Graceful Labeling For bipartite graceful Graphs and related Graphs*

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Abstract: The concept of graceful labels was proposed by Rosa, scholars began to study graceful labels of various graphs and obtained relevant results. In this paper, let $G$ is a bipartite graceful graph, we proved that $S(t \ast G), P_s/(n \ast G), C(t \ast G)$ and $G^*$ are graceful labelings.

Key words: Graceful labeling, Bipartite graceful Graphs, Open star of graphs, Cycle of a graph, Path union of a graph

1. INTRODUCTION AND LEMMEL

The graceful labeling was introduced by A Rosa [1] in 1967. Golomb [2] proved that the complete bipartite graph is graceful. Barrientos [3] proved that union of complete bipartite graphs is also graceful. Vaidya [4] introduced a star of cycle. Kaneria and Makadia [5] proved that star of a cycle is graceful. Kaneria [6] proved that join sum of path union and star are graceful graphs. The detail survey of graph labeling refer Gallian [7].

Let $G$ is a bipartite graceful graph. In this paper, we proved that $S(t \ast G), P_s/(n \ast G), C(t \ast G)$ and $G^*$ are graceful graphs. We assume $G$ is a simple undirected, finite graph, with vertices and edges. For all terminology and notations we follow Harary [8]. We shall give brief summary of definitions which are useful in this paper.

Definition 1.1 A function $f$ is called graceful labeling of a $(p, q)$-graph $G$ if $f: V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ is injective and induced function $f^*: E(G) \rightarrow \{0, 1, 2, \ldots, q\}$ defined as $f(e)=|f(u)-f(v)|$ is bijective for every edge $e=uv$. A graph $G$ is called graceful graph if it admits a graceful labeling.

Definition 1.2 Let $(p, q)$-graph $G$ is a bipartite graph with the bipartition $(X, Y)$, $f$ is a graceful labeling of $G$, if $\text{max}\{f(x)\mid x \in X\} < \text{min}\{f(y)\mid y \in Y\}$, then $f$ is called as a bipartite graceful labeling of $G$. If $G$ admits a bipartite graceful labeling, then call $G$ is a bipartite graceful graph.

Definition 1.3 Let $G_1, G_2, \ldots, G_n (n \geq 2)$ be $n$ graph, Then the graph obtained by adding an edge from $G_i$ to $G_{i+1} (1 \leq i \leq n-1)$ is called path union of graph $G_1, G_2, \ldots, G_n$. We shall denote such graph by $P(G_1, G_2, \ldots, G_n)$.

If $G_1=G_2=\ldots=G_n=G$, we shall denote by $P(n \ast G)$.

Definition 1.4 Let $G_1, G_2, \ldots, G_n (n \geq 2)$ be $n$ graph, Then the graph obtained by replacing each vertex of $K_1, n$ except the apex vertex by the graphs $G_1, G_2, \ldots, G_n$ is known as open star of graphs. We shall denote such graph by $S(G_1, G_2, \ldots, G_n)$.

If we replace each vertices of $K_1, n$ except the apex vertex by a graph $G$, i.e. $G=G_1=G_2=\ldots=G_n$, such open star of a graph, we shall denote by $S(t \ast G)$.

Definition 1.5 Let $G_1, G_2, \ldots, G_n (n \geq 2)$ be $n$ graph, Then the graph obtained by replacing each edge of $K_1, n$ by a path of $P_n$ length $n$ on $n+1$ vertices is called one point union for $t$ copies of path $P_n$. We shall denote such graph $G$ by $P^t_n(G_1, G_2, \ldots, G_n)$.

Definition 1.6 Let $G_1, G_2, \ldots, G_n (n \geq 2)$ be $n$ graph, Then the graph obtained by replacing each vertices of except the central vertex by the graphs $G_1, G_2, \ldots, G_n$ is known as one point union for path of graphs. We shall denote such graph $G$ by $P^t_n(G_1, G_2, \ldots, G_n)$, where $P^t_n$ is the one point union of $t$ copies of path $P_n$. If we replace each vertices of $P^t_n$ except the central vertex by a graph $G$, i.e. $G=G_1=G_2=\ldots=G_n$ such one point union of path graph, we shall denote it by $P^t_n(tn \ast G)$.

Definition 1.7 For a cycle $C_n$, each vertices of $C_n$ is replace by connected graphs $G_1, G_2, \ldots, G_n$ is known as cycle of graphs and we shall denote it by $C(G_1, G_2, \ldots, G_n)$. If we replace each vertices by graph $G$, i.e. $G=G_1=G_2=\ldots=G_n$, such cycle of a graph $G$, we shall denote it by $C(n \ast G)$.

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Definition 1.8 A graph obtained by replacing each vertex of star $K_{1,n}$ by a graph $G$ of $n$ vertices is called star of $G$ and it is denoted by $G^*$. The graph $G$ which replaced at the center of $K_{1,n}$ we call the central copy of $G^*$.

Let $G$ is a bipartite graceful graph, In this paper, We also proved that $P(n \bullet G), S(r \bullet G)$ $P_n \times (m \bullet G)$, $C(t \bullet G)$ and $G^*$ are graceful graphs.

Obviously, there are following conclusions:

Lemma 1.1 $P_n$ is bipartite graceful graph.

Lemma 1.2 $C_n$ ($n \equiv 0 \pmod{4}$) is bipartite graceful graph.

Lemma 1.3 $K_{m,n}$ $(m, n \in \mathbb{N})$ is bipartite graceful graph.

Lemma 1.4 $P_n \times P_n$ $(m, n \in \mathbb{N})$ is bipartite graceful graph.

2 MAIN RESULTS

Theorem 2.1 Let $G$ is bipartite graceful graph and $G(i=1,2,\cdots,t)$ be $t$ copies of graph $G$, Then $P(n \bullet G)$ is graceful.

Proof: Let $(p_0,q_0)$-graph $G$ is a bipartite graph with the bipartition $(V_1,V_2)$, and $V_1 = \{u_i | i \in [1,m]\}$, $V_2 = \{v_j | j \in [1,r]\}$, $f_0$ is a bipartite graceful labeling of $G$, $f_0(u_i) = f_0(v_j), i \in [1,m], f_0(u_i) < f_0(v_j), j \in [1,r-1]$, and $f_0(u_i) < f_0(v_j)$. Let $u_i, v_j (i \in [1,m], j \in [1,r])$ be the vertices of $G$ which is $i^{th}$ copy of the path union of $n$ copies of $G$. We shall join $v_{i,1}$ with $u_{i,1}$.

We shall define labeling function of $P(n \bullet G)$ as follows:

\[ f: P(n \bullet G) \rightarrow \{0,q\}, \text{where } q = m(q_0+1)-1 \text{ defined by,} \]

\[ f(u_i) = f_0(u_i), \quad i \in [1,m]; \]

\[ f(v_j) = q - q_0 + f_0(v_j), \quad j \in [1,r]; \]

\[ f(u_i) = f_0(u_{i,1}) + f_0(u_1) + 1, \quad i \in [1,m], l \in [2,n]; \]

\[ f(v_j) = f(v_{j,1}) + f_0(v_1) - q_0 - 1, \quad j \in [1,r], l \in [2,n]. \]

Above labeling pattern give rise graceful labeling to the path union graph $P(n \bullet G)$ of $n$ copies of $G$.

![Figure 1](image1.png)

Figure 1. Graph $P_4$ and its bipartite graceful labeling

![Figure 2](image2.png)

Figure 2. Graph $P(5 \bullet P_4)$ and its bipartite graceful labeling

By theorem 2.1 and lemma 1.1–1.4, respectively, have the following consequences.

Corollary 2.1.1 $P(t \bullet P_n)$ is graceful.

Corollary 2.1.2 $P(t \bullet C_n)$ is graceful graph, where $n \equiv 0 \pmod{4}$.

Corollary 2.1.3 $P(t \bullet K_{m,n})$ is graceful.

Corollary 2.1.4 $P(t \bullet P_n \times P_n)$ is graceful.

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Theorem 2.2 Let $G$ be bipartite graceful graph and $G(i=1,2,\cdots,t)$ be $t$ copies of graph $G$, then $S(t \cdot G)$ is graceful.

Proof: Let $(p_0,q_0)$-graph $G$ is a bipartite graph with the bipartition $(V_1,V_2)$, and $V_1=[u_i|i \in [1,m]}$, $V_2=[v_j|j \in [1,n]}$, $f_0$ is a bipartite graceful labeling of $G$, $f_0(u_i)\neq f_0(u_{i+1})$, $i \in [1,m-1]$, $f_0(v_j)\neq f_0(v_{j+1})$, $j \in [1,n-1]$, and $f_0(u_0)\neq f_0(v_1)$. Let $S(t \cdot G)$ be a graph obtained by replacing each vertices of $K_{1,t}$ except the apex vertex by the graph $G$. Let $v_0$ is the apex vertex of $K_{1,t}$. i.e. it is central vertex of the graph $S(t \cdot G)$. Let $u_0(1 \leq m)$, $v_0(1 \leq n)$ be the vertices of $l^{th}$ copy of $G^0$ of $G$ in $S(t \cdot G), \forall l \in [1,t]$.

We shall join $v_{lm}$ with the vertex $v_0$ by an edge to form the open star of graphs $S(t \cdot G), \forall l \in [1,t]$, where $q=\pi(q_0+1)$. We define labeling function $f: V(G) \rightarrow [1,q]$ as follows:

\[
\begin{align*}
f(v_0) &= 0; \\
f(u_{i,j}) &= f_0(u_i)+1, \quad i \in [1,m]; \\
f(v_{1,j}) &= q-q_0+f_0(v_j), \quad j \in [1,n]; \\
f(u_{2,j}) &= f(u_{1,j})+q-q_0-1, \quad i \in [1,m]; \\
f(v_{2,j}) &= f(v_{1,j})-q+q_0+1, \quad j \in [1,n]; \\
f(u_{l,j}) &= f(u_{2,j})-(-1)q_0+1), \quad i \in [1,m], l \in [3,t]; \\
f(v_{l,j}) &= f(v_{l-1,j})-(-1)q_0+1), \quad j \in [1,n], l \in [3,t]
\end{align*}
\]

Above labeling pattern give rise graceful labeling to the graph and so it is a graceful graph.

![Graph Q and its bipartite graceful labeling](image1.png)

(a) Graph Q and its bipartite graceful labeling

![Graph $K_{4,3}$ and its bipartite graceful labeling](image2.png)

(b) Graph $K_{4,3}$ and its bipartite graceful labeling

Figure 3

![Figure 4. A graph obtained by open star of $Q$ and its graceful labeling](image3.png)

Figure 4. A graph obtained by open star of $Q$ and its graceful labeling.

![Figure 5. A graph obtained by open star of $K_{4,3}$ and its graceful labeling](image4.png)

Figure 5. A graph obtained by open star of $K_{4,3}$ and its graceful labeling.
By theorem 2.2 and lemma 1.1-1.4, there are following consequences respectively.

**Corollary 2.2.1** \( S(t \cdot P_n) \) is graceful.

**Corollary 2.2.2** \( S(t \cdot C_n) \) is graceful, where \( n \equiv 0 \pmod{4} \).

**Corollary 2.2.3** \( S(t \cdot K_{m,n}) \) is graceful.

**Corollary 2.2.4** \( S(t \cdot P_n \times P_n) \) is graceful.

**Theorem 2.3** Let \( G \) be a bipartite graceful graph and \( G_i \) \( (i = 1, 2, \cdots, t) \) be \( t \) copies of graph \( G \), then \( P_s(t \cdot G) \) is graceful.

**Proof** Let \( (p_0,q_0) \)-graph \( G \) is a bipartite graph with the bipartition \( (V_1,V_2) \), and \( V_i = \{ u_i | i \in [1,m] \} \), \( V_2 = \{ v_j | j \in [1,r] \} \), \( f_0 \) is a bipartite graceful labeling of \( G \), \( f_0(u_i) < f_0(u_{i+1}) \), \( i \in [1,m-1] \), \( f_0(v_j) < f_0(v_{j+1}) \), \( j \in [1,r-1] \), and \( f_0(u_m) < f_0(v_1) \). Let \( G \) be a graph obtained by replacing each vertices of \( P_n \) except the central vertex by the graph \( G \); i.e. \( P_s(t \cdot G) \) is the graph obtained by replacing each vertices of \( K_{1,t} \) except the apex vertex by the path union of \( n \) copies of the graph \( G \). Let \( u_0 \) be the central vertex for the graph \( P_s(t \cdot G) \). Let \( u_{i,t} (i \in [1,m]) \), \( v_{j,r} (j \in [1,r]) \) be the vertices of \( G \) which is \( P_b \) copy of the path union of \( n \) copies of \( G \) lies in \( s \)-th branch of the graph \( P_s(t \cdot G) \), \( l \in [1,n] \) and \( s \in [1,t] \).

First we shall join \( v_{i,t} \) with the vertex \( u_{i,1} \) by an edge to form the path union of \( n \) copies of \( G \) for \( s \)-th branch of \( P_s(t \cdot G) \), \( l \in [1,n] \) and \( s \in [1,t] \). Now we shall join \( u_{i,1} \) with the vertex \( u_0 \) by an edge to form the one point union graph \( P_s(t \cdot G) \), \( s \in [1,t] \). We shall define labeling function \( f \) for the first copy(branch) of the path union of \( n \) copies of \( G \) as follows:

\[
f: P(t \cdot G) \rightarrow [0,q] \text{, where } q = n(q_0+1) - 1 \text{ defined by,}
\]

\[
f(u_{i,1}) = f_0(u_i), \quad i \in [1,m];
\]

\[
f(v_{1,j}) = q - q_0 + f_0(v_j), \quad j \in [1,r];
\]

\[
f(u_{l,1}) = f_0(u_{l-1,m}) + f_0(u_0) + 1, \quad i \in [1,m], l \in [2,n];
\]

\[
f(v_{l,j}) = f_0(v_{j,1}) + f_0(v_0) - q_0 - 1, \quad j \in [1,r], l \in [2,n].
\]

Above labeling pattern give rise graceful labeling to the path union of \( n \) copies of \( G \) which lies in first branch of \( P_s(t \cdot G) \). Now we shall define labeling function \( g \) for \( P_s(t \cdot G) \), where \( Q = |E(P_s(t \cdot G))| = m(q_0+1) \) as follows:

\[
g(u_0) = 0;
\]

\[
g(u_{i,1}) = f(u_{i,1}) + 1, \quad i \in [1,m], l \in [1,n];
\]

\[
g(v_{1,j}) = f(v_{1,j}) + Q - q, \quad j \in [1,r], l \in [1,n];
\]

\[
g(u_{2,1}) = g(u_{1,1}) + Q - q - 1, \quad i \in [1,m], l \in [1,n];
\]

\[
g(v_{2,1}) = g(v_{1,1}) + Q - q + 1, \quad j \in [1,r], l \in [1,n];
\]

\[
g(u_{i,1}) = g(u_{i-1,1}) - (q_0+1), \quad i \in [1,m], l \in [1,n], s \in [3,t];
\]

\[
g(v_{i,1}) = g(v_{i-1,1}) + (q_0+1), \quad j \in [1,r], l \in [1,n], s \in [3,t].
\]

Above labeling pattern give rise graceful labeling to the graph \( P_s(t \cdot G) \) and so it is a graceful graph.

![Figure 6](image-url)

Figure 6: A graph \( Q \) and its bipartite graceful labeling.
By theorem 2.3 and lemma 1.1, respectively, have following consequences.

Corollary 2.3.1 $P_n(m \cdot P_i)$ is graceful.

Corollary 2.3.2 $P_n(t \cdot P_i)$ is graceful.

Corollary 2.3.3 $P_n^t(m \cdot C_m)$ is graceful, where $m \equiv 0 \pmod{4}$. 

Corollary 2.3.4 $P_n(t \cdot K_{m,m})$ is graceful.

Corollary 2.3.5 $P_n(t \cdot P_i \cdot P_j)$ is graceful.

Theorem 2.4 Cycle $C(t \cdot G)$ ($t \equiv 0 \pmod{2}$) of bipartite graceful graph $G$ is graceful graphs.

Proof Let $(p_0,q_0)$-graph $G$ is a bipartite graph with the bipartition $(V_1, V_2)$, and $V_1 = \{u_i | i \in [1,m]\}, \ V_2 = \{v_j | j \in [1,n]\},$ $f_0$ is a bipartite graceful labeling of $G$, $f_0(u_i) < f_0(u_{i+1}), \ i \in [1,m-1]$, $f_0(v_j) < f_0(v_{j+1}), \ j \in [1,n-1]$, and $f_0(u_m) < f_0(v_1).$ $C(t \cdot G)$ be a graph which contains $t$ copies of the bipartite graceful graph $G$, where $t \equiv 0 \pmod{2}$. Let $u_i(j \in [1,m])$ and $v_j(j \in [1,n])$ be vertices of $i^{th}$ copy of $G, i \in [1,t]$. Now join $v_{i,1}$ with $u_{i+1}, (i \in [1, (t/2)-1]),$ join $v_{i,1}$ with $v_{(t/2) + 1},$ join $u_{i,1}$ with $v_{1,1},$ and $v_{i,1}$ with $u_{i,1}$ by an edge to form cycle of graphs $C(t \cdot G)$.

We define the labeling function $f: V(C(tG)) \rightarrow [0,q], \ where \ q = t(q_0 + 1)$ as follows:

$f(u_{i,j}) = f_0(u_i), \ j \in [1,m];$

$f(v_{1,j}) = q - f_0(v_{1}) + f_0(v_j), \ j \in [1,n];$

$f(u_i,j) = f(u_{i-1,m}) + 1 + f_0(u_i), \ j \in [1,m], \ i \in [2, t/2];$

$f(v_{1,i}) = f(v_{1,1}) - 1 - f_0(v_1) + f_0(v_{i+1}), \ j \in [1,n], \ i \in [2, t/2];$

$f(u_{i,2t-1}) = f(v_{2,1}) - 2 - f_0(u_m) + f_0(u_{i+1}), \ j \in [1,m];$
Above labeling pattern give rise graceful labeling to the graph $C(tG)$ and so it is a graceful graph.

Figure 9. $G_0$ and its bipartite graceful labeling

By theorem 2.4 and lemma 1.1-1.4, respectively, have following consequences.

**Corollary 2.4.1** Cycle of path graph $C(t \cdot P_n)$ is graceful.

**Corollary 2.4.2** Cycle of $C_{4m}$ graphs $C(t \cdot C_{4m})$, $i \equiv 0 \pmod{2}$ is graceful graphs.

**Corollary 2.4.3** Cycle of complete bipartite graphs $C(t \cdot K_{m,n})$, $t \equiv 0 \pmod{2}$, $m,n \in N$ is graceful graphs.

**Corollary 2.4.4** Cycle of grid graph $C(t \cdot P_r \times P_s)$ is graceful.

**Theorem 2.5.** $G^*$ the star of a bipartite graceful graph $G$ is graceful graphs.

**Proof** Let $G$ be a bipartite graph with the bipartition $(V_1, V_2)$, and $V_1 = \{u_i \mid i \in [1,m] \}$, $V_2 = \{v_j \mid j \in [1,n] \}$, $f_0$ is a bipartite graceful labeling of $G$, $f_0(u_i) \neq f_0(v_j)$, $i \in [1,m-1]$, $f_0(v_j) \neq f_0(v_{j+1})$, $j \in [1,n-1]$, and $f_0(u_m) \neq f_0(v_1)$. Let $u_{0j}$ ($1 \leq i \leq m$), $v_{0j}$ ($1 \leq j \leq n$) be vertices of central copy of $G$ and $u_{ij}$ ($1 \leq i \leq m$), $v_{ij}$ ($1 \leq j \leq n$) be vertices of other copies of $G$, $\forall$ $l = 1, 2, \cdots, m + n$. We define labeling function $f: V \rightarrow \{0,1, \cdots, q\}$, where $q = (m+n+1)q_0 + m + n$ as follows:

- $f(u_0) = f_0(u_i), \quad i \in [1,m]$;
- $f(v_0) = q - f_0(v_j) + f_0(v_i), \quad j \in [1,n]$;
- $f(u_i) = f(v_{i-1}) - f_0(u_i) + f_0(v_i), \quad i \in [1,m]$;
- $f(v_i) = f_0(v_i), \quad j \in [1,n]$;
- $f(u_{ij}) = f(u_{i-1}) + q_0 + 1, \quad i \in [1,m], l \in [1,m+n]$ and $l$ even;
- $f(v_{ij}) = f(v_{i-1}) - q_0 - 1, \quad j \in [1,n], l \in [1,m+n]$ and $l$ even;
- $f(u_{ij}) = f(u_{i-1}) - q_0 - 1, \quad i \in [1,m], l \in [1,m+n]$ and $l$ odd;
- $f(v_{ij}) = f(v_{i-1}) + q_0 + 1, \quad j \in [1,n], l \in [1,m+n]$ and $l$ odd.
Above defined labeling function \( f \) give rise edge labels \( 1, 2, \cdots, q_0, q_0 + 2, q_0 + 3, \cdots, 2q_0 + 1, 2q_0 + 3, 2q_0 + 4, \cdots, 3q_0 + 2, \cdots, (q_0 + 1)(m + n + 1), \cdots, q \) to all the copies of \( G \). To make \( G^\ast \) as graceful graph \( q_0 + 1, 2(q_0 + 1), \cdots, (m + n)(q_0 + 1) \) edge labels to be require. Now we see that the difference of vertex labels for the central copy \( G^{(0)} \) with its other copies \( G^{(i)} \), \( 1 \leq i \leq m + n \) is precisely the following sequence.

\[
\begin{align*}
(m + n)(q_0 + 1) \\
q_0 + 1 \\
(m + n - 1)(q_0 + 1) \\
2(q_0 + 1) \\
\cdots \\
\end{align*}
\]

\( (m + n)(q_0 + 1)/2 \) when \( m + n \equiv 0 \mod 2 \) or \( (m + n - 1)(q_0 + 1)/2 \) when \( m + n \equiv 1 \mod 2 \)

Using above sequence we can produce required edge labels by joining corresponding vertex of \( G^{(0)} \) with its other copies of \( G^{(i)} \), \( \forall i = 1, 2, \cdots, m + n \). Thus \( G^\ast \) admits a graceful labeling and so it is a graceful graph.

By theorem 2.5 and lemma 1.1-1.4 respectively, have following consequences.

**Corollary 2.5.1** \( P_n^\ast \) of path graph \( P_n \) is graceful

**Corollary 2.5.2** \( C_{4m}^\ast \) of \( C_{4m} \) graphs is graceful graphs.

**Corollary 2.5.3** \([6]\) \( K_{m,n}^\ast \) of complete bipartite graphs is graceful graphs.

**Corollary 2.5.4** \([12]\) \( (P_r \times P_s)^\ast \) of grid graph \( P_r \times P_s \) is graceful.

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