Quantum motional state Dicke squeezing by cavity self-organization of ultracold atoms

Ivor Krešić,1,2, * Gordon R. M. Robb,3 Gian-Luca Oppo,3 and Thorsten Ackemann3

1Institute for Theoretical Physics, Vienna University of Technology (TU Wien), Vienna, A–1040, Austria
2Institute of Physics, Bijenička cesta 46, 10 000 Zagreb, Croatia
3SUPA and Department of Physics, University of Strathclyde, Glasgow G4 0NG, Scotland, UK
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We study the transverse optomechanical self-organization of an ultracold bosonic gas in a ring cavity driven by an external laser. By modeling the light-matter interaction with a many-body Hamiltonian, we demonstrate the spontaneous generation of Dicke squeezing and many-particle entanglement of atomic motional states, occurring due to self-organization of photons and atoms in the stripe phase. Once generated, these squeezed states are maintained after a sudden switch-off of the laser pump in a dissipative cavity. Our results highlight the potential of using self-organization of atomic motion as a tool for quantum metrology.

The study of ultracold atom self organization in optical resonators is a well-established research field, with many notable experimental and theoretical results [1, 2]. Following the pioneering works on self-organization of cold [3, 4] and ultracold [5, 6] atoms coupled to a single longitudinal mode of a Fabry-Perot cavity, the multimode aspects of optomechanical self-organization in cold and ultracold atoms have recently started to generate significant interest [7–25].

Although the majority of these works have studied the nonequilibrium phase diagrams in the mean field limit, where quantum correlations can be neglected, a number of works have shown that the quantum nature of light and matter can play an important role for self-organization [6, 26–35].

Quantum correlated squeezed and entangled states can be used for quantum enhanced measurements, which go beyond the precision achievable in classical metrology [36, 37]. In this context, squeezing and entanglement of the internal atomic degrees of freedom have been recognized as attractive tools for such metrological applications [38–44].

Here, we focus on the external degrees of freedom and demonstrate numerically the spontaneous generation of Dicke squeezed (DS) states [45] in the atomic motion, by transverse optomechanical self-organization of a Bose-Einstein condensate (BEC) in a laser pumped ring cavity. The ring cavity setup, studied theoretically in [46–48] and experimentally in [49, 50], is schematically depicted in Fig. 1a). A prolate shaped zero-temperature BEC is placed in an effectively planar ring cavity of effective length L with one lossy and three perfectly reflecting mirrors (κ - cavity photon decay rate), which is pumped by a coherent electric field with pump strength η at frequency ω. A strong confinement along the y and z axes allows to restrict the analysis to 1D structures. The pump drive excites on-axis running waves with spatial profile eik0z, with the spontaneously generated sidebands having the profile eik0z e±iql, where k0 = 2π/λ0 is the cavity longitudinal mode wavenumber, qL = 2π/Λc and Λc is the pattern lengthscale, tunable via Fourier filtering of intracavity light [51, 52].

Following Refs. [26–28], we take the electric field modes as:

\[ g(\mathbf{r}) = a_0 e^{ik_0 z} + a_+ e^{ik_0 z} e^{i\mathbf{q}_L \cdot \mathbf{x}} + a_- e^{-ik_0 z} e^{-i\mathbf{q}_L \cdot \mathbf{x}}, \]

(1)

where aj are the photonic annihilation operators in the j-th mode. The atomic field momentum operator is...
given by:
\[ \psi(\mathbf{r}) = \frac{1}{\lambda_0 \sqrt{N}} \left( b_0 + b_+ e^{i q_+ \cdot \mathbf{r}} + b_- e^{-i q_- \cdot \mathbf{r}} \right), \]  
(2)

where \( b_j \) is the bosonic annihilation operator of the \( j \)-th transverse atomic momentum mode. The effective many-body Hamiltonian for the photons and atomic motional states can be derived from the Jaynes-Cummings model (see e.g. [1, 10, 53, 54]). For two-level atoms, using the dipole and rotating wave approximations in the low saturation (far-detuned) limit, we therefore have:

\[ H = -\hbar \Delta_0 n_0 - \hbar \Delta'_0 (n_+ + n_-) + i\hbar (\eta a_0^\dagger - \eta^* a_0) \]
\[ + \int_V d^3 r \psi(\mathbf{r}) \left[ \frac{p^2}{2\hbar} + \hbar U_{0g}^\dagger (\mathbf{r}) \eta \right] \psi(\mathbf{r}), \]

(3)

where \( \Delta_0 = \omega - \omega_0 \), \( \Delta'_0 = \omega - \omega_0' \) are the pump detunings from the on-axis and sideband cavity modes, respectively, \( n_0 = \alpha_0^\dagger \alpha_0 \), \( n_\pm = \alpha_\pm^\dagger \alpha_\pm \) are the single photon light shift, \( \Delta_0 = \omega - \omega_0 \) is the laser-atom detuning, and \( g_0 \) is the atom-cavity coupling strength. The pump rate \( \eta \) is taken as real-valued in the remainder of this letter. In writing Eq. (3) we have neglected collisions between the atoms as we are here interested on highlighting the consequences of light-matter interaction. The number of atoms is fixed and given by \( N = N_0 + N_+ + N_- \), where \( N_0 = \alpha_0^\dagger \alpha_0 \) and \( N_\pm = \alpha_\pm^\dagger \alpha_\pm \).

Many-body Hamiltonian and dynamics: We insert Eqs. (1) and (2) into Eq. (3) and integrate over the volume \( V = \Delta_c \times \lambda_0 \times \lambda_0 \) to get the total Hamiltonian

\[ H = H_0 + H_{FWM}^{(1)} + H_{FWM}^{(2)}, \]

where:

\[ H_0 = -\hbar \Delta_c n_0 - \hbar \Delta'_c (n_+ + n_-) \]
\[ + \hbar g_0 (N_+ + N_-) + i\hbar (a_0^\dagger - a_0), \]

(4)

and the four wave mixing terms are:

\[ H_{FWM}^{(1)} = \hbar U_0 (a_\alpha^\dagger b_\alpha + a_\alpha^\dagger b_\alpha^\dagger) a_\alpha b_0 + \text{H.c.}, \]
\[ H_{FWM}^{(2)} = \hbar U_0 (a_\alpha^\dagger b_\alpha + a_\alpha^\dagger b_\alpha^\dagger) b_0 a_\alpha + \text{H.c.}, \]

(5)

(6)

where \( \Delta_c = \Delta_c - NU_0 \), \( \Delta'_c = \Delta'_c - NU_0 \), \( n = n_0 + n_+ + n_- \), and \( \hbar g_0 = (bq_\perp^2) / 2m \) is the transverse recoil energy. Following [5, 6], we here concentrate on the system dynamics for \( \Delta'_c < 0 \) (and \( \Delta_c > 0 \)), where \( \omega_0' \) is tunable via Fourier filtering of intracavity light [52].

Generation of transverse sidebands via the four wave mixing terms of the Hamiltonian can be explained by the momentum conserving processes illustrated in Fig. 1b). Two opposite atomic momentum sidebands are created by the term \( a_\alpha^\dagger a_\alpha^\dagger b_0 + b_0 a_\alpha^\dagger a_\alpha \) by scattering an on-axis photon into the mode with wavenumber \( q_r \), which excites an atomic sideband with \( p_r = -\hbar q_r \), and the term \( a^\dagger_\alpha a_\beta b^\dagger_\alpha b_0 \), which scatters an on-axis photon into the \( -q_c \) mode and excites an atomic sideband with \( p_c = -\hbar q_c \).

The \( a_\alpha^\dagger (b^\dagger_\alpha a_\alpha + b^\dagger_\alpha b^\dagger_\alpha a_\alpha) b_0 \) term of Eq. (5) scatters a photon with \( \pm q_c \) into the on-axis mode and excites an atomic sideband with \( p_c = \pm \hbar q_c \). The \( H_{FWM}^{(2)} \) describes the secondary wave mixing process for stripe patterns, in which a scattering of a photon sideband with \( \pm q_c \) into the mode with \( \mp q_r \) leads to a transition of an atom from the state \( p_r = \pm \hbar q_r \) into the state \( p_r = \pm \hbar q_r \). This process leads to saturation of the sideband mode population far above threshold [25].

The spontaneously generated transverse lattice potential attracts atoms towards the maxima of the light intensity for \( U_0 < 0 \), or their minima for \( U_0 > 0 \). Such ordering of atoms in turn increases the light diffraction and thus also the depth of the lattice, which turns into a runway process when the pump rate \( \eta \) is sufficiently strong to compensate for the kinetic energy cost and the dissipation. Below we study the relationship of this light-induced DS BEC self-organization to the DS states, schematically depicted in Fig. 1c). To this end we define the sideband operators: \( \delta n = n_+ - n_- \), \( \delta N = N_+ - N_- \), \( j_x = (b^\dagger_\alpha b_\alpha - b^\dagger_\alpha b_\alpha^\dagger b^\dagger_\alpha b_\alpha) / 2i \), \( J_x = \delta N / 2 \) and \( J_{x,f} = J_x^2 + J_y^2 \).

Along with coherent unitary temporal evolution governed by the Schrödinger equation, where photon decay out of the cavity is neglected, we study the irreversible (dissipative) evolution governed by the master equation for the density matrix \( \rho \):

\[ \frac{d\rho}{dt} = -i \hbar [H, \rho] + \kappa \sum_{j=\alpha,\beta} \{ 2a_j \rho a_j^\dagger - a_j^\dagger a_j \rho - \rho a_j^\dagger a_j \}, \]

(7)

where the dissipation comes from the cavity photon decay at a rate \( \kappa \), and we neglect the fluctuations in the BEC atom number.

Mean-field dissipative evolution: We start by investigating the system dynamics in the mean-field limit of the dissipative evolution for the Hamiltonian \( H \). In Fig. 2 we plot the square root of the diffracted photon number \( |\alpha^\dagger_0|^2 = \langle |n_0^\dagger|^2 / \langle N_0 \rangle \rangle \) vs. the pump rate \( \eta \). This is given by the steady state of the dynamical mean field equations, as detailed in [52]. For all of the simulations in this letter the system starts at \( t = 0 \) with no photons in the cavity and all atoms in the motional ground state.

The mean field patterns appear for the pump rate \( \eta > \eta_c \), where \( \eta_c = \sqrt{-\omega_0 (\Delta^2_c + \kappa^2) / (4NU_0^2 \Delta^2_c)} \) vs. the pump rate \( \eta \). The initial sharp increase in the transverse excitations, seen in Fig. 2 for \( \eta > \eta_c \), gives way to saturation for larger \( \eta \)’s. In the inset we...
We have observed that for our simulations the atomic density \( n(x,t) \) and the electric field intensity \( I(x,t) \), normalized to the steady state \( \langle \alpha_0^2 \rangle \), denoted by \( I_0 \).

The system starts with \( a_0(0) = \alpha_{x}(0) = 0 \), with \( \alpha_{x}(t) \) rising rapidly on the scale \( 1/\kappa \), which is not detectable on the plot since the \( t \) range is too large. This homogeneous state becomes unstable after around \( 10/\omega_R \), and the stripe state starts to form. Before reaching the steady state, the \( |\alpha_{\pm}(t)| \) and \( |\beta_{\pm}(t)| \) oscillate at a frequency of a few \( \omega_R \). This oscillation of the sideband populations is a signature of the sloshing dynamics, i.e. the continuous oscillation between the bunched and homogeneous atomic structure in the optical lattice (studied e.g. in [6, 55]). The stripes in \( n(x,t) \) and \( I(x,t) \) are complementary, which is a consequence of the optical dipole potential repulsing the atoms away from the intensity peaks for \( U_0 > 0 \) [12, 13].

**Dicke squeezed states in transverse atomic momentum.** The entanglement of many identical atoms has proven to be a useful resource for quantum metrology [39, 44, 56, 57]. Recently it was shown that a new class of quantum correlated states, the DS states, schematically depicted in Fig. 1c), can be used to improve measurement precision beyond the standard quantum limit [58]. The detection criterion for such states is given by [45, 58]:

\[
\xi_D = \frac{N \left( \langle (\Delta J_z)^2 \rangle + 0.25 \right)}{\langle J_{eff}^2 \rangle} < 1, \tag{8}
\]

where \( \langle (\Delta J_z)^2 \rangle \) is the \( J_z \) variance. To detect the presence of Dicke squeezing in our system, we follow the approach of Refs. [59, 60] and project the three mode atomic density matrix onto the two-state subspace of the transverse sidebands, while also tracing out the photonic degrees of freedom. This leaves the detection criterion \( \xi_D < 1 \) unchanged, as projection cannot increase many-particle entanglement. Note that a similar detection criterion for DS state entanglement exists:

\[
\xi_{DS}^2 = \frac{(N - 1) \langle (\Delta J_z)^2 \rangle}{\langle J_{eff}^2 \rangle} - N/2 < 1 \tag{60, 61}.
\]

However, we here concentrate on the \( \xi_D < 1 \) condition, since it more clearly separates the states providing a metrological advantage.

From the parity \((+ + -)\) symmetry of the system Hamiltonian, the expectation value \( \langle J_z \rangle \) must vanish. We have observed that for our simulations the \( \langle J_1 \rangle \) also vanish, indicating that the relative phase between the sideband modes is undetermined, which can be seen as a consequence of the translational invariance of the ideal system along the \( x \) axis [27]. Additionally, since \( [H, \delta n + \delta N] = 0 \) (see [52]), we expect that the main reduction of variance will occur for the \( J_z \) spin component. Indeed, a strong reduction of the \( \delta n \) variance was already predicted to occur in the quantum description of photonic transverse pattern formation [26–28].

These observations imply that the self-organized states are close to the DS states in the sideband subspace [44, 58, 60, 62, 63]. Such states are characterized by a low \( \langle (\Delta J_z)^2 \rangle \) and large \( \langle J_{eff}^2 \rangle \) [60]. One can easily show that \( J_{eff}^2 = N_+ N_- + (N_+ + N_-)/2 \), such that \( \langle J_{eff}^2 \rangle = \langle N_+ N_- \rangle + \langle N_+ + N_- \rangle/2 \). The maximally entangled state, for which \( \langle (\Delta J_z)^2 \rangle = 0 \) and \( \langle J_{eff}^2 \rangle = N(N + 2)/4 \), reaches the minimal \( \xi_D \) in \( 1/(N + 2) \) [58]. In the atomic many-body basis, this maximally many-particle entangled state is \( |0\rangle |N/2\rangle + |N/2\rangle \). This is precisely the state where the atoms are maximally self-organized, with highest possible population in the transverse momentum sidebands.

**Generating motional Dicke squeezing via optomechanical self-organization.** We now look at the full photon-atom dynamics for both the coherent unitary evolution and for modelling the cavity photon decay via Eq. (7). The corresponding evolution equations were solved numerically by using the open-source framework QuantumOptics.jl [64].

Due to \( [H, \delta n + \delta N] = 0 \), the variance of \( \delta N \propto J_z \) is equal to the variance of \( \delta n \) for the unitarily evolving system [52]. This indicates that a reduction of \( \xi_D \) will benefit from lower \( \delta n \) variances, which in general occur when \( \langle n_0 \rangle \) and \( \langle n_\pm \rangle \) are lower, e.g. at larger \( |\Delta_1|, |\Delta_2| \), as long as \( \langle J_{eff}^2 \rangle \) is large.
In Figs. 3a-c we plot the temporal evolution of the relevant atomic observables, for varying the pump strength $\eta$, in the case when the dissipation of photons out of the cavity is present. For the dissipative case the equality of $\delta N$ and $\delta m$ variances no longer holds and the dissipation of photons out of the cavity makes $\langle (\Delta J_z)^2 \rangle$ increase almost linearly with time. However, decreasing the pump amplitude $\eta$ also decreases $\langle (\Delta J_z)^2 \rangle$. This indicates that there is still some correlation between $\delta N$ and $\delta m$ even in the dissipative case, i.e. the low $\delta m$ variances obtained for lower $\eta$ and $\langle n_z \rangle$ still lead to lower $\delta N$ variances.

The $\langle J_{z,\text{eff}}^2 \rangle$, which is a measure of the atomic self-organization, initially rises, and then starts to oscillate in time, indicating sloshing dynamics (see also Fig. 2). Increasing the pump $\eta$, both the growth rate and maximum value of $\langle J_{z,\text{eff}}^2 \rangle$ are increased. From Eq. (8), this leads to a decrease in $\xi_D$ for larger $\eta$. For continuous wave (cw) pumping with constant $\eta$, the $\xi_D$ reaches values below 1, indicating the presence of DS states, however only in a relatively short time span.

In Fig. 3d we plot $\eta$ scans of the minimal $\xi_D$ reached during the simulation time, given by $\xi_m = \min_{t \in [0,3]} \xi_D(t)$, for different $\kappa$ values. Decreasing the $\kappa$ value down to zero, corresponding to the unitary evolution case, the $\xi_m$ decreases to the lowest value of $\xi_m = 0.29$, at $\eta = 400\omega_R$, while for the dissipative evolution the lowest value is $\xi_m = 0.33$, at $\eta = 500\omega_R$ and $\kappa = 5\omega_R$. The lowest attainable value of $\xi_D$ for $N = 8$ is $\xi_D^{\text{min}} = 0.1$. We note that the number of atoms and intracavity photons considered are limited by the dimensionality of Hilbert space tractable numerically [52]. The lowest $\xi_{\text{gen}}^2$ values for these results are $\xi_{\text{gen}}^2 = 0.03$ for the unitary and $\xi_{\text{gen}}^2 = 0.18$ for the dissipative case, which confirms the presence not only of significant DS, but also of many-particle entanglement in our system (see [52] for more details).

In order to generate metrologically useful atomic motional DS states in the steady state, we move away from cw operation and instead apply a temporally tailored pump, given by a single square pulse starting instantaneously at $t = 0$ and switching off instantaneously at $t_{\text{OFF}} = 0.75/\omega_R$, as schematically depicted in Fig. 4b). This switch off time $t_{\text{OFF}}$ is chosen since the minimum of $\xi_D$ is reached around that value. After switching off the pump, the light field inside the cavity drops to zero in a time $\sim 1/\kappa$ (see Fig. 4c,d). In contrast, the atoms are left with a given momentum state distribution, and their kinetic energy at a constant value, as the light-matter interaction vanishes and there are no other channels.
for energy exchange in the system, since interatomic collisions and atom losses are neglected in our model. Indeed, $\tilde{\xi}_p$ stays at a steady-state value of 0.33.

The tentative analysis in [52] suggests that at experimentally available $\kappa$ values of $2\pi \times 0.13$ MHz [65] or $2\pi \times 1$ MHz [5], squeezing with $\tilde{\xi}_p = 0.1$ could be be reached for $N = 3 \times 10^4 - 8 \times 10^4$ atoms at $\Lambda_c = 1$ $\mu$m and $N = 7 \times 10^7 - 2.3 \times 10^8$ atoms at $\Lambda_c = 10$ $\mu$m. These $N$ values are within reach in state-of-the-art experiments.

**Conclusion.** We have theoretically and numerically demonstrated a method for generating momentum state Dicke squeezing via nonlinear self-organization of ultracold atoms in a laser pumped ring cavity. The squeezed degrees of freedom are the transverse atomic motional states of a BEC, which are more robust to noise from the environment as compared to internal atomic degrees of freedom, used commonly in previous ultracold atom based entanglement schemes [44]. Once generated, the squeezing can be maintained by suddenly switching off the laser pump. Going to 2D geometries, e.g. four modes in transversely pumped cavities [5] or six hexagonal transverse modes in longitudinally pumped cavities [25], will enable the generation of squeezing and many-particle entanglement in situations with more than two atomic modes, a highly attractive prospect for quantum technological applications.

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As in the main text, we here use the Hamiltonian while the nonunitary evolution, including the dissipation of photons from the cavity, can be described by the Lindblad-type evolution via the equation:

\[
\frac{dO}{dt} = \frac{i}{\hbar} [H, O] + \kappa \sum_{j=0,\pm} (2a_j^\dagger Oa_j - a_j^\dagger a_j O - O a_j^\dagger a_j).
\]  

(S2)

As in the main text, we here use the Hamiltonian \( H = H_0 + H^{(1)}_{\text{FWM}} + H^{(2)}_{\text{FWM}} \), with the three parts given by:

\[
\begin{align*}
H_0 &= -\hbar \Delta \omega n_0 - \hbar \Delta \omega (n_+ + n_-) + \hbar \omega (n_+ + n_-) + i\hbar \eta (a_0^\dagger - a_0), \\
H^{(1)}_{\text{FWM}} &= \hbar U_0 [(a_0^\dagger b_0 + a_0 b_0^\dagger) a_0 b_0 + \text{H.c.}] + \hbar U_0 [a_0^\dagger (b_0^\dagger a_+ + b_+ a_0) b_0 + \text{H.c.}], \\
H^{(2)}_{\text{FWM}} &= \hbar U_0 (a_0^\dagger a_+ b_0 + \text{H.c.}),
\end{align*}
\]

(S3-S5)

and the commutation relations of the bosonic modes for photons and atoms: \([a_j, a_k^\dagger] = \delta_{j,k}, [a_j^\dagger, a_k] = [a_j, a_k^\dagger] = 0\), and \([b_j, b_k^\dagger] = \delta_{j,k}, [b_j^\dagger, b_k] = [b_j, b_k^\dagger] = 0\), respectively, where \(j, k = 0, +, -\). For the photonic modes, the Eq. (S2) now gives:

\[
\begin{align*}
\dot{a}_0 &= (i\Delta - \kappa) a_0 + i\hbar U_0 [(b_0^\dagger a_+ + b_0^\dagger a_-) b_0 + b_0^\dagger (a_+ b_- + a_- b_+)] + \eta^*, \\
\dot{a}_+ &= (i\Delta - \kappa) a_+ - i\hbar U_0 [(b_0^\dagger b_0 + b_0^\dagger b_+) a_0 + a_- b_0^\dagger b_+], \\
\dot{a}_- &= (i\Delta - \kappa) a_- - i\hbar U_0 [(b_0^\dagger b_0 + b_0^\dagger b_-) a_0 + a_+ b_0^\dagger b_-].
\end{align*}
\]

(S6-S8)

while for the atomic momentum modes, the Eq. (S1) gives:

\[
\begin{align*}
\dot{b}_0 &= -i\hbar U_0 [(a_0^\dagger a_+ + a_0^\dagger a_-) b_0] + (a_0^\dagger b_0 + a_0 b_0^\dagger) a_0, \\
\dot{b}_+ &= -i\hbar \eta b_+ - i\hbar U_0 [(a_0^\dagger a_+ + a_0^\dagger a_-) b_0 + a_- b_0^\dagger a_+], \\
\dot{b}_- &= -i\hbar \eta b_- - i\hbar U_0 [(a_0^\dagger a_+ + a_0^\dagger a_-) b_0 + a_+ b_0^\dagger a_-].
\end{align*}
\]

(S9-S11)

Taking now the expectation values of the right- and left-hand sides, writing \( \langle O_1, O_2, O_3 \rangle \rightarrow \langle O_1 \rangle \langle O_2 \rangle \langle O_3 \rangle \), and using \( a_j \rightarrow \langle a_j \rangle = \sqrt{N} \alpha_j(t) \), \( b_j \rightarrow \langle b_j \rangle = \sqrt{N} \beta_j(t) \), we get the mean field dynamical equations:

\[
\begin{align*}
\dot{\alpha}_0 &= (i\Delta - \kappa) \alpha_0 - i\hbar U_0 [(b_0^\dagger \alpha_+ + b_0^\dagger \alpha_-) \beta_0 + b_0^\dagger (\alpha_+ \beta_- + \alpha_- \beta_+)] + \eta^*, \\
\dot{\alpha}_+ &= (i\Delta - \kappa) \alpha_+ - i\hbar U_0 [(b_0^\dagger \alpha_0 + b_0^\dagger \alpha_0^\dagger) b_0 + a_- \alpha_+ b_0^\dagger], \\
\dot{\alpha}_- &= (i\Delta - \kappa) \alpha_- - i\hbar U_0 [(b_0^\dagger \alpha_0 + b_0^\dagger \alpha_0^\dagger) b_0 + a_+ \alpha_- b_0^\dagger].
\end{align*}
\]

(S12-S14)

where \( \eta_0 = NU_0, \eta = \eta/\sqrt{N} \), and

\[
\begin{align*}
\dot{\beta}_0 &= -i\hbar U_0 [(\alpha_0^\dagger \alpha_+ + \alpha_0^\dagger \alpha_-) \beta_0 + \beta_0^\dagger (\alpha_+ \beta_- + \alpha_- \beta_+ \alpha_0)], \\
\dot{\beta}_+ &= -i\hbar \eta \beta_+ - i\hbar U_0 [(\alpha_0^\dagger \alpha_0 + \alpha_0^\dagger \alpha_0^\dagger) \beta_0 + \alpha_- \alpha_+ \beta_-], \\
\dot{\beta}_- &= -i\hbar \eta \beta_- - i\hbar U_0 [(\alpha_0^\dagger \alpha_0 + \alpha_0^\dagger \alpha_0^\dagger) \beta_0 + \alpha_+ \alpha_- \beta_+].
\end{align*}
\]

(S15-S17)

For the results in Fig. 2 of the main text, the equations (S12)-(S17) are solved numerically and the steady state values are plotted for different \( \eta^* \)'s. In the inset we plot the temporal evolution of the roll patterns in the atomic density, given by \( n(x,t) = |\tilde{\beta}_0(t) + \beta_0(t) \exp(iq_0 x) + \beta_+(t) \exp(-iq_0 x)|^2 \), and in the electric field intensity, given by \( I(x,t) = |\alpha_0(t) + \alpha_+(t) \exp(iq_0 x) + \alpha_-(t) \exp(-iq_0 x)|^2 \), normalized to the steady state \( |\alpha_0^0|^2 \), denoted by \( I_0 \).
MEAN FIELD THRESHOLD

We now look at the steady state limit of the above mean field dynamical equations. Writing now $\alpha_j(t) \to \alpha_j$ and $\beta_j(t) \to \beta_j$, we get the equations:

$$ 0 = (i\Delta_x - \kappa) \alpha_0 - i u_0 [(\beta^*_+ \alpha_+ + \beta^*_\alpha_+) \beta_0 + \beta_0^* (\alpha_+ \beta_- + \alpha_- \beta_+)] + y^*, \quad (S18) $$

$$ 0 = (i\Delta_x - \kappa) \alpha_+ - i u_0 [(\beta^*_0 \beta_0 + \beta_0^* \alpha_0 + \alpha_+ \beta_- \beta_+)], \quad (S19) $$

$$ 0 = (i\Delta_x - \kappa) \alpha_- - i u_0 [(\beta^*_0 \beta_0 + \beta_0^* \alpha_0 + \alpha_+ \beta_- \beta_+)], \quad (S20) $$

and

$$ 0 = u_0 [\alpha_0^* (\alpha_+ \beta_- + \alpha_- \beta_+) + (\alpha_+^* \beta_- + \alpha_-^* \beta_+)] \alpha_0, \quad (S21) $$

$$ 0 = \omega R \beta_+ + u_0 [(\alpha_+ \alpha_0 + \alpha_0^* \alpha_+ \beta_0^* \beta_+)], \quad (S22) $$

$$ 0 = \omega R \beta_- + u_0 [(\alpha_+ \alpha_0 + \alpha_0^* \alpha_+ \beta_0^* \beta_+)]. \quad (S23) $$

At threshold, we neglect all terms square or higher order in the sidebands. We choose a real-valued $\beta_0 = \sqrt{1 - |\beta_+|^2 - |\beta_-|^2} \approx 1$ and find the homogeneous field amplitude of the on-axis mode to be

$$ \alpha_0 = \frac{y^*}{\sqrt{\Delta_x^2 + \kappa^2}} e^{i \arctan(\Delta_x / \kappa)}. \quad (S24) $$

One can then easily calculate that in this approximation:

$$ \beta_{\pm} = -\frac{i u_0}{\omega R} (\alpha_+^* \alpha_0 + \alpha_-^* \alpha_0^*). \quad (S25) $$

Inserting this relation in the equations for the fields, we get:

$$ 0 = (i\Delta_x^* - \kappa) \alpha_+ + 2i \frac{u_0^2}{\omega R} (\alpha_+ \alpha_0^* \alpha_0^* \alpha_0), \quad (S26) $$

$$ 0 = (i\Delta_x^* - \kappa) \alpha_- + 2i \frac{u_0^2}{\omega R} (\alpha_+ \alpha_0^* \alpha_0^* \alpha_0). \quad (S27) $$

We here take the condition $\Delta_x^* = 0$ used in the main text. By inserting the complex conjugate of

$$ \alpha_- \alpha_0^* + \alpha_+^* \alpha_0 = -\frac{(i\Delta_x^* - \kappa)}{2i \frac{u_0}{\omega R}} \alpha_+ $$

into the second equation of the above, we get:

$$ \alpha_- = \frac{i\Delta_x^* + \kappa}{i\Delta_x^* - \kappa} \alpha_0^*. \quad (S28) $$

As the sidebands have equal amplitudes, we can write:

$$ \alpha_{\pm} = \Lambda e^{i\chi_{\pm}}, \quad \beta_{\pm} = B e^{i \psi_{\pm}}. \quad (S29) $$

From (S25), we then get that at threshold $\psi = \psi_+ + \psi_- = 0$, while Eq. (S29) gives $\chi = \chi_+ + \chi_- = 2[\arctan(\Delta_x / \kappa) + \arctan(\Delta_x^* / \kappa) - \arg y]$. Inserting the complex conjugate of Eq. (S29) into the Eq. (S26), we get for the critical intracavity field:

$$ |\alpha_0^*|^2 = -\frac{\omega R (\Delta_x^2 + \kappa^2)}{4 u_0^2 \Delta_x}, \quad (S30) $$

which gives the threshold for the real-valued critical input electric field amplitude $y_c$:

$$ y_c^2 = -\frac{\omega R (\Delta_x^2 + \kappa^2)(\Delta_x^2 + \kappa^2)}{4 u_0^2 \Delta_x^2}, \quad (S31) $$

which leads to

$$ \eta_c = \sqrt{-\frac{\omega R (\Delta_x^2 + \kappa^2)(\Delta_x^2 + \kappa^2)}{4 u_0^2 \Delta_x^2}}. \quad (S32) $$
TEMPORAL EVOLUTION OF THE VARIANCES

The full system Hamiltonian commutes with the combined difference operator for the photon intensities and the atomic momentum sideband populations, which has the form \( \delta n + \delta N \), where \( \delta n = n_+ - n_- \), \( \delta N = N_+ - N_- \), with \( n_\pm = a_\pm^\dagger a_\pm \) and \( N_\pm = b_\pm^\dagger b_\pm \). This can be seen from \( [H_0, \delta n + \delta N] = 0 \) and:

\[
[H^{(1)}_{FWM}, n_\pm] = i\hbar U_0 (-a_\pm^\dagger b_\mp a_0 b_0 + a_0^\dagger b_\pm^\dagger a_\mp b_- - a_\pm^\dagger b_\mp^\dagger a_\mp b_-),
\]

\[
[H^{(2)}_{FWM}, n_\pm] = i\hbar U_0 (+a_\pm^\dagger b_\mp a_\mp b_-, \pm a_\pm^\dagger b_\mp^\dagger a_\pm b_-),
\]

\[
[H^{(1)}_{FWM}, N_\pm] = i\hbar U_0 (-a_\pm^\dagger b_\mp a_0 b_0 + a_0^\dagger b_\pm^\dagger a_\mp b_- - a_\pm^\dagger b_\mp^\dagger a_\mp b_- + a_\pm^\dagger b_\mp^\dagger a_\pm b_-),
\]

\[
[H^{(2)}_{FWM}, N_\pm] = i\hbar U_0 (\pm a_\pm^\dagger b_\mp a_\mp b_- + a_\pm^\dagger b_\mp^\dagger a_\pm b_-),
\]

leading to \( [H, \delta n + \delta N] = 0 \). Note that the Hamiltonian \( H \) does not commute with the individual operators \( \delta n \) and \( \delta N \).

We now discuss qualitatively the consequence of \( [H, \delta n + \delta N] = 0 \) on the behavior of the variances of the \( \delta n + \delta N \), \( \hat{\delta}n \) and \( \delta N \) operators by looking at their temporal evolution in the Heisenberg picture, described by Eqs. (S1) and (S2). The operator \( \delta n + \delta N \) is given by a zero matrix at \( t = 0 \). In the unitary case, \( \langle \delta n + \delta N \rangle \) remains zero for all time, which can be seen from taking the expectation value of:

\[
\frac{d\langle \delta n + \delta N \rangle}{dt} = \frac{i}{\hbar} [H, \delta n + \delta N] = 0.
\]

Using the identity \([A, BC] = [A, B]C + B[A, C]\) cyclically, the unitary temporal evolution gives also zero values for the powers of the \( \delta n + \delta N \) operator at all \( t \), which means that all moments of this operator are also zero at all \( t \). If one includes the cavity photon dissipation into the picture, the Lindblad-type evolution gives:

\[
\frac{d\langle \delta n + \delta N \rangle}{dt} = -2\kappa\delta n.
\]

As \( \langle \delta n \rangle = 0 \) for all \( t \) due to the parity symmetry of the Hamiltonian, the \( \langle \delta n + \delta N \rangle \) will also vanish in the dissipative case. For the variance, we look at the evolution of \( \langle (\delta n + \delta N)^2 \rangle \), described by:

\[
\frac{d\langle (\delta n + \delta N)^2 \rangle}{dt} = 2\kappa(n_+ + n_-) - 4\kappa\langle \delta n + \delta N \rangle\delta n.
\]

The expectation values of the operators on the right hand side no longer vanish, which means \( \langle (\delta n + \delta N)^2 \rangle \) no longer vanishes, leading to a nonvanishing variance of \( \delta n + \delta N \) in the dissipative case.

Fig. S1 shows the behavior of the \( \delta n + \delta N \) variances in the case of unitary and dissipative evolution. In the unitary case, the variance vanishes (see Eq. (S38)), while in the dissipative case the variance increases almost linearly, indicating that the random dissipation of photons from the cavity increases the overall noise for both the photonic and atomic variables.

The temporal evolution of the \( \delta n \) and \( \delta N \) operators is correlated due to \( [H, \delta n] = -[H, \delta N] \). In the unitary case the evolution of the \( \delta n \) and \( \delta N \) operators is related by:

\[
\frac{d\delta n}{dt} = \frac{i}{\hbar} [H, \delta n] = \frac{i}{\hbar} [H, \delta N] = -\frac{d\delta N}{dt},
\]

which, upon integration over \( t \), leads to \( \delta n = -\delta N \) for all \( t \). This in turn means that \( \langle \delta n \rangle^2 = \langle \delta N \rangle^2 \) for all \( t \), leading to equality of the variances of \( \delta n \) and \( \delta N \) for all \( t \) in the unitary case (see Fig. S1b)). Smaller \( \eta \)’s lead to smaller \( \langle n_0 \rangle, \langle n_\pm \rangle \) and \( \delta n \), meaning that the \( J_L \) variance in the unitary case will be reduced for a smaller number of photons in the cavity.

For the dissipative case, the \( \delta n \) evolves as:

\[
\frac{d\delta n}{dt} = \frac{i}{\hbar} [H, \delta n] - 2\kappa\delta n,
\]
FIG. S1. Temporal evolution of the variances for the unitary and dissipative case with $\kappa = 5 \omega_R$. (a) Variance of $\delta n + \delta N$ for the unitary (orange) and $\kappa = 5 \omega_R$ case (blue), along with the variance of $\delta N$ for the unitary (light blue) and $\kappa = 5 \omega_R$ (dark blue) case. Simulation parameters: $N = 8$, $(\eta, \Delta_c, \Delta'_c, U_0) = (40, 110, -45, 10) \omega_R$, with $\bar{h} = 1$.

while

$$\frac{d \delta n}{dt} = \frac{i}{\hbar} [H, \delta n] = -2 \kappa \delta n. \quad (S43)$$

For the variances in the dissipative case we look at the evolution of $(\delta n)^2$, given by:

$$\frac{d (\delta n)^2}{dt} = \frac{i}{\hbar} [H, (\delta n)^2] + 2 \kappa (n_+ + n_-) - 4 \kappa (\delta n)^2, \quad (S44)$$

and the evolution of $(\delta N)^2$, given by

$$\frac{d (\delta N)^2}{dt} = \frac{i}{\hbar} [H, (\delta N)^2] = -2 \left[ \frac{d \delta n}{dt} + 2 \kappa \delta n \right] \delta N. \quad (S45)$$

where we have again used $[A, BC] = [A, B] C + B [A, C]$. The variances of $\delta n$ and $\delta N$ will no longer be equal at all $t$ in the dissipative case, although the temporal evolution of these operators is still coupled.

In Fig. S1b) we plot the evolution of $\delta n$ and $\delta N$ variances for the unitary and dissipative cases. As expected from Eq. (S41), the variances of $\delta n$ and $\delta N$ are equal for the unitary case, and not equal for the dissipative case, with the variance of $\delta N$ increasing almost linearly with time, and the $\delta n$ variance fluctuating around a small constant value. The reason for a relatively low $\delta n$ variance at the used parameters is the low intracavity photon number (see e.g. Fig. S4), occurring due to the laser-cavity detunings $\Delta_c, \Delta'_c$ being much larger in absolute value than the cavity linewidth.

From Eq. (S45) it is also clear that when $\delta n$ is a zero matrix, which happens e.g. when there is no light in the cavity, the variance of $\delta N$ stays constant. As shown in Fig. 4 of the main text, this fact can be used to create steady state motional Dicke squeezing by turning off the pump drive at a suitable time.

**ESTIMATION OF $\xi_m$ FOR EXPERIMENTALLY AVAILABLE PARAMETERS**

To estimate the $\xi_m = \min_{t \in [0.3/\eta R]} \xi_D(t)$ at experimentally available parameters we study its scaling with $N$ and $\kappa$ at a fixed $\eta/\eta_c$ ratio of 3. The simulation results and the corresponding fits are shown in Fig. S2.

The fits give $\xi_m = 0.71/N^{0.23}$ at $\kappa = 5 \omega_R$ and $\xi_m = 0.22 \kappa^{0.43}$ at $N = 8$. To estimate the values of $\xi_m$ for realistic parameters, we use the experimentally available $\kappa$ values of $2\pi \times 0.13$ MHz [2] and $2\pi$ MHz [3].
The frequencies of the on-axis cavity mode $\omega_0$ and the sidebands $\omega'_0$ have the dispersion relations $\omega_0 = c k_0$ and $\omega'_0 = c (k_0^2 + q_0^2)^{1/2}$, leading to $\omega'_0 > \omega_0$. The critical wavenumber $q_c = 2\pi/\Lambda_c$, and thus also $\omega'_0$, can in experimental situations be tuned by Fourier filtering [4]. Experimentally interesting values for the lattice period are $\Lambda_c \sim 1 - 10 \mu m$. On the short side, $\Lambda_c$ is limited by the numerical aperture of the light collection system, while on the long side it is limited by the requirement of having at least a few periods of the transverse pattern in the ultracold cloud. For $\Lambda_c = 1 \mu m$, $\omega_R = 2\pi \times 2.29$ kHz, while for $\Lambda_c = 10 \mu m$, $\omega_R = 2\pi \times 0.02$ kHz. This gives $\kappa/\omega_R = 58 - 438$ at $\Lambda_c = 1 \mu m$ and $\kappa/\omega_R = 5930 - 45000$ at $\Lambda_c = 10 \mu m$.

Extrapolating from the $\kappa$ scaling $\xi_m = 0.22\kappa^{0.43}$ at $N = 8$, the $\xi_m$ will be $1.3 - 3$ for $\Lambda_c = 1 \mu m$, and $9 - 22$ for $\Lambda_c = 10 \mu m$. Considering that the $\xi_m$ will reduce by a factor of $N^{0.25}$ for $N$ atoms, the value of $\xi_m = 0.1$ may be reached for $N = 30000 - 810000$ atoms for $\Lambda_c = 1 \mu m$ and $N = 7 \times 10^7 - 2.3 \times 10^9$ atoms for $\Lambda_c = 10 \mu m$. These numbers are within experimental reach. Taking into account that we presented only rough estimates, with $N$ and $\kappa$ scalings which are dependent on the ratio $\eta/\eta_c$, such that the $\eta$ values can be further increased and $\Delta_c$, $\Delta'_c$ tuned, these results suggest that significant motional squeezing could be generated in a state-of-the-art cavity QED setup.

**ENTANGLEMENT DETECTION**

A criterion for the existence of many-particle entanglement applicable to Dicke squeezed states is given in Ref. [5]:

$$\xi^2_{gen} = \frac{(N - 1)(\langle \Delta J_z \rangle^2)}{(J_{eff}^2) - N/2} < 1.$$  \[(S46)\]

In Fig. S3 we plot the temporal evolution of $\xi^2_{gen}$ for the best unitary and dissipative case with cw and pulsed operation. The lowest value for the unitary case is $\xi^2_{gen} = 0.03$, while for the dissipative case it is $\xi^2_{gen} = 0.18$. This is clear confirmation of many-particle entanglement in our system.

The depth of many-particle entanglement is more difficult to assess from these quantities. A criterion given in Ref. [6] states that the lower bound of entanglement depth is given by $[\xi^2_{D}] - 2$, where $[\xi^2_{D}]$ is the minimum integer no less than $\xi^2_{D}$. This gives for the unitary case with $\xi_{D} = 0.29$ and the dissipative case with $\xi_{D} = 0.33$ the lower bound of 2 for the entanglement depth. However, precisely assessing the entanglement depth is a more involved problem [7], which is a subject for future analysis.

FIG. S2. Scaling of $\xi_m$ with (a) $N$ at fixed $\kappa = 5\omega_R$, $\eta = 3\eta_c$ and $\Delta_c = 110\omega_R$, $\Delta'_c = -45\omega_R$ and (b) $\kappa$ at fixed $N = 8$, $\eta = 3\eta_c$ and $\Delta_c = 110\omega_R$, $\Delta'_c = -45\omega_R$. The lines depict the fits to all data points (orange), with the fit exponent written next to each line. Simulation parameters: $U_0 = 10\omega_R$, with $\hbar = 1$. 

$\xi^2_{gen} = \frac{(N - 1)(\langle \Delta J_z \rangle^2)}{(J_{eff}^2) - N/2} < 1.$
FIG. S3. Temporal evolution of the $\xi_{gen}^2$ for the unitary (blue), dissipative cw (yellow) and pulsed (orange) cases with highest $\xi_D$. The $\xi_{gen}^2$ reaches negative values for $\langle J_{eff}^2 \rangle < N/2$. Simulation parameters: $N = 8$, $(\bar{\Delta}_c, \bar{\Delta}_c', U_0) = (110, -45, 10)\omega_R$, with $\hbar = 1$. Unitary case: $\eta = 40\omega_R$, dissipative case: $\eta = 50\omega_R$, $\kappa = 5\omega_R$. The pulse starts at $t = 0$ and is turned off instantaneously at $t_{OFF} = 0.75/\omega_R$.

DETAILS ON THE NUMERICAL SIMULATIONS

When solving the Schrödinger and master equations for our problem numerically, we truncate the infinite-dimensional Fock space of the photonic degrees of freedom into a finite-sized Fock space. This limits the maximal pump strength $\eta$ that can be used in our simulations, as higher pump rates will naturally lead to larger $\langle n_0 \rangle$ and $\langle n_{\pm} \rangle$ values, such that higher dimensional photonic Fock spaces are needed to correctly capture the system dynamics. The limits of the calculations are in our case set by the size of the working memory of the computational nodes.

To demonstrate the numerical accuracy of the plots in the main text, we have compared the results for the dynamical evolution of the relevant observables for the case where the maximal photon number state in all modes is $|4\rangle_0 |3\rangle_3 + |3\rangle_0$, to the case with the same total maximal photon number but where the state with the maximal photon number in all modes is $|5\rangle_0 |3\rangle_3 + |2\rangle_2$, both at $N = 8$ atoms, and at largest $\eta$ used in the unitary and dissipative cases.

In Fig. S4 we compare the two Fock space truncations by plotting the temporal evolution of relevant variables for the maximum $\eta$ values used in the unitary ($\eta = 40\omega_R$) and dissipative ($\eta = 50\omega_R$) cases of the main text. The plots confirm that the truncation of Fock space where the state with the maximal number of photons in each mode is $|4\rangle_0 |3\rangle_3 + |3\rangle_0$, accurately describes the dynamics in the simulated time interval.

In the unitary case the evolution of both the atomic and photonic variables exhibit fast oscillatory motion, a signature of Vacuum Rabi oscillations occurring in the quantum electrodynamic treatment of light-matter interaction of atoms in a cavity [8]. In the dissipative case with $\kappa = 5$, these fast oscillations are averaged out. However, the slow oscillations, a signature of the sloshing dynamics where atoms slosh around the minima of the dynamical potential and periodically amplify the transverse patterns [9], still persist.

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FIG. S4. Comparison of the temporal evolution of relevant system observables for the case when the state with maximal photon number in all modes is $|4\rangle_0|3\rangle_3$ (red, solid) and the case when the state with maximal photon number in all modes is $|5\rangle_0|3\rangle_3|2\rangle_2$ (blue, dashed), see text for details. The evolution of observables for the unitary case with $\eta = 40\omega_R$: (a) $\langle (\Delta J_z)^2 \rangle$, (b) $\langle J^2_{\text{eff}} \rangle$, (c) $\langle n_0 \rangle$, (d) $\langle n_{\pm} \rangle$ and (e) $\xi_D$. The evolution of observables for the dissipative case with $\eta = 50\omega_R$: (f) $\langle (\Delta J_z)^2 \rangle$, (g) $\langle J^2_{\text{eff}} \rangle$, (h) $\langle n_0 \rangle$, (i) $\langle n_{\pm} \rangle$ and (j) $\xi_D$. Simulation parameters: $N = 8$, $(\Delta \epsilon, \Delta \epsilon', U_0, \kappa) = (110, -45, 10, 5)\omega_R$, with $\hbar = 1$.

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