The bottom-charmed meson spectrum from a QCD approach
based on Tamm–Dancoff Approximation

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Abstract

The bottom-charmed meson spectrum is studied in this work via an effective version of the
Coulomb gauge QCD Hamiltonian. The Tamm-Dancoff approximation is employed to esti-
mate the energies of the low-lying and radial-excited $B_c$ states with quantum numbers $J^P =
0^-, 0^+, 1^-, 1^+, 2^+, 2^-$. In particular, we analyze the effects of incorporating an effective transverse
hyperfine interaction and spin mixing. The Regge trajectories and hyperfine splitting of both $S$-
and $P$-wave states are also examined. The numerical results are compared with available experimental data and theoretical predictions of other models.
I. INTRODUCTION

Despite the enormous experimental developments on heavy-hadron physics in recent decades, bottom-charmed \((B_c)\) spectroscopy remains much less known than the charmonium and bottomonium sectors. The reason comes from the fact that since the \(B_c\) is a quarkonium bound-state consisting of heavy-quarks with different flavors \((c\bar{b}\) or \(b\bar{c})\), the production mechanism demands factories of \(\sigma\) and \(b\bar{b}\) pairs, which results in a small production rate. On the other hand, the different quark-flavor content denies its annihilation into gluons, engendering uniquely weak decays for the pseudoscalar ground-state \(B_c(1S)\) and hadronic or radiative transitions for excited states which are below the strong-decay \(BD\) threshold. These aspects suggest that \(B_c\)-states are more stable than their analogs in charmonium and bottomonium families, and therefore are pretty valuable to study heavy-quark dynamics and understand the dynamics of the strong interaction in a deeper level.

The first observation of \(B_c\) meson was performed by CDF Collaboration more than two decades ago \([1]\), with the detection of the pseudoscalar ground-state \(B_c(1S)^+\). It was confirmed later by other Collaborations \([2, 3]\), and is the only state considered as an established particle with recognized quantum numbers, according to Particle Data Group (PDG) \([4]\), with mass \((6274.9 \pm 0.8)\) MeV. The other state present in PDG with mass \((6871.0 \pm 1.7)\) MeV and identified as \(B_c(2S)^+\) has its quantum numbers not confirmed. This is due to the controversy raised by the results from the ATLAS \([5]\), CMS \([6]\) and LHCb \([7]\) Collaborations. ATLAS \([5]\) reported the mass \((6842 \pm 4 \pm 5)\) MeV of an observed state consistent with a first radially excited pseudoscalar; while very recently CMS \([6]\) and LHCb \([7]\) detected two signals consistent with the \(2^1S_0\) and \(2^3S_1\): for the \(2^1S_0\) the LHCb and CMS found the mass being respectively \((6872.1 \pm 1.3 \pm 0.1 \pm 0.8)\) MeV and \((6871.0 \pm 1.2 \pm 0.8 \pm 0.8)\) MeV. Besides, for the \(2^3S_1\), LHCb obtained the mass \((6841 \pm 0.6 \pm 0.1 \pm 0.8)\) MeV, whereas CMS observed the mass difference \(m[B_c(2^1S_0)] - m[B_c^*(2^3S_1)] = (29 \pm 1.5 \pm 0.7)\) MeV. Thus, it can be remarked two intriguing features from these reports. The first one is the apparent disagreement between the ATLAS and CMS, LHCb outcomes for the \(B_c^*(2^3S_1)\) meson. One possible explanation is that the peak observed by ATLAS could be the superposition of the \(B_c(2^1S_0)\) and \(B_c^*(2^3S_1)\) states, quite narrowly spaced with respect to the resolution of the measurement. The second one is that the \(B_c(2^1S_0)\) state emerges as heavier than the mass \(B_c^*(2^3S_1)\), which is in conflict with theoretical estimations. The plausible justification
is that the observed $B^*_c(2S)$ peak has a mass lower than the true value, which remains unknown due to the impossibility of reconstruction of the low-energy photon emitted in the $B_{c}^{*+} \rightarrow B_{c}^{+} \gamma [6]$. Hence, more observations on $B_c$-meson family are expected in the nearest possible future in order to get a detailed characterization of heavy meson spectroscopy.

On theoretical grounds, different perspectives have been consecrated to investigate the $B_c$ meson spectrum as well as to understand its properties. For example, it can be found studies in the context of non-relativistic quark models [8–15], relativistic constituent quark models [16–21], Quantum Chromodynamics (QCD) sum rules [22, 23], lattice QCD [24–26] and Dyson-Schwinger and the Bethe-Salpeter equations approaches [27]. The point here is that this miscellany of distinct approaches produces a frame to be contrasted with available and future experimental results, which in the end makes possible a compelling comprehension of the $B_c$ phenomenology.

That being so, the present study intends to contribute to the discussion and characterization of the $B_c$ meson spectrum, by employing a different formalism with respect to the preceding analyses mentioned in the previous paragraph. The framework to be utilized is also known as Coulomb gauge QCD model [28–45]. This formulation is based on the exact QCD Hamiltonian in the Coulomb gauge, which is replaced by an effective Hamiltonian where the original non-perturbative confining and hyperfine interactions can be rearranged into calculable effective potentials between color densities as well as currents. The current quark and gluon field operators are dressed via Bogoliubov-Valatin method. This provides the possibility of using relativistic field theory and many-body techniques such as Tamm-Dancoff and Random Phase approximations. The vacuum is represented as a coherent BCS ground state with quark and gluon Cooper pairs (condensates), and the hadrons interpreted as quasiparticle excitations. This approach has been successfully applied to the description of properties of some types of light and heavy mesons, glueballs, glueumps, hybrids and tetraquarks [28–45]. So, these reports demonstrate that this model is efficient in retrieving the essential aspects of QCD with a minimal number of free parameters (current quark masses and dynamical constants) and yielding reasonable predictions.

Here we extend the range of applications of the Coulomb gauge QCD model by studying the basic features of $B_c$ mesons within an unified scheme. The interactions between quarks and antiquarks will be treated through an improved confining potential and a transverse hyperfine interaction, whose kernel is a Yukawa–type potential. Estimations for the energies of
the low-lying and radial-excited $B_c$ states with quantum numbers $J^P = 0^-, 0^+, 1^-, 1^+, 2^+, 2^-$ are obtained. Also, the Regge trajectories are constructed, and a discussion about the hyperfine splittings of the $S$- and $P$-wave spectroscopy is done. The comparison of our results with other works is performed as well.

The paper is organized as follows. In Section II we present the Coulomb gauge QCD model within Tamm-Dancoff approximation. Section III is devoted to show and analyze the numerical calculations of the bottom-charmed meson spectrum, the Regge trajectories and the hyperfine splittings. Concluding remarks are in Section IV. In Appendix we present explicitly the $B_c$ meson spin-orbital wave functions and kernels of the TDA equation of motion used.

II. THE MODEL

Let us start by introducing the formalism to be used in the analysis of the $B_c$ meson spectrum. It is a Coulomb gauge QCD-inspired model, whose effective Hamiltonian is given by [30–36, 39, 40, 44, 45],

$$ H_{\text{eff}} = \int d\mathbf{x} \Psi^\dagger (\mathbf{x}) \left[ -i \mathbf{\alpha} \cdot \nabla + \beta m \right] \Psi (\mathbf{x}) + H_C + H_T, $$

where $\Psi$ and $m$ are the current quark field and mass, respectively. The terms $H_C$ and $H_T$ are the effective couplings associated to the Coulomb and quark hyperfine interactions, i.e.

$$ H_C = -\frac{1}{2} \int d\mathbf{x} dy \rho^a (\mathbf{x}) \hat{V} (|\mathbf{x} - \mathbf{y}|) \rho^a (\mathbf{y}), $$

$$ H_T = \frac{1}{2} \int d\mathbf{x} dy J^a_i (\mathbf{x}) \hat{U}_{ij} (\mathbf{x}, \mathbf{y}) J^a_j (\mathbf{y}), $$

where $\rho^a (\mathbf{x}) = \Psi^\dagger (\mathbf{x}) T^a \Psi (\mathbf{x})$ are the color densities and $J^a (\mathbf{x}) = \Psi^\dagger (\mathbf{x}) \mathbf{\alpha} T^a \Psi (\mathbf{x})$ the quark color currents, with $T^a (a = 1, 2, \ldots, 8)$ being the $SU_c(3)$ generators. In the equations above the flavor indices are not explicitly displayed to simplify the notation. Also, it should be mentioned that pure gluonic contributions have been excluded due to the fact that our interest is devoted to the $q\bar{q}$ states.

We write down below the kernels of the effective couplings in Eq. (2) used in the calculations. For the Coulomb longitudinal interaction $H_C$, the kernel is assumed to be an
improved confining potential based on Yang-Mills dynamics, which in momentum space is represented as [34],
\[ V(p) = \begin{cases} 
-12.25 \frac{m_g^{1.93}}{p^{3.93}}, & p < m_g, \\
8.07 \ln \left( \frac{p^2}{m_g^2} + 0.82 \right)^{-0.62}, & p > m_g,
\end{cases} \]

where \( m_g \) is a parameter. Although we are not directly dealing with dynamical gluons in our model, we interpret them as responsible for \( V(p) \), obtained from a self-consistent method of the nonabelian degrees of freedom in the presence of static quarks, as noticed by the authors of Ref. [31]. Viewed in this way, \( m_g \) can be interpreted as a dynamical mass scale for the constituent gluons.

Turning to the term \( H_T \), it is associated to the quark hyperfine interaction of type \( \vec{\alpha} \cdot \vec{\alpha} \) from the second-order coupling between quarks and transverse gluons after integrating out gluonic degrees of freedom. In this sense, the effective transverse hyperfine potential carries the kernel \( \hat{U}_{ij} \) which keeps the structure of transverse gauge condition,
\[ \hat{U}_{ij}(\mathbf{x}, \mathbf{y}) = \left( \delta_{ij} - \nabla_i \nabla_j \right) \hat{U}(|\mathbf{x} - \mathbf{y}|), \]

with \( \hat{U} \) being chosen to mimic one-gluon exchange potential. Following the analysis done in Ref. [34], in which a Yukawa-type potential appears as the preferred one for reasonable meson descriptions, we choose
\[ U(p) = C_h \begin{cases} 
(-24.57) \frac{1}{p^2 + m_g^2}, & p < m_g, \\
-8.07 \ln \left( \frac{p^2}{m_g^2} + 0.82 \right)^{-0.62}, & p > m_g,
\end{cases} \]

with the constant \( C_h \) standing for the global strength, and the factor \((-24.57)\) being determined by matching the high and low momentum ranges at the scale \( m_g \).

Next, we apply an appropriate quark basis in which calculations for meson states are most conveniently made. Following the standard Bogoliubov-Valatin method (see for example Ref. [32]), we perform the Bogoliubov transformation from the current quark basis
to a improved quasiparticle quark basis represented by quasiparticle and antiquasiparticle \( B_{\lambda c}^{(\dagger)}(k), D_{\lambda c}^{(\dagger)}(k) \) operators, which allows us to write the quark field as

\[
\Psi(x) = \sum_{\lambda i} \int \frac{d^3k}{(2\pi)^3} \left[ U_{\lambda}(k) B_{\lambda}(k) + V_{\lambda}(k) D_{\lambda}^{(\dagger)}(-k) \right] e^{i k \cdot \hat{x} \epsilon_i},
\]

where \( \lambda \) and \( i \) denote the helicity and color indices \((i = 1, 2, 3)\), respectively; \( \{\hat{\epsilon}_c\} \) is the color vector basis; \( U \) and \( V \) are Dirac spinors forming a complete basis,

\[
U_{\lambda}(k) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1 + \sin \phi(k)} \chi_{\lambda} \\ \sqrt{1 - \sin \phi(k)} \sigma \cdot \mathbf{k} \chi_{\lambda} \end{pmatrix},
\]

\[
V_{\lambda}(k) = \frac{1}{\sqrt{2}} \begin{pmatrix} -\sqrt{1 - \sin \phi(k)} \sigma \cdot \mathbf{k} i\sigma_2 \chi_{\lambda} \\ \sqrt{1 + \sin \phi(k)} i\sigma_2 \chi_{\lambda} \end{pmatrix},
\]

with \( \chi_{\lambda} \) being the Pauli spinors.

The Bogoliubov angle \( \phi(|k|) \equiv \phi_k \) connecting the current and quasiparticle quark bases is obtained by the variational minimization of the quasiparticle vacuum energy \( \delta \langle \Omega | H | \Omega \rangle = 0 \), yielding the gap equation

\[
k s_k - m c_k = \int_0^\infty \frac{q^2}{6\pi^2} \left[ s_k c_q (V_1 + 2W_0) - s_q c_k (V_0 + U_0) \right],
\]

where the functions \( s_k \equiv \sin \phi_k \) and \( c_k \equiv \cos \phi_k \) are related to the running quark mass \( M(k) \) through the relationship \( M(k) = k \tan \phi_k \). We identify \( M(k) \to m \) at high \( k \), while at low \( k \) the constituent quark mass is extracted, \( M(0) \to M \). The functions \( V_0, V_1 \) and \( U_0 \) denote angular integrals of longitudinal and transverse potentials in the form

\[
F_n(k, q) \equiv \int_{-1}^1 dx \ x^n \ F(|k - q|),
\]

with \( x = \hat{k} \cdot \hat{q} \); and the \( W \)-function is defined as

\[
W(|k - q|) \equiv U(|k - q|) \frac{x(k^2 + q^2) - s_k c_q (V_0 + 2U_0) - c_k s_q (V_1 + W_0)}{|k - q|^2}.
\]

Also, the expectation value of the effective Hamiltonian with respect to the one-quasiparticle state \( |q\rangle \equiv B_{\lambda c}^{\dagger}(k)|\Omega\rangle \) engenders the expression that can be identified as the self-energy of the quasiparticle,

\[
\epsilon_k = \langle q | H_{eff} | q \rangle
\]

\[
= ms_k + kc_k - \int_0^\infty \frac{q^2}{6\pi^2} \left[ s_k s_q (V_0 + 2U_0) + c_k c_q (V_1 + W_0) \right].
\]
It must be observed that a meson in this framework is supposed to be an excited state consisting of a bound state of the quasiparticle and antiquasiparticle. Then, it is useful to introduce the meson creation operator in the TDA scheme, which is a bosonization method that has been revealed to be a good approximation for a large number of meson families, excluding only the case of the pions. Accordingly, the quasiparticle–antiquasiparticle operator is given by

\[ Q^\dagger_{nJP} = \sum_{\lambda\lambda'} \int \frac{d^3k}{(2\pi)^3} \Psi^{(nJP)}_{\lambda\lambda'}(k) B^\dagger_{\lambda}(k) D^\dagger_{\lambda'}(-k), \]

where \( \Psi^{(nJP)}_{\alpha\beta} \) means the wavefunction corresponding to an open-flavor meson state with total angular momentum \( J \), parity \( P \) and radial quantum number \( n \) (we have omitted the color and flavor indices).

Now the method of calculating the energy levels of mesonic bound states can be expressed. The energies are obtained via the TDA equation of motion for an open-flavor meson, defined by

\[ \langle \Psi^{(nJP)} | [H_{\text{eff}}, Q^\dagger_{nJP}] | \Omega \rangle = (E_{nJP} - E_0) \langle \Psi^{(nJP)} | Q^\dagger_{nJP} | \Omega \rangle. \]

This equation can be recast into a more convenient form, by profiting from the rotational invariance of \( H_{\text{eff}} \) and constructing the wavefunctions via multiplication of Pauli \( \sigma \) matrices by powers of orbital momentum \( \hat{k}^l \) to get partial waves. Concerning this last procedure, we indicate to the reader the Appendix A of Ref. [44], in which the specific case of axial mesons is discussed. Notwithstanding, for completeness we express in detail the wavefunctions exploited in this work, which can be written as (again omitting the flavor indices):

\[ \Psi^{(nJP)}_{\lambda\lambda'}(k) = \frac{S_{ij}^{(color)}}{\sqrt{3}} R^{(nJP)}(k) \psi^{(JP)}_{\lambda\lambda'}(k), \]

where \( R^{(nJP)}(k) \) is the radial wavefunction; \( \psi^{(JP)}_{\lambda\lambda'}(k) \) carries the angular-momentum dependence, and assumes a distinct form according to the nature of the meson state described by the quantum numbers \( L, S, J \), which specify the parity \( P = (-1)^{L+1} \) and also the charge conjugation \( C = (-1)^{L+S} \), if the quark and antiquark have the opposite flavor (equal mass). These wavefunctions are given explicitly in Appendix A. After that, we perform the diagonalization of the effective Hamiltonian in the TDA representation, which is undertaken by the computation of the trace of spinor products coming from commutators in the left-hand...
side of Eq. (14). The final expression for the TDA equation of motion is

\[ M_{nJP} R^{(nJP)}(k) = (\epsilon_k^b + \epsilon_k^c) R^{(nJP)}(k) + \int_0^{\infty} \frac{q^2 dq}{12\pi^2} K^{(JP)}(k,q) R^{(nJP)}(q), \]  

(16)

where \( M_{nJP} \equiv E_{nJP} - E_0 \) is the energy of the \( B_c \) meson state; \( \epsilon_k^b(\epsilon_k^c) \) is the self-energy of the (anti)quasiparticle associated to the \( b(c) \) quark; and \( K^{(JP)}(k,q) \) is the kernel bearing the potential terms, which is dependent on the meson quantum numbers. We should remark that several versions of kernels are accessible in literature, written using different basis as well as distinct interaction terms. Until now, the tensor cases with both longitudinal and transverse potentials, however, are not available (at least to our knowledge). In view of these considerations, the relevant kernels obtained for the mesons described by the wavefunctions given by Eqs. (A1)-(A8) are expressed in Appendix B (Eqs. (B2)-(B8)).

As a final comment in this Section, we must note that open–flavor mesons, like the \( B_c \) mesons, are not eigenstates of charge conjugation, since they have the quark and antiquark with different flavor. Therefore, the total spin \( (S) \) is no longer a good quantum number, and spin-singlet and spin-triplet states with \( J = L \) can mix. This is the case of axial \( (n^3P_1 \text{ and } n^1P_1) \) and pseudotensor \( (n^3D_2 \text{ and } n^1D_2) \) states reported above. A simple mixing prescription for these \( J = L \) states is:

\[
|nL'_{L}⟩ = \cos \theta_{nL}|n^1L_L⟩ + \sin \theta_{nL}|n^3L_L⟩,
\]

\[
|nL_{L}⟩ = -\sin \theta_{nL}|n^1L_L⟩ + \cos \theta_{nL}|n^3L_L⟩,
\]

(17)

where \( \theta_{nL} \) is the mixing angle and \( nL'_{L}, nL_{L} \) are the physical states. Supposing that the masses of \( b \) and \( c \)-quarks satisfy the limit \( m_b \gg m_c \), this leads to the extreme heavy–light expression: \( \theta_{nL} \rightarrow \tan^{-1}\sqrt{L/(L + 1)} \), giving \( \theta_{nJ} \rightarrow 35.3^o, \theta_{nD} \rightarrow 39.2^o \). Here we adopt the following relation between the masses of \( (n^3L_L - n^1L_L) \) and \( (nL'_{L} - nL_{L}) \) pairs [46],

\[
M(nL_L) = M(n^1L_L) \cos^2 \theta_{nL} + M(n^3L_L) \sin^2 \theta_{nL} - [M(n^3L_L) - M(n^1L_L)] \frac{\sin^2 \theta_{nL}}{2 \cos \theta_{nL}},
\]

\[
M(nL'_{L}) = M(n^1L_L) \sin^2 \theta_{nL} + M(n^3L_L) \cos^2 \theta_{nL} + [M(n^3L_L) - M(n^1L_L)] \frac{\sin^2 \theta_{nL}}{2 \cos \theta_{nL}},
\]

(18)
III. NUMERICAL RESULTS

In this Section are exhibited the results for the spectrum of the $B_c$ mesons, generated with the model sketched out above. Briefly, the strategy consists in solving the gap equation (Eq. (9)) for each flavor, in order to get the $k$-dependent gap angles $\phi_b^k$ and $\phi_c^k$; they supply the values of functions $c_{k(q)}^{b(c)}$ and $s_{k(q)}^{b(c)}$ to generate $M_{n,JP}$ that solve numerically the TDA equation of motion in Eq. (16). It should be emphasized the adoption in the calculations of kernels with interactions represented by an improved confining potential and a transverse Yukawa-type potential playing the role of the exchange of a constituent gluon.

In the Coulomb gauge QCD model the input parameters to be fitted to the experimental data are the dynamical mass of the constituent gluon $m_g$, the current quark masses of the $b$ and $c$ quarks, $m_b$ and $m_c$, and the magnitude of the transverse potential $C_h$. However, as discussed in the Introduction, data for the $B_c$-meson families are scarce at present, despite recent results from the ATLAS [5], CMS [6] and LHCb [7] Collaborations. According to PDG [4], until now there are two $B_c$ mesons observed: the ground pseudoscalar state is the only one considered as an established particle, with mass $M[B_c(1S)^+] \approx (6274.9 \pm 0.8)$ MeV; the other one with mass $(6871.0 \pm 1.7)$ MeV is consistent with a first radially excited pseudoscalar, but quantum numbers are not confirmed. Nevertheless, it should be also mentioned that ATLAS and LHCb Collaborations reported the observations of peaks at $(6842 \pm 4 \pm 5)$ MeV and $(6841 \pm 0.6 \pm 1 \pm 0.8)$ MeV, respectively, which are consistent with the $B_c^*(2S_1)$. Remarking that the goal here is to extract the basic picture of the $B_c$ meson spectrum, the values of the parameters $(m_b, m_c, m_g, C_h)$ are adjusted to reproduce approximately these reported states, in particular the confirmed $B_c(1S)^+$.

We start by showing in Table I the values of constituent quark masses $M_{b,c}$ engendered by the current quark masses $m_b = 4000$ MeV, $m_c = 950$ MeV used as inputs in this subsection. The remaining parameters $C_h$ and $m_g$ are taken with different but near values in order to evaluate their impact on the constituent quark masses $M_{b,c}$, extracted from the limit $M_{b,c}(k \to 0) \equiv M_{b,c}$. They are chosen obviously keeping in mind the range that better matches the physical states. It can be seen that the growth of $C_h$ and $m_g$ yields greater values of $M_{b,c}$, because of the modification of the gap angles coming from solutions of the gap equation. We stress that the values of current and constituent quark masses are smaller than in some quark models, due to the contributions from interaction potentials
in gap equation (9) and in the self-energy $\epsilon^{b,c}_k$ (Eq. (12)). For a detailed discussion we refer the reader to Refs. [34, 44]. On this regard, it deserves to be cited that very recent ($n_f = 2 + 1 + 1$) - lattice QCD calculations obtained estimations for the charm quark mass by about 980-995 MeV [47, 48], which are close to the one we utilize.

TABLE I: The constituent quark masses $M_{b,c}$ engendered by the current quark masses $m_b = 4000$ MeV, $m_c = 950$ MeV used as inputs in this subsection. $M_{b,c}$ are obtained from the gap angles $\phi^b_k$ and $\phi^c_k$ that solve the gap equation (Eq. (9)), through the relationship $\lim_{k \to 0} M_{b,c}(k) = \lim_{k \to 0} k \tan \phi^{b,c}_k \equiv M_{b,c}$. The column “Set” denotes the set of parameters $(m_g, C_h)$ used. All quantities are given in MeV, except the value of $C_h$, which is adimensional.

| Set $(m_g, C_h)$ | $M_c$ | $M_b$ |
|------------------|-------|-------|
| I (600, 0.4)     | 1208  | 4343  |
| II (650, 0.4)    | 1222  | 4362  |
| III (700, 0.4)   | 1236  | 4380  |
| IV (700, 0.5)    | 1288  | 4452  |
| Other            | 1000-1600 | 4600-5100 |

For the sake of completeness, we briefly discuss the overall momentum-dependence of the Bogoliubov angles for the different flavors obtained from the solutions of the gap equation (Eq. (9)). To this end, in Fig. I the solutions $\phi^b_k$ and $\phi^c_k$ are plotted as a function of $k$. At higher values of $k$, the solutions exhibit a decreasing exponential behavior, with the c-flavor case experiencing a faster lessening. Particularly, the obtention of $\phi^{b,c}_k \to 0$ in the limit $k \to \infty$ implies the finiteness of the vacuum energy. At small values of $k$, the solutions present a linear behavior with a negative slope and a sharp peak, yielding $\phi^{b,c}_k \to \pi/2$ at $k \to 0$, which also assures the finite-energy density of the vacuum. Although the specific curves for $\phi^{b,c}_k$ obviously depend on the potentials and parameters considered, the point to be stressed is that this formalism yields well-behaved solutions of the gap equation that will be used as inputs in the obtention of the meson spectrum.
FIG. 1: Bogoliubov angles $\phi^b_k$ and $\phi^c_k$, obtained from the solutions of the gap equation (Eq. (9)), as a function of the modulus of the momentum ($k$). We have used the current quark masses $m_b = 4000$ MeV, $m_c = 950$ MeV and the set III for the parameters ($m_g, C_h$), in conformity with Table I.

A. Mass Spectrum

Now we report our predictions for the energy levels for the $B_c$ states, extracted from the numerical solutions of the TDA equation in Eq. (16) considering the different quantum numbers. In Table II are listed the computed masses for ground and radially excited states of $b\bar{c}$ considering the different sets of input parameters of Table I. It gives an overall view of the behavior of computed masses as the parameters $C_h$ and $m_g$ change. In the region of parameter space considered, the augmentation of constituent gluon mass by 100 MeV increases the estimates by about 100-200 MeV, as well as the strengthening of magnitude of transverse potential by 0.1 yields greater masses by about 100-150 MeV.
TABLE II: TDA masses of lowest-lying and radially excited $B_c$ states, obtained for $m_c = 950$ MeV and $m_b = 4000$ MeV. The column “Set” denotes the set of parameters $(m_g, C_h)$ used in conformity with Table I. The masses are given in MeV. Our calculated masses are rounded to 1 MeV. The mixing angles used are: $\theta_{1P} - \theta_{5P} = 35.3^o, \theta_{1D} = 42.5^o, \theta_{2D} = 42.2^o, \theta_{3D} = 33.2^o, \theta_{4D} = 21.1^o, \theta_{5D} = 5.2^o$. The results that better fit to the observed states are in boldface.

| State ($J^P$) | Set | $E_1$ | $E_2$ | $E_3$ | $E_4$ | $E_5$ |
|---------------|-----|-------|-------|-------|-------|-------|
| $0^-$         | I   | 6146  | 6619  | 6986  | 7298  | 7573  |
|               | II  | 6212  | 6733  | 7136  | 7477  | 7779  |
|               | III | **6277** | **6845** | 7284  | 7656  | 7983  |
|               | IV  | 6417  | 6977  | 7411  | 7779  | 8103  |
| $0^+$         | I   | 6449  | 6852  | 7187  | 7478  | 7740  |
|               | II  | 6545  | 6989  | 7356  | 7675  | 7961  |
|               | III | 6639  | 7123  | 7523  | 7871  | 8181  |
|               | IV  | 6786  | 7264  | 7659  | 8002  | 8309  |
| $1^-$         | I   | 6154  | 6625  | 6990  | 7302  | 7576  |
|               | II  | 6222  | 6739  | 7141  | 7482  | 7782  |
|               | III | 6288  | **6853** | 7290  | 7661  | 7988  |
|               | IV  | 6431  | 6986  | 7418  | 7785  | 8108  |
| $1^+$         | I   | 6423  | 6821  | 7153  | 7443  | 7703  |
|               | II  | 6516  | 6955  | 7318  | 7635  | 7920  |
|               | III | 6606  | 7088  | 7488  | 7836  | 8148  |
|               | IV  | 6744  | 7216  | 7608  | 7950  | 8254  |
| $1^{++}$      | I   | 6456  | 6845  | 7171  | 7458  | 7715  |
|               | II  | 6552  | 6979  | 7339  | 7653  | 7935  |
|               | III | 6.656 | 7121  | 7513  | 7856  | 8164  |
|               | IV  | 6783  | 7243  | 7629  | 7968  | 8272  |
On experimental grounds, the set of parameters III seems to generate findings that better fit to the observed states. Although fine tuning of the parameters can give even better outcomes, we believe that set III seems sufficient to generate findings in good conformity with observed states. The spectrum generated for this set is shown schematically in Fig. 2.

It is also noteworthy to evaluate our predictions in light of other works existent in the literature. To this end, in Table III our calculated $B_c$ masses with the set of parameters III are compared with other theoretical results and available experimental data. Stated explicitly, the theoretical frameworks employed in these other studies are: relativized constituent quark model with the presence of a linear confining potential and a color Coulomb interaction [16]; constituent quark model in heavy quark symmetry limit with scalar confining and vector Coulomb potentials [17]; non-relativistic quark model (NRQM) consisting of a confinement potential and one gluon exchange potential [9]; nonrelativistic linear potential model with a spin-dependent interaction [13].
FIG. 2: $B_c$ spectrum generated for the set of parameters III.

TABLE III: Last Column: TDA masses of lowest-lying and radially excited $B_c$ states obtained for the set of parameters III in Table I. Other columns: predictions of other works existent in the literature and available experimental data.

The $B_c(1^1S_0)$ is the only established particle and its mass has been taken from PDG [4]. The states observed by ATLAS [5], CMS [6] and LHCb [7] are not considered as well-established by PDG; quantum numbers of the so-called $2^1S_0$ need to be confirmed. The masses are given in MeV.

| State       | $J^P$ | Ref. [16] | Ref. [17] | Ref. [9] | Ref. [11] | Ref. [13] | Exp. Data    | Our Results |
|-------------|-------|-----------|-----------|----------|-----------|-----------|--------------|-------------|
| $B_c(1^3S_1)$ | $1^-$ | 6338      | 6340      | 6357     | 6314      | 6326      | $\cdots$    | 6288        |
| $B_c(1^1S_0)$ | $0^-$ | 6271      | 6260      | 6275     | 6274      | 6271      | 6275 (PDG)  | 6277        |
| $B_c(2^3S_1)$ | $1^-$ | 6887      | 6900      | 8697     | 8655      | 6890      | 6842$^1$; 6841$^2$ | 6853        |
We stress that our findings, specially for $B_c(1S)$ and $B_c^*(2S)$, present a very good fit with the measured data when the experimental errors are bore in mind. But this comparison must be done with care, because the observed $B_c^*(2S)$ peak has a mass lower than the true value, which remains unknown due to the impossibility of reconstruction of the low-energy photon emitted in the $B_c^+ \rightarrow B_c^+ \gamma$, as pointed out in Ref. [6]. Moreover, the mass of the first radial excitation $B_c(2S)$ is heavier than the ground state $B_c(1S)$ by about 557 MeV, which is fairly good in light of experimental observations, keeping the fact that $B_c(2S)$ is not yet well-established according to PDG [4].
Furthermore, it can be seen that our outcomes get the $B_c$ spectrum in reasonable concor-
dance with other potential model predictions. In general, the masses predicted by us for the
low-lying states have a difference with respect to previous works ranging from a few MeV up
to tens of MeV. The exceptions having larger mass deviations are the $1^3P_2, 1P_1, 1P_1'$ states.
For higher mass states, bigger discrepancies among the predictions are evident, but most of
our results are between the lower and upper values reported in Table III. Particularly, our
results for $1S, 2S, 3S, 2P$–wave states are up to a few tens of MeV discrepant with those
with relativized constituent quark model from Ref. [16], while for $1P_2, 1P_1^{(0)}, 1D$–wave states
are about $100 - 140$ MeV smaller.

B. Regge Trajectories

In addition, the energy levels listed in Tables II and III allow us to obtain the mass
relation between the ground states and their radial and angular excited states, and therefore
construct the Regge trajectories in the $(n, M^2)$ and $(J, M^2)$ planes. They are then plotted
in Fig. 3. In these plots, we assume that the Regge slopes are independent of charge
conjugation, in accordance with the $C$-invariance of QCD [49], and also that the slopes
of the parity partner trajectories coincide.

It can be remarked that the behavior of squared masses with radial quantum number
and $J^P$ (top and middle panels) is not exactly linear. This fact is clearly pronounced in
$(n, M^2)$ plane, due to the high excitation number. More precisely, the daughter trajectories
(incorporating both radially and orbitally excited states) manifest extrapolations closer to
a linear fit. However, the parent trajectories (beginning from the ground states) reveal a
nonlinear nature, mostly in the region of smaller mass. This is in qualitative accordance
with other works that investigated heavy quarkonia states; see for instance Refs. [49–53].
Notwithstanding, using the linear approximation for the Regge trajectories through the
laws [50, 54–56],

\[
M^2(J) = \alpha_0 + J \alpha, \\
M^2(n) = \beta_0 + n \beta, \tag{19}
\]

where $\alpha_0, \beta_0$ are the intercepts and $\alpha, \beta$ the slopes of each corresponding trajectory on
which the meson lies. Now applying this hypothesis in our scenario we can extract the
FIG. 3: Top Panel: Parent and Regge trajectories in \((J, M^2)\) plane for \(B_c\) states with unnatural parity \(P = (-1)^J\), for \(n = 1, \ldots, 5\) (from bottom to top). Middle Panel: Regge trajectories in \((n, M^2)\) plane for \(S\)-wave vector, \(P\)-wave vector and \(D\)-wave tensor \(B_c\) states. Bottom Panel: Nonlinear trajectories in \((J, M^2)\) plane starting from vector, pseudoscalar and scalar \(B_c\) states (from bottom to top), with lines indicating the regions of \(1S, 1P, 1D\) states. Circles represent the predicted masses shown in Table III taking the values of the set of parameters III.
parameters $\alpha_0, \beta_0, \alpha, \beta$ from the linear fits displayed in the top and middle panels of Fig. 3. The estimated values are listed in Table IV. These results are in reasonable accordance with the existing literature, when compared for example with Ref. [50].

**TABLE IV:** Fitted parameters for the linear fit in Eq. (19) of parent and daughter Regge trajectories in the top and middle panels of Fig. 3. The quantities are given in GeV$^2$.

| (J, $M^2$) plane | Trajectory | $\alpha_0$ | $\alpha$ |
|------------------|------------|------------|----------|
| Parent           | 39.620     | 4.243      |
| 1st Daughter     | 46.956     | 3.548      |
| 2nd Daughter     | 53.137     | 3.147      |
| 3rd Daughter     | 58.689     | 2.885      |
| 4th Daughter     | 63.801     | 2.705      |

| (n, $M^2$) plane |
|------------------|
| State            | $\beta_0$ | $\beta$ |
| 1$^-$             | 34.349    | 6.027   |
| 1$^+$'            | 39.253    | 5.570   |
| 2$^-$             | 43.060    | 5.294   |

C. Hyperfine splittings

Additionally, another relevant feature to be noticed is the hyperfine splitting of $B_c$ states. We start with $S$-wave states. The hyperfine splittings $\Delta_{nS}^{HFS}$ are listed in Table V for the set of parameters III chosen as well as other sets in order to see the influence of their change. Despite the small values obtained for these sets, as expected these splittings decrease for higher excited states, and get larger as the parameter $C_h$ grows. Our calculations yield the mass for $B_c(2S)$ heavier than for $B_c^*(2S)$, which coincides with the other theoretical expectations. Nonetheless, our estimations engenders smaller $\Delta_{2S}^{HFS} = M(2^3S_1) - M(2^1S_0)$ hyperfine splitting, as already remarked in Table V, but not too different from the finding

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TABLE V: The hyperfine splittings for the $B_c$ system states, obtained for the set of parameters in Table II. We use the respective definition for the hyperfine splitting: $\Delta_{nS}^{HFS} = M(n^3S_1) - M(n^1S_0)$. The set used in Table III is in boldface. The quantities are given in MeV.

| Set | $\Delta_{1S}^{HFS}$ | $\Delta_{2S}^{HFS}$ | $\Delta_{3S}^{HFS}$ | $\Delta_{4S}^{HFS}$ | $\Delta_{5S}^{HFS}$ |
|-----|---------------------|---------------------|---------------------|---------------------|---------------------|
| I   | 8                   | 6                   | 4                   | 4                   | 3                   |
| II  | 10                  | 6                   | 5                   | 5                   | 3                   |
| III | 11                  | 8                   | 6                   | 5                   | 5                   |
| IV  | 14                  | 9                   | 7                   | 6                   | 5                   |

Now we devote our attention to the hyperfine splitting of the $P$-wave states. For a systematic discussion of the hyperfine splitting for $P$-wave states in the context of hidden-flavor quarkonia, see for example Ref. [57]. In the case of bottom-charmed quarkonia, let us follow as motivation the discussion done in Ref. [14]. Experimentally, it can be remarked that the spin-singlet $P$-wave states almost coincide with the spin-averaged centroid of the triplet [4] for $c\bar{c}$ and $b\bar{b}$ systems, yielding ideally

$$E(n^1P_1) = \frac{1}{9} \left[ 5E(n^3P_2) + 3E(n^3P_1) + E(n^3P_0) \right].$$  \hfill (20)

Since for the $B_c$ system the $C$-parity is no longer a good quantum number, the states $n^3P_1 - n^1P_1$ can mix according to Eq. (17), and therefore Eq. (20) cannot be directly used. Nevertheless, assuming that the relation $E(n^3P_1) \approx E(n^1P_1)$ holds for $B_c$ mesons, then Eq. (20) gives [58]

$$E(n^3P_0) + 5E(n^3P_2) = 3 \left[ E(nP_1) + E(nP'_1) \right].$$  \hfill (21)

In order to test the validity of the relation above, the ratio among the masses can be introduced [14]:

$$r = \frac{E(n^3P_0) + 5E(n^3P_2)}{3 \left[ E(nP_1) + E(nP'_1) \right]}.$$  \hfill (22)

in Ref. [13].
So, the deviation from \( r = 1 \) accounts for how much Eq. (21) is being violated. In this way, the estimations of \( r \) for our calculations reported above are listed in Table VI. We see that the relative theoretical errors for the lowest-lying and radially-excited states are below 0.5%, being even smaller for the excited states. Thus, the relation in Eq. (21) holds to a fair precision. Another test that can be done is that when we augment the strength of the transverse potential, which is associated to the hyperfine interaction within the formalism used, the ratio \( r \) increases (see for example the calculated masses for the set IV in Table II). This suggests that any deviation from \( r = 1 \) depends on the hyperfine interaction, in consonance with the conclusions of Ref. [14].

| \( n \) | 1   | 2   | 3   | 4   | 5   |
|-------|-----|-----|-----|-----|-----|
| \( r \) | 1.0047 | 1.0031 | 1.0021 | 1.0014 | 1.0008 |

As a final remark, this effective approach with a small number of parameters allows us construct the general aspects of the bottom-charmed spectrum. Our predicted energy levels are in fair agreement with other predictions, providing a guide to the experimental search for the unobserved \( B_c \) mesons.

**IV. CONCLUDING REMARKS**

The purpose of this work has been the investigation of the \( B_c \) meson spectrum by employing a different formalism with respect to the preceding analyses. The framework employed has been an effective version of the Coulomb gauge QCD and many-body techniques associated to the Tamm-Dancoff approximation (TDA). The interactions between quarks (quasiparticles) and antiquarks (antiquasiparticles) have been given by the sum of an improved confining potential and a transverse hyperfine interaction with an Yukawa-type kernel, being interpreted as the exchange of a constituent gluon.

Making use of a small number of parameters (dynamical mass of a constituent gluon \( m_g \), current quark masses \( m_b, m_c \), and the magnitude of the transverse potential \( C_h \)), this approach has allowed us to analyze the basic features of \( B_c \) mesons. The calculated masses
have been optimized in order to fit them to the observed states by means of tuning of the parameters. Besides, the estimations of expected but yet-unobserved states are approximately in accordance with other findings in the literature using distinct formalisms. In particular, our calculations yield the mass for $B_c(2S)$ lighter than for $B^*_c(2S)$, which coincides with the other theoretical expectations, but not with the CMS and LHCb results at the present moment. One expects that future findings will show that the true $B^*_c(2S)$ peak must be at a higher mass as the photon emitted in the $B^+_c \rightarrow B^+_c \gamma$ radiative transition can be reconstructed.

Another aspect regarded has been the mass relation between the ground states and their radial excited states in the $(n, M^2)$ and $(J, M^2)$ planes. The nonlinearity is more pronounced in the $(n, M^2)$ plane, due to the high excitation number used. The parent trajectories (beginning from the ground states) reveal a nonlinear nature more evident than the daughter trajectories (incorporating both radially and orbitally excited states), which is in qualitative accordance with other works exploring quarkonia states and mesons.

Further, the hyperfine splitting of both $S$ and $P$-wave states has been studied. In both cases, we found that these splittings decrease for higher excited states but become larger as the parameter $C_h$ grows, as expected, since it drives the strength of the term associated to the hyperfine interaction. In the case of $P$-wave states, the mass relation in Eq. (21) has relative theoretical errors for the lowest-lying and radially-excited states lower than 0.5%. So, this framework engenders a reasonable precision for this mass relation involving the $P$-wave states.

Hence, we believe that this effective approach with a minimal number of parameters is capable of offering the general aspects of the bottom-charmed spectrum, in fair agreement with other predictions. At last, all these works provide a guide to the experimental search for the unobserved $B_c$ mesons. Some obvious extensions that deserve future studies are calculations and predictions on radiative and strong transitions and also on hybrid meson spectrum.

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Appendix A: Wave functions of TDA equation of motion

In this Appendix we present explicitly the $B_s$ meson spin-orbital wave functions, in terms of $L, S, J$ (helicity indices are not displayed):

- **pseudoscalar** ($L = 0; S = 0; J = 0$),
  
  \[ \psi(0^-) = \sqrt{\frac{1}{4\pi}} \frac{i\sigma^2}{\sqrt{2}}. \]  
  \[ (A1) \]

- **scalar** ($L = 1; S = 1; J = 0$),

  \[ \psi(0^+) = -\sqrt{\frac{1}{4\pi}} \left( \sigma \cdot \hat{k} \right) \frac{i\sigma^2}{\sqrt{2}}, \]  
  \[ (A2) \]

- **vector** ($L = 0; S = 1; J = 1$),

  \[ \psi(1^-) = \sqrt{\frac{3}{4\pi}} \sigma \frac{i\sigma^2}{\sqrt{2}}, \]  
  \[ (A3) \]

- **axial** ($L = 1; S = 0; J = 1$),

  \[ \psi(1^+) = i\sqrt{\frac{3}{4\pi}} \hat{k} \frac{i\sigma^2}{\sqrt{2}}, \]  
  \[ (A4) \]

- **axial** ($L = 1; S = 1; J = 1$),

  \[ \psi(1^+)' = -i\sqrt{\frac{3}{8\pi}} \left( \sigma \times \hat{k} \right) \frac{i\sigma^2}{\sqrt{2}}, \]  
  \[ (A5) \]

- **Tensor** ($L = 1; S = 1; J = 2$),

  \[ \psi(2^+) = \sqrt{\frac{3}{4\pi}} \sigma \hat{k} \frac{i\sigma^2}{\sqrt{2}}, \]  
  \[ (A6) \]

- **Pseudotensor** ($L = 2; S = 0; J = 2$),

  \[ \psi(2^-) = \sqrt{\frac{5}{4\pi}} \hat{k} \frac{i\sigma^2}{\sqrt{2}}, \]  
  \[ (A7) \]

- **Pseudotensor** ($L = 2; S = 1; J = 2$),

  \[ \psi(2^-)' = -\sqrt{\frac{5}{4\pi}} \left( \sigma \times \hat{k} \right) \hat{k} \frac{i\sigma^2}{\sqrt{2}}, \]  
  \[ (A8) \]

The factor $(i\sigma^2)$ is introduced in order to use the same convention of Ref. [44].
Appendix B: Kernels of TDA equation of motion

Here we set out the relevant kernels obtained for the mesons described by the wavefunctions given by Eqs. (A1)-(A8):

- pseudoscalar \((L = 0; S = 0; J = 0)\),
  \[K^{(0^-)} (k, q) = V_1 (a_5 + a_6) + V_0 (a_7 + a_8) + 2U_0 (a_1 + a_2) - 2W_0 (a_3 + a_4)\]; (B1)

- scalar \((L = 1; S = 1; J = 0)\),
  \[K^{(0^+)} (k, q) = V_0 (a_5 + a_6) + V_1 (a_7 + a_8) - 2U_0 (a_3 + a_4) + (U_1 + W_0 - kqZ_0) (a_1 + a_2)\]; (B2)

- vector \((L = 0; S = 1; J = 1)\),
  \[K^{(1^-)} (k, q) = \frac{1}{3} [3V_1 (a_5 + a_6) + a_8 (4V_2 - V_0) + 3a_7V_0 - 2 (a_1 + a_2) U_0 + 2 (a_3 + a_4) U_1 + 2qk (a_3 + a_4) Z_0 + 4 (a_1 k^2 + a_2 q^2) Z_0]\]; (B3)

- axial \((L = 1; S = 0; J = 1)\),
  \[K^{(1^+)} (k, q) = (a_5 + a_6) V_2 + (a_7 + a_8) V_1 + 2 (a_1 + a_2) U_1 - 2 (a_3 + a_4) W_1\]; (B4)

- axial \((L = 1; S = 1; J = 1)\),
  \[K^{(1^+)} (k, q) = \frac{1}{2} (V_0 + V_2) (a_5 + a_6) + \frac{1}{2} (U_0 + U_2 - 2W_1) (a_3 + a_4) + V_1 (a_7 + a_8) + Z_1 (a_1 k^2 + a_2 q^2) + Z_0 \frac{1}{2} (k^2 - q^2) (a_4 - a_3)\]. (B5)

- Tensor \((L = 1; S = 1; J = 2)\),
  \[K^{(2^+)} (k, q) = \frac{1}{2} (3V_2 - V_0) (a_5 + a_6) + V_1 a_7 + \frac{5}{12} (12V_3 - 7V_1) a_8 + \frac{1}{2} (U_1 - 5W_0 - 10kqZ_0) (a_1 + a_2) + \frac{12}{5} Z_1 (a_1 k^2 + a_2 q^2) + \frac{1}{10} (27U_2 - 15W_1 - 9kqZ_1 - 4U_0) (a_3 + a_4)\]; (B6)
• Pseudotensor \((L = 2; S = 0; J = 2)\),

\[
K^{(2-)}(k, q) = \frac{1}{2} (3V_3 - V_1) (a_5 + a_6) + \frac{1}{2} (3V_2 - V_0) (a_7 + a_8) \\
+ (3U_2 - U_0) (a_1 + a_2) + (3W_2 - W_0) (a_3 + a_4);
\]  

\[\text{(B7)}\]

• Pseudotensor \((L = 2; S = 1; J = 2)\),

\[
K^{(2-)}(k, q) = V_3 (a_5 + a_6) + \frac{1}{2} (3V_2 - V_0) (a_7 + a_8) \\
+ kqZ_1 (a_1 + a_2) + \frac{1}{2} (2Z_2 - Z_0) (a_3 + a_4);
\]  

\[\text{(B8)}\]

where the coefficients \(a_i\) are given by

\[
a_1 = \sqrt{1 + s_b^b k} \sqrt{1 + s_c^c q} \sqrt{1 - s_q^b k} \sqrt{1 - s_q^c q},
\]

\[
a_2 = \sqrt{1 - s_b^b k} \sqrt{1 - s_c^c q} \sqrt{1 + s_q^b k} \sqrt{1 + s_q^c q},
\]

\[
a_3 = \sqrt{1 + s_b^b k} \sqrt{1 - s_c^c q} \sqrt{1 - s_q^b k} \sqrt{1 + s_q^c q},
\]

\[
a_4 = \sqrt{1 - s_b^b k} \sqrt{1 + s_c^c q} \sqrt{1 + s_q^b k} \sqrt{1 - s_q^c q},
\]

\[
a_5 = \sqrt{1 + s_b^b k} \sqrt{1 - s_c^c q} \sqrt{1 + s_q^b k} \sqrt{1 - s_q^c q},
\]

\[
a_6 = \sqrt{1 - s_b^b k} \sqrt{1 + s_c^c q} \sqrt{1 - s_q^b k} \sqrt{1 + s_q^c q},
\]

\[
a_7 = \sqrt{1 + s_b^b k} \sqrt{1 + s_c^c q} \sqrt{1 - s_q^b k} \sqrt{1 + s_q^c q},
\]

\[
a_8 = \sqrt{1 - s_b^b k} \sqrt{1 + s_c^c q} \sqrt{1 - s_q^b k} \sqrt{1 + s_q^c q}.
\]  

\[\text{(B9)}\]

The functions \(s_b^{b(q)}\) and \(s_c^{c(q)}\) are dependent of the respective gap angle obtained by solving the gap equation for the \(b\) and \(c\) quarks, respectively. Besides the functions \(V_n, U_n\) and \(W_n\), defined in Eqs. \[10\] and \[11\], the auxiliary \(Z\)-function has been also introduced:

\[
Z(|k - q|) \equiv U(|k - q|) \frac{1 - x^2}{|k - q|^2}.
\]  

\[\text{(B10)}\]

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[58] In Ref. 14 is argued that there should be possible hidden symmetry or an underlying principle that makes this relation valid for the $P$-level of charmonium and bottomonium systems, but also for the case of unequal-flavor case, like the the $B_c$ family.