Trigonometric Splines for Oscillator Simulation

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Abstract. We investigate the effects of numerical damping for oscillator simulation with spline methods. Numerical damping results in an artificial loss of energy and leads therefore to unreliable results in the simulation of autonomous systems, as e.g. oscillators. We show that the negative effects of numerical damping can be eliminated by the use of trigonometric splines. This will be in particular important for spline based adaptive methods.

Keywords

Oscillator simulation, splines, trigonometric splines

1. Introduction

The simulation of oscillators suffers often from a loss of energy due to numerical damping. On the other hand numerical damping is often required to cancel out oscillations which occur due to numerical noise. In the recent years, the expansion of the signal waveforms by wavelets or splines was investigated [13, 12, 7, 6, 5, 2, 3]. This has been motivated by the fact that trigonometric basis (originally used in many RF-simulations) are not suited for the representation of pulse-shaped signals due to a slow convergence and the Gibbs’ phenomenon.

In Fig. 1 we see numerical solutions for the periodic steady state of a 3MHz Colpitz-Quartz oscillator (see Fig. 2) computed by a spline collocation method on various grids. Although one expects an increase of the approximation error on coarser grids, we can observe here an additional effect, which is a smaller amplitude for the coarser grid. This artificial loss of energy is due to an effect called numerical damping, which is caused not by the physical system or its mathematical model, but by properties of the numerical method.

Here, we want to investigate the effects of numerical damping on a oscillator simulation by a spline collocation method on uniform grids. Numerical experiments suggest that the effect described and investigated here do also occur on nonuniform grids, which may appear in an adaptive method. However, we restrict to uniform grids, since this restriction permits the theoretical understanding of the observed effects.

In Sect. 2 we will investigate the numerical damping in a spline collocation method. We will overcome the negative effects of this numerical damping by the application of trigonometric splines in Sect. 3.

Fig. 1. Spline solutions for a 3MHz Colpitz-Quartz oscillator.

In Sect. 2 we will investigate the numerical damping in a spline collocation method. We will overcome the negative effects of this numerical damping by the application of trigonometric splines in Sect. 3.
2. Numerical Damping for Splines

We consider the cardinal B-spline \( N_m \) of order \( m \) defined by the recursion

\[
N_1(t) := \chi_{(0,1]}(t) := \begin{cases} 1, & \text{for } t \in (0,1], \\ 0 & \text{otherwise}, \end{cases}
\]

\[
N_m(t) := \frac{t N_{m-1}(t) + (m-t) N_{m-1}(t-1)}{m-1},
\]

For a detailed introduction to splines we refer to [11]. It is well known that the family \( \{ N_m(\frac{t}{m} - k) : k \in \mathbb{Z} \} \) constitutes a stable basis for the spline space

\[
S_{m,h} = \{ f \in C^{m-2}(\mathbb{R}) : f|_{(hk,h(k+1))} \in \Pi_{m-1} \}
\]

of piecewise polynomials of degree less than \( m \), which are \( m - 2 \) times differentiable. Here, \( \Pi_n \) denotes the space of polynomials of degree up to \( n \).

We consider now how the differential operator is approximated by a spline collocation method. For the simplicity of the presentation and without loss of generality we restrict ourselves to 1-periodic functions (i.e. with unity period). The statements can be generalized to other period lengths by scaling. First the function \( z(t) \) is interpolated at the collocation points \( t_k = \frac{k}{m} + \frac{x}{2} \) by a spline \( s \in S_{m,1} \), i.e., we have to determine coefficients \( c_k \) such that

\[
y_k := z(t_k) = s(t_k) = \sum c_k N_m(k - \ell + \frac{x}{2} + \sigma). \tag{1}
\]

Here, the parameter \( \sigma \) describes the deviation of the collocation points from the center of the B-splines in relation to the mesh size. The choice of \( \sigma \) influences the stability of the spline interpolation, but also properties of a collocation scheme for differential equations, as we will see in the sequel.

Applying the discrete Fourier transform

\[
\hat{s}_k = \sum_{\ell=0}^{n-1} x_\ell e^{i 2\pi k \ell / n}
\]

to (1) we obtain

\[
\hat{c}_k = \frac{\hat{y}_k}{\phi_m(\sigma, \frac{k}{m})},
\]

where

\[
\phi_m(x, \xi) = \sum_{k \in \mathbb{Z}} N_m(x + \frac{m}{2} + k) e^{2\pi i k \xi}
\]

is Schoenberg’s exponential Euler spline [9]. Analogously, we obtain the values of the derivatives as

\[
y'_k := s'(t_k) = \sum_{\ell} c_\ell N'_m(k - \ell + \frac{x}{2} + \sigma),
\]

which yields

\[
\hat{y}'_k = \psi_m(\sigma, \frac{k}{m}) \hat{y}_k, \tag{2}
\]

with

\[
\psi_m(x, \xi) = \frac{\partial}{\partial \xi} \phi_m(x, \xi).
\]

Obviously \( \psi_m(x, \xi) \) is 1-periodic in both arguments.

Fig. 3 shows a plot of the real part of \( \psi_3(x, \xi) \). Due to a zero of \( \phi_m \) at \( \frac{m}{2}, \frac{1}{2} \) we have a singularity of \( \psi_m(x, \xi) \) at this point. This means that the numerical derivative becomes instable for \( |\sigma| \approx \frac{1}{2} \) and we restrict ourselves to \( |\sigma| < \frac{1}{2} \) sufficiently small. Furthermore, \( \phi_m(x, 0) = 1 \), which implies \( \psi_m(x, 0) = 0 \). Since \( N_m(x + \frac{m}{2}) \) is even we conclude that \( \phi_m(0, \xi) \) is real, while \( \frac{\partial}{\partial \xi} \phi_m(0, \xi) \) is purely imaginary, i.e., \( \Re \psi_m(0, \xi) = 0 \).

Apparentely \( \Re \psi_m(\sigma, \xi) > 0 \) for \( \sigma \in (-\frac{1}{2}, 0) \), which causes a numerical damping. For the fundamental frequency the damping depends on the size of \( \Re \psi_m(\sigma, \frac{1}{2}) > 0 \), where \( n \) is the grid size. In Fig. 4 we see the corresponding values for the simulations in Fig. 1 where \( \sigma = -\frac{1}{4} \) was used. Obviously the grid size has to be chosen sufficiently large, in order to avoid a loss of energy by numerical damping.

![Fig. 3. Plot of \( \psi_3(x, \xi) \).](image)

![Fig. 4. Plot of \( \psi_3(-\frac{1}{4}, \xi) \).](image)

The numerical damping is reduced for \( \sigma \to 0, \sigma > 0 \). This can be seen in Fig. 5 where almost no damping of the
fundamental frequency can be observed. However, there occurs also no damping of the high frequencies, which is necessary in a simulation to eliminate numerical noise. Thus, one can observe ringing artifacts in the solution.

For \( \sigma \in \left[ 0, \frac{1}{2} \right] \) we have \( \Re \psi_m(\sigma, \xi) \leq 0 \), i.e., we have no numerical damping. That is numerical noise is not damped (for positive \( \sigma \) there is even an amplification). This effect usually causes convergence problems of the applied numerical methods, as e.g. Newton's method for nonlinear problems. In practice stable behavior was observed for the range \( \sigma \in [-0.3, -0.1] \). However, for oscillators we have the problem of numerical damping in this range.

3. Trigonometric splines for elimination of numerical damping

For the simulation of autonomous systems we need a method, with numerical damping at high frequencies, while low frequencies (in particular the fundamental) frequency are not damped at all. This can be achieved if the numerical differentiation works exactly for the low frequencies (cf. [8, 9, 10, 4]). This can be achieved if splines are replaced by function spaces, which contain the low frequencies. In order to preserve the useful properties of spline functions, trigonometric splines seem to be an interesting choice.

For an exhaustive description of trigonometric splines, even on non-uniform grids, we refer to [11, Sect. 10.8]. The space of trigonometric splines of order \( m \) and mesh size \( h \) is given as

\[
\tilde{S}_{m,h} = \left\{ f \in C^{m-2}(\mathbb{R}) : f|_{(h(k+1),h(k+1))] \in T_m \right\},
\]

with the space of trigonometric polynomials

\[
T_m = \left\{ \sum_{k=0}^{m} c_k e^{2\pi i (k - \frac{m}{2}) t} : c_k \in \mathbb{C} \right\}.
\]

Obviously, for even order \( T_{2\mu} \) is a space of anti-periodic functions, i.e. \( f(t) = -f(t + 1) \), and is not suited for our purpose. In particular, constants and the fundamental frequency are not contained in \( T_{2\mu} \) and \( \tilde{S}_{2\mu,h} \). For odd order the space \( T_{2\mu+1} \) contains the real valued trigonometric polynomials

\[
\sum_{k=0}^{\mu} a_k \cos(2\pi k \omega t) + \sum_{k=1}^{\mu} a_k \sin(2\pi k \omega t),
\]

which are therefore also contained in \( \tilde{S}_{2\mu+1,h} \). A stable basis for \( \tilde{S}_{m,h} \) (if \( h m < 1 \)) is given by the translates \( Q_{m,h}(t - h \ell) \) of the trigonometric B-spline \( Q_{m,h}(t) \) defined by the recursion

\[
Q_{1,h}(t) := \chi_{(0,h)}(t), \quad Q_{m,h}(t) := \frac{\sin(\pi t) Q_{m-1,h}(t + \frac{m-1}{2} h)}{\sin \left( \pi (m-1) h \right)}.
\]

One can see easily that \( Q_{m,h}(t) \) is supported on \([0,m h]\) and from \( \sin(t) = t + \mathcal{O}(t^3) \) we conclude \( Q_{m,h}(t) = N(t/h) + \mathcal{O}(h^3) \), i.e., on fine grids the trigonometric splines behave similar to the classical polynomial splines. The above formulation allows us also to take advantage of several spline algorithms with only moderate extra computational effort for the computation of the sine function.

Analogously to the numerical differentiation in a collocation method can be described by

\[
\tilde{y}_k = \tilde{\psi}_{m,h}(\sigma, \frac{k}{m}) \tilde{y}_k,
\]

where

\[
\tilde{\psi}_{m,h}(x,\xi) := \frac{\varphi_{m,h}(x,\xi)}{\varphi_{m,h}(x,\xi)}
\]

and \( \varphi_{m,h}(x,\xi) = \sum_k Q_{m,h}(h(x + \frac{m}{2} + k)) e^{2\pi i k \xi} \).

However, for \( m = 2\mu + 1 \) the interpolation of \( e^{2\pi i k t} \), \( k \in \mathbb{Z}, |k| < \mu \) is exact, i.e., there are uniquely determined coefficients \( c_{k,\ell} \in \mathbb{C} \) such that

\[
e^{2\pi i k t} = s_{k}(t) := \sum_{\ell} c_{k,\ell} Q_{m,h}(\frac{\ell t}{m}), \quad |k| < \mu.
\]

This implies in turn that the derivatives satisfy \( s'_{k}(t) = 2\pi i k s_k(t) \) or \( \tilde{\psi}_{m,h}(\sigma, \frac{k}{m}) = 2\pi i k, |k| < \mu \). That is, there is no damping of low frequencies (for \( m = 3, 5, \ldots \)).

Figure 6 shows the results of a simulation using trigonometric splines of order \( m = 3 \), with the same parameters as for Fig. 4. The numerical damping of the fundamental frequency is eliminated, and the results behave in the range of the usual approximation error for the chosen grid.

4. Conclusion

We have shown that using trigonometric splines instead of the classical polynomial splines can eliminate the
negative effects of numerical damping for oscillator simulation, while we can still take advantage of many useful properties of B-splines.

The methods can easily modified to nonuniform grids used for adaptive methods described in [1, 2, 3]. Here we may have grids with a locally high resolution in one area and low resolution in other areas. Thus several negative effects studied in this article may occur simultaneously. In particular, for adaptive grid refinement the differences in numerical damping between different grids can have serious effects on the performance of the method, which do not occur if trigonometric splines are used. Thus, the introduction of trigonometric splines is an important contribution to oscillator simulation.

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