Infinite-Horizon Offline Reinforcement Learning with Linear Function Approximation: Curse of Dimensionality and Algorithm

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Abstract

In this paper, we investigate the sample complexity of policy evaluation in infinite-horizon offline reinforcement learning (also known as the off-policy evaluation problem) with linear function approximation. We identify a hard regime $d \gamma^2 > 1$, where $d$ is the dimension of the feature vector and $\gamma$ is the discount rate. In this regime, for any $q \in [\gamma^2, 1]$, we can construct a hard instance such that the smallest eigenvalue of its feature covariance matrix is $q/d$ and it requires $\Omega\left(\frac{d \gamma^2}{(q-\gamma^2)\epsilon^2} \exp(\Theta(d \gamma^2))\right)$ samples to approximate the value function up to an additive error $\epsilon$. Note that the lower bound of the sample complexity is exponential in $d$. If $q = \gamma^2$, even infinite data cannot suffice.

Under the low distribution shift assumption, we show that there is an algorithm that needs at most $O\left(\max\left\{\frac{||\theta^\pi||_2^2}{\epsilon^4} \log \frac{d}{\delta}, \frac{1}{\epsilon^2} (d + \log \frac{1}{\delta})\right\}\right)$ samples ($\theta^\pi$ is the parameter of the policy in linear function approximation) and guarantees approximation to the value function up to an additive error of $\epsilon$ with probability at least $1 - \delta$.

1 Introduction

In offline reinforcement learning (also known as batch reinforcement learning) [15, 2, 9], we are interested in evaluating a strategy and making sequential decisions when the algorithm has access to a batch of offline data (for example, watching StarCraft game videos and reading click logs of users of Amazon) rather than interacts directly with the environment, which is modeled by a Markov decision process (MDP). Research on offline reinforcement learning has gained increasing interest because of the following reasons. First, exploration can be expensive and even risky. For example, while a robot explores the environment, in addition to the time and economic costs, it can damage its own hardware as well as objects around

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and even hurt people. Second, we can use offline reinforcement learning to pre-train an agent efficiently using existing data and evaluate the exploitation performance of an algorithm.

To handle large-scale and even continuous states, researchers introduced function approximation to approximate the value of states and state-action pairs [19, 10, 22, 1, 26]. Linear function approximation assumes that every state-action pair is assigned a (hand-craft or learned) feature vector and that the value function is the inner product of the feature vector and an unknown parameter that depends on the policy [5, 16, 17, 24, 11]. [13, 23] considered online and offline episodic finite-horizon reinforcement learning with linear function approximation, respectively. Our work considers infinite-horizon offline reinforcement learning with linear function approximation. We investigate the sample complexity of approximating the value function up to an additive error bound \( \varepsilon \) under a given policy (this problem is also known as the off-policy evaluation). Our results consist of a lower bound and an upper bound. Throughout this paper, let \( d \) denote the dimension of the feature vector and \( \gamma \) the discount rate.

**Lower Bound** Recall that the assumption of linear function approximation means that the value function is linear in the unknown policy-specific parameter \( \theta^\pi \). For the feature vectors of the state-action pairs in the dataset, we call their covariance matrix the feature covariance matrix. We identify a hard regime \( d\gamma^2 > 1 \). In this regime, inspired by [23, 3], we construct a hard instance whose value function satisfies the assumption of linear function approximation and feature covariance matrix is well- or even best-conditioned possible. To be precise, for any \( q \in [\gamma^2, 1] \), we can construct a hard instance whose feature covariance matrix has the smallest eigenvalue \( q/d \). To approximate the value of a state in this instance up to an additive error \( \varepsilon \), with high probability we need \( \Omega \left( \frac{d}{\gamma^2(q-\gamma^2)\varepsilon^2} \exp \left( \Theta (d\gamma^2) \right) \right) \) samples. We see that the sample complexity depends exponentially in \( d \) and suffers from the curse of dimensionality. In fact, \( q = 1 \) represents the best-conditioned feature covariance matrix because the smallest eigenvalue has a \( 1/d \) upper bound. If one chooses \( q = \gamma^2 \), even infinite data cannot guarantee good approximation and we recover the result of [3]. We would like to remark that the result of [3] is a special case of ours. The smallest eigenvalue is \( \gamma^2/d \) in the construction of [3]. We can make it \( 1/d \) in our construction at a cost of degrading the sample complexity lower bound from infinity to being exponential in \( d \). This agrees with our intuition that a problem with a better-conditioned feature covariance matrix (which indicates better feature coverage) is easier to solve. In addition, our result fills the gap from \( \gamma^2/d \) to the best possible condition \( 1/d \).

**Upper Bound** Under the low distribution shift assumption, we show that the Least-Squares Policy Evaluation (LSPE) algorithm needs at most \( O \left( \max \left\{ \frac{\|\theta^\pi\|^2}{\varepsilon^2} \log \frac{d}{\delta}, \frac{1}{\varepsilon^2} \left( d + \log \frac{1}{\delta} \right) \right\} \right) \) samples (\( \theta^\pi \) is the parameter of the policy in linear function approximation) and guarantees approximation to the value function up to an additive error of \( \varepsilon \) with probability at least \( 1 - \delta \). If we also assume \( \|\theta^\pi\|_2 \leq O(\sqrt{d}) \) as in [13, 23], the sample complexity becomes \( O \left( \frac{d^2}{\varepsilon^6} \log \frac{d}{\delta} \right) \). In addition, we show that our hard instance does not satisfy the low
distribution shift assumption and therefore the upper bound does not contradict the lower bound.

**Paper Organization** The rest of the paper is organized as follows. Section 2 presents related work. We introduce notation and preliminaries in Section 3. We show the lower bound in Section 4 and upper bound in Section 5. Section 6 concludes the paper.

## 2 Related Work

There is a large body of work on policy evaluation in offline reinforcement learning (also known as the off-policy evaluation) with function approximation [8, 25, 27, 14, 6, 7, 28, 23, 3, 21]. The seminal work [6] studied offline infinite-horizon reinforcement learning whose value function is approximated by a finite function class. Assuming both low distribution shift and policy completeness, they showed an upper bound on the sample complexity. The upper bound depends polynomially in $1/\varepsilon$ and $1/(1-\gamma)$ (in their paper, $\varepsilon$ denotes the multiplicative approximation error bound), logarithmically in the size of the function class, and linearly in the concentratability coefficient that quantifies distribution shift. If low distribution shift is not assumed, they showed a lower bound that excludes polynomial sample complexity if the MDP dynamics are unrestricted. [7] studied episodic finite-horizon off-policy evaluation with linear function approximation. They assumed that the function class is closed under the conditional transition operator and that the data consists of i.i.d. episode samples, each being a trajectory generated by some policy. Under these two assumptions, they determined the minimax-optimal error of evaluating a policy.

The papers closest to ours are probably [28, 23, 3]. [23] studied offline episodic finite-horizon reinforcement learning with linear function approximation. They proved an $\Omega \left( \left( d/2 \right)^H \right)$ sample complexity lower bound in order to achieve a constant additive approximation error with high probability, where $H$ is the planning horizon. In their hard instance, the smallest eigenvalue of the feature covariance matrix is $1/d$. Under the low distribution shift assumption, they proved an upper bound on the sample complexity. In particular, they showed that the squared additive error is at most $\prod_{h=1}^{H} C_h \cdot \text{poly}(d, H)/\sqrt{N}$, where $N$ is the number of samples and $C_h$ are positive constants coming from their low distribution shift assumption. However, this additional assumption does not exclude their hard instance. It is possible that $C_h = \Theta(d)$ and their upper bound still gives an upper bound exponential in $H$. [3] considered the same problem as in this paper. They presented a hard instance such that the smallest eigenvalue of the feature covariance matrix is $\gamma^2/d$ and any algorithm must have $\Omega(1)$ additive approximation error, even with infinite data. We have compared our work to [3] in Section 1. [28] investigated a different setting where data is obtained via policy-free queries and policy-induced queries. [28] did not consider the condition number of the feature covariance matrix.
3 Preliminaries

We use the shorthand notation $[N] \triangleq \{1, 2, \ldots, N\}$. If $S$ is a set, write $\text{Unif}(S)$ for the uniform distribution on $S$. If $A$ is a matrix, write $\|A\|_2$ for its spectral norm, which equals its largest singular value. If $A$ is a vector, $\|A\|_2$ agrees with its Euclidean norm. For two square matrices $A$ and $B$ of the same size, we write $A \preceq B$ if $B - A$ is a positive semidefinite matrix.

**Infinite-Horizon Reinforcement Learning** We consider the infinite-horizon Markov decision process (MDP) [18]. It is defined by the tuple $(S, A, P, R, \gamma)$, where $S$ is the set of states, $A$ is the set of actions that an agent can choose and play, $P(s | s, a)$ and $R(r | s, a)$ are probability distributions on $S$ and $\mathbb{R}$ respectively given a state-action pair $(s, a) \in S \times A$, and $\gamma \in (0, 1)$ is the discount factor. We assume that the reward $r \sim R(\cdot | s, a)$ takes values from $[-1, 1]$. We will also denote this random variable by $R(s, a)$, i.e., $R(s, a) \sim R(\cdot | s, a)$ (we overload the notation $R$). A policy $\pi(a | s)$ is a probability distribution on $A$ given a state $s$. If $\pi$ is deterministic, we will abuse the notation and write $a = \pi(s)$ if $\pi(\cdot | s)$ is a delta distribution at $a$. Given a policy $\pi$ as well as an initial state $s_0$, it induces a random trajectory $\{(s_i, a_i, r_i) | i \geq 0\}$, where $a_i \sim \pi(\cdot | s_i)$, $r_i \sim R(\cdot | s_i, a_i)$ and $s_{i+1} \sim P(\cdot | s_i, a_i)$.

The value of a state $s$ and the $Q$-function of a state-action pair $(s, a)$ are given by

$$V^\pi(s) = \mathbb{E} \left[ \sum_{i \geq 0} \gamma^i r_i \mid s_0 = s \right], \quad Q^\pi(s, a) = \mathbb{E} \left[ \sum_{i \geq 0} \gamma^i r_i \mid s_0 = s, a_0 = a \right].$$

Since we assume that the absolute value of rewards is at most 1, we have $|V^\pi(s)| \leq \frac{1}{1 - \gamma}$ and $|Q^\pi(s, a)| \leq \frac{1}{1 - \gamma}$.

**Linear Function Approximation** The following Assumption 1 assumes that the $Q$-function is the inner product of the feature vector $\phi(s, a)$ of a state-action pair and the unknown policy-specific parameter $\theta^\pi$. This assumption was also assumed in [16, 3]. Although it was not directly assumed in [13], their linear MDP assumption (Assumption A) implies our Assumption 1 (see Proposition 2.3 in [13]) and they stated it in the context of episodic finite-horizon reinforcement learning.

**Assumption 1** ([16, 13]). For every state-action pair $(s, a)$ and every policy $\pi$, there is a feature vector $\phi(s, a) \in \mathbb{R}^d$ and a parameter $\theta^\pi \in \mathbb{R}^d$ such that

$$Q^\pi(s, a) = \phi(s, a)^\top \theta^\pi.$$

**Assumption 2** ([23]). Since $|Q^\pi(s, a)| = |\phi(s, a)^\top \theta^\pi| \leq \frac{1}{1 - \gamma}$, without loss of generality, we assume $\|\phi(s, a)\|_2 \leq 1$ for every $(s, a) \in S \times A$.

In fact, if $\max_{s,a} \phi(s, a) > 1$, we can use the normalized feature vectors $\frac{\phi(s, a)}{\max_{s',a'} \phi(s', a')}$, and the new parameter for the policy $\pi$ becomes $\max_{s',a'} \phi(s', a') \theta^\pi$, where $\theta^\pi$ is the original policy parameter.
Offline Reinforcement Learning  In offline reinforcement learning, rather than interacts with the MDP directly, the agent has access to a batch of samples \(\{(s_i, a_i, r_i, s'_i) \mid i \in [N]\}\), where \((s_i, a_i)\) are i.i.d. samples from a distribution \(\mu\) on \(S \times A\), \(r_i \sim R(. \mid s_i, a_i)\), and \(s'_i \sim P(\cdot \mid s_i, a_i)\). Given a policy \(\pi\), we are interested in evaluating the value \(V^\pi(s)\) of a state under this policy approximately, using samples from \(\mu\). If our problem satisfies Assumption 1, the feature covariance matrix of \(\mu\) [23, 3] is defined by

\[
\Lambda \triangleq \mathbb{E}_{(s,a) \sim \mu} \left[ \phi(s,a)\phi(s,a)'^\top \right].
\]

We require that the feature covariance matrix be well-conditioned (the smallest eigenvalue of \(\Lambda\) is lower bounded), which indicates that \(\mu\) has a good feature coverage. In our hard instance to be presented in Section 4, the smallest eigenvalue satisfies \(\lambda_{\min}(\Lambda) = q/d\), where \(q\) can be any value on \([\gamma^2, 1]\). Note that under Assumption 2, \(\lambda_{\min}(\Lambda)\) is at most \(1/d\). To see this, we compute the trace \(\text{tr}(\Lambda) = \mathbb{E}_{(s,a) \sim \mu} \left[ \text{tr} \left( \phi(s,a)\phi(s,a)'\right) \right] \leq 1\). Since \(\text{tr}(\Lambda) \geq d\lambda_{\min}(\Lambda)\), we get \(\lambda_{\min}(\Lambda) \leq 1/d\). In other words, \(\lambda_{\min}(\Lambda) = 1/d\) is the best possible condition.

4 Lower Bound

In this section, we present our lower bound on the sample complexity of infinite-horizon offline reinforcement learning with linear function approximation. Recall that \(d\) is the dimension in linear function approximation and \(\gamma\) is the discount rate. Inspired by [3, 23], we can construct a hard instance provided that \(d\gamma^2 > 1\). In the assumption of our lower bound theorem below (Theorem 1), we require that \(d\) be a multiple of \([b/\gamma^2]\) for some constant \(b > 1\). If \(d\gamma^2 > 1\), there exists \(b > 1\) such that \(d \geq b/\gamma^2\). Then Theorem 1 gives an at least exponential, and potentially infinite, lower bound of sample complexity, depending on the condition number (the smallest eigenvalue of the feature covariance matrix, i.e., \(\lambda_{\min}(\Lambda)\)) that we would like to achieve. In other words, we suffer from the curse of dimensionality. Therefore, we can say that the regime where \(d\gamma^2 > 1\) is a hard regime.

**Theorem 1.** Let \(I_d\) denote the set of all infinite-horizon MDPs that satisfy Assumption 1 and Assumption 2 and whose feature vectors have dimension \(d\), rewards lie in \([-1, 1]\). Let \(M(\lambda, S, A)\) denote the set of all probability measures on \(S \times A\) such that the feature covariance matrix \(\Lambda\) has smallest eigenvalue at least \(\lambda\). Fix \(b > 1\) and \(q \in [\gamma^2, 1]\). For any dimension \(d\) which is a multiple of \([b/\gamma^2]\) (thus \(d_{b, \gamma} \triangleq \frac{d}{[b/\gamma^2]}\) is a positive integer), if \(0 < \delta < 1/4\), we have

\[
\sup_{(S,A,P,R,\gamma) \in I_d} \inf_{\bar{V}, \pi} \left| \bar{V} - V^\pi(s) \right| \geq \Omega \left( \sqrt{\frac{1 + \gamma}{(q - \gamma^2)(1 - \gamma)^2 N} d_{b, \gamma} b d_{b, \gamma} \ln \left( \frac{1}{8\delta(1 - 2\delta)} \right)} \right).
\]

(1)

with probability at least \(\delta\), where \(N\) is the number of samples from \(\mu\) and \(\bar{V}\) is a real-valued function with \(N\) samples as input.
Remark 1 (Sample complexity). In the proof of Theorem 1, we present a hard instance with only one action such that the smallest eigenvalue of the feature covariance matrix $\mathbb{E}_{s \sim \mu} [\phi(s)\phi(s)^\top]$ is $\frac{q}{d}$. For this instance, any algorithm requires
\[
\Omega \left( \frac{1 + \gamma}{(q - \gamma^2)(1 - \gamma)\gamma^2 \varepsilon^2 d_{b,\gamma} b^{d_{b,\gamma}} \ln \left( \frac{1}{8\delta(1 - 2\delta)} \right)} \right)
\]
samples in order to approximate the value of a state up to an additive error $\varepsilon$ with probability at least $1 - \delta$. This lower bound for sample complexity follows directly from Equation (1).

Remark 2. Our result subsumes [3] as a special case. Recall that the smallest eigenvalue of the feature covariance matrix is at most $1/d$. Therefore, the parameter $q$ is at most 1. If $q = \gamma^2$, no algorithm can approximate the value of a state up to a constant additive error even provided with an arbitrarily large dataset. In this case, the smallest eigenvalue of the feature covariance matrix is $\gamma^2/d$. We recover the impossibility result of [3].

Remark 3. If $q = 1$ and $0 < \delta < 1/4$, we have
\[
\sup_{(S,A,P,R,\gamma) \in \mathcal{I}_d} \inf_{\pi \in \mathcal{M}_{q/d}(S,A)} \mathbb{E}_s [V - V^\pi(s)] \geq \Omega \left( \frac{1}{\gamma(1 - \gamma)} \sqrt{\frac{1}{Nd} d_{b,\gamma} b^{d_{b,\gamma}} \ln \left( \frac{1}{8\delta(1 - 2\delta)} \right)} \right).
\]
The sample complexity lower bound becomes
\[
\Omega \left( \frac{1}{\gamma^2(1 - \gamma)^2 \varepsilon^2 d_{b,\gamma} b^{d_{b,\gamma}} \ln \left( \frac{1}{8\delta(1 - 2\delta)} \right)} \right).
\]

Proof. Fix integers $m \geq 1/\gamma^2$ and $L \geq 1$. We will set $r_0$ to either 0 or $\frac{2\varepsilon}{\gamma L - 1/m L/2}$. Our hard instance has three groups of states. Each state has one single action. Therefore, we omit the action in $R(s,a)$ and $Q(s,a)$ and write $R(s)$ and $Q(s)$, respectively (in this case, $Q(s) = V(s)$ is the value of state $s$). All transitions are deterministic. Group A contains $mL$ states $G_A \triangleq \{s'_{l,i} \mid l \in [0, L - 1] \cap \mathbb{Z}, m \in [m]\}$. Group B contains $mL$ states $G_B \triangleq \{s_{l,i} \mid l \in [0, L - 1] \cap \mathbb{Z}, i \in [m]\}$. Group C contains $L$ states $G_C \triangleq \{s_{l,0} \mid l \in [0, L - 1] \cap \mathbb{Z}\}$. The total number of states in all three groups is $(2m + 1)L$. Every state $s'_{l,i}$ in group A transitions to the corresponding state $s_{l,i}$ in group B. All states $s_{l,i}$ in group B and C on level $l \geq 1$ transition to state $s_{l-1,0}$. All states $s_{0,i}$ in group B and C on level 0 have a self-loop and transition to themselves. All states in group A have zero reward. All states in group B on level $l > 0$ have zero reward and those in group C on level $l > 0$ have reward $R(s_{l,0}) = r_0(\sqrt{m}\gamma)^l(\sqrt{m} - 1)$. Moreover, define the reward of the state in group C on level 0 to be $R(s_{0,0}) = r_0\sqrt{m}(1 - \gamma)$. The reward of the states in group B on level 0 is a random variable taking values from $\{-1, 1\}$:
\[
R(s_{0,i}) = \begin{cases} 1 & \text{with probability } \frac{1+r_0(1-\gamma)}{2}, \\ -1 & \text{with probability } \frac{1-r_0(1-\gamma)}{2}. \end{cases}
\]
Figure 1: There are three groups of states in the hard instance. Each state has one single action. Therefore, we omit the action in $R(s,a)$ and $Q(s,a)$ and write $R(s)$ and $Q(s)$, respectively (in this case, $Q(s) = V(s)$ is the value of state $s$). All transitions are deterministic and denoted by arrows in the figure. All states in group A have zero reward. In group B, the states on level $l > 0$ have zero reward. The states in group B on level 0 satisfy $R(s_{0,i}) = 1$ with probability $\frac{1+r_0(1-\gamma)}{2}$ and $R(s_{0,i}) = -1$ with probability $\frac{1-r_0(1-\gamma)}{2}$. As a result, they have an expected reward of $\mathbb{E}[R(s_{0,i})] = r_0(1 - \gamma)$. The value of a state $s_{l,i}$ in group B is $Q(s_{l,i}) = r_0(\sqrt{m}\gamma)^l$. In group C, states on level $l > 0$ have reward $R(s_{l,0}) = r_0(\sqrt{m}\gamma)^l(\sqrt{m} - 1)$ and the states on level 0 have reward $R(s_{0,0}) = r_0\sqrt{m}(1 - \gamma)$. Thus the value of state $s_{l,0}$ is $Q(s_{l,0}) = r_0(\sqrt{m}\gamma)^l\sqrt{m}$. 
We illustrate our hard instance in Figure 1. We set the distribution \( \mu \) to the mixture of uniform distributions on \( A \) and \( B \), i.e., \( \mu = p \operatorname{Unif}(G_B) + (1-p) \operatorname{Unif}(G_A) \), where \( p = \frac{q-\gamma^2}{1-\gamma^2} \in [0,1] \).

First, we check that all rewards lie in \([-1, 1]\). Recall that all states in group A have zero reward. In group B, the reward of \( s_{l,i} \) (\( l > 0, i \in [m] \)) is zero and the reward of \( s_{0,i} \) (\( i \in [m] \)) is either \(-1\) or \(1\). In group C, if \( r_0 = 0 \), the reward of \( s_{l,0} \) (\( l \geq 0 \)) is zero. If \( r_0 = \frac{2\varepsilon}{\gamma L-1} \), recalling \( \varepsilon \leq 1/2 \), we have

\[
R(s_{0,0}) = r_0 \sqrt{m}(1-\gamma) \leq \frac{\sqrt{m}(1-\gamma)}{2} = \frac{1-\gamma}{(m\gamma^2)(L-1)/2} \leq 1
\]

and

\[
R(s_{l,0}) = r_0 (\sqrt{m\gamma})^l (\sqrt{m} - 1) \leq R(s_{L-1,0}) \leq \frac{(\sqrt{m\gamma})^{L-1}}{\gamma L-1} \leq (\sqrt{m} - 1) \leq 1.
\]

The second step is to compute the value of each state. We will show \( Q(s_{l,0}) = r_0 (\sqrt{m\gamma})^l \sqrt{m} \) for \( l \geq 0 \) by induction. It holds for \( l = 0 \) because \( Q(s_{0,0}) = \frac{R(s_{0,0})}{1-\gamma} = r_0 \sqrt{m} \). Assume that it holds for some \( l \geq 0 \). We have

\[
Q(s_{l+1,0}) = R(s_{l+1,0}) + \gamma Q(s_{l,0}) = r_0 (\sqrt{m\gamma})^l (\sqrt{m} - 1) + \gamma r_0 (\sqrt{m\gamma})^l \sqrt{m} = r_0 (\sqrt{m\gamma})^l \sqrt{m}.
\]

Then for \( i \in [m] \) and \( l > 0 \), we have

\[
Q(s_{l,i}) = \gamma Q(s_{l-1,0}) = \gamma r_0 (\sqrt{m\gamma})^l \sqrt{m} = r_0 (\sqrt{m\gamma})^l.
\]

Finally, for \( i \in [m] \), we obtain \( Q(s_{0,i}) = \frac{r_0(1-\gamma)}{1-\gamma} = r_0 \). In group A, we have \( Q(s_{l,i}) = r_0 (\sqrt{m\gamma})^l \).

Let \( \{e_{l,i} \mid i \in [m], 0 \leq l \leq L-1\} \) be the standard basis vectors of \( \mathbb{R}^d \), where \( d = mL \). Recall \( Q(s_{l,i}) = r_0 (\sqrt{m\gamma})^l \) for \( i \in [m] \) and \( Q(s_{l,0}) = r_0 (\sqrt{m\gamma})^l \sqrt{m} \). Define \( \phi(s_{l,i}) = e_{l,i} \) and \( \phi(s_{l,0}) = \frac{1}{\sqrt{m}} \sum_{i \in [m]} e_{l,i} \), and

\[
\theta^\pi = \sum_{i \in [m]} \sum_{l=0}^{L-1} r_0 (\sqrt{m\gamma})^l e_{l,i}.
\]

For \( i \in [m] \), we have

\[
\phi(s_{l,i})^\top \theta^\pi = r_0 (\sqrt{m\gamma})^l = Q(s_{l,i}),
\]

\[
\phi(s_{l,0})^\top \theta^\pi = \gamma r_0 (\sqrt{m\gamma})^l = Q(s_{l,0}),
\]

\[
\phi(s_{0,0})^\top \theta^\pi = \frac{1}{\sqrt{m}} \sum_{i \in [m]} r_0 (\sqrt{m\gamma})^l = r_0 (\sqrt{m\gamma})^l \sqrt{m} = Q(s_{0,0}).
\]

The feature vectors of the states in group B and C have unit norm: \( \|\phi_{l,i}\|_2 = 1 \) for \( i \in [m] \) and \( \|\phi_{l,0}\|_2 = \frac{1}{\sqrt{m}} \cdot \sqrt{m} = 1 \). Those in group A have norm \( \|\phi(s_{l,i})\|_2 = \gamma < 1 \). We are in a
Using Lemma 5.1 in [4], we have any algorithm outputs an incorrect $\varepsilon$ of $r$ position to compute the feature covariance matrix

$$
\mathbb{E}_{s \sim \mu} \left[ \phi(s)\phi(s)^\top \right] = p \left( \frac{1}{d} \sum_{i \in [m]} \sum_{l=0}^{L-1} e_{l,i} e_{l,i}^\top \right) + (1-p) \left( \frac{\gamma^2}{d} \sum_{i \in [m]} \sum_{l=0}^{L-1} e_{l,i} e_{l,i}^\top \right) = \frac{q}{d} I_d ,
$$

where $I_d$ is the $d \times d$ identity matrix.

Set $m = \left\lceil \frac{b}{\gamma^2} \right\rceil$ and $L = d/m = d_{b,\gamma}$. In this case, our requirement $m \geq 1/\gamma^2$ is satisfied. Next, we consider an algorithm evaluating the value of $s_{L-1,0}$. If $r_0 = 0$, $Q(s_{L-1,0}) = 0$. If $r_0 = \frac{2\varepsilon}{\gamma L - 1 m \gamma^2}$, $Q(s_{L-1,0}) = 2\varepsilon$. To approximate the value of $s_{L-1,0}$ up to an additive error of $\varepsilon$, the algorithm has to distinguish $r_0 = 0$ and $r_0 = \frac{2\varepsilon}{\gamma L - 1 m \gamma^2}$. The only way that the algorithm obtains the information of $r_0$ is to sample the reward of $s_{0,i} (i \in [m])$ because the other states in the support of $\mu$ have reward 0. Recall the two possible reward distributions of $s_{0,i} (i \in [m])$:

$$
R(s_{0,i}) = \begin{cases} 
1 & \text{with probability } \frac{1}{2} \\
-1 & \text{with probability } \frac{1}{2}
\end{cases}, \\
R(s_{0,i}) = \begin{cases} 
1 & \text{with probability } \frac{1}{2} \left( 1 + \frac{2\varepsilon (1 - \gamma)}{\gamma L - 1 m \gamma^2} \right) \\
-1 & \text{with probability } \frac{1}{2} \left( 1 - \frac{2\varepsilon (1 - \gamma)}{\gamma L - 1 m \gamma^2} \right)
\end{cases}.
$$

Using Lemma 5.1 in [4], we have any algorithm outputs an incorrect $Q(s_{L-1,0})$ from the two choices 0 and $2\varepsilon$ with probability at least

$$
\frac{1}{4} \left( 1 - \sqrt{1 - \exp \left( -\Theta \left( N_0 \left( \frac{2\varepsilon (1 - \gamma)^2}{\gamma L - 1 m \gamma^2} \right)^2 \right) \right)} \right),
$$

where $N_0$ is the number of samples of $s_{0,i} (i \in [M])$. Since only $p/L$ of samples from $\mu$ are $s_{0,i}$, any algorithm outputs an incorrect $Q(s_{L-1,0})$ with probability at least

$$
\frac{1}{4} \left( 1 - \sqrt{1 - \exp \left( -\Theta \left( \frac{p N_0 \varepsilon^2 (1 - \gamma)^2 \gamma^2}{L (m \gamma^2)^L} \right) \right)} \right) \geq \frac{1}{4} \left( 1 - \sqrt{1 - \exp \left( -\Theta \left( \frac{p N_0 \varepsilon^2 (1 - \gamma)^2 \gamma^2}{d_{b,\gamma} b_{b,\gamma}} \right) \right)} \right) = \frac{1}{4} \left( 1 - \sqrt{1 - \exp \left( -\Theta \left( \frac{(q - \gamma^2)(1 - \gamma) \gamma^2}{1 + \gamma} \cdot \frac{N_0 \varepsilon^2}{d_{b,\gamma} b_{b,\gamma}} \right) \right)} \right),
$$

where $N$ is the number of samples from $\mu$ and the inequality follows from $m \gamma^2 \geq b$ and $L = d_{b,\gamma}$. If $\delta = \frac{1}{4} \left( 1 - \sqrt{1 - \exp \left( -\Theta \left( \frac{(q - \gamma^2)(1 - \gamma) \gamma^2}{1 + \gamma} \cdot \frac{N_0 \varepsilon^2}{d_{b,\gamma} b_{b,\gamma}} \right) \right)} \right)$ and $0 < \delta < 1/4$, we can
Therefore, if

\[
\varepsilon = \Theta \left( \sqrt{\frac{1 + \gamma}{(q - \gamma^2)(1 - \gamma)^2 \gamma^2 N d_{b,\gamma} b_{d,b,\gamma} \ln \left( \frac{1}{8\delta(1 - 2\delta)} \right)}} \right).
\]

\[\square\]

5 Upper Bound

In this section, we show that under the low distribution shift assumption, the Least-Squares Policy Evaluation approximates the value function up to any given additive error bound \(\varepsilon\) with \(O \left( \max \left\{ \frac{\|\theta\|_2^4}{\gamma^2}, \frac{d}{\gamma^4} \right\} \right)\) samples. Suppose that the samples that the agent has access to are \(\{(s_i, a_i, r_i, \tilde{s}_i) \mid i \in [N]\}\), where \((s_i, a_i) \sim \mu, r_i \sim R(\cdot \mid s_i, a_i)\) and \(\tilde{s}_i \sim P(\cdot \mid s_i, a_i)\). We would like to approximate the value of state \(s_0\). Recall the feature covariance matrix \(\Lambda \triangleq \mathbb{E}_{(s,a) \sim \mu} \left[ \phi(s, a) \phi(s, a)^\top \right]\). Define \(\bar{\phi}(s) \triangleq \mathbb{E}_{a \sim \pi(\cdot | s)} \phi(s, a)\), \(\bar{\Lambda}_0 \triangleq \bar{\phi}(s_0) \bar{\phi}(s_0)^\top\), and

\[
\bar{\Lambda} \triangleq \mathbb{E}_{(s,a) \sim \mu, \tilde{s} \sim P(\cdot | s, a)} \left[ \bar{\phi}(\tilde{s}) \bar{\phi}(\tilde{s})^\top \right] = \mathbb{E}_{(s,a) \sim \mu, \tilde{s} \sim P(\cdot | s, a), \bar{a} \sim \pi(\cdot | s)} \left[ \phi(\tilde{s}, \bar{a}) \phi(\tilde{s}, \bar{a})^\top \right].
\]

**Assumption 3** (Low distribution shift). There exists \(C \in (0, 1/\gamma^2)\) and \(C_0 > 0\) such that \(\bar{\Lambda} \preceq C \Lambda\) and \(\bar{\Lambda}_0 \preceq C_0 \Lambda\).

**Remark 4.** Assumption 3 rules out the hard instance in Theorem 1. Specifically, there is no \(C \in (0, 1/\gamma^2)\) such that \(\bar{\Lambda} \preceq C \Lambda\) in the hard instance.

**Proof of Remark 4.** In the proof of Theorem 1, we show that \(\Lambda = \frac{q}{d} I_d\). In the sequel, we compute the matrix \(\bar{\Lambda}\). Recall the data distribution \(\mu = p \text{Unif}(G_B) + (1 - p) \text{Unif}(G_A)\), where \(p \triangleq \frac{\eta \gamma^2}{1 - \gamma^2} \in [0, 1]\), \(G_A \triangleq \{s_{l,i} \mid l \in [0, L - 1] \cap \mathbb{Z}, m \in [m]\}\) and \(G_B \triangleq \{s_{l,i} \mid l \in [0, L - 1] \cap \mathbb{Z}, i \in [m]\}\). Suppose that \(\tilde{s} \sim P(\cdot \mid s)\) is the next state for \(s\). Every state in group \(A\) transitions to the corresponding state in group \(B\), i.e., \(\mathbb{P}(\tilde{s} = s_{l,i} \mid s = s_{l,i}) = 1\). Therefore, if \(s \sim \text{Unif}(G_A)\), we have \(\tilde{s} \sim \text{Unif}(G_B)\) and

\[
\mathbb{E}_{s \sim \text{Unif}(G_A), \tilde{s} \sim P(\cdot | s)} \left[ \phi(\tilde{s}) \phi(\tilde{s})^\top \right] = \frac{1}{mL} \sum_{i \in [m]} \sum_{l=0}^{L-1} e_{l,i} e_{l,i}^\top = \frac{1}{d} I_d \triangleq \bar{\Lambda}_A.
\]

(2)

Every state in group \(B\) on level \(l > 0\) transition to state \(s_{l-1,0}\) in group \(C\). All states in group \(B\) on level \(0\) have a self-loop. As a result, if \(s \sim \text{Unif}(G_B)\), we have \(\mathbb{P}(\tilde{s} = s_{0,i}) = \frac{1}{mL}\) (for all \(i \in [m]\)) and \(\mathbb{P}(\tilde{s} = s_{l,0}) = \frac{1}{L}\) (for all \(l \in [0, L - 2] \cap \mathbb{Z}\)). Therefore, we deduce

\[
\mathbb{E}_{s \sim \text{Unif}(G_B), \tilde{s} \sim P(\cdot | s)} \left[ \phi(\tilde{s}) \phi(\tilde{s})^\top \right] = \frac{1}{mL} \sum_{i \in [m]} e_{0,i} e_{0,i}^\top + \frac{1}{L} \sum_{l=0}^{L-2} \left( \frac{1}{\sqrt{m}} \sum_{i \in [m]} e_{l,i} \right) \left( \frac{1}{\sqrt{m}} \sum_{i \in [m]} e_{l,i} \right)^\top.
\]

(3)
Algorithm 1 Least-Squares Policy Evaluation

1: \( \hat{V}_0(\cdot) \leftarrow 0 \)
2: Take samples \( \{(s_i, a_i, r_i, s_i) \mid i \in [N]\} \)
3: \( \hat{\lambda} \leftarrow \sum_{i \in [N]} \phi(s_i, a_i) \phi(s_i, a_i)^\top + \lambda I_d \)
4: for \( t = 1, 2, 3, \ldots, T \) do
5: \( \hat{\theta}_t \leftarrow \hat{\lambda}^{-1} \left( \sum_{i \in [N]} \phi(s_i, a_i) \cdot \left( r_i + \gamma \hat{V}_{t-1}(s_i) \right) \right) \)
6: \( \hat{Q}_t(s, a) \leftarrow \phi(s, a)^\top \hat{\theta}_t \) for all \( s \in S \) and \( a \in A \)
7: \( \hat{V}_t(s) \leftarrow \mathbb{E}_{a \sim \pi(s)} \hat{Q}_t(s, a) \) for all \( s \in S \)
8: end for

Let \( \hat{\Lambda}_B \in \mathbb{R}^{d \times d} \) denote the matrix in Equation (3). For an index in \([d]\), we denote it by two indices \((l, i) \in ([0, L - 1] \cap \mathbb{Z}) \times [m]\). Then \( M_{(0, i), (0, j)} = \frac{1}{d} (1 + \delta_{ij}) \) for \( i \in [m] \) (\( \delta \) is the Kronecker delta such that \( \delta_{ij} = 1 \) if \( i = j \) and it is zero otherwise) and \( [\hat{\Lambda}_B]_{(l, i), (l, j)} = \frac{1}{d} \) for \( i \in [m] \) and \( l \in [L - 2] \). We see that \( \hat{\Lambda}_B \) is a block diagonal matrix. The matrix \( \frac{1}{d} (1_{m \times m} + I_m) \) is one of the blocks, where \( 1_{m \times m} \in \mathbb{R}^{d \times d} \) is an all-one matrix. Recall \( \hat{\Lambda}_A \) in Equation (2) is \( \frac{1}{d} I_d \). Then the matrix \( \hat{\Lambda} = \mathbb{E}_{s \sim \mu, \hat{s} \sim P(\cdot|s)} \left[ \phi(s) \phi(s)^\top \right] = p \hat{\Lambda}_B + (1 - p) \hat{\Lambda}_A \) is also a block diagonal matrix. The matrix \( \frac{p}{d} (1_{m \times m} + I_m) + \frac{1 - p}{d} I_m = \frac{1}{d} \left( p 1_{m \times m} + I_m \right) \) is one of its blocks and its eigenvalues are \( p + \frac{1}{d} \) (with multiplicity 1) and \( \frac{1}{d} \) (with multiplicity \( d - 1 \)). These eigenvalues are also eigenvalues of \( \hat{\Lambda} \). Therefore \( \lambda_{\text{max}}(\hat{\Lambda}) \geq p + 1/d = \frac{q - \gamma^2}{1 - \gamma^2} + \frac{1}{d} \). Consider the function \( f(q) = \frac{1}{q} \cdot \frac{q}{d} - \frac{q - \gamma^2}{1 - \gamma^2} \). We will show that \( f(q) \leq 1/d \) for all \( q \in [\gamma^2, 1] \). Notice that it is a linear function. It suffices to check \( f(1), f(\gamma^2) \leq 1/d \). We have \( f(1) = \frac{1}{d \gamma^2} - 1 < \frac{1}{d} \) because \( (d + 1) \gamma^2 > 1 \) (we use the assumption \( d \gamma^2 > 1 \)). At \( q = \gamma^2 \), we have \( f(\gamma^2) = 1/d \). We conclude that \( \frac{\lambda_{\text{max}}(\hat{\Lambda})}{\lambda_{\text{max}}(\hat{\Lambda})} \geq \left( \frac{q - \gamma^2}{(1 - \gamma^2)} \right) + \frac{1}{d} \geq 1/\gamma^2 \). Hence there is no \( C \in (0, \gamma^2) \) such that \( \hat{\Lambda} \preceq C \Lambda \).

If Assumption 3 is fulfilled, the following theorem presents an upper bound on the sample complexity of approximating the value of a state up to additive error bound \( \epsilon \). See our discussion in Remark 5. Recall that in Theorem 1, we show that there is an instance with \( \lambda_{\text{min}}(\Lambda) = \gamma^2/d \) for which evaluating a state up to a constant additive error is impossible (see also [3]). This suggests that if \( \lambda_{\text{min}}(\Lambda) \leq \gamma^2 \), it is generally impossible to approximate the value of a state. If \( \lambda_{\text{min}}(\Lambda) > \gamma^2 \), there exists \( C \in (0, 1/\gamma^2) \) such that \( 1/\gamma^2 > C > 1/\lambda_{\text{min}}(\Lambda) \). As a result, we have \( C \Lambda \succeq I_d \succeq \hat{\Lambda} \) and \( C \Lambda \succeq I_d \succeq \hat{\Lambda}_0 \) and therefore Assumption 3 holds (with \( C_0 = C \)). Thus our upper bound covers all cases in the regime \( \lambda_{\text{min}}(\Lambda) > \gamma^2 \). Note that Assumption 3 may also cover some cases in the regime \( \lambda_{\text{min}}(\Lambda) \leq \gamma^2 \).

Theorem 2. Suppose that \( C \in (0, 1/\gamma^2) \) and \( C_0 > 0 \) are constants. Let \( \mathcal{I}_d \) denote the set of all infinite-horizon MDPs that satisfy Assumption 1 and Assumption 2 and whose feature vectors have dimension \( d \), rewards lie in \([-1, 1]\). Let \( \mathcal{M}(S, A, P, s_0, C, C_0) \) denote the set of all probability measures on \( S \times A \) such that Assumption 3 holds with constants \( C \) and \( C_0 \). Let \( \eta \in (0, 1] \) be such that \( C < \frac{1}{\gamma^2} - \eta \), and \( \beta = \gamma \sqrt{C + \eta} < 1 \). With probability at least \( 1 - \delta \),
we have

\[
\inf_{\hat{V}} \sup_{(S,A,P,R,\gamma) \in \mathcal{I}_{d,\pi}} |\hat{V} - V^\pi(s_0)| \leq \frac{2\sqrt{C_0}}{1 - \beta} \sqrt{\frac{C_2 (d + \log \frac{3}{\beta})}{(1 - \gamma)^2 N}} + \frac{C_1}{\eta} \sqrt{\frac{1}{N} \log \frac{6d}{\delta} \|\theta^\pi\|_2^2},
\]

where \(C_1 = 4\sqrt{2}, C_2 = 12\), \(N\) is the number of samples from \(\mu\), and \(\hat{V}\) is a real-valued function with \(N\) samples as input. Suppose that Assumption 3 holds and that \(C \in (0, 1/\gamma^2)\) and \(C_0 > 0\) are the constants in Assumption 3. Particularly, if the sample distribution \(\mu\) satisfies Assumption 3 with constants \(C\) and \(C_0\) and we set \(\lambda = \frac{C_0}{\eta} \sqrt{N \log \frac{6d}{\delta}}\) in Algorithm 1, the following upper bound holds with probability at least \(1 - \delta\)

\[
\left( V^\pi(s_0) - \hat{V}_t(s_0) \right)^2 \leq 2C_0 \left[ \frac{2C_2 (d + \log \frac{3}{\beta})}{N (1 - \gamma)^2} + \beta^2 T \right] + \frac{2C_0C_1}{\eta} \sqrt{\frac{1}{N} \log \frac{6d}{\delta} \|\theta^\pi\|_2^2 \left( \frac{2}{(1 - \gamma)^2} + \beta^2 T \right)}. \]

**Remark 5.** If we hide the dependency on the constants \(\gamma, \beta,\) and \(\eta\) and focus on the rate with respect to \(N\) and \(T\), we have

\[
\left( V^\pi(s_0) - \hat{V}_t(s_0) \right)^2 \lesssim \frac{d + \log \frac{1}{\delta}}{N} + \beta^2 T + \sqrt{\frac{1}{N} \log \frac{d}{\delta} \|\theta^\pi\|_2^2 (1 + \beta^2 T)}. \]

If the sample size \(N \gtrsim \max \left\{ \frac{\|\theta^\pi\|_2^4}{\epsilon^2}, \frac{d}{\epsilon^2}, d + \log \frac{1}{\delta} \right\}\), after \(T \gtrsim \log \frac{1}{\epsilon}\) rounds, the additive error \(|Q^\pi(s_0,\pi(s_0)) - \hat{Q}_T(s_0,\pi(s_0))|\) is at most \(\epsilon\). [13, 23] assumed \(\|\theta^\pi\|_2 \leq O\left(\sqrt{d}\right)\) (see Assumption A in [13] and Theorem 5.2 in [23]). Under this additional assumption, we have \(\left( V^\pi(s_0) - \hat{V}_t(s_0) \right)^2 \lesssim \frac{d + \log \frac{1}{\delta}}{N} + \beta^2 T + d \sqrt{\frac{1}{N} \log \frac{d}{\delta} (1 + \beta^2 T)}\). To guarantee the additive error \(|Q^\pi(s_0,\pi(s_0)) - \hat{Q}_T(s_0,\pi(s_0))|\) \(\leq \epsilon\), we need \(N \gtrsim \frac{d^2}{\epsilon^2} \log \frac{d}{\delta}\). In fact, the assumption \(\|\theta^\pi\|_2 \leq O\left(\sqrt{d}\right)\) can be fulfilled if there exists a constant \(c > 0\) such that \(\sup_{\nu} \lambda_{\min} \left( \mathbb{E}(s,a) \sim \nu \phi(s,a)\phi(s,a)^\top \right) \geq c/d\), where the supremum is taken over all distributions on the state-action pairs. Since

\[
\frac{1}{(1 - \gamma)^2} \geq \sup_{\nu} \mathbb{E}(s,a) \sim \nu Q(s,a)^2 \geq \sup_{\nu} \mathbb{E}(s,a) \sim \nu \left( \phi(s,a)^\top \theta^\pi \right)^2 \\
\geq \sup_{\nu} \mathbb{E}(s,a) \sim \nu \lambda_{\min} \left( \mathbb{E}(s,a) \sim \nu \phi(s,a)\phi(s,a)^\top \right) \|\theta^\pi\|_2^2 \geq c/d \cdot \|\theta^\pi\|_2^2,
\]

it follows that \(\|\theta^\pi\|_2 \leq \frac{1}{\sqrt{c}} \cdot \frac{\sqrt{d}}{1 - \gamma}\). [23] justified this assumption using John’s theorem (see the footnote in Theorem 5.2).
5.1 Proof of Theorem 2

Prior to presenting the proof, we introduce some notation. Define $\xi_i \triangleq r_i + \gamma V^\pi(s_i) - Q^\pi(s_i, a_i)$, $\xi \triangleq (\xi_1, \ldots, \xi_N)^\top \in \mathbb{R}^N$, $\Phi^\top \triangleq (\phi(s_1, a_1), \ldots, \phi(s_N, a_N)) \in \mathbb{R}^{d \times N}$ and $\Phi^\top \triangleq (\phi(\bar{s}_1), \ldots, \phi(\bar{s}_N)) \in \mathbb{R}^{d \times N}$. Therefore, $\Phi$ and $\Phi^\top$ are $N \times d$ matrices. We have

$$
\hat{V}_t(s) = \mathbb{E}_{a \sim \pi(\cdot|s)} Q_t(s, a) = \mathbb{E}_{a \sim \pi(\cdot|s)} \phi(s, a)^\top \hat{\theta}_t = \bar{v}(s)^\top \hat{\theta}_t,
$$

$$
V^\pi(s) = \mathbb{E}_{a \sim \pi(\cdot|s)} Q^\pi(s, a) = \mathbb{E}_{a \sim \pi(\cdot|s)} \phi(s, a)^\top \theta^\pi = \bar{v}(s)^\top \theta^\pi.
$$

**Lemma 1.** Define $v \triangleq \hat{\Lambda}^{-1} (\Phi^\top \xi - \lambda \theta^\pi)$, and $B \triangleq \hat{\Lambda}^{-1} \gamma \Phi^\top \Phi$. The following equation holds

$$
\left( V^\pi(s_0) - \hat{V}_t(s_0) \right)^2 = \left\| \sum_{i=0}^{T-1} B^i v - B^T \theta^\pi \right\|_{\hat{\Lambda}^{-1}}^2.
$$

**Proof.** Define $Q_0(\cdot) = 0$ and $\hat{\theta}_0 = 0$. We have

$$
\hat{\theta}_{t+1} = \hat{\Lambda}^{-1} \left( \sum_{i \in [N]} \phi(s_i, a_i) \cdot \left( r_i + \gamma \hat{V}_t(s_i) \right) \right)
$$

$$
= \hat{\Lambda}^{-1} \left( \sum_{i \in [N]} \phi(s_i, a_i) \cdot \left( r_i + \gamma \mathbb{E}_{a \sim \pi(\cdot|s_i)} Q_t(s_i, a) \right) \right)
$$

$$
= \hat{\Lambda}^{-1} \left( \sum_{i \in [N]} \phi(s_i, a_i) \cdot \left( r_i + \gamma \mathbb{E}_{a \sim \pi(\cdot|s_i)} \phi(s_i, a)^\top \hat{\theta}_t \right) \right).
$$

Recall $\bar{v}(s) \triangleq \mathbb{E}_{a \sim \pi(\cdot|s)} \phi(s, a)$. Thus we obtain

$$
\hat{\theta}_{t+1} = \hat{\Lambda}^{-1} \left( \sum_{i \in [N]} \phi(s_i, a_i) \cdot \left( r_i + \gamma \bar{v}(s_i)^\top \hat{\theta}_t \right) \right)
$$

$$
= \hat{\Lambda}^{-1} \left( \sum_{i \in [N]} \phi(s_i, a_i) \cdot \left( r_i + \gamma \bar{v}(s_i)^\top \theta^\pi \right) \right) + \hat{\Lambda}^{-1} \left( \sum_{i \in [N]} \phi(s_i, a_i) \cdot \gamma \bar{v}(s_i)^\top \left( \hat{\theta}_t - \theta^\pi \right) \right).
$$

(5)
We compute the first term:
\[
\hat{\Lambda}^{-1} \left( \sum_{i \in [N]} \phi(s_i, a_i) \cdot \left( r_i + \gamma \tilde{\phi}(\bar{s}_i)^\top \theta^\pi \right) \right)
\]
\[
= \hat{\Lambda}^{-1} \left( \sum_{i \in [N]} \phi(s_i, a_i) \cdot \left( r_i + \gamma V^\pi(\bar{s}_i) \right) \right)
\]
\[
= \hat{\Lambda}^{-1} \left( \sum_{i \in [N]} \phi(s_i, a_i) \cdot (Q^\pi(s_i, a_i) + \xi_i) \right)
\]
\[
= \hat{\Lambda}^{-1} \left( \sum_{i \in [N]} \phi(s_i, a_i) \cdot \xi_i \right) + \hat{\Lambda}^{-1} \sum_{i \in [N]} \phi(s_i, a_i) \cdot \phi(s_i, a_i)^\top \theta^\pi
\]
\[
= \hat{\Lambda}^{-1} \Phi^\top \xi + \theta^\pi - \lambda \hat{\Lambda}^{-1} \theta^\pi,
\]
where the last equality is because \( \sum_{i \in [N]} \phi(s_i, a_i) \cdot \phi(s_i, a_i)^\top = \hat{\Lambda} - \lambda I_d \). Plugging Equation (6) into Equation (5) gives
\[
\hat{\theta}_{t+1} - \theta^\pi = \hat{\Lambda}^{-1} \Phi^\top \xi - \lambda \hat{\Lambda}^{-1} \theta^\pi + \hat{\Lambda}^{-1} \sum_{i \in [N]} \phi(s_i, a_i) \cdot \gamma \tilde{\phi}(\bar{s}_i)^\top \left( \hat{\theta}_t - \theta^\pi \right).
\]
If we define \( \Delta_t = \hat{\theta}_t - \theta^\pi \), we rewrite Equation (7)
\[
\Delta_{t+1} = \hat{\Lambda}^{-1} \left( \Phi^\top \xi - \lambda \theta^\pi \right) + \hat{\Lambda}^{-1} \gamma \Phi^\top \Phi \Delta_t = v + B \Delta_t.
\]
By induction, we deduce
\[
\Delta_T = \sum_{i=0}^{T-1} B^i v + B^T \Delta_0 = \sum_{i=0}^{T-1} B^i v - B^T \theta^\pi.
\]
Therefore, we conclude
\[
\left( V^\pi(s_0) - \hat{V}_T(s_0) \right)^2 = \left\| \theta^\pi - \hat{\theta}_T \right\|_{\hat{\Lambda}_0}^2 = \left\| \sum_{i=0}^{T-1} B^i v - B^T \theta^\pi \right\|_{\hat{\Lambda}_0}^2.
\]

**Lemma 2** (Matrix Hoeffding [20]). Consider a finite sequence \( \{X_k\} \) of independent, random, self-adjoint matrices with dimension \( d \), and let \( \{A_k\} \) be a sequence of fixed self-adjoint matrices. Assume that each random matrix satisfies \( \mathbb{E} X_k = 0 \) and \( X_k^2 \leq A_k^2 \) almost surely. Then, for all \( t \geq 0 \),
\[
P \left[ \lambda_{\text{max}} \left( \sum_k X_k \right) \geq t \right] \leq d \cdot e^{-t^2/8\sigma^2} \text{ where } \sigma^2 \triangleq \left\| \sum_k A_k^2 \right\|_2.
Corollary 1. Under the assumptions of Lemma 2 and further assuming $X_k$ are real symmetric, we have for all $t \geq 0$,

$$
P \left[ \left\| \sum_k X_k \right\|_2 \geq t \right] \leq 2d \cdot e^{-t^2/8\sigma^2}.
$$

Proof. If the matrix $A$ is real symmetric, we have

$$
\|A\|_2 = \sqrt{\lambda_{\text{max}}(A^2)} = \max_i |\lambda_i(A)| = \max\{\lambda_{\text{max}}(A), \lambda_{\text{max}}(-A)\}.
$$

Therefore,

$$
P \left[ \left\| \sum_k X_k \right\|_2 \geq t \right] \leq P \left[ \lambda_{\text{max}} \left( \sum_k X_k \right) \geq t \right] + P \left[ \lambda_{\text{max}} \left( - \sum_k X_k \right) \geq t \right] \leq 2d \cdot e^{-t^2/8\sigma^2}.
$$

Lemma 3 (Matrix concentration). There exists a universal constant $C_1 = 4\sqrt{2}$ such that with probability $1 - 2\delta$, we have

$$
\left\| \frac{1}{N} \Phi^\top \Phi - \Lambda \right\|_2 \leq C_1 \sqrt{\frac{1}{N} \log \frac{2d}{\delta}},
$$

$$
\left\| \frac{1}{N} \tilde{\Phi}^\top \tilde{\Phi} - \Lambda \right\|_2 \leq C_1 \sqrt{\frac{1}{N} \log \frac{2d}{\delta}}.
$$

Proof. To simplify the notation, write $\phi_i \triangleq \phi(s_i, a_i)$ and $\bar{\phi}_i \triangleq \bar{\phi}(\bar{s}_i)$. Moreover, write $\phi \triangleq \phi(s, a)$, where $(s, a) \sim \mu$. Therefore, $\phi$ is a random vector. Recall $\frac{1}{N} \Phi^\top \Phi = \frac{1}{N} \sum_{i=1}^N \phi_i \phi_i^\top$ and $\Lambda \triangleq \mathbb{E} [\phi \phi^\top]$. Let $X_i = \frac{1}{N} (\phi_i \phi_i^\top - \Lambda)$. We have $\mathbb{E} X_i = 0$ and

$$
\lambda_{\text{max}}(X_i^2) = \|X_i\|_2 \leq \frac{1}{N^2} \left( \|\phi_i \phi_i^\top\|_2 + \|\Lambda\|_2 \right)^2 \leq \frac{4}{N^2}.
$$

The last inequality is because $\|\phi_i \phi_i^\top\|_2 = \|\phi_i^\top \phi_i\|_2 = \|\phi_i\|_2^2 \leq 1$ and similarly $\|\Lambda\|_2 = \|\mathbb{E} [\phi \phi^\top]\|_2 \leq \mathbb{E} \|\phi \phi^\top\|_2 = \mathbb{E} \|\phi\|_2^2 \leq 1$. Therefore, if $A_i \triangleq \frac{2}{N} I$, we have $X_i^2 \preceq A_i^2 = \frac{4}{N^2} I$ and

$$
\sigma^2 = \left\| \sum_i A_i^2 \right\|_2 \leq \sum_i \|A_i^2\|_2 = \frac{4}{N}.
$$

By Corollary 1, we have

$$
P \left[ \left\| \frac{1}{N} \Phi^\top \Phi - \Lambda \right\|_2 \geq t \right] = P \left[ \left\| \frac{1}{N} \sum_i \phi_i \phi_i^\top - \Lambda \right\|_2 \geq t \right] \leq 2d \cdot e^{-Nt^2/32}.
$$
Therefore, with probability $1 - \delta$, we have

$$\left\| \frac{1}{N} \sum_i \phi_i \phi_i^\top - \Lambda \right\|_2 < 4\sqrt{\frac{2}{N} \log \frac{2d}{\delta}}.$$  

Similarly, we can show that with probability $1 - \delta$,

$$\left\| \frac{1}{N} \Phi^\top \Phi - \bar{\Lambda} \right\|_2 \leq 4\sqrt{\frac{2}{N} \log \frac{2d}{\delta}}.$$

\[\square\]

**Lemma 4** (Theorem 2.1 and Remark 2.2 [12]). Let $A \in \mathbb{R}^{m \times n}$ be a matrix, and let $\Sigma = A^\top A$. Suppose that $x \in \mathbb{R}^n$ is a random vector such that $\mathbb{E}[x] = 0$ and $\text{cov}(x) \preceq \sigma^2 I$. Then we have

$$\mathbb{P} \left[ \|Ax\|_2^2 > \sigma^2 \left( \text{tr}(\Sigma) + 2\sqrt{\text{tr}(\Sigma^2)}t + 2\|\Sigma\|_2 t \right) \right] \leq e^{-t}.$$  

**Proof of Theorem 2.** Recall $\hat{\Lambda} = \Phi^\top \Phi + \lambda I_d$. Conditioned on the event in Lemma 3, we have

$$\left\| \frac{1}{N} (\hat{\Lambda} - \lambda I_d) - \Lambda \right\|_2 \leq C_1 \sqrt{\frac{1}{N} \log \frac{2d}{\delta}},$$

where $C_1 = 4\sqrt{2}$. Because the spectral norm $\|\cdot\|_2$ of a matrix is greater than or equal to the absolute value of any eigenvalue, it follows that

$$\left| \lambda_{\min}(\hat{\Lambda} - \lambda I_d - N\Lambda) \right| \leq \left\| \hat{\Lambda} - \lambda I_d - N\Lambda \right\|_2 \leq C_1 \sqrt{N \log \frac{2d}{\delta}}.$$  

As a result, we get

$$\lambda_{\min}(\hat{\Lambda} - \lambda I_d - N\Lambda) \geq -C_1 \sqrt{N \log \frac{2d}{\delta}},$$

which implies

$$\hat{\Lambda} - N\Lambda \succeq \left( \lambda - C_1 \sqrt{N \log \frac{2d}{\delta}} \right) I_d \succeq 0. \quad (8)$$

Therefore, by Assumption 3, we deduce

$$\hat{\Lambda} \succeq N\Lambda \succeq \frac{N}{C} \hat{\Lambda}.$$  

In addition, conditioned on the event in Lemma 3, we have

$$\left\| \Phi^\top \Phi - N\bar{\Lambda} \right\|_2 \leq C_1 \sqrt{N \log \frac{2d}{\delta}}.$$  

It follows that

$$\left\| \hat{\Lambda}^{-1/2} (\Phi^\top \Phi - N\bar{\Lambda}) \hat{\Lambda}^{-1/2} \right\|_2 \leq \left\| \hat{\Lambda}^{-1/2} \right\|_2^2 \left\| \Phi^\top \Phi - N\bar{\Lambda} \right\|_2 = \left\| \hat{\Lambda}^{-1} \right\|_2 \left\| \Phi^\top \Phi - N\bar{\Lambda} \right\|_2 \leq \frac{C_1}{\lambda} \sqrt{N \log \frac{2d}{\delta}}.$$
Thus we obtain
\[ \| \hat{\Lambda}^{-1/2} \Phi \hat{\Phi} \hat{\Lambda}^{-1/2} \|_2 \leq N \| \hat{\Lambda}^{-1/2} \hat{\Lambda} \hat{\Lambda}^{-1/2} \|_2 + \| \hat{\Lambda}^{-1/2} (\hat{\Phi}^\top \hat{\Phi} - N \hat{\Lambda}) \hat{\Lambda}^{-1/2} \|_2 \leq C + \frac{C_1}{\lambda} \sqrt{N \log \frac{2d}{\delta}}. \]

By Equation (8) and Assumption 3, we have \( \hat{\Lambda} \succeq \frac{N}{C_0} \Lambda_0 \) and thereby
\[ \| \hat{\Lambda}^{-1/2} \Lambda_0 \hat{\Lambda}^{-1/2} \|_2 \leq C_0/N. \]

Using the Sherman–Morrison–Woodbury formula and writing \( \Phi \hat{\Phi}^\top = V D V^\top \) (\( D \) is a diagonal matrix with non-negative diagonal entries and \( V \) is orthogonal), we have
\[
\Phi \hat{\Lambda}^{-1} \Phi^\top = \Phi \left( \Phi^\top \Phi + \lambda I_d \right)^{-1} \Phi^\top \\
= \frac{1}{\lambda} \Phi \left( I_d - \Phi^\top \left( \lambda I_N + \Phi \Phi^\top \right)^{-1} \Phi \right)^{-1} \Phi^\top \\
= \frac{1}{\lambda} \left( V D V^\top - V D V^\top \left( \lambda I_N + V D V^\top \right)^{-1} V D V^\top \right)^{-1} V D V^\top \\
= \frac{1}{\lambda} V \left( D - D \left( \lambda I_N + D \right)^{-1} D \right) V^\top \\
= V D \left( \lambda I_N + D \right)^{-1} V^\top.
\]

The final equality is because
\[
D - D \left( \lambda I_N + D \right)^{-1} D = D \left( I_N - \left( \lambda I_N + D \right)^{-1} D \right) \\
= D \left( I_N - \left( \lambda I_N + D \right)^{-1} \left( \lambda I_N + D - \lambda I_N \right) \right) = \lambda D \left( \lambda I_N + D \right)^{-1}.
\]

Therefore we get \( \| \Phi \hat{\Lambda}^{-1} \Phi^\top \|_2 = \| D \left( \lambda I_N + D \right)^{-1} \|_2 \leq 1 \). For any \( v \in \mathbb{R}^d \), if \( B \triangleq \hat{\Lambda}^{-1} \gamma \Phi^\top \Phi \), it follows
\[
\begin{align*}
\| B^i \hat{\Lambda}^{-1/2} v \|_{\Lambda_0}^2 &= \| \hat{\Lambda}^{1/2} B^i \hat{\Lambda}^{-1/2} v \|_2^2 \\
&= \gamma^{2i} \| \hat{\Lambda}^{1/2} \left( \hat{\Lambda}^{-1/2} \Phi \hat{\Phi} \hat{\Lambda}^{-1/2} \right) i \hat{\Lambda}^{-1/2} v \|_2^2 \\
&= \gamma^{2i} \| \hat{\Lambda}^{1/2} \hat{\Lambda}^{-1/2} \left( \hat{\Lambda}^{-1/2} \Phi \hat{\Phi} \hat{\Lambda}^{-1/2} \right) i v \|_2^2 \\
&\leq \gamma^{2i} \| \hat{\Lambda}^{1/2} \hat{\Lambda}^{-1/2} \|_2^2 \| \hat{\Lambda}^{-1/2} \Phi \|_2^2 \| \hat{\Phi} \hat{\Lambda}^{-1/2} \|_2^2 \| v \|_2^2 \\
&= \gamma^{2i} \| \hat{\Lambda}^{-1/2} \Lambda_0 \hat{\Lambda}^{-1/2} \|_2 \| \Phi \hat{\Lambda}^{-1} \Phi^\top \|_2^i \| \hat{\Lambda}^{-1/2} \Phi \hat{\Phi} \hat{\Lambda}^{-1/2} \|_2 \| v \|_2^2 \\
&\leq \gamma^{2i} \frac{C_0}{\lambda^i} \left( C + \frac{C_1}{\lambda} \sqrt{N \log \frac{2d}{\delta}} \right) \| v \|_2^2
\end{align*}
\]
where Equation (9) is because for any matrix $A$, $\|A\|_2^2 = \|A^T A\|_2$ (in this equality, $A$ is $\hat{\Lambda}^{-1/2} \Phi^T$ or $\Phi \hat{\Lambda}^{-1/2}$) and the final inequality is because $\|\Phi \hat{\Lambda}^{-1} \Phi^T\|_2 \leq 1$. Recalling $\beta \triangleq \gamma \sqrt{C + \eta}$ and $\lambda \triangleq \frac{C_1}{\eta} \sqrt{N \log \frac{2d}{\delta}}$, we have $\gamma^{2i} \left(C + \frac{C_1}{\lambda} \sqrt{N \log \frac{2d}{\delta}}\right)^i = \gamma^{2i} (C + \eta)^i = \beta^{2i}$. As a result, we obtain

$$\left\| B^i \hat{\Lambda}^{-1/2} v \right\|_{\Lambda_0}^2 \leq \beta^{2i} \frac{C_0}{N} \|v\|_2^2$$

(10)

Since $\hat{\Lambda} = \Phi^T \Phi + \lambda I_d \geq \Phi^T \Phi$, we get $\hat{\Lambda}^{-1/2} \Phi^T \Phi \hat{\Lambda}^{-1/2} \leq I_d$ and $\left(\hat{\Lambda}^{-1/2} \Phi^T \Phi \hat{\Lambda}^{-1/2}\right)^2 \leq I_d$. Therefore, $\text{tr} \left(\Phi \hat{\Lambda}^{-1} \Phi^T\right) = \text{tr} \left(\hat{\Lambda}^{-1/2} \Phi^T \Phi \hat{\Lambda}^{-1/2}\right) \leq \text{tr} (I_d) = d$. Similarly, it follows that $\text{tr} \left(\left(\Phi \hat{\Lambda}^{-1} \Phi^T\right)^2\right) = \text{tr} \left(\left(\hat{\Lambda}^{-1/2} \Phi^T \Phi \hat{\Lambda}^{-1/2}\right)^2\right) \leq \text{tr} (I_d) = d$. Moreover, we have $\text{cov} (\xi) \leq \frac{4}{(1-\gamma)^2} I_N$ because each $\xi_i = r_i + \gamma V^\pi (s_i) - Q^\pi (s_i, a_i)$ is independent, $\mathbb{E} \xi_i = 0$, and $|\xi_i| \leq \frac{2}{1-\gamma}$.

Moreover, recall $\left\| \Phi \hat{\Lambda}^{-1} \Phi^T \right\|_2 \leq 1$. Using Lemma 4 gives

$$\mathbb{P} \left[ \left\| \hat{\Lambda}^{-1/2} \Phi^T \xi \right\|_2^2 > \frac{4}{(1-\gamma)^2} \cdot 3 (d + \tau) \right] \leq \mathbb{P} \left[ \left\| \hat{\Lambda}^{-1/2} \Phi^T \xi \right\|_2^2 > \frac{4}{(1-\gamma)^2} \left( d + 2\sqrt{d\tau} + 2\tau \right) \right] \leq e^{-\tau}.$$  

Let $\tau = \log \frac{1}{\delta}$. There exists $C_2 = 12$ such that with probability at least $1 - \delta$,

$$\left\| \hat{\Lambda}^{-1/2} \Phi^T \xi \right\|_2^2 \leq \frac{C_2 \left( d + \log \frac{1}{\delta} \right)}{(1-\gamma)^2},$$

On the other hand, we bound $\left\| \hat{\Lambda}^{-1/2} \lambda \theta^\pi \right\|_2^2$ as follows

$$\left\| \hat{\Lambda}^{-1/2} \lambda \theta^\pi \right\|_2^2 = \lambda^2 \left\| \theta^\pi \right\|_{\hat{\Lambda}^{-1}}^2 \leq \lambda \left\| \theta^\pi \right\|_2^2.$$  

Define $v \triangleq \hat{\Lambda}^{-1} (\Phi^T \xi - \lambda \theta^\pi)$ and recall $B \triangleq \hat{\Lambda}^{-1} \gamma \Phi^T \Phi$. We are in a position to bound
\[ \| B^i v \|_{\Lambda_0}^2 : \]

\[
\| B^i v \|_{\Lambda_0}^2 = \left\| B^i \hat{\Lambda}^{-1/2} \hat{\Lambda}^{-1/2} (\Phi^\top \xi - \lambda \theta^\pi) \right\|_{\hat{\Lambda}_0}^2 \\
\leq \beta^2 \frac{C_0}{N} \left\| \hat{\Lambda}^{-1/2} (\Phi^\top \xi - \lambda \theta^\pi) \right\|_2^2 \\
\leq \beta^2 \frac{2C_0}{N} \left( \left\| \hat{\Lambda}^{-1/2} \Phi^\top \xi \right\|_2^2 + \left\| \hat{\Lambda}^{-1/2} \lambda \theta^\pi \right\|_2^2 \right) \\
\leq \beta^2 \frac{2C_0}{N} \left( \frac{C_2 (d + \log \frac{1}{\delta})}{(1 - \gamma)^2} + \lambda \left\| \theta^\pi \right\|_2^2 \right),
\]

where the first inequality is because of Equation (10) and the second inequality is because \( \|a + b\|_2 \leq 2 (\|a\|_2^2 + \|b\|_2^2) \) for any vector \( a \) and \( b \). It follows that

\[
\| B^i v \|_{\Lambda_0} \leq \beta^i \sqrt{ \frac{2C_0}{N} \left( \frac{C_2 (d + \log \frac{1}{\delta})}{(1 - \gamma)^2} + \lambda \left\| \theta^\pi \right\|_2^2 \right)}. 
\]

As a result, we get

\[
\left\| \sum_{i=0}^{T-1} B^i v \right\|_{\Lambda_0}^2 \leq \left( \sum_{i=0}^{T-1} \| B^i v \|_{\Lambda_0} \right)^2 \leq \frac{2C_0}{N} \left( \frac{C_2 (d + \log \frac{1}{\delta})}{(1 - \gamma)^2} + \lambda \left\| \theta^\pi \right\|_2^2 \right) \left( \sum_{i=0}^{T-1} \beta^i \right)^2 \\
= \frac{2C_0}{N} \left( \frac{C_2 (d + \log \frac{1}{\delta})}{(1 - \gamma)^2} + \lambda \left\| \theta^\pi \right\|_2^2 \right) \left( \frac{1 - \beta^T}{1 - \beta} \right)^2.
\]

Since

\[
\left\| \hat{\Lambda}^{1/2} \theta^\pi \right\|_2^2 = \left[ \theta^\pi \right]^\top \hat{\Lambda} \theta^\pi = \left[ \theta^\pi \right]^\top \left( \sum_{i \in [N]} \phi(s_i, a_i) \phi(s_i, a_i)^\top + \lambda I_d \right) \theta^\pi \\
= \sum_{i \in [N]} Q(s_i, a_i)^2 + \lambda \left\| \theta^\pi \right\|_2^2 \leq \frac{N}{(1 - \gamma)^2} + \lambda \left\| \theta^\pi \right\|_2^2,
\]

using Equation (10) again, we have

\[
\| B^T \theta^\pi \|_{\Lambda_0}^2 = \left\| B^T \hat{\Lambda}^{-1/2} \hat{\Lambda}^{1/2} \theta^\pi \right\|_{\Lambda_0}^2 \leq \frac{C_0}{N} \beta^{2T} \left\| \hat{\Lambda}^{1/2} \theta^\pi \right\|_2^2 \leq \frac{C_0}{N} \beta^{2T} \left( \frac{N}{(1 - \gamma)^2} + \lambda \left\| \theta^\pi \right\|_2^2 \right).
\]
In light of Lemma 1, we have
\[
\left( V^\pi(s_0) - \hat{V}_t(s_0) \right)^2 \\
\leq 2 \left\| \sum_{i=0}^{T-1} B_i v \right\|_{\Lambda_0}^2 + 2 \left\| B^T \theta^\pi \right\|_{\bar{\Lambda}_0}^2 \\
\leq \frac{2C_0}{N} \left[ 2 \left( \frac{C_2 (d + \log \frac{1}{\delta})}{(1-\gamma)^2} + \lambda \left\| \theta^\pi \right\|_2^2 \right) \left( \frac{1 - \beta^T}{1-\beta} \right)^2 + \beta^{2T} \left( \frac{N}{(1-\gamma)^2} + \lambda \left\| \theta^\pi \right\|_2^2 \right) \right] \\
\leq \frac{2C_0}{N} \frac{2}{(1-\beta)^2} \left( \frac{C_2 (d + \log \frac{1}{\delta})}{(1-\gamma)^2} + \lambda \left\| \theta^\pi \right\|_2^2 \right) + \beta^{2T} \left( \frac{N}{(1-\gamma)^2} + \lambda \left\| \theta^\pi \right\|_2^2 \right). 
\]

We use \(1 - \beta^T \leq 1\) in the last inequality. Plugging in \(\lambda = \frac{C_1}{\eta} \sqrt{N \log \frac{2d}{\delta}}\), we deduce that with probability at least \(1 - 3\delta\),
\[
\left( V^\pi(s_0) - \hat{V}_t(s_0) \right)^2 \\
\leq \frac{2C_0}{(1-\gamma)^2} \left[ \frac{2C_2 (d + \log \frac{1}{\delta})}{N (1-\gamma)^2} + \beta^{2T} \right] + \frac{2C_0C_1}{\eta} \sqrt{\frac{1}{N} \log \frac{2d}{\delta} \left\| \theta^\pi \right\|_2^2 \left( \frac{2}{(1-\beta)^2} + \beta^{2T} \right)}.
\]

Therefore, with probability \(1 - \delta\), if \(\lambda = \frac{C_1}{\eta} \sqrt{N \log \frac{6d}{\delta}}\), we have
\[
\left( V^\pi(s_0) - \hat{V}_t(s_0) \right)^2 \\
\leq \frac{2C_0}{(1-\gamma)^2} \left[ \frac{2C_2 (d + \log \frac{3}{\delta})}{N (1-\gamma)^2} + \beta^{2T} \right] + \frac{2C_0C_1}{\eta} \sqrt{\frac{1}{N} \log \frac{6d}{\delta} \left\| \theta^\pi \right\|_2^2 \left( \frac{2}{(1-\beta)^2} + \beta^{2T} \right)}.
\]

Equation (4) is obtained by taking \(T \to +\infty\).

\[\square\]

6 Conclusion

In this work we study the sample complexity of offline infinite-horizon reinforcement learning with linear function approximation. We identify a hard regime \(d\gamma^2 > 1\). In this regime, we show a lower bound on the sample complexity, which is exponential in the dimension \(d\) and potentially infinite, depending on the desired condition number of the hard instance. Assuming low distribution shift, we show that there exists an algorithm that can approximate the value of a state up to arbitrary precision and requires at most polynomially many samples.

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