Universal Features of the Time Evolution of Evanescent Modes in a Left-Handed Perfect Lens.

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The time evolution of evanescent modes in Pendry’s perfect lens proposal for ideally lossless and homogeneous, left-handed materials is analyzed. We show that time development of sub-wavelength resolution exhibits universal features, independent of model details. This is due to the unavoidable near-degeneracy of surface electromagnetic modes in the deep sub-wavelength region. By means of a mechanical analog, it is shown that an intrinsic time scale (missed in stationary studies) has to be associated with any desired lateral resolution. A time-dependent cut-off length emerges, removing the problem of divergences claimed to invalidate Pendry’s proposal.

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Long ago, Veselago \(^1\) pointed out that very unusual properties, such as negative refraction, would be exhibited by materials with negative refraction index. Recently, Pendry \(^2\) has claimed that those so called left-handed (LH) materials with \(\varepsilon(\omega_0) = \mu(\omega_0) = -1\), can act as perfect lenses with, ideally, arbitrary sub-wavelength resolution. In addition to its genuine conceptual importance, this proposal has attracted much attention due to the practical realization of man-made materials expected to be left-handed \(^3\), where negative refraction has been claimed to be observed \(^4\). The challenge to conventional ideas conveyed in Pendry’s work has fueled a heated debate that has contributed to sharpen the issue, if not to settle it \(^5, 6, 7, 8\).

The criticism expressed by García and N-Vesperinas \(^7\) (see also Refs. \(^7, 8\)) seems of particular importance, for it would imply that some sort of fundamental violation of physical laws is unavoidable in Pendry’s perfect lens. The idea is that the necessary amplification of evanescent modes would turn a square-integrable incoming wave into a non normalizable signal, something deemed unacceptable by the authors of Refs. \(^7, 8\). Pendry \(^8\) has replied that losses, unavoidable in the real world, would provide a natural cut-off, forbidding the amplification of large wavevectors (though degrading the perfect resolution). Haldane \(^10\) has put the blame of divergences on the role played by surface modes (polaritons) \(^11\) localized at the interfaces of the LH material: amplification of evanescent modes is a gift of those modes \(^2\). For a homogeneous material, those surface modes exist and become dispersionless for increasing wavevectors, being the unphysical absence of an intrinsic cut-off length which causes the pathologies \(^10\). Any realization of a LH material (think of the composite nature of proposed LH materials \(^4\) or photonic crystals \(^12\)) must have a natural length below which the homogeneous description fails, providing the necessary cure to divergences at the price of lower resolution.

Although Pendry’s and Haldane’s escapes from divergences are certainly safe, they seem to suggest that the textbook idealization of lossless and homogeneous media, that works so well otherwise \(^13\), is fundamentally flawed when applied to a material that happens to satisfy \(\varepsilon(\omega_0) = \mu(\omega_0) = -1\), at some frequency \(\omega_\text{0}\). Apparently, only after the inclusion on the real world constraints (losses and/or small-distance structure) could the proposal be made physically acceptable. I find this state of affairs very unsatisfactory, and it is the purpose of this paper to show that, even within the self-imposed idealizations of a lossless (for \(\omega = \omega_0\)) and purely homogeneous, left-handed material, Pendry’s perfect lens proposal is correct.

To this end, we will consider the time evolution \(^7, 14, 15\) of sub-wavelength features. Time development requires the study of the frequency dispersion of a particular model (or material), apparently preventing us from drawing general conclusions. However, it will be shown that the structure of the relevant magnitudes in the immediate vicinity of the target frequency is dominated by the surface polariton modes \(^10, 11\). These modes show universal features for any LH material at long wavevectors, therefore allowing us to extract general conclusions. A key point will be the identification of a time scale for any intended length resolution. This time scale, linked to the near-degeneracy of surface modes and missed in stationary studies, will provide the clue for overcoming the problem of divergences.

Consider a transverse current oscillating at frequency \(\omega\), located in a source plane at \(x = x_0\) (see Fig. \(\text{a}\)). The Fourier component \((k)\) of the current leads to an electric field given by \((S\text{ polarization, for simplicity)}\):

\[
E = [0, E_k(x, t), 0] e^{ikz} \tag{1}
\]

In vacuum and for evanescent modes, \(E_k(x, t) \sim \rho e^{−x |x| c} e^{−i\omega t}\), where \(\rho = \sqrt{k^2 − \omega^2}\), in units with light velocity \(c = 1\) (the connection between current and field need not concern us here \(^15\)). In the presence of a slab of LH material between \(x = ±a\), the field acquires the usual reflected and transmitted
components (only the vacuum part explicit and for arbitrary normalization of the incoming wave):
\[
E_k(x,t) = [e^{-i(x-x_0)} + r_k(\omega) e^{i(x-x_0)}] e^{-i\omega t}, \quad x < -a \\
E_k(x,t) = t_k(\omega) e^{-i(x-x_0)} e^{-i\omega t}, \quad x > a
\]

Pendry \[2\] has shown that, for a frequency \(\omega_0\) such that \(\epsilon(\omega_0) = \mu(\omega_0) = -1\), the reflection and transmission coefficients are:
\[
r_k(\omega_0) = 0, \quad t_k(\omega_0) = \exp(4\rho_0 a) \tag{3}
\]
with
\[
\rho_0 = \sqrt{k^2 - \omega_0^2} \tag{4}
\]

This leads to a perfect restoration at the focal plane \(x_1 = 2a\) of a source field at \(x_0 = -2a\), as shown in Fig. 1.

The exponential amplification inside the LH material, necessary for image restoration, is at the root of the divergences pointed out before \[2,4\]. Notice that a square integrable field with Fourier components \(E_k\) at the source plane \(x_0 = -2a\), will emerge at the plane \(x = a\) with norm:
\[
N = \int d^2r ||E(x = a, r)||^2 \propto \sum_k |E_k e^{\rho_0 a}|^2 \tag{5}
\]

This norm certainly diverges for large wavevectors, unless unjustified constraints are put on the source field.

So far we have described the stationary solution, but our aim is the time development of such a situation. Therefore, the frequency behavior of \(r(\omega)\) and \(t(\omega)\) is needed. In the vicinity of \(\omega = \omega_0\), and in the deep subwavelength region, the reflection and transmission coefficients exhibit a highly singular behavior controlled by the appearance of simple poles. This singular behavior can be described by:
\[
r_k(\omega) = e^{4\rho_0 a} \frac{(\omega^2 - \omega_0^2)}{(\omega^2 - \omega_{k,+}^2)(\omega^2 - \omega_{k,-}^2)} \tag{6}
\]
and is depicted in Fig. 2 (left panel). One can see that, for instance, \(t_k(\omega)\) is basically the sum of four simple poles located at frequencies \(\pm \omega_{k,\pm}\), corresponding to the surface modes (right panel of Fig. 2). These are the even and odd combinations of the fundamental mode of an isolated interface, that would take place exactly at \(\omega = \omega_0\). The frequencies \(\omega_{k,\pm}\) are solutions of the equations \[11\]:
\[
|\tan(\rho_0 a)|^2 = -\mu(\omega) \rho / \rho_m, \quad \rho_m = \sqrt{k^2 - \epsilon(\omega) \mu(\omega) \omega^2}
\]
with an exponentially small coupling \[10\] for large wavevectors
\[
\eta_k^2 \simeq 8 C \omega_0^2 e^{-2\rho_0 a} \tag{7}
\]
where \((C\omega_0)^{-1} = 2\mu'_0 + (\epsilon'_0 + \mu'_0)(k^2/\omega_0^2 - 1)^{-1}\), with \(\mu'_0 = d\mu(\omega_0)/d\omega\) and \(\epsilon'_0 = d\epsilon(\omega_0)/d\omega\). It is important to realize that the expressions of Eq. 6 though only valid in the neighborhood of \(\omega_0\), do contain the singular structure associated with the surface modes. Therefore, the relevant dynamics of these modes is expected to be well reproduced. Ruppin \[11\] has studied these modes thoroughly, and their importance for the amplifying process has been noticed before \[2,10,12\].

We can now study, for instance, the time evolution of the transmitted field at the right interface.
Now we force the left oscillator with the external force \( E_{\text{trans}}(x = a, t) \), in terms of the incident field at the left interface \( E_{\text{inc}}(x = -a, t) \). This is enough to understand the amplification issue. We will assume a pure sinusoidal wave at the frequency \( \omega_0 \), but with a well defined time origin, chosen to coincide with the signal arrival at the left interface:

\[
E_{\text{inc}}(x = -a, t) = \theta(t)(E_0 e^{-i\omega_0 t} + \text{c.c.}),
\]

\( \theta(t) \) being Heaviside’s unit step function.

The evaluation of \( E_{\text{trans}}(x = a, t) \) is now trivial but, given the unusual and controversial nature of this subject, we choose to change the language in the hope of bringing the results to a far more familiar situation. Let us say that we have two identical (left and right) oscillators, \((x_l, x_r)\), with natural frequency \( \omega_0 \) and a weak coupling \( \eta_0^2 \) that splits the degeneracy \( \omega_0^2 = \omega_0^2 \pm \eta_0^2 \). Now we force the left oscillator with the external force \( f(t) \) and watch the dynamics of the right oscillator. The equations of motion are:

\[
\begin{align*}
\ddot{x}_l + \gamma \dot{x}_l + \omega_0^2 x_l + \eta_0^2 x_r &= f(t) \\
\ddot{x}_r + \gamma \dot{x}_r + \omega_0^2 x_r + \eta_0^2 x_l &= 0
\end{align*}
\]

(10)

where a damping \( \gamma \) has been added for later convenience but, in accordance with our idealization, is supposed to be \( \gamma = 0 \), for the moment. Upon identifying the external force, left, and right oscillators with the incident, reflected \( (E_{\text{ref}}) \), and transmitted fields in the following manner:

\[
\begin{align*}
f(t) &\leftrightarrow \eta_0^2 e^{2\rho_0 a} E_{\text{inc}}(x = -a, t) \\
x_l &\leftrightarrow E_{\text{ref}}(x = -a, t) \\
x_r &\leftrightarrow E_{\text{trans}}(x = +a, t),
\end{align*}
\]

(11)

this simple problem is entirely equivalent to our original one. The interpretation of Eq. (11) is direct: the incoming wave plays the role of an external force hitting the left interface, and exciting a reflected (left oscillator) and a transmitted (right oscillator) wave. This mapping allows us to understand the physics of the original situation in simpler terms. For instance, forcing the system (left oscillator) with frequency \( \omega_0 \), the stationary solution tells us that, surprisingly, only the right oscillator moves. This corresponds (through Eq. (11)) to the absence of a reflected wave.

The time development of the field corresponding to the incoming perturbation of Eq. (3) is now easily obtained, with the following result:

\[
E_{\text{trans}}(x = +a, t) = A(t) e^{2\rho_0 a} \theta(t) E_0 e^{-i\omega_0 t} + \text{c.c.}
\]

(12)

where a characteristic time-dependent modulation amplitude \( A(t) \) appears:

\[
A(t) = 1 - (e^{-i\Delta\omega_{+} t} + e^{-i\Delta\omega_{-} t}) / 2
\]

(13)

where \( \Delta\omega_{\pm} = \omega_{k,\pm} - \omega_0 \simeq \pm \Delta\omega_k / 2 \) for \( |\Delta\omega_k| < \omega_0 \), with the splitting of polariton modes \( \Delta\omega_k \) given by:

\[
\Delta\omega_k = \omega_{k, +} - \omega_{k, -} \simeq 8 C \omega_0 e^{-2\rho_0 a}
\]

(14)

in the deep sub-wavelength region. This beating of modes, familiar from the mechanical analog, is sketched in Fig. 3. Notice that the total response is the superposition of the stationary solution \( (\omega_0) \), which is Pendry’s solution, with the normal modes \( (\omega_{\pm}) \). The latter are unavoidably excited and do not decay in time owing to the absence of losses around \( \omega_0 \).

The relevance of this behavior for the problem of divergences should be clear by now. For a fixed lapse of time \( t \) (large in units of the bare period: \( \omega_0 t >> 1 \), but otherwise arbitrary), short wavelengths corresponding to splittings smaller than \( \Delta\omega_k \sim t^{-1} \), have barely begun to emerge at the right interface. Therefore, we can identify a time-dependent crossover wavevector \( k_{\text{eff}}(t) \), satisfying the condition:

\[
t \Delta\omega_{k_{\text{eff}}} = 1
\]

(15)

such that, for \( k >> k_{\text{eff}} \), then

\[
A(t) \sim (\Delta\omega_k t)^2 \sim (C\omega_0 t)^2 e^{-4\rho_0 a}
\]

(16)

The norm of Eq. (3) is now replaced by

\[
N \propto (C\omega_0 t)^4 \sum_k |\mathcal{E}_k e^{-3\rho_0 a}|^2,
\]

(17)

for \( k >> k_{\text{eff}} \). Therefore, amplification has been replaced by decay, solving the problem of divergences. Of course, progressively shorter details take longer and longer time (and more energy from the current source) to develop [14, 15], but nothing pathological affects the physics of the system. I emphasize again that this picture is unavoidable in the deep sub-wavelength limit, irrespective of the model details of \( \varepsilon(\omega) \) and \( \mu(\omega) \). It amounts to recognizing that, for a given wavevector \( k \), the system’s dynamics has a characteristic time scale given by the (inverse) of the frequency splitting between the two surface polaritons corresponding to that wavevector (Eq. (14)). The relevance of this time scale would remain hidden if only stationary solutions were analyzed. It is a curious though rewarding fact that, in a system that has been idealized to be homogeneous down to arbitrary small scales (absence of intrinsic length scale), time takes up the responsibility of providing the necessary cut-off length in order to prevent divergent catastrophes.

It could be argued that the solution provided here to the problem of divergences is an artifact of the ideal lossless nature of the problem. Real systems would show losses that will eventually kill any transient regime, perhaps leaving again the divergent stationary solution. This is not the case, as can be seen by the explicit inclusion of a finite life-time \( (\tau = \gamma^{-1}) \) in Eq. (10). A stationary solution is indeed established, but with amplitude \( A_{\text{stat}} \) greatly reduced from the lossless limit \( A_{\text{stat}}(\infty) \) at large wavevectors:

\[
\frac{A_{\text{stat}}(\tau)}{A_{\text{stat}}(\infty)} = \frac{1}{1 + (\tau \Delta\omega_k)^2}
\]

(18)
radiation of wavelength \( \lambda \)

\( \Delta \tau \gg (\Delta \omega_k = \infty) \), weak \( (\Delta \omega_k = 10) \), and strong \( (\Delta \omega_k = 0.5) \) damping.

Notice that, for small values of \( \tau \Delta \omega_k \), the stationary amplitude is what would have been expected if the lossless evolution were suddenly stopped at a time of the order of \( \tau \). Therefore, we see exactly the same physics as in the stationary limit approaches the lossless case. This reinforces the view \cite{12} that both problems (left-handed amplification and extraordinary transmission through hole arrays \cite{19}) are probably different manifestations of the same physical behavior of surface modes.

In spite of potential difficulties in the road to a practical left-handed amplifier, the essential physics described in the paper may have been observed in a different system. The role of coherence between surface modes (and the required time scale associated with it) in the explanation of the extraordinary transmission through hole arrays \cite{14} is strikingly similar to our treatment. This reinforces the view \cite{2} that both problems (left-handed amplification and extraordinary transmission through hole arrays) are probably different manifestations of the same physical behavior of surface modes.

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![Graph showing the time evolution of the modulation amplitude](image)

**Fig. 3:** Time evolution of the modulation amplitude (in units of the stationary, nearly lossless limit) for three values of surface life-time corresponding to zero \( (\tau \Delta \omega_k = \infty) \), weak \( (\tau \Delta \omega_k = 10) \), and strong \( (\tau \Delta \omega_k = 0.5) \) damping.

\[ \tau \Delta \omega_{k_{eff}} = 1 \]  \hspace{1cm} (19)

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such that, for wavenumbers larger than this cut-off, exponential decay rather than amplification is observed at the exit of the slab. The norm of field at the exit is now 

\[ A(0)/A_{stat} (\tau \to \infty) \]

\( \tau \Delta \omega_{k_{eff}} = 1 \)  \hspace{1cm} (19)

The transient behavior for the two regimes \( \tau \gg (\Delta \omega_k)^{-1} \) and \( \tau \leq (\Delta \omega_k)^{-1} \) is illustrated in Fig. 3. The picture is simple: when the life-time of surface polaritons is much longer than the characteristic time scale (inverse splitting of modes) for the wavevector \( k \), the stationary limit approaches the lossless case. This means that surface modes have had enough time to build the final response before dying away. On the other hand, if the surface modes do not live long enough, no amplification is possible.

In the deep sub-wavelength limit, Eq. 19 allows us to provide an explicit expression for the minimum life-time \( (\tau_{min}) \) required to get a resolution \( l_{res} = 2\pi/k_{eff} \) with radiation of wavelength \( \lambda_0 = 2\pi/\omega_0 \):

\[ \tau_{min} \sim \frac{1}{8 C^2 \omega_0} \exp(4\pi a \sqrt{l_{res}^2 - \lambda_0^2}) \]  \hspace{1cm} (20)

Similar results have been obtained before for the resolution \cite{19}, further reinforcing the correctness of our restriction to the polariton dynamics. Notice the characteristic exponential dependence on resolution. Although mainly concerned with matters of principle in this paper, this demanding result clearly shows that the polariton life-time may well be a major limiting factor for a practical realization of perfect lenses. I believe this vulnerability \cite{13,14} of the ideal situation, a fingerprint of the near-degeneracy of surface modes, is at the root of similar results found before \cite{17,18}.

\[ A(t)/A_{stat} (\tau \to \infty) \]

\( \tau \Delta \omega_{k_{eff}} = 1 \)  \hspace{1cm} (19)

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