Interacting Electrons and Localized Spins: Exact Results from Conformal Field Theory

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Introduction

Much of traditional condensed matter physics falls under the Fermi liquid paradigm. The notion goes back to Landau and implies that a liquid of fermions (such as \textsuperscript{3}He or conduction electrons in a metal) can be treated as a system of essentially free particles \cite{1}. Certain conditions apply: symmetries must remain unbroken and the energies probed should be “small.” But with these provisos the only price to pay for removing the interactions is to keep track on how certain parameters (quasiparticle masses, lifetimes, etc.) renormalize. The Fermi-liquid picture has proven enormously successful and explains how we can get away with single-particle quantum mechanics when doing elementary solid state physics. Fortunately — to keep us busy! — experimentalists have found a number of lab systems that violate the standard Fermi liquid picture. (An outstanding example is the metallic phase of the high-$T_c$ superconductors.) The design of concepts and methods to handle this intriguing and growing class of problems — often nicknamed “strongly correlated systems” — is a major challenge for the theorist. By the very nature of the problem one here confronts matter nonperturbative. Lacking a universal paradigm, the most sensible thing to do is to practice on simple models of non-Fermi liquids, work out the consequences, compare with experiments, build up intuition, and collect that “critical mass” of knowledge necessary for the emergence of an effective, unified approach to correlated systems.

Two particularly clean realizations of non-Fermi liquids are the Luttinger liquid and (certain generalized versions of) the Kondo effect. “Luttinger liquid” is a code name for the low-energy, long-wavelength physics of interacting electrons in one dimension (1D) \cite{2}. The topology of the 1D Fermi surface for free electrons (it consists of two distinct points!) yields dramatic effects when interactions are included: The single-particle poles of the propagators get wiped out and are replaced by branch cuts. The resulting spectral density develops a two-peak structure, with a broad band in between: A single electron added to the system “decays” into two distinct collective excitations, one carrying the charge, the other the spin. Remarkably, the two excitations propagate with different velocities, leading to a spatial spin-charge separation. The Kondo effect is a different story \cite{3}. Here one considers the spin exchange interaction between a localized magnetic impurity and a band of free conduction electrons in a metal. By symmetry, the problem is identical to that of 1D free fermions coupled to a local spin. Surprisingly, this mundane setup leads to quite spectacular physics. As one scales to low temperatures and large distances, the electron-impurity coupling flows to a strong-coupling fixed point where

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electrons are effectively interacting with each other via the impurity. In certain versions of the problem, where the impurity couples to several degenerate bands of conduction electrons (multi-channel Kondo effect), the induced interaction among electrons cannot be “renormalized away,” and one finds a new kind of low-temperature critical behavior. Yet another breakdown of the simple Fermi-liquid picture!

The theories of Luttinger liquids and the multi-channel Kondo effect are important landmarks in the physics of correlated systems. Their construction did not come easy, though, and have required decades of shrewd work by many people, exploiting some of the most powerful tools of theoretical physics: Bosonization, the Renormalization Group, Bethe’s Ansatz, and most recently Conformal Field Theory (CFT).

Boundary Conformal Field Theory

The use of CFT to study the Kondo effect was pioneered by Affleck and Ludwig in a series of remarkable papers [4]. The basic idea goes back to Nozières [5]. In the simplest case — with a spin-$\frac{1}{2}$ impurity coupled antiferromagnetically to a single channel of conduction electrons — Nozières showed that the impurity may be traded for a boundary condition: At the strong-coupling fixed point the impurity traps an electron into a bound singlet state, with the remaining electrons being excluded from entering the impurity site. In other words, the impurity disappears from the problem and is replaced by a boundary condition on the free electron wave functions. Affleck and Ludwig argued that all quantum impurity problems essentially work this way, and showed how to take advantage of CFT once the problem is reformulated in terms of a new boundary condition.

By introducing a boundary in CFT, say at $x = 0$, one imposes constraints on any field $\varphi(t, x)$ [6]. The left and right moving pieces get identified via analytic continuation beyond the boundary, such that

$$\varphi(t, x) = \varphi_L(t, x)\varphi_R(t, x) \rightarrow \varphi_L(t, x)\varphi_L(t, -x).$$

Hence, the boundary has effectively turned the field nonlocal. As a consequence, the one-point function in the presence of a boundary develops a profile given by the corresponding two-point function in the bulk,

$$\langle \varphi(t, x) \rangle \rightarrow \langle \varphi_L(t + x)\varphi_L(t - x) \rangle \sim x^{-\Delta_{\text{bulk}}},$$

with $\Delta_{\text{bulk}}$ the ordinary scaling dimension of $\varphi$. Correlation functions are also altered. In particular, the autocorrelation function for $\varphi$ close to the boundary is governed by a boundary-condition dependent exponent $\Delta$ called the boundary scaling dimension of $\varphi$:

$$\langle \varphi(t, x)\varphi(0, x) \rangle - \langle \varphi(t, x) \rangle\langle \varphi(0, x) \rangle \sim t^{-2\Delta}, \quad |t| >> |x|. \tag{3}$$

This sets the strategy for treating quantum impurity problems: Identify first the particular boundary condition that plays the role of the impurity interaction. (In most cases this is a highly nontrivial task. The CFT scheme gives some guidance, though, and it has turned out that conformal fusion often corresponds to such a change of boundary condition.) Given that the new boundary condition is indeed scale invariant (together with the original bulk theory), one can then use the machinery of CFT to extract the corresponding boundary scaling dimensions. As these determine the asymptotic autocorrelation functions, the finite-temperature properties due to the presence of the boundary (alias the impurity!) are easily accessed from standard finite-size scaling by treating (Euclidean) time as an inverse temperature.
Kondo effect in a Luttinger Liquid

What happens if one puts a magnetic impurity into a Luttinger liquid (LL)? As for the Kondo effect in a Fermi liquid we expect that the impurity also here will induce effective interactions among the electrons. But in a LL, the electrons are already strongly correlated by the mutual Coulomb interactions. Then, what happens? Does the interplay between “induced” and “direct” correlations lead to novel effects? Or do we recover the same Kondo physics as in a Fermi liquid? This is not an easy problem, but again, CFT can be put to productive use.

The “standard model” for low-energy electrons in 1D is the Tomonaga-Luttinger (TL) Hamiltonian

\[ H_{\text{TL}} = \frac{1}{2\pi} \int dx \left\{ v_F :\psi_{L,\sigma}^\dagger(x)i \frac{d}{dx}\psi_{L,\sigma}(x) : - :\psi_{R,\sigma}^\dagger(x)i \frac{d}{dx}\psi_{R,\sigma}(x) : \right\} + g \sum_{k,l=L,R} :\psi_{k,\sigma}(x)\psi_{k,\sigma}(x) : :\psi_{-L,-\sigma}(x)\psi_{L,-\sigma}(x) : \]

+ \left\{ g :\psi_{R,\sigma}^\dagger(x)\psi_{L,\sigma}(x)\psi_{L,-\sigma}(x)\psi_{R,-\sigma}(x) : \right\}, \quad g > 0. \tag{4} \]

Here \( \psi_{L/R,\sigma}(x) \) are the left/right moving components of the electron field \( \Psi_\sigma(x) \), expanded about the Fermi points \( \pm k_F \), and we implicitly sum over repeated indices for spin \( \sigma = \uparrow, \downarrow \). The first term in (4) is that of free relativistic fermions, while the second and third terms describe forward and backward electron-electron scattering, respectively. Normal ordering is carried out w.r.t. the filled Dirac sea.

The TL Hamiltonian is conveniently written on diagonal Sugawara form, using the charge and spin currents

\[ j_{L/R}(x) = \cosh \theta :\psi_{L/R,\sigma}^\dagger(x)\psi_{L/R,\sigma}(x) : + \sinh \theta :\psi_{R/L,\sigma}^\dagger(x)\psi_{R/L,\sigma}(x) :, \tag{5} \]

\[ J_{L/R}(x) = :\psi_{L/R,\sigma}^\dagger(x)\tfrac{1}{2}\sigma \psi_{L/R,\mu}(x) :, \tag{6} \]

with \( \tanh 2\theta = g/(v_F + g) \). Dropping a marginally irrelevant term \(- (g/\pi)J_L \cdot J_R\), one obtains the critical bulk Hamiltonian

\[ H_{\text{TL}}^* = \frac{1}{2\pi} \int_{-\ell}^\ell dx \sum_{i=1,2} \left\{ \frac{v_c}{4} :j_i^L(x)j_i^L(x) : + \frac{v_s}{3} :J_i^L(x) \cdot J_i^L(x) : \right\}, \tag{7} \]

where we have confined the system to \( x \in [-\ell, \ell] \) and replaced the right-moving currents with a second channel of left-moving currents. The last step involves folding the interval in half to \([0, \ell]\) and analytically continue the left-handed currents back to the full interval. Note that the spin and charge separation in (4) yields two dynamically independent theories, each Lorentz invariant with a characteristic velocity, \( v_c = v_F (1 + 2g/v_F)^{\frac{1}{2}} \) and \( v_s = v_F - g \). The charge and spin currents satisfy the \( U(1) \) and (level-1) \( SU(2)_1 \) Kac-Moody algebras, respectively, and it follows that \( H_{\text{TL}}^* \) is invariant under the chiral symmetry \( U(1) \times U(1) \times SU(2)_1 \times SU(2)_1 \).

We now couple a localized spin-\( \frac{1}{2} \) impurity \( S \) to the electrons. As a warm-up let us first consider the simple case of only forward scattering off the impurity:

\[ H_F = \sum_{k=L,R} \lambda :\psi_{k,\sigma}^\dagger(0)\frac{1}{2}\sigma_{\sigma \mu} \psi_{k,\mu}(0) : \cdot S. \tag{8} \]
Using the spin currents in (8), and replacing the right-handed one by a second left-handed current, Eq. (8) turns into

$$\mathcal{H}_F = \lambda_F [\mathbf{J}_L^1(0) + \mathbf{J}_L^2(0)] \cdot \mathbf{S}. \quad (9)$$

As the two currents are coupled via $\mathbf{S}$, $\mathcal{H}_F$ breaks the $SU(2) \times SU(2)$ symmetry of $\mathcal{H}_{TL}^*$ down to the diagonal level-2 subalgebra $SU(2)_2$ spanned by $\mathbf{J}(x) = \mathbf{J}_L^1(x) + \mathbf{J}_L^2(x)$. To adopt to this fact we use the coset construction [8] to write the spin part of the Hamiltonian as a sum of an SU(2) Sugawara Hamiltonian and a free Majorana fermion ($\leftrightarrow$ 2D Ising model). We then absorb $\mathcal{H}_F$ into $\mathcal{H}_{TL}^*$ by the canonical transformation $\mathbf{J}(x) \rightarrow \mathbf{J}'(x) \equiv \mathbf{J}(x) + S\delta(x)$, $\mathbf{J}'(x)$ being the spin current of electrons and impurity. The impurity thus disappears from the Hamiltonian and is replaced by a nontrivial boundary condition on $\mathcal{H}_{TL}^*$.

In CFT, a boundary condition is equivalent to a selection rule for quantum numbers of a conformal embedding. In our case, the conformal embedding is $U(1) \times U(1) \times SU(2)_2 \times Ising$, with quantum numbers $Q, \Delta Q \in \mathbb{Z}$ (sum and difference of net charge in the two channels), $j = 0, \frac{1}{2}, 1$ (spin of primary states), and $\phi = 1, \sigma, \epsilon$ (Ising primary fields). Without the impurity, i.e. with a trivial boundary condition at $x = 0$, the eigenstates of $\mathcal{H}_{TL}^*$ organize into products of four conformal towers, labeled by these quantum numbers. The allowed combinations can be described by a selection rule that reproduces the spectrum of interacting fermions. With the impurity (nontrivial boundary condition), the spin current gets redefined as that of electrons and impurity: Effectively, a spin-$\frac{1}{2}$ degree of freedom is added to the $SU(2)_2$ towers, which get shifted according to the CFT fusion rule $j = 0 \rightarrow \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \rightarrow 0$ or $1, 1 \rightarrow \frac{1}{2}$. A new spectrum emerges, with the $U(1) \times U(1), SU(2)_2$, and Ising conformal towers combined differently according to a new selection rule. It describes interacting fermions + impurity and yields immediate information about the boundary scaling dimensions (which are really what we are after!). This follows from a well-known result by Cardy [8]: The eigenstates of a Hamiltonian $\mathcal{H}$ on a finite interval are in 1-1 correspondence to the boundary scaling dimensions of the same theory boosted to Euclidean space-time (with the time axis defining the boundary).

Having thus obtained the spectrum of boundary scaling dimensions, we identify the leading correction-to-scaling boundary operator (LCBO) $\mathcal{O}$ that respects all symmetries of the problem. This operator is added to $\mathcal{H}_{TL}^*$ to give an effective Hamiltonian in the scaling region of the new fixed point:

$$\mathcal{H}_{scaling} = \mathcal{H}_{TL}^* + \mu \mathcal{O}(0), \quad (10)$$

with $\mu$ the scaling field conjugate to $\mathcal{O}$. Unlike for ordinary critical phenomena, $\mathcal{O}$ produces the leading scaling in temperature although it is in fact an irrelevant boundary operator, given by the first Kac-Moody descendant in the $j = 1$ conformal tower: $\mathbf{J}_{-1} \cdot \phi$, of dimension $\frac{3}{2}$. This is the same operator that drives critical scaling in the two-channel Kondo effect for noninteracting electrons [11]. Specifically, the impurity contributions to the specific heat $\delta C$ and spin susceptibility $\delta \chi$ are given to leading order by

$$\delta C = \frac{\mu^2 9\pi^2}{v_s^3} T \ln\left(\frac{1}{\tau_0 T}\right), \quad \delta \chi = \frac{\mu^2 18}{v_s^3} T \ln\left(\frac{1}{\tau_0 T}\right), \quad T \rightarrow 0, \quad (11)$$

with $\tau_0$ a short-time cutoff. With the known bulk response for the Tomonaga-Luttinger model, $C = \pi (v_c^{-1} + v_s^{-1}) T / 3$ and $\chi = 1/2\pi v_s$ we predict a Wilson ratio

$$R_W = \frac{\delta \chi / \chi}{\delta C / C} = \frac{4}{3}(1 + \frac{v_s}{v_c}). \quad (12)$$
For $g \to 0$ ($v_c, v_s \to v_F$), this reduces to the universal number $8/3$ characterizing the ordinary two-channel Kondo effect [10].

Let us now turn to the more realistic case with forward ($F$) and backward ($B$) scattering off the impurity. To keep things simple we assume the $F$ and $B$ amplitudes to be the same, and hence replace $H_F$ by

$$H_K \equiv H_F + H_B = \lambda \sum_{k,l=L,R} :\psi_{k,\sigma}^\dagger(0) \sigma^\mu_{\sigma \mu} \psi_{l,\sigma}(0) : S.$$ \hspace{1cm} (13)

The extra terms, mixing left and right, break the chiral $SU(2)$ and chiral $U(1)$ invariance of $H_{TL}$, and this produces a boundary operator of dimension $\leq \frac{1}{2}$ at the forward scattering fixed point. Back scattering is thus a relevant perturbation and drives the system to a new fixed point describing Kondo interaction in a Luttinger liquid.

With no electron-electron interaction ($g = 0$ in [10]), we have a free bulk Hamiltonian $H_0$ together with $H_K$. Passing to a basis spanned by definite-parity fields $\psi_{\pm,\sigma}(x) = [\psi_{L,\sigma}(x) \pm \psi_{R,\sigma}(-x)]/\sqrt{2}$, $H_0 + H_K$ transforms into a two-channel theory, but with the impurity coupled to the electrons in only one of the channels. This renormalizes to a local Fermi liquid (like the ordinary 3D Kondo problem), with response functions scaling analytically with temperature. However, a different approach must be used for the interacting problem since $H_{TL}$ is non-local in this basis. Here we exploit the expectation that any local impurity interaction, including the Kondo interaction $H_K$, can be substituted by a renormalized boundary condition on the critical bulk theory [10]. The equivalent selection rule defines a fixed point, and by demanding that any associated LCBO must respect the symmetries of the problem and correctly reproduce the non-interacting limit as $g \to 0$, the possible critical theories can be deduced. (Note that a selection rule here defines a boundary fixed point, and is valid for all values of the marginal bulk coupling $g$.

Hence, given a selection rule, Fermi liquid scaling must emerge in the limit $g \to 0$.)

To have a generally applicable formalism we replace $Q$ and $\Delta Q$ by new quantum numbers, treating the two diagonalized $U(1)$ towers as independent. (In the previous case these two towers were tangled up via the labeling by $Q$ and $\Delta Q$, as an implicit part of the corresponding selection rule for combining conformal towers.) This allows us to formulate a general criterion for selecting operators that respect global, but not necessarily chiral, $U(1)$ invariance.

Together with invariance under channel exchange ($1 \leftrightarrow 2$), this leaves only certain possibilities for the $U(1) \times U(1)$ part of the candidate LCBOs. The complete set of possible scaling dimensions are then obtained by also including the $SU(2)$ and Ising conformal towers, as allowed by symmetry.

By requiring Fermi-liquid scaling for the impurity specific heat $\delta C$ and susceptibility $\delta \chi$ as $g \to 0$, we find that there are only two possible types of critical behavior. Either Fermi-liquid scaling persists for $g \neq 0$, or there is a non-Fermi liquid behavior given by

$$\delta C = c_1((1/K_\rho) - 1)^2 T^{(1/K_\rho)-1} + c_2 T, \hspace{1cm} (14)$$

$$\delta \chi = c_3 T^0, \hspace{1cm} (15)$$

as $T \to 0$. Here $K_\rho = (1 + 2g/v_F)^{-1/2}$ and $c_{1,2,3}$ are amplitudes depending on the scaling fields and velocities. The LCBO driving the anomalous scaling in (14) and (15) is given by the composite operator $O_{LCBO} = [V_{2,0}^1 \times V_{2,0}^2 + V_{1,0}^1 \times V_{2,0}^2] \times \epsilon$ where $V_{C,D}^i$ is an $U(1)$ primary (vertex) operator in channel $i$, and $\epsilon$ the Ising energy density.

The scaling in (14) and (15) agrees exactly with a conjecture by Furusaki and Nagaosa [11], in support of the non-Fermi liquid scenario. However, a simplified model (neglecting
backward spin diagonal and forward spin off-diagonal Kondo scattering) suggests that in fact the other scenario (Fermi liquid) may be realized [12]. Whatever is the case, our exact CFT analysis shows that no other fixed point theories are possible. Note that in none of the two cases does the electron-electron interaction influence \( \delta \chi \): the impurity remains completely screened for \( g \neq 0 \).

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References

[1] For a classic treatise, see A. A. Abrikosov, L. P. Gorkov and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Dover, New York, 1963).

[2] For a recent review, see H. J. Schulz, “Fermi liquids and non-Fermi liquids,” cond-mat/9503150, in *1994 Les Houches lecture notes*, to appear.

[3] For a review, see e.g. A. C. Hewson, *The Kondo Problem to Heavy Fermions* (Cambridge University Press, Cambridge, 1993).

[4] I. Affleck, in *Correlation Effects in Low-Dimensional Electron Systems*, edited by A. Okiji and N. Kawakami (Springer-Verlag, Berlin Heidelberg, 1994), and references therein.

[5] P. Nozières, in *Proceedings of the 14th International Conference on Low-Temperature Physics*, edited by M. Krusius and M. Vuorio (North-Holland, Amsterdam, 1974).

[6] J. L. Cardy, Nucl. Phys. B240, 514 (1984); in *Infinite Lie Algebras and Conformal Invariance in Condensed Matter and Particle Physics*, edited by K. Dietz and V. Rittenberg (World Scientific, Singapore, 1986).

[7] P. Fröjdh and H. Johannesson, cond-mat/9505100, Phys. Rev. Lett., in press (1995).

[8] P. Goddard, A. Kent and D. Olive, Commun. Math. Phys. 103, 105 (1986).

[9] J. L. Cardy, J. Phys. A: Math. Gen. 17, L385 (1984); Nucl. Phys. B275, 200 (1986).

[10] I. Affleck and A. W. W. Ludwig, Nucl. Phys. B360, 641 (1991).

[11] A. Furusaki and N. Nagaosa, Phys. Rev. Lett. 72, 892 (1994).

[12] A. Schiller and K. Ingersent, Phys. Rev. B 51, 4676 (1995).