INTERACTING AGENT FEEDBACK FINANCE MODEL

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We consider a financial market model which consists of a financial asset and a large number of interacting agents classified into many types. Different types of agents are heterogeneous in their price expectations. Each agent can change its type based on the current empirical distribution of the types and the equilibrium price, and the equilibrium price follows a recursive price mechanism based on the previous price and the current empirical distribution of the types. The interaction among the agents, and the interaction between the agents and the equilibrium price, feedback, are modeled. We analyze the asymptotic behavior of the empirical distribution of the types and the equilibrium price when the number of agents goes to infinity. We give a case study of a simple example, and also investigate the fixed points of empirical distribution and equilibrium price of the example.

1. Introduction. Stochastic models of interacting systems play an important role in population biology and statistical physics, c.f. [17] and [3]. In recent years a number of leading thinkers have expressed the need for developing economic models that incorporate interactions between agents and evolutionary mechanisms, see [22] and [6]. Agent-based models (ABMs), which arise from many areas of science and social sciences such as ecology, artificial intelligence, communication networks, sociology, economics (see [8], [7], [21], [27]) are the ideal choice to attain this goal. The following features of ABMs (see [10]) are fundamental. Firstly, precise mathematical formulation can be described by ABMs, which make clear, quantitative and objective predictions possible. Secondly, the explanations that link the analysis of the individual agent level and the analysis of the emergent aggregate level can be bridged by ABMs.

In this paper we will concentrate on the agent based modeling in a financial market. At first, we give an account of some aspects of the related works done by other authors. Black [2] classified traders as information traders and noise traders. Föllmer and Schweizer [9] considered an interacting agent financial model in which they used Black’s classification of traders. They assumed the number of traders being countable and introduced an individ-

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ual excess demand function which takes a simple log-linear form. Lux (see [18], [19] and [20]) assumed the number of agents being finite, and divided the traders into three types: chartists, optimists, and pessimists. In Lux’s model, the types of traders can be changed probabilistically, according to the profitability of each type; there are new entrants into the market and exits of current traders from the market. Horst (see [12], [13], [14] and [15]) considered the interacting agent models with local and global interactions. Horst classified the traders into three types: fundamental optimistic traders, fundamental pessimistic traders and noise traders. Let $A$ be a countable set of active agents and $A_n$ be a sequence of finite sets satisfying $\lim_{n \to \infty} A_n = A$. At each period $t \geq 0$, each fundamental trader has its mood, e.g., $x^a_t = +1$ for being optimistic or $x^a_t = -1$ for being pessimistic. Let $C$ be a fixed set of individual states, i.e., $x^a_t \in C$ for each $a \in A$ and $t \geq 0$. Let $x_t = \{x^a_t\}_{a \in A}$. Horst defined the empirical distribution as follows:

$$
(1.1) \quad \rho_t = \rho(x_t) = \lim_{n \to \infty} \frac{1}{|A_n|} \sum_{a \in A_n} \delta_{x^a_t}(\cdot).
$$

$\rho(x_t)$ is called the mood of the market at time $t$. The market mood drives the market price in the following way:

$$
(1.2) \quad p_{t+1} = G(\rho_{t+1}, p_t),
$$

where $p_t$ is the market price in period $t$ and $G$ is a certain function. Assuming a simple log-linear structure for the excess demand function, Horst got the recursive log price formula of the following form: for each period $t \geq 0$,

$$
(1.3) \quad \log p_{t+1} = f(\rho_{t+1}) \log p_t + g(\rho_{t+1}).
$$

The mood for each individual $a \in A$ evolves as follows:

$$
(1.4) \quad \pi_a(x^a_{t+1} = s|x_t, e_{t+1}, h_t) = \pi(x^a_{t+1} = s|x^a_t, e_{t+1}, h_t),
$$

where $s \in C$, and $e_{t+1} \sim Q(\rho(x_t); \cdot)$ is the signal of the market mood $\rho(x_t)$ and $h_t$ is some (exogenous) economic fundamentals revealed in period $t$. Therefore, in Horst’s models, traders can change their types during the evolution of the models and there are interactions among traders. But there is no feedback of the price $p_t$ on the evolution of individual state, i.e., the current market price has no impact on the change of types for the next time period.

As was illustrated above, the empirical distribution of the types of agents can link the behavior of individual agent level, the emergent laws of aggregate level, and the equilibrium price of certain financial asset. This paper is
the second attempt of a systematic study of the interacting agent financial system. Another working paper of the author focuses on the multiagent models evolving in time-varying and random environment, see [29]. We construct the interacting agent feedback finance model (IAFFM) by using agent based modeling. The most general assumptions about the mechanism of IAFFM are as follows:

1. The time is in nonnegative integer units, denoted by $k \geq 0$.
2. There are fixed $N \geq 2$ agents in the financial market at all times. There are no entries of new agents into the market or exits of current agents from the market.
3. There is a financial asset in certain market whose price $S_N(k)$ varies with time $k$.
4. Each agent has one and only one internal state from an internal system with $r \geq 2$ states denoted by $1, \cdots, r$. The internal system does not change with time $k$, i.e., at each time period $k \geq 0$, there is no new state added to it and no existing state removed from it. The agents are classified into $r$ types according to their internal states, and $n^N(k) = (n^N_1(k), \cdots, n^N_r(k))$ is the distribution of these agents among the $r$ types. It follows that $n^N_1(k) + \cdots + n^N_r(k) = N$ by the third assumption, and $\frac{n^N(k)}{N}$ is the empirical distribution of the types of agents at $k \geq 0$.
5. There exists a log price mechanism for the financial asset. Let $Z_+ = \{0,1,\cdots\}$, $K_N = \{N^{-1}\alpha : \alpha \in (Z_+)^r, \sum_{i=1}^r \alpha_i = N\}$, and $g_N$ be defined on $[0,\infty) \times K_N \times R$. At time $k \geq 1$, $\tilde{q}_N(k) = \log S_N(k)$ is determined by $\tilde{q}_N(k) = \log S_N(k)$ and $\frac{n^N(k)}{N}$ through the following recursive formula:

$$
(1.5) \quad \tilde{q}_N \left( k \frac{1}{N} \right) = \tilde{q}_N \left( k - 1 \frac{1}{N} \right) + \frac{1}{N} g_N \left( k \frac{1}{N}, n^N \left( \frac{1}{N} \right) \right).
$$

6. Assume that $P_N(\cdots, \cdot) = (p_{N,i,j}(\cdot, \cdot))_{r \times r}$ is a deterministic stochastic-matrix valued function defined on $Z_+ \times K_N \times R$ which represents the external environment of the multiagent system.
7. Based on all the information of the agents’ types, the equilibrium prices of the financial asset, and the external environment up to time $k$, each agent has an independent strategy of probabilistically choosing its type for the next time unit $k + 1$. The strategy of an agent is realized by keeping or changing its type. The agents of the same type have a common strategy. That is to say, from time $k$ to $k + 1$, the agents of type $i$ switch to type $j$ with probability $p_{N,i,j}(k, \frac{n^N(k)}{N}, \tilde{q}_N(k))$. This
process of changing types occurs locally among agents of the same type.

Note that $S_N$ and $g_N$ depend on $N$, which means that the price mechanism does reflect the influence of the size of the market. The gross performance of an economy consists of an external environment. The economic fundamentals of the financial asset can be reflected in $g_N$. An example will be given in Section 4 to illustrate these. The assumptions 1-7 above will be used to mathematically formulate IAFFM. When we assume that $P_N(\cdot, \cdot, \cdot)$ or $g_N$ are random, we can construct IAFFM evolving in random environment.

In this paper we will mainly study the asymptotic behaviors of $\left\{ \left( \frac{N^N \cdot (N+i)}{N}, \tilde{q}^N \left( \frac{N+i}{N} \right) \right), t \geq 0 \right\}$ as $N \to \infty$. One feature of our model is that the transition structure of the types and log equilibrium price is time-inhomogeneous. Another feature is that the equilibrium price of the financial asset has feedback on the transition of the agent’s types, instead of just being driven by the empirical distribution of the agents’ types. Therefore, we modeled two kinds of interactions: the interaction among the agents, and the interaction between the agents and the equilibrium price.

This paper is organized as follows. In Section 2, we formulate IAFFM and state the main Theorem (Theorem 2.1). In Section 3, we give a complete case study of a simple IAFFM example. We make assumptions for this example, verify its conditions required by Theorem 2.1, and discuss its fixed point problem. We also make connections with the classical stock price formula for this example. In Section 4, we give the proof of Theorem 2.1. In Appendix A, B, and C, we give the proof of Lemmas 3.1, 4.2, and 4.3, respectively.

2. Formulation of IAFFM and Main Theorem. Let $R_+ = [0, \infty)$ and $K = \left\{ \alpha : \alpha \in (R_+)^r, \sum_{i=1}^r \alpha_i = 1 \right\}$.

**Condition 2.1.** $A(t, x, q) = (a_{i,j}(t, x, q))_{r \times r}$ is a $r \times r$ matrix-valued càdlàg function on $[0, \infty) \times K \times R$, which satisfies the conditions as follows:

1) $A(\cdot, x, q) \in D_{R^r \times [0, \infty)}$;
2) $A(t, x, q) e = 0_{r \times 1}$ for $t \geq 0$, where $e = [1, \cdots , 1]^t$;
3) $a_{i,j}(t, x, q) \geq 0$ for $t \geq 0$, and $1 \leq i, j \leq r$, $i \neq j$.

**Condition 2.2.** For each $N \geq 1$, $A_N(t, x, q) = (a_{N,i,j}(t, x, q))_{r \times r}$ is a $r \times r$ matrix-valued function on $[0, \infty) \times K_N \times R$, which satisfies the following slightly different conditions: for each $(x, q) \in K_N \times R$,

1) $A_N(\cdot, x, q) \in D_{R^r \times [0, \infty)}$;
2) \( A_N(t, x, q)e = 0_{r \times 1}, \ t \geq 0; \)
3) \( a_{N, i,j}(t, x, q) \geq 0 \) for \( t \geq 0 \) and \( 1 \leq i, j \leq r, \ i \neq j. \)
4) \( A_N(t, x, q) \) is a constant on \( t \in \left[ \frac{k}{N}, \frac{k+1}{N} \right] \) for \( k \geq 0. \)

Next, we specify the functions which are related to the log price mechanism. For each \( N \geq 1, \) we have defined \( g_N(t, x, q) \) on \( [0, \infty) \times K_N \times R. \) We also define a real valued function \( g(t, x, q) \) on \( [0, \infty) \times K \times R. \) We are only concerned with functions which are ‘linearly’ growing in \( q, \) i.e.

\[
\begin{align*}
(2.1) \quad g(t, x, q) &= \varphi(t, x, q)q + \psi(t, x, q), \\
(2.2) \quad g_N(t, x, q) &= \varphi_N(t, x, q)q + \psi_N(t, x, q),
\end{align*}
\]

where \( \varphi(t, x, q) \) and \( \psi(t, x, q) \) are functions on \( [0, \infty) \times K \times R, \) and \( \varphi_N(t, x, q) \) and \( \psi_N(t, x, q) \) are functions on \( [0, \infty) \times K_N \times R. \)

Let \( X^N(t) = \frac{n^N(Nt)}{N}, \) \( q^N(t) = \tilde{q}^N\left(\frac{Nt}{N}\right) \) and \( Y^N(t) = (X^N(t), q^N(t)), \)

\[(1.5)\text{ represented by the new notations is for } t \geq \frac{1}{N},
\]

\[(2.3) \quad q^N(t) = q^N(t - \frac{1}{N}) + \frac{1}{N} g_N\left(\frac{Nt}{N}, X^N(t), q^N(t - \frac{1}{N})\right).\]

Now we formulate the transition structure of IAFFM as follows. For \( k \geq 0, \)

\[(2.4) \quad P_N[k, x, q] = I + \frac{1}{N} A_N\left(\frac{k}{N}, x, q\right),\]

where \( I \) is the identity matrix of order \( r. \) For each fixed \( k \geq 0, \) \( (x, q) \in K_N \times R, \) \( P_N[k, x, q] \) is a stochastic matrix for large enough \( N. \) Assume that \( n^N(0) \) and \( \tilde{q}^N(0) \) are given. We define the time inhomogeneous Markov chain \( \{(n^N(k), \tilde{q}^N(k))\}_{k=0}^\infty \) by mathematical induction. Assume that for \( k \geq 0, \)

\( (n^N(k), \tilde{q}^N(k)) \) is defined or given, we want to define \( (n^N(k+1), \tilde{q}^N(k+1)) \).

By the assumption 7 in Section 1, for \( 1 \leq i \leq r, \) each agent of type \( i \) can change its type to \( j, \) with probability \( p_{N, i,j}(k, X^N(k), \tilde{q}^N(k)) \) \( (1 \leq j \leq r). \)

Since the \( n^N(k) \) agents of type \( i \) independently make their transitions, the distribution of these \( n^N(k) \) agents among the \( r \) types at time \( k+1 \) is a random vector denoted by \( \Xi_{N,k,i} = (\xi_{N,k,i,1}, \cdots, \xi_{N,k,i,r}) \) on some probability space, which satisfies

\[(2.5) \quad \Xi_{N,k,i} \sim \text{multinomial}\left(n^N(i, k), P_{N,i}, [k, X^N(k), \tilde{q}^N(k)]\right),
\]

where \( P_{N,i}[k, X^N(k), \tilde{q}^N(k)] \) is the \( i \)-th row of matrix \( P_N[k, X^N(k), \tilde{q}^N(k)] \).

Since the agents with different type change their types independently, \( \Xi_{N,k,1}, \cdots, \Xi_{N,k,r} \) are independent. Then, we define

\[(2.6) \quad n^N(k+1) \equiv \Xi_{N,k,1} + \cdots + \Xi_{N,k,r},\]
and \(\tilde{q}^N(k+\frac{1}{N})\) by formula (1.5).

For each \(N \geq 1\), we define the transition operators of \(\{(X^N(k), \tilde{q}^N(k+\frac{1}{N}))\}, k \geq 0\) as follows: for each \(k \geq 0\) and \(f \in \mathcal{C}(K_N \times R)\), the set of bounded continuous functions on \(K_N \times R\), for \((x, q) \in K_N \times R\),

\[
S_{N,k}f(x, q) = E[f(X^N(k+\frac{1}{N}), \tilde{q}^N(k+\frac{1}{N}))) | X^N(k) = x, \tilde{q}^N(k) = q].
\]

We expect that the transpose \(Y'\) of any limit \(Y\) of \(Y^N\) is a solution of the following differential equations:

\[
\frac{dx(t)}{dt} = A(t, x(t), q(t))'x(t),
\]

\[
\frac{dq(t)}{dt} = g(t, x(t), q(t)),
\]

which satisfy the initial conditions \(x(0) = X(0)'\) and \(q(0) = Q(0)\).

Let \(K \times R\) be the state space for the limit process and denote by \(\mathcal{C}(K \times R)\) the set of bounded continuous functions on \(K \times R\), and define \(C_c(K \times R), C^1_c(K \times R), C^2_c(K \times R)\) by

\[
C_c(K \times R) = \{f \in \mathcal{C}(K \times R), f\text{ has compact support on } K \times R\},
\]

\[
C^1_c(K \times R) = \{f \in C_c(K \times R), \frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial q}\text{ are continuous on } K \times R\},
\]

\[
C^2_c(K \times R) = \{f \in C^1_c(K \times R), \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial q}, \text{ and } \frac{\partial^2 f}{\partial q^2}\text{ are continuous on } K \times R\}.
\]

We define the time-dependent generator \(\{G_A(s), 0 \leq s < \infty\}\) on \(C^1_c(K \times R)\): for each \(f \in C^1_c(K \times R)\), and \(s \geq 0\),

\[
G_A(s)f(x, q) = xA(s, x, q)\frac{\partial f'}{\partial x} + g(s, x, q)\frac{\partial f}{\partial q}, \quad (x, q) \in K \times R.
\]

\(\mathcal{D}(G_A) = C^1_c(K \times R)\) is the common domain of the generator \(\{G_A(s), 0 \leq s < \infty\}\), and \(D = C^2_c(K \times R)\), is a subalgebra contained in \(\mathcal{D}(G_A)\).

**Condition 2.3.** For each \(f \in C^2_c(K \times R)\) and \(T > 0\) there exist measurable sets \(\{F_N\} \subset R\) such that

\[
\lim_{N \to \infty} \sup_{q \in F_N} \sup_{x \in K_N} d_U(A_N(\cdot, x, q), A(\cdot, x, q)) = 0,
\]

\[
\lim_{N \to \infty} P[q^N(t) \in F_N, 0 \leq t \leq T] = 1,
\]
where $d_U$ is the uniform metric on $D_{R^r \times [0, \infty)}$ defined by
\[
d_U(u, v) = \int_0^\infty e^{-s} \sup_{0 \leq t \leq s} \|u(t) - v(t)\| \, ds, \quad u, v \in D_{R^r \times [0, \infty)},
\]
and $\| \cdot \|$ is the matrix norm.

**Condition 2.4.** For any $T > 0$,
\[
\lim_{N \to \infty} \sup_{0 \leq t \leq T} \sup_{(x, q) \in K_N \times R} |\varphi_N(t, x, q) - \varphi(t, x, q)| = 0,
\]
\[
\lim_{N \to \infty} \sup_{0 \leq t \leq T} \sup_{(x, q) \in K_N \times R} |\psi_N(t, x, q) - \psi(t, x, q)| = 0.
\]

**Condition 2.5.** For any compact subset $\tilde{K}$ of $K \times R$, there exist $C > 0$ and $\lambda > 0$, such that $A(t, x, q), \varphi(t, x, q)$ and $\psi(t, x, q)$ satisfy that
\[
\sup_{(x, q) \in \tilde{K}} \|A(t, x, q)\| \leq Ce^{\lambda t},
\]
\[
\sup_{(x, q) \in \tilde{K}} |\varphi(t, x, q)| \leq Ce^{\lambda t},
\]
\[
\sup_{(x, q) \in \tilde{K}} |\psi(t, x, q)| \leq Ce^{\lambda t}.
\]

We introduce the following notations:
\[
b_i(t, y) = \sum_{j=1}^r x_j a_{ji}(t, y), \quad 1 \leq i \leq r, \text{ and } y = (x, q)
\]
\[
b_{r+1}(t, y) = g(t, y),
\]
\[
b(t, y) = (b_1(t, y), \cdots, b_r(t, y)) \text{ and } \hat{b}(t, y) = (b_1(t, y), \cdots, b_r(t, y)).
\]

It is clear that (2.8) and (2.9) with the initial conditions $x(0) = X(0)$ and $q(0) = Q(0)$ are equivalent to the integral equations
\[
y(t) = y(0) + \int_0^t b(s, y(s))' \, ds, \quad t \geq 0
\]
or
\[
x(t) = x(0)' + \int_0^t \hat{b}(s, x(s), q(s))' \, ds,
\]
\[
q(t) = q(0) + \int_0^t g(s, x(s), q(s)) \, ds.
\]

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where \( y(0) = (x(0), q(0)) \). (2.25) and (2.26) are nonlinear Volterra equations of the second kind.

(2.25) and (2.26) are assumed to satisfy the semi-Lipschitz conditions, which are guaranteed by the following condition. The definition and property of semi-Lipschitz condition are included in Section 4.

**Condition 2.6.** For any \( T > 0 \), there exists a nonnegative \( L^2([0, T], R) \) function \( C_T(t) \), such that for any \( 0 \leq t \leq T \) and \( y_1 = (x_1, q_1), y_2 = (x_2, q_2) \in K \times R \),

\[
(2.27) \quad \| \tilde{b}(t, y_1) - \tilde{b}(t, y_2) \| \leq C_T(t) \| y_1 - y_2 \|.
\]

g satisfies that for fixed \( T > 0 \) and any \( K \)-valued continuous function \( x(t) \) on \([0, T] \), (2.26) determines a unique solution \( q_\xi(t) \) on \([0, T] \), and for any two solutions \( (x(t), q_\xi(t)), (\tilde{x}(t), q_\tilde{\xi}(t)) \) of (2.25) and (2.26) on \([0, T] \), there exists a nonnegative measurable function \( B(s) \) on \([0, T] \), such that \( q_\xi(t) \) and \( q_\tilde{\xi}(t) \) satisfy

\[
(2.28) \quad \int_0^s (q_\xi(t) - q_\tilde{\xi}(t))^2 dt \leq B(s) \int_0^s \| x(t) - \tilde{x}(t) \|^2 dt \quad \text{for} \ 0 \leq s \leq T
\]

and

\[
(2.29) \quad \int_0^T \left[ \int_0^s C_T(t)^2 dt (1 + B(s)) \right] ds < \infty.
\]

**Theorem 2.1.** Assume that \( A(t, x, q) = (a_{i,j}(t, x, q))_{r \times r} \) satisfies the condition 2.1 and \( \{ A_N(t, x, q) = (a_{N,i,j}(t, x, q))_{r \times r} \} \) satisfies the condition 2.2. Assume that \( g(t, x, q) \) and \( \{ g_N(t, x, q) \} \) satisfy (2.1) and (2.2). Assume that \( \varphi, \psi, \frac{\partial \varphi}{\partial x}, \frac{\partial \psi}{\partial x} \) are continuous. Assume that the condition 2.3 holds for \( \{ A_N \} \{ q^N \} \), and \( A \), and the condition 2.4 holds for \( \{ \varphi_N \} \varphi \) and \( \{ \psi_N \} \psi \). Assume that \( A(t, x, q), \varphi(t, x, q) \) and \( \psi(t, x, q) \) satisfy the condition 2.5 and are bounded on \([0, T] \times K \times R \) for any \( T > 0 \). Define \( b(t, y) \) and \( \tilde{b}(t, y) \) by (2.21), (2.22) and (2.23). Assume that either \( b(t, y) \) satisfies the Lipschitz condition or the condition 2.6 holds for (2.25) and (2.26). Suppose that \( P(Y^N(0))^{-1} \Rightarrow \mu \) for some \( \mu \in \mathcal{P}(K \times R) \), then there exists on some probability space \( (\Omega, \mathcal{F}, P) \) a stochastic process \( Y \) satisfying \( P(Y(0))^{-1} = \mu \), which is the unique solution of the \( C_{K \times R}(0, \infty) \)-martingale problem for \( (G_A, \mu) \) restricted to \( C_c^2(K \times R) \) and for \( P \)-a.s. \( \omega \in \Omega \), \( Y(t, \omega) \) is the unique solution of the integral equations (2.25) and (2.26), such that \( Y^N \Rightarrow Y \).
3. Case Study of a Simple Example.

3.1. A simple example of IAFFM. There are \( N \) agents in a financial market which consist of fundamentalists, optimists, and pessimists, see [20]. At each time \( k \geq 0 \), the number of fundamentalists, optimists and pessimists are \( n^1_N(k) \), \( n^2_N(k) \), and \( n^3_N(k) \) respectively. This interacting agent system is closed, i.e., there are no new entrants into the market or quits of current traders from the market. Therefore \( n^1_N(k) + n^2_N(k) + n^3_N(k) = N \) for all \( k \geq 0 \), and \( n^N(k) = (n^1_N(k), n^2_N(k), n^3_N(k)) \) is the vector of types.

Each type of agents have their own excess demand functions. The excess demand functions have the form of Föllmer and Schweizer, see [9]. Assume that there exists a probability space \((\Omega, \mathcal{F}, P)\). At time unit \( k \geq 1 \), for each \( \omega \in \Omega \), and a proposed price \( p \), each agent \( a \) has an excess demand function \( e^N_a(k,p,\omega) \) which is given by

\[
e^N_a(k,p,\omega) = \alpha^N_a(\frac{k}{N},\omega) \log(\hat{S}^N_{a,k}(\omega)/p) + \frac{\delta^N_a(\frac{k}{N},\omega)}{N},
\]

Here \( \delta^N_a(\frac{k}{N},\omega) \) can be viewed as the total liquidity demand observed by agent \( a \) at time \( k \geq 0 \) and \( \frac{\delta^N_a(\frac{k}{N},\omega)}{N} \) is the average liquidity demand, and \( \hat{S}^N_{a,k}(\omega) \) denotes an individual reference level of agent \( a \) at time \( k \). The time scale for \( \alpha^N_a \) and \( \delta^N_a \) is \( \frac{1}{N} \), instead of 1.

Denote the individual reference level \( \hat{S}^N_{a,k}(\omega) \), the coefficients \( \alpha^N_a(\frac{k}{N},\omega) \) and \( \delta^N_a(\frac{k}{N},\omega) \) for fundamentalist, optimist, and pessimist by \( \hat{S}^N_i(k,\omega), \alpha^N_i(\frac{k}{N},\omega) \) and \( \delta^N_i(\frac{k}{N},\omega) \) \((1 \leq i \leq 3)\) respectively. We assume individual reference levels as follows:

\[
\log \hat{S}^N_1(k,\omega) = \log S_N(k - 1) + \frac{\beta^N_1(\frac{k}{N},\omega)}{N}(\log S_N(k - 1) - \log F^N(\frac{k}{N},\omega)),
\]

\[
\log \hat{S}^N_2(k,\omega) = \log S_N(k - 1) + \frac{\beta^N_2(\frac{k}{N},\omega)}{N}(\log S_N(k - 1) - \log p),
\]

\[
\log \hat{S}^N_3(k,\omega) = \log S_N(k - 1) + \frac{\beta^N_3(\frac{k}{N},\omega)}{N}(\log S_N(k - 1) - \log p),
\]

where random coefficients \( \beta^N_i(\frac{k}{N},\omega) \leq 0 \) \((1 \leq i \leq 3)\), \( F^N(\frac{k}{N},\omega) \) is the fundamental value of the asset at time \( k \). Note that only \( \beta^N_1(\frac{k}{N},\omega) \) in the fundamentalists’ reference level is divided by \( N \), the market size of the agents. One reasonable explanation for it is that fundamentalists know the evolution...
of fundamental value \( F^N(k, \omega) \) of the asset, they value less in their excess demand functions the difference between \( \log S_N(k - 1) \) and \( \log F^N(k, \omega) \). Note also that the time scale for \( \beta^N_1, \beta^N_2, \beta^N_3 \) and \( \log F^N \) is also of \( \frac{1}{N} \).

For each \( k \geq 1 \), if we assume the price \( S_N(k - 1) \) and \( n^N(k) \) are known, then the equilibrium log price \( \log S_N(k) \) is determined by the market clearing condition of zero excess demand:

\[
(3.5) \quad \sum_a e^N_a(k, S_N(k), \omega) = 0,
\]

i.e.,

\[
(3.6) \quad \sum_{i=1}^3 n^N_i(k) \left[ \alpha^N_i \left( \frac{k}{N} \right) (\log \hat{S}^N_i(k) - \log S_N(k)) + \frac{\delta^N_i(k)}{N} \right] = 0.
\]

We have omitted \( \omega \) in the random variables in the above equation and in the rest of this subsection. Substituting (3.2), (3.3), (3.4) into (3.6), and solving for \( \log p \) as \( \log S_N(k) \), we get

\[
(3.7) \quad \log S_N(k) - \log S_N(k - 1) = \frac{1}{N} \cdot \frac{n^N_1(k) \alpha^N_1 \left( \frac{k}{N} \right) \beta^N_1 \left( \frac{k}{N} \right) (\log S_N(k - 1) - \log F^N \left( \frac{k}{N} \right))}{n^N_1(k) \alpha^N_1 \left( \frac{k}{N} \right) + n^N_2(k) \alpha^N_2 \left( \frac{k}{N} \right) (1 + \beta^N_2 \left( \frac{k}{N} \right)) + n^N_3(k) \alpha^N_3 \left( \frac{k}{N} \right) (1 + \beta^N_3 \left( \frac{k}{N} \right))} + \frac{1}{N} \cdot \frac{n^N_1(k) \delta^N_1 \left( \frac{k}{N} \right) + n^N_2(k) \delta^N_2 \left( \frac{k}{N} \right) (1 + \beta^N_2 \left( \frac{k}{N} \right)) + n^N_3(k) \delta^N_3 \left( \frac{k}{N} \right) (1 + \beta^N_3 \left( \frac{k}{N} \right))}{n^N_1(k) \alpha^N_1 \left( \frac{k}{N} \right) + n^N_2(k) \alpha^N_2 \left( \frac{k}{N} \right) (1 + \beta^N_2 \left( \frac{k}{N} \right)) + n^N_3(k) \alpha^N_3 \left( \frac{k}{N} \right) (1 + \beta^N_3 \left( \frac{k}{N} \right))}.
\]

(3.7) is a recursive log price formula.

Let \( K^*_N = \{ N^{-1} \mathbf{x} : \mathbf{x} \in (Z^+)^3, \sum_{i=1}^3 x_i = N \} \). Define \( g_N \) on \([0, \infty) \times K^*_N \times R \) as follows:

\[
(3.8) \quad g_N(t, \mathbf{x}, q) = \frac{x_1 \alpha^N_1(t) \beta^N_1(t) q}{x_1 \alpha^N_1(t) + x_2 \alpha^N_2(t) (1 + \beta^N_2(t)) + x_3 \alpha^N_3(t) (1 + \beta^N_3(t))} + \frac{x_1 \delta^N_1(t) + x_2 \delta^N_2(t) + x_3 \delta^N_3(t) - x_1 \alpha^N_1(t) \beta^N_1(t) \log F^N(t)}{x_1 \alpha^N_1(t) + x_2 \alpha^N_2(t) (1 + \beta^N_2(t)) + x_3 \alpha^N_3(t) (1 + \beta^N_3(t))}.
\]

Then (3.7) can be represented by \( g_N(t, \mathbf{x}, q) \) as

\[
(3.9) \quad \log S_N(k) = \log S_N(k - 1) + \frac{1}{N} g_N \left( \frac{k}{N}, \frac{n^N(k)}{N}, \log S_N(k - 1) \right).
\]

Note that \( g_N(t, \mathbf{x}, q) \) defined by (3.8) is a random function. We have actually made preparations for the IAFFM evolving in a random environment.
We also define \( A_N(t, x, q) = (a_{N,i,j}(t, x, q))_{3 \times 3} \) on \([0, \infty) \times K^3_N \times R\) satisfying the Condition 2.2. Then we can specify the transition structure for \((\{n^N(k), \log S_N(k)\})_{k=0}^{\infty}\) same way as that in Section 2.

In this example, \( \alpha^N_i, \beta^N_i, \delta^N_i \) \((1 \leq i \leq 3)\), and \( F^N \) consist of the external environment of the interacting agent financial system. \( F^N \) is the economic fundamental of the financial asset.

3.2. Assumptions and verifications of the example. In this subsection, we make assumptions for the example and verify the conditions of Theorem 2.1 for this example.

Assume that \( \alpha^N_i(t), \beta^N_i(t), \) and \( \delta^N_i(t) \) for \(1 \leq i \leq 3\) and \( \log F^N(t) \) are real valued functions defined on \([0, \infty)\). Assume also that for any \( x \in K^3_N \) and \( t \geq 0 \), \( x_1 \alpha^N_1(t) + x_2 \alpha^N_2(t)(1 + \beta^N_2(t)) + x_3 \alpha^N_3(t)(1 + \beta^N_3(t)) \neq 0 \), which justify the definition of \( g_N \) in (3.8). Then we can define real valued functions \( \varphi_N, \psi_N \) on \([0, \infty) \times K^3_N\) as follows:

\[
\varphi_N(t, x) = \frac{x_1 \alpha^N_1(t) \beta^N_1(t)}{x_1 \alpha^N_1(t) + x_2 \alpha^N_2(t)(1 + \beta^N_2(t)) + x_3 \alpha^N_3(t)(1 + \beta^N_3(t))},
\]

\[
\psi_N(t, x) = \frac{x_1 \delta^N_1(t) + x_2 \delta^N_2(t) + x_3 \delta^N_3(t) - x_1 \alpha^N_1(t) \beta^N_1(t) \log F^N(t)}{x_1 \alpha^N_1(t) + x_2 \alpha^N_2(t)(1 + \beta^N_2(t)) + x_3 \alpha^N_3(t)(1 + \beta^N_3(t))}.
\]

\( g_N, \varphi_N \) and \( \psi_N \) so defined satisfy the relation (2.2).

Now we make the general assumptions which justify the description of the limit process \( \{y(t) = (x(t), q(t)), 0 \leq t < \infty\} \).

Let \( K^3 = \{x : x \in (R_+)^3, \sum_{i=1}^3 x_i = 1\} \). Let \( \alpha_i(t), \beta_i(t), \delta_i(t) \) \((1 \leq i \leq 3)\), and \( F(t) \) are continuous real valued functions defined on \([0, \infty)\), where \( F(t) > 0 \) for each \( t \geq 0 \). For any \( x \in K^3 \) and \( t \geq 0 \), \( x_1 \alpha_1(t) + x_2 \alpha_2(t)(1 + \beta_2(t)) + x_3 \alpha_3(t)(1 + \beta_3(t)) \neq 0 \). \( \alpha_i(t) \) and \( \beta_i(t) \) \((1 \leq i \leq 3)\) are bounded on \([0, \infty)\). There exist constants \( C > 0 \) and \( \lambda > 0 \) such that for \( t \in [0, \infty) \),

\[
|\delta_i(t)| \leq Ce^{\lambda t} \text{ for } 1 \leq i \leq 3 \text{ and } |\log F(t)| \leq Ce^{\lambda t}.
\]

\( A(t, x, q) \) satisfies the Condition 2.1 and is bounded on \([0, T] \times K^3 \times R\) for any \( T > 0 \). For any compact subset \( \hat{K} \) of \( R \), there exist \( C > 0 \) and \( \lambda > 0 \) such that for any \( t \geq 0 \),

\[
\sup_{(x, q) \in K^3 \times \hat{K}} \|A(t, x, q)\| \leq Ce^{\lambda t}.
\]

\( A(t, y) \) also satisfies the Lipschitz condition.

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Based on the above assumptions, we can define real valued functions \( g, \varphi, \psi \) on \([0, \infty) \times K^3 \times R\) as follows:

\[
g(t, x, q) = \frac{x_1\alpha_1(t)\beta_1(t)q}{x_1\alpha_1(t) + x_2\alpha_2(t)(1 + \beta_2(t)) + x_3\alpha_3(t)(1 + \beta_3(t)) + x_1\delta_1(t) + x_2\delta_2(t) + x_3\delta_3(t) - x_1\alpha_1(t)\beta_1(t)\log F(t)}.
\]  

(3.14)

\[
\varphi(t, x) = \frac{x_1\alpha_1(t)\beta_1(t)}{x_1\alpha_1(t) + x_2\alpha_2(t)(1 + \beta_2(t)) + x_3\alpha_3(t)(1 + \beta_3(t))},
\]

(3.15)

\[
\psi(t, x) = \frac{x_1\delta_1(t) + x_2\delta_2(t) + x_3\delta_3(t) - x_1\alpha_1(t)\beta_1(t)\log F(t)}{x_1\alpha_1(t) + x_2\alpha_2(t)(1 + \beta_2(t)) + x_3\alpha_3(t)(1 + \beta_3(t))}.
\]  

(3.16)

Then \( g, \varphi \) and \( \psi \) satisfy the relation (2.1).

**Remark 3.1.** We make the following immediate comments based on the above general assumptions.

1. (3.13) implies that (2.18) holds for \( A(t, x, q) \). 3.12 implies that (2.19) holds for \( \varphi \) and (2.20) holds for \( \psi \). Therefore, the Condition 2.5 holds. As a result, \( \varphi(t, x) \) and \( \psi(t, x) \) are bounded for \( (t, x, q) \in [0, T] \times K^3 \times R \) for any \( T > 0 \), viewing \( q \) as a dummy variable of \( \varphi(t, x) \) and \( \psi(t, x) \).

2. The boundedness of \( \alpha_i(t) \ (1 \leq i \leq 3) \) on \([0, \infty)\) is just assumed for convenience, since the numerators and denominators of \( g, \varphi, \) and \( \psi \) are linear in \( \alpha_i \ (1 \leq i \leq r) \).

3. (2.26) is equivalent to

\[
\frac{dg(t)}{dt} = g(t, x(t), q(t))
\]  

with initial condition \( q(0) = y_A(0) \). By the form of \( g \), fix \( T > 0 \), we know that for any given \( K^3 \) valued function \( x(t) \) on \([0, T] \), there exists a unique solution \( q_x(t) \) of (3.17) on \([0, T] \).

4. By Remark 4.2, \( b(t, y) = x A(t, y) \) satisfies the Lipschitz condition since \( A(t, y) \) is bounded on \([0, T] \times K^3 \times R \) for any \( T > 0 \).

We prove in the next lemma that (2.28) and (2.29) are satisfied and it follows the uniqueness of the \( C_{K^3 \times R}[0, \infty) \)-martingale problem for \( (G_A, \mu) \).

**Lemma 3.1.** Make the general assumptions on \( \alpha_i(t), \beta_i(t), \) and \( \delta_i(t) \ (1 \leq i \leq 3), F(t) \) and \( A(t, x, q) \) above. Then for fixed \( T > 0 \) and any two...
solutions \((x(t), q_x(t)), (\bar{x}(t), q_{\bar{x}}(t))\) of (2.25) and (2.26) on \([0, T]\), there exists \(M > 0\), such that \(q_x(t)\) and \(q_{\bar{x}}(t)\) satisfy

\[
(3.18) \quad \int_0^s (q_x(t) - q_{\bar{x}}(t))^2 dt \leq M \int_0^s \|x(t) - \bar{x}(t)\|^2 dt \text{ for } 0 \leq s \leq T.
\]

We conclude that the \(C_{K^3 \times R}[0, \infty)-\)martingale problem for \((G_A, \mu)\) restricted to \(C_c^2(K^3 \times R)\) has at most one solution.

We give in the next corollary the conditions which can guarantee the weak convergence of \(\{Y_N(t), 0 \leq t < \infty\}\) to \(\{y(t), 0 \leq t < \infty\}\).

**Corollary 3.1.** In addition to the general assumptions, we assume that \(\{\varphi_N\}, \varphi\) and \(\{\psi_N\}, \psi\) satisfy that for any \(T > 0\),

\[
(3.19) \quad \lim_{N \to \infty} \sup_{0 \leq t \leq T} \sup_{x \in K_N^3} |\varphi_N(t, x) - \varphi(t, x)| = 0,
\]

\[
(3.20) \quad \lim_{N \to \infty} \sup_{0 \leq t \leq T} \sup_{x \in K_N^3} |\psi_N(t, x) - \psi(t, x)| = 0.
\]

Suppose that

\[
(3.21) \quad \lim_{N \to \infty} \sup_{q \in R} d_U(A_N(\cdot, x, q), A(\cdot, x, q)) = 0
\]

and that \(P(Y_N(0))^{-1} = \mu\) for some \(\mu \in \mathcal{P}(K^3 \times R)\), then there exists some probability space \((\Omega, \mathcal{F}, P)\) a stochastic process \(Y\) satisfying \(P(Y(0))^{-1} = \mu\), which is the unique solution of the \(C_{K^3 \times R}[0, \infty)-\)martingale problem for \((G_A, \mu)\) restricted to \(C_c^2(K^3 \times R)\) and for \(P\)-a.s. \(\omega \in \Omega\), \(Y(t, \omega)'\) is the unique solution of (2.25) and (2.26), such that \(Y^N \Rightarrow Y\).

**Proof.** Note that (3.19) and (3.20) imply the Condition 2.4, and (3.21) implies the Condition 2.3.

**Remark 3.2.** The conditions in (3.19) and (3.20) are very general. One sufficient condition to guarantee (3.19) and (3.20) is to assume that for any \(T > 0\), \(\alpha_i^N(t), \beta_i^N(t), \delta_i^N(t) \quad (1 \leq i \leq 3)\), and \(F^N(t)\) converges uniformly to \(\alpha_i(t), \beta_i(t), \delta_i(t)\), and \(F(t)\) on \([0, T]\).

3.3. Fixed points of the example. In this subsection, we consider the fixed point problem for the example. Make assumptions for \(A(x, q) = (a_{ij}(x, q))_{3 \times 3}\) as follows: for each \((x, q) \in K^3 \times R,

1) \(A(x, q)\mathbf{e} = 0_{3 \times 1}\), where \(\mathbf{e} = [1, 1, 1]'\);
2) \(0 \leq a_{ij}(x, q) \leq 1\), for \(1 \leq i, j \leq 3\), \(i \neq j\).
3) $1 - \sum_{j \neq i} a_{ij}(x, q) \geq 0$ for $1 \leq i \leq 3$.

The above assumptions imply that for each $(x, q) \in K^3 \times R$, $E_3 + A(x, q)$ is a stochastic matrix. Assume also that $A(x, q)$ satisfies the Lipschitz condition and is bounded on $K^3 \times R$. Then all the conditions related to $A(x, q)$ required by Lemma 3.1 are satisfied, if viewing $A(x, q)$ as a matrix-valued function on $[0, \infty) \times K^3 \times R$.

Next, we assume that in (3.15) and (3.16), $\alpha_i(t)$, $\beta_i(t)$, and $\delta_i(t)$ ($1 \leq i \leq 3$) and $\log F(t)$ are constants which are denoted by $\alpha_i$, $\beta_i$, $\delta_i$ ($1 \leq i \leq 3$) and $\log F$. Assume also that for any $x \in K^3$, $x_1\alpha_1 + x_2\alpha_2(1 + \beta_2) + x_3\alpha_3(1 + \beta_3) \neq 0$. Then all the conditions related to $\varphi(x)$, $\psi(x)$ and $g(x, q)$ required by Lemma 3.1 are satisfied.

To get the fixed points of the following system,

\begin{align}
\frac{dx(t)}{dt} &= A(x(t), q(t))'x(t), \\
\frac{dq(t)}{dt} &= g(x(t), q(t)),
\end{align}

we need to solve the following equations:

\begin{align}
A(x, q)'x &= 0, \\
g(x, q) &= 0.
\end{align}

Note that we use $x \in K^3$ as a column vector.

At first, we assume that $\alpha_1 \neq 0$, $\beta_1 \neq 0$, $\delta_2\delta_3 > 0$. We know that $g(x, q)$ is of the following form:

\begin{equation}
g(x, q) = \frac{x_1\alpha_1\beta_1 q}{x_1\alpha_1 + x_2\alpha_2(1 + \beta_2) + x_3\alpha_3(1 + \beta_3)} + \frac{x_1\delta_1 + x_2\delta_2 + x_3\delta_3 - x_1\alpha_1\beta_1 \log F}{x_1\alpha_1 + x_2\alpha_2(1 + \beta_2) + x_3\alpha_3(1 + \beta_3)}.
\end{equation}

Then (3.25) implies that for $x \in K^3$ with $x_1 \neq 0$, we have

\begin{equation}
q_x = \frac{x_1\alpha_1\beta_1 \log F - x_1\delta_1 - x_2\delta_2 - x_3\delta_3}{x_1\alpha_1\beta_1}.
\end{equation}

Note that for any $x \in K^3$ with $x_1 = 0$, (3.25) implies that $x_2 = x_3 = 0$ by the assumption $\delta_2\delta_3 > 0$, which contradicts with $x_1 + x_2 + x_3 = 1$. Therefore, any $x \in K^3$ with $x_1 = 0$ does not solve (3.25).

(3.27) defines a function $q_x$: $K^3 \setminus \{x \in K^3, x_1 = 0\} \mapsto R$. It is clear that for any $\bar{x} \in \{x \in K^3, x_1 = 0\}$, $\lim_{x_1 \neq 0, x \to \bar{x}} q_x = \infty$ or $-\infty$. We can actually define $q_x$: $K^3 \mapsto R \cup \{-\infty, \infty\}$, where for $\bar{x} \in \{x \in K^3, x_1 = 0\}$,

\begin{equation}
q_x = \lim_{x_1 \neq 0, x \to \bar{x}} q_x.
\end{equation}
The function $q_x$ with the extended definition is a continuous function on $K^3$.

Since $A(x,q)$ satisfies the Lipschitz condition, $A(x,q)$ is a continuous function on $K^3 \times R$. We make more assumptions on $A(x,q)$, such that we can extend $A(x,q)$ to $K^3 \times (R \cup \{-\infty, \infty\})$. Assume that for any $\bar{x} \in K^3$, $\lim_{x \to \bar{x}, q \to \pm \infty} A(x,q)$ exists. Then we can extend $A(x,q)$ to $K^3 \times (R \cup \{-\infty, \infty\})$ by defining

\begin{equation}
A(\bar{x}, \pm \infty) = \lim_{x \to \bar{x}, q \to \pm \infty} A(x,q),
\end{equation}

where $\bar{x} \in K^3$. The extended function $A(x,q)$ is continuous on $K^3 \times (R \cup \{-\infty, \infty\})$.

Define for $x \in K^3$,

\begin{equation}
T(x) = A(x, q_x)'x + x.
\end{equation}

Since $E_3 + A(x, q_x)$ is a stochastic matrix, $T(x)$ is a map from $K^3$ to $K^3$. Since $q_x$ is a continuous function from $K^3$ to $R \cup \{-\infty, \infty\}$ and $A(x,q)$ is a continuous function from $K^3 \times (R \cup \{-\infty, \infty\})$ to $R^{3 \times 3}$, we have that $T(x)$ is a continuous map from $K^3$ to $K^3$. By Brouwer’s fixed point theorem [24], there exists fixed points $x^0 \in K^3$ such that $T(x^0) = x^0$. It then follows that

\begin{equation}
A(x^0, q_{x^0})'x^0 = 0.
\end{equation}

The fixed point of this example might be unique.

We have to make further assumptions on $A(x,q)$ to exclude the case $x^1_0 = 0$. Each condition as follows guarantees $x^1_0 > 0$: for arbitrary $\bar{x} \in \{x \in K^3, x_1 = 0\}$,

1. $a_{21}(\bar{x}, \pm \infty) > 0$ and $a_{31}(\bar{x}, \pm \infty) > 0$;
2. $a_{21}(\bar{x}, \pm \infty) > 0$ and $a_{32}(\bar{x}, \pm \infty) > 0$;
3. $a_{31}(\bar{x}, \pm \infty) > 0$ and $a_{23}(\bar{x}, \pm \infty) > 0$.

The fixed point $x^0$ determined by (3.30) with $x^1_0 > 0$ satisfies that $(x^0, q_{x^0})$ is the solution of the system (3.24) and (3.25), i.e. $(x^0, q_{x^0})$ is the fixed point of (3.22) and (3.23).

Remark 3.3. The set of fixed points is a subset of the set \{(x,q), x \in K^3, x_1 \alpha_1 + \sum_{i=2}^{3} x_i \alpha_i (1 + \beta_i) = 0, x_1 \alpha_1 \beta_1 (q - \log F) + \sum_{i=1}^{3} x_i \delta_i = 0\}. The latter set contains the stationary solutions or explosive solutions of our financial system.
3.4. Connection with classical stock price formula. We make the same assumptions on \(\alpha^N_i(t), \beta^N_i(t), \alpha_i(t)\) and \(\beta_i(t)\) \((1 \leq i \leq 3)\) as those in subsection 3.2. But we assume that \(\delta^N_i(t), \(1 \leq i \leq 3\),\) \(F^N(t)\) defined on some probability space \((\Omega_N, \mathcal{F}_N, P_N)\), with sample paths satisfying the conditions in subsection 3.2 \(P_N\)-a.s.; and that \(\delta_i(t) \(1 \leq i \leq 3\)\) are Brownian motions, and \(F(t)\) is a geometric Brownian motion. Assume that \((\delta^N_1, \delta^N_2, \delta^N_3, F^N) \rightarrow (\delta_1, \delta_2, \delta_3, F)\) in the sense of weak convergence. Assume also all other conditions for \(A_N(t, x, q)\) and \(A(t, x, q)\) used in subsection 3.2.

**Remark 3.4.** Duffie and Protter (see [4]) justified the weak convergence of properly scaled liquidity demand to Brownian motions. It is also usual to assume the fundamental value of a financial asset to be a geometric Brownian motion. The liquidity demands \(\delta_i(t) \(1 \leq i \leq 3\)\) and the fundamental value \(F(t)\) constitute the random environment for the limit process \(Y\).

We can follow the procedures in subsection 3.1 and 3.2 to define \(\{Y^N(t, \omega_N), 0 \leq t \leq T\}\) and determine \(\{Y(t, \omega), 0 \leq t < \infty\}\) by the *nonlinear Volterra Equations of the second kind*. As to the determination of \(\{Y(t, \omega), 0 \leq t < \infty\}\) by the nonlinear Volterra Equations of the second kind, we note that the Brownian paths \(\delta_i(t, \omega) \(1 \leq i \leq 3\)\) and \(\log F(t, \omega)\) satisfy (3.12) almost surely by the Law of the iterated logarithm. Then we can prove that \((\delta^N_1, \delta^N_2, \delta^N_3, F^N, Y^N) \Rightarrow (\delta_1, \delta_2, \delta_3, F, Y)\), see the method used in [29]. As usual, the Skorohod Representation theorem is the basic tool to establish this result. Therefore, without loss of generality, we can assume that \((\delta^N_1, \delta^N_2, \delta^N_3, F^N)\) and \((\delta_1, \delta_2, \delta_3, F)\) are defined in the same probability space \((\Omega, \mathcal{F}, P)\) and \((\delta^N_1, \delta^N_2, \delta^N_3, F^N)\) converges to \((\delta_1, \delta_2, \delta_3, F)\) almost surely. Note that for \(1 \leq i \leq 3\), \(\lim_{N \rightarrow \infty} d_U(\delta^N_i(\omega), \delta_i(\omega)) = 0\) implies that \(\lim_{N \rightarrow \infty} d_U(\delta^N_i(\omega), \delta_i(\omega)) = 0\) and hence \(\delta^N_i(\omega)\) converges to \(\delta_i(\omega)\) uniformly on \([0, T]\) for any \(T > 0\). Same thing holds for \(F^N(\omega)\) and \(F(\omega)\). Then we actually assumed or verified all the conditions required by Remark 3.2.

Note that the log price function \(q(t)\) in random environment is determined pathwisely by the *nonlinear Volterra Equations of the second kind*. That is, \(F\)-a.s for any sample point \(\omega \in \Omega\), the nonlinear Volterra Equations of the second kind determines a unique log price function \(q(t, \omega)\) on \([0, \infty)\).

This is similar to the classical assumption of the stock price formula which was suggested by Samuelson [23] in 1964: \(\log S_t = \mu t + \sigma W_t\), where \(W_t\) is a Brownian motion. Therefore, the interacting agent feedback financial system in random environment generalizes the classical stock price formula by incorporating the interaction between different types of agents and the interaction between the stock price and the empirical distribution of the types of agents.
4. Proof of Theorem 2.1. At first, we consider the moments for \( \{n^N(k), k \geq 0\} \). Let \( V = (v_1, \cdots, v_r)' \) be a positive vector. For any \( k \geq 1 \), by (2.5), (2.6), and the independence of \( \Xi_{N,k,i}'s, we have
\[
E \left[ \prod_{i=1}^{r} v_i^{n^N(k+1)}(n^N(k), \hat{q}^N(k)N) \right] = \prod_{i=1}^{r} \left( P_{N,i}[k, X^N(k), \hat{q}^N(k)N] V \right)^{n^N(k)}.
\]
It follows by (4.1) that
\[
E \left[ n^N(k+1)(n^N(k), \hat{q}^N(k)N) \right] = n^N(k)P_N[k, X^N(k), \hat{q}^N(k)N],
\]
and for \( 1 \leq i \leq r, \)
\[
E \left[ n_i^N(k+1)(n_i^N(k+1) - 1)(n^N(k), \hat{q}^N(k)N) \right] = \left( n^N(k)P_{N,i}[k, X^N(k), \hat{q}^N(k)N] \right)^2
- \sum_{j=1}^{r} \left( P_{N,j}[k, X^N(k), \hat{q}^N(k)N] P_{N,i}[k, X^N(k), \hat{q}^N(k)N] \right)^2 n_j^N(k),
\]
where \( P_{N,i}[k, X^N(k), \hat{q}^N(k)N] \) is \( i \)-th column of \( P_N[k, X^N(k), \hat{q}^N(k)N] \). Then we can get
\[
E[(n_i^N(k+1) - n_i^N(k))^2(n^N(k), \hat{q}^N(k)N)] = \left( n^N(k)P_{N,i}[k, X^N(k), \hat{q}^N(k)N] \right)^2
+ n^N(k)P_{N,i}[k, X^N(k), \hat{q}^N(k)N]
- \sum_{j=1}^{r} \left( P_{N,j}[k, X^N(k), \hat{q}^N(k)N] P_{N,i}[k, X^N(k), \hat{q}^N(k)N] \right)^2 n_j^N(k)
- 2n^N(k)P_{N,i}[k, X^N(k), \hat{q}^N(k)N]n_i^N(k) + (n_i^N(k))^2.
\]
Next, we prove that \( \{Y^N\} \) satisfies the compact containment condition under certain conditions on \( \{Y^N(0)\}, \{g_N\} \) and \( g \).

**Lemma 4.1.** Assume that \( P(Y^N(0))^{-1} \Rightarrow \mu \) for some \( \mu \in \mathcal{P}(K \times R) \), and \( g(t, x, q) \), \( g_N(t, x, q) \) satisfy (2.1) and (2.2). Assume also that for any \( T > 0, \varphi(t, x, q) \) and \( \psi(t, x, q) \) are bounded on \([0, T] \times K \times R \) and the condition
holds. Then \( \{Y^N\} \) satisfies the compact containment condition, i.e., for every \( \eta > 0 \) and \( T > 0 \), there exists a compact set \( \tilde{K}_{\eta,T} \subset K \times R \) for which

\[
\inf_N P\{Y^N(t) \in \tilde{K}_{\eta,T}, \text{ for } 0 \leq t \leq T \} \geq 1 - \eta.
\]

**Proof.** Since \( P(Y^N(0))^{-1} = \mu, \{P(q^N(0))^{-1}\} \) is tight. For any \( \eta > 0 \), there exists \( b > 0 \) such that

\[
\inf_N P\{q^N(0) \in I\} \geq 1 - \eta,
\]

where \( I = [-b,b] \).

Fix \( T > 0 \), let \( C_T > 0 \) be a bound of \( \varphi(t,x,q) \) and \( \psi(t,x,q) \) on \( [0,T] \times K \times R \). By (1.5), (2.1), (2.2), (2.16) and (2.17), there exists \( N_0 \), such that for \( N > N_0 \), we have for any \( 0 \leq k \leq [NT] - 1 \),

\[
|\bar{q}^N(\frac{k+1}{N}) - \bar{q}^N(\frac{k}{N})| \leq \frac{1}{N}C_T[|\bar{q}^N(\frac{k}{N})| + 1],
\]

which implies that for \( 0 \leq k \leq [NT] \),

\[
|\bar{q}^N(\frac{k}{N})| \leq (1 + \frac{1}{N}C_T)^k[|\bar{q}^N(0)| + 1] - 1.
\]

Let \( I_{\eta,T} = [-e^{TC_T}(b+1), e^{TC_T}(b+1)] \). Since \((1 + \frac{1}{N}C_T)^N \) increases as \( N \) does, with the limit \( e^{C_T} \), we have for \( N > N_0 \)

\[
P\{q^N(t) \in I_{\eta,T}, 0 \leq t \leq T\} = P\{q^N(0) \in I\}.
\]

Let \( \tilde{K}_{\eta,T} = K \times I_{\eta,T} \), then for \( N > N_0 \)

\[
P\{Y^N(t) \in \tilde{K}_{\eta,T}\} = P\{q^N(0) \in I\}
\]

and (4.5) holds. \( \square \)

Next, we state Lemma 4.2 which verifies (3.10) in condition (e) of Corollary 3.5 for \( \{Y^N\} \), see [30]. The proof of Lemma 4.2 is given in Appendix A.

**Lemma 4.2.** Assume that \( A(t,x,q) = (a_{i,j}(t,x,q))_{r \times r} \) satisfies the condition 2.1 and \( \{A_N(t,x,q) = (a_{N,i,j}(t,x,q))_{r \times r}\} \) satisfies the condition 2.2. Assume that \( A(t,x,q) \) is bounded on \( [0,T] \times K \times R \) for any \( T > 0 \). Assume that \( g(t,x,q) \) and \( \{g_N(t,x,q)\} \) satisfy (2.1) and (2.2). Assume that \( \varphi, \psi, \frac{\partial \varphi}{\partial x}, \frac{\partial \psi}{\partial x} \) are continuous, and the condition 2.4 holds for \( \{\varphi_N\}, \varphi \) and \( \{\psi_N\}, \psi \). If for each \( f \in C^1_r(K \times R) \) and \( T > 0 \) there exist measurable sets \( \{F_N\} \subset R \) such that (2.14) holds, then

\[
\lim_{N \to \infty} \sup_{0 \leq t \leq T} \sup_{q \in F_N} \sup_{x \in K_N} |N[S_{N,[Nt]} - I]f(x,q) - G_A(t)f(x,q)| = 0.
\]
Remark 4.1. As a special case, we can assume that \( g_N(t, x, q) = g(N, x, q) \), i.e., \( \varphi_N(t, x, q) = \varphi(N, x, q) \) and \( \psi_N(t, x, q) = \psi(N, x, q) \), for \( (t, x, q) \in [0, \infty) \times K_N \times R \). Then we do not need to assume the Condition 2.4 to get the conclusions of Lemma 4.1 and Lemma 4.2. But the Condition 2.4 holds under the stronger condition that \( \frac{\partial \varphi}{\partial t} \) and \( \frac{\partial \psi}{\partial t} \) are continuous and bounded on \([0, T] \times K \times R\) for any \( T > 0 \).

Corollary 4.1. If we assume in Lemma 4.1 that \( \{ Y_N \} \) satisfies the compact containment condition. Let \( \tilde{F}_N = K_N \times F_N \), by (2.15) and Lemma 4.2, we know that (3.9) and (3.10) in the condition (e) of Corollary 3.5 [30], hold for \( \{ \tilde{F}_N \} \). Note that

\[
\| G_A(t)f \| \leq \| A(t, x, q) \frac{\partial f}{\partial x} \| + \| \frac{\partial f}{\partial q} g(t, x, q) \|,
\]

where \( \| \cdot \| \) above is the supnorm with respect to \( (x, q) \), and that \( A(t, x, q) \) is bounded on \([0, T] \times K \times R\) for any \( T > 0 \), \( g \) is continuous on \([0, \infty) \times K \times R\), and \( f \) has a compact support, we have that \( \| G_A(t)f \| \) is bounded on \([0, T] \times K \times R\) for any \( T > 0 \). Then by the proof of Corollary 3.5 [30], we know that the equations (2.5) and (2.6) of [30] are verified. Then it follows that \( \{ Y^N \} \) is relatively compact.

Let \( K \) be the support of \( f \), then by (2.1) we get

\[
\| G_A(t)f \| \leq \max_{1 \leq i \leq r} \sup_{(x, q) \in K} \| \frac{\partial f}{\partial x_i} \| \sup_{(x, q) \in K} \| A(t, x, q) \| + \sup_{(x, q) \in K} \| \frac{\partial f}{\partial q} \| \sup_{(x, q) \in K} | \varphi(t, x, q) | + \sup_{(x, q) \in K} \| \frac{\partial f}{\partial q} \| \sup_{(x, q) \in K} | \psi(t, x, q) |,
\]

where \( \| G_A(t)f \| \) is the supnorm of \( G_A(t)f \) with respect to \( (x, q) \), and the norms on the right side of (4.9) are matrix norm and Euclidean norm used in the proof of Lemma 4.2. By (2.18), (2.19) and (2.20), we know that (2.7) in the condition (b) of Proposition 2.1 [30] is verified. Then by the proof of Corollary 3.5 [30], any limit point \( Y \) of \( Y^N \) is a solution of the \( D_{K \times R}[0, \infty) \)-martingale problem for \( (A, \mu) \) restricted to \( C^2(K \times R) \).

Now we state Lemma 4.3 and give its proof in Appendix B.
Lemma 4.3. If we assume all the assumptions made in Lemma 4.2 and Corollary 4.1, then any limit point \( Y \) of \( Y^N \) is a solution of the \( C_{K \times R}[0, \infty) \)-martingale problem for \((G_A, \mu)\) restricted to \( C_0^2(K \times R)\).

It is clear that any solution of the \( C_{K \times R}[0, \infty) \)-martingale problem for \((G_A, \mu)\) restricted to \( C_0^2(K \times R)\) is a solution of the integral equations (2.25) and (2.26).

Remark 4.2. If \( A(t,y) \) satisfies the Lipschitz condition, then \( \hat{b}(t,y) = xA(t,y) \) satisfies the Lipschitz condition under the condition that \( A(t,y) \) is bounded on \([0,T] \times K \times R\) for any \( T > 0 \). But even if \( \varphi(t,y) \) and \( \psi(t,y) \) are assumed to be bounded on \([0,T] \times K \times R\) for any \( T > 0 \) and satisfy the Lipschitz condition, we can not claim that \( b_{r+1}(t,y) = g(t,y) \) satisfies the Lipschitz condition. This is the reason that we need to expand the standard theory of nonlinear Volterra equations of the second kind for \( g(t,y) \) that does not satisfy the Lipschitz condition.

We need to introduce notations and property related to the nonlinear Volterra equations of the second kind with semi-Lipschitz conditions. Let \( V(s,t,y) \) be a measurable \( R^n \)-valued function on \([a,b] \times [a,b] \times R^n\) and \( f \) an \( L^2([a,b], R^n) \) function. Let \( m = n + 1 \). We give different notations for the first \( n \) components of \( y \), \( V(s,t,y) \) and \( f(s) \). For any vector \( y = (y_1, \cdots , y_n, y_{n+1})' \in R^{n+1} \), we let \( x = (y_1, \cdots , y_n)' \). Similarly, we let \( \hat{V}(s,t,y) = (V_1(s,t,y), \cdots , V_n(s,t,y))' \) and \( \hat{f}(s) = (f_1(s), \cdots , f_n(s))' \). Then we can represent the integral equations with kernel \( V(s,t,y) \) and a function \( f \) as follows: for \( a \leq s \leq b \),

\[
\begin{align*}
(4.10) & \quad x(s) = \hat{f}(s) + \int_a^s \hat{V}(s,t,x(t),y_{n+1}(t))dt \\
(4.11) & \quad y_{n+1}(s) = f_{n+1}(s) + \int_a^s V_{n+1}(s,t,x(t),y_{n+1}(t))dt
\end{align*}
\]

The semi-Lipschitz conditions on (4.10) and (4.11) are as follow:

(I) If \( (\hat{x}(t)', \hat{y}_{m+1}(t))' \) is any solution of (4.10) and (4.11), then \( \hat{y}_{m+1}(t) \) is the unique solution of (4.11) if we plug into (4.11) \( x(t) = \hat{x}(t) \). That is to say, there exists a dependence, which is based on (4.11), between the first \( n \) components and the last component of any solution. Then we can actually write \( \hat{y}_{m+1}(t) = \xi(t) \).

(II) \( \hat{V}(s,t,y) \) satisfies the Lipschitz condition, i.e. there exists a nonnegative \( L^2([a,b] \times [a,b], R) \) function \( \hat{V}_0(s,t) \), such that for any \( a \leq t < s \leq b \) and \( y_1 = (x_1, y_{n+1})', y_2 = (x_2, y_{n+1})' \in R^{n+1}, \)

\[
\|\hat{V}(s,t,y_1) - \hat{V}(s,t,y_2)\| \leq \hat{V}_0(s,t)\sqrt{\|x_1 - x_2\|^2 + (y_{n+1}^{(1)} - y_{n+1}^{(2)})^2}.
\]
Based on (I), if \((x(t)'', \xi(x(t))'')\) and \((\tilde{x}(t)'', \tilde{\xi}(x(t))'')\) are two solutions of (4.10) and (4.11), then there exists a nonnegative measurable function \(B(s)\) on \([a, b]\) such that

\[
\int_a^s (\xi(t) - \tilde{\xi}(t))^2 \, dt \leq B(s) \int_a^s \|x(t) - \tilde{x}(t)\|^2 \, dt \quad \text{for } a \leq s \leq b,
\]

and

\[
\int_a^b \int_a^s V_0(s, t)^2 \, dt \, ds < \infty.
\]

**Proposition 4.1.** Assume that \(V\) is a measurable \(R^{n+1}\)-valued function on \([a, b] \times [a, b] \times R^{n+1}\) satisfying the semi-Lipschitz conditions (I)-(III) above. Then the nonlinear Volterra equations of the second kind (4.10) and (4.11) have at most one \(L^2([a, b], R^{n+1})\) solution.

**Proof.** Assume that \(V_0(s, t)\) is a nonnegative \(L^2([a, b] \times [a, b], R)\) function which satisfies (4.12). Assume that \((x(t)'', \xi(x(t)))'\) and \((\tilde{x}(t)'', \tilde{\xi}(x(t)))'\) are two solutions of (4.10) and (4.11) and \(B(s)\) is a nonnegative measurable function on \([a, b]\) which satisfies (4.13) and (4.14). Let \(\varphi(s)^2 = \int_a^s V_0(s, t)^2 \, dt \, (1 + B(s))\) for \(a \leq s \leq b\) and let \(h^2 = \varphi(b)^2\). Then by (4.12), Cauchy-Schwartz inequality and (4.14), we have for \(a \leq s \leq b\),

\[
\|x(s) - \tilde{x}(s)\|^2 \leq \int_a^s \|V(s, t, x(t), \xi(x(t))) - \tilde{V}(s, t, \tilde{x}(t), \tilde{\xi}(x(t)))\| \, dt^2 \\
\leq \int_a^s V_0(s, t) \int_a^s [\|x(t) - \tilde{x}(t)\|^2 + (\xi(x(t) - \tilde{\xi}(x(t)))^2 \, dt \\
\leq \int_a^s V_0(s, t) dt \left[ \int_a^s \|x(t) - \tilde{x}(t)\|^2 \, dt + B(s) \int_a^s \|x(t) - \tilde{x}(t)\|^2 \, dt \right] \\
= \varphi(s)^2 \int_a^s \|x(t) - \tilde{x}(t)\|^2 \, dt.
\]

Put \(k^2 = \int_a^b \|x(t) - \tilde{x}(t)\|^2 \, dt\). By successive substitutions into (4.15), we can get

\[
\int_a^s \|x(t) - \tilde{x}(t)\|^2 \, dt \leq k^2 \frac{1}{1!} \left[ \int_a^s \varphi(t)^2 \, dt \right] \leq k^2 \frac{h^2 t}{l!}
\]

for any integer \(l \geq 1\) and \(a \leq s \leq b\). Let \(l \to \infty\), we conclude that \(x(s) \equiv \tilde{x}(s)\) for \(s \in [a, b]\) in the sense of \(L^2([a, b], R^n)\). By condition (I), we also get that \(\xi(x(s) \equiv \tilde{\xi}(x)\) for \(s \in [a, b]\) in the sense of \(L^2([a, b], R)\).
Lemma 4.4. Assume that \( A(t,x,q) = (a_{i,j}(t,x,q))_{\times r} \) satisfies the Condition 2.1 and is bounded on \([0,T] \times K \times R\) for any \( T > 0 \). Assume that \( g(t,x,q) \) satisfy (2.1). Assume also that \( \varphi(t,x,q) \) and \( \psi(t,x,q) \) are bounded on \([0,T] \times K \times R\) for any \( T > 0 \) and that \( \varphi, \psi, \frac{\partial \varphi}{\partial x}, \frac{\partial \psi}{\partial x} \) are continuous. Define \( b(t,y) \) and \( \hat{b}(t,y) \) by (2.21), (2.22) and (2.23). Assume that either \( b(t,y) \) satisfies the Lipschitz condition or that the condition 2.6 holds for (2.25) and (2.26). Then the \( C_{K \times R}[0,\infty) \)-martingale problem for \((G_A, \mu)\) restricted to \( C^2_c(K \times R) \) has at most one solution.

Proof. First, we prove the case when \( b(t,y) \) satisfies the Lipschitz condition. We just need to prove that the deterministic integral equation (2.24) has at most one solution. Define

\[
V(s,t,y) = \begin{cases} 
  b(t,y)' & \text{if } 0 \leq t \leq s < T, \\
  0 & \text{if } s,t \in [0,T] \text{ and } t > s.
\end{cases}
\]

Note that any solution of the integral equation (2.24) is continuous. By the Lipschitz condition on \( b(t,y) \), the nonlinear Volterra equations of the second kind (2.24) has only one continuous solution, see [26].

Second, we prove the case when (2.25) and (2.26) satisfy the semi-Lipschitz condition. This proof is similar to the first case. We just need to prove that the deterministic integral equations (2.25) and (2.26) have at most one solution. Define

\[
\hat{V}(s,t,y) = \begin{cases} 
  \hat{b}(t,y)' & \text{if } 0 \leq t \leq s < T, \\
  0 & \text{if } s,t \in [0,T] \text{ and } t > s;
\end{cases}
\]

and

\[
V_{r+1}(s,t,y) = \begin{cases} 
  g(t,y) & \text{if } 0 \leq t \leq s < T, \\
  0 & \text{if } s,t \in [0,T] \text{ and } t > s.
\end{cases}
\]

Note that any solution of the integral equations (2.25) and (2.26) is continuous. We can check easily that the semi-Lipschitz conditions (I)-(III) for \( V(s,t,y) \) required by Proposition 4.1 are satisfied. Then (2.25) and (2.26) have at most one continuous solution.

Remark 4.3. The requirement on \( g \) that for fixed \( T > 0 \) and any given continuous \( x(t) \) on \([0,T]\), (2.26) determines a unique solution \( q_x(t) \) on \([0,T]\) looks like a awkward one, but it can be easily satisfied for certain kind of functions. For example, \( g \) satisfies this requirement if we assume that the variable \( q \) in \( \varphi(t,x,q) \) and \( \psi(t,x,q) \) is a dummy variable.

Proof of Theorem 2.1 The conclusion follows by Corollary 4.1, Lemma 4.3, and Lemma 4.4. 

\[
\text{imsart-aap ver. 2005/10/19 file: IAFFM.tex date: September 17, 2018}
\]
Appendix A: Proof of Lemma 3.1. For fixed $T > 0$ and any given continuous $K^3$-valued function $x(t)$ on $[0, T]$, we define $P_x(t)$ and $Q_x(t)$ as follows:

\begin{align*}
(4.19) & \quad P_x(t) = \frac{x_1(t)\alpha_1(t)\beta_1(t)}{x_1(t)\alpha_1(t) + x_2(t)\alpha_2(t)(1 + \beta_2(t)) + x_3(t)\alpha_3(t)(1 + \beta_3(t))}, \\
(4.20) & \quad Q_x(t) = \frac{x_1(t)\delta_1(t) + x_2(t)\delta_2(t) + x_3(t)\delta_3(t) - x_1(t)\alpha_1(t)\beta_1(t)\log F(t)}{x_1(t)\alpha_1(t) + x_2(t)\alpha_2(t)(1 + \beta_2(t)) + x_3(t)\alpha_3(t)(1 + \beta_3(t))}.
\end{align*}

Then (3.17) becomes

\begin{equation}
(4.21) \quad \frac{dq(t)}{dt} = P_x(t)q + Q_x(t).
\end{equation}

The unique solution of (4.21) with initial condition $q(0) = q_x(0)$ is given by

\begin{equation}
(4.22) \quad q_x(t) = e^{\int_0^t P_x(u)du} \left[ \int_0^t Q_x(u)e^{-\int_0^u P_x(v)du} du + q_x(0) \right].
\end{equation}

Let $h_x(t) = x_1(t)\alpha_1(t) + x_2(t)\alpha_2(t)(1 + \beta_2(t)) + x_3(t)\alpha_3(t)(1 + \beta_3(t))$, the denominator of $P_x(t)$. Since $\alpha_i(t)$, $\beta_i(t)$ (1 ≤ $i$ ≤ 3) are continuous and for any $x \in K^3$ and $t \geq 0$, $x_1\alpha_1(t) + x_2\alpha_2(t)(1 + \beta_2(t)) + x_3\alpha_3(t)(1 + \beta_3(t)) \neq 0$, it follows that

\begin{equation}
(4.23) \quad B_L \leq |h_x(t)| \leq B_U, \text{ for all } t \in [0, T],
\end{equation}

where $B_L$, $B_U > 0$ do not depend on $x(t)$.

We assume that $(x(t), q_x(t))$ and $(\tilde{x}(t), q_{\tilde{x}}(t))$ are two solutions of (2.25) and (2.26) on $[0, T]$. It follows that $x(0) = \tilde{x}(0)$ and $q_x(0) = q_{\tilde{x}}(0)$. Then it is clear from (4.22) that

\begin{equation}
(4.24) \quad q_x(t) - q_{\tilde{x}}(t) = q_x(0) \left[ e^{\int_0^t P_x(u)du} - e^{\int_0^t P_{\tilde{x}}(u)du} \right] + \left[ e^{\int_0^t P_x(u)du} \int_0^t Q_x(u)e^{-\int_0^u P_x(v)du} du \right. \\
\left. - e^{\int_0^t P_{\tilde{x}}(u)du} \int_0^t Q_{\tilde{x}}(u)e^{-\int_0^u P_{\tilde{x}}(v)du} du \right].
\end{equation}

Let $M_1 = \sup_{0 \leq t \leq T} e^{\int_0^t P_x(u)du} \vee e^{\int_0^t P_{\tilde{x}}(u)du} < \infty$, then for $0 \leq t \leq T$,

\begin{equation}
(4.25) \quad |e^{\int_0^t P_x(u)du} - e^{\int_0^t P_{\tilde{x}}(u)du}| \leq M_1 |\int_0^t P_x(u) - P_{\tilde{x}}(u)du|.
\end{equation}
Observe that for \(0 \leq u \leq T\),

\[
(4.26) \quad |P_x(u) - P_{\tilde{x}}(u)| = \left| \frac{x_1(u)\alpha_1(u)\beta_1(u)}{h_x(u)} - \frac{\dot{x}_1(u)\alpha_1(\beta_1(u))}{h_{\tilde{x}}(u)} \right|
\]

\[
= \left| \alpha_1(u)\alpha_2(u)\beta_1(u)(1 + \beta_2(u))[x_1(u)\dot{x}_2(u) - \dot{x}_1(u)x_2(u)] \right|
\]

\[
\leq \frac{\alpha_1(u)\alpha_3(u)\beta_1(u)(1 + \beta_3(u))[x_1(u)\dot{x}_3(u) - \dot{x}_1(u)x_3(u)]}{h_x(u)h_{\tilde{x}}(u)}
\]

\[
\leq M_2\|\dot{x}(u) - \dot{\tilde{x}}(u)\|
\]

where \(M_2\) depends on \(B_L, B_U\) in (4.23) and \(\alpha_i\) and \(\beta_i\) (\(1 \leq i \leq 3\)). Therefore, for \(0 \leq s \leq T\),

\[
(4.27) \quad \int_0^s \left| e^{\int_0^t P_x(u)du} - e^{\int_0^t P_{\tilde{x}}(u)du} \right|^2 dt \leq \int_0^s M_1^2 \left[ \int_0^t |P_x(u) - P_{\tilde{x}}(u)|du \right]^2 dt
\]

\[
\leq \frac{M_1^2 M_2^2}{2} \int_0^s \int_0^t \|\dot{x}(u) - \dot{\tilde{x}}(u)\|^2 dudt
\]

Next we consider \(\int_0^s [e^{\int_0^t P_x(u)du} \int_0^t Q_x(u)du - \int_0^s P_x(\nu)dv du - e^{\int_0^t P_{\tilde{x}}(u)du} \int_0^t Q_{\tilde{x}}(u)du] \times e^{-\int_0^u P_{\tilde{x}}(\nu)dv du} du dt\). Notice that

\[
(4.28) \quad |e^{\int_0^t P_x(u)du} \int_0^t Q_x(u)e^{-\int_0^u P_x(\nu)dv du} du - e^{\int_0^t P_{\tilde{x}}(u)du} \int_0^t Q_{\tilde{x}}(u)e^{-\int_0^u P_{\tilde{x}}(\nu)dv du} du|
\]

\[
\leq e^{\int_0^t P_x(u)du} \int_0^t Q_x(u)e^{-\int_0^u P_x(\nu)dv du} du - e^{\int_0^t P_{\tilde{x}}(u)du} \int_0^t Q_{\tilde{x}}(u)e^{-\int_0^u P_{\tilde{x}}(\nu)dv du} du
\]

\[
+ e^{\int_0^t P_x(u)du} \int_0^t Q_x(u)e^{-\int_0^u P_x(\nu)dv du} du - e^{\int_0^t P_{\tilde{x}}(u)du} \int_0^t Q_{\tilde{x}}(u)e^{-\int_0^u P_{\tilde{x}}(\nu)dv du} du
\]

\[
= I_1(t) + I_2(t) + I_3(t)
\]
where $I_i(t)$ denotes the $i$-term on the right hand side of the inequality in (4.28). Since $P_x(t)$ and $Q_x(t)$ are continuous functions, there exists $M_3 > 0$, such that for any $0 \leq t \leq T$, $|\int_0^t Q_x(u) e^{-\int_0^u P_x(v) dv} du| \leq M_3$. Then it follows by (4.27) that for $0 \leq s \leq T$,

$$\int_0^s I_1(t)^2 dt \leq M_3^2 \int_0^s |e^\int_0^t P_x(u) du - e^\int_0^t P_x(u) du|^2 dt \leq \frac{M_3^2 M_2^2 M_3 T^2}{2} \int_0^s \|x(u) - \bar{x}(u)\|^2 du. \tag{4.29}$$

As to $I_2(t)$, we have

$$I_2(t)^2 = e^{2 \int_0^t P_x(u) du} \left| \int_0^t [Q_x(u) - Q_x(u)] e^{-\int_0^u P_x(v) dv} du \right|^2 \leq M_4^2 t \int_0^t [Q_x(u) - Q_x(u)]^2 du, \tag{4.30}$$

where $M_4 > 0$ satisfies that for any $0 \leq t \leq T$, $e^{\int_0^t P_x(u) du} \leq M_4$ and $e^{-\int_0^t P_x(u) du} \leq M_4$. Notice that for $0 \leq u \leq T$,

$$|Q_x(u) - Q_x(u)| \leq \left| \left[ (\alpha_1(u) \delta_2(u) - \alpha_2(u) \delta_1(u)(1 + \beta_2(u))) + \alpha_1(u) \alpha_2(u) \beta_1(u)(1 + \beta_2(u)) \log F(u) \right] \right| h_x(u) h_\bar{x}(u)$$

$$\times \left| \left[ x_2(u) \bar{x}_1(u) - x_1(u) \bar{x}_2(u) \right] \right| h_x(u) h_\bar{x}(u)$$

$$+ \left| \left[ x_3(u) \bar{x}_1(u) - x_1(u) \bar{x}_3(u) \right] \left[ (\alpha_1(u) \delta_3(u) - \alpha_3(u) \delta_1(u)(1 + \beta_3(u)) \right] \right| h_x(u) h_\bar{x}(u)$$

$$+ \left| \left[ x_3(u) \bar{x}_2(u) - x_2(u) \bar{x}_3(u) \right] \left[ (\alpha_2(u) \delta_3(u)(1 + \beta_2(u)) - \alpha_3(u) \delta_2(u)(1 + \beta_3(\bar{u})) \right] \right| h_x(u) h_\bar{x}(u)$$

$$\leq M_5 \|x(u) - \bar{x}(u)\|,$$

where $M_5$ depends on $B_L, B_U$ in (4.23) and $\alpha_i, \beta_i, \delta_i$ ($1 \leq i \leq 3$) and log $F$.

Then it follows by (4.30) and (4.31) that for $0 \leq s \leq T$,

$$\int_0^s I_2(t)^2 dt \leq M_4^2 M_5^2 \int_0^s t \int_0^t \|x(u) - \bar{x}(u)\|^2 du dt \leq \frac{M_4^2 M_5^2 T^2}{2} \int_0^s \|x(u) - \bar{x}(u)\|^2 du. \tag{4.32}$$
As to $I_3(t)$, we have

$$I_3(t)^2 = e^2 \int_0^t P_x(u) du \left| \int_0^t Q_x(u) [e^{-\int_0^u P_x(v) dv} - e^{-\int_0^u P_x(v) dv}] du \right|^2 \leq M_6^2 M_2^2 t \int_0^t [e^{-\int_0^u P_x(v) dv} - e^{-\int_0^u P_x(v) dv}]^2 du,$$

(4.33)

where $M_6 > 0$ satisfies that for any $0 \leq t \leq T$, $|Q_x(t)| \leq M_6$. Similar to (4.27), we can prove that there exists $M_7 > 0$, for any $0 \leq t \leq T$, such that

$$\int_0^t [e^{-\int_0^u P_x(v) dv} - e^{-\int_0^u P_x(v) dv}]^2 du \leq M_7^2 \int_0^t \|x(u) - \tilde{x}(u)\|^2 du.$$

Then it follows that for $0 \leq s \leq T$,

$$\int_0^s I_3(t)^2 dt \leq \frac{M_4^2 M_2^2 M_7^2 T^2}{2} \int_0^s \|x(u) - \tilde{x}(u)\|^2 du.$$

(4.35)

Thus, by (4.28), (4.29), (4.32), (4.35), we get

$$\int_0^s \left| e^{t} \int_0^t P_x(u) du \int_0^t Q_x(u) [e^{-\int_0^u P_x(v) dv} - e^{-\int_0^u P_x(v) dv}] du - e^{t} \int_0^t P_x(u) du \int_0^t Q_x(u) [e^{-\int_0^u P_x(v) dv} - e^{-\int_0^u P_x(v) dv}] du \right|^2 dt \leq \frac{3(M_4^2 M_2^2 M_7^2 T^2 + M_4^2 M_6^2 T^2 + M_3^2 M_5^2 M_6^2 T^2)}{2} \int_0^s \|x(u) - \tilde{x}(u)\|^2 du.$$

(4.36)

By (4.24), (4.27) and (4.36), (3.18) is true for some $M > 0$.

The uniqueness of the $C_{K^3 \times R}[0, \infty)$-martingale problem for $(G_A, \mu)$ follows by (3.18) and Lemma 4.4. \hfill \Box

Appendix B: Proof of Lemma 4.2.

PROOF. Let $f \in C^2_0(K \times R)$ and fix $T > 0$. Assume that $\tilde{K}$ is the support of $f$. For $0 \leq t \leq T$, and $y = (x, q) \in K_N \times F_N$, by Taylor’s expansion, (4.2) and (2.3)

$$[S_{N,N} - I] f(y)$$

(4.37)

$$= E[(X^N(t + \frac{1}{N}) - x) \frac{\partial f}{\partial x}(y)'|Y^N(t) = y] + E[(q^N(t + \frac{1}{N}) - q) \frac{\partial f}{\partial q}(y)'|Y^N(t) = y]$$

$$+ E[(Y^N(t + \frac{1}{N}) - y) \frac{\partial^2 f}{\partial y^2}(y)'(Y^N(t) - y)'|Y^N(t) = y]$$

$$\leq \frac{1}{N} [A_N \frac{[Nt]}{N}, y] \frac{\partial f}{\partial x}(y)' + \frac{1}{N} E\left\{ g_N \left( \frac{[Nt]}{N}, X^N(t + \frac{1}{N}), q \right) \frac{\partial f}{\partial q}(y)'|Y^N(t) = y \right\}$$

$$+ E[(Y^N(t + \frac{1}{N}) - y) \frac{\partial^2 f}{\partial y^2}(y)'(Y^N(t) - y)'|Y^N(t) = y],$$
where $Y^N_*(t) = y + \theta^N_t(Y^N(t + \frac{1}{N}) - y)$, for some $\theta^N_t \in (0, 1)$.

Notice that

$$g_N\left(\frac{[Nt]+1}{N}, X^N(t + \frac{1}{N}), q\right) - g(t, x, q)$$

$$= \left[ g_N\left(\frac{[Nt]+1}{N}, X^N(t + \frac{1}{N}), q\right) - g\left(\frac{[Nt]+1}{N}, X^N(t + \frac{1}{N}), q\right) \right]$$

$$+ \left[ g\left(\frac{[Nt]+1}{N}, X^N(t + \frac{1}{N}), q\right) - g\left(\frac{[Nt]+1}{N}, x, q\right) \right]$$

and

$$g\left(\frac{[Nt]+1}{N}, X^N(t + \frac{1}{N}), q\right) - g\left(\frac{[Nt]+1}{N}, x, q\right)$$

$$= (X^N(t + \frac{1}{N}) - x)\frac{\partial g}{\partial x}\left(\frac{[Nt]+1}{N}, X^N_*(t + \frac{1}{N}), q\right)' ,$$

where $\frac{\partial g}{\partial x} = [\frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_N}]$, $X^N_*(t + \frac{1}{N}) = x + \zeta^N_{t+\frac{1}{N}}(X^N(t + \frac{1}{N}) - x)$, for some $\zeta^N_{t+\frac{1}{N}} \in (0, 1)$.

Then it follows that

$$\left| N[S_N, [Nt] - I]f(y) - G_A(t)f(y) \right|$$

$$\leq \left| x[A_N\left(\frac{[Nt]}{N}, y\right) - A(t, y)]\frac{\partial f}{\partial x}(y)^\prime \right| + E\left[ \left( g_N\left(\frac{[Nt]+1}{N}, X^N(t + \frac{1}{N}), q\right) \right) \frac{\partial f}{\partial q}(y) \left| Y^N(t) = y \right| \right]$$

$$+ E\left[ \left( (X^N(t + \frac{1}{N}) - x)\frac{\partial g}{\partial x}\left(\frac{[Nt]+1}{N}, X^N_*(t + \frac{1}{N}), q\right)' \frac{\partial f}{\partial q}(y) \left| Y^N(t) = y \right| \right) \right]$$

$$+ E\left[ \left( g\left(\frac{[Nt]+1}{N}, x, q\right) - g(t, x, q) \right)\frac{\partial f}{\partial q}(y) \left| Y^N(t) = y \right| \right]$$

$$+ N\left[ (Y^N(t + \frac{1}{N}) - y)\frac{\partial^2 f}{\partial y^2}(Y^N_*(t))(Y^N(t + \frac{1}{N}) - y)^\prime \left| Y^N(t) = y \right| \right]$$

$$= I_1(N, y, t) + I_2(N, y, t) + I_3(N, y, t) + I_4(N, y, t) + I_5(N, y, t).$$

Notice that for each $(x, q) \in K_N \times R$, $A_N(t, x, q)$ is a constant on $[\frac{k}{N}, \frac{k+1}{N})$ for each $k \geq 0$, we have $A_N(t, x, q) = A_N\left(\frac{[Nt]}{N}, x, q\right)$ for $t \geq 0$. The norm of matrices is defined as follows: for real-valued matrix $B = (b_{i,j})_{n \times r}, \|B\| = \ldots$
\[ I_1(N, y, t) \leq \| A_N(t, y) - A(t, y) \| \cdot \| \frac{\partial f}{\partial x}(y) \|. \]

Then it follows by (2.14) that

\[ \lim_{N \to \infty} \sup_{0 \leq t \leq T} \sup_{q \in F_N x \in K_N} I_1(N, x, q, t) = 0. \]

Next, we consider \( I_5(N, y, t) \). Let \( \| \frac{\partial^2 f}{\partial y^2} \| = \max_{1 \leq i,j \leq r+1} \| \frac{\partial^2 f}{\partial y_i \partial y_j} \| \) and define \( g_{N,K} \) as follows: \( g_{N,K}(t, y) = g_N(t, y) \), if \( (t, y) \in [0, \infty) \times (K \cap (K_N \times R)) \); \( g_{N,K}(t, y) = 0 \) otherwise. By Hölder inequality and Cauchy-Schwartz inequality,

\[ I_5(N, y, t) \leq \| \frac{\partial^2 f}{\partial y^2} \| N \left\{ \sum_{i=1}^{r} \sum_{j=1}^{r} E \left[ \left| (X_i^N(t + \frac{1}{N}) - x_i)(X_j^N(t + \frac{1}{N}) - x_j) \right| \cdot Y^N(t) = y \right] \right. \\
+ \frac{2}{N} \sum_{i=1}^{r} E \left[ \left| (X_i^N(t + \frac{1}{N}) - x_i)g_{N,K} \left( \frac{[Nt] + 1}{N}, X^N(t + \frac{1}{N}), q \right) \right| \cdot Y^N(t) = y \right] \\
+ \left. \frac{1}{N^2} E \left[ \left| g_{N,K} \left( \frac{[Nt] + 1}{N}, X^N(t + \frac{1}{N}), q \right) \right|^2 \cdot Y^N(t) = y \right] \right\} \]

where \( \| g_{N,K} \|_{T+1} = \sup_{0 \leq t \leq T+1} \sup_{y \in K \cap (K_N \times R)} \| g_N(t, y) \| \). By (2.16), (2.17) and the assumption that \( \varphi(t, x, q) \) and \( \psi(t, x, q) \) are continuous on \([0, T] \times K \times R\) for any \( T > 0 \), it follows that

\[ \sup_{N \geq 1} \| g_{N,K} \|_{T+1} < \infty. \]
For fixed $1 \leq i \leq r$, by (4.4), we get
\begin{equation}
N^2E[(X_i^N(t + \frac{1}{N}) - x_i)^2 | Y^N(t) = y]
= xA_{N,i}(\frac{[Nt]}{N}, y) + (xA_{N,i,i}(\frac{[Nt]}{N}, y))^2 - 2x_ia_{N,i,i}(\frac{[Nt]}{N}, y) \sum_{k=1}^r \frac{x_k}{N}a_{N,k,i}(\frac{[Nt]}{N}, y)
\leq \sum_{l=1}^r |A_{N,i,i}(t, y)| + (\sum_{l=1}^r |A_{N,i,i}(t, y)|)^2 + 2|a_{N,i,i}(t, y)|.
\end{equation}

Then it follows by (4.43), (4.44), (4.45), (2.14) and the assumption that $A(t, x, q)$ is bounded on $[0, T] \times K \times R$, that
\begin{equation}
\lim_{N \to \infty} \sup_{0 \leq t \leq T} \sup_{q \in Q} \sup_{x \in K_N} I_5(N, x, q, t) = 0.
\end{equation}

Similarly, by (2.16) and (2.17), we can get
\begin{equation}
\lim_{N \to \infty} \sup_{0 \leq t \leq T} \sup_{q \in Q} \sup_{x \in K_N} I_2(N, x, q, t) = 0,
\end{equation}
and by (4.45), (2.14) and some calculations, we can get that
\begin{equation}
\lim_{N \to \infty} \sup_{0 \leq t \leq T} \sup_{q \in Q} \sup_{x \in K_N} I_3(N, x, q, t) = 0.
\end{equation}

Notice that for any $(x, q) \in K \times R$ and $0 \leq t \leq T$,
\begin{equation}
I_4(N, x, q, t) \leq \sup_{0 \leq t \leq T} \sup_{(x, q) \in K} \left| g\left(\frac{[Nt]}{N}, x, q\right) - g(t, x, q) \right| \cdot \sup_{y \in R} \left| \frac{\partial f}{\partial q}(y) \right|,
\end{equation}
then it follows by the uniform continuity of $g(t, x, q)$ on the compact set $[0, T + 1] \times K$ that
\begin{equation}
\lim_{N \to \infty} \sup_{0 \leq t \leq T} \sup_{q \in Q} \sup_{x \in K_N} I_4(N, x, q, t) = 0,
\end{equation}
and (4.8) is proved. \hfill \Box

**Appendix C: Proof of Lemma 4.3.** Assume that $Y = (X, Q)$, where $X = (X_1, \cdots, X_r)$, is a limit point of $Y^N$. By Corollary 4.1, $Y$ is a solution of the $D_{K \times R}[0, \infty)$-martingale problem for $(G_A, \mu)$ restricted on $C^2(K \times R)$. We just need to show that $Y$ is continuous almost surely.

Fix $T > 0$. Since $A(t, x, q), \varphi(t, x, q)$ and $\psi(t, x, q)$ are bounded on $[0, T] \times K \times R$, there exists $C_T > 0$, such that for $1 \leq i \leq r$,
\begin{equation}
\sup_{0 \leq t \leq T} \sup_{y \in K \times R} |b_i(t, y)| \leq C_T, \quad \sup_{0 \leq t \leq T} \sup_{y \in K \times R} |\varphi(t, y)| \leq C_T, \quad \sup_{0 \leq t \leq T} \sup_{y \in K \times R} |\psi(t, y)| \leq C_T.
\end{equation}
By (2.16) and (2.17), there exists $N_0$ such that for $N > N_0$, we have
(4.52)
$$
sup_{0 \leq t \leq T} \sup_{y \in K_N \times R} |\varphi_N(t,y)| \leq C_T + 1, \quad sup_{0 \leq t \leq T} \sup_{y \in K_N \times R} |\psi_N(t,y)| \leq C_T + 1.
$$

Let $f_i(x,q) = x_i$, $1 \leq i \leq r$. It follows that $f_i \in C^1_0(K \times R)$ for $1 \leq i \leq r$ and $1 \leq j \leq 4$. We assume that $\{\mathcal{F}_s, 0 \leq s < \infty\}$ is the filtration to which the $D_{K \times R}[0, \infty)$-martingale problem for $(G, \mu)$ referred. Then $X_i(t) - j \int_0^t (X_i(u))^j - 1 X(u) \delta(i(u,Y(u))) du = X_i^j(t) - j \int_0^t (X_i(u))^j - 1 b_i(u,Y(u))du$ is an $\{\mathcal{F}_s\}$-martingale for $1 \leq i \leq r$ and $1 \leq j \leq 4$. Let $0 \leq s < t \leq T$, and fix $1 \leq i \leq r$, it follows that
(4.53)
$$
E[(X_i(t) - X_i(s))^2] = j E[\int_s^t (X_i(u) - X_i(s))^j - 1 b_i(u, Y(u)) du]
$$
for $1 \leq j \leq 4$. Then by (4.53), for $j = 2$, we have
(4.54)
$$
E[(X_i(t) - X_i(s))^2] \leq 2E[\int_s^t |X_i(u) - X_i(s)| |b_i(u, Y(u))| du] \\
\leq 2C_T(t - s).
$$

By (4.53) and (4.54), for $j = 4$,
(4.55)
$$
E[(X_i(t) - X_i(s))^4] \leq 4C_T E[\int_s^t |X_i(u) - X_i(s)|^3 du] \\
\leq 4C_T E[\int_s^t |X_i(u) - X_i(s)|^2 du] \\
\leq 4C_T \int_s^t 2C_T(u - s) du \\
\leq 4(C_T)^2(t - s)^2.
$$

Then by Kolmogorov’s Criterion, we proved for $1 \leq i \leq r$ that $X_i$ is continuous almost surely.

Next, we prove that $Q$ is also continuous almost surely. We introduce the notations in Chapter 3, Section 10 [5]. Let $(E, r)$ be a metric space. For $x \in D_E[0, \infty)$, define
(4.56)
$$
J(x) = \int_0^\infty e^{-u}[J(x,u) \wedge 1] du,
$$
where $J(x,u)$ is defined by
(4.57)
$$
J(x,u) = \sup_{0 \leq t \leq u} r(x(t), x(t-)).
$$
Since $Y$ is a limit point of $Y^N$, we have that a subsequence $\{q_{N_k}\}$ of $\{q^N\}$ converges weakly to $Q$. To prove that $Q$ is continuous almost surely, by Chapter 3, Theorem 10.2 [5], it suffices to prove that $J(q_{N_k}) \to 0$ as $k \to \infty$.

In this case $E = R$ and $r$ is the Euclidean metric. It is enough to show that $\lim_{k \to \infty} E[J(q_{N_k})] = 0$.

By Lemma 4.1, $\{q_{N_k}\}$ satisfies the compact containment condition, i.e. for any $\eta > 0$ and $T > 0$, there exists $B_T > 0$, such that

$$\inf_k P\{|q_{N_k}(t)| \leq B_T, 0 \leq t \leq T\} \geq 1 - \eta.$$  

By the construction of $q^N$, (1.5) and (4.52), for fixed $N_k \geq N_0$ and $0 \leq u \leq T$,

$$J(q_{N_k}, u) = \max_{0 \leq j \leq \lfloor N_k u \rfloor} |\tilde{q}_{N_k}(\frac{j}{N_k}) - \tilde{q}_{N_k}(\frac{j-1}{N_k})|$$

$$\leq \frac{1}{N_k} \max_{0 \leq j \leq \lfloor N_k u \rfloor} \left[|\varphi_{N_k}(\frac{j}{N_k}, X(\frac{j}{N_k}), \tilde{q}_{N_k}(\frac{j-1}{N_k}))\tilde{q}_{N_k}(\frac{j-1}{N_k})| + |\psi_{N_k}(\frac{j}{N_k}, X(\frac{j}{N_k}), \tilde{q}_{N_k}(\frac{j-1}{N_k}))|\right]$$

$$\leq \frac{1}{N_k} (C_T + 1)(B_T + 1),$$

on the event $F_{k,T} = \{|q_{N_k}(t)| \leq B_T, 0 \leq t \leq T\}$. It follows by (4.58) and (4.59) that

$$E[J(q_{N_k})] \leq e^{-T} + E\left[\int_0^T e^{-u}(J(q_{N_k}, u) \land 1)\,du\right]$$

$$\leq e^{-T} + E\left[\int_0^T e^{-u}(J(q_{N_k}, u) \land 1)\,du\,\chi_{F_{k,T}}\right] + P(F_{k,T}^c)$$

$$\leq e^{-T} + \frac{T}{N_k} (C_T + 1)(B_T + 1) + \eta.$$  

Let $k \to \infty$ and then let $T \to \infty$, $\eta \to 0$, we proved that $\lim_{k \to \infty} E[J(q_{N_k})] = 0$.  

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