Twisted Frobenius-Schur indicators for Hopf algebras

Maria Vega

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TWISTED FROBENIUS–SCHUR INDICATORS
FOR HOPF ALGEBRAS

A Dissertation
Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy
in
The Department of Mathematics

by
Maria Vega
B.A., California State University, Fullerton, 2004
M.S., Louisiana State University, 2006
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I dedicate this dissertation to my family.
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Abstract

The classical Frobenius–Schur indicators for finite groups are character sums defined for any representation and any integer \( m \geq 2 \). In the familiar case \( m = 2 \), the Frobenius–Schur indicator partitions the irreducible representations over the complex numbers into real, complex, and quaternionic representations. In recent years, several generalizations of these invariants have been introduced. Bump and Ginzburg in 2004, building on earlier work of Mackey from 1958, have defined versions of these indicators which are twisted by an automorphism of the group. In another direction, Linchenko and Montgomery in 2000 defined Frobenius–Schur indicators for finite dimensional semisimple Hopf algebras. In this dissertation, we construct twisted Frobenius–Schur indicators for semisimple Hopf algebras; these include all of the above indicators as special cases and have similar properties.
Chapter 1
Introduction

Let $G$ be a compact group and $V$ be a $G$-module with character $\chi$. Let $\text{Irr}(G)$ be the set of irreducible characters of $G$. The classical Frobenius–Schur indicator for finite groups is given by

$$\nu_2(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2).$$

This is a special case of more general character sums

$$\nu_m(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^m),$$

where $m$ is a natural number, see [Isa76].

It is a well known result that the classical indicators partition the irreducible representations into real, quaternionic, and complex irreducible representations, [FH91] and [Ser77]. We have $\nu_2(\chi) = 0$ if $V$ is complex (i.e. not self dual), 1 if $V$ is real and $-1$ if $V$ is quaternionic.

Mackey in [Mac58] introduced a twisted version of the Frobenius–Schur indicators in order to study multiplicity-free permutation representations. If $G$ is a finite group with an anti-involution $i$, then the character formula

$$\tilde{\nu}_2(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(gi(g^{-1}))$$

provides the twisted Frobenius–Schur indicators. It is again the case that these are virtual characters which take the value of $1, 0$, or $-1$ on irreducible representations, thus providing a twisted version of the partition of irreducible representations into real, complex and quaternionic. If the involution $i$ is taken to be the inverse map (i.e., $i(g) = g^{-1}$), then the twisted Frobenius–Schur indicator coincides with the
classical one. These twisted indicators have a similar interpretation to that of the classical ones. There are also higher order versions of these twisted indicators. In 2004, Bump and Ginzburg in [BG04], introduced twisted higher order Frobenius–Schur indicators using a group automorphism of order $m$. The indicators discussed above are the case when $m = 2$.

The twisted Frobenius–Schur indicators have useful applications. For example, Vinroot has recently shown that for finite symplectic groups (over fields of odd characteristic), there exists an involution with respect to which all irreducible representations are twisted real, and he has used this result to obtain formulas for the sums of the degrees of the irreducible complex characters for finite symplectic groups [Vin05].

The group algebra of a finite group is a special case of a Hopf algebra and it is natural to expect that the theory of Frobenius–Schur indicators should generalize to appropriate Hopf algebras. In the late 1990’s, Linchenko and Montgomery showed that this was indeed the case by defining Frobenius–Schur indicators for a finite-dimensional semisimple Hopf algebra $H$ over an algebraically closed field of characteristic 0 in [LM00]. Linchenko and Montgomery proved that the second indicator $\nu_2$ also satisfies $\nu_2(\chi) = 0, 1, \text{ or } -1$ for all $\chi \in \text{Irr}(H)$ and that if $V$ is the $H$-module corresponding to $\chi$ then $\nu_2(\chi) \neq 0$ if and only if $V \cong V^*$. Furthermore, $\nu_2(\chi) = 1$ (respectively $-1$) if and only if $V$ admits a symmetric (resp. skew-symmetric) nondegenerate bilinear $H$-invariant form. In the case of group algebras, their construction reduces to the classical Frobenius–Schur indicators for finite groups. In 2006, Kashina, Sommerh"{a}user and Zhu in [KSZ06], studied the higher Frobenius–Schur indicators $\nu_m(\chi)$ for Hopf algebras. They found several formulas for $\nu_m(\chi)$ and used these to prove various results.
Frobenius–Schur indicators are one of the few useful invariants for Hopf algebras. In recent years they have been used extensively to study semisimple Hopf algebras. For example, they have been used by Kashina in [Kas03] and by Ng and Schauenburg in [NS08] to study the dimensions of representations of $H$. The indicators have also been used in the classification of Hopf algebras by Kashina, Sommerhäuser and Zhu in [KSZ02].

In this dissertation we construct a twisted version of Frobenius–Schur indicators for semisimple Hopf algebras that includes all of the above versions as special cases and have similar properties.

We now outline the organization of this dissertation.

In Chapter 2, we give the definitions and preliminary results that serve as foundation for this work. We will focus on the aspects that apply to our setting. In particular, we establish some Hopf algebra notation. Furthermore, we include a description of the classical Frobenius–Schur (FS) indicators for groups as well as the FS indicators for Hopf algebras.

In Chapter 3, we construct twisted FS indicators for Hopf algebras. In Section 3.1, we define the twisted FS indicators, and in Section 3.2, we study the second twisted FS indicators providing a partition of the irreducible representations. Furthermore, in Section 3.3 we focus on the higher order twisted FS indicators, we provide a trace formula and a closed formula for the regular representation.

Lastly, in Chapter 4, we give explicit calculations of the twisted FS indicators for a Hopf algebra that is neither commutative nor cocommutative.
Chapter 2
Preliminaries

In this chapter, we will provide some of the definitions and results from the literature needed for this dissertation. These can be found in the following references [DNR00], [Kas95] and [Mon93]. Before we formally define a Hopf algebra, we will look at some of the reasoning behind its definition. Hopf algebras have representation theory similar to that of groups, that is, tensor products, duals, etc., so we will now provide some motivation for its definition.

Let $A$ be an algebra with unit and let $V$ and $W$ be two representations of $A$. Our goal is to construct a tensor representation, that is, we want $V \otimes W$ to be a representation of $A$. It is well known that $V \otimes W$ is a representation of $A \otimes A$ where $(a \otimes b) \cdot (v \otimes w) = av \otimes bw$, so $V \otimes W$ would be a representation of $A$ if there were a map $\Delta$ such that

$$A \xrightarrow{\Delta} A \otimes A \longrightarrow V \otimes W,$$

where

$$a \cdot (v \otimes w) = \Delta(a)(v \otimes w),$$

for all $a \in A, v \in V$ and $w \in W$. This map $\Delta$ would be a dual map to the multiplication of $A$, that is, a comultiplication on $A$.

In addition to the tensor representation, we are interested in constructing a trivial representation, that is, we want our base field $k$ to be a representation of $A$. For this we would need a map

$$A \xrightarrow{\varepsilon} k,$$
such that

\[ a \cdot x = \varepsilon(a)x, \]

for all \( a \in A \) and \( x \in k \). Such a map \( \varepsilon \) would be a counit for \( A \).

Furthermore, we are also interested in the notion of a dual representation. That is, if \( V \) is a representation of \( A \), we want \( V^* \) to be a representation. To this end, we need a map \( S \) such that

\[ A \xrightarrow{S} A \]

where

\[ (a \cdot f)(v) = f(S(a)v) \]

for all \( f \in V^*, a \in A \) and \( v \in V \). In order for this to make sense, \( S \) must be an antimorphism. The map \( S \) is the antipode for \( A \).

If \( A \) is an algebra with such maps, then \( A \) is called a Hopf algebra.

## 2.1 Hopf Algebras

In this section, we will formally provide the definition of a Hopf algebra. To this end, we will first establish some notation. The map \( \text{Id}_H \) is used to denote the identity map on \( H \), that is, \( \text{Id}_H(h) = h \), for all \( h \in H \). We will suppress the subscript when it is clear what space we are dealing with. For the reminder of this dissertation, we will let \( k \) be an algebraically closed field of characteristic 0, unless otherwise specified.

**Definition 2.1.** An algebra over \( k \) (associative with 1) is a vector space \( A \) over \( k \) together with two linear maps

\[ \mu : A \otimes A \rightarrow A \]

and

\[ \eta : k \rightarrow A \]
such that the following diagrams commute:

\[
\begin{array}{ccc}
A \otimes A \otimes A & \xrightarrow{\mu \otimes \text{Id}} & A \otimes A \\
\text{Id} \otimes \mu & & \mu \\
A \otimes A & \xrightarrow{\mu} & A
\end{array}
\]

and

\[
\begin{array}{ccc}
A \otimes A & \xrightarrow{\eta \otimes \text{Id}} & A \\
\text{Id} \otimes \eta & & \text{Id} \\
A & \xrightarrow{\text{Id}} & A \otimes k \simeq A \\
\end{array}
\]

These are the diagrams for **associativity** and **unit**, respectively. The algebra \( A \) can be viewed as a triple \((A, \mu, \eta)\), where \( A \) is a vector space, \( \mu \) is used to denote **multiplication** and \( \eta \) is used to denote the **unit**. Furthermore, we will write

\[
\mu(a \otimes b) = ab
\]

and

\[
\eta(1) = 1_A
\]

for all \( a, b \in A \) where \( 1_A \) is the unit in \( A \).

We now dualize the notion of an algebra to obtain a coalgebra. A basic way to think about this is to view it as reversing the arrows in the diagrams of the definition of an algebra.

**Definition 2.2.** A coalgebra is a vector space \( C \) over \( k \) together with two maps

\[
\Delta : C \to C \otimes C
\]

and

\[
\varepsilon : C \to k
\]
such that the following diagrams commute:

\[
\begin{array}{ccc}
C & \xrightarrow{\Delta} & C \otimes C \\
\downarrow{\Delta} & & \downarrow{\Delta \otimes \text{Id}} \\
C \otimes C & \xrightarrow{\text{Id} \otimes \Delta} & C \otimes C \otimes C
\end{array}
\]

and

\[
\begin{array}{ccc}
C & \xleftarrow{\eta \otimes \text{Id}} & k \otimes C \\
\uparrow{\text{Id} \otimes \eta} & & \uparrow{\sim} \\
C \otimes k & \xleftarrow{\sim} & C
\end{array}
\]

These are the diagrams for coassociativity and counit, respectively. The coalgebra \( C \) can also be viewed as a triple \((C, \Delta, \varepsilon)\), where \( \Delta \) is called comultiplication and \( \varepsilon \) is called counit. The maps unlabeled are the canonical isomorphisms.

In general, the comultiplication of \( x \in C \) can be denoted by

\[
\Delta(x) = \sum_{(i)} x^1_i \otimes x^2_i.
\]

If

\[
\Delta(x) = \sum_{(i)} x^1_i \otimes x^2_i = \sum_{(i)} x^2_i \otimes x^1_i,
\]

then the coalgebra \( C \) is said to be cocommutative. We often need to apply iterations of the comultiplication \( \Delta \) on the elements of the coalgebra, however the notation used above is not very practical for these type of situations, so to simplify this and other processes, we will use Sweedler’s notation.

**Notation 2.3.** Let \( C \) be any coalgebra with comultiplication \( \Delta \), we will denote

\[
\Delta(x) = \sum_{(i)} x^1_i \otimes x^2_i
\]
by

\[ \Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)} \]

for all \( x \in C \). This notation is called *Sweedler’s notation*. The subscripts (1) and (2) are symbolic and do not indicate particular elements of \( C \).

Sweedler’s notation is often referred to as *Sweedler’s sigma notation* or simply as *sigma notation*. In this dissertation, we will further simplify the notation by suppressing the parenthesis in the subscripts, that is we will write

\[ \Delta(x) = \sum_{(x)} x_1 \otimes x_2. \]

The coassociativity diagram can be written as,

\[ \sum_{(x)} x_{1_1} \otimes x_{1_2} \otimes x_2 = \sum_{(x)} x_1 \otimes x_{2_1} \otimes x_{2_2}, \]

which in turn can be written as

\[ \Delta_2(x) = \sum_{(x)} x_1 \otimes x_2 \otimes x_3 \]

using Sweedler’s notation. In a similar way we can apply coassociativity more than once. This translates to applying the comultiplication \( n \) times to obtain

\[ \Delta_n(x) = \sum_{(x)} x_1 \otimes \cdots \otimes x_n \otimes x_{n+1}, \]

for all \( x \in C \). The counit diagram can be expressed as,

\[ \sum_{(x)} \varepsilon(x_1)x_2 = x = \sum_{(x)} \varepsilon(x_2)x_1 \]

for all \( x \in C \).

**Definition 2.4.** Let \((C, \Delta_C, \varepsilon_C)\) and \((D, \Delta_D, \varepsilon_D)\) be any two coalgebras with the given structural maps. Then a map \( f : C \to D \) is a *coalgebra morphism* if

\[ \Delta_D \circ f = (f \otimes f) \Delta_C \]
\[ \varepsilon_C = \varepsilon_D \circ f. \]

**Definition 2.5.** A vector space \( B \) over \( k \) is a bialgebra if there exist maps \( \mu, \eta, \Delta \) and \( \varepsilon \) such that \( (B, \mu, \eta) \) is an algebra and \( (B, \Delta, \varepsilon) \) is a coalgebra. Furthermore, the maps \( \Delta \) and \( \varepsilon \) must be algebra morphisms or equivalently, \( \mu \) and \( \eta \) must be coalgebra morphisms.

For any \( k \)-space \( V \), let \( V^* \) denote the linear dual of \( V \), that is \( V^* = \text{Hom}(V, k) \).

**Example 2.6.** If \( (A, \mu, \eta) \) is a finite-dimensional algebra, then its dual, \( A^* \), is a coalgebra with comultiplication \( \Delta = \mu^* \) and counit \( \varepsilon = \eta^* \). Explicitly, if \( f \in A^* \), then
\[
\Delta(f)(a \otimes b) = \sum_{(f)} f_1(a) f_2(b) = f(ab)
\]
for all \( a \in A \). The counit can also be written explicitly, \( \varepsilon(f) = f(1_A) \). Analogously, if \( (C, \Delta, \varepsilon) \) is a coalgebra, then \( C^* \) is an algebra with multiplication map \( \mu = \Delta^* \) and unit \( \eta = \varepsilon^* \). This means, that for \( f, g \in C^* \), we have
\[
(fg)(x) = \sum_{(x)} f(x_1)g(x_2)
\]
for all \( x \in C \). Furthermore, \( 1_{C^*} = \varepsilon \).

Let \( (A, \mu, \eta) \) be an algebra and \( (C, \Delta, \varepsilon) \) be coalgebra. Then the set of \( k \)-linear maps from \( C \) to \( A \) is denoted by \( \text{Hom}(C, A) \).

**Definition 2.7.** If \( f \) and \( g \) are linear maps from a coalgebra \( C \) to and algebra \( A \), that is \( f, g \in \text{Hom}(C, A) \), then the convolution product \( f \ast g \) is the composition of the maps
\[
C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{\mu} A.
\]
In Sweedler’s sigma notation, we have

\[(f \ast g)(x) = \sum_{(x)} f(x_1)g(x_2)\]

for all \(x \in C\).

**Example 2.8.** Let \(A\) and \(C\) be an algebra and a coalgebra, respectively. Then \(\text{Hom}(C, A)\) is an algebra with the convolution product \(\ast\) and the unit is \(\eta \circ \varepsilon\).

Let \((B, \mu, \eta, \Delta, \varepsilon)\) be a bialgebra, then \(\text{Hom}(B, B)\) is an algebra with the structure just described in Example 2.8. Furthermore, the map \(\text{Id}_B \in \text{Hom}(B, B)\) is an invertible element in \(\text{Hom}(B, B)\) if and only if there exists a map \(S \in \text{Hom}(B, B)\) such that

\[S \ast \text{Id}_B = \eta \circ \varepsilon = \text{Id}_B \ast S\]

that is,

\[x = \sum_{(x)} S(x_1)x_2 = \varepsilon(x)1_B = \sum_{(x)} x_1S(x_2),\]

for all \(x \in B\).

**Definition 2.9.** A *Hopf algebra* is a vector space \(H\) endowed with an algebra structure (i.e. has multiplication \(\mu\), unit map \(\eta\), a coalgebra structure, (i.e. it has a comultiplication \(\Delta\), counit \(\varepsilon\)) and antipode \(S\), where the two structures are compatible. That is, the maps \(\Delta\) and \(\varepsilon\) are algebra morphisms and there exists a map \(S : H \rightarrow H\) such that \(S\) is the inverse to the identity map \(\text{Id}_H\) with respect to the convolution product in \(\text{Hom}(H, H)\). In Sweedler’s notation:

\[\sum_{(h)} S(h_1)h_2 = \sum_{(h)} h_1S(h_2) = \varepsilon(h)1,\]

where

\[\Delta(h) = \sum_{(h)} h_1 \otimes h_2.\]

The map \(S\) is called an *antipode* for \(H\).
The antipodal map $S$ is defined as the inverse to the identity $\text{Id}_H$ under the convolution product, so if it exists it is unique. The map $S$ is an antimorphism. For the reminder of this dissertation, $S$ will be used to denote the antipodal map, unless otherwise specified. Furthermore, a Hopf algebra can also be defined as a a bialgebra with an antipode.

2.1.1 Examples

We will provide some examples of Hopf algebras.

Example 2.10 (Group Algebra). Let $G$ be a group and $k[G]$, the associated group algebra. The elements of the group $G$ form a basis for $k[G]$, so it is enough to define the structure maps for these elements and then extend linearly to $k[G]$. The set $k[G]$ is an algebra with multiplication map $\mu$, where

$$\mu : k[G] \otimes k[G] \to k[G]$$

is such that $\mu(g \otimes h) = gh$ for all $g, h \in G$. The unit map $\eta$ is given by

$$\eta : k \to k[G]$$

where $\eta(1) = 1_{k[G]} = 1_G$. Furthermore, the algebra $k[G]$ has a coalgebra structure. This structure is given by the comultiplication map $\Delta$ and counit $\varepsilon$,

$$\Delta : k[G] \to k[G] \otimes k[G],$$

such that $\Delta(g) = g \otimes g$ and

$$\varepsilon : k[G] \to k,$$

where $\varepsilon(g) = 1$, for all $g \in G$. 

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The algebra and coalgebra structures are compatible so $k[G]$ is a bialgebra. We can define the antipodal map $S$ on $k[G]$ by

$$S : k[G] \to k[G],$$

such that $S(g) = g^{-1}$ for $g \in G$. If we endow $k[G]$ with the structures described above, then $k[G]$ is a Hopf algebra. The group algebra $k[G]$ is cocommutative. Furthermore, the dimension of $k[G]$ is the order of the group, $|G|$. This means that $k[G]$ is a finite dimensional Hopf algebra when $G$ is a finite group.

In general, for any Hopf algebra $H$, if

$$\Delta(h) = h \otimes h$$

and

$$\varepsilon(h) = 1,$$

then we say that $h$ is a group like element of $H$. This will be denoted as $h \in G(H)$, where $G(H)$ denotes the set of group like elements in $H$.

The next example is a Hopf algebra that is not finite dimensional.

**Example 2.11** (Universal Enveloping Algebra $U(g)$ of a Lie Algebra $g$). Let $g$ be a Lie algebra with bracket $[\ , \ ]$. The universal enveloping algebra is defined as:

$$U(g) = T(g)/I,$$

where $T(g)$ is the tensor algebra of the $k$-vector space $g$, and $I$ is the ideal of $T(g)$ generated by $[x, y] - x \otimes y + y \otimes x$ with $x, y \in g$. The multiplication and unit maps arise from the usual tensor algebra multiplication. Let

$$\Delta : U(g) \to U(g) \otimes U(g),$$

where

$$\Delta(x) = x \otimes 1 + 1 \otimes x,$$
for all \( x \in \mathfrak{g} \). Then \( \Delta \) is a comultiplication for \( U(\mathfrak{g}) \). The counit map is given by

\[
\varepsilon : U(\mathfrak{g}) \to k,
\]

such that

\[
\varepsilon(x) = 0,
\]

where \( x \in \mathfrak{g} \). Furthermore, define

\[
S : U(\mathfrak{g}) \to U(\mathfrak{g}),
\]

where

\[
S(x) = -x,
\]

for \( x \in \mathfrak{g} \), then \( S \) is the antipode map. The universal enveloping algebra \( U(\mathfrak{g}) \), is a Hopf algebra with the maps given above. Furthermore, \( U(\mathfrak{g}) \) is cocommutative.

**Example 2.12.** Let \( H_8 \) be the Hopf algebra which is generated as an algebra by \( x, y \) and \( z \), with relations:

\[
x^2 = y^2 = 1,
\]

\[
z^2 = \frac{1}{2} (1 + x + y - xy),
\]

\[
xy = yx,
\]

\[
xz = zy,
\]

and

\[
yz = zx.
\]

The coalgebra structure of \( H_8 \) is given by the following:

\[
\Delta(x) = x \otimes x, \varepsilon(x) = 1, \text{ and } S(x) = x,
\]

\[
\Delta(y) = y \otimes y, \varepsilon(y) = 1, \text{ and } S(y) = y,
\]
\[ \Delta(z) = \frac{1}{2} (1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z), \varepsilon(z) = 1, \text{ and } S(z) = z. \]

It follows that \( H_8 \) is a semisimple 8-dimensional Hopf algebra over an algebraically closed field \( k \) of characteristic not equal to 2. The basis elements for \( H_8 \) are: \( 1, x, y, xy, z, xz, yz, \) and \( xyz \). Furthermore, \( H_8 \) is neither commutative nor cocommutative. This example was first introduced by Kac and Paljutkin in [KP66] and revisited later by Masuoka in [Mas95].

We will use Example 2.12 extensively in this dissertation, particularly in Chapter 4.

### 2.2 Hopf Algebra Representations

In this section, we review the notion of Hopf modules summarizing the definitions on [Mon93] and [DNR00]. Let \( H \) be a Hopf algebra with structure maps \( \mu, \eta, \Delta, \varepsilon \) and antipode \( S \). When we refer to a linear map, this implies a \( k \)-linear map.

#### 2.2.1 Modules and Comodules

**Definition 2.13.** Let \( A \) be an algebra over \( k \). Then a (left) \( A \)-module is a \( k \)-space \( M \) with a linear map \( \gamma : A \otimes M \rightarrow M \) such that the following diagrams commute:

\[
\begin{array}{ccc}
A \otimes A \otimes M & \xrightarrow{\mu \otimes \text{Id}} & A \otimes M \\
\text{Id} \otimes \gamma & & \gamma \\
A \otimes M & \xrightarrow{\gamma} & M
\end{array}
\]

and

\[
\begin{array}{ccc}
k \otimes M & \xrightarrow{\eta \otimes \text{Id}} & A \otimes M \\
\sim & & \gamma \\
& & M
\end{array}
\]

We now dualize these diagrams to obtain the notion of a comodule.
Definition 2.14. Let $C$ be a $k$-coalgebra. Then a (right) $C$-comodule is a $k$-space $N$ with a linear map $\rho : N \to N \otimes C$ such that the following diagrams commute:

\[
\begin{array}{ccc}
N & \xrightarrow{\rho} & N \otimes C \\
\downarrow{\rho} & & \downarrow{\text{Id} \otimes \Delta} \\
N \otimes C & \xrightarrow{\rho \otimes \text{Id}} & N \otimes C \otimes C
\end{array}
\]

and

\[
\begin{array}{ccc}
N & \xrightarrow{\rho} & N \otimes C \\
\sim & & \downarrow{\text{Id} \otimes \varepsilon} \\
N \otimes k & & 
\end{array}
\]

Similarly, one can define left comodules over a coalgebra $C$, the difference is that the structure map of a left $C$-comodule $N$ is of the form $\rho : N \to C \otimes N$. The map $\rho$ must satisfy the conditions that $(\Delta \otimes \text{Id})\rho = (\text{Id} \otimes \rho)\rho$ and that $(\varepsilon \otimes \text{Id})\rho$ is the canonical isomorphism.

2.2.2 Hopf Modules

Definition 2.15. A $k$-vector space $M$ is called a (right) $H$-Hopf module if $H$ has a right $H$-module structure (the action of an element $h \in H$ on an element $m \in M$ will be denoted by $mh$, and a right $H$-comodule structure, given by the map $\rho : M \to M \otimes H$, where $\rho(m) = \sum m_{(0)} \otimes m_{(1)}$ such that for any $m \in M$ and $h \in H$,

\[
\rho(mn) = \sum m_{(0)} h_1 \otimes m_{(1)} h_2.
\]

In this dissertation when we write $H$-module we mean a left $H$-module, and $\text{Irr}(H)$ will be used to denote the set of irreducible representations of $H$. We will now consider the invariants of a Hopf algebra $H$.

Definition 2.16. Let $M$ be a left $H$-module. Then the invariants of $H$ on $M$ are

\[
M^H = \{m \in M | h \cdot m = \varepsilon(h)m, \forall h \in H\}.
\]
Proposition 2.17. Let $V$ and $W$ be left $H$-modules. Then, $V \otimes W$ is also a left $H$-module, where

$$h \cdot (v \otimes w) = \sum_{(h)} h_1 \cdot v \otimes h_2 \cdot w,$$

for all $h \in H, v \in V$ and $w \in W$.

Lemma 2.18 (Schur’s Lemma). Let $V$ and $W$ be simple left $H$-modules. Then,

$$\dim(V \otimes W^*)^H = \begin{cases} 1 & \text{if } V \simeq W \\ 0 & \text{if } V \not\simeq W. \end{cases}$$

Proposition 2.19 (Schneider p. 42, [Sch95]). Let $V$ and $W$ be left $H$-modules, then

$$\text{Hom}_H(V, W) = \text{Hom}(V, W)^H,$$

the isotypic component of trivial type in $\text{Hom}(V, W)$.

## 2.3 Integrals

Let $H$ be a finite dimensional, semisimple Hopf algebra over an algebraically closed field $k$ of characteristic 0. When dealing with semisimple Hopf algebras we will think of $H$ semisimple as having every (left) $H$-module completely reducible. Furthermore, $H$ finite dimensional and semisimple implies that the antipode map is of order 2.

Definition 2.20. An element $\Lambda \in H$ is called a left integral in $H$ if $h\Lambda = \varepsilon(h)\Lambda$ for all $h \in H$. Similarly, $\Lambda \in H$ is a right integral if $\Lambda h = \varepsilon(h)\Lambda$, for all $h \in H$. If $\Lambda \in H$ is both a left and a right integral we say that $\Lambda$ is an integral in $H$.

Remark 2.21. There is a Hopf algebra version of Maschke’s Theorem that implies that there is a unique two sided integral $\Lambda \in H$ such that $\varepsilon(\Lambda) = 1$. This unique integral $\Lambda$ is cocommutative, that is

$$\Delta(\Lambda) = \sum_{(\Lambda)} \Lambda_1 \otimes \Lambda_2 = \sum_{(\Lambda)} \Lambda_2 \otimes \Lambda_1.$$
Example 2.22. Let $G$ be a finite group, then

$$\Lambda' = \sum_{g \in G} g$$

is an integral in $k[G]$. Furthermore, if we normalize this integral, that is, let

$$\Lambda = \frac{1}{|G|} \sum_{g \in G} g,$$

then $\varepsilon(\Lambda) = 1$.

Example 2.23. Let $H_8$ be the Hopf algebra described in Example 2.12. Then,

$$\Lambda = \frac{1}{8} (1 + x + y + xy + z + xz + yz + xyz)$$

is a two sided integral in $H_8$ and $\varepsilon(\Lambda) = 1$.

2.4 Frobenius–Schur Indicators for Groups

In this section we review the classical Frobenius–Schur (FS) indicator for groups as in [Isa76], [FH91] and [Ser77]. Let $G$ be a compact group, and $V$ be a $G$-module with character $\chi$. Then, the Frobenius–Schur indicator $\nu_2(\chi)$ of a character of a representation of $G$ is defined as the virtual character

$$\nu_2(\chi) = \int_G \chi(g^2)dg.$$  \hfill (2.1)

For finite groups, (2.1) is just

$$\nu_2(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2).$$

In 1906, Frobenius and Schur in [FS06] proved the following theorem.

Theorem 2.24 (Partition of irreducible representations). Let $V$ be an irreducible representation of $G$ with character $\chi$. Then,

$$\nu_2(\chi) = \begin{cases} 1 & \text{if } V \text{ is real}, \\ 0 & \text{if } V \text{ is complex (i.e. not self dual)}, \\ -1 & \text{if } V \text{ is quaternionic}. \end{cases}$$
The subscript 2 is in fact a special case of a more general definition, however the case 2 is the one that was originally considered. If we let \( m \) be any natural number greater than or equal 2, then (2.1) can be defined as:

\[
\nu_m(\chi) = \int_G \chi(g^m) dg. \tag{2.2}
\]

In the case of a finite group \( G \), (2.2) becomes:

\[
\nu_m(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^m). \tag{2.3}
\]

### 2.4.1 Twisted Frobenius–Schur Indicators for Groups

Mackey in 1958, introduced a twisted version of the classical Frobenius–Schur indicator for finite groups. If \( G \) is a finite group with an anti-involution \( i \), then the character formula

\[
\tilde{\nu}_2(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(i(g^{-1})) \tag{2.4}
\]

provides the twisted Frobenius–Schur indicator. If we let \( i \) be the inverse map, then equation (2.4) matches up with equation (2.1).

It is again the case, that these are virtual characters which take the value of 1, 0, or \(-1\), thus providing a twisted version of the partition of irreducible representations into real, complex and quaternionic. The main use was to study multiplicity free representations. Just as in the case of the classical FS indicators, there is a notion twisted higher order FS indicators for groups. This has been done in [BG04] Bump and Ginzburg in 2004 using a group automorphism of order \( m \).

### 2.5 Frobenius–Schur Indicators for Hopf Algebras

In 2000, Linchenko and Montgomery in [LM00] generalized the notion of FS indicators to Hopf algebras. In this section we follow their construction and state some of the main results.
Let $k$ be an algebraically closed field of characteristic 0, and let $H$ be a finite dimensional, semisimple Hopf algebra over $k$ with comultiplication $\Delta$, counit $\varepsilon$, and antipode $S$. The Hopf algebra contains a unique two sided integral $\Lambda$ such that $\varepsilon(\Lambda) = 1$.

**Definition 2.25.** Let $V$ be a representation of $H$ with character $\chi$. Then for any natural number $m$, define the $m$-th Frobenius–Schur indicator of $V$ as

$$
\nu_m(\chi) = \sum_{(\Lambda)} \chi(\Lambda_1 \Lambda_2 \cdots \Lambda_m),
$$

(2.5)

where $\Delta^{m-1}(\Lambda) = \sum_{(\Lambda)} \Lambda_1 \otimes \Lambda_2 \otimes \cdots \otimes \Lambda_m$.

**Example 2.26.** For $H = k[G]$, the group algebra of a finite group $G$. Recall, $\Lambda = \frac{1}{|G|} \sum g$, then $\Delta(\Lambda) = \frac{1}{|G|} \sum g \otimes g$. Thus,

$$
\nu_m(\chi) = \frac{1}{|G|} \sum \chi(g^m).
$$

This coincides with equation (2.3), the indicators for finite groups. It is again the case that the FS indicators provide a partition of the irreducible representations of $H$ as shown in the theorem below.

**Theorem 2.27** (Theorem 3.1 on [LM00]). Let $H$, $\Lambda$ and $\nu_2(\chi)$ as above. If $V$ is the $H$-module corresponding to $\chi \in \text{Irr}(H)$, then the following properties hold:

1. $\nu_2(\chi) = 0, 1, \text{ or } -1, \forall \chi \in \text{Irr}(H)$.

2. $\nu_2(\chi) \neq 0$ if and only if $V \cong V^*$. Moreover, $\nu_2(\chi) = 1$ (resp. $-1$) if and only if $V$ admits a symmetric (resp. skew-symmetric) nondegenerate bilinear $H$-invariant form.
2.6 Higher Order Frobenius–Schur Indicators for Hopf Algebras

In 2006, Kashina, Sommerhäuser and Zhu in [KSZ06], have studied the higher order Frobenius–Schur indicators $\nu_m(\chi)$ where $m > 2$ as defined by Linchenko and Montgomery in [LM00]. They have found several formulas for $\nu_m(\chi)$ and used these to prove various results dealing with FS indicators. We illustrate their work by describing their first formula.

Let $\rho : H \rightarrow \text{End}(V)$ be the corresponding representation to $\chi$ and let

$$\alpha : V^\otimes m \rightarrow V^\otimes m, v_1 \otimes v_2 \otimes \cdots \otimes v_m \mapsto v_2 \otimes v_3 \otimes \cdots \otimes v_m \otimes v_1.$$ 

Denote by $\rho^m : H \rightarrow \text{End}(V^\otimes m)$ the representation obtained from the diagonal action of $H$ on the $m$-th tensor power. They show that

$$\nu_m(\chi) = \text{tr}_{V^\otimes m} (\alpha \circ \rho^m(\Lambda)).$$

The action of $\Lambda$ projects $V^\otimes m$ into its $H$-invariants and $\alpha$ preserves this subspace. They obtain a trace formula to compute the indicators given by the following theorem.

**Theorem 2.28.**

$$\nu_m(\chi) = \text{tr} \left( \alpha_{|_{(V^\otimes m)^H}} \right).$$ (2.6)
Chapter 3
Twisted Frobenius–Schur (FS) Indicators

In this chapter, we provide a construction of twisted FS indicators for Hopf algebras generalizing the results for groups and Hopf algebras illustrated in Chapter 2. We construct twisted Frobenius–Schur indicators for semisimple Hopf algebras over an algebraically closed field of characteristic zero. Given an automorphism of order $n$ of such a Hopf algebra, we define the $m$-th twisted Frobenius–Schur indicator for any positive multiple of $n$. This definition is given in Section 3.1. In Section 3.2, we consider the case of automorphisms of order at most two. We show that the second twisted Frobenius–Schur indicator gives rise to a partition of the simple modules into three classes; this partition involves the relationship between the module and its “twisted dual” (Theorem 3.4). In Section 3.3, we show that the $m$-th twisted Frobenius–Schur indicator can be realized as the trace of an endomorphism of order $m$ (Theorem 3.19), so that the indicator is a cyclotomic integer.

3.1 Twisted FS Indicators for Hopf Algebras

Let $k$ be an algebraically closed field of characteristic 0, and let $H$ be a finite dimensional, semisimple Hopf algebra over $k$ with comultiplication $\Delta$, counit $\varepsilon$, and antipode $S$. The Hopf algebra contains a unique two sided integral $\Lambda$ such that $\varepsilon(\Lambda) = 1$. All $H$-modules considered will be finite-dimensional. We are now ready to define the twisted indicators.

Let $\tau$ be an automorphism of $H$ such that $\tau^m = \text{Id}$ for some $m \in \mathbb{N}$. Let $(V, \rho)$ be an $H$-module with corresponding character $\chi$. 

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**Definition 3.1.** The *m*-th twisted Frobenius–Schur indicator of \((V, \rho)\) (or \(\chi\)) is defined to be the character sum

\[
\nu_m(\chi, \tau) = \sum_{(\Lambda)} \chi \left( \Lambda_1 \tau (\Lambda_2) \cdots \tau^{m-1} (\Lambda_m) \right).
\]  

We note that this is only defined for \(m\) divisible by the order of \(\tau\). We will write \(\tilde{\nu}_m(\chi)\) instead of \(\nu_m(\chi, \tau)\) when this does not cause confusion.

If \(\tau = \text{Id}\), this formula coincides with the definition of Linchenko and Montgomery [LM00]. Moreover, suppose \(H = k[G]\) for a finite group \(G\). In this case, \(\Lambda = \frac{1}{|G|} \sum_{g \in G} g\), and we recover Bump and Ginzburg’s twisted Frobenius–Schur indicators for groups [BG04].

### 3.2 Second-Order Twisted FS Indicators

In this section, we will show that the second twisted Frobenius–Schur indicator gives rise to a partition of the irreducible \(H\)-modules into three classes, depending on the relationship between the module and its twisted dual.

#### 3.2.1 Twisted Duals and the Partition of the Simple Modules

Let \(\tau\) be an automorphism such that \(\tau^2 = \text{Id}\). We will let \(T = \tau S\) denote the corresponding anti-involution. Let \((V, \rho)\) be a finite dimensional left \(H\)-module with character \(\chi\). Using (3.1) for \(m = 2\), we have

\[
\tilde{\nu}_2(\chi) = \sum_{(\Lambda)} \chi \left( \Lambda_1 TS(\Lambda_2) \right).
\]

**Definition 3.2.** The *twisted dual* \(H\)-module of \(V\) is the dual space \(V^*\) equipped with an \(H\)-module structure given by

\[
(h \cdot f)(v) = f(T(h) \cdot v),
\]

for all \(h \in H, f \in V^*\) and \(v \in V\). We will denote it by \((^*V, \tilde{\rho})\).
Proposition 3.3. The twisted dual $H$-module $(\ast V, \tilde{\rho})$ satisfies $\tilde{\rho}(h) = \rho(T(h))^t$.

Proof. For $h \in H, f \in \ast V$ and $v \in V$ we have,

$$\rho(T(h))^t(f)(v) = f(\rho(T(h))(v)) = \tilde{\rho}(h)f(v)$$

Thus, $\rho(T(h))^t = \tilde{\rho}(h)$, as required. \qed

We will now state the main theorem for the second twisted FS indicators.

Theorem 3.4. Let $V$ be an irreducible representation with character $\chi$. Then the following properties hold:

1. $\tilde{\nu}_2(\chi) = 0, 1, \text{ or } -1, \forall \chi \in \text{Irr}(H)$.

2. $\tilde{\nu}_2(\chi) \neq 0$ if and only if $V \cong \ast V$. Moreover, $\tilde{\nu}_2(\chi) = 1$ (resp. $-1$) if and only if there is a symmetric (resp. skew-symmetric) nonzero intertwining map $V \to \ast V$.

If we let $T = S$, that is $\tau = \text{Id}$, then Theorem 3.4 coincides with Theorem 3.1 in [LM00]. On the other hand, if $H$ is a group algebra, then is a theorem of Sharp [Sha60] and Kawanaka-Matsuyama [KM90]. See also [KS08]. We will prove some preliminary results, before proceeding to the proof of Theorem 3.4.

Let $\Lambda^*$ be an integral for $H^*$, then $\Lambda^* \in H^{**} \simeq H$. This implies that $\Lambda^* = \text{ev}(\Lambda)$ for some $\Lambda \in H$, which can be chosen such that $\varepsilon(\Lambda) = 1$, where $\text{ev}$ is the evaluation map. From Theorem 7.5.6 in [DNR00] we have,

$$\Lambda^* ((\chi_i) (S^* \chi_j)) = \delta_{ij} \Lambda^* (\varepsilon),$$

where $\chi_1, \ldots, \chi_n$ denote the irreducible characters of $H$. The equation above can be rewritten as,

$$\sum \chi_i (\Lambda_1) \chi_j (S (\Lambda_2)) = \delta_{ij}. \quad (3.2)$$
Proposition 3.5. There is a canonical isomorphism of $H$-modules $V \to \ast \ast V$.

Proof. Let $\Psi : V \to \ast \ast V$ be the usual evaluation map, $\Psi(v)(f) = f(v)$. Then, $\Psi$ is a linear isomorphism. It remains to show that $\Psi$ is an $H$-map. So for all $h \in H, f \in \ast V$ and $v \in V$ we have,

$$
(h \cdot \Psi(v))(f) = \Psi(v)(T(h) \cdot f) \\
= (T(h) \cdot f)(v) \\
= f(T^2(h) \cdot v) \\
= f(h \cdot v) \\
= \Psi(h \cdot v)(f).
$$

Therefore, $(h \cdot \Psi(v)) = \Psi(h \cdot v)$.  

We now provide a definition for twisted module morphisms.

Definition 3.6. If $f : V \to W$ is a morphism of finite dimensional left $H$-modules, define $^*f : ^*W \to ^*V$ by $(^*f(\beta))(v) = \beta(f(v))$, for $\beta \in ^*W, v \in V$.

Proposition 3.7. Twisted duality is an involutory auto-equivalence of the category of $H$-modules.

Proof. Let us first show that the map $^*f$ is a morphism of left $H$-modules. Given $\alpha \in ^*V, h \in H$, and $v \in V$, then it suffices to show that $^*f(h \cdot \alpha) = h \cdot ^*f(\alpha)$. We have

$$
^*f(h \cdot \alpha)(v) = (h \cdot \alpha)(f(v)) \\
= \alpha(T(h) \cdot f(v)) \\
= \alpha(f(T(h) \cdot v)) \\
= ^*f(\alpha)(T(h) \cdot v) \\
= (h \cdot ^*f(\alpha))(v).
$$
So, $f^*$ is a morphism of left $H$-modules as desired. It is obvious that twisted duality is a functor. The fact that it is an involutory auto-equivalence follows from Proposition 3.5.

**Proposition 3.8.** Let $V$ be a simple left $H$-module. Then $V^*$ is also simple.

**Proof.** Let $X$ be a submodule of $V^*$, and $f : X \to V^*$ be the inclusion map. Then, $f^* : V^* \to X$ is a surjective morphism of left $H$-modules. We know that $V \cong V^*$, so since $V$ is simple, it follows that $V$ is also simple. This implies that $f^* = 0$ or $f^*$ is an isomorphism. If $f^* = 0$, then $X = 0$, so $X = 0$. In the latter case, it follows that $f = \text{Id}$ and $X = V$. Thus, $X = 0$ or $X = V^*$, so $V^*$ is simple. □

**Lemma 3.9.** If $f \in \text{Hom}_H(V, V)$, then $f^t = \Psi \circ f^* : V^* \to V$ is an $H$-map.

**Proof.** Recall that $\Psi : V \to V^*$ is the canonical isomorphism. We know that $f^*$ and $\Psi$ are both $H$-maps, so it follows that so is $f^t$. □

The standard decomposition of $\text{Hom}(V, V)$ into symmetric and anti-symmetric linear maps is no longer an $H$-decomposition. In fact, this is not true even when $\tau = \text{Id}$ unless $H$ is cocommutative. However, the corresponding decomposition does hold for $H$-invariant maps.

It is well known (for example, see [Sch95, p. 42]) that for left $H$-modules $V$ and $W$, then

$$\text{Hom}_H(V, W) = \text{Hom}(V, W)^H,$$

the isotypic component of trivial type in $\text{Hom}(V, W)$. This implies that in particular,

$$\text{Hom}_H(V^*, V) = \text{Hom}(V^*, V)^H.$$

Since $\text{Hom}(V^*, V)^H$ is the isotypic component of trivial type in $\text{Hom}(V^*, V)$ then it follows that $\text{Sym}_H(V^*, V)$ and $\text{Alt}_H(V^*, V)$ are $H$-submodules of $\text{Hom}_H(V^*, V)$,
where
\[ \text{Sym}_H(*V, V) = \{ f \in \text{Hom}_H(*V, V) | f^t = f \} \]
and
\[ \text{Alt}_H(*V, V) = \{ f \in \text{Hom}_H(*V, V) | f^t = -f \} \]
where \( f^t = \Psi \circ *f \).

**Proposition 3.10.** Let \( V \) be a finite dimensional simple left \( H \)-module, then
\[ \text{Hom}_H(*V, V) = \text{Sym}_H(*V, V) \oplus \text{Alt}_H(*V, V). \]

**Proof.** We know that \( \text{Hom}_H(*V, V) = \text{Hom}(*V, V)^H \) and
\[ \text{Sym}_H(*V, V) \cap \text{Alt}_H(*V, V) = \{0\}. \]

Take \( f \in \text{Hom}_H(*V, V) \). The previous lemma shows that \( f^t \) is an \( H \)-map, so
\[ f = \frac{f + f^t}{2} + \frac{f - f^t}{2}, \]
with \( \frac{f + f^t}{2} \in \text{Sym}_H(*V, V) \) and \( \frac{f - f^t}{2} \in \text{Alt}_H(*V, V) \). Furthermore, \( \text{Sym}_H(*V, V) \) and \( \text{Alt}_H(*V, V) \) are \( H \)-submodules of \( \text{Hom}_H(*V, V) \), since \( \text{Hom}_H(*V, V) \) is the isotypic component of trivial type in \( \text{Hom}(*V, V) \).

**Corollary 3.11.** If \( V \) is a simple \( H \)-module, then
\[ \dim \text{Sym}_H(*V, V) - \dim \text{Alt}_H(*V, V) \in \{1, 0, -1\}. \]
Moreover, it takes each value according to the conditions given in Theorem 3.4.

**Proof.** By Schur’s Lemma, we have that
\[ \dim \text{Sym}_H(*V, V) = \begin{cases} 1 & \text{if } *V \simeq V \\ 0 & \text{if } *V \not\simeq V \end{cases} \]
and
\[
\dim \text{Alt}_H(*V,V) = \begin{cases} 
1 & \text{if } *V \simeq V \\
0 & \text{if } *V \not\simeq V.
\end{cases}
\]
These imply that \( \dim \text{Sym}_H(*V,V) - \dim \text{Alt}_H(*V,V) \in \{1, 0, -1\} \).

Theorem 3.4 will now follow from the following proposition.

**Proposition 3.12.** Let \( V \) be a simple \( H \)-module with character \( \chi \). Then
\[
\tilde{\nu}_2(\chi) = \dim \text{Sym}_H(*V,V) - \dim \text{Alt}_H(*V,V). \tag{3.3}
\]

**Proof.** We will compute matrix elements in terms of a fixed basis for \( V \) and the dual basis for \(*V\). This means that elements of \( \text{Sym}(*V,V) \) (resp. \( \text{Alt}(*V,V) \)) will be symmetric (resp. skew-symmetric). We temporarily denote the expression on the right side of (3.3) by \( q(\chi) \).

Writing out \( \tilde{\nu}_2(\chi) \) gives
\[
\tilde{\nu}_2(\chi) = \sum_{(\Lambda)} \chi(\Lambda_1 T(S(\Lambda_2)))
= \sum_{(\Lambda)} \text{tr}(\rho(\Lambda_1 T(S(\Lambda_2))))
= \sum_{(\Lambda)} \text{tr}(\rho(\Lambda_1) T(S(\Lambda_2)))
= \sum_{m, m'} \sum_{(\Lambda)} \rho(\Lambda_1)_{mm'} T(S(\Lambda_2))_{m'm}
= \sum_{m, m'} \sum_{(\Lambda)} \rho(\Lambda_1)_{mm'} T(S(\Lambda_2))_{m'm}^t
= \sum_{m, m'} \sum_{(\Lambda)} \rho(\Lambda_1)_{mm'} \tilde{\rho}(S(\Lambda_2))_{mm'}.
\]

If \( V \not\simeq *V \), then this expression is 0 by the orthogonality relations (3.2), hence coincides with \( q(\chi) \). Otherwise, there exists a nonzero intertwiner \( \varphi \in \text{Hom}_H(*V,V) \), so that \( \tilde{\rho}(h) = \varphi^{-1} \rho(h) \varphi \). By Proposition 3.10, \( \varphi \) is symmetric or skew-symmetric, and in fact, \( \varphi_{nm} = q(\chi) \varphi_{nm} \) for all \( n, m \). Applying these two facts to the previous equation, we have
\[ \tilde{\nu}_2(\chi) = \sum_{m,m'} \sum_{(\Lambda)} \rho(\Lambda_1)_{mm'} \tilde{\rho}(S(\Lambda_2))_{mm'} \]

\[ = \sum_{m,m'} \sum_{(\Lambda)} \rho(\Lambda_1)_{mm'} (\varphi^{-1} \rho(S(\Lambda_2)) \varphi)_{mm'} \]

\[ = \sum_{m,m',n,n'} \sum_{(\Lambda)} \rho(\Lambda_1)_{mm'} (\varphi^{-1})_{mn} \rho(S(\Lambda_2))_{nn'} (\varphi)_{n'm'} \]

\[ = \sum_{m,m',n,n'} (\varphi^{-1})_{mn} (\varphi)_{n'm'} \sum_{(\Lambda)} \rho(\Lambda_1)_{mm'} \rho(S(\Lambda_2))_{nn'} \]

\[ = \sum_{m,m',n,n'} (\varphi^{-1})_{mn} (\varphi)_{n'm'} \frac{\delta_{n'n} \delta_{m,n'}}{\dim V} \]

\[ = \sum_{m,n} (\varphi^{-1})_{mn} (\varphi)_{mn} \frac{1}{\dim V} \]

\[ = \frac{q(\chi)}{\dim V} \sum_{m,n} (\varphi^{-1})_{mn} \varphi_{nm} = q(\chi), \]

as desired. The fifth equality follows from the orthogonality relations for matrix elements given in [Lar71]. The second to last equality follows from the orthogonal relations for matrix elements given in [Lar71].

Thus,

\[ \tilde{\nu}_2(\chi) = \dim \text{Sym}_H(\star V, V) - \dim \text{Alt}_H(\star V, V), \]

as desired. \[\square\]

### 3.3 Higher-Order FS Indicators

We now return to the general case. When \( m > 2 \), it is no longer true that the higher order twisted Frobenius–Schur indicators are integers. However, it is true that they are cyclotomic integers. We will show this by realizing \( \tilde{\nu}_m(\chi) \) as the trace of an endomorphism of order \( m \). We will also compute a closed formula for the regular representation.
3.3.1 A Trace Formula

Let $\tilde{V} \otimes m$ be the vector space $V \otimes m$, with action given by

$$\tilde{\rho}^m : H \to \text{End} \left( V \otimes m \right)$$

where

$$\tilde{\rho}^m (h) (v_1 \otimes v_2 \otimes \cdots \otimes v_m) = \sum_{(h)} \rho(h_1) v_1 \otimes \rho(h_2) v_2 \otimes \cdots \otimes \rho(\tau^{m-1}(h_m)) v_m.$$ 

Furthermore, let $\alpha : V \otimes m \to V \otimes m$ be defined by

$$\alpha(v_1 \otimes v_2 \otimes \cdots \otimes v_m) = v_2 \otimes \cdots \otimes v_m \otimes v_1.$$ 

**Lemma 3.13.** If $\tilde{\nu}_m(\chi)$ and $\alpha$ are as defined above, then

$$\tilde{\nu}_m(\chi) = \text{tr}_{V \otimes m} \left( \alpha \circ \tilde{\rho}^m(\Lambda) \right).$$

**Proof.** Let us first consider $\tilde{\nu}_m(\chi)$, we have

$$\tilde{\nu}_m(\chi) = \sum_{(\Lambda)} \chi \left( \Lambda_1 \tau(\Lambda_2) \cdots \tau^{m-1}(\Lambda_m) \right)$$

$$= \sum_{(\Lambda)} \text{tr}_V \left( \rho(\Lambda_1) \rho(\tau(\Lambda_2)) \cdots \rho(\tau^{m-1}(\Lambda_m)) \right)$$

$$= \text{tr}_{V \otimes m} \left( \alpha \circ \left( \rho \otimes \rho \tau \otimes \cdots \otimes \rho(\tau^{m-1}) \right) \Lambda \right)$$

$$= \text{tr}_{V \otimes m} \left( \alpha \circ \tilde{\rho}^m(\Lambda) \right).$$

The third equality uses Lemma 2.3 from [KSZ02].

It is well known that the integral $\Lambda$ in $H$ is cocommutative, i.e.

$$\Delta(\Lambda) = \sum_{(\Lambda)} \Lambda_1 \otimes \Lambda_2 = \sum_{(\Lambda)} \Lambda_2 \otimes \Lambda_1. \quad (3.4)$$

More generally, we have
Proposition 3.14. For any $m \in \mathbb{N}$, $\Delta^m(\Lambda)$ is invariant under cyclic permutations:

$$\Delta^m(\Lambda) = \sum_{(\Lambda)} \Lambda_1 \otimes \Lambda_2 \otimes \cdots \otimes \Lambda_{m+1} = \sum_{(\Lambda)} \Lambda_2 \otimes \cdots \otimes \Lambda_m \otimes \Lambda_{m+1} \otimes \Lambda_1.$$  

Proof. Let $m = 1$, is equation (3.4). Suppose the result is true for $m - 1$, that is,

$$\Delta^{m-1}(\Lambda) = \sum_{(\Lambda)} \Lambda_1 \otimes \Lambda_2 \otimes \cdots \otimes \Lambda_m = \sum_{(\Lambda)} \Lambda_2 \otimes \cdots \otimes \Lambda_m \otimes \Lambda_1.$$  

Then,

$$\Delta^m(\Lambda) = (\Delta \otimes I^{\otimes (m-1)}) (\Delta^{m-1}(\Lambda))$$

$$= (\Delta \otimes I^{\otimes (m-1)}) \left( \sum_{(\Lambda)} \Lambda_2 \otimes \Lambda_3 \otimes \cdots \otimes \Lambda_m \otimes \Lambda_1 \right)$$

$$= \sum_{(\Lambda)} \Delta(\Lambda_2) \otimes \Lambda_3 \otimes \cdots \otimes \Lambda_m \otimes \Lambda_1$$

$$= \sum_{(\Lambda)} \Lambda_2 \otimes \Lambda_3 \otimes \cdots \otimes \Lambda_m \otimes \Lambda_{m+1} \otimes \Lambda_1.$$  

\qed

Proposition 3.15. Let $\sigma$ be an automorphism of $H$. If $h \in H$ is a left integral, then so is $\sigma(h)$.

Proof. If $h \in H$ is a left integral, then

$$xh = \varepsilon(x)h$$

for all $x \in H$. Applying $\sigma$, we get

$$\sigma(xh) = \sigma(\varepsilon(x)h) = \varepsilon(x)\sigma(h),$$

(3.5)

for all $x \in H$. Let $y = \sigma(x)$, so that $x = \sigma^{-1}(y)$. Applying this change of variables, we can rewrite equation (3.5) as

$$y\sigma(h) = \varepsilon(\sigma^{-1}(y))\sigma(h) = \varepsilon(y)\sigma(h)$$

for all $y \in H$. Thus, $\sigma(h)$ is a left integral. \qed
An analogous proof can be used to show that if $h$ is a right integral, then so is $\sigma(h)$.

**Corollary 3.16.** If $\sigma$ is an automorphism of $H$, then $\sigma(\Lambda) = \Lambda$.

**Proof.** We know that $\Lambda$ is the unique integral such that $\varepsilon(\Lambda) = 1$. However, $\sigma(\Lambda)$ is another integral satisfying $\varepsilon(\sigma(\Lambda)) = \varepsilon(\Lambda) = 1$. \qed

**Lemma 3.17.**

$$
\sum_{(\Lambda)} \Lambda_1 \otimes \tau(\Lambda_2) \otimes \cdots \otimes \tau^{m-1}(\Lambda_m) = \sum_{(\Lambda)} \tau(\Lambda_2) \otimes \cdots \otimes \tau^{m-1}(\Lambda_m) \otimes \Lambda_1.
$$

**Proof.** By the previous corollary, $\Delta^{m-1}(\Lambda) = \Delta^{m-1}(\tau^{m-1}(\Lambda))$. Since $\tau^{m-1}$ is a coalgebra morphism, we get

$$
\sum_{(\Lambda)} \Lambda_1 \otimes \cdots \otimes \Lambda_m = \sum_{(\Lambda)} \tau^{-1}(\Lambda_1) \otimes \cdots \otimes \tau^{-1}(\Lambda_m)
$$

$$
= \sum_{(\Lambda)} \tau^{-1}(\Lambda_1) \otimes \cdots \otimes \tau^{-1}(\Lambda_m).
$$

Combining this equation with Proposition 3.14, we get

$$
\sum_{(\Lambda)} \Lambda_2 \otimes \Lambda_3 \otimes \cdots \otimes \Lambda_m \otimes \Lambda_1 = \sum_{(\Lambda)} \tau^{-1}(\Lambda_1) \otimes \tau^{-1}(\Lambda_2) \otimes \cdots \otimes \tau^{-1}(\Lambda_m).
$$

Applying $(\tau \otimes \tau^2 \otimes \cdots \otimes \tau^m)$, we obtain

$$
\sum_{(\Lambda)} \tau(\Lambda_2) \otimes \cdots \otimes \tau^{m-1}(\Lambda_m) \otimes \Lambda_1 = \sum_{(\Lambda)} \Lambda_1 \otimes \tau(\Lambda_2) \otimes \cdots \otimes \tau^{m-1}(\Lambda_m),
$$

as desired. \qed

It is well-known that the action of $\Lambda$ on an $H$-module $W$ gives a projection onto its invariants. Let $\pi: \widehat{V \otimes^m} \to \left( \widehat{V \otimes^m} \right)^H$ defined by $\pi(\Lambda) = \Lambda \cdot w$ be this projection for $W = \widehat{V \otimes^m}$.  

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Proposition 3.18. The endomorphism $\alpha$ restricts to an endomorphism of $((\tilde{V} \otimes m)^H)$. 

**Proof.** It is enough to show that $(\pi \circ \alpha)(w) = (\alpha \circ \pi)(w)$ for $w = v_1 \otimes \cdots \otimes v_m$.

Computing gives
\begin{align*}
(\pi \circ \alpha)(w) &= (\pi \circ \alpha)(v_1 \otimes \cdots \otimes v_m) \\
&= \pi(v_2 \otimes \cdots \otimes v_m \otimes v_1) \\
&= \sum_{(\Lambda)} \rho(\Lambda_1)v_2 \otimes \rho(\tau(\Lambda_2))v_3 \otimes \cdots \otimes \rho(\tau^{m-1}(\Lambda_m))v_1,
\end{align*}
and
\begin{align*}
(\alpha \circ \pi)(v) &= \alpha(\Lambda \cdot (v_1 \otimes \cdots \otimes v_m)) \\
&= \alpha\left(\sum_{(\Lambda)} \rho(\Lambda_1)v_1 \otimes \rho(\tau(\Lambda_2))v_2 \otimes \cdots \otimes \rho(\tau^{m-1}(\Lambda_m))v_m\right) \\
&= \sum_{(\Lambda)} \rho(\tau(\Lambda_2))v_2 \otimes \cdots \otimes \rho(\tau^{m-1}(\Lambda_m))v_m \otimes \rho(\Lambda_1)v_1.
\end{align*}

By Lemma 3.17, these two expressions are equal. \qed

**Theorem 3.19.**
\[ \tilde{\nu}_m(\chi) = \text{tr}\left(\alpha\big|_{(\tilde{V} \otimes m)^H}\right). \]

**Proof.** By Proposition 3.18, the image of $\alpha \circ \tilde{\nu}(h)$ is contained in $((\tilde{V} \otimes m)^H)$. Moreover, its restriction to $((\tilde{V} \otimes m)^H)$ coincides with the restriction of $\alpha$. The result now follows by Lemma 3.13. \qed

**Corollary 3.20.** Let $\zeta_m$ be a primitive $m$-th root of 1, then
\[ \tilde{\nu}_m(\chi) \in \mathbb{Z}[\zeta_m]. \]

**Proof.** The operator $\alpha$ is of order $m$, so its eigenvalues are $m^{th}$ roots of unity. It is now immediate from the theorem that the twisted indicators are cyclotomic integers. \qed
3.3.2 Closed Formula for the Regular Representation

We now realize the twisted Frobenius–Schur indicators of the regular representation as the trace of an explicit linear endomorphism of $H$. Let $\chi_R$ denote the character of the left regular representation.

Let $\omega^\tau_m : H \to H$ be the linear map defined by

$$
\omega^\tau_m(h) = \sum_{(h)} S(\tau^{m-1}(h^1)\tau^{m-2}(h^2)\cdots\tau^2(h_{m-2})\tau(h_{m-1})).
$$

**Theorem 3.21.**

$$
\tilde{\nu}_m(\chi_R) = \text{tr}(\omega^\tau_m).
$$

Before we proceed to the proof of this theorem, we will need two lemmas.

**Lemma 3.22.** If

$$
h^1 \otimes h^2 \otimes \cdots \otimes h^{m-1} \in H^{\otimes(m-1)}
$$

and

$$
\Delta^{(m)}(h^1) = \sum_{(h^1)} (h^1)_1 \otimes \cdots \otimes (h^1)_{m-1}
$$

then

$$
\sum \Lambda_1 h^1 \otimes \tau(\Lambda_2) h^2 \otimes \cdots \otimes \tau^{m-2}(\Lambda_{m-1}) h^{m-1} \otimes \tau^{m-1}(\Lambda_m)
$$

$$
= \sum \Lambda_1 \otimes \tau(\Lambda_2 S(h^1_{m-1})) h^2 \otimes \cdots \otimes \tau^{m-2}(\Lambda_{m-1} S(h^1_{m-2})) h^{m-1} \otimes \tau^{m-1}(\Lambda_m S(h^1_1)).
$$

**Proof.** From Lemma 1.2(b) p. 270 in [LR88] we have

$$
\sum \Lambda_1 h^1 \otimes \tau(\Lambda_2) = \sum \Lambda_1 \otimes \Lambda_2 S(h^1).
$$

Applying $\text{Id} \otimes \Delta^{m-1}$ to both sides of the equation above, we get

$$
\sum \Lambda_1 h^1 \otimes \Lambda_2 \otimes \cdots \otimes \Lambda_{m-1} \otimes \Lambda_m
$$

$$
= \sum \Lambda_1 \otimes \Lambda_2 S(h^1_{m-1}) \otimes \Lambda_2 S(h^1_{m-2}) \otimes \cdots \otimes \Lambda_{m-1} S(h^1_2) \otimes \Lambda_m S(h^1_1).
We then apply $\text{Id} \otimes \tau \otimes \tau^2 \otimes \cdots \otimes \tau^{m-1}$ to both sides to get

$$\sum \Lambda_1 h^1 \otimes \tau(\Lambda_2) \otimes \cdots \otimes \tau^{m-2}(\Lambda_{m-1}) \otimes \tau^{m-1}(\Lambda_m)$$

$$= \sum \Lambda_1 \otimes \tau(\Lambda_2 S(h^1_{m-1})) \otimes \cdots \otimes \tau^{m-2}(\Lambda_{m-1} S(h^1_2)) \otimes \tau^{m-1}(\Lambda_m S(h^1_1)).$$

Finally, by right multiplying by $1 \otimes h^2 \otimes \cdots \otimes h^{m-1} \otimes 1$ on both sides we get

$$\sum \Lambda_1 h^1 \otimes \tau(\Lambda_2) h^2 \otimes \cdots \otimes \tau^{m-2}(\Lambda_{m-1}) h^{m-1} \otimes \tau^{m-1}(\Lambda_m)$$

$$= \sum \Lambda_1 \otimes \tau(\Lambda_2 S(h^1_{m-1})) h^2 \otimes \cdots \otimes \tau^{m-2}(\Lambda_{m-1} S(h^1_2)) h^{m-1} \otimes \tau^{m-1}(\Lambda_m S(h^1_1)),$$

as desired.

Next, define a linear map $\psi : \tilde{H}^{\otimes (m-1)} \rightarrow \tilde{H}^{\otimes (m-1)}$ by

$$\psi \left( h^1 \otimes h^2 \otimes \cdots \otimes h^{m-1} \right)$$

$$= \sum \tau(S(h^1_{m-1})) h^2 \otimes \tau^2(S(h^1_{m-2})) h^3 \otimes \cdots \otimes \tau^{m-2}(S(h^1_2)) h^{m-1} \otimes \tau^{m-1}(S(h^1_1)).$$

**Lemma 3.23.** Let $\psi$ be the linear map defined above, then

$$\text{tr}(\psi) = \text{tr} \left( \alpha \big|_{\tilde{V}^{\otimes m}} \right).$$

**Proof.** To prove the lemma, it suffices to find a linear isomorphism

$$\varphi : \tilde{H}^{\otimes (m-1)} \rightarrow \left( H \otimes \tilde{H}^{\otimes (m-1)} \right)^H$$

making the diagram

$$\begin{array}{ccc}
\tilde{H}^{\otimes (m-1)} & \xrightarrow{\psi} & \tilde{H}^{\otimes (m-1)} \\
\downarrow \varphi & & \downarrow \varphi \\
\left( H \otimes \tilde{H}^{\otimes (m-1)} \right)^H & \xrightarrow{\alpha} & \left( H \otimes \tilde{H}^{\otimes (m-1)} \right)^H
\end{array}$$

commute. Recall that for any $H$-module $W$, there is a linear isomorphism

$$W \rightarrow (H \otimes W)^H$$
given by

\[ w \mapsto \sum_{(\Lambda)} \Lambda_1 \otimes \Lambda_2 w. \]

Let \( \varphi \) be this isomorphism for \( W = \tilde{H}^{\otimes (m-1)} \). That is,

\[ \varphi(h^1 \otimes h^2 \otimes \cdots \otimes h^{m-1}) = \sum_{(\Lambda)} \Lambda_1 \otimes \tau(\Lambda_2) h^1 \otimes \tau^2(\Lambda_3) h^2 \otimes \cdots \otimes \tau^{m-1}(\Lambda_m) h^{m-1}, \]

where \( (h^1 \otimes h^2 \otimes \cdots \otimes h^{m-1}) \in \tilde{H}^{\otimes (m-1)} \). Calculating gives

\[
\begin{align*}
(\alpha \circ \varphi) (h^1 \otimes h^2 \otimes \cdots \otimes h^{m-1}) & = \sum_{(\Lambda)} \tau(\Lambda_2) h^1 \otimes \tau^2(\Lambda_3) h^2 \otimes \cdots \otimes \tau^{m-1}(\Lambda_m) h^{m-1} \otimes \Lambda_1 \\
& = \sum_{(\Lambda)} \tau(\Lambda_1) h^1 \otimes \tau^2(\Lambda_2) h^2 \otimes \cdots \otimes \tau^{m-1}(\Lambda_{m-1}) h^{m-1} \otimes \Lambda_m \\
& = \sum_{(\Lambda)} \Lambda_1 h^1 \otimes \tau(\Lambda_2) h^2 \otimes \cdots \otimes \tau^{m-2}(\Lambda_{m-1}) h^{m-1} \otimes \tau^{m-1}(\Lambda_m) \\
& = \sum_{(\Lambda)} \Lambda_1 \otimes \tau(\Lambda_2 S(h_{m-1}^1)) h^2 \otimes \cdots \otimes \tau^{m-2}(\Lambda_{m-1} S(h_{2}^1)) h^{m-1} \otimes \tau^{m-1}(\Lambda_m S(h_{1}^1)) \\
& = \sum_{(\Lambda)} \Lambda_1 \otimes \tau(\Lambda_2) \tau(S(h_{m-1}^1)) h^2 \otimes \cdots \otimes \tau^{m-2}(\Lambda_{m-1}) \tau^{m-2}(S(h_{2}^1)) h^{m-1} \otimes \tau^{m-1}(\Lambda_m) \tau^{m-1}(S(h_{1}^1)) \\
& = (\varphi \circ \psi) (h^1 \otimes h^2 \otimes \cdots \otimes h^{m-1}).
\end{align*}
\]

Here, the second and third equalities use Lemmas 3.17 and 3.22 respectively.

We now can proceed to the proof of the theorem.

**Proof of Theorem 3.21.** By the previous lemma, we need only show that

\[ \text{tr}(\psi) = \text{tr}(\Omega^r_m). \]
Choose a basis $b^1, \ldots b^n \in H$ with dual basis $b^*_1, \ldots b^*_n \in H^*$. Writing out $\operatorname{tr}(\psi)$ in terms of the induced basis on $H^\otimes m$, we obtain

\[
\operatorname{tr}(\psi) = \sum_{i_1, \ldots, i_{m-1}=1}^{n} \langle b^*_{i_1} \otimes \cdots \otimes b^*_{i_{m-1}}, \psi (b^{i_1} \otimes \cdots \otimes b^{i_{m-1}}) \rangle \\
= \sum_{i_1, \ldots, i_{m-1}=1}^{n} b^*_{i_1} (\tau(S(b^{i_1}_{m-1})) b^{i_2}) b^*_{i_2} (\tau^2(S(b^{i_1}_{m-2})) b^{i_3}) \cdots b^*_{i_{m-2}} (\tau^{m-2}(S(b^{i_1}_2)) b^{i_{m-1}}) b^*_{i_{m-1}} (\tau^{m-1}(S(b^{i_1}_1))) \\
= \cdots = \sum_{i_1=1}^{n} b^*_{i_1} (\tau(S(b^{i_1}_{m-1})) \tau^2(S(b^{i_1}_{m-2})) \cdots \tau^{m-2}(S(b^{i_1}_2)) \tau^{m-1}(S(b^{i_1}_1))) \\
= \sum_{i=1}^{n} b^*_{i} (\tau(S(b^{i}_{m-1})) \tau^2(S(b^{i}_{m-2})) \cdots \tau^{m-2}(S(b^{i}_2)) \tau^{m-1}(S(b^{i}_1))) \\
= \sum_{i=1}^{n} b^*_{i} (S(\tau(b^{i}_{m-1}))) S(\tau^2(b^{i}_{m-2})) \cdots S(\tau^{m-2}(b^{i}_2)) S(\tau^{m-1}(b^{i}_1))) \\
= \sum_{i=1}^{n} b^*_{i} (S(\tau^{m-1}(b^{i}_1)) \tau^{m-2}(b^{i}_2)) \cdots \tau^2(b^{i}_{m-2}) \tau(b^{i}_{m-1})) \\
= \operatorname{tr}(\Omega^*_m).
\]

as desired.
Chapter 4
A Hopf Algebra of Dimension 8

In this chapter, we have explicit computations for twisted Frobenius–Schur indicators for the Hopf algebra $H_8$ defined in Example 2.12. In Section 4.1, we compute the twisted FS indicators for all the automorphisms and in Section 4.2 we calculate the twisted FS indicator of the regular representation using the linear map $\Omega^r_m$ from Section 3.3.2.

Recall, that smallest semisimple Hopf algebra which is neither commutative nor cocommutative has dimension 8 and is the Hopf algebra $H_8$ described in Example 2.12. As an algebra, $H_8$ is generated by elements $x$, $y$ and $z$, with relations:

$$
\begin{align*}
x^2 &= y^2 = 1, \\
z^2 &= \frac{1}{2} (1 + x + y - xy), \\
xy &= yx, \\
xz &= zx, \\
yz &= zx.
\end{align*}
$$

The coalgebra structure of $H_8$ is given by the following:

$$
\begin{align*}
\Delta(x) &= x \otimes x, \quad \varepsilon(x) = 1, \quad \text{and} \quad S(x) = x, \\
\Delta(y) &= y \otimes y, \quad \varepsilon(y) = 1, \quad \text{and} \quad S(y) = y, \\
\Delta(z) &= \frac{1}{2} (1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x) (z \otimes z), \\
\varepsilon(z) &= 1, \quad \text{and} \quad S(z) = z.
\end{align*}
$$

The normalized integral is given by

$$
\Lambda = \frac{1}{8} (1 + x + y + xy + z + xz + yz + xyz).
$$

4.1 The Second Twisted FS Indicators

It is well known that the Hopf algebra $H_8$ has 4 one-dimensional representations and a single two-dimensional simple module. We computed the characters for the irreducible representations of $H_8$ and we listed them in Table 4.1.
TABLE 4.1. Characters for the Irreducible Representations of $H_8$

| $\chi$ | 1 | $x$ | $y$ | $xy$ | $z$ | $xz$ | $yz$ | $xyz$ |
|--------|---|-----|-----|------|----|------|------|-------|
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | 1 | 1 | 1 | -1 | -1 | -1 | -1 |
| $\chi_3$ | 1 | -1 | -1 | 1 | $i$ | $-i$ | $-i$ | $i$ |
| $\chi_4$ | 1 | -1 | -1 | 1 | $-i$ | $i$ | $i$ | $-i$ |
| $\chi_5$ | 2 | 0 | 0 | -2 | 0 | 0 | 0 | 0 |

TABLE 4.2. Automorphisms of $H_8$

| $\tau$ | 1 | $x$ | $y$ | $z$ |
|--------|---|-----|-----|-----|
| $\tau_1 = \text{Id}$ | 1 | $x$ | $y$ | $z$ |
| $\tau_2$ | 1 | $x$ | $y$ | $xyz$ |
| $\tau_3$ | 1 | $y$ | $x$ | $\frac{1}{2}(z + xz + yz - xyz)$ |
| $\tau_4$ | 1 | $y$ | $x$ | $\frac{1}{2}(-z + xz + yz + xyz)$ |

Let $\tau$ be any of the four automorphisms of $H_8$ given in Table 4.2. Since $\tau^2 = \text{Id}$, we have $m = 2$. All four automorphisms satisfy $\tau^2 = \text{Id}$, so the second twisted FS indicator is defined for all of them. Our goal is to compute

$$\nu_2(\chi, \tau) = \sum_{(\Lambda)} \chi(\Lambda_1 \tau(\Lambda_2))$$

for all 5 characters of the irreducible representations of $H_8$ given in Table 4.1. We need to determine the $\Lambda_1$ and $\tau(\Lambda_2)$, for

$$\Lambda = \frac{1}{8} (1 + x + y + xy + z + xz + yz + xyz).$$

We apply $\Delta$ to $\Lambda$ and use the fact that $\Delta$ is an algebra morphism,

$$\Delta(\Lambda) = \frac{1}{8} (1 \otimes 1 + x \otimes x + y \otimes y + xy \otimes xy + \frac{1}{2}z \otimes z + \frac{1}{2}z \otimes xz + \frac{1}{2}yz \otimes z - \frac{1}{2}yz \otimes xz + \frac{1}{2}xz \otimes xz + \frac{1}{2}xz \otimes z + \frac{1}{2}xyz \otimes xz - \frac{1}{2}xyz \otimes z + \frac{1}{2}yz \otimes yz + \frac{1}{2}yz \otimes xyz + \frac{1}{2}z \otimes yz - \frac{1}{2}z \otimes xyz + \frac{1}{2}xyz \otimes yz + \frac{1}{2}xyz \otimes xyz - \frac{1}{2}xz \otimes yz)$$

$$= \sum_{j=1}^{20} \Lambda_{1j} \otimes \Lambda_{2j}.$$
Recall:

$$\nu_2(\chi, \tau) := \sum_{j=1}^{20} \chi(\Lambda_1 j \tau(\Lambda_2 j)).$$

Applying $\tau$ to to all $\Lambda_2 j$ we have:

$$\nu_2(\chi, \tau) = \frac{1}{8}\left(\chi(1(1)) + \chi(x\tau(x)) + \chi(y\tau(y)) + \chi(xy\tau(xy))
+ \frac{1}{2}\chi(z\tau(z)) + \frac{1}{2}\chi(z\tau(xz)) + \frac{1}{2}\chi(yz\tau(z)) - \frac{1}{2}\chi(yz\tau(xz))
+ \frac{1}{2}\chi(xz\tau(xz)) + \frac{1}{2}\chi(xz\tau(z)) + \frac{1}{2}\chi(xy\tau(xz)) - \frac{1}{2}\chi(xy\tau(z))
+ \frac{1}{2}\chi(zy\tau(z)) + \frac{1}{2}\chi(zy\tau(zy)) + \frac{1}{2}\chi(zy\tau(xy)) - \frac{1}{2}\chi(zy\tau(xy))\right).$$

Let us consider the automorphism $\tau = \tau_1 = \text{Id}_{H_8}$. Then using the linearity of characters and the multiplication relations, we get

$$\nu_2(\chi, \tau_1) = \frac{1}{8}\left(\chi(1) + \chi(1) + \chi(1) + \chi(1)
+ \frac{1}{2}\chi(z^2) + \frac{1}{2}\chi(yz^2) + \frac{1}{2}\chi(yz^2) - \frac{1}{2}\chi(z^2)
+ \frac{1}{2}\chi(xyz^2) + \frac{1}{2}\chi(xz^2) + \frac{1}{2}\chi(xz^2) - \frac{1}{2}\chi(xyz^2)
+ \frac{1}{2}\chi(xyz^2) + \frac{1}{2}\chi(xz^2) + \frac{1}{2}\chi(xz^2) - \frac{1}{2}\chi(xyz^2)
+ \frac{1}{2}\chi(z^2) + \frac{1}{2}\chi(yz^2) + \frac{1}{2}\chi(yz^2) - \frac{1}{2}\chi(z^2)\right).$$

Further simplifying yields,

$$\nu_2(\chi, \tau_1) = \frac{1}{8}\left(4\chi(1) + 2\chi(yz^2) + 2\chi(xz^2)\right)
= \frac{1}{8}\left(6\chi(1) + 2\chi(xy)\right).$$

If $\chi_n$ is any irreducible character of $H_8$, then

$$\nu_2(\chi_n, \tau_1) = 1.$$
We now consider the automorphism \( \tau_2 : H_8 \rightarrow H_8 \) such that:

\[
\begin{align*}
\tau_2(1) &= 1, & \tau_2(z) &= xyz \\
\tau_2(x) &= x, & \tau_2(xz) &= yz \\
\tau_2(y) &= y, & \tau_2(yz) &= xz \\
\tau_2(xy) &= xy, & \tau_2(xyz) &= z.
\end{align*}
\]

We use the linearity of characters and the relations in \( H_8 \), to we get

\[
\nu_2(\chi, \tau_2) = \frac{1}{8} (\chi(1) + \chi(1) + \chi(1) + \chi(1) \\
+ \frac{1}{2} \chi(xy^2) + \frac{1}{2} \chi(xz^2) + \frac{1}{2} \chi(xz^2) - \frac{1}{2} \chi(xy^2) \\
+ \frac{1}{2} \chi(z^2) + \frac{1}{2} \chi(yz^2) + \frac{1}{2} \chi(yz^2) - \frac{1}{2} \chi(z^2) \\
+ \frac{1}{2} \chi(z^2) + \frac{1}{2} \chi(yz^2) + \frac{1}{2} \chi(yz^2) - \frac{1}{2} \chi(z^2) \\
+ \frac{1}{2} \chi(xy^2) + \frac{1}{2} \chi(xz^2) + \frac{1}{2} \chi(xz^2) - \frac{1}{2} \chi(xy^2)).
\]

Simplifying we get

\[
\nu_2(\chi, \tau_2) = \frac{1}{8} (4\chi(1) + 2\chi(xz^2) + 2\chi(yz^2)) \\
= \frac{1}{8} (6\chi(1) + 2\chi(xy)).
\]

If \( \chi_n \) is any irreducible character of \( H_8 \), then

\[
\nu_2(\chi_n, \tau_2) = 1.
\]

Now, let \( \tau_3 : H_8 \rightarrow H_8 \) be the automorphism of \( H_8 \) such that:

\[
\begin{align*}
\tau_3(1) &= 1, & \tau_3(z) &= \frac{1}{2} (-z + xz + yz + xyz) \\
\tau_3(x) &= y, & \tau_3(xz) &= \frac{1}{2} (z + xz - yz + xyz) \\
\tau_3(y) &= x, & \tau_3(yz) &= \frac{1}{2} (z - xz + yz + xyz) \\
\tau_3(xy) &= xy, & \tau_3(xyz) &= \frac{1}{2} (+z + xz + yz - xyz).
\end{align*}
\]
If we apply this map to our definition of the twisted FS indictors and we use the linearity of characters, as well as, the multiplication relations, we get

$$\nu_2(\chi, \tau_3) = \frac{1}{8} \left( 2\chi(1) + 2\chi(xy) + 4\chi(z^2) \right).$$

Further simplifying yields,

$$\nu_2(\chi, \tau_3) = \frac{1}{8} \left( 4\chi(1) + 2\chi(x) + 2\chi(y) \right)$$

$$= \frac{1}{4} \left( 2\chi(1) + \chi(x) + \chi(y) \right).$$

Let $\chi_n$ is an irreducible character of $H_8$. If $n \in \{1, 2, 5\}$, then $\nu_2(\chi_n, \tau_3) = 1$. If $n \in \{3, 4\}$, then $\nu_2(\chi_n, \tau_3) = 0$.

Finally, let $\tau_4 : H_8 \to H_8$ be the automorphism such that:

$$\tau_4(1) = 1, \quad \tau_4(z) = \frac{1}{2} (z + xz + yz - xyz)$$

$$\tau_4(x) = y, \quad \tau_4(xz) = \frac{1}{2} (z - xz + yz + xyz)$$

$$\tau_4(y) = x, \quad \tau_4(yz) = \frac{1}{2} (z + xz - yz + xy)$$

$$\tau_4(xy) = xy, \quad \tau_4(xyz) = \frac{1}{2} (-z + xz + yz + xyz).$$

Then, it follows from the properties of characters and the relations on $H_8$ that

$$\nu_2(\chi, \tau_4) = \frac{1}{4} \left( \chi(x) + \chi(y) + 2\chi(xy) \right).$$

Let $\chi_n$ is an irreducible character of $H_8$. If $n \in \{1, 2\}$, then $\nu_2(\chi_n, \tau_4) = 1$. If $n \in \{3, 4\}$, then $\nu_2(\chi_n, \tau_4) = 0$. In the other hand, if $n = 5$, then $\nu_2(\chi_n, \tau_4) = -1$. This implies that for this automorphism, the second twisted FS indicator of the 2 dimensional irreducible representation of $H_8$ is $-1$. For this Hopf algebra, this is the only negative indicator.

We have now calculated all the second twisted FS indicators. These indicators are given in Table 4.3.
### Table 4.3. Twisted Frobenius–Schur indicators for $H_8$

|        | $\chi_1$ | $\chi_2$ | $\chi_3$ | $\chi_4$ | $\chi_5$ |
|--------|----------|----------|----------|----------|----------|
| $\nu_2(\chi, \tau_1)$ | 1        | 1        | 1        | 1        | 1        |
| $\nu_2(\chi, \tau_2)$ | 1        | 1        | 1        | 1        | 1        |
| $\nu_2(\chi, \tau_3)$ | 1        | 1        | 0        | 0        | 1        |
| $\nu_2(\chi, \tau_4)$ | 1        | 1        | 0        | 0        | $-1$     |

### Table 4.4. The linear maps $\Omega_2^\tau$ for $H_8$

|        | $\Omega_2^1$ | $\Omega_2^2$ | $\Omega_2^3$ | $\Omega_2^4$ |
|--------|--------------|--------------|--------------|--------------|
| 1      | 1            | 1            | 1            | 1            |
| $x$    | $x$          | $x$          | $y$          | $y$          |
| $y$    | $y$          | $y$          | $x$          | $x$          |
| $xy$   | $xy$         | $xy$         | $xy$         | $xy$         |
| $z$    | $xyz$        | $\frac{1}{2}(z + xz + yz - xyz)$ | $\frac{1}{2}(-z + xz + yz + xyz)$ | $\frac{1}{2}(-z + xz + yz + xyz)$ |
| $xz$   | $yz$         | $\frac{1}{2}(z + xz - yz + xyz)$ | $\frac{1}{2}(z - xz + yz + xyz)$ | $\frac{1}{2}(z - xz + yz + xyz)$ |
| $yz$   | $xz$         | $\frac{1}{2}(z - xz + yz + xyz)$ | $\frac{1}{2}(z + xz - yz + xyz)$ | $\frac{1}{2}(z + xz - yz + xyz)$ |
| $xyz$  | $z$          | $\frac{1}{2}(-z + xz + yz + xyz)$ | $\frac{1}{2}(z - xz + yz + xyz)$ | $\frac{1}{2}(z - xz + yz + xyz)$ |

#### 4.2 The Regular Representation

The goal of this section is to compute the regular representation of $H_8$ using the closed formula from Theorem 3.21. We list the linear maps $\Omega_2^\tau$ in Table 4.4. Computing the traces, one obtains the twisted Frobenius–Schur indicators for the regular representation: $\nu_2(\chi_R, \tau_1) = 6$, $\nu_2(\chi_R, \tau_2) = 6$, $\nu_2(\chi_R, \tau_3) = 4$, and $\nu_2(\chi_R, \tau_4) = 0$. These can, of course, also be calculated from the information in Table 4.3.
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Vita

Maria D. Vega was born in August 1981, in Zacatecas, Mexico. She completed her undergraduate studies at California State University, Fullerton, in May 2004. In August 2004, Maria came to Louisiana State University in Baton Rouge, Louisiana, to pursue graduate studies in mathematics. She earned a master of science degree in mathematics from Louisiana State University in May 2006. She is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in December 2011.