Conformal Minimal Foliations on Semi-Riemannian Lie Groups

Victor Ottosson

Master’s thesis
2020:E87

Lund University
Faculty of Science
Centre for Mathematical Sciences
Mathematics
Abstract

In this thesis we investigate semi-Riemannian Lie groups \((G, g)\) and conformal foliations \(\mathcal{F}\) of codimension two on these \(G\) generated by a subgroup \(K\). We are interested in finding out exactly when these foliations admit minimal leaves, these are particularly interesting as these foliations will in turn produce locally defined complex-valued harmonic morphisms on \(G\). The particular choice of Lie groups we study are specifically chosen to investigate whether or not a conjecture by Sigmundur Gudmundsson formulated for Riemannian Lie groups can be extended to also include the more general semi-Riemannian case.

Throughout this work it has been my firm intention to give reference to the stated results and credit to the work of others. All theorems, propositions, lemmas and examples left unmarked are assumed to be too well known for a reference to be given.
Acknowledgments

I would like to express my sincerest gratitude towards my supervisor Sigmundur Gudmundsson for being a great source of inspiration and support, it has been a pleasure to learn about this subject and the many interesting aspects of it. I would also like to sincerely thank my co-supervisor Elsa Ghandour, without whom the contents of this text would be considerably less clear and full of mistakes.

Victor Ottosson
# Contents

1 Introduction 1  
1.1 Semi-Riemannian Geometry 1  
1.2 Harmonic Morphisms 3  
1.3 Foliations 5  
1.4 Conformal Foliations on Lie Groups $G^{n+2}$ 7  

2 Lie Groups $G^5$ Foliated by their Subgroup SU(2) 11  

3 Lie Groups $G^5$ Foliated by their Subgroup SL$_2(\mathbb{R})$ 17  

4 Lie Groups $G^8$ Foliated by their Subgroup SU(2) $\times$ SU(2) 19  

5 Lie Groups $G^8$ Foliated by their Subgroup SU(2) $\times$ SL$_2(\mathbb{R})$ 25  

6 Lie Groups $G^8$ Foliated by their Subgroup SL$_2(\mathbb{R})$ $\times$ SL$_2(\mathbb{R})$ 29  

7 Lie Groups $G^6$ Foliated by their Subgroup SU(2) $\times$ SO(2) 33  

8 Lie Groups $G^6$ Foliated by their Subgroup SL$_2(\mathbb{R})$ $\times$ SO(2) 41  

A A Maple Program for $K = \text{SU}(2)$ 45  

Bibliography 51
Chapter 1

Introduction

In this text we assume that the reader is familiar with and has a basic understanding of some of the concepts in Riemannian geometry. For an introduction to this subject we recommend [7]. We will also assume that all manifolds and maps considered are smooth, i.e. of type $C^\infty$.

1.1 Semi-Riemannian Geometry

The study of semi-Riemannian geometry generalizes that of Riemannian geometry. The classical reference on this subject is O’Neill’s book [13] which is also the inspiration for the presentation of the material in this section.

Remark 1.1. Though we will not put much focus on it, it may be important to note that semi-Riemannian manifolds, and thus semi-Riemannian geometry, are of great importance in certain parts of the field of physics; in particular it is heavily used in the general theory of relativity. The influence this has had can be seen in the naming of certain concepts and objects.

Our main objective in this section is to define what a semi-Riemannian manifold is, informally these are a generalization of the familiar Riemannian manifolds where we remove the assumption of positivity on the metric. For us to be able to achieve this we require bilinear forms.

Definition 1.2. A bilinear form $b$ on a real vector space $V$ is a function $b: V \times V \to \mathbb{R}$ which is linear in both of its arguments. If $b$ satisfies $b(v, w) = b(w, v)$ for all $v, w \in V$ we say that $b$ is symmetric.

Definition 1.3. A symmetric bilinear form $b$ on $V$ is said to be

(i) non-degenerate if, there exists no non-zero $v$ such that $b(v, w) = 0$ for all $w \in V$;
(ii) positive (negative) definite if, for all non-zero $v$, $b$ satisfies $b(v, v) > 0$ ($b(v, v) < 0$).

If the strictness is removed from the inequalities in (ii) we say that $b$ is positive (negative) semidefinite.

Definition 1.4. For a symmetric bilinear form $b$ on a real vector space $V$ we define the index of $b$ on a subspace $U$, denoted $\text{ind}_U b$, to be the largest number $v$ such that $v = \dim W$ where $W$ is a subspace of $U$ on which $b|_W$ is negative definite.
**Definition 1.5.** A symmetric non-degenerate tensor field $g$ of type $(0, 2)$ on a manifold $M$ is said to be a **metric tensor** if it is of constant index.

**Remark 1.6.** We will, when it is clear from context what is meant, use the word metric to refer to a metric tensor.

We are now ready to give a formal definition of semi-Riemannian manifolds.

**Definition 1.7.** A **semi-Riemannian manifold** $(M, g)$, in some texts referred to as a **pseudo-Riemannian manifold**, is a differentiable manifold $M$ equipped with a metric tensor $g$.

This gives us a way to define Riemannian manifolds as a special case of the more general semi-Riemannian manifolds.

**Definition 1.8.** A **Riemannian manifold** $(M, g)$ is a semi-Riemannian manifold such that $\text{ind}_M g = 0$.

Another important special case, especially in physics, are Lorentzian manifolds.

**Definition 1.9.** A **Lorentzian manifold** $(M, g)$ is a semi-Riemannian manifold such that $\dim M \geq 2$ and $\text{ind}_M g = 1$.

**Definition 1.10.** Let $(M, g)$ be a semi-Riemannian manifold. For an element $X$ in any orthonormal basis of $M$ we define $\varepsilon_X$ to be

$$\varepsilon_X = g(X, X).$$

The above definition turns out to be a helpful tool that enables us to easily communicate the metric under consideration.

**Definition 1.11.** Let $(M^n, g)$ be an $n$-dimensional semi-Riemannian manifold and let $\{X_1, X_2, \ldots, X_n\}$ be an orthonormal basis of $M$ with respect to $g$. Then the $n$-tuple $(\varepsilon_{X_1}, \varepsilon_{X_2}, \ldots, \varepsilon_{X_n})$ is called the **signature** of $g$. Reordering the basis by sign such that the $p$ first elements of the signature are equal to $-1$ and the remaining $n-p$ elements are equal to $1$ we can express the signature more concisely as $(p, n-p)$. If $0 < p < n$ we say that the metric has **mixed** signature.

**Example 1.12.** The metric of any arbitrary $n$-dimensional Riemannian manifold has signature $(0, n)$.

**Example 1.13.** An $n$-dimensional Lorentzian manifold is equipped with a metric that has signature $(1, n-1)$.

**Example 1.14.** The semi-Euclidean space $\mathbb{R}^n_v$ is the vector space $\mathbb{R}^n$ where the metric is given by

$$\sum_{i=1}^{v} -dx_i^2 + \sum_{i=v+1}^{n} dx_i^2,$$

the signature of this metric is then $(v, n-v)$. A special case of this is the Minkowski space $\mathbb{R}^4_1$ which serves as a simple model of spacetime.

**Definition 1.15.** Let $(M, g)$ be a semi-Riemannian manifold and let $X \in TM$, then $X$ is said to be

(i) **spacelike** if $g(X, X) > 0$ or $X = 0$, 

(ii) **timelike** if $g(X, X) < 0$ and $X \neq 0$, 

(iii) **lightlike** if $g(X, X) = 0$ and $X \neq 0$.
(ii) timelike if $g(X, X) < 0$,

(iii) null if $g(X, X) = 0$ and $X \neq 0$.

The category a tangent vector falls into is known as its causal character.

**Definition 1.16.** Let $(M, g)$ be a semi-Riemannian manifold and let $p$ be a point in $M$. A subspace $U$ of $T_pM$ is said to be degenerate if for some non-zero $X$ in $U$ it holds that $g(X, Y) = 0$ for all $Y$ in $U$. If no such $X$ exists we say that $U$ is non-degenerate.

**Definition 1.17.** The Levi-Civita connection $\nabla$ on the tangent bundle of a semi-Riemannian manifold $(M, g)$ is the unique metric and torsion-free connection, that is, it is the only connection satisfying the conditions

(i) $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$

(ii) $[X, Y] = \nabla_X Y - \nabla_Y X$

for all $X, Y, Z \in TM$.

Just as in the Riemannian setting $\nabla$ is given by the Koszul formula,

$$g(\nabla_X Y, Z) = \frac{1}{2} \{X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y))$$

$$+ g([Z, X], Y) + g([Z, Y], X) + g(Z, [X, Y])\}.$$

When the vector fields $X, Y$ and $Z$ are left-invariant this takes on the simplified form

$$g(\nabla_X Y, Z) = \frac{1}{2} \{g([Z, X], Y) + g([Z, Y], X) + g(Z, [X, Y])\}.$$

This version proves to be useful for us since we will be investigating semi-Riemannian Lie groups in the coming chapters.

### 1.2 Harmonic Morphisms

The notion of a harmonic morphism, though not known as such at the time, originates from Jacobi’s 1848 study [12] of solutions to Laplace’s equation in $\mathbb{R}^3$. Jacobi wanted to find functions $\phi(x_1, x_2, x_3)$ from an open subset $U$ of $\mathbb{R}^3$ into $\mathbb{C}$ such that the composition $f \circ \phi$ is harmonic for any holomorphic function $f(z)$.

**Proposition 1.18** ([12]). A function $\phi: U \subset \mathbb{R}^3 \to \mathbb{C}$ is a harmonic morphism if and only if it is

(i) horizontally (weakly) conformal, i.e. $\langle \nabla \phi, \nabla \phi \rangle = 0$;

(ii) harmonic, i.e. $\Delta \phi = 0$.

The following result was proven by Fuglede in his 1978 article [4] and concurrently, although published a year later, by Ishihara in [11] independently of each other and it characterizes the harmonic morphisms between Riemannian manifolds.

**Theorem 1.19** ([4, 11]). A map $\phi: (M, g) \to (N, h)$ between Riemannian manifolds is a harmonic morphism if and only if it is harmonic and horizontally (weakly) conformal.
Naturally, one would hope that this could be generalized further to also include semi-Riemannian manifolds, and in his 1996 article [5] Fuglede could affirm that this was indeed the case. But in order for us to understand the result we need to define exactly what it means for a map \( \phi \) between semi-Riemannian manifolds to be harmonic and horizontally conformal.

**Definition 1.20.** The *Laplace-Beltrami operator* \( \Delta_M \), often referred to simply as the *Laplacian*, on a semi-Riemannian manifold \((M, g)\) is given by

\[
\Delta_M = \sum_{i,j} g^{ij} \left( \frac{\partial^2}{\partial x^i \partial x^j} - \sum_k \Gamma^k_{ij} \frac{\partial}{\partial x^k} \right)
\]

where the \( x^i \) are local coordinates on \( M \) and \( \Gamma \) is the Christoffel symbol on \( M \).

**Remark 1.21.** While the Laplacian on a Riemannian manifold is always elliptic, the same is not necessarily true for semi-Riemannian manifolds. For example, it can be easily shown that if \((M^n, g)\) is a Lorentzian manifold, then \(-\Delta_M\) is the wave operator on \( M \).

We define the *harmonic functions* to be those \( C^2 \) functions defined on an open subset \( U \) of \( M \) that solve the homogeneous equation \( \Delta_M f = 0 \) locally.

**Definition 1.22.** The *tension field* \( \tau \) of a map \( \phi : (M, g) \to (N, h) \) between semi-Riemannian manifolds is given by

\[
\tau(\phi) = \text{div grad } \phi.
\]

By introducing the local coordinates \((x^k)\) and \((y^\alpha)\) on \( M \) and \( N \), respectively, we can express the components of \( \tau \) in a more explicit manner.

\[
\tau^\gamma(\phi) = \sum_{i,j} g^{ij} \left( \frac{\partial^2 \phi^\gamma}{\partial x^i \partial x^j} - \sum_k \hat{\Gamma}^{kij} \frac{\partial \phi^\gamma}{\partial x^k} + \sum_{\alpha, \beta} (\Gamma^\gamma_{\alpha\beta} \circ \phi) \frac{\partial \phi^\alpha}{\partial x^i} \frac{\partial \phi^\beta}{\partial x^j} \right)
\]

\[
= \Delta_M \phi^\gamma + \sum_{i,j} g^{ij} \left( \sum_{\alpha, \beta} (\Gamma^\gamma_{\alpha\beta} \circ \phi) \frac{\partial \phi^\alpha}{\partial x^i} \frac{\partial \phi^\beta}{\partial x^j} \right),
\]

here \( \phi^\alpha = y^\alpha \circ \phi \) and \( \hat{\Gamma}, \Gamma \) denote the Christoffel symbols on \( M \) and \( N \), respectively.

In the case that \( \tau(\phi) = 0 \) we say that the map \( \phi \) is *harmonic*.

**Definition 1.23.** A map \( \phi : (M, g) \to (N, h) \) between semi-Riemannian manifolds is called a *harmonic morphism* if the composition \( f \circ \phi \) is harmonic in \( \phi^{-1}(V) \) for any harmonic function \( f : V \to \mathbb{R} \), defined on any open subset \( V \) of \( N \) such that \( \phi^{-1}(V) \) is non-empty.

**Definition 1.24.** Let \( \phi : (M, g) \to (N, h) \) be a map between semi-Riemannian manifolds and \( x \) a point in \( M \). We then define the *vertical space* at \( x \) denoted \( \mathcal{V}_x \) to be the kernel of \( d\phi_x \). We also define the *horizontal space* at \( x \) by

\[
\mathcal{H}_x = \mathcal{V}_x^\perp = \{ X \in T_x M \mid g_x(X, Y) = 0 \text{ for all } Y \in \mathcal{V}_x \}.
\]
One important thing to note here is that if \( V_x \) happens to be degenerate then the tangent space at \( x \) is not a direct sum of the vertical and horizontal spaces, i.e. \( T_x M \neq V_x \oplus H_x \). This is different from how it is in the Riemannian setting and we have to take this into consideration when defining what it means for a map to be horizontally weakly conformal.

**Definition 1.25** ([5]). A map \( \phi : (M, g) \to (N, h) \) between semi-Riemannian manifolds is said to be **horizontally weakly conformal** if:

(i) For any \( x \in M \) at which \( V_x \) (or equivalently \( H_x \)) is non-degenerate and \( d\phi_x \neq 0 \), the restriction of \( d\phi_x \) to \( H_x \) is surjective and conformal in the sense that there is a unique real number \( \lambda(x) \neq 0 \) such that

\[
h_{\phi(x)}(d\phi(X), d\phi(Y)) = \lambda(x)g_x(X, Y) \quad \text{for every } X, Y \in H_x.
\]

(ii) For any \( x \in M \) at which \( V_x \) is degenerate we have \( H_x \subset V_x \), that is,

\[
g_x(X, Y) = 0 \quad \text{for every } X, Y \in H_x.
\]

For the sake of completeness we now state the result of Fuglede.

**Theorem 1.26** ([5]). A map \( \phi : (M, g) \to (N, h) \) between semi-Riemannian manifolds is a harmonic morphism if and only if it is harmonic and horizontally weakly conformal.

### 1.3 Foliations

**Definition 1.27.** Let \( (M, g) \) be semi-Riemannian manifold. An \( n \)-dimensional sub-bundle \( \mathcal{V} \) of the tangent bundle \( TM \) is said to be an **\( n \)-dimensional distribution** on \( M \).

**Definition 1.28.** Let \( (M, g) \) be a semi-Riemannian manifold and \( TM = \mathcal{V} \oplus \mathcal{H} \) be an orthogonal decomposition of its tangent bundle \( TM \). Then the second fundamental forms \( B^V \) and \( B^H \) of \( \mathcal{V} \) and \( \mathcal{H} \) are given by

\[
B^V(U, V) = \frac{1}{2} \mathcal{H}(\nabla_U V + \nabla_V U), \quad U, V \in \mathcal{V}
\]

and

\[
B^H(X, Y) = \frac{1}{2} \mathcal{V}(\nabla_X Y + \nabla_Y X), \quad X, Y \in \mathcal{H},
\]

respectively.

It will, at times, be convenient to view the second fundamental form of an \( n \)-dimensional distribution as an \( n \times n \)-matrix containing the values at the elements of an orthonormal basis, e.g.

\[
B^V = \begin{pmatrix}
B^V(V_1, V_1) & \cdots & B^V(V_1, V_n) \\
\vdots & \ddots & \vdots \\
B^V(V_n, V_1) & \cdots & B^V(V_n, V_n)
\end{pmatrix}.
\]

Because of the symmetry of the second fundamental form it suffices to compute only the values of the upper triangle when determining the form.
Definition 1.29. Let \((M, g)\) be a semi-Riemannian manifold. We say that a distribution \(\mathcal{V}\) on \(M\) is integrable if the Lie bracket satisfies \([\mathcal{V}, \mathcal{V}] \subset \mathcal{V}\).

The above definition seems to suggest that there is some sort of connection between the Lie subalgebras \(\mathfrak{k}\) of a Lie algebra \(\mathfrak{g}\) and integrable distributions \(\mathcal{V}\) on the Lie group \(G\) corresponding to \(\mathfrak{g}\), and indeed there is.

Let \((G, g)\) be a semi-Riemannian Lie group and let \(\mathfrak{g}\) be its Lie algebra. Suppose now that \(\mathfrak{k}\) is a Lie subalgebra of \(\mathfrak{g}\). Then, since \(T_eG \cong \mathfrak{g}\) and \([\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}\), we easily see that \(\mathfrak{k}\) induces an integrable distribution \(\mathcal{V}\) on \(G\).

Definition 1.30 ([3]). A foliation \(\mathcal{F}\) of codimension \(n\) is a partition of a semi-Riemannian manifold \((M, g)\) into connected subsets, the leaves of the foliation, \(\{L_\alpha\}\) satisfying: For every point \(p\) in \(M\) there exists an open neighborhood \(U\) of \(p\) and a submersion \(\phi: U \rightarrow N\) to an \(n\)-dimensional manifold such that, for each \(\alpha\), the connected components of \(U \cap L_\alpha\) are the fibers of \(\phi\).

In the subsequent chapters we are mainly interested in finding conformal foliations that are minimal. It is thus of great importance that we define what we mean by these terms.

Definition 1.31. Let \((M, g)\) be a semi-Riemannian manifold. A foliation \(\mathcal{F}\) tangent to an integrable distribution \(\mathcal{V}\) on \(M\) is said to be minimal if the second fundamental form \(B_\mathcal{V}\) is traceless, i.e., \(\text{trace } B_\mathcal{V} = 0\). If \(B_\mathcal{V} \equiv 0\) we say that \(\mathcal{F}\) is totally geodesic.

Definition 1.32. A foliation \(\mathcal{F}\) of the semi-Riemannian manifold \((M, g)\) tangent to an integrable distribution \(\mathcal{V}\) is said to be conformal if it satisfies

\[ B_\mathcal{H} = g \otimes V \]

for some \(V \in \mathcal{V}\). We say that \(\mathcal{F}\) is semi-Riemannian if \(V = 0\).

To give a better understanding of what conformality means for the manifolds we are interested in investigating we present the following remark.

Remark 1.33. Let \((G, g)\) be a semi-Riemannian Lie group, \(K\) be a subgroup of codimension two and \(\mathcal{F}\) be the Lie foliation generated by the left-invariant translations of \(K\). Let \(\mathcal{V}\) be the integrable distribution on \(G\), tangent to the fibers of \(\mathcal{F}\), and \(\mathcal{H}\) be its complementary orthogonal distribution. Since the foliation \(\mathcal{F}\) is conformal there exists a left-invariant vector field \(V\) on \(G\) such that the second fundamental form \(B_\mathcal{H}: \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{V}\) of the horizontal distribution \(\mathcal{H}\) satisfies

\[ B_\mathcal{H}(E, F) = g(E, F) \otimes V \quad \text{for all } E, F \in \mathcal{H}. \]

We shall now determine \(B_\mathcal{H}\) explicitly in terms of an orthonormal basis \(\{X, Y\}\) for \(\mathcal{H}\). We note that

\[ B_\mathcal{H}(X, Y) = g(X, Y) \otimes V = 0, \]

\[ B_\mathcal{H}(X, X) - B_\mathcal{H}(Y, Y) = (g(X, X) - g(Y, Y)) \otimes V = (\varepsilon_X - \varepsilon_Y) \otimes V, \]

\[ B_\mathcal{H}(X, X) + B_\mathcal{H}(Y, Y) = (g(X, X) + g(Y, Y)) \otimes V = (\varepsilon_X + \varepsilon_Y) \otimes V. \]

Case A \((\varepsilon_X = \varepsilon_Y)\): We can identify the vector field \(V\) to be

\[ V = \frac{1}{\varepsilon_X + \varepsilon_Y} \cdot (B_\mathcal{H}(X, X) + B_\mathcal{H}(Y, Y)), \]
so the second fundamental form $B^H$ satisfies

$$B^H(E, F) = \frac{g(E, F)}{\varepsilon_X - \varepsilon_Y} \otimes \left( B^H(X, X) + B^H(Y, Y) \right)$$

for all $E, F \in \mathcal{H}$.

**Case B** ($\varepsilon_X = -\varepsilon_Y$): Here we see that the vector field $V$ is given by

$$V = \frac{1}{\varepsilon_X - \varepsilon_Y} \cdot \left( B^H(X, X) - B^H(Y, Y) \right),$$

so for the second fundamental form $B^H$ we have

$$B^H(E, F) = \frac{g(E, F)}{\varepsilon_X - \varepsilon_Y} \otimes \left( B^H(X, X) - B^H(Y, Y) \right).$$

The following result is the main motivation for studying minimal and conformal foliations of codimension two. It was first proved by Baird and Eells in [1] for Riemannian manifolds $(M, g), (N, h)$ while the more general version given here was proved in [9] by Gudmundsson.

**Theorem 1.34.** Let $(M^m, g)$ be a semi-Riemannian manifold and let $(N^n, h)$ be a Riemannian manifold. Suppose that $\phi: M \to N$ is a horizontally conformal submersion. Then

(i) if $n = 2$ then $\phi$ is harmonic if and only if $\phi$ has minimal fibers,

(ii) if $n \geq 3$ then any two of the following imply the third,

(a) $\phi$ is a harmonic map,

(b) $\phi$ has minimal fibers,

(c) $\phi$ is horizontally homothetic.

Since $(N^n, h)$ is Riemannian in the above we are mainly interested in the cases where the basis vector fields $X, Y$ of $\mathcal{H}$ are of the same causal character, i.e. $\varepsilon_X = \varepsilon_Y$.

### 1.4 Conformal Foliations on Lie Groups $G^{n+2}$

Let $(G, g)$ be an $(n + 2)$-dimensional semi-Riemannian Lie group and let $K$ be an $n$-dimensional subgroup of $G$ equipped with the induced left-invariant semi-Riemannian metric. Then the Lie algebra $\mathfrak{g}$ of $G$ has an orthogonal decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, here $\mathfrak{k}$ is the Lie subalgebra corresponding to $K$ and $\mathfrak{m}$ its orthogonal complement. Let $V_1, \ldots, V_n \in \mathfrak{k}$ and $X, Y \in \mathfrak{m}$ be left-invariant vector fields forming orthonormal bases for $\mathfrak{k}$ and $\mathfrak{m}$, respectively. We identify $\mathfrak{k}$ with an integrable vertical distribution $\mathcal{V}$ and $\mathfrak{m}$ with the horizontal distribution $\mathcal{H}$. Furthermore, we denote by $\mathcal{F}$ the foliation tangent to the distribution $\mathcal{V}$.

In the case that the subgroup $K$ is semisimple the following fact turns out to be useful.

**Proposition 1.35.** Let $\mathcal{F}$ be the foliation of codimension two on $G$ generated by the left-translations of the semisimple subgroup $K$. Then the following are equivalent:

(i) $\mathcal{F}$ is semi-Riemannian,

(ii) $\mathcal{F}$ is conformal,
(iii) $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{k}$.

**Proof.** (i) $\implies$ (ii): This is by definition.

(ii) $\implies$ (iii): Assume now that $\mathcal{F}$ is conformal. From the fact that $K$ is semisimple it follows that $[\mathfrak{k}, \mathfrak{k}] = \mathfrak{k}$. Repeating the arguments of Remark 3.2 in [10] or the beginning of section 3 in [8], we have that for any $V \in \mathcal{V}$ the adjoint action of $V$ satisfies

$$g(\text{ad}_V X, Y) + g(X, \text{ad}_V Y) = \lambda \cdot g(X, Y),$$

for all $X, Y$ in $\mathcal{H}$. In other words, $\mathcal{H} \text{ ad}_V |_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$ is in the conformal algebra

$$\text{co}(\mathcal{H}) = \mathbb{R} \cdot \text{Id}_{\mathcal{H}} \oplus \mathfrak{so}(\mathcal{H}).$$

This follows from the conformality of $\mathcal{F}$. Furthermore, from the Jacobi identity and the integrability of the distribution $\mathcal{V}$ it follows that

$$\mathcal{H} \text{ ad}_{[V_1, V_2]} |_{\mathcal{H}} = [\mathcal{H} \text{ ad}_{V_1} |_{\mathcal{H}}, \mathcal{H} \text{ ad}_{V_2} |_{\mathcal{H}}].$$

Thus, since $\text{co}(\mathcal{H})$ is abelian when $\dim \mathcal{H} = 2$, we can see that the adjoint action of $[\mathcal{V}, \mathcal{V}]$ has no $\mathcal{H}$-component. The statement now follows from this.

(iii) $\implies$ (i): If we now assume that $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{k}$ then, for all $Z, W \in \mathcal{H}$,

$$B^\mathcal{H}(Z, W) = \frac{1}{2} \mathcal{V}(\nabla_Z W + \nabla_W Z)$$

$$= \frac{1}{2} \sum_{i=1}^{n} \varepsilon_i (g(\nabla_Z W, V_i) + g(\nabla_W Z, V_i)) V_i$$

$$= \frac{1}{2} \sum_{i=1}^{n} \varepsilon_i (g([V_i, Z], W) + g([V_i, W], Z)) V_i$$

$$= 0.$$  

Note that the semisimplicity of $K$ is only used in order to show that (ii) implies (iii). This simple observation leads to the following result.

**Corollary 1.36.** Let $\mathcal{F}$ be the foliation of codimension two on $G$ generated by the left-translations of the subgroup $K$. If $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{k}$ then $\mathcal{F}$ is semi-Riemannian.

In the following chapters we are going to be studying the $(n+2)$-dimensional Lie groups $G$ equipped with a semi-Riemannian metric $g$ containing the $n$-dimensional subgroups listed in the table below. Four of these were previously investigated in the Riemannian setting in [6, 15].

The subgroups that we are going to study can be categorized as in the table below.

| $K$       | compact          | non-compact       |
|-----------|------------------|-------------------|
| simple    | $\text{SU}(2)$   | $\text{SL}_2(\mathbb{R})$ |
| semisimple| $\text{SU}(2) \times \text{SU}(2)$ | $\text{SU}(2) \times \text{SL}_2(\mathbb{R})$ |
| non-semisimple | $\text{SU}(2) \times \text{SO}(2)$ | $\text{SL}_2(\mathbb{R}) \times \text{SO}(2)$ |
The motivation for this categorization is the following conjecture for Riemannian Lie groups by Gudmundsson.

**Conjecture 1.37.** Let $K$ be a subgroup of the Riemannian Lie group $G$ generating a left-invariant conformal foliation $\mathcal{F}$ of $G$ of codimension two. If $K$ is semisimple the foliation $\mathcal{F}$ is minimal. If $K$ is also compact then $\mathcal{F}$ is totally geodesic.

We will see that for a general semi-Riemannian Lie group the second statement does not necessarily hold. In the cases considered it does however hold up when the basis elements of $K$ are all of the same causal character, i.e. when $K$ is Riemannian.
Chapter 2

Lie Groups $G^5$ Foliated by their Subgroup SU(2)

Let $(G, g)$ be a five-dimensional semi-Riemannian Lie group containing the three-dimensional simple and compact Lie group SU(2) as a subgroup. Let $g = su(2) \oplus m$ be an orthogonal decomposition of the Lie algebra of $G$ and let $\{A, B, C, X, Y\}$ be an orthonormal basis consisting of left-invariant vector fields such that $A, B$ and $C$ generate the subalgebra $su(2)$. Then the Lie bracket relations for $su(2)$ are given by

$[A, B] = 2C, \quad [C, A] = 2B, \quad [B, C] = 2A.$

Let $\mathcal{F}$ be a left-invariant conformal foliation induced by SU(2). Using that SU(2) is simple, the conformality of $\mathcal{F}$ and Proposition 1.35 we see that $\mathcal{H}[V, H] = 0$ for all $V \in \mathcal{V}, H \in \mathcal{H}$. The remaining bracket relations for $g$ are thus given by

$[A, X] = a_{11}A + a_{12}B + a_{13}C, \quad [A, Y] = a_{21}A + a_{22}B + a_{23}C,$
$[B, X] = b_{11}A + b_{12}B + b_{13}C, \quad [B, Y] = b_{21}A + b_{22}B + b_{23}C,$
$[C, X] = c_{11}A + c_{12}B + c_{13}C, \quad [C, Y] = c_{21}A + c_{22}B + c_{23}C,$
$[X, Y] = \rho X + \theta_1 A + \theta_2 B + \theta_3 C,$

for some, not necessarily independent, constant coefficients. By using the Jacobi identity we can determine some of the coefficients and simplify this system.

**Proposition 2.1.** Let $(G, g)$ be a five-dimensional semi-Riemannian Lie group containing the subgroup SU(2). Let $g = su(2) \oplus m$ be an orthogonal decomposition of the Lie algebra of $G$ and let $\{A, B, C, X, Y\}$ be an orthonormal basis for $g$ such that $A, B$ and $C$ generate the subalgebra $su(2)$. If the foliation $\mathcal{F}$, tangent to $su(2)$, is conformal then the Lie bracket relations for $g$ are given by

$[A, B] = 2C, \quad [C, A] = 2B, \quad [B, C] = 2A$
$[A, X] = -b_{11}B - c_{11}C, \quad [A, Y] = -b_{21}B - c_{21}C,$
$[B, X] = b_{11}A - c_{12}C, \quad [B, Y] = b_{21}A - c_{22}C,$
$[C, X] = c_{11}A + c_{12}B, \quad [C, Y] = c_{21}A + c_{22}B,$
$[X, Y] = \rho X + \theta_1 A + \theta_2 B + \theta_3 C,$
where \( b_{11}, b_{21}, c_{11}, c_{12}, c_{21}, c_{22}, \rho \) are real, arbitrary coefficients and 

\[
\begin{pmatrix}
\theta_1 \\
\theta_2 \\
\theta_3
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
-\rho c_{12} + b_{11} c_{21} - b_{21} c_{11} \\
\rho c_{11} + b_{11} c_{22} - b_{21} c_{12} \\
-\rho b_{11} + c_{11} c_{22} - c_{21} c_{12}
\end{pmatrix}.
\]

**Proof.** Starting by considering the vector fields \( A, B, C \) and \( X \), we yield the following

\[
\begin{align*}
[[A, B], X] + [[X, A], B] + [[B, X], A] &= 2((a_{13} + c_{11}) A + (b_{13} + c_{12}) B - (a_{11} + b_{12} - c_{13}) C), \\
[[A, C], X] + [[X, A], C] + [[C, X], A] &= 2(-(a_{12} + b_{11}) A + (a_{11} - b_{12} + c_{13}) B - (b_{13} + c_{12}) C), \\
[[B, C], X] + [[X, B], C] + [[C, X], B] &= 2((a_{11} - b_{12} - c_{13}) A + (a_{12} + b_{11}) B + (a_{13} + c_{11}) C).
\end{align*}
\]

Since these all need to be zero in order for \( \mathfrak{g} \) to be a Lie algebra we obtain the following system of equations

\[
\begin{pmatrix}
a_{11} + b_{12} - c_{13} \\
a_{11} - b_{12} + c_{13} \\
a_{11} - b_{12} - c_{13} \\
a_{12} + b_{11} \\
a_{13} + c_{11} \\
b_{13} + c_{12}
\end{pmatrix} = 0,
\]

the solution to this system is given by

\[
\begin{pmatrix}
a_{12} \\
a_{13} \\
b_{13}
\end{pmatrix} = -\begin{pmatrix}
a_{11} \\
c_{11} \\
c_{12}
\end{pmatrix}, \quad \begin{pmatrix}
a_{11} \\
b_{12} \\
c_{13}
\end{pmatrix} = 0.
\]

Since the Lie bracket relations are symmetric in \( X \) and \( Y \) with respect to \( A, B \) and \( C \) we get an equivalent system of equations when we replace \( X \) by \( Y \) in the calculations above,

\[
\begin{pmatrix}
a_{21} + b_{22} - c_{23} \\
a_{21} - b_{22} + c_{23} \\
a_{21} - b_{22} - c_{23} \\
a_{22} + b_{21} \\
a_{23} + c_{21} \\
b_{23} + c_{22}
\end{pmatrix} = 0.
\]

Solving this system gives us

\[
\begin{pmatrix}
a_{22} \\
a_{23} \\
b_{23}
\end{pmatrix} = -\begin{pmatrix}
b_{21} \\
c_{21} \\
c_{22}
\end{pmatrix}, \quad \begin{pmatrix}
a_{21} \\
b_{22} \\
c_{23}
\end{pmatrix} = 0.
\]

We now look at the Jacobi identities involving both \( X \) and \( Y \) while keeping in mind the knowledge gained so far, doing so we obtain

\[
[[A, X], Y] + [[Y, A], X] + [[X, Y], A] = (\rho b_{11} - c_{11} c_{22} + c_{21} c_{12} + 2\theta_3) B
\]

\[12\]
Solving for \( \theta \) to get the simplified relations given above.

This produces the following system of equations

\[
\begin{align*}
& (\rho c_{12} - b_{11}c_{21} + b_{21}c_{11} + 2\theta_1) \\
& (\rho c_{11} + b_{11}c_{22} - b_{21}c_{12} - 2\theta_2) \\
& (\rho b_{11} - c_{11}c_{22} + c_{21}c_{12} + 2\theta_3) = 0.
\end{align*}
\]

Solving for \( (\theta_1, \theta_2, \theta_3) \) we yield

\[
\begin{pmatrix}
\theta_1 \\
\theta_2 \\
\theta_3
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
-\rho c_{12} + b_{11}c_{21} - b_{21}c_{11} \\
\rho c_{11} + b_{11}c_{22} - b_{21}c_{12} \\
-\rho b_{11} + c_{11}c_{22} - c_{21}c_{12}
\end{pmatrix}.
\]

Following these calculations we can thus remove or equate some of the constants to get the simplified relations given above. \( \square \)

Consider now the conformal foliation \( \mathcal{F} \) tangent to the vertical distribution \( \mathcal{V} \) generated by \( \mathfrak{su}(2) \). We are interested in finding out under what conditions \( \mathcal{F} \) is minimal and even totally geodesic.

**Theorem 2.2.** Let \((G, g)\) be a five-dimensional semi-Riemannian Lie group containing \( \mathfrak{su}(2) \) as a subgroup and let \( \mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{m} \) be an orthogonal decomposition of the Lie algebra of \( G \). Furthermore, let \( \{A, B, C, X, Y\} \) be an orthonormal basis for \( \mathfrak{g} \) such that the vector fields \( A, B, C \) generate the Lie subalgebra \( \mathfrak{su}(2) \). If the foliation \( \mathcal{F} \), tangent to the vertical distribution \( \mathcal{V} \) generated by \( \mathfrak{su}(2) \), is conformal then it is also minimal. Moreover, \( \mathcal{F} \) is totally geodesic if and only if one of the following conditions holds

1. \( \varepsilon_A = \varepsilon_B = \varepsilon_C \),
2. \( c_{11} = c_{12} = c_{21} = c_{22} = 0, \quad \varepsilon_A = \varepsilon_B \),
3. \( b_{11} = b_{21} = c_{12} = c_{22} = 0, \quad \varepsilon_A = \varepsilon_C \),
4. \( b_{11} = b_{21} = c_{11} = c_{21} = 0, \quad \varepsilon_B = \varepsilon_C \),
5. \( b_{11} = b_{21} = c_{11} = c_{12} = c_{21} = c_{22} = 0 \).

Before proving this we would like to note that for a Riemannian metric \( g \) the first condition would automatically be satisfied and thus the foliation \( \mathcal{F} \) is always totally geodesic when the Lie group \((G, g)\) is Riemannian.

**Proof.** By definition \( \mathcal{F} \) is minimal whenever \( \text{trace } B^\mathcal{V} = 0 \), we therefore compute the values along the diagonal of the matrix representation of \( B^\mathcal{V} \) to obtain

\[
B^\mathcal{V}(A, A) = \frac{1}{2} (\varepsilon_X \cdot (g([X, A], A) + g([X, A], A)) X + \varepsilon_Y \cdot (g([Y, A], A) + g([Y, A], A)) Y)
\]

13
Remark 2.3. From the result above we see that the values of \( \epsilon \) play a role in determining whether or not \( g \) is minimal or even totally geodesic. Consequently we see that \( F \) is minimal by default, we do not need to put any restrictions on the metric \( g \) or the structure coefficients of the Lie algebra.

We now evaluate \( B^V(Z,W) \) for \( Z,W \in \{ A,B,C \} \), \( Z \neq W \) to find when \( F \) is totally geodesic,

\[
B^V(A,B) = \frac{1}{2}(\epsilon_X \cdot (g([X,A], B) + g([X,B], A))X + \epsilon_Y \cdot (g([Y,A], B) + g([Y,B], A))Y)
\]

\[
= \frac{1}{2}(\epsilon_X \cdot (g(b_{11}B + c_{11}C, B) + g(-b_{11}A + c_{12}C, A))X + \epsilon_Y \cdot (g(b_{21}B + c_{21}C, B) + g(-b_{21}A + c_{22}C, A))Y)
\]

\[
= \frac{1}{2}(\epsilon_X \cdot (b_{11}\epsilon_B - b_{11}\epsilon_A)X + \epsilon_Y \cdot (b_{21}\epsilon_B - b_{21}\epsilon_A)Y),
\]

\[\text{(2.1)}\]

\[
B^V(A,C) = \frac{1}{2}(\epsilon_X \cdot (g([X,A], C) + g([X,C], A))X + \epsilon_Y \cdot (g([Y,A], C) + g([Y,C], A))Y)
\]

\[
= \frac{1}{2}(\epsilon_X \cdot (g(b_{11}B + c_{11}C, C) + g(-c_{11}A - c_{12}B, A))X + \epsilon_Y \cdot (g(b_{21}B + c_{21}C, C) + g(-c_{21}A - c_{22}B, A))Y)
\]

\[
= \frac{1}{2}(\epsilon_X \cdot (c_{11}\epsilon_C - c_{11}\epsilon_A)X + \epsilon_Y \cdot (c_{21}\epsilon_C - c_{21}\epsilon_A)Y),
\]

\[\text{(2.2)}\]

\[
B^V(B,C) = \frac{1}{2}(\epsilon_X \cdot (g([X,B], C) + g([X,C], B))X + \epsilon_Y \cdot (g([Y,B], C) + g([Y,C], B))Y)
\]

\[
= \frac{1}{2}(\epsilon_X \cdot (g(-b_{11}A + c_{12}C, C) + g(-c_{11}A - c_{12}B, B))X + \epsilon_Y \cdot (g(-b_{21}A + c_{22}C, C) + g(-c_{21}A - c_{22}B, B))Y)
\]

\[
= \frac{1}{2}(\epsilon_X \cdot (c_{12}\epsilon_C - c_{12}\epsilon_B)X + \epsilon_Y \cdot (c_{22}\epsilon_C - c_{22}\epsilon_B)Y).
\]

\[\text{(2.3)}\]

From (2.1) we see that either \( b_{11} = b_{21} = 0 \) or \( \epsilon_A = \epsilon_B \) has to hold. Similarly, from (2.2) we get that either \( c_{11} = c_{21} = 0 \) or \( \epsilon_A = \epsilon_C \) and from (2.3) we obtain that one of \( c_{12} = c_{22} = 0 \) and \( \epsilon_B = \epsilon_C \) necessarily holds. Combining these different options we get exactly the conditions of the statement. 

\[\square\]

**Remark 2.3.** From the result above we see that the values of \( \epsilon_X \) and \( \epsilon_Y \) do not play a role in determining whether or not \( F \) is minimal or even totally geodesic. Furthermore, we observe the following:

1. If condition 1 holds we have a seven-dimensional family of Lie groups where the variables \( \{b_{11}, b_{21}, c_{11}, c_{12}, c_{21}, c_{22}, \rho\} \) are free.
2. From conditions 2, 3 and 4 we obtain three-dimensional families of Lie groups. Here the free variables are \( \{b_{11}, b_{21}, \rho\} \), \( \{c_{11}, c_{21}, \rho\} \) and \( \{c_{12}, c_{22}, \rho\} \), respectively.

3. In the last case, when condition 5 holds, we obtain a one-dimensional family of Lie groups where \( \rho \) is the only free variable. For each \( \rho \) the Lie group \( G \) is the direct product of \( SU(2) \) and a complete surface of constant curvature diffeomorphic to the plane.
Chapter 3

Lie Groups $G^5$ Foliated by their Subgroup $\text{SL}_2(\mathbb{R})$

We now consider a five-dimensional semi-Riemannian Lie group $(G, g)$ containing the non-compact simple subgroup $\text{SL}_2(\mathbb{R})$. Let $\mathfrak{g}$ be the Lie algebra of $G$ and let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{m}$ be an orthogonal decomposition of this algebra. Let $\{A, B, C, X, Y\}$ be an orthonormal basis of left-invariant vector fields of $\mathfrak{g}$ such that $A, B, C$ generate the algebra of $\mathfrak{sl}_2(\mathbb{R})$. If the foliation $\mathcal{F}$, tangent to $\mathfrak{sl}_2(\mathbb{R})$, is conformal then, by Proposition 1.35, $[\mathcal{V}, H]$ has no horizontal part and thus the structure equations for $\mathfrak{g}$ are of the form

\[
\begin{align*}
[A, B] &= 2C, \quad [C, A] = 2B, \quad [B, C] = -2A, \\
[A, X] &= a_{11}A + a_{12}B + a_{13}C, \quad [A, Y] = a_{21}A + a_{22}B + a_{23}C, \\
[B, X] &= b_{11}A + b_{12}B + b_{13}C, \quad [B, Y] = b_{21}A + b_{22}B + b_{23}C, \\
[C, X] &= c_{11}A + c_{12}B + c_{13}C, \quad [C, Y] = c_{21}A + c_{22}B + c_{23}C, \\
[X, Y] &= \rho X + \theta_1 A + \theta_2 B + \theta_3 C.
\end{align*}
\]

These relations can be simplified by going through the same procedure as was done in Chapter 2.

Proposition 3.1. Let $(G, g)$ be a five-dimensional semi-Riemannian Lie group containing the subgroup $\text{SL}_2(\mathbb{R})$. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{m}$ be an orthogonal decomposition of the Lie algebra of $G$ and let $\{A, B, C, X, Y\}$ be an orthonormal basis such that $A, B, C$ generate the Lie subalgebra $\mathfrak{sl}_2(\mathbb{R})$. If the foliation $\mathcal{F}$, tangent to the vertical distribution $\mathcal{V}$ induced by $\mathfrak{sl}_2(\mathbb{R})$, is conformal then the Lie bracket relations of $\mathfrak{g}$ are given by

\[
\begin{align*}
[A, B] &= 2C, \quad [C, A] = 2B, \quad [B, C] = -2A, \\
[A, X] &= b_{11}A + c_{12}B + c_{13}C, \quad [A, Y] = b_{21}A + c_{22}B + c_{23}C, \\
[B, X] &= b_{11}A - c_{12}C, \quad [B, Y] = b_{21}A - c_{22}C, \\
[C, X] &= c_{11}A + c_{12}B, \quad [C, Y] = c_{21}A + c_{22}B, \\
[X, Y] &= \rho X + \theta_1 A + \theta_2 B + \theta_3 C,
\end{align*}
\]

where $b_{11}, b_{21}, c_{11}, c_{12}, c_{21}, c_{22}, \rho$ are real, arbitrary coefficients and

\[
\begin{pmatrix}
\theta_1 \\
\theta_2 \\
\theta_3
\end{pmatrix} = \frac{1}{2}
\begin{pmatrix}
-\rho c_{12} - c_{21}b_{11} + c_{11}b_{21} \\
-\rho c_{11} - c_{22}b_{11} + c_{12}b_{21} \\
\rho b_{11} - c_{22}c_{11} + c_{12}c_{21}
\end{pmatrix}.
\]
We are again interested to find what restrictions we need to put on the coefficients in the Lie algebra $\mathfrak{g}$ of $G$ in order for the conformal foliation $\mathcal{F}$, tangent to the vertical distribution $\mathcal{V}$, to be minimal and totally geodesic.

**Theorem 3.2.** Let $(G, g)$ be a five-dimensional semi-Riemannian Lie group containing $\text{SL}_2(\mathbb{R})$ as a subgroup and let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{m}$ be an orthogonal decomposition of the Lie algebra of $G$. Furthermore, let $\{A, B, C, X, Y\}$ be an orthonormal basis for $\mathfrak{g}$ such that $A, B, C$ generate the Lie subalgebra $\mathfrak{sl}_2(\mathbb{R})$. If the foliation $\mathcal{F}$, tangent to the vertical distribution $\mathcal{V}$ generated by $\mathfrak{sl}_2(\mathbb{R})$, is conformal then it is also minimal. Moreover, $\mathcal{F}$ is totally geodesic if and only if one of the following hold

1. $-\varepsilon_A = \varepsilon_B = \varepsilon_C$,
2. $c_{11} = c_{12} = c_{21} = c_{22} = 0$, $\varepsilon_A = -\varepsilon_B$,
3. $b_{11} = b_{21} = c_{12} = c_{22} = 0$, $\varepsilon_A = -\varepsilon_C$,
4. $b_{11} = b_{21} = c_{11} = c_{21} = 0$, $\varepsilon_B = \varepsilon_C$,
5. $b_{11} = b_{21} = c_{11} = c_{12} = c_{21} = c_{22} = 0$.

**Proof.** We start by proving that $\mathcal{F}$ is always minimal. If we look at

$$B^\mathcal{V}(A, A) = \varepsilon_X \cdot g([X, A], A)X + \varepsilon_Y \cdot g([Y, A], A)Y$$

we can easily see that this must be zero since $[X, A]$ has no $A$ component and the same is also true of $[Y, A]$. Similar reasoning shows that both $B^\mathcal{V}(B, B)$ and $B^\mathcal{V}(C, C)$ are zero. Thus $\mathcal{F}$ is minimal independent of the choice of $g$.

Next we check the claim that $\mathcal{F}$ is totally geodesic if and only if it satisfies one of the above conditions, we do this by computing the values $B^\mathcal{V}(A, B)$, $B^\mathcal{V}(A, C)$ and $B^\mathcal{V}(B, C)$.

$$B^\mathcal{V}(A, B) = -\varepsilon_X \cdot (b_{11} \varepsilon_B + b_{11} \varepsilon_A)X - \varepsilon_Y \cdot (b_{21} \varepsilon_B + b_{21} \varepsilon_A)Y,$$
$$B^\mathcal{V}(A, C) = -\varepsilon_X \cdot (c_{11} \varepsilon_C + c_{11} \varepsilon_A)X - \varepsilon_Y \cdot (c_{21} \varepsilon_C + c_{21} \varepsilon_A)Y,$$
$$B^\mathcal{V}(B, C) = \varepsilon_X \cdot (c_{12} \varepsilon_C - c_{12} \varepsilon_B)X + \varepsilon_Y \cdot (c_{22} \varepsilon_C + c_{22} \varepsilon_B)Y,$$

these are clearly all zero exactly when we satisfy one of the stated criteria. \(\square\)

In view of this result we can make the same observations as we did in Remark 2.3, the only change is that in the last case we replace $\text{SU}(2)$ by $\text{SL}_2(\mathbb{R})$ in the direct product.
Chapter 4

Lie Groups $G^8$ Foliated by their Subgroup $\text{SU}(2) \times \text{SU}(2)$

Let $(G,g)$ be an eight-dimensional semi-Riemannian Lie group containing $K = \text{SU}(2) \times \text{SU}(2)$ as a subgroup and let $\mathcal{F}$ be a conformal left-invariant foliation tangent to the vertical distribution $\mathcal{V}$ induced by $K$. Then the Lie algebra, $\mathfrak{g}$, of $G$ has an orthogonal splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Furthermore, let $\{A, B, C, R, S, T, X, Y\}$ be an orthonormal basis for $\mathfrak{g}$ such that the Lie subalgebra $\mathfrak{k} = \mathfrak{su}(2) \times \mathfrak{su}(2)$ is generated by the vector fields $A, B, C, R, S, T$. The Lie bracket relations for this subalgebra are given by

\[
[A, B] = 2C, \quad [C, A] = 2B, \quad [B, C] = 2A, \\
[R, S] = 2T, \quad [T, R] = 2S, \quad [S, T] = 2R, 
\]

i.e. the vector fields in the two sets $\{A, B, C\}$ and $\{R, S, T\}$ both generate a copy of the standard Lie algebra $\mathfrak{su}(2)$ of $\text{SU}(2)$. To avoid confusion we will, when necessary, denote these copies by $\mathfrak{su}_1$ and $\mathfrak{su}_2$, respectively. Then the remaining relations for $\mathfrak{g}$ are given by

\[
[A, X] = a_{11}A + a_{12}B + a_{13}C + a_{14}R + a_{15}S + a_{16}T, \\
[A, Y] = a_{21}A + a_{22}B + a_{23}C + a_{24}R + a_{25}S + a_{26}T, \\
[B, X] = b_{11}A + b_{13}B + b_{13}C + b_{14}R + b_{15}S + b_{16}T, \\
[B, Y] = b_{21}A + b_{23}B + b_{23}C + b_{24}R + b_{25}S + b_{26}T, \\
[C, X] = c_{11}A + c_{12}B + c_{13}C + c_{14}R + c_{15}S + c_{16}T, \\
[C, Y] = c_{21}A + c_{22}B + c_{23}C + c_{24}R + c_{25}S + c_{26}T, \\
[R, X] = r_{11}A + r_{13}B + r_{13}C + r_{14}R + r_{15}S + r_{16}T, \\
[R, Y] = r_{21}A + r_{23}B + r_{23}C + r_{24}R + r_{25}S + r_{26}T, \\
[S, X] = s_{11}A + s_{12}B + s_{13}C + s_{14}R + s_{15}S + s_{16}T, \\
[S, Y] = s_{21}A + s_{22}B + s_{23}C + s_{24}R + s_{25}S + s_{26}T, \\
[T, X] = t_{11}A + t_{13}B + t_{13}C + t_{14}R + t_{15}S + t_{16}T, \\
[T, Y] = t_{21}A + t_{23}B + t_{23}C + t_{24}R + t_{25}S + t_{26}T, \\
[X, Y] = \rho X + \theta_1 A + \theta_2 B + \theta_3 C + \theta_4 R + \theta_5 S + \theta_6 T.
\]

These can in turn be specified further by looking at the Jacobi identities for the involved vector fields.
Let \( (G, g) \) be an eight-dimensional semi-Riemannian Lie group containing \( K = SU(2) \times SU(2) \) as a subgroup. Let \( g = \mathfrak{k} \oplus \mathfrak{m} \) be an orthogonal decomposition of the Lie algebra \( g \) of \( G \). Furthermore, let \( \{ A, B, C, R, S, T, X, Y \} \) be an orthonormal basis for \( g \) such that the Lie subalgebra \( \mathfrak{k} = su(2) \times su(2) \) is generated by the vector fields \( A, B, C, R, S, T \). If the foliation \( \mathcal{F} \), tangent to the vertical distribution \( \mathcal{V} \) induced by \( \mathfrak{k} \), is conformal then the Lie bracket relations for \( g \) can be written as

\[
\begin{align*}
[A, B] &= 2C, \quad [C, A] = 2B, \quad [B, C] = 2A, \\
[R, S] &= 2T, \quad [T, R] = 2S, \quad [S, T] = 2R, \\
[A, X] &= -b_{11}B - c_{11}C, \quad [A, Y] = -b_{21}B - c_{21}C, \\
[B, X] &= b_{11}A - c_{11}C, \quad [B, Y] = b_{21}A - c_{21}B, \\
[C, X] &= c_{11}A + c_{11}B, \quad [C, Y] = c_{21}A + c_{21}B, \\
[R, X] &= -s_{14}S - t_{14}T, \quad [R, Y] = -s_{24}S - t_{24}T, \\
[S, X] &= s_{14}R - t_{14}T, \quad [S, Y] = s_{24}R - t_{24}T, \\
[T, X] &= t_{14}R + t_{15}S, \quad [T, Y] = t_{24}R + t_{25}S, \\
[X, Y] &= \rho X + \theta_1A + \theta_2B + \theta_3C + \theta_4R + \theta_5S + \theta_6T
\end{align*}
\]

where \( b_{11}, b_{21}, c_{11}, c_{12}, c_{21}, c_{22}, s_{14}, s_{24}, t_{14}, t_{15}, t_{24}, t_{25}, \rho \) are real, arbitrary coefficients and

\[
\begin{pmatrix}
\theta_1 \\
\theta_2 \\
\theta_3 \\
\theta_4 \\
\theta_5 \\
\theta_6
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
-\rho c_{12} + b_{11}c_{21} - b_{21}c_{11} \\
\rho c_{11} + b_{11}c_{22} - b_{21}c_{12} \\
-\rho b_{11} + c_{11}c_{22} - c_{12}c_{21} \\
-\rho t_{15} + s_{14}t_{24} - s_{24}t_{14} \\
\rho t_{14} + s_{14}t_{25} - s_{24}t_{15} \\
-\rho s_{14} + t_{14}t_{25} - t_{15}t_{24}
\end{pmatrix}.
\]

For the full details of proving this we refer the reader to [15].

**Theorem 4.2.** Let \( (G, g) \) be an eight-dimensional semi-Riemannian Lie group containing \( K = SU(2) \times SU(2) \) as a subgroup and let \( g = \mathfrak{k} \oplus \mathfrak{m} \) be an orthogonal decomposition of the Lie algebra of \( G \) such that \( \mathfrak{k} = su(2) \times su(2) \). Furthermore, let \( \{ A, B, C, R, S, T, X, Y \} \) be an orthonormal basis for \( g \) such that the vector fields \( A, B, C, R, S, T \) generate the Lie subalgebra \( \mathfrak{k} \). If the foliation \( \mathcal{F} \), tangent to the vertical distribution \( \mathcal{V} \) generated by \( \mathfrak{k} \), is conformal then it is also minimal. Moreover, \( \mathcal{F} \) is totally geodesic if and only if one of the following conditions is satisfied

1. \( \varepsilon_A = \varepsilon_B = \varepsilon_C, \quad \varepsilon_R = \varepsilon_S = \varepsilon_T, \)
2. \( t_{14} = t_{15} = t_{24} = t_{25} = 0, \quad \varepsilon_A = \varepsilon_B = \varepsilon_C, \quad \varepsilon_R = \varepsilon_S = \varepsilon_T, \)
3. \( s_{14} = s_{24} = t_{15} = t_{25} = 0, \quad \varepsilon_A = \varepsilon_B = \varepsilon_C, \quad \varepsilon_R = \varepsilon_T, \)
4. \( s_{14} = s_{24} = t_{14} = t_{24} = 0, \quad \varepsilon_A = \varepsilon_B = \varepsilon_C, \quad \varepsilon_S = \varepsilon_T, \)
5. \( c_{11} = c_{12} = c_{21} = c_{22} = 0, \quad \varepsilon_A = \varepsilon_B, \quad \varepsilon_R = \varepsilon_S = \varepsilon_T, \)
6. \( b_{11} = b_{21} = c_{12} = c_{22} = 0, \)
\[ \varepsilon_A = \varepsilon_C, \quad \varepsilon_R = \varepsilon_S = \varepsilon_T, \]
7. \( b_{11} = b_{21} = c_{11} = c_{21} = 0, \)
\[ \varepsilon_B = \varepsilon_C, \quad \varepsilon_R = \varepsilon_S = \varepsilon_T, \]
8. \( s_{14} = s_{24} = t_{14} = t_{15} = t_{24} = t_{25} = 0, \)
\[ \varepsilon_A = \varepsilon_B = \varepsilon_C, \]
9. \( b_{11} = b_{21} = c_{11} = c_{12} = c_{21} = c_{22} = 0, \)
\[ \varepsilon_R = \varepsilon_S = \varepsilon_T, \]
10. \( c_{11} = c_{12} = c_{21} = c_{22} = t_{14} = t_{15} = t_{24} = t_{25} = 0, \)
\[ \varepsilon_A = \varepsilon_B = \varepsilon_R = \varepsilon_S, \]
11. \( c_{11} = c_{12} = c_{21} = c_{22} = s_{14} = s_{24} = t_{15} = t_{25} = 0, \)
\[ \varepsilon_A = \varepsilon_B = \varepsilon_R = \varepsilon_S, \]
12. \( c_{11} = c_{12} = c_{21} = c_{22} = s_{14} = s_{24} = t_{14} = t_{24} = 0, \)
\[ \varepsilon_A = \varepsilon_B = \varepsilon_S = \varepsilon_T, \]
13. \( b_{11} = b_{21} = c_{12} = c_{22} = t_{14} = t_{15} = t_{24} = t_{25} = 0, \)
\[ \varepsilon_A = \varepsilon_C = \varepsilon_R = \varepsilon_S, \]
14. \( b_{11} = b_{21} = c_{12} = c_{22} = s_{14} = s_{24} = t_{15} = t_{25} = 0, \)
\[ \varepsilon_A = \varepsilon_C = \varepsilon_R = \varepsilon_T, \]
15. \( b_{11} = b_{21} = c_{12} = c_{21} = s_{14} = s_{24} = t_{14} = t_{24} = 0, \)
\[ \varepsilon_A = \varepsilon_C = \varepsilon_S = \varepsilon_T, \]
16. \( b_{11} = b_{21} = c_{11} = c_{21} = t_{14} = t_{15} = t_{24} = t_{25} = 0, \)
\[ \varepsilon_B = \varepsilon_C = \varepsilon_R = \varepsilon_S, \]
17. \( b_{11} = b_{21} = c_{11} = c_{21} = s_{14} = s_{24} = t_{15} = t_{25} = 0, \)
\[ \varepsilon_B = \varepsilon_C = \varepsilon_R = \varepsilon_S, \]
18. \( b_{11} = b_{21} = c_{11} = c_{21} = s_{14} = s_{24} = t_{14} = t_{24} = 0, \)
\[ \varepsilon_B = \varepsilon_C = \varepsilon_S = \varepsilon_T, \]
19. \( c_{11} = c_{12} = c_{22} = c_{14} = s_{24} = t_{14} = t_{15} = t_{24} = t_{25} = 0, \)
\[ \varepsilon_A = \varepsilon_B, \]
20. \( b_{11} = b_{21} = c_{12} = c_{22} = s_{14} = s_{24} = t_{14} = t_{15} = t_{24} = t_{25} = 0, \)
\[ \varepsilon_A = \varepsilon_C, \]
21. \( b_{11} = b_{21} = c_{11} = c_{21} = s_{14} = s_{24} = t_{14} = t_{15} = t_{24} = t_{25} = 0, \)
\[ \varepsilon_B = \varepsilon_C, \]
22. \( b_{11} = b_{21} = c_{11} = c_{12} = c_{21} = c_{22} = t_{14} = t_{15} = t_{24} = t_{25} = 0, \)
\[ \varepsilon_R = \varepsilon_S, \]
23. \( b_{11} = b_{21} = c_{11} = c_{12} = c_{21} = c_{22} = s_{14} = s_{24} = t_{14} = t_{25} = 0, \)
\[ \varepsilon_R = \varepsilon_T, \]
24. \( b_{11} = b_{21} = c_{11} = c_{12} = c_{21} = c_{22} = s_{14} = s_{24} = t_{14} = t_{24} = 0, \)
\[ \varepsilon_S = \varepsilon_T, \]
25. \( b_{11} = b_{21} = c_{11} = c_{12} = c_{21} = c_{22} = s_{14} = s_{24} = t_{14} = t_{15} = t_{24} = t_{25} = 0 \).

**Proof.** Beginning with minimality, we compute

\[
B^V(A, A) = \frac{1}{2} (\varepsilon_X \cdot (g([X, A], A) + g([X, A], A))X \\
+ \varepsilon_Y \cdot (g([Y, A], A) + g([Y, A], A))Y) \\
= \varepsilon_X \cdot g([X, A], A)X + \varepsilon_Y \cdot g([Y, A], A)Y = 0,
\]

and similarly,

\[
B^V(B, B) = \varepsilon_X \cdot g([X, B], B)X + \varepsilon_Y \cdot g([Y, B], B)Y = 0, \\
B^V(C, C) = \varepsilon_X \cdot g([X, C], C)X + \varepsilon_Y \cdot g([Y, C], A)Y = 0.
\]

Using that the Lie group associated with \( \mathfrak{su}_1 \) has the same group structure as the Lie group associated with \( \mathfrak{su}_2 \) we have analogous calculations for \( B^V(R, R), B^V(S, S) \) and \( B^V(T, T) \). From this we see that the minimality of \( \mathcal{F} \) is always guaranteed.

We now find under what assumptions \( \mathcal{F} \) is totally geodesic. By computation we obtain

\[
B^V(A, B) = \frac{1}{2} (\varepsilon_X \cdot (g([X, A], B) + g([X, B], A))X \\
+ \varepsilon_Y \cdot (g([Y, A], B) + g([Y, B], A))Y) \\
= \frac{1}{2} (\varepsilon_X \cdot (g(b_{11}B + c_{11}C, B) + g(-b_{11}A + c_{12}C, A))X \\
+ \varepsilon_Y \cdot (g(b_{21}B + c_{21}C, B) + g(-b_{21}A + c_{22}C, A))Y) \\
= \frac{1}{2} (\varepsilon_X \cdot (b_{11}\varepsilon_B - b_{11}\varepsilon_A)X + \varepsilon_Y \cdot (b_{21}\varepsilon_B - b_{21}\varepsilon_A)Y),
\]

\[
B^V(A, C) = \frac{1}{2} (\varepsilon_X \cdot (g([X, A], C) + g([X, C], A))X \\
+ \varepsilon_Y \cdot (g([Y, A], C) + g([Y, C], A))Y) \\
= \frac{1}{2} (\varepsilon_X \cdot (g(b_{11}B + c_{11}C, C) + g(-c_{11}A - c_{12}B, A))X \\
+ \varepsilon_Y \cdot (g(b_{21}B + c_{21}C, C) + g(-c_{21}A - c_{22}B, A))Y) \\
= \frac{1}{2} (\varepsilon_X \cdot (c_{11}\varepsilon_C - c_{11}\varepsilon_A)X + \varepsilon_Y \cdot (c_{21}\varepsilon_C - c_{21}\varepsilon_A)Y),
\]

\[
B^V(B, C) = \frac{1}{2} (\varepsilon_X \cdot (g([X, B], C) + g([X, C], B))X \\
+ \varepsilon_Y \cdot (g([Y, B], C) + g([Y, C], B))Y) \\
= \frac{1}{2} (\varepsilon_X \cdot (g(-b_{11}A + c_{12}C, C) + g(-c_{11}A - c_{12}B, B))X \\
+ \varepsilon_Y \cdot (g(-b_{21}A + c_{22}C, C) + g(-c_{21}A - c_{22}B, B))Y) \\
= \frac{1}{2} (\varepsilon_X \cdot (c_{12}\varepsilon_C - c_{12}\varepsilon_B)X + \varepsilon_Y \cdot (c_{22}\varepsilon_C - c_{22}\varepsilon_B)Y),
\]

These calculations give the different conditions for the constants related to \( \mathfrak{su}_1 \), i.e. the elements \( b_{11}, b_{21}, c_{11}, c_{12}, c_{21}, c_{22}, \varepsilon_A, \varepsilon_B \) and \( \varepsilon_C \). Once again, we use that \( \mathfrak{su}_1 \) gives a group with the same group structure as that given by \( \mathfrak{su}_2 \) to see that we get equivalent requirements for \( s_{14}, s_{24}, t_{14}, t_{15}, t_{24}, t_{25}, \varepsilon_R, \varepsilon_S \) and \( \varepsilon_T \). Combining
these conditions, with the minimal amount necessary in each case, we obtain the conditions of the statement.

We note that computing $B^V(Z,W)$ where $Z \in \{A, B, C\}$ and $W \in \{R, S, T\}$, or vice versa, is unnecessary; they are clearly zero by the bracket relations.

**Remark 4.3.** The above result gives us the following different families of semi-Riemannian Lie groups:

1. In case \( t \) the variables \( \{b_{11}, b_{21}, c_{11}, c_{12}, c_{21}, c_{22}, s_{14}, s_{24}, t_{14}, t_{15}, t_{24}, t_{25}, \rho\} \) are free, i.e. we have a 13-dimensional family of Lie groups.

2. The conditions 2-7 give us nine-dimensional families of Lie groups. The free variables for these families are as follows:
   2. \( \{b_{11}, b_{21}, c_{11}, c_{12}, c_{21}, c_{22}, s_{14}, s_{24}, \rho\} \),
   3. \( \{b_{11}, b_{21}, c_{11}, c_{12}, c_{21}, c_{22}, t_{14}, t_{24}, \rho\} \),
   4. \( \{b_{11}, b_{21}, c_{11}, c_{12}, c_{21}, c_{22}, t_{15}, t_{25}, \rho\} \),
   5. \( \{b_{11}, b_{21}, s_{14}, s_{24}, t_{14}, t_{15}, t_{24}, t_{25}, \rho\} \),
   6. \( \{c_{11}, c_{21}, s_{14}, s_{24}, t_{14}, t_{15}, t_{24}, t_{25}, \rho\} \),
   7. \( \{c_{12}, c_{22}, s_{14}, s_{24}, t_{14}, t_{15}, t_{24}, t_{25}, \rho\} \).

3. When either 8 or 9 holds we obtain a seven-dimensional family of Lie groups. The free variable for these families are \( \{b_{11}, b_{21}, c_{11}, c_{12}, c_{21}, c_{22}, \rho\} \) and \( \{s_{14}, s_{24}, t_{14}, t_{15}, t_{24}, t_{25}, \rho\} \), respectively.

4. The conditions 10-18 all give five-dimensional families of Lie groups. For these families we have the following free variables:
   10. \( \{b_{11}, b_{21}, s_{14}, s_{24}, \rho\} \)
   11. \( \{b_{11}, b_{21}, t_{14}, t_{24}, \rho\} \)
   12. \( \{b_{11}, b_{21}, t_{15}, t_{25}, \rho\} \)
   13. \( \{c_{11}, c_{21}, s_{14}, s_{24}, \rho\} \)
   14. \( \{c_{11}, c_{21}, t_{14}, t_{24}, \rho\} \)
   15. \( \{c_{11}, c_{21}, t_{14}, t_{24}, \rho\} \)
   16. \( \{c_{12}, c_{22}, s_{14}, s_{24}, \rho\} \)
   17. \( \{c_{12}, c_{22}, t_{14}, t_{24}, \rho\} \)
   18. \( \{c_{12}, c_{22}, t_{15}, t_{25}, \rho\} \).

5. From conditions 19-24 we yield the three-dimensional families of Lie groups whose free variables are:
   19. \( \{b_{11}, b_{21}, \rho\} \)
   20. \( \{c_{11}, c_{21}, \rho\} \)
   21. \( \{c_{12}, c_{22}, \rho\} \)
   22. \( \{s_{14}, s_{24}, \rho\} \)
   23. \( \{t_{14}, t_{24}, \rho\} \)
   24. \( \{t_{15}, t_{25}, \rho\} \).

6. Condition 25 defines a one-dimensional family of Lie groups where the free variable is \( \rho \). For each \( \rho \) we have that \( G \) is the direct product of \( SU(2) \times SU(2) \) and a complete surface of constant curvature diffeomorphic to the plane.
Chapter 5

Lie Groups $G^8$ Foliated by their Subgroup $\text{SU}(2) \times \text{SL}_2(\mathbb{R})$

We will now investigate the case when $K$ is the semisimple, non-compact group $K = \text{SU}(2) \times \text{SL}_2(\mathbb{R})$.

Let $(G, g)$ be an eight-dimensional semi-Riemannian Lie group containing the subgroup $K$ which generates a left-invariant conformal foliation $\mathcal{F}$. Let $g = \mathfrak{k} \oplus \mathfrak{m}$ be an orthogonal decomposition of the Lie algebra of $G$ such that the vector fields \{A, B, C, R, S, T, X, Y\} constitute an orthonormal basis. Here $\mathfrak{k} = \text{su}(2) \times \text{sl}_2(\mathbb{R})$ and $\text{su}(2)$ is generated by the elements $A, B, C$ and $\text{sl}_2(\mathbb{R})$ has $R, S, T$ as a basis. Then $g$ is given by the relations

\[
[A, B] = 2C, \quad [C, A] = 2B, \quad [B, C] = 2A, \\
[R, S] = 2T, \quad [T, R] = 2S, \quad [S, T] = -2R, \\
[A, X] = a_{11}A + a_{12}B + a_{13}C + a_{14}R + a_{15}S + a_{16}T, \\
[A, Y] = a_{21}A + a_{22}B + a_{23}C + a_{24}R + a_{25}S + a_{26}T, \\
[B, X] = b_{11}A + b_{12}B + b_{13}C + b_{14}R + b_{15}S + b_{16}T, \\
[B, Y] = b_{21}A + b_{22}B + b_{23}C + b_{24}R + b_{25}S + b_{26}T, \\
[C, X] = c_{11}A + c_{12}B + c_{13}C + c_{14}R + c_{15}S + c_{16}T, \\
[C, Y] = c_{21}A + c_{22}B + c_{23}C + c_{24}R + c_{25}S + c_{26}T, \\
[R, X] = r_{11}A + r_{12}B + r_{13}C + r_{14}R + r_{15}S + r_{16}T, \\
[R, Y] = r_{21}A + r_{22}B + r_{23}C + r_{24}R + r_{25}S + r_{26}T, \\
[S, X] = s_{11}A + s_{12}B + s_{13}C + s_{14}R + s_{15}S + s_{16}T, \\
[S, Y] = s_{21}A + s_{22}B + s_{23}C + s_{24}R + s_{25}S + s_{26}T, \\
[T, X] = t_{11}A + t_{12}B + t_{13}C + t_{14}R + t_{15}S + t_{16}T, \\
[T, Y] = t_{21}A + t_{22}B + t_{23}C + t_{24}R + t_{25}S + t_{26}T, \\
[X, Y] = \rho X + \theta_1 A + \theta_2 B + \theta_3 C + \theta_4 R + \theta_5 S + \theta_6 T.
\]

Proposition 5.1. Let $(G, g)$ be an eight-dimensional semi-Riemannian Lie group containing the six-dimensional subgroup $K = \text{SU}(2) \times \text{SL}_2(\mathbb{R})$. Let $g = \mathfrak{k} \oplus \mathfrak{m}$ be an orthogonal decomposition of the Lie algebra of $G$. Further let \{A, B, C, R, S, T, X, Y\} be an orthonormal basis of $g$ such that the subalgebra $\mathfrak{k} = \text{su}(2) \times \text{sl}_2(\mathbb{R})$ is generated by the vector fields $A, B, C, R, S, T$. Then the Lie bracket relations are given by

\[
[A, B] = 2C, \quad [C, A] = 2B, \quad [B, C] = 2A, 
\]
\[[R,S] = 2T, \quad [T,R] = 2S, \quad [S,T] = -2R,\]
\[[A,X] = -b_{11}B - c_{11}C, \quad [A,Y] = -b_{21}B - c_{21}C,\]
\[[B,X] = b_{11}A - c_{11}C, \quad [B,Y] = b_{21}A - c_{22}C,\]
\[[C,X] = c_{11}A + c_{12}B, \quad [C,Y] = c_{21}A + c_{22}B,\]
\[[R,X] = s_{14}S + t_{14}T, \quad [R,Y] = s_{24}S + t_{24}T,\]
\[[S,X] = s_{14}R - t_{15}T, \quad [S,Y] = s_{24}R - t_{25}T,\]
\[[T,X] = t_{14}R + t_{15}S, \quad [T,Y] = t_{24}R + t_{25}S,\]
\[[X,Y] = \rho X + \theta_1 A + \theta_2 B + \theta_3 C + \theta_4 R + \theta_5 S + \theta_6 T\]

where \(b_{11}, b_{21}, c_{11}, c_{12}, c_{21}, c_{22}, s_{14}, s_{24}, t_{14}, t_{15}, t_{24}, t_{25}, \rho\) are real, arbitrary coefficients and

\[
\begin{pmatrix}
\theta_1 \\
\theta_2 \\
\theta_3 \\
\theta_4 \\
\theta_5 \\
\theta_6
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
-\rho c_{12} + b_{11}c_{21} - b_{21}c_{11} \\
\rho c_{11} + b_{11}c_{22} - b_{21}c_{12} \\
-\rho b_{11} + c_{11}c_{22} - c_{12}c_{21} \\
-\rho t_{15} - s_{14}t_{24} + s_{24}t_{14} \\
-\rho t_{14} - s_{14}t_{25} + s_{24}t_{15} \\
\rho s_{14} - t_{14}t_{25} + t_{15}t_{24}
\end{pmatrix}.
\]

**Proof.** The Lie bracket does not depend on the metric \(g\), thus the proof is exactly that of Proposition 3.1 in [15]. \(\square\)

**Theorem 5.2.** Let \((G, g)\) be an eight-dimensional semi-Riemannian Lie group containing \(K = SU(2) \times SL_2(\mathbb{R})\) as a subgroup and let \(\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}\) be the Lie algebra of \(G\) such that \(\mathfrak{k} = su(2) \times sl_1(\mathbb{R})\). Furthermore, let \(\{A, B, C, R, S, T, X, Y\}\) be an orthonormal basis for \(\mathfrak{g}\) such that the vector fields \(A, B, C, R, S, T\) generates the Lie subalgebra \(\mathfrak{k}\). If the foliation \(\mathcal{F}\), tangent to the vertical distribution \(\mathcal{V}\) generated by \(\mathfrak{k}\), is conformal then it is also minimal. Moreover, \(\mathcal{F}\) is totally geodesic if and only if one of the following conditions holds

1. \(\varepsilon_A = \varepsilon_B = \varepsilon_C, \quad -\varepsilon_R = \varepsilon_S = \varepsilon_T,\)
2. \(t_{14} = t_{15} = t_{24} = t_{25} = 0, \quad \varepsilon_A = \varepsilon_B = \varepsilon_C, \quad \varepsilon_R = -\varepsilon_S,\)
3. \(s_{14} = s_{24} = t_{15} = t_{25} = 0, \quad \varepsilon_A = \varepsilon_B = \varepsilon_C, \quad \varepsilon_R = -\varepsilon_T,\)
4. \(s_{14} = s_{24} = t_{14} = t_{24} = 0, \quad \varepsilon_A = \varepsilon_B = \varepsilon_C, \quad \varepsilon_S = \varepsilon_T,\)
5. \(c_{11} = c_{12} = c_{21} = c_{22} = 0, \quad \varepsilon_A = \varepsilon_B, \quad -\varepsilon_R = \varepsilon_S = \varepsilon_T,\)
6. \(b_{11} = b_{21} = c_{12} = c_{22} = 0, \quad \varepsilon_A = \varepsilon_C, \quad -\varepsilon_R = \varepsilon_S = \varepsilon_T,\)
7. \(b_{11} = b_{21} = c_{11} = c_{21} = 0, \quad \varepsilon_B = \varepsilon_C, \quad -\varepsilon_R = \varepsilon_S = \varepsilon_T,\)
8. \(s_{14} = s_{24} = t_{14} = t_{15} = t_{24} = t_{25} = 0, \quad \varepsilon_A = \varepsilon_B = \varepsilon_C,\)
9. $b_{11} = b_{21} = c_{11} = c_{12} = c_{21} = c_{22} = 0$, 
   $-\varepsilon_R = \varepsilon_S = \varepsilon_T$, 
10. $c_{11} = c_{12} = c_{21} = c_{22} = t_{14} = t_{15} = t_{24} = t_{25} = 0$, 
    $\varepsilon_A = \varepsilon_B$, $\varepsilon_R = -\varepsilon_S$, 
11. $c_{11} = c_{12} = c_{21} = c_{22} = s_{14} = s_{24} = t_{15} = t_{25} = 0$, 
    $\varepsilon_A = \varepsilon_B$, $\varepsilon_R = -\varepsilon_T$, 
12. $c_{11} = c_{12} = c_{21} = c_{22} = s_{14} = s_{24} = t_{14} = t_{24} = 0$, 
    $\varepsilon_A = \varepsilon_B$, $\varepsilon_S = \varepsilon_T$, 
13. $b_{11} = b_{21} = c_{12} = c_{22} = t_{14} = t_{15} = t_{24} = t_{25} = 0$, 
    $\varepsilon_A = \varepsilon_C$, $\varepsilon_R = -\varepsilon_S$, 
14. $b_{11} = b_{21} = c_{12} = c_{22} = s_{14} = s_{24} = t_{15} = t_{25} = 0$, 
    $\varepsilon_A = \varepsilon_C$, $\varepsilon_R = -\varepsilon_T$, 
15. $b_{11} = b_{21} = c_{12} = c_{22} = s_{14} = s_{24} = t_{14} = t_{24} = 0$, 
    $\varepsilon_B = \varepsilon_C$, $\varepsilon_S = \varepsilon_T$, 
16. $b_{11} = b_{21} = c_{11} = c_{21} = t_{14} = t_{15} = t_{24} = t_{25} = 0$, 
    $\varepsilon_B = \varepsilon_C$, $\varepsilon_R = -\varepsilon_S$, 
17. $b_{11} = b_{21} = c_{11} = c_{21} = s_{14} = s_{24} = t_{15} = t_{25} = 0$, 
    $\varepsilon_B = \varepsilon_C$, $\varepsilon_R = -\varepsilon_T$, 
18. $b_{11} = b_{21} = c_{11} = c_{21} = s_{14} = s_{24} = t_{14} = t_{23} = 0$, 
    $\varepsilon_B = \varepsilon_C$, $\varepsilon_S = \varepsilon_T$, 
19. $c_{11} = c_{12} = c_{21} = c_{22} = s_{14} = s_{24} = t_{14} = t_{15} = t_{24} = t_{25} = 0$, 
    $\varepsilon_A = \varepsilon_B$, 
20. $b_{11} = b_{21} = c_{12} = c_{22} = s_{14} = s_{24} = t_{14} = t_{15} = t_{24} = t_{25} = 0$, 
    $\varepsilon_A = \varepsilon_C$, 
21. $b_{11} = b_{21} = c_{11} = c_{21} = s_{14} = s_{24} = t_{14} = t_{15} = t_{24} = t_{25} = 0$, 
    $\varepsilon_B = \varepsilon_C$, 
22. $b_{11} = b_{21} = c_{11} = c_{12} = c_{21} = c_{22} = s_{14} = s_{24} = t_{15} = t_{25} = 0$, 
    $\varepsilon_R = -\varepsilon_T$, 
23. $b_{11} = b_{21} = c_{11} = c_{12} = c_{21} = c_{22} = t_{14} = t_{15} = t_{24} = t_{25} = 0$, 
    $\varepsilon_R = -\varepsilon_S$, 
24. $b_{11} = b_{21} = c_{11} = c_{12} = c_{21} = c_{22} = s_{14} = s_{24} = t_{14} = t_{24} = 0$, 
    $\varepsilon_S = \varepsilon_T$, 
25. $b_{11} = b_{21} = c_{11} = c_{12} = c_{21} = c_{22} = s_{14} = s_{24} = t_{14} = t_{15} = t_{24} = t_{25} = 0$. 

**Proof.** We once again proceed by looking at the second fundamental form $B^V$ of the vertical distribution $\mathcal{V}$. We begin with checking the minimality of $\mathcal{F}$, noting that the calculations for $B^V(A, A), B^V(B, B)$ and $B^V(C, C)$ are exactly the same as in the previous case we do not need to calculate them here. We further note that since the values of the Lie bracket relations $[S, X], [S, Y], [T, X]$ and $[T, Y]$ are identical
to those in the previous case, we know that $B^V(S, S)$ and $B^V(T, T)$ have the same values as in that case, specifically, they are zero. For $B^V(R, R)$ we have

$$B^V(R, R) = g([X, R], R)X + g([Y, R], R)Y = 0.$$ 

Thus we have confirmed the minimality of $\mathcal{F}$.

We now look at when $\mathcal{F}$ is totally geodesic. We once again remark that the Lie bracket relations involving $A, B$ and $C$ are the same as those in the case considered in Chapter 4, hence we immediately obtain the conditions for the constants related to these. It now remains to compute $B^V(R, S)$ and $B^V(R, T)$, with $B^V(S, T)$ already being known by repeating previous arguments, doing so yields

$$B^V(R, S) = \frac{1}{2}(\varepsilon_X \cdot (g([X, R], S) + g([X, S], R))X$$
$$+ \varepsilon_Y \cdot (g([Y, R], S) + g([Y, S], R))Y)$$
$$= \frac{1}{2}(\varepsilon_X \cdot (g(-s_{14}S - t_{14}T, S) + g(-s_{14}R + t_{15}T, R))X$$
$$+ \varepsilon_Y \cdot (g(-s_{24}S - t_{24}T, S) + g(-s_{24}R + t_{25}T, R))Y)$$
$$= \frac{1}{2}(\varepsilon_X \cdot (-s_{14}\varepsilon_S - s_{14}\varepsilon_R)X + \varepsilon_Y \cdot (s_{24}\varepsilon_S - s_{24}\varepsilon_R)Y),$$

$$B^V(R, T) = \frac{1}{2}(\varepsilon_X \cdot (g([X, R], T) + g([X, T], R))X$$
$$+ \varepsilon_Y \cdot (g([Y, R], T) + g([Y, T], R))Y)$$
$$= \frac{1}{2}(\varepsilon_X \cdot (g(-s_{14}S - t_{14}T, T) + g(-s_{14}R - t_{15}S, R))X$$
$$+ \varepsilon_Y \cdot (g(-s_{24}S - t_{24}T, T) + g(-s_{24}R - t_{25}S, R))Y)$$
$$= \frac{1}{2}(\varepsilon_X \cdot (-t_{14}\varepsilon_T - t_{14}\varepsilon_R)X + \varepsilon_Y \cdot (t_{24}\varepsilon_T - t_{24}\varepsilon_R)Y),$$

these together give the remaining conditions. Combining these such that the resulting collection of conditions are necessary and sufficient we obtain exactly the cases that appear in the statement of the result. \(\square\)

We note that the observations in Remark 4.3 hold here if we replace $\text{SU}(2) \times \text{SU}(2)$ by $\text{SU}(2) \times \text{SL}_2(\mathbb{R})$. 

28
Chapter 6

Lie Groups $G^8$ Foliated by their Subgroup $\text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})$

Let $(G, g)$ be an eight-dimensional semi-Riemannian Lie group containing the semi-simple, non-compact Lie subgroup $K = \text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})$ and let $\mathcal{F}$ be a conformal foliation induced by $K$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be an orthogonal decomposition of the Lie algebra of $G$ where $\mathfrak{k} = \text{sl}_2(\mathbb{R}) \times \text{sl}_2(\mathbb{R})$ is the subalgebra corresponding to $K$. Furthermore, let $\{A, B, C, R, S, T, X, Y\}$ be an orthonormal basis for $\mathfrak{g}$ such that the elements $A, B, C$ generate one of the copies of $\text{sl}_2(\mathbb{R})$ and $R, S, T$ generate the second copy. The structure of $\mathfrak{g}$ is then given by

$$
\begin{align*}
[A, B] &= 2C, \quad [C, A] = 2B, \quad [B, C] = -2A, \\
[R, S] &= 2T, \quad [T, R] = 2S, \quad [S, T] = -2R, \\
[A, X] &= a_{11}A + a_{12}B + a_{13}C + a_{14}R + a_{15}S + a_{16}T, \\
[A, Y] &= a_{21}A + a_{22}B + a_{23}C + a_{24}R + a_{25}S + a_{26}T, \\
[B, X] &= b_{11}A + b_{12}B + b_{13}C + b_{14}R + b_{15}S + b_{16}T, \\
[B, Y] &= b_{21}A + b_{22}B + b_{23}C + b_{24}R + b_{25}S + b_{26}T, \\
[C, X] &= c_{11}A + c_{12}B + c_{13}C + c_{14}R + c_{15}S + c_{16}T, \\
[C, Y] &= c_{21}A + c_{22}B + c_{23}C + c_{24}R + c_{25}S + c_{26}T, \\
[R, X] &= r_{11}A + r_{12}B + r_{13}C + r_{14}R + r_{15}S + r_{16}T, \\
[R, Y] &= r_{21}A + r_{22}B + r_{23}C + r_{24}R + r_{25}S + r_{26}T, \\
[S, X] &= s_{11}A + s_{12}B + s_{13}C + s_{14}R + s_{15}S + s_{16}T, \\
[S, Y] &= s_{21}A + s_{22}B + s_{23}C + s_{24}R + s_{25}S + s_{26}T, \\
[T, X] &= t_{11}A + t_{12}B + t_{13}C + t_{14}R + t_{15}S + t_{16}T, \\
[T, Y] &= t_{21}A + t_{22}B + t_{23}C + t_{24}R + t_{25}S + t_{26}T, \\
[X, Y] &= \rho X + \theta_1A + \theta_2B + \theta_3C + \theta_4R + \theta_5S + \theta_6T.
\end{align*}
$$

These relations can, of course, also be simplified by proceeding in the same way as we have done in the previous chapters.

**Proposition 6.1.** Let $(G, g)$ be an eight-dimensional semi-Riemannian Lie group containing the subgroup $K = \text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be the Lie algebra of $G$ and let $\{A, B, C, R, S, T, X, Y\}$ be an orthonormal basis for this algebra such that $\mathfrak{k} = \text{sl}_2(\mathbb{R}) \times \text{sl}_2(\mathbb{R})$ is generated by the vector fields $A, B, C, R, S, T$. Then the Lie
decomposition of the Lie algebra of $G$.

Theorem 6.2. Let $K = \text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})$ be a
subgroup and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be an orthogonal
decomposition of the Lie algebra of $G$ such that $\mathfrak{k} = \mathfrak{sl}_2(\mathbb{R}) \times \mathfrak{sl}_2(\mathbb{R})$. Furthermore let
$\{A, B, C, R, S, T, X, Y\}$ be an orthonormal basis for $\mathfrak{g}$ such that the elements
$A, B, C, R, S, T$ generate a copy of $\mathfrak{sl}_2(\mathbb{R})$.

The bracket relations can be written as

$$
\begin{align*}
[A, B] &= 2C, \quad [C, A] = 2B, \quad [B, C] = -2A, \\
[R, S] &= 2T, \quad [T, R] = 2S, \quad [S, T] = -2R, \\
[A, X] &= b_{11}B + c_{11}C, \quad [A, Y] = b_{21}B + c_{21}C, \\
[B, X] &= b_{11}A - c_{11}C, \quad [B, Y] = b_{21}A - c_{22}C, \\
[C, X] &= c_{11}A + c_{12}B, \quad [C, Y] = c_{21}A + c_{22}B, \\
[R, X] &= s_{14}S + t_{14}T, \quad [R, Y] = s_{24}S + t_{24}T, \\
[S, X] &= s_{14}R - t_{15}T, \quad [S, Y] = s_{24}R - t_{25}T, \\
[T, X] &= t_{14}R + t_{15}S, \quad [T, Y] = t_{24}R + t_{25}S, \\
[X, Y] &= \rho X + \theta_1 A + \theta_2 B + \theta_3 C + \theta_4 R + \theta_5 S + \theta_6 T,
\end{align*}
$$

where $b_{11}, b_{21}, c_{11}, c_{12}, c_{21}, c_{22}, s_{14}, s_{24}, t_{14}, t_{15}, t_{24}, t_{25}, \rho$ are real, arbitrary coefficients
and

$$
\begin{pmatrix}
\theta_1 \\
\theta_2 \\
\theta_3 \\
\theta_4 \\
\theta_5 \\
\theta_6
\end{pmatrix} = \frac{1}{2}
\begin{pmatrix}
-\rho c_{12} - b_{11}c_{21} + b_{21}c_{11} \\
-\rho c_{11} - c_{11}c_{22} + b_{21}c_{12} \\
\rho b_{11} - c_{11}c_{22} + c_{12}c_{21} \\
-\rho t_{15} - s_{14}t_{24} + s_{24}t_{14} \\
-\rho t_{14} - s_{14}t_{25} + s_{24}t_{15} \\
\rho s_{14} - t_{14}t_{25} + t_{15}t_{24}
\end{pmatrix}.
$$

Theorem 6.2. Let $(G, g)$ be an eight-dimensional semi-Riemannian Lie group containing
$K = \text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})$ as a subgroup and let $g = \mathfrak{k} \oplus \mathfrak{m}$ be an
orthogonal decomposition of the Lie algebra of $G$ such that $\mathfrak{k} = \mathfrak{sl}_2(\mathbb{R}) \times \mathfrak{sl}_2(\mathbb{R})$. Furthermore
let $\{A, B, C, R, S, T, X, Y\}$ where the triples $(A, B, C)$ and $(R, S, T)$ each generate a
vertical distribution $\mathcal{V}$ generated by $\mathfrak{k}$, is conformal then it is also minimal. Moreover, $\mathcal{F}$ is totally geodesic if and only
if one of the following conditions holds

1. $-\varepsilon_A = \varepsilon_B = \varepsilon_C$, $-\varepsilon_R = \varepsilon_S = \varepsilon_T,$

2. $t_{14} = t_{15} = t_{24} = t_{25} = 0,$

3. $s_{14} = s_{24} = t_{15} = t_{25} = 0,$

4. $s_{14} = s_{24} = t_{14} = t_{24} = 0,$

5. $c_{11} = c_{12} = c_{21} = c_{22} = 0,$

6. $b_{11} = b_{21} = c_{12} = c_{22} = 0,$

7. $b_{11} = b_{21} = c_{11} = c_{21} = 0,$

8. $s_{14} = s_{24} = t_{14} = t_{15} = t_{24} = t_{25} = 0,$
9. \( b_{11} = b_{21} = c_{11} = c_{12} = c_{21} = c_{22} = 0, \)
   \(- \varepsilon_R = \varepsilon_S = \varepsilon_T, \)

10. \( c_{11} = c_{12} = c_{21} = c_{22} = t_{14} = t_{15} = t_{24} = t_{25} = 0, \)
    \( \varepsilon_A = -\varepsilon_B, \varepsilon_R = -\varepsilon_S, \)

11. \( c_{11} = c_{12} = c_{21} = c_{22} = s_{14} = s_{24} = t_{15} = t_{25} = 0, \)
    \( \varepsilon_A = -\varepsilon_B, \varepsilon_R = -\varepsilon_S, \)

12. \( c_{11} = c_{12} = c_{21} = c_{22} = s_{14} = s_{24} = t_{14} = t_{24} = 0, \)
    \( \varepsilon_A = -\varepsilon_B, \varepsilon_S = \varepsilon_T, \)

13. \( b_{11} = b_{21} = c_{12} = c_{22} = t_{14} = t_{15} = t_{24} = t_{25} = 0, \)
    \( \varepsilon_A = -\varepsilon_C, \varepsilon_R = -\varepsilon_S, \)

14. \( b_{11} = b_{21} = c_{12} = c_{22} = s_{14} = s_{24} = t_{15} = t_{25} = 0, \)
    \( \varepsilon_A = -\varepsilon_C, \varepsilon_R = -\varepsilon_S, \)

15. \( b_{11} = b_{21} = c_{12} = c_{22} = s_{14} = s_{24} = t_{14} = t_{24} = 0, \)
    \( \varepsilon_A = -\varepsilon_C, \varepsilon_S = \varepsilon_T, \)

16. \( b_{11} = b_{21} = c_{11} = c_{21} = t_{14} = t_{15} = t_{24} = t_{25} = 0, \)
    \( \varepsilon_B = \varepsilon_C, \varepsilon_R = -\varepsilon_S, \)

17. \( b_{11} = b_{21} = c_{11} = c_{21} = s_{14} = s_{24} = t_{15} = t_{25} = 0, \)
    \( \varepsilon_B = \varepsilon_C, \varepsilon_R = -\varepsilon_S, \)

18. \( b_{11} = b_{21} = c_{11} = c_{21} = s_{14} = s_{24} = t_{14} = t_{23} = 0, \)
    \( \varepsilon_B = \varepsilon_C, \varepsilon_S = \varepsilon_T, \)

19. \( c_{11} = c_{12} = c_{21} = c_{22} = s_{14} = s_{24} = t_{14} = t_{15} = t_{24} = t_{25} = 0, \)
    \( \varepsilon_A = -\varepsilon_B, \)

20. \( b_{11} = b_{21} = c_{12} = c_{22} = s_{14} = s_{24} = t_{14} = t_{15} = t_{24} = t_{25} = 0, \)
    \( \varepsilon_A = -\varepsilon_C, \)

21. \( b_{11} = b_{21} = c_{11} = c_{21} = s_{14} = s_{24} = t_{14} = t_{15} = t_{24} = t_{25} = 0, \)
    \( \varepsilon_B = \varepsilon_C, \)

22. \( b_{11} = b_{21} = c_{11} = c_{12} = c_{21} = c_{22} = s_{14} = s_{24} = t_{15} = t_{25} = 0, \)
    \( \varepsilon_R = -\varepsilon_T, \)

23. \( b_{11} = b_{21} = c_{11} = c_{12} = c_{21} = c_{22} = t_{14} = t_{15} = t_{24} = t_{25} = 0, \)
    \( \varepsilon_R = -\varepsilon_S, \)

24. \( b_{11} = b_{21} = c_{11} = c_{12} = c_{21} = c_{22} = s_{14} = s_{24} = t_{14} = t_{24} = 0, \)
    \( \varepsilon_S = \varepsilon_T, \)

25. \( b_{11} = b_{21} = c_{11} = c_{12} = c_{21} = c_{22} = s_{14} = s_{24} = t_{14} = t_{15} = t_{24} = t_{25} = 0. \)

This is proved in the same way as the previous two chapters and, perhaps unsurprisingly, Remark 4.3 applies here as well if we change the mention of SU(2) × SU(2) to SL₂(ℝ) × SL₂(ℝ).
Chapter 7

Lie Groups $G^6$ Foliated by their Subgroup $\text{SU}(2) \times \text{SO}(2)$

In this chapter we investigate the case when the subgroup $K$ is compact but not semi-simple.

Let $(G, g)$ be a six-dimensional semi-Riemannian Lie group containing the subgroup $K = \text{SU}(2) \times \text{SO}(2)$. Let $g = k \oplus m$ be an orthogonal decomposition of the Lie algebra of $G$ such that $k = \text{su}(2) \times \text{so}(2)$. Furthermore, let $\{A, B, C, T, X, Y\}$ be an orthonormal basis for $g$ such that $A, B, C$ generate $\text{su}(2)$ and $T$ generates $\text{so}(2)$. Let the foliation $F$, tangent to the vertical distribution $V$ generated by $k$, be conformal. We know from the proof of Proposition 1.35 that when $F$ is conformal then $H[[V, V], H] = 0$ and since $\text{su}(2)$ is semisimple it follows from this that $H[V, H] = 0$ for all $V \in \text{su}(2)$. The structure of $g$ is then of the form

\[
\begin{align*}
[A, B] &= 2C, \quad [C, A] = 2B, \quad [B, C] = 2A, \\
[A, X] &= a_{11}A + a_{12}B + a_{13}C + a_{14}T, \\
[A, Y] &= a_{21}A + a_{22}B + a_{23}C + a_{24}T, \\
[B, X] &= b_{11}A + b_{12}B + b_{13}C + b_{14}T, \\
[B, Y] &= b_{21}A + b_{22}B + b_{23}C + b_{24}T, \\
[C, X] &= c_{11}A + c_{12}B + c_{13}C + c_{14}T, \\
[C, Y] &= c_{21}A + c_{22}B + c_{23}C + c_{24}T, \\
[T, X] &= x_1X + y_1Y + t_{11}A + t_{12}B + t_{13}C + t_{14}T, \\
[T, Y] &= x_2X + y_2Y + t_{21}A + t_{22}B + t_{23}C + t_{24}T, \\
[X, Y] &= \rho X + \theta_1A + \theta_2B + \theta_3C + \theta_4T,
\end{align*}
\]

for some constant coefficients such that the Jacobi identities are satisfied.

**Theorem 7.1.** Let $(G, g)$ be a six-dimensional semi-Riemannian Lie group containing $K = \text{SU}(2) \times \text{SO}(2)$ as a subgroup. Let $g = k \oplus m$ be an orthogonal decomposition of the Lie algebra of $G$ such that $k = \text{su}(2) \times \text{so}(2)$. Furthermore, let $\{A, B, C, T, X, Y\}$ be an orthonormal basis for $g$ such that $A, B, C$ generate $\text{su}(2)$ and $T$ generates $\text{so}(2)$. Then the foliation $F$, tangent to the vertical distribution $V$ generated by $k$, is conformal if and only if

\[
x_1 = y_2 \text{ and } x_2\varepsilon X + y_1\varepsilon Y = 0.
\]

If we also have $y_2 = 0$ then $F$ is semi-Riemannian and conversely.
Proof. By definition $F$ is conformal if and only if there exists a $V \in \mathcal{V}$ such that $B^H(H_1, H_2) = g(H_1, H_2) \otimes V$ is satisfied for all $H_1, H_2 \in \mathcal{H}$. We thus calculate the values of $B^H$.\[ B^H(X, X) = x_1 \varepsilon T \varepsilon X T, \quad B^H(Y, Y) = y_2 \varepsilon T \varepsilon Y T, \quad (7.1) \]
\[ B^H(X, Y) = \frac{1}{2} \varepsilon T (x_2 \varepsilon X + y_1 \varepsilon Y) T. \quad (7.2) \]

From (7.1) we see that $V = x_1 \varepsilon T T = y_2 \varepsilon T T$, therefore $x_1 = y_2$, and from (7.2) we obtain $x_2 \varepsilon X + y_1 \varepsilon Y = 0$. The second statement now follows immediately from the definition of a semi-Riemannian foliation. \[ \square \]

Since $\varepsilon_Z = \pm 1$ for all basis elements $Z$ we note that $x_2 \varepsilon X + y_1 \varepsilon Y = 0$ can be rewritten as $x_2 = -y_1 \varepsilon X \varepsilon Y$.

We now want to find expressions for the coefficients in the algebra $\mathfrak{g}$. To accomplish this we shall once again use that a Lie algebra has to satisfy the Jacobi identity, however, this time around we find that there are multiple families of solutions.

**Proposition 7.2.** Let $(G, g)$ be a six-dimensional semi-Riemannian Lie group containing $K = \text{SU}(2) \times \text{SO}(2)$ as a subgroup. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be an orthogonal decomposition of the Lie algebra of $G$ such that $\mathfrak{k} = \text{su}(2) \times \text{so}(2)$. Furthermore, let \{A, B, C, T, X, Y\} be an orthonormal basis for $\mathfrak{g}$ such that the elements $A, B, C$ generate $\text{su}(2)$ and $T$ generates $\text{so}(2)$. Suppose that the foliation $F$, tangent to the vertical distribution $\mathcal{V}$ induced by $\mathfrak{k}$, is conformal. Then the structure of $\mathfrak{g}$ is given by

\[ [A, B] = 2C, \quad [C, A] = 2B, \quad [B, C] = 2A, \]
\[ [A, X] = -b_{11} B - c_{11} C, \quad [A, Y] = -b_{21} B - c_{21} C, \]
\[ [B, X] = b_{11} A - c_{12} C, \quad [B, Y] = b_{21} A - c_{22} C, \]
\[ [C, X] = c_{11} A + c_{12} B, \quad [C, Y] = c_{21} A + c_{22} B, \]
\[ T, X] = y_2 X + y_1 Y - \frac{1}{2} ((y_2 c_{12} + y_1 c_{22}) A + (y_2 b_{11} + y_1 b_{21}) C) + t_{14} T, \]
\[ T, Y] = -y_1 \varepsilon X \varepsilon Y X + y_2 Y + \frac{1}{2} ((y_1 \varepsilon X \varepsilon Y c_{12} - y_2 c_{22}) A + (y_1 \varepsilon X \varepsilon Y b_{11} - y_2 b_{21}) C) + t_{24} T, \]
\[ [X, Y] = \rho X + \theta_1 A + \theta_2 B + \theta_3 C + \theta_4 T, \]

where $b_{11}, b_{21}, c_{11}, c_{12}, c_{21}, c_{22}, t_{14}, t_{24}, y_1, y_2, \rho, \theta_4$ are real, arbitrary coefficients and

\[ \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -\rho c_{12} + b_{11} c_{21} - b_{21} c_{11} \\ \rho c_{11} + b_{11} c_{22} - b_{21} c_{12} \\ -\rho b_{11} + c_{11} c_{22} - c_{12} c_{21} \end{pmatrix}. \]

**Proof.** In order to keep it concise we are only looking at when the Jacobi identity is not being satisfied. We begin with the terms containing either $X$ or $Y$, but not both.

\[ [[A, B], X] + [[X, A], B] + [[B, X], A] = 2((a_{13} + c_{11}) A + (b_{13} + c_{12}) B - (a_{11} + b_{12} - c_{13}) C + c_{14} T), \quad (7.3) \]
\[ [[A, B], Y] + [[Y, A], B] + [[B, Y], A] = 2((a_{23} + c_{21})A + (b_{23} + c_{22})B - (a_{21} + b_{22} - c_{23})C + c_{24}T), \]  \hspace{1cm} (7.4)  
\[ [[A, C], X] + [[X, A], C] + [[C, X], A] = 2(-a_{12} + b_{11})A + (a_{11} - b_{12} + c_{13})B - (b_{13} + c_{12})C - b_{14}T), \]  \hspace{1cm} (7.5)  
\[ [[A, C], Y] + [[Y, A], C] + [[C, Y], A] = 2(-a_{22} + b_{21})A + (a_{21} - b_{22} + c_{23})B - (b_{23} + c_{22})C - b_{24}T), \]  \hspace{1cm} (7.6)  
\[ [[A, T], X] + [[X, A], T] + [[T, X], A] = -((y_{2a11} + y_{1a21})A + (y_{2a12} + y_{1a22} - 2t_{13})B + (y_{2a13} + y_{1a23} + 2t_{12})C + (y_{2a14} + y_{1a24})T), \]  \hspace{1cm} (7.7)  
\[ [[A, T], Y] + [[Y, A], T] + [[T, Y], A] = ((y_{1c11} - y_{2a21})A + (y_{1c12} + y_{1c21} + 2t_{13})B + (y_{1c13} + y_{1c23} + 2t_{12})C + (y_{1c14} + y_{1c24})T), \]  \hspace{1cm} (7.8)  
\[ [[B, C], Y] + [[Y, B], C] + [[C, Y], B] = 2((a_{13} - b_{12} - c_{13})A + (a_{12} + b_{11})B + (a_{13} + c_{11})C + a_{14}T), \]  \hspace{1cm} (7.9)  
\[ [[B, C], Y] + [[Y, B], C] + [[C, Y], B] = 2((a_{23} - b_{22} - c_{23})A + (a_{22} + b_{21})B + (a_{23} + c_{21})C + a_{24}T), \]  \hspace{1cm} (7.10)  
\[ [[B, T], X] + [[X, B], T] + [[T, X], B] = -((y_{2b11} + y_{1b21} + 2t_{13})A + (y_{2b12} + y_{1b22})B + (y_{2b13} + y_{1b23} - 2t_{11})C + (y_{2b14} + y_{1b24})T), \]  \hspace{1cm} (7.11)  
\[ [[B, T], Y] + [[Y, B], T] + [[T, Y], B] = ((y_{1b11} - y_{2b21} - 2t_{13})A + (y_{1b12} - y_{2b22})B + (y_{1b13} - y_{2b23} - 2t_{12})C + (y_{1b14} - y_{2b24})T), \]  \hspace{1cm} (7.12)  
\[ [[C, T], X] + [[X, C], T] + [[T, X], C] = -((y_{2c11} + y_{1c21} - 2t_{12})A + (y_{2c12} + y_{1c22})B + (y_{2c13} + y_{1c23})C + (y_{2c14} + y_{1c24})T), \]  \hspace{1cm} (7.13)  
\[ [[C, T], Y] + [[Y, C], T] + [[T, Y], C] = ((y_{1c11} - y_{2c21} + 2t_{12})A + (y_{1c12} - y_{2c22} - 2t_{12})B + (y_{1c13} - y_{2c23})C + (y_{1c14} - y_{2c24})T), \]  \hspace{1cm} (7.14)  

Since the \( T \) parts of equations (7.3–7.6), (7.9) and (7.10) only have one term each those terms all have to be zero, thus we get  
\[ a_{14} = a_{24} = b_{14} = b_{24} = c_{14} = c_{24} = 0. \]

From the \( A \) parts of equations (7.3–7.6) and (7.11–7.14), the \( B \) parts of equations (7.3), (7.4), (7.9), (7.10), (7.13) and (7.14) and the \( C \) parts of equations (7.5), (7.6), (7.9) and (7.10) we obtain  
\[ a_{12} = -b_{11}, \quad a_{22} = -b_{21}, \quad a_{13} = -c_{11}, \quad a_{23} = -c_{21}, \quad b_{13} = -c_{12}, \quad b_{23} = -c_{22}, \]  
\[ \begin{pmatrix} t_{11} & t_{21} \\ t_{12} & t_{22} \\ t_{13} & t_{23} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -(y_{2c12} + y_{1c22}) & (y_{1c12} - y_{2c22}) \\ (y_{2c11} + y_{1c21}) & -(y_{1c11} - y_{2c21}) \\ -(y_{2b11} + y_{1b21}) & (y_{1c11} - y_{2b21}) \end{pmatrix}. \]
From equations (7.3), (7.5) and (7.9) it follows that
\[ a_{11} = b_{12} = c_{13} = 0. \]
Similarly, equations (7.4), (7.6) and (7.10) together imply
\[ a_{21} = b_{22} = c_{23} = 0. \]

Using the information gained thus far we also examine the Jacobi identities involving both \( X \) and \( Y \),
\[
[[A, X], Y] + [[Y, A], X] + [[X, Y], A] = (\rho b_{11} - c_{22}c_{11} + c_{12}c_{21} + 2\theta_3)B + (\rho c_{11} + c_{22}b_{11} - c_{12}b_{21} - 2\theta_2)C, \tag{7.15}
\]
\[
[[B, X], Y] + [[Y, B], X] + [[X, Y], B] = -(\rho b_{11} - c_{22}c_{11} + c_{12}c_{21} + 2\theta_3)A + (\rho c_{12} - c_{21}b_{11} + c_{11}b_{21} + 2\theta_1)C, \tag{7.16}
\]
\[
[[C, X], Y] + [[Y, C], X] + [[X, Y], C] = -(\rho c_{11} + c_{22}b_{11} - c_{12}b_{21} - 2\theta_2)A - (\rho c_{12} - c_{21}b_{11} + c_{11}b_{21} + 2\theta_1)B, \tag{7.17}
\]
In this last equation we do not insert all of the values we have found so far, this is to make it somewhat more pleasing to read.
\[
[[T, X], Y] + [[Y, T], X] + [[X, Y], T] =
- (\rho t_{11} + t_{22}b_{11} - t_{12}b_{21} + t_{23}c_{11} - t_{13}c_{21} + t_{11}t_{24} - t_{14}t_{21} - y_2\theta_1 - y_2\theta_2)A
- (\rho t_{12} - t_{21}b_{11} + t_{11}b_{21} + t_{23}c_{12} - t_{13}c_{22} + t_{13}t_{24} - t_{14}t_{22} - y_2\theta_2 - y_2\theta_3)B
- (\rho t_{13} - t_{21}c_{11} - t_{22}c_{12} + t_{11}c_{21} + t_{12}c_{22} + t_{13}t_{24} - t_{14}t_{23} - y_2\theta_3 - y_2\theta_3)C
- (\rho t_{14} - y_2\theta_1 - y_2\theta_3)T + (\rho y_2 - y_1\varepsilon X \varepsilon Y t_{14} - y_2t_{24} Y).
\]

We can not draw any conclusions from the last calculation without splitting into different cases, but from equations (7.15–7.17) we get the values for \( \theta_1, \theta_2 \) and \( \theta_3 \).

Thus, the coefficients we have found are ones that all six-dimensional semi-Riemannian Lie groups containing \( K \) have in common. These are the ones that can be found through only the Jacobi identity. However, since we are interested in finding minimal foliations we can also assert that \( t_{14} \) and \( t_{24} \) both have to equal zero, this is shown in Theorem 7.3 below. \( \square \)

**Theorem 7.3.** Let \( (G, g) \) be a six-dimensional semi-Riemannian Lie group containing \( K = SU(2) \times SO(2) \) as a subgroup and let \( g = \mathfrak{k} \oplus \mathfrak{m} \) be an orthogonal decomposition of the Lie algebra of \( G \) such that \( \mathfrak{k} = su(2) \times so(2) \). Furthermore, let \( \{A, B, C, T, X, Y\} \) be an orthonormal basis for \( g \) such that the vector fields \( A, B, C, T \) generate the Lie subalgebra \( \mathfrak{k} \), where \( A, B \) and \( C \) generate \( su(2) \) and \( T \) generates \( so(2) \). If the foliation \( F \), tangent to the vertical distribution \( V \) generated by \( \mathfrak{k} \), is conformal then it is minimal if and only if \( t_{14} = t_{24} = 0 \).

**Proof.** We have
\[
\text{trace} \ B^V = \sum_{V \in \{A, B, C, T\}} B^V(V, V)
= -(\varepsilon_X \cdot (a_{11}\varepsilon_A + b_{12}\varepsilon_B + c_{13}\varepsilon_C + t_{14}\varepsilon_T)X
+ \varepsilon_Y \cdot (a_{21}\varepsilon_A + b_{22}\varepsilon_B + c_{23}\varepsilon_C + t_{24}\varepsilon_T)Y).
\]
Since all $G$ satisfying the assumptions of the statement have

$$a_{11} = a_{21} = b_{12} = b_{22} = c_{13} = c_{23} = 0$$

in common we see that trace $B^V = t_{14}e_Te_XX + t_{24}e_Te_YY$. The result now follows immediately from this. \hfill \Box

Using Theorem 7.3 we can thus rewrite Proposition 7.2 in the more specific case that $\mathcal{F}$ is both conformal and minimal.

**Proposition 7.4.** Let $(G, g)$ be as in Proposition 7.2 and let the foliation $\mathcal{F}$, tangent to $\mathfrak{t}$, be conformal and minimal. Then $g$ is of the form

$$[A, B] = 2C, \quad [C, A] = 2B, \quad [B, C] = 2A,$$

$$[A, X] = -b_{11}B - c_{11}C, \quad [A, Y] = -b_{21}B - c_{21}C,$$

$$[B, X] = b_{11}A - c_{12}C, \quad [B, Y] = b_{21}A - c_{22}C,$$

$$[C, X] = c_{11}A + c_{12}B, \quad [C, Y] = c_{21}A + c_{22}B,$$

$$[T, X] = y_2X + y_1Y - \frac{1}{2}((y_2c_{12} + y_1c_{22})A - (y_2b_{11} + y_1b_{21})C),$$

$$[T, Y] = -y_1e_XX + y_2Y + \frac{1}{2}((y_1e_YYc_{12} - y_2c_{22})A - (y_1e_YYb_{11} - y_2b_{21})C),$$

$$[X, Y] = \rho X + \theta_1A + \theta_2B + \theta_3C + \theta_4T,$$

where $b_{11}, b_{21}, c_{11}, c_{12}, c_{21}, c_{22}, t_{14}, t_{24}, y_1, y_2, \rho, \theta_4$ are real, arbitrary coefficients and

$$\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -\rho c_{12} + b_{11}c_{21} - b_{21}c_{11} \\ \rho c_{11} + b_{11}c_{22} - b_{22}c_{11} \\ -\rho b_{11} + c_{11}c_{22} - c_{12}c_{21} \end{pmatrix}.$$ 

The structure given above does not yet fully satisfy the Jacobi identity, specifically for certain choices of coefficients we have

$$[[[T, X], Y] + [[Y, T], X] + [[X, Y], T]] \neq 0.$$

Solving this will divide our Lie groups into three families. It is clear from the conditions we obtain that the pairwise intersections of these are non-trivial. The classifications and their intersections only become more complicated once we start looking at when $\mathcal{F}$ is totally geodesic.

**Proposition 7.5.** If we assume that $\rho = 0$ in Proposition 7.4 then at least one of the following has to hold

(a) $\theta_4 = 0$,

(b) $y_2 = 0$.

If we instead assume that $\rho \neq 0$ then

(c) $y_1 = y_2 = 0$

must be satisfied.
Remark 7.6. In the last case it is not a necessity for us to assume that $\rho$ is non-zero. For instance, if $\rho = 0$ then we see that $y_2 = 0$ is a sufficient condition and $y_1$ can be chosen freely; therefore we may choose, for instance, $y_1 = 0$.

Proof. Rewriting equation (7.18) using Proposition 7.4 we obtain
\[
[[T, X], Y] + [[Y, T], X] + [[X, Y], T] = \frac{\rho}{2}((y_1c_{22} - y_2c_{12})A - (y_1c_{21} - y_2c_{11})b + (y_1b_{21} - y_2b_{11})C)
+ 2y_2\theta_4T + \rho(y_2X - y_1Y).
\]
The cases when $\rho = 0$ follow from the $T$ part of this while the case when $\rho \neq 0$ follows from the $X$ and $Y$ components.

Theorem 7.7. Let $(G, g)$ be a semi-Riemannian Lie group containing the subgroup $K = SU(2) \times SO(2)$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be an orthogonal decomposition of the Lie algebra of $G$ such that $\mathfrak{k} = \mathfrak{su}(2) \times \mathfrak{so}(2)$. Furthermore, let $\{A, B, C, T, X, Y\}$ be an orthonormal basis for $\mathfrak{g}$ such that $A, B, C$ generate $\mathfrak{su}(2)$ and $T$ generates $\mathfrak{so}(2)$. If the foliation $\mathcal{F}$, tangent to $\mathfrak{k}$, is conformal and minimal then $\mathcal{F}$ is totally geodesic if and only if the structure coefficients of $\mathfrak{g}$ satisfy one of the following conditions

1. $y_1 = y_2 = 0, \quad \varepsilon_A = \varepsilon_B = \varepsilon_C$,
2. $b_{11} = b_{21} = c_{11} = c_{12} = c_{21} = c_{22} = 0$,
3. $c_{11} = c_{12} = c_{21} = c_{22} = y_1 = y_2 = 0, \quad \varepsilon_A = \varepsilon_B$,
4. $b_{11} = b_{21} = c_{12} = c_{22} = y_1 = y_2 = 0, \quad \varepsilon_A = \varepsilon_C$,
5. $b_{11} = b_{21} = c_{11} = c_{21} = y_1 = y_2 = 0, \quad \varepsilon_B = \varepsilon_C$,
6. $c_{11} = c_{12} = c_{21} = c_{22} = 0, \quad y_1^2 = y_2^2 \neq 0$,
   \[b_{11} = \frac{y_1}{y_2}b_{21}, \quad \varepsilon_A = \varepsilon_B, \quad \varepsilon_X = -\varepsilon_Y,
   \]
7. $b_{11} = b_{21} = c_{12} = c_{22} = 0, \quad y_1^2 = y_2^2 \neq 0$,
   \[c_{11} = \frac{y_1}{y_2}c_{21}, \quad \varepsilon_A = \varepsilon_C, \quad \varepsilon_X = -\varepsilon_Y,
   \]
8. $b_{11} = b_{21} = c_{11} = c_{21} = 0, \quad y_1^2 = y_2^2 \neq 0$,
   \[c_{12} = \frac{y_1}{y_2}c_{22}, \quad \varepsilon_B = \varepsilon_C, \quad \varepsilon_X = -\varepsilon_Y,
   \]
9. $y_1^2 = y_2^2 \neq 0, \quad b_{11} = \frac{y_1}{y_2}b_{21}, \quad c_{11} = \frac{y_1}{y_2}c_{21}, \quad c_{12} = \frac{y_1}{y_2}c_{22}, \quad \varepsilon_A = \varepsilon_B = \varepsilon_C, \quad \varepsilon_X = -\varepsilon_Y$.

Proof. We begin by calculating the values of $B^V$. If we write out only the non-zero equations we have

\[
B^V(A, B) = -\frac{1}{2}(b_{11} \cdot \varepsilon_X \cdot (\varepsilon_A - \varepsilon_B)X + b_{21} \cdot \varepsilon_Y \cdot (\varepsilon_A - \varepsilon_B)Y), \quad (7.19)
\]
\[
B^V(A, C) = -\frac{1}{2}(c_{11} \cdot \varepsilon_X \cdot (\varepsilon_A - \varepsilon_C)X + c_{21} \cdot \varepsilon_Y \cdot (\varepsilon_A - \varepsilon_C)Y), \quad (7.20)
\]
\[
B^V(B, C) = -\frac{1}{2}(c_{12} \cdot \varepsilon_X \cdot (\varepsilon_B - \varepsilon_C)X + c_{22} \cdot \varepsilon_Y \cdot (\varepsilon_B - \varepsilon_C)Y), \quad (7.21)
\]
\[
B^V(A, T) = -\frac{1}{4}(\varepsilon_A \cdot (y_2c_{12} + y_1c_{22})X - \varepsilon_Y \cdot (y_1\varepsilon_X\varepsilon_Yc_{12} - y_2c_{22})Y), \quad (7.22)
\]
\[ B^V(B, T) = -\frac{1}{4} \cdot \varepsilon_B \cdot (\varepsilon_X \cdot (y_2c_{11} + y_1c_{21})X - \varepsilon_Y \cdot (y_1\varepsilon_X\varepsilon_Yc_{11} - y_2c_{21})Y), \quad (7.23) \]
\[ B^V(C, T) = \frac{1}{4} \cdot \varepsilon_C \cdot (\varepsilon_X \cdot (y_2b_{11} + y_1b_{21})X - \varepsilon_Y \cdot (y_1\varepsilon_X\varepsilon_Yb_{11} - y_2b_{21})Y). \quad (7.24) \]

From these we get

(7.19) \implies \begin{cases} b_{11} = b_{21} = 0, \\
\varepsilon_A = \varepsilon_B, \end{cases}

(7.20) \implies \begin{cases} c_{11} = c_{21} = 0, \\
\varepsilon_A = \varepsilon_C, \end{cases}

(7.21) \implies \begin{cases} c_{12} = c_{22} = 0, \\
\varepsilon_B = \varepsilon_C, \end{cases}

(7.22) \implies \begin{cases} c_{12} = c_{22} = 0, \\
y_1 = y_2 = 0, \\
y_1^2 = y_2^2 \neq 0, c_{12} = \frac{y_1}{y_2}c_{22}, \varepsilon_X = -\varepsilon_Y, \end{cases}

(7.23) \implies \begin{cases} c_{11} = c_{21} = 0, \\
y_1 = y_2 = 0, \\
y_1^2 = y_2^2 \neq 0, c_{11} = \frac{y_1}{y_2}c_{21}, \varepsilon_X = -\varepsilon_Y, \end{cases}

(7.24) \implies \begin{cases} b_{11} = b_{21} = 0, \\
y_1 = y_2 = 0, \\
y_1^2 = y_2^2 \neq 0, b_{11} = \frac{y_1}{y_2}b_{21}, \varepsilon_X = -\varepsilon_Y. \end{cases}

If we combine these conditions in all the unique and compatible ways, without adding superfluous conditions, we get exactly the nine conditions of the statement. \qed

**Remark 7.8.** We first note that for the final four conditions, 6-9, the metric on the horizontal distribution \( H \) is not Riemannian. As a consequence of this Theorem 1.34 cannot be applied to these cases. We also make the following observations regarding the families we yield from these conditions:

1. The family we obtain from condition 1 is eight-dimensional and the free variables in this case are \( \{b_{11}, b_{21}, c_{11}, c_{12}, c_{21}, c_{22}, \rho, \theta_4\} \). Note that since we require \( y_1 = y_2 = 0 \) this is in fact a subfamily of the family corresponding to condition (c) in Proposition 7.5.

2. From condition 2 we obtain three two-dimensional families of semi-Riemannian Lie groups depending on which of the conditions in Proposition 7.5 is satisfied. The free variables for the corresponding conditions of Proposition 7.5 are
   \[ (a) \ \{y_1, y_2\}, \quad (b) \ \{y_1, \theta_4\}, \quad (c) \ \{\rho, \theta_4\}. \]

3. Conditions 3, 4 and 5 give four-dimensional subfamilies of the family obtained from condition (c) in Proposition 7.5 since all of them require that \( y_1 \) and \( y_2 \) to be zero. The following variables are free:
4. The conditions 6, 7 and 8 give two-dimensional families of semi-Riemannian Lie groups. Since they all require that $y_1^2 = y_2^2 
eq 0$ we see that (a) is the only condition in Proposition 7.5 compatible with these. In these cases the variables that are free are:

6. $\{b_{11}, y_1\}$,  
7. $\{c_{11}, y_1\}$,  
8. $\{c_{12}, y_1\}$.

5. Condition 9 also has the requirement that $y_1^2 = y_2^2 
eq 0$ thus (a) is once again the only compatible condition in Proposition 7.5. However, in this case we obtain a four-dimensional family where the variables $\{b_{11}, c_{11}, c_{12}, y_1\}$ are free.
Chapter 8

Lie Groups $G^6$ Foliated by their Subgroup $\text{SL}_2(\mathbb{R}) \times \text{SO}(2)$

Let $(G,g)$ be a six-dimensional semi-Riemannian Lie group containing the non-compact, non-semisimple Lie subgroup $K = \text{SL}_2(\mathbb{R}) \times \text{SO}(2)$ and let $g = \mathfrak{k} \oplus \mathfrak{m}$ be an orthogonal decomposition of the Lie algebra of $G$ such that $\mathfrak{k} = \mathfrak{sl}_2(\mathbb{R}) \times \mathfrak{so}(2)$. Let \{A, B, C, T, X, Y\} be vector fields in $TG$ serving as an orthonormal basis for $g$ such that $A, B, C, T$ generate the subalgebra $\mathfrak{k}$ with the standard Lie bracket relations

\[[A, B] = 2C, \quad [C, A] = 2B, \quad [B, C] = -2A.\]

Let the foliation $F$, tangent to the vertical distribution $V$ induced by $k$, be conformal. Then the rest of the relations are given by

\[[A, X] = a_{11}A + a_{12}B + a_{13}C + a_{14}T,\]
\[[A, Y] = a_{21}A + a_{22}B + a_{23}C + a_{24}T,\]
\[[B, X] = b_{11}A + b_{12}B + b_{13}C + b_{14}T,\]
\[[B, Y] = b_{21}A + b_{22}B + b_{23}C + b_{24}T,\]
\[[C, X] = c_{11}A + c_{12}B + c_{13}C + c_{14}T,\]
\[[C, Y] = c_{21}A + c_{22}B + c_{23}C + c_{24}T,\]
\[[T, X] = x_1X + y_1Y + t_{11}A + t_{12}B + t_{13}C + t_{14}T,\]
\[[T, Y] = x_2X + y_2Y + t_{21}A + t_{22}B + t_{23}C + t_{24}T,\]
\[[X, Y] = \rho X + \theta_1A + \theta_2B + \theta_3C + \theta_4T.\]

**Theorem 8.1.** Let $(G,g)$ be a six-dimensional semi-Riemannian Lie group containing the subgroup $K = \text{SL}_2(\mathbb{R}) \times \text{SO}(2)$. Furthermore, let $g = \mathfrak{k} \oplus \mathfrak{m}$ be an orthonormal decomposition of the Lie algebra $g$ of $G$ such that $\mathfrak{k} = \mathfrak{sl}_2(\mathbb{R}) \times \mathfrak{so}(2)$ and let \{A, B, C, T, X, Y\} be an orthonormal basis for this Lie algebra such that the elements $A, B$ and $C$ generate $\mathfrak{sl}_2(\mathbb{R})$ and $T$ generates $\mathfrak{so}(2)$. Then the foliation $F$, tangent to the vertical distribution $V$ induced by $\mathfrak{k}$, is conformal if and only if

\[x_1 = y_2 \quad \text{and} \quad x_2 \varepsilon_X + y_1 \varepsilon_Y = 0.\]

If we also have $y_2 = 0$ then $F$ is semi-Riemannian and conversely.

The proof of this is identical to that of Theorem 7.1.

We now simplify the Lie bracket relations given above in the same way as we did in Proposition 7.2.
Proposition 8.2. Let \((G, g)\) be a six-dimensional semi-Riemannian Lie group containing \(K = \text{SL}_2(\mathbb{R}) \times \text{SO}(2)\) as a subgroup. Let \(g = \mathfrak{t} \oplus \mathfrak{m}\) be an orthogonal decomposition of the Lie algebra of \(G\) such that \(\mathfrak{t} = \text{sl}_2(\mathbb{R}) \times \text{so}(2)\). Furthermore, let \(\{A, B, C, T, X, Y\}\) be an orthonormal basis for \(g\) such that the elements \(A, B, C\) generate \(\text{su}(2)\) and \(T\) generates \(\text{so}(2)\). If the foliation \(\mathcal{F}\), tangent to the vertical distribution \(\mathcal{V}\) induced by \(\mathfrak{t}\), is conformal then the Lie algebra \(g\) of \(G\) satisfies the bracket relations

\[
[A, B] = 2C, \quad [C, A] = 2B, \quad [B, C] = -2A,
\]
\[
[A, X] = b_{11}B + c_{11}C, \quad [A, Y] = b_{21}B + c_{21}C,
\]
\[
[B, X] = b_{11}A - c_{12}C, \quad [B, Y] = b_{21}A - c_{22}C,
\]
\[
[C, X] = c_{11}A + c_{12}B, \quad [C, Y] = c_{21}A + c_{22}B,
\]
\[
[T, X] = y_2X + y_1Y - \frac{1}{2}((y_2c_{12} + y_1c_{22})A
\]
\[
+ (y_2c_{11} + y_1c_{21})B - (y_2b_{11} + y_1b_{21})C) + t_4T,
\]
\[
[T, Y] = -y_1\varepsilon_X\varepsilon_YX + y_2Y + \frac{1}{2}((y_1\varepsilon_X\varepsilon_Yc_{12} - y_2c_{22})A
\]
\[
+ (y_1\varepsilon_X\varepsilon_Yc_{11} - y_2c_{21})B - (y_1\varepsilon_X\varepsilon_Yb_{11} - y_2b_{21})C) + t_4T,
\]
\[
[X, Y] = \rho X + \theta_1A + \theta_2B + \theta_3C + \theta_4T,
\]
where \(b_{11}, b_{21}, c_{11}, c_{12}, c_{21}, c_{22}, t_{14}, t_{24}, y_1, y_2, \rho, \theta_4\) are real, arbitrary coefficients and

\[
\begin{pmatrix}
\theta_1 \\
\theta_2 \\
\theta_3
\end{pmatrix}
= \frac{1}{2}
\begin{pmatrix}
-\rho c_{12} - b_{11}c_{21} + b_{21}c_{11} \\
-\rho c_{11} - b_{11}c_{22} + b_{21}c_{12} \\
\rho b_{11} - c_{11}c_{22} + c_{12}c_{21}
\end{pmatrix}.
\]

Theorem 8.3. Let \((G, g)\) be a six-dimensional semi-Riemannian Lie group containing \(K = \text{SL}_2(\mathbb{R}) \times \text{SO}(2)\) as a subgroup and let \(g = \mathfrak{t} \oplus \mathfrak{m}\) be an orthogonal decomposition of the Lie algebra of \(G\) such that \(\mathfrak{t} = \text{sl}_2(\mathbb{R}) \times \text{so}(2)\). Furthermore, let \(\{A, B, C, T, X, Y\}\) be an orthonormal basis for \(g\) such that the vector fields \(A, B, C, T\) generate the Lie subalgebra \(\mathfrak{t}\). If the foliation \(\mathcal{F}\), tangent to the vertical distribution \(\mathcal{V}\) generated by \(\mathfrak{t}\), is conformal then it is minimal if and only if \(t_{14} = t_{24} = 0\).

The proof of this is equivalent to that of Theorem 7.3. Putting these conditions and the conditions for conformality into our Lie algebra we yield a simplified system.

Proposition 8.4. Let \((G, g)\) be as in Proposition 8.2 and let the foliation \(\mathcal{F}\), tangent to the vertical distribution \(\mathcal{V}\) induced by \(\mathfrak{t}\), be conformal and minimal. Then the Lie algebra \(\mathfrak{g}\) of \(G\) satisfies the bracket relations

\[
[A, B] = 2C, \quad [C, A] = 2B, \quad [B, C] = -2A,
\]
\[
[A, X] = b_{11}B + c_{11}C, \quad [A, Y] = b_{21}B + c_{21}C,
\]
\[
[B, X] = b_{11}A - c_{12}C, \quad [B, Y] = b_{21}A - c_{22}C,
\]
\[
[C, X] = c_{11}A + c_{12}B, \quad [C, Y] = c_{21}A + c_{22}B,
\]
\[
[T, X] = y_2X + y_1Y - \frac{1}{2}((y_2c_{12} + y_1c_{22})A
\]
\[
+ (y_2c_{11} + y_1c_{21})B - (y_2b_{11} + y_1b_{21})C),
\]
\[
[T, Y] = -y_1\varepsilon_X\varepsilon_YX + y_2Y + \frac{1}{2}((y_1\varepsilon_X\varepsilon_Yc_{12} - y_2c_{22})A
\]
\[
+ (y_1\varepsilon_X\varepsilon_Yc_{11} - y_2c_{21})B - (y_1\varepsilon_X\varepsilon_Yb_{11} - y_2b_{21})C),
\]
\[ [X, Y] = \rho X + \theta_1 A + \theta_2 B + \theta_3 C + \theta_4 T, \]

where \( b_{11}, b_{21}, c_{11}, c_{12}, c_{21}, c_{22}, \ell_{14}, \ell_{24}, y_1, y_2, \rho, \theta_4 \) are real, arbitrary coefficients and

\[
\begin{pmatrix}
\frac{\theta_1}{2} \\
\frac{\theta_2}{2} \\
\frac{\theta_3}{2}
\end{pmatrix} = 
\begin{pmatrix}
-\rho c_{12} - b_{11} c_{21} + b_{21} c_{11} \\
-\rho c_{11} - b_{11} c_{22} + b_{21} c_{12} \\
\rho b_{11} - c_{11} c_{22} + c_{12} c_{21}
\end{pmatrix}.
\]

As in Chapter 7 the only triple of vector fields that does not satisfy the Jacobi identity at this point is \( (T, X, Y) \). Solving this we obtain the following result.

**Proposition 8.5.** If we assume that \( \rho = 0 \) in Proposition 8.4 then at least one of the following has to hold

(a) \( \theta_4 = 0 \),

(b) \( y_2 = 0 \).

If we instead assume that \( \rho \neq 0 \), then the structure coefficients satisfy

(c) \( y_1 = y_2 = 0 \).

This result and its proof is identical to Proposition 7.5 and its proof. The remarks made in view of Proposition 7.5 also hold in this case.

**Theorem 8.6.** Let \( (G, g) \) be a six-dimensional semi-Riemannian Lie group containing \( K = \text{SL}_2(\mathbb{R}) \times \text{SO}(2) \) as a subgroup. Let \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \) be an orthogonal decomposition of the Lie algebra of \( G \) and let \( \{ A, B, C, T, X, Y \} \) be an orthonormal basis for this algebra such that \( A, B, C, T \) generate \( \mathfrak{k} \). If the foliation \( \mathcal{F} \), tangent to \( K \), is conformal and minimal then it is totally geodesic if and only if \( \mathfrak{g} \) satisfies one of the following conditions

1. \( y_1 = y_2 = 0, \quad -\varepsilon_A = \varepsilon_B = \varepsilon_C, \)

2. \( b_{11} = b_{21} = c_{11} = c_{12} = c_{21} = c_{22} = 0, \)

3. \( c_{11} = c_{12} = c_{21} = c_{22} = y_1 = y_2 = 0, \quad \varepsilon_A = -\varepsilon_B, \)

4. \( b_{11} = b_{21} = c_{12} = c_{22} = y_1 = y_2 = 0, \quad \varepsilon_A = -\varepsilon_C, \)

5. \( b_{11} = b_{21} = c_{11} = c_{21} = y_1 = y_2 = 0, \quad \varepsilon_B = \varepsilon_C, \)

6. \( c_{11} = c_{12} = c_{21} = c_{22} = 0, \quad y_1^2 = y_2^2 \neq 0, \)

\[
b_{11} = -\frac{y_1^2}{y_2} b_{21}, \quad \varepsilon_A = -\varepsilon_B, \quad \varepsilon_X = -\varepsilon_Y,
\]

7. \( b_{11} = b_{21} = c_{12} = c_{22} = 0, \quad y_1^2 = y_2^2 \neq 0, \)

\[
c_{11} = -\frac{y_1^2}{y_2} c_{21}, \quad \varepsilon_A = -\varepsilon_C, \quad \varepsilon_X = -\varepsilon_Y,
\]

8. \( b_{11} = b_{21} = c_{11} = c_{21} = 0, \quad y_1^2 = y_2^2 \neq 0, \)

\[
c_{12} = -\frac{y_1^2}{y_2} c_{22}, \quad \varepsilon_B = \varepsilon_C, \quad \varepsilon_X = -\varepsilon_Y,
\]

9. \( y_1^2 = y_2^2 \neq 0, \quad b_{11} = -\frac{y_1^2}{y_2} b_{21}, \quad c_{11} = -\frac{y_1^2}{y_2} c_{21}, \)

\[
c_{12} = -\frac{y_1^2}{y_2} c_{22}, \quad -\varepsilon_A = \varepsilon_B = \varepsilon_C, \quad \varepsilon_X = -\varepsilon_Y.
\]

The proof of this is obtained by proceeding as in the proof of Theorem 7.7. The observations made in Remark 7.8 continue to hold with some minor modifications.
Appendix A

A Maple Program for $K = \text{SU}(2)$

We present here the Maple program used to check the calculations in Chapter 2.

We begin by importing the modules that contain the objects and some of the procedures we will use.

```maple
with (DifferentialGeometry): with (LieAlgebras):
with (Tensor): with (Tools):
```

After this we set a global environment variable to force the program to show all solutions without having to pass `allsolutions` as an option everytime we use the `solve` command.

```maple
_EEnvAllSolutions := true:
```

Next we set up the structure of the Lie group we are working with, in this case this is a five-dimensional Lie group containing $\text{SU}(2)$ as a subgroup.

```maple
Basis := [A,B,C,X,Y]:
StructureEquations := [
    [A,B] = 2*C,
    [C,A] = 2*B,
    [B,C] = 2*A,
    [A,X] = a[11]*A + a[12]*B + a[13]*C,
    [A,Y] = a[21]*A + a[22]*B + a[23]*C,
    [B,X] = b[11]*A + b[12]*B + b[13]*C,
    [B,Y] = b[21]*A + b[22]*B + b[23]*C,
    [C,X] = c[11]*A + c[12]*B + c[13]*C,
    [C,Y] = c[21]*A + c[22]*B + c[23]*C,
    [X,Y] = rho*X + theta[1]*A + theta[2]*B + theta[3]*C
]:
LAD := LieAlgebraData (StructureEquations, Basis, G):
DGsetup (LAD):```

Next we create the procedures that are use to check if the foliation induced by $K$ is conformal, minimal and totally geodesic. To help us out with this task we first implement some basic procedures that are used within these.

It will at times be useful for us to obtain a representation of a vector field $V$ in terms of the basis elements. However, we cannot make use of the list `Basis` that
we created previously as when we used the LieAlgebraData procedure we obtained new names for the basis elements. We will use the following lines to account for this.

\[
\begin{align*}
N &:= \text{numelems}(\text{Basis}) : \\
\text{NewBasis} &:= [\text{cat}(e,1..N)] :
\end{align*}
\]

Here we have used the \text{numelems} command to get the number of elements in the basis and stored this in the variable \(N\), in this case we clearly know that \(N\) is five but this allows us to reuse the same commands for other groups of arbitrary dimensions. After this we used \text{cat} to concatenate \(e\) and the numbers ranging from 1 to \(N\) and stored these in a list we have chosen to call \text{NewBasis}, in other words we assign the list \([e_1, e_2, e_3, e_4, e_5]\), these are the new names for the basis elements, to the variable \text{NewBasis}.

We now need to create a semi-Riemannian metric \(g\) to equip the group \(G\) with, this is accomplished by creating a metric tensor that we call \(g_1\), after this we create a procedure \(g\) that will be shorthand for the command to calculate the inner product with respect to this tensor.

\[
\begin{align*}
\text{evalDG} &\left( \sum_{i=1}^{N} e[i] \theta_i \circ \theta_i \right) \\
g &:= \text{proc}(X,Y) \text{ option inline:} \\
\text{return TensorInnerProduct}\left(g_1,X,Y\right)
\end{align*}
\]

In the above \(e[Basis[i]]\) and \(\theta_i\) are the causal character and differential one-form corresponding to the \(i\)th basis element, respectively, e.g. \(e[Basis[1]] = e[A] = \varepsilon_A\) and \(\theta_4 = \theta X\). The \text{evalDG} command we are using here is used to evaluate expressions containing objects from the \text{DifferentialGeometry} module.

The next line of code tells Maple that we assume the causal characters to be non-zero.

\[
\text{assuming op(map(x->e[x], Basis) <>~ 0)}:
\]

This will add a tilde to the end of the names of the variables we have put assumptions on. If this is unwanted we can execute the following command:

\[
\text{interface(showassumed=0)}:
\]

Next we implement the Koszul formula:

\[
\begin{align*}
\text{Koszul} &:= \text{proc}(X,Y,Z): \\
&\quad \text{return evalDG(1/2*(g(LieBracket(Z,X),Y) + g(LieBracket(Z,Y),X) + g(Z,LieBracket(X,Y)))):}
\end{align*}
\]

We then use this procedure to calculate the second fundamental forms \(B^\nu\) and \(B^H\) in the procedures \text{SecondFFV} and \text{SecondFFH}, respectively.

\[
\begin{align*}
\text{SecondFFV} &:= \text{proc}(E,F): \\
&\quad \text{return evalDG(1/2*add((Koszul(E,F,e||i)) \cdot e||i, i=N-1..N)):}
\end{align*}
\]

\[
\begin{align*}
\text{SecondFFH} &:= \text{proc}(X,Y): \\
&\quad \text{return evalDG(1/2*add((Koszul(X,Y,e||i)) \cdot e||i, i=1..N-2)):}
\end{align*}
\]


We are now in possession of all the ingredients needed for us to implement the procedures Conformal, Minimal and TotallyGeodesic.

Beginning with the procedure that checks the conditions for conformality we have:

```maple
Conformal := proc(details := true):
local a1, a2, a1test, a2test:

a1 := evalDG(SecondFFH(o(N-1), o(N-1))*e['X']
- SecondFFH(o(N), o(N))*e['Y']):
a2 := evalDG(SecondFFH(o(N-1), o(N))):

try:
a1test := evalb(GetComponents(a1, NewBasis) =
GetComponents(DGzero("vector"), NewBasis)):
catch:
a1test := true:
end try:
try:
a2test := evalb(GetComponents(a2, NewBasis) =
GetComponents(DGzero("vector"), NewBasis)):
catch:
a2test := true:
end try:

if details then:
print(simplify(a1), simplify(a2)):
end if:

if a1test then:
return a2test,
elseif(a2test,

{},
solve(GetComponents(a2, NewBasis),
useassumptions)):
else:
return false,
elseif(a2test,

{simplify(GetComponents(a1, NewBasis),
useassumptions)},

{simplify({op(GetComponents(a2, NewBasis))},
{op(GetComponents(a1, NewBasis))},
useassumptions)})
end if:
end proc:
```

In this procedure we first assign the value of $\varepsilon_X \cdot B^H(X, X) - \varepsilon_Y \cdot B^H(Y, Y)$ to the local variable `a1`. Similarly, `a2` is assigned the value of $B^H(X, Y)$. These both need to be zero for $F$ to be conformal, thus the next step is to check if this holds. We have to handle this differently depending on how the value gets stored in Maple, however, we will not go into the details of how this works. If $a1 = a2 = 0$ we will return the value of `true` and the empty set `{}`, the reason behind returning the empty set is to handle errors. If we instead have $a1 = 0 \neq a2$ we return false and the set of solutions to the equation $a2 = 0$. The remaining cases, $a1 \neq 0 = a2$ and $a1 \neq 0 \neq a2$, are treated similarly.

Note that this procedure takes one argument `details` which by default is set to `true`, setting this to `false` disables the output of the values of `a1` and `a2`. 

47
Moving on we have the procedure that checks if $F$ is minimal.

```plaintext
Minimal := proc():
    local a := evalDG(add(SecondFFV(e||i,e||i),i=1..N-2)):
    return a,
        ifelse(a=0,
            {},
            {solve(GetComponents(a, NewBasis),
                useassumptions)}):
end proc:
```

This procedure returns the value of trace $B^V$ and the set of solutions to the equation trace $B^V = 0$, if $F$ is minimal this will be the empty set.

The final procedure we create is the one that finds when $F$ is totally geodesic.

```plaintext
TotallyGeodesic := proc(details := true):
    local i, j, System := []:
    for i from 1 to (N-2) do:
        for j from i to (N-2) do:
            local a := evalDG(1/2*(SecondFFV(NewBasis[i], NewBasis[j])
                +SecondFFV(NewBasis[j], NewBasis[i]))):
                if details then:
                    print(cat("B^V(", Basis[i], ",", Basis[j], ") = ",
                        simplify(a)));
                end if:
                if a <> 0 then:
                    System := [op(System),op(GetComponents(a, NewBasis))]:
                end if:
        end do:
    end do:
    return {solve(System, useassumptions)}:
end proc:
```

Just like the Conformal procedure this takes an argument details which controls the amount of information the user receives. Since $B^V$ has to be identically zero for $F$ to be totally geodesic we see that the major difference in this procedure to the previous two is that we have to solve a system of equations.

Note that from this point forward the exact process of how to proceed will differ from group to group and some experimentation may be needed. What we present here works for the semisimple groups that are covered in Chapters 2 to 6.

What remains now is to run the calculations. We begin with finding the structure coefficients for which the Jacobi identity is satisfied. This is done by running the Query procedure from the LieAlgebras module with the set of variables we wish to solve for and the string "Jacobi" passed as arguments to the procedure.

```plaintext
TF, EQ, SOLN, LA := Query(
    {a[11], a[12], a[13],
     a[21], a[22], a[23],
     b[11], b[12], b[13],
     b[21], b[22], b[23],
     c[11], c[12], c[13],
     c[21], c[22], c[23],
     rho, theta[1], theta[2], theta[3]},
    "Jacobi"):
```
The variable TF is assigned the value true if it is able to find a solution and false otherwise, EQ is the system of equations that are being solved, SOLN is the solution or solutions to the system EQ and LA contains the data for the Lie algebras corresponding to the solutions.

Our next step is to use the information we obtained from this query, specifically we make use of SOLN.

```cpp
if numelems(SOLN) = 1 then:
    print(SOLN);
    assign(SOLN):
else:
    CommonCoefficients := SOLN[1];
    for i in SOLN do:
        CommonCoefficients := CommonCoefficients intersect i;
    end do;
    print("Found more than one solution, assigning only the common coefficients.");
    print(CommonCoefficients);
    assign(CommonCoefficients):
end if;
```

If the solution is unique, which it is in this case, we assign the new values to the structure coefficients. If there are multiple solutions we instead find what all these solutions have in common and assign these common values to the structure coefficients, e.g. all solutions might have the condition \( \rho = 0 \) and we thus assign \( \rho \) the value of zero. In the second case we also inform the user of this fact.

Finally we use the procedures we previously wrote to check under what conditions \( \mathcal{F} \) is conformal, minimal and totally geodesic.

```cpp
Conformal();
Minimal();
TotallyGeodesic();
```

If we are interested in how the Lie bracket relations look at the end we run the command

```cpp
MultiplicationTable();
```
Bibliography

[1] P. Baird, J. Eells, A conservation law for harmonic maps, Geometry Symposium Utrecht 1980, Lecture Notes in Math. 894, Springer (1981), 1-25.

[2] P. Baird, J. C. Wood, Bernstein theorems for harmonic morphisms from $\mathbb{R}^3$ and $S^3$, Math. Ann. 280 (1988), 579-603.

[3] P. Baird, J. C. Wood, Harmonic morphisms between Riemannian manifolds, London Math. Soc. Monogr. (N.S.) 29, Oxford Univ. Press (2003).

[4] B. Fuglede, Harmonic morphisms between Riemannian manifolds, Ann. Inst. Fourier 28 (1978), 107-144.

[5] B. Fuglede, Harmonic morphisms between semi-Riemannian manifolds, Ann. Acad. Sci. Fenn. Math. 21 (1996), 31-50.

[6] E. Ghandour, S. Gudmundsson, T. Turner, Conformal foliations on Lie groups and complex-valued harmonic morphisms, J. Geom. Phys. 159 (2021). https://doi.org/10.1016/j.geomphys.2020.103940

[7] S. Gudmundsson, An Introduction to Riemannian Geometry, Lecture Notes in Mathematics, University of Lund (2020). http://www.matematik.lu.se/matematiklu/personal/sigma/Riemann.pdf

[8] S. Gudmundsson, Harmonic morphisms from five-dimensional Lie groups, Geom. Dedicata 184 (2016), 143-157.

[9] S. Gudmundsson, On the existence of harmonic morphisms from symmetric spaces of rank one, Manuscripta Math. 93 (1997), 421-433.

[10] S. Gudmundsson, M. Svensson, Harmonic morphisms from four-dimensional Lie groups, J. Geom. Phys. 83 (2014), 1-11.

[11] T. Ishihara, A mapping of Riemannian manifolds which preserves harmonic functions, J. Math. Kyoto Univ. 19 (1979), 215-229.

[12] C. G. J. Jacobi, Über eine particuläre Lösung der partiellen Differentialgleichung $\frac{\partial^2(V)}{\partial y^2} + \frac{\partial^2(V)}{\partial z^2} = 0$, J. Reine Angew. Math. 36 (1848), 113-134.

[13] B. O’Neill, Semi-Riemannian geometry, Academic Press (1983).

[14] V. Ottosson, Conformal foliations on Lie groups producing harmonic morphisms, Bachelor’s thesis, Lund University (2019). http://www.matematik.lu.se/matematiklu/personal/sigma/students/Victor-Ottosson-BSc.pdf

[15] T. B. Turner, Minimal and conformal foliations of codimension two on Riemannian Lie groups, Master’s thesis, Lund University (2020). http://www.matematik.lu.se/matematiklu/personal/sigma/students/Thomas-Turner-MSc.pdf
