Constructing compact manifolds with exceptional holonomy

Dominic Joyce
Lincoln College, Oxford, OX1 3DR
dominic.joyce@lincoln.ox.ac.uk
March 2002

1 Introduction

In the theory of Riemannian holonomy groups, perhaps the most mysterious are the two exceptional cases, the holonomy group $G_2$ in 7 dimensions and the holonomy group Spin(7) in 8 dimensions. This is a survey paper on the construction of examples of compact 7- and 8-manifolds with holonomy $G_2$ and Spin(7).

All of the material described can be found in the author’s book [9]. Some, but not all, is also in the papers [5, 6, 7, 8, 10, 11]. In particular, the most complicated and powerful form of the construction of compact manifolds with exceptional holonomy by resolving orbifolds $T^n/\Gamma$, and many of the examples, are given only in [9] and not in any published paper.

The rest of this section introduces the holonomy groups $G_2$, Spin(7) and SU($m$), and the relations between them. Section 2 discusses constructions for compact 7-manifolds with holonomy $G_2$. Most of the section explains how to do this by resolving the singularities of orbifolds $T^7/\Gamma$, but in §2.5 we briefly discuss two other methods starting from Calabi–Yau 3-folds.

Section 3 explains constructions for compact 8-manifolds with holonomy Spin(7). One way to do this is to resolve orbifolds $T^8/\Gamma$, but as this is very similar in outline to the $G_2$ material of §2 we say little about it. Instead we describe a second construction which begins with a Calabi–Yau 4-orbifold.

1.1 Riemannian holonomy groups

Let $M$ be a connected $n$-dimensional manifold, $g$ a Riemannian metric on $M$, and $\nabla$ the Levi-Civita connection of $g$. Let $x, y$ be points in $M$ joined by a smooth path $\gamma$. Then parallel transport along $\gamma$ using $\nabla$ defines an isometry between the tangent spaces $T_x M, T_y M$ at $x$ and $y$.

Definition 1.1 The holonomy group $\text{Hol}(g)$ of $g$ is the group of isometries of $T_x M$ generated by parallel transport around piecewise-smooth closed loops based at $x$ in $M$. We consider $\text{Hol}(g)$ to be a subgroup of $O(n)$, defined up to
conjugation by elements of $O(n)$. Then $\text{Hol}(g)$ is independent of the base point $x$ in $M$.

The classification of holonomy groups was achieved by Berger [1] in 1955.

**Theorem 1.2** Let $M$ be a simply-connected, $n$-dimensional manifold, and $g$ an irreducible, nonsymmetric Riemannian metric on $M$. Then either

(i) $\text{Hol}(g) = \text{SO}(n)$,
(ii) $n = 2m$ and $\text{Hol}(g) = \text{SU}(m)$ or $U(m)$,
(iii) $n = 4m$ and $\text{Hol}(g) = \text{Sp}(m)$ or $\text{Sp}(m)\text{Sp}(1)$,
(iv) $n = 7$ and $\text{Hol}(g) = \text{G}_2$, or
(v) $n = 8$ and $\text{Hol}(g) = \text{Spin}(7)$.

Now $\text{G}_2$ and $\text{Spin}(7)$ are the exceptional cases in this classification, so they are called the *exceptional holonomy groups*. For some time after Berger’s classification, the exceptional holonomy groups remained a mystery. In 1987, Bryant [2] used the theory of exterior differential systems to show that locally there exist many metrics with these holonomy groups, and gave some explicit, incomplete examples. Then in 1989, Bryant and Salamon [3] found explicit, complete metrics with holonomy $\text{G}_2$ and $\text{Spin}(7)$ on noncompact manifolds.

In 1994-5 the author constructed the first examples of metrics with holonomy $\text{G}_2$ and $\text{Spin}(7)$ on *compact* manifolds [5, 6, 7]. These, and the more complicated constructions developed later by the author [8, 9] and by Kovalev [12], are the subject of this article.

### 1.2 The holonomy group $\text{G}_2$

Let $(x_1, \ldots, x_7)$ be coordinates on $\mathbb{R}^7$. Write $dx_{ij\ldots l}$ for the exterior form $dx_i \wedge dx_j \wedge \cdots \wedge dx_l$ on $\mathbb{R}^7$. Define a metric $g_0$, a 3-form $\varphi_0$ and a 4-form $\ast \varphi_0$ on $\mathbb{R}^7$ by $g_0 = dx_1^2 + \cdots + dx_7^2$,

\[
\begin{align*}
\varphi_0 &= dx_{123} + dx_{145} + dx_{167} + dx_{246} - dx_{257} - dx_{347} - dx_{356} \\
\ast \varphi_0 &= dx_{4567} + dx_{2367} + dx_{2345} + dx_{1357} - dx_{1346} - dx_{1256} - dx_{1247}.
\end{align*}
\]

The subgroup of $\text{GL}(7, \mathbb{R})$ preserving $\varphi_0$ is the *exceptional Lie group $\text{G}_2$*. It also preserves $g_0, \ast \varphi_0$ and the orientation on $\mathbb{R}^7$. It is a compact, semisimple, 14-dimensional Lie group, a subgroup of $\text{SO}(7)$.

A $\text{G}_2$-structure on a 7-manifold $M$ is a principal subbundle of the frame bundle of $M$, with structure group $\text{G}_2$. Each $\text{G}_2$-structure gives rise to a 3-form $\varphi$ and a metric $g$ on $M$, such that every tangent space of $M$ admits an isomorphism with $\mathbb{R}^7$ identifying $\varphi$ and $g$ with $\varphi_0$ and $g_0$ respectively. By an abuse of notation, we will refer to $(\varphi, g)$ as a $\text{G}_2$-structure.

**Proposition 1.3** Let $M$ be a 7-manifold and $(\varphi, g)$ a $\text{G}_2$-structure on $M$. Then the following are equivalent:
Theorem 1.4 Let $M$ be a compact 7-manifold, and suppose that $(\varphi, g)$ is a torsion-free $G_2$-structure on $M$. Then $\text{Hol}(g) = G_2$ if and only if $\pi_1(M)$ is finite. In this case the moduli space of metrics with holonomy $G_2$ on $M$, up to diffeomorphisms isotopic to the identity, is a smooth manifold of dimension $b^3(M)$.

1.3 The holonomy group $\text{Spin}(7)$

Let $\mathbb{R}^8$ have coordinates $(x_1, \ldots, x_8)$. Define a 4-form $\Omega_0$ on $\mathbb{R}^8$ by

$$
\Omega_0 = dx_{1234} + dx_{1256} + dx_{1278} - dx_{1368} - dx_{1458} - dx_{1467} - dx_{2358} - dx_{2367} - dx_{2457} + dx_{2468} + dx_{3456} + dx_{3478} + dx_{5678}.
$$

The subgroup of $\text{GL}(8, \mathbb{R})$ preserving $\Omega_0$ is the holonomy group $\text{Spin}(7)$. It also preserves the orientation on $\mathbb{R}^8$ and the Euclidean metric $g_0 = dx_1^2 + \cdots + dx_8^2$. It is a compact, semisimple, 21-dimensional Lie group, a subgroup of $\text{SO}(8)$.

A Spin(7)-structure on an 8-manifold $M$ gives rise to a 4-form $\Omega$ and a metric $g$ on $M$, such that each tangent space of $M$ admits an isomorphism with $\mathbb{R}^8$ identifying $\Omega$ and $g$ with $\Omega_0$ and $g_0$ respectively. By an abuse of notation we will refer to the pair $(\Omega, g)$ as a Spin(7)-structure.

Proposition 1.5 Let $M$ be an 8-manifold and $(\Omega, g)$ a Spin(7)-structure on $M$. Then the following are equivalent:

(i) $\text{Hol}(g) \subseteq \text{Spin}(7)$, and $\Omega$ is the induced 4-form,

(ii) $\nabla \Omega = 0$ on $M$, where $\nabla$ is the Levi-Civita connection of $g$, and

(iii) $d\Omega = 0$ on $M$.

We call $\nabla \Omega$ the torsion of the Spin(7)-structure $(\Omega, g)$, and $(\Omega, g)$ torsion-free if $\nabla \Omega = 0$. A triple $(M, \Omega, g)$ is called a Spin(7)-manifold if $M$ is an 8-manifold and $(\Omega, g)$ a torsion-free Spin(7)-structure on $M$. If $g$ has holonomy $\text{Hol}(g) \subseteq \text{Spin}(7)$, then $g$ is Ricci-flat.

Here is a result on compact 8-manifolds with holonomy Spin(7).

Theorem 1.6 Let $(M, \Omega, g)$ be a compact Spin(7)-manifold. Then $\text{Hol}(g) = \text{Spin}(7)$ if and only if $M$ is simply-connected, and $b^3(M) + b^1_-(M) = b^2(M) + b^4_-(M) + 25$. In this case the moduli space of metrics with holonomy Spin(7) on $M$, up to diffeomorphisms isotopic to the identity, is a smooth manifold of dimension $1 + b^1_-(M)$. 

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1.4 The holonomy groups \( SU(m) \)

Let \( \mathbb{C}^m \cong \mathbb{R}^{2m} \) have complex coordinates \((z_1, \ldots, z_m)\), and define the metric \( g_0 \), Kähler form \( \omega_0 \) and complex volume form \( \theta_0 \) on \( \mathbb{C}^m \) by

\[
g_0 = |dz_1|^2 + \cdots + |dz_m|^2, \quad \omega_0 = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + \cdots + dz_m \wedge d\bar{z}_m),
\]

and \( \theta_0 = dz_1 \wedge \cdots \wedge dz_m \).

The subgroup of \( GL(2m, \mathbb{R}) \) preserving \( g_0, \omega_0 \) and \( \theta_0 \) is the special unitary group \( SU(m) \). Manifolds with holonomy \( SU(m) \) are called Calabi–Yau manifolds.

Calabi–Yau manifolds are automatically Ricci-flat and Kähler, with trivial canonical bundle. Conversely, any Ricci-flat Kähler manifold \((M, J, g)\) with trivial canonical bundle has \( \text{Hol}(g) \subseteq SU(m) \). By Yau’s proof of the Calabi conjecture \( [16] \), we have:

**Theorem 1.7** Let \((M, J)\) be a compact complex \( m \)-manifold admitting Kähler metrics, with trivial canonical bundle. Then there is a unique Ricci-flat Kähler metric \( g \) in each Kähler class on \( M \), and \( \text{Hol}(g) \subseteq SU(m) \).

Using this and complex algebraic geometry one can construct many examples of compact Calabi–Yau manifolds. The theorem also applies in the orbifold category, yielding examples of Calabi–Yau orbifolds.

1.5 Relations between \( G_2, \text{Spin}(7) \) and \( SU(m) \)

Here are the inclusions between the holonomy groups \( SU(m), G_2 \) and \( \text{Spin}(7) \):

\[
\begin{array}{ccc}
SU(2) & \longrightarrow & SU(3) \longrightarrow G_2 \\
\downarrow & & \downarrow & & \downarrow \\
SU(2) \times SU(2) & \longrightarrow & SU(4) \longrightarrow \text{Spin}(7).
\end{array}
\]

We shall illustrate what we mean by this using the inclusion \( SU(3) \hookrightarrow G_2 \). As \( SU(3) \) acts on \( \mathbb{C}^7 \), it also acts on \( \mathbb{R} \oplus \mathbb{C}^3 \cong \mathbb{R}^7 \), taking the \( SU(3) \)-action on \( \mathbb{R} \) to be trivial. Thus we embed \( SU(3) \) as a subgroup of \( GL(7, \mathbb{R}) \). It turns out that \( SU(3) \) is a subgroup of the subgroup \( G_2 \) of \( GL(7, \mathbb{R}) \) defined in \( \ref{SU3} \).

Here is a way to see this in terms of differential forms. Identify \( \mathbb{R} \oplus \mathbb{C}^3 \) with \( \mathbb{R}^7 \) in the obvious way in coordinates, so that \((x_1, (x_2 + ix_3, x_4 + ix_5, x_6 + ix_7))\) in \( \mathbb{R} \oplus \mathbb{C}^3 \) is identified with \((x_1, \ldots, x_7)\) in \( \mathbb{R}^7 \). Then \( \varphi_0 = dx_1 \wedge \omega_0 + \text{Re} \theta_0 \), where \( \varphi_0 \) is defined in \( \ref{SU3} \) and \( \omega_0, \theta_0 \) in \( \ref{SU3} \). Since \( SU(3) \) preserves \( \omega_0 \) and \( \theta_0 \), the action of \( SU(3) \) on \( \mathbb{R}^7 \) preserves \( \varphi_0 \), and so \( SU(3) \subseteq G_2 \).

It follows that if \((M, J, h)\) is Calabi–Yau 3-fold, then \( \mathbb{R} \times M \) and \( S^1 \times M \) have torsion-free \( G_2 \)-structures, that is, are \( G_2 \)-manifolds.

**Proposition 1.8** Let \((M, J, h)\) be a Calabi–Yau 3-fold, with Kähler form \( \omega \) and complex volume form \( \theta \). Let \( x \) be a coordinate on \( \mathbb{R} \) or \( S^1 \). Define a metric \( g = dx^2 + h \) and a 3-form \( \varphi = dx \wedge \omega + \text{Re} \theta \) on \( \mathbb{R} \times M \) or \( S^1 \times M \). Then \((g, \varphi)\) is a torsion-free \( G_2 \)-structure on \( \mathbb{R} \times M \) or \( S^1 \times M \), and \( \ast \varphi = \frac{1}{2} \omega \wedge dx \wedge \text{Im} \theta \).
Similarly, the inclusions $SU(2) \hookrightarrow G_2$ and $SU(4) \hookrightarrow \text{Spin}(7)$ give:

**Proposition 1.9** Let $(M, J, h)$ be a Calabi–Yau 2-fold, with Kähler form $\omega$ and complex volume form $\theta$. Let $(x_1, x_2, x_3)$ be coordinates on $\mathbb{R}^3$ or $T^3$. Define a metric $g = dx_1^2 + dx_2^2 + dx_3^2 + h$ and a 3-form

$$\varphi = dx_1 \wedge dx_2 \wedge dx_3 + dx_1 \wedge \omega + dx_2 \wedge \text{Re} \theta - dx_3 \wedge \text{Im} \theta \quad (4)$$

on $\mathbb{R}^3 \times M$ or $T^3 \times M$. Then $(\varphi, g)$ is a torsion-free $G_2$-structure on $\mathbb{R}^3 \times M$ or $T^3 \times M$, and

$$\ast \varphi = \frac{1}{2} \omega \wedge \omega + dx_2 \wedge dx_3 \wedge \omega - dx_1 \wedge dx_3 \wedge \text{Re} \theta - dx_1 \wedge dx_2 \wedge \text{Im} \theta. \quad (5)$$

**Proposition 1.10** Let $(M, J, g)$ be a Calabi–Yau 4-fold, with Kähler form $\omega$ and complex volume form $\theta$. Define a 4-form $\Omega$ on $M$ by

$$\Omega = \frac{1}{2} \omega \wedge \omega + \text{Re} \theta.$$

Then $(\Omega, g)$ is a torsion-free $\text{Spin}(7)$-structure on $M$.

### 2 Constructing $G_2$-manifolds from orbifolds $T^7/\Gamma$

We now explain the method used in [5, 6] and [9, §11–§12] to construct examples of compact 7-manifolds with holonomy $G_2$. It is based on the Kummer construction for Calabi–Yau metrics on the $K3$ surface, and may be divided into four steps.

**Step 1.** Let $T^7$ be the 7-torus and $(\varphi_0, g_0)$ a flat $G_2$-structure on $T^7$. Choose a finite group $\Gamma$ of isometries of $T^7$ preserving $(\varphi_0, g_0)$. Then the quotient $T^7/\Gamma$ is a singular, compact 7-manifold, an orbifold.

**Step 2.** For certain special groups $\Gamma$ there is a method to resolve the singularities of $T^7/\Gamma$ in a natural way, using complex geometry. We get a nonsingular, compact 7-manifold $M$, together with a map $\pi : M \to T^7/\Gamma$, the resolving map.

**Step 3.** On $M$, we explicitly write down a 1-parameter family of $G_2$-structures $(\varphi_t, g_t)$ depending on $t \in (0, \epsilon)$. They are not torsion-free, but have small torsion when $t$ is small. As $t \to 0$, the $G_2$-structure $(\varphi_t, g_t)$ converges to the singular $G_2$-structure $\pi^*(\varphi_0, g_0)$.

**Step 4.** We prove using analysis that for sufficiently small $t$, the $G_2$-structure $(\varphi_t, g_t)$ on $M$, with small torsion, can be deformed to a $G_2$-structure $(\tilde{\varphi}_t, \tilde{g}_t)$, with zero torsion. Finally, we show that $\tilde{g}_t$ is a metric with holonomy $G_2$ on the compact 7-manifold $M$.

We will now explain each step in greater detail.
2.1 Step 1: Choosing an orbifold

Let \((\varphi_0, g_0)\) be the Euclidean \(G_2\)-structure on \(\mathbb{R}^7\) defined in §1.2. Suppose \(\Lambda\) is a lattice in \(\mathbb{R}^7\), that is, a discrete additive subgroup isomorphic to \(\mathbb{Z}^7\). Then \(\mathbb{R}^7/\Lambda\) is the torus \(T^7\), and \((\varphi_0, g_0)\) pushes down to a torsion-free \(G_2\)-structure on \(T^7\). We must choose a finite group \(\Gamma\) acting on \(T^7\) preserving \((\varphi_0, g_0)\). That is, the elements of \(\Gamma\) are the push-forwards to \(T^7\) of affine transformations of \(\mathbb{R}^7\) which fix \((\varphi_0, g_0)\), and take \(\Lambda\) to itself under conjugation.

Here is an example of a suitable group \(\Gamma\), taken from [9, §12.2].

**Example 2.1** Let \((x_1, \ldots , x_7)\) be coordinates on \(T^7 = \mathbb{R}^7/\mathbb{Z}^7\), where \(x_i \in \mathbb{R}/\mathbb{Z}\). Let \((\varphi_0, g_0)\) be the flat \(G_2\)-structure on \(T^7\) defined by (1). Let \(\alpha, \beta\) and \(\gamma\) be the involutions of \(T^7\) defined by

\[
\alpha : (x_1, \ldots , x_7) \mapsto (x_1, x_2, -x_3, -x_4, -x_5, -x_6, -x_7),
\]
\[
\beta : (x_1, \ldots , x_7) \mapsto (x_1, -x_2, -x_3, x_4, x_5, \frac{1}{2} - x_6, -x_7),
\]
\[
\gamma : (x_1, \ldots , x_7) \mapsto (-x_1, x_2, -x_3, x_4, \frac{1}{2} - x_5, x_6, \frac{1}{2} - x_7).
\]

By inspection, \(\alpha, \beta\) and \(\gamma\) preserve \((\varphi_0, g_0)\), because of the careful choice of exactly which signs to change. Also, \(\alpha^2 = \beta^2 = \gamma^2 = 1\), and \(\alpha, \beta\) and \(\gamma\) commute. Thus they generate a group \(\Gamma = \langle \alpha, \beta, \gamma \rangle \cong \mathbb{Z}_2^3\) of isometries of \(T^7\) preserving the flat \(G_2\)-structure \((\varphi_0, g_0)\).

Having chosen a lattice \(\Lambda\) and finite group \(\Gamma\), the quotient \(T^7/\Gamma\) is an orbifold, a singular manifold with only quotient singularities. The singularities of \(T^7/\Gamma\) come from the fixed points of non-identity elements of \(\Gamma\). We now describe the singularities in our example.

**Lemma 2.2** In Example 2.1 \(\beta \gamma, \gamma \alpha, \alpha \beta\) and \(\alpha \beta \gamma\) have no fixed points on \(T^7\). The fixed points of \(\alpha, \beta, \gamma\) are each 16 copies of \(T^3\). The singular set \(S\) of \(T^7/\Gamma\) is a disjoint union of 12 copies of \(T^3\), 4 copies from each of \(\alpha, \beta, \gamma\). Each component of \(S\) is a singularity modelled on that of \(T^3 \times \mathbb{C}^2/\{\pm 1\}\).

The most important consideration in choosing \(\Gamma\) is that we should be able to resolve the singularities of \(T^7/\Gamma\) within holonomy \(G_2\). We will explain how to do this next.

2.2 Step 2: Resolving the singularities

Our goal is to resolve the singular set \(S\) of \(T^7/\Gamma\) to get a compact 7-manifold \(M\) with holonomy \(G_2\). How can we do this? In general we cannot, because we have no idea of how to resolve general orbifold singularities with holonomy \(G_2\). However, suppose we can arrange that every connected component of \(S\) is locally isomorphic to either

(a) \(T^3 \times \mathbb{C}^2/G\), for \(G\) a finite subgroup of \(\text{SU}(2)\), or
(b) \(S^1 \times \mathbb{C}^3/G\), for \(G\) a finite subgroup of \(\text{SU}(3)\) acting freely on \(\mathbb{C}^3 \setminus \{0\}\).
One can use complex algebraic geometry to find a crepant resolution $X$ of $\mathbb{C}^2/G$ or $\mathbb{C}^3/G$. Then $T^3 \times X$ or $S^1 \times Y$ gives a local model for how to resolve the corresponding component of $S$ in $T^7/\Gamma$. Thus we construct a nonsingular, compact 7-manifold $M$ by using the patches $T^3 \times X$ or $S^1 \times Y$ to repair the singularities of $T^7/\Gamma$. In the case of Example 2.1, this means gluing 12 copies of $T^3 \times X$ into $T^7/\Gamma$, where $X$ is the blow-up of $\mathbb{C}^2/\{\pm 1\}$ at its singular point.

Now the point of using crepant resolutions is this. In both case (a) and (b), there exists a Calabi–Yau metric on $X$ or $Y$ which is asymptotic to the flat Euclidean metric on $\mathbb{C}^2/G$ or $\mathbb{C}^3/G$. Such metrics are called Asymptotically Locally Euclidean (ALE). In case (a), the ALE Calabi–Yau metrics were classified by Kronheimer [10, 12], and exist for all finite $G \subset SU(2)$. In case (b), crepant resolutions of $\mathbb{C}^3/G$ exist for all finite $G \subset SU(3)$ by Roan [15, 16, 8] proved that they carry ALE Calabi–Yau, using a noncompact version of the Calabi Conjecture.

By Propositions 1.8 and 1.9, we can use the Calabi–Yau metrics on $X$ or $Y$ to construct a torsion-free $G_2$-structure on $T^3 \times X$ or $S^1 \times Y$. This gives a local model for how to resolve the singularity $T^3 \times \mathbb{C}^2/G$ or $S^1 \times \mathbb{C}^3/G$ with holonomy $G_2$. So, this method gives not only a way to smooth out the singularities of $T^7/\Gamma$ as a manifold, but also a family of torsion-free $G_2$-structures on the resolution which show how to smooth out the singularities of the $G_2$-structure.

The requirement above that $S$ be divided into connected components of the form (a) and (b) is in fact unnecessarily restrictive. There is a more complicated and powerful method, described in [11–12], for resolving singularities of a more general kind. We require only that the singularities should locally be of the form $\mathbb{R}^3 \times \mathbb{C}^2/G$ or $\mathbb{R} \times \mathbb{C}^3/G$, for $G$ a finite subgroup of $SU(2)$ or $SU(3)$, and when $G \subset SU(3)$ we do not require that $G$ act freely on $\mathbb{C}^3 \setminus \{0\}$.

If $X$ is a crepant resolution of $\mathbb{C}^3/G$, where $G$ does not act freely on $\mathbb{C}^3 \setminus \{0\}$, then the author shows [11, 9, 1] that $X$ carries a family of Calabi–Yau metrics satisfying a complicated asymptotic condition at infinity, called Quasi-ALE metrics. These yield the local models necessary to resolve singularities locally of the form $\mathbb{R} \times \mathbb{C}^3/G$ with holonomy $G_2$. Using this method we can resolve many orbifolds $T^7/\Gamma$, and prove the existence of large numbers of compact 7-manifolds with holonomy $G_2$.

### 2.3 Step 3: Finding $G_2$-structures with small torsion

For each resolution $X$ of $\mathbb{C}^2/G$ in case (a), and $Y$ of $\mathbb{C}^3/G$ in case (b) above, we can find a 1-parameter family $\{h_t : t > 0\}$ of metrics with the properties

(a) $h_t$ is a $\mathbb{K}$ähler metric on $X$ with $\text{Hol}(h_t) = SU(2)$. Its injectivity radius satisfies $\delta(h_t) = O(t^3)$, its Riemann curvature satisfies $\|R(h_t)\|_{C^0} = O(t^{-2})$, and $h_t = h + O(t^4 r^{-4})$ for large $r$, where $h$ is the Euclidean metric on $\mathbb{C}^2/G$, and $r$ the distance from the origin.

(b) $h_t$ is $\mathbb{K}$ähler on $Y$ with $\text{Hol}(h_t) = SU(3)$, where $\delta(h_t) = O(t)$, $\|R(h_t)\|_{C^0} = O(t^{-2})$, and $h_t = h + O(t^6 r^{-6})$ for large $r$. 
In fact we can choose $h_t$ to be isometric to $t^2 h_1$, and the properties above are easy to prove.

Suppose one of the components of the singular set $S$ of $T^7 / \Gamma$ is locally modelled on $T^3 \times \mathbb{C}^2 / G$. Then $T^3$ has a natural flat metric $h_{T^3}$. Let $X$ be the crepant resolution of $\mathbb{C}^2 / G$ and let $\{ h_t : t > 0 \}$ satisfy property (a). Then Proposition 1.5 gives a 1-parameter family of torsion-free $G_2$-structures $(\hat{\varphi}, \hat{g}_t)$ on $T^3 \times X$ with $\hat{g}_t = h_{T^3} + h_t$. Similarly, if a component of $S$ is modelled on $\mathbb{S}^3 \times \mathbb{C}^3 / G$, using Proposition 1.8 we get a family of torsion-free $G_2$-structures $(\hat{\varphi}_t, \hat{g}_t)$ on $\mathbb{S}^4 \times Y$.

The idea is to make a $G_2$-structure $(\varphi_t, g_t)$ on $M$ by gluing together the torsion-free $G_2$-structures $(\hat{\varphi}, \hat{g}_t)$ on the patches $T^3 \times X$ and $\mathbb{S}^3 \times Y$, and $(\hat{\varphi}_0, g_0)$ on $T^7 / \Gamma$. The gluing is done using a partition of unity. Naturally, the first derivative of the partition of unity introduces ‘errors’, so that $(\varphi_t, g_t)$ is not torsion-free. The size of the torsion $\nabla \varphi_t$ depends on the difference $\hat{\varphi}_t - \hat{\varphi}_0$ in the region where the partition of unity changes. On the patches $T^3 \times X$, since $h_t - h = O(t^4 r^{-3})$ and the partition of unity has nonzero derivative when $r = O(1)$, we find that $\nabla \varphi_t = O(t^4)$. Similarly $\nabla \varphi_t = O(t^6)$ on the patches $\mathbb{S}^4 \times Y$, and so $\nabla \varphi_t = O(t^2)$ on $M$.

For small $t$, the dominant contributions to the injectivity radius $\delta(g_t)$ and Riemann curvature $R(g_t)$ are made by those of the metrics $h_t$ on $X$ and $Y$, so we expect $\delta(g_t) = O(t)$ and $\| R(g_t) \|_{C^0} = O(t^{-2})$ by properties (a) and (b) above. In this way we prove the following result [8 Th. 11.5.7], which gives the estimates on $(\varphi_t, g_t)$ that we need.

**Theorem 2.3** On the compact 7-manifold $M$ described above, and on many other 7-manifolds constructed in a similar fashion, one can write down the following data explicitly in coordinates:

- Positive constants $A_1, A_2, A_3$ and $\epsilon$.
- A $G_2$-structure $(\varphi_t, g_t)$ on $M$ with $d\varphi_t = 0$ for each $t \in (0, \epsilon)$, and
- A 3-form $\psi_t$ on $M$ with $d^* \psi_t = d^* \varphi_t$ for each $t \in (0, \epsilon)$.

These satisfy three conditions:

(i) $\| \psi_t \|_{L^2} \leq A_1 t^4$, $\| \psi_t \|_{C^0} \leq A_1 t^3$ and $\| d^* \psi_t \|_{L^{14}} \leq A_1 t^{16/7}$,

(ii) the injectivity radius $\delta(g_t)$ satisfies $\delta(g_t) \geq A_2 t$,

(iii) the Riemann curvature $R(g_t)$ of $g_t$ satisfies $\| R(g_t) \|_{C^0} \leq A_3 t^{-2}$.

Here the operator $d^*$ and the norms $\| \cdot \|_{L^2}$, $\| \cdot \|_{L^{14}}$ and $\| \cdot \|_{C^0}$ depend on $g_t$.

Here one should regard $\psi_t$ as a first integral of the torsion $\nabla \varphi_t$ of $(\varphi_t, g_t)$. Thus the norms $\| \psi_t \|_{L^2}$, $\| \psi_t \|_{C^0}$ and $\| d^* \psi_t \|_{L^{14}}$ are measures of $\nabla \varphi_t$. So parts (i)–(iii) say that the torsion $\nabla \varphi_t$ must be small compared to the injectivity radius and Riemann curvature of $(M, g_t)$.
2.4 Step 4: Deforming to a torsion-free $G_2$-structure

We prove the following analysis result.

**Theorem 2.4** Let $A_1, A_2, A_3$ be positive constants. Then there exist positive constants $\kappa, K$ such that whenever $0 < t \leq \kappa$, the following is true.

Let $M$ be a compact 7-manifold, and $(\varphi, g)$ a $G_2$-structure on $M$ with $d\varphi = 0$. Suppose $\psi$ is a smooth 3-form on $M$ with $d^*\psi = d^*\varphi$, and

(i) $\|\psi\|_{L^2} \leq A_1 t^4$, $\|\psi\|_{C^0} \leq A_1 t^{1/2}$ and $\|d^*\psi\|_{L^4} \leq A_1$,

(ii) the injectivity radius $\delta(g)$ satisfies $\delta(g) \geq A_2 t$, and

(iii) the Riemann curvature $R(g)$ satisfies $\|R(g)\|_{C^0} \leq A_3 t^{-2}$.

Then there exists a smooth, torsion-free $G_2$-structure $(\tilde{\varphi}, \tilde{g})$ on $M$ with $\|\tilde{\varphi} - \varphi\|_{C^0} \leq K t^{1/2}$.

Basically, this result says that if $(\varphi, g)$ is a $G_2$-structure on $M$, and the torsion $\nabla \varphi$ is sufficiently small, then we can deform to a nearby $G_2$-structure $(\tilde{\varphi}, \tilde{g})$ that is torsion-free. Here is a sketch of the proof of Theorem 2.4, ignoring several technical points. The proof is that given in §11.6–§11.8, which is an improved version of the proof in [5].

We have a 3-form $\varphi$ with $d\varphi = 0$ and $d^*\varphi = d^*\psi$ for small $\psi$, and we wish to construct a nearby 3-form $\tilde{\varphi}$ with $d\tilde{\varphi} = 0$ and $d^*\tilde{\varphi} = 0$. Set $\tilde{\varphi} = \varphi + d\eta$, where $\eta$ is a small 2-form. Then $\eta$ must satisfy a nonlinear p.d.e., which we write as

$$d^*d\eta = -d^*\psi + d^*F(d\eta), \quad (9)$$

where $F$ is nonlinear, satisfying $F(d\eta) = O(|d\eta|^2)$.

We solve (9) by iteration, introducing a sequence $\{\eta_j\}_{j=0}^{\infty}$ with $\eta_0 = 0$, satisfying the inductive equations

$$d^*d\eta_{j+1} = -d^*\psi + d^*F(d\eta_j), \quad d^*\eta_{j+1} = 0. \quad (10)$$

If such a sequence exists and converges to $\eta$, then taking the limit in (10) shows that $\eta$ satisfies (9), giving us the solution we want.

The key to proving this is an inductive estimate on the sequence $\{\eta_j\}_{j=0}^{\infty}$. The inductive estimate we use has three ingredients, the equations

$$\|d\eta_{j+1}\|_{L^2} \leq \|\psi\|_{L^2} + C_1\|d\eta_j\|_{L^2}\|d\eta_j\|_{C^0}, \quad (11)$$

$$\|\nabla d\eta_{j+1}\|_{L^4} \leq C_2\left(\|d^*\psi\|_{L^4} + \|\nabla d\eta_j\|_{L^4}\|d\eta_j\|_{C^0} + t^{-4}\|d\eta_{j+1}\|_{L^2}\right), \quad (12)$$

$$\|d\eta_j\|_{C^0} \leq C_3\left(t^{1/2}\|\nabla d\eta_j\|_{L^4} + t^{-7/2}\|d\eta_j\|_{L^4}\right). \quad (13)$$

Here $C_1, C_2, C_3$ are positive constants independent of $t$. Equation (12) is obtained from (10) by taking the $L^2$-inner product with $\eta_{j+1}$ and integrating by parts. Using the fact that $d^*\psi = d^*\varphi$ and $\|\psi\|_{L^2} = O(t^4)$, $|\psi| = O(t^{1/2})$ we get a powerful estimate of the $L^2$-norm of $d\eta_{j+1}$.

Equation (13) is derived from an elliptic regularity estimate for the operator $d + d^*$ acting on 3-forms on $M$. Equation (13) follows from the Sobolev embedding
Theorem, since $L^4_1(M)$ embeds in $C^0(M)$. Both (12) and (13) are proved on small balls of radius $O(t)$ in $M$, using parts (ii) and (iii) of Theorem 2.3, and this is where the powers of $t$ come from.

Using (11), (13) and part (i) of Theorem 2.3 we show that

$$
\|d\eta_j\|_{L^2} \leq C_4 t^4, \quad \|\nabla d\eta_j\|_{L^{14}} \leq C_5, \quad \text{and} \quad \|d\eta_j\|_{C^0} \leq K t^{1/2},
$$

where $C_4, C_5$ and $K$ are positive constants depending on $C_1, C_2, C_3$ and $A_1$, and if $t$ is sufficiently small, then the same inequalities (14) apply to $d\eta_{j+1}$. Since $\eta_0 = 0$, by induction (14) applies for all $j$ and the sequence $\{d\eta_j\}_{j=0}^\infty$ is bounded in the Banach space $L^4_1(\Lambda^p T^*M)$. One can then use standard techniques in analysis to prove that this sequence converges to a smooth limit $d\eta$. This concludes the proof of Theorem 2.4.

Figure 1: Betti numbers $(b^2, b^3)$ of compact $G_2$-manifolds

From Theorems 2.3 and 2.4 we see that the compact 7-manifold $M$ constructed in Step 2 admits torsion-free $G_2$-structures $(\hat{\varphi}, \hat{g})$. Theorem 1.4 then shows that $\text{Hol}(\hat{g}) = G_2$ if and only if $\pi_1(M)$ is finite. In the example above $M$ is simply-connected, and so $\pi_1(M) = \{1\}$ and $M$ has metrics with holonomy $G_2$, as we want.

By considering different groups $\Gamma$ acting on $T^7$, and also by finding topologically distinct resolutions $M_1, \ldots, M_k$ of the same orbifold $T^7/\Gamma$, we can construct many compact Riemannian 7-manifolds with holonomy $G_2$. A good number of examples are given in [8, §12]. Figure 1 displays the Betti numbers of compact, simply-connected 7-manifolds with holonomy $G_2$ constructed there. There are 252 different sets of Betti numbers.

Examples are also known [8, §12.4] of compact 7-manifolds with holonomy $G_2$ with finite, nontrivial fundamental group. It seems likely to the author that the Betti numbers given in Figure 1 are only a small proportion of the Betti numbers of all compact, simply-connected 7-manifolds with holonomy $G_2$. 

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2.5 Other constructions of compact $G_2$-manifolds

Here are two other methods, taken from [3 §11.9], that may be used to construct compact 7-manifolds with holonomy $G_2$. The first method was outlined by the author in [3 §4.3], and is being studied by the author’s student Ben Stephens.

**Method 1.** Let $(Y, J, h)$ be a Calabi–Yau 3-fold, with Kähler form $\omega$ and holomorphic volume form $\theta$. Suppose $\sigma : Y \to Y$ is an involution, satisfying $\sigma^*(h) = h$, $\sigma^*(J) = -J$ and $\sigma^*(\theta) = \theta$. We call $\sigma$ a real structure on $Y$. Let $N$ be the fixed point set of $\sigma$ in $Y$. Then $N$ is a real 3-dimensional submanifold of $Y$, and is in fact a special Lagrangian 3-fold.

Let $S^1 = \mathbb{R}/\mathbb{Z}$, and define a torsion-free $G_2$-structure $(\varphi, g)$ on $S^1 \times Y$ as in Proposition 1.8. Then $\varphi = dx \wedge \omega + \text{Re}\theta$, where $x \in \mathbb{R}/\mathbb{Z}$ is the coordinate on $S^1$. Define $\sigma : S^1 \times Y \to S^1 \times Y$ by $\sigma((x, y)) = (-x, \sigma(y))$. Then $\sigma$ preserves $(\varphi, g)$ and $\sigma^2 = 1$. The fixed points of $\sigma$ in $S^1 \times Y$ are $\{\mathbb{Z}, \frac{1}{2} + \mathbb{Z}\} \times N$. Thus $(S^1 \times Y)/\sigma$ is an orbifold. Its singular set is 2 copies of $N$, and each singular point is modelled on $\mathbb{R}^3 \times \mathbb{R}^4/(\pm 1)$.

We aim to resolve $(S^1 \times Y)/\sigma$ to get a compact 7-manifold $M$ with holonomy $G_2$. Locally, each singular point should be resolved like $\mathbb{R}^3 \times X$, where $X$ is an ALE Calabi–Yau 2-fold asymptotic to $\mathbb{C}^2/\{\pm 1\}$. There is a 3-dimensional family of such $X$, and we need to choose one member of this family for each singular point in the singular set.

Calculations by the author indicate that the data needed to do this is a closed, coclosed 1-form $\alpha$ on $N$ that is nonzero at every point of $N$. The existence of a suitable 1-form $\alpha$ depends on the metric on $N$, which is the restriction of the metric $g$ on $Y$. But $g$ comes from the solution of the Calabi Conjecture, so we know little about it. This may make the method difficult to apply in practice.

The second method is studied by Alexei Kovalev [12], and is based on an idea due to Simon Donaldson.

**Method 2.** Let $X$ be a projective complex 3-fold with canonical bundle $K_X$, and $s$ a holomorphic section of $K_X^{-1}$ which vanishes to order 1 on a smooth divisor $D$ in $X$. Then $D$ has trivial canonical bundle, so $D$ is $T^3$ or $K3$. Suppose $D$ is a $K3$ surface. Define $Y = X \setminus D$, and suppose $Y$ is simply-connected.

Then $Y$ is a noncompact complex 3-fold with $K_Y$ trivial, and one infinite end modelled on $D \times S^1 \times [0, \infty)$. Using a version of the proof of the Calabi Conjecture for noncompact manifolds one constructs a complete Calabi–Yau metric $h$ on $Y$, which is asymptotic to the product on $D \times S^1 \times [0, \infty)$ of a Calabi–Yau metric on $D$, and Euclidean metrics on $S^1$ and $[0, \infty)$. We call such metrics *Asymptotically Cylindrical*.

Suppose we have such a metric on $Y$. Define a torsion-free $G_2$-structure $(\varphi, g)$ on $S^1 \times Y$ as in Proposition 1.8. Then $S^1 \times Y$ is a noncompact $G_2$-manifold with one end modelled on $D \times T^2 \times [0, \infty)$, whose metric is asymptotic to the product on $D \times T^2 \times [0, \infty)$ of a Calabi–Yau metric on $D$, and Euclidean metrics on $T^2$ and $[0, \infty)$.
Donaldson and Kovalev’s idea is to take two such products $S^1 \times Y_1$ and $S^1 \times Y_2$ whose infinite ends are isomorphic in a suitable way, and glue them together to get a compact 7-manifold $M$ with holonomy $G_2$. The gluing process swaps round the $S^1$ factors. That is, the $S^1$ factor in $S^1 \times Y_1$ is identified with the asymptotic $S^1$ factor in $Y_2 \sim D_2 \times S^1 \times [0, \infty)$, and vice versa.

3 Compact Spin(7)-manifolds from Calabi–Yau 4-orbifolds

In a very similar way to the $G_2$ case, one can construct examples of compact 8-manifolds with holonomy Spin(7) by resolving the singularities of torus orbifolds $T^8/\Gamma$. This is done in [6] and [6, §13–§14]. In [6, §14], examples are constructed which realize 181 different sets of Betti numbers. Two compact 8-manifolds with holonomy Spin(7) and the same Betti numbers may be distinguished by the cup products on their cohomologies (examples of this are given in [6, §3.4]), so they probably represent rather more than 181 topologically distinct 8-manifolds.

The main differences with the $G_2$ case are, firstly, that the technical details of the analysis are different and harder, and secondly, that the singularities that arise are typically more complicated and more tricky to resolve. One reason for this is that in the $G_2$ case the singular set is made up of 1 and 3-dimensional pieces in a 7-dimensional space, so one can often arrange for the pieces to avoid each other, and resolve them independently.

But in the Spin(7) case the singular set is typically made up of 4-dimensional pieces in an 8-dimensional space, so they nearly always intersect. There are also topological constraints arising from the $\hat{A}$-genus, which do not apply in the $G_2$ case. The moral appears to be that when you increase the dimension, things become more difficult.

Anyway, we will not discuss this further, as the principles are very similar to the $G_2$ case above. Instead, we will discuss an entirely different construction of compact 8-manifolds with holonomy Spin(7) developed by the author in [8] and [9, §15], a little like Method 1 of §2.5. In this we start from a Calabi–Yau 4-orbifold rather than from $T^8$. The construction can be divided into five steps.

Step 1. Find a compact, complex 4-orbifold $(Y, J)$ satisfying the conditions:

(a) $Y$ has only finitely many singular points $p_1, \ldots, p_k$, for $k \geq 1$.
(b) $Y$ is modelled on $\mathbb{C}^4/\langle i \rangle$ near each $p_j$, where $i$ acts on $\mathbb{C}^4$ by complex multiplication.
(c) There exists an antiholomorphic involution $\sigma : Y \to Y$ whose fixed point set is $\{p_1, \ldots, p_k\}$.
(d) $Y \setminus \{p_1, \ldots, p_k\}$ is simply-connected, and $h^{2,0}(Y) = 0$.

Step 2. Choose a $\sigma$-invariant Kähler class on $Y$. Then by Theorem [4.7] there exists a unique $\sigma$-invariant Ricci-flat Kähler metric $g$ in this Kähler class. Let $\omega$ be the Kähler form of $g$. Let $\theta$ be a holomorphic volume
form for \((Y,J, g)\). By multiplying \(\theta\) by \(e^{i\phi}\) if necessary, we can arrange that \(\sigma^*(\theta) = \overline{\theta}\).

Define \(\Omega = \frac{1}{2} \omega \wedge \omega + \text{Re}\theta\). Then \((\Omega, g)\) is a torsion-free Spin(7)-structure on \(Y\), by Proposition 1.10. Also, \((\Omega, g)\) is \(\sigma\)-invariant, as \(\sigma^*(\omega) = -\omega\) and \(\sigma^*(\theta) = \overline{\theta}\). Define \(Z = Y/\langle \sigma \rangle\). Then \((\Omega, g)\) pushes down to a torsion-free Spin(7)-structure \((\Omega, g)\) on \(Z\).

Step 3. \(Z\) is modelled on \(\mathbb{R}^8/G\) near each \(p_j\), where \(G\) is a certain finite subgroup of Spin(7) with \(|G| = 8\). We can write down two explicit, topologically distinct ALE Spin(7)-manifolds \(X_1, X_2\) asymptotic to \(\mathbb{R}^8/G\). Each carries a 1-parameter family of homothetic ALE metrics \(h_t\) for \(t > 0\) with \(\text{Hol}(h_t) = \mathbb{Z}_2 \ltimes \text{SU}(4) \subset \text{Spin}(7)\).

For \(j = 1, \ldots, k\) we choose \(i_j = 1\) or \(2\), and resolve the singularities of \(Z\) by gluing in \(X_{i_j}\) at the singular point \(p_j\) for \(j = 1, \ldots, k\), to get a compact, nonsingular 8-manifold \(M\), with projection \(\pi: M \to Z\).

Step 4. On \(M\), we explicitly write down a 1-parameter family of Spin(7)-structures \((\Omega_t, g_t)\) depending on \(t \in (0, \epsilon)\). They are not torsion-free, but have small torsion when \(t\) is small. As \(t \to 0\), the Spin(7)-structure \((\Omega_t, g_t)\) converges to the singular Spin(7)-structure \(\pi^*(\Omega_0, g_0)\).

Step 5. We prove using analysis that for sufficiently small \(t\), the Spin(7)-structure \((\Omega_t, g_t)\) on \(M\), with small torsion, can be deformed to a Spin(7)-structure \((\Omega_t, \tilde{g}_t)\), with zero torsion. It turns out that if \(i_j = 1\) for \(j = 1, \ldots, k\) we have \(\pi_1(M) \cong \mathbb{Z}_2\) and \(\text{Hol}(\tilde{g}_t) = \mathbb{Z}_2 \ltimes \text{SU}(4)\), and for the other \(2^k - 1\) choices of \(i_1, \ldots, i_k\) we have \(\pi_1(M) = \{1\}\) and \(\text{Hol}(\tilde{g}_t) = \text{Spin}(7)\). So \(\tilde{g}_t\) is a metric with holonomy Spin(7) on the compact 8-manifold \(M\) for \((i_1, \ldots, i_k) \neq (1, \ldots, 1)\).

Once we have completed Step 1, Step 2 is immediate. Steps 4 and 5 are analogous to Steps 3 and 4 of §2, and can be done using the techniques and analytic results developed by the author for the first \(T^8/\Gamma\) construction of compact Spin(7)-manifolds, [3], [4], §13. So the really new material is in Steps 1 and 3, and we will discuss only these.

### 3.1 Step 1: An example

We do Step 1 using complex algebraic geometry. The problem is that conditions \((a)-(d)\) above are very restrictive, so it is not that easy to find any \(Y\) satisfying all four conditions. All the examples \(Y\) the author has found are constructed using weighted projective spaces, an important class of complex orbifolds.

**Definition 3.1** Let \(m \geq 1\) be an integer, and \(a_0, a_1, \ldots, a_m\) positive integers with highest common factor 1. Let \(\mathbb{C}^{m+1}\) have complex coordinates on
(z_0, \ldots, z_m), and define an action of the complex Lie group C^* on C^{m+1} by

\[(z_0, \ldots, z_m) \mapsto (u^{a_0}z_0, \ldots, u^{a_m}z_m), \quad \text{for } u \in C^*.\]

The weighted projective space CP^n_{a_0, \ldots, a_m} is \((C^{m+1} \setminus \{0\})/C^*.\) The C^*-orbit of \((z_0, \ldots, z_m)\) is written \([z_0, \ldots, z_m].\)

Here is the simplest example the author knows.

**Example 3.2** Let \(Y\) be the hypersurface of degree 12 in \(CP^5_{1,1,1,1,1,4,4,4}\) given by

\[Y = \{[z_0, \ldots, z_5] \in CP^5_{1,1,1,1,4,4,4} : z_0^{12} + z_1^{12} + z_2^{12} + z_3^{12} + z_4^{12} + z_5^{12} = 0\}.\]

Calculation shows that \(Y\) has trivial canonical bundle and three singular points \(p_1 = [0,0,0,0,1,-1], \) \(p_2 = [0,0,0,0,1,e^{\pi i/3}], \) and \(p_3 = [0,0,0,0,1,e^{-\pi i/3}],\) all modelled on \(C^4/\langle i \rangle.\)

Now define a map \(\sigma : Y \to Y\) by

\[\sigma : [z_0, \ldots, z_5] \mapsto \left[\bar{z}_1, -\bar{z}_0, \bar{z}_3, -\bar{z}_2, \bar{z}_5, \bar{z}_4\right].\]

Note that \(\sigma^2 = 1,\) though this is not immediately obvious, because of the geometry of \(CP^5_{1,1,1,1,4,4,4}.\) It can be shown that conditions (a)-(d) of Step 1 above hold for \(Y\) and \(\sigma.\)

More suitable 4-folds \(Y\) may be found by taking hypersurfaces or complete intersections in other weighted projective spaces, possibly also dividing by a finite group, and then doing a crepant resolution to get rid of any singularities that we don’t want. Examples are given in [8, 15, §15].

### 3.2 Step 3: Resolving \(R^8/G\)

Define \(\alpha, \beta : R^8 \to R^8\) by

\[\alpha : (x_1, \ldots, x_8) \mapsto (-x_2, x_1, -x_4, x_3, -x_6, x_5, -x_8, x_7),\]

\[\beta : (x_1, \ldots, x_8) \mapsto (x_3, -x_4, -x_1, x_2, x_7, -x_8, -x_5, x_6).\]

Then \(\alpha, \beta\) preserve \(\Omega_0\) given in [3], so they lie in Spin(7). Also \(\alpha^4 = \beta^4 = 1,\) \(\alpha^2 = \beta^2\) and \(\alpha \beta = \beta \alpha^3.\) Let \(G = \langle \alpha, \beta \rangle.\) Then \(G\) is a finite nonabelian subgroup of Spin(7) of order 8, which acts freely on \(R^8 \setminus \{0\}.\) One can show that if \(Z\) is the compact Spin(7)-orbifold constructed in Step 2 above, then \(T_{p_j}Z\) is isomorphic to \(R^8/G\) for \(j = 1, \ldots, k,\) with an isomorphism identifying the Spin(7)-structures \((\Omega, g)\) on \(Z\) and \((\Omega_0, g_0)\) on \(R^8/G,\) such that \(\beta\) corresponds to the \(\sigma\)-action on \(Y.\)

In the next two examples we shall construct two different ALE Spin(7)-manifolds \((X_1, \Omega_1, g_1)\) and \((X_2, \Omega_2, g_2)\) asymptotic to \(R^8/G.\)

**Example 3.3** Define complex coordinates \((z_1, \ldots, z_4)\) on \(R^8\) by

\[(z_1, z_2, z_3, z_4) = (x_1 + ix_2, x_3 + ix_4, x_5 + ix_6, x_7 + ix_8),\]

\[14\]
Then $g_0 = |dz_1|^2 + \cdots + |dz_4|^2$, and $\Omega_0 = \frac{1}{2}\omega_0 \wedge \omega_0 + \text{Re}(\theta_0)$, where $\omega_0$ and $\theta_0$ are the usual K"ahler form and complex volume form on $\mathbb{C}^4$. In these coordinates, $\alpha$ and $\beta$ are given by

$$
\alpha : (z_1, \ldots, z_4) \mapsto (iz_1, iz_2, iz_3, iz_4),
\beta : (z_1, \ldots, z_4) \mapsto (z_2, -z_1, z_4, -z_3).
$$

Now $\mathbb{C}^4/\langle \alpha \rangle$ is a complex singularity, as $\alpha \in \text{SU}(4)$. Let $(Y_1, \pi_1)$ be the blow-up of $\mathbb{C}^4/\langle \alpha \rangle$ at 0. Then $Y_1$ is the unique crepant resolution of $\mathbb{C}^4/\langle \alpha \rangle$. The action of $\beta$ on $\mathbb{C}^4/\langle \alpha \rangle$ lifts to a free antiholomorphic map $\beta : Y_1 \to Y_1$ with $\beta^2 = 1$. Define $X_1 = Y_1/\langle \beta \rangle$. Then $X_1$ is a nonsingular 8-manifold, and the projection $\pi_1 : Y_1 \to \mathbb{C}^4/\langle \alpha \rangle$ pushes down to $\pi_1 : X_1 \to \mathbb{R}^8/G$.

There exist ALE Calabi–Yau metrics $g_1$ on $Y_1$, which were written down explicitly by Calabi [2, p. 285], and are invariant under the action of $\beta$ on $Y_1$. Let $\omega_1$ be the K"ahler form of $g_1$, and $\theta_1 = \pi_1^*(\theta_0)$ the holomorphic volume form on $Y_1$. Define $\Omega_1 = \frac{1}{2}\omega_1 \wedge \omega_1 + \text{Re}(\theta_1)$. Then $(\Omega_1, g_1)$ is a torsion-free Spin(7)-structure on $Y_1$, as in Proposition 1.10.

As $\beta^*(\omega_1) = -\omega_1$ and $\beta^*(\theta_1) = \theta_1$, we see that $\beta$ preserves $(\Omega_1, g_1)$. Thus $(\Omega_1, g_1)$ pushes down to a torsion-free Spin(7)-structure $(\Omega_1, g_1)$ on $X_1$. Then $(X_1, \Omega_1, g_1)$ is an ALE Spin(7)-manifold asymptotic to $\mathbb{R}^8/G$.

**Example 3.4** Define new complex coordinates $(w_1, \ldots, w_4)$ on $\mathbb{R}^8$ by

$$(w_1, w_2, w_3, w_4) = (-x_1 + ix_3, x_2 + ix_4, -x_5 + ix_7, x_6 + ix_8).$$

Again we find that $g_0 = |dw_1|^2 + \cdots + |dw_4|^2$ and $\Omega_0 = \frac{1}{2}\omega_0 \wedge \omega_0 + \text{Re}(\theta_0)$. In these coordinates, $\alpha$ and $\beta$ are given by

$$
\alpha : (w_1, \ldots, w_4) \mapsto (\bar{w}_2, -\bar{w}_1, \bar{w}_4, -\bar{w}_3),
\beta : (w_1, \ldots, w_4) \mapsto (iw_1, iw_2, iw_3, iw_4).
$$

Observe that (15) and (16) are the same, except that the rôles of $\alpha, \beta$ are reversed. Therefore we can use the ideas of Example 3.3 again.

Let $Y_2$ be the crepant resolution of $\mathbb{C}^4/\langle \beta \rangle$. The action of $\alpha$ on $\mathbb{C}^4/\langle \beta \rangle$ lifts to a free antiholomorphic involution of $Y_2$. Let $X_2 = Y_2/\langle \alpha \rangle$. Then $X_2$ is nonsingular, and carries a torsion-free Spin(7)-structure $(\Omega_2, g_2)$, making $(X_2, \Omega_2, g_2)$ into an ALE Spin(7)-manifold asymptotic to $\mathbb{R}^8/G$.

We can now explain the remarks on holonomy groups at the end of Step 5. The holonomy groups $\text{Hol}(g_1)$ of the metrics $g_1, g_2$ in Examples 3.3 and 3.4 are both isomorphic to $\mathbb{Z}_2 \ltimes \text{SU}(4)$, a subgroup of Spin(7). However, they are two different inclusions of $\mathbb{Z}_2 \ltimes \text{SU}(4)$ in Spin(7), as in the first case the complex structure is $\alpha$ and in the second $\beta$.

The Spin(7)-structure $(\Omega, g)$ on $Z$ also has holonomy $\text{Hol}(g) = \mathbb{Z}_2 \ltimes \text{SU}(4)$. Under the natural identifications we have $\text{Hol}(g_1) = \text{Hol}(g)$ but $\text{Hol}(g_2) \neq \text{Hol}(g)$ as subgroups of Spin(7). Therefore, if we choose $i_j = 1$ for all $j = 1, \ldots, k$, then $Z$ and $X_1$ all have the same holonomy group $\mathbb{Z}_2 \ltimes \text{SU}(4)$, so they combine to give metrics $\tilde{g}_l$ on $M$ with $\text{Hol}(\tilde{g}_l) = \mathbb{Z}_2 \ltimes \text{SU}(4)$.
However, if $i_j = 2$ for some $j$ then the holonomy of $g$ on $Z$ and $g_{i_j}$ on $X_{i_j}$ are different $\mathbb{Z}_2 \ltimes \text{SU}(4)$ subgroups of $\text{Spin}(7)$, which together generate the whole group $\text{Spin}(7)$. Thus they combine to give metrics $\tilde{g}_t$ on $M$ with $\text{Hol}(\tilde{g}_t) = \text{Spin}(7)$.

3.3 Conclusions

The author was able in [8] and [9, Ch. 15] to construct compact 8-manifolds with holonomy $\text{Spin}(7)$ realizing 14 distinct sets of Betti numbers, which are given in Table 1. Probably there are many other examples which can be produced by similar methods.

| Table 1: Betti numbers $(b^2, b^3, b^4)$ of compact Spin(7)-manifolds |
|-------------------|-------------------|-------------------|-------------------|-------------------|
| (4, 33, 200)      | (3, 33, 202)      | (2, 33, 204)      | (1, 33, 206)      | (0, 33, 208)      |
| (1, 0, 908)       | (0, 0, 910)       | (1, 0, 1292)      | (0, 0, 1294)      | (1, 0, 2444)      |
| (0, 0, 2446)      | (0, 6, 3730)      | (0, 0, 4750)      | (0, 0, 11662)     |

Comparing these Betti numbers with those of the compact 8-manifolds constructed in [9, Ch. 14] by resolving torus orbifolds $T^8/\Gamma$, we see that these examples the middle Betti number $b^4$ is much bigger, as much as 11662 in one case.

Given that the two constructions of compact 8-manifolds with holonomy Spin(7) that we know appear to produce sets of 8-manifolds with rather different 'geography', it is tempting to speculate that the set of all compact 8-manifolds with holonomy Spin(7) may be rather large, and that those constructed so far are a small sample with atypical behaviour.

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