Limiting case Hardy inequalities on the sphere

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Abstract
We give sharp limiting case Hardy inequalities on the sphere $S^2$ and show that their optimal constants are unattainable by any $f \in H^1(S^2) \setminus \{0\}$. The singularity of the problem is related to the geodesic distance from a point on the sphere.

Keywords: critical Hardy inequality, sharp constant, 2-sphere, Sobolev spaces

1. Introduction

The classical Hardy inequality

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx \quad (1)$$

is valid in dimensions $n \geq 3$ for all functions $u \in H^1(\mathbb{R}^n)$ (1). It obviously fails on $\mathbb{R}^2$ as the right hand side of (1) no longer makes sense. In order to obtain a version of (1) in the critical case $n = 2$ on bounded domains, a logarithmic weight can be introduced to tame the singularity. In [3, 4, 5, 6, 7, 8, 9], for instance, inequalities of the type

$$\int_{B} |\nabla u|^n dx \geq C_n(\Omega) \int_{B} \frac{|u|^n}{|x|^n \left(\log \frac{1}{|x|}\right)^{\gamma}} dx$$

were analysed for $u \in W^{1,n}_0(B)$ where $B$ is the unit ball in $\mathbb{R}^n$.

Let $n \geq 3$ and $S^n$ be the unit sphere equipped with its Lebesgue surface measure $\sigma_n$ in $\mathbb{R}^{n+1}$. Denote by $d(.,p) : S^n \to [0, \pi]$ the geodesic distance
from $p \in \mathbb{S}^n$, and by $\nabla_{S^2}$ the gradient on $\mathbb{S}^n$. Recently, Xiao [10] proved that if $f \in C^\infty(\mathbb{S}^2)$ then

$$
\bar{c}_n \int_{\mathbb{S}^n} f^2 d\sigma_n + \int_{\mathbb{S}^n} |\nabla_{S^2} f|^2 d\sigma_n \geq c_n^2 \int_{\mathbb{S}^n} \left( \frac{f^2}{d(x,p)^2} + \frac{f^2}{(\pi - d(x,p))^2} \right) d\sigma_n \tag{2}
$$

with $\bar{c}_n = \left( \frac{2}{3} + \frac{1}{\pi^2} \right) c_n^2 + c_n$, $c_n = \frac{n-2}{2}$. It was also shown in [10] that the constant $c_n$ in (2) is sharp in the sense that

$$
c_n^2 = \inf_{f \in C^\infty(\mathbb{S}^n) \setminus \{0\}} \frac{D_n(f)}{\int_{\mathbb{S}^n} \frac{f^2}{d(x,p)^2} d\sigma_n} = \inf_{f \in C^\infty(\mathbb{S}^n) \setminus \{0\}} \frac{D_n(f)}{\int_{\mathbb{S}^n} \frac{f^2}{(\pi - d(x,p))^2} d\sigma_n}
$$

where

$$D_n(f) := c_n \int_{\mathbb{S}^n} f^2 d\sigma_n + \int_{\mathbb{S}^n} |\nabla_{S^2} f|^2 d\sigma_n, \quad f \in C^\infty(\mathbb{S}^n).
$$

We prove $L^2$ Hardy inequalities with optimal constants on the sphere $\mathbb{S}^2$ in $\mathbb{R}^3$. This is a critical exponent case as the integral $\int_{\mathbb{S}^2} \theta^{-1+\lambda} d\sigma_2$, where $\theta$ is the polar angle, diverges for $\lambda \leq -1$. We also argue the lack of maximizers for our inequalities. Our approach denies the possibility of an equality in Xiao’s inequality (2) as well.

2. Preliminaries

A point on the sphere $\mathbb{S}^2$ will have the standard spherical coordinate parametrization $(\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$ where $\theta \in [0, \pi]$ refers to the polar angle and $\varphi \in [0, 2\pi]$ is the azimuthal angle. Then the surface measure induced by the Lebesgue measure on $\mathbb{R}^3$ is $d\sigma_2 = \sin \theta d\theta d\varphi$, the gradient and the Laplace-Beltrami operator, respectively, are given by

$$\nabla_{S^2} = \dot{\theta} \frac{\partial}{\partial \theta} + \dot{\varphi} \frac{1}{\sin \theta} \frac{\partial}{\partial \varphi}, \quad \Delta_{S^2} = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}.
$$

The Sobolev space $H^1(\mathbb{S}^2)$ is the completion of $C^\infty(\mathbb{S}^2)$ in the norm

$$\| f \|_{H^1(\mathbb{S}^2)} := \left( \| f \|_{L^2(\mathbb{S}^2)}^2 + \| \nabla f \|_{L^2(\mathbb{S}^2)}^2 \right)^{\frac{1}{2}}.
$$

In order to find the geodesic distance $d(x,p)$ from a point $x \in \mathbb{S}^2$ to a given a point $p \in \mathbb{S}^2$, we rotate the axes, if necessary, to put $p$ on the zenith direction then place the great circle passing through $p$ and $x$ in the azimuthal reference direction so that we have $d(x,p) = \theta$.

For simplicity, we henceforth denote $d\sigma_2$, $\nabla_{S^2}$ and $\Delta_{S^2}$ by $d\sigma$, $\nabla$ and $\Delta$, respectively.
3. Main results

Let \( \phi : [0, \pi] \to [1, \infty] \) be defined by \( \phi(t) := \log(\pi e/t) \), \( \psi : [0, \pi] \to [1 + \log \pi, \infty] \) be such that \( \psi(t) := \phi(\sin t) \), and \( \rho_\phi(t) := t \phi(t) \). Let \( A > 0 \).

Denote by \( S, T_A, \) and \( Q (\cdot; \phi) \) the positive nonlinear functionals on \( H^1(S^2) \) given by

\[
S(f) := \int_{S^2} |\hat{\theta}, \nabla f|^2 d\sigma + \frac{1}{2\pi^2} \int_{S^2} f^2 d\sigma,
\]

\[
T_A(f) := \int_{S^2} |\nabla f|^2 d\sigma_2 + \frac{A}{4} \int_{S^2} f^2 d\sigma_2,
\]

\[
Q(f; \phi) := \frac{1}{4} \int_{S^2} \left( \frac{f^2}{\rho_\phi^2(d(x, p))} + \frac{f^2}{\rho_\phi^2(\pi - d(x, p))} \right) d\sigma_2.
\]

**Theorem 1.** Assume that \( f \in H^1(S^2) \). Then there exists constants \( A, B > 0 \), independent of \( f \), such that

\[ Q(f; \phi) \leq T_A(f), \quad Q(f; \psi) \leq T_B(f). \]  

Both inequalities (3) and (4) are optimal, but an equality is impossible in either one:

**Theorem 2.**

\[ \sup_{f \in H^1(S^2) \setminus \{0\}} \frac{Q(f; \phi)}{T_A(f)} = 1, \quad (5) \]

\[ \sup_{f \in H^1(S^2) \setminus \{0\}} \frac{Q(f; \psi)}{T_B(f)} = 1. \quad (6) \]

**Theorem 3.** There does not exist \( f \in H^1(S^2) \setminus \{0\} \) such that \( Q(f; \phi) = T_A(f) \), or \( Q(f; \psi) = T_B(f) \).

A variant of the abovementioned results follows via a different approach:

**Theorem 4.** Let \( f \in H^1(S^2) \). Then

\[
\frac{1}{4} \int_{S^2} \frac{f^2}{\rho_\phi^2(d(x, p))} d\sigma \leq S(f) + \frac{1}{2\pi} \int_{S^2} \frac{f^2}{\pi - d(x, p)} d\sigma,
\]

\[
\frac{1}{4} \int_{S^2} \frac{f^2}{\rho_\phi^2(\pi - d(x, p))} d\sigma \leq S(f) + \frac{1}{2\pi} \int_{S^2} \frac{f^2}{d(x, p)} d\sigma.
\]
Moreover
\[ \sup_{f \in H^1(S^2) \setminus \{0\}} \frac{\frac{1}{4} \int_{S^2} \frac{f^2}{\rho_2^2(d(x,p))} \, d\sigma}{S(f)} + \frac{\frac{1}{2\pi} \int_{S^2} \frac{f^2}{\pi - d(x,p)} \, d\sigma}{S(f)} = \sup_{f \in H^1(S^2) \setminus \{0\}} \frac{\frac{1}{4} \int_{S^2} \frac{f^2}{\rho_2^2(\pi - d(x,p))} \, d\sigma}{S(f)} + \frac{\frac{1}{2\pi} \int_{S^2} \frac{f^2}{\pi - d(x,p)} \, d\sigma}{S(f)} = 1, \]
and the suprema in (9) are not attained in \( H^1(S^2) \setminus \{0\} \).

4. Proof of Theorem 1

Proof. Let \( f \in C^\infty(S^2) \). Notice that \( \psi > 1 \) and write \( f(\theta,\varphi) = \psi^{\frac{1}{4}} g(\theta,\varphi) \).

We have
\[
|\nabla f|^2 = \psi^{\frac{1}{4}} \nabla g + g \nabla \psi^{\frac{1}{4}}
\]
\[
= \psi|\nabla g|^2 + \left( \psi^{\frac{1}{4}} \nabla g, g \psi^{-\frac{1}{4}} \nabla \psi \right) + \left( \frac{1}{2} \psi^{-\frac{1}{4}} \nabla \psi \right)^2 g^2
\]
\[
= \psi|\nabla g|^2 + \frac{1}{2} \langle \nabla \psi, \nabla g^2 \rangle + \frac{1}{4} \psi^{\frac{1}{4}} |\nabla \psi|^2 g^2. \tag{10}
\]

Integrating both sides of (10) over \( S^2 \) we get
\[
\int_{S^2} |\nabla f|^2 \, d\sigma = \int_{S^2} \left( \psi|\nabla g|^2 + \frac{1}{2} \langle \nabla \psi, \nabla g^2 \rangle + \frac{1}{4} \psi |\nabla \psi|^2 g^2 \right) \, d\sigma
\]
\[
\geq \frac{1}{4} \int_{S^2} \psi |\nabla \psi|^2 g^2 \, d\sigma + \frac{1}{2} \int_{S^2} \langle \nabla \psi, \nabla g^2 \rangle \, d\sigma \tag{11}
\]
\[
= \frac{1}{4} \int_{S^2} \psi |\nabla \psi|^2 g^2 \, d\sigma - \frac{1}{2} \int_{S^2} g^2 \Delta \psi \, d\sigma \tag{12}
\]
by partial integration over the closed manifold \( S^2 \). Calculating, we find
\[
\Delta \psi = \frac{1}{\sin \theta \, \partial \theta} \left( \sin \theta \, \frac{\partial}{\partial \theta} \psi \right) = 1. \tag{13}
\]

Returning \( g \) to \( f/\sqrt{\psi} \) and substituting for \( \Delta \psi \) from (13) into (12), we obtain
\[
\int_{S^2} |\nabla f|^2 \, d\sigma \geq \frac{1}{4} \int_{S^2} \frac{f^2 \cos^2 \theta}{\psi^2 \sin^2 \theta} \, d\sigma - \frac{1}{2} \int_{S^2} \frac{f^2}{\psi} \, d\sigma. \tag{14}
\]
Adding the finite integral \( \frac{1}{4} \int_{S^2} \left( \frac{1}{\theta^2 \phi^2 (\theta)} + \frac{1}{(\pi - \theta)^2 \phi^2 (\pi - \theta)} \right) f^2 d\sigma \) to both sides of (14) transforms it into the inequality

\[
\frac{1}{4} \int_{S^2} \left( \frac{1}{\theta^2 \phi^2 (\theta)} + \frac{1}{(\pi - \theta)^2 \phi^2 (\pi - \theta)} \right) f^2 d\sigma \leq \int_{S^2} |\nabla f|^2 d\sigma + \frac{1}{4} \int_{S^2} F(\theta) f^2 d\sigma,
\]

(15)

where

\[
F(t) := \frac{1}{t^2 \phi^2 (t)} + \frac{1}{(\pi - t)^2 \phi^2 (\pi - t)} - \frac{\cos^2 t}{\sin^2 t \phi^2 (\sin t)} + \frac{1}{\phi (\sin t)}.
\]

Obviously, \( F \) is continuous on \([0, \pi]\) and, as expected from the facts that \( \phi(t) \to +\infty \) when \( t \to 0^+ \), \( \sin t = t + o(t) \) as \( t \to 0 \), it turns out

\[
\lim_{t \to 0^+} F(t) = \lim_{t \to \pi^-} F(t) = \frac{1}{\pi^2}.
\]

Hence, \( F \) can be extended to a uniformly continuous, consequently a bounded, function on \([0, \pi]\). Noting this in (15) implies (3). Direct computation also shows

\[
A = \sup_{[0, \pi]} |F| = F(\frac{\pi}{2}) = \frac{2}{1 + \log \pi} + \frac{8}{(1 + \log 2)^2 \pi^2}.
\]

To prove (4), we add to both sides of (14) the well-defined integral

\[
\frac{1}{4} \int_{S^2} \left( \frac{1}{\theta^2 + \frac{1}{(\pi - \theta)^2}} \right) f^2 \frac{d^2}{\psi^2 (\theta)} d\sigma. \]

We then obtain the following analogue of (15):

\[
\frac{1}{4} \int_{S^2} \left( \frac{1}{\theta^2 + \frac{1}{(\pi - \theta)^2}} \right) f^2 \frac{d^2}{\psi^2 (\theta)} d\sigma \leq \int_{S^2} |\nabla f|^2 d\sigma + \frac{1}{4} \int_{S^2} G(\theta) f^2 d\sigma,
\]

(16)

where

\[
G(t) := \frac{M(t)}{\psi^2 (t)} + \frac{2}{\psi (t)},
\]

\[
M(t) := \frac{t^2}{t^2 + \frac{1}{(\pi - t)^2}} - \frac{\cos^2 t}{\sin^2 t}.
\]

(17)
Once the boundedness of $G$ is ensured, we see that (16) yields the inequality (14). Evidently, $G$ has the same features as $F$. Since

\[ \lim_{\theta \to 0^+} M(\theta) = \lim_{\theta \to \pi^-} M(\theta) = \frac{2}{3} + \frac{1}{\pi^2}, \quad \lim_{\theta \to 0^+} \psi(t) = \lim_{\theta \to \pi^-} \psi(t) = +\infty \] (18)

then $M \in C[0, \pi]$, and $\lim_{t \to 0^+} G(t) = \lim_{t \to \pi^-} G(t) = 0$, which makes $G$ bounded on $[0, \pi]$. Moreover

\[ B = \sup_{[0, \pi]} |G| = G\left(\frac{\pi}{2}\right) = \frac{2}{1 + \log \pi} + \frac{8}{(1 + \log \pi)^2} \frac{1}{\pi^2}. \]

5. Proof of Theorem 2

Proof. First, we would like to define the weak Laplace-Beltrami gradient of a function $f \in L^1(S^2)$. Suppose $f \in C^\infty(S^2)$ and $v(\theta, \varphi) = v_\theta(\theta, \varphi)\hat{\theta} + v_\varphi(\theta, \varphi)\hat{\varphi}$ with $v_\theta, v_\varphi \in C^\infty(S^2)$. Then

\[ \int_{S^2} \frac{\partial f}{\partial \theta} v_\theta d\sigma = \int_{S^2} \nabla f \cdot \hat{\theta} v_\theta d\sigma = -\int_{S^2} f \nabla \cdot (v_\theta \hat{\theta}) d\sigma, \]
\[ \int_{S^2} \frac{1}{\sin \theta} \frac{\partial f}{\partial \varphi} v_\varphi d\sigma = \int_{S^2} \nabla f \cdot \hat{\varphi} v_\varphi d\sigma = -\int_{S^2} f \nabla \cdot (v_\varphi \hat{\varphi}) d\sigma. \]

Adding these identities we get

\[ \int_{S^2} \nabla f \cdot V d\sigma = -\int_{S^2} f \nabla \cdot V d\sigma \] (19)

for any vector field $V \in C^\infty(S^2 \to T(S^2))$ where $T(S^2)$ is the tangent bundle of the smooth manifold $S^2$. Motivated by (19), $f$ is weakly differentiable if there is a vector field $\theta f \in L^1(S^2 \to T(S^2))$ such that

\[ \int_{S^2} \theta f \cdot V d\sigma = -\int_{S^2} f \nabla \cdot V d\sigma, \quad \forall V \in C^\infty(S^2 \to T(S^2)). \] (20)

This, unique up to a set of zero measure, vector field $\theta f$ is the weak surface gradient of $f$. According to ([2], Proposition 3.2., page 15)

\[ H^1(S^2) = W^{1,2}(S^2) := \{ f \in L^2(S^2) : |\theta f| \in L^2(S^2) \}. \]
We start with (5). By Theorem 1, it suffices to prove the existence of a sequence \( \{f_n\}_{n \geq 1} \) in \( H^1(S^2) \) such that

\[
\lim_{n \to \infty} \frac{Q(f_n; \phi)}{T_A(f_n)} = 1. \tag{21}
\]

Consider the functions

\[
f_n(\theta, \varphi) := \phi(\theta)^{\frac{1}{2} - \frac{1}{n}}. \tag{22}
\]

The functions \( f_n \) are independent of \( \varphi \), hence

\[
\frac{Q(f_n; \phi)}{T_A(f_n)} = \frac{\int_0^\pi f_n^2 \sin \theta d\theta + \int_0^\pi \frac{f_n^2 \sin \theta}{(\pi - \theta)^2 \phi^2(\pi - \theta)} d\theta}{4 \int_0^\pi (\frac{\partial f_n}{\partial \theta})^2 \sin \theta d\theta + A \int_0^\pi f_n^2 \sin \theta d\theta} \tag{23}
\]

where the derivative \( \frac{\partial f_n}{\partial \theta} \) is understood in the week sense discussed above. Since \( \phi \in L^1_{\text{loc}}(\mathbb{R}) \) and \( \phi \geq 1 \) on \([0, \pi]\), then

\[
\int_0^\pi f_n^2 \sin \theta d\theta = \int_0^\pi \phi(\theta)^{1 - \frac{2}{n}} \sin \theta d\theta \leq \int_0^\pi \phi(\theta) d\theta \approx 1. \tag{24}
\]

Thus \( f_n \in L^2(S^2) \) for all \( n \geq 1 \). Notice also that \( f_n \) is smooth on \([0, \pi]\) \( \backslash \{0\} \) and its weak derivative

\[
\frac{\partial f_n}{\partial \theta} = \frac{1}{n} - \frac{1}{\theta \phi^{\frac{1}{2} + \frac{\pi}{2}}}. \tag{25}
\]

Therefore

\[
\int_0^\pi \left( \frac{\partial f_n}{\partial \theta} \right)^2 \sin \theta d\theta = \frac{a_n}{4} \int_0^\pi \frac{1}{\theta \phi^{1 + \frac{\pi}{2}}} \sin \theta d\theta, \quad a_n := \left( 1 - \frac{2}{n} \right)^2.
\]

And since

\[
\int_0^\pi \frac{d\theta}{\theta \phi^{1 + \frac{\pi}{2}}} = \frac{n}{2}, \quad \sin \theta \leq \theta, \quad \text{then} \quad \frac{\partial f_n}{\partial \theta} \in L^2(S^2) \quad \text{for all} \quad n \geq 1.
\]

Substituting for \( f_n \) from (22) and for \( \frac{\partial f_n}{\partial \theta} \) from (25) into (23) implies

\[
\frac{Q(f_n; \phi)}{T_A(f_n)} = \frac{\alpha_n + \beta_n}{\alpha_n \alpha_n + \gamma_n} = \frac{1}{\alpha_n} \left( 1 + \frac{\beta_n - \gamma_n/\alpha_n}{\alpha_n + \gamma_n/\alpha_n} \right) \tag{26}
\]
where

\[ \alpha_n := \int_0^\pi \frac{1}{\theta \phi^{1+\frac{2}{\pi}}} \sin \theta \, d\theta, \]
\[ \beta_n := \int_0^\pi \frac{\phi^{1-\frac{2}{\pi}}(\theta) \sin \theta}{(\pi - \theta)^2 \phi^2(\pi - \theta)} \, d\theta, \]
\[ \gamma_n := A \int_0^\pi \phi^{1-\frac{2}{\pi}} \sin \theta \, d\theta. \]

Observe that \( \lim_{n \to +\infty} \alpha_n = 1 \). We shall show that, while \( \lim_{n \to +\infty} \alpha_n = +\infty \), the sequences \( \{\beta_n\}_{n \geq 1} \) and \( \{\gamma_n\}_{n \geq 1} \) are both convergent. Using this in (26) proves (21).

Exploiting the continuity and positivity of \( \sin \theta / \left( \theta^2 \phi^{1+\frac{2}{\pi}} \right) \) on \([\pi/2, \pi]\), then applying the inequality \( \sin \theta / \theta \geq 2 / \pi \) when \( 0 \leq \theta \leq \pi/2 \), we obtain

\[
\alpha_n = \int_0^{\pi/2} \frac{1}{\theta \phi^{1+\frac{2}{\pi}}} \sin \theta \, d\theta + \int_{\pi/2}^\pi \frac{\sin \theta}{\theta^2 \phi^{1+\frac{2}{\pi}}} \, d\theta
\geq \frac{2}{\pi} \int_0^{\pi/2} \frac{1}{\theta \phi^{1+\frac{2}{\pi}}} \, d\theta = \frac{n}{\pi(1 + \log(2))^{\frac{2}{\pi}}}. \tag{27}
\]

This proves the divergence of \( \{\alpha_n\} \). Next, by the dominated convergence theorem and (24) we readily find

\[
\lim_{n \to +\infty} \gamma_n = A \lim_{n \to +\infty} \int_0^\pi \phi^{1-\frac{2}{\pi}}(\theta) \sin \theta \, d\theta = \int_0^\pi \phi(\theta) \sin \theta \, d\theta \lesssim 1.
\]

Finally, since \( \theta \mapsto \sin \theta / \left( (\pi - \theta)^2 \phi^2(\pi - \theta) \right) \in C\([0, \pi/2]\), then using the local integrability of \( \phi \) and the dominated convergence theorem again implies

\[
\lim_{n \to +\infty} \int_0^{\pi/2} \frac{\phi^{1-\frac{2}{\pi}}(\theta) \sin \theta}{(\pi - \theta)^2 \phi^2(\pi - \theta)} \, d\theta = \int_0^{\pi/2} \frac{\phi(\theta) \sin \theta}{(\pi - \theta)^2 \phi^2(\pi - \theta)} \, d\theta \lesssim 1. \tag{28}
\]

Furthermore, since \( \phi \in C\([\pi/2, \pi]\), and \( \frac{\sin \theta}{\pi - \theta} = \frac{\sin (\pi - \theta)}{\pi - \theta} \leq 1 \), on \([\pi/2, \pi]\), then

\[
\int_{\pi/2}^\pi \frac{\phi^{1-\frac{2}{\pi}}(\theta) \sin \theta}{(\pi - \theta)^2 \phi^2(\pi - \theta)} \, d\theta \lesssim \int_{\pi/2}^\pi \frac{d\theta}{(\pi - \theta)^2 \phi^2(\pi - \theta)} \approx 1. \tag{29}
\]
The convergence of \( \{\beta_n\} \) follows from (28) together with (29).

The proof of (6) shares the main idea of (5). The functions \( g_n(\theta, \varphi) := \psi(\theta)^{\frac{1}{2} - \frac{1}{n}} \) are in \( L^2(S^2) \), \( n \geq 1 \), and satisfy \( \lim_{n \to \infty} \frac{Q(g_n; \psi)}{T_B(g_n)} = 1 \). Indeed, we have

\[
Q(g_n; \psi) = \frac{\int_0^\pi g_n^2 \sin \theta \, d\theta + \int_0^\pi \frac{g_n^2 \sin \theta}{(\pi - \theta)^2} \psi^2(\pi - \theta) \, d\theta}{4 \int_0^\pi \left( \frac{\partial g_n}{\partial \theta} \right)^2 \sin \theta \, d\theta + B \int_0^\pi g_n^2 \sin \theta \, d\theta}
= \frac{\bar{\alpha}_n}{a_n \bar{\alpha}_n + \tilde{\beta}_n} = \frac{1}{a_n} \left( 1 - \frac{\tilde{\beta}_n/a_n}{\bar{\alpha}_n + \tilde{\beta}_n/a_n} \right)
\]

where

\[
\bar{\alpha}_n := \int_0^\pi \frac{\sin \theta \, d\theta}{\theta^2 \psi^{1+\frac{n}{2}}} + \int_0^\pi \frac{\sin \theta \, d\theta}{(\pi - \theta)^2 \psi^{1+\frac{n}{2}}} = 2 \int_0^\pi \frac{\sin \theta \, d\theta}{\theta^2 \psi^{1+\frac{n}{2}}},
\]

\[
\tilde{\beta}_n := B \int_0^\pi \psi^{1-\frac{n}{2}} \sin \theta \, d\theta - a_n \int_0^\pi \frac{M(\theta) \sin \theta \, d\theta}{\psi^{1+\frac{n}{2}}}
\]

Similarly to (27), we have

\[
\bar{\alpha}_n \geq 2 \int_0^1 \frac{\sin \theta \, d\theta}{\theta^2 \psi^{1+\frac{n}{2}}} = 2 \int_0^1 \frac{\sin \theta \, d\theta}{\theta^2 \psi^{1+\frac{n}{2}}}
\geq 2 \int_0^1 \frac{\sin \theta \, d\theta}{\theta^2 \psi^{1+\frac{n}{2}}}
\geq \frac{8}{\pi^2} \int_0^1 \frac{1}{\psi^{1+\frac{n}{2}}} \cos \theta \, d\theta = 4 \int_0^1 \frac{1}{\psi^{1+\frac{n}{2}}} \cos \theta \, d\theta
\]

Hence \( \lim_{n \to \infty} \bar{\alpha}_n = \infty \). Recall from (17) and (18) that \( M \in C([0, \pi]) \). Also, since \( \psi \in L^1_{\text{loc}}(\mathbb{R}) \), \( \psi > 1 \) uniformly, then \( \lim_{n \to \infty} \tilde{\beta}_n \) exists by the dominated convergence theorem.

**6. Proof of Theorem 3**

**Proof.** The transition to the inequalities (3) and (4) from their respective stronger versions, (13) and (16), comes from the bounds

\[
\int_{S^2} F(\theta) f^2 \, d\sigma \leq A \int_{S^2} f^2 \, d\sigma, \quad \int_{S^2} G(\theta) f^2 \, d\sigma \leq B \int_{S^2} f^2 \, d\sigma
\]
where the bounded functions \( F \) and \( G \) are both positive and independent of \( f \). Interestingly, as seen in Section 5 the size of \( 0 < A, B < \infty \) played no role in optimising (3) and (4).

Up to the inequality (15) or (16) an equality relation persists except for the only inequality (11). So a sufficient and necessary condition for an equality in (15) or (16) (and a necessary condition for an equality in (3) and (4)) is an equality in (11). But an equality in (11) occurs if and only if

\[
\int_{S^2} |\nabla g|^2 d\sigma = 0. \quad (30)
\]

Recalling that \( g = f/\sqrt{\psi} \), we compute

\[
\psi |\nabla g|^2 = \psi \left| \frac{\nabla f}{\sqrt{\psi}} - \frac{f}{2 \psi^2} \frac{\partial \psi}{\partial \theta} \right|^2
\]

\[
= |\nabla f|^2 - \frac{f}{\psi} \frac{\partial \psi}{\partial \theta} \nabla f \cdot \hat{\theta} + \frac{1}{4 \psi^2} \left( \frac{\partial \psi}{\partial \theta} \right)^2
\]

\[
= |\nabla f|^2 - \left( \frac{\partial f}{\partial \theta} \right)^2 + \left( \frac{\partial f}{\partial \theta} - \frac{1}{2} \frac{f}{\psi} \frac{\partial \psi}{\partial \theta} \right)^2
\]

\[
= |\nabla f|^2 - \left( \frac{\partial f}{\partial \theta} \right)^2 + \left( \frac{\partial f}{\partial \theta} - \frac{1}{2} \frac{f}{\psi} \frac{\partial \psi}{\partial \theta} \right)^2. \quad (31)
\]

Since

\[
|\nabla f|^2 - \left( \frac{\partial f}{\partial \theta} \right)^2 = \frac{1}{\sin^2 \theta} \left( \frac{\partial f}{\partial \phi} \right)^2 \geq 0,
\]

then, by (31), the equality (30) is equivalent to

\[
\int_{S^2} |\nabla f|^2 - \left( \frac{\partial f}{\partial \theta} \right)^2 d\sigma = \int_{S^2} \left( \frac{\partial f}{\partial \theta} - \frac{1}{2} \frac{f}{\psi} \frac{\partial \psi}{\partial \theta} \right)^2 d\sigma = 0. \quad (32)
\]

The equalities (32) are, in their turn, equivalent to

\[
\frac{1}{\sin \theta} \left| \frac{\partial f}{\partial \phi} \right| = \left| \frac{\partial f}{\partial \theta} - \frac{1}{2} \frac{f}{\psi} \frac{\partial \psi}{\partial \theta} \right| = 0. \quad (33)
\]

Suppose that \( f \) is not the zero function. Then (33) are possible if and only if

\[
f = f(\theta), \quad \frac{df}{f} = \frac{d\psi}{2 \psi}.
\]
That is $f = c\sqrt{\psi}$, $c$ is a constant. But such $f \notin H^1(S^2)$ because

$$\int_{S^2} |\nabla f|^2 d\sigma = 2\pi \int_0^\pi \left( \frac{\partial f}{\partial \theta} \right)^2 d\theta \gtrsim \int_0^1 \frac{\cos^2 \theta}{\sin \theta} \frac{1}{\psi} d\theta$$

$$\gtrsim \int_0^1 \frac{d\theta}{\sin \theta \phi(\sin \theta)} \approx \int_0^1 \frac{d\theta}{\theta \phi(\theta)} = +\infty.$$ 

7. Proof of Theorem 4

Proof. Write

$$\frac{1}{\theta} \frac{1}{\phi^2(\theta)} = \nabla \left( \frac{1}{\phi(\theta)} \right) \cdot \hat{\theta}.$$  

Assume that $f$ is smooth. Then integrating by parts w.r.t. the surface measure $\sigma$ we get

$$\int_{S^2} \frac{f^2}{\theta^2 \phi^2(\theta)} d\sigma = \int_{S^2} \nabla \left( \frac{1}{\phi(\theta)} \right) \cdot \frac{f^2}{\theta} d\sigma$$

$$= - \int_{S^2} \frac{1}{\phi(\theta)} \nabla \left( \frac{f^2}{\theta} \right) d\sigma$$

$$= - 2 \int_{S^2} \frac{f}{\theta \phi(\theta)} \nabla \cdot \frac{f}{\phi(\theta)} d\sigma + \int_{S^2} \frac{f^2}{\theta^2 \phi^2(\theta)} d\sigma +$$

$$- \int_{S^2} \frac{f^2}{\theta^2 \phi^2(\theta)} \cos \theta \sin \theta d\sigma. \quad (34)$$

Observe here that each of the last two integrals on the right hand side of (34) can diverge. They suffer nonintegrable singularities at $\theta = 0$. The reality is, put together, their sum

$$I := \int_{S^2} \frac{f^2}{\theta^2 \phi(\theta)} d\sigma - \int_{S^2} \frac{f^2}{\theta \phi(\theta)} \frac{\cos \theta}{\sin \theta} d\sigma = \int_{S^2} \frac{1}{\theta \phi(\theta)} \left( \frac{1}{\theta} - \frac{\cos \theta}{\sin \theta} \right) f^2 d\sigma \quad (35)$$

is convergent. In fact

$$\lim_{\theta \to 0^+} \frac{1}{\theta \phi(\theta)} \left( \frac{1}{\theta} - \frac{\cos \theta}{\sin \theta} \right) = 0.$$
Also, \( \theta \mapsto 1/ (\theta^2 \phi (\theta)) \) is continuous on a neighborhood of \( \theta = \pi \). Furthermore, if we fix \( \delta > 0 \) and let \( D := \{ x(\theta, \varphi) \in S^2 : 0 \leq \theta < \delta \} \), then the integral \( \int_{S^2 \setminus D} f^2 \cos \theta \sin \theta \, d\sigma \) does exist. Unfortunately, we can not control the integral \( I \) by \( \int_{S^2} f^2 \, d\sigma \), up to a constant factor. The reason is

\[
\lim_{\theta \to \pi^-} \frac{1}{\theta \phi (\theta)} \cos \theta \sin \theta = \infty.
\]

But since

\[
\lim_{\theta \to \pi^-} \left( \frac{1}{\theta \phi (\theta)} \cos \theta \sin \theta + \frac{1}{\pi} \frac{1}{\pi - \theta} \right) = 0
\]

then, we may introduce the convergent integral \( J := \frac{1}{\pi} \int_{S^2} \frac{f^2}{\pi - \theta} \, d\sigma \) to the integral \( I \) to get

\[
I = I - J + J = \int_{S^2} K(\theta) f^2 \, d\sigma + J
\]

where

\[
K(\theta) := \frac{1}{\theta \phi (\theta)} \left( \frac{1}{\theta} - \frac{\cos \theta}{\sin \theta} \right) - \frac{1}{\pi} \frac{1}{\pi - \theta}.
\]

By the continuity of \( K \) on \( [0, \pi] \) and since

\[
\lim_{\theta \to 0^+} K(\theta) = - \lim_{\theta \to \pi^-} K(\theta) = - \frac{1}{\pi^2}
\]

then \( K \) is bounded on \([0, \pi]\). Actually, \( K \) is monotonically increasing. Thus

\[
\sup_{[0, \pi]} |K| = \frac{1}{\pi^2}.
\]  

(37)

Using (37) in (36) we deduce that

\[
I \leq \frac{1}{\pi^2} \int_{S^2} f^2 \, d\sigma + J.
\]  

(38)

Returning with (38) to the inequality (34) in the light of (35) we obtain

\[
\int_{S^2} \frac{f^2}{\theta^2 \phi^2 (\theta)} \, d\sigma \leq -2 \int_{S^2} f \nabla f. \hat{\theta} \, d\sigma + \frac{1}{\pi^2} \int_{S^2} f^2 \, d\sigma + \frac{1}{\pi} \int_{S^2} \frac{f^2}{\pi - \theta} \, d\sigma.
\]  

(39)
Applying Cauchy’s inequality with an $\epsilon$ we find
\[-2 \int_{S^2} \frac{f \nabla f \hat{\theta}}{\theta \phi(\theta)} d\sigma \leq 2\epsilon \int_{S^2} \frac{f^2}{\theta^2 \phi^2(\theta)} d\sigma + \frac{1}{2\epsilon} \int_{S^2} \hat{\theta} \nabla f \nabla f^2 d\sigma. \tag{40}\]
Therefore, it follows from (39) and (40) that
\[2\epsilon(1 - 2\epsilon) \int_{S^2} f^2 \theta \phi^2(\theta) d\sigma \leq \int_{S^2} \hat{\theta} \nabla f \nabla f^2 d\sigma + \frac{2\epsilon}{\pi^2} \int_{S^2} f^2 d\sigma + \frac{2\epsilon}{\pi} \int_{S^2} \frac{f^2}{\pi - \theta} d\sigma, \quad 0 < \epsilon < \frac{1}{2}. \tag{41}\]
The choice $\epsilon = 1/4$ maximizes the factor $2\epsilon(1 - 2\epsilon)$ and, consequently, the left hand side of (41). This proves (7). The inequality (8) can be obtained analogously.

In the fashion of the proof of Theorem 2, the sequence $f_n = \phi^{1 - \frac{1}{n}}$ clearly satisfies
\[\lim_{n \to \infty} \frac{\frac{1}{4} \int_0^\pi f_n^2 \rho_\phi^2(\theta) \sin \theta d\theta}{\int_0^\pi \rho_\phi^2(\theta) \sin \theta d\theta} = \lim_{n \to \infty} \frac{\frac{1}{4} \int_0^\pi f_n^2 \rho_\phi^2(\theta - \pi) \sin \theta d\theta}{\int_0^\pi \rho_\phi^2(\theta - \pi) \sin \theta d\theta} = 1\]
where
\[U(f) = \int_0^\pi \left( \frac{\partial f}{\partial \theta} \right)^2 \sin \theta d\theta + \frac{1}{2\pi^2} \int_0^\pi f^2 \sin \theta d\theta.\]
One only needs to inspect the convergence of $\int_0^\pi \left( \phi^{1 - \frac{2}{n}} \sin \theta / (\pi - \theta) \right) d\theta$ as $n \to \infty$. This is obvious from the bound $\sin \theta \leq \min\{\theta, \pi - \theta\}$ on $[0, \pi]$ and the fact $\phi \in L^1([0, \pi])$.

Finally, careful review of the proof of (7) above reveals that a necessary condition for a function $f \in H^1(S^2) \setminus \{0\}$ to achieve an equality in (7) is that it yields an equality in (10). This is equivalent to
\[\nabla f \hat{\theta} = -\frac{1}{2\theta \phi(\theta)} f. \tag{42}\]
Suppose (42) was true. Then by (34) and (35) we must have
\[\int_{S^2} h(\theta) f^2 \theta \phi(\theta) d\sigma = 0 \tag{43}\]
where
\[ h(\theta) := \frac{1}{\theta} - \frac{\cos \theta}{\sin \theta}. \]

On the other hand
\[ \lim_{\theta \to 0^+} h(\theta) = 0, \quad h'(\theta) = \frac{\theta^2 - \sin^2 \theta}{\theta^2 \sin^2 \theta} > 0, \quad 0 < \theta < \pi. \]

This shows \( h \) is strictly positive on \( [0, \pi] \) and since \( \theta \phi(\theta) \geq 0 \) then (43) is a contradiction.

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