INCOMPRESSIBLE MAGNETOHYDRODYNAMIC LIMIT OF THE VLASOV-MAXWELL-BOLTZMANN EQUATIONS

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Abstract. The hydrodynamic limit of the Vlasov-Maxwell-Boltzmann equations is considered for weak solutions. Using relative entropy estimate about an absolute Maxwellian, an incompressible Electron-Magnetohydrodynamics-Fourier limit for solutions of the Vlasov-Maxwell-Boltzmann equations over any periodic spatial domain in $\mathbb{R}^3$ is studied. It is shown that any properly scaled sequence of renormalized solutions of the Vlasov-Maxwell-Boltzmann equations has fluctuations that (in the weak $L^1$ topology) converge to an infinitesimal Maxwellian with fluid variables that satisfy the incompressibility and Boussinesq relations. It is also shown that the limits of the velocity, the electric field, and the magnetic field are governed by a weak solution of an incompressible electron-magnetohydrodynamics system for all time.

1. Introduction

The hydrodynamic models such as the Euler or Navier-Stokes equations were first established by applying Newton’s second law of motion to infinitesimal volume elements of the fluid under consideration; while the kinetic equations are the mathematical models used to describe the dilute particle gases at an intermediate scale between microscopic and macroscopic level with applications in a variety of sciences such as plasma, astrophysics, aerospace engineering, nuclear engineering, particle-fluid interactions, semiconductor technology, social sciences, and biology. If the particles interact only through a repulsive conservative interparticle force with finite range, then at low enough densities this range will be much smaller than the interparticle spacing. In that regime, the evolution of the density of particles $F = F(x, \xi, t)$ is governed by the classical Vlasov-Maxwell-Boltzmann equations (VMB) [12, 19, 21]:

\[
\begin{align*}
\frac{\partial F}{\partial t} + \xi \cdot \nabla_x F + e (E + \xi \times B) \cdot \nabla_\xi F &= Q(F,F), \quad x \in \mathbb{R}^3, \quad \xi \in \mathbb{R}^3, \quad t \geq 0, \\
\frac{1}{c^2} \frac{\partial E}{\partial t} - \nabla \times B &= -\mu_0 j, \quad \text{div} B = 0, \quad \text{on} \quad \mathbb{R}^3 \times (0, \infty), \\
\frac{\partial B}{\partial t} + \nabla \times E &= 0, \quad \text{div} E = \frac{\rho}{\eta_0}, \quad \text{on} \quad \mathbb{R}^3 \times (0, \infty), \\
\rho &= e \int_{\mathbb{R}^3} F d\xi, \quad j = e \int_{\mathbb{R}^3} F \xi d\xi, \quad \text{on} \quad \mathbb{R}^3 \times (0, \infty),
\end{align*}
\]

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where the nonnegative function \( F(t,x,\xi) \) is the density of particles with velocity \( \xi \) at time \( t \) and position \( x \) under the effect of the Lorentz force
\[
E + \xi \times B,
\]
\( E \) is the electric field, and \( B \) is the magnetic field. The function \( j \) is called the current density, while the function \( \rho \) is the charge density. The constant \( e \) is the charge of the electron. The constant \( c \) is the speed of light. The coefficients \( \mu_0 \) and \( \eta_0 \) are the magnetic permeability and the electric permittivity of the plasma in the vacuum (see \[5, 18\]), satisfying \( \mu_0 \eta_0 c^2 = 1 \).

The collision operator \( Q(F,F) \) is defined as
\[
Q(F,F) = \int_{\mathbb{R}^3} d\xi^* \int_{S^2} d\omega b(\xi - \xi^*,\omega)(F'F^* - FF^*),
\]
where the nonnegative function \( b(\xi,\omega) \) given for \( \xi \in \mathbb{R}^3 \) and \( \omega \in S^2 \) (the unit sphere in \( \mathbb{R}^3 \)) is called the collision kernel, and
\[
F_s = F(t,x,\xi_s), \quad F' = F(t,x,\xi'), \quad F' = F(t,x,\xi'_s),
\]
with
\[
\xi' = \xi - (\xi - \xi_s)\omega,
\]
\[
\xi'_s = \xi_s + (\xi - \xi_s)\omega,
\]
yielding one convenient parametrization of the set of solutions to the law of elastic collisions:
\[
\begin{cases}
\xi' + \xi'_s = \xi + \xi_s, \\
|\xi'|^2 + |\xi'_s|^2 = |\xi|^2 + |\xi_s|^2.
\end{cases}
\]

The interpretation of \( \xi, \xi_s, \xi', \xi'_s \) is the following: \( \xi, \xi_s \) are the velocities of two colliding molecules immediately before collision while \( \xi', \xi'_s \) are the velocities immediately after the collision. We will consider the initial value problem of system (1.1) with the initial condition:
\[
(F,E,B)|_{t=0} = (F^0(x,\xi),E^0(x),B^0(x)) \quad \text{for} \quad x \in \mathbb{R}^3, \quad \xi \in \mathbb{R}^3.
\]

On the macroscopic level, the incompressible Electron-Magnetohydrodynamics-Fourier equations describe the evolution of the velocity field \( u = u(t,x) \) of an idealized fluid over a given spatial domain in \( \mathbb{R}^3 \) under the magnetic field \( B = B(t,x) \) and the electronic field \( E = E(t,x) \), and take the form (cf. \[4\])
\[
\begin{align*}
\partial_t u + u \cdot \nabla u - \mu \Delta u + \nabla p - \alpha eE &= (\nabla \times B) \times B, \\
\partial_t B + \nabla \times E &= 0, \\
\partial_t \theta + u \cdot \nabla \theta &= \kappa \Delta \theta, \\
\text{div}u &= 0, \quad \text{div}B = 0,
\end{align*}
\]
with
\[
\alpha = \frac{1}{3(2\pi)^3} \int_{\mathbb{R}^3} |\xi|^2 \exp\left(-\frac{|\xi|^2}{2}\right) d\xi,
\]
where \( p, \theta, E, h \) denote the pressure, temperature, electric field, and density respectively. The initial value problem will also be considered for system (1.4) with the initial data:
\[
(u, B, \theta)|_{t=0} = (u_0(x), B_0(x), \theta_0(x)), \quad x \in \mathbb{R}^3.
\]
where \( u_0, B_0 \in \{ v \in L^2(\mathbb{R}^3) : \text{div} v = 0 \text{ in } D' \} \) and \( \theta_0 \in L^2(\mathbb{R}^3) \).

We call \((u, p, B, E, \theta)\) a weak solution to \((1.4)-(1.5)\) if \((u, p, B, E)\) is a Leray’s solution of the incompressible electron-magnetohydrodynamic equation \((1.4a)-(1.4b)\) under the constraints \((1.4d)\) with initial data in \((1.5)\), while \(\theta\) is a weak solution in the sense of distributions to \((1.4c)\) with the initial data in \((1.5)\).

The motivation of this paper is to find a scaling and verify mathematically the transition from the microscopic model \((1.1)\) to the macroscopic model \((1.4)\) as some parameter vanishes. One of the main objectives is to connect the DiPerna-Lions theory of global renormalized solutions of the Boltzmann equation with the Leray theory of global weak solutions of the incompressible fluid equations in a periodic spatial domain \(T = [0, 1]^3 \subset \mathbb{R}^3\). More precisely, we consider the hydrodynamic limit of the Vlasov-Maxwell-Boltzmann equations for weak solutions in this paper. Using relative entropy estimate about an absolute Maxwellian, an incompressible Electron-Magnetohydrodynamics-Fourier limit for solutions of the Vlasov-Maxwell-Boltzmann equations over periodic spatial domains in \(\mathbb{R}^3\) is studied. It is shown that any properly scaled sequence of renormalized solutions of the Vlasov-Maxwell-Boltzmann equations has fluctuations that (in the weak \(L^1\) topology) converge to an infinitesimal Maxwellian with fluid variables that satisfy the incompressibility and Boussinesq relations. It is also shown that the limits of the velocity, the electric field, and the magnetic field are governed by a weak solution of an incompressible electron-magnetohydrodynamics system for all time.

We note that if the local conservation laws of momentum and energy are assumed, the similar result for the Navier-Stokes limit from the Boltzmann equation was verified in \([1, 2]\). Later, without making any nonlinear weak compactness hypothesis, Golse and Saint-Raymond in their breakthrough paper \([16]\) established the incompressible Navier-Stokes limit of Boltzmann equations with bounded kernels, where they used the entropy dissipation rate to decompose the collision operator in a new way and used a new \(L^1\) averaging theory to prove the compactness assumption. Recently, Levermore and Masmoudi \([23]\) extended those results to a much wider class of collision kernels. As the collision is neglected, the Vlasov-Maxwell-Boltzmann equations become the Vlasov-Maxwell equations. For Vlasov-Maxwell equations, the mathematical verification from the weak solution of the Vlasov-Maxwell equations to the incompressible models has been conducted in \([4, 6, 15, 27]\). When the solution is smooth, authors in \([3, 22]\) considered the Hilbert expansion of solutions to the Vlasov-Maxwell-Boltzmann equations. For other related results on the Boltzmann equation, see \([7, 13, 17, 23, 28]\) and the references therein. Also for the large-time behavior, stability, and regularity of solutions to the Vlasov-Maxwell-Boltzmann equations, see \([9, 10, 20, 21]\) and the references therein.

This paper is organized as follows. In Section 2, we state the formal scalings, the relative entropy, the technical assumptions, and the main result. Section 3 is devoted to a list of \textit{a priori} estimates on the fluctuations of the density from the relative entropy. In Section 4, we consider the limit of the Maxwell equations. Section 5 will concentrate on the vanishing of conservation defects. And finally, in Section 6 we will give the detailed proof of our main Theorem \([21]\).
2. Dimensionless Analysis, Preliminary, and Main Results

In this section, we first introduce the scaling of (1.1), then deduce the relative entropy, and finally state the main result.

To begin with, we now focus on the nondimensional form of the Vlasov-Maxwell-Boltzmann equations. This form is motivated by the fact that the incompressible Electron-Magnetohydrodynamics-Fourier system (1.4) can be formally derived from the Vlasov-Maxwell-Boltzmann equations through a scaling, when the density $F$ is close to a spatially homogeneous Maxwellian $M = M(\xi)$ that has the same total mass, momentum, and energy as the initial data. To this end, we introduce

\[ t = t_* \hat{t}, \quad x = x_* \hat{x}, \quad \xi = \xi_* \hat{\xi}, \]

\[ F = \frac{1}{\mu_0 \xi_*^3 x_*^2} \hat{F}, \quad E = \frac{\xi_*}{t_*} \hat{E}, \quad B = \frac{1}{t_*} \hat{B}, \]

and

\[ b = \frac{x_*}{\eta_0 \xi_*} \hat{b}, \]

where the constants $t_*, x_*, \xi_*$ are the characteristic time, characteristic distance, and characteristic speed; see [5] for more physical interpretations of these constants. The nonrelativistic effect requires

\[ \xi_* = \frac{x_*}{t_*}, \quad \text{and} \quad \varepsilon = \left( \frac{\xi_*}{c} \right)^2 \ll 1. \]

Substituting those new variables back to (1.1), and dropping hats, we obtain

\[
\begin{align*}
    \frac{\partial F}{\partial t} + \xi \cdot \nabla_x F + \epsilon (E + \xi \times B) \cdot \nabla_\xi F &= \frac{1}{\varepsilon} Q(F, F), \\
    \varepsilon \frac{\partial E}{\partial t} - \nabla \times B &= -j, \\
    \frac{\partial B}{\partial t} + \nabla \times E &= 0, \\
    \text{div} B &= 0, \quad \text{div} E = \frac{\rho}{\varepsilon},
\end{align*}
\]

where the coefficient $\varepsilon$ is usually refereed as the dimensionless mean free path or Knudsen number.

Since the incompressible flow is the large-scale low-frequency fluid-like behavior of a plasma system ([5, 18]), we need to further scale the time to the order of $\varepsilon^{-1}$. For this purpose, in the system (2.1), we further introduce the scaling as

\[ \hat{t} = \varepsilon t, \quad \hat{x} = \varepsilon x, \quad \hat{\xi} = \varepsilon \xi, \]

\[ \hat{F} = \frac{1}{\varepsilon^3} F, \quad \hat{E} = \frac{1}{\varepsilon} E, \quad \hat{B} = \frac{1}{\varepsilon} B, \]

and

\[ \hat{b} = \varepsilon^2 b. \]

Then substituting the above scaling back into (2.1), and dropping tildes, we obtain
\[
\begin{align*}
\varepsilon \frac{\partial F}{\partial t} + \xi \cdot \nabla_x F + e\varepsilon (\xi E + \xi \times B) \cdot \nabla \xi F &= \frac{1}{\varepsilon} Q(F, F), \\
\varepsilon \frac{\partial E}{\partial t} - \nabla \times B &= -\frac{j}{\varepsilon}, \\
\frac{\partial B}{\partial t} + \nabla \times E &= 0, \\
\text{div} B &= 0, \quad \text{div} E = \frac{\rho}{\varepsilon},
\end{align*}
\]

The incompressible Electron-Magnetohydrodynamics-Fourier equations will be obtained when \( F \) is close to the absolute Maxwellian \( M \) with order \( \varepsilon \). Motivated by \([2, 16, 23]\), we set \( F = MG \). Recasting the system \((2.2)\) for \( G \) yields

\[
\begin{align*}
\varepsilon \frac{\partial G}{\partial t} + \xi \cdot \nabla_x G + e\varepsilon (\xi E + \xi \times B) \cdot \nabla \xi G - e\varepsilon^2 E \cdot \xi G &= \frac{1}{\varepsilon} Q(G, G), \\
\varepsilon \frac{\partial E}{\partial t} - \nabla \times B &= -\frac{j}{\varepsilon}, \\
\frac{\partial B}{\partial t} + \nabla \times E &= 0, \\
\text{div} B &= 0, \quad \text{div} E = \frac{\rho}{\varepsilon},
\end{align*}
\]

where the collision operator is now given by

\[
Q(G, G) = \int_{\mathbb{R}^3} \int_{S^2} (G^\prime G' - G_s G) b(\xi - \xi, \omega) d\omega M_\ast d\xi_\ast,
\]

where \( M_\ast = M(\xi_\ast) \).

### 2.1. Relative Entropy.

For any pair of measurable functions \( f \) and \( g \) defined a.e. on \( \mathbb{R}^3 \times \mathbb{R}^3 \) and satisfying \( f \geq 0 \) and \( g > 0 \) a.e., we use the following notation for the relative entropy

\[
H(f|g) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left[ f \ln \left( \frac{f}{g} \right) - f + g \right] d\xi dx \in [0, \infty],
\]

which is a way to measure how far \( f \) is away from \( g \). We are interested in the evolution of

\[
\mathcal{H}_\varepsilon(t) = \varepsilon H(F_\varepsilon|M) + \varepsilon^3 \frac{3}{2} \int_{\mathbb{T}} (|E_\varepsilon|^2 + |B_\varepsilon|^2) dx,
\]

where \( (F_\varepsilon, E_\varepsilon, B_\varepsilon)_{\{\varepsilon>0\}} \) are renormalized solutions (see definition in Section 2.4) of Vlasov-Maxwell-Boltzmann equations \((2.2)\). This quantity contains the information from the standard (rescaled) \( L^2 \) norm of the electro-magnetic field and from the relative entropy between the renormalized solution \( F_\varepsilon(t, x, \xi) \) and the absolute Maxwellian \( M \).

\footnote{The absolute Maxwellian is given as

\[
M(\xi) = \frac{1}{(2\pi)^{3/2}} \exp \left( -\frac{1}{2} |\xi|^2 \right)
\]

and corresponds to the spatially homogeneous fluid state with its density and temperature equal to 1, bulk velocity equal to 0 and no effect from the electric field and the magnetic field.}
The following lemma is devoted to the study of the evolution of the relative entropy, deduced from
\[
\frac{d}{dt} \mathcal{H}_\varepsilon = \varepsilon \int_T \int_{\mathbb{R}^3} \partial_t F_\varepsilon (\ln F_\varepsilon - \ln M) d\xi dx + \frac{\varepsilon^3}{2} \frac{d}{dt} \int_T (\varepsilon |E_\varepsilon|^2 + |B_\varepsilon|^2) dx. \tag{2.7}
\]

Lemma 2.1. Let \((F_\varepsilon, E_\varepsilon, B_\varepsilon)\) be a renormalized solution (refer to the definition in Section 2.2.1 below) to (2.2). Then \(\mathcal{H}_\varepsilon(t)\) satisfies the differential inequality:
\[
\frac{d}{dt} \mathcal{H}_\varepsilon(t) + \frac{1}{4\varepsilon} \int_T \int_{\mathbb{R}^3} \ln \left( \frac{F_\varepsilon^t F_\varepsilon'}{F_\varepsilon^t F_\varepsilon'} \right) (F_\varepsilon^t F_\varepsilon' - F_\varepsilon^t F_\varepsilon) d\xi dx \leq 0. \tag{2.8}
\]

Proof. In view of \([21]\), the inequality will follow from the lower semi-continuity of the weak convergence and an equality version when solutions are smooth. Thus, we will assume that those solutions are smooth. Observing that
\[
\partial_t F_\varepsilon \ln F_\varepsilon = \partial_t (F_\varepsilon \ln F_\varepsilon) - \partial_t F_\varepsilon,
\]
we obtain
\[
\varepsilon \int_T \int_{\mathbb{R}^3} \partial_t F_\varepsilon \ln F_\varepsilon d\xi dx = - \frac{1}{4\varepsilon} \int_T \int_{\mathbb{R}^3} \ln \left( \frac{F_\varepsilon^t F_\varepsilon'}{F_\varepsilon^t F_\varepsilon'} \right) (F_\varepsilon^t F_\varepsilon' - F_\varepsilon^t F_\varepsilon) d\xi dx,
\]
and, by (2.2)
\[
\varepsilon \int_T \int_{\mathbb{R}^3} \partial_t F_\varepsilon \ln M d\xi dx = -\varepsilon^2 e \int_T \int_{\mathbb{R}^3} F_\varepsilon E_\varepsilon \cdot \xi d\xi dx.
\]
Here, we used the following identity twice (see [16])
\[
\int_T Q(f, f) \zeta(\xi) d\xi = \frac{1}{4} \int_T \int_{\mathbb{R}^3} d\zeta d\xi s \int_{S^2} \int_{\mathbb{R}^3} d\omega B(f f' - f f') \zeta + \zeta' - \zeta'' - \zeta'''.
\]
Hence,
\[
\varepsilon \int_T \int_{\mathbb{R}^3} \partial_t F_\varepsilon (\ln F_\varepsilon - \ln M) d\xi dx = - \frac{1}{4\varepsilon} \int_T \int_{\mathbb{R}^3} \ln \left( \frac{F_\varepsilon^t F_\varepsilon'}{F_\varepsilon^t F_\varepsilon'} \right) (F_\varepsilon^t F_\varepsilon' - F_\varepsilon^t F_\varepsilon) d\xi dx
\]
\[
+ \varepsilon^2 e \int_T \int_{\mathbb{R}^3} F_\varepsilon E_\varepsilon \cdot \xi d\xi dx.
\]
On the other hand, multiplying equation (2.2b) by \(E_\varepsilon\), equation (2.2c) by \(B_\varepsilon\), integrating them in \(x\) over \(\mathbb{R}^3\) and then summing them together, we obtain,
\[
\frac{d}{dt} \int_T (\varepsilon |E_\varepsilon|^2 + |B_\varepsilon|^2) dx = - \frac{1}{\varepsilon} \int_T E_\varepsilon \cdot j_\varepsilon dx = - \frac{1}{\varepsilon} \int_T \int_{\mathbb{R}^3} E_\varepsilon \cdot \xi F_\varepsilon d\xi dx. \tag{2.10}
\]
Substituting (2.10) back into (2.9), we obtain
\[
\varepsilon \int_T \int_{\mathbb{R}^3} \partial_t F_\varepsilon (\ln F_\varepsilon - \ln M) d\xi dx = - \frac{1}{4\varepsilon} \int_T \int_{\mathbb{R}^3} \ln \left( \frac{F_\varepsilon^t F_\varepsilon'}{F_\varepsilon^t F_\varepsilon'} \right) (F_\varepsilon^t F_\varepsilon' - F_\varepsilon^t F_\varepsilon) d\xi dx
\]
\[
- \varepsilon^2 e \int_T \int_{\mathbb{R}^3} F_\varepsilon E_\varepsilon \cdot \xi d\xi dx,
\]
which is exactly an equality version of (2.8). \(\square\)
Notations. In order to avoid unnecessary constants in the sequel, we will assume that the nondimensionalization has the following normalizations:

\[
\int_{S^2} d\omega = 1, \quad \int_{\mathbb{R}^3} Md\xi = 1,
\]

associated with the domain \(S^2\), and \(\mathbb{R}^3\) respectively;

\[
\int_T \int_{\mathbb{R}^3} G^0 Md\xi dx = 1, \quad \int_T \int_{\mathbb{R}^3} \xi G^0 Md\xi dx = 0,
\]

\[
\int_T \int_{\mathbb{R}^3} \frac{1}{2} |\xi|^2 G^0 Md\xi dx = \frac{3}{2},
\]

associated with the initial data; and

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} b(\xi_* - \xi, \omega) d\omega M_* d\xi_* Md\xi = 1,
\]

associated with the Boltzmann kernel.

Since \(Md\xi\) is a positive unit measure on \(\mathbb{R}^3\), we denote by \(<\eta>\) the average over this measure of any integrable function \(\eta = \eta(\xi)\),

\[
<\eta> = \int_{\mathbb{R}^3} \eta Md\xi.
\]

Since

\[
dM = b(\xi_* - \xi, \omega) d\omega M_* d\xi_* Md\xi
\]

is a non-negative unit measure on \(\mathbb{R}^3 \times \mathbb{R}^3 \times S^2\), we denote by \(\ll <\tau> \gg\) the average over this measure of any integrable function \(\tau = \tau(\xi, \xi_*, \omega)\),

\[
\ll <\tau> \gg = \int_{\mathbb{R}^3} \tau dM.
\]

The collision measure \(dM\) is invariant under the transformations

\[
(\omega, \xi_*, \xi) \to (\omega, \xi, \xi_*), \quad (\omega, \xi_*, \xi) \to (\omega, \xi', \xi'_*),
\]

which are called collisional symmetries (cf. [2, 16]).

Now, we can explain Lemma 2.1 in terms of \(G_\varepsilon\) as follows:

\[
\varepsilon \frac{d}{dt} \int_T \langle G_\varepsilon \ln G_\varepsilon - G_\varepsilon + 1 \rangle dx + \frac{\varepsilon^3}{2} \frac{d}{dt} \int_T (\varepsilon |E_\varepsilon|^2 + |B_\varepsilon|^2) dx
+ \frac{1}{4\varepsilon} \int_T \left\langle \ln \left( \frac{G_{\varepsilon'} G_{\varepsilon'}'}{G_{\varepsilon} G_{\varepsilon}} \right) (G_{\varepsilon'} G_{\varepsilon'}' - G_{\varepsilon} G_{\varepsilon}) \right\rangle dx \leq 0.
\]

(2.11)

If \(G_\varepsilon\) solves the VMB equations (2.4), then inequality (2.11) implies

\[
\mathcal{H}_\varepsilon(t) + \frac{1}{\varepsilon} \int_0^t \mathcal{R}(G_\varepsilon(s)) ds = \mathcal{H}_\varepsilon(0),
\]

(2.12)

where \(\mathcal{H}_\varepsilon(t)\) is the entropy functional

\[
\mathcal{H}_\varepsilon(t) = \varepsilon \int_{\mathbb{R}^3} <G_\varepsilon \ln G_\varepsilon - G_\varepsilon + 1> dx + \frac{\varepsilon^3}{2} \int_{\mathbb{R}^3} (\varepsilon |E_\varepsilon|^2 + |B_\varepsilon|^2) dx,
\]

(2.13)
and $R(G)$ is the entropy dissipation rate functional
\[ R(G) = \int_{\mathbb{R}^3} \frac{1}{4} \left\langle \ln \left( \frac{G_{\varepsilon} G_{\varepsilon}'}{G_{\varepsilon} G_{\varepsilon}'} \right) (G_{\varepsilon} G_{\varepsilon}' - G_{\varepsilon} G_{\varepsilon}') \right\rangle \, dx. \] (2.14)

This choice of $\mathcal{H}_\varepsilon$ as the entropy functional (2.13) is based on the fact that its integrand is a non-negative strictly convex function of $G$ with a minimum value of zero at $G = 1$. Indeed for any $G$,
\[ H(G) \geq 0, \quad \text{and} \quad H(G) = 0 \quad \text{if and only if} \quad G = 1. \] (2.15)

Here $H(G)$ is called the relative entropy with respect to the absolute equilibrium $G = 1$ which provides a natural measure of the proximity of $G$ to that equilibrium.

We can expect that, the terms involving the entropy $H_{\varepsilon}$ measure the proximity of $G_{\varepsilon}$ and $G_0^\varepsilon$ to the absolute equilibrium value of 1. On the other hand, the terms involving the dissipation rate $R$, can be understood to measure the proximity of $G_{\varepsilon}$ to any Maxwellian through their characterization.

### 2.2. Global Solutions

In order to mathematically justify the incompressible Electron-Magnetohydrodynamics-Fourier limit of the Vlasov-Maxwell-Boltzmann equations, we must make precise:

- the notion of solutions for the Vlasov-Maxwell-Boltzmann equations;
- the notion of solutions for the incompressible Electron-Magnetohydrodynamics-Fourier system (1.4).

Ideally, these solutions should be global while the bounds are physically natural. We therefore work in the setting of DiPerna-Lions renormalized solutions for the Vlasov-Maxwell-Boltzmann equations, and in the setting of Leray solutions for the incompressible Electron-Magnetohydrodynamics-Fourier system. These theories have the virtues of considering physically natural classes of initial data.

#### 2.2.1. Renormalized solutions to the Vlasov-Maxwell-Boltzmann equations

In the spirit of the DiPerna-Lions theory for the Boltzmann equation and the idea in Hu-Wang [21], modified slightly for the periodic box, it is possible to show the weak stability of global weak solutions to a whole class of formally equivalent initial-value problems. More precisely, let $G_{\varepsilon} \geq 0$ be a sequence of DiPerna-Lions renormalized solutions to the scaled Vlasov-Maxwell-Boltzmann initial-value problem (2.4) with

\[ G_{\varepsilon}(0, x, \xi) = G_0^\varepsilon(x, \xi) \geq 0, \quad E_{\varepsilon}(0, x) = E_0^\varepsilon(x), \quad B_{\varepsilon}(0, x) = B_0^\varepsilon(x). \]

A Renormalized Solution Relative to $M$ of (2.2) is a triplet $(F_{\varepsilon}, E_{\varepsilon}, B_{\varepsilon})$ such that

\[ F_{\varepsilon} \in C(\mathbb{R}^+; L_{loc}^1(\mathbb{R}^3), L^1(\mathbb{R}^3)), \quad E_{\varepsilon}, B_{\varepsilon} \in C_w(\mathbb{R}^+; L^2(\mathbb{R}^3)), \]

and satisfies
\[ \Gamma' \left( \frac{F_{\varepsilon}}{M} \right) Q(F_{\varepsilon}, F_{\varepsilon}) \in L_{loc}^1(\mathbb{R}^+ \times \mathbb{R}^3 \times T) \] (2.16)

for all $\Gamma \in C^1(\mathbb{R}^+)$ such that
\[ \Gamma(0) = 0, \quad \text{and} \quad z \mapsto (1 + z)\Gamma'(z) \quad \text{is bounded on} \quad \mathbb{R}^+, \] (2.17)
has finite relative entropy for all positive time:

\[ \mathcal{H}_\varepsilon(t) + \frac{1}{\varepsilon} \int_0^t \mathcal{R}(G(s)) ds \leq \mathcal{H}_\varepsilon(0), \]

and finally satisfies

\[
\int_0^\infty \int_T \Gamma \left( \frac{F_\varepsilon}{M} \right) \left( \partial_t \chi + \frac{1}{\varepsilon} \xi \cdot \nabla_x \chi \right) M d\xi dx dt \\
+ e\varepsilon \int_0^\infty \int_T \int_{\mathbb{R}^3} \Gamma \left( \frac{F_\varepsilon}{M} \right) (\varepsilon E_\varepsilon + \xi \times B_\varepsilon) \cdot \nabla_\varepsilon \chi M d\xi dx dt \\
- e\varepsilon \int_0^\infty \int_T \int_{\mathbb{R}^3} \Gamma \left( \frac{F_\varepsilon}{M} \right) \varepsilon E_\varepsilon \cdot \xi \chi M d\xi dx dt \\
+ e\varepsilon \int_0^\infty \int_T \int_{\mathbb{R}^3} \Gamma' \left( \frac{F_\varepsilon}{M} \right) \varepsilon E_\varepsilon \cdot \xi F_\varepsilon \chi M d\xi dx dt \\
+ \frac{1}{\varepsilon^2} \int_0^\infty \int_T \int_{\mathbb{R}^3} \Gamma' \left( \frac{F_\varepsilon}{M} \right) Q(F_\varepsilon, F_\varepsilon) \chi M d\xi dx dt = 0
\]

(2.19)

for each test function \( \chi \in C_0^\infty((0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3) \).

Throughout the rest of this paper, we assume that the initial data \( G_\varepsilon^0 \) satisfies the normalizations and the entropy bound

\[ \mathcal{H}_\varepsilon(0) \leq C\varepsilon^3, \]

(2.20)

for some fixed \( C > 0 \).

### 2.2.2. Weak formulation of the limiting system (1.4)

Inspired by [29], for the limiting system (1.4) with mean zero initial data, the Leray theory is set in the following Hilbert spaces of vector- and scalar-valued functions:

\[
\mathbf{H}_v = \left\{ w \in L^2(dx; \mathbb{R}^3) : \text{div} w = 0, \int_T w dx = 0 \right\}, \\
\mathbf{H}_s = \left\{ \chi \in L^2(dx; \mathbb{R}) : \int_T \chi dx = 0 \right\}, \\
\mathbf{V}_v = \left\{ w \in \mathbf{H}_v : \int_T |\nabla w|^2 dx < \infty \right\}, \\
\mathbf{V}_s = \left\{ \chi \in \mathbf{H}_s : \int_T |\nabla \chi| dx < \infty \right\}.
\]

Let \( \mathbf{H} = \mathbf{H}_v \oplus \mathbf{H}_v \oplus \mathbf{H}_s \) and \( \mathbf{V} = \mathbf{V}_v \oplus \mathbf{V}_v \oplus \mathbf{V}_s \). Leray’s theory yields: given any \( (u_0, B_0, \theta_0) \in \mathbf{H} \), there exists a \( (u, B, \theta) \in C([0, \infty); w - \mathbf{H}) \cap L^2_{loc}(0, \infty; \mathbf{V}) \) which equals initially \( (u_0, B_0, \theta_0) \in \mathbf{H} \) and satisfies the incompressible system (1.4) in the sense that,
for all \((\phi, \psi, \chi) \in H \cap C^1(T)\),
\[
\int_T \phi \cdot u(t) \, dx - \int_T \phi \cdot u(s) \, dx - \int_s^t \int_T \nabla_x \phi : (u \otimes u) \, dx \, d\tau
= -\mu \int_s^t \int_T \nabla_x \phi : \nabla_x u \, dx \, d\tau - \alpha \int_s^t \int_T \nabla \phi : \nabla (B \otimes B) \, dx \, d\tau;
\]
\[
\int_T \phi \cdot B(t) \, dx - \int_T \phi \cdot B(s) \, dx + \int_s^t \int_T \nabla \cdot (\nabla \times \psi) \, dx \, d\tau = 0;
\]
\[
\int_T \chi \cdot u(t) \, dx - \int_T \chi \cdot u(s) \, dx - \int_s^t \int_T \nabla_x \chi \cdot (u \theta) \, dx \, d\tau
= -\kappa \int_s^t \int_T \nabla_x \chi \cdot \nabla_x \theta \, dx \, d\tau,
\]
for every \(0 \leq s < t\). Moreover, \((u, B, \theta)\) satisfies the dissipation inequalities
\[
\int_T \frac{1}{2} \left( |u(t)|^2 + \alpha |B(t)|^2 \right) + \int_0^t \mu |\nabla u|^2 \, dx \, ds \leq \int_T \frac{1}{2} \left( |u_0|^2 + \alpha |B_0|^2 \right) \, dx,
\]
\[
\int_T \frac{1}{2} \left( \theta(t) \right)^2 \, dx + \int_0^t \int_T \kappa |\nabla \theta|^2 \, dx \, ds \leq \int_T \frac{1}{2} \theta_0^2 \, dx,
\]
for every \(t > 0\).

A global existence theory, similar to Leray’s theory of incompressible Navier-Stokes equations, can be established via Galerkin’s method, the dissipation inequalities \((2.21)\) and Ohm’s law which expresses the electric field \(E\) in terms of the magnetic field and the velocity as, see \[5, 18\]
\[
\mathbf{j} = \sigma (E + \mathbf{u} \times \mathbf{B}),
\]
where \(\sigma > 0\) is the electrical conductivity. To obtain the dissipation inequality \((2.21a)\), we first multiply \((1.4a)\) by \(\mathbf{u}\) to obtain, using \((1.4b)\),
\[
\frac{1}{2} \frac{d}{dt} \|u\|_{L^2(T)}^2 + \mu \|\nabla u\|_{L^2(T)}^2 - \alpha \int_T \mathbf{E} \cdot (\nabla \times \mathbf{B}) \, dx = 0.
\]
Here, we used the identity
\[
\mathbf{B} \times (\nabla \times \mathbf{B}) \cdot \mathbf{u} = \frac{1}{e} \left( \mathbf{B} \times (\nabla \times \mathbf{B}) \right) \cdot \mathbf{j} = \frac{1}{e} (\mathbf{B} \times (\nabla \times \mathbf{B})) \cdot (\nabla \times \mathbf{B}) = 0,
\]
according to \((1.4b)\). Then, we multiply \((1.4b)\) by \(\alpha \mathbf{B}\) to obtain
\[
\frac{\alpha}{2} \frac{d}{dt} \|\mathbf{B}\|_{L^2(T)}^2 + \alpha \int_T \mathbf{E} \cdot (\nabla \times \mathbf{B}) = 0.
\]
Adding \((2.22)\) and \((2.23)\), and then integrating it over \((0, T)\) yield the energy inequality \((2.21a)\).

In summary, we have the following existence theory for the incompressible system \((1.4)\).

**Proposition 2.1.** For each \(u_0, B_0 \in \{ f \in L^2(\mathbb{R}^3) : \text{div} f = 0 \ \text{in} \ D' \} \) and \(\theta_0 \in L^2(\mathbb{R}^3)\), there exists at least one weak solution \((u, B, \theta)\) of \((1.4)-(1.5)\) that satisfies the energy inequalities.
inequality
\[ \frac{1}{2} \int_T \left( |u(t,x)|^2 + \alpha|B(t,x)|^2 + \frac{5}{2} \theta(t,x)^2 \right) dx + \int_0^t \int_T \left( \mu |\nabla u|^2 + \frac{5}{2} \kappa |\nabla \theta|^2 \right) dx ds \]
\[ \leq \frac{1}{2} \int_T \left( |u_0|^2 + \alpha|B_0|^2 + \frac{5}{2} \theta_0^2 \right) dx \]
for all \( t > 0 \).

2.3. Assumptions. In this subsection, we state our technical assumptions. To begin with, we define
\[ A(\xi) = \int_{S^2} b(\xi, \omega) d\omega. \]

Our assumptions regarding the collision kernel \( b \) are stated as follows:

- **(H0)** \( b \in L^1(B_R \times S^2) \) for all \( R \in (0, \infty) \), where \( B_R = \{ z \in \mathbb{R}^3 : |z| < R \} \), and
\[
\left\{ \begin{array}{l}
\frac{1}{2} \int |b(z, w)|^2 \leq 1 + |\xi|^2 \int_{|z| + B_R} A(\xi) d\xi \\
(1 + |\xi|^2)^{-1} (\int_{|z| + B_R} A(\xi) d\xi) & \to 0, \quad |z| \to \infty, \quad \text{for all } R \in (0, \infty).
\end{array} \right.
\]

- **(H1)** \( \frac{1}{\infty} \leq b(z, \omega) \leq b_{\infty}, \quad z \in \mathbb{R}^3, \quad \omega \in S^2, \quad \text{for some } b_{\infty} > 0; \)

The assumption **(H0)** is assumed to make possible the global existence of renormalized solutions to the Vlasov-Maxwell-Boltzmann equations, see [8, 21]. The class of collision kernels satisfying **(H0)**, **(H1)** is not empty since it contains at least all collision kernels of the form \( b(z, \omega) = b(|\cos(z, \omega)|) \) satisfying **(H0)**.

Next, we impose one more technical assumption on the sequence of fluctuations \( \{g_\varepsilon\}_{\varepsilon > 0} \) (see (2.25) below).

- **(H2)** The family \( (1 + |\xi|^2) \frac{\partial}{\partial N_\xi} \) is relatively compact in \( w - L^1(dtMd\xi dx) \), where \( N_\varepsilon = 1 + \frac{\varepsilon}{3} g_\varepsilon \).

This assumption is the same as \( (A2) \) of Lions-Masmoudi [24] and similar to \( (H2) \) of [2], with the only difference being that we had to add the time variable, since we are dealing with the nonstationary case, when compared with the stationary case in [2].

2.4. Main Result. We consider a sequence of solutions \( G_\varepsilon \) to the scaled Vlasov-Maxwell-Boltzmann equations
\[
\varepsilon \partial_t G_\varepsilon + \xi \cdot \nabla_x G_\varepsilon + \varepsilon (\varepsilon E_\varepsilon + \xi \times B) \cdot \nabla_\xi G_\varepsilon - \varepsilon^2 E_\varepsilon \cdot \xi G_\varepsilon = \frac{1}{\varepsilon} Q(G_\varepsilon, G_\varepsilon), \tag{2.24}
\]
in the form
\[
G_\varepsilon = 1 + \varepsilon g_\varepsilon. \tag{2.25}
\]
We expect that as \( \varepsilon \) tends to zero, the leading behavior of the fluctuations \( g_\varepsilon \) is formally consistent with the incompressible Electron-Magnetohydrodynamics-Fourier equations. Indeed, formally, substituting (2.25) into (2.24), we obtain
\[
\varepsilon \partial_t g_\varepsilon + \xi \cdot \nabla_x g_\varepsilon + \varepsilon (\varepsilon E + \xi \times B) \cdot \nabla_\xi g_\varepsilon - \varepsilon E_\varepsilon \cdot \xi - \varepsilon^2 E \cdot \xi g_\varepsilon + \frac{1}{\varepsilon} Lg_\varepsilon = Q(g_\varepsilon, g_\varepsilon), \tag{2.26}
\]
where \( L \), the linearized collision operator, is given by
\[
Lg = -2Q(1, g) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (g + g_* - g'_*) b d\omega M_* d\xi_*.
\]
Repeated applications of the $d\mathcal{M}$-symmetries yield the identity
\[
<vLg> = \langle v(g + g_+ - g' - g'_+),
\]
\[
\frac{1}{4} \langle (v + v_+ - v' - v'_+)(g + g_+ - g' - g'_+),
\]
for every $v = v(\xi)$ and $g = g(\xi)$ for which the integral makes sense. This shows that $L$ is formally self-adjoint and has a non-negative Hermitian form. Furthermore, using the $d\mathcal{M}$-characterization, it can be shown that for any $g = g(\xi)$ in the domain of $L$, the following statements are equivalent:

\[
Lg = 0; \quad \text{(2.27a)}
\]

\[
g = \alpha + \beta \cdot \xi + \frac{1}{2} \gamma |\xi|^2, \quad \text{for some } (\alpha, \beta, \gamma) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}. \quad \text{(2.27b)}
\]

This characterizes $N(L)$, the null space of $L$, as the set obtained by linearizing about $(\alpha, \beta, \gamma) = (0, 0, 0)$. From (2.26), we deduce formally that the limit of $Lg_\varepsilon$ is zero and it can be expected that the limit of $g_\varepsilon$ will belong to $N(L)$. Indeed, it was proved by Grad (see [11, 16]) that for any collision kernel $b$ satisfying (H1), $L$ is a bounded nonnegative self-adjoint Fredholm operator on $L^2(Md\xi)$ with null space
\[
\text{Ker}L = \text{span}\{1, \xi_1, \xi_2, \xi_3, |\xi|^2\}.
\]

Notice that since each entry of the tensor $\xi \otimes \xi - \frac{1}{3} |\xi|^2 I$ and of the vector $\frac{1}{2} \xi(|\xi|^2 - 5)$ is orthogonal to $\text{Ker}L$, there exist a unique tensor $\Phi$ and a unique vector $\Psi$ such that

\[
L\Phi = \xi \otimes \xi - \frac{1}{3} |\xi|^2 I, \quad \Phi \in (\text{Ker}L)^\perp \subset L^2(Md\xi); \quad \text{(2.28)}
\]

\[
L\Psi = \frac{1}{2} |\xi|^2 - 5), \quad \Psi \in (\text{Ker}L)^\perp \subset L^2(Md\xi). \quad \text{(2.29)}
\]

Now, our main result can be stated as follows.

**Theorem 2.1.** Under the hypotheses (H0)-(H2), let $G_\varepsilon(t, x, \xi)$, with the form (2.25), be a sequence of non-negative renormalized solutions to the scaled Vlasov-Maxwell-Boltzmann equations (2.4) satisfying the initial condition (2.20). Then,

- The sequence $g_\varepsilon$ converges in the sense of distributions and almost everywhere to a function $g$ as $\varepsilon$ tends to zero, and $g$ is an infinitesimal Maxwellian,

\[
g = h + u \cdot \xi + \theta \left(\frac{1}{2} |\xi|^2 - \frac{3}{2}\right), \quad \text{(2.30)}
\]

where the velocity $u$ satisfies the incompressibility relation, while the density and temperature functions, $h$ and $\theta$, satisfy the Boussinesq relation:

\[
\text{div}u = 0, \quad \nabla_x(h + \theta) = 0. \quad \text{(2.31)}
\]

- As $\varepsilon \to 0$, $E_\varepsilon$ and $B_\varepsilon$ converge to $E$ and $B$ in the sense of distributions and $L^\infty_t(L^2(T))$ respectively.

- Moreover, the functions $h$, $u$, $\theta$, $B$, and $E$ are weak solutions of (1.4) with

\[
\mu = \frac{1}{10} \langle \Phi : L\Phi \rangle, \quad \kappa = \frac{2}{15} \langle \Psi : L\Psi \rangle. \quad \text{(2.32)}
\]
3. Implications of the Entropy Inequality

In this section, we first recall some results in [2, 16] which were established in the greatest possible generality, and relied only on the a priori estimates and in particular have nothing to do with the equations. To this end, from now on, we assume that the initial data $G^0_\varepsilon$ satisfies the entropy bound:

$$\varepsilon \int_T \langle G^0_\varepsilon \ln G^0_\varepsilon - G^0_\varepsilon + 1 \rangle \, dx + \frac{\varepsilon^3}{2} \int_T (\varepsilon |E^0_\varepsilon|^2 + |B^0_\varepsilon|^2)\, dx \leq C\varepsilon^3$$

(3.1)

with $C > 0$. From the relative entropy, we can obtain the uniform bound $\|B_\varepsilon\|_{L^\infty_t(L^2(dx))}$, and hence we can assume

$$B_\varepsilon \to B \text{ weakly* in } L^\infty_t(L^2(dx)),$$

(3.2)

with $\text{div} B = 0$ in $\mathcal{D}'$. Furthermore, from the relative entropy, $\varepsilon^{\frac{3}{2}} \|E_\varepsilon\|_{L^\infty_t(L^2(dx))}$ is uniformly bounded, and hence, we can assume that

$$\varepsilon^{\frac{3}{2}}E_\varepsilon \to \Omega, \text{ weakly* in } L^\infty_t(L^2(dx))$$

(3.3)

for some function $\Omega \in L^\infty_t(L^2(dx))$. Then the results in [2, 16], combining with (2.11) and (3.1) imply the following convergence.

**Theorem 3.1.** Under assumptions (H0)-(H2), let $F_\varepsilon$ be a family of renormalized solutions to (2.2) with initial data $(F^0_\varepsilon, E^0_\varepsilon, B^0_\varepsilon)$ satisfying (2.20), and define the associated family of fluctuations by

$$g_\varepsilon = \frac{F_\varepsilon - M}{\varepsilon M}.$$ 

Then

- $g_\varepsilon$ is relatively compact in $w - L^1_{\text{loc}}(dt dx; L^1((1 + |\xi|^2)M d\xi))$, and for almost every $t \in [0, \infty)$, $g$ satisfies

$$\int_T \frac{1}{2} \langle g^2(t) \rangle \, dx \leq \liminf_{\varepsilon \to 0} \int \langle \frac{1}{\varepsilon^2} h(\varepsilon g_\varepsilon(t)) \rangle \, dx \leq C;$$

(3.4)

moreover, for almost every $(t, x)$, $g(t, x, \cdot) \in N(L)$, which means that $g$ is of the form

$$g(t, x, \cdot) = h(t, x) + u(t, x) \cdot \xi + \theta(t, x) \left( \frac{1}{2} |\xi|^2 - \frac{3}{2} \right),$$

(3.5)

where $(h, u, \theta) \in L^\infty(dt; L^2(dx; \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R})).$

- the rescaled collision integrands

$$q_\varepsilon = \frac{1}{\varepsilon^2} (G'_\varepsilon G_{\varepsilon*} - G_\varepsilon G_{\varepsilon*})$$

(3.6)

satisfy that $\gamma(G_\varepsilon)q_\varepsilon$ is relatively compact in $w - L^1_{\text{loc}}(dt dx; L^1((1 + |\xi|^2)dM))$; furthermore, any of the limit points $q$ of $\gamma(G_\varepsilon)q_\varepsilon$ as $\varepsilon \to 0$ satisfies the $dM$-symmetry relations

$$\langle \langle \phi(q) \rangle \rangle = \left\langle \frac{1}{4} (\tilde{\phi} + \phi - \phi' - \tilde{\phi}') q \right\rangle,$$

(3.7)

and, $q \in L^2(0, T; L^2(dM dx)).$
We decompose $g$ into a class of bump functions

Proof. Taking the limit inf on both sides of the entropy inequality (2.11), we obtain

Substituting (3.10) and (3.11) back into (3.9), we finish the proof of (3.8).

Lemma 3.1. Under the same conditions as Theorem 3.1, for almost every $t \in [0, \infty)$ the function $g$ and $q$ satisfy

$$\int_T \frac{1}{2} \langle g^2(t) \rangle \, dx + \frac{1}{2} \int_T (|\Omega|^2 + |B|^2) \, dx + \int_0^t \int_T \frac{1}{4} \langle q^2 \rangle \, dxds$$

$$\leq \liminf_{\varepsilon \to 0} \int_T \frac{1}{\varepsilon^2} h(\varepsilon g_\varepsilon(t)) \, dx + \frac{1}{2} \liminf_{\varepsilon \to 0} \int_T (\varepsilon |E_\varepsilon|^2 + |B_\varepsilon|^2) \, dx$$

while, from the second assertion of Proposition 3.1 in [2], we have

$$\int_T \frac{1}{2} \langle g^2(t) \rangle \, dx + \int_0^t \int_T \frac{1}{4} \langle q^2 \rangle \, dxds$$

$$\leq \liminf_{\varepsilon \to 0} \int_T \frac{1}{\varepsilon^2} h(\varepsilon g_\varepsilon(t)) \, dx + \liminf_{\varepsilon \to 0} \int_0^t \int_T \frac{1}{4} \varepsilon^2 \langle q^2 G_{\varepsilon G_\varepsilon} \rangle \, dxds$$

Due to the lower semi-continuity of the weak convergence, we deduce that

$$\int_T (|\Omega|^2 + |B|^2) \, dx \leq \liminf_{\varepsilon \to 0} \int_T (\varepsilon |E_\varepsilon|^2 + |B_\varepsilon|^2) \, dx,$$

Substituting (3.10) and (3.11) back into (3.9), we finish the proof of (3.8).

To better understand the behavior of the fluctuation $\{g_\varepsilon\}_{\varepsilon > 0}$, as in [10] we introduce a class of bump functions

$$\mathcal{Y} = \left\{ \gamma : \mathbb{R}^+ \to [0, 1] | \gamma \in C^1, \gamma \left( \frac{3}{4}, \frac{5}{4} \right) = 1, \text{supp} \gamma \subset \left[ \frac{1}{2}, \frac{3}{2} \right] \right\}.$$

We decompose $g_\varepsilon$ as

$$g_\varepsilon = g_\varepsilon^b + \varepsilon g_\varepsilon^c.$$
with
\[ g^b_\varepsilon = \frac{1}{\varepsilon}(G_\varepsilon - 1)\gamma(G_\varepsilon), \quad g^c_\varepsilon = \frac{1}{\varepsilon^2}(G_\varepsilon - 1)(1 - \gamma(G_\varepsilon)), \]
where \( \gamma \in \Upsilon \). The following entropy controls (Proposition 2.1 and Proposition 2.7 in [16]) will be very useful:

**Lemma 3.2 (Entropy controls).** Assume that the bump function \( \gamma \in \Upsilon \) as in (3.12).

The relative fluctuation \( g_\varepsilon \) of the density satisfies the following estimates:

- \( \varepsilon |g^b_\varepsilon| \leq \frac{1}{2} \) and \( g^b_\varepsilon = O(1) \) in \( L^\infty_t(L^2(Md\xi dx)) \);
- \( (1 - \gamma(G_\varepsilon)) \leq 4\varepsilon^2|g^c_\varepsilon| \), which implies that \( \frac{1}{\varepsilon}(1 - \gamma(G_\varepsilon)) \leq 2|g^c_\varepsilon|^\frac{1}{2} \), and \( g^c_\varepsilon = O(1) \) in \( L^\infty_t(L^1(Md\xi dx)) \);
- \( (1 - \gamma(G_\varepsilon))G_\varepsilon \leq 5\varepsilon^2|g^c_\varepsilon| \), and \( (1 - \gamma(G_\varepsilon)) \leq 4\varepsilon^2|g^c_\varepsilon| \).

4. Implications of the Maxwell Equations

For the asymptotic behavior of the solutions under the hypothesis \( H_\varepsilon(0) \leq C\varepsilon^3 \), one of the difficulties when we deal with the magnetic field and the electric field comes from the fact that the relative entropy does not provide useful information on the electric field \( E_\varepsilon \) due to the \( \varepsilon \) in the front of the electric field in the definition of the relative entropy \( H_\varepsilon \). Fortunately, the uniform estimate from the relative entropy is enough to ensure that \( \Omega = 0 \). Indeed, from (3.3),

\[ \varepsilon \frac{\partial E_\varepsilon}{\partial t} \rightarrow 0, \quad \text{in} \quad D'(\mathbb{R}_+ \times \mathbb{R}^3). \]

Next, since \( g_\varepsilon \) converges to \( g \) in \( w - L^1_{loc}(dtdx; L^1((1 + |\xi|^2)Md\xi dx)) \), by the Cauchy-Schwarz inequality, we can deduce that \( g_\varepsilon \) converges to \( g \) in \( w - L^1_{loc}(dtdx; L^1(|\xi|Md\xi dx)) \). Due to the fact \( <\xi> = 0 \), \( \frac{\xi}{\varepsilon} = <\xi g_\varepsilon> \), \( \frac{\xi}{\varepsilon} \) converges to \( j \) in \( w - L^1_{loc}(dtdx) \). Then we take the limit as \( \varepsilon \rightarrow 0 \) in the equation (2.2b) to get

\[ \nabla \times B = j \quad (4.1) \]

in the sense of distributions. Furthermore,

\[ \frac{||j_\varepsilon||}{\varepsilon} \leq ||<\xi g_\varepsilon>||_{L^\infty_t(L^2(\mathbb{T}))} \leq ||g_\varepsilon||_{L^\infty_t(L^2(Md\xi dx))} \left( ||\xi|^2 \right)^{\frac{1}{2}} < \infty. \]

This implies that \( \frac{j_\varepsilon}{\varepsilon} \) converges weakly* to \( j \) in \( L^\infty_t(L^2(\mathbb{T})) \).

On the other hand, for the electric field \( E_\varepsilon \), we have

**Lemma 4.1.** The family \( \{E_\varepsilon\}_{\varepsilon > 0} \) formally satisfies

\[ E_\varepsilon = \partial_t(\varepsilon E_\varepsilon \times B_\varepsilon) - (\nabla \times B_\varepsilon) \times B_\varepsilon + \frac{j_\varepsilon}{\varepsilon} \times B_\varepsilon + \varepsilon \text{div}(E_\varepsilon \otimes E_\varepsilon) \]

\[ - \varepsilon \frac{1}{2} \nabla |E_\varepsilon|^2 - \varepsilon E_\varepsilon \int_{\mathbb{R}^3} g_\varepsilon Md\xi, \quad (4.2) \]
in the sense of distributions. Hence, \( \{ E_\varepsilon \}_{\varepsilon > 0} \) is uniformly bounded in \( (W_0^{1,\infty}((0, T) \times T))' \).

**Proof.** Indeed, multiplying (2.2b) by \( B_\varepsilon \), multiplying (2.2c) by \( \varepsilon E_\varepsilon \), and adding them together to yield
\[
\partial_t (\varepsilon E_\varepsilon \times B_\varepsilon) - (\nabla \times B_\varepsilon) \times B_\varepsilon + \varepsilon E_\varepsilon \times (\nabla \times E_\varepsilon) = -\frac{j_\varepsilon}{\varepsilon} \times B_\varepsilon. \tag{4.3}
\]

Note that
\[
E \text{div} E + (\nabla \times E) \times E = \text{div}(E \otimes E) - \frac{1}{2} \nabla |E|^2. \tag{4.4}
\]
The identity (4.3) can be rewritten as, using (2.2d)
\[
E_\varepsilon \rho_\varepsilon = \varepsilon E_\varepsilon \text{div} E_\varepsilon = \partial_t (\varepsilon E_\varepsilon \times B_\varepsilon) - (\nabla \times B_\varepsilon) \times B_\varepsilon + j_\varepsilon \times B_\varepsilon + \varepsilon \text{div}(E_\varepsilon \otimes E_\varepsilon) - \frac{1}{2} \nabla |E_\varepsilon|^2. \tag{4.5}
\]

Because
\[
\rho_\varepsilon = \int_{\mathbb{R}^3} (1 + \varepsilon g_\varepsilon) M d\xi = \int_{\mathbb{R}^3} M d\xi + \varepsilon \int_{\mathbb{R}^3} g_\varepsilon M d\xi = 1 + \varepsilon \int_{\mathbb{R}^3} g_\varepsilon M d\xi,
\]
one obtains, according to (4.5),
\[
E_\varepsilon = \partial_t (\varepsilon E_\varepsilon \times B_\varepsilon) - (\nabla \times B_\varepsilon) \times B_\varepsilon + j_\varepsilon \times B_\varepsilon + \varepsilon \text{div}(E_\varepsilon \otimes E_\varepsilon) - \frac{1}{2} \nabla |E_\varepsilon|^2. \tag{4.6}
\]

Next, due to the uniform bounds
\[
\|\sqrt{\varepsilon} E_\varepsilon\|_{L^\infty(0, T; L^2(\mathbb{R}^3))} \leq C, \quad \|B_\varepsilon\|_{L^\infty(0, T; L^2(\mathbb{R}^3))} \leq C,
\]
we have
\[
\partial_t (\varepsilon E_\varepsilon \times B_\varepsilon) \to 0
\]
in \( (W^{1,\infty}((0, T) \times T))' \) as \( \varepsilon \to 0 \), and \( - (\nabla \times B_\varepsilon) \times B_\varepsilon + j_\varepsilon \times B_\varepsilon \) is uniformly bounded in \( (W^{1,\infty}((0, T) \times T))' \) by using the identity (4.4) for \( B \).

Also, we can control the term \( \varepsilon E_\varepsilon \int_{\mathbb{R}^3} g_\varepsilon M d\xi \) as follows
\[
\left\| \varepsilon E_\varepsilon \int_{\mathbb{R}^3} g_\varepsilon M d\xi \right\|_{L^1((0, T) \times T)} \\
\leq \sqrt{\varepsilon} \|\sqrt{\varepsilon} E_\varepsilon\|_{L^2((0, T) \times T)} \left( \int_{\mathbb{R}^3} M d\xi \right)^{1/2} \| g_\varepsilon \|_{L^1((0, T) \times T)}^{1/2} \\
\leq C \sqrt{\varepsilon} \to 0
\]
as \( \varepsilon \to 0 \). Hence, according to (4.6), we deduce that \( \{ E_\varepsilon \}_{\varepsilon > 0} \) is uniformly bounded in \( (W_0^{1,\infty}((0, T) \times T))' \).

As a direct consequence of Lemma 4.1, we have

**Lemma 4.2.** \( E_\varepsilon \to E \) weakly in \( (W_0^{2,p})' \), for some function \( E \in (W_0^{2,p})' \) with \( p > 4 \), and \( (E,B) \) satisfies
\[
\partial_t B + \nabla \times E = 0 \tag{4.7}
\]
in \( (W_0^{2,p})' \).
Proof. Indeed, the uniform bound on $E_\varepsilon$ in $(W_0^{1,\infty})'$ and the Sobolev embedding
\[
W_0^{2,p}((0,T) \times \mathcal{T}) \hookrightarrow W_0^{1,\infty}((0,T) \times \mathcal{T})
\]
for any $4 < p < \infty$ imply that $E_\varepsilon$ is uniformly bounded in $(W_0^{2,p}((0,T) \times \mathcal{T}))'$ and hence is weakly convergent in $(W_0^{2,p}((0,T) \times \mathcal{T}))'$ since $(W_0^{2,p}((0,T) \times \mathcal{T}))'$ with $4 < p < \infty$ is a reflexive space.

Next, since
\[
\frac{\partial B_\varepsilon}{\partial t} + \nabla \times E_\varepsilon = 0,
\]
holds in $\mathcal{D}'(\mathbb{R}_+ \times \mathcal{T})$, we take an arbitrarily test function $\phi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^3)$ to obtain
\[
-\int_0^t \int_\mathcal{T} E_\varepsilon \cdot \nabla \times \phi dx ds + \int_0^t \int_\mathcal{T} B_\varepsilon \cdot \phi dx ds = 0. \quad (4.8)
\]
Hence, from (4.8), we obtain
\[
\int_0^t \int_\mathcal{T} E \cdot \nabla \times \phi dx ds = \lim_{\varepsilon \to 0} \int_0^t \int_\mathcal{T} E_\varepsilon \cdot \nabla \times \phi dx ds = \lim_{\varepsilon \to 0} \int_0^t \int_\mathcal{T} B_\varepsilon \cdot \frac{\partial \phi}{\partial t} dx ds \quad (4.9)
\]
Hence, from (4.9), we deduce that the limits $(E, B)$ satisfy (4.7).

Observe that, since $E_\varepsilon$ is convergent at least in the sense of distributions, we can conclude that $\Omega = 0$.

5. Vanishing of Conservation Defects

Before stating the main result of the present section, we introduce a new class of bump functions as in \([16]\). For each $C > 0$, set
\[
\Upsilon_C = \{ \gamma \in \Upsilon : \| \gamma \|_{L^\infty} \leq C \}.
\]
Consider the transformation $\mathfrak{T}$ defined by $\mathfrak{T} \gamma = 1 - (1 - \gamma)^2$; clearly $\mathfrak{T}$ maps $\Upsilon_C$ into $\Upsilon_{2C}$.

Define
\[
\bar{\Upsilon} = \mathfrak{T} \Upsilon_8 \subset \Upsilon_{16}, \quad (5.1)
\]
and notice that $\bar{\Upsilon} \neq \emptyset$ since $\Upsilon_8 \neq \emptyset$. For each $\gamma \in \bar{\Upsilon}$, define
\[
\hat{\gamma}(z) = \gamma(z) + (z - 1) \frac{d\gamma}{dz}. \quad (5.2)
\]
Notice that
\[
\text{supp}\hat{\gamma} \subset \left[ \frac{1}{2}, \frac{3}{2} \right], \quad \hat{\gamma}\left( \left[ \frac{3}{4}, \frac{5}{4} \right] \right) = \{1\}. \quad (5.3)
\]
On the other hand, let $\tilde{\gamma} \in \Upsilon_8$ be such that $\gamma = \mathfrak{T} \tilde{\gamma}$. One has
\[
1 - \hat{\gamma}(z) = (1 - \tilde{\gamma}) \left[ (1 - \tilde{\gamma}) - 2(z - 1) \frac{d\tilde{\gamma}}{dz} \right], \quad z \geq 0
\]
so that
\[
|1 - \hat{\gamma}| \leq 9(1 - \tilde{\gamma}), \quad z \geq 0. \quad (5.4)
\]
Theorem 5.1 (Vanishing of conservation defects). Let $\gamma \in \tilde{\Upsilon}$, and denote by $\eta \equiv \eta(\xi)$ any collision invariant (i.e., $\eta(\xi) = 1$ or $\eta(\xi) = \xi_1, \ldots, \xi_3$ or else $\eta(\xi) = |\xi|^2$) or any linear combination thereof. Then

$$\partial_t \left( \eta g^b_\epsilon \right) + \frac{1}{\epsilon} \nabla_x \cdot \left( \xi \eta g^b_\epsilon \right) + eB_\epsilon \cdot \left( \xi \times \nabla_\xi \eta g^b_\epsilon \right) - eE_\epsilon \cdot \left( \xi \eta \right) \to 0,$$

in $L^1_{\text{loc}}(\mathbb{R}^+ \times T)$ as $\epsilon \to 0$.

Proof. We begin with the renormalized form (2.19) of the Vlasov-Maxwell-Boltzmann equations (2.2) with $\Gamma(z) = (z-1)\gamma(z)$

$$\left( \frac{\partial_t}{\epsilon} + \frac{\xi}{\epsilon} \cdot \nabla_x \right) (M g^b_\epsilon) + e\left( \epsilon E_\epsilon + \xi \times B_\epsilon \right) \cdot \nabla_\xi (M g^b_\epsilon)$$

$$+ e\epsilon E_\epsilon \cdot \xi M g^b_\epsilon - e \left( \gamma(G_\epsilon) + (G_\epsilon - 1) \frac{d\gamma}{dz}(G_\epsilon) \right) E_\epsilon \cdot \xi F_\epsilon$$

$$= \frac{1}{\epsilon^3} \int_{S^2} \int_{\mathbb{R}^3} \left( F^0 \tilde{F}^0_x - F_x \right) \left( \gamma(G_\epsilon) + (G_\epsilon - 1) \frac{d\gamma}{dz}(G_\epsilon) \right) b d\omega M_x d\xi_x.$$ 

Here, we used the decomposition (3.13). From (5.6), we deduce that

$$\partial_t \left( \eta g^b_\epsilon \right) + \frac{1}{\epsilon} \nabla_x \cdot \left( \xi \eta g^b_\epsilon \right) + e \int_{\mathbb{R}^3} \left( \epsilon E_\epsilon + \xi \times B_\epsilon \right) \cdot \nabla_\xi (M g^b_\epsilon) \eta d\xi$$

$$+ e \int_{\mathbb{R}^3} \epsilon E_\epsilon \cdot \xi M g^b_\epsilon \eta d\xi - e \int_{\mathbb{R}^3} \tilde{\gamma}_\epsilon E_\epsilon \cdot \xi F_\epsilon \eta d\xi$$

$$= \frac{1}{\epsilon} \ll q_\epsilon \tilde{\gamma}_\epsilon \eta \gg,$$

where

$$\tilde{\gamma}_\epsilon = \gamma(G_\epsilon),$$

and the function $\tilde{\gamma}$ is defined in terms of $\gamma$ by (5.2). Observing that

$$(X \times Y) \cdot Z = Y \cdot (Z \times X) = X \cdot (Y \times Z),$$

we have

$$\int_{\mathbb{R}^3} \left( \epsilon E_\epsilon + \xi \times B_\epsilon \right) \cdot \nabla_\xi (M g^b_\epsilon) \eta d\xi$$

$$= - \left( \epsilon E_\epsilon \cdot \nabla_\xi \eta g^b_\epsilon \right) + \int_{\mathbb{R}^3} \left( \xi \times B_\epsilon \right) \cdot \nabla_\xi \eta g^b_\epsilon M d\xi$$

$$= - \left( \epsilon E_\epsilon \cdot \nabla_\xi \eta g^b_\epsilon \right) - B_\epsilon \cdot \left( \xi \times \nabla_\xi \eta g^b_\epsilon \right).$$

Notice that following the same line of the argument of Proposition 4.1 in [16], it can be shown that

$$\left\| \frac{1}{\epsilon} \ll q_\epsilon \tilde{\gamma}_\epsilon \eta \gg \right\|_{L^1_{\text{loc}}(\mathbb{R}^+ \times \mathcal{T})} \to 0$$

as $\epsilon \to 0$.

In order to estimate the $L^1$-norm of the conservation defects, for the last two terms on the left-hand side of (5.7), we claim

$$\left\| \epsilon E_\epsilon \cdot \nabla_\xi \eta g^b_\epsilon \right\|_{L^1_{\text{loc}}(\mathbb{R}^+ \times \mathcal{T})} \to 0;$$

(5.10)
\[
\left\| \int_{\mathbb{R}^3} \varepsilon E_\varepsilon \cdot \xi M g_\varepsilon^b \eta d\xi \right\|_{L^1_{\text{loc}}(\mathbb{R}^3 \times T)} \to 0; \quad (5.11)
\]

and
\[
\left\| \int_{\mathbb{R}^3} \gamma_\varepsilon E_\varepsilon \cdot \xi F_\varepsilon \eta d\xi - \int_{\mathbb{R}^3} E_\varepsilon \cdot \xi \eta M d\xi \right\|_{L^1_{\text{loc}}(\mathbb{R}^3 \times T)} \to 0 \quad (5.12)
\]
as \( \varepsilon \to 0 \). Indeed, using the elementary bounds
\[
|\gamma_\varepsilon| \leq 9, \quad |1 - \gamma_\varepsilon| \leq 9, \quad 0 \leq G_\varepsilon |\gamma_\varepsilon| \leq \frac{27}{2}, \quad (5.13)
\]
for the inequality (5.10), we have,
\[
\left| \left\langle \nabla \xi \eta g_\varepsilon^b \right\rangle \right| \leq \left( \int_{\mathbb{R}^3} (\nabla \xi \eta)^2 M d\xi \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} (g_\varepsilon^b)^2 M d\xi \right)^{\frac{1}{2}} \leq C \left( \int_{\mathbb{R}^3} (g_\varepsilon^b)^2 M d\xi \right)^{\frac{1}{2}},
\]
since
\[
\int_{\mathbb{R}^3} (\nabla \xi \eta)^2 M d\xi \leq C
\]
for all \( \eta \in N(L) \) and where \( C \) is a positive constant. Hence, by the Cauchy-Schwarz inequality and the first statement in Lemma 3.2, one has
\[
\left\| \varepsilon E_\varepsilon \cdot \left\langle \nabla \xi \eta g_\varepsilon^b \right\rangle \right\|_{L^1_{\text{loc}}(\mathbb{R}^3 \times T)} \leq C \left\| \varepsilon |E_\varepsilon| \left( \int_{\mathbb{R}^3} (g_\varepsilon^b)^2 M d\xi \right)^{\frac{1}{2}} \right\|_{L^1_{\text{loc}}(\mathbb{R}^3 \times T)}
\leq C \varepsilon^{\frac{1}{2}} \|E_\varepsilon\|_{L^1_{\text{loc}}(L^2(T))} \|g_\varepsilon^b\|_{L^\infty_{\text{loc}}(L^2(M \eta d\xi))} \to 0,
\]
as \( \varepsilon \to 0 \). Similarly, for the inequality (5.11), we have,
\[
\left| \int_{\mathbb{R}^3} \xi \eta g_\varepsilon^b M d\xi \right| \leq \left( \int_{\mathbb{R}^3} (\xi \eta)^2 M d\xi \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} (g_\varepsilon^b)^2 M d\xi \right)^{\frac{1}{2}} \leq C \left( \int_{\mathbb{R}^3} (g_\varepsilon^b)^2 M d\xi \right)^{\frac{1}{2}},
\]
since
\[
\int_{\mathbb{R}^3} (\xi \eta)^2 M d\xi \leq C
\]
for all \( \eta \in N(L) \), where \( C \) is a positive constant. Hence, by the Cauchy-Schwarz inequality and the first statement in Lemma 3.2, one has
\[
\left\| \int_{\mathbb{R}^3} \varepsilon E_\varepsilon \cdot \xi \eta g_\varepsilon^b M d\xi \right\|_{L^1_{\text{loc}}(\mathbb{R}^3 \times T)} \leq C \left\| \varepsilon |E_\varepsilon| \left( \int_{\mathbb{R}^3} (g_\varepsilon^b)^2 M d\xi \right)^{\frac{1}{2}} \right\|_{L^1_{\text{loc}}(\mathbb{R}^3 \times T)}
\leq C \varepsilon^{\frac{1}{2}} \|E_\varepsilon\|_{L^1_{\text{loc}}(L^2(T))} \|g_\varepsilon^b\|_{L^\infty_{\text{loc}}(L^2(M \eta d\xi))} \to 0,
\]
as \( \varepsilon \to 0 \).
It remains to deal with (5.12). To this end, we rewrite
\[
\int_{\mathbb{R}^3} \tilde{\gamma}_\varepsilon E_\varepsilon \cdot \xi F_\varepsilon \eta d\xi - \int_{\mathbb{R}^3} E_\varepsilon \cdot \xi \eta M d\xi = \int_{\mathbb{R}^3} (\tilde{\gamma}_\varepsilon - 1) E_\varepsilon \cdot \xi F_\varepsilon \eta d\xi + \varepsilon \int_{\mathbb{R}^3} E_\varepsilon \cdot \xi \eta g_\varepsilon M d\xi
\]
\[
= I_1 + I_2.
\]
(5.14)

Notice that from (5.4), we have
\[
|\tilde{\gamma}_\varepsilon - 1| \leq 9(1 - \tilde{\gamma}(G_\varepsilon)) \leq 9(1 - \tilde{\gamma}(G_\varepsilon))^{1/2}
\]
for some $\tilde{\gamma} \in \mathbb{Y}_8$ and hence we can control $I_1$ as, using $F_\varepsilon = MG_\varepsilon$, Lemma 4.2 and the fact $0 \leq 1 - \tilde{\gamma}(G_\varepsilon) \leq 1$,
\[
\|I_1\|_{L^1_{\text{loc}}(\mathbb{R}^+ \times T)} \leq 9 \left\| \int_{\mathbb{R}^3} |E_\varepsilon| \|\xi\| \|1 - \tilde{\gamma}(G_\varepsilon)\|^{1/2} G_\varepsilon M d\xi \right\|_{L^1_{\text{loc}}(\mathbb{R}^+ \times T)}
\]
\[
\leq 9 \left\| \int_{\mathbb{R}^3} |E_\varepsilon| \|\xi\| \|1 - \tilde{\gamma}(G_\varepsilon)\|^{1/2} M d\xi \right\|_{L^1_{\text{loc}}(\mathbb{R}^+ \times T)}
\]
\[
+ 9\varepsilon \left\| \int_{\mathbb{R}^3} |E_\varepsilon| \|\xi\| \|1 - \tilde{\gamma}(G_\varepsilon)\|^{1/2} g_\varepsilon M d\xi \right\|_{L^1_{\text{loc}}(\mathbb{R}^+ \times T)}
\]
\[
\leq 18\sqrt{\varepsilon} \left\| \sqrt{\varepsilon} \int_{\mathbb{R}^3} |E_\varepsilon| \|\xi\| \|g_\varepsilon\|^{1/2} M d\xi \right\|_{L^1_{\text{loc}}(\mathbb{R}^+ \times T)}
\]
\[
+ 9\varepsilon \left\| \int_{\mathbb{R}^3} |E_\varepsilon| \|\xi\| \|g_\varepsilon M d\xi \right\|_{L^1_{\text{loc}}(\mathbb{R}^+ \times T)}
\]
\[
\leq 18\sqrt{\varepsilon} \left\| \sqrt{\varepsilon} E_\varepsilon \right\|_{L^2_{\text{loc}}(\mathbb{R}^+ \times T)} \left\| \|\xi\|^{1/2} \right\|_{L^1_{\text{loc}}(\mathbb{R}^+ \times T)} \left\| \|g_\varepsilon\|^{1/2} \right\|_{L^2_{\text{loc}}(\mathbb{R}^+ \times T)}
\]
\[
+ 9\varepsilon \left\| \int_{\mathbb{R}^3} |E_\varepsilon| \|\xi\| \|g_\varepsilon M d\xi \right\|_{L^1_{\text{loc}}(\mathbb{R}^+ \times T)}
\]
\[
\leq C\sqrt{\varepsilon} + C\varepsilon \to 0
\]
as $\varepsilon \to 0$. For $I_2$, we have
\[
\|I_2\|_{L^1_{\text{loc}}(\mathbb{R}^+ \times T)} \leq \sqrt{\varepsilon} \left\| \sqrt{\varepsilon} E_\varepsilon \right\|_{L^2_{\text{loc}}(\mathbb{R}^+ \times T)} \left\| \|\xi\|^{1/2} \right\|_{L^1_{\text{loc}}(\mathbb{R}^+ \times T)} \left\| \|g_\varepsilon\|^{1/2} \right\|_{L^2_{\text{loc}}(\mathbb{R}^+ \times T; L^2(Md\xi))}
\]
\[
\leq C \varepsilon \to 0
\]
as $\varepsilon \to 0$. Adding (5.14), (5.15) and (5.16) together gives (5.12). Combining (5.7)–(5.12), the proof of (5.5) is finished. \square

**Remark 5.1.** According to Theorem 5.1, if $\eta = 1$ or $\eta = |\xi|^2$, then the last term on the left hand side of (5.5) will vanish; that is,
\[
E_\varepsilon \cdot \langle \xi \rangle = E_\varepsilon \cdot \langle |\xi|^2 \rangle = 0,
\]
because
\[
\langle \xi \rangle = \langle |\xi|^2 \rangle = 0.
\]
This implies that the term $E_\varepsilon \cdot \langle \xi \eta \rangle$ will only possibly appear in the conservation law of momentum. Hence,
\[
\partial_t \langle g^b_\varepsilon \xi_k \rangle + \frac{1}{\varepsilon} \nabla_x \cdot \langle \xi \xi_k g^b_\varepsilon \rangle + eB_\varepsilon \cdot \langle \xi \times \nabla_\xi \xi_k g^b_\varepsilon \rangle - \alpha c(E_\varepsilon)_k \to 0,
\]
(5.17)
in $L^1_{loc}(\mathbb{R}^+ \times T)$ for all $1 \leq k \leq 3$, since $\langle \xi_k^2 \rangle = \alpha = \frac{1}{3} \langle |\xi|^2 \rangle$.

6. PROOF OF THE MAIN RESULT: THEOREM 2.1

In this section, we will finish the proof of Theorem 2.1 via three steps.

6.1. The Incompressibility and Boussinesq Relations. Let us start with considering the renormalized form of the first equation in (2.2):

$$\varepsilon \partial_t h_\varepsilon + \mathbf{\xi} \cdot \nabla \mathbf{x} h_\varepsilon + e\varepsilon (\varepsilon E_\varepsilon + \mathbf{\xi} \times B_\varepsilon) \cdot \nabla \mathbf{\xi} h_\varepsilon - e\varepsilon E_\varepsilon \cdot \xi G_\varepsilon = \frac{1}{\varepsilon^2} \frac{1}{N_\varepsilon} Q(G_\varepsilon, G_\varepsilon),$$

where

$$h_\varepsilon = \frac{3}{\varepsilon} \ln \left(1 + \frac{1}{3} \varepsilon g_\varepsilon \right) = \frac{3}{\varepsilon} \ln N_\varepsilon.$$

Since $h_\varepsilon$ formally behaves like $g_\varepsilon$ for small $\varepsilon$, it should be thought of as the normalized form of the fluctuations $g_\varepsilon$. This means that, for every $\chi \in C^1(T; L^\infty(Md\xi))$ and every $0 \leq s \leq t < \infty$, one has,

$$\varepsilon \int_T < h_\varepsilon(t) \chi > dx - \varepsilon \int_T < h_\varepsilon(s) \chi > dx - \int_s^t \int_T < h_\varepsilon \cdot \nabla \chi > dx d\tau$$

$$+ e \int_s^t \int_T \varepsilon^2 E_\varepsilon \cdot < \xi h_\varepsilon \chi > dx d\tau - e \int_s^t \int_T \varepsilon (\varepsilon E_\varepsilon + \mathbf{\xi} \times B_\varepsilon) \cdot \nabla \chi h_\varepsilon Md\xi dx d\tau$$

$$- e \int_s^t \int_T \varepsilon E_\varepsilon \cdot \langle \xi G_\varepsilon N_\varepsilon \rangle dx d\tau$$

$$= \int_s^t \int_T \langle q_\varepsilon N_\varepsilon \chi \rangle dx d\tau.$$  

Due to the fact

$$\frac{G_\varepsilon}{N_\varepsilon} \leq 3$$

and the entropy control

$$\| \varepsilon^{\frac{1}{2}} E_\varepsilon \|_{L^\infty_t(L^2(dx))} \leq C,$$

one obtains

$$\int_s^t \int_T \varepsilon E_\varepsilon \cdot \langle \xi G_\varepsilon N_\varepsilon \rangle dx d\tau \to 0,$$

as $\varepsilon \to 0$. On the other hand, since as stated in the last statement of Theorem 3.1 (cf. also Corollary 3.2 in [2]) that $h_\varepsilon$ has the same limit $g$ as the sequence $g_\varepsilon$ in $w - L^2_{loc}(dt; w^2 L(Md\xi dx))$, one deduces that

$$\varepsilon \int_T < h_\varepsilon(t) \chi > dx - \varepsilon \int_T < h_\varepsilon(s) \chi > dx \to 0;$$

$$\int_s^t \int_T \varepsilon^2 E_\varepsilon \cdot \langle \xi h_\varepsilon \chi \rangle dx d\tau \to 0;$$

and

$$\int_s^t \int_T \int_{\mathbb{R}^3} \varepsilon (\varepsilon E_\varepsilon + \mathbf{\xi} \times B_\varepsilon) \cdot \nabla \chi h_\varepsilon Md\xi dx d\tau \to 0,$$
as $\varepsilon \to 0$, thanks to the uniform bounds

$$\|\varepsilon^{2}E_{\varepsilon}\|_{L^{\infty}_{r,s}(L^{2}(T))} \leq C, \quad \|B_{\varepsilon}\|_{L^{\infty}_{r,s}(L^{2}(T))} \leq C.$$ 

Taking the limit in (6.2) as $\varepsilon$ tends to zero while using Theorem 3.1 to establish the limits of the terms involving $h_{\varepsilon}$ and $q_{\varepsilon}$ respectively yields

$$- \int_{s}^{t} \int_{T} \langle g\xi \cdot \nabla_{x}x \rangle \, dx \, d\tau = \int_{s}^{t} \int_{T} \langle q\chi \rangle \, dx \, d\tau;$$

hence, the limiting form of (6.1) is

$$\xi \cdot \nabla_{x}g = \int \int q_{b}(\xi, \omega) d\omega M_{s} d\xi_{s}. \quad (6.3)$$

Since $q$ is in $L^{2}(dM d\xi)$, then for every $\eta = \eta(\xi)$ in $L^{2}(dM)$, an application of the Cauchy-Schwarz inequality shows that $\langle \langle \eta q \rangle \rangle$ is in $L^{2}(dx)$. By a repeated application of the $dM$-symmetries in Theorem 3.1, one has that, for any $\eta$ in $L^{2}(dM)$,

$$\langle \langle \eta q \rangle \rangle = \frac{1}{4} \langle \langle \eta + \eta_{s} - \eta_{s} - \eta' \rangle \rangle \quad (6.4)$$

Successively apply the identity (6.4) for $\eta = 1, \xi, \frac{1}{2}|\xi|^{2}$ and use the microscopic conservation laws (1.2) to obtain

$$\langle \langle q \rangle \rangle = 0, \quad \langle \langle \xi q \rangle \rangle = 0, \quad \langle \langle \frac{1}{2}|\xi|^{2}q \rangle \rangle = 0. \quad (6.5)$$

Since these $\eta$ are also in $L^{2}(Md\xi)$, it then follows from the limiting Vlasov-Maxwell-Boltzmann equation (6.3) that $g$ satisfies the local conservation laws of mass, momentum, and energy:

$$\text{div}_{x} \langle \xi g \rangle = 0, \quad \text{div}_{x} \langle \xi \otimes \xi g \rangle = 0, \quad \text{div}_{x} \left\langle \frac{1}{2}|\xi|^{2}g \right\rangle = 0. \quad (6.5)$$

Theorem 3.1 states that $g$ has the form of the infinitesimal Maxwellian

$$g = h + u \cdot \xi + \theta \left( \frac{1}{2}|\xi|^{2} - \frac{3}{2} \right).$$

Substituting this into (6.5), the local mass and energy conservation laws yield the incompressibility relation for the velocity field $u$ while that of momentum yields the Boussinesq relation between $h$ and $\theta$:

$$\text{div}_{x}u = 0, \quad \nabla_{x}(h + \theta) = 0.$$

6.2. **Proof of Convergence to Incompressible Electron-Magnetohydrodynamic-Fourier Equations.** Throughout this subsection, it is assumed that the bump function $\gamma$ belongs to $\bar{\Upsilon}$ (defined by (5.1)). Using Theorem 5.1 the classical Sobolev embedding theorems, and the continuity of pseudo-differential operators of order 0 on $W^{s,p}$ for $1 < p < \infty$, one sees that, for all $s > 0$,

$$\partial_{t} P \left\langle \xi g_{b}^{l} \right\rangle + P \nabla_{x} \cdot \frac{1}{\varepsilon} \left\langle \left( \xi \otimes \xi - \frac{1}{3}|\xi|^{2}I \right) g_{b}^{l} \right\rangle + eP \left( B_{\varepsilon} \cdot \left\langle \xi \times \nabla_{x}g_{\varepsilon}^{l} \right\rangle \right) - \alpha eP E_{\varepsilon} \quad (6.6)$$

$$\to 0.$$
in $L^1_{loc}(dt; W^{-s,1}_{loc}(\mathbb{R}^3))$, and
\[
\partial_t \left( \frac{1}{5} |\xi|^2 - 1 \right) g^b_\xi + \nabla_x \cdot \frac{1}{\varepsilon} \left( \xi \left( \frac{1}{5} |\xi|^2 - 1 \right) g^b_\xi \right) \to 0
\]  
(6.7)
in $L^1_{loc}(dtdx)$ as $\varepsilon \to 0$. Here, the operator $P$ is the Leray projection, i.e. the $L^2(dx)$-orthogonal projection on the space of divergence-free vector fields. In (6.7), we used
\[\xi \times \nabla_\xi \left( \frac{1}{5} |\xi|^2 - 1 \right) = 0.\]

By Theorem 3.1 and Proposition 3.2, pick any sequence $\varepsilon_n \to 0$ such that
\[
g^b_{\varepsilon_n} \to g \quad \text{in} \quad w^* - L^\infty_t (L^2(Md\xi dx)),
\]
(6.8)
\[
\gamma^b_{\varepsilon_n} \to q \quad \text{in} \quad w - L^1_{loc}(dtdx; L^1(1 + |\xi|^2)dM).
\]
(6.9)
In this section, we deal exclusively with such extracted sequences, drop the index $n$ and abuse the notations $g_\varepsilon$, $g^b_\varepsilon$, $g^c_\varepsilon$, $q_\varepsilon$ and so on to designate the subsequences $g_{\varepsilon_n}$, $g^b_{\varepsilon_n}$, $g^c_{\varepsilon_n}$, $q_{\varepsilon_n}$. Set $u$ and $\theta$ the limiting fluctuations of velocity and temperature fields defined by
\[
\left\langle \xi g^b_\varepsilon \right\rangle \to u, \quad \text{in} \quad w^* - L^\infty_t (L^2_x);
\]
(6.10)
\[
\left\langle \frac{1}{3} |\xi|^2 - 1 \right\rangle g^b_\varepsilon \to \theta, \quad \text{in} \quad w^* - L^\infty_t (L^2_x).
\]
(6.11)

The second entropy control in Proposition 3.2 implies that $g^b_\varepsilon$ and $g_\varepsilon$ have the same limit $g$ in $w - L^1_{loc}(dtdx; L^1(Md\xi))$; hence the Boussinesq relation and the incompressibility condition hold:
\[\text{div}_x u = 0, \quad \theta + <g> = 0.\]
(6.12)

Denote by $\varsigma$ either the tensor $\Phi$ or the vector $\Psi$. Since $L$ is self-adjoint on $L^2(Md\xi)$ so that
\[
\frac{1}{\varepsilon} \left\langle (L \varsigma) g^b_\varepsilon \right\rangle = \frac{1}{\varepsilon} \left\langle \varsigma (L g^b_\varepsilon) \right\rangle = \frac{1}{\varepsilon} \left\langle \varsigma (g^b_\varepsilon + g^{b*}_\varepsilon - g^{b'}_\varepsilon - g^{b*'}_\varepsilon) \right\rangle
\]
\[
= \left\langle \varsigma \left[ \frac{1}{\varepsilon} (g^b_\varepsilon + g^{b*}_\varepsilon - g^{b'}_\varepsilon - g^{b*'}_\varepsilon) + (g^{b\varepsilon}_\varepsilon g^{b*}_\varepsilon - g^{b\varepsilon'}_\varepsilon g^{b*'}_\varepsilon) \right] \right\rangle
\]
\[
+ \left\langle \varsigma Q(g^b_\varepsilon, g^{b*}_\varepsilon) \right\rangle.
\]
(6.13)
The first term on the last right hand side of (6.13) converges to the diffusion term while the second term converges to the convection term in the incompressible system (1.4). These limits are analyzed in the next two lemmas. The convergence to the diffusion term is obtained by an argument that closely follows [16], except that the present work should pay additional attention to the Maxwell effect. This apparently minor difference makes our analysis slightly more difficult than that in [16].

**Lemma 6.1.** Define
\[
\nu = \frac{1}{10} \left\langle \Phi : L \Phi \right\rangle, \quad \kappa = \frac{2}{15} \left\langle \Psi : L \Psi \right\rangle.
\]
(6.14)
Then, as $\varepsilon \to 0$,
\[
\frac{1}{\varepsilon} \left\langle (L \Phi) g^b_\varepsilon \right\rangle - \left\langle \Phi Q(g^b_\varepsilon, g^{b*}_\varepsilon) \right\rangle \to -\nu (\nabla_x u + (\nabla_x u)^\top);
\]
\[
\frac{1}{\varepsilon} \langle (L\Psi)g^b_\varepsilon \rangle - \langle \Psi Q(g^b_\varepsilon, g^b_\varepsilon) \rangle \to -\frac{5}{2} \kappa \nabla_x \theta
\]
in \(w - L^1_{lo}(dtdx)\).

The convection term is the nonlinear part of the limiting system and its convergence is therefore the most difficult to establish. The analysis below rests not only on all \textit{a priori} estimates and the arguments in [16], but also the compactness of the moment of \(g^b_\varepsilon\) in \(\xi\) which is stated in Lemma 6.3 below.

**Lemma 6.2.** The following convergence hold in the sense of distributions on \(\mathbb{R}^+ \times \mathbb{R}^3\):

\[
P \nabla_x \cdot \langle \Phi Q(g^b_\varepsilon, g^b_\varepsilon) \rangle \to P \nabla_x \cdot (u \otimes u),
\]
\[
\nabla_x \langle \Psi Q(g^b_\varepsilon, g^b_\varepsilon) \rangle \to \frac{5}{2} \nabla_x \cdot (u \theta),
\]
as \(\varepsilon \to 0\).

### 6.3. The Lorentz Force Term.

The key result of this subsection is to deal with the convergence of the Lorentz force term. To this end, we first state the following compactness about the moment of \(g^b_\varepsilon\) in \(\xi\).

**Lemma 6.3.** Let \(\gamma \in \mathcal{Y}\) be the same as in (3.12) and the hypothesis (H2) hold. Then, the family \(g^b_\varepsilon\) has the following property: for each sequence \(\varepsilon_n \to 0\), each function \(\chi = \chi(\xi)\) such that \(|\chi(\xi)| \leq 1\) as \(|\xi| \to \infty\), each \(T > 0\), there exists a function \(\eta : \mathbb{R}^+ \to \mathbb{R}^+\) such that

\[
\lim_{z \to 0^+} \eta(z) = 0
\]
and

\[
\int_0^T \int_{\mathbb{R}^3} \left( |\langle g^b_{\varepsilon_n, \chi \rangle (t, x + y) - \langle g^b_{\varepsilon_n, \chi \rangle (t, x)|^2 \leq \eta(|y|)\right) dxdt \leq \eta(|y|)
\]
for each \(y \in \mathbb{R}^3\) such that \(|y| \leq 1\), uniformly in \(n\).

**Proof.** For any \(\gamma \in \mathcal{Y}\), since \(F^\varepsilon\) is a renormalized solution of (2.4) relatively to \(M\), using the nonlinear function \(\Gamma(z) = (z - 1)\gamma(z)\) in the renormalized formulation (2.19), we obtain

\[
(\varepsilon \partial_t + \xi \cdot \nabla_x)g^b_\varepsilon = \int_{\mathbb{R}^3} \int_{S^2} q_\varepsilon \hat{\gamma} \varepsilon bd\omega M_\varepsilon d\xi_\varepsilon - \text{div}_x \left( \varepsilon (\varepsilon E_\varepsilon + \xi \times B_\varepsilon)g^b_\varepsilon \right) + \varepsilon \hat{\gamma} \varepsilon eE_\varepsilon \cdot \xi G_\varepsilon,
\]
with \(\hat{\gamma}\) defined in terms of the truncation \(\gamma\) by (5.2). Denoting

\[
f \wedge L = \begin{cases} 
  f, & \text{if } |f| \leq L; \\
  L, & \text{if } f \geq L; \\
  -L, & \text{if } f \leq -L
\end{cases}
\]
for every \(L \geq 1\), we deduce from (6.15) that

\[
(\varepsilon \partial_t + \xi \cdot \nabla_x)g^b_\varepsilon \wedge L = \int_{\mathbb{R}^3} \int_{S^2} q_\varepsilon \hat{\gamma} \varepsilon bd\omega M_\varepsilon d\xi_\varepsilon \cdot 1_{\{|\theta_\varepsilon| \leq L\}} - \text{div}_x \left( \varepsilon (\varepsilon E_\varepsilon + \xi \times B_\varepsilon) \left( g^b_\varepsilon \wedge L \right) \right) + \varepsilon \hat{\gamma} \varepsilon eE_\varepsilon \cdot \xi G_\varepsilon 1_{\{|\theta_\varepsilon| \leq L\}},
\]

with \(\hat{\gamma}\) defined in terms of the truncation \(\gamma\) by (5.2).
Furthermore, for every $N > 1$, we decompose $g^b_\varepsilon \wedge L$ as

$$g^b_\varepsilon \wedge L = \overline{g^b_\varepsilon} + \gamma^b_\varepsilon, \quad \overline{g^b_\varepsilon} = 0,$$

with

$$(\varepsilon \partial_t + \xi \cdot \nabla_x)\overline{g^b_\varepsilon} = \left( \int_{\mathbb{R}^3} \int_{S^2} q_\varepsilon \gamma^b_\varepsilon b \omega M_s d\xi_s \right) 1_{\{|\gamma^b_\varepsilon| \leq L\}} 1_{\{|A^b_\varepsilon| > N\}}, \quad (6.17)$$

and

$$(\varepsilon \partial_t + \xi \cdot \nabla_x)\gamma^b_\varepsilon = \left( \int_{\mathbb{R}^3} \int_{S^2} q_\varepsilon \gamma^b_\varepsilon b \omega M_s d\xi_s \right) 1_{\{|\gamma^b_\varepsilon| \leq L\}} 1_{\{|A^b_\varepsilon| \leq N\}} - ediv_x \left( \varepsilon (\varepsilon E_\varepsilon + \xi \times B_\varepsilon) \left( g^b_\varepsilon \wedge L \right) \right) + e\gamma^b_\varepsilon \varepsilon E_\varepsilon \cdot \xi G_\varepsilon 1_{\{|\gamma^b_\varepsilon| \leq L\}}, \quad (6.18)$$

where

$$A^b_\varepsilon = \int_{\mathbb{R}^3} \int_{S^2} q_\varepsilon \gamma^b_\varepsilon b \omega M_s d\xi_s.$$

**Step 1: Control of $\overline{g^b_\varepsilon}$.** From (6.17), if we denote

$$S_\varepsilon = \left( \int_{\mathbb{R}^3} \int_{S^2} q_\varepsilon \gamma^b_\varepsilon b \omega M_s d\xi_s \right) 1_{\{|A^b_\varepsilon| > N\}} 1_{\{|\gamma^b_\varepsilon| \leq L\}},$$

then we obtain

$$\overline{g^b_\varepsilon}(t, x, \xi) = \frac{1}{\eta} \int_0^\tau S_\varepsilon(t - \varepsilon s, x + s\xi, \xi) ds. \quad (6.19)$$

Notice that, since $|\gamma^b_\varepsilon| \leq 9$ and $q_\varepsilon$ is weakly compact in $L^1(dt dx d\xi)$, $S_\varepsilon$ is uniformly bounded in $L^1(dt dx M d\xi)$. Therefore,

$$\|\overline{g^b_\varepsilon}(t, x, \xi)\|_{L^1(dt dx M d\xi)} \leq \|S_\varepsilon\|_{L^1(dt dx M d\xi)} \leq \|S_\varepsilon\|_{L^1(dt dx M d\xi)}. \quad (6.20)$$

**Step 2: Compactness of $\gamma^b_\varepsilon$.** Setting

$$\hat{S}_\varepsilon = \left( \int_{\mathbb{R}^3} \int_{S^2} q_\varepsilon \gamma^b_\varepsilon b \omega M_s d\xi_s \right) 1_{\{|A^b_\varepsilon| \leq N\}} 1_{\{|\gamma^b_\varepsilon| \leq L\}} - ediv_x \left( \varepsilon (\varepsilon E_\varepsilon + \xi \times B_\varepsilon) \left( g^b_\varepsilon \wedge L \right) \right) + e\gamma^b_\varepsilon \varepsilon E_\varepsilon \cdot \xi G_\varepsilon$$

Notice that $|\gamma^b_\varepsilon G_\varepsilon| \leq \frac{27}{2}$, and hence, by the interpolation between $L^1$ and $L^\infty$, we have

$$\left( \int_{\mathbb{R}^3} \int_{S^2} q_\varepsilon \gamma^b_\varepsilon b \omega M_s d\xi_s \right) 1_{\{|A^b_\varepsilon| \leq N\}} 1_{\{|\gamma^b_\varepsilon| \leq L\}} + e\gamma^b_\varepsilon \varepsilon E_\varepsilon \cdot \xi G_\varepsilon \in L^2(dt dx M dx)$$

and

$$div_x \left( \varepsilon (\varepsilon E_\varepsilon + \xi \times B_\varepsilon) \left( g^b_\varepsilon \wedge L \right) \right) \in L^2(dt dx; H^{-1}(d\xi)).$$

Thus, from (6.18), we obtain

$$(\varepsilon \partial_t + \xi \cdot \nabla_x)g^b_\varepsilon = \hat{S}_\varepsilon \in L^2(dt dx M dx) + L^2(dt dx; H^{-1}(d\xi)). \quad (6.21)$$
Applying the averaging theorem in [8, 14], we deduce from (6.21) that, for all \( \chi(\xi) \) such that \( \frac{\chi(\xi)}{1 + |\xi|^2} \to 0 \) as \( |\xi| \to \infty \),
\[
\left\| \left\langle \hat{g}_\varepsilon^b \chi \right\rangle \right\|_{L^2(0,T;H^1_0(T))} \leq C_{N,L}, \tag{6.22}
\]
where \( C_{N,L} \) depends only on \( N, L \). This yields the compactness of \( \hat{g}_\varepsilon^b \chi \) in space; namely, there exists a function \( \eta : \mathbb{R} \to \mathbb{R} \) such that \( \lim_{\varepsilon \to 0^+} \eta(z) = 0 \)
\[
\left\| \left\langle \hat{g}_\varepsilon^b \chi \right\rangle (t, \cdot + y) - \left\langle \hat{g}_\varepsilon^b \chi \right\rangle (t, \cdot) \right\|_{L^2((0,T) \times T)} \leq \eta(|y|). \tag{6.23}
\]

**Step 3: Compactness of \( g^b_\varepsilon \wedge L \).** From (6.21) and the weak compactness of \( q_\varepsilon \) in \( L^1(dt dx d\xi) \), we have, for large enough \( N \), \( \left\| \hat{g}_\varepsilon^b(t, x, \xi) \right\|_{L^1(dt dx d\xi)} \) can be small as we like. Thus, by (6.23), there exists a function \( \eta : \mathbb{R} \to \mathbb{R} \) with \( \lim_{\varepsilon \to 0^+} \eta(z) = 0 \)
\[
\left\| \left\langle \left( g_\varepsilon^b \wedge L \right) \right\rangle (t, \cdot + y) - \left\langle \left( g_\varepsilon^b \wedge L \right) \right\rangle (t, \cdot) \right\|_{L^1((0,T) \times T)} \leq \eta(|y|). \tag{6.24}
\]
Using the hypothesis that \( \left\{ (g_\varepsilon^b)^2 \right\}_{\varepsilon > 0} \) is relatively compact in \( w - L^1(dt(1 + |\xi|^2) d\xi dx) \), we deduce easily that there exists a function \( \eta : \mathbb{R} \to \mathbb{R} \) with \( \lim_{\varepsilon \to 0^+} \eta(z) = 0 \)
\[
\left\| \left\langle \left( g_\varepsilon^b \wedge L \right) \right\rangle (t, \cdot + y) - \left\langle \left( g_\varepsilon^b \wedge L \right) \right\rangle (t, \cdot) \right\|_{L^2((0,T) \times T)} \leq \eta(|y|). \tag{6.25}
\]

**Step 4: Compactness of \( g^b_\varepsilon \).** Due to the hypothesis that \( \left\{ (g_\varepsilon^b)^2 \right\}_{\varepsilon > 0} \) is relatively compact in \( w - L^1(dt(1 + |\xi|^2) d\xi dx) \), for every \( \beta > 0 \), there exists an integer \( L > 1 \) such that
\[
\left\| \left\langle \left( g_\varepsilon^b \wedge L \right) \right\rangle (t, \cdot) - \left\langle \left( g_\varepsilon^b \right) \right\rangle (t, \cdot) \right\|_{L^2((0,T) \times T)} \leq C_{\beta},
\]
uniformly in \( \varepsilon \). Thus, for such \( \beta \) and \( L \), we have
\[
\left\| \left\langle \left( g_\varepsilon^b \wedge L \right) \right\rangle (t, \cdot + y) - \left\langle \left( g_\varepsilon^b \right) \right\rangle (t, \cdot + y) \right\|_{L^2((0,T) \times T)} \leq C_{\beta},
\]
and
\[
\left\| \left\langle \left( g_\varepsilon^b \wedge L \right) \right\rangle (t, \cdot) - \left\langle \left( g_\varepsilon^b \right) \right\rangle (t, \cdot) \right\|_{L^2((0,T) \times T)} \leq C_{\beta},
\]
uniformly in \( \varepsilon \). Hence, the above two inequalities, combining together with (6.25), imply there exists a function \( \eta : \mathbb{R} \to \mathbb{R} \) such that \( \lim_{\varepsilon \to 0^+} \eta(z) = 0 \) and
\[
\int_0^T \int_T \left\| \left\langle \left( g_\varepsilon^b \right) \right\rangle (t, x + y) - \left\langle \left( g_\varepsilon^b \right) \right\rangle (t, x) \right\|^2 dt dx \leq \eta(|y|)
\]
for each \( y \in \mathbb{R}^3 \) such that \( |y| \leq 1 \), uniformly in \( n \). \( \square \)

Now, we are ready to prove the convergence of the term of Lorentz force.

**Lemma 6.4.** The following convergence holds in the sense of distributions on \( \mathbb{R}^+ \times \mathbb{R}^3 \):
\[
B_\varepsilon \cdot \left\langle \xi \times \nabla \xi \varepsilon^b \right\rangle \rightarrow (B \times (\nabla \times B))_k,
\]
as \( \varepsilon \to 0 \), for all \( 1 \leq k \leq 3 \). The notation \( a_k \) stands for the \( i \)-th component of the vector \( a \). Further, we have \( j = \varepsilon u \).
Proof. For any \(1 \leq k \leq 3\), \(\nabla \xi \xi_k = e_k\), where \(\{e_k\}_{k=1}^3\) is the standard basis for \(\mathbb{R}^3\). This implies
\[
\left\langle \xi \times \nabla \xi_k e_k^b \right\rangle = \left\langle \xi g_k^b \right\rangle \times e_k.
\]
Then, we can rewrite \(B_\varepsilon < \xi \times \nabla \xi_k e_k^b >\) as
\[
B_\varepsilon \cdot \left\langle \xi \times \nabla \xi_k e_k^b \right\rangle = B_\varepsilon \cdot \left( \left\langle \xi g_k^b \right\rangle \times e_k \right).
\]
Defining
\[
j_\varepsilon^b = e \left( \frac{\langle 1 + \varepsilon g_k^b \rangle}{\varepsilon} \right) = e \left( \langle \xi g_k^b \rangle \right),
\]
since \(\langle \xi \rangle = 0\). Then, we have
\[
\left\| j_\varepsilon^b - \frac{j_\varepsilon}{\varepsilon} \right\|_{L_1^\infty(L^1(dxMd\xi))} \to 0,
\]
as \(\varepsilon \to 0\). Indeed, from the definition of \(g_k^b\), we know that \(\varepsilon g_k^b\) is uniformly bounded in \(L_1^\infty(L^2(dxMd\xi))\) while from the second statement of Lemma 3.2 \(g_k^b\) is uniformly bounded in \(L_1^\infty(L^1(dxMd\xi))\). Thus, by the interpolation between \(L^2\) and \(L^1\), we deduce that
\[
\left\| \varepsilon g_k^b \right\|_{L_1^\infty(L^\frac{2}{3}(dxMd\xi))} \leq C \varepsilon^\frac{1}{2},
\]
for some constant \(C > 0\). Therefore, we have
\[
\left\| j_\varepsilon^b - \frac{j_\varepsilon}{\varepsilon} \right\|_{L_1^\infty(L^1(dxMd\xi))} = \left\| g_k^b \xi - g_k \xi \right\|_{L_1^\infty(L^1(dxMd\xi))}
\]
\[
= \left\| \varepsilon g_k^b \xi \right\|_{L_1^\infty(L^1(dxMd\xi))}
\]
\[
\leq \left\| \varepsilon g_k^b \right\|_{L_1^\infty(L^\frac{2}{3}(dxMd\xi))} < |\xi|^3 >^\frac{1}{2}
\]
\[
\leq C \varepsilon^\frac{1}{2} \to 0,
\]
as \(\varepsilon \to 0\). Hence, (6.27), combining with the weak convergence of \(\left\{ j_\varepsilon^b \right\}\) in \(L_1^\infty(L^2(dxMd\xi))\) and the uniform bound of \(\{ j_\varepsilon \} \) in \(L_1^\infty(L^2(dxMd\xi))\), implies that \(j_\varepsilon^b\) converges weakly to \(j\) in \(L_1^\infty(L^2(dxMd\xi))\). Note that \(\hat{\omega}_\varepsilon = e \langle \xi g_k \rangle\), we have \(j = e u\).

Notice that, (2.2a) implies
\[
\partial_t B_\varepsilon = -\nabla \times E_\varepsilon \in L^\infty(0, T; W^{-4,2}(T)) \subset L^1(0, T; W^{-s,1}(T)),
\]
for some \(s > 4\) large enough, and is bounded in \(L^1(0, T; W^{-s,1}(T))\) uniformly in \(\varepsilon\). On the other hand, Lemma 6.3 with \(\chi(\xi) = \xi_k\) implies that for each \(T > 0\),
\[
\int_T \int_T |\langle \xi g_k^b \rangle(t, x + y) \times e_k - \langle \xi g_k^b \rangle(t, x) \times e_k|^2 dx dt \leq \eta(|y|),
\]
for each \(y \in \mathbb{R}^3\) such that \(|y| \leq 1\), uniformly in \(\varepsilon\), where \(\eta\) is a function \(\mathbb{R}_+ \to \mathbb{R}_+\) satisfying \(\lim_{z \to 0^+} \eta(z) = 0\). Hence, by Lemma 5.1 in [26], one has
\[
(B_\varepsilon) \cdot \left( \langle \xi g_k^b \rangle \times e_k \right) \to B \cdot (e u \times e_k) = B \cdot (j \times e_k) = (B \times j)_k,
\]
in the sense of distributions. The proof is complete. \qed
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