Spontaneous symmetry breaking in the $gl(N)$-NLS hierarchy on the half line

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Abstract

We describe an algebraic framework for studying the symmetry properties of integrable quantum systems on the half line. The approach is based on the introduction of boundary operators. It turns out that these operators both encode the boundary conditions and generate integrals of motion. We use this direct relationship between boundary conditions and symmetry content to establish the spontaneous breakdown of some internal symmetries, due to the boundary.
1 Introduction

This paper describes a framework for studying the symmetry properties of integrable quantum systems on the half line. As it can be expected on general grounds, the presence of a boundary in this case has a strong impact on the dynamics and the symmetry content of the systems. It gives rise to a variety of boundary-related phenomena with direct applications to impurity problems in condensed matter, dissipative quantum mechanics, open string theory and brane physics. The recent efforts to gain a deeper insight in these phenomena stimulated a series of investigations [1]-[9] in the subject. Among others, one should mention the attempts to develop an algebraic approach. The new strategy there is to introduce [1, 2] the so called boundary operators, which translate in algebraic terms the associated quantum boundary value problem. It is far from being a surprise that if possible, an algebraic treatment of the boundary value problem turns out to be simpler than the standard analytic one. In the context of integrable systems, the algebraic framework provides a further advantage - the search for general integrability preserving boundary conditions and their implementation become straightforward.

The concept of boundary algebra, we will be dealing below, has been introduced in [10] and is inspired by Cherednik’s scattering theory [11] for integrable models on the half line. An essential feature of this algebra, denoted in what follows by $\mathcal{B}_R$, is that it enables one to reconstruct [10] not only the scattering matrix, but captures also the off-shell dynamics (correlation functions) of the system [12, 13]. Therefore, one can apply $\mathcal{B}_R$ in the study of symmetries as well. This observation is the basic starting point of our investigation. The most interesting property emerging from it is that, besides encoding the boundary conditions, the boundary operators generate also integrals of motion. The main goal of the present work is to investigate this remarkable feature and to explore the consequences from it.

The paper is organized as follows. In the next section we summarize those basic features of the boundary algebra $\mathcal{B}_R$ and its Fock representations, which are needed for our analysis. The general structure involved in the discussion is illustrated by means of the $\mathfrak{gl}(N)$-invariant nonlinear Schrödinger model on the half line. The quantization of the model in terms of $\mathcal{B}_R$ is briefly described in Section 3. Section 4 is devoted to a purely algebraic analysis of the symmetry content of the NLS model. In Section 5 we derive a large
class of boundary conditions which preserve integrability. The phenomenon of spontaneous symmetry breaking is established in Section 6. Section 7 contains a detailed study of the case \( N = 2 \). Finally, Section 8 collects our conclusions.

## 2 Boundary algebras

We recall first the basic structure of a boundary algebra, referring for details to [10]. \( \mathcal{B}_R \) is an associative algebra with identity element 1 and two types of generators,

\[
\{a_i(k), a^*_i(k) : i = 1, ..., N, \ k \in \mathbb{R}\}
\]  

and

\[
\{b^i_j(k) : i, j = 1, ..., N, \ k \in \mathbb{R}\}.
\]  

called bulk and boundary generators respectively. \( N \) is the number of internal degrees of freedom, whereas \( k \) stands for the momentum in the non relativistic case or the rapidity in the relativistic one. (2.1) and (2.2) are subject to the following constraints:

(i) bulk exchange relations

\[
a_i(1) a_j(2) = R_{i_1 j_2}(k_2, k_1) a_j(2) a_i(1) = 0,
\]  

\[
a^*_i(1) a^*_j(2) = a^*_j(2) a^*_i(1) R_{j_2 i_1}(k_2, k_1) = 0, 
\]  

\[
a_i(1) a^*_j(2) - a^*_j(2) a_i(1) R_{j_2 i_1}(k_2, k_1) = 2\pi \delta(k_1 - k_2) \delta^{i_2}_{i_1} 1 + 2\pi \delta(k_1 + k_2) b^i_j(1) ;
\]  

(ii) boundary exchange relations

\[
R^{i_1}_{i_1 i_2}(k_1, k_2) b^{m_1}_{i_1}(k_1) R^{j_2 m_2}_{j_2 i_1}(k_2, -k_1) b^{j_2}_{m_2}(k_2) = 
\]  

\[
b^{i_2}_{i_2}(k_2) R^{m_2 m_1}_{i_1 i_2}(k_1, -k_2) b^{i_1}_{m_1}(k_1) R^{j_1 j_2}_{m_2 i_1}(-k_2, -k_1) ;
\]  

(iii) mixed exchange relations

\[
a_i(1) b^i_j(2) = R^{i_1 i_2}_{i_2 i_1}(k_2, k_1) b^{m_2}_{i_2}(k_2) R^{j_2 m_1}_{j_2 i_1}(k_1, -k_2) a_{m_1}(k_1),
\]  

\[
b^{i_1}_{i_1}(k_1) a^*_j(2) = a^*_m(2) R^{m_2 m_1}_{i_1 m_2}(k_1, k_2) b^{i_1}_{m_1}(k_1) R^{j_1 j_2}_{i_2 i_1}(k_2, -k_1).
\]
Here and in what follows the summation over repeated upper and lower indices is always understood. Finally, using already standard notations, the exchange factor \( R \) obeys

\[
R_{12}(k_1, k_2) R_{12}(k_2, k_1) = 1, \tag{2.9}
\]

and the spectral quantum Yang-Baxter equation (in its braid form)

\[
R_{12}(k_1, k_2) R_{23}(k_1, k_3) R_{12}(k_2, k_3) = R_{23}(k_2, k_3) R_{12}(k_1, k_3) R_{23}(k_1, k_2). \tag{2.10}
\]

These compatibility conditions on \( R \), together with eqs. (2.3-2.8) define the algebra \( \mathcal{B}_R \). Let us observe also that the generators \( \{ b_i^j(k) \} \) close a subalgebra \( \mathcal{S}_R \subset \mathcal{B}_R \), which has been introduced by Sklyanin [14] and which will play a distinguished role in our discussion below.

It is worth mentioning that if one formally takes \( b_i^j(k) \to 0 \), the relations (2.6-2.8) trivialize, while (2.3-2.5) reproduce the defining relations of a Zamolodchikov-Faddeev algebra [13, 16], which is known to describe factorized scattering of 1+1 dimensional integrable systems on the whole line. The situation changes on the half line, where in order to reproduce Cherednik’s scattering amplitudes one needs [10] a reflection algebra, i.e. a \( \mathcal{B}_R \) with the additional constraint

\[
b_i^j(k) b_i^j(-k) = \delta_i^j, \tag{2.11}
\]

which obviously prevents the vanishing of all boundary operators. The condition (2.11) has important consequences; it implies that the mapping

\[
\varrho : a_i(k) \mapsto b_i^j(k) a_j(-k), \tag{2.12}
\]

\[
\varrho : a^*_i(k) \mapsto a^*_j(-k) b_i^j(-k), \tag{2.13}
\]

\[
\varrho : b_i^j(k) \mapsto b_i^j(k), \tag{2.14}
\]

leaves invariant the constraints (2.3-2.8) and extends therefore to an automorphism on \( \mathcal{B}_R \). \( \varrho \), which is called in what follows reflection automorphism, has a direct physical interpretation in scattering theory: it provides a mathematical description of the intuitive physical picture that bouncing back from a wall, particles change the sign of their momenta (rapidities). In fact, the two elements \( a^*_i(-k) \) and \( a^*_j(k) b_i^j(k) \) are \( \varrho \)-equivalent,

\[
a^*_i(-k) \sim a^*_j(k) b_i^j(k). \tag{2.15}
\]
Combined with (2.6), eq.(2.11) turns out to be essential also in the description of symmetries.

For constructing representations of $\mathcal{B}_R$, it is essential to recognize an involution on it. Let us consider for this purpose the mapping $I$ defined by

$$I: a^*i(k) \mapsto a_i(k), \quad a_i(k) \mapsto a^*i(k), \quad b^j_i(k) \mapsto b^j_i(-k).$$  \hspace{1cm} (2.16)

When extended as an antilinear antihomomorphism on $\mathcal{B}_R$, $I$ defines an involution, provided that $R$ satisfies

$$R^j_{ij_2} (k_1, k_2) = R^j_{ij_2} (k_2, k_1).$$  \hspace{1cm} (2.17)

Here and in what follows the dagger stands for matrix Hermitian conjugation, i.e.

$$R^j_{ij_2} (k_1, k_2) \equiv R^j_{ji_2} (k_1, k_2),$$  \hspace{1cm} (2.18)

the bar indicating complex conjugation. The condition (2.17) is known in the literature on factorized scattering as Hermitian analyticity. We observe in passing that a larger class of involutions with the appropriate generalization of (2.17) have been introduced in [10].

Since $\mathcal{B}_R$ is an infinite algebra, for the moment our considerations are a bit formal. In order to give them a precise mathematical meaning, one can construct a class of Fock representations of the reflection algebra with involution $\{\mathcal{B}_R, I\}$, characterized by the following requirements:

1. The representation space $\mathcal{H}$ is a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$;
2. The generators (2.1, 2.2) are operator valued distributions with common and invariant dense domain $\mathcal{D} \subset \mathcal{H}$, where eqs.(2.3-2.8) hold. The involution $I$ defined by eq.(2.16) is realized as a conjugation with respect to $\langle \cdot, \cdot \rangle$;
3. There exists a vector (vacuum state) $\Omega \in \mathcal{D}$ which is annihilated by $a_i(k)$. $\Omega$ is cyclic with respect to $\{a^*i(k)\}$ and $\langle \Omega, \Omega \rangle = 1$.

These requirements imply that the c-number distributions

$$B^j_i (k) \equiv \langle \Omega, b^j_i (k) \Omega \rangle,$$  \hspace{1cm} (2.19)

satisfy

$$B^{ij}_i (k) = B^j_i (-k),$$  \hspace{1cm} (2.20)

which, being an analog of (2.17), is called boundary Hermitian analyticity. One can show moreover, that the vacuum vector $\Omega$ is unique (up to a phase factor) and satisfies

$$b^j_i (k) \Omega = B^j_i (k) \Omega.$$  \hspace{1cm} (2.21)
Therefore, taking the vacuum expectation value of eqs. (2.6,2.11), one finds

\[
R_{1i12}^{l_1}(k_1, k_2) B_{l_1}^{m_1}(k_1) R_{2j2m_1}^{l_2}(k_2, -k_1) B_{m_2}^{j_2}(k_2) = \\
B_{l_2}^{i_2}(k_2) R_{1i12}^{m_2m_1}(k_1, -k_2) B_{l_2}^{j_2}(k_2) R_{m_1}^{j_2}(k_1) R_{m_2}^{j_2}(-k_2, -k_1),
\]

(2.22)

\[
B_{l}^{i}(k) B_{l}^{j}(-k) = \delta_{i}^{j}.
\]

(2.23)

We thus recover at the level of Fock representation the boundary Yang-Baxter equation (2.22), originally derived in [11]. Because of (2.23), we will refer to \(B\) as reflection matrix.

Given a reflection algebra \(\{B_R, I\}\), its Fock representations are classified by all possible reflection matrices \(B\), which actually parametrize the boundary conditions. More precisely, any \(B\)-matrix satisfying eqs. (2.20,2.22,2.23), defines a Fock representation \(\mathcal{F}_R\), whose explicit construction is given in [11]. Each \(\mathcal{F}_B\) uniquely defines in turn a unitary scattering operator \(S\), corresponding to the integrable model described by \(\{B_R, I\}\). The emerging picture is therefore the following. The mere fact that we are dealing with an integrable system with boundary shows up at the algebraic level, turning the Zamolodchikov-Faddeev algebra into reflection algebra. The details of the boundary condition enter at the representation level through the reflection matrix \(B\). We stress at this point another sharp difference between reflection and Zamolodchikov-Faddeev algebras, the latter admitting a unique (up to unitary equivalence) Fock representation.

Let us turn finally to the question of symmetries, considering for instance the simplest non relativistic Hamiltonian coming in mind, namely

\[
H = \int_{-\infty}^{\infty} dp \, p^2 a^*(p) a_i(p).
\]

(2.24)

Using eqs. (2.7,2.8), one easily verifies that

\[
[H, b_{l}^{i}(k)] = 0.
\]

(2.25)

Therefore, besides capturing the presence of boundaries, \(b_{l}^{i}(k)\) generate integrals of motion, whose algebra is encoded in the boundary exchange relations (2.4).

It will be our main objective in what follows to examine thoroughly this observation on both purely algebraic (section 4) and Fock representation (section 5) levels. In order to gain some intuition from a concrete example,
we find it useful to illustrate first the above abstract setup by means of the $gl(N)$-invariant nonlinear Schrödinger ($gl(N)$-NLS) model on the half line.

Note also that the property (2.25) can be extended to a wide class of operators $H_n$:

$$[H_n, b_i^j(k)] = 0 \quad \text{with} \quad H_n = \int_{-\infty}^{\infty} dp \, p^n a^* (p) a_i (p)$$

leading to the notion of hierarchy. However, we want to stress that one has to check the vanishing of the commutators between different $H_n$ to get a true hierarchy. A direct calculation shows:

$$[H_n, H_m] = ((-1)^n - (-1)^m) \int dp \, p^{m+n} a^* (-p) b_i^j (-p) a_j (p)$$

Thus, it is only the $H_n$ of same parity that commute one with each other, and the hierarchy should be associated to the "even" Hamiltonians $H_{2n}$. We reject the hierarchy associated to the "odd" Hamiltonians $H_{2n+1}$, because they satisfy

$$H_{2n+1} = 0 \quad \text{in the Fock representation}$$

In what follows, we will concentrate on the Hamiltonian $H_2$, but one has to keep in mind that the properties will apply to the whole hierarchy $\{H_{2n}, \, n \in \mathbb{N}\}$.

3 The NLS model on the half line

The dynamics of the $gl(N)$-NLS model can be described by a $N$-component field $\Psi_i (t,x)$, satisfying

$$i \partial_t + \partial_x^2 \Psi_i (t,x) = 2g \Psi^j(t,x) \Psi_j(t,x) \Psi_i(t,x), \quad g > 0,$$

on the half line $\mathbb{R}_+ = \{ x \in \mathbb{R} : x > 0 \}$. One must fix in addition the boundary conditions. We start by requiring

$$\lim_{x \rightarrow 0} (\partial_x - \eta) \Psi_i (t,x) = 0, \quad \eta \geq 0;$$

$$\lim_{x \rightarrow \infty} \Psi_i (t,x) = 0.$$
respectively. We will focus in this section on the boundary value problem (3.1-3.3), postponing the consideration of more general integrability preserving boundary conditions to section 4.

In spite of the fact that the NLS model is among the most studied integrable systems, to our knowledge there exist only few papers addressing the problem on \( \mathbb{R}_+ \). References [14] and [17] deal essentially with the proof of integrability, whereas [12] and [13] are concerned with the construction of the quantum field \( \Psi \) and the relative off-shell correlation functions. We will follow below [12, 13], recalling first the basic structures which enter the second quantization of the boundary value problem (3.1-3.3):

1. A Hilbert space \( \{ \mathcal{H}_{g,\eta}, \langle \cdot, \cdot \rangle \} \) describing the states of the system;
2. An operator valued distribution \( \Psi(t,x) \) which acts on an appropriate dense domain \( \mathcal{D} \subset \mathcal{H}_{g,\eta} \) and satisfies:
   (a) the canonical commutation relations
   \[
   \left[ \Psi_i(t,x), \Psi_j(t,y) \right] = \left[ \Psi^*_i(t,x), \Psi^*_j(t,y) \right] = 0, \quad (3.4)
   \left[ \Psi_i(t,x), \Psi^*_j(t,y) \right] = \delta^*_i \delta(x-y), \quad (3.5)
   \]
   where \( \Psi^* \) is the \( \langle \cdot, \cdot \rangle \)-Hermitian conjugate of \( \Psi \);
   (b) the equation of motion
   \[
   (i\partial_t + \partial^2_x) \langle \varphi, \Psi_i(t,x)\psi \rangle = 2g \langle \varphi, : \Psi_j \Psi^*_j \Psi_i : (t,x)\psi \rangle, \quad \forall \varphi, \psi \in \mathcal{D}, \quad (3.6)
   \]
   where : \ldots : denotes a suitably defined normal product;
   (c) the boundary conditions
   \[
   \lim_{x\to 0} (\partial_x - \eta) \langle \varphi, \Psi_i(t,x)\psi \rangle = 0, \quad \forall \varphi, \psi \in \mathcal{D}, \quad (3.7)
   \]
   \[
   \lim_{x\to \infty} \langle \varphi, \Psi_i(t,x)\psi \rangle = 0, \quad \forall \varphi, \psi \in \mathcal{D}; \quad (3.8)
   \]
3. A fundamental state \( \Omega \in \mathcal{D} \), which is cyclic with respect to \( \Psi^* \).

It has been shown in [13] that all these structures can be explicitly realized in terms of the reflection algebra \( \mathcal{B}_R \) defined by the exchange matrix
\[
R_{ij}^{ij}(k_1, k_2) = \frac{1}{k_1 - k_2 + ig} \left[ ig \delta^j_i \delta^j_i + (k_1 - k_2) \delta^j_i \delta^j_i \right], \quad (3.9)
\]
and the associated Fock representation \( \mathcal{F}_R \) with reflection matrix
\[
B_i^j(k) = \frac{k - i\eta}{k + i\eta} \delta^j_i. \quad (3.10)
\]
One has to identify for this purpose $\mathcal{H}_{g,\eta}$ with the Fock space, $\Omega$ with the Fock vacuum and $\langle \cdot, \cdot \rangle$ with the scalar product of the representation $\mathcal{F}_B$. Moreover, the quantum field $\Psi$ admits the expansion

$$\Psi_i(t, x) = \sum_{n=0}^{\infty} (-g)^n \Psi_i^{(n)}(t, x), \quad (3.11)$$

where

$$\Psi_i^{(n)}(t, x) = \int_{-\infty}^{\infty} \prod_{j=1}^{n} dp_j dq_j \frac{e^{i \sum_{j=0}^{n}(xq_j - tq_j^2) - i \sum_{i=1}^{n}(xp_i - tq_i^2)}}{2\pi^n \prod_{i=1}^{n} [(p_i - q_{i-1} - i\epsilon)(p_i - q_i - i\epsilon)]} a^{*j_1}(p_1) \cdots a^{*j_n}(p_n) a_{j_n}(q_n) \cdots a_{j_1}(q_1) a_i(q_0). \quad (3.12)$$

$a_i$ and $a^{*j}$ representing the $B_R$-generators in $\mathcal{F}_B$. By construction, the domain $\mathcal{D}$ involves only vectors with finite, although arbitrary large, particle number. Combining this property with the normal ordered form of $\Psi^{(n)}$, one concludes that the series (3.11) converges in mean value on $\mathcal{D}$.

Eq. (3.12) is the quantum inverse scattering transform for the $gl(N)$-NLS model on $\mathbb{R}_+$. The field $\Psi$ admits (strong) asymptotic limits, giving raise to the following asymptotic states:

$$|k_1, i_1; \ldots; k_n, i_n\rangle^{in} = a^{*i_1}(k_1) \cdots a^{*i_n}(k_n)\Omega, \quad k_1 < \ldots < k_n < 0, \quad (3.13)$$

$$|p_1, j_1; \ldots; p_m, j_m\rangle^{out} = a^{*j_1}(p_1) \cdots a^{*j_m}(p_m)\Omega, \quad p_1 > \ldots > p_n > 0. \quad (3.14)$$

The vectors (3.13) and (3.14) generate the asymptotic in- and out-spaces respectively. Both of these spaces are dense in $\mathcal{H}_{g,\eta}$ and the model is asymptotically complete in the range $g > 0$ and $\eta \geq 0$. The connection with Cherednik's scattering theory [11] is obtained on the level of scattering amplitudes

$$^{out}\langle p_1, j_1, \ldots; p_m, j_m| k_1, i_1, \ldots; k_n, i_n\rangle^{in}. \quad (3.15)$$

The explicit form of (3.15) is easily derived and involves a product of $R$ and $B$-factors: any $R$-factor describes a two-body scattering in the bulk $\mathbb{R}_+$, while the $B$-factors take into account the reflection from the boundary.

It is worth stressing that the time evolution of the field $\Psi(t, x)$ is generated precisely by the Hamiltonian (2.24). In fact, by means of eq. (3.12) one gets

$$\Psi_i(t, x) = e^{itH} \Psi_i(0, x) e^{-itH}. \quad (3.16)$$
Therefore, according to our discussion at the end of the previous section, the boundary operators $b_j^i(k)$ generate integrals of motion for the $gl(N)$-NLS model on $\mathbb{R}_+$. The relative algebra, which follows from eq. (2.6) by inserting the specific exchange factor (3.9), will be investigated in the next section.

In conclusion, we point out that the quantum field (3.11,3.12) solves (3.6) for any reflection matrix $B$ satisfying eq.(2.22) with $R$ given by (3.9). One can also demonstrate [13] that for any $\varphi, \psi \in \mathcal{D}$, there exist $N$ square integrable functions $\chi_i(k)$, such that

$$
\langle \varphi, \Psi_i(t,x) \psi \rangle = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-itk^2} \left[ \delta^j_i e^{ixk} + B^j_i(k)e^{-ixk} \right] \chi_j(k). \quad (3.17)
$$

This identity implies not only (3.7) and (3.8), but allows to analyze (see section 5) the boundary condition at $x = 0$ for general $B$-factors.

## 4 The NLS symmetry algebra

Now we bring our attention to the $S_R$ subalgebra defined from the boundary generators (2.2) and exchange relation (2.6), when the R-matrix is chosen as in (3.9). In the standard notations, the R-matrix takes the form:

$$
R_{12}(k) = \frac{1}{k + ig} (k I \otimes I + ig P_{12}) \quad (4.1)
$$

where $I$ is the $N \times N$ identity matrix, $P_{12} \equiv P = \sum_{i,j=1}^{N} E_{ij} \otimes E_{ji}$ is the flip operator, and $E_{ij}$ is the $N \times N$ matrix with 1 at position $(i,j)$.

### 4.1 Calculation of the $S_R$ algebra

Let us rewrite (2.6) as:

$$
R(k_1 - k_2) b_1(k_1) R(k_1 + k_2) b_2(k_2) = b_2(k_2) R(k_1 + k_2) b_1(k_1) R(k_1 - k_2) \quad (4.2)
$$

or as commutation relation

$$
[b_1(k_1), b_2(k_2)] = \frac{ig}{k_1 - k_2} \left( b_2(k_2)b_1(k_1)P - Pb_1(k_1)b_2(k_2) \right) + \\
\frac{ig}{k_1 + k_2} \left( b_2(k_2)Pb_1(k_1) - b_1(k_1)Pb_2(k_2) \right) + \\
- \frac{g^2}{k_1^2 - k_2^2} \left[ b_2(k_2), b_2(k_1) \right] \quad (4.3)
$$
where

\[ b_1(k) = \sum_{i,j=1}^{N} b_{ij}(k) E_{ij} \otimes \mathbb{I} \quad \text{and} \quad b_2(k) = \sum_{i,j=1}^{N} b_{ij}(k) \mathbb{I} \otimes E_{ij} \quad (4.4) \]

Note that it enters in the algebras of type $ABCD$, introduced in [18], and as such is well-defined, since it obeys the corresponding consistency conditions (see [18] for details).

We develop $b(k)$ as a formal power series in $k^{-1}$:

\[ b(k) = \sum_{n \geq 0} k^{-n} b^{(n)} = \sum_{n \geq 0} \sum_{i,j=1}^{N} k^{-n} b_{ij}^{(n)} E_{ij} \quad (4.5) \]

Plugging this expression into (4.2) leads to the following commutation relations:

\[ \sum_{m,n=0}^{\infty} [b_1^{(m)}, b_2^{(n)}] k_1^{-m} k_2^{-n} = \sum_{p,q,r=0}^{\infty} \left\{ ig \left( b_2^{(q)} b_1^{(p)} P - P b_1^{(p)} b_2^{(q)} + \right. \right. \]
\[ + (-1)^r \left( b_2^{(q)} P b_1^{(p)} - b_1^{(p)} P b_2^{(q)} \right) \left. \right\} k_1^{-p-r-1} k_2^{r-q} - g^2 \left[ b_2^{(q)}, b_1^{(p)} \right] k_1^{-p-2r-2} k_2^{2r-q} \right\} \quad (4.6) \]

In order the $S_R$ algebra to be well-defined, no positive power of $k$ can be admitted on the r.h.s. of the above expression. A direct computation shows that spurious terms are avoided by the necessary and sufficient conditions:

\[ b(k)b(-k) = f(k) \mathbb{I} \quad \text{with} \quad f(k) = 1 + \sum_{m=1}^{\infty} f_{2m} k^{-2m} \quad \text{even function} \quad (4.7) \]

\[ [b_1^{(0)}, b_2^{(k)}] = 0 \quad \text{i.e. all the } b_{ij}^{(0)} \text{ generators are central} \quad (4.8) \]

Remark that the condition (4.7) is a natural generalization of the physical condition (2.11) obtained in section [4]. In the following, we will take $f(k)$ as a pure $\mathbb{C}$-function. The normalization of $f(k)$ is fixed by $\lim_{k \to \infty} f(k) = 1$.

Before commenting on these constraints, let us note that the commutation relations can then be rewritten as:

\[ [b_1^{(m)}, b_2^{(n)}] = ig \sum_{r=0}^{m-1} \left( b_2^{(n+r)} b_1^{(m-1-r)} - b_2^{(m-1-r)} b_1^{(n+r)} \right) P + \]
\[ (+1)^r \left( b_2^{(n+r)} P b_1^{(m-1-r)} - b_1^{(m-1-r)} P b_2^{(n+r)} \right) + \]

\[ -g^2 \sum_{r=0}^{\mu} \left[ b_2^{(n+2r)}, b_2^{(m-2-2r)} \right] \quad \text{where} \quad \mu = \left[ \frac{m-2}{2} \right] \quad (4.9) \]

that is

\[ [b^{(m)}_{ij}, b^{(n)}_{kl}] = ig \sum_{r=0}^{m-1} \left\{ \left( b^{(n+r)}_{kj} b^{(m-1-r)}_{il} - b^{(m-1-r)}_{kj} b^{(n+r)}_{il} \right) + \right. \]

\[ +(-1)^r \left( \delta_{il} b^{(n+r)}_{ka} b^{(m-1-r)}_{al} - \delta_{ka} b^{(m-1-r)}_{il} b^{(n+r)}_{al} \right) + \]

\[ \left. -g^2 \sum_{r=0}^{\mu} \delta_{ij} \left( b^{(n+2r)}_{ka} b^{(m-2-2r)}_{al} - b^{(m-2-2r)}_{ka} b^{(n+2r)}_{al} \right) \right\} \quad (4.10) \]

and also:

\[ [b_{ij}(k_1), b_{kl}(k_2)] = \frac{ig}{k_1 - k_2} \left( b_{kj}(k_2) b_{il}(k_1) - b_{kj}(k_1) b_{il}(k_2) \right) + \]

\[ + \frac{ig}{k_1 + k_2} \left( \delta_{il} b_{ka}(k_2) b_{aj}(k_1) - \delta_{ka} b_{il}(k_1) b_{aj}(k_2) \right) + \]

\[ -\frac{g^2}{k_1^2 - k_2^2} \delta_{ij} \left( b_{ka}(k_2) b_{al}(k_1) - b_{ka}(k_1) b_{al}(k_2) \right) \quad (4.11) \]

In particular, for \( m = 1 \), we get

\[ [b^{(1)}_{ij}, b^{(n)}_{kl}] = ig \left( b^{(n)}_{kj} b^{(0)}_{il} + \delta_{il} b^{(n)}_{ka} b^{(0)}_{aj} - b^{(0)}_{kj} b^{(n)}_{il} - \delta_{kj} b^{(0)}_{ia} b^{(n)}_{al} \right) \quad (4.12) \]

which shows that in each representation where \( b^{(0)} \) is a constant, the \( b^{(1)} \) generators form a Lie subalgebra and the \( b^{(n)} \) generators (for any given \( n \)) fall into representations of this Lie subalgebra. In what follows, we will consider only this type of representations.

### 4.1.1 Commutative subalgebras of the \( S_R \)-algebra

We give in this short section some commutative subalgebras which will be used in the following.
Property 4.1 Let us introduce

\[ t(k) = tr \left( b(k) \right) = \sum_{n=0}^{\infty} k^{-n} t^{(n)} \]  \hspace{1cm} (4.13)

\( t(k) \) defines a commutative subalgebra of \( S_R \): \([t(k_1), t(k_2)] = 0\).

**Proof:** We take the trace in the auxiliary space 1 of the relation \((1.3)\):

\[ [t(k_1), b(k_2)] = \frac{2ik_1 g}{k_1^2 - k_2^2} \left[ b(k_2), b(k_1) \right] \]  \hspace{1cm} (4.14)

Then, taking again the trace we get

\[ [t(k_1), t(k_2)] = \frac{2ik_1 g}{k_1^2 - k_2^2} tr \left( \left[ b(k_2), b(k_1) \right] \right) \]  \hspace{1cm} (4.15)

From \([t(k_1), t(k_2)] = -[t(k_2), t(k_1)]\), we deduce \( tr \left( \left[ b(k_2), b(k_1) \right] \right) = 0\), which implies \([t(k_1), t(k_2)] = 0\). \( \blacksquare \)

Property 4.2 Let \( M \) be any constant matrix such that \( M^2 = I \) and correspondingly

\[ \tilde{t}_M(k) = tr \left( M b(k) \right) = \sum_{n=0}^{\infty} k^{-n} \tilde{t}_M^{(n)} \]  \hspace{1cm} (4.16)

\( \tilde{t}_M(k) \) defines a commutative subalgebra of \( S_R \) which commutes with the one given in proposition 4.1:

\[ [\tilde{t}_M(k_1), \tilde{t}_M(k_2)] = 0 \quad \text{and} \quad [t(k_1), \tilde{t}_M(k_2)] = 0 \]  \hspace{1cm} (4.17)

**Proof:** We start again with \((1.3)\), multiply it by \( M_1 \equiv M \otimes I \), and then take the trace in the auxiliary space 1:

\[ [\tilde{t}_M(k_1), b(k_2)] = \frac{ig}{k_1 - k_2} \left( b(k_2) M b(k_1) - b(k_1) M b(k_2) \right) + \]

\[ + \frac{ig}{k_1 + k_2} \left[ b(k_2)b(k_1), M \right] + \]

\[ - \frac{g^2 tr(M)}{k_1^2 - k_2^2} \left[ b(k_2), b(k_1) \right] \]  \hspace{1cm} (4.18)
Then, after a product on the left by $M$ and the trace, we get

$$\left[ \tilde{t}_M(k_1), \tilde{t}_M(k_2) \right] = \frac{ig}{k_1 - k_2} trM \left( b(k_2)M b(k_1) - M b(k_1) M b(k_2) \right) +$$

$$- \frac{g^2 tr(M)}{k_1^2 - k_2^2} tr \left( M [b(k_2), b(k_1)] \right)$$

The left hand side of the above expression is skew-symmetric in $(k_1, k_2)$, while the right hand side is symmetric, so that each sides are identically zero. This proves the first part of the assertion.

Now, taking the trace of (4.18), one gets

$$\left[ \tilde{t}_M(k_1), t(k_2) \right] = \frac{ig}{k_1 - k_2} tr \left( b(k_2)M b(k_1) - b(k_1) M b(k_2) \right)$$

(4.19)

which proves that $\left[ \tilde{t}_M(k_1), t(k_2) \right]$ is symmetric under the exchange $u \leftrightarrow v$. On the other hand, if one multiplies (4.14) by $M$ and then take the trace, one obtains

$$\left[ t(k_1), \tilde{t}_M(k_2) \right] = \frac{2ik_1g}{k_1^2 - k_2^2} tr \left( M [b(k_2), b(k_1)] \right)$$

(4.20)

The antisymmetric part in $(k_1, k_2)$ of the above expression shows that we have $tr(M [b(k_2), b(k_1)]) = 0$, which then proves the second assertion.

**4.2 Analysis of the consistency relations**

We now turn to the consequences of the constraints (4.7-4.8). As far as $b^{(0)}$ is concerned, we have already seen that it must be central, but the consistency relation (4.7) also implies

$$(b^{(0)})^2 = \mathbb{I}$$

(4.21)

In the Fock representations $\mathcal{F}_B$ we consider, $b^{(0)}$ is a constant matrix and (4.21) shows that it can be diagonalized (by a constant $gl(N)$ matrix). Thus, up to a conjugation, one can suppose that we have

$$b^{(0)} = \mathbb{E} = \sum_{i=1}^N \epsilon_i E_{ii} \text{ with } \epsilon_i = \pm 1$$

(4.22)
For convenience, we will use the notation
\[ A = \{ \alpha \text{ such that } \epsilon_\alpha = +1 \} \subset [1, N] ; \dim A = M \]
\[ \bar{A} = \{ \bar{\alpha} \text{ such that } \epsilon_{\bar{\alpha}} = -1 \} \subset [1, N] ; \dim \bar{A} = N - M \]

(4.23)

In the basis (4.22), the commutation relations with the Lie subalgebra generators read
\[ [b_{ij}^{(1)}, b_{kl}^{(n)}] = ig(\epsilon_i + \epsilon_j) \left( \delta_{il} b_{kj}^{(n)} - \delta_{kj} b_{il}^{(n)} \right) \]

(4.24)

which indicates that the Lie subalgebra depends on the choice of \( E \). Indeed, the consistency relation (4.7) together with the choice of \( b^{(0)} = E \) lead to
\[ b_{\alpha \bar{\beta}}^{(1)} = 0 = b_{\bar{\alpha} \bar{\beta}}^{(1)} \]

(4.25)

so that the Lie subalgebra in \( S_R \) is a \( gl(M) \oplus gl(N - M) \) one.

More generally, the analysis of the consistency relations shows that \( b_{\alpha \bar{\beta}}^{(2n+1)} \), \( b_{\alpha \bar{\beta}}^{(2n+1)} \), \( b_{\alpha \bar{\beta}}^{(2n+1)} \) and \( b_{\alpha \bar{\beta}}^{(2n+1)} \) can be expressed in terms of the other generators, so that \( S_R \) is generated by:
\[ \mathcal{S}_R^{(2n)} = \left\{ b_{\alpha \bar{\beta}}^{(2n)}, b_{\bar{\alpha} \beta}^{(2n)} \right\}_{(\alpha \in A, \bar{\beta} \in \bar{A})} = (M, N - M) + (\bar{M}, N - M) \]

(4.26)

\[ \mathcal{S}_R^{(2n+1)} = \left\{ b_{\alpha \bar{\beta}}^{(2n+1)}, b_{\bar{\alpha} \bar{\beta}}^{(2n+1)} \right\}_{(\alpha, \bar{\beta} \in A, \alpha, \bar{\beta} \in \bar{A})} = (M^2, 0) + (0, (N - M)^2) \]

(4.27)

where we have indicated the decomposition in \( gl(M) \oplus gl(N - M) \) representations.

5 The NLS reflection matrices

Now, we come to the explicit construction of the reflection matrices as defined by (2.19-2.20). As already expressed in section 2, to each allowed \( B \) matrix (defined up to a \( gl(N) \) conjugation, see below) is associated a Fock space representation \( \mathcal{F}_B \) of the reflection algebra \( (\mathcal{B}_R, I) \). Indeed the value of the operator \( b(k) \) on the Fock space vacuum \( \Omega \) is directly given by the matrix \( B(k) \): see (2.21).

5.1 Classification of the \( B \)-reflection matrices

Expliciting the condition (2.22) with \( R \) again defined by (4.1) leads to the equations:
\[ [B(k_1), B(k_2)] = 0 \]

(5.1)
\[ B_1(k_1)B_1(k_2) - B_2(k_2)B_2(k_1) = \frac{k_1 + k_2}{k_1 - k_2} \left( B_2(k_2)B_1(k_1) - B_2(k_1)B_1(k_2) \right) \]  

(5.2)

From (5.1), one immediately deduces that the \( B(k) \) matrices (\( \forall k \)) can simultaneously put into a triangular form using a constant \( Gl(N) \) matrix:

\[ B(k) = UV(k)U^{-1} \quad \text{with} \quad [T(k_1),T(k_2)] = 0 \]  

(5.3)

Moreover, since we have

\[ R_{12}(k_1 - k_2)U_1U_2 = U_2U_1R_{12}(k_1 - k_2) \]  

(5.4)

the transformation (5.3) defines an automorphism of the whole algebra \( B_R \), as well as of the condition (2.11). Hence, we can suppose without any restriction that the matrices \( B(k) \) are triangular.

Imposing \( k_2 = -k_1 \) in (5.2) implies

\[ B(k)B(-k) = B(-k)B(k) = \rho(k)I \]  

(5.5)

where \( \rho \) is an even function which must be real in order to satisfy (2.20).

Then, a detailed study of (5.2) lead to the following classification of \( B \) reflections matrices:

**Case** \( \rho(k) \neq 0 \)

\[ B(k) = \beta(k) \frac{I + iakE}{1 + iak} \]  

(5.6)

\[ B(k) = \beta(k)E \]  

(5.7)

\[ B(k) = \beta(k)(I + akJ) \]  

(5.8)

**Case** \( \rho(k) = 0 \)

\[ B(k) = \beta(k)J \]  

(5.9)

with the conditions

\[ E^2 = I \quad ; \quad J^2 = 0 \quad ; \quad a \in \mathbb{R} \]  

(5.10)

Note that because of the \( Gl(N) \) invariance, \( E \) can be taken diagonal:

\[ E = \sum_{i=1}^{N} \epsilon_i E_{ii} \quad \text{with} \quad (E_{ii})_{kl} = \delta_{ik}\delta_{il} \]  

(5.11)

and \( J \) can be fixed to its Jordanian form:

\[ J = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_r \end{pmatrix} \quad \text{with} \quad J_i = (0) \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad 1 \leq i \leq r \]  

(5.12)
Requiring the matrix $B(k)$ to obey the condition (2.20) leads to the supplementary relation:

$$\beta(k)^* = \beta(-k) \quad (5.13)$$

Let us remark that the solution (5.8) obey to (2.22), but not to the development (4.4), and as such should be rejected in this context. The other solutions can all be expanded in series of $k^{-1}$ (provided $\beta(k)$ can be).

### 5.2 Boundary conditions associated to $B(k)$

From the algebraic study of the $S_R$ algebra and its Fock representations, one can determine the boundary conditions obeyed by the physical fields $\Psi(x, t)$. For such a purpose we associate to each reflection matrix $B(k)$ a differential operator (in the variable $x$) $D_B$ that will act on $\Psi(x, t)$. From (3.17), one infers that this operator will be fixed by its value on the eigenstates

$$\lambda_k(x) = e^{ikx} + B(k)e^{-ikx} \quad (5.14)$$

through the equation

$$D_B \lambda_k(x) \big|_{x=0} = 0 \quad (5.15)$$

We restrict our analysis to the physical condition

$$B(k) \cdot B(-k) = \mathbb{I} \quad (5.16)$$

In all cases, the boundary condition for the physical field $\Psi(x, t)$ takes the form:

$$D_B \Psi(x, t) \big|_{x=0} = 0 \quad (5.17)$$

We specify below the operator $D_B$ according to the classification (5.6-5.9).

#### 5.2.1 Case of $B(k) = \beta(k) \mathbb{I}$

From the condition $\beta(k)\beta(-k) = 1$, one deduces that $\beta(k)$ can be rewritten as

$$\beta(k) = \frac{A(k) + i}{A(k) - i} \quad \text{with} \quad A(k) \quad \text{real odd function} \quad (5.18)$$

Defining the differential operator $D_A$ through

$$(D_A f)(y) = \int dx \hat{A}(x) f(x + y) \quad \forall f \quad \text{C-function} \quad (5.19)$$
where $\hat{A}$ is the inverse Fourier transform of $A$. By definition of $\hat{A}$, we have

$$D_A e_p = A(p) e_p \quad \text{where} \quad e_p(y) = \exp(ipy) \quad (5.20)$$

This implies

$$(D_A + i)\lambda_k = (A(k) + i)e_k + \beta(k)(A(-k) + i)e_{-k} = (A(k) + i)(e_k - e_{-k}) \quad (5.21)$$

In other words, one gets $(D_A + i)\lambda_k(x)|_{x=0} = 0$, and $D_B = D_A + i$ is the required operator.

Let us remark that in the particular case evoked in (3.10)

$$\beta(k) = \frac{k - i\eta}{k + i\eta} \quad \text{i.e.} \quad A(k) = \frac{k}{\eta} \quad (5.22)$$

we get $\hat{A}(x) = \frac{i}{\eta} \frac{\partial}{\partial x}$ and we recover the boundary condition (3.2), using the fact that $\partial_x f(x + y) = \partial_y f(x + y)$.

5.2.2 Case of $B(k) = \beta(k)E$

As in the previous case, we introduce the odd function $A$ defined in (5.18). The calculation is similar to the previous one, except for the matrix dependence. We define the differential operator $D_A$ through (5.19). Then, one can choose

$$D_B = \frac{I + E}{2}(D_A + i) + \frac{I - E}{2}(D_A + i)\partial_x \quad (5.23)$$

This operator will obey the condition (5.15), because of the properties:

$$D_A e_k = A(k)e_k \quad \text{and} \quad D_A \partial_x e_k = ikA(k)e_k \quad (5.24)$$

$$(I + E)E = I + E \quad \text{and} \quad (I - E)E = -(I - E) \quad (5.25)$$

which lead to

$$D_B \lambda_k = \frac{I + E}{2}(A(k) + i)(e_k - e_{-k}) + \frac{I - E}{2}i k(A(k) + i)(e_k - e_{-k}) \quad (5.26)$$

It is then obvious that $D_B \lambda_k|_{x=0} = 0$. 

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5.2.3 Case of $B(k) = \beta(k) \frac{1 + iak \mathbb{E}}{1 + iak}$

This case comes as a mixing of the two previous ones. We first rewrite $B$ as:

$$B(k) = \beta(k) \frac{1 + \mathbb{E}}{2} + \tilde{\beta}(k) \frac{1 - \mathbb{E}}{2} \quad \text{with} \quad \tilde{\beta}(k) = \beta(k) \frac{1 - iak}{1 + iak} \quad (5.27)$$

and define $A(k)$ by $(5.18)$ and $\tilde{A}(k)$ by

$$\tilde{\beta}(k) = \frac{\tilde{A}(k) + i}{\tilde{A}(k) - i} \quad \text{i.e.} \quad \tilde{A}(k) = \frac{A(k) + ak}{1 - akA(k)} \quad (5.28)$$

The differential operators $D_A$ and $D_{\tilde{A}}$ will be constructed as above, and it is straightforward to check that

$$D_B = \frac{1 + \mathbb{E}}{2} (D_A + i) + \frac{1 - \mathbb{E}}{2} (D_{\tilde{A}} + i) \quad (5.29)$$

obeys the relation $(5.13)$.

6 Spontaneous symmetry breaking

In the Fock space representation $\mathcal{F}_R$ of the reflection algebra $\{B_R, I\}$, once the matrix $B(k)$ is fixed, one knows all the operators $b^{(n)}$ which have non-vanishing value on $\Omega$. Since the $S_R$-algebra constitutes the symmetry algebra of our problem, we are exactly faced with a mechanism of spontaneous symmetry breaking for our reflection algebra, itself part of our $B_R$-algebra. We present a generating function for the broken generators, and show that these generators form a commutative subalgebra of $S_R$.

6.1 Case of $B(k) = \beta(k) \mathbb{I}$

We start with the reflection matrix:

$$B(k) = \beta(k) \mathbb{I} \quad \text{with} \quad \beta(k) = \sum_{n \in J} \beta_n k^{-n} \quad ; \quad \beta_n \neq 0 \ \forall n \in J \subset \mathbb{N} \quad (6.30)$$

The vacuum expectation value of $b(k)$ is thus

$$< b^{(p)}_{ij} > = \beta_p \delta_{ij} \Rightarrow < b^{(p)}_{ii} - b^{(p)}_{jj} > = 0 \ \forall p \quad (6.31)$$

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Thus, the broken generators are all contained in $b^{(p)}$

$$\sum_{i=1}^{N} b^{(p)}_{ii} = tr(b^{(p)}) \forall p \in J$$  \hfill (6.32)

We first gather them into

$$t_B(u) = \sum_{p \in J} u^{-p} tr(b^{(p)})$$  \hfill (6.33)

Let us define the operator (depending on the new variable $u$ and on the variable $x = k^{-1}$):

$$D^k_u = \sum_{p \in J} \frac{1}{p!} (u^{-1}\partial)^p \equiv d(u^{-1}\partial) \quad \text{with} \quad \partial = \frac{\partial}{\partial(k^{-1})} ; \quad d(x) = \sum_{p \in J} \frac{x^p}{p!}$$  \hfill (6.34)

Then, from an obvious calculation, we have:

**Property 6.1** The generating function $t_B(u)$ is given by

$$t_B(u) = tr(D^k_u b(k))\big|_{k=\infty} = D^k_u t(k)\big|_{k=\infty}$$  \hfill (6.35)

where $t(k)$ has been defined in (4.13). It satisfies

$$t_B(u)\Omega = B(u)\Omega$$  \hfill (6.36)

**6.2 Case of $B(k) = \beta(k)E$**

Similarly, for

$$B(k) = \beta(k)E$$

with

$$\beta(k) = \sum_{n \in J} \beta_n k^{-n} ; \quad \beta_n \neq 0 \ \forall n \in J \subset \mathbb{N}$$

$$E = \sum_{\alpha=1}^{M} E_{\alpha\alpha} - \sum_{\bar{\alpha}=M+1}^{N} E_{\bar{\alpha}\bar{\alpha}} = \sum_{i=1}^{N} \epsilon_i E_{ii}$$  \hfill (6.37)

\footnote{We will ignore in our general approach the possibility for two broken generators of different grade, i.e. $trb^{(n)}$ and $trb^{(m)}$, to combine and form a third element, i.e. $\beta_n trb^{(n)} - \beta_m trb^{(m)}$, for which the symmetry is restored. A way of determining whether this last element (which algebraically annihilates the vacuum) would be to construct, when possible, a compensating transformation in the momentum space which preserves the correlation functions.}
we get the conditions:

\[ < b_{ij}^{(p)} > = \beta_p \epsilon_i \delta_{ij} \Rightarrow < \epsilon_i b_{ii}^{(p)} - \epsilon_j b_{jj}^{(p)} > = 0 \ \forall p \quad (6.38) \]

Then, the broken generators are of the form:

\[ \sum_{i=1}^{N} \epsilon_i b_{ii}^{(p)} = tr(\mathbb{E}b^{(p)}) \ \forall p \in J \quad (6.39) \]

With the same calculation, as in previous section, we obtain:

**Property 6.2** The generating function for the broken generators is given by:

\[ t_B(u) = \sum_{p \in J} u^{-p} tr(\mathbb{E}b^{(p)}) = tr(\mathbb{E}D_u^k b(k)) \bigg|_{k=\infty} = D_u^k \tilde{t}_E(k) \bigg|_{k=\infty} \quad \forall u \quad (6.40) \]

with the same definition \([6.34]\) of \(D_u^k = d(u^{-1}\partial)\). \(\tilde{t}_E(k)\) has been defined in \([7.14]\), with here \(\mathbb{M} = \mathbb{E}\). \(t_B(u)\) satisfies

\[ t_B(u)\Omega = B(u)\Omega \quad (6.41) \]

### 6.3 General case

Up to a redefinition of \(\beta\) and \(a\), one can rewrite \(B(k)\) defined in \([5.6]\) as:

\[ B(k) = \frac{\beta(k)}{1 + a}(\mathbb{E} + ak^{-1}\mathbb{I}) \quad (6.42) \]

so that the two previous cases are given by the limits \(a \to 0\) or \(\infty\). We have

\[ < b_{ij}(k) > = \frac{\beta(k)}{1 + a}(\epsilon_i + ak^{-1})\delta_{ij} \quad (6.43) \]

which shows that

\[ < b_{\alpha\alpha}(k) - b_{\beta\beta}(k) > = 0, \ \forall \alpha, \beta = 1, \ldots, M \quad (6.44) \]

\[ < b_{\bar{\alpha}\bar{\alpha}}(k) - b_{\bar{\beta}\bar{\beta}}(k) > = 0, \ \forall \bar{\alpha}, \bar{\beta} = M + 1, \ldots, N \quad (6.45) \]

Once again, the broken generators are gathered in a generating function \(t_B(u)\), and we get
Property 6.3  The generating function for broken generators is given by

\[ t_B(u) = \text{tr} \left( D^k u b(k) \right) \big|_{k=\infty} = \frac{1}{1 + a} \left( d(u^{-1} \partial)t(k) + a \tilde{d}(u^{-1} \partial) \tilde{t}_E(k) \right) \big|_{k=\infty} \]  

where now the matrix differential operator reads:

\[ D^k_u = \frac{1}{1 + a} \left( E d(u^{-1} \partial) + \mathbb{I} a \tilde{d}(u^{-1} \partial) \right) \quad \text{with} \quad \beta(k) = \sum_{p \in J} \beta_p k^{-p} \]  

As above
\[ d(x) = \sum_{p \in J} \frac{1}{p!} x^p \quad \text{and} \quad \tilde{d}(x) = \sum_{p \in J} \frac{1}{(p+1)!} x^{p+1} \]  

\[ t_B(u) \text{ satisfies} \quad t_B(u) \Omega = B(u) \Omega \]  

Corollary 6.4  The "broken generators" form a commutative algebra:

\[ [t_B(u), t_B(v)] = 0 \]  

Proof: Direct consequence of properties (4.1) and (4.2).

Remark: In all cases, \( t_B(u) \) is a scalar matrix, which means that we have at most only one broken generator at each level.

6.4  Example:  \( B(k) = \mathbb{I} \)

We are in the particular case \( \eta = 0 \) of the conditions given in (3.2-3.3), so that the boundary conditions are here

\[ \lim_{x \to \infty} \Phi(x,t) = 0 \quad \text{and} \quad \lim_{x \to 0} \partial_x \Phi(x,t) = 0 \]  

Due to the form of \( B(k) \), it is obvious that none of the generators is broken. Hence, we are in the very specific situation where no spontaneous symmetry breaking occurs.

However, remark that the boundary has an effect on the symmetry algebra: indeed, the Yangian symmetry which appears in the NLS hierarchy on the full line [19] is reduced here down to the ‘smaller’ \( S_R \)-algebra.

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7 The particular case \( N = 2 \)

7.1 Characteristic of \( N = 2 \)

The distinctive feature of \( N = 2 \) lies in the fact that the Pauli matrices, together with the identity matrix, form a basis of \( 2 \times 2 \) matrices and that all these matrices obey to the property 4.2. Indeed, one has:

**Property 7.1** For \( N = 2 \), \( t(k) \) is central in the \( S_R \)-algebra, and a generating system of this algebra is given by

\[
t(k) = tr b(k) = b_{11}(k) + b_{22}(k) \quad \text{and} \quad \tilde{t}_a(k) = tr(\sigma_a b(k)), \ a = 1, 2, 3 \quad (7.51)
\]

or more explicitly

\[
\tilde{t}_1(k) = b_{12}(k) + b_{21}(k) \ ; \ \tilde{t}_2(k) = i\left(b_{12}(k) - b_{21}(k)\right) \ ; \ \tilde{t}_3(k) = b_{11}(k) - b_{22}(k)
\]

**Proof:** \( \tilde{t}_a(k) \) are all generators obeying the property 4.2, which proves that \( t(k) \) commutes with these generators. It is then enough to show that these generators form a generating system (together with \( t(k) \)) to proves that \( t(k) \) is central (using also property 4.1). Using

\[
\sigma_a \sigma_b = \delta_{ab} \mathbb{I} + i\epsilon_{abc} \sigma_c \quad (7.52)
\]

and \( tr\sigma_a = 0 \), one gets

\[
b(k) = \frac{1}{2} \left(t(k) + \sum_{a=1}^{3} \tilde{t}_a(k) \sigma_a\right) \quad (7.53)
\]

showing therefore that \( \{t(k), \tilde{t}_a(k), \ a = 1, 2, 3\} \) is a generating system (as \( b_{ij}(k) \)).

The above property allows us to give simpler commutation relation for the \( S_R \)-algebra:

**Property 7.2** The commutation relations of the \( S_R \) algebra, for \( N = 2 \), are given by

\[
[t(k_1), t(k_2)] = 0 ; \quad [t(k_1), \tilde{t}_a(k_2)] = 0 \quad (7.54)
\]

\[
[\tilde{t}_a(k_1), \tilde{t}_b(k_2)] = \frac{-2g\epsilon_{abc} k_1 t(k_1)\tilde{t}_c(k_2) - k_2 t(k_2)\tilde{t}_c(k_1)}{k_1 - k_2} \quad k_1 + k_2 + ig \quad (7.55)
\]

\[
\tilde{t}_a(k_1)\tilde{t}_b(k_2) = \tilde{t}_a(k_2)\tilde{t}_b(k_1) \quad (7.56)
\]
The equations (7.54) are just the rephrasing of previous property. It remains to proves (7.54). We start with (4.18), for $M = \sigma_a$, multiply it by $\sigma_b$ and take the trace:

$$
[\tilde{t}_a(k_1), \tilde{t}_b(k_2)] = \frac{ig}{k_1 - k_2} \left( \sigma_b b(k_2) \sigma_a b(k_1) - \sigma_b b(k_1) \sigma_a b(k_2) \right) + \\
+ \frac{ig}{k_1 + k_2} \left( \sigma_b [b(k_2) b(k_1)] , \sigma_a \right) \tag{7.57}
$$

Proof: The equations (7.54) are just the rephrasing of previous property. It remains to proves (7.54). We start with (4.18), for $M = \sigma_a$, multiply it by $\sigma_b$ and take the trace:

Using the expression (7.53), this can be rewritten as

$$
4[\tilde{t}_a(k_1), \tilde{t}_b(k_2)] = \frac{ig}{k_1 - k_2} \left( \sigma_b b[k_2] \sigma_a \sigma_c \right) \left( \tilde{t}_d(k_2) \tilde{t}_c(k_1) - \tilde{t}_d(k_1) \tilde{t}_c(k_2) \right) + \\
+ \frac{ig}{k_1 + k_2} \left( \sigma_b [\sigma_c, \sigma_d, \sigma_a] \right) \tilde{t}_c(k_2) \tilde{t}_d(k_1) + \\
+ 4i\epsilon_{abc} \left( \frac{ig}{k_1 + k_2} (t(k_1) \tilde{t}_c(k_2) + t(k_2) \tilde{t}_c(k_1)) + \\
+ \frac{ig}{k_1 - k_2} (t(k_1) \tilde{t}_c(k_2) - t(k_2) \tilde{t}_c(k_1)) \right) \tag{7.58}
$$

with summation over repeated indices. A direct calculation shows

$$
tr\left( \sigma_b [\sigma_c, \sigma_d, \sigma_a] \right) = 4 \left( \delta_{bc} \delta_{ad} - \delta_{ac} \delta_{bd} \right) \tag{7.59}
$$

$$
tr\left( \sigma_b [\sigma_c, \sigma_d, \sigma_a] \right) = 2 \left( \delta_{bd} \delta_{ac} + \delta_{ad} \delta_{bc} - \delta_{ab} \delta_{cd} \right) \tag{7.60}
$$

so that we get

$$
[\tilde{t}_a(k_1), \tilde{t}_b(k_2)] = -\frac{1}{2} \frac{ig}{k_1 - k_2} \left( [\tilde{t}_a(k_1), \tilde{t}_b(k_2)] - [\tilde{t}_a(k_2), \tilde{t}_b(k_1)] \right) + \\
+ \frac{ig}{k_1 + k_2} \left( \tilde{t}_b(k_2) \tilde{t}_a(k_1) - \tilde{t}_a(k_2) \tilde{t}_b(k_1) \right) + \\
+ \frac{i\epsilon_{abc}}{k_1^2 - k_2^2} \left( 2igk_1 t(k_1) \tilde{t}_c(k_2) - 2igk_2 t(k_2) \tilde{t}_c(k_1) \right)
$$

Exchanging $(a, k_1) \leftrightarrow (b, k_2)$, the above equality leads to

$$
[\tilde{t}_b(k_2), \tilde{t}_a(k_1)] = -\frac{1}{2} \frac{ig}{k_1 - k_2} \left( [\tilde{t}_a(k_1), \tilde{t}_b(k_2)] - [\tilde{t}_a(k_2), \tilde{t}_b(k_1)] \right) + \\
+ \frac{ig}{k_1 + k_2} \left( \tilde{t}_a(k_1) \tilde{t}_b(k_2) - \tilde{t}_b(k_1) \tilde{t}_a(k_2) \right) + \\
- \frac{i\epsilon_{abc}}{k_1^2 - k_2^2} \left( 2igk_1 t(k_1) \tilde{t}_c(k_2) - 2igk_2 t(k_2) \tilde{t}_c(k_1) \right)
$$
so that
\[
2[\tilde{t}_a(k_1), \tilde{t}_b(k_2)] = [\tilde{t}_a(k_1), \tilde{t}_b(k_2)] - [\tilde{t}_b(k_2), \tilde{t}_a(k_1)]
\]
\[
= \frac{ig}{k_1 + k_2} \left( [\tilde{t}_b(k_2), \tilde{t}_a(k_1)] + [\tilde{t}_b(k_1), \tilde{t}_a(k_2)] \right) + \frac{i\epsilon_{abc}}{k_1^2 - k_2^2} \left( 4igk_1 t(k_1) \tilde{t}_c(k_2) - 4igk_2 t(k_2) \tilde{t}_c(k_1) \right)
\]

Computing \([\tilde{t}_b(k_2), \tilde{t}_a(k_1)] + [\tilde{t}_b(k_1), \tilde{t}_a(k_2)]\) leads to
\[
\frac{k_1 - k_2 + ig}{k_1 - k_2} \left( [\tilde{t}_a(k_1), \tilde{t}_b(k_2)] + [\tilde{t}_a(k_2), \tilde{t}_b(k_1)] \right) = \frac{4i\epsilon_{abc}}{k_1^2 - k_2^2} \left( igk_1 t(k_1) \tilde{t}_c(k_2) - igk_2 t(k_2) \tilde{t}_c(k_1) \right)
\]
which proves (7.53).

On the other hand, starting from (7.58), one computes
\[
[\tilde{t}_a(k_1),\tilde{t}_b(k_2)] - [\tilde{t}_a(k_2),\tilde{t}_b(k_1)] = \frac{ig}{k_1 + k_2} \left( \{\tilde{t}_a(k_1),\tilde{t}_b(k_2)\} - \{\tilde{t}_a(k_2),\tilde{t}_b(k_1)\} \right)
\]
Then, using (7.55), one gets
\[
[\tilde{t}_a(k_1),\tilde{t}_b(k_2)] = \tilde{t}_a(k_2),\tilde{t}_b(k_1)]
\]
\[
\{\tilde{t}_a(k_1),\tilde{t}_b(k_2)\} = \{\tilde{t}_a(k_2),\tilde{t}_b(k_1)\}
\]
which is equivalent to the second relation.

**Corollary 7.3** The commutation relations with the Lie subalgebra generators read

\[
[\tilde{t}_a^{(1)}, \tilde{t}_b^{(2)}] = -2g\epsilon_{abc} t^{(0)} \tilde{t}_c(k_2)
\]
\[
[t^{(1)}, \tilde{t}_b^{(2)}] = 0
\]

**Proof:** One picks up the term \(k_1^{-1}\) in the previous commutators.

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7.2 Vacuum preserving algebra when $B(k) = \beta(k)I$

In this case, the broken generators are all gathered in $t(k)$, and we can replace $t(k)$ (which is central) by its value $2\beta(k)$.

Thus, the vacuum preserving algebra is just the one generated by $\tilde{t}_a(k)$, $a = 1, 2, 3$. Its commutation relations read

$$\left[\tilde{t}_a(k_1), \tilde{t}_b(k_2)\right] = \frac{-4g\epsilon_{abc}}{(k_1 + k_2 + ig)(k_1 - k_2)} \left( k_1 \beta(k_1)\tilde{t}_c(k_2) - k_2 \beta(k_2)\tilde{t}_c(k_1) \right)$$

(7.65)

Specifying to $m = n = 1$, we get

$$\left[\tilde{t}_a^{(1)}, \tilde{t}_b^{(1)}\right] = -4g\epsilon_{abc} \tilde{t}_c^{(1)}$$

(7.66)

which is an $sl_2$ Lie subalgebra.

Then, at the Lie subalgebra level, the spontaneous symmetry breaking is a $gl_2 \to sl_2$ one.

7.3 Vacuum preserving algebra when $B(k) = \beta(k)\sigma_3$

The broken generator is now $\tilde{t}_3(k)$, and its value is $2\beta(k)$.

Thus, the vacuum preserving algebra is now the one generated by $t(k)$, $\tilde{t}_1(k)$ and $\tilde{t}_2(k)$. It is indeed an algebra, as can be seen from the commutators:

$$\left[t(k_1), t(k_2)\right] = 0, \left[\tilde{t}_a(k_1), \tilde{t}_a(k_2)\right] = 0, \ a = 1, 2$$

$$\left[\tilde{t}_1(k_1), \tilde{t}_1(k_2)\right] = 0, \left[\tilde{t}_2(k_1), \tilde{t}_2(k_2)\right] = 0$$

$$\left[\tilde{t}_1(k_1), \tilde{t}_2(k_2)\right] = \frac{-4g}{(k_1 + k_2 + ig)(k_1 - k_2)} \left( k_1 \beta(k_2)t(k_1) - k_2 \beta(k_1)t(k_2) \right)$$

However, since we are in the case $E = \sigma_3 \neq I$, the condition $b(k)b(-k) = I$ implies in particular $b_{12}^{(1)} = b_{12}^{(1)} = 0$, or in our generating system $\tilde{t}_1^{(1)} = \tilde{t}_2^{(1)} = 0$. Thus, for the values $m = n = 1$, we get only one generator $t^{(1)}$, and one recognizes a $gl_1$ Lie subalgebra.

Then, at the Lie subalgebra level, the spontaneous symmetry breaking is a $gl_1 \oplus gl_1 \to gl_1$ one.

8 Outlook and Conclusions

We proposed in this paper an algebraic approach for studying the symmetry content and the phenomenon of spontaneous symmetry breaking in integrable
systems on the half line. The main tool is the quantum inverse scattering transform, where the familiar Zamolodchikov-Faddeev (ZF) algebra is replaced by the boundary algebra $B_R$. The latter reflects the breakdown of translation invariance on the half line, which is codified by a set of boundary generators. These generators close a Sklyanin type subalgebra $S_R \subset B_R$ and commute with the Hamiltonians of the whole integrable hierarchy under consideration. For this reason $S_R$ represents the central point of our investigation.

The basic and actually unique input of the scheme is the $R$-matrix, which describes the two-body scattering and determines, via the boundary Yang-Baxter equation, all reflection matrices respecting integrability. Each reflection matrix in turn fixes a boundary condition and therefore the dynamics and the symmetry of the system. The concrete example, we have focused on, is the $gl(N)$-NLS model, whose $R$-matrix is given by (4.1). If one considers this model on the whole line, the underlying symmetry algebra is the Yangian $Y(sl(N))$ [20]. The corresponding generators can be constructed [19] in terms of the associated ZF algebra. The latter admits unique (up to unitary equivalence) Fock representation, whose vacuum is annihilated by all Yangian generators, i.e. the whole Yangian $Y(sl(N))$ is an exact symmetry of the theory. On the half line the situation is quite different. The counterpart of $Y(sl(N))$ is the Sklyanin algebra $S_R$. The integrability preserving boundary conditions are encoded in the reflection matrices, which at the same time parametrize the inequivalent Fock representations of the underlying boundary algebra $B_R$. The existence of such representations is crucial. It has no analog in the ZF algebra and allows to describe the physically inequivalent phases of the system on the half line. The generators of $S_R$, which do not annihilate the vacuum of a given phase, are spontaneously broken in that phase.

The framework developed in this paper can be applied in a more general context to the trigonometric and the elliptic series of $R$-matrices as well. It will be interesting to investigate the structure of the associated $S_R$ algebras, which will shed new light on the symmetry content of the corresponding integrable models on the half line.

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