Wormholes and naked singularities in Brans–Dicke cosmology

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Abstract

We perform an analytical and numerical study of static spherically symmetric solutions in the context of the Brans–Dicke-like cosmological model by Elizalde et al (2004 Phys. Rev. D \textbf{70} 043539) with an exponential potential. In this model the phantom regime arises without the appearance of any ghost degree of freedom due to the specific form of coupling. For certain parameter ranges the model contains a regular solution that we interpret as a wormhole in an otherwise de Sitter Universe. We put several bounds on the parameter values: $\omega < 0$, $\alpha^2/|\omega| < 10^{-5}$, $22.7 \lesssim \phi_0 \lesssim 25$. The numerical solution could mimic the Schwarzschild one, so the original model is consistent with astrophysical and cosmological observational data. However, differences between our solution and the Schwarzschild one can be quite large, so black hole candidate observations could probably place further limits on the $\phi_0$ value.

Keywords: Brans–Dicke, wormholes, cosmology, naked singularities, numerical solution
1. Introduction

Astrophysical data (especially from recent decades) ranging from high-redshift surveys of supernovae to WMAP observations indicate that our Universe is experiencing an accelerating phase of expansion now [2]. A possible interpretation of this expansion in terms of general relativity (GR) states that about 70% of the total energy of our Universe is attributed to dark energy with large and negative pressure (the cosmological constant is the best fit nowadays, but see [3] for a comprehensive review on dark energy or [4] for a brief one). One of the simplest possible theoretical descriptions of dark energy, beyond the cosmological constant, is to add a scalar field to the GR action (a scalar–tensor theory).

Brans–Dicke (BD) scalar–tensor theory was one of the first attempts to modify gravity [5]. This theory introduces a scalar field $\phi$ which manifests itself as variation in the gravitational constant. The action of BD theory has the form [5]

$$ S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ \phi \mathcal{R} - \frac{\omega}{\phi} \partial_\mu \phi \partial^\mu \phi \right], \quad (1) $$

Through the years the theory passed multiple observational and theoretical tests. A variety of physical phenomena can be interpreted in terms of Brans–Dicke (BD) theory, ranging from the flat Galaxy rotation curves [6] to inflation [7] and the cosmological constant problem (vacuum catastrophe) [8]. It is now also well established that BD theory can be used for dark energy modelling [9–11].

The current observational bound on the BD parameter $|\omega| > 50000$ is imposed in terms of the parametrized post-Newtonian (PPN) expansion of the theory [12]. Such a large value causes debates on the theory’s adequacy since its PPN expansion reaches general relativity (GR) in the limit $\omega \to \infty$ (see [13] for detailed information on the connection between BD and GR). Nevertheless, papers [14–17] show that most of the scalar–tensor theories in their cosmological evolution reach the state where the contribution of the scalar part to gravity virtually vanishes. So although $\omega$ is large nowadays, it could demonstrate more interesting values (from the observational point of view) in the past. On the other hand, regardless of the value of $\omega$ (paper [18] proposes $|\omega| > 10^{40}$) for $|\omega| < \infty$ the theory can describe bouncing cosmology: considering the evolution of the Universe backwards in time, one can see that the scale factor does not become zero at any moment. Its value decreases to a minimum (bounce) and then starts increasing again. All the characteristic functions remain regular at the bounce [18], contrary to the presence of the initial singularity in the classical GR cosmology. Hence the theory is of scientific interest since large $\omega$ simply provides the agreement with observational data and also helps to avoid the cosmological initial singularity.

In the BD theory a possible way to take dark energy into account is to add a scalar field potential to the action [19, 20]. As was shown in [11] for an arbitary form of the scalar field potential, the evolution of the Universe could mimic the $\Lambda$CDM model (i.e. observational data) in BD theory. In this paper we consider spherically symmetric solutions in the BD-like model arising in the cosmological context in the paper by Elizalde et al [1]. The action has the form:

$$ S = \frac{1}{16\pi} \int d^4x \sqrt{-g} e^{\alpha \phi} \left[ \mathcal{R} - \omega \partial_\mu \phi \partial^\mu \phi - V_0 e^{\phi/\omega} \right], \quad (2) $$

Here $\alpha$ and $\phi_0$ are constants. In such a model the phantom regime arises in a BD-type scenario without the appearance of any ghost degree of freedom due to the specific form of the coupling. It is also shown in [1] that quantum gravity effects may prevent (or, at least, delay or soften) the otherwise unavoidable finite-time future singularity (Big Rip), associated with
the phantom. Although this action can seem quite ad hoc, it can naturally emerge in the low-energy limit of string/M-theory [1, 21], which means that the cosmological phantom can emerge naturally within the multidimensional unification theories. All this makes the consideration of this model very attractive from the theoretical point of view.

Elizalde et al consider late-time cosmology with an equation-of-state parameter value close to $-1$ (this value is favored by observations [2]). Under these circumstances an exact, spatially flat Friedman–Robertson–Walker (FRW) cosmology is constructed admitting acceleration phases for the current Universe. The model agrees with the observational data in a wide range of parameters. Such behaviour is not spoiled at the perturbation level since the scenario is free of perturbative instabilities [1]. All the above arguments make this model promising for further research.

Negative values of $\omega$ are often ruled out because of the wrong sign before the kinetic term in the action leading to a ghost. Nevertheless many researchers consider $\omega < 0$ for the following reasons. The latest observational data indicate that a phantom nature of dark energy is more likely [2]. Combining WMAP + eCMB + BAO + $H_0 + SNe$ data yields $w_{DE} = -1.17(\pm 0.13 - 0.12)$ for the flat Universe at the significance level of 68%. Such a trend also occurs for a non-flat Universe model and for different dataset combinations. Finally, consideration of a scalar field as an effective description of the theory with positive defined energy could eliminate the quantum contradictions [22, 23]. All these arguments make the consideration of BD theory with $\omega < 0$ reasonable, since it leads to a phantom cosmology. Modern observational bounds limit the absolute value of $\omega$. Modelling the gravitational collapse does not rule out negative $\omega$ values as well [24]. So the range $\omega < 0$ is of great interest from the phenomenological point of view, since it can provide no-ghost but phantom cosmology in agreement with modern observations.

In this paper we explore static, spherically symmetric solutions of (2). Cosmology studies the time dependence of the scalar field $\phi = \phi(t)$; this approximation stands for cosmological homogeneity. Understanding the local effects $\phi = \phi(r)$ yields a picture of field behaviour at different scales and checks for model consistency. Exploring the cosmological model from another point of view is a step towards a better understanding of dark energy.

The most well-known static, spherically symmetric solution in BD was obtained by Campanelli and Lousto [25]. This solution was independently rediscovered by Agnese and La Camera [26] and correctly reinterpreted as a naked singularity for $\gamma < 1$ and a wormhole for $\gamma > 1$ (here $\gamma$ is a PPN parameter). The scalar field plays the role of exotic matter at the wormhole’s throat and ensures its traversability not only for $\omega < 0$ but also for large positive $\omega$ [27]. Astrophysical properties were studied by Alexeyev et al [28] and found to be in agreement with modern observations.

The purpose of this paper is to explore the properties of the static, spherically symmetric solutions in the framework of Elizalde et al. To do this we analytically define the parametrization for the pure Agnese and La Camera solution and obtain a new analytical solution for the specific metric ansatz, which can be called a ‘stealth Schwarzschild solution’ (analogously to [29]). The presence of the scalar field potential modifies the field equations, so new types of solutions can emerge. We perform a numerical exploration of these solutions and discuss their properties. Elizalde et al construct a model in such a way that the ghosts can be absent, so it is interesting to check whether regular local solutions can appear in the ghost-free sector of the model.

Spherically symmetric solutions for similar actions are present in the literature. An asymptotically Lifshitz black hole solution for a power-law potential in a BD setting was found in [30] but the allowed $\omega$ parameter range appears to be too narrow to fit the observations. Another wormhole solution for the potential of the form $V(\phi) = \Lambda \phi$ is given in [31].
The latter is obtained for \( \phi = \phi(t) \); this wormhole is not static—its throat radius will increase as the cosmic time increases. To the best of our knowledge these are the only solutions in the BD setting with a scalar filed potential. The paper \([32]\) considers quantum wormhole solutions for BD gravity with \( \Lambda \) in the absence of matter perturbatively, from the field theory point of view. Although the framework of \([32]\) is similar to ours and can be obtained via \( V(\phi) = 2\Lambda \), the results of \([32]\) cannot be applied to astrophysics. Wormholes in solutions of the GR setting with exotic matter and \( \Lambda \) representing similar behaviour were found in \([33, 34]\). The cosmological constant affects the wormhole and its properties; the wormhole geometry will be dominated by de Sitter space far from the throat. Our solution is new and differs from those of \([33, 34]\) since the role of the exotic matter is played by the scalar field.

The outline of our paper is the following. In section 2 we study the generic properties of discussed BD-like solutions, section 3 is devoted to specific analytical solutions, in section 4 we obtain some new limitations on model parameters, section 5 is devoted to modifications to the inverse-square law and in section 6 we show the results of numerical study. Section 7 contains discussion and conclusions.

2. General properties

The corresponding Einstein and Klein–Gordon equations for the action (2) have the following form:

\[
\alpha \mathcal{R} + \omega \alpha \partial_\mu \phi \partial^\mu \phi - V_0 e^{\phi/\phi_0} \left( \alpha + \frac{1}{\lambda_0} \right) + 2 \omega \partial_\mu \phi = 0, \tag{3}
\]

\[
G_{\mu \nu} = \omega \left( \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \eta_{\mu \nu} \partial_\mu \phi \partial_\nu \phi \right) + e^{-\phi} \left( \nabla_\mu \nabla_\nu e^{\phi} - g_{\mu \nu} \Box e^{\phi} \right) - \frac{1}{2} g_{\mu \nu} V(\phi). \tag{4}
\]

Hawking’s theorem on BD theory \([35]\) is usually taken to mean that BD static, spherically symmetric solutions are exactly the same as those of GR. We would like to note that this is an overstatement. For example, the black hole solution of Campanelli and Lousto \([25]\) is spherically symmetric and static but does not match the Schwarzschild one. The proof of Hawking’s theorem relies on the equivalence between Jordan and Einstein frames and the fact that the corresponding conformal transformation is well defined. As is also pointed out in \([35]\) the advantage of using the Einstein frame is that in the Einstein frame the BD scalar has a canonical kinetic energy density and obeys the weak (WEC) and null (NEC) energy conditions. If the conformal transformation to the Einstein frame becomes ill defined at the horizon (as occurs in \([25]\)) then its variables \( g_{\mu \nu}, \phi_E \) (subscript \( E \) stands for Einstein frame) cannot be used on such a surface \([36]\). The last statement invalidates the proof of Hawking’s theorem. One could use, of course, the Jordan frame instead of the Einstein one. In this case the scalar \( \phi \) violates the WEC and NEC because its stress–energy tensor has a non-canonical structure containing second derivatives of \( \phi \) instead of being quadratic in the first derivatives.

So, if our Einstein frame scalar does not obey the WEC or the Jordan frame scalar diverges at the horizon then the Hawking theorem does not apply and the solution is not forced to be a Schwarzschild one.

One can perform the transformation to the Einstein frame \( g_{\mu \nu} e^{\phi} = g_{\mu \nu} \Omega^2 = g_{\mu \nu} \).

\[
S_E = \frac{1}{16\pi} \int d^4 x \sqrt{-g_E} \left[ \mathcal{R}_E - 2 \partial_\mu \phi \partial^\mu \phi - V(\phi) \right]. \tag{5}
\]
\[\begin{align*}
dx^2_E &= e^{\alpha x} dx^2_d - e^{\lambda x} dy^2_d - d\theta^2 + \sin^2 \theta \, d\varphi^2, \\
\tilde{\omega} &= (3\alpha^2/2 + \omega), \quad \tilde{V}(\phi) = V e^{-\alpha \phi} = V_{E0} e^{-\phi/\alpha_{E0}}.
\end{align*}\]

It follows from the expression for \(\tilde{\omega}\) (7) that, even if \(\omega\) is negative, in the case when

\[3\alpha^2/2 + \omega > 0\]

the effective kinetic energy of \(\phi\) becomes positive, similarly to the usual scalar field, and therefore the ghost does not appear, although the cosmology is phantom \([1]\). Introducing the four-velocity \(U^a = (e^{-\chi/2}, 0, 0, 0)\) we obtain the WEC in the form

\[\tilde{\omega} \phi' \phi' + \tilde{V}(\phi) \geq 0,\]

which is generally satisfied everywhere if \(V_{E0} \leq 0\) and \(3\alpha^2 + 2\omega > 0\).

Via redefinition \(\Phi = e^{\alpha \phi}\) one can obtain the theory in the form of the standard BD one plus potential:

\[S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ \phi R - \frac{\omega}{\alpha^2} \phi \phi^{\mu\nu} \phi_{\mu\nu} \right] - V_0 \Phi^{1+1/\alpha_0}.\]

Having in mind that we are now working with the ‘usual’ BD we can directly apply its well-known results in the weak field limit to our model. We obtain that \(G \sim \Phi^{-1} = e^{-\alpha \phi}\), where \(G\) is the effective gravitational constant.

### 3. Analytical solutions

Let us rewrite the field equations as in \([25]\). The Klein–Gordon equation (3) can be rewritten as

\[\alpha R - \omega \alpha \partial_\mu \phi \partial^\mu \phi - V_0 e^{\phi/\phi_0} \left( \alpha + \frac{1}{\phi_0} \right) + \frac{2\omega}{\alpha} e^{-\alpha \phi} \Box e^{\alpha \phi} = 0.\]

Multiplying (4) by \(g^{\mu\nu}\) we obtain

\[R - \omega \partial_\mu \phi \partial^\mu \phi - 2V_0 e^{\phi/\phi_0} + 3e^{-\alpha \phi} \Box e^{\alpha \phi} = 0.\]

Combining equations (12) and (11) we get

\[- \frac{2\omega}{\alpha} + 3\alpha \Box e^{\alpha \phi} - V_0 e^{\phi/\phi_0} \left( \alpha - \frac{1}{\phi_0} \right) e^{\alpha \phi} = 0.\]

Substituting (12) and (13) into the field equation (4) we arrive at the following equation:

\[R_{\mu\nu} = \omega \partial_\mu \phi \partial_\nu \phi + e^{-\alpha \phi} \nabla_\mu \nabla_\nu e^{\alpha \phi} + \frac{1}{2} g_{\mu\nu} D e^{\phi/\phi_0},\]

\[D = V_0 \left[ 1 - \left( \alpha - \frac{1}{\phi_0} \right) \frac{\alpha}{2\omega + 3\alpha^2} \right].\]

Further we obtain specific analytic solutions for fixed parameter relations or specific metric structure. Of course, the general analytical solution remains unknown. In section 6 we investigate the general solution numerically.
3.1. The case $D = 0$

When $D = 0$, or equivalently $2^{26}$, one has the well-known solution of Agnese and La Camera (ALC) [26], since the field equations match in this case. Within the framework of Elizalde et al this solution looks like

$$\phi_0 = \frac{\alpha}{2(\alpha^2 + \omega)},$$

(16)

one has the well-known solution of Agnese and La Camera (ALC) [26], since the field equations match in this case. Within the framework of Elizalde et al this solution looks like

$$\phi = \frac{n-m}{2\alpha^2} \ln \left[ F \left( \frac{1-\frac{2\eta}{r}}{r} \right) \right], \quad A(r) = 1 - \frac{2\eta}{r}$$

(18)

$$\eta = M\sqrt{(1 + \gamma)/2}, \quad m = \sqrt[2]{(1 + \gamma)}, \quad n = \gamma\sqrt[2]{(1 + \gamma)},$$

(19)

$$\frac{\omega}{\alpha^2} = \frac{m - 2n}{n - m}, \quad G = e^{-\omega \eta} \frac{2}{1 + \gamma},$$

(20)

with $\gamma$ being the PPN parameter, $\eta$ plays the role of the mass parameter, $G$ is the effective gravitational constant and $F$ can be fixed from the Newtonian limit. The parameter $\alpha$ comes from the coupling of Elizalde et al action (2). This solution corresponds to a naked singularity if $\gamma < 1$, to a Schwarzschild black hole for $\gamma = 1$ and to a wormhole if $\gamma > 1$. The scalar field diverges at the horizon and this is the reason for the violation of the Hawking theorem. The factor $\sqrt{(1 + \gamma)/2}$ is usually absorbed in the mass definition, which results in a correction of order of $10^{-6}$ for $|\gamma - 1| \sim 10^{-5}$.

It is worth noticing that the condition $D = 0$ preserves the potential term in the action but effectively removes it from the Einstein equations. If we suppose that the scalar field potential acts as the cosmological constant, we can arrive at the conclusion that the accelerated expansion of the Universe does not contribute to local dynamics. This effect is indifferent to the value of the cosmological constant.

To estimate the constant $F$ we must set up the Newtonian limit—in other words, get the model linearised. This work is shown in the appendix. Comparing (A11) with (18) for large distances, we arrive at

$$\phi_\infty = \frac{1}{\alpha} \ln \left( \frac{1}{G_0} \frac{2\omega + 4\alpha^2}{2\omega + 3\alpha^2} \right), \quad F = \left( \frac{1}{G_0} \frac{2\omega + 4\alpha^2}{2\omega + 3\alpha^2} \right)^{\frac{2n}{2m}}$$

(21)

For the power-law generalization of Schwarzschild (17) the equations (14) can only be solved for $m = n$, which leads to $D = 0$. So there are no other power-law solutions for the ansatz (17) except ALC here.

3.2. Schwarzschild-like solution

The other solution can be obtained if we suppose that we measure exactly the Schwarzschild metric

$$ds^2 = e^{\mu(r)} dt^2 - e^{\lambda(r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta \, d\varphi^2),$$

(22)
\[ \nu = \ln(1 - r_h/r) \quad \lambda = - \ln(1 - r_h/r). \tag{23} \]

This leads to the system:

\[ \alpha r^2 \phi'' + \gamma \left( \omega + \phi^2 \right) \phi' - r \phi' + X = 0 \tag{24} \]

\[ e^\lambda + \frac{1}{2} r \left( X + \phi' \right) + \frac{1}{2} r^2 \phi'' + \frac{1}{4} r^2 \phi'^2 - \frac{1}{4} r^2 \phi' X = 1 = 0 \tag{25} \]

\[ \frac{\omega}{2} r^2 \phi'^2 - \alpha r^2 \left( \frac{2}{r} + \frac{\phi' - X}{2} \right) \phi' + e^\lambda - r \phi' - 1 + \frac{1}{2} r^2 V_0 e^{\phi'/\omega} e^\lambda = 0. \tag{26} \]

Equation (24) can be straightforwardly integrated as

\[ \phi = \frac{\alpha}{\alpha^2 + \omega^2} \ln|C_1 r + C_2|, \tag{27} \]

Having in mind the redefinition (10), we see that \( G \) is a decreasing function of \( r \):

\[ G \sim \left( r + C_1 \right)^{-1 + \omega'/\omega}. \tag{28} \]

This solution can be called a ‘stealth Schwarzschild solution’, analogously to the solution of [29] (obtained within the Horndeski/Galilean framework) representing a Schwarzschild metric with non-trivial scalar field.

To estimate the constants in (28) we use the Newtonian limit—in other words, we linearise the model. The scalar \( \phi \) increases indefinitely with \( r \), so the theory can only be formally linearised as \( \phi = \phi_0 + \epsilon \) in the region outside the object \( |r| < C_1 \) where the Taylor expansion of the logarithm exists. The constant \( C_1 \) represents then the length scale up to which the solution admits Newtonian behaviour.

The obvious condition preserving (to a certain extent) the observational adequacy of the solution is \( |\omega| \gg |\alpha| \). It makes the power of \( r \) small and ensures that \( \phi \) does not change abruptly in space and varies less and less with the distance. This condition can also be reached from (30). The spacetime is asymptotically flat, but the gravitational constant vanishes at infinity.

The discovery of the discussed solution under the assumption that it is realized in nature seems tricky, since the metric is exactly the Schwarzschild one. It only manifests itself as a gravitational constant decreasing with distance. One could naively suggest that for astrophysical scales this effect would simply hide itself in the mass definition, since accurate mass measurements are done mostly for several binary systems due to their orbital motion [37]. Therefore a lower but very slowly changing value of the gravitational constant would just be absorbed into calculated masses: the less the distance, the larger the object masses would seem to us. This situation contradicts the dark matter effect if we suppose it to be dynamical and not caused by real matter.

4. Parameter estimations

From the paper by Elizalde et al [1] we can get the cosmologically imposed condition, providing the accelerated expansion of the Universe:

\[ \phi_0^2 > \frac{4}{3 \alpha^2 + 2 \omega}. \tag{29} \]
For $|\omega| \gg |\alpha^2|$ and $\omega$ negative this condition is always satisfied and is also in agreement with (16). This means that the ALC solution can represent a wormhole embedded in an expanding Universe, but not affected by this expansion at all.

The other possible parameter estimation comes from the PPN expansion. The observational bound is $[12] \gamma - 1 = (2.1 \pm 2.3) \times 10^{-5}$. Following [38] we derive the PPN parameters $\gamma$ and $\beta$:

$$\gamma - 1 = -\frac{\alpha^2}{2\alpha^2 + \omega}, \quad \beta - 1 \equiv 0. \quad (30)$$

When $|\omega| \gg |\alpha^2|$, taking into account $|\gamma - 1| < 10^{-5}$ we obtain

$$\alpha^2/|\omega| < 10^{-5}. \quad (31)$$

This requirement disagrees with no-ghost condition (8) for negative $\omega$. In fact the inequality above leaves no room for the no-ghost sector of the model. This result is a bit disappointing, but is not critical in view of the reasons regarding negative $\omega$ given in the introduction.

5. Modifications to the inverse-square law

The power-law modification of the inverse-square law (ISL) in the general form reads

$$\mathcal{M}(r) = \frac{G_0M}{r} \left[ 1 + a_N \left( \frac{\eta}{r} \right)^{N-1} \right]. \quad (32)$$

Expanding $g_{00}$ of the ALC solution (17)--(20) in binomial series and going to physical units, we get

$$\mathcal{M} = \sqrt{\frac{2}{1 + \gamma}} \frac{\eta \alpha^2}{r} \left[ 1 + \left( 1 - \sqrt{\frac{2}{1 + \gamma}} \right) \frac{\eta}{r} \right] = \frac{G_0M}{r} \left[ 1 + a_2 \frac{\eta}{r} \right], \quad (33)$$

$$a_2 = \left( 1 - \sqrt{\frac{2}{1 + \gamma}} \right), \quad \eta = \sqrt{\frac{1 + \gamma}{2}} \frac{MG_0}{c^2} \quad (34)$$

for $N = 2.6$. We are interested in the case when $\gamma > 1$, so $a_2 > 0$. It is worth noticing that the correction is mass-dependent. Papers [40--42] claim the following 1σ limits for power-law corrections:

$$N = 2 \quad a_N \eta^{N-1} < 1.3 \times 10^{-6} [m]. \quad (35)$$

For the conditions of the experiment [41] ($M \approx 10$ kg) we estimate

$$a_2 \eta = \frac{MG_0}{c^2} \left( \sqrt{\frac{1 + \gamma}{2}} - 1 \right) = \left( \sqrt{\frac{1 + \gamma}{2}} - 1 \right) \times 10^{-16} = \epsilon \times 10^{-16}. \quad (36)$$

The interesting paper [39] is worth mentioning here. The authors calculate advances in the perihelia of planets in the solar system by treating the power-law correction as a small disturbance and then connect them with the data of INPOP10a (IMCCE, France) and EPM2011 (IAA RAS, Russia) ephemerides (by means of minimizing the sum of squared deviations). They take the uncertainty in the Sun’s quadrupole moment into account and estimate it along with the parameters of the power-law correction. The result is $N = 0.605$ for the exponent. However, from EPM2011, they find that, although it yields $N = 3.001$, the estimated uncertainty in the Sun’s quadrupole moment is much larger than the value of $\pm 10\%$ given by current observations. So, no certain conclusions on the most plausible value of $N$ can be drawn.
Together with (35) this gives us $\epsilon < 10^{10}$. For the PPN parameter $|\gamma - 1| < 2.5 \times 10^{-3}$ we can estimate $\epsilon < 2.5 \times 10^{-6}$, which is a sufficiently stronger bound. This means that one needs to increase the experimental accuracy by 16 orders of magnitude to approach the PPN-based limit. The motion of a satellite in Earth’s gravitational field or the motion of a planet around the Sun could represent a more interesting case, where $M_\oplus \sim 10^{30}$ kg, $M_\odot \sim 10^{24}$ kg.

Following [43] we estimate the additional relative frequency shift $(\delta f / f)_p$ for the potential above and create bounds using the frequency measurement accuracy for the Galileo Navigation Satellite System (GNSS)$^7$.

$$
\left( \frac{\delta f}{f} \right)_p \approx \frac{M_\oplus G_0}{c^2} a_2 \eta \left( \frac{1}{R_\oplus^2} - \frac{1}{(R_\oplus + h)^2} \right) = \frac{M_\odot^2 G_0^2}{c^4} \epsilon \left( 1 - \frac{1}{(R_\odot + h)^2} \right),
$$

(37)

Considering the accuracy of the frequency measurement of the Galileo constellation $\varepsilon_f = 10^{-12}$ analogously to (36), we obtain

$$
\delta f / f \approx 5 \epsilon \times 10^{-13} < \varepsilon_f.
$$

(38)

The resulting bound appears to be $\epsilon < 2$. It is also much weaker than the PPN-based one. This result seems strange, since the experimental techniques that provide the PPN and GNSS bounds are similar. This peculiarity requires additional research.

6. Numerical results

Now we intend to analyse the general solution. For numerical calculations we use the following ansatz:

$$
\mathrm{d}s^2 = \Delta \mathrm{d}r^2 - \frac{\sigma^2}{\Delta} \mathrm{d}r^2 - R(r)^2 \mathrm{d}\Omega^2.
$$

(39)

Without loss of generality we set $\sigma = 1$ since this is equivalent to the radial coordinate transformation. From equation (14) we obtain the system of differential equations, which can be resolved with respect to highest derivatives, as

$$
\Delta''(r) = \frac{D R(r) \phi''(r)}{R(r)} - 2 \Delta'(r) R'(r),
$$

(40)

$$
R''(r) = \frac{-2 R(r) \Delta'(r) R'(r) - 2 \Delta(r) R'(r)^2 + D R(r) \phi'(r) + 2}{2 \Delta(r) R(r)},
$$

(41)

$$
\phi''(r) = - \frac{(\alpha^2 + \omega) \phi'(r)^2}{\alpha} + \frac{2 \left( \Delta(r) R'(r)^2 - 1 \right)}{\alpha \Delta(r) R(r)^2} - \frac{D R(r) \phi'(r)}{\alpha \Delta(r) R(r)} - \frac{2 \Delta(r) R'(r)}{\alpha \Delta(r) R(r)}. \tag{42}
$$

The most interesting scales from the phenomenological point of view are the astrophysical ones, such as neutron stars and binary systems. Hence we will consider a mass of the order of the solar one.

Under the requirement for the potential term in the action to transform to a cosmological constant in the limit $r \to \infty$, $\phi \to \phi_\infty$ we obtain that

$h = 23.222 \times 10^3$ km, $M_\odot = 5.972 \times 10^{24}$ kg, $R_\odot = 6371$ km, $G_0 = 6.673 \times 10^{-11}$ m$^3$ kg$^{-1}$ c$^{-2}$.
where $\Lambda$ is the cosmological constant. The asymptotic value of the scalar field is derived from (A11); $\phi_0$ is a free parameter. The initial values for the numerical integration are taken from the ALC solution. Summarizing the known bounds we can state that $0 < w < 10^{-3}$ (naked singularity in ALC) and $\alpha^2 < 10^{-4}|\omega|$ (PPN bound). For the sake of illustration we take $\omega = -10000, \alpha = 0.1, M = M_e^8$.

Since the values of $\omega$ and $\alpha$ under consideration provide WEC violation, and we require the correspondence with the Agnese and La Camera solution for small potential influence, we claim that the solution represents a wormhole embedded in an otherwise de Sitter Universe (see figure 1). The wormhole geometry is dominant near the throat while the structure of dS reveals itself as the distance increases.

The spacetime approaches the Schwarzschild metric when one continues increasing $\phi_0$ (this corresponds to reducing the potential influence) and differs slightly from Schwarzschild.

\[ V_0 = 2\Lambda e^{-\phi_0}/\phi_i, \quad \phi_\infty = \frac{1}{\alpha} \ln \left( \frac{1 - 2\omega + 4\alpha^2}{G\phi_0 2\omega + 3\alpha^2} \right). \]  

Figure 1. The solution represents a wormhole embedded in an otherwise de Sitter Universe ($\phi_0 = 24.5$). The wormhole geometry is dominant near the throat while the structure of dS reveals itself as the distance increases.

Figure 2. Figure represents the enumeration of $\phi_0$ values. Approximately at $\phi_0 \approx 22.7$ the solution turns from de Sitter to wormhole-like. The de Sitter solution describes the expanding Universe. The wormhole-like solution describes an astrophysical object. Thus the solution can describe both cosmological and astrophysical scales.

$\text{c} = 2.998 \times 10^8 \text{ m s}^{-1}, \ M_0 = 1.988435 \times 10^{30} \text{ kg}$. 

$8$
for $\phi_0 > 25$. For $22.7 \lesssim \phi_0 \lesssim 25$ the solution represents a dS wormhole. Lowering $\phi_0$ further (strengthening the potential influence), we see that spacetime approaches dS. For $\phi_0 < 22.7$ the wormhole throat does not form, suppressed by the influence of the potential term (dS case, see figure 2). The de Sitter solution describes the expanding Universe. The wormhole-like solution describes an astrophysical object. Thus the solution can describe both cosmological and astrophysical scales.

Figures 3 and 4 show that the given numerical solution is a wormhole analogue of the Schwarzschild–dS solution. The wormhole geometry is dominant near the throat (figure 3) while the structure of dS reveals itself as the distance increases (figure 4). The threshold on $\phi_0$ between dS and wormhole-like geometry appears near $\phi_0 = 22.7$ (see figure 3).

Solutions representing similar behaviour were found in [33, 34]. The papers [33, 34] consider the GR setting with exotic matter and $\Lambda$. The cosmological constant affects the wormhole and its properties; the geometry will be dominated by the wormhole near the throat and by the de Sitter space far from the throat, as it is for our solution. Our results are in agreement with [33, 34], but the role of exotic matter is played in our case by the scalar field.
Resetting the metric in a form

\[ ds^2 = e^{2P(r)} dr^2 - \frac{dr^2}{1 - b(r)/r} - r^2 d\Omega^2 , \]  

we can plot the shape function \( b(r) \) and the embedded surface \( z(r) \) \([33]\)

\[ \frac{dz}{dr} = \pm \left( \frac{r}{b(r)} - 1 \right)^{-1/2} . \]  

To be a solution of a wormhole, the geometry has to have a minimum radius, \( r = b(r) = r_0 \), i.e. the throat, at which the embedded surface is vertical, \( dz/dr \to \infty \) (see figure 5). For a wormhole to be traversable one must demand the absence of horizons identified as surfaces with \( e^{2P} \to 0 \), so that the function \( P(r) \) has to be finite everywhere. By the proper choice of
it is possible to avoid such a divergence at the throat. The numerical calculation shows
that for $\phi_0 = 24.5$ the throat radius is around $r = 5927$ m; this is definitely large enough for a
microscopic object to pass through. The corresponding ALC throat is at $r = 3014$ m, so the
potential influence is significant.

The difference between the Schwarzschild solution and numerical solution at the distance
around the innermost stable orbit $r = 6\eta$ is
\begin{align}
\Delta_{\text{num}}/\Delta_{\text{Schw}} &\approx 1.031 \quad \phi_0 = 24.5, \\
\Delta_{\text{num}}/\Delta_{\text{Schw}} &\approx 1.471 \quad \phi_0 = 23.5.
\end{align}

These values appear to be quite large, so black hole observations could probably place some
limits on $\phi_0$. There are observational indications that such a deviation might in fact
correspond to reality: the paper [44] claims that the apparent size of the source at the Galactic
centre is less than the expected apparent size of the event horizon of the presumed Sgr A
black hole. If this is not due to modelling issues, it could be thought of as an indication of GR
violation. To explore the properties of a general solution in more detail one requires the
analytical form of the solution, and we treat this as a task for the future.

We need to distinguish between the cases of a black hole, a wormhole and a naked
singularity. We check different parameter combinations and compare values of the
Kretschmann curvature invariant (see figure 6) and metric function $\Delta$ (39). The analysis of
multiple plots shows that only the case $\omega < 0$, $22.7 \lesssim \phi_0 \lesssim 25$ represents a regular solution:
the Kretschmann scalar is finite at $r > 0$. Therefore this case can represent a traversable
wormhole.

All other parameter ranges seem to represent a naked singularity for different values of $r$
(see figure 6 also). The cosmic censorship conjecture states that naked singularities do not
emerge in nature. In terms of wormhole research naked singularities can be interpreted as not
traversable ones, so they are of limited interest. However, the cosmic censorship conjecture
remains unproven and many researchers suggest naked singularities as candidates for astro-
physical objects [45, 46]. In this sense it would be interesting to check for the astrophysical
properties of the solution in corresponding parameter regions. We leave this for the next step
of our investigation.

7. Discussion and conclusions

We have explored analytical and numerical properties of static, spherically symmetric solu-
tions in the context of a BD-like cosmological model. For a certain parameter range
$\omega < 0$, $22.7 \lesssim \phi_0 \lesssim 25$ the model contains a regular wormhole solution. The spacetime
geometry is analogous to the Schwarzschild–dS one, representing a wormhole in the dS
Universe. The influence of the potential term on the wormhole is shown to be significant. The
properties of this newly obtained numerical wormhole solution require further research,
which is a task for our next paper.

As was mentioned in [47] the Jordan and Einstein frames for the theory are not
equivalent from the cosmological point of view. It was also argued in [48] (for BD theory
without the scalar field potential) that wormhole solutions in the Einstein frame are absent
unless $\bar{\omega} = 3a^2/2 + \omega < 0$ (in our notation). As follows from equation (9), the WEC can
be violated due to the potential alone and one does not need to violate it by setting $\bar{\omega} < 0$ by
‘brute force’ [48]. So, unlike the case of [48], wormholes are in principle possible in both
frames, although a detailed discussion is only possible with the analytical solution in hand.
However, the observational bound (31) implies that the WEC is violated.
From the beginning the discussed model had four free parameters. The original paper [1] provides one useful limitation on \( \phi_0 \) (29). We estimate the parameter \( V_0 \) using the cosmological constant and currently measured values of the gravitational constant in (43), reducing the number of free parameters by one. We also constrain \( \alpha \) and \( \omega \) parameters by (31). When considered together, the equations \( \omega < 0, \ 22.7 \lesssim \phi_0 \lesssim 25 \), (29), (31) and (43) put significant constraints on the parameter values compared with [1].

We demonstrate that when the equations (29), (31) and (43) are taken into account the model gives rise to a potentially traversable wormhole or a naked singularity. Moreover, since the numerical solutions can mimic the Schwarzschild one, the original model [1] is consistent with the astrophysical observational data. The last statement proves that the model could be successfully applied on both cosmological and astrophysical scales. However, differences between the numerical solution and the Schwarzschild one can be quite large (an example is the factor 1.471 in (46)–(47)), so observations of black hole candidates could probably place additional limits on \( \phi_0 \) if an analytical solution were obtained. The question is whether the proposed parameters can provide deviations visible to the Event Horizon Telescope (for example) and this requires further investigations. The numerical plotting does not allow one to explore all possible parameter combinations, so there are probably other parameter ranges where regular solutions exist, and our result represents just one example of such solutions.

This PPN-based requirement (31) disagrees with the no-ghost condition (8) for negative \( \omega \) and in fact leaves no room for the no-ghost sector of the model. This result is a bit disappointing, but is not critical because of the reasons regarding negative \( \omega \) given in the Introduction.

We also derive the power-law correction to the gravitational potential for the solution of Agnese and La Camera [26] and check whether it can be observed in ground-based experiments or via measuring relative frequency shift for the Galileo Navigation Satellite System (GNSS). We conclude that one needs to increase the experimental accuracy to approach the largest possible correction value.

We find that the ALC solution can represent the wormhole embedded in an expanding Universe but not affected by this expansion.

In summary, we conclude that BD theory, being an effective limit of \( f(R) \) gravity [49], describes a wide range of observed phenomena (ranging from cosmological to astrophysical scales), fits the observational data quite well, removes the cosmological initial singularity and provides a scale factor bounce in the early Universe. So taking into account all the other results from studies of BD theory, we would like to state that it is one of the finest candidates for the foundation stone of modern cosmology and extended gravity. Of course the scalar–tensor theories are not free of problems, especially when they are directly considered as dark energy candidates. Nevertheless the attention to phantom models is driven by the lack of a good theoretical understanding of the present Universe in terms of more usual theories. The phenomenology that emerges from these models is rich and pertinent at times, so these models deserve to be investigated comprehensively.

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Appendix

Equation (14) in the presence of matter transforms into

\[
R_{\mu\nu} = \omega \partial_{\mu} \phi \partial_{\nu} \phi + \frac{\nabla_{\mu} \nabla_{\nu} \phi}{e^{\alpha\phi}} + \frac{1}{2} \eta_{\mu\nu} \partial_{\nu} \phi \partial_{\mu} \phi + \frac{8\pi}{e^{\alpha\phi}} \left[ T_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \zeta \right]
\]

(A1)

In the weak field limit we can write

\[
\phi = \phi_{c} + \psi(r), \quad \frac{\psi}{\phi_{c}} \ll 1, \quad e^{\alpha\phi} \approx e^{\alpha\phi_{c}}, \quad \Box e^{\alpha\phi} \approx e^{\alpha\phi_{c}} \partial_{\mu} \partial^{\mu} \psi, \quad \eta_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}
\]

(A2)

where \(\eta_{\mu\nu}\) is the inkowski metric and \(h_{\mu\nu}\) are perturbations. Up to the first expansion order one obtains

\[
R_{\mu\nu}^{(1)} = \alpha \nabla_{\mu} \nabla_{\nu} \psi + \eta_{\mu\nu} \partial_{\nu} \phi \partial_{\mu} \phi + \frac{8\pi}{e^{\alpha\phi}} \left[ T_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \zeta \right]
\]

(A3)

\[
V_{1} = \frac{1}{2} \left( \partial_{\mu} \phi \right) \partial_{\mu} \phi
\]

(A4)

On the other hand

\[
2 \eta_{\mu\nu}^{(1)} = \Box h_{\mu\nu} + h_{\mu\nu} = h_{\mu\nu}^{\alpha} - h_{\mu\nu}^{\alpha} - h_{\nu\mu}^{\alpha}.
\]

(A5)

To simplify (A3) we impose the following conditions, using the weak field version of harmonic coordinate conditions in the case of a constant scalar field:

\[
h_{\mu\nu}^{\alpha} - h_{\mu\nu}^{\alpha} - h_{\mu\nu}^{\alpha} = -2\alpha \nabla_{\mu} \nabla_{\nu} \psi.
\]

(A6)

Equation (A3) can be rewritten as

\[
\frac{1}{2} \Box h_{\mu\nu} = \eta_{\mu\nu} V_{1} + \frac{8\pi}{e^{\alpha\phi}} \left[ T_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \zeta \right]
\]

(A7)

Let us take the spacetime and the scalar field generated by a static point-mass \(M\). Therefore the corresponding stress–energy tensor is given by

\[
T_{\mu\nu} = \text{diag}(M\delta(r), 0, 0, 0), \quad T_{\mu\nu}^{\alpha} = M\delta(r).
\]

(A8)

Therefore the metric perturbation reads as

\[
\Box h_{00} = 2V_{1} + \frac{16\pi e^{-\alpha\phi_{c}} \omega + 2\alpha^{2}}{2\omega + 3\alpha^{2}} M\delta(r)
\]

(A9)

\[
h_{00} = \frac{2G_{0}M}{r} + \frac{V_{1}r^{2}}{3},
\]

(A10)

where \(G_{0} = e^{-\alpha\phi_{c}} \frac{2\omega + 4\alpha^{2}}{2\omega + 3\alpha^{2}}\).

(A11)

Finally we obtain the weak field limit for the considered metric in the form

\[
g_{00} = 1 - \frac{2G_{0}m}{r} + \frac{V_{1}r^{2}}{3}.
\]

(A12)

This result agrees with the fact that one can consider the model [1] in the form of the standard BD (10) where \(G_{0}\) plays the role of the effective gravitational constant. Such a
solution reduces to the usual BD one when \( \alpha = 1, \varphi = e^f, V_0 = 0 \). After comparing it with the de Sitter–Schwarzschild metric we conclude that \( V_1 \) plays the role of the cosmological constant.

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