GLOBAL OPTIMIZATION VIA SCHRÖDINGER-FÖLLMER DIFFUSION

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ABSTRACT. We study the problem of finding global minimizers of $V(x) : \mathbb{R}^d \to \mathbb{R}$ approximately via sampling from a probability distribution $\mu_\sigma$ with density $p_\sigma(x) = \frac{\exp(-V(x)/\sigma)}{\int_{\mathbb{R}^d} \exp(-V(y)/\sigma) dy}$ with respect to the Lebesgue measure for $\sigma \in (0, 1]$ small enough. We analyze a sampler based on the Euler-Maruyama discretization of the Schrödinger-Föllmer diffusion processes with stochastic approximation under appropriate assumptions on the step size $s$ and the potential $V$. We prove that the output of the proposed sampler is an approximate global minimizer of $V(x)$ with high probability at cost of sampling $O(d^3)$ standard normal random variables. Numerical studies illustrate the effectiveness of the proposed method and its superiority to the Langevin method.

1. INTRODUCTION

In this paper we study a challenging problem of finding the global minimizers of a non-convex smooth function $V : \mathbb{R}^d \to \mathbb{R}$. Suppose $N := \{x_1^*, \ldots, x_\kappa^*\} \subset B_R$ is the set of the global minima of $V$ with finite cardinality, i.e.,

$$x_i^* \in \text{argmin}_x V(x), \quad \text{for any } i = 1, \ldots, \kappa,$$

where $B_R$ denotes the ball centered at origin with radius $R > 0$. Precisely speaking, we have $\mathcal{L}(V(x) < V(x_1^*) - \varepsilon) = 0$ and $\mathcal{L}(V(x) < V(x_1^*) + \varepsilon) > 0$ for any $\varepsilon > 0$, where $\mathcal{L}$ denotes the Lebesgue measure on $\mathbb{R}^d$. Without loss of generality, we can assume $V(x) = \|x\|^2/2$ outside the ball $B_R$ without changing the global minimizers of $V$. For any given $\sigma \in (0, 1]$, we define a constant $C_\sigma$ and a probability density function $p_\sigma(x)$ on $\mathbb{R}^d$ as

$$p_\sigma(x) := \frac{\exp(-V(x)/\sigma)}{C_\sigma}, \quad \text{for } C_\sigma := \int_{\mathbb{R}^d} \exp(-V(x)/\sigma) dx < \infty.$$

Let $\mu_\sigma$ be the probability distribution corresponding to the density function $p_\sigma$. If the function $V$ is twice continuously differentiable, we have that the measure $\mu_\sigma$ converges weakly to a probability measure with weights proportional to

$$\left(\det \nabla^2 V(x_i^*)\right)^{-\frac{1}{2}}$$

at $x_i^*$ as $\sigma$ goes to 0, i.e.,

$$\lim_{\sigma \to 0} \mu_\sigma = \frac{\sum_{i=1}^\kappa \left(\det \nabla^2 V(x_i^*)\right)^{-\frac{1}{2}} \delta_{x_i^*}}{\sum_{j=1}^\kappa \left(\det \nabla^2 V(x_j^*)\right)^{-\frac{1}{2}}}.$$

Therefore, solving the optimization problem (1) can be converted into sampling from the probability distribution measure $\mu_\sigma$ for sufficiently small $\sigma$.

An efficient method sampling from $\mu_\sigma$ is based on the overdamped Langevin stochastic differential equation (SDE) which is given by

$$dZ_t = -\nabla V(Z_t) dt + \sqrt{2\sigma} dB_t,$$

where $(B_t)_{t \geq 0}$ is a $d$-dimensional Brownian motion. Under some certain conditions, Langevin SDE (2) admits the unique invariant measure $\mu_\sigma$ (Bakry et al., 2008). Hence the Langevin sampler is generated by applying Euler-Maruyama discretization on this process to achieve the purpose of sampling from $\mu_\sigma$. Convergence properties of the Langevin sampler under the strongly convex potential assumption have been established by Durmus and Moulines (2016); Dalalyan (2017a,b); Cheng and Bartlett (2018); Durmus and Moulines (2019); Dalalyan and Karagulyan (2019). Moreover, the strongly convex potential assumption can be replaced by...
different conditions to guarantee the log-Sobolev inequality for the target distribution, including the dissipativity condition for the drift term (Raginsky et al., 2017; Zhang et al., 2019; Mou et al., 2022) and the local convexity condition for the potential function outside a ball (Durmus and Moulines, 2017; Ma et al., 2019; Bou-Rabee et al., 2020).

An alternative to Langevin sampler is the class of algorithms based on the Schrödinger-Föllmer diffusion process (3). This process has been proposed for sampling and generative modelling (Tzen and Raginsky, 2019; Huang et al., 2021; Jiao et al., 2021; Wang et al., 2021). Subsequently, Ruzayqat et al. (2022) studied the problem of unbiased estimation of expectations based on the Schrödinger-Föllmer diffusion process. Vargas et al. (2021) applied this process to Bayesian inference in large datasets, and the related posterior is reached in finite time. Zhang and Chen (2021) proposed a new Path Integral Sampler (PIS), a generic sampler that generates samples through simulating a target-dependent SDE which can be trained with free-form architecture network design. The PIS is built on Schrödinger-Föllmer diffusion process to reach the terminal distribution. Different from these existing works, the purpose of this paper is to solve the non-convex smooth optimization problems. To that end, we need to rescale the Schrödinger-Föllmer diffusion process to sample from the target distribution \( \mu_\sigma \).

To be precise, the Schrödinger-Föllmer diffusion process associated to \( \mu_\sigma \) is defined as

\[
dX_t = \sigma b(X_t, t) dt + \sqrt{\sigma} dB_t, \quad t \in [0, 1], \quad X_0 = 0, \quad (3)
\]

where the drift function is

\[
b(x, t) = \frac{\mathbb{E}_{Z \sim N(0, \mu_\sigma)} [\nabla f_\sigma(x + (1-t)Z)]}{\mathbb{E}_{Z \sim N(0, \mu_\sigma)} [f_\sigma(x + (1-t)Z)]} : \mathbb{R}^d \times [0, 1] \to \mathbb{R}^d
\]

with the density ratio \( f_\sigma(\cdot) = \frac{\mu_\sigma(\cdot)}{\phi_\sigma(\cdot)} \) and \( \phi_\sigma(\cdot) \) being the density function of a normal distribution \( N(0, \sigma I_\delta) \). According to Léonard (2014) and Eldan et al. (2020), the process \( \{X_t\}_{t \in [0, 1]} \) in (3) was first formulated by Föllmer (Föllmer, 1985, 1986, 1988) when studying the Schrödinger bridge problem (Schrödinger, 1932). The main feature of the above Schrödinger-Föllmer diffusion process is that it interpolates \( b_0 \) and \( \mu_\sigma \) in time \([0, 1]\), i.e., \( X_1 \sim \mu_\sigma \), see Proposition 2.3. Then we can solve the optimization problem (1) by sampling from \( \mu_\sigma \) via the following Euler-Maruyama discretization of (3),

\[
Y_{t_{k+1}} = Y_{t_k} + \sigma s b_m(Y_{t_k}, t_k) + \sqrt{\sigma} \epsilon_{k+1}, \quad Y_{t_0} = 0, \quad k = 0, 1, \ldots, K - 1,
\]

where \( s = 1/K \) is the step size, \( t_k = ks, \) and \( \{\epsilon_k\}_{k=1}^K \) are independent and identically distributed from \( N(0, I_\delta) \). If the expectations in the drift term \( b(x, t) \) do not have analytical forms, one can use Monte Carlo method to evaluate \( b(Y_{t_k}, t_k) \) approximately, i.e., one can sample from \( \mu_\sigma \) according to

\[
\tilde{Y}_{t_{k+1}} = \tilde{Y}_{t_k} + \sigma s b_m(\tilde{Y}_{t_k}, t_k) + \sqrt{\sigma} \epsilon_{k+1}, \quad \tilde{Y}_{t_0} = 0, \quad k = 0, 1, \ldots, K - 1,
\]

where \( \tilde{b}_m(\tilde{Y}_{t_k}, t_k) := \frac{1}{m} \sum_{j=1}^m [\nabla f_\sigma(\tilde{Y}_{t_k} + (1-t_k)Z_j)] \) with \( Z_1, \ldots, Z_m \) i.i.d. \( N(0, I_\delta) \).

The main result of this paper is summarized in the following.

**Theorem 1.1.** (Informal) Under condition (A), \( \forall 0 < \delta \ll 1 \), with probability at least \( 1 - \sqrt{\delta} \), \( \tilde{Y}_{t_k} \) is a \( \tau \)-global minimizer of \( V \), i.e., \( V(\tilde{Y}_{t_k}) \leq \tau + \inf x V(x) \), if number of iterations \( K \geq O \left( \frac{d^2}{\delta} \right) \), number of Gaussian samples per iteration \( m \geq O \left( \frac{d}{\delta} \right) \) and \( \sigma \leq O \left( \frac{\tau}{\log(1/\delta)} \right) \).

The rest of this paper is organized as follows. In Section 2, we give the motivation and details of approximation method, i.e., Algorithm 1. In Section 3, we present the theoretical analysis for the proposed method. In Section 4, a numerical example is given to validate the efficiency of the method. We conclude the manuscript in Section 5. Proofs for all the propositions and theorems are provided in Appendix 7.
2. Methodology Description

In this section we first provide some background on the Schrödinger-Föllmer diffusion. Then we propose Algorithm 1 to solve the minimization problem (1) based on the Euler-Maruyama discretization scheme of the Schrödinger-Föllmer diffusion.

2.1. Background on Schrödinger-Föllmer diffusion. We first recall the Schrödinger bridge problem, then introduce the Schrödinger-Föllmer diffusion.

2.1.1. Schrödinger bridge problem. Let \( \Omega = C([0, 1], \mathbb{R}^d) \) be the space of \( \mathbb{R}^d \)-valued continuous functions on the time interval \([0,1]\). Denote \( Z = (Z_t)_{t \in [0,1]} \) as the canonical process on \( \Omega \), where \( Z_t(\omega) = \omega_t, \omega = (\omega_s)_{s \in [0,1]} \in \Omega \). The canonical \( \sigma \)-field on \( \Omega \) is then generated as \( \mathcal{F} = \sigma(Z_t, t \in [0,1]) = \{ \{ \omega: (Z_t(\omega))_{t \in [0,1]} \in H \} : H \in \mathcal{B}(\mathbb{R}^d) \} \), where \( \mathcal{B}(\mathbb{R}^d) \) denotes the Borel \( \sigma \)-algebra of \( \mathbb{R}^d \). Denote \( \mathcal{P}(\Omega) \) as the space of probability measures on the path space \( \Omega \), and \( \mathcal{W}^\sigma = \mathcal{P}(\Omega) \) as the Wiener measure with variance \( \sigma \) whose initial marginal is \( \delta_x \).

The law of the reversible Brownian motion is then defined as \( \mathbb{P}_\sigma = \int \mathcal{W}^\sigma d\mathbf{x} \), which is an unbounded measure on \( \Omega \). One can observe that \( \mathbb{P}_\sigma \) has a marginal distribution coinciding with the Lebesgue measure \( \mathcal{L} \) at each \( t \). Schrödinger (1932) studied the problem of finding the most likely random evolution between two probability distributions \( \bar{\nu}, \bar{\mu} \in \mathcal{P}(\mathbb{R}^d) \). This problem is referred to as the Schrödinger bridge problem (SBP). SBP can be further formulated as seeking a probability law on the path space that interpolates between \( \bar{\nu} \) and \( \bar{\mu} \), such that the probability law is close to the prior law of the Brownian diffusion with respect to the relative entropy (Jamison, 1975; Léonard, 2014), i.e., finding a path measure \( Q^* \in \mathcal{P}(\Omega) \) with marginal \( Q^*_t = (Z_t)_\# Q^* = Q^* \circ Z_t^{-1}, t \in [0,1] \) such that

\[
Q^* \in \arg\min_{Q_0 = \bar{\nu}, Q_1 = \bar{\mu}} \mathbb{D}_{KL}(Q||\mathbb{P}_\sigma), \quad \text{with} \quad \mathbb{D}_{KL}(Q||\mathbb{P}_\sigma) = \int \log \left( \frac{dQ}{d\mathbb{P}_\sigma} \right) dQ \text{ if } Q \ll \mathbb{P}_\sigma \quad \text{(i.e., } Q \text{ is absolutely continuous w.r.t. } \mathbb{P}_\sigma), \quad \text{and } \mathbb{D}_{KL}(Q||\mathbb{P}_\sigma) = +\infty \text{ otherwise.}
\]

The following theorem characterizes the solution of SBP.

**Theorem 2.1** (Léonard (2014)). If measures \( \bar{\nu}, \bar{\mu} \ll \mathcal{L} \), then SBP admits a unique solution \( dQ^* = f^*(Z_0)g^*(Z_1) d\mathbb{P}_\sigma \), where \( f^* \) and \( g^* \) are \( \mathcal{L} \)-measurable non-negative functions satisfying the Schrödinger system

\[
\begin{aligned}
&f^*(x)\mathbb{E}_{\mathbb{P}_\sigma} \left[ g^*(Z_1) \mid Z_0 = x \right] = \frac{\partial \mathbb{E}_{\mathbb{P}_\sigma} \left[ \log g^*(Z_1) \mid Z_0 = x \right]}{\partial x}, \quad \mathcal{L} \text{-a.e..} \\
g^*(y)\mathbb{E}_{\mathbb{P}_\sigma} \left[ f^*(Z_0) \mid Z_1 = y \right] = \frac{\partial \mathbb{E}_{\mathbb{P}_\sigma} \left[ \log f^*(Z_0) \mid Z_1 = y \right]}{\partial y}, \quad \mathcal{L} \text{-a.e..}
\end{aligned}
\]

Furthermore, the pair \((Q^*_t, v^*_t)\) with

\[ v^*_t(x) = \nabla_x \log \mathbb{E}_{\mathbb{P}_\sigma} \left[ g^*(Z_1) \mid Z_t = x \right] \]

solves the minimum action problem

\[
\min_{\mu_t, v_t} \int_0^1 \mathbb{E}_{\mathbb{P}_\sigma} \left[ (v_t(z))^2 \right] dt
\]

such that

\[
\begin{cases}
\partial_t \mu_t = -\nabla \cdot (\mu_t v_t) + \frac{\sigma \Delta \mu_t}{2}, \quad \text{on } (0,1) \times \mathbb{R}^d, \\
\mu_0 = \bar{\nu}, \mu_1 = \bar{\mu}.
\end{cases}
\]

Let \( K_\sigma(s, x, t, y) = [2\pi \sigma(t-s)]^{-d/2} \exp \left( -\frac{||x-y||^2}{2\sigma(t-s)} \right) \) be the transition density of the Wiener process with variance \( \sigma \), \( \tilde{q}(x) \) and \( \tilde{p}(y) \) be the density of \( \bar{\nu} \) and \( \bar{\mu} \), respectively. Denote by

\[
\begin{aligned}
f_0(x) &= f^*(x), \quad g_1(y) = g^*(y), \\
f_1(y) &= \mathbb{E}_{\mathbb{P}_\sigma} \left[ f^*(Z_0) \mid Z_1 = y \right] = \int K_\sigma(0, x, 1, y) f_0(x) dx, \\
g_0(x) &= \mathbb{E}_{\mathbb{P}_\sigma} \left[ g^*(Z_1) \mid Z_0 = x \right] = \int K_\sigma(0, x, 1, y) g_1(y) dy.
\end{aligned}
\]
Then the Schrödinger system in Theorem 2.1 can also be characterized by
\[ \tilde{q}(x) = f_0(x)g_0(x), \quad \tilde{p}(y) = f_1(y)g_1(y), \]
with the following forward and backward time harmonic equations (Chen et al., 2021),
\[
\begin{align*}
\partial_t f_i(x) &= \frac{\sigma^2}{2} \Delta f_i(x), \\
\partial_t g_i(x) &= -\frac{\sigma^2}{2} g_i(x),
\end{align*}
\] on \((0, 1) \times \mathbb{R}^d\).

Let \( q_i \) denote marginal density of \( Q^*_t \), i.e., \( q_i(x) = \frac{dQ^*_t}{d\mathbb{P}}(x) \), then it can be represented by the product of \( q_i \) and \( f_i \) (Chen et al., 2021). Let \( V \) consist of admissible Markov controls with finite energy. Then, the vector field
\[ \nu^*_I = \sigma \nabla_x \log g_I(x) = \sigma \nabla_x \log \int K_\sigma(t, x, y)g_1(y)dy \] solves the following stochastic control problem.

**Theorem 2.2** (Dai Pra (1991)).
\[ \nu^*_I(x) \in \arg \min_{\nu \in \mathcal{V}} \mathbb{E} \left[ \int_0^1 \frac{1}{2\sigma} \|\nu_t\|^2 dt \right] \]
such that
\[
\begin{align*}
\mathsf{dx}_t = \nu_t dt + \sqrt{\sigma} dB_t, \\
\mathsf{x}_0 &\sim \tilde{q}(x), \quad \mathsf{x}_1 \sim \tilde{p}(x).
\end{align*}
\] (5)

According to Theorem 2.2, the dynamics determined by the SDE in (5) with a time-varying drift term \( \nu^*_I \) in (4) will drive the particles sampled from the initial distribution \( \tilde{\nu} \) to evolve to the particles drawn from the target distribution \( \tilde{\mu} \) on the unit time interval. This nice property is what we need in designing samplers: we can sample from the underlying target distribution \( \tilde{\mu} \) via pushing forward a simple reference distribution \( \tilde{\nu} \). In particular, if we take the initial distribution \( \tilde{\nu} \) to be \( \delta_0 \), the degenerate distribution at 0, then the Schrödinger-Föllmer diffusion process (7) defined below is a solution to (5), i.e., it will transport \( \delta_0 \) to the target distribution.

### 2.1.2. Schrödinger-Föllmer diffusion process
From now on, without loss of generality, we can assume that the minimum value of \( V \) is 0, i.e., \( V(x^*_I) = 0, i = 1, \ldots, \kappa \), otherwise, we consider \( V \) replaced by \( V - \min V(x) \). Since \( \mu_\sigma \) is absolutely continuous with respect to the \( d \)-dimensional Gaussian distribution \( N(0, \sigma I_d) \), then we denote the Radon-Nikodym derivative of \( \mu_\sigma \) with respect to \( N(0, \sigma I_d) \) as follows:
\[ f_\sigma(x) := \frac{d\mu_\sigma}{dN(0, \sigma I_d)}(x) = \frac{p_\sigma(x)}{\phi_\sigma(x)}, \quad x \in \mathbb{R}^d. \]

Let \( Q_t \) be the heat semigroup defined by
\[ Q_t f_\sigma(x) := \mathbb{E}_{Z \sim N(0, I_d)}[f_\sigma(x + \sqrt{\sigma}Z)], \quad t \in [0, 1]. \]
The Schrödinger-Föllmer diffusion process \( \{X_t\}_{t \in [0, 1]} \) (Föllmer, 1985, 1986, 1988) is defined as
\[ \mathsf{d}X_t = \sigma b(X_t, t) dt + \sqrt{\sigma} dB_t, \quad X_0 = 0, \quad t \in [0, 1], \]
where \( b(x, t) : \mathbb{R}^d \times [0, 1] \to \mathbb{R}^d \) is the drift term given by
\[ b(x, t) = \nabla \log Q_{1-t} f_\sigma(x). \]
This process \( \{X_t\}_{t \in [0, 1]} \) defined in (7) is a solution to (5) with \( \tilde{\nu} = \delta_0, \tilde{\mu} = \mu_\sigma \), and \( \nu_t(x) = b(x, t) \) (Dai Pra, 1991; Lehec, 2013; Eldan et al., 2020). Let
\[ \hat{f}_\sigma(x) := C_\sigma \gamma \frac{d}{d\sigma} f_\sigma(x). \]
Since the drift term \( b(x, t) \) is scale-invariant with respect to \( f_\sigma \) in the sense that \( b(x, t) = \nabla \log Q_{1-t} f_\sigma(x) \) for any \( C > 0 \). Therefore, the Schrödinger-Föllmer diffusion can be used for sampling from an unnormalized distribution \( \mu_\sigma \), that is, the normalizing constant \( C_\sigma \) of \( \mu_\sigma \) does not need to be known. We have \( \mu_\sigma(dx) = \exp(-V(x)/\sigma) dx/C_\sigma \) with the normalized constant \( C_\sigma \), then \( f_\sigma(x) = \left( \frac{\sqrt{2\pi}^d}{C_\sigma} \right) \exp(-V(x)/\sigma + \|x\|^2/2\sigma) \) and \( \hat{f}_\sigma(x) = \exp \left( -\frac{V(x)}{\sigma} + \frac{\|x\|^2}{2\sigma} \right) \). To ensure that the SDE (7) admits a unique strong solution, we assume that
(A) $V(x)$ is twice continuous differentiable on $\mathbb{R}^d$ and $V(x) = \|x\|^2/2$ outside a ball $B_R$.

As we mentioned earlier, we can make Assumption (A) by smoothing and it does not change the the global minimizers of $V$. Under Assumption (A), we have the following two properties which further imply the SDE (7) admits a unique strong solution.

(P1) For each $\sigma \in (0,1]$, $\hat{f}_\sigma, \nabla \hat{f}_\sigma$ are Lipschitz continuous with a constant $\gamma_\sigma > 0$, where
\[
\gamma_\sigma := \max_{\|x\| \leq R} \exp \left( -\frac{V(x)}{\sigma} + \frac{\|x\|^2}{2\sigma} \right) \left\{ \left\| \frac{x}{\sigma} - \nabla V(x) \right\|_2^2 + \left\| \frac{\partial}{\partial \sigma} \nabla^2 V(x) \right\|_2 \right\},
\]
and
\[
M_{1,R} := \max_{\|x\| \leq R} \left\{ -\frac{V(x)}{\sigma} + \frac{\|x\|^2}{2\sigma} \right\}, M_{2,R} := \max_{\|x\| \leq R} \| x - \nabla V(x) \|_2, M_{3,R} := \max_{\|x\| \leq R} \left\| \frac{\partial}{\partial \sigma} \nabla^2 V(x) \right\|_2.
\]

(P2) For each $\sigma \in (0,1]$, there exists a constant $\xi_\sigma > 0$ such that $\hat{f}_\sigma \geq \xi_\sigma$, where
\[
\xi_\sigma := \exp \left( m_{1,R}/\sigma \right) \text{ with } m_{1,R} := \min_{\|x\| \leq R} \left\{ -\frac{V(x)}{\sigma} + \frac{\|x\|^2}{2\sigma} \right\}.
\]

Properties (P1)-(P2) are shown in Section 7.1 in Appendix. We should mention here (P1)-(P2) are used as assumptions in Lehec (2013); Tzen and Raginsky (2019).

Thanks to (P1) and (P2), some calculations show that
\[
\sup_{x \in \mathbb{R}^d, t \in [0,1]} \| \nabla \hat{f}_\sigma \|_2 \leq \gamma_\sigma, \quad \| \nabla^2 \hat{f}_\sigma \|_2 \leq \gamma_\sigma,
\]
\[
\sup_{x \in \mathbb{R}^d, t \in [0,1]} \| \nabla^2 Q_{1-t} \hat{f}_\sigma(x) \|_2 \leq \gamma_\sigma, \quad \sup_{x \in \mathbb{R}^d, t \in [0,1]} \| \nabla^2 (Q_{1-t} \hat{f}_\sigma(x)) \|_2 \leq \gamma_\sigma,
\]
and
\[
\left( Q_{1-t} \hat{f}_\sigma(x) \right) = \frac{\nabla Q_{1-t} \hat{f}_\sigma(x)}{Q_{1-t} \hat{f}_\sigma(x)}, \quad \left( \nabla b(x,t) \right) = \frac{\nabla^2 (Q_{1-t} \hat{f}_\sigma(x))}{Q_{1-t} \hat{f}_\sigma(x)} - \left( \nabla b(x,t) \right) b(x,t) \right)^\top.
\]

We conclude that
\[
\sup_{x \in \mathbb{R}^d, t \in [0,1]} \| b(x,t) \|_2 \leq \gamma_\sigma/\xi_\sigma, \quad \sup_{x \in \mathbb{R}^d, t \in [0,1]} \| \nabla b(x,t) \|_2 \leq \gamma_\sigma/\xi_\sigma + \gamma_\sigma^2/\xi_\sigma^2.
\]
Furthermore, we can also easily deduce that the drift term $b$ satisfies a linear growth condition and a Lipschitz continuity condition (Revuz and Yor, 2013; Pavliotis, 2014), that is,
\[
\| b(x,t) \|_2 \leq C_{0,\sigma}(1 + \|x\|^2), \quad x \in \mathbb{R}^d, t \in [0,1]
\]
\[
(C1)
\]
and
\[
\| b(x,t) - b(y,t) \|_2 \leq C_{1,\sigma} \| x - y \|_2, \quad x, y \in \mathbb{R}^d, t \in [0,1],
\]
\[
(C2)
\]
where $C_{0,\sigma}$ and $C_{1,\sigma}$ are two finite positive constants that only depend on $\sigma$. The linear growth condition (C1) and Lipschitz continuity condition (C2) ensure the existence of the unique strong solution of Schrödinger-Föllmer SDE (7). We summarize the above discussion in the following Proposition, whose proof are shown in Appendix 7.

**Proposition 2.3.** Under assumption (A) the Schrödinger-Föllmer SDE (7) has a unique strong solution $\{X_t\}_{t \in [0,1]}$ with $X_0 \sim \delta_{0}$ and $X_1 \sim \mu_\sigma$.

2.2. Euler-Maruyama discretization for Schrödinger-Föllmer Diffusion. Proposition 2.3 shows that the Schrödinger-Föllmer diffusion will transport $\delta_0$ to the probability distribution measure $\mu_\sigma$ on the unite time interval. Because the drift term $b(x,t)$ is scale-invariant with respect to $f_\sigma$ in the sense that $b(x,t) = \nabla \log Q_{1-t} f_\sigma(x)$, $\forall C > 0$, the Schrödinger-Föllmer diffusion can be used for sampling from $\mu_\sigma(dx) = \exp(-V(x)/\sigma)dx/C_\sigma$, where the normalizing constant $C_\sigma$ may not be known. To this end, we use the Euler-Maruyama method to discretize the Schrödinger-Föllmer diffusion (7). Let
\[
t_k = k \cdot s, \quad k = 0, 1, \ldots, K, \quad \text{with} \quad s = 1/K, \quad Y_{t_0} = 0,
\]
the Euler-Maruyama discretization scheme reads
\[Y_{t_{k+1}} = Y_{t_k} + \sigma sb(Y_{t_k}, t_k) + \sqrt{\sigma} \varepsilon_{k+1}, \quad k = 0, 1, \ldots, K - 1,
\]
where \(\{\varepsilon_k\}_{k=1}^K\) are i.i.d. \(N(0, I_d)\) and
\[
b(Y_{t_k}, t_k) = \frac{\mathbb{E}_Z[\nabla f_\sigma(Y_{t_k} + \sqrt{(1-t_k)\sigma Z})]}{\mathbb{E}_Z[f_\sigma(Y_{t_k} + \sqrt{(1-t_k)\sigma Z})]} = \frac{\mathbb{E}_Z[Zf_\sigma(Y_{t_k} + \sqrt{(1-t_k)\sigma Z})]}{\mathbb{E}_Z[f_\sigma(Y_{t_k} + \sqrt{(1-t_k)\sigma Z})] \sqrt{(1-t_k)\sigma}},
\]
where the second equality follows from Stein’s lemma (Stein, 1972, 1986; Landsman and Nešlehová, 2008). From the definition of \(b(Y_{t_k}, t_k)\) in (11) we may not get its explicit expression. We then consider a estimator \(\bar{b}_m\) of \(b\) by replacing \(\mathbb{E}_Z\) in the drift term \(b\) with \(m\)-samples mean, i.e.,
\[
\bar{b}_m(Y_{t_k}, t_k) = \frac{1}{m} \sum_{j=1}^m \left[ \nabla f_\sigma(Y_{t_k} + \sqrt{(1-t_k)\sigma Z_j}) \right], \quad k = 0, \ldots, K - 1,
\]
or
\[
\bar{b}_m(Y_{t_k}, t_k) = \frac{1}{m} \sum_{j=1}^m \left[ Z_j f_\sigma(Y_{t_k} + \sqrt{(1-t_k)\sigma Z_j}) \right] \cdot \sqrt{(1-t_k)\sigma}, \quad k = 0, \ldots, K - 1,
\]
where \(Z_1, \ldots, Z_m\) are i.i.d. \(N(0, I_d)\). The detailed description of the proposed method is summarized in the following Algorithm 1 below.

**Algorithm 1** Solving (1) via Euler-Maruyama discretization of Schrödinger-Föllmer Diffusion

1: Input: \(\sigma, m, K\). Initialize \(s = 1/K\), \(\tilde{Y}_{t_0} = 0\).
2: for \(k = 0, 1, \ldots, K - 1\) do
3: Sample \(\varepsilon_{k+1} \sim N(0, I_d)\).
4: Sample \(Z_i, i = 1, \ldots, m\), from \(N(0, I_d)\).
5: Compute \(\bar{b}_m\) according to (12) or (13),
6: \(\tilde{Y}_{t_{k+1}} = \tilde{Y}_{t_k} + \sigma sb_m(\tilde{Y}_{t_k}, t_k) + \sqrt{\sigma} \varepsilon_{k+1}\).
7: end for
8: Output: \(\{\tilde{Y}_{t_k}\}_{k=1}^K\).

In the next section, we establish the probability bound of \(\tilde{Y}_{t_k}\) being a \(\tau\)-global minimizer (Theorem 3.5), and derive the bound in the Wasserstein-2 distance between the law of \(\tilde{Y}_{t_k}\) generated from Algorithm 1 and the probability distribution measure \(\mu_\sigma\) under some certain conditions (Theorem 3.7).

### 3. Theoretical Property

In this section, we show that the Gibbs measure \(\mu_\sigma\) weakly converges to a multidimensional distribution concentration on the optimal points \(\{x_1^*, \ldots, x_n^*\}\). Since the minimum value of \(V\) is 0, then we estimate the probabilities of \(V(X_1) > \tau\) and \(V(\tilde{Y}_{t_k}) > \tau\) for any \(\tau > 0\), and establish the non-asymptotic bounds between the law of the samples generated from Algorithm 1 and the target distribution \(\mu_\sigma\) in the Wasserstein-2 distance. Recall that the linear growth condition \((C1)\) and Lipschitz continuity \((C2)\) hold under conditions \((P1)\) and \((P2)\), which make the Schrödinger-Föllmer SDE (7) have the unique strong solution. Besides, we assume that the drift term \(b(x, t)\) is Lipschitz continuous in \(x\) and \(1/2\)-Hölder continuous in \(t\),
\[
\|b(x, t) - b(y, s)\|_2 \leq C_{2, \sigma} \left( \|x - y\|_2 + |t - s|^{1/2} \right),
\]
where \(C_{2, \sigma}\) is a positive constant depending on \(\sigma\).

**Remark 3.1.** \((C1)\) and \((C2)\) are the essentially sufficient conditions such that the Schrödinger-Föllmer SDE (7) admits the unique strong solution. (C3) has been introduced in Theorem 4.1 of Tzen and Raginsky (2019) and it is also similar to the condition \((H2)\) of Chau et al. (2021) and Assumption 3.2 of Barkhagen et al. (2021). Obviously, (C3) implies \((C2)\), and \((C1)\) holds if the drift term \(b(x, t)\) is bounded over \(\mathbb{R}^d \times [0, 1]\).
Firstly, we show that the Gibbs measure $\mu_\sigma$ weakly converges to a multidimensional distribution. This result can be traced back to the 1980s. For the overall continuity of the article, we combine the Laplace’s method in Hwang (1980, 1981) to give a detailed proof of the result. The key idea is to prove that for each $\delta' > 0$ small enough, $\mu_\sigma(\{ x ; \| x - x_i^* \| < \delta' \})$ converges to $\left( \det \nabla^2 V(x_i^*) \right)^{-\frac{1}{2}} \sum_{j=1}^\kappa \left( \det \nabla^2 V(x_j^*) \right)^{-\frac{1}{2}}$ as $\sigma \downarrow 0$.

Next, we want to estimate the probabilities of $V(X_1) > \tau$ and $V(Y_{t\kappa}) > \tau$ for any $\tau > 0$. However, the second analysis is more complicated due to the discretization, and the main idea comes from Dalalyan (2017b); Cheng et al. (2018), which constructs a continuous-time interpolation SDE for the Euler-Maruyama discretization. In their works, the relative entropy is controlled via using the Girsanov’s theorem to estimate Radon-Nikodym derivatives.

Another method of controlling the relative entropy is proposed by Mou et al. (2022). By direct calculations, the time derivative of the relative entropy between the interpolated and the original SDE (7) is controlled by the mean squared difference between the drift terms of the Fokker-Planck equations for the original and the interpolated processes. Compared to the bound obtained from Lemma 7.2, the bound in Mou et al. (2022) has an additional backward conditional expectation inside the norm. It becomes a key reason for obtaining higher precision orders. But it must satisfy the dissipative condition to the drift term of the SDE and initial distribution smoothness.

The concrete results are showed in the following Theorems 3.2, 3.5, 3.7. See Appendix 7 for detailed proofs.

**Theorem 3.2.** Let $V : \mathbb{R}^d \rightarrow \mathbb{R}$ be a twice continuously differentiable function. Suppose there exists a finite set $N := \{ x \in \mathbb{R}^d ; V(x) = \inf_x V(x) \} = \{ x_1^*, \ldots, x_\kappa^* \}$, $\kappa \geq 1$ and $\int_{\mathbb{R}^d} \exp(-V(x))dx < \infty$, then

$$
\lim_{\sigma \downarrow 0} \frac{\sum_{i=1}^\kappa \left( \det \nabla^2 V(x_i^*) \right)^{-\frac{1}{2}} \delta_{x_i^*}}{\sum_{j=1}^\kappa \left( \det \nabla^2 V(x_j^*) \right)^{-\frac{1}{2}}} = 1,
$$

as $\sigma \downarrow 0$.

Under Theorem 3.2, a natural question is to consider the convergence rate about the measure $\mu_\sigma$ converging to the multidimensional distribution. To that end, we apply the tools from the large deviation of the Gibbs measure which is absolutely continuous with respect to Lebesgue measure, see Chiang et al. (1987); Holley et al. (1989); Márquez (1997). Therefore we can obtain the following property.

**Proposition 3.3.** Assume that the conditions of Theorem 3.2 hold, then for all $\tau > 0$,

$$
\lim_{\sigma \downarrow 0} \sigma \log \mu_\sigma \left( V(x) - \min_x V(x) \geq \tau \right) = -\tau.
$$

**Remark 3.4.** Although we can directly use the large deviation principle to obtain Proposition 3.3, further, we can conclude that the Gibbs measure $\mu_\sigma$ weakly converges to the global minimum points of the potential function $V$ and obtain the corresponding convergence rate. However, we cannot directly obtain the specific limit distribution form Proposition 3.3.

**Theorem 3.5.** Suppose (A) holds. Then, for each $\varepsilon \in (0, \tau), \sigma \in (0, 1)$, there exists a constant $C_{\tau,\varepsilon,d}$ (depending on $\tau, \varepsilon, d$) given in (A.15) such that

$$
\mathbb{P}(V(X_1) > \tau) \leq C_{\tau,\varepsilon,d} \exp \left( \frac{-\tau - \varepsilon}{\sigma} \right),
$$

$$
\mathbb{P}(V(Y_{t\kappa}) > \tau) \leq C_{\tau,\varepsilon,d} \exp \left( \frac{-\tau - \varepsilon}{\sigma} \right) + C_{1,\sigma}^2 \sqrt{d(2d + 3)\varepsilon} + C_{2,\sigma}^2 \sqrt{\frac{4d}{m}},
$$

where

$$
C_{1,\sigma}^2 := \frac{\gamma_{\sigma}}{\xi_{\sigma}} + \frac{\gamma_{\sigma}^3}{\xi_{\sigma}^3}, \quad \frac{\gamma_{\sigma}}{\xi_{\sigma}} = \left\{ \left( \frac{M_2}{\sigma} \right)^2 + \frac{M_4}{\sigma} \right\} \frac{1}{\sigma} \exp \left( \frac{M_{1,R} - m_{1,R}}{\sigma} \right),
$$

and

$$
C_{2,\sigma}^2 := \frac{\gamma_{\sigma}}{\xi_{\sigma}} \frac{M_{3,\sigma}}{\xi_{\sigma}}.
$$
The latter two terms are caused by Euler discretization of SDE (2019) proposed the Schrödinger bridge sampler by applying the iterative proportional fitting Garbuno-Inigo et al. (2020); Wang and Li (2022) for more details. However, the Langevin Gibbs sampling and optimization, which is determined by the simulated annealing method itself. 

Error bounds depend on \( d \) polynomially and \( \sigma \) exponentially. This result is contrary to the some Langevin methods, that is, the related convergence chain to obtain a process whose marginal distribution at terminal time is approximate to \( \mu \) of dimensionality. 

Assumption (Durmus and Moulines, 2016, 2017, 2019; Dalalyan, 2017a,b; Cheng and Bartlett, 2014). However, only a few recent works use this tool for statistical sampling. Bernton et al. 2018; Dalalyan and Karagulyan, 2019). Also, there are some mean-field Langevin methods, see Garbuno-Inigo et al. (2020); Wang and Li (2022) for more details. However, the Langevin diffusion process tends to the target distribution as time \( t \) goes to infinity while the Schrödinger bridge achieve this in unit time. The Schrödinger bridge has been shown to have close connections with a number of problems in statistical physics, optimal transport and optimal control (Léonard, 2014). However, only a few recent works use this tool for statistical sampling. Bernton et al. (2019) proposed the Schrödinger bridge sampler by applying the iterative proportional fitting method (Sinkhorn algorithm (Sinkhorn, 1964; Peyré and Cuturi, 2019)). For the Gibbs measure \( \mu_\sigma \), Schrödinger bridge samplers iteratively modifies the transition kernels of the reference Markov chain to obtain a process whose marginal distribution at terminal time is approximate to \( \mu_\sigma \), via regression-based approximations of the corresponding iterative proportional fitting recursion.

### 4. Simulation studies

In this section, we conduct numerical simulations to evaluate the performance of the proposed method (Algorithm 1), and compare it with the Langevin method. We consider the following

\[
C_{2,\sigma}^d := \frac{\gamma_a^2}{\xi_a^2} + \frac{\gamma_a \zeta_a}{\xi_a^2}, \quad \frac{\gamma_a}{\xi_a} = \left\{ \left( \frac{M_{2,R}}{\sigma} \right)^2 + \frac{M_{3,R}}{\sigma} \right\} \exp\left( \frac{M_{1,R} - m_{1,R}}{\sigma} \right),
\]

where \( M_{1,R}, M_{2,R}, M_{3,R} \) and \( m_{1,R} \) are given in (8)-(9), and \( \zeta_a = \exp(M_{1,R}/\sigma) \).

**Remark 3.6.** The first term in the right hand side of (14) originates from the difference between Gibbs sampling and optimization, which is determined by the simulated annealing method itself. 

The latter two terms are caused by Euler discretization of SDE (7) and Monte Carlo estimation of drift coefficient \( b(x,t) \) in (11). The latter two terms of (14) depend on \( d \) polynomially and \( \sigma \) exponentially. This result is contrary to the some Langevin methods, that is, the related convergence.

At last, we establish the non-asymptotic bounds between the law of the samples generated from Algorithm 1 and the distribution \( \mu_\sigma \) in the Wasserstein-2 distance. We introduce the definition of Wasserstein distance. Let \( D(\nu_1, \nu_2) \) be the collection of coupling probability measures on \( (\mathbb{R}^{2d}, \mathcal{B}(\mathbb{R}^{2d})) \) such that its respective marginal distributions are \( \nu_1 \) and \( \nu_2 \). The Wasserstein distance of order \( p \geq 1 \) measuring the discrepancy between \( \nu_1 \) and \( \nu_2 \) is defined as

\[
W_p(\nu_1, \nu_2) = \inf_{\nu \in D(\nu_1, \nu_2)} \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \| \theta_1 - \theta_2 \|^p \, d\nu(\theta_1, \theta_2) \right)^{1/p}.
\]

**Theorem 3.7.** Assume (A) and (C3) hold, then

\[
W_2(\text{Law}(\tilde{Y}_{1,K}), \mu_\sigma) \leq C_{3,\sigma}^d \sqrt{s} + C_{2,\sigma}^d \sqrt{\frac{16d}{m}},
\]

where

\[
C_{2,\sigma}^d := \exp(1/2 + 8C_{2,\sigma}^d), \quad C_{3,\sigma}^d := 2C_{1,\sigma} + 4C_{2,\sigma} \left\{ 1 + C_{0,\sigma} \exp\left( 2\sqrt{C_{0,\sigma} + 1} \right) \right\}^{1/2},
\]

with \( C_{0,\sigma} = C_{1,\sigma} = \gamma_a/\xi_a + \frac{\gamma_a^2}{\xi_a^2} \).

**Remark 3.8.** Langevin sampling method has been studied under the (strongly) convex potential assumption (Durmus and Moulines, 2016, 2017, 2019; Dalalyan, 2017a,h; Cheng and Bartlett, 2018; Dalalyan and Karagulyan, 2019). Also, there are some mean-field Langevin methods, see Garbuno-Inigo et al. (2020); Wang and Li (2022) for more details. However, the Langevin diffusion process tends to the target distribution as time \( t \) goes to infinity while the Schrödinger bridge achieve this in unit time. The Schrödinger bridge has been shown to have close connections with a number of problems in statistical physics, optimal transport and optimal control (Léonard, 2014). However, only a few recent works use this tool for statistical sampling. Bernton et al. (2019) proposed the Schrödinger bridge sampler by applying the iterative proportional fitting method (Sinkhorn algorithm (Sinkhorn, 1964; Peyré and Cuturi, 2019)). For the Gibbs measure \( \mu_\sigma \), Schrödinger bridge samplers iteratively modifies the transition kernels of the reference Markov chain to obtain a process whose marginal distribution at terminal time is approximate to \( \mu_\sigma \), via regression-based approximations of the corresponding iterative proportional fitting recursion.
non-convex smooth function (Carrillo et al., 2021), which maps from $\mathbb{R}^d$ to $\mathbb{R}$,

$$V(x) = \frac{1}{d^2} \sum_{i=1}^{d} \left[ (x_i - B)^2 - 10 \cos(2\pi(x_i - B)) \right] + C,$$

with $B = \arg\min_{x \in \mathbb{R}^d} V(x)$ and $C = \min_{x \in \mathbb{R}^d} V(x)$. Figure 1 depicts this target function $V$ with setting $B = 0, C = 0, d = 1$, and we can see that the number of local minimizers in $\mathbb{R}$ is infinite and only one global minimizer exists, i.e., $x = 0$. In the numerical experiment, we consider the case $d = 2, B = 0, C = 0$ in $V$. We set $K = 200, m = 1000, \sigma = 0.01$ in Algorithm 1, and Langevin method is implemented by R package yuima (Iacus and Yoshida, 2018). As shown in Proposition 2.3, the target distribution can be exactly obtained at time one. Thus, we only need to keep the last data in the Euler-Maruyama discretization of Schrödinger-Föllmer diffusion in each iteration and repeat this scheme such that the desired sample size is derived, i.e., we get one sample when running Algorithm 1 one time (it costs $200 \times 1000$ Gaussian samples at each run). In comparison, the Langevin diffusion (3) tends to the target distribution as time $t$ goes to infinity, then it should burn out sufficient data points empirically in each Euler-Maruyama discretization. To make the comparison fair, we run the Langevin method 50 times. At each run we generate $200 \times 1000$ samples and keep the one with the best function value to show. Figure 2 shows the simulation results of 50 independent runs, where the red line yields the global minimizer (GM) $(x_1 = 0, x_2 = 0)$. From Figure 2, we can conclude that our proposed method obtains the global minimizer approximately and performs better than the Langevin method.

Figure 1. $B = 0, C = 0, d = 1$.

Figure 2. Proposed method VS Langevin method.
5. Conclusion

We study the problem of finding global minimizers of $V(x) : \mathbb{R}^d \to \mathbb{R}$ approximately via sampling from a probability distribution $\mu_\sigma$ with density $p_\sigma(x) = \frac{\exp(-V(x)/\sigma)}{\int_{\mathbb{R}^d} \exp(-V(y)/\sigma) dy}$ with respect to the Lebesgue measure for $\sigma \in (0, 1]$ small enough. We analyze a sampler based on the Euler discretization scheme of the Schrödinger-Föllmer diffusion processes with stochastic approximation under appropriate assumptions on the step size $s$ and the potential $V$. We prove that the output of the proposed sampler is an approximate global minimizer of $V(x)$ with high probability. Moreover, simulation studies verify the effectiveness of the proposed method on solving non-convex smooth optimization problems and it performs better than the Langevin method.

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7. Appendix

In the appendix, we will show (P1)-(P2) and prove the all Propositions and Theorems, i.e., Propositions 2.3, 3.3, Theorems 3.2, 3.5, 3.7.

7.1. Verify (P1)-(P2).

Proof. Recall the definition of $f_\sigma$ in (6) and the assumption $V(x) = \|x\|_2^2/2$ when $\|x\|_2 > R$. Through some simple calculations, we have

$$
\nabla f_\sigma(x) = \exp\left(\frac{\|x\|_2^2}{2\sigma} - \frac{V(x)}{\sigma}\right) \cdot \left(\frac{x}{\sigma} - \nabla V(x)\sigma\right),
$$

$$
\nabla^2 f_\sigma(x) = \exp\left(\frac{\|x\|_2^2}{2\sigma} - \frac{V(x)}{\sigma}\right) \cdot \left(\frac{x}{\sigma} - \nabla V(x)\sigma\right) \cdot \left(\frac{x}{\sigma} - \nabla V(x)\sigma\right)^\top + \exp\left(\frac{\|x\|_2^2}{2\sigma} - \frac{V(x)}{\sigma}\right) \cdot \left(\frac{I_d}{\sigma} - \nabla^2 V(x)\sigma\right).
$$

Then, the properties (P1)-(P2) hold if for each $\sigma \in (0, 1]$, 

$$
\lim_{R \to \infty} \sup_{\|x\|_2 \geq R} \exp\left(-\frac{V(x)}{\sigma} + \frac{\|x\|_2^2}{2\sigma}\right) \cdot \left\|\frac{I_d - \nabla^2 V(x)\sigma}{\sigma}\right\|_2 < \infty,
$$

and

$$
\lim_{R \to \infty} \sup_{\|x\|_2 \geq R} \exp\left(-\frac{V(x)}{\sigma} + \frac{\|x\|_2^2}{2\sigma}\right) \cdot \left\|\frac{x - \nabla V(x)\sigma}{\sigma}\right\|_2 < \infty,
$$

where we use the fact that $a \mapsto a \exp(-ax)$ is non-increasing for $a \geq 1, x \in \mathbb{R}_+$. 

□
7.2. Proof of Proposition 2.3.

Proof. By (P1) and (P2), it yields that for all \( x \in \mathbb{R}^d \) and \( t \in [0,1] \),
\[
\|b(x,t)\|_2 = \left\| \frac{\nabla Q_{1-t}\hat{f}_\sigma(x)}{Q_{1-t}\hat{f}_\sigma(x)} \right\|_2 \leq \frac{\gamma_\sigma}{\xi_\sigma}. \tag{A.1}
\]
Then, by (P1)-(P2) and (A.1), for all \( x, y \in \mathbb{R}^d \) and \( t \in [0,1] \),
\[
\|b(x,t) - b(y,t)\|_2 = \left\| \frac{\nabla Q_{1-t}\hat{f}_\sigma(x) - \nabla Q_{1-t}\hat{f}_\sigma(y)}{Q_{1-t}\hat{f}_\sigma(y)} \right\|_2
\]
\[
= \left\| \frac{\nabla Q_{1-t}\hat{f}_\sigma(x) - \nabla Q_{1-t}\hat{f}_\sigma(y)}{Q_{1-t}\hat{f}_\sigma(y)} + \frac{\nabla Q_{1-t}\hat{f}_\sigma(y) (Q_{1-t}\hat{f}_\sigma(y) - Q_{1-t}\hat{f}_\sigma(x))}{Q_{1-t}\hat{f}_\sigma(x)Q_{1-t}\hat{f}_\sigma(y)} \right\|_2
\]
\[
\leq \left\| \frac{\nabla Q_{1-t}\hat{f}_\sigma(x) - \nabla Q_{1-t}\hat{f}_\sigma(y)}{Q_{1-t}\hat{f}_\sigma(y)} \right\|_2 + \frac{\|b(x,t)\|_2 \cdot \left\| \frac{Q_{1-t}\hat{f}_\sigma(x) - Q_{1-t}\hat{f}_\sigma(y)}{Q_{1-t}\hat{f}_\sigma(y)} \right\|_2}{Q_{1-t}\hat{f}_\sigma(y)}
\]
\[
\leq \left( \frac{\gamma_\sigma}{\xi_\sigma} + \frac{\gamma_\sigma^2}{\xi_\sigma^2} \right) \|x - y\|_2.
\]
Setting \( C_{1,\sigma} := \frac{\gamma_\sigma}{\xi_\sigma} + \frac{\gamma_\sigma^2}{\xi_\sigma^2} \) yields the Lipschitz continuous condition (C2). Combining (A.1) and (C2) with the triangle inequality, we have
\[
\|b(x,t)\|_2 \leq \|b(0,t)\|_2 + \|x\|_2 \leq \frac{\gamma_\sigma}{\xi_\sigma} + C_{1,\sigma} \|x\|_2.
\]
Let \( C_{0,\sigma} := \max \left\{ \frac{\gamma_\sigma}{\xi_\sigma}, C_{1,\sigma} \right\} \), then (C1) holds. Therefore, the drift term \( b(x,t) \) satisfies the linear grow condition (C1) and Lipschitz condition (C2), then the Schrödinger-Föllmer diffusion SDE (7) has the unique strong solution (Revuz and Yor, 2013; Pavliotis, 2014).

Moreover, Schrödinger-Föllmer diffusion process \( \{X_t\}_{t \in [0,1]} \) defined in (7) admits the transition probability density
\[
p_{s,t}(x,y) := \tilde{p}_{s,t}(x,y) \frac{Q_{1-t}\sigma(y)}{Q_{1-s}\sigma(x)},
\]
where
\[
\tilde{p}_{s,t}(x,y) = \frac{1}{(2\pi\sigma(t-s))^{d/2}} \exp \left( -\frac{1}{2\sigma(t-s)} \|x - y\|^2_2 \right)
\]
is the transition probability density of a \( d \)-dimensional Brownian motion \( \sqrt{\sigma}B_t \). See Dai Pra (1991); Lehec (2013) for details. It follows that for any Borel measurable set \( A \in \mathcal{B}(\mathbb{R}^d) \),
\[
\mathbb{P}(X_1 \in A) = \int_A p_{0,1}(0,y)dy
\]
\[
= \int_A \tilde{p}_{0,1}(0,y) \frac{Q_{1}\sigma(y)}{Q_{1}\sigma(0)}dy
\]
\[
= \mu_\sigma(A),
\]
where the last equality follows from \( Q_{1}\sigma(0) = \mu_\sigma(\mathbb{R}^d) = 1 \) and \( Q_{0}\sigma(y) = \sigma(y) \). Therefore, \( X_1 \) is distributed as the probability distribution \( \mu_\sigma \). This completes the proof. \( \square \)

7.3. Proof of Proposition 3.3.

Proof. Under the Theorem 3.2, then \( C_\sigma = \int_{\mathbb{R}^d} \exp (-V(x)/\sigma) dx \leq \int_{\mathbb{R}^d} \exp (-V(x)) dx < \infty \) for all \( \sigma \in (0,1] \). According to the Varadhan’s theorem 1.2.3 in Dupuis and Ellis (2011), it follows that the family \( \{\mu_\sigma\}_{\sigma \in [0,1]} \) on \( \mathcal{B}(\mathbb{R}^d) \) satisfies large deviation principle with rate function \( V(x) - \min_x V(x) \), that is, for every function \( F \in C_b(\mathbb{R}^d) \), the bounded continuous function space on \( \mathbb{R}^d \),
\[
\lim_{\sigma \to 0} \sigma \log \int_{\mathbb{R}^d} C_\sigma^{-1} \exp \left( \frac{F(x) - V(x)}{\sigma} \right) dx = \sup_x \{F(x) - I(x)\}, \tag{A.2}
\]
where the rate function \( I(x) \) is defined by

\[
I(x) := V(x) - \min_x V(x).
\]

Next, we only need to prove (A.2). On the one hand,

\[
\log \int_{\mathbb{R}^d} C^{-1}_\sigma \exp \left( \frac{F(x) - V(x)}{\sigma} \right) dx = - \log \int_{\mathbb{R}^d} \exp \left( - \frac{V(x)}{\sigma} \right) dx + \log \int_{\mathbb{R}^d} \exp \left( \frac{F(x) - V(x)}{\sigma} \right) dx.
\]

By Lemma 7.1, we have

\[
- \sigma \int_{\mathbb{R}^d} \exp \left( - \frac{V(x)}{\sigma} \right) dx = - \sigma \int_{\mathbb{R}^d} \exp \left( - \frac{V(x) - \min_x V(x)}{\sigma} \right) dx + \min_x V(x) + \frac{d}{2} \lim_{\sigma \to 0} \sigma \log \sigma = \min_x V(x), \quad \text{as } \sigma \downarrow 0. \tag{A.3}
\]

On the other hand, for any positive \( K > 0 \), combining Lemma 2.2 in Varadhan (1966) and the dominated convergence theorem yields that

\[
\lim_{\sigma \to 0} \sigma \log \int_{\mathbb{R}^d} \exp \left( \frac{F(x) - V(x)}{\sigma} \right) dx = \lim_{K \to \infty} \lim_{\sigma \to 0} \sigma \log \int_{\mathbb{R}^d} \exp \left( \frac{F(x) - V(x) \wedge K}{\sigma} \right) dx = \lim_{K \to \infty} \sup_x \{F(x) - V(x) \wedge K\} = \sup_x \{F(x) - V(x)\}. \tag{A.4}
\]

Hence, by (A.3) and (A.4), we get

\[
\lim_{\sigma \to 0} \sigma \log \int_{\mathbb{R}^d} C^{-1}_\sigma \exp \left( \frac{F(x) - V(x)}{\sigma} \right) dx = \sup_x \{F(x) - V(x) + \min_x V(x)\}.
\]

Since the measure \( \mu_\sigma \) satisfies the large deviation principle with rate function \( I \), if we take the closed set \( F := \{x \in \mathbb{R}^d; V(x) - \min_x V(x) \geq \tau\} \), then

\[
\lim_{\sigma \to 0} \sigma \log \mu_\sigma(F) = \lim_{\sigma \to 0} \sigma \log \mu_\sigma \left( V(x) - \min_x V(x) \geq \tau \right) = - \inf_{x \in F} I(x) = -\tau.
\]

\[ \square \]

7.4. Preliminary lemmas for Theorem 3.2 and Theorem 3.5. In order to prove that the Gibbs measure \( \mu_\sigma \) weakly converges to a multidimensional distribution and estimate the probabilities of \( V(X_1) > \tau \) and \( V(\tilde{Y}_t_k) > \tau \) for any \( \tau > 0 \), we first need to prove the following Lemmas 7.1-7.2.

**Lemma 7.1.** Assume \( V \) is twice continuously differentiable and satisfies \( V(x_0) = 0, \nabla V(x_0) = 0 \), and the Hessian matrix \( \nabla^2 V(x_0) \) is positive definite. When \( \delta > 0 \) is sufficiently small, then

\[
\lim_{t \to +\infty} t^{d/2} \int_{U_\delta(x_0)} e^{-t V(x_1, \ldots, x_d)} dx_1 \cdots dx_d = \frac{(2\pi)^{d/2}}{(\det \nabla^2 V(x_0))^{1/2}}
\]

for any \( x \in U_\delta(x_0) := \{x \in \mathbb{R}^d; \|x - x_0\|_2 < \delta\} \).

**Proof.** For each \( \varepsilon \in (0, 1) \), there exists \( \delta > 0 \) such that

\[
\frac{(1 - \varepsilon)(x - x_0)^T \nabla^2 V(x_0)(x - x_0)}{2} \leq V(x) \leq \frac{(1 + \varepsilon)(x - x_0)^T \nabla^2 V(x_0)(x - x_0)}{2}
\]

holds for any \( x \in U_\delta \). Moreover, we have

\[
\int_{U_\delta(x_0)} \exp(-t V(x)) dx \leq \int_{U_\delta(x_0)} \exp \left( -\frac{t}{2} (1 - \varepsilon)(x - x_0)^T \nabla^2 V(x_0)(x - x_0) \right) dx. \tag{A.5}
\]

There is an orthogonal matrix \( P \) such that \( P^T \nabla^2 V(x_0) P = \text{diag}(\lambda_1, \ldots, \lambda_d) \), where \( \lambda_1, \ldots, \lambda_d \) are the eigenvalues of Hessian matrix \( \nabla^2 V(x_0) \). And we denote \( (y_1, \ldots, y_d) := P(x - x_0) \), then

\[
(x - x_0)^T \nabla^2 V(x_0)(x - x_0) = \sum_{i=1}^d \lambda_i(y_i)^2.
\]
Hence
\[
\int_{U_d(x_0)} \exp \left( -\frac{t}{2} (1 - \varepsilon)(x - x_0)^\top \nabla^2 V(x_0)(x - x_0) \right) \, dx = \int_{U_d(0)} \exp \left( -\frac{t}{2} (1 - \varepsilon) \sum_{i=1}^d \lambda_i(y_i)^2 \right) \, dy.
\] (A.6)

Further, we can get
\[
\int_{U_d(0)} \exp \left( -\frac{t}{2} (1 - \varepsilon) \sum_{i=1}^d \lambda_i(y_i)^2 \right) \, dy = \left( \frac{2}{t(1 - \varepsilon)} \right)^{\frac{d}{2}} \int_{\|z\| < \sqrt{\frac{2(1 - \varepsilon)}{t}}} \exp \left( -\sum_{i=1}^d \lambda_i(z_i)^2 \right) \, dz,
\] (A.7)

where the equality holds by setting \( z = \sqrt{\frac{2(1 - \varepsilon)}{t}} y \). By (A.5), (A.6) and (A.7), it follows that
\[
\limsup_{t \to +\infty} t^{\frac{d}{2}} \int_{U_d(x_0)} e^{-t V(x)} \, dx \leq \left( \frac{2}{1 - \varepsilon} \right)^{\frac{d}{2}} \limsup_{t \to +\infty} \int_{\|z\| < \sqrt{\frac{2(1 - \varepsilon)}{t}}} \exp \left( -\sum_{i=1}^d \lambda_i(z_i)^2 \right) \, dz \\
\leq \left( \frac{2}{1 - \varepsilon} \right)^{\frac{d}{2}} \int_{\mathbb{R}^d} \exp \left( -\sum_{i=1}^d \lambda_i(z_i)^2 \right) \, dz_1 \cdots dz_d \\
= \left( \frac{2\pi}{1 - \varepsilon} \right)^{\frac{d}{2}} \left( \prod_{i=1}^d \frac{1}{\sqrt{\lambda_i}} \right) = \left( \frac{2\pi}{1 - \varepsilon} \right)^{\frac{d}{2}} \frac{1}{(\det \nabla^2 V(x_0))^\frac{d}{2}}.
\]

Similarly, we have
\[
\liminf_{t \to +\infty} t^{\frac{d}{2}} \int_{U_d(x_0)} e^{-t V(x)} \, dx \geq \left( \frac{2}{1 + \varepsilon} \right)^{\frac{d}{2}} \liminf_{t \to +\infty} \int_{\|z\| < \sqrt{\frac{2(1 + \varepsilon)}{t}}} \exp \left( -\sum_{i=1}^d \lambda_i(z_i)^2 \right) \, dz \\
\geq \left( \frac{2}{1 + \varepsilon} \right)^{\frac{d}{2}} \int_{\mathbb{R}^d} \exp \left( -\sum_{i=1}^d \lambda_i(z_i)^2 \right) \, dz_1 \cdots dz_d \\
= \left( \frac{2\pi}{1 + \varepsilon} \right)^{\frac{d}{2}} \left( \prod_{i=1}^d \frac{1}{\sqrt{\lambda_i}} \right) = \left( \frac{2\pi}{1 + \varepsilon} \right)^{\frac{d}{2}} \frac{1}{(\det \nabla^2 V(x_0))^\frac{d}{2}}.
\]

Therefore, letting \( \varepsilon \to 0^+ \), we get
\[
\lim_{t \to +\infty} t^{\frac{d}{2}} \int_{U_d(x_0)} e^{-t V(x)} \, dx = (2\pi)^{\frac{d}{2}} \left( \prod_{i=1}^d \frac{1}{\sqrt{\lambda_i}} \right) = \frac{(2\pi)^{\frac{d}{2}}}{(\det \nabla^2 V(x_0))^\frac{d}{2}}.
\]

\[\square\]

**Lemma 7.2.** Let \( X = (X_t, \mathcal{F}_t), Y = (Y_t, \mathcal{F}_t) \) be strong solutions of the following two stochastic differential equations
\[
dX_t = a(X_t, t) \, dt + \sigma dB_t, \quad t \in [0, 1] \\
dY_t = b(Y_t, t) \, dt + \sigma dB_t, \quad Y_0 = X_0, \quad t \in [0, 1],
\]
and \( X_0 \) is a \( \mathcal{F}_0 \)-measurable random variable. In addition, if drift terms \( a(X_t, t) \) and \( b(X_t, t) \) satisfy \( \mathbb{E} \left[ \exp \left( \int_0^1 \|a(X_t, t)\|_2^2 + \|b(X_t, t)\|_2^2 \, dt \right) \right] < \infty \), then we have
\[
\frac{d\mathbb{P}_Y}{d\mathbb{P}_X}(X) = \exp \left( \sigma^{-1} \int_0^1 \langle b(X_t, t) - a(X_t, t), dB_t \rangle - \frac{1}{2\sigma^2} \int_0^1 \|b(X_t, t) - a(X_t, t)\|_2^2 \, dt \right), \quad (A.8)
\]
and the relative entropy of \( \mathbb{P}_X \) with respect to \( \mathbb{P}_Y \) satisfies
\[
\mathbb{D}_{KL}(\mathbb{P}_X || \mathbb{P}_Y) = \frac{1}{2\sigma^2} \int_0^1 \mathbb{E} \left[ \|b(X_t, t) - a(X_t, t)\|_2^2 \right] \, dt,
\]
where probability distributions \( \mathbb{P}_X, \mathbb{P}_Y \) are induced by process \((X_t, 0 \leq t \leq 1)\) and \((Y_t, 0 \leq t \leq 1)\), respectively.
Proof. By the Novikov condition (Revuz and Yor, 2013), we know that
\[ M_t := \exp \left( \sigma^{-1} \int_0^t (b(X_u, u) - a(X_u, u), dB_u) - \frac{1}{2\sigma^2} \int_0^t \|b(X_u, u) - a(X_u, u)\|^2 du \right) \]
is exponential martingale and \( \mathbb{E}M_t = 1 \) for all \( t \in [0, 1] \). We can denote a new probability measure \( \mathbb{Q} \) such that \( d\mathbb{Q} = M_1 d\mathbb{P} \). By Girsanov’s theorem (Revuz and Yor, 2013), under the new probability measure \( \mathbb{Q} \), we can conclude that
\[ \tilde{B}_t := B_t - \sigma^{-1} \int_0^t (b(X_u, u) - a(X_u, u)) du \]
is a \( \mathbb{Q} \)-Brownian motion. Hence, under the new probability measure \( \mathbb{Q} \),
\[ b(X_t, t)dt + \sigma dB_t = b(X_t, t)dt + \sigma dB_t - (b(X_t, t) - a(X_t, t))dt = a(X_t, t)dt + dB_t = dX_t. \]
Thus, we have \( \mathbb{Q}_X = \mathbb{P}_Y \), where \( \mathbb{Q}_X \) is the distribution of \( X \) under the measure \( \mathbb{Q} \). Furthermore, we can obtain (A.8). On the other hand, by the definition of relative entropy of \( \mathbb{P}_X \) with respect to \( \mathbb{P}_Y \), we have
\[ \mathbb{D}_{KL}(\mathbb{P}_X || \mathbb{P}_Y) = \mathbb{E} \left[ -\log \left( \frac{d\mathbb{P}_Y}{d\mathbb{P}_X}(X) \right) \right] = \frac{1}{2\sigma^2} \int_0^1 \mathbb{E} \left[ \|b(X_t, t) - a(X_t, t)\|^2 \right] dt. \]
Therefore, the proof of Lemma 7.2 is completed. \( \square \)

7.5. Proof of Theorem 3.2.

Proof. The result can be traced back to the 1980s. For the overall continuity of the article, we use the Laplace’s method in Hwang (1980, 1981) to give a detailed proof of the result. The key idea is to prove that for each \( \delta' > 0 \) small enough, \( \mu_\sigma(\{x; \|x - x_i^*\|_2 < \delta'\}) \) converges to
\[ \frac{(\det \nabla^2 V(x_i^*))^{-\frac{1}{2}}}{\sum_{j=1}^\kappa (\det \nabla^2 V(x_j^*))^{-\frac{1}{2}}} \]as \( \sigma \downarrow 0 \). We firstly introduce the following notations
\[ a(\delta') := \inf \{V(x); \|x - x_i^*\|_2 \geq \delta', 1 \leq i \leq \kappa\}, \]
\[ \bar{m}_i(\sigma, \delta') := \int_{\|x - x_i^*\|_2 < \delta'} \exp \left(-\frac{V(x)}{\sigma}\right) dx, \quad 1 \leq i \leq \kappa, \]
\[ \bar{m}(\sigma, \delta') := \int_{\mathbb{R}^d} \exp \left(-\frac{V(x)}{\sigma}\right) dx. \]
Hence, we have
\[ \mu_\sigma(\{x; \|x - x_i^*\|_2 < \delta'\}) = \frac{\int_{\|x - x_i^*\|_2 < \delta'} \exp \left(-\frac{V(x)}{\sigma}\right) dx}{\int_{\mathbb{R}^d} \exp \left(-\frac{V(x)}{\delta}\right) dx} = \sum_{j=1}^\kappa \bar{m}_j(\sigma, \delta') + \bar{m}(\sigma, \delta'). \quad (A.9) \]
On the other hand, \( \nabla^2 V(x_i^*) \) is a positive definite symmetric matrix. For any \( \varepsilon \in (0, 1) \), we can choose \( 0 < \delta' < \varepsilon \) such that
\[ \frac{(x - x_i^*)^T (\nabla^2 V(x_i^*) - \varepsilon I_d)(x - x_i^*)}{2} \leq V(x) - V(x_i^*) \leq \frac{(x - x_i^*)^T (\nabla^2 V(x_i^*) + \varepsilon I_d)(x - x_i^*)}{2} \]
holds for any \( \|x - x_i^*\|_2 < \delta' \). Thus, for any \( i = 1, \ldots, \kappa \), we obtain
\[ (2\pi\sigma)^{-\frac{d}{2}} e^{-\frac{V(x_i^*)}{\sigma}} \bar{m}_i(\sigma, \delta') \leq (2\pi\sigma)^{-\frac{d}{2}} \int_{\|x - x_i^*\|_2 < \delta'} \exp \left(-\frac{(x - x_i^*)^T (\nabla^2 V(x_i^*) - \varepsilon I_d)(x - x_i^*)}{2\sigma}\right) dx, \]
\[ (2\pi\sigma)^{-\frac{d}{2}} e^{-\frac{V(x_i^*)}{\sigma}} \bar{m}_i(\sigma, \delta') \geq (2\pi\sigma)^{-\frac{d}{2}} \int_{\|x - x_i^*\|_2 < \delta'} \exp \left(-\frac{(x - x_i^*)^T (\nabla^2 V(x_i^*) + \varepsilon I_d)(x - x_i^*)}{2\sigma}\right) dx. \]
By Lemma 7.1 and letting \( \sigma \rightarrow 0 \), we have
\[ (\det(\nabla^2 V(x_i^*) + \varepsilon I_d))^{-\frac{1}{2}} \leq \liminf_{\sigma \rightarrow 0} (2\pi\sigma)^{-\frac{d}{2}} e^{-\frac{V(x_i^*)}{\sigma}} \bar{m}_i(\sigma, \delta') \]
We can choose sufficiently large $\varepsilon$ such that
\[
\limsup_{\sigma \to 0} (2\pi\sigma)^{-\frac{d}{2}} e^{V(x_1^+) - \frac{V(x_1^+)}{\sigma}} \tilde{m}_i(\sigma, \delta') \leq \left( \det (\nabla^2 V(x_1^+))^{-\frac{1}{2}} \right),
\]
As $\varepsilon \downarrow 0$, we get
\[
\lim_{\sigma \to 0} (2\pi\sigma)^{-\frac{d}{2}} e^{V(x_1^+) - \frac{V(x_1^+)}{\sigma}} \tilde{m}_i(\sigma, \delta') = \left( \det (\nabla^2 V(x_1^+))^{-\frac{1}{2}} \right). \tag{A.10}
\]
On the other hand, we have
\[
(2\pi\sigma)^{-\frac{d}{2}} e^{V(x_1^+) - \frac{V(x_1^+)}{\sigma}} \tilde{m}(\sigma, \delta') = (2\pi\sigma)^{-\frac{d}{2}} e^{\left( - \frac{a(\delta') - V(x_1^+)}{\sigma} \right)} \times \int_{\bigcup_{i=1}^n \|x-x_i^\| \geq \delta'} \exp \left( - \frac{V(x) - a(\delta')}{\sigma} \right) dx.
\]
Since $a(\delta') := \inf \{V(x); \|x-x_i^\| \geq \delta', 1 \leq i \leq \kappa \} > V(x_i^\ast)$ and $V(x) \geq a(\delta')$ holds in $\{x \in \mathbb{R}^d; \|x-x_i^\| \geq \delta'\}$, then for any $\sigma \in (0, 1)$, we have
\[
\int_{\bigcup_{i=1}^n \|x-x_i^\| \geq \delta'} \exp \left( - \frac{V(x) - a(\delta')}{\sigma} \right) dx \leq \int_{\bigcup_{i=1}^n \|x-x_i^\| \geq \delta'} \exp \left( - (V(x) - a(\delta')) \right) dx \leq \exp(a(\delta')) \int_{\mathbb{R}^d} \exp(-V(x))dx < \infty.
\]
Also, it follows that
\[
(2\pi\sigma)^{-\frac{d}{2}} e^{\left( - \frac{a(\delta') - V(x_1^+)}{\sigma} \right)} \to 0 \quad \text{as} \quad \sigma \downarrow 0.
\]
Thus we get
\[
\lim_{\sigma \to 0} (2\pi\sigma)^{-\frac{d}{2}} e^{V(x_1^+) - \frac{V(x_1^+)}{\sigma}} \tilde{m}(\sigma, \delta') = 0. \tag{A.11}
\]
Taking limit $\sigma \downarrow 0$ in (A.9) and applying (A.10), (A.11), we get
\[
\mu_\sigma(\{x; \|x-x_i^\| \leq \delta\}) \to \frac{(\det (\nabla^2 V(x_1^+))^{-\frac{1}{2}}}{\sum_{j=1}^\kappa (\det (\nabla^2 V(x_j^+))^{-\frac{1}{2}})} \quad \text{as} \quad \sigma \downarrow 0.
\]
Therefore, the proof Theorem 3.2 is completed. \qed

7.6. Proof of Theorem 3.5.

Proof. Note that
\[
P(V(X_1) > \tau) = \frac{\int_{V(x) > \tau} \exp(-V(x)/\sigma)dx}{\int_{\mathbb{R}^d} \exp(-V(x)/\sigma)dx}. \tag{A.12}
\]
According to $V(x) = \|x\|^2/2$ for any $\|x\| \geq R$, then $V$ has at least linear growth at infinity, that is, there exists a constant $C > 0$ such that for $R^\ast$ large enough
\[
V(x) \geq \min_{\|y\| = R^\ast} V(y) + C(\|x\| - R^\ast) \quad \text{for} \quad \|x\| > R^\ast.
\]
We can choose sufficiently large $R^\ast$ such that $\min_{\|y\| = R^\ast} V(y) > \tau$. Hence,
\[
\int_{V(x) > \tau} \exp(-V(x)/\sigma)dx = \int_{V(x) \geq \tau, \|x\| \leq R^\ast} \exp(-V(x)/\sigma)dx + \int_{V(x) \geq \tau, \|x\| > R^\ast} \exp(-V(x)/\sigma)dx \leq \exp \left( - \frac{\tau}{\sigma} \right) \text{Vol}(B_{R^\ast}) + \int_{V(x) \geq \tau, \|x\| > R^\ast} \exp \left( - \frac{\tau + C(\|x\| - R^\ast)}{\sigma} \right) dx \leq \exp \left( - \frac{\tau}{\sigma} \right) \left( \text{Vol}(B_{R^\ast}) + d\text{Vol}(B_1) \int_{R^\ast}^{\infty} \rho^{d-1} \exp \left\{ -C(r - R^\ast) \right\} dr \right) \leq 2\text{Vol}(B_{R^\ast}) \exp \left( - \frac{\tau}{\sigma} \right). \tag{A.13}
\]
where Vol$(B_{R^*})$ is the volume of a ball with radius $R^*$. On the other hand, since $\min_x V(x) = 0$, there then exists $r > 0$ such that $V(x) < \varepsilon$ when $\|x\|_2 < r$, we have
\[
\int_{\mathbb{R}^d} \exp(-V(x)/\sigma) dx \geq \int_{\|x\|_2 < r} \exp(-V(x)/\sigma) dx > \exp\left(-\frac{\varepsilon}{\sigma}\right) \text{Vol}(B_r).
\]  
(A.14)

By injection (A.13), (A.14) into (A.12), we get
\[
\mathbb{P}(V(X_1) > \tau) \leq C_{\tau, \varepsilon, d} \exp\left(-\frac{\tau - \varepsilon}{\sigma}\right),
\]
where
\[
C_{\tau, \varepsilon, d} := \frac{2 \text{Vol}(B_{R^*})}{\text{Vol}(B_r)}.
\]  
(A.15)

Next, we will prove that the second conclusion holds in the discrete case. Recall that $s = 1/K$ is the step size, $t_k := ks$ is the cumulative step size up to iteration $k$, and $\{X_t\}_{t \in [0,1]}$ is the Schrödinger-Föllmer diffusion process (7). Let $\mu_{t_k}$ be the probability measure of $Y_{t_k}$ defined by (10). Then for fixed $\tau > 0$, we have
\[
\mathbb{P}(V(\tilde{Y}_{t_k}) > \tau) \leq \mathbb{P}(V(\tilde{Y}_{t_k}) > \tau) + |\mathbb{P}(V(\tilde{Y}_{t_k}) > \tau) - \mathbb{P}(V(Y_{t_k}) > \tau)|
\leq \mathbb{P}(V(\tilde{Y}_{t_k}) > \tau) + \|\tilde{Y}_{t_k} - Y_{t_k}\|_{TV}
\leq \mathbb{P}(V(X_{t_k}) > \tau) + \|X_{t_k} - Y_{t_k}\|_{TV} + \|\tilde{Y}_{t_k} - Y_{t_k}\|_{TV}
\leq \mathbb{P}(V(X_{t_k}) > \tau) + \sqrt{2 \text{KL}(\mu_{t_k}||\mu_\sigma)} + \sqrt{2 \text{KL}(\text{Law}(\tilde{Y}_{t_k})||\mu_{t_k})},
\]  
(A.16)

where we use Pinsker’s inequality (Bakry et al., 2014) in the last inequality and the second inequality holds due to the fact that letting $g(x) := 1_{V(x) > \tau}$, then
\[
|\mathbb{P}(V(\tilde{Y}_{t_k}) > \tau) - \mathbb{P}(V(Y_{t_k}) > \tau)| = \left| \mathbb{E} \left( g(\tilde{Y}_{t_k}) - g(Y_{t_k}) \right) \right|
= \left| \int_{\mathbb{R}^d} g(x) d \left( \mathbb{P}_{\tilde{Y}_{t_k}}(x) - \mathbb{P}_{Y_{t_k}}(x) \right) \right|
\leq \|\mathbb{P}_{\tilde{Y}_{t_k}} - \mathbb{P}_{Y_{t_k}}\|_{L^1} = \|\tilde{Y}_{t_k} - Y_{t_k}\|_{TV},
\]
where the total variation metric between two probability measures $\mu, \nu$ on $\mathbb{R}^d$ is defined by $\|\mu - \nu\|_{TV} := \sup_{A \subseteq \mathbb{R}^d} |\mu(A) - \nu(A)|$.

Firstly, from the first part of proof, we can get a bound for the first term on the right hand side of (A.16). That is, for each $\varepsilon \in (0, \tau)$, there exists a constant $C_{\tau, \varepsilon, d}$ defined in (A.15) such that
\[
\mathbb{P}(V(X_{t_k}) > \tau) \leq C_{\tau, \varepsilon, d} \exp\left(-\frac{\tau - \varepsilon}{\sigma}\right).
\]  
(A.17)

Secondly, we estimate the boundness of $\text{KL}(\mu_{t_k}||\mu_\sigma)$. To make use of continuous-time tools, we construct a continuous-time interpolation for the discrete-time algorithm (10). In particular, we define a stochastic process $\{Y_t\}_{t \in [0,1]}$ via SDE
\[
dY_t = \sigma \tilde{b}(Y_t, t) dt + \sqrt{\sigma} dB_t, \quad t \in [0, 1], Y_0 = 0,
\]  
(A.18)

with the non-anticipative drift $\tilde{b}(Y_t, t) := \sum_{k=0}^{K-1} b(Y_{t_k}, t_k) \mathbb{1}_{[t_k, t_{k+1})}(t)$. Meanwhile, because of Proposition 2.3, the process $\{X_t\}_{t \in [0,1]}$ (7) satisfies $X_1 \sim \mu_\sigma$. We also denote by $\mu_\sharp_t$ and $\nu_\flat$ the marginal distributions on $C([0, 1], \mathbb{R}^d)$ of $\{X_t, Y_t\}_{t \in [0,1]}$. Thus, combining (10), (A.18) and Lemma 7.2, we obtain
\[
\text{KL}(\mu_{t_k}||\mu_\sigma) \leq \text{KL}(\nu_\flat||\mu_\sharp_t) = \frac{\sigma}{2} \int_0^1 \mathbb{E} \left( \|\tilde{b}(Y_t, t) - \tilde{b}(Y_t, t)\|^2 \right) dt
= \frac{\sigma}{2} \sum_{k=0}^{K-1} \int_{t_k}^{t_{k+1}} \mathbb{E} \left( \|b(Y_t, t) - b(Y_{t_k}, t_k)\|^2 \right) dt.
\]
Therefore, combining (A.18), (A.20), Lemma 7.2 and Lemma 7.5, we get with the non-anticipative drift Lemma 7.3.

Preliminary lemmas for Theorem 3.7.

□

By (12) or (13). Denote by the relative entropy inequality follows from Remark 4.1 in Huang et al. (2021) and the fact that the second inequality holds due to Remark 4.1 in Huang et al. (2021) and the fact that the fourth inequality follows from 

From the definition of \( Y_t \), we have 

By injecting (A.17), (A.19), and (A.21) into (A.16), we can get the marginal distributions on \( C([0,1], \mathbb{R}^d) \) of \( (Y_t, \tilde{Y}_t)_{t \in [0,1]} \). Therefore, combining (A.18), (A.20), Lemma 7.2 and Lemma 7.5, we get

with the non-anticipative drift \( \tilde{b}_m(\tilde{Y}_t, t) := \sum_{k=0}^{K-1} b_m(\tilde{Y}_{tk}, t_k) \mathbb{1}_{[t_k, t_{k+1})}(t) \), where \( b_m(\tilde{Y}_t, t_k) \) is defined by (12) or (13). Denote by \( \nu^Y_t \) and \( \tilde{\nu}^Y_t \) the marginal distributions on \( C([0,1], \mathbb{R}^d) \) of \( (Y_t, \tilde{Y}_t)_{t \in [0,1]} \). Therefore, combining (A.18), (A.20), Lemma 7.2 and Lemma 7.5, we get

where

the second inequality holds due to Remark 4.1 in Huang et al. (2021) and the fact that \((a+b)^2 \leq 2(a^2 + b^2)\), the second equality holds by the continuous-time interpolation (A.18), and the fourth inequality follows from 

So it remains to estimate the relative entropy \( D_{KL}(\text{Law}(\tilde{Y}_t)||\mu_{t,R}) \). Similar to the proof of the relative entropy \( D_{KL}(\mu_{t,R}||\mu_{t}) \), we need to construct a continuous-time interpolation process \( \{Y_t\}_{t \in [0,1]} \) defined by

with \( C_{2,*} := \frac{\gamma_\sigma^4}{\xi_\sigma^4} + \frac{\gamma_\sigma^2}{\xi_\sigma^4} + \gamma_\sigma \frac{\gamma_\sigma}{\xi_\sigma} \left\{ \left( \frac{M_{2,R}}{\sigma} \right)^2 + \frac{M_{3,R}}{\sigma} \right\} \exp \left( \frac{M_{1,R} \cdot m_{1,R}}{\sigma} \right) \),

with \( \zeta_\sigma = \exp \left( M_{1,R} / \sigma \right) \). By injecting (A.17), (A.19), and (A.21) into (A.16), we can get the desired results.

7.7. **Preliminary lemmas for Theorem 3.7.** First, we introduce Lemmas 7.3-7.5 preparing for the proofs of Theorem 3.7.

**Lemma 7.3.** Assume (P1) and (P2) hold, then

\[
\mathbb{E} \left[ \|X_t\|_2^2 \right] \leq 2(C_{0,*} + d) \exp(2C_{0,*}t).
\]

**Proof.** From the definition of \( X_t \) in (7), we have \( \|X_t\|_2 \leq \sigma \int_0^t \|b(X_u, u)\|_2 du + \sqrt{\sigma} \|B_t\|_2 \). Then, we can get

\[
\|X_t\|_2^2 \leq 2\sigma^2 \left( \int_0^t \|b(X_u, u)\|_2 du \right)^2 + 2\sigma \|B_t\|_2^2
\]
we deduce that
\[ \mathbb{E}[\|X_t\|_2^2] \leq 2t \int_0^t C_{0,\sigma} \mathbb{E}[\|X_u\|_2^2] \, du + 2\|B_t\|_2^2, \]
where the first inequality holds by the inequality \((a + c)^2 \leq 2a^2 + 2c^2\), the last inequality holds by (C1). Thus,
\[ \mathbb{E}[\|X_t\|_2^2] \leq 2t \int_0^t C_{0,\sigma} \left(\mathbb{E}[\|X_u\|_2^2] + 1\right) \, du + 2\|B_t\|_2^2, \]
By the Grönwall inequality, we have
\[ \mathbb{E}[\|X_t\|_2^2] \leq 2C_{0,\sigma} \int_0^t \mathbb{E}[\|X_u\|_2^2] \, du + 2(C_{0,\sigma} + d). \]

Lemma 7.4. Assume (P1) and (P2) hold, then for any \(0 \leq t_1 \leq t_2 \leq 1\),
\[ \mathbb{E}[\|X_{t_2} - X_{t_1}\|_2^2] \leq 2(t_2 - t_1) \left\{ 1 + C_{0,\sigma} \exp(2\sqrt{C_{0,\sigma}} + 1) \right\}. \]

Proof. Using (C1) and the elementary inequality \(2ab \leq a^2 + b^2\), one can derive that for any \(\varepsilon > 0\),
\[ 2\langle x, b(x, t) \rangle \leq \varepsilon \|x\|_2^2 + \frac{\|b(x, t)\|_2^2}{\varepsilon} \leq \varepsilon \|x\|_2^2 + \frac{C_{0,\sigma}}{\varepsilon} (1 + \|x\|_2^2). \]
Letting \(\varepsilon = \sqrt{C_{0,\sigma}}\) yields
\[ \langle x, b(x, t) \rangle \leq \sqrt{C_{0,\sigma}}(1 + \|x\|_2^2). \quad (A.22) \]
From the definition of \(X_t\) in (7), we have
\[ X_t = \sigma \int_0^t b(X_s, s) \, ds + \sqrt{\sigma} \int_0^t dB_s, \quad \forall t \in [0, 1]. \]
On the one hand, by the Itô formula and (A.22), for any \(t \in [0, 1]\), we have
\[
\begin{align*}
1 + \|X_t\|_2^2 &= 1 + 2\sigma \int_0^t \langle X_s, b(X_s, s) \rangle \, ds + \int_0^t \sigma \, ds + 2 \int_0^t \langle X_s, \sqrt{\sigma} dB_s \rangle \\
&\leq 1 + 2 \int_0^t \left( X_s^\top b(X_s, s) + \frac{1}{2} \right) \, ds + 2 \int_0^t \langle X_s, dB_s \rangle \\
&\leq 1 + 2\alpha \int_0^t \left( 1 + \|X_s\|_2^2 \right) \, ds + 2 \int_0^t \langle X_s, dB_s \rangle,
\end{align*}
\]
where \(\alpha := \sqrt{C_{0,\sigma}} + 1/2\). Furthermore, we have
\[ \mathbb{E}[1 + \|X_t\|_2^2] \leq 1 + 2\alpha \int_0^t \mathbb{E}[1 + \|X_s\|_2^2] \, ds. \]
The Grönwall inequality yields
\[ \mathbb{E}[1 + \|X_t\|_2^2] \leq \exp(2\alpha t) = \exp \left( 2\sqrt{C_{0,\sigma}} + 1 \right), \quad \forall t \in [0, 1]. \quad (A.23) \]
On the other hand, by the elementary inequality \((a + b)^2 \leq 2(a^2 + b^2)\) then we get
\[ \mathbb{E}[\|X_{t_2} - X_{t_1}\|_2^2] \leq 2\mathbb{E} \left[ \left( \int_{t_1}^{t_2} \sigma b(X_s, s) \, ds \right)^2 \right] + 2\mathbb{E} \left[ \int_{t_1}^{t_2} \sqrt{\sigma} dB_s \right]^2. \]
Thus, using the Cauchy-Schwarz inequality, Burkholder-Davis-Gundy inequality, (C1) and (A.23), we deduce that
\[
\begin{align*}
\mathbb{E}[\|X_{t_2} - X_{t_1}\|_2^2] &\leq 2\sigma^2 (t_2 - t_1) \int_{t_1}^{t_2} \mathbb{E}[\|b(X_s, s)\|_2^2] \, ds + 2\sigma (t_2 - t_1) \\
&\leq 2C_{0,\sigma} \int_{t_1}^{t_2} \mathbb{E}[1 + \|X_s\|_2^2] \, ds + 2(t_2 - t_1)
\end{align*}
\]
Lemma 7.5. Assume (P1) and (P2) hold, then
\[
\sup_{x \in \mathbb{R}^d, t \in [0,1]} \mathbb{E} \left[ \|b(x, t) - \tilde{b}_m(t, x)\|_2^2 \right] \leq \frac{4d}{m} C^*_{2, \sigma},
\]
where
\[
C^*_{2, \sigma} = \frac{\gamma^4_\sigma}{\xi^4_\sigma} + \frac{\gamma^2_\sigma}{\xi^2_\sigma}, \quad \gamma_\sigma = \left( \frac{M_{2,R}}{\sigma} \right)^2 + \frac{M_{3,R}}{\sigma} \exp \left( \frac{M_{1,R} - m_1 R}{\sigma} \right),
\]
with \( \zeta_\sigma = \exp \left( \frac{M_{1,R}}{\sigma} \right) \).

Proof. Denote two independent sequences of independent copies of \( Z \sim N(0, I_d) \), that is, \( Z = \{Z_1, \ldots, Z_m\} \) and \( Z' = \{Z'_1, \ldots, Z'_m\} \). For notation convenience, we also denote
\[
h := \mathbb{E}_Z \left[ \nabla \tilde{f}_\sigma (x + \sqrt{(1-t)\sigma} Z) \right], \quad h_m := \frac{\sum_{i=1}^m \nabla \tilde{f}_\sigma (x + \sqrt{(1-t)\sigma} Z_i)}{m},
\]
\[
e := \mathbb{E}_Z \left[ \nabla \tilde{f}_\sigma (x + \sqrt{(1-t)\sigma} Z) \right], \quad e_m := \frac{\sum_{i=1}^m \nabla \tilde{f}_\sigma (x + \sqrt{(1-t)\sigma} Z_i)}{m},
\]
\[
h'_m := \frac{\sum_{i=1}^m \nabla \tilde{f}_\sigma (x + \sqrt{(1-t)\sigma} Z'_i)}{m}, \quad e'_m := \frac{\sum_{i=1}^m \nabla \tilde{f}_\sigma (x + \sqrt{(1-t)\sigma} Z'_i)}{m}.
\]
Since \( h - h_m = \mathbb{E} [h'_m - h_m | Z] \), then \( \|h - h_m\|_2 \leq \mathbb{E} [\|h'_m - h_m\|_2^2 | Z] \). Moreover, we have
\[
\mathbb{E} [\|h - h_m\|_2^2] \leq \mathbb{E} \left\{ \mathbb{E} \left[ \|h'_m - h_m\|_2^2 | Z \right] \right\} = \mathbb{E} [\|h'_m - h_m\|_2^2]
\]
\[
= \mathbb{E}_{Z_1, Z'_1} \left[ \| \nabla \tilde{f}_\sigma (x + \sqrt{(1-t)\sigma} Z_1) - \nabla \tilde{f}_\sigma (x + \sqrt{(1-t)\sigma} Z'_1) \|_2^2 \right]
\]
\[
\leq \frac{\sigma (1-t) \gamma^2_\sigma}{m} \mathbb{E}_{Z_1, Z'_1} \left[ \| Z_1 - Z'_1 \|_2^2 \right]
\]
\[
\leq 2d \gamma^2_\sigma \frac{m}{m}, \quad \text{(A.24)}
\]
where the second inequality holds by (P1) and the last inequality follows from the fact that \( Z_1 \) and \( Z'_1 \) are independent standard normal distribution. Similarly, we also have
\[
\mathbb{E} [\|e - e_m\|_2^2] \leq \mathbb{E} [\|e'_m - e_m\|_2^2]
\]
\[
= \mathbb{E}_{Z_1, Z'_1} \left[ \| \tilde{f}_\sigma (x + \sqrt{(1-t)\sigma} Z_1) - \tilde{f}_\sigma (x + \sqrt{(1-t)\sigma} Z'_1) \|_2^2 \right]
\]
\[
\leq \frac{\sigma (1-t) \gamma^2_\sigma}{m} \mathbb{E}_{Z_1, Z'_1} \left[ \| Z_1 - Z'_1 \|_2^2 \right]
\]
\[
\leq 2d \gamma^2_\sigma \frac{m}{m}, \quad \text{(A.25)}
\]
where the second inequality holds due to (P1). Thus, by (A.24) and (A.25), it follows that
\[
\sup_{x \in \mathbb{R}^d, t \in [0,1]} \mathbb{E} [\|h - h_m\|_2^2] \leq \frac{2d \gamma^2_\sigma}{m}, \quad \text{(A.26)}
\]
\[
\sup_{x \in \mathbb{R}^d, t \in [0,1]} \mathbb{E} [\|e - e_m\|_2^2] \leq \frac{2d \gamma^2_\sigma}{m}. \quad \text{(A.27)}
\]
Then, by (P1) and (P2), some simple calculations yield that
\[
\|b(x, t) - \tilde{b}_m(x, t)\|_2 = \left\| \frac{h}{e} - \frac{h_m}{e_m} \right\|_2
\]
Then, by (A.30) and 

\[ \frac{|e_m|}{|e_{m_1}|} \leq \frac{\|h\|_2 |e_m - e| + \|h - h_m\|_2 |e|}{|e_m|} \leq \frac{\gamma \sigma |e_m - e| + \|h - h_m\|_2 |e|}{\xi^2} \]  

(A.28)

Recall that \( \zeta \sigma = \exp(M_{1, R}/\sigma) \) with

\[ M_{1, R} := \max_{|x| \leq R} \left\{ -V(x) + \frac{\|x\|_2^2}{2} \right\} , \]

then \( \tilde{f}_\sigma(x) \leq \zeta \sigma \) for any \( x \in B_R \). Further, by (A.28), it follows that for all \( x \in \mathbb{R}^d \) and \( t \in [0, 1] \),

\[ \|b(x, t) - \tilde{b}_m(x, t)\|_2^2 \leq 2 \frac{\gamma^2 \sigma |e_m - e|^2 + \xi^2 \|h - h_m\|_2^2}{\xi^2} \]  

(A.29)

Then, combining (A.26)-(A.27) and (A.29), it follows that

\[ \sup_{x \in \mathbb{R}^d, t \in [0, 1]} \mathbb{E} \left[ \|b(x, t) - \tilde{b}_m(t, x)\|_2^2 \right] \leq \frac{4d}{m} C_{2, \sigma}^* \]

where

\[ C_{2, \sigma}^* := \frac{\gamma^4}{\xi^4} + \frac{\gamma \sigma^2}{\xi}, \quad \frac{\gamma \sigma}{\xi} = \left\{ \left( \frac{M_{2, R}}{\sigma} \right)^2 + \frac{M_{3, R}}{\sigma} \right\} \exp \left( \frac{M_{1, R} - m_{1, R}}{\sigma} \right), \quad \zeta \sigma = \exp \left( \frac{M_{1, R}}{\sigma} \right) . \]

Lemma 7.6. Assume (P1) and (P2) hold, then for any \( k = 0, 1, \ldots, K \),

\[ \mathbb{E} \left[ \|\tilde{Y}_{tk}\|_2^2 \right] \leq \frac{6 \gamma^2}{\xi^2} + 3d. \]

Proof. Define \( \Theta_{k, t} := \tilde{Y}_{tk} + \sigma(t - t_k) \tilde{b}_m(\tilde{Y}_{tk}, t_k) \), hence, we get \( \tilde{Y}_t = \Theta_{k, t} + \sqrt{\sigma}(B_t - B_{tk}) \), where \( t_k \leq t \leq t_{k+1} \) with \( k = 0, 1, \ldots, K - 1 \). By (P1) and (P2), it follows that for all \( x \in \mathbb{R}^d \) and \( t \in [0, 1] \),

\[ \|b(x, t)\|_2^2 \leq \frac{\gamma^2}{\xi^2}, \quad \|\tilde{b}_m(x, t)\|_2^2 \leq \frac{\gamma^2}{\xi^2} \]  

(A.30)

Then, by (A.30) and \( s = 1/K \), we have

\[ \|\Theta_{k, t}\|_2^2 = \|\tilde{Y}_{tk}\|_2^2 + \sigma^2(t - t_k)^2 \|\tilde{b}_m(\tilde{Y}_{tk}, t_k)\|_2^2 + 2\sigma(t - t_k) \langle \tilde{Y}_{tk}, \tilde{b}_m(\tilde{Y}_{tk}, t_k) \rangle \leq (1 + s\sigma)\|\tilde{Y}_{tk}\|_2^2 + \frac{\gamma^2(s\sigma + s^2\sigma^2)}{\xi^2}. \]

Further, we can get

\[ \mathbb{E} \left[ \|\tilde{Y}_t\|_2^2 \right] = \mathbb{E} \left[ \|\Theta_{k, t}\|_2^2 \right] \|\tilde{Y}_{tk}\|_2^2 + \sigma(t - t_k) \]  

\[ \leq (1 + s\sigma)\|\tilde{Y}_{tk}\|_2^2 + \frac{\gamma^2(s\sigma + s^2\sigma^2)}{\xi^2} + s\sigma d. \]

Therefore,

\[ \mathbb{E} \left[ \|\tilde{Y}_{tk+1}\|_2^2 \right] \leq (1 + s\sigma)\mathbb{E} \left[ \|\tilde{Y}_{tk}\|_2^2 \right] + \frac{\gamma^2}{\xi^2} + s\sigma d. \]

Since \( \tilde{Y}_{t_0} = 0 \), then by induction, we have

\[ \mathbb{E} \left[ \|\tilde{Y}_{tk+1}\|_2^2 \right] \leq e^{(k+1)s\sigma} \left( d + \frac{(1 + s\sigma)\gamma^2}{\xi^2} \right) \leq e \left( d + \frac{2\gamma^2}{\xi^2} \right) \leq \frac{6\gamma^2}{\xi^2} + 3d. \]

\[ \square \]
7.8. Proof of Theorem 3.7.

**Proof.** From the definitions of $\bar{Y}_{t_k}$ and $X_{t_k}$, we have

\[
\|\bar{Y}_{t_k} - X_{t_k}\|^2 \leq \|\bar{Y}_{t_k} - X_{t_k}\|^2 + \left( \int_{t_{k-1}}^{t_k} \sigma \|b(X_u, u) - \tilde{b}_m(\bar{Y}_{t_k}, t_{k-1})\| \, du \right)^2
\]

\[
+ 2\sigma \|\bar{Y}_{t_k} - X_{t_k}\| \left( \int_{t_{k-1}}^{t_k} \|b(X_u, u) - \tilde{b}_m(\bar{Y}_{t_k}, t_{k-1})\| \, du \right)
\]

\[
\leq (1 + s) \|\bar{Y}_{t_k} - X_{t_k}\|^2 + (1 + s) \int_{t_{k-1}}^{t_k} \|b(X_u, u) - \tilde{b}_m(\bar{Y}_{t_k}, t_{k-1})\|^2 \, du
\]

\[
\leq (1 + s) \|\bar{Y}_{t_k} - X_{t_k}\|^2 + 2(1 + s) \int_{t_{k-1}}^{t_k} \|b(X_u, u) - \tilde{b}_m(\bar{Y}_{t_k}, t_{k-1})\|^2 \, du
\]

\[
+ 2s(1 + s) \|b(\bar{Y}_{t_k}, t_{k-1}) - \tilde{b}_m(\bar{Y}_{t_k}, t_{k-1})\|^2
\]

\[
\leq (1 + s) \|\bar{Y}_{t_k} - X_{t_k}\|^2 + 4C_{2,\sigma}^2 s(1 + s) \int_{t_{k-1}}^{t_k} \|X_u - X_{t_k}\|^2 \, du
\]

\[
+ 4C_{2,\sigma}^2 s(1 + s) \|\bar{Y}_{t_k} - X_{t_k}\|^2 + 4C_{2,\sigma}^2 s(1 + s)^2
\]

\[
+ 2s(1 + s) \|b(\bar{Y}_{t_k}, t_{k-1}) - \tilde{b}_m(\bar{Y}_{t_k}, t_{k-1})\|^2
\]

\[
\leq (1 + s + C_{2,\sigma}^2 s(s + s^2)) \|\bar{Y}_{t_k} - X_{t_k}\|^2 + 4C_{2,\sigma}^2 (1 + s) \int_{t_{k-1}}^{t_k} \|X_u - X_{t_k}\|^2 \, du
\]

\[
+ 4C_{2,\sigma}^2 (s^2 + s^3) + 2s(1 + s) \|b(\bar{Y}_{t_k}, t_{k-1}) - \tilde{b}_m(\bar{Y}_{t_k}, t_{k-1})\|^2
\]

\[
\leq (1 + s + C_{2,\sigma}^2 s(s + s^2)) \|\bar{Y}_{t_k} - X_{t_k}\|^2 + H(s, \sigma) + 4C_{2,\sigma}^2(s^2 + s^3)
\]

\[
+ 2s(1 + s) \|b(\bar{Y}_{t_k}, t_{k-1}) - \tilde{b}_m(\bar{Y}_{t_k}, t_{k-1})\|^2
\]

where $H(s, \sigma) := 16C_{2,\sigma}^2(s^2 + s^3) \{1 + C_{0,\sigma}\exp(2\sqrt{C_{0,\sigma}} + 1)\} + 4C_{2,\sigma}^2(s^2 + s^3)$ follows from Lemma 7.4, and the last inequality holds by Lemma 7.5. Owing to $\bar{Y}_{t_0} = X_{t_0} = 0$, we can conclude that there exists a constant $C_\sigma > 0$ such that

\[
\mathbb{E} \left[ \|\bar{Y}_{t_K} - X_{t_K}\|^2 \right] \leq \left( \frac{1 + s + 8C_{2,\sigma}^2(s + s^2)K - 1}{s + 8C_{2,\sigma}^2(s + s^2)} \right) \left[ H(s, \sigma) + \frac{8s(1 + s)d}{m} C_{2,\sigma}^\ast \right]
\]

\[
\leq C_\sigma \left( sC_{3,\sigma}^\ast + \frac{16d}{m} C_{2,\sigma}^\ast \right),
\]

where

\[
C_\sigma := \exp(1 + 16C_{2,\sigma}^2), \quad C_{3,\sigma}^\ast := 40C_{2,\sigma}^2 + 32C_{2,\sigma}^2 C_{0,\sigma} \exp(2\sqrt{C_{0,\sigma}} + 1).
\]
Therefore,

\[ W_2^2(\text{Law}(\tilde{Y}_{t_K}), \mu_\sigma) \leq C_\sigma \left( sC_3^* + \frac{16d}{m}C_2^* \right). \]

It completes the proof. \( \square \)

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