Hilbert schemes, wreath products, and the McKay correspondence

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Abstract

Various algebraic structures have recently appeared in a parallel way in the framework of Hilbert schemes of points on a surface and respectively in the framework of equivariant K-theory [N1, Gr, S2, W], but direct connections are yet to be clarified to explain such a coincidence. We provide several non-trivial steps toward establishing our main conjecture on the isomorphism between the Hilbert quotient of the affine space $\mathbb{C}^2n$ by the wreath product $\Gamma_n = \Gamma \sim S_n$ and Hilbert schemes of points on the minimal resolution of a simple singularity $\mathbb{C}^2/\Gamma$. We discuss further various implications of our main conjecture. We obtain a key ingredient toward a direct isomorphism between two forms of McKay correspondence in terms of Hilbert schemes [N1, Gr, N2] and respectively of wreath products [FJW]. We in addition establish a direct identification of various algebraic structures appearing in two different setups of equivariant K-theory [S2, W].

Introduction

Nakajima [N1] constructed a Heisenberg algebra using correspondence varieties which acts on the direct sum over all $n$ of homology groups $H(X^{[n]})$ with complex coefficient of Hilbert schemes $X^{[n]}$ of $n$ points on a quasi-projective surface $X$. This representation is irreducible thanks to Göttsche’s earlier work [G]. Similar results have been independently obtained by Grojnowski [Gr]. We refer to [N2] for an excellent account of Hilbert schemes and related works. In the case when $X$ is the minimal resolution $\mathbb{C}^2/\Gamma$ of the simple singularity $\mathbb{C}^2/\Gamma$ associated to a finite subgroup $\Gamma$ of $SL_2(\mathbb{C})$, this together with some additional simple data provides a geometric realization of the Frenkel-Kac-Segal vertex construction of the basic representation of an affine Lie algebra [FK, S1]. We may view this as a geometric McKay correspondence [Mc] which provides a bijection between finite subgroups of $SL_2(\mathbb{C})$ and affine Lie algebras of ADE types.

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In [W] we realized the important role of wreath products $\Gamma_n = \Gamma \sim S_n$ associated to a finite group $\Gamma$ in equivariant K-theory. Various algebraic structures were constructed on the direct sum over all $n$ of the topological $\Gamma_n$-equivariant K-theory $\mathcal{K}_{s_1}^{top}(X^n) \otimes \mathbb{C}$ of $X^n$ for a $\Gamma$-space $X$. The results of [W] generalized the work of Segal [S2] (also see [WW, Gr]) which corresponds to our special case when $\Gamma$ is trivial (i.e. $\Gamma$ is the one-element group) and the wreath product $\Gamma_n$ reduces to the symmetric group $S_n$.

The wreath product approach obtains further significance in light of the conjectural equivalence of various algebraic structures in the following three spaces:

$$
\begin{align*}
\text{I} & \quad \bigoplus_{n \geq 0} H(Y^{[n]}) \\
\text{II} & \quad \bigoplus_{n \geq 0} \mathcal{K}_{s_1}^{top}(Y^n) \otimes \mathbb{C} \\
\text{III} & \quad \bigoplus_{n \geq 0} \mathcal{K}_{s_1}^{top}(X^n) \otimes \mathbb{C}
\end{align*}
$$

Here one assumes that $X$ is a quasi-projective surface acted upon by a finite group $\Gamma$ and $Y$ is a suitable resolution of singularities of $X/\Gamma$ such that there exists a canonical isomorphism between $K_r(X)$ and $K(Y)$. For $\Gamma$ trivial III reduces to II. The graded dimensions of the three spaces have been shown to coincide [W]. The complexity of the geometry involved decreases significantly from I to II, and then to III. In each of the three setups various algebraic structures have been constructed in [Gr, N2, S2, W] such as Hopf algebra, vertex operators, and Heisenberg algebra, etc. We remark that there has been a construction of an additive isomorphism between the spaces in I and II due to de Cataldo and Migliorini [CM].

In a most important case when $X$ is $\mathbb{C}^2$ acted upon by a finite subgroup $\Gamma \subset SL_2(\mathbb{C})$ and $Y$ is the minimal resolution $\mathbb{C}^2/\Gamma$ of $\mathbb{C}^2/\Gamma$, the above diagram reduces to the following one:

$$
\begin{align*}
\text{I} & \quad \bigoplus_{n \geq 0} H(\mathbb{C}^2/\Gamma^{[n]}) \\
\text{II} & \quad \bigoplus_{n \geq 0} \mathcal{K}_{s_1}^{top}(\mathbb{C}^2/\Gamma^n) \otimes \mathbb{C} \\
\text{III} & \quad \bigoplus_{n \geq 0} R(\Gamma_n)
\end{align*}
$$

by using the Thom isomorphism between $\mathcal{K}_{s_1}^{top}(\mathbb{C}^{2n})$ and the representation ring $R_\mathbb{Z}(\Gamma_n)$ of the wreath product. Here $R(\Gamma_n) = R_\mathbb{Z}(\Gamma_n) \otimes \mathbb{C}$ in our notation. It was pointed out in [W] that the Frenkel-Kac-Segal homogeneous vertex representation can be realized in terms of representation rings of such wreath products. Such a finite group theoretic construction, which can be viewed as a new form of McKay correspondence, has been firmly established recently in our work [FJW] jointly.
with I. Frenkel and Jing. It remains a big puzzle however why there are many parallel algebraic structures in these different setups.

In the present paper we propose a coherent approach to fill in the gap (at least in the setup of diagram (2) above) and present several canonical ingredients in our approach. More explicitly, we provide direct links from wreath products to Hilbert schemes. We find a natural interpretation of a main ingredient (the so-called weighted bilinear form) in [FJW], which brings us one step closer to a direct isomorphism of the two forms of McKay correspondence respectively in terms of Hilbert schemes and wreath products. We also establish isomorphisms of various algebraic structures in II and III. Let us discuss in more detail.

Given a finite subgroup $\Gamma$ of $SL_2(\mathbb{C})$, we observe that there is a natural identification between $\mathbb{C}^2/\Gamma_n$ and the $n$-th symmetric product $(\mathbb{C}^2/\Gamma)^{(n)}$ of the simple singularity $\mathbb{C}^2/\Gamma$. The following commutative diagram $\[\]$

$$
\xymatrix{
\widetilde{\mathbb{C}^2/\Gamma}^{[n]} \ar[r]^{\pi_n} \ar[d]_{\tau_n} & (\mathbb{C}^2/\Gamma)^{(n)}/S_n \ar[d]^{\tau_{(n)}} \\
\mathbb{C}^{2n}/\Gamma_n \ar@{=}[r] & (\mathbb{C}^2/\Gamma)^{(n)}/S_n,
}$$

defines a resolution of singularities $\tau_n : \widetilde{\mathbb{C}^2/\Gamma}^{[n]} \rightarrow \mathbb{C}^{2n}/\Gamma_n$, where $\tau_{(n)}$ is naturally induced from the minimal resolution $\mathbb{C}^2/\Gamma \rightarrow \mathbb{C}^2/\Gamma$. We show that $\tau_n$ is a semismall crepant resolution, which provides the first direct link between wreath products and Hilbert schemes. We show that the fiber of $\tau_n$ over $[0] \in \mathbb{C}^{2n}/\Gamma_n$ (associated to the origin of $\mathbb{C}^{2n}$) is of pure dimension $n$ and we give an explicit description of its irreducible components.

We conjecture that there exists a canonical isomorphism between the Hilbert quotient $\mathbb{C}^{2n}/\Gamma_n$ of $\mathbb{C}^{2n}$ by $\Gamma_n$ (see [Ka] or Subsection 1.3 for the definition of Hilbert quotient) and the Hilbert scheme $\widetilde{\mathbb{C}^2/\Gamma}^{[n]}$, and provide several nontrivial steps toward establishing this conjecture. More explicitly, we first single out a distinguished nonsingular subvariety $X_{\Gamma,n}$ of $(\mathbb{C}^2)^{[nN]}$ and construct a morphism $\varphi$ from $\mathbb{C}^{2n}/\Gamma_n$ to $X_{\Gamma,n}$, where $N$ is the order of the group $\Gamma$. We use here a description of a set of generators for the algebra of $\Gamma_n$ invariant regular functions on $\mathbb{C}^{2n}$ which is a generalization of a theorem of Weyl [Wey]. It follows by construction that our morphism from $\mathbb{C}^{2n}/\Gamma_n$ to $X_{\Gamma,n}$ when restricted to a certain Zariski open set is indeed an isomorphism. We give a quiver variety description of $X_{\Gamma,n}$ and $\mathbb{C}^{2n}/\Gamma_n$ in the sense of Nakajima [N, N3]. Such an identification follows easily from Nakajima’s quiver identification of Hilbert scheme of points on $\mathbb{C}^2$ (cf. [N2] and Varagnolo-Vasserot [VV]). According to Nakajima [N4], it can be shown essentially by using Kronheimer-Nakajima [KN] that the Hilbert scheme $\widetilde{\mathbb{C}^2/\Gamma}^{[n]}$ is a quiver variety associated to the same quiver data as $X_{\Gamma,n}$ but with a different stability condition. It follows that $X_{\Gamma,n}$ and $\widetilde{\mathbb{C}^2/\Gamma}^{[n]}$ are diffeomorphic by Corollary 4.2 in [N]. In this way we have obtained a second direct link between $\Gamma_n$ and $\widetilde{\mathbb{C}^2/\Gamma}^{[n]}$. 

One plausible way to establish our main conjecture will be to establish that \( \varphi \) is an isomorphism between \( C^{2n}/\Gamma_n \) and \( X_{\Gamma,n} \) and that the diffeomorphism between \( X_{\Gamma,n} \) and \( \widetilde{C^2}/[\Gamma] \) is indeed an isomorphism as complex varieties.

Our construction contains two distinguished cases which have been studied by others. The morphism above for \( n = 1 \) becomes an isomorphism due to Ginzburg-Kapranov (unpublished) and independently Ito-Nakumura [IN]. Our morphism above also generalizes Haiman’s construction [H] of a morphism from \( C^{2n}/S_n \) to \( (C^2)^{\lfloor n \rfloor} \) which corresponds to our special case for \( \Gamma \) trivial (where no passage to the quiver variety description is needed). Haiman [H] has in addition shown that the morphism being an isomorphism is equivalent to the validity of the remarkable \( n! \) conjecture due to Garsia and Haiman [GH]. (We remark that there has been also attempt by Bezrukavnikov and Ginzburg [BG] in establishing this conjectural isomorphism for \( \Gamma \) trivial.) Very recently a proof of the \( n! \) conjecture (and this isomorphism conjecture) has been announced by Haiman in his homepage by establishing a Cohen-Macaulay property of a certain universal scheme which was conjectured in [H]. It is natural for us to conjecture similarly a Cohen-Macaulay property of a certain universal scheme in our setup (Conjecture 3), which is sufficient to imply that \( \varphi \) is an isomorphism.

A distinguished virtual character of \( \Gamma_n \) has been used to construct a semipositive definite symmetric bilinear form (called a weighted bilinear form) on \( RZ(\Gamma_n) \) which plays a fundamental role in the wreath product approach to McKay correspondence [FJW]. Indeed it is given by the \( n \)-th tensor of the McKay virtual character \( \lambda(C^2) \) of \( \Gamma \). On the other hand, the virtual character \( \lambda(C^{2n}) \) of \( \Gamma_n \) induced from the Koszul-Thom class defines a canonical bilinear form on the Grothendieck group \( K^0_n(C^{2n}) \) of the bounded derived category \( D^0_n(C^{2n}) \) consisting of \( \Gamma_n \)-equivariant coherent sheaves whose cohomology sheaves are concentrated on the origin. Although they are defined very differently, these two virtual characters of \( \Gamma_n \) are shown to coincide. This establishes an isometry between \( K^0_n(C^{2n}) \) and \( RZ(\Gamma_n) \) endowed with the weighted bilinear form, and thus provides a natural explanation of the weighted bilinear form introduced from a purely group theoretic consideration. (Actually we establish the coincidence of virtual characters for more general \( \Gamma \subset GL_k(C) \) and the induced \( \Gamma_n \)-action on \( C^{kn} \).) We regard this isometry as an important ingredient toward a direct isomorphism of the two forms of McKay correspondence realized respectively in terms of Hilbert schemes [N2, Gr] and of wreath products [FJW].

While our motivation is quite different, our main conjecture fits into the scheme of Reid [R] who asks for what finite subgroup \( G \subset SL_k(C) \) the Hilbert quotient \( C^K/G \) (also called \( G \)-Hilbert scheme on \( C^K \)) is a crepant resolution of \( C^K/G \). Note that the notion of McKay correspondence is meant in the strict sense in this paper while the McKay correspondence in [R] is in a generalized sense. Our work provides supporting evidence for an affirmative answer of the McKay correspondence in the sense of [R] for \( C^{2n} \) acted upon by \( \Gamma_n \) which in turn is a key step to a direct
isomorphism of the two form of McKay correspondence mentioned above.

By applying a remarkable theorem of Bridgeland-King-Reid [BKR] to our situation, our main conjecture on the isomorphism between Hilbert quotients and Hilbert schemes implies that the equivalence of bounded derived categories among $D_{\tau_n}(\mathbb{C}^{2n})$, $D_{S_n}(\mathbb{C}^2/\Gamma_n^n)$, and $D(\mathbb{C}^2/\Gamma)[n]$). For $n = 1$ this is a theorem due to Kapranov-Vasserot [KV]. Such an equivalence can be viewed as a direct connection among the objects in the diagram (4), where K-groups of topological vector bundles are replaced by K-groups of sheaves and connection between K-group and homology is made via Chern character.

In the end we establish a direct isomorphism of various algebraic structures in II and III in the diagram (1). More explicitly, we construct Schur bases of the equivariant K-groups in II and III and show that a canonical one-to-one correspondence between these two bases gives the desired isomorphism for various algebraic structures such as Hopf algebras, $\lambda$-rings, and Heisenberg algebra, etc.

The plan of the paper goes as follows. In Sect. 1 we study the resolution of singularities $\tau_n$ and provide various steps toward establishing our conjecture on the isomorphism between the Hilbert quotient $\mathbb{C}^{2n}//\Gamma_n$ and the Hilbert scheme $\mathbb{C}^2/\Gamma_n$. In Sect. 2 we show that two virtual characters of $\Gamma_n$ arising from different setups coincide with each other and discuss various implications of this and our main conjecture. In Sect. 3 we establish isomorphisms of various algebraic structures in II and III of diagram (1).

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1 Wreath products and Hilbert schemes

In this section, we establish some direct connections between wreath products and Hilbert schemes, i.e. between I and III in the diagram (2). In particular for $\Gamma$ trivial it reduces to relating I and II.

1.1 The wreath products

Given a finite group $\Gamma$, we denote by $\Gamma^*$ the set of all the inequivalent complex irreducible characters $\{\gamma_0, \gamma_1, \ldots, \gamma_r\}$ and by $\Gamma_*$ the set of conjugacy classes. We denote by $\gamma_0$ the trivial character and by $\Gamma_0^*$ the set of non-trivial characters $\{\gamma_1, \ldots, \gamma_r\}$. 


The $\mathbb{C}$-span of $\gamma \in \Gamma^*$, denoted by $R(\Gamma)$, can be identified with the space of class functions on $\Gamma$. We denote by $R_\mathbb{Z}(\Gamma)$ the integral span of irreducible characters of $\Gamma$.

Let $\Gamma^n = \Gamma \times \cdots \times \Gamma$ be the $n$-th direct product of $\Gamma$. The symmetric group $S_n$ acts on $\Gamma^n$ by permutations: $\sigma(g_1, \cdots, g_n) = (g_{\sigma^{-1}(1)}, \cdots, g_{\sigma^{-1}(n)})$. The wreath product of $\Gamma$ with $S_n$ is defined to be the semi-direct product

$$\Gamma_n = \{(g, \sigma) | g = (g_1, \cdots, g_n) \in \Gamma^n, \sigma \in S_n\}$$

with the multiplication given by $(g, \sigma) \cdot (h, \tau) = (g \sigma(h), \sigma \tau)$. Note that $\Gamma_n$ is a normal subgroup of $\Gamma_n$.

Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ be a partition: $\lambda_1 \geq \cdots \geq \lambda_l \geq 1$. The integer $|\lambda| = \lambda_1 + \cdots + \lambda_l$ is called the weight, and $l(\lambda) = l$ is called the length of the partition $\lambda$. We will often make use of another notation for partitions: $\lambda = (1^{m_1}2^{m_2} \cdots)$, where $m_i$ is the number of parts in $\lambda$ equal to $i$.

Given a family of partitions $\rho = (\rho(x))_{x \in S}$ indexed by a finite set $S$, we define the weight of $\rho$ to be

$$\|\rho\| = \sum_{x \in S} |\rho(x)|.$$ 

Sometimes it is convenient to regard $\rho = (\rho(x))_{x \in S}$ as a partition-valued function on $S$. We denote by $\mathcal{P}(S)$ the set of all partitions indexed by $S$ and by $\mathcal{P}_n(S)$ the set of all partitions $\rho$ in $\mathcal{P}(S)$ of weight $n$.

The conjugacy classes of $\Gamma_n$ can be described in the following way. Let $x = (g, \sigma) \in \Gamma_n$, where $g = (g_1, \cdots, g_n) \in \Gamma^n$, $\sigma \in S_n$. The permutation $\sigma$ is written as a product of disjoint cycles. For each such cycle $y = (i_1i_2 \cdots i_k)$ the element $g_{i_1}g_{i_2} \cdots g_{i_k} \in \Gamma$ is determined up to conjugacy in $\Gamma$ by $g$ and $y$, and will be called the cycle-product of $x$ corresponding to the cycle $y$. For any conjugacy class $c$ and each integer $i \geq 1$, the number of $i$-cycles in $\sigma$ whose cycle-product lies in $c$ will be denoted by $m_i(c)$. Denote by $\rho(c)$ the partition $(1^{m_1(c)}2^{m_2(c)} \cdots)$. Then each element $x = (g, \sigma) \in \Gamma_n$ gives rise to a partition-valued function $\rho(c))_{c \in \Gamma_n} \in \mathcal{P}(\Gamma_n)$ such that $\sum_{i,c} im_i(c) = n$. The partition-valued function $\rho = (\rho(c))_{c \in \Gamma_n}$ is called the type of $x$. It is well known (cf. [M, Z]) that any two elements of $\Gamma_n$ are conjugate in $\Gamma_n$ if and only if they have the same type.

### 1.2 A resolution of singularities

Let $X$ be a smooth complex algebraic variety acted upon by a finite group $\Gamma$ of order $N$. We denote by $X^{[n]}$ the Hilbert scheme of $n$ points on $X$ and denote by $X^{(n)} = X^n/S_n$ the $n$-th symmetric product. Both $X^{[n]}$ and $X^{(n)}$ carry an induced $\Gamma$-action from $X$.

Now assume $X$ is a quasi-projective surface. A beautiful theorem of Fogarty [Fog] states that $X^{[n]}$ is non-singular of dimension $2n$. It is well known (cf. e.g.
that the Hilbert-Chow morphism $X^{[n]} \to X^{(n)}$ is a resolution of singularities. Given a partition $\nu$ of $n$ of length $l$: $\nu_1 \geq \nu_2 \geq \ldots \geq \nu_l > 0$, we define

$$X^{(n)}_{\nu} = \left\{ \sum_{i=1}^{l} \nu_i x_i \in X^{(n)} | x_i \neq x_j \text{ for } i \neq j \right\}.$$ 

A natural stratification of $X^{(n)}$ is given by

$$X^{(n)} = \bigsqcup_{\nu} X^{(n)}_{\nu}.$$

In the remainder of this section, we let $\Gamma$ be a finite subgroup of $SL_2(C)$ unless otherwise specified. The classification of finite subgroups of $SL_2(C)$ is well known. The following is a complete list of them: the cyclic, binary dihedral, tetrahedral, octahedral and icosahedral groups. We denote by $\tau : C^2/\Gamma \to C^2/\Gamma$ the minimal resolution of the simple singularity.

A canonical identification between $C^{2n}/\Gamma_n$ and $(C^2/\Gamma)^n/S_n$ is given as follows: given a $\Gamma_n$-orbit, say $\Gamma_n.(x_1, \ldots, x_n)$ for some $(x_1, \ldots, x_n) \in (C^2)^n = C^{2n}$, we obtain a point $[\Gamma.x_1] + \ldots + [\Gamma.x_n]$ in $(C^2/\Gamma)^n/S_n$, where $\Gamma.x_i$ denotes the $\Gamma$-orbit of $x_i$, i.e. a point in $C^2/\Gamma$. It is easy to see that this map is independent of the choice of the representative $(x_1, \ldots, x_n)$ in the $\Gamma_n$-orbit and it is one-to-one. The following commutative diagram

\[
\begin{array}{ccc}
\tilde{C^2/\Gamma}^{[n]} & \xrightarrow{\pi_n} & (\tilde{C^2/\Gamma})^n/S_n \\
\downarrow \tau_n & & \downarrow \tau^{(n)} \\
C^{2n}/\Gamma_n & \cong & (C^2/\Gamma)^n/S_n.
\end{array}
\]

defines a morphism $\tau_n : \tilde{C^2/\Gamma}^{[n]} \to C^{2n}/\Gamma_n$.

**Proposition 1** The morphism $\tau_n : \tilde{C^2/\Gamma}^{[n]} \to C^{2n}/\Gamma_n$ is a semismall crepant resolution of singularities.

**Proof.** It is clear by definition that $\tau_n$ is a resolution of singularities. We now describe a stratification of $C^{2n}/\Gamma_n$.

The simple singularity $C^2/\Gamma$ has a stratification given by the singular point $o$ and its complement denoted by $C^2_o/\Gamma$. It follows that a stratification of $(C^2/\Gamma)^n$ is given by $n + 1$ strata $(C^2/\Gamma)^n[i] (0 \leq i \leq n)$, where $(C^2/\Gamma)^n[i]$ consisting of points in the Cartesian product $(C^2/\Gamma)^n$ which has exactly $n-i$ components given by the singular point. Then a stratification of $C^{2n}/\Gamma_n = (C^2/\Gamma)^n/S_n$ is given by

$$(C^2/\Gamma)^n/S_n = \bigsqcup_{i=0}^{n} (C^2/\Gamma)^n[i]/S_n \cong \bigsqcup_{i=0}^{n} (C^2_o/\Gamma)^i/S_i \times \{(n-i)o\}.$$
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\[ \cong \bigcup_{i=0}^{n} (C_0^2/\Gamma)^{(i)} \bigcup_{i=0}^{n} \bigcup_{|\mu|=i} (C_0^2/\Gamma)^{(i)}_{\mu}. \]

The codimension of the strata \((C_0^2/\Gamma)^{(i)}_{\mu}\) is \(2n - 2l(\mu)\). Clearly we also have

\[ \tau_n^{-1}((C_0^2/\Gamma)^{(i)}_{\mu} \times \{(n-i)\alpha\}) = (C_0^2/\Gamma)^{(i)}_{\mu} \times \tau_n^{-1}(0)^{n-i}. \]

It follows that the dimension of a fiber over a point in this strata is equal to \((i - l(\mu)) + (n-i) = n - l(\mu)\) which is half of the codimension of the strata above. Thus \(\tau_n\) is semismall.

The canonical bundles over \(C^{2n}/\Gamma_n\) and \(\widetilde{C^2/\Gamma}^{[n]}\) is trivial due to the existence of holomorphic symplectic forms (note that \(\Gamma_n\) preserves the symplectic form on \(C^{2n}\)). Thus \(\tau_n\) is crepant.

\[\square\]

Remark 1

1. \(\tau_n\) is a symplectic resolution in the sense that the pullback of the holomorphic symplectic form on \(C^{2n}/\Gamma_n\) outside of the singularities can be extended to a holomorphic symplectic form on \(\widetilde{C^2/\Gamma}^{[n]}\). When \(n=1\) this becomes the minimal resolution \(\tau = \tau_1 : C^2/\Gamma \to C^2/\Gamma\).

2. It follows from the semismallness of \(\tau_n\) that \(\tau_n\) is also semismall. Then the diagram (3) is remarkable in that all three maps \(\tau_n, \pi_n\) and \(\tau(n)\) are semismall.

3. The resolution \(\tau_n : \widetilde{C^2/\Gamma}^{[n]} \to C^{2n}/\Gamma_n\) is one-to-one over the non-singular locus of the orbifold \(C^{2n}/\Gamma_n\) corresponding to regular \(\Gamma_n\)-orbits in \(C^{2n}\).

We denote by \(\widetilde{C^2/\Gamma}^{[n],0}\) the fiber of \(\tau_n\) over \([0]\), where \([0]\) denotes the image in \(C^{2n}/\Gamma_n\) of the origin of \(C^{2n}\). We assume that \(\Gamma\) is not trivial. By the diagram (3), we have

\[ \widetilde{C^2/\Gamma}^{[n],0} = \pi_n^{-1}(\tau_n^{-1}(0)) = \pi_n^{-1}(D^{(n)}), \]

where \(D = \tau^{-1}(0)\) is the exceptional divisor in \(\widetilde{C^2/\Gamma}\). It is well known that the irreducible components of \(D\) are projective lines \(\Sigma_\gamma\) parameterized by the set of non-trivial characters \(\gamma \in \Gamma_0^*\) (cf. e.g. \([3SV]\)).

Recall that given an irreducible curve \(\Sigma \subset \widetilde{C^2/\Gamma}\), the variety

\[ L^n\Sigma := \{ I \in \widetilde{C^2/\Gamma}^{[n]} | \text{Supp}(O/I) \subset \Sigma \} = \pi_n^{-1}(\Sigma^{(n)}). \]
introduced by Grojnowski [Gr] (also see [N2]) plays an important role in understanding the middle-dimensional homology groups of Hilbert schemes and in connection with symmetric functions. One can show (cf. loc. cit.) that the irreducible components of $L^n\Sigma$ are parameterized by partitions $\nu$ of $n$ and given by

$$L^\nu \Sigma = \pi_n^{-1}(\Sigma^{(n)}_\nu),$$

where $\Sigma^{(n)}_\nu$ is the stratum of the symmetric product $\Sigma^{(n)}$ associated to $\nu$.

It is interesting to observe that the fiber $\tilde{C}^2/\Gamma \begin{bmatrix} n \end{bmatrix}, 0$ is a natural generalization of the above construction. Given $\rho = (\rho(\gamma))_{\gamma \in \Gamma_0} \in \mathcal{P}_n(\Gamma_0^*)$, we set $n_i = |\rho(\gamma_i)|$ and define

$$L^\rho D = \pi_n^{-1}\left((\Sigma_{\gamma_1})^{(n_1)}_{\rho(\gamma_1)} \times \cdots \times (\Sigma_{\gamma_r})^{(n_r)}_{\rho(\gamma_r)}\right).$$

**Proposition 2** Let $\Gamma$ be non-trivial. The fiber $\tilde{C}^2/\Gamma \begin{bmatrix} n \end{bmatrix}, 0$ is of pure dimension $n$, and its irreducible components are given by $L^\rho D, \rho \in \mathcal{P}_n(\Gamma_0^*)$.

**Proof.** The component $L^\rho D$ is irreducible since the fiber of $\pi_n$ is so. $L^\rho D$ is of dimension $n$ since the dimension of $(\Sigma_\gamma)^{\rho(\gamma)}_{\rho(\gamma)} (\gamma \in \Gamma_0^*)$ is $l(\rho(\gamma))$ and the dimension of the fiber of $\pi_n$ is equal to

$$\sum_{i=1}^r (|\rho(\gamma_i)| - l(\rho(\gamma_i))) = n - \sum_i l(\rho(\gamma_i)).$$

Therefore the dimension of $L^\rho D$ is $(n - \sum_i l(\rho(\gamma_i))) + \sum_i l(\rho(\gamma_i)) = n$. 

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1.3 Hilbert quotient and a subvariety of $(C^2)^{[nN]}$

Let $X$ be a smooth complex algebraic variety acted upon by a finite group $\Gamma$ of order $N$. A regular $\Gamma$-orbit can be viewed as an element in the Hilbert scheme $X^{[N]}$ of $N$ points in $X$. The Hilbert quotient is the closure $X/\Gamma$ of the set of regular $\Gamma$-orbits in $X^{[N]}$ (cf. [Ka]). It follows that there exists a tautological vector bundle over $X/\Gamma$ of rank $N$. The group $\Gamma$ acts on the tautological bundle fiberwise and each fiber is isomorphic to the regular representation of $\Gamma$.

Note that the wreath product $\Gamma_n$ acts faithfully on the affine space $C^{2n} = (C^2)^n$. We make the following conjecture which provides a key connection between I and III in the diagram (2). We remark that a more general conjecture can be easily formulated in the setup of the diagram (4).

**Conjecture 1** There exists a canonical isomorphism between the Hilbert quotient $C^{2n}/\Gamma_n$ and the Hilbert scheme $\tilde{C}^2/\Gamma^{[n]}$ of $n$ points on $\tilde{C}^2/\Gamma$. 

Remark 2 1. Conjecture [1] for \( n = 1 \) reduces to a theorem due to Ginzburg-Kapronov (unpublished) and independently Ito-Nakamura [IN] which says 
\[ \mathbb{C}^2/\Gamma \cong \mathbb{C}^2/\Gamma. \]

2. When \( \Gamma \) is trivial and so \( \Gamma_n \) is the symmetric group \( S_n \), Haiman [H] has shown that Conjecture [1] for \( \Gamma \) trivial is equivalent to a remarkable \( n! \) conjecture due to Garsia and Haiman [GH]. A proof of the \( n! \) conjecture has been very recently posted by Haiman in his UCSD webpage.

3. Conjecture [1] implies an isomorphism of Hilbert quotients:
\[ \mathbb{C}^{2n}/\Gamma_n \cong (\mathbb{C}^2/\Gamma)^n/S_n, \]
which seems to be in a more symmetric form. On the other hand, assuming the \( n! \)-conjecture, one can show that an isomorphism of Hilbert quotients above implies Conjecture [1].

Since \( \tau_n : \mathbb{C}^2/\Gamma^n \rightarrow \mathbb{C}^{2n}/\Gamma_n \) is a resolution of singularities (and thus is proper) and \( \mathbb{C}^{2n}/\Gamma_n \) is a normal variety, the algebra of regular functions on \( \mathbb{C}^2/\Gamma^n \) is isomorphic to the algebra of \( \Gamma_n \) invariants on the regular functions on \( \mathbb{C}^{2n} \). The following lemma gives a description of the algebra of \( \Gamma_n \) invariants in \( \mathbb{C}[x, y] \), where we denote by \( x \) (resp. \( y \)) the \( n \)-tuple \( x_1, \ldots, x_n \) (resp. \( y_1, \ldots, y_n \)). It generalizes Weyl’s theorem [Wey] for symmetric groups and the proof is similar.

Lemma 1 The algebra of invariants \( \mathbb{C}[x, y]^\Gamma_n \) is generated by
\[ \tilde{f}(x, y) = f(x_1, y_1) + f(x_2, y_2) + \ldots + f(x_n, y_n), \]
where \( f \) runs over an arbitrary linear basis \( B \) for the space of invariants \( \mathbb{C}[x, y]^\Gamma \).

Proof. We prove the lemma by induction on \( n \). When \( n = 1 \) it is evident. Assume now that we have established the lemma for \( n - 1 \). We use \( x' \) and \( y' \) to denote \( x_2, \ldots, x_n \) and respectively \( y_2, \ldots, y_n \). The space \( \mathbb{C}[x', y'] \) is acted on by the wreath product subgroup \( \Gamma_{n-1} \subset \Gamma_n \).

Given any \( F(x, y) \in \mathbb{C}[x, y]^\Gamma_n \), we can write it as a linear combination of \( x_1^\alpha y_1^\beta F_{\alpha\beta}(x', y') \), where \( F_{\alpha\beta}(x', y') \) is some \( \Gamma_{n-1} \)-invariant polynomial. By induction assumption, we can write \( F_{\alpha\beta}(x', y') \) as a polynomial in terms of \( \tilde{f}(x', y') \) where \( f \in B \), and in turn as a polynomial in terms of \( \tilde{f}(x, y) \) and \( x_1, y_1 \). Therefore we can write \( F(x, y) \) as a linear combination of polynomials of the form \( x_1^\alpha y_1^\beta G_{\alpha\beta}(x, y) \), where \( G_{\alpha\beta}(x, y) \) is a polynomial in terms of \( \tilde{f}(x', y') \) where \( f \in B \). Since both \( F_{\alpha\beta}(x, y) \) and \( G_{\alpha\beta}(x, y) \) are \( \Gamma_n \)-invariant and thus in particular invariant with respect to the symmetric group \( S_n \) and the first factor \( \Gamma \) in \( \Gamma^n \subset \Gamma_n \), \( F(x, y) \) becomes a linear combination of \( p_{\alpha\beta}(x, y) G_{\alpha\beta}(x, y) \). Here \( p_{\alpha\beta}(x, y) \) denotes \( \frac{1}{n!} \sum_{i=1}^{n} \sum_{g \in \Gamma} (g.x_i)^\alpha (g.y_i)^\beta \), the average of \( x_1^\alpha y_1^\beta \) over \( \Gamma \times S_n \) (which is the
same as the average over $\Gamma_n$). We complete the proof by noting that $p_{\alpha\beta}(x, y)$ is a linear combination of $f(x, y)$ where $f \in B$. \qed

Given $J \in \mathbb{C}^{2n}/\Gamma_n$ we regard it as an ideal in $\mathbb{C}[x, y]$ of colength $N^n n!$ (which is the order of $\Gamma_n$). Then the quotient $\mathbb{C}[x, y]/J$ affords the regular representation $R$ of $\Gamma_n$, and its only $\Gamma_n$-invariants are constants. Thus we have $f(x, y) = c_f \mod J$ for some constant $c_f$. Recall that $\Gamma_{n-1}$ acts on $\mathbb{C}[x', y']$. By Lemma [4] the space $\mathbb{C}[x, y]^{\Gamma_{n-1}}$ is generated by $x_1, y_1$ and $f(x_2, y_2) + \ldots + f(x_n, y_n)$. The latter is equal to $c_f - f(x_1, y_1) \mod J$. Thus $(\mathbb{C}[x, y]/J)^{\Gamma_{n-1}}$ is generated by $x_1, y_1$ and $c_f - f(x_1, y_1), f \in \mathbb{C}[x, y]$\Gamma. It follows that
\begin{equation}
\mathbb{C}[x_1, y_1]/(J \cap \mathbb{C}[x_1, y_1]) \equiv (\mathbb{C}[x, y]/J)^{\Gamma_{n-1}},
\end{equation}
which has dimension $nN = |\Gamma_n|/|\Gamma_{n-1}|$ because $(\mathbb{C}[x, y]/J)^{\Gamma_{n-1}}$ can be identified with the space of $\Gamma_{n-1}$-invariants in the regular representation of $\Gamma$.

The first copy of $\Gamma$ in the Cartesian product $\Gamma^n \subset \Gamma_n$ commutes with $\Gamma_{n-1}$ above. It follows from [4] that the quotient $\mathbb{C}[x_1, y_1]/(J \cap \mathbb{C}[x_1, y_1])$ as a $\Gamma$-module is isomorphic to $R^n$, a direct sum of $n$ copies of the regular representation $R$ of $\Gamma_n$.

The $\Gamma$-action on $\mathbb{C}^2$ induces a $\Gamma$-action on the Hilbert scheme $(\mathbb{C}^2)^{[nN]}$ and the symmetric product $(\mathbb{C}^2)^{(nN)}$. The Hilbert-Chow morphism $(\mathbb{C}^2)^{[nN]} \to (\mathbb{C}^2)^{(nN)}$ induces one between the set of $\Gamma$-fixed points $(\mathbb{C}^2)^{[nN], \Gamma} \to (\mathbb{C}^2)^{(nN), \Gamma}$. As the fixed point set of a non-singular variety by the action of a finite group, $(\mathbb{C}^2)^{[nN], \Gamma}$ is non-singular. Denote by $X_{\Gamma,n}$ the set of $\Gamma$-invariant ideals $I$ in the Hilbert scheme $(\mathbb{C}^2)^{[nN]}$ such that the quotient $\mathbb{C}[x, y]/I$ is isomorphic to $R^n$ as a $\Gamma$-module. Since the quotient $\mathbb{C}[x, y]/I$ are isomorphic as $\Gamma$-modules for all $I$ in a given connected component of $(\mathbb{C}^2)^{[nN], \Gamma}$, the variety $X_{\Gamma,n}$ is a union of components of $(\mathbb{C}^2)^{[nN], \Gamma}$. In particular $X_{\Gamma,n}$ is non-singular (we shall see that $X_{\Gamma,n}$ is indeed connected of dimension $2n$).

Therefore by sending the ideal $J$ to the ideal $J \cap \mathbb{C}[x_1, y_1]$, we have defined a map $\varphi$ from $\mathbb{C}^{2n}/\Gamma_n$ to $(\mathbb{C}^2)^{[nN]}$, whose image lies in $X_{\Gamma,n}$. We also denote $\varphi : \mathbb{C}^{2n}/\Gamma_n \to X_{\Gamma,n}$.

The map $\varphi$ can be also understood as follows. Let $\mathcal{U}_{\Gamma,n}$ be the universal family over the Hilbert quotient $\mathbb{C}^{2n}/\Gamma_n$ which is a subvariety of the Hilbert scheme $(\mathbb{C}^{2n})^{[n!N^n]}$:
\begin{equation}
\begin{array}{ccc}
\mathcal{U}_{\Gamma,n} & \longrightarrow & \mathbb{C}^{2n} \\
\downarrow & & \\
\mathbb{C}^{2n}/\Gamma_n & \end{array}
\end{equation}
It has a natural $\Gamma_n$-action fiberwise such that each fiber carries the regular representation of $\Gamma_n$. Then $\mathcal{U}_{\Gamma,n}/\Gamma_{n-1}$ is flat and finite of degree $nN$ over $\mathbb{C}^{2n}/\Gamma_n$, and thus can be identified with a family of subschemes of $\mathbb{C}^2$ as above. Then $\varphi$ is the morphism given by the universal property of the Hilbert scheme $(\mathbb{C}^2)^{[nN]}$ for the family $\mathcal{U}_{\Gamma,n}/\Gamma_{n-1}$.

Thus we have established the following.
Theorem 1 We have a natural morphism $\varphi : \mathbb{C}^{2n} / \Gamma_n \to X_{\Gamma,n}$ defined as above.

Remark 3 For $\Gamma$ trivial, $\Gamma_n$ reduces to $S_n$ and $X_{\Gamma,n}$ becomes the Hilbert scheme $(\mathbb{C}^2)[n]$. In this case the above morphism $\mathbb{C}^{2n} / S_n \to (\mathbb{C}^2)[n]$ was earlier constructed by Haiman [H]. We plan to elaborate further on generalizations of [H] to our setup in the future.

Conjecture 2 The morphism $\varphi : \mathbb{C}^{2n} / \Gamma_n \to X_{\Gamma,n}$ is an isomorphism.

Observe that $(\mathbb{C}^2)^{(nN)},\Gamma$ is the subset of $(\mathbb{C}^2)^{(nN)}$ consisting of points

$$\sum_{\gamma \in \Gamma} [\gamma \cdot x_1] + \ldots + \sum_{\gamma \in \Gamma} [\gamma \cdot x_a] + (n - a)N[0], \quad 1 \leq a \leq n, x_1, \ldots, x_a \in \mathbb{C}^2 \backslash 0,$$

which can be thought as the $\Gamma_n$-orbit of $(x_1, \ldots, x_a, 0, \ldots, 0) \in (\mathbb{C}^2)^n = \mathbb{C}^{2n}$. In this way $(\mathbb{C}^2)^{(nN)},\Gamma$ is identified with $\mathbb{C}^{2n}/\Gamma_n$. Thus we have proved the following proposition by noting the inclusion $X_{\Gamma,n} \subset (\mathbb{C}^2)^{(nN)},\Gamma$.

Proposition 3 The $\Gamma$-fixed point set $(\mathbb{C}^2)^{(nN)},\Gamma$ of the symmetric product $(\mathbb{C}^2)^{(nN)}$ can be canonically identified with $\mathbb{C}^{2n}/\Gamma_n$. The Hilbert-Chow morphism $(\mathbb{C}^2)^{(nN)} \to (\mathbb{C}^2)^{(nN)}$, when restricted to the $\Gamma$-fixed point set, induces a canonical morphism $X_{\Gamma,n} \to \mathbb{C}^{2n}/\Gamma_n$.

Remark 4 Take an unordered $n$-tuple $T$ of distinct $\Gamma$-orbits in $\mathbb{C}^2 \backslash 0$. Such an $n$-tuple defines a set of $nN$ distinct points in $\mathbb{C}^2$, and thus can be regarded as an ideal $I(T)$ in the Hilbert scheme $(\mathbb{C}^2)^{(nN)}$. This ideal is clearly $\Gamma$-invariant and as a $\Gamma$-module $\mathbb{C}[x, y]/I(T)$ is isomorphic to $R^n$. On the other hand observe that such an $n$-tuple $T$ can be canonically identified with a regular $\Gamma_n$-orbit in $\mathbb{C}^{2n}$. In this way the sets of generic points in $X_{\Gamma,n}$ and $\mathbb{C}^{2n}/\Gamma_n$ coincide. It is easy to see that the morphism $X_{\Gamma,n} \to \mathbb{C}^{2n}/\Gamma_n$ above is surjective and it is one-to-one over the set of generic points in $\mathbb{C}^{2n}/\Gamma_n$ consisting of regular $\Gamma_n$-orbits.

We define the reduced universal scheme $W_{\Gamma,n}$ as the reduced fibered product

$$W_{\Gamma,n} \longrightarrow \mathbb{C}^{2n} \quad \downarrow \quad \downarrow \quad \downarrow$$

$$X_{\Gamma,n} \quad \tau_n \quad \mathbb{C}^{2n}/\Gamma_n.$$

It is known that a finite surjective morphism from $Z$ to a non-singular variety is flat if and only if $Z$ is Cohen-Macaulay.

Conjecture 3 $W_{\Gamma,n}$ is Cohen-Macaulay.
Under the assumption of the validity of Conjecture 3, the universal properties of the Hilbert scheme \((C^{2n})^{[nN^n]}\) induces a morphism \(\psi : X_{\Gamma, n} \rightarrow (C^{2n})^{[nN^n]}\), whose image lies in \(C^{2n} // \Gamma_n\). By Remark 5 and the fact that the set of generic points of \(C^{2n} // \Gamma_n\) and \(C^{2n} / \Gamma_n\) coincide, the two morphisms \(\varphi\) and \(\psi\) are mutually inverse to each other over generic points. Then it follows that they are inverse everywhere, establishing Conjecture 4.

**Remark 5** The above conjecture 3 for \(\Gamma\) trivial was first conjectured by Haiman [14]. The proof announced very recently by Haiman in his UCSD homepage of \(n!\) conjecture is based on a proof of this conjecture of his.

### 1.4 A quiver variety description

We first recall (cf. [12]) that the Hilbert scheme \((C^2)^{[K]}\) of \(K\) points in \(C^2\) admits a description in terms of a quiver consisting of one vertex and one arrow starting from the vertex and returning to the same vertex itself. More explicitly, we denote

\[
\widetilde{H}(K) = \left\{ (B_1, B_2, i, j) \mid i[B_1, B_2] + ij = 0 \right\},
\]

where \(B_1, B_2 \in \text{End}(C^K), i \in \text{Hom}(C, C^K), j \in \text{Hom}(C^K, C)\). Then we have an isomorphism

\[
(C^2)^{[K]} \cong \widetilde{H}(K)/GL_K(C),
\]

where the action of \(GL_K(C)\) on \(\widetilde{H}(K)\) is given by

\[
g \cdot (B_1, B_2, i, j) = (gB_1g^{-1}, gB_2g^{-1}, gi, gj^{-1}).
\]

It is also often convenient to regard \((B_1, B_2)\) to be in \(\text{Hom}(C^K, C^2 \otimes C^K)\). We remark that one may drop \(j\) in the above formulation because one can show by using the stability condition that \(j = 0\) (cf. [12]).

The bijection in (3) is given as follows. For \(I \in (C^2)^{[K]}\), i.e. an ideal in \(C[x, y]\) of colength \(K\), the multiplication by \(x, y\) induces endomorphisms \(B_1, B_2\) on the \(K\)-dimensional quotient \(C[x, y]/I\), and the homomorphism \(i \in \text{Hom}(C, C^K)\) is given by letting \(i(1) = 1 \mod I\). Conversely, given \((B_1, B_2, i)\), we define a homomorphism \(C[x, y] \rightarrow C^K\) by \(f \mapsto f(B_1, B_2)i(1)\). One can show by the stability condition that the kernel \(I\) of this homomorphism is an ideal of \(C[x, y]\) of colength \(K\). One easily checks that the two maps are inverse to each other.

Set \(K = nN\), where \(N\) is the order of \(\Gamma\). We may identify \(C^K\) with \(R^n\), \(C^2\) with the defining representation \(Q\) of \(\Gamma\) by the embedding \(\Gamma \subset SL_2(C)\), and \(C\) with the trivial representation of \(\Gamma\). Denote by

\[
M(n) = \text{Hom}(R^n, Q \otimes R^n) \bigoplus \text{Hom}(C, R^n) \bigoplus \text{Hom}(R^n, C).
\]
By definition \( \widetilde{H}(nN) \subset M(n) \). Let \( GL_{\Gamma}(R) \) be the group of \( \Gamma \)-equivariant automorphisms of \( R \). Then the group \( G \equiv GL_{\Gamma}(R^n) \cong GL_n(\mathbb{C}) \times GL_{\Gamma}(R) \) acts on the \( \Gamma \)-invariant subspace \( M(n)^{\Gamma} \). We have the following description of \( X_{\Gamma,n} \) as a quiver variety. This result is known to Nakajima [4] (also cf. Theorem 1 of Varagnolo-Vasserot [VV]).

**Theorem 2** The variety \( X_{\Gamma,n} \) admits the following description:

\[
X_{\Gamma,n} \cong (\widetilde{H}(nN) \cap M(n)^{\Gamma})/GL_{\Gamma}(R^n).
\]

It particular \( X_{\Gamma,n} \) is non-singular of pure dimension \( 2n \).

**Remark 6** Consider the \( \Gamma \)-module decomposition \( Q \otimes V_{\gamma} = \bigoplus_{ij} a_{ij} V_{ij} \), where \( a_{ij} \in \mathbb{Z}_+ \), and \( V_{\gamma} (i = 0, \ldots , r) \) are irreducible representations corresponding to the characters \( \gamma_i \) of \( \Gamma \). Set \( \dim V_{\gamma} = n_i \). Then

\[
M(n)^{\Gamma} = \text{Hom}_{\Gamma}(R^n, Q \otimes R^n) \bigoplus \text{Hom}_{\Gamma}(\mathbb{C}, R^n) \bigoplus \text{Hom}_{\Gamma}(R^n, \mathbb{C}) \quad (7)
\]

\[
= \text{Hom}_{\Gamma}(\sum_i C^{n_i} \otimes V_{\gamma_i} \otimes C^2 \otimes \sum_i C^{n_i} \otimes V_{\gamma_i})
\]

\[
\bigoplus \text{Hom}_{\Gamma}(\mathbb{C}, R^n) \bigoplus \text{Hom}_{\Gamma}(R^n, \mathbb{C})
\]

\[
= \sum_{ij} a_{ij} \text{Hom}(C^{n_i} \otimes C^{n_j}) \bigoplus \text{Hom}(\mathbb{C}, V_{\gamma}^n) \bigoplus \text{Hom}(V_{\gamma}^n, \mathbb{C}).
\]

where \( \text{Hom}_{\Gamma} \) stands for the \( \Gamma \)-equivariant homomorphisms. In the language of quiver varieties as formulated by Nakajima [4], [3], the above description of \( X_{\Gamma,n} \) identifies \( X_{\Gamma,n} \) with a quiver variety associated to the following data: the graph consists of the same vertices and edges as the McKay quiver which is an affine Dynkin diagram associated to a finite subgroup \( \Gamma \) of \( SL_2(\mathbb{C}) \); the vector space \( V_i \) associated to the vertex \( i \) is isomorphic to the direct sum of \( n \) copies of the \( i \)-th irreducible representation \( V_{\gamma_i} \); the vector space \( W_i = 0 \) for nonzero \( i \) and \( W_0 = \mathbb{C} \).

**Proof of Theorem 3** Our proof is modeled on the proof of Theorem 1.9 and Theorem 4.4 in [N2] which are special cases of our isomorphism for \( \Gamma \) trivial and for \( n = 1 \) respectively. We sketch below for the convenience of the reader.

One shows \( j = 0 \) by using the stability condition. The isomorphism statement follows directly from the description of \( \mathbb{C}^{2[nN]} \) given by (8), the definition of \( X_{\Gamma,n} \), and Eq. (7). We have seen earlier that \( X_{\Gamma,n} \) is nonsingular by construction.

One shows by a direct check that \( [B_1, B_2] \) is \( \Gamma \)-invariant endomorphism in \( R^n \) for \( (B_1, B_2) \in \text{Hom}_{\Gamma}(R^n, C^2 \otimes R^n) \). The cokernel of the differential of the map \( (B_1, B_2, i) \mapsto [B_1, B_2] \) from \( M(n)^{\Gamma} \) to \( \text{End}_{\Gamma}(R^n) \) consists of the \( \Gamma \)-endomorphisms in \( R^n \) which commute with \( B_1 \) and \( B_2 \). By sending \( f \mapsto f(i(1)) \) we define a map from the cokernel to the \( n \)-dimensional space \( \text{Hom}_{\Gamma}(\mathbb{C}, R^n) \cong V_{\gamma_0}^n \). Conversely,

\[\footnote{I. Frenkel informed us that he also noticed this recently.} \]
given \( v \in V_n^\gamma \), an endomorphism \( f \) in \( R^n \) is uniquely determined by the equation 
\[
f(B_1^a B_2^b v) = B_1^a B_2^b v
\]
by the stability condition ii) in the definition of \( \tilde{H}(nN) \).
One further checks that \( f \) lies in the cokernel. These two maps are inverse to each other. Thus the cokernel has constant dimension \( n \).

The dimension of \( \tilde{H}(nN) \cap M(n)^\Gamma \) is equal to \( \dim M(n) + n - \dim GL_\Gamma(R^n) \) since the dimension of the cokernel is \( n \). Thus the quotient description of \( X_{\Gamma,n} \) implies that the dimension of \( X_{\Gamma,n} \) near \((B_1, B_2, i)\) is given by
\[
dim M(n)^\Gamma + n - 2 \dim GL_\Gamma(R^n)
= (n^2 \dim \text{Hom}_\Gamma(R, \mathbb{C}^2 \otimes R) + n) + n - 2n^2 \dim GL_\Gamma(R)
= n^2(2N) + n + n - 2n^2N
= 2n,
\]
which is independent of which component a point \((B_1, B_2, i)\) is in. Here we have used the fact that the (complex) dimension of \( \text{Hom}_\Gamma(R, \mathbb{C}^2 \otimes R) \) is equal to \( N \) (cf. Kronheimer [Kr]).

**Remark 7** Recall that the minimal resolution \( \hat{C}_2^2/\Gamma \) endowed with certain hyper-Kahler structures are called an ALE spaces [Kr]. According to Nakajima [N4], one can show that the Hilbert scheme \( \hat{C}_2^2/\Gamma[n] \) over an ALE space admits a quiver variety description in terms of the same quiver data as specified in Remark 6 but with a different stability condition, by a modification of the proof for the description of the moduli space of vector bundles over an ALE space [KN]. It follows by Corollary 4.2 of [N] that \( \hat{C}_2^2/\Gamma[n] \) and \( X_{\Gamma,n} \) is diffeomorphic. We conjecture that they are indeed isomorphic as complex varieties. In this way we would have obtained a morphism \( \varphi : \mathbb{C}^{2n} // \Gamma_n \to \hat{C}_2^2/\Gamma[n] \) by combining with Theorem 1.

**Remark 8** It follows readily from the affine algebro-geometric quotient description of the symmetric product \( (\mathbb{C}^2)^{(nN)} \) (Proposition 2.10, [N2]) that the \( \Gamma \)-fixed-point set \( (\mathbb{C}^2)^{(nN)} \) or rather the orbifold \( \mathbb{C}^{2n}/\Gamma_n \) (see Proposition 3) has the following description (also cf. [VV], Theorem 1):
\[
\mathbb{C}^{2n}/\Gamma_n \cong \{(B_1, B_2, i, j) \in M(n)^\Gamma | [B_1, B_2] + ij = 0 \} // GL_\Gamma(R^n).
\]
It follows from the general theory of quiver varieties that there is a natural projective morphism \( \hat{C}_2^2/\Gamma[n] \to \mathbb{C}^{2n}/\Gamma_n \) which is a semismall resolution. We expect that this is the same as the semismall resolution \( \tau_n : \hat{C}_2^2/\Gamma[n] \to \mathbb{C}^{2n}/\Gamma_n \) explicitly constructed by the diagram (6). We also expect that the intermediate variety \( (\mathbb{C}^2/\Gamma)^n/S_n \) (see (3)) can also be identified with a quiver variety associated with
the same quiver data (as specified in Remark 3) but with a new stability condition. In this case it follows from Corollary 4.2 in [N] that the fiber \( \widetilde{\mathbb{C}^2/\Gamma}^{[n],0} \) is a lagrangian subvariety in \( \widetilde{\mathbb{C}^2/\Gamma}^{[n]} \) and it is homotopy equivalent to \( \widetilde{\mathbb{C}^2/\Gamma}^{[n]} \) (compare with Proposition 2).

**Remark 9** Quiver varieties are connected [N, N3]. Thus the variety \( X_{\Gamma,n} \) can also be defined as the closure of the set of ideals \( I(T) \) in \( (\mathbb{C}^2)^{[nN]} \) associated to unordered \( n \)-tuples \( T \) of distinct \( \Gamma \)-orbits in \( \mathbb{C}^2 \setminus 0 \).

### 1.5 Canonical vector bundles

Since \( \widetilde{\mathbb{C}^2/\Gamma} \) is isomorphic to the Hilbert quotient \( \mathbb{C}^2//\Gamma \), there exists the tautological vector bundle \( \mathcal{R} \) on \( \widetilde{\mathbb{C}^2/\Gamma} \) of rank \( N \), whose fiber affords the regular representation of \( \Gamma \) (cf. [GSV, N2]). It decomposes as follows:

\[
\mathcal{V} \cong \bigoplus_{\gamma \in \Gamma^*} \mathcal{R}_\gamma \otimes V_\gamma,
\]

where \( V_\gamma \) is the irreducible representation of \( \Gamma \) associated to \( \gamma \) and \( \mathcal{R}_\gamma \) is a vector bundle over \( \widetilde{\mathbb{C}^2/\Gamma} \) of rank equal to \( \deg \gamma \) (by definition \( \deg \gamma = \dim V_\gamma \)).

One can associate a vector bundle \( E^{[n]} \) of rank \( dn \) on the Hilbert scheme \( \widetilde{\mathbb{C}^2/\Gamma}^{[n]} \) to a rank \( d \) vector bundle \( E \) over \( \widetilde{\mathbb{C}^2/\Gamma} \) as follows: let \( U \subset \widetilde{\mathbb{C}^2/\Gamma}^{[n]} \times \mathbb{C}^2/\Gamma \) be the universal family

\[
\begin{array}{ccc}
U & \xrightarrow{p_1} & \widetilde{\mathbb{C}^2/\Gamma} \\
p_2 \downarrow & & \downarrow \mathbb{C}^2/\Gamma^{[n]} \\
\end{array}
\]

\( U \) is flat and finite of degree \( n \) over \( \mathbb{C}^2/\Gamma^{[n]} \). Then \( E^{[n]} \) is defined to be \( (p_2)_* p_1^* E \). In this way we obtain canonical vector bundles \( \mathcal{R}^{[n]} \) and \( \mathcal{R}_\gamma^{[n]} \) (\( \gamma \in \Gamma^* \)) over \( \mathbb{C}^2/\Gamma^{[n]} \) associated to \( \mathcal{R} \) and \( \mathcal{R}_\gamma \) above.

There exists a tautological vector bundle \( \mathcal{R}^{(n)} \) of rank \( nN \) over \( X_{\Gamma,n} \) (and thus over \( \mathbb{C}^2/\Gamma^{[n]} \)) induced from the inclusion \( X_{\Gamma,n} \subset (\mathbb{C}^2)^{[nN]} \). The group \( \Gamma \) acts on \( \mathcal{R}^{(n)} \) fiberwise such that each fiber as a \( \Gamma \)-module is isomorphic to \( R^n \). Then we have a decomposition:

\[
\mathcal{R}^{(n)} = \bigoplus_{\gamma \in \Gamma^*} \mathcal{R}_\gamma^{(n)} \otimes V_\gamma,
\]

where \( \mathcal{R}_\gamma^{(n)} \) is a vector bundle over \( \mathbb{C}^2/\Gamma^{[n]} \) of rank equal to \( n \deg \gamma \).
The principle bundle

\[ \tilde{H}(K) \cap M^G(n) \xrightarrow{G} X_{\Gamma,n} \]

(which follows from Theorem 2) gives rise to various canonical vector bundles associated to canonical representations of \( G \equiv GL_\Gamma(R^n) \cong GL_n(\mathbb{C}) \times GL_\Gamma(R) \). For example, the vector bundle associated to the representation \( GL_\Gamma(R^n) \hookrightarrow GL(R^n) \) is exactly the above tautological vector bundle \( \mathbb{C}^n \) over \( X_{\Gamma,n} \) (or rather over \( \tilde{C^2}/\Gamma^{[n]} \)); the one associated to \( GL_\Gamma(R^n) \to GL(\gamma^n) \) is \( \mathbb{C}^n \).

In the remainder of this section we assume the validity of Conjecture 1 that \( C^2n/\Gamma_n \) is isomorphic to \( \tilde{C^2}/\Gamma^{[n]} \). An immediate corollary is the existence of a tautological bundle \( \mathcal{V} \) on \( \tilde{C^2}/\Gamma^{[n]} \) whose fiber affords the regular representation of \( \Gamma_n \). This comes from the tautological bundle over the Hilbert quotient \( C^2n/\Gamma_n \).

It is well known (cf. e.g. [M, Z]) that the irreducible representations \( S_\rho \) of \( \Gamma_n \) are parameterized by the set \( \mathcal{P}_n(\Gamma^*) \) of partition-valued functions \( \rho \) of weight \( n \) on the set \( \Gamma^* \) of irreducible characters of \( \Gamma \). It follows that one has a decomposition

\[ \mathcal{V} = \bigoplus_{\rho \in \mathcal{P}_n(\Gamma^*)} S_\rho \otimes \mathcal{V}_\rho, \quad (8) \]

where \( \mathcal{V}_\rho \) is a vector bundle on \( \tilde{C^2}/\Gamma^{[n]} \) of rank equal to \( \dim S_\rho \). The vector bundles \( \mathcal{V}_\rho \) are expected to be a basis for the \( K \)-group of \( \tilde{C^2}/\Gamma^{[n]} \).

We denote by \( \mathcal{R}^{(n)} \) the subbundle of \( \mathcal{V} \) which is given by the (fiberwise) \( \Gamma_{n-1} \) invariants of \( \mathcal{V} \). Since the first copy \( \Gamma \in \Gamma_n \subset \Gamma_n \) commutes with \( \Gamma_{n-1} \), \( \mathcal{R}^{(n)} \) has a \( \Gamma \) action fiberwise such that each fiber affords the \( \Gamma \)-module \( \mathcal{R}^{[n]} \) (cf. Subsect. 1.3). It decomposes as

\[ \mathcal{R}^{(n)} = \bigoplus_{\gamma \in \Gamma^*} \mathcal{R}^{(n)}_{\gamma} \otimes \mathcal{V}_\gamma, \]

where \( \mathcal{R}^{(n)}_{\gamma} \) is a vector bundle over \( \tilde{C^2}/\Gamma^{[n]} \) of rank \( n \deg \gamma \).

We define the reduced universal scheme \( U_{\Gamma,n} \) as the reduced fibered product

\[ \begin{array}{ccc}
U_{\Gamma,n} & \rightarrow & \mathbb{C}^{2n} \\
\downarrow & & \downarrow \\
\tilde{C^2}/\Gamma^{[n]} & \rightarrow & \mathbb{C}^{2n}/\Gamma_n, \\
\end{array} \quad (9) \]

Under the isomorphism \( \varphi : \mathbb{C}^{2n}/\Gamma_n \to \tilde{C^2}/\Gamma^{[n]} \), the universal schemes \( U_{\Gamma,n}, U_{\Gamma,n} \) defined respectively by (3) and (4) can be identified. The following proposition follows now from the way we define \( \varphi \) (cf. Subsect. 1.3).

**Proposition 4** There is a natural identification between the vector bundles \( \mathcal{R}^{[n]} \) and \( \mathcal{R}^{(n)} \), respectively \( \mathcal{R}^{[n]}_{\gamma} \) and \( \mathcal{R}^{(n)}_{\gamma} \).
2 On the equivalence of two forms of McKay correspondence

2.1 A weighted bilinear form

In this subsection we recall the notion of a weighted bilinear form on $R(\Gamma_n)$ introduced in [FJW].

The standard bilinear form on $R(\Gamma)$ is defined as follows:

$$\langle f, g \rangle_{\Gamma} = \frac{1}{|\Gamma|} \sum_{x \in \Gamma} f(x)g(x^{-1}).$$

We will write $\langle \cdot, \cdot \rangle$ for $\langle \cdot, \cdot \rangle_{\Gamma}$ when no ambiguity may arise.

Let us fix a virtual character $\xi \in R(\Gamma)$. The multiplication in $R(\Gamma)$ corresponding to the tensor product of two representations will be denoted by $\ast$. Recall that $\gamma_0, \gamma_1, \ldots, \gamma_r$ are all the inequivalent irreducible characters of $\Gamma$ and $\gamma_0$ denotes the trivial character. We denote by $a_{ij} \in \mathbb{Z}$ the (virtual) multiplicities of $\gamma_j$ in $\xi \ast \gamma_i$, i.e. $\xi \ast \gamma_i = \sum_{j=0}^{r} a_{ij} \gamma_j$. We denote by $A$ the $(r+1) \times (r+1)$ matrix $(a_{ij})_{0 \leq i,j \leq r}$.

We introduce the following weighted bilinear form on $R(\Gamma)$:

$$\langle f, g \rangle_{\xi} = \langle \xi \ast f, g \rangle_{\Gamma}, \quad f, g \in R(\Gamma).$$

It follows that $\langle \gamma_i, \gamma_j \rangle_{\xi} = a_{ij}$.

Throughout this paper we will always assume that $\xi$ is a self-dual, i.e. $\xi(x) = \xi(x^{-1}), x \in \Gamma$. The self-duality of $\xi$ implies that $a_{ij} = a_{ji}$, i.e. $A$ is a symmetric matrix.

Given a representation $V$ of $\Gamma$ with character $\gamma \in R(\Gamma)$, the $n$-th outer tensor product $V \otimes^n$ of $V$ can be regarded naturally as a representation of the wreath product $\Gamma_n$ whose character will be denoted by $\eta_n(\gamma)$: the direct product $\Gamma_n$ acts on $\gamma \otimes^n$ factor by factor while $S_n$ by permuting the $n$ factors. Denote by $\varepsilon_n$ the (1-dimensional) sign representation of $\Gamma_n$ on which $\Gamma_n$ acts trivially while $S_n$ acts as sign representation. We denote by $\varepsilon_n(\gamma) \in R(\Gamma_n)$ the character of the tensor product of $\varepsilon_n$ and $V \otimes^n$.

We may extend naturally $\eta_n$ to a map from $R(\Gamma)$ to $R(\Gamma_n)$. In particular, if $\beta$ and $\gamma$ are characters of $\Gamma$, then

$$\eta_n(\beta - \gamma) = \sum_{m=0}^{n} (-1)^m \text{Ind}_{\Gamma_{n-m} \times \Gamma_m}^{\Gamma_n} [\eta_{n-m}(\beta) \otimes \varepsilon_m(\gamma)]. \quad (10)$$

We define a weighted bilinear form on $R(\Gamma_n)$ by letting

$$\langle f, g \rangle_{\xi, \Gamma_n} = \langle \eta_n(\xi) \ast f, g \rangle_{\Gamma_n}, \quad f, g \in R(\Gamma_n).$$
One can show that the bilinear form $\langle \cdot, \cdot \rangle_{\xi,\Gamma_n}$ is symmetric. A symmetric bilinear form on $R_\Gamma = \bigoplus_n R(\Gamma_n)$ is then given by

$$\langle u, v \rangle_\xi = \sum_{n \geq 0} \langle u_n, v_n \rangle_{\xi,\Gamma_n},$$

where $u = \sum_n u_n$ and $v = \sum_n v_n$ with $u_n, v_n \in \Gamma_n$.

We further specialize to the case when $\Gamma$ is a finite subgroup of $SL_2(\mathbb{C})$ by an embedding $\pi$, and fix the virtual character $\xi$ of $\Gamma$ to be

$$\lambda(\pi) \equiv \sum_{i=0}^2 (-1)^i \Lambda^i \pi = 2\gamma_0 - \pi,$$

where $\Lambda^i$ denotes the $i$-th exterior power. We construct a diagram with vertices corresponding to elements $\gamma_i$ in $\Gamma^*$ and we draw one edge (resp. two edges) between the $i$-th and $j$-th vertices if $a_{ij} = -1$ (resp. $-2$). According to McKay [Mc], the associated diagram can be identified with affine Dynkin diagram of ADE type and the matrix $A$ is the corresponding affine Cartan matrix. It is shown in [FJW] that the weighted bilinear form on $R_\pi(\Gamma_n)$ is semipositive definite symmetric.

### 2.2 Identification of two virtual characters

In this subsection we set $\Gamma$ to be an arbitrary (not necessarily finite) subgroup of $GL_k(\mathbb{C})$ unless otherwise specified. We denote by $\pi$ the $k$-dimensional defining representation of $\Gamma$ for the embedding.

The wreath product $\Gamma_n$ acts naturally on $\mathbb{C}^k = (\mathbb{C}^k)^n$ by letting $\Gamma_n$ act factor-wise and $S_n$ act as permutations of $n$-factors. We denote by $\lambda(\mathbb{C}^k)$ the virtual character $\sum_{i=0}^n (-1)^i \Lambda^i \mathbb{C}^k$ of $\Gamma_n$, where $\Lambda^i \mathbb{C}^k$ carries an induced $\Gamma_n$-action. The geometric significance of $\lambda(\mathbb{C}^k)$ will become clear later.

Let $\eta_n(\lambda(\pi))$ be the virtual character of $\Gamma_n$ built on the $n$-th tensor of the virtual character $\lambda(\pi) = \sum_{i=0}^k (-1)^i \Lambda^i \mathbb{C}^k$ of $\Gamma$.

**Theorem 3** The virtual characters $\lambda(\mathbb{C}^k)$ and $\eta_n(\lambda(\pi))$ of $\Gamma_n$ are equal.

**Proof.** Given $x \times y \in \Gamma_n$, where $x \in \Gamma_m$ and $y \in \Gamma_{n-m}$ for some $m$, we have by definition

$$\eta_n(\lambda(\pi))(x \times y) = \eta_m(\lambda(\pi))(x) \eta_{n-m}(\lambda(\pi))(y),$$

$$\lambda(\mathbb{C}^k)(x \times y) = \lambda(\mathbb{C}^{km})(x) \lambda(\mathbb{C}^{k(n-m)})(y). \tag{11}$$

Thus it suffices to show that the character values $\eta_n(\lambda(\pi))(\alpha, s)$ and $\lambda(\mathbb{C}^k)(\alpha, s)$ are equal for $(\alpha, s) \in \Gamma_n$, where $\alpha = (g, 1, \ldots, 1) \in \Gamma^n$ and $s$ is an $n$-cycle, say $s = (12 \ldots n)$. It is known (cf. [FJW]) that the character value of $\eta_n(\xi)(\alpha, s)$ is $\xi(c)$, where $\xi$ is a class function of $\Gamma$ and $c$ is the conjugacy class of $g$. In particular

$$\eta_n(\lambda(\pi))(\alpha, s) = \lambda(\pi)(c).$$
Denote by $x^i_a, a = 1, \ldots, n, i = 1, \ldots, k$ the coordinates in $\mathbb{C}^{kn} = (\mathbb{C}^k)^n$. The $a$-th factor $\Gamma$ in $\Gamma^n \subset \Gamma_n$ acts on $\mathbb{C}^k$ with coordinates $x^i_a, i = 1, \ldots, k$.

Consider the exterior monomial basis for $\Lambda^i \mathbb{C}^{kn}$. Given such a monomial $X$, we first observe that the coefficient of $X$ in $(\alpha, s) X$ is 0 unless there are equal numbers of lower subscripts $1, 2, \ldots, n$ for $x^i_a$ appearing in $X$. It follows that

$$\Lambda^i \mathbb{C}^{kn}(\alpha, s) = \operatorname{trace} (\alpha, s) \big|_{\Lambda^i \mathbb{C}^{kn}} = 0, \quad \text{if } i \text{ is not divisible by } n. \quad (12)$$

For $i = mn, 1 \leq m \leq k$, we further observe that the coefficient of $X$ in $(\alpha, s) X$ is 0 unless the monomial $X$ is of the form

$$X(i_1 \ldots i_m) = x_1^{i_1} \wedge x_2^{i_2} \wedge \ldots \wedge x_n^{i_n} \wedge x_1^{i_{1}} \wedge x_2^{i_{2}} \wedge \ldots \wedge x_n^{i_{m}} \wedge x_1^{i_1} \wedge \ldots \wedge x_n^{i_m},$$

where $\{i_1, \ldots, i_m\}$ is an unordered $m$-tuple of distinct numbers among $1, 2, \ldots, k$. Write $\pi(g) x_1^i = \sum_j b_{ij} x_1^j$ and denote by $B$ the $k \times k$ matrix $(b_{ij})$. Then

$$(\alpha, s) X(i_1 \ldots i_m) = x_2^{i_2} \wedge x_3^{i_3} \wedge \ldots \wedge x_n^{i_n} \wedge \sum_j b_{i_1 j} x_1^j \wedge x_2^{i_2} \wedge x_3^{i_3} \wedge \ldots \wedge x_n^{i_n} \wedge \sum_j b_{i_2 j} x_1^j \wedge \ldots \wedge x_2^{i_2} \wedge x_3^{i_3} \wedge \ldots \wedge x_n^{i_n} \wedge \sum_j b_{i_m j} x_1^j.$$

It follows that the coefficient of $X(i_1 \ldots i_m)$ in $(\alpha, s) X(i_1 \ldots i_m)$ is equal to

$$\sum_{\sigma} (-1)^{(n-1)m} (-1)^{l(\sigma)} b_{i_1 \sigma(i_1)} b_{i_2 \sigma(i_2)} \ldots b_{i_m \sigma(i_m)} = (-1)^{(n-1)m} \det B(i_1 \ldots i_m), \quad (13)$$

where the summation runs over all permutations $\sigma$ of $i_1, \ldots, i_m, l(\sigma)$ is the length of $\sigma$, and $\det B(i_1 \ldots i_m)$ denotes the determinant of the $m \times m$ minor of $A$ consisting of the rows and columns $i_1, \ldots, i_m$.

By (12) and (13) we calculate that

$$\lambda(\mathbb{C}^{kn})(\alpha, s) = \sum_{m=0}^{k} (-1)^{mn} \Lambda^{mn}(\mathbb{C}^{kn})(\alpha, s)$$

$$= \sum_{m=0}^{k} (-1)^{mn} (-1)^{m(n-1)} \sum_{\{i_1, \ldots, i_m\}} \det B(i_1 \ldots i_m)$$

$$= \sum_{m=0}^{k} (-1)^{m} e_m(t_1, \ldots, t_k)$$

$$= \sum_{m=0}^{k} (-1)^{m} (\Lambda^{m}(\pi))(g)$$

$$= \lambda(\pi)(c),$$
where the summation runs over all unordered \( r \)-tuples \( \{i_1, \ldots, i_m\} \) of distinct numbers among \( 1, 2, \ldots, k \), and \( e_m \) denotes the \( m \)-th elementary symmetric polynomial of the eigenvalues \( t_1, \ldots, t_k \) of the matrix \( \pi(g) \in SL_k(\mathbb{C}) \). The two identities involving \( e_r \) used above are well known.

By comparing the character values \( \eta_n(\lambda(\pi))(\alpha, s) \) and \( \lambda(\mathcal{C}^{kn})(\alpha, s) \) calculated above, we see that

\[
\eta_n(\lambda(\pi))(\alpha, s) = \lambda(\mathcal{C}^{kn})(\alpha, s) = \lambda(\pi)(c).
\]

Therefore if \( x \in \Gamma_n \) is of type \( \rho \in \mathcal{P}_n(\Gamma_*) \) then by using (11) we obtain that

\[
\eta_n(\lambda(\pi))(x) = \prod_{c \in \Gamma_\ast} \lambda(\pi)(c)^{l(\rho(c))} = \lambda(\mathcal{C}^{kn})(x).
\]

This completes the proof. \( \square \)

**Remark 10** The identification of the two virtual characters can be also seen alternatively as follows:

\[
\lambda(\mathcal{C}^{kn}) = \sum_{i=0}^{kn} (-1)^i \Lambda((\mathcal{C}^k)^n)
= \sum_{i_1 + \cdots + i_n = m} (-1)^{i_1 + \cdots + i_n} \Lambda^i_1(\mathcal{C}^k) \otimes \cdots \otimes \Lambda^i_n(\mathcal{C}^k)
= \sum_{\{n_0, \ldots, n_k\}} \left( \sum_{i} (-1)^i \Lambda^i(\mathcal{C}^k) \right) \mathbb{E}^{n_0} \otimes \cdots \otimes \mathbb{E}^{n_k} \tag{14}
\]

\[
= \left( \sum_{i} (-1)^i \Lambda^i(\mathcal{C}^k) \right) \mathbb{E}^n \tag{15}
\]

where \( \{n_0, \ldots, n_k\} \) ranges over the \( (k+1) \)-tuple of non-negative integers such that \( \sum_i n_i = n \). Eq. (14) above basically follows from the definition of an induction functor. Eq. (14) is a generalization of (10) which can be established with some effort.

When \( \Gamma \) is trivial then \( \lambda(\pi) = 0 \in R(\Gamma) \) and so \( \eta_n(\lambda(\pi)) = 0 \). We have an immediate corollary.

**Corollary 1** When \( \Gamma \) is trivial and \( \Gamma_n \) becomes the symmetric group \( S_n \), the virtual \( S_n \)-character \( \lambda(\mathcal{C}^{kn}) \) is zero.
2.3 Derived categories and Grothendieck Groups

In this subsection we let $\Gamma$ be a finite subgroup of $SL_2(\mathbb{C})$ unless otherwise specified.

We denote by $D_{\Gamma_n}(\mathbb{C}^{2n})$ the bounded derived category of $\Gamma_n$-equivariant coherent sheaves on $\mathbb{C}^{2n}$, and denote by $D(\mathbb{C}^2/\Gamma^{[n]})$ the bounded derived category of coherent sheaves on $\mathbb{C}^2/\Gamma^{[n]}$. Define two functors $\Phi : D(\mathbb{C}^2/\Gamma^{[n]}) \to D_{\Gamma_n}(\mathbb{C}^{2n})$ and $\Psi : D_{\Gamma_n}(\mathbb{C}^{2n}) \to D(\mathbb{C}^2/\Gamma^{[n]})$ by

$$\Phi(-) = R\pi_* (\mathcal{O}_{U_{\Gamma_n}} \otimes q^*(-)) \Gamma_n,$$

$$\Psi(-) = (R\pi_* R\mathcal{H}om(\mathcal{O}_{U_{\Gamma_n}}, p^*(-))) \Gamma_n,$$

where $U_{\Gamma_n}$ is the universal scheme defined in (9), and $p, q$ denote the projections $\tilde{\mathbb{C}}^2/\Gamma^{[n]} \times \mathbb{C}^{2n}$ to $\mathbb{C}^{2n}$ and $\tilde{\mathbb{C}}^2/\Gamma^{[n]}$ respectively.

Take a basis of $R(\Gamma_n)$ given by the irreducible characters $s_\rho, \rho \in \mathcal{P}_n(\Gamma^*)$ of $\Gamma_n$ (cf. [M, Z]). We denote by $\mathcal{O}_{\mathbb{C}^{2n}}$ the structure sheaf over $\mathbb{C}^{2n}$. Recall that the vector bundle (i.e. locally free sheaf) $\mathcal{V}_\rho$ is defined in (8). The following theorem can be derived by using a general theorem due to Bridgeland, King and Reid (Theorem 1.2, [BKR]) since $\tau_n : \mathbb{C}^2/\Gamma^{[n]} \to \mathbb{C}^{2n}/\Gamma_n$ is a crepant resolution and $\Gamma_n$ preserves the symplectic structure of $\mathbb{C}^{2n}$.

**Theorem 4** Under the assumption of the validity of Conjecture [B], $\Phi$ is an equivalence of categories and $\Psi$ is its adjoint functor. In particular, $\Psi$ sends $\mathcal{O}_{\mathbb{C}^{2n}} \otimes s_\rho^\vee$ to $\mathcal{V}_\rho$.

**Remark 11** When $n = 1$, Conjecture [B] is known to be true (cf. Remark [B]) and the above theorem was established by Kapranov and Vasserot [KV].

**Remark 12** By assuming the validity of the $n!$ conjecture (cf. [GH]), we have by Remark [B] that $\mathbb{C}^2/\Gamma^{[n]} // S_n \cong \tilde{\mathbb{C}}^2/\Gamma^{[n]}$. Thus we may replace $\mathbb{C}^2/\Gamma^{[n]}$ by $\mathbb{C}^2/\Gamma^{[n]} // S_n$ in the crepant Hilbert-Chow resolution $\mathbb{C}^2/\Gamma^{[n]} \to \mathbb{C}^2/\Gamma^{[n]} // S_n$. Clearly, $S_n$ preserves the holomorphic symplectic structure of the Cartesian product $\mathbb{C}^2/\Gamma^{[n]}$. Then one can apply again Theorem 1.2 in [BKR] to show that there is an equivalence of derived categories between $D(\mathbb{C}^2/\Gamma^{[n]})$ and $D_{S_n}(\mathbb{C}^2/\Gamma^{[n]})$ of $S_n$-equivariant coherent sheaves of $\mathbb{C}^2/\Gamma^{[n]}$.

Below we further specialize and apply the general results of Bridgeland, King and Reid [BKR] to our setup. We denote by $D_{\Gamma_n}^0(\mathbb{C}^{2n})$ the full subcategory of $D_{\Gamma_n}(\mathbb{C}^{2n})$ consisting of objects whose cohomology sheaves are concentrated on the
origin of \( \mathbb{C}^{2n} \). We denote by \( D_{S_n}^0(\mathbb{C}^2/\Gamma^n) \) the full subcategory of \( D_{S_n}(\mathbb{C}^2/\Gamma^n) \) consisting of objects whose cohomology sheaves are concentrated on the \( n \)-th Cartesian product of the exceptional divisor \( D \subset \mathbb{C}^2/\Gamma \). Denote by \( D^0(\mathbb{C}^2/\Gamma[n]) \) the full subcategory of \( D(\mathbb{C}^2/\Gamma[n]) \) consisting of objects whose cohomology sheaves are concentrated on the fiber \( \mathbb{C}^2/\Gamma[n,0] \) of \( \mathbb{C}^2/\Gamma[n] \). Recall that \( \mathbb{C}^2/\Gamma[n,0] \) is described in Subsection 1.2.

The equivalence between the derived categories \( D_{\Gamma_n}(\mathbb{C}^{2n}) \) and \( D(\mathbb{C}^2/\Gamma[n]) \) induces an equivalence between \( D_{\Gamma_n}^0(\mathbb{C}^{2n}) \) and \( D^0(\mathbb{C}^2/\Gamma[n]) \). Thus we have the following commutative diagram (under the assumption of the validity of Conjecture [2]):

\[
\begin{array}{ccc}
D_{\Gamma_n}^0(\mathbb{C}^{2n}) & \xrightarrow{\sim} & D^0(\mathbb{C}^2/\Gamma[n]) \\
\downarrow & & \downarrow \\
D_{\Gamma_n}(\mathbb{C}^{2n}) & \xrightarrow{\sim} & D(\mathbb{C}^2/\Gamma[n]).
\end{array}
\]

Given objects \( E, F \) in \( D_{\Gamma_n}(\mathbb{C}^{2n}) \) and \( D_{\Gamma_n}^0(\mathbb{C}^{2n}) \) respectively, we define the Euler characteristic

\[
\chi_{\Gamma_n}(E, F) = \sum_i (-1)^i \dim \text{Hom}_{D_{\Gamma_n}(\mathbb{C}^{2n})}(E, F[i]).
\]

This gives a natural bilinear pairing between \( D_{\Gamma_n}(\mathbb{C}^{2n}) \) and \( D_{\Gamma_n}^0(\mathbb{C}^{2n}) \). Similarly we can define the Euler characteristic \( \chi(A, B) \) for objects \( A, B \) in \( D(\mathbb{C}^2/\Gamma[n]) \) and \( D^0(\mathbb{C}^2/\Gamma[n]) \) respectively, which gives rise to a bilinear pairing between \( D(\mathbb{C}^2/\Gamma[n]) \) and \( D^0(\mathbb{C}^2/\Gamma[n]) \). We further have \( \chi_{\Gamma_n}(E, F) = \chi(\Psi(E), \Psi(F)) \), cf. [BKR].

We denote by \( K_{\Gamma_n}(\mathbb{C}^{2n}), K_{\Gamma_n}^0(\mathbb{C}^{2n}), K(\mathbb{C}^2/\Gamma[n]), K^0(\mathbb{C}^2/\Gamma[n]), K_{S_n}^0(\mathbb{C}^2/\Gamma[n]) \) and \( K_{S_n}(\mathbb{C}^2/\Gamma^n) \) the Grothendieck groups of the corresponding derived categories. It is well known that \( K_{\Gamma_n}(\mathbb{C}^{2n}) \) and \( K_{\Gamma_n}^0(\mathbb{C}^{2n}) \) are both isomorphic to the representation ring \( R_x(\Gamma_n) \). The bilinear pairings mentioned above together with the embeddings of categories induces a bilinear form on \( K_{\Gamma_n}^0(\mathbb{C}^{2n}) \) and respectively on \( K^0(\mathbb{C}^2/\Gamma[n]) \).

Let \( \mathcal{O}_0 \) be the skyscraper sheaf at the origin 0 on \( \mathbb{C}^{2n} \). The \( \Gamma_n \)-bundles \( \mathcal{O} \otimes s_\rho, \rho \in \mathcal{P}_n(\Gamma^*) \) form a basis for \( K_{\Gamma_n}(\mathbb{C}^{2n}) \) while the modules \( s_\rho \otimes \mathcal{O}_0, \rho \in \mathcal{P}_n(\Gamma^*) \) form the dual basis for \( K_{\Gamma_n}^0(\mathbb{C}^{2n}) \).

**Theorem 5** The map by sending \( s_\rho \) to \( \mathcal{O}_0 \otimes s_\rho, \rho \in \mathcal{P}_n(\Gamma^*) \), is an isometry between \( R_x(\Gamma_n) \) endowed with the weighted bilinear form and \( K_{\Gamma_n}^0(\mathbb{C}^{2n}) \) endowed with the bilinear form defined above. In particular the bilinear form on \( K_{\Gamma_n}^0(\mathbb{C}^{2n}) \) is semipositive definite symmetric.

**Proof.** Following a similar argument as in Gonzalez-Sprinberg and Verdier [GSV] (which is for \( n = 1 \)), we obtain the following commutative diagram by using the
Koszul resolution of $O_0$ on $\mathbb{C}^{2n}$:

\[
R_{Z}(\Gamma_n) \xrightarrow{\sim} K_{\Gamma_n}^0(\mathbb{C}^{2n}) \xrightarrow{\sim} K_{\Gamma_n}(\mathbb{C}^{2n})
\]

Here the horizontal maps are isomorphisms given by sending $s_\rho$ to $O_0 \otimes s_\rho$ and respectively to $O_{\mathbb{C}^{2n}} \otimes s_\rho$, $\rho \in P_n(\Gamma^*)$. The left vertical map $j$ is given by multiplication by the virtual character $\lambda(\mathbb{C}^{2n})$ of $\Gamma_n$, and the right vertical one is induced from the natural embedding of the corresponding categories. Now the theorem follows from the definition of the weighted bilinear form on $R(\Gamma_n)$, Theorem 3, and the fact that the basis $O_0 \otimes s_\rho, \rho \in P_n(\Gamma^*)$ for $K_{\Gamma_n}^0(\mathbb{C}^{2n})$ is dual to the basis $O_{\mathbb{C}^{2n}} \otimes s_\rho, \rho \in P_n(\Gamma^*)$ for $K_{\Gamma_n}(\mathbb{C}^{2n})$. \hfill \blacksquare

**Remark 13** The main results in [FJW] (see Theorem 7.2 and Theorem 7.3 in *loc. cit.* ) can be now formulated by using the space

\[
\mathcal{F}_\Gamma = \bigoplus_{n \geq 0} K_{\Gamma_n}^0(\mathbb{C}^{2n}) \otimes \mathbb{C}[K_{\Gamma}^0(\mathbb{C}^{2})]
\]

with its natural bilinear form induced from the Koszul-Thom class. Here $\mathbb{C}[-]$ denotes the group algebra. Roughly speaking, $\mathcal{F}_\Gamma$ affords a vertex representation of the toroidal Lie algebra and a distinguished subspace of $\mathcal{F}_\Gamma$ affords the basic representation of the affine Lie algebra $\hat{g}$ whose associated affine Dynkin diagram corresponds to $\Gamma$ in the sense of McKay. This may be viewed as a form of McKay correspondence relating finite subgroups of $SL_2(\mathbb{C})$ to affine and toroidal Lie algebras.

Now we have the following commutative diagram (assuming the validity of Conjecture [1]):

\[
R_{Z}(\Gamma_n) \xrightarrow{\sim} K_{\Gamma_n}^0(\mathbb{C}^{2n}) \xrightarrow{\sim} K^0(\mathbb{C}^2/\Gamma^[[n]])
\]

By Remark 12 and a similar argument as above, we have another commutative diagram (assuming the validity of Conjecture [1]):

\[
K_{S_n}^0(\mathbb{C}^2/\Gamma^[[n]]) \xrightarrow{\sim} K^0(\mathbb{C}^2/\Gamma^[[n]])
\]

Combining the two diagrams above, we have obtained the following theorem.
Theorem 6 Under the assumption of the validity of Conjecture \[\text{[1]},\] the isomorphisms among \((R_\mathbb{Z}(\Gamma_n), \langle-,-\rangle_{\lambda(c^2)})\), the K-groups \(K^0_{\mathbb{S}_n}(\mathbb{C}^2/\Gamma^n)\) and \(K^0(\mathbb{C}^2/\Gamma^n)\) are isometries.

Remark 14 All the algebraic structures on \(\bigoplus_n R(\Gamma_n)\) (cf. \[\text{[W, FJW]}\]) and thus on \(\bigoplus_n K_{\Gamma_n}(\mathbb{C}^{2n})\) can now be carried over to \(\bigoplus_n K^0(\mathbb{C}^2/\Gamma^n)\). However it remains to match these with the Grojnowski-Nakajima construction on \(\bigoplus_n H(\mathbb{C}^2/\Gamma^n)\) in terms of correspondence varieties.

Remark 15 Given a finite subgroup \(G\) of \(SL_K(\mathbb{C})\), one asks whether there is a crepant resolution \(Y\) of the affine orbifold \(\mathbb{C}^K/G\) so that there exists a canonical isomorphism between \(K_G(\mathbb{C}^K) = K(Y)\); one further asks whether or not the answer can be provided by the Hilbert quotient \(\mathbb{C}^K//G\), cf. Reid \[\text{[R]}\]. The answer is affirmative for \(K = 2\), known as the McKay correspondence \[\text{[GSV]}\] (also compare \[\text{[IN, KV]}\]). For \(K = 3\), there has been much work by various people, cf. \[\text{[IN, Ro, R, Nr, IN]}\] and references therein, and it is settled by Bridgeland-King-Reid \[\text{[BKR]}\]. However not much is known in general (see however \[\text{[BKR]}\]) and there has been counterexamples. Our work provides strong evidence for an affirmative answer in the case of \(\mathbb{C}^{2n}\) acted upon by the wreath product \(\Gamma_n\) associated to a finite subgroup \(\Gamma \subset SL_2(\mathbb{C})\).

3 A direct isomorphism of algebraic structures on equivariant K-theory

In this section we assume that the reader is familiar with \[\text{[W]}\]. For shortness of notations we will use \(K^\text{top}(-)\) and \(K^\text{top}_G(-)\) to denote the complexified (G-equivariant) topological K-group. We further assume that \(X\) is a quasi-projective surface acted upon by a finite subgroup \(\Gamma\) and \(Y\) is a resolution of singularities of \(X/\Gamma\) such that there exists a canonical isomorphism \(\theta\) between \(K^\text{top}_\Gamma(X)\) and \(K^\text{top}(Y)\).

Let \(C = \{V_1, \ldots, V_l\}\) be a basis for \(K^\text{top}_\Gamma(X)\). Without loss of generality we may assume they are genuine \(\Gamma\)-vector bundles on \(X\). We denote by \(W_i = \theta(V_i)\). The set \(\{W_1, \ldots, W_l\}\) is a basis for \(K(Y)\). We remark that representatives of \(W_i\)’s can again be chosen as certain canonical vector bundles in favorable cases, including the important case when \(X = \mathbb{C}^2\) and \(\Gamma \subset SL_2(\mathbb{C})\).

Let \(S\lambda\) be the irreducible representation of the symmetric group \(S_n\) associated to the partition \(\lambda\) of \(n\). Define

\[S\lambda(V_i) = S\lambda \bigotimes V_i^{S_n}.\]
Endowed with the diagonal action of $S_n$ on the two tensor factors and the action of $\Gamma^n$ on the second factor, $S_\lambda(V_i)$ is an $\Gamma_n$-equivariant vector bundle.

Given a partition-valued function $\lambda = (\lambda_i)_{1 \leq i \leq l} \in \mathcal{P}_n(\mathcal{C})$, we define the $\Gamma_n$-equivariant vector bundle

$$S^X_\lambda = \text{Ind}_{\Gamma[\lambda_i] \times \ldots \times \Gamma[\lambda_l]}^{\Gamma_n} S_{\lambda_1}(V_1) \times \ldots \times S_{\lambda_l}(V_l).$$

In a parallel way, we can define the $S_n$-equivariant bundle $S^Y_\lambda$ associated to $\lambda \in \mathcal{P}_n(\mathcal{C})$ as

$$S^Y_\lambda = \text{Ind}_{S[\lambda_i] \times \ldots \times S[\lambda_l]}^{S_n} S_{\lambda_1}(\theta(V_1)) \times \ldots \times S_{\lambda_l}(\theta(V_l)).$$

Recall that we constructed in $[W]$ various algebraic structures such as Hopf algebra, $\lambda$-ring, Heisenberg algebra on $\bigoplus_{n \geq 0} K^{top}_{\Gamma_n}(X^n)$ for a $\Gamma$-space $X$, generalizing the results of Segal $[S2]$ for $\Gamma$ trivial. The following proposition can be proved using $[W]$ in a way as Macdonald $[M]$ did when $X$ is a point.

**Proposition 5** The $\Gamma_n$-bundles $S^X_\lambda, \lambda \in \mathcal{P}_n(\mathcal{C})$ form a basis of $K^{top}_{\Gamma_n}(X^n)$. The $S_n$-bundles $S^Y_\lambda, \lambda \in \mathcal{P}_n(\mathcal{C})$ form a basis of $K_{S_n}(Y^n)$.

These bases will be referred to as Schur bases, generalizing the usual one for $R(\Gamma_n) = K^{top}_{\Gamma_n}(pt)$.

**Theorem 7** The map $\Theta$ from $\bigoplus_{n \geq 0} K^{top}_{\Gamma_n}(X^n)$ to $\bigoplus_{n \geq 0} K_{S_n}(Y^n)$ by sending $S^X_\lambda$ to $S^Y_\lambda$ is an isomorphism of Hopf algebras, $\lambda$-rings, and representations over the Heisenberg algebra.

**Proof.** We use $[W]$ as a basic reference. We follow the notations there with an additional use of $X, Y$ as subscripts to specify the space we are referring to.

As graded vector spaces $\bigoplus_{n \geq 0} K^{top}_{\Gamma_n}(X^n)$ and $\bigoplus_{n \geq 0} K_{S_n}(Y^n)$ have the same graded dimension due to the isomorphism between $K^{top}_{\Gamma}(X)$ and $K^{top}(Y)$, cf. Theorem 3 in $[W]$. Thus the map $\Theta$ given by matching the Schur basis is an additive isomorphism.

Recall that the Adam’s operations $\varphi^m_X$ on the space $\bigoplus_{n \geq 0} K^{top}_{\Gamma_n}(X^n)$ and $\varphi^m_Y$ on $\bigoplus_{n \geq 0} K_{S_n}(Y^n)$ satisfy the identities (where $q$ is a formal parameter), cf. Proposition 4 in $[W]$:

$$\bigoplus_{n \geq 0} q^n V^{\otimes n}_i = \exp \left( \sum_{m > 0} \frac{1}{m} \varphi^m_X(V_i) q^m \right),$$

$$\bigoplus_{n \geq 0} q^n W^{\otimes n}_i = \exp \left( \sum_{m > 0} \frac{1}{m} \varphi^m_Y(W_i) q^m \right).$$

It follows that $\varphi^m_X(V_i)$ and $\varphi^m_Y(W_i)$ are uniquely determined by $V^{\otimes n}_i$ ($n \geq 0$) and respectively $W^{\otimes n}_i$ in the same way (by taking logarithms of the above identities).

Since the isomorphism $\Theta$ sends $V^{\otimes n}_i$ to $W^{\otimes n}_i$, the Adams operations $\phi^m_X(V_i)$ and
\[ \phi^*_X(V_i) \] matches under \( \Theta \) and so does the \( \lambda \)-ring structures on \( \bigoplus_{n \geq 0} K_{\Gamma_n}^{top}(X^n) \) and \( \bigoplus_{n \geq 0} K_{S_n}(Y^n) \).

Recall that Heisenberg algebras \( \mathcal{H}_X \) and \( \mathcal{H}_Y \) constructed in terms of K-theory maps act irreducibly on \( \bigoplus_{n \geq 0} K_{\Gamma_n}^{top}(X^n) \) and \( \bigoplus_{n \geq 0} K_{S_n}(Y^n) \) respectively, cf. Theorem 4 in [W]. The Heisenberg algebra generators are essentially defined in terms of Adams operations, induction functors and restriction functors such as \( \text{Ind}^{\Gamma_n}_{\Gamma_m \times \Gamma_{n-m}} \), \( \text{Res}^{\Gamma_n}_{\Gamma_m \times \Gamma_{n-m}} \), etc. Since the Adams operations, induction and restriction functors are compatible with the Schur bases and thus with the map \( \Theta \), the Heisenberg algebras acting on \( \bigoplus_{n \geq 0} K_{\Gamma_n}^{top}(X^n) \) and \( \bigoplus_{n \geq 0} K_{S_n}(Y^n) \) also matches under \( \Theta \).

\[ \blacksquare \]

References

[BG] R. Bezrukavnikov and V. Ginzburg, *Hilbert schemes and reductive groups*, in preparation.

[BKR] T. Bridgeland, A. King and M. Reid, *Mukai implies McKay*, preprint, [math.AG/9908027](https://arxiv.org/abs/math.AG/9908027).

[CM] M. de Cataldo and L. Migliorini, *The Douady space of a complex surface*, [math.AG/9811159](https://arxiv.org/abs/math.AG/9811159), to appear in Adv. in Math.

[Fo] J. Fogarty, *Algebraic families on an algebraic surface*, Amer. J. Math. 90 (1968) 511–521.

[FJW] I. B. Frenkel, N. Jing and W. Wang, *Vertex representations via finite groups and the McKay correspondence*, [math.QA/9907166](https://arxiv.org/abs/math.QA/9907166), to appear in IMRN.

[FK] I. B. Frenkel and V. G. Kac, *Basic representations of affine Lie algebras and dual resonance models*, Invent. Math. 62 (1980) 23–66.

[GH] A. Garsia and M. Haiman, *A graded representation model for Macdonald’s polynomials*, Proc. Nat. Acad. USA 903 (1993) 3607–3610.

[GSV] G. Gonzalez-Sprinberg and J.-L. Verdier, *Construction géométrique de la correspondance de McKay*, Ann. Sci. École Norm. Sup. 16 (1983) 409–449.

[G] L. Göttscbe, *The Betti numbers of the Hilbert scheme of points on a smooth projective surface*, Math. Ann. 286 (1990) 193–207.

[Gr] I. Grojnowski, *Instantons and affine algebras I: the Hilbert scheme and vertex operators*, Math. Res. Lett. 3 (1996) 275–291.

[H] M. Haiman, *Macdonald polynomials and Hilbert schemes*, UCSD preprint.

[IN] Y. Ito and H. Nakajima, *McKay correspondence and Hilbert schemes in dimension three*, [math.AG/9803120](https://arxiv.org/abs/math.AG/9803120), to appear in Topology.
Weiqiang Wang

[INr] Y. Ito and I. Nakamura, *McKay correspondence and Hilbert schemes*, Proc. Japan Acad. Ser. A 72 (1996) 135–138.

[Ka] M. Kapranov, *Chow quotients of Grassmannians*, in Gelfand Seminar 1 (eds. S. Gelfand, S. Gindikin), Adv. in Soviet Math. 16 (1993) 29–110, Amer. Math. Soc. Providence RI.

[KV] M. Kapranov and E. Vasserot, *Kleinian singularities, derived categories and Hall algebras*, preprint, math.AG/9812016.

[Kr] P. Kronheimer, *The construction of ALE spaces as hyper-Kähler quotients*, J. Diff. Geom. 28 (1989) 665–683.

[KN] P. Kronheimer and H. Nakajima, *Yang-Mills instantons on ALE gravitational instantons*, Math. Ann. 288 (1990) 263–307.

[M] I. G. Macdonald, *Polynomial functors and wreath products*, J. Pure Appl. Alg. 18 (1980) 173–204.

[Mc] J. McKay, *Graphs, singularities and finite groups*, Proc. Sympos. Pure Math. 37, Amer. Math. Soc, Providence, RI (1980) 183–186.

[N] H. Nakajima, *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras*, Duke Math. J. 76 (1994) 365–416.

[N1] H. Nakajima, *Heisenberg algebra and Hilbert schemes of points on projective surfaces*, Ann. Math. 145 (1997) 379–388.

[N2] H. Nakajima, *Lectures on Hilbert schemes of points on surfaces*, to be published by AMS.

[N3] H. Nakajima, *Quiver varieties and Kac-Moody algebras*, Duke Math. J. 91 (1998) 515–560.

[N4] H. Nakajima, *Private communications*.

[Nr] I. Nakamura, *Hilbert schemes of abelian group orbits*, to appear in J. Alg. Geom.

[R] M. Reid, *McKay correspondence*, in Proc. alg. geom sympos. (Kinosaki, Nov. 1996), T. Katsura (ed.), 14-41; alg-geom/9702016.

[Ro] S.-S. Roan, *Minimal resolutions of Gorenstein orbifolds in dimension three*. Topology 35 (1996) 489–508.

[S1] G. Segal, *Unitary representations of some infinite dimensional groups*, Commun. Math. Phys. 80 (1981) 301–342.
[S2] G. Segal, *Equivariant K-theory and symmetric products*, 1996 preprint (unpublished).

[VV] M. Varagnolo and E. Vasserot, *On the K-theory of the cyclic quiver variety*, preprint, [math.AG/9902091](http://arxiv.org/abs/math.AG/9902091).

[VW] C. Vafa and E. Witten, *A strong coupling test of S-duality*, Nucl. Phys. B 431 (1994) 3–77.

[W] W. Wang, *Equivariant K-theory and wreath products*, MPI preprint # 86, August 1998; *Equivariant K-theory, wreath products and Heisenberg algebra*, [math.QA/9907051](http://arxiv.org/abs/math.QA/9907051) to appear in Duke Math. J.

[Wey] H. Weyl, *The classical groups, their invariants and representations*, Princeton University Press, 1946.

[Z] A. Zelevinsky, *Representations of finite classical groups. A Hopf algebra approach*, Lect. Notes in Math. 869, Springer-Verlag, Berlin-New York, 1981.

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