CERTAIN GEOMETRIC PROPERTIES OF CLOSE-TO-CONVEX HARMONIC MAPPINGS

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Abstract. In this article, we introduce a new family of sense preserving harmonic mappings \( f = h + \overline{g} \) in the open unit disk and prove that functions in this family are close-to-convex. We give some basic properties such as coefficient bounds, growth estimates, convolution and determine the radius of convexity for the functions belonging to this family. In addition, we construct certain harmonic univalent polynomials belonging to this family.

1. Introduction

Let \( H \) denote the class of complex valued harmonic functions \( f \) in \( \mathbb{D} \) normalized by \( f(0) = f_{\overline{z}}(0) - 1 = 0 \). Each such function \( f \) can be expressed uniquely as \( f = h + \overline{g} \), where \( h \) and \( g \) have the following power series representations:

\[
\begin{align*}
    h(z) &= z + \sum_{n=2}^{\infty} a_n z^n \\
    g(z) &= \sum_{n=1}^{\infty} b_n z^n.
\end{align*}
\]

A result of Lewy [13], shows that \( f \in H \) is locally univalent in \( \mathbb{D} \) if and only if \( J_f(z) = |f_z(z)|^2 - |f_{\overline{z}}(z)|^2 \) is non-zero in \( \mathbb{D} \), and is sense preserving if \( J_f(z) > 0 \) \( (z \in \mathbb{D}) \), or equivalently, if the dilatation \( w = g'/h' \) is analytic and satisfies \( |w| < 1 \) in \( \mathbb{D} \). Observe that, the class \( H \) reduces to the class \( A \) of normalized analytic functions if the co-analytic part is zero. Let \( S_H \) be the subclass of \( H \) consisting of univalent and sense-preserving harmonic mappings in \( \mathbb{D} \). The classical family \( S \) of normalized analytic univalent functions is subclass of \( S_H \) as \( S = \{ f = h + \overline{g} \in S_H : g \equiv 0 \ \text{in} \ \mathbb{D} \} \). Also, we denote by \( H^0 = \{ f \in H : f_{\overline{z}}(0) = 0 \} \) and \( S_H^0 = \{ f \in S_H : f_{\overline{z}}(0) = 0 \} \). It is well known that the class \( S_H^0 \) is compact and normal, whereas the class \( S_H \) is normal but not compact. In 1984, Clunie and Sheil-Small [2] investigated the class \( S_H \), together with some of its geometric subclasses.

A function \( h \in A \) is called close-to-convex in \( \mathbb{D} \), if the complement of \( h(\mathbb{D}) \) can be written as the union of non-intersecting half lines. Let \( C \) denote the class of close-to-convex functions in \( \mathbb{D} \). By \( C_H \), we denote the class of close-to-convex harmonic mappings \( f = h + \overline{g} \) for which \( f(\mathbb{D}) \) is close-to-convex in \( \mathbb{D} \). An analytic function \( h \in A \) is close-to-convex in \( \mathbb{D} \), if there exists an convex function \( \phi \) (not necessarily normalized) in \( \mathbb{D} \) such that

\[
\Re \left( \frac{h'(z)}{\phi'(z)} \right) > 0 \quad (z \in \mathbb{D}).
\]
If $\phi(z) = z$, then functions $h \in \mathcal{A}$ which satisfy $\Re(h'(z)) > 0$, are close-to-convex in $\mathbb{D}$. A function $h \in \mathcal{A}$ is said to be close-to-convex function of order $\beta$ ($0 \leq \beta < 1$), if it satisfies $\Re(h'(z)) > \beta$ ($z \in \mathbb{D}$). Let $\mathcal{W}(\alpha, \beta)$ denote a class of functions $h \in \mathcal{A}$ such that $\Re(h'(z) + \alpha z h''(z)) > \beta$ ($\alpha \geq 0, 0 \leq \beta < 1$). The class $\mathcal{W}(\alpha, \beta)$ was studied by Gao and Zoh [6] for $\beta < 1$ and $\alpha > 0$. They determined the extreme points of $\mathcal{W}(\alpha, \beta)$ and obtained a number $\beta(\alpha)$ such that $\mathcal{W}(\alpha, \beta) \subset \mathcal{S}^*$ for fixed $\alpha \in [1, \infty)$. The class $\mathcal{W}(\alpha, \beta)$ is generalization of class $\mathcal{W}(\alpha) \equiv \mathcal{W}(\alpha, 0)$, which was studied by Chichra [1]. In [20], Singh and Singh proved that functions in $\mathcal{W}(1, 0)$ are starlike in $\mathbb{D}$.

A harmonic function $f \in \mathcal{H}$ is said to be convex in $\mathbb{D}$, if $f(\mathbb{D})$ is convex in $\mathbb{D}$. We denote by $\mathcal{K}_\mathcal{H}$ the class of functions in $\mathcal{H}$ which are convex in $\mathbb{D}$. A sense preserving harmonic mapping $f = h + \overline{g} \in \mathcal{H}$ is known to be convex in $\mathbb{D}$, if $\frac{\partial}{\partial \theta}(\arg(\frac{\partial}{\partial \theta} f(re^{i\theta}))) > 0$ for all $z = re^{i\theta} \in \mathbb{D}/\{0\}$. Hence, $f = h + \overline{g} \in \mathcal{H}$ is convex in $\mathbb{D}$, if $f(z) \neq 0$ for all $z \in \mathbb{D}/\{0\}$ and condition

$$\Re \left\{ \frac{z(h'(z) + z h''(z)) + z(g'(z) + zg''(z))}{zh'(z) - zg'(z)} \right\} > 0$$

is satisfied for all $z \in \mathbb{D}/\{0\}$.

Let $h \in \mathcal{S}$ be given by $h(z) = \sum_{n=0}^{\infty} a_n z^n$. Then the $n^\text{th}$ partial sum (or section) of $h(z)$ is defined by

$$s_n(h) = \sum_{k=0}^{n} a_k z^k \quad \text{for} \quad n \in \mathbb{N},$$

where $a_0 = 0$ and $a_1 = 1$. One of the classical results of Szeg"o [21] shows that if $h \in \mathcal{S}$, then the partial sum $s_n(h)(z) = \sum_{k=0}^{n} a_k z^k$ is univalent in disk $|z| < 1/4$ for all $n \geq 2$, and number $1/4$ can not be replaced by larger one. In [19], Robertson proved that $n^\text{th}$ partial sum of the Koebe function $k(z) = z/(1 - z)^2$ is starlike in the disk $|z| < 1 - 3n^{-1}\log n$ ($n \geq 5$), and number $3$ can not be replaced by smaller constant. It is known by a result [4, p. 256, 273], that $s_n(h)$ is convex, starlike, or close-to-convex in the disk $|z| < 1 - 3n^{-1}\log n$ ($n \geq 5$), whenever $h$ is convex, starlike or close-to-convex in $\mathbb{D}$. The largest radius $r_n$ of univalence of $s_n(h)$ ($h \in \mathcal{S}$) is not yet known. However, Jenkins [11] (see also [4, Section 8.2]) observed that $r_n \geq 1 - (4 + \varepsilon)n^{-1}\log n$ for each $\varepsilon$ ($|\varepsilon| = 1$) and for large $n$. There exists a considerable amount of results in the literature for partial sums of functions in the class $\mathcal{S}$ and some of its geometric subclasses.

Analogously in the harmonic case, the $(p, q)$-th partial sum of a harmonic mapping $f = h + \overline{g} \in \mathcal{H}$ is defined by

$$s_{p,q}(f) = s_p(h) + s_q(h),$$

where $s_p(h) = \sum_{k=1}^{p} a_k z^k$ and $s_q(g) = \sum_{k=1}^{q} b_k z^k$, $p, q \geq 1$ with $a_1 = 1$, $p \geq 1$ and $q \geq 2$. In [15], Li and Ponnusamy studied the radius of univalency of partial sums of functions in the class $\mathcal{P}_\mathcal{H}^0 = \{ f = h + \overline{g} \in \mathcal{H}^0 : \Re(h'(z)) > |g'(z)| \quad (z \in \mathbb{D}) \}$. Further, in [14], Li and Ponnusamy studied partial sums of functions in the class $\mathcal{P}_\mathcal{H}^0(\alpha) = \{ f = h + \overline{g} \in \mathcal{H}^0 : \Re(h'(z) - \alpha) > |g'(z)| \quad (\alpha < 1, \quad z \in \mathbb{D}) \}$. Recently, Ghosh and Vasudevarao [7] studied a class of harmonic mappings $\mathcal{W}_\mathcal{H}^0(\alpha) = \{ f = h + \overline{g} \in \mathcal{H}^0 : \Re(h'(z) + \alpha z h''(z)) > |g'(z) + \alpha zg''(z)| \quad (z \in \mathbb{D}) \}$ and gave some results concerning growth, convolution and convex combination for the members of the class $\mathcal{W}_\mathcal{H}^0(\alpha)$.
For two analytic functions $\psi_1(z) = \sum_{n=0}^{\infty} a_n z^n$ and $\psi_2(z) = \sum_{n=0}^{\infty} b_n z^n$, the convolution (or Hadamard product) is defined by $(\psi_1 * \psi_2)(z) = \sum_{n=0}^{\infty} a_n b_n z^n$ ($z \in \mathbb{D}$). Analogously in the harmonic case, for two harmonic mappings $f_1 = h_1 + \overline{g_1}$ and $f_2 = h_2 + \overline{g_2}$ in $\mathcal{H}$ with the power series of the form

$$f_1(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n \quad \text{and} \quad f_2(z) = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} B_n z^n,$$

we define the harmonic convolution as follows:

$$f_1 * f_2 = h_1 * h_2 + \overline{g_1} * \overline{g_2} = z + \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} b_n B_n z^n.$$

Clearly, the class $\mathcal{H}$ is closed under the convolution, i.e. $\mathcal{H} \ast \mathcal{H} \subset \mathcal{H}$. In the case of conformal mappings, the literature about convolution theory is exhaustive. Unfortunately, most of these results do not necessarily carry over to the class of univalent harmonic mappings in $\mathbb{D}$. We refer [3, 12, 16], for more information about convolution of harmonic mappings.

We now define a new class of close-to-convex harmonic mappings as follows:

**Definition 1.1.** For $\alpha \geq 0$ and $0 \leq \beta < 1$, let $\mathcal{W}_H^0(\alpha, \beta)$ denote the class of harmonic mappings $f = h + \overline{g}$, which is defined by

$$\mathcal{W}_H^0(\alpha, \beta) = \{ f = h + \overline{g} \in \mathcal{H}^0 : \Re(h'(z) + \alpha z h''(z) - \beta) > |g'(z) + \alpha z g''(z)| \quad (z \in \mathbb{D}) \}.$$

We observe that, the class $\mathcal{W}_H^0(\alpha, \beta)$ generalizes several previously studied classes of harmonic mappings, as $\mathcal{W}_H^0(\alpha, 0) \equiv \mathcal{W}_H^0(\alpha)$ (see [7]), $\mathcal{W}_H^0(0, \beta) \equiv \mathcal{P}_H^0(\beta)$ (see [14]), $\mathcal{W}_H^0(1, 0) \equiv \mathcal{W}_H^0$ (see [17]), and $\mathcal{W}_H^0(0, 0) \equiv \mathcal{P}_H^0$ (see [15]).

In this article, we establish that functions in the class $\mathcal{W}_H^0(\alpha, \beta)$ are close-to-convex $\mathbb{D}$. In section 3, we obtain certain coefficient inequalities and growth results for the functions in $\mathcal{W}_H^0(\alpha, \beta)$. In section 4, we prove that the functions in $\mathcal{W}_H^0(\alpha, \beta)$ are closed under convex combinations and establish certain convolution results. In section 5, we determine the radius of convexity of partial sums $s_{p,q}(f)$ of functions in $\mathcal{W}_H^0(\alpha, \beta)$. Finally, in section 6, we consider the harmonic mappings which involve the hypergeometric function and obtain conditions on its parameters such that it belongs to the class $\mathcal{W}_H^0(\alpha, \beta)$. Further we construct the univalent harmonic polynomials belonging to $\mathcal{W}_H^0(\alpha, \beta)$. The following results will be needed in our investigation.

**Lemma 1.1.** (see, [9]). Let $p \in \mathcal{P}$, where $\mathcal{P}$ denotes the class of Carathéodory functions in $\mathbb{D}$. Then

$$|p'(z)| \geq \frac{1 - |z|}{1 + |z|} \quad \text{and} \quad \left| \frac{p''(z)}{p'(z)} \right| \leq \frac{2}{1 - |z|^2} \quad (z \in \mathbb{D}).$$

These inequalities are sharp. Equality occurs for suitable $z \in \mathbb{D}$ if and only if $p(z) = -z - 2e^{i\theta} \log(1 - ze^{i\theta})$ ($0 \leq \theta \leq 2\pi$).

**Lemma 1.2** (see [2]). If the harmonic mapping $f = h + \overline{g} : \mathbb{D} \to \mathbb{C}$ satisfies $|g'(0)| < |h'(0)|$ and the function $F_\epsilon = h + \epsilon g$ is close-to-convex for every $|\epsilon| = 1$, then $f$ is close-to-convex function.
The first result provides a one-to-one correspondence between the classes $\mathcal{W}^0_H(\alpha, \beta)$ of harmonic mappings and the class $\mathcal{W}(\alpha, \beta)$ of analytic functions.

**Theorem 2.1.** The harmonic mapping $f = h + \overline{g} \in \mathcal{W}^0_H(\alpha, \beta)$ if and only if $F_{\epsilon} = h + \epsilon g \in \mathcal{W}(\alpha, \beta)$ for each $|\epsilon| = 1$.

**Proof.** Let $f = h + \overline{g} \in \mathcal{W}^0_H(\alpha, \beta)$. Then for each $|\epsilon| = 1$, we have

$$\Re(F_{\epsilon}'(z) + \alpha z F_{\epsilon}''(z)) = \Re(h'(z) + \epsilon g'(z) + \alpha z h''(z) + \epsilon g''(z))$$
$$= \Re(h'(z) + \alpha z h''(z) + \epsilon(g'(z) + \alpha g''(z))]$$
$$> \Re(h'(z) + \alpha z h''(z)) - |g'(z) + \alpha z g''(z)| > \beta \quad (z \in \mathbb{D}).$$

Hence $F_{\epsilon} \in \mathcal{W}(\alpha, \beta)$ for each $|\epsilon| = 1$. Conversely, let $F_{\epsilon} \in \mathcal{W}(\alpha, \beta)$. Then

$$\Re(h'(z) + \alpha z h''(z)) > \Re(-\epsilon(g'(z) + \alpha z g''(z))) + \beta \quad (z \in \mathbb{D}).$$

As $\epsilon(|\epsilon| = 1)$ is arbitrary, then for an appropriate choice of $\epsilon$, we obtain

$$\Re(h'(z) + \alpha z h''(z) - \beta) > |g'(z) + \alpha z g''(z)| \quad (z \in \mathbb{D}),$$

and hence we conclude that $f \in \mathcal{W}^0_H(\alpha, \beta)$. \qed

To establish the next result, we need to establish that functions in the class $\mathcal{W}(\alpha, \beta)$ are close-to-convex in $\mathbb{D}$, and to prove this, we shall need the following result.

**Lemma 2.1.** (Jack's Lemma [10]) Let $\omega(z)$ be analytic in $\mathbb{D}$ with $\omega(0) = 0$. If $|\omega(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in \mathbb{D}$, then we have $z_0 \omega'(z_0) = k\omega(z_0)$ for a real number $k \geq 1$.

**Lemma 2.2.** If $f \in \mathcal{W}(\alpha, \beta)$, then $\Re(f'(z)) > \beta (0 \leq \beta < 1)$, and hence $f$ is close-to-convex in $\mathbb{D}$.

**Proof.** If $f \in \mathcal{W}(\alpha, \beta)$, then $\Re(\psi(z)) > 0$, where $\psi(z) = f'(z) + \alpha z f''(z) - \beta$. Let $w$ be an analytic function in $\mathbb{D}$ such that $w(0) = 0$ and

$$f'(z) = \frac{1 + (1 - 2\beta)w(z)}{1 - w(z)}.$$

To prove the result, we need to show that $|w(z)| < 1$ for all $z \in \mathbb{D}$. If not, then by Lemma 2.1, we could find some $\xi(|\xi| < 1)$, such that $|w(\xi)| = 1$ and $\xi w'(\xi) = k\omega(\xi)$, where $k \geq 1$. A computation gives

$$\Re\{\psi(\xi)\} = \Re\left\{\frac{1 + (1 - 2\beta)w(\xi)}{1 - w(\xi)} + \frac{2ak(1 - \beta)w(\xi)(1 - w(\xi))^2 - \beta}{\left(1 - w(\xi)\right)^2}\right\}$$
$$= \Re\left\{\frac{2ak(1 - \beta)w(\xi)}{(1 - w(\xi))^2}\right\} = -\frac{4ak(1 - \beta)(1 - \Re(w(\xi))}{|1 - w(\xi)|^4} \leq 0$$

for $|w(\xi)| = 1$. This contradicts the hypotheses. Hence, $|w(z)| < 1$, which lead to $\Re(f'(z)) > \beta (0 \leq \beta < 1)$. \qed

**Theorem 2.2.** The functions in the class $\mathcal{W}^0_H(\alpha, \beta)$ are close-to-convex in $\mathbb{D}$. \hfill \qed
Proof. From Lemma 2.2, we find that functions $F_\epsilon = h + \epsilon g \in \mathcal{W}(\alpha, \beta)$ are close-to-convex in $\mathbb{D}$ for each $\epsilon(|\epsilon| = 1)$. Now in view of Lemma 1.2 and Theorem 2.1, we obtain that functions in $\mathcal{W}_H^0(\alpha, \beta)$ are close-to-convex in $\mathbb{D}$. □

### 3. Coefficient Inequalities and Growth Estimates

The following results provides sharp coefficient bounds for the functions in $\mathcal{W}_H^0(\alpha, \beta)$.

**Theorem 3.1.** Let $f = h + \overline{g} \in \mathcal{W}_H^0(\alpha, \beta)$ be of the form (1.1) with $b_1 = 0$. Then we have

$$|b_n| \leq \frac{1 - \beta}{n(1 + \alpha(n-1))}. \tag{3.1}$$

The result is sharp and equality in (3.1) is obtained by $f(z) = z + \frac{1 - \beta}{n(1 + \alpha(n-1))} z^n$.

**Proof.** Since $f = h + \overline{g} \in \mathcal{W}_H^0(\alpha, \beta)$, then using the series representation of $g$, we have

$$r^{n-1} n(1 + \alpha(n-1)) |b_n| \leq \frac{1}{2\pi} \int_0^{2\pi} |g'(r e^{i\theta}) + \alpha r e^{i\theta} g''(r e^{i\theta})| \, d\theta$$

$$< \frac{1}{2\pi} \int_0^{2\pi} \{\Re(h'(r e^{i\theta}) + \alpha r e^{i\theta} h''(r e^{i\theta})) - \beta\} \, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \{1 - \beta + n(1 + \alpha(n-1)) a_n r^{n-1} e^{i(n-1)\theta}\} \, d\theta = 1 - \beta.$$

Now $r \to 1^-$ gives the desired bound. Further, it is easy to see that the equality in (3.1) is obtained for the function $f(z) = z + \frac{1 - \beta}{n(1 + \alpha(n-1))} z^n$. □

**Theorem 3.2.** Let $f = h + \overline{g} \in \mathcal{W}_H^0(\alpha, \beta)$ be of the form (1.1) with $b_1 = 0$. Then for $n \geq 2$, we have

(i) $|a_n| + |b_n| \leq \frac{2(1 - \beta)}{n(1 + \alpha(n-1))},$

(ii) $|a_n| - |b_n| \leq \frac{2(1 - \beta)}{n(1 + \alpha(n-1))},$

(iii) $|a_n| \leq \frac{2(1 - \beta)}{n(1 + \alpha(n-1))}.$

All these results are sharp for the function $f(z) = z + \sum_{n=2}^{\infty} \frac{2(1 - \beta)}{n(1 + \alpha(n-1))} z^n$.

**Proof.** (i) Since $f = h + \overline{g} \in \mathcal{W}_H^0(\alpha, \beta)$, then Theorem 2.1 implies that $F_\epsilon = h + \epsilon g \in \mathcal{W}(\alpha, \beta)$ for each $\epsilon(|\epsilon| = 1)$. Thus for each $|\epsilon| = 1$, we have

$$\Re((h + \epsilon g)'(z) + \alpha z(h + \epsilon g)'(z)) > \beta \quad \text{for} \quad z \in \mathbb{D}.$$ 

This implies that there exists a Carathéodory function of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, with $\Re(p(z)) > 0$ in $\mathbb{D}$, such that

$$h'(z) + \alpha z h''(z) + \epsilon (g'(z) + \alpha z g''(z)) = \beta + (1 - \beta) p(z). \tag{3.2}$$
Comparing coefficients on both sides of (3.2), we obtain
\begin{equation}
(3.3) \quad n(1 + \alpha(n - 1))(a_n + \epsilon b_n) = (1 - \beta)p_{n-1} \quad \text{for} \quad n \geq 2.
\end{equation}
Since $|p_n| \leq 2$ for $n \geq 1$ (see [4, p. 41]), and $\epsilon(\epsilon = 1)$ is arbitrary, therefore the result follows from (3.3). Part (ii) and (iii) follows from part (i).
\hfill \Box

The following result gives a sufficient condition for a function to be in the class $W^0_{\mathcal{H}}(\alpha, \beta)$.

**Theorem 3.3.** Let $f = h + \bar{g} \in \mathcal{H}^0$, where $h$ and $g$ are of the form (1.1). If
\begin{equation}
(3.4) \quad \sum_{n=2}^{\infty} n(1 + \alpha(n - 1))(|a_n| + |b_n|) \leq 1 - \beta,
\end{equation}
then $f \in W^0_{\mathcal{H}}(\alpha, \beta)$.

**Proof.** If $f = h + \bar{g} \in \mathcal{H}^0$, then using (3.4), we have
\[
\Re(h'(z) + \alpha z h''(z)) = \Re\left(1 + \sum_{n=2}^{\infty} n(1 + \alpha(n - 1)) a_n z^{n-1}\right)
\geq 1 - \sum_{n=2}^{\infty} n(1 + \alpha(n - 1)) |a_n| \geq \sum_{n=2}^{\infty} n(1 + \alpha(n - 1)) |b_n| + \beta
\geq |\sum_{n=2}^{\infty} n(1 + \alpha(n - 1)) b_n| + \beta = |g'(z) + \alpha z g''(z)| + \beta,
\]
and so $f \in W^0_{\mathcal{H}}(\alpha, \beta)$.
\hfill \Box

The following theorem gives sharp inequalities in the class $B^0_{\mathcal{H}}(\alpha, \beta)$.

**Theorem 3.4.** If $f = h + \bar{g} \in W^0_{\mathcal{H}}(\alpha, \beta)$, then
\begin{equation}
(3.5) \quad |z| - 2 \sum_{n=2}^{\infty} \frac{(-1)^{n-1}(1 - \beta)|z|^n}{\alpha n^2 + n(1 - \alpha)} \leq |f(z)| \leq |z| + 2 \sum_{n=2}^{\infty} \frac{(1 - \beta)|z|^n}{\alpha n^2 + n(1 - \alpha)}.
\end{equation}
Both the inequalities are sharp when $f(z) = z + \sum_{n=2}^{\infty} \frac{2(1 - \beta)}{\alpha n^2 + n(1 - \alpha)} z^n$, or its rotations.

**Proof.** Let $f = h + \bar{g} \in W^0_{\mathcal{H}}(\alpha, \beta)$. Then $F_{\epsilon} = h + \epsilon g \in \mathcal{W}(\alpha, \beta)$ for each $\epsilon (|\epsilon| = 1)$. Thus there exists an analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$ in $\mathbb{D}$, such that
\begin{equation}
(3.6) \quad F_{\epsilon}(z) + \alpha z F_{\epsilon}''(z) = \frac{1 + (1 - 2\beta)w(z)}{1 - w(z)}.
\end{equation}
Simplifying (3.6), we get
\[
z^{1/\alpha} F_{\epsilon}'(z) = \frac{1}{\alpha} \int_0^{|z|} (\frac{1}{\xi^{1/\alpha}} - 1 + (1 - 2\beta)w(\xi)) \xi^{1/\alpha - 1} \frac{1 + (1 - 2\beta)w(\xi^{1/\alpha})}{1 - w(\xi^{1/\alpha})} e^{i\theta} d\xi.
\]
Therefore using Schwarz Lemma, we have
\[
|z^{1/\alpha} F_{\epsilon}'(z)| = \left|\frac{1}{\alpha} \int_0^{|z|} (\frac{1}{t^{1/\alpha}} - 1 + (1 - 2\beta)w(t^{1/\alpha}) \frac{1 + (1 - 2\beta)w(t^{1/\alpha})}{1 - w(t^{1/\alpha})} e^{i\theta} dt\right| \leq \frac{1}{\alpha} \int_0^{|z|} \frac{1}{t^{1/\alpha - 1}} \frac{1 + (1 - 2\beta)t}{1 - t} dt,
\]
and

\[|z^{1/\alpha}F'(z)| = \left| \frac{1}{\alpha} \int_0^{|z|} \left( te^{i\theta}\right)^{\frac{1}{\alpha}-1} 1 + (1 - 2\beta)w(te^{i\theta}) \frac{1 - w(te^{i\theta})}{1 - w(te^{i\theta})} \, dt \right| \]
\[\geq \frac{1}{\alpha} \int_0^{\frac{|z|}{t^{\frac{1}{\alpha}}}} \left( 1 + (1 - 2\beta)w(te^{i\theta}) \right) \, dt \]
\[\geq \frac{1}{\alpha} \int_0^{\frac{|z|}{t^{\frac{1}{\alpha}}}} 1 + (1 - 2\beta)t \, dt.\]

Further computation gives

\[(3.7)\quad |F'(z)| = |h'(z) + \epsilon g'(z)| \leq 1 + 2(1 - \beta) \sum_{n=1}^{\infty} \frac{|z|^n}{1 + \alpha n},\]

and

\[|F'(z)| = |h'(z) + \epsilon g'(z)| \geq 1 + 2(1 - \beta) \sum_{n=1}^{\infty} \frac{(-1)^n|z|^n}{1 + \alpha n}.\]

Since \(\epsilon(|\epsilon| = 1)\) is arbitrary, it follows from (3.7) that

\[|h'(z)| + |g'(z)| \leq 1 + 2(1 - \beta) \sum_{n=1}^{\infty} \frac{|z|^n}{1 + \alpha n},\]

and

\[|h'(z)| - |g'(z)| \geq 1 - 2(1 - \beta) \sum_{n=1}^{\infty} \frac{(-1)^n|z|^n}{1 + \alpha n}.\]

Let \(\Gamma\) be the radial segment from 0 to \(z\), then

\[|f(z)| = \left| \int_{\Gamma} \frac{\partial f}{\partial \xi} d\xi + \frac{\partial f}{\partial \xi^*} d\xi^* \right| \leq \int_{\Gamma} (|h'(\xi)| + |g'(\xi)|) d\xi \]
\[\leq \int_0^{|z|} \left( 1 + 2(1 - \beta) \sum_{n=1}^{\infty} \frac{|t|^n}{1 + \alpha n} \right) dt = |z| + 2(1 - \beta) \sum_{n=2}^{\infty} \frac{|z|^n}{\alpha n^2 + (1 - \alpha)n},\]

and

\[|f(z)| = \int_{\Gamma} \left| \frac{\partial f}{\partial \xi} d\xi + \frac{\partial f}{\partial \xi^*} d\xi^* \right| \geq \int_{\Gamma} (|h'(\xi)| - |g'(\xi)|) d\xi \]
\[\geq \int_0^{|z|} \left( 1 - 2(1 - \beta) \sum_{n=1}^{\infty} \frac{(-1)^n|t|^n}{1 + \alpha n} \right) dt = |z| + 2(1 - \beta) \sum_{n=2}^{\infty} \frac{(-1)^{n-1}|z|^n}{\alpha n^2 + (1 - \alpha)n}.\]

Equality in (3.5) holds for the function \(f(z) = z + \sum_{n=2}^{\infty} \frac{2(1 - \beta)}{\alpha n^2 + (1 - \alpha)n} z^n\) or its rotations.

\[\square\]

4. Convex combinations and convolutions

In this section, we prove that the class \(W^0_{\alpha}(\alpha, \beta)\) is closed under convex combinations and convolutions. A sequence \(\{c_n\}_{n=0}^{\infty}\) of non-negative real numbers is said to be a convex null sequence, if \(c_n \to 0\) as \(n \to \infty\), and \(c_0 - c_1 \geq c_1 - c_2 \geq c_2 - c_3 \geq \ldots \geq c_{n-1} - c_n \geq \ldots \geq 0\). To prove results for convolution, we shall need the following Lemma 4.1 and 4.2.
Lemma 4.1. [5] If \( \{c_n\}_{n=0}^{\infty} \) be a convex null sequence, then function \( q(z) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n z^n \) is analytic and \( \Re(q(z)) > 0 \) in \( \mathbb{D} \).

Lemma 4.2. [20] Let the function \( p \) be analytic in \( \mathbb{D} \) with \( p(0) = 1 \) and \( \Re(p(z)) > 1/2 \) in \( \mathbb{D} \). Then for any analytic function \( f \) in \( \mathbb{D} \), the function \( p \ast f \) takes values in the convex hull of the image of \( \mathbb{D} \) under \( f \).

Theorem 4.1. The class \( \mathcal{W}_R^0(\alpha, \beta) \) is closed under convex combinations.

Proof. Let \( f_i = h_i + \overline{g_i} \in \mathcal{W}_R^0(\alpha, \beta) \) for \( i = 1, 2, \ldots, n \) and \( \sum_{i=1}^{n} t_i = 1 (0 \leq t_i \leq 1) \). Write the convex combination of \( f_i \)'s as

\[
 f(z) = \sum_{i=1}^{n} t_i f_i(z) = h(z) + \overline{g(z)}, 
\]

where \( h(z) = \sum_{i=1}^{n} t_i h_i(z) \) and \( g(z) = \sum_{i=1}^{n} t_i g_i(z) \). Clearly both \( h \) and \( g \) are analytic in \( \mathbb{D} \) with \( h(0) = g(0) = h'(0) - 1 = g'(0) = 0 \). A simple computation yields

\[
 \Re(h'(z) + \alpha z h''(z)) = \Re \left( \sum_{i=1}^{n} t_i (h'(z) + \alpha z h''(z)) \right) \geq \sum_{i=1}^{n} t_i (|g'_i(z) + \alpha z g''_i(z)| + \beta) 
\]

This shows that \( f \in \mathcal{W}_R^0(\alpha, \beta) \). \( \square \)

Lemma 4.3. If \( F \in \mathcal{W}(\alpha, \beta) \), then \( \Re \left( \frac{F(z)}{z} \right) > \frac{1}{2 - \beta} \).

Proof. If \( F \in \mathcal{W}(\alpha, \beta) \) be given by \( F(z) = z + \sum_{n=2}^{\infty} A_n z^n \), then

\[
 \Re \left( 1 + \sum_{n=2}^{\infty} n(1 + \alpha(n - 1)) A_n z^{n-1} \right) > \beta \quad (z \in \mathbb{D}),
\]

which is equivalent to \( \Re(p(z)) > \frac{1}{2 - \beta} \geq \frac{1}{2} \) in \( \mathbb{D} \), where \( p(z) = 1 + \frac{1}{2 - \beta} \sum_{n=2}^{\infty} n(1 + \alpha(n - 1)) A_n z^{n-1} \). Now consider a sequence \( \{c_n\}_{n=0}^{\infty} \) defined by \( c_0 = 1 \) and \( c_{n-1} = \frac{2 - \beta}{n(1 + \alpha(n - 1))} \) for \( n \geq 2 \). We can easily see that the sequence \( \{c_n\}_{n=0}^{\infty} \) is convex null sequence and hence in view of Lemma 4.1, the function \( q(z) = \frac{1}{2} + \sum_{n=2}^{\infty} \frac{2 - \beta}{n(1 + \alpha(n - 1))} z^{n-1} \) is analytic and \( \Re(q(z)) > 0 \) in \( \mathbb{D} \). Further

\[
 \frac{F(z)}{z} = p(z) \ast \left( 1 + \sum_{n=2}^{\infty} \frac{2 - \beta}{n(1 + \alpha(n - 1))} z^{n-1} \right).
\]

Hence an application of Lemma 4.2 gives that \( \Re \left( \frac{F(z)}{z} \right) > \frac{1}{2 - \beta} \) for \( z \in \mathbb{D} \). \( \square \)

Lemma 4.4. Let \( F_1 \) and \( F_2 \) belong to \( \mathcal{W}(\alpha, \beta) \), then \( F_1 \ast F_2 \in \mathcal{W}(\alpha, \beta) \).
Theorem 4.3. Let $f \in \mathcal{W}_H^0(\alpha, \beta)$ and $\phi \in \mathcal{A}$ be such that $\Re\left(\frac{\phi(z)}{z}\right) > \frac{1}{2}$ for $z \in \mathbb{D}$, then $f \ast \phi \in \mathcal{W}_H^0(\alpha, \beta)$.

Proof. Let $f = h + \overline{g} \in \mathcal{W}_H^0(\alpha, \beta)$. To prove that $f \ast \phi$ belongs to $\mathcal{W}_H^0(\alpha, \beta)$, it suffices to prove that $F_{\epsilon} = h \ast \phi + \epsilon(g \ast \phi)$ belongs to $\mathcal{W}(\alpha, \beta)$ for each $\epsilon(|\epsilon| = 1)$. Since $f = h + \overline{g} \in \mathcal{W}_H^0(\alpha, \beta)$, then $F_{\epsilon} = h + \epsilon g$ belongs to $\mathcal{W}(\alpha, \beta)$ for each $\epsilon(|\epsilon| = 1)$. Therefore

$$\frac{1}{1 - \beta} (F'_{\epsilon}(z) + \alpha z F''_{\epsilon}(z)) = \frac{1}{1 - \beta} (F'_{\epsilon}(z) + \alpha z F''_{\epsilon}(z) - \beta) \ast \frac{\phi(z)}{z}.$$  

Since $\Re\left(\frac{\phi(z)}{z}\right) > \frac{1}{2}$ and $\Re(F'_{\epsilon}(z) + \alpha z F''_{\epsilon}(z)) > \beta$ in $\mathbb{D}$, then in view of Lemma 4.2, we obtain that $F_{\epsilon} \in \mathcal{W}(\alpha, \beta)$. \qed
Proof. It is well known that, if \( \phi \) is convex then \( \Re \left( \frac{\phi(z)}{z} \right) > \frac{1}{2} \) for \( z \in \mathbb{D} \). Hence result follows from Theorem 4.3. \( \square \)

5. Partial sums

In this section, we determine the value of \( r \) such that the partial sums of \( f \in \mathcal{W}_H^0(\alpha, \beta) \) are convex in the disk \( |z| < r \).

Theorem 5.1. Let \( f = h + \overline{g} \in \mathcal{W}_H^0(\alpha, \beta) \). If \( p \) and \( q \) satisfies one of the following conditions:

(i) \( 1 = p < q \)
(ii) \( 3 \leq p < q \)
(iii) \( 3 \leq q < p \),

then \( s_{p,q}(f)(z) \) is convex in \( |z| < 1/4 \).

Proof. (i) By assumption, we know that

\[
\begin{align*}
\sum_{n=2}^{q} b_n z^n.
\end{align*}
\]

Since

\[
\Re \left\{ \frac{z + (z \phi'(q)(z))'}{z - z \phi'(q)(z)} \right\} = \Re \left\{ \frac{z + \sum_{n=2}^{q} n^2 b_n z^n}{z - \sum_{n=2}^{q} n b_n z^n} \right\} \quad \text{and} \quad \lim_{z \to 0} \frac{z + \sum_{n=2}^{q} n^2 b_n z^n}{z - \sum_{n=2}^{q} n b_n z^n} = 1,
\]

it suffices to prove

\[
A := \Re \left\{ \left( z + \sum_{n=2}^{q} n^2 b_n z^n \right) \left( z - \sum_{n=2}^{q} n b_n z^n \right) \right\} > 0 \quad \text{for} \quad |z| = 1/4.
\]

Now, we find that

\[
A = |z|^2 + \Re \left( \sum_{n=2}^{q} n^2 b_n z^n + 1 - \sum_{n=2}^{q} n b_n z^{n+1} \right) - \Re \left\{ \left( \sum_{n=2}^{q} n^2 b_n z^n \right) \left( \sum_{n=2}^{q} n b_n z^n \right) \right\}
\]

\[
\geq |z|^2 - \sum_{n=2}^{q} n(n-1)|b_n||z|^{n+1} - \left( \sum_{n=2}^{q} n^2|b_n||z|^n \right) \left( \sum_{n=2}^{q} n|b_n||z|^n \right).
\]

Further, using Theorem 3.1, we obtain

\[
A \geq |z|^2 - \sum_{n=2}^{q} \frac{(1-\beta)(n-1)}{1+\alpha(n-1)}|z|^{n+1} - \left( \sum_{n=2}^{\infty} \frac{n(1-\beta)}{1+\alpha(n-1)}|z|^n \right) \left( \sum_{n=2}^{\infty} \frac{(1-\beta)}{1+\alpha(n-1)}|z|^n \right)
\]

\[
\geq |z|^2 - (1-\beta) \sum_{n=2}^{q} (n-1)|z|^{n+1} - (1-\beta)^2 \left( \sum_{n=2}^{\infty} n|z|^n \right) \left( \sum_{n=2}^{\infty} |z|^n \right)
\]

\[
= |z|^2 - (1-\beta)|z|^3 \frac{1-q|z|^{q-1}+(q-1)|z|^q}{(1-|z|)^2} - (1-\beta)^2 |z|^4 \frac{(2-|z|-(q+1)|z|^{q-1}+q|z|^q)(1-|z|^{q-1})}{(1-|z|)^3}.
\]
Thus, for $|z| = 1/4$, we have
\[
\frac{A(1 - |z|)^3}{|z|^2} \geq (1 - |z|)^3 - (1 - \beta)|z|(1 - |z|)(1 - q)|z|^{q-1} + (q - 1)|z|^q
\]
\[-(1 - \beta)^2|z|^2(2 - |z| - (q + 3)|z|^{q-1} + (q + 1)|z|^q + (q + 1)|z|^{2q-2} - q|z|^{2q-1})
\]
\[\geq \frac{27}{64} - \frac{3}{16} \left(1 - \frac{q}{4^q} - \frac{q - 1}{q^q}\right) - \frac{1}{16} \left(\frac{7}{4} - \frac{q + 3}{4^q} + \frac{q + 1}{4^2q - 1} + \frac{q + 1}{4^2(q - 1)} - \frac{q}{4^2q - 1}\right)
\]
\[= \frac{1}{8} + \frac{12q + 14}{4^{2q+2}} \frac{3q + 4}{4^{2q-1}} = \frac{1}{8} + \frac{12q(4^q - 1) + 14 \times 4^q - 16}{4^{2q+2}} > 0.
\]
Hence the result follows.

(ii) Let $\sigma_p(h)(z) = \sum_{n=p+1}^{\infty} a_n z^n$ and $\sigma_q(g)(z) = \sum_{n=q+1}^{\infty} b_n z^n$, so that $h(z) = s_p(h)(z) + \sigma_p(h)(z)$ and $g(z) = s_q(g)(z) + \sigma_q(g)(z)$. Thus for each $|\varepsilon| = 1$, we may write
\[(5.1) \quad 1 + z \frac{s_p''(h)(z) + \varepsilon s_q''(g)(z)}{s_p'(h)(z) + \varepsilon s_q'(g)(z)} = 1 + \phi(z) + \psi(z),
\]
where
\[\phi(z) = \frac{z(h''(z) + \varepsilon g''(z))}{h'(z) + \varepsilon g'(z)} \quad \text{and} \quad \psi(z) = \frac{\phi(z)(\sigma_p'(h)(z) + \varepsilon \sigma_q'(g)(z)) - z(\sigma_p''(h)(z) + \varepsilon \sigma_q''(g)(z))}{h'(z) + \varepsilon g'(z) - (\sigma_p'(h)(z) + \varepsilon \sigma_q'(g)(z))}.
\]
Since $h + \varepsilon g \in \mathcal{P}$, using Lemma 1.1, we have
\[(5.2) \quad |\phi(z)| \leq \frac{2|z|}{1 - |z|^2} \quad \text{and} \quad |h'(z) + \varepsilon g'(z)| \geq \frac{1 - |z|}{1 + |z|}.
\]
Now, if $p \leq q$, then Theorem 3.1, yields that
\[(5.3) \quad |\sigma_p'(h)(z) + \varepsilon \sigma_q'(g)(z)| = \left| \sum_{n=p+1}^{q} n a_n z^{n-1} + \sum_{n=q+1}^{\infty} n(a_n + \varepsilon b_n) z^{n-1} \right| \leq \sum_{n=p+1}^{\infty} \frac{2(1 - \beta)}{1 + \alpha(n - 1)} |z|^{n-1} \leq 2(1 - \beta) \sum_{n=p+1}^{\infty} |z|^{n-1}
\]
\[= 2(1 - \beta) \frac{|z|^p}{1 - |z|}.
\]
Similarly,
\[(5.4) \quad |z(\sigma_p''(h)(z) + \varepsilon \sigma_q''(g)(z))| = \left| \sum_{n=p+1}^{q} n(n - 1) a_n z^{n-1} + \sum_{n=q+1}^{\infty} n(n - 1)(a_n + \varepsilon b_n) z^{n-1} \right| \leq \sum_{n=p+1}^{\infty} \frac{2(1 - \beta)(n - 1)}{1 + \alpha(n - 1)} |z|^{n-1} \leq 2(1 - \beta) \sum_{n=p+1}^{\infty} (n - 1)|z|^{n-1}
\]
\[= 2(1 - \beta) \left( \frac{p|z|^p}{1 - |z|} + \frac{|z|^{p+1}}{(1 - |z|)^2} \right).
\]
Using estimates (5.2) - (5.4), by the triangle inequality we deduce that
\[
|\psi(z)| \leq \frac{2(1-\beta)|z|^p \{3|z| + |z|^2 + p(1-|z|^2)\}}{(1 - |z|)(1 - |z|)^2 - 2(1-\beta)|z|^p(1 + |z|)}.
\]
Thus
\[
\Re(1 + \phi(z) + \psi(z)) \geq 1 - |\phi(z)| - |\psi(z)| \\
\geq 1 - \frac{2|z|}{1 - |z|^2} - \frac{2(1-\beta)|z|^p \{3|z| + |z|^2 + p(1-|z|^2)\}}{(1 - |z|)(1 - |z|)^2 - 2(1-\beta)|z|^p(1 + |z|)} \\
= \frac{1 - |z|^2 - 2|z|}{1 - |z|^2} - \frac{2(1-\beta)|z|^p \{3|z| + |z|^2 + p(1-|z|^2)\}}{(1 - |z|)(1 - |z|)^2 - 2(1-\beta)|z|^p(1 + |z|)},
\]
which for \(|z| = 1/4\) gives
\[
\Re(1 + \phi(z) + \psi(z)) \geq \frac{1}{3} \left( \frac{7}{5} - \frac{2(1-\beta)(13 + 15p)}{9 \times 4^n - 10(1-\beta)} \right) = B(p, \beta).
\]
Since the function \(B(p, \beta)\) is monotonically increasing with respect to \(p\) for \(p \geq 3\), the least estimate shows that \(\Re(1 + \phi(z) + \psi(z)) \geq A(p) \geq A(3) > 0\). Thus (5.1) implies for each \(|\epsilon| = 1\), that the section \(s_p(h) + \epsilon s_q(g)\) is convex in \(|z| \leq 1/4\) for \(3 \leq p \leq q\). As \(\epsilon\) is arbitrary, this shows that \(s_{p,q}(f)\) is convex in \(|z| < 1/4\), for \(3 \leq p \leq q\).

(iii) If \(p > q\), then using Theorem 3.1, we have
\[
|\sigma_p'(h)(z) + \epsilon \sigma_q'(g)(z)| = \left| \sum_{n=p+1}^{\infty} n(a_n + \epsilon b_n)z^{n-1} \right| \\
\leq \sum_{n=q+1}^{\infty} \frac{1 - \beta}{1 + \alpha(n - 1)} |z|^{n-1} + \sum_{n=p+1}^{\infty} \frac{2(1-\beta)|z|^p}{1 + \alpha(n - 1)} |z|^{n-1} \\
\leq (1 - \beta) \left( \sum_{n=q+1}^{\infty} |z|^{n-1} + 2 \sum_{n=p+1}^{\infty} |z|^{n-1} \right) = \frac{(1 - \beta)(|z|^p + |z|^q)}{1 - |z|},
\]
and
\[
|z(\sigma_p''(h)(z) + \epsilon \sigma_q''(g)(z))| = \left| \sum_{n=p+1}^{\infty} n(n-1)(a_n - \epsilon b_n)z^{n-1} \right| \\
\leq \sum_{n=q+1}^{\infty} \frac{(n-1)(1-\beta)}{1 + \alpha(n - 1)} |z|^{n-1} + \sum_{n=p+1}^{\infty} \frac{2(n-1)(1-\beta)}{1 + \alpha(n - 1)} |z|^{n-1} \\
\leq (1 - \beta) \left( \sum_{n=q+1}^{\infty} (n - 1)|z|^{n-1} + \sum_{n=p+1}^{\infty} 2(n-1)|z|^{n-1} \right) \\
= \frac{(1 - \beta)\{p|z|^p + q|z|^q - (p - 1)|z|^{p+1} - (q - 1)|z|^{q+1}\}}{(1 - |z|)^2}.
\]
Using estimates (5.2), (5.5) and (5.6), we obtain that
\[
|\psi(z)| \leq \frac{(1 - \beta)}{(1 - |z|)} \left( \frac{p|z|^p + q|z|^q + 3|z|^{p+1} + 3|z|^{q+1} - (p - 1)|z|^{p+2} - (q - 1)|z|^{q+2}}{1 - 2|z| + |z|^2 - (1-\beta)(1 + |z|)(|z|^p + |z|^q)} \right).
\]
Thus \( \Re (1 + \phi(z) + \psi(z)) \geq 1 - |\phi(z)| - |\psi(z)| \), which for \(|z| = 1/4\) reduces to
\[
\Re (1 + \phi(z) + \psi(z)) \geq \frac{4}{3} \left( \frac{7}{20} - \frac{(1 - \beta)\{4^p(15q + 13) + 4^{q}(15p + 13)\}}{9 \times 4^{p+q} - 20(1 - \beta)(4^p + 4^q)} \right)
\]
\[
> \frac{4}{3} \left( \frac{7}{20} - \frac{4^p(15q + 13) + 4^{q}(15p + 13)}{9 \times 4^{p+q} - 20(4^p + 4^q)} \right).
\]
Moreover, for \(p > q \geq 3\), we have
\[
\Re (1 + \phi(z) + \psi(z)) > \frac{4}{3} \left( \frac{7}{20} - \frac{305}{2204} \right) > 0,
\]
which implies that for each \(\epsilon\) with \(|\epsilon| = 1\), \(s_p(h) + \epsilon s_q(g)\) is convex in \(|z| < 1/4\), for \(3 \leq q \leq p\), and thus each section \(s_{p,q}(f)\) is convex in \(|z| < 1/4\) for \(3 \leq q \leq p\). \(\square\)

**Theorem 5.2.** Let \(f = h + \overline{g} \in \mathcal{W}_H^q(\alpha, \beta)\). Then

(i) For \(q > 2\), \(s_{2,q}(f)(z)\) is convex in the disk \(|z| < R_1\), where \(R_1\) is smallest positive root of the equation
\[
1 - 4r + (6\beta - 2)r^2 - 8(1 - \beta)r^3 + (1 - 2\beta)r^4 + 4(1 - \beta)r^5 = 0
\]
in \((0, 1)\).

(ii) For \(p > 2\), \(s_{p,2}(f)(z)\) is convex in the disk \(|z| < R_2\), where \(R_2\) is smallest positive root of the equation
\[
1 - 4r + (1 - \beta)r^2 - (8 - 3\beta)r^3 - (5 - 2\beta)r^4 - (4 - 3\beta)r^5 + 3(1 - \beta)r^6 = 0
\]
in \((0, 1)\).

**Proof.** (i) Let \(f = h + \overline{g} \in \mathcal{W}_H^q(\alpha, \beta)\), and suppose that \(p = 2 < q\). Then for each \(|\epsilon| = 1\), it is sufficient to show that
\[
X = \Re \left( 1 + \frac{z(s_p^\prime(h)(z) + \epsilon s_q^\prime(g)(z))}{s_p^\prime(h)(z) + \epsilon s_q^\prime(g)(z)} \right) > 0
\]
in the disk \(|z| < R_1\). For \(2 = p < q\), the estimates in (5.2)-(5.4) are continue to hold. Therefore, we deduce that
\[
(1 - |z|)X \geq \frac{1 - |z|^2 - 2|z|}{1 + |z|} - \frac{2(1 - \beta)|z|^2\{3|z| + 2(1 - |z|^2) + |z|^2\}}{1 - 2|z| + |z|^2 - 2(1 - \beta)|z|^2(1 + |z|)}
= \frac{1 - |z|^2 - 2|z|}{1 + |z|} - \frac{2(1 - \beta)|z|^2\{2 + 3|z| - |z|^2\}}{1 - 2|z| + (1 - 2(1 - \beta))|z|^2 - 2(1 - \beta)|z|^3}
= \frac{1 - 4|z| + \{4 - 6(1 - \beta)\}|z|^2 - 8(1 - \beta)|z|^3 + \{2(1 - \beta) - 1\}|z|^4 + 4(1 - \beta)|z|^5}{(1 + |z|)\{1 - 2|z| + (1 - 2(1 - \beta))|z|^2 - 2(1 - \beta)|z|^3\}}
= \frac{1 - 4|z| + (6\beta - 2)|z|^2 - 8(1 - \beta)|z|^3 + (1 - 2\beta)|z|^4 + 4(1 - \beta)|z|^5}{(1 + |z|)\{1 - 2|z| + (1 - 2(1 - \beta))|z|^2 - 2(1 - \beta)|z|^3\}}.
\]
which is greater then zero in \(|z| < R_1\), where \(R_1\) is the smallest positive root of the equation (5.7) in \((0, 1)\).
(ii) Let \( f = h + \overline{g} \in \mathcal{W}_R^\alpha(\alpha, \beta) \) and suppose that \( q = 2 < p \). Then for each \( |\epsilon| = 1 \), it is sufficient to show that
\[
Y = \Re \left( 1 + \frac{z(s''_h(h)(z) + \epsilon s''_g(g)(z))}{s''_h(h)(z) + \epsilon s''_g(g)(z)} \right) > 0
\]
in the disk \( |z| < R_2 \). Since for \( 2 = q < p \), the estimates in equations (5.2), (5.5) and (5.6) continue to hold. Therefore we deduce that
\[
(1 - |z|)Y \geq \frac{1 - |z|^2 - 2|z|}{1 + |z|} - \frac{(1 - \beta)(p|z|^p + 2|z|^2 + 3|z|^{p+1} + |z|^{p+2} - |z|^4)}{1 - 2|z| + |z|^2 - (1 - \beta)(|z|^p + |z|^2)(1 + |z|)}
\]
which is greater than zero in \( |z| < R_2 \), where \( R_2 \) is the smallest positive root of (5.8) in \((0, 1)\). \( \square \)

**Theorem 5.3.** If \( f = h + \overline{g} \in \mathcal{W}_R^\alpha(\alpha, \beta) \), then \( s_{2,2}(f)(z) \) is convex in \(|z| < (1 + \alpha)/4(1 - \beta)\).

**Proof.** Let \( s_{2,2}(f)(z) \in \mathcal{W}_R^\alpha(\alpha, \beta) \). Then for each \( |\epsilon| = 1 \), it is sufficient to show that
\[
\Re \left( 1 + \frac{z(s''_h(h)(z) + \epsilon s''_g(g)(z))}{s''_h(h)(z) + \epsilon s''_g(g)(z)} \right) > 0
\]
in the disk \( |z| < \frac{1 + \alpha}{4(1 - \beta)} \). In the view of Theorem 3.2, we have
\[
\Re \left( 1 + \frac{z(s''_h(h)(z) + \epsilon s''_g(g)(z))}{s''_h(h)(z) + \epsilon s''_g(g)(z)} \right) \geq 1 - \frac{z(s''_h(h)(z) + \epsilon s''_g(g)(z))}{s''_h(h)(z) + \epsilon s''_g(g)(z)}
\]
\[
= 1 - \frac{2(a_2 + \epsilon b_2)z}{1 + 2(a_2 + \epsilon b_2)z} \geq 1 - \frac{2|a_2 + \epsilon b_2||z|}{1 - 2|a_2 + \epsilon b_2||z|}
\]
\[
= 1 - \frac{4(1 - \beta)}{1 + \alpha} |z| > 0.
\]
Hence the result follows. \( \square \)

6. **Applications**

In this section, we consider the harmonic mappings whose co-analytic part involve the Gaussian hypergeometric function \( {}_2F_1(a, b; c; z) \), which is defined by
\[
{}_2F_1(a, b; c; z) = F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n \quad (z \in \mathbb{D}),
\]
where \(a, b, c \in \mathbb{C}, c \neq 0, -1, -2, \ldots\) and \((a)_n\) is the Pochhammer symbol defined by \((a)_n = a(a + 1)(a + 2) \cdots (a + n - 1)\) and \((a)_0 = 1\) for \(n \in \mathbb{N}\). The series (6.1) is absolutely convergent in \(\mathbb{D}\). Moreover, if \(\Re(c - a - b) > 0\), then the series (6.1) is convergent in \(|z| \leq 1\). Further, for \(z = 1\), we have the following well-known Gauss formula [22]

\[
F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} < \infty.
\]

We shall use the following Lemma to prove our results in this section:

**Lemma 6.1.** [18] Let \(a, b > 0\). Then the following holds:

(i) For \(c > a + b + 1\),

\[
\sum_{n=0}^{\infty} \frac{(n + 1)(a)_n(b)_n}{(c)_n n!} = \frac{\Gamma(c)\Gamma(c - a - b - 1)}{\Gamma(c - a)\Gamma(c - b)}(ab + c - a - b - 1).
\]

(ii) For \(c > a + b + 2\),

\[
\sum_{n=0}^{\infty} \frac{(n + 1)^2(a)_n(b)_n}{(c)_n n!} = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \left( \frac{(a)_2(b)_2}{(c - a - b - 2)_2} + \frac{3ab}{c - a - b - 1} + 1 \right).
\]

(iii) For \(a, b, c \neq 1\) with \(c > \max\{0, a + b + 1\}\),

\[
\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(n + 1)!} = \frac{1}{(a - 1)(b - 1)} \left[ \frac{\Gamma(c)\Gamma(c - a - b - 1)}{\Gamma(c - a)\Gamma(c - b)} - (c - 1) \right].
\]

**Theorem 6.1.** Let \(f_1(z) = z + z^2F(a, b; c; z)\), \(f_2(z) = z + z(F(a, b; c; z) - 1)\) and \(f_3(z) = z + z \int_0^z F(a, b; c; t)dt\), where \(a, b, c\) are positive real numbers such that \(c > a + b + 2\). Then the following holds:

(i) If

\[
\frac{\Gamma(c)\Gamma(c - a - b - 1)}{\Gamma(c - a)\Gamma(c - b)} \left[ \frac{\alpha(a)_2(b)_2}{c - a - b - 2} + (1 + 4\alpha)ab + 2(1 + \alpha)(c - a - b - 1) \right] \leq 1 - \beta,
\]

then \(f_1 \in \mathcal{W}_{\alpha, \beta}^0\).

(ii) If

\[
\frac{\Gamma(c)\Gamma(c - a - b - 2)}{\Gamma(c - a)(c - b)} [\alpha ab(ab + c - 1) + (1 + \alpha)ab(c - a - b - 2) + 1] \leq 2 - \beta,
\]

then \(f_2 \in \mathcal{W}_{\alpha, \beta}^0\).

(iii) If \(a, b, c \neq 1\) and \(c > \max\{0, a + b + 1\}\),

\[
\frac{\Gamma(c)\Gamma(c - a - b - 1)}{\Gamma(c - a)\Gamma(c - b)} \left[ \alpha ab + (1 + 2\alpha)(c - a - b - 1) + \frac{1}{(a - 1)(b - 1)} \right] \leq 1 - \beta,
\]

\[
-\frac{(c - 1)}{(a - 1)(b - 1)} \leq 1 - \beta,
\]

then \(f_3 \in \mathcal{W}_{\alpha, \beta}^0\).
Proof. (i) Let \( f_1(z) = z + z^2 F(a, b; c; z) = z + \sum_{n=2}^{\infty} C_n z^n \), where

\[
C_n = \frac{(a)_{n-2}(b)_{n-2}}{(c)_{n-2}(n-2)!} \quad \text{for} \quad n \geq 2.
\]

Therefore, we have

\[
\sum_{n=2}^{\infty} n(1 + \alpha(n - 1))|C_n| = \sum_{n=2}^{\infty} n(1 + \alpha(n - 1)) \frac{(a)_{n-2}(b)_{n-2}}{(c)_{n-2}(n-2)!}
\]

\[
= (1 + \alpha) \sum_{n=0}^{\infty} (n + 1) \frac{(a)n(b)_n}{(c)n!} + \alpha \sum_{n=0}^{\infty} (n + 1)^2 \frac{(a)n(b)_n}{(c)n!} + \sum_{n=0}^{\infty} \frac{(a)n(b)_n}{(c)n!}.
\]

Now, using Lemma 6.1 and Gauss formula (6.2), we have

\[
\sum_{n=2}^{\infty} n(1 + \alpha(n - 1))|C_n| = \frac{\Gamma(c)\Gamma(c - a - b - 1)}{\Gamma(c - a)\Gamma(c - b)} \left[ \frac{(a)2(b)_2}{c - a - b - 2} + (1 + 4\alpha)ab + 2(1 + \alpha)(c - a - b - 1) \right].
\]

If (6.3) holds, then \( \sum_{n=2}^{\infty} n(1 + \alpha(n - 1))|C_n| \leq 1 - \beta \). Hence the result follows.

(ii) Let \( f_2(z) = z + z(F(a, b; c; z - 1) = z + \sum_{n=2}^{\infty} D_n z^n \), where

\[
D_n = \frac{(a)n-1(b)_{n-1}}{(c)n-1(n-1)!} \quad \text{for} \quad n \geq 2.
\]

Therefore, we have

\[
\sum_{n=2}^{\infty} n(1 + \alpha(n - 1))|D_n| = \sum_{n=2}^{\infty} n(1 + \alpha(n - 1)) \frac{(a)n-1(b)_{n-1}}{(c)n-1(n-1)!}
\]

\[
= \sum_{n=0}^{\infty} (\alpha(n + 1)^2 + (1 + \alpha)(n + 1) + 1) \frac{(a)n+1(b)_{n+1}}{(c)n+1(n+1)!}.
\]

Now using the identity \( (\gamma)_{n+1} = \gamma(\gamma + 1)_n \), we have

\[
\sum_{n=2}^{\infty} n(1 + \alpha(n - 1))|D_n| = \frac{ab}{c} \sum_{n=0}^{\infty} (n + 1) \frac{(a+1)n(b+1)_n}{(c+1)n!}
\]

\[
+ \frac{ab}{c} \left[ (1 + \alpha) \sum_{n=0}^{\infty} \frac{(a+1)n(b+1)_n}{(c+1)n!} + \sum_{n=0}^{\infty} \frac{(a+1)n(b+1)_n}{(c+1)n(n+1)!} \right].
\]

Further, using Lemma 6.1 and Gauss formula (6.2), we obtain

\[
\sum_{n=2}^{\infty} n(1 + \alpha(n - 1))|D_n| = \frac{\Gamma(c)\Gamma(c - a - b - 2)}{\Gamma(c - a)\Gamma(c - b)} [aab(ab + c - 1) + (1 + \alpha)ab(c - a - b - 2) + 1] - 1.
\]

Now, if (6.4) holds, then \( \sum_{n=2}^{\infty} n(1 + \alpha(n - 1))|D_n| \leq 1 - \beta \), hence the result follows.

(iii) Let \( f_3(z) = z + \int_0^z F(a, b; c; t)dt = z + \sum_{n=2}^{\infty} E_n z^n \), where

\[
E_n = \frac{(a)n-2(b)_{n-2}}{(c)n-2(n-1)!} \quad \text{for} \quad n \geq 2.
\]
Therefore,
\[
\sum_{n=2}^{\infty} n(1 + \alpha(n - 1))|E_n| = \sum_{n=2}^{\infty} n(1 + \alpha(n - 1)) \frac{(a)_{n-2}(b)_{n-2}}{(c)_{n-2}(n - 1)!}.
\]
\[
= \alpha \sum_{n=0}^{\infty} (n + 1) \frac{(a)_{n}(b)_{n}}{n!} + (1 + \alpha) \sum_{n=0}^{\infty} \frac{(a)_{n}(b)_{n}}{n!n^2} + \sum_{n=0}^{\infty} \frac{(a)_{n}(b)_{n}}{(c)_{n}(n + 1)!}.
\]

Now using Lemma 6.1 and Gauss formula (6.2), we obtain
\[
\sum_{n=2}^{\infty} n(1 + \alpha(n - 1))|E_n| = \frac{\Gamma(c)\Gamma(c - a - b - 1)}{\Gamma(c - a)\Gamma(c - b)} \left[ aab + (1 + 2\alpha)(c - a - b - 1) + \frac{1}{(a - 1)(b - 1)} \right] - \frac{(c - 1)}{(a - 1)(b - 1)}.
\]

Further, if (6.5) holds, then the result follows. \(
\square
\)

Note that for \(\eta \in \mathbb{C}/\{-1, -2, \cdots \}\) and \(n \in \mathbb{N} \cup \{0\}\), we have
\[
\frac{(-1)^n(-\eta)_n}{n!} = \frac{\Gamma(\eta + 1)}{n!\Gamma(\eta - n + 1)}.
\]

In particular, when \(\eta = m(m \in \mathbb{N}, m \geq n)\), we have
\[
(-m)_n = \frac{(-1)^nm!}{(m - n)!}.
\]

Using above relations in Theorem 6.1, we get harmonic univalent polynomials which belongs to the class \(\mathcal{W}_0^0(\alpha, \beta)\). Setting \(a = b = -m(m \in \mathbb{N})\), we get

**Corollary 6.1.** Let \(m \in \mathbb{N}\), \(c\) be a positive real number and
\[
F_1(z) = z + \sum_{n=0}^{m} \frac{(m)}{n} \frac{(m - n + 1)}{(c)_n} z^{n+2},
\]
\[
F_2(z) = z + \sum_{n=0}^{m} \frac{(m)}{n} \frac{(m - n + 1)}{(c)_n} z^{n+1},
\]
\[
F_3(z) = z + \sum_{n=0}^{m} \frac{(m)}{n} \frac{(m - n + 1)}{(n + 1)(c)_n} z^{n+2}.
\]

Then the following holds:

(i) If
\[
\frac{\Gamma(c)\Gamma(c + 2m - 1)}{[\Gamma(c + m)]^2} \left[ \frac{cm^2(m - 1)^2}{c + 2m - 2} + (1 + 4\alpha)m^2 + 2(1 + \alpha)(c + 2m - 1) \right] \leq 1 - \beta,
\]
then \(F_1 \in \mathcal{W}_0^0(\alpha, \beta)\).

(ii) If
\[
\frac{\Gamma(c)\Gamma(c + 2m - 1)}{[\Gamma(c + m)]^2} \left[ \frac{cm^2 + c - 1}{c + 2m - 2} + (1 + \alpha)m^2(c + 2m - 2) + 1 \right] \leq 2 - \beta,
\]
then \(F_2 \in \mathcal{W}_0^0(\alpha, \beta)\).
If
\[
\Gamma(c)\Gamma(c + 2m - 1) \left[ \alpha m^2 + (1 + 2\alpha)(c + 2m - 1) + \frac{1}{m + 1)^2} \right] - \frac{(c - 1)}{(m + 1)^2} \leq 1 - \beta,
\]
then \( F_3 \in W_0^H(\alpha, \beta) \).

Further setting \( m = 2 \) and \( c = 1 \) in Corollary 6.1, we get

**Corollary 6.2.** If \( G_1(z) = z + z^2 + 4z^3 + z^4 \), \( G_2(z) = z + 4z^2 + z^3 \), and \( G_3(z) = z + z^2 + 2z^3 + \frac{1}{3}z^4 \), then the following holds:

(i) If \( 2(19\alpha + 9) \leq 1 - \beta \), then \( G_1(z) \in W_0^H(\alpha, \beta) \).

(ii) If \( 28\alpha + 13 \leq 2(2 - \beta) \), then \( G_2(z) \in W_0^H(\alpha, \beta) \).

(iii) If \( 108\alpha + 37 \leq 6(1 - \beta) \), then \( G_3(z) \in W_0^H(\alpha, \beta) \).

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