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PSEUDODIFFERENTIAL ARITHMETIC AND A FAILED ATTEMPT ON THE RIEMANN HYPOTHESIS

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Abstract. A criterion for the validity of the Riemann hypothesis reduced the problem to the search for a certain estimate, relative to a hermitian form associated by means of the Weyl symbolic calculus of operators to a distribution in the plane of an arithmetic nature. One can reduce the question further to an algebraic question. After completion of the calculation of the hermitian form obtained, this attempt does not seem to lead to a genuinely new method of analysis of the conjecture. A better cooperation between usual analysis and congruence arithmetic may be called for, and some possible hints are given at the end.

1. Closing a previous attempt

The present version is identical to the previous one (v7). Only, I should have pointed towards what was incorrect in the previous version (v6): at the very end, I was in too much of a hurry to say that Lemma 3.5 would apply as well in the situation of interest. This is not correct, which kills the proof.

2. Reminders

The two ingredients of the present paper are the Weyl symbolic calculus of operators [5] and the theory of Eisenstein distributions, a chapter in the theory of modular distributions. The novel developments start with Lemma 3.1: all that precedes consists of reminders of results published at least 4 years ago, and amply verified on many occasions.

We normalize the Weyl symbolic calculus as follows. Given $\mathcal{S} \in \mathcal{S}'(\mathbb{R}^2)$, the operator with symbol $\mathcal{S}$ is the linear operator $\Psi(\mathcal{S})$ from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$ weakly defined by the equation

$$ (\Psi(\mathcal{S}) u)(x) = \frac{1}{2} \int_{\mathbb{R}^2} \mathcal{S}\left(\frac{x + y}{2}, \xi\right) e^{i\pi(x-y)\xi} u(y) \, dy \, d\xi, \quad (2.1) $$
or \((v \mid \Psi(\mathcal{G}) u) = \langle \mathcal{G}, W(v, u) \rangle\) with

\[
W(v, u)(x, \xi) = \int_{-\infty}^{\infty} \overline{v}(x + t) u(x - t) e^{2i\pi t\xi} \, dt.
\] (2.2)

The Weyl calculus has been for more than half a century, under the name of "pseudodifferential analysis", one of the main tools in the study of partial differential equations: but the methods used in the present context – for the most, congruence arithmetic – do not intersect the ones experienced there. In [3], the symbolic calculus \(\Psi\) was denoted as \(\text{Op}_2\) and connected by a pair of rescalings [3, (2.1.10)] to the calculus \(\text{Op}\) used in most of the book. We shall quote results of [3] in their \(\Psi\)-version, rather than \(\text{Op}\)-version: it would be a trivial if lengthy job to make the transformations explicit, or we may rely on the fact that we have remade all calculations in [4].

With \(a(r) = \prod_{p\mid r}(1 - p)\) for \(r = 1, 2, \ldots\), consider the distribution

\[
\mathfrak{T}_N(x, \xi) = \sum_{j, k \in \mathbb{Z}} a((j, k, N)) \delta(x - j)\delta(\xi - k),
\] (2.3)

where \((j, k, N) = \text{g.c.d.}(j, k, N)\). Setting \(2i\pi \xi = 1 + x \frac{\partial}{\partial x} + \xi \frac{\partial}{\partial \xi}\), and \((t^{2i\pi \xi} \mathcal{G})(x, \xi) = t \mathcal{G}(tx, t\xi)\) for \(t > 0\), one has ([3, (3.1.9)] or [4, (4.11)])

\[
\mathfrak{T}_N = \prod_{p\mid N} (1 - p^{-2i\pi \xi}) \, \text{Dir},
\] (2.4)

where \(\text{Dir}(x, \xi)\) is the Dirac comb \(\sum_{j, k \in \mathbb{Z}} \delta(x - j)\delta(\xi - k)\). The distribution \(\mathfrak{T}_N\) depends only on the "squarefree version" of \(N\).

We consider also the distribution \(\mathfrak{T}_\infty\) defined as the limit as \(N \to \infty\) along a sequence of integers such that any given squarefree integer eventually divides \(N\), of the distribution \(\mathfrak{T}_N\) obtained from \(\mathfrak{T}_N\) by dropping the term such that \(j = k = 0\): this amounts to replacing \(a((j, k, N))\) by \(a((j, k))\) in the remaining terms. The prime 2 was recognized in [3] as a minor plague. This is the main reason for our having decided to use the version \(\Psi\) of the Weyl calculus: doing so makes it possible to use squarefree odd integers only. If taking \(N\) odd, we shall obtain in the limit a distribution to be denoted as \(\mathfrak{T}_\infty\).

The distributions \(\mathfrak{T}_N\) and \(\mathfrak{T}_\infty\) are automorphic, i.e., invariant under the linear changes of coordinates in \(\mathbb{R}^2\) with matrices in \(SL(2, \mathbb{Z})\). Automorphic distributions homogeneous of some degree refine modular forms of the non-holomorphic type, though they were truly introduced under the influence of the Lax-Phillips scattering theory [1], and [2] was devoted to an exposition of the theory obtained. A special case consists of Eisenstein distributions: if \(\nu \in \mathbb{C}, \text{Re}\nu > 1\), the Eisenstein distribution \(\mathfrak{E}_{-\nu}\) is the modular distribution
homogeneous of degree $-1 + \nu$ defined by the equation [2, p.11], valid for every $h \in \mathcal{S}(\mathbb{R}^2)$,

$$\langle E_{-\nu}, h \rangle = \sum_{|j|+|k|\neq 0} \int_0^\infty t^\nu h(jt, kt) dt.$$  \hspace{1cm} (2.5)

One has the decomposition [3, p.19], valid in the weak sense in $\mathcal{S}'(\mathbb{R}^2)$,

$$T_\infty = \frac{1}{2i\pi} \int_{\text{Re}\nu=c} \frac{E_{-\nu}}{\zeta(\nu)} d\nu, \quad c > 1$$  \hspace{1cm} (2.6)

and, inserting an extra factor $(1 - 2^{-\nu})^{-1}$, one would obtain a similar decomposition for the distribution $T_\infty^2$. One has also if $c > 1$

$$T_N = \frac{1}{2i\pi} \int_{\text{Re}\nu=c} \frac{E_{-\nu}}{\zeta_N(\nu)} d\nu \quad \text{with} \quad \zeta_N(\nu) = \prod_{p\mid N} (1 - p^{-\nu})^{-1}. \hspace{1cm} (2.7)$$

The Eisenstein distribution $E_{-\nu}$ extends as a meromorphic function of $\nu$ in the complex plane, with simple poles at $\nu = 1$ and $-1$, the residues at which are 1 and the negative $-\delta$ of the unit mass at 0. Since $(2i\pi \xi) E_{-\nu} = \nu E_{-\nu}$, one has if R.H. does hold, for every $\varepsilon > 0$, the bound $Q^{2i\pi \xi} \mathcal{T}_\infty = O(Q^{1/2+\varepsilon})$ in the topology of $\mathcal{S}'(\mathbb{R}^2)$. The converse is true, but one can, and one must, do much better than that, combining the question with the Weyl symbolic calculus. A rephrasing of [3, Lemma 2.2.5], to be reproved later, is the following. Let $\nu \in \mathbb{C}$, $\nu \neq \pm 1$, and let $v, u \in C^\infty(\mathbb{R})$ be two functions, the interiors of the supports of which are disjoint and both contained in $[0, \beta]$ with $\beta > 2$ and $b^2 - a^2 < 8$. One has

$$\left( v \mid \Psi(E_{-\nu}) u \right) = 2 \int_1^\infty t^{\nu-1} \pi(t + t^{-1}) u(t - t^{-1}) dt.$$  \hspace{1cm} (2.8)

It follows that if $\nu \neq \pm 1$ and $\beta > 2$ are given, one can find a pair $v, u$ of $C^\infty$ functions with disjoint supports contained in $[0, \beta]$ such that $(v \mid \Psi(E_{-\nu}) u) \neq 0$: assuming $\beta < 2^{\frac{3}{2}}$ and taking $u$ supported in $[0, \sqrt{\beta^2 - 4}]$ and $v$ in $[2, \beta]$, this is easy to ascertain.

In [3, Prop. 3.4.2], we proved the following (necessary and) sufficient condition for the Riemann hypothesis to hold. That, for some $\beta > 2$ and every function $w \in C^\infty$ supported in $[0, \beta]$, there should exist for every $\varepsilon > 0$ a constant $C > 0$ such that

$$\left| \left( w \mid \Psi(Q^{2i\pi \xi} \mathcal{T}_\infty) w \right) \right| \leq C Q^{\frac{1}{2}+\varepsilon}$$  \hspace{1cm} (2.9)
for every squarefree integer $Q$. The trick consists in using the function, holomorphic for $\Re s > \frac{3}{2}$,

$$F(s) = \sum_{Q \text{ squarefree}} Q^{-s} \left( w | \Psi \left( Q^{2i\pi E} \mathcal{S}_\infty \right) w \right) = \frac{1}{2i\pi} \int_{\Re \nu = c} h(\nu) f(s - \nu) \, d\nu,$$

(2.10)

where $c > 1$ and

$$h(\nu) = (\zeta(\nu))^{-1} \left( w | \Psi \left( \mathcal{E}_- \nu \right) w \right), \quad f(s - \nu) = \sum_{Q \text{ sqf}} Q^{-s+\nu} = \frac{\zeta(s - \nu)}{\zeta(2(s - \nu))}.$$

(2.11)

Assuming that $\rho_0$ is a “bad” zero of zeta, choosing $c$ such that $1 < c < \frac{1}{2} + \Re \rho_0$ and changing the line $\Re \nu = c$ to a contour $\gamma$ enclosing $\rho_0$ but no other zero, such that $c - \frac{1}{2} < \Re \nu \leq c$ for $\nu \in \gamma$, one obtains from the theorem of residues, with the help of the special case of (2.6) for which $v = u = w$, that $F(s)$ is singular at $s = 1 + \rho_0$, a contradiction. Using (2.8), it is easy to generalize this criterion to the case when the pair $w, w$ is replaced by any pair $v, u$ with the property that there exist $x$ in the support of $v$ and $y$ in that of $u$ such that $x^2 - y^2 > 4$.

Then, we observe (this is a consequence of (2.2) together with the fact that $W(v, u)(x, \xi) = 0$ unless $0 < x < \beta$ if the algebraic sum of the supports of $v$ and $u$ is contained in $[0, 2\beta]$) that one has $\left( v | \Psi \left( Q^{2i\pi E} \mathcal{S}_\infty \right) u \right) = \left( v | \Psi \left( Q^{2i\pi E} \mathcal{S}_N \right) u \right)$ under this support condition if $N$ is a squarefree multiple of $Q$ divisible by all primes $< \beta Q$. Finally, using $\mathcal{S}_\infty$ to drop the prime 2, one obtains the following criterion for R.H. That, for some $\beta > 2$ and every pair $v, u$ of $C^\infty$ functions satisfying the above support condition, there should exist for every $\varepsilon > 0$ a constant $C > 0$ with the following property: that, for every squarefree odd integer $Q$, one should be able to find a squarefree odd integer $N = RQ$ divisible by all odd primes $< \beta Q$ such that

$$\left| \left( v | \Psi \left( Q^{2i\pi E} \mathcal{S}_N \right) u \right) \right| \leq C Q^{\frac{1}{4} + \varepsilon}.$$

(2.12)

The benefit is that this expression is amenable to a fully algebraic treatment, as shown in [3, Prop. 4.1.3] (and reconsidered in detail in [4] with the new normalization of the Weyl calculus). Indeed, introducing the linear space $E[2N^2]$ of complex-valued functions on $\mathbb{Z}/(2N^2)\mathbb{Z}$ and, for every function $u \in \mathcal{S}(\mathbb{R})$, the function $\theta_N u \in E[2N^2]$ defined by the equation

$$(\theta_N u)(n) = \sum_{n_1 \in \mathbb{Z}} u \left( \frac{n_1}{N} \right), \quad n \mod 2N^2,$$

(2.13)
one obtains an identity

\[(v | \Psi (Q^{2i\pi \mathcal{E}} \mathcal{I}_N) u) = \sum_{m,n \mod 2N^2} c_{R,Q}(m,n) \overline{\vartheta_N v(m)} \vartheta_N u(n). \tag{2.14}\]

The coefficients \(c_{R,Q}(m,n)\) are fully explicit, and the symmetric matrix defining this hermitian form has a Eulerian structure.

If, under the isomorphism \(\mathbb{Z}/(2N^2)\mathbb{Z} \sim \mathbb{Z}/R^2\mathbb{Z} \times \mathbb{Z}/(2Q^2)\mathbb{Z}\), \(n\) identifies with a pair \((n',n'')\), let us denote as \(\nabla\) the class that identifies with the pair \((n',-n'')\). Set, if \(u \in \mathcal{S}(\mathbb{R})\) and \(n \in \mathbb{Z}/(2N^2)\mathbb{Z}\), \((\Lambda_{R,Q} \vartheta_N u)(n) = (\vartheta_N u)(\nabla)\).

There exists a transformation \(\Lambda^\sharp_{R,Q}\) of \(\mathcal{S}(\mathbb{R})\) such that, for any \(u \in \mathcal{S}(\mathbb{R})\), the transfer formula \(\Lambda_{R,Q} \vartheta_N u = \vartheta_N \Lambda^\sharp_{R,Q} u\) should hold. One has then the identity \([3, \text{Theor. 4.2.2}]\) and \([3, \text{Cor. 4.2.7}]\)

\[(v | \Psi (Q^{2i\pi \mathcal{E}} \mathcal{I}_N) u) = \mu(Q) (v | \Psi (\mathcal{I}_N) \Lambda^\sharp_{R,Q} u) \tag{2.15}\]

involving the M"obius indicator \(\mu\). The reflection \(\Lambda^\sharp_{R,Q}\) is given explicitly \([3, \text{Prop. 4.2.3}]\) by the formula

\[
\left(\Lambda^\sharp_{R,Q} w\right)(x) = \frac{1}{Q^2} \sum_{0 \leq \sigma, \tau < Q^2} w\left(x + \frac{2R\tau}{Q}\right) \exp\left(2\pi i \frac{\sigma(Nx + R^2 \tau)}{Q^2}\right). \tag{2.16}\]

We have given three different proofs of the identity (2.15): the first has already been cited, and a second proof was given in \([3, \text{p.62-64}]\). A third, shorter, verification was given in \([4, \text{Theorem 8.2}]\).

3. The Algebra

**Lemma 3.1.** Let \(N\) be a squarefree odd integer, and let \(v, u\) be two functions in \(\mathcal{S}(\mathbb{R})\). One has for every squarefree odd integer \(N\) the identity

\[(v | \Psi (\mathcal{I}_N) u) = 2 \sum_{T|N} \mu(T) \sum_{j,k \in \mathbb{Z}} \overline{v(Tj + k)} u(Tj - k) \tag{3.1}\]

**Proof.** Together with the operator \(2i\pi \mathcal{E}\), let us introduce the operator \(2i\pi \mathcal{E}^2 = r \frac{\partial}{\partial r} - s \frac{\partial}{\partial s}\) when the coordinates \((r,s)\) are used on \(\mathbb{R}^2\). One has if \(\mathcal{F}_2^{-1}\) denotes the inverse Fourier transformation with respect to the second variable \(\mathcal{F}_2^{-1}[(2i\pi \mathcal{E}) \mathcal{G}] = (2i\pi \mathcal{E}^2) \mathcal{F}_2^{-1} \mathcal{G}\) for every tempered distribution
From the relation (2.4) between $\mathfrak{T}_N$ and the Dirac comb, and Poisson’s formula, one obtains

$$\mathcal{F}_{-1}^{-1} \mathfrak{T}_N = \prod_{p|N} \left( 1 - p^{-2i\pi \xi} \right) \mathcal{F}_{-1}^{-1} \text{Dir} = \sum_{T|N} \mu(T) T^{-2i\pi \xi} \text{Dir}, \quad (3.2)$$

explicitly

$$\left( \mathcal{F}_{-1}^{-1} \mathfrak{T}_N \right)(r, s) = \sum_{T|N} \mu(T) \sum_{j,k} \delta \left( \frac{T}{T} - j \right) \delta(Ts - k). \quad (3.3)$$

The integral kernel of the operator $\Psi (\mathfrak{T}_N)$ is

$$K(x, y) = \left( \mathcal{F}_{-1}^{-1} \mathfrak{T}_N \right) \left( \frac{x + y}{2}, \frac{x - y}{2} \right) = 2 \sum_{T|N} \mu(T) \sum_{j,k} \delta \left( x - Tj - \frac{k}{T} \right) \delta \left( y - Tj + \frac{k}{T} \right). \quad (3.4)$$

The equation (3.1) follows.

**Remark 2.1.** It is part of the folklore to conceive of R.H. as originating from a duality between discrete measures on the line, one of which would be carried in some spectral sense by the zeros of zeta and the other by the set of primes. Actually, using (2.6) and replacing (3.1) by its limit as $N \rightarrow \infty$, this notation meaning that any given set of primes (or of primes $\neq 2$) ultimately divides $N$, we have obtained an identity which certainly goes into this direction, with the following differences: it is a two-dimensional version of it, and primes have to be replaced by squarefree integers. The duality expresses the relation between the symbol and the integral kernel of the same operator. This being said, we are still far, at this point, from having completed the algebraic-arithmetic part.

**Lemma 3.2.** Let $Q$ be a squarefree odd positive integer and let $\beta > 0$ be given. There exists $R > 0$, with $N = RQ$ squarefree odd divisible by all odd primes $< \beta Q$, such that $R \equiv 1 \mod 2Q^2$.

**Proof.** Choose $R_1$ positive, odd and squarefree, relatively prime to $Q$, divisible by all odd primes $< \beta Q$ relatively prime to $Q$, and $\overline{R_1}$ such that $\overline{R_1}R_1 \equiv 1 \mod 2Q^2$ and $\overline{R_1} \equiv 1 \mod R_1$. Choose (Dirichlet’s theorem) a prime $r$ such that $r \equiv 1 \mod R_1$ and $r \equiv \overline{R_1} \mod 2Q^2$. The number $R = R_1r$ satisfies the desired condition.

□
Proposition 3.3. Let $N = RQ$ be a squarefree odd integer such that $R \equiv 1 \mod 2Q^2$: given $T_1|R$, let $S = \frac{R}{T_1}$. Given $v, u \in S(\mathbb{R})$, one has the identity

$$
(v \mid \Psi (Q^{2i\pi x} \zeta_N) u) = 2 \sum_{Q_1 Q_2 = Q} \mu(Q_1) \sum_{T_1|R} \sum_{j,k \in \mathbb{Z}} \mu(T_1) \overline{v} \left( T_1 Q_2 j + \frac{k}{T_1 Q_2} \right) u \left( T_1 Q_2 j - \frac{k}{T_1 Q_2} + \frac{2R\omega}{Q_2} \right),
$$

(3.5)

where $\omega$ is the integer characterized by the conditions $0 \leq \omega < Q_1 Q_2^2$ and $\omega \equiv Sk - T_1 Q_2^2 j \mod Q_1 Q_2^2$.

Proof. Combining Lemma 3.1 and (2.16), one has

$$
(v \mid \Psi (Q^{2i\pi x} \zeta_N) u) = \frac{2\mu(Q)}{Q^2} \sum_{T|N} \mu(T) \sum_{0 \leq \sigma, \tau < Q^2} \sum_{j,k \in \mathbb{Z}} \overline{v} \left( T j + \frac{k}{T} \right) u \left( T j - \frac{k}{T} + \frac{2R\tau}{Q} \right) \exp \left( \frac{2i\pi\sigma}{Q^2} \left[ N \left( T j - \frac{k}{T} \right) + R^2\tau \right] \right).
$$

(3.6)

With $\lambda = Q^{-2} \left[ N \left( T j - \frac{k}{T} \right) + R^2\tau \right]$, so that $Q^2\lambda \in \mathbb{Z}$ since $T|N$, one has

$$
\frac{1}{Q^2} \sum_{0 \leq \sigma < Q^2} e^{2i\pi\sigma\lambda} = \text{char}(\lambda \in \mathbb{Z}) = \text{char} \left( NT j - \frac{Nk}{T} + R^2\tau \equiv 0 \mod Q^2 \right)
$$

(3.7)

Let $(T, Q) = Q_2$, and $T = T_1 Q_2$, $Q = Q_1 Q_2$. One has $(T_1, Q) = 1$, $T_1|R$ and $\frac{N}{T} = \frac{Q R}{T_1 Q_2} = \frac{Q_1 R}{T_1}$. The condition $\lambda \in \mathbb{Z}$, or $NT j - \frac{Q_1 R k}{T_1} + R^2\tau \equiv 0 \mod Q^2$ implies $\tau \equiv 0 \mod Q_1$: we set $\tau = Q_1 \omega$. Since $R \equiv 1 \mod 2Q^2$, the quotient $S = \frac{R}{T_1}$ is a representative of the inverse of the class of $T_1$ in $(\mathbb{Z}/(2Q^2)\mathbb{Z})^\times$. Using repeatedly the fact that $R \equiv 1 \mod Q^2$, the condition $NT j - \frac{Q_1 R k}{T_1} + R^2\tau \equiv 0 \mod Q^2$, which expresses that $\lambda \in \mathbb{Z}$, can be rewritten as $\omega \equiv Sk - T_1 Q_2^2 j \mod Q_1 Q_2^2$.

Regarding $Q_1, Q_2$ as fixed, we now rewrite the corresponding part of (3.6), taking as summation variables $T_1, j, k$. One has $T = T_1 Q_2$, one has already performed the summation with respect to $\sigma$, and $\tau \equiv Q_1 \omega$ is determined by the condition $\omega \equiv Sk - T_1 Q_2^2 j \mod Q_1 Q_2^2$, together with
0 ≤ ω < Q_1 Q_2^2.

□

Lemma 3.4. Assume that Q ≥ 3, that u is supported in [−2, 2] and v in [2, 2^{1/2}]. The identity (3.5) can be rewritten as

\[
\left( v \mid \Psi \left( Q^{2 \pi i E} \tau_N \right) u \right) = 2 \sum_{Q_1 Q_2 = Q} \mu(Q_1) \sum_{T_1 \in R} \mu(T_1) \sum_{j,k \in \mathbb{Z}} \nu \left( T_1 Q_2 j + \frac{k}{T_1 Q_2} \right) u \left( T_1 Q_2 j + \frac{k}{T_1 Q_2} - \frac{2aQ_2}{T_1} \right) .
\]

(3.8)

One has \((2^{1/2} - 1) Q_2 < T_1 < (2^{1/2} + 1) Q_2\) for all nonzero terms of this sum.

Proof. Denoting as x and y the arguments of \(\nu\) and u, one has for every nonzero term of the sum (3.5) the inequalities

\[
2 < x = T_1 Q_2 j + \frac{k}{T_1 Q_2} < 2^{1/2}, \quad -2 < y = T_1 Q_2 j - \frac{k}{T_1 Q_2} + \frac{2R \omega}{Q_2} < 2.
\]

(3.9)

Multiplying by \(\frac{Q_2}{T_1}\) the inequality obtained by taking the half-sum of the two equations (3.9) and using \(R = ST_1\), one obtains \(0 < Q_2^2 j + S \omega < \frac{(1+2^{1/2}) Q_2}{T_1}\), from which it follows that \(T_1 < (1 + 2^{1/2}) Q_2\). One has \(\omega \equiv S k \mod Q_2^2\) and

\[
S(k - RT_1 \omega) = Sk - R^2 \omega \equiv Sk - \omega \equiv 0 \mod Q_2^2,
\]

so that \(k - RT_1 \omega \equiv 0 \mod Q_2^2\). Noting that \(\frac{k - RT_1 \omega}{T_1 Q_2} = \frac{x - y}{2}\), one obtains that

\[
0 < k - RT_1 \omega < (2^{1/2} + 1) T_1 Q_2 < (2^{1/2} + 1)^2 Q_2^2.
\]

It follows on one hand that

\[
k - RT_1 \omega = a Q_2^2
\]

(3.11)

with \(a = 1, 2, 3, 4\) or 5, on the other hand that

\[
\frac{Q_2}{T_1} \leq \frac{a Q_2}{T_1} = \frac{k - RT_1 \omega}{T_1 Q_2} < 2^{1/2} + 1,
\]

(3.12)

from which \(T_1 > (2^{1/2} - 1) Q_2\).

Using the equation \(\frac{R \omega}{Q_2} = \frac{k}{T_1 Q_2} - \frac{a Q_2}{T_1}\), one can write the argument of u as

\[
y = T_1 Q_2 j + \frac{k}{T_1 Q_2} - \frac{2aQ_2}{T_1}.
\]

(3.13)

Still, there is a constraint to be careful for, to wit the one that defined \(\omega\) in Proposition 3.3. Indeed, since \(k\) and \(\omega\) are linked by the equation...
We have obtained the identity

\[
(v \mid \Psi (Q^{2\pi \xi} \Sigma_N) u) = \sum_{Q_1, Q_2} \mu(Q_1) \sum_{1 \leq a \leq 5} \sum_{T_1} \mu(T_1) \mu(\Omega) \sum_{j, k \in \mathbb{Z}} \sum_{\nu, \mu} \nu \left( T_1 Q_2 j + \frac{k}{T_1 Q_2} \right) u \left( T_1 Q_2 j + \frac{k}{T_1 Q_2} - 2a Q_2 \right).
\]

(3.14)

What remains to be proved is that \( a = 1 \).

The sum (3.8) is finite, which implies that the hermitian form \((v \mid \Psi (Q^{2\pi \xi} \Sigma_N) u)\) coincides, for \( v \) and \( u \) in \( C^\infty(\mathbb{R}) \) satisfying the support conditions in Lemma 3.4, with \( \langle K, \varphi \otimes u \rangle \), where the integral kernel \( K(x, y) \) is a certain linear combination of point masses \( \delta(x - x_m)\delta(y - y_n) \) with \( x_m, y_n \in \mathbb{Q} \). On the other hand, from the decomposition (2.7) and from (2.8), the hermitian form under consideration admits an integral kernel \( K_1 \in S'(\mathbb{R}^2) \) (cf. the first equation (3.4)), which must coincide with \( K \) when tested on functions \( \varphi \otimes u \) of the species indicated, hence also when tested on functions \( w \in C^\infty(\mathbb{R}^2) \) supported in \([2, 2\frac{2}{3}] \times [-2, 2]\). But, from (2.7) and (2.8), the hermitian form under study does not change if one multiplies the integral kernel \( K_1(x, y) \) by any function \( \phi(x^2 - y^2 - 4) \) with \( \phi \in C^\infty(\mathbb{R}) \), compactly supported, such that \( \phi(0) = 1 \). This implies that, in the sum (3.14), we may keep only the terms such that, \( x \) and \( y \) denoting the arguments of \( \varphi \) and \( u \) there, one has \( x = \sqrt{y^2 + 4} \).

With \( a = 1, \ldots, 5 \) as it occurs in (3.14), we have already used in the proof the equation \( \frac{x-y}{2} = \frac{k-RT_1 \omega}{T_1 Q_2} = \frac{a Q_2}{T_1} \). Since \( x^2 - y^2 = 4 \), one has \( x + y = \frac{2T_1}{a Q_2}, \) and

\[
x = \frac{T_1}{a Q_2} + \frac{a Q_2}{T_1}, \quad y = \frac{T_1}{a Q_2} - \frac{a Q_2}{T_1}.
\]

(3.15)

Then, \((T_1 Q_2)x = a^{-1}T_1^2 + a Q_2^2\) while, on the other hand,

\[
(T_1 Q_2)x = (T_1 Q_2)^2 j + k = (T_1 Q_2)^2 j + a Q_2^2 + RT_1 \omega.
\]

(3.16)
It follows that
\[(T_1Q_2)^2j + RT_1\omega = a^{-1}T_1^2 \quad \text{and} \quad Q_2j + S\omega = a^{-1}, \quad (3.17)\]
hence \(a = 1\).

\[\Box\]

**Proposition 3.5.** Under the support assumptions of Lemma 3.4, one has
\[
(v | \Psi(Q^{2\pi \xi \tau_N})u) = 2 \sum_{Q_2 | Q} \mu(Q/Q_2) \sum_{T_1 | R} \mu(T_1) \nu(\frac{T_1}{Q_2}, \frac{Q_2}{T_1}) u(\frac{T_1}{Q_2} - \frac{Q_2}{T_1}).
\]

\[
(3.18)
\]

**Proof.** Since \(a = 1\), it follows from (3.15) that, in any nonzero term of (3.8), the arguments \(x\) and \(y\) of \(\nu\) and of \(y\) make up the pair \((\frac{T_1}{Q_2} + \frac{Q_2}{T_1}, \frac{T_1}{Q_2} - \frac{Q_2}{T_1})\).

On the other hand, from (3.8), the first term of this pair is \(T_1Q_2j + k\).

Hence, \((T_1Q_2)^2j + k = Q_2^2 + T_1^2\); besides, \(S\omega = 1 - Q_2^2j\).

This implies \(\omega \equiv T_1 \mod Q_2^2\). Then, as \(j \equiv S^2 \mod Q_1\), one has
\[
S\omega = 1 - Q_2^2j \equiv 1 - S^2Q_2^2 \mod Q_1, \quad \omega \equiv T_1 - RQS_2^2 \mod Q_1. \quad (3.19)
\]

Assuming that \(T_1\) is known, the solution of this pair of congruences is \(\omega \equiv T_1 - SQ_2^2 \mod Q_1Q_2^2\), which determines \(\omega\) as \(0 \leq \omega < Q_1Q_2^2\). The knowledge of \(k = RT_1\omega + Q_2^2\) and that of \(j = \frac{1-S\omega}{Q_2}\) follow. With \(\ell \in \mathbb{Z}\) such that \(\omega = T_1 - SQ_2^2 + \ell Q_1Q_2^2\), one has necessarily
\[
j = \frac{1 - S(T_1 - SQ_2^2 + \ell Q_1Q_2^2)}{Q_2^2}, \quad k = RT_1(T_1 - SQ_2^2 + \ell Q_1Q_2^2). \quad (3.20)
\]

Conversely, given \(T_1\), if one defines \(j\) and \(k\) by this pair of equations, with any choice of \(\ell\) (but there is truly no choice since the integer \(T_1 - SQ_2^2 + \ell Q_1Q_2^2\) must lie in \([0, Q_1Q_2^2 - 1]\)), one has
\[
T_1Q_2j + \frac{k}{T_1Q_2} = A + \ell B \quad (3.21)
\]
with
\[
B = \frac{T_1}{Q_2} \times (-SQ_1Q_2^2) + \frac{RT_1}{T_1Q_2}(T_1 - SQ_2^2 + \ell Q_1Q_2^2) = -RQ_1Q_2 + RQ_1Q_2 = 0 \quad (3.22)
\]
and
\[
A = \frac{RT_1^2 - RST_1Q_2^2}{T_1Q_2} + \frac{T_1}{Q_2}(1 - ST_1 + S^2Q_2^2) + \frac{Q_2}{T_1}
= \frac{RT_1}{Q_2} - \frac{R^2Q_2}{T_1} + \frac{T_1}{Q_2} - \frac{ST_1^2 Q_2}{Q_2} + RQS_2 + \frac{Q_2}{T_1} = \frac{T_1}{Q_2} + \frac{Q_2}{T_1}, \quad (3.23)
\]
A FAILED ATTEMPT ON THE RIEMANN HYPOTHESIS

From what has been recalled in the section on reminders, the validity for at least one pair \(v, u\) such that \(v | \Psi \left( Q^{2\pi i \xi} \chi_N \right) u \neq 0\) for some \(y\) of the estimate

\[
\left( v | \Psi \left( Q^{2\pi i \xi} \chi_N \right) u \right) = O(Q^{\frac{1}{2} + \varepsilon})
\]

would suffice to imply R.H. Since the number of divisors of \(Q\) is, for every \(\varepsilon > 0\), a \(O(Q^\varepsilon)\), the collection of estimates

\[
\sum_{T_1 \mid R} \mu(T_1) \psi \left( \frac{T_1}{Q_2} + \frac{Q_2}{T_1} \right) u \left( \frac{T_1}{Q_2} - \frac{Q_2}{T_1} \right) = O(Q^{\frac{1}{2} + \varepsilon}),
\]

or the estimate

\[
\sum_{(T,Q)=1} \mu(T) \psi \left( \frac{T}{Q} + \frac{Q}{T} \right) u \left( \frac{T}{Q} - \frac{Q}{T} \right) = O(Q^{\frac{1}{2} + \varepsilon}),
\]

would suffice.

4. Autopsy of the attempt

Introducing the function \(F(t) = \psi (t + t^{-1}) u (t - t^{-1})\), thus reducing the sum in (3.26) to \(\sum_{(T,Q)=1} \mu(T) F \left( \frac{Q}{T} \right)\), it seems that, after much useless work, we have reduced the criterion based on pseudodifferential arithmetic to a one-dimensional (no operators) criterion. Then, one could start from the measure

\[
\sum_{k \neq 0} \mu(k) \delta(t - k) = \frac{1}{2i\pi} \int_{\text{Re} \nu = c} \frac{|t|^{\nu-1}}{\zeta(\nu)} \, d\nu, \quad c > 1,
\]

and follow exactly, in this one-dimensional context, the proof of the criterion indicated in (2.10), (2.11), dispensing with all the algebra that precedes.

We owe the reader an explanation of our interest in operator theory in connection with the problem under consideration: it had to do with the beautiful structure of a hermitian form closely related to the one which has kept us busy in this paper. In [3, Theorem 4.3.4], we proved the following, which we quote in its \(\Psi\)-version.

**Theorem 4.1.** Given a squarefree odd integer \(Q\), let \(\Lambda_Q\) be the set \(\{\lambda \mod Q : \lambda^2 \equiv 1 \mod Q\}\), in other words the set of \(\lambda \in (\mathbb{Z}/Q\mathbb{Z})^\times\) such that \(\lambda \equiv \pm 1 \mod p\) for every \(p|Q\). Let \(\chi_Q : \Lambda_Q \to \{\pm 1\}\) be the character defined by setting
\( \chi_Q(\lambda) = -1 \) if one has \( \lambda \equiv 1 \pmod{p} \) except for an odd number of \( p \)'s. Given \( \tau \in (\mathbb{Z}/Q\mathbb{Z})^\times \), define the measure on the line

\[
\mathcal{Q}_{Q, \tau}(x) = \sum_{\lambda \in \Lambda_Q} \chi_Q(\lambda) \sum_{\ell \in \mathbb{Z}} \delta \left( x - \frac{Q\ell + \tau\lambda}{Q^2} \right).
\]

(4.2)

Then, given \( w \in S(\mathbb{R}) \), one has

\[
(w \mid \Psi(Q^{i\pi E} \mathfrak{T}_Q) w) = 2^{-i(Q)} \sum_{\tau \in (\mathbb{Z}/Q\mathbb{Z})^\times} \left| \langle \mathcal{Q}_{Q, \tau} \mid w \rangle \right|^2,
\]

(4.3)

where \( i(Q) \) is the number of distinct prime factors of \( Q \).

Note that the rescaling factor \( Q^{2i\pi E} \) has been replaced by \( Q^{i\pi E} \). The operator \( \Psi(Q^{i\pi E} \mathfrak{T}_Q) \) is thus an (unbounded in any classical Hilbert sense) non-negative symmetric operator. There is in connection to this operator a map \( \theta_{Q^2} \) from \( S(\mathbb{R}) \) to \( E[Q^2] \) (cf. [3, (4.1.15)] similar to the map \( \theta_N \) in (2.14) and, after transfer under \( \theta_{Q^2} \), the operator \( \Psi(Q^{i\pi E} \mathfrak{T}_Q) \) becomes the orthogonal projection onto the space of “totally odd” functions of \( \tau \), i.e., functions of \( (\tau_p)_{p \mid Q} \) which are separately odd with respect to each variable. The left-hand side is well understood in a classical sense too. One has

\[
(w \mid \Psi(Q^{i\pi E} \mathfrak{T}_Q) w) = \frac{a(Q)}{Q} \times Q^{\frac{1}{2}} \left[ \|w\|^2 + \mu(Q) \left( w \mid \bar{w} \right) \right] + O(Q^\varepsilon)
\]

(4.4)

([3, Prop.34.1] and [3, Prop.4.5.1], noting that the coefficient of the main term has been erroneously changed to its limit as \( q \to \infty \) in the second reference). The main term would disappear if replacing the pair \( w, w \) by a pair \( v, u \) such that neither the supports of \( v \) and \( u \) nor those of \( \bar{v} \) and \( \bar{u} \) intersect (as in Lemma 3.4 and Proposition 3.5). Note that when using \( Q^{i\pi E} \) in place of \( Q^{2i\pi E} \), R.H. would require bounding \( (w \mid \Psi(Q^{i\pi E} \mathfrak{T}_N) w) \) by a \( O \left( Q^{\frac{1}{2} + \varepsilon} \right) \); but doing this only when \( N = Q \), or when \( N \) is bounded by some power of \( Q \) [3, Prop.3.4.6] would be far from sufficient.

Our excitement about all this originated from the belief that the identity (4.3) could be the starting point of a Hilbert space structure in which arithmetic and usual analysis would equally participate. This belief may have been well-founded or not, but we are still convinced that R.H. depends on reaching some coherence between the arithmetic and analytic aspects of the same object. Here is another argument in favor of this point of view.
The Riemann hypothesis is back to the question (essentially an old criterion by Littlewood) whether, given any function $F \in C^\infty(\mathbb{R})$ with a compact support, the expression

$$\sum_{k \neq 0} \mu(k)F\left(\frac{k}{Q}\right) = \frac{1}{2i\pi} \int_{\text{Re } \nu = c > 1} Q^\nu \frac{\langle |x|^{\nu-1}, F \rangle}{\zeta(\nu)} \frac{d\nu}{\zeta(2\nu)}$$

(4.5)

is for every $\varepsilon > 0$, as $Q \to \infty$, $O(Q^{\frac{1}{2} + \varepsilon})$: one could let $Q$ go to infinity through squarefree integral values only.

Clearly, some regularity has to be demanded from $F$ to make this meaningful: otherwise, it would suffice to take $F$ real, the sign of $F\left(\frac{k}{Q}\right)$ being that of $\mu(k)$, to disprove R.H., making use of the identity

$$\sum_{k \neq 0} |\mu(k)| \delta(x - k) = \frac{1}{2i\pi} \int_{\text{Re } \nu = c > 1} \frac{\zeta(\nu)}{\zeta(2\nu)} |x|^{\nu-1} d\nu$$

(4.6)

and taking advantage of the pole of the integrand at $\nu = 1$.

At this point, the estimate (3.26) still seems to present over the estimate (4.5) the advantage that, since a hermitian form is the object of the analysis, Hilbert space methods and spaces similar to Sobolev spaces may help. We believe it to be so: but spaces of any classical type would be inappropriate, since they would fail the definite test of piecing analysis and arithmetic together. The Hilbert space defined by the right-hand side of (4.3) might be a good starting point.

References

[1] P.D.Lax, R.S. Phillips, *Scattering Theory for Automorphic Functions*, Ann.Math.Studies 87, Princeton Univ.Press, 1976.

[2] A.Unterberger, *Pseudodifferential operators with automorphic symbols*, Pseudodifferential Operators 11, Birkhäuser, Basel–Boston–Berlin, 2015.

[3] A.Unterberger, *Pseudodifferential methods in number theory*, Pseudodifferential Operators 13, Birkhäuser, Basel-Boston-Berlin, 2018.

[4] A.Unterberger, *Pseudodifferential arithmetic and the Riemann hypothesis: reminders*, arXiv # 2111.02792.

[5] H.Weyl, *Gruppentheorie und Quantenmechanik*, reprint of 2nd edition, Wissenschaftliche Buchgesellschaft, Darmstadt, 1977.