WIMAN-VALIRON THEORY FOR A POLYNOMIAL SERIES
BASED ON THE WILSON OPERATOR

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Abstract. We establish a Wiman-Valiron theory for a polynomial series based
on the Wilson operator $D_W$. For an entire function $f$ of order smaller than
$\frac{1}{3}$, this theory includes (i) an estimate which shows that $f$ behaves locally
like a polynomial consisting of the terms near the maximal term in its Wilson
series expansion, and (ii) an estimate of $D^n_W f$ compared to $f$. We then apply
this theory in studying the growth of entire solutions to difference equations
involving the Wilson operator.

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1. Introduction

In this paper, we will first show that every entire function $f$ of small order admits
a Wilson series expansion

$$f(x) = \sum_{k=0}^{\infty} a_k \tau_k(x; 0).$$

The main purpose of this paper is then to show that every entire function $f$ of small
order is mostly contributed by just a few terms in this Wilson series expansion which
are near the maximal term $a_N \tau_N(x; 0)$. With $D_W$ denoting the Wilson divided-
difference operator and $D^n_W$ denoting that we apply $D_W$ for $n$ times, we will deduce

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from this result that
\[(D_W^n f)(x)\quad\text{is asymptotically similar to}\quad \left(\frac{N}{x}\right)^n f(x),\]
which can be applied in analyzing the growth of entire solutions to difference equations involving the Wilson operator. This is parallel to the classical Wiman-Valiron theory for the Maclaurin series \(f(z) = \sum a_k z^k\) having maximal term \(a_N z^N\), from which one can deduce that
\[f^{(n)}(z)\quad\text{is asymptotically similar to}\quad \left(\frac{N}{z}\right)^n f(z).\]

In the following introduction, we briefly review the history of the study of the Wilson operator as well as the classical Wiman-Valiron theory, and we slightly clarify the terminologies stated in the previous paragraph. Classical hypergeometric orthogonal polynomials have been an active research topic in the recent decades. The Wilson polynomials \(W_n(\cdot; a, b, c, d)\), in particular, are hypergeometric orthogonal polynomials located at the top level of the Askey scheme [23], and are widely recognized as the most general hypergeometric orthogonal polynomials that contain all the known classical hypergeometric orthogonal polynomials as special cases [1].

With four complex parameters \(a, b, c, d\), these polynomials are defined, via a terminating Gauss hypergeometric series, by
\[
W_n(x; a, b, c, d) = 4F_3\left(-n, n + a + b + c + d - 1, a + i\sqrt{x}, a - i\sqrt{x}; 1\right),
\]
where the notation \((a)_k := a(a + 1) \cdots (a + k - 1)\) denotes the rising \(k\)-step factorial of a complex number \(a\). The Wilson operator \(D_W\) was first considered by Wilson [2, p. 34] to study these polynomials. This operator acts on Wilson polynomials in a similar manner as the usual differential operator acts on monomials, except with a shift in the four parameters:
\[
(D_W W_n)(x; a, b, c, d) = C_n W_{n-1}\left(x; a + \frac{1}{2}, b + \frac{1}{2}, c + \frac{1}{2}, d + \frac{1}{2}\right),
\]
where \(C_n = -n(n + a + b + c + d - 1)\).

Research work regarding the Wilson operator so far has mainly focused on its interactions with Wilson polynomials. Therefore the author and Chiang have recently looked into this operator in a broader function-theoretic context, and established some results about its interaction with meromorphic functions, including a Nevanlinna theory and a Picard-type theorem [8]. In this paper, we will turn our focus to entire functions and their series expansions. In 1926, Nørlund has written a memoir on interpolation series [27], a large part of which was devoted on investigating how to expand entire functions into various interpolation series, e.g., Newton series and Stirling series. As opposed to the Taylor series which converges on discs, the Newton series has its regions of convergence to be right half-planes, while that for the Stirling series is effectively the whole complex plane. Stirling series converge faster than Newton series, and the interpolating polynomials in Stirling series are obtained by slightly modifying the Wilson polynomials. Thus it is natural to develop a function theory for an interpolation series using the Wilson polynomials
\{W_n\} as a basis, instead of the usual basis \{(x - x_0)^n\} of the Taylor series. In practice we simply use the basis \{\tau_n(\cdot; x_0)\}, where

$$\tau_n(x; a^2) := \prod_{k=0}^{n-1} [(a + ki)^2 - x]$$

is modified from the key factors \((a + i\sqrt{x})_n(a - i\sqrt{x})_n\) in \(W_n(x; a, b, c, d)\). In fact, following the classical idea as in [27], we will show in this paper that every entire function \(f\) satisfying the growth condition

$$\limsup_{r \to \infty} \frac{\ln M(r, f)}{\sqrt{r}} < 2 \ln 2$$

admits, for each \(a \in \mathbb{C}\), a Wilson series expansion

$$\sum_{k=0}^{\infty} a_k \tau_k(x; a)$$

which converges uniformly to \(f\) on any compact subset of \(\mathbb{C}\), where the coefficients \(a_k\) are generated by the values of \(f\) at a sequence of interpolation points.

In the 1910s, Wiman [33, 34] introduced a theory which relates the maximum modulus \(M(r, f)\) of an entire function \(f\) on a circle \(\partial D(0; r)\) to the maximal term \(\mu(r; f)\) of its Maclaurin series expansion on the circle. Here given an entire function \(f(z) = \sum_{k=0}^{\infty} a_k z^k\) of finite order, one defines the maximal term and the central index of \(f\) to be respectively the functions \(\mu(\cdot; f) : [0, +\infty) \to [0, +\infty)\) and \(\nu(\cdot; f) : (0, +\infty) \to \mathbb{N}_0\), given by

$$\mu(r; f) := \max_{n \in \mathbb{N}_0} |a_n| r^n$$

and

$$\nu(r; f) := \max\{n \in \mathbb{N}_0 : |a_n| r^n = \mu(r; f)\},$$

so that for every \(r > 0\) one simply has \(\mu(r; f) = |a_N| r^N \) where \(N = \nu(r; f)\). In the following decades, Wiman’s theory was then developed more extensively by Valiron [29, 33, 34], Saxer [28], Clunie [10, 11] and Kövari [24, 25]. It turned out that using this theory one can describe the local behavior of an entire function using its maximal term. The main idea, in Clunie’s form [10], is that given any entire function \(f(z) = \sum_{k=0}^{\infty} a_k z^k\) of finite order, the quantity

$$\left| \sum_{k:|k-N|>\kappa} a_k z^k \right|,$$

which is the modulus of the tail of its Maclaurin series expansion, is small relative to \(\mu(r; f)\) as \(r = |z| \to \infty\) outside a certain small exceptional set, where \(\kappa\) is a small positive integer that depends on the central index \(N\). Moreover, one can even obtain from this theory a local relationship between \(f\) and its derivative \(f'\), given by

$$\frac{z}{N} f'(z) = f(z)(1 + o(1))$$

as \(|z| = r \to \infty\) outside a certain small exceptional set. With different expressions of the \(o(1)\), this holds for all \(z \in \partial D(0; r)\) in Clunie [11], and just for those \(z \in \partial D(0; r)\)
taking values $|f(z)|$ close to the maximum $M(r; f)$ in Valiron [31], Clunie [10] and Saxer [28]. Fenton [13] has extended the theory to entire functions of finite lower order. Apart from the chapter in Valiron’s book [32] which summarized much of his work on this topic, Hayman has written a comprehensive survey [18] on the theory, and Fenton has also written a short summary [14] that has made the theory easy to access. One can also see [22] and [19] for more modern references of this theory.

Recently, the Wiman-Valiron theory has been extended by Ishizaki and Yanagihara [21] to the case of Newton series and the ordinary difference operator, and Chiang and Feng [9] have also given a Wiman-Valiron estimate of successive ordinary differences. Ishizaki and Yanagihara’s result [21] is that for any entire function $f(z) = \sum_{k=0}^{\infty} a_k z(z-1) \cdots (z-k+1)$ of order smaller than $\frac{1}{2}$, the quantity

$$\sum_{k:|k-N|>\kappa} k^n |a_k| r(r+1) \cdots (r+k-1)$$

is small relative to $N^\mu(r; f)$ as $r \to \infty$ outside a certain small exceptional set, where $\mu^*(r; f)$ and $N = \nu^*(r; f)$ are Newton series analogues of the maximal term and the central index of $f$, and $\kappa$ is a small positive integer that depends on $N$. This result enables us to deduce a local behavior of successive ordinary differences of $f$, namely that

$$(\Delta^nf)(z) \text{ is asymptotically similar to } \left(\frac{N}{z}\right)^n f(z)$$

as $r = |z| \to \infty$ outside the same small exceptional set.

Now in this paper we aim to develop an analogous theory for the Wilson series, i.e., to investigate how the local behavior of an entire function is controlled by the terms in its Wilson series expansion and to look into the behavior of successive Wilson differences of entire functions. In particular, the main result (Theorem 3.2) is that for any entire function $f(x) = \sum_{k=0}^{\infty} a_k \tau_k(x; 0)$ of order smaller than $\frac{1}{3}$, the quantity

$$\sum_{k:|k-N|>\kappa} k^n |a_k \tau_k(r; 0)|$$

is small relative to $N^\mu_W(r; f)$ as $r \to \infty$ outside a certain small exceptional set, where $\mu^*_W(r; f)$ and $N = \nu^*_W(r; f)$ are Wilson series analogues of the maximal term and the central index of $f$, and $\kappa$ is again a small positive integer that depends on $N$. This gives a local behavior of successive Wilson differences of $f$ (see Theorem 3.3), which is that

$$(\mathcal{D}^n_W f)(x) \text{ is asymptotically similar to } \left(\frac{N}{x}\right)^n f(x)$$

as $r = |x| \to \infty$ outside the same small exceptional set. Along the development of this theory, we have also obtained some new results. These include a Leibniz rule for the Wilson operator (see Theorem 2.3) and a result analogous to Lindelöf-Pringsheim Theorem (see Theorem 4.4) which relates the order of the maximal term of a Wilson series and its coefficients. These results on the Wilson calculus
may potentially have independent interest in combinatorics or number theory. We remark here that part of the results obtained in this paper is contained in the PhD thesis of the author [7].

This paper is organized as follows. In §2, we will first give the definition and some basic properties of the Wilson operator $D_W$, including its Leibniz rule. We will also prove the Wilson Series Theorem, which gives the aforementioned sufficient growth condition on the functions that admits a Wilson series expansion that converges uniformly on compact sets to the function itself. In the subsequent sections, we will then develop the Wiman-Valiron theory of $D_W$. We will state in §3 our main results which include two theorems. The first one asserts that the local behavior of an entire function is mainly contributed by those terms in its Wilson series expansion that are near the maximal term; and the second one gives an estimate of $D^n_W f$ compared to $f$. Before proving these two theorems in §5, we will establish some properties of the Wilson maximal term and central index in §4. Finally, the main results will be applied in §6 to give estimates on the growth of transcendental entire solutions to a certain type of Wilson difference equations and on the growth of its Wilson maximal term.

In this paper, we adopt the following notations:

(i) $\mathbb{N}$ denotes the set of all natural numbers excluding 0, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

(ii) For every positive real number $r$ and every complex number $a$, $D(a; r)$ denotes the open disk of radius $r$ centered at $a$ in the complex plane.

(iii) A complex function always means a function in one complex variable, and an entire function always means a holomorphic function from $\mathbb{C}$ to $\mathbb{C}$, unless otherwise specified.

(iv) A summation notation of the form $\sum_{k:S_k}$ denotes a sum running over all the $k$’s such that the statement $S_k$ is true.

(v) For any two functions $f, g : [0, \infty) \to \mathbb{R}$, we write

- $g(r) = O(f(r))$ as $r \to \infty$ if and only if there exist $C > 0$ and $M > 0$ such that $|g(r)| \leq C|f(r)|$ whenever $r > M$;
- $g(r) = o(f(r))$ as $r \to \infty$ if and only if for every $C > 0$, there exists $M > 0$ such that $|g(r)| \leq C|f(r)|$ whenever $r > M$;
- $f(r) \sim g(r)$ as $r \to \infty$ if and only if for every $\varepsilon > 0$, there exists $M > 0$ such that $\left| \frac{f(r)}{g(r)} - 1 \right| < \varepsilon$ whenever $r > M$.

2. The Wilson Operator and the Wilson Basis

In this section, we give the definition of the Wilson operator and a few of its properties.

**Definition 2.1.** Let $\sqrt{\cdot}$ be a branch of the complex square-root with the imaginary axis as the branch cut. For each $x \in \mathbb{C}$ we denote

$$x^+ := \left( \sqrt{x + \frac{i}{2}} \right)^2 \quad \text{and} \quad x^- := \left( \sqrt{x - \frac{i}{2}} \right)^2.$$ 

We also denote $x^{+(0)} := x, x^{+m} := (x^{+(m-1)})^+ \text{ and } x^{+(m)} := x^{+(m)}$ for every positive integer $m$. Then we define the **Wilson operator** $D_W$, which acts
on all complex functions, as follows:

\[(2.1) \quad (D_W f)(x) := \frac{f(x^+) - f(x^-)}{x^+ - x^-} = \frac{f((\sqrt{x} + \frac{i}{2})^2) - f((\sqrt{x} - \frac{i}{2})^2)}{2i\sqrt{x}}.\]

We also define the Wilson averaging operator \(A_W\) by

\[(2.2) \quad (A_W f)(x) := \frac{f(x^+) + f(x^-)}{2} = \frac{f((\sqrt{x} + \frac{i}{2})^2) + f((\sqrt{x} - \frac{i}{2})^2)}{2}.\]

According to (2.1) and (2.2), although there are two choices of \(\sqrt{x}\) for each \(x \neq 0\), \(D_W\) and \(A_W\) are independent of the choice of \(\sqrt{x}\) and are thus always well-defined. Moreover, the value of \(D_W f\) at 0 should be defined as

\[(D_W f)(0) := \lim_{x \to 0} (D_W f)(x) = f'\left(-\frac{1}{4}\right),\]

in case \(f\) is differentiable at \(-\frac{1}{4}\). We sometimes write \(z := \sqrt{x}\). We consider branches of square-root with the imaginary axis as the branch cut because of the shift of \(\frac{i}{2}\) in the definition of \(D_W\).

We first look at some algebraic properties of the Wilson operator. First of all, it is apparent that the Wilson operator has the following product and quotient rules.

**Lemma 2.2.** (Wilson product and quotient rules) For every pair of complex functions \(f\) and \(g\), we have

(i) \((D_W(fg))(x) = (A_W f)(x)(D_W g)(x) + (D_W f)(x)(A_W g)(x),\) and

(ii) \((D_W \frac{f}{g})(x) = \frac{(D_W f)(x)(A_W g)(x) - (A_W f)(x)(D_W g)(x)}{g(x^+)g(x^-)}\) whenever \(g \neq 0\).

More generally, we have the following Leibniz rule for the Wilson operator.

**Theorem 2.3.** (Wilson Leibniz rule) For every pair of complex functions \(f\) and \(g\) and every \(n \in \mathbb{N}_0\), we have

\[(2.3) \quad D^n_W(fg) = \sum_{k=0}^{n} C(n, k) \sum_{j=0}^{n-k} \binom{n-k}{j} A_W^{n-k-j} D_W^{k+j} f A_W^{j} D_W^{n-j} g,\]

where

\[(2.4) \quad C(n, k) = \left(-\frac{1}{4}\right)^k \frac{(n - 1 + k)!}{(n - 1 - k)! k!}\]

for every pair of integers \(0 \leq k \leq n\). In (2.4) we adopt the convention that \((-1)^{n} := 1\) and \((-1)^{n+1} := 0\).
We need the following two lemmas in proving Theorem 2.3.

Lemma 2.4.

\[ A_W D_W - D_W A_W = \frac{1}{2} D_W^2. \]

Proof. For every complex function \( f \), we have

\[
((A_W D_W - D_W A_W)f(x)) = \frac{1}{2} \left( \frac{f(x^+) - f(x)}{2iz - 1} + \frac{f(x) - f(x^-)}{2iz + 1} \right)
- \frac{1}{2iz} \left( \frac{f(x^+) + f(x)}{2} - \frac{f(x) + f(x^-)}{2} \right)
= \frac{1}{2} \left( (f(x^+) - f(x)) \left( \frac{1}{2iz - 1} - \frac{1}{2iz + 1} \right) - (f(x) - f(x^-)) \left( \frac{1}{2iz - 1} - \frac{1}{2iz + 1} \right) \right)
= \frac{1}{2} \left( f(x^+) - f(x) \right) \left( \frac{1}{2iz - 1} - \frac{1}{2iz + 1} \right) - f(x) - f(x^-) \left( \frac{1}{2iz - 1} - \frac{1}{2iz + 1} \right)
= \frac{1}{2} (D_W^2 f)(x).
\]

Lemma 2.5. The numbers \( C(n, k) \) as defined in (2.4) satisfy the recurrence

\[ C(n, 0) = 1 \text{ for all } n \in \mathbb{N}_0, \ C(n, n) = 0 \text{ for all } n \in \mathbb{N}, \text{ and} \]

\[ C(n, k) = \sum_{j=0}^{k} \left( -\frac{1}{2} \right)^{k-j} \frac{(n-1-j)!}{(n-1-k)!} C(n-1, j) \]

for every pair of integers \( 0 < k < n \).

Proof. We first prove that the identity

\[ \sum_{j=0}^{k} \frac{1}{2^j} \frac{n-j}{n+j} \binom{n+j}{j} = \frac{1}{2^k} \binom{n+k}{k} \]

holds for every \( n \in \mathbb{N} \) and every \( k \in \mathbb{N}_0 \). To see this, we note that the left-hand side of (2.6) can be written as a telescoping sum:

\[
\sum_{j=0}^{k} \frac{1}{2^j} \frac{n-j}{n+j} \binom{n+j}{j} = \sum_{j=0}^{k} \frac{1}{2^j} \left( 1 - \frac{2j}{n+j} \right) \binom{n+j}{j}
= 1 + \sum_{j=1}^{k} \left[ \frac{1}{2^j} \binom{n+j}{j} - \frac{1}{2^{j-1}} \binom{n+j-1}{j-1} \right]
= \frac{1}{2^k} \binom{n+k}{k}.
\]
Now using (2.6), we have for every pair of integers $0 < k < n$,

$$
\sum_{j=0}^{k} \left( -\frac{1}{2} \right)^{k-j} \frac{(n-1-j)!}{(n-1-k)!} C(n-1, j) \\
= \sum_{j=0}^{k} \left( -\frac{1}{2} \right)^{k-j} \frac{(n-1-j)!}{(n-1-k)!} \left( -\frac{1}{4} \right)^{j} \frac{(n-2+j)!}{(n-2-j)!} \\
= \frac{(-\frac{1}{4})^{k}(n-1)!}{(n-1-k)!} \sum_{j=0}^{k} 2^{k-j} \frac{n-1-j}{n-1+j} \binom{n-1+j}{j} \\
= \frac{(-\frac{1}{4})^{k}(n-1)!}{(n-1-k)!} \binom{n-1+k}{k} = \left( -\frac{1}{4} \right)^{k} \frac{(n-1+k)!}{(n-1-k)!} k = C(n, k)
$$

which is (2.5). □

**Proof of Theorem 2.3** We prove (2.3) by induction on $n$. (2.3) is obviously true for $n = 0$ and $n = 1$. Assuming that it is true for some $n \geq 1$ and noting that Lemma 2.4 implies that

$$
D_{W}A_{W} = \sum_{l=0}^{j} \left( j \atop l \right) \left( -\frac{1}{2} \right)^{l} l!A_{W}^{l-1}D_{W}^{l+1}
$$

for every $j \in \mathbb{N}_{0}$, we also have

$$
\begin{align*}
D_{W}^{n+1}(fg) \\
= D_{W}D_{W}^{n}(fg) \\
= \sum_{k=0}^{n} C(n, k) \sum_{j=0}^{n-k} \binom{n-k}{j} D_{W}(A_{W}^{n-k-j}D_{W}^{j+k}fA_{W}^{j}D_{W}^{n-j}g) \\
= \sum_{k=0}^{n} C(n, k) \sum_{j=0}^{n-k} \binom{n-k}{j} D_{W}A_{W}^{n-k-j}D_{W}^{j+k}fA_{W}^{j}D_{W}^{n-j}g \\
+ \sum_{k=0}^{n} C(n, k) \sum_{j=0}^{n-k} \binom{n-k}{j} A_{W}^{n+1-k-j}D_{W}^{j+k}fD_{W}A_{W}^{j}D_{W}^{n-j}g \\
= \sum_{k=0}^{n} C(n, k) \sum_{j=0}^{n-k} \binom{n-k}{j} \left( \sum_{l=0}^{j} \binom{n-k-j}{l} \left( -\frac{1}{2} \right)^{l} l!A_{W}^{n-k-j-l}D_{W}^{l+1} \right) D_{W}^{j+k}fA_{W}^{j}D_{W}^{n-j}g \\
+ \sum_{k=0}^{n} C(n, k) \sum_{j=0}^{n-k} \binom{n-k}{j} A_{W}^{n+1-k-j}D_{W}^{j+k} \left( \sum_{l=0}^{j} \binom{j}{l} \left( -\frac{1}{2} \right)^{l} l!A_{W}^{j-l}D_{W}^{l+1} \right) D_{W}^{n-j}g \\
= \sum_{k=0}^{n} C(n, k) \sum_{p=1}^{n-k+1} \sum_{l=0}^{n-k-p+1} \binom{n-k-p+1}{l} \binom{n-k-p+1}{l} \left( -\frac{1}{2} \right)^{l} l!A_{W}^{n-k-p-l+1}D_{W}^{p+k+l}fA_{W}^{p}D_{W}^{n-p+1}g \\
+ \sum_{k=0}^{n} C(n, k) \sum_{j=0}^{n-k} \sum_{p=0}^{j} \binom{n-k}{j-p} \binom{j-p}{p} \left( -\frac{1}{2} \right)^{j-p} (j-p)!A_{W}^{n+1-k-j}D_{W}^{j+k}fA_{W}^{p}D_{W}^{n-p+1}g
\end{align*}
$$
\[ = \sum_{k=0}^{n} C(n, k) \sum_{p=1}^{n-k+1} \sum_{j=p}^{n-k+1} \binom{n-k}{j-p} \left( \sum_{p=0}^{n-k-j+1} D_W^{n-k-j+1} f A_W^{n-p} D_W^{n-p+1} \right) \]

\[ = \sum_{k=0}^{n} C(n, k) \sum_{p=0}^{n-k+1} \binom{n-k}{p} \left( \sum_{j=p+1}^{n-k+1} \left( \frac{-1}{2} \right)^{j-p} (j-p)! A_W^{n-k-j+1} D_W^{n-j+1} f A_W^{n-p} D_W^{n-p+1} \right) \]

\[ = \sum_{k=0}^{n} C(n, k) \sum_{p=0}^{n-k+1} \left( \frac{-1}{2} \right)^{j-p} \binom{n-k}{j-p} \left( \sum_{p=0}^{n-k-j+1} D_W^{n-k-j+1} f A_W^{n-p} D_W^{n-p+1} \right) \]

\[ = \sum_{k=0}^{n} C(n, k) \sum_{p=0}^{n-k+1} \left( \frac{-1}{2} \right)^{j-p} \binom{n-k}{j-p} \frac{(n-k)!}{(n-k-j+1)!} \sum_{p=0}^{n-k-j+1} D_W^{n-k-j+1} f A_W^{n-p} D_W^{n-p+1} \]

\[ = \sum_{k=0}^{n} C(n, k) \sum_{p=0}^{n-k+1} \left( \frac{-1}{2} \right)^{j-p} \binom{n-k}{j-p} \frac{(n-k)!}{(n-k-j+1)!} \sum_{p=0}^{n-k-j+1} D_W^{n-k-j+1} f A_W^{n-p} D_W^{n-p+1} \]

\[ = \sum_{m=0}^{n+1} \sum_{k=0}^{m-1} \left( \frac{-1}{2} \right)^{m-k} \binom{n-k}{m-k} \frac{(n-k)!}{(n-k-m+1)!} \sum_{p=0}^{n-k-m+1} D_W^{n-k-m+1} f A_W^{n-p} D_W^{n-p+1} \]

\[ = \sum_{m=0}^{n+1} \sum_{k=0}^{m-1} \left( \frac{-1}{2} \right)^{m-k} \binom{n-k}{m-k} \frac{(n-k)!}{(n-k-m+1)!} \sum_{p=0}^{n-k-m+1} D_W^{n-k-m+1} f A_W^{n-p} D_W^{n-p+1} \]

where the last step follows from Lemma 2.5.

There is a simpler recurrence formula generating the numbers \( C(n, k) \) than (25), which is

\[ C(n, k) = C(n-1, k) - \frac{n+k-2}{2} C(n-1, k-1) \]

for every pair of integers \( 0 < k < n \), in which only three terms in the array are involved. In fact, \( C(n, k) \) is \( (-\frac{1}{2})^k \) times the coefficient of \( x^k \) in the Bessel polynomial \( y_{n-1} \) of degree \( n-1 \) [17]. Table 1 below shows the numbers \( C(n, k) \) for small values of \( n \) and \( k \).

| \( C(n, k) \) | \( k = 0 \) | \( k = 1 \) | \( k = 2 \) | \( k = 3 \) | \( k = 4 \) | \( k = 5 \) |
|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| \( n = 0 \)  | 1           | 0           | 0           | 0           | 0           | 0           |
| \( n = 1 \)  | 1           | 0           | 0           | 0           | 0           | 0           |
| \( n = 2 \)  | 1           | -\frac{1}{2} | 0           | 0           | 0           | 0           |
| \( n = 3 \)  | 1           | -\frac{1}{2} | \frac{1}{4}  | 0           | 0           | 0           |
| \( n = 4 \)  | 1           | -3          | \frac{15}{4} | -\frac{13}{2} | 0           | 0           |
| \( n = 5 \)  | 1           | -5          | \frac{25}{4} | -\frac{65}{2} | 10         | 0           |

Table 1. Values of \( C(n, k) \) for small \( n \) and \( k \)
Now we turn to some analytic properties of the Wilson operator. We can check easily from the definition of the Wilson operator $D_W$ that it sends polynomials to polynomials. In fact, we have the following.

**Proposition 2.6.** Let $f$ be a complex function. Then

(i) if $f$ is entire, then $D_W f$ and $A_W f$ are also entire;

(ii) if $f$ is meromorphic, then $D_W f$ is also meromorphic; and

(iii) if $f$ is rational, then $D_W f$ is also rational.

After introducing the Wilson operator and some of its useful properties, we will look at a series expansion of entire functions in a polynomial basis based on the Wilson operator.

**Definition 2.7.** Let $x_0 \in \mathbb{C}$. Then the Wilson basis $\{\tau_k(x;x_0) : k \in \mathbb{N}_0\}$ is defined as

$$\tau_0(x;x_0) := 1,$$

and

$$\tau_k(x;x_0) := \prod_{j=0}^{k-1} (x_0^{(2j)} - x) = \prod_{j=0}^{k-1} [(z_0 + ji)^2 - x]$$

for $k \in \mathbb{N}$, where $z_0 := \sqrt{x_0}$ with the branch of square root $\arg z_0 \in (-\pi/2, \pi/2]$ if $x_0 \neq 0$. We emphasize that whenever the Wilson basis $\{\tau_k(x;x_0) : k \in \mathbb{N}_0\}$ is mentioned, the symbol $z_0$ automatically takes the aforesaid meaning.

Note that the choice of branch of square root in Definition 2.7 ensures that $2z_0 \not\in \mathbb{N}$, so that $x_0, x_0^+, x_0^{+(2)}, \ldots$ are distinct points.

The Wilson operator interacts with the Wilson basis in a similar way as the ordinary differential operator $\frac{d}{dx}$ does with the basis $\{(x-x_0)^k : k \in \mathbb{N}_0\}$. The following formula is the Wilson counterpart of the ordinary differentiation formula $\frac{d}{dx} x^k = kx^{k-1}$ for every positive integer $k$. Its $q$-analogue was discussed in [2].

**Proposition 2.8.** For every $x_0 \in \mathbb{C}$ and every $k \in \mathbb{N}$, we have

$$(D_W \tau_k)(x;x_0) = -k \tau_{k-1}(x;x_0^+).$$

The following theorem from Gelfond’s book [16, p. 172] implies that the domain of a function defined by a Wilson series is “either all or nothing”.

**Theorem 2.9.** Let $\{a_n\}_{n\in\mathbb{N}_0}$ and $\{x_n\}_{n\in\mathbb{N}_0}$ be sequences of complex numbers such that

$$\sum_{k=0}^{\infty} \frac{1}{|x_k|} < +\infty.$$

If the polynomial series

$$\sum_{k=0}^{\infty} a_k (x-x_0) \cdots (x-x_{k-1})$$

converges at a point $a \in \mathbb{C} \setminus \{x_0, x_1, x_2, \ldots\}$, then it converges uniformly on every compact subset of $\mathbb{C}$. In particular, given $x_0 \in \mathbb{C}$ and a sequence $\{a_n\}_{n\in\mathbb{N}_0}$ of complex numbers, the Wilson series

$$\sum_{k=0}^{\infty} a_k \tau_k(x;x_0)$$
either converges nowhere except at the points $x_0, x_0^+, x_0^{+(4)}, \ldots$ or converges uniformly on every compact subset of $\mathbb{C}$.

We next investigate the Wilson series expansion of an entire function. We will find out the coefficients in the expansion and look for a condition for uniform convergence of such a series on compact subsets of $\mathbb{C}$.

**Theorem 2.10.** (Wilson series expansion) Let $x_0 \in \mathbb{C}$ and let $f$ be an entire function satisfying

\[(2.7) \quad \limsup_{r \to \infty} \frac{\ln M(r, f)}{\sqrt{r}} < 2 \ln 2.\]

Then there exists a unique sequence of complex numbers \(\{a_n\}_{n \in \mathbb{N}_0}\), given by

\[(2.8) \quad a_n = \frac{1}{n!} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \frac{1}{(-2z_0i + j)(-2z_0i + 2j + 1)n-j} f(x_0^{+(2j)}),\]

such that the Wilson series

\[\sum_{k=0}^{\infty} a_k \tau_k(x; x_0)\]

converges uniformly to $f$ on every compact subset of $\mathbb{C}$.

The uniqueness statement in Theorem 2.10 together with the fact that

\[a_n = \frac{(-1)^n}{n!} (D^0_W f)(x_0^{+(n)})\]

imply that a Wilson series can only represent functions that satisfy (2.7). Thus if $f \neq 0$, $f \in \ker D_W$ and $f$ has at least one zero, then

\[\limsup_{r \to \infty} \frac{\ln M(r, f)}{\sqrt{r}} \geq 2 \ln 2,\]

i.e. $f$ is an entire function of order at least $\frac{1}{2}$, and of type at least $2 \ln 2$ in case the order is exactly $\frac{1}{2}$.

The following corollary relates the $n$th Wilson difference of $f$ at a point and the values taken by $f$ at the nearby interpolation points. It was first introduced by Cooper [12].

**Corollary 2.11.** Let $f$ be an entire function satisfying (2.7). Then at each point $x_0 \in \mathbb{C}$, we have

\[(D^0_W f)(x_0^{+(n)}) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{1}{(-2z_0i + j)(-2z_0i + 2j + 1)n-j} f(x_0^{+(2j)})\]

for every non-negative integer $n$. Replacing $z_0$ by $(z_0 - \frac{n}{2})$, we also have

\[(D^n_W f)(x_0) = (-1)^n \sum_{j=0}^{n} \binom{n}{j} \frac{1}{(-2z_0i - n + j)(2z_0i - j)n-j} f(x_0^{+(2j-n)})\]

for every non-negative integer $n$. 
The following proposition is an example of applying Corollary 2.11.

**Proposition 2.12.** For every pair of non-negative integers \( k \) and \( n \), we let

\[
T(k, n) := \frac{(-1)^{n+k}}{n!} D_n^k x^k |_{x=0+(n)}.
\]

Then \( T(0, 0) = 1, T(0, n) = 0 \) for all \( n \in \mathbb{N} \), \( T(k, 0) = 0 \) for all \( k \in \mathbb{N} \), and

\[
T(k, n) = \sum_{j=1}^{n} \frac{2(-1)^{n+j} j^{2k}}{(n-j)! (n+j)!}
\]

for every pair of positive integers \( k \) and \( n \). In particular, we have the followings:

(i) \( T(k, n) \) satisfy the recurrence

\[
T(k, n) = T(k-1, n-1) + n^2 T(k-1, n)
\]

for every pair of positive integers \( k \) and \( n \).

(ii) For each \( n \in \mathbb{N}_0 \),

\[
T(k, n) \sim \frac{2}{(2n)!} n^{2k}
\]

as \( k \to \infty \).

The numbers \( T(k, n) \) in Proposition 2.12 are called the Carlitz-Riordan central factorial numbers or the Chebyshev-Stirling numbers of the second kind. In fact \( T(k, n) \) is the number of partitions of the set \( \{1, 1', 2, 2', ..., k, k'\} \) into \( n \) disjoint nonempty subsets \( V_1, \ldots, V_n \) such that, for each \( 1 \leq j \leq n \), if \( i \) is the smallest integer such that either \( i \in V_j \) or \( i' \in V_j \) then \( \{i, i'\} \subseteq V_j \). Therefore \( T(k, n) \) can be regarded as a “two-colored” version of the Stirling numbers of the second kind. These numbers are first investigated by Carlitz and Riordan in [4], and their properties and asymptotics are studied in [5], [15], [3], etc. Matsumoto and Novak [26] gave another combinatorial interpretation for these numbers, which is the number of primitive factorizations of a full cycle.

After studying the Askey-Wilson series expansion of entire functions, we will develop a Wiman-Valiron theory for this series expansion in the following sections.

### 3. Main Results

We have seen in Theorem 2.10 that an entire function of order smaller than \( \frac{1}{2} \) has a Wilson series expansion at 0, which converges uniformly to itself on compact subsets of \( \mathbb{C} \). Such a Wilson series expansion must in particular converge at each positive real number \( r \), so we are able to make the following definition.

**Definition 3.1.** Let \( f \) be an entire function of order smaller than \( \frac{1}{2} \) with Wilson series expansion \( f(x) = \sum_{k=0}^{\infty} a_k \tau_k(x; 0) \). The **Wilson maximal term** and **Wilson central index** of \( f \) are respectively the functions \( \mu_W(\cdot; f) : [0, +\infty) \to [0, +\infty) \) and \( \nu_W(\cdot; f) : (0, +\infty) \to \mathbb{N}_0 \) defined by

\[
\mu_W(r; f) := \max_{n \in \mathbb{N}_0} \max_{x \in \partial D(0,r)} |a_n \tau_n(x; 0)|
\]

\[
= \max_{n \in \mathbb{N}_0} |a_n| r(r+1)^2 \cdots (r+(n-1)^2)
\]

and

\[
\nu_W(r; f) := \max \{ n \in \mathbb{N}_0 : |a_n| r(r+1)^2 \cdots (r+(n-1)^2) = \mu_W(r; f) \}.
\]
The following is the main theorem of this paper, which is about an entire function of order smaller than $\frac{1}{3}$. In the case $h = 0$, it says that outside a small exceptional set, the terms in the Wilson series expansion of such an entire function that are far away from the maximal term, are small. In other words, the local behavior of such an entire function is mainly contributed by those terms in its Wilson series expansion that are near the maximal term.

**Theorem 3.2.** Let $f(x) = \sum_{k=0}^{\infty} a_k \tau_k(x; 0)$ be a transcendental entire function of order $\sigma < \frac{1}{3}$, $\gamma \in (3, \frac{1}{\sigma})$, and $\delta > 0$. Then there exists a set $E \subset [1, \infty)$ of finite logarithmic measure such that for every $h \in \mathbb{R}$, $\beta > 0$ and $\omega \in (0, \beta)$, we have

$$k^h |a_k \tau_k(r; 0)| = o(\mu_W(r; f) N^h b(N)^{\frac{\beta}{\omega}})$$

as $r \to \infty$ and $r \in [0, \infty) \setminus E$, where $N = \nu_W(r; f)$, $b(N) := \frac{1}{N \ln N (\ln \ln N)^{1 + \rho}}$ and $\kappa = \left[ \frac{\beta}{\omega} \ln \frac{1}{\rho} \right]$.

Applying Theorem 3.2, we obtain the following asymptotic behavior of successive Wilson differences of a transcendental entire function of order smaller than $\frac{1}{3}$.

**Theorem 3.3.** Let $f(x) = \sum_{k=0}^{\infty} a_k \tau_k(x; 0)$ be a transcendental entire function of order $\sigma < \frac{1}{3}$, $\gamma \in (3, \frac{1}{\sigma})$, and $\delta > 0$. Then there exists a set $E \subset [1, \infty)$ of finite logarithmic measure such that for every $n \in \mathbb{N}$, we have

$$\left( \frac{x}{N} \right)^n (D_W^n f)(x) = f(x) + O\left( \frac{N}{x} \right) M(r; f)$$

as $r \to \infty$ and $r \in [0, \infty) \setminus E$, where in the above formula $r = |x|$, $N = \nu_W(r; f)$ and $\kappa = \left[ \sqrt{N (\ln N)^2 (\ln \ln N)^{1 + \rho}} \right]$.

4. **Properties of the Wilson maximal term and central index**

We start by stating three lemmas which are about some useful properties of the functions $\mu_W(\cdot; f)$ and $\nu_W(\cdot; f)$. All the results in this section hold for entire functions of order smaller than $\frac{1}{3}$.

**Lemma 4.1.** Let $f$ be a non-constant entire function of order smaller than $\frac{1}{3}$. Then

(i) $\mu_W(\cdot; f)$ is continuous on $(0, +\infty)$ and strictly increasing on $[R, +\infty)$ for some $R > 0$. Also $\lim_{r \to \infty} \mu_W(r; f) = +\infty$.

(ii) $\nu_W(\cdot; f)$ is a right-continuous non-decreasing piecewise-constant function. If $f$ is transcendental, then $\lim_{r \to \infty} \nu_W(r; f) = +\infty$.

**Proof.** We write $f(x) = \sum_{k=0}^{\infty} a_k \tau_k(x; 0)$.

(i) We first prove that $\mu_W(\cdot; f)$ is continuous on $(0, +\infty)$. Since $f$ is non-constant, there exists $n \in \mathbb{N}$ such that $a_n \neq 0$. Now let $r_0 \in (0, +\infty)$. Since the Wilson series of $f$ converges absolutely at $2r_0$, we have

$$\lim_{k \to \infty} \left| a_k \right| (2r_0)(2r_0 + 1^2) \cdots (2r_0 + (k - 1)^2) = 0,$$
and so there exists $K > n$ such that
\[ |a_k|(2r_0)(2r_0 + 1^2) \cdots (2r_0 + (k-1)^2) < |a_n|r_0(r_0 + 1^2) \cdots (r_0 + (n-1)^2) \]

for every $k > K$. Then for every $r \in [r_0, 2r_0]$ we have
\[ |a_k|r(r + 1^2) \cdots (r + (k-1)^2) \leq |a_k|(2r_0)(2r_0 + 1^2) \cdots (2r_0 + (k-1)^2) \]
\[ < |a_n|r_0(r_0 + 1^2) \cdots (r_0 + (n-1)^2) \]
\[ \leq |a_n|r(r + 1^2) \cdots (r + (n-1)^2) \]

for every $k > K$, so
\[ \mu_W(r; f) = \max_{k \in \{0, 1, \ldots, K\}} |a_k|r(r + 1^2) \cdots (r + (k-1)^2). \]

Being the maximum of finitely many polynomials on $[r_0, 2r_0]$, $\mu_W(\cdot; f)$ is continuous on $[r_0, 2r_0]$. Since $r_0 \in (0, +\infty)$ was arbitrary, $\mu_W(\cdot; f)$ is continuous on $(0, +\infty)$.

Next, since $f$ is non-constant, there exists $R > 0$ such that $\nu_W(r; f) \geq 1$ for every $r > R$. Now for every $r_1 > r_2 \geq R$, denoting $N := \nu_W(r_2; f)$, we have $|a_N| \neq 0$ and
\[ \mu_W(r_2; f) = |a_N|r_2(r_2 + 1^2) \cdots (r_2 + (N-1)^2) \]
\[ < |a_N|r_1(r_1 + 1^2) \cdots (r_1 + (N-1)^2) \]
\[ \leq \mu_W(r_1; f), \]

so $\mu_W(\cdot; f)$ is strictly increasing on $[R, +\infty)$.

Finally, since $f$ is non-constant, there exists $n \in \mathbb{N}$ such that $a_n \neq 0$. Thus we have
\[ \mu_W(r; f) \geq |a_n|r(r + 1^2) \cdots (r + (n-1)^2) \]
for every $r \in [0, +\infty)$, and so $\lim_{r \to +\infty} \mu_W(r; f) = +\infty$.

(ii) We prove that $\nu_W(\cdot; f)$ is non-decreasing. Let $r_1 > r_2 > 0$ and denote $N_1 := \nu_W(r_1; f)$ and $N_2 := \nu_W(r_2; f)$. If on the contrary $N_1 < N_2$, then
\[ \frac{|a_{N_2}|r_2(r_2 + 1^2) \cdots (r_2 + (N_2-1)^2)}{|a_{N_1}|r_2(r_2 + 1^2) \cdots (r_2 + (N_1-1)^2)} = \frac{|a_{N_2}|}{|a_{N_1}|} \frac{(r_2 + N_2^2) \cdots (r_2 + (N_2-1)^2)}{N_1^2} \]
\[ < \frac{|a_{N_2}|}{|a_{N_1}|} \frac{(r_1 + N_2^2) \cdots (r_1 + (N_2-1)^2)}{N_1^2} \]
\[ = \frac{|a_{N_2}|r_1(r_1 + 1^2) \cdots (r_1 + (N_2-1)^2)}{|a_{N_1}|r_1(r_1 + 1^2) \cdots (r_1 + (N_1-1)^2)} \]
\[ \leq 1, \]

which is a contradiction. So $\nu_W(\cdot; f)$ is non-decreasing.

Next we prove that $\nu_W(\cdot; f)$ is right-continuous and piecewise constant. Let $r_0 > 0$ and denote $N_0 := \nu_W(r_0; f)$. Similar to (i), there exists $K > N_0$ such that
\[ (4.1) \quad |a_k|(2r_0)(2r_0 + 1^2) \cdots (2r_0 + (k-1)^2) < \mu_W(r_0; f) \]

for every $k > K$. Also, by the definition of $N_0$ we must have
\[ (4.2) \quad |a_k|r_0(r_0 + 1^2) \cdots (r_0 + (k-1)^2) < \mu_W(r_0; f) \]
for every \( k > N_0 \). Since each polynomial \(|a_k| r(r+1^2) \cdots (r+(k-1)^2)\) is continuous, (4.2) implies that there exists \( \delta \in (0, r_0) \) such that for every \( r \in [r_0, r_0 + \delta] \) and for those (finitely many) \( k \in \{N_0 + 1, \ldots, K\} \) we have

\[
|a_k| r(r+1^2) \cdots (r+(k-1)^2) < \mu_W(r_0; f).
\]

(4.3)

On the other hand, for every \( r \in [r_0, r_0 + \delta] \subseteq [r_0, 2r_0] \) and every \( k > K \), (4.1) gives

\[
|a_k| r(r+1^2) \cdots (r+(k-1)^2) \leq |a_k|(2r_0)(2r_0 + 1^2) \cdots (2r_0 + (k-1)^2) < \mu_W(r_0; f).
\]

(4.4)

So combining (4.3) and (4.4), for every \( r \in [r_0, r_0 + \delta] \) and every \( k > N_0 \) we have

\[
|a_k| r(r+1^2) \cdots (r+(k-1)^2) < \mu_W(r_0; f) = |a_{N_0}| r_0(r_0+1^2) \cdots (r_0 + (N_0-1)^2),
\]

which implies that \( \nu_W(r; f) \leq N_0 \) for every \( r \in [r_0, r_0 + \delta] \). But since \( \nu_W(r; f) \) is non-decreasing, we must have \( \nu_W(r; f) = N_0 \) for every \( r \in [r_0, r_0 + \delta] \). Since \( r_0 \in (0, +\infty) \) was arbitrary, it follows that \( \nu_W(\cdot; f) \) is right-continuous and piecewise constant.

Finally we let \( a := \max_{n \in N_0} |a_n| \). Then for every \( n \in N_0 \) and every \( r > 0 \) we have

\[
|a_n| r(r+1^2) \cdots (r+(n-1)^2) \leq a r(r+1^2) \cdots (r+(\nu_W(r; f) - 1)^2).
\]

So for all those \( n \in N_0 \) with \( a_n \neq 0 \) we have

\[
n \leq \liminf_{r \to +\infty} \nu_W(r; f).
\]

(4.5)

If \( f \) is transcendental, then there are infinitely many \( n \in N_0 \) with \( a_n \neq 0 \). (4.5) now holds for infinitely many \( n \), so we have \( \lim_{r \to +\infty} \nu_W(r; f) = +\infty \).

\( \Box \)

**Lemma 4.2.** Let \( f(x) = \sum_{k=0}^{\infty} a_k \tau_k(x; 0) \) be an entire function of order smaller than \( \frac{1}{2} \), and let \( \gamma > 3 \). Then for each \( n \in N_0 \), there exists \( K_n > 1 \) such that

\[
|a_n \tau_n(r; 0)| \leq K_n M(r, f)
\]

for every \( r \geq \max\{4n^2, n^\gamma\} \), and the sequence \( \{K_n\}_{n \in N_0} \) decreases to 1.

**Proof.** We let \( K_0 = K_1 = 3 \) and \( K_n := \left(1 + \frac{1}{n^{\gamma/2}}\right)^n \left(1 - \frac{1}{n^{\gamma/2}}\right)^{-n} \) for \( n \geq 2 \). Then the sequence \( \{K_n\}_{n \in N_0} \) decreases to 1. Now for each \( n \in N_0 \) and each \( r \geq \max\{4n^2, n^\gamma\} \), applying Cauchy’s Residue Theorem as in Theorem 2.10 we have

\[
a_n = -\frac{1}{2\pi i} \int_{\partial D(0; r)} \frac{f(\xi)}{\tau_{n+1}(\xi; 0)} d\xi,
\]

where \( \partial D(0; r) \) is the boundary of the disk of radius \( r \) centered at the origin.
so
\[ |a_n \tau_n(r; 0)| \leq \frac{1}{2\pi r} \frac{2\pi r M(r, f)}{r(r - 1)^2(r - 2^2) \cdots (r - n^2)} r(r + 1^2) \cdots (r + (n - 1)^2) \]
\[ = \frac{(1 + \frac{\sigma^2}{r})(1 + \frac{1^2}{r}) \cdots (1 + \frac{(n - 1)^2}{r})}{(1 - \frac{1^2}{r})(1 - \frac{2^2}{r}) \cdots (1 - \frac{n^2}{r})} M(r; f) \]
\[ \leq \left(1 + \frac{n^2}{r}\right)^n \left(1 - \frac{n^2}{r}\right)^{-n} M(r; f) \]
\[ \leq K_n M(r, f). \]

\[ \square \]

**Lemma 4.3.** Let \( f \) be an entire function of order \( \sigma < \frac{1}{2} \). Then

(i) \( \sigma_{\mu W(\cdot; f)} = \limsup_{r \to \infty} \frac{\ln \nu W(r; f)}{\ln r} \leq \sigma. \)

(ii) In particular, for every \( \gamma < \frac{1}{\sigma_{\mu W(\cdot; f)}} \), we have

\[ \nu W(r; f) \gamma \leq r \]

for every sufficiently large \( r \in (0, +\infty) \).

The inequality in Lemma 4.3 (i) can be improved to an equality for an entire function of order \( \sigma < \frac{1}{3} \)(see Remark 5.5), but for \( \sigma \in \left[\frac{1}{3}, \frac{1}{2}\right) \) we are currently unsure about whether this can be improved.

**Proof.** Let the Maclaurin series expansion of \( f \) be \( f(x) = \sum_{k=0}^{\infty} b_k x^k \). Since \( \sigma < \frac{1}{2} \), Theorem 2.10 implies that there exists a sequence \( \{a_n\}_{n \in \mathbb{N}_0} \) of complex numbers such that

\[ f(x) = \sum_{k=0}^{\infty} a_k \tau_k(x; 0), \]

and it follows that

\[ a_n = \frac{(-1)^n}{n!} \sum_{k=n}^{\infty} b_k D^n_{\nu W} x^k \bigg|_{x=0^{+}(n)} = \sum_{k=n}^{\infty} (-1)^k b_k T(k, n) \]

where the notation \( T(k, n) \) is as in Proposition 2.12. Next let \( f^* \) be the function defined by the Wilson series

\[ f^*(x) := \sum_{k=0}^{\infty} (-1)^k |a_k| \tau_k(x; 0) \]

and let \( \{b^*_n\}_{n \in \mathbb{N}_0} \) be the sequence of real numbers such that \( f^*(x) = \sum_{k=0}^{\infty} b^*_k x^k \). Note that for each pair of positive integers \( k \geq n \) we have

\[ \frac{1}{n!} \left. \frac{d^n}{dx^n} (-1)^k \tau_k(x; 0) \right|_{x=0} = \text{coefficient of } x^{k-n} \text{ in } x(x + 1^2) \cdots (x + (k - 1)^2) \]

\[ \leq \left(\frac{(k - 1)!}{(n - 1)!}\right)^2 \frac{k!}{n!(k-n)!}. \]
Now we prove (i), and we divide the proof into the following four steps. The proof of (ii) is essentially step (3).

(1) We first show that $f^*$ is entire and $\sigma_{f^*} \leq \sigma$. Let $\gamma \in (2, \frac{1}{\sigma})$ be arbitrary. By Lindelöf-Pringsheim theorem we have

$$\sigma = \limsup_{k \to \infty} \frac{k \ln k}{-\ln |b_k|},$$

so $|b_k| < k^{-\gamma}$ for every sufficiently large $k$. Applying this together with Proposition 2.12 (ii) and Stirling’s approximation to (4.6), we see that there exist positive constants $K_0$, $K_1$ and $K_2$ such that for every $n \in \mathbb{N}$,

$$|a_n| \leq \sum_{k=n}^{\infty} |b_k| T(k, n) \leq \frac{K_0}{(2n)!} \sum_{k=n}^{\infty} \frac{n^{2k}}{k^{k\gamma}} \leq K_1 \frac{e^{2n}}{n^{\gamma n}} \sum_{k=n}^{\infty} \frac{n^{2k}}{k^{k\gamma}} \leq K_2 \frac{e^{2n}}{n^{\gamma n}}.$$ 

Applying (4.7) and Stirling’s approximation again, we see that there exist positive constants $K_3$ and $K_4$ such that for every $n \in \mathbb{N}$,

$$b_n^* = \frac{1}{n!} \sum_{k=n}^{\infty} |a_k| \frac{d^n}{dx^n} (-1)^k T_k(x; 0) \bigg|_{x=0} \leq K_2 \sum_{k=n}^{\infty} \frac{e^{2k}}{k^{k\gamma}} \frac{(k-1)!}{(n-1)!} \frac{k!}{(k-n)!} \leq K_3 \frac{e^{2n}}{n^{\gamma n}} \sum_{k=n}^{\infty} \frac{1}{k^{(k-2)k\gamma}} \leq K_4 \frac{e^{3n}}{n^{\gamma n}}.$$ 

This shows that $f^*$ is an entire function of order $\sigma_{f^*} \leq \frac{1}{\gamma}$.

Since $\gamma \in (2, \frac{1}{\sigma})$ was arbitrary, we have $\sigma_{f^*} \leq \sigma$.

(2) We next show that $\sigma_{\mu_W(\cdot; f)} \leq \sigma_{f^*}$. For every $r > 0$, writing $N := \nu_W(r; f)$ we have

$$\mu_W(r; f) = |a_N|r\left(r + 1\right)^2 \cdots \left(r + (N - 1)\right)^2 \leq f^*(r) \leq M(r; f^*),$$

so we immediately obtain

$$\sigma_{\mu_W(\cdot; f)} = \limsup_{r \to \infty} \frac{\ln \ln \mu_W(r; f)}{\ln r} \leq \limsup_{r \to \infty} \frac{\ln \ln M(r; f^*)}{\ln r} = \sigma_{f^*}.$$
(3) Now we show that \( \limsup_{r \to \infty} \frac{\ln \nu_W(r; f)}{\ln r} \leq \sigma_{\mu_W(f)} \). For every \( r > 1 \) and every \( R > r \), writing \( N := \nu_W(r; f) \), we have

\[
\begin{align*}
\left[ \frac{R + (N - 1)^2}{r + (N - 1)^2} \right]^N & \leq \frac{R}{r} \cdot \frac{R + 1^2}{r + 1^2} \cdots \frac{R + (N - 1)^2}{r + (N - 1)^2} \\
& = \frac{|\tau_N(R; 0)|}{|\tau_N(r; 0)|} \\
& \leq \frac{\mu_W(R; f)}{\mu_W(r; f)}.
\end{align*}
\]

By Lemma 4.1 (i) we have \( \mu_W(r; f) \geq 1 \) for every sufficiently large \( r \), so

\[
(4.8) \quad N \ln \frac{R + (N - 1)^2}{r + (N - 1)^2} \leq \ln \mu_W(R; f).
\]

In particular putting \( R = 2r + (N - 1)^2 \) in (4.8), taking natural logarithms and dividing by \( \ln r \) on both sides, we arrive at

\[
\frac{\ln N + \ln 2}{\ln r} \leq \frac{\ln \mu_W(2r + (N - 1)^2; f)}{\ln (2r + (N - 1)^2)} - \ln \frac{2}{\ln r}.
\]

Now for every \( \gamma < \frac{1}{\sigma_{\mu_W(f)}} \) and every sufficiently large \( r \), we have

\[
(4.9) \quad \frac{\ln N + \ln 2}{\ln r} \leq \frac{1}{\gamma} \frac{\ln (2r + (N - 1)^2)}{\ln r}.
\]

We claim that \( N^2 \leq r \) for every sufficiently large \( r \), so that (4.9) will give

\[
\frac{\ln N}{\ln r} \leq \frac{1}{\gamma} \frac{3r}{\ln r} - \frac{\ln 2}{\ln r}
\]

for every sufficiently large \( r \), which implies the desired inequality on taking limit superior as \( r \to \infty \). To prove this claim, we observe that if on the contrary there exists some sequence \( \{r_n\}_{n \in \mathbb{N}} \) of positive real numbers increasing to \( \infty \) such that \( \nu_W(r_n; f)^2 > r_n \) for every \( n \in \mathbb{N} \), then (4.9) gives

\[
\frac{1}{2} < \frac{\ln \nu_W(r_n; f)}{\ln r_n} \leq \frac{1}{\gamma} \frac{2 \ln \nu_W(r_n; f) + \ln 3}{\ln r_n} - \frac{\ln 2}{\ln r_n},
\]

and thus

\[
\frac{1}{2} \left( 1 - \frac{2}{\gamma} \right) < \left( 1 - \frac{2}{\gamma} \right) \frac{\ln \nu_W(r_n; f)}{\ln r_n} \leq \frac{\ln 3}{\gamma} - \frac{\ln 2}{\ln r_n}
\]

for every \( n \in \mathbb{N} \). Since \( \sigma < \frac{1}{2} \) enables one to choose \( \gamma > 2 \), this implies that \( 0 \geq \frac{1}{2} (1 - \frac{2}{\gamma}) > 0 \), a contradiction.
Finally we show that \( \sigma \mu W(\cdot; f) \leq \limsup_{r \to \infty} \frac{\ln \mu W(r; f)}{\ln r} \). Write \( N := \nu W(r; f) \) for every \( r > 0 \). Since \( |a_N| \leq 1 \) for every sufficiently large \( r \), we have

\[
\ln \mu W(r; f) = \ln |a_N| + \ln r + \ln(r + 1^2) + \cdots + \ln(r + (N - 1)^2)
\]

\[
= \ln |a_N| + N \ln r + \sum_{k=1}^{N-1} \ln \left( 1 + \frac{k^2}{r} \right)
\]

\[
\leq N \ln r + \sum_{k=1}^{N-1} \ln(1 + k^2)
\]

\[
= N(\ln r + O(N^2))
\]

as \( r \to \infty \). This proves the desired inequality.

\[\square\]

The following is a new Wilson series analogue of the Lindelöf-Pringsheim theorem. It relates the order of the maximal term of a Wilson series and its coefficients. In fact one can apply the same technique to obtain a similar result for Newton series under the setting in Ishizaki and Yanagihara’s [21].

**Theorem 4.4.** Let \( f(x) := \sum_{k=0}^{\infty} a_k \tau_k(x; 0) \) be an entire function of order \( \sigma < \frac{1}{2} \). Then

\[
\sigma \mu W(\cdot; f) = \limsup_{n \to \infty} \frac{n \ln n}{-\ln |a_n|}.
\]

In particular if \( \sigma < \frac{1}{3} \), then \( \sigma = \limsup_{n \to \infty} \frac{n \ln n}{-\ln |a_n|} \).

**Proof.** We denote \( L := \limsup_{n \to \infty} \frac{n \ln n}{-\ln |a_n|} \).

(1) We first show that \( L \leq \sigma \mu W(\cdot; f) \). By Lemma 1.3 (i) we have \( \sigma \mu W(\cdot; f) < \frac{1}{2} < +\infty \), so we let \( \alpha \in (\sigma \mu W(\cdot; f), +\infty) \) be arbitrary. Then for every sufficiently large \( n \), we have

\[
|a_n|n^{\frac{1}{\alpha}}(n^{\frac{1}{\alpha}} + 1^2) \cdots (n^{\frac{1}{\alpha}} + (n - 1)^2) \leq \mu W(n^{\frac{1}{\alpha}}; f) \leq e^n.
\]

So

\[
\ln |a_n| \leq n - \ln n^{\frac{1}{\alpha}} - \ln(n^{\frac{1}{\alpha}} + 1^2) - \cdots - \ln(n^{\frac{1}{\alpha}} + (n - 1)^2)
\]

\[
\leq n - \frac{n \ln n}{\alpha}
\]

\[
= n - \frac{n \ln n}{\alpha}(1 + o(1))
\]

as \( n \to \infty \). This gives

\[
\frac{-\ln |a_n|}{n \ln n} \geq \frac{1}{\alpha}(1 + o(1))
\]

as \( n \to \infty \), and so \( L \leq \alpha \). Since \( \alpha \in (\sigma \mu W(\cdot; f), +\infty) \) was arbitrary, we have \( L \leq \sigma \mu W(\cdot; f) \).
(2) Next we show that $\sigma_{\mu_W(\cdot; f)} \leq L$. By the last paragraph and Lemma 4.3 (i), we have $L < \frac{1}{2}$, so we let $\beta \in (2, \frac{1}{2})$ be arbitrary. Then $|a_n| \leq n^{-\beta n}$ for every sufficiently large $n$. Now for each $r > 0$, since $\beta > 2$, we have $n^\beta - n^2 \geq \frac{1}{2} n^\beta \geq r$ for every sufficiently large $n$, so for these $n$ we have

$$
|a_n|r + 1^2) \cdots (r + (n - 1)^2) \leq n^{-\beta n}(r + 1^2) \cdots (r + (n - 1)^2)
$$

$$
\leq (r + n^2)^{-n}(r + 1^2) \cdots (r + (n - 1)^2)
\leq 1.
$$

Let $a := \max_{n \in \mathbb{N}_0} |a_n|$. Then for every sufficiently large $r > 0$, we have

$$
\mu_W(r; f) = \max\{|a_n|r + 1^2) \cdots (r + (n - 1)^2) : n \leq (2r)^{\frac{1}{\beta}}\}
\leq ar^{(2r)^{\frac{1}{\beta}} + 1},
$$

and so

$$
\sigma_{\mu_W(\cdot; f)} = \limsup_{r \to \infty} \frac{\ln \mu_W(r; f)}{\ln r} \leq \frac{1}{\beta}.
$$

Since $\beta \in (2, \frac{1}{2})$ was arbitrary, we have $\sigma_{\mu_W(\cdot; f)} \leq L$. If $\sigma < \frac{1}{2}$, then $\sigma_{\mu_W(\cdot; f)} = \sigma$ by Remark 5.5 so the final statement follows. \hfill \Box

5. PROOFS OF THE MAIN RESULTS

In the remainder of this paper, we will focus on entire functions of order smaller than $\frac{1}{2}$ and follow an approach that is similar to [21], which deals with Newton series expansions. We will show that an entire function $f$ behaves locally like a polynomial consisting of the few terms around the maximal term in its Wilson series expansion. To do this, we write $N := \nu_W(r; f)$ and aim to show that those terms $a_n \tau_n(x; 0)$ in the Wilson series that are far away from the maximal term $a_N \tau_N(x; 0)$ are small, by defining comparison sequences $\{\alpha_n\}_n$ and $\{\rho_n\}_n$ and comparing the ratio $\frac{a_n \tau_n(r; 0)}{a_N \tau_N(r; 0)}$ with $\frac{\alpha_n \rho_n}{\alpha_N \rho_N}$, whose growth can be controlled. The construction of the comparison sequences follows from Kővari [25].

Definition 5.1. In the remainder of this chapter, we pick a $\delta > 0$ and define comparison sequences $\{\alpha_n\}_{n \in \mathbb{N}_0}$ and $\{\rho_n\}_{n \in \mathbb{N}_0}$ by

$$
\alpha_n := e^{\int_0^t \alpha(t) \, dt} \quad \text{and} \quad \rho_n := e^{-\alpha(n)},
$$

where $\alpha : [0, \infty) \to \mathbb{R}$ is a $C^1$ function which is linear on $[0, t_0]$ and satisfies that $\alpha'(t) = e^{-t} / \ln(t \ln t)$ on $[t_0, +\infty)$, and $t_0 \geq e^e$ is a number such that the range of $\alpha$ is contained in $[-\ln 2, 0]$.

We immediately have $\rho_0 \in (1, \frac{\alpha(t_0)}{\alpha_1})$ and $\rho_n \in (\frac{\alpha_{n+1}}{\alpha_n}, \frac{\alpha_n}{\alpha_{n-1}})$ for every $n \in \mathbb{N}$, so that $\{\rho_n\}_{n \in \mathbb{N}_0}$ is an increasing sequence bounded above by 2.
We are interested in only those radii \( r \) on which \( \frac{\alpha_n \tau_n(r;0)}{\alpha_N \tau_N(r;0)} \) can be controlled by \( \frac{\alpha_n^\alpha}{\alpha_N^\alpha} \), so we give them a name.

**Definition 5.2.** Let \( f(x) := \sum_{k=0}^{\infty} a_k \tau_k(x;0) \) be an entire function of order \( \sigma < \frac{1}{3} \) and let \( \gamma \in (3, \frac{1}{3}) \). A number \( r \geq 0 \) is said to be \( \tau \)-**normal** (for the Wilson series \( f \), with respect to \( \gamma \) and the test sequences \( \{\alpha_n\}_{n \in \mathbb{N}_0} \) and \( \{\rho_n\}_{n \in \mathbb{N}_0} \) if there exists \( N \in \mathbb{N} \) such that for every \( n \in \mathbb{N}_0 \),

\[
|a_n \tau_n(r;0)| \leq |a_N \tau_N(r;0)| \frac{\alpha_n}{\alpha_N} \rho_n^{-\gamma} \quad \text{if } n \geq N,
\]

\[
|a_n \tau_n(r;0)| \leq |a_N \tau_N(r;0)| (1 + \varepsilon_{n,N}) \frac{\alpha_n}{\alpha_N} \rho_n^{-\gamma} \quad \text{if } n < N,
\]

where \( \varepsilon_{n,N} := \frac{r^2}{N^2} + \cdots + \frac{(N-1)^2}{N^2} < \frac{1}{N^2} \). Non-negative numbers that are not \( \tau \)-normal are said to be \( \tau \)-**exceptional**.

The inequality requirements in Definition 5.2 are motivated by the following theorem, which asserts that most non-negative numbers are \( \tau \)-normal.

**Theorem 5.3.** Let \( f \) be an entire function of order \( \sigma < \frac{1}{3} \) and let \( \gamma \in (3, \frac{1}{3}) \). Then the set

\[ E := \{ r \in [1, \infty) : r \text{ is } \tau \text{-exceptional for the Wilson series } f \} \]

has finite logarithmic measure.

**Proof.** We write \( f(x) = \sum_{k=0}^{\infty} a_k \tau_k(x;0) \). Since \( \nu_W(\cdot;f) \) is integer-valued, non-decreasing and right-continuous by Lemma 4.1 we let \( \{r_n\}_{n \in \mathbb{N}_0} \) be the monotonic increasing sequence of non-negative numbers such that \( r_0 := 0 \) and \( \nu_W(r;f) = n \) for every \( r \in [r_n, r_{n+1}) \). (If \( n \) is not in the range of \( \nu_W(\cdot;f) \), then \( r_n = r_{n+1} \).

Now by the choice of \( \{\mu_n\}_{n \in \mathbb{N}_0} \) and the continuity of \( \mu_W(\cdot;f) \) by Lemma 4.1 for every \( j \in \mathbb{N}_0 \) and \( k \in \mathbb{N} \) satisfying \( r_j < r_{j+1} = \cdots = r_{j+k} \), we have

\[
|a_j| r_{j+k} (r_{j+k} + 1^2) \cdots (r_{j+k} + (j - 1)^2) \\
\leq \mu_W(r_{j+k};f) = \mu_W(r_{j+1};f) \\
= \lim_{r \to r_{j+1}} \mu_W(r;f) = \lim_{r \to r_{j+1}} |a_j| r (r + 1^2) \cdots (r + (j - 1)^2) \\
= |a_j| r_{j+1} (r_{j+1} + 1^2) \cdots (r_{j+1} + (j - 1)^2) \\
= |a_j| r_{j+k} (r_{j+k} + 1^2) \cdots (r_{j+k} + (j - 1)^2).
\]

This gives

\[
\frac{|a_j|}{|a_j|} \leq \frac{1}{(r_{j+k} + j^2) \cdots (r_{j+k} + (j - 1)^2)} = \frac{1}{(r_{j+1} + j^2) \cdots (r_{j+k} + (j - 1)^2)}
\]

whenever \( r_j < r_{j+1} = \cdots = r_{j+k} \). So for every \( n \in \mathbb{N}_0 \), taking products for the appropriate \( j \)'s we get

\[
|a_n| \leq \frac{1}{r_1 (r_2 + 1^2) \cdots (r_n + (n - 1)^2)}.
\]
Since \( \rho_n \in (a_{n-1}/a_n, a_n/a_{n+1}) \) for every \( n \in \mathbb{N} \), we have

\[
(5.2) \quad \frac{\alpha_n}{\alpha_0} \geq \frac{1}{\rho_1 \rho_2 \cdots \rho_n}
\]

and so combining (5.1) and (5.2) we obtain

\[
\frac{|a_n|}{\alpha_n} \leq \frac{|a_0|}{\alpha_0} \frac{\rho_1 \rho_2 \cdots \rho_n}{r_1 r_2 + 1^2 \cdots r_n + (n-1)^2}.
\]

Lemma 4.3 (ii) implies that \( r_n > n^\gamma \) for every sufficiently large \( n \), so there exists \( K_0 > 0 \) such that

\[
(5.3) \quad A_n := \frac{|a_n|}{\alpha_n} \leq K_0 \frac{|a_0|}{\alpha_0} \frac{2^n}{(n!)^\gamma}
\]

for every sufficiently large \( n \). For every \( r > 0 \), (5.3) and Stirling’s approximation imply that there exist \( K_1 > 0 \) and \( K_2(r) > 0 \) such that

\[
\sum_{k=1}^{N} |A_k \tau_k(x; 0)| \leq K_0 \frac{|a_0|}{\alpha_0} \sum_{k=1}^{N} \frac{2^k}{(k!)^\gamma} \gamma r(r+1^2) \cdots (r+(k-1)^2)
\]

\[
\leq K_0 \frac{|a_0|}{\alpha_0} \sum_{k=1}^{N} \frac{2^k}{(k!)^\gamma} \gamma \sqrt{r} \gamma (\sqrt{r} + 1)^2 \cdots (\sqrt{r} + (k-1))^2
\]

\[
\leq K_1 \sum_{k=1}^{N} (2e^\gamma)^k \left( \sqrt{r} (\sqrt{r} + 1) \cdots (\sqrt{r} + (k-1))^2 \right)
\]

\[
\leq K_2(r) \sum_{k=1}^{N} e^{\gamma k} k^{(2-\gamma)k}
\]

for every \( x \in D(0; r) \) and every sufficiently large \( N \), so the function \( F \) defined by the Wilson series

\[
F(x) := \sum_{k=0}^{\infty} A_k \tau_k(x; 0)
\]

is an entire function of order at most \( \frac{1}{\gamma} \).

Now suppose that \( \rho > 0 \) and that \( M = \nu_W(\rho; F) \geq 1 \). Then noting that \( 1 < \rho_M < 2 \), for every \( n > M \) we have

\[
\frac{|a_n \tau_n(\rho M; 0)|}{|a_M \tau_M(\rho M; 0)|} = \frac{\alpha_n A_n}{\alpha_M A_M} |\tau_n(\rho M; 0)| = \frac{\alpha_n A_n}{\alpha_M A_M} (\rho M + M^2) \cdots (\rho M + (n-1)^2)
\]

\[
\leq \frac{\alpha_n A_n}{\alpha_M A_M} (\rho + M^2) \cdots (\rho + (n-1)^2) \rho_M^{n-M} = \frac{\alpha_n A_n |\tau_n(\rho; 0)|}{\alpha_M A_M |\tau_M(\rho; 0)|} \rho_M^{n-M}
\]

\[
\leq \frac{\alpha_n}{\alpha_M} \rho_M^{n-M} < 1,
\]
while for every \( n < M \), since

\[
\sum_{k=n}^{M-1} \frac{k^2}{M^2} < \frac{1}{3M^{\gamma-3}} < 1,
\]

we have

\[
\left| a_n \tau_n(\rho M; 0) \right| = \left| \frac{\alpha_n A_n}{\alpha_M A_M} \right| = \frac{\alpha_n A_n}{\alpha_M A_M} \left( \rho M + n^2 \right) \left( \rho M + (M-1)^2 \right) 
\]

\[
= \frac{\alpha_n A_n}{\alpha_M A_M} \left( \rho M + n^2 \right) \left( \rho M + n^2 \right) \cdots \left( \rho M + (M-1)^2 \right) 
\]

\[
\leq \frac{\alpha_n}{\alpha_M} \rho_M n^{-M} \left( 1 + \frac{\rho^2}{\rho M} \right) \cdots \left( 1 + \frac{(M-1)^2}{\rho M} \right) \leq \frac{\alpha_n}{\alpha_M} \rho_M n^{-M} \left( 1 + \frac{k^2}{2M^2} \right) 
\]

where in the third last step we have used the inequality \( \frac{1 + \frac{2a}{1 + a}}{1 + a} \leq 1 + a \) which holds for every \( a > 0 \), in the second last step we have used the inequality \( \rho > M^\gamma \) which follows from Lemma 5.3 (ii), and in the last step we have used the inequality

\[
\prod_k \left( 1 + \frac{\lambda_k}{2} \right) \leq 1 + \sum_k \lambda_k \text{ which holds for every sequence } \{ \lambda_k \}_k \text{ of non-negative numbers with } \sum_k \lambda_k < 1. \]

We have thus shown that \( r \) is \( \tau \)-normal for \( f \) if there exists \( \rho > 0 \) such that \( r = \rho_M \) where \( M = \nu_W(p; R) \), i.e. if there exists \( M \in \mathbb{N}_0 \) such that \( \nu_W(\frac{p}{\rho M}; F) = M \). Therefore if we let \( \{ R_n \}_{n \in \mathbb{N}} \) be the monotonic increasing sequence such that \( \nu_W(R; F) = n \) for every \( R \in [R_n, R_{n+1}] \), then

\[
E \subseteq \bigcup_{k \in \mathbb{N}} [R_{k+1}, R_k].
\]

Now for every \( r \in [R_n \rho_n, R_{n+1} \rho_n] \), we have \( r = R \rho_n \) for some \( R \in [R_n, R_{n+1}] \) and so \( \nu_W(r; F) = n \) by the above computations. So by the definition of \( \{ r_n \}_{n \in \mathbb{N}_0} \) we have \( r_n \leq R_n \rho_n \). Therefore whenever \( r \in [r_n, r_{n+1}] \), i.e. \( \nu_W(r; F) = n \), we must have \( r < R_{n+1} \rho_{n+1} \), and so

\[
E \cap [1, r] \subseteq E \cap [1, R_{n+2} \rho_{n+1}] \subseteq \bigcup_{k=1}^{n+1} [R_k \rho_{k-1}, R_k \rho_k],
\]

which implies that

\[
\log \text{mea} \left( E \cap [1, r] \right) \leq \sum_{k=1}^{n+1} \int_{R_k \rho_{k-1}}^{R_k \rho_k} \frac{dt}{t} = \ln \frac{\rho_{n+1}}{\rho_0}.
\]

Since \( \{ r_n \}_{n \in \mathbb{N}_0} \) is unbounded and \( \{ \rho_n \}_{n \in \mathbb{N}_0} \) is bounded above, it follows that \( \log \text{mea} E \leq +\infty \). \( \square \)

We call the set \( E \) in Theorem 5.3 the \( \tau \)-exceptional set for \( f \). We note that \( E \) depends not only on \( f \), but also on the choice of \( \gamma \) as well as the construction of the test sequences \( \{ \alpha_n \} \) and \( \{ \rho_n \} \) (which depends on the choice of \( \delta \)).
Lemma 5.4. Let \( f(x) = \sum_{k=0}^{\infty} a_k \tau_k(x; 0) \) be an entire function of order \( \sigma < \frac{1}{\gamma}, \gamma \in (3, \frac{1}{\sigma}) \), and \( E \) be the \( \tau \)-exceptional set for \( f \). Then for every \( r \in [0, \infty) \setminus E \) we have

\[
\frac{|a_{N+k} \tau_{N+k}(r; 0)|}{\mu_W(r; f)} \leq e^{-\frac{1}{2}k^2b(N+k)}
\]

for every \( k \in \mathbb{N} \) and

\[
\frac{|a_{N-k} \tau_{N-k}(r; 0)|}{\mu_W(r; f)} \leq \left( 1 + \frac{1}{3N^{\gamma-3}} \right) e^{-\frac{1}{2}k^2b(N)}
\]

for every \( k \in \{0, 1, \ldots, N-1\} \), where \( N = \nu_W(r; f) \) and \( b(N) := \frac{1}{N \ln N (\ln \ln N)^{1+\varepsilon}} \).

Proof. From the definition of the comparison sequences \( \{\alpha_n\}_{n \in \mathbb{N}_0} \) and \( \{\rho_n\}_{n \in \mathbb{N}_0} \), we have

\[
\frac{\alpha_{N+k}}{\alpha_N} \rho_N = e^{f_N^k(a(t) - \alpha(N))} dt \leq e^{f_N^k(t-N)\alpha'(t) dt} \\
\leq e^{-\frac{1}{2}k^2 \min \{|\alpha'(t)| : t \in [N,N+k]\}} = e^{-\frac{1}{2}k^2b(N+k)}
\]

for every \( k \in \mathbb{N} \), and

\[
\frac{\alpha_{N-k}}{\alpha_N} \rho_N = e^{-f_N^k(a(t) - \alpha(N))} dt \leq e^{-f_N^k(t-N)\alpha'(t) dt} \\
\leq e^{-\frac{1}{2}k^2 \min \{|\alpha'(t)| : t \in [N-N-k] \}} = e^{-\frac{1}{2}k^2b(N)}
\]

for every \( k \in \{0, 1, \ldots, N-1\} \). So the result follows from Definition 5.2.

\( \square \)

Remark 5.5. Let \( f(x) = \sum_{k=0}^{\infty} a_k \tau_k(x; 0) \) be an entire function of order \( \sigma < \frac{1}{\gamma}, \gamma \in (3, \frac{1}{\sigma}) \), and \( E \) be the \( \tau \)-exceptional set for \( f \). Then for every \( \varepsilon > 0 \), one can deduce that

\[
\mu_W(r; f) \leq K(r)M(r; f) \leq \mu_W(r; f)\ln \mu_W(r; f)^{\frac{1}{2}\varepsilon}
\]

for every sufficiently large \( r \in [0, \infty) \setminus E \), where \( K(r) := K_{\nu_W(r; f)} \) as defined in the proof of Lemma 4.2, so that \( K(r) \) decreases to 1 as \( r \to \infty \). The first inequality here follows from Lemma 4.3(ii) and Lemma 4.2, while the second inequality follows from Lemma 5.4 and similar arguments as in [18], pp. 330–334. These two inequalities together show that the inequality \( \sigma_{\mu_W(f)} \leq \sigma \) in Lemma 4.3(i) can be improved to an equality if \( \sigma < \frac{1}{\gamma} \).

Proof of Theorem 3.3. The proof is similar to the one of [21, Theorem 3.3]. We take \( E \) to be the \( \tau \)-exceptional set for \( f \). Then we let \( \eta \in (0, \frac{1}{\gamma}) \) be a number to be determined later, and divide the sum into four parts

\[
\sum_{k: |k-N| \geq \kappa} k^h|a_k \tau_k(r; 0)| \\
= \left( \sum_{k: k \leq (1-\eta)N} + \sum_{k: (1-\eta)N < k \leq N - \kappa} + \sum_{k: N - \kappa < k < (1+\eta)N} + \sum_{k: k \geq (1+\eta)N} \right) k^h|a_k \tau_k(r; 0)|.
\]
(i) For $k \geq (1 + \eta)N$ and $r \not\in E$, let $p := k - N$. Lemma 5.4 gives
\[
\frac{k^h|a_k\tau_k(r; 0)|}{|a_N\tau_N(r; 0)|} \leq e^{-\frac{1}{2}p^2 b(N + p) + h \ln(N + p)}.
\]
Since $\lim_{r \to \infty} N = +\infty$ by Lemma 4.1 and since $p \geq \eta N$, we have
\[
-\frac{1}{2}p^2 b(N + p) + h \ln(N + p) \leq -\sqrt{p} \quad \text{for every sufficiently large } r.
\]
Therefore
\[
\frac{1}{|a_N\tau_N(r; 0)|} \sum_{k : k \geq (1 + \eta)N} k^h|a_k\tau_k(r; 0)| \leq \sum_{p \in \mathbb{N}} e^{-\sqrt{p}} \leq \int_{\eta N - 1}^{\infty} e^{-\sqrt{\tau}} d\tau = O(e^{-\frac{1}{2}\sqrt{\eta N}})
\]
as $r \to \infty$ and $r \in [0, \infty) \setminus E$.

(ii) For $k \leq (1 - \eta)N$ and $r \not\in E$, let $p := N - k$. Lemma 5.4 gives
\[
\frac{k^h|a_k\tau_k(r; 0)|}{|a_N\tau_N(r; 0)|} \leq \left(1 + \frac{1}{3N^{\gamma - 3}}\right) e^{-\frac{1}{2}p^2 b(N) + h \ln(N - p)}.
\]
Since $\lim_{r \to \infty} N = +\infty$ and since $p \geq \eta N$, we have
\[
-\frac{1}{2}p^2 b(N) + h \ln(N - p) \leq -\frac{1}{2}\eta^2 \ln N (\ln \ln N)^{1+\delta} + h \ln p + h \ln \left(\frac{1}{\eta} - 1\right) \leq -\sqrt{p}
\]
for every sufficiently large $r$. Therefore similar to the last paragraph we also have
\[
\frac{1}{|a_N\tau_N(r; 0)|} \sum_{k : k \leq (1 - \eta)N} k^h|a_k\tau_k(r; 0)| = O(e^{-\frac{1}{2}\sqrt{\eta N}})
\]
as $r \to \infty$ and $r \in [0, \infty) \setminus E$.

(iii) In the remaining case, we let $\varepsilon \in (0, \frac{1}{\eta N - 1})$ be arbitrary. Then by the continuity of the function $b$, the number $\eta \in (0, \frac{1}{2}]$ can be chosen small enough so that
\[
(1 - \eta)^{-|h|} < 1 + \varepsilon
\]
and
\[
\frac{b(N + |p|)}{b(N)} > 1 - \varepsilon \quad \text{for every } p \in [-\eta N, \eta N].
\]
Now for \( k \in ((1 - \eta)N, (1 + \eta)N) \) and \( r \notin E \), let \( p := k - N \). Both estimates in Lemma 5.5 give

\[
\frac{k^h|a_k \tau_k(r; 0)|}{|a_{N+1} \tau_N(r; 0)|} \leq N^h \left( 1 + \frac{p}{N} \right)^{\frac{h}{2}} \left( 1 + \frac{1}{3N^{\gamma - \frac{3}{2}}} \right) e^{-\frac{1}{2} b^*(1 - \varepsilon)h N} e^{-b^* p^2} \leq N^h \left( 1 + \frac{1}{3N^{\gamma - \frac{3}{2}}} \right) e^{-\frac{1}{2} b^*(1 - \varepsilon)h N} e^{-b^* p^2},
\]

where \( b^* := \frac{1}{2}(1 - \varepsilon)b(N) \).

Now combining the above three paragraphs, we see that for every \( \varepsilon \in (0, \frac{1}{3N^{\gamma - \frac{3}{2}}} \) and every \( \kappa \in \mathbb{N} \), there exists \( \eta \in (0, \frac{1}{2}] \) such that

\[
\sum_{k \in \mathbb{N}} k^h|a_k \tau_k(r; 0)| \leq 2 \left( 1 + \frac{1}{3N^{\gamma - \frac{3}{2}}} \right)^2 N^h \mu_w(r; f) \left[ \sum_{p=\eta}^{\infty} e^{-b^* p^2} + O(e^{-\frac{1}{2} \sqrt{b(N)}}) \right]
\]
as \( r \to \infty \) and \( r \in [0, \infty) \setminus E \). Note that

\[
\sum_{p=\eta}^{\infty} e^{-b^* p^2} \leq \int_{\kappa-1}^{\infty} e^{-b^* t^2} dt = \frac{1}{\sqrt{b^*}} \left( e^{-y_0^2} \right) \left( \int_{y_0}^{\infty} e^{-y^2} dy \right)
\]

where \( y_0 := (\kappa - 1) \sqrt{b^*} \). So given any \( \beta > 0 \), if we take \( \kappa = \left[ \sqrt{\frac{\beta}{b(N)}} \ln \frac{1}{b(N)} \right] \), then for every \( \omega \in (0, \beta) \), the number \( \varepsilon \) can be chosen so small that

\[
\sum_{p=\eta}^{\infty} e^{-b^* p^2} = O \left( \frac{e^{-y_0^2}}{y_0 \sqrt{b^*}} \right) = O \left( \frac{e^{-b^*(1 - \varepsilon) \beta \ln b(N)}}{\sqrt{b(N)} \ln \frac{1}{b(N)}} \right) = o(b(N)^{\frac{d-1}{d}})
\]
as \( r \to \infty \) and \( r \in [0, \infty) \setminus E \). \( \square \)

**Lemma 5.6.** Let \( r > \frac{1}{2} \) and \( p \) be a polynomial of degree \( d \). Then for every \( R \geq r \) and every \( x \in \partial D(0; R) \), we have

\[
(5.5) \quad |(A_w p)(x)| \leq \frac{(\sqrt{R} + \frac{1}{2})^{2d}}{r^d} M(r; p)
\]

and

\[
(5.6) \quad |(D_w p)(x)| \leq \frac{2 \varepsilon d (\sqrt{R} + \frac{1}{2})^{2d-1}}{r^d} M(r; p).
\]

**Proof.** Applying maximum principle to \( \frac{p(x)}{r^d} \) on \( \hat{C} \setminus D(0; r) \), we have

\[
|p(z)| \leq \frac{|z|^d}{r^d} M(r; p)
\]

whenever \( |z| > r \). Now let \( R \geq r \).

(i) Applying maximum principle to \( p \) on \( D(0; (\sqrt{R} + \frac{1}{2})^2) \), we have

\[
|p(x)| \leq \frac{(\sqrt{R} + \frac{1}{2})^{2d}}{r^d} M(r; p)
\]

for every \( x \in D(0; (\sqrt{R} + \frac{1}{2})^2) \), and so (5.5) follows for every \( x \in \partial D(0; R) \).
(ii) Applying Cauchy’s inequality we have
\[ |p'(x)| \leq \frac{cd(\sqrt{R} + \frac{1}{2})^{2(d-1)}}{r^d} M(r; p) \]
for every \( x \in D(0; (\sqrt{R} + \frac{1}{2})^2) \) [18, Lemma 7, p. 337]. Hence for every \( x \in \partial D(0; R) \),
\[
|\langle Dw^p \rangle (x)| = \left| \frac{p(x^+) - p(x^-)}{2i\sqrt{x}} \right|
\leq \int_{-\frac{i}{2}}^{\frac{i}{2}} \left| p'((\sqrt{x} + it)^2) \frac{2i(\sqrt{x} + it)}{2i\sqrt{x}} \right| \, dt
\leq 2cd(\sqrt{R} + \frac{1}{2})^{2(d-1)} \frac{1}{r^d} M(r; p)
\]
which is (5.6).
Since \( Dw^p \) and \( Aw^p \) are polynomials, both (5.5) and (5.6) still hold for every \( x \in D(0; R) \) by the maximum principle. \( \square \)

With Theorem [3.2] and Lemma [5.6], we can now prove Theorem [3.3], which is an estimate on the behavior of successive Wilson differences of a transcendental entire function of order smaller than \( \frac{1}{4} \).

**Proof of Theorem [3.3]**: At each \( x \in \mathbb{C} \), we let \( b(N) := \frac{1}{\sqrt{\frac{1}{N^2} \ln N}} \) and \( \kappa := \sqrt{\frac{10}{b(N)^2 \ln \frac{1}{b(N)}}} \), and let
\[
\phi(x) := \sum_{k:|k-N|>\kappa} a_k \tau_k(x; 0) \quad \text{and} \quad p(x) := \sum_{k=N-\kappa}^{N+\kappa} a_k \frac{\tau_k(x; 0)}{\tau_{N-k}(x; 0)}.
\]
Then locally \( p \) is a polynomial of degree at most \( 2\kappa \) and
\[ f(x) = \phi(x) + \tau_{N-\kappa}(x; 0)p(x). \]
We take \( E \) to be the \( \tau \)-exceptional set for \( f \).

(i) Applying Theorem [5.2] with \( h = n, \beta = 10 \) and \( \omega = 9 \), we have
\[
\begin{align*}
&\ r^n \left| \langle Dw^n \phi \rangle (x) \right| = r^n \left| \sum_{k:|k-N|>\kappa} a_k (-1)^n k(k - 1) \cdots (k - n + 1) \tau_{k-n}(x; 0^+(n)) \right| \\
&\ \leq \sum_{k:|k-N|>\kappa} k^n |a_k \tau_k(r; 0)| \frac{r^n}{r(r + 1)^2 \cdots (r + (n - 1)^2)} \\
&\ \leq \sum_{k:|k-N|>\kappa} k^n |a_k \tau_k(r; 0)| \\
&\ = o(\mu_W(r; f)N^n b(N)^4) = o(\mu_W(r; f)N^{-4})
\end{align*}
\]
as \( r \to \infty \) and \( r \in [0, \infty) \setminus E \). In particular, we have
\[
(5.7) \quad |\phi(x)| = o(\mu_W(r; f)N^{-4}) = o \left( \frac{K}{N} \right) M(r; f)
\]
as \( r \to \infty \) and \( r \in [0, \infty) \setminus E \).
(ii) On the other hand, since \(\sum_{k=1}^{N-2-1} \frac{k^2}{r^2} < \frac{(N-\kappa)(N-\kappa-1)(2N-2\kappa-1)}{6N^2} < 1\), we have

\[
|\tau_{N-\kappa}(x; 0)| \geq r(r - 1^2) \cdots (r - (N - \kappa - 1)^2)
\]

\[
= r^{N-\kappa} \left(1 - \frac{1^2}{r}\right) \cdots \left(1 - \frac{(N - \kappa - 1)^2}{r}\right)
\]

\[
\geq r^{N-\kappa} \left(1 - 2 \sum_{k=1}^{N-2-1} \frac{k^2}{r}\right)
\]

\[
\geq r^{N-\kappa} \left(1 - \frac{(N - \kappa)(N - \kappa - 1)(2N - 2\kappa - 1)}{3N^2}\right)
\]

\[
= r^{N-\kappa}(1 - \varepsilon)
\]

where \(\varepsilon \to 0\) as \(r \to \infty\). This together with (5.7) give

\[
|p(x)| = \frac{1}{|\tau_{N-\kappa}(x; 0)|} |f(x) - \phi(x)| \leq r^{\kappa-N}(1 + \varepsilon')M(r; f),
\]

where \(\varepsilon' \to 0\) as \(r \to \infty\) and \(r \in [0, \infty) \setminus E\). Setting \(M_0 := r^{\kappa-N}(1 + \varepsilon')M(r; f)\) and applying (5.6) in Lemma 5.6 we have

\[
|\langle DWP \rangle (x) | \leq \frac{4\kappa\sqrt{r} + \frac{1}{2})^{4\kappa-2}}{r^{2\kappa}} M_0 = O \left( \frac{\kappa}{r^j} \right) M_0
\]

as \(r \to \infty\) and \(r \in [0, \infty) \setminus E\), and inductively for each \(j \in \mathbb{N}_0\) we have

\[
|\langle DWP \rangle (x) | = O \left( \frac{\kappa}{r^j} \right) M_0
\]

as \(r \to \infty\) and \(r \in [0, \infty) \setminus E\). We also note that for each \(j \in \{0, 1, \ldots, n\}\),

\[
D_{\kappa}^{-j} \tau_{N-\kappa}(x; 0) = (-1)^{n-j} \frac{(N - \kappa)!}{(N - \kappa - n + j)!} \tau_{N-\kappa-n+j}(x; 0^{+(n-j)})
\]

\[
= (-1)^n \frac{(N - \kappa)!}{(N - \kappa - n)!} \tau_{N-\kappa-n}(x; 0^{+(n)})
\]

\[
\cdot \frac{(N - \kappa - n)!}{(N - \kappa - n + j)!} (-1)^j \tau_{N-\kappa-n+j}(x; 0^{+(n-j)}) \frac{(r + N^2)^j}{N^j}
\]

\[
= (-1)^n \frac{(N - \kappa)!}{(N - \kappa - n)!} \tau_{N-\kappa-n}(x; 0^{+(n)}) O \left( \frac{r + N^2}{N^j} \right)
\]

\[
= (-1)^n \frac{(N - \kappa)!}{(N - \kappa - n)!} \tau_{N-\kappa-n}(x; 0^{+(n)}) O \left( \frac{r^j}{N^j} \right)
\]
as \( r \to \infty \), where the last step followed from Lemma \([4,3]\) (ii). These together with Theorem \([2,3]\) \([5,3]\) in Lemma \([5,0]\) and \([5,7]\) yield

\[
\begin{align*}
\mathcal{D}_W^n(\tau_{N-\kappa}(x;0)p(x)) &= \sum_{k=0}^{n} C(n,k) \sum_{j=0}^{n-k} \binom{n-k}{j} A_W^{n-k-j} \mathcal{D}_W^j p(x) A_W^{\kappa} \mathcal{D}_W^{n-j} \tau_{N-\kappa}(x;0) \\
&= A_W^n p(x) \mathcal{D}_W^n \tau_{N-\kappa}(x;0) + \sum_{j=1}^{n} \binom{n}{j} A_W^{n-j} \mathcal{D}_W^j p(x) A_W^{\kappa} \mathcal{D}_W^{n-j} \tau_{N-\kappa}(x;0) \\
&\quad + \sum_{k=1}^{n} C(n,k) \sum_{j=0}^{n-k} \binom{n-k}{j} A_W^{n-k-j} \mathcal{D}_W^j p(x) A_W^{\kappa} \mathcal{D}_W^{n-j} \tau_{N-\kappa}(x;0) \\
&= (-1)^n \frac{(N-\kappa)!}{(N-\kappa-n)!} \tau_{N-\kappa-n}(x;0)^{(n)} \left( A_W^n p(x) + O \left( \frac{\kappa}{N} \right) M_0 \right) \\
&= (-1)^n \frac{(N-\kappa)!}{(N-\kappa-n)!} \tau_{N-\kappa-n}(x;0)^{(n)} \left( p(x) + O \left( \frac{\kappa}{N} \right) M_0 \right) \\
&= (-1)^n \frac{(N-\kappa)!}{(N-\kappa-n)!} \frac{\tau_{N-\kappa-n}(x;0)^{(n)}}{\tau_{N-\kappa}(x;0)} \left( f(x) - \phi(x) + \tau_{N-\kappa}(x;0)O \left( \frac{\kappa}{N} \right) M_0 \right) \\
&= (-1)^n \frac{(N-\kappa)!}{(N-\kappa-n)!} \frac{\tau_{N-\kappa-n}(x;0)^{(n)}}{\tau_{N-\kappa}(x;0)} \left( f(x) + O \left( \frac{\kappa}{N} \right) M(r;f) \right)
\end{align*}
\]

as \( r \to \infty \) and \( r \in [0,\infty) \setminus E \). The above paragraphs imply that

\[
\left( \frac{x}{N} \right)^n (\mathcal{D}_W^n f)(x) = \left( \frac{x}{N} \right)^n (\mathcal{D}_W^n \phi)(x) + \left( \frac{x}{N} \right)^n \mathcal{D}_W^n (\tau_{N-\kappa}(x;0)p(x)) \\
= f(x) + O \left( \frac{\kappa}{N} \right) M(r;f)
\]

as \( r \to \infty \) and \( r \in [0,\infty) \setminus E \). In this final conclusion we may replace \( \kappa \) by \( \kappa = \sqrt{N(\ln N)^2 (\ln \ln N)^{1+\delta}} \).  

\[\square\]

6. Applications

Our Wilson version of the Wiman-Valiron theory can be applied when studying difference equations involving the Wilson operator. One can refer to Z. Chen’s book \([6]\) for a comprehensive study on basics of complex difference equations. An (ordinary) Wilson difference equation is an equation involving an unknown complex function and its Wilson differences, i.e. an equation of the form

\[
F(x, y, \mathcal{D}_W y, \mathcal{D}_W^2 y, \mathcal{D}_W^3 y, \ldots) = 0,
\]

in which \( y \) is an unknown function of the complex variable \( x \). It is said to be linear if \( F \) is linear in \( y \) and its Wilson differences, i.e. if it is of the form

\[
a_n \mathcal{D}_W^n y + \cdots + a_1 \mathcal{D}_W y + a_0 y = 0,
\]

where \( a_0, \ldots, a_n \) are given functions of \( x \).
Example 6.1. (Eigenvectors of $D_W$) It can be readily verified that the simplest linear first-order Wilson difference equation

$$D_W y = y$$

has two linearly independent entire solutions given by

$$f_1(x) = I_{2i\sqrt{x}}(2) + I_{-2i\sqrt{x}}(2)$$
$$f_2(x) = K_{2i\sqrt{x}}(-2) + K_{-2i\sqrt{x}}(-2),$$

where $I_\alpha$ and $K_\alpha$ are respectively the modified Bessel function of the first and the second kind with order $\alpha \in \mathbb{C}$, which are defined by

$$I_\alpha(z) := \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(k + \alpha + 1)} \left(\frac{z}{2}\right)^{2k+\alpha}$$

and

$$K_\alpha(z) := \frac{\pi}{2 \sin \alpha \pi} [I_{-\alpha}(z) - I_\alpha(z)].$$

$f_1$ and $f_2$ in (6.1) are eigenfunctions of $D_W$ corresponding to the eigenvalue 1, so they can be regarded as Wilson analogues of the exponential function. In fact, for every $\lambda \in \mathbb{C} \setminus \{0\}$, the functions

$$f_1(x) = I_{2i\sqrt{x}}\left(\frac{2}{\lambda}\right) + I_{-2i\sqrt{x}}\left(\frac{2}{\lambda}\right)$$
$$f_2(x) = K_{2i\sqrt{x}}\left(-\frac{2}{\lambda}\right) + K_{-2i\sqrt{x}}\left(-\frac{2}{\lambda}\right)$$

are linearly independent entire solutions to the Wilson difference equation

$$D_W y = \lambda y.$$

Theorem 3.3 can be used to obtain the following result about linear Wilson difference equations. The classical analogue of this result about linear differential equations can be found in [32, §4.5].

Theorem 6.2. Let $f$ be a transcendental entire solution of order $\sigma < \frac{1}{3}$ to the Wilson difference equation

$$a_n D_n^a W y + \cdots + a_1 D_W y + a_0 y = 0,$$

where $a_0, \ldots, a_n$ are polynomials and $a_n \neq 0$. Then there exists $L > 0$ such that

$$\ln M(r; f) = L r^\chi (1 + o(1))$$

as $r \to \infty$, where $\chi = \sigma$ is the slope of some edge of the Newton polygon for the Wilson difference equation (6.2), i.e. the convex hull of

$$\bigcup_{k=0}^{n} \{(x, y) \in \mathbb{R}^2 : x \geq k \text{ and } y \leq (\deg a_{n-k}) - (n - k)\}.$$

In particular, we have $\sigma > 0$.

Proof. Given a solution $f$ of (6.2), we let

$$S := \{x \in \mathbb{C} : |f(x)| = M(|x|; f)\}.$$
Then $S$ has non-empty intersection with $\partial D(0; r)$ for every $r > 0$. Substituting $f$ into (6.2) and applying Theorem 3.3 to $f$, we have

$$\left( a_n \frac{N^n}{x^n} + \cdots + a_1 \frac{N}{x} + a_0 \right) f(x)(1 + o(1)) = 0$$

uniformly on $S$ as $r = |x| \to \infty$ and $r \in [0, +\infty) \setminus E$, where $N = \nu_W(r; f)$ and $E$ is the $\tau$-exceptional set for $f$. So denoting $c_k$ as the leading coefficient of the polynomial $a_k$ for each $k$, we have

$$\sum_{k=0}^{n} c_k N^k x^{\deg a_k} - k(1 + o(1)) = 0$$

uniformly on $S$ as $r \to \infty$ and $r \in [0, +\infty) \setminus E$. This implies that

$$(6.3) \quad N = L r^\chi (1 + o(1))$$

as $r \to \infty$ and $r \in [0, +\infty) \setminus E$ for some $L > 0$ and some positive rational number $\chi$ which is the slope of some edge of the Newton polygon for (6.2). We have

$$\chi = \limsup_{r \to \infty} \frac{\ln \nu_W(r; f)}{\ln r} = \sigma_W(r; f) = \sigma$$

by Lemma 4.3 (i) and by (5.4).

Next, since

$$\ln \mu_W(r; f) = \ln |a_N| + \ln r + \ln(r + 1^2) + \cdots + \ln(r + (N-1)^2)$$

$$= \ln |a_N| + N \ln r + \sum_{k=1}^{N-1} \ln \left( 1 + \frac{k^2}{r} \right),$$

if we let $\{r_j\}_{j \in \mathbb{N}}$ be the monotonic increasing sequence of positive real numbers such that $\nu_W(r; f) = j$ for all $r \in [r_j, r_{j+1})$, then

$$\frac{d}{dr} \ln \mu_W(r; f) = \frac{j}{r} + \frac{d}{dr} \sum_{k=1}^{j-1} \ln \left( 1 + \frac{k^2}{r} \right)$$

for all $r \in (r_j, r_{j+1})$, and so

$$\ln \mu_W(r; f) - \ln \mu_W(r_j; f) = \int_{r_j}^{r} \frac{j}{t} dt + \sum_{k=1}^{j-1} \ln \left( 1 + \frac{k^2}{r_j} \right) - \sum_{k=1}^{j-1} \ln \left( 1 + \frac{k^2}{r_j} \right)$$
for all \( r \in [r_j, r_{j+1}] \). Now for all \( r \in [r_{j+1}, r_{j+2}] \) we have

\[
(6.4) \quad \ln \mu_W(r; f) = \ln \mu_W(r_{j+1}; f) + \int_{r_{j+1}}^{r} \frac{j+1}{t} \, dt + \sum_{k=1}^{j} \ln \left( 1 + \frac{k^2}{r} \right) - \sum_{k=1}^{j} \ln \left( 1 + \frac{k^2}{r_{j+1}} \right)
\]

\[
= \ln \mu_W(r_j; f) + \int_{r_{j+1}}^{r} \frac{j+1}{t} \, dt + \sum_{k=1}^{j} \ln \left( 1 + \frac{k^2}{r} \right) - \sum_{k=1}^{j} \ln \left( 1 + \frac{k^2}{r_{j+1}} \right)
\]

Now by choosing any \( \gamma \in (3, \frac{1}{3}) \), we see that for all \( r \in [r_{j+1}, r_{j+2}] \), the first sum on the right-hand side of (6.4) satisfies

\[
\sum_{k=1}^{j} \ln \left( 1 + \frac{k^2}{r} \right) \leq \sum_{k=1}^{j} \ln \left( 1 + \frac{k^2}{r_{j+1}} \right) \leq \sum_{k=1}^{j} \frac{k^2}{(j+1)\gamma} \leq \frac{1}{3(j+1)^{1-3}},
\]

and the second sum on the right-hand side of (6.4) satisfies

\[
\sum_{k=1}^{j} \ln \left( 1 + \frac{k^2}{r_{k+1}} \right) \leq \sum_{k=1}^{j} \ln \left( 1 + \frac{k^2}{r_{k+1}} \right) \leq \sum_{k=1}^{j} \frac{1}{(k+1)^{1-2}} \leq \ln(j+1).
\]

So we have

\[
\sum_{k=1}^{j} \ln \left( 1 + \frac{k^2}{r} \right) = O(N^{3-\gamma}) \quad \text{and} \quad \sum_{k=1}^{j} \ln \left( 1 + \frac{k^2}{r_{k+1}} \right) = o(r^{\chi})
\]

as \( r \to \infty \) and \( r \in [0, +\infty) \setminus E \) by (6.3). Therefore as \( r \to \infty \) and \( r \in [0, +\infty) \setminus E \), (6.4) becomes

\[
\ln \mu_W(r; f) = \ln \mu_W(r_{j+1}; f) + \int_{r_{j+1}}^{r} \frac{\nu_W(t; f)}{t} \, dt + O(N^{3-\gamma}) + o(r^{\chi}) = \frac{L}{\chi} r^{\chi}(1 + o(1)).
\]
Applying (5.4) to this asymptotic, we also have
\[ \ln M(r; f) = \frac{L}{\chi} r^\chi (1 + o(1)) \]
as \( r \to \infty \) and \( r \in [0, +\infty) \setminus E \). Since \( \log \mea E < +\infty \) by Theorem 5.3, one can show by the same arguments as in [20, pp. 259–261] that the same asymptotic holds as \( r \to \infty \) without exceptional set. □

In [9], Chiang and Feng have obtained a result for difference equations which works for entire solutions of order smaller than 1. The result was established via a direct comparison between \( \Delta f \) and \( f' \), which was done without using Newton series at all. This estimate is more general compared with Ishizaki and Yanagihara’s result in [21], which only works for entire solutions of order smaller than \( \frac{1}{2} \). Although one can potentially obtain a better result than Theorem 6.2 by following Chiang and Feng’s approach, we follow Ishizaki and Yanagihara’s approach in this paper because the Wiman-Valiron theory for Wilson series established here is of function theoretic importance.

7. Discussion

In this paper, we have investigated an interpolation series expansion of entire functions with respect to a polynomial basis related to the Wilson divided-difference operator. A convergent series expansion of this type exists for any entire function of order smaller than \( \frac{1}{2} \). Moreover, as Ishizaki and Yanagihara have done for the Newton series expansion [21], we have developed in this paper a Wiman-Valiron theory for this type of interpolation series expansions. A key estimate for those terms which are far away from the maximal term has been established for entire functions of order smaller than \( \frac{1}{3} \), and this estimate shows that the local behavior of these functions is mainly contributed by those terms which are near the maximal term. This key estimate also gives rise to a growth relation between an entire function \( f \) and its \( n \)th Wilson difference \( D_n^W f \), which can be applied to study difference equations involving the Wilson operator. Along the way of proving the estimate, we have also got various properties of the Wilson maximal term and central index, and have obtained a Wilson series version of the Lindelöf-Pringsheim theorem which compares the coefficients of the series with the growth of the maximal term. Combined with the Nevanlinna theory for the Wilson operator established in [8], we have got better understanding in the function theory behind the Wilson operator. There are corresponding versions of residue calculus for the Wilson operator as well as the Askey-Wilson divided-difference operator acting on meromorphic functions, and these may provide natural ways to better understand the corresponding special functions. These issues will be discussed in subsequent papers.

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