Nonlinear evolution of bichromatic waves based on the fifth order solution of benjamin bona mahony equation

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Abstract. This research concerns on the influence of higher order solution to nonlinear evolution of bichromatic waves based on Benjamin Bona Mahony (BBM) equation. Asymptotic method was applied to determine the solution expanded to the fifth order. In previous researches, this method was applied on the same equation to find its solution up to the third order and it was obtained that the waves underwent deformation and amplitude amplification affected by the occurrence of a pair of side bands during their propagation. In this study, the effect of the higher order solution on the increasing amplitude was observed. A quantity called as Maximal Temporal Amplitude (MTA) was measured to find the maximum position where the highest amplitude occurred and the Amplitude Amplification Factor (AAF). MTA gives information about the waves' highest elevation at every spatial position during the observation time. The analysis includes comparison between the third and fifth order solutions in their contribution to the rise of the amplitude and the maximum position.

1. Introduction
Waves generally occur on the surface of water bodies open to the atmosphere. The waves are a manifestation of forces acting on fluids which tend to violate the gravitation and surface tension trying to preserve the surface state [1]. Many unexpected phenomena that occur in the ocean and become a threat to offshore activities, one of which is the phenomenon of extreme waves. Waves are considered extreme if the wave height is more than 2.2 times the height of the significant waveform [2]. There are several types of causes for the formation of waves such as wind, earth plate activity, tides and currents. But waves often occur due to wind, especially if strong winds occur. These conditions can cause high waves that hamper marine transportation activities and offshore facilities. According to Dean and Dalrymple [1], the highest wave ever observed was 34 m and had attacked countless ships. This shows that extreme waves are dangerous for sea transportation and offshore facilities.

Surface wave behavior can be described in partial differential equations, namely the Korteweg de Vries (KdV), Boussinesq, Kadomtsev-Petviashvili (KP), Benjamin Bona Mahony (BBM) equations, and other equations which are all in nonlinear form. These equations are
simplification of Laplace full water equation. Simplification is done by investigating the direction of the wave propagation under review.

The KdV equation is a wave equation that describes one-way wave propagation on shallow water surfaces with small wave amplitudes and large wavelengths. The KdV equation was discovered in 1895, this equation can be applied to explain some physics problems, including studies of hydromagnetic waves, internal waves, acoustic-ion waves, and plasma physics [4]. However, the KdV equation has a weakness, the KdV equation is only able to model waves with small wave numbers so that the resulting waves have large wavelengths, whereas it is expected that in wave generation experiments in hydrodynamics laboratories, waves must have short wavelengths. Thus to obtain waves with dispersive properties in accordance with these equations, a very long water tank and a very long time are needed [5].

To correct this weakness, the BBM Equation was developed by Benjamin, Bona and Mahony in 1972. The BBM Equation was an improvement on the weakness of the KdV equation in modeling waves that have large wave numbers. The BBM Equation form has a dispersion relation and short wave properties that can explain its physical phenomena better than the KdV [6] equation.

Most of wave equations are presented in nonlinear form which are often difficult solve. One method of solving nonlinear equation is asymptotic method [8]. According to Mashuri and Marwan [9], the asymptotic method can be applied to determine the approximate solution of the KdV equation where the amplitude can be expanded using a power series of the third order. The asymptotic expansion method was also applied by Ramli [10] to determine the solution of Boussinesq equation in studying extreme waves. This method was also applied by Mashuri et. al. [11] to determine the solution of Kadomsevt Petviashvili equation which describes the two dimensions of the KdV equation.

Marwan [12] conducted a research on the KdV Equation using asymptotic power series up to the third order side band to observe the maximum height of the amplitude that can be produced, it turned out that the amplitudes of the side bands in the third order term exceeded the ones of second order term. However the height of the amplitude was lower than the results of experiments using the HUBRIS software which was developed from full water equation. This is allegedly due to inadequate expansion of the power series. Therefore Ramli et. al [13] conducted a study of the KdV Equation to find a solution up to fifth order side bands using asymptotic expansion and to compare the amplitude heights of third-order solutions with fifth-order solutions. Afriadi et. al [14] continued the research to find a fifth-order solution and determine the maximum position. However, the amplitude obtained from the fifth order solution was not able to match the experimental results using the HUBRIS software. Because the KdV equation has not been able to match the HUBRIS experimental results in reaching extreme positions, Halfiani and Ramli [15] conducted research on the BBM equation up to third-order side band terms to find solutions and obtained that waves experiences deformation and amplification amplitude caused by the appearance of a pair of side bands in propagation.

In this study, the effect of a higher order on increasing wave amplitude using the BBM equation is observed. The solution of the BBM equation is calculated using the asymptotic expansion method of the fifth order. Finally, the comparison of the results and the third-order BBM solution is presented.

2. Benjamin Bona Mahony (BBM) Equation

The BBM equation is a wave equation that models a one directional long wave propagation that has a small amplitude on the surface. The equation reads [6]

$$\eta_t + \eta_x + \eta\eta_x - \eta_{xxx} = 0 \quad (1)$$
where $\eta$ represents the surface wave elevation, $x$ is the space scale, and $t$ is the time scale. The linear form of the BBM equation is

$$\eta_t + \eta_x - \eta_{xxx} = 0$$

(2)

with the dispersion relation as follows [7].

$$\omega = \frac{k}{1 + k^2},$$

(3)

where $\omega$ is the frequency and $k$ is the wave number. $\eta_t + \eta_x$ describes wave propagation on shallow water surfaces that have small amplitudes and large wavelengths. Propagation occurs in one direction and moves in the positive $x$ direction. The term $\eta_x$ describes the nonlinear effect affecting wave propagation and $\eta_{xxx}$ describes the dispersive nature of the wave. From the dispersion relation phase velocity is obtained:

$$\frac{\omega}{k} = \frac{1}{1 + k^2}$$

(4)

and a group velocity:

$$\frac{d\omega}{dk} = \frac{1 - k^2}{(1 + k^2)^2}$$

(5)

which has a boundary value to the value of $k$ and is close to zero for a large value of $k$. This means that for large scale $k$, BBM waves tend not to propagate. In other words, the model has desirable properties related to short wavelengths and large wave numbers.

3. The Fifth Order Solution of BBM Equation and Extreme Positions of BBM Bichromatic signals

The asymptotic method emerged in the early 1800s because of the desire to develop formulas for evaluating specific functions. The asymptotic expansion method is a method for finding an approximate solution to a problem of differential equations, where the function is expressed in power series of a small parameter value [8]. The smaller the value of the parameter chosen, the more accurate the approach obtained.

To solve the BBM equation using the asymptotic expansion method, $\eta(x,t)$ will be expanded to the fifth order and written in the following form

$$\eta(x,t) = \varepsilon \eta^{(1)} + \varepsilon^2 \eta^{(2)} + \varepsilon^3 \eta^{(3)} + \varepsilon^4 \eta^{(4)} + \varepsilon^5 \eta^{(5)} + \ldots$$

(6)

The wave number ($k$) is also expanded and written in the following form

$$k = k^{(0)} + \varepsilon k^{(1)} + \varepsilon^2 k^{(2)} + \varepsilon^3 k^{(3)} + \varepsilon^4 k^{(4)} + \ldots$$

(7)

where $\varepsilon$ is a small valued parameter. Equations 6 and 7 will be substituted into the BBM Equation 5 and separated by the order of $\varepsilon$. By collecting the polynomial coefficients for each order of $\varepsilon$, it is obtained first order:

$$\eta_t + \eta_x - \eta_{xxx} = 0$$

(8)

Equation 8 is in the form of a linear equation, an ansatz is chosen as in the following equation.

$$\eta^{(1)}(x,t) = \sum_{p=1}^{2} a_p e^{i\theta_p} + c.c$$

(9)
where $a$ is the amplitude, $\theta_p = k_p x - \omega_p t$, $k$ is the wave number, $\omega$ is the frequency and $c.c$ is the complex conjugate. Equation 9 shows the form of bichromatic signals. By substituting the ansatz into the linear form of the BBM equation 9, we get the following dispersion relations.

$$\Omega(k^{(0)}_p) = \omega_p = \frac{k^{(0)}_p}{1 + (k^{(0)}_p)^2}$$

where

$$A_{\pm,pq} = \frac{k^{(0)}_p \pm k^{(0)}_q}{2(\omega_p \pm \omega_q) - (k^{(0)}_p \pm k^{(0)}_q) + (\omega_p \pm \omega_q)(k^{(0)}_p \pm k^{(0)}_q)^2}$$

The third order solution is

$$\eta^{(3)}(x,t) = \sum_{p=1}^{2} \sum_{q=1}^{2} \sum_{r=1}^{2} a_p a_q a_r B_{+pq} e^{i(\theta_p + \theta_q + \theta_r)} + \sum_{p \neq q} \sum_{r=1}^{2} a_p a_q a_r B_{-pq} e^{i(\theta_p + \theta_q - \theta_r)} + c.c$$

where

$$B_{+pq} = \frac{(k^{(0)}_p + k^{(0)}_q + k^{(0)}_r)A_{+pq}}{(\omega_p + \omega_q + \omega_r) - (k^{(0)}_p + k^{(0)}_q + k^{(0)}_r)A_{+pq} + (\omega_p + \omega_q + \omega_r)(k^{(0)}_p + k^{(0)}_q + k^{(0)}_r)^2}$$

$$B_{-pq} = \frac{(k^{(0)}_p + k^{(0)}_q - k^{(0)}_r)A_{+pq} + A_{-pq} + A_{-qr}}{(\omega_p + \omega_q - \omega_r) - (k^{(0)}_p + k^{(0)}_q - k^{(0)}_r) + (\omega_p + \omega_q - \omega_r)(k^{(0)}_p + k^{(0)}_q - k^{(0)}_r)^2}$$

The fourth order solution reads

$$\eta^{(4)}(x,t) = \sum_{p=1}^{2} \sum_{q=1}^{2} \sum_{r=1}^{2} \sum_{s=1}^{2} a_p a_q a_r a_s C_{+pqrs} e^{i(\theta_p + \theta_q + \theta_r + \theta_s)} + \sum_{p \neq q} \sum_{r} \sum_{s} a_p a_q a_r a_s C_{-pqrs} e^{i(\theta_p + \theta_q + \theta_r - \theta_s)}$$

$$+ \sum_{p \neq q} \sum_{r \neq s} a_p a_q a_r a_s D_{pqrs} e^{i(\theta_p + \theta_q - \theta_r - \theta_s)}$$

$$+ \sum_{p \neq q} \sum_{r \neq s} a_p a_q a_r a_s E_{+pqrs} e^{i(\theta_p + \theta_q + \theta_r - \theta_s)}$$

$$+ \sum_{p \neq q} \sum_{r \neq s} a_p a_q a_r a_s E_{-pqrs} e^{i(\theta_p + \theta_q - \theta_r - \theta_s)} + c.c$$

where

$$C_{+pqrs} = \frac{b_{+}B_{+}A_{+} + A_{+}A_{-}r_{s}}{c_{+}}, \quad C_{-pqrs} = \frac{b_{-}B_{-}A_{-} + A_{+}A_{-}p_{s}}{c_{-}}$$
Finally, the fifth order solution is

\[
\begin{align*}
D_{+pqrs} &= \sum_{e} \left( b_{+}(B_{+pq} + f_{+} + B_{-pq} + A_{+pq}A_{-qr}) + (k_{p}^{(2)} + k_{q}^{(2)} + k_{r}^{(2)} - k_{s}^{(2)})(1 + A_{+qr}) - \right. \\
E_{+pqrs} &= \left. \frac{(\omega_{p} + \omega_{q} + \omega_{r} - \omega_{s})(2k_{p}^{(0)} + 2k_{q}^{(0)} + k_{r}^{(0)} + k_{r}^{(0)})A_{+qr}}{e^{-}} \right), p \neq q \\
E_{-pqrs} &= \frac{(d(B_{+qrst} + b_{-} + B_{-qrs} + A_{+pq} + A_{pq} + A_{pq}A_{+qr}) + (k_{p}^{(2)} - k_{q}^{(2)}))}{(1 + A_{+qr}) - (\omega_{p} + \omega_{q} + \omega_{r} - \omega_{s})(2k_{p}^{(0)} - 2k_{q}^{(0)} + k_{r}^{(0)} + k_{r}^{(0)})A_{-qr}} \\
b_{\pm} &= (k_{p}^{(0)} + k_{q}^{(0)} + k_{r}^{(0)} \pm k_{s}^{(0)}), \ d = (k_{p}^{(0)} + k_{q}^{(0)} - k_{r}^{(0)} - k_{s}^{(0)}) \\
c_{\pm} &= (\omega_{p} + \omega_{q} + \omega_{r} \pm \omega_{s}) - (k_{p}^{(0)} + k_{q}^{(0)} + k_{r}^{(0)} \mp k_{s}^{(0)}) + (\omega_{p} + \omega_{q} + \omega_{r} \pm \omega_{s})(k_{p}^{(0)} + k_{q}^{(0)} + k_{r}^{(0)} \pm k_{s}^{(0)})^{2} \\
e &= (\omega_{p} + \omega_{q} - \omega_{r} - \omega_{s}) - (k_{p}^{(0)} + k_{q}^{(0)} - k_{r}^{(0)} - k_{s}^{(0)}) + (\omega_{p} + \omega_{q} - \omega_{r} - \omega_{s})(k_{p}^{(0)} + k_{q}^{(0)} - k_{r}^{(0)} - k_{s}^{(0)})^{2} \\
f_{\pm} &= B_{+pq} + B_{-pq} + B_{pqr}, f_{-} = B_{+pq} + B_{-pq} + B_{+qr}
\end{align*}
\]

Finally, the fifth order solution is

\[
\eta^{(5)}(x, t) = \sum_{p=1}^{2} \sum_{q=1}^{2} \sum_{r=1}^{2} \sum_{s=1}^{2} \sum_{u=1}^{2} \sum_{e=1}^{2} a_{p}a_{q}a_{r}a_{s}a_{u}F_{+pqrs}e^{i(\theta_{p} + \theta_{q} + \theta_{r} + \theta_{s} + \theta_{u})} + \sum_{p=q=r}^{2} \sum_{q=s}^{2} \sum_{r=u}^{2} a_{p}a_{q}a_{r}a_{s}a_{u}F_{-pqrs}e^{i(\theta_{p} + \theta_{q} + \theta_{r} + \theta_{s} + \theta_{u})} \\
+ \sum_{p=1}^{2} \sum_{q=r}^{2} \sum_{q=s}^{2} \sum_{r=u}^{2} a_{p}a_{q}a_{r}a_{s}a_{u}G_{+pqrs}e^{i(\theta_{p} + \theta_{q} + \theta_{r} + \theta_{s} + \theta_{u})} \\
+ \sum_{p=1}^{2} \sum_{q=r}^{2} \sum_{q=s}^{2} \sum_{r=u}^{2} a_{p}a_{q}a_{r}a_{s}a_{u}G_{-pqrs}e^{i(\theta_{p} + \theta_{q} + \theta_{r} + \theta_{s} + \theta_{u})} \\
+ \sum_{p=q=r}^{2} \sum_{q=s}^{2} \sum_{r=u}^{2} a_{p}a_{q}a_{r}a_{s}a_{u}H_{pqrs}e^{i(\theta_{p} + \theta_{q} + \theta_{r} + \theta_{s} + \theta_{u})} + c.c
\]

where

\[
F_{+pqrs} = \frac{g_{+}(C_{+pqrs} + A_{+pq}B_{-pq})}{h_{+}}, \quad F_{-pqrs} = \frac{g_{-}(C_{-pqrs} + C_{-pqrs} + A_{-pq}B_{-pq} + A_{-pq}B_{+pq})}{h_{-}}
\]
terms have the same frequency as the bound wave terms. So the fifth-order solution of BBM
\[ p, q, r, s, t, u, v, w, x, y, z \]

Because the \( (k) \) wave number is expanded to eliminate the resonance terms that appear in the third and fifth order, values of \( k \) for each order are
\[ \Omega(k_{p}^{(0)}) = \frac{k_{p}^{(0)}}{1 + k_{p}^{(0)}}, k_{p}^{(1)} = 0 \]
\[ k_{p}^{(2)} = -\frac{a_{p}^{2}k_{p}^{(0)} + a_{0}^{2}k_{p}^{(0)}(A_{+,pq} + A_{-,pq} + A_{-,pqr})}{1 - 2a_{p}k_{p}^{(0)}}, k_{p}^{(3)} = 0 \]
\[ k_{p}^{(4)} = \left( -\frac{a_{p}^{4}k_{p}^{(0)}(E_{+pq} + E_{-,pq} + E_{+pqr} + E_{-,pqr})}{A_{+,pqr}B_{+,pqr} + B_{+,pr}B_{+,qr} + B_{+,qrs} + A_{+,qrs} + A_{-,pts} + A_{-,pt}A_{-,pts} + A_{-,pts}} + A_{+,pq}B_{+,pqr} + A_{+,pqr}B_{+,qr} + A_{+,qrs}B_{+,pq} + B_{+,pqr}B_{+,qr} + B_{+,qrs}B_{+,pqr} \right) \]

for \( p, r = 1, 2 \) then \( q = s = (pmod2) + 1 \). To meet the boundary conditions and obtain the form of bifurcation signals when \( x = 0 \), it is necessary to generate signals called free waves from the second to fifth order terms. Free waves have the same amplitude and frequency as the boundwaves, but different wave number which is \( K = \Omega^{k} - 1(\omega_{p}) \) for each term. The free wave terms have the same frequency as the bound wave terms. So the the fifth-order solution of BBM Equation can be written as
\[ \eta(x, t) = \eta^{(1)}(x, t) + \eta^{(2)}(x, t) - \eta^{(3)}(x, t) - \eta^{(4)}(x, t) - \eta^{(5)}(x, t) - \eta^{(6)}(x, t) - \eta^{(7)}(x, t) \]
then Equation 8 can be written as

\[
\eta^{(1)} = 2a(\cos \theta_1 + \cos \theta_2) = 4a \cos(\bar{k}x - \bar{\omega}t \cos(\kappa x - \nu t))
\]  

(17)

where \( \bar{k} = \frac{k_1^{(0)} + k_2^{(0)}}{2} \), \( \kappa = \frac{k_1^{(0)} - k_2^{(0)}}{2} \), \( \bar{\omega} = \frac{\omega_1 + \omega_2}{2}, \) \( \nu = \frac{\omega_1 - \omega_2}{2} \). The calculation of \( x_{max} \) only involves the side band terms of odd orders, even and non-side band terms are not involved because they only affect the increase in wave amplitude. The side band terms of the odd orders have frequencies that is close to the frequency of first order solution which is bichromatic waves. Therefore, They will interact with the first order terms and form a wave group and cause changes in shape along their propagation.

For some calculation \( x_{max} \) can be written as

\[
x_{max} = \frac{\pi}{|\left(\bar{K} - 3\kappa\right) + (\bar{K}' - 5\kappa)|}
\]

(18)

with \( \bar{k} = \frac{k_1 + k_2}{2} \), \( \kappa = \frac{k_1 - k_2}{2} \), \( \bar{K} = \frac{K(\bar{\omega} + 3\nu) + K(\bar{\omega} - 3\nu)}{2}, \) \( \bar{K}' = \frac{K(\bar{\omega} + 5\nu) - K(\bar{\omega} - 5\nu)}{2} \), and \( \bar{K}^\prime = \frac{K(\bar{\omega} + 5\nu) - K(\bar{\omega} - 5\nu)}{2} \)

4. Maximal Temporal Amplitude

In this section, the changes in wave amplitude throughout the wave propagation are observed. Variables that are used must first be transformed into a laboratory scale.

\[
\eta = \frac{3}{2} \eta_{lab} \quad x = \sqrt{\frac{6}{g}} \frac{x_{lab}}{h} \quad t = \sqrt{\frac{6}{g}} \frac{t_{lab}}{h} \quad \omega = \frac{1}{6} \sqrt{\frac{6h}{g \omega_{lab}}} \quad k = \frac{\sqrt{6}}{6} k_{lab}
\]

Extreme position and amplitude amplification is examined using Maximal Temporal Amplitude (MTA). The MTA equation is defined as follows [16].

\[
m(x) = \max_t \eta(x, t)
\]

Amplitude Amplification Factor (AAF) is defined as follows. [10, 12]

\[
AAF = \frac{m(x_{max})}{m(0)}
\]

The chosen parameter values are based on the research of Halfiani and Ramli (2017) [15]. The depth of the water is 5 m, the gravitational force is 9.8m/s² and \( a = 0.2m \). Whereas the frequency values of \( \omega_1 \) and \( \omega_2 \) are not based on previous studies because the dispersion relation is a bounded function that has upper and lower bounds so that the selection of values is based on its boundary values as seen in Figure 1. In the figure, it can be seen that the frequency \( \omega \) has a maximum value at 0.5. Therefore the value of \( \omega \) can not exceed the value of this limit. So in this study, the values are assigned as \( \omega_1 = 0.099 \) and \( \omega_2 = 0.090 \) which give the physical scales \( \omega_{1lab} = 0.34rad/s, \) and \( \omega_{2lab} = 0.31rad/s \).
Figure 1. The values of frequency ($\omega$) with respect to wave number ($k$).

Figure 2. MTA of the bichromatics signals with $a = 0.2m$, $\omega_{1,\text{lab}} = 0.34\,\text{rad/s}$, dan $\omega_{2,\text{lab}} = 0.31\,\text{rad/s}$.

Figure 3. Bichromatic waveform of the fifth (blue) and third (red) order solutions at (a) $x = 0m$ (b) $x = 15m$ (c) $x = x_{\text{max}}$ (d) $x = 70m$ with $a = 0.2m$, $\omega_{1,\text{lab}} = 0.34\,\text{rad/s}$, and $\omega_{2,\text{lab}} = 0.31\,\text{rad/s}$.

Figure 2 presents the MTA comparison between the third-order solution obtained from the results of the research of Halfiani and Ramli (2017) and the fifth-order solutions obtained from
this study. In Figure 2 it can be seen that the increase in the amplitude of the bichromatic wave generated using the third-order and fifth-order approaches shows a similar pattern. Initially, at x = 0, the amplitude of bichromatic signals is 0.8 m for both orders. At the extreme position, the wave of the third order solution has a smaller amplitude than that of the fifth order solution. The third order reaches an extreme position at x = 38 m with $\eta_{\text{max}} = 1.09\text{m}$ while the fifth order reaches an extreme position at x = 43.4 m with $\eta_{\text{max}} = 1.28\text{m}$. This gives a fifth-order AAF of 1.6, which in general shows the changes in wave elevation. Meanwhile, the AAF of third order amplitude is 1.36. The difference in amplitude height of the third order and fifth order began to appear at x = 11 m. At this position, the third order amplitude increases to 0.85 m while the fifth order amplitude reaches 0.92 m and both waves continue to increase to their maximum heights and descend slowly before returning to rise again. These results give an indication that the approximate solution using higher order effect on increasing the maximum amplitude of bichromatic waves. Figure 3 exhibits the form of bichromatic signals from the fifth and third order solutions at several positions including extreme positions. From the figure, at position x = 0 m the signal is in the form of bichromatic waves with an amplitude of 0.8 m. In other positions the wave groups are deformed and the amplitudes continue to increase until it reaches its maximum and slowly decreases before returning to rise. Waves that originally had a symmetrical shape become asymmetrical as the position x changes and the steepness on the front increases.

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5. Conclusion
From the results of the study it can be concluded that high order gives a considerable influence on the increase in the amplitude of bichromatic waves. An increase in amplitude of the fifth order appears to be higher than amplitude of the third order. For $a = 0.2 \text{ m}$, $\omega_{1(\text{lab})} = 0.34\text{rad/s}$ and $\omega_{2(\text{lab})} = 0.31\text{rad/s}$, the amplitude of the fifth order wave increases to 1.28 m compared to the third order which only reaches 1.09 m with an initial amplitude of 0.8 m for both orders. The amplitude of the fifth order solution increases by 1.6 times the initial amplitude while the increase of the amplitude of the third order solution is 1.36 times the initial amplitude. But the position achieved by the fifth order is not classified as extreme because the increase in maximum height does not reach 2.2 times the initial height of the wave. In addition, the two orders show different maximum peaking positions. The third order solution experienced the maximum peaking faster than the maximum peaking occurred in the fifth order solution.

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