CANTOR POLYNOMIALS FOR SEMIGROUP SECTORS

MELVYN B. NATHANSON

Abstract. A packing function on a set Ω in \( R^n \) is a one-to-one correspondence between the set of lattice points in Ω and the set \( N_0 \) of nonnegative integers. It is proved that if \( r \) and \( s \) are relatively prime positive integers such that \( r \) divides \( s - 1 \), then there exist two distinct quadratic packing polynomials on the sector \( \{(x, y) \in R^2 : 0 \leq y \leq sx/r \} \). For the rational numbers \( 1/s \), these are the unique quadratic packing polynomials. Moreover, quadratic quasi-polynomial packing functions are constructed for all rational sectors.

1. The Cantor polynomial packing problem

A packing function on a set Ω in \( R^n \) is a function that is a one-to-one correspondence between the set of lattice points in Ω and the set \( N_0 \) of nonnegative integers. A packing polynomial on Ω is a packing function that is a polynomial. Packing polynomials are used in computer science to store and retrieve multi-dimensional arrays in linear memory. In this paper we consider only \( n = 2 \).

Define \( e_{2,s} = (s, 1) \) for all \( s \in N_0 \). Let \( e_1 = (1, 0) \) and \( e_2 = e_{2,0} = (0, 1) \). The standard basis for \( R^2 \) is \( \{e_1, e_2\} \). We define \( 1/0 = \infty \) and \( 1/\infty = 0 \).

For every positive real number \( \alpha \) we consider the real sector
\[
S(\alpha) = \{(x, y) \in R^2 : 0 \leq y \leq \alpha x\}
\]
and the integer sector
\[
I(\alpha) = S(\alpha) \cap N_0^2 = \{(x, y) \in N_0^2 : 0 \leq y \leq \alpha x\}.
\]
If \( (a, b) \in S(\alpha) \setminus \{(0, 0)\} \), then \( b/a \leq \alpha \). A sector is called rational if \( \alpha \) is a rational number and irrational if \( \alpha \) is an irrational number. The real sector \( S(\alpha) \) is the cone with vertex at \( (0, 0) \) generated by the points \( (1, 0) \) and \( (1, \alpha) \). Equivalently, \( S(\alpha) \) is the convex hull of the rays \( \{(x, 0) : x \geq 0\} \) and \( \{(x, \alpha x) : x \geq 0\} \).

We also define the real and integer sectors
\[
S(\infty) = \{(x, y) \in R^2 : x \geq 0 \text{ and } y \geq 0\}
\]
and
\[
I(\infty) = S(\infty) \cap N_0^2 = N_0^2.
\]
The real sector \( S(\infty) \) is the cone with vertex at \( (0, 0) \) generated by the points \( (1, 0) \) and \( (0, 1) \).

Date: May 14, 2013.
2010 Mathematics Subject Classification. 05A15, 11B34, 11B75, 03D15, 03D20.
Key words and phrases. Cantor polynomial, packing polynomial, lattice point enumeration, cone semigroup, multi-dimensional arrays, recursion theory, quasi-polynomial.
Cantor proved that the sector \( I(\infty) \) has the same cardinality as the set \( \mathbb{N}_0 \) by constructing two explicit packing polynomials from \( I(\infty) \) to \( \mathbb{N}_0 \):

\[
F_\infty(x, y) = \frac{(x + y)^2}{2} + \frac{x + 3y}{2}
\]

and

\[
G_\infty(x, y) = \frac{(x + y)^2}{2} + \frac{3x + y}{2}
\]

These are called the Cantor polynomials. Using methods from analytic number theory, Fueter and Pólya [2] proved in 1923 that these are the unique quadratic polynomials that are bijections from \( I(\infty) \) to \( \mathbb{N}_0 \). Recently, Vsemirnov [14] gave an elementary proof using only quadratic reciprocity. The survey paper [11] contains an exposition of Vsemirnov’s proof and other results on packing polynomials.

It has been conjectured since the work of Fueter and Pólya that the two Cantor polynomials are the only packing polynomials on \( I(\infty) \), or, equivalently, that every packing polynomial on \( I(\infty) \) is quadratic. Lew and Rosenberg [6] proved that there is no cubic or quartic packing polynomial on \( I(\infty) \). Despite considerable work (see references), the Cantor polynomial problem is still unsolved. In this paper we construct two quadratic packing polynomials on \( I(\alpha) \) for certain rational numbers \( \alpha \), and apply the Fueter-Pólya theorem to prove that these are the only quadratic packing polynomials on these sectors. We also construct quadratic quasi-polynomial packing functions for all rational sectors. The final section lists some related open problems.

2. SECTORS AND SEMIGROUPS

A subset \( W \) of an additive abelian semigroup is a generating set for the semigroup if every element of the semigroup can be represented as a finite nonnegative integral linear combination of elements of \( W \). Such additive representations are not necessarily unique. An additive abelian semigroup is free of rank \( k \) if it contains a generating set \( W = \{ w_1, \ldots, w_k \} \) such that every element in the semigroup has a unique representation in the form \( \sum_{i=1}^{k} x_i w_i \) with \( x_i \in \mathbb{N}_0 \) for \( i = 1, \ldots, k \). The set \( W = \{ w_1, \ldots, w_k \} \) is called a free basis for the semigroup. If \( w \) is an element of a free basis, then \( w \neq 0 \).

For every real number \( \alpha > 0 \) and for \( \alpha = \infty \), the real sector \( S(\alpha) \) and the integer sector \( I(\alpha) \) are additive abelian semigroups. The following lemma identifies the numbers \( \alpha \) such that \( I(\alpha) \) is a free semigroup.

**Theorem 1.** The additive abelian semigroup \( I(\alpha) \) is free of rank \( k \) if and only if \( k = 2 \) and \( \alpha \in \{1/s : s \in \mathbb{N}_0 \} \). Moreover, \( \{(1,0),(s,1)\} \) is the unique free basis of \( I(1/s) \) for all \( s \in \mathbb{N}_0 \).

**Proof.** Let \( \alpha \) be a positive real irrational number, and let \( W \) be a finite subset of \( I(\alpha) \setminus \{(0,0)\} \). Let \( W = \{w_1, \ldots, w_k\} \), where \( w_i = (a_i, b_i) \in I(\alpha) \setminus \{(0,0)\} \) and \( a_i \neq 0 \) for \( i = 1, \ldots, k \). We define

\[
\lambda = \min \left\{ \frac{b_i}{a_i} : i = 1, \ldots, k \right\}
\]

and

\[
\mu = \max \left\{ \frac{b_i}{a_i} : i = 1, \ldots, k \right\}.
\]
The cone generated by the nonnegative rays \( y = \lambda x \) and \( y = \mu x \) is
\[
C = \{(x, y) \in S(\infty) : \lambda x \leq y \leq \mu x \}.
\]
This cone contains \( W \), and so contains the additive subgroup generated by \( W \).

Because \( b_i/a_i \leq \alpha \) and \( \alpha \) is irrational, we have \( b_i/a_i < \alpha \) for all \( i = 1, \ldots, k \), and so \( \mu < \alpha \). There exist positive integers \( c \) and \( d \) such that
\[
\mu \leq \frac{c}{d} \leq \alpha
\]
and the lattice point \((c, d)\) belongs to \( I(\alpha) \) but not to \( C \). Thus, the semigroup \( I(\alpha) \) is not finitely generated. In particular, \( I(\alpha) \) is not a free abelian semigroup of rank \( k \) for any \( k \in \mathbb{N} \).

Let \( \Gamma \neq \{(0, 0)\} \) be an additive abelian semigroup contained in \( \mathbb{N}_0^2 \) or, more generally, in \( \mathbb{Q}_0^2 \), where \( \mathbb{Q}_0 \) is the set of nonnegative rational numbers. If \( \Gamma \) is free of rank \( k \geq 3 \), then there exist \( w_1, w_2, w_3 \in \Gamma \) that are elements of a free basis for \( \Gamma \). Because
\[
w_i \in \mathbb{Q}_0^2 \setminus \{(0, 0)\} \subseteq \mathbb{Q}^2 \setminus \{(0, 0)\}
\]
for \( i = 1, 2, 3 \), it follows that \( w_1, w_2, w_3 \) are \( \mathbb{Q} \)-linearly dependent, and there exist rational numbers \( t_1, t_2, t_3 \), not all 0, such that \( t_1w_1 + t_2w_2 + t_3w_3 = (0, 0) \). The vectors \( w_i \) have nonnegative coordinates, and so \( t_i > 0 \) for some \( i \) and \( t_j < 0 \) for some \( j \). Thus, we can assume that \( t_1 > 0, t_2 \geq 0, \) and \( t_3 < 0 \). We have
\[
t_1w_1 + t_2w_2 = -t_3w_3.
\]
Multiplying by a common multiple of the denominators of the fractions \( t_1, t_2, t_3 \), we obtain nonnegative integers \( x_1, x_2, x_3 \) with \( x_1 \geq 1 \) and \( x_3 \geq 1 \) such that
\[
x_1w_1 + x_2w_2 = x_3w_3.
\]
This is impossible if \( w_1, w_2, w_3 \) belong to a free basis for \( \Gamma \). Thus, \( k \leq 2 \). If \( k = 0 \), then \( \Gamma = \{(0, 0)\} \), which is absurd. Thus, if \( \Gamma \) is a free abelian semigroup of rank \( k \), then \( k = 1 \) or 2.

If \( \alpha > 0 \) or \( \alpha = \infty \), then \((x, 0) \in I(\alpha)\) for all \( x \in \mathbb{N}_0 \), and \((x, 1) \in I(\alpha)\) for all \( x \geq 1/\alpha \). Thus, \( I(\alpha) \) is an abelian semigroup in \( \mathbb{Q}_0^2 \) that does not lie on a line, and so, if \( I(\alpha) \) is free of rank \( k \), then \( k = 2 \).

Let \( r \) and \( s \) be relatively prime positive integers with \( r \geq 2 \). The semigroup \( I(r/s) \) is contained in the convex hull of the nonnegative rays \( y = 0 \) and \( y = rx/s \). The lattice points \((1, 0)\) and \((s, r)\) are in \( I(r/s) \) and lie on the rays \( y = 0 \) and \( y = rx/s \), respectively. Let \( W = \{w_1, w_2\} \subseteq I(r/s) \setminus \{(0, 0)\} \). If \( W \) generates \( I(r/s) \), then there exist nonnegative integers \( x_1 \) and \( x_2 \) such that such that \((1, 0) = x_1w_1 + x_2w_2 \) for \( x_1, x_2 \in \mathbb{N}_0 \). This is possible only if \( w_1 = (1, 0) \) for \( i = 1 \) or 2. Let \( w_1 = (1, 0) \). Because \( W \) generates \( I(r/s) \), then there exist nonnegative integers \( y_1 \) and \( y_2 \) such that such that \((s, r) = y_1(1, 0) + y_2w_2 \). It follows that \( y_2w_2 = (s, r) - y_1(1, 0) = (s - y_1, r) \in I(r/s) \). However, if \((a, r) \in I(r/s) \), then \( a \geq s \) and so \( y_1 = 0 \) and \((s, r) = y_2w_2 \). Because \( r \) and \( s \) are relatively prime, we conclude that \( y_2 = 1 \) and \( w_2 = (s, r) \). Thus, every element generated by \( W \) is of the form
\[
x_1(1, 0) + x_2(s, r) = (x_1 + x_2s, x_2r)
\]
with \( x_1, x_2 \in \mathbb{N}_0 \). Because \((x, 1) \in I(r/s) \) for all \( x \geq s/r \), it follows that \( x_2r = 1 \) for some \( x_2 \in \mathbb{N}_0 \), and this is possible only if \( x_2 = r = 1 \). Thus, if the integer sector \( I(r/s) \) is generated by a set \( W \) of cardinality 2, then \( r = 1 \) and \( W = \{(1, 0), (s, 1)\} \).
If \((x, y) \in I(1/s)\), then \(y \leq x/s\) or, equivalently, \(x - sy \in \mathbb{N}_0\), and
\[
(x, y) = ((x - sy) + sy, y) = (x - sy, 0) + (sy, y) = (x - sy)(1, 0) + y(s, 1).
\]
This representation of \((x, y)\) is unique because the vectors \((1, 0)\) and \((s, 1)\) are linearly independent in \(\mathbb{R}^2\). It follows that \(I(1/s)\) is free of rank 2, and that \(\{(1, 0), (s, 1)\}\) is the unique basis for \(I(1/s)\).

Finally, the semigroup \(I(\infty) = \mathbb{N}_0^2\) is free, and \(\{(1, 0), (0, 1)\}\) is the unique free basis for \(I(\infty)\). This completes the proof. \(\square\)

**Theorem 2.** There exists a linear transformation from \(\mathbb{R}^2\) to \(\mathbb{R}^2\) whose restriction to \(I(\infty)\) is a one-to-one function onto \(I(\alpha)\) if and only if \(\alpha \in \{1/s : s \in \mathbb{N}_0\}\). Moreover, for each \(s \in \mathbb{N}_0\) there are exactly two such linear transformations, whose matrices with respect to the standard basis are

\[
(1) \quad \Lambda_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad M_s = \begin{pmatrix} s & 1 \\ 1 & 0 \end{pmatrix}.
\]

**Proof.** Let \(T : \mathbb{R}^2 \to \mathbb{R}^2\) be a linear transformation. Let \(e_1 = (1, 0)\) and \(e_2 = (0, 1)\), and let \(w_1 = T(e_1)\) and \(w_2 = T(e_2)\). Then \(I(\infty)) = \{xe_1 + ye_2 : x, y \in \mathbb{N}_0\}\) and
\[
T(I(\infty)) = \{xT(e_1) + yT(e_2) : x, y \in \mathbb{N}_0\} = \{xw_1 + yw_2 : x, y \in \mathbb{N}_0\}.
\]
Thus, \(T(I(\infty))\) is the additive abelian semigroup generated by the set \(\{w_1, w_2\}\). If the restriction of \(T\) to \(I(\infty)\) is a one-to-one function, then \(T(I(\infty))\) is a free semigroup of rank 2, and \(\{w_1, w_2\}\) is a free basis for \(T(I(\infty))\). If the restriction of \(T\) to \(I(\infty)\) is a one-to-one function onto \(I(\alpha)\), then \(I(\alpha)\) is a free semigroup of rank 2, and \(\alpha \in \{1/s : s \in \mathbb{N}_0\}\) by Theorem 1.

The unique free basis for \(I(\infty)\) is \(\{e_1, e_2\}\). If the restriction of \(T\) to \(I(\infty)\) is a one-to-one function onto \(I(\infty)\), then \(T = \Lambda_0 = \text{id}\) or \(T = M_0\). For every positive integer \(s\), the unique free basis for \(I(1/s)\) is \(\{e_1, e_2, s\}\). If the restriction of \(T\) to \(I(\infty)\) is a one-to-one function onto \(I(1/s)\), then \(T = \Lambda_s\) or \(T = M_s\). This completes the proof. \(\square\)

For all nonnegative integers \(s\) and \(t\), we have \(\Lambda_{s+t} = \Lambda_s \Lambda_t\), and so \(\Lambda_s = \Lambda_s^t\) and \(\Lambda_s^{-1} = \Lambda_{-s}\). Moreover, \(\Lambda_s\) is a bijection from \(I(1/t)\) to \(I(1/(s + t))\), and \(M_s = \Lambda_s M_0\).

**Theorem 3.** For every nonnegative integer \(s\), the linear transformation
\[
(2) \quad \Phi_s = \begin{pmatrix} s & 1 - s^2 \\ 1 & -s \end{pmatrix}
\]
is the unique non-identity linear transformation whose restriction to \(I(1/s)\) is a bijection onto \(I(1/s)\). Moreover, \(\Phi_s\) is an involution.

For every positive integer \(r\), the linear transformation
\[
(3) \quad \Psi_r = \begin{pmatrix} 1 & 0 \\ r & -1 \end{pmatrix}
\]
is an involution whose restriction to \(I(r)\) is a bijection onto \(I(r)\).

Note that \(\Phi_1 = \Psi_1 = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}\).
Proof. By Theorem 2 the unique free basis for $I(1/s)$ is $\{e_1, e_{2,s}\}$. If $\Phi_s$ is a non-identity linear transformation that induces a bijection on $I(1/s)$, then $\Phi_s(e_1) = e_{2,s}$ and $\Phi_s(e_{2,s}) = e_1$. Because $e_2 = e_{2,s} - se_1$, we have

$$\Phi_s(e_2) = \Phi_s(e_{2,s}) - s\Phi_s(e_1) = (1, 0) - s(s, 1) = (1 - s^2, -s).$$

This determines the matrix for $\Phi_s$ with respect to the standard basis. Moreover, $\Phi_s^2$ is the identity transformation.

Similarly, $\Psi_{r}^2 = I$ and $\Psi_r(x, y) = (x, rx - y)$. If $0 \leq y \leq rx$, then $0 \leq rx - y \leq rx$, and so the restriction of $\Psi_r$ to $I(r)$ is a bijection. This completes the proof. □

3. Packing polynomials on rational sectors

Let $\alpha > 0$ or $\alpha = \infty$. For every positive integer $d$, let $P_d(\alpha)$ denote the set of packing polynomials of degree $d$ on the integer sector $I(\alpha)$. The Fueter-Pólya theorem states that $P(\infty) = \{F_\infty, G_\infty\}$. We also have $P_3(\infty) = P_3(\infty) = \emptyset$ by the Lew-Rosenberg theorem.

Theorem 4. For $\alpha > 0$ and $\alpha = \infty$, there exists no linear packing polynomial on $I(\alpha)$. Equivalently, $P_1(\alpha) = \emptyset$.

Proof. This is a simple counting argument. Let $\alpha > 0$. For every positive integer $n$, let

$$I_n(\alpha) = \{(x, y) \in I(\alpha) : x \leq n\}.$$

Then

$$|I_n(\alpha)| = \sum_{j=0}^{n} (\alpha j + 1) > \sum_{j=0}^{n} \alpha j = \frac{\alpha n(n + 1)}{2} > \frac{\alpha n^2}{2}.$$

Let $f(x, y) = ax + by + c$ be a linear polynomial such that $f(x, y) \in N_0$ for all $(x, y) \in I(\alpha)$. If $(x, y) \in I_n(\alpha)$, then $x \leq n$ and $y \leq \alpha n$, and so

$$0 \leq f(x, y) = ax + by + c \leq (|a| + |b| + |c|) n.$$

Thus, the linear function $f$ maps a set with more than $(\alpha/2)n^2$ elements into a set with at most $Cn$ elements, where $C = |a| + |b| + |c| + 1$, and this function cannot be one-to-one for $n > 2C/\alpha$. It follows that there is no linear packing polynomial on $I(\alpha)$.

There is a similar counting argument for $I(\infty)$ by considering the finite set $I_\infty(\infty) = \{(x, y) \in I(\infty) : x \leq n \text{ and } y \leq n\}$. □

Theorem 5. For $s \in N_0$, let $\Lambda_s$ and $M_s$ be the matrices defined by $[1]$. The functions from $P_d(1/s)$ to $P_d(0)$ defined by $F \mapsto F \circ \Lambda_s$ and $F \mapsto F \circ M_s$ are bijections with inverses $F \mapsto F \circ \Lambda_s^{-1}$ and $F \mapsto F \circ M_s^{-1}$, respectively.

Proof. This follows immediately from Theorem 2 and the observation that if $F = F(x, y)$ is a polynomial of degree $d$, then $F \circ \Lambda_s$ and $F \circ M_s$ are polynomials of degree $d$. □

Theorem 6. For $s \in N_0$, let $\Phi_s$ be the matrix defined by $[2]$. The function from $P_d(1/s)$ to $P_d(1/s)$ defined by $F \mapsto F \circ \Phi_s$ is an involution.

Proof. This follows immediately from Theorem 3. □
Theorem 7. Let $r$ be a positive integer. The polynomials
\[
F_r(x, y) = \frac{rx(x-1)}{2} + x + y
\]
and
\[
G_r(x, y) = \frac{rx(x+1)}{2} + x - y.
\]
are packing polynomials on the integer sector $I(r) = \{(x, y) \in \mathbb{N}_0^2 : 0 \leq y \leq rx\}$.

Proof. Consider the finite set $J_a = \{(a, b) \in \mathbb{N}_0^2 : 0 \leq b \leq ra\}$. We have $|J_a| = ra + 1$, and
\[
I(r) = \bigcup_{a \in \mathbb{N}_0} J_a.
\]
Moreover, if $a, a' \in \mathbb{N}_0$ and $a \neq a'$, then $J_a \cap J_{a'} = \emptyset$.

We construct the function $F_r(x, y)$ by enumerating the lattice points in $I(r)$ as follows: For $a \geq 1$, count the elements of $J_a$ after the elements of $J_{a-1}$, and count the elements of $J_a$ from bottom $(a, 0)$ to top $(a, ra)$. Thus, $F_r(0, 0) = 0$ and, for $x \geq 1$,
\[
F_r(x, 0) = 1 + \sum_{a=1}^{x-1} (ra + 1) = \frac{rx(x-1)}{2} + x.
\]
For $1 \leq y \leq rx$,
\[
F_r(x, y) = F_r(x, 0) + y = \frac{rx(x-1)}{2} + x + y.
\]

We construct the function $G_r(x, y)$ by enumerating the lattice points in $I(r)$ as follows: For $a \geq 1$, count the elements of $J_a$ after the elements of $J_{a-1}$, and count the elements of $J_a$ from top $(a, ra)$ to bottom $(a, 0)$. Thus, $G_r(0, 0) = 0$ and, for $a \geq 1$,
\[
G_r(x, rx) = 1 + \sum_{a=1}^{x-1} (ra + 1) = \frac{rx(x-1)}{2} + x.
\]
For $1 \leq y \leq x$,
\[
G_r(x, y) = G_r(x, rx) + rx - y = \frac{rx(x+1)}{2} + x - y.
\]
Thus, $F_r(x, y)$ and $G_r(x, y)$ are quadratic packing polynomials on $I(r)$. This completes the proof.

There is another way to construct the polynomial $G_r$. By Theorem 3, the linear transformation $\Psi_r : \mathbb{R}^2 \to \mathbb{R}^2$ that sends $(x, y)$ to $(x, rx - y)$ is a bijection on $I(r)$. Composing $F_r$ with $\Psi_r$, we obtain
\[
F_r \circ \Psi_r(x, y) = F_r(x, rx - y) = \frac{rx(x-1)}{2} + x + (rx - y) = G_r(x, y).
\]

Theorem 8. $F_1(x, y)$ and $G_1(x, y)$ are the unique quadratic packing polynomials on $I(1)$.

Proof. Applying Theorem 7 with $r = 1$, we obtain the quadratic packing polynomials
\[
F_1(x, y) = \frac{x(x+1)}{2} + y
\]
Moreover, $s$ is a positive integer. We have
\[ G_1(x, y) = \frac{x(x + 1)}{2} + x - y \]
on the integer sector $I(1)$. We also have
\[ F_\infty \circ \Lambda^{-1}_1(x, y) = F_\infty(x - y, y) = \frac{x^2}{2} + \frac{x + 2y}{2} = F_1(x, y) \]
and
\[ G_\infty \circ \Lambda^{-1}_1(x, y) = G_\infty(x - y, y) = \frac{x^2}{2} + \frac{3(x - y) + y}{2} = G_1(x, y). \]
By Theorem 5 composition with $\Lambda^{-1}_1$ is a bijection from $\mathcal{P}_2(0)$ to $\mathcal{P}_2(1)$. By the Fueter-Pólya theorem, $\mathcal{P}_2(0) = \{F_\infty, G_\infty\}$ and so
\[ \mathcal{P}_2(1) = \{F_\infty \circ \Lambda^{-1}_1, G_\infty \circ \Lambda^{-1}_1\} = \{F_1, G_1\}. \]
This completes the proof. \hfill $\square$

**Theorem 9.** Let $r$ and $s$ be relatively prime positive integers such that $1 \leq r < s$ and $r$ divides $s - 1$. Let $d = (s - 1)/r$. For $a \in \mathbb{N}_0$, let
\[ J_a = \{(a + dj, j) : j = 0, 1, \ldots, ra\}. \]
Then
\[ I(r/s) = \bigcup_{a \in \mathbb{N}_0} J_a. \]
Moreover, if $a \neq a'$, then $J_a \cap J_{a'} = \emptyset$.

**Proof.** If
\[ f_a(x) = \frac{x - a}{d} \]
then $f_a(a + dj) = j$ and
\[ J_a = \{(a + dj, f_a(a + dj)) : j = 0, 1, \ldots, ra\}. \]
If $a \neq a'$, the graphs of $y = f_a(x)$ and $y = f_{a'}(x)$ are distinct parallel lines, and so $J_a \cap J_{a'} = \emptyset$.

If $(x, y) \in J_a$, then $x = a + dj$ and $y = j$ for some $j \in \{0, 1, \ldots, ra\}$. In particular, if $j = 0$, then $(x, y) = (a, 0) \in S(r/s)$ and if $j = ra$, then $(x, y) = (a + dra, ra) = (sa, ra) \in S(r/s)$. It follows by convexity that the line segment between $(a, 0)$ and $(sa, ra)$ lies in $S(r/s)$, and this line segment is the graph of $y = f_a(x)$ for $a \leq x \leq sa$. We conclude that $J_a \subseteq I(r/s)$ for all $a \in \mathbb{N}_0$, and so $\bigcup_{a \in \mathbb{N}_0} J_a \subseteq I(r/s)$.

Conversely, let $(x, y) \in I(r/s)$. If $x = 0$, then $y = 0$ and $(x, y) = (0, 0) \in J_0$. If $x \geq 1$, then
\[ y \leq \frac{rx}{s} < \frac{rx}{s - 1} = \frac{x}{d} \]
and so
\[ a = x - dy \]
is a positive integer. We have
\[ (x, y) = (a + dy, y). \]
Moreover, $sy \leq rx$ implies that
\[ y \leq rx - (s - 1)y = r(x - dy) = ra \]
and so $(x, y) \in J_a$. It follows that $I(r/s) \subseteq \bigcup_{a \in \mathbb{N}_0} J_a$. This completes the proof. \hfill $\square$
Theorem 10. Let \( r \) and \( s \) be relatively prime positive integers such that \( 1 \leq r < s \) and \( r \) divides \( s - 1 \). Let \( d = (s - 1)/r \). The polynomials

\[
F_{r,s}(x, y) = \frac{r(x - dy)^2}{2} + \frac{(2 - r)x + (dr - 2d + 2)y}{2}
\]

and

\[
G_{r,s}(x, y) = \frac{r(x - dy)^2}{2} + \frac{(r + 2)x - (2d + s + 1)y}{2}
\]

are quadratic packing polynomials for the sector \( I(r/s) \).

Proof. Using Theorem 9, we have

\[
|J_a| = ra + 1
\]

and

\[
1 + \sum_{a=1}^{x-1} |J_a| = 1 + \sum_{a=1}^{x-1} (ra + 1) = \frac{rx(x-1)}{2} + x.
\]

We construct the function \( F_{r,s}(x, y) \) by enumerating the lattice points in \( I(r/s) \) as follows: For \( a \geq 1 \), count the elements of \( J_a \) after the elements of \( J_{a-1} \), and count the elements of \( J_a \) from the bottom lattice point \((a, 0)\) to the top lattice point \((ra, sa)\). Thus, \( F_{r,s}(0, 0) = 0 \) and, for \( x \geq 1 \),

\[
F_{r,s}(x, 0) = 1 + \sum_{a=1}^{x-1} (ra + 1) = \frac{rx(x-1)}{2} + x.
\]

If \((x, y) \in I(r/s) \setminus \{(0, 0)\}\), then \( 0 \leq y \leq rx/s \) and \((x, y) = ((x - dy) + dy, y) = (a + dy, y)\), where, as in the proof of Theorem 9, the integer \( a = x - dy \) is positive, and so \((x, y) \in J_a\). We have

\[
F_{r,s}(x, y) = F_{r,s}(x - dy, 0) + y
= \frac{r(x - dy)(x - dy - 1)}{2} + x - (d - 1)y
= \frac{r(x - dy)^2}{2} - \frac{r(x - dy)}{2} + x - (d - 1)y.
\]

We construct the function \( G_{r,s}(x, y) \) by enumerating the lattice points in \( I(r/s) \) as follows: For \( a \geq 1 \), count the elements of \( J_a \) after the elements of \( J_{a-1} \), and count the elements of \( J_a \) from the top lattice point \((ra, sa)\) to the bottom lattice point \((a, 0)\). Thus, \( G_{r,s}(0, 0) = 0 \) and, for \( x \geq 1 \),

\[
G_{r,s}(sx, rx) = 1 + \sum_{a=1}^{x-1} (ra + 1) = \frac{rx(x-1)}{2} + x.
\]

If \((x, y) \in I(r/s) \setminus \{(0, 0)\}\), then \( 0 \leq y \leq rx/s \) and \( a = x - dy \) is a positive integer. We have

\[
(x, y) = ((x - dy) + dy, y) = (a + dy, y) = (a + dy, ra - z)
\]
with \( z = ra - y \). Then

\[
G_{r/s}(x, y) = G_{r/s}(a + dy, ra - z)
\]

\[
= G_{r/s}(sa, ra) + z
\]

\[
= \frac{ra(a - 1)}{2} + a + z
\]

\[
= \frac{r(x - dy)(x - dy - 1)}{2} + (x - dy) + r(x - dy) - y
\]

\[
= \frac{r(x - dy)(x - dy - 1)}{2} + (r + 1)x - (d + s)y
\]

\[
= \frac{r(x - dy)^2}{2} + \frac{(r + 2)x}{2} - \frac{(2d + s + 1)y}{2}.
\]

This completes the proof. \(\square\)

**Theorem 11.** For every integer \( s \geq 2 \), the polynomials

\[
F_{1/s}(x, y) = \frac{(x - (s - 1)y)^2}{2} + \frac{x + (3 - s)y}{2}
\]

and

\[
G_{1/s}(x, y) = \frac{(x - (s - 1)y)^2}{2} + \frac{3x + (1 - 3s)y}{2}
\]

are the unique quadratic packing polynomials on the sector \( I(1/s) \).

**Proof.** Applying Theorem 10 with \( r = 1 \) and \( d = s - 1 \), we obtain the polynomials \( F_{1/s}(x, y) \) and \( G_{1/s}(x, y) \). We also have

\[
F_{\infty} \circ \Lambda_{s}^{-1}(x, y) = F_{\infty}(x - sy, y)
\]

\[
= \frac{(x - (s - 1)y)^2}{2} + \frac{(x - sy) + 3y}{2}
\]

\[
= F_{1/s}(x, y).
\]

and

\[
G_{\infty} \circ \Lambda_{s}^{-1}(x, y) = G_{\infty}(x - sy, y)
\]

\[
= \frac{(x - (s - 1)y)^2}{2} + \frac{3(x - sy) + y}{2}
\]

\[
= G_{1/s}(x, y).
\]

By Theorem 5 the function \( F \mapsto F \circ \Lambda_{s}^{-1} \) is a bijection from \( P_2(0) \) onto \( P_2(1/s) \), and so \( P_2(1/s) = \{ F_{1/3}, G_{1/3} \} \). Thus, \( F_{1/3} \) and \( G_{1/3} \) are the unique quadratic packing polynomials on \( I(1/s) \). This completes the proof. \(\square\)

### 4. Quasi-polynomials

Let \( H \) be a function whose domain is a set of integers. The function \( H \) is a **quasi-polynomial** with period \( m \) if

\[
H(x) = \sum_{i=0}^{n} c_i(x)x^i
\]

where the value of \( c_i(x) \) depends only on the congruence class of \( x \) modulo \( m \). The coefficient functions \( c_i(x) \) are not necessarily integer-valued. A quasi-polynomial
restricted to a congruence class modulo $m$ is simply a polynomial. Every polynomial is a quasi-polynomial with period $m$ for every positive integer $m$.

Similarly, if $H$ is a function whose domain is a set of lattice points in $\mathbb{Z}^2$, then $H$ is a quasi-polynomial with period $m$ if

$$H(x,y) = \sum_{i=0}^{d_1} \sum_{j=0}^{d_2} c_{i,j}(x,y)x^iy^j$$

where the value of $c_{i,j}(x,y)$ depends only on the congruence classes of $x$ and $y$ modulo $m$. Again, a quasi-polynomial in two variables, restricted to fixed congruence classes modulo $m$, is an ordinary polynomial. To compute the value of a quasi-polynomial $H(x,y)$ is as fast as computing the value of a polynomial:

First determine the congruence classes of $x$ and $y$ modulo $m$, and then evaluate the appropriate polynomial.

The following theorem describes the construction of quasi-polynomial packing functions on every integer sector.

**Theorem 12.** Let $r$ and $s$ be relatively prime positive integers. For $\ell \in \{0,1,2,\ldots,s-1\}$, let

$$u_{\ell} = \left\lfloor \frac{r\ell}{s} \right\rfloor$$

and

$$h_{r/s}^{(\ell)}(x,y) = \frac{r(x-\ell)(x-\ell-s)}{2s^2} + \frac{(u_{\ell}+1)(x-\ell)}{s} + y.$$ 

The function $H_{r/s}(x,y)$ defined by

$$H_{r/s}(x,y) = sh_{r/s}^{(\ell)}(x,y) + \ell \quad \text{if } x \equiv \ell \pmod{s}$$

is a quasi-polynomial packing function with period $s$ on the integral sector $I(r/s)$.

**Proof.** For $\ell \in \{0,1,2,\ldots,s-1\}$, let

$$\Omega_{\ell} = \{(x,y) \in I(r/s) : x \equiv \ell \pmod{s}\}.$$ 

Then $\Omega_{\ell} \cap \Omega_{\ell'} = \emptyset$ if $\ell \neq \ell'$, and $\bigcup_{\ell=0}^{s-1} = I(r/s)$. We shall construct a packing polynomial $h_{r/s}^{(\ell)}(x,y)$ on each of the sets $\Omega_{\ell}$, and then dilate and translate these polynomials to obtain a quasi-polynomial $H_{r/s}(x,y)$ with period $s$ and domain $I(r/s)$.

Fix an integer $\ell \in \{0,1,2,\ldots,s-1\}$. For every nonnegative integer $a$, let

$$J_a = \{(sa+\ell,y) : y = 0,1,\ldots,ra+u_{\ell}\}.$$ 

Then $\bigcup_{a\in\mathbb{N}_0} J_a = \Omega_{\ell}$ and $J_a \cap J_{a'}$ if $a \neq a'$. We construct the polynomial $h_{r/s}^{(\ell)}(x,y)$ by enumerating the lattice points in $\Omega_{\ell}$ as follows: For $(\ell,y) \in J_0$, let $h_{r/s}^{(\ell)}(\ell,y) = y$ for $y = 0,1,\ldots,u_{\ell}$. For $a \geq 1$, count the elements of $J_a$ after the elements of $J_{a-1}$, and count the elements of $J_a$ from the bottom lattice point $(sa+\ell,0)$ to the top.
lattice point \((sa + \ell, ra + u_\ell)\). If \(x = sx' + \ell\), then

\[
h_{r/s}^{(\ell)}(x, y) = h_{r/s}^{(\ell)}(sx' + \ell, y)
\]

\[
= \sum_{a=0}^{x'-1} |J_a| + y
\]

\[
= \sum_{a=0}^{x'-1} (ra + u_\ell + 1) + y
\]

\[
= \frac{rx'(x' - 1)}{2} + (u_\ell + 1)x' + y
\]

\[
= \frac{r(x-\ell)(x-\ell-s)}{2s^2} + \frac{(u_\ell + 1)(x-\ell)}{s} + y.
\]

The polynomial \(h_{r/s}^{(\ell)}(x, y)\) is a packing polynomial on the set \(\Omega_\ell\), and the quasi-polynomial \(H_{r/s}(x, y)\) defined by (6) is a packing function on the integral sector \(I(r/s)\). This completes the construction. \(\square\)

For example, if \(r = 3\) and \(s = 2\), then

\[
h_{3/2}^{(0)}(x, y) = \frac{3x^2}{8} - \frac{x}{4} + y
\]

and

\[
h_{3/2}^{(1)}(x, y) = \frac{3x^2}{8} - \frac{x}{2} + y + \frac{1}{8}.
\]

The quasi-polynomial

\[
H_{3/2}(x, y) = \begin{cases} 
\frac{3x^2}{4} - \frac{x}{2} + 2y & \text{if } x \equiv 0 \pmod{2} \\
\frac{3x^2}{4} - x + 2y + \frac{5}{4} & \text{if } x \equiv 1 \pmod{2}.
\end{cases}
\]

is a packing function on the integer sector \(I(3/2)\).

5. Open problems

(1) Is there a quadratic packing polynomial for the sector \(I(3/5)\)? Is there a quadratic packing polynomial for the sector \(I(3/2)\)? For what rational numbers \(\alpha\) do there exist quadratic packing polynomials?

(2) In Theorem 7 we constructed two quadratic packing polynomials on the integer sector \(I(r)\) for every integer \(r \geq 2\)? Are these the only quadratic packing polynomials on \(I(r)\)?

(3) Can any rational sector have more than two quadratic packing polynomials?

(4) Can any rational sector have more than two packing polynomials?

(5) Is there a rational sector with a packing polynomial of degree greater than two?

(6) Prove that there is no packing polynomial for any irrational sector.

Acknowledgements: I wish to thank for Dick Bumby and Tim Susse for helpful discussions.
References

[1] H. L. Fetter, J. H. Arredondo R., and L. B. Morales, The diagonal polynomials of dimension four, Adv. in Appl. Math. 34 (2005), no. 2, 316–334.
[2] R. Fueter and G. Pólya, Rationale Abzählung der Gitterpunkte, Vierteljschr Naturforsch. Gesellsch. Zurich 58 (1923), 380–386.
[3] J. S. Lew, L. B. Morales, and A. Sánchez-Flores, Diagonal polynomials for small dimensions, Math. Systems Theory 29 (1996), no. 3, 305–310.
[4] J. S. Lew, Polynomial enumeration of multidimensional lattices, Math. Systems Theory 12 (1978/79), no. 3, 253–270.
[5] J. S. Lew and A. L. Rosenberg, Polynomial indexing of integer lattice-points. I. General concepts and quadratic polynomials, J. Number Theory 10 (1978), no. 2, 192–214.
[6] Polynomial indexing of integer lattice-points. II. Nonexistence results for higher-degree polynomials, J. Number Theory 10 (1978), no. 2, 215–243.
[7] L. B. Morales, Diagonal polynomials and diagonal orders on multidimensional lattices, Theory Comput. Syst. 30 (1997), no. 4, 367–382.
[8] L. B. Morales and J. S. Lew, An enlarged family of packing polynomials on multidimensional lattices, Math. Systems Theory 29 (1996), no. 3, 293–303.
[9] L. B. Morales and A. Sánchez-Flores, Erratum: “Diagonal polynomials and diagonal orders on multidimensional lattices” [Theory Comput. Syst. 30 (1997), no. 4, 367–382; MR1450861 (98h:06009)], Theory Comput. Syst. 33 (2000), no. 1, 107.
[10] L. B. Morales and J. H. Arredondo R., A family of asymptotically $e(n - 1)!$ polynomial orders of $\mathbb{N}^n$, Order 16 (1999), no. 2, 195–206 (2000).
[11] M. B. Nathanson, Cantor packing polynomials, arXiv, 2013.
[12] A. Sánchez-Flores, A family of $(n - 1)!$ diagonal polynomial orders of $\mathbb{N}^n$, Order 12 (1995), no. 2, 173–187.
[13] C. Smoryński, Logical Number Theory. I: An Introduction, Universitext, Springer-Verlag, Berlin, 1991.
[14] M. A. Vsemirnov, Two elementary proofs of the Fueter-Pólya theorem on matching polynomials, Algebra i Analiz 13 (2001), no. 5, 1–15.
[15] Erratum: “Two elementary proofs of the Fueter-Pólya theorem on matching polynomials” (Russian) [Algebra i Analiz 13 (2001), no. 5, 1–15; MR1882861 (2003a:11021)], Algebra i Analiz 14 (2002), no. 5, 240.

Department of Mathematics, Lehman College (CUNY), Bronx, NY 10468
E-mail address: melvyn.nathanson@lehman.cuny.edu