Variation and $\lambda$-jump inequalities on $H^p$ spaces

Sakin Demir
Agri Ibrahim Cecen University
Faculty of Education
Department of Basic Education
04100 Ağrı, Turkey
E-mail: sakin.demir@gmail.com

September 7, 2022

Abstract
Let $\phi \in \mathcal{S}$ with $\int \phi(x) \, dx = 1$, and define

$$\phi_t(x) = \frac{1}{2^n} \phi\left(\frac{x}{t}\right),$$

and denote the function family $\{\phi_t \ast f(x)\}_{t>0}$ by $\Phi \ast f(x)$. Suppose that there exists a constant $C_1$ such that

$$\sum_{t>0} |\hat{\phi}_t(x)|^2 < C_1$$

for all $x \in \mathbb{R}^n$. Then

(i) There exists a constant $C_2 > 0$ such that

$$\| \mathcal{V}_2(\Phi \ast f) \|_{L^p} \leq C_2 \|f\|_{H^p}, \quad \frac{n}{n+1} < p \leq 1$$

for all $f \in H^p(\mathbb{R}^n)$, $\frac{n}{n+1} < p \leq 1$. 

\textit{2020 Mathematics Subject Classification:} Primary 42B25; Secondary 42B30.

\textit{Key words and phrases:} Hardy space, variation operator, $\lambda$-jump operator.
(ii) The $\lambda$-jump operator $N_\lambda(\Phi * f)$ satisfies

$$\|\lambda[N_\lambda(\Phi * f)]^{1/2}\|_{L^p} \leq C_3 \|f\|_{H^p}, \quad \frac{n}{n+1} < p \leq 1,$$

uniformly in $\lambda > 0$ for some constant $C_3 > 0$.

Variation, oscillation and $\lambda$-jump inequalities on $L^p$ spaces have long been the research subjects of many mathematicians in probability, harmonic analysis and ergodic theory (see [7], [3], [1], [4], and [5]). When $0 < p \leq 1$ the Hardy space $H^p$ does no longer behave like $L^p$, that’s why proving variation and oscillation inequalities on $H^p$ spaces for $0 < p \leq 1$ requires a completely different work than what one does when working on an $L^p$ space. It has been proved in the author’s Ph.D thesis (see [2]) that several operators including the ergodic square function of differences of averages over lacunary sequences map ergodic $H^1$ to $L^1$.

Let $\phi \in \mathcal{S}$ with $\int \phi(x) \, dx = 1$, and define

$$\phi_t(x) = \frac{1}{t^n} \phi\left(\frac{x}{t}\right),$$

and denote the function family $\{\phi_t * f(x)\}_{t > 0}$ by $\Phi * f(x)$. Let $0 < p < \infty$. A distribution $f$ belongs to $H^p(\mathbb{R}^n)$ if the maximal function

$$Mf(x) = \sup_{t > 0} |(\phi_t * f(x)|$$

is in $L^p$.

**Definition 1.** Let $\mathcal{J}$ be a subset of $\mathbb{R}$ (or more generally an ordered index set). We consider real or complex valued functions $t \mapsto a_t$ defined on $\mathcal{J}$ and define their $\rho$-variation as

$$\|a\|_{v_\rho} = \sup \left( \sum_{i \geq 1} |a_{t_{i+1}} - a_{t_i}|^\rho \right)^{1/\rho},$$

where the supremum runs over all finite decreasing sequences $(t_i)$ in $\mathcal{J}$. 

2
We can define 

\[ \mathcal{V}_\rho(\Phi \ast f)(x) = \| \{ \phi_t \ast f(x) \}_{t > 0} \|_{v_\rho} \]

\[ = \sup \left( \sum_{i \geq 1} |\phi_{t_{i+1}} \ast f(x) - \phi_{t_i} \ast f(x)|^\rho \right)^{1/\rho} \]

for any \( x \in \mathbb{R}^n \).

Let \( F = \{ F_t : t \in J \} \) be a family of Lebesgue measurable functions defined on \( \mathbb{R}^n \).

The \( \lambda \)-jump function \( N_\lambda(F) \) is defined as the supremum of all integers \( N \) for which there is an increasing sequence \( 0 < s_1 < t_1 \leq s_2 < t_2 \leq \cdots \leq s_N < t_N \) such that

\[ |F_{t_k}(x) - F_{s_k}(x)| > \lambda \]

for each \( k = 1, 2, 3, \ldots, N \).

H. Liu [8] studied the variation and \( \lambda \)-jump inequalities for the function family \( \Phi \ast f \) when \( \rho > 2 \), and obtained the following results:

**Theorem 1.** For any \( \rho > 2 \), there exists a constant \( C_\rho > 0 \) such that

\[ \| \mathcal{V}_\rho(\Phi \ast f) \|_{L^p} \leq C_\rho \| f \|_{H^p}, \quad \frac{n}{n + 1} < p \leq 1 \]

for all \( f \in H^p(\mathbb{R}^n) \).

**Theorem 2.** If \( \rho > 2 \), then the \( \lambda \)-jump operator \( N_\lambda(\Phi \ast f) \) satisfies

\[ \| \lambda [N_\lambda(\Phi \ast f)]^{1/\rho} \|_{L^p} \leq C_\rho \| f \|_{H^p}, \quad \frac{n}{n + 1} < p \leq 1, \]

uniformly in \( \lambda > 0 \), for all \( f \in H^p(\mathbb{R}^n) \).

In this article we prove the above results for the case \( \rho = 2 \) with an additional assumption on the Fourier transform of \( \phi_t \).

We first need to present some known facts related to the atomic decomposition of \( H^p(\mathbb{R}^n) \) that will be used when proving our results.

**Definition 2.** Let \( 0 < p \leq 1 \leq q \leq \infty, \ p \neq q \). We say that a function \( a \in L^q(\mathbb{R}^n) \) is a \((p, q)\)-atom with the center at \( c \), if the following conditions are satisfied:
(i) supp $a \subset B(c, r)$;
(ii) $\|a\|_q \leq |B(c, r)|^{1/q-1/p}$;
(iii) $\int a(x) \, dx = 0$.

The following lemma and its proof can be found in R. H. Latter [6].

**Lemma 1.** A distribution $f$ is in $H^p(\mathbb{R}^n)$, $0 < p \leq 1$, if and only if there exists a sequence of $(p, q)$-atoms with $1 \leq q \leq \infty$ and $q \neq p$, $\{a_j\}$, and a sequence of scalars $\{\lambda_j\}$ such that

$$f = \sum_{j=0}^{\infty} \lambda_j a_j$$

in the sense of distributions and

$$C_1\|f\|_{H^p}^p \leq \sum_{j=0}^{\infty} \lambda_j^p \leq C_2\|f\|_{H^p}^p$$

where $C_1$ and $C_2$ are constants which depend only on $n$ and $p$.

We will also need the following Lemma when proving our results.

**Lemma 2.** Note that for $\rho \geq 2$ we have the following inequalities

$$\|a\|_{v_\rho} \leq \|a\|_{v_2} \leq 2 \cdot S(a)$$

where

$$S(a) = \left( \sum_{t \in J} |a_t|^2 \right)^{1/2}.$$

**Proof.** First note that

$$\|a\|_{v_\rho} \leq \|a\|_{v_2}$$

since $\rho \geq 2$. Since in the definition of $\|a\|_{v_2}$ the supremum runs over all finite decreasing sequence $(t_j) \in J$, and $|x - y|^2 \leq 2|x|^2 + 2|y|^2$, $2S(a)$ is an upper bound for the sequence

$$\left\{ \left( \sum_{i \geq 1} |a_{t_{i+1}} - a_{t_i}|^2 \right)^{1/2} \right\},$$

thus the result follows. \qed
We can now state and prove our main result.

**Theorem 3.** Suppose that there exists a constant $C_1$ such that
\[ \sum_{t>0} |\hat{\phi}_t(x)|^2 < C_1 \]
for all $x \in \mathbb{R}^n$. Then there exists a constant $C_2 > 0$ such that
\[ \| \mathcal{V}_2(\Phi \ast f) \|_{L^p} \leq C_2 \| f \|_{H^p}, \quad \frac{n}{n+1} < p \leq 1 \]
for all $f \in H^p(\mathbb{R}^n)$.

**Proof.** First we have
\[ \| \mathcal{V}_2(\Phi \ast f) \|_{L^2}^2 \leq 4 \int \sum_{t>0} |\hat{\phi}_t \ast f(x)|^2 \, dx \quad \text{(by Lemma 2)} \]
\[ = 4 \int \sum_{t>0} |\hat{\phi}_t \ast f(x)|^2 \, dx \quad \text{(by Plancherel’s theorem)} \]
\[ = 4 \int \sum_{t>0} |\hat{\phi}_t(x) \hat{f}(x)|^2 \, dx \]
\[ = 4 \int \sum_{t>0} |\hat{\phi}_t(x)|^2 |\hat{f}(x)|^2 \, dx \]
\[ \leq 4C_1 \int |\hat{f}(x)|^2 \, dx \quad \text{(by the hypothesis)} \]
\[ = 4C_1 \int |f(x)|^2 \, dx \quad \text{(by Plancherel’s theorem)} \]
\[ = 4C_1 \| f \|_{L^2}^2. \]

Let now $a$ be a $(p, 2)$ atom and $\frac{n}{n+1} < p \leq 1$. Suppose that $a$ is supported in a cube $Q$, $c$ is the center of $Q$. By Hölder’s inequality and our previous observation we have
\[ \int_{4Q} \mathcal{V}_2(\Phi \ast a)^p(x) \, dx \leq |4Q|^{1-p/2} \| \mathcal{V}_2(f) \|_{L^2}^p \]
\[ \leq 2\sqrt{C_1} |4Q|^{1-p/2} \| a \|_{L^2}^p \]
\[ \leq 2\sqrt{C_1}. \]
By the Minkowski inequality and the mean value theorem, we see that for $y, \xi \in Q$ and $x \in \mathbb{R}^n\setminus 4Q$,

$$\|\{\phi_t(x - y) - \phi_t(x - \xi)\}_{t > 0}\|_{\mathbb{R}^N} \leq \int_0^\infty \left| \frac{\partial}{\partial t} (\phi_t(x - y) - \phi_t(x - \xi)) \right| dt \leq C_2 |y - \xi| \int_0^\infty \frac{1}{t^{n+2}} \left( 1 + \frac{|x - \xi|}{t} \right)^{-n-2} dt \leq C_2 \frac{|y - \xi|}{|x - \xi|^{n+1}}.$$

Since we also have

$$\mathcal{V}_2(\Phi * a)(x) \leq \int_Q |a(y)| \|\{\phi_t(x - y) - \phi_t(x - c)\}_{t > 0}\|_{\mathbb{R}^N} dy,$$

we obtain

$$\int_{\mathbb{R}^n\setminus 4Q} \mathcal{V}_2(\Phi * a)^p(x) dx \leq C_2 \int_{|x - c| \geq 4|Q|} \frac{|Q|^p}{|x - c|^{(n+1)p}} dx \left( \int_Q |a(y)| dy \right)^p \leq C_2,$$

and this completes our proof.

**Corollary 4.** Suppose that there exists a constant $C_1$ such that

$$\sum_{t > 0} |\hat{\phi}_t(x)|^2 < C_1$$

for all $x \in \mathbb{R}^n$. Then the $\lambda$-jump operator $N_\lambda(\Phi * f)$ satisfies

$$\|\lambda[N_\lambda(\Phi * f)]^{1/2}\|_{L^p} \leq C_2 \|f\|_{H^p}, \quad \frac{n}{n+1} < p \leq 1,$$

uniformly in $\lambda > 0$, for all $f \in H^p(\mathbb{R}^n)$.

**Proof.** It has been proven in [5] and easy to see that

$$\lambda[N_\lambda(\Phi * f)]^{1/2} \leq \mathcal{V}_p(\Phi * f)$$

holds for any $\rho$ uniformly in $\lambda$. Thus the proof follows from Theorem 2. □

**Remark 1.** Note that Corollary 4 answers a conjecture of H. Liu [8] affirmatively.
References

[1] J. Bourgain, *Pointwise ergodic theorems for arithmetic sets*, Inst. Hautes Études Sci. Publ. Math. 69 (1989) 5-45.

[2] S. Demir, *$H^p$ spaces and inequalities in ergodic theory*, Ph.D Thesis, University of Illinois at Urbana-Champaign, USA, May 1999.

[3] S. Demir, *Inequalities for the variation operator*, Bull. of Hellenic Math. Soc. 64 (2020) 92-97.

[4] R. L. Jones, R. Kaufman, J. M. Rosenblatt and Máté Wierdl, *Oscillation in ergodic theory*, Ergodic Th. & Dynam. Sys 18 (1998) 889-935.

[5] R. L. Jones, A. Seeger and J. Wright, *Strong variational and jump inequalities in harmonic analysis*, Trans. AMS 360 2 (2008) 6711-6742.

[6] R. H. Latter, *A characterization of $H^p(\mathbb{R}^n)$ in terms of atoms*, Studia Math. 62 1 (1978) 93-101.

[7] D. Lépingle, *La variation d’ordre $p$ des semi-martingales*, Z. Wahrscheinlichkeitstheorie Verw. Gebiete, 36 (1976) 295-316.

[8] H. Liu, *Variational characterization of $H^p$*, Proc. of the Royal Soc. of Edinburgh, 149 (2019) 1123-1134.