Electron mass anomalous dimension at $O(1/N_f^2)$ in quantum electrodynamics.

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Abstract. The critical exponent corresponding to the renormalization of the composite operator $\bar{\psi}\psi$ is computed in quantum electrodynamics at $O(1/N_f^2)$ in arbitrary dimensions and covariant gauge at the non-trivial zero of the $\beta$-function in the large $N_f$ expansion and the exponent corresponding to the anomalous dimension of the electron mass which is a gauge independent object is deduced. Expanding in powers of $\epsilon = 2 - d/2$ we check it is consistent with the known three loop perturbative structure and determine the subsequent coefficients in the coupling constant expansion to \textit{all} orders in \textit{MS}.
The fundamental functions of the renormalization group equation, such as the $\beta$-function and mass anomalous dimension, are crucial ingredients in determining the properties of the Green’s functions of a renormalizable quantum field theory and how they depend on the energy scale. They are ordinarily determined by renormalizing Green’s functions at successive orders in perturbation theory where the coupling constant, $g$, is assumed to be small. Invariably one finds that calculations become exceedingly difficult at a certain order, due to the large number of Feynman graphs to be evaluated, which is therefore tedious, and the intricate nature of some of the integrals which can arise. Thus the renormalization group functions are only known to the first few orders for most theories. For example, the $\beta$-function of QED has recently been calculated to fourth order in $\overline{\text{MS}}$ in [1] by use of computer algebra packages and the wave function and mass renormalization functions are known to third order, by restricting the QCD results of [2] to the abelian case, which extended the earlier two loop calculations of [3, 4]. Clearly it is important to have methods which circumvent these calculational difficulties and allows one to gain an insight into the perturbative structure of these functions to very high orders in the coupling constant.

One way to achieve this is by computing the functions in terms of a different expansion parameter. For example, in theories which possess $N$ fundamental fields, such as $N_f$ electrons in QED, one can use $1/N$ as an alternative expansion parameter. In the earlier work of [5] and [6] the $\beta$-function and mass renormalization, which are gauge independent quantities [7, 8], were computed at leading order in large $N_f$ in QED. This was achieved by explicitly computing the simple poles of the leading order Green’s functions, in dimensional regularization, where the photon propagator was replaced by a chain of one loop bubble graphs. The simple poles were then absorbed minimally into a renormalization constant and the $(4-2\epsilon)$-dimensional perturbation series of $\beta(g)$ and $\gamma_m(g)$ were deduced, which gave the $O(1/N_f)$ coefficients of each function to all orders in $g$. Whilst this was successful in giving agreement with what had previously been calculated in $\overline{\text{MS}}$ and also provided new coefficients, it is certainly not easy to extend that analysis to probe the perturbation series more deeply.

An alternative approach is to use the large $N_f$ self consistency method introduced in [9, 10] where $d$-dimensional critical exponents are computed at the non-trivial zero of the $\beta$-function. As the theory is finite and possesses a conformal symmetry there, the renormalization group equation simplifies to the extent that the critical exponents one computes are related to the critical renormalization group functions which also allows one to extract all orders
information. The beauty of applying the method to a four dimensional gauge theory such as we do here for QED is that by using dressed propagators one substantially reduces the number of Feynman diagrams to be analysed and crucially it is possible to achieve results at $O(1/N_f^2)$. Previously using these techniques in QED the electron wave function exponent $\eta$ was deduced at $O(1/N_f)$ in the Landau gauge in [11] and later at $O(1/N_f^2)$ in [12]. The original $\beta$-function calculation of [6] was reproduced more concisely in [13] at $O(1/N_f)$ and the gauge independence was checked in [14] by computing in an arbitrary covariant gauge. The significance of the agreement of [13] with [6] is that it establishes that the exponents calculated by the self consistency method of [9, 10] encode MS information.

In this letter, we extend the work of [6] by computing the electron mass anomalous dimension, $\gamma_m(g)$, to $O(1/N_f^2)$ by determining the relevant critical exponent. Its calculation is necessary as an independent check on the three loop result of [2] as well as providing the remainder of the perturbation series for $\gamma_m(g)$ at $O(1/N_f^2)$. This is required for future four loop QED and QCD calculations and will also serve as the foundation for finding the $O(1/N_f^2)$ quark mass anomalous dimensions. Simultaneously we deduce a result for three dimensional QED which is important since it will provide improved estimates for a gauge independent quantity which can be measured on the lattice. This will give a useful check for the Monte Carlo algorithms used to simulate problems such as the existence or otherwise of a chirally symmetric phase. Finally, we draw attention to the fact that the procedure we follow here is based very much on the $O(1/N^2)$ calculation of the mass operator in the $O(N)$ Gross Neveu model, [15].

To compute critical exponents in large $N_f$ in QED we use the lagrangian in the form, [11],

$$L = i\bar{\psi}^i \slashed{D} \psi^i + A_\mu \bar{\psi}^i \gamma^\mu \psi^i - \frac{F_{\mu\nu}^2}{4e^2} - \frac{(\partial_\mu A_\mu)^2}{2\xi e^2} \quad (1)$$

where $\psi^i$ is the electron field, $1 \leq i \leq N_f$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\xi$ is the covariant gauge parameter and $A_\mu$ is the $U(1)$ gauge field. The electron coupling constant, $e$, has been rescaled into the gauge field kinetic term to ensure that the vertex has the correct structure for applying the method of uniqueness of [16] which is the main technique for computing the massless Feynman integrals which arise. The location of the critical coupling, $g_c$, in whose neighbourhood we will solve (1) is defined as the non-trivial zero of the $d$-dimensional $\beta$-function. It has been calculated in MS to fourth order
in (4−2\(\epsilon\))-dimensions using dimensional regularization as
\[
\beta(g) = (d-4)g + \frac{2N_f}{3}g^2 + \frac{N_f}{2}g^3 - \frac{N_f(22N_f + 9)}{144}g^4
- \frac{N_f}{64} \left[ \frac{616N_f^2}{243} + \left( \frac{416\zeta(3)}{9} - \frac{380}{27} \right) N_f + 23 \right] g^5 + O(g^6) \tag{2}
\]
where we use the conventions of [17] but have defined \(g = (\epsilon/2\pi)^2\). Thus to deduce information on the (4−2\(\epsilon\))-dimensional perturbation series from the exponent we will calculate, (2) implies that then
\[
g_c \sim \frac{3\epsilon}{N_f} - \frac{27\epsilon^2}{4N_f^2} + \frac{99\epsilon^3}{16N_f^2} + \frac{77\epsilon^4}{16N_f^2} \tag{3}
\]
In the critical region the field theory has a conformal symmetry and so the propagators and Green’s functions obey a simple power law behaviour. For instance, in momentum space, as \(k^2 \to \infty\), the propagators of (1) satisfy,
\[
\psi(k) \sim \frac{\tilde{A}^k}{(k^2)^{\mu-\alpha}} , \quad A_{\nu\sigma}(k) \sim \frac{\tilde{B}}{(k^2)^{\mu-\beta}} \left[ \eta_{\nu\sigma} - (1-\xi)k_{\nu}k_{\sigma}k^2 \right] \tag{4}
\]
where \(\tilde{A}\) and \(\tilde{B}\) are the respective \(k\)-independent amplitudes and \(\alpha\) and \(\beta\) are the critical exponents of each field and their canonical and anomalous dimensions are defined to be, [11],
\[
\alpha = \mu - 1 + \frac{1}{2} \eta , \quad \beta = 1 - \eta - \chi \tag{5}
\]
where \(\eta\) is the electron anomalous dimension and \(\chi\) is the anomalous dimension of the 3-vertex. Each is \(O(1/N_f)\) and depends on \(d = 2\mu\). The former corresponds through an examination of the critical point RGE to the electron wave function renormalization which is known to third order in [2] and \(\eta\) has been deduced at \(O(1/N_f^2)\) in the Landau gauge in [12]. Although it is a gauge dependent quantity it must always be calculated first within the self consistency formalism of [4] as it plays a central role in the computation of other exponents.

To compute the electron mass exponent, \(\gamma_m(g_c)\), we follow the same strategy as the two and three loop perturbative approaches of [2-4]. There \(\gamma_m(g)\) was calculated indirectly by first renormalizing the composite operator \(\bar{\psi}\psi\) and then using
\[
\gamma_m(g) = \gamma_{\bar{\psi}\psi}(g) + \gamma_2(g,\xi) \tag{6}
\]
where $\gamma_2(g, \xi)$ is the wave function renormalization constant which differs in overall sign to the definition given in [2]. In exponent language (6) translates into

$$\gamma_m(g_c) = \eta + \gamma_{\bar{\psi}\psi}(g_c)$$

(7)

Since both $\eta$ and $\gamma_{\bar{\psi}\psi}(g_c)$ are gauge dependent the cancellation of the parameter $\xi$ when we calculate with an arbitrary covariant gauge will provide a very stringent check on our computation before checking with [2]. For completeness, in the same notation as (2), [2],

$$\gamma_m(g) = -\frac{3}{2} g + \frac{(20N_f - 9)}{48} g^2$$

$$+ \left[ \frac{140N_f^2}{27} + (46 - 48\zeta(3))N_f - \frac{129}{2} \right] \frac{g^3}{32} + O(g^4)$$

(8)

We will now detail the leading order analysis to reproduce the result of [6] but in the critical point self consistency approach which will also illustrate the simplicity of the method and will allow us to elaborate on several technical points in preparation for the $O(1/N_f^2)$ calculation. First, we have to introduce a regularization to handle the infinities which will arise in the calculation. This is achieved by shifting the exponent $\chi$ by $\chi \rightarrow \chi + \Delta$ where $\Delta$ is an infinitesimal quantity, [12]. Once this is introduced the Green’s functions which one calculates in $d$-dimensions will have a three part structure, [18, 19]. It will involve terms which have poles in $\Delta$ which are removed by a conventional renormalization, [18]. This leaves $O(1)$ terms which are multiplied by $\ln p^2$, where $p$ is the external momentum, and $O(\Delta)$ terms which vanish as the regularization is lifted. As we are working at a critical point where $p^2 \rightarrow \infty$ the presence of $\ln p^2$ terms will spoil the scaling behaviour. General arguments, however, show that their sum will be of the form $(p^2)^{1/2}\gamma_{\bar{\psi}\psi}(g_c)$, [18, 19]. Therefore, expanding this in powers of $1/N_f$ and choosing $\gamma_{\bar{\psi}\psi}(g_c)$ appropriately at each order will remove these $\ln p^2$ terms to leave a Green’s functions with sensible scaling behaviour. Indeed this observation which was discussed originally in [18] was also used to renormalize the 2-point function of QED in [12, 20]. In other words to deduce $\gamma_{\bar{\psi}\psi}(g_c)$ at each order in $1/N_f$ one computes the Feynman diagrams with dressed propagators which contribute to the inclusion of $\bar{\psi}\psi$ as a composite operator in some Green’s function, which for this letter will be $\langle \bar{\psi} [\bar{\psi}\psi] \psi \rangle$. It isolates the $\ln p^2$ terms after $\Delta$-renormalization and sums their coefficients.

At leading order there is only one Feynman graph to consider which is given in fig. 1. The circle with a cross indicates a $\bar{\psi}\psi$ insertion. Substituting
the asymptotic scaling forms of the propagators of (4) for the lines of fig. 1 and computing the trivial one loop integral, it is

\[ z(2\mu - 1 + \xi)a(\mu - 1 + \Delta)a(\mu - \Delta)(p^2)^{-\Delta} \]  
(9)

where we have defined \( a(\alpha) = \Gamma(\mu - \alpha)/\Gamma(\alpha) \) for all \( \alpha \) and set \( z = \tilde{A}^2 \tilde{B} \).

Expanding (9) in powers of \( \Delta \) one obtains

\[ \frac{(2\mu - 1 + \xi)z_1}{\Delta \Gamma(\mu)N_f} \left[ 1 - \Delta \ln p^2 + \frac{\Delta}{(\mu - 1)} + O(\Delta^2) \right] \]  
(10)

where \( z = \sum_{i=1}^{\infty} z_i/N_f^i \) and \( z_1 \) is deduced from the leading order computation of \( \eta \) in [11] as

\[ z_1 = \frac{\Gamma(2\mu)}{8\Gamma^2(\mu)\Gamma(2 - \mu)} \]  
(11)

Thus,

\[ \gamma_{\bar{\psi}\psi,1}(g_c) = -\frac{\mu(2\mu - 1 + \xi)\eta_1^0}{(\mu - 2)(2\mu - 1)} \]  
(12)

where

\[ \eta_1^0 = \frac{(2\mu - 1)(\mu - 2)\Gamma(2\mu)}{4\Gamma^2(\mu)\Gamma(\mu + 1)\Gamma(2 - \mu)} \]  
(13)

which corresponds to the Landau gauge expression for \( \eta_1 \). It is a trivial exercise to repeat the computation of \( \eta_1 \) for non-zero \( \xi \) to deduce

\[ \eta_1^\xi = \frac{(2\mu - 1)(\mu - 2) + \xi\mu\eta_1^0}{(2\mu - 1)(\mu - 2)} \]  
(14)

which from (7) implies

\[ \gamma_{m,1}(g_c) = -\frac{2\eta_1^0}{(\mu - 2)} \]  
(15)

in agreement with [6], but deduced by considering only one Feynman graph.

The way to proceed to \( O(1/N_f^2) \) is now clear and involves first of all computing the contributions from fig. 1 since its lines are represented by propagators which possess \( N_f \) dependent exponents as well as the obvious counterterm graphs. However, the main effort lies in considering the higher order graphs of fig. 2. The fourth graph can be determined by a simple extension of the one loop integral of fig. 1 and it contributes

\[ -\frac{3\mu^2(2\mu - 1 + \xi)^2(\eta_1^0)^2}{2(\mu - 1)(\mu - 2)^2(2\mu - 1)^2} \]  
(16)
to $\gamma_{\bar{\psi}\psi,2}(g_c)$. To calculate the remaining graphs one first maps each integral into coordinate space through the Fourier transform

$$\frac{1}{(x^2)^\alpha} = \frac{a(\alpha)}{2^{2\alpha}\pi^\mu} \int_k \frac{e^{ikx}}{(k^2)\mu - \alpha}$$

(17)

Since the third graph of fig. 2 then corresponds to an insertion on the completely internal electron line of the second order correction to the electron self energy it is $\Delta$-finite in coordinate space, [20]. Also, the values of the first two graphs are equivalent and after mapping to coordinate space they correspond to a self energy graph where the insertion is now on an electron line joining to an external vertex. It is not instructive to reproduce the tedious algebra connected with the calculation of these graphs aside from remarking that the techniques of integration by parts, [14], subtractions, [10], and uniqueness, [16], we used have already been discussed in the computation of the electron self energy at second order, [12, 20]. After a substantial amount of algebra we found that the first graph of fig. 2 contributes

$$-\frac{\mu(\eta_0^0)^2}{(2\mu - 1)^2(\mu - 2)^2}\left[\frac{(2\mu - 1 + \xi)(2\mu - 1)(\mu - 2) + \xi\mu}{(\mu - 1)} - \frac{(2\mu - 1)(\mu - 1)(1 - \xi)}{\mu}\right]$$

(18)

to $\gamma_{\bar{\psi}\psi,2}(g_c)$, whilst the third gave

$$-\frac{\mu^2(\eta_0^0)^2}{2(\mu - 1)(2\mu - 1)^2(\mu - 2)^2}[(2\mu - 1 + \xi)(2\mu - 3 - \xi) - 4\xi(\mu - 1)]$$

(19)

The contributions from the graph of fig. 1 which involves the various counterterm pieces are

$$\frac{\mu(2\mu - 1 + \xi)(\eta_0^0)^2}{(\mu - 1)(2\mu - 1)^2(\mu - 2)^2}\left[\frac{(2\mu - 1)(3\mu - 4)}{(\mu - 1)} + 3\xi\mu - 3\mu(\mu - 1)^2\left(\hat{\Theta} - \frac{1}{(\mu - 1)^2}\right)\right]$$

(20)

where $\hat{\Theta}(\mu) = \psi'(-1) - \psi'(-1)$, $\psi(x)$ is the logarithmic derivative of the $\Gamma$-function and we have used $\chi_1 = -\eta_1$, [4], and

$$z_2 = \frac{3\mu^2(\mu - 1)\Gamma(\mu)(\eta_0^0)^2}{2(\mu - 2)^2(2\mu - 1)^2}\left[\hat{\Theta} - \frac{1}{(\mu - 1)^2}\right]$$

(21)
from the $x$-space calculation of $\eta_2$, [12]. Thus, we have

$$
\gamma_{\bar{\psi},2}(g_c) = \frac{\mu(\eta_1^0)^2}{(2\mu - 1)^2(\mu - 2)^2} \left[ \frac{2(1 - \xi)(2\mu - 1)(\mu - 1)}{\mu} - 2\mu(2\mu - 1) - 3\mu(2\mu - 1 + \xi)(\mu - 1)\left(\hat{\Theta} - \frac{1}{(\mu - 1)^2}\right) \right] \tag{22}
$$

Finally, to obtain $\gamma_{m,2}(g_c)$ we need to add $\eta_2$ in an arbitrary gauge to (21). We have deduced this by adapting the Landau gauge computation of [12] to find

$$
\eta_2^\xi = \frac{(\mu - 1)(\eta_1^0)^2}{(\mu - 2)^2(2\mu - 1)^2} \left[ \frac{2(2\mu - 1)}{\mu}[(\mu - 1)(\mu - 3) + \xi\mu] + 3\mu[(2\mu - 1)(\mu - 2) + \xi\mu] \left(\hat{\Theta} - \frac{1}{(\mu - 1)^2}\right) \right] \tag{23}
$$

From (7), (22) and (23) then we have simply

$$
\gamma_{m,2}(g_c) = -\frac{6(\eta_1^0)^2}{(\mu - 2)^2(2\mu - 1)} \left[ \mu(\mu - 1)\hat{\Theta} - \frac{\mu}{(\mu - 1)} + \frac{1}{\mu} + \frac{4\mu}{3} - 2 \right] \tag{24}
$$

where it is reassuring to note that, as in (15), all $\xi$-dependence has cancelled, which is the first check on our result.

The subsequent check is to set $\mu = 2 - \epsilon$ in (24) and expand in powers of $\epsilon$ to $O(\epsilon^3)$ in order to compare with (8) at the value of (3). We find that both (24) and (8) are in agreement and note that the $\zeta(3)$ of (8) arises from the expansion of $\hat{\Theta}$ in (24). Thus the gauge independent result (24) corresponds to the $O(1/N_f^2)$ correction to (15). Moreover, one can now deduce new coefficients which will appear in $\gamma_m(g)$ at higher orders in $g$ by expanding $\gamma_{m,2}(g_c)$ to the subsequent orders in $\epsilon$. The $O(1/N_f^2)$ corrections for $g_c$ are required for this and they are encoded in the $\beta$-function exponent $\lambda = -\frac{1}{2}\beta'(g_c)$ of [3, 4],

$$
\lambda = \mu - 2 - \frac{(2\mu - 3)(\mu - 3)\eta_1^0}{N_f} + O\left(\frac{1}{N_f^2}\right) \tag{25}
$$

in the present notation. We record, for example, that in $\overline{MS}$ to fourth order in $g$, (8) will become

$$
\gamma_m(g) = -\frac{3g}{2} + \left[ \frac{10N_f}{3} - \frac{3}{2} \right] \frac{g^2}{8} + \left[ \frac{140}{27} N_f^2 + (46 - 48\zeta(3))N_f - \frac{129}{2} \right] \frac{g^3}{32} + \left[ \frac{N_f^3}{9} \left( \frac{83}{144} - \zeta(3) \right) + \frac{N_f^2}{4} \left( 5\zeta(3) - 3\zeta(4) - \frac{19}{54} \right) + pN_f + q \right] g^4 \tag{26}
$$
where the unknown coefficients \( p \) and \( q \) can only be determined by an explicit 4-loop calculation or respectively knowledge of \( O(1/N_f^3) \) and \( O(1/N_f^4) \) exponents. However, it is worth pointing out that one can now make use of (26) to substantially reduce the amount of work one would have to do in the explicit calculation. By isolating only those Feynman graphs with one and no electron loops the residues of the simple poles in \( \epsilon \) respectively determine \( p \) and \( q \).

Finally, since (24) is valid in \( d \)-dimensions we can restrict to \( \mu = \frac{3}{2} \), to obtain

\[
\gamma_m(g_c) = -\frac{32}{3\pi^2 N_f} - \frac{64}{9\pi^4 N_f^2} [3\pi^2 - 28] + O\left(\frac{1}{N_f^3}\right)
\]

(27)

As this also is a gauge independent quantity it can be used to determine exponents for relatively low values of \( N_f \) to compare with numerical work.

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Figure Captions.

Fig. 1. Leading order graph for $\gamma_{\bar{\psi}\psi}(g_c)$.

Fig. 2. Higher order graphs for $\gamma_{\bar{\psi}\psi}(g_c)$. 