Classical Motives : Motivic $L$-functions

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The exposition here follows the lecture delivered at the summer school, and hence, contains neither precision, breadth of comprehension, nor depth of insight. The goal rather is the curious one of providing a loose introduction to the excellent introductions that already exist, together with scattered parenthetical commentary. The inadequate nature of the exposition is certainly worst in the third section. As a remedy, the article of Schneider [37] is recommended as a good starting point for the complete novice, and that of Nekovar [34] might be consulted for more streamlined formalism. For the Bloch-Kato conjectures, the paper of Fontaine and Perrin-Riou [19] contains a very systematic treatment, while Kato [25] is certainly hard to surpass for inspiration. Kings [28], on the other hand, gives a nice summary of results (up to 2003).

1 Motivation

Given a variety $X$ over $\mathbb{Q}$, it is hoped that a suitable analytic function

$$\zeta(X, s),$$

a $\zeta$-function of $X$, encodes important arithmetic invariants of $X$. The terminology of course stems from the fundamental function

$$\zeta(\mathbb{Q}, s) = \sum_{n=0}^{\infty} n^{-s}$$

named by Riemann, which is interpreted in this general context as the zeta function of $\text{Spec}(\mathbb{Q})$. A general zeta function should generalize Riemann’s function in a manner similar to Dedekind’s extension to number fields. Recall that the latter can be defined by replacing the sum over positive integers by a sum over ideals:

$$\zeta(F, s) = \sum_{I} N(I)^{-s}$$

where $I$ runs over the ideals of the ring of integers $\mathcal{O}_F$ and $N(I) = |\mathcal{O}_F/I|$, and that $\zeta(F, s)$ has a simple pole at $s = 1$ (corresponding to the trivial motive factor of $\text{Spec}(F)$, as it turns out) with

$$(s - 1)\zeta(F, s)|_{s=1} = \frac{2^r(2\pi)^{r_2}h_FR_F}{w_FR_F\sqrt{|D_F|}}$$

By the unique factorization of ideals, $\zeta(F, s)$ can also be written as an Euler product

$$\prod_{P}(1 - N(P)^{-s})^{-1}$$

as $P$ runs over the maximal ideals of $\mathcal{O}_F$, that is, the closed points of $\text{Spec}(\mathcal{O}_F)$. Now, if a scheme $\mathcal{Y}$ is of finite type over $\mathbb{Z}$, then for any closed point $y \in \mathcal{Y}$, its residue field $k(y)$ is finite. Write $N(y) := |k(y)|$. We can then form an Euler product [38]

$$Z(\mathcal{Y}, s) := \prod_{y \in \mathcal{Y}_0} (1 - N(y)^{-s})^{-1},$$

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where \((\cdot)_0\) denotes the set of closed points for any scheme \((\cdot)\). In the case when the map

\[ \mathcal{Y} \to \text{Spec}(\mathbb{Z}) \]

factors through \(\text{Spec}(\mathbb{F}_p)\), \(Z(\mathcal{Y}, s)\) reduces to Weil’s zeta function for a variety over a finite field (with the substitution \(p^{-s} \mapsto t\) if a formal variable has intervened as in [35], section 1.6).

When we are starting with \(X/\mathbb{Q}\), a straightforward imitation of Dedekind’s definition might involve taking an integral model \(X\) of \(X\), which is a proper flat scheme of finite-type over \(\mathbb{Z}\) with \(X\) as generic fiber, and defining

\[ \zeta(X, s) := \prod_{x \in X_0} (1 - N(x)^{-s})^{-1} \]

The problem with this approach is that the function thus obtained will depend on the model, and there is no general method for choosing a canonical one. However, there will be some set \(S\) of primes such that there is a model \(X_S\) over \(\text{Spec}(\mathbb{Z}[1/S])\) which is furthermore smooth. Even though such a \(\mathbb{Z}[1/S]\)-model need be no more canonical, it does turn out that the incomplete zeta function

\[ \zeta_S(X, s) := \prod_{x \in (X_S)_0} (1 - N(x)^{-s})^{-1} \]

is independent of the model. (More on this point below.) So there are good elementary generalizations of incomplete zeta functions. We note in this connection that

\[ Z(X, s) = \prod_p Z(X_p, s) \]

where

\[ X_p = X \otimes \mathbb{F}_p \]

is the reduction of \(X\) modulo \(p\), so that that

\[ \zeta_S(X, s) = \prod_{p \notin S} Z(X_p, s) \]

is the result of deleting a few Euler factors. Thus, the problem of defining a canonical zeta function becomes one of inserting canonical factors for the primes of bad reduction. It is not impossible that there is a theory of integrals models that isolates a class that is canonical enough to yield a good definition of \(\zeta(X, s)\). But the current approach proceeds instead to break up partial zeta function into natural factors

\[ \zeta_S(X, s) = \prod L_S(M_i, s)^{\pm 1}, \]

according to the way \(X\) is decomposed into constituent motives \(\{M_i\}\) in a suitable category. (It is not much of an exaggeration to say that the decomposition of zeta functions is the main empirical phenomenon leading to the hypothesis of a category of motives.) The incomplete \(L\)-functions \(L_S(M_i, s)\) of the \(M_i\) should then encode arithmetic invariants of the \(M_i\), which, in turn, refine the arithmetic invariants of \(X\). It is believed that good analytic properties must be established to access the invariants efficiently, including functional equations. This, in turn, requires us to complete the \(L\)-functions using cohomological machinery in general. The completed \(L\)-functions then will lead to a completed zeta function.

A simple illustration is provided by the elementary example of an elliptic curve \(E/\mathbb{Q}\) with affine equation

\[ y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \]

Let \(E_S\) be a smooth and proper \(\mathbb{Z}[1/S]\) model. Then

\[ \zeta_S(E, s) := Z(E_S, s) \]
It is not very hard to check that
\[ \zeta_S(E, s) = \zeta_S(\mathbb{Q}, s)\zeta_S(\mathbb{Q}, s - 1)/L_S(H^1(E), s) \]
([40], V.2.4) illustrating the kind of decomposition alluded to above. Here
\[ \zeta_S(\mathbb{Q}, s) = \sum_{(n, p)=1, \forall p \in S} n^{-s} \]
is a standard incomplete zeta function and
\[ L_S(H^1(E), s) = \prod_{p \notin S} L_p(H^1(E), s) \]
is the incomplete L-function of E with factors defined by
\[ L_p(E, s) = \frac{1}{1 - a_p p^{-s} + p^{1-2s}} \]
Here \( a_p = p + 1 - N_p \) and \( N_p \) is the number of points on \( E \) mod \( p \). \( L_S(E, s) \) turns out to be the partial L-function corresponding to the motivic factor \( H^1(E) \) of \( E \).

We can put in Euler factors for \( p \in S \). It is obvious how to do it for \( \zeta_S(\mathbb{Q}, s) \) and \( \zeta_S(\mathbb{Q}, s - 1) \) giving us the Riemann zeta function \( \zeta(\mathbb{Q}, s) \) and its shift \( \zeta(\mathbb{Q}, s - 1) \) respectively. For the incomplete \( L_S(H^1(E), s) \), we put in the factors according to a recipe determined by the reduction of \( E \) at \( p \):
\[ L_p(H^1(E), s) = \begin{cases} 
1/(1 - p^{-*}) & \text{split multiplicative;} \\
1/(1 + p^{-*}) & \text{non-split multiplicative;} \\
1 & \text{additive.}
\end{cases} \]
([41], II.10) and define
\[ L(H^1(E), s) := \prod_p L_p(H^1(E), s) \]
Here we have used the breakdown of the incomplete zeta function into three factors as an aid in defining the full zeta function of \( E \). However, this case is somewhat misleading in that there is a canonical model that could have been used instead, namely, the Weierstrass minimal model
\[ E \]
that appears in basic textbooks. In fact, one can check that
\[ \zeta(E, s) = \zeta(\mathbb{Q}, s)\zeta(\mathbb{Q}, s - 1)/L(H^1(E), s) = Z(E, s) \]
as follows from the trace formula ([40], V.2) for the Frobenius map on elliptic curves for \( p \notin S \), and a much easier counting argument for \( p \in S \). So this would seem to be an instance where the naive extension of Dedekind’s method works out. Nevertheless, we explain how the bad factors can be obtained without reference to the model, starting at this point to use the language of étale cohomology [52]. In the sequel, we fix an algebraic closure \( \bar{\mathbb{Q}} \) of \( \mathbb{Q} \), closures \( \bar{\mathbb{Q}}_p \) of \( \mathbb{Q}_p \), and embeddings \( \bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p \). Therefore, we have embeddings of Galois groups
\[ G_p := \text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to G := \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \]
The residue field of \( \bar{\mathbb{Q}}_p \) is an algebraic closure \( \bar{\mathbb{F}}_p \) of \( \mathbb{F}_p \), and we have an exact sequence
\[ 0 \to I_p \to G_p \to \text{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p) \to 0 \]
defining the inertia subgroup \( I_p \). Denote by \( F_{r_p} \) the generator of \( \text{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p) \) that takes \( x \) to \( x^{1/p} \). Finally, \( \mathcal{E}_p \) denotes the basechange of \( \mathcal{E} \) to \( \mathbb{F}_p \) and \( \mathcal{E} \) the basechange of \( E \) to \( \bar{\mathbb{Q}} \). We need the étale cohomology
\[ H^1(\mathcal{E}, \mathbb{Q}_l) \]
for primes $l$, and

$$H^1(\mathcal{E}_p, \mathbb{Q}_l)$$

for $l \neq p$. By the Lefschetz trace formula ([32], VI.12.3),

$$Z(\mathcal{E}_p, s) = \frac{\det([I - p^{-s}Fr_p]|H^1(\mathcal{E}_p, \mathbb{Q}_l))}{\det([I - p^{-s}Fr_p]|H^0(\mathcal{E}_p, \mathbb{Q}_l)) \det([I - p^{-s}Fr_p]|H^2(\mathcal{E}_p, \mathbb{Q}_l))}$$

But for each $i = 0, 1, 2$,

$$H^i(\mathcal{E}_p, \mathbb{Q}_l) \simeq H^i(\tilde{E}, \mathbb{Q}_l)^{I_p}$$

the superscript referring to the subspace of elements fixed by the inertia action. (For $H^0$ and $H^2$, this is an easy exercise. The $H^1$ case is slightly harder. See [32], proof of theorem V.3.5. Although the discussion there is given for smooth surfaces fibered over ‘geometric’ curves, it is rather straightforward to adapt it to the present situation.) For $p \notin S$, any pair $X \hookrightarrow \mathcal{X}$ as above satisfies

$$H^i(X_p, \mathbb{Q}_l) \simeq H^i(\tilde{X}, \mathbb{Q}_l)$$

where the $I_p$-action must be trivial, and provides the reason that the incomplete zeta function is independent of the model ([32], VI.4.1). In any case, it ends up that the bad factor could have been written

$$Z(\mathcal{E}_p, s) = \frac{\det([I - p^{-s}Fr_p]|H^1(\tilde{E}, \mathbb{Q}_l)^{I_p})}{\det([I - p^{-s}Fr_p]|H^0(\tilde{E}, \mathbb{Q}_l)^{I_p}) \det([I - p^{-s}Fr_p]|H^2(\tilde{E}, \mathbb{Q}_l)^{I_p})}$$

in a way that refers only to $E$. It is this formula that generalizes to arbitrary motives.

Since we have thus far been entirely cavalier about convergence, we note in passing that the estimate $|a_p| \leq 2 \sqrt{p}$ ([30], V.II) implies that the Euler product converges for $\text{Re}(s) > 3/2$.

To control fine analytic properties, one establishes a relation to automorphic $L$-functions. For elliptic curves such a relation can be made explicit by computing the conductor

$$N_E := \prod_{p \in S} p^{f_p}$$

Here

$$f_p = \text{ord}_p(\Delta_E) + 1 - m_E$$

where $\Delta_E$ is the discriminant of $E$ and $m_E$ is the number of geometric components (that is, components over $\mathbb{F}_p$) of the special fiber of the Neron model of $E$. Even though this formula for $f_p$ again refers to the model, it can be defined purely in terms of the Galois action on $H^1(\tilde{E}, \mathbb{Q}_l)$ ([11], IV.10).

The well-known and deep fact, established through the work of Wiles, Taylor-Wiles, and Breuil-Conrad-Diamond-Taylor ([17], [16], [10]), is that $L$ has an analytic continuation to the complex plane. More precisely,

$$L(E, s) = L(f_E, s) = \frac{1}{(2\pi)^s \Gamma(s)} \int_0^\infty f_E(iy)y^{s-1}dy$$

$$= \frac{1}{(2\pi)^s \Gamma(s)} \left[ \int_{1/\sqrt{N_E}}^\infty f_E(iy)y^{s-1}dy + \int_{1/\sqrt{N_E}}^\infty f_E(iy)y^{1-s}dy \right]$$

for a normalized weight 2 new cusp form $f_E$ of level $N_E$ which is an eigenvector for the Hecke operators, determined by a $q$ expansion

$$f_E = q + a_2 q^2 + \cdots$$

where the $a_p$ have to be the same as those for $E$ and the general coefficient is determined by those with prime index. Because the weight 2 cusp forms of level $N_E$ form a finite dimensional space, it is easy to see that $f_E$ is completely determined by a finite computation of the $a_p$’s, and that this integral formula can then be used to compute $L$-values.
The celebrated conjecture of Birch and Swinnerton-dyer (BSD) [5] says

$$\text{ord}_{s=1} L(E, s) = \text{rank} E(\mathbb{Q})$$

The equality is known if \(\text{ord}_{s=1} L(E, s) \leq 1\) by the work of Gross-Zagier and Kolyvagin ([20], [29]).

Recall that

$$E(\mathbb{Q}) = \mathbb{Z}^r \times E(\mathbb{Q})_{\text{tor}}$$

where the finite abelian group \(E(\mathbb{Q})_{\text{tor}}\) is easily computed using the Nagell-Lutz theorem ([40], VIII.I.7).

This conjecture promises to give an analytic approach to understanding the elusive rank \(r\). However, it must be admitted that even though the \(L\)-function is computable, the utility of this equality for actually computing the rank of an elliptic curve is somewhat ambiguous. This is because the order of zero of an analytic function might not be possible to determine using a finite computation. We will discuss below how the vanishing itself can be computably determined using the refined version of this conjecture. On the other hand, an extremely useful viewpoint on the order of vanishing arises from the functional equation. That is to say, one inserts a gamma factor, determined by the Hodge theory of \(E\) and viewed as the contribution of the prime at infinity. With another correction factor contributed by the conductor, we arrive at a further completion:

$$\Lambda(E, s) := (2\pi)^s \Gamma(s) N_E^{s/2} L(E, s)$$

which then satisfies a functional equation

$$\Lambda(E, 2 - s) = w_E \Lambda(E, s)$$

where \(w_E = \pm 1\) depends on the curve \(E\). In fact, \(w_E\) can be expressed as a product of local terms

$$w_E = \prod_p w_{E,p}$$

each of which can be computed in a straightforward fashion ([6], section 6).

A significant corollary is that the parity of the order is determined by the sign of \(w_E\), usually referred to as the sign of the functional equation. For example, if \(w_E = -1\), then clearly

$$L(E, 1) = 0$$

Suppose you can check \(L'(E, 1) \neq 0\) using the equality with \(L'(f, 1)\) (non-vanishing \(can\) be verified!). Then we conclude that \(E(\mathbb{Q})\) has rank one. Thus, one can produce many examples where the refined analysis of the \(L\)-function, including the functional equation and computation, gives us the complete structure of \(E(\mathbb{Q})\). (For deeper developments in this direction, see [31].)

The BSD conjecture continues as follows. If \(r\) is the order of vanishing, then

$$(s - 1)^{-r} L(E, s)|_{s=1} = |\text{Sha}(E)| R_E \Omega \prod_p c_p / |E(\mathbb{Q})_{\text{tor}}|^2.$$
rational term is the order of \( \text{Sha}(E) \), the Tate-Shafarevich group of \( E \), conjectured to be finite. It is defined as the kernel

\[
0 \to \text{Sha}(E) \to H^1(G, E(\bar{\mathbb{Q}})) \to \prod_p H^1(G_p, E(\bar{\mathbb{Q}}))
\]

of the localization map on classifying spaces of torsors for \( E \) in the \( \text{étale} \) topology of \( \text{Spec}(\mathbb{Q}) \) as \( p \) runs over all primes of \( \mathbb{Q} \).

Then there are the transcendental terms: \( \Omega \), the real period (or twice that), defined as an integral

\[
\Omega = \int_{E(\mathbb{R})} |\omega|
\]

where \( \omega = dx/(2y + a_1x + a_3) \). The period can be easily computed, but the inaccessible part is the regulator \( R_E \). This is the covolume of the lattice \( E(\mathbb{Q})/E(\mathbb{Q})_{\text{tor}} \) inside the inner product space

\[
([E(\mathbb{Q})/E(\mathbb{Q})_{\text{tor}}] \otimes \mathbb{R}, <\cdot, \cdot>)
\]

where \( <\cdot, \cdot> \) is defined by the Neron-Tate canonical height. Thus, if \( \{P_1, P_2, \ldots, P_r\} \) is a basis for

for \( E(\mathbb{Q})/E(\mathbb{Q})_{\text{tor}} \), then

\[
R_E := |\det(<P_i, P_j>)|
\]

Obviously, computation of \( R_E \) would require knowledge of the Mordell-Weil group. On the other hand, since the formula gives a computable bound for the denominator of \( L(E, 1)/\Omega \) when \( R_E = 1 \), assuming its validity allows us to verify the vanishing of \( L(H^1(E), 1) \) after a finite computation. We refer the reader to [36] for an accessible report on the BSD conjecture, covering work up to 2002.

The known relations between \( L \)-functions and arithmetic are expected to generalize vastly. As indicated above, \( L \)-functions are defined using Galois actions on \( \text{étale} \) cohomology and completed using Hodge theory.

Before we summarize the relevant definitions, we recall the big picture represented by the following conjectures:

(1) Hasse-Weil: the completed \( L \)-function has a meromorphic continuation to the complex plane and satisfies a functional equation. This conjecture is supposed to be addressed by Langlands’ program, which says ‘Motivic \( L \)-functions are automorphic \( L \)-functions’ ([30]).

(2) Conjectures about values:

(a) Deligne’s conjecture gives a general definition of periods (in non-vanishing case) using comparison of rational De Rham and topological cohomologies ([12]):

(b) The Beilinson conjectures continue the discussion of orders of vanishing and the regulator using the rank and covolume of motivic cohomology ([1], [2], [3], [4]).

(c) The Bloch-Kato conjectures generalize the discussion of the rational part using Tamagawa numbers (or determinants) for Galois representations via \( p \)-adic Hodge theory ([3], [25], [19]).

2 Definitions

Let \( X/\mathbb{Q} \) be a smooth projective variety as before. Associated to \( X \) is a well-known collection of cohomology groups, the \textit{realizations} of the motive(s) of \( X \).

\[
H^n_l(X) = H^n_{\text{ét}}(\bar{X}, \mathbb{Q}_l)
\]

for each prime \( l \), the \( \mathbb{Q}_l \)-coefficient \( \text{étale} \) cohomology of degree \( n \).

This carries a natural action of \( G = \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \).

\[
H^n_{DR}(X) := H^n(X, \Omega_X),
\]

the algebraic De Rham cohomology equipped with a Hodge filtration given by

\[
F^i H^n_{DR}(X) = H^n(X, \Omega_X^{\geq i}) \to H^n_{DR}(X)
\]

for each \( i \).

\[
H^n_{B}(X) := H^n(X(\mathbb{C}), \mathbb{Q}),
\]

the \( \mathbb{Q} \)-coefficient singular cohomology of the complex manifold \( X(\mathbb{C}) \) equipped with a continuous action \( F_\infty \) of complex conjugation.
The completed $L$-function of $H^n(X)$ uses all these structures. These cohomology groups are bound together by an intricate system of canonical comparison isomorphisms. For example,

$$H^n_B(X) \otimes \mathbb{Q}_l \simeq H^n_l(X)$$

preserving the action of $F_\infty$, the complex conjugation. And then,

$$H^n_B(X) \otimes \mathbb{C} \simeq H^n_{DR}(X) \otimes \mathbb{C}.$$  

This isomorphism endows the pair $(H^n_B(X), H^n_{DR}(X) \otimes \mathbb{R})$ with a rational Hodge structure of weight $n$ ‘defined over $\mathbb{R}$.’ That is, we have a direct sum decomposition

$$H^n_B(X) \otimes \mathbb{C} \simeq \bigoplus H^{p,q}(X)$$

where

$$H^{p,q} := F^p \cap \tilde{F}^q$$

and

$$F_\infty(H^{p,q}) = H^{q,p}.$$  

If we denote by $c$ the complex conjugation on $\mathbb{C}$ then

$$(H^n_B(X) \otimes \mathbb{C})^{F_\infty \otimes c} = H^n_{DR} \otimes \mathbb{R}.$$  

At non-archimedean places, there is an important analogue. For any prime $p$, we have

$$D_{DR}(H^n_p(X)) := (H^n_p(X) \otimes B_{DR})^{G_p} \simeq H^n_{DR}(X) \otimes \mathbb{Q}_p$$

where $B_{DR}$ is Fontaine’s ring of $p$-adic periods $\mathbb{B}$. These structures taken together motivate the following observation. Regardless of its precise definition, a motive $M$ should have associated to it a collection of objects as above that we call a pure system of realizations that make up a category $R$. This is a collection

$$R(M) = \{\{M_l\}, M_{DR}, M_B\}$$

where each $M_l$ is a representation of $G$ on a (finite-dimensional) $\mathbb{Q}_l$-vector space, $M_{DR}$ is a filtered $\mathbb{Q}$-vector space, and $M_B$ is a $\mathbb{Q}$-vector space with an involution $F_\infty$. These vector spaces should all have the same dimension and be equipped with a system of comparison isomorphisms as above. The data must be subject to further subtle constraints having to do with local Galois representations.

That is to say recall the exact sequence:

$$0 \to I_p \to G_p \xrightarrow{v} \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \to 0$$

For $l \neq p$, $I_p$ has a tame $l$-quotient

$$t_l : I_p \to I_{p,l}$$

with the structure

$$I_{p,l} \simeq \hat{\mathbb{Z}}_l(1) \simeq \lim_{\leftarrow} \mu_{p^n}$$

as a module for $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. Define

$$W_p := v^{-1}(\mathbb{Z}) \subset G_p,$$

the Weil group at $p$. It is convenient to analyze the data of $M_l$ using an associated Weil-Deligne (W-D) representation $WD_p(M_l)$ for each $p$, consisting of a representation $r$ of $W_p$ such that $r|I_p$ has finite image and a nilpotent operator $N_p$ acting on the representation.
These satisfy a compatibility condition
\[ r(\phi_p)N_p r(\phi_p^{-1}) = p^{-1} N_p \]
for any lift \( \phi_p \in W_p \) of \( Fr_p \).

The construction of \( WD_p(M_l) \) for \( p \neq l \) uses the fact that the action of \( G_p \) when restricted to some finite index subgroup \( G'_p \) is semi-stable, i.e., its inertia subgroup \( I'_p \) acts unipotently. Hence, the action of \( I'_p \) can be expressed as
\[ \sigma \mapsto \exp(t_l(\sigma)N_p) \]
for a nilpotent \( N_p \). Then the representation \( r \) on \( I_p \) is given by
\[ r(\phi_p^n \sigma) = \phi_p^n \sigma \exp(-t_l(\sigma)N_p) \]
for some choice of \( \phi_p \). In fact, since the data \( (\phi_p, N_p) \) determine the Weil-Deligne representation, it is usual to identify the representation with the such a pair.

For \( p = l \), we use the fact that any De Rham representation is potentially semi-stable \(^{[11]}\), and hence, gives us a filtered \( (\phi_p, N_p) \) module via
\[ M_p \mapsto (M_p \otimes B_{st})^G'_p \]
If \( G'_p = G_p \) (that is, if the representation itself is semi-stable), then this gives us a Weil-Deligne representation in an obvious way by defining
\[ r(g) = \phi_p^n \]
if \( g \mapsto Fr_p^n \in \text{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p) \). In \(^{[14]}\), I.1, it is explained in detail how one extracts a Weil-Deligne representation from the data in the general case.

The viewpoint of the Weil-Deligne representation allows us to parametrize the information of the Galois representations in a form that does not use the topology of \( \mathbb{Q}_l \). It provides, thereby, a suitable framework for comparing the representations as \( l \) varies, and makes natural the connection to complex automorphic forms \(^{[45]}\). Furthermore, one creates thereby a rather precise analogy with the theory of limit mixed Hodge structures \(^{[22]}\).

Now define the Frobenius semi-simplification \( WD_p(M_l)^{ss} \) of \( WD_p(M_l) \) by replacing \( \phi_p \) with its semi-simple part. With the terminology thus introduced, here are the constraints we impose on our pure system of realizations:

‘Good reduction almost everywhere’: We assume that there exists a finite set \( S \) of primes such that \( WD_p(M_l) \) is unramified for all \( p \not\in S \), i.e., \( N_p = 0 \) and \( I_p \) acts trivially.

‘Algebraicity and independence of \( l \)’: There exists a Frobenius semi-simple W-D representation \( WD_p(M) \) over \( \bar{\mathbb{Q}} \) such that
\[ WD_p(M) \otimes \bar{\mathbb{Q}}_l \simeq WD_p^{ss}(M_l) \otimes \bar{\mathbb{Q}}_l \]

Subject to these conditions, the collection \( \{M_l\} \) is then referred to as a strongly compatible system of \( l \)-adic representations.

‘Weil conjecture’: There should exist an integer \( n \), called the weight of \( M \), such that the eigenvalues of \( Fr_p \) acting on \( WD_p(M) \) for \( p \not\in S \) have all Archimedean absolute values equal to \( p^{n/2} \). Furthermore, the Hodge structure \( M_B \) should be pure of weight \( n \).

‘Purity of the monodromy filtration’: If we denote by \( M_n \) the unique increasing filtration on \( WD_p(M) \) such that \( M_{n-k} = 0 \), \( M_{n_k} = WD_p(M) \) for sufficiently large \( k \), and
\[ N(M_{n_k}) \subset M_{n_{k-2}}, \]
then the associated graded piece
\[ Gr_k^{MN}(WD_p(M)) \]
has all Frobenius eigenvalues of archimedean absolute value \( p^{(n+k)/2} \).

It should be remarked that in general, we need to allow coefficients in \( E_\lambda \) for the representations where \( E \) is a number field and \( E_\lambda \) are completions. Such coefficient systems arise naturally when considering direct summands of \( \mathbb{Q}_l \) representations or motives with extra endomorphisms, e.g., abelian varieties with CM. We will omit this generality in this summary. Another interesting view is that the bi-grading
\[ M_B \otimes C \simeq \oplus M^{p,q}, \]
which is compatible with the complex conjugation of coefficients, corresponds to a representation of the group
\[ Res^C_R(G_m). \]
Together with the action of
\[ F_\infty \circ C \]
it can be viewed as a representation of the real Weil group \( W_\mathbb{R} \) (citate1) with points given by
\[ W_\mathbb{R}(\mathbb{R}) = \mathbb{C}^* \cup \mathbb{C}^* j \]
where \( j^2 = -1 \) and \( jzj^{-1} = \bar{z} \). Here, \( C \) is the Weil operator defined by
\[ C|M^{p,q} = i^{q-p} \]

It is conjectured that the realizations
\[ H^n(X) = (\{H^n_l(X)\}, H^n_B(X), H^n_{DR}(X)) \]
coming from a smooth projective variety \( X \) satisfy the algebraicity, independence of \( l \), and purity conditions even for \( p \in S \).

The category of pure motives should be comprised of objects in \( R \) of geometric origin, a notion without an entirely precise interpretation [13]. For example, we need to admit at every stage duals (homology) and tensor products of all objects considered. Objects that are not generated in an obvious way from those of the form
\[ H^n(X) \]
arise via images (or kernels) under pull-backs and push-forwards in cohomology induced by maps of varieties, as well as \( \mathbb{Q} \)-linear combinations of geometric maps. We should also be able to compose pull-backs with push-forwards. Such compositions give rise to the idea of using correspondences modulo homological equivalence as morphisms [12]. Once morphisms are constructed in this manner, we naturally obtain new objects using the decomposition of
\[ End(H^n(X)), \]
which is a semi-simple \( \mathbb{Q} \)-algebra subject to one of the standard conjectures saying that numerical equivalence and homological equivalence coincide [24].

One can consider also a category of mixed systems of realizations by requiring a weight filtration
\[ \cdots \subset W_{n-1}M \subset W_nM \subset W_{n+1}M \subset \]
compatible with all the comparisons and such that each graded quotient
\[ Gr^W_k(M) \]
is a pure system of realizations of weight \( n \). Mixed motives should be those of geometric origin such as the cohomology of varieties that are not necessarily smooth or proper. But then, we need to include objects like (finite-dimensional quotients of)

\[ \mathbb{Q}[\pi_1] \]

or the (co-)homology of (co-)simplicial varieties \([23]\).

Given a pure system \( M \) of realizations we can define its \( L \)-function \( L(M, s) \) as an Euler product

\[
L(M, s) = \prod_p L_p(M, s)
\]

with

\[
L_p(M, s) = \frac{1}{\det(1 - p^{-s} Fr_p)(WD_p(M))_{I_p=1,N_p=0}}
\]

If \( M \) is of weight \( n \), then the product converges (and hence is non-zero) for \( \Re(s) > n/2 + 1 \). For some conceptual motivation for this definition based on duality in the function field case, see \([16]\). (The point is that the inertia fixed part is the stalk of the intermediate extension of the étale sheaf corresponding to \( M \).)

There is also a factor at \( \infty \) depending upon the representation \( M_B \otimes \mathbb{C} \) of \( W_{\mathbb{R}} \). Define

\[
\Gamma_R := \pi^{-s/2} \Gamma(s/2)
\]

\[
\Gamma_C := 2(2\pi)^{-s/2} \Gamma(s)
\]

\[
h^{pq} := \dim M^{pq}
\]

\[
h^{p,\pm} := \dim M^{pp,\pm 1}
\]

where the signs in the superscript refer to the \( \pm 1 \) eigenspaces of the \( F_{\infty} \)-action. Then

\[
L_{\infty}(M, s)
\]

is defined as

\[
\prod_{p<q} \Gamma_C(s - p)^{h^{pq}}
\]

for odd \( n \), and

\[
\prod_{p<q} \Gamma_C(s - p)^{h^{pq}} \Gamma_R(s - n/2)^{h^{n/2+}} \Gamma_R(s - n/2 + 1)^{h^{n/2-}}
\]

for \( n \) even \([39]\).

It is conjectured that

\[
\Lambda(M, s) = L_{\infty}(M, s)L(M, s)
\]

has a meromorphic continuation to \( \mathbb{C} \) and satisfies a functional equation

\[
\Lambda(M, s) = \epsilon(M, s)\Lambda(M^*, 1 - s)
\]

where the epsilon factor has the form \( \epsilon(M, s) = ba^s \). Note that the contribution of the conductor has also been incorporated into this factor. (For a precise discussion of the factor in the case \( M = H^n(X) \), see \([39]\).) As alluded to above, the general expectation is that this conjecture will be dealt with by the Langlands’ program.
3 Conjectures on zeros, poles, and values

A list of overall references to this section should include the papers mentioned in the introduction as well as the original articles [7], [12], [1], [2], [3], [4], and [9]. Having mentioned thus the sources, we will then proceed to be somewhat sloppy with specific citation.

Here is some convenient notation:

\[ Q_0 : \text{trivial system of realizations.} \]
\[ Q(1) := H^2(P^1)^* \]
\[ Q(i) = Q(1)^{\otimes i} \text{ for } i \geq 0 \]
\[ Q(i) = \text{Hom}(Q(-i), Q) \text{ for } i < 0. \]

For a system \( M \) of realizations we define its Tate twists by tensor products with \( Q(1) \):

\[ M(i) := M \otimes Q(i) \]

Then for any smooth projective variety of dim \( d \), we have (32, VI.6)

\[ H^{2d}(X) \simeq Q(-d) \]

and a perfect pairing

\[ H^i(X) \times H^{2d-i}(X) \rightarrow H^{2d}(X) \]

Repeated cup product with the cohomology class of a hyperplane gives us the hard Lefschetz theorem [14]

\[ H^i(X) \simeq H^{2d-i}(X)(d-i). \]

The effect of the twisting on realizations is such that \( M(n)_l \) is the tensor product of \( M_l \) with the \( n \)-th power of the \( Q_l \) cyclotomic character giving the action of \( G \) on the \( l \)-power roots of unity, and

\[ F^i(M(n)_{DR}) = F^{n+i}M_{DR} \]

with a corresponding shift in Hodge numbers \( h^{pq} \). Furthermore,

\[ F_{\infty}|M(n)_B = (F_{\infty}|M_B) \otimes (-1)^n \]

and for the \( L \)-functions,

\[ L(M(n), s) = L(M, s + n). \]

These facts all follow in an elementary way from the structure of \( H^2(P^1) \).

To state the main conjectures without spending time on categorical preliminaries, we will focus on the case where \( M \) is

\[ H^n(X) = (\{H^n_l(X), H^n_{DR}(X), H^n_B(X)\}) \]

for a smooth projective variety \( X \) of dimension \( d \). Assume that \( H^n(X) \) is a pure system of realizations so that the analytic continuation and functional equation hold true. Since we have

\[ H^n(X)^* \simeq H^{2d-n}(X)(d) \simeq H^n(X)(n) \]

the functional equation relates

\[ L(H^n(X), s) \]

and

\[ L(H^n(X)(n), 1 - s) = L(H^n(X), n + 1 - s) \]

with center of reflection

\[ (n + 1)/2. \]
We will therefore confine interest mostly to

\[ m \geq \frac{n + 1}{2} \]

or, equivalently,

\[ n + 1 - m \leq \frac{n + 1}{2}, \]

and refer to \( \text{Re}(s) = \frac{n + 1}{2} \) as the critical line. (The reader should consult Nekovar’s elegant article [34] for a careful discussion of how to relate the points addressed here to those on the right of the critical line.) In fact, it is conceptually convenient to parametrize by the letter \( m \) the integers \( \geq \frac{n + 1}{2} \), but to focus then on the values at the points \( n + 1 - m \). In the discussion of orders, however, we will make explicit the case of \( m = n/2 \), and hence, the possible pole at \( n + 1 - m = n/2 + 1 \) (just to the right of the critical line), whose importance is evident from the classical example of Dedekind’s zeta functions. As we will explain below, the general geometric importance of this pole is related to Tate’s conjecture on the cohomology classes of algebraic cycles [44]. We remark also that the study of \( L(H^n(X), s) \) near \( m \) corresponds to the study of \( L(H^n(X)(m), s) \) near \( s = 0 \). Thus, when the conjectures are formulated in terms of values at zero in the literature, one encounters the assumption that the weight \( n - 2m \) is negative.

We start then with the conjecture on orders. The simple case arising from an elliptic curve was reviewed already where \( n = 1 \) and \( m = n + 1 - m \) is the reflection point \( (n + 1)/2 = 1 \). The conjecture of Birch and Swinnerton-dyer says

\[ \text{ord}_{s=1} L(H^1(E), s) = \text{rank} E(\mathbb{Q}) \]

Now, an element

\[ x \in E(\mathbb{Q}) \]

gives rise to an extension in the category \( \mathcal{R} \) of realizations

\[ \delta(x) \in \text{Ext}^1_{\mathcal{R}}(\mathbb{Q}, H^1(E)(1)) \]

via Kummer theory. It is conjectured that when \( \mathcal{R} \) is replaced by a suitable category of motives, this is the only way to construct such extensions. This notion conveys the basic flavor of conjectures on orders in the case of odd weight.

For an example in the even weight case, let \( X = \text{Spec}(F) \) for a number field \( F \), assumed for simplicity to be Galois over \( \mathbb{Q} \). Then

\[ \zeta(F, s) = L(H^0(\text{Spec}(F)), s) \]

which breaks up as into a product of Artin L-functions

\[ \zeta(F, s) \prod_{\rho} L(\rho, s) \]

as \( \rho : \text{Gal}(F/\mathbb{Q}) \to \text{Aut}(V) \) runs over finite-dimensional representations on \( \mathbb{Q} \)-vector spaces. This is the most basic example of a motivic decomposition. In fact, any finite-dimensional \( \mathbb{Q} \)-representation defines an Artin motive \( M_\rho \) and it is a theorem that

\[ \text{ord}_{s=1} L(\rho, s) = -\dim \text{Hom}_{AM}(\mathbb{Q}, M_\rho), \]

where the Hom occurs inside the category of Artin motives [12]. Note that \( s = 1 \) in this case is the integer point \( n/2 + 1 \) just to the right of the critical line mentioned above.

The general conjecture about orders says

\[ \text{ord}_{s=n+1-m} L(H^n(X), s) \]
The Hom and Ext should occur inside a conjectural category of mixed motives over \( \mathbb{Z} \) with \( \mathbb{Q} \)-coefficients. For weight reasons, the Hom term should vanish unless \( n = 2m \) in which case the Ext term should vanishes. That is, in the pure situation we are considering, only one term or the other is supposed to occur.

The point just to the right of the critical line is of interest in the even weight situation when \( n = 2m \), where the formula predicts

\[
\text{ord}_{x=m+1} L(H^{2m}(X), s) = -\dim \text{Hom}_{\text{Mot}_{\mathbb{Z}}}(\mathbb{Q}, H^{2m}(X)(m))
\]

generalizing the pole of the Artin \( L \)-function \( (m = 0) \). As for an explicit connection to arithmetic geometry, it is expected that

\[
\text{Hom}_{\text{Mot}_{\mathbb{Z}}}(\mathbb{Q}, H^{2m}(X)(m)) \cong [CH^m(X)/CH^m(X)^0] \otimes \mathbb{Q}
\]

Of course the isomorphism should arise via a cycle map

\[
CH^m(X) \to H^{2m}(X)(m)
\]

killing the cycles \( CH^m(X)^0 \) homologically equivalent to zero.

As we move to the left, we encounter the point \( m = (n + 1)/2 \) for \( n \) odd \( (n = 2m - 1) \), and the conjecture predicts the order of vanishing at the central critical point:

\[
\text{ord}_{x=m} L(H^{2m-1}(X), s) = \dim \text{Ext}_{\text{Mot}_{\mathbb{Z}}}(\mathbb{Q}, H^{2m-1}(X)(m))
\]

It is then expected that

\[
\text{Ext}_{\text{Mot}_{\mathbb{Z}}}(\mathbb{Q}, H^{2m-1}(X)(m)) \cong CH^m(X)^0 \otimes \mathbb{Q}
\]

The map from cycles to extensions generalizes Kummer theory: given a representative \( Z \) for a class in \( CH^m(X)^0 \), we get an exact sequence

\[
0 \to H^{2m-1}(X)(m) \to H^{2m-1}(X \setminus Z)(m) \to H^{2m}(X)(m)
\]

There is a local cycle class

\[
cl(Z) \in H^{2m}_Z(X)(m)
\]

that maps to zero in \( H^{2m}(X)(m) \), giving rise to the desired extension:

\[
0 \to H^{2m-1}(X)(m) \to \delta^{-1}(cl(Z)) \to \mathbb{Q} \to 0
\]

These two classical points, central critical, \( n + 1 - m = m = (n + 1)/2 \) for \( n \) odd, and just right of it, \( n + 1 - m = n/2 + 1 \), for \( n \) even, are somewhat exceptional. As \( n + 1 - m \) moves further left \( (m > n/2 + 1) \), one expects

\[
\text{Ext}_{\text{Mot}_{\mathbb{Z}}}(\mathbb{Q}, H^n(X)(m)) = H^{n+1}_{\text{Mot}_{\mathbb{Z}}}(X, \mathbb{Q}(m))
\]

with the last group, often referred to as motivic cohomology, defined using \( K \)-theory:

\[
\text{Im}[\{(K_{2m-n-1}(X))^{(m)} \to (K_{2m-n-1}(X))^{(m)}\}]
\]

(\( \mathcal{X} \) is a proper flat regular \( \mathbb{Z} \)-model for \( X \)) or Bloch’s higher Chow groups \[S\]:

\[
\text{Im}[CH^{n+1}(\mathcal{X}, 2m - n - 1) \otimes \mathbb{Q} \to CH^{n+1}(\mathcal{X}, 2m - n - 1) \otimes \mathbb{Q}]
\]
The latter interpretation, carrying with it the hope of representing motivic cohomology classes quite explicitly, is more popular lately \[17\]. In fact, when \( m > \frac{n}{2} + 1 \), the conjectured functional equation implies

\[
\text{ord}_{s = n+1-m} L(H^n(X), s) = \dim \Ext^1_{\text{Mot}_\mathbb{Z}}(\mathbb{Q}, H^n(X)(m) \otimes \mathbb{R})
\]

where the extension occurs inside the category of real mixed Hodge structures defined over \( \mathbb{R} \). So the statement on the order of vanishing follows from the conjecture that the Hodge realization functor induces an isomorphism

\[
\Ext^1_{\text{Mot}_\mathbb{Z}}(\mathbb{Q}, H^n(X)(m)) \otimes \mathbb{R} \simeq \Ext^1_{\text{Mot}_{\text{HS}}}(\mathbb{R}, H^n(X)(m) \otimes \mathbb{R})
\]

It has been emphasized by Deligne that the regulator map (discussed below) is this realization functor.

Our outline thus far should already make clear that the conceptual structure of the conjectures falls into two parts:

1. Relation between \( L \) functions and \( \Ext \) and \( \text{Hom} \) groups in category of motives;
2. geometric interpretation of \( \Ext \) and \( \text{Hom} \) groups.

That is to say, in addition to the difficult problem of drawing the lower edge of the following triangle

```
Interpretation in the category of motives
```

```
Order of \( L \)-function
```

```
Arithmetic-geometric rank
```

the further problem has been created of constructing a category that realizes the upper vertex.

There is a computation, convenient in practice, of the real \( \Ext \) group via Deligne cohomology:

\[
\Ext^1_{\text{Mot}_{\text{HS}}}(\mathbb{R}, H^n_B(X)(m) \otimes \mathbb{R}) \simeq H^{n+1}_D(X, \mathbb{R}(m))
\]

Using the explicit nature of Deligne cohomology, one can construct regulator maps

\[
H^{n+1}_{\text{Mot}_\mathbb{Z}}(X, \mathbb{Q}(m)) \to \Ext^1_{\text{Mot}_{\text{HS}}}(\mathbb{R}, H^n_B(X)(m) \otimes \mathbb{R})
\]

that can be studied independently of a category of motives. For example, in essentially all known cases of the Beilinson conjectures (e.g. [2],[15]), one constructs subgroups

\[
L \subset H^{n+1}_{\text{Mot}_\mathbb{Z}}(X, \mathbb{Q}(m)),
\]

conjecturally of full rank, and studies their images in Deligne cohomology.

As we move on to the conjectures on values, we start with the central critical ones due to Bloch and Beilinson. Thus, \( m = n + 1 - m = (n + 1)/2 \) (and \( n = 2m - 1 \)). We then have an isomorphism

\[
F^m H^{2m-1}_{\text{DR}}(X) \otimes \mathbb{R} \simeq [H^{2m-1}_{B}(X)(m-1)]^{(-1)^{m-1}} \otimes \mathbb{R}
\]

which gives rise to the period isomorphism

\[
p = p(H^{2m-1}(X, m)) : \det(F^m H^{2m-1}_{\text{DR}}(X)) \otimes \mathbb{R} \simeq \det([H^{2m-1}_{B}(X)(m-1)]^{(-1)^{m-1}}) \otimes \mathbb{R}
\]

of real vector spaces of dimension one, where the \( \det \) refers to top exterior powers. That is to say,

\[
\det(F^m H^{2m-1}_{\text{DR}}(X))
\]
and
\[\det[[H_B^{2m-1}(X)(m-1)]^{(-1)^{m-1}}]\]
are viewed as two \(\mathbb{Q}\)-lines sitting inside the same real line. An additional transcendental contribution comes from the **height pairing**, conjectured to be non-degenerate:
\[CH^n(X)^0 \times CH^{\dim(X)+1-m}(X)^0 \to \mathbb{R}\]
whose determinant gives us a regulator
\[r = r(H^{2m-1}(X)(m)) \in \mathbb{R}^*/\mathbb{Q}^*
\]
Recall that conjecturally
\[d_m := \text{ord}_{s=m} L(H^{2m-1}(X), s) = \dim CH^m(X)^0 \otimes \mathbb{Q}\]
For any motive \(M\), denote by
\[L^*(M, m)\]
the leading coefficient of \(L(M, s)\) at \(s = m\). So
\[L^*(H^{2m-1}(X), m) = \lim_{s \to m} (s - m)^{-d_m} L(H^{2m-1}(X), s)\]
The general conjecture on central critical values then says
\[r \cdot p[\det(F^m H_{DR}^{2m-1}(X))] = L^*(H^{2m-1}(X), m) \det[[H_B^{2m-1}(X)(m-1)]^{(-1)^{m-1}}]\]
inside \(\det[[H_B^{2m-1}(X)(m-1)]^{(-1)^{m-1}}] \otimes \mathbb{R}\).
Moving left to the values at \(n + 1 - m \leq n/2\) \((m \geq n/2 + 1)\) we point out first that the period isomorphism is replaced by an exact sequence:
\[
0 \to F^m H_{DR}^n(X) \otimes \mathbb{R} \to [H_B^n(X)(m-1)]^{(-1)^{m-1}} \otimes \mathbb{R} \\
\to \text{Ext}^1_{MHS^S_{\mathbb{R}}} (\mathbb{R}, H_B^n(X)(m) \otimes \mathbb{R}) \to 0
\]
Thus, the transcendental part should incorporate a \(\mathbb{Q}\)-structure on
\[\text{Ext}^1_{MHS^S_{\mathbb{R}}} (\mathbb{R}, H_B^n(X)(m) \otimes \mathbb{R}).\]
We will skip for a moment the classically interesting point \(n + 1 - m = n/2\) for \(n\) even and assume \(n + 1 - m < n/2\). Therefore, \(m > n/2 + 1\) lies in the region of convergence of the \(L\)-function. Then the \(\mathbb{Q}\) structure is expected to come from the conjectured **regulator isomorphism**:
\[H^{n+1}_{M, \mathbb{Z}}(X, \mathbb{Q}(m))) \otimes \mathbb{R} \simeq \text{Ext}^1_{MHS^S_{\mathbb{R}}} (\mathbb{R}, H_B^n(X)(m) \otimes \mathbb{R})\]
We are led thus to an isomorphism
\[c = c(H^n(X)(m)) : [\det(H^{n+1}_{M, \mathbb{Z}}(X, \mathbb{Q}(m))))] \otimes \mathbb{R} \simeq [\det F^m H_{DR}^n(X)]^{-1} \otimes \det([H_B^n(X)(m-1)]^{(-1)^{m-1}}) \otimes \mathbb{R}\]
and Beilinson’s conjecture says
\[c[\det(H^{n+1}_{M, \mathbb{Z}}(X, \mathbb{Q}(m))))] = L^*(H^n(X), n + 1 - m)[\det F^m H_{DR}^n(X)]^{-1} \otimes \det([H_B^n(X)(m-1)]^{(-1)^{m-1}})\]
Finally, we return to the value at \(n + 1 - m = n/2\) \((m = n/2 + 1)\) for \(n = 2m - 2\) even. Here, the conjecture is identical to the situation further left except the regulator involves maps from both motivic cohomology
\[H^{n+1}_{M, \mathbb{Z}}(X, \mathbb{Q}(m))\]
and 

\[ CH^{m-1}(X) \]

in a manner similar to how the central critical value incorporates periods and a height pairing. That is, we have a map

\[ [CH^{m-1}(X)/CH^{m-1}(X)^0] \rightarrow [H_B^{2m-2}(X)(m-1)]^{(-1)^{m-1}} \]

coming from the cycle map that induces an injection

\[ [CH^{m-1}(X)/CH^{m-1}(X)^0] \rightarrow Ext^1_{MHS}(\mathbb{R}, H_B^n(X)(m) \otimes \mathbb{R}) \]

via the quotient map

\[ [H_B^n(X)(m-1)]^{(-1)^{m-1}} \otimes \mathbb{R} \rightarrow Ext^1_{MHS}(\mathbb{R}, H_B^n(X)(m) \otimes \mathbb{R}) \]

We remark then only that the conjectured isomorphism is

\[ H_{M,\mathbb{Z}}^{n+1}(X, m) \otimes \mathbb{R} \otimes [CH^{m-1}(X)/CH^{m-1}(X)^0] \otimes \mathbb{R} \]

\[ \simeq Ext^1_{MHS}(\mathbb{R}, H_B^n(X)(m) \otimes \mathbb{R}) \]

The point is that in order to give the correct order of zero (dimension of motivic cohomology) to the left of the critical strip, the poles to the right of the critical strip must be canceled out.

In the Bloch-Kato conjectures, isomorphisms are normalized more carefully, comparing certain integral structures one prime at a time. More precisely, the conjecture concerns the left of the critical strip, the poles to the right of the critical strip must be canceled out.

We give here just a flavor of the conjecture by discussing a range of cases that doesn’t involve too many definitions. There is a theory of determinants of perfect complexes over a principal ring \( R \) that goes with the conjectures \([19]\), where the useful facts are:

1. If \( M \) is a finitely generated free module, then \( \det(M) \) is the top exterior power.
2. If \( K \) is the field of fractions of \( R \), then for any finitely generated module \( M \), the map \( M \rightarrow M \otimes K \) induces

\[ \det(M) \otimes_R K \simeq \det(M \otimes_R K) \]

canonically.
3. If \( M \) is finitely generated and \( M' \) is its torsion-free quotient, then the natural isomorphism

\[ \det(M) \otimes K \simeq \det(M') \otimes K \]

takes a generator of \( \det(M) \) to \( 1/r \) times a generator of \( \det(M') \), where \( r \) is a characteristic element for the torsion submodule of \( M \).

We note in regard to these facts that the determinant module is always free of rank one, so it is only the morphisms that are the relevant data. Now assume that \( m > \min\{n, \dim(X)\} \) so that \( F^mH^n_{DR} = 0 \) and

\[ [H_B^n(X)(m-1)]^{(-1)^{m-1}} \otimes \mathbb{R} \simeq Ext^1_{MHS}(\mathbb{R}, H_B^n(X)(m) \otimes \mathbb{R}) \]

Therefore, we should have

\[ H_{M,\mathbb{Z}}^{n+1}(X, \mathbb{Q}(m)) \otimes \mathbb{R} \simeq Ext^1_{MHS}(\mathbb{R}, H_B^n(X)(m) \otimes \mathbb{R}) \]

\[ \simeq [H^n(X(\mathbb{C}), \mathbb{Z}(m-1))]^{(-1)^{m-1}} \otimes \mathbb{R} \]

Let \( \gamma \) be a generator of

\[ \det(H^n(X(\mathbb{C}), \mathbb{Z}(m-1))]^{(-1)^{m-1}}) \]
and let
\[ \omega_\gamma \in \det(H^{n+1}_{M,\mathbb{Z}}(X,\mathbb{Q}(m))) \]
be an element mapping to
\[ L^*(H^n(X)(m), n + 1 - m)\gamma. \]
As before, let \( S \) be a set of primes including those of bad reduction for \( X \), the Archimedean prime, and a fixed prime \( p \). We denote by \( G_S \) the Galois group of the maximal extension of \( \mathbb{Q} \) unramified outside the primes in \( S \). There is a Chern-class map
\[ H^{n+1}_{M,\mathbb{Z}}(X,\mathbb{Q}(m)) \rightarrow H^1(G_S, H^n_p(X)(m)) \]
that conjecturally induces an isomorphism
\[ H^{n+1}_{M,\mathbb{Z}}(X,\mathbb{Q}(m)) \otimes \mathbb{Q}_p \simeq H^1(G_S, H^n_p(X)(m)) \]
That is to say, there are two maps
\[
\begin{array}{ccc}
H^{n+1}_{M,\mathbb{Z}}(X,\mathbb{Q}(m)) & \longrightarrow & \Ext^1_{MHS_S}(\mathbb{R}, H^n_B(X)(m) \otimes \mathbb{R}) \\
& \downarrow & \\
H^1(G_S, H^n_p(X)(m)) & \longrightarrow & H^1(G_S, H^n_p(X)(m))
\end{array}
\]
that are both supposed to induce isomorphisms upon changing coefficients, and with which the rational part will be controlled. Let
\[ z_\gamma \in \det(H^1(G_S, H^n_p(X)(m))) \]
be the image of \( \omega_\gamma \) so that we have the diagram
\[
\begin{array}{ccc}
\omega_\gamma & \longrightarrow & L^*(H^n(X)(m), n + 1 - m)\gamma \\
& \downarrow & \\
z_\gamma
\end{array}
\]
Then the conjecture is that
\[
[\det(H^1(G_S, H^1(G, H^n_p(X)(m)))) : \mathbb{Z}_p] = [H^0(G_S, H^1(G, H^n_p(X)(m))))][H^2(G_S, H^1(G, H^n_p(X)(m)))]
\]
It is easy to see that the \( H^0 \) term is finite, but the finiteness of the \( H^2 \), like that of \( Sha \) in the case of elliptic curves, must also be conjectured. In the uniform formalism, the desired equality is interpreted as the assertion that \( z_\gamma \) is a \( \mathbb{Z}_p \)-basis of the determinant of the perfect \( \mathbb{Z}_p \)-complex \( R\Gamma(G_S, H^n_p(X)(m)) \).
Extraction of the rational part is supposed to lead eventually to a \( p \)-adic \( L \)-function
\[ \mathcal{L}^{(p)}(H^n(X)) \]
that exercises control over Galois cohomology (i.e., Selmer groups) and Diophantine invariants. This \( p \)-adic theory appears so far to be the best strategy for applying the theory of \( L \)-functions to the elucidation of Diophantine structures (\cite{35, 20, 12}).
Remark

We conclude with the warning that there is a conspicuous deficiency in theory of motives: This is that even in the best of possible worlds (ours), only abelian invariants are accessible, such as

$$CH^m(X).$$

These abelian invariants do not yield in general information about

$$X(\mathbb{Q})$$

and leave thereby untouched the most basic questions of Diophantine geometry. This is an artifact of the fact that the theory of motives as presently developed is implicitly modeled on the theory of abelian varieties and $H_1$. Attempts to redress this deficiency for certain varieties are contained in Grothendieck’s anabelian program

([21], [33]) that concerns itself with the theory of pro-finite $\pi_1$’s. The technology of motives ends up contributing here as well because the Diophantine aspect of this theory [27] assigns an interesting role to motivic fundamental groups [13], where $Ext$ groups are replaced by

"classifying spaces for non-abelian torsors"

However, what is entirely missing as yet is an analogue of the $L$-function.

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