FLAT COMMUTATIVE RING EPIMORPHISMS
OF ALMOST KRULL DIMENSION ZERO

LEONID POSITSELSKI

Abstract. We consider flat epimorphisms of commutative rings $R \twoheadrightarrow U$ such that, for every ideal $I \subset R$ for which $IU = U$, the quotient ring $R/I$ is semilocal of Krull dimension zero. Under these assumptions, we show that the projective dimension of the $R$-module $U$ does not exceed 1. We also describe the Geigle–Lenzing perpendicular subcategory $U_{0,1}$ in $R$–Mod. Assuming additionally that the ring $U$ and all the rings $R/I$ are perfect, we show that all flat $R$-modules are $U$-strongly flat. Thus we obtain a generalization of some results of the paper [6], where the case of the localization $U = S^{-1}R$ of the ring $R$ at a multiplicative subset $S \subset R$ was considered.

Contents

Introduction 1
1. $F$-h-locality 4
2. $I$-Contramodule $R$-Modules 6
3. Projective Dimension 1 Theorem 9
4. Divisible Modules 10
5. $u$-Contramodule $R$-Modules 12
6. $U$-Strongly Flat $R$-Modules 14
References 16

Introduction

Let $R$ be a commutative integral domain with the field of fractions $Q$. It was discovered by Matlis [16] that certain commutative and homological algebra constructions behave much better when the projective dimension of the $R$-module $Q$ does not exceed 1. Commutative domains $R$ with this property are now known as Matlis domains [9]. More generally, the same observation applies to the case when $R$ is a commutative ring and $Q$ is its full ring of quotients, that is $Q = S^{-1}R$, where $S \subset R$ is the set of all regular elements in $R$ [17].

Even more generally, the case of a multiplicative subset $S \subset R$ consisting of (some) regular elements in $R$ was considered by Angeleri Hügel, Herbera, and Trlifaj in the paper [1], where a number of conditions equivalent to $\text{pd}_R S^{-1}R \leq 1$ was listed. In particular, the projective dimension of the $R$-module $Q = S^{-1}R$ does not exceed 1 if and only if $Q \oplus Q/R$ is a 1-tilting $R$-module, if and only if two natural notions
of \(S\)-divisibility for \(R\)-modules coincide, and if and only if \(Q/R\) is a direct sum of countably presented \(R\)-modules. The next step, towards the study of localizations \(S^{-1}R\) where a multiplicative subset \(S \subset R\) is allowed to contain some zero-divisors in \(R\), was made in the present author’s papers [19, Section 13] and [20, 23, 6].

From the contemporary point of view, it appears that the maximal natural generality in this line of thought is achieved by considering ring epimorphisms \(u: R \to U\). In this context, the condition that the projective dimension of the (left or right) \(R\)-module does not exceed 1 continues to play an important role. In particular, for an injective homological ring epimorphism \(u\) (of associative, not necessarily commutative rings) a number of homologically relevant conditions equivalent to \(\text{pd}_RU \leq 1\) were listed in the paper of Angeleri Hügel and Sánchez [3, Theorem 3.5].

As a general rule, all the above-mentioned results allow to obtain nice corollaries of the projective dimension 1 condition, but none of them provide a workable technology for proving this condition. How does one show that \(\text{pd}_RS^{-1}R \leq 1\), for a specific multiplicative subset \(S\) in a commutative ring \(R\)? Or more generally that \(\text{pd}_RU \leq 1\), for a specific ring epimorphism \(u: R \to U\)?

Notice first of all that \(S^{-1}R\) is always a flat \(R\)-module, while for a ring epimorphism \(R \to U\) the (left or right) \(R\)-module \(U\) does not have to be flat. The following condition is usually imposed on a ring epimorphism, however: a ring epimorphism \(u: R \to U\) is said to be homological if \(\text{Tor}_n^R(U, U) = 0\) for all \(n \geq 1\). In the context of commutative rings, the following recent result of Bazzoni and the present author is relevant in this connection: if \(R \to U\) is an epimorphism of commutative rings such that \(\text{Tor}_1^R(U, U) = 0\) and \(\text{pd}_RU \leq 1\), then \(U\) is a flat \(R\)-module [7, Theorem 5.2]. In a sense, this means that one can restrict oneself to flat epimorphisms when studying the projective dimension 1 condition for epimorphisms of commutative rings.

Flat ring epimorphisms can be described in terms of Gabriel filters (otherwise known as Gabriel topologies) [26, Chapters VI and IX–XI]. To every epimorphism of associative rings \(u: R \to U\) such that \(U\) is a flat left \(R\)-module, one assigns the set \(G\) of all right ideals \(I \subset R\) such that \(R/I \otimes_R U = 0\), or equivalently, \(IU = U\). The ring \(U\) then can be recovered as the ring of quotients \(U = R_G\) with respect to the Gabriel filter \(G\). In particular, if \(R\) is commutative and \(U = S^{-1}R\), then \(G\) is the set of all ideals in \(R\) intersecting \(S\).

The simplest class of ring epimorphisms \(R \to U\) for which the projective dimension of the \(R\)-module \(U\) does not exceed 1 is the following one. Let \(S \subset R\) be a countable multiplicative subset. Then \(\text{pd}_RS^{-1}R \leq 1\). A far-reaching generalization of this elementary observation was obtained in the paper [21]. Specifically, if \(u: R \to U\) is a left flat ring epimorphism and the related Gabriel filter of right ideals in \(R\) has a countable base, then the projective dimension of the flat left \(R\)-module \(U\) does not exceed 1 [21, Theorem 8.5]. The proof of this result given in [21] uses the fundamental concept of the abelian category of contramodules over a topological ring (specifically, over the completion of \(R\) with respect to \(G\)).

Another basic observation is that \(\text{pd}_RS^{-1}R \leq 1\) whenever \(R\) is a Noetherian commutative ring of Krull dimension 1. In fact, any flat module over a Noetherian
commutative ring of Krull dimension \( \leq 1 \) has projective dimension \( \leq 1 \) [25, Corollaire II.3.2.7]. A contramodule-based proof of this assertion was suggested in [19, Corollary 13.7 and Remark 13.10]. Building up on the techniques of [19], Bazzoni and the present author proved the following result in [6, Theorem 6.13]: for any commutative ring \( R \) with a multiplicative subset \( S \subseteq R \) such that \( R/sR \) is a semilocal ring of Krull dimension zero for every \( s \in S \), one has \( \text{pd}_R S^{-1}R \leq 1 \).

In this paper, we generalize some results of [6] to the case of a flat epimorphism of commutative rings \( u: R \rightarrow U \). Let \( G \) denote the Gabriel filter of ideals in \( R \) related to \( u \). We show that \( \text{pd}_R U \leq 1 \) whenever the quotient ring \( R/I \) is semilocal of Krull dimension zero for every ideal \( I \in G \). The argument is based on the notion of \( I \)-contramodule \( R \)-modules for an ideal \( I \) in a commutative ring \( R \). This result of ours has already found its uses in the work of Bazzoni and Le Gros on envelopes and covers in the tilting cotorsion pairs related to \( 1 \)-tilting modules over commutative rings; see [4, Remark 8.6 and Theorem 8.7] and [5, Remark 8.2 and Theorem 8.17].

The proofs of the \( \text{pd}_R U \leq 1 \) theorems in the papers [21], [6], and the present paper follow a single common approach, which can be described in very general terms as follows. Given a ring epimorphism \( u: R \rightarrow U \), one considers the two-term complex \( K_u^\bullet = (R \rightarrow U) \); when \( R \) and \( U \) are associative rings, \( K_u^\bullet \) is a complex of \( R \)-\( R \)-bimodules; in the commutative case, it is simply a complex of \( R \)-modules. To any (left) \( R \)-module \( B \), one assigns the sequence of (left) \( R \)-modules \( \text{Ext}^n_R(K_u^\bullet, B) = \text{Hom}_{D(\text{Mod}_R)}([K_u^\bullet, B[n]]) \) of homomorphisms in the derived category \( D(\text{Mod}_R) \), indexed by the integers \( n \geq 0 \). In particular, for \( n \geq 2 \) there is a natural \( R \)-module isomorphism \( \text{Ext}^n_R(K_u^\bullet, B) \cong \text{Ext}^n_R(U, B) \) [7, Lemma 2.1(b)].

In each of the three settings mentioned in the italicized phrases in the previous paragraphs where the results of the papers [21], [6], and the present paper are discussed, the argument concerning the projective dimension \( \text{pd}_R U \) proceeds as follows. In order to prove that \( \text{Ext}^n_R(U, B) = 0 \) for \( n \geq 2 \), one shows that the two classes of \( R \)-modules of the form \( \text{Ext}^n_R(K_u^\bullet, B) \) and \( \text{Ext}^n_R(U, B) \) (where \( n \geq 0 \) and \( B \in \text{Mod}_R \)) only intersect at zero. The two kinds of \( R \)-modules just have incompatible properties. In fact, \( \text{Ext}^n_R(U, B) \) is the underlying \( R \)-module of a \( U \)-module. The \( R \)-modules \( \text{Ext}^n_R(K_u^\bullet, B) \) are described in terms of some kind of contramodules (it depends on the setting which specific contramodule category is more convenient to work with). One wants to show that no nonzero \( R \)-module of the form \( C = \text{Ext}^n_R(K_u^\bullet, B) \) admits an extension of its \( R \)-module structure to a \( U \)-module structure. In fact, one usually proves that \( \text{Hom}_R(U, C) = 0 \); this is certainly enough.

The Geigle–Lenzing perpendicular subcategory \( U_{\perp 0.1} \) to the \( R \)-module \( U \) in the category of \( R \)-modules \( R \text{-Mod} \) consists all \( R \)-modules \( C \) such that \( \text{Hom}_R(U, C) = 0 = \text{Ext}^1_R(U, C) \) [11]. We use the name \( u \)-contramodules for \( R \)-modules \( C \in U_{\perp 0.1} \), and the notation \( R \text{-Mod}_{u\text{-contra}} = U_{\perp 0.1} \subseteq R \text{-Mod} \) for the full subcategory formed by such modules. Another result of this paper is a description of the abelian category \( R \text{-Mod}_{u\text{-contra}} \) in terms of the abelian categories of \( \mathfrak{m} \)-contramodule \( R \)-modules, where \( \mathfrak{m} \) ranges over the maximal ideals of \( R \) belonging to \( G \). The assumption that \( R/I \) is semilocal of Krull dimension zero for all \( I \in G \) is made here.
Finally, let us recall that a module $F$ over a commutative domain $R$ is said to be strongly flat if it is a direct summand of an $R$-module $G$ appearing in a short exact sequence of $R$-modules $0 \to V \to G \to W \to 0$ in which $V$ is a free $R$-module and $W$ is a $Q$-vector space. A series of generalizations of this concept was developed in the papers [10, 6, 23, 21]. We refer to the introduction to [23] for a detailed discussion with further references.

In particular, let $R$ be a commutative ring and $S \subset R$ be a multiplicative subset. Then an $R$-module $F$ is called $S$-strongly flat if it is a direct summand of an $R$-module $G$ appearing in a short exact sequence of $R$-modules $0 \to V \to G \to W \to 0$ in which $V$ is a free $R$-module and $W$ is a free $S^{-1}R$-module. In the terminology of [6], a ring $R$ is said to be $S$-almost perfect if $S^{-1}R$ is a perfect ring and $R/sR$ is a perfect ring for every $s \in S$. According to [6, Theorem 7.9], $R$ is $S$-almost perfect if and only if all flat $R$-modules are $S$-strongly flat.

More generally, given a left flat epimorphism of associative rings $u: R \to U$, a left $R$-module $F$ is called $U$-strongly flat [21, Section 9] if it is a direct summand of a left $R$-module $G$ appearing in a short exact sequence of left $R$-modules $0 \to V \to G \to W \to 0$ in which $V$ is a free left $R$-module and $W$ is a free left $U$-module.

Let $u: R \to U$ be a flat epimorphism of commutative rings and $\mathcal{G}$ be the related Gabriel filter of ideals in $R$. In the terminology of [5, Sections 6 and 8], a ring $R$ is said to be $\mathcal{G}$-almost perfect if $U = R_{\mathcal{G}}$ is a perfect ring and $R/I$ is a perfect ring for all $I \in \mathcal{G}$. We show that all flat $R$-modules are $U$-strongly flat whenever $R$ is $\mathcal{G}$-almost perfect (the converse assertion also holds). More generally, in the assumption that $R/I$ is semilocal of Krull dimension zero for all $I \in \mathcal{G}$, we characterize $U$-strongly flat $R$-modules $F$ as flat $R$-modules for which the $U$-module $U \otimes_R F$ is projective and the $R/I$-modules $F/IF$ are projective.

Acknowledgement. I am grateful to Silvana Bazzoni and Michal Hrbek for numerous very helpful discussions and communications. The author is supported by the GACR project 20-13778S and research plan RVO: 67985840.

1. $\mathcal{F}$-h-locality

Recall that a commutative ring $T$ is said to be semilocal if the set of all maximal ideals in $T$ is finite. The ring $T$ is said to have Krull dimension zero if all the prime ideals in $T$ are maximal. The following description is standard and easy.

**Lemma 1.** (a) A commutative ring $T$ is isomorphic to a finite product of local rings if and only if $T$ is semilocal with the additional property that every prime ideal of $T$ is contained in a unique maximal ideal.

(b) A commutative ring $T$ is semilocal of Krull dimension zero if and only if it is isomorphic to a finite product of local rings $T_1 \times \cdots \times T_m$ such that, for every $1 \leq j \leq m$, the maximal ideal in $T_j$ consists of nilpotent elements. □

A filter $\mathcal{F}$ of ideals (or a “linear topology”) in a commutative ring $R$ is a set of ideals in $R$ such that $R \in \mathcal{F}$, and $K \supseteq I \cap J$, $I, J \in \mathcal{F}$ implies $K \in \mathcal{F}$. 
Let \( R \) be a commutative ring with a filter \( \mathcal{F} \) of ideals. An \( R \)-module \( N \) is said to be \( \mathcal{F} \)-torsion if, for every element \( x \in N \), the annihilator of \( x \) in \( R \) belongs to \( \mathcal{F} \).

We will say that the ring \( R \) is \( \mathcal{F} \)-h-local \([9, \text{Section IV.3}], [5, \text{Section 7}]\) if every ideal \( I \in \mathcal{F} \) is contained only in finitely many maximal ideals of \( R \) and every prime ideal of \( R \) belonging to \( \mathcal{F} \) is contained in a unique maximal ideal. In other words, \( R \) is \( \mathcal{F} \)-h-local if and only if for every \( I \in \mathcal{F} \) the quotient ring \( R/I \) is semilocal and every prime ideal of \( R/I \) is contained in a unique maximal ideal. Equivalently, this means that the ring \( R/I \) is a finite product of local rings (by Lemma 1(a)).

As usually, we denote by \( \text{Max} R \) the set of all maximal ideals of the ring \( R \). Given a maximal ideal \( m \subset R \) and an \( R \)-module \( N \), one can consider the local ring \( R_m \) and the \( R_m \)-module \( N_m \), that is, the localizations of \( R \) and \( N \) at \( m \). Conversely, any \( R_m \)-module can be considered as an \( R \)-module. Notice that if \( N \) is \( \mathcal{F} \)-torsion and \( m/\mathcal{F} \in \mathcal{F} \), then \( N_m = 0 \) (indeed, for any \( x \in N \), the annihilator of \( x \) in \( R \) is not contained in \( m \)).

**Lemma 2.** Let \( R \) be a commutative ring with a filter \( \mathcal{F} \) of ideals such that the ring \( R \) is \( \mathcal{F} \)-h-local. Given an \( \mathcal{F} \)-torsion \( R \)-module \( N \), consider all the maximal ideals \( m \) of the ring \( R \) such that \( m \in \mathcal{F} \). Then the natural map \( N \to \prod_m N_m \) whose components are the localization maps \( N \to N_m \) factorizes through the submodule \( \bigoplus_m N_m \subset \prod_m N_m \) and induces an isomorphism \( N \cong \bigoplus_m N_m \).

Conversely, if \( \mathcal{F} \) is a filter of ideals in \( R \) such that for every \( \mathcal{F} \)-torsion \( R \)-module \( N \) there is an isomorphism of \( R \)-modules \( N \cong \bigoplus_m N_m \), then the ring \( R \) is \( \mathcal{F} \)-h-local.

**Proof.** This is [5, Proposition 7.4]. \( \square \)

It follows from Lemma 2 that, for any \( \mathcal{F} \)-h-local ring \( R \), the category of \( \mathcal{F} \)-torsion \( R \)-modules is equivalent to the Cartesian product of the categories of \( \mathcal{F} \)-torsion \( R_m \)-modules, taken over the maximal ideals \( m \) of the ring \( R \) belonging to the filter \( \mathcal{F} \). The direct sum functor

\[
(N(m))_{m \in \mathcal{F}\cap \text{Max } R} \to \bigoplus_{m \in \mathcal{F}\cap \text{Max } R} N(m)
\]

establishes the equivalence, with an inverse equivalence provided by the localization functor

\[
N \mapsto (N_m)_{m \in \mathcal{F}\cap \text{Max } R}.
\]

In particular, the decomposition of an \( R \)-module \( N \) into a direct sum of \( \mathcal{F} \)-torsion \( R_m \)-modules \( N(m) \) is unique and functorial if it exists (and it exists if and only if \( N \) is \( \mathcal{F} \)-torsion). All these assertions can be found in the discussion in [5, Section 7]. Alternatively, they can be deduced directly from Lemma 2 using the dual version of [7, Lemma 8.6].

Let \( R \) be a commutative ring with a filter \( \mathcal{F} \) of ideals. We will say that \( R \) is \( \mathcal{F} \)-h-nil \([6, \text{Section 6}], [5, \text{Section 7}]\) if every ideal \( I \in \mathcal{F} \) is contained only in finitely many maximal ideals of \( R \) and all the prime ideals of \( R \) belonging to \( \mathcal{F} \) are maximal. In other words, \( R \) is \( \mathcal{F} \)-h-nil if and only if for every \( I \in \mathcal{F} \) the quotient ring \( R/I \) is semilocal of Krull dimension zero. Equivalently, this means that \( R/I \) is a finite product of
local rings whose maximal ideals consist of nilpotent elements (by Lemma 1(b)). Obviously, any \( \mathcal{F} \)-h-nil ring is \( \mathcal{F} \)-h-local.

We refer to the book [26, Chapter VI] for the definition of a Gabriel filter of ideals (also known as a Gabriel topology) in a ring \( R \). A filter \( \mathcal{F} \) of ideals in a ring \( R \) is said to have a base of finitely generated ideals if for any ideal \( J \in \mathcal{F} \) there exists a finitely generated ideal \( I \subset R \) such that \( I \in \mathcal{F} \) and \( I \subset J \).

In the rest of this section (and partly in the next one) we will be working in the following setup.

**Setup 3.** Let \( R \) be a commutative ring and \( \mathcal{G} \) be a Gabriel filter of ideals in \( R \) with a base of finitely generated ideals. We assume that the ring \( R \) is \( \mathcal{G} \)-h-nil.

Given a commutative ring \( R \) and an ideal \( I \subset R \), we say that an \( R \)-module \( N \) is \( I \)-torsion if for every \( s \in I \) and \( x \in N \) there exists an integer \( n \geq 1 \) such that \( s^n x = 0 \) in \( N \).

**Lemma 4.** Assume Setup 3, and let \( m \) be a maximal ideal of \( R \) belonging to \( \mathcal{G} \). Then an \( R \)-module \( N \) is a \( \mathcal{G} \)-torsion \( R_m \)-module if and only if \( N \) is an \( m \)-torsion \( R \)-module.

**Proof.** “If”: clearly, any \( m \)-torsion \( R \)-module \( N \) is an \( R_m \)-module. To show that \( N \) is \( \mathcal{G} \)-torsion, choose a finitely generated ideal \( I \) such that \( I \subset m \) and \( I \in \mathcal{G} \). Then for every \( x \in N \) there exists \( n \geq 1 \) such that \( I^n x = 0 \) in \( N \). Since \( I^n \in \mathcal{G} \) [26, Lemma VI.5.3], it follows that the annihilator of \( x \) in \( R \) belongs to \( \mathcal{G} \).

“Only if”: let \( N \) be a \( \mathcal{G} \)-torsion \( R_m \)-module and \( x \in N \) be an element. Choose an ideal \( I \in \mathcal{G} \) annihilating \( x \). Then the cyclic submodule \( R_m x \subset N \) is a module over the ring \( R_m/R_mI = (R/I)_m \). The ring \( (R/I)_m \) is local and, by Setup 3 and Lemma 1(b), its maximal ideal \( R_m/R_mI \) consists of nilpotent elements. Hence every module over \( (R/I)_m \) is \( m \)-torsion. \( \square \)

**Corollary 5.** Assuming Setup 3, an \( R \)-module \( N \) is \( \mathcal{G} \)-torsion if and only if it is isomorphic to a direct sum of some \( m \)-torsion \( R \)-modules \( N(m) \) over the maximal ideals \( m \in \mathcal{G} \cap \text{Max} \, R \),

\[
N \simeq \bigoplus_{m \in \mathcal{G} \cap \text{Max} \, R} N(m).
\]

Such a direct sum decomposition is unique and functorial when it exists, and the \( R \)-modules \( N(m) \) can be recovered as the localizations \( N(m) = N_m \).

**Proof.** The ring \( R \) in Setup 3 is \( \mathcal{G} \)-h-local (as \( \mathcal{G} \)-h-nil implies \( \mathcal{G} \)-h-local). Hence the assertion follows from Lemmas 2 and 4. \( \square \)

### 2. \( I \)-Contramodule \( R \)-Modules

Let \( R \) be a commutative ring. Given an element \( t \in R \), an \( R \)-module \( C \) is said to be a \( t \)-contramodule if \( \text{Hom}_R(R[t^{-1}], C) = 0 = \text{Ext}_R^1(R[t^{-1}], C) \). Given an ideal \( I \subset R \), an \( R \)-module \( C \) is said to be an \( I \)-contramodule if \( C \) is a \( t \)-contramodule for
every \( t \in I \). The class of all \( I \)-contramodule \( R \)-modules is closed under the kernels, cokernels, extensions, and infinite products in \( R\Mod \) [6, Lemma 5.1(1)]. The full subcategory of \( I \)-contramodule \( R \)-modules is denoted by \( R\Mod_{I\text{-ctra}} \subset R\Mod \).

**Lemma 6.** Let \( R \) be a commutative ring and \( \mathfrak{m} \subset R \) be a maximal ideal. Then any \( \mathfrak{m} \)-contramodule \( R \)-module is an \( R_{\mathfrak{m}} \)-module.

*Proof.* This is [6] Lemma 5.1(2)]. \( \square \)

**Lemma 7.** Assuming Setup 3, for any \( \mathfrak{G} \)-torsion \( R \)-module \( N \), any \( R \)-module \( B \), and any integer \( n \geq 0 \), the \( R \)-module \( P = \text{Ext}^n_R(N, B) \) can be presented as a product of some \( \mathfrak{m} \)-contramodule \( R \)-modules \( P(\mathfrak{m}) \) over the maximal ideals \( \mathfrak{m} \in \mathfrak{G} \cap \text{Max} R \),

\[
\text{Ext}^n_R(N, B) \simeq \prod_{\mathfrak{m} \in \mathfrak{G} \cap \text{Max} R} P(\mathfrak{m}).
\]

*Proof.* Follows from Corollary 5 and [6, Lemma 6.5(1)]. \( \square \)

**Lemma 8.** Let \( R \) be a commutative ring, \( t \in R \) be an element, \( D \) be an \( R[t^{-1}] \)-module, and \( Q \) be a \( t \)-contramodule \( R \)-module. Then \( \text{Ext}^i_R(D, Q) = 0 \) for all integers \( i \geq 0 \).

*Proof.* The element \( t \) acts invertibly in the \( R \)-module \( E = \text{Ext}^i_R(D, Q) \), since it acts invertibly in \( D \). On the other hand, the \( R \)-module \( E \) is a \( t \)-contramodule, since the \( R \)-module \( Q \) is [6, Lemma 6.5(2)]. Thus \( E \simeq \text{Hom}_R(R[t^{-1}], E) = 0 \). \( \square \)

Given a filter \( \mathcal{F} \) of ideals in a commutative ring \( R \), we will say that the ring \( R \) is \( \mathcal{F} \)-h-semilocal if the ring \( R/I \) is semilocal for all \( I \in \mathcal{F} \). This captures “a half of” the conditions from the definition of an \( \mathcal{F} \)-h-local ring in Section 1.

**Proposition 9.** Let \( \mathcal{F} \) be a filter of ideals in a commutative ring \( R \) such that \( \mathcal{F} \) has a base of finitely generated ideals and the ring \( R \) is \( \mathcal{F} \)-h-semilocal. Let \( P(\mathfrak{m}) \) and \( Q(\mathfrak{m}) \) be two families of \( \mathfrak{m} \)-contramodule \( R \)-modules, indexed over the maximal ideals \( \mathfrak{m} \in \mathcal{F} \cap \text{Max} R \). Then, for every \( i \geq 0 \), the natural map

\[
\prod_{\mathfrak{m} \in \mathcal{F} \cap \text{Max} R} \text{Ext}^i_R(P(\mathfrak{m}), Q(\mathfrak{m})) \longrightarrow \text{Ext}^i_R \left( \prod_{\mathfrak{m} \in \mathcal{F} \cap \text{Max} R} P(\mathfrak{m}), \prod_{\mathfrak{m} \in \mathcal{F} \cap \text{Max} R} Q(\mathfrak{m}) \right)
\]

is an isomorphism.

*Proof.* Fix \( \mathfrak{n} \in \mathcal{F} \cap \text{Max} R \), and denote by \( P \) the product of \( P(\mathfrak{m}) \) over all \( \mathfrak{m} \in \mathcal{F} \cap \text{Max} R \), \( \mathfrak{m} \neq \mathfrak{n} \). We have to show that \( \text{Ext}^i_R(P, Q(\mathfrak{n})) = 0 \).

Let \( I \) be a finitely generated ideal in \( R \) such that \( I \subset \mathfrak{n} \) and \( I \in \mathcal{F} \). Since \( R \) is \( \mathcal{F} \)-h-semilocal, the set of all maximal ideals \( \mathfrak{m} \) of the ring \( R \) containing the ideal \( I \) is finite.

Let \( s_1, \ldots, s_m \) be a finite set of generators of the ideal \( I \). For any ideal \( \mathfrak{m} \in \mathcal{F} \cap \text{Max} R \) such that \( \mathfrak{m} \) does not contain \( I \), there exists an integer \( 1 \leq j \leq m \) such that \( \mathfrak{m} \) does not contain \( s_j \). Denote by \( P_j \) the product of the \( R \)-modules \( P(\mathfrak{m}) \) over all the maximal ideals \( \mathfrak{m} \in \mathcal{F} \cap \text{Max} R \) such that \( s_j \notin \mathfrak{m} \).

By Lemma 6, \( P(\mathfrak{m}) \) is an \( R_{\mathfrak{m}} \)-module. Hence \( s_j \) acts invertibly in \( P_j \). On the other hand, \( Q(\mathfrak{n}) \) is an \( s_j \)-contramodule. By Lemma 6, it follows that \( \text{Ext}^i_R(P_j, Q(\mathfrak{n})) = 0 \).
It remains to show that \( \text{Ext}_i^R(P(m), Q(n)) = 0 \) for each one among the finite set of maximal ideals \( m \in \mathcal{F} \cap \text{Max} R \) such that \( I \subset m, \ m \neq n \). Let \( s \in R \) be an element such that \( s \in n \) and \( s \notin m \). Then \( s \) acts invertibly in \( P(m) \), while \( Q(n) \) is an \( s \)-contramodule. Once again, it follows that \( \text{Ext}_i^R(P(m), Q(n)) = 0 \).

Corollary 10. For any filter \( \mathcal{F} \) of ideals in a commutative ring \( R \) such that \( \mathcal{F} \) has a base of finitely generated ideals and the ring \( R \) is \( \mathcal{F} \)-\( h \)-semilocal, the full subcategory of all \( R \)-modules of the form \( \prod_{m \in \mathcal{F} \cap \text{Max} R} C(m) \), where \( C(m) \) are \( m \)-contramodule \( R \)-modules, is closed under the kernels, cokernels, extensions, and infinite products in \( R\text{-Mod} \).

Proof. Follows from Proposition 9 (for \( i = 0 \) and 1).

Given a complex of \( R \)-modules \( M^\bullet \), an \( R \)-module \( B \), and an integer \( n \geq 0 \), we will denote by \( \text{Ext}_n^R(M^\bullet, B) \) the derived category Hom module \( \text{Hom}_{\text{D}(R\text{-Mod})}(M^\bullet, B[n]) \). Here \( \text{D}(R\text{-Mod}) \) is the derived category of \( R \)-modules.

The following proposition is the key technical assertion of this paper, on which the proofs of the main results are based.

Proposition 11. Assume Setup 3. Let \( M^\bullet = (M^{-1} \to M^0) \) be a two-term complex of \( R \)-modules whose cohomology modules \( H^{-1}(M^\bullet) \) and \( H^0(M^\bullet) \) are \( G \)-torsion. Then for any \( R \)-module \( B \) and any integer \( n \geq 0 \), the \( R \)-module \( C = \text{Ext}_R^n(M^\bullet, B) \) can be presented as the product of some \( m \)-contramodule \( R \)-modules \( C(m) \) over the maximal ideals \( m \in G \cap \text{Max} R \),

\[
\text{Ext}_R^n(M^\bullet, B) \simeq \prod_{m \in G \cap \text{Max} R} C(m).
\]

Proof. By Lemma 7, the \( R \)-modules \( \text{Ext}_R^n(H^{-1}(M^\bullet), B) \) and \( \text{Ext}_R^n(H^0(M^\bullet), B) \) can be decomposed as direct products of \( m \)-contramodule \( R \)-modules for all \( n \geq 0 \). The existence of a similar direct product decompositions of the \( R \)-modules \( \text{Ext}_R^n(M^\bullet, B) \) follows from the natural long exact sequence of \( R \)-modules

\[
\cdots \to \text{Ext}^{n-2}(H^{-1}(M^\bullet), B) \to \text{Ext}_R^n(H^0(M^\bullet), B) \to \text{Ext}_R^n(M^\bullet, B) \to \text{Ext}^{n+1}(H^{-1}(M^\bullet), B) \to \cdots
\]

in view of Corollary 10 (cf. [6, proof of Lemma 6.11] or [21, proof of Lemma 8.2]).

An obvious generalization of Proposition 11 holds for any bounded above complex of \( R \)-modules \( M^\bullet \) with \( G \)-torsion cohomology modules and any bounded below complex of \( R \)-modules \( B^\bullet \), as one can see from the related spectral sequence. We do not include the details, as we do not need them.
3. Projective Dimension 1 Theorem

Let \( u : R \rightarrow U \) be a flat epimorphism of commutative rings. This means that \( u \) is a homomorphism of commutative rings such that \( U \) is a flat \( R \)-module and the induced ring homomorphisms \( U \Rightarrow U \otimes_R U \rightarrow U \) are isomorphisms.

Denote by \( \mathcal{G} \) the set of all ideals \( I \subset R \) such that \( R/I \otimes_R U = 0 \). Then \( \mathcal{G} \) is a Gabriel filter of ideals in \( R \), and the ring \( U \) can be recovered as the ring of quotients \( U = R_\mathcal{G} \) of the ring \( R \) with respect to the Gabriel filter \( \mathcal{G} \).

A Gabriel filter is said to be perfect if it corresponds to a flat epimorphism of rings in this way [26, Chapter XI]. Notice that any perfect Gabriel filter \( \mathcal{G} \) has a base of finitely generated ideals [26, Section XI.3].

The main results of this paper presume the following setup.

**Setup 12.** Let \( u : R \rightarrow U \) be a flat epimorphism of commutative rings, and let \( \mathcal{G} \) be the related perfect Gabriel filter of ideals in \( R \). We assume that, for every \( I \in \mathcal{G} \), the quotient ring \( R/I \) is semilocal of Krull dimension zero (in other words, the ring \( R \) is \( \mathcal{G} \)-h-nil in our terminology).

Following [19, Section 13], [20, Section 1], [23, Section 3], [6, Section 4], [21, Section 8], [7, Section 2], etc., we will denote by \( K_u^* \) the two-term complex of \( R \)-modules \( R \rightarrow U \), with the term \( R \) placed in the cohomological degree \(-1\) and the term \( U \) placed in the cohomological degree 0. The next corollary is the particular case of Proposition 11 which we will use.

**Corollary 13.** Assuming Setup 12, for any \( R \)-module \( B \) and any integer \( n \geq 0 \), the \( R \)-module \( C = \text{Ext}^n_R(K_u^*, B) \) can be presented as the product of some \( m \)-contramodule \( R \)-modules \( C(m) \) over the maximal ideals \( m \in \mathcal{G} \cap \text{Max } R \),

\[
\text{Ext}^n_R(K_u^*, B) \simeq \prod_{m \in \mathcal{G} \cap \text{Max } R} C(m).
\]

**Proof.** The complex of \( U \)-modules \( U \otimes_R K_u^* \) is contractible, hence the cohomology modules \( H^{-1}(K_u^*) \) and \( H^{0}(K_u^*) \) of the complex \( K_u^* \) are \( \mathcal{G} \)-torsion \( R \)-modules. Therefore, Proposition 11 is applicable. \( \square \)

The aim of this section is to prove the following theorem, which is our most important result.

**Theorem 14.** Assuming Setup 12, the projective dimension of the \( R \)-module \( U \) cannot exceed 1.

The argument is based on Corollary 13. We need several more lemmas before proceeding to prove the theorem.

**Lemma 15.** Let \( R \) be a commutative ring, \( t \in R \) be an element, and \( C \) be a \( t \)-contramodule \( R \)-module. Then \( C/tC \neq 0 \) whenever \( C \neq 0 \).

**Proof.** Assume that \( C = tC \) for an \( R \)-module \( C \). Then one easily observes that the natural \( R \)-module map \( \text{Hom}_R(R[t^{-1}], C) \rightarrow C \) is surjective. If \( C \) is a \( t \)-contramodule, then \( \text{Hom}_R(R[t^{-1}], C) = 0 \), and it follows that \( C = 0 \). \( \square \)
Lemma 16. Let $R$ be a commutative ring and $I = (s_1, \ldots, s_m) \subseteq R$ be a finitely generated ideal. Let $C$ be an $I$-contramodule $R$-module. Then $C/IC \neq 0$ whenever $C \neq 0$.

Proof. Assume that $C \neq 0$. Then, by Lemma 15, we have $C/s_1C \neq 0$. Furthermore, $C/s_1C$ is still an $I$-contramodule, since it is the cokernel of the $R$-module morphism $s_1: C \rightarrow C$ between two $I$-contramodules. Applying Lemma 15 again, we see that $C/(s_1C + s_2C) \neq 0$, etc. □

Lemma 17. Let $u: R \rightarrow U$ be a flat epimorphism of commutative rings, and let $\mathbb{G}$ be the related perfect Gabriel filter of ideals in $R$. Suppose that $D$ is a $U$-module and $C$ is an $I$-contramodule $R$-module, where $I \in \mathbb{G}$. Suppose further that there is a surjective $R$-module morphism $D \rightarrow C$. Then $C = 0$.

Proof. Without loss of generality we can assume the ideal $I \in \mathbb{G}$ to be finitely generated (because the filter $\mathbb{G}$ has a base of finitely generated ideals). We have $D/IID = 0$, since $R/I \otimes_R U = 0$. As the morphism $D/IID \rightarrow C/IC$ is surjective, it follows that $C/IC = 0$. By Lemma 16, we can conclude that $C = 0$. □

Proof of Theorem 14. Let $B$ be an $R$-module. Then, for every integer $n \geq 2$, there is a natural isomorphism of $R$-modules $\text{Ext}_R^n(K^*u, B) \simeq \text{Ext}_R^n(U, B)$ (see [7, Lemma 2.1(b)]; cf. [6, Lemma 4.8(3)]). By Corollary 13, we have $\text{Ext}_R^n(K^*u, B) \simeq \prod_{m \in \mathbb{G} \cap \text{Max } R} C(m)$, where $C(m)$ are some $m$-contramodule $R$-modules. On the other hand, $D = \text{Ext}_R^n(U, B)$ is a $U$-module. For every $m$, the $R$-module $C(m)$ is a quotient (in fact, a direct summand) of the $R$-module $D$. By Lemma 17, it follows that $C(m) = 0$, hence $\text{Ext}_R^n(U, B) \simeq \text{Ext}_R^n(K^*u, B) = 0$. □

4. Divisible Modules

The following definitions, generalizing the classical notions of $S$-divisible and $S$-h-divisible modules for a multiplicative subset $S \subseteq R$ and the related ring epimorphism $R \rightarrow S^{-1}R$ [20, Section 1], are quite natural.

Definition 18. Given a filter $\mathbb{F}$ of ideals in a commutative ring $R$ and an $R$-module $D$, we say that $D$ is $\mathbb{F}$-divisible [26, Section VI.9] if $R/I \otimes_R D = 0$ for all $I \in \mathbb{F}$.

Definition 19. Let $u: R \rightarrow U$ be an epimorphism of commutative rings. We say that an $R$-module $D$ is $u$-divisible (or $u$-h-divisible) [7, Remark 1.2(1)] if $D$ is a quotient $R$-module of a $U$-module.

For a flat epimorphism $u$ and the related Gabriel filter $\mathbb{G}$, one can immediately see that any $u$-divisible $R$-module is $\mathbb{G}$-divisible. The converse is not true in general, even when $U = S^{-1}R$ [11, Proposition 6.4], [20, Lemma 1.8(b)].

Theorem 20. Let $u: R \rightarrow U$ be a flat epimorphism of commutative rings, and let $\mathbb{G}$ be the related perfect Gabriel filter of ideals in $R$. Assume that the projective dimension of the $R$-module $U$ does not exceed 1. Then the classes of $u$-divisible and $\mathbb{G}$-divisible $R$-modules coincide.
Proof. The following argument based on the results of the paper [2] was communicated to the author by S. Bazzoni. Denote the $R$-module $\text{coker}(u) = U/u(R)$ simply by $U/R$. Then, by [15, Example 6.5], the direct sum $U \oplus U/R$ is a silting $R$-module. The related silting class is the class of all quotient modules of direct sums of copies of $U \oplus U/R$, which means the class of all $u$-divisible $R$-modules. By [2, Theorem 4.7], for any silting class over a commutative ring $R$ there exists a Gabriel filter $G$ (with a base of finitely generated ideals) in $R$ such that the silting class consists of all the $G$-divisible $R$-modules. So the class of all $u$-divisible $R$-modules coincides with the class of all $G$-divisible $R$-modules for some Gabriel filter $G$. It follows immediately that $G$ is the perfect Gabriel filter related to $u$, as desired. □

Remark 21. For injective flat epimorphisms of commutative rings $u: R \rightarrow U$ (i.e., when the map $u$ is injective), the following converse assertion to Theorem 20 holds. If all $G$-divisible $R$-modules are $u$-divisible, then the projective dimension of the $R$-module $U$ does not exceed 1. This is a part of [13, Theorem 5.4].

However, there do exist noninjective flat commutative ring epimorphisms $u$ of projective dimension more than 1 for which the class of $u$-divisible modules coincides with that of $G$-divisible ones. In fact, such examples exist already among the maps of localization by multiplicative subsets $u: R \rightarrow S^{-1}R = U$. Consequently, neither [11, Proposition 6.4] nor [20, Lemma 1.8(b)] hold true for multiplicative subsets $S \subset R$ containing zero-divisors. The following transparent construction of counterexamples was communicated to the author by M. Hrbek.

Let $R$ be a von Neumann regular commutative ring and $I \subset R$ be an ideal. Then $I$ is generated by some set of idempotent elements $e_j \in R$. Let $S \subset R$ be the multiplicative subset generated by the complementary idempotents $f_j = 1 - e_j$. Then $S^{-1}R \simeq R/I$. Furthermore, an $R$-module $M$ is annihilated by $e_j$ for a given index $j$ if and only if $f_j$ acts invertibly in $M$, and if and only if $f_j$ acts by a surjective endomorphism of $M$. Hence an $R$-module $M$ is annihilated by the whole ideal $I$ if and only if all the elements of $S$ act invertibly in $M$, and if and only if all the elements of $S$ act in $M$ by surjective maps.

Consequently, all $S$-divisible $R$-modules are $S$-$h$-divisible (moreover, all of them are $S^{-1}R$-modules). In other words, if $u: R \rightarrow S^{-1}R = R/I = U$ is the natural surjective flat epimorphism of rings and $G$ is the related perfect Gabriel filter in $R$ (i.e., the filter of all ideals intersecting $S$), then the classes of $u$-divisible and $G$-divisible $R$-modules coincide. (Cf. the discussion of silting and cosilting classes over von Neumann regular commutative rings in [14, Section 1.5.3].)

On the other hand, it is well-known that there exist von Neumann regular commutative rings of arbitrary homological dimension (see, e.g., [18]). So one can choose $R$ and $I$ so as to make the projective dimension of the $R$-module $R/I$ to be equal to any chosen nonnegative integer or infinity.

Corollary 22. Assuming Setup 12, an $R$-module is $u$-divisible if and only if it is $G$-divisible.

First proof. Follows immediately from Theorems 14 and 20. □
In order to illustrate the workings of contramodule techniques, in the rest of this section we present an alternative proof of Corollary 22. For this purpose, we need the following strengthening of the lemmas from Section 3.

**Proposition 23.** Let \( R \) be a commutative ring and \( I = (s_1, \ldots, s_m) \subset R \) be a finitely generated ideal. Let \( B \) be an \( R \)-module such that \( B/IB = 0 \) and \( C \) be an \( I \)-contramodule \( R \)-module. Then \( \text{Hom}_R(B, C) = 0 \).

**Proof.** In other words, the proposition says that an \( I \)-contramodule \( R \)-module \( C \) has no \( R \)-submodules \( D \) for which \( ID = D \). This is provable using infinite summation operations, as hinted in [19, Lemma 4.2].

Alternatively, one can use the reflector \( \Delta_I: R\text{-Mod} \to R\text{-Mod}_{I\text{-ctra}} \) onto the full subcategory of \( I \)-contramodule \( R \)-modules \( R\text{-Mod}_{I\text{-ctra}} \subset R\text{-Mod} \) (that is, the left adjoint functor to the inclusion \( R\text{-Mod}_{I\text{-ctra}} \to R\text{-Mod} \)). The functor \( \Delta_I \) was constructed in [19, Theorem 7.2]. Any \( R \)-module morphism \( B \to C \) into an \( I \)-contramodule \( R \)-module \( C \) factorizes uniquely as \( B \to \Delta_I(B) \to C \), where \( B \to \Delta_I(B) \) is the adjunction morphism.

The functor \( \Delta_I \) is \( R \)-linear and right exact, so applying it to the morphism \( (s_1, \ldots, s_m): B^m \to B \), which is surjective by the assumption \( IB = B \), produces a surjective morphism \( (s_1, \ldots, s_m): \Delta_I(B)^m \to \Delta_I(B) \). Hence \( B/IB = 0 \) implies \( \Delta_I(B)/I\Delta_I(B) = 0 \). But \( \Delta_I(B) \) is an \( I \)-contramodule, so by Lemma 16 it follows that that \( \Delta_I(B) = 0 \). Thus any \( R \)-module morphism \( B \to C \) vanishes. \( \square \)

**Second proof of Corollary 22.** For any \( R \)-module \( B \), there is a natural 5-term exact sequence of \( R \)-modules

\[
0 \to \text{Ext}^0(R_u, B) \to \text{Hom}_R(U, B) \to B \\
\to \text{Ext}^1(R_u, B) \to \text{Ext}_R(U, B) \to 0
\]

produced by applying the functor \( \text{Hom}_{D(R\text{-Mod})}(-, B) \) to the distinguished triangle

\[
R \to U \to K_u \to R[1]
\]

in \( D(R\text{-Mod}) \) (cf. [3] formula (**) in Section 4, [21] formula (8.2), or [7] formula (9))). By Corollary [13] we have an isomorphism of \( R \)-modules \( \text{Ext}_R^1(K_u, B) \simeq \prod_{m \in \mathbb{G} \cap \text{Max} R} C(m) \), where \( C(m) \) are some \( m \)-contramodule \( R \)-modules.

Now assume that the \( R \)-module \( B \) is \( \mathbb{G} \)-divisible. Given a maximal ideal \( m \in \mathbb{G} \cap \text{Max} R \), choose a finitely generated ideal \( I \in \mathbb{G} \) such that \( I \subset m \). Then \( C(m) \) is an \( I \)-contramodule \( R \)-module and \( B/IB = 0 \). By Proposition 23 it follows that \( \text{Hom}_R(B, C(m)) = 0 \). As this holds for all \( m \in \mathbb{G} \cap \text{Max} R \), we can conclude that the map \( B \to \text{Ext}_R^1(K_u, B) \) vanishes. Hence the map \( \text{Hom}_R(U, B) \to B \) is surjective and the \( R \)-module \( B \) is \( u \)-divisible. \( \square \)

### 5. \( u \)-Contramodule \( R \)-Modules

In this section we obtain a description of the Geigle–Lenzing perpendicular subcategory \( R\text{-Mod}_{u\text{-ctra}} = U_{\leq 0,1} \) in the category of \( R \)-modules \( R\text{-Mod} \).
An $R$-module $C$ is said to be a $u$-contramodule [7 Section 1] if $C \in U^{1,0,1}$, that is $\text{Hom}_R(U, C) = 0 = \text{Ext}^1_R(U, C)$. Assuming Setup 12, the $R$-module $U$ has projective dimension at most 1 by Theorem 14. By [11 Proposition 1.1] or [19 Theorem 1.2(a)], it follows that the full subcategory of $u$-contramodule $R$-modules $R\text{-Mod}_{u\text{-contra}}$ is closed under the kernels, cokernels, extensions, and infinite products in $R\text{-Mod}$.

The functor $\Delta_u = \text{Ext}^1_R(K_u \cdot, -)$ plays an important role as the reflector onto the full subcategory $R\text{-Mod}_{u\text{-contra}} \subset R\text{-Mod}$ [7 Proposition 3.2(b)].

**Lemma 24.** Let $u : R \rightarrow U$ be a flat epimorphism of commutative rings and $I \subset R$ be an ideal such that $IU = U$. Then any $R/I$-module is a $u$-contramodule.

**Proof.** Since $U$ is a flat $R$-module, by [24 Lemma 4.1(a)] we have $\text{Ext}^i_R(U, D) \simeq \text{Ext}^i_{R/I}(U/IU, D) = 0$ for any $R/I$-module $D$ and all $i \geq 0$. □

**Proposition 25.** Let $R \rightarrow U$ be a flat epimorphism of commutative rings, and let $G$ be the related perfect Gabriel filter of ideals in $R$. Assume that the projective dimension of the $R$-module $U$ does not exceed 1. Then, for any ideal $I \in G$, any $I$-contramodule $R$-module is a $u$-contramodule.

**Proof.** Without loss of generality we can assume the ideal $I$ to be finitely generated. Let $C$ be an $I$-contramodule $R$-module. Then $\text{Hom}_R(U, C) = 0$ by Proposition 23.

There are several ways to prove that $\text{Ext}^i_R(U, C) = 0$. One can observe that any $I$-contramodule $R$-module is obtainable as an extension of two quotseparated $I$-contramodule $R$-modules [24 Section 5.5], [22 Section 1]; any quotseparated $I$-contramodule $R$-module is the cokernel of an injective morphism between two $I$-adically separated and complete modules; and finally any $I$-adically separated and complete module is the kernel of a morphism of $R$-modules of the form $\text{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z})$, where $N$ ranges over the $G$-torsion $R$-modules [21 Proposition 5.6]. Since $R$-modules of the latter form belong to $R\text{-Mod}_{u\text{-contra}}$, it follows that all the $I$-contramodule $R$-modules do.

Alternatively, one can say that all $I$-contramodule $R$-modules $C$ are simply right obtainable from $R/I$-modules (i. e., obtainable using the passages to extensions, cokernels of monomorphisms, infinite products, and infinitely iterated extensions in the sense of the projective limit [24 Section 3], [23 Section 2]). This is the assertion of [24 Lemma 8.2] (see also [19 proof of Theorem 9.5]). Since $\text{Ext}^i_R(U, D) = 0$ for any $R/I$-module $D$ and all $i > 0$ by Lemma 24, it follows that $\text{Ext}^i_R(U, C) = 0$ for $i > 0$ by [24 Lemma 3.4]. □

**Theorem 26.** Assuming Setup 12, an $R$-module $C$ is a $u$-contramodule if and only if it is isomorphic to a product of some $m$-contramodule $R$-modules $C(m)$ over the maximal ideals $m \in G \cap \text{Max} R$,

$$C \simeq \prod_{m \in G \cap \text{Max} R} C(m).$$

Such an infinite product decomposition is unique and functorial when it exists, and the $R$-modules $C(m)$ can be recovered as the colocalizations $C(m) = \text{Hom}_R(R_m, C)$. 13
Proof. The assertion “if” is provided by Proposition 25. To prove the “only if”, notice that the natural morphism $C \rightarrow \text{Ext}^1_R(K^\bullet, C)$ is an isomorphism for any $u$-contramodule $R$-module $C$ (in view of the exact sequence (*) from Section 4). Now Corollary 13 provides the desired direct product decomposition.

Let us compute the colocalizations. Any $m$-contramodule $R$-module $C_m$ is an $R_m$-module by Lemma 6, hence $\text{Hom}_R(R_m, C_m) = C(m)$. On the other hand, for any maximal ideal $n \neq m$ of the ring $R$, we have $\text{Hom}_R(R_n, C_m) = 0$ by Lemma 8 (choose any element $t \in m \setminus n$). Finally, our direct product decomposition is unique and functorial by Proposition 9 (for $i = 0$). □

Thus, assuming Setup 12, the category of $u$-contramodule $R$-modules is equivalent to the Cartesian product of the categories of $m$-contramodule $R$-modules, taken over the maximal ideals $m$ of the ring $R$ belonging to the filter $\mathcal{G}$. The product functor

$$(C(m))_{m \in \mathcal{G} \cap \text{Max } R} \mapsto \prod_{m \in \mathcal{G} \cap \text{Max } R} C(m)$$

establishes the equivalence, with an inverse equivalence provided by the colocalization functor

$$C \mapsto (C^m = \text{Hom}_R(R_m, C))_{m \in \mathcal{G} \cap \text{Max } R}.$$ 

6. $U$-Strongly Flat $R$-Modules

We refer to [21, beginning of Section 9] for a general discussion of $U$-weakly cotorsion and $U$-strongly flat left $R$-modules for a left flat epimorphism of associative rings $u: R \rightarrow U$. In this section, as in the rest of this paper, we restrict ourselves to the commutative case.

Let $u: R \rightarrow U$ be a flat epimorphism of commutative rings. A left $R$-module $C$ is said to be $U$-weakly cotorsion if $\text{Ext}^1_R(U, C) = 0$. A left $R$-module $F$ is said to be $U$-strongly flat if $\text{Ext}^1_R(F, C) = 0$ for all $U$-weakly cotorsion $R$-modules $C$.

By [11, Theorem 4.4], we have $\text{Ext}^i_R(U, U^{(X)}) \simeq \text{Ext}^i_U(U, U^{(X)}) = 0$ for all $i > 0$ and any set $X$; in particular, $\text{Ext}^1_R(U, U^{(X)}) = 0$ (where $U^{(X)}$ denotes the direct sum of $X$ copies of $U$). Hence [12, Corollary 6.13] provides the following description of $U$-strongly flat $R$-modules. An $R$-module $F$ is $U$-strongly flat if and only if it is a direct summand of an $R$-module $G$ appearing in a short exact sequence of $R$-modules

$$0 \rightarrow V \rightarrow G \rightarrow W \rightarrow 0,$$

where $V$ is a free $R$-module and $W$ is a free $U$-module.

The notion of a simply right obtainable module (from a given class of modules) was already mentioned in the proof of Proposition 25. We refer to [24, Section 3] or [23, Section 2] for a detailed discussion.

Proposition 27. Assuming Setup 12, the class of all $u$-contramodule $R$-modules $R^{\text{-Mod}}_{u\text{-ctra}} \subset R^{\text{-Mod}}$ coincides with the class of all $R$-modules simply right obtainable from $R/I$-modules, where $I$ ranges over the Gabriel filter $\mathcal{G}$.
Proof. By Theorem 26, any $u$-contramodule $R$-module is obtainable as a product of $m$-contramodules over the maximal ideals $m \in \mathcal{G} \cap \text{Max } R$. For any $m$, there exists a finitely generated ideal $I \subset m$ such that $I \in \mathcal{G}$. Then any $m$-contramodule is an $I$-contramodule. Finally, all $I$-contramodule $R$-modules are simply right obtainable from $R/I$-modules by [23] Lemma 8.2] or [19] proof of Theorem 9.5).

Conversely, since by Theorem 14 we have $\text{pd}_R U \leq 1$, the class of all $u$-contramodule $R$-modules is closed under the kernels, cokernels, extensions, and infinite products in $R \text{-Mod}$ by [11] Proposition 1.1] or [19] Theorem 1.2(a)]. Since all the $R/I$-modules with $I \in \mathcal{G}$ belong to $R \text{-Mod}_{u \text{-ctra}}$ by Lemma 23, it follows that all the $R$-modules simply right obtainable from $R/I$-modules, $I \in \mathcal{G}$, belong to $R \text{-Mod}_{u \text{-ctra}}$. □

Proposition 28. Assuming Setup 12 the class of all $U$-weakly cotorsion $R$-modules coincides with the class of all $R$-modules simply right obtainable from $U$-modules and $R/I$-modules with $I \in \mathcal{G}$.

Proof. For any $R$-module $B$, it is clear from Corollary 13 and Proposition 25 that both the $R$-modules $\text{Ext}_R^i(K_u^\bullet, B)$ and $\text{Ext}_R^1(K_u^\bullet, B)$ are $u$-contramodules (cf. [7] Lemma 2.6(a) and (c)]. Now it follows from the exact sequence $(*)$ from Section 4 that any $U$-weakly cotorsion $R$-module $C$ is obtainable as an extension of a $u$-contramodule $\text{Ext}_R^1(K_u^\bullet, C)$ and the cokernel of an injective morphism from a $u$-contramodule to a $U$-module $\text{Ext}_R^0(K_u^\bullet, C) \twoheadrightarrow \text{Hom}_R(U, C)$. It remains to use Proposition 27 for obtainability of $u$-contramodules.

Conversely, all $R/I$-modules are $U$-weakly cotorsion $R$-modules by Lemma 24 and all $U$-modules $D$ are $U$-weakly cotorsion $R$-modules, since $\text{Ext}_R^i(U, D) \simeq \text{Ext}_R^1(U, D) = 0$ for $i > 0$ by [11] Theorem 4.4]. By virtue of [24] Lemma 3.4, it follows that all the $R$-modules simply right obtainable from $U$-modules and $R/I$-modules are $U$-weakly cotorsion. □

The following theorem can be viewed as confirming a version of [23] Optimistic Conjecture 1.1] for flat epimorphisms of commutative rings (cf. [21] Corollary 9.7)). It is also a generalization of [6] Proposition 7.13].

Theorem 29. Assume Setup 12 and let $F$ be a flat $R$-module. Then $F$ is $U$-strongly flat if and only if it satisfies the following two conditions:

(i) the $U$-module $U \otimes_R F$ is projective;

(ii) for every ideal $I \in \mathcal{G}$, the $R/I$-module $F/IF$ is projective.

Proof. The “only if” assertion holds for any flat epimorphism of commutative rings $u: R \twoheadrightarrow U$ and the related perfect Gabriel filter $\mathcal{G}$. It is clear from the description of $U$-strongly flat $R$-modules $F$ as the direct summands of the $R$-modules $G$ appearing in short exact sequences of $R$-modules $0 \rightarrow V \rightarrow G \rightarrow W \rightarrow 0$ with a free $R$-module $V$ and a free $U$-module $W$ that conditions (i–ii) are satisfied for $F$.

Our proof of the “if” depends on the assumption of Setup 12. Let $F$ be a flat $R$-module satisfying (i–ii), and let $C$ be a $U$-weakly cotorsion $R$-module. We need to show that $\text{Ext}_R^1(F, C) = 0$, but it will be more convenient for us to prove that $\text{Ext}_R^1(F, C) = 0$ for all $i > 0$. By Proposition 28, $C$ is simply right obtainable from
modules and $R/I$-modules with $I \in \mathcal{G}$. According to [24, Lemma 3.4], it suffices to consider the cases when $C$ is either a $U$-module or an $R/I$-module.

For any $U$-module $D$, we have $\text{Ext}^i_R(F, D) \simeq \text{Ext}^i_U(U \otimes_R F, D)$ by [24, Lemma 4.1(a)]. In view of condition (i), $\text{Ext}^i_U(U \otimes_R F, D) = 0$. Similarly, for any $R/I$-module $D$, we have $\text{Ext}^i_R(F, D) \simeq \text{Ext}^i_{R/I}(F/IF, D)$ by [24, Lemma 4.1(a)]. In view of condition (ii), $\text{Ext}^i_{R/I}(F/IF, D) = 0$. □

Let $u : R \to U$ be a flat epimorphism of commutative rings and $\mathcal{G}$ be the related perfect Gabriel filter of ideals in $R$. Following [5, Sections 6 and 8], we will say that the ring $R$ is $\mathcal{G}$-almost perfect if the ring $U = R_\mathcal{G}$ is perfect and the rings $R/I$ are perfect for all $I \in \mathcal{G}$. Clearly, the ring epimorphism $u : R \to U$ satisfies Setup 12 whenever $R$ is $\mathcal{G}$-almost perfect for the related Gabriel filter $\mathcal{G}$ (as all perfect commutative rings are semilocal of Krull dimension zero).

**Corollary 30.** Let $\mathcal{G}$ be the perfect Gabriel filter of ideals related to a flat epimorphism of commutative rings $u : R \to U$. Then the ring $R$ is $\mathcal{G}$-almost perfect if and only if all the flat $R$-modules are $U$-strongly flat.

**Proof.** The “if” assertion is provable similarly to [6, Lemma 7.8]. One shows that all flat $U$-modules are projective whenever all flat $R$-modules are $U$-strongly flat, and all Bass flat $R/I$-modules are projective whenever all Bass flat $R$-modules are $U$-strongly flat. We skip the (straightforward) details.

The implication “only if” is a simple corollary of Theorem 29. Assume that the ring $R$ is $\mathcal{G}$-almost perfect; then Setup 12 is satisfied. Let $F$ be a flat $R$-module. Then the $U$-module $U \otimes_R F$ is flat, and the $R/J$-module $F/IF$ is flat for every ideal $J \subset R$. Since the ring $U$ is perfect by assumption, it follows that the $U$-module $U \otimes_R F$ is projective. Since the ring $R/I$ is perfect for all $I \in \mathcal{G}$, it follows that the $R/I$-module $F/IF$ is projective, too. Thus conditions (i–ii) hold, and the $R$-module $F$ is $U$-strongly flat by Theorem 29. □

**References**

[1] L. Angeleri Hügel, D. Herbera, J. Trlifaj. Divisible modules and localizations. *Journ. of Algebra* 294, #2, p. 519–551, 2005.

[2] L. Angeleri Hügel, M. Hrbek. Slitting modules over commutative rings. *Internat. Math. Research Notices* 2017, #13, p. 4131–4151. arXiv:1602.04321 [math.RT]

[3] L. Angeleri Hügel, J. Sánchez. Tilting modules arising from ring epimorphisms. *Algebras and Represent. Theory* 14, #2, p. 217–246, 2011. arXiv:0804.1313 [math.RT]

[4] S. Bazzoni, G. Le Gros. A characterisation of enveloping 1-tilting classes over commutative rings. *Journ. of Pure and Appl. Algebra* 226, #1, article ID 106813, 29 pp., 2022. arXiv:1901.07921 [math.AC]

[5] S. Bazzoni, G. Le Gros. Covering classes and 1-tilting cotorsion pairs over commutative rings. *Forum Math.* 33, #3, p. 601–629, 2021. arXiv:2006.01176 [math.AC]

[6] S. Bazzoni, L. Positselski. $S$-almost perfect commutative rings. *Journ. of Algebra* 532, p. 323–356, 2019. arXiv:1801.04820 [math.AC]

[7] S. Bazzoni, L. Positselski. Matlis category equivalences for a ring epimorphism. *Journ. of Pure and Appl. Algebra* 224, #10, article ID 106398, 25 pp., 2020. arXiv:1907.04973 [math.RA]
[8] S. Bazzoni, L. Salce. Strongly flat covers. *Journ. of the London Math. Society* **66**, #2, p. 276–294, 2002.

[9] L. Fuchs, L. Salce. Modules over non-Noetherian domains. Mathematical Surveys and Monographs 84, American Math. Society, 2001.

[10] L. Fuchs, L. Salce. Almost perfect commutative rings. *Journ. of Pure and Appl. Algebra* **222**, #12, p. 4223–4238, 2018.

[11] W. Geigle, H. Lenzing. Perpendicular categories with applications to representations and sheaves. *Journ. of Algebra* **144**, #2, p. 273–343, 1991.

[12] R. Göbel, J. Trlifaj. Approximations and endomorphism algebras of modules. Second Revised and Extended Edition. De Gruyter Expositions in Mathematics 41, De Gruyter, Berlin–Boston, 2012.

[13] M. Hrbek. One-tilting classes and modules over commutative rings. *Journ. of Algebra* **462**, p. 1–22, 2016. arXiv:1507.02811 [math.AC]

[14] M. Hrbek. Tilting theory of commutative rings. Ph. D. Thesis, Faculty of Mathematics and Physics, Department of Algebra, Charles Univ. in Prague, 2017.

[15] F. Marks, J. Šťovíček. Universal localizations via silting. *Proc. of the Royal Soc. of Edinburgh, Sect. A* **149**, #2, p. 511–532, 2019. arXiv:1605.04222 [math.RT]

[16] E. Matlis. Cotorsion modules. *Memoirs of the American Math. Society* **49**, 1964.

[17] E. Matlis. 1-dimensional Cohen–Macaulay rings. *Lecture Notes in Math.* **327**, Springer, 1973.

[18] R. S. Pierce. The global dimension of Boolean rings. *Journ. of Algebra* **7**, #1, p. 91–99, 1967.

[19] L. Positselski. Contraadjusted modules, contramodules, and reduced cotorsion modules. *Moscow Math. Journ.* **17**, #3, p. 385–455, 2017. arXiv:1605.03934 [math.CT]

[20] L. Positselski. Triangulated Matlis equivalence. *Journ. of Algebra and its Appl.* **17**, #4, article ID 1850067, 2018. arXiv:1605.08018 [math.CT]

[21] L. Positselski. Flat ring epimorphisms of countable type. *Glasgow Math. Journ.* **62**, #2, p. 383–439, 2020. arXiv:1808.00937 [math.RA]

[22] L. Positselski. Remarks on derived complete modules and complexes. Electronic preprint arXiv:2002.12331 [math.AC], to appear in *Math. Nachrichten*.

[23] L. Positselski, A. Slávik. On strongly flat and weakly cotorsion modules. *Math. Zeitschrift* **291**, #3–4, p. 831–875, 2019. arXiv:1708.06833 [math.AC]

[24] L. Positselski, A. Slávik. Flat morphisms of finite presentation are very flat. *Annali di Matem. Pura ed Appl.* **199**, #3, p. 875–924, 2020. arXiv:1708.00846 [math.AC]

[25] M. Raynaud, L. Gruson. Critères de platitude et de projectivité: Techniques de “platification” d’un module. *Inventiones Math.* **13**, #1–2, p. 1–89, 1971.

[26] B. Stenström. Rings of quotients. An introduction to methods of ring theory. Springer-Verlag, Berlin–Heidelberg–New York, 1975.

[27] J. Trlifaj. Cotorsion theories induced by tilting and cotilting modules. In: Abelian groups, rings and modules (Perth, 2000), *Contemp. Math.* **273**, AMS, Providence, 2001, p. 285–300.

Institute of Mathematics of the Czech Academy of Sciences, Žitná 25, 115 67 Praha 1 (Czech Republic); and Laboratory of Algebra and Number Theory, Institute for Information Transmission Problems, Moscow 127051 (Russia)

Email address: positselski@math.cas.cz