Abstract. Let $G$ be any connected graph on $n$ vertices, $n \geq 2$. Let $k$ be any positive integer. Suppose that a fire breaks out on some vertex of $G$. Then in each turn $k$ firefighters can protect vertices of $G$ — each can protect one vertex not yet on fire; next a fire spreads to all unprotected neighbours.

The $k$-surviving rate of $G$, denoted by $\rho_k(G)$, is the expected fraction of vertices that can be saved from the fire by $k$ firefighters, provided that the starting vertex is chosen uniformly at random. In this paper, it is shown that for any planar graph $G$ we have $\rho_3(G) \geq \frac{2}{21}$. Moreover, 3 firefighters are needed for the first step only; after that it is enough to have 2 firefighters per each round. This result significantly improves known solutions to a problem of Cai and Wang (there was no positive bound known for surviving rate of general planar graph with only 3 firefighters). The proof is done using the separator theorem for planar graphs.

1. Introduction

The following Firefighter Problem was introduced by Hartnell [5]. Consider any connected graph, say $G$, on $n$ vertices, $n \geq 2$. Let $k$ be any positive integer. Suppose that a fire breaks out on some vertex $v \in V(G)$. Then in each turn firefighters can protect $k$ vertices of $G$, not yet on fire and the protection is permanent. Next a fire spreads to all unprotected neighbours of vertices that are already on fire. The goal is to save as much as possible and the question is how many vertices can be saved. We would like to refer the reader to the survey of Finbow and MacGillivray [3] for more information on the background of the problem and directions of its consideration.

In this paper we will focus on the following aspect of the problem. Let $sn_k(G, v)$ denote the maximum number of vertices of $G$ that $k$ firefighters can save when the fire breaks out at vertex $v$. This parameter may depend heavily on the choice of starting vertex $v$, for example when graph $G$ is a star. Therefore Cai and Wang [1] introduced the following graph parameter: the $k$-surviving rate $\rho_k(G)$ is the expected fraction of vertices that can be saved by $k$ firefighters, provided that the starting vertex is chosen uniformly at random. Namely

$$\rho_k(G) = \frac{1}{|V(G)|^2} \sum_{v \in V(G)} sn_k(G, v).$$

While discussing the surviving rate let us mention recent results by Prałat [9, 10] which have provided a threshold for average degree of general graphs which guarantees positive surviving rate with a given number of firefighters. To be more precise, for $k \in \mathbb{N}^+$ let us
define
\[ \tau_k = \begin{cases} 
\frac{30}{k+2} - \frac{1}{k+2} & \text{for } k = 1 \\
11 & \text{for } k \geq 2.
\end{cases} \]

Then there exist constants \( c > 0 \) and \( d > 0 \), such that for any \( \epsilon \), with \( d > \epsilon > 0 \), any \( n \in \mathbb{N}^+ \) and any graph \( G \) on \( n \) vertices and at most \((\tau_k - \epsilon)n/2\) edges one has \( \rho_k(G) > c \cdot \epsilon > 0 \). Moreover, there exists a family of graphs with average degree tending to \( \tau_k \) and \( k \)-surviving rate tending to 0.

The \( k \) surviving rate is investigated for many particular families of graphs — we focus here on planar graphs. Cai and Wang [1] asked about the minimum number of firefighters \( k \) such that \( \rho_k(G) > c \) for some positive constant \( c \) and any planar graph \( G \). It is easy to see that \( \rho_1(K_{2,n}) \xrightarrow{n \to \infty} 0 \), hence at least two firefighters are necessary. This is also the upper bound for triangle-free planar graphs [2] and planar graphs without 4-cycles [7].

So far, for general planar graphs the best known upper bound for the number of firefighters is 4: Kong, Wang and Zhu [6] have shown that \( \rho_4(G) > \frac{1}{7} \) for any planar graph \( G \). Esperet, Heuvel, Maffray and Sipma [2] have shown that using 4 firefighters only in the first round and just 3 in subsequent rounds is also possible to save positive fraction of any planar graph \( G \), namely \( \rho_{4,3}(G) > \frac{1}{2712} \). We use the notation of \( \rho_{k,l} \) and \( sn_{k,l} \) to describe the model with \( k \) firefighters in the first round and \( l \) firefighters in subsequent rounds.

In this paper we improve bounds by the following theorem:

**Theorem 1.1.** Let \( G \) be any planar graph. Then:

(i) \( \rho_{4,2}(G) > \frac{2}{9} \).

(ii) \( \rho_3(G) \geq \rho_{3,2}(G) > \frac{2}{21} \).

In other words, we show that with 3 firefighters in the first round and just 2 in subsequent rounds we can save at least \( \frac{2}{21} \) vertices of a planar graph, and with one extra firefighter in the first round we can increase the saved fraction to \( \frac{2}{9} \).

### 2. The proof

The proof is done using the lemma given by Lipton and Tarjan to prove the separator theorem for planar graphs [8]. The key lemma in their proof slightly reformulated to use in the firefighter problem is quoted below.

**Lemma 2.1.** Let \( G \) be any \( n \)-vertex plane triangulation and \( T \) be any spanning tree of \( G \). Then there exists an edge \( e \in E(G) \setminus E(T) \) such that the only cycle \( C \) in \( T + e \) has the property that the number of vertices inside \( C \) as well as outside \( C \) is lower than \( \frac{2}{7}n \).

Similar approach (using the above lemma to the firefighter problem) with announcement of even stronger result, however not in terms of the surviving rate, was given in [4]. Unfortunately, there is a serious error in the proof: it implicitly assumed that adding some edge joining two vertices of some induced subgraph of the planar graph which preserves planarity of the subgraph should also preserve planarity of the whole graph. In our opinion the proof given in [4] is uncorrectable.

The proof of Theorem 1.1 is presented in two steps — first we show that \( \rho_{4,2}(G) > \frac{2}{9} \) for any planar graph \( G \), then that \( \rho_{3,2}(G) > \frac{2}{21} \). At first let us note that surviving rate is monotone (non-increasing) by the operation of adding edges to the graph. Hence, it is enough to prove bounds given by Theorem 1.1 only for plane triangulations. Moreover,
the first step with respectively 3 or 4 firefighters is enough to get desired bounds for any planar graph on not more than 18 vertices.

Let $G$ be any $n$-vertex plane triangulation, where $n > 18$. Suppose that fire breaks out at vertex $r$. Consider a tree $T$ obtained by the breadth first search algorithm starting from vertex $r$. By Lemma 2.1 there is an edge $e$ and the cycle $C \subseteq T + e$ such that $|C \cup \text{in}C| > \frac{1}{3}n$ and $|C \cup \text{out}C| > \frac{1}{3}n$, where in$C$ and out$C$ denote sets of vertices inside the cycle $C$ and outside the cycle $C$ respectively. Note that in the cycle $C$ there are at most 2 vertices on any given distance from $r$. Firefighters strategy depends on the cycle $C$. When vertex $r$ does not belong to the cycle then firefighters protect vertices of $C$ in order given by the distance from vertex $r$, but it may be not enough, as the fire may spread through the neighbours of $r$ inside and outside the cycle $C$. Because either in$C$ or out$C$ contains not more than $\left\lfloor \frac{\deg r - 2}{2} \right\rfloor$ neighbours of $r$ we get immediately:

**Lemma 2.2.** Let $G$ be any $n$-vertex plane triangulation, where $n > 18$. Suppose that fire breaks out at a vertex $r$. Then using $2 + \left\lceil \frac{\deg r - 2}{3} \right\rceil$ firefighters at the first step and 2 at subsequent steps one can save more than $1/3n - 1$ vertices.

Using 4 firefighters at the first step one can save $n - 1$ vertices if vertex $r$ has degree 3 or 4, more than $n/3 - 1$ vertices if $r$ has degree 5, 6 or 7 and at least 4 vertices if $r$ has degree 8 or more. But for plane triangulation there is

$$\sum_{v \in V(G)} \deg v = 6n - 12.$$

Simple calculation shows now that $\rho_{4,2}(G) > \frac{2}{3}$.

Let us start the proof for 3 firefighters with a simple observation derived from Lemma 2.2.

**Observation 2.3.** Let $G$ be any $n$-vertex plane triangulation, where $n > 18$. Suppose that fire breaks out at a vertex $r$ of degree at most 5. Then

$$s_{n,3,2}(G, r) > \begin{cases} n - 1 & \text{for } \deg(r) \leq 3 \\ n/3 - 1 & \text{for } \deg(r) \in \{4, 5\} \end{cases}$$

Dealing with vertices of degree higher than 5 is a bit more complicated. Of course firefighters still can save at least 3 vertices, but frequently enough it is possible to save more.

**Lemma 2.4.** Let $G$ be any $n$-vertex plane triangulation, where $n > 18$. Suppose that fire breaks out at a vertex $r$ of degree 6 or 7. Then either

$$s_{n,3,2}(G, r) > \begin{cases} n/4 - 1 & \text{for } \deg(r) = 6 \\ n/6 - 1 & \text{for } \deg(r) = 7 \end{cases}$$

or vertex $r$ has at least 2 adjacent neighbours, say $u$ and $v$, such that both $s_{n,3,2}(G, u) > n/3$ and $s_{n,3,2}(G, v) > n/3$.

**Proof.** Let $G$ be any $n$-vertex plane triangulation, where $n > 18$. Let $r \in V(G)$ be the vertex of degree 6 or 7. Consider a tree $T$ obtained by the breadth first search algorithm starting from vertex $r$. By Lemma 2.1 there is an edge $e$ and the cycle $C \subseteq T + e$ such that $|C \cup \text{in}C| > \frac{1}{3}n$ and $|C \cup \text{out}C| > \frac{1}{3}n$. If vertex $r$ has no more than one neighbour either inside the cycle $C$ or outside the cycle, then $s_{n,3,2}(G, r) \geq n/3 - 1$. 
Assume then that vertex \( r \) has at least 2 neighbours inside the cycle \( C \) as well as outside. When vertex \( r \) has degree 6 assume without loss of generality that number of vertices inside \( C \) is not lower than number of vertices outside, it is \(|C \cup \text{in}C| > \frac{1}{4}n\) and of course \(|N(r) \cap \text{in}C| = 2\). When \( \deg(r) = 7 \) — assume without loss of generality that vertex \( r \) has 2 neighbours inside the cycle \( C \).

Let \( u \) and \( v \) be the neighbours of vertex \( r \) inside the cycle. Then one of two cases occur:

Case 1. In the graph \( G \) there exists a path from \( u \) or \( v \) to a vertex on a cycle containing of vertices in increasing distance from \( r \).

Case 2. There is no such path.

The path described in the first case divides the cycle \( C \) into two cycles \( C' \) and \( C'' \) (see Figure), both of which has the properties that there are at most 2 vertices on any given distance from \( r \) and there is at most one neighbour of vertex \( r \) inside the cycle. So, with 3 firefighters in the first round and 2 in subsequent rounds it is possible to save every vertex except \( r \) from \( C' \cup \text{in}C'' \) as well as from \( C'' \cup \text{in}C'' \). Hence, it is possible to save at least half of vertices from the set \( C \cup \text{in}C \).

Considering the second case note that as \( G \) is a triangulation then \( u \) and \( v \) are adjacent. Suppose now that fire breaks out at vertex \( u \) or \( v \) instead of vertex \( r \). Firefighters can save all vertices in \( C \cup \text{out}C \) by protecting in the first round vertex \( r \) and its neighbours on the cycle \( C \), and vertices of \( C \) ordered in increasing distance from \( r \) in subsequent rounds. \(\square\)

Let now partition the vertex set of \( G \) into 4 subsets defined by the conditions:

\[
\begin{align*}
X &= \{v \in V(G): sn_{3,2}(G, v) > \frac{n}{3} - 1\}, \\
Y &= \{v \in V(G): \deg(v) \leq 7 \wedge sn_{3,2}(G, v) \leq \frac{2n}{21}\}, \\
Z &= \{v \in V(G): \deg(v) \geq 8 \wedge sn_{3,2}(G, v) \leq \frac{2n}{21}\}, \\
W &= V(G) \setminus (X \cup Y \cup Z).
\end{align*}
\]
Note that $X$ contains every vertex of $G$ with degree lower than 6. As average degree of vertex in a plane triangulation is lower than 6 then the set $X$ is nonempty and average degree of vertex in $X$ is also lower than 6. Every vertex $v \in W$ has $\deg(v) \geq 6$ and $sn_{3,2}(G, v) < \frac{2n}{21}$. By Lemma 2.4 every vertex in the set $Y$ has at least 2 adjacent neighbours in the set $X$, hence it is
\[ |Y| < \sum_{x \in X} \frac{\deg x}{2} - \frac{|X|}{2}. \]
As any vertex in the set $Z$ has degree at least 8, while average degree in $G$ is lower than 6, we have
\[ |Z| < \sum_{x \in X} \frac{(6 - \deg(x))}{2}. \]
By adding these inequalities we get that $|Y| + |Z| < \frac{5}{2}|X|$. Hence,
\[ \rho_{3,2}(G) > \frac{1}{n^2} \left( |X|(n/3 - 1) + 3|Y| + 3|Z| + |W|2n/21 \right) > \frac{|X|}{3} + \frac{2|W|}{21} = \frac{2}{21}. \]

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