On an $L^2$ extension theorem from log-canonical centres with log-canonical measures

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Abstract
With a view to prove an Ohsawa–Takegoshi type $L^2$ extension theorem with $L^2$ estimates given with respect to the log-canonical (lc) measures, a sequence of measures each supported on lc centres of specific codimension defined via multiplier ideal sheaves, this article is aiming at providing evidence and possible means to prove the $L^2$ estimates on compact Kähler manifolds $X$. A holomorphic family of $L^2$ norms on the ambient space $X$ is introduced which is shown to “deform holomorphically” to an $L^2$ norm with respect to an lc-measure. Moreover, the latter norm is shown to be invariant under a certain normalisation which leads to a “non-universal” $L^2$ estimate on compact $X$. Explicit examples on $\mathbb{P}^3$ with detailed computation are presented to verify the expected $L^2$ estimates for extensions from lc centres of various codimensions and to provide hint for the proof of the estimates in general.

Keywords $L^2$ extension · Ohsawa–Takegoshi extension · lc centres

Mathematics Subject Classification Primary 32J25 · Secondary 32Q15 · 14E30

1 Introduction and preparation

1.1 Background and main results

In [5], an Ohsawa–Takegoshi type $L^2$ extension theorem for holomorphic sections on compact Kähler manifolds with estimates with respect to 1-lc-measures is proved. With only minor generalisations to the assumptions on the potential $\varphi_L$ and the global function $\psi$ (see Sect. 1.3 for details), the claim in [5, Thm. 1.4.5] is essentially the same as the quantitative extension in [6, Thm. (2.12)] on compact Kähler manifolds (see also [3]) but with the Ohsawa measure in the estimate replaced by the 1-lc-measure defined in [5]. The proof of [5, Thm. 1.4.5] works...
only when the section to be extended is defined on log-canonical (lc) centres of codimension 1 but vanishing on those of codimension 2 and higher. However, in view of the conjecture of dlt extension ([7, Conj. 1.3]), it is crucial to have some sort of estimates for holomorphic extensions of sections on lc centres of codimension 2 or higher (see [5, §1.1] for a brief account).

While the result in [5, Thm. 1.4.5] is no better than the known results from all previous studies on the same topic, it is a precursor of the study of $L^2$ estimates of holomorphic extensions of sections on lc centres of higher codimensions in terms of lc-measures. The example of Berndtsson (see [5, Example 2.3.2] or [4, Appendix A.3]) hints that expressing the $L^2$ estimates in terms of various lc-measures is possible. The goal of this paper is to further investigate in this direction.

Let $X$ be a compact Kähler manifold, $(L, e^{-\psi L})$ a holomorphic line bundle on $X$ endowed with a singular hermitian metric $e^{-\psi L}$, and $\psi$ a global negative function with poles on $X$. Assuming that both $\varphi_L$ and $\psi$ have neat analytic singularities and that there is a $\delta > 0$ such that $\varphi_L + (1 + \beta)\psi$ is plurisubharmonic (psh) for all $\beta \in [0, \delta]$. Let $S \subset (\psi)^{-1}(-\infty)$ be the subvariety which is the scheme-theoretic difference between the subvarieties defined by the multiplier ideal sheaves $I(\varphi_L + \psi)$ and $\mathcal{I}(\varphi_L)$ and suppose that $S$ is reduced (i.e. $I(\varphi_L + m\psi)$ does not “jump” more than once along the same subvariety when $m$ increases within $[0, 1]$; see Sect. 1.3 for more precise assumptions on $\varphi_L$ and $\psi$ and definition of $S$). Denote the defining ideal sheaf of the union of lc centres $\mathcal{I}_{X}(S) = \mathcal{I}_{X}(\varphi_L, \psi)$ of $(X, \varphi_L, \psi)$ of codimension $\sigma$ by $\mathcal{I}_{\mathcal{I}_{X}(S)}$ (see Definition 1.13).

Following the analysis of Berndtsson ([4, Appendix A.3] or [5, Example 2.3.2]), set

$$\mathcal{H}_\sigma := H^0 \left( X, K_X \otimes L \otimes \mathcal{I}(\varphi_L) \cdot \mathcal{I}_{\mathcal{I}_{X}(S)} \right),$$

the space of holomorphic sections of $K_X \otimes L \otimes \mathcal{I}(\varphi_L)$ vanishing on $\mathcal{I}_{\mathcal{I}_{X}(S)}$. One then obtains the filtration

$$H^0(X, K_X \otimes L \otimes \mathcal{I}(\varphi_L)) = \mathcal{H}_{\sigma_{\text{mlc}} + 1} \supset \mathcal{H}_{\sigma_{\text{mlc}}} \supset \cdots \supset \mathcal{H}_1 = H^0(X, K_X \otimes L \otimes \mathcal{I}(\varphi_L + \psi)), \quad (1)$$

where $\sigma_{\text{mlc}} \leq n := \dim X$ is the codimension of the minimal lc centres (mlc) of $(X, \varphi_L, \psi)$. By the “qualitative extension” of Demailly ([6], see also [3]), it follows that, under the psh assumption on $\varphi_L + (1 + \beta)\psi$, existence of extension is guaranteed and thus

$$\mathcal{H}_{\sigma_{\text{mlc}} + 1}/\mathcal{H}_1 \cong H^0 \left( S, K_X \otimes L \otimes \frac{\mathcal{I}(\varphi_L)}{\mathcal{I}(\varphi_L + \psi)} \right).$$

Indeed, even if the isomorphism does not hold true (e.g. when $\varphi_L + (1 + \beta)\psi$ is not psh), one can still discuss about the $L^2$ estimates for “extendible” holomorphic sections, i.e. sections in $\mathcal{H}_{\sigma_{\text{mlc}} + 1}/\mathcal{H}_1$. The strategy to obtain an $L^2$ estimate for some holomorphic extension of $f \in \mathcal{H}_{\sigma_{\text{mlc}} + 1}/\mathcal{H}_1$ is to consider the orthogonal decomposition

$$\mathcal{H}_{\sigma + 1} = \mathcal{H}_\sigma \oplus \mathcal{E}_\sigma.$$

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1 Strictly speaking, since $\varphi_L$ is allowed to have poles along any lc centres of $(X, S)$ and thus sections of $\mathcal{I}(\varphi_L)$ can vanish along those centres, the precise description should be “the space of holomorphic sections of $K_X \otimes L \otimes \mathcal{I}(\varphi_L)$ with an extra vanishing order along $\mathcal{I}_{\mathcal{I}_{X}(S)}$.”

2 Notice that the inclusions need not be strict in general. Moreover, the equality $\mathcal{I}(\varphi_L) \cdot \mathcal{I}_S = \mathcal{I}(\varphi_L + \psi)$ is due to the assumption that $\mathcal{I}(\varphi_L + m\psi)$ for $m \in [0, 1]$ “jumps” along $S$ exactly when $m = 1.$
with respect to certain $L^2$ norm for each $\sigma = 1, 2, \ldots, \sigma_{\text{mlc}}$ and to obtain a minimal element $F_\sigma \in \mathcal{E}_\sigma$ for each $\sigma$ such that each $F_\sigma$ comes with an $L^2$ estimate and

$$
\sum_{\sigma=1}^{\sigma_{\text{mlc}}} F_\sigma \equiv f \mod \mathcal{I}(\varphi_L + \psi) \text{ on } X
$$

(see Notation 1.12).

The goal is to express the $L^2$ estimate on $F_\sigma$ in terms of $\sigma$-lc-measure defined in [5]. The definition of $\sigma$-lc-measure is recalled in Definition 1.14 for reader’s convenience. Instead of considering a single norm on all subspaces $\mathcal{H}_{\sigma+1}$, the following norms are introduced to each of the subspaces.

**Definition 1.1** Fix any number $\ell > 0$ such that $|\ell \psi| > 1$ on $X$. For any $\varepsilon > 0$ and any smooth $L \otimes \overline{L}$-valued $(n, n)$-form $G$ on $X$, set

$$
\delta^G_{(\varepsilon)_\sigma} := \delta^G_{(\varepsilon)} := \varepsilon \int_X \frac{G e^{-\varphi_L - \psi}}{|\psi|^\sigma (\log|\ell \psi|)^{1+\varepsilon}}.
$$

It is denoted by $\delta(f^2)$ when $G = |f|^2$ for some $f \in \mathcal{H}_{\sigma_{\text{mlc}}+1}$ and $\delta_{(\varepsilon)_\sigma}^{(f, \psi)}$ when $G = c_n f \wedge \overline{g}$ (see Notation 1.9) for some $f, g \in \mathcal{H}_{\sigma_{\text{mlc}}+1}$. (Here, $\delta_{(\varepsilon)_\sigma}^G$, $\delta_{(\varepsilon)_\sigma}^{f^2}$ or $\delta_{(\varepsilon)_\sigma}^{(f, \psi)}$ may diverge if they come without further restrictions.) For convenience,

the value $\delta_{(0)_\sigma}^{f^2} := \lim_{\varepsilon \to 0^+} \delta_{(\varepsilon)_\sigma}^{f^2}$ and the function $\varepsilon \mapsto \delta_{(\varepsilon)_\sigma}^{f^2}$

are respectively named the residue (squared) norm and the residue function of $f$ for the lc centres of $(X, S)$ of codimension $\sigma$ (the naming can be justified by Theorem 1.2).

It turns out that one has the following Theorem 1.2, which is proved in this paper.

**Theorem 1.2** (ref. Prop. 2.1, Thm. 2.3 and Cor. 2.5) For any $f \in \mathcal{H}_{\sigma+1}$,

- the integral $\delta_{(\varepsilon)_\sigma}^{f^2}$ is convergent for any $\varepsilon > 0$,
- the function $\varepsilon \mapsto \delta_{(\varepsilon)_\sigma}^{f^2}$ can be analytically continued to an entire function,\(^3\) and
- $\delta_{(0)_\sigma}^{f^2} = \int_{\mathcal{L}C} \delta_{(0)_\sigma}^{f^2} d\text{lc}\sigma_{\varphi_L, \varphi_L} [\psi]$, the squared norm of $f$ on $\text{lc}_X(S)$ with respect to the $\sigma$-lc-measure $d\text{lc}\sigma_{\varphi_L, \varphi_L} [\psi]$, and its value is independent of the normalisation of $|\ell \psi|$.

One can then see that $\mathcal{H}_\sigma = \left\{ f \in \mathcal{H}_{\sigma+1} \bigg| \delta_{(0)_\sigma}^{f^2} = 0 \right\}$ by the computation of $\sigma$-lc-measure in [5, Prop. 2.2.1 and Remark 2.2.3].

Now, equipped $\mathcal{H}_{\sigma+1}$ with the squared $L^2$ norm $\delta_{(1)_\sigma}^{f^2}$ such that $\mathcal{H}_{\sigma+1} = \mathcal{H}_\sigma \oplus \mathcal{E}_\sigma$ is the orthogonal decomposition with respect to it. One then has the following conjecture which is the goal of the current research.

**Conjecture 1.3** There exists a constant $b \geq 1$ which is independent of $\varphi_L$ and $\psi$ given in Sect. 1.3 such that the following holds true. Given the normalisation $\log|\ell \psi| \geq b$ on $X$ (by adding suitable constant to $\psi$ and/or varying $\ell > 0$ suitably), suppose that $\mathcal{H}_{\sigma+1} = \mathcal{H}_\sigma \oplus \mathcal{E}_\sigma$

\(^3\) Analytic continuation of the residue function $\varepsilon \mapsto \delta_{(\varepsilon)_\sigma}^{f^2}$ across 0 is suggested already by the study of residue currents in [2], [9] and [1]. See [5, §1.4] for a discussion.
is the orthogonal decomposition with respect to $\mathfrak{F}(1)_\sigma$. For any $f \in \mathcal{H}\sigma_{\text{mlc}}+1/\mathcal{H}_1$, one can find $F_\sigma \in \mathcal{E}_\sigma$ for $\sigma = 1, \ldots, \sigma_{\text{mlc}}$ inductively (starting from $\sigma = \sigma_{\text{mlc}}$) such that, for every $\sigma = 1, \ldots, \sigma_{\text{mlc}},$

$$\sum_{j=\sigma}^{\sigma_{\text{mlc}}} F_j \equiv f \mod \mathcal{J}(\varphi_L) \cdot \mathcal{T}_{Ic^\sigma(S)} \text{ on } X^4$$

and

$$|F| \leq \mathfrak{F}(1) \sigma \leq \mathfrak{F}(0) \sigma = \mathfrak{F}(0) \sigma. \quad \text{(since } \sum_{\sigma=1}^{\sigma_{\text{mlc}}-1} F_\sigma \in \mathcal{H}_{\sigma_{\text{mlc}}} = (\mathcal{E}_{\sigma_{\text{mlc}}})^+)$$

Consequently, $F := \sum_{\sigma=1}^{\sigma_{\text{mlc}}} F_\sigma$ is an holomorphic extension of $f$ which, if further assume that $|\psi| \geq 1$ on $X$, comes with the estimate

$$\mathfrak{F}(1) \sigma \leq \mathfrak{F}(0) \sigma = \mathfrak{F}(0) \sigma$$

under the normalisation $\log|\psi| \geq b \geq 1$.

**Remark 1.4** The result in [5, Thm. 1.4.5] states in particular that, under the setting in Conjecture 1.3, any $f \in \mathcal{H}_2/\mathcal{H}_1$ has a holomorphic extension $F$ satisfying the estimate

$$\int_X \frac{|F|^2 e^{-\varphi_L-\psi}}{|\psi|((\log|\psi|)^2 + 1)} \leq \mathfrak{F}(0)$$

for some suitably normalised $\log|\psi|$. Notice that, given $\log|\psi| > 0$ on $X$, one has

$$\frac{1}{(\log|\ell \psi|)^2} \leq \frac{1}{(\log|\psi|)^2 + 1}. \quad \text{for some } \ell \geq 0.$$ 

Therefore, [5, Thm. 1.4.5] indeed implies that Conjecture 1.3 is true when $\sigma_{\text{mlc}} = 1$.

Although the above conjecture is not proved yet in this paper, one can prove relatively easily the following “non-universal” estimate.

**Theorem 1.5** (ref. Theorem 3.1) On a compact Kähler manifold $X$ with $\varphi_L$ and $\psi$ given as above, there exists a constant $C := C(X, \varphi_L + \psi) > 0$ such that, when $\psi$ and $\ell > 0$ are chosen to satisfy the normalisation $\log|\ell \psi| \geq C$ (by varying $\ell > 0$ or adding suitable constant to $\psi$) while $\varphi_L$ is adjusted accordingly so that $\varphi_L + \psi$ is kept unchanged, the estimate

$$\mathfrak{F}(1) \sigma \leq \mathfrak{F}(0) \sigma$$

holds for all $F \in \mathcal{E}_\sigma$. 

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4 Here, $f$ is abused to mean its image under the map $\mathcal{H}_{\sigma_{\text{mlc}}+1}/\mathcal{H}_1 \rightarrow \mathcal{H}_{\sigma_{\text{mlc}}+1}/\mathcal{H}_\sigma$. 

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Note that the proof of the above “non-universal” estimate does not make use of the psh assumption on $\varphi_L + (1 + \beta) \psi$. It indeed follows from the compactness of $X$ and the invariant property of the residue norm $\tilde{\mathcal{F}}(0)_{\sigma}$.

Now it makes sense to talk of the “minimal normalisation” on $\log |e^\ell\psi|$ with respect to $(\varphi_L, \psi)$, i.e. there is a minimal normalising constant $C_{\min} = C_{\min}(\varphi_L, \psi) > 0$ such that under the normalisation $\log |e^\ell\psi| \geq C_{\min}$ on $X$, the inequality $\tilde{\mathcal{F}}(0)_{\sigma} \leq \tilde{\mathcal{F}}(1)_{\sigma}$ holds true for all $F \in \mathcal{E}_\sigma$. Conjecture 1.3 claims that the constant $b := \sup C_{\min}(\varphi_L, \psi)$, where the supremum is taken over all possible $\varphi_L$ and $\psi$ with suitable curvature assumption on $X$, should be finite.

The claim on the normalising constant $b$ being no less than 1 in the conjecture is verified in the explicit examples with $X = \mathbb{P}^3$ and $K_{\mathbb{P}^3} \otimes L$ isomorphic to $\mathcal{O}$ or $\mathcal{O}(1)$ given in Sect. 3.2. They indeed satisfy the stronger inequality $\tilde{\mathcal{F}}(\varepsilon)_{\sigma} \leq \tilde{\mathcal{F}}(0)_{\sigma}$ for all $\varepsilon \geq 0$ for every $F \in \mathcal{E}_\sigma$, under the normalisation $\log |e^\ell\psi| \geq 1$. Example 3.4 even provides an instance that the $L^2$ estimates hold true even when the usual curvature assumption, i.e. $\varphi_L + (1 + \beta) \psi$ being psh for $\beta \in [0, \delta]$ with $\delta > 0$, is not satisfied. The computation in the examples may hopefully provides hints, as well as difficulties, on proving Conjecture 1.3.

The rest of Sect. 1 provides the notation and the basic setup used in this article. Discussion on the properties of $\tilde{\mathcal{F}}(\varepsilon)_{\sigma}$ starts from Sect. 2. Comparison of the (squared) norm $\tilde{\mathcal{F}}(\varepsilon)_{\sigma}$ with the sup-norm and $L^2$ norms with smooth weights for $p > 1$ is also provided in addition to the proof of Theorem 1.2. Section 3 provides the proof of the “non-universal” estimates in Theorem 1.5 and detailed computations on the particular examples on $\mathbb{P}^3$.

### 1.2 Notation

In this paper, the following notations are used throughout.

**Notation 1.6** Set $i := \frac{\sqrt{-1}}{2\pi}$.

**Notation 1.7** Each potential $\varphi$ (of the curvature of a metric) on a holomorphic line bundle $L$ in the following represents a collection of local functions $\{\varphi_V\}_V$ with respect to some fixed local coordinates and trivialisation of $L$ on each open set $V$ in a fixed open cover $\{V\}_V$ of $X$. The functions are related by the rule $\varphi_V = \varphi_{V'} + 2 \text{Re } h_{VV'}$ on $V \cap V'$, where $e^{h_{VV'}}$ is a (holomorphic) transition function of $L$ on $V \cap V'$ (such that $s_V = s_{V'} e^{h_{VV'}}$, where $s_V$ and $s_{V'}$ are the local representatives of a section $s$ of $L$ under the trivialisations on $V$ and $V'$ respectively). Inequalities between potentials is meant to be the inequalities under the chosen trivialisations over open sets in the fixed open cover $\{V\}_V$.

**Notation 1.8** For any prime (Cartier) divisor $E$, let

- $\varphi_E := \log |s_E|^2$, representing the collection $\{\log |s_{E,V}|^2\}_V$, denote a potential (of the curvature of the metric) on the line bundle associated to $E$ given by the collection of

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5 The notation is chosen by mimicking the reduced Planck constant $\hbar = \frac{\hbar}{2\pi}$. It is typeset with the code \raisebox{-0.9ex}{\textbackslash{}mathchar’26$\textbackslash{}mkern=-6.7mu i$}. Springer
local representations \( \{ s_{E,Y} \} \) of some canonical section \( s_E \) (thus \( \Phi_E \) is uniquely defined up to an additive constant);
- \( \Phi^m_E \) denote a smooth potential on the line bundle associated to \( E \);
- \( \Psi_E := \Phi_E - \Phi^m_E \), which is a global function on \( X \), when both \( \Phi_E \) and \( \Phi^m_E \) are fixed.

All the above definitions are extended to any \( \mathbb{R} \)-divisor \( E \) by linearity. For notational convenience, the notations for a \( \mathbb{R} \)-divisor and its associated \( \mathbb{R} \)-line bundle are used interchangeably.

**Notation 1.9** For any \( (n, 0) \)-form (or \( K_X \)-valued section) \( f \), define \( |f|^2 := c_n f \wedge \overline{f} \), where \( c_n := (-1)^{n(n-1)/2} (\pi i)^n \). For any Kähler metric \( \omega = \pi i \sum_{1 \leq j, k \leq n} h^j_k dz^j \wedge d \overline{z}^k \) on \( X \), set \( d \text{ vol}_{X, \omega} := \omega^n/n! \). Set also \( |f|^2 \omega \text{ vol}_{X, \omega} = |f|^2 \).

**Notation 1.10** For any two non-negative functions \( u \) and \( v \), write \( u \lesssim v \) (equivalently, \( v \gtrsim u \)) to mean that there exists some constant \( C > 0 \) such that \( u \leq C v \), and \( u \sim v \) to mean that both \( u \lesssim v \) and \( u \gtrsim v \) hold true. For any functions \( \eta \) and \( \phi \), write \( \eta \lesssim \log \phi \) if \( e^\eta \lesssim e^\phi \).

Define \( \sim \log \) and \( \lesssim \log \) accordingly.

1.3 Basic setup

The same setup as in [5, §1.3] is considered in this paper.

Let \( (X, \omega) \) be a compact Kähler manifold of complex dimension \( n \), and let \( \mathcal{I}(\phi) := \mathcal{I}_X(\phi) \) be the multiplier ideal sheaf of the potential \( \phi \) on \( X \) given at each \( x \in X \) by

\[
\mathcal{I}(\phi)_x := \left\{ f \in \mathcal{O}_X \mid f \text{ is defined on a coord. neighbourhood } V_x \ni x \text{ and } \int_{V_x} |f|^2 e^{-\psi} d\lambda_{V_x} < +\infty \right\},
\]

where \( d\lambda_{V_x} \) is the Lebesgue measure on \( V_x \). Throughout this paper, the following are assumed on \( X \):

1. \( (L, e^{-\psi_L}) \) is a hermitian line bundle with an analytically singular metric \( e^{-\psi_L} \), where \( \psi_L \) is locally equal to \( \psi_1 - \psi_2 \), where each of the \( \psi_i \)'s is a quasi-psh local function with neat analytic singularities, i.e. locally

\[
\psi_i = c_i \log \left( \sum_{j=1}^N |g_{ij}|^2 \right) \mod \mathcal{O}_X^\infty,
\]

where \( c_i \in \mathbb{R}_{\geq 0} \) and \( g_{ij} \in \mathcal{O}_X \);

2. \( \psi \) is a global function on \( X \) such that it can also be expressed locally as a difference of two quasi-psh functions with neat analytic singularities;

3. \( \psi < 0 \) on \( X \) (which implies that \( \psi \) is quasi-psh after some blow-ups as it has only neat analytic singularities);

4. \( \psi_L + (1 + \beta) \psi \) is a plurisubharmonic (psh) potential for all \( \beta \in [0, \delta] \) for some \( \delta > 0 \);

5. 1 is a jumping number of the family \( \{ \mathcal{I}(\psi_L + m\psi) \}_{m \in \mathbb{R}_{\geq 0}} \) such that

\[
\mathcal{I}(\psi_L) = \mathcal{I}(\psi_L + m\psi) \supset \mathcal{I}(\psi_L + \psi) \quad \text{for all } m \in [0, 1)
\]

(the jumping numbers exist on compact \( X \) by the openness property of multiplier ideal sheaves as \( \psi \) is quasi-psh after suitable blow-ups);
(6) \( S \subset \psi^{-1}(-\infty) \) is a reduced subvariety defined by the annihilator
\[
\mathcal{I}_S := \text{Ann}_{\mathcal{O}_X} \left( \frac{\mathcal{I}(\varphi_L)}{\mathcal{I}(\varphi_L + \psi)} \right)
\]
(see [6, Lemma 4.2] for the proof that \( \mathcal{I}_S \) is reduced).

When it helps in the computation, one can make the following assumption.

**Snc assumption 1.11** (See [5, §2.1] for details) By considering a suitable log-resolution of \((X, \varphi_L, \psi)\), one can assume that

- \( S \) is a reduced divisor, and
- the polar ideal sheaves of \( \varphi_L \) and \( \psi \) respectively are principal and the corresponding divisors have only simple normal crossings (snc) with each other.

Under such assumption, one may define \( \tilde{\varphi}_L \) by
\[
\tilde{\varphi}_L + \psi_S := \varphi_L + \psi,
\]
where \( \psi_S := \varphi_S - \varphi_S^{\text{sm}} < 0 \) (see Notation 1.8), for convenience.

**Notation 1.12** Given a set \( V \subset X \), a section \( f \) of \( \tilde{\mathcal{I}}(\varphi_L) \) on \( V \) (which is supported in \( S \cap V \)), and a section \( F \) of \( \tilde{\mathcal{I}}(\varphi_L) \) on \( V \), the notation
\[
F \equiv f \mod \tilde{\mathcal{I}}(\varphi_L + \psi) \quad \text{on} \quad V
\]
is set to mean that, for all \( x \in V \), if \( (F)_x \) and \( (f)_x \) denote the germs of \( F \) and \( f \) at \( x \) respectively, one has
\[
((F)_x \mod \tilde{\mathcal{I}}(\varphi_L + \psi)_x) = (f)_x.
\]

If such a relation between \( F \) and \( f \) holds, \( F \) is said to be an extension of \( f \) on \( V \). If the set \( V \) is not specified, it is assumed to be the whole space \( X \). Such notation is also applied to cases with a slight variation of the sheaf \( \tilde{\mathcal{I}}(\varphi_L + \psi) \) (for example, with \( \tilde{\mathcal{I}}(\varphi_L + \psi) \) replaced by \( \mathcal{C}_X^\infty \otimes \tilde{\mathcal{I}}(\varphi_L + \psi) \)).

**Definition 1.13** If \( S \) is a reduced divisor with snc on \( X \), an lc centre of \((X, S)\) of codimension \( \sigma \) in \( X \) is an irreducible component of any intersections of \( \sigma \) irreducible components of \( S \) in \( X \) (see [8, Def. 4.15] for the general definition of lc centres when \( S \) is a divisor). Define \( \text{lc}_X^\sigma(S) \) to be the union of all lc centres of \((X, S)\) of codimension \( \sigma \) in \( X \). For a general reduced subvariety \( S \) (which may not even be a divisor) in \( X \) defined above, define \( \text{lc}_X^\sigma(S) \) (or, more precisely, \( \text{lc}_X^\sigma(\varphi_L, \psi) \)) as
\[
\text{lc}_X^\sigma(S) := \pi \left( \text{lc}_{\tilde{X}}^\sigma(\tilde{S}) \right),
\]
where \( \pi : \tilde{X} \to X \) is a log-resolution of \((X, \varphi_L, \psi)\) and \( \tilde{S} \) is the reduced divisor with snc described in [5, §2.1] (which satisfies \( \pi(\tilde{S}) = S \)). Moreover, an lc centre of \((X, S)\) (or, more precisely, lc centre of \((X, \frac{\mathcal{I}(\varphi_L)}{\mathcal{I}(\varphi_L + \psi)}\)) or \((X, \varphi_L, \psi)\)) of codimension \( \sigma \) is meant to be the image under \( \pi \) of an lc centre of \((\tilde{X}, \tilde{S})\) of codimension \( \sigma \) in \( \tilde{X} \).

**Definition 1.14** The lc-measure supported on the lc centres of \((X, S)\) of codimension \( \sigma \) (or \( \sigma \)-lc-measure for short) with respect to \( f \in \mathcal{H}_{\sigma+1}/\mathcal{H}_1 \), denoted as \( |f|_{\omega}^2 \text{lc}_\omega^\sigma(\varphi_L, \psi) \), is defined by
\[
\mathcal{C}_0^\infty(\varphi_L) \ni g \mapsto \int_{\text{lc}_X^\sigma(S)} g|f|_{\omega}^2 \text{lc}_\omega^\sigma(\varphi_L, \psi) := \lim_{\varepsilon \to 0^+} \varepsilon \int_{\tilde{X}} \tilde{g} |f|^2 e^{-\varphi_L - \psi \frac{\sigma + \varepsilon}{|\psi|}}.
\]
where
• \( \tilde{f} \) is a smooth extension of \( f \) to an \( L \)-valued \((n, 0)\)-form on \( X \) such that \( \tilde{f} \in \mathcal{C}^\infty \otimes \mathcal{I}(\phi^L) \otimes \mathcal{I}(\psi^L) \);
• \( \tilde{g} \) is any smooth extension of \( g \) to a function on \( X \).

### 1.4 Technical preparation

Here are a specific open cover of \( X \) and an inequality which are referred to frequently in the proofs.

#### 1.4.1 Admissible open covers

Under the snc assumption 1.11 on \( \phi^L \) and \( \psi^L \), let \( \{V_\gamma\}_{\gamma \in I} \) and \( \{V'_\gamma\}_{\gamma \in I} \) be finite open covers of \( X \) such that

- each \( V'_\gamma \) lies in some (fixed) coordinate chart of \( X \) on which \( L \) is trivialised;
- \( V_\gamma \subset V'_\gamma \subset X \) for each \( \gamma \in I \) where \( V_\gamma = \Delta^n(0; 1) \) and \( V'_\gamma = \Delta^n(0; 2) \) are concentric polydiscs centred at the origin with polyradii 1 and 2 respectively in the coordinate system on \( V'_\gamma \);
- if the polar sets \( P_{\phi^L} \) and \( P_{\psi^L} \) of respectively \( \phi^L \) and \( \psi^L \) have non-empty intersection with \( V'_\gamma \), each irreducible component of \( (P_{\phi^L} \cup P_{\psi^L}) \cap V'_\gamma \) must lie inside a coordinate plane and pass through the origin.

#### 1.4.2 \( x \log x \)-inequality

It can be shown via calculus that

\[
x^\varepsilon |\log x|^s \leq \frac{s^s}{e^s e^{\frac{s}{\varepsilon}}} \tag{2}
\]

for all \( x \in [0, 1], \varepsilon > 0 \) and \( s \geq 0 \) (where \( 0^0 \) is treated as 1). Indeed, the function \( x \mapsto x^\varepsilon |\log x|^s \) on \([0, 1]\) has its unique maximum at \( x = e^{-\frac{s}{\varepsilon}} \).

### 2 Properties of the residue function \( \mathfrak{R} \)

In this section, the number \( \sigma \) is fixed and \( \mathfrak{R}_{(\varepsilon)}^{1/2} \) is written as \( \tilde{\mathfrak{R}}_{(\varepsilon)}^{1/2} \). Moreover, the snc assumption 1.11 is assumed by passing to a log-resolution of \((X, \phi^L, \psi)\) if necessary.

#### 2.1 Comparison with sup-norm and \( L^2 \) norm with smooth weight

Under the snc assumption 1.11, let \( V_\gamma \subset V'_\gamma \) be members of the admissible open covers of \( X \) given in Sect. 1.4.1. Recall that \( \phi^L + \psi \) is psh and thus locally bounded from above. Pick any smooth potential \( \phi_{sm}^L \) on \( L \) and normalise it (by adding a suitable constant) such that \( \phi^L - \phi_{sm}^L + \psi \leq 0 \) on \( X \). Then, for any \( \varepsilon > 0 \), Cauchy’s integral formula for holomorphic functions infers that
\[
\sup_{V_{\gamma}} |f|^2 \lesssim \left( \int_{V_{\gamma}'} |f| e^{-\frac{1}{2} \phi_{L_{\gamma}}^m} \right)^2 \leq \int_{V_{\gamma}'} \frac{|f|^2 e^{-\phi_{L_{\gamma}}^m}}{|\psi|^\sigma (\log |\ell \psi|)^{1+\varepsilon}} \cdot \int_{V_{\gamma}'} |\psi|^\sigma (\log |\ell \psi|)^{1+\varepsilon} \, d\text{vol}_{X,\omega} \\
\leq \frac{1}{\varepsilon} \|f\|_\gamma^2 \int_{V_{\gamma}'} |\psi|^\sigma (\log |\ell \psi|)^{1+\varepsilon} \, d\text{vol}_{X,\omega},
\]

where the constant involved in \( \lesssim \) depends only on \( \phi_{L_{\gamma}}^m, V_{\gamma} \) and \( V_{\gamma}' \). Notice that the last integral on the right-hand side is convergent for any \( V_{\gamma}' \) since \( \psi \) has at worst logarithmic poles along coordinate planes. This shows that convergence in \( \tilde{\mathcal{F}}_{(\varepsilon)} \) for any \( \varepsilon > 0 \) implies locally uniform convergence.

Similarly, the squared norm \( \tilde{\mathcal{F}}_{(\varepsilon)} \) for \( \varepsilon > 0 \) can also be compared with \( L_2^p \) norms for \( p > 1 \), under the psh assumption on (or, more precisely, the local-upper-boundedness of) \( \phi_{L_{\gamma}} + \psi \). By invoking Hölder’s inequality, one obtains, for any \( f \in \mathcal{H}_{\sigma+1} \), any \( \varepsilon > 0 \) and any number \( p > 1 \),

\[
\int_X |f|^2 e^{-\frac{1}{p} \phi_{L_{\gamma}}^m} \leq \left( \int_X |f|^2 e^{-\phi_{L_{\gamma}}^m} \right)^\frac{1}{p} \cdot \left( \int_X |\psi|^{\frac{q}{p}} (\log |\ell \psi|)^{1+\varepsilon} \, d\text{vol}_{X,\omega} \right)^\frac{1}{q},
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \), and the last integral on the right-hand side is convergent for the same reason as before together with the assumption that \( X \) being compact. Therefore, convergence in \( \tilde{\mathcal{F}}_{(\varepsilon)} \) for any \( \varepsilon > 0 \) implies convergence in \( L_2^p \) norm for \( p > 1 \).

If one insists in comparing \( \tilde{\mathcal{F}}_{(\varepsilon)} \) with an \( L^2 \) norm with smooth weight (with a somewhat more controllable constant in the estimate instead of the one in the comparison with the sup-norm), an extra assumption, namely, \( \phi_{L_{\gamma}} \) (or at least \( \phi_{L_{\gamma}} + \alpha \psi \) for some \( \alpha \in (0, 1) \)) being locally bounded from above, is needed. In that case, since

\[
e^{\psi} |\psi|^\sigma (\log |\ell \psi|)^{1+\varepsilon} = e^{-|\psi|} |\psi|^\sigma + 1 \cdot \frac{\ell}{|\ell \psi|} (\log |\ell \psi|)^{1+\varepsilon} \leq \left( \frac{\sigma + 1}{e} \right)^{\sigma + 1} \left( \frac{1 + \varepsilon}{e} \right)^{1+\varepsilon} =: C,
\]

it follows that, with the normalisation \( \phi_{L_{\gamma}} = \phi_{L_{\gamma}}^m \leq 0 \) on \( X \), one has

\[
\int_X |f|^2 e^{-\phi_{L_{\gamma}}^m} \leq \int_X |f|^2 e^{-\phi_{L_{\gamma}}} \leq C \int_X \frac{|f|^2 e^{-\phi_{L_{\gamma}} - \psi}}{|\psi|^\sigma (\log |\ell \psi|)^{1+\varepsilon}} = \frac{C}{\varepsilon} \int_X |f|^2 \, d\text{vol}_{X,\omega}.
\]

This is how it is done in [7] to obtain [7, Thm. 4.1, eq. (24)], which requires the extra assumption on \( \phi_{L_{\gamma}} \). In [5, §4] (mainly [5, Lemma 4.4.1]), the extra assumption on \( \phi_{L_{\gamma}} \) is removed using an argument essentially the same as the comparison between \( \tilde{\mathcal{F}}_{(\varepsilon)} \) and an \( L^1 \) norm given above.

### 2.2 Identity of \( \tilde{\mathcal{F}} \) via integration by parts

Recall that, under the snc assumption 1.11, the potential \( \tilde{\phi}_{L_{\gamma}} \) is defined by

\[
\tilde{\phi}_{L_{\gamma}} + \psi_S := \phi_{L_{\gamma}} + \psi,
\]
where \( \psi_S := \phi_S - \varphi_S^{\mathrm{sm}} < 0 \) (see Notation 1.8 for the meaning of \( \phi_S \) and \( \varphi_S^{\mathrm{sm}} \)).

Let \( \{ \rho_{V_y} \}_{y \in I} \) be a smooth partition of unity subordinated to the admissible open cover \( \{ V_y \}_{y \in I} \) defined in Sect. 1.4.1. On an open set \( V_y \) with \( S \cap V_y \neq \emptyset \), let \( z_1, \ldots, z_n \) be the holomorphic coordinates such that

\[
S \cap V_y = \left\{ z_1 \cdots z_{j_S} = 0 \right\} \quad \text{and} \quad \phi_S = \sum_{j=1}^{j_S} \log |z_j|^2
\]

for some integer \( j_S \in [1, n] \). Then, it follows from the snc assumption that \( \psi \) and \( \tilde{\varphi}_L \) can be written on \( V_y \) as

\[
\psi|_{V_y} = \sum_{j=1}^{n} v_j \log |z_j|^2 + \alpha \quad \text{and} \quad \tilde{\varphi}_L|_{V_y} = \sum_{j=1}^{n} c_j \log |z_j|^2 + \beta,
\]

where \( v_j \)'s and \( c_j \)'s are non-negative numbers and \( \alpha \) and \( \beta \) are smooth functions on \( V_y \). By passing to a refinement of the covering \( \{ V_y \}_{y \in I} \) if necessary, assume without loss of generality that \( \sup_{V_y} \frac{r_j}{2v_j} \frac{\partial}{\partial r_j} \alpha > -1 \) for \( j = 1, \ldots, j_S \), where \( r_j \)'s are the radial components of the polar coordinate \( (r_j, \theta_j) \) such that \( z_j = r_j e^{\sqrt{-1} \theta_j} \).

For any integer \( \sigma' \in [1, j_S] \), one has

\[
\text{lc}'^\sigma_{X}(S) \cap V_y = \bigcup_{p \in \mathcal{E}_{\sigma}/(\mathcal{E}_{\sigma'} \times \mathcal{E}_{\tau})} \left\{ z_p(1) = z_p(2) = \cdots = z_p(\sigma') = 0 \right\} =: \bigcup_{p} \text{lc}'^\sigma_{X}(S)^{p}_{\sigma'}
\]

where \( p \) is a choice of \( \sigma' \) elements from the set \( \{ 1, 2, \ldots, j_S \} \) and is abused to mean a corresponding permutation. Using the above notations, for any \( f \in \mathcal{H}_{\sigma+1} \), after the cancellation between the poles of \( e^{-\tilde{\varphi}_L} \) with the zeros of \( |f|^2 \), one obtains

\[
\rho_{V_y} |f|^2 e^{-\tilde{\varphi}_L} = \rho_{V_y} |\tilde{f}|^2 e^{-\beta} \sum_{j=1}^{j_S} (\pi i \, d z_j \wedge d \bar{z}_j) \wedge \prod_{k=j_{S}+1}^{n} \left( \frac{\pi i \, d z_k \wedge d \bar{z}_k}{|z_k|^{2\ell_k}} \right)
\]

for some holomorphic function \( \tilde{f} \in \mathcal{I}_{\text{lc}'^\sigma_{X}(S)} \) on \( V_y \) and, furthermore,

\[
\rho_{V_y} |f|^2 e^{-\tilde{\varphi}_L - \psi_S} = \sum_{p, p' \in \mathcal{E}_{\sigma}/(\mathcal{E}_{\sigma'} \times \mathcal{E}_{\tau})} F_{p, p'} := \rho_{V_y} \tilde{f}_p \tilde{f}_{p'} e^{-\beta + \psi_S^{\mathrm{sm}}} \prod_{j=1}^{j_S} (\pi i \, d z_j \wedge d \bar{z}_j) \wedge \prod_{k=j_{S}+1}^{n} \left( \frac{\pi i \, d z_k \wedge d \bar{z}_k}{|z_k|^{2\ell_k}} \right)
\]

where \( \sigma' \) is either \( \sigma \) when \( \text{lc}'^\sigma_{X}(S) \cap V_y \neq \emptyset \) or \( j_S \) when \( \text{lc}'^\sigma_{X}(S) \cap V_y = \emptyset \) (thus \( j_S < \sigma \)), and \( \text{lc}'_{X}(S) \cap V_y \neq \emptyset \) but \( \text{lc}'_{X+1}(S) \cap V_y = \emptyset \), and \( \tilde{f}_p \)'s and \( \tilde{f}_{p'} \)'s are some holomorphic functions on \( V_y \) which thus infer that \( F_{p, p'} \)'s are smooth. The numbers \( \ell_k \) are \( < 1 \) and can possibly be zero or negative. This shows explicitly the fact that \( \rho_{V_y} |f|^2 e^{-\tilde{\varphi}_L - \psi_S} \) has lc singularities along (some of) the irreducible components of \( S \cap V_y \), has at worst klt singularities along other coordinate planes and is smooth elsewhere.

Let \( \text{sym}^s(x_1, x_2, \ldots, x_s) \) be the elementary symmetric polynomial of degree \( s \) in \( \sigma \) variables. Define

\[
\text{sym}^s_{\sigma-1} := \text{sym}^s \left( 1, \frac{1}{2}, \ldots, \frac{1}{\sigma-1} \right)
\]

for convenience.
Proposition 2.1 (Theorem 1.2) For any \( \varepsilon > 0 \), any \( f \in H_\sigma + 1 \) and on any \( V_\gamma \) such that \( \text{lc}_X(S) \cap V_\gamma \neq \emptyset \), the integral \( \mathfrak{S}_\varepsilon f^2 \) converges and one has

\[
\mathfrak{S}_\varepsilon f^2 + \varepsilon \sum_{s=1}^{\sigma-1} \prod_{k=1}^{s-1} (k + \varepsilon) \cdot \text{sym}_{s-1} \cdot \mathfrak{S}_\varepsilon (s + \varepsilon) = \frac{(-1)^\sigma}{(\sigma - 1)!} \int_X \frac{G_\sigma}{(\log|\ell \psi|)^\varepsilon},
\]

where \( G_\sigma \) is an \((n, n)\)-form with at worst klt singularities along the coordinate planes in \( V_\gamma \) and being smooth elsewhere. The coefficients of \( G_\sigma \) contain derivatives of \( \rho_\varepsilon f^2 \) (of order at most \( \sigma \)) in the normal directions of the irreducible components of \( S \). Notice also that the usual conventions \( \sum_{s=1}^0 = 0 \) and \( \prod_{k=1}^0 = 1 \) are used here.

Proof By the linearity of integrals, it suffices to prove the statement for each summand in the decomposition of \( \mathfrak{S}_\varepsilon f^2 \) according to (4). Write \( \mathfrak{J}_{\varepsilon} F_{p, p'} \) as the summand containing \( F_{p, p'} \) in the decomposition of \( \mathfrak{S}_\varepsilon f^2 \), and thus

\[
\mathfrak{S}_\varepsilon f^2 = \sum_{p, p'} \mathfrak{J}_{\varepsilon} F_{p, p'}.
\]

The main procedure in the proof is to get rid of the lc singularities by applying integration by parts.

First the summands \( \mathfrak{J}_{\varepsilon} F_{p, p'} \) (i.e. \( p = p' \)) are considered. It suffices to consider only the term with \( p = p' = \text{id} \), the identity permutation, and write \( F_0 := F_{\text{id}, \text{id}} \). Note that, when all variables but \( r_j \) are fixed (where \( z_j = r_j e^{\sqrt{-1} \theta_j} \), one has \( d|\psi| = -d\psi = -\nu_j (1 + \frac{r_j}{2v_j} \frac{\partial}{\partial r_j} \alpha) d\log r_j^2 \), in which the right-hand-side is nowhere zero on \( V_\gamma \) by assumption. Set

\[
F_j := \frac{\partial}{\partial r_j} \left( \frac{F_{j-1}}{1 + \frac{r_j}{2v_j} \frac{\partial}{\partial r_j} \alpha} \right)
\]

for \( j = 1, \ldots, \sigma \). Making the variables with only klt singularities implicit in view of Fubini’s theorem (i.e. variables \( z_{\sigma+1}, \ldots, z_n \) are hidden), the claim of this proposition is reduced to the (absolute) convergence of the integral \( \mathfrak{J}_{\varepsilon} F_0 \) and the equality

\[
\mathfrak{J}_{\varepsilon} F_0 + \varepsilon \sum_{s=1}^{\sigma-1} \prod_{k=1}^{s-1} (k + \varepsilon) \cdot \text{sym}_{s-1} \cdot \mathfrak{J}_{\varepsilon} (s + \varepsilon) = \frac{(-1)^\sigma}{(\sigma - 1)!} \int_{V_\gamma} \frac{F_\sigma}{(\log|\ell \psi|)^\varepsilon} \prod_{j=1}^{\sigma} \frac{dr_j d\theta_j}{2v_j}
\]

for any smooth function \( F_0 \), any integer \( \sigma \geq 1 \) and any real number \( \varepsilon > 0 \). This is proved via an induction on \( \sigma \) as follows.
Recall that $V_\gamma = \Delta^n(0; 1)$ in the coordinate system $(z_j)$. Then, when $\sigma > 1$,

$$\mathcal{J}_{(\varepsilon)}^{F_0} = \varepsilon \int \frac{F_0}{|\psi|^{\sigma} (\log|\ell\psi|)^{1+\varepsilon}} \prod_{j=1}^{\sigma} \frac{\pi i \, dz_j \wedge d\bar{z}_j}{|z_j|^2}$$

$$= \varepsilon \int_{V_\gamma} \frac{F_0}{|\psi|^{\sigma} (\log|\ell\psi|)^{1+\varepsilon}} \sum_{j=1}^{\sigma} d \log r_j^2 \cdot \prod_{j=1}^{\sigma} \frac{d\theta_j}{2}$$

$$= -\frac{\varepsilon}{v_1(\sigma - 1)} \int_{V_\gamma} \frac{F_0}{1 + \frac{r_1}{2v_1} \frac{\partial}{\partial r_1} \alpha} \frac{|\psi|^{\sigma} (\log|\ell\psi|)^{1+\varepsilon}}{\psi} \prod_{j=2}^{\sigma} d \log r_j^2$$

$$= -\frac{\varepsilon}{\sigma - 1} \mathcal{J}_{(\varepsilon)}^{F_0} - \frac{1}{v_1(\sigma - 1)} \mathcal{J}_{(\varepsilon)}^{F_1} \sigma - 1,$$

and when $\sigma = 1$,

$$\mathcal{J}_{(\varepsilon)}^{F_0} = -\frac{\varepsilon}{v_1} \int_{V_\gamma} \frac{F_0}{1 + \frac{r_1}{2v_1} \frac{\partial}{\partial r_1} \alpha} \frac{|\psi| (\log|\ell\psi|)^{1+\varepsilon}}{\psi} \frac{d|\psi|}{v_1} = \frac{1}{v_1} \int_{V_\gamma} \frac{F_0}{1 + \frac{r_1}{2v_1} \frac{\partial}{\partial r_1} \alpha} \frac{d}{(\log|\ell\psi|)^\varepsilon}$$

$$= -\int_{V_\gamma} \frac{F_1}{(\log|\ell\psi|)^\varepsilon} \frac{d\gamma}{v_1}.$$

Notice that the boundary terms from the integration by parts in both cases vanish, and the equalities hold for any $\varepsilon > 0$ and any smooth $F_0$, assuming convergence of the constituent integrals.

The claim on the convergence and the equality (6) is proved for $\sigma = 1$ as seen from (**), in which the integral on the right-hand-side is absolutely convergent as $F_1$ is smooth on a neighbourhood of $V_\gamma$ and $\frac{1}{(\log|\ell\psi|)^\varepsilon}$ is bounded from above.

For the case $\sigma > 1$, make the inductive assumption that $\mathcal{J}_{(\varepsilon)}^{F_1}$ converges and satisfies (6) (with $F_1$ in place of $F_0$ and $\sigma - 1$ in place of $\sigma$). The equality (*) can still be obtained from the integration by parts on $\mathcal{J}_{(\varepsilon)}^{F_1}$. Since both integrals $\mathcal{J}_{(\varepsilon)}^{F_0}$ and $\mathcal{J}_{(\varepsilon)}^{F_1}$ are $\geq 0$, both of them converge thanks to the finiteness of $\mathcal{J}_{(\varepsilon)}^{F_1}$. Now applying the inductive assumption on the equality (*) to $\mathcal{J}_{(\varepsilon)}^{F_1}$, one obtains
\[ J_{(\varepsilon)}(\sigma) = \begin{cases} 0 & \text{by (*)} \\
\frac{1}{v_1(\sigma - 1)} J_{(\varepsilon)}(\sigma - 1) - \frac{\varepsilon}{\sigma - 1} J_{(1 + \varepsilon)}(\sigma) \\
= -\varepsilon \sum_{s=1}^{\sigma-2} \prod_{k=1}^{s} (k + \varepsilon) \cdot \text{sym}_{s}^{s-2} \cdot \frac{(-1)^{s}}{v_1(\sigma - 1)} J_{(s + \varepsilon)}(\sigma - 1) \\
- \frac{1}{v_1(\sigma - 1)} \cdot \frac{(-1)^{s-1}}{(\sigma - 2)!} \int_{V_\gamma} \frac{F_\sigma}{(\log |\ell_\psi|)^{\varepsilon}} \prod_{j=1}^{\sigma} \frac{dr_j}{v_j} + \frac{\varepsilon}{\sigma - 1} J_{(1 + \varepsilon)}(\sigma) \\
\end{cases} \]

where the identities \( \text{sym}_{s}^{0} = \text{sym}_{s-2}^{s-2} \cdot \text{sym}_{s-2}^{0} \) for \( s = 1, \ldots, \sigma - 1 \) (with the convention \( \text{sym}_{s-2}^{0} = 1 \) and \( \text{sym}_{s-2}^{0} = 0 \)) are used in the last equality above. Note that \( \prod_{j=1}^{\sigma} \frac{\partial \psi_j}{\partial z} \) is again made implicit.

The claim (6) for \( F_0 = F_{p,p} \) is thus proved by induction.

For the summands \( J_{(\varepsilon)}(\sigma) \) with \( p \neq p' \), since the variables with only klt singularities can simply be ignored in view of Fubini’s theorem, the integral can be rewritten and handled like \( J_{(\varepsilon)}(\sigma) \) (i.e. \( p = p' \)). Indeed, if, for example, \( p(j) = p'(j) \) for \( j = 1, \ldots, \sigma - 1 \) and \( p(\sigma) \neq p'(\sigma) \), then the corresponding term with \( F_{p,p'} \) in (4) can be written as (noting that \( p'(\sigma) \in \{p(\sigma + 1), \ldots, p(j_s)\} \))

\[
F_{p,p'} = \frac{\prod_{j=1}^{j_s} (\pi i \cdot d z_j \wedge d \bar{z}_j)}{\prod_{j=1}^{j_s-1} |z_{p(j)}|^2 \cdot z_{p(\sigma)} z'_{p'(\sigma)}} \wedge \frac{\prod_{k=1}^{n} \pi i \cdot d z_k \wedge d \bar{z}_k}{|z_{p'(\sigma)}|^2 \cdot z_{p(j_s)} z'_{p'(j_s)}}
\]

\[
\frac{F_{p,p'}}{\text{smooth}} = \frac{\prod_{j=1}^{\sigma} (\pi i \cdot d z_{p(j)} \wedge d \bar{z}_{p(j)})}{|z_{p(j)}|^2} \wedge \frac{\prod_{k=1}^{n} \pi i \cdot d z_k \wedge d \bar{z}_k}{|z_{p'(\sigma)}|^2 \cdot z_{p(j_s)} z'_{p'(j_s)}},
\]

\[ \square \text{ Springer} \]
which can then be handled like the term with $F_{p,p}$. When convergence of the integral $\mathcal{J}_{\rho}^{F_{p,p}}$ or its derived integrals occurring in the inductive process of integration by parts is concerned, the smooth factor $\pi_{p}(\sigma)F_{p,p}'$ (or its derivatives resulted from (5) with $\pi_{p}(\sigma)F_{p,p}'$ in place of $F_{0}$) is replaced by its absolute value and the estimate $|\pi_{p}(\sigma)F_{p,p}'| \lesssim \rho_{\gamma}$ is considered.

The argument that proves the convergence of $\mathcal{J}_{\rho}^{F_{p,p}}$ also proves the convergence of $\mathcal{J}_{\rho_{\gamma}}^{F_{p,p}}$, hence the convergence of $\mathcal{J}_{\rho_{\gamma}}^{\mathcal{J}_{\rho}^{F_{p,p}}}$ in $\mathcal{J}_{\rho}^{\mathcal{J}_{\rho}^{F_{p,p}}}$, The formula of the form (6) for $\mathcal{J}_{\rho}^{\mathcal{J}_{\rho}^{F_{p,p}}}$ is then obtained via the inductive procedure described above. The other choices of $p$ and $p'$ can also be treated similarly.

Consequently, the proof of this proposition is completed.

**Remark 2.2** Suppose $V_{\gamma}$ is an open set such that $\text{lc}_{\rho_{\gamma}}^{\sigma}(S) \cap V_{\gamma} = \emptyset$ and $\sigma' = j_{S} < \sigma$ such that $\text{lc}_{\rho_{\gamma}}^{\sigma'}(S) \cap V_{\gamma} = \emptyset$ as in (4). Note that the sum $\sum_{p,p'}$ in (4) is reduced to a single term with $p = p' = \text{id}$ in this case. A similar argument as in the proof of Proposition 2.1 yields

\[
\mathcal{J}_{\rho_{\gamma}}^{\mathcal{J}_{\rho}^{\mathcal{J}_{\rho}^{F_{p,p}}}} = (-1)^{\sigma'} \frac{(\sigma - \sigma' - 1)!}{(\sigma - 1)!} \varepsilon \int_{X} \frac{G_{\sigma'}}{|\psi|^{\sigma-\sigma}(\log|\ell\psi|)^{1+\varepsilon}}
\]

for any $\varepsilon > 0$ and $f \in \mathcal{H}_{\sigma+1}$, where $G_{\sigma'}$ has the same meaning as in Proposition 2.1 and all the integrals involved converge.

2.3 $\mathfrak{S}$ as an entire function and the value of the residue norm $\mathfrak{S}(0)$

For any $f \in \mathcal{H}_{\sigma+1}$ and any $\varepsilon \in \mathbb{C}$ such that $\text{Re} \varepsilon > 0$, it follows from the inequality

\[
|\mathfrak{S}_{\rho}^{f}(\varepsilon)| = \varepsilon \int_{X} \frac{|f|^{2} e^{-\varphi_{L} - \psi}}{|\psi|^{\sigma}(\log|\ell\psi|)^{1+\varepsilon}} \leq |\varepsilon| \int_{X} \frac{|f|^{2} e^{-\varphi_{L} - \psi}}{|\psi|^{\sigma}(\log|\ell\psi|)^{1+\varepsilon}} \leq |\varepsilon| \frac{|f|^{2}}{\text{Re} \varepsilon} \mathfrak{S}(\text{Re} f)
\]

that the function $\varepsilon \mapsto \mathfrak{S}_{\rho}^{f}(\varepsilon)$ is well-defined on the right-half-plane $\{\varepsilon \in \mathbb{C} \mid \text{Re} \varepsilon > 0\}$ in $\mathbb{C}$. Since, when $\text{Re} \varepsilon > 0$, one has

\[
\frac{\partial}{\partial \varepsilon} \left( \frac{\varepsilon |f|^{2} e^{-\varphi_{L} - \psi}}{|\psi|^{\sigma}(\log|\ell\psi|)^{1+\varepsilon}} \right) = \frac{|f|^{2} e^{-\varphi_{L} - \psi}}{|\psi|^{\sigma}(\log|\ell\psi|)^{1+\varepsilon}}(1 - \varepsilon \log \log |\ell\psi|) \in L^{1}(X),
\]

the function $\varepsilon \mapsto \mathfrak{S}_{\rho}^{f}(\varepsilon)$ is thus holomorphic on the right-half-plane. Proposition 2.1 infers that the function can be continued to the whole complex plane $\mathbb{C}$ analytically.

**Theorem 2.3** (Theorem 1.2) Given any $f \in \mathcal{H}_{\sigma+1}$, the function $\varepsilon \mapsto \mathfrak{S}_{\rho}^{f}(\varepsilon)$ can be analytically continued to an entire function.

**Proof** It suffices to show that, under the snc assumption 1.11, $\varepsilon \mapsto \mathfrak{S}_{\rho_{\gamma}}^{\mathcal{J}_{\rho}^{F_{p,p}}} f(\varepsilon)$ is an entire function for each $\gamma \in I$, which corresponds to $V_{\gamma}$ in the open cover $\{ V_{\gamma} \}_{\gamma \in I}$.

First consider the case when $\text{lc}_{\rho_{\gamma}}^{\sigma}(S) \cap V_{\gamma} \neq \emptyset$. In view of the decomposition (4) and using the reduction argument as well as the notation in the proof of Proposition 2.1, it suffices to
show that $\mathcal{I}^{F_0}_{(\epsilon)} = \mathcal{I}^{F_0}_{(\epsilon)} \sigma$ is an entire function (the subscript $\sigma$ is made implicit as there is no induction on $\sigma$ required).

The proof starts by showing that the integral on the right-hand-side of (6) in the proof of Proposition 2.1 is an entire function in $\epsilon$. It suffices to check that the integral converges absolutely when $\epsilon = -R$ for any number $R \geq 0$. Indeed, up to a multiple constant, the integral is of the form

$$\int_{V_{\gamma}} F_\sigma \left( \log|\ell \psi| \right)^R \prod_{j=1}^n \frac{dr_j^2}{r_j^2}$$

when $\epsilon = -R$, where $\ell_j < 1$ for all $j = 1, \ldots, n$. Recall that $v_j$’s are the coefficients in the local expression of $\psi$ on $V_{\gamma}$ in (3). Take a sufficiently small number $\delta > 0$ such that $\ell_j + \delta v_j < 1$ for all $j = 1, \ldots, n$. Then,

$$(\log|\ell \psi|)^R \prod_{j=1}^n \frac{1}{r_j^{2\epsilon}} = |\psi| \left( \frac{\log|\ell \psi|}{|\psi|} \right)^R \prod_{j=1}^n \frac{r_j^{2\delta v_j}}{r_j^{2(\ell_j + \delta v_j)}} = e^{-\delta|\psi| - \delta \alpha} |\psi| \left( \frac{\log|\ell \psi|}{|\psi|} \right)^R \prod_{j=1}^n \frac{1}{r_j^{2(\ell_j + \delta v_j)}}$$

by (2)

$$\leq \frac{1}{\delta} \ell \left( \frac{R}{e} \right)^R \prod_{j=1}^n \frac{e^{-\delta \alpha}}{r_j^{2(\ell_j + \delta v_j)}}$$

where the right-hand-side is integrable with respect to $\prod dr_j^2$ for any $R \geq 0$ (under the convention $0^0 = 1$). Since $F_\sigma$ is smooth on a neighbourhood of $V_{\gamma}$, the integral on the right-hand-side of (6) converges absolutely for every $\epsilon \in \mathbb{R}$, and consequently every $\epsilon \in \mathbb{C}$. It is also easy to check that the integral is entire in $\epsilon$.

Now, $\mathcal{I}^{F_0}_{(\epsilon)}$ can be analytically continued via (6). First, (6) holds for all $\epsilon \in \{w \in \mathbb{C} \mid \text{Re } w > 0\}$ by the identity theorem. Then, $\mathcal{I}^{F_0}_{(\epsilon)}$ for $\epsilon$ with $-1 < \text{Re } \epsilon \leq 0$ can be defined and shown to be holomorphic on the region via (6) as all terms in (6) other than $\mathcal{I}^{F_0}_{(\epsilon)}$ are already well-defined and holomorphic. The same argument can be applied to define $\mathcal{I}^{F_0}_{(\epsilon)}$ and to show its holomorphicity on the regions $\{\epsilon \in \mathbb{C} \mid -\mu - 1 < \text{Re } \epsilon \leq -\mu\}$ for $\mu = 1, 2, 3, \ldots$ successively via an induction on $\mu$. This concludes that $\epsilon \mapsto \mathcal{I}^{F_0}_{(\epsilon)}$, and consequently $\epsilon \mapsto \mathcal{I}^{\rho \psi |f|^2}_{(\epsilon)}$ with $\text{lc}_X(S) \cap V_{\gamma} \neq \emptyset$, is an entire function.

To prove $\mathcal{I}^{\rho \psi |f|^2}_{(\epsilon)}$ being entire for the case $\text{lc}_X(S) \cap V_{\gamma} = \emptyset$, consider the equation in Remark 2.2 in place of (6). The argument is easier since that $\epsilon \int_X \frac{G_\alpha'}{|\psi|^{\sigma'-\sigma} (\log|\ell \psi|)^{1+\epsilon}}$ being absolutely convergent even when $\epsilon = -R$ for any $R \geq 0$ can be seen from the inequality

$$\frac{(\log|\ell \psi|)^{R-1}}{|\psi|^{\sigma'-\sigma}} \leq \frac{(\log|\ell \psi|)^R}{|\psi|^{\sigma'-\sigma}} \leq e^{\sigma'-\sigma} \left( \frac{R}{e(\sigma'-\sigma)} \right)^R.$$

The rest of the arguments are the same as the previous case.

As a result, the function $\epsilon \mapsto \mathcal{I}^{\rho \psi |f|^2}_{(\epsilon)}$ is entire.

Remark 2.4 The proof of $\epsilon \mapsto \mathcal{I}^{\rho \psi |f|^2}_{(\epsilon)}$ being entire for the case $\text{lc}_X(S) \cap V_{\gamma} = \emptyset$ indeed verifies that the integral

$$\frac{1}{\epsilon} \mathcal{I}^{\rho \psi |f|^2}_{(\epsilon)} = \int_{V_{\gamma}} \frac{\rho \psi |f|^2 e^{-\psi \psi S}}{|\psi|^{\sigma} (\log|\ell \psi|)^{1+\epsilon}}$$

is
Corollary 2.5 (Theorem 1.2) For any \( f \in \mathcal{H}_{\sigma+1} \), the value \( \tilde{S}(0)^2 \) is the squared norm of \( f \) on \( \text{lc}^\sigma_X(S) \) with respect to the lc-measure \( d \text{lc}^{\sigma,\varphi_L}_{\omega}[\psi] \), i.e.
\[
\tilde{S}(0)^2 = \int_{\text{lc}^\sigma_X(S)} |f|_{\omega}^2 d \text{lc}^{\sigma,\varphi_L}_{\omega}[\psi].
\]

Moreover, the value \( \tilde{S}(0)^2 \) is invariant even if \( \ell \psi \) is replaced by another function \( \ell' \psi' < 0 \) on \( X \) (but without changing \( \varphi_L + \psi_S = \varphi_L + \psi \)) as long as \( |\ell' \psi'| > 1 \) and \( \psi - \psi' \) is smooth on \( X \).

Proof Assume the snc assumption 1.11 without loss of generality. It suffices to evaluate \( \mathcal{S}(0)^2 \) for each \( \gamma \in I \), which corresponds to \( V_\gamma \) in the open cover \( \{V_\gamma\}_{\gamma \in I} \).

For \( \gamma \) such that \( \text{lc}^\sigma_X(S) \cap V_\gamma = \emptyset \), as Remark 2.4 remarks that 1.2 gives
\[
\mathcal{S}(0)^2 = 0. (This equality can also be obtained by substituting \( \varepsilon = 0 \) into the equation in Remark 2.2.)
\]

It remains to consider the case \( \text{lc}^\sigma_X(S) \cap V_\gamma \neq \emptyset \). It suffices to evaluate \( \mathcal{S}(0)^2 = \frac{(-1)^\sigma}{(\sigma-1)!} \int_X G_\sigma \) according to Proposition 2.1. In view of the decomposition (4) and using the notation in the proof of Proposition 2.1, it suffices to evaluate each summand \( \mathcal{J}(0)^\sigma = \mathcal{J}(0)^\sigma \) of the decomposition of \( \mathcal{S}(0)^2 \). When \( p = p' \), (6) gives
\[
\mathcal{J}(0)^\sigma = \frac{(-1)^\sigma}{(\sigma-1)!} \int_{V_\gamma} F_\sigma \prod_{j=1}^\sigma d\rho_{p(j)}d\theta_{p(j)} \quad \text{(where } F_0 := F_{p,p} \text{ and } F_\sigma \text{ given by (5))}
\]

\[
= \frac{\pi^\sigma}{(\sigma-1)!} \prod_{\gamma} \mathcal{J}(0)^\sigma \quad \text{(where } \mathcal{J}(0)^\sigma \text{ is smooth)}
\]

after a successive application of the fundamental theorem of calculus with respect to the variables \( r_{p(\sigma)}, \ldots, r_{p(2)}, r_{p(1)} \). Note that the last expression on the right-hand-side is independent of the number \( \ell \) and the function \( \alpha \) in \( \psi|_{V_\gamma} \), which leads to the last claim in this corollary.

When \( p \neq p' \), as discussed in the proof of Proposition 2.1, the integral \( \mathcal{J}(0)^\sigma \) can be handled like \( \mathcal{J}(0)^\sigma \), but the role of \( F_0 = F_{p,p} \) is replaced by the product of \( F_{p,p} \) with some coordinate functions vanishing on \( \text{lc}^\sigma_X(S) \) (which is the function \( z_{p(\sigma)}F_{p,p} \) in the example in the proof of Proposition 2.1). It follows from the above computation that \( \mathcal{J}(0)^\sigma = 0 \).

As a result,
\[
\mathcal{S}(0)^2 = \sum_{\gamma} \sum_{p} \mathcal{J}(0)^\sigma = \sum_{\gamma} \sum_{p} \prod_{j=1}^\sigma \frac{\pi^\sigma}{(\sigma-1)!} \prod_{\gamma} \mathcal{J}(0)^\sigma = \int_{\text{lc}^\sigma_X(S)} |f|_{\omega}^2 d \text{lc}^{\sigma,\varphi_L}_{\omega}[\psi],
\]

where the last equality follows from [5, Prop. 2.2.1 and Remark 2.2.3].
3 $L^2$ estimate for the extension problem

3.1 A non-universal estimate

Let $\mathcal{H}_{\sigma+1} = \mathcal{E}_\sigma \oplus \mathcal{H}_\sigma$ be the orthogonal decomposition of $\mathcal{H}_{\sigma+1}$ with respect to the squared norm $\mathcal{F}(1) = \mathcal{F}(1)_\sigma$. Then elements in $\mathcal{E}_\sigma$ are the holomorphic extensions of elements in $\mathcal{H}_{\sigma+1}/\mathcal{H}_\sigma$ (which are sections on $\mathcal{L}(S)$) with minimal norm with respect to the squared norm $\mathcal{F}(1)$.

**Theorem 3.1** *(Theorem 1.5)* On a compact Kähler manifold $X$ with the potential $\varphi_L$ of a line bundle $L$ over $X$ and the functions $\psi$ on $X$ given as above, there exists a constant $C := C(X, \varphi_L + \psi) > 0$ such that, when $\psi$ and $\ell > 0$ are chosen to satisfy the normalisation $\log|\ell\psi| \geq C$ (by varying $\ell > 0$ or adding suitable constant to $\psi$ but without changing $\varphi_L + \psi$), the estimate

$$\frac{|f|^2}{\mathcal{F}(1)} \leq \frac{|f|^2}{\mathcal{F}(0)}$$

holds for all $f \in \mathcal{E}_\sigma$.

**Proof** Notice that both $\mathcal{F}(\cdot, \cdot)$ and $\mathcal{F}(1)$ are positive definite hermitian inner product on $\mathcal{E}_\sigma$ (and $\mathcal{F}(0)$ is trivial on $\mathcal{H}_\sigma$). Take a $\mathbb{C}$-basis $\{\Phi_k\}_{k=1}^{N_{\mathcal{E}_\sigma}}$ of $\mathcal{E}_\sigma$ such that it is orthonormal with respect to $\mathcal{F}(\cdot, \cdot)$.

For any $f, g \in \mathcal{H}_{\sigma+1}$, the value $\mathcal{F}(f, g)$ converges to 0 when $\log|\ell\psi|$ is increased to $+\infty$ by increasing $\ell$ or adding some (negative) constant to $\psi$, while $\mathcal{F}(0)$ does not change according to Corollary 2.5. Therefore, as the basis $\{\Phi_k\}_{k=0}^{N_{\mathcal{E}_\sigma}}$ of $\mathcal{E}_\sigma$ is finite, there exists a normalisation of $\log|\ell\psi|$ such that the inequality

$$\left[\frac{\langle \Phi_j, \Phi_k \rangle}{\mathcal{F}(1)}\right]_{1 \leq j, k \leq N_{\mathcal{E}_\sigma}} \leq \left[\frac{\langle \Phi_j, \Phi_k \rangle}{\mathcal{F}(0)}\right]_{1 \leq j, k \leq N_{\mathcal{E}_\sigma}} = I_{N_{\mathcal{E}_\sigma}} \quad \text{(identity matrix)}$$

between $(N_{\mathcal{E}_\sigma} \times N_{\mathcal{E}_\sigma})$-hermitian matrices holds true. The claim thus follows.

3.2 Explicit examples

Here are examples of the function $\mathcal{F}(\cdot, \cdot)$ which can be shown to satisfy $\mathcal{F}(\cdot, \cdot) \leq \frac{|f|^2}{\mathcal{F}(0)}$, or indeed $\mathcal{F}(\cdot, \cdot) \leq \frac{|f|^2}{\mathcal{F}(0)}$ for all $\varepsilon \geq 0$, under the normalisation $\log|\ell\psi| \geq 1$.

**Example 3.2** On the $n$-dimensional complex projective space $\mathbb{P}^n$, the canonical bundle is $K_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-n - 1) = \mathcal{O}(-n - 1)$. Let $X_0, \ldots, X_n$ be the homogeneous coordinates and let $U_j := \{X_j \neq 0\}$ for $j = 0, \ldots, n$ be the open sets which constitute a finite cover of $\mathbb{P}^n$. Consider the line bundle $L = \mathcal{O}(n + 1)$ endowed with a smooth metric $e^{-\varphi_L}$ whose potential $\varphi_L \in \{\varphi_L, U_j\}_{j=0,\ldots,n}$ is given by

$$\varphi_L, U_j := (n + 1) \log \left( \frac{|X_0|^2 + \cdots + |X_n|^2}{|X_j|^2} \right) + 1.$$
Define also

\[ \psi := \sigma \log \left| \frac{X_j}{X_0} \right|^2 - \sigma \log \left( 1 + \sum_{j=1}^{n} \left| \frac{X_j}{X_0} \right|^2 \right) - 1 \]

for some integer \( \sigma \in \{1, n\} \). Notice that \( \psi \) is a well-defined function defined on \( \mathbb{P}^n \) with \( \psi < -1 \) on \( \mathbb{P}^n \) and \( \psi^{-1}(\{-\infty\}) = \bigcup_{j=1}^{\sigma} \{X_j = 0\} \). Indeed, the family \( \{ \mathcal{I}(\mathcal{L} + m\psi) \}_{m \in \mathbb{R}_{>0}} \) of multiplier ideal sheaves has the first jumping number \( m = 1 \) and the annihilator of \( \frac{\mathcal{I}(\mathcal{L} + \psi)}{\mathcal{I}(\mathcal{L})} \) is given by

\[ \operatorname{Ann}_\sigma \left( \frac{\mathcal{I}(\mathcal{L} + \psi)}{\mathcal{I}(\mathcal{L})} \right) = \mathcal{I}_{\sum_{j=1}^{\sigma} \{X_j = 0\}} = \mathcal{I}_S, \]

where the right-hand-side is the defining ideal sheaf of the reduced divisor \( S := \sum_{j=1}^{\sigma} \{X_j = 0\} \). The mlc of \( (\mathbb{P}^n, S) \) has codimension \( \sigma \) in \( \mathbb{P}^n \).

Notice also that \( K_{\mathbb{P}^n} \otimes L \cong \mathcal{O} \). Let \( x_1, \ldots, x_n \) be the inhomogeneous coordinates on \( U_0 \). Then, \( f := dx_1 \wedge \cdots \wedge dx_n \) can be viewed as a global section of \( K_{\mathbb{P}^n} \otimes L \) which spans the vector space \( H^0(\mathbb{P}^n, K_{\mathbb{P}^n} \otimes L) \cong \mathbb{C} \). Let \( x_j = r_j e^{\sqrt{-1} \theta_j} \) for \( j = 1, \ldots, n \) be the expression of \( x_j \) in polar coordinates. Set also \( |x|^2 = \sum_{j=1}^{n} |x_j|^2 = \sum_{j=1}^{n} r_j^2 \) for convenience.

The residue function \( \mathcal{R}(\mathcal{L}_\sigma) \) of \( f \) for the lc centres of \( (\mathbb{P}^n, S) \) of codimension \( \sigma \) is then given by

\[ \mathcal{R}(\mathcal{L}_\sigma) = \mathcal{L} \int_{\mathbb{P}^n} \frac{|f|^2 e^{-\psi} - \psi}{|\psi|^\sigma (\log|\ell\psi|)^{1+\varepsilon}} = \varepsilon \int_{U_0} \frac{|dx_1 \wedge \cdots \wedge dx_n|^2}{|x_1|^2 \left( 1 + |x|^2 \right)^3 |\psi| (\log|\ell\psi|)^{1+\varepsilon}}. \]

For simplicity, only the cases with \( n = 3 \) and \( \sigma = 1 \) and \( 2 \) are considered.

For the case \( n = 3 \) and \( \sigma = 1 \), the mlc of \( (\mathbb{P}^3, S) \) has codimension 1. Taking

\[ \ell := e^b \quad \text{for some constant } b \geq 1, \]

the function \( \mathcal{R}(\mathcal{L}_1) \) is given by

\[ \mathcal{R}(\mathcal{L}_1) = \mathcal{L} \int_{\mathbb{R}^3} \frac{\pi^3 e^b \frac{r_1^2}{(1 + r_1^2)^2}}{r_1^2 (1 + r_1^2)^2 |\psi| (\log|\ell\psi|)^{1+\varepsilon}} = \mathcal{L} \int_{\mathbb{R}^3} \frac{\pi^3 \frac{r_1^2}{(1 + r_1^2)^2}}{(1 + r_1^2)^2 |\psi| (\log|\ell\psi|)^{1+\varepsilon}}. \]

Int. by parts

\[ \int_{\mathbb{R}^3} \left[ \frac{\pi^3}{(\log|\ell\psi|)^{\varepsilon}} (1 + r_1^2 + r_2^2)(1 + r_2^2)^2 \right]_{r_1 = 0}^{r_1 = \infty} dr_2^2 dr_3^2 \]

\[ + \int_{\mathbb{R}^3} \frac{2\pi^3 r_1^2}{(\log|\ell\psi|)^{\varepsilon}} (1 + r_1^2 + r_3^2)(1 + r_3^2)^2 \]

\[ = \int_{\mathbb{R}^3} \frac{2\pi^3 r_1^2}{(\log|\ell\psi|)^{\varepsilon}} (1 + r_1^2 + r_2^2)(1 + r_2^2)^2. \]
Notice that the last expression is a decreasing function in $\varepsilon$ as $\log |\ell \psi| \geq b \geq 1$. Therefore, one has
\[
\mathfrak{D}_f^{(1)1} \leq \mathfrak{D}_f^{(0)1}.
\]
Indeed, even $\mathfrak{D}_f^{(1)1} \leq \mathfrak{D}_f^{(0)1}$ holds true for all $\varepsilon \geq 0$.

For the case $n = 3$ and $\sigma = 2$, the mlc of $(P^3, S)$ has codimension 2. Noticing that $r_j^2 \psi_j = 1 - \frac{2r_j^2}{1 + \varepsilon^2}$ for $j = 1, 2$ have zeros in $U_0$, the computation has to be adjusted a bit.

Under the same notation, the function $\mathfrak{D}_f^{(1)2}$ is given by
\[
\mathfrak{D}_f^{(1)2} = \varepsilon \int_{U_0} \frac{|dx_1 \wedge dx_2 \wedge dx_3|^2}{|x_1|^2|x_2|^2\left(1 + \frac{|x|^2}{r^2}\right)^2} |\psi|^2 (\log |\ell \psi|)^{1+\varepsilon} = \int_{\mathbb{R}^3_{\geq 0}} \frac{\pi^3 \varepsilon^2 dr^2 dr_3 dr_2}{r_2^2 r_2^2 \left(1 + \varepsilon^2\right)^2} |\psi|^2 (\log |\ell \psi|)^{1+\varepsilon}
\]
\[
= \int_{r_1 < r_2} + \int_{r_2 < r_1} \text{ by symmetry}
\]
\[
= 2 \int_{\mathbb{R}^2_{\geq 0}} \int_{r_1 = 0}^{r_1 = r_2} \frac{2\pi^3 \varepsilon^2 dr_1^2 dr_2 dr_3}{r_2^2 \left(1 + \frac{2r_1^2}{1 + \varepsilon^2}\right)^2 (1 + \varepsilon^2)^2 (\log |\ell \psi|)^{1+\varepsilon}}
\]
\[
= \int_{\mathbb{R}^2_{\geq 0}} \int_{r_1 = 0}^{r_1 = r_2} \frac{2\pi^3 \varepsilon^2 \left(\frac{dr_2^2}{1 + \varepsilon^2} - \frac{dr_2^2}{1 + \varepsilon^2 - 2\varepsilon^2} - \frac{(1 + \varepsilon^2) dr_1}{|\psi| (\log |\ell \psi|)}\right)}{r_2^2 \left(1 + \frac{2r_1^2}{1 + \varepsilon^2}\right)^2 (1 + \varepsilon^2)^2 (\log |\ell \psi|)^{1+\varepsilon}} dr_2^2 dr_3
\]
\[
= \frac{2\pi^3 \varepsilon^2 \left(\frac{dr_2^2}{1 + \varepsilon^2} - \frac{dr_2^2}{1 + \frac{2r_1^2}{1 + \varepsilon^2}}\right)}{r_2^2 \left(1 + \frac{2r_1^2}{1 + \varepsilon^2}\right)^2 (1 + \varepsilon^2)^2 (\log |\ell \psi|)^{1+\varepsilon}} \left(\psi_{r_1 = r_2}\right)
\]
\[
= \int_{\mathbb{R}^3_{\geq 0} \cap \left(r_1 < r_2\right)} \frac{4\pi^3 \varepsilon^2 dr_1^2 dr_2^2 dr_3}{r_2^2 \left(1 + \frac{2r_1^2}{1 + \varepsilon^2}\right)^2 (1 + \varepsilon^2)^2 (\log |\ell \psi|)^{1+\varepsilon}} = I(\varepsilon)
\]
\[
= \frac{2\pi^3 \varepsilon^2 \left(1 + \varepsilon\right)^2}{r_2^2 r_2^2 \left(1 + \varepsilon^2\right)^2} |\psi|^2 (\log |\ell \psi|)^{2+\varepsilon}
\]
\[
(\varepsilon = b) \quad \frac{\pi^3}{b^2} \int_{\mathbb{R}^2_{\geq 0}} \frac{dr_2^2}{(1 + \varepsilon^2)^2} - I(\varepsilon) - \varepsilon \mathfrak{D}_f^{(1)2} + \mathfrak{D}_f^{(1)2} = \frac{\pi^3}{b^2} - I(\varepsilon) - \varepsilon \mathfrak{D}_f^{(1)2}.
\]

Therefore, one obtains
\[
\mathfrak{D}_f^{(1)2} + \varepsilon \mathfrak{D}_f^{(1)2} + I(\varepsilon) = \frac{\pi^3}{b^2}.
\]
The integral $I(\epsilon)$ is non-negative and can be expressed as

$$I(\epsilon) = 4\pi^3 \int_{\mathbb{R}^2_{\geq 0}} \int_{r_2=r_1}^{r_2=\infty} \frac{r_1^2 \, dr_1^2}{r_2^2 (1 + r_2^2 - 2r_1^2)^2 (1 + r_2^2)^2 |\psi|(|\log|\ell\psi||)^{1+\epsilon}} \, dr_1 \, dr_2^2.$$ 

Notice that the integral

$$\epsilon \int_{r_2=r_1}^{r_2=\infty} \frac{dr_2^2}{r_2^2 (1 + r_2^2 - 2r_1^2)^2 (1 + r_2^2)^2 |\psi|(|\log|\ell\psi||)^{1+\epsilon}}$$

is convergent for every $(r_1, r_3) \in \mathbb{R}_{\geq 0}^2$ even when $r_1 = 0$,

- as a function in $\epsilon$ is continuous on $(0, +\infty)$ and can be analytically continued across 0 in view of Proposition 2.1 and Theorem 2.3 for every $(r_1, r_3) \in \mathbb{R}_{\geq 0}^2$ even when $r_1 = 0$, and

- converges to 0 as $\epsilon \to 0^+$ whenever $r_1 > 0$.

Therefore, after taking into account the analytic continuation of the function $\epsilon \mapsto I(\epsilon)$ across 0, the dominated convergence theorem infers that

$$I(0) = \lim_{\epsilon \to 0^+} I(\epsilon) = 0.$$

As a result, since $\frac{1}{b^\epsilon}$ is a decreasing function in $\epsilon$ with the choice $b \geq 1$, one obtains, for all $\epsilon \geq 0$, that

$$\Delta_{(\epsilon)2} \leq \Delta_{(\epsilon)2} + \epsilon \Delta_{(1 + \epsilon)2} + I(\epsilon) = \frac{\pi^3}{b^\epsilon} \leq \pi^3 = \Delta_{(0)2} + 0 \cdot \Delta_{(1)2} + I(0) = \Delta_{(0)2},$$

as desired.

**Example 3.3** This example has the same setup as in Example 3.2, except that now the line bundle $L$ is $\mathcal{O}(n+2)$ over $\mathbb{P}^n$, endowed with the potential $\varphi_L = \{\varphi_L, U_j\}_{j=0,\ldots,n}$ given by

$$\varphi_L, U_j := (n + 2) \log \left( \frac{|X_0|^2 + \cdots + |X_n|^2}{|X_j|^2} \right) + 1.$$

Then, $K_{\mathbb{P}^n} \otimes L \cong \mathcal{O}(1)$ and its global sections $f_j := x_j \, dx_1 \wedge \cdots \wedge dx_n$ for $j = 0, \ldots, n$ (where $x_0 = 1$), expressed by their representatives on $U_0$, form a basis of $H^0(\mathbb{P}^n, K_{\mathbb{P}^n} \otimes L) \cong \mathbb{C}^{n+1}$.

With the same choice of $\psi$ and using same notation as in Example 3.2, the case of $n = 3$ and $\sigma = 2$ is considered here. In this case, $S = \{X_1 = 0\} + \{X_2 = 0\}$ and the mlc of $(\mathbb{P}^3, S)$ has codimension 2. It is clear that both $f_1$ and $f_2$ vanish on $\operatorname{lc}_{\mathbb{P}^3}(S)$ while $f_0$ and $f_3$ are non-trivial there.

Since the weight $e^{-\varphi_L \psi} |\psi|^\epsilon (|\log|\ell\psi||)^{1+\epsilon}$ in the integral $\Delta_{(\epsilon)\sigma}$ is independent of $\theta_j$’s for all $\sigma \in \mathbb{N}$ and $\epsilon \in \mathbb{R}$, it is easy to see that $f_0, \ldots, f_3$ are orthogonal with respect to $\Delta_{(\epsilon)2}$ for all $\epsilon \geq 0$. Therefore, the section $f_3$, for example, is the minimal holomorphic extension of $f_3|_{\operatorname{lc}_{\mathbb{P}^3}(S)}$ with respect to $\Delta_{(\epsilon)2}$ for any $\epsilon > 0$.

As an illustration, under the same normalisation of $\log|\ell\psi|$ as in Example 3.2 (i.e. $\ell = e^b$ with $b \geq 1$), consider the function $\Delta_{f_1 f_2}$, which gives
\[
\delta_{(f^2)} \leq \delta_{(f^2)} = \int_{U_0} |x_3^2| |dx_1 \wedge dx_2 \wedge dx_3|^2 = \int_{\mathbb{R}^3_{\leq 0}} \pi^3 \varepsilon r_1^2 r_2^2 r_3^2 \ dr_1^2 \ dr_2^2 \ dr_3^2 =: I_1(\varepsilon) - I_2(\varepsilon).
\]

Notice that, in view of Fubini’s theorem, the integral \(I_1(\varepsilon)\) can be handled exactly as in Example 3.2 (with \(n = 3, \sigma = 2\)), and thus

\[
I_1(\varepsilon) \leq I_1(0)
\]

for all \(\varepsilon \geq 0\). Since \(I_2(\varepsilon)\) is non-negative for all \(\varepsilon > 0\) and \(I_2(0) = 0\) by Corollary 2.5 and [5, Prop. 2.2.1], it follows that

\[
\delta_{(f^2)} - \delta_{(f^2)} + I_2(\varepsilon) = I_1(\varepsilon) \leq I_1(0) = \delta_{(f^2)} - I_2(0) = \delta_{(f^2)}
\]

for all \(\varepsilon \geq 0\), as desired.

Using the same trick, one can also show that \(\delta_{(f^2)} \leq \delta_{(f^2)}\) for all \(\varepsilon \geq 0\) (with \(\psi\) remaining the same such that \(S = \{X_1 = 0\} \cup \{X_2 = 0\}\)) by reducing the computation to the situation in Example 3.2 (with \(n = 3, \sigma = 1\)).

The above examples all satisfy the usual curvature assumption for \(L^2\) extension, i.e. there is some number \(\delta > 0\) such that \(\varphi_L + (1 + \beta)\psi\) being psh for all \(\beta \in [0, \delta]\).

The following example satisfies the assumption \(\varphi_L + \psi\) being psh, but there exists no \(\delta > 0\) such that so is for \(\varphi_L + (1 + \delta)\psi\).

**Example 3.4** This example has the same setup as in Example 3.2 such that the line bundle \(L = \mathcal{O}(n + 1)\) over \(\mathbb{P}^n\) endowed with the potential \(\varphi_L = \{\varphi_L, U_j\}_{j=0, \ldots, n}\) given by

\[
\varphi_L, U_j := (n + 1) \log \left(\frac{|X_0|^2 + \cdots + |X_n|^2}{|X_j|^2}\right) + 1,
\]

but the function \(\psi\) is chosen to be

\[
\psi := \sum_{j=0}^n \log \left|\frac{X_j}{X_0}\right|^2 - (n + 1) \log \left(1 + \sum_{j=1}^n \left|\frac{X_j}{X_0}\right|^2\right) - 1.
\]

Then, the mlc of \((\mathbb{P}^n, S) = (\mathbb{P}^n, \varphi_L, \psi)\) are \(n + 1\) (reduced) points each located at the origin given by the inhomogeneous coordinates on \(U_j\) for \(j = 0, \ldots, n\).

The global section \(f := dx_1 \wedge \cdots \wedge dx_n\) (its representative on \(U_0\)) of \(K_{\mathbb{P}^n} \otimes L \cong \mathcal{O}\) is considered. The case \(n = 3\) is computed here.

Use the same notation as in Example 3.2 and choose \(\ell := e_b\) for some \(b \geq 1\) as before. Note that \(\mathbb{P}^3\) is the closure of the (disjoint) union of the unit polydiscs \(\Delta_j \subset U_j\) centred at the origin in each of their coordinate charts for \(j = 0, 1, 2, 3\). Let \(y_1, y_2\) and \(y_3\) be the inhomogeneous
coordinates on $U_1$, where $y_1 = \frac{1}{x_1}t^1_2$, $y_2 = \frac{x_2}{x_1}$ and $y_3 = \frac{x_3}{x_1}$. Set $|y|^2 := |y_1|^2 + |y_2|^2 + |y_3|^2$. On $U_1$, the function $\psi$ is given by

$$\psi|_{U_1} = \log \left( \frac{|y_1|^2 |y_2|^2 |y_3|^2}{(1 + |y|^2)^4} \right) - 1.$$ 

Indeed, $\psi$ has the same formula on all of the open sets $U_j$ for $j = 0, 1, 2, 3$. Recall from Corollary 2.5 that the residue norm $\frac{|f|^2}{\delta(0)}$ does not change if $\psi$ is only altered by a smooth function. Then, one can make use of the estimates

$$|\psi|_{\Delta_0} = \left| \log \frac{|x_1|^2 |x_2|^2 |x_3|^2}{(1 + |x|^2)^4} \right| + 1 \geq -\log(|x_1|^2 |x_2|^2 |x_3|^2) + 1 =: |\psi_0| \geq 1.$$ 

From the symmetry of the integrand in $\frac{|f|^2}{\delta(\varepsilon)}$, one obtains, for any $\varepsilon > 0$,

$$\frac{|f|^2}{\delta(\varepsilon)} = \int_{U_0} \frac{\varepsilon |dx_1 \wedge dx_2 \wedge dx_3|^2}{|x|^2 |x_1|^2 |x_2|^2 |x_3|^2} = 4 \int_{U_0} \frac{|dx_1 \wedge dx_2 \wedge dx_3|^2}{|x|^2 |x_1|^2 |x_2|^2 |x_3|^2}$$

$$\leq 4\pi^3 \varepsilon \int \left[ \frac{dr_1^2 dr_2^2 dr_3^2}{r_1^2 r_2^2 r_3^2 |\psi_0|^3 (\log |\psi_0|)^{1+\varepsilon}} \right] = 2\pi^3 \varepsilon \int \left[ \frac{dr_1^2 dr_2^2 dr_3^2}{r_1^2 r_2^2 r_3^2 (\log |\psi_0|)^{1+\varepsilon}} \right] - \frac{\varepsilon}{2} F(1 + \varepsilon)$$

$$= 2\pi^3 \varepsilon \int \left[ \frac{dr_1^2 dr_2^2 dr_3^2}{r_1^2 r_2^2 r_3^2 (|\psi_0|^3 (\log |\psi_0|)^{1+\varepsilon})} \right] - \frac{\varepsilon}{2} \left( F(1 + \varepsilon) + \frac{1+\varepsilon}{2} F(2 + \varepsilon) \right) - \frac{\varepsilon}{2} F(1 + \varepsilon)$$

$$= \frac{2\pi^3}{b^2} - \frac{3\varepsilon}{2} F(1 + \varepsilon) - \frac{\varepsilon(1+\varepsilon)}{2} F(2 + \varepsilon).$$

The first term on the right-hand-side is decreasing when $b \geq 1$ and the remaining terms are negative and vanish when $\varepsilon = 0$. It follows that

$$\frac{|f|^2}{\delta(\varepsilon)} \leq F(\varepsilon) \leq F(0) = \frac{|f|^2}{\delta(0)}$$

for all $\varepsilon \geq 0$.

**Remark 3.5** In all of the above examples, the computation of $\frac{|f|^2}{\delta(\varepsilon)}$ is simply reproving Proposition 2.1 and Remark 2.2 without using any partition of unity. However, even in such simple cases, one still has to partition the region of integration according to the zero loci of derivatives of $\psi$ in order to apply integration by parts. Indeed, when the subregion of integration does not contain an lc centre of codimension $\sigma$ (when $\frac{|f|^2}{\delta(\varepsilon)}$ is under consideration), positive summand which is not decreasing in $\varepsilon$ may arise (for example, $\frac{\pi^3}{2^{1+\varepsilon}}$ instead of $\frac{\pi^3}{2^\sigma}$ may show up). It is then difficult to claim in general that the sum of the computation of the integral $\frac{|f|^2}{\delta(\varepsilon)}$ on different subregions yields only decreasing function in $\varepsilon$, an argument essential for proving the $L^2$ estimates in the above examples. The difficulty is avoided in those examples by using the symmetry of the integrand on different subregions.
The same difficulty persists when partition of unity is used which results in the identities in Proposition 2.1 and Remark 2.2. Moreover, the form $G_\sigma$ in those identities may not have a definite sign, which puts an extra hurdle to the analysis.

References

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