Compressible Navier–Stokes–Fourier flows at steady-state

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Abstract
The heat conducting compressible viscous flows are governed by the Navier–Stokes–Fourier (NSF) system. In this paper, we study the NSF system accomplished by the Newton law of cooling for the heat transfer at the boundary. On one part of the boundary, we consider the Navier slip boundary condition, while in the remaining part the inlet and outlet occur. These boundary effects are the unique sink/source to the problem under study, and others effects such as the gravity and dissipation are neglected. The existence of a weak solution is proved via a new fixed point argument. With this new approach, the weak solvability is possible in Lipschitz domains, by making recourse to $L^q$-Neumann problems with $q > n$. Thus, standard existence results can be applied to auxiliary problems and the claim follows by compactness techniques. Quantitative estimates are established.

Keywords Compressible Navier–Stokes–Fourier system · Navier slip boundary conditions · Newton law of cooling · Inlet/outlet flows · Helmholtz decomposition

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1 Introduction
The heat conductive flows are described by a coupled system consisting of the equations of continuity, motion and energy. The study of compressible flows depends on the knowledge of solving the continuity equation, because this equation has its shortcomings. Due to fact that the continuity equation provides additional estimates

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Dedicated to my coauthor and beloved father Victor Consiglieri.

Communicated by Claudio Gorodski.

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on the density that are not possible in the stationary regime, the problem firstly had been studied in the non-stationary regime [13, 19]. We refer to [2] for the existence of stationary solutions if the transport coefficients are, at least, of class $W^{2,p}(\Omega)$ with $p > n$.

Several works deal with barotropic flows, where the pressure is a function of the density only. To cover the physical point of view, namely, the adiabatic exponent $\gamma = 5/3$ for the monoatomic gases or $\gamma = 7/5$ for the diatomic gases at ordinary temperature 150 K to 600 K, the imposed assumption on the pressure has being studied in function of the adiabatic exponent $\gamma$. To deal with this, the renormalized bounded energy weak solutions, in the context of the theory introduced by Lions [22], are proved for $\gamma \geq 5/3$ if $n = 3$. Since then the adiabatic exponent is becoming realistic. In [12], the renormalized bounded energy weak solutions are proved under the assumption that the adiabatic exponent satisfies $\gamma > 4/3$. We refer the existence of renormalized weak solutions for $\gamma > (3 + \sqrt{41})/8$ to [3], for the flows powered by volume potential forces in a rectangular domain with periodic boundary conditions, and recently, for $\gamma > 1$ to [28], in a bounded domain with no-slip boundary condition. For a general case, the existence of a fixed point to the Navier–Stokes system is applied in [31] by using the Schauder theorem under smallness of the $H^3$-norm for the velocity field if providing the system by smooth coefficients. The higher order derivatives are essential in establishing the estimate of $\text{div } u$. We remind that a fluid that flows at low velocity is described by the Stokes equations and not by the Navier–Stokes equations.

Nonisothermal steady state studies are well known and there exists a vast literature under the Dirichlet condition, for instance, on optimal control of low Mach number [18] and on uniqueness [26] and the literature cited therein. The better regularity of solutions by introducing the effective viscous flux $G = p - (2\mu + \lambda)\text{div } u$ is only possible under constant viscosities $\mu$ and $\lambda$ (see [12], and the references therein). With this assumption, the authors in [25] prove the existence of weak solutions by replacing, in NSF system, the energy equation by the total energy equation. This new system has the particularity of adding the equations, the pressure and the dissipation disappear, in the establishment of the crucial estimates.

Here, we consider the transport coefficients as temperature and spatial dependent. The behavior of the transport coefficients do not allow standard techniques [9] as, for instance, the use of either the above $G$ or the inverse of the Stokes operator.

The inhomogeneous boundary value problems are, in contrast, less common. We refer to [27] to the existence of continuous strong solutions to NSF problem under the assumptions that the Reynolds number and the inverse viscosity ratio are small and the Mach number $Ma \ll 1$.

The study of the NSF system that the source/sink is the heat transfer at the boundary, which is given by the Newton law of cooling, can be applicable to the physical situations such that come from biomedical engineering (as, for instance, thermal ablation for the treatment of thyroid nodules [4, 29]) as well as geological engineering (as, for instance, the natural gas flow in wells at the region that a single phase occurs).

A priori estimates are the core in a fixed point argument. However, they are usually deduced from the boundedness propriety of the operators. Then, there exist a
universal constant that is abstract, that is, it does not reflect the data dependency. To fill this gap, additional attention is paid in the determination of quantitative estimates in which the dependence on the data is explicit.

The outline of this paper is as follows. Next section is concerned for modeling of the problem under study and the description of the model itself. Section 3 is devoted to the mathematical framework, the establishment of the data assumptions, and the statement of the main theorems. In Sect. 4, we delineate the fixed point argument. The following sections (Sects. 5, 6 and 7) concentrate on the wellposedness of three auxiliary problems, namely a Dirichlet–Navier problem for the velocity field, a inlet/outlet problem for the density scalar and a Dirichlet–Robin problem for the temperature. The remaining sections (Sects. 8 and 9) are devoted to the proofs of the main theorems, respectively, Theorems 1 and 2.

2 Statement of the problem

Let $\Omega$ be a bounded domain (connected open set) of $\mathbb{R}^n$, $(n = 2, 3)$, with Lipschitz boundary. The boundary $\partial \Omega$ consists of three pairwise disjoint relatively open $(n - 1)$-dimensional submanifolds, $\Gamma_{in}, \Gamma_{out}$ and $\Gamma$, with positive Lebesgue measures, whose verify

$$\text{cl}(\Gamma_{in}) \cup \text{cl}(\Gamma_{out}) \cup \text{cl}(\Gamma) = \partial \Omega,$$

where cl stands for the set closure.

The heat conducting fluid at steady-state is governed by the Navier–Stokes–Fourier equations

$$\nabla \cdot (\rho \mathbf{u}) = 0$$

$$\rho(\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla \cdot \sigma = \rho \mathbf{g}$$

$$\rho \mathbf{u} \cdot \nabla e - \nabla \cdot (k(\theta)\nabla \theta) = \sigma : D\mathbf{u} \text{ in } \Omega.$$  

Here, the unknown functions are the density $\rho$, the velocity field $\mathbf{u}$, and the specific internal energy $e$. We denote $\zeta : \zeta = \xi_{ij}$ taking into account the convention on implicit summation over repeated indices. The gravitational force $\mathbf{g}$ and the dissipation $\sigma : D\mathbf{u}$ may be negligible, while the gravity has no significant contribution in gas dynamics, because gases have low densities, the Joule effect has no significant contribution because the transformation of the kinetic energy into heat may be neglected in case of small viscosities [23]. Notice that the neglecting the external force fields does not imply that the fluid is at rest. Indeed, the fluid flow is driven both by inlet and outlet flows and by heat transfer on the boundary.

In the case of ideal gases, the specific internal energy $e$ is related with the absolute temperature $\theta$ by the linear relationship $e = c_v \theta$, where $c_v$ denotes the specific heat capacity of the fluid at constant volume. Thus, the energy equation (3) can be written in terms of the temperature. Assuming that the thermal conductivity $k$ is a
function dependent on both temperature and space variable, the smoothness of the temperature depends on this coefficient.

The Cauchy stress tensor \( \sigma \), which is temperature dependent, obeys the constitutive law

\[
\sigma = -pI + \mu(\theta)D\mathbf{u} + \lambda(\theta)\text{tr}(D\mathbf{u})I, \quad \text{tr}(D\mathbf{u}) = I : D\mathbf{u} = \nabla \cdot \mathbf{u},
\]

(4)

where \( I \) denotes the identity \((n \times n)\)-matrix, \( D = (\nabla + \nabla^T)/2 \) the symmetric gradient, and \( \mu \) and \( \lambda \) are the viscosity coefficients in accordance with the second law of thermodynamics

\[
\mu(\theta) > 0, \quad \nu(\theta) := \frac{\lambda(\theta) + \mu(\theta)}{n} \geq 0,
\]

(5)

with \( \nu \) denoting the bulk (or volume) viscosity and \( \mu/2 \) being the shear (or dynamic) viscosity.

The pressure \( p \) in the case of ideal gases obeys to the Boyle–Marriott law

\[
p = R_{\text{specific}}\rho \theta
\]

(6)

where \( R_{\text{specific}} = R/M \) is the specific gas constant, with \( R = 8.314 \text{ J mol}^{-1}\text{K}^{-1} \) being the gas constant and \( M \) denoting the molar mass.

To understand the range of values we are talking to about, we exemplify some well known values for the dry air. For the air (assumed to be at the atmospheric pressure \( p = 101.325 \text{ kPa} \)), the molar mass of dry air is \( M = 28.96 \text{ kg kmol}^{-1} \) at temperature \( \theta = 298.15 \text{ K} \) (= 25 °C), then the density \( \rho = 1.184 \text{ kg m}^{-3} \). Thus, we have \( R_{\text{specific}} = 287 \text{ J kg}^{-1}\text{K}^{-1} \). The dry air can be assumed as diatomic, \( f = 5 \), then \( c_v = \frac{5R}{2} \). The dynamic viscosity \( \mu/2 = 0.018 \text{ mPa s} \) and the bulk viscosity \( \nu = 0.8\mu/2 \) [17]. Similar values are known for \( \text{O}_2 \) (see Table 1).

The triple point of the air is reached at temperature of 59.75 K (= −213.4 °C) and a correlated pressure (which value varies from author to author because how it is assumed the air composition). Thus, a minimum temperature \( \theta_0 \) is admissible. The values for the velocity, however, range from that the flow has Reynolds number \( \text{Re} \ll 1 \), in which case is described by the Stokes equation, until the flow behaves in the turbulent regime of \( \text{Re} \geq 10^6 \). This means \( \text{Re} > 6.5 \times 10^4\nu L \), with \( \nu \) standing for an average velocity and \( L \) the maximum length of the cross-section of the domain, in the above conditions.

### Table 1 Parameters at the atmospheric pressure [20, 23]

| \( \theta \) (K) | \( \mu/2 \) (Air) (10^{-5} \text{ Pa s}) | \( \mu/2 \) (O\(_2\)) (10^{-5} \text{ Pa s}) | \( k \) (Air) (10^{-2} \text{ W m}^{-1}\text{K}^{-1}) | \( k \) (O\(_2\)) (10^{-2} \text{ W m}^{-1}\text{K}^{-1}) |
|---|---|---|---|---|
| 100 | 0.7 | 0.8 | 0.9 | 1.0 |
| 200 | 1.3 | 1.5 | 1.8 | 1.8 |
| 300 | 1.9 | 2.0 | 2.6 | 2.7 |
| 500 | 2.7 | 3.0 | 4.0 | 4.3 |
| 800 | 3.7 | 4.2 | 5.7 | 6.6 |
| 1000 | 4.3 | 4.9 | 6.8 | 8.0 |
We notice that, in this work, we only assume as constant the specific heat capacity. This assumption is essential to leave the thermal conductivity as space variable dependent, by replacing the specific internal energy by the temperature as an unknown to seek. We leave all the remaining coefficients dependent on the temperature (see, for instance, Table 1) and on the space variable.

On the Dirichlet boundary \( \Gamma_D = \text{int}(\text{cl}(\Gamma_{\text{in}}) \cup \text{cl}(\Gamma_{\text{out}})) \), we assume inhomogeneous Dirichlet boundary condition

\[
\rho = \rho_\infty \quad \text{and} \quad u = u_D.
\]

This concise condition represents both

- the inflow \( u_{\text{in}} := u_D \cdot n < 0 \) and the density \( \rho_{\text{in}} \) are given;
- the outflow \( u_{\text{out}} := u_D \cdot n > 0 \) and the density \( \rho_{\text{out}} \) are given.

**Remark 1** Assuming smoothness, the density can be determined along the pathlines by \( \rho = \rho_{\text{in}} \exp[- \int \text{div} u] \). Then, the density on the outlet part of the boundary is determined by the density on the inlet part. In this work, however, the density is only measurable and has no well-defined trace. By this reason, the density should be imposed in both the outlet and inlet parts of the boundary, which is related with the existence of an auxiliary scalar potential \( (\psi) \).

On the remaining boundary \( \Gamma \), the fluid do not penetrate the solid wall, and it obeys the Navier slip boundary condition

\[
u_N := u \cdot n = 0, \quad \tau_T = -\gamma(\theta) u_T,
\]

where \( n \) stands for the unit outward vector to \( \Gamma \), \( u_N, u_T \) are the normal and tangential components of the velocity vector, respectively, \( \tau_T = \tau \cdot n - \tau_N n \) and \( \tau_N = (\tau \cdot n) \cdot n \) are the tangential and normal components of the deviator stress tensor \( \tau = \sigma + p I \), respectively, and \( \gamma \) denotes the friction coefficient.

For the heat transfer conditions, it is admissible to assume prescribed temperature in the inlet, that is, we consider the Dirichlet condition

\[
\theta = \theta_{\text{in}} \quad \text{on} \quad \Gamma_{\text{in}}.
\]

For the sake of simplicity, we assume \( \theta_{\text{in}} \) as a positive constant. Alternatively, we might assume that \( \theta_{\text{in}} \) may be extended to a function \( \hat{\theta}_{\text{in}} \in H^1(\Omega) \).

On the boundary \( \Gamma_N = \partial \Omega \setminus \text{cl}(\Gamma_{\text{in}}) \), we assume the Newton law of cooling

\[
k(\theta) \nabla \theta \cdot n + h_c(\theta)(\theta - \theta_e) = 0,
\]

where \( h_c \) denotes the heat transfer coefficient and \( \theta_e \) represents a given (eventually nonconstant) external temperature. This condition is mathematically known as the Robin condition. The heat source/sink is completely driven from the boundary and we denote
\[ \theta_0 = \begin{cases} \theta_{in} & \text{on } \Gamma_{in} \\ \theta_c & \text{on } \Gamma \\ \theta_{out} & \text{on } \Gamma_{out}. \end{cases} \]

3 Main results

We assume that \( \Omega \subset \mathbb{R}^n \) is a bounded domain with its boundary \( \partial \Omega \in C^{0,1} \). The standard notation of Lebesgue and Sobolev spaces is used. Let us define the Hilbert spaces endowed with the norms, respectively,

\[ H^{1}_{in}(\Omega) := \{ v \in H^{1}(\Omega) : v = 0 \text{ on } \Gamma_{in} \}; \]
\[ V := \{ v \in H^{1}(\Omega) : v = 0 \text{ on } \Gamma_{D}, \ v \cdot n = 0 \text{ on } \Gamma \}, \]

endowed with the norms, respectively,

\[ \| v \|_{1,2,\Omega} = \left( \| \nabla v \|^2_{2,\Omega} + \| v \|^2_{2,\Gamma} \right)^{1/2}; \]
\[ \| v \|_{V} = \left( \| Dv \|^2_{2,\Omega} + \| v \|^2_{2,\Gamma} \right)^{1/2}. \]

The meaning of the condition \( v \cdot n = 0 \) on \( \Gamma \) should be understood as

\[ \langle v \cdot n, v \rangle_{\Gamma} = 0, \quad \forall v \in H^{1/2}_{00}(\Gamma) = \{ v \in H^{1/2}(\partial \Omega) : v = 0 \text{ on } \Gamma_{D} \}, \]

where the symbol \( \langle \cdot, \cdot \rangle_{\Gamma} \) stands for the duality pairing \( \langle \cdot, \cdot \rangle_{Y \times Y} \), where \( Y = H^{1/2}_{00}(\Gamma) \).

**Definition 1** (NSF problem) We say that the triplet \((\rho, u, \theta)\) is a weak solution to the NSF problem if it satisfies the integral identities

\[ \int_{\Omega} \rho u \cdot \nabla v dx = \int_{\Gamma_D} \rho_{\infty} u_D \cdot n v ds, \quad \forall v \in W^{1,q'}(\Omega); \quad (11) \]

\[ \int_{\Omega} \rho (u \cdot \nabla) u \cdot v dx + \int_{\Omega} \mu(\theta) Du \cdot Dv dx + \int_{\Omega} \lambda(\theta) \nabla \cdot u \nabla \cdot v dx \]
\[ + \int_{\Gamma} \gamma(\theta) u_T \cdot v_T ds = \int_{\Omega} p \nabla \cdot v dx, \quad \forall v \in V; \quad (12) \]

\[ c_v \int_{\Omega} \rho u \cdot \nabla \theta v dx + \int_{\Omega} k(\theta) \nabla \theta \cdot \nabla v dx + \int_{\Gamma_N} h_c(\theta) \theta v ds = \int_{\Gamma_N} h(\theta) v ds, \quad \forall v \in H^{1}_{in}(\Omega), \quad (13) \]

subject to (6), (7) and (9). Here, \( q' \) stands for the conjugate exponent of \( q \), i.e. \( 1/q' + 1/q = 1 \), and \( h = h_c \theta_c \).
Remark 2 The variational formulations (11)-(13) are standardly derived from the NSF system (1)-(3) by the Green formula. We point out that the general formula
\[ \langle \rho (\mathbf{u} \cdot \nabla) \mathbf{u}, \mathbf{v} \rangle = \langle \tau, \mathbf{v} \rangle |_{\Gamma} - \langle p, \mathbf{n} \cdot \mathbf{v} \rangle |_{\Gamma} - \int_{\Omega} \sigma : D \mathbf{v} \, dx \]
holds for any \( \mathbf{v} \in \mathbf{V} \), under \( \nabla \cdot \sigma \in \mathbf{V}' \) [7].

The following assertions on the physical parameters appearing in the equations are assumed:

(H1) The viscosities \( \mu \) and \( \lambda \) are Carathéodory functions from \( \Omega \times \mathbb{R} \) into \( \mathbb{R} \) such that
\[ \exists \mu_# > 0 : \mu(x, e) \geq \mu_# > 0; \]
\[ \exists \mu_# > 0 : \mu(x, e) \leq \mu_#; \]
\[ \exists \lambda_# > 0 : |\lambda(x, e)| \leq \lambda_# , \]
for a.e. \( x \in \Omega \) and for all \( e \in \mathbb{R} \).

(H2) The thermal conductivity \( k \) is a Carathéodory function from \( \Omega \times \mathbb{R} \) into \( \mathbb{R} \) such that
\[ \exists k_# > 0, k_# : k_# \leq k(x, e) \leq k_# , \]
for a.e. \( x \in \Omega \) and for all \( e \in \mathbb{R} \).

(H3) The friction coefficient \( \gamma \) is a continuous function from \( \mathbb{R} \) into \( \mathbb{R} \) such that
\[ \exists \gamma_#, \gamma_# > 0 : \gamma_# \leq \gamma(e) \leq \gamma_# , \forall e \in \mathbb{R} . \]

(H4) The heat transfer coefficient \( h_c \) is a Carathéodory function from \( \Gamma_N \times \mathbb{R} \) into \( \mathbb{R} \) such that
\[ \exists h_# > 0 : h_c(e) \leq h_# \text{ a.e. on } \Gamma_N ; \]
\[ \exists h_# > 0 : h_c(e) \geq h_# \text{ a.e. on } \Gamma ; \]
\[ h_c(e) \geq 0 \text{ a.e. on } \Gamma_{\text{out}} , \]
for all \( e \in \mathbb{R} \). Moreover, \( h = \theta_h c \) with the function \( \theta_c \in L^\infty(\Gamma_N) \).

(H5) The boundary term \( \rho_{\infty} \mathbf{u}_D \cdot \mathbf{n} \in L^q(\Gamma_D) \), for some \( q > n \), satisfy the compatibility condition
\[ \int_{\Gamma_D} \rho_{\infty} \mathbf{u}_D \cdot \mathbf{n} ds = 0. \]
There exists \( \mathbf{u}_D \in H^1(\Omega) \) such that its trace \( \tilde{\mathbf{u}}_D = \mathbf{u}_D \) on \( \Gamma_D \) and the normal component of trace vanishes on \( \Gamma \). Indeed, the trace operator has a continuous right inverse operator, and in particular it is surjective from \( W^{1,q}(\Omega) \) onto \( W^{1-1/q,q}(\partial \Omega) \).

**Remark 3** We denote by \( p^* = pn/(n - p) \) the critical Sobolev exponent related to the embedding \( W^{1,q}(\Omega) \hookrightarrow L^{p^*}(\Omega) \), if \( p < n \). For the sake of simplicity, we also denote by \( p^* \) any real value greater than one, if \( p = n \). The Rellich–Kondrachov embedding stands for any exponent between 1 and the critical Sobolev exponent \( p^* \). Notice that the Morrey embedding \( W^{1,q}(\Omega) \hookrightarrow C^{0,1/n/q}(\Omega) \) holds for \( q > n \).

**Remark 4** All terms are meaningful in the integral identities (11)–(13). The nonlinear terms, the convective term in (12) and the advective term in (13), are justified in Lemma 1, with \( \mathbf{m} = \rho \mathbf{u} \in L^q(\Omega), q > n \), i.e. \( \rho \in L^r(\Omega) \) and \( \mathbf{u} \in H^1(\Omega) \), with

\[
\frac{1}{q} = \frac{1}{r} + \frac{1}{p} \quad \text{if} \quad r = \frac{2p}{p - 4} > \frac{2n}{4 - n}, \quad 4 < p < 2^* \ (n = 2, 3).
\]

We observe that \( r > 2n/(4 - n) \) follows from \( 1/n > 1/q = 1/r + 1/p > 1/r + 1/2^* \), while \( r = 2p/(p - 4) \) follows from \( 1/r + 1/p = 1/q \) altogether to \( 1/q + 1/p = 1/2 \).

Let us state our first main theorem, where the density function is only defined a.e. in \( \Omega \).

**Theorem 1** Let the assumptions (H1)–(H5) be fulfilled. For any \( M \in \mathbb{N} \), there exists a triplet \((\rho, \mathbf{u}, \theta)\) such that

- \( \rho \) is a measurable function satisfying \( \rho \mathbf{u} \in L^q(\Omega) \), with \( n < q < n + \varepsilon \), for some \( \varepsilon \) depending on \( \Omega \);
- \( \mathbf{u} \in \mathbf{u}_D + \mathbf{V} \);
- \( \theta \in (\theta^* + H^1(\Omega)) \cap L^\infty(\Omega) \),

which is a weak solution to the NSF problem, with (6) replaced by

\[
p_M = T_M(\rho)R_{\text{specific}} \theta.
\]

Here, \( T_M \) stands for the truncation, i.e. \( T_M(z) = z \) for \( 0 \leq z \leq M \) and \( T_M \equiv M \) otherwise.

Let us state our second main theorem, where the density function is assumed to have \( L^r \)-regularity, for some \( r > 2n/(4 - n) \) \((n = 2, 3)\).

**Theorem 2** Under the conditions of Theorem 1, the NSF problem admits at least one solution in \( L^r(\Omega) \times (\mathbf{u}_D + \mathbf{V}) \times H^1(\Omega) \) if provided by \( \rho \in L^r(\Omega) \) satisfying

\[
\| \rho \|_{r,\Omega} \leq \mathcal{R},
\]

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for some positive constant $R$ independent on $M$ and $r$ verifying (23). Moreover, the following quantitative estimates

$$
\|u - \tilde{u}_D\|_V \leq \max\left\{ \frac{n}{(n-1)\mu\gamma}, \frac{1}{\gamma} \right\} (R_4|\Omega|^{1/2-1/r} + R_1\|\tilde{u}_D\|_{p,\Omega} \\
+ \mu^\#\|D\tilde{u}_D\|_{2,\Omega} + \lambda^\#\|\nabla \cdot \tilde{u}_D\|_{2,\Omega}) + \sqrt{\min\left\{ \frac{n-1}{n} H\gamma, \gamma \right\}}\|\tilde{u}_D\|_{2,\Gamma};
$$

(26)

$$
\|\theta\|_{1,2,\Omega} \leq R_2
$$

(27)

hold, where $R_1$, $R_2$ and $R_4$ are defined in (50), (51) and (61), respectively.

Remark 5 The quantitative estimate (26) may be simplified if, for instance, in the assumption (H5) we assume the existence of $\tilde{u}_D \in H^1(\Omega)$ having the trace

$$
\tilde{u}_D = \begin{cases} 
  u_D & \text{on } \Gamma_D \\
  0 & \text{on } \Gamma
\end{cases}
$$

instead.

4 Strategy

Our strategy is based on the fact that the velocity field is not admissible to use for the fixed point argument, because the velocity field is not directly measurable and the linear momentum is easier to be physically determined.

For fixed $q > n$ and $r > 2$, we define the closed set

$$
K_{q,r} := \{ m \in L^q(\Omega) : (29) \text{ holds} \} \times H^1(\Omega) \times L'(\Omega)
$$

(28)
in the reflexive Banach space $L^q(\Omega) \times H^1(\Omega) \times L'(\Omega)$. The set $K_{q,r}$ is nonempty (cf. Lemma 2).

For fixed $M \in \mathbb{N}$, we build an operator $T$

$$
T : (m, \xi, \pi) \in K_{q,r} \mapsto w = w(m, \xi, \pi) \quad \text{(Dirichlet–Navier problem)}
$$

$$
\mapsto u = w + \tilde{u}_D
$$

$$
\mapsto \rho = \rho(u) \quad \text{(Inlet/outlet problem)}
$$

$$
\mapsto \theta = \theta(m, \xi) \quad \text{(Dirichlet–Robin problem)}
$$

$$
\mapsto (\rho u, \theta, \rho_M)
$$

with $m \in L^q(\Omega)$ satisfying
Here, we consider three auxiliary problems.

(Dirichlet–Navier problem) The auxiliary velocity \( u_{1D} \) is the unique solution to the Dirichlet–Navier problem defined by

\[
- \int_{\Omega} \mathbf{m} \otimes w : \nabla v dx + \int_{\Omega} \mu(\xi) Dv : Dv dx + \int_{\Omega} \lambda(\xi) \nabla \cdot w \nabla \cdot v dx + \int_{\Gamma} \gamma(\xi) w_T \cdot v_T ds = \int_{\Omega} \pi \nabla \cdot v dx + G(m, \xi, \tilde{u}_D, v), \quad \forall v \in V,
\]

with

\[
G(m, \xi, \tilde{u}_D, v) := \int_{\Omega} \mathbf{m} \otimes \tilde{u}_D : \nabla v dx - \int_{\Gamma} \gamma(\xi) \tilde{u}_D \cdot v_T ds - \int_{\Omega} (\mu(\xi) D\tilde{u}_D : Dv + \lambda(\xi) \nabla \cdot \tilde{u}_D \nabla \cdot v) dx.
\]

(Inlet/outlet problem) The auxiliary density \( \rho \) is a unique solution to the inlet/outlet problem defined by

\[
\int_{\Omega} \rho u \cdot \nabla v dx = \int_{\Gamma_D} \rho_{\infty} u_D \cdot n v ds, \quad \forall v \in W^{1,q}(\Omega).
\]

(Dirichlet–Robin problem) The auxiliary temperature \( \theta - \theta_{\text{in}} \in H^1_{\text{in}}(\Omega) \) is the unique weak solution to the Dirichlet–Robin problem defined by

\[
c_v \int_{\Omega} \mathbf{m} \cdot \nabla \theta v dx + \int_{\Omega} k(\xi) \nabla \theta \cdot \nabla v dx + \int_{\Gamma_N} h_\xi(\xi) \theta v ds = \int_{\Gamma_N} h_\xi(\xi) v ds, \quad \forall v \in H^1_{\text{in}}(\Omega).
\]

Finally, the auxiliary pressure is given by (24).

Let us establish some properties of the linearized convective and advective terms, which are the key points of this paper.

Lemma 1 Let \( \Omega \subset \mathbb{R}^n \) be a bounded Lipschitz domain. For each \( \mathbf{m} \in L^q(\Omega), \ q > n \), which verifies (29), the following functionals are well defined and continuous:

(1) \( \mathbf{u} \in H^1(\Omega) \leftrightarrow \langle B \mathbf{u}, v \rangle := \int_{\Omega} \mathbf{m} \otimes \mathbf{u} : \nabla v dx, \quad \forall v \in V \). Moreover, \( B \) is skew-symmetric in the sense

\[
\langle B \mathbf{u}, v \rangle = -\int_{\Omega} \langle \mathbf{m} \cdot \nabla \rangle \mathbf{u} \cdot v dx, \quad \forall \mathbf{u} \in H^1(\Omega) \forall v \in V
\]

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and, in particular, \( \langle Bv, v \rangle = 0 \) holds for all \( v \in V \).

(advective) \( e \in H^1(\Omega) \mapsto \int_\Omega \mathbf{m} \cdot \nabla ev \, dx \), for all \( v \in H^1(\Omega) \). Assuming (H5), the relation

\[
\int_\Omega \mathbf{m} \cdot \nabla ev = \int_{\Gamma_D} \rho_\infty \mathbf{u}_D \cdot \mathbf{n} ev \, ds - \int_\Omega \mathbf{m} \cdot \nabla vedx
\]

(34)

holds for any \( e, v \in H^1(\Omega) \).

**Proof** The wellposedness of each functional is consequence of the Hölder inequality, with exponents \( q, p \) and 2 such that

\[
\frac{1}{2^*} < \frac{1}{p} = \frac{1}{2} - \frac{1}{q} \Leftrightarrow q > n,
\]

and the Rellich–Kondrachov embedding \( H^1(\Omega) \hookrightarrow L^p(\Omega) \) (cf. Remark 3).

The skew symmetry of \( B \), (33), follows from the relation

\[
\langle Bu, v \rangle + \int_\Omega (\mathbf{m} \cdot \nabla) u \cdot v \, dx = \int_\Omega \mathbf{m} \cdot \nabla(u \cdot v) \, dx = \int_{\Gamma_D} \rho_\infty \mathbf{u}_D \cdot \mathbf{n}(u \cdot v) \, ds
\]

by using (29) with \( u \cdot v \in W^{1,q'}(\Omega), 1 < q' < n/(n - 1) \).

In (34), the wellposedness of the boundary integral follows from the Hölder inequality, with exponents

\[
\frac{1}{t} + 2 \frac{n - 2}{2(n - 1)} = 1 \Leftrightarrow t = \begin{cases} n - 1 & \text{if } n = 3, 4 \\ \text{arbitrary} & \text{if } n = 2 \end{cases}
\]

and considering the embedding \( H^1(\Omega) \hookrightarrow L^{2(n-1)/(n-2)}(\partial \Omega) \) and \( \rho_\infty \mathbf{u}_D \cdot \mathbf{n} \in L^q(\Gamma_D) \), where \( q > t \).

\[ \square \]

5 Wellposedness of the Dirichlet–Navier problem

The following properties are well known in the fluid mechanics theory. However, the quantitative estimate is essential in the fixed point argument and we will fix it.

**Proposition 1** Let the assumptions (H1), (H3) and (H5) be fulfilled. For each \( (\mathbf{m}, \xi, \pi) \in K_{q,2} \), with \( q > n \), let \( w \in V \) be a solution to the problem (30). Then, the following quantitative estimate

\[
\min \left\{ \frac{n-1}{n} \mu_\pi \gamma_\pi \right\} \| w \|_V^2 \leq \frac{n}{(n - 1) \mu_\pi} \left( \| \pi \|_{2,\Omega} + \| \mathbf{m} \|_{q,\Omega} \| \mathbf{u}_D \|_{p,\Omega} \right.
\]

\[
+ \mu_\pi \| D\mathbf{\tilde{u}}_D \|_{2,\Omega} + \lambda_\pi \| \nabla \cdot \mathbf{\tilde{u}}_D \|_{2,\Omega} \right)^2 + \gamma_\pi \| \mathbf{\tilde{u}}_D \|_{2,\Omega}^2
\]

(35)

\( \square \)
holds, with $1 \leq p < 2^*$ being such that $1/q + 1/p = 1/2$.

**Proof** This proof is standard, but we sketch it because its quantitative expression. Choose $v = w \in V$ as a test function in (30), and use Lemma 1 to find

$$
\int_{\Omega} \mu(\xi)|Dw|^2 \, dx + \int_{\Omega} \lambda(\xi)|\nabla \cdot w|^2 \, dx + \int_{\Gamma} \gamma(\xi)|w_T|^2 \, ds
\leq \left( \|\pi\|_{2,\Omega} + \|m\|_{q,\Omega}\|\tilde{u}_D\|_{p,\Omega} + 2\|\mu(\xi)Du\|_{2,\Omega} + \|\lambda(\xi)\nabla \cdot \tilde{u}_D\|_{2,\Omega} \right) \|\nabla w\|_{2,\Omega}
+ \frac{1}{2} \|\sqrt{\gamma(\xi)}\tilde{u}_D\|_{2,\Gamma}^2 + \frac{1}{2} \|\sqrt{\gamma(\xi)}w_T\|_{2,\Gamma}^2
$$

taking the Hölder and Young inequalities into account and using the fact that $(\nabla \cdot w)^2 \leq |\nabla w|^2$. Since $n\lambda(\xi) + \mu(\xi) \geq 0$, applying (14) and (18) we have

$$
\frac{n-1}{n}\mu_\#\|Dw\|_{2,\Omega}^2 + \frac{\gamma_\#}{p}\|w_T\|_{2,\Gamma}^2
\leq \int_{\Omega} \mu(\xi)(|Dw|^2 - \frac{1}{n}|\nabla \cdot w|^2) \, dx + \frac{1}{2} \int_{\Gamma} \gamma(\xi)|w_T|^2 \, ds
\leq \frac{n-1}{2n} \mu_\#\|Dw\|_{2,\Omega}^2 + \frac{n}{2(n-1)}\mu_\# \left( \|\pi\|_{2,\Omega} + \|m\|_{q,\Omega}\|\tilde{u}_D\|_{p,\Omega} \right)
+ \mu_\#\|Du\|_{2,\Omega} + \lambda_\#\|\nabla \cdot \tilde{u}_D\|_{2,\Omega}
+ \frac{\gamma_\#}{2}\|\tilde{u}_D\|_{2,\Gamma}^2.
$$

Then, readjusting the above estimate we conclude Proposition 1.  \(\square\)

The following proposition asserts the existence and uniqueness of auxiliary velocity field.

**Proposition 2** Let the assumptions (H1), (H3) and (H5) be fulfilled. For each $(\mathbf{m}, \xi, \pi) \in K_{q,2}$, with $q > n$, the problem (30) admits a unique solution $w \in V$.

**Proof** Let $a$ be the (non symmetric) bilinear form on $V \times V$, which is associated to the energy functional $J : V \to \mathbb{R}$, defined by

$$
J(v) = \int_{\Omega} \left( \mu(\xi)\frac{|Dv|^2}{2} + \lambda(\xi)\frac{|\nabla \cdot v|^2}{2} - \pi \nabla \cdot v \right) \, dx + \int_{\Gamma} \gamma(\xi)\frac{|v_T|^2}{2} \, ds - G(\mathbf{m}, \xi, \tilde{u}_D, v).
$$

The existence of $J'$ is well-defined [30, Appendix C] as the Fréchet derivative of a Nemytskii operator $F : \Omega \times \mathbb{R}^n \times M_{sym}^{n \times n} \to \mathbb{R}$, and the form $a$ is sum of the non symmetric and symmetric parts

$$
a(w, v) = -\int_{\Omega} \mathbf{m} \otimes w : \nabla v \, dx + \langle J'(w), v \rangle.
$$
Then, the existence and uniqueness of solution are consequence of the Lax–Milgram Lemma [21].

We finalize this section by proving the continuous dependence.

**Proposition 3** (Continuous dependence) Let \( \{ (m_m, \xi_m, \pi_m) \}_{m \in \mathbb{N}} \) be a sequence weakly convergent in \( K_{q,r} \) for some \( q > n \) and \( r > 2 \). Then, the corresponding solutions \( w_m = w(m_m, \xi_m, \pi_m) \) to the problem \( (30)_m \) for each \( m \in \mathbb{N} \), weakly converge to \( w = w(m, \xi, \pi) \) in \( V \), which is the solution to the problem \( (30) \) corresponding to the weak limit \( (m, \xi, \pi) \).

**Proof** Let us take the sequences

\[
m_m \rightharpoonup m \quad \text{in} \quad L^q(\Omega) \quad \text{for some} \quad q > n;
\]

\[
\xi_m \rightharpoonup \xi \quad \text{in} \quad H^1(\Omega);
\]

\[
\pi_m \rightharpoonup \pi \quad \text{in} \quad L^r(\Omega) \quad \text{for some} \quad r > 2.
\]

The Rellich–Kondrachov embeddings \( H^1(\Omega) \hookrightarrow \hookrightarrow L^2(\Omega) \) and \( H^1(\Omega) \hookrightarrow \hookrightarrow L^2(\partial \Omega) \) yield \( \xi_m \rightharpoonup \xi \) in \( L^2(\Omega) \) and \( L^2(\partial \Omega) \). The continuity of the Nemytskii operators, \( \mu, \lambda \) and \( \gamma \), and Lebesgue dominated convergence theorem imply

\[
\mu(\xi_m)Dv \rightharpoonup \mu(\xi)Dv \quad \text{in} \quad [L^2(\Omega)]^{n \times n};
\]

\[
\lambda(\xi_m)\nabla \cdot v \rightharpoonup \lambda(\xi)\nabla \cdot v \quad \text{in} \quad L^2(\Omega);
\]

\[
\gamma(\xi_m)v_T \rightharpoonup \gamma(\xi)v_T \quad \text{in} \quad L^2(\Gamma).
\]

Let \( w_m = w(m_m, \xi_m, \pi_m) \) be the corresponding solution to the problem \( (30)_m \), for each \( m \in \mathbb{N} \). The uniform estimate \( (35) \) allows to extract at least one subsequence, still denoted by \( w_m \), of the solutions \( w_m = w(m_m, \xi_m, \pi_m) \) weakly convergent for some \( w \in V \). Consequently, we have

\[
\nabla w_m \rightharpoonup \nabla w \quad \text{in} \quad [L^2(\Omega)]^{n \times n};
\]

\[
w_m \rightharpoonup w \quad \text{in} \quad L^p(\Omega) \quad \text{and on} \quad L^2(\partial \Omega),
\]

for \( p < 2^* \).

The above convergences allow to pass to the limit as \( m \) tends to infinity in \( (30)_m \), concluding that \( w \) satisfies the system \( (30) \). □

### 6 Existence and uniqueness of density solution

In this section, our objective is not to apply the artificial viscosity technique that approximates the continuity equation by an elliptic equation through a vanishing viscosity (also known as elliptic approximation) as it has being usual. Our argument goes out in the spirit of the Helmholtz decomposition.
\[ a = a_\omega + a_\psi, \]

where

- \( a_\omega = \nabla \times \omega \) stands for the solenoidal (divergence-free) component, i.e. it satisfies \( \nabla \cdot a_\omega = 0 \) in \( \Omega \).
- \( a_\psi = \nabla \psi \) stands for the irrotational component (curl-free), i.e. it satisfies \( \nabla \times a_\psi = 0 \) in \( \Omega \).

We refer to [14] for the weak \( L^q \)-solution to the Dirichlet–Laplace problem being motivated by the Weyl decomposition.

On the one hand, we consider the Neumann–Laplace problem

\[
\Delta \psi = 0 \quad \text{in } \Omega \tag{36}
\]

\[
\nabla \psi \cdot n = \rho_\infty u_D \cdot n \text{ on } \Gamma_D \tag{37}
\]

\[
\nabla \psi \cdot n = 0 \quad \text{on } \Gamma, \tag{38}
\]

with the zero mean value datum \( g := \rho_\infty u_D \cdot n \mathbf{1}_{\Gamma_D} \in L^q(\partial \Omega) \mapsto \left( B^{q'}_{1/q}(\partial \Omega) \right)' \), where the Besov space under the usual notation \( B^{1/q}_{1/q}(\partial \Omega) \) is in fact the Slobodetskii space \( W^{1/q}_{1/q}(\partial \Omega) \) for \( 0 < 1/q < 1/n \) and \( 1 < q' < n/(n-1) \). Thanks to potential theory [11, 15], the problem (36)–(38), with the zero mean value datum \( g \in B^{-1/q}_{1/q}(\partial \Omega) = \left( B^{q'}_{1/q}(\partial \Omega) \right)' \), admits the unique solution \( \psi \in W^{1/q}(\Omega) \) represented by

\[
\psi(x) = \int_{\partial \Omega} G_N(x, y)g(y)\text{d}s_y + \overline{\psi}
\]

where \( G_N(x, y) = E(x - y) + \phi(y) \) is the Green function of the second type, i.e. it solves the Neumann–Poisson boundary value problem \( \Delta G_N(x, \cdot) = \delta_x + 1/|\Omega| \) in \( \Omega \) and \( \nabla G_N \cdot n = 0 \) on \( \partial \Omega \) [6]. Here, \( \delta_x \) is the Dirac delta function at the point \( x \). The Green function \( E \), being the fundamental solution for \( \Delta \) in \( \mathbb{R}^n \) with pole at the origin, is given by

\[
E(x) = \begin{cases} 
\frac{1}{2\pi} \ln |x| & \text{if } n = 2; \\
\frac{1}{4\pi} \frac{1}{|x|} & \text{if } n = 3.
\end{cases}
\]

The function \( \phi \) solves \( \Delta \phi = 1/|\Omega| \) in \( \Omega \) and \( \nabla(E(x - \cdot) + \phi) \cdot n = 0 \) on \( \partial \Omega \). The uniqueness of the Neumann problem is possible by the compatibility condition (22), up to the additive constant \( \overline{\psi} = |\Omega|^{-1} \int_{\partial \Omega} \psi \text{d}x \), where \( |\Omega| \triangleq \text{meas}(\Omega) \).

The solution \( \psi \), the so called scalar potential, satisfies the estimate

\[
\| \nabla \psi \|_{q, \Omega} \leq C_q \| \rho_\infty u_D \cdot n \|_{q, \partial \Omega}, \tag{39}
\]
for any $1 < q < \infty$ if $\Omega$ is of class $C^1$, for the sharp ranges $4/3 - \varepsilon < q < 4 + \varepsilon$ if $n = 2$ or $3/2 - \varepsilon < q < 3 + \varepsilon$ if $n = 3$ and $\Omega$ is bounded Lipschitz, with $\varepsilon > 0$ depending on $\Omega$ and $C_q > 0$ depending on $n, q$, and the Lipschitz character of $\Omega$ [15, 16, 24]. Some specific results are known for convex domains for $1 < q < \infty$ if $n = 2$ and for $1 < q < 4$ if $n = 3$ [8, 16].

On the other hand, we find the corresponding vector that makes possible the decomposition.

First, let us establish in the two dimensional space the existence of our auxiliary density function.

**Proposition 4** $(n = 2)$ Let $(m, \xi, \pi) \in L^q(\Omega) \times H^1(\Omega) \times L^r(\Omega)$ and let $u \in H^1(\Omega)$ be the corresponding solution to $(30)$ obtained in Sect. 5. Then, there exists a unique function $\rho$ verifying

$$\rho u = \nabla \psi + \nabla \times \omega \text{ a.e. in } \Omega,$$

with $\psi$ being the unique solution to $(36)-(38)$ and for some $\omega$ in $W^{1, q}(\Omega)$. In particular, it is non-negative. Moreover, $(31)$ holds.

**Proof** Let $\psi \in W^{1, q}(\Omega)$ be the unique solution to $(36)$-$(38)$, which verifies $(39)$, for $1 < q < 2 + \varepsilon$, with $\varepsilon > 0$ depending on $\Omega$ and $C_q > 0$ depending on $n, q$, and the Lipschitz character of $\Omega$.

By the potential theory [1, 24], it suffices to seek for a unique non-negative density function that satisfies

$$\rho u = \nabla \psi + z,$$

with $z$ belonging to

$$H_q := \{ v \in L^q(\Omega) : \nabla \cdot v = 0 \text{ in } \Omega, \ v \cdot n = 0 \text{ on } \partial \Omega \}.$$

Taking in $(41)$ the inner product with $u$, we obtain

$$\rho = \frac{1}{|u|^2} (\nabla \psi + z) \cdot u \text{ in } \Omega[|u| \neq 0],$$

otherwise, we define $\rho = \rho_0$ in $\Omega[|u| = 0]$ (cf. Remark 6).

Hereafter, the set $A[S]$ means $\{ x \in A : S(x) \}$, with $S$ denoting a sentence to be pointwisely (a.e.) satisfied in $A$, which may represent either $\Omega$ or $\Gamma$.

Taking in $(41)$ the inner product with $u_\perp = (-u_2, u_1) \in L^p(\Omega)$, for any $1 < p < \infty$, we find the relation

$$z \cdot u_\perp = u_1 \partial_2 \psi - u_2 \partial_1 \psi := \nabla \times \psi \cdot u,$$

taking $\psi = (0, 0, \psi)$ into account. We emphasize that the absolute value of the real number $\nabla \times \psi \cdot u/|u| = |\nabla \psi \cdot u_\perp/|u| \in L^q(\Omega)$ is the magnitude of the vector rejection of $\nabla \psi$ from $u$, which is defined as
\[ \nabla \psi - \frac{\nabla \psi \cdot \mathbf{u}_\perp}{|\mathbf{u}|^2} \mathbf{u}_\perp = \left( \frac{\nabla \psi \cdot \mathbf{u}}{|\mathbf{u}|} \right) \frac{\mathbf{u}}{|\mathbf{u}|}, \]  

(43)

where the right hand side stands for the vector projection of \( \nabla \psi \) onto \( \mathbf{u} \). In the following, for the sake of simplicity, we assume that \( \nabla \psi \cdot \mathbf{u}_\perp \geq 0 \), otherwise we may similarly argue by redefining \( \mathbf{u}_\perp = (u_2, -u_1) \).

Let us consider the following two cases.

**Case 1.** If \( \nabla \times \psi \cdot \mathbf{u} = 0 \), it means that

\[ \nabla \psi = \pm |\nabla \psi| \frac{\mathbf{u}}{|\mathbf{u}|}. \]

If \( \angle(\nabla \psi, \mathbf{u}) = 0 \), we may take \( \mathbf{z} = \mathbf{0} \) in (41). Then, we obtain \( \rho = |\nabla \psi|/|\mathbf{u}| \). In particular, \( \rho \) is unique and non-negative.

Notice that the case \( \angle(\nabla \psi, \mathbf{u}) = \pi \) does not occur a.e. in \( \Omega \), because it leads to the contradiction

\[ -|\nabla \psi|^* / |\mathbf{u}_D| \mathbf{u}_D \cdot \mathbf{n} = \rho_\infty \mathbf{u}_D \cdot \mathbf{n} \text{ on } \Gamma_D, \]

where \( (\nabla \psi)^* \) denotes the nontangential maximal function of \( \nabla \psi \) [15, 16]. If \( \angle(\nabla \psi, \mathbf{u}) = \pi \) in an open ball \( B \subset \subset \Omega \), we may take \( \mathbf{z} = -2 \nabla \psi \) that fulfills (41) in \( B \). Then, we obtain \( \rho = |\nabla \psi|/|\mathbf{u}| \), which is unique and non-negative.

**Case 2.** If \( \nabla \times \psi \cdot \mathbf{u} \neq 0 \), it means that \( \cos(\angle(\nabla \psi, \mathbf{u})) < 1 \). Next, we will need three auxiliary functions, denoted by \( \mathbf{a}, \varphi \) and \( \mathbf{F} \).

In accordance with the latter case, we define the vector \( \mathbf{a} \in L^q(\Omega) \) as follows

\[ \nabla \psi + \mathbf{a} = |\nabla \psi| \frac{\mathbf{u}}{|\mathbf{u}|} := \rho_1 \mathbf{u}. \]  

(44)

This definition captures both cases (a) \( \nabla \psi \cdot \mathbf{u} > 0 \) and (b) \( \nabla \psi \cdot \mathbf{u} \leq 0 \) (see Fig. 1).

By the \( L^q - \text{Helmholtz--Weyl} \) decomposition (see e.g. [16, 24]), the vector \( \mathbf{a} \) may be decomposed as \( \mathbf{a} = \nabla \varphi + \mathbf{F} \). Here, \( \varphi \in W^{1,q}(\Omega) \) is the unique (up to additive constants) solution to the variational problem

\[ \int_\Omega \nabla \varphi \cdot \nabla v \, dx = \int_\Omega \mathbf{a} \cdot \nabla v \, dx, \quad \forall v \in W^{1,q}(\Omega). \]  

(45)

The existence and uniqueness of this scalar potential in the quotient space \( W^{1,q}(\Omega)/\mathbb{R} \) is guaranteed by the range values of \( q \) for Lipchitz domains. Setting

\[ \mathbf{F} = \mathbf{a} - \nabla \varphi, \]

it is unique and it belongs to \( H^q_q \). Moreover, we have

\[ \max \{ ||\nabla \varphi||_{q,\Omega}, ||\mathbf{F}||_{q,\Omega} \} \leq C_q ||\mathbf{a}||_{q,\Omega}, \]  

(46)

where \( C_q \) depends only on \( q \) and the Lipschitz character of \( \Omega \).

Taking (44) and the decomposition (43) for \( \varphi \), we have
It remains to evaluate the last term of the above relation, namely

$$
\rho_1 \mathbf{u} - \left( \nabla \varphi \cdot \frac{\mathbf{u}}{|\mathbf{u}|} \right) \frac{\mathbf{u}}{|\mathbf{u}|} = \nabla \psi + \mathbf{F} + \left( \nabla \varphi \cdot \frac{\mathbf{u}_\perp}{|\mathbf{u}|} \right) \frac{\mathbf{u}_\perp}{|\mathbf{u}|},
$$

$$
\rho_2 = \begin{cases} 
\rho_1 - \nabla \varphi \cdot \frac{\mathbf{u}}{|\mathbf{u}|}^2 & \text{if } \rho_1 - \nabla \varphi \cdot \frac{\mathbf{u}}{|\mathbf{u}|}^2 > 0 \\
0 & \text{otherwise}
\end{cases}
$$

(47)

It remains to evaluate the last term of the above relation, namely

$$
f = \left( \nabla \varphi \cdot \frac{\mathbf{u}_\perp}{|\mathbf{u}|} \right) \frac{\mathbf{u}_\perp}{|\mathbf{u}|}.
$$

Or, equivalently

$$
f_i(t, n) = 0
$$

$$
f_n(t, n) = \nabla \varphi \cdot \mathbf{e}_n,
$$

by taking the change of coordinates

$$
\begin{bmatrix}
\mathbf{e}_1 \\
\mathbf{e}_n
\end{bmatrix} = \begin{bmatrix}
\frac{u_1}{|\mathbf{u}|} & \frac{u_2}{|\mathbf{u}|} \\
\frac{-u_2}{|\mathbf{u}|} & \frac{u_1}{|\mathbf{u}|}
\end{bmatrix} \begin{bmatrix}
\mathbf{e}_1 \\
\mathbf{e}_2
\end{bmatrix}
$$

into account. Hence, we define $z_t$ and $z_n$ such that

$$
z_n = f_n \quad \text{and} \quad z_t = - \int \partial_n z_n \, dt + \rho_3(n),
$$

FIG. 1 Graphical representation of $\nabla \psi$ and $\mathbf{u}$, in black and blue solid lines, respectively, relative to the coordinate system $(\mathbf{u}/|\mathbf{u}|, \mathbf{u}_\perp/|\mathbf{u}|)$. Cases (a) $\nabla \psi \cdot \mathbf{u} > 0$ and (b) $\nabla \psi \cdot \mathbf{u} \leq 0$. Red small dashed line represents the rejection, while green long dashed lines stand for vectors $\nabla \varphi$ and $\mathbf{F}$ (Color figure online)
with \( \rho_3 \) being such that \( z_t < 0 \). Finally, we choose the vector \( z \in H_q \) such that

\[
z = F + z_t e_t + z_n e_n,
\]

by recalling the vector \( F \) from (47).

From (40), the function \( \rho \) verifies the variational formulation (31), which concludes the proof of Proposition 4.

**Remark 6** We call by \( \rho_0 \) the constant density at STP (standard temperature and pressure). Notice that the velocity may be zero, the so called stagnation. The no upper boundedness of the density is related to that \( |u| \to 0 \) means \( \rho \to \infty \). However, neither the velocity function nor the density function are continuous. It suggests that some upper boundedness will be possible, but it is still an open problem. We recall that different approaches have been showing up, namely the assumption of barotropic flows which the parameter \( \gamma \) determines the quality of density estimates from the renormalized continuity equation [3], the concept of oscillation defect measure in isothermal regime [28], the so-called effective viscous pressure identity, which measures oscillations of a sequence of approximate densities in the problem under constant viscosities [12], by the recourse of the total energy balance and the entropy function in the non-isothermal regime [25], or the presence of the material derivative of the density in the non-stationary regime turns on the construction of a solution by means the method of characteristics [13, 19].

Next, we study the three dimensional space.

**Proposition 5** \((n = 3)\) Let \( (m, \xi, \pi) \in L^q(\Omega) \times H^1(\Omega) \times L^q(\Omega) \) and let \( u \) be the corresponding solution to (30) obtained in Sect.5. Then, there exists a unique function \( \rho \) verifying

\[
\rho u = \nabla \psi + \nabla \times \omega \text{ a.e. in } \Omega,
\]

with \( \psi \) being the unique solution to (36)–(38) and for some \( \omega \) in \( W^{1,q}(\Omega) \). In particular, it is non-negative. Moreover, (31) holds.

**Proof** Let \( \psi \in W^{1,q}(\Omega) \) be the unique solution to (36)–(38), which verifies (39), for \( 3/2 - \varepsilon < q < 3 + \varepsilon \), with \( \varepsilon > 0 \) depending on \( \Omega \) and \( C_q > 0 \) depending on \( n, q \), and the Lipschitz character of \( \Omega \).

In the three dimensional space, arguing as in the two dimensional Proposition 4 we seek for

\[
\rho = (\nabla \psi + \nabla \times \omega) \cdot u/|u|^2 \text{ in } \Omega[|u| \neq 0]
\]

otherwise, we define \( \rho = \rho_0 \) if \( \Omega[|u| = 0] \) (cf. Remark 6). As the objective is to find a scalar function, the argument of the proof of Proposition 4 may be repeated in the plane formed by the vectors \( u \) and \( \nabla \psi \), i.e. we consider the local coordinate system \((e_t, e_n, 0)\), where \( e_t = u/|u| \) and \( e_n = \nabla \psi \times u/|\nabla \psi \times u|\).
Therefore, there exists a vector potential $\omega$ such that $\nabla \times \omega = \rho u - \nabla \psi$, i.e. (48), which may be given unique \[1\].

**Remark 7** The identities (40) and (48) do not define $\rho$ when $\nabla \psi$ is not colinear with the velocity field $u$, but they indeed are consequence of its existence. However, this unique function suffices our proposes. It is still an open problem the finding of a unique function solving (31).

Finally, we are in conditions to determine the estimate for the linear momentum.

**Corollary 1** Let $\Omega$ be Lipschitz. For $n = 2, 3$, let $\rho$ be the unique function given at Propositions 4 and 5. Then, the estimate

$$\|\rho u\|_{q, \Omega} \leq 3C_q \|\rho_{\infty} u_D \cdot n\|_{q, \partial\Omega} := R_1$$

holds, for any $n < q < n + \epsilon$ and $n = 2, 3$, with $\epsilon$ depending on $\Omega$ and $C_q$ depending on $n, q$, and the Lipschitz character of $\Omega$.

**Proof** To prove the estimate (50), we apply (39) in the inequality

$$\|\rho u\|_{q, \Omega} \leq 3\|\nabla \psi\|_{q, \Omega},$$

by using (46) if $|u| \neq 0$. Otherwise, the estimate is clearly verified. \[\Box\]

### 7 Wellposedness of the Dirichlet–Robin problem

The existence of the solution $\theta \in H^1(\Omega)$, which satisfies (9), to the problem (32) is stated in the following proposition.

**Proposition 6** (Existence and uniqueness) Let the assumptions (H2) and (H4)–(H5) be fulfilled. For each $\mathbf{m} \in L^q(\Omega) \times H^1(\Omega)$, which verifies (29), the problem (32) admits a unique solution $\theta \in H^1(\Omega)$ such that $\theta = \theta_{in}$ on $\Gamma_{in}$. Moreover, the estimate

$$\|\nabla \theta\|_{2, \Omega}^2 + \|\theta\|_{2, \Gamma}^2 \leq \frac{h^\#}{\min \{2k^\#, h^\#\}} \|\theta_{in} + \theta_{e}\|_{2, \Gamma_N}^2 := R_2^2$$

holds.

**Proof** The existence and uniqueness of $\theta = u + \theta_{in}$, with $u \in H^1_{in}(\Omega)$, solving (32) is standard by the Lax–Milgram Lemma. The problem (32) reads

$$a(u, v) = \int_{\Gamma_N} h_N(\xi)(\theta_{e} - \theta_{in}) v ds, \quad \forall v \in H^1_{in}(\Omega),$$

where the continuous bilinear form $a$ from $H^1_{in}(\Omega) \times H^1_{in}(\Omega)$ into $\mathbb{R}$, is defined by
Moreover, using the assumptions (17) and (20)–(21), the form $a$ is coercive:

$$ a(u, u) = c_v \int_{\Omega} m \cdot \nabla u^2/2 \, dx + \int_{\Omega} k(\xi) |\nabla u|^2 \, dx + \int_{\partial \Omega} h_c(\xi) u^2 \, ds $$

$$ \geq \min\{k_\#, h_\#\} \left( \|\nabla u\|_{2,\Omega}^2 + \|u\|_{2,\Gamma}^2 \right), $$

taking $u^2 \in W^{1,q}(\Omega)$ into account, that is, (29) reads

$$ \int_{\Omega} m \cdot \nabla u^2/2 \, dx = \int_{\partial \Omega} \rho_\infty u_D \cdot nu^2/2 \, ds \geq 0. $$

The estimate (51) follows by choosing $v = \theta - \theta_m$ as a test function in (32), arguing as above, considering that $\nabla u = \nabla \theta$ and

$$ k_\# \|\nabla \theta\|_{2,\Omega}^2 + \frac{1}{2} \|\sqrt{h_c(\xi)} \theta\|_{2,\Gamma_N}^2 \leq \frac{1}{2} \left( \|\sqrt{h_c(\xi)}(\theta_m + \theta_c)\|_{2,\Gamma_N}^2 \right) $$

after routine computations. \qed

The following minimum-maximum principle is standard, its proof argument differs on the advective and boundary terms. For reader convenience, we provide the proof.

**Proposition 7** (Minimum-maximum principle) Let $\theta \in H^1(\Omega)$ be a solution to the problem (32). Then, the lower and upper bounds

$$ \text{ess inf}_{\partial \Omega} \theta_0 \leq \theta \leq \text{ess sup}_{\partial \Omega} \theta_0 \text{ a.e. in } \Omega $$

hold.

**Proof** Let us define $T_{\text{min}} = \text{ess inf}\{\theta_0(x) : x \in \partial \Omega\}$. Let us choose $\phi(\theta) = (\theta - T_{\text{min}})^- = \min\{\theta - T_{\text{min}}, 0\} \in H^1_{\text{in}}(\Omega)$ as a test function in (32). Applying the assumptions (17) and (20)-(21), we have

$$ \int_{\Omega} m \cdot \nabla \phi(\theta) \, dx + k_\# \|\nabla \theta\|_{2,\Omega[\theta < T_{\text{min}}]}^2 + h_\# \|\theta - T_{\text{min}}\|_{2,\Gamma[\theta < T_{\text{min}}]}^2 \leq 0. $$

Since the advective term verifies

$$ \int_{\Omega} m \cdot \phi(\theta) \, dx = \int_{\Omega[\theta < T_{\text{min}}]} m \cdot \nabla(\phi^2(\theta)/2) \, dx $$

$$ = \int_{\Gamma_D} \rho_\infty u_D \cdot n \phi^2(\theta)/2 \, ds = \int_{\Gamma_{\text{out}}} \rho_\infty u_{\text{out}} \phi^2(\theta)/2 \, ds \geq 0. $$
taking (29) and next (20)–(21) into account, we deduce
\[ k_\# \| \nabla \phi(\theta) \|^2_{2, \Omega} + h_\# \| \phi(\theta) \|^2_{2, \Gamma} \leq 0. \] (53)

Then, we conclude that \( \phi(\theta) = 0 \) in \( \Omega \), which means that the lower bound is proved.

The upper bound is analogously proved, by defining \( T_{\text{max}} = \text{ess sup}\{\theta_0(x) : x \in \partial \Omega\} \) and choosing \( \phi(\theta) = (\theta - T_{\text{max}})^+ = \max\{\theta - T_{\text{max}}, 0\} \in H^1_{\text{in}}(\Omega) \) as a test function in (32).

We finalize this section by proving the continuous dependence.

**Proposition 8** (Continuous dependence) Let \( \{\mathbf{m}_m, \xi_m\}_{m \in \mathbb{N}} \) be a weakly convergent sequence in \( L^q(\Omega) \times H^1(\Omega) \), for some \( q > n \). Then, the corresponding solutions \( \theta_m = \theta(\mathbf{m}_m, \xi_m) \in H^1(\Omega) \) to the problem (32), for each \( m \in \mathbb{N} \), weakly converge to \( \theta = \theta(\mathbf{m}, \xi) \), which is the solution to the problem (32) corresponding to the weak limit \( (\mathbf{m}, \xi) \).

**Proof** Let us take the sequences
\[
\mathbf{m}_m \rightharpoonup \mathbf{m} \text{ in } L^q(\Omega);
\xi_m \rightharpoonup \xi \text{ in } H^1(\Omega).
\]
The Rellich–Kondrachov embeddings \( H^1(\Omega) \hookrightarrow L^2(\Omega) \) and \( H^1(\Omega) \hookrightarrow L^2(\partial \Omega) \) yield \( \xi_m \rightharpoonup \xi \) in \( L^2(\Omega) \) and \( L^2(\partial \Omega) \).

By the one hand, from \( \xi_m \rightharpoonup \xi \) in \( L^1(\Omega) \) and a.e. in \( \Omega \), and the assumption (17), the continuity property of the Nemytskii operator associated to the leading coefficient \( k \) implies that
\[
k(\cdot, \xi_m) \to k(\cdot, \xi) \text{ a.e. in } \Omega;
k(\xi_m)\nabla v \to k(\xi)\nabla v \text{ in } L^2(\Omega).
\]

By the other hand, from \( \xi_m \rightharpoonup \xi \) in \( L^1(\partial \Omega) \) and a.e. on \( \partial \Omega \), and the assumptions (19)-(21), the continuity property of the Nemytskii operator associated to the boundary coefficient \( h_c \) implies that
\[
h_c(\cdot, \xi_m) \to h_c(\cdot, \xi) \text{ a.e. on } \Gamma_N;
h_c(\xi_m)v \to h_c(\xi)v \text{ in } L^2(\Gamma_N).
\]

For each \( m \in \mathbb{N} \), let \( \theta_m = \theta(\mathbf{m}_m, \xi_m) \) be the corresponding solution to the problem (32). The uniform estimate (51) allows to extract at least one subsequence, still denoted by \( \theta_m \), of the solutions \( \theta_m = \theta(\mathbf{m}_m, \xi_m) \) weakly convergent for some \( \theta \in H^1(\Omega) \).

The above convergences do not be sufficient to the passage to the limit, as \( m \) tends to infinity, in (32). It remains to pass the advective term to the limit. To this aim, we prove the following strong convergence \( \nabla \theta_m \to \nabla \theta \) in \( L^2(\Omega) \). Arguing as in [5], we apply the assumption (17) and we decompose to obtain
Next, to prove that \( I_1 \) also tends to zero, we take \( v = \theta_m - \theta \) as a test function in (32). Hence, we obtain

\[
k\# \int_{\Omega} |\nabla (\theta_m - \theta)|^2 \, dx \leq \int_{\Omega} (k(\xi_m)\nabla \theta_m - k(\xi_m)\nabla \theta) \cdot \nabla (\theta_m - \theta) \, dx = I_1 - I_2,
\]

with

\[
I_1 = \int_{\Omega} k(\xi_m)\nabla \theta_m \cdot \nabla (\theta_m - \theta) \, dx
\]

\[
I_2 = \int_{\Omega} k(\xi_m)\nabla \theta \cdot \nabla (\theta_m - \theta) \, dx \to 0 \text{ as } m \to \infty.
\]

Next, to prove that \( I_1 \) also tends to zero, we take \( v = \theta_m - \theta \) as a test function in (32). Hence, we obtain

\[
\int_{\Omega} m \cdot \nabla \frac{(\theta_m - \theta)^2}{2} \, dx + I_1 = \int_{\Omega} m \cdot \nabla \theta (\theta_m - \theta) \, dx
\]

\[
+ \int_{\partial \Omega} (h(\xi_m) - h_c(\xi_m)\theta_m)(\theta_m - \theta) \, ds \to 0 \text{ as } m \to \infty,
\]

taking the Rellich–Kondrachov embeddings \( H^1(\Omega) \hookrightarrow L^p(\Omega) \), with \( p < 2^* \), and \( H^1(\Omega) \hookrightarrow L^2(\partial \Omega) \) into account for \( n = 2, 3 \). Then, applying the relation (29) into the left hand side of the above equality, we find the claim, i.e. the strong convergence.

Then, the passage to the limit yields that \( \theta \) satisfies (32), concluding Proposition 8. \( \square \)

8 Existence of a fixed point to the problem (Proof of Theorem 1)

We will apply the following Tychonoff extension to weak topology of the Schauder fixed point theorem [10, pp. 453-456 and 470].

Theorem 3 Let \( K \) be a nonempty weakly sequential compact convex subset of a locally convex linear topological vector space \( V \). Let \( T : K \to K \) be a weakly sequential continuous operator. Then \( T \) has at least one fixed point.

Let \( V = L^q(\Omega) \times H^1(\Omega) \times L^r(\Omega) \) and \( K_{q,r} \) be the nonempty convex set defined in (28). We define \( K = K_{q,r} \cap B \), where \( B \) is the closed (bounded) ball, with radius \( R_1, R_2, R_3 > 0 \) defined in (50), (51) and (57), respectively. In the reflexive Banach space \( V \), the closed, convex and bounded set \( K \) is compact for the weak topology \( \sigma(V, V') \), i.e. it is weakly sequential compact. The nonemptiness of the set \( K \) is stated in the following lemma.

Lemma 2 Let the assumption (H5) be fulfilled. For \( n < q < n + \varepsilon \), with \( \varepsilon \) depending on \( \Omega \), there exists \( m \in L^q(\Omega) \) satisfying (29). Moreover, the following estimate

\[ k^\# \int_{\Omega} |\nabla (\theta_m - \theta)|^2 \, dx \leq \int_{\Omega} (k(\xi_m)\nabla \theta_m - k(\xi_m)\nabla \theta) \cdot \nabla (\theta_m - \theta) \, dx = I_1 - I_2, \]
\[ \| \mathbf{m} \|_{q, \Omega} \leq R_1 / 2. \]

holds.

**Proof** The \( L^q \)-Neumann problem

\[
\nabla \cdot \mathbf{m} = 0 \text{ in } \Omega;
\]

\[
\mathbf{m} \cdot \mathbf{n} = g \text{ on } \partial \Omega
\]

subject to the constraint

\[
\int_{\partial \Omega} g \, ds = 0,
\]

is uniquely solvable for solutions in the form \( \mathbf{m} = \nabla u \), which means the Neumann–Laplace problem (36)-(38), for \( n < q < n + \varepsilon \), where \( \varepsilon \) depends on \( \Omega \), which satisfies the estimate (39).

Observe that the existence of \( \mathbf{m} \) is not unique. \( \Box \)

Let \( T \) be the operator defined in Section 4. The fixed point argument (cf. Theorem 3) guarantees the existence of the required solution, by proving the following two propositions, namely, Propositions 9 and 10.

**Proposition 9** Let the assumptions (H1)–(H5) be fulfilled. Then, the operator \( T \) is well defined and it maps \( K \) into itself.

**Proof** The well-definiteness of \( T \) is consequence of Proposition 2, Corollary 1, and Proposition 6. In order to prove that \( T \) maps \( K \) into itself, let \((\mathbf{m}, \xi, \pi) \in K \) and

\[
T(\mathbf{m}, \xi, \pi) = (\rho \mathbf{u}, \theta, p_M).
\]

That is, we seek for \( R_1, R_2, R_3 > 0 \) such that

\[
\| \mathbf{m} \|_{q, \Omega} \leq R_1, \quad \| \xi \|_{1,2, \Omega} \leq R_2, \quad \| \pi \|_{r, \Omega} \leq R_3;
\]

\[
\| \rho \mathbf{u} \|_{q, \Omega} \leq R_1, \quad \| \theta \|_{1,2, \Omega} \leq R_2, \quad \| p_M \|_{r, \Omega} \leq R_3.
\]

Thanks to Corollary 1, the quantitative estimate (50) guarantees the existence of \( R_1 \), for \( q \) depending on the smoothness of the domain \( \Omega \). Thanks to Proposition 6, the quantitative estimate (51) guarantees the existence of \( R_2 \).

The existence of \( R_3 \) is due to the definition (24), we concretely have

\[
\| p_M \|_{r, \Omega} \leq M |\Omega|^{1/r} R \text{specific ess sup}_{\partial \Omega} \theta_0 := R_3,
\]

by considering the estimate (52). \( \Box \)
Proposition 10 Let the assumptions (H1)–(H5) be fulfilled. Then, the operator $T$ is weakly sequential continuous.

Proof Let $\{(m_m, \xi_m, \pi_m)\}_{m \in \mathbb{N}}$ be a sequence of $V$ weakly convergent to $(m, \xi, \pi)$, namely

\[ m_m \rightharpoonup m \text{ in } L^q(\Omega); \]
\[ \xi_m \rightharpoonup \xi \text{ in } H^1(\Omega); \]
\[ \pi_m \rightharpoonup \pi \text{ in } L^r(\Omega). \]

Thanks to Proposition 3, the corresponding solutions $w_m = w(m_m, \xi_m, \pi_m) \in V$ to the problem $(30)_m$ for each $m \in \mathbb{N}$, weakly converge to the solution $w = w(m, \xi, \pi)$ to the problem $(30)$. Thus, we get

\[ u_m \rightharpoonup u \text{ in } H^1(\Omega). \]

Consequently, we get $u_m \to u$ a.e. in $\Omega$. Notice that $u$ satisfies

\[-\int_\Omega m \times u : \nabla v \, dx + \int_\Omega \mu(\xi) D u : D v \, dx + \int_\Omega \lambda(\xi) \nabla \cdot u \nabla \cdot v \, dx + \int_\Gamma \gamma(\xi) u_T \cdot v_T \, ds = \int_\Omega \pi \nabla \cdot v \, dx,\]

for all $v \in V$, and the convective term verifies (33).

Let $\rho_m$ be the unique solution given at Propositions 4 and 5, for $n = 2, 3$, respectively. Then, it follows that $\rho_m$ a.e. converges to $\rho$ in $\Omega$. Thanks to Corollary 1, we have $\rho_m u_m \rightharpoonup \rho u$ in $L^q(\Omega)$, which limit satisfies (31).

Thanks to Proposition 8, the corresponding solutions $\theta_m = \theta(m_m, \xi_m)$ to the problem $(32)_m$ for each $m \in \mathbb{N}$, weakly converge to the solution $\theta = \theta(m, \xi)$ in $H^1(\Omega)$. Thus, $\theta_m$ strongly converges to $\theta$ in $L^p(\Omega)$, for $1 < p < 2^*$. Thanks to (57) and the Lebesgue dominated convergence theorem, we have

\[ T_M(\rho_m) \theta_m \rightharpoonup T_M(\rho) \theta \text{ in } L^r(\Omega). \]

Then, the operator $T$ is weakly sequential continuous, which finishes the proof of Proposition 9. \hfill \qed

Therefore, we are in condition to obtain the fixed point

\[ (m, \xi, \pi) = (\rho u, \theta, \rho_M), \]

which is the required solution. Finally, the argument of Proposition 7, with the auxiliary problem (32) being replaced by the variational problem (13), can be applied to obtain the $L^\infty$-regularity of the temperature $\theta$, and the proof of Theorem 1 is concluded.
9 Passage to the limit as $M \to \infty$ (Proof of Theorem 2)

The proof of the main result is due to compactness arguments.

Under the assumption (25), the solution $(\rho_M, u_M, \theta_M)$ determined in Theorem 1 satisfies

$$
\|\rho_M\|_{r,\Omega} \leq \mathcal{R}; \\
\|\rho_M u_M\|_{q,\Omega} \leq R_1; \\
\|	heta_M\|_{1,2} \leq R_2,
$$

(58)
(59)
(60)

considering $R_1$ and $R_2$ from (50) and (51), respectively. Arguing as in (57) with $\mathcal{R}$ replacing $M|\Omega|^{1/r}$, we get

$$
\|p_M\|_{r,\Omega} \leq \mathcal{R} \text{ess sup}_{\partial \Omega} \theta_0 := R_4.
$$

(61)

Hence, we can extract a subsequence of $p_M$, still labeled by $p_M$, weakly convergent to $p$ in $L'(\Omega)$.

Considering (59) and (61), the estimate (35) reads

$$
\min \left\{ \frac{n-1}{n} \mu, \gamma \right\} \|w\|_{V}^2 \leq \frac{n}{(n-1)\mu}(R_4|\Omega|^{1/2-1/r} + R_1\|\bar{u}_D\|_{p,\Omega}) \\
+ \mu \|D\bar{u}_D\|_{2,\Omega} + \lambda \|\nabla \cdot \bar{u}_D\|_{2,\Omega}^2 + \gamma \|\bar{u}_D\|_{2,\Gamma}^2.
$$

(62)

Then, the convergences

$$
\rho_M \rightharpoonup \rho \text{ in } L'(\Omega); \\
u_M \rightharpoonup u \text{ in } H^1(\Omega); \\
\theta_M \rightharpoonup \theta \text{ in } H^1(\Omega),
$$

hold, as $M$ tends to infinity. From the above convergences, we identify the limit

$$
p = p R \text{specific } \theta.
$$

The quantitative estimates (26)-(27) are established from the estimates (62) and (51), respectively. Therefore, the proof of Theorem 2 is concluded.

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Declarations

Conflict of interest The author declares that there is no conflict of interest.
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