EMERGENT DYNAMICS OF AN ORIENTATION FLOCKING MODEL FOR MULTI-AGENT SYSTEM

SEUNG-YEAL HA
Department of Mathematical Sciences and Research Institute of Mathematics
Seoul National University, Seoul 08826, Republic of Korea and
Korea Institute for Advanced Study, Hoegiro 85, Seoul, 02455, Republic of Korea

DOHYUN KIM
National Institute for Mathematical Sciences,
70, Yuseong-daero 1689 beon-gil, Yuseong-gu, Daejeon 34047, Republic of Korea

JAeseung Lee
The Research Institute of Basic Sciences,
Seoul National University, Seoul 08826, Republic of Korea

Se Eun Noh*
Department of Mathematics, Myongji University,
116 Myongjiro, Yong-In 17058, Republic of Korea

(Communicated by José A. Carrillo)

Abstract. We study the orientation flocking for the deterministic counterpart of a stochastic agent-based model introduced by Degond, Frouvelle and Merino-Aceituno in 2017, where the orientation is defined as a SO(3) matrix. Their proposed model can be reduced to the other collective dynamics models such as the Lohe matrix model and the Viscek-type model as special cases. In this work, we study the emergent dynamics of the orientation flocking model in two frameworks. First, we present sufficient conditions leading to the orientation flocking when the natural frequency matrices are identical. To be precise, we prove that all orientation matrices asymptotically converge to the common one, and the spatial position diameter remains uniformly bounded. Second, we show the emergence of orientation-locked states for non-identical natural frequency matrices, that is, the difference of any two orientation matrices tends to the definite constant matrix. On the other hand, we establish the finite-in-time stability with respect to initial data of the proposed orientation flocking model. We also present the numerical results consistent with our rigorous analysis. Our work remains valid even for dimensions greater than three.

2010 Mathematics Subject Classification. Primary: 82C10, 82C22, 35B37, 34C15.
Key words and phrases. Body attitude coordination, emergence, Lohe model, orientation flocking, stability.

The work of S.-Y. Ha was supported by the National Research Foundation of Korea (NRF-2017R1A5A1015626), the work of D. Kim was supported from the National Institute for Mathematical Sciences (NIMS) grant funded by the Korea government (MIST) (No.B19610000), and the work of S. E. Noh was supported by the National Research Foundation of Korea (NRF-2017R1C1B5018312).

* Corresponding author: Se Eun Noh.
1. Introduction. Collective dynamics of self-propelled agents in biological complex systems can be easily found in our nature, for instance, swarming of fish, aggregation of bacteria, flashing of fireflies, etc. Indeed, there are many agent-based models describing the flocking and synchronization phenomena, such as the Kuramoto dynamics [30, 31], the Cucker-Smale model [10], the Vicsek model [38], the swarm sphere model [33], etc, and these models have been extensively studied in literature [1, 22, 34]. In this paper, we are mainly concerned with an agent-based model for alignment of body attitudes which was proposed in [11] where the states of agents are described by the positions of their center of mass and body attitudes. For simplicity of modeling, other detailed internal structures are ignored at the level of agents are described by the positions of their center of mass and body attitudes. The skew-symmetric matrix $H_i$ denotes a generalized frequency-like matrix, $\kappa$ measures the coupling strength between the agents, and the communication function $\psi = \psi(r)$ describes the capacity of channel between agents determined by the relative spatial distances. In addition, we also assume that there exist two positive constants $\psi_m$ and $\psi_M$ such that

$$0 < \psi_m \leq \psi(r) \leq \psi_M, \quad r \geq 0. \quad (2)$$

Note that the condition (2) might appear to be a strong restriction from the modeling point of view, since any two particles can strongly interact even if their relative distance is large. In fact, one can consider the channel capacity in the Cucker-Smale model whose form is $\psi(s) = (1 + s^2)^{-\beta}$ for a long-ranged interaction $\beta \in (0, 1/2)$ and a short-ranged interaction $\beta \in [1/2, \infty)$. In [11], the authors proposed two types of body attitude models on $SO(3)$. For the first one, the dynamics of $A_i$ is given by the following gradient flow of the weighted average $\tilde{M}_i$:

$$\begin{aligned}
\dot{x}_i &= v_0 A_i e_1, \\
\dot{A}_i &= \nu \nabla_A (\tilde{M}_i \cdot A) \bigg|_{A = A_i} = \nu \mathbb{P}_{T_{A_i}} (\tilde{M}_i), \\
M_i &= \frac{1}{N} \sum_{k=1}^{N} K(|x_i - x_k|) A_k,
\end{aligned} \quad (3)$$

where $(\nu, K)$ have a same role of $(2\kappa, \psi)$ in (1), and $\mathbb{P}_{T_{A_i}}$ denotes the projection operator on the tangent space $T_{A_i}$ so that $A_i$ always belongs to $SO(d)$ for all time for $A \in SO(d)$ and $M \in \mathbb{R}^{d \times d}$,

$$\mathbb{P}_{T_{A}} (M) := \frac{1}{2} (M - AM^t A) \quad (4)$$
The second one is also formulated by the gradient flow, however using the polar decomposition of $M_i$:

$$
\begin{align*}
\dot{x}_i &= v_0 A_i e_1, \\
\dot{A}_i &= \nu P_{T_{A_i}} (PD(M_i)), \quad M_i = \frac{1}{N} \sum_{k=1}^{N} K(|x_i - x_k|) A_k,
\end{align*}
$$

(5)

where $PD(M)$ denotes the orthogonal matrix which comes from the polar decomposition of the matrix $M$. Compared to the model (3), our flocking model (1) has an extra intrinsic component $H_i$ which plays the role of natural frequency matrix in the analogy with the Kuramoto model. We also refer the reader to [12] for an equivalent description of (5) using quaternions. For our model, we choose (1) instead of (5), since the polar decomposition process contains an implicit process. In order to provide explicit calculation, we only consider (1). See Proposition A.4 in [11] in which they showed that the trajectory can follow a geodesic in some special case. We also refer the reader to [35, 37] for attitude synchronization models which contain the dynamics of $SO(d)$ and [5] for a system of interacting orientable agents.

On the other hand, the authors in [24] introduced a new class of the matrix Lie groups, namely Lohe group, which can be generated by the continuous matrix-valued ODE system:

$$
\dot{X}_i X_1^{-1} = H_i + \frac{\kappa}{2N} \sum_{k=1}^{N} \left( X_k X_1^{-1} - X_i X_1^{-1} \right), \quad t > 0, \quad i = 1, \cdots, N.
$$

(6)

For a general matrix Lie group $G$, $H_i$ belongs to the corresponding Lie algebra $g$, and we impose the following structure condition on $G$:

$$
X - X^{-1} \in g, \quad \text{for all } X \in G,
$$

so that a solution of (6) stays on the given Lie group $G$ for all time. In particular, if we choose $G = SO(d)$, then (6) becomes (1) with $\psi(r) \equiv 1$ and $A_i = X_i$. See also [13] for the generalized model of (6).

In this paper, we give a link between two aforementioned works [11] and [24]. The authors of [11] proposed the ODE systems (3) and (5) with a noise effect, formally derived their corresponding kinetic and hydrodynamic models, and studied the self-organized hydrodynamics for body attitude coordination. Hence, the analysis of the ODE model is left as a blank and we thus focus on the emergent dynamics of the agent-based model (1). Indeed, since system (1) is a matrix-valued ODE system, the techniques developed in [24, 29] can be adopted throughout the paper.

We classify our system (1) into two types:

(i) (Identical case): $H_i \equiv H$ for all $i = 1, \cdots, N$.

(ii) (Non-identical case): $H_i \neq H_j$ for some $i \neq j \in \{1, \cdots, N\}$.

For the Lohe matrix model [32], $H_i$ plays the role of constant Hamiltonian associated with $i$-th quantum subsystem. In what follows, we will use the following notation:

$$
X = (x_1, \cdots, x_N), \quad A = (A_1, \cdots, A_N) \quad \text{and} \quad H = (H_1, \cdots, H_N).
$$

The main results of this paper are two-fold. First, we describe some of the emergent dynamics of system (1) with both identical and non-identical cases. For the
identical case, we present a sufficient framework leading to the orientation flocking (see Definition 2.1): there exist positive numbers $x_{\infty} > 0$ and $C > 0$ such that
$$D(X(t)) < x_{\infty} \quad \text{and} \quad D(A(t)) \leq O(1)e^{-Ct}, \quad t > 0,$$
where $D(X(t))$ and $D(A(t))$ are maximal diameters:
$$D(X(t)) := \max_{1 \leq i,j \leq N} |x_i(t) - x_j(t)| \quad \text{and} \quad D(A(t)) := \max_{1 \leq i,j \leq N} \|A_i(t) - A_j(t)\|.$$
Here, $\|\cdot\|$ and $\|\cdot\|_{\infty}$ are $\ell_2$-norm and Frobenius norm in $\mathbb{R}^d$ and $\text{SO}(d)$, respectively (see the end of this section). For the non-identical case, we show that the orientation-locked states (see Definition 2.1) emerge asymptotically in a large coupling regime. In this case, we cannot guarantee that the diameter of position variable is uniformly bounded. Second, we present the finite-in-time stability estimate which is valid on any finite time interval. To be more specific, for a given $T \in (0, \infty)$, we can find a positive constant $\bar{G} = \bar{G}(T)$ which does not depend on the number of particles $N$ and initial data such that for any two solutions $(x_i, A_i)$ and $(\tilde{x}_i, \tilde{A}_i)$, the following estimate holds for $t \in [0, T)$:
$$\max_{1 \leq i \leq N} \left( |x_i(t) - \tilde{x}_i(t)| + \|A_i(t) - \tilde{A}_i(t)\| \right) < \bar{G}(T) \max_{1 \leq i \leq N} \left( |x_i^0 - \tilde{x}_i^0| + \|A_i^0 - \tilde{A}_i^0\| \right).$$

As a direct application of this stability estimate, we can derive the corresponding kinetic model (39) of (1) and obtain a global measure-valued solution to (39) as in the Cucker-Smale model [17] using a particle-in-cell method.

The rest of paper is organized as follows. In Section 2, we study basic properties of our system, such as an invariance property and present connections between our model and other collective models, e.g., the generalized Lohe matrix model and the Visccek-type flocking model. In Section 3, we present orientation flocking estimates with both identical and non-identical cases. In Section 4, we show that our system is stable with respect to the initial data in any finite time interval. In Section 5, we provide several numerical simulations consistent with the theoretical analysis obtained in Section 3. Finally, Section 6 is devoted to a brief summary of our main results and discussion of future works.

**Notation:** Throughout this paper, we use the $\ell_2$-norm and Frobenius norm for $x_i = (x_i^{(1)}, \cdots, x_i^{(d)}) \in \mathbb{R}^d$ and a $d \times d$ matrix $A = (a_{ij})$:

$$|x_i| := \left( \sum_{j=1}^{d} |x_i^{(j)}|^2 \right)^{\frac{1}{2}}, \quad ||A|| := \left( \text{tr}[AA^t] \right)^{\frac{1}{2}} = \left( \sum_{i,j=1}^{d} |a_{ij}|^2 \right)^{\frac{1}{2}}. \quad (7)$$

2. **Preliminaries.** In this section, we briefly review basic properties of system (1) and study relations between our system (1) and other existing flocking models such as the Lohe matrix model and the the Visccek-type flocking model. First, we introduce several concepts related to orientation flocking as follows.

**Definition 2.1.** Let $(X, A)$ be a global solution to (1).

1. (Group formation): System (1) achieves the group formation, if the position fluctuation is uniformly bounded in time:

$$\sup_{0 \leq t < \infty} \max_{1 \leq i,j \leq N} |x_i - x_j| < \infty.$$
2. (Orientation alignment): System (1) achieves the (asymptotic) orientation alignment, if the orientation fluctuation converges to zero as time goes to infinity:
\[
\lim_{t \to \infty} \max_{1 \leq i, j \leq N} \| A_i - A_j \| = 0.
\]

3. (Orientation locking): System (1) achieves the (asymptotic) orientation locking, if there exists a constant matrix \( F_{ij}^\infty \) for each \( i, j \in \{1, \cdots, N\} \) such that
\[
\lim_{t \to \infty} (A_i(t)A_j(t)^{-1}) = F_{ij}^\infty.
\]

4. (Orientation flocking): We say that system (1) exhibits orientation flocking if and only if (1) and (2) hold.

2.1. Basic properties of the OFM. In this subsection, we study the basic properties of (1). First, we begin with the following simple properties of the Frobenius norm without proofs.

**Lemma 2.2.** Let \( A \) and \( B \) be complex valued \( d \times d \)-matrices. Then, the following relations hold:
\[
\| \text{tr}(AB) \| \leq \| A \| \| B \|, \quad \| AB \| \leq \| A \| \| B \|, \quad |Ae| \leq \| A \|,
\]
where \( e \) is any unit vector in \( \mathbb{R}^d \).

**Remark 1.** For \( A_i, A_j \in \text{SO}(d) \), the following relations hold:
\[
\| A_i - A_j \|_2 = \text{tr}((A_i - A_j)(A_i^t - A_j^t)) = \text{tr}((A_i - A_j)(A_i^{-1} - A_j^{-1}))
\]
\[
= \text{tr}(2I_d - A_iA_i^{-1} - A_jA_j^{-1}) = 2\text{tr}(I_d - A_iA_i^{-1}).
\]

Below, we recall the Grönwall-type lemma in [24]. For the convenience of the reader, we provide its proof.

**Lemma 2.3.** Suppose that \( X = X(t) \) and \( Y = Y(t) \) are \( d \times d \) matrix-valued functions satisfying
\[
\frac{dX}{dt} = -\gamma X + Y, \quad t > 0,
\]
where \( \gamma = \gamma(t) \) is a time-dependent function. Then, \( \| X \| \) satisfies
\[
\frac{d}{dt}\| X \| \leq -\gamma(t)\| X \| + \| Y \|, \quad \text{if} \quad \| X \| > 0.
\]

**Proof.** We use Lemma 2.2 to see
\[
\frac{d}{dt}\| X \|^2 = \frac{d}{dt}\text{tr}(XX^t) = 2\text{tr}(X\dot{X}^t) = 2\text{tr}(-\gamma XX^t + YY^t)
\]
\[
\leq -2\gamma\| X \|^2 + 2\| X \|\| Y \|.
\]
Hence, if \( \| X \| > 0 \), then one has
\[
\frac{d}{dt}\| X \| \leq -\gamma\| X \| + \| Y \|, \quad t > 0.
\]

Next, we show the positive invariance of (1). More precisely, we prove that if \( A_0 \in \text{SO}(d) \), then \( A_t \in \text{SO}(d) \) for all \( t \geq 0 \).

**Proposition 1.** Let \( (X, A) \) be a global solution to (1). Then, \( \text{SO}(d) \) is positively invariant under the flow (1).
Proof. First, we recall the definition of $G = \text{SO}(d)$ and its corresponding Lie algebra $\mathfrak{g} = \text{so}(d)$:

$$
G = \{ X \in \text{M}_d(\mathbb{R}) : XX^t = Id \}, \quad \mathfrak{g} = \{ A \in \text{M}_d(\mathbb{R}) : A^t = -A \}.
$$

If the right-hand side of $(1)_2$ belongs to $\mathfrak{g}$, then we see that

$\dot{A}_i A_i^{-1} \in \mathfrak{g}$ or $\dot{A}_i \in \mathfrak{g} A_i = T_{A_i} G$.

On the other hand, since $H_i$ is a skew-symmetric matrix and the following relation holds:

$$(A_k A_i^{-1} - A_i A_k^{-1}) + (A_k A_i^{-1} - A_i A_k^{-1})^t = A_k A_i^{-1} - A_i A_k^{-1} + A_i A_k^{-1} - A_k A_i^{-1} = O_d,$$

we see that the right-hand side of $(1)_2$ indeed belongs to $\mathfrak{g}$. Hence, we conclude that $A_i \in G$ for all $t \geq 0$.

Remark 2. (i) In [11], the authors formulate model (1) in the following way:

$$
\frac{d}{dt} A_i = \kappa \nabla A (M_i \cdot A)|_{A=A_i} = \kappa \mathbb{P}_{T_{A_i} G} M_i, \quad M_i := \frac{1}{N} \sum_{k=1}^{N} \psi(|x_i - x_k|) A_k,
$$

where $\mathbb{P}_{T_{A_i} G}$ defined in (4) denotes the projection operator onto the tangent space $T_{A_i} G$. Hence, the solution $\mathcal{A}$ to (1) lies on $\text{SO}(d)^N$ for all time. Moreover in [25], they showed that (6) on the unitary group or the special orthogonal group also can be represented as a gradient flow.

(ii) For the global well-posedness of system (1), we just note that $\text{SO}(d)$ is a compact manifold. Thus, global well-posedness directly follows from Remark 2.4 in [24].

Below, we study the solution splitting property of system (1) with identical $H_i$’s:

$$
H_i \equiv H, \quad i = 1, \ldots, N. \tag{8}
$$

In this case, we consider two subsystems on $\text{SO}(d)$ and $\mathbb{R}^d \times \text{SO}(d)$, respectively:

$$
\begin{cases}
\dot{B}_i B_i^{-1} = H, \quad \text{or equivalently,} \quad \dot{B}_i = H B_i, \\
B_i(0) = B_i^0,
\end{cases} \tag{9}
$$

and

$$
\begin{cases}
\dot{C}_i C_i^{-1} = \frac{\kappa}{N} \sum_{k=1}^{N} \psi(|x_i - x_k|) (C_k C_i^{-1} - C_i C_k^{-1}), \\
C_i(0) = C_i^0, 
\end{cases} \tag{10}
$$

Then, we introduce two solution operators $R_H(t)$ and $L(t)$ of two subsystems (9) and (10), respectively. For $B_i \in \text{SO}(d)$, $R_H(t)$ is a solution operator corresponding to (9):

$$
R_H(t) B_i^0 := e^{H t} B_i^0,
$$

and for $C_i \in \text{SO}(d)$, $L(t)$ is a solution operator corresponding to (10):

$$
L(t) C_i^0 := C_i(t).
$$

Then, the solution $\mathcal{A}$ to (1) with (8) can be represented via a composition of two solution operators $R_H(t)$ and $L(t)$ as follows.
Proposition 2. Suppose that the initial data and $H_i$ satisfy
\[ A_i^0 \in \text{SO}(d), \quad H_i \equiv H, \quad \text{for all } i = 1, \cdots, N, \]
and let $(X, A)$ be a global solution to (1). Then, we have
\[ A_i(t) = R_H(t) \circ L(t) A_i^0, \quad t > 0, \quad i = 1, \cdots, N. \] (11)

Proof. Although the proof can be found in Proposition 4.1 in [24], we provide it for the convenience of readers. We denote
\[ Y_i := e^{Ht} A_i, \quad \dot{Y}_i = H e^{Ht} A_i + e^{Ht} \dot{A}_i, \quad Y_i^{-1} = A_i^{-1} e^{-Ht}. \]
Then, we check that $Y_i$ satisfies
\[ \dot{Y}_i Y_i^{-1} = H + \frac{\kappa}{N} \sum_{k=1}^N \psi(|x_i - x_k|)(Y_k Y_i^{-1} - Y_i Y_k^{-1}). \]
Thus, the relation (11) holds.

Remark 3. Note that our solution splitting does not hold for the spatial variable $x_i$. The only candidate is $(y_i, Y_i) := (e^{-Ht} x_i, e^{-Ht} A_i)$, if it does. When we differentiate $y_i$, however, we have
\[ \dot{y}_i = e^{-Ht} \dot{x}_i - H e^{-Ht} x_i = e^{-Ht} v_0 A_i e_1 - H e^{-Ht} x_i = v_0 Y_i e_1 - H y_i. \]
Since the additional term $H y_i$ appears in the relation above, our solution splitting property cannot be extended to the variables $(x_i, A_i)$.

Next, we introduce Lyapunov functionals for the flocking estimate and study their rates of changes over time. First, we set
\[ \|H\|_\infty := \max_{1 \leq i \leq N} \|H_i\|, \quad D(H) := \max_{1 \leq i, j \leq N} \|H_i - H_j\|, \quad D(X) := \max_{1 \leq i, j \leq N} |x_i - x_j|, \]
\[ D(A) := \max_{1 \leq i, j \leq N} \|A_i A_j^{-1} - I\|, \quad D(A, \dot{A}) := \max_{1 \leq i, j \leq N} \|A_i A_j^{-1} - \dot{A}_i \dot{A}_j^{-1}\|, \]
where $\|\cdot\|$ is the Frobenius norm defined in (7) and we use the relation $\|A_i - A_j\| = \|A_i A_j^{-1} - I\|$. Below, we derive differential inequalities for $D(A)$ and $D(A, \dot{A})$.

Lemma 2.4. Let $(X, A)$ and $(\tilde{X}, \tilde{A})$ be any two global solutions to (1)–(2). Then, the following estimates hold:

(i) The diameter functional $D(A)$ satisfies
\[ \frac{d}{dt} D(A) \leq \left( -\kappa(3\psi_m - \psi_M) + 2\|H\|_\infty \right) D(A) + 2\kappa\psi_M D(A)^2 + D(H), \quad t > 0. \]

(ii) The functional $D(A, \dot{A})$ satisfies
\[ \frac{d}{dt} D(A, \dot{A}) \leq \left( -\kappa(3\psi_m - \psi_M) + 2\|H\|_\infty \right) D(A, \dot{A}) + 4\kappa\psi_M D(A) D(A, \dot{A}), \quad t > 0. \]

Proof. Since the proofs are rather long, we present them in Appendix A.

2.2. Previous results. We here provide previous results on the generalized Lohe matrix model (6) on $G = \text{SO}(d)$ which can be regarded as a space homogeneous case of the OFM (1). Thus, it is worthwhile to mention the previous results of (6). For this, we denote the maximal quantities:
\[ D(X(t)) := \max_{1 \leq i, j \leq N} \|X_i(t) - X_j(t)\|, \quad t \geq 0. \]
Below, we state the main result without the proof and it can be found in Theorems 4.2 and 5.1 of [24].
Theorem 2.5. [24] Let $X_i$ be a solution to (6). Then, the following assertions hold.

(i) Suppose that system parameters and initial data satisfy

$$H_i \equiv H, \quad i = 1, \cdots, N, \quad \kappa > 2\|H\|_{\infty}, \quad D(X^0) < 1.$$ 

Then, the following estimate holds:

$$D(X(t)) \leq D(X^0)e^{-(\kappa - 2\|H\|_{\infty})t}, \quad t > 0.$$ 

(ii) Suppose that system parameters and initial data satisfy

$$H_i \neq H_j \quad \text{for} \quad i \neq j \in \{1, \cdots, N\}, \quad \kappa > (6 + 4\sqrt{2})\|H\|_{\infty}, \quad D(X^0) \ll 1.$$ 

Then, we verify

$$\lim_{t \to \infty} X_i(t)X_j^{-1}(t) \text{ exists.}$$

2.3. Relations with other collective models. In this subsection, we briefly discuss the connections between our model and other synchronization models in literature.

For the case where the communication weight $\psi_{ik} = \psi(|x_i - x_k|)$ is space-homogeneous, i.e., $\psi_{ik} \equiv a_{ik}$ is a constant, system (1) reduces to the generalized Lohe matrix model [24] with a network topology $(a_{ik})$:

$$\dot{A}_iA_i^{-1} = H_i + \frac{\kappa}{N} \sum_{k=1}^{N} a_{ik}(A_kA_k^{-1} - A_iA_i^{-1}), \quad t > 0, \quad i = 1, \cdots, N.$$ 

Now, we consider a two-dimensional case $d = 2$. In this case, we can parameterize $H_i$ and $A_i \in \text{SO}(2)$ as

$$H_i \equiv 0, \quad A_i := \begin{pmatrix} \cos \theta_i & -\sin \theta_i \\ \sin \theta_i & \cos \theta_i \end{pmatrix}, \quad i = 1, \cdots, N.$$ 

Then, we substitute the above ansatz into system (1) to recover the Viscenk-type flocking model [19]:

$$\frac{dx_i}{dt} = (\cos \theta_i, \sin \theta_i), \quad t > 0, \quad i = 1, \cdots, N,$$

$$\frac{d\theta_i}{dt} = \frac{\kappa}{N} \sum_{k=1}^{N} \psi(|x_i - x_k|)\sin(\theta_k - \theta_i).$$

For $\psi \equiv 1$, (12) is exactly the same as the Kuramoto model with zero natural frequencies which has been extensively studied in [2, 3, 4, 8, 15, 16, 14, 18, 21, 22, 23, 28, 36]. On the other hand, (12) can be obtained from Cucker-Smale flocking model with unit speed constraint in [7, 26].

Before we end this section, we present the Grönwall-type estimates to be used later.

Lemma 2.6. Let $y = y(t)$ be a nonnegative $C^1$-function satisfying the following Riccati-type differential inequality:

$$\begin{cases} 
\dot{y} \leq -py + qy^2 + r, \quad t > 0, \\
y(0) = y_0,
\end{cases}$$

where $p, q$ and $r$ are positive constants.
(i) Suppose that $r = 0$. Then, one has
\[ y(t) \leq \frac{1}{\left(\frac{1}{y(0)} - \frac{q}{p}\right)} e^{pt} + \frac{q}{p}, \quad t \geq 0. \]

(ii) Suppose that $r > 0$ and $p^2 - 4qr > 0$, and let $y_-$ and $y_+$ be two distinct positive roots of a quadratic equation $qy^2 - py + r = 0$:
\[ y_- := \frac{p - \sqrt{p^2 - 4qr}}{2q}, \quad y_+ := \frac{p + \sqrt{p^2 - 4qr}}{2q}. \]
Furthermore, if we assume that $y(0) < y_+$, then there exists a finite positive entrance time $T_*$ such that
\[ y(t) < y_-, \quad t > T_. \]

Proof. Although the proof can be found in Lemma 3.7 of [6], we provide it for the consistency of the paper.

(i) We set $u = 1/y$ to derive the inequality for $u$ and then directly integrate the resulting relation to obtain the desired estimate.

(ii) We split the proof into two cases.

- **Case A**: suppose that $y(0) \leq y_-$. For $t = T$ such that $y(T) = y_-$, it follows from (13) that
\[ \frac{d}{dt} y(t) \bigg|_{t=T} \leq 0. \]
Hence, $y(t)$ is restricted in the interval $[0, y_-]$ for all time, i.e., $y(t) \leq y_-$ for all $t \geq 0$.

- **Case B**: suppose that $y_- < y(0) < y_+$. Then, it follows from (13) that $y(t)$ starts to decrease strictly. It follows from Proposition 3.1 in [8] that there exists a finite entrance time $T_*$ such that
\[ y(t) < y_-, \quad t \geq T_. \]

Finally, we combine the cases A and B to obtain the desired result.

3. Emergence of orientation flocking. In this section, we present the orientation flocking estimate for (1). In what follows, we consider both identical and non-identical cases. Before we state the main result, we set
\[ \Delta(\psi) := 3\psi_m - \psi_M, \]
where $\psi_m$ and $\psi_M$ are defined in (2) and $\Delta(\psi)$ is assumed to be positive from the condition (14).

3.1. Identical case. In this section, we consider the identical case, i.e., $H_i \equiv H$. Then, we state our first main result on the emergence of orientation flocking of system (1).

**Theorem 3.1.** Suppose that system parameters and initial data satisfy
\[ (i) \quad \Delta(\psi) > 0 \quad \text{and} \quad \|H\|_\infty < \frac{\kappa}{2} \Delta(\psi). \]
\[ (ii) \quad D(A^0) \quad \text{such that} \quad y(t) < y_-, \quad t \geq T_. \]

Finally, we combine the cases A and B to obtain the desired result. \qed

3.1. Idem. In this section, we consider the identical case, i.e., $H_i \equiv H$. Then, we state our first main result on the emergence of orientation flocking of system (1).

**Theorem 3.1.** Suppose that system parameters and initial data satisfy
\[ (i) \quad \Delta(\psi) > 0 \quad \text{and} \quad \|H\|_\infty < \frac{\kappa}{2} \Delta(\psi). \]
\[ (ii) \quad D(A^0) \quad \text{such that} \quad y(t) < y_-, \quad t \geq T_. \]
and let \((X, A)\) be a global solution to (1)–(2). Then, system (1) exhibits orientation flocking: there exists a positive constant \(x_\infty\) such that
\[
D(X(t)) < x_\infty \quad \text{and} \quad \lim_{t \to \infty} D(A(t)) = 0.
\]

**Proof.** (i) (Exponential decay of \(D(A)\)): Since we assume \(H_i \equiv H\), we have \(D(H) \equiv 0\). Then, we use Lemma 2.4 (i) and the fact that \(D(H) \equiv 0\) to obtain
\[
\frac{d}{dt} D(A) \leq \left( -\kappa \Delta(\psi) + 2 \|H\|_\infty \right) D(A) + 2\kappa \psi M D(A)^2 + \frac{2\kappa \psi M}{\alpha} D(A)^2.
\]
Note that the condition (14) yields \(\alpha > 0\). Thus, we apply Lemma 2.6(i) to find the desired exponential decay of \(D(A)\):
\[
D(A(t)) \leq \left( \frac{1}{D(A^0)} - \frac{2\kappa \psi M}{\alpha} \right) e^{\alpha t} + \frac{2\kappa \psi M}{\alpha} D(A^0)^{-1}, \quad t > 0.
\]
(ii) (Uniform boundedness of \(D(X)\)): For the group formation condition, we roughly estimate (16) as
\[
D(A(t)) \leq \left( \frac{1}{D(A^0)} - \frac{2\kappa \psi M}{\alpha} \right) e^{-\alpha t} =: \Lambda_0 e^{-\alpha t}.
\]
On the other hand, we have
\[
\frac{d}{dt} D(X) \leq v_0 D(A), \quad t > 0.
\]
Now, we use (17) and (18) to obtain
\[
D(X(t)) \leq D(X^0) + v_0 \int_0^t D(A(s)) ds \leq D(X^0) + v_0 \Lambda_0 / \alpha.
\]
Thus, the group formation holds.

3.2. Non-identical case. In this subsection, we study the non-identical case, i.e., \(D(H) > 0\). Although we cannot guarantee the orientation alignment, we here show that orientation-locked states can emerge from some well-prepared initial data. Before we begin our discussion, we introduce the handy notations:
\[
\Lambda(\kappa, \psi, H) := \Delta(\psi)^2 \kappa^2 - 2\kappa(2\|H\|_\infty \Delta(\psi) + 4\psi M D(H)) + 4\|H\|_\infty^2,
\]
\[
\beta_- := \frac{\kappa \Delta(\psi) - 2\|H\|_\infty}{4\kappa \psi M}, \quad \beta_+ := \frac{\kappa \Delta(\psi) - 2\|H\|_\infty + \sqrt{\Lambda(\kappa, \psi, H)}}{4\kappa \psi M}.
\]
Note that \(\Lambda(\kappa, \psi, H)\) can be positive for large \(\kappa > 0\) and this can be guaranteed under the assumption (20). Below, we show that if the coupling strength \(\kappa\) is sufficiently large and initial diameter \(D(A^0)\) is small enough, then \(D(A)\) can be made sufficiently small.

**Proposition 3.** Suppose that system parameters and initial data satisfy
\[
\Delta(\psi) > 0, \quad \kappa > \frac{2\Delta(\psi)\|H\|_\infty + 4\psi M D(H)}{\Lambda(\psi)^2}, \quad D(A^0) < \beta_+,
\]
and let \((X, A)\) be a global solution to (1)–(2). Then, there exists a finite entrance time \(T_*\) such that
\[
D(A(t)) < \beta_-, \quad t > T_*. \tag{21}
\]
Proof. Recall Lemma 2.4(i):
\[
\frac{d}{dt} D(A) \leq \left( -\kappa \Delta(\psi) + 2\|H\|_\infty \right) D(A) + 2\kappa \psi_M D(A)^2 + D(H), \quad t > 0, \quad (22)
\]
and we set
\[
y := D(A), \quad -p := -\kappa \Delta(\psi) + 2\|H\|_\infty, \quad q := 2\kappa \psi_M, \quad r := D(H).
\]
Then, (22) can be rewritten as
\[
\dot{y} \leq -py + qy^2 + r. \quad (23)
\]
On the other hand, the condition (20) implies,
\[
\Delta(\psi)^2 \kappa^2 - 2(2\|H\|_\infty \Delta(\psi) + 4\psi_M D(H))\kappa > 0 \implies p^2 - 4qr > 0.
\]
From (20) and (21), we apply Lemma 2.6(ii) to conclude that there exists a finite entrance time \(T^*\) such that
\[
y(t) < y_0 \quad \text{or} \quad D(A(t)) < \beta, \quad t > T^*.
\]

Remark 4. (i) The condition (20) is necessary when we apply the Grönwall-type inequality Lemma 2.6 (ii). More precisely, the conditions (20) guarantee the existence of two positive roots of \(-py + qy^2 + r = 0\) in (23) and the negativity of \(-p\), respectively.

(ii) For \(\beta_-\) in (19), one has
\[
\beta_- = \frac{\kappa \Delta(\psi) - 2\|H\|_\infty - \sqrt{\Lambda(\kappa, \psi, H)}}{4\kappa \psi_M} \leq \frac{\kappa \Delta(\psi) - 2\|H\|_\infty}{4\kappa \psi_M} - \frac{\Lambda(\kappa, \psi, H)}{4\kappa \psi_M D(H)} \leq \frac{8\kappa \psi_M D(H)}{4\kappa \psi_M \left( \kappa \Delta(\psi) - 2\|H\|_\infty + \sqrt{\Lambda(\kappa, \psi, H)} \right)} = O\left( \frac{1}{\kappa} \right).
\]
Hence, if the coupling strength is sufficiently large, then we can make \(\beta_-\) sufficiently small and in turn, \(D(A(t))\) can be also made sufficiently small.

We now arrive at our second main result which states that system (1) tends to the orientation-locked states under the constant communication weight and large coupling strength regime.

Theorem 3.2. Suppose that system parameters and initial data satisfy
\[
\begin{align*}
(\ i) & \quad \Delta(\psi) > 4\psi_M \beta_-, \\
(\ ii) & \quad \kappa > \max \left\{ \frac{2\|H\|_\infty}{\Delta(\psi) - 4\psi_M \beta_-}, \frac{2\Delta(\psi)\|H\|_\infty + 4\psi_M D(H)}{\Delta(\psi)^2} \right\}, \quad (24) \\
(\ iii) & \quad D(A_0) < \beta_+ \quad (25)
\end{align*}
\]
and let \((X, A)\) and \((\tilde{X}, \tilde{A})\) be any two global solutions to (1)–(2). Then, there exists a constant limit matrix \(F_{ij}^\infty \in \text{SO}(d)\) such that
\[
\lim_{t \to \infty} A_i A_j^{-1}(t) = F_{ij}^\infty.
\]
Proof. We split the proof into two steps.

- **(Step A: convergence of \(D(A, \tilde{A})\) towards zero):** it follows from the condition (25) and Proposition 3 that there exists \(T_0 > 0\) such that \(D(A(t)) < \beta_-\), \(t \geq T_0\).

Then, we use Lemma 2.4(i): for \(t > T_0\),
\[
\frac{d}{dt} D(A, \tilde{A}) \leq \left( -\kappa (3\psi_m - \psi_M) + 2\|H\|_\infty \right) D(A, \tilde{A}) + 4\kappa \psi_M \beta_- D(A, \tilde{A}) \]
\[
= \left( -\kappa (3\psi_m - \psi_M) + 2\|H\|_\infty + 4\kappa \psi_M \beta_- \right) D(A, \tilde{A}) \quad (26)
\]
\[
=: -\bar{\gamma} D(A, \tilde{A}).
\]

The coefficient \(-\bar{\gamma}\) in (26) is negative due to (24). Hence, we find the relation which yields the zero convergence of \(D(A, \tilde{A})\):
\[
D(A, \tilde{A}) \leq D(A, \tilde{A})(T_0) e^{-\bar{\gamma}(t-T_0)}, \quad t > T_0.
\]

- **(Step B: convergence of \(A_i A_i^{-1}\)):** since we are only interested in the large-time behavior, without loss of generality, we may assume \(T_0 = 0\). For any \(T \geq 0\), \(\{A_i(t + T)\}\) is also a solution with the shifted initial data \(\{A_i(T)\}\) due to system (1) being autonomous. Hence, we set \(\tilde{A}_i(t) = A_i(t + T)\) to apply the result of Step A:
\[
\|A_i A_i^{-1}(t + T) - A_i A_i^{-1}(t)\| \leq \max_{1 \leq i, j \leq N} \|A_i A_j^{-1}(T) - A_i A_j^{-1}(0)\| e^{-\bar{\gamma}T}.
\]

To verify the convergence, we discretize (27) by setting \(t = n\) and \(T = 1\). Then, (27) can be rewritten as
\[
\|A_i A_j^{-1}(n + 1) - A_i A_j^{-1}(n)\| \leq \max_{1 \leq i, j \leq N} \|A_i A_j^{-1}(1) - A_i A_j^{-1}(0)\| e^{-\bar{\gamma}n},
\]
and by the induction argument for \(m \in \mathbb{N}^\ast\),
\[
\|A_i A_j^{-1}(n + m) - A_i A_j^{-1}(n)\| \leq \max_{1 \leq i, j \leq N} \|A_i A_j^{-1}(1) - A_i A_j^{-1}(0)\| \frac{e^{-\bar{\gamma}n}}{1 - e^{-\bar{\gamma}n}},
\]
hence \(\{A_i A_j^{-1}(n)\}\) is a Cauchy sequence in the compact set \(\{D(A(t)) \leq \beta_-\}\). Therefore, there exists a limit \(F_{ij}^\infty \in \text{SO}(d)\) such that
\[
\lim_{t \to \infty} A_i A_j^{-1}(t) = F_{ij}^\infty,
\]
and its convergence rate is exponential due to (27).\(\square\)

4. **Finite-in-time stability.** In this section, we study the finite-in-time stability of system (1) with respect to initial data. Before we proceed further, we present definition of stability of (1) with respect to the initial data. For \(Z_i := (x_i, A_i) \in \mathbb{R}^d \times \text{SO}(d)\), we set \(Z := (Z_1, \cdots, Z_N)\) and define an \(\ell_2, \infty\)-like norm denoted by \(|\cdot|_{2, \infty}\):
\[
|Z_i| := |x_i| + \|A_i\|, \quad \|Z\|_{2, \infty} := \max_{1 \leq i \leq N} |Z_i|.
\]

**Definition 4.1.** For any \(T \in (0, \infty)\), there exists a positive constant \(G = G(T)\) which does not depend on the number of oscillators and the initial data such that for any two global solutions \(Z\) and \(\tilde{Z}\), if the following estimate hold, then we say that system (1) is finite-in-time stable with respect to the initial data.
\[
\sup_{0 \leq t < T} \|Z(t) - \tilde{Z}(t)\|_{2, \infty} \leq G \|Z_0 - \tilde{Z}_0\|_{2, \infty}, \quad t \in [0, T).
\]

(28)
Remark 5. If $T$ can take on $\infty$, then we call the estimate (28) as the uniform-in-time stability with respect to initial data. In fact, the uniform-in-time stability has been established in literature, for instance, [20] for the swarm sphere model and [27] for the Cucker-Smale model.

Before we begin a lengthy estimate, we briefly outline our strategy to derive the finite-time stability estimate (28) for any two solutions $Z$ and $\hat{Z}$ of (1).

- **(Estimate of $\| A_i - \hat{A}_i \|$:)** To estimate the rate of change of $\| A_i - \hat{A}_i \| = \| A_i \hat{A}_i^{-1} - I_d \|$, we estimate $\dot{A}_i \hat{A}_i^{-1} - I_d$ in Lemmas 4.2 and 4.3 and show that for a given $T \in (0, \infty)$, there exists a positive constant $C = C(T)$ such that

$$\frac{d}{dt} \| A_i \hat{A}_i^{-1} - I_d \| \leq C(T) \| A_i \hat{A}_i^{-1} - I_d \|, \quad t \in [0, T). \quad (29)$$

- **(Estimate of $|x_i - \hat{x}_i|$):** We use (1) and (29) to derive

$$|x_i(t) - \hat{x}_i(t)| \leq |x_0^i - \hat{x}_0^i| + v_0 \int_0^t \| A_i(s) - \hat{A}_i(s) \| ds \leq |x_0^i - \hat{x}_0^i| + v_0 T C(T) \max_{1 \leq k \leq N} \| A_0^i - \hat{A}_0^i \|, \quad t \in [0, T),$$

which yields the stability of spatial differences.

Now, let us begin our stability estimate. Let $Z$ and $\hat{Z}$ be two solutions to (1). Then, we use (40) to find

$$\dot{A}_i = H_i A_i + \frac{K}{N} \sum_{k=1}^N \psi_{ik} \left( A_k A_i^{-1} A_i \right),$$

$$\dot{A}_i^{-1} = -\hat{A}_i^{-1} H_i - \frac{K}{N} \sum_{k=1}^N \tilde{\psi}_{ik} \left( \hat{A}_k^{-1} - \hat{A}_i^{-1} \hat{A}_k \hat{A}_i^{-1} \right), \quad (30)$$

where $\psi_{ik} := \psi(|x_i - x_k|)$ and $\tilde{\psi}_{ik} := \psi(|\hat{x}_i - \hat{x}_k|)$. For notational simplicity, we simply write

$$P_i := A_i \hat{A}_i^{-1} - I_d, \quad \hat{P}_i := \hat{A}_i A_i^{-1} - I_d, \quad Q_{ik} := A_i A_k^{-1} - I_d, \quad \tilde{Q}_{ik} := \hat{A}_i \hat{A}_k^{-1} - I_d.$$

Note that $\dot{P}_i^1 = P_i$ and $Q_{ik}^1 = Q_{ki}$. Then, we calculate (30)$_1 \times \hat{A}_i^{-1} + A_i \times (30)_2$ to find the dynamics of $P_i$:

$$\frac{dP_i}{dt} = H_i A_i \hat{A}_i^{-1} - A_i \hat{A}_i^{-1} H_i$$

$$+ \frac{K}{N} \sum_{k=1}^N \psi_{ik} \left( A_k \hat{A}_i^{-1} - A_i A_k^{-1} A_i \hat{A}_i^{-1} \right) + \frac{K}{N} \sum_{k=1}^N \tilde{\psi}_{ik} \left( A_i \hat{A}_k^{-1} - A_i \hat{A}_i^{-1} \hat{A}_k \hat{A}_i^{-1} \right)$$

$$= H_i A_i \hat{A}_i^{-1} - A_i \hat{A}_i^{-1} H_i$$

$$+ \frac{K}{N} \sum_{k=1}^N \psi_{ik} \left( A_k A_i^{-1} A_i - A_i A_k^{-1} A_i \right)$$

$$+ \frac{K}{N} \sum_{k=1}^N \tilde{\psi}_{ik} \left( A_i \hat{A}_k^{-1} \hat{A}_i - A_i \hat{A}_i^{-1} \hat{A}_k \hat{A}_i^{-1} \right)$$

$$=: \mathcal{I}_{11} + \mathcal{I}_{12} + \mathcal{I}_{13}. \quad (31)$$
Below, we estimate the terms $I_{1k}, k = 1, 2, 3$, separately.

**Lemma 4.2.** Let $Z$ and $\tilde{Z}$ be two solutions to (1). Then, the terms $I_{1k}, k = 1, 2, 3$ can be rewritten as follows.

(i) $I_{11} = H_i P_i - P_i H_i$.

(ii) $I_{12} = \frac{N}{N} \sum_{k=1}^{N} \psi_{ik} \left( Q_{ki} P_i - Q_{ki} P_i + Q_{ki} - Q_{ik} \right)$.

(iii) $I_{13} = \frac{N}{N} \sum_{k=1}^{N} \tilde{\psi}_{ik} \left( P_i \tilde{Q}_{ik} - P_i \tilde{Q}_{ki} + \tilde{Q}_{ik} - \tilde{Q}_{ki} \right)$.

**Proof.**

(i) By direct calculation, we see

\[ I_{11} = H_i A_i \tilde{\lambda}^{-1} - A_i \tilde{\lambda}^{-1} H_i = H_i P_i - P_i H_i. \]

(ii) After tedious algebraic manipulation, we have

\begin{align*}
I_{12} &= \frac{N}{N} \sum_{k=1}^{N} \psi_{ik} \left( (A_k A_i^{-1} - I)(A_i \tilde{\lambda}^{-1} - I) + A_k A_i^{-1} + A_i \tilde{\lambda}^{-1} - I \\
&\quad - (A_i A_k^{-1} - I)(A_i \tilde{\lambda}^{-1} - I) - A_i A_k^{-1} - A_i \tilde{\lambda}^{-1} + I \right) \\
&= \frac{N}{N} \sum_{k=1}^{N} \psi_{ik} \left( (A_k A_i^{-1} - I)(A_i \tilde{\lambda}^{-1} - I) - (A_i A_k^{-1} - I)(A_i \tilde{\lambda}^{-1} - I) \\
&\quad + A_k A_i^{-1} - A_i A_k^{-1} \right) \\
&= \frac{N}{N} \sum_{k=1}^{N} \psi_{ik} \left( Q_{ki} P_i - Q_{ki} P_i + Q_{ki} - Q_{ik} \right).
\end{align*}

(iii) By the same manipulation as in (ii), we have

\begin{align*}
I_{13} &= \frac{N}{N} \sum_{k=1}^{N} \tilde{\psi}_{ik} \left( (A_i \tilde{\lambda}^{-1} - I)(A_i \tilde{\lambda}^{-1} - I) - (A_i \tilde{\lambda}^{-1} - I)(A_i \tilde{\lambda}^{-1} - I) \\
&\quad + \tilde{\lambda}_i \tilde{\lambda}_k^{-1} - \tilde{\lambda}_k \tilde{\lambda}_i^{-1} \right) \\
&= \frac{N}{N} \sum_{k=1}^{N} \tilde{\psi}_{ik} \left( P_i \tilde{Q}_{ik} - P_i \tilde{Q}_{ki} + \tilde{Q}_{ik} - \tilde{Q}_{ki} \right).
\end{align*}

Next, we derive Grönwall’s inequalities for $|x_i(t) - \tilde{x}_i(t)|$ and $\|P_i(t)\| = \|A_i(t) - \tilde{A}_i(t)\|$.

**Lemma 4.3.** Let $Z$ and $\tilde{Z}$ be two solutions to (1)–(2). Then, the following Grönwall inequalities hold:

(i) $|x_i(t) - \tilde{x}_i(t)| \leq |x_i^0 - \tilde{x}_i^0| + v_0 \int_0^t \|A_i(s) - \tilde{A}_i(s)\| ds$.

(ii) \( \frac{d\|P_i\|^2}{dt} \leq \frac{N}{N} \sum_{k=1}^{N} \left[ \|P_i\| \left( \|P_i\| \|\tilde{S}_{ik}\| \|P_k\| + \|P_i\| \|\tilde{S}_{ik}\| + \|\tilde{S}_{ik}\| \|P_k\| \right) \right] \)
\[ + \psi_M \| P_i \| + \psi_M \| P_k \| + \psi_M \| P_k \| \]
\[ + 2 \| P_i \| \left( \| P_k \| \| S_{ki} \| + \| P_k \| \| S_{ki} \| + \| S_{ki} \| \| P_i \| \right) \]
\[ + \psi_M \| P_k \| \| P_i \| + \psi_M \| P_k \| + \psi_M \| P_i \| \right). \]

**Proof.** (i) Since
\[
\frac{d}{dt}(x_i - \tilde{x}_i) = v_0(A_i - \tilde{A}_i)e_1,
\]
it can be directly integrated to yield that
\[
|x_i(t) - \tilde{x}_i(t)| \leq |x_i^0 - \tilde{x}_i^0| + v_0 \int_0^t \| A_i(s) - \tilde{A}_i(s) \| ds, \quad t \in [0, T).
\]
(ii) In (31), we use Lemma 4.2 to obtain
\[
\frac{dP_i}{dt} = H_i P_i - P_i H_i + \frac{\kappa}{N} \sum_{k=1}^N \psi_{ik} \left( Q_{ki} P_i - Q_{ik} P_i + Q_{ki} - Q_{ik} \right)
\]
\[+ \frac{\kappa}{N} \sum_{k=1}^N \tilde{\psi}_{ik} \left( \tilde{P}_i \tilde{Q}_{ik} - P_i \tilde{Q}_{ki} + \tilde{Q}_{ik} - \tilde{Q}_{ki} \right). \tag{32}\]

We change the roles of \( A_i \) and \( \tilde{A}_i \) to see
\[
\frac{d\tilde{P}_i}{dt} = H_i \tilde{P}_i - \tilde{P}_i H_i + \frac{\kappa}{N} \sum_{k=1}^N \tilde{\psi}_{ik} \left( \tilde{Q}_{ki} \tilde{P}_i - \tilde{Q}_{ik} \tilde{P}_i + \tilde{Q}_{ki} - \tilde{Q}_{ik} \right)
\]
\[+ \frac{\kappa}{N} \sum_{k=1}^N \psi_{ik} \left( P_i \tilde{Q}_{ik} - \tilde{P}_i Q_{ki} + Q_{ik} - Q_{ki} \right), \tag{33}\]

which also can be obtained by taking tilde notation into (32) if we allow to abuse the tilde notation. We add (32) and (33) to obtain
\[
\frac{d}{dt} \left( P_i + \tilde{P}_i \right) = H_i (P_i + \tilde{P}_i) - (P_i + \tilde{P}_i) H_i
\]
\[+ \frac{\kappa}{N} \sum_{k=1}^N \psi_{ik} \left( Q_{ki} P_i - Q_{ik} P_i + \tilde{P}_i Q_{ki} - \tilde{P}_i Q_{ki} \right) \tag{34}\]
\[+ \frac{\kappa}{N} \sum_{k=1}^N \tilde{\psi}_{ik} \left( P_i \tilde{Q}_{ik} - \tilde{P}_i Q_{ki} + \tilde{Q}_{ik} \tilde{P}_i - \tilde{Q}_{ki} \tilde{P}_i \right) \]

On the other hand, since \( \tilde{P}_i = P_i^t \) and the following relation holds:
\[
(P_i + I_d)(\tilde{P}_i + I_d) = (A_i \tilde{A}_i^{-1}) \cdot (\tilde{A}_i A_i^{-1}) = I_d, \quad \text{or equivalently,} \quad P_i \tilde{P}_i = -P_i - \tilde{P}_i,
\]
we take a trace on both sides of (34) to see
\[
-\frac{d}{dt} \| P_i \|^2 = 2 \frac{\kappa}{N} \sum_{k=1}^N \psi_{ik} \text{tr} \left( P_i (Q_{ki} - Q_{ik}) \right) + \tilde{\psi}_{ik} \text{tr} \left( \tilde{P}_i (\tilde{Q}_{ki} - \tilde{Q}_{ik}) \right)
\]
\[= 2 \frac{\kappa}{N} \sum_{k=1}^N \psi_{ik} \text{tr} (P_i (Q_{ki} - Q_{ik})) + \tilde{\psi}_{ik} \text{tr} (P_i (\tilde{Q}_{ik} - \tilde{Q}_{ki})). \tag{35}\]
Since $\psi_{ik} = \psi_{ki}$ and $\tilde{\psi}_{ik} = \tilde{\psi}_{ki}$ hold, we write $S_{ik} := \psi_{ik}Q_{ik}$. Then, (35) can be rewritten as

$$
\frac{d\|P_i\|^2}{dt} = \frac{2\kappa}{N} \sum_{k=1}^{N} \text{tr} \left( P_i (S_{ik} - S_{ki}) \right) + \text{tr} \left( \tilde{P}_i (\tilde{S}_{ik} - \tilde{S}_{ki}) \right)
$$

(36)

Note that

$$
Q_{ik} = A_i A_k^{-1} - I_d = A_i \tilde{A}_i^{-1} \tilde{A}_k^{-1} - I_d = (P_i + I_d)(\tilde{Q}_{ik} + I_d)(\tilde{P}_k + I_d) - I_d
$$

$$
P_i \tilde{Q}_{ik} \tilde{P}_k + P_i \tilde{Q}_{ik} \tilde{P}_k + P_i \tilde{P}_k + P_i + \tilde{P}_k + \tilde{Q}_{ik}.
$$

(37)

Hence, we have

$$
S_{ik} - \tilde{S}_{ik} = P_i \tilde{S}_{ik} \tilde{P}_k + P_i \tilde{S}_{ik} \tilde{P}_k + \psi_{ik}(P_i \tilde{P}_k + P_i + \tilde{P}_k).
$$

We obtain from (36) that

$$
\frac{d\|P_i\|^2}{dt} = \frac{\kappa}{N} \sum_{k=1}^{N} \text{tr} \left[ (P_i - \tilde{P}_i)(P_i \tilde{S}_{ik} \tilde{P}_k + P_i \tilde{S}_{ik} \tilde{P}_k + \psi_{ik}(P_i \tilde{P}_k + P_i + \tilde{P}_k)) 
- (P_i - \tilde{P}_i)(P_k \tilde{S}_{ki} \tilde{P}_i + P_k \tilde{S}_{ki} \tilde{P}_i + \psi_{ik}(P_k \tilde{P}_i + P_k + \tilde{P}_i)) \right].
$$

Then, we give a rough estimate as

$$
\frac{d\|P_i\|^2}{dt} \leq \frac{\kappa}{N} \sum_{k=1}^{N} \left[ 2\|P_i\| \left( \|P_i\| \|\tilde{S}_{ik}\| \|P_k\| + \|P_i\| \|\tilde{S}_{ki}\| \|P_i\| + \|\tilde{S}_{ik}\| \|P_k\| 
+ \psi_M\|P_i\| \|P_k\| + \psi_M\|P_i\| \|\tilde{S}_{ki}\| \|P_i\| 
+ 2\|P_i\| \left( \|P_k\| \|\tilde{S}_{ki}\| \|P_i\| + \|P_k\| \|\tilde{S}_{ki}\| + \|\tilde{S}_{ki}\| \|P_i\| 
+ \psi_M\|P_k\| \|P_i\| + \psi_M\|P_k\| \|P_i\| \right) \right].
$$

Finally, we use Lemmas 4.2 and 4.3 to obtain the following theorem.

**Theorem 4.4.** For any $T \in (0, \infty)$, there exists a positive constant $G = G(T)$ such that for any solutions $Z$ and $\tilde{Z}$ to (1)-(2), the following estimate holds.

$$
\sup_{0 \leq t < T} \|Z(t) - \tilde{Z}(t)\|_{2, \infty} \leq G \|Z^0 - \tilde{Z}^0\|_{2, \infty}, \quad t \in [0, T).
$$

**Proof.** Note that there exists a uniform constant $C > 0$ such that

$$
\|\tilde{S}_{ik}\| \leq C, \quad \|\tilde{S}_{ki}\| \leq C, \quad t > 0,
$$

(38)

where $S_{ik} = \psi_{ik}Q_{ik} = \psi_{ik}(A_i A_k^{-1} - I_d)$. We write $\|P\| := \max_{1 \leq k \leq N} \|P_k\|$ and use Lemma 4.3(ii) and (38) to obtain

$$
\frac{d\|P\|^2}{dt} \leq 4\kappa(1 + \delta)(C + \psi_M)\|P\| =: \delta\|P\|, \quad t > 0.
$$

Then, Grönwall’s lemma yields that for a given $T \in (0, \infty)$,

$$
\|P(t)\| \leq e^{\delta T}\|P_0\| =: g(T)\|P_0\|, \quad t \in [0, T),
$$
where the projection matrix is defined as $P_i$.

On the other hand, for spatial variables, we use Lemma 4.3 (i) to obtain

$$|x_i(t) - \tilde{x}_i(t)| \leq |x_i^0 - \tilde{x}_i^0| + v_0 \int_0^t \|A_i(s) - \tilde{A}_i(s)\|ds$$

$$\leq |x_i^0 - \tilde{x}_i^0| + v_0 T g(T) \max_{1 \leq i \leq N} \|A_i^0 - \tilde{A}_i^0\|.$$

Hence, we can find a constant $G(T) := \max\{1, (1 + v_0)g(T)\}$ such that

$$\max_{1 \leq i \leq N} \left(|x_i(t) - \tilde{x}_i(t)| + \|A_i - \tilde{A}_i(t)\|\right) \leq G(T) \max_{1 \leq i \leq N} \left(|x_i^0 - \tilde{x}_i^0| + \|A_i^0 - \tilde{A}_i^0\|\right).$$

Remark 6. The issue when one concerns with the (uniform-in-time or finite-in-time) stability is whether the maximal time $T$ can be $\infty$ or not. In addition, if two orientation matrices are not the same asymptotically, that is, if $\|A_i - A_j\|$ does not converge to zero as time $t$ goes to infinity, then $|x_i - x_j|$ tends to infinity. Thus, when we deal with the stability with respect to the initial data, we always consider the case of the orientation alignment. In fact, Theorem 4.4 fails to give uniform-in-time stability estimate due to the following technical reason. If we apply Lemma 2.3 in (32), then we obtain the following Grönwall-type inequality:

$$\dot{y} \leq Cy^{\alpha t} + Cy^{-\alpha t}, \quad t > 0.$$ 

This, however, does not guarantee the uniform stability not only for whole time but also for finite time. This failure comes from the zeroth order terms such as $Q_{1k}$ and $Q_{ki}$ in the right-hand side of (32). To overcome this, we split $Q_{ik}$ into higher order terms of $P_i$ as in (37).

Since the Frobenius norm of $M_d(\mathbb{R})$ corresponds to the $\ell_2$-norm of $\mathbb{R}^{d^2}$, (1) can be regarded as a system defined on $\mathbb{R}^d \times \mathbb{R}^{d^2}$. Hence, if we adopt the Wasserstein framework in [27] to apply the finite-in-time stability, then we can identify the kinetic model of (1), which is valid on any finite time interval, as follows: for the one-particle distribution function $f = f(x, A, H, t)$,

$$\partial_t f + v_0 A e_1 \cdot \nabla_x f + \nabla_A \cdot (L[f]f) = 0,$$

$$L[f] = HA + \int_{\mathbb{R}^d \times SO(d)} \psi(|x - x_*|)\mathbb{P}_A(A_*)f(x_*, A_*, H_*, t)dx_*dA_*dH_*,$$  

where the projection matrix is defined as $\mathbb{P}_A(A_*) = \frac{1}{2}(A_* - AA^t_* A)$. We refer the reader to [11] for a detailed calculation of $\nabla_A$ and integration with respect to $A$.

5. Numerical experiments. In this section, we provide several numerical simulations for the emergent dynamics of our system (1) for both identical case ($H_i \equiv H$) and the non-identical case ($D(H) > 0$). We used the fourth-order Runge-Kutta method for numerical integrations.
Figure 1. Identical case: $H_i \equiv H$

5.1. **Identical case.** The initial data and the parameters have been chosen to satisfy the assumptions (14)–(15) in Theorem 3.1:

- $N = 100$, $v_0 = 1$, $\kappa = 5$, $\psi \equiv 1$, $x_i^0 \in (-2, 2)$,
- $D(A^0) \approx 0.7749$, $H_{mn} \in (-0.5, 0.5)$, $1 \leq m, n \leq 3$.

Here, the initial data and the entries of skew-symmetric matrices $H_i$ are chosen randomly. Note that since we only consider a constant communication function $\psi$, the second equation becomes autonomous system in terms of the orientation $A_i$. Thus, the initial data of position is only used to indicate relative positions of particles.

Figure 1(a) describes an agent moving with velocity $A_i e_1$ (red arrow) and the orientation determined by two vectors $A_i e_2$ and $A_i e_3$. Note the positivity of orientation. In Figures 1(b)-(d), we can observe the orientation flocking of agents. In particular, Figure 1(d) exhibits the comparison of the decay of $D(A(t))$ obtained numerically and the analytical result in Theorem 3.1 which supports (16).

5.2. **Non-identical case.** For the non-identical case, we chose random initial data and the parameters, with no constraint on $A_i^0 \in SO(3)$, such that

- $N = 100$, $v_0 = 1$, $\kappa = 1$, $\psi \equiv 1$, 


\[
A_i \in SO(3)
\]
The orientation alignment fails in this case, as can be seen in Figures 2(a)-(b). Figures 2(c)-(d) illustrate the orientation-locking of the system where we can observe the evolution of each entries of the matrix $A_1A_2^{-1}$. We also check numerically that there exists a limit matrix $F_{12}^\infty$ such that

$$
F_{12}^\infty \approx \begin{bmatrix}
0.7738 & -0.5864 & 0.2395 \\
0.5465 & 0.8092 & 0.2157 \\
-0.3203 & -0.0360 & 0.9466
\end{bmatrix} \in \text{SO}(3), \quad \lim_{t \to \infty} \|A_1A_2^{-1} - F_{12}^\infty\| = 0,
$$

which confirm Theorem 3.2.

6. Conclusion. In this paper, we have studied flocking behaviors of the agent-based model [11] for the orientation flocking of a multi-agent system. We present sufficient frameworks leading to the emergence of orientation flocking and orientation-locked state. To be more precise, orientation flocking can emerge for the identical case $H_i \equiv H$ for all $i = 1, \cdots, N$, that is, all orientations become eventually the same, as time goes to infinity. On the other hand, the orientation-locked state can occur for the non-identical case in a large coupling regime. In this case, the difference between any two orientation matrices converges to the definite constant nonzero matrix, whereas it converges to zero matrix for the case of orientation flocking. In addition, we show that our system is finite-in-time stable with respect
to the initial data in any finite time interval, and as a corollary, we can identify its kinetic model in a Wasserstein framework. We also provided several numerical simulations supporting our analysis. Of course, there are still lots of interesting open questions such as the extension of stability estimate to the whole time interval, global-wellposedness, regularity and emergent dynamics of the kinetic model. These issues will be addressed in future works.

Appendix A. Proof of lemma 2.4. In this appendix, we provide the proof of Lemma 2.4.

(i) (Derivation of Gröndall’s differential inequality for \( D(A) \)): Note that

\[
D(A) = \max_{1 \leq i, j \leq N} \| A_i A_j^{-1} - I_d \|.
\]

We first observe that

\[
0 = \frac{d}{dt} (A_i A_j^{-1}) = \dot{A}_i A_j^{-1} + A_i \dot{A}_j^{-1}, \quad \text{or equivalently,} \quad \dot{A}_j^{-1} = -A_i^{-1} \dot{A}_i A_j^{-1}
\]

to obtain the dynamics for inverse matrix \( A_j^{-1} \):

\[
\dot{A}_j^{-1} = -A_i^{-1} H_i - \frac{K}{N} \sum_{k=1}^{N} \psi_{ik} (A_i^{-1} A_k A_j^{-1} - A_k^{-1} A_i).
\]

Let \( i, j \) be indices in \( \{1, \cdots, N\} \). Then, we have

\[
\begin{align*}
\frac{d}{dt} (A_i A_j^{-1} - I_d) &= H_i (A_i A_j^{-1} - I_d) - (A_i A_j^{-1} - I_d) H_j + H_i - H_j \\
&= \frac{K}{N} \sum_{k=1}^{N} \psi_{ik} (A_k A_j^{-1} - A_i A_k^{-1} A_j^{-1}) + \psi_{jk} (A_j A_k^{-1} - A_i A_j^{-1} A_k)
\end{align*}
\]

\[
= \frac{K}{N} \sum_{k=1}^{N} \psi_{ik} \left( A_k A_j^{-1} - I_d + 2I_d - A_i A_j^{-1} - A_i A_k^{-1} - (A_i A_k^{-1} - I_d)(A_i A_j^{-1} - I_d) \right)
\]

\[
+ \frac{K}{N} \sum_{k=1}^{N} \psi_{jk} \left( A_j A_k^{-1} - I_d + 2I_d - A_i A_j^{-1} - A_k A_j^{-1} - (A_i A_j^{-1} - I_d)(A_k A_j^{-1} - I_d) \right).
\]

On the other hand, \( Q_{ij} = A_i A_j^{-1} - I_d \) satisfies

\[
\begin{align*}
\frac{d}{dt} Q_{ij} &= H_i Q_{ij} - Q_{ij} H_j + H_i - H_j \\
&\quad + \frac{K}{N} \sum_{k=1}^{N} \psi_{ik} (Q_{kj} - Q_{ij} - Q_{ik} Q_{ij}) + \psi_{jk} (Q_{ik} - Q_{ij} - Q_{kj} - Q_{ij} Q_{kj})
\end{align*}
\]

\[
= H_i Q_{ij} - Q_{ij} H_j + H_i - H_j - \frac{K}{N} \sum_{k=1}^{N} (\psi_{ik} + \psi_{jk}) Q_{ij}
\]

\[
+ \frac{K}{N} \sum_{k=1}^{N} (\psi_{ik} - \psi_{jk}) (Q_{kj} - Q_{ik}) - \frac{K}{N} \sum_{k=1}^{N} (\psi_{ik} Q_{ik} Q_{ij} + \psi_{jk} Q_{ij} Q_{kj}).
\]

(41)
We apply Lemma 2.3 with $X = Q_{ij}$ to obtain

$$\frac{d}{dt} \|Q_{ij}\| \leq 2\|H\|_\infty \|Q_{ij}\| + D(H) - \frac{\kappa}{N} \sum_{k=1}^{N} (\psi_{ik} + \psi_{jk}) \|Q_{ij}\|$$

$$+ \frac{\kappa}{N} \sum_{k=1}^{N} |\psi_{ik} - \psi_{jk}| \|Q_{ik} - Q_{kj}\|$$

$$+ \frac{\kappa}{N} \sum_{k=1}^{N} (\psi_{ik} \|Q_{ik}\| \|Q_{ij}\| + \psi_{jk} \|Q_{ij}\| \|Q_{kj}\|).$$

For given $t \geq 0$, we choose the index $(i_t, j_t)$ such that

$$D(A) = \|Q_{i_t,j_t}\| = \|A_{i_t} A_{j_t}^{-1} - I_d\|.$$ 

Then, $D(A)$ satisfies

$$\frac{d}{dt} D(A) \leq 2\|H\|_\infty D(A) + D(H)$$

$$+ \frac{\kappa}{N} \sum_{k=1}^{N} \left( 2|\psi_{i_t,k} - \psi_{j_t,k}| - (\psi_{i_t,k} + \psi_{j_t,k}) \right) D(A)$$

$$+ \frac{\kappa}{N} \sum_{k=1}^{N} (\psi_{i_t,k} + \psi_{j_t,k}) D(A)^2.$$  \hspace{1cm} (42)

Note that for $a, b \in \mathbb{R}$,

$$-a - b + 2|a - b| = \max\{a, b\} - 3 \min\{a, b\}. \hspace{1cm} (43)$$

Hence, (42) becomes

$$\frac{d}{dt} D(A) \leq 2\|H\|_\infty D(A) + D(H) + \kappa \left( \max\{\psi_{i_t,k}, \psi_{j_t,k}\} - 3 \min\{\psi_{i_t,k}, \psi_{j_t,k}\} \right) D(A)$$

$$+ \frac{\kappa}{N} \sum_{k=1}^{N} (\psi_{i_t,k} + \psi_{j_t,k}) D(A)^2$$

$$\leq -\kappa (3\psi_m - \psi_M + 2\|H\|_\infty) D(A) + 2\kappa \psi_M D(A)^2 + D(H).$$

This completes the derivation of the differential inequality (i) in Lemma 2.4.

(ii) (Derivation of Grönwall’s inequality for $D(A, \tilde{A})$): It follows from (41) that $\tilde{Q}_{ij} = \tilde{A}_i \tilde{A}_j^{-1} - I$ satisfies

$$\frac{d}{dt} \tilde{Q}_{ij} = -\frac{\kappa}{N} \sum_{k=1}^{N} (\tilde{\psi}_{ik} + \tilde{\psi}_{jk}) \tilde{Q}_{ij} + \frac{\kappa}{N} \sum_{k=1}^{N} (\tilde{\psi}_{ik} - \tilde{\psi}_{jk}) (\tilde{Q}_{kj} - \tilde{Q}_{ik})$$

$$- \frac{\kappa}{N} \sum_{k=1}^{N} (\psi_{ik} \tilde{Q}_{ik} Q_{ij} + \psi_{jk} \tilde{Q}_{ij} Q_{kj}) + (H_i - H_j) + H_i \tilde{Q}_{ij} - \tilde{Q}_{ij} H_j.$$
Since \(A_i A_j^{-1} - \tilde{A}_i \tilde{A}_j^{-1} = Q_{ij} - \tilde{Q}_{ij}\), we have
\[
\frac{d}{dt}(Q_{ij} - \tilde{Q}_{ij}) := H_i(Q_{ij} - \tilde{Q}_{ij}) - (Q_{ij} - \tilde{Q}_{ij})H_j
\]
\[
- \frac{\kappa}{N} \sum_{k=1}^{N} (\psi_{ik} + \psi_{jk})(Q_{ij} - \tilde{Q}_{ij}) + \frac{\kappa}{N} \sum_{k=1}^{N} (\tilde{\psi}_{ik} + \tilde{\psi}_{jk} - \psi_{jk})\tilde{Q}_{ij}
\]
\[
+ \frac{\kappa}{N} \sum_{k=1}^{N} \mathcal{M}_{ij}^{kj},
\]
where \(\mathcal{M}_{ij}^{kj}\) is defined as
\[
\mathcal{M}_{ij}^{kj} := \psi_{ik}(Q_{kj} - \tilde{Q}_{kj}) - \tilde{\psi}_{ik}(\tilde{Q}_{kj} - \tilde{Q}_{kj}) + \psi_{jk}(Q_{ik} - \tilde{Q}_{ik}) - \tilde{\psi}_{jk}(\tilde{Q}_{ik} - \tilde{Q}_{ij}) + \psi_{ik}(Q_{kj} - \tilde{Q}_{kj})\tilde{Q}_{kj} - \tilde{\psi}_{ik}(\tilde{Q}_{kj} - \tilde{Q}_{kj})Q_{ij}
\]
Moreover, we rewrite \(\mathcal{M}_{ij}^{kj}\) in terms of differences between \(Q_{mn}\) and \(\tilde{Q}_{mn}\):
\[
\mathcal{M}_{ij}^{kj} = \psi_{ik}(Q_{kj} - \tilde{Q}_{kj}) + (\psi_{ik} - \tilde{\psi}_{ik})\tilde{Q}_{kj} + \psi_{jk}(Q_{ik} - \tilde{Q}_{ik}) + (\psi_{jk} - \tilde{\psi}_{jk})\tilde{Q}_{ik}
\]
\[
- \psi_{ik}(Q_{ik} - \tilde{Q}_{ik}) - (\psi_{ik} - \tilde{\psi}_{ik})\tilde{Q}_{ik} + \psi_{jk}(Q_{kj} - \tilde{Q}_{kj}) + (\psi_{jk} - \tilde{\psi}_{jk})\tilde{Q}_{kj}
\]
\[
- \psi_{ik}Q_{ik}(Q_{kj} - \tilde{Q}_{ij}) - \psi_{jk}(Q_{ik} - \tilde{Q}_{ik})Q_{kj} + (\psi_{jk} - \tilde{\psi}_{jk})Q_{ik}\tilde{Q}_{kj}
\]
\[
- \psi_{jk}Q_{ij}(Q_{kj} - \tilde{Q}_{ij}) + \psi_{jk}(Q_{ij} - \tilde{Q}_{ij})\tilde{Q}_{kj} + (\psi_{jk} - \tilde{\psi}_{jk})\tilde{Q}_{ij}\tilde{Q}_{kj}.
\]
Since we assume that \(\psi_{ik}\) is a constant, we have
\[
\psi_{ij} - \tilde{\psi}_{ij} = 0.
\]
In (44), we use the relation (45) to obtain
\[
\frac{d}{dt}(Q_{ij} - \tilde{Q}_{ij})
\]
\[
= - \frac{\kappa}{N} \sum_{k=1}^{N} (\psi_{ik} + \psi_{jk})(Q_{ij} - \tilde{Q}_{ij}) + H_i(Q_{ij} - \tilde{Q}_{ij}) - (Q_{ij} - \tilde{Q}_{ij})H_j
\]
\[
+ \frac{\kappa}{N} \sum_{k=1}^{N} (\psi_{ik}Q_{kj} - \tilde{Q}_{kj}) + \psi_{jk}(Q_{ik} - \tilde{Q}_{ik}) - \psi_{ik}(Q_{ik} - \tilde{Q}_{ik})Q_{kj} + (\psi_{jk} - \tilde{\psi}_{jk})\tilde{Q}_{kj}
\]
\[
- \psi_{jk}Q_{ij}(Q_{kj} - \tilde{Q}_{ij}) + \psi_{jk}(Q_{ij} - \tilde{Q}_{ij})\tilde{Q}_{kj} + (\psi_{jk} - \tilde{\psi}_{jk})\tilde{Q}_{ij}\tilde{Q}_{kj}.
\]
By a similar argument as in (i), we choose the indices \((i_1, j_1)\) such that
\[
D(A, \tilde{A}) = \|Q_{ij} - \tilde{Q}_{ij}\|.
\]
Then, $D(A, \tilde{A})$ satisfies

$$\frac{d}{dt}D(A, \tilde{A}) \leq -\frac{\kappa}{N} \sum_{k=1}^{N} (\psi_{i,k} + \psi_{j,k})D(A, \tilde{A}) + 2\|H\|_{\infty}D(A, \tilde{A})$$

$$+ \frac{\kappa}{N} \sum_{k=1}^{N} \left( 2|\psi_{i,k} - \psi_{j,k}|D(A, \tilde{A}) + 4\psi_{M}D(A)D(A, \tilde{A}) \right).$$

Finally, we use the relation (43) to obtain the desired inequality for $D(A, \tilde{A}):$

$$\frac{d}{dt}D(A, \tilde{A}) \leq \left( -\kappa(3\psi_{m} - \psi_{M}) + 2\|H\|_{\infty} \right)D(A, \tilde{A}) + 4\kappa\psi_{M}D(A)D(A, \tilde{A}).$$

REFERENCES

[1] J. A. Acebrón, L. L. Bonilla, C. J. Pérez Vicente, F. Ritort and R. Spigler, The Kuramoto model: A simple paradigm for synchronization phenomena, Rev. Mod. Phys., 77 (2005), 137–185.

[2] D. Benedetto, E. Caglioti and U. Montemagno, On the complete phase synchronization for the Kuramoto model in the mean-field limit, Commun. Math. Sci., 13 (2015), 1775–1786.

[3] D. Benedetto, E. Caglioti and U. Montemagno, Exponential dephasing of oscillators in the kinetic Kuramoto model, J. Stat. Phys., 162 (2016), 813–823.

[4] J. C. Bronski, L. DeVille and M. J. Park, Fully synchronous solutions and the synchronization phase transition for the finite-N Kuramoto model, Chaos, 22 (2012), 033133, 17 pp.

[5] S. Chandra, M. Girvan and E. Ott, Complexity reduction ansatz for systems of interacting orientable agents: Beyond the Kuramoto model, Chaos, 29 (2019), 053107, 8 pp.

[6] S.-H. Choi and S.-Y. Ha, Complete entrainment of Lohe oscillators under attractive and repulsive couplings, SIAM J. Appl. Dyn. Syst., 13 (2014), 1417–1441.

[7] S.-H. Choi and S.-Y. Ha, Emergence of flocking for a multi-agent system moving with constant speed, Commun. Math. Sci., 14 (2016), 953–972.

[8] Y.-P. Choi, S.-Y. Ha, S. Jung and Y. Kim, Asymptotic formation and orbital stability of phase-locked states for the Kuramoto model, Phys. D, 241 (2012), 735–754.

[9] N. Chopra and M. W. Spong, On exponential synchronization of Kuramoto oscillators, IEEE Trans. Automat. Control, 54 (2009), 355–357.

[10] F. Cucker and S. Smale, Emergent behavior in flocks, IEEE Trans. Automat. Control, 52 (2007), 852–862.

[11] P. Degond, A. Frouvelle and S. Merino-Aceituno, A new flocking model through body attitude coordination, Math. Models Methods Appl. Sci., 27 (2017), 1005–1049.

[12] P. Degond, A. Frouvelle, S. Merino-Aceituno and A. Trescases, Quaternions in collective dynamics, Multiscale Model. Simul., 16 (2018), 28–77.

[13] L. DeVille, Synchronization and stability for quantum kuramoto, J. Stat. Phys., 174 (2019), 160–187.

[14] J.-G. Dong and X. P. Xue, Synchronization analysis of Kuramoto oscillators, Commun. Math. Sci., 11 (2013), 465–480.

[15] F. Dörfler and F. Bullo, Synchronization in complex networks of phase oscillators: A survey, Automatica J. IFAC, 50 (2014), 1539–1564.

[16] F. Dörfler and F. Bullo, On the critical coupling for Kuramoto oscillators, SIAM. J. Appl. Dyn. Syst., 10 (2011), 1070–1099.

[17] S.-Y. Ha and J.-G. Liu, A simple proof of Cucker-Smale flocking dynamics and mean field limit, Commun. Math. Sci., 7 (2009), 297–325.

[18] S.-Y. Ha, T. Ha and J.-H. Kim, On the complete synchronization for the globally coupled Kuramoto model, Phys. D, 239 (2010), 1692–1700.

[19] S.-Y. Ha, M.-J. Kang and E. Jeong, Emergent behaviour of a generalized Viscek-type flocking model, Nonlinearity, 23 (2010), 3139–3156.

[20] S.-Y. Ha, D. Kim, J. Lee and S. E. Noh, Particle and kinetic models for swarming particles on a sphere and stability properties, J. Stat. Phys., 174 (2019), 622–655.

[21] S.-Y. Ha, H. K. Kim and J. Park, Remarks on the complete synchronization of Kuramoto oscillators, Nonlinearity, 28 (2015), 1441–1462.
[22] S.-Y. Ha, D. Ko, J. Park and X. T. Zhang, Collective synchronization of classical and quantum oscillators, EMS Surv. Math. Sci., 3 (2016), 209–267.

[23] S.-Y. Ha, H. K. Kim and S. W. Ryoo, Emergence of phase-locked states for the Kuramoto model in a large coupling strength, Discrete Contin. Dyn. Syst., 35 (2015), 3417–3436.

[24] S.-Y. Ha, D. Ko and S. W. Ryoo, Emergent dynamics of a generalized Lohe model on some class of Lie groups, J. Stat. Phys., 168 (2017), 171–207.

[25] S.-Y. Ha, D. Ko and S. W. Ryoo, On the relaxation dynamics of Lohe oscillators on some Riemannian manifolds, J. Stat. Phys., 172 (2018), 1427–1478.

[26] S.-Y. Ha, D. Ko and Y. L. Zhang, Remarks on the critical coupling strength for the Cucker-Smale model with unit speed, Discrete Contin. Dyn. Syst., 38 (2018), 2763–2793.

[27] S.-Y. Ha, J. Kim and X. T. Zhang, Uniform stability of the Cucker-Smale model and its application to the mean-field limit, Kinet. Relat. Mod., 11 (2018), 1157–1181.

[28] S.-Y. Ha, Z. C. Li and X. P. Xue, Formation of phase-locked states in a population of locally interacting Kuramoto oscillators, J. Differential Equations, 255 (2013), 3053–3070.

[29] S.-Y. Ha and S. W. Ryoo On the emergence and orbital stability of phase-locked states for the Lohe model, J. Stat. Phys., 163 (2016), 411–439.

[30] Y. Kuramoto, Chemical Oscillations, Waves and Turbulence. Springer Series in Synergetics, 19. Springer-Verlag, Berlin, 1984.

[31] Y. Kuramoto, Self-entrainment of population of coupled non-linear oscillators, International Symposium on Mathematical Problems in Theoretical Physics, Lecture Notes in Phys., Springer, Berlin, 39 (1975), 420–422.

[32] M. A. Lohe, Non-Abelian Kuramoto model and synchronization, J. Phys. A: Math. Theor., 42 (2009), 395101, 25 pp.

[33] R. Olfati-Saber, Swarms on sphere: A programmable swarm with synchronous behaviors like oscillator networks, Proc. of the 45th IEEE conference on Decision and Control, (2006), 5060–5066.

[34] A. Pikovsky, M. Rosenblum and J. Kurths, Synchronization: A Universal Concept in Nonlinear Sciences, Cambridge Nonlinear Science Series, 12. Cambridge University Press, Cambridge, 2001.

[35] A. Sarlette, R. Sepulchre and N. E. Leonard, Autonomous rigid body attitude synchronization, Automatica 45 (2009), 572–577.

[36] S. H. Strogatz, From Kuramoto to Crawford: Exploring the onset of synchronization in populations of coupled oscillators, Physica D., 143 (2000), 1–20.

[37] R. Tron, B. Afsari and R. Vidal, Intrinsic consensus on SO(3) with almost-global convergence, 2012 IEEE Conference on Decision and Control, (2012), 2052–2058.

[38] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen and O. Shochet, Novel type of phase transition in a system of self-driven particles, Phys. Rev. Lett., 75 (1995), 1226–1229.