Periodic Solutions of Non-Autonomous Second Order Hamiltonian Systems

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Abstract

We try to generalize a result of M. Willem on forced periodic oscillations which required the assumption that the forced potential is periodic on spatial variables. In this paper, we only assume its integral on the time variable is periodic, and so we extend the result to cover the forced pendulum equation. We apply the direct variational minimizing method and Rabinowtz’s saddle point theorem to study the periodic solution when the integral of the potential on the time variable is periodic.

Keywords

Forced second order Hamiltonian systems, the forced pendulum equation, variational minimizers, Saddle Point Theorem.

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1 Introduction and Main Results

In [10] and [5], M. Willem and Mawhin studied the following second order Hamiltonian system

\[ \ddot{u}(t) = \nabla F(t, u(t)) = F'(t, u(t)) \] (1.1)

where \( F : [0, T] \times \mathbb{R}^N \to \mathbb{R}, \nabla F(t, u(t)) = F'(t, u(t)) \) is the gradient of \( F(t, u(t)) \) with respect to \( u \).

We assume \( F(t, u(t)) \) satisfies the following assumption:

(A). \( F(t,x) \) is measurable in \( t \) for each \( x \in \mathbb{R}^N \), continuously differentiable in \( x \) for a.e. \( t \in [0, T] \), and there exist \( a \in C((0, +\infty); \mathbb{R}^+)) \) and \( b \in L^1(0, T; \mathbb{R}^+) \) such that

\[ |F(t,x)| \leq a(|x|)b(t), \]

\[ |\nabla F(t,x)| \leq a(|x|)b(t) \]

for all \( x \in \mathbb{R}^N \) and a.e. \( t \in [0, T] \).

M. Willem ([10]) got the following theorem :

**Theorem 1.1** ([10] and [5]) Assume \( F \) satisfies condition (A) and for the canonical basis \( \{e_i|1 \leq i \leq N\} \) of \( \mathbb{R}^N \), there exist \( T_i > 0 \) such that for \( \forall x \in \mathbb{R}^N \) and a.e. \( t \in [0, T], \)

\[ F(t,x + T_ie_i) = F(t,x), \quad 1 \leq i \leq N \] (1.2)
Then (1.1) has at least one solution which minimizes
\[ f(u) = \int_0^T \frac{1}{2} |\dot{u}(t)|^2 + F(t, u(t)) \, dt \]
on $H^1_T = \{u|u, \dot{u} \in L^2[0, T], u(t + T) = u(t)\}$.

In order to cover the forced pendulum equation:
\[ \ddot{u}(t) = -a \sin u + e(t), \quad (1.3) \]
Mawhin-Willem [5] also study the following forced equation:
\[ \ddot{u}(t) = \nabla F(t, u(t)) + e(t) = F'(t, u(t)) + e(t) \quad (1.4) \]
they got the following Theorem:

**Theorem 1.2** ([10] and [5]) Assume $F$ satisfies the conditions of Theorem 1.1, and $e(t) \in L^1(0, T; R^N)$ verifying
\[ \int_0^T e(t) \, dt = 0, \]
then (1.4) has at least one solution which minimizes on $H^1_T$ the following functional:
\[ f(u) = \int_0^T \frac{1}{2} |\dot{u}(t)|^2 + F(t, u(t)) + e(t)u(t) \, dt \]
We notice that the potential $F(t, x) = a \cos x + e(t)x$ does not satisfy (1.2). But if $\int_0^T e(t) \, dt = 0$, then $F(t, x) = a \cos x + e(t)x$ does satisfy
\[ \int_0^T F(t, x + 2\pi) \, dt = \int_0^T F(t, x) \, dt. \] (1.5)
So instead of (1.2) we only assume the weaker integral condition:
\[ \int_0^T F(t, x + Ti e_i) \, dt = \int_0^T F(t, x) \, dt \quad i = 1, 2, ..., N \] (1.6)
We obtain the following results:

**Theorem 1.3** Assume $F : R \times R^N \rightarrow R$ satisfies condition (A) and
\begin{itemize}
  \item[(F1).] $F(t + T, x) = F(t, x)$, $\forall (t, x) \in R \times R^N$,
  \item[(F2).] $F$ satisfies (1.6),
  \item[(F3).] There exist $0 < C_1 < \frac{1}{2}(2\pi)^2$, $C_2 > 0$ such that
    \[ |F(t, x)| \leq C_1 |x|^2 + C_2 \]
\end{itemize}
Then (1.1) has at least one $T$-periodic solution.

**Corollary 1.1** (J. Mawhin, M. Willem [11]) For the pendulum equation (1.3), the potential $F(t, x) = a \cos x + e(t)x$ satisfies all conditions in Theorem 1.3 provided $e(t + T) = e(t)$ and $\int_0^T e(t) \, dt = 0$. In this case, (1.3) has at least one $T$-periodic solution.

**Theorem 1.4** Suppose $F : R \times R^N \rightarrow R$ satisfies conditions (A), (F1), (F2) and
\begin{itemize}
  \item[(F4).] $\exists \mu_1 < 2, \mu_2 \in R$ such that
    \[ F'(t, x) \cdot x \leq \mu_1 F(t, x) + \mu_2, \]
  \item[(F5).] $\exists \delta > 0$ such that $\forall t \in R$, $F(t, x) > \delta$, $|x| \rightarrow +\infty$,
  \item[(F6).] $F(t, x) \leq \delta |x|^2$.
\end{itemize}
Then $T < \frac{\pi}{\sqrt{\frac{2}{\delta} + 1}}$, (1.1) has a $T$-periodic solution; furthermore, if $\forall x \in R^N$, $\int_0^T F(t, x) \, dt \leq 0$, then (1.3) has a non-constant $T$-periodic solution.
2 Some Important Lemmas

Lemma 2.1 (Eberlin-Smulian[1]) A Banach space $X$ is reflexive if and only if any bounded sequence in $X$ has a weakly convergent subsequence.

Lemma 2.2 ([1], [5], [12]) Let $q \in W^{1,2}(R/TZ, R^n)$ and $\int_0^T q(t) dt = 0$, then

(i) We have Poincare-Wirtinger’s inequality

$$\int_0^T |\dot{q}(t)|^2 dt \geq \left( \frac{2\pi}{T} \right)^2 \int_0^T |q(t)|^2 dt$$

(ii) We have Sobolev’s inequality

$$\max_{0 \leq t \leq T} |q(t)| = \|q\|_\infty \leq \sqrt{\frac{T}{12}} \left( \int_0^T |\dot{q}(t)|^2 dt \right)^{1/2}$$

We define the equivalent norm in $H^1_0 = H^1 = W^{1,2}(R/TZ, R^n)$:

$$\|q\|_{H^1} = (\int_0^T |\dot{q}(t)|^2)^{1/2} + |\int_0^T q(t) dt|$$

Lemma 2.3([3]) Let $X$ be a reflexive Banach space, $M \subset X$ a weakly closed subset, and $f : M \to R \cup \{+\infty\}$ weakly lower semi-continuous. If the minimizing sequence for $f$ on $M$ is bounded, then $f$ attains its infimum on $M$.

Definition 2.1([3]) Suppose $X$ is a Banach space and $\{q_n\} \subset X$ satisfies

$$f(q_n) \to C, \quad (1 + \|q_n\|) f'(q_n) \to 0.$$ 

Then we say $\{q_n\}$ satisfies the $(CPS)_C$ condition.

Lemma 2.4 (Rabinowitz’s Saddle Point Theorem[9], Mawhin-Willem[5]) Let $X$ be a Banach space with $f \in C^1(X, R)$. Let $X = X_1 \oplus X_2$ with

$$\dim X_1 < +\infty$$

and

$$\sup_{S_2^1} f < \inf_{X_2} f,$$

where $S_2^1 = \{u \in X_1 ||u|| = R\}$.

Let $B_R^1 = \{u \in X_1, ||u|| \leq R\}$, $M = \{g \in C(B_R^1, X) | g(s) = s, s \in S_R^1\}$

$$C = \inf_{g \in M} \max_{s \in B_R^1} f(g(s)).$$

Then $C > \inf_{X_2} f$, and if $f$ satisfies $(CPS)_C$ condition, then $C$ is a critical value of $f$.

3 The Proof of Theorem 1.3

Lemma 3.1 (Morrey [7], M-W [10]) Let $L : [0, T] \times R^N \times R^N \to R, (t, x, y) \to L(t, x, y)$ be measurable in $t$ for each $(x, y) \in R^N \times R^N$ and continuously differentiable in $(x, y)$ for a.e. $t \in [0, T]$. Suppose there exists $a \in C(R^+, R^+)$, $b \in L^1(0, T; R^+)$ and $c \in L^q(0, T; R^+)$, $1 < q < \infty$, such that for a.e. $t \in [0, T]$ and every $(x, y) \in R^N \times R^N$ one has

$$|L(t, x, y)| \leq a(|x|)(b(t) + |y|^q),$$

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$$|D_x L(t, x, y)| \leq a(|x|)(b(t) + |y|^p),$$
$$|D_y L(t, x, y)| \leq a(|x|)(c(t) + |y|^{p-1}).$$

where $\frac{1}{p} + \frac{1}{q} = 1$, then the functional

$$\varphi(u) = \int_0^T L(t, u(t), \dot{u}(t)) dt$$

is continuously differentiable on the Sobolev space

$$W^{1,p} = \{u \in L^p(0, T), \dot{u} \in L^p(0, T)\}$$

and

$$< \varphi'(u), v > = \int_0^T [< D_x L(t, u, \dot{u}), v > + D_y L(t, u, \dot{u}) \cdot \dot{v}] dt$$

From Lemma 3.1 and the assumptions (A), we know that the variational functional

$$f(u) = \int_0^T \left[ \frac{1}{2} |\dot{u}|^2 + F(t, u(t)) \right] dt$$

is $C^1$ on $W^{1,2}_\Gamma = H^1_\Gamma$, and the critical point is just the periodic solution for the system (1.1).

Furthermore, if (F1) and (F2) are satisfied, we will prove the functional $f(u)$ attains its infimum on $H^1_\Gamma$; in fact,

$$H^1_\Gamma = X \oplus R^N,$$

where

$$X = \{x \in H^1_\Gamma : \bar{x} \triangleq \frac{1}{T} \int_0^T x(t) dt = 0\}$$

and $\forall u \in H^1_\Gamma$, we have $\tilde{u} \in X$ and $\overline{u} \in R^N$, such that $u = \tilde{u} + \overline{u}$.

By Poincare-Wirtinger’s inequality,

$$f(\tilde{u}) \geq \frac{1}{2} \int_0^T |\dot{\tilde{u}}|^2 dt - C_1 \int_0^T |\tilde{u}|^2 dt - C_2 T \int_0^T |\dot{\tilde{u}}|^2 dt - C_2 T$$

hence, $f$ is coercive on $X$.

Let $\{u_k\}$ be a minimizing sequence for $f(u)$ on $H^1_\Gamma$, $u_k = \tilde{u}_k + \overline{u}_k$, where $\tilde{u}_k \in X$, $\overline{u}_k \in R^N$,

then by (3.6) we have

$$||\tilde{u}_k||_{H^1_\Gamma} \leq C.\tag{3.7}$$

By condition (F2), we have

$$f(u + T_i e_i) = f(u), \quad \forall u \in H^1_\Gamma, \quad 1 \leq i \leq N.\tag{3.8}$$

So if $\{u_k\}$ is a minimizing sequence for $f$, then

$$(\tilde{u}_k \cdot e_1 + \overline{u}_k \cdot e_1 + k_1 T_1, ..., \tilde{u}_k \cdot e_N + \overline{u}_k \cdot e_N + k_N T_N)$$

is also a minimizing sequence of $f(u)$, and so we can assume

$$0 \leq \overline{u}_k \cdot e_i \leq T_i, \quad 0 \leq i \leq N.\tag{3.9}$$

By (3.7) and (3.9), we know $\{u_k\}$ is a bounded minimizing sequence in $H^1_\Gamma$, and it has a weakly convergent subsequence; furthermore, $f$ is weakly lower semi-continuous since $f$ is the sum of a convex continuous function and a weakly continuous function. We can conclude that $f$ attains its infimum on $H^1_\Gamma$. The corresponding minimizer is a periodic solution of (1.1).
4 The Proof of Theorem 1.4

Lemma 4.1: If conditions (A), (F1), (F2) and (F4) in Theorem 1.4 hold, then \( f(q) \) satisfies the \((CPS)_C\) condition on \( H^1 \).

**Proof:** For any \( C \), let \( \{u_n\} \subset H^1 \) satisfy
\[
f(u_n) \to C, \quad (1 + \|u_n\|)f'(u_n) \to 0.
\]
We claim \( \|\dot{u}_n\|_{L^2} \) is bounded; in fact, by \( f(u_n) \to C \), we have
\[
\frac{1}{2}\|\dot{u}_n\|^2_{L^2} - \int_0^T F(t, u_n)dt \to C.
\]
By (F4) we have
\[
< f'(u_n), u_n > = \|\dot{u}_n\|^2_{L^2} - \int_0^T (\langle F'(t, u_n), u_n \rangle)dt
\]
By (4.6) we know
\[
\|\dot{u}_n\|^2_{L^2} - \int_0^T [\mu_2 + \mu_1 F(t, u_n)]dt.
\]
We have shown that \( \|\dot{u}_n\|_{L^2} \) is bounded, and so we can assume
\[
0 \leq \|\dot{u}_n\|_{L^2} \leq T_i, \quad 0 \leq i \leq N.
\]
Thus, if \( \{u_k\} \) is a \((CPS)_C\) sequence for \( f \), then
\[
(\tilde{u}_k \cdot e_1 + \overline{u}_k \cdot e_1 + k_1, \ldots, \tilde{u}_k \cdot e_N + \overline{u}_k \cdot e_N + k_N T_N)
\]
is also a \((CPS)_C\) sequence of \( f(u) \), so we can assume
\[
0 \leq \overline{u}_k \cdot e_i \leq T_i, \quad 0 \leq i \leq N.
\]
By (4.3), we know \( |\tilde{u}_k| \) is bounded, and so \( \|u_n\| = \|\dot{u}_n\|_{L^2} + \int_0^T u_n(t)dt \) is bounded.

The rest of the lemma can be completed in a now standard fashion.

We finish the proof of Theorem 1.4. In Rabinowitz’s Saddle Point Theorem, we take
\[
X_1 = R^N, \quad X_2 = \{ u \in W^{1,2}(R/\pi Z, R^N), \int_0^T u dt = 0 \}.
\]
For \( u \in X_2 \), we may use the Poincare-Wirtinger inequality, and so by Lemma 2.2 and (F6), we have
\[
f(u) \geq \frac{1}{2} \int_0^T |\dot{u}|^2dt - b \int_0^T |u|^2dt
\]
\[
\geq \left[ \frac{1}{2} - b(2\pi)^{-2}T^2 \right] \int_0^T |\dot{u}|^2dt
\]
\[
\geq 0.
\]
On the other hand, if \( u \in R^N \), then by (F5) we have
\[
f(u) = -\int_0^T F(t, u)dt \leq -\delta, \quad |u| = R \to +\infty.
\]
The proof of Theorem 1.4 is concluded by calling upon Rabinowitz’s Saddle Point Theorem. In fact, there is a critical point \( \tilde{u} \) such that \( f(\tilde{u}) = C > \inf_{X_2} f(u) \geq 0 \), which is nonconstant since otherwise \( f(\tilde{u}) = \int_0^T F(\tilde{u}, t)dt \leq 0 \).
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