Variational Approach to Tunneling Dynamics.
Application to Hot Superfluid Fermi Systems.
Spontaneous and Induced Fission.

S. Levit

Department of Condensed Matter Physics
Weizmann Institute of Science
Rehovot 76100, Israel
(Dated: January 9, 2022)

We introduce a general variational framework to address the tunneling of hot Fermi systems. We use the representation of the trace of the imaginary time $\tau = it$ propagator as a functional integral type of a sum over complete sets of states at intermediate propagation slices. We assume that these states are $\tau$-dependent and generated by an arbitrary trial Hamiltonian $H_0(\tau)$. We then use the convexity inequality to derive $H_0(\tau)$ controlled variational bound for a trial action functional. This functional has a general structure consisting of two parts - statistically weighted quantum penetrability and dynamical tunneling entropy. We examine how this structure incorporates the basic physics of tunneling of hot Fermi systems. Using the variational inequality one can optimise the dynamical parameters controlling the action functional for any choice of the trial problem.

As an application we take $H_0(\tau)$ to describe imaginary time dynamics of non interacting Bogoliubov-de Gennes (BdG) quasiparticles. Optimising its dynamical parameters we extend the tunneling theory of hot Fermi systems to the Hartree-Fock-Bogoliubov (HFB) frame and derive the corresponding generalisation of imaginary time temperature dependent BdG mean field equations.

As in the trial action the prominent feature of these equations is an inseparable interplay between quantum dynamical and entropic statistical effects. In the zero temperature limit these equations describe the "false ground state" tunneling decay of superfluid Fermi systems (spontaneous fission in nuclear physics). With increasing excitation energy (effective temperature) the decay process is gradually evolving from pure quantum tunneling to statistical "bottle neck" escape mechanism. Correspondingly the statistically weighted dynamical penetrability part in the action gradually decreases while the tunneling entropy increases with increasing effective temperature.

PACS numbers: 03.75.Kk, 03.75.Ss, 05.30.Fk, 21.10.Tg, 24.75.+i, 24.10.Cn, 25.85.Ca, 25.85.Ec, 31.15.Ne, 67.57.?z
Keywords: variational approach, many fermion tunneling, hot superfluid Fermi systems, Hartree-Fock-Bogoliubov, Bogoliubov-de Gennes, spontaneous and induced fission
1. **MOTIVATION. STATISTICALLY AVERAGED PENETRABILITY AND TUNNELING ENTROPY.**

Continuation of classical equations to imaginary time domain have long been understood as a way to describe quantum mechanical tunneling phenomena, cf., Ref.[1]. By extending the field theoretical description of "false vacuum decay", Ref.[2], the fermion mean field tunneling description of spontaneous fission, i.e. decay of a "false ground state" was proposed in Refs.[3, 4]. This formalism essentially amounts to imaginary time vacuum decay", Ref.[2], the fermion mean field tunneling description of spontaneous fission, i.e. decay of a "false quantum mechanical tunneling phenomena, cf., Ref.[1]. By extending the field theoretical description of "false quantum mechanical tunneling phenomena, cf., Ref.[1].

The resulting dynamical equations describe statistically averaged tunneling bounce of Slater determinants. This theory was based on auxiliary field functional integral formalism with the resulting equations of the Hartree-Fock type. The ways to extend this to the Hartree-Fock frame were outlined in Refs.[12] and [13].

Here we introduce a more general variational approach to imaginary time dynamics of equilibrated Fermi systems. We use the representation of the trace of the imaginary time propagator $\text{Tr}e^{-\beta(\hat{H}-\mu \hat{N})}$ as a functional integral type of a sum over complete sets of states at the intermediate propagation slices. We assume that these states are $\tau$-dependent and generated by a trial Hamiltonian $H_0(\tau)$. Apart of the symmetry condition $H_0(\tau) = H_0(-\tau)$ for $-\beta/2 \leq \tau \leq \beta/2$ this Hamiltonian can be arbitrary. We then use the convexity inequality to derive a variational bound for the trial action functional which has the following form

$$- \ln \text{Tr}e^{-\beta(\hat{H}-\mu \hat{N})} \leq \sum_K W_K(0) \int_{-\beta/2}^{\beta/2} d\tau \langle \Psi_K^{(0)}(\tau) | \frac{\partial}{\partial \tau} + \hat{H} - \mu \hat{N} | \Psi_K^{(0)}(\tau) \rangle + \sum_K W_K(0) \ln W_K(0)$$

where $\hat{H}$ is the exact Hamiltonian, $|\Psi_K^{(0)}(\tau)\rangle$’s are solutions of the quasienergy problem defined by the trial $\hat{H}_0(\tau)$ on the interval $-\beta/2 \geq \tau \geq \beta/2$, cf., Eq.(16) below and $W_K(0)$ are statistical weights defined in terms of the quasienergies $\Lambda_K$, cf., Eqs.(13, 8).

The first part of the functional on the right hand side of (1) has the form of statistically averaged action while the second part can be interpreted as dynamical entropy. Using the above inequality one can optimise the parameters of the trial $H_0(\tau)$ controlling this functional for a given $\beta$ and $\mu$ and any choice of $H_0(\tau)$ symmetric in $\tau$. In the application below we will illustrate this procedure and the use of its results to estimate the tunneling probability, cf., Eq. (24).

In order to understand the physics of the trial action in (1) let us consider tunneling of a hot Fermi system like e.g. induced fission of an equilibrated (compound) nucleus having fixed excitation energy $E^*$ and a given particle number $N$. Such a system must be described by a micro-canonical ensemble which means that the values of $\beta$ and $\mu$ will be fixed by $E^*$ and $N$. After the optimization the terms with $\hat{H}$ and $\hat{N}$ in the trial action will be removed by the appropriate Legendre transformations or alternatively by the saddle point approximation to the integral relation between the grand canonical and the micro-canonical formulations. As a result the first part of the trial action in Eq. (1) would consist of only the $\partial/\partial \tau$ term which should clearly be viewed as statistically weighted quantum penetrability. The second part has the form of the statistical entropy which however depends on the tunneling dynamics via quasi-energies. We shall call it tunneling entropy. The first term vanishes for a static trial problem while the last tends to zero at zero temperature, $\beta \to \infty$ with the statistical weights becoming stepwise 1 to 0 function.

To understand the role and the interplay of the averaged penetrability and tunneling entropy let us consider the following schematic semiclassical picture of tunneling of such systems. Imagine a fly caught in a bottle with a narrow opening on the side. The bottle is the phase space of an (equilibrated) system (all its $6N$ degrees of freedom). The fly is the point representing the state of the system in the ensemble. At $E^* = 0$ the phase space is just one point and the only way for the fly to escape is via tunneling through the bottle’s "thick walls”. This is the spontaneous fission, $\beta \to \infty$ limit.

As $E^*$ goes up the phase space for the fly grows. It can still tunnel from the points where the bottle wall is the
thinnest (at this $E^*$) i.e. have the largest quantum tunneling probability or from other places (phase space volume). The wall is thicker there (smaller tunneling probability) but there are many more such points (larger entropy). So the tunneling becomes a mixture of QM tunneling and entropic effects.

The important point is that even as $E^*$ is above the bottle opening (i.e. above the $T = 0$ barrier energy) the same competition is going on - the fly can either go through the opening (where no quantum tunneling is needed) from a small number (volume) of points “near the opening” or tunnel from other, much larger in number (high entropy) points but pay the price of quantum tunneling.

This is consistent with the recent study \[22\] of the excitation energy $E^*$ dependence of isentropic fission barriers. The barrier height $E_B(E^*)$ is not a fixed (zero temperature) parameter. As $E^*$ grows $E_B(E^*)$ grows with it and although the difference $E_B(E^*) - E^*$ decreases it vanishes only asymptotically at large values of $E^*$.

As a first application of our formalism we derive the generalisation of the Bogoliubov-de Gennes (BdG) mean field equations to describe imaginary time tunneling bounce at zero and finite effective temperatures/excitation energies. Such a generalisation is needed since significant changes of the mean field shape during tunneling lead to multiple crossings of single particle levels with different symmetry, Ref. \[6, 8\]. Effective switching between such levels requires presence of terms in the mean field Hamiltonian which are absent in the Hartree-Fock approximation. Theoretical considerations supported by growing body of numerical simulations clearly indicate that the pairing interaction is often the missing component in the HF theories of collective tunneling dynamics \[14\] - \[21\]. The Hartree-Fock-Bogoliubov (HFB) extension appears to be the most suitable framework.

Constraining $H_0(\tau)$ to have the HFB form describing non interacting Bogoliubov-de Gennes (BdG) quasiparticles we optimise the dynamical parameters of $H_0(\tau)$. As a result we derive an extension of the BdG equations with both imaginary time and temperature dependent mean field. The quasiparticle eigenfunctions are bouncing in the imaginary time inverse temperature interval while the self-consistent density and pairing matrices depend on the thermal Fermi occupations which in turn depend on the quasiparticle quasienergies, cf., Eq. \(23\). Thus as in the trial action an inseparable interplay exists between quantum dynamical and entropic statistical effects. With increasing excitation energy (effective temperature) the decay process is gradually evolving from pure quantum tunneling to statistical ”bottle neck” escape mechanism. Correspondingly the statistically weighted dynamical penetrability part in the action gradually decreases while the tunneling entropy increases with increasing effective temperature.

Theory of nuclear fission is a self evident field of application of our development. Impressive recent improvements of numerical techniques and computer resources should help to apply our results for improving the microscopic description of spontaneous and low energy induced fission. It should also be useful in providing novel microscopic insights in the existing phenomenological theories of the fission phenomena.

2. IMAGINARY TIME PROPAGATION IN AN ARBITRARY MANY BODY BASIS

Assume many fermion Hamiltonian

\[
\hat{H} = \hat{T} + \hat{V} = \sum_{ij} t_{ij} \hat{a}_i^+ \hat{a}_j + \frac{1}{2} \sum_{ijkl} V_{ijkl} \hat{a}_i^+ \hat{a}_j^+ \hat{a}_l^+ \hat{a}_k
\]

(2)

in terms of standard Fermi operators $\hat{a}_i$ and $\hat{a}_j^+$ in an arbitrary single particle basis \{${\phi}_k(r\sigma) \equiv \phi_k(x)$\} with one body part $t_{ij}$ representing kinetic energy and possibly external potential and two body interaction $V_{ijkl}$.

The most common Fermi systems which exhibit tunneling are heavy nuclei undergoing fission. The most common initial state is the so called compound nucleus which can statistically be described as an approximately equilibrated system with a given excitation energy and a particle number. Formally this is a microcanonical ensemble with the partition function $Tr\delta(E - \hat{H})\delta(N - \hat{N})$. In a standard way it can be related to the grand canonical partition function

\[
Z(T, \mu) = Tr e^{-\beta(\hat{H} - \mu \hat{N})} = \sum_K \langle \Psi_K | e^{-\beta(\hat{H} - \mu \hat{N})} | \Psi_K \rangle , \quad \beta = 1/T
\]

(3)

where \{${\Psi}_K$\} denotes a complete set of many fermion states in the Fock space of the system and $\beta$ and $\mu$ are determined in a standard way by $E$ and $N$, cf., Ref. \[10\].
Let us slice the exponential under the trace in (4) 

\[ T r e^{-\beta(\hat{H} - \mu \hat{N})} = \lim_{\epsilon \to 0} \sum_K \langle \Psi_K | \prod_{m=1}^M \left( 1 - \beta \epsilon(\hat{H} - \mu \hat{N}) \right) | \Psi_K \rangle, \quad \epsilon = \frac{1}{M} \]  

and insert complete sets of (many fermion) states at each slice in the expression \( \text{(4)} \). We label these sets by imaginary time \( \{ \Psi_K(\tau_m) \} \) with \( -\beta/2 \leq \tau \leq \beta/2 \) and moreover choose them not identical but changing from slice to slice and take them obeying 

\[ \{ \Psi_K(\tau_m) \} = \{ \Psi_K(-\tau_m) \} ; \{ \Psi_K^*(\tau_m) \} = \{ \Psi_K^*(-\tau_m) \} \equiv \{ \bar{\Psi}_K(\tau_m) \} \]  

In a standard way we obtain in the limit \( M \to \infty \) a functional integral like sum 

\[ Z(T, \mu) = \sum_{\{ \Psi_{K}^{(0)}(\tau) \}} \exp \left\{ - \int_{-\beta/2}^{\beta/2} d\tau \left[ \langle \Psi_K(\tau) | \hat{H} - \hat{H}_0(\tau) | \Psi_K(\tau) \rangle \right] \right\} \]  

where the condition \( \text{(5)} \) assures that each term in this sum is real and positive.

There are several ways to proceed from this exact expression to devise a suitable mean field theory. These include the superfluid density functional approach or the use of functional integral techniques. We will describe them elsewhere, Ref.\([24]\). Here we will follow a variational approach.

3. VARIATIONAL FRAMEWORK - USING THE CONVEXITY INEQUALITY

Let us assume some trial Hamiltonian \( \hat{H}_0(\tau) \) which is in general \( \tau \) dependent. At the moment there is no need to specify the exact form of \( \hat{H}_0(\tau) \) apart of it being symmetric \( \hat{H}_0(\tau) = \hat{H}_0(-\tau) \). In the following we will consider a specific example of \( \hat{H}_0(\tau) \) leading to finite temperature imaginary time dependent Hartree-Fock-Bogoliubov(HFB) mean field formalism.

Let us add and subtract \( \hat{H}_0(\tau) \) in the exponential of \( \text{(4)} \) and write the result as 

\[ Z(T, \mu) \equiv T r e^{-\beta(\hat{H} - \mu \hat{N})} = Z_0(T, \mu) \sum_{\{ \Psi_{K}^{(0)}(\tau) \}} W_{K}^{(0)} \exp \left\{ - \int_{-\beta/2}^{\beta/2} d\tau \left[ \langle \Psi_K(\tau) | \hat{H} - \hat{H}_0(\tau) | \Psi_K(\tau) \rangle \right] \right\} \]  

where we defined the corresponding probabilities 

\[ W_{K}^{(0)} = \frac{1}{Z_0(T, \mu)} \exp \left\{ - \int_{-\beta/2}^{\beta/2} d\tau \langle \Psi_K(\tau) | \hat{H} - \hat{H}_0(\tau) | \Psi_K(\tau) \rangle \right\} \]  

The symmetry \( \hat{H}_0(\tau) = \hat{H}_0(-\tau) \) and Eq.\([14]\) assure \( W_{K}^{(0)} \geq 0 \) and \( Z_0(T, \mu) \) normalises \( \sum_K W_{K}^{(0)} = 1 \).

Using \( \langle e^{-X} \rangle \geq e^{-\langle X \rangle} \), Ref.\([23]\), in Eq.\([7]\) we arrive at a variational inequality 

\[ \Omega(T, \mu) = -T \ln Z(T, \mu) \leq \Omega_{\text{trial}}(T, \mu) = \Omega_0(T, \mu) + \frac{1}{\beta} \int_{-\beta/2}^{\beta/2} d\tau \langle \hat{H} - \hat{H}_0(\tau) \rangle \]  

with the notation \( \Omega_0(T, \mu) = -T \ln Z_0(T, \mu) \) and \( \langle \hat{O}(\tau) \rangle = \sum_K W_{K}^{(0)} \langle \bar{\Psi}_K(\tau) | \hat{O}(\tau) | \Psi_K^{(0)}(\tau) \rangle \).

To evaluate \( \Omega_{\text{trial}}(T, \mu) \) it is useful to introduce the imaginary time evolution operator for \( \hat{H}_0(\tau) - \mu \hat{N} \)

\[ \hat{U}_0(\tau, -\beta/2) = T \exp \left\{ - \int_{-\beta/2}^{\beta/2} d\tau' \langle \hat{H}_0(\tau') - \mu \hat{N} \rangle \right\} \]  

It is straightforward to show via time slicing that 

\[ Z_0(T, \mu) = T r \hat{U}_0(\beta/2, -\beta/2) \rightarrow \Omega_0(T, \mu) = -T \ln T r \hat{U}_0(\beta/2, -\beta/2) \]
Let us consider the eigenvalue equation
\[ \hat{U}_0(\beta/2, -\beta/2)\ket{\Phi_K} = e^{-\beta\Lambda_K} \ket{\Phi_K} \] (12)

The operator \( \hat{U}_0(\beta/2, -\beta/2) \) is Hermitian and moreover positive for the symmetric \( \hat{H}_0(\tau) = \hat{H}_0(-\tau) \) so one can choose \( \Lambda_K \) as real. We shall refer to them as quasienergies. We can write
\[ Z_0 = \sum_K e^{-\beta\Lambda_K} \rightarrow W_K^{(0)} = \frac{1}{Z_0} e^{-\beta\Lambda_K} \] (13)

A practical way to work with \( \hat{U}_0(\tau, -\beta/2) \) is to introduce the \( \tau \) dependent
\[ \ket{\Phi_K(\tau)} \equiv \hat{U}_0(\tau, -\beta/2)\ket{\Phi_K} \] (14)
in terms of which the eigenvalue equation (12) is equivalent to solving the boundary value problem
\[ \left[ \frac{\partial}{\partial \tau} + \hat{H}_0(\tau) - \mu \hat{N} \right] \ket{\Phi_K(\tau)} = 0 \quad \ket{\Phi_K(\beta/2)} = e^{-\beta\Lambda_K} \ket{\Phi_K(-\beta/2)} \] (15)

It is furthermore convenient to introduce \( \ket{\Psi_K(\tau)} = e^{\Lambda_K(\tau+\beta/2)} \ket{\Phi_K(\tau)} \) which satisfies
\[ \left[ \frac{\partial}{\partial \tau} + \hat{H}_0(\tau) - \mu \hat{N} \right] \ket{\Psi_K^{(0)}(\tau)} = \Lambda_K \ket{\Psi^{(0)}(\tau)} \quad \text{with} \quad \ket{\Psi^{(0)}(\beta/2)} = \ket{\Psi^{(0)}(-\beta/2)} \] (16)

These equations together with the identity \( -\ln Z_0 = \beta \sum_K \Lambda_K W_K^{(0)} + \sum_K W_K^{(0)} \ln W_K^{(0)} \) allow to write
\[ \beta \Omega_0 = \sum_K W_K^{(0)} \int_{-\beta/2}^{\beta/2} d\tau \left[ \hat{\Phi}^{(0)}_K(\tau) \left( \frac{\partial}{\partial \tau} + \hat{H}_0(\tau) - \mu \hat{N} \right) \hat{\Phi}^{(0)}_K(\tau) \right] + \sum_K W_K^{(0)} \ln W_K^{(0)} \] (17)

Inserting this into the inequality (11) allows to write it in a form which was quoted in the Introduction, Eq. (1).

As a last remark let us indicate that the transformation from the equations for \( \ket{\Phi_K(\tau)} \) to the boundary problem introduces (gauge like) ambiguities in the definitions of \( \Lambda_K \) and \( \ket{\Psi^{(0)}(\tau)} \). We refer the reader to Ref. [8] for the discussion of how to deal with this issue.

### 4. THERMAL TUNNELING BOUNCE - IMAGINARY TIME DEPENDENT BOGOLIUBOV-DE GENNES EQUATIONS

We take the trial Hamiltonian \( \hat{H}_0 \) to have the form
\[ \hat{H}_0(\tau) = \sum_{ij} \left[ (t_{ij} + \gamma_{ij}(\tau)) \hat{a}_i^\dagger \hat{a}_j + \frac{1}{2} \Delta_{ij}(\tau) \hat{a}_i^\dagger \hat{a}_j^\dagger + \frac{1}{2} \Delta_{ij}^*(\tau) \hat{a}_i \hat{a}_j \right] \] (18)
with
\[ \gamma_{ij}(\tau) = \sum_{kl} V_{ikjl}^* \sigma_{jk}(\tau) \quad \Delta_{ij}(\tau) = \sum_{kl} V_{ijkl} \eta_{jk}(\tau) \] (19)
and \( \sigma(\tau) \) and \( \eta(\tau) \) matrices obeying \( \sigma_{ij}(\tau) = \sigma^*_{ij}(\tau), \eta_{ij}(\tau) = -\eta_{ij}(\tau) \) considered as variational parameters.

One needs to solve Eq. (18) with \( \hat{H}_0(\tau) \) given by Eq. (18), use the solutions to express \( \Omega_{\text{trial}}(T, \mu) \) of Eq. (1) as a functional of \( \sigma_{ij}(\tau) \)'s and \( \eta_{ij}(\tau) \)'s and then minimise it. One can do this generalising the derivation of the static finite temperature Hartree-Fock-Bogoliubov formalism. The details will be published elsewhere [24]. Here is the summary.

Let us define (\( \tau \)-dependent) quasiparticle operators
\[ \hat{d}^\dagger_{\nu}(\tau) = \sum_i \left[ u_{\nu}(i, \tau) \hat{a}_i^\dagger + v_{\nu}(i, \tau) \hat{a}_i \right] \quad \hat{d}_{\nu}(\tau) = \sum_i \left[ \bar{u}_{\nu}(i, \tau) \hat{a}_i^\dagger + \bar{v}_{\nu}(i, \tau) \hat{a}_i \right] \] (20)
where \( \bar{u}_\nu(\tau) \equiv u_\nu^*(\tau) \), \( \bar{v}_\nu(\tau) \equiv v_\nu^*(\tau) \). The functions \( u_\nu(\tau), v_\nu(\tau) \) satisfy the eigenvalue equations

\[
\left( \frac{\partial}{\partial \tau} + \mathcal{H}(\tau) \right) w_\nu(\tau) = \lambda_\nu w_\nu(\tau) \quad , \quad w_\nu(\beta/2) = w_\nu(-\beta/2)
\]

(21)

where denoted

\[
\mathcal{H}(\tau) = \begin{pmatrix} h(\tau) - \mu & \Delta(\tau) \\ -\Delta^*(\tau) & -h^*(\tau) + \mu \end{pmatrix}
\]

(21)

With such \( u_\nu(\tau), v_\nu(\tau) \) the operators \( \hat{d}_\nu(\tau), \hat{d}_\nu^*(\tau) \) satisfy the Fermi commutation relations and one can write the solutions of Eq. (16) as

\[
|\Psi_K(\tau)\rangle \equiv |\Psi_{n_1, n_\nu, \ldots}(\tau)\rangle = \prod_{\nu > 0} |\hat{d}_\nu^*(\tau)|^{n_\nu} |\Psi_0(\tau)\rangle \quad \text{with} \quad n_\nu = 0 \text{ or } 1
\]

(22)

This provided that the quasiparticle vacuum \(|\Psi_0(\tau)\rangle\) is annihilated by all \( \hat{d}_\nu(\tau) \)'s. The corresponding \( \Lambda_K = \Lambda_0 + \sum_\nu \lambda_\nu \).

Using such \(|\Psi_K(\tau)\rangle\)'s in \( \Omega_{\text{trial}}(T, \mu) \), Eq. (14), and minimising the result with respect to \( \sigma_{ij}(\tau) \)'s and \( \eta_{ij}(\tau) \)'s leads to the self consistency conditions

\[
\sigma_{ji}(\tau) = \sum_{\nu > 0} [(1 - f_\nu) \bar{v}_\nu(j, \tau) v_\nu(i, \tau) + f_\nu u_\nu(j, \tau) \bar{u}_\nu(i, \tau)]
\]

\[
\eta_{ji}(\tau) = \sum_{\nu > 0} [(1 - f_\nu) u_\nu(i, \tau) \bar{v}_\nu(j, \tau) + f_\nu \bar{u}_\nu(i, \tau) v_\nu(j, \tau)]
\]

\[
f_\nu = \frac{1}{1 + e^{\lambda_\nu}}
\]

(23)

It is instructive to examine several limits of the above equations. In the static, \( \tau \) independent limit one readily finds the standard, temperature dependent HFB expressions. Disregarding the pairing field \( \Delta \) in the trial Hamiltonian \( \{15\} \) leads to the imaginary time dependent HF form of the equations with thermal averaged mean field. These were first derived in the Hartree form in Ref. [10] where it was argued that they provide a mean field description of tunneling in induced fission. At zero temperature the HF limit of the above equations reduce to the HF equations describing spontaneous fission, cf., Refs. [3, 4, 11, 15]. The zero temperature limit of our equations provide the HFB generalisation of that theory.

Solution of the equations Eqs. (21 - 23) generate an (approximate) contribution to the partition function given by the exponential of \( -\beta \Omega_{\text{trial}}(T, \mu) \) as given by Eq. (14). As was outlined in the discussion after Eq. (1) the contribution of such solution to microcanonical partition function is simply obtained by dropping the terms \( \hat{H}(\tau) - \mu \hat{N} \) from the expression for \( \Omega_{\text{trial}} \) and expressing the inverse temperature \( \beta \) and chemical potential \( \mu \) via the excitation energy \( E^* \) and the particle number \( N \).

Taking into account the contribution from the static mean field one can follow Ref. [10] and write for the tunneling decay width (inverse of the tunneling probability per unit time)

\[
\Gamma(E^*, A) = \frac{D(E^*, A)}{2\pi} e^{-\mathcal{R}(E^*, N) + \mathcal{S}(E^*, N)}
\]

(24)

with

\[
\mathcal{R} = \sum_K W_K^{(0)} \int_{\beta/2}^{\beta/2} d\tau \langle \tilde{\Psi}_K^{(0)}(\tau) \big| \frac{\partial}{\partial \tau} |\tilde{\Psi}_K^{(0)}(\tau)\rangle \quad , \quad \mathcal{S} = -\sum_K W_K^{(0)} \ln W_K^{(0)} = -\sum_\nu [f_\nu \ln f_\nu + (1 - f_\nu) \ln(1 - f_\nu)]
\]

(25)

and \( D(E^*, N) = \rho(E^*, N)^{-1} \) - the inverse level density of the decaying system (compound nucleus). Both terms in the exponential have statistical (via \( W_K^{(0)} \)’s) and imaginary time dynamic (via Eqs. (21 - 23)) contents.
1. S. Coleman, The uses of instantons. In "Aspects of Symmetry" (Cambridge University Press, 1985)
2. S. Coleman, Phys. Rev. D15, 2929 (1977); C. Callan and S. Coleman, ibid, 1762 (1977).
3. S. Levit, J. W. Negele, and Z. Paltiel, Phys. Rev. C22, 1979 (1980)
4. H. Reinhardt, Nucl. Phys. A367(1981) 269
5. H. Reinhardt and H. Schulz, Nucl. Phys. A391 (1982) 36
6. G. Puddu and J.W. Negele, Phys. Rev. C35, 1007 (1987)
7. P. Arve, G.F. Bertsch, J.W. Negele and G. Puddu, Phys. Rev. C36, 2018 (1987)
8. J.W. Negele, Nucl. Phys. A502, 371c (1989)
9. M. Baranger, M. Strayer, and Jian-Shi Wu, Phys. Rev. C67, 014318 (2003)
10. A. K. Kerman and S. Levit, Phys. Rev. C24, 1029 (1981)
11. N. Schunck and L. M. Robledo, Rep. Prog. Phys. 79 116301 (2016)
12. J. P. Blaizot and H. Orland, Phys. Rev. C24, 1740 (1981)
13. A. K. Kerman, S. Levit and T. Troudet, Ann. Phys., 148, 436 (1983)
14. D. M. Brink, R. A. Broglia, Nuclear Superfluidity: Pairing in Finite Systems, Cambridge University Press, 2005.
15. G.F. Bertsch, Nucl. Phys. A574 (1994) 169c
16. A. Bulgac, Annu. Rev. Nucl. Part. Sci. 2013, 63:97
17. Jhilam Sadhukhan, J. Dobaczewski, W. Nazarewicz, J. A. Sheikh and A. Baran, Phys. Rev. C90, 061304(R) (2014)
18. C. Simenel and A.S. Umar, Progress in Particle and Nuclear Physics 103, 19 (2018)
19. A. Bulgac, S. Jin and I. Stetcu, Front. Phys. March 2020
20. G. Scamps, C. Simenel and D. Lacroix, Phys. Rev. C92, 011602(R) (2015)
21. Jhilam Sadhukhan, J. Dobaczewski, W. Nazarewicz, J. A. Sheikh and A. Baran, Phys. Rev. C90, 061304(R) (2014)
22. J. C. Pei, W. Nazarewicz, J. A. Sheikh and A. K. Kerman, Phys. Rev. Lett. 102, 192501 (2009)
23. R. Feynman, Statistical Mechanics: A Set Of Lectures, Taylor and Francis, 1998. To remind - every convex function lies above any of its tangents, i.e. for any $X_0$, $f(X) \geq f(X_0) + (X - X_0)f'(X_0)$. Averaging with positive $w(X)$ and choosing $X_0 = \langle X \rangle$ gives $\langle f(X) \rangle \geq f(\langle X \rangle)$.
24. S. Levit, in preparation.