Mathematical modeling of the magnetic field of red blood cells in narrow capillaries

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Abstract. The red blood cell (RBC) moves in a narrow capillary and asymmetric of its shape lead to tank-treading motion of the membrane. Electric charges on the RBC membrane generate a magnetic field in the surrounding space. A mathematical model imitating the motion of RBC through capillaries with a diameter less than 8 \( \mu m \) is developed in order to estimate the distribution of plasma pressure, plasma velocity and magnetic field strength in the vicinity of RBC. It is assumed that plasma flow between erythrocytes in the capillary satisfies the Poiseuille law, and in the gap between the capillary wall and the RBC membrane satisfies the Reynolds system of equations for the lubricating layer. Calculations made on a computer allow us to estimate the shape of the RBC, the speed of rotation of the RBC membrane, the plasma pressure, the plasma velocity and the magnetic field strength in the vicinity of the RBC. It is shown that the distribution of the magnetic field strength in the vicinity of the erythrocyte is non-uniform and depends on the capillary diameter, RBC speed, RBC charge, volume and surface area of RBC.

1. Introduction
A mathematical model imitating the motion of red blood cells (RBCs) through capillaries with a diameter less than 8 \( \mu m \) is developed in order to analyze blood flow in the microcirculatory system. In this model, the size of capillaries, the volume and the surface area of red blood cells as well as their visco-elastic properties, and blood plasma viscosity are considered. The volume and the surface area of red blood cells are supposed to be constant, and the cell membrane is assumed to be perfectly flexible, but inextensible [1]. Plasma flow in the gaps between the cell surface and the capillary wall is supposed to be governed by lubrication theory. The model is formulated as a system of partial differential equations which can be solved by a computer using the method of finite differences.

The distribution of red blood cells and the velocity with which they move through capillaries, both determining the conditions of respiratory gases transfer the tissue microdomains, depend upon the forces influencing blood flow, the shape of circulatory channel, the size and mechanical properties of blood cells, and plasma viscosity. In order to investigate the influence of a complex of all these factors upon red blood cells in microvessels, mathematical models are generally used [2], [3]. Most of them imitate an axisymmetric, parachute-shaped red blood cell. However, there also exist more complicated
two-dimensional models considering cell asymmetry [4], [5], [6], [7]. For further approximation to the realistic conditions, the development of a model imitating the motion of a three-dimensional, asymmetric red blood cell through capillaries with a diameter less than 8 μm seems to be important. In such narrow capillaries, the influence performed by the size of blood cells and their mechanical properties as well as by plasma viscosity upon the resistance to blood flow is expressed most prominently. Here we develop such a model and use it to analyze the relations between main parameters which determine the microhemodynamic conditions, i.e. the red blood cell and plasma velocities, the width of a gap between the red blood cell and the capillary wall, and the frequency of a tank-treading motion of the cell membrane.

The magnetic field generated by the charges located on the red blood cell membrane can influence the processes occurring both outside the erythrocyte and inside it. In particular, the magnetic field of the red blood cell can affect the movement of other elements of the blood flow with a charge, in particular, erythrocytes, platelets, artificial magnetic particles used in the targeted transport of drugs [8]. In addition, the magnetic field of the red blood cell can affect the processes occurring inside the erythrocyte, in particular, hemoglobin molecules containing iron atoms. Therefore, the study of the magnitude and distribution of the magnetic field created by the red blood cell at short distances is an important task of the microcirculation system.

2. Mathematical model

Blood flow in a capillary may be regarded as a motion of an elastic body (red blood cell) together with surrounding viscous fluid (plasma) through a cylindrical tube, due to a hydrostatic pressure gradient. For the imitation of this motion a system of equations can be used which describes both the hydrodynamics phenomena in plasma and the deformation of a red blood cell, taken as an elastic body. Red blood cell is assumed to be a bullet in shape and to move through a capillary with definite velocity. Its surface represents a truncate cylinder which is limited at one side by a rotation hemiellipsoid with hemiaxes a and b.

Plasma pressure in a gap between the red blood cell and the capillary wall is supposed to depend on two parameters x and φ. Then, in the cylindrical system of coordinates (x, r, φ), pressure P(x, φ), axial u, radial v and azimuth w component of plasma velocity in the gap between the red blood cell and the capillary wall, with the width h(x, φ), satisfy equations [5]

\[
\begin{align*}
\frac{\partial P}{\partial x} &= \mu \frac{\partial}{\partial r} \left[ r \frac{\partial u}{\partial r} \right], \\
\frac{\partial P}{\partial \phi} &= \mu \frac{\partial}{\partial r} \left[ r \frac{\partial w}{\partial r} \right], \\
\frac{\partial P}{\partial r} &= 0, \\
\frac{\partial w}{\partial \phi} - \frac{\partial (ru)}{\partial x} - \frac{\partial (rv)}{\partial r} &= 0,
\end{align*}
\]

where μ is plasma viscosity.

While solving the system of equations, it is convenient to assume that the cell is motionless and that the capillary wall moves with velocity U. In this case, the boundary conditions are given by

\[
\begin{align*}
u = W_1, \quad w = 0, \text{ given } r = R, \\
u = U_1, \quad w = 0, \text{ given } r = R+h,
\end{align*}
\]

where U₁ and W₁ are the projections of U and W on the axis X, R (x, φ) is the distance from the axis X to the erythrocyte membrane, h (x, φ) is the width of the gap between the capillary wall and the red blood cell.
The radial component of plasma velocity \( v \) at both boundaries satisfies kinematic conditions

\[
v = w \frac{\partial r}{\partial r} + u \frac{\partial r}{\partial \varphi}.
\]

\( P, u, v, w \) are the \( 2\pi \)-periodical functions of \( \varphi \). \( R(x, \varphi) \) is the red blood cell radius, \( h(x, \varphi) \) is the gap width, \( U_1 \) and \( V_1 \) are projection of \( U \) and \( V \) (velocities of points on the cell membrane) on the axis \( X \). The value of \( V \), considering that the points on the cell membrane move in the plane which is parallel to that of cell symmetry with equal frequency \( f \), is estimated by \[4\]

\[
W = n \times \nabla(f \ F(z)),
\]

where \( n \) is a single normal to the cell surface, and \( F(x) \) is a function which satisfies equation

\[
F_1(z) = T(z),
\]

where \( T(x) \) are the lengths of close guidelines which indicate the motion of the cell membrane points.

Using equations and considering the boundary conditions, plasma velocity components, \( u \) and \( w \), can be estimated.

\[
u = \frac{1}{4\mu} \frac{\partial P}{\partial x} \left[ r^2 - R^2 - \frac{\ln(r/R)(h^2 + 2Rh)}{\ln(1 + h/R)} \right] + W_l + (U_1 - W_1) \frac{\ln(r/R)}{\ln(1 + h/R)},
\]

\[
w = \frac{1}{\mu} \frac{\partial P}{\partial \varphi} \left[ r - R - h \frac{\ln(r/R)}{\ln(1 + h/R)} \right].
\]

Integrating of equations according to the gap width and filling in \( u \) and \( w \) into resulting equation give differential equation of elliptical type \[5\]

\[
A_1 \frac{\partial^2 P}{\partial x^2} + A_2 \frac{\partial P}{\partial x} + A_3 \frac{\partial^2 P}{\partial \varphi^2} + A_4 \frac{\partial P}{\partial \varphi} + A_5 = 0,
\]

whose coefficients are functions of \( R, h, \mu, x, \varphi, U_1, W_1 \).

Quotients of equation, with given values for \( U, D \) (capillary diameter), \( \mu, S \) (red blood cell surface area), and \( V \) (red blood cell volume), are determined by the cells shape \((a, b)\), its position in the capillary \( \beta \) is the angle between the axis \( X \) and the capillary axis, \( l \) is the distance between the point at which the axis \( X \) intersects the cell surface and the capillary axis), and the frequency \( f \). Equation can be solved using the network procedure and considering condition of \( 2\pi \)-periodicity of \( P \) according to the angle \( \varphi \) and a boundary condition that plasma pressure before the front of the red blood cell is zero.

While estimating the red blood cells shape and its position in the capillary, it is assumed that its motion is rectilinear and uniform, and that its membrane rotates uniformly. That means that the integral sums of forces and their moments which act upon the red blood cell are zero. It is also considered that the resultant of tangent forces attached to the membrane from without end from within the cell is equal to zero. In addition to these conditions, a relationship which characterized the cells elastic properties can be obtained.

To an approximation,

\[
P_1 - P_2 = E \frac{\Delta c}{c},
\]
where $\Delta c$ is the increment of $c$ (the cell length as measured along the axis $X$), $P_1$ and $P_2$ are the frontal and the lateral tensions in the cell, which both result from the external forces, and $E$ is Young’s module of the cell. Thus, there is a system of 5 relationships including 5 unknown parameters, $a$, $b$, $\beta$, $I$, and $f$, which can be estimated with an aid of computer.

On the erythrocyte membrane there is a finite number of discrete charges and each of these charges at some pre-selected point generates a magnetic field. A charge $Q$ moving with velocity $V$ at a distance $R$ generates a magnetic field of intensity $H$ (in the SI system)

$$H = \frac{QV \sin \alpha}{4\pi R^2},$$

where $\alpha$ is the angle between the radius vector $R$ and the direction of the velocity $V$ [9], [10].

The magnetic field strength of several charges at some point in space is defined as the vector sum of the strengths generated by the charges at this point.

3. Results and discussion
Computations using the present model allow to estimate the shape of the red blood cell and its position in the capillary, the gap width, the frequency of the cell membrane rotation, the distribution of plasma pressure and velocity in the vicinity of the cell, and the pressure gradient over the endpoints of the cell, which causes the cells motion through the capillary.

The calculations were carried out on a computer with the following parameters: erythrocyte charge $Q_{RBC}=3.2 \times 10^{-12}$ C, the number of closed trajectories $N=201$ on the surface of the erythrocyte, on which $N_z=149624$ charges are located, the erythrocyte volume $V_{RBC}=94 \mu m^3$, the erythrocyte surface area $S_{RBC}=135 \mu m^2$, erythrocyte radius $R=2 \mu m$, forming a truncated cylinder $L_2=11.5 \mu m$ and $L_1=3.4 \mu m$, the speed of the erythrocyte in the capillary is $V=100 \mu m/sec$.

The figures show the distribution of the magnetic field strength $H$ (A/m) at a distance of 1 $\mu m$ from the front of the erythrocyte in the plane perpendicular to the erythrocyte axis $X$ with no membrane rotation (figure 1) and with a membrane rotation speed of 1 revolution per second (figure 2).

![Figure 1. The distribution of the magnetic field strength $H$ (A/m). The membrane does not rotate.](image_url)
It can be seen from the figures that with an increase in the speed of rotation of the erythrocyte membrane, the distribution of the magnetic field strength becomes more asymmetric. So, at a rotational speed \( w = 1 \) revolution per second, the difference between the maxima of \( 7.61 \times 10^{-7} \) A/m and \( 6.42 \times 10^{-7} \) A/m is equal to \( 1.19 \times 10^{-7} \) A/m, and at speed rotation \( w = 3 \) revolutions per second, the difference between the maxima of \( 8.70 \times 10^{-7} \) A/m and \( 5.39 \times 10^{-7} \) A/m is equal to \( 3.31 \times 10^{-7} \) A/m, i.e. increases by 2.8 times.

4. Conclusion
Thus, a mathematical model of the distribution of the magnetic field of the red blood cell is developed. It is shown that the distribution of the red blood cell magnetic field strength depends on the red blood cell charge, the volume and surface area of the red blood cell, the red blood cell velocity in the capillary, the capillary diameter and, consequently, the red blood cell diameter. At large distances from the red blood cell, the magnetic field of the red blood cell approximately coincides with the field of the dipole. At small distances from the red blood cell (smaller capillary diameter), the distribution of the magnetic field strength has large drops and has a significant decrease in front of the forward part of the red blood cell. This can influence, for example, the trajectories of magnetic microparticles of drugs used in the targeted transport.

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