Economic Production Quantity Model With Generalized Pareto Rate of Production and Weibull Decay Having Selling Price Dependent Demand

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Abstract

Economic production quantity models usually assume that the production is fixed and finite. However, due to various random factors effecting the production, the production process becomes random. This paper deals with the development and analysis of economic production quantity model in which the production is random and follows a generalized Pareto distribution. The generalized Pareto distribution is capable of including different types of production rates. Here it is further assumed that the lifetime of the commodity is random and follows a two parameter Weibull distribution. The Weibull decay includes constant, increasing and decreasing rates of deterioration. It is also assumed that the demand is dependent on selling price. Assuming that shortages are allowed and fully backlogged the instantaneous state of on hand inventory is derived. With suitable cost considerations the total cost function and profit rate function are obtained and minimized with respect to the production uptime and downtime. The optimal production uptime, downtime, production quantity and selling price are derived. A numerical illustration demonstrating the solution procedure of the model is presented. The sensitivity analysis of the model revealed that the production and deteriorating distributions parameters have significant influence on the optimal production schedule and production quantity. This model is extended to the case of without shortages. This model also includes some of the earlier models as particular cases for specific or limiting values of the parameters.

Key words : EPQ model, Generalized Pareto rate of production, Selling price dependent demand, Random production, Weibull decay.

Subject Classification Code: 90 59- Operations Research

1 Introduction

In classical production inventory models it is customary to assume that the demand is constant. But,
in many production systems the demand is a function of selling price. Recently much work has been reported in literature regarding selling price dependent demand. In addition to demand another important factor of production level inventory model or EPQ models for deteriorating items is a common phenomenon for scheduling production systems.

Begum et al.\textsuperscript{2} and Tripathy et al.\textsuperscript{15} considered Weibull rate of deterioration. Srinivasa Rao et al.\textsuperscript{13} developed and analyzed an inventory model for deteriorating items with generalized Pareto decay. Srinivasa Rao et al.\textsuperscript{14} considered the case of additive exponential lifetime. Roy et al.\textsuperscript{10} introduced an order level inventory model for a deteriorating item, taking the demand rate to be dependent on the sale price of the item and incorporating the concept of the special sale campaign by way of price of reduction is into the model. Madhavi et al.\textsuperscript{7} developed an inventory model with the assumption that demand is a function of selling price. lifetime of the item is random and follows two parameter exponential distribution and that the deteriorated items are kept in the inventory for second sale. Inventory models for deteriorating items with multivariate demand functions were studied by some authors. Chen et al.\textsuperscript{3} studied an inventory model with a multivariate demand function of price and time. Khanra et al.\textsuperscript{4} studied models for deteriorating items having stock level and selling price dependent demand rate. Urban et al.\textsuperscript{16} studied an inventory model in which demand is a function of price, time and inventory level. Kousar Jaha Begum et al.\textsuperscript{5} developed an E.P.Q model with the assumptions that the lifetime of commodity is random and follow a Generalized Pareto Distribution and is assumed that demand is a function of both the time and selling price. Ajay Kumar Agarwal et al.\textsuperscript{1} studied an inventory model with variable demand rate for deteriorating items under Permissible Delay in Payments. Varsha Sharma et al.\textsuperscript{17} studied an inventory model for deteriorating products having demand which is function of selling price. Maragatham et al.\textsuperscript{8} and Raman Patel et al.\textsuperscript{9} developed an inventory model with time and price dependent demand.

In all these models it is assumed that the production is finite and constant rate. But in many practical situations the production process is random due to various random factors such as availability of raw material, manpower, power supply, breakdowns etc. Hence recently Sridevi et al.\textsuperscript{11}, Srinivasa Rao et al.\textsuperscript{12} and Lakshmana Rao et al.\textsuperscript{6} have developed production level inventory models with the assumption that the production process is random and follows Weibull distribution. Very little work has been reported regarding EPQ models with selling price dependent demand having Weibull rate of decay and generalized Pareto rate of production. Hence in this paper we fill the gap in this area of research by developing and analyzing an EPQ model with generalized Pareto rate of production and Weibull decay having selling price dependent demand. The generalized Pareto rate of production includes constant and time dependent rates of productions as particular cases.

Using the differential equations the instantaneous state of inventory is derived. The total cost function and profit rate function under suitable cost considerations assuming shortages which are fully backlogged are derived. The optimal production uptime and downtime are obtained by minimizing the profit rate function and optimal production quantity and optimal selling price are derived. The sensitivity analysis of the model is included to study the changes in input variables and costs. This model is extended to the case of without shortages.

2 Assumptions:
For developing the model the following assumptions are made:

i) The demand rate is selling price dependent demand. Say \( \lambda(s) = a - bs \) \hspace{1cm} (1)
where ‘a’ and ‘b’ are constants and ‘s’ is selling price.

ii) The production is finite and follows a Generalized Pareto distribution. The instantaneous rate of production is
\[ K(t) = \frac{1}{\alpha - \gamma t} ; 0 < t < \frac{\alpha}{\gamma} \]  

(2)

iii) Lead time is zero.

iv) Cycle length is T. It is known and fixed.

v) Shortages are allowed and fully backlogged.

vi) A deteriorated unit is lost.

vii) The lifetime of the item is random and follows a two parameter Weibull distribution with probability density function

\[ f(t) = \theta \eta t^{\eta-1} e^{-\theta t^\eta} ; \theta, \eta > 0, \ t > 0 \]

Therefore the instantaneous rate of deterioration is

\[ h(t) = \frac{f(t)}{1-F(t)} = \theta \eta t^{\eta-1} ; \theta, \eta > 0, \ t > 0 \]  

(3)

The following notations are used for developing the model.

Q: Production quantity.

A: Setup cost.

C: Cost per unit.

h: Inventory holding cost per unit per unit time.

\( \pi \): Shortages cost per unit per unit time.

3 EPQ Model with Shortages:

Consider a production system in which the stock level is zero at time \( t = 0 \). The stock level increases during the period \((0, t_1)\), due to production after fulfilling the demand and deterioration. The production stops at time \( t_1 \) when stock level reaches \( S \). The inventory decreases gradually due to demand and deterioration in the interval \((t_1, t_2)\). At time \( t_2 \) the inventory reaches zero and back orders accumulate during the period \((t_2, t_3)\). At time \( t_3 \), the replenishment again starts and fulfils the backlog after satisfying the demand. During \((t_3, T)\) the backorders are fulfilled and inventory level reaches zero at the end of the cycle \( T \). The schematic diagram representing the instantaneous state of inventory is given in Figure 1.

\[
\frac{d}{dt} I(t) + h(t)I(t) = \frac{1}{\alpha - \gamma t} - (a - bs) ; \quad 0 \leq t \leq t_1
\]  

(4)
\[
\frac{d}{dt} I(t) + h(t) I(t) = -(a - bs) \quad ; \quad t_1 \leq t \leq t_2
\]  
(5)

\[
\frac{d}{dt} I(t) = -(a - bs) \quad ; \quad t_2 \leq t \leq t_3
\]  
(6)

\[
\frac{d}{dt} I(t) = \frac{1}{a - \eta t} - (a - bs) \quad ; \quad t_3 \leq t \leq T
\]  
(7)

where, \(h(t)\) is as given in equation (3), with the initial conditions \(I(0) = 0, I(t_1) = S, I(t_2) = 0\) and \(I(T) = 0\).

Substituting \(h(t)\) in equations (4) and (5) and solving the differential equations, the on hand inventory at time 't' is obtained as

\[
I(t) = S e^{\theta(t_1^\eta - t^\eta)} - e^{-\theta t^\eta} \int_t^{t_1} \left(\frac{1}{a - \eta u} - (a - bs)\right) e^{\theta u^\eta} \, du \quad ; \quad 0 \leq t \leq t_1
\]  
(8)

\[
I(t) = S e^{\theta(t_1^\eta - t^\eta)} - (a - bs)e^{-\theta t^\eta} \int_{t_1}^{t} e^{\theta u^\eta} \, du \quad ; \quad t_1 \leq t \leq t_2
\]  
(9)

\[
I(t) = (a - bs)(t_2 - t) \quad ; \quad t_2 \leq t \leq t_3
\]  
(10)

\[
I(t) = \frac{1}{\gamma} \log \left(\frac{a}{a - \eta t_3}\right) + (a - bs)(T - t) \quad ; \quad t_3 \leq t \leq T
\]  
(11)

Production quantity \(Q\) in the cycle of length \(T\) is

\[
Q = \frac{1}{\gamma} \log \left(\frac{a}{a - \eta t_3}\right)
\]  
(12)

From equation (8) and using the initial condition \(I(0) = 0\), we obtain the value of 'S' as

\[
S = e^{-\theta t_1^\eta} \int_0^{t_1} \left(\frac{1}{a - \eta u} - (a - bs)\right) e^{\theta u^\eta} \, du
\]  
(13)

When \(t = t_1\), then equations (10) and (11) become

\[
I(t_3) = (a - bs)(t_2 - t_3) \quad and
\]  
(14)

\[
(t_3) = \frac{1}{\gamma} \log \left(\frac{a - \eta T}{a - \eta t_3}\right) + (a - bs)(T - t_3) \quad respectively.
\]  
(15)

Equating the equations (14) and (15) and on simplification, one can get

\[
t_2 = T + \frac{1}{\gamma(a - bs)} \log \left(\frac{a - \eta T}{a - \eta t_3}\right) = \chi(t_3, s) \quad (say)
\]  
(16)

Let \(K(t_1, t_2, t_3, s)\) be the total production cost per unit time. Since the total production cost is the sum of the set up cost, cost of the units, the inventory holding cost. Hence the total production cost per unit time becomes

\[
K(t_1, t_2, t_3, s) = \frac{A}{T} + \frac{C}{\gamma T} \log \left(\frac{a}{a - \eta t_1}\right) (a - \eta T)
\]

\[
+ \frac{h}{T} \left[ S e^{\theta(t_1^\eta - t^\eta)} - e^{-\theta t^\eta} \int_t^{t_1} \left(\frac{1}{a - \eta u} - (a - bs)\right) e^{\theta u^\eta} \, du \right] dt
\]

\[
+ \int_{t_1}^{t_2} \left[ S e^{\theta(t_1^\eta - t^\eta)} - (a - bs)e^{-\theta t^\eta} \int_t^{t_1} e^{\theta u^\eta} \, du \right] dt
\]

\[
+ \int_{t_1}^T \left[ S e^{\theta(t_1^\eta - t^\eta)} - (a - bs)e^{-\theta t^\eta} \int_t^{t_1} e^{\theta u^\eta} \, du \right] dt
\]
\[ P(t_1, t_3, s) = s(a - bs) - \frac{A}{T} = \frac{C}{\gamma T} \log \left( \frac{\alpha (a - \gamma t_3)}{(a - \gamma t_1)(a - \gamma T)} \right) \]

\[-\frac{h}{T} \left[ \int_0^{x(t_1, s)} e^{-\theta t\gamma} \int_0^{t_1} \left( \frac{1}{\alpha - \gamma u} - (a - bs) \right) e^{\theta u\gamma} du \right] dt \]

\[-\int_0^{t_1} \left[ e^{-\theta t\gamma} \int_t^{t_1} \left( \frac{1}{\alpha - \gamma u} - (a - bs) \right) e^{\theta u\gamma} du \right] dt \]

\[-(a - bs) \int_{t_1}^{x(t_3, s)} \left[ e^{-\theta t\gamma} \int_{t_1}^t e^{\theta u\gamma} du \right] dt \]

\[-\frac{\pi}{Ty} \left[ t_3 - T + \left( T - t_3 - \frac{1}{\gamma} (a - \gamma t_3) + \frac{1}{2\gamma (a - bs)} \log \left( \frac{a - \gamma T}{a - \gamma t_1} \right) \right) \log \left( \frac{a - \gamma T}{a - \gamma t_3} \right) \right] \]

4 Optimal Ordering Policies of the Model with Shortages:

In this section we obtain the optimal policies of the system under study. To find the optimal values of \( t_1, t_3 \) and \( s \), we obtain the first order partial derivatives of \( P(t_1, t_2, t_3, s) \) given in equation (18) with respect to \( t_1, t_3 \) and \( s \) and equate them to zero. The condition for minimization of \( P(t_1, t_3, s) \) is

\[ D = \begin{bmatrix}
\frac{\partial^2 P(t_1, t_3, s)}{\partial t_1^2} & \frac{\partial^2 P(t_1, t_3, s)}{\partial t_1 \partial t_3} & \frac{\partial^2 P(t_1, t_3, s)}{\partial t_1 \partial s} \\
\frac{\partial^2 P(t_1, t_3, s)}{\partial t_1 \partial t_3} & \frac{\partial^2 P(t_1, t_3, s)}{\partial t_3^2} & \frac{\partial^2 P(t_1, t_3, s)}{\partial t_3 \partial s} \\
\frac{\partial^2 P(t_1, t_3, s)}{\partial t_1 \partial s} & \frac{\partial^2 P(t_1, t_3, s)}{\partial t_3 \partial s} & \frac{\partial^2 P(t_1, t_3, s)}{\partial s^2}
\end{bmatrix} < 0 \]

Where, \( D \) is the Hessian matrix.

Differentiating \( P(t_1, t_3, s) \) with respect to \( t_1 \) and equating to zero, we get

\[ \frac{C}{\alpha - \gamma t_1} + he^{\theta t_1} \left[ \left( \frac{1}{\alpha - \gamma t_1} - (a - bs) \right) \int_0^{x(t_1, s)} e^{-\theta t\gamma} dt - \int_0^{t_1} e^{-\theta t_1\gamma} dt \right] \]

\[ + (a - bs) \int_{t_1}^{x(t_3, s)} e^{-\theta t\gamma} dt = 0 \]
Differentiating $P(t_1, t_3, s)$ with respect to $t_3$ and equating to zero, we get
\[
C(\alpha - \gamma T) - h e^{-\theta(x(t_3, s))} \left[ \frac{1}{(a - bs)} \int_{0}^{t_1} \left( \frac{1}{(a - bs)} - (a - bs) \right) e^{\theta u} du \right. \\
\left. + \frac{a}{y} \left[ (a - \gamma t_3)(1 - \alpha + \gamma T) - \gamma(T - t_3) \right] - \frac{1}{a - bs} \log \left( \frac{a - \gamma T}{a - \gamma t_3} \right) \right] = 0
\]  
(20)

Differentiating $P(t_1, t_3, s)$ with respect to $s'$ and equating to zero, we get
\[
a - 2bs - \frac{b}{T} \left[ y(t_3, s) e^{-\theta(x(t_3, s))} \int_{0}^{t_1} \left( \frac{1}{(a - bs)} - (a - bs) \right) e^{\theta u} du \right. \\
\left. + \int_{t_1}^{t_2} e^{-\theta u} \left[ \int_{0}^{t_1} e^{\theta u} du \right] dt - b \int_{0}^{t_1} e^{-\theta u} \left[ \int_{t_1}^{t_2} e^{\theta u} du \right] dt + \\
\int_{t_1}^{t_2} e^{-\theta u} \left[ \int_{0}^{t_1} e^{\theta u} du \right] dt - (a - bs) y(t_3, s) e^{-\theta(x(t_3, s))} \int_{t_1}^{x(t_3, s)} e^{\theta u} du \right] \\
- \frac{\pi b}{2T(y(a - bs))^2} \left( \log \left( \frac{a - \gamma T}{a - \gamma t_3} \right) \right)^2 = 0
\]

where, $x(t_3, s) = t_2 = T + \frac{1}{y(a - bs)} \log \left( \frac{a - \gamma T}{a - \gamma t_3} \right)$ and $y(t_3, s) = \frac{b}{y(a - bs)} \log \left( \frac{a - \gamma T}{a - \gamma t_3} \right)$

Solving the equations (19), (20) and (21) simultaneously, we obtain the optimal time at which production is stopped $t_1^*$, the optimal time $t_3^*$ at which the production is restarted after accumulation of backorders and the optimal selling price $s^*$.

The optimum production quantity $Q^*$ in the cycle of length $T$ is obtained by substituting the optimal values of $t_1^*$ and $t_3^*$ in equation (3.9) as
\[
Q^* = \frac{1}{y} \log \left( \frac{a (a - \gamma t_3^*)}{(a - \gamma t_1^*) (a - \gamma T)} \right)
\]  
(22)

5 Results and Discussion on EPQ Model with Shortages:

Sensitivity analysis is carried to explore the effect of changes in model parameters and costs on the optimal policies, by varying each parameter (-15%, -10%, -5%, 0%, 5%, 10%, 15%) at a time for the model under study. The results are presented in Table 1(a) and Table 1(b). The relationship between the parameters and the optimal values of the replenishment schedule is shown in Figure 2.

It is observed that the costs are having a significant influence on the optimal production quantity and replenishment schedules. As the setup cost ‘A’ decreases, the optimal production downtime $t_1^*$, the optimal production quantity $Q^*$ and the profit rate $P^*$ are increasing and the
### Table 1(a) Sensitivity Analysis of the Model – With Shortages

| Parameters | Optimal Policies | -15% | -10% | -5%  | 0%  | 5%  | 10% | 15% |
|------------|------------------|------|------|------|-----|-----|-----|-----|
| $\theta$  | $t_1$            | 7.7887 | 7.7871 | 7.7855 | 7.7839 | 7.7822 | 7.7806 | 7.779 |
|           | $t_2$            | 9.4573 | 9.4596 | 9.4619 | 9.4642 | 9.4665 | 9.4688 | 9.4711 |
|           | $s^*$            | 17.7875 | 17.7906 | 17.7936 | 17.7966 | 17.7997 | 17.8027 | 17.8057 |
| $Q^*$     | 29.2339 | 29.22 | 29.2061 | 29.1922 | 29.1783 | 29.1644 | 29.1505 |
| $P^*$     | 74.1892 | 72.9544 | 71.7197 | 70.4849 | 69.2501 | 68.0154 | 66.7806 |
| $c$       | $t_1$            | 7.8472 | 7.8262 | 7.8051 | 7.7839 | 7.7625 | 7.7411 | 7.7201 |
|           | $t_2$            | 9.3525 | 9.3895 | 9.4267 | 9.4642 | 9.502 | 9.54 | 9.5782 |
|           | $s^*$            | 17.7874 | 17.7905 | 17.7936 | 17.7966 | 17.7996 | 17.8025 | 17.805 |
| $Q^*$     | 29.8175 | 29.6113 | 29.4028 | 29.1922 | 28.9794 | 28.7645 | 28.5475 |
| $P^*$     | 74.3954 | 73.0689 | 71.7653 | 70.4849 | 69.2281 | 67.9952 | 66.7842 |
| $h$       | $t_1$            | 7.7907 | 7.7884 | 7.7861 | 7.7839 | 7.7816 | 7.7794 | 7.7771 |
|           | $t_2$            | 9.4531 | 9.4568 | 9.4605 | 9.4642 | 9.4679 | 9.4716 | 9.4753 |
|           | $s^*$            | 17.8367 | 17.8233 | 17.81 | 17.7966 | 17.7833 | 17.77 | 17.7566 |
| $Q^*$     | 29.2563 | 29.2349 | 29.2135 | 29.1922 | 29.1709 | 29.1496 | 29.1283 |
| $P^*$     | 70.0055 | 70.1646 | 70.3244 | 70.4849 | 69.2281 | 67.9952 | 66.7842 |
| $\pi$     | $t_1$            | 7.7836 | 7.7837 | 7.7838 | 7.7839 | 7.784 | 7.7841 | 7.7842 |
|           | $t_2$            | 9.4992 | 9.4875 | 9.4759 | 9.4642 | 9.4526 | 9.441 | 9.4293 |
|           | $s^*$            | 17.7994 | 17.7985 | 17.7976 | 17.7966 | 17.7957 | 17.7948 | 17.7938 |
| $Q^*$     | 29.0537 | 29.1 | 29.1461 | 29.1922 | 29.2381 | 29.284 | 29.3297 |
| $P^*$     | 70.1435 | 70.2557 | 70.3695 | 70.4849 | 70.6019 | 70.7205 | 70.8407 |
| $\alpha$  | $t_1$            | 7.7584 | 7.7594 | 7.7713 | 7.7839 | 7.7951 | 7.8049 | 7.8132 |
|           | $t_2$            | 9.7335 | 9.5758 | 9.5052 | 9.4642 | 9.4372 | 9.418 | 9.4036 |
|           | $s^*$            | 17.9335 | 17.8586 | 17.8205 | 17.7966 | 17.78 | 17.7677 | 17.7581 |
| $Q^*$     | 50.5859 | 39.452 | 33.3184 | 29.1922 | 26.1418 | 23.7575 | 21.8239 |
| $P^*$     | 45.0981 | 58.2576 | 65.5364 | 70.4849 | 74.1849 | 77.1077 | 79.5008 |
| $\gamma$  | $t_1$            | 7.7995 | 7.7945 | 7.7892 | 7.7839 | 7.7787 | 7.7746 | 7.7734 |
|           | $t_2$            | 9.3791 | 9.4054 | 9.4333 | 9.4642 | 9.5008 | 9.5477 | 9.6162 |
|           | $s^*$            | 17.7706 | 17.7777 | 17.7862 | 17.7966 | 17.8102 | 17.8288 | 17.8569 |
| $Q^*$     | 24.6423 | 25.8791 | 27.3622 | 29.1922 | 31.5418 | 34.7442 | 39.567 |
| $P^*$     | 76.4693 | 74.7751 | 72.8208 | 70.4849 | 67.5602 | 63.6472 | 57.8171 |
Table 1(b) Sensitivity Analysis of the Model – With Shortages

| Parameters | Optimal Policies | -15% | -10% | -5% | 0%  | 5%  | 10% | 15% |
|------------|-----------------|------|------|-----|-----|-----|-----|-----|
| $\theta$  | $t_1^*$         | 7.783| 7.7833| 7.7836| 7.7839| 7.7842| 7.7844| 7.7847 |
|           | $t_3^*$         | 9.4645| 9.4644| 9.4643| 9.4642| 9.4641| 9.464| 9.4639 |
|           | $s^*$           | 17.7955| 17.7958| 17.7962| 17.7966| 17.797| 17.7974| 17.7978 |
|           | $Q^*$           | 29.191| 29.1914| 29.1918| 29.1922| 29.1926| 29.193| 29.1934 |
|           | $P^*$           | 70.5001| 70.495| 70.49| 70.4849| 70.4799| 70.4748| 70.4698 |
| $\eta$    | $t_1^*$         | 7.8026| 7.7959| 7.7896| 7.7839| 7.7787| 7.774| 7.7699 |
|           | $t_3^*$         | 9.4387| 9.4479| 9.4565| 9.4642| 9.4712| 9.4774| 9.4827 |
|           | $s^*$           | 18.647| 18.3754| 18.0917| 17.7966| 17.4912| 17.1762| 16.8526 |
|           | $Q^*$           | 29.3485| 29.2921| 29.2399| 29.1922| 29.1491| 29.1108| 29.0776 |
|           | $P^*$           | 87.7922| 81.7651| 75.9933| 70.4849| 65.2453| 60.2771| 55.5807 |
| $b$       | $t_1^*$         | 7.7561| 7.7634| 7.7728| 7.7839| 7.7962| 7.8097| 7.8241 |
|           | $t_3^*$         | 9.4989| 9.4907| 9.4789| 9.4642| 9.4474| 9.4288| 9.4087 |
|           | $s^*$           | 16.7352| 17.1049| 17.4642| 17.7966| 18.1048| 18.3871| 18.6424 |
|           | $Q^*$           | 28.9724| 29.0268| 29.1015| 29.1922| 29.2954| 29.4088| 29.5303 |
|           | $P^*$           | 36.6672| 47.5851| 58.8665| 70.4849| 82.4099| 94.6082| 107.044 |

Variations in $t_1^*$

Variations in $t_3^*$

Variations in $s^*$
optimal production uptime $t_3^*$ and the optimal selling price $s^*$ are decreasing. As the cost per unit ‘C’ decreases, the optimal production downtime $t_1^*$, the optimal production quantity $Q^*$ and the profit rate $P^*$ are increasing and the optimal production uptime $t_3^*$ and the optimal selling price $s^*$ are decreasing. As the holding cost ‘h’ decreases the optimal production downtime $t_1^*$, the optimal selling price $s^*$ and the optimal production quantity $Q^*$ are increasing and the optimal production uptime $t_3^*$ and the profit rate $P^*$ are decreasing. As the shortage cost ‘$\pi$’ decreases, the optimal production downtime $t_3^*$ and the optimal selling price $s^*$ are increasing and the optimal production downtime $t_1^*$, the optimal production quantity $Q^*$ and the profit rate function $P^*$ are decreasing.

As the production rate parameter ‘$\gamma$’ decreases, the optimal production downtime $t_1^*$ and the profit rate function $P^*$ are increasing and the optimal production downtime $t_3^*$, the optimal selling price $s^*$ and the optimal ordering quantity $Q^*$ are decreasing. As production parameter ‘$\alpha$’ decreases, the optimal production downtime $t_3^*$, the optimal selling price $s^*$ and the optimal production quantity $Q^*$ are increasing and the optimal production downtime $t_1^*$ and the profit rate $P^*$ are decreasing. As deteriorating parameter $\theta$ decreases, the optimal production uptime $t_3^*$, the profit rate $P^*$ are increasing and the optimal production downtime $t_1^*$, the optimal selling price $s^*$ and the optimal production quantity $Q^*$ are decreasing. As demand parameter $\alpha$ decreases, the optimal production up time $t_3^*$ increases and the optimal production downtime $t_1^*$, the optimal selling price $s^*$, the optimal production quantity $Q^*$ and the profit rate $P^*$ are decreasing. As another demand parameter $b$ decreases the optimal production downtime $t_1^*$, the optimal selling price $s^*$ the optimal...
production quantity $Q^*$ and the profit rate $P^*$ are increasing and the optimal production up time $t^*_3$ decreases.

6 EPQ Model without Shortages:
In this section the inventory model for deteriorating items without shortages is developed and analyzed. Here, it is assumed that shortages are not allowed and the stock level is zero at time $t = 0$. The stock level increases during the period $(0, t_1)$ due to excess production after fulfilling the demand and deterioration. The production stops at time $t_1$ when the stock level reaches $S$. The inventory decreases gradually due to demand and deterioration in the interval $(t_1, T)$. At time $T$ the inventory reaches zero. The schematic diagram representing the instantaneous state of inventory is given in Figure 3.

Let $I(t)$ be the inventory level of the system at time $t$ ($0 \leq t \leq T$). Then the differential equations governing the instantaneous state of $I(t)$ over the cycle of length $T$ are

$$\frac{d}{dt}I(t) + h(t)I(t) = \frac{1}{\alpha - \gamma t} - (a - bs) ; \quad 0 \leq t \leq t_1$$

$$\frac{d}{dt}I(t) + h(t)I(t) = -(a - bs) ; \quad t_1 \leq t \leq T$$

where, $h(t)$ is as given in equation (3), with the initial conditions $I(0) = 0$, $I(t_1) = S$ and $I(T) = 0$.

Substituting $h(t)$ in equations (23) and (24) and solving the differential equations, the on hand inventory at time $t$ is obtained as

$$I(t) = Se^{\theta(t_1 - t)} - e^{-\theta t} \int_{t}^{t_1} \left( \frac{1}{\alpha - \gamma u} - (a - bs) \right) e^{\theta u} du ; \quad 0 \leq t \leq t_1$$

$$I(t) = Se^{\theta(t_1 - t)} - (a - bs) e^{-\theta t} \int_{t_1}^{t} e^{\theta u} du ; \quad t_1 \leq t \leq T$$

Production quantity $Q$ in the cycle of length $T$ is

$$Q = \frac{1}{\gamma} \log \left( \frac{a}{\alpha - \gamma t_1} \right)$$
From equation (25) and using the initial condition I(0) = 0, we obtain the value of ‘S’ as

\[ S = e^{-\theta t_1} \int_0^{t_1} \left( \frac{1}{\alpha - \gamma u} - (a - bs) \right) e^{\theta u} du \]  

(28)

Let \( K(t_1, s) \) be the total production cost per unit time. Since the total production cost is the sum of the setup cost, cost of the units, the inventory holding cost. Therefore, the total production cost per unit time becomes

\[ K(t_1, s) = \frac{A}{T} + \frac{C}{\gamma T} \log \left( \frac{\alpha}{\alpha - \gamma t_1} \right) \]

\[ + \frac{1}{T} \left[ \int_0^{t_1} \left( Se^{\theta (t_1 - t_\eta)} - e^{\theta t_\eta} \int_t^{t_1} \left( \frac{1}{\alpha - \gamma u} - (a - bs) \right) e^{\theta u} du \right) dt \right] \]

\[ + \int_{t_1}^T \left[ Se^{\theta t_\eta} - (a - bs) e^{\theta t_\eta} \int_t^{t_1} e^{\theta u} du \right] dt \quad (29) \]

Let \( P(t_1, s) \) be the profit rate function. Since the profit rate function is the total revenue per unit minus total production cost per unit time. We have,

\[ P(t_1, s) = s(a - bs) - \frac{A}{T} - \frac{C}{\gamma T} \log \left( \frac{\alpha}{\alpha - \gamma t_1} \right) \]

\[ \quad - \frac{1}{T} \left[ \int_0^{t_1} e^{-\theta t_\eta} \int_0^{t_1} \left( \frac{1}{\alpha - \gamma u} - (a - bs) \right) e^{\theta u} du \right] dt \]

\[ \quad - \int_0^{t_1} \left[ e^{-\theta t_\eta} \int_t^{t_1} \left( \frac{1}{\alpha - \gamma u} - (a - bs) \right) e^{\theta u} du \right] dt - (a - bs) \int_{t_1}^T \left[ e^{-\theta t_\eta} \int_t^{t_1} e^{\theta u} du \right] dt \quad (30) \]

7 Optimal ordering policies of the model without shortages:

In this section we obtain the optimal policies of the inventory system under study. To find the optimal values of \( t_1 \) and \( s \), we equate the first order partial derivatives of \( P(t_1, s) \) with respect to \( t_1 \) and \( s \) and equate them to zero. The condition for minimum of \( P(t_1, s) \) is

\[ D = \begin{vmatrix} \frac{\partial^2 P(t_1, s)}{\partial t_1^2} & \frac{\partial^2 P(t_1, s)}{\partial t_1 \partial s} \\ \frac{\partial^2 P(t_1, s)}{\partial t_1 \partial s} & \frac{\partial^2 P(t_1, s)}{\partial s^2} \end{vmatrix} < 0 \]

Differentiating \( P(t_1, s) \) with respect to \( t_1 \) and equating to zero, we get

\[ \frac{C}{\alpha - \gamma t_1} + he^{\theta t_1} \left[ \left( \frac{1}{\alpha - \gamma t_1} - (a - bs) \right) \int_0^{t_1} e^{-\theta t_\eta} dt - \int_0^{t_1} e^{-\theta t_\eta} dt \right] + (a - bs) \int_{t_1}^T e^{-\theta t_\eta} dt = 0 \quad (31) \]

Solving the equations (31), we obtain the optimal time \( t_1^{*} \) of \( t_1 \) at which the production is to be stopped.

Differentiating \( P(t_1, s) \) with respect to \( s \) and equating to zero, one can get
\[ T(a - 2bs) - hb \left[ \int_0^T e^{-\theta t}\int_0^{t_1} e^{\theta u} du \right] dt - \int_0^{t_1} e^{-\theta t} \int_0^{t_1} e^{\theta u} du \right] dt \\
+ \int_{t_1}^{T} \left[ e^{-\theta t} \int_{t_1}^{t} e^{\theta u} du \right] dt = 0 \]  

(32)

Solving the equations (32), we obtain the optimal value of \( s \) of \( t_1 \) at which the production is to be stopped. The optimal production quantity \( Q^* \) in the cycle of length \( T \) is obtained by substituting the optimal values of \( t_1 \) in equation (27) as

\[ Q^* = \frac{1}{r} \log \left( \frac{a}{(a-\gamma t_1^*)} \right) \]  

(33)

8 Results and Discussion of the model without shortages:

Sensitivity analysis is carried to explore the effect of changes in model parameters and costs on the optimal policies, by varying each parameter (-15%, -10%, -5%, 0%, 5%, 10%, 15%) at a time for the model under study. The results are presented in Table 2(a) and Table 2(b). The relationship between the parameters and the optimal values of the production schedule is shown in Figure 4.

It is observed that the costs are having a significant influence on the optimal production quantity and production schedules. As the setup cost ‘A’ decreases, the optimal production downtime \( t_1^* \), the optimal selling price \( s^* \), the optimal production quantity \( Q^* \) and the profit rate \( P^* \) are increasing. As the cost per unit ‘C’ decreases, the optimal production downtime \( t_1^* \), the optimal selling price \( s^* \), the optimal production quantity \( Q^* \) and the profit rate \( P^* \) are increasing. As the holding cost ‘h’ decreases, the optimal production quantity \( Q^* \) is increasing and the profit rate \( P^* \) decreases. As the production rate parameter ‘\( \gamma \)’ decreases, the optimal production downtime \( t_1^* \), the optimal selling price \( s^* \) and the profit rate function \( P^* \) are increasing and the optimal production quantity \( Q^* \) decreases. As production rate parameter ‘\( \alpha \)’ decreases, the optimal production quantity \( Q^* \)

| Variation Parameters | Optimal policies | -15% | -10% | -5% | 0% | 5% | 10% | 15% |
|----------------------|-----------------|------|------|-----|----|----|-----|-----|
| A                    | \( t_1^* \)     | 7.8306 | 7.8292 | 7.8277 | 7.8262 | 7.8248 | 7.8233 | 7.8218 |
|                      | \( s^* \)       | 16.594 | 16.5927 | 16.5914 | 16.5901 | 16.5888 | 16.5875 | 16.5862 |
|                      | \( Q^* \)       | 16.0302 | 16.0257 | 16.0213 | 16.0168 | 16.0123 | 16.0079 | 16.0034 |
|                      | \( P^* \)       | 47.8934 | 46.6495 | 45.4056 | 44.1617 | 42.9178 | 41.6739 | 40.4301 |
| C                    | \( t_1^* \)     | 7.8873 | 7.8767 | 7.8466 | 7.8262 | 7.8058 | 7.7852 | 7.7647 |
|                      | \( s^* \)       | 16.5924 | 16.5916 | 16.5908 | 16.5901 | 16.5893 | 16.5886 | 16.5879 |
|                      | \( Q^* \)       | 16.2024 | 16.1405 | 16.0787 | 16.0168 | 15.9549 | 15.8931 | 15.8312 |
|                      | \( P^* \)       | 46.3622 | 45.6224 | 44.889 | 44.1617 | 43.4408 | 42.726 | 42.0175 |
| |  | -15% | -10% | -5% | 0% | 5% | 10% | 15% |
|---|---|---|---|---|---|---|---|---|
| **b** | $s^*$ | 16.6335 | 16.619 | 16.6046 | 16.5901 | 16.5756 | 16.5612 | 16.5467 |
| | $Q^*$ | 16.0567 | 16.0434 | 16.0301 | 16.0168 | 16.0035 | 15.9903 | 15.9771 |
| | $P^*$ | 43.82 | 43.933 | 44.0469 | 44.1617 | 44.2774 | 44.394 | 44.5114 |
| **a** | $s^*$ | 16.5838 | 16.5863 | 16.5883 | 16.5901 | 16.5916 | 16.5929 | 16.5941 |
| | $Q^*$ | 20.4351 | 18.7126 | 17.2589 | 16.0168 | 14.9435 | 14.0035 | 13.1821 |
| | $P^*$ | 38.0591 | 40.4352 | 42.4433 | 44.1617 | 45.6483 | 46.9473 | 48.0924 |
| **γ** | $s^*$ | 16.5914 | 16.591 | 16.5906 | 16.5901 | 16.5896 | 16.589 | 16.5884 |
| | $Q^*$ | 15.054 | 15.3232 | 15.6597 | 16.0168 | 16.3966 | 16.8013 | 17.2335 |
| | $P^*$ | 45.5209 | 45.0933 | 44.6411 | 44.1617 | 43.6523 | 43.1101 | 42.5318 |
| **θ** | $s^*$ | 16.5863 | 16.5876 | 16.5888 | 16.5901 | 16.5913 | 16.5925 | 16.5937 |
| | $Q^*$ | 16.016 | 16.0163 | 16.0165 | 16.0168 | 16.0171 | 16.0173 | 16.0176 |
| | $P^*$ | 44.1887 | 44.1796 | 44.1706 | 44.1617 | 44.1529 | 44.1442 | 44.1355 |
| **η** | $s^*$ | 16.5862 | 16.5875 | 16.5888 | 16.5901 | 16.5914 | 16.5926 | 16.5939 |
| | $Q^*$ | 16.016 | 16.0163 | 16.0165 | 16.0168 | 16.0171 | 16.0173 | 16.0177 |
| | $P^*$ | 44.1894 | 44.1801 | 44.1709 | 44.1617 | 44.1527 | 44.1436 | 44.1347 |

Table 2(a) Sensitivity Analysis of the Model – Without Shortages
increases and the optimal production downtime $t_1^*$, the optimal selling price $s^*$ and the profit rate $P^*$ are decreasing. As deteriorating parameter $\theta$ decreases, the profit rate increases and the optimal production downtime $t_1^*$, the optimal production quantity $Q^*$ and the optimal selling price $s^*$ are decreasing. As another deteriorating parameter $\eta$ decreases, the profit rate $P^*$ increases and the optimal production downtime $t_1^*$, the optimal production quantity $Q^*$ and the optimal selling price $s^*$ are decreasing. As demand parameter $a$ decreases, the optimal production downtime $t_1^*$, the optimal selling price $s^*$, the optimal production quantity $Q^*$ and the profit rate $P^*$ are decreasing. As another demand parameter $b$ decreases, the optimal production downtime $t_1^*$, the optimal selling price $s^*$, the optimal production quantity $Q^*$ and the profit rate $P^*$ are increasing.

Conclusions

This paper addresses the derivation of optimal ordering policies of an EPQ model with the assumption that the production process is random and follows a generalized Pareto distribution. Further it is assumed that the life time of the commodity is random and follows a Weibull distribution. The generalized Pareto distribution characterizes the production process more close to the reality. The Weibull rate deterioration can include
increasing/decreasing/constant rates of deterioration for different values of parameters. The sensitivity analysis of the model reveals that the production distribution parameters have significant influence on the optimal values of the production uptime, production downtime, production quantity and profit. The deterioration distribution parameter also influencing the optimal values of the model. The production and deterioration distribution parameters can be estimated by using historical data. The production manager can obtain the optimal production downtime and uptime, by estimating the parameters and costs. It is also observed that the model with shortages has less production cost per a unit time than that of without shortages. This model also includes some of the earlier models as particular cases for specific or limiting values of the parameters. This model can be extended for the cases of changing money value (inflation) and multicommodity production systems which will be taken elsewhere.

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