Cosmic microwave background bispectrum of tensor passive modes induced from primordial magnetic fields

Maresuke Shiraishi,¹ Daisuke Nitta,¹ Shuichiro Yokoyama,¹ Kiyotomo Ichiki,¹ and Keitaro Takahashi²

¹Department of Physics and Astrophysics, Nagoya University, Aichi 464-8602, Japan
²Graduate School of Science and Technology Kumamoto University 2-39-1 Kurokami, Kumamoto 860-8555, Japan

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If the seed magnetic fields exist in the early Universe, tensor components of their anisotropic stresses are not compensated prior to neutrino decoupling and the tensor metric perturbations generated from them survive passively. Consequently, due to the decay of these metric perturbations after recombination, the so-called integrated Sachs-Wolfe effect, the large-scale fluctuations of CMB radiation are significantly boosted. This kind of CMB anisotropy is called the “tensor passive mode.” Because these fluctuations deviate largely from the Gaussian statistics due to the quadratic dependence on the strength of the Gaussian magnetic field, not only the power spectrum but also the higher-order correlations have reasonable signals. With these motives, we compute the CMB bispectrum induced by this mode. When the magnetic spectrum obeys a nearly scale-invariant shape, we obtain an estimation of a typical value of the normalized reduced bispectrum as \( \ell_1 (\ell_1 + 1) F_{\ell_1} \sim (130 - 6) \times 10^{-16} (B_{1\text{Mpc}}/4.7\text{nG})^6 \) depending on the energy scale of the magnetic field production from \( 10^{14}\text{GeV} \) to \( 10^{8}\text{GeV} \). Here, \( B_{1\text{Mpc}} \) is the strength of the primordial magnetic field smoothed on 1Mpc. From the above estimation and the current observational constraint on the primordial non-Gaussianity, we get a rough constraint on the magnetic field strength as \( B_{1\text{Mpc}} < 2.6 - 4.4\text{nG} \).

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I. INTRODUCTION

Cosmological observations have suggested the existence of micro-Gauss strength magnetic fields in galaxies and clusters of galaxies at the present Universe. As their origin, many researchers have discussed the possibility of generating the seed fields in the early Universe (e.g. [1, 2]). These scenarios have been verified by constraining the strength of the primordial magnetic fields (PMFs) through the effect on CMB fluctuations.

Conventional studies have provided upper bounds on PMFs with the two point correlations (power spectra) of the CMB temperature and polarization anisotropies [3, 4]. On the other hand, taking into account the CMB three-point correlations (bispectra), which have a nonzero value because the CMB fluctuations are sourced from the quadratic (non-Gaussian) terms of the stochastic (Gaussian) PMFs, some new consequences have been obtained. In Refs. [5, 6], the authors evaluated the contribution of the scalar modes at large scale with several approximations, such as the Sachs-Wolfe limit, and roughly estimated the upper limit on the PMF strength. In our previous papers [8, 9], we computed the effect of the vector modes without neglecting the complicated angular dependence, and obtained tighter bounds due to the dominant contribution at small scale induced by the Doppler and the integrated Sachs-Wolfe (ISW) effects [10, 11]. However, if the gravitational waves are generated from the PMF anisotropic stresses uncompensated prior to neutrino decoupling, these superhorizon modes survive passively and the decay of their modes after recombination amplifies the CMB anisotropies through the ISW effect [12]. This type of fluctuation is called the “tensor passive mode” and it is expected that the CMB bispectrum of this mode has the most dominant signal at large scales, as inferred from the power spectrum [13]. Therefore, in this paper, we investigate the exact CMB bispectrum of tensor passive modes induced from PMFs and place a new constraint on the strength of PMFs. In the calculation, because there are complicated angular integrals as there are in the vector mode case, we apply our computation approach, as discussed in Ref. [6].

This paper is organized as follows. In the next section, we formulate the CMB bispectrum of tensor passive modes induced from PMFs. In Sec. III, we show our result for the CMB bispectrum and the limit on the strength of PMFs, and give a discussion.

II. FORMULATION OF TENSOR BISPECTRUM INDUCED FROM PMFS

Let us consider the stochastic PMFs \( B^b(x, \tau) \) on the Friedmann-Robertson-Walker and small perturbative metric as

\[
ds^2 = a(\tau)^2 [-d\tau^2 + 2h_{0b}d\tau dx^b + (\delta_{bc} + h_{bc})dx^b dx^c],
\]

Here \( a \) is a scale factor and \( \tau \) is a conformal time. In this space-time, the PMF evolves as \( B^b(x, \tau) = B^b(x)/a^2 \). Then the spatial components of the PMF’s energy mo-
momentum tensors are given by
\[ T_{\mu}^c = \frac{1}{4\pi a^4} \left[ \frac{B^2(x)}{2} \delta^c_{\mu} - B^b(x)B_c(x) \right] \]
\[ \equiv \rho_\gamma (\Delta B \delta^c_{\mu} + \Pi^b_{\mu c}) \] (2)
where \( B^2 = B^\mu B_\mu \) and we use the photon energy density \( \rho_\gamma \propto a^{-4} \) for normalization. In the following discussion, the index is lowered by \( \delta_{bc} \), and the summation is implied for repeated indices.

A. Bispectrum of the tensor anisotropic stress fluctuations

The Fourier component of \( \Pi_{\mu \nu \rho} \) is given by the convolution of the PMFs as
\[ \Pi_{\mu \nu \rho} = -\frac{1}{4\pi \rho_\gamma} \int \frac{d^3k'}{(2\pi)^3} B_\mu(k')B_\nu(k-k') \] (3)
where \( \rho_\gamma \equiv \rho_\gamma a^4 \) denotes the present energy density of photons. If \( B_\mu(x) \) obeys the Gaussian statistics, the power spectrum of the PMFs \( P_B(k) \) is defined by
\[ \langle B_\mu(k)B_\nu(p) \rangle = (2\pi)^3 \frac{P_B(k)}{2} P_{ab}(k)\delta(k+p) \] (4)
with a projection tensor
\[ P_{ab}(k) = \sum_{\sigma = \pm 1} \epsilon^a_\sigma \epsilon^b_\sigma = \delta_{ab} - \hat{k}_a \hat{k}_b \] (5)
which comes from the divergenceless of the PMF. Here \( \hat{k} \) denotes a unit vector, \( \epsilon^{(\pm 1)} \) is a normalized divergenceless polarization vector satisfying the orthogonal condition, \( \hat{k}_a^{(\pm 1)} = 0 \), and \( \sigma = \pm 1 \) expresses the helicity of the polarization vector. In general, the magnetic power spectrum should contain an asymmetric helical term \( (14-16) \). However, we assume the magnetic fields are isotropic and homogeneous, for simplicity; hence, this effect is neglected in Eq. (4). Because the production mechanism of PMFs remains to be done, we use a simple power-law form as the power spectrum:
\[ P_B(k) = \frac{(2\pi)^{n_B+5}}{\Gamma(n_B/2 + 3/2)k_{1\text{Mpc}}^2} B_{1\text{Mpc}}^2 \left( \frac{k}{k_{1\text{Mpc}}} \right)^{n_B} \] (6)
where \( B_{1\text{Mpc}} \) denotes the magnetic field strength smoothed on a scale 1Mpc, \( k_{1\text{Mpc}} \equiv 2\pi \text{Mpc}^{-1} \), and \( n_B \) is a spectral index.

With a transverse and traceless polarization tensor \( \epsilon^{(\pm 2)}_{ab} \equiv \sqrt{2} \epsilon^{(\pm 1)}_{ab} \), the anisotropic stress fluctuation is decomposed into two helicity states of the tensor mode as
\[ \Pi_{\mu \nu \rho}(k) = \sum_{\lambda = \pm 2} \Pi^{(\lambda)}_{\mu \nu \rho} \epsilon^{(\lambda)}_{ab}(k) \] (7)
which is inversely converted into
\[ \Pi^{(\pm 2)}_{\mu \nu \rho}(k) = \frac{1}{2} \epsilon^{(\mp 2)}_{ab}(k) \Pi_{\mu \nu \rho}(k) \] (8)

From the above equations, the bispectrum of \( \Pi^{(\pm 2)}_{\mu \nu \rho} \) is symmetrically formed as

\[
\left\{ \prod_{n=1}^{3} \Pi^{(\lambda_n)}_{\mu \nu \rho}(k_n) \right\} = (-16\pi \rho_\gamma)^{-3} \left[ \prod_{n=1}^{3} \int d^3k_n P_B(k_n) \right] \delta(k_1 - k_1') \delta(k_2 - k_2') \delta(k_3 - k_3 + k_2') 
\times e^{(-\lambda_1)}_{ab}(k_1) e^{(-\lambda_2)}_{cd}(k_2) e^{(-\lambda_3)}_{ef}(k_3) [P_{ad}(k_1') P_{be}(k_3') P_{cf}(k_2') + \{a \leftrightarrow b \ or \ c \leftrightarrow d \ or \ e \leftrightarrow f \}] \] (9)

where \( \lambda \) means two helicities: \( \lambda_1, \lambda_2, \lambda_3 = \pm 2 \), and the curly brackets denote the symmetric 7 terms under the permutations of indices: \( a \leftrightarrow b \), \( c \leftrightarrow d \), or \( e \leftrightarrow f \). Because the anisotropic stress fluctuation depends quadratically on the Gaussian magnetic fields as shown in Eq. (3), the statistics of their tensor modes given by [8] is highly non-Gaussian. Hence, the bispectrum of Eq. (9) also has a nonzero value and induces the finite CMB bispectrum.

B. CMB temperature bispectrum of tensor passive modes

As is well known, the gravitational potential of tensor modes can be generated from anisotropic stresses via the Einstein equation. If PMFs exist, the anisotropic stresses, as mentioned in the previous subsection, also behave as a source. In general, after neutrino decoupling, the anisotropic stresses of PMFs vanish via the compensation of those of neutrinos. However, prior to this epoch, there is no compensation process due to the absence of the neutrino anisotropic stresses. Hence, from
the Einstein equation, we find the evolution equation of the tensor-mode metric perturbations as

where $\tau_\nu$ and $\tau_B$ are the conformal times at neutrino decoupling and the generation of the PMF, respectively, and $'$ denotes a derivative of conformal time. Here $h^{(\pm 2)}$ is given by \[ h^{(\pm 2)}(k, \tau) = \frac{1}{2} e^{i(\tau_\nu)} \langle k \rangle h_{ab}(k, \tau). \] (11)

From Eq. (11), we find a superhorizon solution of the tensor metric perturbation as \[ h^{(\pm 2)}(k) \approx 6 R_\gamma \ln \left( \frac{\tau_\nu}{\tau_B} \right) \Pi^{(\pm 2)}(k), \] (12)

where $R_\gamma \sim 0.6$ is the ratio by the energy density of photons to all relativistic particles for $\tau < \tau_\nu$.

The CMB temperature fluctuation is expanded into spherical harmonics as \[ \Delta T(v) = \sum_{\ell m} \alpha_{\ell m} Y_{\ell m}(\hat{\mathbf{n}}). \] The $\alpha_{\ell m}$ sourced from the initial tensor perturbations can be expressed as \[ \alpha_{\ell m} = (-i)^\ell \int \frac{k^2 dk}{2\pi^2} \mathcal{T}_\ell(k) \sum_{\lambda = \pm 2} h^{(\lambda)}_{\ell m}(k), \] (13)

where $\mathcal{T}_\ell(k)$ denotes the transfer function of tensor modes. Because the solution of the magnetic passive mode is equal to $2H_T$ of Refs. [12, 13].

The CMB angle-averaged bispectrum is given by \[ B_{\ell_1 \ell_2 \ell_3} = \sum_{m_1 m_2 m_3} \left( \frac{\ell_1}{m_1} \frac{\ell_2}{m_2} \frac{\ell_3}{m_3} \right) \sum_{n=1}^{\ell_1} \alpha_{\ell n} \alpha_{m n} \right) \] (15)

where the bracket denotes the Wigner-3nj symbol.

In order to calculate the bispectrum of $\alpha_{\ell m}$ given by Eq. (13), we rewrite all angular dependencies in Eq. (14) in terms of the spin-weighted spherical harmonics with the notation as \[ \epsilon^{(\pm 1)}_a(\hat{\mathbf{n}}) = \epsilon^{(\mp 1)*}_a(\hat{\mathbf{n}}) = \mp \sum_m \alpha^{m}_a \pm 1 Y_{\ell m}(\hat{\mathbf{n}}), \] (16)

where $\epsilon^{m}_a$ are the Wigner-J3 symbols over the azimuthal quantum numbers in the same manner as in Ref. [14]. Then, we obtain the final form of the bispectrum as

\[ B_{\ell_1 \ell_2 \ell_3} = (-4\pi \rho_\gamma T^3)_n \left[ \prod_{n=1}^{\ell} \int \frac{k^2 \, dk}{2\pi^2} \mathcal{T}_n(k) \sum_{\lambda = \pm 2} \int_0^{k_D} k^2 \, dk \, P_B(k') \right] \times \sum_{LL'LL''} \sum_{SS'SS''} \left\{ \begin{array}{ccc} \ell_1 & \ell_2 & \ell_3 \\ L & L' & L'' \end{array} \right\} f^{SS'LL}_{LL'}(k_3', k_1, k_1) f^{SS'LL'}(k_1', k_2) f^{SS''LL''}(k_2', k_3), \]

\[ f^{SS'LL}_{LL'}(r_3, r_2, r_1) \equiv -4(8\pi)^{3/2} R_\gamma \ln \left( \frac{\tau_\nu}{\tau_B} \right) \sum_{L_1 L_2 L_3} \int_0^\infty y^2 dy j_L_1(r_3 y) j_L_2(r_2 y) j_L_3(y) \]

\[ \times (1)^{\ell+2} L_2 L_3 (1)^{L_1+L_2+L_3}/2 f_{L_1 L_2 L_3}^{00} f_{L_1 L_2 L_3}^{00} f_{L_1 L_2 L_3}^{00} f_{L_1 L_2 L_3}^{00} \left| \begin{array}{cccc} L' & L & L_1 \\ L_2 & L_3 & 1 & 2 \end{array} \right| , \]

where $j_L(x)$ is the spherical Bessel function, $k_D$ is the Alfvén-wave damping length scale, the 2 × 3 and 3 × 3 matrices in the curly brackets denote the Wigner-6j and
9j symbols, respectively, and

\[ f_{s_1 s_2 s_3}^{l_1 l_2 l_3} = \frac{(2l_1 + 1)(2l_2 + 1)(2l_3 + 1)}{4\pi} \left( \frac{s_1 s_2 s_3}{l_1 l_2 l_3} \right). \]

As shown in Eq. (17), the bispectrum depends on \( \tau_B \). Although the production mechanism of PMFs is unclear and still being discussed, we assume that PMFs arise sometime between the energy scale of any grand unification theory and the electroweak transition. Hence, in the computation of the CMB bispectrum, we consider two corresponding values: \( \tau_\sigma/\tau_B \sim 10^{-17}, 10^6 \). This leads to a factor of 23 difference in the amplitude of the CMB bispectrum due to the logarithmic dependence on \( \tau_\sigma/\tau_B \). Therefore, due to the sextuplicate dependence of the CMB bispectrum on the magnetic strength, there is a model-dependent factor 23^{1/6} \sim 1.7 in bounds with the PMF strength.

### III. NUMERICAL RESULTS AND DISCUSSION

Following the final expression (17), we compute the CMB temperature bispectrum of tensor passive modes numerically.\(^2\)

In Fig. 1 we describe the reduced bispectra of temperature fluctuations induced by the PMFs defined as \( f_{s_1 s_2 s_3}^{l_1 l_2 l_3} \). The PMF parameters are fixed to \( B_{1\text{Mpc}} = 4.7\text{nG}, n_B = -2.9 \), and \( \tau_\sigma/\tau_B = 10^{-17} \) (upper line) and \( 10^6 \) (lower line), and the other cosmological parameters are equal to the mean values from the Wilkinson Microwave Anisotropy Probe 7-yr data.\(^3\)

In this paper, we study the CMB temperature bispectrum generated from the tensor anisotropic stresses of PMFs and find a new constraint on the magnetic field magnitude when the PMF spectrum is close to a scale-invariant shape. Although there is a touch of uncertainty in the production epoch of PMFs, this bound is tighter than ones obtained by the CMB power spectra.\(^3\) This value is consistent with a simple prediction from the cosmic variance.\(^3\)

Using the above equation, we can obtain the upper bound on the PMF strength. As shown in Fig. 1 because the tensor bispectrum is highly damped for \( \ell \gtrsim 100 \), we should use an upper bound on \( f_{\text{NL}}^{\text{local}} \) obtained by the current observational data for \( \ell < 100 \), namely \( f_{\text{NL}}^{\text{local}} < 100 \). This value is consistent with a simple prediction from the cosmic variance.\(^3\)

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\(^2\) Unlike the case of the vector mode bispectrum calculation,\(^3\) we do not use the thin LSS approximation because the temperature anisotropies from tensor modes are nonlocal. To check our numerical calculation, we computed the CMB power spectrum of the tensor magnetic passive mode using the same method described in the main text, namely, by expanding the nonlinear convolution of magnetic anisotropic stress with the spin-weighted spherical harmonics. We observe that our results are consistent with the previous results.\(^3\)

\(^3\) The PMF parameters are fixed to \( B_{1\text{Mpc}} = 4.7\text{nG}, n_B = -2.9 \), and \( \tau_\sigma/\tau_B = 10^{-17} \) (upper line) and \( 10^6 \) (lower line), and the other cosmological parameters are equal to the mean values from the Wilkinson Microwave Anisotropy Probe 7-yr data.
amplitude at large scales will have a drastic impact on the precise calculation of the limit on PMFs, including the scalar, vector, and tensor-mode contributions.

In our previous studies and the above analysis, we find that tensor (vector) modes dominate at large (small) scale, not only in the power spectrum but also in the bispectrum. It is also expected that the scalar mode dominates at the intermediate scale. Therefore, using this scale-dependent property, we will also constrain a spectral index of the PMF spectrum in addition to the magnetic strength. These reasonable bounds will be obtained by considering the CMB temperature and polarization bispectrum of autocorrelations and cross-correlations between scalar, vector, and tensor modes in the estimation of the signal-to-noise ratio.

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