Holographic dark energy through Tsallis entropy

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Abstract. In order to apply holography and entropy relations to the whole universe, which is a gravitational and thus nonextensive system, for consistency one should use the generalized definition for the universe horizon entropy, namely Tsallis nonextensive entropy. We formulate Tsallis holographic dark energy, which is a generalization of standard holographic dark energy quantified by a new dimensionless parameter $\delta$, possessing the latter as a particular sub-case. We provide a simple differential equation for the dark energy density parameter, as well as an analytical expression for its equation-of-state parameter. In this scenario the universe exhibits the usual thermal history, namely the successive sequence of matter and dark-energy epochs, before resulting in a complete dark energy domination in the far future. Additionally, the dark energy equation-of-state parameter presents a rich behavior and, according to the value of $\delta$, it can be quintessence-like, phantom-like, or experience the phantom-divide crossing before or after the present time. Finally, we confront the scenario with Supernovae type Ia and Hubble parameter observational data, and we show that the agreement is very good, with $\delta$ preferring a value slightly larger than its standard value 1.

Keywords: dark energy theory, modified gravity

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1 Introduction

According to the concordance paradigm of cosmology the universe experienced an early-time accelerated phase, followed by the sequence of radiation and matter dominated eras, before resulting in the current, late-time, accelerated epoch. The two accelerated phases cannot be easily described through general relativity and standard model of particle physics, since extra degree(s) of freedom seem to be necessarily required. From one hand we can attribute these extra degrees of freedom to new, exotic forms of matter, collectively named as dark energy [1–3]. On the other hand we can consider them to be of gravitational origin, namely to arise from a modified theory of gravity that includes general relativity as a low-energy limit [4–8].

One interesting alternative for the explanation of dark energy origin and nature can be acquired applying the holographic principle [9–11] at a cosmological framework [12–14]. In particular, one takes advantage of the connection between the Ultraviolet cutoff of the (quantum field) theory, which is related to the vacuum energy, with the (necessary for the applicability of the quantum field theory at large distances) largest distance of the theory [15]. In this way the resulted vacuum energy will be a form of dark energy of holographic origin, named holographic dark energy [16] (see [17] for a review). Holographic dark energy leads to interesting cosmological phenomenology [16–27] and it has been also extended through various ways [28–47]. Additionally, holographic dark energy can be shown to be in agreement with observational data [48–53].

A crucial ingredient of the cosmological application of holography is the fact that the entropy of the whole universe, considered as a system with radius the aforementioned largest distance, is proportional to its area, similarly to a black hole. However, already at 1902 Gibbs had pointed out that in systems in which the partition function diverges the Boltzmann-Gibbs theory cannot be applied, and we now know that gravitational systems lie within this class. As it was shown by Tsallis, in such cases the usual Boltzmann-Gibbs additive entropy (which is founded on the hypothesis of weak probabilistic correlations and their connections to ergodicity) must be generalized to the non-additive entropy (i.e the entropy of the whole system is not necessarily the sum of the entropies of its sub-systems), know as Tsallis entropy [54–57]. In particular, this nonextensive Tsallis entropy can be written in compact form as [58]

\[ S_T = \gamma A^\delta, \]  

(1.1)
where $A \propto L^2$ is the area of the system with characteristic length $L$. The parameters $\gamma$ and $\delta$ under the hypothesis of equal probabilities are related to the dimensionality of the system $d$, and specifically the important one is $\delta = d/(d - 1)$ for $d > 1$ [58], however in the general case they remain as completely free parameters. Obviously, in the case where $\delta = 1$ and $\gamma = 2\pi M_p^2$ (in units where $\hbar = k_B = c = 1$), with $M_p$ the Planck mass, we obtain the usual additive entropy.

Having these in mind, we deduce that in order to apply holography and entropy relations to the whole universe, which is a gravitational and thus nonextensive system, one should use the above generalized definition of the universe horizon entropy. Hence, in the case of holographic dark energy, which is obtained from the inequality $\rho_{DE} L^4 \leq S$ with $S \propto A \propto L^2$ [17], the consistent scenario will arise if we use the Tsallis entropy (1.1) in this inequality, resulting to

$$\rho_{DE} = BL^{2\delta - 4},$$

with $B$ a parameter with dimensions $[L]^{-2\delta}$. As mentioned above, for $\delta = 1$ the above expression gives the usual holographic dark energy $\rho_{DE} = 3c^2 M_p^4 L^{-2}$, with $B = 3c^2 M_p^2$ and $c^2$ the model parameter. Additionally, it is worth mentioning that in the special case $\delta = 2$ the above relation gives the standard cosmological constant case $\rho_{DE} = \text{const.} = \Lambda$.

In the present work we are interested in formulating Tsallis holographic dark energy, which is characterized by energy density (1.2), and investigate its cosmological implications. Although relation (1.2) has been also extracted in a recent work too [59], its cosmological application has the serious disadvantage that it does not possess standard entropy and standard holographic dark energy as a sub-case. The reason behind this failure is the fact that it was the Hubble horizon that was used as $L$ (see also [60, 61] where the same inconsistency appears), which is well known that cannot lead to realistic cosmology in the case of usual holographic dark energy [62]. Hence, in the present paper we proceed to a consistent formulation of Tsallis holographic dark energy, taking as $L$ the future event horizon, namely the same length that is used in standard holographic dark energy scenario. In this way Tsallis holographic dark energy is indeed a consistent generalization of standard holographic dark energy, possessing it as a particular limit, namely for $\delta = 1$.

The plan of the manuscript is the following. In section 2 we formulate Tsallis holographic dark energy in a consistent way, extracting the corresponding cosmological equations. In section 3 we investigate the cosmological behavior of the scenario, focusing on the evolution of the dark-energy density and equation-of-state parameters, and we confront it with Supernovae type Ia observational data. Finally, section 5 is devoted to the conclusions.

## 2 Tsallis holographic dark energy

In this section we present the basic expressions of holographic dark energy based on Tsallis nonextensive entropy. Throughout this work we consider a flat homogeneous and isotropic Friedmann-Robertson-Walker (FRW) geometry with metric

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j,$$

with $a(t)$ the scale factor.

As we mentioned in the Introduction, the starting point for Tsallis holographic dark energy is expression (1.2). In the formulation of holographic dark energy one needs to consider a particular IR cutoff, namely the largest length of the theory $L$ that appears in the
expression of holographic dark energy density. It is well known that in the case of standard holographic dark energy models $L$ cannot be the Hubble horizon $H^{-1}$ (with $H \equiv \dot{a}/a$ the Hubble parameter), since such a choice leads to inconsistencies [62]. Hence, it was the future event horizon that was finally used [16], namely

$$R_h \equiv a \int_t^{\infty} \frac{dt}{a} = a \int_a^{\infty} \frac{da}{Ha^2}.$$  \hspace{1cm} (2.2)$$

In a recent attempt to construct Tsallis holographic dark energy the authors used the extended relation (1.2) but they chose $L$ to be the Hubble horizon [59]. Thus, the resulted model does not have standard holographic dark energy and standard thermodynamics as a sub-case, which is a serious disadvantage. On the contrary, in the present work we desire to formulate Tsallis holographic dark energy in a consistent way, and hence we use as $L$ the future event horizon (2.2). In this way, as we will see, standard holographic dark energy is included as a sub-case, and can be obtained for $\delta = 1$.

According to the above discussion, and using (1.2) with $L$ the $R_h$, the energy density of Tsallis holographic dark energy writes as

$$\rho_{DE} = B R_h^{2\delta-4}.$$  \hspace{1cm} (2.3)$$

In the following we focus on the general case $\delta \neq 2$, since as we mentioned for $\delta = 2$ the model gives the standard cosmological constant $\rho_{DE} = \Lambda$. The Friedmann equations in a universe containing the dark energy and matter perfect fluids are

$$3M_p^2 H^2 = \rho_m + \rho_{DE}$$  \hspace{1cm} (2.4)$$

and

$$-2M_p^2 \dot{H} = \rho_m + p_m + \rho_{DE} + p_{DE},$$  \hspace{1cm} (2.5)$$

with $p_{DE}$ the pressure of Tsallis holographic dark energy, and $\rho_m$ and $p_m$ respectively the energy density and pressure of the matter sector. The equations close by considering the matter conservation equation

$$\dot{\rho}_m + 3H(\rho_m + p_m) = 0.$$  \hspace{1cm} (2.6)$$

It proves convenient to introduce the dark energy and matter density parameters through

$$\Omega_m \equiv \frac{1}{3M_p^2 H^2} \rho_m$$  \hspace{1cm} (2.7)$$

and

$$\Omega_{DE} \equiv \frac{1}{3M_p^2 H^2} \rho_{DE}.$$  \hspace{1cm} (2.8)$$

Using these definitions, relations (2.2), (2.3) lead to

$$\int_x^{\infty} \frac{dx}{Ha} = \frac{1}{a} \left( \frac{B}{3M_p^2 H^2 \Omega_{DE}} \right)^{\frac{1}{1-2\delta}},$$  \hspace{1cm} (2.9)$$

where $x \equiv \ln a$. In the following we focus on the dust matter case, namely we consider the matter equation-of-state parameter to be zero, and thus (2.6) gives $\rho_m = \rho_{m0}/a^3$, with $\rho_{m0}$ the value of the matter energy density at the present scale factor $a_0 = 1$ (from now on the subscript “0” marks the present value of a quantity). Therefore, inserting into (2.7) gives
\[ \Omega_m = \Omega_{m0} \frac{H_0^2}{a^3 H^2}, \]

which, using that the Friedmann equation (2.4) becomes \( \Omega_m + \Omega_{DE} = 1 \), leads to

\[ \frac{1}{Ha} = \sqrt{\frac{a(1 - \Omega_{DE})}{H_0 \sqrt{\Omega_{m0}}}}. \quad (2.10) \]

Inserting (2.10) into (2.9) we obtain

\[ \int_x^\infty \frac{dx}{H_0 \sqrt{\Omega_{m0}}} \sqrt{a(1 - \Omega_{DE})} = \frac{1}{a} \left( \frac{B}{3M_p^2 H^2 \Omega_{DE}} \right)^{\frac{1}{3(2 - \delta)}}. \quad (2.11) \]

It proves convenient to use \( x = \ln a \) as the independent variable, and thus for every quantity \( f \) we have \( \dot{f} = f'H \), where primes denote derivatives with respect to \( x \). Thus, differentiating (2.11) with respect to \( x \) we finally acquire

\[ \frac{\Omega'_{DE}}{\Omega_{DE}(1 - \Omega_{DE})} = 2\delta - 1 + Q(1 - \Omega_{DE})^{\frac{\delta - 4}{2(\delta - 2)}} (\Omega_{DE})^{\frac{1}{2(\delta - 2)}} e^{\frac{3(1 - \delta)}{2(\delta - 2)} x}, \quad (2.12) \]

where

\[ Q \equiv 2(2 - \delta) \left( \frac{B}{3M_p^2} \right)^{\frac{1}{2(\delta - 2)}} \left( H_0 \sqrt{\Omega_{m0}} \right)^{\frac{1}{3(2 - \delta)}}. \quad (2.13) \]

Equation (2.12) is the differential equation that determines the evolution of Tsallis holographic dark energy, in a flat universe and for dust matter, as a function of \( x = \ln a \). In the case where \( \delta = 1 \) this equation does not have an explicit \( x \)-dependence and it coincides with the one of usual holographic dark energy \([16]\), namely \( \Omega'_{DE}|_{\delta=1} = \Omega_{DE}(1 - \Omega_{DE}) \left( 1 + 2 \sqrt{\frac{3M_p^2 \Omega_{DE}}{B}} \right) \) (complete coincidence is acquired under the identification \( B = 3c^2 M_p^2 \)), which accepts an analytic solution in an implicit form \([16]\). Nevertheless, in the case where \( \delta \neq 1 \), differential equation (2.12) exhibits an explicit \( x \)-dependence and cannot accept an analytical solution. Hence, in the following we will elaborate it numerically.

Let us now determine the other important observable, namely the Tsallis holographic dark energy equation-of-state parameter \( w_{DE} \equiv p_{DE}/\rho_{DE} \). Since the matter sector is conserved, namely eq. (2.6) holds, the two Friedmann equations (2.4), (2.5) imply that the dark energy sector is conserved too, namely

\[ \dot{\rho}_{DE} + 3H \rho_{DE}(1 + w_{DE}) = 0. \quad (2.14) \]

Differentiating the basic relation (2.3) we obtain that \( \dot{\rho}_{DE} = 2(\delta - 2) B R_h^{(\delta - 5)} \dot{R}_h \), where \( \dot{R}_h = H R_h - 1 \), and where \( \dot{R}_h \) can be straightforwardly found from (2.2) to be \( \dot{R}_h = H R_h - 1 \), and where \( \dot{R}_h \) can be eliminated in terms of \( \rho_{DE} \) through \( R_h = \left( \rho_{DE}/B \right)^{1/(2\delta - 4)} \), according to (2.3). Inserting these into (2.14) we obtain

\[ 2(\delta - 2) B \left( \frac{\rho_{DE}}{B} \right)^{\frac{2(\delta - 5)}{\delta - 2}} \left[ H \left( \frac{\rho_{DE}}{B} \right)^{\frac{1}{2(\delta - 2)}} - 1 \right] + 3H \rho_{DE}(1 + w_{DE}) = 0. \quad (2.15) \]

Finally, substituting \( H \) from (2.10), and using the dark energy density parameter definition (2.8) we result to

\[ w_{DE} = \frac{1 - 2\delta}{3} - \frac{Q}{3} (\Omega_{DE})^{\frac{1}{2(\delta - 2)}} (1 - \Omega_{DE})^{\frac{\delta - 1}{2(\delta - 2)}} e^{\frac{3(1 - \delta)}{2(\delta - 2)} x}. \quad (2.16) \]
Thus, this expression provides \( w_{DE} \) as a function of \( \ln a \), as long as \( \Omega_{DE} \) is known from the solution of (2.12). As expected, for \( \delta = 1 \) (2.16) does not have an explicit \( x \)-dependence and it gives the usual holographic dark energy equation-of-state parameter, namely \( w_{DE}|_{\delta=1} = -\frac{1}{3} - \frac{2}{3} \sqrt{\frac{3M^2_{DE}}{B}} \) [17], where complete coincidence is acquired under the identification \( B = 3c^2M^2_p \).

We close this section by introducing the convenient deceleration parameter \( q \), which reads as

\[
q \equiv -1 - \frac{\dot{H}}{H^2} = \frac{1}{2} + 3\left(w_m\Omega_m + w_{DE}\Omega_{DE}\right). \tag{2.17}
\]

Hence, in the case of dust matter \((w_m = 0)\), \( q \) is straightforwardly known as long as \( \Omega_{DE} \) is known from (2.12).

3 Cosmological evolution

In this section we proceed to the investigation of the cosmological behavior in a universe where the dark energy sector is the Tsallis holographic dark energy. The basic differential equation that determines the evolution of \( \Omega_{DE} \) as a function of \( x = \ln a \) is eq. (2.12). Unfortunately, this equations can be analytically solved in an implicit form only in the case \( \delta = 1 \) [16], since in the \( \delta \neq 1 \) case it acquires an explicit \( x \)-dependence that does not allow for an analytical solution. Hence, one should resort to numerical elaboration in order to extract its solution. As long as the solution for \( \Omega_{DE}(x) \) is obtained, its behavior in terms of the redshift \( z \) can be straightforwardly obtained through \( x \equiv \ln a = -\ln(1 + z) \) (having set \( a_0 = 1 \)).

We elaborate eq. (2.12) numerically, imposing that \( \Omega_{DE}(x = -\ln(1 + z) = 0) \equiv \Omega_{DE0} \approx 0.7 \) and thus \( \Omega_m(x = -\ln(1 + z) = 0) \equiv \Omega_{m0} \approx 0.3 \) as required by observations [63]. In the upper graph of figure 1 we present \( \Omega_{DE}(z) \) and \( \Omega_{m}(z) = 1 - \Omega_{DE}(z) \). In the middle graph we depict the corresponding behavior of \( w_{DE}(z) \) as it arises from (2.16). And in the lower graph we draw the deceleration parameter from (2.17). Additionally, note that in the graphs we have extended the evolution into the future, namely for \( z < 0 \), since \( z \to -1 \) corresponds to \( t \to \infty \).

From the upper graph of figure 1 we observe that we can acquire the usual thermal history of the universe, namely the sequence of matter and dark energy eras, while the universe asymptotically results in a complete dark-energy domination. Furthermore, from the third graph of figure 1 we can see that the transition from deceleration to acceleration happens at \( z \approx 0.5 \) as required from observations. Finally, from the middle graph of figure 1 we can see that the current value of \( w_{DE} \) is around \(-1\) in agreement with observations. We mention that in this explicit example in the future \( w_{DE} \) enters slightly in the phantom regime, which according to relation (2.16) is allowed in the model at hand, which is an advantage showing the enhanced capabilities.

Let us now investigate the effect of \( \delta \) on \( w_{DE} \). In figure 2 we present \( w_{DE}(z) \) for various values of \( \delta \), including the value \( \delta = 1 \) which corresponds to standard holographic dark energy. As we observe, for increasing \( \delta \) the \( w_{DE}(z) \) evolution, as well as its present value \( w_{DE}(z = 0) \), tend to lower values. Note that for \( \delta \gtrsim 1.2 \) the value of \( w_{DE}(z = 0) \) is in the phantom regime. Hence, according to the value of \( \delta \), the dark energy sector can be quintessence-like, phantom-like, or experience the phantom-divide crossing before or after the present time.

We mention here that although the scenario at hand has two parameters, namely the new exponent \( \delta \) and the constant \( B \), in the above examples we preferred to fix \( B = 3 \), which is required in order to obtain exact coincidence with standard holographic dark energy when \( \delta = 1 \), and explore the role of \( \delta \) in a pure way. Nevertheless, as we showed, changing \( \delta \) is adequate in order to obtain a cosmology in agreement with observations, without the need
Figure 1. Upper graph: the evolution of Tsallis holographic dark energy density parameter $\Omega_{\text{DE}}$ (black-solid) and of the matter density parameter $\Omega_m$ (red-dashed), as a function of the redshift $z$, for $\delta = 1.1$ and $B = 3$, in units where $M_p^2 = 1$. Middle graph: the evolution of the corresponding dark energy equation-of-state parameter $w_{\text{DE}}$. Lower graph: the evolution of the corresponding deceleration parameter $q$. In all graphs we have imposed $\Omega_{\text{DE}}(x = -\ln(1 + z) = 0) \equiv \Omega_{\text{DE}0} \approx 0.7$ at present in agreement with observations, and we have added a vertical dotted line denoting the present time $z = 0$.

Figure 2. The evolution of the equation-of-state parameter $w_{\text{DE}}$ of Tsallis holographic dark energy, as a function of the redshift $z$, for $B = 3$, and for $\delta = 0.8$ (red-dashed), $\delta = 1$ (black-solid), $\delta = 1.2$ (blue-dotted), and $\delta = 1.3$ (green-dashed-dotted), $\delta = 1.4$ (magenta-dashed-dot-dotted), in units where $M_p^2 = 1$. In all graphs we have imposed $\Omega_{\text{DE}}(x = -\ln(1 + z) = 0) \equiv \Omega_{\text{DE}0} \approx 0.7$ at present in agreement with observations.

to change the constant $B$. This is a significant advantage of Tsallis holographic dark energy comparing to standard holographic dark energy, since in the latter one needs to use a value of the constant $c^2$ different than 1 in order to obtain satisfying observational fittings, which has then difficulties to be theoretically justified (the essence of holographic dark energy is that the total energy in a region of size $L$ should not exceed the mass of a black hole of the same size, and not the latter multiplied by a tuned constant). Definitely, changing additionally the value of $B$ would significantly enhanced the capabilities of Tsallis holographic dark energy.
4 Observational constraints

In this section we confront the scenario with cosmological data from Supernovae type Ia observations as well as direct $H(z)$ Hubble data. In particular, we desire to extract the constraints on the free parameters of the model, through the maximum likelihood analysis. Assuming Gaussian errors this can be obtained by minimizing the $\chi^2$ function in terms of the model free parameters $a_m$. Since we utilize SNIa and Hubble rate data, the total $\chi^2$ reads

$$\chi^2_{\text{tot}} = \chi^2_H + \chi^2_{\text{SN}}, \quad (4.1)$$

where $\chi^2_H, \chi^2_{\text{SN}}$ will be defined in the following. In the case of holographic dark energy through Tsallis entropy the statistical vector of the free parameters is $a_m = (\Omega_{m0}, B, \delta, h)$, with $h = H_0/100$. Note that, as we mentioned above, we fix $B = 3$, i.e. to its standard holographic dark energy value, in order to explore the role of $\delta$ in a pure way. We use the Markov Chain Monte Carlo (MCMC) algorithm within the Python package emcee [64], in order to minimize $\chi^2$ with respect to $a_m$. Finally, the algorithm convergence is checked with auto-correlation time considerations, while we also employ the Gelman-Rubin criterion [65] for completeness.

4.1 Type Ia Supernovae

Type Ia Supernovae (SNIa) are widely used in cosmological confrontation, since they can be handled as “standard candles”, offering a way to measure cosmic distances. In these data sets one measures the apparent luminosity as function of redshift or equivalently the apparent magnitude. The $\chi^2$ that corresponds to the fit is given by

$$\chi^2_{\text{SN}} = \mu C^{-1}_{\text{SN,cov}} \mu^T, \quad (4.2)$$

where $\mu = \{\mu_{\text{obs}} - \mu_{\text{th}}(z_1; a_m), \ldots, \mu_{\text{obs}} - \mu_{\text{th}}(z_N; a_m)\}$ and $N = 40$. In the above expression $\mu_{\text{obs}}$ is the observed distance modulus, which for every supernova is defined as the difference between its absolute and apparent magnitude. We use the binned SNIa data points and the corresponding inverse covariance matrix $C^{-1}_{\text{SN,cov}}$ from [66]. Moreover, in the statistical vector $a_m$ we include the quantity $\mathcal{M}$ that quantifies errors of astrophysical origins in the observed distance moduli. The theoretically calculated distance modulus $\mu_{\text{th}}$ has a dependence on the model parameters $a_m$ which reads as

$$\mu_{\text{th}}(z) = 42.38 - 5 \log_{10} h + 5 \log_{10} [D_L(z; a_m)], \quad (4.3)$$

where the dimensionless luminosity distance $D_L(z; a_m)$ is given by

$$D_L(z; a_m) \equiv (1 + z) \int_0^z dz' \frac{H_0}{H(z'; a_m)}. \quad (4.4)$$

Note that the quantity $H(z'; a_m)$ in the scenario at hand cannot be obtained analytically and thus it is extracted from (2.10) after the numerical elaboration of eq. (2.12).

4.2 Cosmic chronometer Hubble data

The direct measurements of the Hubble constant is a very powerful implementation in cosmological confrontation, introduced first in [67] with the method determining the Hubble parameter data through the differential age evolution of the passively evolving early-type
galaxies. In particular, since the Hubble function can be expressed as $H = -(1 + z)^{-1}dz/dt$, by measuring $dz/dt$ one can directly measure $H(z)$ data (see [68] for a detailed discussion).

In this work we use the most recent Hubble data from [69]. The corresponding dataset contains $N = 36$ measurements of $H(z)$ in the $0.07 \leq z \leq 2.33$ range. Amongst these, there are 5 data points that are based on Baryon Acoustic Oscillations (BAO), nevertheless for all the remaining points the Hubble constant is measured through the passive evolving galaxies differential age. The $\chi^2$ that corresponds to the fit is given by

$$\chi^2_H(a_m) = \mathcal{H} C_{H,\text{cov}}^{-1} \mathcal{H}^T,$$

where $\mathcal{H} = \{H_1 - H_0 E(z_1, a_m), \ldots, H_N - H_0 E(z_N, a_m)\}$, $H_i$ is the observed Hubble values at redshifts $z_i (i = 1, \ldots, N)$, and with $C$ being the involved covariance matrix (see [70, 71] for more details). Note that the theoretical quantity $E(z_i, a_m) \equiv H(z_i, a_m)/H_0$ in the scenario at hand has to be extracted numerically through (2.10) and (2.12).

In figure 3 we present the contour plots for several combinations of various quantities of Tsallis holographic dark energy scenario, using SNIa and $H(z)$ data. Additionally, we depict the corresponding 1-dimensional (1D) marginalized posterior distributions and the mean values of the parameters corresponding to the $1\sigma$ area of the MCMC chain. As we can see the agreement with the data is very good, and the matter energy density as well as the current value of the Hubble parameter coincide with those of Planck within $1\sigma$ [63]. Concerning the new physical parameter of the present work, namely the exponent $\delta$, there is a tendency for a slight deviation from its standard value 1, however the value 1 is included within $2\sigma$. Moreover, note that these results could be improved allowing $B$ to change too (in this case in the end one has to use the AIC [72] and BIC [73] criteria in order to consistently weight the effect of the additional model parameters). We mention here that the incorporation of other datasets such as Cosmic Microwave Background (CMB), although necessary, would require a special treatment of the $H(z)$ form, which in the current scenario cannot be obtained analytically in general. This complicated elaboration lies beyond the scope of the present work and is left for future investigation, along with the perturbation analysis and the use of Large Scale Structure data.

5 Conclusions

In the present work we formulated Tsallis holographic dark energy, which is a generalization of standard holographic dark energy. In particular, in order to apply holography and entropy relations to the whole universe, which is a gravitational and thus nonextensive system, for consistency one should use the generalized definition of the universe horizon entropy, namely Tsallis nonextensive entropy, quantified by a new dimensionless parameter $\delta$. Although a similar idea appeared in a recent work too [59], its cosmological application had the serious disadvantage that it did not possess standard entropy and standard holographic dark energy as a sub-case, due to the fact that it was the Hubble horizon that was used as the IR cutoff, which is well known that cannot lead to realistic cosmology in case of usual holographic dark energy. On the other hand, in the present investigation we presented a consistent formulation of Tsallis holographic dark energy, taking the IR cutoff to be the future event horizon, namely the same length that is used in standard holographic dark energy scenario. In this way Tsallis holographic dark energy is indeed a consistent generalization of standard holographic dark energy, possessing it as a particular limit, namely for $\delta = 1$. 

\[ -8 - \]
Figure 3. The $1\sigma$, $2\sigma$, and $3\sigma$ 2-dimensional contour plots for several combinations of various quantities of Tsallis holographic dark energy scenario, using SNIa and $H(z)$ data. Additionally, we depict the corresponding 1-dimensional (1D) marginalized posterior distributions and the mean values of the parameters corresponding to the $1\sigma$ area of the MCMC chain. The parameter $\mathcal{M}$ is the usual free parameter of SNIa data that quantifies possible systematic errors of astrophysical origin [66]. For these fittings we obtain $\chi^2_{\text{min}}/\text{dof} = 43.248/76$.

In order to study the cosmological applications of Tsallis holographic dark energy we first provided a simple differential equation for the holographic dark energy density parameter $\Omega_{\text{DE}}$. Additionally, we extracted an analytical expression for the holographic dark energy equation-of-state parameter $w_{\text{DE}}$ as a function of $\Omega_{\text{DE}}$. Although in the case $\delta = 1$ the above differential equation can be solved analytically in an implicit form, in the general case it does not accept an analytical solution and thus one has to elaborate it numerically.

The scenario of Tsallis holographic dark energy leads to interesting cosmological phenomenology. Firstly, the universe exhibits the usual thermal history, namely the successive sequence of matter and dark-energy epochs, with the transition from deceleration to acceleration happening at $z \approx 0.5$ in agreement with observations, before it results in a complete dark energy domination in the far future. Furthermore, the corresponding dark energy equation-of-state parameter presents a rich behavior, and according to the value of $\delta$, it can be quintessence-like, phantom-like, or experience the phantom-divide crossing before or after the present time.
Additionally, we confronted the scenario with Supernovae type Ia and $H(z)$ observational data, we constructed the corresponding contour plots, and we saw that the agreement is very good. Concerning the new physical parameter of the present work, namely the exponent $\delta$, there is a tendency for a slight deviation from its standard value 1, however the value 1 is included within 2$\sigma$.

We mention that the above behaviors were obtained changing only the value of $\delta$ and keeping the second parameter (the one that is present in holographic dark energy models) $B$ fixed. This is a significant advantage comparing to standard holographic dark energy, since in the latter one needs to use a value of this parameter different than the straightforward one in order to obtain satisfying observational fittings, which has then difficulties to be theoretically justified. Definitely, changing additionally the value of $B$ enhances significantly the capabilities of Tsallis holographic dark energy.

In summary, as we can see, the scenario of Tsallis holographic dark energy exhibits richer behavior comparing to standard holographic dark energy, quantified by the present of the new parameter $\delta$, while due to its consistent formulation one can still obtain as a sub-case the scenario of standard holographic dark energy, namely for $\delta = 1$. There are additional studies that need to be performed before the scenario can be considered as a successful candidate for the description of nature. Firstly, one should perform a joint observational analysis at both the background and perturbation levels, using data from Cosmic Microwave Background (CMB) and Large Scale Structure (such as f$\sigma$8), in order to constrain the model parameters. Moreover, one should perform a detailed phase-space analysis in order to extract the global features of the scenario at late times, independently of the initial conditions and the specific evolution. These necessary investigations lie beyond the scope of this work and are left for future projects.

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