A note on the reduction of the $AdS_4 \times \mathbb{CP}^3$ string \( \sigma \)-model

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Abstract

We study the reduction of the (bosonic) string sigma model on $AdS_4 \times \mathbb{CP}^3$ background. We give a brief review of the known results for the AdS part and apply an explicit reduction scheme to the $\mathbb{CP}^3$ part of the model. A brief discussion on the reduced model is presented.

1 Introduction

The correspondence between the large N limit of gauge theories and string theory was on the focus of intensive promising research for more than thirty years and in different periods it showed different faces. One of the most promising explicit realizations of this correspondence was provided by the Maldacena conjecture about AdS/CFT correspondence [1]. Being an excellent example of exact duality between type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super Yang-Mills theory [1, 2, 3], this subject has became a major research and many fascinating developments have been done.

A cornerstone in the current understanding of the duality between gauge theories and strings (M-theory) is the world-volume dynamics of the branes. Recently there has been a considerable amount of work focused on the understanding of the worldvolume dynamics of multiple M2-branes - an interest inspired by Bagger, Lambert and Gustavsson [8] and their investigations based on the structure of Lie 3-algebra.

Recently, inspired by the study of the Bagger-Lambert-Gustavson theory on $N$ membranes and motivated by the possible description of the worldvolume dynamics of coin-
cident membranes in M-theory, a new class of conformal invariant, maximally supersymmetric field theories in 2+1 dimensions has been found \[9,10\]. The main feature of these theories is that they contain gauge fields with Chern-Simons kinetic terms. Based on this development, Aharony, Bergman, Jafferis and Maldacena proposed a new gauge/string duality between an \( \mathcal{N} = 6 \) super-conformal Chern-Simons theory (ABJM theory) coupled to bi-fundamental matter, describing \( N \) M2 branes on \( \mathbb{R}^8/\mathbb{Z}_k \). This model is believed to be dual to M-theory on \( AdS_4 \times S^7/\mathbb{Z}_k \).

The ABJM theory actually has two Chern-Simons gauge fields with opposite levels, \( k \) and \(-k\) correspondingly, each with gauge group \( SU(N) \) (or \( U(N) \)). The two pairs of chiral superfields transform in the bi-fundamental representations of \( SU(N) \times SU(N) \) and the R-symmetry is \( SU(4) \) as it should be for \( \mathcal{N} = 6 \) supersymmetry of the theory. It was observed in \[10\] that there exists a natural definition of a ’t Hooft coupling \(-\lambda = N/k\). It was observed that in the ’t Hooft limit \( N \to \infty \) with \( \lambda \) held fixed, one has a continuous coupling \( \lambda \) and the ABJM theory is weakly coupled for \( \lambda \ll 1 \). The ABJM theory is conjectured to be dual to M-theory on \( AdS_4 \times S^7/\mathbb{Z}_k \) with \( N \) units of four-form flux. In the scaling limit \( N,k \to \infty \) with \( k \ll N \ll k^5 \) the theory reduces to type IIA string theory on \( AdS_4 \times \mathbb{C}P^3 \). Thus, the AdS/CFT correspondence, which has led to many exciting developments in the duality between type IIB string theory on \( AdS_5 \times S^5 \) and \( \mathcal{N} = 4 \) super Yang-Mills theory, is now being extended to the \( AdS_4/CFT_3 \) and is expected to constitute a new example of exact gauge/string theory duality.

Semi-classical strings have played, and still play, an important role in studying various aspects of \( AdS_5/SYM_4 \) correspondence \[4\]-\[5\]. The development in this subject gave a strong hint about how the new emergent \( AdS_4/CFT_3 \) duality can be investigated. An important role in these studies is played by integrability. The superstrings on \( AdS_4 \times \mathbb{C}P^3 \) as a coset was first studied in \[11\] which opens the door for investigation of the integrable structures in the theory. Various properties on the gauge theory side and tests on string theory side as rigid rotating strings, pp-wave limit, relation to spin chains, as well as pure spinor formulation have been considered \[11\]-\[46\]. In these intensive studies many properties were uncovered and impressive results obtained, but still the understanding of this duality is far from complete.

**ABJM and strings on \( AdS_4 \times \mathbb{C}P^3 \)** To find the ABJM theory one starts with analysing M2-brane dynamics governed by eleven-dimensional supergravity action \[10\]

\[
S = \frac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{-g} \left( R - \frac{1}{2} \cdot \frac{1}{4!} F_{\mu
u\rho\sigma} F^{\mu
u\rho\sigma} \right) - \frac{1}{12\kappa_{11}^2} \int C^{(3)} \wedge F^{(4)} \wedge F^{(4)}, \tag{1.1}
\]

where \( \kappa_{11}^2 = 2^7\pi^8 l_p^9 \). Solving for the equations of motions

\[
R^\mu_\nu = \frac{1}{2} \left( \frac{1}{3!} F^{\mu\alpha\beta\gamma} F_{\nu\alpha\beta\gamma} - \frac{1}{3} \cdot \frac{1}{4!} \delta^\mu_\nu F_{\alpha\beta\rho\sigma} F^{\alpha\beta\rho\sigma} \right), \tag{1.2}
\]

\[^1\text{See also } [12] \]
\[
\partial_\sigma (\sqrt{-g} F^{\sigma \mu \nu \xi}) = \frac{1}{2 \cdot (4!)^2} \epsilon^{\mu \nu \xi \alpha_1 \ldots \alpha_8} F_{\alpha_1 \ldots \alpha_4} F_{\alpha_5 \ldots \alpha_8},
\]

(1.3)

one can find the M2-brane solutions whose near horizon limit becomes \(AdS_4 \times S^7\)

\[
ds^2 = \frac{R^2}{4} ds^2_{AdS_4} + R^2 ds^2_{S^7}.
\]

(1.4)

In addition we have \(N'\) units of four-form flux

\[
F^{(4)} = \frac{3 R^3}{8} \epsilon_{AdS_4}, \quad R = l_p (2^5 N' \pi^2)^{\frac{1}{6}}.
\]

(1.5)

Now one proceeds with considering the quotient \(S^7/\mathbb{Z}_k\) acting as \(z_i \rightarrow e^{i \frac{2\pi}{k}} z_i\). It is convenient first to write the metric on \(S^7\) as

\[
ds^2_{S^7} = (d\varphi' + \omega)^2 + ds^2_{\mathbb{C}P^3},
\]

(1.6)

where

\[
ds^2_{\mathbb{C}P^3} = \sum_i dz_i d\bar{z}_i - \frac{1}{r^4} \sum_i |z_i|^2,
\]

\[
d\varphi' + \omega \equiv \frac{i}{2r^2} \sum_i (z_i d\bar{z}_i - \bar{z}_i dz_i), \quad d\omega = J = i \sum_i d\left(\frac{z_i}{r}\right) d\left(\frac{\bar{z}_i}{r}\right).
\]

(1.7)

and then to perform the \(\mathbb{Z}_k\) quotient identifying \(\varphi' = \varphi/k\) with \(\varphi \sim \varphi + 2\pi\) (\(J\) is proportional to the Kähler form on \(\mathbb{C}P^3\)). The resulting metric becomes

\[
ds^2_{S^7/\mathbb{Z}_k} = \frac{1}{k^2} (d\varphi + k \varphi)^2 + ds^2_{\mathbb{C}P^3}.
\]

(1.8)

One observes that the first volume factor on the right hand side is divided by factor of \(k\) compared to the initial one. In order to have consistent quantized flux one must impose \(N' = kN\) where \(N\) is the number of quanta of the flux on the quotient. One should note that the spectrum of the supergravity fields of the final theory is just the projection of the initial \(AdS_4 \times S^7\) onto the \(\mathbb{Z}_k\) invariant states. In this setup there is a natural definition of 't Hooft coupling \(\lambda \equiv N/k\). Decoupling limit should be taken as \(N,k \rightarrow \infty\) while \(N/k\) is kept fixed.

One can follow now [10] to make reduction to type IIA with the following final result

\[
ds^2_{\text{string}} = \frac{R^3}{k} \left( \frac{1}{4} ds^2_{AdS_4} + ds^2_{\mathbb{C}P^3} \right),
\]

(1.9)

\[
e^{2\phi} = \frac{R^3}{k^3} \sim \frac{N^{1/2}}{k^{5/2}} = \frac{1}{N} \left( \frac{N}{k} \right)^{5/2},
\]

(1.10)

\[
F_4 = \frac{3}{8} R^3 \epsilon_4, \quad F_2 = kd\omega = kJ,
\]

(1.11)
We end up then with $AdS_4 \times \mathbb{CP}^3$ compactification of type IIA string theory with $N$ units of $F_4$ flux on $AdS_4$ and $k$ units of $F_2$ flux on the $\mathbb{CP}^1 \subset \mathbb{CP}^3$ 2-cycle.

The radius of curvature in string units is $R_{str}^2 = \frac{k^3}{k} = 2^{5/2} \pi \sqrt{\lambda}$. It is important to note that the type IIA approximation is valid in the regime where $k \ll N \ll k^5$.

To fix the notations, we write down the explicit form of the metric on $AdS_4 \times \mathbb{CP}^3$ in spherical coordinates. The metric on $AdS_4 \times \mathbb{CP}^3$ can be written as 

$$
    ds^2 = R^2 \left\{ \frac{1}{4} [-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\Omega_2^2] + d\mu^2 + \sin^2 \mu \left[ d\alpha^2 + \frac{1}{4} \sin^2 \alpha (\sigma_1^2 + \sigma_2^2 + \cos^2 \alpha \sigma_3^2) + \frac{1}{4} \cos^2 \mu (d\chi + \sin \mu \sigma_3)^2 \right] \right\}. 
$$

Here $R$ is the radius of the $AdS_4$, and $\sigma_{1,2,3}$ are the $SU(2)$ left-invariant 1-forms, parameterized by $(\theta, \phi, \psi)$,

$$
    \sigma_1 = \cos \psi \, d\theta + \sin \psi \sin \theta \, d\phi, \\
    \sigma_2 = \sin \psi \, d\theta - \cos \psi \sin \theta \, d\phi, \\
    \sigma_3 = d\psi + \cos \theta \, d\phi.
$$

The range of the coordinates is

$$
    0 \leq \mu, \alpha \leq \frac{\pi}{2}, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi, \quad 0 \leq \chi, \psi \leq 4\pi.
$$

2 Reduction of $AdS_4 \times \mathbb{CP}^3$ sigma models

2.1 The reduction of $AdS_4$

In this Section we present the reduction of the $AdS_4$ part of the string sigma model. Although most of the results in this section are known, it would be useful to review some methods of reduction which were proved to be useful in the context of strings. First we consider a string moving only in the $AdS_4$ part of spacetime and then assume that the motion is not constrained to that part. The difference is as follows. Considering the dynamics of the string on $AdS_4 \times \mathbb{CP}^3$ we will always assume that the Virasoro constraints are satisfied, i.e.

$$
    T_{\pm \pm}^{AdS} + T_{\pm \pm}^{\mathbb{CP}^3} = 0, \quad T_{\pm \pm}^{AdS} = -\kappa^2, \quad T_{\pm \pm}^{\mathbb{CP}^3} = \kappa^2. \quad (2.1)
$$

When the dynamics is confined only to the $AdS_4$ part of the geometry $T_{\pm \pm}^{\mathbb{CP}^3} = 0$ and thus, one can distinguish two cases. We present below both of them using slightly different approaches.
Reduction of pure $AdS_4$.

To apply the reduction scheme one must represent the $AdS_n$ space as a coset space $SO(n,1)/SO(n-1,1)$. Then the string sigma model can be thought of as sigma model on the above symmetric space. Here we follow the approach developed in [47, 48, 55, 50, 51].

Let us restrict our attention to the specific case of $AdS_4$ spacetime and consider it as a hyperboloid embedded into five-dimensional Euclidean space:

$$\vec{Y} \cdot \vec{Y} \equiv -Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 + Y_3^2 = -1 \tag{2.2}$$

Writing the action in these variables (plus the above constraint), one finds the equations of motion

$$\vec{Y}_{\xi\eta} - (\vec{Y}_{\xi} \cdot \vec{Y}_{\eta}) \vec{Y} = 0, \tag{2.3}$$

where $Y_{\xi} \equiv \partial_{\xi} Y$ etc. as usual $\xi = \frac{1}{2}(\sigma + \tau), \eta = \frac{1}{2}(\sigma - \tau)$.

The corresponding Virasoro constraints have the explicit form

$$\vec{Y}_{\xi} \cdot \vec{Y}_{\xi} = \vec{Y}_{\eta} \cdot \vec{Y}_{\eta} = 0. \tag{2.4}$$

To explicitly carry out the reduction we introduce a basis

$$\{ \vec{e}_i \} = \{ \vec{Y}, \vec{Y}_{\xi}, \vec{Y}_{\eta}, \vec{e}_4, \vec{e}_5 \} \tag{2.5}$$

where for $i = 1, 2, 3$ the properties of the basis vectors are dictated by the embedding (2.2) and equations of motion (2.3) while for $i = 4, 5$ we require the following conditions to be satisfied

$$\vec{e}_i^2 = 1, \quad \vec{e}_i \cdot \vec{Y} = \vec{e}_i \cdot \vec{Y}_{\xi} = \vec{e}_i \cdot \vec{Y}_{\eta} = 0. \tag{2.6}$$

The Virasoro constraints take the form

$$T_{\xi\xi} = \vec{Y}_{\xi}^2 = 0 \quad T_{\eta\eta} = \vec{Y}_{\eta}^2 = 0 \tag{2.7}$$

and $T_{\xi\eta} \equiv 0$ automatically (note also that $\vec{Y}_{\xi}$ and $\vec{Y}_{\eta}$ are null-vectors).

Now we define the angle $\alpha(\xi, \eta)$ through (Liouville mode for the case of $AdS_2$)

$$\vec{Y}_{\xi} \cdot \vec{Y}_{\eta} = e^{\alpha(\xi, \eta)}. \tag{2.8}$$

One can find some useful relations from the above definitions

$$\vec{Y} \cdot \vec{Y}_{\xi} = \vec{Y}_{\xi} \cdot \vec{Y}_{\eta} = 0, \quad \vec{Y}_{\xi} \cdot \vec{Y}_{\xi\xi} = \vec{Y}_{\eta} \cdot \vec{Y}_{\eta\eta} = 0, \tag{2.9}$$

\footnote{From now on the subscripts $\xi$ and $\eta$ denote derivatives with respect to the corresponding variable.}
Next we want to express the second derivatives expanded over the basis \( (2.5) \). One of them follows immediately from the definitions and the properties above

\[
\tilde{Y}_{\xi\eta} = e^{\alpha(\xi,\eta)}\tilde{Y},
\]

(2.10)

but we want also to find the other second derivatives of \( \tilde{Y} \). To find them we expand over the basis \( \{ \tilde{e}_k \} \) \( (2.5) \)

\[
\tilde{Y}_{\xi\xi} = A\tilde{Y}_{\xi} + B\tilde{Y}_{\eta} + (a_4\tilde{e}_4 + a_5\tilde{e}_5) \quad (2.11)
\]

\[
\tilde{Y}_{\eta\eta} = C\tilde{Y}_{\xi} + D\tilde{Y}_{\eta} + (b_4\tilde{e}_4 + b_5\tilde{e}_5) \quad (2.12)
\]

To obtain the coefficients \( a_i \) one must multiply \( (2.11) \) with \( \tilde{e}_i \) and take into account the orthogonality conditions \( (2.6) \) (analogously for \( b_i \)). The result is

\[
a_i = \tilde{Y}_{\xi\xi}\tilde{e}_i, \quad b_i = \tilde{Y}_{\eta\eta}\tilde{e}_i \quad (2.13)
\]

We want to obtain the coefficients \( A, B, C, D \). Using the orthogonality of the basis and \( (2.8) \) one finds

\[
B = C = 0. \quad (2.14)
\]

From

\[
\tilde{Y}_{\eta},\tilde{Y}_{\xi\xi} = Ae^\alpha, \quad \tilde{Y}_{\xi},\tilde{Y}_{\eta\eta} = De^\alpha \quad \text{and} \quad \alpha_\eta e^\alpha = \tilde{Y}_{\xi},\tilde{Y}_{\eta\eta} \quad (2.15)
\]

we find

\[
A = \alpha_\xi(\xi,\eta) \quad (2.16)
\]

\[
D = \alpha_\eta(\xi,\eta) \quad (2.17)
\]

The final form of the second derivatives of \( \tilde{Y} \) is

\[
\tilde{Y}_{\xi\xi} = \alpha_\xi(\xi,\eta)\tilde{Y}_{\xi} + (a_4\tilde{e}_4 + a_5\tilde{e}_5) \quad (2.18)
\]

\[
\tilde{Y}_{\eta\eta} = \alpha_\eta(\xi,\eta)\tilde{Y}_{\eta} + (b_4\tilde{e}_4 + b_5\tilde{e}_5) \quad (2.19)
\]

To obtain the equation for \( \alpha(\xi,\eta) \) we must eliminate all the \( \tilde{Y} \) and its derivatives. First, differentiating \( (2.8) \) with respect to \( \eta \) one finds (using also that \( \tilde{Y}_{\xi\eta} = e^\alpha\tilde{Y} \) and \( \tilde{Y}_{\xi},\tilde{Y}_{\eta\eta} = 0 \))

\[
\alpha_\eta(\xi,\eta) = e^{-\alpha(\xi,\eta)}\tilde{Y}_{\xi},\tilde{Y}_{\eta\eta} \quad (2.20)
\]

Differentiating the above equation with respect to \( \xi \) we get

\[
\alpha_{\xi\eta}(\xi,\eta) = e^{-\alpha} \left[ -\alpha_\xi\tilde{Y}_{\xi\eta} + \tilde{Y}_{\xi\xi},\tilde{Y}_{\eta\eta} + \tilde{Y}_{\xi},\tilde{Y}_{\eta\eta}\xi \right] \quad (2.21)
\]

Combining the properties of the orthogonal basis from \( (2.21) \) we find

\[
\alpha_{\xi\eta}(\xi,\eta) - e^{\alpha(\xi,\eta)} - e^{-\alpha(\xi,\eta)}(a_4b_4 + a_5b_5) = 0 \quad (2.22)
\]

\(^3\)Since \( \tilde{Y},\tilde{Y}_{\xi\xi} = 0 \) there is no term proportional to \( \tilde{Y} \).
It is a simple exercise to cast the resulting equations in linear form

\[ \frac{d}{d\xi} \vec{e}_i(\xi, \eta) = A_{ij}(\xi, \eta)\vec{e}_j(\xi, \eta) \]  
\[ \frac{d}{d\eta} \vec{e}_i(\xi, \eta) = B_{ij}(\xi, \eta)\vec{e}_j(\xi, \eta), \]  
(2.23, 2.24)

with compatibility condition

\[ \partial_{\eta} A - \partial_{\xi} B + [A, B] = 0. \]  
(2.25)

To obtain the entries of the matrices \(A\) and \(B\) one has to use the orthogonality conditions and the properties discussed above. Skipping the details, for the matrix \(A\) we find

\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & \alpha_{\xi} & 0 & a_4 & a_5 \\
e^{\alpha} & 0 & 0 & 0 & 0 \\
0 & 0 & -a_4 e^{-\alpha} & 0 & (\vec{e}_{4\xi}, \vec{e}_5) \\
0 & 0 & -a_5 e^{-\alpha} & 0 & (\vec{e}_{5\xi}, \vec{e}_4)
\end{pmatrix}, \]  
(2.26)

while for the other matrix, \(B\), we find

\[
B = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
e^{\alpha} & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha_{\eta} & b_4 & b_5 \\
0 & 0 & -b_4 e^{-\alpha} & 0 & (\vec{e}_{4\eta}, \vec{e}_5) \\
0 & 0 & -b_5 e^{-\alpha} & 0 & (\vec{e}_{5\eta}, \vec{e}_4)
\end{pmatrix}. \]  
(2.27)

One can also find the equations for the coefficients \(a_i\) and \(b_j\)

\[ a_{4\eta} = a_5(\vec{e}_5, \vec{e}_{4\eta}), \quad a_{5\eta} = a_4(\vec{e}_4, \vec{e}_{5\eta}) \]  
(2.28)
\[ b_{4\xi} = b_5(\vec{e}_5, \vec{e}_{4\xi}), \quad b_{5\xi} = b_4(\vec{e}_4, \vec{e}_{5\xi}). \]  
(2.29)

Having in mind that \((\vec{e}_4, \vec{e}_{5\eta}) = -(\vec{e}_{4\eta}, \vec{e}_5)\) and analogously \((\vec{e}_4, \vec{e}_{5\xi}) = -(\vec{e}_{4\xi}, \vec{e}_5)\), one find

\[ \partial_{\eta} [a_4^2 + a_5^2] = 0, \quad \partial_{\xi} [b_4^2 + b_5^2] = 0, \]  
(2.30)

or,

\[ a_4 = P(\eta) \cos \delta(\xi, \eta), \quad a_5 = P(\eta) \sin \delta(\xi, \eta), \]  
(2.31)
\[ b_4 = Q(\xi) \cos \gamma(\xi, \eta), \quad b_5 = Q(\xi) \sin \gamma(\xi, \eta). \]  
(2.32)

The compatibility condition for the equations (2.23, 2.24) is the well-known zero curvature condition

\[ \partial_{\eta} A - \partial_{\xi} B + [A, B] = 0. \]  
(2.33)
According to (2.31) and (2.32)
\[(\vec{e}_4, \vec{e}_5) = \gamma_\xi, \quad (\vec{e}_4, \vec{e}_5_\eta) = \delta_\eta,\] (2.34)
which combined with the compatibility condition (2.25), \(\partial_\eta A - \partial_\xi B + [A, B] = 0\), gives the equations for the dynamical variables
\[
\alpha_{\xi\eta}(\xi, \eta) - e^{\alpha(\xi, \eta)} - (a_4 b_4 + a_5 b_5) e^{-\alpha(\xi, \eta)} = 0
\]
\[
\beta_{\xi\eta}(\xi, \eta) - (a_4 b_5 - a_5 b_4) e^{-\alpha(\xi, \eta)} = 0, \quad \beta(\xi, \eta) = \gamma - \delta.
\] (2.35, 2.36)

One can use the explicit expressions for \(a_i\) and \(b_i\) to obtain
\[
\alpha_{\xi\eta}(\xi, \eta) - e^{\alpha(\xi, \eta)} - Q(\xi) P(\eta) e^{-\alpha(\xi, \eta)} \cos \beta = 0
\]
\[
\beta_{\xi\eta}(\xi, \eta) - Q(\xi) P(\eta) e^{-\alpha(\xi, \eta)} \sin \beta = 0.
\] (2.37, 2.38)

Let us make the transformations
\[
\alpha(\xi, \eta) = \dot{\alpha}(x, y) + \log[F(\xi) G(\eta)].
\] (2.39)
Choosing
\[
\frac{d x}{d \xi} = F(\xi), \quad \frac{d y}{d \eta} = G(\eta),
\]
\[
F^2(\xi) G^2(\eta) = -Q(\xi) P(\eta)
\] (2.40)
we find
\[
\dot{\alpha}_{xy}(x, y) - e^{\dot{\alpha}(x, y)} + e^{-\dot{\alpha}(x, y)} \cos \beta(x, y) = 0
\]
\[
\dot{\beta}_{xy}(x, y) + e^{-\dot{\alpha}(x, y)} \sin \beta(x, y) = 0.
\] (2.41, 2.42)

One must note that the derivation of the above result relies on the Virasoro constraints (2.4) defining \(\vec{Y}_\xi\) and \(\vec{Y}_\eta\) as null-vectors. This, however, is not the general case. We proceed with the more general case in the next paragraph.

Reduction of \(AdS_4\) part of string sigma model.

Let us consider the string sigma model on \(AdS_4 \times \mathbb{CP}^3\). As we already discussed above, this case is different because although the total energy-momentum is vanishing, the energy-momentum of each of the two parts in the product space is a non-vanishing constant (2.1). In this case we will shortly present the method of [50, 51, 49] applied to this concrete case.
We start with the parametrization of the Lax connection. The linear problem associated with our sigma model is defined by

\[
\partial_\xi \phi(\xi, \eta, \zeta) = L^a(\xi, \eta, \zeta) \phi(\xi, \eta, \zeta),
\]
\[
\partial_\eta \phi(\xi, \eta, \zeta) = M^a(\xi, \eta, \zeta) \phi(\xi, \eta, \zeta).
\] (2.43)

The general dependence on \( \zeta \) is as follows

\[
L^a = \zeta A^a + C^a, \quad M^a = \zeta^{-1} B^a + D^a.
\] (2.44)

The consistency condition of the above defined linear problem reads off

\[
\partial_\eta L^a - \partial_\xi M^a + [L^a, M^a] = 0
\] (2.45)

and it splits into

\[
\partial_\eta A^a + [A^a, D^a] = 0 \quad (2.46)
\]
\[
\partial_\xi B^a + [B^a, C^a] = 0 \quad (2.47)
\]
\[
\partial_\eta C^a - \partial_\xi D^a + [A^a, B^a] + [C^a, D^a] = 0
\] (2.48)

The general form of the matrices \( A^a \) and \( B^a \) is

\[
A^a = \begin{pmatrix}
0 & \bar{\psi}_1 & \bar{\psi}_2 & \bar{\psi}_3 \\
-\bar{\psi}_1 & 0 & 0 & 0 \\
-\bar{\psi}_2 & 0 & 0 & 0 \\
-\bar{\psi}_3 & 0 & 0 & 0
\end{pmatrix}
\] (2.49)

and

\[
B^a = \kappa \begin{pmatrix}
0 & Y_1 & Y_2 & Y_3 & Y_4 \\
-Y_1 & 0 & 0 & 0 & 0 \\
-Y_2 & 0 & 0 & 0 & 0 \\
-Y_3 & 0 & 0 & 0 & 0 \\
-Y_4 & 0 & 0 & 0 & 0
\end{pmatrix}.
\] (2.50)

Here \( Y_i \) satisfy the relation

\[
Y_1^2 - \sum_{k=2}^{2} Y_k^2 = 1.
\] (2.51)

With the help of certain gauge transformations one can make \( D^a = 0 \). Then the matrices \( L^a \) and \( M^a \) can be brought into the form

\[
L^a = \zeta A^a + C^a, \quad M^a = \zeta^{-1} B^a,
\] (2.52)

where

\[
A^a = \kappa \begin{pmatrix}
0 & -1 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\] (2.53)
The matrix $C^a$ belongs to the centralizer of $SO(2, 5)$ and has the form

$$C^a = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -c_2 & -c_3 & c_4 \\ 0 & c_2 & 0 & 0 & 0 \\ 0 & c_3 & 0 & 0 & 0 \\ 0 & c_4 & 0 & 0 & 0 \end{pmatrix}.$$  \hfill (2.54)

Eliminating $c_k$, we end up with the equations

$$\frac{\partial \xi}{\partial \eta} + \frac{\partial \eta Y_k}{\sqrt{1 + \sum_{l=2}^{4} Y_l^2}} = -\kappa^2 Y_k, \quad k = 2, 3, 4. \hfill (2.55)$$

One can use the following convenient parametrization \[50, 51\]

$$Y_1 = \cosh 2\phi, \quad Y_k = r_k \sinh 2\phi, \quad \sum_{j=2}^{4} r_l^2 = 1, \quad k = 2, 3, 4. \hfill (2.56)$$

Substituting into (2.55) and using (2.51) we find

$$\frac{\partial \xi \partial \eta \phi - \frac{1}{2} \tanh 2\phi \sum_{l=2}^{4} \partial \eta r_l \partial \xi r_l + \frac{\kappa^2}{2} \sinh 2\phi = 0} \hfill (2.57)$$

$$\frac{\partial \xi \partial \eta r_l + \left( \sum_{k=2}^{4} \partial \xi r_k \partial \eta r_k \right) r_l + \frac{2}{\sinh 2\phi} \left( \cosh 2\phi \partial \eta \phi \partial \xi r_l + \frac{1}{\cosh 2\phi} \partial \xi \phi \partial \eta r_l \right) = 0, \quad l = 2, 3, 4. \hfill (2.58)$$

We must note that to further reduce the degrees of freedom one may need to fix the residual conformal symmetry. For instance, in the lower dimensional $AdS_3$ case such a gauge fixing relates the above approach to the sinh-Gordon equation for a single dynamical field \[51\]

### 2.2 Reduction of $\mathbb{CP}^3$

In this Section we investigate the reduction of the string sigma model on $\mathbb{CP}^3$.

#### General remarks

Let us briefly review the basic properties of $\mathbb{CP}^n$. The most convenient way to define $n$-dimensional complex projective space $\mathbb{CP}^n$ is as the family of one-dimensional subspaces in $\mathbb{C}^{n+1}$, i.e. this is the quotient $\mathbb{C}^{n+1}/(\mathbb{C} \setminus \{0\})$. The equivalence relation is defined as

$$\alpha Z_1 : \cdots : \alpha Z_{n+1} = Z : \cdots : Z_{n+1}.$$
The space $\mathbb{C}P^n$ itself is covered by patches $U_i : \{Z_1 : \cdots : Z_{n+1} \in \mathbb{C}P^n \mid Z_i \neq 0\}, i = 1, \cdots, n+1$. One can see that each patch $U_i$ is isomorphic to $\mathbb{C}P^n$, where the isomorphism is defined by $W_j^{(i)} = Z_j/Z_i, j \neq i$. One can choose local coordinates $W = (W_1, W_2, \cdots, W_n)^t \in \mathbb{C}^{n+1}$ with $W_j \equiv W_j^{(n+1)}$. The Fubini-Study metric then is given by the line element

$$ds^2 = \frac{(1 + |W|^2)|dW|^2 - |W^\dagger dW|^2}{(1 + |W|^2)^2}. \quad (2.59)$$

One can think of $\mathbb{C}P^n$ as the homogeneous space $\mathbb{C}P^n = U(n+1)/(U(n) \times U(1))$. The $u(n+1)$ Lie algebra $\mathfrak{f}$ can be realized as anti-hermitian matrices and splits into two parts: $\mathfrak{p} = u(n) \oplus u(1)$ and its orthogonal completion $\mathfrak{cp}(n)$ with respect to the $U(n+1)$ Killing form

$$\mathfrak{p} = u(n) \oplus u(1) = \{iM \in u(n+1) \mid [\Gamma, M] = 0\}
\mathfrak{cp}(n) = \{iM \in u(n+1) \mid \{\Gamma, M\} = 0\}, \quad (2.60)$$

where $M$ is traceless and hermitian and

$$\Gamma = \begin{pmatrix} -1 \\ \text{1}_n \end{pmatrix}.$$

Using diagonal embedding of $\mathfrak{p}$, $\mathbb{C}P^n$ can be though as an orbit in the coset with a generator of $\mathfrak{cp}(n)$ part, $B$ then is given by

$$B = \begin{pmatrix} -W & \text{W}^\dagger \end{pmatrix}. \quad (2.61)$$

Then one can write schematically

$$\mathfrak{f} = \mathfrak{p} \oplus \mathfrak{cp}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{cp}] \subset \mathfrak{cp}, \quad [\mathfrak{cp}, \mathfrak{cp}] \subset \mathfrak{cp}. \quad (2.62)$$

The linear problem associated with our sigma model is defined by

$$\partial_\zeta \phi_\zeta, \eta, \zeta) = L(\xi, \eta, \zeta) \phi_{\xi, \eta, \zeta}
\partial_\eta \phi_\xi, \eta, \zeta) = M(\xi, \eta, \zeta) \phi_{\xi, \eta, \zeta}. \quad (2.63)$$

The general dependence on $\zeta$ is as follows

$$L = \zeta A + C, \quad M = \zeta^{-1}B + D. \quad (2.64)$$

The consistency condition of the above defined linear problem reads off

$$\partial_\eta L - \partial_\zeta M + [L, M] = 0 \quad (2.65)$$

and it splits into

$$\partial_\eta A + [A, D] = 0 \quad (2.66)
\partial_\zeta B + [B, C] = 0 \quad (2.67)
\partial_\eta C - \partial_\zeta D + [A, B] + [C, D] = 0. \quad (2.68)$$
The general form of the matrices $A$ and $B$ is

\[
A = \begin{pmatrix}
0 & \tilde{\psi}_1 & \tilde{\psi}_2 & \tilde{\psi}_3 \\
-\psi_1 & 0 & 0 & 0 \\
-\psi_2 & 0 & 0 & 0 \\
-\psi_3 & 0 & 0 & 0
\end{pmatrix}
\] (2.69)

and

\[
B = \begin{pmatrix}
0 & W_1 & W_2 & W_3 \\
-W_1 & 0 & 0 & 0 \\
-W_2 & 0 & 0 & 0 \\
-W_3 & 0 & 0 & 0
\end{pmatrix}
\] (2.70)

By making use of the gauge transformations one can make $D=0$ and bring the matrices $L$ and $M$ into the form

\[
L = \zeta A + C, \quad M = \zeta^{-1} B,
\] (2.71)

where

\[
A = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\] (2.72)

\[
C = \begin{pmatrix}
c_1 & 0 & 0 & 0 \\
0 & -c_1 & -\bar{c}_2 & -\bar{c}_3 \\
0 & c_2 & 0 & 0 \\
0 & c_3 & 0 & 0
\end{pmatrix}
\] (2.73)

Since we are dealing with CP$^3$ sigma model, one can always normalize the fields as

\[
|W_1|^2 + |W_2|^2 + |W_3|^2 = 1.
\] (2.74)

Using this normalization, the entries of the matrix $C$ are given by

\[
c_1 = \frac{W_1 \partial_\xi \bar{W}_1 + \partial_\xi W_2 \bar{W}_2 + \partial_\xi W_3 \bar{W}_3}{3|W_1|^2 - 1},
\] (2.75)

\[
c_k = \frac{\partial_\xi W_k}{W_1} + \frac{W_k}{W_1} c_1, \quad k = 2, 3.
\] (2.76)

The condition (2.68) gives

\[
\partial_\eta c_1 + W_1 - W_1 = 0
\]
\[
\partial_\eta c_k + W_k = 0, \quad k = 2, 3.
\] (2.77)

The reduced system involves 5 independent real scalar fields.
Equations of motion for the reduced model

We will use the following parametrization

\[ W_i = r_i e^{i \phi_i}, \quad i = 1, 2, 3, \]  

(2.78)

which implies

\[ r_1^2 + r_2^2 + r_3^2 = 1. \]  

(2.79)

The matrix \( B \) takes the form

\[
B = \begin{pmatrix}
0 & r_1 e^{-i \phi_1} & r_2 e^{-i \phi_2} & r_3 e^{-i \phi_3} \\
-r_1 e^{i \phi_1} & 0 & 0 & 0 \\
-r_2 e^{i \phi_2} & 0 & 0 & 0 \\
-r_3 e^{i \phi_3} & 0 & 0 & 0 \\
\end{pmatrix}.
\]  

(2.80)

The entries of the matrix \( C \) are

\[ c_1 = \frac{r_2^2 \partial_\xi \phi_2 + r_3^2 \partial_\xi \phi_3 - r_1^2 \partial_\xi \phi_1}{3r_1^2 - 1} := \frac{i}{2} \partial_\xi \beta \]  

(2.81)

\[ c_k = \frac{e^{i(\phi_k - \phi_1)}}{r_1} \left[ \partial_\xi r_k + i r_k (\partial_\xi \phi_k + \frac{1}{2} \partial_\xi \beta) \right], \quad k = 2, 3. \]  

(2.82)

Let us determine one of the angles, say the angle \( \phi_2 \), through the other angles. Using

\[ r_2^2 \partial_\xi \phi_2 + r_3^2 \partial_\xi \phi_3 - r_1^2 \partial_\xi \phi_1 = \frac{1}{2}(3r_1^2 - 1) \partial_\xi \beta \]  

(2.83)

and the equations (2.74)-(2.77) one finds that

\[ r_2 \partial_\xi (\phi_2 + \frac{1}{2} \beta) = \frac{r_2^2}{r_2} \partial_\xi \phi_1 - \frac{r_3^2}{r_2} \partial_\xi + \frac{1}{2r_2} (2r_1^2 - r_2^2) \partial_\xi \beta, \]  

(2.84)

and

\[ \partial_\eta (\phi_2 - \phi_1) = -\frac{1}{r_2^2} \left[ \partial_\eta \beta + (r_1^2 + r_2^1) \partial_\eta \phi_1 + r_3^2 \partial_\eta \phi_3 \right]. \]  

(2.85)

Then

\[ c_2 = e^{i(\phi_2 - \phi_1)} \left[ \partial_\xi r_2 + i \left( \frac{r_2^2}{r_1} \partial_\xi \phi_1 - \frac{r_3^2}{r_2} \partial_\xi \phi_3 \right) + \frac{2r_1^2 - r_3^2}{2r_1 r_2} \partial_\xi \beta \right], \]  

(2.86)

\[ c_3 = e^{i(\phi_3 - \phi_1)} \left[ \partial_\xi r_3 + i \frac{r_3}{r_1} \partial_\xi (\phi_3 + \frac{1}{2} \beta) \right]. \]  

(2.87)

After long and tedious, but straightforward calculations one finds the equation of motion of the dynamical variables of the system. They are found to be

\[
\Box r_2 + \frac{\partial_\xi r_2 \partial_\eta r_2}{r_2} - \frac{\partial_\eta (r_1 r_2) \partial_\xi r_2}{r_1 r_2} \nonumber \\
+ \frac{(\partial_\eta \beta + (r_1^2 + r_2^1) \partial_\eta \phi_1 + r_3^2 \partial_\eta \phi_3)(r_1^2 \partial_\xi \phi_1 - r_3^2 \partial_\xi (\phi_3 + \frac{1}{2} \beta) + r_1^2 \partial_\xi \beta)}{r_3^2} \\
+ r_1 r_2 \cos \phi_1 = 0,
\]  

(2.88)
\[ \partial_\phi_1 + \frac{r_3[\partial_\xi r_3 \partial_\eta (\phi_3 - \phi_1) - \partial_\eta r_3 \partial_\xi (\phi_3 + \frac{1}{2} \beta)]}{r_1^2} + \frac{r_3^2 \partial_\xi r_1 \partial_\xi (\phi_3 + \frac{1}{2} \beta)}{r_1^3} + \frac{2 \partial_\eta r_1 \partial_\xi (\phi_1 + \beta)}{r_1^2} = \frac{\partial_\xi (r_1 + \beta) + r_2^2 \partial_\eta (\phi_3 - \phi_1)}{r_1^2} + \frac{2(2r_1^2 + r_3^2) + r_1 r_2}{r_1} \sin \phi_1 = 0 \quad (2.89) \]

\[ \Box \beta - 4r_1 \sin \phi_1 = 0, \quad (2.90) \]

\[ \Box r_3 - \frac{\partial_\xi r_3 \partial_\eta r_1}{r_1} - r_3 \partial_\eta (\phi_3 - \phi_1) \partial_\xi (\phi_3 + \frac{1}{2} \beta) + r_1 r_3 \cos \phi_1 = 0, \quad (2.91) \]

\[ \Box \phi_3 + \frac{\partial_\eta r_3 \partial_\xi (\phi_3 + \frac{1}{2} \beta) + \partial_\xi r_3 \partial_\eta (\phi_3 - \phi_1)}{r_3} = \frac{\partial_\eta r_1 \partial_\xi (\phi_3 + \frac{1}{2} \beta)}{r_1} + 3r_1 \sin \phi_1 = 0 \quad (2.92) \]

Note that \( r_1^2 + r_2^2 + r_3^2 = 1 \) so that the number of dynamical variables is five.

To reduce the \( \mathbb{CP}^3 \) system to the \( \mathbb{CP}^2 \) case one has (carefully) to set \( r_3 = \phi_3 = 0 \). It is easy then (using the parametrization \( r_1 = \cos \alpha, r_2 = \sin \alpha \)) to see that the equations reduce to the known ones.

3 Conclusions

This study is inspired by the recent breakthrough in our understanding of membrane dynamics and its application to \( AdS_4/CFT_3 \) correspondence [10]. The hopes are that this is another example of exact duality between gauge theory and strings/M-theory. The wide range of possible applications make the subject even more attractive. This strongly motivates the intensive research on various aspects of string theory on \( AdS_4 \times \mathbb{CP}^3 \) background. The experience from the well studied \( AdS_5 \times S^5 \) case teaches us that the techniques coming from integrable systems and the study of integrable structures play an important role.

In this note we initiated a more detailed analysis of the reduction of string sigma model on the \( AdS_4 \times \mathbb{CP}^3 \) background. We presented a short analysis of the Pohlmeyer reduction of the \( AdS_4 \) part of the sigma model action and its extension to the string sigma models action. After we performed a reduction of the \( \mathbb{CP}^3 \) part of the Polyakov string action. As a result we found the equations of motion for the dynamical degrees of freedom of the reduced model. Certainly this study is far from complete, nevertheless it seems to be good basis to proceed further. For instance, it would be interesting to use Bäcklund transformations (or dressing method) to generate nontrivial solutions. Even restricted to some subspaces such investigations would be very useful.

Note added. After this paper was sent to the Arxiv another interesting and related to our study paper appeared [56]. It presents a systematic study of Pohlmeyer reduction.
with emphasis to the Lagrangian formulation. Although there is a partial overlap, in general the results in the two papers are complimentary and may be useful in further study of AdS/CFT correspondence.

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Appendix: Reduction of $AdS_4 \times \mathbb{CP}^3$ sigma model

The two seemingly unrelated parts of $AdS_4 \times \mathbb{CP}^3$ sigma model actually are related through the Virasoro constraints

$$T_{\pm \pm} (AdS_4) + T_{\pm \pm} (\mathbb{CP}^3) = 0. \quad (A.1)$$

The reduction above then have to be modified. Actually one can show (see for instance [54, 38]) that

$$T_{\pm \pm} (\mathbb{CP}^3) = \mu^2 = -T_{\pm \pm} (AdS_4), \quad (A.2)$$

which causes small but important modifications in the above reduction. One must note that there are cases when the string dynamics is confined in one part of the geometry, $S^5$ or $\mathbb{CP}^3$. In this case we have $T_{\pm \pm} (\mathbb{CP}^3) = 0 = T_{\pm \pm} (AdS_4)$ and these cases should be derivable taking the limit $\mu \to 0$ and therefore these cases are covered by the analysis above.

We find for the matrix $A$ the expression

$$A = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
-\mu^2 & \frac{\alpha e^\alpha}{\Upsilon} & \frac{\alpha \mu^2}{\Upsilon} & a_4 & a_5 \\
e^\alpha & 0 & 0 & 0 & 0 \\
0 & -\frac{\mu^2 a_4 e^{-\alpha}}{\Upsilon} & -\frac{a_4}{\Upsilon} & 0 & (\vec{e}_4, \vec{e}_5) \\
0 & -\frac{\mu^2 a_5 e^{-\alpha}}{\Upsilon} & -\frac{a_5}{\Upsilon} & (\vec{e}_5, \vec{e}_4) & 0
\end{pmatrix} \quad (A.3)$$

where $\Upsilon = e^\alpha - \mu^4 e^{-\alpha}$. Analogously one can find for the matrix $B$

$$B = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
e^\alpha & 0 & 0 & 0 & 0 \\
-\mu^2 & \frac{\alpha e^\alpha}{\Upsilon} & \frac{a_4 e^\alpha}{\Upsilon} & b_4 & b_5 \\
0 & -\frac{b_4}{\Upsilon} & -\frac{\mu^2 b_4 e^{-\alpha}}{\Upsilon} & 0 & (\vec{e}_4, \vec{e}_5) \\
0 & -\frac{b_5}{\Upsilon} & -\frac{\mu^2 b_5 e^{-\alpha}}{\Upsilon} & (\vec{e}_5, \vec{e}_4) & 0
\end{pmatrix} \quad (A.4)$$
The compatibility condition (2.33) gives the following equations for the entries of A and B:

\[ \Box \alpha - \frac{\mu^4 \alpha \xi e^{-\alpha}}{e^\alpha - \mu^4 e^{-\alpha}} - e^\alpha + \mu^4 e^{-\alpha} - (a_4 b_4 + a_5 b_5) e^{-\alpha} = 0 \]  
(A.5)

\[ \partial_\eta a_4 + \frac{\mu^2 \alpha \xi}{e^\alpha - \mu^4 e^{-\alpha}} b_4 + a_5 (\bar{a}_5, \bar{a}_4) = 0, \quad \partial_\eta a_5 + \frac{\mu^2 \alpha \xi}{e^\alpha - \mu^4 e^{-\alpha}} b_5 + a_4 (\bar{a}_4, \bar{a}_5) = 0 \]  
(A.6)

\[ \partial_\xi b_4 + \frac{\mu^2 \alpha \eta}{e^\alpha - \mu^4 e^{-\alpha}} a_4 + b_5 (\bar{b}_5, \bar{b}_4) = 0, \quad \partial_\xi b_5 + \frac{\mu^2 \alpha \eta}{e^\alpha - \mu^4 e^{-\alpha}} a_5 + b_4 (\bar{b}_4, \bar{b}_5) = 0. \]  
(A.7)

From these one can derive

\[ \partial_\eta (a_4^2 + a_5^2) = -(a_4 b_4 + a_5 b_5) \frac{\mu^2 \alpha \xi}{\Upsilon} \]  
(A.8)

\[ \partial_\xi (b_4^2 + b_5^2) = -(a_4 b_4 + a_5 b_5) \frac{\mu^2 \alpha \eta}{\Upsilon}, \]  
(A.9)

or,

\[ \frac{\mu^4 \alpha \xi \alpha \eta}{\Upsilon^2} = \frac{\partial_\xi (b_4^2 + b_5^2) \partial_\eta (a_4^2 + a_5^2)}{(a_4 b_4 + a_5 b_5)^2}. \]  
(A.10)

Then one can write

\[ \Box \alpha - \frac{\partial_\xi (b_4^2 + b_5^2) \partial_\eta (a_4^2 + a_5^2)}{(a_4 b_4 + a_5 b_5)^2} e^{-\alpha} (e^\alpha - \mu^4 e^{-\alpha}) - (e^\alpha - \mu^4 e^{-\alpha}) - (a_4 b_4 + a_5 b_5) e^{-\alpha} = 0, \]  
(A.11)

or

\[ \Box \alpha - \frac{\partial_\xi (b_4^2 + b_5^2) \partial_\eta (a_4^2 + a_5^2)}{(a_4 b_4 + a_5 b_5)^2} (1 - \mu^4 e^{-2\alpha}) - (e^\alpha - \mu^4 e^{-\alpha}) - (a_4 b_4 + a_5 b_5) e^{-\alpha} = 0. \]  
(A.12)

When \( \mu \to 0 \), the r.h.s. of (A.8) and (A.9) vanishes and the equation for \( \alpha(\xi, \eta) \) reduces to that in (2.41) (2.42). Note that the vanishing of the r.h.s. of (A.8) and (A.9) means \( \partial_\eta \alpha_k = 0 \) and \( \partial_\xi b_k = 0 \) (k=1,2) which is related to the fixing of the conformal symmetry.

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