The Painlevé test for $\mathbb{C}P^{N-1}$ sigma models

P P Goldstein$^1$ and A M Grundland$^{2,3,4}$

$^1$Theoretical Physics Division, National Centre for Nuclear Research, Hoza 69, 00–681 Warsaw, Poland
$^2$Department of Mathematics and Computer Science, Université du Québec, Trois-Rivières. CP500 (QC) G9A 5H7, Canada
$^3$Centre de Recherches Mathématiques. Université de Montréal. Montréal CP6128 (QC) H3C 3J7, Canada
$^4$The author to whom any correspondence should be addressed.

E-mail: Piotr.Goldstein@ncbj.gov.pl and grundlan@crm.umontreal.ca

Keywords: sigma models, integrability, singularity analysis, painlevé property

Abstract

We test the $\mathbb{C}P^{N-1}$ sigma models for the Painlevé property. While the construction of finite action solutions ensures their meromorphicity, the general case requires testing. The test is performed for the equations in the homogeneous variables, with their first component normalised to one. No constraints are imposed on the dimensionality of the model or the values of the initial exponents. This makes the test nontrivial, as the number of equations and dependent variables are indefinite. The $\mathbb{C}P^{N-1}$ system proves to have a $(4N - 5)$-parameter family of solutions whose only movable singularities are poles, while the order of the investigated system is $4N - 4$. The remaining degree of freedom, connected with an extra negative resonance, may correspond to a branching movable essential singularity. An example of such a solution is provided.

1. Introduction

Numerous physical applications of models with effective Lagrangians, in particular the $\mathbb{C}P^{N-1}$ sigma models [4, 5, 14, 27, 29, 36, 39], make these models an interesting subject of study [15–17]. In particular, these models have been shown to play an essential role in several applications to nonlinear phenomena in such areas of physics as quantum field theory, string theory (both bosonic [26, 35] and superstrings) [6], statistical physics, e.g. the Ising model [31], gauge field theories, e.g. the reduction of the self-dual Yang–Mills equations to the Ernst model for cylindrical gravitational waves [1, 3], phase transitions (e.g. the growth of crystals, deformations of membranes [13, 33]), coherent states obtained via the $\mathbb{C}P^{N-1}$ sigma models [18, 23], fluid dynamics e.g. the motion of boundaries between regions of different densities and viscosities [9]. In biochemistry and biology there are applications such as biological membranes and vesicles, for example long protein molecules [14, 28, 34] and the Canham–Helfrich–Evans membrane models [24, 27]. These macroscopic models can be derived from microscopic ones and allow us to explain basic features and equilibrium shapes both for biological membranes and for liquid interfaces [37]. In mathematics, $\mathbb{C}P^N$ models have been shown to play a pertinent role for the systematic description of surfaces immersed in Lie algebras and isomonodromic deformations in connection with surfaces and Painlevé type equations [7, 19]. The question of the integrability of the equations governing these models has found an apparently positive answer in the works of Din and Zakrzewski [17]. Moreover, the linear spectral problem is known for them, so (in principle) the initial problem may be solved by the inverse scattering method. However the above results only concern systems with finite action. On the other hand, if we are interested in the dynamics of the systems, we start from the corresponding Euler–Lagrange (EL) equations, which allow for a much larger class of solutions. Singularities are an intrinsic property of nonlinear equations and some of them may make the action infinite. A natural question arises, as to whether the equations remain integrable if we remove the assumption of finite action. In the present paper we will discuss this question and provide a self-contained approach to the subject.

The first approach which we try when testing a system of equations for integrability is usually the Painlevé test in the form introduced in [2], or its generalisation to partial differential equations (PDEs) [38], with possible further refinements (as discussed in [10, 11, 32], which provide a comprehensive review of the method).
In our case, the Painlevé test entails extra difficulties due to the fact that the dimensionality of the \(\mathbb{CP}^{N-1}\) model and the number of equations are arbitrary. Nevertheless, the test can be carried out (see section 3).

In what follows, section 2 contains a short summary of \(\mathbb{CP}^{N-1}\) models and various methods of their description. We conclude that section with selecting the description (system of PDEs) suitable for the Painlevé test. In section 3 we perform the test, obtaining a ‘nearly general’ local solution in the form of a Laurent series. By ‘nearly general’ we mean that our solution provides \(4N - 5\) out of the \(4(N - 1)\) first integrals in the general solution, i.e. one integral fewer than the order of the system. Section 4 contains a discussion of the missing first integrals. A counterexample, i.e. an example of the non-Painlevé behaviour, is given in the form of a solution which has an essential singular manifold with branching. The manifold depends on four parameters (although not on an arbitrary function), which means that the position of the singularity depends on the initial conditions.

2. \(\mathbb{CP}^{N-1}\) sigma models

Sigma models describe complex systems by a simple Lagrangian defined in terms of an effective field which lies in an appropriate space, while the complexity remains in the metrics of the space.

\[
\mathcal{L} = \sum_{i,j=0}^{\infty} g_{i,j} dz_i dz_j,
\]

where \(z_i, \bar{z}_i\) represent the field variables in \(\mathbb{C}^N\), while \(g_{ij}\) is the metric tensor. A bar over a symbol denotes its complex conjugate.

The models prove to be rich in interesting properties provided that the metric depends on the fields, i.e. the model is nonlinear. Even simple nonlinear cases, like the \(\mathbb{CP}^{N-1}\) models, have many applications, from two dimensional gravity to biological membranes \([8, 20, 27]\). In these models the independent variables \(\xi^1, \xi^2\) take values in the Riemann sphere or in a 2D Minkowski space, \(z \in \mathbb{R}^N\), while the differential in (1) is expressed in terms of the \(z\)-dependent covariant derivatives \(D_{\mu}\) by

\[
D_{\mu}z = \partial_{\mu}z - (z^\dagger \cdot \partial_{\mu}z)z, \quad \partial_{\mu} = \partial_{\xi^\mu}, \quad \mu = 1, 2.
\]

producing a Lagrangian density of the form

\[
\mathcal{L} = \frac{1}{4}(D_{\mu}z)^\dagger \cdot (D_{\mu}z),
\]

where the convention of summation over repeating Greek indices is assumed, and \(z^\dagger\) and \(z^\dagger\) are complex unit vectors in \(\mathbb{C}^N\), a dagger denotes the Hermitian conjugate, while \(\partial\) and \(\bar{\partial}\) are the derivatives with respect to \(\xi = \xi^1 + i\xi^2\) and \(\bar{\xi} = \xi^1 - i\xi^2\) respectively. The normalisation of \(z\) requires that

\[
z^\dagger \cdot z = 1, \quad z = (z_0, \ldots, z_{N-1}).
\]

The EL equations corresponding to the Lagrangian (3)

\[
D_{\mu}D_{\mu}z + (D_{\mu}z)^\dagger \cdot (D_{\mu}z) = 0,
\]

are simple, but they are not suitable for testing the Painlevé property; due to the normalisation (4), a pole of \(z\) has to correspond to a zero of \(z^\dagger\), at least for real \(\xi^N\). For the same reason, we do not analyse even simpler equations satisfied by the rank-1 projectors \(P = z \otimes z^\dagger\), namely

\[
[\partial \bar{\partial} P, P] = 0,
\]

The necessary freedom is achieved if we use the homogeneous unnormalised field variables \(f\), such that

\[
z = f / (f^\dagger \cdot f)^{1/2}, \quad \mathbb{C} \ni \xi \mapsto f(\xi, \bar{\xi}) = (f_0(\xi, \bar{\xi}), \ldots, f_{N-1}(\xi, \bar{\xi})) \in \mathbb{C}^N \setminus \{0\},
\]

whose dynamics are governed by the unconstrained EL equations

\[
\left[ 1 - \frac{f \otimes f^\dagger}{f^\dagger \cdot f} \right] \left[ \partial \bar{\partial} f - \frac{1}{f^\dagger \cdot f} \left( (f^\dagger \cdot \bar{\partial} f) \partial f + (f^\dagger \cdot \partial f) \bar{\partial} f \right) \right] = 0,
\]

The way in which these vector functions are constructed makes them elements of a Grassmannian space \(\text{Gr}(1, \mathbb{C}^N)\) \([39]\) and suggests that equations (8) are invariant under multiplication of \(f^\dagger\) by any scalar function (which may easily be checked by direct calculation). This property leaves too much freedom for the shape of possible singularities. However if we normalise the homogeneous variables in such a way that the first component \(f_0\) is equal to 1, we eventually obtain a system of equations suitable for the Kovalevsky–Gambier analysis, commonly known as the Painlevé test. The equations in terms of the affine variables \(w_i = (w_1, \ldots, w_{N-1})\), such that

\[
w_i = f_i / f_0, \quad i = 1, \ldots, N - 1 \quad (\text{generically } f_0 \neq 0).
\]
read
\[
\left( 1 + \sum_{l=1}^{N-1} \bar{w}_i w_l \right) \partial_\xi \bar{w}_i - \sum_{l=1}^{N-1} \left( \bar{w}_i \partial_\xi w_l \bar{w}_i + \bar{w}_i \partial_\xi w_l \bar{w}_i \right) = 0,
\]
\[
\left( 1 + \sum_{l=1}^{N-1} w_i \bar{w}_l \right) \partial_\xi \bar{w}_i - \sum_{l=1}^{N-1} \left( \bar{w}_i \partial_\xi \bar{w}_l \bar{w}_i + w_i \partial_\xi \bar{w}_l \bar{w}_i \right) = 0,
\]
where the complex conjugates of (10a) have been written separately as (10b) because the complex conjugation will no longer link the variables \(w_i\) with \(\bar{w}_i\) when we extend the independent variables analytically to the double complex plane \(\mathbb{C}^2\) (as it is done in the Painlevé test). Therefore, in what follows, we put ‘complex conjugation’ in quotation marks while naming the symmetry which turns the unbarred quantities into the barred ones and vice versa.

Equations (10a) (10b) will be the subject of further analysis. They constitute a system of \(2(N - 1)\) second-order PDEs, which requires \(4(N - 1)\) first integrals to build the general solution.

3. The Painlevé test

To perform the test, we look for the solution of system (10a), (10b), extended to the double complex plane \((\xi, \bar{\xi}) \in \mathbb{C}^2\), in the form of a Laurent series about a movable non-characteristic singularity manifold
\[
\Phi(\xi, \bar{\xi}) = \bar{\xi} - \varphi(\xi) \quad \text{(Kruskal's simplification),}
\]
where the function \(\varphi\) defining the singularity manifold is a holomorphic function of \(\xi\), while the coefficients of the expansion are analytic in their arguments \((\xi, \bar{\xi})\).

The condition of being non-characteristic excludes the surfaces \(\xi = 0\) and \(\bar{\xi} = 0\), which in turn eliminates locally holomorphic and locally antiholomorphic functions \(w, \bar{w}\), including the solutions of Din and Zakrzewski [15, 17]. On the other hand, the selection of non-characteristic singularity manifolds makes possible both the Kruskal simplification [25] and the assumption \(\varphi'(\xi) \neq 0\).

In the series below, we adopt the notation in which a superscript for \(\Phi\) is simply an exponent, while a superscript for a dependent variable, e.g. \(w_i^n\) denotes the \(n\)-th order coefficient in the Laurent expansion of \(w_i\). Additionally, it is convenient to extend the notation to negative \(n\), assuming
\[
w_i^0 = 0 \quad \text{whenever} \quad n < 0, \quad \text{for all} \quad i = 1, \ldots, N - 1.
\]
We do not limit the number of dependent variables \(w_i\) and allow \(a\) priori the possibility that the initial exponents of each \(w_i\) may be different. Thus the Laurent expansion has the form
\[
w_i = \sum_{n=0}^{\infty} \sum_{\alpha_i=0}^{\infty} w_i^n (\xi) \Phi^{n-\alpha_i},
\]
where for all \(i\) we have \(w_i^0 = 0\) and \(\bar{w}_i^0 = 0\) (otherwise we would start from higher-order terms). We also assume that \(\alpha_i > 0\) and \(\beta_i > 0\) for all \(i\).

Let us substitute (13) into our equations (10a) (10b). As these equations are of 3rd degree, the resulting equations contain quadruple sums (a sum over the components of \(w\) and products of 3 sums of the Laurent series). We first rearrange the latter sums \(\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \rightarrow \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty}\), where \(k = m + n + p\). Then we shift the dummy index \(k\) in such a way that all terms of the same \(k\) become proportional to the same powers of \(\Phi\) (the range of \(k\) remains unchanged thanks to our convention (12)). Next we require that the coefficients of all powers in \(\Phi\) vanish.

Under the assumption that \(\alpha_i > 0\), \(\beta_i > 0\) for all \(i = 0, \ldots, N - 1\), there is no balance of terms with exponents of different form in the lowest order. Therefore, the initial exponents are obtained from the equations satisfied by the coefficients of the lowest-order terms \(k = 0\) rather than those satisfied by the exponents of those terms in the lowest-order, we obtain for \(i = 1, \ldots, N - 1\)
\[
w_i^0 \varphi' \alpha_i \sum_{l=1}^{N-1} \bar{w}_l^0 w_l^0 (2 \alpha_l - \alpha_i - 1) = 0
\]
where the prime denotes the derivative with respect to $\xi$ (we have omitted the $\xi$-dependence of $\varphi$ and all the $w$'s).

Equations (14), divided by $w_1^0 \varphi' + \beta_i$ or $w_1^{0} \varphi' + \beta_i$ constitute two separate systems of linear equations: one for $\alpha_1, \ldots, \alpha_{N-1}$ and a similar one for $\beta_1, \ldots, \beta_{N-1}$. It is evident that

$$\alpha_1 = \ldots = \alpha_{N-1} = \beta_1 = \ldots = \beta_{N-1} = 1$$

(15)
solves both systems (14). A question arises as to whether this is the only solution in which all $\alpha_i$ and $\beta_i$ are positive. We will show that this is indeed the case. The proof below is performed for the system (14a). The proof for (14b) is identical.

**Proof.** Suppose that our movable singularity manifold intersects the plane $\bar{\xi} = \xi^s$ (the plane of real $\xi^s$ and $\bar{\xi}^s$), where the asterisk denotes the actual complex conjugation. Consider the matrix of coefficients, $B_n = \sum_{i=1}^{N-1} w_i^{0} w_i^{0}$, where the prime denotes the derivative with respect to $w_i^{0}$. For each $j = 1, \ldots, N - 1$, all elements of the $j$-th column are identical, with the exception of the diagonal element. If we add all the columns to the first one and then subtract the first row of the resulting matrix from each of the other rows, we obtain a triangular matrix, with zeros everywhere except for the first row and the main diagonal. The diagonal has $\sum_{i=1}^{N-1} w_i^{0} w_i^{0}$ in the first row and minus this sum in all other rows. Hence the determinant of the system can be calculated explicitly

$$\det B = -\left( \sum_{i=1}^{N-1} w_i^{0} w_i^{0} \right)^{N-1}.$$  

(16)

On the complex plane $\bar{\xi} = \xi^s$ the barred $w_i^{0}$'s are indeed the complex conjugates of their unbarred counterparts. Hence all the components of the sum in (16) are positive on this plane, as we have assumed that $w_i^{0}$ and $\bar{w}_i^{0}$ are nonzero for all $i$. Continuity of these coefficients ensures that they remain positive in some neighbourhood of this plane. Thus the determinant (16) is nonzero (negative) and the solution (15) is unique in some domain. However the initial exponents have to be independent of $(\xi^s, \bar{\xi}^s)$, hence the solution (15) is unique in the neighbourhood of the whole singularity manifold.

Putting all $\alpha_i$ and $\beta_i$ equal to 1, as in (15), we obtain the recurrence relations at $k > 0$. For each $k = 1, 2, \ldots$ they have the form of a system of $2(N - 1)$ linear algebraic equations with the unknowns $w_1^k, \ldots, w_{N-1}^k$ and $\bar{w}_1^k, \ldots, \bar{w}_{N-1}^k$

$$(k - 1) k \varphi \left( \sum_{i=1}^{N-1} w_i^{0} w_i^{0} w_i^{k} + 2 \sum_{i=1}^{N-1} w_i^{0} w_i^{0} w_i^{k} \right)$$

$$= -\varphi \sum_{i=1}^{N-1} k - n \sum_{p=0}^{k-n} (n - 1)(n - 2p) w_i^{0} w_i^{k-n-p} w_i^{p} + 2 n w_i^{0} w_i^{k-n} w_i^{n}$$

$$+ \sum_{i=1}^{N-1} \sum_{n=1}^{k-n} \sum_{p=0}^{k-n} \bar{w}_i^{0} \bar{w}_i^{k-n-p} \left( (n - p) w_i^{0} (w_i^{p}) + (n - 1)(w_i^{p})' w_i^{0} \right)$$

$$+ (k - 3) (k - 4) \varphi \bar{w}_i^{k-2} - (k - 4) (w_i^{k-3})'$$

(17)

and a similar set of $N - 1$ equations for the ‘complex conjugates’ $w_i^k, \ldots, w_{N-1}^k$. Note that the unknowns $\bar{w}_1^k, \ldots, \bar{w}_{N-1}^k$ are absent from the left-hand sides (lhs) of (17) and similarly, the unknowns $w_1^k, \ldots, w_{N-1}^k$ are absent from the lhs of the conjugate system (although the systems remain coupled with each other through the right-hand sides (rhs)). This absence means that the matrix of coefficients of the complete linear system is a direct sum of two square matrices and its determinant is a product of their determinants. The Fuchs indices or resonances (we use the second name to avoid misunderstandings in our multi-index notation) are calculated from the requirement that the determinant vanish.

The first matrix has the elements

$$A_{ij}^k = \varphi k \left( (k - 1) \delta_{ij} \left( \sum_{i=1}^{N-1} w_i^{0} w_i^{0} \right) - 2 w_i^{0} w_i^{0} \right).$$

(18)

The second component of the direct sum is its ‘complex conjugate’ (further abbreviated to ‘c.c.’). The instances in which the determinant of their direct sum vanishes are listed in table 1.
(Continued.)

| \( k = -1 \) | 2 | If each of the \( i \)-th rows, where \( i = 1, \ldots, N - 1 \), is multiplied by \( \tilde{w}_i^0/\tilde{w}_1^0 \) and the products are added together, the result is zero. The second component of the direct sum is the ‘c.c.’ of the first one. |

Altogether we have \( 4N - 4 \) zeros. This number is equal to the total order of the system of PDEs. Hence there are no more resonances.

We now test the compatibility of the resonances by checking whether the rhs of equations (17) have the same linear dependence between rows as their lhss.

For \( k = 0 \), all terms on the rhs contain \( w \) with a negative superscript, which according to our convention (12) means that they are equal to zero. Hence the whole rhs is equal to zero, as it should be. Consequently, this leaves room for \( 2(N - 1) \) arbitrary functions of \( \xi \) (first integrals).

For \( k = 1 \), the rhs of the \( i \)-th equation, \( i = 1, \ldots, N - 1 \), reduces to \( w_i^0 \sum_{i=1}^{N-1} \tilde{w}_i^0 (w_i^0)' \), i.e. all rows are proportional. For instance, we may take the first row and write each of the rows \( i = 2, \ldots, N - 1 \) as equal to the first row multiplied by \( w_i^0 /w_1^0 \). This satisfies the linear dependence condition of table 1 and leaves room for another \( 2(N - 2) \) arbitrary functions.

The above verification of compatibility cannot be performed for negative zeros. One of the two zeros \( k = -1 \) corresponds to the arbitrariness of \( \varphi(\xi) \). The compatibility of the other zero at \( k = -1 \) remains unknown.

The verified zeros allow us to introduce a total of \( 4N - 6 \) arbitrary functions of \( \xi \). These are \( w_i^0 \) and \( \tilde{w}_i^0 \) for \( i = 1, \ldots, N - 1 \) and \( w_1^0 \) and \( \tilde{w}_1^0 \) for \( i = 2, \ldots, N - 1 \). Together with the arbitrary singularity manifold \( \varphi \) (corresponding to one of the two zeros \( k = -1 \)), they constitute a set of \( 4N - 5 \) first integrals. There remains the second zero \( k = -1 \), which is the cause of the missing \( (4N - 4) \)-th first integral. This problem will be addressed in the next section.

4. The question of the double resonance at \( k = -1 \)

The negative resonances, except for a single \( k = -1 \) resonance, correspond to essential singular points. In the \( \mathbb{CP}^{N-1} \) model, they are connected with the coupling between \( w \) and \( \tilde{w} \) (if not for the coupling we would have two separate systems, each possessing a single resonance \( k = -1 \)). A singularity connected with the phase may indeed be essential. A question arises: does the essential singularity introduce multivaluedness in the solution or not.

The authors tried to apply the perturbative Painlevé analysis of [12] to the \( \mathbb{CP}^1 \) model. Up to the third order in the perturbation of the Laurent series (13) all the resonances are compatible. However the order at which an incompatibility may occur is difficult to predict. Being unable to prove the Painlevé or non-Painlevé property by any systematic method, we limit ourselves to a counterexample.

An example of a solution (an envelope solitary wave) which has branching at a point dependent on the initial conditions has been derived by Lie group analysis and the corresponding symmetry reduction of the \( \mathbb{CP}^1 \) model in [21, 22]. A typical solution of the kind reads

\[
\begin{align*}
 w(\xi, \tilde{\xi}) &= R \exp[i(\xi/a - f)], \\
 \tilde{w}(\xi, \tilde{\xi}) &= R \exp[-i(\xi/a - f)], \quad \text{where} \\
 R &= \pm \sqrt{(p - 1) \cosh \theta + p + 1}, \\
 f &= \arctan \left( \frac{p + 1}{2 \sqrt{-p}} \tanh g \right) + \frac{(p + 2 \sqrt{-p} - 1) \chi - 2 \sqrt{-p} \chi_0}{2(p - 1)} + d, \\
 g &= \frac{(p + 1) (\chi - \chi_0)}{2(p - 1)}, \quad \text{where} \quad \chi = \frac{\xi}{a} - \frac{\tilde{\xi}}{b}. \tag{19}
\end{align*}
\]

To ensure that \( w \) and \( \tilde{w} \) are complex conjugates of each other when the remaining quantities are real, it is usually assumed that \( p < 1 \). However the solution is valid for any \( p \).

This solution (as well as several other solutions in the form of elliptic functions) is associated with multileaf surfaces [21, 22]. It is singular for \( \chi = \chi_0 = (k + 1/2)i\pi, \quad k \in \mathbb{Z} \). For these values of \( \chi \), the argument of \( \arctan \) in (19) becomes infinite, which results in branching (i.e. the multivaluedness of the \( \arctan \) function). These singularities do not lie on characteristics (\( \xi = \text{const} \) and \( \tilde{\xi} = \text{const} \)), which makes them proper for the analysis. The position of the singularities depends on four parameters: \( p, \ a, \ b, \ \chi_0 \), and thus also on the initial conditions, which contradicts the usual understanding of the Painlevé property. However, the authors are aware
that a more constructive answer to the question of compatibility at the negative resonance would be provided by a non-Painlevé solution with its position dependent on an arbitrary function rather than a few parameters. We do not have such a solution.

The action integral for the example (19) is not finite, hence it is compatible with the theorem of Din and Zakrzewski [15, 17, 39]. Neither does it contradict the classical result of [2], because (19) cannot be obtained as a solution of a Gelfand-Levitan–Marchenko equation [30] with a finite integral kernel.

The $\mathbb{CP}^1$ model is a limit case of $\mathbb{CP}^{N-1}$ models, where all but one affine coordinates (and all but one ‘complex conjugates’) tend to zero. Thus the absence of the Painlevé property in the $\mathbb{CP}^1$ model infers its absence for all $\mathbb{CP}^{N-1}$ models.

Conclusion

We have shown that the equations governing the behaviour of $\mathbb{CP}^{N-1}$ models, without the constraint of finite action, may have solutions with movable singularities in the form of pole manifolds. The order of the poles is equal to one for all dependent variables (the calculation, based on the usual assumption of the Painlevé test, i.e. negative initial exponents, has eliminated poles of other orders). For the $\mathbb{CP}^{N-1}$ model equations, the Laurent series about a pole manifold is consistent at all $4N − 5$ nonnegative resonances. In this way, it provides a family of solutions with $4N − 5$ parameter functions (first integrals) within the domain of convergence of the series. However, branching may still occur at essential singular points. We have provided an example of a solution which is multivalued in the neighbourhood of a sequence of non-characteristic movable singular manifolds. Their position depends on the initial conditions through four parameters. It would be desirable to find a deformation of such solutions turning them into solutions depending on an arbitrary function. The result shows that the $\mathbb{CP}^{N-1}$ models admit solutions which would branch (multifurcate) at the points at which the action integral diverges. It seems to extend the range of applicability of these models to physical and biological phenomena.

The Painlevé analysis is nontrivial for these models due to the indefinite number of equations and dependent variables.

Acknowledgments

AMG’s work was supported by a research grant from NSERC of Canada. P P G wishes to thank the Centre de Recherches Mathématiques (Université de Montréal) for the NSERC financial support provided for his visit to the CRM.

ORCID iDs

P P Goldstein © https://orcid.org/0000-0002-0236-5332
A M Grundland © https://orcid.org/0000-0003-4457-7656

References

[1] Ablowitz M J, Chakravaty S and Halburd R 2003 Integrable systems and reduction to self-dual Yang–Mills J. Math. Phys. 44 3147–73
[2] Ablowitz M J, Ramani A and Segur H 1980 A connection between nonlinear evolution equations and ordinary differential equations of P-type J. Math. Phys. 21 715–21 1006–1015
[3] Amit D 1978 Field Theory, The Renormalization Group and Critical Phenomena (New York: McGraw-Hill)
[4] Babelon O 2007 A Short Introduction to Classical and Quantum Integrable Systems (Paris: Univ. Paris 6)
[5] Babelon O, Bernard D and Talon M 2003 Introduction to Classical Integrable Systems (Cambridge: Cambridge University Press)
[6] Barret J, Gibbons G W, Perry M J and Ruback P 1994 Kleinian geometry and N = 2 superstring Int. J. Mod. Phys. A 9 1457–93
[7] Bobenko A and Eitner U 2000 Painlevé equations in differential geometry of surfaces Lect. Notes Math. 1753 (Berlin: Springer)
[8] Carrol R and Konopelchenko B 1996 Generalized Weierstrass–Enneper inducing conformal immersions and gravity Int. J. Mod. Phys. A 11 1183–216
[9] Chavolin J, Joanny J F and Zinn-Justin J 1989 Liquides et Interfaces (Amsterdam: Elsevier)
[10] Conte R 1999 The Painlevé approach to nonlinear ordinary differential equations The Painlevé Property One Century Later ed R Conte (New York: Springer Verlag) 77–180 ch 3
[11] Conte R and Musette M 2008 The Painlevé Handbook (Dordrecht: Springer)
[12] Conte R, Fordy A P and Pickering A 1993 A perturbative Painlevé approach to nonlinear differential equations Physica D 69 33–58
[13] David F, Ginsparg P and Zinn-Justin Y 1996 Fluctuating Geometries in Statistical Mechanics and Field Theory (Amsterdam: Elsevier)
[14] Davydov A 1999 Solitons in Molecular Systems (New York: Kluver)
[15] Din M A and Zakrzewski W J 1981 Interpretation and further properties of general classical $\mathbb{CP}^{N-1}$ solutions Nucl. Phys. B 182 151–7
[16] Din M A, Horvath Z and Zakrzewski W J 1984 The Riemann–Hilbert problem and finite action $\mathbb{CP}^{N-1}$ solutions Nucl. Phys. B 233 269–88
[17] Din M A and Zakrzewski W J 1980 General classical solutions in the $\mathbb{CP}^{N-1}$ model Nucl. Phys. B 174 397–406
[18] Gazeau J-P 2009 Coherent States in Quantum Physics (Weinheim: J Willey–VCH)
[19] Goldstein P P and Grundland A M 2010 Invariant recurrence relations for $CP^{N-1}$ models J. Phys. A Math. Theor. 43 265206
[20] Gross D G, Piran T and Weinberg S 1992 Two-Dimensional Quantum Gravity and Random Surfaces (Singapore: World Scientific)
[21] Grundland A M and Šnobl L 2006 Description of surfaces associated with $CP^{N-1}$ sigma models on Minkowski space J Geom Phys. 56 512–31
[22] Grundland A M and Šnobl L 2006 Surfaces associated with sigma models Stud Appl Math 117 335–51
[23] Grundland A M, Strasburger A and Dziewa-Dawidczyk D 2017 $CP^N$ sigma models via the SU(2) coherent states approach, Banach Center Publ Polish Acad. Sc., 50th seminar ‘Sophus Lie’ 113 169–91
[24] Jensen G R, Mussoaud E and Nicolodi L 2014 The geometric Cauchy problem for the membrane shape equation J. Phys. A: Math. Theor. 47 495201
[25] Jimbo M, Krukal M D and Miwa T 1982 Painlevé test for the self-dual Yang-Mills equation Phys. Lett. A 92 59–60
[26] Konopelchenko B and Landol G 1997 On classical string configuration Mod. Phys. Lett. 12 3161–8
[27] Landolfi G 2003 New results on the Canham–Helfrich membrane model via the generalized Weierstrass representation J. Phys. A: Math. Gen. 36 11937–54
[28] Lipowski R and Sackman E 1995 Handbook of Biological Physics Structure and Dynamics of Membranes vol 1 (Amsterdam: Elsevier)
[29] Manton N and Sutcliffe P 2004 Topological Solitons (Cambridge: Cambridge University Press)
[30] Marchenko V A 2011 Sturm-Liouville Operators and Applications Revised edition (Providence RI: AMS-Chelsea)
[31] McCoy B and Wu T 1973 The Two-Dimensional Ising model (Harvard: Havard Univ. Press)
[32] Musette M 1999 Painlevé Analysis for Nonlinear Partial Differential Equations in: The Painlevé Property One Century Later ed R Conte (New York: Springer Verlag) 517–72
[33] Nelson D, Piran T and Weinberg S 1992 Statistical Mechanics of Membranes and Surfaces (Singapore: World Scientific)
[34] Ou-Yang Z, Lui J and Xie Y 1999 Geometric Methods in Elastic Theory of Membranes in Liquid Crystal Phases (Singapore: World Scientific)
[35] Rajaraman R 1992 String Theory (Cambridge: Cambridge Univ. Press)
[36] Rajaraman R 2002 CP_Nsolitons in quantum hall systems Europhys J B 28 157–62
[37] Safran S A 1994 Statistical Thermodynamics of Surfaces, Interfaces and Membranes (New York: Addison-Wesley)
[38] Weiss J, Tabor M and Carnevale G 1983 The Painlevé property for partial differential equations J. Math. Phys. 24 522–6
[39] Zakrzewski W J 1989 Low Dimensional Sigma Models (Bristol: Adam Hilger) 8–11 ch. 4