THE SKEW GROWTH FUNCTIONS $N_{M,\text{deg}}(t)$ FOR THE
MONOID OF TYPE $B_{ii}$ AND OTHERS.

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Abstract. Let $M$ be a positive homogeneously finitely presented monoid $\langle L \mid R \rangle_{m_o}$ that satisfies the cancellation condition and is equipped with the degree map $\text{deg} : M \to \mathbb{Z}_{\geq 0}$ defined by assigning to each equivalence class of words the length of the words, and let $P_{M,\text{deg}}(t) := \sum_{w \in M} t^{\text{deg}(w)}$ be its generating series, called the (spherical) growth function. If $M$ satisfies the condition that any subset $J$ of $I_0$ (:= the image of the set $L$ in $M$) admits either the least right common multiple $\Delta_J$ or no common multiple in $M$, then the inversion function $P_{M,\text{deg}}(t)^{-1}$ is given by the polynomial $\sum_{J \subset I_0} (-1)^{\#J} t^{\text{deg}(\Delta_J)}$, where the summation index $J$ runs over all subsets of $I_0$ whose least right common multiple exists. Since a monoid $M$, in general, may not admit the least right common multiple $\Delta_J$ for a given subset $J$ of it, if we attempt to generalize the formula, the consideration to obtain the above formula is invalid. In order to resolve this obstruction, we will examine the set $\text{mcm}(J)$ of minimal common right multiples of elements of $M$ and denote the set of all the towers in $M$ by $T_{\text{mcm}}(M)$. By considering the structure of the set $T_{\text{mcm}}(M)$ of all the towers, K. Saito has proved the inversion formula

$$P_{M,\text{deg}}(t) \cdot N_{M,\text{deg}}(t) = 1,$$

where the second factor in LHS is a suitably signed generating series

$$N_{M,\text{deg}}(t) := 1 + \sum_{T \in T_{\text{mcm}}(M)} (-1)^{\#J_1 + \cdots + \#J_n} \sum_{\Delta \in \text{mcm}(J_n)} t^{\text{deg}(\Delta)},$$

called the skew growth function.

In this article, we will present several explicit calculations of examples of the skew growth functions for the monoid of type $B_{ii}$ and others whose towers do not stop on the first stage $J_1$. For monoids of this kind, on one hand, we generally find difficulty in calculating the skew growth functions, since, for any tower $T = (I_0, J_1, J_2, \cdots, J_n)$, we need to calculate the set $\text{mcm}(J_n)$ explicitly for each $J_i$. On the other hand, we find difficulty in showing the cancellativity of them, since the pre-existing technique is far from perfect. By improving the technique, we will show the cancellativity of the above examples successfully.

1. Introduction

Let $M$ be a positive homogeneously finitely presented monoid $\langle L \mid R \rangle_{m_o}$ that satisfies the cancellation condition (i.e. $axb = ayb$ implies $x = y$). Due to the homogeneity, we naturally define a map $\text{deg} : M \to \mathbb{Z}_{\geq 0}$ defined by assigning to each equivalence class of words the length of the words. By analogy to the (spherical) growth function for a group, the (spherical) growth function for a monoid $M$ is defined as $P_{M,\text{deg}}(t) := \sum_{w \in M} t^{\text{deg}(w)}$ and has been discussed and calculated by several authors ([B], [X], [B-B], [D], [A], [S], [S3], [S4], [S5], [X]). In these studies except the [A], the monoid $M$ satisfies the condition $L$ that any subset $J$ of $I_0$ (:= the image of the set $L$ in $M$) admits either the least right common multiple $\Delta_J$ or no common
function, called the skew growth function, $N_{M,\deg}(t) := \sum_{J \subseteq I_0} (-1)^{\#J} t^{\deg(\Delta_J)}$, where the summation index $J$ runs over all subsets of $I_0$ whose least right common multiple exists. Thus, the growth function $P_{M,\deg}(t)$ can be calculated through the skew growth function $N_{M,\deg}(t)$. We remark that, in this case, $N_{M,\deg}(t)$ is a polynomial.

In general, a monoid $M$ may not admit the least right common multiple $\Delta_J$ for a given subset $J$ of it. Indeed, the monoid, called the type Bii, that was discussed in \[1\], does not satisfy the condition $\mathcal{L}$. Nevertheless, the author has made a success in calculating the spherical growth function for it. The spherical growth function can be expressed as a rational function $\frac{1-t+t^2}{(1-t)^4}$. Since the form of the numerator polynomial $1 - t + t^2$ is suggestive, a generalization of the inversion formula to a wider class of monoids which do not always satisfy the condition $\mathcal{L}$ is naturally expected. Instead of considering the least common multiple $\Delta_J$, we study the set $\text{mcm}(J)$ of minimal common right multiples of $J$. However, the datum $\{\text{mcm}(J)\}_{J \subseteq I_0}$ is still not sufficient to recover the inversion formula, since a subset $J'$ of $\text{mcm}(J)$ in general may have common right multiples. Thus, we need to consider the set $\text{mcm}(J')$ for a subset $J'$ of $\text{mcm}(J)$. Then, we may again need to consider $\text{mcm}(J'')$ for a subset $J''$ of $\text{mcm}(J')$, and so on. Repeating this process, we are naturally led to consider a notion of tower: a finite sequence $I_0 \supset J_1, J_2, \ldots, J_n$ of subsets of $M$ such that $J_2 \subseteq \text{mcm}(J_1), \ldots, J_n \subseteq \text{mcm}(J_{n-1})$.

In \[SI\], K. Saito has made a success in generalizing the inversion formula for a rather wider class of monoids. Namely, for a cancellative monoid $M$ equipped with a discrete degree map $\deg : M \to \mathbb{R}_{\geq 0}$ (see \[SI4\]), he defined the (spherical) growth function $P_{M,\deg}(t)$ for $M$ with respect to $\deg$, and, by considering the set $\text{Tmcm}(M)$ of all towers $T = (I_0, J_1, J_2, \ldots, J_n)$ in $M$ (we do not explain here what the set $I_0$ is about, cf. \[SI\]), he defined the skew growth function

$$N_{M,\deg}(t) := 1 + \sum_{T \in \text{Tmcm}(M)} (-1)^{\#J_1 + \cdots + \#J_n - n + 1} \sum_{\Delta \in \text{mcm}(J_n)} t^{\deg(\Delta)}.$$ 

Then, he has shown the inversion formula for $M$ with respect to $\deg$

$$P_{M,\deg}(t) \cdot N_{M,\deg}(t) = 1.$$ 

Thus, when we put $h(M, \deg) := \max\{n \mid T = (I_0, J_1, J_2, \ldots, J_n) \in \text{Tmcm}(M)\}$, the inversion formula covers all the cases $\infty \geq h(M, \deg) \geq 0$.

In this article, for a positive homogeneously finitely presented cancellative monoid $M = \langle L \mid R \rangle_{\text{ho}}$ that do not satisfy the condition $\mathcal{L}$, we will present several explicit calculations of examples of the skew growth functions. For a non-abelian monoid $M = \langle L \mid R \rangle_{\text{ho}}$ whose $h(M, \deg)$ is equal to $\infty$, one may think that calculations of the skew growth functions are not practicable. However, in $\S 5$, we succeed in the non-trivial calculation for the monoid of type Bii, partially because, for any tower $T = (I_0, J_1, J_2, \ldots, J_n)$, we can calculate the set $\text{mcm}(J_i)$ explicitly for each $J_i$ due to the Lemma 5.5. For the same reason, for the monoids $G_n^+$ and $H_n^+$ whose $h(M, \deg)$ is equal to $2$ and the abelian monoid $M_{\text{abel}}$ whose $h(M, \deg)$ is equal to $\infty$, we can calculate the skew growth functions in $\S 5$. As far as we know, for non-abelian monoids that do not satisfy the condition $\mathcal{L}$, there are few examples

multiple in $M$. Then, the inversion function $P_{M,\deg}(t)^{-1}$ is given by the following function, called the skew growth function,
for which the cancellativity of them has been shown, since the pre-existing technique to show the cancellativity is far from perfect (G[B-S][Deh1][Deh2]). For calculations of the skew growth functions, improvement of the technique to show the cancellativity is expected. In [Deh1], [Deh2], if presentation of a positive homogeneously presented monoid satisfies some condition, called completeness, the cancellativity of it can be trivially checked. However, in general, the presentation of a monoid is not complete. When the presentation is not complete, in order to obtain a complete presentation, some procedure, called completion, is carried out. From our experience, for most of non-abelian monoids that do not satisfy the condition $L$, these procedures do not finish in finite steps. For monoids of this kind, nothing is discussed in [Deh1], [Deh2]. Thus, we attempt improving the technique for this class of monoids. On the other hand, the presentations of the examples $G_{B_n}^+(11)$, $G_{m,n}^+(12)$, and $H_n^+(3)$ are not complete and the procedures do not finish in finite steps. Nevertheless, we show the cancellativity of them successfully (11, 12), by improving the technique. In §4, by the improved technique, we also show the cancellativity. In [S2][S3][S4][S5][K-T-Y], the distribution of the zeroes of the denominator polynomials of the growth functions are investigated from the viewpoint of limit functions ([S4]). In §6, we will explore another viewpoint from an analysis of the three examples. Namely, we will present three examples that suggest the relationship between the form of the spherical growth function for a positive homogeneously finitely presented cancellative monoid $\langle L \mid R \rangle_{m_0}$ and properties of the corresponding group $\langle L \mid R \rangle$.

2. Positive homogeneous presentation

In this section, we first recall from [S-I], [B-S] some basic definitions and notations. Secondly, for a positive homogeneously finitely presented group

$$G = \langle L \mid R \rangle,$$

we associate a monoid defined by it. We will prepare basic definitions in a positive homogeneously presented monoid. Lastly, we define two operations on the set of subsets of a monoid.

First, we recall from [S-I] basic definitions on a monoid $M$.

**Definition 2.1.** 1. A monoid $M$ is called cancellative, if a relation $AXB = AY B$ for $A, B, X, Y \in M$ implies $X = Y$.

2. For two elements $u, v$ in $M$, we denote

$$u \mid v$$

if there exists an element $x$ in $M$ such that $v = ux$. We say that $u$ divides $v$ from the left, or, $v$ is a right-multiple of $u$.

3. We say that $G^+$ is conical, if 1 is the only invertible element in $G^+$.

Next, we recall from [B-S] some terminologies and concepts. Let $L$ be a finite set. Let $F(L)$ be the free group generated by $L$, and let $L^*$ be the free monoid generated by $L$ inside $F(L)$. We call the elements of $F(L)$ words and the elements of $L^*$ positive words. The empty word $\varepsilon$ is the identity element of $L^*$. If two words $A, B$ are identical letter by letter, we write $A \equiv B$. Let $G = \langle L \mid R \rangle$ be a positive homogeneously presented group (i.e. the set $R$ of relations consists of those of the form $R_i = S_i$ where $R_i$ and $S_i$ are positive words of the same length), where $R$ is the set of relations. We often denote the images of the letters and words under the
we define a growth function $P$.

**Definition**

Recall from [S1] the inversion formula for the growth function of an equivalence relation $\sim$.

ii) $\text{mcm}(J_i)$

iii) $\text{mo}(J_i)$

Remark

We associate a monoid $G$ generated by $L$ with $\text{deg}$ : $G^+ \to \mathbb{Z}_{\geq 0}$ by assigning to each equivalence class of words the length of the words.

Remark

A relation of two words $U$ and $V$ in $L^*$ are called elementarily equivalent if either $U \equiv V$ or $V$ is obtained from $U$ by substituting a substring $R_i$ of $U$ by $S_i$ where $R_i=S_i$ is a relation of $R$ ($S_i = S_i$ is also a relation if $R_i = S_i$ is a relation).

Due to the homogeneity of the relations, we define a homomorphism:

$$\deg : G^+ \to \mathbb{Z}_{\geq 0}$$

by assigning to each equivalence class of words the length of the words.

**Remark 1.** For a positive homogeneously finitely presented group $G = \langle L \mid R \rangle$, the associated monoid $G^+$ is conical.

**Remark 2.** In [S1], for a monoid $M$, the quotient set $M/\sim$ is considered, where the equivalence relation $\sim$ on $M$ is defined by putting $u \sim v \iff \deg u = \deg v$. Due to the conicity, if $M = \langle L \mid R \rangle$, then we say that $M/\sim = M$.

Lastly, we consider two operations on the set of subsets of a monoid $M$. For a subset $J$ of $M$, we put

$$\text{cm}_r(J) := \{ u \in M \mid j \vdash u, \forall j \in J \}$$

$$\text{min}_r(J) := \{ u \in J \mid \exists v \in J \text{ s.t. } \vdash u \Rightarrow v = u \}$$

and their composition: the set of minimal common multiples of the set $J$ by

$$\text{mcm}(J) := \text{min}_r(\text{cm}_r(J))$$

**3. Generating functions $P_{M, \deg}$ and $N_{M, \deg}$**

In this section, for a positive homogeneous presented cancellative monoid $M = \langle L \mid R \rangle$, we define a growth function $P_{M, \deg}$ and a skew growth function $N_{M, \deg}$. Next, we recall from [S1] the inversion formula for the growth function of $M$.

First, we introduce a concept of towers of minimal common multiples in $M$.

**Definition 3.1.** A tower of $M$ of height $n \in \mathbb{Z}_{\geq 0}$ is a sequence

$$T := (I_0, J_1, J_2, \ldots, J_n)$$

of subsets of $M$ satisfying the followings.

i) $I_0$ is the image of the set $L$ in $M$.

ii) $\text{mcm}(J_k) \neq \emptyset$ and we put $I_k := \text{mcm}(J_k)$ for $k = 1, \ldots, n$.

iii) $J_k \subset I_{k-1}$ such that $1 < \sharp J_k < \infty$ for $k = 1, \ldots, n$. 
Here, we call $I_0$, $J_k$ and $I_k$, the ground, the $k$th stage and the set of minimal common multiples on the $k$th stage of the tower $T$, respectively. In particular, the set of minimal common multiples on the top stage is denoted by $|T| := I_n$.

The set of all towers of $M$ shall be denote by $\text{Tmcm}(M)$. We put $h(M, \deg) := \max\{\text{height of } T \in \text{Tmcm}(M)\}$.

**Remark 3.** i) It is clear that $M$ is a free monoid if and only if $h(M, \deg) = 0$.

ii) All of the monoids discussed in [A-N], [B-S], [S2], [S3] have $h(M, \deg) \leq 1$.

iii) For the following cancellative monoid

$$G_{B_4}^+ := \left< a, b, c \mid \begin{array}{l} cb = bba, \\ ab = bc, \\ ac = ca \end{array} \right>_{mo}$$

we have $h(G_{B_4}^+, \deg) = \infty$ (§5).

iv) For the following cancellative monoids

$$G_n^+ := \left< a, b, c \mid \begin{array}{l} cb^n = b^n a, \\ ab = bc, \\ ac = ca \end{array} \right>_{mo} (n = 3, 4, \ldots),$$

$$H_n^+ := \left< a, b, c \mid \begin{array}{l} b(ab)^n ba = cb(ab)^n b, \\ ab = bc, \\ ac = ca \end{array} \right>_{mo} (n = 1, 2, \ldots),$$

we have $h(G_n^+, \deg) = 2$ and $h(H_n^+, \deg) = 2$ (§5).

v) For the following abelian cancellative monoid

$$M_{\text{abel}, m} := \left< a, b \mid \begin{array}{l} a^m = b^m, \\ ab = ba \end{array} \right>_{mo} (m = 2, 3, \ldots),$$

we have $h(M_{\text{abel}, m}, \deg) = \infty$ (§5).

Secondly, we define a growth function $P_{M, \deg}$ and a skew growth function $N_{M, \deg}$.

In the previous section, we have fixed a degree map $\deg$ on $M$. Then, we define the *growth function of the monoid* $(M, \deg)$ by

$$P_{M, \deg} := \sum_{u \in M} t^{\deg(u)} = \sum_{d \in \mathbb{Z}_{\geq 0}} \#(M_d) t^d,$$

where we put $M_d := \{u \in M \mid \deg(u) = d\}$. And we define the *skew growth function of the monoid* $(M, \deg)$ by

$$N_{M, \deg}(t) := 1 + \sum_{T \in \text{Tmcm}(M)} (-1)^{\#J_1 + \cdots + \#J_n - n + 1} \sum_{\Delta \in [T]} t^{\deg(\Delta)}.$$

**Remark 4.** In the definition (3.1), we can write down the coefficient of the term $t$ directly. Namely, we write

$$N_{M, \deg}(t) = 1 - \#(I_0) t + \sum_{\text{height of } T \geq 1} (-1)^{\#J_1 + \cdots + \#J_n - n + 1} \sum_{\Delta \in [T]} t^{\deg(\Delta)}.$$

**Remark 5.** Therefore, if $M$ is a free monoid of rank $n$, then we have $N_{M, \deg}(t) = 1 - nt$.

Lastly, we recall from [S1] the inversion formula for the growth function of $(M, \deg)$. 

Theorem 3.2. We have the inversion formula
\[ P_{M, \deg}(t) \cdot N_{M, \deg}(t) = 1. \]

4. Cancellativity of \( G_n^+ \) and \( H_n^+ \)

In this section, for a preparation for calculations of the skew growth functions for the monoids \( G_n^+ \) and \( H_n^+ \) in §5, we prove the cancellativity of them.

First, we show the cancellativity of \( G_n^+ \).

Theorem 4.1. The monoid \( G_n^+ \) is a cancellative monoid.

Proof. First, we remark the following.

Proposition 4.2. The left cancellativity on \( G_n^+ \) implies the right cancellativity.

Proof. Consider a map \( \varphi : G_n^+ \to G_n^+ \), \( W \mapsto \varphi(W) := \sigma(\text{rev}(W)) \), where \( \sigma \) is a permutation \((a, b, c)\) and \( \text{rev}(W) \) is the reverse of the word \( W = x_1 x_2 \cdots x_k \) (\( x_i \) is a letter) given by the word \( x_k \cdots x_2 x_1 \). In view of the defining relation of \( G_n^+ \), \( \varphi \) is well-defined and is an anti-isomorphism. If \( \beta \alpha = \gamma \alpha \), then \( \varphi(\beta \alpha) = \varphi(\gamma \alpha) \), i.e., \( \varphi(\alpha) \varphi(\beta) = \varphi(\alpha) \varphi(\gamma) \). Using the left cancellativity, we obtain \( \varphi(\beta) = \varphi(\gamma) \) and, hence, \( \beta = \gamma \).

The following is sufficient to show the left cancellativity on \( G_n^+ \).

Proposition 4.3. Let \( Y \) be a positive word in \( G_n^+ \) of length \( r \in \mathbb{Z}_{\geq 0} \) and let \( X^{(h)} \) be a positive word in \( G_n^+ \) of length \( r - h \in \{ r - n + 1, \ldots, r \} \).

(i) If \( vX^{(0)} = vY \) for some \( v \in \{a, b, c\} \), then \( X^{(0)} = Y \).

(ii) If \( aX^{(0)} = bY \), then \( X^{(0)} = bZ \), \( Y = cZ \) for some positive word \( Z \).

(iii) If \( aX^{(0)} = cY \), then \( X^{(0)} = cZ \), \( Y = aZ \) for some positive word \( Z \).

(iv) If \( bX^{(0)} = aY \), then there exist an integer \( k \) (\( 0 \leq k \leq r - n \)) and a positive word \( Z \) such that \( X^{(0)} = c^k b^{n-1-a} \cdot Z \) and \( Y = a^k b^{n-1} \cdot Z \).

(iv-1-a) There does not exist words \( X^{(1)} \) and \( Y \) that satisfy an equation \( ba \cdot X^{(1)} = cY \).

(iv-1-b) If \( bb \cdot X^{(1)} = cY \), then \( X^{(1)} = b^{n-2-a} \cdot Z \) and \( Y = b^n \cdot Z \) for some positive word \( Z \).

(iv-1-c) If \( bc \cdot X^{(1)} = cY \), then there exist an integer \( k \) (\( 0 \leq k \leq r - n - 1 \)) and a positive word \( Z \) such that \( X^{(1)} = c^k b^{n-1-a} \cdot Z \) and \( Y = a^{k-1} b^{n} \cdot Z \).

If \( n \geq 4 \), then, for \( 2 \leq h \leq n - 2 \), we have to prepare the following propositions

(iv-h-a) (iv-h-b) and (iv-h-c)

(iv-h-a) There does not exist positive words \( X^{(h)} \) and \( Y \) that satisfy an equation \( b^h a \cdot X^{(h)} = cY \).

(iv-h-b) If \( b^{h+1} \cdot X^{(h)} = cY \), then \( X^{(h)} = b^{n-h-1-a} \cdot Z \) and \( Y = b^n \cdot Z \) for some positive word \( Z \).

(iv-h-c) There does not exist positive words \( X^{(h)} \) and \( Y \) that satisfy an equation \( b^h c \cdot X^{(h)} = cY \).

(iv-(n-1)-a) If \( b^{n-1-a} \cdot X^{(n-1)} = cY \), then \( X^{(n-1)} = ba \cdot Z \) and \( Y = b^n \cdot c \cdot Z \) for some positive word \( Z \).

(iv-(n-1)-b) If \( b^a \cdot X^{(n-1)} = cY \), then \( X^{(n-1)} = aZ \) and \( Y = b^n \cdot Z \) for some positive word \( Z \).
(iv.-(n - 1)-c) There does not exist positive words $X^{(n-1)}$ and $Y$ that satisfy an equation $b^{n-1}c \cdot X^{(n-1)} = cY$.

Proof: We will show the general theorem, by referring to the triple induction (see [12]). The theorem for a positive word $Y$ of word-length $r$ and $X^{(h)}$ of word-length $r - h \in \{ r - n + 1, \ldots , r \}$ will be referred to as $H_{r,h}$. It is easy to show that, for $r = 0, 1$, $H_{r,h}$ is true. If a positive word $U_1$ is transformed into $U_2$ by using $t$ single applications of the defining relations of $G_n^+$, then the whole transformation will be said to be of chain-length $t$. For induction hypothesis, we assume

(A) $H_{s,h}$ is true for $s = 0, \ldots , r$ and arbitrary $h$ for transformations of all chain-lengths,

and

(B) $H_{r+1,h}$ is true for $0 \leq h \leq n - 1$ for all chain-lengths $\leq t$.

We will show the theorem $H_{r+1,h}$ for chain-lengths $t + 1$. For the sake of simplicity, we devide the proof into two steps.

**Step 1.** $H_{r+1,h}$ for $h = 0$

Let $X, Y$ be of word-length $r + 1$, and let

$$v_1X \equiv v_2W_2 \equiv \cdots \equiv v_{t+1}W_{t+1} \equiv v_{t+2}Y,$$

be a sequence of single transformations of $t + 1$ steps, where $v_1, \ldots , v_{t+2} \in \{ a, b, c \}$ and $W_2, \ldots , W_{t+1}$ are positive words of length $r + 1$. By the assumption $t > 1$, there exists an index $\tau \in \{ 2, \ldots , t + 1 \}$ such that we can decompose the sequence into two steps

$$v_1X \equiv v_\tau W_\tau \equiv v_{t+2}Y,$$

in which each step satisfies the induction hypothesis (B).

If there exists $\tau$ such that $v_\tau$ is equal to either to $v_1$ or $v_{t+2}$, then by induction hypothesis, $W_{\tau}$ is equivalent either to $X$ or to $Y$. Hence, we obtain the statement for the $v_1X \equiv v_{t+2}Y$. Thus, we assume from now on $v_\tau \neq v_1, v_{t+2}$ for $1 < \tau \leq t + 1$.

Suppose $v_1 = v_{t+2}$. If there exists $\tau$ such that $(v_1 = v_{t+2}, v_\tau) \neq (b, c), (c, b)$, then each of the equivalences says the existence of $\alpha, \beta \in \{ a, b, c \}$ and positive words $Z_1, Z_2$ such that $X \equiv \alpha Z_1$, $W_\tau \equiv \beta Z_1 \equiv \beta Z_2$ and $Y \equiv \alpha Z_2$. Applying the induction hypothesis (A) to $\beta Z_1 \equiv \beta Z_2$, we get $Z_1 \equiv Z_2$. Hence, we obtain the statement $X \equiv \alpha Z_2 \equiv \alpha Z_1 \equiv \alpha Z_2 \equiv Y$. Thus, we exclude these cases from our considerations. Next, we consider the case $(v_1 = v_{t+2}, v_\tau) = (b, c)$. However, because of the above consideration, we say $v_2 \equiv \cdots = v_{t+1} = c$. Hence, we consider the following case

$$bX \equiv cW_1 \equiv \cdots \equiv cW_{t+1} \equiv bY.$$

Applying the induction hypothesis (B) to each step, we say that there exist positive words $Z_3$ and $Z_4$ such that

$$X \equiv b_1^{n-1}a \cdot Z_3$$

and

$$W_{t+1} \equiv b_1^nZ_3,$$

$$W_{t+1} \equiv b_1^n \cdot Z_4,$$

$$Y \equiv b_1^{n-1}a \cdot Z_4.$$

Since an equation $W_1 \equiv W_{t+1}$ holds, we say that

$$b_1^n \cdot Z_3 \equiv b_1^n \cdot Z_4.$$

By induction hypothesis, we have $X \equiv Y$.

In the case of $(v_1 = v_{t+2}, v_\tau) = (c, b)$, we can prove the statement in a similar
manner.
Suppose $v_1 \neq v_{t+2}$. We consider the following three cases.

Case 1: $(v_1, v_{\tau}, v_{t+2}) = (a, b, c)$

Because of the above consideration, we consider the case $\tau = t + 1$, namely

$$aX = bW_{t+1} = cY.$$  

Applying the induction hypothesis to each step, we say that there exist positive words $Z_1$ and $Z_2$ such that

$$X = bZ_1, \quad W_{t+1} = cZ_1,$$

$$W_{t+1} = b^{n-1}a \cdot Z_2, \quad Y = b^n \cdot Z_2.$$  

Thus, we say that $c \cdot Z_1 = b^{n-1}a \cdot Z_2$. Applying the induction hypothesis (A) to this equation, we say that there exists a positive word $Z_3$ such that

$$Z_1 = b^n \cdot Z_3, \quad Z_2 = ba \cdot Z_3.$$  

Hence, we have $X = cb^{n+1} \cdot Z_3$ and $Y = ab^{n+1} \cdot Z_3$.

Case 2: $(v_1, v_{\tau}, v_{t+2}) = (a, c, b)$

We consider the case $\tau = t + 1$, namely

$$aX = cW_{t+1} = bY.$$  

Applying the induction hypothesis to each step, we say that there exist positive words $Z_1$ and $Z_2$ such that

$$X = cZ_1, \quad W_{t+1} = aZ_1,$$

$$W_{t+1} = b^n \cdot Z_2, \quad Y = b^{n-1}a \cdot Z_2.$$  

Thus, we say that $aZ_1 = b^n \cdot Z_2$. Applying the induction hypothesis (A) to this equation, we say that there exists a positive word $Z_3$ such that

$$Z_1 = b^{n+1} \cdot Z_3, \quad Z_2 = ba \cdot Z_3.$$  

Hence, we have $X = b \cdot b^n \cdot Z_3$ and $Y = c \cdot b^n \cdot Z_3$.

Case 3: $(v_1, v_{\tau}, v_{t+2}) = (b, a, c)$

Then, we consider the following case

$$bX = bW_{\tau} = cY.$$  

Applying the induction hypothesis to each step, we say that there exist positive words $Z_1$ and $Z_2$ such that

$$X = cZ_1, \quad W_{\tau} = bZ_1,$$

$$W_{\tau} = cZ_2, \quad Y = aZ_2.$$  

Moreover, we say that there exist a positive word $Z_3$ and an integer $k \in \mathbb{Z}_{\geq 0}$ such that

$$Z_1 = c^k b^{n-1} \cdot Z_3, \quad Z_2 = a^k b^n \cdot Z_3.$$  

Thus, we have

$$X = c^{k+1} b^{n-1}a \cdot Z_3, \quad Y = a^{k+1} b^n \cdot Z_3.$$  

**Step 2.** $H_{r+1,h}$ for $0 \leq h \leq n - 1$

We will show the general theorem $H_{r+1,h}$ by induction on $h$. The case $h = 0$ is proved in Step 1. First, we show the case $h = 1$. Let $X^{(1)}$ be of word-length $r$, and
let $Y$ be of word-length $r+1$. We consider a sequence of single transformations of $t+1$ steps

$$V \cdot X^{(1)} = \cdots = cY,$$

where $V$ is a positive word of length 2. We discuss three cases.

Case 1: $V = ba$.

We consider the following case

(4.1) \[ ba \cdot X^{(1)} = \cdots = cY. \]

By the result of Step 1, we say that there exist a positive word $Z_1$ and an integer $k \in \mathbb{Z}_{\geq 0}$ such that

$$a X^{(1)} = c^k b^{n-1} a \cdot Z_1,\; Y = a^k b^n \cdot Z_1.$$  

Applying the induction hypothesis (A), we say that there exists a positive word $Z_2$ such that

$$X^{(1)} = c^k \cdot Z_2,\; b^{n-1} a \cdot Z_1 = aZ_2.$$  

Moreover, we say that there exists a positive word $Z_3$ such that

$$b^{n-2} a \cdot Z_1 = cZ_3,\; Z_2 = bZ_3.$$  

By the induction hypothesis, we have a contradiction. Hence, there does not exist positive words $X^{(1)}$ and $Y$ that satisfy the equation (4.1).

Case 2: $V = bb$.

We consider the following case

$$bb \cdot X^{(1)} = V_2 \cdot W_2 = \cdots = V_{t+1} \cdot W_{t+1} = cY,$$

where $V_2$ and $V_{t+1}$ are positive words. It is enough to discuss the case $(V_2, V_{t+1}) = (bcb^n, ac)$. Applying the induction hypothesis (A) to the equation

(4.2) \[ bcb^n \cdot W_2 = ac \cdot W_{t+1}, \]

we say that there exists a positive word $Z_1$ such that $cW_{t+1} = bZ_1$. Applying the induction hypothesis, we say that there exist a positive word $Z_2$ and an integer $k \in \mathbb{Z}_{\geq 0}$ such that

(4.3) \[ W_{t+1} = a^k b^n \cdot Z_2,\; Z_1 = c^k b^{n-1} a \cdot Z_2. \]

Applying (4.3) to the equation (4.2), we have

$$bcb^n \cdot W_2 = ac \cdot a^k b^n \cdot Z_2.$$  

Moreover, we say

(4.4) \[ b^n \cdot W_2 = c^k b^{n-1} a \cdot Z_2. \]

We consider the following two cases.

Case 2 - 1: $k = 0$

There exists a positive word $Z_3$ such that

$$W_2 = cZ_3,\; Z_2 = bZ_3.$$  

Thus, we have

$$X^{(1)} = b^{n-1} a \cdot cZ_3 = b^{n-2} a \cdot ba \cdot Z_3,$$

$$Y = ab^n b \cdot Z_3 = b^n \cdot baZ_3.$$
Case 2 – 2: \( k \geq 1 \)

Applying the induction hypothesis to the equation (4.4), we say that there exists a positive word \( Z_3 \) such that

\[ W_2 \equiv a^k \cdot Z_3. \]

Thus, we consider the equation \( b^n \cdot Z_3 \equiv b^{n-1} a \cdot Z_2 \). We say that there exists a positive word \( Z_4 \) such that

\[ Z_2 \equiv bZ_4, \; Z_3 \equiv cZ_4. \]

Thus, we have

\[ X^{(1)} \equiv b^{n-1} a \cdot a^k c \cdot Z_3 \equiv b^{n-2} a \cdot ba^{k+1} Z_3, \]
\[ Y \equiv a a^k b^n b \cdot Z_3 \equiv b^n \cdot ba^{k+1} Z_3. \]

Case 3: \( V = bc \).

Then, we consider the following case

\[ bc \cdot X^{(1)} \equiv \cdots \equiv cY. \]

By the induction hypothesis, we say that there exist a positive word \( Z_1 \) and an integer \( k \in \mathbb{Z} \geq 0 \) such that

\[ cX^{(1)} \equiv c^k b^{n-1} a \cdot Z_1, \; Y \equiv a^k b^n \cdot Z_1. \]

We consider the following two cases.

Case 3 – 1: \( k = 0 \)

By the induction hypothesis, we say that there exists a positive word \( Z_2 \) such that

\[ X^{(1)} \equiv b^n c \cdot Z_2, \; Z_1 \equiv ba \cdot Z_2. \]

Thus, we have

\[ X^{(1)} \equiv b^n a \cdot bZ_2, \; Y \equiv b^n ba \cdot Z_2 \equiv ab^n \cdot bZ_2. \]

Case 3 – 2: \( k \geq 1 \)

Then, we have

\[ X^{(1)} \equiv c^{k-1} b^{n-1} a \cdot Z_1, \; Y \equiv a^k b^n \cdot Z_1. \]

Second, when \( n \geq 4 \), we show the theorem \( H_{r+1,h} \) \((2 \leq h \leq n - 2)\) by induction on \( h \). We assume \( h = 1, 2, \ldots, j \) \((j \leq n - 3)\). The case \( h = 1 \) has been proved. Let \( X^{(j+1)} \) be of word-length \( r - j \), and let \( Y \) be of word-length \( r + 1 \). We consider a sequence of single transformations of \( t + 1 \) steps

\[ (4.5) \quad V \cdot X^{(j+1)} \equiv \cdots \equiv cY, \]

where \( V \) is a positive word of length \( j + 2 \). We discuss the following three cases.

Case 1: \( V \equiv bb'a. \)

Applying the induction hypothesis, we say that there exists a positive word \( Z_1 \) such that

\[ aX^{(j+1)} \equiv b^{n-j-1} a \cdot Z_1, \; Y \equiv b^n \cdot Z_1. \]

By the induction hypothesis, we say that there exists a positive word \( Z_2 \) such that

\[ X^{(j+1)} \equiv bZ_2, \; b^{n-j-2} a \cdot Z_1 \equiv cZ_2. \]

By the induction hypothesis, we have a contradiction. Hence, there does not exist positive words \( X^{(j+1)} \) and \( Y \) that satisfy the equation (4.5).
Case 2: $V = b^{j+1}$.

Applying the induction hypothesis, we say that there exists a positive word $Z_1$ such that

\[ bX^{(j+1)} = b^{n-j-1}a \cdot Z_1, \quad Y = b^n \cdot Z_1. \]

Thus, we have $X^{(j+1)} = b^{n-j-2}a \cdot Z_1$.

Case 3: $V = b^j c$.

Applying the induction hypothesis, we say that there exists a positive word $Z_1$ such that

\[ cX^{(j+1)} = b^{n-j-1}a \cdot Z_1, \quad Y = b^n \cdot Z_1. \]

By the induction hypothesis, we have a contradiction. Hence, there does not exist positive words $X^{(j+1)}$ and $Y$ that satisfy the equation (4.5).

Lastly, we show the theorem $H_{r+1,n-1}$. Let $X^{(n-1)}$ be of word-length $r - n + 2$, and let $Y$ be of word-length $r + 1$. We consider a sequence of single transformations of $t + 1$ steps

\[ V \cdot X^{(n-1)} \equiv \cdots \equiv cY, \]

where $V$ is a positive word of length $n$. We discuss the following three cases.

Case 1: $V = b^{n-1}a$.

By the above result, we say that there exists a positive word $Z_1$ such that

\[ aX^{(n-1)} = ba \cdot Z_1, \quad Y = b^n \cdot Z_1. \]

By the induction hypothesis, we say that there exists a positive word $Z_2$ such that

\[ X^{(n-1)} = ba \cdot Z_2, \quad Z_1 = cZ_2. \]

Thus, we have $Y = b^n \cdot Z_2$.

Case 2: $V = b^{n-1}b$.

By the above result, we say that there exists a positive word $Z_1$ such that

\[ bX^{(n-1)} = ba \cdot Z_1, \quad Y = b^n \cdot Z_1. \]

Thus, we have $X^{(n-1)} = aZ_1$.

Case 3: $V = b^{n-1}c$.

By the above result, we say that there exists a positive word $Z_1$ such that

\[ cX^{(n-1)} = ba \cdot Z_1, \quad Y = b^n \cdot Z_1. \]

We have a contradiction. Hence, there does not exist positive words $X^{(n-1)}$ and $Y$ that satisfy the equation (4.6).

This completes the proof of Theorem 4.1.

Secondly, we show the cancellativity of $H_n^+$.

**Theorem 4.4.** The monoid $H_n^+$ is a cancellative monoid.

**Proof.** First, we remark the following.

**Proposition 4.5.** The left cancellativity on $H_n^+$ implies the right cancellativity.

**Proof.** Consider a map $\varphi : H_n^+ \to H_n^+$, $W \mapsto \varphi(W) := \sigma(\text{rev}(W))$, where $\sigma$ is a permutation $(a,b,c)$. By following the proof in Proposition 4.2, we can show the statement.\]
The following is sufficient to show the left cancellativity of the monoid $H_n^+$. 

**Proposition 4.6.** Let $Y$ be a positive word in $H_n^+$ of length $r \in \mathbb{Z}_{\geq 0}$ and let $X^{(h)}$ be a positive word in $H_n^+$ of length $r - h \in \{2n, \ldots, r\}$.

(i) If $vX^{(0)} = vY$ for some $v \in \{a, b, c\}$, then $X^{(0)} = Y$.

(ii) If $aX^{(0)} = bY$, then $X^{(0)} = bZ, Y = cZ$ for some positive word $Z$.

(iii) If $aX^{(0)} = cY$, then $X^{(0)} = cZ, Y = aZ$ for some positive word $Z$.

(iv) If $bX^{(0)} = cY$, then there exist an integer $k$ $(0 \leq k \leq r - 2n - 2)$ and a positive word $Z$ such that $X^{(0)} = c^k(ab)^nba \cdot Z$ and $Y = a^k(b(ab)^n)^b \cdot Z$.

(v) If $bb \cdot X^{(1)} = cY$, then $X^{(1)} = c(ab)^{n-1}ba \cdot Z, Y = b(ab)^{n-1}b \cdot Z$ for some positive word $Z$.

For $2 \leq h \leq r - 2n$, we prepare the following propositions.

(vi) If $c^{h-1}bb \cdot X^{(h)} = bY$, then $X^{(h)} = c(ab)^nba \cdot Z$ and $Y = (ab)^nba^{h-1} \cdot Z$ for some positive word $Z$.

**Proof.** By referring to the double induction (see [G, BS] for instance), we show the general theorem. The theorem for a positive word $Y$ of word-length $r$ and $X^{(h)}$ of word-length $r - h \in \{r - 2n, \ldots, r\}$ will be referred to as $H_{r-h}$. It is easy to show that, for $r = 0, 1$, $H_{r-h}$ is true. For induction hypothesis, we assume

(A) $H_{s,h}$ is true for $s = 0, \ldots, r$ and arbitrary $h$ for transformations of all chain-lengths,

and

(B) $H_{r+1,h}$ is true for $0 \leq h \leq \max\{0, r+1 - 2n\}$ for all chain-lengths $\leq t$.

We will show the theorem $H_{r+1,h}$ for chain-lengths $t + 1$. For the sake of simplicity, we divide the proof into two steps.

**Step 1.** $H_{r+1,h}$ for $h = 0$

Let $X, Y$ be of word-length $r + 1$, and let

\[ v_1X = v_2W_2 = \cdots = v_{t+1}W_{t+1} = v_{t+2}Y \]

be a sequence of single transformations of $t + 1$ steps, where $v_1, \ldots, v_{t+2} \in \{a, b, c\}$ and $W_2, \ldots, W_{t+1}$ are positive words of length $r + 1$. By the assumption $t > 1$, there exists an index $\tau \in \{2, \ldots, t + 1\}$ such that we can decompose the sequence into two steps

\[ v_1X = v_\tau W_\tau = v_{t+2}Y \]

in which each step satisfies the induction hypothesis (B).

If there exists $\tau$ such that $v_\tau$ is equal to either to $v_1$ or $v_{t+2}$, then by induction hypothesis, $W_\tau$ is equivalent either to $X$ or to $Y$. Hence, we obtain the statement for the $v_1X = v_{t+2}Y$. Thus, we assume from now on $v_\tau \neq v_1, v_{t+2}$ for $1 < \tau \leq t + 1$.

Suppose $v_1 = v_{t+2}$. If there exists $\tau$ such that $(v_1 = v_{t+2}, v_\tau) \neq (b,c), (c,b)$, then each of the equivalences says the existence of $\alpha, \beta \in \{a, b, c\}$ and positive words $Z_1, Z_2$ such that $X = \alpha Z_1, W_\tau = \beta Z_1 = \beta Z_2$ and $Y = \alpha Z_2$. Applying the induction hypothesis (A) to $\beta Z_1 = \beta Z_2$, we get $Z_1 = Z_2$. Hence, we obtain the statement $X = \alpha Z_1 = \alpha Z_2 = Y$. Thus, we exclude these cases from our considerations. Next, we consider the case $(v_1 = v_{t+2}, v_\tau) = (b,c)$. However, because of the above consideration, we say $v_2 = \cdots = v_{t+1} = c$. Hence, we consider the following case

\[ bX = cW_1 = \cdots = cW_{t+1} = bY. \]
Applying the induction hypothesis (B) to each step, we say that there exist positive words $Z_3$ and $Z_4$ such that

$$X = (ab)^n ba \cdot Z_3, \quad W_1 = b(ab)^n b \cdot Z_3, \quad W_{t+1} = b(ab)^n b \cdot Z_4, \quad Y = (ab)^n ba \cdot Z_4.$$

Since an equation $W_1 = W_{t+1}$ holds, we say that $X = Y$.

In the case of $(v_1 = v_{t+2}, v_1) = (c; b)$, we can prove the statement in a similar manner.

Suppose $v_1 \neq v_{t+2}$. It suffices to consider the following two cases.

Case 1: $(v_1, v_r, v_{t+2}) = (a, b, c)$

Because of the above consideration, we consider the case $\tau = t + 1$, namely

$$aX = bW_{t+1} = cY.$$

Applying the induction hypothesis to each step, we say that there exist positive words $Z_1$ and $Z_2$ such that

$$X = bZ_1, \quad W_{t+1} = cZ_1, \quad W_{t+1} = (ab)^n ba \cdot Z_2, \quad Y = b(ab)^n b \cdot Z_2.$$

Thus, we say that $cZ_1 = (ab)^n ba \cdot Z_2$. Applying the induction hypothesis (A) to this equation, we say that there exists a positive word $Z_3$ such that

$$Z_1 = aZ_3, \quad b(ab)^n ba \cdot Z_2 = cZ_3.$$

Hence, we have $abc(ab)^n ba \cdot Z_2 = cZ_3$. Applying the induction hypothesis (A) to this equation, there exists a positive word $Z_4$ such that

$$c(ab)^n ba \cdot Z_2 = (ab)^b b \cdot Z_4, \quad Z_3 = b(ab)^n b \cdot Z_4.$$

Hence, we have $ba \cdot Z_2 = abba \cdot Z_4$. Moreover, we say that there exists a positive word $Z_5$ such that

$$Z_2 = cba \cdot Z_5, \quad Z_4 = cZ_5.$$

Thus, we have

$$X = bab(ab)^n bc \cdot Z_5 = c \cdot b(ab)^n bcb \cdot Z_5, \quad Y = b(ab)^n bcb \cdot Z_5 = a \cdot b(ab)^n bcb \cdot Z_5.$$

Case 2: $(v_1, v_r, v_{t+2}) = (a, c, b)$

We consider the case $\tau = t + 1$, namely

$$aX = cW_{t+1} = bY.$$

Applying the induction hypothesis to each step, we say that there exist positive words $Z_1$ and $Z_2$ such that

$$X = cZ_1, \quad W_{t+1} = aZ_1, \quad W_{t+1} = b(ab)^n b \cdot Z_2, \quad Y = (ab)^n ba \cdot Z_2.$$

Thus, we say that $aZ_1 = b(ab)^n b \cdot Z_2$. Applying the induction hypothesis (A) to this equation, we say that there exists a positive word $Z_3$ such that

$$Z_1 = bZ_3, \quad (ab)^n b \cdot Z_2 = cZ_3.$$

Hence, there exists a positive word $Z_4$ such that

$$b(ab)^n b \cdot Z_2 = cZ_4, \quad Z_3 = aZ_4.$$
We have \( bbc(ab)^{n-2}b \cdot Z_2 \equiv cZ_4 \). Applying the induction hypothesis (A) to this equation, we say that there exists a positive word \( Z_5 \) such that

\[
c(ab)^{n-2}b \cdot Z_2 \equiv c(ab)^{n-1}ba \cdot Z_5, \quad Z_4 \equiv b(ab)^n b \cdot Z_5.
\]

Hence, we have \( Z_2 \equiv cba \cdot Z_5 \). Thus, we have

\[
X \equiv cb(ab)^n b \cdot Z_5 \equiv b(ab)^n bacb \cdot Z_5,
\]

\[
Y \equiv (ab)^n bacb \cdot Z_5 \equiv c(ab)^n bacb \cdot Z_5.
\]

**Step 2.** \( H_{r+1,h} \) for \( 1 \leq h \leq r + 1 - 2n \)

We will show the general theorem \( H_{r+1,h} \). First, we show the case \( h = 1 \). Then, we consider the following case

\[
bb \cdot X^{(1)} \equiv \cdots \equiv cY.
\]

By the result of Step 1, we say that there exist a positive word \( Z_1 \) and an integer \( k \in \mathbb{Z}_{\geq 0} \) such that

\[
bX^{(1)} \equiv c^k(ab)^n ba \cdot Z_1, \quad Y \equiv a^k b(ab)^n b \cdot Z_1.
\]

Thus, we have \( bX^{(1)} \equiv ac^k b(ab)^{n-1} ba \cdot Z_1 \). Applying the induction hypothesis (A), we say that there exists a positive word \( Z_2 \) such that

\[
X^{(1)} \equiv cZ_2, \quad bZ_2 \equiv c^k b(ab)^{n-1} ba \cdot Z_1 \equiv c^{k+1} b(ab)^{n-2} ba \cdot Z_1.
\]

We consider the case \( k \geq 1 \). By the induction hypothesis, we say that there exists a positive word \( Z_3 \) such that

\[
Z_2 \equiv (ab)^n ba^k \cdot Z_3, \quad c(ab)^{n-2} ba \cdot Z_1 \equiv c(ab)^{n-1} b \cdot Z_3.
\]

Hence, we have \( ba \cdot Z_1 \equiv abb \cdot Z_3 \). Then, we have \( aZ_1 \equiv cb \cdot Z_3 \). By the induction hypothesis, there exists a positive word \( Z_4 \) such that

\[
Z_1 \equiv cb \cdot Z_4, \quad Z_3 \equiv cZ_4.
\]

Thus, we have

\[
X^{(1)} \equiv c(ab)^n ba^k c \cdot Z_4 \equiv c(ab)^{n-1} ba \cdot cba^k \cdot Z_4,
\]

\[
Y \equiv a^k b(ab)^n bcb \cdot Z_4 \equiv b(ab)^n b \cdot cba^k \cdot Z_4.
\]

Next, we consider the case \( 2 \leq k \leq r + 1 - 2n \). We consider the following case

\[
(4.7) \quad c^{h-1}bb \cdot X^{(h)} \equiv \cdots \equiv bY.
\]

By the result of Step 1, we say that there exist a positive word \( Z_1 \) and an integer \( k_1 \in \mathbb{Z}_{\geq 0} \) such that

\[
c^{h-2} bb \cdot X^{(h)} \equiv a^{k_1} b(ab)^n b \cdot Z_1, \quad Y \equiv c^{k_1} (ab)^n ba \cdot Z_1.
\]

By repeating the same process \( h - 1 \) times, there exist integers \( k_2, \ldots, k_{h-1} \in \mathbb{Z}_{\geq 0} \) and positive word \( Z_{h-1} \) such that

\[
bb \cdot X^{(h)} \equiv a^{k_{h-1}} \cdot b(ab)^n b \cdot Z_{h-1}.
\]

Then, we have \( b \cdot X^{(h)} = c^{k_{h-1}} \cdot (ab)^n b \cdot Z_{h-1} \equiv ac^{k_{h-1}} \cdot b(ab)^{n-1} b \cdot Z_{h-1} \). By the induction hypothesis, there exists a positive word \( Z_h \) such that

\[
X^{(h)} = cZ_h, \quad c^{k_{h-1}} \cdot b(ab)^{n-1} b \cdot Z_{h-1} \equiv bZ_h.
\]
Hence, we have \( bZ_h \equiv c^{k_h-1} \cdot babc(ab)^{n-2}b \cdot Z_{h-1} \). By the induction hypothesis, there exists a positive word \( Z_0 \) such that

\[
c(ab)^{n-2}b \cdot Z_{h-1} \equiv c(ab)^{n-1}b \cdot Z_0, \quad Z_h \equiv (ab)^n ba^{k_h-1} \cdot Z_0.
\]

Thus, we have \( bZ_{h-1} \equiv abb \cdot Z_0 \). We have \( Z_{h-1} \equiv cb \cdot Z_0 \). Then, we have

\[
X^{(h)} \equiv c(ab)^n ba^{k_h-1} \cdot Z_0 \equiv c(ab)^{n-1}b \cdot cbab^{k_h-1} \cdot Z_0.
\]

Applying this result to (4.7), we have

\[
bY \equiv c^{h-1}bb \cdot c(ab)^{n-1}b \cdot cbab^{k_h-1} \cdot Z_0 \equiv b(ab)^n ba^{h-1} \cdot cbab^{k_h-1} \cdot Z_0.
\]

Hence, we have \( Y \equiv (ab)^n ba^{h-1} \cdot cbab^{k_h-1} \cdot Z_0 \).

This completes the proof of Theorem 4.4.

5. Calculations of the skew growth functions

In this section, we will calculate the skew growth functions for the monoids \( G_B^+ \), \( G_n^+ \), \( H_n^+ \) and \( M_{ab, m} \).

First, we present an explicit calculation of the skew growth function for the monoid \( G_B^+ \). In [11], we have made a success in calculating the growth function \( P_{G_B^+} \). By the inversion formula, we can calculate the skew growth function \( N_{G_B^+, \deg} \). Nevertheless, we present an explicit calculation, because, in spite of the fact that the monoid is non-abelian and the height of it is infinite, we succeed in the non-trivial calculation.

**Example. 1.** First of all, we recall a fact from [5] §7.

**Lemma 5.1.** Let \( X \) and \( Y \) be positive words in \( G_B^+ \) of length \( r \in \mathbb{Z}_{\geq 0} \).

(i) If \( vX \equiv vY \) for some \( v \in \{a, b, c\} \), then \( X \equiv Y \).

(ii) If \( aX \equiv bY \), then \( X \equiv bZ \), \( Y \equiv cZ \) for some positive word \( Z \).

(iii) If \( aX \equiv cY \), then \( X \equiv cZ \), \( Y \equiv aZ \) for some positive word \( Z \).

(iv) If \( bX \equiv cY \), then there exist an integer \( k \in \mathbb{Z}_{\geq 0} \) and a positive word \( Z \) such that \( X \equiv c^k ba \cdot Z \), \( Y \equiv a^k bb \cdot Z \).

Thanks to the Lemma 5.1, we have proved the cancellativity in [5]. And we prove the following Lemma.

**Lemma 5.2.** If an equation \( bb \cdot X \equiv cY \) in \( G_B^+ \) holds, then \( X \equiv aZ \), \( Y \equiv bb \cdot Z \) for some positive word \( Z \).

**Proof.** Due to the Lemma 5.1, we say that there exist an integer \( k \in \mathbb{Z}_{\geq 0} \) and a positive word \( Z_0 \) such that

\[
bX \equiv c^k ba \cdot Z_0, \quad Y \equiv a^k bb \cdot Z_0.
\]

We consider the case \( k \geq 1 \). Due to the Lemma 5.1, we say that there exist an integer \( i_1 \in \mathbb{Z}_{\geq 0} \) and a positive word \( Z_1 \) such that

\[
X \equiv c^{i_1} ba \cdot Z_1, \quad c^{k-1} ba \cdot Z_0 \equiv a^{i_1} bb \cdot Z_1.
\]

Moreover, we say that there exists a positive word \( Z_0^{(1)} \) such that

\[
Z_0 \equiv c^{i_1} \cdot Z_0^{(1)}, \quad c^{k-1} ba \cdot Z_0^{(1)} \equiv bb \cdot Z_1.
\]
Repeating the same process $k$-times, there exist integers $i_2, \ldots, i_k \in \mathbb{Z}_{\geq 0}$ and positive words $Z_0^{(k)}$ and $Z_k$ such that

$$Z_0 = c^{i_1+i_2+\cdots+i_k} \cdot Z_0^{(k)}, \quad ba \cdot Z_0^{(k)} = bb \cdot Z_k.$$  

Moreover, we say that there exists a positive word $Z'$ such that

$$Z_0^{(k)} = bZ', \quad Z_k = cZ'.$$

Applying this result to (5.1), we have

$$bX = c^k ba c^{i_1+i_2+\cdots+i_k} \cdot Z', \quad Y = a^k bounding.$$

Thus, we have $X = a \cdot c^{i_1+i_2+\cdots+i_k} ba^k \cdot Z'$. \hfill \Box

As a consequence of Lemma 5.2, we obtain the followings.

Corollary 5.3. If an equation $bb \cdot X = c^l \cdot Y$ in $G_{B_1}^+$ holds for some positive integer $l$, then $X = a^l \cdot Z$, $Y = bb \cdot Z$ for some positive word $Z$.

Due to the Corollary 5.3, we can solve the following equation.

Proposition 5.4. If, for $0 \leq i < j$, an equation $c^i b \cdot X = c^j b \cdot Y$ in $G_{B_1}^+$ holds, then there exist an integer $k \in \mathbb{Z}_{\geq 0}$ and a positive word $Z$ such that

$$X = c^k ba^{j-i} \cdot Z, \quad Y = c^k b \cdot Z.$$

Proof. Due to the cancellativity, we show $c^j b \cdot X = c^j b \cdot Y \iff bX = c^{j-i} b \cdot Y$. Thanks to the Lemma 5.1, we say that there exists an integer $k \in \mathbb{Z}_{\geq 0}$ and a positive word $Z_1$ such that

$$X = c^k ba \cdot Z_1, \quad c^{j-i} b \cdot Y = a^k bb \cdot Z_1.$$

Moreover, we say that there exist $Y'$

$$Y = c^k \cdot Y', \quad c^{j-i-1} b \cdot Y' = bb \cdot Z_1.$$

Due to the Corollary 5.3, there exists a positive word $Z_2$ such that

$$bY' = bb \cdot Z_2, \quad Z_1 = a^{j-1} \cdot Z_2.$$

Thus, we have

$$X = c^k ba^{j-i} \cdot Z_2, \quad Y = c^k b \cdot Z_2.$$ \hfill \Box

As a corollary of the Proposition 5.4, we show the following lemma.

Lemma 5.5. For $0 \leq \kappa_1 < \kappa_2 < \cdots < \kappa_m$,\n
$$\text{mcm}\{c^{\kappa_1} b, c^{\kappa_2} b, \ldots, c^{\kappa_m} b\} = \{c^{\kappa m} b \cdot c^k b \mid k = 0, 1, \ldots\}.$$\n
By using the Lemma 5.5, we easily show the following.

Proposition 5.6. We have $h(G_{B_1}^+, \text{deg}) = \infty$.

Proof. Due to the Proposition 5.1, we show

$$\text{mcm}\{b, c\} = \{cb \cdot c^k b \mid k = 0, 1, \ldots\}.$$\n
Due to the Lemma 5.1, for $0 \leq \kappa_1 < \kappa_2 < \cdots < \kappa_m$, we say

$$\text{mcm}\{cb \cdot c^{\kappa_1} b, cb \cdot c^{\kappa_2} b, \ldots, cb \cdot c^{\kappa_m} b\} = \{cb \cdot c^{\kappa m} b \cdot c^k b \mid k = 0, 1, \ldots\}.$$\n
By using the Lemma 5.5 repeatedly, we show $h(G_{B_1}^+, \text{deg}) = \infty$. \hfill \Box
By using the Lemma 5.5, we calculate the skew growth function. We have to consider four cases \( J_1 = \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\} \). The set \( \text{Tmcm}(G^+_\mathbb{B}_n, J_1) \) denotes the set of all the towers starting from a fixed \( J_1 \). If \( J_1 = \{a, b\}, \{a, c\} \), due to the Lemma 5.1, then \( \text{mcm}(\{a, b\}) \) and \( \text{mcm}(\{a, c\}) \) consist of only one element, respectively. Next, we consider the case \( J_1 = \{b, c\} \). For a fixed tower \( T \), if there exists an element \( \Delta \in |T| \) such that \( \deg(\Delta) = l + 2 \), then, from the Lemma 5.5, we say the uniqueness. For any fixed \( l \in \mathbb{Z}_{>0} \), we calculate the coefficient of the term \( t^{l+2} \) which is denoted by \( a_l \), by counting all the signs \( (-1)^{\#J_1+\cdots+\#J_n-n+1} \) in the definition (3.1) associated with the towers \( T = (I_0, J_1, J_2, \cdots, J_n) \) for which \( \deg(\Delta) \) can take a value \( l + 2 \). In order to calculate the \( a_l \), we consider the set

\[
\mathcal{T}^l_{G^+_{\mathbb{B}_n}} := \{ T \in \text{Tmcm}(G^+_\mathbb{B}_n, J_1) \mid \Delta \in |T| \text{ s.t. } \deg(\Delta) = l + 2 \}.
\]

By using the Lemma 5.5 repeatedly, we show

\[
\max\{\text{height of } T \in \mathcal{T}^l_{G^+_{\mathbb{B}_n}} \} = [(l + 1)/2].
\]

For \( u \in \{1, \ldots, [(l + 1)/2]\} \), we define the set

\[
\mathcal{T}^l_{G^+_{\mathbb{B}_n}, u} := \{ T \in \text{Tmcm}(G^+_\mathbb{B}_n, J_1) \mid \text{height of } T = u, \Delta \in |T| \text{ s.t. } \deg(\Delta) = l + 2 \}.
\]

From here, we write \( \mathcal{T}^l_{G^+_{\mathbb{B}_n}} \) (resp. \( \mathcal{T}^l_{G^+_{\mathbb{B}_n}, u} \)) simply by \( \mathcal{T}^l \) (resp. \( \mathcal{T}^l_u \)). Thus, we have the decomposition:

\[
(5.2) \quad \mathcal{T}^l = \bigcup_u \mathcal{T}^l_u.
\]

**Claim 1.** For any \( u \), we show the following equality

\[
(-1)^{u-1} l-u C_{u-1} = \sum_{T \in \mathcal{T}^l_u} (-1)^{\#J_1+\cdots+\#J_n-u+1}.
\]

**Proof.** For the case of \( u = 1 \), the equality holds. For the case of \( u = 2 \), we calculate the sum \( \sum_{T \in \mathcal{T}^l_2} (-1)^{\#J_2-1} \). By indices \( 0 \leq \kappa_1 < \kappa_2 < \cdots < \kappa_m \), the set \( J_2 \) is generally written by \( \{ cb \cdot c^{\kappa_1} b, cb \cdot c^{\kappa_2} b, \ldots, cb \cdot c^{\kappa_m} b \} \). Due to the Lemma 5.5, we show that the maximum index \( \kappa_m \) can range from 1 to \( l - 2 \). For a fixed index \( \kappa_m = \kappa \in \{1, \ldots, l-2\} \), we easily show

\[
\sum_{T \in \mathcal{T}^l_2 : \kappa_m = \kappa} (-1)^{\#J_2-1} = -1.
\]

Therefore, we show that the sum \( \sum_{T \in \mathcal{T}^l_2} (-1)^{\#J_2-1} = -(l - 2) = -(l-2)C_{2-1} \). We show the case for \( 3 \leq u \leq [(l + 1)/2] \) by induction on \( u \). We assume the case \( u = j \). For the case of \( u = j + 1 \), we focus our attention to the set \( J_2 \). If we write the set \( J_2 \) by \( \{ cb \cdot c^{\kappa_1} b, cb \cdot c^{\kappa_2} b, \ldots, cb \cdot c^{\kappa_m} b \} \), due to the the Lemma 5.5, we show that the maximum index \( \kappa_m \) can range from 1 to \( l - 2j \). By induction hypothesis, it suffices to show the following equality

\[
\sum_{k=1}^{l-2j} l-j-k-1 C_{j-1} = l-j-1 C_j.
\]

Therefore, we have shown the case \( u = j + 1 \). This completes the proof. \( \square \)
By the decomposition (5.2), we show the following equality.

**Claim 2.** \( a_l = \sum_{k=0}^{(l-1)/2} (-1)^k l_{-k-1} C_k. \)

Then, we easily show the following.

**Claim 3.** \( a_{l+2} - a_{l+1} + a_l = 0. \)

**Proof.** Since an equality \( n + 1 C_k - n C_k = n C_{k-1} \) holds, we can show our statement. \( \Box \)

We easily show \( a_1 = a_2 = 1 \). Hence, the sequence \( \{a_l\}_{l=1}^\infty \) has a period 6. Lastly, we consider the case \( J_1 = \{a, b, c\} \). For any fixed \( l \in \mathbb{Z}_{>0} \), we calculate the coefficient of the term \( t^{l+3} \) which is denoted by \( b_l \). Since \( \text{mcm}(\{a, b, c\}) = \{cb \cdot cb \mid k = 1, 2, \ldots \} \), we can reuse the Lemma 5.5. In a similar manner, we have the following conclusion.

**Claim 4.** \( b_{l+2} - b_{l+1} + b_l = 0. \)

Since \( b_1 = b_2 = 1 \), we also show that the sequence \( \{b_l\}_{l=1}^\infty \) has a period 6. After all, we can calculate the skew growth function for the monoid \( G_{b_l}^+ \):

\[
N_{G_{b_l}^+}^{\text{deg}}(t) = 1 - 3t + 2t^2 + \frac{t^3}{1 - t + t^2} - \frac{t^4}{1 - t + t^2} = \frac{(1 - t)^4}{1 - t + t^2}.
\]

Secondly, we present an explicit calculation of the skew growth function for the monoid \( G_n^+ \).

**Example. 2.** First of all, we show the following proposition.

**Proposition 5.7.** If, for \( 0 < i < j \), an equation \( c^i b^{n-1} \cdot X = c^j b^{n-1} \cdot Y \) in \( G_n^+ \) holds, then there exists a positive word \( Z \) such that

\[
X \equiv b a^{j-i} \cdot Z, \quad Y \equiv bZ.
\]

**Proof.** Since we have shown the cancellativity in §4, we show \( c^i b^{n-1} \cdot X = c^j b^{n-1} \cdot Y \Leftrightarrow b^{n-1} \cdot X = c^{j-i} b^{n-1} \cdot Y \). Thanks to the Proposition 4.3(iv - (n - 2) - b), we say that there exists a positive word \( Z \) such that

\[
X \equiv b a^{j-i} \cdot Z, \quad Y \equiv bZ.
\]

\( \Box \)

As a corollary of the Proposition 5.7, we show the following lemma.

**Lemma 5.8.** For \( 0 \leq \kappa_1 < \kappa_2 < \cdots < \kappa_m \),

\[
\text{mcm}(\{c^{\kappa_1} b^{n-1}, c^{\kappa_2} b^{n-1}, \ldots, c^{\kappa_m} b^{n-1}\}) = \{c^{\kappa_m} b^n\}
\]

Thus, we obtain the following proposition.

**Proposition 5.9.** We have \( h(G_{b_l}^+, \text{deg}) = 2. \)

By using the Lemma 5.8, we calculate the skew growth function. We have to consider four cases \( J_1 = \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\} \). The set \( \text{Tmcm}(G_{b_l}^+, J_1) \) denotes the set of all the towers starting from a fixed \( J_1 \). If \( J_1 = \{a, b\}, \{a, c\} \), due to the Proposition 4.8, then \( \text{mcm}(\{a, b\}) \) and \( \text{mcm}(\{a, c\}) \) consist of only one element, respectively. Next, we consider the case \( J_1 = \{b, c\} \). For any fixed \( l \in \mathbb{Z}_{>0} \), we calculate the coefficient of the term \( t^{n+l} \) which is denoted by \( c_l \). In order to calculate the \( c_l \), we consider the set

\[
T^l_{G_n^+} := \{ T \in \text{Tmcm}(G_n^+, J_1) \mid \Delta \in |T| \text{ s.t. } \deg(\Delta) = n + l \}.
\]
For \( u \in \{1, 2\} \), we define the set
\[
T_{G_n^+, u}^l := \{ T \in \text{Mcm}(G_n^+, J_1) \mid \text{height of } T = u, \Delta \in |T| \text{ such that } \Delta = n + l \}.
\]
Since \( \text{mcm}(\{b, c\}) = \{ cb \cdot c^k b^{n-1} \mid k = 0, 1, \ldots \} \), we easily show \( c_1 = c_2 = 1 \).
Moreover, we show the following.

**Proposition 5.10.** We have \( c_l = 0 \) (\( l = 3, 4, \ldots \)).

**Proof.** From the consideration in Claim 1 of Example 1, for \( u = 2 \), we also show
\[
\sum_{T \in T_{G_n^+, u}^l} (-1)^{\#J_1 + \cdots + \#J_{u-1} + 1} = -1.
\]

Thus, we have \( c_l = 0 \) (\( l = 3, 4, \ldots \)). \( \square \)

Lastly, we consider the case \( J_1 = \{a, b, c\} \). For any fixed \( l \in \mathbb{Z}_{>0} \), we calculate the coefficient of the term \( t^{n+l+1} \) which is denoted by \( d_l \). In a similar way, we show \( d_1 = d_2 = 1 \) and \( d_l = 0 \) (\( l = 3, 4, \ldots \)). After all, we calculate the skew growth function for the monoid \( G_n^+ \):
\[
N_{G_n^+, \text{deg}}(t) = 1 - 3t + 2t^2 + (t^{n+1} + t^{n+2}) - (t^{n+2} + t^{n+3}) = (1-t)(t^{n+2} + t^{n+1} - 2t + 1).
\]

**Remark 6.** By the inversion formula, we can calculate the growth function \( P_{G_n^+, \text{deg}}(t) \).
As far as we know, it is difficult to calculate \( P_{G_n^+, \text{deg}}(t) \) directly.

Thirdly, we present an explicit calculation of the skew growth function for the monoid \( H_n^+ \).

**Example 3.** First of all, we show the following proposition.

**Proposition 5.11.** If, for \( 0 \leq i < j \), an equation \( c^i b(ab)^{n-1} ba \cdot X = c^j b(ab)^{n-1} ba \cdot Y \) in \( H_n^+ \) holds, then there exists a positive word \( Z \) such that
\[
X = cba^{j-i} \cdot Z, \quad Y = cb \cdot Z.
\]

**Proof.** Since we have shown the cancellativity in \( \S 4 \), we show \( c^i b(ab)^{n-1} ba \cdot X = c^j b(ab)^{n-1} ba \cdot Y \Leftrightarrow b(ab)^{n-1} ba \cdot X = c^{j-i} b(ab)^{n-1} ba \cdot Y \). Thanks to the Proposition 4.6 (vi - h), we say that there exists a positive word \( Z_1 \) such that
\[
(ab)^{n-1} ba \cdot X = (ab)^{n-1} ba \cdot Z_1, \quad c(ab)^{n-2} ba \cdot Y = c(ab)^{n-1} b \cdot Z_1.
\]
Therefore, we say that there exists a positive word \( Z_2 \) such that
\[
X = cba^{j-i} \cdot Z_2, \quad Y = cb \cdot Z_2.
\]
\( \square \)

As a corollary of the Proposition 5.11, we show the following lemma.

**Lemma 5.12.** For \( 0 \leq \kappa_1 < \kappa_2 < \cdots < \kappa_m \),
\[
\text{mcm}(\{ c^{\kappa_1} b(ab)^{n-1} ba, c^{\kappa_2} b(ab)^{n-1} ba, \ldots, c^{\kappa_m} b(ab)^{n-1} ba \}) = \{ c^{\kappa_m} b(ab)^{n-1} bab \}
\]
Thus, we obtain the following proposition.

**Proposition 5.13.** We have \( h(H_n^+, \text{deg}) = 2 \).
Thanks to the Lemma 5.12, we can calculate the skew growth function. We have to consider four cases \(J_1 = \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\). The set \(\operatorname{mcm}(H^+_n, J_1)\) denotes the set of all the towers starting from a fixed \(J_1\). If \(J_1 = \{a, b\}, \{a, c\}\), due to the Proposition 4.6, then \(\operatorname{mcm}(\{a, b\})\) and \(\operatorname{mcm}(\{a, c\})\) consist of only one element, respectively. Next, we consider the case \(J_1 = \{b, c\}\). For any fixed \(l \in \mathbb{Z}_{>0}\), we calculate the coefficient of the term \(t^{2n+3+l}\) which is denoted by \(e_l\). In order to calculate the \(e_l\), we consider the set

\[
T^t_{H^+_n} := \{ T \in \operatorname{mcm}(H^+_n, J_1) \mid \Delta \in |T| \text{ s.t. } \deg(\Delta) = 2n + 3 + t \}.
\]

For \(u \in \{1, 2\}\), we define the set

\[
T^t_{H^+_n, u} := \{ T \in \operatorname{mcm}(H^+_n, J_1) \mid \text{ height of } T = u, \Delta \in |T| \text{ s.t. } \deg(\Delta) = 2n + 3 + t \}.
\]

Since \(\operatorname{mcm}(\{b, c\}) = \{bc^n(ab)^nba \mid k = 0, 1, \ldots\}\), we easily show \(e_1 = e_2 = e_3 = 1\). Moreover, we show the following.

**Proposition 5.14.** We have \(e_l = 0 (l = 4, 5, \ldots)\).

**Proof.** From the consideration in Claim 1 of Example 1, for \(u = 2\), we also show

\[
\sum_{T \in T^t_{H^+_n, u}} (-1)^{#J_1 + \cdots + #J_u - u + 1} = -1.
\]

Thus, we have \(e_l = 0 (l = 4, 5, \ldots)\). \(\square\)

Lastly, we consider the case \(J_1 = \{a, b, c\}\). For any fixed \(l \in \mathbb{Z}_{>0}\), we calculate the coefficient of the term \(t^{2n+4+l}\) which is denoted by \(f_l\). In a similar way, we show \(f_1 = f_2 = f_3 = 1\) and \(f_1 = 0 (l = 4, 5, \ldots)\). After all, we calculate the skew growth function for the monoid \(H^+_n\):

\[
N_{H^+_n, \deg}(t) = 1 - 3t + 2t^2 + (t^{2n+3} + t^{2n+4} + t^{2n+5} - (t^{2n+4} + t^{2n+5} + t^{2n+6})
\]

\[
= (1 - t)(t^{2n+5} + t^{2n+4} + t^{2n+3} - 2t + 1).
\]

**Remark 7.** By the inversion formula, we can calculate the growth function \(P_{H^+_n, \deg}(t)\).

As far as we know, it is difficult to calculate \(P_{H^+_n, \deg}(t)\) directly.

Lastly, we calculate the skew growth function for the monoid \(M_{\text{abel}, m}\).

**Example 4.** First of all, we easily show the following proposition.

**Proposition 5.15.** Let \(X\) and \(Y\) be positive words in \(M_{\text{abel}, m}\) of length \(r \in \mathbb{Z}_{\geq 0}\).

(i) If \(vX \equiv vY\) for some \(v \in \{a, b\}\), then \(X \equiv Y\).

(ii) If \(aX \equiv bY\), then either \(X \equiv a^{m-1}, Z_1, Y \equiv b^{m-1}, Z_1\) for some positive word \(Z_1\) or \(X \equiv bZ_2, Y \equiv aZ_2\) for some positive word \(Z_2\).

**Lemma 5.16.** There exists a unique tower \(T_n = (I_0, J_1, J_2, \ldots, J_n)\) of height \(n \in \mathbb{Z}_{>0}\) with the ground set \(I_0 = \{a, b\}\) such that

\[
J_{2k-1} = \{a^{(k-1)m+1}, a^{((k-1)m+1)l}\} (k = 1, \ldots, [(n + 1)/2]),
\]

\[
J_{2k} = \{a^{kn}, a^{(km+1)l}\} (k = 1, \ldots, [n/2]).
\]

**Proof.** We easily show \(J_1 = \{a, b\}\) and \(J_0 = \{a^m, ab\}\). Thanks to the Proposition 5.15, we show our statement by induction on \(k\). \(\square\)
Therefore, we immediately show \( h(M_{\text{abel},m}, \deg) = \infty \). And, from the definition (3.1), we can calculate the skew growth function
\[
N_{M_{\text{abel},m}, \deg}(t) = (1 - 2t + t^2)(1 + t^m + t^{2m} + \cdots) = \frac{(1 - t)^2}{1 - t^m}.
\]

6. Appendix

In this section, we present three examples that suggest the relationship between the form of the spherical growth function for a monoid \( \langle L \mid R \rangle_{mo} \) and properties of the corresponding group \( \langle L \mid R \rangle \). For the three examples, by observing the distribution of the zeroes of the denominator polynomials of the growth functions for them, we conjecture that the corresponding groups contain free abelian subgroups of finite index.

**Example. 1.** We recall an example, the monoid \( G_{B_{ii}}^+ \), from [II]. By using the normal form of the monoid \( G_{B_{ii}}^+ \), the author has calculated the spherical growth function \( P_{G_{B_{ii}}^+, \deg}(t) \) for the monoid. The spherical growth function \( P_{G_{B_{ii}}^+, \deg}(t) \) can be expressed as a rational function \( \frac{1 - t + t^2}{(1 - t)^3} \). Since the zeroes of the denominator polynomial of \( P_{G_{B_{ii}}^+, \deg}(t) \) only consists of 1 with multiplicity 4, it is conjectured that the corresponding group \( G_{B_{ii}} \) contains a free abelian subgroup of rank 4 of finite index. Indeed, the author has shown the following.

**Proposition 6.1.** The following (i), (ii) and (iii) hold.

(i) The group \( G_{B_{ii}} \) contains a subgroup of index three isomorphic to \( \mathbb{Z}^4 \).
(ii) The group \( G_{B_{ii}} \) has a polynomial growth rate.
(iii) The group \( G_{B_{ii}} \) is a solvable group.

**Example. 2.** We consider the following example
\[
\left\langle a, b, c \mid \begin{array}{l}
\text{cb = ba,} \\
\text{ab = bc,} \\
\text{ac = ca}
\end{array} \right\rangle_{mo}.
\]

We easily show the cancellativity of it by referring to the double induction (see [I]). The spherical growth function can be expressed as a rational function \( \frac{1 - t + t^2}{(1 - t)^3} \). It is also conjectured that the corresponding group contains a free abelian subgroup of rank 3 of finite index. Indeed, we can show that corresponding group contains a subgroup of index two isomorphic to \( \mathbb{Z}^3 \). Moreover, we show that the group has a polynomial growth rate and is a solvable group.

**Example. 3.** We consider the following example
\[
\left\langle a, b, c, d \mid \begin{array}{l}
\text{ab = bc, ac = ca,} \\
\text{cb = ba, bd = db,} \\
\text{ad = dc, cd = da}
\end{array} \right\rangle_{mo}.
\]

We easily show the cancellativity of it by referring to the double induction (see [I]). The spherical growth function can be expressed as a rational function \( \frac{1 - t + t^2}{(1 - t)^4} \). It is also conjectured that the corresponding group contains a free abelian subgroup of rank 4 of finite index. Indeed, we can show that corresponding group contains a subgroup of index three isomorphic to \( \mathbb{Z}^4 \). Moreover, we show that the group has a polynomial growth rate and is a solvable group.
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