Abstract. We prove finiteness of the number of smooth blow-downs on Fano manifolds and boundedness results for the geometry of non projective Fano-like manifolds. Our proofs use properness of Hilbert schemes and Mori theory.

Introduction

In this Note, we say that a compact complex manifold \( X \) is a Fano-like manifold if it becomes Fano after a finite sequence of blow-ups along smooth connected centers, i.e if there exist a Fano manifold \( \tilde{X} \) and a finite sequence of blow-ups along smooth connected centers \( \pi : \tilde{X} \to X \). We say that a Fano-like manifold \( X \) is simple if there exists a smooth submanifold \( Y \) of \( X \) (\( Y \) may not be connected) such that the blow-up of \( X \) along \( Y \) is Fano. If \( Z \) is a projective manifold, we call smooth blow-down of \( Z \) (with an \( s \)-dimensional center) a map \( \pi \) and a manifold \( Z' \) such that \( \pi : Z \to Z' \) is the blow-up of \( Z' \) along a smooth connected submanifold (of dimension \( s \)). We say that a smooth blow-down of \( Z \) is projective (resp. non projective) if \( Z' \) is projective (resp. non projective).

It is well-known that any Moishezon manifold becomes projective after a finite sequence of blow-ups along smooth centers. Our aim is to bound the geometry of Moishezon manifolds becoming Fano after one blow-up along a smooth center, i.e the geometry of simple non projective Fano-like manifolds.

Our results in this direction are the following, the simple proof of Theorem 1 has been communicated to us by Daniel Huybrechts.

Theorem 1. Let \( Z \) be a Fano manifold of dimension \( n \). Then, there is only a finite number of smooth blow-downs of \( Z \).

Let us recall here that the assumption \( Z \) Fano is essential: there are projective smooth surfaces with infinitely many \(-1\) rational curves, hence with infinitely many smooth blow-downs.

Since there is only a finite number of deformation types of Fano manifolds of dimension \( n \) (see [KMM92] and also [Deb97] for a recent survey on Fano manifolds) and since smooth blow-downs are stable under deformations [Kod63], we get the following corollary (see section 1 for a detailed proof):

Corollary 1. There is only a finite number of deformation types of simple Fano-like manifolds of dimension \( n \).

The next result is essentially due to Wiśniewski ([Wis91], prop. (3.4) and (3.5)). Before stating it, let us define

\[
\text{d}_n = \max \{ (-K_Z)^n \mid Z \text{ is a Fano manifold of dimension } n \}
\]
and

\[ \rho_n = \max \{ \rho(Z) := \text{rk}(\text{Pic}(Z)/\text{Pic}^0(Z)) \mid Z \text{ is a Fano manifold of dimension } n \} . \]

The number \( \rho_n \) is well defined since there is only a finite number of deformation types of Fano manifolds of dimension \( n \) and we refer to [Deb97] for an explicit bound for \( d_n \).

**Theorem 2.** Let \( X \) be an \( n \)-dimensional simple non projective Fano-like manifold, \( Y \) a smooth submanifold such that the blow-up \( \pi : \tilde{X} \to X \) of \( X \) along \( Y \) is Fano, and \( E \) the exceptional divisor of \( \pi \). Then

(i) if each component of \( Y \) has Picard number equal to one, then each component of \( Y \) has ample conormal bundle in \( X \) and is Fano. Moreover \( \deg_{-K_{\tilde{X}}}(E) \leq (\rho_n - 1)d_{n-1} \).

(ii) if \( Y \) is a curve, then (each component of) \( Y \) is a smooth rational curve with normal bundle \( O_{\mathbb{P}^1}(-1)^{n-1} \).

Finally, we prove here the following result:

**Theorem 3.** Let \( Z \) be a Fano manifold of dimension \( n \) and index \( r \). Suppose there is a non projective smooth blow-down of \( Z \) with an \( s \)-dimensional center. Then

\[ r \leq (n - 1)/2 \text{ and } s \geq r. \]

Moreover,

(i) if \( r > (n - 1)/3 \), then \( s = n - 1 - r \);

(ii) if \( r < (n - 1)/2 \) and \( s = r \), then \( Y \simeq \mathbb{P}^r \).

Recall that the index of a Fano manifold \( Z \) is the largest integer \( m \) such that \( -K_Z = mL \) for \( L \) in the Picard group of \( Z \).

**Remarks.**

a) For a Fano manifold \( X \) of dimension \( n \) and index \( r \) with second Betti number greater than or equal to 2, it is known that \( 2r \leq n + 2 \) [Wi91], with equality if and only if \( X \simeq \mathbb{P}^{r-1} \times \mathbb{P}^{r-1} \).

b) Fano manifolds of even dimension (resp. odd dimension \( n \)) and middle index (resp. index \( (n + 1)/2 \)) with \( b_2 \geq 2 \) have been intensively studied, see for example [Wis93]. Our Theorem 3 shows that there are no non projective smooth blow-down of such a Fano manifold, without using any explicit classification.

c) The assumption that there is a *non projective* smooth blow-down of \( Z \) is essential in Theorem 3: the Fano manifold obtained by blowing-up \( \mathbb{P}^{2r-1} \) along a \( \mathbb{P}^{r-1} \) has index \( r \).

1. **Proof of Theorem 1 and Corollary 1. An example.**

1.1. **Proof of Theorem 1.** Thanks to D. Huybrechts for the following proof.

Let \( Z \) be a Fano manifold and \( \pi : Z \to Z' \) a smooth blow-down of \( Z \) with an \( s \)-dimensional connected center. Let \( f \) be a line contained in a non trivial fiber of \( \pi \). Then, the Hilbert polynomial \( P_{-K_Z}(m) \) := \( \chi(f, m(-K_Z)f) \) is determined by \( s \) and \( n \) since \( -K_Z \cdot f = n-s-1 \) and \( f \) is a smooth rational curve. Since \( -K_Z \) is ample, the Hilbert scheme \( \text{Hilb}_{-K_Z} \) of curves in \( Z \) having \( P_{-K_Z} \) as Hilbert polynomial is a projective scheme, hence has a finite number of irreducible components. Since each curve being in the component \( \mathcal{H} \) of \( \text{Hilb}_{-K_Z} \) containing \( f \) is contracted by \( \pi \), there is only a finite number of smooth blow-downs of \( Z \) with an \( s \)-dimensional center. \[ \blacksquare \]
1.2. Proof of Corollary 1. Let us first recall ([Deb97] section 5.2) that there exists an integer $\delta(n)$ such that every Fano $n$-fold can be realized as a smooth submanifold of $\mathbb{P}^{2n+1}$ of degree at most $\delta(n)$. Let us denote by $T$ a closed irreducible subvariety of the disjoint union of Chow varieties of $n$-dimensional subvarieties of $\mathbb{P}^{2n+1}$ of degree at most $\delta(n)$, and by $\pi : X_T \rightarrow T$ the universal family.

**Step 1 : Stability of smooth blow-downs.** Fix $t_0$ in the smooth locus $T_{\text{smooth}}$ of $T$ and suppose that $X_{t_0} := \pi^{-1}(t_0)$ is a Fano $n$-fold and there exists a smooth blow-down of $X_{t_0}$ (denote by $E_{t_0}$ the exceptional divisor, $P$ its Hilbert polynomial with respect to $O_{\mathbb{P}^{2n+1}}(1)$). Let $S$ be the component of the Hilbert scheme of $(n-1)$-dimensional subschemes of $\mathbb{P}^{2n+1}$ with Hilbert polynomial $P$ and $u : \mathcal{E}_S \rightarrow S$ the universal family. Finally, let $I$ be the following subscheme of $T \times S$:

$$I = \{(t, s) \mid u^{-1}(s) \subset X_t\}$$

and $p : I \rightarrow T$ the proper algebraic map induced by the first projection. Thanks to the analytic stability of smooth blow-downs due to Kodaira (see [Kod63], Theorem 5), the image $p(I)$ contains an analytic open neighbourhood of $t_0$ hence it also contains a Zariski neighbourhood of $t_0$. Moreover, since exceptional divisors are rigid, the fiber $p^{-1}(t)$ is a single point for $t$ in a Zariski neighbourhood of $t_0$. Finally, we get algebraic stability of smooth blow-downs (the $\mathbb{P}^r$-fibered structure of exceptional divisor is also analytically stable - [Kod63], Theorem 4 - hence algebraically stable by the same kind of argument).

**Step 2 : Stratification of $T$ by the number of smooth blow-downs.** For any integer $k \geq 0$, let us define

$$U_k(T) = \{t \in T_{\text{smooth}} \mid X_t \text{ is a Fano manifold and there exists at least } k \text{ smooth blow-downs of } X_t\};$$

and $U_{-1}(T) = T_{\text{smooth}}$. Thanks to Step 1, $U_k(T)$ is Zariski open in $T$, and thanks to Theorem 1,

$$\bigcap_{k \geq -1} U_k(T) = \emptyset.$$ 

Since $\{U_k(T)\}_{k \geq -1}$ is a decreasing sequence of Zariski open sets, by noetherian induction, we get that there exists an integer $k$ such that $U_k(T) = \emptyset$ and we can thus define

$$k(T) := \max\{k \geq -1 \mid U_k(T) \neq \emptyset\}, \quad U(T) := U_{k(T)}(T).$$

Finally, we have proved that $U(T)$ is a non empty Zariski open set of $T_{\text{smooth}}$ such that for every $t \in U(T)$, $Z_t$ is a Fano $n$-fold with exactly $k(T)$ smooth blow-downs ($k(T) = -1$ means that for every $t \in T_{\text{smooth}}$, $X_t$ is not a Fano manifold).

Now let $T_0 = T$, and $T_1$ be any closed irreducible component of $T_0 \setminus U(T_0)$. We get $U(T_1)$ as before and denote by $T_2$ any closed irreducible component of $T_1 \setminus U(T_1)$, and so on. Again by noetherian induction, this process terminates after finitely many steps and we get a finite stratification of $T$ such that each strata corresponds to an algebraic family of Fano $n$-folds with the same number of smooth blow-downs.

**Step 3 : Conclusion.** Since there is only a finite number of irreducible components in the Chow variety of Fano $n$-folds, each being finitely stratified by Step 2, we get a finite number of deformation types of simple Fano-like $n$-folds.

As it has been noticed by Kodaira, it is essential to consider only smooth blow-downs. A $-2$ rational smooth curve on a surface is, in general, not stable under deformations of the surface.

1.3. An example. Before going further, let us recall the following well known example. Let $Z$ be the projective 3-fold obtained by blowing-up $\mathbb{P}^3$ along a smooth curve of type $(3, 3)$ contained in a smooth quadric $\mathcal{Q}$ of $\mathbb{P}^3$. Let $\pi$ denotes the blow-up $Z \rightarrow \mathbb{P}^3$. Then $Z$ is a Fano manifold of index one and there are at least three smooth blow-downs of $Z : \pi$, which is projective, and two non projective smooth blow-downs consisting in contracting...
the strict transform $Q'$ of the quadric $Q$ along one of its two rulings (the normal bundle of $Q'$ in $Z$ is $O(-1,-1)$).

**Lemma 1.** There are exactly three smooth blow-downs of $Z$.

**Proof:** the Mori cone $\text{NE}(Z)$ is a 2-dimensional closed cone, one of its two extremal rays being generated by the class of a line $f_\pi$ contained in a non trivial fiber of $\pi$, the other one, denoted by $[R]$, by the class of one of the two rulings of $Q'$ (the two rulings are numerically equivalent, the corresponding extremal contraction consists in contracting $Q'$ to a singular point in a projective variety, hence is not a smooth blow-down). If $E$ is the exceptional divisor of $\pi$, we have

$$E \cdot [f_\pi] = -1, \ E \cdot [R] = 3, \ Q' \cdot [f_\pi] = 1, \ Q' \cdot [R] = -1.$$ 

Now suppose there exists a smooth blow-down $\tau$ of $Z$ with a 1-dimensional center, which is not one of the three previously described. Let $L$ be a line contained in a non trivial fiber of $\tau$, then since $-K_Z \cdot [L] = 1$, we have $[L] = a[f_\pi] + b[R]$ for some strictly positive numbers such that $a + b = 1$. Since we have moreover

$$Q' \cdot [L] = a - b = 2a - 1 \in Z \text{ and } E \cdot [L] = 3b - a = 3 - 4a,$$

we get $a = b = 1/2$. Therefore $Q' \cdot [L] = 0$ hence $L$ is disjoint from $Q'$ (it can not be contained in $Q'$ since $Q'_{|Q'} = O(-1,-1)$). It implies that there are two smooth blow-downs of $Z$ with disjoint exceptional divisors, which is impossible since $\rho(Z) = 2$.

Finally, if there is a smooth blow-down $\tau : Z \to Z'$ of $Z$ with a 0-dimensional center, then $Z'$ is projective and $\tau$ is a Mori extremal contraction, which is again impossible since we already met the two Mori extremal contractions on $Z$. ■

2. **Non projective smooth blow-downs on a center with Picard number 1.**

**Proof of Theorem 2.**

The proof of Theorem 2 we will give is close to Wişniewski’s one but we give two intermediate results of independant interest.

2.1. **On the normal bundle of the center.** Let us recall that a smooth submanifold $A$ in a complex manifold $W$ is contractible to a point (i.e. there exists a complex space $W'$ and a map $\mu : W \to W'$ which is an isomorphism outside $A$ and such that $\mu(A)$ is a point) if and only if $N_{A/W}^*$ is ample (Grauert’s criterion [Gra62]).

The following proposition was proved by Campana [Cam89] in the case where $Y$ is a curve and $\dim(X) = 3$.

**Proposition 1.** Let $X$ be a non projective manifold, $Y$ a smooth submanifold of $X$ such that the blow-up $\pi : \tilde{X} \to X$ of $X$ along $Y$ is projective. Then, for each connected component $Y'$ of $Y$ with $\rho(Y') = 1$, the conormal bundle $N_{Y'/X}^*$ is ample.

Before the proof, let us remark that $Y$ is projective since the exceptional divisor of $\pi$ is.

**Proof of Proposition 1:** (following Campana) we can suppose that $Y$ is connected.

Let $E$ be the exceptional divisor of $\pi$ and $f$ a line contained in a non trivial fiber of $\pi$. Since $E \cdot f = -1$, there is an extremal ray $R$ of the Mori cone $\text{NE}(\tilde{X})$ such that $E \cdot R < 0$.

Since $E \cdot R < 0$, $R$ defines an extremal ray of the Mori cone $\text{NE}(E)$ which we still denote by $R$ (even if $\text{NE}(E)$ is not a subcone of $\text{NE}(\tilde{X})$ in general!). Since $\rho(Y) = 1$, we have $\rho(E) = 2$, hence $\text{NE}(E)$ is a 2-dimensional closed cone, one of its two extremal rays being generated by $f$. Then:

- either $R$ is not generated by $f$ and $E|_E$ is strictly negative on $\text{NE}(E) \setminus \{0\}$. In that case, $-E|_E = O_E(1)$ is ample by Kleiman’s criterion, which means that $N_{Y'/X}^*$ is ample.
- or, $R$ is generated by $f$. In that case, the Mori contraction $\varphi_R: \tilde{X} \to Z$ factorize through $\pi$:

$$\begin{array}{c}
\tilde{X} \\
\downarrow \varphi_R \\
Z \\
\downarrow \pi \\
X \\
\psi
\end{array}$$

where $\psi: X \to Z$ is an isomorphism outside $Y$. Since the variety $Z$ is projective and $X$ is not, $\psi$ is not an isomorphism and since $\rho(Y) = 1$, $Y$ is contracted to a point by $\psi$, hence $N_{Y/X}$ is ample by Grauert’s criterion.

Let us prove the following consequence of Proposition 1:

**Proposition 2.** Let $X$ be a non projective manifold, $Y$ a smooth submanifold of $X$ such that the blow-up $\pi: \tilde{X} \to X$ of $X$ along $Y$ is projective with $-K_{\tilde{X}}$ numerically effective (nef). Then, each connected component $Y'$ of $Y$ with $\rho(Y') = 1$ is a Fano manifold.

**Proof:** we can suppose that $Y$ is connected. Let $E$ be the exceptional divisor of $\pi$. Since $-E|_E$ is ample by Proposition 1, the adjunction formula $-K_E = -K_{\tilde{X}|E} - E|_E$ shows that $-K_E$ is ample, hence $E$ is Fano. By a result of Szurek and Wiśniewski [SzW90], $Y$ is itself Fano.

### 2.2. Proof of Theorem 2.

For the first assertion, we only have to prove that

$$\deg -K_{\tilde{X}}(E) \leq (\rho_n - 1)d_{n-1}.$$  

Let $Y'$ be a connected component of $Y$ and $E' = \pi^{-1}(Y')$. Then, since $-E|_{E'}$ is ample:

$$\deg -K_{\tilde{X}}(E') = (-K_{\tilde{X}|E'})^{n-1} = (-K_{E'} + E|_{E'})^{n-1} \leq (-K_{E'})^{n-1} \leq d_{n-1}.$$  

Now, if $m$ is the number of connected components of $Y$, then

$$\rho(\tilde{X}) = m + \rho(X) \geq m + 1.$$  

Putting all together, we get

$$\deg -K_{\tilde{X}}(E) \leq (\rho_n - 1)d_{n-1},$$  

which ends the proof of the first point.

We refer to [Wis91] prop. (3.5) for the second point.

### 3. On the Dimension of the Center of Non Projective Smooth Blow-Downs.

**Proof of Theorem 3.**

Theorem 3 is a by-product of the more precise following statement and of Proposition 3 below:

**Theorem 4.** Let $Z$ be a Fano manifold of dimension $n$ and index $r$, $\pi: Z \to Z'$ be a non projective smooth blow-down of $Z$, $Y \subset Z'$ the center of $\pi$. Let $f$ be a line contained in a non trivial fiber of $\pi$, then

(i) if $f$ generates an extremal ray of $\text{NE}(Z)$, then $\dim(Y) \geq (n - 1)/2$.

(ii) if $f$ does not generate an extremal ray of $\text{NE}(Z)$, then $\dim(Y) \geq r$. Moreover, if $\dim(Y) = r$, then $Y$ is isomorphic to $\mathbb{P}^r$.

In both cases (i) and (ii), $Y$ contains a rational curve.

The proof relies on Wiśniewski’s inequality (see [Wis91] and [AnW95]), which we recall now for the reader’s convenience: let $\varphi: X \to Y$ be a Fano-Mori contraction (i.e $-K_X$ is $\varphi$-ample) on a projective manifold $X$, $\text{Exc}(\varphi)$ its exceptional locus and

$$l(\varphi) := \min\{-K_X \cdot C; C \text{ rational curve contained in } \text{Exc}(\varphi)\}$$
its length, then for every non trivial fiber $F$:
\[
\dim \text{Exc}(\varphi) + \dim(F) \geq \dim(X) - 1 + l(\varphi).
\]

**Proof of Theorem 4.** The method of proof is taken from Andreatta’s recent paper [And99] (see also [Bon96]).

*First case:* suppose that a line $f$ contained in a non trivial fiber of $\pi$ generates an extremal ray $R$ of $\text{NE}(Z)$. Then the Mori contraction $\varphi_R : Z \to W$ factorizes through $\pi$:

\[
Z \xrightarrow{\varphi_R} W \xrightarrow{\psi} Z',
\]

where $\psi$ is an isomorphism outside $Y$. In particular, the exceptional locus $E$ of $\pi$ is equal to the exceptional locus of the extremal contraction $\varphi_R$.

Let us now denote by $\psi_Y$ the restriction of $\psi$ to $Y$, $s = \dim(Y)$, $\pi_E$ and $\varphi_{R,E}$ the restriction of $\pi$ and $\varphi_R$ to $E$. Since $Z'$ is not projective, $\psi_Y$ is not a finite map. Since $\varphi_R$ is birational, $W$ is $\mathbb{Q}$-Gorenstein, hence $K_W$ is $\mathbb{Q}$-Cartier and $K_{Z'} = \psi^*K_W$. Therefore, $K_{Z'}$ is $\psi$-trivial, hence $K_Y + \det N_{Y/Z'}^*\varphi$ is $\psi_Y$-trivial. Moreover, $O_E(1) = -E|_E$ is $\varphi_{R,E}$-ample by Kleiman’s criterion, hence $N_{Y/Z'}^*\varphi$ is $\psi_Y$-ample. Finally, $\psi_Y$ is a Fano-Mori contraction, of length greater or equal to $n - s = \text{rk}(N_{Y/Z}^*)$. Together with Wiśniewski’s inequality applied on $Y$, we get that for every non trivial fiber $F$ of $\psi_Y$

\[
2s \geq \dim(F) + \dim(\text{Exc}(\psi_Y)) \geq n - s + s - 1
\]

hence $2s \geq n - 1$. Moreover, $\text{Exc}(\psi_Y)$ is covered by rational curves, hence $Y$ contains a rational curve.

*Second case:* suppose that a line $f$ contained in a non trivial fiber of $\pi$ does not generate an extremal ray $R$ of $\text{NE}(Z)$. In that case, since $E \cdot f = -1$, there is an extremal ray $R$ of $\text{NE}(Z)$ such that $E \cdot R < 0$. In particular, the exceptional locus $\text{Exc}(R)$ of the extremal contraction $\varphi_R$ is contained in $E$, and since $f$ is not on $R$, we get for any fiber $F$ of $\varphi_R$:

\[
\dim(F) \leq s = \dim(Y).
\]

By the adjunction formula, $-K_E = -K_{Z|E} - E|_E$, the length $l_E(R)$ of $R$ as an extremal ray of $E$ satisfies

\[
l_E(R) \geq r + 1,
\]

where $r$ is the index of $Z$. Together with Wiśniewski’s inequality applied on $E$, we get :

\[
r + 1 + (n - 1) - 1 \leq s + \dim(\text{Exc}(R)) \leq s + n - 1.
\]

Finally, we get $r \leq s$, and since the fibers of $\varphi_R$ are covered by rational curves, there is a rational curve in $Y$. Suppose now (up to the end) that $r = s$. Then $E$ is the exceptional locus of the Mori extremal contraction $\varphi_R$. Moreover, $K_Z + r(-E)$ is a good supporting divisor for $\varphi_R$, and since every non trivial fiber of $\varphi_R$ has dimension $r$, $\varphi_R$ is a smooth projective blow-down. In particular, the restriction of $\pi$ to a non trivial fiber $F \simeq \mathbb{P}^r$ induces a finite surjective map $\pi : F \simeq \mathbb{P}^r \to Y$ hence $Y \simeq \mathbb{P}^r$ by a result of Lazarsfeld [Laz83].

This ends the proof of Theorem 4. ■

The proof of Theorem 4 does not use the hypothesis $Z$ Fano in the first case. We therefore have the following:

**Corollary 2.** Let $Z$ be a projective manifold of dimension $n$, $\pi : Z \to Z'$ be a non projective smooth blow-down of $Z$, $Y \subset Z'$ the center of $\pi$. Let $f$ be a line contained in a non trivial fiber of $\pi$ and suppose $f$ generates an extremal ray of $\text{NE}(Z)$. Then $\dim(Y) \geq (n - 1)/2$. Moreover, if $\dim(Y) = (n - 1)/2$, then $Y$ is contractible on a point.
We finish this section by the following easy proposition, which combined with Theorem 4 implies Theorem 3 of the Introduction:

**Proposition 3.** Let $Z$ be a Fano manifold of dimension $n$ and index $r$, $\pi : Z \to Z'$ be a smooth blow-down of $Z$, $Y \subset Z'$ the center of $\pi$. Then $n - 1 - \dim(Y)$ is a multiple of $r$.

**Proof.** Write

$$-K_Z = rL \quad \text{and} \quad -K_Z = -\pi^*K_{Z'} - (n - 1 - \dim(Y))E$$

where $E$ is the exceptional divisor of $\pi$. Let $f$ be a line contained in a fiber of $\pi$. Then $rL \cdot f = n - 1 - \dim(Y)$, which ends the proof. ■

**Proof of Theorem 3.** Let $Z$ be a Fano manifold of dimension $n$ and index $r$ and suppose there is a non projective smooth blow-down of $Z$ with an $s$-dimensional center. By Proposition 3, there is a strictly positive integer $k$ such that $n - 1 - kr = s$. By Theorem 4, either $n - 1 - kr \geq (n - 1)/2$ or $n - 1 - kr \geq r$. In both cases, it implies that $r \leq (n - 1)/2$ and therefore $s \geq r$. If $r > (n - 1)/3$, since $n - 1 \geq (k+1)r > (k+1)(n-1)/3$, we get $k = 1$ and $s = n - 1 - r$. ■

4. Rational curves on simple Moishezon manifolds.

The arguments of the previous section can be used to deal with the following well-known question: does every non projective Moishezon manifold contain a rational curve? The answer is positive in dimension three (it is due to Peternell [Pet86], see also [CKM88] p. 49 for a proof using the completion of Mori’s program in dimension three).

**Proposition 4.** Let $Z$ be a projective manifold, $\pi : Z \to Z'$ be a non projective smooth blow-down of $Z$. Then $Z'$ contains a rational curve.

**Proof.** With the notations of the previous section, it is clear in the first case where a line $f$ contained in a non trivial fiber of $\pi$ generates an extremal ray $R$ of $\text{NE}(Z)$ (in that case, the center of $\pi$ contains a rational curve). In the second case, since $f$ is not extremal and $K_Z$ is not nef, there is a Mori contraction $\varphi$ on $Z$ such that any rational curve contained in a fiber of $\varphi$ is mapped by $\pi$ to a non constant rational curve in $Z'$. ■

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