GERSTENHABER STRUCTURE ON HOCHSCHILD COHOMOLOGY OF TOUPIE ALGEBRAS

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Abstract. We study homological properties of a family of algebras called toupie algebras. Our main objective is to obtain the Gerstenhaber structure of their Hochschild cohomology, with the purpose of describing the Lie algebra structure of the first Hochschild cohomology space, together with the Lie module structure of the whole Hochschild cohomology.

Keywords: Hochschild cohomology, Gerstenhaber algebra.

1. Introduction

In this article we study homological properties of toupie algebras, first defined in [CDHL]. Toupie algebras combine features of canonical algebras with monomial algebras. Canonical algebras were introduced by Ringel in [Ri], see also [BKL] for historical references about canonical algebras.

An algebra is toupie if it is a quotient of the path algebra of a finite quiver $Q$ which has a source 0, a sink $\omega$ and branches going from 0 to $\omega$ by an ideal $I \subseteq Q_{\geq 2}$ generated by a set containing two types of relations: monomial ones, which involve arrows of one branch each, and linear combinations of branches.

Canonical algebras are part of a more general class, the concealed-canonical algebras, see [LP]. Since one of the properties distinguishing canonical algebras within the class of concealed-canonical algebras is that their quiver has only one sink and only one source [Ri], we conclude that toupie algebras and concealed-canonical algebras only share the subfamily of canonical algebras.

Almost all toupie algebras are of wild representation type, see [Art]. Toupie algebras are also special multiserial algebras, see [GSc] for the definition, which are usually of wild representation type too. As a consequence, modules over toupie algebras are multiserial, that is non necessarily direct finite sums of uniserial modules. The Hochschild cohomology of special multiserial algebras is still unknown except for some particular examples.

The Hochschild cohomology $HH^*(A)$ of an algebra $A$, together with its associative algebra structure given by the cup product and its Gerstenhaber algebra structure, is a derived invariant. Even having an explicit description of $HH^*(A)$, the cup product and the Gerstenhaber bracket are not easy to compute. Bustamente proved in [Bus] that the cup product of $HH^*(A)$ is trivial for any triangular quadratic string algebra $A$, and the Gerstenhaber bracket vanishes for elements in cohomological degrees greater than 1 when $A$ is a gentle triangular algebra. These are tame algebras with a particularly easy resolution. Also, Redondo and Román computed in [RR1] the Gerstenhaber structure of the Hochschild cohomology of a triangular string algebra, showing that

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it is trivial in degrees greater than 1. Subsequently, they computed in \[RR2\] the Gerstenhaber structure of the Hochschild cohomology of string quadratic algebras. In this case they gave conditions on the quiver associated to the string quadratic algebra in order to get non trivial cup product and Gerstenhaber bracket in degrees greater than 1. The first Hochschild cohomology space of an algebra \(A\) is always a Lie algebra with the Gerstenhaber bracket. Strametz \[S\] described the Lie algebra structure of \(HH^{1}(A)\) for \(A\) a finite dimensional monomial algebra. Moreover, Sánchez-Flores made explicit in \[S\] the Lie module structure of higher cohomology spaces \(HH^{n}(A)\) over the Lie algebra \(HH^{1}(A)\) when \(A \cong \mathbb{k} Q / \langle Q_2 \rangle\) –that is, radical square zero– and \(Q\) is either an oriented cycle of length \(n\) or a finite quiver with no cycles.

In this article we describe the Lie algebra structure of \(HH^{1}(A)\), when \(A\) is a toupie algebra, as well as the Lie module structure of \(HH^{n}(A)\) over \(HH^{1}(A)\). For this, we construct a resolution of \(A\) as \(A\)-bimodule, using technics of \[CS\]. We shall see that the existence of non monomial relations will only have an effect in degrees 0, 1, 2 of the resolution but this difference will considerably change the Lie structure of the first cohomology group.

Even if the dimensions of the \(k\)-vector spaces \(HH^{n}(A)\) are already known \[GL\], an explicit computation of this is needed for the description of the Gerstenhaber structure.

The contents of the article are as follows. In Section 2 we fix notations and prove some preliminary results. Section 3 is devoted to the computation of a \(k\)-basis of each Hochschild cohomology space, while in Section 4 we obtain the comparison morphisms between the reduced bar resolution and ours.

In Section 5 we prove that the Gerstenhaber bracket is zero in degrees greater that 1, and we compute it when restricted to \(HH^{1}(A)\).

The description of \(HH^{1}(A)\) as a Lie algebra is given in Section 6, where we find necessary and sufficient conditions for it to be abelian and to be semisimple. We describe its centre and we prove that when \(k = \mathbb{C}\), it has a Lie subalgebra isomorphic to \(sl_{n}(\mathbb{C})\), where \(n\) is the number of arrows from 0 to \(\omega\) in the quiver \(Q\). Finally, we prove Theorem 6.5, one of our main theorems.

In Sections 7.1 and 7.2 we describe the Lie module structure of \(HH^{n}(A)\) for \(n \geq 2\). The main results are Theorem 7.2 and Theorem 7.3. We end the article with an example.

2. Preliminaries

Let \(k\) be a field of characteristic zero.

In this section we will recall some definitions, as well as some preliminary results.

### 2.1. \(E\)-reduced Bar resolution

Given a finite dimensional \(k\)-algebra \(A\) with radical \(r\) such that \(A = E \oplus r\) with \(E\) a separable \(k\)-subalgebra \(E \subset A\), Cibils proved in \[GL\] Lemma 2.1 that the following complex is a projective \(A\)-bimodule resolution of \(A\), see also \[CS\]:

\[
\cdots A \otimes E \overline{A}^{\otimes 3} \otimes E A \xrightarrow{\delta_3} A \otimes E \overline{A}^{\otimes 2} \otimes E A \xrightarrow{\delta_2} A \otimes E \overline{A} \otimes E A \xrightarrow{\delta_1} A \otimes E \overline{A} \to 0
\]

where \(\overline{A} = A/E\) and \(\delta_i\) is such that \(\delta_i(a_0 \otimes E \overline{a}_1 \otimes E \overline{a}_2) = \delta_i(a_0 \otimes a_1 \otimes \cdots \otimes a_{i-1} \otimes a_{i+2})\).

From now on we will simply call it \(\delta_i\) instead of \(\delta_i\).

We will give a contracting homotopy \(t = \{t_i\}_{i \geq 1}\) that we are going to use later to define the comparison maps between this resolution and the minimal one:

\[
t_i : A \otimes E \overline{A}^{\otimes i} \otimes E A \to A \otimes E \overline{A}^{\otimes i+1} \otimes E A
\]

such that \(t_{i+1} \circ t_{i+1} + t_i \circ \delta_i = id\). We define \(t_i(a_0 \otimes E \overline{a}_1 \otimes E \cdots \otimes E \overline{a}_i \otimes E a_{i+1}) = 1 \otimes E \overline{a}_0 \otimes E \overline{a}_1 \otimes E \cdots \otimes E \overline{a}_i \otimes E a_{i+1}\).

This projective resolution will be called the \(E\)-reduced Bar resolution of \(A\). \(C_{\text{Bar}E}(A)\) To simplify notation we will still denote by \(\delta_i\) the differentials of this complex. It is possible to define the reduced cup product \(\cup_\text{red}\) and the reduced Gerstenhaber bracket \([\ , \ ]_\text{red}\) in terms
of this resolution. Applying the functor $\text{Hom}_{A^e}(\cdot, A)$ to the $E$-reduced Bar resolution and considering the isomorphism

$$ F_X : \text{Hom}_{A^e}(A \otimes E^o X \otimes E, A) \to \text{Hom}_{E^o}(X, A) $$

natural in $X$ given by

$$ F_X(\varphi)(x) = \varphi(1 \otimes x \otimes 1), $$

we obtain the following complex:

$$ 0 \to \text{Hom}_{E^o}(E, A) \to \text{Hom}_{E^o}(A, A) \to \text{Hom}_{E^o}(A \otimes E, A) \to \text{Hom}_{E^o}(A \otimes E^o, A) \to \cdots $$

The reduced cup product in $\bigoplus_{i=0}^\infty \text{Hom}_{E^o}(\overline{A}^{\otimes E^o}, A)$ is as follows: consider $\varphi \in \text{Hom}_{E^o}(\overline{A}^{\otimes E^o}, A)$ and $\phi \in \text{Hom}_{E^o}(\overline{A}^{\otimes E}, A)$, given $a_1, \ldots, a_{n+m} \in A$

$$ \varphi \circ_{\text{red}} \phi([a_1] \otimes \cdots \otimes [a_{n+m}]) = \varphi([a_1] \otimes \cdots \otimes [a_m])\phi([a_{m+1}] \otimes \cdots \otimes [a_{n+m}]). $$

It is easy to prove that $\circ_{\text{red}}$ induces a product in Hochschild cohomology that coincides with the usual cup product, see remarks 2.3.14 and 2.3.19 of [Arten].

2.2. Toupie algebras.

**Definition 1.** We recall some definitions from [Arten]. A finite quiver is toupie if it has a unique source and a unique sink and the target of exactly one arrow. The source and the sink will be denoted 0 and $\omega$, respectively.

Given a toupie quiver $Q$ and any admissible ideal $I \subset \mathbb{k}Q$, $A = \mathbb{k}Q/I$ will be called a toupie algebra, the paths from 0 to $\omega$ will be called branches. The $j$-th branch will be denoted $\alpha^{(j)}$. The length of a branch $\alpha^{(j)}$ – denoted by $|\alpha^{(j)}|$ – is the number of arrows in it.

We have already observed that canonical algebras are toupie algebras. Besides canonical algebras there are many examples of toupie algebras, let us just mention the $n$-Kronecker algebras.

From now on, $A = \mathbb{k}Q/I$ will always denote a toupie algebra. There are four possible kinds of branches in $Q$ and we will consider the following order within the branches. First, let $Z = \{\alpha^{(1)}, \ldots, \alpha^{(n)}\}$ be the set of arrows from 0 to $\omega$. The branches $\alpha^{(a+1)}, \ldots, \alpha^{(a+n)}$ will be those of length greater than or equal to 2 not involved in any relation. Next, $\alpha^{(n+1)}, \ldots, \alpha^{(n+m)}$ will be the branches containing monomial relations and finally $\alpha^{(a+n+m+1)}, \ldots, \alpha^{(a+n+m+n)}$ will be the branches involved in non monomial relations.

Denoting by $e_x$ the idempotent of $A$ corresponding to the vertex $x$, we define:

$$ D := \dim_{\mathbb{k}} e_0 A e_\omega = a + l + m - \#(\text{linearly independent non monomial relations}). $$

Given a finite set of $s$ equations generating the non monomial relations and having fixed an order of the branches, let $C = (c_{i,j}) \in \mathbb{k}^{s \times n}$ be the matrix whose rows are the coefficients of each of these equations. Replacing the given relations by those obtained from the reduced row echelon form of the matrix gives of course the same algebra; from now on we will always suppose that this matrix is already reduced. Every non monomial relation will be of the form:

$$ \rho_i = \alpha^{(k_i)} + \sum_{j=k_i}^s b_{ij}\alpha^{(j)}. $$

We will call $W_{\rho_i} = \alpha^{(k_i)}$ and $f_{\rho_i} = -\sum_{j=k_i} b_{ij}\alpha^{(j)}$, keeping in mind the idea that the word $W_{\rho_i}$ will be replaced in $A$ by $f_{\rho_i}$. Let $R$ be a minimal set of generators of $I$ containing $\rho_i$ for all $i$. The set $R$ will be the disjoint union of $R_{\text{nonon}}$, consisting of monomial relations and $R_{\text{nonon}} = \{\rho_i\}$, the set of non monomial relations.

Since we are going to compute the Hochschild cohomology spaces of $A$, the first thing we need is a useful projective resolution of $A$ as $A$-bimodule; if possible a minimal one.
2.3. The resolution. We will next recall the definition of $n$-ambiguity from [SK].

**Definition 2.** Given $n \geq 2$,

1. the path $p \in Q$ is a left $n$-ambiguity if there exist $u_0 \in Q_1$ and $u_1, \ldots, u_n$ paths not in $I$ such that
   - (i) $p = u_0 u_1 \cdots u_n$,
   - (ii) for all $i$, and for any proper left divisor $d$ of $u_{i+1}$, the path $u_i u_{i+1}$ belongs to $I$ but $u_d$ does not belong to $I$.
2. the path $p \in Q$ is a right $n$-ambiguity if there exist $v_0 \in Q_1$ and $v_1, \ldots, v_n$ paths not in $I$ such that
   - (i) $p = v_n \cdots v_0$,
   - (ii) for all $i$, and for any proper right divisor $d$ of $v_{i+1}$, the path $v_i v_{i+1}$ belongs to $I$ but $dv_i$ does not belong to $I$.

As we have already mentioned, the subalgebra $E = kQ_0$ of $A$ is separable over $k$, so we can compute the resolution relative to $E$. This resolution will come from a monomial order, see [CS]. Let us fix an order in $Q_0 \cup Q_1$. For this, let us draw the toupie quiver $Q$ as follows

![Diagram of the toupie quiver](image)

and for each $i$ denote $\{e^{(i)}_j / 1 \leq j \leq n_i \}$ the set of vertices in the branch $\alpha^{(i)}$ such that $e^{(i)}_j \neq 0$ and $e^{(i)}_j = \omega$. Fix $\omega = \min Q_0$, $0 = \max Q_0$ and $e^{(i)}_j < e^{(i)}_k$ if $l < i$ or if $i = l$ and $k < j$. For the arrows, fix $\alpha^{(i)}_j < \alpha^{(i)}_l$ if $l < i$ or $i = l$ and $k < j$. Let $\leq$ be the order in $kQ/I$ which is compatible with concatenation and extends $<$. Choose $S = \text{Mintip} (I)$ as in [CS] Def. 2.8 for example. It is then known that $I$ is the two sided ideal generated by $\{s - f_s\}_{s \in S}$ and that the reduction system $\mathcal{R} = \{(s, f_s)\}_{s \in S}$ is such that every path is reduction unique, see for example [CS] Lemmas 2.4 and 2.10. Explicitly, $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$, where $\mathcal{R}_1 = \{(\sigma_i, 0) / \sigma_i$ a monomial relation$\}$ and $\mathcal{R}_2 = \{((\alpha^{(i)}, \Sigma_{j<k}k_j^{(i)}\alpha^{(j)})/a + l + m + 1 \leq k, \leq a + l + m + n\}$.

In our situation, it is easy to see that all possible ambiguities arise from $\mathcal{R}_1$, and since they correspond to the monomial part, they reduce to 0. We denote by $B$ the basis of $A$ obtained from the classes in $kQ/I$ of the irreducible paths. From now on we will fix the reduction system. We are now ready to construct the resolution.

**Notation 2.0.1.** Given a path $u \in Q$, whenever we write $\sum u^{(1)} \otimes u^{(2)} \otimes u^{(3)} \in A \otimes_E A \otimes_E A$ we are considering the sum of all the possible factorizations of $u$ as product $u^{(1)} u^{(2)} u^{(3)} = u$. If $u^{(2)}$ is an arrow we write $\sum u^{(1)} \otimes u^{(2)} \otimes u^{(3)}$. If $\rho_i$ is a non monomial relation, we know that it is of the form $\alpha^{(k_i)} + \sum_{j<k_i} b_{ij} \alpha^{(j)}$, then when we write $\sum[\rho_i]^{(1)} \otimes [\rho_i]^{(2)} \otimes [\rho_i]^{(3)}$ we refer to $\sum[\alpha^{(k_i)}]^{(1)} \otimes [\alpha^{(k_i)}]^{(2)} \otimes [\alpha^{(k_i)}]^{(3)} + \sum_{j<k_i} b_{ij} \sum[\alpha^{(j)}]^{(1)} \otimes [\alpha^{(j)}]^{(2)} \otimes [\alpha^{(j)}]^{(3)}$. 
In low degrees, we already know that the extended minimal resolution is
\[ A \otimes_E kR \otimes_E A \to A \otimes_E kQ_1 \otimes_E A \to A \otimes_E A \to A \to 0 \]
As usual, \( \mu, d_0 \) and \( d_1 \) are the \( A \)-bimodule maps defined by:
\[
\mu(a \otimes b) = ab \text{ for } a, b \in A,
\]
\[
d_0(1 \otimes v \otimes 1) = v \otimes 1 - 1 \otimes v \text{ for } v \in Q_1,
\]
\[
d_1(1 \otimes \sigma \otimes 1) = \sigma^{(1)} \otimes \sigma^{(2)} \otimes \sigma^{(3)} \text{ for } \sigma \in \mathcal{R}.
\]
For \( n \geq 2 \), let us denote \( P_n = A \otimes_E kA_n \otimes_E A \), where \( A_n \) is the set of \( n \)-ambiguities.
Given \( v = v_0 \ldots v_n = w_0 \ldots w_0 \in A_n \), the differential \( d_n : P_n \to P_{n-1} \) will be defined –using the same notation as before– as follows:
\[
d_n(1 \otimes v \otimes 1) = \left\{ \begin{array}{ll}
\sum v^{(1)} \otimes v^{(2)} \otimes v^{(3)}, & \text{with } v^{(2)} \in A_{n-1} \\
 w_n \otimes w_{n-1} \ldots w_0 \otimes 1 - 1 \otimes u_0 \ldots u_{n-1} \otimes u_n, & \text{if } n \text{ is even}
\end{array} \right.
\]
Let us call \( C_{\min}(A) \) the sequence
\[
\ldots \to P_n \xrightarrow{d_2} P_{n-1} \ldots \to P_2 \xrightarrow{d_2} A \otimes_E kR \otimes_E A \xrightarrow{d_1} A \otimes_E kQ_1 \otimes_E A \xrightarrow{d_0} A \otimes_E A \to 0
\]
\textbf{Lemma 2.1.} The sequence \( C_{\min}(A) \) is a complex of projective \( A \)-bimodules.
\textbf{Proof.} For \( n > 2 \), the only relations involved in \( P_n \) are monomial and the proof that \( d_n \circ d_{n+1} = 0 \) is in [SG]. We have to verify that \( d_1 \circ d_2 = 0 \) and \( d_2 \circ d_3 = 0 \). Let us first prove that \( d_1 \circ d_2 = 0 \).
Given a 2-ambiguity \( v = v_0 v_1 v_2 = w_2 w_1 w_0 \),
\[
d_1 \circ d_2(1 \otimes v \otimes 1) = d_1(1 \otimes w_2 \otimes v_0 \otimes 1 - 1 \otimes v_0 v_1 \otimes v_2)
\]
\[
= \sum w_2(1 \otimes w_0)^{(1)} \otimes (w_1 \otimes w_0)^{(2)} \otimes (v_1 \otimes v_0)^{(3)}
\]
\[
- \sum (v_0 \otimes v_1)^{(1)} \otimes (v_0 v_1)^{(2)} \otimes (v_0 v_1)^{(3)} v_2.
\]
In case there is a term that appears appearing in the second one; then \( (w_1 w_0)^{(3)} \) must be a proper divisor of \( v_2 \), which implies that \( (w_1 w_0)^{(2)} (w_1 w_0)^{(3)} \) is a divisor of \( v_2 \) and so \( w_2(w_1 w_0)^{(1)} \) would be divisible by \( v_0 v_1 \), which is 0 in \( A \). A similar argument is used to prove that every non zero term that appears in the second sum appears in the first one too.

Let us now prove that \( d_2 \circ d_3 = 0 \). Given a 3-ambiguity \( v = u_0 u_1 u_2 u_3 = w_3 w_2 w_1 w_0 \),
\[
d_2 \circ d_3(v) = d_2(\sum v^{(1)} \otimes v^{(2)} \otimes v^{(3)})
\]
with \( v^{(2)} \in A_2 \). The set of 2-ambiguities appearing in the decomposition of \( v \) is finite, so we can order the summands by length of the first term \( v^{(1)} \). Notice that two different terms cannot have \( v^{(1)} \)'s of the same length: if this happens, then one of the \( v^{(2)} \)'s would be strictly shorter that the other, and this contradicts Lemma 3.1.5 of [Arten].

Let us order the summands as
\[
v_1^{(1)} \otimes v_3^{(2)} \otimes v_1^{(3)} < \ldots < v_n^{(1)} \otimes v_n^{(2)} \otimes v_n^{(3)}
\]
with \( |v_i^{(1)}| < |v_{i+1}^{(1)}| \) for \( i = 1, \ldots, n - 1 \).

Given \( t, i, t \leq i \leq n \), we know after the second item of Lemma 3.1.6 of [Arten] that \( v_t^{(2)} \) and \( v_{t+1}^{(2)} \) both have only two divisors in \( \mathcal{R} \), respectively \( v_t' \), \( v_t'' \) and \( v_{t+1}' \), \( v_{t+1}'' \). The first part of Lemma 3.1.6 implies that \( v_{t+1}' \) equals \( v_t' \), because if not \( v_t^{(2)} \) and \( v_{t+1}'' \) would share a 2-ambiguity, which is impossible.
Using the notation $v_i^{(2)} = v_i' u_{2,i} = w_{2,i} v_{i+1}'$, and observing that $u_{2,1} = u_2$, since $v_1' = u_0 u_1$, and that $w_{2,n} = w_2$ since $v_{n+1}' = w_1 u_0$, we get

$$d_2 \circ d_3 (1 \otimes v \otimes 1) = -\sum_{i=1}^{n} v_i^{(1)} w_{2,i} \otimes v_i' \otimes v_{i}^{(3)} + \sum_{i=1}^{n} v_i^{(1)} \otimes v_i' \otimes u_{2,i} v_i^{(3)}$$

$$= -w_3 w_2 \otimes v_{n+1}' \otimes 1 + 1 \otimes v_1' \otimes u_2 u_3$$

$$= 0.$$

Notice that, since $E$ is separable, $E \otimes_k E^{op}$ is semisimple, thus, all the $E$-bimodules appearing in the complex are projective, implying that the $A$-bimodules $P_n$ are projective. \hfill \Box

**Proposition 2.2.** The complex $C_{min}(A)$ is a projective resolution of $A$ as $A^e$-module.

**Proof.** The only thing left to prove is the exactness of the complex, and this follows from [CS, Theorem 4.1]. Nevertheless, we shall construct a contracting homotopy $\{s_i\}_{i \geq 1}$ that will be useful in Section 3.1. For $i = -1$ and $a \in A$, let

$$s_{-1}(a) = a \otimes e_{\epsilon(a)}.$$

For $i \geq 0$, let $a$ be an element of the $k$-basis of irreducible paths $B$. It is sufficient to define $s_i$ on the elements $e_{\epsilon(w)} \otimes w \otimes a \in P_i$ and extend it to a morphism of left $A$-modules. Set

$$s_0(e_{\epsilon(a)} \otimes a) = - \sum a^{(1)} \otimes a^{(2)} \otimes a^{(3)},$$

$$s_1(e_{\epsilon(a)} \otimes a) = \begin{cases} e_0 \otimes \rho_1 \otimes e_\omega, & \text{if } \alpha a = W_{\rho_1} , \\ e_{\epsilon(a)} \otimes \sigma_j \otimes b, & \text{if } \alpha a = \sigma j b \text{ in } Q \text{ with } \sigma_j \text{ monomial}, \\ 0, & \text{otherwise.} \end{cases}$$

For $i = 2$, given $r \in R$, set

$$s_2(e_{\epsilon(r)} \otimes r \otimes a) = - \sum (ra)^{(1)} \otimes (ra)^{(2)} \otimes (ra)^{(3)}$$

with $(ra)^{(2)} \in A_2$ if $ra$ contains a $2$-ambiguity and zero otherwise. Observe that if $r = \rho_i$ is a non monomial relation, then $s_2(e_{\epsilon(r)} \otimes r \otimes a) = 0$.

For $i > 2$, given $w \in A_{i-1}$,

$$s_i(e_{\epsilon(w)} \otimes w \otimes a) = (-1)^{i+1} \sum (wa)^{(1)} \otimes (wa)^{(2)} \otimes (wa)^{(3)}$$

with $(wa)^{(2)} \in A_i$ if $wa$ contains an $i$-ambiguity, and zero otherwise. It is straightforward to verify that $\{s_i\}_{i \geq 1}$ is a contracting homotopy. \hfill \Box

**Remark 3.** To check that the projective resolution is minimal consider $r^e$, the Jacobson radical of $A^e$. This radical is generated by the elements of the form $\alpha \otimes e_j$ and $e_j \otimes \beta$ with $i, j \in Q_0$ and $\alpha, \beta \in Q_1$ since $\text{rad}(A \otimes A^{op}) = A \otimes \text{rad}(A^{op}) + \text{rad}(A) \otimes A^{op}$. It is easy to see that $\text{Im}(d_i) \subset P_{i-1} r^e$, so we conclude that $C_{min}(A)$ is minimal.

3. Computation of Hochschild Cohomology

In this section we will construct an explicit basis for each cohomology space of $A$. The knowledge of such bases will be useful for the computation of the deformations of toupie algebras and for description of the Gerstenhaber structure. Applying the functors $\text{Hom}_{A^e}(-, A)$ to the minimal resolution $C_{min}(A)$ and using again the canonical isomorphism $F_X : \text{Hom}_{A^e}(A \otimes_E X \otimes_E A, A) \rightarrow \text{Hom}_{E^e}(X, A)$, that we will simply denote $F$, we obtain the following complex:

$$0 \rightarrow \text{Hom}_{E^e}(E, A) \xrightarrow{D_0} \text{Hom}_{E^e}(kQ_1, A) \xrightarrow{D_1} \text{Hom}_{E^e}(kQ_2, A) \xrightarrow{D_2} \text{Hom}_{E^e}(kA_2, A) \rightarrow \ldots$$

where $D_i = F \circ \text{Hom}_{A^e}(d_i, A) \circ F^{-1}$ for all $i$. 
For any \( E \)-bimodule \( W \) and any \( f \in \text{Hom}_{E^c}(W, A) \), the equalities \( e_i f(w) e_j = f(e_i w e_j) \), for every \( i, j \in Q_0 \) and \( w \in W \) mean that \( w \) and \( f(w) \) share source and target. Using Cibils’ notation in [Cib], we will write \( w \mid f(w) \) for the morphism in \( \text{Hom}_{E^c}(W, A) \) sending a basis element \( w \) of \( W \) to \( f(w) \) and the other basis elements to zero. Also, given sets \( H \) and \( G \), we will denote \( \text{k}(H) \parallel G \) the \( \text{k} \)-span of \( \{h \parallel g \}_{h \in H, g \in G} \).

Let us denote \( \mathcal{B}_\omega \), the subset of all branches belonging to \( \mathcal{B} \).

We will next describe explicitly the vector spaces and the differentials appearing in the complex.

Since toupie algebras have no cycles, \( \text{Hom}_{E^c}(E, A) \) is just \( \text{k}(Q_0 \mid Q_0) \). A careful look shows that the spaces appearing in degrees 1 are 2 are respectively

\[
\text{Hom}_{E^c}(\text{k}Q_1, A) = \text{k}(Q_1 \mid Q_1) + \text{k}(Z \parallel (0 \mathcal{B}_\omega - Z)).
\]

\[
\text{Hom}_{E^c}(\text{k}R, A) = \text{k}(R_{\text{nonon}} \parallel (0 \mathcal{B}_\omega) + \text{k}(0 \parallel (R_{\text{nonon}}) \mid 0 \mathcal{B}_\omega).
\]

For \( i \geq 2 \),

\[
\text{Hom}_{E^c}(\text{k}A_i, A) = \text{k}(0 \parallel (A_i) \mid 0 \mathcal{B}_\omega).
\]

The differentials are as follows. Given \( j \in Q_0 \) and \( e_j \mid e_j \in \text{Hom}_{E^c}(E, A), \)

\[
D_0(e_j \mid e_j) = \sum_{\alpha \in (Q_0 \parallel Z) \parallel (0 \mathcal{B}_\omega - Z)} \alpha \parallel \beta - \sum_{\beta \in (Q_0 \parallel Z) \parallel (0 \mathcal{B}_\omega - Z)} \beta \parallel \beta.
\]

Given \( \alpha_r \in Q_1, \)

\[
D_1(\alpha_r \mid \alpha_r) = \begin{cases} \sum_{i \in (Q_0 \parallel Z) \parallel (0 \mathcal{B}_\omega - Z)} \rho_i |b_{ij} \alpha^{(j)}, & \text{if } \alpha_r \text{ is in } \alpha^{(j)} \text{ and } \alpha^{(j)} \text{ is a branch of } \rho_i, \\ 0, & \text{otherwise}. \end{cases}
\]

For \( \alpha^{(h)} \parallel \alpha^{(j)} \in Z \parallel (0 \mathcal{B}_\omega - Z), \)

\[
D_1(\alpha^{(h)} \parallel \alpha^{(j)}) = 0.
\]

Finally, \( D_1 = 0 \) for \( i \geq 2 \).

An easy verification shows that, as it is well known when the quiver contains no oriented cycles, \( \text{HH}^0(A) = (\sum_{i \in Q_0} e_i \mid e_i) \).

3.1. Computation of \( \text{HH}^1(A) \). To compute \( \text{HH}^1(A) \) we first obtain a basis of \( \text{Ker}D_1 \) that we modify afterwards so that it contains a basis of \( \text{Im}D_0 \). For that purpose, using the branches of the quiver of length greater than 1 that do not contain monomial relations, we will construct a non oriented graph \( Q_\rho \) as follows.

**Definition 4.** The graph \( Q_\rho \) is the following:

1. its vertices are in bijection with the branches in \( Q \) that are not arrows and do not contain monomial relations, labeled under the name of the corresponding branch,
2. there is an edge between the vertices \( \alpha^{(k)} \) and \( \alpha^{(p)} \) if there exists a relation that involves both branches.

We will denote \( Q_\rho^k, 1 \leq k \leq r, \) the connected components of \( Q_\rho \).

**Remark 5.** We recall the dimension of \( \text{HH}^1(A) \), computed in [GL]:

\[
dim_k \text{HH}^1(A) = r + m + Da - 1.
\]

Let us define the following four subsets of \( k(Q_1 \parallel \mathcal{B}) \):

1. \( C_1 = \{ \alpha_j \parallel \alpha_j : \alpha^{(i)} \text{ is a branch which contains monomial relations} \} \).
2. \( C_2 = (Z \parallel \mathcal{B}_\omega \).
3. \( C_3 = \{ \alpha_j \parallel \alpha_j : \alpha_0^{(j)} \parallel \alpha_0^{(j)} : j \neq 0, |\alpha^{(j)}| > 1 \text{ and } \alpha^{(i)} \text{ does not contain monomial relations} \}.
4. \( C_4 = \{ \sum_{\alpha \in (Q_0 \parallel \mathcal{B}_\omega) \parallel (0 \mathcal{B}_\omega) : k = 1 \ldots, r \} \).

Lemma 3.1. The set $U = \bigcup_{i=1}^{4} C_i$ is a basis of $\text{Ker} D_1$.

Proof. Since $\dim_k \text{HH}^1(A) = r + m + Da - 1$, our knowledge of the dimension of $\text{Im}(D_0)$ implies that $\dim_k \text{Ker} D_1 = r + m + Da + \#Q_0 - 2$. By construction, $U$ is linearly independent. Observe that the disjoint union $C_1 \cup C_3$ is in bijection with the set consisting of all the arrows that do not start in 0 and the arrows of the monomial part that start in 0, so $\#C_1 + \#C_3 = \#Q_0 - 2 + m$. Since clearly $\#C_2 = Da$ and $\#C_4 = r$ we conclude that $\#U = r + m + Da + \#Q_0 - 2$. \hfill \Box

Next we will replace $U$ by another basis containing a basis of $\text{Im}D_0$; we will proceed as follows.

1. Replace each element $\alpha_j^i |\alpha_k^i$ in $C_1$ with $j \neq 0$ by $\alpha_j^i |\alpha_j^i - \alpha_k^i |\alpha_k^i$. Let us call $C'_1$ the set consisting of the modified elements and

$$C''_1 = \{\alpha_0^i |\alpha_0^i : \alpha^{(i)} \text{ is a branch that contains monomial relations}\},$$

its complement in $C_1$.

2. If $a > 0$, first replace the elements in $C_2$ of the form $\alpha^{(i)} |\alpha^{(i)}$ with $i \neq 1$ by $\alpha^{(i)} |\alpha^{(i)} - \alpha^{(i)} |\alpha^{(i)}$ and then replace $\alpha^{(i)} |\alpha^{(i)}$ by $s = \sum_{\alpha^{(i)} |\alpha_0^i |\alpha_0^i}$ where the sum ranges over all the branches of the quiver.

3. If $a = 0$, let us call $\alpha^{(B)}$ the last branch of the toupie algebra and consider the element of $U$ in $C_1 \cup C_4$ that contains $\alpha_0^B |\alpha_0^B$ as a summand. Replace this element by $s = \sum_{\alpha^{(i)} |\alpha_0^i |\alpha_0^i}$ where the sum ranges over all the branches of the quiver.

Let us call $\bar{U}$ the set obtained from $U$ in this way. After some direct and tedious computations, it turns out that the $k$-vector spaces generated by $U$ and $\bar{U}$ coincide.

Lemma 3.2. The set $K = \{s\} \cup C'_1 \cup C_3$ is a basis of $\text{Im}D_0$.

Proof. We know that $\dim_k \text{Im}D_0 = \#Q_0 - 1$ because on one hand $\dim_k \text{Ker} D_0 = 1$ and on the other hand $\dim_k \text{Hom}_{E^\ast}(E, A) = \#Q_0$. Since $K \subset \text{Im}D_0$, just observe that $\#C'_1 + \#C_3 = \#Q_0 - 2$ and $K$ is linearly independent. \hfill \Box

The proof of the following theorem is now immediate.

Theorem 3.3. The classes of the elements of $\bar{U} - K$ in $\text{Ker}D_1/\text{Im}D_0$ form a basis of $\text{HH}^1(A)$.

3.2. Computation of HH$^1(A)$ for higher degrees. In order to obtain a basis of $\text{HH}^1(A)$ we will follow the same lines as for $\text{HH}^1(A)$. Recall that $D_2 = 0$ and so $\text{Ker}D_2 = \text{Hom}_{E}(kR, A)$.

Remark 6. To compute a basis of $\text{Im}D_1$ we will need to define an order $\prec'$ in $R_{\text{non}}$. Every element in $R_{\text{non}}$ is of the form

$$\rho_i = \alpha^{(k_i)} + \sum_{j > k_i} b_{ij} \alpha^{(j)}$$

so we say that $\rho_i \prec' \rho_j$ if $k_i < k_j$. This order induces an order on the elements of $R_{\text{non}} |\rho_0 B_{\omega}$ as follows,

$$\rho_i |\rho_j \prec' \rho_j |\rho_j \text{ if } \rho_i \prec' \rho_j \text{ or } \rho_i = \rho_j \text{ and } k < 1.$$ 

For each connected component of $Q_\omega$ associated to a non monomial relation there exists a relation containing $\alpha^{(k_i)}$ with $k_i$ maximum. We will call this relation the last one of the component.

Denote by $X$ the set consisting of all the elements in $\rho_0 B_{\omega}$ that are involved in non monomial relations and define $d = \#X$.

Consider the following subsets of $\text{k}(R |\rho_0 B_{\omega})$:

- $B_1 = \{\rho_i |\rho_i : \rho_i \in R_{\text{non}} \text{ such that } \rho_i \text{ is not the last one of } Q^k_\omega, \text{ with } k = 1, \ldots, r\}$,
\[ B_2 = \{ \sum_{i, \alpha^{(i)}_{\text{term}}, h, k} \rho_i b_{ik} \alpha^{(k)} : \alpha^{(k)} \in X \}. \]

Observe that \#B_1 = n - d + l - r and \#B_2 = d.

**Lemma 3.4.** The set \( B_1 \cup B_2 \) is a basis of \( \text{Im} D_1 \).

**Proof.** It is clear that \( B_1 \cup B_2 \subset \text{Im} D_1 \) since \( B_1 \subset \{ D_1(\alpha^{(i)}_0 | \alpha^{(i)}_0) : \alpha^{(i)} = W_{\rho_j} \text{ for some } j \} \) and \( B_2 = \{ D_1(\alpha^{(i)}_0 | \alpha^{(i)}_0) : \alpha^{(i)} \in X \} \). We know that \( \text{dim}_{k} \text{Ker} D_1 = r + m + Da + \#Q_0 - 2 \) and, on the other hand, \( \text{dim}_{k} \text{Hom}_{R^e}(kQ_1, A) = \#Q_1 + (D - 1)a = m + l + n + Da + \#Q_0 - 2 \). From the previous computations we deduce that \( \text{dim}_{k} \text{Im} D_1 = l + n - r \).

Observe that the union \( B_1 \cup B_2 \) is disjoint: an element in the intersection of \( B_1 \) and \( B_2 \) must be of the form \( t = \rho_i \| \alpha^{(k)} \) with \( \alpha^{(k)} \neq \alpha^{(k)} \) the only branch except for \( \alpha^{(k)} \) in \( \rho_i \) and \( c \in k \). Now, since \( t \) has only one summand, the branch \( \alpha^{(k)} \) is not involved in any other relation. On the other hand \( \alpha^{(k)} \), by definition, is only involved in the relation \( \rho_i \). Thus, the branches \( \alpha^{(k)} \) and \( \alpha^{(k)} = W_{\rho_i} \) are the only ones in their connected component of \( Q_{\rho_i} \). So \( \rho_i = \alpha^{(k)} - c\alpha^{(k)} \) is the unique relation of \( Q^{k}_{\rho} \) for some \( k \), and this cannot happen due to the definition of \( B_1 \).

Since \#B_1 = n - d + l - r, \#B_2 = d and the union is disjoint, it follows that \#(B_1 \cup B_2) = \#B_1 + \#B_2 = l + n - r, so it is sufficient to prove that \( B_1 \cup B_2 \) generates \( \text{Im}(D_1) \). By definition, \( D_1(\alpha^{(i)}_0 | \alpha^{(i)}_0) \) coincides with \( D_1(\alpha^{(i)}_0 | \alpha^{(i)}_0) \) for any arrow \( \alpha^{(i)}_0 \) in the branch \( \alpha^{(i)}_0 \), also \( D_1(\sum_{\alpha^{(i)}_0 \in Q_{\rho_i}^{k}} \alpha^{(i)}_0 | \alpha^{(i)}_0) = 0 \), for every connected component \( Q^{k}_{\rho} \) of \( Q_{\rho} \), so the result is proven.

Again, we will modify the basis \( B_1 \cup B_2 \), following the next steps.

1. For every element \( w \in B_2 \) with a summand of the form \( \rho_j \| b_{ik} \alpha^{(k)} \) where \( \alpha^{(k)} \) is the first branch of \( f_{\rho_j} \) and \( \rho_j \) is not the last relation for any connected component of \( Q_{\rho_j} \), subtract from \( w \) the element \( \rho_j \| f_{\rho_j} \) which belongs to \( B_1 \). Let us call \( B' \) the set obtained from \( B_2 \) in this way. The vector spaces spanned by \( B_1 \cup B_2 \) and by \( B_1 \cup B' \) are equal.

2. Consider the order defined in \( R_{\text{nonom}} \) for \( R_{\text{nonom}}^{k} \| B_{\omega} \) and the matrix associated to the equations corresponding to the rows of \( B'_{\omega} \) in its echelon form. Reconstruct the set in \( R_{\text{nonom}}^{k} \| B_{\omega} \) from the echelon form and call it \( B''_{\omega} \).

Since \( (B_1 \cup B''_{\omega}) = (B_1 \cup B'_{\omega}) = (B_1 \cup B_2) \), we conclude that \( B_1 \cup B''_{\omega} \) is a basis of \( \text{Im} D_1 \).

Now we will modify the set
\[ N = \{ \rho_i \| \alpha^{(k)} : \rho_i \in R_{\text{nonom}} \text{ and } \alpha^{(k)} \in 0B_{\omega} \} \cup \{ \sigma_i \| \alpha^{(k)} : \sigma_i \in 0(R_{\text{monom}})_{\omega} \text{ and } \alpha^{(k)} \in 0B_{\omega} \}, \]
which is clearly a basis of \( \text{Hom}_{E^e}(kR, A) \), in order to obtain another basis \( N' \) of \( \text{Hom}_{E^e}(kR, A) \) including \( B_1 \cup B''_{\omega} \). The procedure is as follows.

1. To have the elements of \( B_1 \) in our basis, given \( \rho_i \in R_{\text{nonom}} \) and \( \alpha^{(k)} \in 0B_{\omega} \), with \( k \) the minimum of \( f_{\rho_i} \), replace \( \rho_i \| \alpha^{(k)} \) by \( \rho_i \| f_{\rho_i} \).

2. To get the elements of \( B''_{\omega} \) in our new basis consider the pivot of every element in \( B''_{\omega} \) and replace the element having this pivot in \( N' \) by the corresponding one in \( B''_{\omega} \).

**Theorem 3.5.** The classes of \( N' = B_1 \cup B''_{\omega} \) in \( \text{Hom}_{E^e}(kR, A)/\text{Im}(D_1) \) form a basis of \( \text{HH}^2(A) \).

The last theorem of this section gives a basis of the Hochschild cohomology of \( A \) for higher degrees.

**Theorem 3.6.** Given \( i \geq 3 \), the set \( B_i = \{ w_j \| \alpha^{(k)} : \alpha^{(k)} \in 0B_{\omega} \text{ and } w_j \in 0(A_{i-1})_{\omega} \} \) is a basis of \( \text{HH}^i(A) \).

**Proof.** Since \( D_i = 0 \) for \( i \geq 3 \) it follows that \( \text{HH}^i(A) = \text{Hom}_{E^e}(kA_{i-1}, A) \). The set \( B_i \) is a basis of this \( k \)-vector space so it is also a basis of \( \text{HH}^i(A) \). \( \square \)
4. Comparison morphisms

For the computation of Hochschild cohomology we used a minimal resolution. One possible way of computing the Gerstenhaber structure is via comparison morphisms between this minimal resolution and the $E$-reduced Bar resolution in whose existence is guaranteed since both complexes are projective resolutions of $A$ as $A^e$-module. We will describe them explicitly in the sequel. Let us call $\varphi: C_{\text{min}}(A) \to C_{\text{Bar}E}(A)$ and $\eta: C_{\text{Bar}E}(A) \to C_{\text{min}}(A)$ a pair of comparison morphisms between both resolutions:

\[
\begin{array}{ccccccc}
\vdots A \otimes E kA_2 \otimes E A & \xrightarrow{d_2} & A \otimes E k\mathcal{R} \otimes E A & \xrightarrow{d_1} & A \otimes E kQ_1 \otimes E A & \xrightarrow{d_0} & A \otimes E A & \to 0 \\
\nu_3 \downarrow \eta_3 & & \nu_2 \downarrow \eta_2 & & \nu_1 \downarrow \eta_1 & & \nu_0 \downarrow \eta_0 & \\
\vdots A \otimes E A^3 \otimes E A & \xrightarrow{\delta_2} & A \otimes E A^2 \otimes E A & \xrightarrow{\delta_1} & A \otimes E A \otimes E A & \xrightarrow{\delta_0} & A \otimes E A & \to 0. \\
\end{array}
\]

After applying the functor $\text{Hom}_{A^e}(\cdot, A)$ and considering the natural identifications at the beginning of Section 3 we obtain the maps $\varphi^*$ and $\eta^*$ and the following diagram:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \text{Hom}_{E^e}(E, A) & \xrightarrow{D_0} & \text{Hom}_{E^e}(kQ_1, A) & \xrightarrow{D_1} & \text{Hom}_{E^e}(k\mathcal{R}, A) & \xrightarrow{D_2} & \text{Hom}_{E^e}(kA_k, A) & \cdots \\
\nu_0^* \downarrow \nu_0^* & & \nu_1^* \downarrow \nu_1^* & & \nu_2^* \downarrow \nu_2^* & & \nu_3^* \downarrow \nu_3^* & \\
0 & \longrightarrow & \text{Hom}_{E^e}(E, A) & \xrightarrow{\delta_0^*} & \text{Hom}_{E^e}(A^3, A) & \xrightarrow{\delta_1^*} & \text{Hom}_{E^e}(A^2, A) & \xrightarrow{\delta_2^*} & \text{Hom}_{E^e}(A, A) & \cdots \\
\end{array}
\]

The maps $\varphi^*$ and $\eta^*$ induce isomorphisms at the cohomology level that we will still denote $\varphi^*$ and $\eta^*$. We start the description of the morphisms of complexes $\varphi: C_{\text{min}}(A) \to C_{\text{Bar}E}(A)$ and $\eta: C_{\text{Bar}E}(A) \to C_{\text{min}}(A)$. Firstly, $\varphi_0$ and $\eta_0$ can both be chosen as the identity of $A \otimes E A$. We will construct $\varphi_i$ for $i > 0$ using the homotopy $t_i$ defined in Lemma ??.

Given $i \in \mathbb{N}$ and $1 \otimes E u \otimes E 1 \in \mathcal{P}_{i-1}$, we define:

$$
\varphi_i(1 \otimes E u \otimes E 1) = t_{i-1} \varphi_{i-1} d_{i-1}(1 \otimes E u \otimes E 1).
$$

and the we extend $\varphi_i$ as an $A$-bimodule morphism. Given $\lambda, \mu \in \mathbb{B}$:

$$
\varphi_i(\lambda \otimes E u \otimes E \mu) = \lambda_i \varphi_{i-1} d_{i-1}(1 \otimes E u \otimes E 1) \mu.
$$

Finally we extend it linearly.

**Proposition 4.1.** The family of functions $\varphi = \{\varphi_i\}_{i \in \mathbb{N} \cup \{0\}}$ is a morphism of complexes.

**Proof.** To prove that $\varphi$ is a morphism of complexes we need to verify that the corresponding diagrams commute, that means that $\delta_j \varphi_{i+1} = \varphi_j d_i$ for all $i$. We will prove it by induction. The case $i = 0$ is clear. For the inductive step, let us suppose that $t_{j-1} \varphi_j = t_{j-1} \varphi_{j-1} d_{j-1}$ for any $j > 0$ and prove that $t_j \varphi_{j+1} = t_j \varphi_{j-1} d_j$. Firstly, observe that it is sufficient to prove the result for $1 \otimes E u \otimes E 1 \in \mathcal{P}_i$ since $\varphi_i, d_i$ and $\delta_i$ are morphisms of $A$-bimodules for all $i$.

By definition of $\varphi_j$,

$$
\delta_j \varphi_{j+1}(1 \otimes E u \otimes E 1) = \delta_j t_{j-1} \varphi_{j-1} d_j(1 \otimes E u \otimes E 1)
$$

$$
= (1d_j - t_{j-1} \delta_j \varphi_{j-1} d_{j-1})(1 \otimes E u \otimes E 1).
$$

The inductive hypothesis says that $t_{j-1} \varphi_{j-1} = t_{j-1} \varphi_{j-1} d_{j-1}$, as a consequence

$$
t_{j-1} \delta_{j-1} \varphi_{j-1} d_j(1 \otimes E u \otimes E 1) = t_{j-1} \varphi_{j-1} d_{j-1}(1 \otimes E u \otimes E 1)
$$

which is zero since $d_{j-1} d_j = 0$, and we obtain the desired equality. \qed

**Remark 7.** In lower degrees, using the inductive definition, we obtain the explicit formulas:

$$
\varphi_1(\lambda \otimes E A \otimes E \mu) = \lambda \otimes E A \otimes E \mu.
$$
\[ \varphi_2(\lambda \otimes_E r \otimes_E \mu) = \sum \lambda \otimes_E r(1) \otimes_E r(2) \otimes_E r(3) \mu. \]

\[ \varphi_3(\lambda \otimes_E u \otimes_E \mu) = \sum \lambda \otimes_E u_i \otimes_E (w_1w_0)(1) \otimes_E (w_1w_0)(2) \otimes_E (w_1w_0)(3) \mu \]

where \( u = w_2w_1w_0 \) as a right 2-ambiguity.

For the definition of \( \{ \eta_i \}_{i \in \mathbb{N}} \), we will use a similar procedure, this time with the homotopy \( \{ s_i \}_{i \geq 1} \) constructed in Proposition \( \ref{prop:homotopy} \) for our minimal resolution. Namely, given \( i \in \mathbb{N} \) and \( 1 \otimes_E \overline{a_1} \otimes_E \cdots \otimes_E \overline{a_i} \otimes_E 1 = \lambda \otimes_E \overline{s_1} \otimes_E A \cdot \lambda \otimes_E \overline{s_2} \otimes_E A \cdot \lambda \otimes_E \cdots \otimes_E \overline{s_i} \otimes_E \lambda \otimes_E \overline{a_i} \otimes_E A 

\eta_i(1 \otimes_E \overline{a_1} \otimes_E \cdots \otimes_E \overline{a_i} \otimes_E 1) = s_{i-1} \eta_{i-1}(1 \otimes_E \overline{a_1} \otimes_E \cdots \otimes_E \overline{a_i} \otimes_E 1),

and we extend \( \eta_i \) as a morphism of \( A \)-bimodules and then linearly.

The proof of the following proposition is analogous to the previous one and will be omitted.

**Proposition 4.2.** The family of functions \( \eta = \{ \eta_i \}_{i \in \mathbb{N}} \) is a morphism of complexes.

Let us obtain the explicit formulas of \( \eta_i \) for lower degrees. For \( i = 1 \), and \( 1 \otimes_E \overline{a_1} \otimes_E 1 \in \lambda \otimes_E \overline{a_1} \otimes_E A 

\eta_1(1 \otimes_E \overline{a_1} \otimes_E 1) = s_0 \eta_0(a \otimes_E 1 - 1 \otimes_E a) 

= s_0(a \otimes_E 1 - 1 \otimes_E a) 

= \sum a(1) \otimes_E a(2) \otimes_E a(3),

since \( s_0(a \otimes_E 1) = 0 \).

We continue now with \( \eta_2 \). Given \( 1 \otimes_E a_1 \otimes_E a_2 \otimes_E 1 \in \lambda \otimes_E \overline{a_1} \otimes_E A \cdot \lambda \otimes_E \overline{a_2} \otimes_E A \) with \( a_1, a_2 \in B = \mathcal{B} - \{ e_x : x \in \mathcal{Q}_0 \} \),

\eta_2(1 \otimes_E a_1 \otimes_E a_2 \otimes_E 1) = s_1 \eta_1(1 \otimes_E a_1 \otimes_E a_2 \otimes_E 1) 

= s_1 \eta_1(1 \otimes_E a_1 \otimes_E a_2 \otimes_E 1 - 1 \otimes_E a_1a_2 \otimes E 1 + 1 \otimes_E a_1 \otimes E a_2).

There are three different cases to consider:

1. If \( a_1a_2 \in \mathcal{B} \), then

\[ \eta_1(1 \otimes_E a_1 \otimes_E a_2 \otimes_E 1) = s_1(1) \otimes_E a_1(2) \otimes E a_2(3) 

\begin{array}{c}
- \sum (a_1(1) \otimes_E a_2(2) \otimes E (a_1a_2)(3) 

\begin{array}{c}
+ \sum a(1) \otimes_E a(2) \otimes E a(3)a_2 = 0.
\end{array}
\end{array}

So, if \( a_1a_2 \in \mathcal{B} \); \( \eta_2(1 \otimes_E a_1 \otimes E a_2 \otimes_E 1) = 0 \).

2. If \( a_1a_2 \) contains at least one monomial relation, let \( \{ \sigma_1, \sigma_2, \ldots, \sigma_p \} \) be the set of monomial relations in \( a_1a_2 \), ordered by their starting point in \( a_1 \). For every \( \sigma_i \), we write \( a_1a_2 = (\sigma_i)(\sigma_i)^r \) where \( (\sigma_i)^r \) is the part of \( a_1a_2 \) on the left of \( \sigma_i \) and \( (\sigma_i)^r \) is the right part. By definition of \( \eta_1 \),

\[ \eta_2(1 \otimes_E a_1 \otimes_E a_2 \otimes_E 1) = \sum s_1(a_1a_2(1) \otimes E a_2(2) \otimes E a_2(3)) 

\begin{array}{c}
+ \sum s_1(a_1(1) \otimes E a_2(2) \otimes E a_2(3)a_2).
\end{array}

The first summand is zero since \( a_2 \in \mathcal{B} \) and this implies that no monomial relation in \( a_1a_2 \) starts after the end of \( a_1 \). With respect to the second summand, observe that if
We will need the explicit formula of $\eta_1$, for the purpose of computing it we are going to consider four different cases.

Given $a_1, a_2, a_3 \in B$, let us consider $\lambda \otimes E a_1 \otimes E a_2 \otimes E a_3 \otimes E \mu \in A \otimes_E A \otimes_E A$; the expression

$$\eta_3(\lambda \otimes E a_1 \otimes E a_2 \otimes E a_3 \otimes E \mu) = \lambda(s_2\eta_2(a_1 \otimes E a_2 \otimes E a_3 \otimes E 1) + \eta_2(1 \otimes E a_1 \otimes E a_2 a_3 \otimes E 1)) \mu.$$ 

Observe that in this case $\eta_2(1 \otimes E a_1 \otimes E a_2 a_3 \otimes E 1) = \eta_2(1 \otimes E a_1 \otimes E a_2 a_3 \otimes E 1)$ since, by definition of $\eta_2$ both terms depend on $a_1a_2a_3$.

In conclusion, if $a_1a_2 \in B$ and $a_2a_3 \in B$, then:

$$\eta_3(\lambda \otimes E a_1 \otimes E a_2 \otimes E a_3 \otimes E \mu) = 0.$$

(2) Let us now consider the case where $a_1a_2 \in B$ and $a_2a_3 \in I$. In this case,

$$\eta_3(\lambda \otimes E a_1 \otimes E a_2 \otimes E a_3 \otimes E \mu) = \lambda(s_2\eta_2(a_1 \otimes E a_2 \otimes E a_3 \otimes E 1) + 1 \otimes E a_1a_2a_3 \otimes E 1)) \mu.$$ 

By definition of $\eta_2$ we obtain that

$$\eta_3(\lambda \otimes E a_1 \otimes E a_2 \otimes E a_3 \otimes E \mu) = \lambda(s_2(\sigma_p \otimes E \sigma_p \otimes E (\sigma_p)^\gamma - a_1(\sigma_p \otimes E \sigma_p \otimes E (\sigma_p)^\gamma)) \mu$$

since the last relation of $a_2a_3$ starting from the left coincides with the last one of $a_1a_2a_3$.

In conclusion, if $a_1a_2 \in B$ and $a_2a_3 \in I$, then:

$$\eta_3(\lambda \otimes E a_1 \otimes E a_2 \otimes E a_3 \otimes E \mu) = 0.$$

(3) Let us continue with the symmetric case, this means $a_1a_2 \in I$ and $a_2a_3 \in B$.

$$\eta_3(\lambda \otimes E a_1 \otimes E a_2 \otimes E a_3 \otimes E \mu) = \lambda(s_2\eta_2(1 \otimes E a_1 \otimes E a_2 a_3 \otimes E 1 - 1 \otimes E a_1 \otimes E a_2 a_3)) \mu.$$ 

Using the definition of $\eta_2$ in this case we obtain

$$\eta_3(\lambda \otimes E a_1 \otimes E a_2 \otimes E a_3 \otimes E \mu) = \lambda(s_2((\delta_q \otimes E \delta_q \otimes E (\delta_q)^\gamma - (\sigma_q \otimes E \sigma_q \otimes E (\sigma_q)^\gamma))) \mu$$

where $\delta_q$ is the last relation in $a_1a_2a_3$ and $\sigma_q$ is the last relation of $a_1a_2$ both starting from the left.
Proposition 4.3. Let \( \varphi \) be a representation of \( HH \) with \( \sigma \). For some even \( i \), holds for some even \( i \), we obtain

\[
\eta_i(\lambda \otimes E a_1 \otimes E a_2 \otimes E a_3 \otimes E \mu) = \sum_{i=q}^{p-1} \lambda(w_i) \otimes E w_i \otimes E (w_i') \mu.
\]

Summarising,

\[
\eta_i(\lambda \otimes E a_1 \otimes E a_2 \otimes E a_3 \otimes E \mu) = \begin{cases} 
\lambda \sum_{i=q}^{p-1} (w_i) \otimes E w_i \otimes E (w_i') \mu & \text{if } a_1a_2 \in I \text{ and } \delta_r \cap \sigma_q \neq \emptyset, \\
0 & \text{otherwise},
\end{cases}
\]

where \( \{w_i\}_{i=q}^{p-1} \) is the set of 2-ambiguities contained in \( a_1a_2a_3 \), such that \( s(w_i) \geq s(\sigma_q) \) with \( \sigma_q \) the last relation of \( a_1a_2 \) and \( \delta_r \) the last relation of \( a_1a_2a_3 \).

The explicit formulas of the comparison morphisms in higher degrees are usually very hard to find. In spite of that, the next propositions will help us later to describe \( HH^i(A) \) as Lie representation of \( HH^1(A) \) for \( i \geq 2 \).

**Proposition 4.3.** Let \( i \in \mathbb{N} \) and \( w = w_0 \ldots w_{i-1} = w_{i-1} \ldots w_0 \) an \((i-1)\)-ambiguity. The comparison morphism \( \varphi_i \) is such that

\[
\varphi_i(1 \otimes E w \otimes E 1) = \sum_{a^{(1)} \ldots a^{(i+1)} = w} 1 \otimes E a^{(1)} \otimes E \ldots \otimes E a^{(i)} \otimes E a^{(i+1)}
\]

where \( a^{(j)} \) with \( 1 \leq j \leq i+1 \) are non zero paths in \( A \).

**Proof.** For the cases \( i = 1, 2, 3 \) we use the explicit formulas. Let us now suppose that the result holds for some even \( i-1 \in \mathbb{N} \). Using the formula of \( \varphi_i \),

\[
\varphi_i(1 \otimes E w \otimes E 1) = l_i \varphi_{i-1} + d_i \varphi_{i-1}(1 \otimes E w \otimes E 1)
\]

where \( d_i \) is such that

\[
d_i = \sum_{a^{(1)} \ldots a^{(i)} = w} 1 \otimes E a^{(1)} \otimes E \ldots \otimes E a^{(i+1)}
\]

and \( l_i \) is such that

\[
l_i = \sum_{a^{(1)} \ldots a^{(i)} = w} 1 \otimes E a^{(1)} \otimes E \ldots \otimes E a^{(i+1)}
\]
Observe that the second summand is zero in the \( E \)-reduced Bar resolution due to the inductive hypothesis and the definition of \( t_{i-1} \). Finally, the inductive hypothesis for \( \varphi_{i-1}(1 \otimes E w_{i-2} \ldots w_0 \otimes E 1) \) and the definition of \( t_{i-1} \), give the result.

When \( i-1 \) is odd,
\[
\varphi_i(1 \otimes E w \otimes E 1) = t_{i-1} \varphi_{i-1} d_{i-1}(1 \otimes E w \otimes E 1)
\]
\[
= \sum_w t_{i-1}(w^{(1)} \varphi_{i-1}(1 \otimes E w^{(2)} \otimes E 1)w^{(3)})
\]
\[
= \sum_w \sum_{b^{(1)} \ldots b^{(j)} = w^{(2)}} t_{i-1}(w^{(1)} \otimes E b^{(1)} \otimes E \ldots \otimes E b^{(i-1)} \otimes E b^{(i)} w^{(3)})
\]
\[
= \sum_w \sum_{b^{(1)} \ldots b^{(j)} = w^{(2)}} 1 \otimes E w^{(1)} \otimes E b^{(1)} \otimes E \ldots \otimes E b^{(i-1)} \otimes E b^{(i)} w^{(3)},
\]

observing that \( w^{(1)} b^{(1)} \ldots b^{(j)} w^{(3)} = w \), the proof is complete. \( \square \)

**Proposition 4.4.**

1. For any \( i \geq 3 \) and \( a_1, \ldots, a_i \) in \( B \)
\[
\eta_i(1 \otimes E a_1 \otimes E \cdots \otimes E a_i \otimes E 1) = \sum_a \alpha \eta_{a_1}(a^{(1)} \otimes E a^{(2)} \otimes E a^{(3)}
\]

with \( \alpha \in K \), \( a^{(1)} a^{(2)} a^{(3)} = a = a_1 \ldots a_i \) and \( a^{(2)} \in A_{i-1} \).

In particular, if the path \( a_1 \ldots a_i \) does not contain \((i-1)\)-ambiguities,
\[
\eta_i(1 \otimes E a_1 \otimes E \cdots \otimes E a_i \otimes E 1) = 0.
\]

2. For all \( i \geq 1 \) and \( w = w_0 \ldots w_{i-1} = z_1 \ldots z_0 \) an \((i-1)\)-ambiguity:
\[
\eta_i(1 \otimes E w_0 \otimes E w_{i-1} \otimes E 1) = 1 \otimes E w \otimes E 1.
\]

**Proof.** (1) The case \( i = 3 \) has already been checked. For the inductive step,
\[
\eta_i(1 \otimes E a_1 \otimes E \cdots \otimes E a_i \otimes E 1)
\]
\[
= a_1 s_{i-1} \eta_{i-1}(1 \otimes E a_1 \otimes E a_2 \otimes E \cdots \otimes E a_i \otimes E 1)
\]
\[
+ \sum_{j=1}^{i-1} (-1)^j s_{i-1} \eta_{i-1}(1 \otimes E a_1 \otimes E \cdots \otimes E a_j a_{j+1} \otimes E \cdots \otimes E 1)
\]
\[
+ (-1)^i s_{i-1} \eta_{i-1}(1 \otimes E a_1 \otimes E \cdots \otimes E a_{i-1} \otimes E 1)a_i.
\]

Using the inductive hypothesis and the definition of \( s_{i-1} \) in every summand, the result is obtained.

(2) Firstly observe that since \( w_j, z_k \in B \) for \( 0 \leq j, k \leq i-1 \) it will not be necessary to take classes of \( w_j \) or \( z_k \) in \( \overline{A} \).

The case where \( i = 1 \) is clear using the definition of \( \eta_1 \). For \( i = 2 \), the explicit formula of \( \eta_2 \) gives
\[
\eta_2(1 \otimes E w_0 \otimes E w_1 \otimes E 1) = (\sigma_p)^r \otimes E \sigma_p \otimes E (\sigma_p)^r
\]
where \( \sigma_p \) is the last monomial relation from the left that appears in \( w_0 w_1 \). Since, by definition of ambiguity, \( w_0 w_1 \) cannot strictly contain a monomial relation, \( \sigma_p = w_0 w_1 \) and the result is proven.

For the inductive step,
\[
\eta_i(1 \otimes E w_0 \otimes E \cdots \otimes E w_{i-1} \otimes E 1)
\]
\[
= w_0 s_{i-1} \eta_{i-1}(1 \otimes E w_0 \otimes E \cdots \otimes E w_{i-1} \otimes E 1)
\]
\[
+ \sum_{j=0}^{i-2} (-1)^{j+1} s_{i-1} \eta_{i-1}(1 \otimes E \cdots \otimes E w_j w_{j+1} \otimes E \cdots \otimes E 1)
\]
\[
+ (-1)^{i-1} s_{i-1} \eta_{i-1}(1 \otimes E w_0 \otimes E \cdots \otimes E w_{i-1}).
\]
The summands in the second line vanish since \( w_j w_{j+1} \in I \) for all \( j = 0, \ldots, i - 2 \). Using the inductive hypothesis for the last summand it turns out that it is equal to
\[
(-1)^i s_{i-1}(1 \otimes_E w_0 w_1 \ldots w_{i-2} \otimes_E w_{i-1})
\]
and using the definition of \( s_{i-1} \) the expression equals to \( 1 \otimes_E w \otimes_E 1 \), since \( w \) is the only \((i - 1)\)-ambiguity contained in \( w_0 \ldots w_{i-1} \).

The last step is to prove that the first summand vanishes. Using the first part of this proposition, the term \( \eta_{i-1}(1 \otimes_E w_1 \otimes_E \cdots \otimes_E w_{i-1} \otimes_E 1) \) will be a linear combination of elements of the form \( a^{(1)} \otimes_E a^{(2)} \otimes_E a^{(3)} \) with \( a = a^{(1)} a^{(2)} a^{(3)} \) a path from \( s(w_1) \) to \( t(w_{i-1}) \) and \( a^{(2)} \) an \((i - 2)\)-ambiguity. Since the path \( w_1 \) does not start at 0 and the quiver is toupie, the only path from \( s(w_1) \) to \( t(w_{i-1}) \) is \( w_1 \ldots w_{i-1} \). The result of applying \( s_{i-1} \) to the linear combination vanishes since \( a^{(1)} a^{(2)} a^{(3)} = w_1 \ldots w_{i-1} \) and it cannot contain an \((i - 1)\)-ambiguity. \( \square \)

**Proposition 4.5.** For any toupie algebra \( A \), \( \eta \) is a left inverse of \( \varphi \).

**Proof.** Again, we will prove the result by induction. The cases \( i = 0 \) and \( i = 1 \) are almost immediate by using the corresponding explicit formulas. Consider \( i \geq 2 \). Observe that it is sufficient to verify the equality for the elements of the form \( 1 \otimes_E w \otimes_E 1 \in \mathcal{P}_i \).

Let us check the result for \( i = 2 \). Given a monomial relation \( \sigma_i \),
\[
\eta_2 \varphi_2(1 \otimes_E \sigma_i \otimes_E 1) = \eta_2 \left( \sum_{\alpha_i} 1 \otimes_E \sigma_i^{(1)} \otimes_E \sigma_i^{(2)} \otimes_E \sigma_i^{(3)} \right)
= 1 \otimes_E \sigma_i \otimes_E 1
\]
since, by definition of \( \eta_2 \), the only summand which is not zero is when \( \sigma_i^{(3)} = 1 \).

Let us now consider a non monomial relation of the form
\[
\rho_i = a^{(k_i)} + \sum_{j > k_i} b_{ij} a^{(j)}.
\]
In this case,
\[
\eta_2 \varphi_2(1 \otimes_E \rho_i \otimes_E 1) = \sum_{\alpha_i^{(k_i)}} \eta_2 \left( 1 \otimes_E \left( a^{(k_i)} \right)^{(1)} \otimes_E \left( a^{(k_i)} \right)^{(2)} \otimes_E \left( a^{(k_i)} \right)^{(3)} \right)
+ \sum_{j > k_i} b_{ij} \sum_{\alpha_i^{(j)}} \eta_2 \left( 1 \otimes_E \left( a^{(j)} \right)^{(1)} \otimes_E \left( a^{(j)} \right)^{(2)} \otimes_E \left( a^{(j)} \right)^{(3)} \right).
\]
In the first sum, the only summand which is not zero is when \( \left( a^{(k_i)} \right)^{(3)} = 1 \) and in this case the result is \( 1 \otimes_E \rho_i \otimes_E 1 \). The second sum is zero by definition of \( \eta_2 \), so we obtain the equality.

Now consider the case \( i \geq 3 \). If \( w = w_0 \ldots w_{i-1} \) is an \((i - 1)\)-ambiguity, then
\[
\eta_i \varphi_i(1 \otimes_E w \otimes_E 1) = s_{i-1} \eta_{i-1} \delta_{i-1} \varphi_{i-1} \delta_{i-1} (1 \otimes_E w \otimes_E 1).
\]
The map \( t_* \) is a contracting homotopy for the \( E \)-reduced Bar complex,
\[
\delta_{i-1} t_{i-1} = Id - t_{i-2} \delta_{i-2}
\]
and the right hand side of the previous equality is
\[
s_{i-1} \eta_{i-1} \varphi_{i-1} \delta_{i-1} (1 \otimes_E w \otimes_E 1) - s_{i-1} \eta_{i-1} t_{i-2} \delta_{i-2} \varphi_{i-1} \delta_{i-1} (1 \otimes_E w \otimes_E 1).
\]
Since \( \varphi \) is a morphism of complexes from \( C_{\text{min}}(A) \) to \( C_{\text{Bar}}(E) \) we know that \( \delta_{i-2} \varphi_{i-1} \delta_{i-1} = \varphi_{i-2} \delta_{i-2} \delta_{i-1} = 0 \) and this implies that the second term is zero. Using the inductive hypothesis in the first term it is sufficient to prove that
\[
s_{i-1} \delta_{i-1} (1 \otimes_E w \otimes_E 1) = 1 \otimes_E w \otimes_E 1.
\]
For \( i \) odd,
\[
s_{i-1}d_{i-1}(1 \otimes_E w \otimes_E 1) = u_{i-1}s_{i-1}(1 \otimes_E w_{i-2} \ldots w_0 \otimes_E 1) - s_{i-1}(1 \otimes_E w_0 \ldots w_{i-2} \otimes_E w_{i-1}) = 1 \otimes_E w \otimes_E 1.
\]

For \( i \) even,
\[
s_{i-1}d_{i-1}(1 \otimes_E w \otimes_E 1) = s_{i-1}\left( \sum_{w(\sigma)\vDash (-2)} w^{(1)} \otimes_E w^{(2)} \otimes_E w^{(3)} \right) = 1 \otimes_E w \otimes_E 1
\]
and the result is proven. \( \square \)

5. Gerstenhaber structure of \( HH^*(A) \)

In this section we fix \( \mathbb{k} = \mathbb{C} \). We describe the Lie structure of \( HH^1(A) \), while in the next sections we will provide the structure of \( HH^i(A) \), with \( i \geq 2 \), as Lie representations of \( HH^i(A) \).

5.1. Cup product. The cup product of the Hochschild cohomology of a toupie algebra is trivial, see [GL]. For the convenience of the reader we sketch an alternative proof.

**Proposition 5.1.** Given a toupie algebra \( A \), \( \overline{\mathcal{f}} \in HH^m(A) \) and \( \overline{\mathcal{g}} \in HH^n(A) \) with \( m,n > 0 \), the cup product \( \overline{\mathcal{f}} \cdot \overline{\mathcal{g}} \) vanishes in \( HH^{m+n}(A) \).

**Proof.** We will prove that \( \overline{\mathcal{f}} \cdot \overline{\mathcal{g}} = 0 \) in \( HH^{m+n}(A) \). Since \( \eta^* \) is an isomorphism at the cohomology level, there exist \( F \in \text{Hom}_{E^*}(\mathbb{k}A_{n-1}, A) \) and \( G \in \text{Hom}_{E^*}(\mathbb{k}A_{m-1}, A) \) such that
\[
\overline{\mathcal{f}} = \overline{\eta^*_n F} = \overline{\eta^*_n F},
\]
\[
\overline{\mathcal{g}} = \overline{\eta^*_m G} = \overline{\eta^*_m G}.
\]
We choose \( f' \) and \( g' \) such that \( f' = \eta^*_n F \) and \( g' = \eta^*_m G \).

Let us suppose that \( \overline{f'} \cdot \overline{g'} \neq 0 \), that is \( \overline{f'} \cdot \overline{g'} \) is not a coboundary and in particular there exists \( a_1 \otimes \cdots \otimes a_{n+m} \in \overline{A}^{(m+n)} \) such that
\[
f'(a_1 \otimes \cdots \otimes a_{n+m}) = \eta^*_n(F)(a_1 \otimes \cdots \otimes a_n)g'(a_{n+1} \otimes \cdots \otimes a_{n+m}) \neq 0.
\]
However,
\[
f'(a_1 \otimes \cdots \otimes a_n) = \eta^*_n(F)(a_1 \otimes \cdots \otimes a_n) = \eta^*_n(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1) \]
and
\[
g'(a_{n+1} \otimes \cdots \otimes a_{n+m}) = \eta^*_m(G)(a_{n+1} \otimes \cdots \otimes a_{n+m}) = G(\eta_m(1 \otimes a_{n+1} \otimes \cdots \otimes a_{n+m} \otimes 1)),
\]
so \( f'(a_1 \otimes \cdots \otimes a_n) \) and \( g'(a_{n+1} \otimes \cdots \otimes a_{n+m}) \) must be linear combinations of paths of positive length that start at 0, so their cup product will be zero, which leads us to a contradiction. In conclusion, \( \overline{\mathcal{f}} \cdot \overline{\mathcal{g}} = \overline{f' \cdot g'} = 0 \). \( \square \)
5.2. Gerstenhaber bracket in higher degrees. Next we will compute the Gerstenhaber bracket of two elements in the higher cohomology spaces of $A$. We will prove that the bracket of two cocycles of degree greater than one is always zero due to the particular shape of the quiver.

**Proposition 5.2.** Let $A = kQ/I$ be a toupie algebra, $\overline{f} \in HH^m(A)$ and $\overline{g} \in HH^m(A)$ with $m, n > 1$. The Gerstenhaber bracket $\overline{[f, g]}_{red}$ vanishes in $HH^{m+n-1}(A)$.

**Proof.** We want to prove that $\overline{[f, g]}_{red} = 0$ in $HH^{m+n-1}(A)$. As before we choose $f'$ and $g'$ such that $f' = \eta_n^*F$ and $g' = \eta_m^*G$ with $F$ and $G$ obtained from the minimal resolution. Let us suppose that there exists $a_1 \otimes \cdots \otimes a_{n+m-1} \in A^\otimes_{E} \mathcal{S}^{m+n-1}$ such that $[f', g']_{red}(a_1 \otimes \cdots \otimes a_{n+m-1}) \neq 0$. Recall that $[f', g']_{red} = \phi'(g'(-1)^{(m-1)(n-1)}g' \circ f')$ and that $f' \circ g'(a_1 \otimes \cdots \otimes a_{n+m-1})$ is

$$\sum_{i=1}^n (-1)^{(i-1)(m-1)} f'(a_1 \otimes \cdots \otimes a_{i-1} \otimes g'(a_i \otimes \cdots \otimes a_{n+m-1}) \otimes a_{i+m} \otimes \cdots \otimes a_{n+m-1}).$$

We will verify that each summand is zero. Suppose that $g'(a_1 \otimes \cdots \otimes a_{n+m-1}) \neq 0$ in $A$. Since $g'(a_1 \otimes \cdots \otimes a_{n+m-1}) = G(\eta_m(1 \otimes a_1 \otimes \cdots \otimes a_{n+m-1} \otimes 1))$ we know that it will be a linear combination of paths from 0 to $\omega$ that is $g'(a_1 \otimes \cdots \otimes a_{n+m-1}) = \sum_{c=0}^r \sum_{b_1, b_2, \ldots, b_r} \lambda_{c, b_1, b_2, \ldots, b_r} \alpha_{c}^{b_1 b_2 \cdots b_r}$. If $\alpha_{c}^{b_1 b_2 \cdots b_r} = 0$ then $g' \circ f' = 0$ and we conclude that $\overline{[f, g]}_{red} = 0$ in $HH^{m+n-1}(A)$. □

5.3. Gerstenhaber bracket in $HH^1(A)$. From now on we will identify the elements of $HH^1(A)$ with their classes when there is no confusion. We will use the notation of $\overline{T}$ that we recall. Given $\alpha \parallel h \in Hom_E(kQ_1, A)$ and a path $\overline{a}$ in $A$, we will denote $\overline{a} \alpha \parallel h$ the sum of all the non zero paths obtained replacing every appearance of $\alpha$ in $\overline{a}$ by $h$. If the path $\overline{a}$ does not contain the arrow $\alpha$ or if when we replace $\alpha$ in $\overline{a}$ by $h$ we get $0 \in A$, we define $\overline{a} \alpha \parallel h = 0$. Observe that for toupie algebras any element of type $\overline{a} \alpha \parallel h$ will have at most one summand.

**Lemma 5.3.** Given $\alpha \parallel h \in Hom_E(kQ_1, A)$, $b \in \overline{T}$ and $f \in Hom_E(\overline{A}, A)$, the following equalities hold:

1. $\eta_1^*(\alpha \parallel h)(b) = \bar{b} \alpha \parallel h$.
2. $\varphi_1^*(f) = \sum_{c \in Q_1} \sum_{\alpha \parallel c} \sum_{b \in \overline{T}} \lambda_{\alpha, c} \bar{b}$, if $f(\alpha) = \sum_{c \in \overline{T}} \lambda_{\alpha, c} b$.

**Proof.** (1) Given $\alpha \parallel h \in Hom_E(kQ_1, A)$, we will denote by $(\alpha \parallel h)$ the element in $Hom_A(A \otimes_E kQ_1 \otimes_E A, A)$ obtained from $\alpha \parallel h$ using the canonical identification of $Hom_E(kQ_1, A)$ with $Hom_A(A \otimes_E kQ_1 \otimes_E A, A)$. For $b \in \overline{T}$,

$$\eta_1^*(\alpha \parallel h)(b) = \alpha \parallel h(1 \otimes_E b \otimes_E 1) = \alpha \parallel h(1 \otimes_E b \otimes_E 1).$$

- If $\alpha$ is an arrow contained in $b$, that is $b = \alpha \parallel h \alpha$, then $\eta_1^*(\alpha \parallel h)(b) = b \parallel h \alpha$.
- If not, then $\eta_1^*(\alpha \parallel h)(b) = 0$.

The first equality is proven.

(2) Given $f \in Hom_E(\overline{A}, A)$ and $\alpha \in Q_1$, we will denote by $\bar{f}$ the element in $Hom_A(A \otimes_E \overline{A} \otimes_E A, A)$ associated to $f$ using the canonical identification of $Hom_E(\overline{A}, A)$ with $Hom_A(A \otimes_E \overline{A} \otimes_E A, A)$. Now,

$$\varphi_1^*(f)(\alpha) = \bar{f}(\varphi_1(1 \otimes_E \alpha \otimes_E 1)) = \bar{f}(1 \otimes_E \alpha \otimes_E 1) = f(\alpha).$$

If $f(\alpha) = \sum_{c \in \overline{T}} \lambda_{\alpha, c} \bar{b}$, then we conclude that $\varphi_1^*(f) = \sum_{\alpha \parallel c} \sum_{b \in \overline{T}} \lambda_{\alpha, c} \bar{b}$. □

The next theorem gives the formula of the Gerstenhaber bracket restricted to $HH^1(A)$. The corresponding result for monomial algebras has been proven in [SI]. Even if the elements of $HH^1(A)$ here differ from those studied by Strametz, the formula for the bracket can be written in a similar way.
Theorem 5.4. Let $A$ be a toupie algebra. The Gerstenhaber bracket in $\text{HH}^1(A)$ can be expressed in terms of the minimal resolution as:
\[ [\alpha \parallel h, \beta \parallel b] = \beta \parallel b^{\alpha h} - \alpha \parallel h^{\beta b}, \]
with $\alpha \parallel h, \beta \parallel b \in \text{Hom}_{E^*}(kQ_1, A)$.

Proof. Given $\alpha \parallel h, \beta \parallel b \in \text{Hom}_{E^*}(kQ_1, A)$, let us compute $[\alpha \parallel h, \beta \parallel b] \in \text{Hom}_{E^*}(kQ_1, A)$ using the comparison morphisms. Given $\gamma \in Q_1$,
\[ [\alpha \parallel h, \beta \parallel b](\gamma) = \varphi_1[\eta^*_1(\alpha \parallel h), \eta^*_1(\beta \parallel b)](\gamma) \]
\[ = [\eta^*_1(\alpha \parallel h), \eta^*_1(\beta \parallel b)]\varphi_1(1 \otimes E \gamma \otimes E 1) \]
\[ = [\eta^*_1(\alpha \parallel h), \eta^*_1(\beta \parallel b)](\gamma). \]

Applying now the definition of the Gerstenhaber bracket, we get:
\[ [\eta^*_1(\alpha \parallel h), \eta^*_1(\beta \parallel b)](\gamma) = \eta^*_1(\alpha \parallel h) \parallel \eta^*_1(\beta \parallel b)(\gamma) - \eta^*_1(\beta \parallel b) \parallel \eta^*_1(\alpha \parallel h)(\gamma) \]
\[ = (\gamma^{\beta \parallel b})_{\alpha \parallel h} - (\gamma^{\alpha \parallel h})_{\beta \parallel b}. \]

There are three cases to consider.

- If $\gamma = \beta$, then $\gamma^{\beta \parallel b} = b$, and if also $b$ contains $\alpha$, then we replace $\alpha$ by $h$ in $b$.
- If $\gamma = \alpha$, then $\gamma^{\alpha \parallel h} = h$, and if also $h$ contains $\beta$, then we replace $\beta$ by $b$ in $h$.
- If $\gamma \neq \alpha$ and $\gamma \neq \beta$, then $[\alpha \parallel h, \beta \parallel b](\gamma) = 0$.

In conclusion, $[\alpha \parallel h, \beta \parallel b] = \beta \parallel b^{\alpha h} - \alpha \parallel h^{\beta b}$.

The computation we have just made in terms of the complex induces the formula of the Gerstenhaber bracket in $\text{HH}^1(A)$.

\[\square\]

Remark 9. If $\alpha \parallel \alpha$ and $\beta \parallel \beta$ belong to $\text{HH}^1(A)$, then $[\alpha \parallel \alpha, \beta \parallel \beta] = 0$.

6. Decomposition of $\text{HH}^1(A)$ as a Lie algebra

In this section we will give a description of $\text{HH}^1(A)$ as a Lie algebra. We will first find necessary and sufficient conditions for $A$ to be abelian and next we will describe in detail the centre of $\text{HH}^1(A)$.

We will use the following notation for the explicit basis of $\text{HH}^1(A)$ computed in Subsection 3.3:

- $y_i = \alpha^0_i \alpha^1_i$ with $\alpha^{(i)}$ a branch containing monomial relations.
- $w_{pq} = \alpha^{(p)} \parallel \alpha^{(q)}$ for $p \neq q$ and $p, q = 1, \ldots, a$.
- $z_{us} = \alpha^{(u)} \parallel \alpha^{(s)}$ for $u = 1, \ldots, a$ and $\alpha^{(s)} \in qB_\omega - Z$.
- $x_j = \alpha^{(0)} \parallel \alpha^{(j)} - \alpha^{(1)} \parallel \alpha^{(j)}$ for $j = 2, \ldots, a$.
- $t_k = \sum_{i \in Q_0^+} \alpha^0_i \alpha^1_k$ for $k = 1, \ldots, r$.

Recall that $C^\\alpha_i = \{ y_i : \alpha^{(i)} \text{ is a branch with monomial relations} \}$.

Using Theorem 5.4, we compute the Gerstenhaber brackets of the elements of the basis of $\text{HH}^1(A)$ and obtain the following table:

where

- $A^p_{q'} = [x_j, w_{p'q'}] = \delta_{j,q'} w_{p'q'} - \delta_{p',q'} w_{p'q'} - \delta_{q',1} w_{p'1} + \delta_{1,p'} w_{1q'}$,
- $B^s_{u'} = [x_j, z_{u'w'}] = -\delta_{j,u} z_{u'w'} + \delta_{u',1} z_{1s'}$,
- $C^s_{u'} = [t_k, z_{u'w'}] = \begin{cases} z_{u'w'} & \text{if } \alpha^{(k)} \in Q^0_p \setminus Q^0_p \\ 0 & \text{otherwise}, \end{cases}$
For the converse, suppose that $\alpha \in A$. If $D = 0$, then the enveloping algebra $\mathfrak{u}(\mathfrak{H}^1(A))$, is isomorphic to the polynomial algebra $k[x_1, \ldots, x_{r+m-1}]$. In case $a = D = 1$, we have that $\mathfrak{u}(\mathfrak{H}^1(A))$ is isomorphic to $k[y_1, \ldots, y_m]$.

Proof. For the first statement, recall from Section 3 that:

$$\text{Hom}_{E^r}(kQ_1, A) = k(Q_1 - Z|Q_1 - Z) + k(Z|\omega B_r).$$

If $a = 0$, then $Z$ is empty and $\text{Hom}_{E^r}(kQ_1, A)$ is generated by the elements of the form $\alpha|\alpha$ for some $\alpha \in Q_1$. Using Remark 9 we conclude that $\mathfrak{H}^1(A)$ is nilpotent.

If $D \leq 1$, we distinguish two different cases:

- If $a = 0$, we argue as before,
- If $a = 1$, then $Z = \{\alpha(1)\}$ and $\text{Hom}_{E^r}(kQ_1, A)$ is generated by the elements of the form $\alpha|\alpha$ with $\alpha \in Q_1 - Z$ together with $\{\alpha(1)|\alpha(1)\}$. Using again Remark 9 we conclude that $\mathfrak{H}^1(A)$ is abelian.

For the converse, suppose that $a > 0$ and $D > 1$. Again, we consider two different cases:

1. If $D = a$, then there are at least two elements in $Z$ that we will call $\alpha(1)$ and $\alpha(2)$ and the Lie bracket table gives:

$$[w_{21}, w_{12}] = -x_2 \neq 0.$$  

2. If $D > a > 0$, then there exists at least one element in $Z$ that we will call $\alpha(1)$ and one element in $Q_0 B_r - Z$ that we will call $\alpha(p)$ which belongs to the connected component $Q_r^k$ for some $k$. Consider the elements $z_{1p}$ and $t_k$. Using again the computations in the Lie bracket table:

$$[z_{1p}, t_k] = -z_{1p} \neq 0.$$  

If $a = 0$, Remark 9 implies that $\dim_k \mathfrak{H}^1(A) = r + m - 1$ and since $\mathfrak{H}^1(A)$ is abelian, the enveloping algebra $\mathfrak{u}(\mathfrak{H}^1(A))$ is isomorphic to $k[x_1, \ldots, x_{r+m-1}]$. If $a = D = 1$, then $r = 0$ and $Da = 1$ so again by Remark 9 $\dim_k \mathfrak{H}^1(A) = m$ and $\mathfrak{u}(\mathfrak{H}^1(A)) \cong k[y_1, \ldots, y_m]$. 

From now on we will suppose that $a > 0$ and $D > 1$.

We will next describe the centre of the Lie algebra $\mathfrak{H}^1(A)$.

\begin{table}[h]
\centering
\begin{tabular}{cccccc}
\hline
$y_i$ & $x_j'$ & $w_{pq}$ & $z_{us}'$ & $t_k$ \\
\hline
$y_i$ & $0$ & $0$ & $0$ & $0$ \\
$x_j$ & $0$ & $A^r_{ij}'$ & $B_{ij}'$ & $0$ \\
$w_{pq}$ & $E^r_{pq}'$ & $D^r_{pq}'$ & $0$ & $0$ \\
$z_{us}'$ & $0$ & $-C_{us}'$ \\
$t_k$ & \\
\hline
\end{tabular}
\caption{Lie bracket table}
\end{table}
Proposition 6.2. Let $A$ be a toupie algebra with $a > 0$. The centre of the Lie algebra $\text{HH}^1(A)$ is the $k$-vector space spanned by $C''_i = \{ y_i : i = 1, \ldots, m \}$.

Proof. Using the Lie bracket table we check that $C''_i \in Z(\text{HH}^1(A))$. Next we will prove that every element of the centre of the algebra can be written as a linear combination of elements of $C''_i$.

Let $\Gamma$ be an element in $Z(\text{HH}^1(A))$. Write $\Gamma$ in terms of the given basis of $\text{HH}^1(A)$ as follows:

$$\Gamma = \sum_j A_j x_j + \sum_{p,q} B_{p,q} w_{pq} + \sum_{u,s} C_{u,s} z_{us} + \sum_k D_k t_k + \sum_i E_i y_i.$$ 

Since the bracket of $\Gamma$ with any element of $C''_i$ is 0, we determine the restrictions that this condition imposes on the coefficients of $\Gamma$. For every $k$,

$$0 = [\Gamma, t_k] = [\sum_{u,s} C_{u,s} z_{us}, t_k] = \sum_{u,s} C_{u,s} [z_{us}, t_k] = \sum_{u,s} C_{u,s} z_{us}.$$ 

The $z_{us}$'s are linearly independent, so $C_{u,s}$ vanishes for every $s$ and for every $u = 1, \ldots, a$.

We also know that for every $j = 2, \ldots, a$:

$$0 = [\Gamma, x_j] = [\sum_{p,q} B_{p,q} w_{pq}, \alpha(j)] - [\Gamma, \alpha(j)] \alpha(j)] = [\sum_{p,q} B_{p,q} w_{pq}, \alpha(j)] - [\sum_{p,q} B_{p,q} w_{pq}, \alpha(j)],$$

using Remark 9 and the fact that the coefficients $C''_i$ impose on the coefficients of $\Gamma$. For every $k$,

$$0 = [\Gamma, t_k] = [\sum_{u,s} C_{u,s} z_{us}, t_k] = \sum_{u,s} C_{u,s} [z_{us}, t_k] = \sum_{u,s} C_{u,s} z_{us}.$$ 

The $z_{us}$'s are linearly independent, so $C_{u,s}$ vanishes for every $s$ and for every $u = 1, \ldots, a$.

We will now see that $A_j = 0$ for $j = 2, \ldots, a$. Choosing $w_{pq}$ with $p \neq q$ and $1 \leq p, q \leq a$, it turns out that

$$0 = [\Gamma, w_{pq}] = \sum_j A_j [x_j, w_{pq}],$$

If $p, q \neq 1$ then,

$$0 = [\Gamma, w_{pq}] = (A_q - A_p) w_{pq}$$

so we conclude that $A_p = A_q$. If $p = 1$, then:

$$0 = [\Gamma, w_{1q}] = \sum_j A_j [\alpha(j), w_{1q}] = \sum_j A_j [\alpha(j), w_{1q}] = A_q w_{1q} + \sum_j A_j w_{1q} = a A_q w_{1q},$$

which implies $A_q = 0$ for $q = 2, \ldots, a$.

Finally, given $u$ and $s$ such that $1 \leq u \leq a$ and $\alpha^{(s)} \in \mathfrak{b}_2 \mathfrak{z}_\omega - Z$ we have:

$$0 = [\Gamma, z_{us}] = \sum_k D_k [t_k, z_{us}] = \sum_k D_k [t_k, \alpha^{(s)}] = D_t z_{us}$$

if $\alpha^{(s)}$ belongs to $Q''_i$. As a consequence, $D_t = 0$ for every $k$ and we conclude that

$$\Gamma = \sum_i E_i y_i.$$
Consider the $\alpha$-Kronecker quiver $Q_\alpha$: it has only two vertices, one source and one sink, with $a$ arrows from the source to the sink. In the next proposition we will prove that if the quiver $Q_A$ associated to the toupie algebra $A$ has $Q_\alpha$ as a subquiver, then the Lie algebra $\text{HH}^1(A)$ contains a subalgebra isomorphic to $\text{HH}^1(kQ_\alpha)$.

**Proposition 6.3.** The Lie subalgebra of $\text{HH}^1(A)$ generated by

$$\{x_j, w_{pq} \text{ with } p \neq q; \ p, q = 1, \ldots, a \text{ and } j = 2, \ldots, a\}$$

is isomorphic to $\text{HH}^1(kQ_\alpha)$.

**Proof.** Let $Q_A$ be the quiver of the toupie algebra $A$. First note that $kQ_\alpha$ is a toupie algebra. Consequently, by Theorem 6.5 the space $\text{HH}^1(kQ_\alpha)$ has a basis:

$$\{X_j, W_{pq} \text{ with } p \neq q; \ p, q = 1, \ldots, a \text{ and } j = 2, \ldots, a\}.$$

There is a morphism of $k$-Lie algebras from $\text{HH}^1(kQ_\alpha)$ to $\text{HH}^1(A)$ that sends $X_j$ to $x_j$ and $W_{pq}$ to $w_{pq}$ for all $j, p, q$. This map is injective and induces an isomorphism between $\text{HH}^1(kQ_\alpha)$ and its image. \hfill $\Box$

The proof of the next proposition is in [S, Corollary 2.2.4]. We will just give the explicit isomorphism since we are going to use it later.

From now on, suppose $k = \mathbb{C}$.

**Proposition 6.4.** The Lie subalgebra of $\text{HH}^1(A)$ generated by the set $\{x_j, w_{pq} \text{ with } p \neq q; \ p, q = 1, \ldots, a \text{ and } j = 2, \ldots, a\}$ is isomorphic to $\text{sl}_a(\mathbb{C})$.

**Proof.** The isomorphism sends $w_{pq}$ to the elementary matrix $E_{pq}$, and $x_j$ to the $a \times a$ diagonal matrix $E_{jj} - E_{11}$. \hfill $\Box$

Now we will give a decomposition of $\text{HH}^1(A)$ as a Lie algebra.

**Theorem 6.5.** Let $Q$ be a toupie quiver, $Q_{\rho}$ the quiver described in Definition 4 and $\{Q_{\rho}^h : h = 1, \ldots, r\}$ the set of the connected components of $Q_{\rho}$. The $k$-vector space

$$L = \{t_h, z_{us} : h = 1, \ldots, r; \ u = 1, \ldots, a \text{ and } s \text{ such that } \alpha(s) \in 0B_\omega - Z\}$$

is a solvable Lie ideal of $\text{HH}^1(A)$. Moreover,

1. $S_1 = \{t_h : h = 1, \ldots, r\}$ is an abelian Lie subalgebra; $L_2 = \{z_{us} : u = 1, \ldots, a \text{ and } s \text{ such that } \alpha(s) \in 0B_\omega - Z\}$ is an abelian ideal, and:

$$L = S_1 \times L_2.$$

2. There is an isomorphism of Lie algebras $\text{HH}^1(A) \cong (C_1') \oplus \text{sl}_a(C) \times (S_1 \times L_2)$.

**Proof.** Using the Lie bracket table we know that $[t_h, t_{h'}] = 0$, $[t_k, z_{us}] = z_{us}$ or $0$, and $[z_{us}, z_{u's'}] = 0$, so we conclude that $L$ is a Lie subalgebra of $\text{HH}^1(A)$. Moreover, since $[x_j, t_k] = 0$ and $[w_{pq}, t_k] = 0$, $L$ is also an ideal.

From the previous computations we deduce that $L$ is solvable since $[L, L] \subset L_2$ and $L_2$ is abelian.

We also deduce that $S_1$ is an abelian subalgebra and $L_2$ is an ideal. Besides, since $L_2$ is abelian, it is solvable. Finally, since the projection $\pi' : L \rightarrow L/L_2$ induces an isomorphism between $S_1$ and $L/L_2$ it turns out that

$$L = S_1 \times L_2.$$
On the other hand, consider the Lie algebra $\overline{\text{HH}}^1(A) := \text{HH}^1(A)/\pi(\text{HH}^1(A))$ and the projection $\pi : \overline{\text{HH}}^1(A) \to \text{HH}^1(A)/L$. Propositions 6.3 and 6.4 provide an isomorphism between $\mathfrak{sl}_a(C)$ and $\text{HH}^1(A)/L$, which implies that:

$$\overline{\text{HH}}^1(A) \cong \mathfrak{sl}_a(C) \times L$$

Using this isomorphism and Lemma 6.2, we obtain the decomposition of $\text{HH}^1(A)$. □

**Corollary 6.6.** Let $A$ be a toupie algebra. The radical of the Lie algebra $\text{HH}^1(A)$ decomposes as follows:

$$\text{rad}(\text{HH}^1(A)) = L \oplus (C_1^\omega).$$

**Proof.** Since $L$ is a solvable ideal and $(C_1^\omega)$ is abelian, it follows that $L \oplus (C_1^\omega)$ is a solvable ideal. Besides, the quotient of $\text{HH}^1(A)$ by $L \oplus (C_1^\omega)$ is isomorphic to $\mathfrak{sl}_a(C)$, which is semisimple. □

The next corollary gives a similar decomposition to the one given in Theorem 6.3. It can be easily proven using the previous corollary and the Levi decomposition.

**Corollary 6.7.** Let $A$ be a toupie algebra. The algebra $\text{HH}^1(A)$ decomposes as follows:

$$\text{HH}^1(A) = (L \oplus (C_1^\omega)) \times \mathfrak{sl}_a(C).$$

**Corollary 6.8.** Let $A$ be a toupie algebra such that $\text{HH}^1(A)$ is not abelian. The Lie algebra $\text{HH}^1(A)$ is semisimple if and only if $a = D$ and $m = 0$.

**Proof.** If $\text{HH}^1(A)$ is semisimple, then $\text{rad}(\text{HH}^1(A)) = 0$ and $\#C_1^\omega = m = 0$. Using Corollary 6.6, we know that $\text{HH}^1(A)$ is isomorphic to $\mathfrak{sl}_a(C)$, via the identification of $\mathfrak{sl}_a(C)$ with $(x_j, u_{pq} : p \neq q; p, q = 1, \ldots, a)$ and $j = 2, \ldots, a)$ as in Corollary 6.7. Since $\text{HH}^1(A)$ is not abelian, Theorem 6.4 implies that there exists at least one arrow from 0 to $\omega$ that we will call $\alpha^{(1)} \in Z$. If $a$ was strictly smaller than $D$, then it would exist a branch $\alpha^{(1)} \in q_{R, a} \in Z$. Consequently, the element $z_{1_i} = \alpha^{(1)}|_{\alpha^{(1)}}$ would be non zero in $\text{HH}^1(A)$ and that leads to a contradiction.

Let us prove the converse. We know that if $a = D$ and $m = 0$, then the basis of $\text{HH}^1(A)$ does not contain elements of the form $y_i$, $z_{i\omega}$ or $t_h$ and it follows that $\text{HH}^1(A) \cong \mathfrak{sl}_a(C)$. □

7. $\text{HH}^2(A)$ as representation of $\text{HH}^1(A)$

7.1. $\text{HH}^2(A)$ as a Lie representation of $\text{HH}^1(A)$. In this section we will describe the action of the Lie algebra $\text{HH}^1(A)$ on $\text{HH}^2(A)$. Given a path $c$, we will call $\{c\}$ the first arrow of $c$. Consider $\alpha|b \in \text{HH}^1(A)$, $\rho|c \in \text{HH}^2(A)$ and a monomial relation $\sigma$ belonging to a minimal set of generators of $I$. Let us compute $[\alpha|b, \rho|c] \in \text{HH}^2(A)$:

$$[\alpha|b, \rho|c](\sigma) = \varphi_2^{\sigma}[\eta_1^{\sigma}(\alpha|b), \eta_2^{\sigma}(\rho|c)](\sigma)$$

$$= \sum_{\sigma}[\eta_1^{\sigma}(\alpha|b), \eta_2^{\sigma}(\rho|c)](1 \otimes \sigma^{(1)} \otimes \sigma^{(2)} \otimes \sigma^{(3)})$$

$$= \sum_{\sigma}(\eta_1^{\sigma}(\alpha|b) \circ \eta_2^{\sigma}(\rho|c)\sigma^{(1)} \otimes \sigma^{(2)} \otimes \sigma^{(3)})$$

$$- \sum_{\sigma}\eta_2^{\sigma}(\rho|c)(\eta_1^{\sigma}(\alpha|b)\sigma^{(1)} \otimes \sigma^{(2)} \otimes \sigma^{(3)})$$

$$- \sum_{\sigma}\eta_2^{\sigma}(\rho|c)(\eta_1^{\sigma}(\alpha|b)\otimes \eta_2^{\sigma}(\rho|c)\sigma^{(3)}),$$

where the sums range over all possible decompositions of $\sigma$ with $\sigma^{(2)}$ an arrow. Let us first compute the first summand:
\[
\sum_{\sigma} (\eta_1^\sigma(\alpha\|b) \circ \eta_2^\sigma(\rho\|c)(\sigma_1 \otimes (\sigma_2 \otimes 1)))_{\sigma} = \sum_{\sigma} \eta_1^\sigma(\alpha\|b)((\rho\|c)(\eta_2(1 \otimes \sigma_1 \otimes \sigma_2 \otimes 1)))_{\sigma}.
\]

Since the monomial relation \(\sigma\) does not contain any other relation, the term \(\eta_2(1 \otimes \sigma_1 \otimes \sigma_2 \otimes 1)\) vanishes except for the case where \(\sigma = 1\), thus

\[
\sum_{\sigma} \eta_1^\sigma(\alpha\|b)((\rho\|c)(\eta_2(1 \otimes \sigma_1 \otimes \sigma_2 \otimes 1)))_{\sigma} = \eta_1^\sigma(\alpha\|b)(\rho\|c)(\eta_2(1 \otimes \sigma_1 \otimes \sigma_2 \otimes 1))
\]

\[
= \delta_{\sigma,\sigma_1 \otimes \sigma_2 \otimes 1} \cdot b(\eta_1(1 \otimes \sigma_1 \otimes 1))
\]

\[
= \delta_{\sigma,\sigma} \sum_c b(\eta_1(1 \otimes \sigma_1 \otimes 1))
\]

As for the second summand,

\[
- \sum_{\sigma} \eta_2^\sigma(\rho\|c)(\eta_1^\sigma(\alpha\|b)(\sigma_1 \otimes \sigma_2 \otimes 1))_{\sigma} = \left\{ \begin{array}{ll} -\sum_{\sigma} (\rho\|c)\eta_2(1 \otimes \sigma_1 \otimes \sigma_2 \otimes 1)_{\sigma} & \text{if } \alpha = \frac{1}{\sigma}; \\ 0 & \text{otherwise}, \end{array} \right.
\]

since if \(\alpha \neq \frac{1}{\sigma}\) then \(\eta_1^\sigma(\alpha\|b)(\sigma_1 \otimes 1) = (\alpha\|b)(\eta_1(1 \otimes \sigma_1 \otimes 1)) = 0\). For the computation in the first line observe that if \(\alpha = \frac{1}{\sigma}\), then \(b = \alpha\).

Taking into account that \(\sigma\) does not contain any other relation, we know that \(\eta_2(1 \otimes \sigma_1 \otimes \sigma_2 \otimes 1)\) vanishes except for the case where \(\sigma = 1\); in that case,

\[
- \sum_{\sigma} (\rho\|c)\eta_2(1 \otimes \sigma_1 \otimes \sigma_2 \otimes 1)_{\sigma} = -(\rho\|c)\eta_2(1 \otimes \sigma_1 \otimes \sigma_2 \otimes 1)_{\sigma} = -\delta_{\sigma,0}c.
\]

Finally, we will prove that the last summand vanishes. For this, notice that

\[
- \sum_{\sigma} \eta_2^\sigma(\rho\|c)(1 \otimes \sigma_1 \otimes \sigma_2 \otimes 1)_{\sigma} = \left\{ \begin{array}{ll} -\eta_2^\sigma(\rho\|c)(1 \otimes \sigma_1 \otimes \sigma_2 \otimes 1)_{\sigma} & \text{if } \alpha = \frac{1}{\sigma}; \\ 0 & \text{otherwise}. \end{array} \right.
\]

and observe that \(1 \otimes \sigma_1 \otimes \sigma_2 \otimes 1\) is zero in the \(E\)-reduced Bar resolution.

Given a non monomial relation, which is necessarily of the form

\(\rho_i = \alpha^{(k_i)} + \sum_{j \neq k_i} b_{ij} \alpha^{(j)}\) for some \(i\)

and using the same notations,

\[
[\alpha\|b, \rho\|c](\rho_i) = \varphi_2^\sigma([\eta_1^\sigma(\alpha\|b), \eta_2^\sigma(\rho\|c)](\rho_i)
\]

\[
= \sum_{\alpha^{(k_i)}} [\eta_1^\sigma(\alpha\|b), \eta_2^\sigma(\rho\|c)](1 \otimes (\alpha^{(k_i)})(1) \otimes (\alpha^{(k_i)})(2))_{\sigma} + \sum_{j \neq k_i} b_{ij} \sum_{\alpha^{(j)}} [\eta_1^\sigma(\alpha\|b), \eta_2^\sigma(\rho\|c)](1 \otimes (\alpha^{(j)})(1) \otimes (\alpha^{(j)})(2))_{\sigma} + \sum_{\alpha^{(k_i)}} [\eta_1^\sigma(\alpha\|b), \eta_2^\sigma(\rho\|c)](1 \otimes (\alpha^{(k_i)})(1) \otimes (\alpha^{(k_i)})(2))_{\sigma}
\]

\[
= \sum_{\alpha^{(k_i)}} [\eta_1^\sigma(\alpha\|b), \eta_2^\sigma(\rho\|c)](1 \otimes (\alpha^{(k_i)})(1) \otimes (\alpha^{(k_i)})(2))_{\sigma} + \sum_{j \neq k_i} b_{ij} \sum_{\alpha^{(j)}} [\eta_1^\sigma(\alpha\|b), \eta_2^\sigma(\rho\|c)](1 \otimes (\alpha^{(j)})(1) \otimes (\alpha^{(j)})(2))_{\sigma} + \sum_{\alpha^{(k_i)}} [\eta_1^\sigma(\alpha\|b), \eta_2^\sigma(\rho\|c)](1 \otimes (\alpha^{(k_i)})(1) \otimes (\alpha^{(k_i)})(2))_{\sigma}
\]
Hence, case, by definition of 

\[ \eta \]

Let us verify that the second summand is zero:

\[ \eta \]

Let us compute the first summand:

\[ \eta \]

The term \( \eta_2(1 \otimes (\alpha(k,i))^{(1)} \otimes (\alpha(k,i))^{(2)} \otimes 1) \) vanishes except when \( (\alpha(k,i))^{(3)} = 1 \), and in this case, by definition of \( \eta_2 \), the result equals \( \rho_i \).

Hence,

\[ \sum_{\alpha(k,i)} \eta^*_i(\alpha \| b) \otimes \eta^*_2(\rho \| c)(1 \otimes (\alpha(k,i))^{(1)} \otimes (\alpha(k,i))^{(2)} \otimes (\alpha(j))^{(3)}) = \eta^*_i(\alpha \| b) (\rho \| c)(\rho_i) \]

\[ = \delta_{\rho,\rho_i} \eta^*_i(\alpha \| b)(\rho) \]

\[ = \delta_{\rho,\rho_i} \eta^*_i(\alpha \| b)(\rho_1 \otimes \rho \otimes \eta_1(1 \otimes c \otimes 1)) \]

\[ = \delta_{\rho,\rho_i} \alpha^{(1)b} \]

otherwise.

Let us verify that the second summand is zero:

\[ \sum_{\alpha(k,i)} \eta^*_i(\alpha \| b) \otimes \eta^*_2(\rho \| c)(1 \otimes (\alpha(j))^{(1)} \otimes (\alpha(j))^{(2)} \otimes (\alpha(j))^{(3)}) = \sum_{\alpha(k,i)} \eta^*_i(\alpha \| b)(\rho \| c)(\eta_2(1 \otimes (\alpha(j))^{(1)} \otimes (\alpha(j))^{(2)} \otimes 1))(\alpha(j))^{(3)}) \]

\[ = 0 \]

since \( \eta_2(1 \otimes (\alpha(j))^{(1)} \otimes (\alpha(j))^{(2)} \otimes 1) = 0 \) for every \( j > k_i \).

To deal with the third summand, we need to distinguish two different cases.

(1) If \( \alpha \) is not the first arrow of \( \alpha(k,i) \) then:

\[ - \sum_{\alpha(k,i)} \eta^*_2(\rho \| c)(\eta^*_i(\alpha \| b)(\alpha(k,i))^{(1)} \otimes (\alpha(k,i))^{(2)})(\alpha(k,i))^{(3)} = 0. \]
where $\mathcal{B}$ be a subset of the basis of $HH_1^2(A)$

From the fact that $1 \in HH_1^2(A)$ is zero in the $E$-reduced
sum over a direct sum as follows:

$E = \{ \rho \in \alpha^{(k)} : \rho \text{ is a relation from } 0 \text{ to } \omega \text{ and } \alpha^{(k)} \in \mathcal{B}_\omega - \mathcal{Z} \}$

be a subset of the basis of $HH_1^2(A)$.

The space $HH_1^2(A)$ decomposes, as representation of $s_{\mathcal{A}}(C)$, in a direct sum as follows:

$$HH_1^2(A) = \bigoplus_{\dim C \leq E} V_0 \oplus \bigoplus_{\rho, a_1(\rho) = 0, \omega(\rho) = \omega} V_0 \cong \mathbb{C}$$
Proof. Recall that $D_2 = 0$ and so $\text{Ker}D_2 = \text{Hom}_{k^\ast}(k\mathfrak{R}, A)$. Observe that $HH^2(A)$ is generated by the classes of the elements of the set $\{\rho[\alpha^{(i)}] : \rho$ is a relation from $0$ to $\omega$ and $\alpha^{(i)} \in B_\omega - Z\}$.

Using Proposition 6.3 and Proposition 6.4, we identify the generators of $sl_n(C)$ with the set $\{x_j, w_{pq}$ with $p \neq q; p, q = 1, \ldots, a$ and $j = 2, \ldots, a\}$ in $HH^1(A)$. Now identify the set $\mathfrak{h} = \{b_1\alpha^{(1)}[\alpha^{(1)}] + b_2\alpha^{(2)}[\alpha^{(2)}] + \cdots + b_n\alpha^{(n)}[\alpha^{(n)}] : b_1 + b_2 + \cdots + b_n = 0\} = \{x_j : j = 2, \ldots, a\}$ with the diagonal matrices with trace zero in $sl_n(C)$. Using the invariance of the Jordan decomposition –see Theorem 9.20 in [FH]– we know that $\mathfrak{h}$ acts diagonally on $HH^2(A)$. This means that $HH^2(A) = \oplus V_{\lambda}$ where $\lambda$ runs over $\mathfrak{h}^\ast$ and $V_{\lambda} = \{v \in V : h.v = \lambda(h)v$ for all $h \in \mathfrak{h}\}$ is the weight space of weight $\lambda$. For every $i$, $1 \leq i \leq a$, let $L_i \in \mathfrak{h}^\ast$ be defined by $L_i(b_1\alpha^{(1)}[\alpha^{(1)}] + \cdots + b_n\alpha^{(n)}[\alpha^{(n)}]) = b_i$.

Notice that $\mathfrak{h}^\ast = C\{L_1, \ldots, L_a\}/(L_1 + \cdots + L_a)$.

Given $b_1\alpha^{(1)}[\alpha^{(1)}] + \cdots + b_n\alpha^{(n)}[\alpha^{(n)}] \in \mathfrak{h}$, $\rho$ a relation from $0$ to $\omega$ and $\alpha^{(i)} \in Z$, and $i$ such that $1 \leq i \leq a$, the equality (10) implies that

$$[b_1\alpha^{(1)}[\alpha^{(1)}] + \cdots + b_n\alpha^{(n)}[\alpha^{(n)}], \rho[\alpha^{(i)}]] = b_i\rho[\alpha^{(i)}],$$

therefore $\rho[\alpha^{(i)}]$ belongs to the weight space associated to the weight $L_i$. Let us see that $L_1$ is a maximal weight of $HH^2(A)$. For that, again by (10), it is enough to observe that given $w_{ij} = \alpha^{(j)}[\alpha^{(i)}]$ with $j < i$, the bracket $[w_{ij}, \rho[\alpha^{(i)}]]$ is zero. Using existence and uniqueness theorems –see [Hu] Chapter VI Theorems A and B– and the fact that $L_1$ is the unique maximal weight in the standard representation, we conclude that the irreducible representation generated by $\rho[\alpha^{(1)}]$, that turns out to be $\{\rho[\alpha^{(i)}] : \rho$ is a relation from $0$ to $\omega$ and $\alpha^{(i)} \in Z\}$, is isomorphic to the standard representation for each $\rho$ from $0$ to $\omega$.

On the other hand, if $\alpha^{(i)} \in B_\omega - Z$, then

$$[b_1\alpha^{(1)}[\alpha^{(1)}] + \cdots + b_n\alpha^{(n)}[\alpha^{(n)}], \rho[\alpha^{(i)}]] = 0$$

and this implies that if $\rho[\alpha^{(i)}]$ is not zero in $HH^2(A)$, then it belongs to the weight space associated to the weight $0$. Besides, in this case, the subrepresentation generated by $\rho[\alpha^{(i)}]$, which is $C.\rho[\alpha^{(i)}]$, is isomorphic to the trivial representation and we obtain the decomposition we were looking for.

Remark 11. Given a relation $\rho$ from $0$ to $\omega$, the $k$-vector space $V_{\rho} = \langle \rho[\alpha^{(k)}] : \alpha^{(k)} \in B_\omega\rangle$ is a Lie subrepresentation of $HH^1(A)$ in $HH^2(A)$.

Theorem 7.2. Let $A$ be a toupie algebra such that $a$ is positive. The decomposition of $HH^2(A)$ as Lie representation of $HH^1(A)$ is

$$HH^2(A) = \bigoplus_{\rho: \rho(0) = 0, \rho(\omega) = \omega} V_{\rho},$$

where $V_{\rho} = \langle \rho[\alpha^{(k)}] : \alpha^{(k)} \in B_\omega\rangle$. Besides, for every relation $\rho$ from $0$ to $\omega$, the $HH^1(A)$-module $V_{\rho}$ is indecomposable. Moreover, if $D = a$, then $V_{\rho}$ is irreducible for every $\rho$. The converse holds if there exists a monomial relation from $0$ to $\omega$.

Proof. It is clear that $HH^2(A) = \sum_{\rho: \rho(0) = 0, \rho(\omega) = \omega} V_{\rho}$ and that $V_{\rho} \cap \Sigma_{\rho' \neq \rho} V_{\rho'} = \{0\}$ for every $\rho$ from $0$ to $\omega$. Let us now see that $V_{\rho}$ is indecomposable for every relation $\rho$ from $0$ to $\omega$. We know that, since $a$ is positive, there exists at least one arrow from $0$ to $\omega$ which will be called $\alpha^{(1)}$.

We assert that $\rho[\alpha^{(1)}]$ is not zero in $HH^2(A)$ since $\rho[\alpha^{(1)}]$ does not belong to $\text{Im}(D_1)$. In the particular case where $D = 1$, we have $\text{dim}V_{\rho} = 1$ and $V_{\rho}$ is indecomposable. In case $D > 1$, given $\alpha^{(i)} \in B_\omega$ with $i \neq 1$:

$$[\alpha^{(1)}[\alpha^{(i)}], \rho[\alpha^{(1)}]] = \rho[\alpha^{(i)}].$$
This equality implies that the orbit of the action of $\HH_1^1(A)$ on $\rho|\alpha(1)$ is $V_\rho$ and then $V_\rho$ is indecomposable.

We will next prove that if $D = a$, then $V_\rho$ is irreducible for every relation $\rho$ from 0 to $\omega$. Since $D = a$, using Theorem 4.3, we know that $\HH_1^1(A) = A^\ell_\omega \otimes s_\omega(\C)$. Let us call $V_\rho$ the representation of the Lie subalgebra $s_\omega(\C)$ with underlying vector space $V_\rho$. Using Theorem 7.1 there is an isomorphism $\tilde{V}_\rho \cong V$ where $V$ is the standard representation of $s_\alpha(C)$ and as we already know, it is irreducible. It remains to prove that $V_\rho$ is also irreducible. Any non trivial subrepresentation of $\HH_1^1(A)$ contained in $V_\rho$ would also be a non zero subrepresentation of $s_\omega(\C)$ contained in $\tilde{V}_\rho$ and that is absurd. Finally, we will prove that if there exists a monomial relation $\rho$ from 0 to $\omega$ and $V_\rho$ is irreducible, then $D = a$. Suppose that $D \neq a$. In this case there exists $\alpha(1) \in aB_\omega - Z$ such that $\rho|\alpha(1)$ is not zero in $\HH_1^1(A)$. Observe that $H := \langle \rho|\alpha(1) \rangle$ is a representation of $\HH_1^1(A)$ contained in $V_\rho$, since given $\alpha|c \in \HH_1^1(A)$, Eq. (10) implies that,

$$[\alpha|c, \rho|\alpha(1)] = \delta_{\alpha, i(\alpha(1))} \rho|\alpha(1) - \delta_{\alpha, i\rho(1)} \rho|\alpha(1),$$

but $a$ is positive and $V_\rho$ is irreducible so this leads us to a contradiction and we conclude that $D = a$. \hfill \Box

7.2. $\HH_i^1(A)$, $i \geq 3$ as representation of $\HH_1^1(A)$. Fix $i > 2$. Given $\alpha \in Q_1$ and $b$ a non zero path in $A$ such that $0 = s(\alpha) = s(b)$ and $t(\alpha) = t(b)$, consider $\alpha|b \in \HH_1^1(A)$. Also, given an $(i - 1)$-ambiguity $a$ from 0 to $\omega$ and a path $c$ in $aB_\omega$ consider that, as usual, we will identify with its class $u|c \in \HH_i^1(A)$. Next we will compute the Gerstenhaber bracket of these two elements. Given an $(i - 1)$-ambiguity $w$,

$$[\alpha|b, u|c](w) = \varphi_i^1[\eta_i^1(\alpha|b), \eta_i^1(u|c)](w)$$

$$= \eta_i^1(\alpha|b), \eta_i^1(u|c)](1 \otimes w \otimes 1)$$

$$= \eta_i^1(\alpha|b) \circ \eta_i^1(u|c)(\varphi_i(1 \otimes w \otimes 1)) - \eta_i^1(u|c) \circ \eta_i^1(\alpha|b)(\varphi_i(1 \otimes w \otimes 1)).$$

The first term is equal to

$$\eta_i^1(\alpha|b) \circ \eta_i^1(u|c)(\varphi_i(1 \otimes w \otimes 1)) = \eta_i^1(\alpha|b)((u|c)\eta_i(1 \otimes w \otimes 1))$$

$$= \eta_i^1(\alpha|b)(u|c)(1 \otimes w \otimes 1)$$

$$= \delta_{u,w}\eta_i^1(\alpha|b)(c)$$

$$= \delta_{u,w}(\alpha|b) \sum_c c^{(1)} \otimes c^{(2)} \otimes c^{(3)}$$

$$= \begin{cases} \\
\delta_{w,u}c^{(1)}, & \text{if } \alpha = \frac{1}{w}; \\
0, & \text{otherwise.} 
\end{cases}$$

Observe that in the third equality we have used Proposition 4.5. Let us compute the second term. By Proposition 4.3, we know that $\varphi_i(1 \otimes w \otimes 1)$ has the following form,

$$\sum_{a^{(1)} \ldots a^{(i+1)} = w} 1 \otimes a^{(1)} \otimes \ldots \otimes a^{(i)} \otimes a^{(i+1)}.$$ 

Thus,

$$\eta_i^1(u|c) \circ \eta_i^1(\alpha|b)(\varphi_i(1 \otimes w \otimes 1)) = \sum_{a^{(1)} \ldots a^{(i+1)} = w} \eta_i^1(u|c) \circ \eta_i^1(\alpha|b)(1 \otimes a^{(1)} \otimes \ldots \otimes a^{(i)} \otimes a^{(i+1)})$$

$$= \sum_{a^{(1)} \ldots a^{(i+1)} = w} \eta_i^1(u|c) \sum_j (a^{(1)} \otimes \ldots \otimes a^{(j)} \otimes a^{(i)} \otimes a^{(i+1)}).$$

Since $\alpha$ is the first arrow of some branch, we know that $\eta_i^1(\alpha|b)(a^{(j)}) = 0$ for $j \neq 1$, so the last
expression equals
\[
\sum_{a(1)\ldots a(i+1)\vdash w} \eta^i(u\|c)((\eta^i(\alpha\|b)(a(1)) \otimes \cdots \otimes a(i))a(i+1))
\]
which, by definition of \(\eta^i\) and using again that \(\alpha\) is the first arrow of some branch, is
\[
\delta_{\alpha,\varepsilon^1} \sum_{a(1)\ldots a(i+1)\vdash w} \eta^i(u\|c)((a(1)) \otimes \cdots \otimes a(i+1)).
\]
By Proposition 14 we have that,
\[
\delta_{\alpha,\varepsilon^1} \sum_{a(1)\ldots a(i+1)\vdash w} \eta^i(u\|c)((a(1)) \otimes \cdots \otimes a(i+1)) = \delta_{\alpha,\varepsilon^1}(u\|c)\eta(\sum_{a(1)\ldots a(i+1)\vdash w} (1 \otimes a(1) \otimes \cdots \otimes a(i+1))) = \delta_{u,w}\delta_{\alpha,\varepsilon^1}c.
\]
Summarising, given \(\alpha\|b\in HH^i(A)\) and \(u\|c\in HH^i(A)\):
\[
[\alpha\|b, u\|c] = \delta_{[c,\alpha]}u\|c \delta_{a,u,\alpha} - \delta_{\alpha,\varepsilon^1}c.
\]
Theorems 7.1 and 7.2 can be adapted to the general case, that is, \(i \geq 2\). We state them now and we omit the proofs since they are analogous to the case \(i = 2\).

**Theorem 7.3.** Consider a toupie algebra \(A\) with \(a > 0\). The decomposition of \(HH^i(A)\) as a representation of \(sl_a(\mathbb{C})\) is:
\[
HH^i(A) \cong \bigoplus_{i=1}^{\omega} V_0 \oplus \bigoplus_{u\in\alpha(k)\vdash \omega} V,
\]
where \(V_0 \cong \mathbb{C}\) is the trivial representation and \(V\) is the standard representation of \(sl_a(\mathbb{C})\).

**Remark 12.** Given an \((i-1)\)-ambiguity \(u\) from 0 to \(\omega\), the \(k\)-vector subspace of \(HH^i(A)\) \(V_u = \{u\|\alpha(k)\in \mathcal{B}_\omega\}\) is a Lie subrepresentation of \(HH^i(A)\).

**Theorem 7.4.** Let \(A\) be a toupie algebra with \(a > 0\). The decomposition of \(HH^i(A)\) as a Lie representation of \(HH^i(A)\) is the following:
\[
HH^i(A) = \bigoplus_{u\in\alpha(k)\vdash \omega} V_u
\]
where \(V_u = \{u\|\alpha(k)\mid \alpha(k)\in \mathcal{B}_\omega\}\). Besides, note that for every \((i-1)\)-ambiguity \(u\) from 0 to \(\omega\), the module \(V_u\) is indecomposable. The representation \(V_u\) is irreducible if and only if \(D = a\).

We end this article with an example, for which we compute the whole structure.
**Example 7.4.1.** Consider the toupie algebra $A = kQ/I$ where $Q$ is the quiver bellow with $|Q_0| = 13$ and $|Q_1| = 15$:

Let $\rho_i' = \alpha_i^6 \alpha_i^7 - \alpha_i^6 \alpha_i^6$ and $\sigma_i = \alpha_i^4 \alpha_i^4 \alpha_i^2 \alpha_i^2 \alpha_i^2 \alpha_i^4$ with $i = 0, \ldots, 4$ and $I = (\rho_i', \sigma_i : i = 0, \ldots, 4)$. Note that there are two arrows from 0 to $\omega$, there is one branch not involved in any relation, one branch with monomial relations and two branches involved in a non monomial relation. This means that $D = 4$, $a = 2$, $l = 1$, $m = 1$ and $n = 2$.

There are four 2-ambiguities, $\alpha_i^4 \alpha_i^4 \alpha_i^4 \alpha_i^4$, $\alpha_i^4 \alpha_i^4 \alpha_i^4 \alpha_i^4$, $\alpha_i^4 \alpha_i^4 \alpha_i^4 \alpha_i^4$ and $\alpha_i^4 \alpha_i^4 \alpha_i^4 \alpha_i^4$ and one 3-ambiguity $\alpha^{(4)}$, for $j > 4$ the set of $j$-ambiguities is empty.

The Hochschild cohomology is the following:

- $HH^0(A) = \{ \sum_{i \in Q_0} e_i \| e_i \}$
- $HH^1(A)$ we obtain that $HH^1(A) = \langle y_4, w_{12}, w_{21}, x_2, z_{13}, z_{23}, z_{16}, z_{26}, t_1, t_2 \rangle$, where we have used the previously defined notation.
- $HH^2(A) = \langle \rho_1' | \alpha^{(1)} \rangle, \rho_1' | \alpha^{(2)} \rangle, \rho_1' | \alpha^{(3)} \rangle$
- $HH^3(A) = 0$
- $HH^4(A) = \langle \alpha^{(4)} | \alpha^{(1)} \rangle, \alpha^{(4)} | \alpha^{(2)} \rangle, \alpha^{(4)} | \alpha^{(3)} \rangle, \alpha^{(4)} | \alpha^{(4)} \rangle$
- $HH^5(A) = 0$ for $i > 4$.

The decomposition of the Lie algebra $HH^1(A)$ is the following:

$$HH^1(A) \cong y_4 \oplus sl_2(C) \times \langle (t_1, t_2) \rangle \times \langle (z_{13}, z_{23}, z_{16}, z_{26}) \rangle$$

The non-null Gerstenhaber brackets of $HH^1(A)$ with $HH^2(A)$ are the following:

| $w_{12}$, $\rho_i'$ | $| \alpha^{(1)}$ | $\rho_i' | \alpha^{(2)}$ | $\rho_i' | \alpha^{(3)}$ | $\rho_i' | \alpha^{(4)}$ | $\rho_i' | \alpha^{(5)}$ | $\rho_i' | \alpha^{(6)}$ |
|---------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $x_2$, $\rho_i'$ | $| \alpha^{(1)}$ | $\rho_i' | \alpha^{(2)}$ | $\rho_i' | \alpha^{(3)}$ | $\rho_i' | \alpha^{(4)}$ | $\rho_i' | \alpha^{(5)}$ | $\rho_i' | \alpha^{(6)}$ |
| $z_{13}$, $\rho_i'$ | $| \alpha^{(1)}$ | $\rho_i' | \alpha^{(2)}$ | $\rho_i' | \alpha^{(3)}$ | $\rho_i' | \alpha^{(4)}$ | $\rho_i' | \alpha^{(5)}$ | $\rho_i' | \alpha^{(6)}$ |
| $t_2$, $\rho_i'$ | $| \alpha^{(1)}$ | $\rho_i' | \alpha^{(2)}$ | $\rho_i' | \alpha^{(3)}$ | $\rho_i' | \alpha^{(4)}$ | $\rho_i' | \alpha^{(5)}$ | $\rho_i' | \alpha^{(6)}$ |
The non-null Gerstenhaber brackets of $HH^1(A)$ with $HH^2(A)$ are the following:

\[
\begin{align*}
[y_4, \alpha^{(4)}|\alpha^{(1)}] &= -\alpha^{(4)}|\alpha^{(1)} \\
[w_{12}, \alpha^{(4)}|\alpha^{(1)}] &= -\alpha^{(4)}|\alpha^{(2)} \\
[x_2, \alpha^{(4)}|\alpha^{(1)}] &= -\alpha^{(4)}|\alpha^{(3)} \\
[z_{13}, \alpha^{(4)}|\alpha^{(1)}] &= \alpha^{(4)}|\alpha^{(3)} \\
[z_{16}, \alpha^{(4)}|\alpha^{(1)}] &= \alpha^{(4)}|\alpha^{(6)} \\
[y_4, \alpha^{(4)}|\alpha^{(2)}] &= -\alpha^{(4)}|\alpha^{(2)} \\
[w_{21}, \alpha^{(4)}|\alpha^{(2)}] &= \alpha^{(4)}|\alpha^{(1)} \\
[x_2, \alpha^{(4)}|\alpha^{(2)}] &= \alpha^{(4)}|\alpha^{(3)} \\
[z_{23}, \alpha^{(4)}|\alpha^{(2)}] &= \alpha^{(4)}|\alpha^{(3)} \\
[z_{26}, \alpha^{(4)}|\alpha^{(2)}] &= \alpha^{(4)}|\alpha^{(6)} \\
[y_4, \alpha^{(4)}|\alpha^{(3)}] &= -\alpha^{(4)}|\alpha^{(3)} \\
[t_1, \alpha^{(4)}|\alpha^{(3)}] &= \alpha^{(4)}|\alpha^{(3)} \\
[y_4, \alpha^{(4)}|\alpha^{(6)}] &= -\alpha^{(4)}|\alpha^{(6)} \\
[t_2, \alpha^{(4)}|\alpha^{(6)}] &= \alpha^{(4)}|\alpha^{(6)} \\
\end{align*}
\]

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