Geometrical Conditions for the Existence of a Milnor Vector Field

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Received: 19 February 2020 / Accepted: 7 September 2020 / Published online: 17 September 2020
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Abstract
We introduce several sufficient conditions to guarantee the existence of the Milnor vector field for new classes of singularities of map germs. This special vector field is related with the equivalence problem of the Milnor fibrations for real and complex singularities, if they exit.

Keywords Singularities of real analytic maps · Milnor fibrations · Mixed singularities · Stratification theory · Topology of subanalytic sets

Mathematics Subject Classification 32S55 · 58K15 · 57Q45 · 32C40 · 32S60 · 32B20 · 14D06 · 58K05 · 57R45 · 14P10 · 32S20

1 Introduction

For the holomorphic functions germs $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with $\dim \text{Sing } f \geq 0$, Milnor showed that for any small enough $\epsilon > 0$ the restriction map

$$f / |f| : S^{2n+1}_\epsilon \setminus K_\epsilon \to S^1_1$$

is a locally trivial smooth fibration, where $K_\epsilon := S^{2n+1}_\epsilon \cap f^{-1}(0)$.

It was proved later by Dũng Tráng (1977) that for any small enough $\epsilon > 0$, there exists $0 < \delta \ll \epsilon$, such that the restriction map $f| : B^{2n+2}_\epsilon \cap f^{-1}(D_\delta \setminus \{0\}) \to D_\delta \setminus \{0\}$
is the projection of a locally trivial smooth fibration, where $B_{k}^{2n+2}$ and $D_{\delta}$ are the open balls in the respective spaces. It is well known that the previous fibration induces the smooth fibration

$$f / |f| : B_{k}^{2n+2} \cap f^{-1} \left( S_{\eta}^{1} \right) \to S_{1}^{1}. \quad (2)$$

Moreover, it follows from the Milnor (1968) and from Lê works (Dũng Tráng 1977) that the fibrations (1) and (2) are equivalent.

In the case of real analytic map germ $G : (\mathbb{R}^{m}, 0) \to (\mathbb{R}^{p}, 0)$, $m > p \geq 2$ with $\text{Sing} \ G \subset G^{-1}(0)$ as a set germ, the authors of Massey (2010) and Parameswaran and Tibăr (2018) gave several conditions that ensure the existence of the empty tube fibration

$$G / \|G\| : B_{k}^{m} \cap G^{-1} \left( S_{\eta}^{p-1} \right) \to S_{1}^{p-1}. \quad (3)$$

Under special conditions, in the papers (dos Santos et al. 2013; dos Santos and Tibăr 2008, 2010; Cisneros-Molina et al. 2010; Pichon 2005; Pichon and Seade 2003, 2008) it was proved the existence of the sphere fibration

$$G / \|G\| : S_{k}^{m-1} \setminus K_{k} \to S_{1}^{p-1}. \quad (4)$$

More recently, in dos Santos et al. (2019, 2020), the authors extended all previous results for the case when the singular set $\text{Sing} \ G$ has positive dimension and is not necessarily included in the central fibre $G^{-1}(0)$, i.e., when $\text{Disc} \ G$ has positive dimension. However, even in the case where $\text{Sing} \ G = 0$ it is not known whether or not these two fibrations are equivalent.

This equivalence problem has been approached by several authors in the last years, first in the case $\text{Disc} \ G = \{0\}$, e.g., dos Santos (2012), dos Santos and Tibăr (2008, 2010), Cisneros-Molina et al. (2010), Hansen (2014) and Oka (2010), and more recently in dos Santos et al. (2020) in the more general case where $\text{Disc} \ G$ has positive dimension.

In the paper (dos Santos et al. 2020), this problem was formulated in general as a conjecture, see Conjecture 1, Sect. 3. Moreover, the authors showed the relationship between the equivalence problem and the existence of the so called Milnor vector field. Beside that, it was also shown that under a topological condition the so called Milnor set (see (dos Santos et al. 2020, [Corollary 4.11])) this conjecture is solved.

The aim of this paper is to introduced several new sufficient conditions to ensure the existence of the Milnor vector field and hence to ensure the equivalence between the tube and sphere fibrations. Moreover, we present plenty of new classes of real and complex singularities satisfying our conditions.

The paper is organized as follows. In Sect. 2, we remind the main results about the existence of the local tube and sphere fibration structures in the general setting as proved in dos Santos et al. (2019) and dos Santos et al. (2020). In Sect. 3, we introduce the Milnor vector field as previously defined in dos Santos et al. (2020) and connect it with the equivalence problem. In Sect. 4, we introduce new criteria which ensure the existence of the Milnor vector field. In last section, as application, we introduce new classes of real and complex singularities for which the Conjecture 1 holds true.
Among them, we point out the *mixed simple L-maps* which are a special type of mixed singularities.

## 2 Fibrations Structures

As explained in the paper (dos Santos et al. 2019), given a non-constant analytic map germ $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, $m \geq p > 0$, the set $\text{Sing } G$ is well-defined as a germ of set on the source space. However, in general, in the target space the germ of sets $G(\text{Sing } G)$ and $\text{Im } G$ do not. We make it clear in the next definition.

**Definition 1** (dos Santos et al. 2019) Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, $m \geq p > 0$, be a continuous map germ. We say that the image $G(K)$ of a set $K \subset \mathbb{R}^m$ containing 0 is a well-defined set germ at $0 \in \mathbb{R}^p$ if for any open balls $B_\varepsilon, B_\varepsilon'$ centred at 0, with $\varepsilon, \varepsilon' > 0$, we have the equality of germs $[G(B_\varepsilon \cap K)]_0 = [G(B_\varepsilon' \cap K)]_0$.

Whenever the images $\text{Im } G$ and $G(\text{Sing } G)$ are well-defined as germs, we say that $G$ is a *nice map germ*.

In dos Santos et al. (2019) the authors found sufficient conditions to an analytic map germ to be a nice germ and have introduced a good class of maps with this property, namely the map germ of type $\bar{f} \bar{g} : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ where $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ are holomorphic germs such that the meromorphic function $f/g$ is irreducible. Moreover, they considered the *discriminant* of a nice map germ $G$, $\text{Disc } G$, as the locus where the topology of the fibers may change, which is a closed subanalytic germ of dimension strictly less than $p$.

### 2.1 Existence of the Tube Fibration

**Definition 2** (dos Santos et al. 2019) Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, $m \geq p > 0$, be a non-constant nice analytic map germ. We say that $G$ has *Milnor-Hamm tube fibration* if, for any $\varepsilon > 0$ small enough, there exists $0 < \eta \ll \varepsilon$ such that the restriction:

$$G| : B^m_\varepsilon \cap G^{-1}(B^p_\eta \setminus \text{Disc } G) \to B^p_\eta \setminus \text{Disc } G \quad (3)$$

is a locally trivial fibration over each connected component $C_i \subset B^p_\eta \setminus \text{Disc } G$, such that it is independent of the choices of $\varepsilon$ and $\eta$ up to diffeomorphisms.

Let $U \subset \mathbb{R}^m$ be an open set, $0 \in U$, and let $\rho : U \to \mathbb{R}_{\geq 0}$ be a non-negative proper function which defines the origin, for instance the square of the Euclidean distance.

We consider here the following definition:

**Definition 3** Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ be a non-constant nice analytic map germ. The set germ at the origin:

$$M_\rho(G) := \{x \in U \mid \rho \not\equiv x \}$$

is called the set of $\rho$-nonregular points of $G$, or the *Milnor set of $G$*. 
In this paper we will consider only the Euclidean distance function $\rho_E$, and we write for short $M(G) := M_{\rho_E}(G)$.

The following condition was used in dos Santos et al. (2019) and (dos Santos et al. 2020) to ensure the existence of the Milnor-Hamm fibrations.

$$M(G) \setminus G^{-1}(\text{Disc } G) \cap V_G \subseteq \{0\}. \quad (4)$$

**Proposition 1** (dos Santos et al. 2019, [Lemma 3.3]) Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ be a non-constant nice analytic map germ, $m \geq p > 0$. If $G$ satisfies condition (4), then $G$ has a Milnor-Hamm tube fibration (3).

**Corollary 1** (Existence of the Milnor-Hamm empty tube fibration) Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ be a non-constant nice analytic map germ. If $G$ satisfies condition (4), then for all small enough $\varepsilon > 0$, there exists $0 < \eta \ll \varepsilon$ such that the restriction:

$$G| : B_{\varepsilon}^m \cap G^{-1}(S_{\eta}^{p-1} \setminus \text{Disc } G) \to S_{\eta}^{p-1} \setminus \text{Disc } G \quad (5)$$

is a locally trivial smooth fibration.

### 2.2 Existence of the Sphere Fibration

Several authors have worked on the problem of fibration over spheres in the real settings, for isolated and non-isolated singularities, e.g. dos Santos (2008), dos Santos et al. (2013), dos Santos and Tibăr (2008, 2010), Cisneros-Molina et al. (2010), Pichon (2005), Pichon and Seade (2003), Pichon and Seade (2008), Ruas et al. (2002), Ruas and dos Santos (2005), Seade (1997). In the recent paper (dos Santos et al. 2020) the authors generalized all previous results as we describe below.

**Definition 4** (dos Santos et al. 2020) Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ be a nice analytic map germ. We say that its discriminant $\text{Disc } G$ is radial if, as a set germ at the origin, it is a union of real half-lines or just the origin.

Let $G : U \to \mathbb{R}^p$ be a representative of the nice map germ $G$ for some open set $U \ni 0$. We consider the map:

$$\Psi_G := \frac{G}{\|G\|} : U \setminus V_G \to S^{p-1}. \quad (6)$$

As we have seen in dos Santos et al. (2020), if $\text{Disc } G$ is radial then the restriction map

$$\Psi_G| : S_{\varepsilon}^{m-1} \setminus G^{-1}(\text{Disc } G) \to S_{\eta}^{p-1} \setminus \text{Disc } G \quad (7)$$

is well defined for small enough $\varepsilon > 0$, in this case we say that $G$ has a Milnor-Hamm sphere fibration whenever (7) is a locally trivial smooth fibration which is independent, up to diffeomorphisms, of the choice of $\varepsilon$ provided it is small enough.

We consider the Milnor set $M(\Psi_G)$ of the map (6), i.e. the germ at the origin of the $\rho$-nonregular points of $\Psi_G$. We say that $\Psi_G$ is $\rho$-regular if:

$$M(\Psi_G) \setminus G^{-1}(\text{Disc } G) = \emptyset. \quad (8)$$
Theorem 1 (dos Santos et al. 2020, [Theorem 3.5]) Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, $m > p \geq 2$, be non-constant nice analytic map germ with radial discriminant, satisfying the condition (4). If $\Psi_G$ is $\rho$-regular then $G$ has a Milnor-Hamm sphere fibration.

In the case where Disc $G = \{0\}$ the condition (4) becomes

$$M(G) \setminus V_G \cap V_G \subseteq \{0\}$$

which has been used in dos Santos et al. (2013), dos Santos and Tibăr (2010) and Massey (2010) in order to guarantee the existence of the tube and sphere fibrations. It is also important to point out that under the conditions (4) and Disc $G = \{0\}$ the map germ $G$ is nice, see (Massey 2010,[Corollary 4.7]).

2.3 Fibrations Equivalence Problem

As has been seen in dos Santos et al. (2019), if the discriminant set Disc $G$ has positive dimension it intersects all spheres of small enough radius in the target space. However, the condition (4) ensure that the restriction (5) is also the projection of a locally trivial smooth fibration.

If Disc $G$ is radial, the fibration (5) may be composed with the canonical projection $\pi := s/\|s\| : \mathbb{R}^p \setminus \{0\} \to S^{p-1}$ to get a locally trivial smooth fibration

$$\Psi_{G^1} : B^m \cap G^{-1}(S^p_{\rho} \setminus \text{Disc } G) \to S^{p-1} \setminus \text{Disc } G. \quad (9)$$

Moreover, under additional hypothesis of $\rho$-regularity for $\Psi_G$, the Theorem 1 ensure that the restriction (7) is a Milnor–Hamm sphere fibration. Therefore, the following question becomes natural:

When are the fibrations (7)and (9)equivalent?

In order to study the equivalence of fibrations in this setting, the authors in dos Santos et al. (2020) have adapted the Milnor’s method of “blowing away the tube to the sphere” introduced in Milnor (1968), which uses a special vector field as we remind below.

3 The Milnor Vector Field

Definition 5 (dos Santos et al. 2020) Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, $m > p \geq 2$ be an analytic map germ with radial discriminant. One calls Milnor vector field, abbreviated MVF, a vector field $\nu$ which satisfies the following conditions for any $x \in B^m \setminus G^{-1}(\text{Disc } G)$:

(c1) $\nu(x)$ is tangent to the fiber $X_y = \Psi^{-1}_G(y)$, where $y = \Psi_G(x)$,
(c2) $\langle \nu(x), \nabla \rho(x) \rangle > 0$,
(c3) $\langle \nu(x), \nabla \|G(x)\|^2 \rangle > 0$. 

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Remark 1 The existence of a MVF does not insure the existence of the fibrations (7) or (9). Actually, the Milnor–Hamm fibration may not even be well-defined as shown by the Hansen’s example $G(x, y, z) = (x^2 + y^2, (x^2 + y^2)z)$ which is not nice.

As pointed out in dos Santos et al. (2020), the existence of MVF guarantees the equivalence between the Milnor-Hamm fibrations provided that they exist, as shows the next result.

Theorem 2 (dos Santos et al. 2020, [Theorem 4.2]) Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, $m > p \geq 2$ be an analytic nice map germ with radial discriminant, such that the Milnor-Hamm tube and sphere fibrations exist. If a Milnor vector field exists, then these fibrations are equivalent.

It is important notice, however, that in the real setting the existence of MVF is not insured in general, like the complex case. This existence problem has been approached by several authors in the last years, first in the case Disc $G = \{0\}$ e.g., dos Santos (2012), dos Santos and Tibăr (2008, 2010), Cisneros-Molina et al. (2010), Hansen (2014), Oka (2010), and more recently in dos Santos et al. (2020). In each setting, the authors produced sufficient conditions. However, up to now, there is no clear proof for this problem in its complete generality, and no counterexample has been found yet.\footnote{As have been point out in dos Santos et al. (2020), the existence proof attempt in Cisneros-Molina et al. (2010) for case Disc $G = \{0\}$ appears to contain a non-removable gap. For more details, see (Ribeiro 2018, Chapter7)}

The following conjecture explains what we mean by “complete generality”:

Conjecture 1 (dos Santos et al. 2020) Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, $m > p \geq 2$ be an analytic nice map germ with radial discriminant and such that $\Psi_G$ is $\rho$-regular. If both fibrations (7) and (9) exist for any small enough $\varepsilon > 0$ and $0 < \eta \ll \varepsilon$, then they are equivalent.

For the sake of simplicity, given $x \in B^m_\varepsilon \setminus V_G$ we will assume that $x$ belongs to the open set $\{G_1(x) \neq 0\}$. All next results do not depend on the particular choice of the open set. See Dutertre et al. (2016, Lemma 3.3 and Remark 3.4) for more details.

It is well known that $\text{Sing } G \subset M(G)$ and $M(\Psi_G) \subset M(G) \setminus V_G$. Moreover, we notice that, in the open set $\{G_1(x) \neq 0\}$ one can express the condition $x \in M(G) \setminus G^{-1}(\text{Disc } G)$ by

$$\nabla \rho(x) = a(x)\nabla\|G(x)\|^2 + \sum_{k=2}^p a_k(x)\Omega_k(x), \quad (10)$$

where $\Omega_k = G_1 \nabla G_k - G_k \nabla G_1$, for $k = 2, \ldots, p$ are the generators of the normal space $T_xX_y$ in $\mathbb{R}^m$, with $T_xX_y$ the tangent space of the fiber $X_y$, over a regular value $y = \Psi_G(x)$. Therefore, one has that for any $x \in M(G) \setminus G^{-1}(\text{Disc } G)$, $a(x) = 0$ if and only if $x \in M(\Psi_G) \setminus G^{-1}(\text{Disc } G)$. This shows the importance of coefficient $a(x)$ for study of $\rho$-regularity of $\Psi_G$. On the other hand, the next theorem characterize the existence of a MVF for $G$ in terms of $a(x)$.

Theorem 3 (dos Santos et al. 2020, [Existence of MFV]) Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, $m \geq p \geq 2$, be an analytic map germ. Then, there exists a small positive $\varepsilon$ so that $G$ is defined on $B^m_\varepsilon$ and the following are equivalent:
(i) there exists a MVF for $G$ on $B^m _\varepsilon \setminus G^{-1}(\text{Disc } G)$;
(ii) $a(x) > 0$, for any $x \in (M(G) \setminus G^{-1}(\text{Disc } G)) \cap B^m _\varepsilon$.

In order to prove Theorem 3, the author have considered the following vector fields on $B^m _\varepsilon \setminus G^{-1}(\text{Disc } G)$:

$$v_1 (x) := \text{proj}_{T_x \chi}(\nabla \|G(x)\|^2) \text{ and } v_2 (x) := \text{proj}_{T_x \chi}(\nabla \rho(x)).$$

They have noted that vector field $v_1$ has no zeros since the tube $\|G(x)\|^2 = \text{const}.$ is transversal to $\chi_y$, for $y \notin \text{Disc } G$ and the vector field $v_2$ has no zeros on $B^m _\varepsilon \setminus G^{-1}(\text{Disc } G)$ since $a(x) > 0$ for any $x \in M(G) \setminus G^{-1}(\text{Disc } G)$, i.e., $\Psi_G$ is $\rho$-regular. Consequently, the vector field

$$v(x) = \frac{v_1 (x)}{\|v_1 (x)\|} + \frac{v_2 (x)}{\|v_2 (x)\|} \quad (11)$$

is well-defined on $B^m _\varepsilon \setminus G^{-1}(\text{Disc } G)$ and has no zero on $M(G) \setminus G^{-1}(\text{Disc } G)$. Moreover, the vectors $v_1 (x)$ and $v_2 (x)$ are linearly independent if, and only if $x \notin M(G) \cup G^{-1}(\text{Disc } G)$, (for more details, see (dos Santos et al. 2020)) hence, $v$ has no singularity on $B^m _\varepsilon \setminus G^{-1}(\text{Disc } G)$. Finally, they get that the conditions $(c_1) - (c_3)$ are satisfied. Hence, $v$ is a MVF for $G$ on $B^m _\varepsilon \setminus G^{-1}(\text{Disc } G)$.

Next section will we introduce several sufficient conditions in order to guarantee the existence of such a vector field.

4 Criteria for MVF Existence

Motivated by Theorem 3 in this section we give some descriptions of $a(x)$ to ensure the existence of a MVF for analytic maps $G$. As already pointed out the results does not depend on the particular choice of open set $\{G_1(x) \neq 0\}$. See Dutertre et al. (2016) for details. From now on when needed we will be working on this open set without explicit mention to it.

4.1 A Matricial Criterion

Here we give a matricial criterion for deciding when exists a MVF for a map germ $G$.

Lemma 1 Let $G : (\mathbb{R}^m, 0) \rightarrow (\mathbb{R}^p, 0)$ be smooth map germ. For any $x \in M(G) \setminus (V_G \cup \text{Sing } G)$ one has that:

$$a(x) = \frac{\det D(x)}{\det M(x)} \quad (12)$$

where

$$D(x) = \begin{bmatrix}
\langle \nabla \rho(x), \nabla \|G(x)\|^2 \rangle & \langle \Omega_2(x), \nabla \|G(x)\|^2 \rangle & \cdots & \langle \Omega_p(x), \nabla \|G(x)\|^2 \rangle \\
\langle \nabla \rho(x), \Omega_2(x) \rangle & \langle \Omega_2(x), \Omega_2(x) \rangle & \cdots & \langle \Omega_p(x), \Omega_2(x) \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle \nabla \rho(x), \Omega_p(x) \rangle & \langle \Omega_2(x), \Omega_p(x) \rangle & \cdots & \langle \Omega_p(x), \Omega_p(x) \rangle 
\end{bmatrix}$$
and,

\[
M(x) = \begin{bmatrix}
\langle \nabla \| G(x) \|_2^2, \nabla \| G(x) \|_2^2 \rangle & \langle \Omega_2(x), \nabla \| G(x) \|_2^2 \rangle & \cdots & \langle \Omega_p(x), \nabla \| G(x) \|_2^2 \rangle \\
\langle \nabla \| G(x) \|_2^2, \Omega_2(x) \rangle & \langle \Omega_2(x), \Omega_2(x) \rangle & \cdots & \langle \Omega_p(x), \Omega_2(x) \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle \nabla \| G(x) \|_2^2, \Omega_p(x) \rangle & \langle \Omega_2(x), \Omega_p(x) \rangle & \cdots & \langle \Omega_p(x), \Omega_p(x) \rangle
\end{bmatrix}.
\]

**Proof** Let \( x \in M(G) \setminus (V_G \cup \text{Sing } G) \). From Eq. (10) one gets the matrix equation

\[
T(x) = M(x) \cdot L(x),
\]

where

\[
T(x) = \begin{bmatrix}
\langle \nabla \rho(x), \nabla \| G(x) \|_2^2 \rangle \\
\langle \nabla \rho(x), \Omega_2(x) \rangle \\
\vdots \\
\langle \nabla \rho(x), \Omega_p(x) \rangle
\end{bmatrix},
\]

\[
M(x) = \begin{bmatrix}
\langle \nabla \| G(x) \|_2^2, \nabla \| G(x) \|_2^2 \rangle & \langle \Omega_2(x), \nabla \| G(x) \|_2^2 \rangle & \cdots & \langle \Omega_p(x), \nabla \| G(x) \|_2^2 \rangle \\
\langle \nabla \| G(x) \|_2^2, \Omega_2(x) \rangle & \langle \Omega_2(x), \Omega_2(x) \rangle & \cdots & \langle \Omega_p(x), \Omega_2(x) \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle \nabla \| G(x) \|_2^2, \Omega_p(x) \rangle & \langle \Omega_2(x), \Omega_p(x) \rangle & \cdots & \langle \Omega_p(x), \Omega_p(x) \rangle
\end{bmatrix},
\]

\[
L(x) = \begin{bmatrix}
a(x) \\
\alpha_2(x) \\
\vdots \\
\alpha_p(x)
\end{bmatrix}.
\]

The matrix \( M(x) \) is non-singular because \( x \notin (V_G \cup \text{Sing } G) \). Moreover by Lagrange’s identity its determinant is given by the sum of the squares of all \( p \times p \) sub-determinants of the matrix

\[
\begin{bmatrix}
\nabla \| G(x) \|_2^2 \\
\Omega_2(x) \\
\vdots \\
\Omega_p(x)
\end{bmatrix}.
\]

Hence, \( \det M(x) > 0 \) and one can write \( L(x) = M(x)^{-1} T(x) \). By Cramer’s rule the coefficient \( a(x) \) can be written as

\[
a(x) = \frac{\det D(x)}{\det M(x)}.
\]
**Remark 2** It follows from the previous proof that the rank of the matrix $D(x)$ can be understood as the rank of two matrices $A(x)$ and $B(x)$ below considering $D(x) = A(x) \cdot B(x)$:

$$A(x) = \begin{bmatrix} \nabla \|G\|^2 \\ \Omega_2 \\ \vdots \\ \Omega_p \end{bmatrix} \quad \text{and} \quad B(x) = \begin{bmatrix} \nabla \rho \Omega_2 \cdots \Omega_p \end{bmatrix}_{m \times p}.$$

Hence, the rank of the matrix $D(x)$ is maximal if, and only if the ranks of the matrices $A(x)$ and $B(x)$ are maximal. Nevertheless, under the condition $x \notin (V_G \cup \text{Sing } G)$ one has that the rank of $D(x)$ is maximal if, and only if the rank of $B(x)$ is maximal. In the case of $\text{Disc } G = \{0\}$ it amounts to saying that $x \notin M(\Psi_G)$.

**Theorem 4** Let $G : (G_1, \ldots, G_p) : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ be an analytic map germ. If $\det D(x) > 0$, then $a(x) > 0$ for all $x \in M(G) \setminus (V_G \cup \text{Sing } G)$. In particular, if $\text{Disc } G = \{0\}$, then there exists a MVF for $G$ on $B^m_\varepsilon \setminus V_G$ and $\Psi_G$ is $\rho$-regular.

**Proof** The proof follows from Lemma 1 and Theorem 3. □

**Remark 3** In Massey (2010) the author addressed the case $p = 2$ and considered $G$ satisfying the “Milnor condition (c)” at the origin if

$$\|\omega(x)\|^2 \langle \nabla \rho(x), \nabla \|G(x)\|^2 \rangle - \langle \omega(x), \nabla \|G(x)\|^2 \rangle \cdot \langle \omega(x), \nabla \rho(x) \rangle > 0,$$

where $\omega(x) := \Omega_2(x)$. Using our notations it is equivalent to say that $\det D(x) > 0$. Thus, our condition “$\det D(x) > 0$” can be thought as a kind of generalization of the Milnor condition (c) at the origin.

The result below shows that the Milnor tube expands in the radial direction. Hence, since $\text{Disc } G = \{0\}$, in order to inflate the smooth Milnor tube to the sphere $^2$ one can replace the Milnor vector field by the direction $\nabla \|G(x)\|^2$.

**Proposition 2** Let $G = (G_1, \ldots, G_p) : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ be an analytic map germ, with $m > p \geq 2$. Then for $\varepsilon$ sufficiently small and $x \in B^m_\varepsilon \setminus (V_G \cup \text{Sing } G)$ one has that $\langle \nabla \|G(x)\|^2, \nabla \rho(x) \rangle > 0$.

**Proof** The proof can safely be left as an exercise to the reader. It also can be found in the proof of Theorem 1’ by Fukuda (1985). □

As an application of the matricial criterion one has:

**Corollary 2** Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, $m > p \geq 2$ be an analytic map germ. Suppose that for any $x \in M(G) \setminus (V_G \cup \text{Sing } G)$ one has either:

1. $\langle \nabla \|G(x)\|^2, \Omega_j(x) \rangle = 0$ for all $2 \leq j \leq p$, or

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$^2$ as was made by Milnor in Milnor (1968),
2. \( \{ \nabla \rho(x), \Omega_j(x) \} = 0 \) for all \( 2 \leq j \leq p \).

Then \( a(x) > 0 \) for \( x \in M(G) \setminus (V_G \cup \text{Sing } G) \). In particular, whenever \( \text{Disc } G = \emptyset \), there exists a MVF for \( G \) on \( B_\epsilon^m \setminus V_G \) and \( \Psi_G \) is \( \rho \)-regular.

**Proof** In fact, assume that \( \{ \nabla \|G(x)\|^2, \Omega_j(x) \} = 0 \) for all \( 2 \leq j \leq p \). It follows from Proposition 2 that \( \det D(x) = \langle \nabla \rho(x), \nabla \|G(x)\|^2 \rangle \cdot \det \left( \langle \Omega_i(x), \Omega_j(x) \rangle_{i,j} \right) > 0 \). The case \( \{ \nabla \rho(x), \Omega_j(x) \} = 0 \) for all \( 2 \leq j \leq p \), follows from the same argument. \( \square \)

### 4.2 Other Criteria for the Existence of MVF

Let us consider the action on \( \mathbb{R}^m : t \cdot x := (tx_1, \ldots, tx_m) \) for \( t \in \mathbb{R}_{\geq 0} \). We say that \( G = (G_1, \ldots, G_p) : \mathbb{R}^m \rightarrow \mathbb{R}^p \) is homogeneous of degree \( d > 0 \), if \( G(t \cdot x) = t^d G(x) \).

Our main result in this section is the following.

**Theorem 5** Let \( G := (G_1, \ldots, G_p) : (\mathbb{R}^m, 0) \rightarrow (\mathbb{R}^p, 0) \) be an analytic map germ. Assume that there exists \( \epsilon > 0 \) small enough such that inside the open ball \( B_\epsilon^m \) the map \( G \) satisfies some of conditions below:

(i) \( G \) is homogeneous of degree \( d > 0 \);

(ii) \( \{ \nabla G_i(x), \nabla G_j(x) \} = 0 \), for any \( i, j = 1, \ldots, p \) with \( i \neq j \) and \( x \in (M(G) \setminus (V_G \cup \text{Sing } G)) \cap B_\epsilon^m \).

Then for any \( x \in (M(G) \setminus (V_G \cup \text{Sing } G)) \cap B_\epsilon^m \) the coefficient \( a(x) \) is positive. In particular, if \( \text{Disc } G = \emptyset \) then there exists a MVF for \( G \) and \( \Psi_G \) is \( \rho \)-regular.

**Example 1** Let \( G(x, y, z) = (xy, xz) \). One can show that \( G \) satisfies the condition (4) and \( \text{Disc } G = \emptyset \). Since \( G \) satisfies Theorem 5 (i) above, there exists a MVF for \( G \) and \( \Psi_G \) is \( \rho \)-regular. Therefore, \( G \) has the tube and sphere fibrations and they are equivalent.

For the proof of Theorem 5, one needs the matricial criterion and the Lemmas 2 and 3 stated below. Since we are working with germs of maps and sets in both statements of the Lemmas 2 and 3 we are concerned with the behavior of the representatives of the germs inside an open ball \( B_\epsilon^m \) for \( \epsilon > 0 \) small enough. However, for the sake of simplicity we will omit that.

**Lemma 2** Let \( G(x) = (G_1(x), \ldots, G_p(x)) \) be an analytic map germ. For \( x \in M(G) \setminus (V_G \cup \text{Sing } G) \) one has

\[
a(x) = \frac{\langle \alpha(x), G(x) \rangle}{\|G(x)\|^2},
\]

where \( \alpha(x) = (\alpha_1(x), \ldots, \alpha_p(x)) \in \mathbb{R}^p \) is such that \( \nabla \rho(x) = \Sigma_{k=1}^p \alpha_k(x) \nabla G_k(x) \). In particular, \( x \in M(\Psi_G) \setminus (V_G \cup \text{Sing } G) \) if, and only if, \( \langle \alpha(x), G(x) \rangle = 0 \).

**Proof** It follows from Proposition 1 that, for any \( x \in M(G) \setminus (V_G \cup \text{Sing } G) \) the coefficient

\[
a(x) = \frac{\det[A(x) \cdot B(x)]}{\det M(x)}
\]
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where the matrices $A(x)$ and $B(x)$ are as in the Remark 2 and $M(x) = A(x)^t$. Since $x \notin V_G \cup \text{Sing } G$, one has from Remark 2 that $A(x)$ and $M(x)$ has maximal rank.

On the open set $\{G_1(x) \neq 0\}$ one can rewrite $A(x) \cdot B(x) = L_1(x) \cdot L_2(x) \cdot L_3(x)$ and $M(x) = L_1(x) \cdot L_2(x) \cdot L_1(x)^t$, for the matrices

$$L_1(x) = \begin{bmatrix} G_1(x) & G_2(x) & \cdots & G_p(x) \\ -G_2(x) & G_1(x) & & 0 \\ & \vdots & \ddots & \vdots \\ -G_p(x) & 0 & \cdots & G_1(x) \end{bmatrix},$$

$$L_2(x) = \begin{bmatrix} \nabla G_1(x) \\ \nabla G_2(x) \\ \vdots \\ \nabla G_p(x) \end{bmatrix} \begin{bmatrix} \nabla G_1(x) & \nabla G_2(x) & \cdots & \nabla G_p(x) \end{bmatrix},$$

the Grassmann matrix of $JG(x)$, and

$$L_3(x) = \begin{bmatrix} \alpha_1(x) & -G_2(x) & \cdots & -G_p(x) \\ \alpha_2(x) & G_1(x) & & 0 \\ & \vdots & \ddots & \vdots \\ \alpha_p(x) & 0 & \cdots & G_1(x) \end{bmatrix},$$

where in the matrices $L_1(x)$ and $L_3(x)$ the $(p-1) \times (p-1)$ submatrices obtained by deleting the first column and the first row are diagonal.

Thus, one has that

$$a(x) = \frac{\det L_3(x)}{\det L_1(x)}.$$

On the other hand, on the open set $\{G_1 \neq 0\}$ one has that $\det L_1(x) = (G_1(x))^{p-2} \|G(x)\|^2$. Analogously, applying Laplace’s rule in the first column $\det L_3(x) = (G_1(x))^{p-2} \sum_{k=1}^p \alpha_k(x)G_k(x)$. Therefore,

$$a(x) = \frac{\sum_{k=1}^p \alpha_k(x)G_k(x)}{\|G(x)\|^2} = \frac{\langle a(x), G(x) \rangle}{\|G(x)\|^2}$$

and we finish the proof. \hfill \Box

Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ be a analytic map germ with $G(x) = (G_1(x), \ldots, G_p(x))$. One can write

$$\begin{align*}
G_1(x) &= G_{m_1}^1(x) + G_{m_1+1}^1(x) + \cdots \\
& \vdots \\
G_p(x) &= G_{m_p}^p(x) + G_{m_p+1}^p(x) + \cdots
\end{align*} \quad (13)$$

\[ Springer \]
where $G^i_{m_j}$ is the homogeneous term of degree $m_j$, for $i, j = 1, \ldots, p$. Next, let us consider the Grassmann matrix of $JG(x)$, namely, the $p \times p$ square matrix $JG(x) \cdot JG(x)^t$ that under the condition $x \notin V_G \cup \text{Sing} \ G$ is an invertible matrix. Let us define $K(x) := [JG(x) \cdot JG(x)^t]^{-1}$.

**Lemma 3** Let $G(x) = (G_1(x), \ldots, G_p(x))$ be an analytic map germ. In the set $M(G) \setminus (V_G \cup \text{Sing} \ G)$, one has that $\langle \alpha(x), G(x) \rangle = \langle G(x) \cdot [D(m_i) \cdot K(x)], G(x) \rangle + \langle V(x) \cdot K(x), G(x) \rangle$, where, $V(x) = \left(\sum_{j=1}^{\infty} jG^1_{m_1+j}(x), \ldots, \sum_{j=1}^{\infty} jG^p_{m_p+j}(x)\right)$ and

$$D(m_i) = \begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_p \end{bmatrix}.$$  

**Proof** Since $x \in M(G) \setminus (V_G \cup \text{Sing} \ G)$, by definition there exist $\alpha_1(x), \ldots, \alpha_p(x) \in \mathbb{R}$ such that

$$\nabla \rho(x) = \sum_{j=1}^{p} \alpha_j(x) \nabla G_j(x).$$

Let us define $\alpha(x) := (\alpha_1(x), \ldots, \alpha_p(x))$. After changing $\alpha(x)$ by $\frac{\alpha(x)}{2}$, one can consider the matrix equation $[x] = [\alpha(x)] \cdot [JG(x)]$. Now, multiplying both side of the matrix equation by $[JG(x)^t]$ one gets:

$$[x] \cdot [JG(x)^t] = [\alpha(x)] \cdot [JG(x) \cdot JG(x)^t]. \quad (14)$$

As we have seen, the $p \times p$ square matrix $JG(x) \cdot JG(x)^t$ is invertible under the condition $x \notin V_G \cup \text{Sing} \ G$. So, after multiplying both side of equality (14) by its inverse $K(x)$ one has the following equation $[\alpha(x)] = [x] \cdot [JG(x)^t] \cdot K(x)$. Now, one can write the scalar product (dot product) as

$$\langle \alpha(x), G(x) \rangle = \langle [x] \cdot [JG(x)^t] \cdot K(x), G(x) \rangle. \quad (15)$$

It follows from equations in (13) that:

$$\nabla G_1(x) = \nabla G^1_{m_1}(x) + \nabla G^1_{m_1+1}(x) + \cdots$$

$$\vdots$$

$$\nabla G_p(x) = \nabla G^p_{m_p}(x) + \nabla G^p_{m_p+1}(x) + \cdots.$$  

Then by Euler’s identity one finds

$$\left\langle \nabla G^i_k(x), x \right\rangle = kG^i_k(x).$$
Hence,

\[ \langle \nabla G_i(x), x \rangle = \sum_{j=0}^{\infty} \langle \nabla G_{m_i+j}^i(x), x \rangle \]

\[ = \sum_{j=0}^{\infty} (m_i + j)G_{m_i+j}^i(x) = m_i G_i(x) + \sum_{j=1}^{\infty} j G_{m_i+j}^i(x) \]

and one can decompose

\[ (\langle \nabla G_1(x), x \rangle, \ldots, \langle \nabla G_p(x), x \rangle) = (m_1 G_1(x), \ldots, m_p G_p(x)) \]

\[ + \left( \sum_{j=1}^{\infty} j G_{m_1+j}^1(x), \ldots, \sum_{j=1}^{\infty} j G_{m_p+j}^p(x) \right) \]

(16)

After denote the vector expression \( \left( \sum_{j=1}^{\infty} j G_{m_1+j}^1(x), \ldots, \sum_{j=1}^{\infty} j G_{m_p+j}^p(x) \right) \) by \( V(x) \), it follows from Eq. (15) that

\[ \langle \alpha(x), G(x) \rangle = \langle (m_1 G_1(x), \ldots, m_p G_p(x)) \cdot K(x), G(x) \rangle + \langle V(x) \cdot K(x), G(x) \rangle. \]

Now, it is enough to write the vector \( (m_1 G_1(x), \ldots, m_p G_p(x)) \) as a matrix product \( G(x) \cdot D(m_i) \) and we finish the proof. \( \square \)

**Proof** (Theorem 5) **item (i).** Since \( G \) is homogeneuos of degree \( d \), one has \( d = m_1 = \cdots = m_p > 0 \), the matrix \( D(m_i) = d \cdot I_{p \times p} \) where \( I_{p \times p} \) is the \( p \times p \) identity matrix, and \( V(x) \equiv 0 \).

It follows from Lemma 3 that for any \( x \in M(G) \setminus (V_G \cup \text{Sing } G) \) we have

\[ \langle \alpha(x), G(x) \rangle = d \langle (G(x) \cdot [K(x)], G(x) \rangle \]

(17)

Moreover, the matrix \( K(x) \) is positive definite because \( JG(x) \cdot JG(x)^t \) is.

Therefore, \( \langle \alpha(x), G(x) \rangle > 0 \) for any \( x \in M(G) \setminus (V_G \cup \text{Sing } G) \) and the result follows from Lemma 2.

**item (ii).** It follows from hypothesis that \( \langle \nabla G_i(x), \nabla G_j(x) \rangle = 0 \), for any \( i, j = 1, \ldots, p \) with \( i \neq j \) and \( x \in M(G) \setminus (V_G \cup \text{Sing } G) \). Consequently, one has that \( K(x) = \left( 1/\|\nabla G(x)\|^2 \right) \cdot I_{p \times p} \). Thus,

\[ \langle G(x) \cdot [D(m_i) \cdot K(x)], G(x) \rangle = \left( 1/\|\nabla G(x)\|^2 \right) \cdot \sum_{j=1}^{p} m_j (G_j(x))^2. \]

(18)
On the other hand, one has
\[
\langle V(x) \cdot K(x), G(x) \rangle = \left( 1/\|\nabla G(x)\| \right)^2 \cdot \left( G_1(x) \sum_{j=1}^{\infty} jG_{m_1+j}^1(x) + \ldots + \sum_{j=1}^{\infty} jG_{m_p+j}^p(x) \right) .
\]

(19)

It follows from Eqs. (18) and (19) and Lemma 3 that for any \( x \in M(G) \setminus (V_G \cup \text{Sing } G) \) close enough to the origin the scalar product \( \langle \alpha(x), G(x) \rangle \) must be positive. \( \square \)

5 Applications

In this section we show that for some classes of maps there exist the fibrations (7) and (9) and they are equivalent.

Definition 6 (Massey 2010) Let \( G := (G_1, \ldots, G_p) : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0) \) be an analytic map germ. We say that \( G \) is a Simple Ł-Map, if \( \langle \nabla G_i, \nabla G_j \rangle = 0 \) for any \( i, j = 1, \ldots, p \) with \( i \neq j \) and \( \| \nabla G_i \| = \| \nabla G_j \| \) for any \( i, j = 1, \ldots, p \).

Example 2 Consider the real map germ \( G := (G_1, G_2) : (\mathbb{R}^8, 0) \to (\mathbb{R}^2, 0) \) given by
\[
\begin{align*}
G_1(x, y, z, w, a, b, c, d) &= -w^2x^2 + w^2y^2 + 4wxyz + x^2z^2 - y^2z^2 + ac + bd \\
G_2(x, y, z, w, a, b, c, d) &= -2w^2xy - 2wx^2z + 2wy^2z + 2xyz^2 - ad + bc.
\end{align*}
\]

One can show that \( \langle \nabla G_1, \nabla G_2 \rangle = 0 \) and \( \| \nabla G_1 \|^2 = \| \nabla G_2 \|^2 \). Thus, \( G \) is a Simple Ł-Map.

Proposition 3 If \( G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0) \) be an analytic map germ which is a Simple Ł-Map. Then there exist the fibrations (7) and (9) and they are equivalent.

Proof If \( G \) is a Simple Ł-Map then it satisfies the condition (4) and \( \text{Disc } G = \{0\} \) by (Massey 2010 [Theorem 5.7]). Now by Theorem 5 item (ii) the result follow. \( \square \)

5.1 Mixed Functions and Equivalence

Mixed singularities has been systematically studied by Mutuo Oka in the sequence of papers (Oka 2008, 2010, 2015) published since 2008, thenceforward the term mixed function was coined and it has been used to identify function \( f = (u + iv) : \mathbb{C}^n \to \mathbb{C} \) with \( f(z, \bar{z}) = \sum_{\nu, \mu} c_{\nu, \mu} z^\nu \bar{z}^\mu \), where \( c_{\nu, \mu} \in \mathbb{C}, z^\nu \) and \( \bar{z}^\mu \) are multi-monomials in the variables \( z_j, \bar{z}_j \), with \( j = 1, \ldots, n \).

One may view \( f \) as a real analytic map of \( 2n \) variables \((x, y)\) from \( \mathbb{R}^{2n} \) to \( \mathbb{R}^2 \) by identifying \( \mathbb{C}^n \) with \( \mathbb{R}^{2n}, (z_1, \ldots, z_n) = z \mapsto (x, y) = (x_1, y_1, \ldots, x_n, y_n) \), writing
$z = x + iy \in \mathbb{C}^n$, where $z_j = x_j + iy_j \in \mathbb{C}$ with $x_j, y_j \in \mathbb{R}$ for $j = 1, \ldots, n$. Moreover, one can define the holomorphic and anti-holomorphic gradients of $f$ as follows:

$$df := \left( \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n} \right)$$

and $\bar{df} := \left( \frac{\partial f}{\partial \bar{z}_1}, \ldots, \frac{\partial f}{\partial \bar{z}_n} \right)$.

where,

$$\frac{\partial f}{\partial z_j} = \frac{\partial u}{\partial z_j} + i \frac{\partial v}{\partial z_j} \quad \text{and} \quad \frac{\partial f}{\partial \bar{z}_j} = \frac{\partial u}{\partial \bar{z}_j} + i \frac{\partial v}{\partial \bar{z}_j}.$$

Consequently, one can consider the identifications $\nabla u(x, y) = \bar{d}f(z, \bar{z}) + \bar{df}(z, \bar{z})$ and $\nabla v(x, y) = i \left( \bar{d}f(z, \bar{z}) - \bar{df}(z, \bar{z}) \right)$. Since $u$ and $v$ are vectors in $\mathbb{C}^n$ we denote its hermitian product by $\langle u, v \rangle_\mathbb{C}$. Hence one has that $4\text{Re} \left( \bar{d}f, \bar{df} \right)_\mathbb{C} = \| \nabla u \|^2 - \| \nabla v \|^2$ and $2\text{Im} \left( \bar{d}f, \bar{df} \right)_\mathbb{C} = \langle \nabla u, \nabla v \rangle$. With these notations and definitions, one has the following.

**Corollary 3** A mixed function germ $f : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ is a Simple Ł-Map if and only if $\left( \bar{d}f, \bar{df} \right)_\mathbb{C} = 0$, i.e., $\text{Re} \left( \bar{d}f, \bar{df} \right)_\mathbb{C} = \text{Im} \left( \bar{d}f, \bar{df} \right)_\mathbb{C} = 0$. In particular, if $f$ is holomorphic then it is a Simple Ł-Map and the fibrations (7) and (9) exist and they are equivalent.

In what follows we will consider an algorithm to construct classes of mixed functions which are Simple Ł-Maps, for short, MSL.

**Proposition 4** (Algorithm to MSL) Consider the following algorithm:

1. Fix a copy of $\mathbb{C}^n$, $n \geq 2$, and a coordinate system $z = (z_1, \ldots, z_n)$.
2. For each $1 \leq k < n$ choose natural numbers $i_1, \ldots, i_k \in \{1, 2, \ldots, n\}$ with $i_1 < i_2 < \cdots < i_k$ and fix the coordinates $(z_{i_1}, \ldots, z_{i_k})$. For the complementary ordered list $q_1, \ldots, q_{n-k} \in \{1, 2, \ldots, n\} \setminus \{i_1, \ldots, i_k\}$, $q_1 < \cdots < q_{n-k}$, consider the remaining coordinates $(z_{q_1}, \ldots, z_{q_{n-k}})$.
3. For any natural numbers $j, t$ and $p$, choose arbitrary holomorphic functions $f_j(z_{i_1}, \ldots, z_{i_k}), r_t(z_{q_1}, \ldots, z_{q_{n-k}})$, $g_j(z_{q_1}, \ldots, z_{q_{n-k}})$ and $h_p(z_{q_1}, \ldots, z_{q_{n-k}})$.
4. Define the mixed function germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ by $f(z_1, \ldots, z_n) = \sum_{\alpha = 1}^{j} f_\alpha(z_{i_1}, \ldots, z_{i_k}) g_\alpha(z_{q_1}, \ldots, z_{q_{n-k}}) + \sum_{\beta = 1}^{t} r_\beta(z_{i_1}, \ldots, z_{i_k})$

$$+ \sum_{\gamma = 1}^{p} h_\gamma(z_{q_1}, \ldots, z_{q_{n-k}}).$$

**Claim:** The mixed function germ $f$ is MSL.

**Proof** By simplicity we will choose $j = t = p = 1$, $n = 2k$ for some $k \in \mathbb{N}$ and $i_\eta = \eta$ for $1 \leq \eta \leq k$. In this case,

$$f(z_1, \ldots, z_n) = f_1(z_1, \ldots, z_k) g_1(z_{k+1}, \ldots, z_n) + r_1(z_1, \ldots, z_k) + h_1(z_{k+1}, \ldots, z_n).$$
We notice that
\[
\frac{\partial f}{\partial z_\alpha} = \frac{\partial f_1}{\partial z_\alpha} g_1 + \frac{\partial r_1}{\partial z_\alpha} \text{ and } \frac{\partial f}{\partial z_\alpha} = 0
\]
for \(\alpha = 1, \ldots, k\),
\[
\frac{\partial f}{\partial z_\beta} = 0 \text{ and } \frac{\partial f}{\partial \bar{z}_\beta} = \frac{\partial g_1}{\partial z_\beta} f_1 + \frac{\partial h_1}{\partial z_\beta}
\]
for \(\beta = k + 1, \ldots, n\). Thus, one has
\[
\bar{d}f = \left(\frac{\partial f_1}{\partial z_1} g_1 + \frac{\partial r_1}{\partial z_1}, \ldots, \frac{\partial f_1}{\partial z_k} g_1 + \frac{\partial r_1}{\partial z_k}, 0, \ldots, 0\right)
\]
and
\[
\check{d}f = \left(0, \ldots, 0, \frac{\partial g_1}{\partial z_{k+1}} f_1 + \frac{\partial h_1}{\partial z_{k+1}}, \ldots, \frac{\partial g_1}{\partial z_n} f_1 + \frac{\partial h_1}{\partial z_n}\right).
\]
Therefore, \(\langle \bar{d}f, \check{d}f \rangle_C = 0\) and by Corollary 3 the function \(f\) is MSL.

**Example 3** Let \(G : (\mathbb{C}^4, 0) \to (\mathbb{C}, 0)\) given by \(G(z_1, z_2, z_3, z_4) = z_1^2 z_2^3 + z_3 z_4 + z_4^3 - z_3 + z_2 z_4^3\). Considering \(f_1(z_1, z_3) = z_1^2, f_2(z_1, z_3) = z_3, g_1(z_2, z_4) = z_2^2, g_2(z_2, z_4) = z_4, r(z_1, z_3) = z_1^4 - z_3\) and \(h(z_2, z_4) = z_2 z_4^3\). Then one has that \(G = f_1 g_1 + f_2 g_2 + r - h\). By Proposition 4, \(G\) is a MSL.

We point out that the Proposition 4 also extends the Thom-Sebastiani type result obtained in Corollary 4.2 of Parameswaran and Tibăr (2018).

**Remark 4** If \(f : \mathbb{C}^n \to \mathbb{C}\) is a mixed functions it follows by Theorem 5 that only the condition \(\text{Im} \langle \bar{d}f(z), \check{d}f(z) \rangle_C = 0\) is enough to guarantee that the coefficient \(a(z)\) is positive. In particular, if Disc \(f = \{0\}\) then there exists a MVF for \(f\) and \(\Psi_f\) is \(\rho\)-regular.

### 5.2 Product of Mixed Functions

Let \(F = fg\), where \(f : \mathbb{C}^n \to \mathbb{C}\) and \(g : \mathbb{C}^m \to \mathbb{C}\) are mixed functions in separable variables. If we consider the identifications \(\nabla |f|^2 = 2(f \bar{d}f + \check{d}f)\) and \(\Omega_f = i(f \bar{d}f - \check{d}f)\), where \(\Omega_f\) denote the direction \(u \nabla v - v \nabla u\), one can prove that:
\[
M(F) \setminus V_F \subset \left(M(f) \setminus V_f\right) \times \left(M(g) \setminus V_g\right), M(F/|F|) \subset M(f/|f|) \times M(g/|g|).
\]
In particular, if \(M(f/|f|) = \emptyset\) or \(M(g/|g|) = \emptyset\), then \(M(F/|F|) = \emptyset\).

In another words, for mixed functions in separable variables in order to get \(F/|F|\) \(\rho\)-regular it is enough to ask the \(\rho\)-regularity for \(f/|f|\) or \(g/|g|\). Moreover, \(\text{Sing } F = (\text{Sing } f \times \text{Sing } g) \cup ((V_f \cap \text{Sing } f) \times \mathbb{C}^m) \cup (\mathbb{C}^n \times (V_g \cap \text{Sing } g)) \cup (V_f \times V_g)\). Hence, if either \(\text{Disc } f = \{0\}\), or \(\text{Disc } g = \{0\}\), then \(\text{Disc } F = \{0\}\).
Proposition 5 Let \( f : \mathbb{C}^n \to \mathbb{C} \) and \( g : \mathbb{C}^m \to \mathbb{C} \) be mixed functions in separable variables. Suppose that the mixed function \( F = fg : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C} \) satisfies the condition (4). If either the conditions below holds true:

(i) \( \text{Disc } f = \{0\} \) and on \( B_{\epsilon_1}^{2n} \setminus V_f \) there exists a MVF for \( f \), for some small enough \( \epsilon_1 > 0 \);

(ii) \( \text{Disc } g = \{0\} \) and on \( B_{\epsilon_2}^{2m} \setminus V_g \) there exists a MVF for \( g \), for some small enough \( \epsilon_2 > 0 \),

then both fibrations (7) and (9) exist and they are equivalent.

Proof We will prove the item (i). The item (ii) follows in the same way. By hypothesis \( \text{Disc } F = \{0\} \) thus, for any \( (x, y) \in M(F) \setminus V_F \) one has \( (x, y) \notin \text{Sing } F \). Consequently, there exist \( a(x, y), b(x, y) \in \mathbb{R} \) such that

\[
(x, y) = a(x, y) \nabla |F(x, y)|^2 + b(x, y) \Omega_F(x, y).
\]

Since, \( \nabla |F(x, y)|^2 = (|g(y)|^2 \nabla |f(x)|^2, |f(x)|^2 \nabla |g(y)|^2) \) and \( \Omega_F(x, y) = (|g(y)|^2 \Omega_f(x), |f(x)|^2 \Omega_g(y)) \), one gets that

\[
\begin{align*}
\begin{cases}
  x = a(x, y)|g(y)|^2 \nabla |f(x)|^2 + b(x, y)|g(y)|^2 \Omega_f(x) \\
  y = a(x, y)|f(x)|^2 \nabla |g(y)|^2 + b(x, y)|f(x)|^2 \Omega_g(y)
\end{cases}
\end{align*}
\] (20)

Hence, \( x \in M(f) \setminus V_f \) and \( y \in M(g) \setminus V_g \).

Suppose that there exists a MVF for \( f \). Let \( (x, y) \in (M(F) \setminus V_F) \cap B_{\epsilon_1}^{2(n+m)} \). Since \( x \notin \text{Sing } f \) there exist \( a_1(x), b_1(x) \in \mathbb{R} \) such that

\[
x = a_1(x) \nabla |f(x)|^2 + b_1(x) \Omega_f(x).
\] (21)

Comparing the first equation of (20) and the Eq. (21), one has that

\[
\left( a(x, y)|g(y)|^2 - a_1(x) \right) \nabla |f(x)|^2 + \left( b(x, y)|g(y)|^2 - b_1(x) \right) \Omega_f(x) = 0
\]

Since \( \text{Disc } f = \{0\} \), then \( \{\nabla |f(x)|^2, \Omega_f(x)\} \) are linearly independent on \( \mathbb{R} \). Hence, \( a_1(x) = a(x, y)|g(y)|^2 \) and \( b_1(x) = b(x, y)|g(y)|^2 \).

By hypothesis, \( a_1(x) > 0 \) which implies that \( a(x, y) > 0 \). Then, by Theorem 3 the vector field (11) is a MVF for \( F \). Therefore, by Corollary 1 and Theorem 1 one gets the existence of the Milnor-Hamm tube and sphere fibrations, and by Theorem 2 they are equivalent. □

Corollary 4 Let \( f : \mathbb{C}^n \to \mathbb{C} \) be a holomorphic function and \( g : \mathbb{C}^m \to \mathbb{C} \) be a mixed functions in separable variables. Suppose that the mixed function \( F = fg : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C} \) satisfies the condition (4). Then, there exist the fibrations (7) and (9) and they are equivalent.

Proof It is enough to check that \( f \) satisfies the condition (i) of Proposition 5. In fact, it follows from Corollary 3. □
Example 4 Consider the mixed function $F: \mathbb{C}^2 \to \mathbb{C}$, $F(x, y) = y\|x\|^2$. In dos Santos et al. (2013) the authors have shown that $F$ satisfies the condition (4) but does not have the Thom $a_F$-condition. Let us consider the mixed functions $f: \mathbb{C} \to \mathbb{C}$ and $g: \mathbb{C} \to \mathbb{C}$ given by $f(y) = y$ and $g(x) = x \cdot \overline{x}$, then $F = fg$. By Corollary 4, there exist the fibrations (7) and (9) and they are equivalent.

Acknowledgements The authors would like to thank the anonymous the Professor Mihai Marius Tibăr for the valuable suggestions. They also thank the anonymous referee for his/her time reading our paper and for all valuable comments, suggestions and corrections. The second author would like to thank the partial support from CNPq grant 313780/2017-0, Fapesp grant 2017/20455-3 and the Fapesp Thematic Project grant 2019/21181-0. Thanks a lot!

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