The group structure of the homotopy set whose target is the automorphism group of the Cuntz algebra

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Abstract

We determine the group structure of the homotopy set whose target is the automorphism group of the Cuntz algebra $\mathcal{O}_{n+1}$ for finite $n$ in terms of K-theory. We show that there is an example of a space for which the homotopy set is a non-commutative group, and hence the classifying space of the automorphism group of the Cuntz algebra for finite $n$ is not an H-space. We also make an improvement of Dadarlat’s classification of continuous fields of the Cuntz algebras in terms of vector bundles.

1 Introduction

Dadarlat [9] computed the homotopy set $[X, \text{Aut } A]$ for a Kirchberg algebra $A$ under a mild assumption of a space $X$. He constructed a bijection between $[X, \text{Aut } A]$ and a relevant KK-group, and showed that it is a group homomorphism when $X$ is an H’-space (co-H-space). However, the group structure of $[X, \text{Aut } A]$ for more general $X$ is still unknown.

The Cuntz algebra $\mathcal{O}_{n+1}$ is a typical example of a Kirchberg algebra, and it plays an important role in operator algebraic realization of the mod $n$ K-theory [22]. Dadarlat’s computation shows that $[X, \text{Aut } \mathcal{O}_{n+1}]$ as a set is identified with the mod $n$ K-group $K_1(X; \mathbb{Z}_n)$. One of the main purposes of this paper is to determine the group structure of $[X, \text{Aut } \mathcal{O}_{n+1}]$, and we show that it is indeed different from the ordinary group structure of $K^1(X; \mathbb{Z}_n)$ in general. In particular, we verify that the group $[X, \text{Aut } \mathcal{O}_{n+1}]$ is non-commutative when $X$ is the product of the Moore space $M_n$ and its reduced suspension $\Sigma M_n$. Our computation uses the Cuntz-Toeplitz algebra $E_{n+1}$ in an essential way, for which the homotopy groups of the automorphism group are computed in [24].

The unitary group $U(n+1)$ acts on $\mathcal{O}_{n+1}$ through the unitary transformations of the linear span of the canonical generators, and it induces a map from $[X, BU(n+1)]$ to $[X, B \text{Aut } \mathcal{O}_{n+1}]$. When $X$ is a finite CW-complex with dimension $d$, Dadarlat [12, Theorem 1.6] showed that the map is a bijection provided that $n \geq [(d-3)/2]$ and $H^*(X)$ has no $n$-torsion. Another purpose of this paper is to remove the first condition by a localization trick.

We use the following notation throughout the paper. For a unital C*-algebra $A$, we denote by $U(A)$ the unitary group of $A$, and by $U(A)_0$ the path component of $1_A$ in $U(A)$. For a non-unital C*-algebra $B$, we denote its unitization by $B^\sim$. We denote by $B(H)$ the algebra of bounded operators on a Hilbert space $H$, by $K$ the algebra of compact operators on a separable Hilbert space, and by $M_n$ the algebra of $n \times n$ matrices.

Our standard references for K-theory are [4, 16]. For a projection $p \in A$ (resp. a unitary $u \in U(A)$), we denote by $[p]_0$ (resp. $[u]_1$) its class in the $K$-group $K_0(A)$ (resp. $K_1(A)$). For a compact Hausdorff space $X$, we identify the topological K-groups $K^i(X)$ with $K_i(C(X))$ where $C(X)$ is the C*-algebra of

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the continuous functions on $X$. When moreover $X$ is path connected, we choose a base point $x_0$, and set $K^0(X)$ to be the kernel of the evaluation map $(ev_{x_0})_*: K^0(X) \to K^0(\{x_0\}) = \mathbb{Z}$, which is identified with $K_i(C_0(X,x_0))$ where $C_0(X,x_0)$ is the $C^*$-algebra of the continuous functions on $X$ vanishing at $x_0$. We denote by $\Sigma X$ the reduced suspension of $X$. For two topological spaces $X$ and $Y$, we denote by $\text{Map}(X,Y)$ the set of continuous map from $X$ to $Y$, and by $[X,Y]$, the quotient of $\text{Map}(X,Y)$ by homotopy equivalence.

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## 2 Mod $n$ K-theory

In this section, we summarize the basics of mod $n$ K-theory from the viewpoint of operator algebras.

Recall that the Cuntz algebra $\mathcal{O}_{n+1}$ is the universal C*-algebra generated by $n+1$ isometries $\{S_i\}_{i=0}^n$ with mutually orthogonal ranges whose summation is 1. Its K-groups are

$$K_0(\mathcal{O}_{n+1}) = \mathbb{Z}_n, \quad K_1(\mathcal{O}_{n+1}) = 0,$$

(see [2] Theorem 3.7, 3.8, Corollary 3.11]). The Cuntz Toeplitz algebra $E_{n+1}$ is the universal C*-algebra generated by $n+1$ isometries $\{T_i\}_{i=0}^n$ with mutually orthogonal ranges, and it is KK-equivalent to the complex numbers $\mathbb{C}$. The closed two-sided ideal generated by the minimal projection $e = 1 - \sum_{i=0}^{n} T_i T_{i}^*$ is isomorphic to $K$, which is known to be the only closed non-trivial two-sided ideal. Then the quotient algebra $E_{n+1}/K$ is isomorphic to $\mathcal{O}_{n+1}$ with identification $S_i = \pi(T_i)$, where $\pi$ is the quotient map.

For a natural number $n$, we denote by $M_n$ the Moore space, the mapping cone of the map $n: S^1 \ni z \mapsto z^n \in S^1$:

$$M_n := ([0,1] \times S^1) \sqcup S^1/\sim,$$

where $(0,z) \sim (0,1)$ and $(1,z) \sim z^n$ for every $z \in S^1$. For cohomology and K-groups, we have

$$H^0(M_n) = \mathbb{Z}, \quad H^1(M_n) = 0, \quad H^2(M_n) = \mathbb{Z}_n, \quad H^k(M_n) = 0 \text{ for } k \geq 2,$$

$$\tilde{K}^0(M_n) = \mathbb{Z}_n, \quad \tilde{K}^1(M_n) = 0,$$

(see [15] Theorem 9.10 for example). Since $C_0(M_n, pt)$ and $\mathcal{O}_{n+1}$ have the same K-theory and they are in the bootstrap class, they are KK-equivalent (see [1] Section 22.3]).

The mod $n$ K-group of the pointed space $(X,x_0)$ is originally defined by

$$\tilde{K}^i(X;\mathbb{Z}_n) := \tilde{K}^i(X \wedge M_n), \quad i = 0,1.$$

We refer to [1] for the mod $n$ K-theory, and refer to [22] Section 8] for an operator algebraic aspect of it. The Bott periodicity of the K-theory induces the Bott periodicity of the mod $n$ K-theory. By the KK-equivalence of $C_0(M_n, pt)$ and $\mathcal{O}_{n+1}$, the identification

$$\tilde{K}^i(X \wedge M_n) = K_i(C_0(X,x_0) \otimes C_0(M_n, pt)) \cong K_i(C_0(X,x_0) \otimes \mathcal{O}_{n+1})$$

is natural in the variable $X$ (see [23] Theorem 6.4]). We can identify the Bockstein exact sequence with the 6-term exact sequence

$$
\begin{array}{c}
K^0(X) \xleftarrow{\beta} K^0(X;\mathbb{Z}_n) \\
\downarrow \quad \downarrow \\
K^1(X;\mathbb{Z}_n) \xleftarrow{\rho} K^1(X) \xrightarrow{-n} K^1(X).
\end{array}
$$

arising from the exact sequence

$$0 \to C_0(X,x_0) \otimes \mathbb{K} \to C_0(X,x_0) \otimes E_{n+1} \to C_0(X,x_0) \otimes \mathcal{O}_{n+1} \to 0.$$

The map $\beta$ is called Bockstein map, and $\rho$ is called the reduction map. We frequently identify $\beta$ with the index map or the exponential map in the 6-term exact sequence.
Lemma 2.1. We have the following isomorphisms from the Bockstein exact sequence:

\[ \rho: \tilde{K}^0(M_n) \to \tilde{K}^0(M_n; \mathbb{Z}_n), \quad \beta: \tilde{K}^1(M_n; \mathbb{Z}_n) \to \tilde{K}^0(M_n). \]

The K-theory has a multiplication \( \mu \) defined by the external tensor product of vector bundles:

\[ \mu: K^0(X) \otimes K^0(Y) \to K^0(X \times Y) \]

We denote the diagonal map by \( \Delta_X: X \to X \times X \). This gives the ring structure of \( K^0(X) \):

\[ x \cdot y := \Delta_X \mu(x \otimes y), \quad x, y \in K^0(X). \]

This induce the ring structure of \( \tilde{K}^0(X) \) by \( \Delta_X: X \to X \wedge X \):

\[ x \cdot y := \Delta_X \mu(x \otimes y), \quad x, y \in \tilde{K}^0(X). \]

From [16, Chap.II, Theorem 5.9], the reduced K-group \( \tilde{K}^0(X) \) is the set of nilpotent elements of \( K^0(X) \), and in particular \( \tilde{K}^0(\Sigma X) \cdot \tilde{K}^0(\Sigma X) = \{0\} \).

The multiplication \( \mu \) extends to \( \tilde{K}^i(X) \), \( i = 0, 1 \) by

\[ \mu: \tilde{K}^0(S^i \wedge X) \otimes \tilde{K}^0(S^j \wedge Y) \to \tilde{K}^0(S^{i+j} \wedge X \wedge Y), \]

with the property

\[ T_{X,Y} \mu(y \otimes x) = (-1)^{ij} \mu(x \otimes y), \quad x \in \tilde{K}^i(X), \ y \in \tilde{K}^j(Y). \]

where the map \( T_{X,Y}: X \wedge Y \to Y \wedge X \) is the exchange of the coordinates (see [16, Chap. II section 5.30]). In a similar way, the multiplication \( \mu \) defines the following:

\[ \mu_L: \tilde{K}^i(X) \otimes \tilde{K}^j(Y; \mathbb{Z}_n) \to \tilde{K}^{i+j}(X \wedge Y; \mathbb{Z}_n), \]

\[ \mu_R: \tilde{K}^i(X; \mathbb{Z}_n) \otimes \tilde{K}^j(Y) \to \tilde{K}^{i+j}(X \wedge Y; \mathbb{Z}_n), \]

with the same property (see [11, Section 3]):

\[ T_{X,Y} \mu_R(y \otimes x) = (-1)^{ij} \mu_L(x \otimes y), \ x \in \tilde{K}^i(X), \ y \in \tilde{K}^j(Y). \]

The multiplications \( \mu, \mu_L \) and \( \mu_R \) are compatible with the reduction \( \rho \) and the map \( \delta : \)

\[ \mu_R(\rho \otimes \text{id}) = \rho \mu, \quad \beta(\mu_R(\text{id} \otimes \text{id})) = \mu(\beta \otimes \text{id}), \]

\[ \mu_L(\text{id} \otimes \rho) = \rho \mu, \quad \beta(\mu_L(\text{id} \otimes \text{id})) = \mu(\text{id} \otimes \beta). \]

Since the identification \( \tilde{K}^i(X; \mathbb{Z}_n) \cong K_i(C_0(X, x_0) \otimes \mathcal{O}_{n+1}) \) is natural, it is compatible with the Kasparov product, and the multiplications \( \mu_L \) and \( \mu_R \) extend to

\[ \mu_L: K_i(C(X)) \otimes K_j(C(Y) \otimes \mathcal{O}_{n+1}) \to K_{i+j}(C(X \times Y) \otimes \mathcal{O}_{n+1}) \]

\[ \mu_R: K_i(C(X) \otimes \mathcal{O}_{n+1}) \otimes K_j(C(Y)) \to K_{i+j}(C(X \times Y) \otimes \mathcal{O}_{n+1}). \]

In particular, for \( u \in U((C(X) \otimes \mathcal{O}_{n+1})) \) and a projection \( p \in C(X) \otimes \mathbb{M}_m \), we have

\[ \mu_L([p]_0 \otimes [u]_1) = [p \otimes u + (1_m - p) \otimes 1_{\mathcal{O}_{n+1}}]_1 \in K_1(C(X \times X, \mathbb{M}_m \otimes \mathcal{O}_{n+1})) = K_1(C(X \times X, \mathcal{O}_{n+1})). \]

We also use the Künneth theorem of the reduced K-theory.

Theorem 2.2 ( [11, Theorem 23.1.3]). For pointed spaces \( X \) and \( Y \), we have the following exact sequence

\[ 0 \to \bigoplus_{i=0,1} \tilde{K}^i(X) \otimes \tilde{K}^{i+*}(Y) \to \tilde{K}^*(X \wedge Y) \to \bigoplus_{i=0,1} \text{Tor}(\tilde{K}^i(X), \tilde{K}^{i+1-*}(Y)) \to 0, \]

that splits unnaturally.
We note that the map $\tilde{K}^i(X) \otimes \tilde{K}^j(Y) \to \tilde{K}^{i+j}(X \wedge Y)$ above is given by the multiplication $\mu$. Puppe sequence yields the following lemmas.

**Lemma 2.3** ([13] Section 10, Proposition 3.4). For compact pointed spaces $X$ and $Y$, the sequence $X \vee Y \to X \times Y \to X \wedge Y$ induces a split exact sequence

$$0 \to \tilde{K}^i(X \wedge Y) \to \tilde{K}^i(X \times Y) \to \tilde{K}^i(X) \otimes \tilde{K}^i(Y) \to 0.$$ 

The splitting is given by the projections $\Pr_X : X \times Y \to X$ and $\Pr_Y : X \times Y \to Y$.

We have the diagram below

$$
\begin{array}{ccc}
K^i(X \times Y) & \xrightarrow{\mu(\otimes 1)} & K^i(X) \\
\downarrow & & \downarrow \\
\tilde{K}^i(X \times Y) & \xrightarrow{\Pr^*_X \otimes \Pr^*_Y} & \tilde{K}^i(X)
\end{array}
$$

where $1 \in K^0(\{y_0\})$. So we identify the map $\Pr^*_X$ with the map $\mu(\cdot \otimes 1)$. We also identify $\Pr^*_Y$ with the map $\mu(1 \otimes \cdot)$ where $1 \in K^0(\{x_0\}) = \mathbb{Z}$.

### 3 The group structure of $[X, \text{Aut} \mathcal{O}_{n+1}]$  

#### 3.1 Description of the group structure

Let $(X, x_0)$ be a pointed compact metrizable space. For every $\alpha \in \text{Map}(X, \text{Aut} \mathcal{O}_{n+1})$, we set

$$u_\alpha = \sum_{i=0}^n \alpha(1_{C(X)} \otimes S_i)(1_{C(X)} \otimes S^*_i) \in U(C(X) \otimes \mathcal{O}_{n+1}).$$

By [13] Theorem 7.4, the map

$$[X, \text{Aut} \mathcal{O}_{n+1}] \ni [\alpha] \mapsto [u_\alpha]_1 \in K_1(C(X) \otimes \mathcal{O}_{n+1}) = K^1(X; \mathbb{Z})$$

is a bijection, though it is not a group homomorphism in general as we will see below. From the definition of $u_\alpha$, we have $u_{\alpha_2}(x) = \alpha_x(u_\beta(x))u_\alpha(x)$, and $[u_{\alpha_2}]_1 = [u_\alpha]_1 + [\alpha(u_\beta)]_1$. Thus to determine the group structure of $[X, \text{Aut} \mathcal{O}_{n+1}]$, it suffices to determine the map

$$K_1(\alpha) : K_1(C(X) \otimes \mathcal{O}_{n+1}) \to K_1(C(X) \otimes \mathcal{O}_{n+1}),$$

induced by $u(x) \mapsto \alpha_x(u(x))$.

**Theorem 3.1.** For every $\alpha \in \text{Map}(X, \text{Aut} \mathcal{O}_{n+1})$ and $a \in K_1(C(X) \otimes \mathcal{O}_{n+1})$, we have

$$K_1(\alpha)(a) = a - [u_\alpha]_1 \cdot \delta(a),$$

where $\delta : K_1(C(X) \otimes \mathcal{O}_{n+1}) \to \text{Tor}(\tilde{K}^0(X), \mathbb{Z})$ is the index map.

**Proof.** For a given $b \in \text{Tor}(\tilde{K}^0(X), \mathbb{Z})$, we look for the preimage $\delta^{-1}(b)$ first. We may assume that $b$ is of the form $[p]\otimes [1_{m}]_0$ with a projection $p \in C(X) \otimes M_{2m}$ such that there exists a unitary $v \in C(X) \otimes M_{2m}$ satisfying $v\otimes q = 1 \otimes p$, where $q = \text{Diag}(1_m, 0_m)$.

Identifying $\tilde{K}^0(X)$ with $K_0(C(X) \otimes \mathbb{K})$, we may replace $p$ and $q$ with $e \otimes p$ and $e \otimes q$ respectively, where $e = 1_{E_{n+1}} - \sum_{i=0}^n T_iT^*_i$ is a minimal projection in $K \subset E_{n+1}$. Furthermore, we may adjoin $1_{E_{n+1}}$ to $C(X) \otimes \mathbb{K}$, and

$$e = ([1_{E_{n+1}} - e] \otimes q + e \otimes p) - [1_{E_{n+1}} \otimes q].$$

In what follows, we simply denote $1_E = 1_{E_{n+1}}$ and often denote $1_{2m}$ for $1 \otimes 1_{2m}$.

We will construct a unitary $U \in C(X) \otimes E_{n+1} \otimes M_{10m}$ satisfying

$$U \text{Diag}((1 - e) \otimes q + e \otimes p, 1_{4m}, 0_{4m})U^{-1} = \text{Diag}(q, 1_{4m}, 0_{4m}).$$
Expressing \( v(x) = \sum_{i,j=1}^{n} e_{i,j} \otimes v_{i,j}(x) \), where \( \{e_{i,j}\}_{1 \leq i,j \leq n} \) is a system of matrix units \( M_n \), we let

\[
\tilde{v}(x) = (e + T_0 T_0^*) \otimes 1_{2m} + \sum_{i,j=1}^{n} T_i T_j^* \otimes v_{i,j}(x).
\]

Then \( \tilde{v} \) is a unitary in \( C(X) \otimes E_{n+1} \otimes M_{2m} \) satisfying

\[
\tilde{v}((1 - e) \otimes q + e \otimes p) \tilde{v}^* = T_0 T_0^* \otimes q + (1 - T_0 T_0^*) \otimes p.
\]

Thus if we put

\[
U_1 = \text{Diag}(\tilde{v}, \begin{pmatrix} 0 & 0 & 1_{2m} & 0 \\ 0 & 1_{2m} & 0 & 0 \\ 1_{2m} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{2m} \end{pmatrix}),
\]

we get

\[
U_1 \text{Diag}((1 - e) \otimes q + e \otimes p, 1_{4m}, 0_{4m}) U_1^{-1} = \text{Diag}(T_0 T_0^* \otimes q + (1 - T_0 T_0^*) \otimes p, 0_{2m}, 1_{4m}, 0_{2m}).
\]

Let

\[
U_2 = \text{Diag}((T_0 T_0^* \otimes q + (1 - T_0 T_0^*) \otimes p, 0_{2m}, 1_{4m}, 0_{2m}) U_2^{-1} = \text{Diag}(T_0 T_0^* \otimes q, (1 - T_0 T_0^*) \otimes p, 1_{4m}, 0_{2m}).
\]

Let \( U_3 = \text{Diag}(1_{2m}, V_1, V_2) \) with

\[
V_1 = \begin{pmatrix} 1_{2m} - T_0 T_0^* \otimes p & T_0 T_0^* \otimes p \\ T_0 T_0^* \otimes p & 1_{2m} - T_0 T_0^* \otimes p \end{pmatrix},
\]

\[
V_2 = \begin{pmatrix} T_0 T_0^* \otimes q & 1_{2m} - T_0 T_0^* \otimes q \\ 1_{2m} - T_0 T_0^* \otimes q & T_0 T_0^* \otimes q \end{pmatrix}.
\]

Then

\[
U_3 \text{Diag}(T_0 T_0^* \otimes q, (1 - T_0 T_0^*) \otimes p, 1_{4m}, 0_{2m}) U_3^{-1} = \text{Diag}(T_0 T_0^* \otimes q, 1 \otimes p, 1_{2m} - T_0 T_0^* \otimes p, T_0 T_0^* \otimes q, 1_{2m} - T_0 T_0^* \otimes q).
\]

Let

\[
U_4 = \text{Diag}(1_{2m}, \begin{pmatrix} T_0 \otimes 1_{2m} & 0 & (1 - T_0 T_0^*) \otimes 1_{2m} \\ 0 & 1_{2m} & 0 \\ 0 & 0 & T_0 \otimes 1_{2m} \end{pmatrix}),1_{2m}).
\]

Then

\[
U_4 \text{Diag}(T_0 T_0^* \otimes q, 1 \otimes p, 1_{2m} - T_0 T_0^* \otimes p, T_0 T_0^* \otimes q, 1_{2m} - T_0 T_0^* \otimes q) U_4^{-1} = \text{Diag}(T_0 T_0^* \otimes q, T_0 T_0^* \otimes p, 1_{2m} - T_0 T_0^* \otimes p, q, 1_{2m} - T_0 T_0^* \otimes q)
\]

Let

\[
U_5 = \begin{pmatrix} T_0 T_0^* \otimes q & 0 & 0 & 0 & 1_{2m} - T_0 T_0^* \otimes q \\ 0 & T_0 T_0^* \otimes p & 1_{2m} - T_0 T_0^* \otimes p & 0 & 0 \\ 0 & 1_{2m} - T_0 T_0^* \otimes p & T_0 T_0^* \otimes p & 0 & 0 \\ 0 & 0 & 0 & 1_{2m} & 0 \\ 1_{2m} - T_0 T_0^* \otimes q & 0 & 0 & 0 & T_0 T_0^* \otimes q \end{pmatrix}.
\]
Then
\[ U_5 \text{Diag}(T_0 T_0^* \otimes q, T_0 T_0^* \otimes p, 1_{2m} - T_0 T_0^* \otimes q) U_5^{-1} = \text{Diag}(1_{2m}, 1_{2m}, q, 0_{4m}). \]

Let
\[
U_6 = \begin{pmatrix}
0 & 0 & 0 & 1_{2m} & 0 \\
0 & 1_{2m} & 0 & 0 & 0 \\
1_{2m} & 0 & 0 & 0 & 0 \\
0 & 0 & 1_{2m} & 0 & 0 \\
0 & 0 & 0 & 0 & 1_{2m}
\end{pmatrix}.
\]

Then
\[ U_6 \text{Diag}(1_{2m}, 1_{2m}, q, 0_{4m}) U_6^{-1} = \text{Diag}(q, 1_{4m}, 0_{4m}). \]

Thus if we put \( U = U_0 U_2 U_3 U_2 U_1 \), we get
\[ U \text{Diag}((1 - e) \otimes q + e \otimes p, 1_{4m}, 0_{4m}) U^{-1} = \text{Diag}(q, 1_{4m}, 0_{4m}). \]

Recall that \( \pi : E_{n+1} \to \mathcal{O}_{n+1} \) is the quotient map. Since
\[ (\pi \otimes \text{id}_{M_{1m}})(\text{Diag}((1 - e) \otimes q + e \otimes p, 1_{4m}, 0_{4m})) = \text{Diag}(q, 1_{4m}, 0_{4m}), \]
the unitary \( (\pi \otimes \text{id}_{M_{1m}})(U) \) commutes with \( \text{Diag}(q, 1_{4m}, 0_{4m}) \). Let
\[ W = \text{Diag}(q, 1_{4m}, 0_{4m})(\pi \otimes \text{id}_{M_{1m}})(U^{-1}) \text{Diag}(q, 1_{4m}, 0_{4m}), \]
which we regard as a unitary in \( C(X, \mathcal{O}_{n+1} \otimes M_{5m}) \). Then by the definition of the index map, we get \( \delta([W]_1) = b \).

Let
\[ V(x) = \sum_{i,j=1}^{n} S_i S_j^* \otimes v_{i,j}(x). \]

Direct computation yields
\[ W = \begin{pmatrix}
0 & V^*(S_0^* \otimes p) & S_0 S_0^* \otimes q \\
S_0 \otimes q & 0 & 1_{2m} - S_0 S_0^* \otimes p + S_0 S_0^* \otimes p \\
1_{2m} - S_0 S_0^* \otimes q & 0 & 0
\end{pmatrix}. \]

Let \( \beta = \text{Ad} u_{\alpha}^* \circ \alpha \). Then \( K_1(\alpha) = K_1(\beta) \), and \( \beta(S_0) = S_0 u_{\alpha} \). Now
\[
W^*(\beta \otimes \text{id}_{M_{1m}})(W)
\]
\[
= \left( \begin{array}{ccc} S_0^* \otimes q & 0 & 0 \\
0 & 1_{2m} - S_0 S_0^* \otimes p + S_0 S_0^* \otimes p & 0 \\
1_{2m} - S_0 S_0^* \otimes q & 0 & 0
\end{array} \right)
\times \left( \begin{array}{ccc} 0 & 0 & S_0 S_0^* \otimes q \\
0 & 1_{2m} - S_0 S_0^* \otimes p + S_0 S_0^* u_{\alpha}^{-1} S_0^* \otimes p & 0 \\
S_0 u_{\alpha} \otimes q & 0 & 0
\end{array} \right)
\times \left( \begin{array}{ccc} 0 & 0 & S_0 S_0^* \otimes q \\
0 & 1_{2m} - S_0 S_0^* \otimes p + S_0 S_0^* u_{\alpha}^{-1} S_0^* \otimes p & 0 \\
0 & 0 & 0
\end{array} \right)
\times \left( \begin{array}{ccc} 0 & 0 & S_0 \otimes 1_{2m} \\
0 & 0 & (1 - S_0 S_0^*) \otimes 1_{2m} \\
0 & 0 & (1 - S_0 S_0^*) \otimes 1_{2m}
\end{array} \right).
\]

whose \( K_1 \)-class is
\[ [u_{\alpha}]_1([q]_0 - [p]_0) = -[u_{\alpha}]_1 \cdot b = -[u_{\alpha}]_1 \cdot \delta([W]_1). \]
Thus \[ K_1(\alpha)([W]_1 + a) = [W]_1 + a - [u_\alpha]_1 \cdot \delta([W]_1 + a), \]
which finishes the proof.

Recall that we identify the index map $\delta$ with the Bockstein map $\beta$. By Theorem 3.1, the group 
$[X, \text{Aut } \mathcal{O}_{n+1}]$ is isomorphic to $(K^1(X; \mathbb{Z}_n), \circ)$ with
\[ a \circ b = a + b - \beta(b), \quad a, b \in K^1(X; \mathbb{Z}_n). \]
Note that $(K^1(X; \mathbb{Z}_n), \circ)$ is a group extension
\[ 0 \to K_1(X) \otimes \mathbb{Z}_n \to (K^1(X; \mathbb{Z}_n), \circ) \xrightarrow{\hat{\beta}} (1 + \text{Tor}(\tilde{K}_0(X), \mathbb{Z}_n))^\times \to 0, \]
where $\hat{\beta}(a) = 1 - \beta(a)$. We denote the inverse of an element $a \in (K^1(X; \mathbb{Z}_n), \circ)$ by $a^{\circ(-1)}$.

**Lemma 3.2.** For any $a, b \in (\tilde{K}_1(X; \mathbb{Z}_n), \circ)$, we have
\begin{enumerate}
  \item $a^{\circ(-1)} = -a \cdot (1 - \beta(a))^{-1}$.
  \item $a^{\circ(-1)} \circ b \circ a = b + a \cdot \beta(b) - \beta(a) \cdot b$. In particular, if $b \in K^1(X) \otimes \mathbb{Z}_n = \ker \beta$, we have $a^{\circ(-1)} \circ b \circ a = (1 - \beta(a))b$.
\end{enumerate}

**Proof.** Direct computation yields
\[ (-a \cdot (1 - \beta(a))^{-1} \circ a = -a \cdot (1 - \beta(a))^{-1} + a \cdot (1 - \beta(a))^{-1} \cdot \beta(a) = a + a \cdot (1 - \beta(a))^{-1} \cdot (\beta(a) - 1) = 0, \]
showing the first equation. The second one follows from the first one. \qed

Now we discuss the relationship between the two groups $[X, \text{Aut } E_{n+1}]$ and $[X, \text{Aut } \mathcal{O}_{n+1}]$. Let $H_1$ be the set of vectors of norm 1 in a separable infinite dimensional Hilbert space $H$. Then $H_1$ is contractible. Indeed, we can identify $H_1$ with the set \{f \in L^2[0, 1] \mid \|f\|_2 = 1\}, and define a homotopy $h_t : H_1 \to H_1$ sending $f$ to \((1 - t) f + 1_{[0,1]} ||f||_2 = 1\), where $1_{[a,b]}$ is the characteristic function of $[a, b]$. This gives a deformation retraction of $H_1$ to the set \{1_{[0,1]}\}, and the space $H_1$ is contractible (see [22]). Since the group $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ freely acts on $H_1$ by multiplication, we can adopt $H_1$ as a model of the universal principal $S^1$-bundle $E S^1$ and identify the classifying space $BS^1$ of $S^1$ with the set of all minimal projections. The space $BS^1$ is the Eilenberg-MacLane space $K(\mathbb{Z}, 2)$ and we identifies the homotopy set $[X, BS^1]$ with $H^2(X)$ via the Chern classes of the line bundles.

Let $\eta$ be the map $\text{Aut } E_{n+1} \ni \alpha \mapsto \alpha(e) \in B S^1$. We denote by $\eta_\ast$ the induced map $\eta_\ast : [X, \text{Aut } E_{n+1}] \to H^2(X)$, which is a group homomorphism with image in $\text{Tor}(H^2(X), \mathbb{Z}_n)$ (see [24, Theorem 3.15]). We will show that the two maps $\eta_\ast$ and $\hat{\beta}$ are compatible. Let $\mathcal{P}(K)$ be the set of all projections of $K$. We remark that the map $[X, \mathcal{P}(K)] \ni [p] \mapsto [p]_0 \in K^0(X)$ is well-defined by the definition of the $K_0$-group. Since $\mathcal{O}_{n+1}$ is the quotient of $E_{n+1}$ by a unique non-trivial closed two sided ideal, every element in $\text{Aut } E_{n+1}$ induces an element in $\text{Aut } \mathcal{O}_{n+1}$, which gives a group homomorphism from $\text{Aut } E_{n+1}$ to $\text{Aut } \mathcal{O}_{n+1}$. We denote by $\eta$ the group homomorphism from $[X, \text{Aut } E_{n+1}]$ to $[X, \text{Aut } \mathcal{O}_{n+1}]$ induced by this homomorphism.

**Proposition 3.3.** Let $q$ be as above, and let $l : H^2(X) \to K^0(X)$ be a map induced by the map $BS^1 \to \mathcal{P}(K)$ where we identify $BS^1$ with the set of all minimal projections. Then we have the following commutative diagram
\[
\begin{array}{ccc}
[X, \text{Aut } E_{n+1}] & \xrightarrow{\eta_\ast} & \text{Tor}(H^2(X), \mathbb{Z}_n) \\
\downarrow q & & \downarrow l \\
[X, \text{Aut } \mathcal{O}_{n+1}] & \xrightarrow{\hat{\beta}_\ast} & (1 + \text{Tor}(\tilde{K}_0(X), \mathbb{Z}_n))^\times
\end{array}
\]
Proof. For \( \alpha \in \text{Map}(X, \text{Aut } E_{n+1}) \), we denote by \( \hat{\alpha} \) the map in \( \text{Map}(X, \text{Aut } O_{n+1}) \) induced by \( \alpha \). Then with the identification of \([X, \text{Aut } O_{n+1}] \) and \( K^1(X; \mathbb{Z}_n) \), the map \( \eta \) sends \([\alpha] \) to \([u_\alpha] \).

One has \( l \circ \eta_\alpha([\alpha]) = [\alpha(1 \otimes e) \circ e]_0 \in K_0(C(X)) \) for every \( \alpha \in \text{Map}(X, \text{Aut } E_{n+1}) \) by definition. Since \( \beta \) is given by the index map \( \delta: K^1(C(X) \otimes O_{n+1}) \to K_0(C(X)) \). We compute the index \( \text{ind } [u_\alpha] \). We have a unitary lift \( V \in U(M_2(C(X) \otimes O_{n+1})) \) of the unitary \( u_\alpha \oplus u_\alpha^* : \)

\[
V = \left( \begin{array}{cc} \sum_{i=1}^{n+1} \alpha(1 \otimes T_i) T_i^* & \sum_{i=1}^{n+1} \alpha(1 \otimes e) 1 \otimes \alpha(1 \otimes T_i^*) \\ 1 \otimes e & \sum_{i=1}^{n+1} \alpha(1 \otimes e) 1 \otimes \alpha(1 \otimes T_i^*) \end{array} \right).
\]

Direct computation yields

\[
V(1 \otimes 0)V^* = (1 - \alpha(1 \otimes e)) \oplus (1 \otimes e)
\]

where we write \( 1_{C(X)} \otimes e \) simply by \( 1 \otimes e \). Hence we have

\[
\text{ind } [u_\alpha] = [1 - \alpha(1 \otimes e)]_0 + [1 \otimes e]_0 - [1]_0 = 1 - [\alpha(1 \otimes e)]_0 \in K_0(C(X) \otimes \mathbb{K}).
\]

Now we have \( 1 - \text{ind } [u_\alpha] = [\alpha(1 \otimes e)]_0 \), and this proves the statement. \( \square \)

**Lemma 3.4.** We have the following commutative diagram with exact rows

\[
\begin{array}{cccccc}
K^1(X) & \xrightarrow{\rho} & [X, \text{Aut } E_{n+1}] & \xrightarrow{\eta_*} & \text{Tor}(H^2(X), \mathbb{Z}_n) \\
\downarrow & & \downarrow \delta & & \downarrow \beta & \\
K^1(X) & \xrightarrow{\rho} & [X, \text{Aut } O_{n+1}] & \xrightarrow{\eta_*} & \text{Tor}(H^2(X), \mathbb{Z}_n)^\times.
\end{array}
\]

**Proof.** Let \( \text{End} E_{n+1} \) be the set of unital endomorphisms of \( E_{n+1} \), and let \( \text{End}_0 E_{n+1} \) be its connected component of id. Then the inclusion \( \text{Aut } E_{n+1} \subset \text{End}_0 E_{n+1} \) is a weak homotopy equivalence (see [24, Theorem 3.14]). For \( u \in U(E_{n+1}) \), we denote by \( \rho_u \) the unitary endomorphism of \( E_{n+1} \) defined by \( \rho_u(T_i) = u T_i \). Then the correspondence \([u]_1 \to [\rho_u]_0\) gives the map from \( K^1(X) \) to \([X, \text{Aut } E_{n+1}] \). The exactness follows from [24, Theorem 3.15] and the Bockstein exact sequence. The right square commutes by Proposition 3.3. The left square commutes because the following diagram commutes

\[
\begin{array}{cccc}
u \in U(C(X) \otimes E_{n+1}) & \xrightarrow{\text{Map}(X, \text{End}_0 E_{n+1}) \ni \alpha = \rho_u} & \text{Map}(X, \text{End}_0 E_{n+1}) \ni \alpha = \rho_u \\
\downarrow & & \downarrow & \\
u \in U(C(X) \otimes E_{n+1}) & \xrightarrow{\text{Map}(X, \text{End}_0 E_{n+1}) \ni \alpha = \rho_u} & \text{Map}(X, \text{End}_0 E_{n+1}) \ni \alpha = \rho_u
\end{array}
\]

where \( \rho_u : X \ni x \mapsto \rho_u x \in \text{End}_0 E_{n+1} \) for every \( u \in U(C(X) \otimes E_{n+1}) \). \( \square \)

**3.2 An example of non-commutative \([X, \text{Aut } O_{n+1}]\)**

We first examine the ring structure of \( K^*(M_n \times \Sigma M_n) \) to show that \([M_n \times \Sigma M_n, \text{Aut } O_{n+1}]\) is a non-commutative group. By Lemma 2.2 and Theorem 2.2 we have

\[
\tilde{K}^1(M_n \times \Sigma M_n) \cong 1 \otimes \tilde{K}^1(\Sigma M_n) \oplus \tilde{K}^0(M_n) \oplus \tilde{K}^1(\Sigma M_n),
\]

\[
\tilde{K}^0(M_n \times \Sigma M_n) \cong \tilde{K}^0(M_n) \oplus 1 \oplus \tilde{K}^0(M_n \wedge \Sigma M_n).
\]

Therefore Lemma 2.1 yields \( \tilde{K}^0(M_n \times \Sigma M_n) \cong \mathbb{Z}_n^\oplus 2 \). In particular, the map \( \rho: \tilde{K}^1(M_n \times \Sigma M_n) \to \tilde{K}^1(M_n \times \Sigma M_n; \mathbb{Z}_n) \) is injective by the Bockstein exact sequence.

We determine a generator of \( K^1(M_n; \mathbb{Z}_n) \cong K^0(M_n) \cong \mathbb{Z}_n \). Recall that the canonical gauge action \( \lambda_z: S^1 \to \text{Aut } E_{n+1} \) is a generator of \( \pi_1(\text{Aut } E_{n+1}) = \mathbb{Z}_n \) (see [24, Theorem 2.36, 3.14]). Therefore we have a homotopy

\[
h: [0,1] \times S^1 \to \text{Aut } E_{n+1}
\]

with \( h_0(z) = \text{id}_{E_{n+1}}, h_1(z) = \lambda_z^o \), which extend \( \lambda \) to a map

\[
\lambda: M_n \to \text{Aut } E_{n+1}
\]

satisfying \( \lambda \circ i = \lambda_z \) for the map \( i: S^1 \hookrightarrow M_n \). For the gauge action \( \tilde{\lambda} \) of \( O_{n+1} \), we get an extension \( \tilde{\lambda}: M_n \to \text{Aut } O_{n+1} \) in the same way.

8
Lemma 3.5. We have the following isomorphisms:

\[ i^*: [M_n, \text{Aut } E_{n+1}] \ni [\lambda] \mapsto [\lambda_1] \in [S^1, \text{Aut } E_{n+1}], \]

\[ i^*: [M_n, \text{Aut } O_{n+1}] \ni [\lambda] \mapsto [\lambda_2] \in [S^1, \text{Aut } O_{n+1}]. \]

**Proof.** First, we show that \( i^*: [M_n, \text{Aut } O_{n+1}] \rightarrow [S^1, \text{Aut } O_{n+1}] \) is an isomorphism. By [14], Puppe sequence \( S^n \rightarrow S^1 \rightarrow M_n \rightarrow S^2 \rightarrow \cdots \) gives an exact sequence

\[ \mathbb{Z}_n = \pi_1(\text{Aut } O_{n+1}) \xrightarrow{\mathbb{Z}_1} \pi_1(\text{Aut } O_{n+1}) \xrightarrow{i^*} [M_n, \text{Aut } O_{n+1}] \rightarrow 0. \]

Hence, the map \( i^* \) is an isomorphism of groups.

Similarly, the map \( i_*: [M_n, \text{Aut } E_{n+1}] \rightarrow [S^1, \text{Aut } E_{n+1}] \) is an isomorphism by [23] Theorem 2.36, 3.14.

Lemma 3.6. For every \( \alpha \in \text{Map}(M_n, \text{Aut } O_{n+1}) \), we have \( K_1(\alpha) = \text{id}_{K^0(M_n; \mathbb{Z}_n)}. \) In particular, we have \( K^0(M_n) \cdot K^1(M_n; \mathbb{Z}_n) = K^0(M_n) \cdot K^0(M_n) = \{0\}. \)

**Proof.** By Lemma 3.5, we have the following commutative diagram

\[
\begin{array}{ccc}
[M_n, \text{Aut } O_{n+1}] & \xrightarrow{i^*} & [S^1, \text{Aut } O_{n+1}] \\
\downarrow & & \downarrow \\
(K_1(C_0(M_n, pt) \otimes O_{n+1}), \circ) & \xrightarrow{K_1(r)} & (K_1(C_0(S^1, pt) \otimes O_{n+1}), \circ),
\end{array}
\]

where \( r: C_0(M_n, pt) \rightarrow C_0(S^1, pt) \) is a restriction by \( i: S^1 \hookrightarrow M_n \). Since two vertical maps are group isomorphisms, the map \( K_1(r) \) is a group homomorphism with respect to the multiplication \( \circ \). We have \( K_1(C(S^1) \otimes O_{n+1}) \xrightarrow{\beta} K^0(S^1) = 0 \), and it follows that \( (K^1(S^1; \mathbb{Z}_n), +) = (K^1(S^1; \mathbb{Z}_n), \circ) \) by Theorem 3.4. Therefore two multiplications \( \circ \) and \( + \) coincide in \( K^1(M_n; \mathbb{Z}_n) \), and we have \( K_1(\alpha) = \text{id}_{K^0(M_n; \mathbb{Z}_n)} \) and \( K^1(M_n; \mathbb{Z}_n) \cdot K^0(M_n) = 0 \). Since the map \( \beta \) is compatible with multiplication, we have \( K^0(M_n) \cdot K^0(M_n) = \beta(K^1(M_n; \mathbb{Z}_n) \cdot K^0(M_n)) = \{0\}. \)

We denote by \( a_1 \), the generator \( [\lambda] \in [M_n, \text{Aut } O_{n+1}] = K^1(M_n; \mathbb{Z}_n) \), and denote \( g_1 = \beta(a_1) \).

By Lemma 2.7, two elements \( g \) and \( \rho(g) \) are the generators of \( K^0(M_n) \) and \( K^0(M_n; \mathbb{Z}_n) \) respectively. By Lemma 3.4, we have

\[ g \cdot g = 0 \in K^0(M_n), \]

\[ a_1 \cdot g = 0 \in K^1(M_n; \mathbb{Z}_n). \]

Now, we determine the group \( [M_n \times \Sigma M_n, \text{Aut } E_{n+1}] \). Since the reduction \( \rho: \tilde{K}^1(M_n \times \Sigma M_n) \rightarrow \tilde{K}^1(M_n \times \Sigma M_n; \mathbb{Z}_n) \) is injective, we regard \( \tilde{K}^1(M_n \times \Sigma M_n) \) as a subgroup of \( (\tilde{K}^1(M_n \times \Sigma M_n; \mathbb{Z}_n), \circ) \). From Lemma 3.4, we can regard \( \tilde{K}^1(M_n \times \Sigma M_n) \) as a normal subgroup of the group \( [M_n \times \Sigma M_n, \text{Aut } E_{n+1}] \) too. Consider the map

\[ \lambda: = \lambda \circ \text{Pr}_M: M_n \times \Sigma M_n \rightarrow \text{Aut } E_{n+1}. \]

By definition, we have \( q([\lambda]) = [u_\lambda]_1 = \text{Pr}_M([u_\lambda]_1) = \mu_R(a_\lambda \otimes 1) \in \tilde{K}^1(M_n \times \Sigma M_n; \mathbb{Z}_n). \)

**Proposition 3.7.** The group homomorphism \( q: [M_n \times \Sigma M_n, \text{Aut } E_{n+1}] \rightarrow [M_n \times \Sigma M_n, \text{Aut } O_{n+1}] \) is injective.

**Proof.** Note that the K"unneth formula implies \( H^2(M_n \times \Sigma M_n) \cong \mathbb{Z}_n. \) Since \( \hat{\beta}(\mu_R(a_\lambda \otimes 1)) = 1 - \mu(g \otimes 1) \) has order \( n \), and \( \hat{\beta}(\mu_R(a_\lambda \otimes 1)) = l(\eta([\lambda])), \) the element \( \eta([\lambda]) \) is a generator of \( H^2(M_n \times \Sigma M_n) \), and \( l \) is injective. Thus the statement follows from Lemma 3.4.
Theorem 3.8. With the above notation, the group \([M_n \times \Sigma M_n, \text{Aut } E_{n+1}]\) is isomorphic to the Heisenberg group \(\mathbb{Z}_n^2 \rtimes \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) \mathbb{Z}_n\).

Proof. We already know that the group \([M_n \times \Sigma M_n, \text{Aut } E_{n+1}]\) isomorphic to the subgroup of \((K^1(M_n \times \Sigma M_n; \mathbb{Z}_n), \circ)\) generated by \(K^1(M_n \times \Sigma M_n)\) and \([u_\Lambda]_1\). Since the order of \([u_\Lambda]_1\) is \(n\), the group is a semi-direct product \((\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes \mathbb{Z}_n\). To determine the group structure, it suffices to compute the action of \(\hat{\beta}([u_\Lambda]_1) = 1 - \mu(g \otimes 1)\) on \(K^1(M_n \times \Sigma M_n)\) by multiplication. Since \(K^1(M_n \times \Sigma M_n) = (\mu(1 \otimes u)) \otimes (\mu(g \otimes u)) \cong \mathbb{Z}_n \otimes \mathbb{Z}_n\), and \(g \cdot g = 0\), we get the statement.

\(\square\)

Corollary 3.9. The groups \([M_n \times \Sigma M_n, \text{Aut } E_{n+1}]\) and \([M_n \times \Sigma M_n, \text{Aut } \mathcal{O}_{n+1}]\) are non-commutative for any \(n \geq 2\). In particular, two spaces \(\text{BAut } \mathcal{O}_{n+1}\) and \(\text{BAut } E_{n+1}\) are not H-spaces.

Remark 3.10. If \(n\) is an odd number, we can actually show

\([M_n \times \Sigma M_n, \text{Aut } \mathcal{O}_{n+1}] \cong [M_n \times \Sigma M_n, \text{Aut } E_{n+1}] \times \mathbb{Z}_n\).

4 Continuous fields of Cuntz algebras

We first review Dadarlat’s results on the continuous fields of the Cuntz algebras. We refer to [10] Definition 10.1.2, 10.1.3 for the definition of the continuous fields of C*-algebras. A locally trivial continuous field of a C*-algebra \(A\) is the section algebra of a locally trivial fiber bundle with the fibre \(A\), which is an associated bundle of a principal \(Aut\ A\) bundle. By [13] Theorem 1.1, all continuous fields of \(\mathcal{O}_{n+1}\) over finite CW-complexes are locally trivial. So we identify the continuous fields of \(\mathcal{O}_{n+1}\) over finite CW-complexes with principal \(Aut\ \mathcal{O}_{n+1}\) bundles.

For a compact Hausdorff space \(X\), we denote by \(\text{Vect}_m(X)\) the set of the vector bundles of rank \(m\). Dadarlat investigated continuous fields of \(\mathcal{O}_{n+1}\) over \(X\) arising from \(E \in \text{Vect}_{n+1}(X)\), which are Cuntz-Pimsner algebras. We refer to [17] and [19] for Cuntz-Pimsner algebras. Fixing a Hermitian structure of \(E\), we get a Hilbert \(C(X)\)-module from \(E\), which we regard as a \(C(X)\)-\(C(X)\)-bimodule. Then the Pimsner construction gives the Cuntz-Pimsner algebra \(\mathcal{O}_E\), which is the quotient of \(T_E\) by \(K_E\). The algebra \(\mathcal{O}_E\) is a continuous field of \(\mathcal{O}_{n+1}\) over \(X\). We denote by \(\theta_E : C(X) \to \mathcal{O}_E\) the natural unital inclusion.

Theorem 4.1 ([19] Theorem 4.8]). Let \(X\) be a compact Hausdorff space, and let \(E\) be a vector bundle over \(X\). Then we have the following exact sequence

\[
\begin{array}{cccc}
K_0(C(X)) & \xrightarrow{1-[E]} & K_0(C(X)) & \xrightarrow{\theta_E} & K_0(\mathcal{O}_E) \\
\downarrow & & \downarrow & & \downarrow \\
K_1(\mathcal{O}_E) & \xrightarrow{\theta_E} & K_1(C(X)) & \xrightarrow{1-[E]} & K_1(C(X))
\end{array}
\]

where the map \(\theta_E : C(X) \to \mathcal{O}_E\) is the natural inclusion, and the map \(1-[E]\) is the multiplication by \(1-[E] \in K^0(X)\).

Dadarlat found an invariant to classify the \((C(X))\)-linear isomorphism classes of \(\mathcal{O}_E\).

Theorem 4.2 ([12] Theorem 1.1]). Let \(X\) be a compact metrizable space, and let \(E\) and \(F\) be vector bundles of rank \(\geq 2\) over \(X\). Then there is a unital \(*\)-homomorphism \(\varphi : \mathcal{O}_E \to \mathcal{O}_F\) with \(\varphi \circ \theta_E = \theta_F\) if and only if \((1-[E]) \cdot K^0(X) \subset (1-[F]) \cdot K^0(X)\). Moreover we can take \(\varphi\) to be an isomorphism if and only if \((1-[E]) \cdot K^0(X) = (1-[F]) \cdot K^0(X)\).

The key observation of Dadarlat is that if there is a \((C(X))\)-linear isomorphism \(\varphi : \mathcal{O}_E \to \mathcal{O}_F\), we have \((1-[E]) \cdot K^0(X) = \text{Ker } K_0(\theta_E) = \text{Ker } K_0(\theta_F) = (1-[F]) \cdot K^0(X)\) by the exact sequence of Theorem 4.1.

Dadarlat also estimate the cardinality of the set of the \((C(X))\)-linear isomorphism classes of \(\mathcal{O}_E\). We denote \([x] : = \min\{k \in \mathbb{Z} : k \geq x\}\).
**Theorem 4.3.** Let X be a finite connected CW-complex with Tor(H*(X), Z_n) = 0. Then the following holds.
1. \(|K^0(X) \otimes \mathbb{Z}_n| = |\bar{H}^{even}(X, \mathbb{Z})|\).
2. If \(n \geq \lfloor (\dim X - 3)/3 \rfloor\), the set \(\{|O_E|; E \in \text{Vect}_{n+1}(X)\}\) exhausts all the isomorphism classes of continuous fields of \(O_{n+1}\) over \(X\), and its cardinality is \(|K^0(X) \otimes \mathbb{Z}_n|\).

Our goal in this section is to remove the restriction \(n \geq \lfloor (\dim X - 3)/3 \rfloor\) from the above statement using a localization trick. In fact, all the necessary algebraic arguments for the proof are already in Dadarlat’s paper [12].

Let \(F_n\) be the set of all prime numbers \(p\) with \((n, p) = 1\), and let \(M_{\langle n \rangle}\) be the UHF algebra

\[M_{\langle n \rangle} = \prod_{p \in F_n} M_p\).

This is the unique UHF algebra satisfying \(K_0(M_{\langle n \rangle}) = \mathbb{Z}_{\langle n \rangle}\) where \(\mathbb{Z}_{\langle n \rangle}\) is a localization of \(\mathbb{Z}\) by \(\langle n \rangle\). Assume that \(r\) is a natural number with \((n, r) = 1\). Then the \(K\)-groups of \(O_{nr+1} \otimes M_{\langle n \rangle}\) are

\[K_0(O_{nr+1} \otimes M_{\langle n \rangle}) = \mathbb{Z}_{nr} \otimes \mathbb{Z}_{\langle n \rangle} = \mathbb{Z}_n = \langle 1 \rangle, \quad K_1(O_{nr+1} \otimes M_{\langle n \rangle}) = 0[13].

Therefore Kirchberg and Phillips’ classification theorem [13, Theorem 4.2.4] yields \(O_{nr+1} \otimes M_{\langle n \rangle} \cong O_{n+1}\). Let \(F_r\) be a vector bundle over \(X\) of rank \(nr + 1\). Then we have a continuous field of \(O_{n+1}\) of the form \(O_F \otimes M_{\langle n \rangle}\).

**Definition 4.4.** We denote by \(O(X)_n\) the \(C(X)\)-linear isomorphism classes of continuous fields of the Cuntz algebra \(O_{n+1}\) over \(X\) of the form \(O_F \otimes M_{\langle n \rangle}\) for \(F_r \in \text{Vect}_{nr+1}(X)\) with \((n, r) = 1\).

Note that we have \(K_*(C(X) \otimes M_{\langle n \rangle}) = K^*(X) \otimes \mathbb{Z}_{\langle n \rangle}\). Following Dadarlat’s argument, we consider an ideal \((1 - [F_r])K^0(X) \otimes \mathbb{Z}_{\langle n \rangle}\) of the ring \(K^0(X) \otimes \mathbb{Z}_{\langle n \rangle}\).

**Lemma 4.5.** Let \(X\) be a finite connected CW-complex. Let \(F_r\) and \(F_R\) be vector bundles over \(X\) of rank \(nr + 1\) and \(nR + 1\) respectively, with \((n, r) = (n, R) = 1\). If \(O_F \otimes M_{\langle n \rangle}\) is \(C(X)\)-linearly isomorphic to \(O_{F_R} \otimes M_{\langle n \rangle}\), we have \((1 - [F_r])K^0(X) \otimes \mathbb{Z}_{\langle n \rangle} = (1 - [F_R])K^0(X) \otimes \mathbb{Z}_{\langle n \rangle}\).

**Proof.** Let \(\varphi: O_F \otimes M_{\langle n \rangle} \rightarrow O_{F_R} \otimes M_{\langle n \rangle}\) be a \((C(X))-linear isomorphism. First, we show that the following diagram induces a commutative diagram of \(K_0\)-groups:

\[
\begin{array}{cccc}
C(X) \otimes M_{\langle n \rangle} & \overset{\theta_F \otimes \text{id}}{\longrightarrow} & O_F \otimes M_{\langle n \rangle} & \overset{\text{id} \otimes 1 \otimes \text{id}}{\longrightarrow} & O_{F_r} \otimes M_{\langle n \rangle} \\
\downarrow & & \downarrow & & \downarrow \\
C(X) \otimes M_{\langle n \rangle} & \overset{\theta_F \otimes \text{id} \otimes 1}{\longrightarrow} & O_F \otimes M_{\langle n \rangle} \otimes M_{\langle n \rangle} & \overset{\varphi \otimes \text{id}}{\longrightarrow} & O_{F_R} \otimes M_{\langle n \rangle} \otimes M_{\langle n \rangle} \\
\downarrow & & \downarrow & & \downarrow \\
C(X) \otimes M_{\langle n \rangle} & \overset{\theta_F \otimes \text{id} \otimes 1}{\longrightarrow} & O_F \otimes M_{\langle n \rangle} \otimes M_{\langle n \rangle} & \overset{\text{id} \otimes 1 \otimes \text{id}}{\longrightarrow} & O_{F_R} \otimes M_{\langle n \rangle} \otimes M_{\langle n \rangle} \\
\downarrow & & \downarrow & & \downarrow \\
C(X) \otimes M_{\langle n \rangle} & \overset{\theta_F \otimes \text{id}}{\longrightarrow} & O_F \otimes M_{\langle n \rangle} & \overset{\text{id} \otimes 1 \otimes \text{id}}{\longrightarrow} & O_{F_R} \otimes M_{\langle n \rangle}.
\end{array}
\]

The middle square of the diagram commutes because \(\varphi\) is \(C(X)\)-linear. By [13, Theorem 2.2], two \(*\)-homomorphisms \(1 \otimes \text{id}, \text{id} \otimes 1: M_{\langle n \rangle} \rightarrow M_{\langle n \rangle} \otimes M_{\langle n \rangle}\) are homotopic. So the upper and lower square of the diagram commute up to homotopy, and commutes in the level of \(K\)-groups.

Second, we show the vertical map \(\text{id} \otimes 1 \otimes \text{id}: O_F \otimes M_{\langle n \rangle} \rightarrow O_F \otimes M_{\langle n \rangle} \otimes M_{\langle n \rangle}\) induces an isomorphism of the \(K\)-groups. One has an isomorphism \(\psi: M_{\langle n \rangle} \rightarrow M_{\langle n \rangle} \otimes M_{\langle n \rangle}\). By [13, Theorem 2.2], two maps \(1 \otimes \text{id}\) and \(\psi\) are homotopic. So the map \(K_0(\text{id} \otimes 1 \otimes \text{id}) = K_0(\text{id} \otimes \psi)\) is an isomorphism.

Finally, we show \((1 - [F_r])K^0(X) \otimes \mathbb{Z}_{\langle n \rangle} = (1 - [F_R])K^0(X) \otimes \mathbb{Z}_{\langle n \rangle}\). An exact sequence \(0 \rightarrow K_{F_r} \otimes M_{\langle n \rangle} \rightarrow T_{F_r} \otimes M_{\langle n \rangle} \rightarrow O_F \otimes M_{\langle n \rangle} \rightarrow 0\) gives a 6-term exact sequence, and we have the following exact sequence:

\[K_0(C(X)) \otimes \mathbb{Z}_{\langle n \rangle} \overset{\phi}{\rightarrow} K_0(C(X)) \otimes \mathbb{Z}_{\langle n \rangle} \overset{\phi_{\theta_F \otimes \text{id}}}{\rightarrow} K_0(O_F \otimes M_{\langle n \rangle}).\]
So we have \( \text{Ker} K_0(\theta_{F_r} \otimes \text{id}) = (1 - [F_r])K^0(X) \otimes \mathbb{Z}_{(n)} \). This gives the conclusion because the diagram below commutes by the above argument:

\[
\begin{array}{ccc}
K_0(C(X)) \otimes \mathbb{Z}_{(n)} & \xrightarrow{\text{K}_0(\text{id} \otimes \theta_{F_r})} & K_0\left( \mathcal{O}_{F_r} \otimes M_{(n)} \right) \\
\downarrow & & \downarrow \text{K}_0(\phi) \\
K_0(C(X)) \otimes \mathbb{Z}_{(n)} & \xrightarrow{\text{K}_0(\text{id} \otimes \theta_{F_r})} & K_0\left( \mathcal{O}_{F_r} \otimes M_{(n)} \right).
\end{array}
\]

We define an equivalence relation \( \sim_n \) in \( \tilde{K}^0(X) \otimes \mathbb{Z}_{(n)} \).

**Definition 4.6.** Let \( a \) and \( b \) be elements in \( \tilde{K}^0(X) \otimes \mathbb{Z}_{(n)} \). Then \( a \sim_n b \) if there exists \( z \in \tilde{K}^0(X) \otimes \mathbb{Z}_{(n)} \) satisfying \( (a + b) - (a + z) = (n + b) \).

All elements of \( \tilde{K}^0(X) \otimes \mathbb{Z}_{(n)} \) are nilpotent by [16, Chap.II, Theorem 5.9]. The relation \( \sim_n \) is well-defined because \( (1 - z) \) has the inverse \( \sum_{k=0}^{\infty} z^k \).

**Lemma 4.7.** Let \( X \) be a connected compact Hausdorff space, and let \( F_r \) and \( F_R \) be vector bundles of rank \( n+1 \) and \( nR+1 \) respectively with \( (n, r) = (n, R) = 1 \). If \( (1 - [F_r])K^0(X) \otimes \mathbb{Z}_{(n)} = (1 - [F_R])K^0(X) \otimes \mathbb{Z}_{(n)} \), we have \( [F_r]r^{-1} \sim_n [F_R]R^{-1} \).

**Proof.** By assumption we have \( h \in K^0(X) \otimes \mathbb{Z}_{(n)} \) satisfying \( (nr + [F_r])h = (nR + [F_R]) \). A split exact sequence \( 0 \to K^0(X) \otimes \mathbb{Z}_{(n)} \to K^0(X) \otimes \mathbb{Z}_{(n)} \to K^0(\{pt\}) \otimes \mathbb{Z}_{(n)} \rightarrow 0 \) yields \( h - R/r \in K^0(X) \otimes \mathbb{Z}_{(n)} \). So we have \( (n + [F_r]r^{-1})(1 + \frac{1}{r}(h - R/r)) = (n + [F_R]) \).

By Lemma 4.5 and Lemma 4.7, the map \( I_n : \mathcal{O}(X)_n \ni [\mathcal{O}_{F_r} \otimes M_{(n)}] \mapsto [[F_r]r^{-1}] \in \tilde{K}^0(X) \otimes \mathbb{Z}_{(n)} \sim_n \) is well-defined.

**Lemma 4.8.** Let \( X \) be a finite dimensional compact connected Hausdorff space. Then the map \( I_n \) is surjective, and we have

\[
|X, \text{BAut} \mathcal{O}_{n+1}]| \geq |\mathcal{O}(X)_n| \geq |\tilde{K}^0(X) \otimes \mathbb{Z}_{(n)}| \sim_n .
\]

**Proof.** Every element of \( \tilde{K}^0(X) \otimes \mathbb{Z}_{(n)} \) is of the form \( \frac{1}{r} \otimes x \) where \( (n, r) = 1 \) and \( x \in \tilde{K}^0(X) \). By [15, Section 9, Theorem 1.2], we have \( R \in \mathbb{N} \) satisfying \( \tilde{K}^0(X) = \{ [E] \in \tilde{K}^0(X) \mid \text{rank } E = nR+1 \} \). So we have a vector bundle of rank \( nR+1 \), \( F_{R} \) with \( RX = [F_{R}] \). Therefore we have \( I_n([F_{R}]) = [\frac{1}{R} \otimes x] \). By [14, Theorem 1.4], one has \( \mathcal{O}(X)_n \subseteq [X, \text{BAut} \mathcal{O}_{n+1}] \). This proves the lemma.

Let \( R \) be a commutative algebra. A filtration of \( R \) is a sequence of subalgebras

\[
\cdots R_{k+1} \subseteq R_k \subseteq \cdots \subseteq R_1 = R
\]

with \( R_p \subseteq R_{p+q} \). Let \( X \) be a finite CW-complex. Then the group \( \tilde{K}^0(X) \) is a finitely generated commutative group by induction argument of attaching cells. The algebra \( \tilde{K}^0(X) \) has a filtration

\[
0 = K^0_m(X) \subseteq \cdots \subseteq K^0_0(X) = \tilde{K}^0(X)
\]

by [3, Section 2.1]. Consider a sequence of \( k \)-skeleta \( \{pt\} = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_m = X \). Then we define \( K^0_k(X) \) by \( \text{Ker}(\tilde{K}^0(X) \to K^0(X_k)) \). If the cohomology groups of a finite CW-complex \( X \) have no torsion, one has \( \text{Tor}(K^0_k(X)/K^0_{k+1}(X), \mathbb{Z}_n) = 0 \) by [3, Section 2.3] and [3, Section 2.4]. Moreover Dadarlat shows in his proof of [12, Theorem 5.3] that if the cohomology groups of the space \( X \) have no \( n \)-torsion, one has \( \text{Tor}(K^0_k(X)/K^0_{k+1}(X), \mathbb{Z}_n) = 0, m \geq k \). The proof of the following lemma is the same as in the proof of [12, Lemma 5.2].

**Lemma 4.9.** Let \( R \) be a filtered commutative ring with \( 0 = R_m \subseteq R_{m-1} \subseteq \cdots \subseteq R_1 = R \) and such that \( R \) is finitely generated as an additive group. If \( \text{Tor}(R_k/R_{k+1}, \mathbb{Z}_n) = 0 \) for every \( k \), we have \( |(R \otimes \mathbb{Z}_n)|/ \sim_n \geq |R \otimes \mathbb{Z}_n| \).
Corollary 4.10. Let $X$ be a finite CW-complex. Suppose $\text{Tor}(H^*(X), \mathbb{Z}_n) = 0$. Then we have

$$|\tilde{K}^0(X) \otimes \mathbb{Z}(n)/\sim_n| \geq |\tilde{K}^0(X) \otimes \mathbb{Z}_n|.$$ 

We need the following proposition.

Proposition 4.11 ([12, Proposition 5.1]). Let $X$ be a finite CW-complex. Then we have

$$|\tilde{H}^\text{even}(X, \mathbb{Z}_n)| \geq |X, \text{BAut} \mathcal{O}_{n+1}|,$$

where $\tilde{H}^\text{even}(X, \mathbb{Z}_n) := \prod_{k \geq 1} H^{2k}(X, \mathbb{Z}_n)$.

Now we show the following theorem.

Theorem 4.12. Let $X$ be a finite CW-complex. Suppose $\text{Tor}(H^*(X), \mathbb{Z}_n) = 0$. Then the map $I_n : \mathcal{O}(X)_n \to \tilde{K}^0(X) \otimes \mathbb{Z}(n)/\sim_n$ is bijective, and we have

$$|[X, \text{BAut} \mathcal{O}_{n+1}]| = |\mathcal{O}(X)_n| = |\tilde{H}^\text{even}(X, \mathbb{Z}_n)|.$$ 

Proof. By Corollary 4.10 we have $|\tilde{K}^0(X) \otimes \mathbb{Z}(n)/\sim_n| \geq |\tilde{K}^0(X) \otimes \mathbb{Z}_n|$. By Lemma 4.8 we have $|[X, \text{BAut} \mathcal{O}_{n+1}]| \geq |\tilde{K}^0(X) \otimes \mathbb{Z}(n)/\sim_n|$. From Proposition 4.11 we have $|\tilde{H}^\text{even}(X, \mathbb{Z}_n)| \geq |[X, \text{BAut} \mathcal{O}_{n+1}]|$, and Theorem 4.3 yields $|[X, \text{BAut} \mathcal{O}_{n+1}]| = |\mathcal{O}(X)_n| = |\tilde{H}^\text{even}(X, \mathbb{Z}_n)|$. 

References

[1] S. Araki and H. Toda, Multiplicative structures in mod q cohomology theories. I, Osaka J. Math. 2 (1965), 71–115.

[2] S. Araki and H. Toda, Multiplicative structures in mod q cohomology theories. II, Osaka J. Math. 3 (1966), 81–120.

[3] M. F. Atiyah and F. Hirzebruch, Vector bundles and homogeneous spaces, Proc. Sympos. Pure Math. 3 (1961), 7–38.

[4] B. Blackadar, K-theory for operator algebras, 2nd ed., Math. Sci. Inst. Publ., vol. 5, Cambridge University Press, Cambridge, 1998.

[5] N. P. Brown and N. Ozawa, C*-algebras and finite dimensional approximations, vol 88. Amer. Math. Soc., 2008.

[6] J. Cuntz, K-theory for cerain C*-algebras, Ann. of Math. (2) 113 : 1 (1981), 181–197.

[7] J. Cuntz, On the homotopy groups of the space of endomorphisms of a C*-algebra (with applications to topological Markov chains), Operator algebras and group representations, vol. I (Neptun, 1980), 124–137, Monogr. Stud. Math. 17, Pitman, Boston, MA, 1984.

[8] J. F. Davis and P. Kirk, Lecture notes in algebraic topology, Graduate Studies in Mathematics, 35. Amer. Math. Soc, Providence, RI, 2001.

[9] M. Dadarlat, The homotopy groups of the automorphism groups of Kirchberg algebras, J. Noncommut. Geom. 1 (2007), no. 1, 113–189.

[10] J. Dixmier, Les C*-algebres et leurs representations, 2nd ed., Cahiers Scientifiques 29, Gauthier-Villars, Paris, 1969. Reprinted by Editions Jacques Gabay, Paris, 1996. Translated as C*-algebras, North-Holland, Amsterdam, 1977.

[11] M. Dadarlat and U. Pennig, A Dixmier-Douady theory for strongly self-absorbing C*-algebras, J. Reine Angew. Math. 718 (2016), 153–181.

[12] M. Dadarlat, The C*-algebra of a vector bundle, J. Reine Angew. Math. 670 (2012), 121–143.

[13] M. Dadarlat and W. Winter, On the KK-theory of strongly self-absorbing C*-algebras, Math. Scand. 104 (2009), no. 1, 95–107.
[14] M. Dadarlat, Continuous fields of C*-algebras over finite dimensional spaces, Adv. Math. 222 (2009), no. 5, 1850–1881.
[15] D. Husemoller, Fibre bundles third edition., Grad. Texts Math. 20, Springer-Verlag, New York 1994.
[16] M. Karoubi, K-theory, Grundl. Math. Wiss. 226, Springer-Verlag, Berlin 1978.
[17] T. Katsura, On C*-algebras associated with C*-correspondences, Journal of Functional Analysis. 217 (2004), 366–401.
[18] N. C. Phillips, A classification theorem for nuclear purely infinite simple C*-algebras, Doc. Math. 5 (2000), 49–114.
[19] M. Pimsner, A class of C*-algebras generalizing both Cuntz-Krieger algebras and crossed products by \( \mathbb{Z} \), Fields Inst. Commun. 12 (1997), 189–212.
[20] M. Rørdam, F. Larsen and N. J. Laustsen, An introduction to K-theory for C*-algebras, London Mathematical Society Student Texts, vol. 49, Cambridge University Press, Cambridge, 2000.
[21] I. Raeburn and Dana P. Williams, Morita equivalence and continuous-trace C*-algebras, Mathematical Surveys and Monographs, 60. American Mathematical Society, Providence, RI, 1998.
[22] J. Rosenberg and C. Schochet, The Künneth theorem and the universal coefficient theorem for Kasparov’s generalized K-functor, Duke. Mathematical. Journal. vol. 55, no. 2, (1987), 431–474.
[23] C. Schochet, Topological methods for C*-algebras , IV : Mod p homology, Pacific J. Math. 114 : 2 (1984), 447–469.
[24] T. Sogabe, The homotopy groups of the automorphism groups of Cuntz-Toeplitz algebras, preprint, arXiv:1903.02796
[25] A. S. Toms and W. Winter, Strongly self-absorbing C*-algebras, Trans. Amer. Math. Soc. 359 (2007), no. 8, 3999–4029.
[26] W. Winter, Strongly self-absorbing C*-algebras are \( \mathbb{Z} \)-stable, J. Noncommut. Geom. 5 (2011), no. 2, 253–264.