Some Exact Solutions of the Local Induction Equation for the Motion of a Vortex in a Bose–Einstein Condensate with a Gaussian Density Profile

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The dynamics of a vortex filament in a Bose–Einstein condensate whose equilibrium density in the reference frame rotating at the angular velocity $\Omega$ is Gaussian with the quadratic form $|\beta|^2$ has been considered. It has been shown that the equation of motion of the filament in the local-induction approximation permits a class of exact solutions in the form $v_0 = \mathbf{v} + \mathbf{d}t$, where $\mathbf{v}$ is the longitudinal parameter and $t$ is the time. The vortex slips over the surface of an ellipsoid, which follows from the conservation laws $\mathbf{N} \cdot \mathbf{D} = C_1$ and $\mathbf{M} \cdot \mathbf{D} = C_0 = 0$. The equation of the evolution of the tangential vector $\mathbf{d}$ appears to be closed and has integrals of motion $\mathbf{M} \cdot \mathbf{D} = C_2$ and $(\mathbf{M} \cdot \mathbf{G} \cdot \mathbf{D}) = C$, with the matrix $G = 2(\mathbf{T} \cdot \mathbf{D} - \mathbf{d})^{-1}$. Crossing of the respective isosurfaces specifies trajectories in the phase space.

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INTRODUCTION

Theoretical description of the dynamics of quantized vortex filaments in a rotating Bose–Einstein condensate is a complicated physical problem (see, e.g., [1–3] and numerous references therein). The consideration can be somewhat simplified if the rotation frequency $\Omega$ is much lower than the characteristic transverse frequency $\omega_\perp$ of the trap and the condensate appears in a so-called Thomas–Fermi regime (i.e., $[\mu - V_{\text{min}}] \gg \omega_\perp$, where $\mu$ is the chemical potential and $V_{\text{min}}$ is the minimum of the trap potential). Under such conditions, the density field inside the condensate unperturbed by a vortex, i.e., not too close to the Thomas–Fermi surface $[\mu - V(r)] = 0$, is to a good accuracy given by the expression $\rho_0(r) = \text{const}[\mu - V(r)]$ (under the assumption that the condensate wavefunction obeys the Gross–Pitaevskii equation). In this case, the characteristic size $\tilde{R}$ of the condensate is much larger than the radius $\xi$ of the vortex core and the density is nearly static far from the vortex (in the absence of potential perturbations). The behavior of the vortex filament is described to a good accuracy by classical equations of slow hydrodynamics of an inviscid compressible liquid with a given spatially inhomogeneous density profile. It is worth mentioning two technical difficulties inherent in this problem already at its initial stage. The first one is associated with the necessity of finding the velocity field $v_0(r)$ of the condensate itself unperturbed by a vortex in the reference frame rotating with the trap, which implies solving the set of equations with variable coefficients:

$$v_0 = \nabla \varphi - \Omega \times r, \quad \nabla \cdot (\rho_0 v_0) = 0, \quad (1)$$

where $\Omega$ is the trap rotation frequency vector.

Let there be a single vortex with the circulation $\Gamma = \pi \mu / m_{\text{atom}}$ in the system, the dynamics of which in three-dimensional space is described by the unknown vector function $\mathbf{R}(\beta, t)$, where $\beta$ is an arbitrary parameter along the filament and $t$ is the time. The equation of motion of such a vortex filament in classical hydrodynamics follows from the variational principle with the Lagrangian of the form (for details, see [4, 5]):

$$\mathcal{L} = \Gamma \int (\mathbf{F}(\mathbf{R}) \cdot (\mathbf{R}_{\beta} \times \mathbf{R}_\perp)) d\beta - \mathcal{H}(\mathbf{R}), \quad (2)$$

where the vector function $\mathbf{F}(\mathbf{R})$ satisfies the condition

$$\nabla \cdot \mathbf{F}(\mathbf{R}) = \rho_0(\mathbf{R}). \quad (3)$$

The Hamiltonian $\mathcal{H}(\mathbf{R})$ is the sum

$$\mathcal{H}(\mathbf{R}) = \mathcal{H}_r(\mathbf{R}) + \Gamma \int (\mathbf{A}(\mathbf{R}) \cdot \mathbf{R}_\beta) d\beta, \quad (4)$$

where $\mathcal{H}_r(\mathbf{R})$ is the kinetic energy of the vortex itself and $\mathbf{A}(\mathbf{R})$ is the vector potential of the unperturbed
(mass) flux density, i.e., $\rho_0 v_0 = \text{curl} A$. The kinetic energy functional is given by the formula

$$\mathcal{H}_R = \left(\Gamma/2\right) \int_{S_R} (\rho_0 v_R \cdot dS),$$

(5)

where $S_R$ is the surface stretched on a vortex contour and $v_R$ is the self-consistent velocity field produced by the (quasi) singular vortex filament (the integration is cut at a distance on the order of $\xi$ from the vortex line). The second difficulty is that this integral cannot be expressed in the closed form in terms of the function $\mathbf{R}(\beta, t)$, since the spatial inhomogeneity of the density prevents the analytical calculation of $v_R$. However, the functional $\mathcal{H}_R(R)$ can always be computed to a logarithmic accuracy in the so-called local-induction approximation:

$$\mathcal{H}_R(R) = \left(\Gamma^2/4\pi\right) \int \rho_0(R) |\mathbf{R}_0(R)| d\beta,$$

(6)

where the logarithm $\Lambda = \log(\tilde{R}/\xi) = \log((\mu - V_{\text{min}})/\hbar\omega_1) = \text{const} \gg 1$ is high. The respective variational equation of motion

$$\prod R_\beta \times R_\beta |\rho_0(R)| = \delta \mathcal{H}/\delta R$$

(7)

after resolving it with respect to the time derivative reads [3–5]

$$R_\beta = \frac{\Gamma \Lambda}{4\pi} \left\{ \kappa b + \left[ \frac{\nabla \rho_0(R)}{|\rho_0(R)|} \times \frac{R_\beta}{|R_\beta|} \right] \right\} + v_0(R),$$

(8)

where $\kappa$ is the local curvature of the vortex filament, $b$ is the unit binormal vector, and $|R_\beta|/|R_\beta|$ is the unit tangent vector. The solutions of this equation (mostly in the linearized form) were studied for three-dimensional harmonic traps (see, e.g., [6, 7]) or strictly two-dimensional density profiles [5]. To the best of my knowledge, no exact essentially time-dependent solutions have been found so far. In this work, an exact finite-dimensional integrable reduction of Eq. (8) will be found in the case of an anharmonic trap with a Gaussian equilibrium condensate density profile. It should be mentioned that the Gaussian density distribution was considered in the earlier theoretical investigation of vortex lattices in oblate elliptical quasi-two-dimensional condensates [8]. The Gaussian profile is assumed in the spatial region, where Eq. (8) is applicable and where the vortex energy is essentially concentrated. Deviation from the Gaussian distribution is implied only near the Thomas–Fermi surface and should be disregarded in the leading approximation.

**Simplification in the Case of a Gaussian Density Profile**

The equilibrium density profile was chosen in the form $\rho_0(r) \propto \exp(-r \cdot \hat{D} r)$ with a constant positive-definite symmetric matrix $\hat{D}$ because, first, the gradient of the density logarithm, which appears in the local induction equation, is in this case $-2\hat{D} r$, i.e., linear in $r$. Second, Eqs. (1) for the density profiles, which depend only on the combination $r \cdot \hat{D} r$, have a simple explicit solution and the velocity field $v_0$ turns out to be also linear in $r$:

$$v_0 = -[\nabla \times \hat{D} r], \quad \mathbf{B} = 2(\hat{I} \Gamma \hat{r} \hat{D} - \hat{D})^{-1} \Omega,$$

(9)

where $\hat{I}$ is the identity matrix. Indeed, assuming the solution in the form $v_0 = A \hat{r}$ and substituting it into Eq. (1), we arrive at the set of equations for the matrix $A$:

$$\epsilon_{ijk} A_{jk} = 2\Omega_i, \quad \text{Tr} A = 0, \quad \hat{D} A + A^T \hat{D} = 0.$$  

(10)

Decomposing these conditions in the basis of the fundamental axes of the quadratic form, we find that the components of the matrix $A$ are given by the formula

$$A_{ij} = 2\epsilon_{ijk} \Omega_k \frac{d_j}{d_i + d_j},$$

(11)

where $d_i > 0$ are the eigenvalues of the matrix $\hat{D}$. Next, we note that

$$2\epsilon_{ijk} \frac{\Omega_k}{d_i + d_j} = \epsilon_{ijk} B_k, \quad B_k = -\frac{2\Omega_k}{d_k + \sum_j d_j},$$

(12)

from where Eqs. (9) follow.

Taking into account the found formulas, the local induction equation (in the dimensionless form) in the Gaussian case reads

$$R_\beta = \kappa b + \left[ \frac{R_\beta}{|R_\beta|} \times \hat{D} R \right] - [\mathbf{B} \times \hat{D} R].$$

(13)

**Integrable Reduction**

An interesting observation is that Eq. (13) permits solutions in the form of a time-dependent straight vortex

$$R(\beta, t) = \beta M(t) + N(t).$$

(14)

The curvature of the vortex filament is identically zero in this case, and having substituted Eq. (14) into Eq. (13), we come to the set of ordinary differential equations

$$\mathbf{N} = \left[ \frac{\mathbf{M}}{|\mathbf{M}|} - \mathbf{B} \right] \times \hat{D} \mathbf{N},$$

(15)

$$\mathbf{M} = \left[ \frac{\mathbf{M}}{|\mathbf{M}|} - \mathbf{B} \right] \times \hat{D} \mathbf{M},$$

(16)

One can easily verify the presence of the integrals of motion

$$\mathbf{N} \cdot \hat{D} \mathbf{N} = C_1, \quad \mathbf{M} \cdot \hat{D} \mathbf{N} = C_0, \quad \mathbf{M} \cdot \hat{D} \mathbf{M} = C_2.$$  

(17)
Without the loss of generality, we can set $C_0 = 0$. Thus, it becomes clear that the straight vortex at any time is tangent to the ellipsoid $\mathbf{r} \cdot \dot{\mathbf{r}} = C_1$ at the point $\mathbf{N}$.

We notice now that the equation for $\mathbf{M}$ has a noncanonical Hamiltonian structure resembling the structure of the Landau–Lifshitz equation, yet involving the matrix:

$$
\mathbf{M} = \left[ \frac{\partial H(\mathbf{M})}{\partial \mathbf{M}} \times \dot{\mathbf{M}} \right], \quad H = |\mathbf{M}| - \mathbf{B} \cdot \mathbf{M}. \quad (18)
$$

We did not assume so far that the angular velocity is independent of the time in the reference frame of the principal axes of the ellipsoid: the above equations of motion also hold for time-dependent $\Omega(t)$. However, if $\Omega = \text{const}$ (which will be assumed below), the Hamiltonian $H(\mathbf{M}) = C$ is one more integral of motion.

The length of the vector $\mathbf{M}$ does not have a direct geometrical meaning, only its direction $\mathbf{m} = \mathbf{M}/|\mathbf{M}|$ matters. Hence, it is convenient for further investigation of the dynamic system in question to use the combination of the conservation laws, which involve only the unit tangent vector $\mathbf{m}$:

$$
\gamma(\mathbf{m}) = \frac{1 - \mathbf{B} \cdot \mathbf{m}}{\sqrt{\mathbf{m} \cdot \dot{\mathbf{m}}}} = \text{const}, \quad \mathbf{m}^2 = 1. \quad (19)
$$

It is interesting to mention that the equation of motion for $\mathbf{m}$ has a somewhat different Hamiltonian structure:

$$
\mathbf{m} = \sqrt{(\mathbf{m} \cdot \dot{\mathbf{m}})^3} \left[ \frac{\partial \gamma(\mathbf{m})}{\partial \mathbf{m}} \times \mathbf{m} \right]. \quad (20)
$$

The trajectories of the system are the lines of constant $\gamma(\mathbf{m})$ values on a unit sphere. The phase portraits can be qualitatively different depending on the relation between the quantities $d_i$ and the value of the vector parameter $\mathbf{B}$. We introduce the parametrization

$$
\begin{align*}
    d_1 &= 1 + \alpha, & d_2 &= 1 - \alpha, & d_3 &= \lambda, \quad (21) \\
    \mathbf{m} &= \left( \sqrt{1 - q^2} \cos \phi, \sqrt{1 - q^2} \sin \phi, q \right) \quad (22)
\end{align*}
$$

(where $q$ is the cosine of the polar angle) and consider by two examples how the control parameter $\Omega$ changes the vortex dynamics.
**Example 1.** Let the vector \( \mathbf{\Omega} \) be directed along the \( z \) axis. Then, we have the family of curves corresponding to different \( \gamma \) values

\[
1 - \Omega q = \gamma \sqrt{(1 - q^2)(1 + \alpha \cos 2\phi) + \lambda q^2}.
\]

Figure 1 presents the phase portraits at \( \alpha = 0.3 \) and \( \lambda = 0.6 \) (rotation about the major axis of a triaxial ellipsoid) for \( \Omega = 0.05 \) and \( 0.3 \) (only a quarter of the unit sphere is shown; the curves should be continued symmetrically in the azimuthal direction). As is seen, both “poles” at a sufficiently low rotation frequency are stable singular points of the “center” type. There are two more centers on the \( x \) meridians and two saddle points on the \( y \) meridians. Such a structure of the integral lines is natural, since at zero rotation frequency of the trap we actually deal with classical equations describing the dynamics of a solid. At a faster rotation, the “south pole” becomes a center and two meridian saddle points approach the “north pole,” transforming it to a “saddle” after bifurcation.

Figure 2 corresponds to the parameter \( \lambda = 1.1 \); i.e., the rotation proceeds about the middle axis. At a low angular velocity, both poles are saddle points and there are two poles on each of the \( x \) and \( y \) meridians. With an increase in \( \Omega \), the centers on the \( x \) meridians approach the north pole and transform it to a center.

Figure 3 corresponds to \( \lambda = 1.5 \); i.e., the rotation proceeds about the minor axis. In this case, two saddle points on the \( x \) meridians approach the initially stable south pole with an increase in \( \Omega \) and transform it into a saddle, whereas the north pole remains a center and two more centers persist on the \( y \) meridians.

It is worth mentioning that only two centers (not shown) survive in all cases at an even higher rotation frequency. However, one should bear in mind that in practice two or more vortices interacting with each other penetrate into the condensate at fast rotation.

**Example 2.** Let now the anisotropy be \( \alpha = 0 \) but the vector \( \mathbf{B} \) form an angle with the symmetry axis of the ellipsoid. Then,

\[
1 - B_q q - B_z \sqrt{1 - q^2} \cos \phi = \gamma \sqrt{1 + (\lambda - 1)q^2}.
\]

Figure 4 shows the phase portraits in the case of a prolate ellipsoid of revolution at \( \lambda = 0.5 \) for two \( \mathbf{B} \) vectors of the same direction and distinct magnitudes (only half of the unit sphere is shown; the curves should be continued symmetrically in the azimuthal direction). At a lower rotation frequency, there are two
centers on the $x_-$ meridian near the points $\pm e_x$, one more center on the $x_+$ meridian, and a saddle on the $x_-$ meridian. At a higher rotation frequency, the saddle and center near the north pole annihilate and, as a result, only two centers survive on the sphere.

Figure 5 shows the case of an oblate ellipsoid of revolution at $\lambda = 2.0$. Here, at a lower rotation frequency, there is one center on the $x_-$ meridian and two centers and a saddle on the $x_+$ meridian. At a higher rotation frequency, the saddle and center annihilate near the south pole.

CONCLUSIONS

Thus, it has been shown theoretically that three-dimensional dynamics of a quantum vortex filament in a rotating essentially anisotropic Bose–Einstein condensate with a Gaussian density profile can occur in the regime of a straight vortex, when rotation of the trap and spatial inhomogeneity of the condensate dominate over the effect of the filament curvature. The respective integrable set of ordinary differential equations has been analyzed. The behavior of the straight vortex can be controlled to a certain extent by varying the trap rotation frequency. It is natural to assume that a qualitatively similar regime, when the curvature of the vortex filament is noticeable only near the surface of the condensate, can also occur under the inclusion of nonlocal corrections to the equations of motions and for nearly Gaussian profiles. At least, approximate solutions in the form of a straight vortex passing through the symmetry point of a nearly spherical nonrotating harmonic trap have been found in [3].

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