_1) + q(r-1)(a_2f_1 + a_1f_2) + q(r-1)(f_1b_2 + f_2b_1) + \frac{\phi_1\phi_2}{2} = 0; \quad (B.20) \]

\[-qa_1^2 - rf_1^2 + qb_1^2 +
\]

\[ + 2qa_1b_1 + 2rf_1b_1 - \frac{\phi_1^2}{2} + \frac{h^2}{2} = 0; \quad (B.21) \]

\[-qa_2^2 - rf_2^2 + qb_2^2 +
\]

\[ + 2qa_2b_2 + 2rb_2f_2 - \frac{\phi_2^2}{2} + \frac{\gamma^2}{2} = 0; \quad (B.22) \]

\[-qa_1a_2 - rf_1f_2 + qb_1b_2 +
\]

\[ + q(a_1b_2 + a_2b_1) + r(f_1b_2 + f_2b_1) - \frac{\phi_1\phi_2}{2} = 0; \quad (B.23) \]

\[ \frac{q(q+1)}{2}a_1^2 + \frac{r(r+1)}{2}f_1^2 + \frac{q(q-1)}{2}b_1^2 +
\]

\[ + q(q-1)a_1b_1 + r(q-1)b_1f_1 + rqf_1a_1 + \frac{\phi_1^2}{4} - \frac{h^2}{4} = 0; \quad (B.24) \]

\[ \frac{q(q+1)}{2}a_2^2 + \frac{r(r+1)}{2}f_2^2 + \frac{q(q-1)}{2}b_2^2 +
\]

\[ + q(q-1)a_2b_2 + r(q-1)a_2f_2 + rqb_2f_2 + \frac{\phi_2^2}{4} - \frac{\gamma^2}{4} = 0; \quad (B.25) \]

\[ q(q+1)a_1a_2 + r(r+1)f_1f_2 + q(q-1)b_1b_2 + q(q-1)(a_1b_2 + a_2b_1) \]
\[ + r(q - 1)(b_2 f_1 + b_1 f_2) + rq(f_1 a_2 + f_2 a_1) + \frac{\phi_1 \phi_2}{2} = 0. \]  

(B.26)

The variables \(a_1, a_2, b_1, b_2, f_1, f_2, \phi_1, \phi_2\) are functions of \(h, \alpha, \gamma\). So we have a system of twelve quadratic equations for three unknown parameters \(h, \alpha, \gamma\). We solved this system of equations on MAPLE V. It occurs that the system has solutions only for special value of \(\alpha\),

\[ \alpha = \pm q \sqrt{\frac{2}{D - 2}} \]  

(B.27)

The solution is:

\[ h^2 = \gamma^2 = \frac{2}{q}, \]  

(B.28)

\[ a_1 = \frac{q + r}{q(r + 2q)}, \quad a_2 = \frac{1}{r + 2q}, \]  

(B.29)

\[ b_1 = -\frac{1}{r + 2q}, \quad b_2 = -\frac{q + r}{q(r + 2q)} \]  

(B.30)

\[ f_1 = -\frac{1}{r + 2q}, \quad f_2 = \frac{1}{r + 2q}, \]  

(B.31)

\[ \phi_1^2 = \frac{2}{r + 2q}, \quad \phi_2^2 = \frac{2}{r + 2q}. \]  

(B.32)
C  Boosted three-block metric

It is convinient to rewrite the metric (3.1) in the light-cone coordinates.
\[ v = y_0 + y_1, \ u = y_0 - y_1; \]
\[ ds^2 = e^{2A(x)}[-du dv + K(x)du^2] + e^{2F(x)}dz^m dz^m + e^{2B(x)}dx^\alpha dx^\alpha. \]  
(C.33)

The first two components of the metric and its inverse have the form
\[ g_{uu} = K e^{2A(x)}; \quad g_{uv} = g_{vu} = -\frac{1}{2} e^{2A}; \quad g_{vv} = 0; \]
\[ g^{uu} = 0; \quad g^{uv} = g^{vu} = -2 e^{-2A}; \quad g^{vv} = -4 K e^{-2K} \]  
(C.34)

Non-vanishing components of the Cristoffel symbols are:
\[ \Gamma_{\alpha u u} = \partial_{\alpha} A, \quad \Gamma_{\alpha v v} = \partial_{\alpha} A, \quad \Gamma_{\alpha v u} = \partial_{\alpha} K, \]  
\[ \Gamma_{\alpha u v} = \frac{1}{2} e^{2(A-B)} \left( \partial_{\alpha} K + 2 K \partial_{\alpha} A \right), \quad \Gamma_{\alpha v u} = \frac{1}{2} e^{2(A-B)} \partial_{\alpha} A, \]  
\[ \Gamma_{\alpha m m} = -\delta_{m}^{\alpha} e^{2(F-B)} \partial_{\alpha} F, \]  
\[ \Gamma_{\alpha n n} = \delta_{n}^{\alpha} \partial_{\alpha} F, \]  
\[ \Gamma_{\beta \gamma} = \delta_{\beta}^{\alpha} \partial_{\gamma} B + \delta_{\gamma}^{\alpha} \partial_{\beta} B - \delta_{\beta}^{\gamma} \partial_{\alpha} B. \]  
(C.35) (C.36) (C.37) (C.38) (C.39)

We see that only the Cristoffel symbols containing index \( u \) have changed from those in Appendix A. It is easy to check that only one component of the Ricci tensor has changed
\[ R^{(K)}_{uu} = e^{2(A-B)} \left\{ -\frac{1}{2} \Delta K + K \left[ -\Delta A + \Delta B + (\partial A)^2 + \frac{r(r + 1)}{2} (\partial F)^2 \right. \right. \]
\[ +3(\partial B)^2 + 2(\partial A \partial B) + r(\partial A \partial F) + 2r(\partial F \partial B) \right\} + \partial K (\partial A + \partial B - \frac{r}{2} \partial F) \right\} \]  
(C.40)

The other components do not change
\[ R^{(K)}_{uv} = \frac{1}{2} R^{(K)}_{00} = \frac{1}{2} R_{00}, \]  
(C.41)
\[ R^{(K)}_{mn} = R_{mn}, \quad R^{(K)}_{\alpha \beta} = R_{\alpha \beta}, \quad (C.42) \]

as well as the scalar curvature does not change

\[ R^{(K)} = R. \quad (C.43) \]

In particular, the LHS of the Einstein equations for \( uv \) component does not change and we have

\[
R_{uv} - \frac{1}{2} g_{uv} R = -\frac{1}{2} e^{2(A-B)}[\Delta A + 3\Delta B + r\Delta F + (\partial A)^2 + \frac{r(r+1)}{2}(\partial F)^2 + 3(\partial B)^2 + 2\partial A \partial B + r \partial A \partial F + 2r \partial F \partial B] \quad (C.44)
\]

Here we use that

\[ R_{uv} = \frac{1}{4}(R_{00} - R_{11}) = \frac{1}{2} R_{00}. \]