Low-energy dynamical response of an Anderson insulator with local attraction

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The low-frequency dynamical response of an Anderson insulator is dominated by so-called Mott resonances: hybridization of pairs of states close in energy, but separated spatially. We study the effect of interaction on Mott resonances in the model of spinful fermions (electrons) with local attraction. This model is known to exhibit a so-called pseudogap: a suppression of the low-energy single-particle excitations. Correspondingly, the low-energy dynamical response is also reduced. However this reduction has mostly quantitative character. In particular, the Mott formula for frequency-dependent conductivity preserves its functional asymptotic behavior at low frequencies, but with a small numerical prefactor. This result can be explained in terms of Mott resonances for electron pairs instead of single electrons.

I. INTRODUCTION

Adding interactions to an Anderson insulator\textsuperscript{11} brings in a broad spectrum of new phenomena, from Coulomb gap and related repulsion effects\textsuperscript{2–4} to many-body localization\textsuperscript{5, 6}. Recently, a model of a disordered metal with local attraction was studied in Refs.\textsuperscript{7, 8}. This model exhibits a superconductor-insulator transition (SIT)\textsuperscript{9, 10}, with unusual properties of the superconducting state formed by paired localized electrons. The insulating state on the other side of the SIT is expected to be unusual as well; in particular, it may realize a very good testground for experimental studies of many-body localization phenomena\textsuperscript{10}.

The latter development motivated us to study in more detail the effect of local attraction on the low-energy dynamical response in an Anderson insulator. It is known that low-energy properties of a coherent Anderson insulator are determined by Mott resonances: hybridization of pairs of states close in energy, but separated spatially\textsuperscript{11}. In particular, the dynamical density-response function can be calculated using the statistical distribution of Mott resonances\textsuperscript{12}, and, in 1D, these calculations reproduce the results obtained directly with the Gorkov-Berezinskii method\textsuperscript{13}. In higher dimensions, the statistical properties of Mott resonances are less studied, and similar results can only be conjectured\textsuperscript{12}. This dynamical density-response function can be used to express physically relevant quantities such as frequency-dependent conductivity and polarizability.

In this paper, we study the same dynamical density-response function in the presence of local attraction, similar to the model considered in Refs.\textsuperscript{7, 8}. The interaction introduces an additional energy scale: the pseudogap energy $\Delta_P$. One therefore expects (and it will be confirmed by the calculation below) that the dynamical density-response function is affected by interaction at frequencies below $\Delta_P$ (we put $\hbar = 1$ throughout the paper and do not distinguish between energy and frequency). We consider the regime $\Delta_P \ll \Delta_\xi$, where $\Delta_\xi$ is the level spacing at the localization length $\xi$ (i.e., the energy scale associated with localization). We also consider the zero-temperature limit (temperature much lower than all other energy scales). Finally, we assume the absence of superconductivity (the insulating side of the superconductor-insulator transition\textsuperscript{7, 8}).

Under all these assumptions, we study the dynamical density-response function in the whole range of frequencies $\omega \ll \Delta_\xi$ (without any assumption on the relation between $\omega$ and $\Delta_P$). As expected, at $\omega \gg \Delta_P$, the dynamical density-response function is close to its non-interacting form, but at $\omega \ll \Delta_P$ it is considerably reduced. This reduction can be viewed as two renormalizations: first, the overall numerical prefactor is renormalized; second, the Mott length scale is renormalized (approximately by the factor of one half). This can be interpreted in terms of Mott resonances for pairs of particles.

The paper is organized as follows. In Section\textsuperscript{11}, we define the main object of our study: the dynamical density-response function and discuss its general properties. In Section\textsuperscript{11} we define the model of the disordered system with interaction and its simplified version studied in the paper. In Section\textsuperscript{11} we derive a general formula for the contribution to the dynamical density-response function from Mott resonances. Section\textsuperscript{11} solves two special cases of such resonances, the non-interacting case and the perfect-resonance case, which are further used as a basis for a more general analysis. In Section\textsuperscript{11} we compute the effect of the interaction at frequencies $\omega$ much lower than the pseudogap energy $\Delta_P$. Section\textsuperscript{11} contains numerical results on the renormalization of the density-response function in the more general situation of $\omega$ comparable to $\Delta_P$. Finally, in Section\textsuperscript{11} we discuss conditions of applicability of our results and their physical implications for the frequency-dependent conductivity and polarizability.
II. DYNAMICAL DENSITY-RESPONSE FUNCTION

We define the disorder-averaged dynamical density-response function at zero temperature (more precisely, its dissipative component) as

\[ s(\omega, x) = \frac{1}{\omega} \left\langle \sum_i \langle 0 | \hat{n}_i(0) | i \rangle \langle i | \hat{n}(x) | 0 \rangle \delta(E_i - E_0 - \omega) \right\rangle, \tag{1} \]

where

\[ \hat{n}(x) = \sum_\sigma \hat{\Psi}^\dagger_\sigma(x) \hat{\Psi}_\sigma(x) \tag{2} \]

is the fermionic density operator (\( \hat{\Psi}_\sigma(x) \) and \( \hat{\Psi}^\dagger_\sigma(x) \) are the fermionic creation and annihilation operators, \( \sigma \) is the spin index), |0⟩ and |i⟩ are the ground state and the excited eigenstates with the energies \( E_0 \) and \( E_i \), respectively, \( \langle \ldots \rangle \) denotes disorder averaging.

This response function is related to various physical observables. For example, the frequency-dependent conductivity (the dissipative part) can be expressed as

\[ \sigma_{\alpha\beta}(\omega) = -\frac{\pi}{2} \omega^2 \int dx \, x_n x_{\beta} \, s(\omega, x) \tag{3} \]

and the static polarizability, via the Kramers-Kronig relation, as

\[ \chi_{\alpha\beta} = \frac{2}{\pi} \int_0^\infty \frac{d\omega}{\omega^2} \sigma_{\alpha\beta}(\omega) = -\int d\omega \int d\omega' \, x_n x_{\beta} \, s(\omega, x) \tag{4} \]

(here and below the integral over \( x \) is understood in the \( d \)-dimensional space).

The dimensionality of the so defined response function \( s(\omega, x) \) is \( \text{(Energy-Volume)}^{-2} \). For non-interacting fermions, in the approximation of a constant density of states,

\[ s(\omega, x)_{\text{nonint}} = 2\nu^2 S(\omega, x), \tag{5} \]

where \( \nu \) is the density of states per spin and \( S(\omega, x) \) is the single-particle response function as defined in Ref. [12] (for spinless fermions):

\[ S(\omega, x) = \nu^{-2} \left\langle \sum_{n,m} \delta(E_n - E) \delta(E_m - E - \omega) \psi^*_n(0) \psi_n(x) \psi^*_m(x) \psi_m(0) \right\rangle, \tag{6} \]

where \( \psi_n(x) \) are single-particle eigenstates. The coefficient 2 in Eq. (5) comes from the spin degeneracy.

The qualitative shape of the dynamical density-response function \( s(\omega, x) \) at \( \omega \ll \Delta_\xi \) is shown in Fig. 1. At small \( x \) (\( x \ll \xi \)), \( s(\omega, x) \) is proportional to the correlation function of a single localized state [12]. At large \( x \) (\( x \gg \xi \)), \( s(\omega, x) \) has a negative hump, so that the overall integral obeys the sum rule (following from the orthogonality of eigenstates):

\[ \int dx \, s(\omega, x) = 0. \tag{7} \]

For non-interacting fermions in one dimension, this form of \( s(\omega, x) \) was derived in Ref. [13]. Later, it was reproduced in Ref. [12] based on the Mott-hybridization phenomenology, where it was further conjectured that a similar form should also hold in higher dimensions.

In the non-interacting case, the \( x \) position of the negative hump is given by the Mott length [11–13]

\[ L_M = 2\xi \ln(\Delta_\xi/\omega), \tag{8} \]

and this is the only dependence of \( s(\omega, x) \) on \( \omega \) in the small-\( \omega \) limit.

We argue below that this form of \( s(\omega, x) \) remains valid in a model with local attraction, with the two differences: the \( x \) position and the overall weight are renormalized.

III. MODEL

We adopt the model of Refs. [7–8], which includes electrons with disorder and with local attraction between electrons of opposite spin. Attraction between electrons of the same spin has little effect on the considered phenomena and can be neglected. One possible form of the system Hamiltonian is

\[ \hat{H} = \hat{H}_{\text{kin}} + \hat{H}_{\text{disorder}} + \hat{H}_{\text{int}}, \tag{9} \]

where the kinetic, disorder, and interaction parts of the Hamiltonian are

\[ \hat{H}_{\text{kin}} = \int dx \, \sum_\sigma \hat{\Psi}^\dagger_\sigma(x) \left( -\frac{\nabla^2}{2m} \right) \hat{\Psi}_\sigma(x), \tag{10} \]

\[ \hat{H}_{\text{disorder}} = \int dx \, \sum_\sigma \hat{\Psi}^\dagger_\sigma(x) V_{\text{disorder}}(x) \hat{\Psi}_\sigma(x), \tag{11} \]

\[ \hat{H}_{\text{int}} = -g \int dx \, \hat{\Psi}^\dagger_\uparrow(x) \hat{\Psi}^\dagger_\downarrow(x) \hat{\Psi}_\downarrow(x) \hat{\Psi}_\uparrow(x). \tag{12} \]
The specific form of the kinetic term, the statistics of the disorder potential, and of the short-range attraction is not important for the discussion below, since only the long-distance properties of the tails of the localized states are relevant. In particular, it will be convenient to consider the model in the basis of localized states,

\[ \hat{H} = \sum_i \varepsilon_i \left( \hat{\Psi}_{i \uparrow}^\dagger \hat{\Psi}_{i \uparrow} + \hat{\Psi}_{i \downarrow}^\dagger \hat{\Psi}_{i \downarrow} \right) + \sum_{i \neq j} J_{ij} \left( \hat{\Psi}_{i \uparrow}^\dagger \hat{\Psi}_{j \downarrow} + \hat{\Psi}_{i \downarrow}^\dagger \hat{\Psi}_{j \uparrow} \right) - \sum_i \Delta_i \left( \hat{\Psi}_{i \uparrow}^\dagger \hat{\Psi}_{i \downarrow}^\dagger \hat{\Psi}_{i \downarrow} \hat{\Psi}_{i \uparrow} \right). \]  

(13)

Here the operators \( \hat{\Psi}_{i \sigma} \) correspond to states localized at the localization length \( \xi \), without including long-range Mott hybridization. The small tunneling amplitudes \( J_{ij} \) describe long-range Mott hybridization, and the attraction is included as the one-site energy term \( \Delta_i \) favoring double occupation of localized states. We neglect attraction between different localized states, assuming the system to be far from the SIT. Furthermore, to simplify the calculation, we first assume that the energy gain from a double occupation \( \Delta_i \) is the same for all localized states (and denote it \( \Delta_P \)). In reality, this energy gain is different for different states and depends on the inverse participation ratio for a given state [8]. We discuss the effect of the distribution of \( \Delta_i \) qualitatively after the main calculation.

**IV. LOW-ENERGY CONTRIBUTION FROM MOTT-RESONANT STATES**

Even in the presence of interactions, provided \( \omega \ll \Delta_{\xi} \), the dynamical density-response function \( s(\omega, x) \) at low frequencies \( \omega \) is determined by Mott resonances. Below we calculate it by solving the case of a single Mott resonance [the Hamiltonian (13) with two states only] and then averaging over the statistics of such Mott resonances, following the method of Ref. [12]. A similar approach was used in Refs. [3, 4] to treat the case of Coulomb repulsion. Unlike that calculation where spin played no important role, the case of attraction requires an explicit treatment of the spin degrees of freedom.

Consider a Mott resonance formed by two localized states at initial energies \( \varepsilon_1 + \Delta_P / 2 \) and \( \varepsilon_2 + \Delta_P / 2 \) located at the positions 0 and \( x \), respectively (we include the energy shift of \( \Delta_P / 2 \) for convenience). We assume \( x \gg \xi \). Let the Mott-hybridization amplitude equal \( J \). This hybridization amplitude is a random variable (depending on the disorder realization and on the particular pair of states), whose probability distribution may, in principle, depend on the type of disorder. It was conjectured in Ref. [12] that, in a single-parameter-scaling regime, the probability distribution is approximately log-normal:

\[ dP_x(J) \approx \frac{\xi}{\pi x} \exp \left[ - \frac{(2\xi \ln |J/\Delta_{\xi} + x|)^2}{4x\xi} \right] d\ln |J|. \]  

(14)

For our derivation below, the actual form of the probability distribution \( dP_x(J) \) is not important: the only thing we assume about it is that its width in the logarithmic scale is much larger than unity (we believe that it is a plausible assumption at distances \( x \gg \xi \) for a wide class of disorder models). We further assume \( \omega \ll \Delta_{\xi} \).

The two localized states produce four single-particle states (due to the spin degeneracy), which can be occupied in the ground state by a number of particles between 0 and 4. One can show, however, that, in the case of attractive interaction, the ground state must always have an even number of particles (0, 2, or 4). Indeed, if a one-particle state has a negative energy, one can lower the energy even further by adding one more particle in the same state, but with an opposite spin; the same argument applies for a three-particle state, if one uses holes instead of particles. The state with 0 particles has energy 0, the state with 4 particles has energy 2(\( \varepsilon_1 + \varepsilon_2 \)). If one of those states has energy lower than the lowest two-particle state, this resonance does not contribute, since there are no relevant excitations.

The only contribution to \( s(\omega, x) \) comes from the resonances with the two-particle ground state. Because of the particle-hole symmetry, we can assume \( \varepsilon_1 + \varepsilon_2 > 0 \) and then multiply the result by 2 (to include the sector with \( \varepsilon_1 + \varepsilon_2 < 0 \)). Furthermore, it is easy to see that the relevant two-particle states are in the singlet sector (the triplet sector has the energy \( \varepsilon_1 + \varepsilon_2 + \Delta_P \), which is higher than that of at least one of the states with 0 particles or with 4 particles). There are three such states with the normalized basis

\[ \frac{1}{\sqrt{2}} \left( \hat{\Psi}_{1 \uparrow}^\dagger \hat{\Psi}_{2 \downarrow} \right) \left| \ast \right> \right) + \left. \hat{\Psi}_{1 \downarrow}^\dagger \hat{\Psi}_{2 \uparrow} \right| \ast >, \left. \hat{\Psi}_{1 \uparrow}^\dagger \hat{\Psi}_{1 \downarrow} \right| \ast >, \left. \hat{\Psi}_{2 \uparrow}^\dagger \hat{\Psi}_{2 \downarrow} \right| \ast >, \]  

(15)

where \( |\ast> \) is the vacuum state (without particles), and \( \hat{\Psi}_{i \sigma} \) are the electron creation operators in the localized state \( \sigma \) with the spin \( \sigma \), see Fig. 2. The Hamiltonian restricted to these three states has the form

\[ H = \varepsilon_1 + \varepsilon_2 + H_0 \]  

\[ H_0 = \begin{pmatrix} \Delta_P & \sqrt{2}J & \sqrt{2}J \\ \sqrt{2}J & \varepsilon & 0 \\ \sqrt{2}J & 0 & -\varepsilon \end{pmatrix} \]  

(16)
where $\varepsilon = \varepsilon_1 - \varepsilon_2$. The contribution of such states to the linear-response function \cite{1} can be written as

$$s(\omega, x) = \frac{2\nu^2}{\omega} \int dP_x(J) \int_{\varepsilon_1 + \varepsilon_2 > 0} d\varepsilon_1 d\varepsilon_2 \theta(-(\varepsilon_1 + \varepsilon_2 + E_0)) \sum_{i=1,2} (|0\rangle n_1 |i\rangle (i|n_2\rangle |0\rangle) \delta(E_i - E_0 - \omega).$$

(17)

Here, $\nu$ is the density of states at the Fermi level per spin, $n_1 = \text{diag}(1, 2, 0)$ and $n_2 = \text{diag}(1, 0, 2)$ are the number-of-particles operators in the first and the second localized state, respectively. The state $|0\rangle$ is the lowest of the three eigenstates of the Hamiltonian $H_0$ (with the eigenvalue $E_0$), and the states $|1\rangle$ and $|2\rangle$ are the two excited states.

Integration over $\varepsilon_1 + \varepsilon_2$ may be easily performed. It is further convenient to replace the operators $n_1$ and $n_2$ by the operator $D = (n_1 - n_2)/2 = \text{diag}(0, 1, -1)$, using the orthogonality of the eigenstates. We arrive at

$$s(\omega, x) = -2\nu^2 \int dP_x(J) \int_0^\infty d\varepsilon \frac{(-E_0)}{\omega} \sum_{i=1,2} |\langle 0|D|i\rangle|^2 \delta(E_i - E_0 - \omega).$$

(18)

(here we have folded the integral, without loss of generality, on the half line $\varepsilon > 0$).

V. SPECIAL CASES

A. Non-interacting case

We first check that the above formula reproduces the non-interacting result in the limit $\Delta_p = 0$. In this case, the Hamiltonian $H_0$ can be diagonalized with the eigenvalues

$$E_0 = -\sqrt{\varepsilon^2 + 4J^2}, \quad E_1 = 0, \quad E_2 = \sqrt{\varepsilon^2 + 4J^2}.$$  

(19)

On explicitly computing the eigenvectors and substituting them into (18) (note that, in the non-interacting case, $\langle 0\rangle D |2\rangle = 0$, which simplifies the calculation), we find

$$s(\omega, x)_{\text{nonint}} = -\nu^2 \int_{|J| < \omega} dP_x(J) \frac{4J^2}{\omega \sqrt{\omega^2 - 4J^2}},$$

(20)

which reproduces Eq. (31) of Ref. [12] (up to a factor of 2 due to spin). The limit of integration in $|J|$ in Eq. (20) emerges naturally as a consequence of the delta-function constraint in Eq. (18).

For completeness, we reproduce further calculation from Ref. [12]. Note that the integration measure $dP_x(J)$ is distributed on the logarithmic scale of $J$ with a width much larger than one, while the integrand has the width of order unity on the same logarithmic scale. We may therefore replace the integral (20) by

$$s(\omega, x)_{\text{nonint}} = -C\nu^2 \int dP_x(J) \delta(\ln |J| - \ln \omega),$$

(21)

where the coefficient $C$ is calculated as

$$C = \int_{|J| < \omega} d\ln |J| \frac{4J^2}{\omega \sqrt{\omega^2 - 4J^2}} = 1.$$  

(22)

B. Perfect resonance: $\varepsilon = 0$

A helpful insight in the interaction effect on Mott resonances can be obtained from the case $\varepsilon = 0$ (perfect resonance). In this case, the Hamiltonian $H_0$ can also be easily diagonalized with the eigenvalues

$$E_0 = -\frac{\Delta J - \Delta P}{2}, \quad E_1 = 0, \quad E_2 = \frac{\Delta J + \Delta P}{2},$$

(23)

where

$$\Delta J = \sqrt{\Delta^2 p + 16J^2}.$$  

(24)

Note that in the limit $J \ll \Delta p$, the excitation gap is small: $E_0 \approx -4J^2/\Delta p$, in spite of the single-particle gap.

VI. LOW-ENERGY LIMIT: $\omega \ll \Delta p$

The dynamical density-response function (18) may be calculated analytically in the limit $\omega \ll \Delta p$. In this case, as will be checked self-consistently, the relevant scale of $|J|$ also obeys $|J| \ll \Delta p$. In this regime, the first of the three basis states in the matrix (16) is separated by a large energy gap from the other two states and can be taken into account perturbatively. The effective Hamiltonian for the remaining two states is

$$H_{\text{eff}} = \left(\begin{array}{cc} \varepsilon - J_{\text{eff}} & -J_{\text{eff}} \\ -J_{\text{eff}} & -\varepsilon - J_{\text{eff}} \end{array}\right), \quad J_{\text{eff}} = 2J^2/\Delta p,$$

(25)

and its diagonalization gives the energies

$$E_0 = -J_{\text{eff}} - \sqrt{\varepsilon^2 + J_{\text{eff}}^2}, \quad E_1 = -J_{\text{eff}} + \sqrt{\varepsilon^2 + J_{\text{eff}}^2}.$$  

(26)

Continuing the calculation along the lines of the non-interacting case considered above [specifically, using Eq. (18) with only one excited state and substituting the eigenvectors of the $2 \times 2$ Hamiltonian (25)], we find

$$s(\omega, x) = -C(\omega)\nu^2 \int dP_x(J) \delta(\ln |J| - \ln J_*(\omega)),$$

(27)

where the characteristic overlap scale $J_*(\omega)$ is now

$$J_*(\omega \ll \Delta J) = \sqrt{\omega \Delta p}.$$  

(28)
The coefficient is renormalized. Second, the Mott length scale except for two differences. First, the overall numerical may be estimated from the zero-
tunnelling amplitudes. With logarithmic precision, they are shown in Fig. 3. First, even though $C_2(\omega)$ has a strict cut-off at $\omega = \Delta_P$, its actual contribution needs to be averaged over a distribution of $\Delta_P$, so that no sharp feature is expected in $s(\omega, x)$ as a function of $\omega$. Second, the numerical values of $C_2(\omega)$ are very small, as compared to $C_1(\omega)$, so for most practical purposes the contribution $s_2(\omega, x)$ can be neglected. Third, the renormalization coefficient $C_1(\omega)$ further needs to be averaged over the distribution of $\Delta_P$, which would produce a function qualitatively similar to the left panel of Fig. 3, but quantitatively slightly smoothed (the $\omega \to 0$ limit remains the same).

VII. GENERAL CASE

In the general case ($\omega$ comparable to $\Delta_P$, but much smaller than $\Delta_\xi$), the expression (18) cannot be calculated analytically, and we need to resort to a numerical study. In addition, for a numerically accurate estimate one would need to take into account the statistical distribution of $\Delta_P$, which was neglected in earlier formulas. The main part of the calculation will be performed at a fixed $\Delta_P$, and the averaging over $\Delta_P$ will only be discussed qualitatively afterwards.

As in the previous sections, one can argue that, since $J$ is distributed on a logarithmic scale, one can write in general

$$s(\omega, x) = s_1(\omega, x) + s_2(\omega, x),$$

where $s_1(\omega, x)$ and $s_2(\omega, x)$ are the contributions from the two excited states in Eq. (18):

$$s_1(\omega, x) = -C_1(\omega)\nu^2 \int dP_x(J) \delta(\ln |J| - \ln J_{\text{eff}}^{(1)}(\omega)),$$

where $J_{\text{eff}}^{(1)}(\omega)$ are the characteristic scales of the relevant tunnelling amplitudes. With logarithmic precision, they may be estimated from the zero-$\varepsilon$ case by solving $E_i - E_0 = \omega$, where $E_i$ and $E_0$ are given by Eq. (25). For the main excitation ($i = 1$), this gives (up to a coefficient $1/2$ which is beyond our logarithmic level of precision)

$$J_{\text{eff}}^{(1)}(\omega) \sim \sqrt{\omega(\omega + \Delta_P)}.$$

Alternatively, this formula may be understood as an interpolation between the low-frequency regime [28] and the high-frequency noninteracting limit $J = \omega$ [see eq. (21)]. The second contribution ($i = 2$) has $J_{\text{eff}}^{(2)}(\omega) \sim \sqrt{\omega^2 - \Delta_p^2}$, which is of the same order of magnitude as $J_{\text{eff}}^{(1)}(\omega)$, except in a narrow vicinity of $\Delta_P$ (the second contribution only appears for $\omega > \Delta_P$).

The renormalization coefficients $C_i(\omega)$ are given by

$$C_i(\omega) = 2 \int d\ln |J| \int_0^\infty d\varepsilon \frac{(-E_0 - \langle 0 | D | i \rangle)^2 \delta(E_i - E_0 - \omega)}{\omega}.$$

Results of the numerical calculation of $C_1(\omega)$ and $C_2(\omega)$ are shown in Fig. 3. First, even though $C_2(\omega)$ has a strict cut-off at $\omega = \Delta_P$, its actual contribution needs to be averaged over a distribution of $\Delta_P$, so that no sharp feature is expected in $s(\omega, x)$ as a function of $\omega$. Second, the numerical values of $C_2(\omega)$ are very small, as compared to $C_1(\omega)$, so for most practical purposes the contribution $s_2(\omega, x)$ can be neglected. Third, the renormalization coefficient $C_1(\omega)$ further needs to be averaged over the distribution of $\Delta_P$, which would produce a function qualitatively similar to the left panel of Fig. 3, but quantitatively slightly smoothed (the $\omega \to 0$ limit remains the same).

VIII. PHYSICAL IMPLICATIONS

A. Dynamical response function

Our results can be summarized in a simple relation between the dynamical response function with and without interaction:

$$s(\omega \ll \Delta_\xi, x) = C(\omega/\Delta_P) s(\sqrt{\omega(\omega + \Delta_P)}, x)_{\text{nonint}},$$

where $C(\omega/\Delta_P)$ is a function increasing from 0.446 to 1 on the characteristic scale of $\omega \sim \Delta_P$. Note that even though our derivation above only considers the region $x \gg \xi$, the relation (34) is valid for all $x$, due to the sum rule (7) and to the fact the the correlation function of a single localized state is only weakly modified by interaction as long as $\Delta_P \ll \Delta_\xi$.

The exact form of the function $C(\omega/\Delta_P)$ should depend on the number of dimensions, since it involves averaging $C_1(\omega)$ over the statistics of $\Delta_P$. The latter, in turn, depends on the statistics of the inverse participation ratio, which is dimension dependent; such a statistics in various dimensions was studied in Refs. [8, 14–16]. The results reported there suggest that the width of the distribution of $\Delta_P$ is of the order of $\Delta_P$ itself, but in two and three dimensions it can only be obtained from numerical simulations. Such an averaging would require numerical methods beyond the simple analytical approach of the present paper. As a result, it would only smooth $C_1(\omega)$.
without changing its qualitative behavior (in particular, preserving the $\omega \to 0$ limit). For this reason, we do not perform it here, but only discuss qualitatively.

The function $s(\omega, x)_{\text{nonint}}$ is known analytically in 1D [13], but is only conjectured in higher dimensions [12].

B. Frequency-dependent conductivity

The result (34) implies a similar relation for the frequency-dependent conductivity (3):

$$\sigma(\omega \ll \Delta \xi) = C(\omega / \Delta P) \frac{\omega}{\omega + \Delta P} \sigma(\sqrt{\omega(\omega + \Delta P)})_{\text{nonint}}.$$  (35)

The well-known Mott argument based on the structure of $s(\omega, x)$ shown in Fig. [1] predicts the low-frequency behavior [11]

$$\sigma(\omega) \propto \omega^2 (\ln \omega)^{d+1}$$  (36)

in $d$ dimensions. Our result (35) implies that, while this form is preserved in the presence of interactions at $\omega \ll \Delta P$, the overall proportionality coefficient is strongly reduced by the factor $0.446 \cdot (1/2)^{d+1}$, which is about 0.03 in the three-dimensional case.

C. Polarizability

From the above discussion, one can also estimate an effect of interaction on the static polarizability $\chi$. In the presence of interaction, the integrand in Eq. (4) is strongly reduced below the energy scale $\omega \sim \Delta P$. As a consequence, the polarizability is reduced as compared to the noninteracting case. The magnitude of this interaction correction, in the regime $\Delta P \ll \Delta \xi$, can be estimated, using Eqs. (4) and (36), as

$$\frac{\delta \chi}{\chi} \propto - \frac{\Delta P}{\Delta \xi} \left( \ln \frac{\Delta \xi}{\Delta P} \right)^{d+1},$$  (37)

with a numerical proportionality coefficient of order one. The calculation presented above is insufficient to calculate this proportionality coefficient, as it would require, in particular, the knowledge of the statistical distribution of the interaction energies $\Delta_i$.

D. Universality of results

The main result (34) has a high level of universality: the only assumption used in its derivation was that the probability distribution of the hybridization matrix element $J$ is a slow function of $\ln |J|$. We have not assumed any specific details of this probability distribution. Therefore, we believe that our result is applicable to any type of disorder and in any dimension. However, the exact form of the function $C(\omega / \Delta P)$ may depend on the details of the disorder statistics and on the dimensionality.

In our derivation, we assumed zero temperature. This assumption was crucial for excluding single-particle hopping and keeping only hopping of pairs (Fig. 2). Thus the validity of our results is restricted by the condition $T \ll \Delta P$. Even in this range, some additional $T$-dependent contribution to conductivity due to thermally excited pair-tunnelling processes can be expected, similar to the Austin-Mott law [17]: we leave this subject for a future study.

E. Possible experiments

In terms of measurable quantities, our main result (35) provides a prediction for the frequency dependence of the $ac$ conductivity in the insulating state of materials that possess local attraction between electrons. By now, the best-studied example of this type is the amorphous non-stoichiometric indium oxide. It is well known that InO$_x$ with conduction electron densities of the order of a few $10^{20}$ cm$^{-3}$ is superconducting or insulating depending upon a slight variation of the oxygen content or upon applying a magnetic field of the order of several Teslas [18–20]. A pseudogap in this material was demonstrated, in particular, by scanning tunnelling spectroscopy in the superconducting state with a typical transition temperature around 1-2 K [21]. A relatively weak magnetic field is known to suppress such a fragile superconductivity, leading to an insulating state with a giant resistivity above $10^8$ Ohm at temperatures below 0.1 K [19] [22]. An even stronger insulating state is known to appear in the same InO$_x$ compound with somewhat higher resistivity [18]; a purely activated resistivity proportional to $\exp(T_0/T)$ was reported in Ref. [23] with $T_0$ as high as 15 K. These examples demonstrate that measurements of real and imaginary parts of the dielectric response in the GHz range and at sub-Kelvin temperatures should be able to experimentally test our theoretical predictions.

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