ON THE DUAL POSITIVE CONES AND THE ALGEBRAICITY OF A COMPACT KÄHLER MANIFOLD

by

Hsueh-Yung Lin

Abstract. — We investigate the algebraicity of compact Kähler manifolds admitting a positive rational Hodge class of bidimension (1, 1). We prove that if the dual Kähler cone of a compact Kähler manifold $X$ contains a rational class as an interior point, then its Albanese variety is projective. As a consequence, we answer the Oguiso–Peternell problem for Ricci-flat compact Kähler manifolds. We also study related algebraicity problems for threefolds.

1 Introduction

1.1 Dual statements of the Kodaira embedding theorem

Let $X$ be a compact Kähler manifold of dimension $n$. The celebrated Kodaira embedding theorem asserts that if the Kähler cone $\mathcal{K}(X)$ of $X$ contains a rational cohomology class, then $X$ is projective [26]. Consider now the dual Kähler cone

$$\mathcal{K}(X)^\vee := \left\{ \alpha \in H^{n-1,n-1}(X, \mathbb{R}) \subset H^{2n-2}(X, \mathbb{R}) \mid \langle \alpha, \omega \rangle \geq 0 \text{ for every } \omega \in \mathcal{K}(X) \right\}$$

of $X$, where $\langle \ , \ \rangle : H^{n-1,n-1}(X, \mathbb{R}) \otimes H^{1,1}(X, \mathbb{R}) \to \mathbb{R}$ is the perfect pairing defined by the Poincaré duality. As $\mathcal{K}(X)$ is open in $H^{1,1}(X, \mathbb{R})$, its dual $\mathcal{K}(X)^\vee$ has nonempty interior $\text{Int}(\mathcal{K}(X)^\vee)$ in $H^{n-1,n-1}(X, \mathbb{R})$. The following problem was first asked and studied by Oguiso and Peternell [32, 33] in search of a dual statement of the Kodaira embedding theorem.

Problem 1.1 (Oguiso–Peternell). — Let $X$ be a compact Kähler manifold of dimension $n$ such that $\text{Int}(\mathcal{K}(X)^\vee)$ contains an element of $H^{2n-2}(X, \mathbb{Q})$. How algebraic is $X$? For instance, what are the possible algebraic dimensions of $X$?

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Here is another problem dual to the Kodaira embedding theorem that we can formulate. The closure $\mathcal{N}(X)$ of the Kähler cone is called the nef cone of $X$. Due to Demailly and Păun, the Poincaré dual of $\mathcal{N}(X)$ is the closed convex cone $N(X) \subset H^{n-1,n-1}(X, \mathbb{R})$ generated by the classes of closed positive currents of type $(n-1,n-1)$ \cite[Theorem 2.1]{Demailly-Paun}. The analog of $N(X)$ in $H^{1,1}(X, \mathbb{R})$ is the pseudoeffective cone Psef($X$), defined as the closed convex cone in $H^{1,1}(X, \mathbb{R})$ generated by the classes of closed positive currents of type $(1,1)$. From this point of view, if $\text{Psef}(X)^{\vee} \subset H^{n-1,n-1}(X, \mathbb{R})$ denotes the Poincaré dual of Psef($X$) and let $\text{Int}(\text{Psef}(X)^{\vee})$ be its interior in $H^{n-1,n-1}(X, \mathbb{R})$, then the following can also be considered as a dual problem to the Kodaira embedding theorem.

**Problem 1.2.** — Let $X$ be a compact Kähler manifold of dimension $n$ such that $\text{Int}(\text{Psef}(X)^{\vee})$ contains an element of $H^{2n-2}(X, \mathbb{Q})$. Is $X$ always projective? If not, how algebraic is $X$?

We can compare the positivity assumptions in Problems 1.1 and 1.2 together with the one in the Kodaira embedding theorem as follows. On the one hand if $\omega \in \mathcal{N}(X)$, then $\omega^{n-1} \in \text{Int}(\text{Psef}(X)^{\vee})$, so the assumption in Problem 1.2 is weaker than the one in the Kodaira embedding theorem. On the other hand since $\mathcal{N}(X) \subset \text{Psef}(X)$, the assumption in Problem 1.2 is stronger than the one in Problem 1.1. It is still unknown whether these assumptions are equivalent.

In \cite{Oguiso-Peternell}, a similar problem had also been studied by Oguiso and Peternell.

**Problem 1.3 (Oguiso–Peternell).** — Let $X$ be a compact Kähler manifold of dimension $n$ such that $X$ contains a smooth curve with ample normal bundle. How algebraic is $X$?

The two Oguiso–Peternell Problems 1.1 and 1.3 could be related by the conjecture that if $C \subset X$ is a smooth curve with ample normal bundle, then $[C] \in \text{Int}(\mathcal{N}(X)^{\vee})$ \cite[Conjecture 0.3]{Oguiso-Peternell}. All the problems introduced above aim at understanding the algebraicity of compact Kähler manifolds containing some positive rational Hodge class of bidimension $(1,1)$.

### 1.2 Main results

In this article we study the aforementioned problems. Our first result provides the following partial answer to the Oguiso–Peternell problem (and also to Problem 1.2).

**Theorem 1.4.** — Let $X$ be a compact Kähler manifold of dimension $n$. If $\text{Int}(\mathcal{N}(X)^{\vee}) \cap H^{2n-2}(X, \mathbb{Q})$ is not empty, then the Albanese variety of $X$ is projective.

For a compact Kähler manifold $X$ as in Problem 1.1, Theorem 1.4 gives a lower bound of the algebraic dimension $a(X)$ of $X$, by the dimension of its Albanese image. In particular, if $X$ has maximal Albanese dimension (namely, if the Albanese map is generically finite onto its image), then $X$ is projective.

**Remark 1.5.** — If we replace the nonemptiness of $\text{Int}(\mathcal{N}(X)^{\vee}) \cap H^{2n-2}(X, \mathbb{Q})$ in Theorem 1.4 by the existence of a smooth curve $C \subset X$ such that $N_{C/X}$ is ample (hence in the context of Problem 1.3), then the projectivity of $\text{Alb}(X)$ simply follows from the surjectivity of $\text{Alb}(C) \to \text{Alb}(X)$ \cite[Lemma 12]{Albanese}.
As a consequence of Theorem 1.4 (together with Proposition 6.2 about hyper-Kähler manifolds), we answer Problem 1.1 for Ricci-flat compact Kähler manifolds.

**Corollary 1.6.** — Let \( X \) be a compact Kähler manifold of dimension \( n \) with \( c_1(X) = 0 \in H^2(X, \mathbb{R}) \). If \( \text{Int} (\mathcal{X}(X)^\vee) \cap H^{2n-2}(X, \mathbb{Q}) \) is not empty, then \( X \) is projective.

For a compact Kähler surface \( S \), Huybrechts [20, 22] and independently Oguiso–Peternell [32] proved that if \( \text{Int} (\mathcal{X}(S)^\vee) \) contains a rational cohomology class, then \( S \) is projective. This completely answers Problem 1.1 and Problem 1.2 in dimension 2. In this article, we answer Problem 1.2 in dimension 3 except for simple non-Kummer threefolds, which presumably do not exist (see Remark 1.9).

**Theorem 1.7.** — Let \( X \) be a smooth compact Kähler threefold. Assume that \( X \) is not a simple non-Kummer threefold. If \( \text{Int} (\text{Psef}(X)^\vee) \) contains a rational cohomology class, then \( X \) is projective.

The study of the Oguiso–Peternell problem in the threefold case was initiated by Oguiso and Peternell in [33]. Suppose that \( X \) is a compact Kähler threefold which is not simple non-Kummer. They proved that if \( \text{Int} (\mathcal{X}(X)^\vee) \) contains an element of \( H^{2n-2}(X, \mathbb{Q}) \) which is a curve class, then \( X \) has algebraic dimension \( a(X) \geq 2 \). We improve their result by removing the curve class assumption.

**Theorem 1.8.** — Let \( X \) be a smooth compact Kähler threefold. Assume that \( X \) is not a simple non-Kummer threefold. If \( \text{Int} (\mathcal{X}(X)^\vee) \) contains a rational cohomology class, then \( a(X) \geq 2 \).  

**Remark 1.9.** — In both Theorems 1.7 and 1.8, the existence of simple non-Kummer threefolds would be excluded if the abundance conjecture holds for compact Kähler threefolds (see [24, Proof of Theorem 6.2]), which is still unknown [9].

If we replace the assumption \( \text{Int} (\mathcal{X}(X)^\vee) \cap H^4(X, \mathbb{Q}) \neq \emptyset \) in Theorem 1.8 by the existence of a smooth curve in \( X \) with ample normal bundle (thus in the context of Problem 1.3), then the same conclusion \( a(X) \geq 2 \) holds and was proven by Oguiso and Peternell [33, Theorem 0.5]. Oguiso and Peternell have also outlined a strategy to construct a non-algebraic threefold \( X \) such that \( \text{Int} (\mathcal{X}(X)^\vee) \) contains a curve class (or such that \( X \) contains a smooth curve with ample normal bundle). An explicit construction of such an example is still missing.

While we are still unable to solve Problems 1.1 and 1.3 in dimension 3, we can relate these problems to a problem about 1-cycles in threefolds. We postpone the discussion to §1.4, after we give an outline of the proofs of Theorem 1.7 and Theorem 1.8 about threefolds.

### 1.3 Outline of the proofs of Theorem 1.7 and Theorem 1.8

To prove Theorem 1.8, we have to show that if \( X \) is a compact Kähler threefold \( X \) of algebraic dimension \( a(X) \leq 1 \) which is not simple non-Kummer, then \( \text{Int} (\mathcal{X}(X)^\vee) \cap H^{2n-2}(X, \mathbb{Q}) = \emptyset \). Essentially based on Fujiki’s descriptions of algebraic reductions of compact Kähler threefolds [18], those threefolds are bimeromorphic to one of the following (see Proposition 2.13):

i) A threefold which dominates a surface.
ii) A fibration in abelian varieties over a curve.

iii) A finite quotient of a smooth isotrivial torus fibration over a curve without multi-sections.

iv) A finite quotient of a 3-torus $T$.

According to the bimeromorphic invariance of the emptiness of $\operatorname{Int} (\mathcal{X}(X)) \cap H^{2n-2}(X, \mathbb{Q})$ (see Corollary 3.2), it suffices to prove that $\operatorname{Int} (\mathcal{X}(X)) \cap H^{2n-2}(X, \mathbb{Q}) = \emptyset$ for the above four types of varieties. The first case is an immediate consequence of [33, Proposition 2.6]. Cases ii), iii), and iv) will be carried out in §4, §7, and §5 respectively.

Since $\mathcal{X}(X) \subset \operatorname{Psef}(X)$, Theorem 1.8 also implies that a compact Kähler threefold $X$ as in Theorem 1.7 satisfies $a(X) \geq 2$. Therefore to prove Theorem 1.7 for $X$, it suffices to exclude the possibility that $a(X) = 2$. Compact Kähler threefolds of algebraic dimension 2 are bimeromorphic to elliptic fibrations over a projective surface, and we will prove Theorem 1.7 for elliptic threefolds in §8.

1.4 A question about 1-cycles and the Oguiso–Peternell problem

We now relate Problem 1.1 to a question about 1-cycles in compact Kähler threefolds, which we first formulate.

Let $X$ be a compact Kähler manifold and $Y \subset X$ a complex subvariety of codimension $l$. Let $\alpha \in H^{l,k}(X, \mathbb{Q})$ be a Hodge class which vanishes in $H^{2l}(X \setminus Y, \mathbb{Q})$. Since $H^{2l}(X, \mathbb{Q})$ carries a pure Hodge structure, $\alpha$ belongs to the image of $\iota : H^{2l-2}(\tilde{Y}, \mathbb{Q}) \to H^{2l}(X, \mathbb{Q})$ where $\iota : \tilde{Y} \to X$ is the composition of a Kähler desingularization $\tilde{Y} \to Y$ of $Y$ with $Y \hookrightarrow X$ (see e.g. Lemma 2.7). If moreover we assume that $X$ is projective, then based on the existence of polarization on the underlying $\mathbb{Q}$-Hodge structure of (the summands of the primitive decomposition of) $H^{2k}(X, \mathbb{Q})$, the class $\alpha$ is even the image of a Hodge class $\beta \in H^{k-1,k-1}(\tilde{Y}, \mathbb{Q})$ [40, Remark 2.30]. Without the projectivity assumption of $X$, it is yet unknown whether the later property still holds, especially for 1-cycles in threefolds.

Question 1.10. — Let $X$ be a smooth compact Kähler threefold and let $Y \subset X$ be a surface (possibly singular with several irreducible components). Let $\iota : \tilde{Y} \to X$ be the composition of a desingularization $\tilde{Y} \to Y$ of $Y$ with the inclusion $Y \hookrightarrow X$. Given a Hodge class $\alpha \in H^{2,2}(X, \mathbb{Q})$ which vanishes in $H^4(X \setminus Y, \mathbb{Q})$, does there exist $\beta \in H^{1,1}(\tilde{Y}, \mathbb{Q})$ such that $\iota \beta = \alpha$?

So far, Question 1.10 can be answered in the affirmative if $Y$ is irreducible (see Lemma 8.3). Under the assumption that Question 1.10 has a positive answer, we are able to answer Problems 1.1 and 1.3 in dimension 3 (except for simple non-Kummer threefolds; see Remark 1.9).

Corollary 1.11. — Assume that Question 1.10 has a positive answer. Let $X$ be a smooth compact Kähler threefold which is not simple non-Kummer. If $X$ satisfies the conditions in Problems 1.1 or 1.3, then $X$ is projective.

1.5 The existence of connecting families of curves and Problem 1.2

We finish this introduction by discussing how one could expect Problem 1.2 to have a positive answer. Let $X$ be a compact Kähler manifold. The movable cone $\mathcal{M}(X) \subset H^{n-1,n-1}(X, \mathbb{R})$ of $X$ is defined as the closed convex cone generated by classes of the form $\mu(\alpha_1 \wedge \cdots \wedge \alpha_{n-1})$ where $\mu : \tilde{X} \to X$ is a bimeromorphic morphism
from a compact Kähler manifold $\tilde{X}$ and the $\omega_i$'s are Kähler classes on $\tilde{X}$. Let $\mathcal{M}(X)_{\text{NS}} \subset H^{n-1,n-1}(X, \mathbb{R})$ be the closed convex cone in $H^{n-1,n-1}(X, \mathbb{R})$ generated by $\mathcal{M}(X) \cap H^{n-1,n-1}(X, \mathbb{Q})$. As a consequence of [6, Theorem 2.4], if we assume that $X$ is projective, then $\mathcal{M}(X)_{\text{NS}}$ coincides with the closed convex cone generated by the classes of connecting family of curves $(C_t)_{t \in T}$ (which means that for every general pair of points $x, y \in X$, there exist $t_1, \ldots, t_l \in T$ such that $x, y \in C_{t_1} \cup \cdots \cup C_{t_l}$ and $C_{t_1} \cup \cdots \cup C_{t_l}$ is connected). One can ask whether this property remains true without the projectivity assumption.

**Question 1.12.** Let $X$ be a compact Kähler manifold. Does the Néron-Severi part $\mathcal{M}(X)_{\text{NS}}$ of the movable cone coincide with the closed convex cone generated by classes of connecting family of curves?

Conjecturally $\mathcal{M}(X)$ is the dual cone of $\text{Psef}(X)$ [6, Conjecture 2.3]. If we assume this conjecture and that Question 1.12 has a positive answer, then a compact Kähler manifold $X$ satisfying $\text{Int}(\text{Psef}(X)^\vee) \cap H^{n-1,n-1}(X, \mathbb{Q}) \neq \emptyset$ would be algebraically connected (see §2.6 for the definition). By Campana’s projectivity criterion (Theorem 2.9), a compact Kähler manifold $X$ as in Problem 1.2 would then be projective. Together with the evidence provided by threefolds (Theorem 1.7) and Ricci-flat manifolds (Corollary 1.6), this leads us to conjecture the following.

**Conjecture 1.13.** Let $X$ be a compact Kähler manifold of dimension $n$. If $\text{Int}(\text{Psef}(X)^\vee)$ contains an element of $H^{2n-2}(X, \mathbb{Q})$, then $X$ is projective.

### 1.6 Organization of the article

In the next section, we recall and prove some preliminary results that we need in this article. In §3, we prove some results about the invariance of the conditions in Problem 1.1 or Problem 1.2 under dominant maps. In §4, we study smooth isotrivial torus fibrations, which will be useful to prove the main theorems. We then prove Theorem 1.4 in §5, and Corollary 1.6 in §6. In §7, we study fibrations in abelian varieties over a curve, similar to what we do in §4 for smooth isotrivial torus fibrations. §8 is devoted to elliptic fibrations. We will conclude the proofs of Theorem 1.7 and 1.8 in §9. Finally, we prove Corollary 1.11 in §10, which relates the Oguiso–Peternell problems to Question 1.10.

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### 2 Preliminaries

#### 2.1 Convention and terminology

In this article, compact complex manifolds and varieties are assumed to be irreducible (but subvarieties can be reducible). A *fibration* is a surjective proper holomorphic map $f : X \to B$ with connected fibers.
We say that a compact complex variety $X$ is in the Fujiki class $\mathcal{C}$ if $X$ is meromorphically dominated by a compact Kähler manifold.

Let $X$ be a compact complex manifold. The hypercohomology of a bounded complex of sheaves of abelian groups $\mathcal{F}^\bullet$ on $X$ is denoted by $H^\bullet(X, \mathcal{F}^\bullet)$. For any subring $R \subset \mathbb{C}$, we define $H^{k,k}(X, R)$ as the kernel of the composition

$$H^{2k}(X, R) \to H^{2k}(X, \mathbb{C}) \cong H^{2k}(X, \Omega^*_X) \to H^{2k}(X, \Omega^{2k-1}_X),$$

where $\Omega^*_X$ is the holomorphic de Rham complex. Note that we have

$$H^{k,k}(X, \mathbb{Q}) \cong H^{k,k}(X, \mathbb{Z}) \otimes \mathbb{Q}$$

(which does not hold in general if $\mathbb{Q}$ is replaced by other rings e.g. $\mathbb{R}$). When $X$ is a compact Kähler manifold, $H^{k,k}(X, R)$ is the sub-$R$-module of $H^{2k}(X, R)$ consisting of elements whose image in $H^{2k}(X, \mathbb{C})$ is of type $(k,k)$ with respect to the Hodge decomposition. The weight filtration of a mixed Hodge structure $H$ is denoted by $W_i H$.

We follow [13] for the definition and convention of positive cones, and the reader is referred to loc. cit. for related basic properties.

### 2.2 Kähler forms on normal complex spaces

Let $X$ be a normal complex space (for instance, the quotient of a complex manifold by a finite group). A smooth function on $X$ is a continuous function $f : X \to \mathbb{R}$ such that for some open cover $\{U_i\}$ of $X$, there exist holomorphic embeddings $U_i \hookrightarrow \mathbb{C}^N$ such that each $f|_{U_i}$ extends to a smooth function on a neighborhood of $U_i$ in $\mathbb{C}^N$. The sheaf of germs of smooth functions on $X$ is denoted by $\mathcal{C}^\infty_X$. Similarly, a strictly plurisubharmonic (or psh for short) function on $X$ is an upper semi-continuous function with values in $\mathbb{R} \cup \{-\infty\}$ which extends to a strictly psh function in a neighborhood of a local embedding $X \hookrightarrow \mathbb{C}^N$.

Let $\mathcal{PH}_X = \mathbb{R}\mathcal{O}_X \subset \mathcal{C}^\infty_X$ be the subsheaf of pluriharmonic functions (i.e. the real part of $\mathcal{O}_X$). A Kähler metric on $X$ is a collection of smooth strictly psh functions $\{\phi_i : U_i \to \mathbb{R}\}_{i \in I}$ where $\{U_i\}_{i \in I}$ is an open cover of $X$ such that $\phi_i|_{U_i \cap U_j} - \phi_j|_{U_i \cap U_j} \in \mathcal{PH}_X(U_i \cap U_j)$. In particular, a Kähler metric on $X$ is an element of $H^0(X, \mathcal{C}^\infty_X/\mathcal{PH}_X)$. The short exact sequences

$$0 \longrightarrow \mathcal{PH}_X \longrightarrow \mathcal{C}^\infty_X \longrightarrow \mathcal{C}^\infty_X/\mathcal{PH}_X \longrightarrow 0 \quad (2.1)$$

and

$$0 \longrightarrow \mathbb{R} \xrightarrow{\sqrt{-1}} \mathcal{O}_X \xrightarrow{2\mathbb{R}} \mathcal{PH}_X \longrightarrow 0 \quad (2.2)$$

give the composition

$$[\bullet] : H^0(X, \mathcal{C}^\infty_X/\mathcal{PH}_X) \to H^1(X, \mathcal{PH}_X) \to H^2(X, \mathbb{R})$$

and a Kähler class $[\omega] \in H^2(X, \mathbb{R})$ is the image of a Kähler metric $\omega \in H^0(X, \mathcal{C}^\infty_X/\mathcal{PH}_X)$. The Kähler classes form a convex cone $\mathcal{K}(X) \subset H^2(X, \mathbb{R})$ and elements in the closure $\overline{\mathcal{K}(X)}$ are called nef classes of $X$.

The two following lemmas are presumably well-known, which are direct consequences of Varouchas’ work [38, 39].
Lemma 2.1. — Let \( f : X \to X/G \) be the quotient of a compact Kähler manifold \( X \) by a finite group \( G \). A \( G \)-invariant nef class \([\alpha]\in \text{H}^2(X, \mathbb{R})\) is the pullback \( f^* [\alpha] \) of a nef class \([\alpha'] \in \text{H}^2(X/G, \mathbb{R})\).

Proof. — Since \( f^* : \text{H}^2(X/G, \mathbb{R}) \to \text{H}^2(X, \mathbb{R}) \) is injective (because \( f \) is finite) and nef classes are limits of Kähler classes, it suffices to prove that a \( G \)-invariant Kähler class \( [\alpha] \) in \( \text{H}^2(X, \mathbb{R}) \) is the pullback \( f^*[\alpha'] \) of a Kähler class \([\alpha']\in \text{H}^2(X/G, \mathbb{R})\). Let \( \omega \) be a \( G \)-invariant Kähler form representing \([\alpha]\). Then we can find an open cover \( \{V_i\} \) of \( X/G \) and \( G \)-invariant \( \mathcal{C}^\infty \)-strictly psh functions \( u_i \) defined over \( U_i := f^{-1}(V_i) \) such that \( \alpha|_{U_i} = i\partial\bar{\partial}u_i \). By [39, Lemma II.3.1.2], the pushforwards \( v_i = f_* u_i \) are strictly psh and \( v_j = (v_i - v_j)|_{V_i \cap V_j} \) are pluriharmonic. By [39, Theorem 1]\(^{(1)}\), there exist \( \mathcal{C}^\infty \)-strictly psh functions \( v'_i \) defined over \( V_i \) such that \( v'_i - v'_j = v_{ij} \). Therefore the Čech 1-cocycle \( \frac{1}{| \alpha |} f^* v_{ij} = (u_i - u_j)|_{U_i \cap U_j} \) we have \( f^*[\alpha'] = [\alpha] \). \( \square \)

Lemma 2.2. — Let \( f : X \to Y \) be a finite morphism between compact complex manifolds. If \([\alpha] \in \text{H}^2(X, \mathbb{R})\) is a Kähler class, then \( f_* [\alpha] \in \text{H}^2(Y, \mathbb{R}) \) is also a Kähler class.

Proof. — Let \( \omega \) be a Kähler form which represents \([\alpha]\). Then there exist an open cover \( \{U_i\} \) of \( Y \) and smooth strictly psh functions \( \phi_i : f^{-1}(U_i) \to \mathbb{R} \) such that \( \alpha|_{f^{-1}(U_i)} = \sqrt{-1} \partial\bar{\partial} \phi_i \) for all \( i \). The functions \( \psi_i : U_i \to \mathbb{R} \) defined by \( \psi_i(y) := \sum_{x \in f^{-1}(y)} \phi_i(x) \) (here we regard \( f^{-1}(y) \) as a multiset prescribed by its scheme structure) are strictly psh [39, Lemma II.3.1.2] and \( \psi_{ij} = \psi_i|_{U_i \cap U_j} - \psi_j|_{U_i \cap U_j} \) are pluriharmonic. The pushforward \( f_* [\alpha] \) is thus represented by the image of the Čech 1-cocycle \( \psi_{ij} \) under the connecting morphism induced by (2.2). Again by [39, Theorem 1], there exist \( \mathcal{C}^\infty \)-strictly psh functions \( \psi'_i \) defined over \( U_i \) such that \( \psi'_i - \psi'_j = \psi_{ij} \), so \( \sqrt{-1} \partial\bar{\partial} \psi'_i \) glue to a Kähler form on \( Y \) which represents \( f_* [\alpha] \). \( \square \)

2.3 Gysin morphisms and projection formula for varieties with quotient singularities

Let \( f : X \to Y \) be a proper continuous map between two closed rational homology manifolds\(^{(2)}\) (e.g. complex varieties with at worst quotient singularities [7, Proposition A.1 (iii)]). Then the Poincaré duality holds for \( X \) and \( Y \) (see [25, Proof of V.3.2]), which allows us to define the Gysin morphism

\[
f_* : \text{H}^k(X, \mathbb{Z}) \xrightarrow{\text{PD}} \text{H}^0_{\text{dim}X-k}(X, \mathbb{Z}) \xrightarrow{f^*} \text{H}^k_{\text{dim}X-k}(Y, \mathbb{Z}) \xrightarrow{\text{PD}} \text{H}^{k-r}(Y, \mathbb{Z}),
\]

where \( r = \dim X - \dim Y \) and PD denotes the Poincaré duality. The following is the reformulation of the projection formula (see e.g. [25, IX.3.7]) in terms of Gysin morphism.

Proposition 2.3 (Projection formula). — Given \( \alpha \in \text{H}^k(X, \mathbb{Q}) \) and \( \beta \in \text{H}^l(Y, \mathbb{Q}) \), we have

\[
f_*(\alpha \cdot f^* \beta) = f_* \alpha \cdot \beta.
\]

\(^{(1)}\)Note that since \( X/G \) is reduced, the condition ii) in [39, Theorem 1] holds automatically; see [39, Remark II.2.2].

\(^{(2)}\)A closed rational homology manifold of dimension \( n \) is a compact topological space \( X \) such that for every \( x \in X \), we have \( H_i(X, X\setminus \{x\}, \mathbb{Q}) = \mathbb{Q} \) if \( i = n \) and \( H_i(X, X\setminus \{x\}, \mathbb{Q}) = 0 \) if \( i \neq n \).
2.4 An isomorphism statement about Gysin morphisms

**Lemma 2.4.** — Let \( f : X \to U \) be a surjective proper morphism between complex manifolds with equidimensional connected fibers. Let \( n = \dim X \) and \( m = \dim U \). Assume that \( U \) is affine, then the Gysin morphism

\[
f_\ast : H^{2n-m}(X, \mathbb{C}) \to H^m(U, \mathbb{C}).
\]

is an isomorphism.

**Proof.** — Let \( d := n - m \). Let \( E^{p,q}_r \) be the Leray spectral sequence computing \( H^*(X, \mathbb{C}) \) through \( f \). Since every fiber of \( f \) has dimension \( d \), we have \( R^q f_* \mathbb{C} = 0 \) for every \( q > 2d \). As \( U \) is affine, by Artin vanishing [14, Theorem 4.1.26] \( E^{p,q}_2 = H^p(U, R^q f_* \mathbb{C}) = 0 \) for every \( p > m \). This implies that the only non-vanishing \( E^{p,q}_2 \) is isomorphic to the constant sheaf \( \mathbb{C} \) over \( U \) by Poincaré duality. So \( j_\ast j^\ast (R^d f_* \mathbb{C}) = \mathbb{C} \) and the natural morphism

\[
\Phi : R^d f_* \mathbb{C} \to j_\ast j^\ast (R^d f_* \mathbb{C}) \cong \mathbb{C}
\]

has kernel \( K := \ker \Phi \) supported on \( U \setminus U' \). As \( f \) has equidimensional fibers, \( f \) has local multi-sections around every point of \( U \), which implies that \( \Phi \) is surjective. Since

\[
H^m(U, R^d f_* \mathbb{C}) \cong H^{2n-m}(X, \mathbb{C}) \xrightarrow{f_\ast} H^m(U, \mathbb{C})
\]

is isomorphic to the morphism induced by \( \Phi \), it suffices to prove that \( H^i(U, K) = 0 \) for \( i = m \) and \( i = m + 1 \). Since \( \dim \text{supp} K \leq \dim U - 1 \), by [14, Proposition 5.1.16] we have \( K[\dim U - 1] \in pD^{\geq 0}(U) \) where \( p \) is the middle perversity. It follows from Artin vanishing [14, Corollary 5.2.18] that \( H^i(U, K) = 0 \) for every \( i \geq \dim U = m \).

**Remark 2.5.** — In Lemma 2.4, we need to assume that \( f \) is equidimensional. As a counterexample, let \( C \) be any smooth projective curve and consider the blowup \( \tilde{C} \times \mathbb{C}^2 \) of \( C \times \mathbb{C}^2 \) at one point. Then \( \tilde{C} \times \mathbb{C}^2 \to \mathbb{C}^2 \) does not satisfy the conclusion of Lemma 2.4.

2.5 Maps between cohomology spaces and Hodge classes

We collect some well-known results about maps between cohomology spaces and include the proofs for completeness.

**Lemma 2.6.** — Let \( f : X \to Y \) be a proper surjective morphism from a compact Kähler manifold \( X \). We have

\[
\ker (f^\ast : H^k(Y, \mathbb{Q}) \to H^k(X, \mathbb{Q})) = W_{k-1}H^k(Y, \mathbb{Q})
\]

and dually,

\[
\text{Im} (f_\ast : H^k(X, \mathbb{Q}) \to H^k(Y, \mathbb{Q})) = W_{-k}H^k(Y, \mathbb{Q}).
\]
Proof. — Since \( H^k(X, \mathbb{Q}) \) is a pure Hodge structure of weight \( k \), the first statement follows from [35, Corollary 5.43]\(^3\). Taking the dual, we have

\[
\text{coker}(f_* : H_k(X, \mathbb{Q}) \to H_k(Y, \mathbb{Q})) = H_k(Y, \mathbb{Q})/W_{-k}H_k(Y, \mathbb{Q}),
\]

which proves the second statement.

\[\square\]

Lemma 2.7. — Let \( X \) be a compact Kähler manifold of dimension \( n \) and let \( Y \subset X \) be an irreducible closed subvariety of \( X \). Let \( \iota : \bar{Y} \to Y \to X \) be the composition of a Kähler desingularization of \( Y \) with the inclusion \( \iota : Y \to X \). For every \( k \), we have

\[
\text{Im}(\iota_* : H_{2n-k}(\bar{Y}, \mathbb{Q}) \to H^k(X, \mathbb{Q})) = \ker(H^k(X, \mathbb{Q}) \to H^k(X \setminus Y, \mathbb{Q})).
\]

Proof. — We have the exact sequence

\[
H^k(X, \mathbb{Q}) \to H^k(X, \mathbb{Q}) \to H^k(X \setminus Y, \mathbb{Q}) (2.3)
\]

and by Poincaré duality [25, Proposition II.9.2], \( \iota_* \) is isomorphic to

\[
\iota_* : H_{2n-k}(Y, \mathbb{Q}) \to H_{2n-k}(X, \mathbb{Q}). (2.4)
\]

Since \( H_{2n-k}(X, \mathbb{Q}) \) is a pure Hodge structure of weight \( k - 2n \), we have

\[
\iota_* W_{k-2n}H_{2n-k}(Y, \mathbb{Q}) = \text{Im}(\iota_*) (2.5)
\]

by the strictness of the morphism \( \iota_* \) of mixed Hodge structures. Finally, we have

\[
\text{Im}(\nu_* : H_{2n-k}(\bar{Y}, \mathbb{Q}) \to H_{2n-k}(Y, \mathbb{Q})) = W_{k-2n}H_{2n-k}(Y, \mathbb{Q}) (2.6)
\]

by Lemma 2.6. Combining (2.3), (2.4), (2.5), (2.6) proves Lemma 2.7.

\[\square\]

Lemma 2.8. — Let \( X \) be a compact complex variety in the Fujiki class \( \mathcal{C} \) and let \( \nu : \bar{X} \to X \) be a desingularization of \( X \). If \( X \) has at worst rational singularities, then \( \ker(\nu_* : H_2(\bar{X}, \mathbb{C}) \to H_2(X, \mathbb{C})) \) consists of Hodge classes.

Proof. — It suffices to show that, dually,

\[
\text{coker}(\nu^* : H^2(X, \mathbb{C}) \to H^2(\bar{X}, \mathbb{C}))
\]

consists of Hodge classes. Since \( X \) has at worst rational singularities, the image of \( \nu^* \) contains \( H^{2,0}(\bar{X}) \) by [1, Lemma 2.1]. Hence the above cokernel consists of Hodge classes.

\[\square\]

2.6 Campana’s criterion

A compact complex variety \( X \) is called algebraically connected if for every general pair of points \( x, y \in X \), there exists a connected proper curve \( C \subset X \) such that \( x, y \in C \). The following criterion for a variety in the Fujiki class \( \mathcal{C} \) to be Moishezon is due to Campana.

\[\footnote{The existence of Kähler desingularizations of varieties in the Fujiki class \( \mathcal{C} \) allows one to extend Deligne’s mixed Hodge theory on complex algebraic varieties to Zariski open subvarieties of compact complex varieties in the Fujiki class \( \mathcal{C} \) [17, §1]. The cited result in [35, Chapter 5] proven for complex algebraic varieties generalizes to this larger context.}

\[\square\]
Theorem 2.9 (Campana [11, Corollaire on p.212]). — A compact complex variety $X$ in the Fujiki class $\mathcal{C}$ is Moishezon if and only if it is algebraically connected.

We list some direct consequences of Campana’s criterion.

Corollary 2.10. — Let $X$ be a compact complex variety in the Fujiki class $\mathcal{C}$ and let $f : X \to B$ be a fibration. Assume that a general fiber of $f$ and $B$ are both Moishezon. Then $X$ is Moishezon if and only if $f$ has a multi-section.

Corollary 2.11 ([29, Corollary 2.12]). — Let $X$ be a compact complex variety in the Fujiki class $\mathcal{C}$ and $f : X \to B$ a $\mathbb{P}^1$-fibration. If $B$ is Moishezon, then $X$ is Moishezon.

Lemma 2.12. — Let $X$ be a compact Kähler manifold with $a(X) = \dim X - 1$. The algebraic reduction $f : X \to B$ of $X$ is almost holomorphic whose general fiber is an elliptic curve.

Proof. — We already know by [37, Theorem 12.4] that any resolution of $f$ is an elliptic fibration. It remains to show that $f$ is almost holomorphic. Let $\bar{f} : \bar{X} \to B$ be a resolution of $f$ by a compact Kähler manifold $\bar{X}$. Let $E \subset \bar{X}$ be the exceptional divisor of $\bar{X} \to X$. Since $X$ is not Moishezon and since $B$ is Moishezon and a general fiber of $\bar{f}$ is a curve, Corollary 2.10 implies that $E$ does not dominate $B$. Therefore $f$ is almost holomorphic.

### 2.7 Bimeromorphic models of compact Kähler threefolds with $a \leq 1$

Bimeromorphic models of compact Kähler threefolds of algebraic dimension $a \leq 1$ had essentially been classified by Fujiki [18] (see also [29, Proposition 1.9]). In this subsection, we state this classification result in the following form for our needs.

Proposition 2.13. — Let $X_0$ be a compact Kähler threefold such that $a(X_0) \leq 1$. Assume that $X_0$ is not a simple non-Kummer threefold. Then $X_0$ is bimeromorphic to a compact complex manifold $X$ in the Fujiki class $\mathcal{C}$, satisfying one of the following descriptions.

i) $X$ is the total space of a $\mathbb{P}^1$-fibration $X \to S$ over a smooth compact Kähler surface $S$.

ii) $X = (S \times F)/G$ where $S$ is a non-algebraic smooth Kähler surface, $F$ is a smooth curve, and $G$ is a finite group acting diagonally on $S \times F$.

iii) $X$ is the total space of a fibration $f : X \to B$ over a smooth projective curve $B$ whose general fiber is an abelian variety.

iv) $X = \hat{X}/G$ where $G$ is a finite group and $\hat{X}$ is the total space of a $G$-equivariant smooth isotrivial fibration $\hat{f} : \hat{X} \to \hat{B}$ in non-algebraic 2-tori without multi-sections.

v) $X$ is the quotient $T/G$ of a 3-torus by a finite group.

In i) and iii), we can choose $X$ to be a compact Kähler manifold.

Proof. — First we assume that $X_0$ is uniruled. Since $X_0$ is non-algebraic, the base of the MRC fibration $X_0 \to S_0$ is a surface (see e.g. [8, Proof of Theorem 9.1]). Resolving $X_0 \to S_0$ by some Kähler desingularizations of $X_0$ and $S_0$ gives rise to a $\mathbb{P}^1$-fibration $X \to S$ as in i).
Now assume that $X_0$ is not uniruled. If $a(X_0) = 0$, then by [18, Theorem in p.236], since $X_0$ is not a simple non-Kummer threefold by assumption, necessarily $X_0$ is bimeromorphic to a quotient $T/G$ of a 3-torus $T$ by a finite group $G$ as in $v$).

Assume that $a(X_0) = 1$, then by [18, Theorem in p.236 and Theorem 3], $X_0$ is bimeromorphic to a threefold $X$ such that

- either $X$ satisfies $ii$;
- or $X$ is the total space of a fibration $f : X \to B$ over a smooth projective curve such that a general fiber $F$ of $f$ a 2-torus, and $f$ satisfies "Property (A)" (namely, for any fibration $g : X' \to B$ and any bimeromorphic map $\phi : X \to X'$ over $B$, there exists a Zariski open $U \subset B$ such that $\phi$ induces an isomorphism $f^{-1}(U) \cong g^{-1}(U)$).

Given a fibration $f : X \to B$ as in the second case. By [18, Lemma 11.1], the fact that $f$ has "Property (A)" implies that $f$ has no multi-section. It also implies that if $X' \to X$ is a Kähler desingularization of $X$ then a general fiber of the composition $X' \to X \to B$ is still a 2-torus. Therefore up to replacing $X$ by $X'$, we can assume that $X$ is a compact Kähler manifold. Assume that $F$ is not algebraic, then by [18, Remark 13.1], $X_0$ is bimeromorphic to the quotient the total space of a $G$-equivariant smooth isotrivial torus fiberation $f' : \bar{X} \to \bar{B}$ by $G$. This shows that $f$ satisfies the description $iv$) in Proposition 2.13. Hence the bimeromorphic descriptions of $X$ in Proposition 2.13 is exhaustive. \hfill $\square$

## 3 Dual positive cones under dominant meromorphic maps

### 3.1 Dual Kähler cones

First we prove that the existence of rational classes in the interior of the dual Kähler cone is invariant under bimeromorphic modifications (see Corollary 3.2 for a more general statement). The statement can be reduced to the special case of blow-ups along smooth centers.

**Proposition 3.1.** — Let $X$ be a compact Kähler manifold and let $v : \bar{X} \to X$ be the blow-up of $X$ along a submanifold $Y \subset X$. If $\text{Int}(\mathcal{K}(X)) \cap H^{2n-2}(X, \mathbb{Q}) \neq \emptyset$, then $\text{Int}(\mathcal{K}(\bar{X})) \cap H^{2n-2}(\bar{X}, \mathbb{Q}) \neq \emptyset$.

In dimension 3, Proposition 3.1 was proven by Oguiso and Peternell in [33, Proposition 2.1]. We prove Proposition 3.1 in arbitrary dimension with a different argument, relying on [36, Théorème 1] as a key ingredient.

**Proof of Proposition 3.1.** — Let $E = v^{-1}(Y)$ be the exceptional divisor and let $\ell$ be a line in $v^{-1}(y)$ for some $y \in Y$. Then every element $\bar{\gamma} \in H^{1,1}(\bar{X}, \mathbb{R})$ is of the form $\bar{\gamma} = v'\gamma - rE$ where $\gamma := v_\ast\bar{\gamma} \in H^{1,1}(X, \mathbb{R})$ and $r$ is some real number. Fix $\alpha \in \text{Int}(\mathcal{K}(X)) \cap H^{2n-2}(X, \mathbb{Q})$. We will construct $q \in \mathbb{Q}_{>0}$ satisfying

\[(v'\alpha + q\ell)(v'\gamma - rE) > 0 \quad (3.1)\]

for all $\gamma \in H^{1,1}(X, \mathbb{R})$ and $r \in \mathbb{R}$ such that $v'\gamma - rE \neq 0$ is nef; this implies

\[v'\alpha + q\ell \in \text{Int}(\mathcal{K}(\bar{X})) \cap H^{2n-2}(\bar{X}, \mathbb{Q}).\]
Note that if \( v' \gamma - rE \) is nef, then
\[
  r = (v' \gamma - rE) \cdot \ell \geq 0.
\]
Also note that if moreover \( v' \gamma - rE \neq 0 \), then \( \gamma \neq 0 \), because otherwise \(-rE \in K(X)\) implies \( r = 0 \). For every \( \gamma \in v_* K(X) \subset H^{1,1}(X, \mathbb{R}) \), define
\[
  r_\gamma := \min \{ r \in \mathbb{R} \mid v' \gamma - rE \text{ is nef} \} \geq 0.
\]
Let
\[
  \mathcal{C} := \left\{ \gamma \in v_* K(X) \mid \alpha \cdot \gamma > 0 \right\}.
\]
Fix a norm \( \| \cdot \| \) on \( H^{1,1}(X, \mathbb{R}) \). For every subset \( \Sigma \subset H^{1,1}(X, \mathbb{R}) \), define
\[
  \Sigma_i := \{ \sigma \in \Sigma \mid \|\sigma\| = 1 \}.
\]
**Claim.** — We have
\[
  R := \inf \left\{ r_\gamma \mid \gamma \in \left( v_* K(X) \right)_1 \setminus \mathcal{C} \right\} > 0.
\]
**Proof.** — Since \( r_\gamma \geq 0 \) for each \( \gamma \), we have \( R \geq 0 \). Assume to the contrary that \( R = 0 \). Then there exists a sequence \( \gamma_i \) in \( \left( v_* K(X) \right)_1 \setminus \mathcal{C} \) such that \( \lim_{i \to \infty} r_{\gamma_i} = 0 \). Up to extracting a subsequence, we can assume that \( \gamma := \lim_{i \to \infty} \gamma_i \in H^{1,1}(X, \mathbb{R})_1 \) exists, so
\[
  v' \gamma = \lim_{i \to \infty} (v' \gamma_i - r_{\gamma_i} E)
\]
is nef. Thus \( \gamma \) is nef by [36, Théorème 1], so \( \alpha \cdot \gamma > 0 \) (because \( \alpha \in \text{Int}(K(X')) \)). It follows that \( \alpha \cdot \gamma_i > 0 \) for \( i \gg 0 \), contradicting \( \gamma \notin \mathcal{C} \). Hence \( R > 0 \).

Let \( M < 0 \) such that \( M \leq c \) for all \( c \in H^{1,1}(X, \mathbb{R})_1 \). By the above claim, we can find \( q \in \mathbb{Q}_{\geq 0} \) such that
\[
  M + qR > 0.
\]
Let \( \gamma \in H^{1,1}(X, \mathbb{R}) \) and \( r \in \mathbb{R} \) such that \( v' \gamma - rE \neq 0 \) is nef (so \( \gamma \in v_* K(X) \), \( \gamma \neq 0 \), and \( r \geq 0 \)). If \( \gamma \in \mathcal{C} \), then
\[
  (v' \alpha + q \ell)(v' \gamma - rE) = \alpha \cdot \gamma + qr > 0.
\]
Suppose that \( \gamma \notin \mathcal{C} \), then
\[
  (v' \alpha + q \ell)(v' \gamma - rE) = \|\gamma\| \left( \frac{\gamma}{\|\gamma\|} + \alpha \frac{\ell}{\|\gamma\|} \right)
\]
\[
  \geq \|\gamma\| \left( M + qr_\gamma/\|\gamma\| \right) \geq \|\gamma\| (M + qR) > 0,
\]
where the first inequality follows from the nefness of \( v' \frac{\gamma}{\|\gamma\|} - \frac{rE}{\|\gamma\|} \), and the second inequality from \( \frac{\gamma}{\|\gamma\|} \in \left( v_* K(X) \right)_1 \setminus \mathcal{C} \). Hence (3.1) holds regardless whether \( \gamma \in \mathcal{C} \) or not, which finishes the proof.

**Corollary 3.2.** — Let \( f : X \to Y \) be a dominant meromorphic map between compact Kähler manifolds. If \( \text{Int}(K(X')) \cap H_2(X, \mathbb{Q}) \neq \emptyset \), then \( \text{Int}(K(Y')) \cap H_2(Y, \mathbb{Q}) \neq \emptyset \).
Proof. — Let $X \xrightarrow{\nu} \tilde{X} \xrightarrow{f} Y$ be a resolution of $f$ by a sequence of blow-ups $\nu : \tilde{X} \to X$ along smooth centers. By Proposition 3.1, there exists $\tilde{\alpha} \in \text{Int}(\mathcal{K}(\tilde{X})) \cap H_2(\tilde{X}, \mathbb{Q})$. We conclude by [33, Proposition 2.5] that $f_*\tilde{\alpha} \in \text{Int}(\mathcal{K}(Y)) \cap H_2(Y, \mathbb{Q})$. □

We also have the following for dual Kähler cones.

Lemma 3.3. — Let $f : X \to Y$ be a finite morphism between compact Kähler manifolds. If $\alpha \in \text{Int}(\mathcal{K}(Y))$, then $f^*\alpha \in \text{Int}(\mathcal{K}(X))$.

Proof. — Fix a Kähler class $\omega_Y \in H^2(Y, \mathbb{R})$ on $Y$. Since $f$ is finite, $f^*\omega_Y$ is a Kähler class on $X$. Thus for every nonzero nef class $\omega \in \mathcal{K}(X)$, we have $(f_*\omega) \cdot \omega_Y = f^*f_*\omega_Y > 0$ where $n = \dim X$, so $f_*\omega \neq 0$ in $H^2(Y, \mathbb{R})$.

By Lemma 2.2, we have $f_*\omega \in \mathcal{K}(Y)$. It follows that $f^*\alpha \cdot \omega = \alpha \cdot f_*\omega > 0$. Hence $f^*\alpha \in \text{Int}(\mathcal{K}(X))$. □

3.2 Dual pseudoeffective cones

We start with the easy observation that the interior of the dual pseudoeffective cone is stable under pushforwards by surjective morphisms.

Lemma 3.4. — Let $f : X \to Y$ be a surjective map between compact Kähler manifolds. If $\alpha \in \text{Int}(\text{Psef}(X))$, then $f_*\alpha \in \text{Int}(\text{Psef}(Y))$.

Proof. — For every $\gamma \in \text{Psef}(Y) \setminus \{0\}$, since $\alpha \in \text{Int}(\text{Psef}(X))$ and $f^*\gamma \in \text{Psef}(X) \setminus \{0\}$, we have $(f_*\alpha) \cdot \gamma = \alpha \cdot f^*\gamma > 0$. Hence $f_*\alpha \in \text{Int}(\text{Psef}(Y))$. □

The following result is the analog of Proposition 3.1 for pseudoeffective cone.

Proposition 3.5. — Let $X$ be a compact Kähler manifold and let $\nu : \tilde{X} \to X$ be the blow-up of $X$ along a submanifold $Y \subset X$. If $\text{Int}(\text{Psef}(X)) \cap H_2(X, \mathbb{Q}) \neq \emptyset$, then $\text{Int}(\text{Psef}(\tilde{X})) \cap H_2(\tilde{X}, \mathbb{Q}) \neq \emptyset$.

Proof. — Fix $\alpha \in \text{Int}(\text{Psef}(X)) \cap H^{2n-2}(X, \mathbb{Q})$. Let $E = \nu^{-1}(Y)$ be the exceptional divisor and let $\ell$ be a line in $\nu^{-1}(y)$ for some $y \in Y$. Then every element $\tilde{\gamma} \in H^{1,1}(\tilde{X}, \mathbb{R})$ is of the form $\tilde{\gamma} = \nu^*\gamma + r E$ for some $\gamma \in H^{1,1}(X, \mathbb{R})$ and $r \in \mathbb{R}$. If moreover $\tilde{\gamma} \in \text{Psef}(\tilde{X})$, then $\gamma = \nu_*\tilde{\gamma}$ is also pseudoeffective. Therefore to prove the proposition, it suffices to find $q \in \mathbb{Q}_{>0}$ such that $(\nu^*\alpha - q \ell) \cdot (\nu^*\gamma + r E) > 0$ for every $\gamma \in \text{Psef}(X)$ and $r \in \mathbb{R}$ such that $\nu^*\gamma + r E \in \text{Psef}(\tilde{X}) \setminus \{0\}$.

Fix a norm $\| \cdot \|$ on $H^2(X, \mathbb{R})$ and let

$$\text{Psef}(X)_1 = \text{Psef}(X) \cap \left\{ \gamma \in H^2(X, \mathbb{R}) \mid \| \gamma \| = 1 \right\}.$$

For every $\gamma \in \text{Psef}(X)_1$, let $r_\gamma = \inf \left\{ r \in \mathbb{R} \mid \nu^*\gamma + r E \in \text{Psef}(\tilde{X}) \right\}$. Since $\nu^*\gamma \in \text{Psef}(\tilde{X})$, we have $r_\gamma \leq 0$. As $\alpha \cdot \gamma > 0$ for every $\gamma \in \text{Psef}(X)_1$ and both $\gamma \mapsto \alpha \cdot \gamma$ and $\gamma \mapsto r_\gamma$ are continuous functions defined on the compact set $\text{Psef}(X)_1$, there exists $q \in \mathbb{Q}_{>0}$ such that

$$\alpha \cdot \gamma + qr_\gamma > 0$$

for all $\gamma \in \text{Psef}(X)_1$. 

\[ \text{ON THE DUAL POSITIVE CONES AND THE ALGEBRAICITY OF A COMPACT KÄHLER MANIFOLD} \]
Now let $\gamma \in \text{Psef}(X)$ and $r \in \mathbb{R}$ such that $\nu' \gamma + r E \in \text{Psef}(\tilde{X}) \setminus \{0\}$. If $\gamma = 0$, then $r > 0$, so

$$(\nu' \alpha - q \ell) \cdot (r E) = qr > 0.$$ 

If $\gamma \neq 0$ then $\frac{\nu' \gamma}{\|\gamma\|} \geq r \gamma/\|\gamma\|$, so we also have

$$(\nu' \alpha - q \ell) \cdot (\nu' \gamma + r E) = \alpha \cdot \gamma + qr = \|\gamma\| \left( \alpha \cdot \frac{\gamma}{\|\gamma\|} + q \frac{r}{\|\gamma\|} \right) \geq \|\gamma\| \left( \alpha \cdot \frac{\gamma}{\|\gamma\|} + \frac{qr \gamma}{\|\gamma\|} \right) > 0.$$ 

□

**Remark 3.6.** — In the setting of Proposition 3.5, it is not true that $\alpha \in \text{Int}(\text{Psef}(X)^{\nu})$ implies $\nu' \alpha \in \text{Int}(\text{Psef}(\tilde{X})^{\nu})$ (which is already false when $\nu$ is the blow-up of $\mathbb{P}^2$ along a point). However, since elements of $\text{Psef}(X)^{\nu}$ can be represented by closed positive smooth $(\dim X - 1, \dim X - 1)$-forms [23, Lemma 4.4], if $f : X \to Y$ is a generically finite surjective morphism between compact Kähler manifolds, then $\alpha \in \text{Psef}(Y)^{\nu}$ implies $f' \alpha \in \text{Psef}(X)^{\nu}$.

As an immediate consequence of Lemma 3.4 and Proposition 3.5, we have the following.

**Corollary 3.7.** — Let $f : X \to Y$ be a dominant meromorphic map between compact Kähler manifolds. If $\text{Int}(\text{Psef}(X)^{\nu}) \cap H_2(X, \mathbb{Q}) \neq \emptyset$, then $\text{Int}(\text{Psef}(Y)^{\nu}) \cap H_2(Y, \mathbb{Q}) \neq \emptyset$.

**Proof.** — Let $X \leftarrow \tilde{X} \xrightarrow{\nu} Y$ be a resolution of $f$ by a sequence of blow-ups $\nu : \tilde{X} \to X$ along smooth centers. By Proposition 3.5, there exists $\tilde{\alpha} \in \text{Int}(\text{Psef}(\tilde{X})^{\nu}) \cap H_2(\tilde{X}, \mathbb{Q})$. We conclude by Lemma 3.4 that $f^* \tilde{\alpha} \in \text{Int}(\text{Psef}(Y)^{\nu}) \cap H_2(Y, \mathbb{Q})$. □

### 4 Smooth torus fibrations

In this section we study the Oguiso–Peternell problem for smooth torus fibrations. The argument involves the Deligne cohomology in an essential way, and the reader is referred to e.g. [15, §2 and §3] for a reference. See also [12, §2].

Let $f : X \to B$ be a smooth torus fibration such that $X$ and $B$ are compact Kähler manifolds and let $g = \dim X - \dim B$. Recall that the (absolute) Deligne complex $D_X(g)$ is defined as

$$\cdots \to Z_X \xrightarrow{x(2\pi i \sqrt{-1})} O_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^{g-1} \to 0 \xrightarrow{\cdots},$$

where $Z_X$ is placed at the 0th degree. The Deligne cohomology group of degree $2g$ is defined by $H^{2g}_D(X, Z(g)) = H^{2g}(X, D_X(g))$. We have the short exact sequence of complexes

$$0 \to \Omega_X^{*g-1} \to D_X(g) \to Z_X \to 0 \quad (4.1)$$

which gives rise to the regulator map

$$cl : H^{2g}_D(X, Z(g)) = H^{2g}(X, D_X(g)) \to H^{2g}(X, Z) \quad (4.2)$$
with image equal to 
\[ \ker(H^2g(X, \mathbb{Z}) \to H^2g(X, \Omega^*_{X/B}^{g-1})) = H^{\phi,g}(X, \mathbb{Z}). \]

Similarly, the relative Deligne complex \( \underline{D}_{X/B}(g) \) is defined as
\[ \cdots 0 \to \mathbb{Z}_X \xrightarrow{\times(2n \sqrt{d}')} \mathcal{O}_{X/B}^d \to \Omega^1_{X/B} \to \cdots \to \Omega_{X/B}^{g-1} \to 0 \to \cdots, \]
and we have the short exact sequence of complexes
\[ 0 \to \Omega^*_{X/B}^{g-1}[1] \to \underline{D}_{X/B}(g) \to \mathbb{Z}_X \to 0, \quad (4.3) \]
Applying \( Rf \) to (4.3), together with the vanishing \( R^{2g}f_*\Omega^*_{X/B}^{g-1} = 0 \) (because \( f \) is smooth of relative dimension \( g \)), we obtain a short exact sequence
\[ 0 \to \mathcal{J} \to R^{2g}f_*\underline{D}_{X/B}(g) \to R^{2g}f_*\mathbb{Z}_X \cong \mathbb{Z}_B \to 0 \quad (4.4) \]
where
\[ \mathcal{J} := \text{coker}(R^{2g-1}f_*\mathbb{Z}_X \to R^{2g-1}f_*\Omega^*_{X/B}^{g-1}) \]

The natural map \( \Omega^*_X \to \Omega^*_X \) of de Rham complexes induces a morphism of exact sequences from (4.1) to (4.3), which further induces the commutative diagram
\[ \begin{array}{ccc}
H^2d(X, \mathbb{Z}(g)) & \xrightarrow{\epsilon} & H^{\phi,g}(X, \mathbb{Z}) \\
\downarrow & & \downarrow f \\
H^0(B, R^{2g}f_*\underline{D}_{X/B}(g)) & \xrightarrow{g} & H^0(B, \mathbb{Z}) \to H^1(B, \mathcal{J}) \end{array} \quad (4.5) \]
where the vertical arrow on the left is the composition of \( H^2d(X, \mathbb{Z}(g)) \to H^2d(X, \underline{D}_{X/B}(g)) \) induced by the natural map \( \underline{D}_X(g) \to \underline{D}_{X/B}(g) \) with \( H^2d(X, \underline{D}_{X/B}(g)) \to H^0(B, R^{2g}f_*\underline{D}_{X/B}(g)) \) and the second row is an exact sequence induced by (4.4).

Finally the sheaf \( \mathcal{J} \) is isomorphic to the sheaf of germs of sections of the Jacobian fibration \( p : J \to B \) associated to \( f \). For every \( J \)-torsor \( f : X \to B \), let \( \eta(f) = \delta(1) \in H^1(B, \mathcal{J}) \). This defines a bijection
\[ \eta : \{ \text{Isomorphism classes of } J \text{-torsors} \} \xrightarrow{\sim} H^1(B, \mathcal{J}) \quad (4.6) \]
and \( \eta(f) \) is torsion if and only if \( f \) has a multi-section (which can be chosen \( \text{étale} \) over \( B \)) \cite[Proposition 2.1 and 2.2]{12}.

**Lemma 4.1.** — Let \( f : X \to B \) be a smooth torus fibration of relative dimension \( g \) over a compact complex manifold. If \( f_* : H^{\phi,g}(X, \mathbb{Q}) \to H^0(B, \mathbb{Q}) \) is surjective, then \( f \) has an \( \text{étale} \) multi-section.

**Proof.** — It suffices to show that \( \eta(f) \) is torsion. Since \( f_* : H^{\phi,g}(X, \mathbb{Q}) \to H^0(B, \mathbb{Q}) \) is surjective, there exists \( \alpha \in H^{\phi,g}(X, \mathbb{Z}) \) such that \( f_*\alpha \in H^0(B, \mathbb{Z}) \) is nonzero. By (4.5), \( f_*\alpha \) lifts to an element of \( H^0(B, R^{2g}f_*\underline{D}_{X/B}(g)) \), so \( \delta(f,\alpha) = 0 \). Hence \( \eta(f) = \delta(1) \) is torsion. \( \square \)
Proposition 4.2. — Let $X_0$ be a compact Kähler manifold of dimension $n$ such that $\text{Int}(\mathcal{K}(X_0)^\vee)$ contains a rational class $\alpha \in H^{2n-2}(X_0, \mathbb{Q})$. Assume that $X_0$ is bimeromorphic to $X/G$ where $G$ is a finite group and $X$ is the total space of a $G$-equivariant smooth torus fibration $f : X \to B$ over a smooth curve $B$, then $f$ has an étale multi-section.

Proof. — Let $X_0 \leftarrow \tilde{X} \to X/G$ be a resolution of a bimeromorphic map $X_0 \dasharrow X/G$ by a sequence of blow-ups $\tilde{X} \to X_0$ along smooth centers. By Proposition 3.1, $\text{Int}(\mathcal{K}(\tilde{X})^\vee)$ contains a rational class $\beta$. Let $X_0 \leftarrow \tilde{X} \to X/G$ be a resolution of the rational map $X \to X/G$ by a compact Kähler manifold $\tilde{X}'$. The situation is summarized in the commutative diagram

\[ \begin{array}{ccc}
\tilde{X}' & \xrightarrow{p} & X \\
\downarrow & \ & \downarrow f \\
\tilde{X} & \xrightarrow{\beta} & X/G \\
\downarrow & \ & \downarrow \\
\tilde{X} & \xrightarrow{\tilde{f}} & B/G
\end{array} \]

With the notations therein,

\[ r, f, p, q^* \beta = \tilde{f}, q, q^* \beta = \deg(q) \cdot \tilde{f}, \beta \neq 0 \in H^0(B/G, \mathbb{Q}) \]

where the non-vanishing follows from [33, Proposition 2.5]. In particular if $\alpha := p, q^* \beta$, then $f, \alpha \neq 0$ in $H^0(B, \mathbb{Q})$. Therefore $f_\ast : H^{n-1,n-1}(X, \mathbb{Q}) \to H^0(B, \mathbb{Q})$ is surjective and we apply Lemma 4.1 to conclude. $\square$

5 Algebraicity of the Albanese variety

The aim of this section is to prove Theorem 1.4, answering Problem 1.1 in terms of the Albanese variety for every compact Kähler manifold.

First we study the Oguiso–Peternell problem for complex tori (and their finite quotients for later use).

Proposition 5.1. — Let $T$ be a complex torus and $G$ a finite group acting on $T$. If there exists $\beta \in H^{n-1,n-1}(T, \mathbb{Q})^G$ such that $\beta \cdot \omega \neq 0$ for every $\omega \neq 0 \in \mathcal{K}(T)^G$, then $T$ is projective.

We will need the following lemma.

Lemma 5.2 (Poincaré’s formula). — Let $L$ be a line bundle on the complex torus $T$ of dimension $n$. Assume that

\[ d := \frac{c_1(L)^n}{n!} \neq 0, \]

and let

\[ c_L := \frac{c_1(L)^{n-1}}{(n-1)!} \in H^{2n-2}(T, \mathbb{Q}). \]

Then for every integer $p \in [0, n]$, we have

\[ \frac{c_1(L)^p}{p!} \ast \frac{c_L \ast_{n-p}}{(n-p)!} \in H^{2p}(T, \mathbb{Q}), \]

where $\ast$ is the Pontryagin product on $H^\ast(T, \mathbb{Q})$. 

This lemma is well-known, we provide a proof for the sake of completeness.

Proof. — Let $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n$ be a symplectic basis of $H_1(T, \mathbb{Z})$ for $L$ (see [5, §3.1]) and let $dx_1, \ldots, dx_n, dy_1, \ldots, dy_n$ be the associated dual basis of $H^1(T, \mathbb{Z})$. Assume that $L$ is of type $(d_1, \ldots, d_n)$. By [5, Lemma 3.6.4], we have

$$c_1(L) = -\sum_{i=1}^{n} d_i \cdot dx_i \wedge dy_i, \quad (5.1)$$

and also

$$d = (-1)^d_1 \cdots d_n \quad (5.2)$$

by [5, Theorem 3.6.1], where $s$ is number of negative eigenvalues of the Hermitian form associated to $L$.

Computation thus shows that

$$c_L = \frac{c_1(L)^{n-1}}{(n-1)!d} = \sum_{i=1}^{n} \frac{(-1)^{s+n-1}}{d_i} dx_i \wedge dy_i, \quad (5.3)$$

where

$$dx_i \wedge dy_i := (dx_1 \wedge dy_1) \wedge \cdots \wedge (dx_{i-1} \wedge dy_{i-1}) \wedge (dx_{i+1} \wedge dy_{i+1}) \wedge \cdots \wedge (dx_n \wedge dy_n).$$

Let $PD : H^*(T, \mathbb{Z}) \to H_{2n-*}(T, \mathbb{Z})$ denote the Poincaré duality. For every subset $I$ of $[1, n] \cap \mathbb{Z}$, we deduce from [5, Lemma 3.6.5, Lemma 4.10.1] that

$$PD\left(\bigwedge_{i \in I} (dx_i \wedge dy_i)\right) = (-1)^{s+n} \bigwedge_{i \in I} (\lambda_i \star \mu_i) \quad (5.4)$$

where $I^\circ := [1, n] \cap \mathbb{Z}\setminus I$. It follows from (5.3) and (5.4) that

$$PD\left(\frac{c_L^{n-p}}{(n-p)!} \right) = \frac{PD(c_L)^{n-p}}{(n-p)!} = (-1)^{s+n-p} \sum_{I} \left(\prod_{i \in I} d_i \right)^{-1} \bigwedge_{i \in I} (\lambda_i \star \mu_i),$$

where $I$ in the sum runs through all subsets of $[1, n] \cap \mathbb{Z}$ of cardinal $n-p$, and thus

$$\frac{c_L^{n-p}}{(n-p)!} = (-1)^{s+n-p} \sum_{I} \left(\prod_{i \in I} d_i \right)^{-1} \bigwedge_{i \in I} (dx_i \wedge dy_i),$$

again by (5.4). Hence

$$d \cdot \frac{c_L^{n-p}}{(n-p)!} = (-1)^p \sum_{I} \left(\prod_{i \in I} d_i \right)^{-1} \bigwedge_{i \in I} (dx_i \wedge dy_i) = \frac{c_1(L)^p}{p!}$$

by (5.2) and (5.1).

Proof of Proposition 5.1. — We can assume that the group action is trivial. Indeed, if $\beta$ satisfies the assumption of Proposition 5.1, then the same $\beta$ satisfies

$$\beta \cdot \omega = \frac{1}{|G|} \sum_{g \in G} g^* \beta \cdot \omega = \beta \cdot \frac{1}{|G|} \sum_{g \in G} g^* \omega \neq 0$$

for every $\omega \neq 0 \in \mathcal{H}(T)$. Up to replacing $\beta$ by some nonzero multiple of it, we can also assume that $\beta$ is the image of some element of $H^{n-1,n-1}(T, \mathbb{Z})$ (still denoted by $\beta$).
We first prove Proposition 5.1 assuming \( T \) is a simple torus (in the sense that \( T \) does not contain any nontrivial sub-torus). Let \( n = \dim T \) and let \( \hat{T} \) be the dual of \( T \). Let
\[
\mathcal{F} : H^{2n-\ast}(T, \mathbb{Z}) \sim \to H^\ast(\hat{T}, \mathbb{Z})
\]
be the Fourier transformation. Up to sign, \( \mathcal{F} \) is the composition of the Poincaré duality \( H^{2n-\ast}(T, \mathbb{Z}) \sim \to H^\ast(\hat{T}, \mathbb{Z})^\vee \) with the isomorphism \( H^\ast(\hat{T}, \mathbb{Z})^\vee \sim \to H^\ast(\hat{T}, \mathbb{Z}) \) induced by the natural perfect pairing \( H^1(T, \mathbb{Z}) \otimes H^1(\hat{T}, \mathbb{Z}) \to \mathbb{Z} \) [3, Proposition 1], and it follows that
\[
\mathcal{F}(\gamma_1 \star \gamma_2) = \pm \mathcal{F}(\gamma_1) \cdot \mathcal{F}(\gamma_2) \quad \text{and} \quad \mathcal{F}(\gamma_1 \cdot \gamma_2) = \pm \mathcal{F}(\gamma_1) \star \mathcal{F}(\gamma_2).
\]
The map \( \mathcal{F} \) is an isomorphism of Hodge structures, so there exists a line bundle \( L \) on \( \hat{T} \) such that \( c_1(L) = \mathcal{F}(\beta) \); let \( \phi_L : \hat{T} \to T \) be the homomorphism induced by \( L \). Since \( T \) is simple, \( \ker(\phi_L) \) is either \( \hat{T} \) or finite. In other words, either \( c_1(L) = 0 \) or \( \phi_L \) is finite [5, Lemma 2.47]. As \( \beta \neq 0 \), we have \( c_1(L) = \mathcal{F}(\beta) \neq 0 \). It follows from [5, Corollary 3.6.2, Theorem 3.6.3] that \( (\hat{c}(\beta)^n) = \deg \phi_L \neq 0 \). So \( c_1(L)^n \neq 0 \), and \( \beta^n = \pm \mathcal{F}^{-1}(c_1(L)^n) \neq 0 \). In particular, \( \beta^{n(n-1)} \neq 0 \).

Let \( L' \) be a line bundle over \( T \) such that \( c_1(L') = \beta^{n(n-1)} \). By the same argument, we have \( c_1(L')^n \neq 0 \), so the Hermitian form \( h \) on \( H_1(T, \mathbb{C}) \) which corresponds to \( c_1(L') \) is non-degenerate. Let \( (p, q) \) be the signature of \( h \). There exist \( dz_1, \ldots, dz_n \in H^1(T, \mathbb{C}) \) such that \( dz_1, \ldots, dz_n, d\bar{z}_1, \ldots, d\bar{z}_n \) form a basis of \( H^1(T, \mathbb{C}) \) and
\[
c_1(L') = \sqrt{-1} \sum_{j=1}^{n} cjdz_j \land d\bar{z}_j \in H^1(T, \mathbb{R})
\]
with \( c_1, \ldots, c_p > 0 \) and \( c_{p+1}, \ldots, c_n < 0 \). Define
\[
\omega := \sqrt{-1} \left( \frac{q}{p} \sum_{j=1}^{n} c_j dz_j \land d\bar{z}_j - \sum_{j=p+1}^{n} c_j dz_j \land d\bar{z}_j \right),
\]
which is a Kähler form.

Assume that \( p, q \neq n \). Then \( \omega \cdot c_1(L')^{n-1} = 0 \) by elementary computation. Since \( c_1(L') = \beta^{n(n-1)} \) and \( \beta^{n\ast} \neq 0 \), we have \( \hat{\mathcal{F}}(c_1(L')) = \pm \hat{\mathcal{F}}(\beta)^{n-1} \) and \( \hat{\mathcal{F}}(\beta)^n \neq 0 \) where \( \hat{\mathcal{F}} : H^1(\hat{T}, \mathbb{Z}) \to H^\ast(\hat{T}, \mathbb{Z}) \) denotes the Fourier transform of \( \hat{T} \). So by Lemma 5.2, there exists \( C \in \mathbb{Q} \setminus \{0\} \) such that
\[
\hat{\mathcal{F}}(c_1(L')^{n-1}) = \pm \hat{\mathcal{F}}(c_1(L'))^{n(n-1)} = \pm \left( \hat{\mathcal{F}}(\beta)^{n-1} \right)^{*(n-1)} = C. \hat{\mathcal{F}}(\beta),
\]
and therefore
\[
c_1(L')^{n-1} = C\beta.
\]
It follows that
\[
\omega \cdot c_1(L')^{n-1} = C (\omega \cdot \beta) \neq 0,
\]
where non-vanishing follows from the assumption of \( \beta \) and \( \omega \in \mathcal{X}(T) \). This contradicts \( \omega \cdot c_1(L')^{n-1} = 0 \). Hence either \( p = n \) or \( q = n \), so either \( L' \) or \( L' \vee \) is ample. Thus \( T \) is projective.

Now we prove Proposition 5.1 by induction on \( \dim T \). When \( \dim T = 1 \), \( T \) is always projective. Assume that \( \dim T > 1 \) and that Proposition 5.1 is proven for every complex torus of dimension strictly less than \( \dim T \) endowed with a finite group action. By what we have proven, we can assume that \( T \) is not simple.
Then $T$ is the total space of a smooth isotrivial torus fibration $\pi : T \to T'$ over a simple complex torus $T'$ with $0 < \dim T' < \dim T$. For every $\omega' \in \mathcal{K}(T')\setminus\{0\}$, we have $\pi^*\omega' \in \mathcal{K}(T)\setminus\{0\}$, so

$$\pi_*\beta \cdot \omega' = \beta \cdot \pi^*\omega' \neq 0$$

by assumption. Thus $T'$ is projective by the induction hypothesis. Moreover, $\pi_*\beta \neq 0 \in H_2(T', \mathbb{Z})$ and since $T'$ is simple, previous argument showing that $\beta^{*n} \neq 0$ (when $T$ is simple) also proves that

$$\pi_* (\beta^{* \dim T'}) = (\pi_* \beta)^{* \dim T'} \neq 0 \in H^0(T', \mathbb{Z}),$$

so $\pi$ has an étale multi-section $\Sigma \subset T$ by Lemma 4.1. It follows that there exists a finite étale cover $\tau : \tilde{T} \to T$ together with a surjective homomorphism $\pi' : \tilde{T} \to F$ over a fiber $F \subset T$ of $\pi : T \to T'$. For every $\omega'' \neq 0 \in \mathcal{K}(F)$, we have $\tau_* \pi'' \omega'' \neq 0 \in \mathcal{K}(\tilde{T})$, so

$$\pi'_* \tau_* \beta \cdot \omega'' = \beta \cdot \tau_* \pi'' \omega'' \neq 0.$$

Thus $F$ is projective as well by the induction hypothesis. Since $\pi : T \to T'$ is a fibration with a multi-section such that $T'$ and the fibers of $\pi$ are projective, by Corollary 2.10 $\tilde{T}$ is also projective. \hfill \Box

Before we prove Theorem 1.4, let us prove some immediate consequences of Proposition 5.1.

**Corollary 5.3.** — Let $X$ be a compact Kähler manifold of dimension $n$ which is bimeromorphic to the quotient $T/G$ of a complex torus $T$ by a finite group $G$. If $\operatorname{Int}(\mathcal{K}(X)) \cap H^{2n-2}(X, \mathbb{Q})$ is not empty, then $X$ is projective.

**Proof.** — Let $\alpha \in \operatorname{Int}(\mathcal{K}(X)) \cap H^{2n-2}(X, \mathbb{Q})$. Fix a bimeromorphic map $p : X \to T/G$. Up to resolving $p$ by a successive blow-ups of $X$ along smooth centers, by Proposition 3.1 we can assume that $p$ is holomorphic. Let $q : T \to T/G$ be the quotient map. For every $\omega \neq 0 \in \mathcal{K}(T)$, there exists by Lemma 2.1 a nef class $\omega' \neq 0 \in H^2(T/G, \mathbb{R})$ such that $\omega = q^* \omega'$. As $\omega'$ is nef, the pullback $p^* \omega'$ is also nef. So if we set $\beta := q^* p_* \alpha \in H^{n-1,n-1}(T, \mathbb{Q})^{\alpha}$, then since $\alpha \in \operatorname{Int}(\mathcal{K}(X)) \cap H^{2n-2}(X, \mathbb{Q})$, we have

$$\beta \cdot \omega = q^* p_* \alpha \cdot \omega = q^* (p_\beta \cdot \omega') = q^* p_\beta (p^* \alpha) > 0, \quad (5.5)$$

where the last equality follows from the projection formula (Proposition 2.3). It follows from Proposition 5.1 that $T$ is projective, hence $X$ is also projective. \hfill \Box

**Corollary 5.4.** — Let $X$ be compact Kähler manifold of dimension $n$ such that $\operatorname{Int}(\mathcal{K}(X)) \cap H^{2n-2}(X, \mathbb{Q})$ is not empty. If $a(X) = 0$, then $b_1(X) = 0$.

**Proof.** — It is equivalent to show that $\operatorname{Alb}(X)$ is a point. Since $a(X) = 0$, the Albanese map $X \to \operatorname{Alb}(X)$ is surjective and $a(\operatorname{Alb}(X)) = 0$ [37, Lemma 13.1]. So $\operatorname{Int}(\mathcal{K}(\operatorname{Alb}(X))) \cap H^{2n-2}(X, \mathbb{Q}) \neq \emptyset$ implies that $\operatorname{Int}(\mathcal{K}(\operatorname{Alb}(X)))$ contains a rational class as well [33, Proposition 2.5]. It follows from Proposition 5.1 that $\operatorname{Alb}(X)$ is projective, hence $\operatorname{Alb}(X)$ is a point. \hfill \Box

We finish this section by a proof of Theorem 1.4.

**Proof of Theorem 1.4.** — Let $\operatorname{alb} : X \to T$ be the Albanese map of $X$. 


Claim. — For every \([\alpha] \in \mathcal{X}(T) \backslash \{0\}\), we have \(\text{alb}^*[\alpha] \in \mathcal{X}(X) \backslash \{0\}\).

Proof. — Clearly \(\text{alb}^*[\omega] \in \mathcal{X}(X)\). It remains to show that \(\text{alb}^*[\omega] \neq 0\). As \([\omega]\) is nef, we can assume that \([\omega]\) is represented by \(\omega = \sqrt{-1} \sum c_j dz_j \wedge d\bar{z}_j\) for some complex coordinates \((z_1, \ldots, z_n)\) of \(H_1(T, \mathbb{C})\) and some \(c_j \geq 0\) such that \(c_j \neq 0\) for some \(j = 1, \ldots, n\). It follows that \(\text{alb}^* \omega\) is semi-positive, so it suffices to show that \(\text{alb}^* \omega \neq 0\).

Assume to the contrary that \(\text{alb}^* \omega = 0\). Let \(Y := \text{alb}(X)\) and let \(Y^\circ \subset Y\) be a nonempty Zariski open over which \(\text{alb} : X \to Y\) is smooth. Then for any point \(y \in Y^\circ\) and any vector \(v \in T_{Y,y}\), we have \(\omega(v \wedge \bar{v}) = 0\). Since \(Y\) is the Albanese image of \(X\), regarding each \(T_{Y,y}\) as a subspace of \(T_{T,y}\) by translation where \(o \in T\) is a chosen origin of \(T\), the subspaces \(T_{Y,y}\) generate \(T_{T,o}\) where \(y\) runs through \(Y^\circ\). As \(\omega_o\) is positive semi-definite, we thus have \(\omega_o = 0\), contradicting the assumption that \(\omega \neq 0\). Hence \(\text{alb}^* \omega \neq 0\). \(\square\)

Now let \(\alpha \in \text{Int}(\mathcal{X}(X)^\vee) \cap H^{2n-2}(X, \mathbb{Q})\). By the above claim, we have \([\alpha] \cdot \text{alb} \alpha = \text{alb}^*[\alpha] \cdot \alpha > 0\) for every \([\alpha] \in \mathcal{X}(T) \backslash \{0\}\). We conclude by Proposition 5.1 that \(T\) is projective. \(\square\)

6 Ricci-flat manifolds

In this section we prove Corollary 1.6, answering Problem 1.1 for Ricci-flat compact Kähler manifolds. Let us start with the special case of hyper-Kähler manifolds.

6.1 Hyper-Kähler manifolds

In this article, a compact hyper-Kähler manifold is a simply connected compact Kähler manifold \(X\) such that \(H^0(X, \Omega_X^2)\) is generated by a nowhere degenerate holomorphic 2-form. The reader is referred to [21] for basic results about compact hyper-Kähler manifolds.

Let \(X\) be a compact hyper-Kähler manifold. Let \(q_X\) be the Beauville–Bogomolov–Fujiki quadratic form on \(H^2(X, \mathbb{R})\) and let

\[\Phi_X : H^2(X, \mathbb{R}) \cong H^{2n-2}(X, \mathbb{R})\]

be the isomorphism (of Hodge structures) sending \(\alpha \in H^2(X, \mathbb{R})\) to the Poincaré dual of \(q_X(\alpha, \cdot) \in H^2(X, \mathbb{R})^\vee\). For every cone \(C \subset H^{1,1}(X, \mathbb{R})\), its dual with respect to \(q_X\) is denoted by

\[C^\ast := \{ \alpha \in H^{1,1}(X, \mathbb{R}) \mid q_X(\alpha, \beta) \geq 0 \text{ for every } \beta \in C \} \cdot\]

We have

\[\Phi_X(C^\ast) = C^\vee,\]

where \(C^\vee \subset H^{p-1, n-1}(X, \mathbb{R})\) is the Poincaré dual of \(C\).

For a compact hyper-Kähler manifold \(X\), there are two other natural positive cones that we can define in \(H^{1,1}(X, \mathbb{R})\): the birational Kähler cone \(\mathcal{B} \mathcal{X}(X)\) and the positive cone \(\mathcal{C}(X)\). Recall that

\[\mathcal{B} \mathcal{X}(X) := \bigcup_{f : X \to X'} f^* \mathcal{X}(X') \subset H^{1,1}(X, \mathbb{R})\]

and

\[\mathcal{C}(X) := \{ \beta \in H^{1,1}(X, \mathbb{R}) \mid \beta \geq 0 \text{ for every } \alpha \in \mathcal{X}(X) \}\]
where the union runs through all bimeromorphic maps $f$ from $X$ to another compact hyper-Kähler manifold, and $\mathcal{K}(X)$ is defined to be the connected component of

$$\left\{ \alpha \in H^{1,1}(X, \mathbb{R}) \mid q_X(\alpha, \alpha) > 0 \right\}$$

containing $\mathcal{K}(X)$. The chain of inclusions in the following statement sums up some fundamental results proven by Huybrechts.

**Theorem 6.1 (Huybrechts).** — Let $X$ be a compact hyper-Kähler manifold. We have

$$\mathcal{K}(X) \subset \mathcal{R}(X) = \text{Psef}(X)^* \subset \overline{\mathcal{K}(X)} \subset \overline{\mathcal{R}(X)} \subset \text{Psef}(X) \subset \mathcal{K}(X)^*$$.  \hspace{1cm} (6.1)

If $X$ is a very general compact hyper-Kähler manifold, then all these cones are equal.

**Proof.** — The inclusion $\mathcal{K}(X) \subset \mathcal{R}(X)$ is obvious and $\mathcal{R}(X) = \text{Psef}(X)^*$ follows from [23, Corollary 4.6]. The inclusion $\mathcal{R}(X) \subset \overline{\mathcal{K}(X)}$ follows from $\mathcal{K}(X) \subset \mathcal{R}(X)$ and [20, Lemma 2.6]. It remains to show that $\overline{\mathcal{K}(X)} \subset \overline{\mathcal{R}(X)}$. Assume to the contrary that $\overline{\mathcal{K}(X)} \not\subset \overline{\mathcal{R}(X)}$, then there exist $\alpha, \beta \in \mathcal{K}(X)$ such that $q_X(\alpha, \beta) < 0$. Since $\mathcal{K}(X)$ is convex and $q_X(\alpha, \alpha) > 0$, there exists $\beta' \in \mathcal{K}(X)$ such that $q_X(\alpha, \beta') = 0$. As the signature of $q_X$ on $H^{1,1}(X, \mathbb{R})$ is $(1, h^{1,1}(X))$, we obtain $q_X(\beta', \beta') < 0$ and thus a contradiction.

Finally if $X$ is a very general hyper-Kähler manifold, then $\overline{\mathcal{K}(X)} = \text{Psef}(X)$ by [22, Corollary 1], so (6.1) becomes a chain of equalities.

We are able to answer Problem 1.1 for hyper-Kähler manifolds based on Huybrechts’ description of their nef cones.

**Proposition 6.2.** — Let $X$ be a compact hyper-Kähler manifold of dimension $n$. If $\text{Int}(\mathcal{K}(X)^*) \cap H^{n-1,n-1}(X, \mathbb{Q}) \neq \emptyset$, then $X$ is projective.

**Proof.** — We show that if $X$ is not projective, then $\text{Int}(\mathcal{K}(X)^*) \cap H^{n-1,n-1}(X, \mathbb{Q}) = \emptyset$. Let $V \subset H^{1,1}(X, \mathbb{R})$ be the subspace generated by $\Phi_X^{-1}(H^{n-1,n-1}(X, \mathbb{Q}))$. Since $X$ is assumed to be non-projective, we have $q_X(\alpha, \alpha) \leq 0$ for every $\alpha \in H^{1,1}(X, \mathbb{Q})$ [21, Proposition 26.13]. In particular, $q_{X|V}$ is negative semi-definite. As the signature of $q_X$ on $H^{1,1}(X, \mathbb{R})$ is $(1, h^{1,1}(X))$, there exists $\omega \in H^{1,1}(X, \mathbb{R})\setminus\{0\}$ such that $q_X(\omega, \omega) \geq 0$ and $q_X(\omega, \alpha) = 0$ for every $\alpha \in V$. It follows that $\omega \cdot \beta = 0$ for every $\beta \in H^{n-1,n-1}(X, \mathbb{Q})$.

Since $q_X(\omega, \omega) \geq 0$, up to replacing $\omega$ by $-\omega$ we can assume that $\omega \in \overline{\mathcal{K}(X)}$. For every rational curve $C \subset X$, since $[C] \in H^{n-1,n-1}(X, \mathbb{Q})$, we have $\omega \cdot C = 0$. It follows from [21, Proposition 28.2] that $\omega \in \overline{\mathcal{K}(X)}$. Since $\omega \cdot \beta = 0$ for every $\beta \in H^{n-1,n-1}(X, \mathbb{Q})$, we conclude that $\text{Int}(\mathcal{K}(X)^*) \cap H^{n-1,n-1}(X, \mathbb{Q}) = \emptyset$.

### 6.2 Ricci-flat manifolds

We finish this section by a proof of Corollary 1.6.

**Proof of Corollary 1.6.** — By [4, Théorème 2], there exists a finite étale cover $\tilde{X} \to X$ such that $\tilde{X} = T\times Y \times \prod_i Z_i$ where $T$ is a complex torus, $Y$ is a compact Kähler manifold with $H^0(Y, \Omega_Y^2) = 0$, and each $Z_i$ is a hyper-Kähler...
manifold. By Lemma 3.3, we have $\text{Int}(\mathcal{H}(\mathcal{X})) \cap H^{2n-2}(\tilde{X}, \mathbb{Q}) \neq \emptyset$. It follows from [33, Proposition 2.5] that $\text{Int}(\mathcal{H}(T)) \cap H^{2\dim T-2}(T, \mathbb{Q}) \neq \emptyset$ and $\text{Int}(\mathcal{H}(Z_i)) \cap H^{2\dim Z_i-2}(Z_i, \mathbb{Q}) \neq \emptyset$. Thus $T$ and $Z_i$ are projective by Theorem 1.4 and Proposition 6.2 respectively. Since $Y$ is also projective (because $H^0(Y, \Omega_Y^2) = 0$), we conclude that $X$ is projective.

\section{Fibrations in abelian varieties}

In this section, we study the Oguiso–Peternell problem for fibrations in abelian varieties over a curve. A positive answer to the Oguiso–Peternell problem will be obtained as a consequence of the following result, which is an analog of Lemma 4.1.

**Proposition 7.1.** — Let $f : X \to B$ be a fibration over a smooth projective curve $B$ whose general fiber is an abelian variety of dimension $g$. Assume that $X$ is a compact Kähler manifold and $f$ has local sections (for the Euclidean topology) at every point of $B$. If there exists $\alpha \in H^{p,q}(X, \mathbb{Z})$ such that $f_*\alpha \neq 0 \in H^0(B, \mathbb{Z})$, then $X$ is projective.

In particular, if $\text{Int}(\mathcal{H}(X)) \cap H^{p+1-q}(X, \mathbb{Q}) \neq \emptyset$, then $X$ is projective.

Before we prove Proposition 7.1, let us first recall some properties of the fibration $f : X \to B$ following [30, \S 4], which partially generalize the discussion in \S 4.

Given a fibration $f : X \to B$ as in Proposition 7.1 and assume that every fiber of $f$ is a normal crossing divisor. Since the fibers of $f$ are Moishezon, $f$ is locally projective [10, Theorem 10.1]. Let $j : B^* \hookrightarrow B$ be the inclusion of a non-empty Zariski open of $B$ over which $f$ is smooth and let $i : X^* := f^{-1}(B^*) \hookrightarrow X$. Let $D := f^{-1}(B \setminus B^*)$. The relative Deligne complex $\mathcal{D}_{X/B}(g)$ is defined to be the cone of the composition

$$R_i*Z \xrightarrow{\times[2\pi \sqrt{-1}]^c} R_i*C \cong \Omega^*_X(\log D) \to \Omega^*_{X/B}(\log D)$$

shifted by $-1$. Applying $Rf_!$ to the distinguished triangle

$$\mathcal{D}_{X/B}(g) \to R_i*Z \to \Omega^*_{X/B}(\log D) \to \mathcal{D}_{X/B}(g)[1]$$

and break up the associated long exact sequence at $R^2f_!\mathcal{D}_{X/B}(g)$, we obtain a short exact sequence (see [30, (4.6)])

$$0 \to \mathcal{J} \to R^2f_!\mathcal{D}_{X/B}(g) \to \mathcal{H}^{p,q}(X/B) \to 0$$

where $\mathcal{J}$ is the canonical extension of the sheaf of germs of sections of the Jacobian fibration associated to $X^* \to B^*$ and the quotient $\mathcal{H}^{p,q}(X/B)$ in (7.2) is isomorphic to $R^2f_!(f \circ i)_!\mathcal{Z}$ [30, Lemma 4.6]. By construction, the restriction of (7.2) to $B^*$ is the short exact sequence (4.4) defined for the smooth torus fibration $X^* \to B^*$.

The natural map $\Omega^*_X \to \Omega^*_{X/B}(\log D)$ induces a map $\mathcal{D}_{X}(g) \to \mathcal{D}_{X/B}(g)$, which further induces

$$H^2_D(X, \mathcal{Z}(g)) = H^2(X, \mathcal{D}_{X}(g)) \to H^2(X, \mathcal{D}_{X/B}(g)) \to H^0(B, R^2f_!\mathcal{D}_{X/B}(g)),$$
fitting into the commutative diagram

$$
\begin{array}{ccc}
H^2_{\partial}(X, \mathbb{Z}(g)) & \longrightarrow & H^0(X, \mathbb{Z}) \\
\downarrow & & \downarrow \\
H^0(B, R^{2g}f_*D_{\chi/B}(g)) & \longrightarrow & H^0(B, \mathcal{H}^g(X/B)) \\
\end{array}
$$

(7.3)

where the second row is an exact sequence induced by (7.2), and the vertical arrow on the right is the composition

$$
H^0(X, \mathbb{Z}) \hookrightarrow H^2_{\partial}(X, \mathbb{Z}) \xrightarrow{\delta} H^0(B, \mathcal{H}^g(X/B)).
$$

(7.4)

The commutative diagram (7.3) is the analog of (4.5) with the presence of singular fibers.

Let $\mathbf{H} = R^{2g-1}f_*\mathbb{Z}$ and let $\mathcal{E}(B, \mathbf{H})$ be the set of bimeromorphic classes of fibrations $f : X \to B$ in abelian varieties over $B$ such that

i) $f$ is locally bimeromorphically Kähler over $B$;

ii) $f$ has local sections at every point of $B$;

iii) $f$ is smooth over $B^*$, and the underlying $\mathbb{Z}$-local system over $B^*$ is isomorphic to $\mathbf{H}$.

Elements of $\mathcal{E}(B, \mathbf{H})$ are called bimeromorphic $J$-torsors. There exists a map

$$
\Phi : H^1(B, \mathcal{J}) \to \mathcal{E}(B, \mathbf{H})
$$

which is a generalization of the inverse of (4.6) [30, p.88, 89]. By construction, the map $\Phi$ sends $0 \in H^1(B, \mathcal{J})$ to the bimeromorphic class of a compactification $\bar{J} \to B$ of the Jacobian fibration $J \to B^*$ associated to $f^* = f_{\chi^*} : X^* \to B^*$ such that the closure in $\bar{J}$ of the $0$-section of $J \to B^*$ is a $0$-section of $\bar{J} \to B$.

Finally, for every $m \in \mathbb{Z}$ and $\eta \in H^1(B, \mathcal{J})$, if $f : X \to B$ is a bimeromorphic $J$-torsor representing $\Phi(\eta)$, then there exists a bimeromorphic $f$-torsor $f_m : X_m \to B$ representing $\Phi(m\eta)$ together with a generically finite map $m : X \dashrightarrow X_m$ over $B$ [30, Lemma-Definition 4.12, p.91]. The bimeromorphic $J$-torsor $f_m : X_m \to B$ is called the multiplication-by-$m$ of $f : X \to B$.

**Proof of Proposition 7.1.** — Up to replacing $f$ by a Kähler log-desingularization of $(X, D)$ where $D \subset X$ is the union of singular fibers of $f$, we can assume that every fiber of $f$ is a normal crossing divisor. Since the second row of (7.3) is exact and the horizontal arrow in (7.3) on the top is surjective, we have

$$
\delta(\gamma(f_*\alpha)) = 0 \in H^1(B, \mathcal{J})
$$

where $f_*$ and $\gamma$ are defined in (7.4). It follows from [30, Lemma 4.14] that for some $m \in \mathbb{Z}_{>0}$, the multiplication-by-$m$ of $f : X \to B$ is bimeromorphic to $\bar{J} \to B$. Hence we have a generically finite map $m : X \dashrightarrow \bar{J}$ over $B$, and the pre-image of the $0$-section of $\bar{J} \to B$ under $m$ gives a multi-section of $f : X \to B$. We conclude by Corollary 2.10 that $X$ is Moishezon, hence projective (because $X$ is Kähler).

Finally, if $\alpha \in H^{n-1,n-1}(X, \mathbb{Q}) \cap \text{Int}(\mathcal{X}(X)^*)$, then $f_*\alpha \neq 0 \in H^0(B, \mathbb{Z})$ by [33, Proposition 2.5]. It follows from the main statement of Proposition 7.1 that $X$ is projective. \qed
Corollary 7.2. — Let $X$ be a compact Kähler manifold of dimension $n$ such that $\text{Int}(\mathscr{K}(X)^\vee) \cap H^{n-1,n-1}(X, \mathbb{Q}) \neq \emptyset$. Assume that $X$ is bimeromorphic to a compact Kähler manifold $X'$ which is the total space of a fibration $f : X' \to B$ over a smooth projective curve $B$ whose general fiber is an abelian variety. Then $X$ is projective.

Proof. — As $X'$ is bimeromorphic to $X$ and $\text{Int}(\mathscr{K}(X)^\vee) \cap H^{n-1,n-1}(X, \mathbb{Q}) \neq \emptyset$, there exists $\alpha \in H^{n-1,n-1}(X', \mathbb{Q})$ by Corollary 3.2. Since the fibers of $f$ are Moishezon and $X'$ is Kähler, $f$ is locally projective [10, Theorem 10.1]. In particular $f$ has local multi-sections around every point of $B$. Let $\tilde{X} \to X' \times_B \tilde{B}$ be a Kähler desingularization and let $q : \tilde{X} \to X'$ and $\tilde{f} : \tilde{X} \to \tilde{B}$ be the projections. The situation is summarized in the commutative diagram

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{f} & X' \\
\downarrow f & & \downarrow f \\
\tilde{B} & \xrightarrow{r} & B
\end{array}
$$

Since

$$r_*f_*q^*\alpha = f_*q_*\alpha = \deg(q) \cdot f_*\alpha \neq 0$$

where the non-vanishing follows from [33, Proposition 2.5], we have $f_*q^*\alpha \neq 0$. Since $\tilde{f}$ has local sections around every point of $\tilde{B}$ by construction, we deduce from Proposition 7.1 that $\tilde{X}$ is projective. Hence $X'$, and therefore $X$, are projective. \hfill \square

8 Elliptic fibrations

The main result of this section is the following proposition.

Proposition 8.1. — Let $f : X \to B$ be an elliptic fibration from a smooth compact Kähler threefold $X$. If $\text{Int}(\text{Psef}(X)^\vee)$ contains a rational class, then $X$ is projective.

Let us first prove some auxiliary results, starting with an analogous statement of Proposition 8.1 for surfaces.

Lemma 8.2. — Let $S$ be a smooth connected compact Kähler surface and let $f : S \to B$ be a surjective map onto a projective curve. If there exists $\alpha \in H^1(S, \mathbb{Q})$ such that $f_*\alpha \neq 0 \in H^0(B, \mathbb{Q})$, then $S$ is projective.

Proof. — Up to replacing $\alpha$ by a multiple of it, we can assume that $\alpha = c_1(L)$ for some line bundle $L$ on $S$. Since $\alpha \cdot [F] = f_*\alpha \neq 0 \in H^0(B, \mathbb{Q}) \cong \mathbb{Q}$ where $F$ is a fiber of $f$, we have $c_1(L \otimes O(mF))^2 = c_1(L)^2 + 2m\alpha \cdot [F] > 0$ for $m \gg 0$ or $m \ll 0$. Thus $S$ is projective by [2, Theorem IV.6.2]. \hfill \square

The next lemma concerns 1-cycles vanishing away from an irreducible surface, giving a partial answer to Question 1.10 in the affirmative.

Lemma 8.3. — Let $X$ be a compact Kähler manifold of dimension $n$ and $Y \subset X$ an irreducible surface. Let $i : \tilde{Y} \to X$ be the composition of a desingularization of $Y$ with the inclusion $i : Y \hookrightarrow X$. There exists a sub-$\mathbb{Q}$-Hodge structure
We verify that the restriction of the intersection product to \( Y \)
The last statement of Lemma 2.7, then \( \alpha = 1, \beta \) for some \( \beta \in H^{1,1}(\tilde{Y}, Q) \).

We start with an easy lemma.

**Lemma 8.4.** — Let \( \phi : L \to M \) be a morphism of \( Q \)-Hodge structures. Assume that \( L \) has a pairing \( Q \) such that 
\((L, Q) \otimes R\) is a direct sum of polarized \( R \)-Hodge structures \((L_i, Q_i)\). If \( \ker(\phi) \subset L_i \) for some \( i \), then there exists a sub-\( Q \)-Hodge structure \( L' \subset L \) such that \( \phi_{L'} \) is an isomorphism onto \( \text{Im}(\phi) \).

**Proof.** — As \( Q \otimes R \) is a direct sum of polarizations of \( R \)-Hodge structures, the orthogonal complement 
\( L' := \ker(\phi)\perp \) of \( \ker(\phi) \) with respect to \( Q \) is a sub-\( Q \)-Hodge structure of \( L \), and we have 
\( \dim \ker(\phi) + \dim \ker(\phi)\perp = \dim L \) (because \( Q \otimes R \) is non-degenerate). Thus to prove Lemma 8.4, it suffices to show that 
\( \ker(\phi) \cap \ker(\phi)\perp = 0 \), or equivalently \( \ker(\phi_R) \cap \ker(\phi_R)\perp = 0 \).

Since \((L_i, Q_i)\) is a polarized \( R \)-Hodge structure, the orthogonal complement \( L_i' \) of \( \ker(\phi_R) \) with respect to \( Q_i \) satisfies \( \ker(\phi_R) \cap L_i' = 0 \). Hence
\[
\ker(\phi_R) \cap \ker(\phi_R)\perp = \ker(\phi_R) \cap \left( L_i' \oplus \bigoplus_{j \neq i} L_j \right) = 0.
\]

\( \Box \)

**Proof of Lemma 8.3.** — Fix a Kähler class \( \omega \in H^2(X, R) \) and let 
\( H = (\Gamma \omega)\perp \subset H^2(\tilde{Y}, R) \) where the orthogonal is defined with respect to the intersection product on \( H^2(\tilde{Y}, R) \). Since \( \Gamma \omega \neq 0 \), we have
\[
H^2(\tilde{Y}, R) = R^* \omega \oplus H.
\]

We verify that the restriction of the intersection product to \( H \) is a polarization of the \( R \)-Hodge structure \( H \):
The induced pairing on \( H \) is non-degenerate and the Hodge decomposition \( H \otimes C = H^{2,0} \oplus H^{1,1} \oplus H^{0,2} \) is orthogonal with respect to the Hermitian form \( h(\beta, \gamma) := \beta \cdot \gamma \). The restriction of \( h \) to \( H^{2,0} \oplus H^{0,2} \) is positive definite. Since \( Y \) is irreducible, \( H^{1,1}(\tilde{Y}) \) is of signature \((1, \dim H^{1,1}(\tilde{Y}) - 1)\). As \( \Gamma \omega^2 > 0 \), the restriction of \( h \) to \( H^{1,1} \) is negative definite.

We have \( \ker(\iota) \subset H \). Indeed, let \( \xi \in \ker(\iota) \) and write \( \xi = a \cdot \Gamma \omega + \beta \) with \( a \in R \) and \( \beta \in H \). Then
\[
0 = \iota(a \cdot \Gamma \omega + \beta) \cdot \omega = a[Y] \cdot \omega^2 + \iota(\beta \cdot \Gamma \omega) = a[Y] \cdot \omega^2.
\]
As \( [Y] \cdot \omega^2 \neq 0 \), we have \( a = 0 \), so \( \xi \in H \). It follows from Lemma 8.4 that there exists a sub-\( Q \)-Hodge structure \( L \) of \( H^2(\tilde{Y}, Q) \) such that 
\( H^2(\tilde{Y}, Q) = \ker(\iota) \oplus L \), which proves the main statement of Lemma 8.3.

The last statement of Lemma 8.3 follows from the observation that \( \iota_{\text{id}} : L \to H^{2n-2}(X, Q) \) is an isomorphism of \( Q \)-Hodge structures onto \( \text{Im}(\iota_L) \).

The bimeromorphic models of elliptic threefolds in the following statement will be useful in our proof of Proposition 8.1.
Theorem 8.5 ([31, Theorem A.1]). — Let $f : X \to B$ be an elliptic fibration with $X$ being a compact Kähler threefold. Then $f$ is bimeromorphic to an elliptic fibration $f' : X' \to B'$ over a compact normal surface $B'$ satisfying the following properties:

- $X'$ has at worst terminal singularities;
- $f'$ has equidimensional fibers.

Proof. — Up to replacing $X$ and $B$ by some Kähler desingularizations of them, we can assume that both $X$ and $B$ are smooth, $X$ is Kähler, and the discriminant locus of $f$ is a normal crossing divisor of $B$. Under these assumption, $f$ is locally projective by [31, Theorem 3.3.3]. Theorem 8.5 is then a consequence of [31, Theorem A.1]. □

Now we prove Proposition 8.1.

Proof of Proposition 8.1. — Let $f' : X' \to B'$ be an elliptic fibration bimeromorphic to $f$ as in Theorem 8.5. Note that since $\dim X' = 3$ and $\dim B' = 2$, both $X'$ and $B'$ have at worst isolated singularities. Let

$$Z := \text{Sing}(B') \cup f'(\text{Sing}(X')),$$

which is a finite subset of $B'$.

By Corollary 3.7, we can freely replace $X$ by any smooth bimeromorphic model of it. Up to replacing $f : X \to B$ by a bimeromorphic model of it by resolving the bimeromorphic map $X \rightarrow X'$ and taking Kähler desingularizations, we can assume that $X$ is a compact Kähler manifold, $B$ is a smooth projective surface, and $f$ fits into the commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\nu} & X' \\
\downarrow{f} & & \downarrow{f'} \\
B & \xrightarrow{\mu} & B'
\end{array}
$$

(8.1)

where the horizontal arrows are bimeromorphic morphisms.

Since $\text{Int}(	ext{Psef}(X)^\vee)$ contains a rational class $\alpha$, we have $f_*\alpha \in \text{Int}(	ext{Psef}(B)^\vee)$ by Lemma 3.4. Since $B$ is a smooth projective surface, $f_*\alpha \in H^{1,1}(B, \mathbb{Q})$ is ample by Kleiman’s criterion. Up to replacing $\alpha$ by a positive multiple of it, we can assume that $f_*\alpha = c_1(Z) \in H^{1,1}(B, \mathbb{Q})$ for some very ample line bundle $Z$ on $B$. Up to further replacing $\alpha$ by a multiple of it, we can find a linear system $T \subset |Z|$ such that a general member $C$ of $T$ satisfies the following properties:

- $C$ is smooth and irreducible;
- $C' := \mu(C)$ is a curve containing $Z \subset B'$ (as $Z$ is finite);
- $D' := f'^{-1}(C')$ is irreducible (as $f'$ is equidimensional);
- a general pair of points of $B$ is connected by a chain of general members of $T$. 
Let $C$ be a general member of $T$ (satisfying the above properties). We have the commutative diagram

\[
\begin{array}{ccc}
H_2(D', \mathbb{Q}) & \xrightarrow{\zeta} & H_2(X', \mathbb{Q}) \\
\downarrow{f_{D'}} & & \downarrow{f} \\
H_2(C', \mathbb{Q}) & \rightarrow & H_2(B', \mathbb{Q})
\end{array}
\]

(8.2)

where the rows of (8.2) are part of the long exact sequences of Borel-Moore homology groups induced by the open embeddings [25, IX.2.1], and the vertical arrows between them are induced by $f' : X' \rightarrow B'$.

**Lemma 8.6.** — The map $(f'_{|X'|_{\Delta'}})$, in (8.2) is an isomorphism.

**Proof.** — As $C' = \mu(C)$ contains $Z := \text{Sing}(B') \cup f'(\text{Sing}(X'))$, both $X' \setminus D'$ and $B' \setminus C'$ are smooth. So $(f'_{|X'|_{\Delta'}})$, is identified with

\[(f'_{|X'|_{\Delta'}}) : H^4(X' \setminus D', \mathbb{Q}) \rightarrow H^2(B' \setminus C', \mathbb{Q})\]

through Poincaré duality. Let $\tilde{C} := \mu^{-1}(C') \subset B$. Then $\tilde{C} = C \cup C_1$ for some curve $C_1 \subset B$. As $m(m'C + C_1)$ is very ample for $m, m' \gg 0$, the complement $B' \setminus C' \approx B \backslash (C \cup C_1)$ is affine. As $f'$ has equidimensional connected fibers, it follows from Lemma 2.4 that $(f'_{|X'|_{\Delta'}})$, is an isomorphism. \quad \square

Let $\tilde{D}$ be a desingularization of the proper transform of the divisor $D'$ under $\nu$, and let

\[
\begin{array}{ccc}
\tilde{D} & \xrightarrow{\iota} & X \\
\downarrow & & \downarrow{f'} \\
D' & \xrightarrow{\nu} & X'.
\end{array}
\]

be the induced commutative diagram. Since

\[f'_{\nu, \alpha} = \mu_\ast f_\ast \alpha = [C'] \in H_2(B', \mathbb{Q}),\]

by (8.2) and Lemma 8.6 we have $\nu_\ast \alpha \in \text{Im}(\iota' : H_2(D', \mathbb{Q}) \rightarrow H_2(X', \mathbb{Q}))$. As $\alpha \in H_2(X, \mathbb{Q})$, which is a pure Hodge structure of weight $-2$, we have $\nu_\ast \alpha \in W_{-2}H_2(X, \mathbb{Q})$, so by the strictness of $\nu_\ast$, we have $\nu_\ast \alpha = \iota'_0 \alpha_0$ for some $\alpha_0 \in W_{-2}H_2(D', \mathbb{Q})$. Since $\tilde{D} \rightarrow D'$ is a desingularization of $D'$, $\alpha_0 \in W_{-2}H_2(D', \mathbb{Q})$ can be lifted to $\alpha_1 \in H_2(\tilde{D}, \mathbb{Q})$ by Lemma 2.6, so

\[\nu_\ast \iota_\ast \alpha_1 = \nu_\ast \alpha \in H_2(X', \mathbb{Q}).\]

Since $X'$ has at worst rational singularities, Lemma 2.8 implies that $\iota_\ast \alpha_1 - \alpha \in H_2(X, \mathbb{Q})$ is a Hodge class. As $\alpha$ is a Hodge class, so is $\iota_\ast \alpha_1$. Since $\tilde{D}$ is irreducible, by Lemma 8.3 there exists $\alpha_2 \in H^{1,1}(\tilde{D}, \mathbb{Q})$ such that $\iota_\ast \alpha_2 = \iota_\ast \alpha_1$. By construction, $f(\iota_\ast \tilde{D}) = C$, so we have the factorization

\[f \circ \iota : \tilde{D} \rightarrow \tilde{C} \rightarrow B.\]

We have

\[\mu_\ast f_\ast \iota_\ast \alpha_2 = f'_{\nu, \iota_\ast} \alpha_2 = f'_{\nu, \iota} \alpha_1 = f'_{\nu} \nu_\ast \alpha = \mu_\ast f_\ast \alpha = [C'] \neq 0,\]

so

\[\iota_\ast \nu_\ast \alpha_2 = \iota_\ast \nu_\ast \alpha_1 \neq 0,\]
and thus \( p \cdot a_2 \neq 0 \in H^0(C, \mathbb{Q}) \). It follows from Lemma 8.2 that \( \mathcal{D} \) is a projective surface, so \( D := i(\mathcal{D}) \subset X \) is algebraically connected.

As a general pair of points of \( B \) is connected by a chain of general members \( C \) of \( T \), when \( C \) varies in \( T \), the divisors \( D \) thus connect \( X \) by construction. Hence \( X \) is projective by Theorem 2.9.

Thanks to Proposition 8.1, we can exclude threefolds of algebraic dimension 2 in Problem 1.2.

Corollary 8.7. — Let \( X \) be a smooth compact Kähler threefold. If \( \text{Int}(\text{Psef}(X)') \) contains a rational class, then \( a(X) \neq \frac{2}{n} \).

Proof. — Assume that \( a(X) = 2 \), then the algebraic reduction of \( X \) is bimeromorphic to an elliptic fibration \( f : X' \to B \) [37, Theorem 12.4]. By desingularization, we can assume that \( X' \) is smooth and Kähler. As \( \text{Int}(\text{Psef}(X')') \) contains a rational class, \( \text{Int}(\text{Psef}(X')') \) contains a rational class \( a \) as well by Corollary 3.7. Thus \( X' \) is projective by Proposition 8.1, which contradicts \( a(X) = 2 \). Hence \( a(X) \neq 2 \). □

9 Proof of Theorem 1.7 and Theorem 1.8

Proof of Theorem 1.8. — Let \( X \) be a compact Kähler threefold as in Theorem 1.8. Assume to the contrary that \( a(X) \leq 1 \), then \( X \) is bimeromorphic to a variety \( X' \) satisfying one of the descriptions listed in Proposition 2.13. We will rule out these descriptions case by case, therefore obtain a contradiction. Suppose that \( X' \) is in case \( i \), namely \( X' \) is a \( \mathbb{P}^1 \)-fibration \( X' \to S \) over a smooth compact Kähler surface. Then \( S \) is projective by [33, Proposition 2.6], so \( X' \) is Moishezon by Corollary 2.11, which is impossible. If \( X' \) is in case \( ii \), then the projection \( (S \times B)/G \to S/G \) induces a dominant meromorphic map \( X \to S/G \). Once again, [33, Proposition 2.6] implies that \( S/G \) is projective, contradicting the fact that \( S \) is non-algebraic. Case \( iii \) and case \( iv \) are ruled out by Corollary 7.2 and Proposition 4.2 respectively. Finally we rule out case \( v \) by Corollary 5.3. □

Proof of Theorem 1.7. — Let \( X \) be a compact Kähler threefold as in Theorem 1.7. Since \( \mathcal{X}(X) \subset \text{Psef}(X) \), Theorem 1.8 already implies that \( a(X) \geq 2 \). The case \( a(X) = 2 \) is excluded by Corollary 8.7, hence \( X \) is projective. □

10 One-cycles in compact Kähler threefolds and the Oguiso–Petrernell problem

In this final section, we work under the assumption that Question 1.10 has a positive answer and prove that every compact Kähler threefold \( X \) as in Problem 1.1 or Problem 1.3 is projective (Corollary 1.11), except for simple non-Kummer threefolds which presumably do not exist (see Remark 1.9). Already by Theorem 1.8, we know that such a threefold \( X \) has algebraic dimension \( a(X) \geq 2 \), so the proof consists of excluding the case \( a(X) = 2 \). We will deduce the latter as a consequence of the following result.

Proposition 10.1. — Let \( f : X \to B \) be an elliptic fibration over a smooth projective surface \( B \). Suppose that \( X \) is a smooth compact Kähler threefold and that Question 1.10 has a positive answer for \( X \). If there exists \( \alpha \in H^2(B, \mathbb{Q}) \) such that \( f_* \alpha \in H^2(B, \mathbb{Q}) \) is big, then \( X \) is projective.
Proof. — As \( f.\alpha \in H^2(B, \mathbb{Q}) \) is big, \( f.\alpha \) is the sum of an ample curve class and an effective curve class [27, Corollary 2.2.7]. We have the following more precise statement.

**Lemma 10.2.** — Up to replacing \( \alpha \) by a positive multiple of it, there exist a very ample line bundle \( \mathcal{L} \), integral curves \( E_1, \ldots, E_l \subset B \), and positive integers \( m_1, \ldots, m_l \) such that

\[
f.\alpha = c_1(\mathcal{L}) + \sum_{i=1}^l n_i[E_i] \in H^2(B, \mathbb{Q})
\]

and that \( c_1(\mathcal{L}) \) does not lie in the subspace of \( H^2(B, \mathbb{Q}) \) spanned by \( [E_1], \ldots, [E_l] \).

**Proof.** — Since \( f.\alpha \in H^2(B, \mathbb{Q}) \) is big, up to replacing \( \alpha \) by a positive multiple of it there exist a very ample line bundle \( \mathcal{L}' \), integral curves \( E_1, \ldots, E_l \subset B \), and \( m'_1, \ldots, m'_l \in \mathbb{Z}_{>0} \) such that

\[
f.\alpha = c_1(\mathcal{L}') + \sum_{i=1}^l n'_i[E_i] \in H^2(B, \mathbb{Q}).
\]

Suppose that \( c_1(\mathcal{L}') = \sum_{i=1}^l m_i[E_i] \) for some \( m_i \in \mathbb{Q} \). Then

\[
m_i \cdot f.\alpha = (n'_i + m_i)c_1(\mathcal{L}') + \sum_{i=1}^l (m_i n'_i - n'_i m_i)[E_i]. \quad (10.1)
\]

Up to reordering the indices we can assume that \( \frac{m_i}{n'_i} \geq \frac{m_j}{n'_j} \) (so \( m_i n'_i - n'_i m_i \geq 0 \)) for every \( i = 1, \ldots, l \). As \( \mathcal{L}' \) is ample, we have \( m_i > 0 \) for at least one \( j \), so \( m_i > 0 \). Therefore up to replacing \( \alpha \) by a positive multiple of it, (10.1) gives a new expression

\[
f.\alpha = c_1(\mathcal{L}'') + \sum_{i=1}^{l'} n''_i[E_i] \in H^2(B, \mathbb{Q}) \quad (10.2)
\]

for some \( l' < l \) and \( n_1, \ldots, n_{l'} \in \mathbb{Z}_{>0} \) together with another very ample line bundle \( \mathcal{L}'' \).

We can repeat the same procedure as long as \( c_1(\mathcal{L}'') \in \sum_{i=1}^{l'} \mathbb{Q}[E_i] \). Since the integer \( l' \) in (10.2) decreases strictly, this procedure eventually stops, which gives an expression \( f.\alpha = c_1(\mathcal{L}) + \sum_{i=1}^l n_i[E_i] \) satisfying the properties in Lemma 10.2.

Let \( Z \subset B \) be the subset such that \( \dim f^{-1}(z) > 1 \) for every \( z \in Z \). As \( f \) is a surjective morphism from an irreducible threefold to a surface, \( Z \) is finite.

We write \( f.\alpha = c_1(\mathcal{L}) + \sum_{i=1}^l n_i[E_i] \) as in Lemma 10.2. Up to further replacing \( \alpha \) by a multiple of it, we can find a linear system \( T \subset |\mathcal{L}| \) such that a general member \( C \) of \( T \) satisfies the following properties,

- \( C \) is smooth and irreducible;
- \( C \neq E_i \) for every \( i \) and \( C \) contains \( Z \subset B \) (as \( Z \) is finite);
- a general pair of points of \( B \) is connected by a chain of general members of \( T \).
Let $C$ be a general member of $T$ and let $E := \cup_i E_i$. Let $D := f^{-1}(C)$ and $D' := f^{-1}(E)$. We have the commutative diagram

$$
\begin{array}{c}
H_2(D \cup D', \mathbb{Q}) \xrightarrow{\iota} H_2(X, \mathbb{Q}) \xrightarrow{\iota_*} H_{BM}^2(X \setminus (D \cup D'), \mathbb{Q}) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
H_2(C \cup E, \mathbb{Q}) \xrightarrow{\iota} H_2(B, \mathbb{Q}) \xrightarrow{\iota_*} H_{BM}^2(B \setminus (C \cup E), \mathbb{Q}) \quad (10.3)
\end{array}
$$

where the rows of (10.3) are part of the long exact sequences of Borel-Moore homology groups induced by the open embeddings [25, IX.2.1], and the vertical arrows between them are induced by $f : X \to B$. The same argument proving Lemma 8.6 shows that the map $(f_{\ast X}(D, B'))$, in (10.3) is an isomorphism.

Let $\tilde{D}$ (resp. $\tilde{D}'$) be a desingularization of $D$ (resp. $D'$) and let

$$
\begin{array}{c}
T : \tilde{D} \cup \tilde{D}' & \to & D \cup D' \leftarrow X.
\end{array}
$$

be the composition of the desingularizations and inclusion. Since $f_*\alpha \in H_2(B, \mathbb{Q})$ is supported on $C \cup E$, by (10.3) we have

$$
\alpha \in \text{Im} (\iota_* : H_2(D \cup D', \mathbb{Q}) \to H_2(X, \mathbb{Q})).
$$

As we assume that Question 1.10 has a positive answer for $X$, there exist $\beta \in H^{1,1}(\tilde{D}, \mathbb{Q})$ and $\beta' \in H^{1,1}(\tilde{D}', \mathbb{Q})$ such that $\iota_*(\beta + \beta') = \alpha$. By construction, we have the factorization

$$
f \circ T : \tilde{D} \cup \tilde{D}' \xrightarrow{\pi} C \cup E \xrightarrow{\iota} B,
$$

so

$$
j_*p_*\beta + j_*p'_*\beta' = f_! \iota_!(\beta + \beta') = f_! \iota_* \alpha = c_1(\mathcal{L}) + \sum_{i=1}^r n_i [E_i] \quad (10.4)
$$

in $H^2(B, \mathbb{Q})$. As $j_*p_*\beta$ is supported on $C$ and $j_*p'_*\beta'$ supported on $\cup_i E_i$, we have

$$
j_*p_*\beta \in \mathbb{Q} \cdot [C] = \mathbb{Q} \cdot c_1(\mathcal{L}) \subset H^2(B, \mathbb{Q}) \quad \text{and} \quad j_*p'_*\beta' \in \sum_j \mathbb{Q} \cdot [E_i] \subset H^2(B, \mathbb{Q}).
$$

Since $c_1(\mathcal{L}) \not\in \sum_i \mathbb{Q} \cdot [E_i]$ by Lemma 10.2, it follows from (10.4) that $j_*p_*\beta \neq 0$ and thus $p_*\beta \neq 0 \in H^0(C, \mathbb{Q})$. Therefore $D$ is projective by Lemma 8.2, so $D$ is algebraically connected.

As a general pair of points of $B$ is connected by a chain of general members $C$ of $T$, when $C$ varies in $T$, the divisors $D$ connects $X$ by construction. It follows from Theorem 2.9 that $X$ is projective. \hfill \Box

\textbf{Proof of Corollary 11.1 for Problem 1.1.} — As we mentioned before, Theorem 1.8 implies that $a(X) = 2$, so $X$ is bimeromorphic to an elliptic fibration $f : X' \to B$ over a projective surface. By desingularization, we can assume that both $X'$ and $B$ are smooth and $X'$ is Kähler. As $\text{Int}(\mathcal{X}(X'))$ contains a rational class, $\text{Int}(\mathcal{X}(X'))$ contains a rational class $\alpha$ as well [33, Proposition 2.1], and we have $f_* \alpha \in \text{Int}(\mathcal{X}(B'))$ by [33, Proposition 2.5]. Since $B$ is a smooth projective surface, $f_* \alpha \in H^{1,1}(B, \mathbb{Q})$ is big by Kleiman’s criterion. Applying Proposition 10.1 to the elliptic fibration $f : X' \to B$ shows that $X'$ is projective. Hence $X$ is projective. \hfill \Box
Finally we prove Corollary 1.11 for Problem 1.3. Before we start the proof, let us first recall and prove some statements about subvarieties with ample normal bundles. The first one is a theorem due to Fulton and Lazarsfeld, asserting that a subvariety with ample normal bundle intersects non-negatively with other subvarieties.

**Theorem 10.3 (Fulton-Lazarsfeld)** [28, Corollary 8.4.3]. — Let $X$ be a compact complex manifold and let $Y \subset X$ be a local complete intersection subvariety of dimension $k$. Assume that $N_{Y/X}$ is ample, then for every subvariety $Z \subset X$ of codimension $k$, we have $Y \cdot Z \geq 0$. Moreover, if $Y \setminus Z \neq \emptyset$, then $Y \cdot Z > 0$.

While the pullback of an effective cycle does not necessarily remain effective, in some situations the ampleness of the normal bundle of a subvariety $Y \subset X$ ensures that the pullback of the cycle class of $Y$ is still effective.

**Lemma 10.4.** — Let $\mu : Y \to X$ be a bimeromorphic morphism between smooth compact Kähler threefolds and let $C \subset X$ be a smooth irreducible curve. If $N_{C/X}$ is ample, then $\mu^*[C] \in H^4(Y, Q)$ is an effective curve class. More precisely, there exists an irreducible curve $C'$ on $Y$ such that $\mu(C') = C$ and $m\mu^*[C] = [C] + [C']$ for some $m \in \mathbb{Z}_{>0}$ and some effective curve class $[C'] \in H^4(Y, Q)$.

We need to first prove a technical lemma before proving Lemma 10.4. Let $X$ be a compact complex variety. A curve $C \subset X$ is called displaceable if for every point $x \in X$ and every irreducible component $C'$ of $C$, we can find a curve $C'' \subset X$ such that $C''$ is deformation equivalent to $C'$ and $x \notin C''$. For any subvariety $Y \subset X$, we say that $C$ is displaceable in $Y$ if $C \subset Y$ and in the previous definition, $C''$ is deformation equivalent to $C'$ in $Y$.

**Lemma 10.5.** — Let $\nu : \tilde{X} \to X$ be the blow-up of a compact Kähler threefold $X$ along an ample smooth center $Z \subset X$. Let $C \subset X$ be a displaceable curve. Then $\nu^*[C] \in H^4(\tilde{X}, Q)$ can be represented by a displaceable curve $\tilde{C} \subset \tilde{X}$.

If moreover $C$ has an irreducible component $C_0$ which is displaceable in some surface $S \subset X$, then $\nu^*[C] \in H^4(\tilde{X}, Q)$ can be represented by a displaceable curve $\tilde{C} \subset \tilde{X}$ which contains an irreducible component $\tilde{C}_0$ such that $\nu(\tilde{C}_0)$ is deformation equivalent to $C_0$ in $S$ and $\tilde{C}_0$ is displaceable in the strict transform $\tilde{S} \subset \tilde{X}$ of $S$.

**Proof.** — Let $C_0, \ldots, C_k$ be the irreducible components of $C$. The blow-up center $Z$ is either a point or a curve. If $Z$ is a point (resp. a curve), then since $C$ is displaceable, we can choose a general deformation $C'_i \subset X$ of $C_i$ such that $C'_i \cap Z = \emptyset$ (resp. $C'_i \cap Z = \emptyset$ or dim $C'_i \cap Z = 0$); for the irreducible component $C_0$, we choose a general deformation $C'_0$ of $C_0$ which moreover remains in $S$. By the blow-up formula [19, Theorem 6.7], we have

$$\nu^*[C] = \sum_i \nu^*[C'_i] = \sum_i [\tilde{C}'_i] + m[F] \in H^4(\tilde{X}, Q)$$

for some $m \in \mathbb{Z}_{>0}$ where $\tilde{C}'_i \subset \tilde{X}$ is the strict transform of $C'_i$, and $F$ is a fiber of $\nu^{-1}(Z) \to Z$.

Clearly $F$ is displaceable. Since each $C'_i$ (resp. $C'_0$) is a general deformation of the displaceable curve $C_i$ (resp. $C_0$), the strict transform $\tilde{C}'_i$ is displaceable (resp. displaceable in $\tilde{S}$). Finally, since $\tilde{C}'_0$ is the strict transform of $C'_0$ under $\nu$, the image $\nu(\tilde{C}'_0) = C'_0$ is deformation equivalent to $C_0$ in $S$ by assumption. □
Proof of Lemma 10.4. — Let \( q : X' \to X \) be the blow-up of \( X \) along \( C \). We resolve the bimeromorphic map \( \mu^{-1} \circ q : X' \to Y \) by a sequence of blow-ups along smooth centers \( \nu : \tilde{X} \to X' \) and let \( p : \tilde{X} \to Y \) be the induced bimeromorphic morphism. The following commutative diagram summarizes the situation.

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\nu} & X' \\
\downarrow{\mu} & & \downarrow{q} \\
Y & \xrightarrow{\nu} & X
\end{array}
\]

Since \( \mu^*[C] = p_\nu \nu^* q^*[C] \), it suffices to prove Lemma 10.4 for the bimeromorphic morphism \( q \circ \nu \).

Let \( E = q^{-1}(C) \) and \( g = q|_E : E \to C \). By the blow-up formula [19, Proposition 6.7.(a)], we have

\[
q^*[\nu] = j_*c_1(g^* N_{C/X}) + j_*c_1 (\mathcal{O}_{E/C}(1))
\]

(10.5)

where \( j : E \hookrightarrow X' \) is the inclusion. As \( N_{C/X} \) is ample, \( c_1(g^* N_{C/X}) + c_1(\mathcal{O}_{E/C}(1)) \) is an ample class in \( E \). Let \( m \in \mathbb{Z}_{\geq 0} \) such that \( \mathcal{L} := (\det(g^* N_{C/X}) \otimes \mathcal{O}_{E/C}(1))^{\otimes m} \) is very ample. We can therefore find an irreducible curve \( C_1 \subset [\mathcal{L}] \) which is displaceable in \( E \). As \( \nu \) is a sequence of blow-ups of threefolds along smooth centers, applying Lemma 10.5 to these blow-ups yields an irreducible curve \( \tilde{C} \subset \tilde{X} \) such that \( \nu(\tilde{C}) \) is deformation equivalent to \( C_1 \) in \( E \) and \( \nu^*[C_1] = [\tilde{C}] + [C'] \) for some effective curve class \([C']\) in \( \tilde{X} \). Since \( m \cdot q^*[\nu] = [C_1] \) by (10.5) and the construction of \( C_1 \), we have

\[
m \cdot q^*[\nu] = [\tilde{C}] + [C'].
\]

Finally, since \( \nu(\tilde{C}) \) is deformation equivalent to \( C_1 \) in \( E \) and \( g(C_1) = C \), necessarily \( q(\nu(\tilde{C})) = g(\nu(\tilde{C})) = C \). This proves Lemma 10.4 for the bimeromorphic morphism \( q \circ \nu \).

Proof of Corollary 1.11 for Problem 1.3. — By [33, Corollary 4.8], we have \( a(X) \geq 2 \).

Assume that \( a(X) = 2 \). By Lemma 2.12, the algebraic reduction \( f : X \to B \) of \( X \) is almost holomorphic whose general fiber \( F \) is an elliptic curve. Let \( \Sigma \) be the irreducible component of the Douady space of \( X \) containing \( F \) and let

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{q} & X \\
\downarrow{p} & & \downarrow \\
\Sigma
\end{array}
\]

denote the universal family. Since \( X \) is a compact Kähler manifold, \( \Sigma \) is proper [16] (and so is \( \mathcal{C} \)). As

\[
2 = \dim B \leq \dim \Sigma \leq H^0(F, N_{F/X}) = H^0(F, \mathcal{O}_F^2) = 2,
\]

the generically injective meromorphic map \( \tau : B \to \Sigma \) induced by the almost holomorphic fibration \( X \to B \) is bimeromorphic. Consequently, \( q \) is bimeromorphic.

Let \( \nu : \Sigma' \to \Sigma \) be the normalization of \( \Sigma \) and let \( \mathcal{C}' := \mathcal{C} \times_{\Sigma} \Sigma' \). We have the commutative diagram

\[
\begin{array}{ccc}
\mathcal{C}' & \xrightarrow{\nu'} & \mathcal{C}' \\
\downarrow{\nu} & & \downarrow{q} \\
\Sigma' & \xrightarrow{\nu'} & \Sigma
\end{array}
\]

\[
\begin{array}{ccc}
\hat{\mathcal{C}} & \xrightarrow{\hat{\nu}} & \hat{\mathcal{C}}' \\
\downarrow{\hat{q}} & & \downarrow{q} \\
\hat{\Sigma} & \xrightarrow{\hat{\nu}'} & \hat{\Sigma}'
\end{array}
\]

\[
\begin{array}{ccc}
\hat{\mathcal{C}} & \xrightarrow{\hat{\nu}} & \hat{\mathcal{C}}' \\
\downarrow{\hat{q}} & & \downarrow{q} \\
\hat{\Sigma} & \xrightarrow{\hat{\nu}'} & \hat{\Sigma}'
\end{array}
\]
where \( \nu' \) (resp. \( \tilde{\nu} \)) is a desingularization of \( \Sigma' \) (resp. \( \mathcal{C}' \)). Since \( C \) is a smooth curve and \( N_{C/X} \) is ample, the irreducible components of \( C \) have ample normal bundles as well, so we can assume that \( C \) is irreducible. By Lemma 10.4, there exist \( m \in \mathbb{Z}_{\geq 0} \) and an irreducible curve \( \tilde{C} \subset \mathcal{C}' \) such that \( \tilde{\nu}(\tilde{C}) = C \) and \( m\tilde{\nu}'[C] = [\tilde{C}] + [C'] \in H^4(\mathcal{C}', \mathbb{Q}) \) for some effective curve class \([C']\).

Now we show that \( \tilde{\nu}([\tilde{C}] + [C']) \) is nef. To this end, it suffices to first show that \( \tilde{\nu}([\tilde{C}] + [C']) \) is nef, and then \((\tilde{\nu}([\tilde{C}] + [C']))^2 > 0 \). Since \( N_{C/X} \) is ample, Theorem 10.3 implies that for every curve \( C_0 \subset \Sigma \), we have

\[
[C_0] \cdot \tilde{\nu}([\tilde{C}] + [C']) = m[C_0] \cdot \tilde{\nu}'[C] = m\tilde{\nu}'[C_0] \cdot [C] \geq 0.
\]

So \( \tilde{\nu}([\tilde{C}] + [C']) \) is nef. To show that \((\tilde{\nu}([\tilde{C}] + [C']))^2 > 0\), first we note that if \( F' \) is a fiber of \( p' \), then \( C \not\subset q'(F') \). Indeed, if \( C \subset q'(F') \), then \( p' \) is flat and a general fiber of \( p \) is an elliptic curve, we would have \( g(C) \leq 1 \). So \( X \) would be projective by [33, Corollary 4.6], contradicting \( a(X) = 2 \). It follows from \( C \not\subset q'(F') \) that \( \nu'(\tilde{\nu}(C)) \subset \mathcal{C}' \) is a curve.

**Claim.** \( D := \tilde{\nu}^{-1}(\tilde{\nu}(C)) \) has a divisorial component \( D' \) containing \( C \)

**Proof.** Let \( E \subset \Sigma \) be the exceptional divisor of \( \nu' \) and let \( U := \Sigma \setminus E \). Since \( \nu' \) is a bimeromorphic morphism between normal surfaces and since \( \nu'(\tilde{\nu}(C)) \subset \Sigma' \) is a curve, we have \( C^\circ := \tilde{\nu}(C) \cup U \neq \emptyset \), which is a curve and \( \nu'(\nu'(C')) \) is a curve as well. As \( p : \mathcal{C}' \to \Sigma \) is the universal family of the Douady space \( \Sigma \), the complex subspace

\[
D' := q'(p'^{-1}(\nu'(C'))) = q(p^{-1}(\nu'(C'))) \subset X,
\]

which is the union of curves parameterized by \( \nu'(C') \), is a divisor and \( D' \) contains \( C \). Finally, note that since \( \nu'|_U \) is isomorphic onto its image, we have \( p'^{-1}(\nu'(C')) \subset \tilde{\nu}(p^{-1}(C')) \). Hence \( D' \subset D \).

By the above claim, it follows from Theorem 10.3 that

\[
[C] \cdot \tilde{\nu} \tilde{\nu}( [\tilde{C}] + [C']) \geq [C] \cdot [D] \geq [C] \cdot [D'] > 0,
\]

so

\[
(\tilde{\nu}([\tilde{C}] + [C']))^2 = ([\tilde{C}] + [C']) \cdot \tilde{\nu} \tilde{\nu}( [\tilde{C}] + [C'])
= m\tilde{\nu}'[C] \cdot \tilde{\nu} \tilde{\nu}( [\tilde{C}] + [C']) = m[C] \cdot \tilde{\nu} \tilde{\nu}( [\tilde{C}] + [C']) > 0,
\]

which, together with the nefness of \( \tilde{\nu}([\tilde{C}] + [C']) \), shows that \( \tilde{\nu}([\tilde{C}] + [C']) \) is big.

Since \( \tilde{\nu} \) is bimeromorphic to \( p \), \( \tilde{\nu} \) is an elliptic fibration. As \( \tilde{\nu}([\tilde{C}] + [C']) \) is big, it follows from Proposition 10.1 that \( \tilde{\mathcal{C}} \) is projective, which contradicts the assumption that \( a(X) = 2 \) because \( X \) is bimeromorphic to \( \mathcal{C}' \). Hence \( X \) is projective.

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Department of Mathematics, National Taiwan University, and National Center for Theoretical Sciences, Taipei, Taiwan.

E-mail: hsuehyung1@ntu.edu.tw