Analytical matrix solutions of linear ordinary differential equations with constant coefficients

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Abstract. The article puts forward a modified finite element method based on decomposition and analytical solution techniques. The algorithm is as follows. A complex structure is divided into simple form sub-regions which involve partial differential equations. Next, the equations are decomposed. The decomposed equation solutions are written using analytical solution formulae. Meanwhile, the finite element size of the method proposed is defined only by the value of an averaging interval of required functions, since ordinary differential equation formulae are analytical. The algorithm has been tested by solving rectangular plate and bicurved shallow shell bending problems. The results proved proper convergence to precise values with increasing number of finite elements.

1. Introduction
The applied problem of continuum mechanics involves the analysis of thin structure stresses, e.g. aircraft. Meanwhile, the mathematical problem involves the controllable error construction and analysis of mathematical models of shell and plate deformation mechanics, i.e. in an analytical manner [1-6].

The mathematical models of plate and shell mechanics are generally based on partial differential equations. Such equations reduce to ordinary differential expressions and are solved using the Fourier method (separation of variables) among others. After the separation of variables, the complete system of ordinary differential equations as to the required values describing shell section state is written down as first order equation system

\[
\begin{align*}
\frac{dy_1}{dx} &= a_{11}y_1 + a_{12}y_2 + \ldots + a_{1n}y_n + f_1, \\
\frac{dy_2}{dx} &= a_{21}y_1 + a_{22}y_2 + \ldots + a_{2n}y_n + f_2, \\
\frac{dy_n}{dx} &= a_{n1}y_1 + a_{n2}y_2 + \ldots + a_{nm}y_n + f_n,
\end{align*}
\]

and represented as matrix

\[y'(x) = A(x)y(x) + f(x),\quad (1)\]
where \( \|A(x)_{ij}\| \ (i = 1, 2, \ldots, 8; j = 1, 2, \ldots, 8) \) is a matrix of variable and constant coefficients of the equation system, \( \|y_{ii}\| \ (i = 1, 2, \ldots, 8; j = 1) \) is a shell section properties column.

2. Matrix solution of boundary value problems

To solve the boundary value problem, the given boundary conditions are written down in a matrix form

\[
Uy(x_0) = u \\
Vy(x_n) = v
\]  

(2)

where

\[
\|U_{ij}\| \ (i = 1, 2, \ldots, 4; j = 1, 2, \ldots, 8), \|V_{ij}\| \ (i = 1, 2, \ldots, 4; j = 1), \|u_{ij}\| \ (i = 1, 2, \ldots, 4; j = 1), \|v_{ij}\| \ (i = 1, 2, \ldots, 4; j = 1).
\]

The solution of the boundary value problem (1) includes the general solution of the corresponding homogeneous differential equation and particular solution of non-homogeneous equation. The solution of the corresponding homogeneous equation

\[
y'(x) = A(x)y(x),
\]  

(3)

in \((x_1, x_n)\) is obtained through Picard method and can be generally written as follows

\[
y(x) = K_{x_0}^{x_n}(A(x))y_0,
\]

where \(K_{x_0}^{x_n}(A(x))\) is a Cauchy-Krylov function matrix. Cauchy-Krylov function matrix has a meaning of a matrix exponential, thus, it has the multiplicative property. If the matrix exponential solution interval is divided into sub-intervals, then the product of sub-interval matrix exponentials is equal to a complete interval matrix exponential

\[
K_{x_0}^{x_n}(A(x)) = K_{x_n}^{x_{n-1}}K_{x_{n-2}}^{x_{n-1}}\ldots K_{x_0}^{x_1}.
\]

If the differential equation coefficients are constant (e.g. cylindrical shell), Cauchy-Krylov function matrix is found as follows

\[
K_{x_0}^{x_n}(A) = \sum_{m=0}^{m=\infty} \frac{(A\Delta x)^m}{m!}, \Delta x = x_n - x_0.
\]  

(4)

In case of variable coefficients (conical, spherical shells) the integration segment is divided into \(n\) sub-intervals, where the differential equation coefficients (1) can be averaged. Then, each sub-interval can involve formula (4) followed by the multiplicative property of Cauchy-Krylov function matrix

\[
K_{x_0}^{x_n}(A(x)) = \prod_{i=n}^{i=m} \sum_{m=0}^{m=\infty} \frac{[A(\tau_i)\Delta x_i]^m}{m!},
\]

where \(A(\tau_i)\) are averaged coefficient matrixes of the equation (3).

The partial solution of the differential equation is defined by an integral of so-called Cauchy matrix. In an arbitrary interval \((x_{i-1}, x_i)\) it is defined as follows

\[
y_{x_{i-1}}^{x_i} = K_{x_{i-1}}^{x_i}(A_i) \int_{x_{i-1}}^{x_i} \left[ K_{\tau_{i-1}}^{\tau_i}(A_i) \right]^{-1} f_i d\tau_i.
\]  

(5)
Or:

\[ y_{x_i-1}^{x_i} = e^{A_i \Delta x_i} \int_{x_{i-1}}^{x_i} e^{-A_i \Delta \tau_i} f_i d\tau_i, \quad \left[ e^{A_i \Delta \tau_i} \right]^{-1} = e^{-A_i \Delta \tau_i} \]

\[ \Delta x_i = x_i - x_{i-1}, \quad \Delta \tau_i = \tau_i - x_{i-1}, \tau_i \in [x_{i-1}, x_i] \]

The elements of column \( f_i \) are averaged and the column is taken out of the integral. The matrix exponential under the integral is replaced with a matrix series. Then integration is performed. Then

\[ y_{x_i-1}^{x_i} = e^{A_i \Delta x_i} T_i f_i \Delta x_i, \quad T_i = E - \frac{A_i \Delta x_i}{2!} + \frac{(A_i \Delta x_i)^2}{3!} - \frac{(A_i \Delta x_i)^3}{4!} + \ldots \]  

(6)

By multiplying series of the matrix exponential and matrix \( T_i \), we shall obtain a formula of a particular solution in the arbitrary interval \((x_{i-1}, x_i)\)

\[ y_{x_i-1}^{x_i} = T_i f_i \Delta x_i, T_i = E + \frac{A_i \Delta x_i}{2!} + \frac{(A_i \Delta x_i)^2}{3!} + \frac{(A_i \Delta x_i)^3}{4!} + \ldots \]  

(7)

Formula (7) defines the particular solution of the equation (1).

The simple algorithm of stable calculation dedicated to the boundary value problem (1)(2), lies in the fact that the main interval with intermediate points \(x_1, x_2, \ldots, x_i, \ldots, x_{n-1}\) is divided into stable calculation intervals. The differential equation solution is defined in each interval with a controllable error under one of the formulae. In fact, the solution establishes a link between the interval ends, i.e. between the parameters of she’ll section states

\[ y(x_i) = K_{x_i-1}^{x_i} (A(x)) y(x_{i-1}) + y_{x_i-1}^{x_i}, \]  

(8)

Let us write down the equations (8) for each interval as follows

\[-K_{x_0}^{x_1} (A(x)) y(x_0) + Ey(x_1) = y_{x_0}^{x_1}, \]

\[-K_{x_1}^{x_2} (A(x)) y(x_1) + Ey(x_2) = y_{x_1}^{x_2}, \]

\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]

\[-K_{x_{n-1}}^{x_n} (A(x)) y(x_{n-1}) + Ey(x_n) = y_{x_{n-1}}^{x_n}. \]

The obtained equations are one less than the section number including extremes. Then we supplement the system with boundary conditions (2). The algebraic equation system derived in such a manner is represented as matrix

\[
\begin{bmatrix}
U & 0 & 0 & 0 & \ldots & 0 & 0 \\
-K_{x_0}^{x_1} & E & 0 & 0 & \ldots & 0 & 0 \\
0 & -K_{x_1}^{x_2} & E & 0 & \ldots & 0 & 0 \\
0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & -K_{x_{n-1}}^{x_n} & E \\
0 & 0 & 0 & \ldots & 0 & V
\end{bmatrix}
\times
\begin{bmatrix}
y(x_0) \\
y(x_1) \\
y(x_2) \\
\vdots \\
y(x_{n-1}) \\
y(x_n)
\end{bmatrix}
=
\begin{bmatrix}
u \\
y_{x_0}^{x_1} \\
y_{x_1}^{x_2} \\
\vdots \\
y_{x_{n-1}}^{x_n} \\
y_{x_n}
\end{bmatrix}
\]  

(9)

The boundary value problem solution ends with a solution of the obtained matrix algebraic equation system.
3. Decomposition method

However, the class of problems restricted by the potential of the Fourier method is rather narrow. Besides, the method of solving ordinary differential equations mentioned above can be applied to a wide variety of ordinary differential equations. Thus, the creation of finite element modified method with analytical solution of element-based equations requires a more general method of reducing partial differential equations to ordinary ones. The modification of such method with a high degree of generality is a decomposition method. The application of the decomposition method to solving some particular problems of structural mechanics is described in works by Pshenichnov G.I. Before this the method was described in works by Rozin L.A. named as ‘Split Method’.

\[ L(z) = f(\alpha, \beta), \]

let us represent the differential operator as \( h \) summands

\[ L = \sum_{k=1}^{h} L_k, \]

and represent the right part as a sum \( h \) of unknown functions

\[ f(\alpha, \beta) = \sum_{k=1}^{h} f^{(k)}(\alpha, \beta). \]

By associating the summand of the right part with each summand of the differential operator the original equation can be replaced with a system of simpler equations

\[ L_k \left( z^{(k)} \right) = f^{(k)}(\alpha, \beta), \]

and equations of constraints

\[ z^{(1)}(\alpha, \beta) = z^{(k)}(\alpha, \beta). \] (10)

The decomposition method can be more clearly demonstrated using the plate bending equation

\[ \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q_z}{D}. \] (11)

By representing the right part as three summands (unknown functions)

\[ \frac{q_z}{D} = f^{(1)}(x, y) + f^{(2)}(x, y) + f^{(3)}(x, y), \] (12)

let us associate the summand of the right part with each summand of the differential operator

\[
\begin{align*}
\frac{\partial^4 w}{\partial x^4} &= f^{(1)}(x, y) \\
\frac{\partial^4 w}{\partial x^2 \partial y^2} &= f^{(2)}(x, y) \\
\frac{\partial^4 w}{\partial y^4} &= f^{(3)}(x, y)
\end{align*}
\] (13)

It is easy to show that using the system of equations, by exclusion, the original equation can be easily reached. Next, the system equations can be solved through analytical solution formulae mentioned above. Let us assume that the first equation solution resulted in \( w^{(1)}(f^{(1)}(x, y)) \), the second resulted in \( w^{(2)}(f^{(2)}(x, y)) \) and the third resulted in \( w^{(3)}(f^{(3)}(x, y)) \). Now we can write down the equations of constraint

\[
\begin{align*}
w^{(1)} &= w^{(2)} \\
w^{(2)} &= w^{(3)}
\end{align*}
\] (14)
Equations, appear as a consistent system to define unknown functions
\[ f^{(1)}(x, y), f^{(2)}(x, y), f^{(3)}(x, y). \]

After defining unknown functions and inserting them in \( w^{(1)}, w^{(2)}, w^{(3)} \), the solution of the original equation is determined completely.

However, the constraint equations solution involves a fundamental difficulty. The unknown functions are in the right parts of the equations of the system. Thus, they appear under the integral in the partial solution formula. Example for \( w^{(1)} \):
\[
w^{(1)} = K_{x_0}^x (A_x) w^{(1)}_{x_0} + \int_{x_0}^x K(x, \tau) f^{(1)}(\tau)d\tau.
\]

Thus, the constraint equations are not algebraic but integral equations. The simplest way of overcoming the difficulty is dividing the integration interval \((x_0, x)\) into smaller intervals, where functions \( f^{(1)}(x, y), f^{(2)}(x, y), f^{(3)}(x, y) \), can be averaged as it is done for external load in formula (6). In other words, a piecewise constant approximation of unknown functions should be used. Then the averaged functions can be taken out of the integral, so the integral equations turn to algebraic. A priori errors cannot be controlled in this case.

Improvement of constraint equation solution and development of a priori error estimate method involve the opportunity of improving and developing of the method at issue.

The mentioned method is naturally extended from partial differential equations into partial differential equation systems. Thus, decomposition can be applied to the shell resolving equation systems. The transition to systems does not lead to any crucially new difficulties, it only results in more intricate calculations.

4. Modified method of finite elements

Decomposition and analytical solution methods make the basis for the modified method of finite elements. The method algorithm is as follows: Complex structure is divided into simple form sub-regions with partial differential equations available; Equations are decomposed; After the decomposition the equation solutions are written using analytical solution formulae; Elements are converged; Global system of linear algebraic equations is produced (9). The size of the finite element of the proposed method is defined only by the value of unknown function averaging interval \( f^{(1)}(x, y), f^{(2)}(x, y), f^{(3)}(x, y) \), since ordinary differential equation formulae are analytical. Thus, the improvement of solving constraint equations results in reducing number of finite elements. The algorithm has been tested by solving rectangular plate and bicurved shallow shell bending problems. The results proved proper convergence to precise values with increasing number of finite elements.

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