Approximation by Simple Poles—Part I: Density and Geometric Convergence Rate in Hardy Space

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Abstract—Optimal linear feedback control design is a valuable but challenging problem due to the nonconvexity of the underlying optimization and the infinite dimensionality of the Hardy space of stabilizing controllers. A powerful class of techniques for solving optimal control problems involves using reparameterization to transform the control design into a convex but infinite-dimensional optimization. To make the problem tractable, historical work focuses on Galerkin-type finite-dimensional approximations to Hardy space, especially those involving Lorentz series approximations such as the finite impulse response approximation. However, Lorentz series approximations can lead to infeasibility, difficulty incorporating prior knowledge, deadbeat control in the case of finite impulse response, and increased suboptimality. The goal of this two-part article is to introduce a new Galerkin-type method based on approximation by transfer functions with a selection of simple poles, and to apply this simple pole approximation for optimal control design. In Part I, error bounds for approximating arbitrary transfer functions in Hardy space are provided based on the geometry of the pole selection. It is shown that the space of transfer functions with these simple poles converges to the full Hardy space, and a uniform convergence rate is provided based purely on the geometry of the pole selection. This is then specialized to derive a convergence rate for a particularly interesting pole selection based on an Archimedes spiral. In Part II, the simple pole approximation is combined with system-level synthesis, a recent reparameterization approach, to develop a new control design method with desirable properties and bounded suboptimality.

Index Terms—$H_\infty$ and $H_2$ control, optimal control, system level synthesis.

I. INTRODUCTION

OPTIMAL control design for linear time-invariant systems has been thoroughly studied but remains a difficult problem. Key challenges arise from the infinite dimensionality of the Hardy space of stabilizing controllers as well as the nonconvexity of the resulting optimization problem for the design.

One of the most celebrated approaches for optimal control design is the Youla parameterization [1], which parameterizes all stabilizing controllers and leads to a convex reformulation in terms of the Youla parameter. Recently, new optimal control design techniques have been introduced, known as system level synthesis (SLS) [2], [3] and input-outputparameterization (IOP) [4], which also parameterize all stabilizing controllers and include closed-loop system responses as decision variables. However, they have some advantages over Youla parameterization, including a larger class of optimal control problems that admit convex representations and the ability to preserve structure from the closed-loop transfer functions in the internal controller realizations.

Applying these parameterizations to mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control design results in a convex but infinite-dimensional optimization problem since the decision variables are transfer functions living in an infinite-dimensional Hardy space. In order to solve these problems, historical work has focused on using finite-dimensional approximations of Hardy space based on Lorentz series approximations [5, Ch. 15], [4], [6] to obtain a tractable optimization for the design. In particular, in discrete time, the Lorentz series approximation with all poles at the origin, known as the finite impulse response (FIR) approximation, is the most popular choice, and is commonly used with Youla parameterization [7], [8], IOP [4], and SLS [9].

The rate at which the Lorentz series approximation converges determines the number of poles required to achieve a close approximation and, hence, acceptable suboptimality. For Lorentz approximations, this convergence rate typically depends on the particular optimal transfer function, and therefore may result in large numbers of poles to achieve acceptable performance. This is an important practical concern, as large numbers of poles can lead to high computational complexity for the control design, lack of robustness in the resulting controller, and implementation challenges in practice [10, Ch. 19]. For FIR, the number of poles is equal to the length of the FIR, so this becomes especially problematic when the optimal transfer function has a
long settling time, such as in systems with large separation of time scales, where short sampling times are needed to capture the fast dynamics, which are also coupled with much slower dynamics.

Lorentz series approximations can also lead to the infeasibility of the control design because all the poles in the approximation are at the same location. This occurs, for example, when the optimal transfer function is proportional to the closed-loop transfer function, and the plant has stable but uncontrollable poles. In some cases, it may be possible to recover feasibility, such as in SLS, by introducing a slack variable that enables constraint violation [9, Sec. 4.5.3], but this leads to additional suboptimality [9, Theorem 4.7] and results in a quasi-convex optimization that requires an iterative approach such as golden section search to solve.

Furthermore, since all the poles in the Lorentz series approximation are at the same location, it is unclear how to incorporate prior knowledge about the optimal solution into the control design. For example, prior information about the locations of some of the optimal closed-loop poles may be available, such as for model matching [11], model reference control [12], design based on the internal model principle [13], expensive control [14, Theorem 3.12(b)], etc. In such situations, it would be desirable to use this information to improve performance, but it is not clear how to do so with the Lorentz approximation.

In addition, FIR in particular results in deadbeat control (DBC), which often experiences poorly damped oscillations between discrete sampling times that can even persist in steady state, as well as a lack of robustness to model uncertainty and parameter variations because of the high control gains required to reach the origin in finite time [15]. So, it is often advantageous to avoid deadbeat control when possible.

This two-part work develops a new Galerkin-type method based on approximation by transfer functions with simple poles, i.e., poles with multiplicity no more than one, and uses it to develop a novel optimal control design method to address the limitations of the Lorentz approximation approach. The main contributions of our two-part paper are as follows.

In Part I, the simple pole approximation (SPA) is introduced, which gives the designer the freedom to form an approximation using any finite selection of stable poles in the unit disk that are closed under complex conjugation. It is shown that, for any transfer function in Hardy space, the approximation error from SPA is bounded, and this bound is proportional to the geometric distance between the poles of SPA and the poles of the desired transfer function. Combining these approximation error bounds with the notion of a space-filling sequence of poles in the unit disk, it is shown that the subspace of SPAs converges to the entire Hardy space. In particular, it is possible to approximate any transfer function in Hardy space arbitrarily well with respect to both the $H_2$ and $H_{\infty}$ norms using SPA. Furthermore, if the poles of a space-filling sequence converge to the entire unit disk at a certain rate, which we term a geometric convergence rate, then it is shown that SPA converges to any transfer function in Hardy space at this same uniform convergence rate, which depends purely on the geometry of the pole selection. Therefore, unlike with Lorentz approximations in general, and FIR in particular, the convergence rate of SPA is not reduced for transfer functions with long settling times, such as those resulting from systems with large separation of time scales. Finally, a particularly interesting space-filling sequence of poles is introduced based on the Archimedes spiral, and used to provide an explicit uniform convergence rate for this special case based on the spiral geometry.

In Part II [16], SPA is combined with SLS to develop a new control design method which has a uniform convergence rate, works well for systems with large separation of time scales, automatically ensures feasibility for stabilizable plants, is convex and tractable, and is not FIR, so it does not result in deadbeat control. Suboptimality bounds with a uniform convergence rate are provided for the method, and its superior performance is demonstrated on an example of converter design control.

The main connection between Part I and Part II is that the control design method and suboptimality certificates developed in Part II rely heavily on the SPA method, bounded approximation error results, and convergence certificates developed in Part I. Furthermore, the example of Part II demonstrates the practical use of the SPA method from Part I for control design, and its superior performance compared to the FIR approach for that case. As the SPA method and results provided in Part I are developed in generality, and not specialized to SLS with state feedback until Part II, they may be of independent interest for other control design techniques as well, such as IOP, Youla parameterization, and SLS with output feedback. In turn, the derivation of the suboptimality bounds in Part II may provide a general recipe for deriving similar suboptimality bounds for these other control design methods as well.

The rest of this article is organized as follows. Section II provides some notation and preliminaries. Section III provides the development of the SPA method as well as the main results regarding bounded approximation error, convergence of the SPA to all of Hardy space, uniform convergence rate, and specialization of these to the Archimedes spiral pole selection. Section V gives the proofs of the theoretical results. Finally, Section VI concludes this article.

II. PRELIMINARIES AND NOTATION

We will use the following notation throughout the article. Let $\mathbb{D}$ be the open unit disk in the complex plane, $\partial \mathbb{D}$ the closed unit disk, and $\overline{\mathbb{D}}$ the unit circle. Let $\overline{B}_r$ be the closed ball of radius $r$ centered at the origin. For any set $S \subset \mathbb{C}$ and any $z \in \mathbb{C}$, let $d(z,S) = \inf_{w \in S} |z - w|$. For any sets $S, S' \subset \mathbb{C}$, let $d(S,S') = \inf_{w \in S} \inf_{v \in S'} |w - v|$. For any set $S \subset \mathbb{C}$ and $\epsilon > 0$, define the $\epsilon$ neighborhood of $S$, denoted $S_{\epsilon}$, to be the set $\{z \in \mathbb{C} : d(z,S) \leq \epsilon\}$. For any nonempty sets $S, S' \subset \mathbb{C}$, define the Hausdorff distance $d_H(S,S')$ to be the infimum over $\epsilon > 0$ such that $S \subset S'_{\epsilon}$ and $S' \subset S_{\epsilon}$. Then $d_H$ is a well-defined metric, and we say that a sequence of sets $\{S_n\}_{n=1}^{\infty}$ converges to the set $S^*$, denoted $S_n \to S^*$, if $\lim_{n \to \infty} d_H(S_n,S^*) = 0$. For any $z \in \mathbb{C}$, let $\overline{\mathbb{C}}$ denote the complex conjugate of $z$, and let $\Re(z)$ and $\Im(z)$ denote the real and imaginary components of $z$, respectively. For any positive integer $m$, let $I_m = \{1,2,\ldots,m\}$. For any set $S$,
let \(|S|\) be the cardinality of set \(S\) (i.e., the number of elements it contains).

For a matrix \(M\), let \(|M|_2\) denote the spectral norm of \(M\), and let \(|M|_F\) denote the Frobenius norm of \(M\). Let \(M_{i,j}\) denote the entry of \(M\) at row \(i\) and column \(j\).

Let \(\mathcal{RH}_\infty\) be the Hardy space of real, rational, proper, and stable transfer functions in discrete time (see [10] and [17] for a more detailed discussion). Let \(m \times n\) denote the dimensions of each transfer function in \(\mathcal{RH}_\infty\). Let \(\frac{1}{2}\mathcal{RH}_\infty \subset \mathcal{RH}_\infty\) consist of the strictly proper transfer functions in \(\mathcal{RH}_\infty\). For \(S \in \mathcal{RH}_\infty\), define the norms

\[
||S||_{\mathcal{RH}_\infty} = \sup_{z \in \mathbb{D}} ||S(z)||_2, \quad ||S||^2_{\mathcal{RH}_\infty} = \frac{1}{2\pi} \int_{z \in \mathbb{D}} ||S(z)||^2_2 dz.
\]

For any \(S, S' \in \mathcal{RH}_\infty\), define the metrics

\[
d_{\mathcal{RH}_\infty}(S, S') = ||S - S'||_{\mathcal{RH}_\infty}, \quad d_{\mathcal{RH}'}(S, S') = ||S - S'||_{\mathcal{RH}'},
\]

For any \(S \in \mathcal{RH}_\infty\) and positive integer \(k\), let \(\mathcal{I}(S)(k)\) denote the time-domain impulse response of \(S\) at time \(k\). For \(S \in \mathcal{RH}_\infty\), by Parseval’s theorem it can be shown that

\[
||S||^2_{\mathcal{RH}_\infty} = \sum_{k=1}^{\infty} ||\mathcal{I}(S)(k)||^2_2 =: ||\mathcal{I}(S)||^2_2 = \lim_{T \rightarrow \infty} ||\mathcal{I}(S)(T)||^2_2,
\]

where \(\mathcal{I}(S) := [S(1)^T \ S(2)^T \ \cdots \ S(T)^T]^T\), which can be well approximated numerically using \(T\) finite. For any \(S \in \mathcal{RH}_\infty\) let \(\mathcal{C}(S)\) denote the convolution operator of \(S\) so that for any input signal \(u(k)\) and nonnegative integer \(n\), \(\mathcal{C}(S)(u)[n] = \sum_{k=1}^{\infty} \mathcal{I}(S)(k)u(n-k)\). Note that for \(S \in \mathcal{RH}_\infty\), since \(||\cdot||_{\mathcal{RH}_\infty}\) is the induced \(L_2 \rightarrow L_2\) norm, it can be shown that

\[
||S||_{\mathcal{RH}_\infty} = \sup_{\|u\|_2 \neq 0} ||\mathcal{C}(S)(u)||_2 =: ||\mathcal{C}(S)||_2 = \lim_{T \rightarrow \infty} ||\mathcal{C}(S)(T)||_2
\]

\[
eq [S(1) \ 0 \ 0 \ 0 \\
S(2) \ S(1) \ 0 \ 0 \\
\vdots \ \ \ \ \ \ \vdots \ 0 \\
S(T) \ S(T-1) \ \cdots \ S(1)]
\]

which can be well approximated numerically using \(T\) finite.

For any \(S \in \frac{1}{2}\mathcal{RH}_\infty\), let \(\mathcal{Q}\) be the poles of \(S\). For each pole \(q \in \mathcal{Q}\), let \(m_q\) be its multiplicity in \(S\) and let \(m_{\max}\) be the maximum multiplicity, i.e., \(m_{\max} = \max_{q \in \mathcal{Q}} m_q\). Then the partial fraction decomposition of \(S\) can be written

\[
S(z) = \sum_{q \in \mathcal{Q}} \sum_{j=1}^{m_q} \frac{G_{(q,j)}}{(z - q)^j}
\]

for some constant coefficient matrices \(G_{(q,j)}\).

We consider another notion of set convergence, which will be useful when proving density of the simple pole approximation in Hardy space. Let \(U\) be a metric space with metric \(\rho\), let \(S \subset U\), and let \(\{S_n\}_{n=1}^{\infty}\) be a sequence of subsets of \(U\). Define \(\liminf_{n \rightarrow \infty} S_n\) with respect to \(\rho\) to be the set of points \(x \in U\) such that there exists a sequence \(\{x_n\}_{n=1}^{\infty}\) with \(x_n \in S_n\) for all \(n\) such that \(\liminf_{n \rightarrow \infty} x_n = x\) with respect to \(\rho\). Define \(\limsup_{n \rightarrow \infty} S_n\) with respect to \(\rho\) to be the set of points \(x \in U\) such that there exists a subsequence \(\{S_{n_m}\}_{m=1}^{\infty}\) of \(\{S_n\}_{n=1}^{\infty}\) and a sequence \(\{x_{n_m}\}_{m=1}^{\infty}\) such that \(x_{n_m} \in S_{n_m}\) for all \(m\) and \(\lim_{m \rightarrow \infty} x_{n_m} = x\) with respect to \(\rho\). If \(\liminf_{n \rightarrow \infty} S_n = \limsup_{n \rightarrow \infty} S_n\), say \(\liminf_{n \rightarrow \infty} S_n = \limsup_{n \rightarrow \infty} S_n = S\), then we say that \(\{S_n\}_{n=1}^{\infty}\) converges to \(S\), and write \(\lim_{n \rightarrow \infty} S_n = S\), with respect to \(\rho\). Note that in the special case where \(U\) is compact, \(\lim_{n \rightarrow \infty} S_n = S\) if and only if \(\lim_{n \rightarrow \infty} \rho_H(S_n, S) = 0\) [18, Sec. 28, Statement V], so this notion of set convergence is a natural extension to noncompact metric spaces of convergence with respect to the Hausdorff distance.

### III. MAIN RESULTS

#### A. Approximation by Simple Poles

We begin by showing that transfer functions with only simple poles (i.e., poles with multiplicity no greater than one) can be used to approximate any transfer function in \(\frac{1}{2}\mathcal{RH}_\infty\), including those with repeated poles, to arbitrary accuracy. Toward that end, let \(S \in \frac{1}{2}\mathcal{RH}_\infty\) and let \(\mathcal{Q}\) denote the poles of \(S\). Let \(P\) be a set of simple and distinct poles, hereafter referred to as approximating poles, which will be used to construct a transfer function for approximating \(S\) (i.e., \(S \approx \sum_{p \in P} G_p \frac{1}{z - p}\)). The key idea is that for each pole \(q \in \mathcal{Q}\), an approximating transfer function is constructed to approximate \(q\)’s contribution to the partial fraction decomposition of \(S\). The poles of this approximation are selected to be the \(m_q\) closest poles in \(P\) to \(q\), which we denote by \(P(q)\). Then, the overall approximating transfer function for \(S\) is obtained by summing over the individual approximating transfer functions for each \(q \in \mathcal{Q}\).

To evaluate the accuracy of the approximation, for each \(q \in \mathcal{Q}\) let \(\delta(q)\) be the distance from \(q\) to the furthest of the \(m_q\) simple poles being used to approximate it, i.e., \(\delta(q) = \max_{p \in P(q)}|p - q|\). Let \(D(P)\) be the maximum of these distances over all the poles in \(\mathcal{Q}\), i.e., \(D(P) = \max_{q \in \mathcal{Q}} \delta(q)\). Then \(D(P)\) represents the largest distance between approximating poles in \(P\) and the poles in \(\mathcal{Q}\) they are being used to approximate, so it measures the worst-case geometric error in this pole approximation. Intuitively one might therefore expect that as \(D(P) \rightarrow 0\), the approximating transfer function would approach \(S\). This intuition is formalized in Theorem 1, which provides an approximation error bound in terms of standard Hardy space norms of the simple pole approximation, which is linear in \(D(P)\). Thus, Theorem 1 shows that the simple pole approximating transfer function converges to \(S\) at least linearly with \(D(P)\) and, therefore, that this convergence rate depends purely on the geometry of the pole selection. Before presenting Theorem 1, we make the following assumptions regarding the set of approximating poles \(P\).

(A1) There exists some \(r \in (0, 1)\) such that \(P \subset B_r\).

(A2) \(|P| \geq m_{\max}\).

(A3) \(P\) is closed under complex conjugation (i.e., \(p \in P\) implies that \(\bar{p} \in P\)).

(A4) \(|D(P)| < 1\).

(A5) Let \(\sigma \in \mathbb{C}\) be finite. Then for every \(q \in \mathcal{Q}\) and every \(\lambda \in \sigma\) with \(\lambda \neq q\), \(\lambda \notin P(q)\).

Assumption A1 ensures that \(P\) consists of stable poles, Assumption A2 ensures that the size of \(P\) is at least as large as \(m_{\max}\), Assumption A3 that \(P\) can be used to construct a transfer
function with real coefficients (as will be seen in the proof of Theorem 1), and Assumption A4 that \( P \) is not excessively far from \( Q \). Note that Assumption A4 can be satisfied with only two pairs of complex conjugate poles if \( S \) has only simple poles, so it tends not to be restrictive in practice. Assumption A5 is introduced here in anticipation of its use in [16, Theorem 1], where \( \sigma \) will represent the poles of the plant, and can be easily satisfied in practice. For its use in [16, Theorem 1], define \( \delta = \min_{p \in Q, \lambda \in \sigma, p \neq q} d(\lambda, P(q)) \), and note that \( \delta > 0 \) by Assumption A5. We are now ready to present Theorem 1.

**Theorem 1 (Simple pole approximation):** Let \( S \in \mathbb{C}^{ \infty } \) and let \( P \) be a set of poles satisfying Assumptions A1–A4. Then there exist constants \( c_S = c_S(Q, G^*_S(q,j), r) > 0 \) and \( c'_S = c'_S(Q, G^*_S(q,j), r) > 0 \), and constant matrices \( \{G_p\}_{p \in P} \) such that

\[
\sum_{p \in P} G_p \frac{1}{z - p} - S \mid_{\mathcal{H}_2} \leq cSD(P)
\]

and

\[
\sum_{p \in P} G_p \frac{1}{z - p} - S \mid_{\mathcal{H}_\infty} \leq c'_SD(P).
\]

(1)

This theorem shows that a transfer function with only simple poles, namely, \( \sum_{p \in P} G_p \), can be approximated by any transfer function \( S \) in the Hardy space \( \mathbb{C}^{ \infty } \) to arbitrary accuracy in the \( \mathcal{H}_2 \) and \( \mathcal{H}_\infty \) norms as long as each pole \( q \) in \( S \) has \( m \) poles in \( P \) that are sufficiently close to it. Note that the constants \( c_S \) and \( c'_S \) appearing in Theorem 1 depend only on \( S \in \mathbb{C}^{ \infty } \) and on the radius \( r \) of the closed ball \( B_r \) in which \( P \) is contained, and do not otherwise depend on the specific pole selection \( P \). This feature will play a crucial role in the proofs of the density and uniform convergence results of Section III-B.

**B. Density and Uniform Convergence Rate**

The goals of this section are to show that simple pole approximations using suitable pole selections are dense in the Hardy space \( \mathbb{C}^{ \infty } \), and that a uniform convergence rate of the simple pole approximation to any transfer function in \( \mathbb{C}^{ \infty } \) can be provided, which depends only on the geometry of the pole selection. Toward that end, we define a sequence of poles, denoted \( \{P_n\}_{n=1}^{\infty} \), to be a sequence where for each \( n \), \( P_n \) is a finite collection of poles contained in \( \mathbb{D} \) and closed under complex conjugation. We say that a sequence of poles \( \{P_n\}_{n=1}^{\infty} \) exhibits geometric convergence if, for any \( S \in \mathbb{C}^{ \infty } \), its worst-case geometric approximation error converges to zero, i.e., \( \lim_{n \to \infty} D(P_n) = 0 \). This definition relates naturally to the approximation error bound provided in Theorem 1. However, it is valuable to introduce the related notion of a space-filling sequence of poles. In particular, we say that a sequence of poles \( \{P_n\}_{n=1}^{\infty} \) is space-filling if it is a sequence of pole selections such that \( P_n \to \mathbb{D} \) with respect to the Hausdorff distance, i.e., the pole sequence converges to the entire unit disk. A particularly interesting example of a space-filling sequence based on the Archimedes spiral is provided in Section III-C.

Unlike geometric convergence, the notion of space-filling is more intuitive to visualize and understand, avoids an abstract definition in terms of transfer functions, is easier to establish in practice, and will serve to illuminate the connection between convergence of the pole sequence to the entire unit disk and convergence of the space of simple pole approximations to the entire Hardy space \( \mathbb{C}^{ \infty } \). Therefore, it is natural that Theorem 2, which establishes the convergence of the space of simple pole approximations to \( \mathbb{C}^{ \infty } \), is shown for any space-filling sequence of poles.

**Lemma 1 (Equivalence of space-filling and geometric convergence):** Let \( \{P_n\}_{n=1}^{\infty} \) be a sequence of poles. Then \( \{P_n\}_{n=1}^{\infty} \) is space-filling if and only if it exhibits geometric convergence.

For any sequence of poles \( \{P_n\}_{n=1}^{\infty} \), for each \( n \) let

\[
A_n = \left\{ \sum_{p \in P_n} G_p \frac{1}{z - p} \in \mathbb{C}^{ \infty } : G_p \in \mathbb{C}^{n \times n} \right\}
\]

Then \( A_n \subset \mathbb{C}^{ \infty } \) denotes the space of simple pole approximations resulting from the pole selection \( P_n \). Theorem 2 shows the density of the simple pole approximation using space-filling sequences of poles in the Hardy space \( \mathbb{C}^{ \infty } \). For the statement of Theorem 2, it is useful to recall the notion of set convergence defined in Section II.

**Theorem 2 (Density in Hardy Space):** Let \( \{P_n\}_{n=1}^{\infty} \) be a space-filling sequence of poles. Then

\[
\lim_{n \to \infty} \mathcal{A}_n = \frac{1}{2} \mathbb{C}^{ \infty }
\]

with respect to \( d_{\mathcal{H}_2} \) and \( d_{\mathcal{H}_\infty} \).

For any \( k > 0 \), we say that a sequence of poles \( \{P_n\}_{n=1}^{\infty} \) has a geometric convergence rate \( \frac{1}{n^k} \) if for each \( S \in \frac{1}{2} \mathbb{C}^{ \infty } \), there exists a constant \( c_S > 0 \) such that \( D(P_n) \leq \frac{c_S}{n^k} \) for all positive integers \( n \). Note that this is in fact a uniform convergence rate since the rate \( k \) is independent of the choice of \( S \in \frac{1}{2} \mathbb{C}^{ \infty } \). Theorem 3 shows that if a sequence of poles has geometric convergence rate \( \frac{1}{n^k} \), then for any \( S \in \frac{1}{2} \mathbb{C}^{ \infty } \), the simple pole approximation converges to \( S \) in the Hardy space norms at the rate \( \frac{1}{n^k} \).

**Theorem 3 (Uniform convergence rate):** For some \( k > 0 \), let \( \{P_n\}_{n=1}^{\infty} \) be a sequence of poles with geometric convergence rate \( \frac{1}{n^k} \). Then for any \( S \in \frac{1}{2} \mathbb{C}^{ \infty } \), there exist constants \( c_S = c_S(Q, G^*_S(q,j), r) > 0 \) and \( c'_S = c'_S(Q, G^*_S(q,j), r) > 0 \), and \( N > 0 \) such that for any \( n \geq N \) there exist \( \{G^*_n\}_{p \in P} \) such that...
\[ \sum_{p \in \mathbb{P}_n} G_p^n \frac{1}{z - p} \in \frac{1}{z} \mathcal{H}_\infty \] and

\[ \left\| \sum_{p \in \mathbb{P}_n} G_p^n \frac{1}{z - p} - S \right\|_{\mathcal{H}_2} \leq \frac{c_S}{n^k} \]

\[ \left\| \sum_{p \in \mathbb{P}_n} G_p^n \frac{1}{z - p} - S \right\|_{\mathcal{H}_\infty} \leq \frac{c_S'}{n^k}. \]

Note that \( N \) in Theorem 3 only needs to be chosen to ensure \( D(\mathbb{P}_N) < 1 \) and \( |\mathbb{P}_N| \geq m_{\max} \), which if \( S \) has only simple poles can be satisfied with only two pairs of complex conjugate poles (i.e., \( N = 4 \)), so it tends not to be restrictive in practice.

### C. Archimedes Spiral Pole Selection

As discussed in Section I, many control design approaches to mixed \( \mathcal{H}_2/\mathcal{H}_\infty \) control design (see [16, eq. (2)]) for a detailed formulation) require Galerkin-type finite-dimensional approximations of Hardy space \( \frac{1}{z} \mathcal{H}_\infty \). Such methods implicitly seek a ground-truth optimal transfer function lying in \( \frac{1}{z} \mathcal{H}_\infty \), which is, in general, unknown a priori. By Theorem 2, using the simple pole approximation (SPA) with any space-filling sequence of poles is guaranteed to converge to this optimal transfer function as the number of poles in the approximation increases.

In this section, we provide recommendations for achieving fast convergence rates, which typically leads to improved performance of the control design. Motivated by Section III-B, to use SPA as well as possible the goal is to choose a pole selection whose poles are as close as possible to the poles of an optimal transfer function. In the cases of system-level synthesis (which will be explored further in Part II [16]) and input–output parameterization, these optimal transfer functions are optimal closed-loop responses. Let \( \mathcal{Q} \) denote the poles of the optimal closed-loop transfer function from disturbance to control input (see \( T_{v \rightarrow u}(z) \) in [16, Sec. II.A] for a detailed definition), and let \( \sigma \) denote the stable poles of the plant. When the control synthesis is feasible (i.e., the plant is stabilizable), optimal closed-loop transfer functions exist (see [16, Assumption A6] and the following discussion) and have a finite collection of poles, all of which are contained in \( \mathcal{Q} \cup \sigma \) (by the proof of [16, Lemma 1]). So, in many cases, it is desirable to choose a pole selection as close as possible to \( \mathcal{Q} \cup \sigma \). Therefore, we recommend to first include the stable poles of the plant \( \sigma \) in SPA, as well as any optimal poles that are known a priori, such as discussed in Section I. In case the plant has any unstable poles, these should not be included in the pole selection for system level synthesis or input–output parameterization methods since this can result in closed-loop instability. Further details are available in Part II [16].

Once the prior information about the optimal and plant poles have been incorporated into SPA, if no information about the remaining optimal poles is known in advance, a natural choice for these remaining poles in SPA would be to distribute the poles evenly over the unit disk so as to minimize \( D(\mathbb{P}_n) \) (i.e., finding \( \mathbb{P}_n \) for each \( n \) such that \( D(\mathbb{P}_n) \) is minimized for the given number of poles in \( \mathbb{P}_n \)). However, finding an exactly even pole distribution over the unit disk is equivalent to finding the minimum energy configuration of a collection of identical point charges over a disk. This is a nonconvex optimization problem that has never been solved in the general case, requires high computational effort to solve even for smaller numbers of poles, and for which even when solutions can be obtained, they are not guaranteed to be globally optimal [19]. Therefore, rather than attempting to find exactly even pole distributions, we resort to finding approximate solutions instead. These approximations also have the advantage that, unlike for the exact solution methods, they can be used to derive approximation error bounds with a geometric convergence rate based on Theorem 3.

One class of heuristic techniques that has been used to generate approximately evenly spaced points over a disk, and that has worked well in practice in a variety of different fields, is to select points along a spiral [20], [21], [22], transforming the pole selection problem from a 2-D selection over a disk to a 1-D selection along a spiral curve. After selecting a particular spiral, typically specified in polar coordinates by \( r = r(\theta) \), appropriate points can then be chosen along the spiral to yield an approximately even distribution. Therefore, we choose the poles in SPA along an Archimedes spiral according to the selection proposed in [22, Sec. 5.1] and as shown in Fig. 1. This yields a powerful heuristic for minimizing \( D(\mathbb{P}_n) \) which tends to work well in practice (see, for example, [16, Sec. IV]), and for which it is possible to derive approximation error bounds with a geometric convergence rate of \( \frac{1}{n^{1/2}} \) based on the spiral geometry (see Corollary 1). The spiral is then reflected over the imaginary axis to ensure that \( \mathbb{P}_n \) is closed under complex conjugation. Letting \( m \) be the total number of poles in \( \mathbb{P}_n \), the poles chosen for this selection (as shown in Fig. 1) are given by

\[ \theta_k = k \frac{2\pi}{m} \]

\[ r_k = \sqrt{\frac{k}{m} + 1} \]

\[ p_k = (r_k, \theta_k) \]

\[ p_{-k} = (r_k, -\theta_k) \]

\[ (2) \]
for \( k \in \{1, \ldots, \frac{n}{2}\} \). The poles are selected in this fashion (i.e., choosing \( r_k \) proportional to \( \sqrt{k} \)) in order to obtain more uniform coverage of the disk. The Archimedes spiral was selected because it has constant distance between its windings, and the spacing between consecutive poles chosen along the spiral converges to the distance between consecutive windings as \( n \) goes to infinity, both of which are desirable space-filling properties.

Theorem 4 shows that the Archimedes spiral pole selection of (2) yields a space-filling sequence of poles with a geometric convergence rate \( \frac{1}{n^{1/2}} \).

**Theorem 4 (Spiral geometric convergence rate):** For each integer \( n \geq 2 \), let \( \mathcal{P}_n \) denote the selection of \((2n-2)\) poles along an Archimedes spiral given by \( p_k \) in (2) for \( k \in \{-(n-1), \ldots, -1, 1, \ldots, n-1\} \). Then \( \{\mathcal{P}_n\}_{n=2}^{\infty} \) is a space-filling sequence of poles and has a geometric convergence rate \( \frac{1}{n^{1/2}} \). In particular, for \( S \in \frac{1}{2}\mathcal{R} \mathcal{C}_\infty \) it satisfies

\[
D(\mathcal{P}_n) \leq c_S \frac{n^{1/2}}{n^{1/2}}
\]

\[
c_S = \sqrt{\pi} \left(1 + m_{\text{max}}\right)
\]

for all positive integers \( n \geq 2 \).

As shown in Section IV, the decay rate of Theorem 4 seems to accurately reflect the decay rate found in numerical experiments, indicating that the approximation is rate-optimal, although the constant \( c_S \) is often conservative. Corollary 1 provides a uniform convergence rate of \( \frac{1}{n^{1/2}} \) for the simple pole approximation using the Archimedes spiral pole selection of (2) for any transfer function in \( \frac{1}{2}\mathcal{R} \mathcal{C}_\infty \).

**Corollary 1 (Spiral approximation convergence rate):** Consider the pole selection of Theorem 4. Then for any \( S \in \frac{1}{2}\mathcal{R} \mathcal{C}_\infty \) there exist constants \( c_S = c_S(Q, G_{(q,j)}^*) > 0 \), \( c'_S = c'_S(Q, G_{(q,j)}^*) > 0 \), and \( N > 0 \) such that for each integer \( n \geq N \) there exist \( \{G_p^p\}_{p \in \mathcal{P}_n} \) such that

\[
\left| \sum_{p \in \mathcal{P}_n} G_p^p \frac{1}{z - p} - S \right| \leq c_S \frac{n^{1/2}}{n^{1/2}}
\]

\[
\left| \sum_{p \in \mathcal{P}_n} G_p^p \frac{1}{z - p} - S \right| \leq c'_S \frac{n^{1/2}}{n^{1/2}}.
\]

Note that \( N \) in Corollary 1 is chosen only to satisfy the conditions in the remark after Theorem 3, and so is typically small in practice (i.e., \( N \approx 4 \)).

For comparison, Galerkin-type approximations based on Lorentz series, such as FIR, often have exponential convergence rates of the form \( \frac{1}{p^r} \) for some \( p \in (0, 1) \), which is a faster convergence rate than in Corollary 1. However, this rate \( p \) typically depends on the decay rate of the optimal transfer function, which may be very slow in practice, especially for systems with large separation of time scales. Additionally, for this convergence rate to apply, it may require \( n \) sufficiently large such that \( \rho^n \) has decayed sufficiently (see, [9, Theorem 4.7]), and so may not apply in practice (see, for example, [16, Sec. IV]). In contrast, the convergence rate in Corollary 1 is uniform over all \( S \in \frac{1}{2}\mathcal{R} \mathcal{C}_\infty \), and so does not depend on the decay rate of the optimal transfer function, and therefore tends to work well even for systems with large separation of time scales (see, for example, [16, Sec. IV]). In addition, if some optimal poles can be included in the SPA due to prior knowledge (see Section I), this will typically have the effect of decreasing the constants \( c_S \) and \( c'_S \), as well as reducing \( D(\mathcal{P}_n) \), resulting in faster convergence. For approximations based on the Lorentz series, it is unclear how such prior information could be included to improve the convergence rate. Furthermore, when the optimal transfer function is approximated represents a closed-loop response, such as for system level synthesis and input–output parameterization, Lorentz series approximations are generally unable to include the poles of the plant, which are stable but uncontrollable. In such cases, the control design may be infeasible, so additional slack variables and constraint violations would have to be introduced ([9, eq. (4.36)], further reducing the convergence rate to the optimal transfer function [9, Theorem 4.7]). For SPA, the poles of the plant can be automatically incorporated, so the feasibility of the control design is guaranteed for stabilizable plants, and the reductions in convergence rate resulting from introducing slack variables can be avoided. The numerical experiments shown in Section IV seem to indicate that the convergence rate of the approximation error is faster than shown in Corollary 1 for small numbers of poles, appearing to be exponential initially, but seems to approach the rate given in Corollary 1 asymptotically as the number of poles grows large. The constant \( c_S \) also appears to be conservative.

**IV. NUMERICAL EXAMPLE**

To illustrate the main results of Section III, we consider the simple pole approximation of the following transfer function using the Archimedes spiral:

\[
T_{\text{des}}(z) = G_{(p_1, 2)} \frac{1}{(z - p_1)^2} + G_{(p_2, 1)} \frac{1}{z - p_1^2} + G_{(p_2, 1)} \frac{1}{z - p_2^2} + G_{(p_2, 1)} \frac{1}{z - p_2^2}
\]

\[
G_{(p_1, 2)} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} j
\]

\[
G_{(p_2, 1)} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} j
\]

\[
p_1 = 0.24 + 0.125j
\]

\[
p_2 = -0.05 + 0.15j.
\]

Note that \( T \) is chosen to have repeated poles and multiple poles in order to illustrate the main results. The poles \( \mathcal{P}_n \) for the simple pole approximation are selected along an Archimedes spiral according to (2), and the simple pole approximation is then given by

\[
S(z) = \sum_{p \in \mathcal{P}_n} G_p^p \frac{1}{z - p}
\]
where \( \{G_p\}_{p \in \mathcal{P}_n} \) are variable coefficients. To find the best approximation of \( T_{\text{des}} \) using the simple pole approximation, we solve the following unconstrained finite-dimensional optimization problem:

\[
\min_{G_p} \left\| \sum_{p \in \mathcal{P}_n} G_p \frac{1}{z - p} - T_{\text{des}}(z) \right\|_{2 \mathcal{H}_2}.
\]  
\( (3) \)

To solve this, we use the method detailed in Section II for computing the \( \mathcal{H}_2 \) norm using \( \mathcal{B}_T \) for \( T \) sufficiently large. In this case, we focus on the \( \mathcal{H}_2 \) norm because of favorable numerical properties that allow it to scale well to large numbers of poles for illustration purposes, as shown in the example in Part II [16], but similar results also hold when using the \( \mathcal{H}_\infty \) norm.

Fig. 2 shows a comparison between the worst-case distances \( D(\mathcal{P}_n) \) of the solutions of (3) and their theoretical upper bound from Theorem 4 as a function of the number of approximating poles \( n \). As expected, the actual distances of the optimal simple pole approximations lie beneath their theoretical upper bounds. Furthermore, although there is a gap between the optimal distances and their upper bounds, both exhibit similar decay rates with the number of poles, which suggests that the approximation is rate-optimal and that conservativeness in the bounds comes primarily from the constant \( c_S \) in Theorem 4 rather than the \( \frac{1}{n^{1/2}} \) decay rate.

Fig. 3 shows the approximation error between \( T_{\text{des}} \) and the simple pole approximation, as given by the optimal solution to (3), as well as its theoretical upper bound from Corollary 1, as a function of the number of approximating poles \( n \). While the theoretical upper bound decreases monotonically with the number of approximating poles, the approximation error for the optimal solution of (3) does not because the poles from spirals with a larger \( n \) do not contain the poles from spirals with a smaller \( n \). Thus, it is possible that spirals with smaller \( n \) might yield better approximations than with larger \( n \), although asymptotically as \( n \) goes toward infinity the approximation error must converge to zero. Overall, since the approximation error decays more rapidly than the worst-case distances \( D(\mathcal{P}_n) \), this suggests that the optimal solution to (3) relies on poles beyond the geometrically closest approximating poles to each desired pole in \( T_{\text{des}} \) to approximate the contribution of that desired pole to \( T_{\text{des}} \). However, after the number of poles gets sufficiently large, in this case around \( n = 50 \), the benefit of these more distant poles quickly reduces and the geometrically closest poles appear to dominate again. So, for \( n < 50 \), the approximation error decays approximately exponentially fast, whereas for \( n > 50 \) the decay rate is approximately equal to the decay rate of the theoretical upper bound. The initial mismatch between the approximation error and its theoretical bound likely comes from the conservativeness in the constant \( c_S \) appearing in Corollary 1.

V. PROOFS

A. Proof of Theorem 1

The proof of Theorem 1 proceeds by bounding the error in the simple pole approximation in terms of the distance between the approximating poles and the repeated poles they are approximating. The next two lemmas provide useful identities for developing approximation error bounds for a single repeated pole in the SISO case.

Lemma 2: For any positive integer \( m \), let \( p_1, \ldots, p_m, q \in \mathbb{D} \), and let \( I_m = \{1, 2, \ldots, m\} \). Then

\[
q^m - \prod_{i=1}^{m} p_i \leq \sum_{k=1}^{m} \sum_{S \subset I_m} |q|^{m-k} \prod_{i \in S} |p_i - q|.
\]

Proof of Lemma 2: We compute

\[
\prod_{i=1}^{m} p_i = \prod_{i=1}^{m} ((p_i - q) + q) \quad \text{distributive property} \quad \sum_{k=0}^{m} \sum_{S \subset I_m} |q|^{m-k} \prod_{i \in S} (p_i - q)
\]
which implies that
\[
\left| q^m - \prod_{i=1}^{m} p_i \right| \overset{\text{above identity}}{=} \left| q^m - \sum_{k=0}^{m} \sum_{S \subset I_m} \sum_{|S|=k} q^{m-k} \prod_{i \in S} (p_i - q) \right|
\]
canceling \( q^m \) \( \Longleftrightarrow \)
\[
\sum_{k=1}^{m} \sum_{S \subset I_m} q^{m-k} \prod_{i \in S} (p_i - q) \leq \sum_{k=1}^{m} \sum_{S \subset I_m} \left| q \right|^{m-k} \prod_{i \in S} \left| p_i - q \right|
\]
by the triangle inequality.

**Lemma 3:** For any positive integer \( m \), let \( p_1, \ldots, p_m, q \in \mathbb{D} \), and let \( z \in \partial \mathbb{D} \). Then there exist constants \( c_1, \ldots, c_m \) such that
\[
\sum_{i=1}^{m} c_i \frac{1}{z - p_i} = \frac{1}{\prod_{i=1}^{m} (z - p_i)}.
\]

Multiplying both sides by the product \( \prod_{i=1}^{m} (z - p_i) \) and evaluating at \( z = p_i \) gives
\[
c_i = \frac{1}{\prod_{j \neq i} (p_i - p_j)}
\]
for all \( i \in I_m \), which satisfy (5). So, after choosing constants \( \{c_i\}_{i=1}^{m} \) by (6), to prove the claim it suffices to show that
\[
\left| \prod_{i=1}^{m} \left( \frac{1}{z - p_i} \right) - \left( \frac{1}{z - q} \right)^m \right|
\]
satisfies the inequality of Lemma 3. We compute
\[
\prod_{i=1}^{m} \left( \frac{1}{z - p_i} \right) - \left( \frac{1}{z - q} \right)^m = \left( \frac{z - q}{z - q} \right)^m - \prod_{i=1}^{m} \left( \frac{1}{z - p_i} \right).
\]
Then
\[
(z - q)^m - \prod_{i=1}^{m} (z - p_i)
\]
binomial theorem
\[
= \sum_{k=0}^{m} \binom{m}{k} z^{m-k} (-q)^k - \sum_{k=0}^{m} \sum_{S \subset I_m} \sum_{|S|=k} q^{m-k} \prod_{j \in S} (-p_j)
\]
canceling \( z^m \)
\[
\leq \sum_{k=1}^{m} \sum_{S \subset I_m} \sum_{|S|=k} z^{m-k} \prod_{j \in S} (-p_j)
\]
difference of sums
\[
= \sum_{k=1}^{m} \sum_{S \subset I_m} \sum_{|S|=k} \left( (-q)^k - \prod_{j \in S} (-p_j) \right)
\]
where the last equality follows since the number of terms in \( \sum_{S \subset I_m} \sum_{|S|=k} \) is the number of ways to select \( k \) poles out of \( m \) poles, which is \( \binom{m}{k} \). Then, recalling that \( z \in \partial \mathbb{D} \) so \( |z| = 1 \), and applying Lemma 2 to poles \(-q\) and \( \{p_i\}_{i=1}^{k} \) for each set \( S \) in the sum yields
\[
\left| z - q \right|^m - \prod_{i=1}^{m} \left( z - p_i \right) \leq \sum_{k=1}^{m} \sum_{S \subset I_m} \sum_{|S|=k} \left| q \right|^{k-1} \prod_{j \in S} \left| p_j - q \right|
\]
triangle inequality
\[
\leq \sum_{k=1}^{m} \sum_{S \subset I_m} \sum_{|S|=k} \left| q \right|^{k-1} \prod_{j \in S} \left| p_j - q \right|
\]
Lemma 2
\[
\left| z - q \right|^m \prod_{i=1}^{m} \left( z - p_i \right) \geq d(q, \partial \mathbb{D})^m \prod_{i=1}^{m} d(p_i, \partial \mathbb{D}).
\]

Furthermore, since \( z \in \partial \mathbb{D} \)
\[
\left| z - q \right|^m \prod_{i=1}^{m} \left( z - p_i \right) \geq \frac{d(q, \partial \mathbb{D})^m \prod_{i=1}^{m} d(p_i, \partial \mathbb{D})}{d(q, \partial \mathbb{D})^m (1 - r)^m} \right) d(q).
\]

**Proof of Corollary 2:** To prove the claim, we upper bound the right-hand side of (4). We compute
\[
\sum_{k=1}^{m} \sum_{S \subset I_m} \sum_{|S|=k} \sum_{|T|=1} \left| q \right|^{k-1} \prod_{j \in T} \left| p_j - q \right|
\]
bound by \( d(q) \)
\[
\leq \sum_{k=1}^{m} \sum_{S \subset I_m} \sum_{|S|=k} \sum_{|T|=1} \left| q \right|^{k-1} d(q)^i
\]
Theorem 1 extends the approximation error bound to an arbitrary number of (possibly repeated) poles and to the MIMO case.

Proof of Theorem 1: The proof begins by writing the partial fraction decomposition of $T$, and for each pole $q$ of $T$, using the SISO approximation of Corollary 2 to approximate each SISO term $\frac{1}{z-q}$ with the $j$ nearest poles in $P$. These are then combined with the MIMO coefficients in the partial fraction decomposition of $T$ to yield the approximating transfer function $\sum_{j \in P} G_P \frac{1}{z-p}$. Care must be taken to ensure symmetry between the approximations of complex conjugate poles in $T$ so that the approximating transfer function has real coefficients. Next, using the SISO approximation error bounds of Corollary 2, the main approximation error bounds of the corollary are derived. Finally, it is shown that, by construction, the approximating transfer function does indeed have real coefficients, and therefore belongs to $\mathcal{H}_\infty$.

We begin by writing the partial fraction decomposition of $T$ and constructing the approximating transfer function. Let $\Omega_2 \subset \Omega$ denote the real poles of $T$, and let $\Omega_\mathbb{C} \subset \Omega$ denote the remaining poles. Since $T \in \mathcal{H}_\infty$, we can write its partial fraction decomposition with matrix-valued coefficients $C(q,j)$ as

$$T(z) = \sum_{q \in \Omega_2} \sum_{j=1}^{\infty} c(q,j) \frac{1}{z-q} + \sum_{q \in \Omega_\mathbb{C}} \sum_{j=1}^{\infty} C(q,j) \frac{1}{z-q}.$$  \hspace{1cm} (11)

For each $q \in \Omega$ and each $j \in I_{m_q}$, let $P(q,j) \subset P(q)$ denote the $j$ closest poles in $P$ to $q$ (or at least one choice in case this is not unique) and choose constants $\{c_p(q,j)\}_{p \in P(q,j)}$ as in Corollary 2 for approximating the pole $q$ with multiplicity $j$ by $P(q,j)$. Then, choose $P(q,j) = P(q,j)$ (note that by symmetry, since $P$ and $\Omega$ are both closed under complex conjugation, these are the $j$ closest poles in $P$ to $q$) and choose constants $\{c_p(q,j)\}_{p \in P(q,j)}$ as in Corollary 2 for approximating the pole $q$ with multiplicity $j$ by $P(q,j)$. Note that in the case $q$ is real, for each $j \in I_{m_q}$, this results in two sets of constants for approximating $q$: one for $P(q,j)$ and one for $P(q,j)$ (where we have abused notation for simplicity of presentation).

Then, for each $p \notin P(q,j)$, let $c_p(q,j) = 0$, and for each $p \notin P(q,j)$, let $c_p(q,j) = 0$ for notational convenience. For each $q \in \Omega$ and each $j \in I_{m_q}$, let $\hat{d}(q,j) = \max_{p \in P(q,j)} |p-q|$. Then $\hat{d}(q,j) \leq \hat{d}(q) \leq D(P)$. By Corollary 2, this implies that for each $q \in \Omega$ and each $j \in I_{m_q}$

$$\sum_{p \in P} c_p(q,j) \frac{1}{z-p} \leq \hat{d}(q,j) \cdot D(P).$$  \hspace{1cm} (12)
and similarly for \( \overline{q} \). Then for each \( p \in \mathcal{P} \), define

\[
G_p = \sum_{q \in \mathcal{Q}} m_q \sum_{j=1}^{m_q} c_{(q,j)}^p c_{(q,j)}^p + c_{(q,j)}^p \sum_{q \in \mathcal{Q}} c_{(q,j)}^p c_{(q,j)}^p.
\]

This completes the construction of the approximating transfer function \( \sum_{p \in \mathcal{P}} G_p \frac{1}{z - p} \).

Next we show that the approximating transfer function satisfies the desired approximation error bounds of (1). We compute

\[
\left\| \sum_{p \in \mathcal{P}} G_p \frac{1}{z - p} - T \right\|_{\mathcal{H}_\infty} \leq \sum_{p \in \mathcal{P}} \left\| \sum_{q \in \mathcal{Q}} m_q \sum_{j=1}^{m_q} c_{(q,j)}^p c_{(q,j)}^p + c_{(q,j)}^p \sum_{q \in \mathcal{Q}} c_{(q,j)}^p c_{(q,j)}^p \right\|_{\mathcal{H}_\infty}.
\]

Finally, we show that \( \sum_{p \in \mathcal{P}} G_p \frac{1}{z - p} \in \mathcal{H}_\infty \). Clearly \( \sum_{p \in \mathcal{P}} G_p \frac{1}{z - p} \) is strictly proper, rational, and stable, so it suffices to show it has real coefficients. Since \( T \) has real coefficients it satisfies \( T(z) = \overline{T(\overline{z})} \) which implies, by matching coefficients in the partial fraction decomposition and since \( \mathcal{P}(\mathcal{Q}, j) = \mathcal{P}(\mathcal{Q}, j) \) for all \( q \in \mathcal{Q} \) and \( j \in \mathcal{I}_m \),

\[
C_{(\mathcal{Q}, j)} = \overline{C_{(\mathcal{Q}, j)}}
\]

for all \( q \in \mathcal{Q} \) and \( j \in \mathcal{I}_m \). Furthermore, using the definition of \( \{ c_{(q,j)}^p \}_{p \in \mathcal{P}, q} \) from (6) and since \( \mathcal{P}(\mathcal{Q}, j) = \mathcal{P}(\mathcal{Q}, j) \) for all \( q \in \mathcal{Q} \) and \( j \in \mathcal{I}_m \), it is straightforward to verify that

\[
c_{(q,j)}^p = \overline{c_{(q,j)}^p}
\]

for all \( q \in \mathcal{Q} \) and \( j \in \mathcal{I}_m \). It is also straightforward to verify that for any complex-valued matrix \( M \) and pole \( p \in \mathcal{D} \), \( M \frac{1}{z - p} + M \frac{1}{z - \overline{p}} \) can be expressed as a rational transfer function matrix with real coefficients [we will refer to this later as fact (a)]. Let \( \mathcal{Q}_+ \subset \mathcal{Q}_c \) be the subset of poles with nonnegative imaginary component, and similarly for \( \mathcal{P}_+ \subset \mathcal{P} \). Then

\[
\sum_{p \in \mathcal{P}_+} G_p \frac{1}{z - p}
\]

and similarly for \( \overline{p} \). Then for each \( p \in \mathcal{P} \), define

\[
G_p = \sum_{q \in \mathcal{Q}} m_q \sum_{j=1}^{m_q} c_{(q,j)}^p c_{(q,j)}^p + c_{(q,j)}^p \sum_{q \in \mathcal{Q}} c_{(q,j)}^p c_{(q,j)}^p.
\]

This completes the result for the \( \mathcal{H}_\infty \) case. Let \( s \) be the minimum of the dimensions of \( T \), and note that

\[
\left\| T \right\|_{\mathcal{H}_2} \leq \sqrt{s} \left\| T \right\|_{\mathcal{H}_\infty} \leq \sqrt{s} K_2 D(\mathcal{P}) = K_2 D(\mathcal{P})
\]

where \( K_2 = \sqrt{s} K_\infty \). This completes the result for the \( \mathcal{H}_2 \) case.

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which, by fact (a) above and (15), is a sum of rational transfer functions with real coefficients, hence it is rational with real coefficients.

\[\square\]

### B. Proofs of Theorem 2 and Theorem 3

This section will first use the approximation error bounds of Theorem 1 to prove that the space of simple pole approximations converges to the Hardy space \(1/2 \mathbb{H}_\infty\) in Theorem 2. Then, Theorem 1 is combined with the notion of geometric convergence rate to provide a uniform convergence rate in Theorem 3 for the simple pole approximation to any transfer function in \(1/2 \mathbb{H}_\infty\). The first step is to show the equivalence of space-filling and exhibiting geometric convergence pole sequences in Lemma 1.

**Proof of Lemma 1:** To show that geometric convergence implies space-filling, a finite set of poles with a small Hausdorff distance from the unit disk is chosen. Then, selecting a transfer function whose poles contain this pole selection, and using that the pole sequence exhibits geometric convergence, it is possible to choose a pole in \(P_n\) close to each pole in the pole selection. This will result in the Hausdorff distance between \(P_n\) and the pole selection being small, and hence small Hausdorff distance between \(P_n\) and the unit disk, which will lead to the space-filling property. Next, assuming that the sequence of poles is space-filling, to show that it exhibits geometric convergence we begin by selecting an arbitrary transfer function. Then, letting its maximum multiplicity be \(m_{\max}\), each pole near its \(n\) poles are chosen in the unit disk. Using the space-filling property, poles in \(P_n\) are found close to each of these poles in the unit disk, and hence close to each pole of the transfer function, establishing geometric convergence.

Let \(\{P_n\}_{n=1}^{\infty}\) be a sequence of poles. First, suppose that \(\{P_n\}_{n=1}^{\infty}\) exhibits geometric convergence. To show that \(P_n \subset \overline{D}\), it suffices to show that for every \(\varepsilon > 0\), there exists \(N\) such that for all \(n \geq N\) implies that \(d_H(P_n, \overline{D}) \leq \varepsilon\). So, let \(\varepsilon > 0\), and note that this implies that \(\bigcup_{q \in Q} \{q\} \supset \overline{D}\). Choose \(S \subset 1/2 \mathbb{H}_\infty\) which has simple poles at \(Q\) (and possibly additional poles as well, such as the complex conjugate of \(Q\)). Since \(\{P_n\}_{n=1}^{\infty}\) exhibits geometric convergence, there exists \(N\) such that \(n \geq N\) implies that \(D(P_n) < \frac{\varepsilon}{2}\). In particular, this implies that for any \(n \geq N\) and every \(q \in Q\), there exists \(p^{(q)} \in P_n\) such that \(|p^{(q)} - q| < \frac{\varepsilon}{2}\), which implies that \(\{p^{(q)}\} \supset \{q\} \supset \overline{D}\). Thus, for every \(n \geq N\),

\[
(P_n)_{e} \supset \bigcup_{q \in Q} \{p^{(q)}\} \supset \bigcup_{q \in Q} \{q\} \supset \overline{D},
\]

As \(P_n \subset \overline{D}\) for all \(n\), this implies that for \(n \geq N\), \(d_H(P_n, \overline{D}) \leq \varepsilon\). So, \(\{P_n\}_{n=1}^{\infty}\) is space-filling.

Next suppose that \(\{P_n\}_{n=1}^{\infty}\) is space-filling. Let \(S \subset 1/2 \mathbb{H}_\infty\). Let \(Q\) be the poles of \(S\), and let \(m_{\max} = \max_{q \in Q} m_q\) be their maximum multiplicity, which is finite since \(S\) is rational. To show that \(\lim_{n \to \infty} D(P_n) = 0\), it suffices to show that for every \(\varepsilon > 0\) there exists \(N\) such that \(n \geq N\) implies that \(D(P_n) \leq \varepsilon\). So, let \(\varepsilon > 0\). For each \(q \in Q\), choose \(q^{(1)}, \ldots, q^{(m_{\max})} \subset \overline{D}\) such that for all \(i, j \in \{1, \ldots, m_{\max}\}\) with \(i \neq j\), \(|q^{(i)} - q^{(j)}| \leq \varepsilon\) and \(|q^{(i)} - q^{(j)}| > \frac{\varepsilon}{2m_{\max}}\) (one way to do this is to choose all of the \(q^{(i)}\) along a line passing through \(q\) such that \(|q^{(i)} - q^{(i+1)}| = \frac{\varepsilon}{m_{\max}}\) and \(|q^{(i)} - q^{(i+1)}| = \frac{\varepsilon}{m_{\max}}\) for all \(i \in \{1, \ldots, m_{\max}\}\). Since \(\{P_n\}_{n=1}^{\infty}\) is space-filling, there exists \(N\) such that \(n \geq N\) implies that \(d_H(P_n, \overline{D}) < \frac{\varepsilon}{2m_{\max}}\). Therefore, for each \(n \geq N\), \(q \in Q\), and \(i \in \{1, \ldots, m_{\max}\}\) there exists \(p^{(i)} \in P_n\) such that \(|q^{(i)} - p^{(i)}| < \frac{\varepsilon}{2m_{\max}}\). By the choice of the \(\{q^{(i)}\}_{i=1}^{m_{\max}}\), this implies that \(\{p^{(i)}\}_{i=1}^{m_{\max}} \subset P_n\) consists of \(m_{\max}\) distinct poles, all of which satisfy \(|p^{(i)} - q| \leq \varepsilon\). Thus

\[
\tilde{d}(q) = \max_{p \in P_n(q)} |p - q| \leq \max_{i=1}^{m_{\max}} |p^{(i)} - q| \leq \varepsilon
\]

since \(P_n(q)\) are the \(m_{\max}\) closest poles in \(P_n\) to \(q\). As \(q \in Q\) is arbitrary, this implies that for all \(n \geq N\)

\[
D(P_n) = \max_{q \in Q} \tilde{d}(q) \leq \varepsilon.
\]

So, \(\{P_n\}_{n=1}^{\infty}\) exhibits geometric convergence. \(\square\)

Now we are ready to prove Theorem 2.

**Proof of Theorem 2:** The proof begins by fixing an arbitrary transfer function in \(1/2 \mathbb{H}_\infty\), and uses Theorem 1 to obtain approximation error bounds using the simple pole approximation with poles \(P_n\) for each \(n\). Some initial work is required to ensure the assumptions of Theorem 1 are met uniformly by \(P_n\) for all \(n\) sufficiently large. As the sequence of poles is space filling, by Lemma 1, it exhibits geometric convergence, so taking the limit of the approximation error bounds as \(n \to \infty\) implies convergence of the simple pole approximations to the desired transfer function. This implies that this transfer function is contained in the \(\liminf\) of the simple pole approximation spaces and, since it is arbitrary, that this \(\liminf\) is equal to \(1/2 \mathbb{H}_\infty\). As the \(\limsup\) is contained in the \(\liminf\), this completes the proof.

Let \(S \subset 1/2 \mathbb{H}_\infty\) and let \(Q\) be the poles of \(S\). Let \(r = 1/2(1 + \max_{q \in Q} |q|) \in (0, 1)\). As \(\{P_n\}_{n=1}^{\infty}\) is space-filling, by Lemma 1 it exhibits geometric convergence, so \(\liminf_{n \to \infty} D(P_n) = 0\). Thus, there exists \(N\) such that \(n \geq N\) implies that \(D(P_n) < 1\) and \(D(P_n) < 1/2(1 + \max_{q \in Q} |q|)\). This implies that for \(n \geq N\), the poles from \(P_n\) used to approximate \(Q\) in the simple pole approximation of Theorem 1 all lie within

\[
\overline{B}_{1/2(1 + \max_{q \in Q} |q|)}(0) = B_r.
\]

Furthermore, since \(\{P_n\}_{n=1}^{\infty}\) is space-filling, increasing \(N\) further if necessary implies that for \(n \geq N\), \(P_n \cap B_r\) contains at least \(m_{\max}\) poles. Thus, for every \(n \geq N\), \(P_n \cap B_r\) is contained in \(B_r\) with \(r \in (0, 1)\) (Assumption A1), contains at least \(m_{\max}\) poles (Assumption A2), is closed under complex conjugation (Assumption A3), and satisfies \(D(P_n) < 1\) (Assumption A4). So, by Theorem 1, there exist constants \(c_S, c'_S > 0\) such that for every \(n \geq N\), there exist coefficients \(\{G_p\}_{p \in P_n}\) such that

\[
\left\| \sum_{p \in P_n} G_p \frac{1}{z - p} - S \right\|_{\mathcal{H}^2} \leq c_S D(P_n)
\]

\[
\left\| \sum_{p \in P_n} G_p \frac{1}{z - p} - S \right\|_{\mathcal{H}^\infty} \leq c'_S D(P_n).
\]
Thus, taking the limit as \( n \to \infty \) implies

\[
\lim_{n \to \infty} \left| \sum_{p \in \mathcal{P}_n} \frac{G^n_p}{z-p} - S \right| < c_S \lim_{n \to \infty} D(\mathcal{P}_n) = 0
\]

\[
\lim_{n \to \infty} \left| \sum_{p \in \mathcal{P}_n} \frac{G^n_p}{z-p} - S \right| < c'_S \lim_{n \to \infty} D(\mathcal{P}_n) = 0.
\]

Therefore, \( \lim_{n \to \infty} \sum_{p \in \mathcal{P}_n} G^n_p \frac{1}{z-p} = S \) with respect to the metrics induced by both the \( \mathcal{H}_2 \) and \( \mathcal{H}_\infty \) norms. As \( \sum_{p \in \mathcal{P}_n} G^n_p \frac{1}{z-p} \in A_n \) for all \( n \geq N \), and \( \lim_{n \to \infty} \sum_{p \in \mathcal{P}_n} G^n_p \frac{1}{z-p} = S \), \( S \in \liminf_{n \to \infty} A_n \). As \( S \in \frac{1}{z} \mathcal{H}_\infty \) is arbitrary, \( \liminf_{n \to \infty} A_n = \frac{1}{z} \mathcal{H}_\infty \). By definition

\[
\frac{1}{z} \mathcal{H}_\infty = \liminf_{n \to \infty} A_n \subseteq \limsup_{n \to \infty} A_n \subseteq \frac{1}{z} \mathcal{H}_\infty
\]

so

\[
\frac{1}{z} \mathcal{H}_\infty = \liminf_{n \to \infty} A_n = \limsup_{n \to \infty} A_n = \lim_{n \to \infty} A_n.
\]

Next we prove Theorem 3.

**Proof of Theorem 3:** The proof proceeds by combining the approximation error bounds of Theorem 1 with the definition of geometric convergence rate. Some initial work is required to ensure that the assumptions of Theorem 1 are met uniformly by \( \mathcal{P}_n \) for all \( n \) sufficiently large.

Let \( S \in \frac{1}{z} \mathcal{H}_\infty \), let \( \Omega \) be the poles of \( S \), and let \( m_{\text{max}} \) be their maximum multiplicity. Let \( r_0 = \frac{1}{2} (1 + \max_{q \in \Omega} |q|) \). Since \( \{\mathcal{P}_n\}_{n=1}^\infty \) is a sequence of poles with geometric convergence rate \( \frac{1}{2} \), it exhibits geometric convergence. So, there exists \( N' \) such that \( n \geq N' \) implies that \( D(\mathcal{P}_n) < \frac{1}{2} (1 - \max_{q \in \Omega} |q|) \).

Thus, for \( n \geq N' \), the poles from \( \mathcal{P}_n \) used to approximate \( \Omega \) in the simple pole approximation of Theorem 1 all lie within

\[
\mathcal{B}_{\max_{q \in \Omega} |q| + D(\mathcal{P}_n)} \subset \mathcal{B}_{\frac{1}{2} (1 + \max_{q \in \Omega} |q|)} = \mathcal{B}_{r_0}.
\]

For \( n \in \{1, \ldots, N'-1\} \), let \( r_n = \max_{p \in \mathcal{P}_n} |p| \), so \( \mathcal{P}_n \cap \mathcal{B}_{r_n} = \mathcal{P}_n \). Let \( r = \max_{n=1}^{N'-1} r_n \). Then for all \( n \), the poles from \( \mathcal{P}_n \) used to approximate \( \Omega \) in the simple pole approximation of Theorem 1 all lie within \( \mathcal{P}_n \cap \mathcal{B}_r \) (Assumption A1). Furthermore, \( \mathcal{P}_n \cap \mathcal{B}_r \) is closed under complex conjugation for all \( n \) (Assumption A3).

As \( \{\mathcal{P}_n\}_{n=1}^\infty \) exhibits geometric convergence, there exists \( N \) such that \( n \geq N \) implies that \( D(\mathcal{P}_n) < 1 \) (Assumption A4). As \( \{\mathcal{P}_n\}_{n=1}^\infty \) exhibits geometric convergence, by Lemma 1 it is space-filling. So, increasing \( N \) if necessary implies that for \( n \geq N \), \( |\mathcal{P}_n \cap \mathcal{B}_r| \geq m_{\text{max}} \) (Assumption A2). Note that \( N \) is chosen just to ensure that \( D(\mathcal{P}_n) < 1 \) and that \( \mathcal{P}_n \) has at least \( m_{\text{max}} \) poles. As Assumptions A1–A4 are satisfied for \( \mathcal{P}_n \cap \mathcal{B}_r \), with \( n \geq N \), by Theorem 1 there exist constants \( \hat{c}_S, \hat{c}'_S > 0 \) such that for every \( n \geq N \), there exist coefficients \( \{G^n_p\}_{p \in \mathcal{P}_n} \) such that

\[
\sum_{p \in \mathcal{P}_n} G^n_p \frac{1}{z-p} - S \leq \hat{c}_S D(\mathcal{P}_n)
\]

\[
\sum_{p \in \mathcal{P}_n} G^n_p \frac{1}{z-p} - S \leq \hat{c}'_S D(\mathcal{P}_n).
\]

As \( \{\mathcal{P}_n\}_{n=1}^\infty \) is a sequence of poles with geometric convergence rate \( \frac{1}{n^k} \), there exists a constant \( \hat{c}_S > 0 \) such that for all \( n \)

\[
D(\mathcal{P}_n) \leq \hat{c}_S \frac{n^k}{n^k}.
\]

Combining this with the approximation error bounds above yields

\[
\sum_{p \in \mathcal{P}_n} G^n_p \frac{1}{z-p} - S \leq \frac{c_S}{n^k}
\]

\[
\sum_{p \in \mathcal{P}_n} G^n_p \frac{1}{z-p} - S \leq \frac{c'_S}{n^k}
\]

\[
c_S = \hat{c}_S \hat{c}_S
\]

\[
c'_S = \hat{c}'_S \hat{c}_S
\]

for all \( n \geq N \).

C. **Proofs of Theorem 4 and Corollary 1**

This section will prove Theorem 4 and Corollary 1. To achieve an approximately uniform selection of poles over the unit disk, the basic idea is to choose a spiral whose windings are equally spaced apart, and then to select poles along this spiral such that the distance between any two successive poles is close to but less than the distance between the windings. Such a selection is shown in Fig. 1 and first appeared in [22, Sec. 5.1], although we are not aware of any prior uses for control design. Note that as the number of poles increases, the spiral also changes so that the distance between windings (and, hence, between successive poles) converges towards zero. The main challenge is to show that the chosen pole selection along this spiral does in fact possess these geometric properties. Then, these will be used to show that as the number of poles increases, \( D(\mathcal{P}) \) converges to zero at the same rate at which the distance between windings converges to zero.

Lemma 4 and Corollary 3 provide basic geometric facts about the Archimedes spiral as parameterized in (2). Lemma 4 appears without proof in [22, p. 18].

**Lemma 4:** The Archimedes spiral given in polar coordinates by \( r = \theta \) for a fixed \( c > 0 \) has a constant distance between its windings of \( 2\pi c \).

**Proof of Lemma 4:** Two points on successive windings of the spiral are given by \( p_1 = (c\theta_1, \theta_1) \) and \( p_2 = (c(\theta_1 + 2\pi), \theta_1 + 2\pi) \) for some \( \theta_1 \geq 0 \). Then, using the law of cosines to compute
distances in polar coordinates yields
\[
\begin{align*}
d(p_1, p_2)^2 &= (c\theta_1)^2 + (c(\theta_1 + 2\pi))^2 \\
&= 2(c\theta_1)(c(\theta_1 + 2\pi)) \cos(\theta_1 + 2\pi - \theta_1) \\
&= c^2\theta_1^2 + c^2(\theta_1^2 + 4\pi \theta_1 + 4\pi^2) - 2c^2(\theta_1^2 + 2\pi \theta_1) = 4\pi^2c^2
\end{align*}
\]
so, taking the square root implies \(d(p_1, p_2) = 2\pi c\).

**Corollary 3:** For any \(z \in \mathbb{D}\), there exists a point \(p\) on the Archimedean spiral \(r = c\theta\) for a fixed \(c > 0\) such that \(|p| < |z|\) and \(d(z, p) < 2\pi c\).

**Proof of Corollary 3:** Fix \(z = (r, \theta) \in \mathbb{D}\) with \(\theta \in [0, 2\pi)\). If \(|z| = 0\) or \(z\) is in the spiral, then the distance to the spiral is zero. Therefore, assume that \(z\) is not in the spiral. Then there exists \(\theta_1 \geq 0\) and an integer \(n\) such that \(r \in (c\theta_1, c(\theta_1 + 2\pi))\) and \(\theta_1 = \theta + 2\pi n\). Let \(p_1 = (c\theta_1, \theta_1)\) and note that \(p_1\) is in the spiral and \(|p_1| = c\theta_1 < r = |z|\). From the above, we have
\[
d(z, p_1)^2 = r^2 + (c\theta_1)^2 - 2r(c\theta_1) \cos(\theta + 2\pi n - \theta) \\
= (r - c\theta_1)^2 < (c(\theta_1 + 2\pi) - c\theta_1)^2 = (2\pi c)^2.
\]
So taking the square root implies that \(d(z, p_1) < 2\pi c\). As \(|p_1| < |z|\) and \(p_1\) is in the spiral, the claim follows.

**Corollary 4** and **Corollary 5** above establish basic geometric properties about the particular pole selection along the spiral shown in Fig. 1. Lemma 5 below provides a technical result that is needed for the proof of **Corollary 4**. For any positive even integer \(m\), let \(c_m = \frac{2\sqrt{m\pi}}{m+\pi}\). Then every pole in the selection (2) lies along the Archimedean spiral \(r = c_m\theta\), as shown in Fig. 1.

**Lemma 5:** For \(x \in [1, \infty)\) and a positive even integer, let
\[
g(x) = \frac{1}{\sqrt{2}} \left[2x + 1 - 2\sqrt{x(x+1)} \cos(2\sqrt{\pi} \sqrt{x + 1 - \sqrt{x}})\right].
\]
Then \(\lim_{x \to \infty} g(x) = (2\pi c_m)^2\) and \(g(x)\) is monotonically increasing over \(x \in [1, \infty)\), so \(g(x) < (2\pi c_m)^2\) for all \(x \in [1, \infty)\).

**Proof of Lemma 5:** By [22, Appendix]
\[
\lim_{x \to \infty} g(x) = \frac{\pi}{2} + 1 = (2\pi c_m)^2.
\]
To prove that \(g(x)\) is monotonically increasing over \([1, \infty)\), it suffices to show that \(g' = \frac{dg}{dx} > 0\) over \([1, \infty)\). Taking a derivative of \(g\) leads to
\[
\left(\frac{m}{2} + 1\right) g'(x) = 2 - \frac{2x + 1}{\sqrt{x(x+1)}} \cos(a_x) - a_x \sin(a_x)
\]
where \(a_x = 2\sqrt{\pi} \sqrt{x + 1 - \sqrt{x}} = \frac{2\sqrt{\pi}}{\sqrt{x+1} + \sqrt{x+1}}\).

For \(x \in (0, 1]\), define the function
\[
h(z) = \left(\frac{m}{2} + 1\right) g'\left(\frac{1}{z}\right)
\]
\[
= 2 - \frac{2 + z}{\sqrt{1 + z}} \cos(a_z) - a_z \sin(a_z), a_z = 2\sqrt{\pi} \frac{\sqrt{z}}{\sqrt{1 + z} + 1}
\]
Taking a derivative leads to
\[
h'(z) = \sqrt{\pi} \frac{2 + z - \sqrt{1 + z}}{\sqrt{(1 + z)(\sqrt{1 + z} + 1)}} \sin(a_z)
\]
\[
- \left(\frac{z}{2\sqrt{1 + z}} + \frac{1}{\sqrt{1 + z}(\sqrt{1 + z} + 1)}\right) \cos(a_z).
\]
Taking another derivative leads to \(h''\) as shown in Fig. 4.

**Fig. 4.** Graph of \(h'' = \frac{d^2h}{dx^2}\).
where the final inequality follows from evaluation. Thus, \( \theta \in [\theta_k, \theta_{k+1}] \) implies \( \sin(\theta - \theta_k) \geq 0 \) and \( \cos(\theta - \theta_k) \geq 0 \). So

\[
\frac{1}{c_m} \frac{dg}{d\theta} = 2\theta - 2\theta_k \cos(\theta - \theta_k) + 2\theta_k \sin(\theta - \theta_k) \\
\geq 2\theta - 2\theta_k + 2\theta_k \sin(\theta - \theta_k) \geq 0
\]

since \( \theta > \theta_k \) and the other term is positive. Thus, \( \frac{dg}{d\theta} \geq 0 \) over \([\theta_k, \theta_{k+1}]\), which concludes the proof of Case 1.

**Case 2:** Suppose \( \hat{\theta} < \theta_1 \). If \( \theta_1 - \hat{\theta} > \frac{\pi}{2} \) then

\[
d(p, p_1)^2 = c_m^2 (\hat{\theta}_1^2 + \theta_1^2 - 2\hat{\theta}_1 \cos(\hat{\theta}_1 - \theta_1)) \\
\leq \hat{\theta}_1^2 + \theta_1^2 - 2\hat{\theta}_1 \cos(\hat{\theta}_1 - \theta_1) \\
\leq (\hat{\theta}_1 - \theta_1)^2 (\frac{1}{c_m^2}) \leq \frac{c_m^2}{\pi^2} (\hat{\theta}_1 - \theta_1)^2 \\
\leq \frac{c_m^2}{\pi^2} (\hat{\theta}_1 - \theta_1)^2 \\
\leq \frac{c_m^2}{\pi^2} (\sqrt{2\hat{\theta}_1} - \theta_1)^2 \\
\leq \frac{c_m^2}{\pi^2} (2\pi)^2
\]

since \( \sqrt{2\hat{\theta}_1} = 2\sqrt{2\pi} \leq 2\pi \), so taking the square root implies that \( d(p, p_1) < 2\pi c_m \). Otherwise, if \( \theta_1 - \hat{\theta} < \frac{\pi}{2} \) then \( \cos(\theta_1 - \hat{\theta}) \geq 0 \), so

\[
d(p, p_1)^2 = c_m^2 (\hat{\theta}_1^2 + \theta_1^2 - 2\hat{\theta}_1 \cos(\hat{\theta}_1 - \theta_1)) \\
\leq \theta_1^2 + \hat{\theta}_1^2 - 2\hat{\theta}_1 \cos(\hat{\theta}_1 - \theta_1) \\
\leq (\theta_1 - \hat{\theta})^2 (\frac{1}{c_m^2}) \leq \frac{c_m^2}{\pi^2} (\theta_1 - \hat{\theta})^2 \\
\leq \frac{c_m^2}{\pi^2} (\sqrt{2\theta_1})^2 \\
\leq \frac{c_m^2}{\pi^2} (2\pi)^2
\]

since \( \sqrt{2\theta_1} = 2\sqrt{2\pi} \leq 2\pi \), so taking the square root implies that \( d(p, p_1) < 2\pi c_m \). Thus, either way we have \( d(p, p_1) < 2\pi c_m \), which proves Case 2.

**Proof of Theorem 4:** Let \( S \in \mathcal{P}_\infty \) and let \( \Omega \) be the poles of \( S \). The proof considers the spiral pole selection \( \mathcal{P} \) of Corollary 4. For any pole \( q \) in \( \Omega \), by Corollary 5, the distance between \( q \) and the closest pole in the spiral \( \mathcal{P} \) is bounded by \( 4\pi c_m \). Traversing the spiral along neighboring poles (i.e., \( p_k \) and \( p_{k+1} \)), Corollary 4 implies that the distance from \( q \) to each successive pole in the spiral cannot increase by more than \( 2\pi c_m \). This implies that the \( m_q \) closest poles in the spiral to \( q \) are all within a distance of \( (m_q + 1)(2\pi c_m) \) from \( q \). As \( (m_q + 1)(2\pi c_m) \) is bounded by a constant times \( \frac{1}{\sqrt{m_q+1}} \), this provides a bound on \( D(\mathcal{P}) \). Furthermore, for use of Theorem 4 in the proofs of Corollary 1 and [16, Corollary 1], additional work is required at the beginning of the proof to only choose poles from the spiral within a ball of fixed radius \( r \in (0, 1) \), and at the end of the proof to only choose poles from the spiral that are at least a fixed distance \( \delta > 0 \) away from all poles in \( \sigma \) which are not approximating (see Assumption A5).

We begin by finding the desired radius \( r \in (0, 1) \) of the ball for intersecting with the spiral. Let the poles \( p_k \) and \( p_{k+1} \) be selected along the Archimedes spiral according to Corollary 4 for the even integer \( m \), and define \( \mathcal{P}_m = \bigcup_{k=1}^{m} \{p_k, p_{k+1}\} \). Let \( m_{\text{max}} = \max_{q \in \Omega} m_q \). Let \( r_q = \max_{p \in \mathcal{P}_m} |q - p| \) and let \( r_p = \max_{p \in \mathcal{P}_m} \). Let \( \hat{r} = \max \{r_q, r_p\} \) and note that this implies that \( \mathcal{P}_m \subset B_{\hat{r}} \) and \( p_k \in B_{\hat{r}} \) for any \( m \). Let \( \mathcal{P}_m = \mathcal{P}_m \cap B_{\hat{r}} \). By the definition of the pole selection in Corollary 4, and since \( \mathcal{P}_m \subset B_{\hat{r}} \), for any \( m \geq m_{\text{max}} \) we have \( \mathcal{P}_m = \bigcup_{k=1}^{m_{\text{max}}} \{p_k, p_{k+1}\} \) for some \( k' \in \{m_{\text{max}} + 1, \ldots, \frac{m_{\text{max}}}{\hat{r}} \} \). Thus, \( |\mathcal{P}_m| \geq m_{\text{max}} \).

Next, we consider a pole \( q \) of \( \Omega \), and bound the distance of its \( m_q \) closest approximating poles from the spiral. Let \( q \in \Omega \). By Corollary 5, there exists \( k \in \{1, \ldots, m_q\} \) such that

\[ (0, \frac{\pi}{2}), \cos(2\theta_k - 2\pi) > 0 \]

\[
d(p_1, p_{k+1})^2 = r_1^2 (2 - 2 \cos(2\theta_1)) = \frac{2 - 2 \cos(2\theta_1)}{m_q + 1} \]

\[
< \frac{2}{m_q + 1} < \frac{\pi}{2} = (2\pi c_m)^2.
\]
$d(q,p_k) < 4\pi c_m$ and either $|p_k| \leq |q|$ or $k = 1$. The latter implies that $p_k \in \mathbb{B}_r$, so $p_k \in \mathbb{P}_m$. As $2k' = |\mathbb{P}_m| \geq m_{\text{max}} \geq m_q$, there exists a subset of consecutive integers $S(q,m) \subset \{-k',\ldots,-1,1,\ldots,k'\}$ (where we define $-1$ and $1$ to be consecutive for this purpose) such that $k \in S(q,m)$ and $|S(q,m)| = m_q$. Let $\mathbb{P}_m(q)$ denote the $m_q$ closest poles in $\mathbb{P}_m$ to $q$, and let $i = \arg \max_{i \in S(q,m)} |p_i - q|$. Then, by Corollary 4

$$d(q) = \max_{p \in \mathbb{P}_m(q)} |p - q| \leq \max_{i \in S(q,m)} |p_i - q| = d(q,p_i) \leq d(q,p_k) + d(p_k,p_i) \leq 4\pi c_m + \sum_{i=k}^{i-1} d(p_i,p_{i+1}) \leq 4\pi c_m + |i - k|(2\pi c_m) \leq 4\pi c_m + (m_q - 1)(2\pi c_m) = \frac{(m_q + 1)}{(2\pi c_m)^{-1}} \leq \frac{(m_{\text{max}} + 1)}{(2\pi c_m)^{-1}} = \frac{(m_{\text{max}} + 1)}{\sqrt{2\pi}} \sqrt{\frac{1}{m + 2}}.
$$

So

$$D(\mathbb{P}_m) = \max_{q \in \mathbb{C}} d(q) \leq (m_{\text{max}} + 1) \sqrt{\frac{1}{\sqrt{n}}}$$

Substituting $m = 2n - 2$ as in the statement of Theorem 4 gives

$$D(\mathbb{P}_n) \leq (m_{\text{max}} + 1) \sqrt{\frac{1}{\sqrt{n}}}$$

so $\{p_n\}_{n=2}^{\infty}$ has geometric convergence rate $\frac{1}{\sqrt{n}}$. Thus, $\{p_n\}_{n=2}^{\infty}$ exhibits geometric convergence, so by Lemma 1 it is space-filling.

Now we derive the desired $\delta > 0$ for ensuring the spiral poles are at least distance $\delta$ from all poles of the plant which they are not approximating. As $Q$ and $\sigma$ are finite (the latter by Assumption A5), $\eta = \min_{q \notin Q, \lambda \neq \sigma} d(\lambda, q) > 0$ and $d(\lambda, q) \geq \eta$ for all such $\lambda \neq q$. As $\{p_n\}_{n=2}^{\infty}$ exhibits geometric convergence, there exists $N$ such that $n \geq N$ implies that $D(\mathbb{P}_n) \leq \frac{1}{\eta}$. This implies that for $n \geq N$, $q \in Q$, and $\lambda \in \sigma$ with $\lambda \neq q$, $d(\mathbb{P}_n(q), \lambda) \geq \frac{1}{\eta}$. For $n \in \{2,\ldots,N - 1\}$ let $d_n = \min_{q \notin Q, \lambda \neq \sigma} d(\lambda, q)$. Traversing downwards from $n = N - 1$, let $N$ denote the final value of $n \in \{2,\ldots,N - 1\}$ for which Assumption A5 is satisfied by $\mathbb{P}_n$. Then, for each $n \in \{N,\ldots,N - 1\}$, $d_n > 0$. Let $d_N = \frac{1}{\eta}$, and let $\delta = \min_{n=N} d_n > 0$. Then for each $n \geq N$, $q \in Q$, and $\lambda \in \sigma$ with $\lambda \neq q$, $d(\mathbb{P}_n(q), \lambda) \geq \delta$. The choices of $r$ and $\delta$ in this proof make it possible to apply Theorem 1 uniformly to $\mathbb{P}_n$ for all $n \geq N$ in the proofs of Corollary 1 and [16, Corollary 1].

Proof of Corollary 1: By Theorem 4, $\{p_n\}_{n=1}^{\infty}$ is a sequence of poles with geometric convergence rate $\frac{1}{\sqrt{n}}$. Thus, the result follows from Theorem 3.

VI. CONCLUSION

In this article, SPA was introduced as a new Galerkin-type method for finite-dimensional approximations of Hardy space that is an alternative to Loewner series approximations such as FIR. For any transfer function in Hardy space, approximation error bounds for SPA were provided which bound the Hardy space norms between the SPA and the desired transfer function proportionally to the geometric distance between their poles. These were then used to show that the space of SPAs converges to the entire Hardy space for any space-filling sequence of poles, and to provide a uniform convergence rate that depends purely on the geometry of the SPA pole selection. Unlike with Lorentz series approximations such as FIR, where the convergence rate is often related to the settling time of the optimal transfer function, the uniform convergence of SPA ensures that it works well even when the optimal transfer function has a long settling time, such as in systems with large separation of time scales. Finally, these results were specialized to the particular case of an Archimedes spiral pole selection, for which an explicit uniform convergence rate was provided.

In Part II, these results will be combined with SLS to develop a new control design method with reduced suboptimality, guaranteed feasibility, ability to include prior knowledge, formulation that requires solving only a single SDP, and that avoids deadbeat control. Furthermore, the proofs of the suboptimality certificates rely heavily on the approximation and convergence results for SPA from Part I. An example of optimal control design in Part II will demonstrate superior performance of SPA compared to FIR for that case.

Future work will involve applying SPA to IOP. Youla parameterization, and SLS with output feedback, considerations of different space-filling sequences with favorable geometric properties, extensions to continuous time approximation methods, and extensions to unstable transfer functions using, for example, $L^2$ and $L^\infty$ norms. For example, many different types of spirals exist, which would all yield space-filling sequences of poles [20], [21], [22], but obtaining uniform convergence rates would require bounds on the spacing between different windings of the spiral, as well as between subsequent pole selections along each winding, as a function of the number of poles, which are not trivial in general to derive. In principle, similar results to those obtained here for geometric approximation error bounds may be available for unstable transfer functions as well, although some work is required to overcome the challenge that unstable poles are not constrained to any bounded set. Future work will also consider pole selections using low-discrepancy sequences in the unit disk, which provide a quantitative measure (i.e., the discrepancy) of their favorable space-filling properties that may result in improved convergence rates.

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