WEAK FROBENIUS BIMONADS AND FROBENIUS BIMODULES

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Abstract. As shown by S. Eilenberg and J.C. Moore (1965), for a monad $F$ with right adjoint comonad $G$ on any category $A$, the category of unital $F$-modules $A_F$ is isomorphic to the category of counital $G$-comodules $A^G$. The monad $F$ is Frobenius provided we have $F = G$ and then $A_F \simeq A^G$. Here we investigate which kind of equivalences can be obtained for non-unital monads (and non-counital comonads). For this we observe that the mentioned equivalence is in fact an equivalence between $A_F$ and the category of bimodules $A_F A$ subject to a certain compatibility condition (Frobenius bimodules). Eventually we obtain that for a weak monad $(F, \mu, \eta)$ and a weak comonad $(F, \delta, \varepsilon)$ satisfying $\mu \cdot F \eta = F \varepsilon \cdot \delta$, the category of compatible $F$-modules $A_F$ is equivalent to the category of compatible Frobenius bimodules $A_F A_F$ and the category of compatible $F$-comodules $A^F$.

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Introduction

A monad $(F, \mu, \eta)$ on any category $B$ is called a Frobenius monad provided the functor $F$ is (right) adjoint to itself. Then $F$ also allows for a comonad structure $(F, \delta, \varepsilon)$ and the (Eilenberg-Moore) category $B_F$ of $F$-modules is isomorphic to the category $B^F$ of $F$-comodules. As shown in [3, Theorem 3.13], this isomorphism characterises a functor with monad and comonad structure as Frobenius monad. It is not difficult to see - and will also come out in our constructions - that the categories $B_F$ and $B^F$ are in fact isomorphic to the category $B_F^F$ of what we will call (unital and counital) Frobenius bimodules. In this setting units and counits play a crucial role.

Here we are concerned with the question what is left from these correspondences if we consider weak monads and weak comonads, that is, when the conditions on units and counits are weakened (e.g. [1], [3]). An elementary approach to these notions is offered in [6] and [7] where adjunctions between two functors are replaced by more general relationships between those.
For functors \( L : \mathcal{A} \to \mathcal{B} \) and \( R : \mathcal{B} \to \mathcal{A} \) between any categories \( \mathcal{A} \) and \( \mathcal{B} \) a \textit{pairing} for \((L, R)\) is defined by (natural) maps

\[
\text{Mor}_\mathcal{B}(L(A), B) \xrightarrow{\alpha} \text{Mor}_\mathcal{A}(A, R(B)),
\]

and a pairing for \((R, L)\) is given by maps

\[
\text{Mor}_\mathcal{A}(R(B), A) \xrightarrow{\beta} \text{Mor}_\mathcal{B}(B, L(A)).
\]

If \( \alpha \) is a bijection, then \((L, R)\) is an adjoint pair, if \( \tilde{\alpha} \) is a bijection, then \((R, L)\) is an adjoint pair, and if \( \alpha \) and \( \tilde{\alpha} \) are bijections, then \( LR \) and \( RL \) are Frobenius functors.

The mere existence of \( \alpha \) and \( \tilde{\beta} \), given by two natural transformations \( \eta : I_\mathcal{A} \to RL \) and \( \varphi : RL \to I_\mathcal{B} \), makes \((LR, L\varphi R)\) a non-unital monad and \((LR, L\eta R)\) a non-counital comonad satisfying the Frobenius condition (see (2.1)), allowing for the definition of non-unital modules, non-counital comodules, and Frobenius bimodules (see (2.3)). Furthermore, there is a natural transformation \( \varphi \cdot \eta : I_\mathcal{A} \to I_\mathcal{A} \) inducing a natural transformation \( \theta : LR \to LR \) which is useful for understanding the structures involved. In the standard situation, \( \varphi \cdot \eta = I_{LR} \) means separability of the functor \( L \). The setting here suggests to generalise this to \textit{weak separability} by requiring \( \eta \) (or \( \varphi \)) to be regular, that is, \( \eta = \eta \cdot \varphi \cdot \eta \) (or \( \varphi = \varphi \cdot \eta \cdot \varphi \)) (e.g. Proposition 2.4).

Given a non-counital \( LR \)-comodule \((B, \omega)\), the question arises when it can be extended to a Frobenius bimodule by some \( \rho : LR(B) \to B \). As sufficient condition it turns out that the defining cofork for \( \rho \) is a coequaliser in the category of non-counital comodules (see Proposition 2.8). As is well-known, counital comodules over a comonad have this property and this applies if \((L, R)\) is an adjunction. This case was considered by Böhm and Gómez-Torrecillas in [2].

More generally, if \( \beta \) is given by some \( \varepsilon : LR \to I_\mathcal{B} \), imposing regularity conditions on \( \alpha \) and \( \tilde{\beta} \) and symmetry on \( \beta \) (making \((LR, L\eta R, \varepsilon)\) a weak comonad), we obtain that compatible \( LR \)-comodules also allow for the construction of Frobenius modules: they satisfy a generalised coequaliser condition which is restricted to a certain class of comodule morphisms (see Definition 1.1, Proposition 2.12). Similar results hold for extending compatible modules to a Frobenius bimodule by a suitable comodule structure. For this, a variation of the equaliser condition is needed.

These constructions lead to various functors between (compatible) module, comodule and bimodule categories (see Propositions 2.14, 2.18).

In the final section, the notions developed for functor pairs are written out for weak monads and comonads leading to equivalences between module and comodule categories for weak Frobenius bimonads (see Theorem 3.8).

1. Preliminaries

Throughout \( \mathcal{A} \) and \( \mathcal{B} \) will denote any categories. The composition of two morphisms \( f \) and \( g \) in a category will be written as \( g \cdot f \) or \( gf \). \( I_\mathcal{A} \), \( A \), or just \( I \), will stand for the identity morphism on an object \( A \). \( I_F \) or \( F \) denote the identity transformation on the functor \( F \), and \( I_\mathcal{A} \) means the identity functor of a category \( \mathcal{A} \). The application of a natural transformation \( \xi \) on an object \( A \) is (as usual) denoted
by $\xi_A$. To ease notation in diagrams we will take the liberty to delete the suffix from $\xi_A$ and only write $\xi$ if no ambiguity arises.

The following modification of the notion of (co-)equalisers in categories will be of help.

1.1. Definitions. Let $\mathbb{K}$ be a class of morphisms in a category $\mathbb{A}$ closed under composition. A cofork

$$B \xrightarrow{k} C \xrightarrow{g} D$$

is said to be a $\mathbb{K}$-equaliser provided $k \in \mathbb{K}$ and, for any $h : Q \to C$ in $\mathbb{K}$ with $f \cdot h = g \cdot h$, there exists a unique $q : Q \to B$ in $\mathbb{K}$ such that $h = k \cdot q$. If this holds, then, for morphisms $r, s : X \to B$ in $\mathbb{K}$, $k \cdot r = k \cdot s$ implies $r = s$.

Similarly, a fork

$$B \xleftarrow{g} C \xleftarrow{s} D$$

is said to be a $\mathbb{K}$-coequaliser provided $s \in \mathbb{K}$ and, for any $h : C \to Q$ in $\mathbb{K}$ with $h \cdot f = h \cdot g$, there is a unique $q : Q \to B$ in $\mathbb{K}$ such that $s = q \cdot s$. In this case, for morphisms $t, u : D \to Y$ in $\mathbb{K}$, $t \cdot s = u \cdot s$ implies $t = u$.

A class $\mathbb{K}$ of morphisms in $\mathbb{A}$ is called an ideal class if for morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathbb{A}$, $f$ or $g$ in $\mathbb{K}$ implies $g \cdot f$ is in $\mathbb{K}$.

Taking for $\mathbb{K}$ the class of all morphisms in $\mathbb{A}$, the notions defined above yield the usual equalisers and coequalisers in the category $\mathbb{A}$.

We recall some notions from [6], [7].

1.2. Pairings of functors. For functors $L : \mathbb{A} \to \mathbb{B}$ and $R : \mathbb{B} \to \mathbb{A}$ between any categories $\mathbb{A}$ and $\mathbb{B}$, pairings are defined as maps, natural in $A \in \mathbb{A}$ and $B \in \mathbb{B}$,

$$\text{Mor}_\mathbb{B}(L(A), B) \xrightarrow{\alpha} \text{Mor}_\mathbb{A}(A, R(B)),$$

$$\text{Mor}_\mathbb{A}(R(B), A) \xrightarrow{\beta} \text{Mor}_\mathbb{B}(B, L(A)).$$

These - and their compositions - are determined by natural transformations obtained as image of the corresponding identity morphisms,

| map | natural transformation |
|-----|-----------------------|
| $\alpha$ | $\eta : I_\mathbb{A} \to RL$, |
| $\beta$ | $\varepsilon : LR \to I_\mathbb{B}$, |
| $\beta \cdot \alpha$ | $L \xrightarrow{\eta} LRL \xleftarrow{L} L$ |
| $\alpha \cdot \beta$ | $R \xrightarrow{\eta R} RLR \xrightarrow{R} R$ |

| map | natural transformation |
|-----|-----------------------|
| $\tilde{\alpha}$ | $\psi : I_\mathbb{B} \to LR$, |
| $\tilde{\beta}$ | $\varphi : RL \to I_\mathbb{A}$, |
| $\tilde{\beta} \cdot \tilde{\alpha}$ | $L \xrightarrow{\psi L} LRL \xrightarrow{L \varphi} L$ |
| $\tilde{\alpha} \cdot \tilde{\beta}$ | $R \xrightarrow{R \psi} RLR \xrightarrow{\psi R} R$ |

From this we obtain natural endomorphisms (e.g. [7 2.4])

$$\vartheta : RL \xrightarrow{RLR} RLLR \xrightarrow{RLL} RL$$

$$\tilde{\vartheta} : RL \xrightarrow{R \psi L} RLLR \xrightarrow{L \psi L} RL$$

$$\gamma : LR \xrightarrow{LRL} LRLR \xrightarrow{LRL} LR$$

$$\tilde{\gamma} : RL \xrightarrow{R \psi L} RRLR \xrightarrow{RRL} RL.$$
A pairing \((L, R, \alpha, \beta)\) is said to be **regular** if \(\alpha \cdot \beta \cdot \alpha = \alpha\) and \(\beta \cdot \alpha \cdot \beta = \beta\); in this case, \(\vartheta\) and \(\gamma\) are idempotent, \(\vartheta \cdot \eta = \eta \cdot \vartheta\), and \(\varepsilon \cdot \gamma = \varepsilon = \varepsilon \cdot \gamma\).

\(\beta\) (resp. \(\alpha\)) is said to be **symmetric** if \(\gamma = \gamma\) (resp. \(\vartheta = \vartheta\)) (see [7, Section 3]).

### 1.3. Related comonads

For any natural transformation \(\eta : I_B \rightarrow RL\), \((LR, L\eta R)\) is a non-counital comonad and defines the category \(B \rightarrow LR\) of non-counital \(LR\)-comodules. Given an ideal class \(K\) of morphisms in \(B \rightarrow LR\), a comodule \((B, \omega)\) is called a **K-cofirm comodule** provided the defining cofork

\[
B \xrightarrow{\omega} LR(B) \xrightarrow{L\eta R(B)} LRLR(B)
\]

is a \(K\)-equaliser. If we choose for \(K\) all morphisms in \(B \rightarrow LR\), a \(K\)-cofirm comodule is just called **cofirm**.

### 1.4. Regular pairings and comodules

Now assume \((L, R, \alpha, \beta)\) to be a regular pairing with \(\beta\) symmetric. Then \((LR, L\eta R, \varepsilon)\) is a weak comonad (see [7, Definition 4.3]) on \(B\). A non-counital \(LR\)-comodule \((B, \omega)\) is called a **\(K\)-cofirm comodule** provided the defining cofork

\[
B \xrightarrow{\omega} LR(B) \xrightarrow{L\eta R(B)} LRLR(B)
\]

is a \(K\)-equaliser. If we choose for \(K\) all morphisms in \(B \rightarrow LR\), a \(K\)-cofirm comodule is just called **cofirm**.

### 1.5. Compatible comodule morphisms

Again let \((L, R, \alpha, \beta)\) be a regular pairing with \(\beta\) symmetric and \(\gamma = LR\varepsilon \cdot L\eta R\). We call a morphism \(h\) between \(LR\)-comodules \((B, \omega)\) and \((B', \omega')\) **\(\gamma\)-compatible**, provided it induces commutativity of the triangles in the diagram

\[
\begin{array}{ccc}
B & \xrightarrow{\omega} & LR(B) & \xrightarrow{\varepsilon_B} & B \\
\downarrow h & & \downarrow h & & \downarrow h \\
B' & \xrightarrow{\omega'} & LR(B') & \xrightarrow{\varepsilon_{B'}} & B'.
\end{array}
\]

Clearly, since the outer diagram is always commutative for comodule morphisms, it is enough to require commutativity for one of the triangles. Thus one readily obtains:

1. The class \(K_\gamma\) of all \(\gamma\)-compatible morphisms in \(B \rightarrow LR\) is an ideal class.
2. A morphism \(h : Q \rightarrow LR(B)\) of \(LR\)-comodules is in \(K_\gamma\) if and only if \(\gamma_B \cdot h = h\).
3. A morphism \(h : LR(B) \rightarrow Q\) of \(LR\)-comodules is in \(K_\gamma\) if and only if \(h \cdot \gamma_B = h\).
Evidently, an \( LR \)-comodule \((B, \omega)\) is compatible (as in [1.3]) if and only if \( \omega \in \mathbb{K}_\gamma \), that is, \( \omega = \gamma_B \cdot \omega \).

Notice that \( \gamma = I_{LR} \) implies that every non-counital \( LR \)-comodule is \( \gamma \)-compatible, that is, \( \mathbb{K}^L_{LR} = \mathbb{K}^R_{LR} \); in this case, however, not every \( LR \)-comodule morphism need to be \( \gamma \)-compatible and an \( LR \)-comodule \((B, \omega)\) need not be counital but satisfies \( \omega = \omega \cdot \varepsilon_B \cdot \omega \).

1.6. Proposition. Let \((L, R, \alpha, \beta)\) be a regular pairing with \( \beta \) symmetric. Then any compatible \( LR \)-comodule \((B, \omega)\) is \( \mathbb{K}_\gamma \)-cofirm.

Proof. We have to show that the cofork

\[
\begin{array}{c}
B \xrightarrow{\omega} LR(B) \xrightarrow{L_{R(B)}} LRLR(B)
\end{array}
\]

is a \( \mathbb{K}_\gamma \)-equaliser. Let \((Q, \kappa)\) be an \( LR \)-comodule and \( h : Q \to LR(B) \) a morphism in \( \mathbb{K}_\gamma \) with \( LR(\omega) \cdot h = L_{R(B)} \cdot h \). For the morphism \( \tilde{h} = \varepsilon_B \cdot h : Q \to B \) we have the diagram

\[
\begin{array}{ccc}
Q & \xrightarrow{h} & LR(B) \\
\downarrow{\kappa} & & \downarrow{\varepsilon_B} \\
LR(Q) & \xrightarrow{L_{R(h)}} & LRRLR(B) \xrightarrow{\varepsilon_{LR}} LR(B)
\end{array}
\]

with commutative inner rectangles and commutative outer paths. This shows that \( \tilde{h} \) is an \( LR \)-comodule morphism with

\[
\begin{align*}
\omega \cdot \tilde{h} &= \omega \cdot \varepsilon_B \cdot h = \varepsilon_{LR(B)} \cdot LR(\omega) \cdot h \\
&= \varepsilon_{LR(R)} \cdot L_{R(R)}(\cdot) h = \gamma_B \cdot h = h, \\
\varepsilon_B \cdot \omega \cdot \tilde{h} &= \varepsilon_B \cdot \gamma_B \cdot h = \varepsilon_B \cdot h = \tilde{h},
\end{align*}
\]

thus \( \tilde{h} \in \mathbb{K}_\gamma \). Moreover, for any \( q : Q \to B \) in \( \mathbb{K}_\gamma \) with \( \omega \cdot q = h \), we have \( \varepsilon_B \cdot h = \varepsilon_B \cdot \omega \cdot q = q \), showing uniqueness of \( q \).

Replacing \((Q, h)\) in diagram \((1.1)\) by \((B, \omega)\), we see that \( \varepsilon_B \cdot \omega \) is a comodule morphism and this leads to the following observation.

1.7. Proposition. Let \((L, R, \alpha, \beta)\) be an adjunction. Then a (non-counital) \( LR \)-comodule \((B, \omega)\) is cofirm if and only if it is counital.

Proof. Since we have an adjunction, \( \gamma = I_{LR} \), every \( LR \)-comodule \((B, \omega)\) morphisms is \( \gamma \)-compatible, and \( \omega = \omega \cdot \varepsilon_B \cdot \omega \) (see [1.3]).

If \((B, \omega)\) is cofirm, then \( \omega \) is monomorph in \( \mathbb{K}^L_{LR} \); since \( \varepsilon_B \cdot \omega \) and \( I_B \) are morphisms in \( \mathbb{K}^L_{LR} \) we conclude \( \varepsilon_B \cdot \omega = I_B \), that is, \((B, \omega)\) is counital.

It is folklore that any counital \( LR \)-comodule is cofirm.

1.8. Related monads. Given any natural transformation \( \varphi : RL \to I_R \), \((LR, L\varphi R)\) is a non-unital monad and the category of non-unital \( LR \)-modules is denoted by \( \mathbb{K}^L_{LR} \) (see [7]). For an ideal class \( \mathbb{K} \) of morphisms in \( \mathbb{K}^L_{LR} \), a module \((B, \varrho)\) is called \( \mathbb{K} \)-firm provided the fork

\[
\begin{array}{c}
LRLR(B) \xrightarrow{L_{R(R)}(B)} LR(B) \xrightarrow{\varrho} B
\end{array}
\]
is a \( \mathbb{K} \)-coequaliser (Definitions 1.1).

1.9. Remarks. Following [2, 2.3], an \( LR \)-module \((B, \varrho)\) is called firm provided it is \( \mathbb{K} \)-firm for the class \( \mathbb{K} \) of all morphisms in \( \mathbb{B}_{LR} \) and \( \varrho \) is an epimorphism in \( \mathbb{B} \). The term firm was coined by Quillen for non-unital algebras \( A \) over a commutative ring \( k \) with the property that the map \( A \otimes_A A \to A, a \otimes b \mapsto ab \), is an isomorphism. Then, an \( A \)-module is firm provided it is firm for the monad \( A \otimes_k - \) on the category of \( k \)-modules. In the category of non-unital \( A \)-modules, equalisers are induced by equalisers of \( k \)-modules and hence are epimorph (in fact surjective) as \( k \)-module morphisms (e.g. [2, Proposition 5]).

1.10. Regular pairings and monads. Now assume \((R, L, \tilde{\alpha}, \tilde{\beta})\) to be a regular pairing with \( \tilde{\alpha} \) symmetric. Then \((LR, R\varphi L, \psi)\) is a weak monad (see [7, Definition 3.3]) on \( \mathbb{B} \).

A non-counital \( LR \)-module \((B, \varrho)\) is said to be compatible (see [7, 3.2]) provided

\[
LR(B) \xrightarrow{\varrho} B = LR(B) \xrightarrow{LR\psi} LRLR(B) \xrightarrow{L\varphi R(B)} LR(B) \xrightarrow{\varrho} B
\]

where the two equalities are equivalent conditions on \( \varrho \) and can be written as \( \varrho = \varrho \cdot \tilde{\varrho} \). By \( \mathbb{B}_{LR} \) we denote the full subcategory of \( \mathbb{B}_{LR} \) formed by the compatible \( LR \)-modules.

For any \( B \in \mathbb{B} \), \((LR(B), L\varphi R(B))\) is a compatible \( LR \)-module yielding a functor

\[
\phi_{LR}: \mathbb{B} \to \mathbb{B}_{LR}, \quad B \mapsto (LR(B), L\varphi R(B)).
\]

The forgetful functor \( U_{LR}: \mathbb{B}_{LR} \to \mathbb{B} \) need not be (right) adjoint to \( \phi_{LR} \).

1.11. Compatible module morphisms. Let \((R, L, \tilde{\alpha}, \tilde{\beta})\) be regular with \( \tilde{\alpha} \) symmetric and \( \tilde{\varrho} = L\varphi R \cdot LR\psi \) (see 1.2). A morphism \( h \) between \( LR \)-modules \((B, \varrho)\) and \((B', \varrho')\) is called \( \tilde{\varrho} \)-compatible, provided it induces commutativity of the triangles in the diagram

\[
\begin{array}{ccc}
B & \xrightarrow{\psi_B} & LR(B) & \xrightarrow{\varrho} & B \\
\downarrow{h} & & \downarrow{h} & & \downarrow{h} \\
B' & \xrightarrow{\psi_{B'}} & LR(B') & \xrightarrow{\varrho'} & B'.
\end{array}
\]

Since the outer diagram is always commutative for module morphisms, it is enough to require commutativity for one of the triangles. One easily can show (compare 1.5):

(1) The class \( \mathbb{K}_{\tilde{\varrho}} \) of all \( \tilde{\varrho} \)-compatible morphisms in \( \mathbb{B}_{LR} \) is an ideal class.

(2) A morphism \( h: Q \to LR(B) \) of \( LR \)-modules is in \( \mathbb{K}_{\tilde{\varrho}} \) if and only if \( \tilde{\varrho}_B \cdot h = h \).

(3) A morphism \( h: LR(B) \to Q \) of \( LR \)-modules is in \( \mathbb{K}_{\tilde{\varrho}} \) if and only if \( h \cdot \tilde{\varrho}_B = h \).

Clearly, an \( LR \)-module \((B, \varrho)\) is compatible (see 1.10) if and only if \( \varrho \in \mathbb{K}_{\tilde{\varrho}} \), that is, \( \varrho \cdot \tilde{\varrho}_B = \varrho \).
1.12. Remarks. Given the assumptions in 1.11 one may consider the subcategory of $\mathcal{B}_{LR}$ consisting of the same objects and as morphisms the $\tilde{\vartheta}$-compatible morphisms. Then the identity morphism on a $\tilde{\vartheta}$-compatible module $(B, \varrho)$ is $\varrho \cdot \psi_B : B \to B$. This is also considered in [3, Remark 2.5] in similar situations (but with different terminology). The equalisers in this category are essentially the $K_{\tilde{\vartheta}}$-equalisers.

Dual to the Propositions 1.6 and 1.7 we now have:

1.13. Proposition. Let $(R, L, \tilde{\alpha}, \tilde{\beta})$ be a regular pairing with $\tilde{\alpha}$ symmetric. Then any $\tilde{\vartheta}$-compatible $LR$-module $(B, \varrho)$ is $K_{\tilde{\vartheta}}$-firm.

1.14. Proposition. Let $(R, L, \bar{\alpha}, \bar{\beta})$ be an adjunction. Then a (non-unital) $LR$-module $(B, \varrho)$ is firm if and only if it is unital.

2. Frobenius property and Frobenius bimodules

2.1. Related monads and comonads. In the setting of 1.2, assume $\alpha$ and $\tilde{\beta}$ to be given, that is, there are natural transformations $\eta : I_R \to RL$ and $\varphi : RL \to I_L$.

Then $(LR, L\eta R)$ is a non-counital comonad with comodule category $\mathcal{B}_{LR}$ and $(LR, L\varphi R)$ is a non-unital monad on $\mathcal{B}$ with module category $\mathcal{B}_{LR}$ (see [7]). By naturality we have the commutative diagram (Frobenius property)

\[
\begin{array}{ccc}
LRLR & \xrightarrow{L\eta R} & LR \\
\downarrow L\varphi R & \quad & \downarrow L\eta R \\
LRLR & \xrightarrow{L\varphi R} & LR \\
\end{array}
\]

We are interested in $LR$-modules and $LR$-comodules subject to a reasonable compatibility condition.

2.2. Frobenius bimodules. A triple $(B, \varrho, \omega)$ with an object $B \in \mathcal{B}$ and two morphisms $\varrho : LR(B) \to B$ and $\omega : B \to LR(B)$ is called a Frobenius bimodule provided the data induce commutativity of the diagram

\[
\begin{array}{ccc}
LRLR(B) & \xrightarrow{LR(\varrho)} & LR(B) \\
\downarrow L\varphi R & \quad & \downarrow \varrho & \quad & \downarrow L\varphi R \\
LR(B) & \xrightarrow{\omega} & LR(B) \\
\downarrow L\eta R & \quad & \downarrow \omega & \quad & \downarrow L\eta R \\
LRLR(B) & \xrightarrow{LR(\omega)} & LR(B) \\
\end{array}
\]

This implies that $\varrho : LR(B) \to B$ defines a (non-unital) $LR$-module and $\omega : B \to LR(B)$ a (non-counital) $LR$-comodule; if that is already known, the conditions on Frobenius bimodules reduce to commutativity of the diagrams (II) and (III), that
is commutativity of (Frobenius property for modules)

\[
\begin{array}{ccc}
\text{LR}(B) & \xrightarrow{\eta} & B \\
& \downarrow \phi & \downarrow \omega \\
\text{LR}(B) & \xrightarrow{\omega} & \text{LR}(B)
\end{array}
\]

With the Frobenius bimodules as objects and morphisms, which respect the module as well as the comodule structure, we obtain the category of Frobenius bimodules which we denote by \( \mathbb{B}^{LR} \).

It follows from the commutative diagram (2.1) that, for any \( B \in \mathbb{B} \), \( \text{LR}(B) \) is a Frobenius bimodule with the canonical structures, that is, there is a comparison functor

\[
K_{LR}^{LR} : \mathbb{B} \rightarrow \mathbb{B}^{LR}, \quad B \mapsto (\text{LR}(B), L\varphi_{R(B)}, L\eta_{R(B)}).
\]

2.3. Natural mappings. Assume again \( \eta : I_{\mathbb{A}} \rightarrow RL \) and \( \varphi : RL \rightarrow I_{\mathbb{A}} \) to be given (see 1.2). Then we have maps, natural in the two arguments \( A, A' \in \mathbb{A} \),

\[
\begin{align*}
\Phi_{A, A'} : \text{Mor}_\mathbb{B}(L(A), L(A')) & \rightarrow \text{Mor}_\mathbb{A}(A, A'), \quad g \mapsto \varphi_{A'} \cdot R(g) \cdot \eta_A, \\
L_{A, A'} : \text{Mor}_\mathbb{A}(A, A') & \rightarrow \text{Mor}_\mathbb{B}(L(A), L(A')), \quad f \mapsto L(f), \\
\Phi_{A, A'} \cdot L_{A, A'} : \text{Mor}_\mathbb{A}(A, A') & \rightarrow \text{Mor}_\mathbb{A}(A, A'), \quad f \mapsto f \cdot \varphi_{A} \cdot \eta_{A} = \varphi_{A'} \cdot \eta_{A'} \cdot f.
\end{align*}
\]

In case \( \varphi \cdot \eta = I_{\mathbb{A}} \), \( \Phi \cdot L_{\_, \_} \) is the identity and \( L \) is a separable functor.

If \( \eta \cdot \varphi \cdot \eta = \eta \) (\( \eta \) is regular), then \( \Phi \cdot L_{\_, \_} \) is idempotent, more precisely, \( \Phi \cdot L_{\_, \_} \cdot \Phi = \Phi \), that is, \( \Phi \) is regular, and \( L \) may be called a weakly separable functor.

The natural transformation

\[
\theta : LR \xrightarrow{L\eta} LRLR \xrightarrow{L\varphi} LR
\]

is an \( LR \)-module as well as an \( LR \)-comodule morphism.

2.4. Proposition. Given \( \eta : I_{\mathbb{A}} \rightarrow RL \) and \( \varphi : RL \rightarrow I_{\mathbb{A}} \), let \( (B, \varrho, \omega) \) be a Frobenius \( LR \)-bimodule (see 2.2). Then

\[
\varrho \cdot \omega \cdot \varrho = \varrho \cdot \theta_B \quad \text{and} \quad \omega \cdot \varrho \cdot \omega = \theta_B \cdot \omega.
\]

(1) If \( \varphi \cdot \eta = I_{\mathbb{A}} \), then \( \varrho \cdot \omega \cdot \varrho = \varrho \) and \( \omega \cdot \varrho \cdot \omega = \omega \). Thus, if \( \varrho \) is an epimorphism in \( \mathbb{B}^{LR} \) or \( \omega \) is a monomorphism in \( \mathbb{B}^{LR} \), then \( \varrho \cdot \omega = I_B \).

(2) (i) If \( \eta \cdot \varphi \cdot \eta = \eta \), then \( \varrho \cdot \omega \) is an idempotent morphisms.

(ii) If \( \varphi \cdot \eta \cdot \varphi = \varphi \), then \( \omega \cdot \varrho \) is an idempotent morphisms.
Proof. The equalities claimed and (1) can be derived from the commutative diagram

\[
\begin{array}{cccccccccc}
\ & \ & LRLR(B) & \ \ & LR(B) & \ \ & \ & \ & \\
\ & \ & \ & \ & \ & \ & \ & \ & \\
B & \xrightarrow{\omega} & LR(B) & \xrightarrow{\theta} & B & \xrightarrow{\omega} & LR(B) & \xrightarrow{\varphi} & B \\
\ & \ & \ & \ & \ & \ & \ & \ & \\
\ & \ & LR(B) & \xrightarrow{L\varphi(B)} & LRLR(B) & \ & \ & \ & \\
\end{array}
\]

(2) To show this, extend the above diagram by \(\omega\) on the right or by \(\varphi\) on the left, respectively.

\[\square\]

2.5. Compatible bimodule morphisms. Let \(\eta : I_h \to RL\) and \(\varphi : RL \to I_h\) be given. A morphism \(h\) between Frobenius modules \((B, \omega, \varphi)\) and \((B', \omega', \varphi')\) is called \(\theta\)-compatible, provided it induces commutativity of the diagram

\[
\begin{array}{cccccccccc}
\ & \ & LR(B) & \ \ & LR(B) & \ \ & \ & \ & \\
\ & \ & \ & \ & \ & \ & \ & \ & \\
B & \xrightarrow{\omega} & LR(B) & \xrightarrow{\theta} & B & \xrightarrow{\omega} & LR(B) & \xrightarrow{\varphi} & B \\
\ & \ & \ & \ & \ & \ & \ & \ & \\
\ & \ & LR(B) & \xrightarrow{LR(B)} & LR(B) & \ & \ & \ & \\
\end{array}
\]

One easily obtains the following.

1. The class \(\mathbb{K}_\theta\) of all \(\theta\)-compatible bimodule morphisms in \(\mathbb{B}\) is an ideal class.

2. A morphism \(h : Q \to LR(B)\) of \(LR\)-bimodules is in \(\mathbb{K}_\theta\) if and only if \(\theta_B \cdot h = h\).

3. A morphism \(h : LR(B) \to Q\) of \(LR\)-bimodules is in \(\mathbb{K}_\theta\) if and only if \(h \cdot \theta_B = h\).

4. If \(\varphi \cdot \eta \cdot \varphi = \varphi\), then \(L\varphi R = \theta \cdot L\varphi R\), that is, \(L\varphi R\) is \(\theta\)-compatible.

5. If \(\eta \cdot \varphi \cdot \eta = \eta\), then \(L\eta R = L\eta R \cdot \theta\), that is, \(L\eta R\) is \(\theta\)-compatible.

6. For a Frobenius bimodule \((B, \omega, \varphi)\), \(\omega\) is \(\theta\)-compatible if and only if \(\omega = \omega \cdot \varphi \cdot \omega\) and \(\varphi\) is \(\theta\)-compatible if and only if \(\varphi = \varphi \cdot \omega \cdot \varphi\).

Further observations:

2.6. Proposition. Let \(\eta : I_h \to RL\) and \(\varphi : RL \to I_h\) be given.

1. Consider a Frobenius \(LR\)-bimodule \((B, \omega, \varphi)\).
   \[(i)\] If \(\omega\) is \(\theta\)-compatible, then \((B, \omega)\) is \(\mathbb{K}_\theta\)-cofirm.
   \[(ii)\] If \(\varphi\) is \(\theta\)-compatible, then \((B, \varphi)\) is \(\mathbb{K}_\theta\)-firm.

2. If \((\eta, \varphi)\) is a regular pair, then, for any \(B \in \mathbb{B}\),
   \[(i)\] \((LR(B), L\varphi R(B))\) is a \(\mathbb{K}_\theta\)-firm module,
   \[(ii)\] \((LR(B), L\eta R(B))\) is a \(\mathbb{K}_\theta\)-cofirm comodule.

We add some technical observations for later use.
2.7. Lemma. Refer to the notation in \[1.2\] and \[2.3\]

1. Let \((L, R, \alpha, \beta)\) be any pairing and \(\varphi : RL \to I_A\) a natural transformation satisfying \(\eta \cdot \varphi \cdot \eta = \eta\). Then \(\gamma \cdot \theta = \gamma\).

2. Let \((R, L, \bar{\alpha}, \bar{\beta})\) be any pairing and \(\eta : I_A \to RL\) a natural transformation satisfying \(\varphi \cdot \eta \cdot \varphi = \varphi\). Then \(\bar{\alpha} \cdot \theta = \bar{\alpha}\).

Proof. The assertions follow immediately from the definitions. \(
\)

2.8. Proposition. As in \[2.1\] assume \(\eta : I_A \to RL\) and \(\varphi : RL \to I_A\) to be given. Let \(\mathbb{K}\) be an ideal class of \(LR\)-comodule morphisms and assume \(L\varphi_{R(B)}\) in \(\mathbb{K}\) for any \(B \in \mathbb{B}\).

1. If \((B, \omega)\) in \(\mathbb{B}^{LR}\) is a \(\mathbb{K}\)-cofirm comodule (see \[1.3\]), then there exists a morphism \(\varrho : LR(B) \to B\) in \(\mathbb{K}\) making \((B, \varrho, \omega)\) a Frobenius bimodule.

2. With this module structure, \(LR\)-comodule morphisms between \(\mathbb{K}\)-cofirm \(LR\)-comodules \((B, \omega)\) and \((B', \omega')\) are morphisms of the Frobenius bimodules \((B, \omega, \varrho)\) and \((B', \omega', \varrho')\).

Proof. (1) Consider the diagram

\[
\begin{array}{cccc}
LRLR(B) & \xrightarrow{L^R(\varrho)} & LR(B) & \xrightarrow{L^R(\omega)} & LRLR(B) \\
\downarrow{L_\varphi R} & (I) & \downarrow{\varrho} & (II) & \downarrow{L_\varphi R} \\
LR(B) & \xrightarrow{\varrho} & B & \xrightarrow{\omega} & LR(B) \\
\downarrow{L_\eta R} & (III) & \downarrow{\omega} & (IV) & \downarrow{L_\eta R} \\
LRLR(B) & \xrightarrow{L^R(\varrho)} & LR(B) & \xrightarrow{L^R(\omega)} & LRLR(B),
\end{array}
\]

where \(IV\) is assumed to be a \(\mathbb{K}\)-equaliser. Since

\[
L_\eta_{R(B)} \cdot L\varphi_{R(B)} \cdot LR(\omega) = L\varphi_{RLR(B)} \cdot LRL_\eta_{R(B)} \cdot LR(\omega) = L\varphi_{RLR(B)} \cdot LRL(\omega) \cdot LR(\omega) = LR(\omega) \cdot L\varphi_{R(B)} \cdot LR(\omega)
\]

and \(L\varphi_{R(B)} \cdot LR(\omega)\) is a morphisms in \(\mathbb{K}\), there exists a morphism \(\varrho : LR(B) \to B\) in \(\mathbb{K}\) leading to the commutative diagrams \((II)\) and \((III)\). Moreover,

\[
\omega \cdot \varrho \cdot L\varphi_{R(B)} = LR(\varrho) \cdot L_\eta_{R(B)} \cdot L\varphi_{R(B)} = LR(\varrho) \cdot L\varphi_{RLR(B)} \cdot LRL_\eta_{R(B)} = L\varphi_{R(B)} \cdot LRLR(\varrho) \cdot LRL_\eta_{R(B)} = L\varphi_{R(B)} \cdot LR(\omega) \cdot LR(\varrho) = \omega \cdot \varrho \cdot LR(\varrho),
\]

and hence \(\varrho \cdot L\varphi_{R(B)} = \varrho \cdot LR\varrho\) since \(\omega\) is a \(\mathbb{K}\)-equaliser. This means that the diagram \((I)\) is also commutative.

(2) Now let \(h : B \to B'\) be an \(LR\)-comodule morphism. Then

\[
\omega' \cdot h \cdot \varrho = LR(h) \cdot \omega' \cdot \varrho = LR(h) \cdot L\varphi_{R(B')} \cdot LR(\omega) = L\varphi_{R(B')} \cdot LRLR(h) \cdot LR(\omega) = L\varphi_{R(B')} \cdot LR(\omega') \cdot LR(h) = \omega' \cdot \varrho' \cdot LR(h)
\]
and, since both $h \cdot \varrho$ and $q' \cdot LR(h)$ are in $K$, this implies that they are equal (see Definition 1.1), that is, $h$ is also an $LR$-module morphism. 

Symmetric to Proposition 2.8 we get:

2.9. Proposition. As in 2.1 assume $\eta : I_\mathcal{L} \to RL$ and $\varphi : RL \to I_\mathcal{H}$ to be given. Let $\mathcal{K}'$ be an ideal class of $LR$-module morphisms.

(1) If $(B, \varrho)$ in $\mathcal{K}'$-firm module (see 1.8) and $L\eta_{RL}(B)$ belongs to $\mathcal{K}'$, then there is a morphism $\omega : B \to LR(B)$ in $K'$ making $(B, \varrho, \omega)$ a Frobenius bimodule.

(2) With this comodule structure, $LR$-module morphisms between $K'$-firm modules $(B, \varrho)$ and $(B', \varrho')$ are morphisms of the Frobenius bimodules $(B, \omega, \varrho)$ and $(B', \omega', \varrho')$.

2.10. Regular pairings $(L, R)$ and Frobenius bimodules. Now let $(L, R, \alpha, \beta)$ be a regular pairing with $\beta$ symmetric. Then $\hat{\varrho} : RL \to RL$ (see 1.2) is an idempotent natural transformation with $\eta = \hat{\varrho} \cdot \eta$ and $\hat{\varphi} := \varphi \cdot \hat{\varrho} : RL \to I_\mathcal{L}$ is a natural transformation with $\hat{\varphi} = \hat{\varphi} \cdot \varphi$. Hence $L\hat{\varphi}R$ is the top path in the diagram

\[
\begin{array}{cccc}
LRLR & \xrightarrow{L\eta_{RL}} & LRLRLR & \xrightarrow{LR\varepsilon LR} & LRRLR \\
L\hat{\varphi}R & \downarrow & L\eta_{RL} & \downarrow LRLRLR & \xrightarrow{\varepsilon LR} & L\hat{\varphi}R \\
LR & \xrightarrow{L\eta_{RL}} & LRLR & \xrightarrow{\varepsilon LR} & LR.
\end{array}
\]

The left rectangle is commutative by the Frobenius property and shows that $L\hat{\varphi}R$ is an $LR$-comodule morphism. Since $\gamma = \gamma_{\mathcal{K}}$, $LR\varepsilon$ and $\varepsilon LR$ may be interchanged in the second rectangle which then becomes commutative. Thus we obtain commutativity of the diagram

\[
\begin{array}{cccc}
LRLR & \xrightarrow{\gamma LR} & LRLR \\
L\hat{\varphi}R & \downarrow & L\hat{\varphi}R & \downarrow L\hat{\varphi}R \\
LR & \xrightarrow{\gamma} & LR
\end{array}
\]

showing that $L\hat{\varphi}R$ is in $\mathcal{K}_{\gamma}$. Hence, without loss of generality, we may - and will - assume that $L\varphi R$ is $\gamma$-compatible as $LR$-comodule morphism.

2.11. Proposition. Let $(L, R, \alpha, \beta)$ be a regular pairing with $\beta$ symmetric and assume to have a natural transformation $\varphi : RL \to I_\mathcal{H}$ such that $L\varphi R$ is $\gamma$-compatible in $\mathcal{K}_{LR}$.

(1) $L\varphi R = L\varphi R \cdot L\eta_{RL} = \vartheta$.
(2) if $L\varphi R$ is epimorph in $\mathcal{K}_{\gamma}$, then $L\varphi R \cdot L\eta_{RL} = LR\varepsilon \cdot L\eta_{RL}$, that is, $\vartheta = \gamma$;
(3) if $L\varphi R$ is epimorph in $\mathcal{K}_{LR}$, then $\vartheta = I_{LR}$;
(4) if $\varphi \cdot \eta = I_\mathcal{H}$, then $\gamma = I_{LR}$.

Proof. (1) By our assumptions and Lemma 2.1,

$L\varphi R : L\eta_{RL} = L\varphi R \cdot \gamma LR \cdot \theta LR = L\varphi R \cdot \gamma LR = L\varphi R$;

now the formula can be derived by extending diagram 2.1 with $L\varphi R$ to the right.

(2) This follows by the fact that $L\varphi R \in \mathcal{K}_{\gamma}$ by assumption and $L\eta_{RL} \in \mathcal{K}_{\gamma}$ by regularity.
(3) The claim is a consequence of (1) by properties of epimorphisms.
(4) follows directly from (1) and (2). □

As shown in [1.6], compatible $LR$-comodules are $K_\gamma$-firm in our situation and we use this for our next results.

**2.12. Proposition.** Let $(L, R, \alpha, \beta)$ be a regular pairing with $\beta$ symmetric and assume to have a natural transformation $\varphi : RL \to I_K$ such that $L_\varphi R$ is $\gamma$-compatible.

(1) For any compatible $LR$-comodule $\omega : B \to LR(B)$, there is some $\varrho : LR(B) \to B$ in $K_\gamma$ making $(B, \varrho, \omega)$ a Frobenius module and it is given by

\[
\varrho : LR(B) \xrightarrow{LR(\omega)} LRLR(B) \xrightarrow{L_\varphi R(B)} LR(B) \xrightarrow{\varepsilon_B} B.
\]

(2) Any $LR$-comodule morphism between compatible comodules $(B, \omega)$ and $(B', \omega')$ is an $LR$-bimodule morphism between $(B, \varrho, \omega)$ and $(B', \varrho', \omega')$.

(3) If $\eta \cdot \varphi \cdot \eta = \eta$, then, for any Frobenius module $(B, \varrho, \omega)$ with $\omega$ $\gamma$-compatible, $\omega \cdot \varrho \cdot \omega = \omega$ and $\varrho \cdot \omega = \varepsilon_B \cdot \omega$.

**Proof.** (1) As shown in Proposition 1.6, $(B, \omega)$ is $K_\gamma$-cofirm and hence the existence of $\varrho$ follows by Proposition 2.8. For the Frobenius module $(B, \varrho, \omega)$, we have the commutative diagram

\[
\begin{array}{ccc}
LRLR(B) & \xrightarrow{\varepsilon_{LR}} & LR(B) \\
\downarrow{L\eta R} & & \downarrow{L_{R(\omega)}} \\
LR(B) & \xrightarrow{\varrho} & B \\
\downarrow{LR(\omega)} & & \downarrow{\varepsilon_B} \\
LRLR(B) & & \\
\end{array}
\]

Since $\varrho$ is $\gamma$-compatible, the upper paths yields $\varrho \cdot \gamma = \varrho$. The lower path is the composite given in (1).

(2) Since $K_\gamma$ is an ideal class, the assertion follows by Proposition 2.8.

(3) We know from [2.3] that $\omega \cdot \varrho \cdot \omega = \theta_B \cdot \omega$ and, under the given condition, Lemma 2.7 implies $\gamma = \gamma \cdot \theta$. Combining these equalities yields $\omega \cdot \varrho \cdot \omega = \omega$. By regularity of $(L, R, \alpha, \beta)$, $\omega = \omega \cdot \varepsilon_B \cdot \omega$ and, since $(B, \omega)$ is $K_\gamma$-firm, this implies $\varrho \cdot \omega = \varepsilon_B \cdot \omega$. □

As a special case, we may assume that $(L, R, \alpha, \beta)$ is an adjunction (that is, $\beta = \alpha^{-1}$). This situation is considered in [2 Section 4] and then the outcoming results correspond essentially to [2 Lemma 2] and [2 Corollary 1].

**2.13. Corollary.** Let $(L, R)$ be an adjoint pair and $\varphi : RL \to I_K$ a natural transformation.

(1) For any counital $LR$-comodule $\omega : B \to LR(B)$, there is some $LR$-comodule morphism $\varrho : LR(B) \to B$ - given by (2.5) - making $(B, \varrho, \omega)$ a Frobenius bimodule.

(2) If $(L, R, L_\varphi R)$ allows for a unit, then $(B, \varrho)$ is a unital $LR$-module.

(3) If $\varphi \cdot \eta = I_K$, then, for any Frobenius bimodule $(B, \varrho, \omega)$, $(B, \varrho)$ is a firm $LR$-module.
**Proof.**  (1) follows directly from Proposition 2.12.
(2) can be easily seen from (2.5).
(3) Under the given condition, \( \varrho \cdot \omega = \varepsilon_B \cdot \omega = I_B \). Consider the diagram

\[
\begin{array}{ccc}
LRLR(B) & \xrightarrow{\lambda \varphi R(B)} & LR(B) \\
& \xrightarrow{LR(\varrho)} & B \\
& \downarrow h & \Downarrow q \\
Q & \xrightarrow{q} & B
\end{array}
\]

for some module morphism \( h \) with \( h \cdot L\varphi R(B) = h \cdot LR\varrho \). Then \( q := h \cdot \omega \) is an \( LR \)-module morphism and

\[
q \cdot \varrho = h \cdot \omega \cdot \varrho \\
= h \cdot LR\varrho \cdot L\eta R(B) \\
= h \cdot L\varphi R(B) \cdot L\eta R(B) = h,
\]

so the triangle commutes. Furthermore, for any \( q \) with \( q \cdot \varrho = h \), \( q = q \cdot \varrho \cdot \omega = h \cdot \omega \).
\( \Box \)

2.14. **Functors from comodules.** The construction in Proposition 2.12 yields an equivalence of categories

\[
\Psi : \mathbb{B}^{LR} \rightarrow \mathbb{B}^{LR}, \quad \left( B, \omega \right) \mapsto \left( B, \varrho, \omega \right),
\]

with the forgetful functor \( U_{LR} : \mathbb{B}^{LR} \rightarrow \mathbb{B}^{LR} \) as inverse, where \( \mathbb{B}^{LR} \) denotes the category of Frobenius modules which are compatible as \( LR \)-comodules.

The forgetful functor \( U^{LR} : \mathbb{B}^{LR} \rightarrow \mathbb{A}^{LR} \) yields the functor

\[
U^{LR} \cdot \Psi : \mathbb{B}^{LR} \rightarrow \mathbb{B}^{LR} \rightarrow \mathbb{B}^{LR}, \quad \left( B, \omega \right) \mapsto \left( B, \varrho, \omega \right) \mapsto \left( B, \varrho \right).
\]

Notice that here, for the \( LR \)-module \(( B, \varrho )\), we have \( \varrho = \varrho \cdot \gamma_B \).

2.15. **Regular pairings \(( R, L )\) and Frobenius bimodules.** Referring to the notation in 1.2, let \(( R, L, \tilde{\alpha}, \tilde{\beta} )\) be a regular pairing with \( \tilde{\alpha} \) symmetric, that is, \( \tilde{\alpha} = \tilde{\beta} \), and \( \eta : I_{\tilde{\alpha}} \rightarrow RL \) a natural transformation. Then \( \tilde{\gamma} : RL \rightarrow RL \) is idempotent and \( \tilde{\gamma} := \tilde{\gamma} \cdot \eta : I_{\tilde{\alpha}} \rightarrow RL. \) is a natural transformation with \( I_{\tilde{\eta}} = \tilde{\gamma} \cdot \tilde{\eta} \). Similar to the arguments in 2.10, one obtains that \( L\tilde{\eta}R \) is a \( \tilde{\vartheta} \)-compatible \( LR \)-module morphism, that is, \( L\tilde{\eta}R = \tilde{\vartheta} \cdot L\tilde{\eta}R \) (see 1.10). Hence, without loss of generality, we may and will - assume that \( L\eta R \) is \( \tilde{\vartheta} \)-compatible.

2.16. **Proposition.** Let \(( R, L, \tilde{\alpha}, \tilde{\beta} )\) be a regular pairing with \( \tilde{\alpha} \) symmetric and assume to have a natural transformation \( \eta : I_{\tilde{\alpha}} \rightarrow RL \) such that \( L\eta R \) is \( \tilde{\vartheta} \)-compatible.

1. For any compatible \( LR \)-module \( \varrho : LR(B) \rightarrow B \), there is some \( \omega : B \rightarrow LR(B) \) in \( \mathbb{K}_{\tilde{\eta}} \) making \(( B, \varrho, \omega )\) a Frobenius bimodule and it is given by

\[
\omega : B \xrightarrow{\psi_B} LR(B) \xrightarrow{LR R(B)} LRLR(B) \xrightarrow{LR(\varrho)} LR(B).
\]

2. Any \( LR \)-module morphism between compatible modules \(( B, \varrho )\) and \(( B', \varrho' )\) is an \( LR \)-bimodule morphism between \(( B, \varrho, \omega )\) and \(( B', \varrho', \omega' )\).
2.17. Corollary. Let \((R,L)\) be an adjoint pair and \(\eta : I_\kappa \to RL\) a natural transformation.

1. For any unital \(LR\)-module \(\varrho : LR(B) \to B\), there is some \(LR\)-comodule morphism \(\omega : B \to LR(B)\) – given by (2.6) – making \((B,\varrho,\omega)\) a Frobenius bimodule.

2. If \((LR,L\eta R)\) allows for a counit, then \((B,\omega)\) is a counital \(LR\)-comodule.

2.18. Functors from modules. Proposition 2.16 yields an equivalence

\[
\Phi : B_{LR} \to B_{LR}, \quad (B,\varrho) \mapsto (B,\varrho,\omega),
\]

with the forgetful functor \(U_{LR} : B_{LR} \to B_{-}\) as inverse, where \(B_{LR}\) denotes the category of Frobenius \(LR\)-modules which are compatible as \(LR\)-modules.

The forgetful functor \(U_{LR} : B_{LR} \to B_{-}\) yields the functor

\[
U_{LR} \cdot \Phi : B_{LR} \to B_{LR}, \quad (B,\varrho) \mapsto (B,\varrho,\omega)
\]

Notice that here, for the \(LR\)-comodule \((B,\omega)\), we have \(\omega = \vartheta_B \cdot \omega\).

Summarising we obtain our main result for regular pairings.

2.19. Theorem. In the setting of 1.2, let \((L,R,\alpha,\beta)\) and \((R,L,\tilde{\alpha},\tilde{\beta})\) be regular pairings with \(\beta\) and \(\tilde{\alpha}\) symmetric, and assume \(L\varphi R\) to be \(\gamma\)-compatible and \(L\eta R\) to be \(\tilde{\theta}\)-compatible. Then \(\gamma = \tilde{\theta}\) and the functors constructed above yield category equivalences

\[
\Psi : B^{LR} \to B^{LR}, \quad \Phi : B^{LR} \to B^{LR},
\]

where \(B^{LR}\) denotes the category of Frobenius \(LR\)-bimodules which are compatible as \(LR\)-modules as well as \(LR\)-comodules.

Proof. By the compatibility conditions on \(L\varphi R\) and \(L\eta R\), we have

\[
LR \xrightarrow{L\varphi} LRLR \xrightarrow{L\eta R} LRLR \xrightarrow{L\varphi} LR = LR \xrightarrow{L\eta R} LRLR \xrightarrow{L\varphi} LR = LR \xrightarrow{L\varphi} LRLR \xrightarrow{L\varphi} LR,
\]

that is, \(\gamma = \tilde{\theta}\).

Now consider \((B,\omega) \in B^{LR}\) and construct the Frobenius module \((B,\varrho,\omega)\) as in Proposition 2.12. Then \((B,\varrho)\) is \(\gamma\)-compatible, hence \(\tilde{\theta}\)-compatible as \(LR\)-module and so \(B^{LR}\)-firm by 1.6. This implies that the functor \(\Psi\) is invertible.

Applying Proposition 2.16 similar arguments show that \(\Phi\) is an invertible functor. \(\square\)

For proper adjunctions \((L,R)\) and \((R,L)\), all non-unital \(LR\)-modules are compatible and all non-counital \(LR\)-comodules are compatible, that is, \(B^{LR} = B^{LR}\) and \(B^{LR} = B^{LR}\). Thus we have:

2.20. Corollary. With the notation from 1.2, assume \((L,R)\) and \((R,L)\) to be adjoint pairs. There are category equivalences

\[
\Psi : B^{LR} \to B^{LR}, \quad \Phi : B^{LR} \to B^{LR},
\]

where \(B^{LR}\) denotes the category of non-unital and non-counital Frobenius \(LR\)-bimodules, and

\[
\Psi' : B^{LR} \to B^{LR}, \quad \Phi' : B^{LR} \to B^{LR},
\]

where \(B^{LR}\) denotes the category of unital and counital Frobenius \(LR\)-bimodules.
The latter statement follows from the Corollaries 2.13 and 2.17. It induces an equivalence between the category of unital LR-modules and counital LR-comodules, an observation which was made by Eilenberg and Moore in [4] and was the starting point for the categorical treatment of Frobenius algebras.

3. Weak Frobenius monads

In this section we reformulate our results in terms of monads and comonads.

3.1. Non-counital comonads. Let \((\mathcal{F}, \delta)\) be a non-counital monad on the category \(\mathcal{B}\) and denote the Eilenberg-Moore category of non-counital \(\mathcal{F}\)-modules by \(\mathcal{B}^\mathcal{F}\). There are the free and the forgetful functors

\[
\phi^\mathcal{F} : \mathcal{B} \to \mathcal{B}^\mathcal{F}, \quad B \mapsto (\mathcal{F}(B), \delta_B), \quad U^\mathcal{F} : \mathcal{B}^\mathcal{F} \to \mathcal{B}, \quad (B, \omega) \mapsto B.
\]

An \(\mathcal{F}\)-comodule \((B, \varrho)\) is called cofirm, if the defining cofork is an equaliser in \(\mathcal{B}^\mathcal{F}\).

A triple \((\mathcal{F}, \delta, \varepsilon)\) is said to be a \(q\)-counital monad if \((\mathcal{F}, \delta)\) is a non-counital monad and \(\varepsilon : \mathcal{F} \to I_{\mathcal{B}}\) is a natural transformation. This yields natural transformations

\[
\gamma : \mathcal{F} \mathrel{\delta} FF \mathrel{\varepsilon F} \mathcal{F}, \quad \gamma : \mathcal{F} \mathrel{\delta} FF \mathrel{\varepsilon F} \mathcal{F}.
\]

These data allow for a pairing \((\phi^\mathcal{F}, U^\mathcal{F}, \alpha^\mathcal{F}, \beta^\mathcal{F})\) with the maps, for \(X \in \mathcal{B}\), \((B, \omega) \in \mathcal{B}^\mathcal{F}\),

\[
\alpha^\mathcal{F} : \text{Mor}^\mathcal{F}(U^\mathcal{F}(B), X) \to \text{Mor}^B(B, \phi^\mathcal{F}(X)), \quad f \mapsto \omega \cdot F(f),
\]

\[
\beta^\mathcal{F} : \text{Mor}^B(B, \phi^\mathcal{F}(X)) \to \text{Mor}^\mathcal{F}(U^\mathcal{F}(B), X), \quad g \mapsto \varepsilon_X \cdot g.
\]

3.2. Weak comonads. \((\mathcal{F}, \delta, \varepsilon)\) is said to be a weak comonad provided \((\phi^\mathcal{F}, U^\mathcal{F}, \alpha^\mathcal{F}, \beta^\mathcal{F})\) is a regular pairing with \(\beta^\mathcal{F}\) symmetric. This means that \(\varepsilon\) is regular, \(\gamma\) is symmetric, and \(\delta = \delta \cdot \gamma\) (e.g. [7, Proposition 4.4]).

Morphisms \(h\) between \(\mathcal{F}\)-comodules \((B, \omega)\) and \((B', \omega')\) are \(\gamma\)-compatible if \(h = h \cdot \varepsilon_{B'} \cdot \omega\) and \(\mathcal{K}'\gamma\) stands for the class of all these morphisms.

A non-counital \(\mathcal{F}\)-comodule \((B, \omega)\) is called \(\gamma\)-compatible (or just compatible) if \(\omega\) is \(\gamma\)-compatible (i.e. \(\omega = \gamma_B \cdot \omega\)). The category of these \(\mathcal{F}\)-comodules is denoted by \(\mathcal{B}_\gamma\).

An \(\mathcal{F}\)-comodule \((B, \omega)\) is called \(\mathcal{K}_\gamma\)-firm if the defining cofork

\[
B \xrightarrow{\omega} \mathcal{F}(B) \xrightarrow{\delta_B} FF(B)
\]

is a \(\mathcal{K}_\gamma\)-equaliser. It follows from Proposition 1.6 that any compatible \(\mathcal{F}\)-comodule is \(\mathcal{K}_\gamma\)-cofirm.

3.3. Non-unital monads. Let \((\mathcal{F}, \mu)\) be a non-unital monad on the category \(\mathcal{B}\) and \(\mathcal{B}_F\) the Eilenberg-Moore category of non-unital \(\mathcal{F}\)-modules. There are the free and the forgetful functors

\[
\phi_F : \mathcal{B} \to \mathcal{B}_F, \quad B \mapsto (\mathcal{F}(B), \mu_B), \quad U_F : \mathcal{B}_F \to \mathcal{B}, \quad (B, \varrho) \mapsto B.
\]

An \(\mathcal{F}\)-module \((B, \varrho)\) is said to be firm, if the defining fork is an equaliser in \(\mathcal{B}_F\).

A triple \((\mathcal{F}, \mu, \eta)\) is a \(q\)-unital monad if \((\mathcal{F}, \mu)\) is a non-unital monad and there is some natural transformation \(\eta : I_{\mathcal{B}} \to \mathcal{F}\). This yields natural transformations

\[
\varrho : \mathcal{F} \mathrel{\eta F} FF \mathrel{\mu F} \mathcal{F}, \quad \varrho : \mathcal{F} \mathrel{\eta F} FF \mathrel{\mu F} \mathcal{F}.
\]
These data allow for a pairing \((\phi_F, U_F, \alpha_F, \beta_F)\) with the maps, for \(X \in \mathbb{B}\), \((B, \varrho) \in \mathbb{R}_F\),
\[
\begin{align*}
\alpha_F : \text{Mor}_F(\phi_F(X), B) &\to \text{Mor}_B(X, U_F(B)), \quad f \mapsto f \cdot \eta_B, \\
\beta_F : \text{Mor}_B(X, U_F(B)) &\to \text{Mor}_F(\phi_F(X), B), \quad g \mapsto \varrho \cdot F(g).
\end{align*}
\]

3.4. Weak monads. \((F, \mu, \eta)\) is said to be a weak monad provided \((\phi_F, U_F, \alpha_F, \beta_F)\) is a regular pairing with \(\alpha_F\) symmetric. This means that \(\eta\) is regular, \(\varrho\) is symmetric, and \(\mu = \vartheta \cdot \mu\) (e.g. \cite[Proposition 3.4]{7}).

Morphisms \(h\) between \(F\)-modules \((B, \varrho)\) and \((B', \varrho')\) are \(\vartheta\)-compatible if \(h = h \cdot \varrho \cdot \eta_B\) and \(\mathbb{K}_\vartheta\) denotes the ideal class formed by these morphisms (see \ref{111}).

A non-unital \(F\)-module \((B, \varrho)\) is called \(\vartheta\)-compatible (or just compatible) if \(\varrho\) is \(\vartheta\)-compatible (i.e. \(\varrho = \varrho \cdot \vartheta_B\)). The category of these modules is denoted by \(F\).

An \(F\)-module \((B, \varrho)\) is said to be \(\mathbb{K}_\vartheta\)-firm if the defining fork
\[
\begin{array}{ccc}
F F(B) & \xrightarrow{\mu B} & F(B) \\
\downarrow{\varrho} & & \downarrow{\varrho} \\
F F(B) & \xrightarrow{\varrho} & B
\end{array}
\]
is a \(\mathbb{K}_\vartheta\)-coequaliser. It follows from Proposition \ref{113} that any compatible \(F\)-comodule is \(\mathbb{K}_\varrho\)-cofirm.

3.5. Frobenius bimodules. Let \((F, \mu)\) be a non-unital monad, \((F, \delta)\) a non-
comuntlial comonad, \((B, \varrho) \in \mathbb{R}_F\) and \((B, \omega) \in \mathbb{R}^F\). We say that \((F, \mu, \delta)\) satisfies the Frobenius property and \((B, \varrho, \omega)\) is a Frobenius bimodule, provided they induce commutativity of the diagrams, respectively,

\[
(3.1)
\]

\[
\begin{array}{ccc}
F F & \xrightarrow{\delta} & FF \\
\downarrow{\delta F} & & \downarrow{\delta F} \\
F F & \xrightarrow{\mu} & F F
\end{array}
\]

\[
\begin{array}{ccc}
F F(B) & \xrightarrow{\delta B} & F F(B) \\
\downarrow{\delta F} & & \downarrow{\delta F} \\
F F(B) & \xrightarrow{\mu B} & F F(B)
\end{array}
\]

The Frobenius bimodules as objects and the morphisms, which are \(F\)-module as well as \(F\)-comodule morphisms, form a category which we denote by \(\mathbb{R}^F\). Transferring the Propositions \ref{2.5} and \ref{2.9} yields:

3.6. Proposition. Assume \((F, \mu, \delta)\) to satisfy the Frobenius property and let \(\mathbb{K}'\) (resp. \(\mathbb{K}\)) be an ideal class of (co-)module morphisms.

(i) For any \(\mathbb{K}\)-cofirm \(F\)-comodule \((B, \omega)\), there is a module \((B, \varrho)\) such that \((B, \varrho, \omega)\) is a Frobenius bimodule.

(ii) Any \(F\)-comodule morphism between \(\mathbb{K}\)-cofirm comodules \((B, \omega)\) and \((B', \omega')\) becomes a morphism between the Frobenius bimodules \((B, \varrho, \omega)\) and \((B', \varrho', \omega')\).

(ii) For any \(\mathbb{K}'\)-firm \(F\)-module \((B, \varrho)\), there is a comodule \((B, \omega)\) such that \((B, \varrho, \omega)\) is a Frobenius bimodule.

(ii) Any \(F\)-module morphism between \(\mathbb{K}'\)-firm comodules \((B, \varrho)\) and \((B', \varrho')\) becomes a morphism between the Frobenius bimodules \((B, \varrho, \omega)\) and \((B', \varrho', \omega')\).
3.7. Definition. A quintuple \((F, \mu, \eta; \delta, \varepsilon)\) is called a weak Frobenius bimonad provided

(i) \((F, \mu, \eta)\) is a weak monad,
(ii) \((F, \delta, \varepsilon)\) is a weak comonad,
(iii) \((F, \mu, \delta)\) satisfy the Frobenius property (see (3.1)),
(iv) \(\mu \cdot F\eta = F\varepsilon \cdot \delta\), i.e. \(\vartheta = \gamma\).

Let \(K_\gamma\) denote the class of \(\gamma\)-compatible morphisms in \(B \to F\) and \(K_\vartheta\) the class of \(\vartheta\)-compatible morphisms in \(B \to F\). By (iii), \(\mu\) and \(\delta\) are \(F\)-module as well as \(F\)-comodule morphism and hence, by (iv), we may identify \(K_\gamma\) and \(K_\vartheta\) as classes of morphisms in \(B \to F\). In particular, \(\mu\) is \(\gamma\)-compatible as \(F\)-comodule morphism and \(\delta\) is \(\vartheta\)-compatible as \(F\)-module morphism. So the conditions required in Proposition 2.19 are satisfied.

3.8. Theorem. Let \((F, \mu, \eta; \delta, \varepsilon)\) be a weak Frobenius bimonad. Consider the categories

- \(\mathcal{B}_F\) of compatible \(F\)-comodules,
- \(\mathcal{B}_F\) of compatible \(F\)-modules,
- \(\mathcal{B}_F\) of Frobenius \(F\)-bimodules which are compatible as modules and comodules.

Then there are category equivalences

\[
\begin{align*}
\mathcal{B}_F & \xrightarrow{\Psi} \mathcal{B}_F \\
\mathcal{B}_F & \xrightarrow{\Phi} \mathcal{B}_F \\
\mathcal{B}_F & \xrightarrow{U_F} \mathcal{B}_F
\end{align*}
\]

Proof. Any compatible \(F\)-comodule \((B, \omega)\) is \(K_\gamma\)-cofirm and thus, by Proposition 2.8, there is a \(\vartheta\)-compatible \(F\)-module \((B, \vartheta)\) making \((B, \vartheta, \omega)\) a Frobenius bimodule. This defines the equivalence \(\Psi\) (with inverse the forgetful functor \(U_F\)).

The remaining assertions are seen in a similar way. \(\Box\)

Clearly, if \(F\) has the structure of a proper Frobenius monad, that is, \((F, \mu, \eta)\) is a monad with adjoint comonad \((F, \delta, \varepsilon)\), then all non-(co)unital (co)modules are compatible and the equivalences in the theorem are between non-unital modules and non-counital comodules. They may be (co-)restricted to unital modules and counital comodules to obtain the initial Eilenberg-Moore result (see 2.20).

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