Description of $G$-bundles over $G$-spaces with quasi-free proper action of discrete group II

Morales Meléndez, Quitzeh
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1 The setting of the problem

This problem naturally arises from the Connor-Floyd’s description of the bordisms with the action of a group $G$ using the so-called fix-point construction. This construction reduces the problem of describing the bordisms to two simpler problems: a) description of the fixed-point set (or, more generally, the stationary point set), which happens to be a submanifold attached with the structure of its normal bundle and the action of the same group $G$, however, this action could have stationary points of lower rank; b) description of the bordisms of lower rank with an action of the group $G$. We assume that the group $G$ is discrete.

Let $\xi$ be an $G$-equivariant vector bundle with base $M$.

\[ \xi \]
\[ \downarrow \]
\[ M \]

Where the action of the group $G$ is quasi-free over the base with normal stationary subgroup $H < G$ and there is no more fixed points of the action of the group $H$ in the total space of the bundle $\xi$.

According [1, p.1] the bundle $\xi$ separates as the sum of its $G$-subbundles:

\[ \xi \approx \bigoplus_k \xi_k \]

where the index runs over all (unitary) irreducible representations $\rho_k : H \longrightarrow \mathbb{U}(V_k)$ of the group $H$ and, as a $H$-bundle $\xi_k$ can be presented as the tensor product:

\[ \xi \approx \bigoplus_k \eta_k \otimes V_k, \]

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where the action of the group $H$ over the bundles $\eta_k$ is trivial, $V_k$ denotes the trivial bundle with fiber $V_k$ and with fiberwise action of the group $H$, defined using the linear representation $\rho_k$.

The particular case $\xi = \eta_k \otimes V_k$ was described in the previous article [1]. According [1, p.14] the bundle $\xi_k$ can be obtained as the inverse image of a mapping $f_k : M/G_0 \longrightarrow B\text{Aut}_G (X_k)$

where $G_0 = G/H$, $X_k = G_0 \times (F_k \otimes V_k)$ is the canonical model and $\text{Aut}_G (X_k)$ is the group of equivariant automorphisms of the space $X_k$ as a vector $G$-bundle over the base $G_0$. So, the bundle $\xi$ can be given by a mapping $f : M/G_0 \longrightarrow \prod_k B\text{Aut}_G (X_k)$.

Consider the vector bundle over the discrete base $G_0$

$$X_\rho = G_0 \times \left( \bigoplus_k (F_k \otimes V_k) \right).$$

(1)

Define a fiberwise action $G \times X_\rho \rightarrow X_\rho$ by the formula

$$\phi([g], g_1) : [g] \times \left( \bigoplus_k (F_k \otimes V_k) \right) \rightarrow [g_1 g] \times \left( \bigoplus_k (F_k \otimes V_k) \right)$$

$$\phi([g], g_1) = \bigoplus_k (\text{Id} \otimes \rho_k (u(g_1 g)u^{-1}(g))) = \rho(u(g_1 g)u^{-1}(g)).$$

(2)

**Definition 1** The bundle $X_\rho \longrightarrow G_0$ with the just defined action is called the canonical model for the representation $\rho$.

By $\text{Aut}_G (X_\rho)$ we denote the group of equivariant automorphisms of the canonical model $X_\rho$ as a vector $G$-bundle over the base $G_0$ with fiber $\bigoplus_k (F_k \otimes V_k)$ and canonical action of the group $G$.

**Lemma 1** There exists a monomorphism

$$i : \text{Aut}_G (X_\rho) \longrightarrow \prod_k \text{Aut}_G (X_k)$$

**Proof.** As before, an element of the group $\text{Aut}_G (X_\rho)$ is an equivariant mapping $A^a$ such that the pair $(A^a, a)$ defines a commutative diagram

$$\begin{array}{ccc}
X_\rho & \xrightarrow{A^a} & X_\rho \\
\downarrow & & \downarrow \\
G_0 & \xrightarrow{a} & G_0,
\end{array}$$
By the lemma 1 [1] applied to group of automorphisms \(\text{Aut}_G(X_\rho)\), for \(A^a \in \text{Aut}_G(X_\rho)\), we have
\[A^a|_{X_k} : X_k \longrightarrow X_k.\]
Note that this restriction is \(G\)-equivariant.

Define
\[i : \text{Aut}_G(X_\rho) \longrightarrow \prod_k \text{Aut}_G(X_k)\]
by the formula
\[i(A^a) = (A^a|_{X_k})_k.\]
This is clearly a homomorphism: it is a product of restrictions over invariant subspaces. Let's prove that it is injective. If \(A^a|_{X_k} = \text{Id}_{X_k}\), then \(A^a = \text{Id}_X\) because \(X = \oplus_k X_k\).

In order to prove that the image of \(i\) is closed, note that it coincides with those automorphisms which commute with the inclusion
\[\Delta \times \text{Id} : X_\rho \longrightarrow \prod_k X_k\]
i.e. if an element \((A^a_k)_k \in \prod_k \text{Aut}_G(X_k)\) leaves the image of \(X_\rho\) invariant, then its restriction defines an element in \(\text{Aut}_G(X_\rho)\) and the diagram
\[
\begin{array}{ccc}
X_\rho & \xrightarrow{A^a} & X_\rho \\
\downarrow \quad \quad \quad \quad \downarrow & & \downarrow \\
\prod_k X_k & \xrightarrow{(A^a_k)_k} & \prod_k X_k
\end{array}
\]
commutes, i.e. \(a^k = a \forall k\) and \(A^a|_{X_k} = A^a_k\). In other words
\[
\prod_k pr_k \circ i(A^a) = \Delta(a)
\]
where
\[pr_k : \text{Aut}_G(X_k) \longrightarrow G_0\]
is the epimorphism of lemma 2 [1, p. 9] and
\[\Delta : G_0 \hookrightarrow \prod_k G_0.\]
Then
\[i(\text{Aut}_G(X_\rho)) = \prod_k pr_k^{-1} \Delta(G_0).\]
Corollary 1 It takes place an exact sequence of groups

\[ 1 \to \prod_k GL(F_k) \xrightarrow{\phi} Aut_G(X_\rho) \xrightarrow{pr} G_0 \to 1 \]

Proof. Define \( pr = \Delta^{-1} \prod_k pr_k \circ i \). This is an epimorphism: let \( A_k^a \in pr_k(a) \), then \( (A_k^a)_k \in i(Aut_G(X_\rho)) \).

We define the monomorphism

\[ \phi : \prod GL(F_k) \to Aut_G(X_\rho) \]

using the monomorphisms

\[ \phi_k : GL(F_k) \to Aut_G(X_k) \]

by the formula

\[ \phi = i^{-1} \prod_k \phi_k. \]

Then \( pr \circ \phi = 1 \) and if \( a = 1 \), then \( \prod_k pr_k \circ i(A_1^1) = \Delta(1) \). This means that, for every \( k \), there is a \( B \in GL(F_k) \) such that \( \phi_k(B_k) = A_1^1|_{X_k} \), i.e. \( \phi(B_k)_k = A_1^1 \).

Denote by \( Vect_G(M, \rho) \) the category of \( G \)-equivariant vector bundles \( \xi \) over the base \( M \) with quasi-free action of the group \( G \) over the base and normal stationary subgroup \( H < G \).

Then, by lemma 1 and the observations on p. 13 in [1] in terms of homotopy we have

\[ Vect_G(M, \rho) \approx \prod_k [M, BU(F_k)] \] (3)

Denote by \( Bundle(X, L) \) the category of principal \( L \)-bundles over the base \( X \).

Theorem 1 There exists an inclusion

\[ Vect_G(M, \rho) \to Bundle(M/G_0, Aut_G(X_\rho)). \] (4)

Proof. We already have a monomorphism

\[ Vect_G(M, \rho) \to Bundle(M/G_0, \prod_k Aut_G(X_k)) \]

i.e.

\[ \prod_k Vect_G(M, \rho_k) \to \prod_k Bundle(M/G_0, Aut_G(X_k)) \]

see [1] p. 13

A bundle \( \xi \in Vect_G(M, \rho) \) is given by transition functions

\[ \Psi_{\alpha\beta}(x) \in \prod_k Aut_G(X_k) \]
with the property that there exist $h_{\alpha,k}(x) \in G_0$ such that

$$h_{\alpha,k}^{-1}(x)pr_k \circ \Psi_{\alpha\beta}(x)h_{\alpha,k}(x)$$

does not depend on $k$. Let's show that there can be found transition functions with the property that

$$\prod_k pr_k \Psi_{\alpha\beta}(x) = \Delta(a_{\alpha\beta}(x))$$

for some cocycle $a_{\alpha\beta}(x)$.

Because the group $G_0$ is discrete, for an atlas of connected charts with connected intersections, we can assume that $pr_k \circ \Psi_{\alpha\beta}(x) = a_{\alpha\beta,k}$ and $h_{\alpha,k}(x) = h_{\alpha,k} \in G_0$ do not depend on $x$ and, therefore,

$$h_{\alpha,k}^{-1}(x)pr_k \circ \Psi_{\alpha\beta}(x)h_{\alpha,k}(x) = a_{\alpha\beta}$$

does not depend on $k$ nor $x$. Let $H_{\alpha,k} \in \text{Aut}_G(X_k)$ such that $pr_k(H_{\alpha,k}) = h_{\alpha,k}$. Then,

$$\prod_k pr_k(H_{\alpha,k}^{-1} \Psi_{\alpha\beta}(x)H_{\alpha,k}) = \Delta(a_{\alpha\beta}).$$

\textbf{Theorem 2} If the space $X$ is compact, then

$$\text{Bundle}(X, \text{Aut}_G(X_\rho)) \approx \bigcup_{M \in \text{Bundle}(X,G_0)} \text{Vect}_G(M, \rho).$$

\textbf{Proof.} We will follow the proof theorem 3 in [1, p. 14]. Given a bundle $\xi \in \text{Bundle}(X, \text{Aut}_G(X_\rho))$ with transition functions

$$\Psi_{\alpha\beta}(x) \in \text{Aut}_G(X_\rho)$$

we obtain transition functions

$$pr \circ i \circ \Psi_{\alpha\beta}(x) \in \text{Aut}_G(G_0) \approx G_0,$$

defining an element $M \in \text{Bundle}(X, G_0)$ together with a projection $\xi \rightarrow M$. Changing the fibers $\text{Aut}_G(X_\rho)$ by $X_\rho$, we obtain an action of the group $G$, that reduces over the base to the factor group $G_0$.

Let's rewrite this in terms of homotopy.

\textbf{Corollary 2} If the space $X$ is compact, then

$$[X, \text{Aut}_G(X_\rho)] \approx \bigcup_{M \in [X,BG_0]} \prod_k [M, BU(F_k)].$$

5
2 The case when the subgroup $H < G$ is not normal

Consider an equivariant vector $G$-bundle $\xi$ over the base $M$

$$\xi \xrightarrow{p} M.$$ 

Let $H < G$ be a finite subgroup. Assume that $M$ is the set of fixed points of the conjugation class of this subgroup, more accurately

$$M = \bigcup_{[g] \in G/N(H)} M^{gHg^{-1}},$$

and that there is no more fixed points of the conjugation class of $H$ in the total space of the bundle $\xi$; here we have denoted by $M^H$ the set of fixed points of the action of the subgroup $H$ over the space $M$, $N(H)$ the normalizer of the group $H$ in $G$ and we are using the equality $gM^H = M^{gHg^{-1}}$ and the fact that $lHl^{-1} = gHg^{-1}$ if and only if $g^{-1}l \in N(H)$.

Let $\tilde{\mathcal{F}}_\xi$ be the family of subgroups of $G$ having non-trivial fixed points in the total space of the bundle $\xi$, i.e.

$$\tilde{\mathcal{F}}_\xi = \{ K < G | \xi^K \neq \emptyset \}.$$ 

This is a partial ordered set by inclusions and is closed under the action of the group $G$ by conjugation\footnote{If $\xi^K \neq \emptyset$, then $g\xi^Kg^{-1} = g\xi^K \neq \emptyset$.}. Also, the action

$$G \times \tilde{\mathcal{F}}_\xi \rightarrow \tilde{\mathcal{F}}_\xi$$

$$ (g, K) \mapsto gKg^{-1}$$

preserves the order.

**Definition 2** We will say that $H < G$ is the unique, up to conjugation, maximal subgroup for the $G$-bundle $\xi$ if every conjugate $gHg^{-1}$ is maximal in $\tilde{\mathcal{F}}_\xi$ and there is no more maximal elements in this family.

In this section will assume in any case, that $H < G$ is the unique, up to conjugation, maximal subgroup.

**Lemma 2** If $H \neq gHg^{-1}$, then

$$M^H \cap M^{gHg^{-1}} = \emptyset$$

\textbf{Proof.} If there is an $x \in M^H \cap M^{gHg^{-1}}$ then, the point $x$ is fixed under the action of the subgroup generated by $H$ and $gHg^{-1}$, but this group is not contained in any of the subgroups of the form $lH^{-1}l$, $l \in G$. \hfill $\blacksquare$
Lemma 3 If the condition (7) holds, then the $G$-bundle $\xi$ can be presented as a disjoint union of pair-wise isomorphic bundles with quasi-free action over the base. More precisely

$$\xi = \bigsqcup_{[g] \in G/N(H)} \xi_{[g]},$$

where

$$\xi_{[g]} = p^{-1}(MgHg^{-1})$$

is a vector bundle with quasi-free action of the group $N(gHg^{-1})$ and, for every element $g \in G$ the mapping

$$g : \xi_{[1]} \rightarrow g\xi_{[1]} = \xi_{[g]}$$

defines an equivariant isomorphism of this bundles, i.e. the diagram

$$\begin{array}{ccc}
N(H) \times \xi_{[1]} & \longrightarrow & \xi_{[1]} \\
\downarrow s_g \times g & & \downarrow g \\
N(gHg^{-1}) \times \xi_{[g]} & \longrightarrow & \xi_{[g]}
\end{array} \tag{8}$$

commutes, where

$$s_g : N(H) \rightarrow N(gHg^{-1}) = gN(H)g^{-1}, \quad (g, n) \mapsto gng^{-1}.$$

Proof. From lemma 2 it follows that

$$M = \bigsqcup_{[g] \in G/N(H)} MgHg^{-1}$$

and, therefore,

$$\xi = \bigsqcup_{[g] \in G/N(H)} \xi_{[g]}.$$

Since the action of $G$ is fiberwise, we have $g \cdot \xi_{[1]} = \xi_{[g]}$ for every $g \in G$. Restricting the projection $\xi \rightarrow M$ to the space $\xi_{[g]}$, we obtain the bundle

$$\begin{array}{ccc}
\xi_{[g]} & \longrightarrow & \xi_{[g]} \\
\downarrow p & & \downarrow \\
MgHg^{-1} & \rightarrow & 
\end{array}$$

The bundle $\xi_{[g]}$ has an action of the normalizer $N(gHg^{-1})$:

$$N(gHg^{-1}) \times \xi_{[g]} \rightarrow \xi_{[g]},$$

i.e. $\xi_{[g]}$ is a $N(gHg^{-1})$-bundle for every $g \in G$.

Note that group conjugation $s_g : N(H) \rightarrow N(gHg^{-1})$ defines an isomorphism between these groups that fits into the commutative diagram

$$\begin{array}{ccc}
N(H) \times \xi_{[1]} & \longrightarrow & \xi_{[1]} \\
\downarrow & & \downarrow \\
N(gHg^{-1}) \times \xi_{[g]} & \longrightarrow & \xi_{[g]}
\end{array}.$$
i.e. \(gng^{-1} \cdot gx = g \cdot nx\). This means that the bundles \(\xi_{[1]}\) and \(\xi_{[g]}\) are naturally and equivariantly isomorphic.

Evidently, the mappings on the diagram do not depend on the elements \(n \in N(H)\), but they depend on the element \(g \in G\).

The action of the group \(N(H)\) over the base \(M^H\) reduces to the factor group \(N(H)/H\):

\[
\begin{array}{ccc}
N(H) \times \xi_{[1]} & \longrightarrow & \xi_{[1]} \\
N(H)/H \times M^H & \longrightarrow & M^H
\end{array}
\]

where, considering the maximality of the group \(H\), the action \(N(H)/H \times M \longrightarrow M\) is free and, by hypothesis, there is no more fixed of the action of the subgroup \(H\) in the total space of the bundle \(\xi\), i.e. \(N(H)\) acts quasi-freely over the base and has normal stationary subgroup \(H\).

**Definition 3** If the condition holds, we will say that the group \(G\) acts quasi-freely over the bundle \(\xi\) with (non-normal) stationary subgroup \(H\).

As we will see in theorem for classifying purposes, it is enough to consider bundles with normal stationary subgroup.

Let \(X(\rho)\) be the canonical model for the representation \(\rho : H \longrightarrow GL(F)\) with action of the group \(N(H)\). Define a canonical model \(X(\rho_g)\) for the representation \(\rho_g : gHg^{-1} \longrightarrow H \longrightarrow \rho \longrightarrow GL(F)\), \(s_g(n) = gng^{-1}\). The action of the group \(N(gHg^{-1})\) over \(X(\rho_g)\) is defined using the homomorphism of right \(gHg^{-1}\)-modules

\[
u_g : gHg^{-1} \longrightarrow H \longrightarrow N(H) \longrightarrow N(gHg^{-1})
\]

by the formula.

Let

\[
GX(\rho) := \bigsqcup_{[g] \in G/N(H)} X(\rho_g)
\]

i.e. if \(lHl^{-1} = gHg^{-1}\), then the spaces \(X(\rho_g)\) and \(X(\rho_l)\) coincide.

This notation will be clear after the next lemma.

**Lemma 4** The group \(G\) acts over the space \(GX(\rho)\) quasi-freely with (non-normal) stationary subgroup \(H\) and, under this action, the space \(GX(\rho)\) coincides with the orbit of the subspace \(X(\rho)\). In particular, we have the relations

\[N(H)(X(\rho)) = X(\rho)\]

and

\[(GX(\rho))^{gHg^{-1}} = N(gHg^{-1})/gHg^{-1}.
\]
Proof. The action $G \times GX(\rho) \to GX(\rho)$ is defined in the following way: for a fixed $g \in G$ define the mapping

$$g : X(\rho) \to X(\rho_g)$$

as

$$s_g \times \text{Id} : N(H)_0 \times F \to N(gHg^{-1})_0 \times F$$

$(N(H)_0 = N(H)/H)$ and, if $lHl^{-1} = gHg^{-1}$, then the mapping $l : X(\rho) \to X(\rho_l)$ is chosen to make the diagram

$$\begin{array}{ccc}
X(\rho_g) & \xrightarrow{s_g^{-1}\times\text{Id}} & X(\rho) \\
\parallel & & \downarrow l^{-1}g \\
X(\rho_l) & \xrightarrow{l^{-1}} & X(\rho).
\end{array}$$

(9)

commutative, i.e.

$$l = (s_g \times \text{Id}) \circ (g^{-1}l)$$

where the mapping

$$g^{-1}l : X(\rho) \to X(\rho) = X(\rho_{g^{-1}l})$$

is the canonical left translation by the element $g^{-1}l \in N(H)$.  

Corollary 3 There is an isomorphism

$$g : \text{Aut}_{N(H)}(X(\rho)) \cong \text{Aut}_{N(gHg^{-1})}(X(\rho_g))$$

(11)

that depends only on the class $[g] \in G/N(H)$.  

Proof. We have a diagram (9) for $\xi = GX(\rho)$. Such a diagram always induces an isomorphism

$$\text{Aut}_{N(H)}(X(\rho)) \cong \text{Aut}_{N(gHg^{-1})}(X(\rho_g))$$

by the rule

$$A \mapsto gAg^{-1}$$

and, if $l \in [g] \in G/N(H)$ then $l^{-1}g \in N(H)$ commutes with $A \in \text{Aut}_{N(H)}(X(\rho))$. Therefore

$$gAg^{-1} = g(g^{-1}l)(l^{-1}g)A(g^{-1}l) = g(g^{-1}l)A(l^{-1}g)g^{-1} = lAl^{-1}. \quad \blacksquare$$

Definition 4 The space $GX(\rho)$ is called the canonical model for the case when the subgroup $H < G$ is not normal.

Lemma 5

$$\text{Aut}_G(GX(\rho)) \approx \text{Aut}_{N(H)}(X(\rho))$$

(12)
Proof. By definition, an element of the group $\text{Aut}_G(X)$ is an equivariant mapping $A^a$ such that the pair $(A^a, a)$ defines the commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{A^a} & X \\
\downarrow & & \downarrow \\
G/H & \xrightarrow{a} & G/H,
\end{array}
$$

that commutes with the canonical action, i.e. the mapping $a \in \text{Aut}_G(G/H)$ satisfies the condition

$$a \in \text{Aut}_G(G/H) \approx N(H)/H, \quad a[g] = [ga], \quad [g] \in N(H)/H.$$

Therefore, $A^a = (A^a[g])_{[g] \in N(H)/H} \in \text{Aut}_{N(H)}(X(\rho))$.

The value of the operators $(A^a[g])_{[g] \in G/H}$ can be calculated in terms of the operator $A^a[1]$ as in lemma 2 from [1, p. 9].

Denote by $\widetilde{\text{Vect}}_G(M, \rho)$ the category of vector bundles with quasi-free action of the group $G$ over the base $M$.

Theorem 3 $\widetilde{\text{Vect}}_G(M, \rho) \approx \text{Vect}_{N(H)}(M^H, \rho)$.

Proof. From lemma 3 follows that the bundles $\xi_{[1]}$ and $\xi_{[g]}$ equivariantly isomorphic and are given by mappings

$$M^{gH^{-1}}/N(gHg^{-1}) \to B\text{Aut}_{N(gHg^{-1})}(X(\rho)),$$

and

$$M^H/N(H) \to B\text{Aut}_{N(H)}(X(\rho)),$$

that can be put in the commutative diagram

$$
\begin{array}{ccc}
M^H/N(H) & \to & B\text{Aut}_{N(H)}(X(\rho)) \\
\downarrow{\tilde{g}} & & \downarrow{\tilde{g}} \\
M^{gHg^{-1}}/N(gHg^{-1}) & \to & B\text{Aut}_{N(gHg^{-1})}(X(\rho)).
\end{array}
$$

Here, $g : \xi^H \to \xi^{gHg^{-1}}$ is the action over the bundle $\xi$. The arrow on the right side is induced by the isomorphism [11] and does not depend on the element $g \in [g] \in G/N(H)$.

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