Application of Kolmogorov complexity and universal codes to identity testing and nonparametric testing of serial independence for time series.

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Abstract

We show that Kolmogorov complexity and such its estimators as universal codes (or data compression methods) can be applied for hypotheses testing in a framework of classical mathematical statistics. The methods for identity testing and nonparametric testing of serial independence for time series are suggested.

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1 Introduction.

The Kolmogorov complexity, or algorithmic entropy, was suggested in [7] and was investigated in numerous papers; see for review [8]. Now this notation plays important role in theory of algorithms, information theory, artificial intelligence and many other fields and is closely connected with such deep theoretical issues as definition of randomness, logical basis of probability theory, randomness and complexity (see [8,10,17,18,19,20]). In this paper we show that Kolmogorov complexity can be applied to hypotheses testing in framework of mathematical statistics. Moreover, we suggest using universal codes (or methods of data compression), which are estimations of Kolmogorov complexity, for testing.

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In this paper we consider a stationary and ergodic source (or process), which generates elements from a finite set (or alphabet) $A$ and two problems of statistical testing. The first problem is the identity testing, which is described as follows: a hypotheses $H^0_{id}$ is that the source has a particular distribution $\pi$ and the alternative hypothesis $H^1_{id}$ that the sequence is generated by a stationary and ergodic source, which differs from the source under $H^0_{id}$. One particular case where the source alphabet $A = \{0, 1\}$ and the main hypothesis $H^0_{id}$ is that a bit sequence is generated by the Bernoulli source with equal probabilities of 0’s and 1’s, is applied to the randomness testing of random number and pseudorandom number generators.

The second problem is a generalization of the problem of nonparametric testing for independence of time series. More precisely, we consider two following hypotheses: $H^{ind}_0$ is that the source is Markovian, which memory (or connectivity) is not larger than $m$, $(m \geq 0)$, and the alternative hypothesis $H^{ind}_1$ that the sequence is generated by a stationary and ergodic source, which differs from the source under $H^{ind}_0$. In particular, if $m = 0$, this is the problem of testing for independence of time series. This problem is well known in mathematical statistics and there is an extensive literature dealing with nonparametric independence testing.

In both cases the testing should be based on a sample $x_1 \ldots x_t$ generated by the source.

We suggest statistical tests for identity testing and nonparametric testing of serial independence for time series, which are based on Kolmogorov complexity and such estimates of it as universal codes. It is important to note that practically used so-called archivers can be used for suggested testing, because they can be considered as methods for estimation of Kolmogorov complexity.

The outline of the paper is as follows. The next part contains definitions and necessary information. The parts three and four are devoted to the identity testing and testing of serial independence, correspondingly. The fifth part contains results of experiments, where the suggested method of identity testing is applied to pseudorandom number generators. All proofs are given in Appendix.

2 Definitions and Preliminaries.

First we define stochastic processes (or sources of information). Consider an alphabet $A = \{a_1, \ldots, a_n\}$ with $n \geq 2$ letters and denote by $A^t$ and $A^*$ the set of all words of length $t$ over $A$ and the set of all finite words over $A$, correspondingly ($A^* = \bigcup_{t=1}^{\infty} A^t$). Let $\mu$ be a source which generates letters from $A$. Formally, $\mu$ is a probability distribution on the set of words of infinite length over $A$, or, more simply, $\mu = (\mu^t)_{t \geq 1}$ is a consistent set of probabilities over the sets $A^t$; $t \geq 1$. By $M_\infty(A)$ we denote the set of all stationary and ergodic sources, which generate letters from $A$. Let $M_k(A) \subset M_\infty(A)$ be the set of Markov sources with memory.
(or connectivity) \( k, k \geq 0 \). More precisely, by definition \( \mu \in M_k(A) \) if

\[
\mu(x_{t+1} = a_{i_1}/x_t = a_{i_2}, x_{t-1} = a_{i_3}, \ldots, x_{t-k+1} = a_{i_k+1}, \ldots)
\]

\[
= \mu(x_{t+1} = a_{i_1}/x_t = a_{i_2}, x_{t-1} = a_{i_3}, \ldots, x_{t-k+1} = a_{i_k+1})
\]  

(1)

for all \( t \geq k \) and \( a_{i_1}, a_{i_2}, \ldots \in A \). By definition, \( M_0(A) \) is the set of all Bernoulli (or i.i.d.) sources over \( A \) and \( M^*(A) = \bigcup_{i=0}^{\infty} M_i(A) \) is the set of all finite-memory sources.

Now we define codes and the Kolmogorov complexity. Let \( A^\infty \) be the set of all infinite words \( x_1x_2 \ldots \) over the alphabet \( A \). A data compression method (or code) \( \varphi \) is defined as a set of mappings \( \varphi_n \) such that \( \varphi_n : A^n \to \{0,1\}^* \), \( n = 1, 2, \ldots \) and for each pair of different words \( x, y \in A^n \), \( \varphi_n(x) \neq \varphi_n(y) \). Informally, it means that the code \( \varphi \) can be applied for compression of each message of any length \( n \) over alphabet \( A \) and the message can be decoded if its code is known. It is also required that each sequence \( \varphi_n(u_1)\varphi_n(u_2)\ldots\varphi_n(u_r), r \geq 1, \) of encoded words from the set \( A^n \), \( n \geq 1, \) can be uniquely decoded into \( u_1u_2\ldots \). Such codes are called uniquely decodable. For example, let \( A = \{a, b\} \), the code \( \psi_1(a) = 0, \psi_1(b) = 00 \), obviously, is not uniquely decodable. It is well known that if a code \( \varphi \) is uniquely decodable then the lengths of the codewords satisfy the following inequality (Kraft inequality): \( \sum_{u \in A^n} 2^{-|\varphi_n(u)|} \leq 1 \), see, for ex., [2]. (Here and below \(|v|\) is the length of \( v \), if \( v \) is a word and the number of elements of \( v \) if \( v \) is a set.) It will be convenient to reformulate this property as follows:

**Claim 1.** Let \( \varphi \) be a uniquely decodable code over an alphabet \( A \). Then for any integer \( n \) there exists a measure \( \mu_\varphi \) on \( A^n \) such that

\[
|\varphi(u)| \geq - \log \mu_\varphi(u)
\]  

(2)

for any \( u \) from \( A^n \). (Here and below \( \log \equiv \log_2 \).)

(Obviously, the claim is true for the measure

\[
\mu_\varphi(u) = 2^{-|\varphi(u)|}/\sum_{u \in A^n} 2^{-|\varphi(u)|}.
\]

In this paper we will use the so-called prefix Kolmogorov complexity, whose precise definition can be found in [3] [5]. Its main properties can be described as follows. There exists a uniquely decodable code \( \kappa \) such that i) there is an algorithm of decoding (i.e. there is a Turing machine, which maps \( \kappa(u) \) to \( u \) for any \( u \in A^* \)) and ii) for any uniquely decodable code \( \psi \), whose decoding is algorithmically realizable, there exists a constant \( C_\psi \) that

\[
|\kappa(u)| - |\psi(u)| < C_\psi
\]  

(3)

for any \( u \in A^* \). The prefix Kolmogorov complexity \( K(u) \) is defined as the length of \( \kappa(u) \): \( K(u) = |\kappa(u)| \). The code \( \kappa \) is not unique, but the second property means that codelengths of two codes \( \kappa_1 \) and \( \kappa_2 \), for which i) and ii) is true, are equal up to a constant: \( |\kappa_1(u)| - |\kappa_2(u)| \mid < C_{1,2} \) for any word \( u \) (and the constant \( C_{1,2} \) does not depend on \( u \), see [3].) So, \( K(u) \) is defined up to a constant.
In what follows we call this value "Kolmogorov complexity" and uniquely decodable codes just "codes".

We can see from ii) that the code $\kappa$ is asymptotically (up to the constant) the best method of data compression, but it turns out that there is no algorithm that can calculate the codeword $\kappa(u)$ (and even $K(u)$). That is why the code $\kappa$ (and Kolmogorov complexity) cannot be used for practical data compression directly. On the other hand, so-called universal codes can be realized and, in a certain sense, can be used instead of the optimal code $\kappa$, if they are applied for compression of sequences generated by any stationary and ergodic source. For their description we recall that (as it is known in Information Theory) sequences $x_1...x_t$, generated by a source $p$, can be "compressed" till the length $-\log p(x_1...x_t)$ bits and, on the other hand, there is no code $\psi$ for which the average codeword length $\left( \Sigma_{x_1...x_t} p(x_1...x_t) |\psi(x_1...x_t)| \right)$ is less than $-\Sigma_{x_1...x_t} p(x_1...x_t) \log p(x_1...x_t)$. The universal codes can reach the lower bound $-\log p(x_1...x_t)$ asymptotically for any stationary and ergodic source $p$ with probability 1. The formal definition is as follows: A code $\varphi$ is universal if for any stationary and ergodic source $p$

$$\lim_{t \to \infty} t^{-1}(-\log p(x_1...x_t) - |\varphi(x_1...x_t)|) = 0 \quad (4)$$

with probability 1. So, informally speaking, universal codes estimate the probability characteristics of the source $p$ and use them for efficient "compression".

The main idea of the suggested test is quite natural: compress a sample sequence $x_1...x_n$ by a code $\varphi$. If the length of codeword $|\varphi(x_1...x_n)|$ is significantly less than the value $-\log \pi(x_1...x_n)$, then $H_0^{id}$ should be rejected. The main observation is that the probability of all rejected sequences is quite small for any $\varphi$, that is why the Type I error can be made small. The precise description of the test is as follows: The hypothesis $H_0^{id}$ is accepted if

$$-\log \pi(x_1...x_n) - |\varphi(x_1...x_n)| \leq -\log \alpha. \quad (5)$$

Otherwise, $H_0^{id}$ is rejected. We denote this test by $\Gamma_{\pi,\alpha,\varphi}^{(n)}$.

**Theorem 1.**

i) For each distribution $\pi, \alpha \in (0, 1)$ and a code $\varphi$, the Type I error of the described test $\Gamma_{\pi,\alpha,\varphi}^{(n)}$ is not larger than $\alpha$.  

3 Identity Testing.

Now we consider the problem of testing $H_0^{id}$ against $H_1^{id}$. Let the required level of significance (or a Type I error) be $\alpha$, $\alpha \in (0, 1)$. (By definition, the Type I error occurs if $H_0$ is true, but the test rejects $H_0$). We describe a statistical test which can be constructed based on any code $\varphi$.

The main idea of the suggested test is quite natural: compress a sample sequence $x_1...x_n$ by a code $\varphi$. If the length of codeword $|\varphi(x_1...x_n)|$ is significantly less than the value $-\log \pi(x_1...x_n)$, then $H_0^{id}$ should be rejected. The main observation is that the probability of all rejected sequences is quite small for any $\varphi$, that is why the Type I error can be made small. The precise description of the test is as follows: The hypothesis $H_0^{id}$ is accepted if

$$-\log \pi(x_1...x_n) - |\varphi(x_1...x_n)| \leq -\log \alpha. \quad (5)$$

Otherwise, $H_0^{id}$ is rejected. We denote this test by $\Gamma_{\pi,\alpha,\varphi}^{(n)}$.
ii) If, in addition, \( \pi \) is a finite-memory stationary and ergodic process (i.e. \( \pi \in M^*(A) \)) and \( \varphi \) is a universal code, then the Type II error of the test \( \Gamma_{\pi,\alpha,\varphi}^{\mathrm{ind}} \) goes to 0, when \( n \) tends to infinity.

**Remark.** The Kolmogorov complexity can be used instead of the length of a code. Namely, let \( K^{\mathrm{ind}}_{\pi,\alpha} \) be the following test: the hypothesis \( H_0^{\mathrm{ind}} \) is accepted if 
\[-\log \pi(x_1...x_n) - K(x_1...x_n) \leq -\log \alpha, \text{ otherwise, } H_0^{\mathrm{ind}} \text{ is rejected}.\]

Theorem 1 is valid for this test, too.

## 4 Testing of Serial Independence

We first give some additional definitions. Let \( v \) be a word \( v = v_1...v_k, k \leq t, v_i \in A \). Denote the rate of a word \( v \) occurring in the sequence \( x_1x_2...x_k, x_2x_3...x_{k+1}, x_3x_4...x_{k+2}, ... , x_{t-k+1}...x_t \) as \( \nu^t(v) \). For example, if \( x_1...x_t = 000100 \) and \( v = 00 \), then \( \nu^t(00) = 3 \). Now we define for any \( k \geq 0 \) the so-called empirical Shannon entropy of order \( k \) as follows:

\[
h_k^e(x_1...x_t) = -\frac{1}{(t-k)} \sum_{v \in A^k} \nu^t(v) \sum_{a \in A} (\nu^t(va)/\nu^t(v)) \log(\nu^t(va)/\nu^t(v)), \tag{6}
\]

where \( k < t \) and \( \nu^t(v) = \sum_{a \in A} \nu^t(va) \). In particular, if \( k = 0 \), we obtain

\[
h_0^e(x_1...x_t) = -\frac{1}{t} \sum_{a \in A} \nu^t(a) \log(\nu^t(a)/t),
\]

Let, as before, \( H_0^{\mathrm{ind}} \) be that the source \( \pi \) is Markovian with memory (or connectivity) not grater than \( m \), \( m \geq 0 \), and the alternative hypothesis \( H_1^{\mathrm{ind}} \) be that the sequence is generated by a stationary and ergodic source, which differs from the source under \( H_0^{\mathrm{ind}} \). The suggested test is as follows.

Let \( \psi \) be any code. By definition, the hypothesis \( H_0^{\mathrm{ind}} \) is accepted if

\[
(t - m) h_m^e(x_1...x_t) - |\psi(x_1...x_t)| \leq \log(1/\alpha), \tag{7}
\]

where \( \alpha \in (0, 1) \). Otherwise, \( H_0^{\mathrm{ind}} \) is rejected. We denote this test by \( \Upsilon_{\alpha, \psi, m}^{\mathrm{ind}} \).

**Theorem 2.** i) For any distribution \( \pi \) and any code \( \psi \) the First Type error of the test \( \Upsilon_{\alpha, \psi, m}^{\mathrm{ind}} \) is less than or equal to \( \alpha, \alpha \in (0, 1) \).

ii) If, in addition, \( \pi \) is a stationary and ergodic process over \( A^\infty \) and \( \psi \) is a universal code, then the Type II error of the test \( \Upsilon_{\alpha, \psi, m}^{\mathrm{ind}} \) goes to 0, when \( t \) tends to infinity.

**Comment.** If we use Kolmogorov complexity \( K(x_1...x_n) \) instead of the length of the code \( |\psi(x_1...x_t)| \), the obtained test will have the same properties.

## 5 Experiments

We applied the described method of identity testing to pseudorandom number generators. More precisely, we denote by \( U \) a source, which generates equiprobable and independent symbols from the alphabet \( \{0, 1\} \) and consider the hypothesis \( H_0^{\mathrm{ind}} \) that a sequence is generated by \( U \).
We have taken linear congruent generators (LCG), which are defined by the following equality

\[ X_{n+1} = (A \cdot X_n + C) \mod M, \]

where \( X_n \) is the \( n \)-th generated number \[6\]. Each such generator we will denote by \( LCG(M, A, C, X_0) \), where \( X_0 \) is the initial value of the generator. Such generators are well studied and many of them are used in practice, see \[6\].

In our experiments we extract an eight-bit word from each generated \( X_i \) using the following algorithm. Firstly, the number \( \mu = \lceil M/256 \rceil \) was calculated and then each \( X_i \) was transformed into an 8-bit word \( \hat{X}_i \) as follows:

\[
\begin{align*}
\hat{X}_i &= \lfloor X_i/256 \rfloor \quad \text{if } X_i < 256 \mu \\
\hat{X}_i &= \text{empty word} \quad \text{if } X_i \geq 256 \mu
\end{align*}
\]

(8)

Then a sequence was compressed by the archiver \( ACE v 1.2b \) (see http://www.winace.com/). Experimental data about testing of four linear congruent generators is given in the table.

| parameters / length (bits) | 400 000 | 8 000 000 |
|---------------------------|---------|-----------|
| \( 10^8 + 1, 23, 0, 47594118 \) | 390 240 | 7635936 |
| \( 2^{31}, 2^{16} + 3, 0, 1 \) | extended | 7797984 |
| \( 2^{32}, 134775813, 1, 0 \) | extended | extended |
| \( 2^{32}, 69069, 0, 1 \) | extended | extended |

So, we can see from the first line of the table that the 400000-bit sequence generated by the LCG(\( 10^8 + 1, 23, 0, 47594118 \)) and transformed according to \[8\], was compressed to a 390240-bit sequence. (Here 400000 is the length of the sequence after transformation.) If we take the level of significance \( \alpha \geq 2^{-9760} \) and apply the test \( \Gamma^\alpha_{U, \alpha, \varphi} \) \( (\varphi = ACE v 1.2b) \), the hypothesis \( H_0 \) should be rejected, see Theorem 1 and \[4\]. Analogously, the second line of the table shows that the 8000000-bit sequence generated by LCG(\( 2^{31}, 2^{16} + 3, 0, 1 \)) cannot be considered as random. \( (H_0^{rd} \) should be rejected if the level of significance \( \alpha \) is greater than \( 2^{-202016} \).) On the other hand, the suggested test accepts \( H_0^{rd} \) for the sequences generated by the two latter generators, because the lengths of the “compressed” sequences increased.

The obtained information corresponds to the known data about the generators mentioned above. Thus, it is shown in \[1\] that the first two generators are bad whereas the last two generators were investigated in \[11\] and \[9\] correspondingly, and are regarded as good. So, we can see that the suggested testing is quite efficient.
6 Appendix.

The following well known inequality, whose proof can be found in [2], will be used in proofs of both theorems.

Lemma. Let \( p \) and \( q \) be two probability distributions over some alphabet \( B \). Then \( \sum_{b \in B} p(b) \log(p(b)/q(b)) \geq 0 \) with equality if and only if \( p = q \).

Proof of Theorem 1. Let \( C_\alpha \) be a critical set of the test \( \Gamma_{\pi,\alpha,\varphi}^{(n)} \), i.e., by definition, \( C_\alpha = \{ u : u \in A^k \ \& \ - \log \pi(u) - |\varphi(u)| > - \log \alpha \} \). Let \( \mu_\varphi \) be a measure for which the claim 1 is true. We define an auxiliary set

\[
\hat{C}_\alpha = \{ u : - \log \pi(u) - (- \log \mu_\varphi(u)) > - \log \alpha \}.
\]

We have

\[
1 \geq \sum_{u \in C_\alpha} \mu_\varphi(u) \geq \sum_{u \in \hat{C}_\alpha} \pi(u)/\alpha = (1/\alpha)\pi(\hat{C}_\alpha).
\]

(Here the second inequality follows from the definition of \( \hat{C}_\alpha \), whereas all others are obvious.) So, we obtain that \( \pi(\hat{C}_\alpha) \leq \alpha \). From definitions of \( C_\alpha \), \( \hat{C}_\alpha \) and \( \pi(\hat{C}_\alpha) \leq \alpha \) we immediately obtain that \( \hat{C}_\alpha \supseteq C_\alpha \). Thus, \( \pi(C_\alpha) \leq \alpha \). By definition, \( \pi(C_\alpha) \) is the value of the Type I error. The first statement of the theorem 1 is proven.

Let us prove the second statement of the theorem. Suppose that the hypothesis \( H_1^{id} \) is true. That is, the sequence \( x_1 \ldots x_t \) is generated by some stationary and ergodic source \( \tau \) and \( \tau \neq \pi \). Our strategy is to show that

\[
\lim_{t \to \infty} - \log \pi(x_1 \ldots x_t) - |\varphi(x_1 \ldots x_t)| = \infty
\]

for probability 1 (according to the measure \( \tau \)). First we represent (9) as

\[
- \log \pi(x_1 \ldots x_t) - |\varphi(x_1 \ldots x_t)| = t \left( \frac{1}{t} \log \frac{\tau(x_1 \ldots x_t)}{\pi(x_1 \ldots x_t)} + \frac{1}{t}(- \log \tau(x_1 \ldots x_t) - |\varphi(x_1 \ldots x_t)|) \right).
\]

From this equality and the property of a universal code [1], we obtain

\[
- \log \pi(x_1 \ldots x_t) - |\varphi(x_1 \ldots x_t)| = t \left( \frac{1}{t} \log \frac{\tau(x_1 \ldots x_t)}{\pi(x_1 \ldots x_t)} + o(1) \right) \geq 0.
\]

Now we use some results of the ergodic theory and the information theory, which can be found, for ex., in [1]. Firstly, according to the Shannon-MacMillan-Breiman theorem, there exists the limit \( \lim_{t \to \infty} - \log \tau(x_1 \ldots x_t)/t \) with probability 1 and this limit is equal to the so-called limit Shannon entropy, which we denote as \( h_\infty(\tau) \). Secondly, it is known that for any integer \( k \) the following inequality is true: \( h_\infty(\tau) \leq - \sum_{v \in A^k} \tau(v) \sum_{a \in A} \tau(a/v) \log \tau(a/v) \). (Here the right hand value is called \( m \)-order conditional entropy.) It will be convenient to represent both statements as follows:

\[
\lim_{t \to \infty} - \log \tau(x_1 \ldots x_t)/t \leq - \sum_{v \in A^k} \tau(v) \sum_{a \in A} \tau(a/v) \log \tau(a/v) \quad (11)
\]
for any \( k \geq 0 \) (with probability 1). It is supposed that the process \( \pi \) has a finite memory, i.e. belongs to \( M_k(A) \) for some \( s \). Having taken into account the definition of \( M_s(A) \), we obtain the following representation:

\[
- \log \pi(x_1 \ldots x_t)/t = -t^{-1} \sum_{i=1}^{t} \log \pi(x_i/x_1 \ldots x_{i-1})
\]

\[
= -t^{-1} \sum_{i=1}^{k} \log \pi(x_i/x_1 \ldots x_{i-1}) + \sum_{i=k+1}^{t} \log \pi(x_i/x_{i-k} \ldots x_{i-1})
\]

for any \( k \geq s \). According to the ergodic theorem there exists a limit

\[
\lim_{t \to \infty} t^{-1} \sum_{i=k+1}^{t} \log \pi(x_i/x_{i-k} \ldots x_{i-1}),
\]

which is equal to \(- \sum_{v \in A^k} \tau(v) \sum_{a \in A} \tau(a/v) \log \pi(a/v)\), see \([1, 2]\). So, from the two latter equalities we can see that

\[
\lim_{t \to \infty} (- \log \pi(x_1 \ldots x_t))/t = - \sum_{v \in A^k} \tau(v) \sum_{a \in A} \tau(a/v) \log \pi(a/v).
\]

Taking into account this equality, \([1]\) and \([10]\), we can see that

\[- \log \pi(x_1 \ldots x_t) - |\varphi(x_1 \ldots x_t)| \geq t \left( \sum_{v \in A^k} \tau(v) \sum_{a \in A} \tau(a/v) \log (\tau(a/v)/\pi(a/v)) + o(t) \right)\]

for any \( k \geq s \). From this inequality and the Lemma we can obtain that

\[- \log \pi(x_1 \ldots x_t) - |\varphi(x_1 \ldots x_t)| \geq c t + o(t),\]

where \( c \) is a positive constant, \( t \to \infty \). Hence, \([9]\) is true and the theorem is proven.

**Proof of Theorem 2.** First we show that for any source \( \theta^* \in M_0(A) \) and any word \( x_1 \ldots x_t \in A^t, t > 1 \), the following inequality is valid:

\[
\theta^*(x_1 \ldots x_t) = \prod_{a \in A} (\theta^*(a))^{n(a)} \leq \prod_{a \in A} (\nu^t(a)/t)^{\nu^t(a)} \leq \prod_{a \in A} (\nu^t(a)/t)^{\nu^t(a)} \quad (12)
\]

Here the equality holds, because \( \theta^* \in M_0(A) \). The inequality follows from the Lemma. Indeed, if \( p(a) = \nu^t(a)/t \) and \( q(a) = \theta^*(a) \), then \( \sum_{a \in A} \nu^t(a) \log (\nu^t(a)/\theta^*(a)) \geq 0 \). From the latter inequality we obtain \([12]\).

Let now \( \theta \) belong to \( M_m(A), m > 0 \). We will prove that for any \( x_1 \ldots x_t \)

\[
\theta(x_1 \ldots x_t) \leq \prod_{a \in A^m} \prod_{a \in A} (\nu^t(ua)/\nu^t(u))^{\nu^t(ua)}.
\]

Indeed, we can present \( \theta(x_1 \ldots x_t) \) as

\[
\theta(x_1 \ldots x_t) = \theta(x_1 \ldots x_m) \prod_{a \in A^m} \prod_{a \in A} \theta(a/u)^{\nu^t(u)},
\]

Indeed, we can present \( \theta(x_1 \ldots x_t) \) as

\[
\theta(x_1 \ldots x_t) = \theta(x_1 \ldots x_m) \prod_{a \in A^m} \prod_{a \in A} \theta(a/u)^{\nu^t(u)},
\]

Indeed, we can present \( \theta(x_1 \ldots x_t) \) as

\[
\theta(x_1 \ldots x_t) = \theta(x_1 \ldots x_m) \prod_{a \in A^m} \prod_{a \in A} \theta(a/u)^{\nu^t(u)},
\]
where \( \theta(x_1 \ldots x_m) \) is the limit probability of the word \( x_1 \ldots x_m \). Hence, \( \theta(x_1 \ldots x_t) \leq \prod_{u \in A^m} \prod_{u \in A} \theta(a/u)^{\nu'(ua)} \). Taking into account the inequality (12), we obtain
\[
\prod_{a \in A} \theta(a/u)^{\nu'(ua)} \leq \prod_{a \in A} (\nu'(ua)/\nu(u))^\nu'(ua)
\]
for any word \( u \). So, from the last two inequalities we obtain (13).

It will be convenient to define two auxiliary measures on \( A^t \) as follows.
\[
\pi_m(x_1 \ldots x_t) = \Delta 2^{-th_m^*(x_1 \ldots x_t)}, \quad \sigma(x_1 \ldots x_t) = 2^{-|\bar{\psi}(x_1 \ldots x_t)|}
\]
where \( x_1 \ldots x_t \in A^t \) and \( \Delta = (\sum_{x_1 \ldots x_t \in A^t} 2^{-q_h^*(x_1 \ldots x_t)})^{-1} \). If we take into account that \( 2^{-(t-m)h_m^*(x_1 \ldots x_t)} = \prod_{u \in A^m} \prod_{u \in A} (\nu'(ua)/\nu'(u))^\nu'(ua) \), we can see from (13) and (14) that, for any measure \( \theta \in M_m(A) \) and any \( x_1 \ldots x_t \in A^t \),
\[
\theta(x_1 \ldots x_t) \leq \pi_m(x_1 \ldots x_t)/\Delta.
\]
Let us denote the critical set of the test \( \Upsilon_{\alpha, \sigma, m} \) as \( C_\alpha \), i.e., by definition, \( C_\alpha = \{ x_1 \ldots x_t : (t-m)h_m^*(x_1 \ldots x_t) - |\bar{\psi}(x_1 \ldots x_t)| > \log(1/\alpha) \} \). From (14) we obtain
\[
C_\alpha = \{ x_1 \ldots x_t : (t-m)h_m^*(x_1 \ldots x_t) - (-\log \sigma(x_1 \ldots x_t))) > \log(1/\alpha) \}.
\]
From (15) and (16) we can see that for any measure \( \theta \in M_m(A) \)
\[
\theta(C_\alpha) \leq \pi_m(C_\alpha)/\Delta.
\]
From (16) and (14) we obtain
\[
C_\alpha = \{ x_1 \ldots x_t : 2^{(t-m)h_m^*(x_1 \ldots x_t)} > (\alpha \sigma(x_1 \ldots x_t))^{-1} \} = \{ x_1 \ldots x_t : (\pi_m(x_1 \ldots x_t)/\Delta)^{-1} > (\alpha \sigma(x_1 \ldots x_t))^{-1} \}.
\]
Finally,
\[
C_\alpha = \{ x_1 \ldots x_t : \sigma(x_1 \ldots x_t) > \pi_m(x_1 \ldots x_t)/(\alpha \Delta) \}.
\]
The following chain of inequalities and equalities is valid:
\[
1 \geq \sum_{x_1 \ldots x_t \in C_\alpha} \sigma(x_1 \ldots x_t) \geq \sum_{x_1 \ldots x_t \in C_\alpha} \pi_m(x_1 \ldots x_t)/(\alpha \Delta) = \pi_m(C_\alpha)/(\alpha \Delta) \geq \theta(C_\alpha)/(\alpha \Delta) = \theta(C_\alpha)/\alpha.
\]
(Here both equalities and the first inequality are obvious, the second and the third inequalities follow from (13) and (14), correspondingly.) So, we obtain that \( \theta(C_\alpha) \leq \alpha \) for any measure \( \theta \in M_m(A) \). Taking into account that \( C_\alpha \) is the critical set of the test, we can see that the probability of the First Type error is not greater than \( \alpha \). The first claim of the theorem is proven.
The proof of the second statement of the theorem will be based on some results of Information Theory. The \( t \)-order conditional Shannon entropy is defined as follows:

\[
h_t(p) = - \sum_{x_1 \ldots x_t \in A^t} p(x_1 \ldots x_t) \sum_{a \in A} p(a|x_1 \ldots x_t) \log p(a|x_1 \ldots x_t),
\]

where \( p \in M_\infty(A) \). It is known that for any \( p \in M_\infty(A) \) firstly, \( \log |A| \geq h_0(p) \geq h_1(p) \geq \ldots \), secondly, there exists limit Shannon entropy \( h_\infty(p) = \lim_{t \to \infty} h_t(p) \), thirdly, \( \lim_{t \to \infty} -t^{-1} \log p(x_1 \ldots x_t) = h_\infty(p) \) with probability 1 and, finally, \( h_m(p) \) is strictly greater than \( h_\infty(p) \), if the memory of \( p \) is grater than \( m \), (i.e. \( p \in M_\infty(A) \setminus M_m(A) \)), see, for example, [1, 2].

Taking into account the definition of the universal code (4), we obtain from the above described properties of the entropy that

\[
\lim_{t \to \infty} t^{-1} |\psi(x_1 \ldots x_t)| = h_\infty(p)
\]

with probability 1. It can be seen from [19] that \( h^*_m \) is an estimate for the \( m \)-order Shannon entropy (19). Applying the ergodic theorem we obtain \( \lim_{t \to \infty} h^*_m(x_1 \ldots x_t) = h_m(p) \) with probability 1; see [11 2]. Having taken into account that \( h_m(p) > h_\infty(p) \) and (20) we obtain from the last equality that \( \lim_{t \to \infty} ((t-m) h^*_m(x_1 \ldots x_t) - |\psi(x_1 \ldots x_t)|) = \infty \). This proves the second statement of the theorem.

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