A $q$-ANALOGUE OF AN IDENTITY OF N.WALLACH

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1. Let $n$ be an integer $\geq 2$. For $i \in [1, n-1]$ let $s_i$ be the transposition $(i,i+1)$ in the group $S_n$ of permutations of $[1,n]$. (Given two integers $a, b$ we denote by $[a,b]$ the set of all integers $c$ such that $a \leq c \leq b$.) Consider the following element of $\mathbb{C}[S_n]$ (the group algebra of $S_n$):

$$t = s_1s_2 \ldots s_{n-1} + s_2s_3 \ldots s_{n-1} + \cdots + s_{n-2}s_{n-1} + s_{n-1} + 1.$$ (The sum of an $n$-cycle, an $n-1$-cycle, ..., a 2-cycle and the identity.) Wallach [W] proved the remarkable identity

$$(a) \quad t \prod_{k \in [1,n], k \neq n-1} (t - k) = 0$$

in $\mathbb{C}[S_n]$ and used it to establish a vanishing result for some Lie algebra cohomologies. In particular, left multiplication by $t$ in $\mathbb{C}[S_n]$ has eigenvalues in \{0, 1, 2, \ldots, n-2, n\}. A closely related result appeared later in connection with a problem concerning shuffling of cards in Diaconis, Fill and Pitman [DFP] and also in Phatarfod [Ph].

Let $q$ be an indeterminate. Let $H$ be the $\mathbb{Z}[q]$-algebra with generators $T_1, T_2, \ldots, T_{n-1}$ and relations $(T_i + 1)(T_i - q) = 0$ for $i \in [1, n-1]$, $T_iT_{i+1}T_i = T_{i+1}T iT_{i+1}$ for $i \in [1, n-2]$, $T_iT_j = T_jT_i$ for $i \neq j$ in $[1, n-1]$, a Hecke algebra of type $A_{n-1}$. Set

$$\tau = T_1T_2 \ldots T_{n-1} + T_2T_3 \ldots T_{n-1} + \cdots + T_{n-2}T_{n-1} + T_{n-1} + 1 \in H.$$ Under the specialization $q = 1$, $\tau$ becomes the element $t$ of $\mathbb{C}[S_n]$. Our main result is the following $q$-analogue of (a):

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Proposition 2. The following equality in $H$ holds:

$$\tau \prod_{k \in [1,n], k \neq n-1} (\tau - 1 - q - q^2 - \cdots - q^{k-1}) = 0.$$ 

The proof will be given in Section 4. The proof of the Proposition is a generalization of the proof of 1(a) given in [GW]. However, there is a new difficulty due to the fact that the product of two standard basis elements of $H$ is not a standard basis element (as for $S_n$) but a complicated linear combination of basis elements. To overcome this difficulty we will work in a model of $H$ as a space of functions on a product of two flag manifolds over a finite field.

Let $V$ be a vector space of dimension $n$ over a finite field $\mathbb{F}_q$ of cardinal $q$. Let $\mathcal{F}$ be the set of complete flags

$$V_\ast = (V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_n)$$

in $V$ where $V_k$ is a subspace of $V$ of dimension $k$ for $k \in [0, n]$. Now $GL(V)$ acts on $\mathcal{F}$ by

$$g : V_\ast \mapsto gV_\ast = (gV_0 \subset gV_1 \subset gV_2 \subset \ldots \subset gV_n)$$

and on $\mathcal{F} \times \mathcal{F}$ by $g : (V_\ast, V'_\ast) \mapsto (gV_\ast, gV'_\ast)$. Let $\mathcal{H}$ be the $\mathbb{C}$-vector space of all functions $f : \mathcal{F} \times \mathcal{F} \to \mathbb{C}$ that are constant on the orbits of $GL(V)$. This is an associative algebra with multiplication

$$f, f' \mapsto f \cdot f', \quad (f \cdot f')(W_\ast, V'_\ast) = \sum_{V'_\ast \in \mathcal{F}} f(W_\ast, V'_\ast)f'(V'_\ast, V_\ast).$$

Define $f_1 \in \mathcal{H}$ by

$$f_1(W_\ast, V'_\ast) = 1$$

if there exists $g \in [1, n]$ (necessarily unique) with $W_r = V'_r$ for $r \in [1, g-1]$, $V'_r \neq W_r \subset V'_{r+1}$ for $r \in [g, n-1]$;

$$f_1(W_\ast, V'_\ast) = 0,$$

otherwise.

For $t \in [0, n]$ and any sequence $1 \leq i_1 < i_2 < \cdots < i_{n-t} \leq n$ let $X_t^{i_1, i_2, \ldots, i_{n-t}}$ be the set of all pairs $(V'_\ast, V_\ast) \in \mathcal{F} \times \mathcal{F}$ such that $V'_r \subset V_{i_r}$, $V'_r \not\subset V_{i_{r-1}}$ for $r \in [1, n-t]$.

For $t \in [0, n]$ let $X_t = \bigcup X_t^{i_1, i_2, \ldots, i_{n-t}} \subset \mathcal{F} \times \mathcal{F}$ where the union is taken over all sequences $1 \leq i_1 < i_2 < \cdots < i_{n-t} \leq n$. Clearly, this union is disjoint and $X_0 \subset X_1 \subset X_2 \subset \ldots \subset X_n = \mathcal{F} \times \mathcal{F}$. Also, $X_0$ is the diagonal in $\mathcal{F} \times \mathcal{F}$. Define $f_t \in \mathcal{H}$ by

$$f_t(V'_\ast, V_\ast) = 1$$

if $(V'_\ast, V_\ast) \in X_t$, $f_t(V'_\ast, V_\ast) = 0$, otherwise.

For $t = 1$ this agrees with the earlier definition of $f_1$. Note that $f_0$ is the unit element of $\mathcal{H}$. The following result is a $q$-analogue of a result in [DFP].

Lemma 3. For $t \in [1, n-1]$ we have $f_1 \cdot f_t = (1 + q + q^2 + \cdots + q^{t-1})f_t + q^t f_{t+1}$.

Let $f = f_1 \cdot f_t$. From the definitions we have $f = \sum_{g=1}^n \phi_g$ where $\phi_g \in \mathcal{H}$ is defined as follows: for $(W_\ast, V_\ast) \in \mathcal{F} \times \mathcal{F}$, $\phi_g(W_\ast, V_\ast)$ is the number of $V'_\ast \in \mathcal{F}$ such that
\( V'_r = W_r \) for \( r \in [1, g - 1] \), \( V'_r \neq W_r \subset V'_{r+1} \) for \( r \in [g, n - 1] \) and there exists \( 1 \leq i_1 < i_2 < \cdots < i_{n-t} \leq n \) with \( V'_r \subset V_{i_r}, V'_r \nsubseteq V_{i_r-1} \) for \( r \in [1, n-t] \).

Here \( V'_r \) is uniquely determined for \( r \in [1, g - 1] \) (we have \( V'_r = W_r \)) while for \( r \in [g + 1, n - 1] \), \( V'_r \) is equal to \( V'_g + W_{r-1} \) (this follows by induction from \( V'_r = V'_{r-1} + W_{r-1} \) which holds since \( V'_{r-1}, W_{r-1} \) must be distinct hyperplanes of \( V'_r \)). Hence \( \phi_g(W_*, V_*) \) is the cardinal of the set \( Y_g \) consisting of all \( g \)-dimensional subspaces \( V'_g \) of \( V \) such that

\[
W_{g-1} \subset V'_g,
\]
\[
V'_g + W_{r-1} \neq W_r \text{ for } r \in [g, n - 1] \text{ (or equivalently } V'_g \nsubseteq W_{n-1} \text{),}
\]

and there exists \( 1 \leq i_1 < i_2 < \cdots < i_{n-t} \leq n \) (necessarily unique) with

\[
W_r \subset V_{i_r}, W_r \nsubseteq V_{i_r-1} \text{ if } r \in [1, n-t] \cap [1, g - 1],
\]
\[
V'_r \subset V_{i_r}, V'_r \nsubseteq V_{i_r-1} \text{ if } g \in [1, n-t],
\]
\[
V'_g + W_{r-1} \subset V_{i_r}, V'_g + W_{r-1} \nsubseteq V_{i_r-1} \text{ if } r \in [1, n-t] \cap [g + 1, n - 1].
\]

Assume first that \( g \in [1, n-t] \). If a \( V'_g \in Y_g \) exists and if \( 1 \leq i_1 < i_2 < \cdots < i_{n-t} \leq n \) is as above then, setting \( j_r = i_r \) for \( r \in [1, g - 1] \) and \( j_r = i_{r+1} \) for \( r \in [g, n-t-1] \), we have \( 1 \leq j_1 < j_2 < \cdots < j_{n-t-1} \leq n \) and

(a) \( W_r \subset V_{j_r}, W_r \nsubseteq V_{j_{r-1}} \) for \( r \in [1, n-t-1] \).

(For \( r \in [1, g - 1] \) this is clear. Assume now that \( r \in [g, n-t-1] \). Since \( V'_r + W_r \subset V_{j_r} \), we have \( W_r \subset V_{j_r} \). If \( W_r \subset V_{j_r-1} \) then, since \( V'_g \subset V_{i_g} \subset V_{j_r-1} \) and \( j_r = i_{r+1} \), we would have \( V'_g + W_r \subset V_{r+1-1} \), contradiction.)

We see that \( \phi_g(W_*, V_*) = 0 \) if \( (W_*, V_*) \notin X_{t+1} \). We now assume that \( (W_*, V_*) \in X_{t+1} \). Let \( 1 \leq j_1 < j_2 < \cdots < j_{n-t-1} \leq n \) be such that (a) holds. Then \( \phi_g(W_*, V_*) \) is the number of \( g \)-dimensional subspaces \( V'_g \) of \( V \) such that

(b) \( W_{g-1} \subset V'_g \nsubseteq W_{n-1} \),

and

(c) if \( g = 1 \leq n - t - 1 \) then \( V'_g \subset V_i, V'_g \nsubseteq V_{i-1} \) for some \( i \) with \( 1 \leq i < j_g \);

(d) if \( g \in [2, n-t-1] \) then \( V'_g \subset V_i, V'_g \nsubseteq V_{i-1} \) for some \( i \) with \( j_g_1 < i < j_g \);

(e) if \( g = n - t \geq 2 \) then \( V'_g \subset V_i, V'_g \nsubseteq V_{i-1} \) for some \( i \) with \( j_g-1 < i < n \).

Now conditions (c),(d),(e) can be replaced by:

(e') if \( g = 1 \leq n - t - 1 \) then \( V'_g \subset V_{j_g}, V'_g \nsubseteq V_{j_g-1} \);

(d') if \( g \in [2, n-t-1] \) then \( V'_g \subset V_{j_g}, V'_g \nsubseteq V_{j_g-1} \);

(e') if \( g = n - t \geq 2 \) then \( V'_g \nsubseteq V_{j_g-1} \).

Setting \( L = V'_g/W_{g-1} \) we see that \( \phi_g(W_*, V_*) \) is the number of lines \( L \) in \( V/W_{g-1} \) such that \( L \nsubseteq W_{n-1}/W_{g-1} \) and

- if \( g = 1 \leq n - t - 1 \) then \( L \subset V_{j_g-1}/W_{g-1} \);
- if \( g \in [2, n-t-1] \) then \( L \subset V_{j_g-1}/W_{g-1}, L \nsubseteq V_{j_g-1}/W_{g-1} \);
- if \( g = n - t \geq 2 \) then \( L \nsubseteq V_{j_g-1}/W_{g-1} \).

Since \( W_{n-1}/W_{g-1} \) is a hyperplane in \( V/W_{g-1} \), we see that \( \phi_g(W_*, V_*) \) is given by:

\[
(q^{j_g-g} - q^{j_g-g-1})/(q-1) = q^{j_g-g-1} \text{ if } g = 1 \leq n - t - 1 \text{ and } V_{j_g-1} \nsubseteq W_{n-1},
\]

\[
0 \text{ if } g = 1 \leq n - t - 1 \text{ and } V_{j_g-1} \subset W_{n-1},
\]

\[
(q^{j_g-g} - q^{j_g-g-1} - q^{j_g-1-g+1} + q^{j_g-1-g})/(q-1) = q^{j_g-g-1} - q^{j_g-1-g} \text{ if } g \in [2, n-t-1] \text{ and } V_{j_g-1} \nsubseteq W_{n-1}.
\]
we see that

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Hence

\(\sum_{g ∈ [n−t+1, n]} \phi_g(W_1, V_1) = 1 + q + q^2 + \cdots + q^{t−1}\).

Summarizing, we see that for \((W_1, V_1) ∈ F × F\),

\(f(W_1, V_1) = \sum_{g=1}^n \phi_g(W_1, V_1)\)

is equal to

\[1 + q + q^2 + \cdots + q^t\] if \((W_1, V_1) ∈ X_t\),

\[q^t\] if \((W_1, V_1) ∈ X_{t+1} - X_t\),

\[0\] if \((W_1, V_1) \not∈ X_{t+1}\).

The lemma follows immediately.

4. We show that

(a) \(q^{1+2+\cdots+(t−1)} f_t = f_1 (f_1 - 1) * (f_1 - 1 - q) * \cdots * (f_1 - 1 - q - q^2 - \cdots - q^{t−2})\)

for \(t ∈ [1, n−1]\) by induction on \(t\). For \(t = 1\) this is clear. Assume that \(t ∈ [2, n−1]\)

and that (a) holds when \(t\) is replaced by \(t − 1\). Using Lemma 3 we have

\(q^{t−1} f_t = (f_1 - 1 - q - q^2 - \cdots - q^{t−2}) * f_{t−1}\). Using this and the induction hypothesis we have

\[q^{1+2+\cdots+(t−1)} f_t = (f_1 - 1 - q - q^2 - \cdots - q^{t−2}) * f_1 * (f_1 - 1) * \]

\[\cdots * (f_1 - 1 - q) * \cdots * (f_1 - 1 - q - q^2 - \cdots - q^{t−3})\].
This proves (a).

Next we note that \( X_{n-1} \) is the set of all \( (V'_i, V_i) \in \mathcal{F} \times \mathcal{F} \) such that for some \( i \in [1, n] \) we have \( V'_i \subset V_i, V'_i \not\subset V_{i-1} \). Thus, \( X_{n-1} = \mathcal{F} \times \mathcal{F} = X_n \) so that \( f_{n-1} = f_n \). Using this and Lemma 3 we see that \( f_1 * f_{n-1} = (1 + q + q^2 + \cdots + q^{n-1})f_{n-1} \) that is \( (f_1 - 1 - q - q^2 - \cdots - q^{n-1})f_{n-1} = 0 \). Hence multiplying both sides of (a) (for \( t = n - 1 \)) by \( (f_1 - 1 - q - q^2 - \cdots - q^{n-1}) \) we obtain

\[
f_1 * (f_1 - 1) * (f_1 - 1 - q) * \cdots *
\]

\[
* (f_1 - 1 - q - q^2 - \cdots - q^{n-3}) * (f_1 - 1 - q - q^2 - \cdots - q^{n-1}) = 0.
\]

Thus an identity like that in Proposition 2 holds in \( \mathcal{H} \) instead of \( H \) (with \( f_1, q \) instead of \( \tau, q \)). It is known that the algebra \( \mathcal{H} \) may be identified with \( \mathbb{C} \otimes_{\mathbb{Z}[q]} H \) (where \( \mathbb{C} \) is regarded as a \( \mathbb{Z}[q] \)-algebra via the specialization \( q \mapsto q \)) in such a way that \( 1 \otimes \tau \) is identified with \( f_1 \). Since \( q \) can take infinitely many values, the identity in Proposition 2 follows.

5. Setting \( f_t = 0 \) for \( t > n \), we see that the identity in Lemma 3 remains valid for any \( t \geq 0 \). We see that subspace of \( \mathcal{H} \) spanned by \( \{ f_t; t \geq 0 \} \) coincides with the subspace spanned by \( \{ f'_t; t \geq 0 \} \); in particular it is a commutative subring.

6. Consider the endomorphism of \( \mathbb{Q}(q) \otimes_{\mathbb{Z}[q]} H \) given by left multiplication by \( \tau \). Proposition 2 shows that the eigenvalues of this endomorphism are in

\[
\{ 0, 1, 1 + q, 1 + q + q^2, \ldots, 1 + q + \cdots + q^{n-3}, 1 + q + \cdots + q^{n-1} \}.
\]

The multiplicity of the eigenvalue \( 1 + q + \cdots + q^{k-1} \) is preserved by the specialization \( q = 1 \) hence it is the same as the multiplicity of the eigenvalue \( k \) for the left multiplication by \( t \) on \( C[S_n] \), which by [DFP] is the number of permutations of \( [1, n] \) with exactly \( k \) fixed points.

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