THE INFINITESIMAL-OPERATOR ALGEBRAS OF CONTINUOUS GROUPS WITH ANTILINEAR OPERATIONS

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Abstract. Continuous groups with antilinear operations of the form $G + a_0G$, where $G$ denotes a linear Lie group, and $a_0$ is an antilinear operation which fulfills the condition $a_0^2 = \pm 1$, were defined and their matrix algebras were investigated in [11]. In this paper infinitesimal-operator algebras are defined for any group of the form $G + a_0G$, and their properties are determined.

1 Introduction

Birman’s successful idea [1] of introducing space groups with antilinear operations to the description of lattice vibrations suggested a definition of continuous groups with antilinear operations of the form $G + a_0G$, where $G$ denotes a Lie group, and $a_0$ is an antilinear operation, fulfilling the condition $a_0^2 = \pm 1$. The matrix algebras of these groups were defined, their properties were investigated and examples were given in [11]. That investigation is extended in this paper, by defining in Sections 4 and 5 the algebra of infinitesimal operators connected with the groups $G + a_0G$. These are the operators $J_\sigma$, connected with the subgroup $G$, and the operators $J'_\mu$, connected with the coset $a_0G$. In the choice of the appropriate definitions we will be guided by the requirement that the one-to-one correspondence between the commutators of the infinitesimal operators and the respective commutators of the basis vectors of the respective matrix algebra should hold. There appear three types of commutators: $[J_\sigma, J_\tau]$, $[J_\sigma, J'_\mu]$, and $[J'_\mu, J'_\nu]$. It will be shown that the commutators $[J_\sigma, J_\tau]$ and the commutators $[J'_\mu, J'_\nu]$ yield linear combinations of operators connected with the subgroup $G$, while the commutators $[J_\sigma, J'_\mu]$ yield linear combinations of operators connected with the coset $a_0G$. 
2 Transformations of the coordinate space of corepresentations

This section is based on the knowledge of the coirrep matrices of $a$− and $b$−type coirreps and of their basis functions, in the forms which were derived in [11]. We will denote the original coordinates in the representation space of a corepresentation by $y_j$, $j = 1, \ldots, d, d + 1, \ldots, 2d$. After the Kovalev-Gorbanyuk transformations of the corepresentation matrices in Eqs. (11) and (12) of [11], the respectively transformed coordinates acquire the following forms:

For $a$−type coirreps, after the application of transformation $V_1$ in Eq. (42) of [11], we obtain the coordinate spaces of the reduced corepresentation matrix in the form,

$$x_{i}^{(1)} = \frac{1}{\sqrt{2}} \left( y_i + e^{i \xi} \sum_{j=1}^{d} N_{ij} y_{d+j} \right), \quad x_{i}^{(2)} = \frac{i}{\sqrt{2}} \left( - y_i + e^{i \xi} \sum_{j=1}^{d} N_{ij} y_{d+j} \right) \quad i = 1, \ldots, d \quad (2.1)$$

with $\mu/\lambda = \exp(i \xi)$, with a real $\xi$, where the coordinates $x_{i}^{(1)}$, $i = 1, \ldots, d$ and $x_{i}^{(2)}$, $i = 1, \ldots, d$, are connected with the two $d$−dimensional blocks, respectively, with $N$ in Eqs. (20) and (39) of [11].

For $b$−type coirreps, after the application of the transformation $V_2$ in Eq. (52) of [11], we obtain the coordinates in the space of the transformed $2d$−dimensional matrices in the form:

$$x_i = -i y_i, \quad x_{d+i} = -i \sum_{k=1}^{d} N_{ik} y_{d+k}, \quad i = 1, \ldots, d \quad (2.2)$$

with $N$ in Eqs. (20) and (48) of [11]. In the following we will omit the label ”prime” of the corepresentation matrices $D'(g)$ and $D'(a)$, which was introduced in [11], remembering that they were obtained from the original corepresentation matrices with the help of the transformations in Eqs. (42) and (52) of [11]. The coordinates in Eqs. (2.1) and (2.2) have to undergo another transformation, namely the transformation $S_1$ in Eq. (19) of [11], which introduces the factor $\exp(i \alpha_0)$ in front of the coset $a_0G$ matrices. After that transformation the coordinates acquire the factor $\exp(i \alpha_0/2)$.

We now can write out the results of action of the coirrep matrices in Eqs. (43) (44) in [11] for $a$−type coirreps, and in Eqs. (28), (50) and (51) in [11] for $b$−type coirreps, supplied with the factor $\exp(i \alpha_0)$, on the respective cooordinates in Eqs. (2.1) and (2.2), supplied with the factor $\exp(i \alpha_0/2)$.

Coirreps of $a$−type. Let $x^0$ denote a one-column matrix with the elements $(x_1^0, \ldots, x_d^0)$, of the type $x_{i}^{(1)}$ or $x_{i}^{(2)}$ in Eq. (2.1), and let $x'$ denote a one-column matrix with the elements $x_{i}'$, $x_{d+i}'$. The action of the subgroup $G$ matrices on the coordinates in the representation space is determined by:
\[ \Delta(g) \exp(i\alpha_0/2)x^0 = x \]  

where \( x \) denotes a one-column matrix with the elements \((x_1, ..., x_d)\). The respective action of a single block of the coset matrix \( \exp(i\alpha_0)D(ga_0) \), is determined by:

\[ \exp(i\alpha_0)(\mu/\lambda)\Delta(g)N \exp(i\alpha_0/2)x^0 = x' \]  

where \( x' \) denotes a one-column matrix, with the elements \((x'_1, ..., x'_d)\). An analogous expression is obtained for the action of a single block of the coset matrix \( \exp(i\alpha_0)D(a_0g) \),

\[ \exp(i\alpha_0)(\mu/\lambda)N\Delta^*(g) \exp(i\alpha_0/2)x^0 = x'' \]  

where \( x'' \) denotes a one-column matrix, with the elements \((x''_1, ..., x''_d)\).

**Coirreps of \( b \)-type.** Let

\[ \exp(i\alpha_0/2) \begin{pmatrix} x^0 \\ x^0_d \end{pmatrix} \]  

denote a one-column matrix with the elements \( \exp(i\alpha_0/2)(x^0_1, ..., x^0_d, x^0_d+1, ..., x^0_d) \) in its successive rows, where the first \( d \) elements belong to \( x^0 \) and the remaining \( d \) elements to \( x^0_d \). The transformations of that point with the matrices of the subgroup \( G \) are determined by:

\[ \left( \begin{array}{c|c} \Delta(g) & 0 \\ \hline 0 & \Delta(g) \end{array} \right) \exp(i\alpha_0/2) \begin{pmatrix} x^0 \\ x^0_d \end{pmatrix} = \begin{pmatrix} x \\ x_d \end{pmatrix} \]  

where on the right hand side we have the one-column matrix with the elements of the first \( d \) rows denoted by \( x \), and of the successive \( d \) rows denote by \( x_d \). The transformations of the point in Eq. (2.6) with the matrices \( D(ga_0) \) and \( D(a_0g) \) of the coset \( a_0G \) in Eqs. (50) and (51) of [11], respectively, with the factor \( \exp(i\alpha_0) \), are determined by:

\[ \exp(i\alpha_0) \begin{pmatrix} 0 & \Delta(g)N \\ -\Delta(g)N & 0 \end{pmatrix} \exp(i\alpha_0/2) \begin{pmatrix} x^0 \\ x^0_d \end{pmatrix} = \begin{pmatrix} x' \\ x'_d \end{pmatrix} \]  

and

\[ \exp(i\alpha_0) \begin{pmatrix} 0 & N\Delta^*(g) \\ -N\Delta^*(g) & 0 \end{pmatrix} \exp(i\alpha_0/2) \begin{pmatrix} x^0 \\ x^0_d \end{pmatrix} = \begin{pmatrix} x'' \\ x''_d \end{pmatrix} \]
3 The action of coirrep matrices on points in the coordinate space

Before defining the infinitesimal operators, we firstly have to consider two types of products of corepresentation matrices: (1) the product of two corep matrices of which each belongs to the coset $a_0G$, which is equal to a matrix belonging to the subgroup $G$, and (2) the product of a matrix belonging to the subgroup $G$ with a matrix belonging to the coset $a_0G$, which is equal to a matrix belonging to the coset $a_0G$. The rules of action of the corepresentation matrices on points in the corepresentation spaces have to be established. We begin with the first type of product of two matrices.

\[ x + \delta x = x^0 \exp(i \alpha_0 / 2) \]

\[ D(g(\delta \alpha)) \]

\[ D(g(\alpha + d\alpha)) \]

\[ D(g(\alpha)) \]

\[ x^0 \exp(i \alpha_0 / 2) \]

Fig.1a The customary diagram for the Lie subgroup $G$ of the group $G + a_0G$.

\[ \exp(i \delta \alpha_0) D(g(\delta \alpha) a_0) \]

\[ D(g(\alpha + d\alpha)) \]

\[ x' \]

\[ \exp(-i \delta \alpha_0) D^*(g(\alpha) a_0) \]

\[ x^0 \exp(i \alpha_0 / 2) \]

Fig.1b The diagram for the coset $a_0G$ of the group $G + a_0G$. This diagram is based on the property of the product of two corepresentation matrices, each belonging to the coset $a_0G$, which always is equal to a matrix belonging to the subgroup $G$. 
The diagrams in Figs. 1a,b, refer to the corepresentation spaces in Eqs. (2.1) and (2.2), of $a$- or $b$-type coirreps, respectively.

The diagram in Fig. 1a is the customary diagram for the Lie subgroup $G$ of the group $G + a_0 G$. In the diagram in Fig. 1b we made use of the fact that the product of two corepresentation matrices belonging to the coset $a_0 G$ is equal to a corepresentation matrix belonging to the subgroup $G$.

The following argument is valid for $a$- and $b$-type coirreps. For brevity, the symbols $x^0, x$ and $x'$ will represent the respective column matrices for both $a$-type and $b$-type coirreps.

In Fig. 1a, the point $x + \delta x$ in the corepresentation space is reached starting from the point $x^0 \exp (i \alpha_0 / 2)$ by acting on it either with the transformation $D(g(\alpha + d \alpha))$, or by acting on it successively with the transformations $D(g(\alpha))$ and $D(g(\delta \alpha))$, all belonging to the subgroup $G$. This leads to the equality:

$$D(g(\alpha + d \alpha))x^0 \exp (i \alpha_0 / 2) = D(g(\delta \alpha))D(g(\alpha))x^0 \exp (i \alpha_0 / 2)$$

(3.1)

As it is shown in Fig. 1b, the same point $x + \delta x$ can be obtained from the point $x^0 \exp (i \alpha_0 / 2)$ by acting on it either with the transformation $D(g(\alpha + d \alpha))$, belonging to the subgroup $G$, or successively with the transformations $\exp(-i \delta \alpha_0)D^*(g(\alpha)a_0)$ and $\exp(i \delta \alpha_0)D(g(\delta \alpha)a_0)$, belonging to the coset $a_0 G$. This leads to the equality:

$$D(g(\alpha + d \alpha))x^0 \exp (i \alpha_0 / 2) = \exp(i \delta \alpha_0)D(g(\delta \alpha)a_0)\exp(-i \delta \alpha_0)D^*(g(\alpha)a_0)x^0 \exp (i \alpha_0 / 2)$$

(3.2)

These two equalities are valid for $a$- and $b$-type coirreps.

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Fig 2 The point $x + \delta x$ is obtained by acting on the point $x^0 \exp (i \alpha_0 / 2)$, with the transformation $\exp(i \delta \alpha_0)D(g(\delta \alpha)a_0)$, or with the successive transformations $D^*(g(\alpha))$ and $\exp(i \delta \alpha_0)D(g(\delta \alpha)a_0)$. 

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We next consider the second type of product, which is represented in Fig. 2. In Fig. 2 we start from the point \( x_0 \exp(i\alpha_0/2) \) which is the same as in Figs. 1a,b. Acting on this point with the matrix \( \exp(i\delta_\alpha)D(g(\alpha + d\alpha)a_0) \) we obtain the point \( \pi + \delta \pi \). This point can alternatively be obtained after acting on the point \( x_0 \exp(i\alpha_0/2) \) with the matrix \( D^*(g(\alpha)) \), obtaining the point \( \pi \), and next acting on the later point with the matrix \( \exp(i\delta_\alpha)D(g(\delta\alpha)a_0) \).

We observe that the infinitesimal operators connected with the coset \( a_0G \), which can be defined at the point \( x \) in Fig. 2, are determined on the basis of the same transformation as the respective infinitesimal operators which can be defined at the point \( x' \) in Fig. 1b. This shows that we can define the infinitesimal operators connected with the coset \( a_0G \) at different points in the representation space. In the following calculations we will define the infinitesimal operators connected with the coset \( a_0G \) at the point \( x' \).

We will establish the relations between the points \( x' \) and \( x \). On the basis of Figs. 1a,b we define:

\[
x = D(g(\alpha))x_0 \exp(i\alpha_0/2), \quad x' = \exp(-i\delta_\alpha)D^*(g(\alpha)a_0)x_0 \exp(i\alpha_0/2) \tag{3.3}
\]

With these two definitions, the right hand sides of Eqs. (3.1) and (3.2) yield the relation between points \( x \) and \( x' \):

\[
D(g(\delta\alpha))x = \exp(i\delta_\alpha)D(g(\delta\alpha)a_0)x' \tag{3.4}
\]

The explicit relations between the points \( x' \) and \( x \) for \( a- \) and \( b- \)type coirreps will now be determined.

**Coirreps of \( a- \) type.** These are given by a single block in (43) and (44) in [11] for the subgroup \( G \) and for the coset \( a_0G \), respectively. The respective matrices are given in Eqs. (2.3), (2.4) and (2.5). The points \( x \) and \( x' \) in Figs. 1a and 1b, respectively, are determined by the equalities

\[
x = \Delta(g(\alpha))x_0 \exp(i\alpha_0/2), \quad \text{and} \quad x' = (\mu/\lambda)^*\Delta^*(g(\alpha))N^*x_0 \exp(-i\alpha_0/2) \tag{3.5}
\]

and the relation in Eq. (3.3) between the points \( x' \) and \( x \) takes the form:

\[
\Delta(g(\delta\alpha))x = (\mu/\lambda)\Delta(g(\delta\alpha))N\exp(i\delta_\alpha)x', \quad \text{or} \quad x' = N^{-1}\exp(-i\delta_\alpha)x \tag{3.6}
\]

since the factor \( \mu/\lambda = \exp(i\xi) \), with an arbitrary real \( \xi \), can be absorbed by the factor \( \exp(i\delta_\alpha) \).
Coirreps of $b$-type. The coirrep matrices are given in Eqs. (2.7), (2.8), and (2.9). The points $x$ and $x'$ in Figs. 1a and 1b, respectively, are determined by the equalities:

\[
\left(\frac{x}{x_d}\right) = D(g(\alpha)) \left(\frac{x^0}{x^0_d}\right) \exp(i\alpha_0/2)
\]

(3.7)

and

\[
\left(\frac{x'_d}{x'_d}\right) = \exp(-i\delta\alpha_0) D^*(g(\alpha)a_0) \left(\frac{x^0}{x^0_d}\right) \exp(i\alpha_0/2)
\]

(3.8)

The relation between these two points, according to Eq. (3.4) now takes the form:

\[
D(g(\delta\alpha)) \left(\frac{x}{x_d}\right) = \exp(i\delta\alpha_0) D(g(\delta\alpha)a_0) \left(\frac{x'_d}{x'}\right)
\]

(3.9)

Substituting into this equality the expressions for the matrices $D(g(\delta\alpha))$ and $D(g(\delta\alpha)a_0)$ from Eqs. (2.7) and (2.8), we obtain the equality:

\[
\left(\frac{x'}{x'_d}\right) = \exp(-i\delta\alpha_0) \left(\frac{N^{-1}x}{-N^{-1}x_d}\right)
\]

(3.10)

4 The infinitesimal operators of $a$-type coirreps

We have to consider the infinitesimal operators $J_\sigma$ connected with the subgroup $G$, and $J'_\rho$ connected with the coset $a_0G$. The definitions of the infinitesimal operators connected with the subgroup $G$ and of their commutators are the same as those for the Lie subgroup $G$. We distinguish by a "prime" the infinitesimal operators connected with the coset $a_0G$ from those connected with the subgroup $G$. Two additional types of commutators: $[J'_\mu, J'_\nu], \mu, \nu = 0, 1, \ldots, n$, and $[J_\sigma, J'_\mu], \sigma = 1, \ldots, n, \mu = 0, 1, \ldots, n$ have to be determined.

The importance of the relation between the points $x$ and $x'$ in Eq. (3.4) is connected with the fact that the infinitesimal operators for the subgroup $G$ will be defined at the point $x$, in accordance with Fig. 1a, while the infinitesimal operators for the coset $a_0G$ will be defined at the point $x'$ in accordance with Fig. 1b. We will require the equality of the commutator of two infinitesimal operators connected with the coset to a linear combination of operators connected with the subgroup, and the equality of the commutator of an operator connected with the subgroup with an operator connected with the coset to a linear combination of operators connected with the coset. The operators in the commutator have to be referred to the same point.
4.1 The subgroup \( G \).

The definition of the infinitesimal operators \( J_\sigma, \sigma = 1, \ldots, n \), and the calculation of the structural constants for the linear Lie subgroup \( G \) of the group \( G + a_0G \) carries over exactly from the Lie group theory. The increments of the coordinates at the point \( x \) in Fig. 1a, are given by

\[
dx_i = \left( \frac{\partial}{\partial \alpha_\sigma} \Delta(g(\delta\alpha))_{ij} \right)_{\delta\alpha = 0} x_j \delta \alpha_\sigma = (X_\sigma)_{ij} x_j \delta \alpha_\sigma = u_{i\sigma}(x) \delta \alpha_\sigma = u_{i\sigma}(x) M_{\sigma\lambda}^{-1} \delta \alpha_\lambda
\]

where \( \delta \alpha \equiv (\delta \alpha_1, \ldots, \delta \alpha_n) \) and \( x \) given in Eq. (2.3), where \( (X_\sigma)_{ij} \) is the element \((i,j)\) of the matrix \( X_\sigma \) of the matrix basis of the Lie subgroup \( G \) algebra, and where the matrix \( M_{\sigma\lambda} \) is determined from the product of matrices \( D(g(\alpha + \delta\alpha)) = \Delta(g(\delta\alpha))\Delta(g(\alpha)) \) according to Eq. (3.1). In the customary way, we obtain the equations,

\[
\frac{\partial x_i}{\partial \alpha_\lambda} = u_{i\sigma}(x) M_{\sigma\lambda}^{-1}, \quad i = 1, \ldots, d; \quad \lambda, \sigma = 1, \ldots, n (4.2)
\]

From the integrability condition of these equations we derive the structural constants \( c^\tau_{\sigma\rho} \).

Defining the infinitesimal operators:

\[
J_\sigma = u_{i\sigma}(x) \frac{\partial}{\partial x_i} = (X_\sigma)_{ij} x_j \frac{\partial}{\partial x_i}, \quad i, k = 1, \ldots, d; \quad \sigma = 1, \ldots, n (4.3)
\]

where \( (X_\sigma)_{ij} \) is the \((ij)\)–element of the matrix basis vector \( X_\sigma \), we determine the commutators

\[
[J_\sigma, J_\rho] = c^\tau_{\sigma\rho} J_\tau
\]

as in the Lie groups, \([2, 5, 13]\).

4.2 The coset \( a_0G \).

Since we have \( a_0^2 = \pm 1 \), the matrix \( \Delta(a_0^2) \) in (38) of \([11]\) is equal to \( \pm E \), and then \( \mu/\lambda = \exp(i\xi) \), with a real \( \xi \), which can be absorbed by \( \exp(i\delta\alpha_0) \). Performing an infinitesimal transformation of the point \( x' \), which was defined in Fig.1b, with the matrix \( \exp(i\delta\alpha_0)\Delta(g(\delta\alpha))N \), we obtain:

\[
dx'_i = \left( \frac{\partial}{\partial \alpha_\sigma} \exp(i\delta\alpha_0)\Delta(g(\delta\alpha))_{ij} \right)_{\delta\alpha = 0} N_{jk} x'_k \delta \alpha_\sigma = (X'_\sigma)_{ik} x'_k \delta \alpha_\sigma = u'_{i\sigma}(x') \delta \alpha_\sigma, \quad i, j, k = 1, \ldots, d; \quad \sigma = 0, 1, \ldots, n (4.5)
\]

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where the derivatives are calculated at the point δα₀ = δα₁ = ... = δαₙ = 0, and (Xₜ')ₜk is the (tk)−element of the matrix basis vector Xₜ' connected with the coset a₀G, which was defined in [11].

We next define the infinitesimal operators connected with the coset a₀G: Jₜ', σ = 0, 1, ..., n, at the point x' (see Fig.1b), in the form:

\[ Jₜ' = u'_iσ(x') \frac{∂}{∂x_i'}, \quad \sigma = 0, 1, ..., n; \quad i, k = 1, ..., d \] (4.6)

according to Eq. (4.5).

The commutators [Jₜ', Jₚ'], σ ≠ ρ = 0, 1, ..., n, have the form:

\[ [Jₜ', Jₚ'] = \left( (Xₜ')ₜk \frac{∂}{∂x_i'}, (Xₚ')ₚl \frac{∂}{∂x_j'} \right) \]

\[ = \sum_{i,j,k,l=1}^{d} (Xₜ')ₜk (Xₚ')ₚl \left( x'_k \frac{∂}{∂x_i'} x'_l \frac{∂}{∂x_j'} - x'_l \frac{∂}{∂x_j'} x'_k \frac{∂}{∂x_i'} \right) \] (4.7)

After the transformation to the point x, according to Eq. (4.6), the commutator acquires the form of a linear combination of infinitesimal operators connected with the subgroup G.

For the remaining type of commutators we find:

\[ [Jₜ' , Jₚ'] = \left( (Xₜ')ₜk \frac{∂}{∂x_i'}, (Xₚ')ₚl \frac{∂}{∂x_j'} \right) \]

\[ = \sum_{i,j,k,l=1}^{d} (Xₜ')ₜk (Xₚ')ₚl \left( x_k \frac{∂}{∂x_i} x_l \frac{∂}{∂x_j} - x_l \frac{∂}{∂x_j} x_k \frac{∂}{∂x_i} \right) \]

\[ = \sum_{i,j,k,l=1}^{d} \left( x_k \frac{∂}{∂x_i} x_l \frac{∂}{∂x_j} - x_l \frac{∂}{∂x_j} x_k \frac{∂}{∂x_i} \right) \] (4.8)

After the transformation of the terms depending on the point x to the point x', we obtain a linear combination of infinitesimal operators connected with the coset a₀G.

5 The infinitesimal operators of b-type coirreps.

5.1 The subgroup G.

As the corep matrices are constructed from square blocks of equal dimensions, it will be convenient to label the coordinates from 1 to d and from d + 1 to 2d, where d denotes the dimension of the irrep Γ of the subgroup G, with matrices Δ(g). The transformation of the point (x|xₙ) in Eq. (2.7) with the matrix D(g(δα)) is determined by
The structural constants and the infinitesimal operators connected with the subgroup $G$, are obtained from the infinitesimal increments of the coordinates at the point $(x_1, \ldots, x_d, x_{d+1}, \ldots, x_{2d})$, in the form:

$$dx_i = \left( \frac{\partial}{\partial \alpha} \Delta(g(\delta \alpha)) \right)_{\delta \alpha = 0} x_j \delta \alpha_{\sigma} = \left( \bar{X}_{\sigma} \right)_{ij} x_j \delta \alpha_{\sigma} = u_{i \sigma}(x_w) \delta \alpha_{\sigma} = u_{i \sigma}(x_w) M_{\sigma \lambda}^{-1} d \alpha_{\lambda}$$

and

$$dx_{d+i} = \left( \frac{\partial}{\partial \alpha} \Delta(g(\delta \alpha)) \right)_{\delta \alpha = 0} x_{d+j} \delta \alpha_{\sigma} = \left( \bar{X}_{\sigma} \right)_{ij} x_{d+j} \delta \alpha_{\sigma} = u_{d+i, \sigma}(x_{d+w}) \delta \alpha_{\sigma} = u_{d+i, \sigma}(x_{d+1}, \ldots, x_{2d}) M_{\sigma \lambda}^{-1} d \alpha_{\lambda}$$

where the derivatives with respect to $\alpha_{\sigma}$ are calculated at $\delta \alpha = (\delta \alpha_1, \ldots, \delta \alpha_n) = 0$, and where $(\bar{X}_{\sigma})_{ij}$ is the element $(i, j)$ of the upper non-zero block of the matrix $X_{\sigma}$, belonging to the matrix basis of the subgroup $G$ algebra. The matrix $M_{\lambda \sigma}$ is calculated as in the case of $a$--type coirreps. We obtain the equations,

$$\frac{\partial x_i}{\partial \alpha_\lambda} = u_{i \sigma}(x) M_{\sigma \lambda}^{-1}, \quad \text{and} \quad \frac{\partial x_{d+i}}{\partial \alpha_\lambda} = u_{d+i, \sigma}(x_{d+w}) M_{\sigma \lambda}^{-1}$$

where $\alpha_\lambda = 1, \ldots, n$ (5.4)

From the integrability conditions of these equations the respective structural constants $c^{\tau}_{\sigma \rho}$ are derived. The infinitesimal operators are define by:

$$J_\sigma = u_{i \sigma}(x_1, \ldots, x_d) \frac{\partial}{\partial x_i} + u_{d+i, \sigma}(x_{d+1}, \ldots, x_{2d}) \frac{\partial}{\partial x_{d+i}} =$$

$$\left( \bar{X}_{\sigma} \right)_{ij} x_j \frac{\partial}{\partial x_i} + \left( \bar{X}_{\sigma} \right)_{ij} x_{d+j} \frac{\partial}{\partial x_{d+i}}, \quad i, j = 1, \ldots, d; \quad \sigma = 1, \ldots, n$$

For the commutator of two infinitesimal operators $J_\sigma$ and $J_\rho$, connected with the subgroup $G$, we find in the way analogous to that for the $a$--type coirreps the expression:

$$[J_\sigma, J_\rho] = c^{\tau}_{\sigma \rho} J_\tau; \quad \sigma, \rho, \tau = 1, \ldots, n$$

with the structural constants $c^{\tau}_{\sigma \rho}$.
5.2 The coset \( a_0G \).

The transformation of the point \((x'|x')\) in Eq. (3.8) with the matrix \( \exp(i\delta a_0)D(g(\delta a_0)a_0) \) has the form:

\[
\begin{align*}
\exp(i\delta a_0) & \left( \begin{array}{cc}
0 & \Delta(g(\delta a_0))N \\
-\Delta(g(\delta a_0))N & 0
\end{array} \right) \left( \begin{array}{c}
x'_d \\
x'
\end{array} \right) = \exp(i\delta a_0) \\
\exp(i\delta a_0) & \left( \begin{array}{c}
\Delta(g(\delta a_0))N\,x'_d \\
-\Delta(g(\delta a_0))N\,x'_d
\end{array} \right) \left( \begin{array}{c}
dx' \\
dx'_d
\end{array} \right)
\end{align*}
\]

(5.7)

from which we obtain the expressions:

\[
dx'_i = \left( \frac{\partial}{\partial \delta \alpha} \exp(i\delta a_0)\Delta(g(\delta a_0)) \right)_{\delta \alpha = 0} N_{jk} x'_k \delta \alpha_\sigma = (X'_\sigma)_{ik} x'_k \delta \alpha_\sigma = u'_{i\sigma}(x'_1, ..., x'_d) \delta \alpha_\sigma; \quad i, j, k = 1, ..., d, \quad \lambda, \sigma = 0, 1, ..., n \quad (5.8)
\]

where \( X'_\sigma \) denotes the upper non-zero block of the matrix \( X'_\sigma \) belonging to the matrix basis of the \( b \)-type coirep algebra, and

\[
dx'_{d+i} = -\left( \frac{\partial}{\partial \delta \alpha} \exp(i\delta a_0)\Delta(g(\delta a_0)) \right)_{\delta \alpha = 0} N_{jk} x'_{d+k} \delta \alpha_\sigma = -(X'_\sigma)_{ik} x'_{d+k} \delta \alpha_\sigma = u'_{d+i,\sigma}(x'_{d+1}, ..., x'_{2d}) \delta \alpha_\sigma \quad i, j, k = 1, ..., d, \quad \lambda, \sigma = 0, 1, ..., n \quad (5.9)
\]

where in Eqs. (5.8) and (5.9) the derivatives are calculated at the point: \( \delta \alpha = (\delta \alpha_0, \delta \alpha_1, ..., \delta \alpha_n) = 0 \). For brevity we will define:

\[
(x'_1, ..., x'_d) \equiv x'_w, \quad (x'_{d+1}, ..., x'_{2d}) \equiv x'_{d+w} \\
u'_{i\sigma}(x'_1, ..., x'_d) \equiv u'_{i\sigma}(x'_w), \quad and \quad u'_{d+i,\sigma}(x'_{d+1}, ..., x'_{2d}) \equiv u'_{d+i,\sigma}(x'_{d+w}) \quad (5.10)
\]

The infinitesimal operators \( J'_\sigma \), \( \sigma = 0, 1, ..., n \), connected with the coset \( a_0G \) are defined at the point \((x'|x'_d)\) in Eq. (3.8) in the form:

\[
J'_\sigma = u'_{i\sigma}(x'_w) \frac{\partial}{\partial x'_i} + u'_{d+i,\sigma}(x'_{d+w}) \frac{\partial}{\partial x'_{d+i}} = (X'_\sigma)_{ik} x'_k \frac{\partial}{\partial x'_i} - (X'_\sigma)_{ik} x'_{d+k} \frac{\partial}{\partial x'_{d+i}} \quad i = 1, ..., d; \quad \sigma = 0, 1, ..., n \quad (5.11)
\]
For the commutators \([J'_\sigma, J'_\rho]\), with \(\sigma = 1, \ldots, n, \rho = 0, 1, \ldots, n\) we find the following expression:

\[
[J'_\sigma, J'_\rho] = \left[ (X'_\sigma)_{ik} \left( x_i' \frac{\partial}{\partial x_i} - x_i' \frac{\partial}{\partial x_{i+1}} \right), (X'_\rho)_{jl} \left( x_j' \frac{\partial}{\partial x_j} - x_j' \frac{\partial}{\partial x_{j+1}} \right) \right] = \\
\sum_{i,j,k,l=1}^{d} \left( (X'_\sigma)_{ik} (X'_\rho)_{jl} \left( x'_{d+k} \frac{\partial}{\partial x_{d+i}} - x'_{d+k} \frac{\partial}{\partial x_{d+j}} \right) \right)
\]

After the transformation to the point \(x\), the last expression turns into a linear combination of the infinitesimal operators connected with the subgroup \(G\).

The remaining type of commutators has the form: \([J'_\sigma, J'_\rho]\), \(\sigma = 1, \ldots, n, \rho = 0, 1, \ldots, n\). Introducing for \(J'_\sigma\) and \(J'_\rho\) the expressions in Eqs. (5.13) and (5.11) respectively, we obtain:

\[
[J_\sigma, J'_\rho] = \left[ (X_\sigma)_{ik} \left( x_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial x_{i+1}} \right), (X'_\rho)_{jl} \left( x_j' \frac{\partial}{\partial x_j} - x_j' \frac{\partial}{\partial x_{j+1}} \right) \right] = \\
\sum_{i,j,k,l=1}^{d} \left( (X_\sigma)_{ik} (X'_\rho)_{jl} \left( x_{d+k} \frac{\partial}{\partial x_{d+i}} + x_{d+k} \frac{\partial}{\partial x_{d+j}} \right) \right)
\]

After the transformation to the point \(x'\) in Fig. 1b, the last expression turns into a linear combination of operators defined in Eq. (5.11).

Observation 5.1. When the operators \(J'_\rho\) are linearly independent on the operators \(J_\sigma, \sigma = 1, \ldots, n, \rho = 0, 1, \ldots, n\), which always holds for \(b\)-type coirreps, the real algebra connected with the group \(G + a_0G\) is spanned by the operators \(J_\sigma\) and \(J'_\rho\), and it is \((2n+1)\)-dimensional. When for \(a\)-type coirreps, the operators \(J'_\rho, \rho = 1, \ldots, n,\) linearly depend on the operators \(J_\sigma,\) the respective real algebra is spanned by the operators \(J_\sigma, \sigma = 1, \ldots, n\) together with the operator \(J'_0\), and it is \((n+1)\)-dimensional.

6 Conclusions

The matrix algebras of continuous groups with antilinear operations of the type \(G + a_0G\), where \(G\) denotes a Lie group and the antilinear operation \(a_0\) fulfills the condition \(a_0^2 = \pm 1\),
were determined in [11]. In this paper the infinitesimal operators of these groups were determined, when the coirreps of the groups \( G + a_0G \) are of \( a-\)type or of \( b-\)type. The infinitesimal operators connected with the subgroup \( G \) differ from those connected with the coset \( a_0G \). By definition the commutators of the infinitesimal operators are to be compatible with the commutators of the basis matrices of the matrix algebra which was determined in [11].

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