Huygens’ cycloidal pendulum: an elementary derivation

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Received 14 February 2022, revised 28 April 2022
Accepted for publication 17 June 2022
Published 14 July 2022

Abstract
A pedagogical derivation of the Huygens cycloidal pendulum, suitable for undergraduates, is here presented. Our derivation rests only on simple algebraic and geometrical tricks, without the need of Calculus.

Keywords: Huygens’ pendulum, oscillations, Newton’s laws

1. Introduction

It is a well known fact that the motion of a simple pendulum is far from being rigorously isochrone, being the period of its oscillations an increasing function of their amplitude [1]. For ‘small’ oscillations the swinging time is assumed to be approximately independent of the amplitude, and it is within such a regime that simple pendulums are employed as clocks. The ‘isochronism question’ was known since Galileo’s time, and in 1659 Christiaan Huygens, ‘the most ingenious watchmaker of all time’, to use Sommerfeld’s words [2], was the first to find the solution [3, 4]. Instead of using a circular motion, the pendulum bob was forced to move across a cycloid, i.e., the trajectory drawn by a typical point on the periphery of a bicycle wheel when the latter moves on the ground without slipping.

While the simple pendulum is a central topic in any undergraduate physics course, the same cannot be said as far as the cycloidal pendulum is concerned. The latter is rarely offered to students as an example of application of Newton’s laws of motion, although it was studied already in Newton’s \textit{Principia} [5]. Most treatments of cycloidal pendulum rest on some concepts of differential geometry and Calculus which could not be yet available inside the math toolbox of first-year undergraduates. Cycloid is also associated to the solution of the celebrated brachistochrone problem [6], for which the principles of Calculus of Variation were first developed [2]. During last years a considerable deal of work has been done about elementary treatments

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\textsuperscript{1}To Professor Frank Silvio Marzano, in Memoriam.
of these ‘magic’ properties of cycloid, namely brachistochronism and tautochronism (see for
instance [7–13]).

In the present paper a pedagogical derivation of Huygens’ pendulum isochronism is pro-
posed. The isochronism property is derived from basic mechanical principles and by using
only a trivial algebraic trick that, once translated into the vector language, leads to the very
geometrical definition of cycloids without the need of Calculus.

To help Teachers, what follows will be arranged as a whole didactical unit that could be
proposed to undergraduates within a standard two-hour lecture. Some mathematical steps could
seem a little bit redundant, but this has been done to keep the level as elementary as possible.

2. The statement of the problem

In figure 1 the geometry of the problem is shown.

$P$ is a point having mass $m$, sliding in a vertical plane $Oxy$ along a frictionless surface under
the action of its weight $mg$. The vector $N$ describes the surface reaction which is directed along
the surface normal, due to absence of friction. Symbol $\hat{\tau}$ denotes the tangential unit vector. The

$\text{Figure 1. The isochrone pendulum geometry.}$

\begin{equation}
ma = mg + N,
\end{equation}

with $a$ being the point acceleration. Projection of both sides of equation (1) along the tangential
direction $\hat{\tau}$ gives at once

\begin{equation}
a_\tau = -g \sin \alpha,
\end{equation}

where symbols $a_\tau$ and $\alpha$ denote the tangential component of the acceleration and the incli-
nation angle of the curve with respect the horizontal direction $x$, respectively (see figure 1). Now, for the motion along the trajectory to be rigorously harmonic it is mandatory that $a_\tau$ be
proportional, up to a minus sign, to the curvilinear abscissa $s = OP$,

\begin{equation}
a_\tau = -\omega^2 s,
\end{equation}
where $\omega = 2\pi/T$, with $T$ denoting the oscillation period. On comparing equations (2) and (3), the following equation is obtained:

$$s = \ell \sin \alpha,$$

with $\ell = g/\omega^2$. Equation (4) represents an implicit definition of the trajectory shape in terms of the curvilinear abscissa $s$, the slope $\alpha$, and the characteristic length $\ell$.

The task consists in proving that behind equation (4) a cycloid is hidden. To this end, it is worth first describing a ‘standard solution’, based on a few concept of differential geometry. This has been carried out in section 3. Subsequently, our elementary solution will be presented in section 4.

3. A ‘standard’ solution

Our aim is to extract the parametric representation of the trajectory with respect to $\alpha$, i.e., functions $x = x(\alpha)$ and $y = y(\alpha)$. To this end, both sides of equation (4) are first derived with respect to $\alpha$,

$$\frac{dx}{d\alpha} = \ell \cos \alpha,$$

where

$$ds = \sqrt{dx^2 + dy^2} = dx \sqrt{1 + \tan^2 \alpha} = \frac{dx}{\cos \alpha}.$$

From equations (5) and (6) we then have

$$\frac{dx}{d\alpha} = \frac{dx}{ds} \frac{ds}{d\alpha} = \cos \alpha \frac{ds}{d\alpha} = \ell \cos^2 \alpha,$$

and, similarly for $y$,

$$\frac{dy}{d\alpha} = \frac{dy}{dx} \frac{dx}{d\alpha} = \tan \alpha \frac{dx}{d\alpha} = \ell \sin \alpha \cos \alpha,$$

where in the last step use has been made of equation (7). On using elementary trigonometric, the following system is then obtained:

$$\begin{cases} \frac{dx}{d\alpha} = \frac{\ell}{2}(1 + \cos 2\alpha), \\ \frac{dy}{d\alpha} = \frac{\ell}{2} \sin 2\alpha. \end{cases}$$

Finally, on introducing the parameter $R = \ell/4$ and on integrating the system (9) with the initial condition $\{x(0), y(0)\} \equiv \{0, 0\}$, the required parametric representation of the trajectory is found

$$\begin{cases} x(\alpha) = R(2\alpha + \sin 2\alpha), \\ y(\alpha) = R(1 - \cos 2\alpha), \end{cases}$$

a cycloid, in fact.
4. The elementary solution

The differential quantities $ds$ and $d\alpha$ are first replaced by finite, small quantities $\Delta s$ and $\Delta \alpha$, respectively. Equation (5) can then be introduced as follows: consider a ‘small’ displacement of the point $P$ to another point along the curve, say $Q$, corresponding to the inclination $\alpha + \Delta \alpha$, as sketched in figure 2. The displacement length, say $\Delta s$, coincides with the measure of the arc $\overarc{PQ}$, i.e.,

$$\Delta s = \overarc{OQ} - \overarc{OP} = \ell \left[ \sin(\alpha + \Delta \alpha) - \sin \alpha \right]$$

$$= \ell \left[ \sin \alpha (1 - \cos \Delta \alpha) - \cos \alpha \sin \Delta \alpha \right], \quad (11)$$

where use has been made of equation (4).

For $Q$ sufficiently close to $P$, i.e., for $\Delta \alpha$ sufficiently small, the following approximations can be used:

$$\begin{cases} 
\sin \Delta \alpha \simeq \Delta \alpha, \\
\cos \Delta \alpha \simeq 1,
\end{cases} \quad (12)$$

which, once inserted into equation (11) leadsto

$$\Delta s = \ell \cos \alpha \Delta \alpha, \quad (13)$$

and where the arc $\overarc{PQ}$ will be identified with the segment $\overarc{PQ}$, as shown in figure 2. Needless to say, equation (13) is nothing but equation (5).

The first algebraic trick consists to recast equation (13) as follows:

$$\Delta s = 2 \left( \frac{\ell}{2} \Delta \alpha \right) \cos \alpha, \quad (14)$$

and to introduce the isosceles triangle $\triangle PQR$, built up in such a way that $\overarc{PR} = \overarc{QR} = \frac{\ell}{4} \Delta \alpha$ (see figure 3).

On using simple geometrical considerations it follows that the horizontal inclination of $RQ$ is $\theta = 2\alpha$. Accordingly, we have

$$\overarc{PR} = \overarc{QR} = \frac{\ell}{4} \Delta \theta, \quad (15)$$
where it has been set $\Delta \theta = 2\Delta \alpha$.

Now the second trick: to give suitable orientations to the three sides of the triangle $PRQ$ in order for them to be interpreted as vectors of the Euclidean plane. This, in particular, allows the triangle itself to be represented through an algebraic equation, like for instance

$$\overrightarrow{PQ} = \overrightarrow{PR} + \overrightarrow{RQ}, \quad (16)$$

as sketched in figure 4. From the same figure it also follows that the ratio $\overrightarrow{RQ}/\Delta \theta = \ell/4$ can be interpreted as the radius of a circumference centred at the point $C$.

Then, the final step: the vector $\overrightarrow{DC} = \overrightarrow{PR}$ is introduced in such a way that equation (16) is recast as follows:

$$\overrightarrow{PQ} = \overrightarrow{DC} + \overrightarrow{RQ}. \quad (17)$$

Accordingly, the total displacement $\overrightarrow{PQ}$ along the curve can be viewed as the result of the composition of two rigid motions of the circumference: (i) an horizontal displacement represented by $\overrightarrow{DC}$, plus (ii) a counterclockwise rotation by an angle $\Delta \theta$ around the centre $C$. Furthermore, since $\overrightarrow{DC} = \overrightarrow{RQ}$, it should be immediately recognized that such a composition is equivalent to a perfect (i.e., without slipping) rolling of the circumference along the horizontal line $r$, thus proving that the displacement $\overrightarrow{PQ}$ must lay on a cycloid, Q.E.D.
5. Conclusions: why is Huygens the ‘most ingenious watchmaker of all time’?

The teacher could not resist to explain to his/her audience the Sommerfeld homage to Huygens. Again, this can be done elementarily. To this aim, the concept of evolute of a curve, i.e., the geometric locus of its curvature centres, must first be introduced to students. A possibility is to employ the kinematic strategy used, for instance, in [14] to evaluate the radius of curvature of conics. Accordingly, a point moving across the cycloid is figured out, so that the radius of curvature follows from the classical Huygens formula involving normal component of acceleration and speed. For simplicity, the motion of the circle along the line \( r \) will be supposed to be uniform. In this way it is not difficult to realize that the velocity of \( P \), say \( v \), can be thought of as the sum of two velocities: (i) that pertinent to the (uniform) rectilinear motion of the centre \( C \), say \( v_C \), and (ii) the velocity of \( P \), say \( v_R \), due to its uniform circular motion around \( C \), as sketched into figure 5.

Due to the absence of slipping along \( r \) (pure rolling), it can be set \( v_C = v_R = v \), so that the speed of \( P \) turns out to be

\[
V = 2v \cos \alpha. \tag{18}
\]

Moreover, since the motion of \( P \) results from the composition of a uniform rectilinear and a uniform circular ones, the point acceleration \( a \) will be constantly directed toward the centre \( C \), its modulus being

\[
a = \frac{v^2}{PC}. \tag{19}
\]

The normal component of the acceleration, say \( a_v \), is then obtained by projecting \( a \) along the direction \( PQ \) of figure 5, which is by construction normal to the direction of \( V \),

\[
a_v = \frac{v^2}{PC} \cos \alpha. \tag{20}
\]

On using equations (18) and (20), the radius of curvature of the cycloid at \( P \), say \( \rho \), is then obtained,

\[
\rho = \frac{V^2}{a_v} = \frac{2QPC \cos \alpha}{4PC} = 2QP, \tag{21}
\]
Figure 6. Cycloid’s evolute is a cycloid.

where in the last step use has been made of the fact that the triangle $CPQ$ is isosceles (see figure 5).

Equation (21) implies that the centre of curvature, say $P'$, is placed symmetrically to $P$ with respect to the contact point $Q$ between the cycloid generatrix circle and $r$, as shown in figure 6. On introducing an identical circle (the dashed one in figure 6) in the opposite side with respect $r$, it is not difficult to realize that also $P'$ must belong to an identical cycloid (the dashed one in figure 6). The final step is to prove that the direction $P'P$ is always tangent to such cycloidal evolute (but this could be left to our students as a useful exercise). Accordingly, if a point mass were suspended on an ideal, perfectly flexible and inextensible rope which is forced bending along a solid barrier having the evolute shape, as shown in figure 7, then $m$ will necessarily move across the right cycloidal trajectory. This, together with all has been said and proved above, should be enough to help our students to grasp the way Huygens practically made his isochronous pendulum.

To worthily conclude the present lecture, no words would be better than those employed in [2] by Sommerfeld to masterfully describe Huygens’ genius:

*Just as remarkable as Huygens’ discovery of the isochronism of the cycloidal pendulum is the way in which he actually achieved the frictionless motion of the bob in the cycloid. He availed himself of the rule that the evolute of a cycloid is another cycloid equal to the generating one. If, therefore, we tie a string of length $l = 4R$ to the point $O$ of figure 7 in which the two upper cycloid arcs form a cusp, and if this string be pulled taut so that it rests against the right part of the cycloid (or the left part if deflected to the left), the endpoint $P$ of the string describes the lower cycloidal arc. The guiding of the bob along the lower cycloid effected in this manner is almost as frictionless as the guiding of the simple pendulum along a circular arc.*
Figure 7. Huygens’s isochronous cycloidal pendulum.

Acknowledgments

I wish to thank Turi Maria Spinozzi for his invaluable help during the preparation of the work.

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