Imaginary in all directions: an elegant formulation of special relativity and classical electrodynamics

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Abstract

A suitable parameterization of space-time in terms of one complex and three quaternionic imaginary units allows Lorentz transformations to be implemented as multiplication by complex-quaternionic numbers rather than matrices. Maxwell’s equations reduce to a single equation.

1 Introduction

One of the most established symmetries of our universe is the invariance under Lorentz transformations, or the principle of special relativity \[\Box\]. Lorentz originally discovered that Maxwell’s theory of electromagnetism was invariant under these transformations. Einstein subsequently interpreted them as a principle of relativity, which is by no means specific to electromagnetism but a fundamental and far reaching invariance of our universe. The transformations were formulated in terms of $4 \times 4$ matrices $(\Lambda^\nu_\mu)$ that act on contravariant four vectors denoting space-time $(q^\mu) \equiv (t, x, y, z)$, energy-momentum $(p^\mu) \equiv (E, p_x, p_y, p_z)$, or the like. The introduction of the Minkowski metric $(g_{\mu\nu})$ and another set of covariant vectors such that $(q_\mu) \equiv (t, -x, -y, -z)$
 provided a highly convenient notation for relativistic theories, which has ever since been taught and referred to as relativistic notation. Einstein once even joked that his most important contribution to physics was to introduce this notation.

Apart from the success of the notation, it is taught to students of physics at such an early stage that they hardly question it. They learn in other classes, for example in alternating current circuit theory, the enormous technical advantage that can result if one employs complex numbers to describe the phase shifts in the time dependencies of real but alternating voltages and currents, but nothing suggests that any of this formalism, or a generalization of it, can simplify Lorentz transformations in a similar way. The only obvious application of imaginary units appears to be a rotation from Minkowski to Euclidean space by replacing the time $t$ by an imaginary time $\tau \equiv -it$, with the effect that the Minkowski metric $(g_{\mu\nu})$ is replaced by a simple Euclidean metric $(\delta_{\mu\nu})$.

In this paper, we will show that a suitable imaginary parameterization of space-time or other Lorentz contra- or covariant quantities, using both one complex and three quaternionic imaginary units \[\mathbb{C}\], \[\mathbb{H}\], yields a tremendous simplification of relativistic invariance \[\mathbb{I}\]. Lorentz transformations will no longer require multiplication by matrices, but only multiplication by complex-quaternionic (CQ) numbers. Classical electrodynamics will then provide us with a first orchard in which to harvest the gain in formal elegance \[\mathbb{J}\]. To be specific, Maxwell’s four equations will reduce to a single identity between two CQ numbers.

## 2 Special relativity

Let us now set up the formalism. We introduce a complex algebra with generators $1, \bar{a} \in \mathbb{C}$, such that
\[
\bar{a}^2 = -1, \tag{1}
\]
as well as a quaternionic algebra with generators $1, i, j, k \in \mathbb{H}$, such that
\[
i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j, \tag{2}
\]
which mutually commute:
\[
[i, \bar{a}] = [j, \bar{a}] = [k, \bar{a}] = 0. \tag{3}
\]
We further introduce a complex conjugate operation $^*$, which takes the form

$$\bar{\mathbb{a}} \rightarrow \bar{\mathbb{a}}^* = -\bar{\mathbb{a}}$$

but leaves $i, j, k$ unchanged, as well as a quaternionic conjugate operation $\bar{\cdot}$, which leaves $\mathbb{a}$ unchanged but takes the form

$$i \rightarrow \bar{i} = -i, \quad j \rightarrow \bar{j} = -j, \quad \text{and} \quad k \rightarrow \bar{k} = -k.$$  

Note that if $o_1, o_2 \in \mathbb{C} \otimes \mathbb{H}$ are two CQ numbers, the order of the product $o_1 o_2$ is reversed under quaternionic conjugation only:

$$\left( o_1 o_2 \right)^* = o_1^* o_2^* \quad \text{but} \quad \bar{o_1 o_2} = \bar{o_2} \bar{o_1}. \quad (4)$$

No simple product rule would exist if we had two or more mutually commuting quaternionic algebras.

We label space-time (and other Lorentz contravariant quantities usually denoted by four vectors) by a purely imaginary CQ number,

$$q \equiv \bar{\mathbb{a}} t + ix + jy + kz \quad (5)$$

where $t, x, y, z \in \mathbb{R}$. We identify this subspace of $\mathbb{C} \otimes \mathbb{H}$ using Minkowski space and denote it by $\mathbb{M}$. Complex conjugation $^*$ and quaternionic conjugation $\bar{\cdot}$ correspond within this space to time reversal (T) and parity (P) transformations, respectively. Note that $q^* = -\bar{q}$ for $q \in \mathbb{M}$.

The corresponding covariant quantity is given by its quaternionic conjugate or parity reversed CQ number,

$$\bar{q} = \bar{\mathbb{a}} t - ix - jy - kz, \quad (6)$$

yielding the proper time interval

$$-\bar{q}q = -q\bar{q} = t^2 - x^2 - y^2 - z^2. \quad (7)$$

Let $n = in_x + jn_y + kn_z$ with $nn = n_x^2 + n_y^2 + n_z^2 = 1$ and $n_x, n_y, n_z \in \mathbb{R}$ be a quaternionic imaginary unit vector $(n_x, n_y, n_z)$. Then a Lorentz transformation is given simply by

$$q \rightarrow q' = \omega q \omega^*, \quad (8)$$

with either

$$\omega = e^{\frac{1}{2} n\theta} = \cos \frac{\theta}{2} + n \sin \frac{\theta}{2} \quad (9)$$

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for a rotation by an angle \( \theta \) around \( n \), or

\[
\omega = e^{\frac{i}{2} n \otimes \Lambda} = \cosh \frac{\Lambda}{2} + \otimes n \sinh \frac{\Lambda}{2}
\]

(10)

for a boost by a Lorentz angle \( \Lambda \) in direction \( n \). (8-10) are readily verified using \( n^2 = -1 \) and \( (n \otimes)^2 = 1 \).

Clearly the covariant CQ number \( \bar{q} \) transforms as

\[
\bar{q} \rightarrow \bar{q}' = \omega^* \bar{q} \bar{\omega}.
\]

(11)

With \( \omega \bar{\omega} = \bar{\omega} \omega = 1 \), the Lorentz invariance of the proper time \( -\bar{q}q \) is evident.

At this point we assume that the reader is aware of how awkward it is to write down a rotation around or a boost along an arbitrary axis using \( 4 \times 4 \) matrices, so that there is no need to dwell on the elegance of the formulation proposed here.

It is convenient at this point to use the norm given by the proper time interval to define a scalar product between two imaginary or Minkowski CQ numbers \( M \times \bar{M} \rightarrow \mathbb{R} \), according to

\[
\langle p, q \rangle \equiv \frac{1}{4} \left( (p - q)(p - q) - (p + q)(p + q) \right)
= -\frac{1}{2} (\bar{p}q + \bar{q}p)
= -\frac{1}{2} (\bar{p}q + q\bar{p}).
\]

(12)

The proper time interval \( (7) \) is given by \( \langle q, q \rangle = -\bar{q}q \). With \( p = \otimes E + ip_x + jp_y + kp_z \) and \( q = \otimes t + ix + jy + k z \), we obtain

\[
\langle p, q \rangle = Et - p_x x - p_y y - p_z z.
\]

(13)

Furthermore, it is convenient to define the contravariant differentiation operator

\[
D \equiv \otimes \frac{\partial}{\partial t} - i \frac{\partial}{\partial x} - j \frac{\partial}{\partial y} - k \frac{\partial}{\partial z} = \otimes \partial_t - i \partial_x - j \partial_y - k \partial_z,
\]

(14)

which likewise transforms according to

\[
D \rightarrow D' = \omega D \bar{\omega}^*.
\]

(15)
Note that $-D\bar{q} = -\bar{D}q = 4$, as expected. The covariant differentiation operator is, of course, given by

$$\bar{D} = \mathbf{i} \partial_t + i \partial_x + j \partial_y + k \partial_z,$$

and transforms

$$\bar{D} \rightarrow \bar{D}' = \omega^* \bar{D} \bar{\omega}. \quad (17)$$

### 3 Classical electrodynamics

We proceed by applying this formalism to classical electrodynamics. To begin with, we introduce a contravariant vector (or imaginary CQ field)

$$A \equiv \mathbf{i} \phi + i A_x + j A_y + k A_z, \quad (18)$$

and require the theory to be invariant under electromagnetic gauge transformations

$$A \rightarrow A + D\lambda(q), \quad (19)$$

where $\lambda(q)$ is an arbitrary real-valued scalar function of space-time $q$. We proceed by defining the electromagnetic field strength

$$F \equiv \frac{1}{2} \left( \bar{D}A - \bar{A}D \right). \quad (20)$$

Clearly $F$ is invariant under (19). Note that we cannot replace $\bar{D}A$ by $\bar{A}D$ in the second term in (20), since the derivative operator has to act on $A$. Under a Lorentz transformation, $F = -\bar{F}$ transforms as

$$F \rightarrow F' = \omega^* F \bar{\omega}^*. \quad (21)$$

At this point it is propitious to introduce

$$e_1 \equiv i, \quad e_2 \equiv j, \quad e_3 \equiv k \quad (22)$$

such that

$$e_i e_j = -\delta_{ij} + \epsilon_{ijk} e_k \quad \text{with} \quad i, j, k \in \{1, 2, 3\}, \quad (23)$$

where the indices $i, j, k$ in (23) are not to be confused with the quaternionic generators $i, j, k$ in (22). With summation over repeated indices but no hidden minus signs implied, we write

$$A = \mathbf{i} \phi + e_i A_i, \quad D = \mathbf{i} \partial_t - e_i \partial_i \quad \text{etc.} \quad (24)$$
Writing the field strength \(20\) out in components, we obtain

\[ F = e_i \epsilon_{ijk} \partial_j A_k - \partial_t e_i (\partial_t A_i - \partial_i \phi). \]  

(25)

Note that

\[ DF = (-\partial_t + e_i \partial_i)(-\partial_t A_i - \partial_i \phi) - e_i \epsilon_{ijk} \partial_j \epsilon_{klm} \partial_l A_m \]  

(26)

is a purely imaginary CQ number \((DF \in \mathbb{M})\). This implies

\[ DF + \overline{DF}^* = 0, \]  

(27)

which will prove useful below.

We proceed by defining magnetic and electric field strengths according to

\[ B_i \equiv \epsilon_{ijk} \partial_j A_k \quad \text{and} \quad E_i \equiv -\partial_t A_i - \partial_i \phi, \]  

(28)

and use them to rewrite \(25\) as

\[ F = e_i (B_i - \partial_t E_i). \]  

(29)

We may hence write the Lagrangian density for the electromagnetic field coupled to an external current \((\rho, J_i)\) as

\[ L \equiv \frac{1}{2} \left( E_i^2 - B_i^2 \right) + \rho \phi - J_i A_i \]

\[ = \frac{1}{4} (F^2 + (F^*)^2) + \frac{1}{2} (\bar{J} A + \bar{A} J), \]  

(30)

where we have defined

\[ J \equiv \partial \rho + e_i J_i. \]  

(31)

Clearly, \(L\) and \(F^2 = E_i^2 + 2 \partial \rho B_i + B_i^2\) are Lorentz invariant.

Variation of \(L\) with respect to \(\delta \bar{A}\) yields, after integration by parts,

\[ \frac{1}{4} (DF - \overline{DF}^*) + \frac{1}{2} J = 0. \]  

(32)

Together with \(27\), we obtain

\[ DF + J = 0. \]  

(33)

This is Maxwell’s equation. (In the standard formulation, there is need to speak of equations (plural), but we shall see now that this single equation
replaces all four of them.) Note that (33) transforms contravariantly under Lorentz transformations. Evaluation of $DF$, starting from (29), yields

$$DF = \partial_i B_i - \bar{\partial} \partial_i E_i + e_i (\partial_i E_i - \epsilon_{ijk} \partial_j B_k) + \bar{\partial} e_i (\partial_i B_i + \epsilon_{ijk} \partial_j E_k).$$  \hspace{1cm} (34)$$

Since (33) implies that the coefficients of $DF + J$ in all eight orthogonal directions $\{1, \bar{\partial}, e_i, \bar{\partial} e_i\}$ of $\mathbb{C} \otimes \mathbb{H}$ vanish, substitution of (34) and (31) into (33) immediately yields

$$\begin{align*}
\partial_i B_i &= 0 \\
-\partial_i E_i + \rho &= 0 \\
\partial_i E_i - \epsilon_{ijk} \partial_j B_k + J_i &= 0 \\
\partial_i B_i + \epsilon_{ijk} \partial_j E_k &= 0.
\end{align*}$$  \hspace{1cm} (35)$$

We assume the reader is familiar with these equations.

In Lorentz gauge,

$$\partial_t \phi - \partial_i A_i = -\frac{1}{2} \left( \bar{D} \partial A + \bar{A} \right) = 0.$$  \hspace{1cm} (36)$$

Maxwell’s equation (33) then reduces to

$$D \bar{D} \partial A + J = 0,$$  \hspace{1cm} (37)$$

which is readily recognized as a wave equation.

4 Conclusion

In conclusion, we have shown that a suitable CQ parameterization of Lorentz contra- and covariant quantities can greatly enhance elegance and simplicity of relativistic theories. We believe that this language is likely to yield new perspectives on quantum field theories [9, 10, 11, 12], which we are currently investigating.

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