A Covariant Poisson Deformation Quantization with Separation of Variables up to the Third Order

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Abstract

We give a simple formula for the operator $C_3$ of the standard deformation quantization with separation of variables on a Kähler manifold $M$. Unlike $C_1$ and $C_2$, this operator cannot be expressed in terms of the Kähler-Poisson tensor on $M$. We modify $C_3$ to obtain a covariant deformation quantization with separation of variables up to the third order which is expressed in terms of the Poisson tensor on $M$ and thus can be defined on an arbitrary complex manifold endowed with a Poisson bivector field of type (1,1).

1 Introduction

Let $M$ be a Poisson manifold with the Poisson bivector field $\eta$. Then $\{f, g\} = \langle \eta, df \wedge dg \rangle$ is a Poisson bracket on $M$. If $\eta$ is non-degenerate, its inverse $\omega$ is a symplectic form on $M$. Denote by $C^\infty(M)[[\nu]]$ the space of formal series in $\nu$ with coefficients from $C^\infty(M)$. As introduced in [1], a formal differentiable deformation quantization on $M$ is an associative algebra structure on $C^\infty(M)[[\nu]]$ with the $\nu$-linear and $\nu$-adically continuous product $\star$ (named star-product) given on $f, g \in C^\infty(M)$ by the formula

$$f \star g = \sum_{r=0}^{\infty} \nu^r C_r(f, g),$$  \hspace{1cm} (1)

where $C_r, r \geq 0$, are bidifferential operators on $M$, $C_0(f, g) = fg$ and $C_1(f, g) - C_1(g, f) = i\{f, g\}$. We adopt the convention that the unit of
A star-product is the unit constant. Two differentiable star-products \( \ast, \ast' \) on a Poisson manifold \((M, \{\cdot, \cdot\})\) are called equivalent if there exists an isomorphism of algebras \( B : (C^\infty(M)[[\nu]], \ast) \to (C^\infty(M)[[\nu]], \ast') \) of the form 
\[
B = 1 + \nu B_1 + \nu^2 B_2 + \ldots ,
\]
where \( B_r, r \geq 1, \) are differential operators on \( M. \) The existence and classification problem for deformation quantization was first solved in the non-degenerate (symplectic) case (see \([5], [14], [8]\) for existence proofs and \([9], [12], [6], [2], [16]\) for classification) and then Kontsevich \([11]\) showed that every Poisson manifold admits a deformation quantization and that the equivalence classes of deformation quantizations can be parameterized by the formal deformations of the Poisson structure.

If \( M \) is a Kähler manifold, there exist special deformation quantizations on \( M \) such that the bidifferential operators \( C_r \) defining the star-product differentiate their first argument only in antiholomorphic directions and the second argument only in holomorphic ones. In \([10]\) all the star-products with separation of variables on an arbitrary Kähler manifold \( M \) were completely described and parameterized by the formal deformations of the original Kähler structure on \( M. \) In what follows we will deal with the standard deformation quantization with separation of variables which corresponds to the trivial deformation of the Kähler structure. This standard star-product was independently constructed in \([3]\) with the use of the properly tuned Fedosov’s quantization scheme. Recently it was shown in \([13]\) that all deformation quantizations with separation of variables can be obtained via Fedosov’s method. A star-product with separation of variables on an arbitrary Kähler manifold was also obtained in \([15]\) by interpreting integral formulas of Berezin’s quantization formally.

The coefficients of bidifferential operators \( C_r \) of the standard star-product with separation of variables \( \ast \) on a Kähler manifold \((M, \omega)\) in holomorphic local coordinates are polynomials in partial derivatives of the Kähler metric tensor \( g_{kl} \) and its inverse \( g^{lk}. \) Notice that the tensor \( g^{lk} \) defines a global Poisson bivector field of type \((1,1)\) w.r.t. the complex structure on \( M. \) Since the construction of the standard star-product with separation of variables does not depend on the choice of local holomorphic coordinates, the operators \( C_r \) are ”covariant” or ”geometric”. It follows almost immediately from Fedosov’s method that the operators \( C_r \) can be expressed in terms of the Kähler connection, its curvature and covariant derivatives of the curvature. In \([7]\) Engliš calculated in a covariant form the formal Berezin transform up to the third order of what turns out to be the standard deformation quantization with separation of variables. As it was noticed in \([13]\), unlike the operators \( C_1 \)
and $C_2$, the operator $C_3$ can not be expressed in terms of the Poisson tensor $g^{ik}$ only (there are always terms containing $g_{kl}$). Therefore, the formulas for the operators $C_r$ can not be used to define a deformation quantization with separation of variables on an arbitrary complex manifold endowed with a Poisson bivector field of type (1,1) w.r.t. the complex structure.

In this letter we obtain a fairly simple covariant formula for the operator $C_3$. Then we extract a non-covariant locally defined one-differentiable (i.e., first order in both arguments) summand from $C_3$ such that the rest is expressed in terms of $g^{ik}$ only. Finally, we modify $C_3$ by adding a covariant one-differentiable bidifferential operator such that the sum is expressed in terms of $g^{ik}$ only. Thus we obtain a global covariant associative star-product with separation of variables up to the third order on an arbitrary complex manifold endowed with a Poisson bivector field of type (1,1).

It is quite plausible that Kontsevich Formality can be established in the “separation of variables” setting and a non-covariant Kontsevich-type star product with separation of variables can be defined in terms of a (possibly degenerate) Poisson tensor $g^{ik}$ and then globalized by the methods of [4]. However, the construction of a global star-product on a Poisson manifold from [4] involves non-canonical choices and is unlikely to produce directly a covariant star-product with separation of variables. It would be interesting to show the existence of such covariant star-products.

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2 The standard deformation quantization with separation of variables

A differentiable star-product $\ast$ on $M$ can be localized (restricted) to any open subset $U \subset M$. We shall retain the same notation $\ast$ for its restriction. For $f, g \in C^\infty(U)[[\hbar]]$ denote by $L_f$ and $R_g$ the operators of left star-multiplication by $f$ and of right star-multiplication by $g$ respectively, so that $L_f g = f \ast g = R_g f$. Denote by $L(U)$ and $R(U)$ the algebras of left and right star-multiplication operators respectively. Notice that in order for a product $\ast$ to be associative it is necessary and sufficient that for any local functions
Let \( M \) be a Kähler manifold with the Kähler form \( \omega \). Its inverse \( \eta \) is a Poisson bivector field of type \((1,1)\) w.r.t. the complex structure on \( M \). It determines a Poisson bracket on \( M \). A formal differentiable deformation quantization on \( M \) is called quantization with separation of variables if for any open subset \( U \subset M \) and any functions \( a, b, f \in C^\infty(U) \) where \( a \) is holomorphic and \( b \) antiholomorphic, \( a \ast f = af, f \ast b = bf \), that is, the operators \( L_a \) and \( R_b \) are pointwise multiplication operators, \( L_a = a, R_b = b \).

This means that the bidifferential operators \( C_r \) defining the star-product differentiate their first argument only in antiholomorphic directions and the second argument only in holomorphic ones.

Recall the construction of the standard deformation quantization with separation of variables on \( M \) from [10]. Let \((U, \{z^k\})\) be an arbitrary contractible holomorphic coordinate chart on \( M \). Locally \( \omega = -i g_{k \bar{l}} dz^k \wedge d\bar{z}^l \). Set \( \partial_k = \frac{\partial}{\partial z^k} \) and \( \bar{\partial}_l = \frac{\partial}{\partial \bar{z}^l} \). Pick a Kähler potential \( \Phi \) of the form \( \omega \) on \( U \), so that \( g_{k \bar{l}} = \partial_k \bar{\partial}_l \Phi \). There exists a unique star-product with separation of variables \( \ast \) such that on any local chart \( U \) as above \( L_{\partial_k \Phi} = \partial_k \Phi + \nu \partial_k \) and \( R_{\bar{\partial}_l \Phi} = \bar{\partial}_l \Phi + \nu \bar{\partial}_l \). The algebra of left star-multiplication operators \( L(U) \) is described as the commutant of the operators \( R_{\bar{\partial}_l} = \bar{\partial}_l \) and \( R_{\bar{\partial}_l \Phi} = \bar{\partial}_l \Phi + \nu \bar{\partial}_l \) in the algebra of all formal differential operators on \( U \). Once \( L(U) \) is known, one can recover the product \( f \ast g \) of any two functions \( f, g \in C^\infty(U) \) as follows. Since \( f = f \ast 1 = L_f 1 \), then \( L_f \) is found as the unique operator \( A \in L(U) \) such that \( A 1 = f \). Now \( f \ast g = L_f g \). Recall how to find the operator \( A = L_f \). Since \( L_f \) commutes with \( R_{\bar{\partial}_l} = \bar{\partial}_l \), it does not contain antiholomorphic partial derivatives. Since it commutes with \( R_{\partial_k \Phi} = \partial_k \Phi + \nu \partial_k \), one obtains recursive equations for the components of \( A = A_0 + \nu A_1 + \ldots \),

\[
[A_r, \bar{\partial}_l \Phi] = [\bar{\partial}_l, A_{r-1}], \quad r = 1, 2, \ldots 
\]  

(2)

It was shown in [10] that equations (2) deliver a unique operator \( A \) commuting with antiholomorphic functions, such that \( A 1 = f \). Finally we get that

\[
C_r(f, g) = A_r g.
\]  

(3)

The star-product \( \ast \) is well defined globally on \( M \). It does not depend on the choices of local coordinates and Kähler potentials. The coefficients of the bidifferential operators \( C_r \) of the standard star-product \( \ast \) written in local holomorphic coordinates are polynomials in partial derivatives of the Kähler metric tensor \( g_{k \bar{l}} \) and of its inverse \( g^{k \bar{l}} \) (which determines the Poisson bivector
field $\eta = ig^lk \partial_k \wedge \partial_l$). The operators $C_r$ are coordinate-independent and therefore ”covariant” or ”geometric”. They can be written in a ”covariant” form in terms of the Kähler connection on $M$.

## 3 A covariant formula for the operator $C_3$

First introduce and recall some notations and standard facts from Kähler geometry. Throughout the paper we will use Einstein's summation convention. Let $(U, \{z^k\})$ be an arbitrary contractible holomorphic coordinate chart on a Kähler manifold $(M, \omega)$ and $\Phi$ be a Kähler potential of $\omega$ on $U$. For $A$ and $B$ a holomorphic and antiholomorphic multi-indices respectively, set $g_{AB} = \partial_A \bar{\partial}_B \Phi$. In particular, $g_{kl} = \partial_k \bar{\partial}_l \Phi$. The Christoffel symbols of the Kähler connection $\nabla$ on $U$ are given by the following formulas:

$\Gamma^s_{kp} = g_{kpl} \bar{g}^l s, \ \Gamma^i_{lq} = \bar{g}^i s g_{lqi}.$

(4)

Subsequently covariantly differentiating a function $f$ we obtain the following symmetric tensors: $f_A = \nabla_A f$ and $f_B = \nabla_B f$ (here $A$ and $B$ are a holomorphic and antiholomorphic multi-indices respectively). In particular, $f_k = \partial_k f$ and $f_l = \bar{\partial}_l f$. The tensors $g_{kl}$ and $g^{lk}$ will be used to lower and raise tensor indices. The Jacobi identity for the Poisson tensor $\bar{g}_{lk}$ takes the form

$\bar{g}_{lk}(\partial_k g^{mn}) = g^{nk}(\partial_m g^{lk}), \ \bar{g}_{lk}(\bar{\partial}_l g^{mn}) = g^{mk}(\bar{\partial}_l g^{nk}).$

(5)

The two following tensors obtained from the curvature tensor of the Kähler connection $\nabla$

$R_{pqkl} = g_{p\bar{m}} R^\bar{m}_{qkl}$ and $R_{k\bar{m}} = g^{\bar{m}n} R^n_{mk\bar{l}}$

are given by the formulas

$R_{pqkl} = g^{\bar{m}n} g_{m\bar{l}} g_{pk\bar{m}} - g_{pqkl}$

(6)

and

$R_{k\bar{m}} = \partial_k \bar{\partial}_l g^{\bar{m}p} - (\bar{\partial}_l g^{\bar{m}p})(\partial_k g^{\bar{m}p})g_{m\bar{n}}.$

(7)

Using recursive equations (2), formulas (3), (4), (5), and (6), one can derive the following covariant formulas for the operators $C_1, C_2$ and $C_3$:

$C_1(\phi, \psi) = g^{lk}(\bar{\partial}_l \phi)(\partial_k \psi) = g^{lk} \phi_{l/\kappa} \psi_{/k},$

(8)
\( C_2(\phi, \psi) = \frac{1}{2} (g^{qp} g^{lk} (\bar{\partial}_l \bar{\partial}_q \phi)(\partial_k \partial_p \psi) + g^{qp} (\partial_q g^{lk})(\bar{\partial}_l \bar{\partial}_q \phi)(\partial_k \psi) + \\
(\bar{\partial}_l g^{qp}) g^{lk} (\bar{\partial}_q \phi)(\partial_k \partial_p \psi) + (\bar{\partial}_l g^{qp}) (\partial_q g^{lk})(\bar{\partial}_l \bar{\partial}_q \phi)(\partial_k \psi)) = \frac{1}{2} g^{lk} g^{qp} \phi \frac{\partial_k \psi}{kp}. \)

\( C_3(\phi, \psi) = \frac{1}{6} g^{lk} g^{rs} \phi \frac{\partial_k \psi}{kp} + \frac{1}{4} R^{lkqp} \phi \frac{\partial_k \psi}{kp}. \)

**4 A Poisson deformation quantization up to the third order**

If the formal parameter \( \nu \) is nilpotent so that \( \nu^{N+1} = 0 \) for some natural \( N \), an associative product on a Poisson manifold \( M \) determined by the formula

\[ f \ast g = \sum_{r=0}^{N} \nu^r C_r(f, g) \]

with \( C_0, C_1 \) as in (1) is called a "star-product up to the \( N \)-th order".

It can be seen from formulas (8) and (9) in section 3 that both operators \( C_1 \) and \( C_2 \) for the standard deformation quantization with separation of variables can be expressed in terms of the Poisson tensor \( g^{lk} \) only. However, this is not the case for the operator \( C_3 \). We will call a bidifferential operator regular if its coefficients in local holomorphic coordinates can be written as polynomials in partial derivatives of the tensor \( g^{lk} \). Introduce the following covariant global bidifferential operators on \( M \):

\[ P(\phi, \psi) := g^{lk} g^{qp} \bar{g}^{rs} \phi \frac{\partial_k \psi}{kp}, \quad Q(\phi, \psi) := -R^{lkqp} \phi \frac{\partial_k \psi}{kp}, \quad R(\phi, \psi) := g^{nm} R^{lp}_{mq} R^{nk}_{pq} \phi \frac{\partial_k \psi}{kp}. \]

It turns out that all these operators coincide modulo regular operators. Introduce the following locally defined (non-covariant) bidifferential operator:

\[ S(\phi, \psi) := g_{m\bar{n}} (\bar{\partial}_q g^{\bar{s}l})(\partial_s g^{\bar{r}p})(\bar{\partial}_l g^{\bar{m}l})(\partial_p g^{\bar{k}k})(\partial_l \phi)(\partial_k \psi). \]

**Proposition 1** All the operators \( P, Q, R \) coincide with the operator \( S \) modulo regular operators.
Proof. We will prove only that the operator \( R - S \) is regular. The rest of the proposition can be proved similarly. Using formula (7), rewrite \( R(\phi, \psi) \) as follows:

\[
R(\phi, \psi) = g^{\bar{m}}(\partial_{\bar{m}}g^{\bar{p}} - (\partial_{\bar{m}}g^{\bar{a}})(\partial_{\bar{n}}g^{\bar{b}})g_{\bar{a}\bar{b}})
\]

\[
(\partial_{\bar{p}}\partial_{\bar{n}}g^{\bar{a}} - (\partial_{\bar{n}}g^{\bar{q}})(\partial_{\bar{p}}g^{\bar{k}})g_{\bar{s}})(\partial_{\bar{t}}\phi)(\partial_{\bar{k}}\psi).
\]

Using formula (5), we get

\[
g^{\bar{m}}(\partial_{\bar{q}}g^{\bar{a}})(\partial_{\bar{m}}g^{\bar{b}})g_{\bar{a}\bar{b}} = (\partial_{\bar{q}}g^{\bar{a}})g_{\bar{a}\bar{b}}g^{\bar{m}}(\partial_{\bar{m}}g^{\bar{b}}) = (\partial_{\bar{q}}g^{\bar{m}})(\partial_{\bar{p}}g^{\bar{b}}).
\]

Similarly,

\[
g^{\bar{m}}(\partial_{\bar{n}}g^{\bar{q}})(\partial_{\bar{m}}g^{\bar{k}})g_{\bar{s}} = (\partial_{\bar{n}}g^{\bar{q}})g_{\bar{s}}g^{\bar{m}}(\partial_{\bar{m}}g^{\bar{k}}).
\]

It follows from formulas (14) and (15) that the operator \( R(\phi, \psi) \) coincides modulo regular terms with the operator

\[
\tilde{S}(\phi, \psi) = (\partial_{\bar{q}}g^{\bar{m}})(\partial_{\bar{m}}g^{\bar{a}})(\partial_{\bar{p}}g^{\bar{q}})(\partial_{\bar{n}}g^{\bar{a}})(\partial_{\bar{t}}\phi)(\partial_{\bar{k}}\psi).
\]

We will now show that the operators \( S \) and \( \tilde{S} \) coincide. Since \( (\partial_{\bar{n}}g^{\bar{q}})g_{\bar{s}} = -g^{\bar{s}}g_{\bar{s}}\bar{n} = (\partial_{\bar{t}}g^{\bar{q}})g_{\bar{s}}\bar{n} \), we get from formula (16) that

\[
\tilde{S}(\phi, \psi) = (\partial_{\bar{q}}g^{\bar{m}})(\partial_{\bar{m}}g^{\bar{a}})(\partial_{\bar{p}}g^{\bar{q}})(\partial_{\bar{n}}g^{\bar{q}})(\partial_{\bar{t}}\phi)(\partial_{\bar{k}}\psi).
\]

It remains to swap the indices \( s \) and \( m \) in (17) to obtain formula (12) for the operator \( S \).

In the notation of formulas (11) formula (10) for the operator \( C_3 \) can be rewritten as

\[
C_3 = \frac{1}{6}P - \frac{1}{4}Q.
\]

It follows from Proposition (1) that the operator

\[
\tilde{C}_3(\phi, \psi) = C_3(\phi, \psi) + \frac{1}{12}R(\phi, \psi) = \frac{1}{6}g^{\bar{p}}g^{\bar{q}}g^{\bar{s}}\phi_{/\bar{q}\bar{t}}\psi_{/\bar{p}\bar{s}} + \frac{1}{12}g^{\bar{p}}g^{\bar{q}}g^{\bar{s}}\phi_{/\bar{q}\bar{t}}\psi_{/\bar{p}\bar{s}} + \frac{1}{4}R^{\bar{p}}_{\bar{m}\bar{q}}R^{\bar{q}}_{\bar{p}\bar{n}}\phi_{/\bar{q}\bar{i}}\psi_{/\bar{p}\bar{k}}
\]

is a regular covariant bidifferential operator that differs from the operator \( C_3 \) by the one-differentiable (first order in both arguments) operator \( \frac{1}{12}R \). The following proposition immediately follows from the associativity of the standard star-product \( \ast \) and the fact that one-differentiable operators are Hochschild cocycles.
Proposition 2 The formula
\[ f \ast g = fg + \nu C_1(f, g) + \nu^2 C_2(f, g) + \nu^3 \tilde{C}_3(f, g) \]
defines a regular covariant associative star-product with separation of variables up to the third order on any Kähler manifold. It remains valid on an arbitrary complex manifold endowed with a Poisson bivector field of type (1,1) w.r.t. the complex structure.

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