On the Capacity of Fading Channels with Peak and Average Power Constraints at Low SNR

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Abstract—The capacity of fading channels under peak and average power constraints at the low SNR regime is considered. With full channel side information at the transmitter and the receiver sides, we show that the capacity scales essentially as $C \approx A \text{SNR} \int_{1}^{A \text{SNR}} F^{-1}(t) \, dt$, where $A$ is the peak to average power ratio, and $F(.)$ is the cumulative distribution function of the fading channel. We also prove that an On-Off power scheme is sufficient to asymptotically achieve the capacity. Furthermore, by considering the variable peak to average power ratio scenario, we generalize the scalability of capacity, and derive the asymptotic expression for the capacity at the low SNR regime.

Index Terms—Ergodic capacity, low SNR, peak to average power ratio, Rayleigh fading channel, On-Off signaling.

I. INTRODUCTION

In response to the increasing demand for higher power efficiency in wireless communications, many researchers have been devoted to setting up the theory on performance limits in the power limited systems [1], [2]. Correspondingly, many new practical power allocation schemes have been proposed to approach these limits. For instance, different adaptive schemes take advantage of the feedback channel to increase the power efficiency. [3]. The low SNR framework does not apply only to applications where the power budget is asymptotically low, but also includes applications where the power budget is arbitrary and the available degrees of freedom (DoF) are large enough. For instance, many wideband wireless communication systems, e.g., satellite, deep-space, and sensor networks can achieve a very high total rate by utilizing large DoF.

Our main goal, in this letter, is to better understand the capacity of communication under peak and average power constraints at the low SNR regime. The work in [4] shows that the capacity of Rayleigh fading channels with perfect channel state information (CSI) at the transmitter side (CSI-T) and at the receiver side (CSI-R) scales essentially as $\text{SNR} \log_{10} \frac{1}{\text{SNR}}$. The work in [5] generalizes the channel model to Nakagami-m channels. The capacity is shown to behave as $\frac{\Omega}{m} \text{SNR} \log_{10} \frac{1}{\text{SNR}}$, where $m$ is the Nakagami-m fading parameter and $\Omega$ is the channel mean-square. The main limitation of these previous characterizations lies in their high peak to average power ratio (PAPR). In fact, as SNR goes to zero, the PAPR approaches infinity. In practical communication systems, the PAPR is limited by hardware restrictions. In addition, as the PAPR is larger, the RF power amplifier is required to operate in high back-off region resulting in increasing cost. Consequently, it is important to consider these practical constraints [6].

In this letter, we investigate the capacity of fading channels subject to both peak and average power constraints with perfect CSI at both the transmitter and the receiver (CSI-TR). Although, in some cases, we focus on Rayleigh fading to simplify the presentation, we believe that the proposed framework is applicable to arbitrary continuous fading channels. Our contributions can be summarized as follows:

- Analyzing the effect of considering an additional peak power constraint $A$ on the optimal power profile.
- Proving that under both peak and average power constraints, the capacity scales essentially as $C \approx A \text{SNR} \int_{1}^{A \text{SNR}} F^{-1}(t) \, dt$, where $A$ is the peak to average power ratio, and $F(.)$ is the cumulative distribution function of the fading channel.
- Presenting an On-Off power scheme which is asymptotically optimal. This implies that at the low SNR regime, a 1-bit feedback on the CSI-T is enough to approach the channel capacity, which is very helpful from a signaling design point of view.
- Studying the case where the PAPR constraint depends on SNR, and generalize the scalability of capacity at the low SNR regime. Specializing our result to Rayleigh fading, we derive a simple asymptotic expression of the capacity.

II. SYSTEM MODEL

We consider the continuous complex-baseband model of a flat fading single-input single-output (SISO) communication specified by

$$y = h x + v,$$

where $y$, $h$, and $x$ are complex random variables (r.v.) representing the received signal, the channel gain, and the transmitted signal, respectively. The r.v. $v$ represents a circularly symmetric complex Gaussian noise with mean zero, and unit variance, i.e., $v \sim \mathcal{CN}(0,1)$ and is independent of all other r.v. We assume perfect CSI-TR and that the channel follows an arbitrary continuous fading distributions. The average transmitted power is constrained to $\mathbb{E}[P(h)] \leq \text{SNR}$, and the peak power at the transmitter side is restricted to $\max_h P(h) \leq A \text{SNR}$, where $A \geq 1$ is the PAPR, and the previous averaging is over all input conditioned distributions $p(x|h)$. In the following sections, the notation $f(x) \approx g(x)$ will be used to denote $\lim_{\text{SNR} \to 0} \frac{f(\text{SNR})}{g(\text{SNR})} = 1$. Inequalities $<$ and $>$ are defined similarly.

III. LOW SNR CAPACITY WITH PERFECT CSI-TR

In this section, we briefly recall results on the channel capacity. Then, we present our main result, namely, the asym-
totic behavior of capacity subject to the peak and average power constraints, along with the corresponding proof. Then, we show that when SNR goes to zero, the optimal power allocation can be achieved (asymptotically) by an On-Off power scheme.

A. Capacity Results with no PAPR Constraint

With perfect CSI-TR, the ergodic capacity subject to an average power constraint is maximized by the well-known water-filling scheme, \( P(h) \), given by:

\[
P(h) = \left[ \frac{1}{\lambda_0 \text{SNR}} - \frac{1}{|h|^2} \right]^+,
\]

where \([x]^+ = \max\{0, x\}\) and where \(\lambda_0 \text{ (SNR)}\) is the Lagrange multiplier obtained by satisfying the average power constraint with equality \((7)\), i.e., \(\text{SNR} = \mathbb{E}_{|h|^2} \left[ \left( \frac{1}{\lambda_0 \text{SNR}} - \frac{1}{|h|^2} \right)^+ \right]\).

The capacity is then obtained by averaging \(\log \left(1 + \text{SNR} |h|^2\right)\), and is given by:

\[
C = \int_{\lambda_0}^{\infty} \log \left( \frac{t}{\lambda_0} \right) f_{|h|^2}(t) \, dt.
\]

Note that this result does not account for the peak power constraint. Since the PAPR needs to be taken into consideration, we analyze the PAPR in this classic optimal power allocation. From the expression of \(P(h)\) in \((2)\), the PAPR, denoted as \(A(\text{SNR})\), can be obtained as

\[
A(\text{SNR}) = \frac{1}{\lambda_0 (\text{SNR}) \text{SNR}}.
\]

Thus, \(A(\text{SNR})\) only depends on SNR. To analyze the properties of PAPR, we define the function \(A(x)\) on \((0, \infty)\) as:

\[
A(x) = \frac{1}{x \mathbb{E}_{|h|^2} \left( \left[ \frac{1}{x} - \frac{1}{|h|^2} \right]^+ \right)},
\]

and we summarize its properties in Lemma 1.

**Lemma 1:** The function \(A(x)\) defined by \((5)\) is
i) continuous and strictly monotonically increasing;
ii) when \(x \to 0\), \(A(x) \to 1\);
iii) when \(x \to \infty\), \(A(x) \to 0\).

**Proof:** The proof of Lemma 1 is in Appendix A.

Lemma 1 shows that the PAPR approaches 1 in high SNR, which implies that PAPR is not crucial in this regime and there is little benefit in adapting the power in high SNR in agreement with the results in \((7)\). However, at the low SNR regime, in order to achieve the capacity, as SNR goes to zero, the PAPR goes to infinity, which is difficult to implement in practical communication systems. Therefore, considering the effect of the PAPR constraint on the capacity of channels at the low SNR regime is important when designing low SNR systems.

B. Constant PAPR Constraint

1) Capacity Results: In \([8]\), the optimal power allocation subject to peak and average power constraint. The optimal power, \(P(h)\) is given by:

\[
P(h) = \min \left\{ \left[ \frac{1}{\lambda (\text{SNR})} - \frac{1}{|h|^2} \right]^+, \text{ A SNR} \right\}
\]

where \(\lambda (\text{SNR})\) is the Lagrange multiplier obtained by satisfying the average power constraint with equality:

\[
\text{SNR} = \mathbb{E}_{|h|^2} \left[ \min \left\{ \left[ \frac{1}{\lambda (\text{SNR})} - \frac{1}{|h|^2} \right]^+, \text{ A SNR} \right\} \right].
\]

The corresponding capacity is given by:

\[
C(\lambda) = \int_{\lambda}^{\infty} \min \left\{ \log t, \log (1 + A \text{ SNR} \, t) \right\} f_{|h|^2}(t) \, dt
\]

In Lemma 2 we present some properties of \(\lambda(\text{SNR})\).

**Lemma 2:** The Lagrange multiplier \(\lambda(\text{SNR})\) satisfies the following properties:
i) \(\lambda \in \left[ \frac{F^{-1}(1 - \frac{1}{n})}{1 + A \text{ SNR} F^{-1}(1 - \frac{1}{n})}, F^{-1}(1 - \frac{1}{n}) \right]\).
ii) when \(\text{SNR} \to 0\), \(\lambda \to F^{-1}(1 - \frac{1}{n})\);
iii) when \(\text{SNR} \to 0\), \(\lambda^n \text{SNR} \to 0\), where \(n\) is an arbitrary positive constant.

**Proof:** The proof of Lemma 2 is in Appendix B.

Note that \(1 - \frac{1}{n} \geq 0\) since peak power is greater or equal to the average power. As we can see in Lemma 2 when \(A\) is fixed, \(\lambda\) is upper bounded by \(F^{-1}(1 - \frac{1}{n})\), which does not scale with SNR. Next, we state our main theorem.

**Theorem 1:** For any fading channel in \((1)\), with perfect CSI-TR, the capacity at asymptotically low SNR is:

\[
C \approx A \text{ SNR} \int_{\frac{1}{1 - \frac{1}{n}}}^{1} F^{-1}(t) \, dt
\]

**Proof:** The proof of Theorem 1 is in Appendix C.

As shown in Theorem 1, when the PAPR constraint is adopted, the capacity at the low SNR regime linearly changes with SNR. The PAPR constraint has much effect on the achievable capacity. In the special case of \(A = 1\), the power scheme evolves to be the constant power allocation, \(C \approx \text{SNR}\).

When \(A\) approaches infinity, the power scheme becomes the classic water-filling power allocation. In the scenarios of limited system power, increasing the PAPR is an effective way to improve the achievable capacity.

2) On-Off Power Scheme: We note that an On-Off power scheme archives the capacity as described in the following Corollary.

**Corollary 1:** An On-Off power scheme that transmits when \(|h|^2 \geq F^{-1}(1 - \frac{1}{n})\), with a power \(A \text{ SNR}\), and remains silent otherwise, is asymptotically optimal.

**Proof:** The proof of Corollary 1 is in Appendix D.
the optimality of the On-Off power scheme as follows. The 
average power constraint \[ \text{SNR} = \int_{\lambda}^{\frac{1}{1-\lambda}} \left( 1 - \frac{1}{t} \right) f(t) dt + \int_{\frac{1}{1-\lambda}}^{\infty} A \text{SNR} f(t) dt \]  
(10)
The first term corresponds to the continuous power allocation 
requiring the exact CSI-T. As SNR decreases, it scales as:  
\[ \int_{\lambda}^{\frac{1}{1-\lambda}} \left( 1 - \frac{1}{t} \right) f(t) dt \]  
\[ \leq \max_{t \in \left[ \lambda, \frac{1}{1-\lambda} \text{SNR} \right]} f(t) A \text{SNR} \left( \frac{\lambda}{1 - \lambda A \text{SNR}} - \lambda \right) \]  
\[ \approx f(\lambda) \lambda^2 A^2 \text{SNR}^2 \approx o(\text{SNR}) . \]  
(13)
The second terms belongs to the constant power allocation, which behaves as \[ \int_{\frac{1}{1-\lambda}}^{\infty} A \text{SNR} f(t) dt \approx \int_{0}^{\infty} A \text{SNR} f(t) dt \approx A \text{SNR} [1 - F(\lambda)]. \]  
When SNR \( \rightarrow 0 \), the second part dominates. Therefore, one 
bit of CSI-T is sufficient.

From another side, we apply our results to Rayleigh fading 
channels and we obtain the results in Corollary 2.

Corollary 2: For the Rayleigh fading channel described in \[ \text{SNR} \], the capacity is given by:  
\[ C \approx \text{SNR} (1 + \log A) \]  
(14)

Proof: The proof of Corollary 2 is in Appendix E.  

3) Energy Efficiency: Our framework also characterizes 
the energy efficiency of arbitrary fading channels at low 
SNR. To illustrate this observation, we denote by \( E_n \) the 
transmitted energy in Joules per information nats. Then, we 
have: \( \frac{E_n}{\sigma^2} C^{0}(\text{SNR}) = \text{SNR} \). By using Theorem 1, we obtain:  
\[ \frac{E_n}{\sigma^2} \approx \frac{1}{A f_{\frac{1}{1-\lambda}}^{\infty} F^{-1}(t) dt} \]  
(15)
which indicates that the energy required to communicate one 
nat of information reliably only depends on the PAPR \( A \). One 
can achieve a lower energy efficiency per information bit by 
increasing \( A \). In the special case of Rayleigh fading, we have 
\[ \frac{E_n}{\sigma^2} \approx \frac{1}{1 + \log A} . \]

C. Variable PAPR Constraint

This section considers the case where the PAPR varies with 
the SNR. In practical communications systems, the PAPR may 
not be fixed as SNR decreases. Also, exploring this area offers 
a general formula for the optimal power allocation under peak 
and average power constraint at low SNR. To simplify the 
analysis, we focus on Rayleigh fading channels.

The variable PAPR is denoted as \( A(\text{SNR}) \). Some properties, 
that the effective \( A(\text{SNR}) \) needs to satisfy, are summarized in 
the Lemma 3.

Lemma 3: The effective \( A(\text{SNR}) \) should satisfy:  
i) When \( \text{SNR} \rightarrow 0 \), \( A(\text{SNR}) \) \( \text{SNR} \rightarrow 0 \);  
ii) When \( \text{SNR} \rightarrow 0 \), if \( A(\text{SNR}) \rightarrow \infty \), then \( \lambda \rightarrow \infty \).

Proof: The proof of Lemma 3 is in Appendix F.

To analyze the effect of the variable PAPR constraint on the 
capacity, we assume that \( A(\text{SNR}) \) satisfies all the conditions 
in Lemma 3. Then, we introduce the main theorem of the 
capacity with variable PAPR constraint at the low SNR regime.

Theorem 2: Let \( l(\text{SNR}) = \frac{\lambda}{\lambda - A(\text{SNR})} \) \( \text{SNR} \rightarrow 0 \). For Rayleigh fading channels with 
variable PAPR constraint, the capacity can be expressed as follows:  
\[ C \approx \begin{cases} \text{SNR} \log \left( \frac{1}{\text{SNR}} \right) & \text{if } l_0 > 0. \\ \text{SNR} \log \left( A(\text{SNR}) \right) & \text{if } l_0 = 0. \end{cases} \]  
(16)

Proof: The proof of Theorem 2 is in Appendix G.

Intuitively, the value of \( l \) corresponds to the region where 
the transmit power is adapted with the feedback of channel 
gain. The other regions belong to the silent mode or constant 
transmitting power mode. When \( l \) goes to infinity, the power 
allocated to this region dominates. On the other hand, when \( l \) 
approaches 0, the power allocated to the constant power mode 
dominates.

To better understand the PAPR constraint on the capacity, 
we analyze the case when \( l_0 > 0 \). From the definition of \( l \), we 
can easily obtain the following expression:  
\[ A(\text{SNR}) = \frac{1}{\text{SNR} \lambda} \frac{1}{l + \lambda} \]  
(17)
Based on \[ \text{SNR} \], we have \( \lambda \approx \log \frac{1}{\text{SNR}} \). Hence, Eq. (17) can be 
expressed as:  
\[ \frac{1}{\text{SNR} \left( \log \frac{1}{\text{SNR}} \right)^2} \approx A(\text{SNR}) \approx \frac{1}{\text{SNR} \log \frac{1}{\text{SNR}}} \]  
(18)
From Eq. (18), we can get:  
\[ \log A(\text{SNR}) \approx \log \frac{1}{\text{SNR}} \] 
Therefore, in this case, the capacity also satisfies the following 
expression:  
\[ C \approx \text{SNR} \times A(\text{SNR}) \] 
In fact, the expression for the capacity without peak power 
constraint at the low SNR regime in [5] also satisfies the 
expression. As Lemma 1 shows, the PAPR in [5] is  
\[ A(\text{SNR}) = \frac{1}{\lambda \text{SNR}} \approx \frac{1}{\text{SNR} \log \frac{1}{\text{SNR}}} \]  
(19)
The expression for the capacity in [5] is given by:  
\[ C \approx \text{SNR} \log \frac{1}{\text{SNR}} \] 
\approx \text{SNR} \log \frac{1}{\text{SNR} \log \frac{1}{\text{SNR}}} \] 
\approx \text{SNR} \log A(\text{SNR}) \] 

Hence, we present a new perspective on the capacity of 
Rayleigh fading channels at the low SNR regime. The capacity 
actually are determined by the two parameters: SNR and 
\( A(\text{SNR}) \).

IV. NUMERICAL RESULTS AND DISCUSSION

We first consider the constant PAPR in Rayleigh fading 
channel. Figure 1 depicts the ergodic capacity in nats per
channel use (npcu). The optimal power allocation with and without the PAPR obtained by using standard optimization methods are presented as the benchmark curves. In Fig. 1a, where PAPR=2, we show that the gap between the two curves increases as SNR goes to 0 implying that the PAPR constraint has higher impact on the capacity at low SNR. Also, the asymptotic capacity representing the low SNR characterization given by Theorem 1 is shown in the figure. The asymptotical capacity curve accords well with the capacity curve with the PAPR constraint. Furthermore, the capacity of the proposed On-Off scheme is plotted in Fig. 1a. This rate matches perfectly the exact curve at the low SNR values displayed in Fig. 1.

Figure 1b displays the results when the PAPR=20. The capacity presents similar results to Fig. 1a with remarkable improvement due to the increase in the PAPR. Then we generalize the channel to Nakagami-m fading channel. We have set $\Omega = 1$ for all numerical results.

Figure 2 displays the capacity of an i.i.d. Nakagami-m fading channel for $m = \frac{1}{2}$ and $m = 2$, respectively. As can be seen in both figures, the curves depicting the characterizations in Theorem 1 follow the same shape as the curve obtained by optimal power allocation with the PAPR constraint. In both Fig. 2a and Fig. 2b, the On-Off scheme achievable rate is also depicted and is almost indistinguishable from the capacity curve showing that the proposed suboptimal scheme is accurate at the low SNR regime.

In Fig. 3, we consider the variable PAPR scenario following Lemma 3. In Fig. 3a, we choose $PAPR=\log \left( e + \frac{1}{SNR} \right)$, which satisfies the requirement on trend of the PAPR. Also, the PAPR corresponds to $l_0 = 0$ scenario in Eq. (16). In Fig. 3b, we choose $PAPR=1 + \frac{1}{SNR \left( \log \left( e + \frac{1}{SNR} \right) \right)}$. This belongs to the case $l_0 > 0$. In both Fig. 3a and Fig. 3b, the curve obtained by Eq. ??E400) matches very well the capacity obtained by the optimal power allocation. Furthermore, the On-Off power schemes achieve good approximation for the capacity curve.

**V. CONCLUSION**

In this paper, we studied the capacity of fading channels at low SNR under peak and average power constraints. We presented an approximation of the capacity as function of the SNR and the peak to average power ration (PAPR). We also presented a practical On-Off scheme that achieves asymptoti-
we explicitly express the optimal power, $P(h)$, as:

$$P(h) = \begin{cases} 
0 & \text{if } |h|^2 \leq \lambda \\
\frac{1}{A} - \frac{1}{|h|^2} & \text{if } \lambda < |h|^2 < \left[\frac{1 - \lambda}{A \text{ SNR}}\right]^+ \\
A \text{ SNR} & \text{if } |h|^2 \geq \left[\frac{1 - \lambda}{A \text{ SNR}}\right]^+ \end{cases}$$ (20)

In the case, $\lambda < |h|^2 < \left[\frac{1 - \lambda}{A \text{ SNR}}\right]^+$, we get: $0 < \frac{1}{A} - \frac{1}{|h|^2} < A \text{ SNR}$. Using 0 and $A \text{ SNR}$ to substitute $\frac{1}{A} - \frac{1}{|h|^2}$ respectively, the following inequalities are satisfied:

$$\begin{align*}
&|h|^2 > \frac{1}{1 - \lambda A \text{ SNR}} \\
&\leq \frac{1}{|h|^2} \left[\text{min} \left\{ \frac{1}{\lambda (\text{SNR})} - \frac{1}{|h|^2}, A \text{ SNR} \right\} \right] \\
&\leq \frac{1}{|h|^2} [A \text{ SNR}].
\end{align*}$$

Hence, i) is proved.

For ii), as $\text{SNR} \to 0$, $\frac{F^{-1}(1 - \frac{1}{A})}{1 + A \text{ SNR}F^{-1}(1 - \frac{1}{A})} \to F^{-1}(1 - \frac{1}{A})$, the lower bound will coincide with the upper bound, ii) is proved.

For iii), since $\lambda$ lies in a bounded region, iii) is straightforward.

**APPENDIX C: PROOF OF THEOREM**

Eq. (8) can be explicitly expressed as:

$$C = \int_{\lambda}^{1 - \lambda \text{ SNR}} \log \left( \frac{1}{\lambda} \right) f_{|h|^2}(t) \, dt$$

$$+ \int_{1 - \lambda \text{ SNR}}^{\infty} \log (1 + A \text{ SNR} \, t) f_{|h|^2}(t) \, dt$$

We define $\alpha(\lambda) = \frac{\lambda}{1 - \lambda A \text{ SNR}}$. For the first term in the RHS of (21),

$$\int_{\lambda}^{\alpha(\lambda)} \log \left( \frac{1}{\lambda} \right) f_{|h|^2}(t) \, dt$$

$$\leq \max_{t \in [\lambda, \alpha(\lambda)]} f_{|h|^2}(t) \log \left( \frac{\alpha(\lambda)}{\lambda} \right) (\alpha(\lambda) - \lambda)$$

$$\approx f(\lambda) \lambda A \text{ SNR} \lambda^2 A \text{ SNR}$$

$$= f(\lambda) \lambda^3 A^2 \text{ SNR}^2$$

$$\approx o(\text{SNR})$$

Inequality (22) is derived from substituting the integrand with its maximum value in the integral interval. Eq. (23) uses the properties of $\log (1 + x) \approx x$, and $\frac{1}{1+x} - 1 \approx x$. Eq. (25) and (26) follow from Lemma 2.

The second term in the RHS of (21) corresponds to
\[
\int_{1 - \lambda}^{1 + \lambda} \log (1 + A \text{SNR} \ t) f_{|h|^2} (t) \ dt \\
\approx \int_{1 - \lambda}^{1 + \lambda} \log (1 + A \text{SNR} \ t) f_{|h|^2} (t) \ dt \tag{27}
\]

\[
\approx ASNR \int_{1 - \lambda}^{1 + \lambda} t f_{|h|^2} (t) \ dt \\
= A \text{SNR} \ E_{|h|^2 \geq \lambda} \left[ |h|^2 \right] 	ag{29}
\]

Eq. (27) can be derived from the following proof:

\[
\int_{1 - \lambda}^{1 + \lambda} \log (1 + A \text{SNR} \ t) f (t) \ dt \\
= \int_{1 - \lambda}^{1 + \lambda} \log (1 + A \text{SNR} \ t) f (t) \ dt \\
- \int_{1 - \lambda}^{1 + \lambda} \log (1 + A \text{SNR} \ t) \ dt. \tag{32}
\]

The first term in the RHS of Eq. (32) achieves the target in (27). Therefore, it is sufficient to prove the second term on the RHS of Eq. (32) is \(o(\text{SNR})\) provided (29) is satisfied. Following is the proof of this.

\[
\int_{1 - \lambda}^{1 + \lambda} \log (1 + A \text{SNR} \ t) f (t) \ dt \tag{33}
\]

\[
\leq \max_{t \in [1 - \lambda, 1 + \lambda]} f (t) \left( \frac{\lambda}{1 - \lambda A \text{SNR}} - \lambda \right) \tag{34}
\]

\[
\approx f (\lambda) \lambda^2 A \text{SNR} \lambda \text{SNR} \tag{35}
\]

\[
o(\text{SNR}). \tag{36}
\]

Equations (33), (35) and (36) follow along similar lines as (22), (24) and (25).

In the following, we only need to show that:

\[
\int_{1 - \lambda}^{1 + \lambda} \log (1 + A \text{SNR} \ t) f_{|h|^2} (t) \ dt \approx A \text{SNR} \ E_{|h|^2 \geq \lambda} \left[ |h|^2 \right]. 
\]

Since \(\log (1 + x) \approx x\), we have \(\forall \epsilon > 0\), \(t \in (\lambda, \infty)\), \(\exists \eta_t > 0\), when \(\text{SNR} \in [0, \eta_t]\), \(\left| \log (1 + A \text{SNR}) - 1 \right| \leq \epsilon\). Here \(\eta_t\) differs with the change of \(t\).

Assume \(\eta_t = \min_{t \in (\lambda, \infty)} \eta_t\), when \(\text{SNR} \in (0, \eta_t)\), the following expression can be satisfied:

\[
\int_{1 - \lambda}^{1 + \lambda} (1 - \epsilon) A \text{SNR} \ t f (t) \ dt \leq \int_{1 - \lambda}^{1 + \lambda} \log (1 + A \text{SNR} \ t) f (t) \ dt \\
\leq \int_{1 - \lambda}^{1 + \lambda} (1 + \epsilon) A \text{SNR} \ t f (t) \ dt \tag{37}
\]

Since \(\epsilon\) can be arbitrary,

\[
\int_{1 - \lambda}^{1 + \lambda} \log (1 + A \text{SNR} \ t) f_{|h|^2} (t) \ dt \\
\approx \int_{1 - \lambda}^{1 + \lambda} A \text{SNR} \ t f_{|h|^2} (t) \ dt \tag{28}
\]

Therefore, the second term in the RHS of Eq. (21) dominates the first term. From Lemma 2, we have \(\lambda \approx F^{-1} (1 - \frac{1}{\lambda})\). Hence, Eq. (31) can be written as

\[
A \text{SNR} \ E_{|h|^2 \geq \lambda} \left[ |h|^2 \right] \approx \text{SNR} A \int_{F^{-1} (1 - \frac{1}{\lambda})}^{\infty} t f_{|h|^2} (t) \ dt \tag{38}
\]

\[
= \text{SNR} A \int_{1 - \frac{1}{\lambda}}^{\infty} F^{-1} (t) \ dt. \tag{39}
\]

where (39) is obtained after applying the Leibniz integral rule. Theorem 1 is proved.

**APPENDIX D: PROOF OF COROLLARY 1**

The On-Off power scheme can be explicitly expressed as:

\[
P (h) = \begin{cases} 
A \text{SNR}, & |h|^2 \geq F^{-1} \left( 1 - \frac{1}{\lambda} \right) \\
0, & \text{otherwise}
\end{cases} \tag{40}
\]

It is clear that \(P (h)\) in (40) is an eligible candidate since it satisfies the average and peak power constraint. Following is the proof of the rate achieved by the scheme, which asymptotically approaches the capacity.

According to Lemma 2, \(\lambda \geq \frac{F^{-1} (1 - \frac{1}{\lambda})}{1 + A \text{SNR} F^{-1} (1 - \frac{1}{\lambda})}\) which yields to \(F^{-1} \left( 1 - \frac{1}{\lambda} \right) \leq \frac{1}{1 - \lambda A \text{SNR}}\). The rate achieved by the On-Off power scheme satisfies:

\[
R = \int_{F^{-1} (1 - \frac{1}{\lambda})}^{\infty} \log (1 + A \text{SNR} \ t) f_{|h|^2} (t) \ dt \tag{41}
\]

\[
\geq \int_{1 - \lambda}^{1 + \lambda} \log (1 + A \text{SNR} \ t) f_{|h|^2} (t) \ dt \tag{42}
\]

\[
\approx \int_{1 - \lambda}^{1 + \lambda} \log (1 + A \text{SNR} \ t) f_{|h|^2} (t) \ dt \tag{43}
\]

\[
\approx \text{SNR} A \int_{1 - \frac{1}{\lambda}}^{\infty} F^{-1} (t) \ dt. \tag{44}
\]

**APPENDIX E: PROOF OF COROLLARY 2**

For Rayleigh fading, \(F^{-1} (1 - \frac{1}{\lambda}) = \log A\). Since the rate achieved by the On-Off power scheme proposed in Corollary 1 is asymptotically optimal, the capacity can be derived as:

\[
C \approx R = \int_{\log A}^{\infty} \log (1 + A \text{SNR} \ t) e^{-t} dt \tag{45}
\]

\[
\approx A \text{SNR} \int_{\log A}^{\infty} e^{-t} dt \tag{46}
\]

\[
= \text{SNR} (1 + \log A) \tag{47}
\]

**APPENDIX F: PROOF OF LEMMA 3**

To prove i), from section A, we know that the peak power without PAPR constraint is \(\frac{1}{\eta_0}\). Therefore, to make the PAPR
constraint effective, it should satisfy \( A(\text{SNR}) \text{SNR} < \frac{1}{\lambda_0} \). Since \( \frac{1}{\lambda_0} \to 0 \), when \( \text{SNR} \to 0 \), it is necessary for \( A(\text{SNR}) \text{SNR} \) to approach 0 to satisfy the inequality.

To prove ii), from the average power constraint, we can get: \( \int_{-1}^{1} A(\text{SNR}) \text{SNR} f(h) \frac{2}{\lambda} (t) \, dt \leq \text{SNR} \), which, after multiple simplifications, gives \( \frac{1}{\lambda} < A(\text{SNR}) \text{SNR} + \frac{1}{1-\lambda - A(\text{SNR}) \text{SNR}} \). Since the RHS of the equality goes to 0, when \( A(\text{SNR}) \to \infty \), we have \( \lambda \to \infty \).

**APPENDIX G: PROOF OF THEOREM 2**

Based on Lemma (3) and Eq. (7), the series expansion of the average power constraint at zero can be expressed as:

\[
\text{SNR} \approx e^{-\lambda} \left( \frac{1}{\lambda^2} - \left( \frac{1}{\lambda} - A(\text{SNR}) \text{SNR} \right)^2 e^{-t} + o \left( \frac{1}{\lambda^2} \right) \right) \tag{48}
\]

Similarly, the capacity can be expressed as:

\[
C \approx e^{-\lambda} \left( \frac{1}{\lambda} - \left( \frac{1}{\lambda} - A(\text{SNR}) \text{SNR} - \frac{1}{\lambda^2} \right) e^{-t} \right) \tag{49}
\]

i) If \( l_0 > 0 \), equations (48) and (49) can be expressed as:

\[
\text{SNR} \approx e^{-\lambda} \left( \frac{1}{\lambda^2} - \frac{1}{\lambda} \right) (1 - e^{-l_0}) \tag{50}
\]

\[
C \approx e^{-\lambda} \left( \frac{1}{\lambda} \right) (1 - e^{-l_0}) \tag{51}
\]

Based on [5], the \( \lambda \) in Eq. (50) is:

\[
\lambda \approx \log \frac{1}{\text{SNR}} \tag{52}
\]

The capacity can further be simplified as:

\[
C \approx \text{SNR} \log \frac{1}{\text{SNR}} \tag{53}
\]

ii) If \( l_0 = 0 \),

\[
\text{SNR} \approx A(\text{SNR}) \text{SNR} e^{-\lambda} \tag{53}
\]

\[
C \approx \lambda A(\text{SNR}) \text{SNR} e^{-\lambda} \tag{54}
\]

Equations (53) and (54) utilize the series expansion \( e^{-t} = 1 - t + \frac{t^2}{2} + o(t^2) \) to get the dominating terms.

The capacity can be expressed as:

\[
C \approx \text{SNR} \log A(\text{SNR})
\]

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