Crisis–induced intermittency due to attractor–widening in a buoyancy–driven solar dynamo

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In a recent paper [M. A. J. H. Ossendrijver, A&A 359, 364 (2000)] numerical simulations of a 2D mean–field model where shown to produce grand minima, typical of the long-term behavior of solar magnetic activity. The model consisted of dynamo that features an $\alpha$ effect based on the buoyancy instability of magnetic flux tubes, which gives rise to the switching back and forth from grand minima to “regular” solar behavior. In this Letter, we report evidence from a time-series analysis of the model for the presence of crisis–induced intermittency due to attractor–widening. We support this finding by showing that the average duration of the minima, $\langle \tau \rangle$, follows the theoretically predicted scaling $\langle \tau \rangle \sim (C_{\delta\alpha} - C_{\delta\alpha}^*)^{-\gamma}$, where $C_{\delta\alpha}$ is the bifurcation parameter of interest, together with other statistical evidence. As far as we are aware, this is the first time concrete and detailed evidence has been produced for the occurrence of this type of crisis–induced intermittency – due to attractor widening – for such dynamo models.

The records of past solar magnetic activity reveal an outstanding phenomenon referred to as grand minima. During the latest of such intervals, the so-called Maunder minimum (1645–1715), sunspots were virtually absent. This and earlier grand minima are clearly visible in the records of cosmogenic isotopes, e.g. $^{14}$C and $^{10}$Be [2]. Timing and duration of the known grand minima are irregular. Solar variability is also apparent e.g. in length and amplitude variations of the 11-year sunspot cycle. The spectral and statistical properties of solar variability are still not well-known. Although there is some evidence for various modulations of the solar cycle [3], it remains unclear, due to the lack of a sufficiently long and accurate date set, whether they are truly periodic rather than chaotic or even purely stochastic [4].

Intermittency has been proposed [5] as one of the possible scenarios for the underlying mechanism behind the occurrence of grand minima. Several dynamo models have shown to exhibit quasiperiodic or chaotic intermittent behavior [1]. In [1] a case was made for a stochastically driven 2D mean–field dynamo model showing grand minima [6]. Earlier, the authors in [8] produced grand minima using a 1D model, in which the flux injections were treated as an additive random source term. Of course, mean-field theory cannot replace full MHD calculations [6] and is at best capable of capturing the most important aspects of solar dynamo action. The purpose of the calculations was to illustrate that grand minima are an inherent feature of a solar dynamo based on magnetic flux tubes.

In this Letter, we show concrete evidence that the switching back and forth from grand minima to the “regular” 22–year cycle is a manifestation of a known dynamical process called attractor–widening, resulting in crisis–induced intermittency [1]. This type of intermittency has been found in several experimental and numerical studies [1]. Other types of crisis [10] have also been concretely demonstrated to exist in dynamo models [12, 13].

I. DYNAMO MODEL

The induction equation for the mean magnetic field $\mathbf{B}$ in the first–order smoothing approximation is given by

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \{ \mathbf{u} \times \mathbf{B} + \alpha \mathbf{B} - \beta \nabla \times \mathbf{B} \}. \quad (1)$$

All mean quantities are defined here as longitudinal averages, so that the mean field is the axisymmetric component of the actual field. The $\alpha$ term parametrises the effect of helical motions. In the solar dynamo, its main effect is to create a poloidal field from a toroidal field. The total $\alpha$ effect consists of a buoyancy-driven term in the overshoot layer, which exists only if $|\mathbf{B}|$ exceeds a threshold value, and a small-scale kinematic $\alpha$ effect in the convection zone, which is modeled as a random forcing term. The flow $\mathbf{u}$ consists of differential rotation (chosen to model helioseismological inversion results [14]), one updraft at a fixed low latitude, and downdrafts, with a finite lifetime, placed at random chosen latitudes.

For a detailed description of the model, the reader is referred to [1] (‘model B’). Its main features are as follows. The dynamo equation (1) is solved in a spherical shell between $r = 0.5R_\odot$ and $r = R_\odot$, consisting of a layer with convective overshooting between $r_1 = 0.6R_\odot$ and $r_2 = 0.7R_\odot$, and a convection zone above it. The magnetic field strength is expressed in units of $B_0$, the threshold value for the buoyancy instability. Time is measured in terms of $R^2/\beta_0$, the magnetic diffusion time for the convection zone (see below).
II. RESULTS

The numerical integration is carried out using an implicit 2D code developed by D. Schmitt and T. Prautzsch of Göttingen. The grid size is set to 61 points in the radial direction, and 51 in the latitudinal direction.

Fig. 1 shows a butterfly diagram for the mean toroidal magnetic field of a solution with grand minima (model B). The numerical solution features intervals of ordinary cyclic dynamo action interrupted by intervals of irregular duration without cycles, reminiscent of the grand minima of solar activity. The ordinary solar cycle is characterized by belts of magnetic activity on both hemispheres, which have opposite polarity, and migrate towards the equator during the course of a cycle. At the solar minimum, the old belt disappears, and a new belt of opposite polarity is formed at higher latitudes. Note that the present schematic model is not designed to reproduce all known detailed features of the solar cycle but some of the most important aspects, particularly the regular butterfly diagram and the existence of grand minima.

![Fig. 1: Butterfly diagram of the toroidal field at \( r = 0.625R_\odot \) for parameters \( C_\omega = 2 \times 10^4, C_{\alpha 1} = -0.05, C_{\alpha 2} = -1, C_{\delta 0} = 100, C_b = -0.1, \) and \( C_\delta = 0.014 \) (from Ossendrijver – model B). \( C_\alpha \) measures the strength of the differential rotation, \( C_{\alpha 1} \) the strength of the \( \alpha \) effect on the overshoot layer, \( C_{\alpha 2} \) the strength of the mean \( \alpha \) effect in the convection zone, \( C_{\delta 0} \) the strength of the fluctuating \( \alpha \) effect in the convection zone, \( C_b \) the strength of the downdrafts and \( C_\delta \) the strength of the updrafts. Time is in years.](image)

Fig. 2 shows the evolution of the solutions as one parameter of interest, \( C_{\delta 0} \), is changed. The shutdown periods correspond to grand minima and the active period correspond to the (noisy) solar cycle. As can be easily seen, if \( C_{\delta 0} \) is raised, the bursting becomes more and more frequent. The critical parameter value \( C_{\delta 0}^* \) for which the smaller attractor, corresponding to the minima, starts to burst was found to be around \( C_{\delta 0}^* \sim 59 \). This is indicative of the existence of some form of crisis, which we substantiate below. According to the theory \([10]\), around the critical parameter value \( C_{\delta 0}^* \sim 59 \) the attractor collides in phase space with the stable manifold of some unstable periodic orbit. The excursions or burst phases seen in Fig. 2 result in the widening of the attractor after this crisis.

Phenomenologically, the time series could be interpreted in terms of various different types of intermittency. In particular, the time series could be mislead as solutions of systems exhibiting on–off intermittency \([5]\), or Type I/II/III Pomeau–Manneville intermittency \([6]\). However, we have eliminated this possibility by analysing both the power spectra (see Fig. 3) and the probability distribution of the shutdown phases (see Fig. 3).

It has been shown \([17]\) that the corresponding power spectra for Type I intermittency have a broad-band feature whose shape obeys approximately the inverse-power law, \( P(f) \sim f^{-1}, \) for \( f > f_\epsilon, \) where \( f_\epsilon \) is the saturation frequency. Our power spectra do not seem to show this property. Neither do they conform to the theoretical expectation that the power spectra for an on–off process follows a law \( P(f) \sim f^{-1/2} \) over an intermediate range of frequencies. Furthermore the probability distribution of shutdown phases (Fig. 3) does not follow the scaling \( P(n) \sim n^{-\frac{\gamma}{2}}, \) as predicted for on–off intermittency \([13]\), but the theoretically predicted scaling \([10]\)

\[ \tau \sim e^{-\tau/(\gamma)}. \]

Finally we confirm the agreement with the theoretical predictions for “crisis–induced intermittency” by calculating the average time \( \langle \tau \rangle \) between the bursts which obeys the scaling law \([10]\)

\[ \langle \tau \rangle \sim |C_{\delta 0} - C_{\delta 0}^*|^{-\gamma}, \]

which we confirmed for our model, as can be seen in Fig. 3. The value of \( \gamma \) is found to be \( \gamma = 2.19 \pm 0.04. \) This further reinforces the conclusion that the type of

![Fig. 2: Times series of the toroidal field, in units of \( B_0, \) at \( r = 0.625R_\odot, \) and \( \theta = 0.47\pi. \) Except the bifurcation parameter \( C_{\delta 0}, \) all parameters are as in Fig. 1. As \( C_{\delta 0} \) is increased beyond the critical value, \( C_{\delta 0} \sim 59, \) the frequency of the shutdowns (grand minima) decreases rapidly.](image)
The intermittency observed here is neither Type I (for which \( \gamma = -\frac{1}{2} \)), Type II or III (for which \( \gamma = -1 \)) or on-off intermittency (for which \( \gamma = -1 \)).

The \( \gamma \) coefficient can be calculated also analytically, as shown in [10]. The method involves calculating the stable and unstable manifolds of the unstable orbit mediating the crisis. However, because our attractor seems to be of a high dimension, the method of using the time series to reconstruct the attractor and analyse Poincaré sections of such embeddings was not useful in identifying the mediating unstable orbit. The value of \( \gamma \) seems to confirm that our attractor is indeed of a high dimension, given the theoretically expected range \( (D - 1)/2 \leq \gamma \leq (D + 1)/2 \) [10]. This would imply \( D = 4 \).

We have obtained concrete evidence, in terms of time series signatures, power spectra and dynamical scalings, to demonstrate concretely the presence of crisis–induced intermittency due to attractor widening in a mean–field PDE dynamo model. Despite the presence of simplifications in these models, this is of potential importance since it shows the occurrence of another type of intermittency (in addition to Type I Pomeau–Manneville, attractor merging intermittency [13] and in–out intermittency [19] recently discovered) in these models. This may be taken as an possible indication that more than one type of intermittency may occur in solar and stellar dynamos [20]. Given that the underlying model showed several important features of solar activity, such as minima of activity and a realistic butterfly diagram, this suggest that intermittency, and in particular the form showed here, may be the underlying nonlinear dynamical process responsible for the observed solar magnetic activity.

III. DISCUSSION

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