Statistical Mechanics of Online Learning for Ensemble Teachers

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Abstract

We analyze the generalization performance of a student in a model composed of linear perceptrons: a true teacher, ensemble teachers, and the student. Calculating the generalization error of the student analytically using statistical mechanics in the framework of on-line learning, it is proven that when learning rate $\eta < 1$, the larger the number $K$ and the variety of the ensemble teachers are, the smaller the generalization error is. On the other hand, when $\eta > 1$, the properties are completely reversed. If the variety of the ensemble teachers is rich enough, the direction cosine between the true teacher and the student becomes unity in the limit of $\eta \to 0$ and $K \to \infty$.

keywords: ensemble teachers, on-line learning, generalization error, statistical mechanics, learning rate

1 Introduction

Learning is to infer the underlying rules that dominate data generation using observed data. Observed data are input-output pairs from a teacher and are called examples. Learning can be roughly classified into batch learning and on-line learning [1]. In batch learning, given examples are used repeatedly. In this paradigm, a student becomes to give correct answers after training if the student has adequate freedom. However, it is necessary to have a long amount of time and a large memory in which to store many examples. On the contrary, in online learning examples used once are discarded. In this case, a student cannot give correct answers for all examples used in training. However, there are merits, for example, a large memory for storing many examples isn’t necessary, and it is possible to follow a time variant teacher.

Recently, we [5, 6] analyzed the generalization performance of ensemble learning [2, 3, 4] in a framework of on-line learning using a statistical mechanical method [1, 8]. Using the same method, we also analyzed the generalization performance of a student supervised by a moving teacher that goes around a true teacher [7]. As a result, it was proven that the generalization error of a student can be smaller than a moving teacher, even if the student only uses examples from the moving teacher. In an actual human society, a teacher observed by a student doesn’t always present the correct answer. In many cases, the teacher is learning and continues to change. Therefore, the analysis of such a model is interesting for considering the analogies between statistical learning theories and an actual human society.

On the other hand, in most cases in an actual human society a student can observe examples from two or more teachers who differ from each other. Therefore, we analyze the generalization performance of such a model and discuss the use of imperfect teachers in this paper. That is, we consider a true teacher and $K$ teachers called ensemble teachers who exist around the true teacher. A student uses input-output pairs from ensemble teachers in turn or randomly. In this paper, we treat a model in which all of the true teacher, the ensemble teachers and the student are linear perceptrons [5] with noises. We obtain order

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parameters and generalization errors analytically in the framework of on-line learning using a statistical mechanical method. As a result, it is proven that when student’s learning rate $\eta < 1$, the larger the number $K$ and the variety of the ensemble teachers are, the smaller the student’s generalization error is. On the other hand, when $\eta > 1$, the properties are completely reversed. If the variety of ensemble teachers is rich enough, the direction cosine between the true teacher and the student becomes unity in the limit of $\eta \to 0$ and $K \to \infty$.

2 Model

In this paper, we consider a true teacher, $K$ ensemble teachers and a student. They are all linear perceptrons with connection weights $A, B_k$ and $J$, respectively. Here, $k = 1, \ldots, K$. For simplicity, the connection weight with the true teacher, the ensemble teachers and the student are simply called the true teacher, the ensemble teachers and the student, respectively. True teacher $A = (A_1, \ldots, A_N)$, ensemble teachers $B_k = (B_{k1}, \ldots, B_{kN})$, student $J = (J_1, \ldots, J_N)$ and input $x = (x_1, \ldots, x_N)$ are $N$ dimensional vectors. Each component $A_i$ of $A$ is drawn from $\mathcal{N}(0, 1)$ independently and fixed, where $\mathcal{N}(0, 1)$ denotes a Gaussian distribution with a mean of zero and variance unity. Some components $B_{ki}$ are equal to $A_i$, multiplied by $-1$, the others are equal to $A_i$. Which component $B_{ki}$ is equal to $-A_i$ is independent from the value of $A_i$. Hence, $B_{ki}$ also obeys $\mathcal{N}(0, 1)$. $B_{ki}$ is also fixed. The direction cosine between $B_k$ and $A$ is $R_{Bk}$ and that between $B_k$ and $B_{k'}$ is $q_{kk'}$. Each of the components $J^0_i$ of the initial value $J^0$ of $J$ are drawn from $\mathcal{N}(0, 1)$ independently. The direction cosine between $J$ and $A$ is $R_J$ and that between $J$ and $B_k$ is $R_{Bk,j}$. Each component $x_i$ of $x$ is drawn from $\mathcal{N}(0, 1/N)$ independently. Thus,

$$
\langle A_i \rangle = 0, \quad \langle A_i^2 \rangle = 1, \quad (1)
$$

$$
\langle B_{ki} \rangle = 0, \quad \langle (B_{ki})^2 \rangle = 1, \quad (2)
$$

$$
\langle J^0_i \rangle = 0, \quad \langle (J^0_i)^2 \rangle = 1, \quad (3)
$$

$$
\langle x_i \rangle = 0, \quad \langle x_i^2 \rangle = \frac{1}{N}, \quad (4)
$$

$$
R_{Bk} = \frac{A \cdot B_k}{\|A\| \|B_k\|}, \quad q_{kk'} = \frac{B_k \cdot B_{k'}}{\|B_k\| \|B_{k'}\|}, \quad (5)
$$

$$
R_J = \frac{A \cdot J}{\|A\| \|J\|}, \quad R_{Bk,j} = \frac{B_k \cdot J}{\|B_k\| \|J\|}, \quad (6)
$$

where $\langle \cdot \rangle$ denotes a mean.

Figure 1 illustrates the relationship among true teacher $A$, ensemble teachers $B_k$, student $J$ and direction cosines $q_{kk'}, R_{Bk}, R_J$ and $R_{Bk,j}$.

In this paper, the thermodynamic limit $N \to \infty$ is also treated. Therefore,

$$
\|A\| = \sqrt{N}, \quad \|B_k\| = \sqrt{N}, \quad \|J^0\| = \sqrt{N}, \quad \|x\| = 1. \quad (7)
$$

Generally, norm $\|J\|$ of the student changes as time step proceeds. Therefore, ratios $l^m$ of the norm to $\sqrt{N}$ are introduced and called the length of the student. That is, $\|J^m\| = l^m \sqrt{N}$, where $m$ denotes the time step.

The outputs of the true teacher, the ensemble teachers, and the student are $y^m + n_A^m$, $v_k^m + n_{Bk}^m$ and $u^m l^m + n_J^m$, respectively. Here,

$$
y^m = A \cdot x^m, \quad (8)
$$

$$
v_k^m = B_k \cdot x^m, \quad (9)
$$

$$
u^m l^m = J^m \cdot x^m, \quad (10)
$$

$$
n_A^m \sim \mathcal{N}(0, \sigma_A^2), \quad (11)
$$

$$
n_{Bk}^m \sim \mathcal{N}(0, \sigma_{Bk}^2), \quad (12)
$$

$$
n_J^m \sim \mathcal{N}(0, \sigma_J^2). \quad (13)
$$
That is, the outputs of the true teacher, the ensemble teachers and the student include independent Gaussian noises with variances of $\sigma^2_A$, $\sigma^2_{Bk}$, and $\sigma^2_J$, respectively. Then, $y^m$, $v^m$, and $u^m$ of Eqs. (3)-(10) obey Gaussian distributions with a mean of zero and variance unity.

Let us define error $\epsilon_{Bk}$ between true teacher $A$ and each member $B_k$ of the ensemble teachers by the squared errors of their outputs:

$$
\epsilon_{Bk}^m \equiv \frac{1}{2} (y^m + n_A^m - v_k^m - n_{Bk}^m)^2.
$$

(14)

In the same manner, let us define error $\epsilon_{BkJ}$ between each member $B_k$ of the ensemble teachers and student $J$ by the squared errors of their outputs:

$$
\epsilon_{BkJ}^m \equiv \frac{1}{2} (v_k^m + n_{Bk}^m - u^m l^m - n_J^m)^2.
$$

(15)

Student $J$ adopts the gradient method as a learning rule and uses input $x$ and an output of one of the $K$ ensemble teachers $B_k$ in turn or randomly for updates. That is,

$$
J^{m+1} = J^m - \eta \frac{\partial \epsilon_{BkJ}}{\partial J^m}
$$

(16)

$$
= J^m + \eta (v_k^m + n_{Bk}^m - u^m l^m - n_J^m) x^m,
$$

(17)

where $\eta$ denotes the learning rate of the student and is a constant number. In cases where the student uses $K$ ensemble teachers in turn, $k = \text{mod}(m, K) + 1$. Here, $\text{mod}(m, K)$ denotes the remainder of $m$ divided by $K$. On the other hand, in random cases, $k$ is a uniform random integer that takes one of $1, 2, \ldots, K$.

Generalizing the learning rules, Eq. (17) can be expressed as

$$
J^{m+1} = J^m + f_k x^m
$$

(18)

$$
= J^m + f (v_k^m + n_{Bk}^m, u^m l^m + n_J^m) x^m,
$$

(19)

where $f$ denotes a function that represents the update amount and is determined by the learning rule.

In addition, let us define error $\epsilon_J$ between true teacher $A$ and student $J$ by the squared error of their outputs:

$$
\epsilon_J^m \equiv \frac{1}{2} (y^m + n_A^m - u^m l^m - n_J^m)^2.
$$

(20)
3 Theory

3.1 Generalization error

One purpose of a statistical learning theory is to theoretically obtain generalization errors. Since generalization error is the mean of errors for the true teacher over the distribution of new input and noises, generalization error $\epsilon_{Bkg}$ of each member $B_k$ of the ensemble teachers and $\epsilon_{Jg}$ of student $J$ are calculated as follows. Superscripts $m$, which represent the time steps, are omitted for simplicity unless stated otherwise.

$$
\epsilon_{Bkg} = \int dx dn_A dn_Bk P(x, n_A, n_Bk) \epsilon_{Bk} 
$$

(21)

$$
= \int dy dv_k dn_A dn_Bk P(y, v_k, n_A, n_Bk) \frac{1}{2} (y + n_A - v_k - n_Bk)^2
$$

(22)

$$
= \frac{1}{2} (-2R_{Bk} + 2 + \sigma_A^2 + \sigma_{Bk}^2),
$$

(23)

$$
\epsilon_{Jg} = \int dx dn_A dn_J P(x, n_A, n_J) \epsilon_{J}
$$

(24)

$$
= \int dy du dn_A dn_J P(y, u, n_A, n_J) \frac{1}{2} (y + n_A - u - n_J)^2
$$

(25)

$$
= \frac{1}{2} (-2R_{Jl} + l^2 + 1 + \sigma_A^2 + \sigma_{J}^2).
$$

(26)

Here, integrations have been executed using the following: $y, v_k$ and $u$ obey $N(0, 1)$. The covariance between $y$ and $v_k$ is $R_{Bk}$, that between $v_k$ and $u$ is $R_{BkJ}$, and that between $y$ and $u$ is $R_{J}$. All $n_A, n_Bk$, and $n_J$ are independent from other probabilistic variables.

3.2 Differential equations for order parameters and their analytical solutions

To simplify analysis, the following auxiliary order parameters are introduced:

$$
r_J \equiv R_{Jl},
$$

(27)

$$
r_{BkJ} \equiv R_{BkJl}.
$$

(28)

Simultaneous differential equations in deterministic forms [8], which describe the dynamical behaviors of order parameters, have been obtained based on self-averaging in the thermodynamic limits as follows:

$$
\frac{dr_{BkJ}}{dt} = \frac{1}{K} \sum_{k=1}^{K} \langle f_k v_k \rangle,
$$

(29)

$$
\frac{dr_{J}}{dt} = \frac{1}{K} \sum_{k=1}^{K} \langle f_k y \rangle,
$$

(30)

$$
\frac{dl}{dt} = \frac{1}{K} \sum_{k=1}^{K} \left( \langle f_k u \rangle + \frac{1}{2l} \langle f_k^2 \rangle \right).
$$

(31)

Here, dimension $N$ has been treated to be sufficiently greater than the number of ensemble teachers $K$. Time $t = m/N$, that is, time step $m$ normalized by dimension $N$. Note that the above differential equations are identical whether the $K$ ensemble teachers are used in turn or randomly.

Since linear perceptrons are treated in this paper, the sample averages that appeared in the above equations can be easily calculated as follows:

$$
\langle f_k u \rangle = \eta \left( \frac{R_{BkJ} - l}{l} \right),
$$

(32)

$$
\langle f_k^2 \rangle = \eta^2 \left( l^2 - 2R_{BkJ} + 1 + \sigma_{BkJ}^2 + \sigma_{J}^2 \right),
$$

(33)

$$
\langle f_k y \rangle = \eta \left( R_{Bk} - r_{J} \right),
$$

(34)
\[
\frac{1}{K} \sum_{k'=1}^{K} \langle f_{k'} v_k \rangle = \eta \left( -r_{BkJ} + \frac{1}{K} \sum_{k'=1}^{K} q_{kk'} \right).
\]

Since all components \(A_i, J_0^0\) of true teacher \(A\), and the initial student \(J_0^0\) are drawn from \(\mathcal{N}(0, 1)\) independently and because the thermodynamic limit \(N \to \infty\) is also treated, they are orthogonal to each other in the initial state. That is,

\[
R_J = 0 \text{ when } t = 0.
\]

In addition,

\[
l = 1 \text{ when } t = 0.
\]

By using Eqs. (35), simultaneous differential equations Eqs. (29)–(31) can be solved analytically as follows:

\[
r_{BkJ} = \frac{1}{K} \sum_{k'=1}^{K} q_{kk'} \left( 1 - e^{-\eta t} \right),
\]

\[
r_J = \frac{1}{K} \sum_{k=1}^{K} R_{Bk} \left( 1 - e^{-\eta t} \right),
\]

\[
l^2 = \frac{1}{2 - \eta} \left[ 2 (1 - \eta) \bar{q} + \eta \left( 1 + \sigma_B^2 + \sigma_J^2 \right) \right] + \left[ 1 + \frac{2}{2 - \eta} \eta \left( 1 + \sigma_B^2 + \sigma_J^2 \right) - 2\bar{q} \right] e^{\eta(\eta-2)t} - 2\bar{q} e^{-\eta t},
\]

where

\[
\bar{q} = \frac{1}{K^2} \sum_{k=1}^{K} \sum_{k'=1}^{K} q_{kk'},
\]

\[
\sigma_B^2 = \frac{1}{K} \sum_{k=1}^{K} \sigma_{Bk}^2.
\]

## 4 Results and Discussion

In this section, we treat the case where direction cosines \(R_{Bk}\) between the ensemble teachers and the true teacher, direction cosines \(q_{kk'}\) among the ensemble teachers and variances \(\sigma_{Bk}^2\) of the noises of ensemble teachers are uniform. That is,

\[
R_{Bk} = R_B, \ k = 1, \ldots, K,
\]

\[
q_{kk'} = \begin{cases} q, & k \neq k', \\ 1, & k = k' \end{cases},
\]

\[
\sigma_{Bk}^2 = \sigma_B^2.
\]

In this case, Eqs. (41) and (42) are expressed as

\[
\bar{q} = q + \frac{1-q}{K},
\]

\[
\sigma_B^2 = \sigma_B^2.
\]

The dynamical behaviors of generalization errors \(\epsilon_{Jg}\) have been analytically obtained by solving Eqs. (26), (27) and (38)–(47). Figure 2 shows the analytical results and the corresponding simulation results, where \(N = 2000\). In computer simulations, \(K\) ensemble teachers are used in turn. \(\epsilon_{Jg}\) was obtained by averaging the squared errors for \(10^4\) random inputs at each time step. Generalization error \(\epsilon_{Bg}\) of one of the ensemble teachers is also shown. The dynamical behaviors of \(R\) and \(l\) are shown in Fig. 3.
In these figures, the curves represent theoretical results. The dots represent simulation results. Conditions other than $q$ are common: $\eta = 0.3, K = 3, R_B = 0.7, \sigma_A^2 = 0.0, \sigma_B^2 = 0.1$ and $\sigma_J^2 = 0.2$. Figure 2 shows that the smaller $q$ is, that is, the richer the variety of the ensemble teachers is, the smaller generalization error $\epsilon_{Jg}$ of the student is. Especially in the cases of $q = 0.6$ and $q = 0.49$, the generalization error of the student becomes smaller than a member of the ensemble teachers after $t \approx 5$. This means that the student in this model can become more clever than each member of the ensemble teachers even though the student only uses the input-output pairs of members of the ensemble teachers. Figure 3 shows that the larger the variety of the ensemble teachers is, the larger direction cosine $R_J$ is and the smaller length $l$ of the student is. The reason minimum value 0.49 of $q$ is taken as the squared value of $R_B = 0.7$ in Figs. 2 and 3 is described later.

In Figs. 2 and 3, $\epsilon_{Jg}, R_J$ and $l$ almost seem to reach a steady state by $t = 20$. The macroscopic
behaviors of $t \to \infty$ can be understood theoretically since the order parameters have been obtained analytically. Focusing on the signs of the powers of the exponential functions in Eqs. 35–40, we can see that $\epsilon_{jg}$ and $l$ diverge if $\eta < 0$ or $\eta > 2$. The steady state values of $r_{Bk,l}, r_j$ and $l^2$ in the case of $0 < \eta < 2$ can be easily obtained by substituting $t \to \infty$ in Eqs. 38–40 as follows:

$$r_{Bk,l} \to q + \frac{1 - q}{K},$$

$$r_j \to R_B,$$

$$l^2 \to \frac{1}{2 - \eta} \left( 2(1 - \eta) \left( q + \frac{1 - q}{K} \right) + \eta \left( 1 + \sigma_B^2 + \sigma_T^2 \right) \right),$$

Therefore, the steady value of generalization error $\epsilon_{jg}$ and direction cosine $R_j$ are independent from $K$ and $q$ in this case. In the case of $0 < \eta < 1$, the smaller $q$ is or the larger $K$ is, the smaller the steady values of $l$ and $\epsilon_{jg}$ are and the larger the steady value of $R_j$ is. In the case of $1 < \eta < 2$, on the contrary, the smaller $q$ is or the larger $K$ is, the larger the steady values of $l$ and $\epsilon_{jg}$ are and the smaller the steady value of $R_j$ is. That is, in the case of $\eta < 1$, the more teachers exist and the richer the variety of teachers is, the more clever the student can become. On the contrary, in the case of $\eta > 1$, the number of teachers should be small and the variety of teachers should be low for the student to become clever.

In the right hand side of Eq. 41, since the second and the third terms are positive, the steady value of $l$ is independent from the number $K$ of teachers and direction cosine $q$ among the ensemble teachers. Therefore, the steady value of generalization $\epsilon_{jg}$ and direction cosine $R_j$ are independent from $K$ and $q$ in this case. In the case of $0 < \eta < 1$, the smaller $q$ is or the larger $K$ is, the smaller the steady values of $l$ and $\epsilon_{jg}$ are and the larger the steady value of $R_j$ is. In the case of $1 < \eta < 2$, on the contrary, the smaller $q$ is or the larger $K$ is, the larger the steady values of $l$ and $\epsilon_{jg}$ are and the smaller the steady value of $R_j$ is.

Figures 4–7 show the relationships between learning rate $\eta$ and $\epsilon_{jg}, R_j$. In Figs 4 and 5, $K = 3$ and is fixed. In Figs 6 and 7, $q = 0.49$ and is fixed. Conditions other than $K$ and $q$ are $\sigma_A^2 = \sigma_B^2 = \sigma_T^2 = 0.0$ and $R_B = 0.7$. Computer simulations have been executed using $\eta = 0.3, 0.6, 1.0, 1.4$ and 1.7. The values on $t = 20$ are plotted for the simulations and considered to have already reached a steady state.

These figures show the following: the smaller learning rate $\eta$ is, the smaller generalization error $\epsilon_{jg}$ and the larger direction cosine $R_j$ is. Needless to say, when $\eta$ is small, learning is slow. Therefore, residual generalization error and learning speed are in a relationship tradeoff. The phase transition in which $\epsilon_{jg}$ diverges and $R_j$ becomes zero on $\eta = 2$ is shown. In the case of $\eta < 1$, the larger $K$ is or the smaller $q$ is, that is, the richer the variety of ensemble teachers is, the smaller $\epsilon_{jg}$ is and the larger $R_j$ is. On the contrary, the properties are completely reversed in the case of $\eta > 1$.

As described above, learning properties are dramatically changed with learning rate $\eta$. It is difficult to explain the reason qualitatively. Here, we try to explain the reason intuitively by showing the geometrical meaning of $\eta$. Figures 5(a)–(c) show the updates of $\eta = 0.5, \eta = 1$ and $\eta = 2$, respectively. Here, the noises are ignored for simplicity. Needless to say, teacher $B_k$ itself cannot be observed directly and only output $v$ can be observed when student $J$ is updated. In addition, since the projections from $J^{m+1}$ to $x^m$ and from $B_k$ to $x^m$ are equal in the case of $\eta = 1$, as shown in Fig. 5(b), $\eta = 1$ is a special condition where the student uses up the information obtained from input $x^m$. In the case of $\eta < 1$, the update is short. Since in a sense this fact helps balance the information from the ensemble teachers, the generalization error of the student is improved when the number $K$ of teachers is large and their variety is rich. On the other hand, the update is excessive when $\eta > 1$. Therefore, the student is shaken or swung, and its generalization performance worsens when $K$ is large and the variety is rich. In addition, the reason that learning diverges if $\eta < 0$ or $\eta > 2$ can be understood intuitively from Fig. 5 distance $\|(\eta - 1)(v^m - u^mlm)x^m\|$, measured by the projections to $x^m$ between student $J^{m+1}$ after the update.
Figure 4: Steady value of generalization error $\epsilon_{Jg}$ in the case of $K = 3$. Theory and computer simulations. Conditions other than $K$ and $q$ are $\sigma_A^2 = \sigma_B^2 = \sigma_J^2 = 0.0$ and $R_B = 0.7$.

Figure 5: Steady value of direction cosine $R_J$ in the case of $K = 3$. Theory and computer simulations. Conditions other than $K$ and $q$ are $\sigma_A^2 = \sigma_B^2 = \sigma_J^2 = 0.0$ and $R_B = 0.7$. 
Figure 6: Steady value of generalization error $\epsilon_{Jg}$ in the case of $q = 0.49$. Theory and computer simulations. Conditions other than $K$ and $q$ are $\sigma_A^2 = \sigma_B^2 = \sigma_J^2 = 0.0$ and $R_B = 0.7$.

Figure 7: Steady value of direction cosine $R_J$ in the case of $q = 0.49$. Theory and computer simulations. Conditions other than $K$ and $q$ are $\sigma_A^2 = \sigma_B^2 = \sigma_J^2 = 0.0$ and $R_B = 0.7$. 
and teacher $B_k$, is larger than distance $\| (v^m - u^m l^m)x^m \|$ between student $J^m$ before the update and teacher $B_k$ in the case of $\eta < 0$ or $\eta > 2$. Therefore, the learning diverges.

Figure 8: Geometric meaning of learning rate $\eta$

5 Conclusion

We analyzed the generalization performance of a student in a model composed of linear perceptrons: a true teacher, ensemble teachers, and the student. The generalization error of the student was analytically calculated using statistical mechanics in the framework of online learning, proving that when learning rate $\eta < 1$, the larger the number $K$ and the variety of the ensemble teachers are, the smaller the generalization error is. On the other hand, when $\eta > 1$, the properties are completely reversed. If the variety of ensemble teachers is rich enough, the direction cosine between the true teacher and the student becomes unity in the limit of $\eta \to 0$ and $K \to \infty$.

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A Direction cosine $q$ among ensemble teachers

Let us consider the case where $S$ and $T$ are generated independently satisfying the condition that direction cosines between $S$ and $P$ and between $T$ and $P$ are both $R_0$, as shown in Fig. 9 where $S$, $T$ and $P$ are $N$ dimensional vectors. In this figure, the inner product of $s$ and $t$ is

\[
s \cdot t = \left( S - R_0 \frac{\|S\|}{\|P\|} P \right) \cdot \left( T - R_0 \frac{\|T\|}{\|P\|} P \right) = \|S\||\|P\|\left( q_0 - R_0^2 \right), \tag{52}
\]

where $s$ and $t$ are projections from $S$ to the orthogonal complement $C$ of $X$ and from $T$ to $C$, respectively. $q_0$ denotes the direction cosine between $S$ and $T$.

Incidentally, if dimension $N$ is large and $S$ and $T$ have been generated independently, $s$ and $t$ should be orthogonal to each other. Therefore, $q_0 = R_0^2$. 

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Figure 9: Direction cosine among ensemble teachers

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