An Alternative Basis for the Wigner-Racah Algebra of the Group SU(2)

M. KIBLER\(^1\) and M. DAOUD\(^2\)

\(^1\)Institut de Physique Nucléaire de Lyon
IN2P3-CNRS et Université Claude Bernard
43 bd du 11 novembre 1918, 69622 Villeurbanne Cedex, France

\(^2\)Laboratoire de Physique Théorique
Université Mohammed V
Avenue Ibn Batouta, B.P. 1014, Rabat, Morocco

Abstract

The Lie algebra of the classical group SU(2) is constructed from two quon algebras for which the deformation parameter is a common root of unity. This construction leads to (i) a (not very well-known) polar decomposition of the generators \(J_-\) and \(J_+\) of the SU(2) Lie algebra and to (ii) an alternative to the \(\{J^2, J_3\}\) quantization scheme, viz., the \(\{J^2, U_r\}\) quantization scheme. The key ideas for developing the Wigner-Racah algebra of the group SU(2) in the \(\{J^2, U_r\}\) scheme are given. In particular, some properties of the coupling and recoupling coefficients as well as the Wigner-Eckart theorem in the \(\{J^2, U_r\}\) scheme are briefly discussed.

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1 Motivations and Introduction

In recent years, intermediate statistics and deformed statistics were the object of considerable interest [1-19]. The use of deformed oscillator algebras proved to be useful in parastatistics, anyonic statistics and deformed statistics. In particular, one- and two-parameter deformations of the Bose-Einstein statistics (more precisely, deformations of the relevant second quantization formalism) were studied by several authors [6-19]. A common characteristics of most of these studies is that it is possible to obtain a Bose-Einstein condensation of a free gas of bosons in $D = 2$ and 3 dimensions. However, in $D = 3$ dimensions, the $q$-deformed Bose-Einstein (B-E) temperature is generally greater than the classical (corresponding to $q = 1$) B-E temperature. In the specific case of $^4$He super-fluid in phase II, the usual $q$-deformations, i.e., the à la Biedenharn [20] and à la Macfarlane [21] $q$-deformations, yield the following inequality:

$$ (T_{B-E})_{q \neq 1} > (T_{B-E})_{q = 1} > (T_{B-E})_{exp} $$

so that we do not gain anything when passing from $q = 1$ to $q \neq 1$. On the other hand, by using a à la Rideau [22,23] deformation, it is feasible to lower the critical temperature $(T_{B-E})_{q \neq 1}$ due to the occurrence of a second parameter $\nu'_0$ in addition to the deformation parameter $q$. This result corresponds to the model M$_1$ introduced in ref.[19]. For this model, we can obtain couples $(\nu'_0, q)$ for which $(T_{B-E})_{q \neq 1}$ is in agreement with the experimental value $(T_{B-E})_{exp} \sim 2.17$ K. However, as a drawback, the model M$_1$ depends on two parameters. Although it is possible to find a physical interpretation (in terms of the chemical potential) of the deformation parameter $q$, there is up to now no satisfying interpretation of the phenomenological parameter $\nu'_0$.

The just mentioned difficulty to interpret the parameter $\nu'_0$ was the starting point of an investigation of alternative deformations of the second quantization formalism. More specifically, we investigated the à la Arik and Coon [24] deformation but in the case where $q$ is a root of unity. (In the original work by Arik and Coon, the deformation parameter $q$ is a real number : The reality of $q$ ensures that the creation and annihilation operators are connected via Hermitean conjugation.) Thus, we arrived at the conclusion that it is necessary to simultaneously consider two quon algebras $A_q$ and $A_{\bar{q}}$ for which $q = \exp(2\pi i/k)$ with $k \in \mathbb{N} \setminus \{0, 1\}$. The case $k = 2$ corresponds to fermions and the limiting case $k \rightarrow \infty$ to bosons. Generalized coherent states (connected to $k$-fermionic states) and super-coherent states (involving a $k$-fermionic sector and a purely bosonic sector) were examined. In addition, the operators in the $k$-fermionic algebra were used to find realizations

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of the Dirac quantum phase operator and of the $W_\infty$ Fairlie-Fletcher-Zachos algebra [25]. All these matters were discussed in Bregenz (at the Symposium *Symmetries in Science X*), Dubna (at the VIII International Conference on *Symmetry Methods in Physics*) and Istanbul (at the International Workshop *Quantum Groups, Deformations and Contractions*) and shall be reported elsewhere [26,27].

In the present paper, we would like to deal with a second by-product of our quon approach. Here, instead of considering two non-commuting quon algebras $A_q$ and $A_{\bar{q}}$, we shall consider two realizations of two commuting quon algebras corresponding to the same root of unity $q = \exp(2\pi i/k)$ with $k \in \mathbb{N} \setminus \{0, 1\}$. We shall see how to construct (in Section 2) the Lie algebra of SU(2) from these two quon algebras; how to obtain (in Section 3) an alternative to the \{$J^2, J_z$\} scheme of SU(2); and how to develop (in Section 4) the Wigner-Racah algebra of SU(2) in this new scheme. In a last section (Section 5), we shall indicate some perspectives and briefly discuss some open problems.

## 2 A Quon Approach to SU(2)

We start with two commuting quon algebras $A_i = \{a_i-, a_i+, N_i \}$, with $i = 1$ and 2, for which the generators satisfy

$$a_i-a_i+ - qa_i+a_i- = 1, \quad [N_i, a_i\pm] = \pm a_i\pm \tag{1}$$

where the deformation parameter

$$q = \exp \left(\frac{2\pi i}{k}\right) \quad \text{with} \quad k \in \mathbb{N} \setminus \{0, 1\} \tag{2}$$

(the same for $A_1$ and $A_2$) is a root of unity. As constraint relations, compatible with (1) and (2), we take the nilpotency conditions

$$(a_i\pm)^k = (a_i-)^k = 0 \quad \text{with} \quad k \in \mathbb{N} \setminus \{0, 1\} \tag{3}$$

Grassmannian realizations of eqs.(1) and (3) are obtainable from ref.[26]. In this work, we take the representations of $A_1$ and $A_2$ defined by

$$a_{1+}|n_1) = |n_1 + 1), \quad a_{1+}|k - 1) = 0$$

$$a_{1-}|n_1) = [n_1]_q |n_1 - 1), \quad a_{1-}|0) = 0$$

$$a_{2+}|n_2) = [n_2 + 1]_q |n_2 + 1), \quad a_{2+}|k - 1) = 0$$

$$a_{2-}|n_2) = |n_2 - 1), \quad a_{2-}|0) = 0$$

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\[ N_1 |n_1\rangle = n_1 |n_1\rangle, \quad N_2 |n_2\rangle = n_2 |n_2\rangle \]

on a Fock space \( \mathcal{F} = \{|n_1n_2\rangle = |n_1\rangle \otimes |n_2\rangle : n_1, n_2 = 0, 1, \cdots, k-1\} \) of finite dimension \((\dim \mathcal{F} = k^2)\). We use here the notation
\[ [x]_q = \frac{1-q^x}{1-q} \quad \text{for} \quad x \in \mathbb{R} \]

so that \([n]_q = 1 + q + \cdots + q^{n-1}\) for \(n \in \mathbb{N}^*\).

We now define the two following linear operators
\[ H = \sqrt{N_1(N_2 + 1)} \]

and
\[ U_r = \left[ a_{1+} \exp \left( i \frac{\phi_r}{2} \right) \frac{(a_{1-})^{k-1}}{[k-1]_q!} \right] a_{2-} \exp \left( i \frac{\phi_r}{2} \right) \frac{(a_{2+})^{k-1}}{[k-1]_q!} \]

where the real parameter \(\phi_r\) is taken in the form
\[ \phi_r = \pi (k-1)r \quad \text{with} \quad r \in \mathbb{R} \]

and the \(q\)-deformed factorial is defined by
\[ [n]_q! = [1]_q[2]_q \cdots [n]_q \quad \text{for} \quad n \in \mathbb{N}^* \quad \text{and} \quad [0]_q! = 1 \]

The action of \(U_r\) on \(\mathcal{F}\) is easily found to satisfy
\[ U_r |n_1n_2\rangle = |n_1 + 1, n_2 - 1\rangle \quad \text{for} \quad n_1 \neq k-1 \quad \text{and} \quad n_2 \neq 0 \quad (4) \]

and
\[ U_r |k-1, 0\rangle = \exp(i\phi_r) |0, k-1\rangle \quad (5) \]

while for \(H\) we have
\[ H |n_1n_2\rangle = \sqrt{n_1(n_2 + 1)} |n_1n_2\rangle \quad (6) \]

By using the Schwinger trick
\[ j = \frac{1}{2} (n_1 + n_2), \quad m = \frac{1}{2} (n_1 - n_2) \quad \Rightarrow \quad |n_1n_2\rangle = |j + m, j - m\rangle \equiv |jm\rangle \]

we can rewrite eqs.\((4)\) and \((5)\) as
\[ U_r |jm\rangle = [1 - \delta(m,j)] |j + m + 1\rangle + \delta(m,j) \exp(i\phi_r) |j - m\rangle \]

Similarly, eq.\((6)\) can be rewritten
\[ H |jm\rangle = \sqrt{(j + m)(j - m + 1)} |jm\rangle \]
Furthermore, we have

\[ U_r^\dagger |jm\rangle = [1 - \delta(m, -j)] |j, m - 1\rangle + \delta(m, -j)\exp (-i\phi_r) |jj\rangle \]

where \( U_r^\dagger \) stands for the adjoint of \( U_r \). For a fixed value of \( k \), we take

\[ 2j = k - 1 \quad \text{with} \quad k \in \mathbb{N} \setminus \{0, 1\} \]

We can thus have \( j = \frac{1}{2}, 1, \frac{3}{2}, \ldots \). The case \( j = 0 \) corresponds to the limiting situation where \( k \to \infty \).

It is obvious that the operator \( H \) is Hermitean and the operator \( U_r \) is unitary. The action of \( U_r \) on \( \mathcal{F} \) is cyclic. As a further property of \( U_r \), we have

\[ (U_r)^{2j+1} = \exp(i\phi_r) \]

that reflects the cyclical character of \( U_r \).

Let us introduce the three operators

\[ J_+ = HU_r, \quad J_- = U_r^\dagger H \tag{7} \]

and

\[ J_3 = \frac{1}{2}(N_1 - N_2) \tag{8} \]

It is immediate to check that the action on the state \( |jm\rangle \) of the operators defined by eqs.(7) and (8) is given by

\[ J_\pm |jm\rangle = \sqrt{(j \mp m)(j \pm m + 1)}|j, m \pm 1\rangle \]

and

\[ J_3 |jm\rangle = m |jm\rangle \]

Consequently, we have the commutation relations

\[ [J_3, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_3 \]

which correspond to the Lie algebra of the group SU(2). As a result, the non-deformed Lie algebra su(2) is obtained from two \( q \)-deformed oscillator algebras.

To close this section, it is interesting to note that we can generate the infinite dimensional Lie algebra \( W_\infty \) from the generators of \( A_1 \) and \( A_2 \). Indeed, by putting

\[ U = U_r, \quad V = q^{N_1 - N_2} \]

and

\[ T_{(m_1,m_2)} = q^{m_1m_2}U^{m_3}V^{m_2} \]

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we can prove that
\[ [T_m, T_n] = -2i \sin \left( \frac{2\pi}{k} m \times n \right) T_{m+n} \] (9)
where we use the abbreviations
\[ m = (m_1, m_2), \quad n = (n_1, n_2) \]
and
\[ m + n = (m_1 + n_1, m_2 + n_2), \quad m \times n = m_1 n_2 - m_2 n_1 \]

Equation (9) shows that the operators \( T_\ell \) span the algebra \( W_\infty \) introduced by Fairlie, Fletcher and Zachos [25]. This result parallels a similar result obtained in ref.[26] in the study of \( k \)-fermions and of the Dirac quantum phase operator.

### 3 A New Basis for SU(2)

At this stage, it is important to establish a link with the work by Lévy-Leblond [28]. The decomposition (7), in terms of \( H \) and \( U_r \), coincides with the polar decomposition, described in ref.[28], of the shift operators \( J_+ \) and \( J_- \) of the Lie algebra \( su(2) \). This is easily seen by taking the matrix elements of \( U_r \) and \( H \) and by comparing these elements to the ones of the operators \( \Upsilon \) and \( J_\ell \) in [28]. This yields \( H \equiv J_\ell \); furthermore, by identifying the arbitrary phase \( \phi \) of [28] to \( \phi_r = 2\pi jr = \pi(k - 1)r \), we obtain that \( U_r \) turns out to be identical to the operator \( \Upsilon \) of [28]. Equation (7) constitutes an important original result of ref.[28].

It is easy to prove that the Casimir operator
\[ J^2 = \frac{1}{2} (J_+ J_- + J_- J_+) + J^2_3 = H^2 + J^2_3 - J_3 \]
commutes with \( U_r \) for any value of \( r \). (Note that the commutator \([U_r, U_s]\) is different from zero for \( r \neq s \).) Therefore, for fixed \( r \), the commuting set \( \{J^2, U_r\} \) provides us with an alternative to the familiar commuting set \( \{J^2, J_3\} \) of angular momentum theory. The (complete) set of commuting operators \( \{J^2, U_r\} \) can be easily diagonalized. This leads to the following result.

**Result**: The spectra of the operators \( U_r \) and \( J^2 \) are given by
\[ U_r |j\alpha; r\rangle = q^{\alpha} |j\alpha; r\rangle, \quad J^2 |j\alpha; r\rangle = j(j+1) |j\alpha; r\rangle \] (10)
where
\[ |j\alpha; r\rangle = \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^{j} q^{am} |jm\rangle \] (11)
with the range of values
\[ \alpha = -jr, -jr + 1, \cdots, -jr + 2j, \quad 2j \in \mathbb{N} \]
The parameter $q$ in eqs.(10) and (11) is

$$q = \exp \left( \frac{2\pi i}{2j + 1} \right)$$  \hspace{1cm} (12)

(cf. eq.(2) with $k = 2j + 1$ for $k \in \mathbb{N} \setminus \{0, 1\}$ and $k \to \infty$ for $j = 0$).

It is important to note that in eqs.(10) and (11) the label $\alpha$ goes, by step of 1, from $-jr$ to $-jr + 2j$. (It is only for $r = 1$ that $\alpha$ goes, by step of 1, from $-j$ to $j$.)

The inter-basis expansion coefficients

$$\langle jm|j\alpha; r \rangle = \frac{1}{\sqrt{2j + 1}} q^{\alpha m}$$

(with $m = -j, -j + 1, \ldots, j$ and $\alpha = -jr, -jr + 1, \ldots, -jr + 2j$) in eq.(11) define a unitary transformation that allows to pass from the well-known orthonormal standard basis $\{|jm\rangle : 2j \in \mathbb{N}, m = -j, -j + 1, \ldots, j\}$ of the space $\mathcal{F}$ to the orthonormal non-standard basis $B_r = \{|j\alpha; r \rangle : 2j \in \mathbb{N}, \alpha = -jr, -jr + 1, \ldots, -jr + 2j\}$. Then, the expansion

$$|jm\rangle = \frac{1}{\sqrt{2j + 1}} \sum_{\alpha = -jr}^{-jr+2j} q^{-\alpha m} |j\alpha; r \rangle$$

with

$$m = -j, -j + 1, \ldots, j, \quad 2j \in \mathbb{N}$$

is the inverse of eq.(11).

We thus foresee that it is possible to develop the Wigner-Racah algebra (WRa) of the group SU(2) in the $\{J^2, U_r\}$ scheme. This furnishes an alternative to the WRa of SU(2) in the SU(2) $\supset$ U(1) basis corresponding to the $\{J^2, J_3\}$ scheme.

4 A New Approach to the Wigner-Racah Algebra of SU(2)

In this section, we give the basic ingredients for the WRa of SU(2) in the $\{J^2, U_r\}$ scheme. The Clebsch-Gordan coefficients (CGc’s) adapted to the $\{J^2, U_r\}$ scheme are defined from the SU(2) $\supset$ U(1) CGc’s adapted to the $\{J^2, J_3\}$ scheme. The adaptation to the $\{J^2, U_r\}$ scheme afforded by eq.(11) is transferred to SU(2) irreducible tensor operators. This yields the Wigner-Eckart theorem in the $\{J^2, U_r\}$ scheme.
4.1 Coupling and Recoupling Coefficients in the \{J^2, U_r\} Scheme

The CGc’s or coupling coefficients \((j_1j_2\alpha_1\alpha_2|j\alpha;r)\) in the \(\{J^2, U_r\}\) scheme are simple linear combinations of the SU(2) ⊃ U(1) CGc’s. In fact, we have

\[
(j_1j_2\alpha_1\alpha_2|j\alpha;r) = \frac{1}{\sqrt{(2j_1+1)(2j_2+1)(2j+1)}} \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \sum_{m=-j}^{j} \times q^m q_1^{-\alpha_1 m_1} q_2^{-\alpha_2 m_2} (j_1j_2m_1m_2|jm)
\]

where \(q, q_1\) and \(q_2\) are given by eq.(12) in terms of \(j, j_1\) and \(j_2\), respectively. The symmetry properties of the coupling coefficients \((j_1j_2\alpha_1\alpha_2|j\alpha;r)\) cannot be expressed in a simple way (except the symmetry under the interchange \(j_1\alpha_1 \leftrightarrow j_2\alpha_2\)). Let us introduce the \(f_r\) symbol via

\[
f_r \left( j_1 \atop \alpha_1 \right) \left( j_2 \atop \alpha_2 \right) \left( j_3 \atop \alpha_3 \right) = (-1)^{2j_3} \frac{1}{\sqrt{2j_1+1}} (j_2j_3\alpha_2\alpha_3|j_1\alpha_1;r)^* \quad (13)
\]

where the star indicates the complex conjugation. Its value is multiplied by the factor \((-1)^{j_1+j_2+j_3}\) when its two last columns are interchanged. However, the interchange of two other columns cannot be described by a simple symmetry property. Nevertheless, the \(f_r\) symbol is of central importance for the Wigner-Eckart theorem in the \(\{J^2, U_r\}\) scheme (see eq.(17) below).

Following ref.[29], we define a more symmetrical symbol, namely the \(\bar{f}_r\) symbol, through

\[
\bar{f}_r \left( j_1 \atop \alpha_1 \right) \left( j_2 \atop \alpha_2 \right) \left( j_3 \atop \alpha_3 \right) = \frac{1}{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}} \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \sum_{m_3=-j_3}^{j_3} \times q_1^{-\alpha_1 m_1} q_2^{-\alpha_2 m_2} q_3^{-\alpha_3 m_3} \left( j_1 \atop m_1 \right) \left( j_2 \atop m_2 \right) \left( j_3 \atop m_3 \right) \quad (14)
\]

where the parameters \(q_i\) are given by eq.(12) with \(q \equiv q_i\) and \(j \equiv j_i\) for \(i = 1, 2, 3\). The \(3 - jm\) symbol on the right-hand side of eq.(14) is an ordinary Wigner symbol for the group SU(2) in the SU(2) ⊃ U(1) basis. As a matter of fact, it is possible to pass from the \(f_r\) symbol to the \(\bar{f}_r\) symbol and vice versa by means of a metric tensor. The \(\bar{f}_r\) symbol is more symmetrical than the \(f_r\) symbol. The \(\bar{f}_r\) symbol exhibits the same symmetry properties under permutations of its columns as the \(3 - jm\) Wigner symbol: Its value is multiplied by \((-1)^{j_1+j_2+j_3}\) under an odd permutation and does not change under an even permutation. In addition, the orthogonality properties of the highly symmetrical \(\bar{f}_r\) symbol easily follow from the corresponding properties of the \(3 - jm\) Wigner symbol. Thus, we have

\[
\sum_{j_3\alpha_3} (2j_3+1) \bar{f}_r \left( j_1 \atop \alpha_1 \right) \left( j_2 \atop \alpha_2 \right) \left( j_3 \atop \alpha_3 \right) \bar{f}_r \left( j_1' \atop \alpha_1' \right) \left( j_2' \atop \alpha_2' \right) \left( j_3' \atop \alpha_3' \right) = \delta(\alpha_1',\alpha_1)\delta(\alpha_2',\alpha_2) \quad (15)
\]
\[
\sum_{\alpha_1, \alpha_2} \bar{f}_r \left( \frac{j_1}{\alpha_1}, \frac{j_2}{\alpha_2}, \frac{j_3}{\alpha_3} \right) \bar{f}_r \left( \frac{j_1}{\alpha_1}, \frac{j_2}{\alpha_2}, \frac{j_3'}{\alpha_3'} \right)^* = \frac{1}{2j_3 + 1} \Delta(0|j_1 \otimes j_2 \otimes j_3) \delta(j_3', j_3) \delta(\alpha_3', \alpha_3)
\]

where \( \Delta(0|j_1 \otimes j_2 \otimes j_3) = 1 \) or 0 according to as the Kronecker product \((j_1) \otimes (j_2) \otimes (j_3)\) contains or does not contain the identity irreducible representation \((0)\) of SU(2).

Observe that the real number \( r \) is the same for all the \( \bar{f}_r \) symbols occurring in eqs.(15) and (16).

The values of the SU(2) CGc’s in the \( \{ J^2, U_r \} \) scheme as well as of the \( f_r \) and \( \bar{f}_r \) coefficients are not necessarily real numbers. For instance, we have the following property under complex conjugation

\[
\bar{f}_r \left( \frac{j_1}{\alpha_1}, \frac{j_2}{\alpha_2}, \frac{j_3}{\alpha_3} \right)^* = (-1)^{j_1 + j_2 + j_3} \bar{f}_r \left( \frac{j_1}{\alpha_1}, \frac{j_2}{\alpha_2}, \frac{j_3}{\alpha_3} \right)
\]

Hence, the value of the \( \bar{f}_r \) coefficient is real if \( j_1 + j_2 + j_3 \) is even and pure imaginary if \( j_1 + j_2 + j_3 \) is odd. Then, the behavior of the \( \bar{f}_r \) symbol under complex conjugation is completely different as the one of the ordinary \( 3 - jm \) Wigner symbol.

Finally, it is worth to mention that the recoupling coefficients of the group SU(2) can be expressed in terms of coupling coefficients of SU(2) in the \( \{ J^2, U_r \} \) scheme. For example, the \( 9 - j \) symbol can be expressed in terms \( \bar{f}_r \) symbols by replacing, in its decomposition in terms of \( 3 - jm \) symbols, the \( 3 - jm \) symbols by \( \bar{f}_r \) symbols.

On the other hand, the decomposition of the \( 6 - j \) symbol in terms of \( \bar{f}_r \) symbols requires the introduction of six metric tensors corresponding to the six arguments of the \( 6 - j \) symbol. These matters may be developed by following the approach initiated in refs.[29-32].

### 4.2 Wigner-Eckart Theorem in the \( \{ J^2, U_r \} \) Scheme

From the spherical components \( T^{(k)}_q \) (with \( q = -k, -k + 1, \ldots, k \)) of an SU(2) irreducible tensor operator \( T^{(k)} \), we define the \( 2k + 1 \) components

\[
T^{(k)}_{\alpha}(r) = \frac{1}{\sqrt{2k + 1}} \sum_{m=-k}^{k} q^{\alpha m} T^{(k)}_m
\]

with

\[
\alpha = -kr, -kr + 1, \ldots, -kr + 2k, \quad 2k \in \mathbb{N}
\]

In the \( \{ J^2, U_r \} \) scheme, the Wigner-Eckart theorem reads

\[
\langle \tau_1 j_1 \alpha_1; r|T^{(k)}_{\alpha}(r)|\tau_2 j_2 \alpha_2; r \rangle = \left( \tau_1 j_1||T^{(k)}||\tau_2 j_2 \right) f_r \left( \frac{j_1}{\alpha_1}, \frac{j_2}{\alpha_2}, \frac{k}{\alpha} \right) \quad (17)
\]

where \( \left( \tau_1 j_1||T^{(k)}||\tau_2 j_2 \right) \) denotes an ordinary reduced matrix element. Such an element is basis-independent. Therefore, it does not depend on the labels \( \alpha_1, \alpha_2 \) and \( \alpha \). On the contrary, the \( f_r \) coefficient in eq.(17), defined by eq.(13), depends on the labels \( \alpha_1, \alpha_2 \) and \( \alpha \).
5 Concluding Remarks

In this paper, we have developed a quon approach to the Lie algebra of the classical (not quantum!) group SU(2). Such an approach leads to the polar decomposition of the generators $J_+$ and $J_-$ of SU(2), a decomposition originally introduced by Lévy-Leblond [28].

The familiar $\{J^2, J_3\}$ quantization scheme with the (usual) standard spherical basis $\{|jm\rangle : 2j \in \mathbb{N}, m = -j, -j + 1, \cdots, j\}$, corresponding to the canonical chain of groups SU(2)$\supset$U(1), is thus replaced by the $\{J^2, U_r\}$ quantization scheme with a (new) basis, namely, the non-standard basis $B_r = \{|j\alpha; r\rangle : 2j \in \mathbb{N}, \alpha = -jr, -jr + 1, \cdots, -jr + 2j\}$. We have given the premises of the construction of the Wigner-Racah algebra of the group SU(2) in the $B_r$ basis. Of course, there exists an infinity of $B_r$ bases due to the fact that $r \in \mathbb{R}$. The case $r = 1$ probably deserves a special attention. We shall give elsewhere a complete development of the Wigner-Racah algebra of SU(2) in the $B_1$ basis. In particular, the calculation and the properties, including Regge symmetry properties, of the coupling coefficients ($\hat{f}_1$ and $f_1$ symbols and CGc’s in the $\{J^2, U_1\}$ scheme) shall be the object of a forthcoming paper.

As a further interesting step, it would be interesting to find realizations of the $B_r$ basis (i) on the sphere $S^2$ for $j$ integer and (ii) on the Fock-Bargmann spaces (of entire analytical functions) in 1 and 2 dimensions for $j$ integer or half of an odd integer. In this respect, the problem of finding a differential realization of the operator $U_r$ on $S^2$ and of expressing its eigenfunctions

$$[y_r]_{\ell\alpha}(\theta, \varphi) = \frac{1}{\sqrt{2\ell + 1}} \sum_{m=-\ell}^{\ell} q^{\alpha m} Y_{\ell m}(\theta, \varphi)$$

with

$$\alpha = -\ell r, -\ell r + 1, \cdots, -\ell r + 2\ell, \quad \ell \in \mathbb{N}$$

as special functions is very appealing. (In eq.(18), $Y_{\ell m}$ denotes a spherical harmonic.)

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