ON THE PERIODS OF SOME FEYNMAN INTEGRALS

FRANCIS BROWN

Abstract. We study the related questions: (i) when Feynman amplitudes in massless $\phi^4$ theory evaluate to multiple zeta values, and (ii) when their underlying motives are mixed Tate. More generally, by considering configurations of singular hypersurfaces which fiber linearly over each other, we deduce sufficient geometric and combinatorial criteria on Feynman graphs for both (i) and (ii) to hold. These criteria hold for some infinite classes of graphs which essentially contain all cases previously known to physicists. Calabi-Yau varieties appear at the point where these criteria fail.

0.1. Background. Let $G$ be a connected graph. To each edge $e \in E_G$ associate a variable $\alpha_e$, known as a Schwinger parameter, and consider the graph polynomial

$$\Psi_G = \sum_{T \subset G \in E(T)} \prod_{e \in E} \alpha_e,$$

where the sum is over all spanning trees $T$ of $G$. It is homogeneous of degree $h_1(G)$, the loop number of $G$. When $G$ satisfies $|E_G| = 2h_1(G)$ and is primitive for the Connes-Kreimer coproduct, one can show that the Feynman integral:

$$I_G = \int_0^\infty \cdots \int_0^\infty \prod_{e \in E_G} \frac{d\alpha_e}{\Psi_G^2} \delta\left( \sum_{e \in E_G} \alpha_e - 1 \right),$$

converges and defines a real number. This quantity is renormalization-scheme independent, and it is a general fact that Feynman integrals in tensor quantum field theories can be reduced to scalar integrals at the cost of modifying only the numerator, and not the denominator, of these integrals. Thus the residues (1), and their variants with numerators, capture much of the number-theoretic content of any massless, single-scale quantum field theory in four dimensions.

Broadhurst and Kreimer [2], and recently Schnetz [21] have computed the residues $I_G$ by a variety of impressive numerical and analytic methods for all such graphs in $\phi^4$ theory up to six loops, and for some graphs up to nine loops. They found in all identifiable cases that $I_G$ is a rational linear combination of multiple zeta values

$$\zeta(n_1, \ldots, n_r) = \sum_{1 \leq k_1 < \cdots < k_r} \frac{1}{\prod_{i=1}^r k_i^{n_i}}, \quad n_i \in \mathbb{N}, n_r \geq 2,$$

which are periods of the mixed Tate motives of the unipotent fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. For a long time, a widely held view was consequently that all such residues (1) should evaluate to multiple zeta values, and our original goal was to prove this. However, the methods of this paper have recently led to counterexamples which make this very unlikely, even for planar graphs [6]. Therefore the question we seek to address is to determine for which graphs $G$ is $I_G$ a combination of multiple zeta values. This is the analytic side of the problem.

The algebro-geometric approach to this problem was initiated by Bloch, Esnault and Kreimer in the foundational paper [5], where they interpreted $I_G$ as the period...
of a mixed Hodge structure, as follows. Let $X_G \subset \mathbb{P}^{E_G - 1}$ denote the singular graph hypersurface defined by the zero locus of $\Psi_G$, and let $\Delta \subset \mathbb{P}^{E_G - 1}$ denote the union of the coordinate hyperplanes. Since $X_G$ and $\Delta$ do not cross normally, they constructed a blow-up $P \to \mathbb{P}^{E_G - 1}$, and defined the ‘graph motive’ to be:

\begin{equation}
(2) 
 m_G = H^{E_G - 1}(P \setminus Y, B \setminus (B \cap Y)) ,
\end{equation}

where $B$ denotes the total transform of $\Delta$ and $Y$ the strict transform of $X_G$. They proved that the residue $I_G$ is a period of the mixed Hodge structure underlying $m_G$. The algebraic version of the question raised earlier is to determine for which graphs $G$ is $m_G$ (or some piece of $m_G$ which carries the period $I_G$) of mixed Tate type.

A more accessible problem is instead to consider the point-counting function

$$ R_G : q \mapsto |X_G(\mathbb{F}_q)| $$

which to any graph $G$ associates the number of points of $X_G$ over finite fields $\mathbb{F}_q$, as $q$ ranges over all prime powers. Motivated by the philosophy of mixed Tate motives, Kontsevich informally conjectured in 1997 that $R_G$ should be polynomial in $q$ for all graphs. This was studied by Stanley [25], and Stembridge [26] proved by computer that it holds for all graphs up to 12 edges. Belkale and Brosnan subsequently showed in [1] that the conjecture is in general false, and moreover that the functions $R_G$ are of general type. This result implied that the cohomology of $X_G$ can be very complicated, but did not rule out the possibility that the $m_G$ are always mixed Tate, still less that the $I_G$ evaluate to multiple zetas (especially if the graphs $G$ are constrained to lie in $\phi^4$). Indeed, the first ‘potentially-Tate’ counter-examples to Kontsevich’s conjecture were only discovered very recently (22, 11).

In this paper, we show for certain infinite families of graphs that a variant of the graph motives $2$ are mixed Tate, and that their periods $I_G$ evaluate to multiple zeta values. Our methods also have consequences for the point-counting problem, which are addressed in a separate paper [6]. The idea is that for some graphs, the complement of the graph hypersurfaces can be related to moduli spaces of curves of genus 0 via a sequence of linear fibrations. The existence of such linear fibrations results from the vanishing of certain graph invariants. When these invariants do not vanish, we can extract Calabi-Yau varieties from the graph hypersurfaces which yield explicit (modular) counter-examples [6] to Kontsevich’s conjecture.

0.2. Overview. On the analytic side, the main idea is to choose an order on the $N = |E_G|$ edges of $G$, and consider the partial Feynman integrals

\begin{equation}
(3) 
 I_G(\alpha_{i+1}, \ldots, \alpha_N) = \int_0^\infty \cdots \int_0^\infty \frac{1}{\Psi_G} d\alpha_1 \cdots d\alpha_i .
\end{equation}

These are multivalued functions of the Schwinger parameters $\alpha_{i+1}, \ldots, \alpha_N$ with singularities along a certain discriminant locus we denote $L_i$. We call these the Landau varieties of $G$, by analogy with the case of Feynman integrals which depend on masses and external momenta. For certain graphs, the varieties $L_i$ can be computed using stratified Morse theory, and the monodromy of the functions $I_G$ controlled. When the monodromy is unipotent, the $I_G$ are periods of unipotent fundamental groups. The functions (3) can therefore be expressed in terms of multiple polylogarithms, and this explains the appearance of multiple zeta values.
The algebraic interpretation is to consider the graph hypersurface $X_G$ in $(\mathbb{P}^1)^N$, with coordinates $\alpha_1, \ldots, \alpha_N$, along with the hypercube $B = \bigcup_{i=1}^N \{ \alpha_i = 0, \infty \}$. Together they define a stratification on $(\mathbb{P}^1)^N$. Consider the projection:

$$
\begin{array}{ccc}
(\mathbb{P}^1)^N & \rightarrow & (\alpha_1, \ldots, \alpha_N) \\
\pi_i & \downarrow & \\
(\mathbb{P}^1)^{N-i} & \rightarrow & (\alpha_{i+1}, \ldots, \alpha_N)
\end{array}
$$

and let $L_i \subset (\mathbb{P}^1)^{N-i}$ denote the (reduced) discriminant locus. Thus $L_i$ is the smallest variety such that $\pi_i : (\mathbb{P}^1)^N \setminus \pi_i^{-1}(L_i) \rightarrow (\mathbb{P}^1)^{N-i} \setminus L_i$ is a locally trivial map of stratified varieties, and therefore has topologically constant fibers. When all components of $L_i$ are linear in one of the Schwinger parameters, one can show that the fundamental group of the base acts unipotently on the relative cohomology of the fibers. By induction, this implies that a variant of (2) is mixed Tate.

Most of this paper is therefore devoted to computing the Landau varieties associated to an edge-ordered graph, and relating them to its combinatorics.

0.3. Plan of the paper. In §1 we recall some basic concepts from graph theory, and define the notion of vertex width (§1.4). A connected graph $G$ has vertex width at most $n$ if there exists a filtration of subgraphs $G_i \subset G$, where $G_i$ has $i$ edges:

$$
\emptyset \subset G_1 \subset \ldots \subset G_{N-1} \subset G_N = G
$$

such that the number of vertices in the intersection $G_i \cap (G \setminus G_i)$ is at most $n$, for all $i$. We then show that ‘physical’ graphs (i.e., primitive divergent graphs in $\phi^4$ theory) are in fact completely general from the point of view of their graph minors.

Section 2 is a detailed study of certain polynomials related to graphs. From the matrix-tree theorem, the graph polynomial can be expressed as a determinant

$$
\Psi_G = \det M_G
$$

where $M_G$ is obtained from the incidence matrix of $G$. We define a related set of polynomials (‘Dodgson polynomials’), indexed by any sets of edges $I, J, K$ of $G$, where $|I| = |J|$, and denoted by $\Psi_{I,J}^{G,K}$. These are the determinants of matrix minors of $M_G$ obtained by deleting rows and columns. The key to understanding the periods of Feynman integrals rests in the algebraic relations between the $\Psi_{I,J}^{G,K}$, which are of two types. The first type are completely general identities (due to Dodgson, Jacobi, Plücker, . . . ) relating the minors of matrices, and the second type depend on the combinatorics of $G$, namely the existence of cycles and corollas.

In §3, we prove some properties of Feynman integrals under simple operations of graphs using the parametric representation. The main new result is that there is a natural definition of the set of periods of any (not necessarily convergent) graph, and that this set is minor monotone. In particular, it follows from well-known results in graph minor theory that the set of graphs whose periods lie in a fixed ring (e.g., the ring of multiple zeta values), is determined by a finite set of forbidden minors, i.e., a finite number of ‘critical counterexamples’. This gives some insight into the structural properties of graphs whose periods are multiple zeta values.

In §4 we study the geometry of the graph hypersurface $X_G$, viewed as a subvariety of $(\mathbb{P}^1)^N$, and its intersections with the coordinate hypercube. The reason for this choice of ambient space is because the Schwinger parametrization is naturally
adapted to a cubical representation, and this point of view preserves the symmetry between contracting and deleting edges in graphs. We explain how to blow up linear subspaces contained in \((\mathbb{P}^1)^N\) to obtain a divisor which is normal crossing in the neighbourhood of the hypercube \([0, \infty)^N\), in much the same way as [5], and show that \(I_G\) (in the case when \(G\) is primitively divergent) is a period of the corresponding mixed Hodge structure, or ‘motive’, which we denote by \(M_G\).

In §5 we recall some definitions from stratified Morse theory, and define the Landau varieties \(L(G, \pi_i)\) of a graph \(G\), relative to a projection \(\pi_i : (\mathbb{P}^1)^N \to (\mathbb{P}^1)^{N-i}\). We show, by a standard argument, that the partial Feynman integrals are multivalued functions with singularities along the \(L(G, \pi_i)\). Unfortunately, discriminant varieties are in general difficult to compute, and the standard methods are ill-adapted to the case of Feynman graphs, which are very degenerate. In particular, the divisors \(L(G, \pi_i)\) have large numbers of components.

To this end, in §6 we develop an inductive method to compute an upper bound for the Landau varieties \(L(G, \pi_i)\) by computing iterated resultants of polynomials, and removing spurious components. The method applies to any family of hypersurfaces \(S \subset (\mathbb{P}^1)^N\). Such a family is defined to be linearly reducible if, for all \(i\), all components of the Landau varieties \(L(S, \pi_i) \subset (\mathbb{P}^1)^{N-i}\) are of degree at most 1 in one coordinate. This notion depends on a choice of ordering on the coordinates.

In §7 we apply this method to the graph polynomials \(\Psi_G\). Using all the identities obtained in §2 we show that generically, the Landau varieties \(L(G, \pi_i)\) always contain the zero locus of the Dodgson polynomials \(\Psi_{G,K}^{I,J} = 0\) where \(I \cup J \cup K = \{1, \ldots, i\}\). We define a graph \(G\) to be of matrix type if the converse is true: i.e.,

\[
L(G, \pi_i) \subseteq \{\Psi_{G,K}^{I,J} = 0 : I \cup J \cup K = \{1, \ldots, i\}\} \quad \text{for all } i.
\]

This is the most accessible family of graphs one can define, and in particular, they are linearly reducible since each \(\Psi_{G,K}^{I,J}\) is of degree at most one in all its variables. Unfortunately, not all graphs are of matrix type. For \(i \leq 4\), condition (4) actually holds for all graphs, but in the general case, \(L(G, \pi_5)\) contains a component given by the zero locus of a new polynomial

\[
5\Psi_G(i, j, k, l, m),
\]

which we call the 5-invariant of a set of five edges \(i, j, k, l, m\) in a graph. For general graphs, this is an irreducible polynomial which is quadratic in each Schwinger parameter and gives the first obstruction for a graph to be of matrix type. But under certain combinatorial conditions, it factorizes into a product of Dodgson polynomials and (4) still holds for \(i = 5\). In particular, by studying the effect of triangles and 3-valent vertices in a graph on the degeneration of the 5-invariant, we show that it (and all higher possible obstructions) vanish when \(G\) has small vertex-width. This provides an infinite supply of non-trivial graphs of matrix type.

**Theorem 1.** If \(G\) has vertex-width \(\leq 3\), then \(G\) is of matrix type.

In §8 we return to the more general situation of an arbitrary configuration of hypersurfaces \(S\), and consider what happens when it is linearly reducible. We have a sequence of Landau varieties \(L(S, \pi_i) \subset (\mathbb{P}^1)^{N-i}\), and maps \(\pi_{i+1} : (\mathbb{P}^1)^{N-i} \to (\mathbb{P}^1)^{N-i-1}\) which project out each successive coordinate. The linearity assumption implies that each projection can be completed to a commutative diagram mapping...
to the universal curve of genus $0$ and $m_i$ marked points, for some $m_i$:

$$
(p^1)^{N-i}\backslash L(S, \pi_i) \xrightarrow{\rho} \mathcal{M}_{0,m_i+1} \\
\pi_{i+1} \\
(p^1)^{N-i-1}\backslash L(S, \pi_{i+1}) \xrightarrow{\pi} \mathcal{M}_{0,m_i}.
$$

Thus we relate any linearly reducible configuration to moduli spaces, and do this explicitly for graphs of matrix type in §8.3. Putting these diagrams together for different $i$ yields connecting maps between certain (fiber products) of moduli spaces.

The induced maps on the fundamental groups, plus the integers $m_i$, completely encode all the data about the periods. One application we have in mind is for the algorithmic computation of the periods of Feynman integrals: rather than study the geometry and function theory of each individual graph, one needs only implement the function theory on the moduli spaces $\mathcal{M}_{0,n}$ once and for all.

In §9 we explain how to compute any period integral whose singularities are given by a set of linearly reducible hypersurfaces. For this, refer to the diagram (6) above. The partial Feynman integrals naturally live on the spaces on the left-hand side, but using the maps $\rho$, we show that they can be pulled back to the moduli spaces $\mathcal{M}_{0,m_i+1}$ and computed by working entirely in the de Rham fundamental group of the $\mathcal{M}_{0,n}$’s, which we studied in [8]. In particular, we proved in that the periods one obtains in this way are multiple zeta values. Thus we deduce:

**Theorem 2.** Let $G$ be positive and of matrix type. Then $I_G$ is a rational linear combination of multiple zeta values of weight $\leq N - 3$, where $N = |E_G|$.

The positivity condition means that the coefficients of the polynomials $\Psi_{G,K}^{I,J}$ which occur in $L(G, \pi_i)$ should be positive, but for a general graph of matrix type one should obtain multiple polylogarithms evaluated at roots of unity $\pm 1$. We show (theorem 118) that if $G$ has vertex width $\leq 3$, then it is positive of matrix type. The method gives an algorithm for the computation of any terms in the $\varepsilon$-expansion of such graphs to all orders, and with arbitrary dressings [9].

In §10 we define an iterative procedure (the denominator reduction) to compute the denominators in the partial Feynman integrals. Thus, for a linearly reducible graph, the partial Feynman integral at the $n^{th}$ stage of integration is of the form

$$
\text{Multiple polylogarithms in the } \Psi_{G,K}^{I,J} / P_n,
$$

where $P_n$ is a polynomial which can be computed very easily. The polynomials $P_n$ give the deepest contributions to the Landau varieties, and the first place that non-Tate phenomena appear. We explain how the denominator reduction gives a mechanism for proving if the residue of a graph $I_G$ has transcendental weight drop, and these methods have subsequently been used in [7] to give a complete combinatorial explanation of all known weight-drop phenomena in $\phi^4$ theory.

In §11, we use the iterated fibrations of §7 to prove that:

**Theorem 3.** If $G$ is linearly reducible then the motive of $G$ is mixed Tate.
In particular, graphs of vertex width 3 are mixed Tate.\footnote{We have recently given a constructive proof in \cite{6} that graph hypersurfaces of vertex \( \leq 3 \) have polynomial point counts over finite fields using the methods of \S\S 2, 7.} The idea is that a composition of linear fibrations has global unipotent monodromy, and so one can compute the motive inductively using the Leray spectral sequence. However, in the present paper we do not discuss the rather essential question of ramification, and only touch on the issue of framings in \S11.2.

Finally, in \S12, we gather examples and counter-examples of critical graphs at various loop orders which are related to the discussions of \S\S 1, 2, 7, 10. Using the denominator reduction, we search for examples which lie just outside the scope of the present method, and find that the first serious obstructions occur at 8 loops. Since writing the first versions of this paper, we have shown in \cite{6} that one of these 8 loop examples contains a singular K3 surface which is not Tate. Thus the first non-Tate examples occur as soon as our criteria for linear reducibility fail. Furthermore, there exists a non-Tate example with vertex width 4, which shows that the \( \text{vw}(G) \leq 3 \) condition cannot be improved. In \S12.7 we explain how the denominator reduction associates a Calabi-Yau variety to any ordered graph, and list the modular forms which conjecturally correspond to them, for all graphs up to 8 loops. They are all expressible in terms of the Dedekind eta function. We conclude the paper with some remarks on classification, and a list of open problems.

The paper \cite{8} may serve as an introduction to this one, and in particular contains a worked example in the case of the wheel with 3 spokes.

0.4. Outlook. Here follow some general remarks on possible future developments.

In this paper, we only considered massless, subdivergence-free graphs, but we expect the same methods to work for graphs with subdivergences and trivial dependence on a single external momentum. In the case where there are several kinematic variables, we expect the present method to prove that the Feynman integrals are polylogarithmic functions of the external kinematics, but only at much lower loop orders (examples of elliptic integrals are known to occur).

Secondly, although \( \phi^4 \) theory is considered to be ‘unphysical’, it is the universal scalar quantum field theory from the point of view of its periods. Thus the methods of this paper give upper bounds for the types of numbers which can occur in any such scalar massless quantum field theory. In practice, the periods often tend to cancel when one sums over all graphs in the perturbative expansion at a given loop order, but the new phenomena occurring at 8 loops in \( \phi^4 \) may suggest that the transcendentality is merely postponed to higher loop orders. We also believe that one might gain an understanding of the cancellation phenomenon by studying the interaction between the motivic Galois group acting on the periods of \( \phi^4 \), and the possible symmetries of a given quantum field theory.

Finally, we believe that the motivic approach to the Feynman amplitudes may also have consequences for the possible convergence of the perturbative series of primitive graphs. In particular, it follows from \cite{5} that there is a well-defined Hodge and weight filtration on the perturbative series of primitively-divergent graphs, and we expect from \S12, \cite{6} and \cite{7}, that these filtrations are highly non-trivial. It therefore makes sense to sum this particular perturbative series according to its Hodge and weight filtrations, which might improve its convergence.
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1. Preliminaries on Graphs

1.1. Terminology. Throughout, a graph $G$ will denote an undirected multigraph with no external edges. Such a graph $G$ can be represented by a pair $(V_G, E_G)$ where $V_G$ is a finite set (the vertices of $G$), and $E_G$ is a finite set of unordered pairs \{$v_i, v_j$\} of elements of $V_G$ (the edges of $G$), which may occur with multiplicity.

A subgraph $\gamma$ of $G$, denoted $\gamma \subseteq G$, is defined by specifying a subset of edges $E' \subseteq E_G$, and setting $\gamma = (V', E')$, where $V' \subset V_G$ is the set of vertices which occur in $E'$. A tadpole (following the terminology in physics) is an edge of the form \{$v_i, v_i$\}. The degree or valency of a vertex $v$ is the number of edges in $E$ which are incident to $v$. A graph $G$ is said to be in $\phi^4$ if every vertex has degree at most 4.

We write $e_G = |E_G|$, $v_G = |V_G|$, $h_G = h_1(G)$, and $h_0(G)$ for the number of edges, vertices, independent cycles (the loop number), and connected components of a graph $G$. They are related by Euler’s formula: $h_G - h_0(G) = e_G - v_G$.

Definition 4. A graph $G$ is said to be primitive divergent if $e_G = 2h_G$, and for all strict subgraphs $\gamma \subseteq G$, $e_\gamma > 2h_\gamma$.

Given two disjoint sets of edges $C, D \subset E_G$, we write $\gamma = G\backslash D/ /C$ for the graph obtained by deleting the edges in $D$, and contracting the edges in $C$ (i.e., removing each edge in $C$ and identifying its endpoints). Since the operations of deleting and contracting disjoint edges commute, $\gamma$ is well-defined. We need the convention that the contraction of any $C$ such that $h_C > 0$ is the empty graph. Any graph $\gamma$ obtained from $G$ by contracting and deleting edges is called a minor of $G$ and will be denoted $\gamma \asymp G$.

The complete graph $K_n$ for $n \geq 2$, is the graph with $V_{K_n} = \{v_1, \ldots, v_n\}$ and $E_{K_n} = \{\{v_i, v_j\}\}$ for all $1 \leq i < j \leq n$. The complete bipartite graph $K_{p,q}$ is the graph with vertex set $\{v_1, \ldots, v_p, w_1, \ldots, w_q\}$ and edges $\{v_i, w_j\}$ for $1 \leq i, j \leq q$.

1.2. Standard operations on graphs. The theory of electrical circuits suggests the five basic operations pictured below:

- **E**
- **T**
- **S**
- **P**
- **ST**
They are: External leg, Tadpole, Series, Parallel, and Star-Triangle operations. A graph is said to be \(1PI\) (1-particle irreducible) if \(h_{G_i+} < h_G\) for all edges \(e \in E_G\).

**Definition 5.** We define the simplification of a graph \(G\) to be the smallest minor \(G' \preceq G\) which is obtained from \(G\) by applying operations \(E, T, S, P\) above. We say that a graph \(G\) is simple if it is equal to its simplification.

Now let \(G_1\) and \(G_2\) denote two connected graphs, and let \(v_i \in V_{G_i}\) for \(i = 1, 2\). The one vertex join \(G_1 \bullet G_2\) of \(G_1\) and \(G_2\) is the graph obtained by gluing \(G_1\) and \(G_2\) together by identifying \(v_1\) and \(v_2\). Now let \(v_i \neq w_i\) be two vertices in \(G_i\) for \(i = 1, 2\) which are connected by a single edge \(e_i\). A two vertex join of \(G_1\) and \(G_2\), which we denote by \(G_1 \bowtie G_2\), is obtained by identifying \(v_1\) with \(v_2\), and \(w_1\) with \(w_2\), and deleting the edges \(e_1, e_2\) (see below). We have \(h_{G_1 \bowtie G_2} = h_{G_1} + h_{G_2} - 1\).

![Diagram](https://via.placeholder.com/150)

The connectivity \(\kappa(G)\) of a graph \(G\) is the minimal number of vertices required to disconnect \(G\). We say \(G\) is \(n\)-vertex reducible, or \(n\)VR, if \(\kappa(G) \leq n\). The one (resp. two) vertex join of two graphs is 1VR (resp. 2VR).

1.3. **Forbidden minors and local minors.** We recall some well-known concepts concerning forbidden minors which will be important for the sequel.

**Definition 6.** A set of graphs \(S\) is minor closed if for all \(\Gamma \in S\), \(\gamma \preceq \Gamma \implies \gamma \in S\).

The set of planar graphs or the set of trees are examples of minor closed sets. One way to define a set of minor-closed graphs is to specify a finite set of forbidden minors \(\mathcal{F} = \{\gamma_1, \ldots, \gamma_N\}\). Then if one defines \(\mathcal{F}^c\) to be the set of all graphs which do not contain any of the forbidden minors \(\gamma_i\), then \(\mathcal{F}^c\) is clearly minor-closed. The converse is a celebrated theorem due to Robertson and Seymour.

**Theorem 7.** Let \(S\) be a minor closed set of graphs. Then there exists a finite set of forbidden minors \(\mathcal{F}_S\) such that \(S = \mathcal{F}_S^c\).

Let \(S\) be a minor-closed set of graphs. A critical minor for \(S\) is a graph \(G \notin S\) such that every minor of \(G\) is in \(S\). A minor closed property for graphs is in principle completely determined by its set of critical minors. For example, a theorem due to Wagner states that the forbidden minors for the set of planar graphs is \(\{K_{3,3}, K_5\}\).

In this paper, we will require a related notion of local minors.

**Definition 8.** An ordered graph \((G, O)\) is a graph \(G\) with a total ordering \(O\) on its set of edges \(E_G\). For each \(1 \leq i \leq e_G\), we obtain a partition \(G = L_i \cup R_i\) into disjoint subgraphs, where \(L_i\) is defined by the first \(i\) edges and \(R_i\) by the remaining \(e_G - i\) edges. Let

\[
V_i(G, O) = V_{L_i} \cap V_{R_i}, \quad \text{for } 1 \leq i \leq e_G
\]

be the set of vertices which are common to \(L_i\) and \(R_i\). We define a local \(k\)-minor of \((G, O)\) to be any minor of \(L_i\) which has exactly \(k\) edges. It has a distinguished subset of vertices which come from \(V_i(G, O)\).
Thus for any ordered graph \((G, O)\), we obtain a finite list of local \(k\)-minors which reflects the local structure of the graph after contracting and deleting certain edges according to \(O\). Our main results hold for graphs with forbidden local 5-minors.

1.4. The vertex width of a graph. One simple way to control the local minors which can occur in a graph is with the notion of vertex width.

**Definition 9.** Let \((G, O)\) be an ordered graph, and let \(V_i(G, O)\) be as in definition \(\square\) We define the vertex width of \((G, O)\) to be:

\[ vw(G, O) = \max_{1 \leq i \leq n-1} |V_i(G, O)|. \]

Finally, we define the vertex width of the graph \(G\) to be the smallest vertex width of all possible orderings \(O\) of the set of edges of \(G\):

\[ vw(G) = \min_O vw(G, O). \]

The connectivity of \(G\) is bounded by its vertex width: \(\kappa(G) \leq vw(G)\). It is clear that if \(\gamma \not\subseteq G\) is a graph minor of \(G\), then \(vw(\gamma) \leq vw(G)\). The set of graphs satisfying \(vw(G) \leq n\) is therefore minor closed, for each \(n \geq 1\).

**Example 10.** For all \(m, n \geq 3\), it is easy to check that \(vw(K_n) = n - 1\) and \(vw(K_{m,n}) = \min\{m, n\} + 1\). The square lattice \(B_n\) with \(n^2\) boxes has vertex width \(n + 1\). In particular, the vertex width is unbounded on the set of all planar graphs.

Some simple families of graphs of vertex width \(\leq 3\) are given by the wheel with \(n\) spokes and what are known as zigzag graphs in the physics literature. A simple way to generate infinite families of such graphs is by subdividing triangles.

1.5. Minors of \(\phi^4\) graphs. Restricting oneself to primitive-divergent graphs in \(\phi^4\) theory does not change the minors which can occur, but only delays them to higher loop orders.

First observe that if \(G\) is simple, connected, and in \(\phi^4\), then it only has vertices of degree three and four. It then follows from Euler’s formula that the relation \(2h_G = e_G\) is equivalent to \(G\) having exactly four vertices of valency 3.

**Lemma 11.** Every complete graph \(K_n\) occurs as a minor of a primitive-divergent graph in \(\phi^4\) theory (perhaps at higher loop order).

**Proof.** By embedding \(K_n \to K_{n+1}\), we can assume that \(n\) is large and odd. Consider the graph with vertices arranged in a square lattice with \(n\) elements \(x_1, \ldots, x_n\), arranged in \(n - 1\) rows \(1 \leq i \leq n - 1\). Take vertical edges \(\{x_i^j, x_{i+1}^j\}\) for all \(i, j\), and twisted horizontal edges \(\{x_i^j, x_{i+j}^j\}\), where the lower indices are taken modulo \(n\). Add edges \(\{x_1^{n-1}, \ldots, x_n^{n-1}\}\). In the resulting graph \(G_n\), every vertex is 4-valent except for the four corners \(x_1^n, x_n^1, x_n^n, x_n^n\), which have valency 3. By contracting the vertical edges in each column, one sees that \(G_n\) contains the complete graph \(K_n\) as a minor. To see that it is primitive-divergent, it suffices to consider connected strict subgraphs \(\gamma \subset G_n\) which only have vertices of degree 2, 3, 4. By Euler’s formula, the equation \(2h_\gamma \geq e_\gamma\) is equivalent to \(2v_2 + v_3 \leq 4\), where \(v_2, v_3\) denotes the number of 2 and 3 valent vertices in \(\gamma\). Since every remaining vertex of \(\gamma\) is 4-valent, \(\gamma\) must also contain each of its \(G_n\)-neighbours. Thus by considering the sets of entire rows and columns of \(G_n\) contained in \(\gamma\), one easily verifies that no such divergent subgraphs can exist. Thus \(G_n\) is primitive divergent.
A different, and more loop-efficient way to embed $K_n$ as a minor of a primitive-divergent graph in $\phi^4$ theory, is simply by subdividing its vertices. □

It follows that every graph occurs as a minor of a primitive-divergent graph in $\phi^4$, since one can subdivide multiple edges using operation $S$, until no more remain. The graph one obtains can then be embedded as a subgraph of the complete graph on its set of vertices, and thence into $\phi^4$ using the previous lemma. Conversely, every primitive-divergent graph in $\phi^4$ with $h$ loops is a minor of $K_{h+1}$.

**Lemma 12.** The square lattice $B_n$ occurs as a minor of a planar primitive-divergent graph in $\phi^4$ with the same number of vertices (i.e., $n^2 - 1$ loops).

**Proof.** The graph $B_n$ has $n^2$ vertices, each of which has degree 4 except the ones lying along an outer edge (degree 3) and the four corners, which have degree 2. According to whether $n$ is odd or even, connected the outer vertices as shown in the two cases below. The resulting graph has exactly 4 vertices of degree 3 (highlighted), and is primitive divergent, by a similar argument to the one given in the previous lemma. □

Since every planar graph is a minor of a square lattice, it follows that every planar graph occurs as a minor of a planar primitive-divergent graph in $\phi^4$ (perhaps at higher loop order), and conversely, every planar primitive-divergent graph $G$ is a minor of $B_N$, for some sufficiently large $N$, by the same argument.

In conclusion, any minor-closed property of graphs holds for all graphs in $\phi^4$ (planar $\phi^4$) if and only if it holds for the basic family of graphs $K_n$ (resp. $B_n$), and if and only if it holds for all graphs (resp. all planar graphs). Thus, from the point of view of minors, there is no such thing as an ‘unphysical’ graph.

2. **Graph Polynomials and Determinantal identities**

2.1. **Reminders on graph polynomials.** To each edge $e$ of a connected graph $G$, we associate a variable $\alpha_e$, known as the Schwinger coordinate of $e$.

**Definition 13.** Let $G$ be a connected graph. The *graph polynomial* of $G$ is

$$
\Psi_G = \sum_{T \subseteq G} \prod_{e \notin T} \alpha_e \in \mathbb{Z}[\alpha_e, e \in E_G],
$$

where the sum is over all spanning trees $T$ of $G$, i.e., all connected subgraphs $T \subseteq G$ such that $T$ contains every vertex of $G$ and does not contain a loop. If $G$ is not connected there are no spanning trees, and so we set $\Psi_G = 0$. 
In the case where \( G \) is a tree, \( \Psi_G = 1 \). By convention, \( T \) can be empty, and thus if \( G \) is a tadpole consisting of a single edge \( e \), we have \( \Psi_G = \alpha_e \).

**Lemma 14.** For any edge \( e \) of \( G \), there is the contraction-deletion formula

\[
(8) \quad \Psi_G = \Psi_{G \setminus e} + \Psi_{G / e}.
\]

**Proof.** A spanning tree \( T \) of \( G \) either contains the edge \( e \) or does not contain it. In the first case, \( T \setminus e \) defines a spanning tree of \( G / e \); in the second, \( T \) defines a spanning tree of \( G \setminus e \). The details are left as an exercise. Note that the validity of \( \Psi \) requires the contraction of tadpoles to be zero. \( \square \)

The previous lemma is useful for doing inductions, and gives an alternative definition of \( \Psi_G \). For example, if \( G \) is connected, it follows from Euler’s formula that \( \deg \Psi_G = h_G \). The following corollary follows immediately from the definitions.

**Corollary 15.** (Vanishing condition) Let \( C, D \) be two disjoint sets of edges of \( G \). Then \( \Psi_{G \setminus (C \cup D)} = 0 \) if and only if either \( G \setminus C \) is disconnected, or \( D \) contains a loop. In particular, \( \Psi_{G \setminus C} = 0 \) implies that either \( \Psi_{G / C} = 0 \) or \( \Psi_{C \setminus D} = 0 \).

It follows from the next lemma that the graph polynomial \( \Psi_G \) of a connected graph is a product \( \prod_{i=1}^n \Psi_{G_i} \), of graph polynomials of 1PI subgraphs.

**Lemma 16.** If \( G_1 \) and \( G_2 \) are connected graphs, then \( \Psi_{G_1 \star G_2} = \Psi_{G_1} \Psi_{G_2} \).

**Proof.** The map which takes \( T_1 \subset G_1 \) and \( T_2 \subset G_2 \) to \( T_1 \cup T_2 \subset G \), is a bijection between the set of spanning trees of \( G \) and pairs of spanning trees of \( G_1 \) and \( G_2 \). \( \square \)

Conversely, one can also show that if \( G \) is a 1PI graph, then \( \Psi_G \) is reducible if and only if it is the one-vertex join of two subgraphs \( G_1 \) and \( G_2 \).

**Lemma 17.** Let \( G_1 \) and \( G_2 \) be as in \( \square \) then

\[
(9) \quad \Psi_{G_1 \star G_2} = \Psi_{G_1 \setminus e_1} \Psi_{G_2 \setminus e_2} + \Psi_{G_1 \setminus e_1} \Psi_{G_2 / e_1} \Psi_{G_1 / e_2}.
\]

**Proof.** A subgraph \( T \subset G_1 \star G_2 \) defines a pair of subgraphs \( T_1 \subset G_1 \setminus e_2 \) and \( T_2 \subset G_2 \setminus e_2 \). One checks that \( T \) is a spanning tree of \( G_1 \star G_2 \) if and only if either \( T_1 \) and \( T_2 \cup \{e_2\} \) are spanning trees of \( G_1 \) and \( G_2 \) respectively, or \( T_1 \cup \{e_1\} \) and \( T_2 \) are. \( \square \)

The following lemma is well-known and follows from \( \square \).

**Lemma 18.** Let \( G \) be a connected graph, and let \( G_S \) denote the graph obtained from \( G \) by subdividing an edge \( e \) into two new edges \( e_1, e_2 \) in series, and let \( G_P \) denote the graph obtained by replacing \( e \) with two new edges \( e_1, e_2 \) in parallel. Then

\[
\Psi_{G_S} = \Psi_G(\alpha_{e_1} + \alpha_{e_2}) \quad \text{and} \quad \Psi_{G_P} = (\alpha_{e_1} + \alpha_{e_2}) \Psi_G(\frac{\alpha_{e_1}}{\alpha_{e_1}} + \frac{\alpha_{e_2}}{\alpha_{e_2}}).
\]

The relationship between graph polynomials and the star-triangle operation will be examined later. Finally, suppose that a graph \( G \) has a planar embedding, and let \( G^\circ \) be its planar dual. It is well-known that

\[
(10) \quad \Psi_{G^\circ}(\alpha_e) = \Psi_G(\alpha_e^{-1}) \prod_{e \in E_G} \alpha_e.
\]
2.2. The graph matrix. Graph polynomials can be expressed as determinants of a Laplacian matrix in several ways. The following presentation is the most convenient for our purposes. Let us fix a connected graph $G$, and choose an orientation of its edges. For each edge $e$ and each vertex $v$ of $G$ define:

$$
\varepsilon_{e,v} = \begin{cases} 
1, & \text{if } s(e) = v, \\
-1, & \text{if } t(e) = v, \\
0, & \text{otherwise,}
\end{cases}
$$

where $s(e)$ denotes the source of the oriented edge $e$, and $t(e)$ its target. Let $E_G$ be the $e_G \times (v_G - 1)$ matrix obtained by deleting one of the columns of $(\varepsilon)_{e,v}$.

**Definition 19.** Let $n = e_G + v_G - 1$, and consider the $(n \times n)$ matrix:

$$
M_G = \begin{pmatrix} 
\alpha_1 & \cdots & \alpha_{e_G} \\
\vdots & \ddots & \vdots \\
-\varepsilon_{e_G} & \cdots & 0
\end{pmatrix}
$$

(11)

It is not well-defined, because it depends on the choice of the deleted column in $E_G$, the orientation, and the chosen order of the edges and vertices. Throughout this paragraph, $M_G$ will refer to any such choice of matrix.

The following lemma relating $E_G$ to trees goes back to Kirchhoff.

**Lemma 20.** Let $I$ denote a subset of edges of $G$ such that $|I| = h_G = e_G - v_G + 1$, and let $E_G(I)$ denote the square $(v_G - 1) \times (v_G - 1)$ matrix obtained by deleting the rows of $E_G$ corresponding to elements of $I$. Then

$$
\det(E_G(I)) = \begin{cases} 
\pm 1, & \text{if } I \text{ is a spanning tree of } G, \\
0, & \text{otherwise.}
\end{cases}
$$

**Proposition 21.** If $G$ is connected, then $\Psi_G = \det M_G$. If $G$ is not connected, then $\Psi_G = \det M_G = 0$. Deleting an edge $e$ corresponds to taking the determinant of the minor of $M_G$ obtained by deleting the row and column corresponding to $e$, and contracting $e$ corresponds to setting the variable $\alpha_e$ to 0, i.e.:

$$
\Psi_{G \setminus e} = \det M_G(e, e), \quad \text{and} \quad \Psi_{G \setminus e} = \det M_G|_{\alpha_e = 0}.
$$

**Proof.** It is clear from the shape of the matrix $M_G$ that

$$
\det(M_G) = \sum_{I \subseteq G, \alpha_e \notin I} \prod_{i} \alpha_i \det \left( \frac{-\varepsilon_G(I)}{0} \right) = \sum_{I \subseteq G, |I| = h_G, \alpha_e \notin I} \prod_{i} \alpha_i \det(E_G(I))^2.
$$

In the second expression, the sum is over all subsets of edges $I$ of $G$. In the case when $|I| < h_G$ or $|I| > h_G$, the columns of the matrix are not independent and so its determinant vanishes. This leaves the case $|I| = h_G$, when $E_G(I)$ is a square matrix. The previous lemma implies that $\det(E_G(I))^2 = 1$ if $I$ is a spanning tree and zero otherwise, which gives back formula (13), and therefore $\Psi_G = \det(M_G)$. It follows from the contraction-deletion formula (8) that $\Psi_{G \setminus e}$ is the coefficient of $\alpha_e$ in $\det(M_G)$, which is precisely given by the minor $\det(M_G(e, e))$. Likewise, (8) implies that $\Psi_{G \setminus e}$ is obtained by setting $\alpha_e$ to zero. □
2.3. Dodgson Polynomials. Proposition [21] motivates the following definition.

Definition 22. Let $I, J, K$ denote sets of edges of $G$ such that $|I| = |J|$, and let $M_G(I, J)$ denote the matrix obtained from $M_G$ by deleting the rows $I$ and columns $J$. Then we define the Dodgson polynomial to be:

$$\Psi_{G,K}^{I,J} = \det M_G(I, J)|_{a_\alpha = 0, \epsilon \in K}. \quad (12)$$

Changing the choice of matrix $M_G$ may change $\Psi_{G,K}^{I,J}$ by a sign. When the graph $G$ is clear from the context, we will drop the $G$ from the notation.

It is easy to verify that $\deg(\Psi_{G,K}^{I,J}) = h_G - |I| = h_G - |J|$ if $\Psi_{G,K}^{I,J} \neq 0$, and it is immediate from the definition that $\Psi_{G,J}^{I,J} = \Psi_{G,K}^{I,J}$ for any $A \subseteq I \cup J$, and $\Psi_{G,K}^{I,J} = \Psi_{G,K}^{I,J}$. It follows from the previous proposition that for all $A, B \subseteq E_G$,

$$\Psi_{G\setminus A \setminus B,K}^{I,J} = \Psi_{G,K \cup B}^{I \cup A, J \cup A}. \quad (13)$$

Therefore by passing to a minor we will often assume that $I \cap J = \emptyset$, and $K = \emptyset$.

Also, when $I = J$, we will sometimes write $\Psi_{G,K}^{I,J}$ instead of $\Psi_{G,K}^{I,J}$.

Proposition 23. Let $I, J, K$ be as above. Then

$$\Psi_{G,K}^{I,J} = \sum_{T \subseteq G} \pm \prod_{\epsilon \in T} \alpha_\epsilon. \quad (14)$$

where the sum is over all subgraphs $T \subseteq G$ which are simultaneously spanning trees for both $G \setminus I/(J \setminus (I \cap J))$ and $G \setminus J/(I \setminus (I \cap J))$. In particular, every monomial which occurs in $\Psi_{G,K}^{I,J}$ also occurs in both $\Psi_{G,J \cup K}^{I,J}$ and $\Psi_{G,I \cup K}^{I,J}$.

Proof. By passing to the minor $G \setminus (I \cap J)/K$, we can assume that $I \cap J = \emptyset$, and $K = \emptyset$. As before, it follows from the shape of the matrix $M_G(I, J)$ that

$$\det(M_G(I, J)) = \sum_{U \subseteq G \setminus (I \cup J)} \prod_{\epsilon \notin U} \alpha_\epsilon \det \begin{pmatrix} 0 & \mathcal{E}_G(U \cup I) \\ -\mathcal{E}_G(U \cup J) & 0 \end{pmatrix}.$$

For both $\det(\mathcal{E}_G(U \cup I))$ and $\det(\mathcal{E}_G(U \cup J))$ to be non-zero, it follows from lemma [20] that both $U \cup I$ and $U \cup J$ must be spanning trees in $G$. The tree $U \cup I$ does not involve any edges from $J$ (since, by assumption $I \cap J = \emptyset$), and so it follows that $U$ is a spanning tree in $G \setminus I/J$ and likewise $G \setminus J/I$, by symmetry. Conversely, for any such $U$, lemma [20] implies that $\det(\mathcal{E}_G(U \cup I))$ and $\det(\mathcal{E}_G(U \cup J))$, and hence their product, are equal to $\pm 1$.

Remark 24. A formula for the signs in (14) is given in terms of spanning forests in [7]. We will only need the following fact: if $i,j$ are adjacent edges in $G$ meeting at a vertex $v$, then all the coefficients of $\Psi_{G,K}^{I,J}$ have the same sign. This easily follows from the previous proof on choosing the removed vertex in $M_G$ to be $v$.

A spanning tree $T$ in $G \setminus I/J$ lifts to a subgraph $T \cup I \cup J \subseteq G$ which has $|I| = |J|$ loops. One can therefore view Dodgson polynomials as sums over subgraphs of $G$ containing cycles which satisfy certain properties. The previous proposition implies the following vanishing condition for the Dodgson polynomials.
Corollary 25. Let $I, J, K$ be as above. Then $\Psi_{G,K}^{I,J} = 0$ if and only if there are no subgraphs $T \subset G$ such that $T \cup I$ is a spanning tree in $G \setminus (I \cap J) \parallel K$ and $T \cup J$ is a spanning tree in $G \setminus (I \cap J) \parallel K$.

Examples 26. Consider the wheel with four spokes, and let $I = \{1, 2\}$, $J = \{3, 4\}$ with the numbering of the edges shown below.

![Wheel diagram]

It is clear that the unique common spanning tree of $G \setminus I \parallel J$ and $G \setminus J \parallel I$ is $\{6, 8\}$, and therefore $\Psi_{12,34}^{I,J} = \pm \alpha_5 \alpha_7$. The subgraph $\{1, 2, 3, 4, 6, 8\}$ therefore forms a double cycle. Likewise, $\Psi_{14,23}^{I,J} = \pm \alpha_6 \alpha_8$, and $\Psi_{13,24}^{I,J} = \pm (\alpha_6 \alpha_8 - \alpha_5 \alpha_7)$.

2.4. Plücker identities. From now on, when considering identities between Dodgson polynomials, we will fix a representative matrix $M_G$ once and for all. This also fixes a representative matrix for all minors of $G$, so it makes sense to write, e.g.,

(15) $\Psi_{G,K}^{I,J} = \Psi_{G,K}^{I,J,e} \alpha_e + \Psi_{G,K}^{I,J} = \Psi_{G,K}^{I,J,e} \alpha_e + \Psi_{G,K}^{I,J}.$

Lemma 27. Let $M$ be a $N \times N$ symmetric matrix, and let $i_1, \ldots, i_{2n}$ be distinct indices between $1$ and $N$. Then

(16) $\sum_{k=n}^{2n} (-1)^k \det(M(\{i_1, \ldots, i_{n-1}, i_k\}, \{i_n, \ldots, \hat{i}_k, \ldots, i_{2n}\})) = 0$.

Proof. By doing row expansions with respect to rows not in $\{i_1, \ldots, i_{2n}\}$, it suffices by linearity to consider the case $N = 2n$. Then, after deleting rows $i_1, \ldots, i_n$ from $M$, one sees that (16) reduces to the classical Plücker identity. □

The determinant of the matrix $M_G$ is $\pm 1$ times the determinant of a symmetric matrix. Applying the previous lemma gives linear identities of the form:

(17) $\sum_{k=n}^{2n} (-1)^k \Psi_G^{i_1, \ldots, i_{n-1}, i_k}(i_n, \ldots, \hat{i}_k, \ldots, i_{2n}) = 0$.

We will mainly require the special case $n = 2$ where $i, j, k, l$ are distinct indices:

(18) $\Psi_G^{i,j,k,l} - \Psi_G^{i,k,j,l} + \Psi_G^{i,l,j,k} = 0$.

2.5. Determinantal identities.

Lemma 28. (Jacobi’s determinant formula) Let $M = (a_{ij})$ be an invertible $n \times n$ matrix, and let $\text{adj} M = (A_{ij})$ denote the adjoint of $M$, i.e., the transpose of the matrix of cofactors of $M$. Then for any integer $1 \leq k \leq n$,

(19) $\det(A_{ij})_{k<i,j<n} = \det(M)^{n-k-1} \det(a_{ij})_{1 \leq i,j \leq k}$.

Proof. If $I_n$ denotes the identity matrix of size $n$, we have:

$$(\text{adj} M) M = I_n(\det M).$$
In particular, $\det(\text{adj} \, M) = \det(M)^{n-1}$. If we replace $M$ in this equation with the matrix obtained from $M$ by replacing the last $k$ columns with the corresponding columns of the identity matrix $I_n$, we obtain the equation:

$$
\begin{pmatrix}
A_{ij} & \vdots \\
\vdots & \ddots \\
A_{ij} & \vdots \\
\end{pmatrix}
\begin{pmatrix}
\begin{array}{c}
a_{ij} \\
\vdots \\
a_{ij} \\
\end{array}
\end{pmatrix}
\begin{pmatrix}
0 \\
\vdots \\
0 \\
\end{pmatrix}
= 
\begin{pmatrix}
\frac{\det(M) I_k}{A_{ij}} \\
\vdots \\
\frac{\det(M) I_k}{A_{ij}} \\
\end{pmatrix}.
$$

Taking the determinant of this equation gives

$$(\det M)^{n-1} \det(a_{ij})_{1 \leq i, j \leq k} = (\det M)^k \det(A_{ij})_{k < i, j \leq n}.$$ 

Finally, dividing by $(\det M)^k$ yields $[13]$.

In the special case $k = n - 2$, we can rearrange the indices to give the following quadratic identity, for any $1 \leq p < q \leq n$ and $1 \leq r < s \leq n$:

$$A_{p, r} A_{q, s} - A_{p, s} A_{q, r} = \det(M) \det(M_{pq}) \det(M_{rs}) \det(M_{rs})$$

where $M_{pq}$ denotes the matrix $M$ with rows $p$ and $q$, and columns $r$ and $s$ removed. This identity is usually attributed to C. L. Dodgson.

Now let $G$ be any graph, and let $A, B$ denote two subsets of the set of edges of $G$ with $|A| = |B|$. Applying the previous identity to the matrix obtained from $M_G$ by removing rows $A$ and columns $B$, gives the identity:

$$\Psi^A_{S} A_{pq} B_{rs} - \Psi^A_{S} A_{qs} B_{pr} = \Psi^{A, B}_{S} \Psi^{A, B}_{S}$$

for any $p, q, r, s \notin A \cup B$, and where $S = A \cup B \cup \{p, q, r, s\}$.

**Lemma 29.** Let $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ be sets of edges of $G$ with $A \cap B = \emptyset$. Then for any fixed $1 \leq i \leq n$,

$$\Psi^{A \cup \{a_i\}, B \cup \{b_i\}} = 0 \quad \text{for all} \quad j = 1, \ldots, n \implies \Psi^{A, B} = 0.$$

**Proof.** Let $C = A \setminus \{a_i\}$, $D = B \setminus \{b_i\}$. It follows from (21) that $\Psi^{C, D} = 0$ implies that all terms on the right-hand side are zero, and so

$$\Psi^{A, B} = 0.$$

The Plücker identity gives $\Psi^{A, B} + \sum_{j=1}^{m} \pm \Psi^{A \cup \{b_j\} \setminus \{a_i\}, B \cup \{a_i\} \setminus \{b_j\}} = 0$, which together imply that $\Psi^{A, B} = 0$.

The following two quadratic identities will be crucial for the sequel, and will later be reformulated more simply in terms of resultant in \[19\], \[20\].

**Lemma 30.** Let $I, J$ be subsets of $E_G$ satisfying $|I| = |J|$, and let $a, b, x \notin I \cup J$. If we set $S = I \cup J \cup \{a, b, x\}$, then we have the following identity:

$$\Psi^{I}_{S} \Psi^{J}_{S} I_{ax, Jb} - \Psi^{I}_{S} \Psi^{J}_{S} I_{bx, Jb} = \Psi^{I}_{S} \Psi^{J}_{S} I_{xa, Jb}.$$ 

Now let $I, J$ be two subsets of $E_G$ satisfying $|J| = |I| + 1$, and let $a, b, x \notin I \cup J$. If we set $S = I \cup J \cup \{a, b, x\}$, then we have the following identity:

$$\Psi^{I, J}_{S} \Psi^{J}_{S} I_{ax, Jb} - \Psi^{I, J}_{S} \Psi^{J}_{S} I_{bx, Jb} = \Psi^{I, J}_{S} \Psi^{J}_{S} I_{xa, Jb}.$$
Proof. The first identity (22) follows immediately from (21) on setting \( A = I, B = J \), and \((p, q, r, s) = (a, x, b, x)\) and rearranging terms. For the second identity (23), let \( A = I \), and set \( B = J \cup \{ y \} \), where \( y \notin A \). Set \((p, q, r, s) = (a, b, x, y)\) in (21), and writing \( S = A \cup B \cup \{ a, b, x, y \} \) gives
\[
\Psi_S^{Aa, Bx} \Psi_A^{ab, By} - \Psi_S^{Aa, By} \Psi_A^{ab, Bx} = \Psi_S^A \Psi_A^{ab, Bxy}.
\]
Multiplying this equation through by \( \Psi_S^{Ax, By} \) gives
\[
(24) \quad \Psi_S^{Ax, By} \Psi_S^{Aa, Bx} \Psi_A^{ab, By} - \Psi_S^{Ax, Bx} \Psi_S^{Aa, By} \Psi_A^{ab, By} = \Psi_S^A \Psi_A^{Ax, Bx} \Psi_S^{ab, Bxy}.
\]
Two further applications of (22) gives
\[
\Psi_S^{Aa, Bx} \Psi_A^{ab, By} - \Psi_S^{Aa, By} \Psi_A^{ab, Bx} = \Psi_S^A \Psi_A^{Aa, Bx} \Psi_S^{ab, By}.
\]
Substituting these into (24) and cancelling terms, gives
\[
\Psi_S^A \left( \Psi_S^{Ab, By} \Psi_S^{Ax, Bx} - \Psi_S^{Aa, By} \Psi_S^{Aa, By} \Psi_A^{ab, Bx} \right) = \Psi_S^A \Psi_A^{Ax, By} \Psi_S^{ab, Bxy}.
\]
If \( \Psi_S^{A, B} \) is non-zero, we are done. The opposite case requires that \( \Psi_S^{A, B} = 0 \) for all \( B \). Since we can assume that \( I \cap J = \emptyset \), we can choose the element \( y \) as we wish. But then lemma 29 implies that the identity (23) reduces identically to zero. \( \Box \)

2.6. Graph-specific identities. The Jacobi identity (19) implies some linear relations between Dodgson polynomials in the case when its right-hand side vanishes.

Lemma 31. Let \(|A| = |B|\), and let \( I = \{ i_1, \ldots, i_k \} \), where \( I \cap A = I \cap B = \emptyset \). If \( \Psi^{A \cup J, B \cup J} \neq 0 \) for all \( J \subseteq I \). Applying the Jacobi identity (19) to the matrix \( M_G \) with rows \( A \) and columns \( B \) removed gives:
\[
\det \begin{pmatrix}
\Psi^{A_1, B_{i_1}} & \cdots & \Psi^{A_1, B_{i_k}} \\
\vdots & \ddots & \vdots \\
\Psi^{A_{i_k}, B_{i_1}} & \cdots & \Psi^{A_{i_k}, B_{i_k}}
\end{pmatrix} = (\Psi^{A, B})^{k-1} \Psi^{A \cup I, B \cup J} = 0.
\]
The matrix on the left therefore has rank \( < k \), and so there is a non-trivial linear relation between its columns. In particular, there exist \( \lambda_1, \ldots, \lambda_k \) such that:
\[
(26) \quad \lambda_1 \Psi^{A_1, B_{i_1}} + \lambda_2 \Psi^{A_1, B_{i_2}} + \ldots + \lambda_k \Psi^{A_{i_k}, B_{i_k}} = 0,
\]
where \( \Psi^{A_{i_j}, B_{i_j}} \) should be viewed as a polynomial in the variables \( \alpha_{i_2}, \ldots, \alpha_{i_j}, \ldots, \alpha_{i_k} \) and the \( \lambda_j \) do not depend on these variables. Choosing a monomial which occurs in (26), say, \( \alpha_{i_3} \ldots \alpha_{i_k} \), and taking its coefficient yields
\[
(27) \quad \lambda_1 \Psi^{A_1i_{i_3} \ldots i_k, B_1i_{i_3} \ldots i_k} + \lambda_2 \Psi^{A_1i_{i_3} \ldots i_k, B_2i_{i_3} \ldots i_k} + \ldots = 0.
\]
By assumption, \( \Psi^{A_1i_{i_3} \ldots i_k, B_1i_{i_3} \ldots i_k} \) is non-zero. Using the interpretation of Dodgson polynomials as sums of trees, we know that every monomial in \( \Psi^{A_1i_{i_3} \ldots i_k, B_2i_{i_3} \ldots i_k} \) also occurs in \( \Psi^{A_1i_{i_3} \ldots i_k, B_1i_{i_3} \ldots i_k} \). Taking the coefficient of any monomial in (26) therefore implies that \( \lambda_1 \pm \lambda_2 = 0 \). Similarly, we deduce that \( \lambda_1 = \pm \lambda_j \) for all \( 2 \leq j \leq k \), and therefore (26) implies (25). \( \Box \)
A similar result holds when $\Psi^A_B = 0$, by inverting the variables $\alpha_i$. We will not require the general case for the sequel, only the following special cases.

**Example 32.** Suppose $G$ contains a three-valent vertex, or star:

![Three-valent vertex diagram](image)

Deleting edges 1, 2, and 3 disconnects the central vertex, so $\Psi^{123} = 0$, by corollary 15 and we can therefore apply the previous lemma. Another obvious identity is that $\Psi^1_3 = \Psi^2_1 = \Psi^1_2$, since deleting any two of the three edges and contracting the third always gives rise to the same minor. Let us define:

$$f_0 = \Psi^{0, 123}, f_1 = \Psi^{1, 2}, f_2 = \Psi^{2, 3}, f_3 = \Psi^{1, 2}, f_0 = \Psi^{12} = \Psi^{13} = \Psi^2_1 = \Psi^3_2.$$

The Jacobi identity implies that

$$\begin{vmatrix}
\Psi^1 & \Psi^{1, 2} & \Psi^{1, 3} \\
\Psi^{2, 1} & \Psi^2 & \Psi^{1, 3} \\
\Psi^{3, 1} & \Psi^{3, 2} & \Psi^3
\end{vmatrix} = 0.$$

Applying the previous lemma to the first row gives the equation $\Psi^1 = \Psi^{1, 2} + \Psi^{1, 3}$. The coefficients are all positive, since we know by remark 24 that $\Psi^{i, j}$ for adjacent edges $i, j$ always has positive coefficients. This implies the identity

$$\Psi^{1, 2} \alpha_1 + \Psi^{1, 3} \alpha_2 + \Psi^{2, 3} \alpha_3 = \Psi^{13, 23} \alpha_3 + \Psi^{12} \alpha_2 + \Psi^2_1 \alpha_1 = \Psi^{13} \alpha_1 + \Psi^{12} \alpha_2 + \Psi^2_1 \alpha_1.$$

By symmetry, we deduce that for all $\{i, j, k\} = \{1, 2, 3\}$:

$$f_0 = \Psi^{i, j} = \Psi^{i, j, k} \quad \text{and} \quad \Psi^{i, j} = f_j + f_k \quad (= \Psi^{j, k} + \Psi^{i, j}).$$

It follows that the graph polynomial $\Psi = \Psi_G$ can be written:

$$\Psi = f_{123} + (f_2 + f_3)\alpha_1 + (f_1 + f_2)\alpha_2 + (f_1 + f_2)\alpha_3 + f_0(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3),$$

The Dodgson identity (22) implies that $\Psi^{23}_1 = \Psi^{13}_1 = \Psi^{12}_1 = \Psi^{23}_1 = \Psi^{13}_1 = \Psi^{12}_1$, which, in the new notation, is $f_0f_{123} - (f_1 + f_3)(f_1 + f_2) = f_1^2$. We therefore have:

$$f_0f_{123} = f_1f_2 + f_1f_3 + f_2f_3.$$

**Example 33.** Now suppose that $G$ contains an internal triangle:

![Internal triangle diagram](image)

In this case it is clear, since 1, 2, 3 forms a loop, that $\Psi_{123} = 0$. Define:

$$f^{123} = \Psi^{123}, f^1 = \Psi^{12, 13}, f^2 = \Psi^{12, 23}, f^3 = \Psi^{13, 23}, f^0 = \Psi^{12}_1 = \Psi^{23}_1 = \Psi^{13}_1.$$
By a similar argument to the case of a 3-valent vertex, we have:

\[ f^0 = \Psi_{ij} = \Psi_{j}^{i,k}, \]
\[ \Psi_{ij}^{k} = f^j + f^k, \text{ for all } \{i, j, k\} = \{1, 2, 3\}. \]

We deduce that the graph polynomial \( \Psi \) can be written:

\[ \Psi = f^{123} \alpha_1 \alpha_2 \alpha_3 + (f^1 + f^2)(\alpha_1 \alpha_2) + (f^3)(\alpha_1 \alpha_3) + (f^2 + f^3)(\alpha_2 \alpha_3) + f^0(\alpha_1 + \alpha_2 + \alpha_3), \]

The Dodgson identity implies that

\[ f^0 f^{123} = f^1 f^2 + f^1 f^3 + f^2 f^3. \]

**Corollary 34.** Let \( G \) be a connected graph which contains a 3-valent vertex (resp. triangle). Then with the previous notations, \( f^0 \Psi_G \) (resp. \((f^0)^2 \Psi_G\)) is the graph polynomial of the following graph on the right (resp. left), where a labelled edge means replacing its Schwinger coordinate with that labelling.

**Example 35.** (Star-Triangle duality) Consider a pair of graphs \( G_Y \) and \( G_\Delta \) related by a Star-Triangle transformation. With the notations of the two previous examples, we have \( f^0 = f_{123}, f_0 = f^{123}, \) and \( f^i = f_i \) for \( i = 1, 2, 3. \) It follows that the general star-triangle duality follows from the special case of the two graphs \( K_{2,3} \) and \( K_4 \) above. One can show that for every set of subsets \( I, J, K \subset E_{G_Y} \) such that their union \( I \cup J \cup K \) contains the edges 1, 2, 3, we have

\[ \Psi_{I,J,K}^{G_Y,K} = \Psi_{I,J,K'}^{G_\Delta,K''}, \]

for some \( I', J', K' \subset E_{G_\Delta} \) obtained from \( I, J, K, \) and vice-versa. In other words, there is an (explicit) bijection between the Dodgson polynomials \( \Psi_{I,J,K}^{G_Y,K} \) of \( G_Y \) and \( G_\Delta \) provided that \( I \cup J \cup K \) contains the edges of the 3-valent vertex (resp. triangle).

These examples illustrate the large numbers of identities between the \( \Psi_{G,Y,K}^{I,J,K} \) which arise from constraining the local structure of \( G \). It is these identities which determine the nature of the periods \( I_G \).
3. **Analytic properties of Feynman integrals**

We prove that the periods of Feynman graphs are minor monotone. As a result, any graph invariant which relates to periods should also satisfy this property. We then give parametric proofs of some identities between periods of Feynman integrals which are well-known in momentum or coordinate space.

3.1. **Parametric Feynman integrals for primitive divergent graphs.** Let $G$ be a primitive divergent graph (and so $e_G = 2h_G$ is even), and set

\[ \Omega_G = \sum_{i=1}^{e_G} (-1)^i \alpha_i \, d\alpha_1 \ldots d\alpha_i \ldots d\alpha_{e_G}. \]

The residue of $G$ is defined to be the projective integral [5]:

\[ I_G = \int_\Delta \frac{\Omega_G}{\Psi_G}, \]

where $\Delta = \{ (\alpha_1 : \ldots : \alpha_{e_G}) \in \mathbb{P}^{e_G-1}(\mathbb{R}) | \alpha_i \geq 0 \}$ is the real coordinate simplex. This integral is (absolutely) convergent, and defines a positive real number. In this paper we only consider Feynman integrals whose momentum dependence factors out of the integral (c.f. ‘bpd’ graphs in [9]). In this case one sometimes considers propagators raised to powers $1 + n\varepsilon$ where $n \in \mathbb{Z}$ (the dressed case). In the parametric setting, this corresponds to choosing for each edge $e$ of $G$, an integer $n_e$ such that the following homogeneity condition is satisfied:

\[ \sum_{i=1}^{e_G} n_i = -h_G. \]

The general form of the parametric Feynman integral in dimension $D = 4 - 2\varepsilon$ is:

\[ I_G(n_1, \ldots, n_{e_G}, \varepsilon) = \int_\Delta \prod_{i=1}^{e_G} \alpha_i^{n_i} \frac{\Omega_G}{\Psi_G^{2-\varepsilon}} \in \mathbb{R}[\varepsilon], \]

viewed as a Taylor expansion in $\varepsilon \geq 0$. We are interested in its coefficients of $\varepsilon^k$, for all possible values of decorations $n_e$. Note that these coefficients are given by integrals involving powers of $\log(\alpha_e)$ and $\log(\Psi_G)$ in the numerator [9].

It is convenient to rewrite the integral [80] as an affine integral:

\[ I_G(n_1, \ldots, n_{e_G}, \varepsilon) = \int_{\Delta_H} \prod_{i=1}^{e_G} \alpha_i^{n_i} \frac{\Omega_G}{\Psi_G^{2-\varepsilon}} \delta(H) \prod d\alpha_i, \]

where $H$ is any hyperplane in $\mathbb{A}^{e_G}$ not passing through the origin, and $\Delta_H$ is the subset of points of $H$ which project onto $\Delta$. In the physics literature it is common to take $H$ to be the hyperplane $\sum_e \alpha_e = 1$, but in this paper, we shall always take $H$ to be $\alpha_{e_0} = 1$ for some particular choice of edge $e_0$.

Since the set of all primitive divergent graphs do not form a minor closed set, we need to define the periods for an arbitrary graph.

3.2. **Periods of arbitrary graphs.** For any connected graph $G$, we define two $\mathbb{Q}$-vector spaces of periods, in the following naive sense.
It suffices to prove the formulae in the two cases

Proof. Consider a Feynman integral of the form (39) for the graph

and write it as an affine integral over the hyperplane $H$.

Definition 36. We define the real periods of $G$, denoted $\mathfrak{P}(G)$, to be the $\mathbb{Q}$-vector space spanned by the real numbers

where $n \in \mathbb{N}$ and $P \in \mathbb{Q}[\alpha, e, \varepsilon \in E_G]$ such that the integral converges absolutely. Likewise, we define the real logarithmic periods of $G$, denoted $\mathfrak{P}^{\log}(G)$, to be the $\mathbb{Q}$-vector space spanned by the coefficients in the Taylor expansion at $\varepsilon = 0$ of all absolutely convergent integrals of the form:

where $m, n \in \mathbb{N}, n_i \in \mathbb{Z}$, and $P$ is as above.

Proposition 37. The real periods (resp. real logarithmic periods) of Feynman graphs are minor monotone. More precisely, for every minor $\gamma \leq G$,

$\mathfrak{P}(\gamma) \subseteq \mathfrak{P}(G)$ and $\mathfrak{P}^{\log}(\gamma) \subseteq \mathfrak{P}^{\log}(G)$.

Proof. It suffices to prove the formulae in the two cases $\gamma = G \setminus e$, and $\gamma = G \parallel e$, and the general case will follow by induction. First observe that for all $s > 0$,

where the second equation follows from the first by changing variables $x \mapsto x^{-1}$. Using the fact (5) that $\Psi_G = \Psi_{G \setminus e} \alpha_e + \Psi_{G \parallel e}$, these formulae imply that:

Consider a Feynman integral of the form (39) for the graph $\gamma = G \setminus e$ or $\gamma = G \parallel e$, and write it as an affine integral over the hyperplane $H = \{ \alpha_1 = 1 \}$, for simplicity:

By substituting the appropriate formula from (41) with $s = n + m \varepsilon$, we deduce in both cases $\gamma = G \setminus e$ and $\gamma = G \parallel e$, that $I_\gamma$ can be formally rewritten

where $n_e = m$. We deduce that the coefficients of $\varepsilon^k$ in $I_\gamma$ can be written as linear combinations of coefficients in the Taylor expansion of (39) for the bigger graph $G$. To justify the interchange of integrals, and to show that (42) converges absolutely, observe that any graph polynomial $\Psi_G, \Psi_{G \setminus e}$ or $\Psi_{G \parallel e}$ is a sum of monomials with positive coefficients, and is therefore positive on the domain of integration $(0, \infty)^{\gamma-1}$. The conclusion follows from a standard application of Fatou's
lemma for the Lebesgue integral by taking a compact exhaustion of the domain of integration $[0, \infty)^{n-1}$. □

**Corollary 38.** The set of graphs with periods contained in a fixed subring of $\mathbb{R}$ is minor closed, and thus determined by a finite set of forbidden minors (theorem 7).

One aim of this paper is to establish properties of the set of graphs whose periods are multiple zetas or, more generally, values of polylogarithms.

### 3.3. Simplification

As is well-known, the periods of a graph $G$ and its simplification $G'$ are related in a rather trivial way via Euler’s beta function [24].

**Lemma 39.** For all $0 < \rho, \sigma$ and $u, v \neq 0$:

\[
\int_0^\infty \frac{x^{\rho-1}}{(u x + v)^\sigma} dx = \frac{1}{u^\rho v^{\sigma-\rho}} \frac{\Gamma(\rho) \Gamma(\sigma - \rho)}{\Gamma(\sigma)} .
\]

The graph $G$ can be deduced from $G'$ by applying the operations $S$ and $P$ successively. Consider the case of the operation $S$. Therefore let $G_S$ be a graph obtained from $G$ by subdividing edge $N$ into two edges $N, N + 1$. We have

\[
\Psi_{G_S} = \Psi_G(\alpha_1, \ldots, \alpha_{N-1}, \alpha_N + \alpha_{N+1}).
\]

Now let $r_1, r_2 \in \mathbb{R}$ and let $f$ denote any function such that the following integrals converge absolutely. Then

\[
\int_0^\infty \int_0^\infty x_1^{r_1} x_2^{r_2} f(x_1 + x_2) dx_1 dx_2 = \frac{\Gamma(r_1 + 1) \Gamma(r_2 + 1)}{\Gamma(r_1 + r_2 + 2)} \int_0^\infty x^{r_1 + r_2 + 1} f(x) dx
\]

which is proved by a change of variables $y = x_1, x = x_1 + x_2$ and the definition of Euler’s beta function. We conclude that

\[
\int \prod_{j=1}^{N+1} a_j^{n_j} d\alpha_j = \frac{\Gamma(\rho_1) \Gamma(\rho_2)}{\Gamma(\rho_1 + \rho_2)} \frac{\prod_{i=1}^{N-1} \alpha_i^{n_i} d\alpha_i}{\Psi_G^{d-2e}}
\]

where $\rho_1 = n_N + 1, \rho_2 = n_{N+1} + 1$. Therefore the decorated Feynman integral for $G_S$ is a product of gamma factors with a period integral for $G$. The case of a parallel reduction $P$ is similar. Since the Taylor expansion of Euler’s beta function involves products of zeta values, and the graphs we are interested in typically evaluate to multiple zeta values, we henceforth consider a graph to be equivalent to its simplification from the point of view of its periods (compare also lemma 78).

### 3.4. The two-vertex join of $G_1$ and $G_2$

Let $G_1, G_2$ denote two connected primitive-divergent graphs with edges $e_0, \ldots, e_r$ and $e'_0, \ldots, e'_s$, respectively, and let $G = G_1 \star G_2$ be their two vertex join along the endpoints of $e_0, e'_0$. It follows that $G$ is also primitive-divergent, with edges $E_G = \{e_1, \ldots, e_r, e'_1, \ldots, e'_s\}$. Let $n_1, \ldots, n_r$ be integers corresponding to the edges $e_i$, and let $m_1, \ldots, m_s$ be integers corresponding to the edges $e'_i$ which satisfy the homegeneity condition [34]:

\[
\sum_{i=1}^r n_i + \sum_{j=1}^s m_j = -h_G .
\]

Define integers $n_0, m_0$ by the corresponding conditions for $G_1, G_2$:

\[
n_0 = -(h_{G_1} + \sum_{i=1}^r n_i) \quad \text{and} \quad m_0 = -(h_{G_1} + \sum_{j=1}^s m_j) .
\]

Let us write $\underline{n}$ (respectively $\underline{m}$) for $(n_1, \ldots, n_r)$ (resp. $(m_1, \ldots, m_s)$).
Proposition 40. With these notations, we have

\[ I_G(n, m, \varepsilon) = \frac{\Gamma(-n_0 \varepsilon) \Gamma(-m_0 \varepsilon)}{\Gamma(2 - \varepsilon)} I_G_1(n_0, n, \varepsilon) I_G_2(m_0, m, \varepsilon). \]

Proof. Let us denote the Schwinger parameters corresponding to \( e_0, \ldots, e_r \) (resp. \( e'_0, \ldots, e'_r \)) by \( \alpha_0, \ldots, \alpha_r \) (resp. \( \beta_0, \ldots, \beta_s \)). The decorated Feynman integral of \( G_1 \) along the hyperplane \( \alpha_0 = 1 \) is given by:

\[ I_G_1(n_0, n, \varepsilon) = \int_{\alpha_0=1} \prod_{i=0}^n \frac{\alpha_i^{n_i \varepsilon}}{\Psi_{G_1}^{2-\varepsilon}} \prod_{i=1}^r d\alpha_i. \]

By multiplying each \( \alpha_i \), for \( 1 \leq i \leq r \), by the same parameter \( \lambda > 0 \), we obtain:

\[ I_G_1(n_0, n, \varepsilon) = \frac{\lambda^{-\sum_{i=1}^r n_i \varepsilon}}{\lambda^{2-\varepsilon} \Psi_{G_1}^{-1}} \int_{[0, \infty]^r} \prod_{i=1}^r \frac{\alpha_i^{n_i \varepsilon}}{\Psi_{G_1}^{2-\varepsilon}} \prod_{i=1}^r d\alpha_i. \]

Recall from (11) that \( \Psi_G = \Psi_{G_1 \backslash e_0} \Psi_{G_2 \backslash e'_0} + \Psi_{G_1 \backslash e_0} \Psi_{G_2 \backslash e'_0} \). Therefore, set \( \lambda = \Psi_{G_2} \Psi_{G_2}^{-1} \), which is of homogeneous degree 1. Since \( G_1 \) is primitive divergent, we have \( r_1 + 1 = 2 h_{G_1} \). We deduce that:

\[ I_G_1(n_0, n, \varepsilon) \left( \frac{\Psi_{G_1}^{n_0 \varepsilon}}{\Psi_{G_2}^{2+(n_0-1)\varepsilon}} \right) = \int_{[0, \infty]^r} \prod_{i=1}^r \frac{\alpha_i^{n_i \varepsilon}}{\Psi_{G_2}^{2-\varepsilon}} \prod_{i=1}^r d\alpha_i. \]

Now we can multiply through by \( \prod_{i=1}^r \beta_i^{m_i \varepsilon} d\beta_i \) and integrate over the intersection \( \Delta \) of a suitable hyperplane with \( [0, \infty]^s \). This gives:

\[ I_G_1(n_0, n, \varepsilon) \prod_{i=1}^s \beta_i^{m_i \varepsilon} d\beta_i \left( \frac{\Psi_{G_2}^{n_0 \varepsilon}}{\Psi_{G_2}^{2+(n_0-1)\varepsilon}} \right) = \int_{[0, \infty]^r \times \Delta} \prod_{i=1}^r \frac{\alpha_i^{n_i \varepsilon}}{\Psi_{G_2}^{2-\varepsilon}} \prod_{i=1}^r \beta_i^{m_i \varepsilon} d\beta_i. \]

By (60), this is just:

\[ I_G_1(n_0, n, \varepsilon) \int_{[0, \infty]^s} \prod_{i=1}^s \beta_i^{m_i \varepsilon} d\beta_i \left( \frac{\Psi_{G_2}^{n_0 \varepsilon}}{\Psi_{G_2}^{2+(n_0-1)\varepsilon}} \right) = I_G(n, m, \varepsilon) \]

Now apply (68) with \( \rho = -n_0 \varepsilon \) and \( \sigma = 2 - \varepsilon \), and writing \( \beta_0 \) for \( x \). By the contraction-deletion formula for \( \Psi_{G_2} \), we obtain

\[ I_G_1(n_0, n, \varepsilon) \int_{[0, \infty]} \beta_0^{n_0 \varepsilon - 1} d\beta_0 \prod_{i=1}^s \beta_i^{m_i \varepsilon} d\beta_i = \frac{\Gamma(2 - \varepsilon)}{\Gamma(-m_0 \varepsilon) \Gamma(-n_0 \varepsilon)} I_G(n, m, \varepsilon) \]

We have by (61) and the definitions of \( m_0, n_0 \) that:

\[ m_0 + n_0 = h_G - h_{G_1} - h_{G_2} = -1. \]

This implies that:

\[ I_G_1(n_0, n, \varepsilon) \int_{[0, \infty]} \prod_{i=1}^s \beta_i^{m_i \varepsilon} d\beta_i = \frac{\Gamma(2 - \varepsilon)}{\Gamma(-m_0 \varepsilon) \Gamma(-n_0 \varepsilon)} \]

which is precisely the statement of the proposition. \( \square \)

Setting \( \varepsilon = 0 \), and all \( n_i, m_i \) to zero, we retrieve the well-known result:

\[ I_G = I_G_1 I_G_2, \]

for the leading term of the Feynman integral of a two-vertex join. Note that this induces a drop in the expected transcendental weight of \( I_G \) [2].
3.5. The star-triangle relations. The star-triangle relations have been studied extensively by physicists, but mainly from the point of view of momentum or coordinate space. We give a short parametric proof here.

**Lemma 41.** Let \( G_\Delta, G_Y \) be a pair of graphs related by the Star-Triangle operation. Let \( \alpha_1, \alpha_2, \alpha_3 \) denote the Schwinger parameters corresponding to the edges of the star or triangle. We have the following identities between periods of \( G_\Delta \) and \( G_Y \):

\[
\int \frac{(f^{123})^n \prod_{i=1}^{N} \alpha_i^\lambda_i \, d\alpha_i}{\Psi_{G_\Delta}} \delta(H) = \int \frac{(f^0)^\kappa \prod_{i=1}^{N} \alpha_i^\lambda_i \, d\alpha_i}{\Psi_{G_Y}} \delta(H'),
\]

where \( \lambda_i, \lambda_i', \mu \) are parameters such that the integrals converge which also satisfy \( \lambda_i + \lambda_i' = \mu - 2 \) for \( i = 1, 2, 3 \), and \( \lambda_j = \lambda_j' \) otherwise. Also, \( f^{123} = \Psi(G_\Delta \setminus \{1, 2, 3\}) \) and \( f^0 = \Psi(G_Y \setminus \{1, 2, 3\}) \), and \( \kappa = \sum_{i=1}^3 \lambda_i + 3 - 2\mu \).

**Proof.** Let \( \alpha_1, \alpha_2, \alpha_3 \) denote the Schwinger coordinates of \( G_\Delta \) corresponding to the edges of the triangle, and let \( \beta_1, \beta_2, \beta_3 \) denote the Schwinger coordinates of \( G_Y \) corresponding to the dual 3-valent vertex. We know from examples 32 and 33 that

\[
\Psi_{G_\Delta} = f^{123} \alpha_1 \alpha_2 \alpha_3 + (f^1 + f^2)(\alpha_1 \alpha_2) + (f^1 + f^3)(\alpha_1 \alpha_3) + (f^2 + f^3)(\alpha_2 \alpha_3) + f^0(\alpha_1 + \alpha_2 + \alpha_3).
\]

\[
\Psi_{G_Y} = f_{123} + (f_{1} + f_{2})\beta_1 + (f_{1} + f_{3})\beta_2 + (f_{2} + f_{3})\beta_3 + f_0(\beta_1 \beta_2 + \beta_1 \beta_3 + \beta_2 \beta_3).
\]

where \( f^{123} = f_0, \; f^0 = f_{123} \) and \( f_i = f_i^\prime \) for \( i = 1, 2, 3 \). One checks that

\[
\int_{[0, \infty]^3} \frac{\prod_{i=1}^{N} \alpha_i^\lambda_i \, d\alpha_i}{\Psi_{G_\Delta}(\alpha_1, \alpha_2, \alpha_3)^\mu} = \left( \frac{f_0}{f^{123}} \right)^{\Sigma_{i=1}^3 \lambda_i + 3 - 2\mu} \int_{[0, \infty]^3} \frac{\prod_{i=1}^{N} \beta_i^{\mu - 2\lambda_i} \, d\beta_i}{\Psi_{G_Y}(\beta_1, \beta_2, \beta_3)^\mu},
\]

by making the change of variables \( \alpha_i = f_0(f^{123} \beta_i)^{-1} \). \( \square \)

If \( 2\mu = D - 2\varepsilon \), and say \( D = 3 \) dimensions, then the numerator terms \( (f^0)^\kappa \) and \( (f^{123})^\kappa \) can be made to disappear from the formula to give an identity between the residues of graphs (the uniqueness relations). In 4 spacetime dimensions this fails, but one obtains relations between certain subsets of the set of periods of \( G_\Delta \) and \( G_Y \), which are sometimes known as the almost-uniqueness relations. However, the relationship between the star-triangle relations and the residues (leading terms) of Feynman graphs is not at all clear.

3.6. The complete graph and universal Feynman integral. It follows from proposition 37 that the Feynman integral for the complete graph is the universal Feynman integral for graphs with a given number of vertices.

**Corollary 42.** Let \( G \) be any simple Feynman graph with \( v_G \) vertices. Then since \( G \) is a subgraph of the complete graph \( K_{v_G} \), we have

\[
\Phi(G) \subseteq \Phi(K_{v_G}) \text{ and } \Phi^{\log}(G) \subseteq \Phi^{\log}(K_{v_G}).
\]

It follows from the proof of proposition 37 that for any primitive divergent graph \( \gamma \) with at most \( v \) vertices, there exists a polynomial \( N_\gamma \), such that

\[
I_\gamma = \int_\Delta \frac{\Omega_\gamma}{\Psi^{2}_\gamma} = \int_\Delta \frac{N_\gamma}{\Psi^{r-e_\gamma}_K} \Omega_{K_v},
\]

where \( r = \binom{v}{2} \) is the number of edges of \( K_v \). By repeatedly applying (11), the polynomial \( N_\gamma \) is a product of graph polynomials and powers of Schwinger parameters \( \alpha_\varepsilon \), but is not unique since it depends on the choice of embedding of \( \gamma \) into \( K_v \) and the order in which the edges are removed from \( K_v \) to obtain \( \gamma \). One can make it
unique using the natural action of the symmetric group $\mathfrak{S}_v$ on $v$ letters which acts on $K_v$ by permuting its vertices.

Remark 43. If $X_{K_v} = \{ \Psi_{K_v} = 0 \}$ denotes the graph hypersurface of $K_v$, and writing $r = \binom{v}{2} - 1$ for the number of edges of $K_v$, observe that

$$\mathbb{P}^r \setminus X_{K_v} \cong GL(v - 1)/O(v - 1)$$

can be identified with the symmetric space of symmetric $(v - 1) \times (v - 1)$ non-singular square matrices $^{[25]}$.

For physical applications, one is required to sum the Feynman amplitudes over all graphs in a theory at a given loop order. The previous corollary should allow one in principle to place all such (unrenormalized) integrands over a common denominator, and reduce the sum of the contributions of all graphs to a single integral.

4. The graph hypersurface and blow-ups

Following $^{[5]}$, we blow up coordinate linear subspaces in $(\mathbb{P}^1)^N$ which are contained in the graph hypersurface so that the Feynman integral defines a period of the corresponding mixed Hodge structure.

4.1. The graph hypersurface. Let $G$ be a graph, with edges numbered $1, \ldots, N$, and let $\alpha_i$ denote the corresponding Schwinger parameters, viewed as affine coordinates on each copy of $\mathbb{P}^1$ in $(\mathbb{P}^1)^N$. Equivalently, let

$$\tilde{\psi}_G = \sum_{T \subseteq G} \prod_{e \in T} a_e \prod_{e \in T} b_e ,$$

where the sum is over all spanning trees $T$ of $G$, and $(\alpha_e : 1) = (a_e : b_e)$. The variables $a_e$ and $b_e$ are interchanged on passing to the planar dual graph (matroid).

Let $B, X_G \subset (\mathbb{P}^1)_N$ be the (resp. coordinate, graph) hypersurfaces defined by:

$$B : \prod_{i=1}^N a_i b_i = 0 \quad \text{and} \quad X_G : \tilde{\psi}_G = 0 .$$

Let $D = [0, \infty]^N$ denote the real hypercube in $(\mathbb{P}^1(\mathbb{R}))^N$ with positive coordinates $\alpha_i$. Its boundary is contained in $B$. Since the coefficients of $\psi_G$ are positive, $X_G$ does not meet the interior of $D$. To see how it meets the boundary of $D$, let

$$L_{S,T} : \prod_{s \in S} a_s \prod_{t \in T} b_t = 0 \quad \text{and} \quad F_{S,T} = D \cap L_{S,T} .$$

where $S, T \subseteq \{1, \ldots, N\}$ are disjoint. It is clear that $F_{S,T} = F_{S,\emptyset} \cap F_{\emptyset,T}$.

Lemma 44. (cf $^{[5]}$, lemma 7.1) Let $S, T$ be as above. The following are equivalent:

i). $F_{S,T} \cap X_G \neq \emptyset$.

ii). $F_{S,T} \subset X_G$.

iii). $F_{S,\emptyset} \subset X_G$ or $F_{\emptyset,T} \subset X_G$.

The first case $F_{S,\emptyset} \subset X_G$ occurs if and only if the subgraph of $G$ defined by $S$ contains a loop ($h_1(S) > 0$), and the second case $F_{\emptyset,T} \subset X_G$ occurs if and only if removing the set of edges $T$ from $G$ causes it to disconnect ($h_0(G\setminus T) > 0$).

Proof. It follows from $^{[8]}$ that the restriction of $\tilde{\psi}_G$ to the face $F_{S,T}$ is $\tilde{\psi}_G|_{T/S}$. Since it has positive coefficients, the zero locus $X_G$ meets $F_{S,T}$ if and only if it is identically zero along $F_{S,T}$. The result then follows from corollary $^{[14]}$.
Definition 45. Let $G$ be a graph. For any set of edges $S \subset E_G$, write $\mathbb{P}^1_S$ for $(\mathbb{P}^1)^{|S|}$ whose affine coordinates are the Schwinger parameters of $S$, and let

\[ \pi_S : \mathbb{P}^1_G \to \mathbb{P}^1_S, \]

denote the map $\pi_S : (\alpha_e)_{e \in E_G} \mapsto (\alpha_e)_{e \in E_S}$. If we identify $\mathbb{P}^1_S$ with $\text{Hom}(S, \mathbb{P}^1)$, then $\pi_S$ is the natural map induced by the inclusion $S \subset E_G$.

4.2. Combinatorics of blow ups. As in [3], we can blow up linear spaces $L_{S,T}$ in such a way that the strict transform of the graph hypersurface is moved away from the inverse image of the domain of integration $D$. In general, suppose we are given a set of faces $\mathcal{F} = \{F_{S_i,T_i}\}$, of codimension $|S_i| + |T_i| \geq 2$. Let $\tilde{\mathcal{F}}$ be the set of all intersections of faces in $\mathcal{F}$, and blow-up $\mathbb{P}^1_G$ along $L_{S,T}$ for each $F_{S,T} \in \tilde{\mathcal{F}}$, in order of increasing dimension. More precisely, for each $0 \leq k \leq N - 2$, let $\mathcal{F}^{(k)}$ denote the set of intersections of faces $F_{S,T} = \cap_i F_{S_i,T_i}$ of dimension $k$. As is standard practice, we blow up the coordinate hyperplanes $L_{S,T}$ corresponding to elements $F_{S,T}$ in $\mathcal{F}^{(0)}, \mathcal{F}^{(1)}, \ldots, \mathcal{F}^{(N-2)}$ in turn. Denote the resulting space by $\pi_{\tilde{\mathcal{F}}} : \mathbb{P}_{\tilde{\mathcal{F}}} \to \mathbb{P}^1_G$. It does not depend on the choice of order of the blow-ups, and only depends on $\tilde{\mathcal{F}}$. Let $D_{\tilde{\mathcal{F}}}$ be the (real analytic) closure of the inverse image of $D = (0, \infty)^N$. It is a compact manifold with corners, whose boundary stratification is a polytope whose poset of faces is determined by the following lemma.

Lemma 46. For every face $F_{S,T} \in \tilde{\mathcal{F}} \cup \{F_{\emptyset,i}, F_{i,\emptyset}, 1 \leq i \leq N\}$, let $P_{S,T}$ denote the strict transform of $F_{S,T}$. The resulting map from $\tilde{\mathcal{F}} \cup \{F_{\emptyset,i}, F_{i,\emptyset}, 1 \leq i \leq N\}$ to the set of facets of $D_{\tilde{\mathcal{F}}}$ is a bijection. Two such facets $P_{S,T}$ and $P_{S',T'}$ meet if and only if one of the following holds:

i). $S \subset S'$ and $T \subset T'$,

ii). $S' \subset S$ and $T' \subset T$,

iii). $(S \cup S') \cap (T \cup T') = \emptyset$, and $F_{S \cup S', T \cup T'}$ is not an element of $\tilde{\mathcal{F}}$.

Proof. It is easily verified that truncating the faces of a hypercube in increasing order of dimension leads to the above poset structure. \(\square\)

Definition 47. A set of faces $\mathcal{F}$ is polarized if, for every $F_{S,T} \in \mathcal{F}$, either $S = \emptyset$ or $T = \emptyset$. In this case we write $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_\infty$, where $\mathcal{F}_0 = \{F_{\emptyset,i}, F_{i,\emptyset} : 1 \leq i \leq N\}$ and $\mathcal{F}_\infty = \{F_{\emptyset,T} : F_{\emptyset,T} \in \mathcal{F}\}$.

Now let $G$ be a connected graph, and let $X_G, B$ be as in [4]. The set of faces $F_{S,T}$ which need blowing up are given by lemma [46]. Let

\[ \mathcal{F}_0 = \{ F_{S,\emptyset} : S \subset E_G \text{ minimal s.t. } |S| \geq 2 \text{ and } S \text{ contains a loop} \} \]

\[ \mathcal{F}_\infty = \{ F_{\emptyset,T} : T \subset E_G \text{ minimal s.t. } |T| \geq 2 \text{ and } G \setminus T \text{ is disconnected} \} \]

and consider the polarized set $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_\infty$.

Example 48. Consider the sunset diagram (the graph with vertex set $V = \{1, 2\}$ and edges $E = \{\{1, 2\}, \{1, 2\}, \{1, 2\}\}$, numbered 1, 2, 3). Then \(\mathcal{F}_0 = \{12, 23, 13\}\), and $\mathcal{F}_\infty = \{123\}$. Then $\mathcal{F} = \{F_{12,\emptyset}, F_{23,\emptyset}, F_{13,\emptyset}, F_{123,\emptyset}, F_{\emptyset,123}\}$, so one must first blow up the two points $\alpha_1 = \alpha_2 = \alpha_3 = 0$, and $\alpha_1 = \alpha_2 = \alpha_3 = \infty$, followed by the three lines $\alpha_1 = \alpha_2 = 0, \alpha_1 = \alpha_3 = 0$, and $\alpha_2 = \alpha_3 = 0$.

Definition 49. Write $\mathcal{P}_G$ for $\mathbb{P}_{\tilde{\mathcal{F}}}$, where $\mathcal{F}$ is the polarized set given by (49). Let $B' \subset \mathcal{P}_G$ denote the total transform of $B$, and let $X'_G \subset \mathcal{P}_G$ denote the strict
transform of the graph hypersurface $X_G$. The irreducible components of $B'$ are given in lemma \[40\] Let $D_G$ denote the closure (in the real analytic topology) of the strict transform of the hypercube $(0, \infty)^N$. It is a compact manifold with corners whose boundary is contained in $B'$. Its poset of faces is determined by lemma \[40\]

The following lemma is well-known (implicit in \[5\], §3) and is essentially the Schwinger-parametric interpretation of the renormalization Hopf algebra.

**Lemma 50.** Let $G$ be a connected graph, and let $\gamma \subset G$ be a subgraph, with connected components $\gamma = \gamma_1 \cup \ldots \cup \gamma_k$. Then

\[
(50) \quad \Psi_G = \Psi_{G/\gamma} \prod_{i=1}^k \Psi_{\gamma_i} + R,
\]

where $R$ is a polynomial of total degree $\geq h_{\gamma} + 1$ in the parameters $\{\alpha_e, e \in E(\gamma)\}$. Here, $\Psi_{G/\gamma}$ is the graph polynomial of the graph-theoretic quotient $G/\gamma$ where each component $\gamma_i$ is shrunk to a point (note the shrinking of loops is not zero here).

For any connected graph $G$, define:

\[
\omega_G = \frac{d\alpha_1 \wedge \ldots \wedge d\alpha_N}{\Psi_G^2} \in \Omega^N(P_G^1 \setminus X_G).
\]

**Proposition 51.** \[5\] Let $\pi_G : P_G \to P_G^1$ denote the blow-up defined above. Then $B' \subset P_G$ is a normal crossings divisor, and $D_G$ does not intersect $X'_G$. If $G$ is primitive divergent, then $\pi_G^*\omega_G$ has no poles along components of $B'$.

**Proof.** The fact that $B'$ is a normal crossings divisor follows from general properties of blowing up linear subspaces in projective space. To show that $D_G$ does not intersect $X'_G$ (no further blow-ups are required), is entirely analogous to the proof of proposition 7.3 in \[5\], so we omit the details. The idea is that the restriction of $\Psi_G$ to each face of $B'$ can be expressed in terms of graph polynomials of sub and quotient graphs of $G$ by the contraction-deletion relations, or by \[50\] for the exceptional divisors. The statement follows by an induction on the number of edges.

To see how the primitive divergence comes in, the key remarks are:

(i) The order of the pole of $\omega_G$ along a facet of the form $F_{S,0}$ is $2h_1(S)$.

This follows from \[50\] on setting $\gamma = S$. Writing \[50\] in homogeneous coordinates $a_e, b_e$, where $(a_e : 1) = (a_e : b_e)$, shows that the order of vanishing of $\tilde{\psi}_G$ along a facet $F_{b,T}$, where $T$ is the set of edges of $G$ not in $\gamma$, is $|G/\gamma| - h_1(G/\gamma)$. By Euler’s formula, this is $V(G/\gamma) - 1$, which is 0 if and only if $G\setminus T$ is connected. For any subset of edges $T \subset G$, write $\gamma_T$ for the subgraph of $G$ defined by the complement of the set of edges of $T$ in $G$. We have shown:

(i') The order of the pole of $\omega_G$ along a facet of the form $F_{b,T}$ is $2(V(G/\gamma_T) - 1)$. Now, the order of the poles of $\pi_G^*\omega_G$ along components of the exceptional divisors of $B'$ are computed using the fact that blowing up a linear subspace of codimension $p$ decreases the order of the pole by $p - 1$. This follows from a direct calculation.

That $\pi_G^*\omega_G$ has no poles along $B'$, involves checking, in the case (i), that $(|S| - 1) - 2h_1(S) \geq 0$, which holds precisely because $G$ is primitive divergent.

\[2\] Given that $D_G$ does not intersect $X'_G$, the fact that $\pi_G^*\omega_G$ has no poles along $B'$ is equivalent to the convergence of the residue $t_G$ for $G$ primitive divergent.
corresponding inequality in the case (i') is:

\[
|T| - 2(V(G/\gamma T) - 1) = |G/\gamma T| - 1 - 2(|G| - |\gamma T| - h_1(G/\gamma T))
\]

\[
= |G| - |\gamma T| - 2(|G| - |\gamma T| - h_1(G) + h_1(\gamma T)) - 1
\]

\[
= (2h_1(G) - |G|) + (|\gamma T| - 2h_1(\gamma T) - 1) \geq 0
\]

which is again non-negative since \( G \) is primitive divergent.

\[\square\]

Recall that the domain of integration of a Feynman integral is \( \Delta_H = [0, \infty]^N \cap H \), where \( H \) is a hyperplane in \( \mathbb{P}_G \) such that \( 0 \notin H \). If \( H \) is the hyperplane \( \sum_{i=1}^N \alpha_i = 1 \), then \( H \) intersects the hypercube \([0, \infty[^N\) in faces of the form \( F_{S,0} \) only. After blowing up, the subspace \( \pi^{-1}(H) \cap P_G \) therefore has the identical geometry as the blow-ups constructed in [3], §7. In this paper, we will always fix an edge, say \( e = N \), and choose the hyperplane \( H \subset \mathbb{P}_G \) defined by \( \alpha_N = 1 \). We can therefore define the mixed Hodge structure of any connected graph \( G \):

\[(51) \quad \mathcal{M}_G = H^{N-1}(P_G \setminus X_G', B' \setminus (B' \cap X_G')) \]

where, by abuse of notation, \( P_G, X_G', B' \) actually denotes their intersection with the hyperplane \( \alpha_N = 1 \). In a standard way, the domain of integration \( \pi^{-1}(\Delta_H) \) defines a relative Betti homology class in \( \text{gr}^W H_{N-1}(P_G \setminus X_G', B' \setminus (B' \cap X_G')) \) (see [3], proposition 7.5). When \( G \) is primitive divergent, proposition [31] implies that the differential form

\[
\omega_G \delta(H) = \prod_{i=1}^{N-1} \frac{d\alpha_i}{\Psi_G|_{\alpha_N=1}}
\]

defines a relative de Rham cohomology class in \( H^{N-1}(P_G \setminus X_G', B' \setminus (B' \cap X_G')) \).

**Corollary 52.** If \( G \) is primitively divergent, then the residue

\[
I_G = \int_{\Delta_H} \omega_G \delta(H)
\]

converges, and defines a period of \( \mathcal{M}_G \).

**Remark 53.** Since the boundary components of \( B' \) corresponding to facets of the form \( F_{i,0} \) (resp. \( F_{0,i} \)) intersect \( X_G \) along \( X_{G\setminus i} \) (resp. \( X_{G\setminus i} \)), the inclusion of a facet defines a morphism \( \mathcal{M}_\gamma \to \mathcal{M}_G \) where \( \gamma = G\setminus i \) or \( G\setminus i \). By induction, for any minor \( \gamma \prec G \) (which contains the chosen edge \( N \)) there is a morphism \( \mathcal{M}_\gamma \to \mathcal{M}_G \) of mixed Hodge structures. Proposition [37] computes a representative for the image of the class of the Feynman differential form \( \omega_\gamma \) under this map.
5. Stratification and singularities of integrals

We recall some results from stratified Morse theory [16], and use them to compute the singularities of integrals associated to graph hypersurfaces.

5.1. The Landau varieties of a graph. Let $P, T$ be smooth connected complex analytic manifolds, and consider a smooth proper map

$$\pi : P \rightarrow T.$$ 

Let $S$ be a closed analytic subset of $P$. It gives rise to a stratification of $P$, i.e., there is a sequence of closed analytic submanifolds

$$S^0 = P \supset S^1 \supset S^2 \supset \ldots \supset S^k$$

such that the complements $S^i \setminus S^{i+1}$ are smooth. Furthermore, the irreducible components $A_k$ (the open strata) of $S^i \setminus S^{i+1}$ have the property that the boundary $\partial A_k = A_k \setminus A_k$ is a union of strata of lower dimension, and satisfy Whitney’s conditions A and B. The critical set $cA_i$ of a stratum $A_i$ is defined to be the (analytic) set of points of $A_i$ where $\pi$ fails to be submersive:

$$cA_i = \{ x \in A_i : \text{rank } T_x \pi < \dim T \}.$$ 

In particular, all strata of dimension less than $\dim T$ are critical.

**Definition 54.** The Landau variety $L(S, \pi)$ is the codimension 1 part of $\pi(\cup_i cA_i)$, where the union is taken over all strata of $S$ (cf [20]).

It follows from Thom’s isotopy theorem that $L(S, \pi)$ has the following property: for all open $U \subset T \setminus L(S, \pi)$, the restriction of $\pi : \pi^{-1}(U) \rightarrow U$ is a locally trivial stratified map, i.e., each point $u \in U$ has an open neighbourhood $V$ such that $\pi^{-1}(V)$ is homeomorphic (as a stratified space) to $\pi^{-1}(u) \times V$ ([16], §1.7). In other words, $L(S, \pi)$ is the smallest subvariety of $T$ outside of which $\pi$ has topologically constant (but not necessarily smooth) fibers.

**Definition 55.** Let $G$ be a connected graph, and let $B, X_G$ denote the coordinate and graph hypersurfaces in $\mathbb{P}_G^1$. For any subset $K \subset E_G$, consider the map

$$\pi_K : \mathbb{P}_G^1 \rightarrow \mathbb{P}_K^1.$$ 

We define the Landau variety of $G$ relative to $K$ to be $L(X_G \cup B, \pi_K)$.

Since $L(X_G \cup B, \pi_K)$ is a hypersurface in $\mathbb{P}_K^1$, we will often represent it as the zero locus of a set of polynomials $\{f_1, \ldots, f_n\}$, where $f_i \in \mathbb{Q}[[\alpha_e]_{e \in K}]$.

5.2. Singularities of integrals. Let $\pi : P \rightarrow T$ be as above, let $S = X \cup B \subset P$, and let $L(S, \pi)$ be its Landau variety. For each complex point $t \in T \setminus L$, let $P_t$ denote the fiber of $\pi$ over $t$, and let $\ell = \dim P_t$ denote its dimension. Set $X_t = X \cap P_t$ and $B_t = B \cap P_t$. For all $t$ in a neighbourhood of a fixed complex point $t_0 \in T \setminus L$, suppose that we are given a continuous family of real compact submanifolds $\Delta_t \subset P_t \setminus X_t$ of dimension $\ell$ such that $\partial \Delta_t \subset B_t$, and $\Delta_t \cap X_t = \emptyset$. Suppose that we are given a family $\omega_t \in \Omega^\ell(P_t \setminus X_t)$ of closed differential $\ell$-forms which depend analytically on $t$ for all $t \in T \setminus L$. Let

$$(52) \quad I(t) = \int_{\Delta_t} \omega_t,$$ 

defined in a neighbourhood of $t_0$. It is absolutely convergent since $\omega_t$ has singularities contained in $X_t$, which is disjoint from the compact set $\Delta_t$. 

Theorem 56. \( I(t) \) extends to a multivalued function on \( T \backslash L \).

Proof. (see [20], Chapter X). By repeating the construction of ([20], Appendix A) in the relative case, one can construct a homology sheaf \( \mathcal{F} \) whose stalks at \( t \in T \backslash L \) are isomorphic to:

\[
\mathcal{F}_t \cong H_t(P_t \backslash X_t, B_t \backslash (B_t \cap X_t))
\]

By assumption, the relative homology class \( [\Delta_t] \) defines a continuous section of \( \mathcal{F} \) in a neighbourhood of \( t_0 \). Since \( \pi : P \backslash \pi^{-1}(L) \to T \backslash L \) is a locally trivial fibration, the sheaf \( \mathcal{F} \) is locally constant, and therefore the section \( [\Delta_t] \) extends to a global multivalued section of \( \mathcal{F} \). Now fix a point \( s \in T \backslash L \), and let \( U \) be a simply-connected open neighbourhood of \( s \). For any \( t \in U \), let \( [\Delta'_t] \) denote a fixed local branch of this multivalued section. Since \( \omega_t \) extends to an analytic family \( \omega_t \in \Omega^1(P_t \backslash X_t) \) for all \( t \in T \backslash L \), we can define a (multivalued) continuation of \( I(t) \) by the formula

\[
I(t) = \int_{[\Delta_t]} \omega_t \quad \text{for all } t \in U.
\]

To finish the proof, it suffices to show that the function \( I(t) \) defined in this way is analytic in a neighbourhood of \( s \). Note that by Hartogs’ theorem, a function is analytic at \( t \in T \) if and only if it is analytic at \( t \) along every smooth curve in \( T \), so we can assume that \( T \) is of dimension 1. We can replace \( [\Delta_t] \) with a homology class which does not depend on \( t \) by considering the relative Leray coboundary map:

\[
\delta_t : H_t(P_t \backslash X_t, B_t \backslash (B_t \cap X_t)) \to H_{t+1}(P \backslash (X \cup P_t), B \backslash (B \cap (X \cup P_t))).
\]

The group on the right is locally constant and is therefore constant on the simply-connected set \( U \). Therefore \( \delta_t[\Delta'_t] \) is constant for all \( t \in U \), and let \( h \) denote a representative of this relative homology class. We can assume that \( h \) is a real smooth submanifold of \( X \) such that \( \partial h \subset B \), and \( h \cap A = \emptyset \) (\( h \) is a tubular neighbourhood of a representative of \( [\Delta'_t] \)). By the residue formula we can write the local branch of \( I(t) \) as:

\[
I(t) = \int_{[\Delta_t]} \omega_t = \frac{1}{2 \pi i} \int_h \omega_t \wedge \frac{du}{t-u},
\]

for all \( t \in U \). By doing a Taylor expansion under the integral, we see at once that \( I(t) \) is holomorphic in a neighbourhood of \( s \) in \( T \backslash L \).

Remark 57. The proof extends to the case where \( \omega_t \) is a multivalued function on \( P \backslash (X \cup B) \), since we can pass to a universal cover of \( P \backslash (X \cup B) \) and the projection to \( T \) will still be a locally trivial stratified map on an open subset of \( T \backslash L \). One must add to the initial assumptions, however, that \( \omega_t \) is single-valued along \( \Delta_t \) for some \( t \) in a neighbourhood of \( t_0 \), otherwise ([20]) may not be well-defined.

5.3. Application to partial Feynman integrals. Let \( G \) be a connected graph, \( K \) a subset of edges of \( G \), and consider the coefficient of some power of \( \varepsilon \) in an absolutely convergent Feynman integral of the form ([39]). Integrating only the edges in \( E_G \setminus E_K \) gives a partial integral:

\[
I^K((\alpha_e)_{e \in E_K}) = \int_0^\infty \cdots \int_0^\infty \frac{P((\alpha_e), \log(\alpha_e)), \log(\Psi_G)}{\Psi_G^n} \prod_{e \in E_G \setminus K} d\alpha_e,
\]

where \( P \) is a polynomial with coefficients in \( \mathbb{Q} \), and \( n \in \mathbb{N} \). It is a multivalued function on some open subset of \( \mathbb{P}^1_K \). By Fubini’s theorem, the full Feynman integral ([39]) is the integral of \( I^K \) over a hyperplane \( H \) chosen to lie in \( \mathbb{P}^1_K \).
**Theorem 58.** The integral (5.53) is a multivalued function on \( \mathbb{P}^1_K \setminus L(G, K) \), and has no singularities on the interior of the hypercube \([0, \infty])^{[K]} \subset \mathbb{P}^1_K\).

**Proof.** Consider the blow-up of definition 29. Since the exceptional divisors lie over hyperplanes \( \alpha_i = 0, \infty \) which are critical, the map \( \pi = \pi_K \circ \pi_G : P_G \to \mathbb{P}^1_K \), where \( P_G \) is stratified by \( X'_G \cup B' \), is still a locally trivial fibration outside \( L(G, K) \) (see (11)). By proposition 51 the domain of integration \( D_G \) and \( X'_G \) do not meet, so the argument of theorem 56 still applies in this slightly modified situation. By assumption, (5.53) is absolutely convergent, so the pull-back of the integrand has no polar singularities along \( \pi^{-1}(\Delta_i) \). By remark 57 we can then pass to a universal cover, since the multivalued functions \( \log(\alpha) \) and \( \log(\Psi_G) \) are ramified only along \( B' \) and \( X'_G \). We conclude that \( I(\{\alpha_i\}) \) extends to a multivalued holomorphic function on \( \mathbb{P}^1_K \setminus L(G, K) \), which proves the first part of the theorem. In this case, the germ of the chain of integration \( \Delta_i = [0, \infty]^{[G, K]} \) is constant for all \( t \in [0, \infty]^{[K]} \subset \mathbb{P}^1_K \). Since \( \Psi_G \) is a sum of monomials with positive coefficients, we can differentiate under the integral to deduce that (5.53) is analytic on the interior of \([0, \infty])^{[K]}\). \( \square \)

5.4. Landau varieties of linear hypersurfaces. Let \( k \geq 1 \), and consider the situation where \( P = (\mathbb{P}^1)^k \times T \), and \( T = (\mathbb{P}^1)^2^k \) with coordinates we denote by \( \psi_j^i \), where \( I \cap J = \emptyset \) and \( I \cup J = \{1, \ldots, k\} \). Let \( x_1, \ldots, x_k \) be the coordinates in the fibers \( (\mathbb{P}^1)^k \) of the projection \( \pi_k : P \to T \), and consider a general linear form

\[
\psi = \sum_{I \subset \{1, \ldots, k\}} \psi^i_j \prod_{i \in I} x_i
\]

where \( J \) is the complement of \( I \) in \( \{1, \ldots, k\} \). We omit \( I \) or \( J \) from the notation if they are the empty set. As before, let \( X \) denote the zero locus of \( \psi \) in \( P \), and let \( B \) denote the hypercube \( \cup_{i=1}^k \{x_i = 0, \infty\} \). Note that \( \pi_k \) fails to be submersive at a smooth point \( p \) of the hypersurface \( X \) if and only if:

\[
(5.4) \quad \frac{\partial \psi}{\partial x_i}(p) = 0, \quad \text{for all } 1 \leq i \leq k.
\]

At a singular point \( p \), these partial derivatives will also vanish, but it can happen that a smooth stratum in the singular locus of \( X \) may project submersively onto \( T \). In the following examples, we write down the Landau varieties \( L(X \cup B, \pi_k) \) in the cases \( k = 1, 2, 3 \), and consider a Feynman-type integral of the form:

\[
I^k = \int_{[0, \infty]^k} \frac{dx_1 \ldots dx_k}{\psi^2}.
\]

Strictly speaking, we should first stratify \( X \cup B \), and then compute the critical strata, but a posteriori it will be clear that \( \pi_k \) is locally trivial on \( T \setminus L(X \cup B, \pi_k) \).

**Example 59.** If \( k = 1 \), we have \( \psi = \psi_1 + \psi^1 x_1 \), and \( B = x_1 = 0, \infty \). The critical strata \( X \cap B \) are given by \( \{\psi = 0\} \cap \{x_1 = 0\} \) and \( \{\psi = 0\} \cap \{x_1 = \infty\} \), which project down to \( \{\psi^1 = 0\} \) and \( \{\psi_1 = 0\} \) respectively. Clearly, \( \pi_1 |_X \) is submersive everywhere outside this locus. The Landau variety is therefore \( \psi^1 \psi_1 = 0 \), and indeed \( I^1 = \int_0^\infty \frac{dx}{\psi^2} = \frac{1}{\psi_1 \psi^1} \) has singularities at both \( \psi^1 = 0 \) and \( \psi_1 = 0 \).

**Example 60.** When \( k = 2 \), we have \( \psi = \psi_{12} + \psi^1_2 x_1 + \psi^1_1 x_2 + \psi^2_1 x_1 x_2 \), and \( B \) is the square \( x_1, x_2 = 0, \infty \). The four strata \( \{\psi = 0\} \cap \{x_1 = 0, \infty\} \cap \{x_2 = 0, \infty\} \) are of codimension 3 and are therefore critical. They project down to \( \psi_{12} = \)
0, \psi_1^2 = 0, \psi_1^2 = 0, \psi^{12} = 0. The intersection of \( X \) with a single face of \( B \) defines a stratum which is linear, and therefore projects submersively onto \( T \), by the previous example. The stratum \( X \) fails to be submersive when \( \psi = \frac{\partial \psi}{\partial x_1} = \frac{\partial \psi}{\partial x_2} = 0 \), which can only occur when \( \psi_{12} \psi^{12} - \psi_1^2 \psi_2^1 = 0 \). The Landau variety is therefore

\[ L_2 = \{ \psi^{12} = 0 \} \cup \{ \psi_2^1 = 0 \} \cup \{ \psi_2^1 = 0 \} \cup \{ \psi_1^2 = 0 \} \cup \{ \psi_{12} \psi^{12} - \psi_1^2 \psi_2^1 = 0 \} . \]

Four components of \( L_2 \) correspond to the locus where \( X \) passes through a corner of the square \( B \), and one when \( X \) degenerates into a product of two lines (dashed):

Decomposing \( I^1 \) into partial fractions and integrating gives:

\[
I^2 = \frac{\log(\psi_2^1) + \log(\psi_1^2) - \log(\psi^{12}) - \log(\psi_{12})}{\psi_2^1 \psi_1^2 - \psi^{12} \psi_{12}} ,
\]

which is ramified along \( L_2 \), as expected, and has unipotent monodromy.

**Example 61.** The situation quickly gets out of hand. If \( k = 3 \), then \( B \) is a cube \( x_1, x_2, x_3 = 0, \infty \). The intersection of \( X : \psi = 0 \) with the corners of the cube are critical strata whose projections give the contributions

\[
\{ \psi_{123}, \psi_{13}^2, \psi_{21}^3, \psi_{12}^3, \psi_1^{13}, \psi_2^{13}, \psi_1^{23}, \psi_{12}^{23} \}
\]

to the Landau variety. Next, the intersection of \( \psi = 0 \) with any one-dimensional edge of the cube is always submersive, by the first example. The intersection of \( \psi = 0 \) with any of the six faces of the cube puts us in the situation of the previous example, and gives rise to six quadratic terms

\[
\psi^i_{jk} \psi^j_{ik} - \psi^j_{ik} \psi^i_{jk} \quad \text{and} \quad \psi^{ijk} \psi^k_{ij} - \psi^i_{jk} \psi^j_{ik} \quad \text{where} \quad \{i,j,k\} = \{1,2,3\} .
\]

Finally, the hypersurface itself is not submersive when \( \psi = 0 \) and \( \partial \psi / \partial x_i = 0 \) for all \( 1 \leq i \leq 3 \). These four equations admit a solution when the discriminant:

\[
F = \sum_{\{i,j,k\} = \{1\to3\}} \psi^i_{jk} \psi^j_{ik} \psi^k_{ij}^2 - 2 \psi^i_{jk} \psi^j_{ik} \psi^k_{ij}^2 - 2 \psi^{ijk} \psi^i_{jk} \psi^{ijk} \psi^{jik}
\]

vanishes. Thus the Landau variety \( L_3 \) for a general hypersurface \( \psi \) is the union of the zero loci of the polynomials \( 56, 57 \) and \( 58 \). One can verify that the integral \( I^3 \) can be written as an ugly expression involving the dilogarithm function, and has unipotent monodromy on a degree 2 covering of the complement of the Landau variety \( L_3 \) (it has quadratic ramification around \( F = 0 \)).
Example 62. What saves us in the graph polynomial case are the Dodgson identities. Let us reconsider the previous example when $\psi = \Psi$ is a (generic) graph polynomial. By identity (22), the quadratic terms in (57) factorize as follows:

$$
(59) \quad \Psi_{ij}^k \Psi_{ijk} - \Psi_{i}^j \Psi_{jk}^i - \Psi_{ik}^j \Psi_{jk}^i = (\Psi_{i,j}^k)^2,
$$

$$
(59) \quad \Psi_{ij}^k \Psi_{ijk} - \Psi_{i}^j \Psi_{jk}^i - \Psi_{ik}^j \Psi_{jk}^i = (\Psi_{i,k,j}^j)^2.
$$

The polynomial $F$ in (58) is now identically zero. What happens is that there is a unique solution to the equations $\Psi = 0$ and $\frac{\partial \Psi}{\partial x_i} = 0$ for $i = 1, 2, 3$ given by:

$$
(x_1, x_2, x_3) = (-\Psi_{2,3}^1, \Psi_{1,2}^3, \Psi_{1,3}^2).
$$

It follows that every fiber of $\pi$ has a singular point. These singular points define a stratum which is everywhere submersive over the subvariety of $T$ implied by (59), and meets the boundary of the cube $x_1, x_2, x_3 = 0$ over loci of the form $\Psi_{i,j}^k = 0$ and $\Psi_{i,k,j}^k = 0$, which we have already taken into account in the situation (59).

Corollary 63. The Landau variety obtained by projecting a graph hypersurface $\Psi$ is the zero locus of graph polynomials corresponding to the eight corners of a cube:

$$
\{ \Psi_{123}, \Psi_{121}, \Psi_{213}, \Psi_{231}, \Psi_{132}, \Psi_{123} \},
$$

and Dodgson polynomials corresponding to the six faces of a cube:

$$
\{ \Psi_{3,1,2}, \Psi_{1,3,23}, \Psi_{1,2,3}, \Psi_{1,2,13}, \Psi_{1,3,23}, \Psi_{1,2,13} \}.
$$

Likewise, the integral (55) also simplifies since its denominator $\Psi_{1,2}^1 \Psi_{1,2}^2 - \Psi_{12}^1 \Psi_{12}^2$ is now a perfect square $(\Psi_{1,2}^1)^2$, and can be easily integrated one more time.

Corollary 64. Using both Dodgson identities (22) and (23), we calculate:

$$
I^3 = \frac{\Psi_{123}^{123} \log \Psi_{123}^{123}}{\Psi_{1,2,3}^{1,1,2} \Psi_{1,2,3}^{1,1,3} \Psi_{1,2,3}^{1,2,2}} - \frac{\Psi_{123}^{123} \log \Psi_{123}^{123}}{\Psi_{1,2,3}^{1,1,2} \Psi_{1,2,3}^{1,1,3} \Psi_{1,2,3}^{1,2,2}} + \sum_{\{i,j,k\}} \frac{\Psi_{ij}^{i,j,k} \log \Psi_{ij}^{i,j,k} \Psi_{ij}^{i,j,k}}{\Psi_{ij,k}^{i,j,k} \Psi_{ij,k}^{i,j,k} \Psi_{ij,k}^{i,j,k}} - \frac{\Psi_{ij}^{i,j,k} \log \Psi_{ij}^{i,j,k} \Psi_{ij}^{i,j,k}}{\Psi_{ij,k}^{i,j,k} \Psi_{ij,k}^{i,j,k} \Psi_{ij,k}^{i,j,k}}
$$

where the sum is over all $\{i,j,k\} = \{1, 2, 3\}$, giving 8 terms in total.

Notice that the integral $I^3$ now has unipotent monodromy, and that its weight has dropped by one (the dilogarithms are replaced by logarithms, see §10.3).

Remark 65. Instead of trying to pursue this approach for higher $k$, we will instead approximate the Landau varieties of graph hypersurfaces by a simpler inductive method. The previous example might lead one to think that the difficulties of example 58 went away simply because the graph polynomial $\Psi_G$ is the determinant of a symmetric matrix. This is indeed true for $k \leq 4$, but we will see that the components of $L^1(G, K)$ for general graphs with $|E_G| - |E_K| \geq 5$ are non-linear in the Schwinger parameters. Beyond this point, the structure of the Landau varieties become highly dependent on the combinatorics of $G$. 

6. Genealogy of singularities

We describe a naive inductive method to compute an upper bound for the Landau variety (discriminant) of a configuration of singular hypersurfaces.

6.1. Basic observations. The idea is to approximate the Landau variety of a projection by considering one-dimensional projections at a time.

Lemma 66. Let $P_1, P_2, P_3$ be smooth connected complex analytic manifolds, and consider two smooth proper maps $P_1 \xrightarrow{\pi_1} P_2 \xrightarrow{\pi_2} P_3$. Let $S \subset P_1$ be a closed analytic subset. Then the Landau variety $L(S, \pi_1)$ is a closed analytic subset of $P_2$, and

$$L(S, \pi_2 \circ \pi_1) \subseteq L(L(S, \pi_1), \pi_2) .$$

If $P_1 \xrightarrow{\pi_1} P_2 \xrightarrow{\pi_2} P_3$ are also smooth proper maps of connected complex analytic manifolds satisfying $\pi_2 \circ \pi_1 = \pi_2' \circ \pi_1'$, then it follows that

$$L(S, \pi_2 \circ \pi_1) \subseteq L(L(S, \pi_1), \pi_2) \cap L(L(S, \pi_1'), \pi_2') .$$

Proof. There is a stratification on $L(S, \pi_1)$ such that $\pi_1$ is a locally trivial stratified map on each stratum [16], and similarly for $L(L(S, \pi_1), \pi_2)$ with respect to $\pi_2$. The lemma follows from the fact that the composition of two locally trivial stratified maps is still a locally trivial stratified map. □

By theorem [15] this lemma corresponds to the fact that the composition of two holomorphic functions is holomorphic, and that a function which is locally holomorphic on two open sets is locally holomorphic on their union.

6.2. Resultants. We recall some properties of resultants of polynomials in one variable ([14], Chapter 12). Let $f = \sum_{i=0}^n a_i x^i$, $g = \sum_{i=0}^m b_i x^i$ denote two polynomials with complex coefficients, where $a_n b_m \neq 0$. If we write $f = a_n \prod_{i=1}^n (x - \alpha_i)$, and $g = b_m \prod_{j=1}^m (x - \beta_j)$, then the resultant of $f$ and $g$ is defined by:

$$[f, g]_x = a_n b_m \prod_{i,j} (\alpha_i - \beta_j) .$$

It is clear that the resultant is multiplicative, i.e., $[f_1 f_2, g]_x = [f_1, g]_x [f_2, g]_x$, and that $[g, f]_x = (-1)^{mn} [f, g]_x$. It is an irreducible polynomial in the coefficients of $a_i, b_j$, given by Sylvester’s determinantal formula. We also write

$$D_x(f) := a_n^{-1} [f, f']_x , \quad [0, f]_x := a_0 , \quad [\infty, f]_x := a_n ,$$

for the discriminant of $f$, and for the constant and leading terms of $f$. If $f = f_0 + f_x x$ and $g = g_0 + g_x x$ are both of degree one in $x$, Sylvester’s formula reduces to

$$[f, g]_x = g_x f_0 - g_0 f_x .$$

It follows from definition (62) that $D_x(f g) = D_x(f) D_x(g) [f, g]_x^2$.

6.3. Iterated one dimensional projections. Let $P = \mathbb{P}^1 \times T$ and consider the one-dimensional projection $\pi : P \to T$, $\pi(x, t) = t$, where $x$ is the coordinate on $\mathbb{P}^1$. Let $S = S_1 \cup \ldots \cup S_N$, where the $S_i$ are (possibly singular) distinct, irreducible hypersurfaces in $P$. Then the Landau variety is given by:

$$L(S, \pi) = \bigcup_{1 \leq i < j \leq N} \pi(S_i \cap S_j) \cup \bigcup_{1 \leq i \leq N} \pi(c(S_i)) .$$
Lemma 67. Suppose that $S$ contains the hyperplanes $x = 0$, $x = \infty$, and every other $S_i$ is the zero locus of a polynomial $f_i = \sum_{n \geq 0} a_{i,n} x^n$, where $a_{i,n} : T \to \mathbb{P}^1$. Then $L(S, \pi)$ is given by the zeros of the resultants:

$$[0, f_i]_x, [\infty, f_i]_x, [f_i, f_j]_x, D_x(f_i) .$$

Proof. If $f_i$ is of degree $\geq 2$ in $x$, then $\pi(c(S_i))$ is given by the zero locus of the discriminant $D_x(f_i)$. The terms $\pi(S_i \cap S_j)$ are given by the zero locus of the resultants $[f_i, f_j]_x$, and $[0, f_i]_x$, $[\infty, f_i]_x$. The key remark is that in the degenerate case when $f_i$ is of degree 1, the non-submersive locus $\{f_i = 0\} \cap \{\partial f_i / \partial x = 0\}$ is already contained in both $[0, f_i]_x$ and $[\infty, f_i]_x$. □

Now suppose that $P = (\mathbb{P}^1)^N$, with coordinates $x_1, \ldots, x_N$. By abuse of notation, we let $\pi_i$ denote a projection onto the hyperplane $x_i = 0$ without specifying the source and target space. Thus if $1 \leq k \leq N$, we write

$$\pi_{[1, \ldots, k]} = \pi_1 \circ \ldots \circ \pi_k = \pi_{\sigma(1)} \circ \ldots \circ \pi_{\sigma(k)} ,$$

where $\sigma$ is any permutation of $\{1, \ldots, k\}$. If $S = \bigcup_i S_i$ is a set of irreducible hypersurfaces which contains the coordinate hypercube $\cup_i \{x_i = 0, \infty\}$ we can approximate $L(S, \pi_{[1, \ldots, k]})$ inductively using lemma 66. We can represent $S$ by the set of polynomials $f_i$ which define each non-trivial component $S_i$ of $S$.

Definition 68. Let $S = \{f_1, \ldots, f_N\}$ denote a set of irreducible polynomials in variables $x_1, \ldots, x_N$ as above. For any $1 \leq r \leq N$, let

$$\tilde{S}_{x_r} = \{[0, f_i]_{x_r}, [\infty, f_i]_{x_r}, [f_i, f_j]_{x_r}, D_{x_r}(f_i)\} ,$$

and let $S_{x_r}$ be the set of irreducible factors of elements of $\tilde{S}_{x_r}$. By iterating, we set

$$S_{(x_1, x_2, \ldots, x_k)} = (S_{(x_1, \ldots, x_{k-1})})_{x_k} .$$

Corollary 69. It follows from lemma 66 that the Landau variety $L(S, \pi_{[1, \ldots, k]})$ is contained in the zero locus of $S_{(x_1, \ldots, x_k)}$. In particular, its irreducible components can be expressed as factors of iterated resultants of the defining polynomials of $S$. 

\[ \begin{array}{c}
\xymatrix{
 x = \infty & \xymatrix{ f_1 = 0 & f_3 = 0 } & T \times \mathbb{P}^1 \\
 x = 0 & & T
} \\
\end{array} \]
6.4. Genealogy.

**Definition 70.** We say that an irreducible factor of $[f_1, f_2]_x$ (resp. $D_x(f_i)$), is a *descendent* of $f_i$ and $f_j$ (resp. $f_i$). Conversely, a *set of parents* of an irreducible polynomial factor $c$ for the projection $\pi_x$ is a set $\{f_1, f_2\}$ (resp. $\{f_i\}$) such that $c$ is a descendent of $f_1$ and $f_2$ (resp. $f_i$). Note that $c$ may have several sets of parents, since different polynomials may give rise to the same descendents. Likewise, we define a *set of grandparents* of $c$ for the projection $\pi_{[1,2]}$ to be a set of parents of a set of parents of $c$, and, more generally, a *set of k-ancestors* of $c$ for the projection $\pi_{[1,...,k]}$ to be a set of ancestors going $k$ generations back.

A set of $k$-ancestors has at most $2^k$ elements. However, not all iterated resultants define components which actually occur in the Landau variety, and the upper bound in corollary 70 is a gross over-estimate. In the generic case, it is enough to consider iterated resultants whose $k$-ancestors number at most $k + 1$. In this paper, it is enough to require only the weaker condition that every term has at most 3 grandparents. In other words, we discount all resultants of the form $$[[f_1, f_2], [f_3, f_4]]$$ with $f_1, \ldots, f_4$ distinct.

**Remark 71.** In [9] we eliminated spurious resultants by a different, but equally simple-minded method. Let $S$ be as above, and set $S[x_i] = S_{x_i}$. The ‘Fubini reduction’ of $S$ with respect to $x_1, \ldots, x_k$ was defined by the inductive formula:

$$S_{[x_1, \ldots, x_k]} = \bigcap_{i=1}^{k} (S_{[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k]})_{x_i}.$$

In particular, $S_{[x_1, x_j]} = S_{[x_1, x_j]} \cap S_{[x_j, x_i]}$, and $S_{[x_1, \ldots, x_k]} \subseteq \bigcap_{\sigma \in S_k} S_{(x_{\sigma(1)}, \ldots, x_{\sigma(k)})}$, where the intersection is over all permutations $\sigma$ of $\{1, \ldots, k\}$. It follows from lemma 69 and (65) that $\Lambda(X, \pi_{[1,\ldots,k]})$ is contained in the zero locus of $S_{[x_1,\ldots,x_k]}$.

6.5. Examples and identities in the linear case. For our study of Feynman graphs, it suffices to consider only the case of a two-dimensional projection of degree $(1,1)$ hypersurfaces, generalising example 60. Let $P = (\mathbb{P}^1)^2 \times T$, and consider a projection $\pi : P \to T$, where the coordinates on $(\mathbb{P}^1)^2$ are $x, y$. Let $S = X \cup B \subset P$, where $B = \{x, y = 0, \infty\} = B_1 \cup B_2 \cup B_3 \cup B_4$ is the coordinate square and $X = \bigcup_{i=1}^k X_i$, where $X_i$ is the zero locus of a polynomial

$$f_i = f^i_{0} x + f^i_{1} y + f^i_{0} xy$$

for $1 \leq i \leq k$.

We consider each type of critical stratum which can occur, and relate the corresponding Landau variety to a set of iterated resultants.

1. **Strata of the form $\pi(X_i \cap B_{j_1} \cap B_{j_2})$.** Suppose that we reduce first with respect to $x$, then $y$. The equations of the 4 possible Landau varieties are:

$$[0, [0, f_1]_x], [0, [\infty, f_1]_x], [\infty, [0, f_1]_x], [\infty, [\infty, f_1]_x],$$

corresponding to the fibres where $X_i$ passes through one of the four corners of the square $B$. Clearly, reducing first with respect to $y$ then $x$ gives rise to the same terms.

2. **Strata of the form $\pi(X_i \cap X_j \cap B_k)$.** If we reduce first with respect to $x$ and then $y$, the Landau varieties are given by the four possible cases:

$$[0, [f_1, f_2]_x], [\infty, [f_1, f_2]_x], [[f_1, 0]_x, [f_2, 0]_x], [[f_1, \infty]_x, [f_2, \infty]_x],$$
corresponding to the four sides of the square $B$. We can reverse the order of reduction, by obvious identities such as $[0, [f_i, f_j]_x]_y = [[f_i, 0]_y, [f_j, 0]_y]_x$.

If we write $(X_i \cap X_j) \cap (X_j \cap B_k) = (X_i \cap B_k) \cap (X_j \cap B_k)$, we obtain the following identities between resultants, where $\omega = 0$, $\infty$:

$$[[\omega, f_i]_x, [f_i, f_j]_x]_y = [[\omega, f_i]_x, [\omega, f_j]_x]_y \times [[0, f_i]_x, [\infty, f_i]_x]_y.$$  

(3) *Strata of the form* $\pi(X_i \cap X_j \cap X_k)$. The equation of the Landau variety of a triple intersection is given by a polynomial $\{f_i, f_j, f_k\}$ of degree 6 in the coefficients of $f_i, f_j, f_k$. Using the fact that $X_i \cap X_j \cap X_k = (X_i \cap X_j) \cap (X_j \cap X_k)$, it corresponds to a factor of the iterated resultants:

$$[[f_i, f_j]_x, [f_j, f_k]_x]_y = \{f_i, f_j, f_k\} \times [[0, f_j]_x, [\infty, f_j]_x]_y.$$  

(4) *Non-submersive strata of the form* $X_i$. A single stratum $X_i = \{f_i = 0\}$ is non-submersive when $f_i = \partial f_i / \partial x = \partial f_i / \partial y = 0$. This implies that $f_i^x f_i^y f_i^z f_i^w = 0$, and that $X_i$ degenerates into a product of two lines. This situation corresponds to terms of the form:

$$[[0, f_i]_x, [\infty, f_i]_x]_y = [[0, f_i]_y, [\infty, f_i]_y]_x.$$  

(5) *Non-submersive strata of the form* $X_i \cap X_j$. The final possibility is that two curves $X_i, X_j$ in the fiber meet at a double point. This corresponds to:

$$D_y([f_i, f_j]_x) = D_x([f_i, f_j]_y).$$

In conclusion, every possible iterated resultant can occur, except for the three cases:

$$[[0, f_i]_x, [f_j, \infty]_x]_y, \quad [[0, f_i, \infty]_x, [f_j, f_k]_x]_y, \quad [[f_i, f_j]_x, [f_k, f_\ell]_x]_y,$$

where $i, j, k, \ell$ are distinct. These spurious singularities are precisely the resultants which have four distinct grandparents. Note that we must include terms of the form $[[0, f_i]_x, [f_i, \infty]_x]_y$ even though the three hyperplanes $x = 0, x = \infty, f_i = 0$ do not intersect. This is because we are considering the degenerate degree (1,1) case.

**Remark 72.** The general (non-linear) case is similar, except that (4) is replaced by:

(4') *Non-submersive strata of the form* $X_i$. Given by a double discriminant:

$$D_x D_y(f_i) = D_y D_x(f_i)$$

We need not consider resultants of the form $[D_x f, [f, g]_x]_y$ or $[D_x f, D_x g]_y$.

### 6.6. A reduction algorithm.

We can refine our previous reduction algorithm by keeping track of the number of ancestors with some combinatorial data.

**Definition 73.** Let $S = \{f_1, \ldots, f_k\}$ denote a set of irreducible polynomials in $\mathbb{C}[x_1, \ldots, x_N]$, and let $C$ be a simple graph with $k$ vertices indexed by the elements of $S$. Two polynomials $f_i, f_j$ are *compatible* if there is an edge connecting vertices $i$ and $j$ in $C$. Let $1 \leq m \leq k$. Define a new set of polynomials:

$$\tilde{S}_{xm} = \{D_{xm}(f_i), [0, f_i]_{xm}, [\infty, f_i]_{xm}, [f_i, f_j]_{xm} \text{ for all compatible pairs } f_i, f_j\}.$$  

Let $S_{xm}$ denote the set of irreducible factors of $\tilde{S}_{xm}$. Define a new set of compatibilities $C_{xm}$ between all irreducible factors of resultants of the form $[s_1, s_2]$ and $[s_2, s_3]$, where $s_1, s_2, s_3 \in \{0, \infty, f_1, \ldots, f_k\}$, and also between the irreducible factors of a single discriminant $D_{xm}(f_i)$. We set $(S, C)_{xm} := (S_{xm}, C_{xm}).$

Unfortunately, iterating this reduction is not independent of the chosen order of variables in the case when the resultants are degenerate.
applying operations
Lemma 78. of the star-triangle operations on the linear reducibility of graphs is subtle.

Then it follows by lemma 66 that the Landau variety is contained in the output of the graph is in this sense minor-monotone. In particular, if $G$ is linearly reducible, then the reduction of $G$ never requires computing any grandparents need be considered. Since this holds for all possible orderings of the edges $e_1, \ldots, e_N$ on the set of edges of $G$, we set

\[
(S_{e_1, \ldots, e_k}(G), C_{e_1, \ldots, e_k}(G)) = (S_0(G), C_0(G))_{\alpha_{e_1}, \ldots, \alpha_{e_k}}
\]

for $1 \leq k \leq N$, according to definition 74.

Thus to any graph, and any ordering on its edges, we associate a cascade of polynomials together with compatibility relations between them.

Definition 75. A graph $G$ is linearly reducible if there is an ordering on its edges $e_1, \ldots, e_N$ such that every term in $S_{e_1, \ldots, e_k}(G)$ is of degree at most one in $\alpha_{e_{k+1}}$.

If $G$ is linearly reducible, then the reduction of $G$ never requires computing any discriminants $D_{\alpha}(f)$. The main result of this section is the following theorem.

Theorem 77. Let $G$ be linearly reducible for some ordering $e_1, \ldots, e_N$ of its edges.

For each $1 \leq k \leq N$, the Landau variety $L(G, \{e_1, \ldots, e_k\})$ is contained in the union of the zero locus of the polynomials $S_{e_1, \ldots, e_k}(G)$ and the hyperplanes $\alpha = 0, \infty$ from the set $S_0(G)$.

Proof. The computations in 6.5 show that only resultants with at most three grandparents need be considered. Since this holds for all possible orderings of the edges $e_1, \ldots, e_i$ with respect to which $G$ is linearly reducible, for all $1 \leq i \leq k$, it follows by lemma 76 that the Landau variety is contained in the output of the linear reduction of $G$. $\square$

6.8 Properties of linear reducibility. It is clear that if $\gamma \preceq G$ is a graph minor, and $e_1, \ldots, e_k$ are edges of $\gamma$, then:

\[
(S_{[e_1, \ldots, e_k]}(\gamma), C_{[e_1, \ldots, e_k]}(\gamma)) \subseteq (S_{[e_1, \ldots, e_k]}(G), C_{[e_1, \ldots, e_k]}(G))
\]

This follows immediately from the contraction-deletion relations. The reduction of a graph is in this sense minor-monotone. In particular, if $G$ is linearly reducible, then $\gamma \preceq G$ is linearly reducible for all minors $\gamma$ of $G$.

Lemma 78. $G$ is linearly reducible if and only if its simplification is.

This follows from lemma 74. In particular, all series-parallel graphs obtained by applying operations $S$ and $P$ to the trivial graph are linearly reducible. The effect of the star-triangle operations on the linear reducibility of graphs is subtle.
Remark 79. Suppose that $G_\triangle, G_Y$ are two Feynman graphs related by a star-triangle relation with edges $e_1, e_2, e_3$. Then one can show that

$$(S(G_\triangle), C(G_\triangle))_{|e_1, e_2, e_3|} = (S(G_Y), C(G_Y))_{|e_1, e_2, e_3|}.$$

It is not true in general that $(S(G_\triangle), C(G_\triangle))_{|e_1, ..., e_m|} = (S(G_Y), C(G_Y))_{|e_1, ..., e_m|}$ for all $m \geq 5$, because the definition of reduction involves intersecting over all possible orderings, not just those commencing with the edges $e_1, e_2, e_3$. Indeed, there are counter-examples (e.g., $K_{3,4}$ considered below) to show that linear reducibility is not preserved by the star-triangle relations in general.

6.9. Degeneration. Let $P = (\mathbb{P}^1)^N \times T$, and let $\pi : P \times T \to T$ be the projection. Let $x_1, \ldots, x_N$ denote the coordinates in the fiber $(\mathbb{P}^1)^N$. Let $S \subset P$ be a union of irreducible hypersurfaces $S_i$ as above, which contains the hypercube $B = \cup_i \{x_i = 0, \infty\}$. Suppose that $S$ is linearly reducible, and let $V \subset T$ denote the zero locus of the outcome of the linear reduction algorithm applied to $S$. We know that $L(S, \pi) \subset V$.

**Lemma 80.** Let $R \to T$ be a closed subvariety of $T$, and write $P_R = P \times_T R$, $S_R = S|_R$ and $\pi_R : P_R \to R$. If $S$ is linearly reducible with respect to $\pi$, then so is $S_R$ with respect to $\pi_R$, and $L(S_R, \pi_R) \subset V \times_T R$.

**Proof.** Clearly, linearity is preserved since the restriction to $R$ of a polynomial which is of degree at most one in a reduction variable is still of degree at most one. The linear reduction on $R$ retains triple resultants of polynomials of the form $[[g_1, g_2], [g_2, g_3]]$. By induction these can always be written as the restriction to $R$ of a triple $[[f_1, f_2], [f_2, f_3]]$, where $f_i$ occur in the linear reduction of $S$. \qed

In §7 we will apply the linear reduction to graph hypersurfaces. The previous lemma will be used implicitly, since for certain graphs it may happen that some polynomials defining components in the Landau varieties will vanish identically.
7. Graphs of matrix-type

We study the linear reduction of a graph $G$, and show that the first obstruction to $G$ being matrix type is having a non-trivial 5-invariant. We then prove that graphs of vertex width at most 3 are of matrix type.

7.1. Resultants and Dodgson polynomials. In many cases, we can translate the reductions of §6.5 into identities between Dodgson polynomials.

Notation 81. Let $(G, O)$ be an ordered graph, and suppose that we have reduced $\Psi_G$ with respect to $K_i = \{e_1, \ldots, e_i\}$. Then many of the typical terms in the reduction will be of the form $\Psi_{I,J}^{K_i}$ where $|I| = |J|$, and $I, J \subset K_i$. Since the $K_i$ is implicit in the choice of reduced variables, we will frequently omit the subscripts, and denote $\Psi_{I,J}^{K_i}$ simply by the pair $(I, J)$ (= $(J, I)$).

Firstly, there are trivial identities from the contraction-deletion formulae:

\[
\begin{align*}
\Psi_{I,J}^{K_i}, 0_x = \Psi_{I,J}^{K_i} x \quad \text{and} \quad \Psi_{I,J}^{K_i}, \infty_x = \Psi_{I,J}^{K_i} x
\end{align*}
\]

These can be depicted graphically using notation 81 as:

\[
\begin{array}{c}
(I, J) \quad (I, J) \\
0 \quad x \quad \infty
\end{array}
\]

The reduction variable (here, $x$) is indicated on the left. The reason why linear reduction works is because of the following identities.

Lemma 82. Let $I, J$ be two subsets of $[n]$ such that $|I| = |J|$ and let $a, b, x \notin I \cup J$. Then identity (22) implies that:

\[
\begin{align*}
\Psi_{I,J}^{K_i}, \Psi_{I,J}^{a,J} = \Psi_{I,J}^{Ia,J} x
\end{align*}
\]

Let $I, J$ be two subsets of $[n]$ such that $|J| = |I| + 1$, and let $a, b, x \notin I \cup J$. Then identity (23) implies that:

\[
\begin{align*}
\Psi_{I,J}^{a,J} x \Psi_{I,J}^{b,J} = \Psi_{I,J}^{Ia,J} x
\end{align*}
\]

These two identities can be depicted graphically as follows:

\[
\begin{array}{c}
(I, J) \quad (Ia, Jb) \quad (Ia, J) \quad (Ib, J) \\
(Ix, Jb)(Ia, Jx) \quad (Ix, J)(Iab, Jx)
\end{array}
\]

Definition 83. Given two pairs of subsets $(P, Q)$ and $(R, S)$, define the distance between them by the formula:

\[
||(P, Q), (R, S)|| = \min\{|P \circ R| + |Q \circ S|, |P \circ S| + |Q \circ R|\}
\]

where for two sets $A$ and $B$, $A \circ B = (A \cup B) \setminus (A \cap B)$ denotes their symmetric difference. This takes values in the set of non-negative even integers.

Remark 84. There are two observations which will be important in the sequel:

1. The distance between a parent and its child in (72)-(74) is at most 2.
2. After reducing with respect to $x$, all offspring $(A, B)$ in identities (73) and (74) have the property that exactly one of the two sets $A, B$ contains $x$. 
Using these identities, we can perform the first few linear reductions for any graph \( G \). Writing \( \Psi = \Psi_G \), and reducing with respect to edges numbered 1, 2, 3, 4 in order, we have \( S = \{\emptyset\} \), \( S_{[1]} = \{\emptyset, (1, 1)\} \), and \( S_{[1,2]} = \{\emptyset, (1, 1), (2, 2), (12, 12), (1, 2)\} \), which corresponds to the fact that \( L_2 = \{\Psi_{12}^1\Psi_{2}^2, \Psi_{12,12}^1, \Psi_{1,2}^2, \Psi_{12,12}^1, \Psi_{1,2}^2 = 0\} \), as computed in example 60 which has the following genealogy:

\[
\begin{array}{c}
0 & 0 & \emptyset & (1,1) & \infty \\
\emptyset & (1,1) & (1,2) & (2,2) & (12,12)
\end{array}
\]

The compatibilities between the elements of the sets \( S_{[1]} \) and \( S_{[1,2]} \) can be represented by the following graphs \( C_{[1]} \) and \( C_{[1,2]} \):

At the third stage, the identities (72)-(74) imply that \( S_{[1,2,3]} \) is the union of \( \{\emptyset, (1, 1), (2, 2), (3, 3), (12, 12), (13, 13), (23, 23), (123, 123)\} \) with the six Dodgson terms \( \{1, 2), (1, 3), (2, 3), (12, 13), (12, 23), (13, 23)\} \). We will show (proposition 89) that two pairs \( (A, B) \), \( (C, D) \) are compatible if and only if the distance between them is 2. The compatibility graph can be represented by a cube, with one of the eight terms above at each corner, and a Dodgson term inscribed in the middle of each face (each face is isomorphic to a \( C_{[1,2]} \)). The six Dodgson terms are mutually compatible and form a complete graph \( K_6 \).

A new phenomenon occurs at the fourth stage. The set \( S_{[1,2,3,4]} \) consists of sixteen graph polynomials of minors \( (A, A) \) such that \( A \subset \{1, 2, 3, 4\} \) (corresponding to the vertices of a 4-cube), a further 24 Dodgson terms \( (A, B) \) where \( A, B \subset \{1, 2, 3, 4\} \) such that \( |A| = |B| \) and \( |A \circ B| = 2 \), and three further terms:

\[
(12, 34) , (13, 24) , (14, 23)
\]

which are mutually compatible. There is no determinantal identity to compute the resultant of any two of these terms at the fifth stage of reduction, and in general one obtains a term (the ‘five-invariant’ defined below) which does not factorize.

**Definition 85.** We say that a graph \( G \) is of **matrix type** if there exists an ordering \( O \) of the set of edges of \( G \) such that the only elements which occur in the reduction of \( G \) with respect to \( O \) are Dodgson polynomials \( \Psi_{K_O}^I \).

If \( G \) is of matrix type, then it is linearly reducible.
7.2. The five-invariant. This was first observed implicitly in [5], equation (8.13).

**Definition 86.** Let \( G \) be a graph, and let \( e_1, \ldots, e_5 \) be five distinct edges of \( G \). We define the five-invariant \( 5\Psi_G(e_1, \ldots, e_5) \) to be \( \pm |\Psi_G(e_1, e_2, e_3, e_4, e_5)|_e \).

The five-invariant is the first quadratic term which may occur in a reduction of \( \Psi_G \).

**Lemma 87.** The five-invariant is well-defined up to a sign, i.e., it does not depend on the choice of the ordering of the five edges in the definition above.

**Proof.** The proof follows from the fact that if \( f, g \) are two (generic) polynomials which are of degree at most 1 in two variables, \( x \) and \( y \), then

\[
D_y([f, g]_x) = D_x([f, g]_y) = (\frac{\partial}{\partial y}([f, g]_x) + \frac{\partial}{\partial x}([f, g]_y))^2,
\]

so interchanging any pairs of indices in \( [\Psi_{ijkl}^G, \Psi_{ijkm}^G] \) multiples it by \( \pm 1 \). Since the symmetric group on 5 letters is generated by transpositions, we are done. \( \square \)

For any four edges \( i, j, k, l \) of \( G \), we have the identity \( \Psi_{ijkl}^G - \Psi_{ikjl}^G + \Psi_{iljk}^G = 0 \), (having chosen a representative for \( M_G \) to fix the signs). This implies that:

\[
\Psi_{ijkl}^G - \Psi_{ikjl}^G + \Psi_{iljk}^G = 0,\]

\[
\Psi_{ijkm}^G - \Psi_{ikjm}^G + \Psi_{ilmk}^G = 0.
\]

**Definition 88.** We say that the five-invariant \( 5\Psi(ijklm) \) splits if, for some ordering of the indices, one of the 30 terms in \( (75) \) vanishes, i.e.,

\[
(76) 5\Psi_{ijkl}^G = 0 \quad \text{or} \quad 5\Psi_{ijkm}^G = 0.
\]

In this case, the 5-invariant \( 5\Psi(i, j, k, l, m) \) either vanishes altogether, or after permuting the indices if necessary, can be written in the form

\[
5\Psi(i, j, k, l, m) = \Psi_{ijkl}^G \Psi_{ijklm}^G.
\]

By corollary 25, equation (76) is a minor monotone property which is equivalent to the non-existence of certain cycles in \( G \setminus m \) or \( G/m \).

7.3. Linear reduction. The following result is the key to the main theorem.

**Proposition 89.** In the generic case, the polynomials occurring in the linear reduction of \( \Psi_G \) are either Dodgson polynomials \( \Psi_{ij}^A,B \) or descendents of five-invariants of minors of \( G \). Two Dodgson polynomials \( \Psi_{ij}^A,B \) and \( \Psi_{ij}^C,D \) which are not descendents of any five-invariants can only be compatible if either:

i). \quad ||(A, B), (C, D)|| = 2,

or \quad ii). \quad \{(A, B), (C, D)\} = \{\{M_{ij}, M_{kl}\}, \{M_{ik}, M_{jl}\}\},

where, e.g., \( M_{ij} \) denotes \( M \cup \{i, j\} \). Thus any further compatibilities between Dodgson polynomials only occur if they are descendents of five-invariants.
For reductions of type $[[0|\infty]]$, since we are in the linear case, we need only consider the resultants of polynomials $\Psi^{A,B}$ and $\Psi^{C,D}$ are compatible. If we are in case $i),$ then we must have either $\{(A,B), (C,D)\} = \{(I,J), (Ia,Ja)\}$, where $|I| = |J|$ and $a \notin I \cup J$, or $\{(A,B), (C,D)\} = \{(Ia,Ja), (Ib,Jb)\}$, where $|J| = |I| + 1$ and $a,b \notin I \cup J$. By applying either (73) or (74), we see that the resultant $[\Psi^{A,B}, \Psi^{C,D}]_x$ factorizes as a product of Dodgson polynomials. If we are in case $ii),$ then the resultant $[\Psi^{A,B}, \Psi^{C,D}]_x$ is by definition a five-invariant $\Psi^{G',i,j,k,l,x}$ where $G'$ is a minor of $G$. We must next check the compatibilities for any two Dodgson polynomials $\Psi^{P,Q}, \Psi^{R,S}$ in the new generation.

Firstly, by definition of linear reduction, two such polynomials can only be compatible if they share a common ancestor $\Psi^{U,V}_K$. Furthermore, by remark 83 (1), the distance between parent and child is at most 2, so we have:

$$||\{(P,Q),(R,S)\}|| \leq ||\{(P,Q),(U,V)\}|| + ||\{(R,S),(U,V)\}|| \leq 2 + 2 = 4.$$ 

Since we are in the linear case, we need only consider the resultants of $\{(0,0), [f,\infty]\}$, $\{(0,f), [0,\infty]\}$, $\{(0,f), (0,g)\}$ (and those obtained by interchanging 0 with $\infty$) the distance between offspring is exactly 2: i.e., $||\{(P,Q),(R,S)\}|| = 2$ and we are in case $i)$. Otherwise, the interesting case is when we have a reduction of type $[[f,g], [f,h]]$ and $||\{(P,Q),(R,S)\}|| = 4$, and we must show that we obtain case $ii)$. We first claim that $P \cup Q = R \cup S$. If not, then without loss of generality there exists $y \in P$ such that $y \notin R \cup S$. By changing the order of reduction, we can also assume that $y = x$, the reduction variable. But by remark 83 (2), the index $y$ must occur in either $R$ or $S$, which gives a contradiction. Therefore $P \cup Q = R \cup S$, and we can assume that $\{(P,Q),(R,S)\} = \{(Aij,Bkl), (Aik,Bjl)\}$. By passing to the subgraph $G \backslash (A \cap B)$, we can further assume that $A \cap B = \emptyset$. If $A = B = \emptyset$, then we are in case $ii)$. Therefore suppose in the opposite case that $|A| = |B| \geq 1$, and let $a \in A$ and $b \in B$. We must show that $(Aij,Bkl)$ and $(Aik,Bjl)$ cannot be compatible. Otherwise $(Aij,Bkl)$ and $(Aik,Bjl)$ would have to possess a common ancestor $(U,V)$, which is of distance at least 2 from each. It follows from this that, after interchanging $U$ and $V$ if necessary, we must have $Ai \subseteq U$ and $Bl \subseteq V$. In fact, there are only three possibilities: $(U,V)$ is one of $(Ai,Bl)$, $(Aij,Bjl)$ or $(Aik,Bkl)$. But, if the reduction variable $x$ is $a$ or $b$, then this clearly cannot be the case, since we must have $x \notin U,V$ at the previous generation. It follows that $(Aij,Bkl)$ and $(Aik,Bjl)$ are not compatible.

The proposition states that the first non-trivial obstruction to being of matrix type is the five-invariant, and furthermore, that there can occur no higher obstructions of the form $[\Psi^{123,456}, \Psi^{134,256}]$, and so on (unless these Dodgson polynomials themselves happen to occur as descendents of five-invariants, e.g., lemma 83).

**Definition 90.** We call the compatibilities defined in $i), ii)$ of proposition 83 the **basic compatibilities** between Dodgson polynomials. We call extra compatibilities the new ones induced by the possible descendents of a split 5-invariant.

**Corollary 91.** Let $G$ be a graph, and let $e_1, \ldots, e_n$ denote an ordering on the set of edges in $G$. Suppose that at the 5th stage of reduction there are only basic compatibilities (i.e., the 5-invariant $^5\Psi_G(e_1, \ldots, e_5)$ splits and induces no extra compatibilities), and suppose that for all $i = 1, \ldots, 5$ the minors

$$G \backslash e_i$$

and $G / e_i$.
with the induced ordering of edges are linearly reducible. Then $G$ is linearly reducible with respect to the ordering $e_1, \ldots, e_n$.

The previous corollary will enable us to do inductions. The entire difficulty of the problem is that one has to go all the way down to the fifth level to see the first non-trivial combinatorial phenomena.

7.4. Triangles and 3-valent vertices. We study the effect of the presence of a triangle or 3-valent vertex on the five-invariants of $G$.

**Lemma 92.** Let $a, b, c, i, j$ be any 5 distinct edges of $G$. Suppose that $\{a, b, c\}$ forms a triangle. Then $\Psi_{abc} = 0$ and

$$(77) \quad \Psi^{ab,ij}_c = \Psi^{bc,ij}_a = \Psi^{ac,ij}_b = 0$$

and the five invariant $5\Psi(abcij)$ is the product of any element in the second row with one in the third. If $\{a, b, c\}$ forms a 3-valent vertex, then $\Psi_{abc} = 0$ and

$$(78) \quad \Psi^{abc,aij}_c = \Psi^{abc,bij}_a = \Psi^{abc,acij}_b = 0$$

and the five invariant $5\Psi(abcij)$ is the product of any element in the second row with one in the third.

If $a, b, c, i, j$ contains a 2-valent vertex or a 2-loop, then $5\Psi(abcij)$ vanishes.

**Proof.** Suppose $a, b, c$ forms a triangle, and hence $\Psi_{abc}$ vanishes. By proposition $23$, $\Psi^{ab,ij}_c$ is a sum over all common monomials in $\Psi^{ab,ij}_c$ and $\Psi^{ij,ij}_{abc}$. Since the latter vanishes, so must $\Psi^{ab,ij}_c$. By symmetry, this gives the first row of (77). Now by the Plücker identity $\Psi^{ab,ij}_c - \Psi^{ai,bi}_c + \Psi^{aj,bi}_c = 0$, we deduce that $\Psi^{ai,bi}_c = \Psi^{aj,bi}_c$. We showed in example 33 that $\Psi^{a,b} = \Psi^{a,b}_{hc}$, so we have:

$$[\Psi^{ai,bi}_c, \Psi^{aj,bi}_c] = \Psi^{a,b}_{hc}, \Psi^{a,b}_{hc}$$

The left-hand side is $\Psi^{ai,bi}_c \Psi^{aj,bi}_c = (\Psi^{ai,bi}_c)^2$ and the right-hand side is $(\Psi^{ai,bi}_c)^2$. Since $G\setminus a/bc$ is symmetric in $a, b, c$, we obtain the second row of (77). We can therefore write the five-invariant $5\Psi(abcij) = \pm (\Psi^{ab,ij}_c \Psi^{aci,bcj} - \Psi^{abc,cij} \Psi^{ai,bi}_c)$ in all possible ways $5\Psi(abcij) = \pm \Psi^{ai,bi}_c \Psi^{abc,acij}$ obtained by permuting the indices $\{a, b, c\}$ and $\{i, j\}$. It follows that the third row of (77) must hold too. The case where $a, b, c$ forms a 3-valent vertex is similar.

For the last statement suppose, for example, that $\Psi_{ab} = 0$. Then $\Psi^{ab,ij} = 0$ by the same argument, and hence $5\Psi(abcij) = 0$. $\square$

We need to proceed one step further in the reduction.

**Lemma 93.** At the 5th stage of reduction, the 5-invariant $\Psi(ijklm)$ is only compatible with Dodgson polynomials of the form $\Psi^{ijklm}_{ijklm}$ or $\Psi^{ijklm}_{ijklm}$. We have

$$\Psi_m(ijklm), \Psi^{ijk}_m m(ijklm) = \Psi_m(ijklx), \Psi^{ijklm}_{ijklm}$$

where we write $\Psi_m(ijklx) = \Psi_{G\setminus m}(ijklx)$ and $\Psi^m(ijklx) = \Psi_{G\setminus m}(ijklx)$. 


Proof. For two polynomials to be compatible in the reduction requires them to have a common parent. All parents of five invariants at the fourth generation are of the form \((ij,kl)\), which can only have descendents of the form \((ij,kl)\) or \((ijm,klm)\), or the five invariant \(\Psi_G(ijklm)\). This proves the first statement. Recall that for linear polynomials \(f,g\) in the variables \(x\) and \(m\), we have the identity 
\[
[0,f]_m,[f,g]_m]_x = [0,f]_m,[\infty,f]_m]_x \times [0,f]_m,[0,g]_m]_x.
\]
Applying this in the case 
\[
f = \Psi_{ij,kl} \quad \text{and} \quad g = \Psi_{ik,jl}
\]
gives: 
\[
[\Psi_{ij,kl}^{m,kl}, \Psi(ijklm)]_x = [\Psi_{i,jkl}^{ij,kl}, \Psi_{ijm,klm}^{ijm,klm}]_x \times [\Psi_{i,jkl}^{ik,jl}, \Psi_{m}^{i,jkl}]_x
\]

The second equation is similar, on replacing 0 in the previous formula with \(\infty\). \(\square\)

Lemma 94. Let \(a,b,c,i,j\) be five distinct edges in \(G\). If \(\{a,b,c\}\) forms a triangle in \(G\), then at the 5th stage of its linear reduction, we only have terms of the form 
\[(A,B), \text{ where } |A| = |B| \subset \{a,b,c,i,j\}\]. The only possible extra compatibilities are between \((abc,aij) = (abc,bij)\) and terms of the form \((pq,rs)\) or \((pq,srt)\), where \(\{p,q,r,s,t\} = \{a,b,c,i,j\}\).

If \(\{a,b,c\}\) is a 3-valent vertex in \(G\), the same holds, except that the extra compatibilities are at most between \((ab,ij) = (ac,ij) = (bc,ij)\) and \((pq,rs)\) or \((pq,srt)\).

Proof. Suppose that \(\{a,b,c\}\) forms a triangle in \(G\). The case when \(a,b,c\) forms a 3-valent vertex is similar (and dual to it). By lemma 92, we can write 
\[5\Psi(abcij) = \pm \Psi_{c}^{ai,bj} \Psi_{a}^{c,bc,ij}\]
in all possible ways by permuting the indices \(\{a,b,c\}\) and \(\{i,j\}\). Now consider the reduction of \(\Psi\) with respect to the five edges \(\{a,b,c,i,j\}\). Since it factorizes, the five-invariant \(5\Psi(abcij)\) will be replaced with the factors occurring in the right-hand side of (78). This will introduce extra compatibility conditions between these factors owing to the fact (lemma 93) that the five-invariant is compatible with all terms of the form \(\Psi_{pq,rs}^{apqr}\) and \(\Psi_{pq,rst}^{apqr}\), where \(\{p,q,r,s,t\} = \{a,b,c,i,j\}\). These can be of five different kinds (since terms of type \(\Psi_{c}^{api,bcj}\) vanish):

1. \(i) \quad \Psi_{c}^{ai,bj} \quad ii) \quad \Psi_{a}^{abc,bi} \quad iii) \quad \Psi_{c}^{aci,bcj} \quad iv) \quad \Psi_{a}^{aci,bij} \quad v) \quad \Psi_{c}^{abc,ciij}\)

where \(\{a,b,c\}\) and \(\{i,j\}\) are to be permuted in all possible ways. Each representative of \(i) - iv)\) listed above is already compatible with the left-hand factor \(\Psi_{c}^{ai,bj}\) of (78) by the basic compatibilities of definition 90. Since we can always write \(5\Psi(abcij)\) in many ways (78), it follows that any polynomial obtained from \(i) - iv)\) by permuting \(\{a,b,c\}\) and \(\{i,j\}\) will always be compatible with an appropriate choice of left-hand factor of (78). Therefore, the only new compatibilities that must be taken into account are the ones arising from the right-hand factor of (78), and \(v\), which are both of the form \(\Psi_{c}^{abc,ciij}\).

We will only use this lemma in the following context.

Corollary 95. Suppose that \(e_1,\ldots,e_5\) are five distinct edges in \(G\), three of which form a triangle, and three of which form a 3-valent vertex. Then at the 5th stage of the linear reduction of \(G\), we have only Dodgson polynomials of the form \((A,B)\), where \(|A| = |B| \subset \{e_1,\ldots,e_5\}\). The compatibilities are precisely the basic compatibilities of definition 90 and no others.
Proof. Put the first and second halves of the previous lemma together to deduce that any extra compatibilities that might arise actually reduce to basic compatibilities after rewriting terms using the identifications in lemma 92.

7.5. Graphs of vertex-width at most 3. The most basic family of graphs that are linearly reducible are given by the following theorem.

Theorem 96. Any graph $G$ satisfying $vw(G) \leq 3$ is of matrix type.

Proof. Let $O = (e_1, \ldots, e_N)$ be an ordering on $E_G$ such that $vw_O(G) \leq 3$, and consider the reduction of $\Psi = \Psi_G$ with respect to $e_1, \ldots, e_N$. Let $G_1, G_2$ denote the subgraphs of $G$ spanned by the edges $\{e_1, \ldots, e_n\}$ and $\{e_{n+1}, \ldots, e_N\}$, respectively. By assumption, $G_1, G_2$ meet in at most 3 vertices which we denote by $v_1, v_2, v_3$ (the case where there are fewer than 3 vertices is trivial and left to the reader). Let $n \geq 2$ and suppose by induction that the reduction at the $(n-1)^{th}$ stage consists of Dodgson polynomials of the form $\Psi_{A,B}^{G_1}$, where $|A| = |B|$ and $A, B \subset G_1$, and basic compatibilities given by cases i) and ii) of proposition 89. The compatibilities in case i) give rise to descendents which are products of Dodgson polynomials, and one verifies the induction step exactly as in the proof of proposition 89. The compatibilities in case ii) are between polynomials of the form $(M_{ij}, M_{kl})$ and $(M_{ik}, M_{jl})$. Let $m = e_m \in E_{G_1}$, and consider the minor $G'' = G \setminus M' / N'$, where $N = G_1 \setminus M \cup \{i, j, k, l, m\}$. It suffices to show that the five-invariant $5\Psi_{G'}(ijklm)$ splits and induces no extra compatibilities. The minor $G_1''$ inherits a decomposition $G_1'' \cup G_2$, where $G_1'' = \{i, j, k, l, m\}$ has exactly 5 edges, and $G_1''$ and $G_2$ meet in at most 3 vertices (again we only consider the non-trivial case when there are exactly three: $v_1, v_2, v_3$). There are a limited number of possibilities. Suppose that $G_1''$ has no vertex besides $v_1, v_2, v_3$. Since it has 5 edges, it must contain two double edges or one triple edge. One of these situations is pictured below (left).

It follows from lemma 92 that $5\Psi_{G'}(ijklm)$ vanishes. If $G_1''$ contains three or more vertices different from $v_1, v_2, v_3$, then one of them must have degree 2 or less (e.g. above right), and again we conclude that $5\Psi_{G'}(ijklm) = 0$ by lemma 92.

For $G_1''$ to be simple it must have one or two vertices disjoint from $v_1, v_2, v_3$ and there are exactly two cases (above). In either case, $\{i, j, k, l, m\}$ contains both a
triangle and a 3-valent vertex. We conclude by corollary \[95\] that \(5\Psi_G'(ijklm)\) splits and induces no extra compatibilities. This completes the induction step. \(\square\)

**Remark 97.** The proof consists in analysing the local 5-minors in a graph of vertex width 3, and checking the triviality of the 5-invariant. It is likely that a similar argument will extend to the vertex width 4 case if one excludes the appropriate local minors. Also, instead of appealing to corollary \[95\] it suffices to compute the possible compatibilities for the two situations depicted above. One can show \[7\] that there is a universal formula for the graph polynomial of a 3-vertex join, in the same spirit as corollary \[34\]. In other words, the heart of the previous theorem can be reduced to computing the linear reduction of a few small graphs, the most non-trivial of which is the wheel with four spokes \(W_4\). In conclusion: the fact that \(W_4\) is of matrix type implies that all graphs of vertex width 3 are of matrix type.

The previous theorem is closely related to the star-triangle relations, as the diagram below illustrates. The first row shows a sequence of operations which splits a triangle down the middle, the second splits a 3-valent vertex in two.

The operation of splitting a triangle preserves primitive divergence.

**Theorem 98.** Let \(G\) be a graph with an ordering \(e_1, \ldots, e_N\) on its edges. Suppose that \(e_1, \ldots, e_5\) forms a split triangle or a split vertex as indicated above (far right). Let \(G_\Delta\) and \(G_Y\) denote the graph with \(N-2\) edges obtained by replacing \(e_1, \ldots, e_5\) with a triangle and a star (far left). Then if both \(G_\Delta\) and \(G_Y\) are linearly reducible with respect to the induced orderings, then so is \(G\).

**Proof.** Apply corollaries \[91\] and \[95\] in turn. It suffices to consider successive minors of the form \(G\setminus e_i\) and \(G/ e_i\), for \(1 \leq i \leq 5\). In all cases one eventually obtains a minor of either \(G_\Delta\) or \(G_Y\). \(\square\)

**Example 99.** Let \(W_3\) be the wheel with 3 spokes. Choose a triangle in \(W_3\) and split it. This creates two new triangles. Choose either of these two triangles, and split them. Continuing in this way generates an infinite family of planar graphs which contains the wheels and zig-zags. They are clearly of vertex width 3, and hence by theorem \[96\] of matrix type.
8. Moduli spaces and linearly reducible hypersurfaces

For any set of linearly reducible hypersurfaces, we construct explicit maps to the moduli spaces $\mathcal{M}_{0,n}$ of genus 0 curves with $n$ marked points. This will enable us to compute the periods in the following section using the function theory of $\mathcal{M}_{0,n}$.

8.1. Recap. We briefly recall the situation so far. We begin with a set of irreducible hypersurfaces $S$ contained in $(\mathbb{P}^1)^N$ (typically, the zero locus of a graph polynomial), along with the coordinate hypercube $B$. In this section, we shall work on the open complement of $S$, so it is convenient to replace $\mathbb{P}^1$ with $\mathbb{G}_m = \mathbb{P}^1 \setminus \{0, \infty\}$. Consider the maps $\pi_i : \mathbb{G}_m^N \to \mathbb{G}_m^{N-i}$ given by projecting out the first $i$ coordinates, and write $\pi_i = \pi_i \circ \ldots \circ \pi_1$, where $\pi_i : \mathbb{G}_m^{N-i+1} \to \mathbb{G}_m^{N-i}$ denotes each successive projection.

In §5 we defined the Landau varieties $L_i = L(S, \pi_i)$ which, in particular, have the property that:

$$\pi_i : \mathbb{G}_m^N \setminus (S \cup \pi_i^{-1}(L_i)) \to \mathbb{G}_m^{N-i} \setminus L_i$$

is a locally trivial fibration. We write $L_0 = S$, and define:

$$\tilde{L}_i = L_i \cup \pi_i^{-1}(L_{i+1}) \cup \ldots \cup (\pi_{N-1} \circ \ldots \circ \pi_{i+1})^{-1}(L_{N-1}).$$

The situation is summarized by the following diagram:

\[
\begin{array}{cccc}
\mathbb{G}_m^N \setminus \tilde{L}_0 & \xrightarrow{\pi_1} & \mathbb{G}_m^{N-1} \setminus \tilde{L}_1 & \xrightarrow{\pi_2} \mathbb{G}_m^{N-2} \setminus \tilde{L}_2 & \xrightarrow{\pi_3} \cdots & \mathbb{G}_m^{N-k} \setminus \tilde{L}_k \\
\end{array}
\]

The maps $\pi_k$ are trivial fibrations, the $\pi_k$ in general are not. We say $S$ is linearly reducible if each component of $\tilde{L}_i$ is of degree at most one in the fiber of $\pi_{i+1}$.

8.2. Maps to the moduli spaces $\mathcal{M}_{0,n}$. Consider a set of distinct irreducible hypersurfaces $S$ contained in $\mathbb{G}_m^n$, and let $\pi$ denote the coordinate projection $\mathbb{G}_m^n \to \mathbb{G}_m^{n-1}$. The set $S$ is partitioned into a set of vertical components $S_v$ (of degree 0 in the fiber of $\pi$), and horizontal components $S_h$ (of degree $\geq 1$ in the fiber):

$$S = S_v \cup S_h.$$ 

Suppose that every component of $S_h$ is linear in the fiber of $\pi$, and let $M = |S_h|$ denote the number of horizontal components in $S$. Then there is a natural map:

$$\rho : \mathbb{G}_m^n \setminus S \to \mathcal{M}_{0,M+3}.$$ 

This follows from the universal property of the moduli spaces $\mathcal{M}_{0,M+3}$. The fiber of the map $\pi$ is isomorphic to $\mathbb{G}_m$, and the horizontal components of $S_h$ cut out $M$. 
points on $\mathbb{G}_m$, and hence $M + 2$ points on $\mathbb{P}^1$. This maps to the universal curve $\overline{M}_{0,M+3} \to \overline{M}_{0,M+2}$ with $M + 2$ marked points. More precisely, we have:

**Lemma 100.** There is a commutative diagram mapping to the open moduli spaces:

$$
\begin{array}{ccc}
\mathbb{G}^n_m \setminus (S \cup \pi^{-1}L(S,\pi)) & \xrightarrow{\rho'} & \overline{M}_{0,M+3} \\
\downarrow \pi & & \downarrow f \\
\mathbb{G}^{n-1}_m \setminus L(S,\pi) & \xrightarrow{\overline{\pi}} & \overline{M}_{0,M+2}
\end{array}
$$

where $f$ is the map which forgets one of the marked points.

**Proof.** We write down all the maps explicitly. Let $\alpha_1,\ldots,\alpha_n$ denote coordinates on $\mathbb{G}^n_m$, and let $A$ denote the coordinate ring of $\mathbb{G}^{n-1}_m \setminus S$. Let the set of horizontal components of $S$ be the zeros of irreducible polynomials $f_i = a_i \alpha_i + b_i$, where $a_i, b_i \in \mathbb{Z}[\alpha_2,\ldots,\alpha_n]$ and $a_i b_i \neq 0$, for $1 \leq i \leq M$. Recall from lemma [67] that the Landau variety $L(S,\pi)$ is given by the zeros of the resultants

$$
[f_i, f_j]_{\pi_i} = a_i b_j - a_j b_i, \quad [0, f_i]_{\pi_i} = b_i, \quad [\infty, f_i]_{\pi_i} = a_i.
$$

Define $A^+ = A[a_i^{-1}, b_i^{-1}, (a_i b_j - a_j b_i)^{-1}]$. The lemma follows on taking the Spec of the following commutative diagram:

$$
\begin{array}{ccc}
A^+[a_i^{\pm 1}, f_i^{-1}] & \xrightarrow{\rho^*} & \mathbb{Z}[t_1^{\pm 1}, \ldots, t_M^{\pm 1}, (1-t_i)^{-1}_{1 \leq i \leq M}, (t_i-t_j)^{-1}_{1 \leq i < j \leq M}] \\
\downarrow \pi^* & & \downarrow f^* \\
A^+[a_i^{\pm 1}, f_i^{-1}] & \xrightarrow{\overline{\pi}^*} & \mathbb{Z}[t_1^{\pm 1}, \ldots, t_{M-1}^{\pm 1}, (1-t_i)^{-1}_{1 \leq i \leq M-1}, (t_i-t_j)^{-1}_{1 \leq i < j \leq M-1}]
\end{array}
$$

where $f^*$ is the inclusion, and the horizontal maps $\rho^*, \overline{\pi}^*$ are given by

$$
t_M \mapsto \frac{a_M}{b_M} \alpha_1, \quad \text{and} \quad t_i \mapsto \frac{a_i b_i}{b_M a_i} \quad \text{for} \quad i < M.
$$

The map $\rho^*$ is uniquely determined up to a choice of ordering on the hypersurfaces in $S_h \cup \{0, \infty\}$ (resp. marked points on $\overline{M}_{0,M+3}$).

We wish to apply this lemma to the tower of maps (79). Let $R_i$ denote the coordinate ring of $\mathbb{G}^{N-1}_m \setminus \tilde{L}_i$. We obtain a nested sequence of rings:

$$R_0 \supseteq R_1 \supseteq \ldots \supseteq R_N$$

**Definition 101.** Let $\tilde{L}_i = V_i \cup H_i$ denote the decomposition of $\tilde{L}_i$ into horizontal and vertical components with respect to the projection $\pi_i$. We write

$$M_i = |H_i|$$

for the number of irreducible horizontal components. This is of course equal to the number of horizontal components of $L_i$.

**Remark 102.** If $(G, O)$ is an ordered graph, and $S = X_G$, then the numbers $M_0, \ldots, M_{e_G}$ are interesting invariants of $(G, O)$. In the generic case, when $G$ is of matrix type, these numbers initially coincide with the number of Dobgson polynomials $M_0 = 1, M_1 = 2, M_2 = 5, M_3 = 14, M_4 = 43, \ldots$, given by

$$M_k = \sum_{i=0}^{\lfloor k/2 \rfloor} \binom{k-1}{2i-1} \binom{k}{2i} 2^{k-2i} = 2^{k-1} + \frac{1}{2} \binom{2k}{k}.$$
For any given graph, however, the numbers $M_k$ are typically smaller because many Dodgson polynomials vanish due to lemma 20 and so the $M_i$ tail off for large $i$.

Now suppose that $S$ is linearly reducible, and write the components $H_k$ of $\widetilde{L}_k$ as zeros of linear terms $f_1, \ldots, f_m$ where $f_i = a_i \alpha_k + b_i$, and $a_i, b_i \in R_{k+1}$. We have

$$R_k = R_{k+1} \left[ a_k, \frac{1}{\alpha_k}, \frac{1}{f_i} : f_i \in H_k \right].$$

Since the Landau variety of $\widetilde{L}_k$ with respect to $\pi_k$ contains, but is not in general equal to $\widetilde{L}_k$, we cannot apply the previous lemma directly to each step in the tower (79). But if we set $L_{k+1}^+ = L(\widetilde{L}_k, \pi_{k+1})$, lemma 100 gives:

$${\begin{array}{ccl}
\mathcal{G}_{m}^{N-k} \setminus \widetilde{L}_k & \overset{\pi_k}{\longrightarrow} & \mathcal{G}_{m}^{N-k} \setminus \pi_{k+1}^{-1} L_{k+1}^+ \\
\mathcal{G}_{m}^{N-k-1} \setminus \widetilde{L}_{k+1} & \overset{\pi_k}{\longrightarrow} & \mathcal{G}_{m}^{N-k-1} \setminus L_{k+1}^+ \\
\end{array}}$$

We use the notation $R_{k+1}^+ = R_{k+1}[(a_i b_j - a_j b_i)^{-1}, 1 \leq i < j \leq M_k]$ (see 9). Then the left-hand square of the above diagram is simply the Spec of the diagram:

$$\begin{array}{c}
R_k \\
\downarrow \downarrow \\
R_{k+1} \\
\end{array} \quad \begin{array}{c}
R_k \otimes R_{k+1} \\
\downarrow \downarrow \\
R_{k+1}^+ \\
\end{array}$$

**Lemma 103.** There exists a pair of spaces $\mathcal{M}_{0,M_k+j}^\dagger$, with $j = 2, 3$ which satisfy $\mathcal{M}_{0,M_k+j} \subseteq \mathcal{M}_{0,M_k+j}^\dagger \subseteq \mathcal{G}_{m}^{M_k+j-3}$, such that $\rho, \overline{\rho}$ extend to maps:

$$\begin{array}{c}
\text{Spec } R_k = \mathcal{G}_{m}^{N-k} \setminus \widetilde{L}_k \\
\downarrow \downarrow \\
\text{Spec } R_{k+1} = \mathcal{G}_{m}^{N-k-1} \setminus \widetilde{L}_{k+1} \\
\end{array}$$

and induce an isomorphism:

$$\gamma_k : \text{Spec } R_k \xrightarrow{\sim} \mathcal{M}_{0,M_k+j}^\dagger \times \mathcal{M}_{0,M_k+j+2} \text{ Spec } R_{k+1}.$$

**Proof.** The map $\overline{\rho}$ defined in the proof of lemma 100 gives a map from Spec $R_{k+1}^+$ to $\mathcal{M}_{0,M_k+2} = \text{Spec } \mathbb{Z}[t_{1}^{\pm 1}, \ldots, t_{M_k-1}^{\pm 1}, (1 - t_i)^{-1}, (t_i - t_j)^{-1}]$ such that:

$$t_i \mapsto \frac{a_{M_k} b_i}{b_{M_k} a_i} \quad \text{for} \quad 1 \leq i \leq M_k - 1.$$

Define $\mathcal{M}_{0,M_k+2}^\dagger$ to be the Spec of the ring $Q[t_{1}^{\pm 1}, \ldots, t_{M_k-1}^{\pm 1}]$ by inverting terms $(1 - t_i)$ (respectively $(t_i - t_j)$) if and only if $[f_i, f_{M_k}] = a_{M_k} b_i - a_i b_{M_k}$ (respectively $[f_i, f_j] = a_i b_j - a_j b_i$) is invertible in $R_{k+1}$. Now define

$$\mathcal{M}_{0,M_k+3}^\dagger = \text{Spec } \mathbb{Z}[t_{1}^{\pm 1}, \frac{1}{1 - t_{M_k}}, \frac{1}{t_{M_k} - t_i}, 1 \leq i \leq M_k - 1],$$

along with the natural map from $\mathcal{M}_{0,M_k+3}^\dagger \rightarrow \mathcal{M}_{0,M_k+2}^\dagger$. This is no longer a fibration in general, reflecting the fact that Spec $R_k \rightarrow$ Spec $R_{k+1}$ is also not a fibration. $\square$
The space $\mathcal{M}_{0,M+2}$ is the complement of only those hyperplanes in $\mathbb{G}^M$ which correspond to resultants $[f_1, f_2]$ which survive in the Landau variety $\tilde{L}_{k+1}$. The superscript $\dagger$ reflects the discrepancy between $\tilde{L}_{k+1}$ and $L(L_k, \pi_{k+1})$, or, in other words, the failure of $\pi_{k+1}$ from being a fibration.

**Theorem 104.** Let $S$ be a set of linearly reducible hypersurfaces in $\mathbb{G}^N$, and let $L_i = L(S, \pi_{[i]}) \subset \mathbb{G}^N_i$ denote the corresponding Landau varieties. Let $\tilde{L}_i$ be defined as above, and let $R_i$ denote the coordinate ring of $\mathbb{G}^{N-i}_i \setminus \tilde{L}_i$. Let us write

$$m_i = M_i + 2 \geq 3,$$

where $M_i$ is the number of horizontal components of $L_i$, for $1 \leq i \leq N$. Then for each $1 \leq i \leq N$, there exists a pair of affine schemes $\mathcal{M}_{i,m_i}^\dagger, \mathcal{M}_{i,m_i+1}^\dagger$ satisfying

$$\begin{align*}
\mathcal{M}_{0,m_i+1}^\dagger & \subseteq \mathcal{M}_{0,m_i+1}^\dagger \subseteq \mathbb{G}_m^{m_i-2} \\
\mathcal{M}_{0,m_i}^\dagger & \subseteq \mathcal{M}_{0,m_i}^\dagger \subseteq \mathbb{G}_m^{m_i-3}
\end{align*}$$

and hence defined over $\mathbb{Z}$. There are connecting morphisms $\phi_i$ such that: $\phi_N : \text{Spec } R_N \to \mathcal{M}_{0,m_N-1}^\dagger$, and for all $1 \leq i \leq N - 1$,

$$\phi_i : \mathcal{M}_{0,m_i+1}^\dagger \times \ldots \times \mathcal{M}_{0,m_N-1}^\dagger \to \text{Spec } R_N \to \mathcal{M}_{0,m_i}^\dagger,$$

where the maps in the fiber product are given on the left of each $\times$ by the natural maps $\mathcal{M}_{0,m_i+1}^\dagger \to \mathcal{M}_{0,m_i}^\dagger$, and on the right by the maps $\phi_j$ for $i < j \leq N$. Then for each $1 \leq i \leq N$, there is an isomorphism:

$$\psi_i : \text{Spec } R_i \cong \mathcal{M}_{0,m_i+1}^\dagger \times \ldots \times \mathcal{M}_{0,m_N-1}^\dagger \times \text{Spec } R_N$$

such that the projections between the Spec $R_i$ are induced by the natural maps $\mathcal{M}_{0,m_i+1}^\dagger \times \mathcal{M}_{0,m_i}^\dagger \to \mathcal{M}_{0,m_i}^\dagger \times \mathcal{M}_{0,m_i}^\dagger \to X$, i.e:

$$\begin{align*}
\text{Spec } R_i & \sim \mathcal{M}_{0,m_i+1}^\dagger \times \mathcal{M}_{0,m_i}^\dagger \times \ldots \times \mathcal{M}_{0,m_N-1}^\dagger \\
\pi_i & \sim \mathcal{M}_{0,m_i+1}^\dagger \times \mathcal{M}_{0,m_i+1}^\dagger \times \ldots \times \mathcal{M}_{0,m_N-1}^\dagger
\end{align*}$$

In other words, the schemes $R_i$ can be entirely constructed out of Spec $R_N \subseteq \mathbb{G}_m$, the modified moduli spaces $\mathcal{M}_{i,m_i}^\dagger$, and the connecting maps $\phi_i$.

**Proof.** In the previous lemma we defined morphisms $\beta_k : \text{Spec } R_{k+1} \to \mathcal{M}_{0,m_k}^\dagger$, and isomorphisms $\gamma_k : \text{Spec } R_k \sim \mathcal{M}_{0,m_k+1}^\dagger \times \mathcal{M}_{0,m_k}^\dagger \text{ Spec } R_{k+1}$. Formally set $\gamma_N = \psi_N+1 = 1$, and define, by decreasing induction for $2 \leq i \leq N$:

$$\begin{align*}
\phi_i & = \beta_{i-1} \circ \psi_i^{-1} \\
\psi_i & = (1 \times \psi_{i+1}) \circ \gamma_i
\end{align*}$$

$\square$
8.3. The case when $G$ is of matrix type. Let $(G, \{1, \ldots, e_G\})$ be of matrix type. Let $Q_0 = \{(\emptyset, \emptyset)\}$, and for each $1 \leq i \leq e_G$, denote by $Q_i$ the set of all indices $(\{I,J\}, K)$ where $I, J, K$ are subsets of $\{1, \ldots, i\}$ such that $I \cup J \cup K = \{1, \ldots, i\}$, $|I| = |J|$, and such that the Dodgson polynomial $\Psi_{K}^{I,J}$ does not vanish. Since $G$ is of matrix type, its Landau varieties are contained in the zero locus of the Dodgson polynomials. Therefore, working over $\mathbb{Q}$, we have:

$$R_i = \mathbb{Q}[\alpha_{i+1}^{\pm 1}, \ldots, \alpha_{e_G}^{\pm 1}, (\Psi_{K}^{I,J})^{-1}],$$

for $0 \leq i \leq e_G - 1$, see also (86). Next, we decompose $Q_i$ into its set of horizontal and vertical components: $Q_i = Q_i^h \cup Q_i^v$ where $Q_i^{h}$ is the set of indices $(\{I,J\}, K)$ such that $\Psi_{K}^{I,J} = \Psi_{K}^{I,J} \alpha_i + \Psi_{K}^{I,J}$ satisfies $\Psi_{K}^{I,J} \Psi_{K}^{I,J} \neq 0$ and $Q_i^{v}$ is its complement in $Q_i$. The numbers $M_i = |Q_i|$ are the number of horizontal Dodgson polynomials at the $i$th level.

Next, we define the spaces $M_{0,M_i+3} \subseteq G_{m,3}$ in terms of simplicial coordinates we denote by $t_{I,J}^{K}$, for every $(\{I,J\}, K) \in Q_i^h$, i.e., first set:

$$(86) \quad M_{0,M_i+3} = \text{Spec } \mathbb{Q}[(t_{I,J}^{K})^{\pm 1}, \frac{1}{1-t_{I,J}^{K}}, \frac{1}{t_{I,J}^{K} - t_{L,M}^{K}}].$$

Now, fixing some element $(A,B,C) \in Q_i^h$, we can write down a map $\rho_i : \text{Spec } R_i \to M_{0,M_i+3}$ as in lemma 100 defined on the affine coordinate rings by

$$(\rho_i)_* : t_{I,J}^{K} \mapsto \Psi_{I,J}^{A,B} \frac{\Psi_{I}^{A,B} \Psi_{C}^{A,B}}{\Psi_{K}^{A,B}}, \quad (\rho_i)_* : t_{C}^{A,B} \mapsto -\alpha_i \frac{\Psi_{C}^{A,B}}{\Psi_{I}^{A,B}}.$$ 

The coordinate on the universal curve, i.e., the fiber of the forgetful map $M_{0,M_i+3} \to M_{0,M_i+2}$, is $t_{C}^{A,B}$. The modified space $M_{0,M_i+2}$ defines in a similar manner to (86) except that we denote the coordinates by $s_{K}^{I,J}$ for $(\{I,J\}, K) \neq (\{A,B\}, C)$, and the only terms besides $s_{K}^{I,J}$ that are inverted are terms of the form $(1 - s_{K}^{I,J})$ if $(I,J), (A,B)$ satisfy a basic compatibility (definition 90), and $(s_{K}^{I,J} - s_{N}^{L,M})$ if $(I,J), (L,M)$ do. The map $\beta_i : \text{Spec } R_{i+1} \to M_{0,M_i}^{1}$ of lemma 103 is given by:

$$(\beta_i)_* : s_{K}^{I,J} \mapsto \Psi_{I,J}^{A,B} \frac{\Psi_{I}^{A,B} \Psi_{C}^{A,B}}{\Psi_{K}^{A,B}}.$$ 

To verify that this map is well-defined uses the identities 22 and 23. The map $\rho_i$ defined above also gives the isomorphism

$$\gamma_i : \text{Spec } R_i \cong M_{0,M_i+1} \times \text{Spec } R_{i+1},$$

where the fiber product is defined using $\beta_i$. Thus on the affine rings we have

$$(\gamma_i)_* : R_{i+1}[(s_{C}^{A,B})^{\pm 1}, (1 - s_{C}^{A,B})^{-1}, (s_{C}^{A,B} - s_{K}^{I,J})^{-1}] \cong R_i$$

$$(\gamma_i)_* : s_{C}^{A,B} \mapsto -\alpha_i \frac{\Psi_{C}^{A,B}}{\Psi_{I}^{A,B}}.$$ 

The maps $\phi_i$ and $\psi_i$ are deduced from 85. Concretely, the maps $\phi_i$, send each $s_{K}^{I,J}$ to a product of cross-ratios involving other $s_{R}^{P,Q}$'s.

Remark 105. The upshot of this is to write down changes of variables which essentially turn the graph polynomial into products of cross ratios.
9. Calculation of the Periods

9.1. The bar construction of $\mathfrak{M}_{0,n}$. We briefly recall some properties of the $\mathbb{Q}$-algebra of iterated integrals on the moduli spaces $\mathfrak{M}_{0,n}$ (see also [8], §3).

Definition 106. Let $V(\mathfrak{M}_{0,n}) = H^0(B(\Omega^*_{\mathfrak{M}_{0,n}}(\log(\mathfrak{M}_{0,n}\setminus\mathfrak{M}_{0,n}))))$ denote the zeroth cohomology of Chen’s reduced bar complex on the global logarithmic forms on $\mathfrak{M}_{0,n}$, for $n \geq 3$. It is a graded commutative Hopf algebra defined over $\mathbb{Q}$.

One can also view $V(\mathfrak{M}_{0,n})$ as the de Rham realization of the motivic fundamental group of $\mathfrak{M}_{0,n}$. Suppose we are given a tangential basepoint $t$ on $\mathfrak{M}_{0,n}$ which is defined over $\mathbb{Z}$. Then we obtain an isomorphism ([8], §6.7):

$$\rho_t : V(\mathfrak{M}_{0,n}) \xrightarrow{\sim} L(\mathfrak{M}_{0,n}),$$

where $L(\mathfrak{M}_{0,n}) \otimes_{\mathbb{Q}} \mathbb{C}$ is the graded Hopf algebra of homotopy-invariant iterated integrals on $\mathfrak{M}_{0,n}(\mathbb{C})$. Its $\mathbb{Q}$-structure $L(\mathfrak{M}_{0,n})$ is given by the choice of basepoint $t$. The elements of $L(\mathfrak{M}_{0,n})$ can be expressed as multiple polylogarithms in $n - 3$ variables, which are multivalued functions on $\mathfrak{M}_{0,n}(\mathbb{C})$ with unipotent monodromy.

Definition 107. Let $Z = \mathbb{Q}[\zeta(n_1, \ldots, n_r) : n_1, \ldots, n_r \in \mathbb{N}, n_r \geq 2]$ denote the ring of multiple zeta values. It is filtered by the weight $n_1 + \ldots + n_r$.

Choosing a different tangential base point over $\mathbb{Z}$ changes the $\mathbb{Q}$-structure on $L(\mathfrak{M}_{0,n}) \otimes_{\mathbb{Q}} \mathbb{C}$, but preserves the $\mathbb{Q}$-structure on the filtered algebra $L(\mathfrak{M}_{0,n}) \otimes_{\mathbb{Q}} Z$ (the $\mathbb{Q}$-structure is modified by a product of Drinfel’d associators, which have coefficients in $Z$). By abuse of notation, we will sometimes consider elements of $V(\mathfrak{M}_{0,n})$ as elements of $L(\mathfrak{M}_{0,n}) \otimes_{\mathbb{Q}} Z$, for some unspecified isomorphism $\rho_t$.

We also require a relative version of the bar construction. Let

$$\mathfrak{M}_{0,n+1} \longrightarrow \mathfrak{M}_{0,n}$$

denote the map which forgets a marked point. It comes with $n$ sections $\sigma_1, \ldots, \sigma_n$, and is a fibration with fibers isomorphic to $\mathbb{P}^1 \setminus \{\sigma_1, \ldots, \sigma_n\}$. We define

$$V_{\mathfrak{M}_{0,n}}(\mathfrak{M}_{0,n+1}) = H^0(B(\Omega^*_{\mathfrak{M}_{0,n+1}/\mathfrak{M}_{0,n}}(\log(\sigma_1 \cup \ldots \cup \sigma_n)))),$$

to be the bar construction of the fiber relative to the base, which is again graded by the weight. Let $L_{\mathfrak{M}_{0,n}}(\mathfrak{M}_{0,n+1})$ be its realization in terms of hyperlogarithms ([8] §5, [9], §5.1), which again depends on the choice of a tangential basepoint. The algebraic structure of $V_{\mathfrak{M}_{0,n}}(\mathfrak{M}_{0,n+1})$ is that of a free shuffle algebra on $n - 1$ generators. We will only require the following fact, proved in [8]:

Theorem 108. Let $\omega \in \Omega^1_{\mathfrak{M}_{0,n+1}/\mathfrak{M}_{0,n}} \otimes_{\mathbb{Q}} L_{\mathfrak{M}_{0,n}}(\mathfrak{M}_{0,n+1})$ of weight $k$. For any two distinct sections $\sigma_i, \sigma_j : \mathfrak{M}_{0,n} \rightarrow \mathfrak{M}_{0,n+1}$ the integral in the fiber satisfies

$$\int_{\sigma_i}^{\sigma_j} \omega \in Z \otimes_{\mathbb{Q}} L(\mathfrak{M}_{0,n}),$$

and is of total weight at most $k + 1$. For the integral to make sense, we must assume that in each fiber, the domain of integration is a continuous path from $\sigma_i$ to $\sigma_j$ along which $\omega$ is single-valued.

It is proved in ([8], §3.5) that there is an isomorphism of algebras (which does not respect the coproducts) $V(\mathfrak{M}_{0,n+1}) \cong V_{\mathfrak{M}_{0,n}}(\mathfrak{M}_{0,n+1}) \otimes_{\mathbb{Q}} V(\mathfrak{M}_{0,n})$. The algebraic structure of $V(\mathfrak{M}_{0,n})$ is a product of shuffle algebras and is therefore well suited to algorithmic calculations of Feynman integrals. See [9] for a concrete example.
9.2. Linearly reducible spaces. Let \( S \subset (\mathbb{P}^1)^N \) be a linearly reducible set of hypersurfaces over \( \mathbb{Q} \). Recall from [8.2] that this gives rise to a nested sequence of rings \( R_0 \supset R_1 \supset \ldots \cup R_{N-1} \cup R_N \supset \mathbb{Q} \), along with two families of maps:

\[
\beta_i : \text{Spec } R_{i+1} \rightarrow \mathcal{M}_{0,m_i}^i
\]

and

\[
\gamma_i : \text{Spec } R_i \rightarrow \mathcal{M}_{0,m_i+1}^i \times \text{Spec } R_{i+1},
\]

where \( \mathcal{M}_{0,m_i}^i \supset \mathcal{M}_{0,m_i}^0 \) is isomorphic to an affine complement of hyperplanes, and \( \text{Spec } R_N \) is an open subscheme of \( \mathbb{P}^1 \). We will use the maps \( \beta_i \) and \( \gamma_i \) to construct a space of functions \( V(R_i) \) on \( \text{Spec } R_i \) in which to compute periods.

**Definition 109.** Let \( V(\mathcal{M}_{0,m_i}^i) \subset V(\mathcal{M}_{0,m_i}^0) \) be the zeroth cohomology of the reduced bar construction on \( \mathcal{M}_{0,m_i}^i \supset \mathcal{M}_{0,m_i}^0 \).

By an isomorphism \( \rho_1 \), elements of \( V(\mathcal{M}_{0,m_i}^i) \) correspond to multivalued functions in \( L(\mathcal{M}_{0,m_i}^i) \) which are unramified along boundary components of \( \mathcal{M}_{0,m_i}^i \setminus \mathcal{M}_{0,m_i}^0 \).

**Proposition 110.** Let \( E \) be a complement of hypersurfaces in \( \mathbb{A}^n \times \mathbb{A}^1 \) which are defined over \( \mathbb{Q} \), let \( \pi : \mathbb{A}^n \times \mathbb{A}^1 \rightarrow \mathbb{A}^n \) denote the induced projection, and let \( B = \pi(E) \subset \mathbb{A}^n \). Assume that the components of \( E \) are of degree \( \leq 1 \) in the fiber of \( \pi \), and let \( B' \) be the largest open subvariety of \( B \) such that

\[
\pi : E' \rightarrow B'
\]

is a fibration, where \( E' = \pi^{-1}(B') \). Denote the fiber over the generic point of \( B' \) by \( F \). Thus \( B' \) contains the Landau variety \( L(E,\pi) \) as defined in lemma 77.

For any \( X = E,B,E',B' \), let us write \( b(X) = H^0(\overline{B}(\Omega^*(X))) \) for the zeroth cohomology group of the reduced bar construction on the de Rham complex of \( X \) with coefficients in \( \mathbb{Q} \), and \( b_B(F) \) for the relative version \( H^0(\overline{B}(\Omega^*(F/B'))) \).

(i). There is an isomorphism of algebras \( b(E') \cong b(B') \otimes \mathbb{Q} b_B(F) \).

(ii). The image of the map \( b(E) \rightarrow b(E') \) is contained in \( b(B) \otimes \mathbb{Q} b_B(F) \).

**Proof.** Write \( B = \text{Spec } R, E = \text{Spec } R[(a_i \alpha + b_i)^{-1}], \) where \( a_i, b_i \in R \). We have \( B' = \text{Spec } R^+, \) where \( R^+ = R[(a_i b_i - a_i b_i)^{-1}] \), and \( E' = B' \times_B E \). The isomorphism (i) was proved in [8], corollary 3.24, and is dual to theorem 3.1 in [13].

To prove (ii), note that the natural map \( H^1(E;\mathbb{Q}) \rightarrow H^1(F;\mathbb{Q}) \) is split by lifting each logarithmic form on the fiber to \( d \log(a_i \alpha + b_i) \), and this gives a map \( H^1(E;\mathbb{Q}) \rightarrow H^1(B;\mathbb{Q}) \). In general, it follows from the Eilenberg-Moore spectral sequence that the associated graded of \( b(X) \) (with respect to the length filtration) is a certain subspace of the tensor algebra \( T(H^1(X;\mathbb{Q})) \) on \( H^1(X;\mathbb{Q}) \). Thus

\[
\text{gr}^* b(E) \subset \text{gr}^* b(E') \cong b(B') \otimes \mathbb{Q} b_B(F)
\]

But the map \( b(E) \rightarrow b(E') \rightarrow b(B') \) lands in \( b(B) \), since it is induced by \( H^1(E;\mathbb{Q}) \rightarrow H^1(B;\mathbb{Q}) \), and \( \text{gr}^* b(B) = \text{gr}^* b(B') \cap T(H^1(B)) \), in \( T(H^1(B')) \). This implies (ii). \( \square \)

By [59], this motivates the following inductive definition of the spaces \( V(R_i) \).

**Definition 111.** For \( 1 \leq i \leq N \) define \( V(R_i) \) inductively by the formula:

\[
V(R_i) = V_{\mathcal{M}_{0,m_i}^i}(\mathcal{M}_{0,m_i+1}^i) \otimes \mathbb{Q} V(R_{i+1}).
\]
Thus \( V(R_i) \) is defined to be a tensor product of shuffle algebras on \( m_i - 1 \) elements \( \mathcal{M}_{0,m_i} \). The elements of \( V(R_i) \) can be thought of as multivalued functions on an open subset of \( \text{Spec} R_i(\mathbb{C}) \).

**Lemma 112.** The map \( \beta_i : \text{Spec} R_{i+1} \to \mathcal{M}_{0,m_i} \) induces a map:

\[
(\beta_i)_* : V(\mathcal{M}_{0,m_i}^!) \to V(R_{i+1})
\]

**Proof.** By functoriality, \( \beta_i \) induces a map:

\[
(\beta_i)_* : H^0(\overline{\mathcal{B}}(\Omega^*(\mathcal{M}_{0,m_i}^!))) \to H^0(\overline{\mathcal{B}}(\Omega^*(\text{Spec} R_{i+1})))
\]

It suffices to show that for all \( j \),

\[
H^0(\overline{\mathcal{B}}(\Omega^*(\text{Spec} R_j))) \subset V(R_j)
\]

i.e., our recursive definition of \( V(R_j) \) does indeed contain the iterated integrals on \( \text{Spec} R_j \). To see this, consider the open subscheme \( U_{j+1} \subset \text{Spec} R_{j+1} \) given by \( U_{j+1} = \beta_j^{-1}(\mathcal{M}_{0,m_j}) \). Then the open subscheme \( W_j \subseteq \text{Spec} R_j \) given by

\[
W_j = \gamma_j^{-1}(\mathcal{M}_{0,m_{j+1}} \times_{\mathcal{M}_{0,m_j}} U_{j+1})
\]

fibers linearly over \( U_{j+1} \). The lemma follows from the previous proposition on setting \( E = \text{Spec} R_j, B = \text{Spec} R_{j+1}, B' = U_{j+1}, E' = W_j \).

Therefore lemma 112 defines a transfer map

\[
\text{id} \otimes (\beta_i)_* : V(R_{i+1}) \otimes_{\mathbb{Q}} V(\mathcal{M}_{0,m_i}^!) \to V(R_{i+1})
\]

This is the only data we will require to compute the periods in §7.4.

**9.3. Tangential base points and rational structures.** In order to compute periods, the maps induced by \( \beta_i, \gamma_i \) on the iterated integrals should be defined over \( \mathbb{Q} \). Therefore we must first define a \( \mathbb{Q} \)-structure on the algebra of iterated integrals on \( \text{Spec} R_i(\mathbb{C}) \), which follows from a choice of tangential base-point on \( \text{Spec} R_i \). For this, it is natural to use the Schwingen coordinates \( \alpha_i, \ldots, \alpha_N \), as follows. Recall that there are linear functions \( f_k = a_k \alpha_i + b_k \) of \( \alpha_i \) such that:

\[
R_i = R_{i+1}[\alpha_i^{k+1}, f_k^{-1}, 1 \leq k \leq M_i]
\]

It follows from §3.5 in §8 that \( R_i \) inherits a tangential base-point by induction. Set

\[
L(R_i) = L(R_{i+1}) \otimes_{\mathbb{Q}} \gamma_i^*(L(\mathcal{M}_{0,m_i}(\mathcal{M}_{0,m_{i+1}})))
\]

by induction. The tangential basepoint means that the elements of \( L(R_i) \) are regularized by letting \( \alpha_i \to 0, \ldots, \alpha_N \to 0 \) in that order. Concretely, the functions in \( L(R_i) \) are \( \mathbb{Q} \)-linear combinations of products of elements in \( L(R_{i+1}) \), with functions

\[
f(\alpha_i, \ldots, \alpha_N) = \sum_k \log^k(\alpha_i) f_k(\alpha_i, \ldots, \alpha_N),
\]

where \( f_k \in \gamma_i^*(L(\mathcal{M}_{0,m_i}(\mathcal{M}_{0,m_{i+1}}))) \) is identically zero on \( \alpha_i = 0 \). This defines a \( \mathbb{Q} \)-structure on a space of iterated integrals on \( \text{Spec} R_i \), i.e., the left-hand side of the diagram of lemma 112. The spaces of iterated integrals on the moduli spaces \( \mathcal{M}_{0,m_{i+1}} \) on the right-hand side already have a canonical \( \mathbb{Q} \)-structure after tensoring with \( \mathbb{Z} \). More precisely, if \( L(\mathcal{M}_{0,m_{i+1}}) \subseteq L(\mathcal{M}_{0,m_i}) \) is the \( \mathbb{Q} \)-subalgebra of \( L(\mathcal{M}_{0,m_i}) \) (defined earlier, for some tangential basepoint over \( \mathbb{Z} \)) given by the homotopy-invariant iterated integrals on \( \mathcal{M}_{0,m_i} \), then \( L(\mathcal{M}_{0,m_{i+1}}) \otimes_{\mathbb{Q}} \mathbb{Z} \) is well-defined.
3.1. Compactification of \( \mathfrak{M}^1_{0,m+1} \). Recall that \( \mathfrak{M}^1_{0,m+1} \subset (\mathbb{G}_m)^{m_i-2} \) is the complement of a configuration of hyperplanes of the form \( t_i = t_j \), \( t_i = 1 \), where \( t_i \) are coordinates on each component \( \mathbb{G}_m \). There exists a minimal smooth compactification \( \overline{\mathfrak{M}}^1_{0,m} \) such that \( \mathfrak{M}^1_{0,m+1} \subset \overline{\mathfrak{M}}^1_{0,m+1} \) is the complement of a smooth normal divisor. It is defined over \( \mathbb{Z} \) and there is a map \( \overline{\mathfrak{M}}^1_{0,m+1} \to \overline{\mathfrak{M}}^1_{0,m+1} \) which blows down certain divisors of \( \overline{\mathfrak{M}}^1_{0,m+1} \). For example, if \( \mathfrak{M}^1_{0,m+1} = (\mathbb{G}_m)^{m_i-2} \), then its minimal compactification is \( (\mathbb{P}^1)^{m_i-2} \). There is a stratification on \( \overline{\mathfrak{M}}^1_{0,m+1} \) obtained by intersecting components of boundary divisors, and its deepest stratum is of dimension 0, i.e., a collection of points. By standard techniques, one can write down normal coordinates in the neighbourhood of each such point as quotients of terms \( t_i, t_i - 1 \) and \( t_i - t_j \) corresponding to the hyperplanes in \( \mathbb{A}^{m_i-2} \setminus \mathfrak{M}^1_{0,m+1} \).

3.2. Ramification. In order to determine if the rational structures on the algebras of iterated integrals defined previously are compatible, consider any open path

\[
\gamma : (0, \varepsilon) \to (\text{Spec } R_i)(\mathbb{C})
\]

whose image lies in the sector \( 0 \ll \alpha_1 \ll \alpha_{i+1} \ll \ldots \ll \alpha_N \ll 1 \). By composing with the map \( \rho : \text{Spec } R_i \to \mathfrak{M}^1_{0,m+1} \to \overline{\mathfrak{M}}^1_{0,m+1} \) we obtain a path in \( (\overline{\mathfrak{M}}^1_{0,m+1})(\mathbb{C}) \). By compactness, its limit \( \lim_{t \to 0} (\rho(\gamma(t))) \) defines a point \( p \in \overline{\mathfrak{M}}^1_{0,m+1} \).

**Definition 113.** The map \( \rho \) is unramified if \( \rho^* : W^1 L(\mathfrak{M}^1_{0,m+1}) \to W^1 L(R_i) \otimes \mathbb{C} \) is defined over \( \mathbb{Q} \), and \( p \in \overline{\mathfrak{M}}^1_{0,m+1} \) lies in a deepest possible boundary stratum.

**Proposition 114.** If \( \rho \) is unramified, then \( \beta_i^* : L(\mathfrak{M}^1_{0,m_i}) \otimes _\mathbb{Q} \mathbb{Z} \to L(R_{i+1}) \otimes _\mathbb{Q} \mathbb{Z} \) and \( (\gamma_i^{-1})^* : L_{\mathfrak{M}_{0,m_i}}(\mathfrak{M}^1_{0,m_i}) \otimes _\mathbb{Q} L(R_{i+1}) \to L(R_i) \otimes _\mathbb{Q} \mathbb{Z} \) are defined over \( \mathbb{Q} \).

**Proof.** Every function in \( L(\mathfrak{M}^1_{0,m_i}) \) lifts to a multivalued function on \( \overline{\mathfrak{M}}^1_{0,m_i} \) with logarithmic singularities along the boundary divisors. By choosing appropriate normal coordinates \( z_1, \ldots, z_\ell \) which vanish at the point \( p \), and taking this as tangential base-point, we see that an algebra basis for \( L(\mathfrak{M}^1_{0,m_i}) \otimes _\mathbb{Q} \mathbb{Z} \) is spanned by

\[
\log(z_1), \ldots, \log(z_\ell), f(z_1, \ldots, z_\ell),
\]

where \( f(z_1, \ldots, z_\ell) \in L(\mathfrak{M}^1_{0,m_i}) \otimes _\mathbb{Q} \mathbb{Z} \) vanishes at \( p \). By assumption,

\[
\beta_i^*(\log z_k) \in L(R_{i+1})
\]

is defined over \( \mathbb{Q} \) for \( 1 \leq k \leq \ell \). It suffices to check that \( \beta_i^* f(z_1, \ldots, z_\ell) \) is in \( L(R_{i+1}) \). But this follows from the fact that \( p = \lim_{t \to 0} (\beta_i \circ \gamma(t)) \) and the definition of the \( Q \)-structure on \( L(R_{i+1}) \otimes _\mathbb{Q} \mathbb{C} \). The case \( \gamma_i^{-1} \) is similar. \( \square \)

**Remark 115.** One can show that definition 113 is equivalent to the property that \( \rho^* : W^1 L(\mathfrak{M}^1_{0,m+1}) \to W^1 L(R_i) \otimes _\mathbb{C} \mathbb{C} \) can be defined over \( \mathbb{Q} \).

Definition 113 can be made explicit. Let \( L_i \) be the Landau variety of \( S \) with horizontal components \( f_k = a_k \alpha_i + b_k, a_k b_k \neq 0 \). Recall that the map \( \rho_i : \text{Spec } R_i = (\mathbb{P}^1)^{N-i} \setminus L_i \to \mathfrak{M}_{0,m+1} \) is given in simplicial coordinates by:

\[
\rho_i^*(t_{M_i}) = -\alpha_i \frac{a_{M_i}}{b_{M_i}} \quad \text{and} \quad \rho_i^*(t_k) = \frac{a_{M_i} b_k}{b_{M_i} a_k}
\]
A \( \mathbb{Q} \)-basis of \( W^1V(\mathfrak{m}_{0,n}^i) \) is given by the logs of \( t_i, 1 - t_i, \) and \( t_i - t_j \) for certain pairs \( i, j \), and therefore definition 113 requires that their images in \( W^1L(R_{i+1}) \):

\[
\log \left( \frac{a_i}{b_j} \right) \quad \text{and} \quad \log \left( \frac{a_ib_j - a_jb_i}{a_ib_j} \right)
\]

be defined over \( \mathbb{Q} \), and that certain cross-ratios in the same quantities \( t_i, 1 - t_i, t_i - t_j \) tend to \( \{0, 1, \infty\} \) as \( \alpha_i \to 0, \ldots, \alpha_N \to 0 \) in that order. By linear reducibility, the arguments of the logarithms (92) factorize into terms which are linear in \( \mathbb{Q} \)-basis of \( \mathfrak{m}_{0,n}^i \):

\[
g = g_0 \alpha_i^{s_i} \prod_{k=1}^{m} (\rho_k \alpha_i + 1)^{s_k} \in R_{i+1}
\]

where \( s_i \in \mathbb{Z} \), and \( g_0 \in R_{i+2} \) is the leading non-zero term in the Laurent expansion of \( g \) at \( \alpha_i = 0 \). Thus \( \log g \in V(R_{i+1}) \) is defined over \( \mathbb{Q} \) if and only if \( \log g_0 \in V(R_{i+2}) \) is defined over \( \mathbb{Q} \). Proceeding by induction, we see that it is enough that

\[
\lim_{\alpha_i \to 0} \ldots \lim_{\alpha_i \to 0} \lim g = 1
\]

where \( \lim x \to 0 h(x) \) denotes the first non-zero term in the Laurent expansion of a rational function \( h \) at \( x = 0 \), and where \( g \) ranges over the set

\[
\{ [f_k, 0], [f_k, \infty] \text{ for all } k \} \cup \{ [f_k, f_i] \text{ for all compatible } f_k, f_i \}.
\]

**Definition 116.** Let \( S \) be a linearly reducible set of hypersurfaces. We say that \( S \) is unramified if (93) holds for all \( i = 1, \ldots, N \).

This implies in particular that \( \operatorname{Spec} R_N \supset \mathbb{P}^{1} \setminus \{0, 1, \infty\} \) is the complement of at most 3 marked points. If \( G \) is of matrix type, then there exists an ordering on the edges of \( G \) such that the components of the Landau varieties \( L_i \) are zeros of Daudson polynomials \( \Psi^{I,J}_K \). We say that \( G \) is positive if, for some ordering on its set of edges, every such Dodgson polynomial has only positive coefficients.

**Lemma 117.** If \( G \) is of matrix type and positive then \( \Psi_G = 0 \) is unramified.

**Proof.** Since \( G \) is of matrix type, the terms occuring in (93) are products of polynomials \( (\Psi^{I,J}_K)^{\pm 1} \). By the positivity assumption, each such \( \Psi^{I,J}_K \) is a sum of monomials with coefficient \(+1\) and so the limit in (93) is necessarily 1. \( \square \)

**Theorem 118.** If \( G \) has vertex width \( \leq 3 \), then \( G \) is of matrix type and positive.

**Proof.** We have already shown that vertex width 3 implies matrix type. It suffices to show that \( \Psi^{I,J}_G \) has positive coefficients if \( I \cup J \cup K = \{1, \ldots, i\} \), where \( 1, \ldots, N \) is the ordering on the edges of \( G \). By theorem 23 of \([7]\), there is a universal formula for 3-vertex joins, in terms of the graph obtained by adding a 3-valent vertex to the subgraph spanned by \( \{1, \ldots, i\} \) and connecting it to the three distinguished vertices (see §4.6 of \([7]\)). Then \( \Psi^{I,J}_G \) is a polynomial in the variables \( x, y, z \) corresponding to these three edges, which are themselves spanning forest polynomials of \( G \), and therefore have positive coefficients (proposition 38 of \([7]\)). By proposition 23 there are very few possibilities, and all are checked to be positive. \( \square \)

**Remark 119.** One can also consider the case when the ramification is contained in a set of roots of unity (\([9]\)), by working throughout with the moduli spaces \( \mathfrak{m}_{0,n}^\nu \), which are finite covers of \( \mathfrak{m}_{0,n}, \) and their ring of periods \( \mathbb{Z}^\nu \), which are values of multiple polylogarithms at these roots of unity.
9.4. **Calculation of Feynman integrals.** Let $S$ be linearly reducible and defined over $\mathbb{Q}$, and let $f_0$ be a multivalued function on $\mathbb{C}_m^N \setminus S$. Consider an integral

$$I = \int_{[0, \infty]^N} f_0 \, d\alpha_1 \ldots d\alpha_N .$$

For this to make sense, we must assume that $f_0$ is single-valued along the domain of integration and that the integral converges. If $f_0$ has unipotent monodromy and is defined over $\mathbb{Q}$, then it corresponds to an element in $R_0 \otimes \mathbb{Q} V(R_0)$, and $I$ can be computed by integrating one variable at a time via the diagram:

$$\begin{array}{c}
\mathbb{P}^1 \setminus S \leftarrow \text{Spec } R_0 \\
\downarrow \pi_1 \downarrow \\
\mathbb{P}^{N-1} \setminus L_1 \leftarrow \text{Spec } R_1 \\
\downarrow \downarrow \\
\vdots \\
\end{array}$$

This can be done by working in the rings $V(R_i)$, which correspond to algebras of unipotent functions on $(\text{Spec } R_i)(\mathbb{C})$. More precisely, we define

$$f_{i+1} = \int_0^\infty f_i \, d\alpha_{i+1} ,$$

and $I = f_N$. By theorem 56 and remark 57 we know that the partial integrals $f_i$ are multivalued functions on $(\mathbb{P}^{N-1} \setminus L_i)$, i.e.,

$$\text{Sing}(f_i) \subseteq L_i .$$

**Example 120.** The Feynman case is similar. Let $G$ be a connected graph, and consider a convergent affine Feynman integral of the form:

$$I = \int_{[0, \infty]^N} P(\alpha_i, \log \alpha_i, \log \Psi_G) \frac{\delta(\alpha_N = 1)}{\Psi_G^k} \, d\alpha_1 \ldots d\alpha_N$$

where $P$ is a polynomial with coefficients in $\mathbb{Q}$, and $S = X_G$. The integrand defines an element $f_0 \in R_0 \otimes \mathbb{Q} V(R_0)$, and $I = f_{N-1}\mid_{\alpha_N = 1}$, where the $f_i$ are given by (96).

**Theorem 121.** Let $S$ be a linearly reducible set of hypersurfaces which is unramified. Then $I \in \mathbb{Z}$.

**Proof.** We write $B(R_i) = R_i \otimes \mathbb{Q} L(R_i) \otimes \mathbb{Q} \mathbb{Z}$, equipped with the weight filtration. The elements of $B(R_i)$ can either be seen as elements of the bar construction or as iterated integrals on $\text{Spec } R_i$, and we will use both points of view interchangeably. Likewise, let $B_{\mathbb{Z}_0, m_i} \mathfrak{M}_{0, m_i+1} = \mathcal{O}(\mathfrak{M}_{0, m_i+1}) \otimes \mathbb{Q} L_{\mathbb{Z}_0, m_i} (\mathfrak{M}_{0, m_i+1}) \otimes \mathbb{Q} \mathbb{Z}$.

Recall that there is a commutative diagram:

$$\begin{array}{c}
\text{Spec } R_i \xrightarrow{\gamma_i} \mathfrak{M}_{0, m_i+1} \times \text{Spec } R_{i+1} \\
\downarrow \pi_{i+1} \downarrow \\
\text{Spec } R_{i+1} \xrightarrow{\gamma_{i+1}} \mathfrak{M}_{0, m_i} \times \mathfrak{M}_{0, m_i} \\
\end{array}$$
We can therefore compute the integrals by travelling around the right-hand side of this diagram and working on the moduli spaces $\mathcal{M}_{0,m_i+1}$. More precisely, we have:

\[
\begin{align*}
B(R_i) & \xrightarrow{\sim} B\mathcal{M}_{0,m_i}^\dagger(\mathcal{M}_{0,m_i+1}^\dagger) \otimes_{\mathcal{O}(\mathcal{M}_{0,m_i}^\dagger)} B(R_{i+1}) \\
\pi^\ast_{i+1} & \\
B(R_{i+1}) & \xleftarrow{\sim} B(\mathcal{M}_{0,m_i}^\dagger) \otimes_{\mathcal{O}(\mathcal{M}_{0,m_i}^\dagger)} B(R_{i+1})
\end{align*}
\]

The horizontal isomorphism along the top follows from the definition of $L(R_i)$ as a tensor product, and the horizontal map along the bottom is given by \((\ref{eq:91})\). Suppose by induction that $f_i \in B(R_i)$. Since $\alpha_{i+1}$ corresponds to the coordinate in the fiber of $\mathcal{M}_{0,m_i+1} \to \mathcal{M}_{0,m_i}$, the 1-form $f_i \, d\alpha_{i+1}$ corresponds via $(\gamma_i^{-1})^*$ to an element

\[
\sum_k \eta_k \otimes g_k \in \Omega^1 B\mathcal{M}_{0,m_i}^\dagger(\mathcal{M}_{0,m_i+1}^\dagger) \otimes_{\mathcal{O}(\mathcal{M}_{0,m_i}^\dagger)} B(R_{i+1})
\]

where $\Omega^1 B\mathcal{M}_{0,m_i}^\dagger(\mathcal{M}_{0,m_i+1}^\dagger) = \Omega^1_{\mathcal{M}_{0,m_i+1}^\dagger/\mathcal{M}_{0,m_i}} \otimes_{\mathcal{O}(\mathcal{M}_{0,m_i}^\dagger)} L_{\mathcal{M}_{0,m_i}}(\mathcal{M}_{0,m_i+1}^\dagger) \otimes_{\mathcal{O}} \mathbb{Z}$. We have

\[
\int_{\gamma_i(0,\infty)} (\gamma_i^{-1})^* f_i \, d\alpha_{i+1} = \sum_k \left( \int_{\gamma_i(0)} \eta_k \right) g_k
\]

which by theorem \([108]\) lies in

\[
B(\mathcal{M}_{0,m_i}) \otimes_{\mathcal{O}(\mathcal{M}_{0,m_i}^\dagger)} B(R_{i+1})
\]

and defines a multivalued function on the subset $U_i = \mathcal{M}_{0,m_i} \times_{\mathcal{M}_{0,m_i}} \text{Spec } R_{i+1}$ of $\text{Spec } R_{i+1}$. By theorem \([58]\) or \([97]\), its ramification locus is contained in $L_{i+1}$, and therefore has trivial monodromy around components of $\text{Spec } R_{i+1} \setminus U_i$. In other words, it must actually lie in the subspace:

\[
B(\mathcal{M}_{0,m_i}^\dagger) \otimes_{\mathcal{O}(\mathcal{M}_{0,m_i}^\dagger)} B(R_{i+1})
\]

This maps via $id \otimes (\beta_i)^*$ to $B(R_{i+1})$ by proposition \([114]\). Thus we conclude that

\[
f_{i+1} = \int_0^\infty f_i \, d\alpha_{i+1} \in B(R_{i+1})
\]

and has weight at most one greater than $f_i$. At the penultimate stage we deduce that $f_{N-1} \in B(R_N)$, and hence $I = f_N \in \mathbb{Z}$. 

**Corollary 122.** Let $G$ be a positive graph of matrix type. Then the periods of Feynman integrals associated to $G$ are $\mathbb{Q}$-linear combinations of multiple zeta values.

**Remark 123.** In the case of graphs which are ramified at roots of unity, the same result holds with $\mathbb{Z}$ replaced with $\mathbb{Z}^{[\mu]}$ simply by working throughout with $\mathcal{M}_{0,n}^{[\mu]}$. In practice, for many Feynman graphs at low loop orders, the ramification occurs at the final step, so one can in fact carry out most of the computations in $\mathcal{M}_{0,n}$. 

10. Leading terms and denominators

We study the residue $I_G$ of a primitive divergent graph $G$, its most interesting period. We show that there is a simple iterative way to compute the denominators (or polar singularities) of its partial integrals, and deduce an upper bound for its transcendental weight. We also sketch an integrality result for its coefficients.

10.1. Higher graph invariants. Let $G$ be a connected graph with edges $e_1, \ldots, e_N$.

**Definition 124.** Recall the definition of the 5-invariant $^5Ψ_G(e_1, \ldots, e_5)$ from \[ \text{vanish for all } n \geq 7. \] In the linearly reducible (or nearly linearly reducible) case it is more interesting, therefore, to take irreducible factors of each $^nΨ_G(e_1, \ldots, e_n)$, and repeatedly take discriminants of these terms. This will give a sequence of polynomials which, as we show below, computes the denominators of the partial Feynman integrals, and yields information about the residue.

10.2. Unipotent functions and primitives. Let $n \geq 4$ and consider, as in the previous section, the universal curve of genus 0 with $n$ marked points, given by the fiber of the forgetful map: $M_{0,n+1} \to M_{0,n}$. We identify the fiber with $P^1 \backslash \Sigma$, where $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$, and $\sigma_i$ are sections of the map above. Let $x$ be the coordinate on $P^1 \backslash \Sigma$, and assume that $\sigma_1, \sigma_2, \sigma_3$ correspond to $x = 0, 1, \infty$ respectively. Since it is not known whether the ring of multiple zeta values is graded by the weight, we are obliged to consider $\text{gr}^W Z$, which is the associated graded (for the weight filtration) of the ring of multiple zeta values over $\mathbb{Q}$. Recall that $L(M_{0,n})$ is a $\mathbb{Q}$-structure on the graded algebra of iterated integrals on $M_{0,n}$ corresponding to a (canonical) tangential basepoint. We need a more precise version of theorem [108]

**Theorem 125.** Let $F(x) \in L(M_{0,n+1})$ of pure weight $m$. Then for $\sigma \in \sigma \backslash \{0, 1, \infty\}$,

\[ \int_0^\infty \frac{F(x)}{x-\sigma}dx \in L(M_{0,n}) \otimes_{\mathbb{Q}} \text{gr}^W \mathbb{Z}, \]

where the right-hand side is pure of total weight $m + 1$. Likewise, for $i \neq j$,

\[ \int_0^\infty \frac{F(x)}{(x-\sigma_i)(x-\sigma_j)}dx \in \frac{1}{\sigma_i - \sigma_j}L(M_{0,n}) \otimes_{\mathbb{Q}} \text{gr}^W \mathbb{Z}, \]

is also pure of total weight $m + 1$. By contrast,

\[ \int_0^\infty \frac{F(x)}{(x-\sigma)^2}dx \in \sum_{\tau \in \Sigma \backslash \{0, 1, \infty\}} \frac{1}{\tau - \sigma}L(M_{0,n}) \otimes_{\mathbb{Q}} \text{gr}^W \mathbb{Z}, \]

where the right-hand side can be of any weight up to and including $m$. 

Decompose $L(\mathfrak{M}_{0,n+1})$ as a tensor product of algebras of hyperlogarithms ((6.7) in [8]). The function $F(x)$ can be written as a $Q$-linear combination of hyperlogarithms in the variable $x$ with singularities in $\Sigma$ and it follows from the definitions that there exists a primitive $G(x)$ of $(x-\sigma)^{-1}F(x)dx$ which is in $L(\mathfrak{M}_{0,n+1})$, and is pure of weight $m+1$. It follows from theorem 6.25 in [8] that the regularized limit $\text{Reg}_{\epsilon=0}G(x)-\text{Reg}_{\epsilon=0}G(x)$ is in $L(\mathfrak{M}_{0,n})\otimes_{\mathbb{Q}} W^* Z$, and is pure of total weight $m+1$. This proves (100). The proof of (101) follows by partial fractions, and (102) follows by integration by parts (exact formulae are given in [7], §4).

Note that in the case (102) the denominators on the right-hand side can be any descendents (in the sense of §6) of the denominator $(x-\sigma)^2$ of the integrand. We apply the previous proposition to partial Feynman integrals.

Corollary 126. Let $F$ be an iterated integral of weight $n$ on the universal curve $\mathbb{P}^1\setminus \Sigma$ with coordinate $x$, and single-valued along some path from 0 to $\infty$. In particular, it has at most logarithmic singularities along $\Sigma$, so the integrals below are well-defined and convergent. Then there exists a function $\tilde{F}$ on $\mathfrak{M}_{0,n}(\mathbb{C})$ which is unipotent of weight $n+1$ such that:

\begin{equation}
\int_0^\infty \frac{F(x)}{(\alpha x + \beta)(\gamma x + \delta)} dx = \frac{\tilde{F}}{\alpha \delta - \beta \gamma} \quad \text{if} \quad \alpha \delta - \beta \gamma \neq 0.
\end{equation}

In other words, the denominator after one integration is just the square root of the discriminant of the previous denominator $\sqrt{D_x(\alpha x + \beta)(\gamma x + \delta)}$. Similarly,

\begin{equation}
\int_0^\infty \frac{F(x)}{(\alpha x + \beta)^2} dx \quad \text{and} \quad \int_0^\infty F(x)dx,
\end{equation}

are linear combinations of unipotent functions of weight at most $n$.

Proof. By a change of variables, we reduce to the previous proposition. Equation (1103) is precisely (1104), and the first equation of (1104) is exactly (1102). The case $\int_0^\infty F(x)dx$ follows by integration by parts and is left to the reader.

Definition 127. In either case of (1104) we say that the integral has a weight drop.

10.3. Initial integrations. Using the above, we can compute denominators in partial Feynman integrals inductively. Let $G$ be any primitive-divergent graph with $N$ edges, so that the following integral converges:

\[ I_G = \int_{\alpha_N=1}^{N-1} \prod_{i=1}^{N-1} d\alpha_i. \]

A priori the transcendental weight is bounded above by $N-1$, but the denominator is a perfect square, so we immediately have a weight drop. The only descendents of $\Psi (= \Psi_G)$ are $\Psi^1$ and $\Psi_1$. After a single integration with respect to $\alpha_1$ we have:

\[ I_1 = \int_{\alpha_N=1}^{N-1} \prod_{i=2}^{N-1} d\alpha_i. \]

After a second integration with respect to $\alpha_2$, we had (example 67):

\[ I_2 = \int_{\alpha_N=1}^{N-1} \log(\Psi^1_2) + \log(\Psi^2_1) - \log(\Psi_{12}) - \log(\Psi_{12}) \prod_{i=3}^{N-1} d\alpha_i. \]
The denominator is a perfect square once again, so we get a second weight drop. Recall from example 62 that we had:

\[ I_3 = \int_{\alpha_N=1} (\Psi_{123} \log \Psi_{123} \Psi_{13} \log \Psi_{123} + \sum_{\{i,j,k\}} \frac{\Psi_{ij} \log \Psi_{jk} - \Psi_{ik} \log \Psi_{ij}}{\Psi_{ij} \Psi_{jk} \Psi_{ik}}) \prod_{i=4}^{N-1} d\alpha_i, \]

where the sum is over all \(\{i, j, k\} = \{1, 2, 3\}\). Since the weight can increase by at most one at each integration, the transcendental weight is therefore at most \(N - 3\), which is the generic case. The denominators are the Dodgson polynomials \(\Psi_{ij,k}\) and \(\Psi^{ij,k}\) for \(\{i, j, k\} = \{1, 2, 3\}\), which are precisely the descendents of \(\Psi^{1,2}\).

**Lemma 128.** The integrand at the fourth stage is a sum of three terms:

\[ I_4 = \int_{\alpha_N=1} \left( \frac{A}{\Psi_{12,34,13,24}} + \frac{B}{\Psi_{14,23,13,24}} + \frac{C}{\Psi_{12,34,13,24}} \right) \prod_{i=5}^{N-1} d\alpha_i \]

where \(A, B, C\) are unipotent of weight 2.

**Proof.** The terms \(A, B, C\) can be computed explicitly, but it is instructive to see why the denominators are what they are. Let us illustrate by decomposing only the coefficient of \(\log \Psi_{123}\) of \(I_3\) into partial fractions with respect to \(\alpha_4\). It is:

\[
\frac{\Psi_{123}}{\Psi_{1,2} \Psi_{1,2} \Psi_{1,2}} = \frac{[\Psi_{123}, \Psi_{1,2}]}{[\Psi_{1,2}]^4} \frac{[\Psi_{1,2}]}{[\Psi_{1,2}]^4} \frac{[\Psi_{1,2}]}{[\Psi_{1,2}]^4} \left( \frac{\Psi_{14,24}}{\alpha_4 + \Psi_{1,2}} \right) + 2 \text{ other terms}
\]

\[
= \frac{1}{\Psi_{12,24} \Psi_{13,24}} \left( \frac{\Psi_{14,24}}{\alpha_4 + \Psi_{1,2}} \right) + 2 \text{ other terms}
\]

The second line follows from the first by applying identities (23) and (24) and cancelling terms. The result follows from theorem 125 after integrating out \(\alpha_4\). \(\Box\)

Note that one can show that any unipotent function of weight \(\leq 3\) can be expressed only in terms of logarithms, dilogarithms and trilogarithms.

**Corollary 129.** It follows that at the fifth stage of integration, we have

\[ I_5 = \int_{\alpha_N=1} \frac{F}{\prod_{i=6}^{N-1} d\alpha_i} \]

where \(F\) is a unipotent function of weight 3.

Thus after five integrations there is a single denominator, which is the 5-invariant. Thereafter we can compute the denominators by applying corollary 126.

### 10.4. Denominator reduction

We obtain the following reduction algorithm.

**Proposition 130.** Suppose that \(P_n\) is the denominator at the \(n\)th stage. It is a polynomial of degree at most 2 in \(\alpha_{n+1}\). Suppose that \(P_n\) factorizes into a product of linear factors\(^3\) in \(\alpha_{n+1}\), i.e., the discriminant \(D_{\alpha_{n+1}}(P_n)\) is a perfect square. If it is identically zero, then there is a drop in the transcendental weight. If it is non-zero then the denominator at the \((n+1)\)th stage is

\[ P_{n+1} = \sqrt{D_{\alpha_{n+1}}(P_n)} . \]

\(^3\)This is necessarily the case if \(G\) is linearly reducible
Since we know that the denominator $P_5$ at the fifth stage is just the 5-invariant $\Psi(1,2,3,4,5)$, this gives an algorithm to compute the denominators at each stage, up to the point where a weight drop occurs (if it occurs at all). We call this the \textit{denominator reduction}. This is much easier to compute than the full reduction.

**Corollary 131.** Suppose that $G$ is linearly reducible for an ordering $e_1, \ldots, e_N$ on its edges. The terms $P_n$ occurring in the denominator reduction (proposition 130) are contained in the linear reduction of $G$. In particular, if for some $n \geq 5$, $P_n$ is irreducible and of degree two in every variable $\alpha_{n+1}, \ldots, \alpha_N$, then $G$ is not linearly reducible with respect to that ordering.

This gives a simple way to show if a graph \textit{fails} to be linearly reducible. In this case, we say that a denominator of degree exactly 2 in every (unintegrated) variable is \textit{totally quadratic}. It is these kinds of considerations which led to the discovery of the first non-Tate counter examples (§12).

**Definition 132.** A graph $G$ is \textit{denominator reducible} if there exists an ordering $e_1, \ldots, e_N$ on its edges such that, for all $5 \leq n \leq N$,

$$P_n \in \mathbb{Q}[\alpha_{n+1}, \ldots, \alpha_N].$$

In other words, $D_{\alpha_n}(P_{n-1})$ is a perfect square in $\mathbb{Q}[\alpha_{n+1}, \ldots, \alpha_N]$.

Note that denominator reducibility is not minor monotone. To rectify this, we can say that a graph $G$ is \textit{strongly denominator reducible} if it, and all its minors, are denominator reducible for the induced edge ordering.

**Example 133.** The smallest primitive divergent graph in $\phi^4$ that exhibits a weight-drop is the 5-loop non-planar graph depicted below. After five integrations, the denominator is the five-invariant:

$$\Psi(1,2,3,4,5) = \alpha_7(\alpha_8\alpha_9 + \alpha_8\alpha_{10} + \alpha_9\alpha_{10})(\alpha_8\alpha_6 + \alpha_{10}\alpha_6 + \alpha_7\alpha_{10} + \alpha_6\alpha_7),$$

which is split, since the edges 1, 3, 4 form a 3-valent vertex. Since this is linear in $\alpha_6$, the denominator at the sixth stage is just the coefficient of $\alpha_6$ in this polynomial. Likewise, at the seventh stage, the denominator is:

$$P_7 = (\alpha_8\alpha_9 + \alpha_8\alpha_{10} + \alpha_9\alpha_{10})(\alpha_8 + \alpha_{10})$$

and after one more integration, the denominator $P_8$ is the resultant of both factors with respect to $\alpha_8$, which is simply $\alpha_{10}^2$, a perfect square. Hence we get a weight drop at the 8th stage, and the expected transcendental weight is $10-4=6$. 

10.5. **Split triangles and 3-valent vertices.** The denominator reduction behaves well with respect to split triangles and 3-valent vertices, as expected.

**Theorem 134.** Let \( G \) be a graph containing a triangle (resp. 3-valent vertex), and let \( \hat{G} \) be the graph obtained by subdividing that triangle (resp. vertex). Denote the five (resp. three) distinguished edges in \( \hat{G} \) (resp. \( G \)) by \( e_1, \ldots, e_5 \) (resp. \( e_1, e_2, e_3 \)), and let \( e_a, e_b \) be any other two edges in \( G \). Then

\[
P_{e_1, \ldots, e_5, e_a, e_b}(\hat{G}) = P_{e_1, e_2, e_3, e_a, e_b}(G),
\]

and \( \hat{G} \) has a weight drop if and only if \( G \) does.

The proof is similar to lemma [92]. A more general version of splitting triangles, and a combinatorial interpretation of the \( P_n \)'s will be considered in [7]. Splitting triangles or vertices should be the first in a family of operations on graphs which relate the higher invariants \( ^n \Psi_G(e_1, \ldots, e_n) \) to \( ^m \Psi_\gamma(e_1, \ldots, e_m) \), where \( \gamma \) is a minor of \( G \) and \( m < n \).

10.6. **Purity and weight drops.** Putting all the previous elements together:

**Theorem 135.** Suppose that \( G \) is primitive divergent, linearly reducible, has no weight drop, and is unramified. Then the Feynman integral \( I_G \) is of ‘pure’ transcendental weight \( e_G - 3 \) in \( \text{gr} W \mathbb{Z} \).

**Proof.** Repeat the proof of theorem [121] using theorem [125]. \( \square \)

In the case when there is a weight-drop, then we may expect the mixing of weights, by [102]. This was somewhat unexpected, but, indeed, numerical examples of non-pure residues of \( I_G \) have since been found by Schnetz.

10.7. **Towards integrality of the leading term.** The denominator reduction should also give us a handle on the integrality of the Feynman residue. First of all, one can define a \( \mathbb{Z} \)-structure on \( V(\mathcal{M}_{0,n}) \), and hence on \( L(\mathcal{M}_{0,n}) \). The crucial reason for this is that the Drinfel’d associator has integral coefficients in the multiple zeta values. Then, in theorem [125] all statements should also hold over \( \mathbb{Z} \), although we have not checked that all the proofs in [8] go through unchanged. The reason for this is due to the crucial fact that the power in the denominator of \( I_G \), and all subsequent integrals, is 2. This motivates the following definition.

**Definition 136.** Let \( G \) be denominator-reducible with respect to the ordering \( e_1, \ldots, e_N \), and has no weight drop. The **final denominator** \( P_N \) is a non-zero element \( d_G \in \mathbb{Z} \), defined up to sign. It does not depend on this ordering.

Up to checking all the details of the above, we expect in theorem [135] that the coefficients of the multiple zeta values for \( I_G \) can be taken to lie in \( \frac{1}{d_G} \mathbb{Z} \).

**Remark 137.** Almost all primitive divergent graphs up to 7 loops have final denominator 1. Oliver Schnetz has kindly verified that in fact all known residues in \( \phi^4 \) theory evaluate to integer linear combinations of multiple zeta values (personal communication). It also should be possible, by refining the argument above, to give a bound for the heights of the coefficients which occur by counting the number of terms which can occur in the numerators.
11. Relative cohomology of linearly reducible hypersurfaces

Let $X$ be a linearly reducible set of hypersurfaces in $(\mathbb{P}^1)^N$ defined over $\mathbb{Q}$ and let $B \subset (\mathbb{P}^1)^N$ be the coordinate hypercube. Let $L_i = L(S, \pi_i)$ denote the Landau variety of $S = X \cup B$ with respect to $\pi_i : (\mathbb{P}^1)^N \to (\mathbb{P}^1)^{N-i}$. In particular, the map

$$\pi_i : (\mathbb{P}^1)^N \setminus S = L_i \to (\mathbb{P}^1)^{N-i} \setminus L_i$$

has topologically constant fibers we denote by $F_i$.

**Proposition 138.** $\pi_1((\mathbb{P}^1)^{N-i}\setminus L_i)$ acts trivially on $H^*(F_i)$.

**Proof.** It is proved in [13] that if $p : E \to B$ is a fibration with one-dimensional fibers $F$, then $\pi_1(B)$ acts trivially on $H^*(F)$ whenever $p$ admits a section and the map $p_1(F) \to H_1(E)$ is injective. Both conditions hold for the linear fibrations considered in [8,2]. Therefore for any linear projection $\pi : (\mathbb{P}^1)^N \setminus T \to (\mathbb{P}^1)^{N-k-1} \setminus L(T)$ (recall this means that the components of $T$ are of degree at most 1 in the fiber), the higher direct images $R^d\pi_*$ map constant sheaves to constant sheaves.

Assume by induction that the statement is true for $i = k$. Then we can compare

$$\pi_{k+1} : (\mathbb{P}^1)^N \setminus S \to (\mathbb{P}^1)^{N-k-1} \setminus L_{k+1}$$

with the diagram

$$(\mathbb{P}^1)^N \setminus S \setminus \pi_k^{-1}L_k \xrightarrow{\pi} (\mathbb{P}^1)^N \setminus \pi_k^{-1}L_k \cup (\mathbb{P}^1)^{N-k} \setminus \pi^{-1}L(\pi, L_k) \xrightarrow{\pi} (\mathbb{P}^1)^{N-k-1} \setminus L(\pi, L_k)$$

where $\pi : (\mathbb{P}^1)^N \to (\mathbb{P}^1)^{N-k-1}$ is the projection such that $\pi_{k+1} = \pi \circ \pi_k$. By induction hypothesis, $R^d(\pi_k)_*$ maps constant sheaves to constant sheaves, and likewise $R^d\pi_*$, since it is a linear fibration. The same is true for the restriction maps. It follows from a Grothendieck spectral sequence that the compositum also maps constant sheaves to constant sheaves. For all $n \in \mathbb{N}$, $R^n(\pi_{k+1})_*$ maps constant sheaves to locally constant sheaves on $(\mathbb{P}^1)^{N-k-1} \setminus L_{k+1}$, which by the above have trivial monodromy on the open subset $(\mathbb{P}^1)^{N-k-1} \setminus L(\pi, L_k)$. Therefore $R^n(\pi_{k+1})_*$ also maps constant sheaves to constant sheaves. This proves the proposition. □

Now suppose that $P \to (\mathbb{P}^1)^N$ is a blow up of intersections of components of $B$. Let $X'$ be the strict transform of $X$ and $B'$ the total transform of $B$, where $B'$ is normal crossing, as in the case (43) when $X$ is the graph hypersurface of a connected graph. We are interested in the mixed Hodge structure:

$$M_S = H^N(P \setminus X', B' \setminus B' \cap X').$$

Let $j : P \setminus X' \to P \setminus X'$ be the inclusion. Then $M_S = H^N(P \setminus X', j_!\mathbb{Q})$. We wish to apply the Leray spectral sequence to the composed map

$$\pi : P \setminus X' \to (\mathbb{P}^1)^N \xrightarrow{\pi} (\mathbb{P}^1)^{N-i}.$$

Since $P$ blows up only over intersections of components of $B$ (whose vertical components are critical), this is still a locally trivial fibration over the complement of $L_i$. Thus $R^n\pi_*j_!\mathbb{Q}$ is a local system over $(\mathbb{P}^1)^{N-i} \setminus L_i$, for $n \in \mathbb{N}$. Let $P_i, X'_i, B'_i$ denote the fibers of $P, X', B'$ over the generic point of $(\mathbb{P}^1)^{N-i}$. The stalks of $R^n\pi_*j_!\mathbb{Q}$ are isomorphic to $H^n(P_i \setminus X'_i, B'_i \setminus B'_i \cap X'_i)$.

**Corollary 139.** $\pi_1((\mathbb{P}^1)^{N-i} \setminus L_i)$ acts unipotently (with respect to the weight filtration) on the restriction of $R^n\pi_*j_!\mathbb{Q}$ to $(\mathbb{P}^1)^{N-i} \setminus L_i$. 


Proof. Let \( \{W_j \subset P^1\}_{j \in J} \) be the set of horizontal irreducible components of \( B'_i \setminus X' \cap B'_i \) and write \( W_A = \cap_{j \in A} W_j \) for \( A \subseteq J \), and \( W_\emptyset = P^1 \). The same argument as in proposition 138 shows that \( \pi_1((P^1)^{N-i}\setminus L_i) \) acts trivially on \( H^*(W_A) \) for any \( A \subseteq J \). The mixed Hodge structure on \( H^*(P^1\setminus X'_i, B'_i \setminus (B'_i \cap X'_i)) \) is computed from the spectral sequence with \( E^1_{pq} = \bigoplus_{|A|=q} H^p(W_A) \). Since \( \pi_1((P^1)^{N-i}\setminus L_i) \) acts trivially on all \( E^1_{pq} \) terms, it also acts trivially on \( \text{gr}^W H^*(P^1\setminus X'_i, B'_i \setminus (B'_i \cap X'_i)) \), and therefore acts unipotently on \( H^n(P^1\setminus X'_i, B'_i \setminus (B'_i \cap X'_i)) \). \( \square \)

11.1. Mixed Tate cohomology.

Theorem 140. If \( S \) is linearly reducible, then \( M_S \) is mixed Tate.

Proof. The proof is by induction on the dimension \( N \). If \( N = 1 \), then \( S \cup B \subset P^1 \) is a finite set of points, and therefore \( M_S \) is a sum of Kummer motives. In the case \( N > 1 \), apply the Leray spectral sequence to the map

\[
\pi : P^1 \setminus X' \longrightarrow (P^1)^N \setminus \frac{N-1}{P^1}.
\]

which gives \( E^{pq}_{pq} = H^p(P^1, R^q\pi_*j_!Q) \Rightarrow H^{p+q}(P^1 \setminus X', j_!Q) = M_S \). The map \( \pi \) is a locally trivial fibration on the complement of the Landau variety \( S \). By proposition 138, the local system \( R^q\pi_*j_!Q \) on \( (P^1)^{N-i}\setminus L_i \) has unipotent monodromy. If \( \xi \) is the generic point of \( P^1 \), then \( S_\xi = S \cap \pi^{-1}_N(\xi) \) is also linearly reducible, has dimension \( N-1 \), and by induction hypothesis \( M_{S_\xi} \) is mixed Tate. Therefore \( R^q\pi_*j_!Q \) is a unipotent variation of mixed Hodge-Tate structures over \( P^1 \setminus L_{N-1} \), i.e., \( \text{gr}^W R^q\pi_*j_!Q \) is a constant Hodge-Tate variation on \( P^1 \setminus L_{N-1} \).

Now let \( x \in L_{N-1} \). By lemma 80, \( S_x = S \cap \pi^{-1}_{N-1}(x) \) is linearly reducible of dimension \( N-1 \). The stalk of \( R^q\pi_*j_!Q \) at the point \( x \) will not necessarily coincide with \( M_{S_x} \), because of possible exceptional divisors in the fiber. But these are also projective spaces, and their intersection with \( S \) are linearly reducible also. Thus a similar inductive argument proves that \( R^q\pi_*j_!Q \) is mixed Tate over \( L_{N-1} \) as well.

We have shown that \( \text{gr}^W R^q\pi_*j_!Q \) is mixed Tate and constant on \( P^1 \setminus L_{N-1} \), and mixed Tate over points in \( L_{N-1} \). By §14.4 of [17], there is a mixed Hodge structure on \( H^p(P^1 \setminus L_{N-1}, R^q\pi_*j_!Q) \), which is mixed Tate. We can apply the Leray spectral sequence in the category of mixed Hodge structures to conclude that \( M_S \) is mixed Tate, since its \( E_2 \) terms are.

\( \square \)

Remark 141. The previous proof essentially constructs a mixed Tate stratification of the graph hypersurface complement for linearly reducible graphs. The largest open stratum is \( U = (P^1)^N \setminus X \pi^{-1}_{N-1}L_{N-1} \). To see that \( U \) is mixed Tate, it follows from proposition 138 that \( H^*(U) \cong H^*(P^1 \setminus L_{N-1}) \otimes H^*(F_{N-1}) \), and we know that \( H^*(F_{N-1}) \) is mixed Tate by induction hypothesis. All other strata in \( P \), constructed by blowing-up linear spaces in \( \pi \) are complements of graph hypersurfaces of sub and quotient graphs, which are also linearly reducible, and hence mixed Tate by induction. So it follows that \( P \setminus X' \) has a Tate stratification, and one can deduce that \( M_G \) is mixed Tate by a standard argument.

In other situations, it may be the case that only the subquotient of \( M_G \) which carries the period is mixed Tate. A more sophisticated discussion of this case would require introducing a suitable category of equivalence classes of framed mixed Tate sheaves. We hope to return to this question in the future.
11.2. Denominator reduction revisited. We give a cohomological interpretation of the denominator reduction. There are two cases: the non-weight drop and weight-drop cases. In both cases, consider a projection \( \pi : (\mathbb{P}^1)^n \times \mathbb{P}^1 \to (\mathbb{P}^1)^n \), and let \( x \) denote the coordinate in the fiber. Let \( \Omega_n = dx_1 \ldots dx_n \), where \( x_i \) are coordinates on the base, and let \( \Omega_{n+1} = \Omega_n \wedge dx \).

11.2.1. General case. Let \( R \subset (\mathbb{P}^1)^n \times \mathbb{P}^1 \), where \( R = R_1 \cup R_2 \) has exactly two irreducible components which are of degree one in the fiber, and let \( R_x \subset (\mathbb{P}^1)^n \) be the discriminant. The inclusion of the open subset

\[
U = (\mathbb{P}^1)^{n+1} \setminus (R \cup \pi^{-1}R_x) \xrightarrow{i} (\mathbb{P}^1)^{n+1} \setminus R
\]
gives rise to a map

\[
H^{n+1}((\mathbb{P}^1)^{n+1} \setminus R) \xrightarrow{i_*} H^{n+1}(U) \cong H^n((\mathbb{P}^1)^n \setminus R_x) \otimes H^1(G_m)
\]

\[
\frac{\Omega_{n+1}}{(f^1 x + f_1)(g^1 x + g_1)} \mapsto \frac{\Omega_n}{f^1 g_1 - f_1 g^1} \otimes \left[ -\frac{\Omega_n}{f^1 g_1 - f_1 g^1} \right]^{-1}
\]

Since \( H^1(G_m) \cong \mathbb{Q}(-1) \), we can simply write

\[
(106) \quad i_*\left[ \frac{\Omega_{n+1}}{(f^1 x + f_1)(g^1 x + g_1)} \right] = \left[ \frac{\Omega_n}{f^1 g_1 - f_1 g^1} \right]^{-1}
\]

11.2.2. Weight-drop case. Let \( R \subset (\mathbb{P}^1)^n \times \mathbb{P}^1 \), where \( R \) is irreducible of degree one in the fiber, and let \( B_0 : x = 0, B_\infty : x = \infty \), and \( R_0 = R \cap B_0, R_\infty = R \cap B_\infty \). Write \( B^\circ = (B_0 \cup B_\infty) \setminus (R_0 \cup R_\infty) \), and consider the inclusion

\[
V = (\mathbb{P}^1)^{n+1} \setminus (R \cup \pi^{-1}(R_0 \cup R_\infty)) \xrightarrow{i} (\mathbb{P}^1)^{n+1} \setminus R
\]

which gives rise to a map on relative cohomology

\[
H^{n+1}((\mathbb{P}^1)^{n+1} \setminus R, B^\circ) \xrightarrow{i_*} H^{n+1}(V, B^\circ) \cong H^n((\mathbb{P}^1)^n \setminus (R_0 \cup R_\infty)) \otimes H^1(A^1, \{0, \infty\})
\]

\[
\frac{\Omega_{n+1}}{(f^1 x + f_1)^2} \mapsto \frac{\Omega_n}{f^1 f_1} \otimes \left[ -\frac{dy}{(y + 1)^2} \right]
\]

after changing fiber coordinate \( y = \frac{f^1}{f_1} x \). Since \( H^1(A^1, \{0, \infty\}) \cong \mathbb{Q}(0) \), we write:

\[
(107) \quad i_*\left[ \frac{\Omega_{n+1}}{(f^1 x + f_1)^2} \right] = \left[ \frac{\Omega_n}{f^1 f_1} \right](0)
\]

Note that the form \( \frac{dy}{(y + 1)^2} \) is exact in absolute cohomology.

Now apply this argument to a (primitive-divergent) Feynman integrand

\[
\left[ \frac{\Omega_N}{\Psi^2} \right] \in H^N((\mathbb{P}^1)^N \setminus X_G, (B_2 \cup X_G))
\]

where \( B_2 \) is the part of the hypercube given only by \( \alpha_1, \alpha_3 = 0, \infty \). Applying (106), (107) in turn gives a cohomological version of the denominator reduction algorithm. Heuristically, and dropping all \( i_*'s \) from the notation, we get:

\[
(108) \quad \left[ \frac{\Omega_N}{\Psi^2} \right] \mapsto \left[ \frac{\Omega_{N-1}}{\Psi^1 \Psi_1} \right](0) \mapsto \left[ \frac{\Omega_{N-2}}{\Psi^1 \Psi_1^2} \right]^{-1} \mapsto \left[ \frac{\Omega_{N-3}}{\Psi^1 \Psi_1^3 \Psi_2} \right](-1) \mapsto \left[ \frac{\Omega_{N-4}}{\Psi^1 \Psi_1^4 \Psi_2^2} \right](-2) \mapsto \left[ \frac{\Omega_{N-5}}{\Psi^1 \Psi_1^5 \Psi_2^3 \Psi_3} \right](-3) \mapsto \left[ \frac{\Omega_{N-k}}{\Psi^1 \Psi_1^6 \Psi_2^4 \Psi_3^2 \Psi_4} \right](2-k) \mapsto \ldots
\]
where $P_k$ is the $k^{th}$ term in the denominator reduction. Unfortunately, the maps $i_k$ may lose some information, and this calculus only yields information about the Hodge type of the Feynman integrand after restricting it to some open subset.

More precisely, let $P_k$ be the $k^{th}$ denominator in the sequence \((108)\), for $k \geq 1$, e.g., $P_1 = \Psi^2_1$. As usual, let $\pi : (\mathbb{P}^1)^N \to (\mathbb{P}^1)^{N-i}$ project out the first $i$ Schwinger parameters, and let

$$Y_G = X_G \cup \pi^{-1}V(P_1) \cup \ldots \cup \pi^{-1}V(P_k).$$

The above argument can be used to compute the position of $\left[\frac{\Omega_N}{\Psi^2}\right] \in H^N((\mathbb{P}^1)^N \setminus Y_G, B_2 \setminus (B_2 \cap Y_G))$ in the Hodge and weight filtrations (see §12.7). For example, one can take the denominator reduction of the 8-loop graph computed in [6], §6, and by performing all the reductions and changes of variables therein, one arrives at a Tate twist of the canonical form of a singular K3 surface, which is of type $(2,0)$. It is these kind of considerations which motivate conjecture 149.

Remark 142. This argument, as it stands, is not quite sufficient to deduce general results about the framing in the graph motive $M_G$, but we expect it to be valid in most cases. In the special case of a linearly reducible non-weight drop graph $G$ with $N$ edges, the argument of §11 should suffice to construct a framing

$$Q(3-N) \to gr^W_{N-q}M_G$$

since contributions of lower-dimensional strata only affect the lower weight-graded pieces of the graph motive. For the non-Tate counterexamples of §12.7, one needs only show that the lower-dimensional strata are Tate to deduce that the framing is non-Tate. Again, we expect this to be true for these examples.

Yet another way around this problem, suggested by S. Bloch, is if the denominators $D_k$ all have positive coefficients, for then one can pass to the smaller open subset $(\mathbb{P}^1)^N \setminus Y_G \hookrightarrow (\mathbb{P}^1)^N \setminus X_G$ without affecting the period. But one is still left with the problem that $(\mathbb{P}^1)^N \setminus Y_G$ is only an open in the full blown-up motive. The positivity assumption is in fact possible for certain non-trivial graphs such as $K_{3,4}$.

In short, the question of framings merits a more detailed analysis.

Remark 143. A third interpretation of the denominator reduction is in terms of the point-counts of the graph hypersurface $X_G$ over finite fields $\mathbb{F}_q$. In [6], we show that for any connected graph $G$ satisfying $2h_G \leq N_G$ and $N_G \geq 5$,

$$X_G(\mathbb{F}_q) \equiv \pm q^2 V(P_k)(\mathbb{F}_q) \mod q^3$$

where $q$ is any prime power, and $P_k$ for $k \geq 5$, is any term in the denominator reduction of $G$, with respect to any ordering.

12. SOME CRITICAL GRAPHS

We discuss some examples of graphs in increasing order of complexity.

12.1. SOME CRITICAL MINORS FOR NON-SPLIT 5-INVARINTS. Let $G$ be a connected simple graph, and $i, j, k, l, m \in E_G$. Recall that the splitting of the five-invariant $5\Psi_Gijklm$ is a minor-monotone property, and this occurs if \{i, j, k, l, m\} contains a triangle or 3-valent vertex by lemma 92.
Lemma 144. Let \( i, j, k, l, m, n \) be any 6 edges in a graph \( G \). If \( i, j, n \) forms a triangle, then \( \Psi(ijklm) \) is divisible by \( \alpha_n \). If \( i, j, n \) forms a 3-valent vertex, then \( \Psi(ijklm) \) is of degree at most 1 in \( \alpha_n \).

Proof. Suppose that \( i, j, n \) forms a triangle. Then \( i, j \) forms a loop in the quotient graph \( G/\alpha_n \). By lemma 92 we have \( 5\Psi_G(ijklm)|_{\alpha_n=0} = 5\Psi_G(ijklm) \) = 0. It follows that \( \alpha_n \) divides \( 5\Psi_G(ijklm) \). The other case is similar.

Similar results hold for the higher invariants too. The smallest simple graphs with an irreducible 5-invariant are \( K_5 \) (fewest vertices) and \( K_{3,3} \) (fewest edges):

With these numberings, one can check that:

\[
5\Psi_{K_5}(1, 3, 4, 5, 8) = \alpha_2\alpha_6\alpha_7\alpha_{10}(\alpha_6\alpha_9 + \alpha_9\alpha_2\alpha_6 + \alpha_2\alpha_9\alpha_{10} + \alpha_2\alpha_7\alpha_9 + \alpha_2\alpha_7\alpha_{10})
\]

\[
5\Psi_{K_{3,3}}(1, 2, 4, 6, 8) = \alpha_5\alpha_7^2 + \alpha_3\alpha_5\alpha_9 + \alpha_5\alpha_7\alpha_9 + \alpha_3\alpha_5\alpha_7 - \alpha_3\alpha_7\alpha_9
\]

The only other sets of 5 edges (up to symmetries) which do not contain a triangle or 3-valent vertex are \( 1, 4, 5, 8, 10 \) (for \( K_5 \)), and \( 1, 2, 4, 5, 9 \) (for \( K_{3,3} \)), and these give rise to 5-invariants which do not split, but factorize into terms of degree at most \( 1, \ldots, 1 \) nonetheless. Since \( \{K_5, K_{3,3}\} \) are the critical minors for the set of planar graphs (Wagner’s theorem) we obtain:

Corollary 145. Every non-planar graph contains an irreducible 5-invariant.

Unfortunately, there also exist planar graphs with irreducible 5-invariant. Consider the graph \( G \) with seven vertices, and edges \( e_1, \ldots, e_{12} \) given by:

\[
\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{6, 7\}, \{7, 1\}, \{1, 3\}, \{1, 4\}, \{2, 6\}, \{3, 5\}, \{4, 7\}
\]

where \( \{i, j\} \) denotes an edge connecting vertex \( i \) and vertex \( j \). It is planar primitive divergent with six loops. One can check that the five-invariant \( 5\Psi_G(e_1, e_2, e_3, e_4, e_6) \) is irreducible, and quadratic in the variables \( e_7, e_9, e_{12} \). All other planar primitive divergent graphs at 6 loops or less have five-invariants which either split or factorize into terms which are linear in every variable.

12.2. Linear reducibility in \( \phi^4 \) up to 6 loops. It turns out that all the above examples are in fact in linearly reducible, despite having a non-trivial 5-invariant.

Lemma 146. The vertex-width 4 graphs \( K_5 \) and \( K_{3,3} \) are linearly reducible.

The proof is by calculation, made plausible by the fact that the non-split 5-invariants above are of degree 1 in almost all the Schwinger parameters. The single case \( K_{3,3} \) is enough to prove that all primitive divergent graphs in \( \phi^4 \) theory up to
six loops (with the sole exception of \(K_{3,4}\) considered below) are linearly reducible, since by applying corollaries 95 and 91 we either reduce to the trivial graph or \(K_{3,3}\).

**Corollary 147.** All primitive-divergent graphs in \(\phi^4\) theory up to 6 loops, bar \(K_{3,4}\), are linearly reducible (this is the main result of [9]).

12.3. **The case \(K_{3,4}\).** Consider the bipartite graph \(K_{3,4}\) with the given edge labelling. It has vertex width 4. Since the edges 1, 2, 3 form a three-valent vertex, the first five-invariant \(5\Psi(1,2,3,4,5)\) splits. Likewise, because 4, 5, 6 form a second 3-valent vertex disjoint from the first, all five-invariants \(5\Psi(i,j,k,l,m)\) where \(\{i,j,k,l,m\} \subset \{1,\ldots,6\}\) must split too. The first problem must occur at the 7th reduction, and indeed, \(5\Psi(1,2,4,5,7)\) is irreducible and quadratic. One checks that it is linear in \(\alpha_{10}\), so we can pass to \(K_{3,4}/\alpha_{10}\). Direct computation gives:

\[
7\Psi_{K_{3,4}/10}(1,\ldots,7) = \alpha_{11}\alpha_{12}(\alpha_8^2\alpha_{12} + \alpha_9^2\alpha_{11} + \alpha_8\alpha_{11}\alpha_{12} + \alpha_9\alpha_{11}\alpha_{12}).
\]

By the symmetry of \(K_{3,4}\) which interchanges \((7,8,9) \leftrightarrow (10,11,12)\), we also have:

\[
7\Psi_{K_{3,4}/7}(1,\ldots,6,10) = \alpha_8\alpha_9(\alpha_8\alpha_{12}^2 + \alpha_9\alpha_{11}^2 + \alpha_8\alpha_{11}\alpha_{12} + \alpha_9\alpha_{12}\alpha_{11}).
\]

Thus, after the 8th stage of reduction, we find that \(S_{[1,\ldots,7,10]}(K_{3,4})\) contains two distinct quadratic terms (in fact, one can show that all other terms are Dodgson polynomials), and there remain no more variables with respect to which every term is linear. Thus \(K_{3,4}\) is the first example of a primitive-divergent graph in \(\phi^4\) which is not linearly reducible.

Its motive is surely mixed Tate, since (after setting, say \(\alpha_{12} = 1\)) at the 9th stage we obtain a finite cover of \(\mathbb{P}^1 \times \mathbb{P}^1\setminus L_9\), and this surface can continue to be fibered in curves of genus 0. Thus we expect the methods of §11 to generalize to this, and similar families (but not all) graphs of vertex width 4.

12.4. **Splitting.** There is a completely different way to deal with the non-linear reducibility of \(K_{3,4}\), which should be more practical for computations. The idea is that, since \(109\) and \(110\) are not compatible, one should be able to split \(T = S_{[1,\ldots,7,10]}(K_{3,4})\) into two sets \(T_1 \cup T_2\) such that \(T_1\) contains only Dodgson polynomials and \(109\), and \(T_2\) contains only Dodgson polynomials and \(110\). Then \(T_1, T_2\) are individually linearly reducible (with respect to different orderings). Analytically, this corresponds to the following, in the case of the residue \(I_G\). We easily check that \(K_{3,4}\) is denominator reducible, and that its denominator at the 8th stage is \(P_8 = (\alpha_8\alpha_{12} - \alpha_9\alpha_{11})^2\) (in particular, \(K_{3,4}\) has weight drop and the expected
weight is $12 - 4 = 8$. The period is identified numerically with $\zeta(5,3) - \frac{29}{12}\zeta(8)$ [2]. The idea is to split the integral at the 8th stage into a sum of two pieces

$$I_G = \int \frac{f_1}{P_8} + \int \frac{f_2}{P_8},$$

where $f_i$ only contains one of the quadratic terms above. Then, after regularizing if necessary, each piece of the integral is linearly reducible and can be treated with a different order of integration. We have not carried this out explicitly, but we expect that the idea translates naturally to the context of framed mixed Hodge structures and generalizes to certain families of graphs with non-trivial local 5-invariants.

Thus the first serious obstruction to being non-Tate is likely to come from the failure of denominator reducibility, which first occurs in $\phi^4$ at 7 loops.

12.5. Totally quadratic denominators. By direct computation, every primitive-divergent graph in $\phi^4$ at 7 loops is denominator-reducible, except for a single example. The graph (called $Q48$ in [2]) with vertices 1, . . . , 8 and edges \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{6, 7\}, \{7, 8\}, \{8, 1\}, \{1, 3\}, \{1, 5\}, \{2, 6\}, \{3, 7\}, \{4, 6\}, \{4, 8\}, leads to an irreducible denominator which is of degree two in every variable, for all possible orderings on the set of edges. For example, reducing in the order 1, 2, 3, 4, 9, 13, 5, 10, 11, 7 leads to the denominator

$$P_{10} = \alpha_8^2\alpha_6\alpha_1 + 3\alpha_6\alpha_8\alpha_{12}\alpha_{14} - \alpha_1^2\alpha_2\alpha_4^2.$$ 

This graph is one of two examples at 7 loops whose period remains unidentifiable by numerical methods, at the time of writing. By inspection of $P_{10}$ we suspect that one can show, as for $K_{3,4}$, that this graph defines a mixed Tate motive. Indeed, by changing variables, one easily shows that $P_{10}$ defines a Tate variety. This takes us to the limit of all presently known analytic and experimental numerical data.

12.6. Non-trivial higher invariants and non-Tate examples. It is easy to construct graphs with non-trivial higher invariants $\Psi(e_1, \ldots, e_n)$, for any $n$, by considering complete graphs $K_n$ or bipartite graphs $K_{n, b}$, which can in turn be promoted to $\phi^4$ primitive divergent graphs using the methods from §2. The first interesting 6-invariants in primitive $\phi^4$ theory occur at the 8 loop level. Indeed, the graphs obtained by deleting a vertex in the last 5 graphs $P_{8,37}, \ldots, P_{8,41}$ of the census [21] are not denominator reducible. Since writing the first versions of this paper, we studied in detail the example $G_{8,37}$ obtained by deleting a vertex in $P_{8,37}$ in [6]. It has vertices 1, . . . , 9 and edges

12, 13, 14, 25, 27, 34, 58, 78, 89, 59, 49, 47, 35, 36, 67, 69

where $ij$ denotes an edge connecting vertex $i$ and $j$. Its denominator reduction is carried out explicitly in [6] and leads to an irreducible polynomial which is totally quadratic and not of mixed Tate type. Indeed, we prove that its point-counting function is given by a singular K3 surface with complex multiplication by $\mathbb{Q}(\sqrt{-7})$. This counter-example has vertex weight 4. In [6], we also exhibit a planar example at 9 loops (but higher vertex-width) which has the same point-counting function. Thus, the fact that non-Tate examples occur at exactly the point where the methods of this paper fail suggests that linear reducibility is a complete explanation for mixed Tate motives in $\phi^4$ theory. We expect to see the first non-MZV Feynman amplitudes to start to appear at 8 loops.
12.7. Reduction to Calabi-Yau varieties. Let $G$ be a primitively divergent, ordered graph. The denominator reduction provides a sequence of polynomials

$$P_5 = 5\Psi_G(1, 2, 3, 4, 5), P_6, \ldots, P_k$$

where each $P_i$ has the ‘Calabi-Yau’ property of having degree equal to the number of remaining Schwinger variables, namely $2h_G - i$. Suppose that $G$ does not have weight drop, and is not denominator reducible. Then the denominator reduction terminates with a polynomial $P_k$ which is totally quadratic, and typically in far fewer variables than $\Psi_G$. One then has a hope of counting the points of $P_k$ over $\mathbb{F}_q$ modulo $q$, which is well-defined, since it is independent of $k$ and of the chosen ordering on the $G$ (this is the $c_2(G)$ invariant of $[6]$). In $\phi^4$ up to 7 loops we find that all graphs are denominator reducible or potentially Tate, but at 8 loops we also have point counts determined by the coefficients of the following modular forms:

| 4-regular completed graph | Modular Form | Weight | Level |
|--------------------------|--------------|--------|-------|
| $P_{3,39}$               | $\eta(q)\eta(q^4)\eta(q^7)\eta(q^8)^2$ | 3      | 8     |
| $P_{3,37}$               | $(\eta(q)\eta(q^7))^3$               | 3      | 7     |
| $P_{3,38}$               | $(\eta(q)\eta(q^5))^4$               | 4      | 5     |
| $P_{3,41}$               | $(\eta(q)\eta(q^3))^6$               | 6      | 3     |

The point counts are obtained from the completed graphs on the left (as listed in [21]) by deleting a vertex. Conjecturally ([22], [6]), the choice of deleted vertex does not affect the $c_2$-invariant. The entry for $P_{3,37}$ has been proved [6], but the rest were found experimentally with O. Schnetz by counting points, and are expected to yield counterexamples to Kontsevich’s conjecture. The modular forms should correspond to the framing of the graph motives by the kind of argument sketched in §11.2. We are therefore led to make the following definitions.

**Definition 148.** Let $[\omega_G]$ denote the relative de Rham class of the Feynman integrand in $M_G \otimes \mathbb{Q} \otimes \mathbb{C} = H^{N-1}(P_G \setminus X_G, B' \cap X_G; \mathbb{C})$ (see [51]). The mixed Hodge structure $M_G \otimes \mathbb{Q}$ has Hodge and weight filtrations $F_\ast, W^\ast$. Let

$$w(G) = \min\{k : [\omega_G] \in W_k M_G\}$$
$$b(G) = \max\{k : [\omega_G] \in F^k(M_G \otimes \mathbb{Q} \otimes \mathbb{C})\}$$

We define the (motivic) weight drop to be the quantity

$$wd(G) = 2N_G - w(G) - 6 \geq 0,$$

where $N_G$ denotes the number of edges of $G$, and the Tate defect to be

$$td(G) = 2b(G) - w(G).$$

The results on the transcendental weights [7] suggest that $wd(G)$ can be arbitrarily high, and, in particular, increases for every double triangle contained in a graph. If a Feynman diagram is mixed Tate, then $td(G) = 0$. By §11.2 we expect the non-Tate counter-examples above to satisfy $td(G) > 0$.

**Conjecture 149.** There exists a sequence of primitive-divergent graphs $G_i$ in $\phi^4$ such that $td(G_i) \to \infty$. Furthermore, the $G_i$ can be taken to be planar.

Since our criteria for graphs to be mixed Tate are combinatorially restrictive, we expect that the proportion of primitive-divergent $\phi^4$ graphs $G$ satisfying $wd(G) = td(G) = 0$ at loop order $n$ tends to zero as a function of $n$. In particular, there should be asymptotically few multiple zeta value graphs of maximal weight.
12.8. **Towards a classification.** The following picture summarizes some of the minor-monotone classes of graphs we have considered in this paper.

![Graph Classification Diagram](image)

In conclusion, the hypersurfaces of graphs in the inner onion rings fiber in curves of genus 0 and therefore give mixed Tate motives and multiple zetas. This gives a geometric and combinatorial explanation for the numerical Feynman amplitude computations in the physics literature. Beyond the outer onion ring, we find non-denominator reducible graphs which give non-Tate counterexamples to Kontsevich’s conjecture, and we expect them to have non-MZV amplitudes.

12.9. **Some open questions.**

1. What is the precise relationship between the combinatorics of $G$ and the existence of identities between the polynomials $\Psi_{G,K}^{I,J}$ or the higher invariants $\Psi_{G\setminus I\setminus K}(e_1,\ldots,e_n)$? There is an associated reconstruction problem: to what extent does the vanishing of these polynomials determine the graph?

2. One can show by conformal transformations that two graphs $G_1$, $G_2$ have the same residue if adding an apex to the four 3-valent vertices in $G_1$ and $G_2$ gives isomorphic graphs [21]. Interpret this using graph polynomials.

3. Extend our main results to classes of graphs of vertex width 4 by allowing quadratic terms or by splitting the motives/linear reduction (cf [12,30,12,31]).

4. By splitting triangles, one can generate infinite families of graphs $G$ in $\phi^4$ of vertex width 3. Not counting isomorphisms, these number at least $2^n$. The number of MZVs of weight $n$ grows approximately like $(4/3)^n$, so there must exist many identities between the residues $I_G$. Generate such identities by finding relations between their partial Feynman integrals.

5. Can one compute the motivic coproduct [15] for graphs of matrix type? Certain families of graphs should be highly constrained in the coradical filtration, which would explain the prevalence of certain linear combinations of MZVs in the Feynman integral calculations known to date [2, 21].

6. Use the fibrations (79) to relate the polynomial point counts of $X_G$ over finite fields to the combinatorics of $G$, for $G$ of matrix type.

7. For which (ordered) graphs is it true that its Landau varieties are contained in the zero locus of the Dodgson polynomials $\Psi_{G,K}^{I,J}$ and the higher graph invariants $\Psi_{\gamma}(e_1,\ldots,e_n)$ as $\gamma$ ranges over the minors of $G$?

8. Sum the total contribution in the perturbative expansion of a family of graphs obtained by, say, splitting triangles.
(9) What proportion of graphs in φ⁴ theory have a weight drop or positive Tate defect? What is the physical significance of the ‘maximal Hodge or weight’ part of the perturbative expansion in a quantum field theory? What are the implications for the radius of convergence of the perturbative series generated by primitive graphs?

(10) The zig-zag graphs are known to evaluate to rational multiples of ζ(n) [2], but on the other hand should be integral linear combinations of MZVs. Therefore, let Z ⊂ R denote the Z-module spanned by all multiple zeta values ζ(n₁, ..., n_r). For what prime powers q is ζ(n) ∈ qZ?

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