On Locally Decodable Source Coding

Ali Makhdoumi, Shao-Lun Huang, Muriel Médard, Yury Polyanskiy
Massachusetts Institute of Technology,
Email: {makhdoum, shaolun, medard, yp}@mit.edu,

Abstract—Locally decodable channel codes form a special class of error-correcting codes with the property that the decoder is able to reconstruct any bit of the input message from querying only a few bits of a noisy codeword. It is well known that such codes require significantly more redundancy (in particular have vanishing rate) compared to their non-local counterparts. In this paper, we define a dual problem, i.e. locally decodable source codes (LDSC). We consider both almost lossless (block error) and lossy (bit error) cases. In almost lossless case, we show that optimal compression (to entropy) is possible with $O(\log n)$ queries to compressed string by the decompressor. We also show the following converse bounds: 1) linear LDSC cannot achieve any rate below one, with a bounded number of queries, 2) rate of any source coding with linear decoder (not necessarily local) in one, 3) for 2 queries, any code construction cannot have a rate below one. In lossy case, we show that any rate above rate distortion is achievable with a bounded number of queries. We also show that, rate distortion is achievable with any scaling number of queries. We provide an achievability bound in the finite block-length regime and compare it with the existing bounds in succinct data structures literature.

I. INTRODUCTION
A. Motivation
The basic communication problem may be expressed as transmitting data source with the highest fidelity without exceeding a given bit rate, or expressed as transmitting the source data using the lowest bit rate possible while maintaining a given reproduction fidelity \[ I \]. In either case, a fundamental trade-off is made between bit rate and distortion/error level. Therefore, source coding is primarily characterized by rate and distortion/error of the code. However, in practical communication systems, many issues such as memory access requirements (both updating memory and querying from memory) must be considered. In traditional compression algorithms (both in theory and practice) a small change in one symbol of the source sequence leads to a large change in the encoded sequence. Another issue is that, in order to retrieve one symbol of the source, accessing all the encoded symbols are required. The latter issue is the main topic of this paper.

One way to resolve these issues is to place constraints on the encoder/decoder. In particular, in order to address the issue of memory reading access requirement, we study a class of codes for which the decoder is local. This problem appears in many applications in distributed data management. For instance, assume that a given source is compressed and stored on some storage cells. If we use the traditional source coding, then in order to recover only one bit of the original source, we would need to read the entire encoded data on all data storage cells. Since reading from the storage cells is generally costly, we may want to design a compression scheme, which only need to read part of the storage cells to recover one bit of the source. Assuming a local decoder is one possible solution for this matter. In another example, assume that we encode a source and then store it on some data storage cells. We are asked to reveal information about one symbol/coordinate of the source to some party, but, we do not want to reveal the information about the entire source symbols. If we use a conventional source coding, we may have to reveal all the encoded data. Thus, a honest but curious party may have access to the entire original source sequence. On the other hand, with a local decoder, we only provide a small part of the encoded data, so that the party can only recover a small part of original source symbols without capability of extracting information about the other symbols.

B. Contributions
We introduce locally decodable source coding (LDSC): A source sequence $X^n$ (this denotes the vector $(x_1, \ldots, x_n)$) takes values from the source alphabet $\mathcal{X}$ and is mapped into a sequence $Y_i^n$ of encoded symbols taking values in the alphabet $\mathcal{Y}$. These symbols are then used to generate the reproduction sequence $\hat{X}_i^n$. A scheme is called $t$-locally decodable, if for any $i = 1, \ldots, n$, each reproduced symbol $\hat{x}_i$ is a function of at most $t$ of the symbols $y_{1}, \ldots, y_k$. We shall define this notion formally in Section \[ III \]. The number of queries to decode any source symbol, is called locality and is shown by $t$. This is different from traditional source coding, as we are restricting the way that $y^k$ can be mapped back to the $\hat{x}^n$ sequence. Throughout this paper, $X$ denotes a random variable taking values in $\mathcal{X}$, where $x$ denotes an outcome of $X$. The same notation holds for other letters such as $Y$ and $\hat{X}$. Also, for any subset $S \subset \{1, \ldots, n\}$, $X^S$ is defined as the vector $(X_i : i \in S)$.

We consider almost lossless source coding with local decoder. We provide a converse bound on the rate of linear LDSC and show that, the rate of linear LDSC is one rather than the entropy rate.

We also consider source coding with linear decoder and provide a converse bound on the rate of any code (not necessarily local). We then show the rate of source coding with linear decoder is one, whereas using linear and local encoder we can achieve any rate above entropy rate as it is shown in \[ II \]. Moreover, we consider a general encoder and $2$—local decoder ($t = 2$) and show the rate of which is one.

We provide achievability bound on the rate with scaling number of queries. In particular, with $O(\log n)$ queries, any rate above entropy rate is achievable. Furthermore, we consider lossy source coding with local decoder and provide achievability bound on the rate with both scaling number of queries and
constant number of queries. Scaling number of queries: we show that, with any number of queries scaling with \( n \), rate distortion is achievable. We provide an upper bound on the rate in the finite block-length regime (finite \( n \)). We compare our achievability bound with the existing results in the data structure literature and show that, our achievability bound is tighter than the existing bound in [3].

Constant number of queries: we show that, for any given rate above rate distortion, there exists a constant, \( t \), such that sequences of source symbols can be compressed with the given rate and then decompressed with locality \( t \), without exceeding the distortion constraint.

C. Related Work

A long line of research has addressed a similar problem from a data structure perspective. For example, Bloom filters [4] are data structures for storing a set in a compressed form while allowing membership queries to be answered in constant time. The rank/select problem [5], [6] and dictionary problem in the field of succinct data structures are also examples of problems involving both compression and the ability to recover efficiently a single symbol of the input message. In particular, reference [3] provides a succinct data structure for arithmetic coding that supports efficient recovery of source symbols. Moreover, reference [7] studies both issues simultaneously and introduces a data structure that is efficient in both updating and querying. In most of these works, the efficiency is interpreted in terms of the decoding time, whereas in this work it is interpreted in terms of memory access requirement. We formulate this problem from an information theoretic view and study the fundamental trade-offs between locality and the rate of source coding.

Causal Source Coding is a related topic: the constraint on the decoder is not locality, but, causality [8], [9].

Locally decodable codes (LDC) ([10]) is a counter part in the error-correction world. Another recent variation is Locally repairable codes ([11]).

The problem of source coding with local encoding has been studied in many works in both data structure and information theoretic literatures. This line of research addresses the update efficiency issue. Varshney et al. [12] analyzed continuous source codes from an information theoretic point of view. Also, Mossel and Montanari [13] have constructed source codes with local encoder based on nonlinear graph codes. Sparse linear codes have been studied by Mackay [2], who introduced a class of local linear encoders. Also, Mazumdar et al. [14] have considered update efficient codes, which studies channel coding problem with local encoders.

The organization of the paper is as follows. In Section III we give the problem formulation and the converse bound on the rate of LDSC. We also provide an achievability bound in case of scaling number of queries with block-length. Locally decodable lossy source coding (LDLSC) is defined in Section III where we provide achievability bounds on the rate of LDLSC with both constant and scaling number of queries. We conclude the paper in Section IV.

II. Locally Decodable Source Coding (LDSC)

First, we define LDSC and the fundamental limits of it. We then show converse bounds on the rate of LDSC with linear encoder, linear decoder and general encoder-decoder with locality, \( t = 2 \).

An almost lossless LDSC is defined as a pair, consisting of an encoder, \( f \), and a decoder, \( g \), such that \( f : \mathcal{X}^n \rightarrow \{0,1\}^k \) and \( g : \{0,1\}^k \rightarrow \mathcal{X}^n \). The decoder is called local if each coordinate of the output is affected by a bounded number of input coordinates. Formally, Let \( g_a \), for \( a \in \{1,\ldots,n\} \), be the \( a \)-th component of the decoding function. Assume \( g_a \) depends on \( Y^k = \{0,1\}^k \) only through the vector \( Y_{N_a}^j = \{Y_j : j \in N_a\} \) for some \( N_a \subset \{1,\ldots,k\} \). In other words, we have:

For any \( y^k \) and \( y^k \), \( g_a(y^k) = g_a(y^k) \) if \( y^N_a = y^N_a \).

For any given \( t \), a decoder is called \( t \)-local if \( |N_a| \leq t \) for any \( a \in \{1,\ldots,n\} \). We may represent the decoder \( g \), by \( n \) functions: \( g_a : Y^N_a \rightarrow \mathcal{X} \) for any \( 1 \leq a \leq n \).

**Definition 1.** A \((n,k,t,\epsilon)-\text{LDSC}\) is a pair, consisting of an encoder, \( f : \mathcal{X}^n \rightarrow \{0,1\}^k \), and a \( t \)-local decoder, \( g : \{0,1\}^k \rightarrow \mathcal{X}^n \), such that

\[
\Pr[g(f(X^n)) \neq X^n] \leq \epsilon. \tag{1}
\]

Let

\[
k^*(n,\epsilon,t) = \min\{k : \exists (n,k,\epsilon,t) - \text{LDSC}\}. \tag{2}
\]

For a given \( n \), \( t \), and \( \epsilon \), the best rate of LDSC is given by

\[
R^*(n,\epsilon,t) = \frac{k^*(n,\epsilon,t)}{n}, \tag{3}
\]

\[
R^*(\epsilon,t) = \lim_{n \rightarrow \infty} R(n,\epsilon,t), \tag{4}
\]

and

\[
R^*(t) = \lim_{\epsilon \rightarrow 0} R(\epsilon,t). \tag{5}
\]

**Note 1.** In this paper we assume both encoder and decoder are deterministic. because for a given \((n,k,\epsilon,t) - \text{LDSC}\) with randomized encoder and decoder, there exists an \((n,k,\epsilon,t) - \text{LDSC}\) code with deterministic encoder and decoder: Let \( M \) and \( N \) be two random variables and consider randomized encoder and decoder \( f(M) \) and \( g(N) \), respectively. Equation (1) then becomes

\[
\Pr[g(f(X^n,M),N) \neq X^n] = \mathbb{E}[\Pr[g(f(X^n,M),N) \neq X^n]\mid M,N] \leq \epsilon. \tag{6}
\]

Since the expectation in (6) is less than or equal to \( \epsilon \), there exist \( m, n \) such that

\[
\Pr[g(f(X^n,M),N) \neq X_i]\mid M = m, N = n] \leq \epsilon,
\]

showing that \( f(m) \) and \( g(n) \) are our desired deterministic encoder and decoder, respectively.

Next, we prove a converse bound on the rate of LDSC with linear encoder.
A. Linear Encoder

We focus on binary sources, where \( \mathcal{X} = \{0,1\} \). We show that, using a linear encoder, the rate of LDSC is one rather than the entropy rate.

In order to prove the converse bound we use the following lemma.

**Lemma 1.** Let \( \mathbb{F}_2^n \) be a vector space over \( \mathbb{F}_2 \). Let \( \mathbb{P}_X = \text{Bern}(p) \) and define a probability measure over \( \mathbb{F}_2^n \) according to a \( n \)-fold product of \( \mathbb{P}_X \), i.e. \( \mathbb{P}_X^n \). If \( U \) is a \( k \)-dimensional subspace of \( \mathbb{F}_2^n \), we have

\[
(\max\{p, 1 - p\})^{n-k} \geq \mathbb{P}[U] \geq (\min\{p, 1 - p\})^{n-k}. \tag{7}
\]

**Proof:** We first prove the lower bound. Define \( E = \{ v \in \mathbb{F}_2^n \mid H(v) = 1 \} \), where \( H(v) \) denotes the Hamming weight of \( v \). Since the dimension of \( U \) is \( k \), there exists \( E' \), a subset of \( E \), with \( n-k \) elements such that

\[
U \oplus U' = \mathbb{F}_2^n
\]

\[
U \cap U' = \{0\}, \tag{8}
\]

where \( U' = \text{span}(E') \) and \( \oplus \) denotes the direct sum of two subspaces. For each \( u' \in U' \), define \( U_{u'} = U + u' \). Since \( U \cap U' = \{0\} \), \( U_{u'} \)'s are disjoint for \( u' \in U' \). Next, we shall bound \( \mathbb{P}(U_{u'}) \). Suppose \( \mathbb{P}(U) = r \), then we have

\[
\mathbb{P}[U_{u'}] = \sum_{u \in U_{u'}} \mathbb{P}[u] = \sum_{u \in U} \mathbb{P}[u + u']
\]

\[
\leq \sum_{u \in U} \mathbb{P}[u] \left( \max\{p, 1 - p\} \right)^r
\]

\[
= \mathbb{P}[U] \left( \max\{p, 1 - p\} \right)^r \left( \min\{p, 1 - p\} \right)^{r-k}, \tag{9}
\]

where the inequality holds because adding \( u' \) to \( u \) flips \( r \) of the coordinates of \( u \). Since the elements of \( E' \) have Hamming weight \( 1 \) and \( U' = \text{span}(E') \), we have \( \mathbb{H}(u') \leq n-k \) for any \( u' \in U' \). Thus, the following holds

\[
1 \leq \mathbb{P}[U'] \geq \sum_{u' \in U'} \mathbb{P}[U_{u'}]
\]

\[
\geq \sum_{u' \in U'} \mathbb{P}[U] \left( \max\{p, 1 - p\} \right)^r \left( \min\{p, 1 - p\} \right)^{r-k}
\]

\[
= \mathbb{P}[U] \sum_{r=0}^{n-k} \binom{n-k}{r} \left( \frac{\max\{p, 1 - p\}}{\min\{p, 1 - p\}} \right)^r
\]

\[
= \mathbb{P}[U] \left( 1 + \frac{\max\{p, 1 - p\}}{\min\{p, 1 - p\}} \right)^{n-k}
\]

\[
= \mathbb{P}[U] \left( \frac{1}{\min\{p, 1 - p\}} \right)^{n-k}. \tag{10}
\]

This shows that,

\[\mathbb{P}[U] \geq (\min\{p, 1 - p\})^{n-k}.\]

Modifying \( \Box \) to obtain a lower bound on \( \mathbb{P}[U_{u'}] \) and modifying the third line of \( \Box \), the upper bound is proved similarly. \( \square \)

A linear encoder, \( f : \mathcal{X}^n \to \mathcal{Y}^k \), where \( \mathcal{X} = \{0,1\} \) is defined as:

Let \( G \in \mathbb{F}_2^{n \times k} \) be the generating matrix of the encoder. \( G \) is a mapping \( \{0,1\}^n \to \{0,1\}^k \). The encoding is as following

\[x \mapsto xG,\]

where all the operations are over \( \mathbb{F}_2 \).

**Theorem 1.** Assume \( X \) has a Bernoulli distribution and \( (n,k,\epsilon,t) \) is a LDSC for this source with a linear encoder. If \( \epsilon < (\min\{p, 1 - p\})^t \), then \( k \geq n \).

**Proof:** In this proof, all linear spaces are over \( \mathbb{F}_2 \). Without loss of generality, assume \( X_1 \) is recovered by \( Y_1, \ldots, Y_t \) and the decoder maps \( Y^t = 0^t \) to \( X_1 = 0 \). Consider the induced linear mapping \( \pi : \mathcal{X}^n \to \mathcal{Y}^t \). Since the dimension of the range of \( \pi \) is \( n \) and the dimension of the image of \( \pi \) is at most \( t \), we have \( \dim(\text{ker}(\pi)) \geq n - t \). Note that \( 0^t \in \text{ker}(\pi) \) since \( \pi \) is a linear mapping. If there exists \( x^n \in \text{ker}(\pi) \) such that \( x_1 = 1 \), then half of the vectors in \( \text{ker}(\pi) \) have \( x_1 = 0 \) and half of them have \( x_1 = 1 \) (because \( \text{ker}(\pi) \) is a linear space over \( \mathbb{F}_2 \)). Since the decoder maps \( 0^t \) to \( X_1 = 0 \), then the vectors in \( \text{ker}(\pi) \) with \( x_1 = 1 \) are erroneous. Eliminate the first coordinate and consider all the vectors in \( \text{ker}(\pi) \) such that \( x_1 = 1 \); they will form a subspace of dimension at least \( n - t - 1 \) in a space of dimension \( n - 1 \). Therefore, using Lemma 1, we obtain

\[
\mathbb{P}[X^n \neq X^n] \geq \mathbb{P}[\hat{X}_1 \neq X_1]
\]

\[
\geq (\min\{p, 1 - p\})^{n-1-(n-t-1)} = (\min\{p, 1 - p\})^t, \tag{11}
\]

which contradicts \( \epsilon < (\min\{p, 1 - p\})^t \). Therefore, for any \( x^n \in \text{ker}(\pi) \), \( x_1 = 0 \). This means that, if we look at the sub-matrix of \( G \) of dimension \( n \times t \) consisting of the first \( t \) columns, the first row is not in the span of the rest of rows. This implies that, in the matrix \( G \), the first row is not in the span of the rest of rows. If we apply the same argument for any \( X_i \), we conclude that the rows of the matrix \( G \) are independent, resulting in \( k \geq n \). \( \square \)

**Corollary 1.** For any source \( X \) with Bernoulli distribution and any locality \( t \), the rate of LDSC with a linear encoder is \( R^*(t) = 1 \).

**Proof:** Using Theorem 1 for any \( \epsilon < (\min\{p, 1 - p\})^t \), we have \( k^*(n,\epsilon,t) \geq n \). Thus, \( R^*(\epsilon,t) = \limsup_{n \to \infty} k^*(n,\epsilon,t) \geq 1 \). Therefore, the rate is \( R^*(t) \geq \lim_{n \to \infty} R^*(\epsilon,t) \geq 1 \). On the other hand, without using any encoding-decoding, we obtain the rate 1. This completes the proof. \( \square \)

Corollary 1 implies that, with local decoder and linear encoder, no compression is possible and the rate of best possible scheme is the same as not using any compression scheme.

B. Linear Decoder

In this section, we consider a local and linear decoder. We show that, for a linear decoder, even without locality assumption, the rate of compression is 1. This implies if the decoder is linear, then no compression is possible.
Theorem 2. Let $X$ has a Bernoulli($p$) distribution. Assume $(n,k,\epsilon)$ is a source coding with linear decoder. We have
\[ k \geq n - \frac{\log(1-\epsilon)}{\log(\max\{p, 1-p\})}. \tag{12} \]

Proof: Assume $e_1, \ldots , e_k$ form the canonical basis of $\{0,1\}^k$. Since the decoder is linear, it can only recover $\operatorname{Span}\{g(e_1), \ldots , g(e_k)\}$ without error and the rest of the elements of $\{0,1\}^n$ are erroneous. Note that the dimension of $\operatorname{Span}\{g(e_1), \ldots , g(e_k)\}$ is not greater than $k$. Thus, using Lemma 1 we obtain
\[ P[g(f(X^n)) \neq X^n] \geq 1 - \frac{1}{\max\{p, 1-p\}}. \]
We also know $P[g(f(X^n)) \neq X^n] \leq \epsilon$. Therefore,
\[ k \geq n - \frac{\log(1-\epsilon)}{\log(\max\{p, 1-p\})}. \]

This completes the proof. \hfill \square

Corollary 2. Let $X$ be a Bernoulli($p$) source. Also let $f: \mathcal{X}^n \mapsto \{0,1\}^k$ and $g: \{0,1\}^k \mapsto \mathcal{X}^n$ be the encoder and linear decoder, respectively. For any $t$-local decoder we have $R^*(t) = 1$.

Moreover, without the locality constraint, the rate is still 1.

Proof: Using Theorem 2 if we take minimum over all choices of codes, we obtain
\[ k^*(n,\epsilon) \geq n - \frac{\log(1-\epsilon)}{\log(\max\{p, 1-p\})}. \]
Thus, $R^*(n,\epsilon) \geq 1 - \frac{1}{n\log(\max\{p, 1-p\})}$. Taking $n \to \infty$, we obtain $R^*(\epsilon) \geq 1$ and $R^* \geq 1$ (where $R^*$ denote the rate without any assumption on the locality of decoder). Therefore, the rate is 1, because without using any encoding-decoding we can achieve rate 1.

C. General Encoder-Decoder

We focus on the special case of 2-local decoder with a general encoder ($t=2$).

Theorem 3. Let $X$ be a Bernoulli($p$) source and $f: \mathcal{X}^n \mapsto \{0,1\}^k$ and $g: \{0,1\}^k \mapsto \mathcal{X}^n$ be the encoder and local decoder. Also, assume a $(n,k,\epsilon,t)$-LDSC for this source. For $t=2$, if $\epsilon < (\min\{p, 1-p\})^2$, then $k \geq n$.

Proof: We prove this by contradiction. Without loss of generality assume $p \leq \frac{1}{2}$. For the sake of contradiction, assume $n > k$. We shall show $\epsilon \geq p^2$.

The claim is that if the code can recover $X_{i+1}^{k+1}$ i.i.d. Bernoulli($p$) with a 2-local mapping from $Y_k$ on a set with probability $p(k)$, then $p(k) \leq 1 - p^2$. This implies $\epsilon \geq p^2$.

By induction on $k$ we show $p(k) \leq 1 - p^2$. For $k=1$, by considering all 16 possible encoder functions ($X^2 \mapsto Y_1$), it can be seen that $p(1) \leq 1 - p^2$. Assume $p(k-1) \leq 1 - p^2$. Let $X_1$ be recovered by $Y_1$ and $Y_2$. Without loss of generality assume $g_i(0,0) = 0$, where for any $1 \leq i \leq n$, $g_i$ is a mapping with two inputs, producing $X_i$, the reproduction of $X_i$. We list all the possible cases:

1) $g_1(0,1) = 0$. If we consider the induced mapping from $Y_k^2$ to $X_2^k+1$, by replacing 0 with $Y_1$ in all the mappings that use $Y_1$ as one of their inputs, we obtain a local mapping on a set with maximum probability of $p(k-1)$. Similarly, since $g_1(1,1) = g_1(0,0) = 0$, if we replace $1$ with $Y_1$, we obtain another local mapping on a set with maximum probability $p(k-1)$. Therefore, $p(k) \leq p.p(k-1) + p.p(k-1) = p(k-1) \leq 1 - p^2$.

2) $g_1(1,0) = 0$. In this case, replace 0 with $Y_2$ and construct a mapping from $Y_1, Y_2^k$ to $X_2^{k+1}$. Similarly, it can be shown $p(k) \leq 1 - p^2$.

3) $g_1(1,1) = 0$. In this case, replace $Y_1$ by $Y_2$ in all the mappings that are using $Y_1$ as one of their inputs. Similarly we obtain $p(k) \leq 1 - p^2$.

4) $g_1(0,1) = g_1(0,1) = 0$. In this case, $X_1 = 1$ cannot be decoded correctly. Thus, $p(k) \leq 1 - p \leq 1 - p^2$.

5) $g_1(0,0) = g_1(0,1) = g_1(1,1) = 1$. For a binary variable, $Y$, let $\bar{Y}$ denote its complement ($\bar{Y} = Y+1$, mode 2 ). In this case, $\bar{g}_1(Y_1, Y_2) = \bar{Y}_1, \bar{Y}_2$. In general, we call $Y_1, Y_2, \bar{Y}_1, Y_2, \bar{Y}_1, \bar{Y}_2, \bar{Y}_1, \bar{Y}_2$, and their complements product forms. Next, we consider this case.

Note that if only one of the $k+1$ decoding functions is not of the product form, then considering that mapping and the above argument, by induction we obtain $p(k) \leq 1 - p^2$. Now, assume all the mappings are in the product form. If $Y_i$ is appeared in one of the decoding functions as $X_{i_1} = Y_i, Y_j$ and in another one as its complement, i.e., $X_{i_2} = Y_i, Y_k$, then $X_1 = X_2 = 1$ cannot be recovered and we have $p(k) \leq 1 - p^2$. Therefore, without loss of generality we assume that all the mappings are of the form $Y_i, \bar{Y}_j$ and no complement is used.

Consider a bipartite graph demonstrating the relation between the variables $X_1, \ldots , X_{k+1}$ and $Y_1, \ldots , Y_k$. On the Y-side of it we have $k$ nodes corresponding to $Y_i$s and on X-side of it we have $k+1$ nodes corresponding to $X_i$s. The degree of each node on X-side is 2 indicating the variables on the Y-side that are involved in decoding of that node. If two nodes on the X-side have the same neighbors on the Y-side, then we have $X_{i_1} = Y_i Y_j$ and $X_{i_2} = Y_i Y_j$. Thus, only $X_{i_1} = X_{i_2}$ is recoverable and $p(k) \leq 1 - p^2$. Therefore, there exists nodes such that we have $X_{i_1} = Y_i Y_j, X_{i_2} = Y_i Y_k$, and $X_{i_3} = Y_j Y_i$ (note that $l$ might be equal to $k$). If $X_{i_1} = 1$, then we can find a mapping form $(Y_{i-1}^2, Y_k^2)$ to $(X_{i_1}^2, X_{i_1}^{k+1})$ on a set with maximum probability $p(k-1)$. Also, note that $X_{i_1} = 0, X_{i_2} = X_{i_3} = 1$ is not possible to recover. Therefore, $p(k) \leq pp(k-1) + (1-p)(1-p^2) \leq 1 - p^2$. The proof is complete for $t=2$. \hfill \square

Corollary 3. Let $X$ be a Bernoulli($p$) source. Using a general encoder and 2-local decoder, we have $R^*(t) = 1$.

Proof: It follows from Theorem 3.
D. Scaling Number of Queries

We give an achievability bound on the rate of LDSC with logarithmic number of queries with respect to the source block length. The number of queries, $t$, can be a growing function of $n$. In the conventional source coding (not necessarily local) $t(n)$ is a linear function of $n$. Therefore, interesting locality regime are the sub-linear type. In order to establish an achievability bound on LDSC with scaling queries, we use the following result on the error exponent of source coding. This approach is motivated by the achievability bound given in \[14\].

**Theorem 4. (\[13\])** For a discrete memoryless source with probability measure $P_X$ and a source encoding with rate $R$, we have:

For any $\epsilon > 0$, $\exists \ell, \ell_0$ such that for any $n \geq 0$ there exists an encoding-decoding pair $f_n$ and $g_n$ such that

$$\mathbb{P}[g_n(f_n(X^n)) \neq X^n] \leq \ell_0 2^{-n(E^*_n(R) - \epsilon)},$$

where

$$E^*_n(R) = \min_{Q: H(Q) \geq R} D(Q \parallel P).$$

Moreover, this bound is asymptotically tight.

Now, consider the following construction of an encoder-decoder for a source sequence of length $n$, where the source has a Bern($\rho$) distribution:

Let rate, $R$, be equal to $(1 + \delta)H(X)$. Let $X^n$ be a sequence of source symbols. Divide this sequence into blocks of length $t(n)$ and apply the encoder-decoder pair, found by Theorem\[4\], to each block separately. Form an encoder-decoder for $X^n$ by concatenating these $\frac{n}{t(n)}$ (for the sake of presentation, without loss of generality, we drop ceiling and floor in this analysis) pairs of encoder-decoder. We now analyze the error of the concatenated source coding. Using the union bound, we obtain

$$\mathbb{P}[^X^n \neq X^n] = \mathbb{P}[^X^{t(n)} \neq X^{t(n)}] \leq \frac{n}{t(n)} \mathbb{P}[^X(t(n)) \neq X(t(n))].$$

Using, \[13\] for any $\epsilon$, we obtain

$$\mathbb{P}[^X^n \neq X^n] \leq \frac{n}{t(n)} \ell_0 2^{-n(E^*_n(R) - \epsilon)}.$$

Since $R > H(X)$, $E^*_n(R) = \Delta > 0$. Thus, we have

$$\mathbb{P}[^X^n \neq X^n] \leq \frac{n}{t(n)} \ell_0 2^{-t(n)(\Delta - \epsilon)}.$$

Choosing $\epsilon < \Delta$, this bound goes to zero if

$$t(n) > C \log n,$$

for some constant $C$. Therefore, we have the following result.

**Proposition 1.** Let $X$ be a Bern $(\rho)$ source. Also, let $f : X^n \rightarrow \{0,1\}^k$ and $g : \{0,1\}^k \rightarrow X^n$ be an encoder and $t(n)$-local decoder. For any $\epsilon$ and $R > H(X)$, there exist constants $C$ and $n_0$ such that for any $n > n_0$, there exist a $(n, nR, C \log n, \epsilon)$-LDSC. Moreover, for any $R > H(X)$, there exists a constant $C$ such

$$\lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} k^*(n, \epsilon, C \log n) = R.$$

Proposition\[1\] states that, with a relatively small number of queries ($\log n$), the rate of LDSC approaches the optimal rate $h(p)$.

Using the result of \[16\] on the error exponent of source coding with linear encoder, we have the following analog for codes with linear encoder.

**Corollary 4.** Let $X$ be a Bern($\rho$) source. Also, let $f : X^n \rightarrow \{0,1\}^k$ and $g : \{0,1\}^k \rightarrow X^n$ be a linear encoder and $t(n)$-local decoder, respectively. For any $\epsilon > 0$ and $R > H(X)$, there exist constants $C$ and $n_0$ such that for any $n > n_0$, there exist a $(n, nR, C \log n, \epsilon)$-LDSC. Moreover, for any $R > H(X)$, there exists a constant $C$ such

$$\lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} k^*(n, \epsilon, C \log n) = R,$$

where the encoder is assumed to be linear.

Note that Corollary\[1\] shows the rate of LDSC with linear encoder and constant number of queries is one. However, Corollary\[4\] shows with $O(\log n)$ number of queries we can achieve any rate above the entropy rate.

III. **Locally Decodable Lossy Source Coding (LDLSC)**

We first define LDLSC and the fundamental limits of it. Then we provide achievability bounds on the rate of LDLSC for both scaling and constant number of queries.

Consider a separable distortion metric defined as

$$d(x^n, \hat{x}^n) = \frac{1}{n} \sum_{i=1}^{n} d(x_i, \hat{x}_i),$$

where $d : X \times \hat{X} \rightarrow \mathbb{R}^+$ is a distortion measure.

**Definition 2.** A $(n, k, d, t)$-LDLSC is a pair containing an encoder, $f : X^n \rightarrow Y^k$, and a decoder, $g : Y^k \rightarrow X^n$, where the decoder is $t$-local and the distortion is bounded, $\mathbb{E}[d(X^n, g(f(X^n))) \leq d]$. Let

$$k^*(n, d, t) = \min\{k \text{ such that } \exists (n, k, d, t) \text{-LDLSC}\},$$

and

$$R^*(d, t) = \limsup_{n \rightarrow \infty} \frac{k^*(n, d, t)}{n}.$$

Note 2. We assume a binary source, $F = \mathbb{F}_2$ with $d(x, \hat{x}) = 1\{x \neq \hat{x}\}$. In this case we have

$$\mathbb{E}[d(X^n, g(f(X^n))) \leq 1 \sum_{i=1}^{n} \mathbb{P}[X_i \neq \hat{X}_i] \leq d,$$

which is the same as assuming the bit error rate is bounded (comparing to block error rate in the definition of LDLSC).
A. Scaling Number of Queries

In this section we consider the scaling number of queries. Therefore, let \( t(n) \) be a growing function of \( n \). The following is an achievability bound on the rate for finite block length.

**Theorem 5.** For a Bernoulli source, a distortion level \( d \), and any growing number of queries \( t(n) \), we have

\[
R^*(n, d, t(n)) \leq h(p) - h(d) + \frac{\log t(n)}{t(n)} + o \left( \frac{\log t(n)}{t(n)} \right) \tag{16}
\]

**Proof:** Recall the finite block length results on source coding [17]: For a Bernoulli source, and distortion level \( d \), there exists a code such that

\[
R(n, d) \leq h(p) - h(d) + \frac{\log n}{n} + o \left( \frac{\log n}{n} \right) \tag{17}
\]

Now, divide the sequence \( X^n \) into \( \frac{n}{t(n)} \) blocks of length \( t(n) \) (for the sake of presentation, we drop ceiling and floor in this argument). Apply the encoder-decoder obtained from (17) to each block. Concatenate these \( \frac{n}{t(n)} \) pairs to obtain an encoder-decoder for \( X^n \). The average distortion of the overall code is also bounded by \( d \), and its rate is bounded by

\[
h(p) - h(d) + \frac{\log t(n)}{t(n)} + o \left( \frac{\log t(n)}{t(n)} \right).
\]

Which theorem shows the theorem. \( \square \)

Theorem 5 shows that for any number of queries such as \( t(n) \), if \( \lim_{n \to \infty} t(n) = \infty \), then \( R^*(n, d, t) = h(p) - h(d) \), which is the rate distortion.

**Corollary 5.** For the special case of \( t(n) = t \log n \), we have

\[
R^*(n, d, t \log n) \leq h(p) - h(d) + \frac{\log(t \log n)}{t \log n} + o \left( \frac{\log(t \log n)}{t \log n} \right) \tag{18}
\]

**Proof:** result of Theorem 5 for \( t(n) = t \log n \). \( \square \)

Reference [3] studies the problem of storage of bits with local recovery (with the same definition of locality we use here). Those results are based on a generic transformation of augmented B-trees to succinct data structures. They have shown that:

**Theorem 6 ([3]).** Consider a sequence of length \( n \) from alphabet \( X \). We can represent this sequence with

\[
O(|X| \log n) + nH + \frac{n}{(\log n)^2} + O(n^{3/4}) \tag{19}
\]

many bits, supporting single bit recovery in \( t \log n \) queries, where \( H \) denotes the empirical entropy of the sequence. Moreover, we can represent a binary sequence of length \( u \), with \( n \) ones, using

\[
\log \binom{u}{n} + \frac{u}{(\log u)^2} + O(u^{3/4}) \tag{20}
\]

bits. A decoder exists querying only \( t \log u \) bits to decode any bit of the sequence.

We now compare the bound given in corollary 5 with the bound suggested by Theorem 6.

Using Theorem 6 and identity \( \log \binom{n}{p_n} = nh(p) + O(\log n) \), for any \( d \), we obtain

\[
R^*(n, d, t \log n) \leq h(p) + O \left( \frac{\log n}{n} \right) + \frac{1}{\log n} + O(n^{3/4}) \tag{21}
\]

It is clear that for any fixed \( d \), the bound given by (18) is asymptotically (in \( n \)) better than (21). Note that the bound given in (21) does not gain from the fact that encoding-decoding scheme can tolerate a distortion \( d \). One may consider the case where \( d \) goes to zero as \( n \) goes to infinity. Assume both bounds hold for this case as well. We show that if \( d(n) = O \left( \frac{\log n}{n} \right) \), then (21) is tighter than (18). We omit the last term in both bounds and assume \( t = 1 \). In order to show that (21) is tighter than (18), we need to prove

\[
h(p) + O \left( \frac{\log n}{n} \right) + \frac{1}{\log n} \leq h(p) - h(d(n)) + \frac{\log n}{n}.
\]

This inequality holds if

\[
h(d(n)) \leq \frac{\log \log n}{\log n} - O \left( \frac{\log n}{n} \right) - \frac{1}{\log n}.
\]

It can be seen that for \( d(n) = O \left( \frac{\log n}{n} \right) \), this inequality holds.

B. Constant Number of Queries

For a given number of queries, we show that one can achieve any rate above the rate distortion function with a properly large locality. Consider the following construction:

For a given \( \delta \), we wish to show there exists \( t \), such that a LDLSC with locality \( t \) achieves the rate \( (1 + \delta)(R(d)) \) with average distortion bounded by \( d \). From (17), we can get the bound \( R(t, d) \leq R(d) + \frac{2 \log t}{t} \) for large enough \( t \). Also, let \( t \) be large enough such that \( \frac{2 \log t}{t} \leq \delta R(d) \). Therefore, there exists \( t \) such that

\[
R(t, d) \leq R(d)(1 + \delta).
\]

Thus, there exists an encoder and decoder pair for \( X^t \), such that the rate of the code is less than \( (1 + \delta)R(d) \) and the distortion is bounded by \( d \). Now, consider \( n \) pairs of the same encoder-decoder. Concatenate these encoder-decoder pairs to form an encoder-decoder for \( X^{nt} \). In this way, we obtain a source coding for \( X^{nt} \) with distortion

\[
E \left[ \frac{1}{nt} \sum_{i=1}^{nt} d(x_i, \hat{x}_i) \right] = \frac{1}{n} \sum_{j=1}^{n} E[d(X_{(j-1)t+1}^{jt}, \hat{X}_{(j-1)t+1}^{jt})] \leq d, \tag{22}
\]

and rate

\[
R^*(nt, d, t) \leq R(d)(1 + \delta).
\]

Therefore, for any block length, there exists a \( t \)-local LDLSC with rate \( (1 + \delta)R(d) \) and average distortion bounded by \( d \) for this source.

**Proposition 2.** For any source \( X \) with probability measure \( P_X \) and any distortion measure, and distortion level, \( d \), \( R(d) \) is

\[
\inf \{ R \ : \exists t \text{ and a sequence of } t - \text{LDLSC with rate } R \}.
\]
This proposition states that, in order to achieve the rate $(1 + \delta) R(d)$, one need to choose $t$ to be roughly $\frac{1}{\delta R(d)}$.

**IV. CONCLUSION AND FUTURE WORK**

We introduced locally decodable source coding in both almost lossless and lossy cases. The following summarizes the main results we showed in this work:

- **Almost lossless source coding:**
  - Constant locality: We show that, the rate of linear LDSC is one, meaning that no compression is possible. Moreover, we show that, the rate of source coding with a general encoder and a linear decoder (not necessarily local) is one, meaning that no compression is possible. Also for locality, $t = 2$, the rate of any encoder-decoder is one. A future work is to consider LDSC with a general encoder and $t - local decoder ($ $t > 3$) and study the converses bounds on it.
  - Scaling locality: We can achieve any given rate above the Shannon fundamental entropy rate, with logarithmic locality in the block-length.

- **Lossy source coding:**
  - Constant locality: Any given rate above the Shannon fundamental rate distortion is achievable with a proper constant locality. This locality is proportional to the inverse of the difference between the given rate and rate distortion.
  - Scaling locality: Shannon fundamental rate distortion is achievable with any scaling locality ($\lim_{n \to \infty} t(n) = \infty$) and the rate of convergence is upper bounded as in Theorem 5. We show that, this upper bound is asymptotically tighter than the existing bounds in data structure literature.

**REFERENCES**

[1] C. E. Shannon, “A mathematical theory of communication,” *Bell Syst. Tech. J.*, vol. 27, pp. 379–423, 1948.

[2] D. MacKay, “Good error-correcting codes based on very sparse matrices,” *Information Theory, IEEE Transactions on*, vol. 45, no. 2, pp. 399–431, 1999.

[3] M. Patrascu, “Succincter,” in *Foundations of Computer Science, 2008. FOCS’08. IEEE 49th Annual IEEE Symposium on*. IEEE, 2008, pp. 305–313.

[4] B. H. Bloom, “Space/time trade-offs in hash coding with allowable errors,” *Communications of the ACM*, vol. 13, no. 7, pp. 422–426, 1970.

[5] R. Pagh, “Low redundancy in static dictionaries with constant query time,” *SIAM Journal on Computing*, vol. 31, no. 2, pp. 353–363, 2001.

[6] G. Jacobson, “Space-efficient static trees and graphs,” in *Foundations of Computer Science, 1989., 30th Annual Symposium on*. IEEE, 1989, pp. 549–554.

[7] V. Chandar, D. Shah, and G. Wornell, “A locally encodable and decodable compressed data structure,” in *Communication, Control, and Computing, 2009. Allerton 2009. 47th Annual Allerton Conference on*. IEEE, 2009, pp. 613–619.

[8] D. Neuhoﬀ and R. Gilbert, “Causal source codes,” *Information Theory, IEEE Transactions on*, vol. 28, no. 5, pp. 701–713, 1982.

[9] Y. Kaspi and N. Merhav, “Zero-delay and causal single-user and multi-user lossy source coding with decoder side information,” *CoRR*, vol. abs/1301.0079, 2013.

[10] S. Yekhanin, “New locally decodable codes and private information retrieval schemes,” in *Electronic Colloquium on Computational Complexity, vol. TR06*, 2006, p. 127.

[11] D. Papailiopoulos and A. Dimakis, “Locally repairable codes,” in *Information Theory Proceedings (ISIT), 2012 IEEE International Symposium on*. IEEE, 2012, pp. 2771–2775.

[12] L. R. Varshney, J. Kusuma, and V. K. Goyal, “Malleable coding: Compressed palimpsests,” arXiv preprint arXiv:0806.4722, 2008.

[13] A. Montanari and E. Mossel, “Smooth compression, gallager bound and nonlinear sparse-graph codes,” in *Information Theory, 2008. ISIT 2008. IEEE International Symposium on*. IEEE, 2008, pp. 2474–2478.

[14] A. Mazumdar, G. W. Wornell, and V. Chandar, “Update efficient codes for error correction,” in *Information Theory Proceedings (ISIT), 2012 IEEE International Symposium on*. IEEE, 2012, pp. 1558–1562.

[15] I. Csiszar and J. Korner, *Information theory: Coding theorems for discrete memoryless systems*. Cambridge University Press, 2011.

[16] I. Csiszar, “Linear codes for sources and source networks: Error exponents, universal coding,” *Information Theory, IEEE Transactions on*, vol. 28, no. 4, pp. 585–592, 1982.

[17] Z. Zhang, E. Yang, and V. Wei, “The redundancy of source coding with a fidelity criterion. 1. known statistics,” *Information Theory, IEEE Transactions on*, vol. 43, no. 1, pp. 71–91, 1997.