Dirac operators on quasi-Hamiltonian $G$-spaces

Yanli Song*

March 12, 2015

Abstract

We develop notions of twisted spinor bundle and twisted pre-quantum bundle on quasi-Hamiltonian $G$-spaces. The main result of this paper is that we construct a Dirac operator with index given by positive energy representation of loop group. This generalizes the quantization of Hamiltonian $G$-spaces to quasi-Hamiltonian $G$-spaces.

Contents

1 Introduction 2
2 Loop group and positive energy representation 3
3 Hamiltonian $LG$-spaces and q-Hamiltonian $G$-spaces 5
4 Spin representation and cubic Dirac operator 9
5 Twisted spinor bundle and twisted pre-quantum bundle 13
6 Dirac operators on q-Hamiltonian $G$-spaces 18

*University of Toronto, songyan1@math.utoronto.ca
1 Introduction

Let $G$ be a compact, connected Lie group, and $(M, \omega)$ a compact symplectic manifold with a Hamiltonian $G$-action. By choosing a $G$-invariant $\omega$-compatible almost complex structure on $M$, we obtain a $G$-equivariant $\text{Spin}^c$-structure on $M$ with spinor bundle $S_M$. Coupling with the pre-quantum line bundle $L$, one defines a $\mathbb{Z}_2$-graded Hilbert space

$$\mathcal{H} = \Gamma_{\mathbb{Z}_2}(M, S_M \otimes L)$$

and a $G$-equivariant $\text{Spin}^c$-Dirac operator $D_L$. Attributed to Bott, the quantization of $(M, \omega)$ can be defined as the equivariant index

$$Q(M, \omega) = \text{ind}(D_L) \in \mathbb{R}(G).$$

The goal of this paper is to generalize the quantization process to the quasi-Hamiltonian $G$-space introduced by Alekseev-Malkin-Meinrenken [AMM98]. The q-Hamiltonian $G$-space, arising from infinite-dimensional Hamiltonian loop group space, differs in many respects from Hamiltonian $G$-space.

In particular, the moment map takes values in the group $G$ and the 2-from $\omega$ doesn’t have to be closed or non-degenerate. Consequently, the two key ingredients in defining $Q(M, \omega)$: the spinor bundle $S_M$ and pre-quantum line bundle $L$ might not exist in general.

Given a q-Hamiltonian $G$-space $(M, \omega)$, we construct a twisted spinor bundle $S_{\text{spin}}$ and a twisted pre-quantum bundle $S_{\text{pre}}$ using spin representation of loop group and Hilbert space of Wess-Zumino-Witten model. Being $G$-equivariant bundles of infinite-dimensional Hilbert space over $M$, $S_{\text{spin}}$ and $S_{\text{pre}}$ encode the “twisting” given by $\omega$ (in a different situation when the twisting is a torsion class, the Dirac operator and corresponding index theorem are studied in [MMS06]). We analogously define a Hilbert space

$$\mathcal{H} := \left[ \Gamma_{\mathbb{Z}_2}(M, S_{\text{spin}} \otimes S_{\text{pre}}) \right]^G.$$

One key in the construction of Dirac operators on $\mathcal{H}$ is the algebraically defined cubic Dirac operator. It was introduced by Kostant [Kos99] for finite-dimensional Lie algebras, and by now has been extensively studied. Landweber [Lan01] generalizes the construction to the loop group and Posthuma [Pos11] applies the same technique to a different homogeneous setting. More generally, the cubic Dirac operators for Kac-Moody algebra are discussed in [Mei11] and lecture notes of Wassermann [Was10].

Our strategy is to construct a Dirac operator as a combination of algebraical cubic Dirac operator and geometrical $\text{Spin}^c$-Dirac operator. To be more precise, we choose a $G$-invariant open covering of $M$, so that every
open subset $U$ has the geometric structure:

$$U \cong G \times_H V,$$

where $V$ is submanifold admitting a $H$-equivariant Spin$c$-structure. Accordingly, the tangent bundle $TU$ splits equivariantly into “vertical direction” and “horizontal direction”. We define a Dirac operator on $U$ so that it acts as the Spin$c$-Dirac operator on the vertical part $V$ and the cubic Dirac operator for loop group on the horizontal part. By partition of unity, we obtain a Dirac operator $\mathcal{D}$ on $\mathcal{H}$, and then we show its index is given by positive energy representations of loop group.

Alternative approaches to the quantization of q-Hamiltonian $G$-spaces can be found in [Mei12] using K-homology pushforwards, and [AMW01, CW08] using symplectic reduction in Hamiltonian LG-spaces. In a subsequent paper we will show that the definitions are consistent.

Acknowledgements
The author would like Eckhard Meinrenken and Nigel Higson for many benefited discussions.

2 Loop group and positive energy representation

In this section, we give a brief review on loop groups and their representations. We use [PS86] as our primary reference.

2.1 Loop group

Let $G$ be a compact, simple and simply connected Lie group, and fix a “Sobolev level” $s > 1$. We define $LG$ the loop group as the Banach Lie group consisting of maps $S^1 \to G$ of Sobolev class $s + \frac{1}{2}$ with the group structure given by pointwise multiplication. The Lie algebra $Lg = \Omega^0(S^1, g)$ is given by the space Lie algebra $g$-valued 0-forms of Sobolev class $s + \frac{1}{2}$ and $Lg^* = \Omega^1(S^1, g)$ the space of $g$-valued 1-forms of Sobolev class $s - \frac{1}{2}$. Integration over $S^1$ gives a natural non-degenerate pairing between $Lg$ with $Lg^*$.

To study the loop group, it is natural to consider its central extension:

$$0 \to U(1) \to \hat{LG} \to LG \to 0,$$
with Lie algebra $\widehat{\mathfrak{g}} \cong \mathfrak{g} \oplus \mathbb{R}$. By the assumption that $G$ is simple, there is a unique ad-invariant inner product $\langle \cdot, \cdot \rangle_g$ on $\mathfrak{g}$, rescaled so that the highest root of $\mathfrak{g}$ has norm $\sqrt{2}$. This inner product on $\mathfrak{g}$ induces an inner product on $\mathfrak{Lg}$ by

$$B(X, Y) = \frac{1}{2\pi} \int_0^{2\pi} \langle X(\theta), Y(\theta) \rangle_g d\theta, \quad X, Y \in \mathfrak{Lg}.$$  

The space $\mathfrak{Lg}^*$ can also be identified with the affine space of connections on the bundle $S^1 \times G$. Then the loop group $\mathcal{L}G$ acts on $\mathfrak{Lg}^*$ by gauge transformation

$$\lambda \cdot \gamma = \lambda^{-1} d\lambda + \text{Ad}_\lambda(\gamma), \quad \lambda \in \mathcal{L}G, \gamma \in \mathfrak{Lg}^*.$$  

Note it is an affine action rather than a linear action on the vector space $\mathfrak{Lg}^*$.

Recall that the orbits of coadjoint $G$-action on $\mathfrak{g}^*$ are parametrized by elements of positive Weyl chamber. The set of coadjoint $\mathcal{L}G$-orbits can be described as follow. Fixing a maximal torus $T$, the choice of a set of positive roots $\mathfrak{R}_+$ for $G$ determines a positive Weyl chamber $t_+$. Denote by $\alpha_0$ the highest root and

$$\rho_G = \frac{1}{2} \sum_{\alpha \in \mathfrak{R}_+} \alpha.$$  

The dual Coxeter number of $G$ is defined by

$$h^\vee = 1 + \langle \rho_G, \alpha_0 \rangle_g,$$  

and the fundamental Weyl alcove for $G$ is the simplex

$$\mathfrak{A} = \{ \xi \in t_+ | \langle \alpha_0, \xi \rangle_g \leq 1 \} \subset t \subset \mathfrak{g}.$$  

There is a natural map $\text{Hol} : \mathfrak{Lg}^* \rightarrow G$ sending $\lambda \in \mathfrak{Lg}^*$ to its holonomy around $S^1$. The map sets up a 1-1 correspondence between the set of conjugacy classes and coadjoint $\mathcal{L}G$-orbits. Both of them are parametrized by points in the alcove $\mathfrak{A}$.

### 2.2 Positive Energy Representation

Let $S^1_{\text{rot}}$ be the group of rigid rotation on $S^1$ and $\partial$ its infinitesimal generator. Consider a unitary representation of $S^1_{\text{rot}} \times \widehat{\mathcal{L}G}$ on a complex Hilbert space $V$, on which the central circle acts by scalar multiplication. We denote by $E_\partial$ the action of $\partial$ on $V$.

**Definition 2.1.** We say that $V$ is a positive energy representation if the opera-
tor $E_\theta$ is self-adjoint with spectrum bounded below. Moreover, we say that $V$ has level $k$ if the central circle of $\hat{LG}$ acts with weight $k$.

The positive energy representations of loop groups behave quite analogously to the representation theory of compact Lie groups. For example, every irreducible positive energy representation is uniquely determined by the highest weight. To be more precise, let $T$ be a maximal torus of $G$ and $\Lambda^*$ the weight lattice. We take $S^1_{\text{rot}} \times T \times S^1$ as the maximal torus of $S^1_{\text{rot}} \ltimes \hat{LG}$, where the second $S^1$ factor comes from the central extension. The affine weights of $LG$ are in the forms of $(m, \lambda, k)$, where $m \in \mathbb{Z}$ is the energy, $\lambda \in \Lambda^*$ is the weight of $G$, and $k$ is the level.

The affine Weyl group $W_{\text{aff}} = W \ltimes \Lambda$ acts on affine weights as follow: the Weyl group $W$ acts as usual on $\Lambda^*$ and the action of $z \in \Lambda$ is given by

$$z \cdot (m, \lambda, k) = (m + \langle \lambda, z \rangle + \frac{k}{2} \cdot \|z\|^2, \lambda + k \cdot z, k).$$

The level $k$ is fixed by the affine Weyl group action and the energy is shifted so as to preserve the inner product:

$$\langle m_1, \lambda_1, k_1 \rangle \cdot \langle m_2, \lambda_2, k_2 \rangle = \langle \lambda_1, \lambda_2 \rangle - m_1k_2 - m_2k_1.$$ 

For a fixed level $k$, every irreducible positive energy representation $V$ of $LG$ is uniquely determined by the dominant weight $\lambda$ at the minimum energy $m$. We call $\lambda = (m, \lambda, k)$ the highest weight of $V$. In addition, we can always normalize so that the lowest energy equals zero.

It is well-known by the Borel-Weil theorem that all the irreducible $G$-representations are parameterized by the dominant weights $P_{G,+} = \Lambda^* \cap t^*_+$. Similarly there is an 1-1 correspondence between irreducible positive energy representations at level $k$ and weights in

$$P_{k,+} = k\mathfrak{a} \cap \Lambda^* = \{ \lambda \in \Lambda^* | \frac{\lambda}{k} \in \mathfrak{a} \}.$$ 

The abelian group $R_k(LG)$ generated by irreducible positive energy representations at level $k$ has a finite basis and a ring structure known as fusion product. But we won’t discuss it in this paper.

3 Hamiltonian $LG$-spaces and $q$-Hamiltonian $G$-spaces

The theory of $q$-Hamiltonian $G$-spaces was developed in [AMM98]. One of the motivation is to build finite-dimensional models for the Hamilton-
nnian LG-spaces. In this section, we begin by reviewing the basic definitions, and then study their corresponding cross-section theorems. We assume that $G$ is a compact, simple and simple connected Lie group.

3.1 Basic definitions

Recall that a Hamiltonian $G$-space is a triple $(M, \omega, \mu)$, with $\omega$ the $G$-equivariant symplectic 2-form, and $\mu : M \to g^*$ the moment map satisfying that

$$t_{\xi_M} \omega = d\langle \mu, \xi \rangle, \quad \xi \in g.$$

The above definition can be extended to the loop group setting. Let $M$ be an infinite-dimensional Banach manifold. We say that it is weakly symplectic if it is equipped with a closed 2-form $\omega \in \Omega^2(M)$ so that the induced map

$$\omega^\flat : T_mM \to T^*_mM$$

is injective.

**Definition 3.1.** A Hamiltonian LG-space is a weakly symplectic Banach manifold $(M, \omega)$ together with a LG-action and a LG-equivariant map $\mu : M \to Lg^*$ so that $t_{\xi_M} \omega = d\langle \mu, \xi \rangle$ for all $\xi \in Lg$.

For example, the coadjoint LG-orbit is a Hamiltonian LG-space, with moment map the inclusion.

Let $(M, \omega, \mu)$ be a Hamiltonian LG-space. Since the based loop group

$$\Omega G = \{\lambda \in LG | \lambda(0) = e\}$$

acts freely on $Lg^*$, it acts freely on $M$ as well by the equivariance of $\mu$. We thus obtain a commuting square

$$
\begin{array}{ccc}
M & \xrightarrow{\mu} & Lg^* \\
\downarrow & \searrow \text{Hol} & \\
M/\Omega G & \xrightarrow{\phi} & G
\end{array}
$$

where the quotient $M = M/\Omega G$ is a finite-dimensional compact smooth manifold provided that $\mu$ is proper.

Alekseev-Malkin-Meinrenken [AMM98] give a set of conditions a $G$-space $M$ must satisfy in order to arise from a Hamiltonian LG-space by such a construction.
Choose an invariant inner product \( \langle \cdot, \cdot \rangle_g \) on \( g \) and denote by \( \theta^L, \theta^R \in \Omega^1(G, g) \) the left and right invariant Maurer-Cartan forms on \( G \) and the Cartan 3-form

\[
\chi = \frac{1}{12} \langle \theta^L, [\theta^L, \theta^L] \rangle_g = \frac{1}{12} \langle \theta^R, [\theta^R, \theta^R] \rangle_g \in \Omega^3(G).
\]

**Definition 3.2 ([AMM98]).** A \( q \)-Hamiltonian \( G \)-space is a compact \( G \)-manifold \( M \), together with an equivariant 2-form \( \omega \), and an equivariant map \( \phi : M \to G \) satisfying the following properties:

1. \( d\omega = \phi^*\chi \);
2. \( \iota_{\xi_M} \omega = \frac{1}{2} (\phi^*(\theta^L + \theta^R), \xi)_g \) for all \( \xi \in g \);
3. \( \ker(\omega) \cap \ker(d\phi) = 0 \).

We call \( \phi \) the **group-valued moment map**.

According to [AMM98, Theorem 8.3], there is a 1-1 correspondence between Hamiltonian \( LG \)-spaces with proper moment map and \( q \)-Hamiltonian \( G \)-spaces. One can always choose to work with infinite-dimensional Hamiltonian \( LG \)-spaces with more conventional definitions or to use finite dimensional \( q \)-Hamiltonian \( G \)-spaces. The counterparts of coadjoint orbits for \( q \)-Hamiltonian \( G \)-spaces are conjugacy classes \( C \) in \( G \) with group-valued moment map the embedding \( C \hookrightarrow G \).

### 3.2 Cross-section theorems

The Hamiltonian \( LG \)-spaces and their equivalent finite-dimensional models behave in many respects like the usual Hamiltonian \( G \)-spaces. This is due to the existence of the cross-section theorem we shall now describe.

Let us first introduce a partial order of open faces of \( \mathfrak{A} \) by setting \( \tau \preceq \sigma \) if \( \tau \subseteq \sigma \). The isotropy group \( (LG)_\xi \) of the coadjoint \( LG \)-action on \( g^* \) depends only on the open face \( \sigma \) of \( \mathfrak{A} \) containing \( \xi \) and will be denoted by \( (LG)_\sigma \) (note however \( (LG)_\sigma \) will generally contain non-constant loops). One has that

\[
\sigma \preceq \tau \Rightarrow (LG)_\tau \subseteq (LG)_\sigma.
\]

In particular, \( (LG)_0 = G \) and \( (LG)_{\text{int}\mathfrak{A}} = T \).
We define a \((LG)_\sigma\)-invariant open subset of \((LG)_\sigma^*\) by
\[
A_\sigma = (LG)_\sigma \cdot \bigcup_{\sigma \leq \tau} \tau.
\]
Note that \(A_\sigma\) is a slice for all \(\xi \in \sigma\) for the action of \(LG\) in the sense that
\[
LG \times_{(LG)_\sigma} A_\sigma \rightarrow LG \cdot A_\sigma
\]
is a diffeomorphism of Banach manifolds.

**Theorem 3.3.** Let \((M, \omega, \mu)\) be a Hamiltonian \(LG\)-space with proper moment map. For every open face \(\sigma\) of \(\mathfrak{A}\), the cross-section

\[
V_\sigma = \mu^{-1}(A_\sigma)
\]

is a finite-dimensional symplectic submanifold with Hamiltonian \((LG)_\sigma\)-action. The restriction of \(\mu|_{V_\sigma}\) is a moment map of the \((\hat{LG})_\sigma\)-action (the central circle acts trivially on \(V_\sigma\)).

**Proof.** [MW98, Theorem 4.8].

The symplectic cross-section theorem carries over to q-Hamiltonian \(G\)-spaces.

The centralizer \(G_{\exp(\xi)}\) with \(\xi \in \mathfrak{A}\) is isomorphic to \((LG)_\xi\) and it depends only on the open face \(\sigma\) of \(\mathfrak{A}\) containing \(\xi\). We denoted it by \(G_\sigma\). The subset
\[
A_\sigma = \text{Ad}(G_\sigma) \cdot \exp(\bigcup_{\sigma \leq \tau} \tau) \subset G_\sigma \subset G
\]
is smooth and is a slice for the \(\text{Ad}(G)\)-action at points in \(\sigma\).

**Theorem 3.4.** Let \((M, \omega, \phi)\) be a q-Hamiltonian \(G\)-space. The cross-section
\[
V_\sigma = \phi^{-1}(A_\sigma)
\]
is a smooth \(G_\sigma\)-invariant submanifold and
\[
G \times_{G_\sigma} V_\sigma \cong G \cdot V_\sigma
\]
is a \(G\)-invariant open subset of \(M\). Moreover, \(V_\sigma\) is a q-Hamiltonian \(G_\sigma\)-space with the restriction of \(\phi\) as the group-valued moment map.

**Proof.** [AMM98, Proposition 7.1].

**Remark 3.5.** It is important to point out that by identifying \(G_\sigma\) and \((LG)_\sigma\), the two cross-sections
\[
V_\sigma \subset M, \hspace{1cm} V_\sigma \subset M
\]
are equivariantly diffeomorphic.
4 Spin representation and cubic Dirac operator

The cubic Dirac operator is an algebraically defined operator, introduced by Kostant [Kos99], and is by now a well-studied object in the theory of loop groups. In this section we set the stage by reviewing some basic properties of the cubic Dirac operator in finite dimensional case and then discuss its generalization to the infinite-dimensional case.

4.1 Finite dimensional case

Let $G$ be a compact Lie group and $\mathfrak{g}$ its Lie algebra equipped with an ad-invariant inner product.

**Definition 4.1.** The Clifford algebra $\text{Cliff}(\mathfrak{g})$ is generated by the vectors in $\mathfrak{g}$ subject to the anti-commutator

$$[X, Y] = 2 \cdot \langle X, Y \rangle \mathfrak{g}, \quad X, Y \in \mathfrak{g}.$$  

We equip $\text{Cliff}(\mathfrak{g})$ with the standard $\mathbb{Z}_2$-grading. A spinor module $S_\mathfrak{g}$ is an irreducible complex $\mathbb{Z}_2$-graded $\text{Cliff}(\mathfrak{g})$-module, with Clifford action denoted by

$$c : \mathfrak{g} \to \text{End}(S_\mathfrak{g}).$$

Fix an orthonormal basis

$$X_a, \quad a = 1, \ldots, \text{dim}\mathfrak{g}.$$  

The spin representation $\text{ad}^\mathfrak{g} : \mathfrak{g} \to \text{End}(S_\mathfrak{g})$ is given by the formula

$$\text{ad}^\mathfrak{g}(X) := \frac{1}{4} \sum_{a=1}^{\text{dim}(\mathfrak{g})} c([X, X_a])c(X_a), \quad X \in \mathfrak{g}.$$  

**Definition 4.2.** Let $W$ be a representation of the Lie algebra $\pi : \mathfrak{g} \to \text{End}(W)$. We define the cubic Dirac operator on $W \otimes S_\mathfrak{g}$ by

$$D_\mathfrak{g} = \sum_{a=1}^{\text{dim}(\mathfrak{g})} (\pi(X_a) \otimes c(X_a) + \frac{1}{3} \otimes \text{ad}^\mathfrak{g}(X_a) \cdot c(X_a))$$

As it stands, the operator $D_\mathfrak{g} \in \text{End}(W \otimes S_\mathfrak{g})$ and can also be interpreted as a distinguish element of the non-commutative Weil algebra [AM00, Mei13]:

$$W(\mathfrak{g}) := \mathcal{U}(\mathfrak{g}) \otimes \text{Cliff}(\mathfrak{g}).$$

It has the following interesting property.
Theorem 4.3 ([Kos99]). The square of the cubic Dirac operator equals
\[ D^2_g = 2 \cdot \text{Cas}_g + \frac{1}{12} \text{tr}(\text{Cas}_g). \]

Next, we consider \( H \subset G \) a closed subgroup of the equal rank. Using the inner product we write \( g = h \oplus p \). This decomposition induces isomorphisms:
\[
\text{Cliff}(g) \cong \text{Cliff}(h) \otimes \text{Cliff}(p), \quad S_g \cong S_h \otimes S_p,
\]
where \( S_h, S_p \) are spinor modules of \( \text{Cliff}(h) \) and \( \text{Cliff}(p) \) respectively. Accordingly the spin representation breaks up as a sum
\[
\text{ad}^p(X) = \text{ad}^h(X) + \text{ad}^p(X), \quad X \in h.
\]

Definition 4.4. We define the relative cubic Dirac operator \( D_{g,h} \) acting on \( W \otimes S_p \) by
\[
(4.1) \quad D_{g,h} = \sum_{i=1}^{\dim p} (\pi(X_a) \otimes c(X_a) + \frac{1}{3} \otimes \text{ad}^p(X_a) \cdot c(X_a)),
\]
where \( \{X_a\}_{a=1}^{\dim p} \) is an orthonormal basis for \( p \).

To exhibit the structure of \( D_{g,h} \), we decompose \( W \otimes S_p \) with respect to the \( h \)-action and denote by \( M(\nu) \) the isotypical \( h \)-summand with highest weight \( \nu \).

Theorem 4.5 ([Kos99]). Suppose that \( W_\lambda \) is an irreducible \( g \)-representation with highest weight \( \lambda \). The following formula holds
\[
D^2_{g,h} \big|_{M(\nu)} = \|\lambda + \rho_G\|^2 - \|\nu + \rho_H\|^2.
\]

4.2 Dirac operators on homogeneous spaces
Consider a homogeneous space \( M = G/H \). For simplicity, we assume that the adjoint \( H \)-action on \( p \) lifts to a \( H \)-action on \( S_p \). In this case, the bundle
\[
S_M = G \times_H S_p,
\]
defines a \( G \)-equivariant Spin structure on \( M \). The Hilbert space \( \Gamma_{L^2}(M, S_M) \) can be identified with
\[
\left[ L^2(G) \otimes S_p^* \right]^H \cong \bigoplus_{\lambda \in \mathcal{P}_G^+} W_\lambda \otimes [W_\lambda^* \otimes S_p^*]^H,
\]
where the isomorphism comes from the Peter-Weyl theorem.
We define Dirac operators on $M$ in two different ways. First of all, the Levi-Civita connection $\nabla^{TM}$ lifts to a Hermitian connection $\nabla^{S_{M}}$ on $S_{M}$. One can construct a geometric Spin$^{c}$-Dirac operator by

$$D_{\text{geo}} = \sum_{a=1}^{\dim M} c(X_{a}) \cdot \nabla^{S_{M}}_{X_{a}},$$

where $\{X_{a}\}_{a=1}^{\dim M}$ is an orthonormal basis of $TM$.

On the other hand, since the relative cubic Dirac operator $D_{g,h}$ is $H$-equivariant, it restricts to an operator $D_{\lambda,g,h}$ on

$$[W_{\lambda}^{*} \otimes S_{p}^{*}]^{H}.$$

Tensoring each operator $D_{\lambda,g,h}$ with the identity operator on $V_{\lambda}$, and summing over $V_{\lambda}$, one obtains an operator $D_{\text{alg}}$.

**Lemma 4.6.** The two Dirac operators $D_{\text{alg}}$ and $D_{\text{geo}}$ are homotopic.

**Proof.** The operator $D_{\text{alg}}$ can be written as

$$D_{\text{alg}} = \sum_{i=1}^{\dim M} c(X_{a}) \cdot (\pi(X_{a}) + \frac{1}{3}\text{ad}^{g}_{X_{a}}).$$

Notice that $\pi + \frac{1}{3}\text{ad}^{g}$ resembles an equivariant connection on the spinor bundle $S_{M}$. Thus, the two operators are differed by choices of connections and homotopic. \(\square\)

### 4.3 Infinite dimensional case

The definitions of spin representation and cubic Dirac operator can be extended to the infinite-dimensional loop algebra $L_{g}$.

Let now $G$ be a compact, simple and simple connected Lie group with Lie algebra $\mathfrak{g}$. The loop algebra $L_{g}$ carries an inner product defined in (2.1). As in the finite dimensional case, we can define the Clifford algebra $\text{Cliff}(L_{g})$, spinor module $S_{L_{g}}$ and spin representation $\text{ad}^{L_{g}}: L_{g} \to \text{End}(S_{L_{g}})$.

For explicit construction, one can find an algebraic approach in [KS87] and a geometric one in [PS86]. For the general theory of Clifford algebras and representations for infinite dimensional Hilbert space we refer to [PR94].

A priori, the formal expression defining the cubic Dirac operator $D_{L_{g}}$ is undefined since they involve infinite sums. Nevertheless, it can be fixed
by choosing a “normal ordered” basis. To be precise, fix an orthonormal basis \( \{X_a^\ell\}\) of \( g \). For \( n \in \mathbb{Z} \), we write \( X_a^n \) for the loop
\[
s \mapsto e^{ins} \cdot X_a, \quad s \in \mathbb{R},
\]
and \( g(n) \) the subspace spanned by \( \{X_a^n\}_{a=1}^{\dim g} \). The algebraic direct sum
\[
\bigoplus_{n \in \mathbb{Z}} g(n)
\]
is dense in \( L_g \).

**Definition 4.7.** Let \( V \) be a positive energy representation of \( LG \) at level \( k \), with infinitesimal action \( \pi \). We define the **cubic Dirac operator** \( D_Lg \) on \( V \otimes S_Lg \) by
\[
D_Lg = \sum_{n \in \mathbb{Z}} \sum_{a=1}^{\dim g} \left( \pi(X_a^n) \otimes c(X_a^{-n}) + \frac{1}{3} \otimes \text{ad}^g(\pi(X_a^n) \cdot c(X_a^{-n})) \right),
\]
Being a well-defined element, the above infinite sum was interpreted in [Lan01].

Let \( H \) be an isotropy group of the coadjoint \( LG \)-action on \( L_g^* \), which is compact and has the equal rank of \( G \). We decompose
\[
L_g = L_g/h \oplus h, \quad S_{Lg} = S_{Lg/h} \otimes S_h.
\]
Accordingly, the spin representation
\[
\text{ad}^g(X) = \text{ad}^{Lg/h}(X) \otimes 1 + 1 \otimes \text{ad}^h(X), \quad X \in h.
\]

**Definition 4.8.** Let \( \{X_\alpha\}_\alpha \) be a subset of \( \{X_a^n\}_{a,n} \) so that they form a basis of \( L_g/h \) and \( \{X_\alpha\} \) its dual basis. We define the **relative cubic Dirac operator** by
\[
D_{Lg,h} = \sum_\alpha \left( \pi(X_\alpha) \otimes c(X_\alpha) + \frac{1}{3} \otimes \text{ad}^{Lg/h}(X_\alpha) \cdot c(X_\alpha) \right).
\]
Meinrenken showed in [Mei11] that the above infinite sum is a well-defined, self-adjoint, \( H \)-equivariant operator on \( V \otimes S_{Lg/h} \).

As in the finite dimensional case, we decompose \( V \otimes S_{Lg,h} \) with respect to the \( S_{rot}^1 \times H \times S^1 \)-action and denote by \( M(\nu) \) the isotypical summand labeled by \( \nu \in \mathbb{Z} \times \Lambda^* \times \mathbb{Z} \).

**Theorem 4.9.** Suppose that \( V_\lambda \) is an irreducible positive energy representation with highest weight \( \lambda = (0, \lambda, k) \). One has, restricted to \( M(\nu) \)
\[
D_{Lg,h}^2 \big|_{M(\nu)} = \|\lambda + \rho_G\|^2 - \|\nu + \rho_H\|^2,
\]
where \( \rho_G = (0, \rho_G, h^\vee) \) and \( \rho_H = (0, \rho_H, 0) \).
Proof. [Mei11, Theorem 7.5].

Let \( W \) be an irreducible \( \hat{H} \)-space with highest weight \((\nu, k + h^\vee) \in \Lambda^* \times \mathbb{Z}\). Since the relative cubic Dirac operator \( D_{Lg,h} \) is \( H \)-equivariant, it restricts to an operator \( D_W \) on
\[
\mathcal{H} = [V^*_\Lambda \otimes S^*_{Lg,h} \otimes W]^H.
\]
By definition, \( D_W \) is an essentially self-adjoint unbounded operator. We have the following Corollary for later use.

**Corollary 4.10.** By functional calculus, one has that
\[
(1 + D^2_W)^{-1}
\]
is a compact operator on \( \mathcal{H} \).

**Proof.** Let us decompose \( \mathcal{H} \) into \( S^1_{rot} \)-eigenspaces
\[
\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}(k_n),
\]
where \( \mathcal{H}(k_n) \) are finite-dimensional spaces on which \( S^1_{rot} \) acts with weight \( k_n \in \mathbb{Z} \). By Theorem 4.9, the operator \( D^2_W \) acts on \( \mathcal{H}(k_n) \) as a scalar
\[
\|\lambda + \rho_G\|^2 - \|\nu\|^2 + 2(k + h^\vee)k_n.
\]
We notice that \( \lim_{n \to \infty} k_n = \infty \) and \( \|\lambda + \rho_G\|^2 - \|\nu\|^2 \) remains the same. Therefore, the set of eigenvalues of \( D_W \) has no finite accumulation point and every eigenspace is finite-dimensional. The claim follows.

5 **Twisted spinor bundle and twisted pre-quantum bundle**

Suppose that \((M, \omega)\) is a symplectic manifold. For any \( \omega \)-compatible almost complex structure \( J \), we decompose
\[
T_C M = T^{(1,0)} M \oplus T^{(0,1)} M
\]
into \( \pm \sqrt{-1} \) eigen-bundles of \( J \). Then
\[
S_M = \Lambda^* T^{(0,1)} M = \Lambda^{\text{even}} T^{(0,1)} M \oplus \Lambda^{\text{odd}} T^{(0,1)} M
\]
defines a \( \mathbb{Z}_2 \)-graded spinor bundle for \( M \) with associated canonical line bundle
\[
K_M = \text{det}_C(TM)^*.
\]
If we temporarily assume that $M$ is spin, one can construct the square root
$K^{1/2}_M$, which together with $S_M$ determine a spin structure on $TM$:

\begin{equation}
S_M \otimes K^{1/2}_M \cong S(TM).
\end{equation}

The pre-quantization of $(M, \omega)$ is traditionally defined to be a line bundle $L \to M$ whose first Chern class is an integral lift of $[\omega] \in H^2(M, \mathbb{R})$ of the symplectic 2-form.

We next generalize the notions of spinor bundle and pre-quantum line bundle to $\mathfrak{q}$-Hamiltonian $G$-spaces.

5.1 Construction of the twisted spinor bundle

From now on, we assume that $G$ is a compact, simple, and simply connected Lie group. Let $(\mathcal{M}, \omega)$ be a $\mathfrak{q}$-Hamiltonian $G$-space and $\mathcal{M}$ its corresponding Hamiltonian $LG$-space.

For every vertex $\sigma$ of $\mathfrak{A}$, let $V_\sigma \subset \mathcal{M}$ be the cross section defined in (3.1) and $Y_\sigma \subset V_\sigma$ a relatively compact, $(LG)_\sigma$-invariant open subset. Here we choose the collection of subsets

\[ \{Y_\sigma\}_{\sigma, \dim\sigma=0} \]

so that they cover $\mathcal{M}/LG$. For the open faces $\tau$ of $\mathfrak{A}$ with $\dim\tau > 0$, we define

\[ Y_\tau = \bigcap_{\sigma \preceq \tau, \dim\sigma=0} Y_\sigma. \]

By Theorem 3.3, each $Y_\tau$ admits a $(LG)_\tau$-invariant almost complex structure. Moreover, we have the following.

**Lemma 5.1.** There exists a collection of $(LG)_\sigma$-invariant almost complex structures on the collection of $Y_\sigma$ such that the embedding

\[ Y_\tau \hookrightarrow Y_\sigma, \quad \sigma \prec \tau \]

is almost complex. In addition, any two almost complex structures with the required properties are homotopic.

**Proof.** [MW01] Lemma 3.2. \hfill \square

**Lemma 5.2.** For any $\sigma \prec \tau$, the normal bundle $\nu_\tau^\sigma$ of $Y_\tau \hookrightarrow Y_\sigma$ has a $(LG)_\tau$-equivariant almost complex structure with spinor bundle isomorphic to

\begin{equation}
S_{(LG)_\sigma/(LG)_\tau} \otimes \mathbb{C}_{(\rho_\sigma - \rho_\tau)^*}.
\end{equation}
Proof. By Theorem 3.3, the normal bundle $\nu^\sigma_\tau$ is isomorphic to the trivial bundle $(Lg)_\sigma/(Lg)_\tau$ with equivariant almost complex structure described as follow.

Let us choose compatible sets of positive roots $R_\sigma, R_\tau$ for $(LG)_\sigma$ and $(LG)_\tau$, and define $\rho_\tau, \rho_\sigma$ the half-sums of positive roots. Then

$$(Lg)_\sigma/(Lg)_\tau = \bigoplus_{\alpha \in R_\sigma \setminus R_\tau} C_\alpha$$

and

$$\det_C(\nu^\sigma_\tau) = \bigotimes_{\alpha \in R_\sigma \setminus R_\tau} C_\alpha = C_{2(\rho_\sigma - \rho_\tau)}.$$ 

The lemma follows from equation (5.1). \qed

Since the construction of spin representation is multiplicative, we decompose

$$S_{Lg} \cong S_{Lg}/(Lg)_\sigma \otimes S_{(Lg)_\sigma}.$$ 

The Lie algebra $(\widehat{Lg})_\sigma$ acts on $S_{Lg}/(Lg)_\sigma$ so that the central circle acts with weight $h^\vee$. On the other hand, it is shown in [MW01, Lemma 3.1] that

$$2(\rho_\sigma - \rho_\sigma, h^\vee) \in \Lambda^* \times \mathbb{Z}$$

is fixed under $(\widehat{Lg})_\sigma$. Thus, $C_{(\rho_\sigma - \rho_\sigma, h^\vee)}$ is also a representation of the Lie algebra $(\widehat{Lg})_\sigma$ with the central circle acting by weight $h^\vee$.

**Lemma 5.3.** The tensor product

(5.3) \quad $S^*_{Lg}/(Lg)_\sigma \otimes C_{(\rho_\sigma - \rho_\sigma, h^\vee)}$

is a $(LG)_\sigma$-space.

Proof. Because the central circle acts trivially on the tensor product, it suffices to show that (5.3) restricts to a $T$-representation. The claim follows from the Weyl-Kac formula. \qed

By identifying $G_\sigma \cong (LG)_\sigma$, we can consider $Y_\sigma$ as a $G_\sigma$-manifold and define

$$U_\sigma := G \times_{G_\sigma} Y_\sigma.$$ 

As pointed out in Remark 3.5, the subset $U_\sigma$ is equivariantly diffeomorphic to a $G$-invariant open subset of the $q$-Hamiltonian $G$-space $M$. By the fact that $\{Y_\sigma\}_{\dim \sigma = 0}$ cover $M/LG \cong M/G$, the set of subsets $\{U_\sigma\}_{\dim \sigma = 0}$ gives a $G$-invariant open covering of $M$.  

15
Let us denote by $S_{Y_\sigma}$ the spinor bundle for $Y_\sigma$ induced by the almost complex structure and define

$$S_{U_\sigma}^{\text{spin}} = G \times_{G_\sigma} (S^*_L/\mathbb{L} \sigma \otimes \mathbb{C}_{(\rho G - \rho_\sigma, h^\vee)} \otimes S_{Y_\sigma}) \to U_\sigma,$$

where $G_\sigma$ acts on

$$S^*_L/\mathbb{L} \sigma \otimes \mathbb{C}_{(\rho G - \rho_\sigma, h^\vee)}$$

factoring through the identification $G_\sigma \cong (\mathbb{L} G) \sigma$. In addition, we equip $S_{U_\sigma}^{\text{spin}}$ with a $\mathbb{Z}_2$-grading induced by that on $S^*_L/\mathbb{L} \sigma \otimes \mathbb{C}_{(\rho G - \rho_\sigma, h^\vee)}$. Thus, we obtain a collection of $G$-equivariant bundles of $\mathbb{Z}_2$-graded Hilbert space $\{S_{U_\sigma}^{\text{spin}}\}$.

**Lemma 5.4.** There are canonical isomorphisms

$$\Psi_{\tau, \sigma} : S_{U_\tau}^{\text{spin}} \cong S_{U_\sigma}^{\text{spin}}|_{U_\tau}, \quad \sigma \prec \tau$$

and they automatically satisfy the cocycle condition.

**Proof.** The lemma follows from the definition and the fact that

$$S_{Y_\sigma}|_{Y_\tau} \cong S^*_L/\mathbb{L} \tau \otimes \mathbb{C}_{(\rho G - \rho_\tau)} \otimes S_{Y_\tau}.$$

The lemma above enables us to glue all the $\{S_{U_\sigma}^{\text{spin}}\}$ together.

**Definition 5.5.** The twisted spinor bundle $S^{\text{spin}}$ is the $G$-equivariant bundle of $\mathbb{Z}_2$-graded Hilbert space over $M$ with the property that

$$S^{\text{spin}}|_{U_\sigma} \cong S_{U_\sigma}^{\text{spin}}.$$

One sees that the twisted spinor bundle is determined by the choice of almost complex structures on subsets $\{Y_\sigma\}$. By Lemma 5.4 all the choices are homotopic. Hence, the twisted spinor bundle $S^{\text{spin}}$ is unique up to homotopy.

**Remark 5.6.** The twisted spinor bundle $S^{\text{spin}}$ is closed related to the distinguished twisted Spin$^c$-structure introduced by Alekseev-Meinrenken [AM12].

### 5.2 Construction of the twisted pre-quantum bundle

Recall that the 2-form $\omega$ for a q-Hamiltonian $G$-space $M$ is not closed in general. Instead the condition $d\omega = \phi^* \chi$ and $\chi$ is a closed 3-form mean that the pair $(\omega, \chi)$ defines a cocyle for the relative de Rham theory (see [Mei12, Appendix B] for a reference). We denote by $[(\omega, \chi)] \in H^3(\phi, \mathbb{R})$ its cohomology class.
Definition 5.7 ([Mei12]). We say that a q-Hamiltonian G-space \((M, \omega, \phi)\) is pre-quantizable at level \(k\) if \(k \cdot \langle [\omega, \chi] \rangle\) is integral.

Remark 5.8. The pre-quantization condition can be rephrased in a more conventional way in terms of Hamiltonian \(LG\)-space. In fact, a q-Hamiltonian G-space \(M\) is pre-quantizable at level \(k\) if and only if its corresponding Hamiltonian \(LG\)-space \(M\) has a \(\hat{LG}\)-equivariant line bundle \(\mathcal{L} \to M\) such that the central circle acts with weight \(k\) and the first Chern class \(c_1(\mathcal{L})\) equals to the symplectic 2-form on \(M\). Because \(\mathcal{L}\) is \(\hat{LG}\)-equivariant instead of \(LG\)-equivariant, it doesn’t descend to an actual line bundle on \(M\).

Definition 5.9. The Wess-Zumino-Witten space is a unitary \(\hat{LG} \times \hat{LG}\)-space \(H_{wzw,k} = \bigoplus_{\lambda \in P_k, \ +} V_{\lambda} \otimes V_{\lambda}^*\), where \(V_{\lambda}\) is the irreducible positive energy \(LG\)-representation with highest weight \(\lambda = (0, \lambda, k)\), and \(V_{\lambda}^*\) its dual. (for a physics description we refer to [Gaw00]).

Because the central circle acts on \(V_{\lambda}^*\) with weight \(-k\), the tensor product \(V_{\lambda}^* \otimes \mathcal{L} \to M\) becomes \(LG\)-equivariant. Consequently, it descends to a bundles of Hilbert space on \(M\) by \(S_{\lambda}^{Pre} := [V_{\lambda}^* \otimes \mathcal{L}]^{\Omega G} \to M\).

Definition 5.10. We define the twisted pre-quantum bundle by
\[
S_{\lambda}^{Pre} := [H_{wzw,k} \otimes \mathcal{L}]^{\Omega G} = \bigoplus_{\lambda \in P_k, \ +} V_{\lambda} \otimes S_{\lambda}^{Pre}.
\]

The twisted pre-quantum bundle can be reconstructed using cross-sections as well.

On the cross-section \(y_\sigma\), there exists a \(\hat{LG}_\sigma\)-equivariant line bundle obtained by restriction \(L_{y_\sigma} = \mathcal{L}|_{y_\sigma}\).
on which the central circle acts with weight $k$. The collection of line bundles $\{L_y\}$ satisfy a compatibility condition in the sense that

\[(5.6) \quad (LG)_\sigma \times (LG)_\tau L_y \cong L_y|_{Y_\tau}, \quad \sigma \prec \tau.\]

where

\[y_\tau^\sigma = (LG)_\sigma \times (LG)_\tau y_\tau\]

is a $(LG)_\sigma$-invariant open subset of $y_\sigma$.

Consider

\[(5.7) \quad H_{wzw, k} \otimes L_{y_\sigma} = \bigoplus_{\lambda \in P_{k,+}} V_\lambda \otimes (V_\lambda^* \otimes L_{y_\sigma}),\]

where $\hat{G}_\sigma$ acts on the second factor using the identification $\hat{G}_\sigma \cong \hat{(LG)}_\sigma$.

Then the following

\[(5.8) \quad S_{pre|U_\sigma} = G \times_{G_x} (H_{wzw, k} \otimes L_{y_\sigma})\]

defines a $LG \times G$-equivariant bundle of Hilbert space over $U_\sigma$. By (5.6), there are canonical isomorphisms

\[\Psi_{\tau, \sigma} : S_{pre|U_\tau} \cong S_{pre|U_\sigma}, \quad \sigma \prec \tau,\]

satisfying the cocycle condition. Thus, they can be glued together and the twisted pre-quantum bundle $S_{pre}$ is the unique $G$-equivariant bundle of Hilbert space over $M$ with the property that $S_{pre|U_\sigma} = S_{pre|U_\sigma}$.

## 6 Dirac operators on q-Hamiltonian $G$-spaces

With the twisted spinor bundle and twisted pre-quantum bundle defined in last section, we now proceed to construct Hilbert spaces with Dirac operators acting on. We use the same notations as in last section and assume that $G$ is a compact, simple and simply connected Lie group, and $M$ a pre-quantizable q-Hamiltonian $G$-space at level $k$. In addition, unless otherwise stated, we identify $G_\sigma$ with $(LG)_\sigma$.

### 6.1 Dirac operators on cross-sections

Let $\sigma$ be an open face of $A$ and define a Hilbert space

\[H_\sigma = \left[\Gamma_{L^2(U_\sigma, S_{pre}^{spin} \otimes S_{pre})}\right]^G,\]

equipped with a $\mathbb{Z}_2$-grading induced by that on $S_{pre}^{spin}$. 
Lemma 6.1. We have the following isomorphism
\[
\mathcal{H}_\sigma \cong \bigoplus_{\lambda \in P_{k,+}} V_\lambda \otimes \left[ V^*_\lambda \otimes S^*_{L^G/(L^G)_\sigma} \otimes C_{(\rho_G - \rho_\sigma, h^\vee)} \otimes \Gamma_{L^2} \left( y_\sigma, S_{y_\sigma} \otimes L_{y_\sigma} \right) \right]^{G_\sigma}.
\]

Proof. The lemma follows directly from (5.4) and (5.8). \qed

Let $D_{\sigma,\text{alg}}$ be the algebraical relative cubic Dirac operator on
\[
V^*_\lambda \otimes S^*_{L^G/(L^G)_\sigma}
\]
and $D_{\sigma,\text{geo}}$ an equivariant geometric Spin$^c$-Dirac operator on
\[
\Gamma_{L^2} \left( y_\sigma, S_{y_\sigma} \otimes L_{y_\sigma} \right).
\]

Since the sum
\[
D_{\sigma,\text{alg}} \otimes 1 + 1 \otimes D_{\sigma,\text{geo}}
\]
is equivariant, it restricts to the $G_\sigma$-invariant subspace, giving a collection of operators
\[
\left[ D_{\sigma,\text{alg}} \otimes 1 + 1 \otimes D_{\sigma,\text{geo}} \right]^{G_\sigma}
\]
on
\[
\left[ V^*_\lambda \otimes S^*_{L^G/(L^G)_\sigma} \otimes C_{(\rho_G - \rho_\sigma, h^\vee)} \otimes \Gamma_{L^2} \left( y_\sigma, S_{y_\sigma} \otimes L_{y_\sigma} \right) \right]^{G_\sigma}.
\]

Tensoring with the identity operator on $V_\lambda$, and summing over $V_\lambda$, one obtains an operator $D_\sigma$ on $\mathcal{H}_\sigma$.

Proposition 6.2. For $\sigma \prec \tau$, $\mathcal{H}_\tau$ is a Hilbert sub-space of $\mathcal{H}_\sigma$ and the two operators $D_\tau$ and $D_\sigma|_{\mathcal{H}_\tau}$ are homotopic.

Proof. By Lemma 6.1
\[
\mathcal{H}_\tau \cong \left[ H_{\text{wzw},k} \otimes S^*_{L^G/(L^G)_\tau} \otimes C_{(\rho_G - \rho_\sigma, h^\vee)} \otimes \Gamma_{L^2} \left( y_\sigma, S_{y_\sigma} \otimes L_{y_\sigma} \right) \right]^{G_\sigma}
\]
\[
\cong \left[ H_{\text{wzw},k} \otimes S^*_{L^G/(L^G)_\tau} \otimes C_{(\rho_G - \rho_\sigma, h^\vee)} \otimes \Gamma_{L^2} \left( y_\sigma, S_{y_\sigma} \otimes L_{y_\sigma} \right) \right]^{G_\sigma},
\]

where
\[
y_\tau^\sigma = G_\sigma \times_{G_\tau} y_\tau
\]
is an open subset in $y_\sigma$. We first notice that on the factor
\[
H_{\text{wzw},k} \otimes S^*_{L^G/(L^G)_\sigma},
\]
both $D_\tau$ and $D_\sigma$ act as the relative cubic Dirac operator. Thus, it remains to compare their actions on
\[
\Gamma_{L^2} \left( y_\sigma^\sigma, (S_{y_\sigma} \otimes L_{y_\sigma})|_{y_\tau^\sigma} \right).
\]

By (5.2) and (5.6),
\[
S_{y_\sigma}|_{y_\tau^\sigma} \cong G_\sigma \times_{G_\tau} (S_{y_\tau^\sigma} \otimes C_{\rho_\tau - \rho_\sigma, h^\vee}) \quad \text{and} \quad L_{y_\sigma}|_{y_\tau^\sigma} \cong G_\sigma \times_{G_\tau} L_{y_\tau^\sigma}.
\]
Thus, the Hilbert space (6.3) decomposes into “vertical part” and “horizontal part”:

\[
\left[(L^2(G_\sigma) \otimes S^*_{(Lg)_{\tau}} \otimes \mathbb{C}_{\rho_\sigma-\rho_\tau}) \otimes \Gamma_{L^2}(y_{\tau}, S_{y_{\tau}} \otimes L_{y_{\tau}})\right]^{G_{\tau}}.
\]

On the vertical direction, that is \(\Gamma_{L^2}(y_{\tau}, S_{y_{\tau}} \otimes L_{y_{\tau}})\), the two operators \(D_\tau\) and \(D_\sigma\) coincide, acting as the geometric Spin\(^c\)-Dirac operators; while on the horizontal part

\(L^2(G_\sigma) \otimes S^*_{(Lg)_{\tau}}\),

\(D_\tau\) acts as the relative cubic Dirac operator, and \(D_\sigma\) is a geometric Spin\(^c\)-Dirac operator. By Lemma 4.6, they are homotopic. This completes the proof.

\[\square\]

6.2 Construction of the Dirac operator

Recall that the collection of \(G\)-invariant open subsets

\[\{U_\sigma = G \times_{G_\sigma} \mathcal{Y}_{\alpha,\dim\sigma=0}\}\]

gives an open covering of \(M\). Select a family of \(G\)-invariant smooth functions \(\{f_\sigma\}\) so that \(\{f^2_\sigma\}\) form a partition of unity subordinating to the open covering.

**Definition 6.3.** We define a Hilbert space

\[
\mathcal{H} = \left[\Gamma_{L^2}(M, S^{\text{spin}} \otimes S^{\text{pre}})\right]^G,
\]

equipped with a \(\mathbb{Z}_2\)-grading from that on \(S^{\text{spin}}\), and a Dirac operator

\[
\mathcal{D} = \sum_{\sigma, \dim\sigma=0} f_\sigma \cdot D_\sigma \cdot f_\sigma.
\]

Because both the relative cubic Dirac operator and Spin\(^c\)-Dirac operators are equivariant, odd, essentially self-adjoint, first order differential operators, so is the operator \(\mathcal{D}\).

**Lemma 6.4.** For any open face \(\tau\), the operator \(\mathcal{D}|_{\mathcal{H}_{\tau}}\) is homotopic to \(D_\tau\).

**Proof.** By Proposition 6.2, for any \(\sigma \preceq \tau\) and \(\dim\sigma = 0\),

\[D_\sigma|_{\mathcal{H}_\tau} \sim D_\tau.
\]
Thus,
\[
D|_{\mathcal{G}_\tau} = \sum_{\sigma \leq \tau, \dim \sigma = 0} f_\sigma \cdot (D_{\sigma}|_{\mathcal{G}_\tau}) \cdot f_\sigma \sim \sum_{\sigma \leq \tau, \dim \sigma = 0} f_\sigma \cdot D_\tau \cdot f_\sigma
\]
(6.4)
\[
= \sum_{\sigma \leq \tau, \dim \sigma = 0} D_\tau \cdot f_\sigma + \sum_{\sigma \leq \tau, \dim \sigma = 0} [f_\sigma, D_\tau] \cdot f_\sigma
\]
\[
= D_\tau + \sum_{\sigma \leq \tau, \dim \sigma = 0} [f_\sigma, D_\tau] \cdot f_\sigma.
\]

Notice that
\[
[f_\sigma, D_\tau] = f_\sigma \cdot D_\tau - D_\tau \cdot f_\sigma
\]

is an order-zero operator. Hence, \(D|_{\mathcal{G}_\tau}\) is homotopic to \(D_\tau\).

One consequence of the above lemma is that the Dirac operator \(D\) is independent of the choice of the partition of unity up to homotopy.

As in (5.5), let us decompose
\[
\mathcal{H} = \bigoplus_{\lambda \in \mathcal{P}_{k,+}} V_\lambda \otimes \mathcal{H}_\lambda,
\]
where
\[
\mathcal{H}_\lambda = [\Gamma_{L^2}(M, S^{\text{spin}} \otimes S_\lambda^{\text{pre}})]^G.
\]

We denote by \(D_\lambda\) the restriction of \(D\) to \(\mathcal{H}_\lambda\), which is essentially self-adjoint. To show that \(D_\lambda\) has finite-dimensional kernel, we proceed to prove a stronger result that
\[
(1 + D_\lambda^2)^{-1} \in K(\mathcal{H}_\lambda).
\]

**Lemma 6.5.** Let \(D_1, D_2\) be two essentially self-adjoint operators on Hilbert spaces \(H_1\) and \(H_2\). If
\[
(1 + D_1^2)^{-1} \in K(H_1), \quad (1 + D_2^2)^{-1} \in K(H_2).
\]
then
\[
(1 + D_1^2 + D_2^2)^{-1} \in K(H_1 \otimes H_2).
\]

**Proof.** Notice that
\[
(1 + D_1^2 + D_2^2)^{-1} - (1 + D_1^2)^{-1} \otimes (1 + D_2^2)^{-1}
\]
(6.6)
\[
= [D_1^2 \cdot D_2^2 \cdot (1 + D_1^2 + D_2^2)^{-1}] \cdot [(1 + D_1^2)^{-1} \otimes (1 + D_2^2)^{-1}].
\]
By the assumption that \((1 + D_2^2)^{-1} \in K(H_2)\), we can find a sequence of operators \(K_\eta\) on \(H_2\) of finite rank and
\[
\|K_\eta - (1 + D_2^2)^{-1}\| \to 0,
\]
where \(\| \cdot \|\) is the strong operator norm.
Proposition 6.6. One has that

\[ (1 + D^2) \in \mathcal{K}(H). \]

Proof. Choose a family of G-invariant smooth functions \( \{ g_\sigma \} \) so that they form a partition of unity subordinating to the open covering \( \{ U_\sigma \} \). Let us consider the operators

\[ T_\sigma = \left[(1 + D^2_{\sigma,alg})^{-1} \right] \otimes \left[(1 + D^2_{\sigma,geo})^{-1} \right] : g_\sigma \]

and

\[ F_\sigma = \left[(1 + D^2_{\sigma,alg} + D^2_{\sigma,geo})^{-1} \right] : g_\sigma \]

acting on

\[ H^\lambda_\sigma := \left(V^\lambda_\sigma \otimes S^*_\rho/(L_\rho)_{\lambda_\sigma} \otimes C(\rho_{\rho_\sigma, h_{\lambda_\sigma}^1}) \right) \otimes \Gamma_{L^2}(y, S_{y, \sigma} \otimes L_{y, \sigma}). \]

We denote by \( [T_\sigma]^{G_\sigma}, [F_\sigma]^{G_\sigma} \) their restrictions to the \( G_\sigma \)-invariant subspace \( [H^\lambda_\sigma]^{G_\sigma} \). By Lemma 6.5 it suffices to show

\[ [F_\sigma]^{G_\sigma} \in \mathcal{K}(\{H^\lambda_\sigma\}) \]

it suffices to prove that

\[ [T_\sigma]^{G_\sigma} \in \mathcal{K}(\{H^\lambda_\sigma\}). \]

Because \( g_\sigma \) has compact support in \( y_\sigma \), it follows from the Rellich’s lemma that

\[ (1 + D^2_{\sigma,geo})^{-1} : g_\sigma \in \mathcal{K}(\Gamma_{L^2}(y_\sigma, S_{y, \sigma} \otimes L_{y, \sigma})). \]

Thus, one can find a sequence of equivariant operators of finite rank

\[ K_n \in \mathbb{B}(\Gamma_{L^2}(y_\sigma, S_{y, \sigma} \otimes L_{y, \sigma})), \]

Because

\[ [D^2_1 \cdot D^2_2 \cdot (1 + D^2_1 + D^2_2)^{-1}] \cdot K_n \]

are bounded operators on \( H_1 \otimes H_2 \), the operators

\[ [D^2_1 \cdot D^2_2 \cdot (1 + D^2_1 + D^2_2)^{-1}] \cdot [(1 + D^2_1)^{-1} \otimes K_n] \]

must be compact for all \( n \). And they can be approached by a sequence of finite operators \( \{ K_n \}_{k=1}^\infty \).

By the Cauchy-Schwarz inequality, we see that

\[ \left\| [D^2_1 \cdot D^2_2 \cdot (1 + D^2_1 + D^2_2)^{-1}] \cdot [(1 + D^2_1)^{-1} \otimes (1 + D^2_2)^{-1}] \right\| \]

\[ \leq \left\| [D^2_1 \cdot D^2_2 \cdot (1 + D^2_1 + D^2_2)^{-1}] \cdot [(1 + D^2_1)^{-1} \otimes K_n] - K_n^n \right\| \]

\[ + \left\| [D^2_1 \cdot D^2_2 \cdot (1 + D^2_1 + D^2_2)^{-1}] \cdot (1 + D^2_1)^{-1} \right\| \cdot \left\| K_n - (1 + D^2_2)^{-1} \right\| \rightarrow 0 \]

as \( n \rightarrow \infty \). Therefore, the operator on the right-hand side of (6.6) is compact and the claim follows. \( \square \)
so that
\[ \|K_n - (1 + D_{\sigma,\text{alg}}^2)^{-1} \cdot g_\sigma\| \to 0. \]

Because of Corollary 4.10 and the fact that $K_n$ are operators of finite rank,
\[ [(1 + D_{\sigma,\text{alg}}^2)^{-1} \otimes K_n]^{G_\sigma} \in \mathbb{K}(\mathcal{H}_\sigma^{\lambda} G_\sigma). \]
Thus there exists a sequence of finite operators \( \{K_n\}_{n=1}^\infty \) on \( \mathcal{H}_\sigma^{\lambda} G_\sigma \) so that
\[ \lim_{n \to \infty} \|K_n - [(1 + D_{\sigma,\text{alg}}^2)^{-1} \otimes K_n]^{G_\sigma}\| = 0. \]
By the Cauchy-Schwarz inequality and argument in Lemma 6.5
\[ \lim_{n \to \infty} \|K_n - [T_\sigma]^{G_\sigma}\| = 0. \]
Hence, the operator \( [T_\sigma]^{G_\sigma} \) as well as \( [F_\sigma]^{G_\sigma} \) are compact.

At last, by Lemma 6.4 the two operators
\[ (1 + D_\lambda^2)^{-1} \cdot g_\sigma, \quad [F_\sigma]^{G_\sigma} \]
are homotopic after restricting to \( \mathcal{H}_\sigma^{\lambda} G_\sigma \). This implies that
\[ (1 + D_\lambda^2)^{-1} = \sum_{\dim \mathcal{H}_\sigma^{\lambda} = 0} (1 + D_\lambda^2)^{-1} \cdot g_\sigma \]
is a compact operator. \qed

We can now state the main result of this paper.

**Theorem 6.7.** If $M$ is a pre-quantizable $q$-Hamiltonian $G$-space at level $k$, then
\[ \text{ind}(D) = \text{ker}(D) \cap \mathcal{H}^+ - \text{ker}(D) \cap \mathcal{H}^- \in \mathbb{R}_k(LG). \]
We define the quantization of $M$ to be
\[ Q(M) := \text{ind}(D). \]

**Proof.** By the decomposition \(6.5\),
\[ \text{ind}(D) = \bigoplus_{\lambda \in \mathbb{P}_{k,+}} V_\lambda \otimes \text{ind}(D_\lambda), \]
where
\[ \text{ind}(D_\lambda) := \text{ker}(D_\lambda) \cap \mathcal{H}_\lambda^+ - \text{ker}(D_\lambda) \cap \mathcal{H}_\lambda^- \]
is the multiplicity of $V_\lambda$ in $\text{ind}(D)$. By Proposition 6.6 $D_\lambda$ has finite dimensional kernel. Thus, $\text{ind}(D_\lambda) \in \mathbb{Z}$. \qed

**Remark 6.8.** In [Mei12], Meinrenken develop a quantization from pre-quantized $q$-Hamiltonian $G$-spaces to the equivariant twisted $K$-homology of $G$, which is isomorphic to $\mathbb{R}_k(LG)$ by the work of Freed-Hopkins-Teleman.
(they prove the isomorphism for more general Lie groups). In a forthcoming paper, we will show that $Q(M)$ defined above satisfies the “quantization commutes with reduction” principle in a suitable sense. And it implies the two approaches of quantization are consistent.

### 6.3 Example: conjugacy classes

Suppose that $C$ is a conjugacy class of the element $\exp(\xi)$ with $\xi \in \mathfrak{a}$. It is a pre-quantizable at level $k$ if and only if

$$\nu := B^\flat(k \cdot \xi) \in \Lambda^*.$$ 

Suppose that $C \cong G/H$. The twisted spinor bundle and twisted pre-quantum bundle are given by

$$S^{\text{spin}} = G \times_H (S^*_{Lg,h} \otimes \mathbb{C}_{(\rho_G - \rho_H, h^\vee)})$$

and

$$S^{\text{pre}} = G \times_H (H_{wzw,k} \otimes \mathbb{C}_{(\nu,k)}).$$

The Hilbert space is

$$\mathcal{H} = \left[\Gamma_{L^2}(C, S^{\text{spin}} \otimes S^{\text{pre}})\right]^G$$

$$= \left[H_{wzw,k} \otimes S^*_{Lg,h} \otimes \mathbb{C}_{(\rho_G - \rho_H, h^\vee)} \otimes \mathbb{C}_{(\nu,k)}\right]^H$$

$$= \bigoplus_{\lambda \in P_{k,+}} V_{\lambda} \otimes (V^*_\lambda \otimes S^*_{Lg,h} \otimes \mathbb{C}_{(\rho_G - \rho_H + \nu, h^\vee + k)})^H,$$

and the Dirac operator $\mathcal{D}$ on $\mathcal{H}$ acts as the relative cubic Dirac operator $D_{Lg,h}$ on the factors $V^*_\lambda \otimes S^*_{Lg,h}$. By Theorem 4.9 (or [Mei11, Theorem 7.5] for more details), one can calculate that

$$\ker(\mathcal{D}_\lambda) = \begin{cases} 1\text{-dimensional space} & \lambda = \nu. \\ 0 & \text{otherwise} \end{cases}$$

The above formula shows that

$$\text{ind}(\mathcal{D}) = V_\nu \in R_k(LG),$$

where $\nu = (0, \nu, k)$. Let us finally remark that the quantization of conjugacy is closed related to the Dirac induction for loop group described in [Pos11].
References

[AM00] Anton. Alekseev and Eckhard. Meinrenken. The non-commutative Weil algebra. *Invent. Math.*, 139(1):135–172, 2000.

[AM12] Anton Alekseev and Eckhard Meinrenken. Dirac structures and Dixmier-Douady bundles. *Int. Math. Res. Not. IMRN*, (4):904–956, 2012.

[AMM98] Anton Alekseev, Anton Malkin, and Eckhard Meinrenken. Lie group valued moment maps. *J. Differential Geom.*, 48(3):445–495, 1998.

[AMW01] A. Alekseev, E. Meinrenken, and C. Woodward. The Verlinde formulas as fixed point formulas. *J. Symplectic Geom.*, 1(1):1–46, 2001.

[CW08] Alan L. Carey and Bai-Ling Wang. Fusion of symmetric D-branes and Verlinde rings. *Comm. Math. Phys.*, 277(3):577–625, 2008.

[FHT11] Daniel S. Freed, Michael J. Hopkins, and Constantin Teleman. Loop groups and twisted $K$-theory III. *Ann. of Math. (2)*, 174(2):947–1007, 2011.

[FHT13] Daniel S. Freed, Michael J. Hopkins, and Constantin Teleman. Loop groups and twisted $K$-theory II. *J. Amer. Math. Soc.*, 26(3):595–644, 2013.

[Gaw00] Krzysztof Gawedzki. Conformal field theory: a case study. In *Conformal field theory (Istanbul, 1998)*, volume 102 of *Front. Phys.*, page 55. Adv. Book Program, Perseus Publ., Cambridge, MA, 2000.

[Kos99] Bertram Kostant. A cubic Dirac operator and the emergence of Euler number multiplets of representations for equal rank subgroups. *Duke Math. J.*, 100(3):447–501, 1999.

[KS87] Bertram Kostant and Shlomo Sternberg. Symplectic reduction, BRS cohomology, and infinite-dimensional Clifford algebras. *Ann. Physics*, 176(1):49–113, 1987.
[Lan01] Gregory D. Landweber. Multiplets of representations and Kostant’s Dirac operator for equal rank loop groups. *Duke Math. J.*, 110(1):121–160, 2001.

[Mei11] Eckhard Meinrenken. The cubic Dirac operator for infinite-dimensional Lie algebras. *Canad. J. Math.*, 63(6):1364–1387, 2011.

[Mei12] Eckhard Meinrenken. Twisted K-homology and group-valued moment maps. *Int. Math. Res. Not. IMRN*, (20):4563–4618, 2012.

[Mei13] Eckhard Meinrenken. *Clifford algebras and Lie theory*, volume 58 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer, Heidelberg, 2013.

[MMS06] V. Mathai, R. B. Melrose, and I. M. Singer. Fractional analytic index. *J. Differential Geom.*, 74(2):265–292, 2006.

[MW98] E. Meinrenken and C. Woodward. Hamiltonian loop group actions and Verlinde factorization. *J. Differential Geom.*, 50(3):417–469, 1998.

[MW01] E. Meinrenken and C. Woodward. Canonical bundles for Hamiltonian loop group manifolds. *Pacific J. Math.*, 198(2):477–487, 2001.

[Pos11] Hessel Posthuma. Dirac induction for loop groups. *Lett. Math. Phys.*, 95(1):89–107, 2011.

[PR94] R. J. Plymen and P. L. Robinson. *Spinors in Hilbert space*, volume 114 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1994.

[PS86] Andrew Pressley and Graeme Segal. *Loop groups*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1986. Oxford Science Publications.

[Was10] Antony Wassermann. Kac-moody and virasoro algebras. 2010.