Involution on surfaces with $p_g = q = 1$

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Abstract

In this paper some numerical restrictions for surfaces with an involution are obtained. These formulas are used to study surfaces of general type $S$ with $p_g = q = 1$ having an involution $i$ such that $S/i$ is a non-ruled surface and such that the bicanonical map of $S$ is not composed with $i$. A complete list of possibilities is given and several new examples are constructed, as bidouble covers of surfaces. In particular the first example of a minimal surface of general type with $p_g = q = 1$ and $K^2 = 7$ having birational bicanonical map is obtained.

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1 Introduction

Several authors have studied surfaces of general type with \(p_g = q = 1\) ([Ca1], [Ca2], [CC1], [CC2], [CP], [Po1], [Po2], [Po3], [Pi]), but these surfaces are still not completely understood.

In [Ri2], the author gives several new examples of double planes of general type with \(p_g = q = 1\) having bicanonical map \(\varphi_2\) composed with the corresponding involution. The case \(S/i\) non-ruled and \(\varphi_2\) composed with \(i\) is considered in [Ri1].

In this paper we study the case \(\varphi_2\) not composed with \(i\). More precisely, we consider surfaces of general type \(S\) with \(p_g = q = 1\) having an involution \(i\) such that \(S/i\) is a non-ruled surface and \(\varphi_2\) is not composed with \(i\). A list of possibilities is given and new examples are obtained for each value of the birational invariants of \(S/i\) (only the existence of the case \(\text{Kod}(S/i) = 2, \chi(S/i) = 1\) and \(q(S/i) = 0\) remains an open problem).

The paper is organized as follows. In Section 2.2 we obtain formulas which, for a surface \(S\) with an involution \(i\), relate the invariants of \(S\) and \(S/i\) with the branch locus of the cover \(S \to S/i\), its singularities and the number of nodes of \(S/i\). Section 2.3 contains a description of the action of the involution \(i\) on the Albanese fibration of \(S\). In Section 3 we apply the numerical formulas of Section 2.2 to the case \(p_g = q = 1\), obtaining the classification results, Theorems 7, 8 and 9. Results of Miyaoka and Sakai on the maximal number of disjoint smooth rational or elliptic curves on a surface are also used here. Finally Section 4 contains the construction of examples, as bidouble covers of surfaces: the surfaces constructed in Sections 4.1, 4.2 and 4.3 are Du Val double planes (cf. [Ri2]) which have other interesting involutions; Section 4.4 contains the construction of a surface with \(K^2 = 4\), Albanese fibration of genus \(g = 2\) and \(\text{deg}(\varphi_2) = 2\) (thus it is not the example in [Ca2], for which \(\varphi_2\) is composed with the three involutions associated to the bidouble cover);

in Section 4.5 a new surface with \(K^2 = 8\) is obtained (it is not a standard isotrivial fibration);
Sections 4.6 and 4.7 contain the construction of new surfaces with \(K^2 = 7, 6\) and \(\text{deg}(\varphi_2) = 1\) (it is the first example with \(p_g = q = 1\) and \(K^2 = 7\) having birational bicanonical map);
bidouble covers of irregular ruled surfaces give interesting examples in Sections 4.8 and 4.9.

Some branch curves for these bidouble cover examples are computed in Appendix A.2 using the Computational Algebra System Magma ([BCP]).

Notation and conventions

We work over the complex numbers; all varieties are assumed to be projective algebraic. For a projective smooth surface \(S\), the canonical class is denoted by \(K\), the geometric genus by \(p_g := h^0(S, O_S(K))\), the irregularity by \(q := h^1(S, O_S(K))\) and the Euler characteristic by \(\chi = \chi(O_S) = 1 + p_g - q\).

An \((-n)\)-curve \(C\) on a surface is a curve isomorphic to \(\mathbb{P}^1\) such that \(C^2 = -n\). We say that a curve singularity is negligible if it is either a double point or a triple point which resolves to at most a double point after one blow-up. An \((m_1, m_2, \ldots)\)-point, or point of order \((m_1, m_2, \ldots)\), is a point of multiplicity \(m_1\), which resolves to a point of multiplicity \(m_2\) after one blow-up, etc.
An involution of a surface $S$ is an automorphism of $S$ of order 2. We say that a map is composed with an involution $i$ of $S$ if it factors through the double cover $S \to S/i$.

The rest of the notation is standard in Algebraic Geometry.

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2 Results on involutions

2.1 General facts

Let $S$ be a smooth minimal surface of general type with an involution $i$. Since $S$ is minimal of general type, this involution is biregular. The fixed locus of $i$ is the union of a smooth curve $R''$ (possibly empty) and of $t \geq 0$ isolated points $P_1, \ldots, P_t$. Let $S/i$ be the quotient of $S$ by $i$ and $p : S \to S/i$ be the projection onto the quotient. The surface $S/i$ has nodes at the points $Q_i := p(P_i)$, $i = 1, \ldots, t$, and is smooth elsewhere. If $R'' \neq \emptyset$, the image via $p$ of $R''$ is a smooth curve $B''$ not containing the singular points $Q_i$, $i = 1, \ldots, t$. Let now $h : V \to S$ be the blow-up of $S$ at $P_1, \ldots, P_t$ and set $R' = h^*(R'')$. The involution $i$ induces a biregular involution $\tilde{i}$ on $V$ whose fixed locus is $R := R' + \sum_1^t h^{-1}(P_i)$. The quotient $W := V/\tilde{i}$ is smooth and one has a commutative diagram:

$$
\begin{array}{ccc}
V & \xrightarrow{h} & S \\
\downarrow{\pi} & & \downarrow{p} \\
W & \xrightarrow{g} & S/i
\end{array}
$$

where $\pi : V \to W$ is the projection onto the quotient and $g : W \to S/i$ is the minimal desingularization map. Notice that

$$A_i := g^{-1}(Q_i), \quad i = 1, \ldots, t,$$

are $(-2)$-curves and $\pi^*(A_i) = 2 \cdot h^{-1}(P_i)$.

Set $B' := g^*(B'')$. Since $\pi$ is a double cover with branch locus $B' + \sum_1^t A_i$, it is determined by a line bundle $L$ on $W$ such that

$$2L \equiv B := B' + \sum_1^t A_i.$$  

It is well known that (cf. [BPV, Chapter V, Section 22]):

$$
p_g(S) = p_g(V) = p_g(W) + h^0(W, O_W(K_W + L)),
$$

$$
q(S) = q(V) = q(W) + h^1(W, O_W(K_W + L))
$$

(1)
and

\[ K_S^2 - t = K_V^2 = 2(K_W + L)^2, \]
\[ \chi(O_S) = \chi(O_V) = 2\chi(O_W) + \frac{1}{2}L(K_W + L). \] \hspace{1cm} (2)

Denote by \( \phi_2 \) the bicanonical map of \( S \) (given by \(|2K|\)). From the papers \([CM]\) and \([CCM]\).

\( \phi_2 \) is composed with \( i \) if and only if \( h^0(W, O_W(2K_W + L)) = 0 \).

### 2.2 Numerical restrictions

Let \( P \) be a minimal model of the resolution \( W \) of \( S/i \) and \( \rho : W \to P \) be the corresponding projection. Denote by \( B \) the projection \( \rho(B) \) and by \( \delta \) the "projection" of \( L \).

**Remark 1** If \( B \) is singular, there are exceptional divisors \( E_i \) and numbers \( r_i \in 2\mathbb{N} \) such that

\[
E_i^2 = -1, \quad K_W \equiv \rho'(K_P) + \sum E_i, \quad 2L \equiv B = \rho'(B) - \sum r_i E_i = \rho'(2\delta) - \sum r_i E_i.
\]

**Proposition 2** With the previous notation, if \( S \) is a surface of general type then:

a) \( \chi(O_P) - \chi(O_S) = K_P(K_P + \delta) + \frac{1}{2} \sum (r_i - 2) - h^0(W, O_W(2K_W + L)) \);

b) \( \delta^2 = -2\chi(O_P) - 2K_P^2 - 3K_P\delta + \frac{1}{4} \sum (r_i - 2)(r_i - 4) + 2h^0(W, O_W(2K_W + L)). \)

**Proposition 3** Let \( t \) be the number of nodes of \( S/i \). One has:

a) \( t = K_S^2 + 6\chi(O_W) - 2\chi(O_S) - 2h^0(W, O_W(2K_W + L)) \);

b) \( t = K_S^2P' + 8\chi(O_W) - 4\chi(O_S) \geq 8\chi(O_W) - 4\chi(O_S) \);

c) \( K_S^2 \geq 2\chi(O_W) - 2\chi(O_S) + 2h^0(W, O_W(2K_W + L)). \)

**Proposition 4** With the above notation:

a) \( h^0(W, O_W(2K_W + L)) \leq \frac{1}{4}K_W^2 - \chi(O_W) + \frac{11}{3}\chi(O_S) + \frac{1}{27}K_S^2; \)

b) \( h^0(W, O_W(2K_W + L)) \leq \frac{1}{2}K_W^2 + 5\chi(O_S) + 2q(S) - 3\chi(O_W) - 2q(W). \)

**Proof of Proposition 2** (cf. \([CM]\))

a) From the Kawamata-Viehweg’s vanishing theorem (see e.g. \([EV]\) Corollary 5.12, c)), one has

\[ h^i(W, O_W(2K_W + L)) = 0, \ i = 1, 2. \]

The Riemann-Roch theorem implies

\[ \chi(O_W(2K_W + L)) = \chi(O_W) + \frac{1}{2}L(K_W + L) + K_W(K_W + L), \]
thus, using (2),

\[ h^0(W, \mathcal{O}_W(2K_W + L)) = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_W) + K_W(K_W + L). \]  

(3)

With the notation of Remark 1 we can write

\[ \chi(\mathcal{O}_P) - \chi(\mathcal{O}_S) = \frac{1}{2}K_W(2K_W + 2L) - h^0(W, \mathcal{O}_W(2K_W + L)) = \]

\[ = \frac{1}{2} \left( \rho^*(K_P) + \sum E_i \right) \left( 2\rho^*(K_P + \delta) + \sum (2 - r_i)E_i \right) - h^0(W, \mathcal{O}_W(2K_W + L)) = \]

\[ = K_P(K_P + \delta) + \frac{1}{2} \sum (r_i - 2) - h^0(W, \mathcal{O}_W(2K_W + L)). \]

b) From the proof of a),

\[ h^0(W, \mathcal{O}_W(2K_W + L)) = \chi(\mathcal{O}_W(2K_W + L)) = \chi(\mathcal{O}_W) + \frac{1}{2}(2K_W + L)(K_W + L). \]

Using Remark 1 this means

\[ h^0(W, \mathcal{O}_W(2K_W + L)) = \chi(\mathcal{O}_P) + \]

\[ + \frac{1}{2} \left( \rho^*(2K_P + \delta) + \frac{1}{2} \sum (4 - r_i)E_i \right) \left( \rho^*(K_P + \delta) + \frac{1}{2} \sum (2 - r_i)E_i \right) = \]

\[ = \chi(\mathcal{O}_P) + K_P^2 + \frac{3}{2}K_P\delta + \frac{1}{8}g^2 - \frac{1}{8} \sum (r_i - 2)(r_i - 4). \]

\[ \square \]

**Proof of Proposition 3**

a) From formulas (2) and (3),

\[ t = K_S^2 - 2K_W(K_W + L) - 2L(K_W + L) = \]

\[ = K_S^2 + 2\chi(\mathcal{O}_S) - 2\chi(\mathcal{O}_W) - 2h^0(W, \mathcal{O}_W(2K_W + L)) - 4\chi(\mathcal{O}_S) + 8\chi(\mathcal{O}_W). \]

b) (This is also a consequence of the holomorphic fixed point formula.)

From (2),

\[ 4\chi(\mathcal{O}_S) - 8\chi(\mathcal{O}_W) = 2L(K_W + L) = \left( B' + \sum \frac{t}{1} A_i \right) (K_W + L) = \]

\[ = B'(K_W + L) - t = \frac{1}{2}\pi^*(B')\pi^*(K_W + L) - t = R''K_S - t. \]

Since \( S \) is of general type, \( K_SR'' \geq 0 \), thus

\[ t \geq 8\chi(\mathcal{O}_W) - 4\chi(\mathcal{O}_S). \]
c) This is immediate from a) and b).

Proof of Proposition 4

a) This inequality is given by the following three claims.

Claim 1:
\[ 1 - p_a(B') = 3\chi(O_W) - 3\chi(O_S) - K_W^2 - K_W^2 + 3h^0(W, O_W(2K_W + L)). \]

Proof: Formulas (2) and (3) give
\[ L^2 - K_W^2 = [2\chi(O_S) - 4\chi(O_W) - LK_W] - [h^0(W, O_W(2K_W + L)) - \chi(O_S) + \chi(O_W) - K_W L], \]
thus
\[ L^2 = K_W^2 + 3\chi(O_S) - 5\chi(O_W) - h^0(W, O_W(2K_W + L)). \]  

Now we perform a straightforward calculation using the adjunction formula, (2), Proposition 3, a) and (4):
\[ 2p_a(B') - 2 = 2K_WB' + B'^2 = K_W2L + (2L)^2 + 2t = 2L(K_W + L) + 2t + 2L^2 = 2[2\chi(O_S) - 4\chi(O_W)] + \\
+ 2[K_S^2 + 6\chi(O_W) - 2\chi(O_S) - 2h^0(W, O_W(2K_W + L))] + \\
+ 2[K_W^2 + 3\chi(O_S) - 5\chi(O_W) - h^0(W, O_W(2K_W + L))] = 2K_S^2 + 2K_W^2 + 6\chi(O_S) - 6\chi(O_W) - 6h^0(W, O_W(2K_W + L)). \]

Denote by \( \tau \) the number of rational curves of \( B' \).

Claim 2:
\[ 1 - p_a(B') \leq \tau. \]

Proof: Write
\[ B' = \sum_{i=1}^{\tau} B'_i + \sum_{\tau+1}^{h} B'_i \]
as a decomposition of \( B' \) in (smooth) connected components such that \( B'_i, i \leq \tau \), are the rational ones. The adjunction formula gives
\[ 2p_a(B') - 2 = \sum_{i=1}^{h} (K_WB'_i + B'^2_i) = \sum_{i=1}^{\tau} (2g(B'_i) - 2) + \sum_{\tau+1}^{h} (2g(B'_i) - 2) \geq -2\tau. \]

Claim 3:
\[ \tau \leq 8 \left( \chi(O_S) - \frac{1}{9}K_S^2 \right). \]
Proof: Since $B'$ does not contain $(-2)$-curves and it is contained in the branch locus of the cover $\pi : V \to W$, then each rational curve in $B'$ corresponds to a rational curve in $S$. Now the result follows from Proposition 5 below. 

Therefore $1 - p_a(B') \leq 8\left(\chi(O_S) - \frac{1}{9}K_S^2\right)$ and using Claim 1 we obtain the desired inequality.

b) Proposition 3 a) says that
\[ K_V^2 = K_S^2 - t = 2\chi(O_S) - 6\chi(O_W) + 2h^0(W, O_W(2K_W + L)). \]

The second Betti number $b_2$ of a surface $X$ satisfies
\[ b_2(X) = 12\chi(O_X) - K_X^2 + 4q(X) - 2. \]

Therefore
\[ b_2(V) = 10\chi(O_V) + 6\chi(O_W) + 4q(V) - 2 - 2h^0(W, O_W(2K_W + L)). \]

Since $b_2(V) \geq b_2(W)$, one has the result.

**Proposition 5** ([Mi, Proposition 2.1.1]) Let $X$ be a minimal surface of non-negative Kodaira dimension. Then the number of disjoint smooth rational curves in $X$ is bounded by
\[ 8\left(\chi(O_X) - \frac{1}{9}K_X^2\right). \]

### 2.3 Surfaces with an involution and $q = 1$

Let $S$ be a surface of general type with $q = 1$. Then the Albanese variety of $S$ is an elliptic curve $E$ and the Albanese map is a connected fibration (see e.g. [Be] or [BPV]).

Suppose that $S$ has an involution $i$. Then $i$ preserves the Albanese fibration (because $q(S) = 1$) and so we have a commutative diagram
\[
\begin{array}{ccc}
V & \xrightarrow{h} & S \\
\downarrow{\pi} & & \downarrow{p} \\
W & \longrightarrow & S/i \\
& & \longrightarrow \Delta
\end{array}
\]

where $\Delta$ is a curve of genus $\leq 1$. Denote by
\[ f_A : W \to \Delta \]

the fibration induced by the Albanese fibration of $S$.

Recall that
\[ \rho : W \to \mathbb{P} \]
is the projection of $W$ onto its minimal model $P$ and 
\[ \overline{B} := \rho(B), \]
where $B := B' + \sum_{i} A_i \subset W$ is the branch locus of $\pi$. Let 
\[ \overline{B'} := \rho(B') \quad \text{and} \quad \overline{A}_i = \rho(A_i). \]

When $\overline{B}$ has only negligible singularities the map $\rho$ contracts only exceptional curves contained in fibres of $f_A$. In fact otherwise there exists a $(-1)$-curve $J \subset W$ such that $JB = 2$ and $\pi^*(J)$ is transverse to the fibres of the (genus 1 base) Albanese fibration of $S$. This is impossible because $\pi^*(J)$ is a rational curve. Moreover $\rho$ contracts no curve meeting $\sum A_i$, thus the singularities of $\overline{B}$ are exactly the singularities of $\overline{B'}$, i.e. $\overline{B'} \cap \sum \overline{A}_i = \emptyset$. We denote the image of $f_A$ on $P$ by $\overline{f_A}$.

If $\Delta \cong \mathbb{P}^1$ and $\overline{f_A}$ is ramified over 4 points $p_j$ of $\Delta$, then the branch locus $B' + \sum_{i} A_i$ is contained in 4 fibres $F_j := f_A^{-1}(p_j)$, $j = 1, \ldots, 4$.

of the fibration $f_A$. Hence by Zariski’s Lemma (see e.g. [BPV]) the irreducible components $B'_i$ of $B'$ satisfy $B_i'^2 \leq 0$. If $\overline{B}$ has only negligible singularities then also $\overline{B'}^2 \leq 0$. Since $\pi^*(F_j^A)$ has even multiplicity, each component of $F_j^A$ which is not a component of the branch locus $B' + \sum_{i} A_i$ must be of even multiplicity.

## 3 Classification results

From now on $S$ is a smooth minimal surface of general type with $p_g = q = 1$ having an involution $\iota$ such that the bicanonical map $\varphi_2$ of $S$ is not composed with $\iota$. Notice that then $2 \leq K_S^2 \leq 9$, by the Debarre’s inequality for an irregular surface ($K_S^2 \geq 2p_g$) and by the Miyaoka-Yau inequality ($K_S^2 \leq 9\chi(O_S)$).

Recall from Section 2.2 that 
\[ h^0(W, \mathcal{O}_W(2K_W + L)) \neq 0, \]
where $W$ is the minimal resolution of $S/\iota$ and $L \equiv \frac{1}{2}B$ is the line bundle which determines the double cover $V \rightarrow W$.

Let $P$ be a minimal model of $W$ and $\delta$, $\overline{B} = 2\delta$ and the numbers $r_i$ be as defined in Section 2.2. Recall that $t$ denotes the number of nodes of $S/\iota$. Notice that $p_g(P) \leq p_g(S) = 1$ and $q(P) \leq q(S) = 1$.

In the next sections the following result is useful:

**Proposition 6** (Sa) Let $S$ be a minimal smooth surface of general type and $C \subset S$ be a disjoint union of smooth elliptic curves. Then 
\[ -C^2 \leq 36\chi(O_S) - 4K_S^2. \]

**Proof:** This follows from the inequality of [Sa, Corollary 7.8], using $KC + C^2 = 2p_a(C) - 2 = 0$. \[ \square \]
3.1 The case $\text{Kod}(S/i) = 0$

Here we give a list of possibilities for the case $\text{Kod}(S/i) = 0$.

**Theorem 7** Let $S$ and $P$ be as above. If $\text{Kod}(P) = 0$, only the following cases can occur:

a) $P$ is an Enriques surface and
- $\{ r_i \neq 2 \} = \{ 4 \}$, $B^2 = 0$, $t = 2 = K_S^2 \in \{ 2, \ldots, 7 \}$, or
- $\{ r_i \neq 2 \} = \{ 4, 4 \}$, $B^2 = 8$, $t = K_S^2 \in \{ 4, \ldots, 8 \}$, or
- $\{ r_i \neq 2 \} = \{ 6 \}$, $B^2 = 16$, $t = K_S^2 \in \{ 4, \ldots, 8 \}$;

b) $P$ is a bielliptic surface and
- $\{ r_i \neq 2 \} = \emptyset$, $B^2 = 8$, $t = 0$, $K_S^2 = 4$, or
- $\{ r_i \neq 2 \} = \{ 4 \}$, $B^2 = 16$, $t = K_S^2 = 6$ or $7$, or
- $\{ r_i \neq 2 \} = \{ 4, 4 \}$, $B^2 = 24$, $t = 0$, $K_S^2 = 8$, or
- $\{ r_i \neq 2 \} = \{ 6 \}$, $B^2 = 32$, $t = 0$, $K_S^2 = 8$.

Furthermore, there are examples for

- a) with $K_S^2 = 8$;
- b) with $K_S^2 = 4, 6, 7$ or $8$.

**Proof:** It is easy to see that $P$ cannot be a K3 surface: in this case we get from Proposition 4 b) that

$$K_W^2 \geq 2h^0(W, \mathcal{O}_W(2K_W + L)) - 2,$$

which implies $h^0(W, \mathcal{O}_W(2K_W + L)) = 1$ and $K_W^2 = 0$. This contradicts the fact $\sum(r_i - 2) = 4 \neq 0$, given by Proposition 2 a).

So, from the classification of surfaces (see e.g. Be or BPV), $p_g(P) = q(P) = 0$ or $p_g(P) = 0$, $q(P) = 1$ (notice that $p_g(P), q(P) \leq 1$), i.e. $P$ is an Enriques surface or a bielliptic surface.

a) Suppose $P$ is an Enriques surface: Proposition 4 a) implies that $h^0(W, \mathcal{O}_W(2K_W + L)) \leq 3$, with equality holding only if $K_W^2 = 0$. In this case the branch locus $B$ is smooth, i.e. $\sum(r_i - 2) = 0$, which contradicts Proposition 2 a). Therefore $h^0(W, \mathcal{O}_W(2K_W + L)) = 1$ or $2$.

Now the only possibilities allowed by Propositions 2 and 3 a), b) are:

1) $\sum(r_i - 2) = 2$, $B^2 = 0$, $t = K_S^2 + 2 \geq 4$;
2) $\sum(r_i - 2) = 4$, $B^2 = 8$ or $16$, $t = K_S^2 \geq 4$.

Moreover, if a nodal curve $A_i \subset B$ is not contracted to a point, then it is mapped onto a nodal curve of the Enriques surface $P$. Indeed, from the adjunction formula, $K_W A_i = 0$, which means that $A_i$ does not intersect any $(-1)$-curve of $W$. 

An Enriques surface has at most 8 disjoint \((-2)\)-curves. In case 1), the non-negligible singularities of \(\overline{B}\) are a 4-uple or \((3,3)\)-point, hence \(t \leq 9\). In case 2), \(t = 9\) only if \(\overline{B}\) has a \((3,3)\)-point, which implies that \(S\) has an elliptic curve with negative self-intersection. Since in this case \(K^2_S = 9\), this is impossible from Proposition 5, therefore \(t \leq 8\).

\[\sum (r_i - 2) = 2h^0(W, \mathcal{O}_W(2K_W + L)) - 2 = 6 \neq 0,\]

which contradicts \(K^2_W = 0\). Hence \(h^0(W, \mathcal{O}_W(2K_W + L)) \leq 3\).

As in a), if a \((-2)\)-curve \(A_i \subset B\) is not contracted to a point, then it is mapped onto a \((-2)\)-curve of \(P\). But a bielliptic surface has no \((-2)\)-curves (from Proposition 5), thus the nodal curves of \(B\) are contracted to singularities of \(\overline{B}\).

Using Propositions 2 and 3, a) one obtains the following possibilities:

1) \(\sum (r_i - 2) = 0, \overline{B}^2 = 8, K^2_S = t + 4;\)
2) \(\sum (r_i - 2) = 2, \overline{B}^2 = 16, K^2_S = t + 6;\)
3) \(\sum (r_i - 2) = 4, \overline{B}^2 = 24, K^2_S = t + 8;\)
4) \(\sum (r_i - 2) = 4, \overline{B}^2 = 32, K^2_S = t + 8.\)

In case 1), \(t = 0\), because \(\overline{B}\) has only negligible singularities. In case 2), \(\overline{B}\) can have a \((3,3)\)-point, thus \(t = 0\) or 1. In case 3), \(t = 1\) only if \(\overline{B}\) has a \((3,3)\)-point, but then \(K^2_S = 9\) and \(S\) has an elliptic curve, which is impossible from Proposition 6. Finally, in case 4), the only non-negligible singularity of \(\overline{B}\) is a point of multiplicity 6 (from Proposition 2 b)), thus \(t = 0\).

The examples are constructed in Sections 4.3, 4.5, 4.6, 4.7 and 4.9.

3.2 The case \(\text{Kod}(S/i) = 1\)

Now we give a list of possibilities for the case \(\text{Kod}(S/i) = 1\).

**Theorem 8** Let \(S\) and \(P\) be as above. If \(\text{Kod}(P) = 1\), only the following cases can occur:

a) \(\chi(\mathcal{O}_P) = 2, q(P) = 0\) and
- \(\{r_i\} = \emptyset, K_P \overline{B} = 4, \overline{B}^2 = -32, t - 8 = K^2_S \in \{4, \ldots, 8\};\)

b) \(\chi(\mathcal{O}_P) = 1, q(P) = 0\) and
- \(\{r_i \neq 2\} = \emptyset, K_P \overline{B} = 2, \overline{B}^2 = -12, t - 2 = K^2_S \in \{2, 3, 4\},\) or
- \(\{r_i \neq 2\} = \emptyset, K_P \overline{B} = 4, \overline{B}^2 = -16, t = K^2_S \in \{4, \ldots, 8\},\) or
\[ \{ r_i \neq 2 \} = \{ 4 \}, \ K_P B = 2, \ B^2 = -4, \ t = K_S^2 \in \{ 4, \ldots, 8 \}; \]

\textbf{c)} \( \chi(O_P) = 1, \ q(P) = 1 \) and

\[ \{ r_i \neq 2 \} = \emptyset, \ K_P B = 2, \ B^2 = -12, \ t - 2 = K_S^2 \in \{ 2, \ldots, 6 \}, \] or

\[ \{ r_i \} = \emptyset, \ K_P B = 4, \ B^2 = -16, \ t = K_S^2 \in \{ 4, \ldots, 8 \}. \]

\textbf{d)} \( \chi(O_P) = 0, \ q(P) = 1 \) and

\[ \{ r_i \neq 2 \} = \emptyset, \ K_P B = 2, \ B^2 = 4, \ t = 0, \ K_S^2 = 6, \] or

\[ \{ r_i \neq 2 \} = \emptyset, \ K_P B = 4, \ B^2 = 0, \ t = 0, \ K_S^2 = 8, \] or

\[ \{ r_i \neq 2 \} = \{ 4 \}, \ K_P B = 2, \ B^2 = 12, \ t = 0, \ K_S^2 = 8. \]

Furthermore, there exist examples for
- a) with \( K_S^2 = 8; \)
- b) with \( K_S^2 = 4, 6 \) or 7;
- c) with \( K_S^2 = 8; \)
- d) with \( K_S^2 = 6 \) or 8.

\textbf{Proof:} Since \( p_g(P), q(P) \leq 1, \) we have the following cases:

\textbf{a)} \( \chi(O_P) = 2, \ q(P) = 0. \)

From Proposition 4, b) it is immediate that \( h^0(W, O_W(2K_W + L)) = 1 \) and \( K_W^2 = 0 \) (thus \( B \) is smooth). Proposition 2 gives \( K_P B = 4 \) and \( B^2 = -32. \)

If \( K_S^2 = 9, \) then the number of nodal curves of \( B \) is \( t = K_S^2 + 8 = 17, \) from Proposition 4, a). This is impossible because Proposition 5 implies \( t \leq 16. \) Proposition 3, c) gives \( K_S^2 \geq 4. \)

\textbf{b)} \( \chi(O_P) = 1, \ q(P) = 0. \)

Proposition 4, a) implies \( h^0(W, O_W(2K_W + L)) \leq 3, \) with equality only if \( K_S^2 = 9 \) and \( K_W^2 = 0 \) (hence \( \sum (r_i - 2) = 0 \) and \( W = P \)). In this case Proposition 2, a) implies \( K_W B' = 6 \) and then \( B' \neq 0. \) Now \( p_a(B') = 1 \) (see Claim 1 in the proof of Proposition 4), thus \( B' \) is an union of elliptic components. But Proposition 5 implies that a minimal surface of general type with \( \chi = 1 \) and \( K^2 = 9 \) contains no elliptic curves. Therefore \( h^0(W, O_W(2K_W + L)) \leq 2. \)

Since \( \text{Kod}(P) = 1, \ K_P B = 0 \) implies that \( B \) is contained in the elliptic fibration of \( P \) and then \( S \) has an elliptic fibration, which is impossible because \( S \) is of general type.

So \( K_P B' = 0. \) Now Propositions 2 and 3, a) give the following possibilities:

1) \( \sum (r_i - 2) = 0, \ K_P B = 2, \ B^2 = -12, \ t = K_S^2 + 2; \)

2) \( \sum (r_i - 2) = 0, \ K_P B = 4, \ B^2 = -16, \ t = K_S^2; \)

3) \( \sum (r_i - 2) = 2, \ K_P B = 2, \ B^2 = -4, \ t = K_S^2. \)
In case 1), $t > 6$ implies $B^2 = B^2 + 2t > 0$, a contradiction (see Section 2.3).

Similarly $t \leq 8$, in case 2). Proposition 3, c) gives $K_S^2 \geq 4$, in this case.

In case 3), the quadruple or $(3, 3)$-point of $B$ gives rise to an elliptic curve in $S$, thus $K_S^2 \neq 9$, from Proposition 3. Again Proposition 3, c) implies $K_S^2 \geq 4$.

c) $\chi(O_P) = 1$, $q(P) = 1$.

This is analogous to the proof of b): just notice that Proposition 3, b) excludes case 3) and implies $K_P^2 = 0$ in case 2); in case 1) is no longer true that $t \leq 6$, instead use Proposition 5 to obtain $t \leq 8$ (thus $K_S^2 \leq 6$).

d) $\chi(O_P) = 0$, $q(P) = 1$.

As in b), one shows that $h^0(W, O_W(2K_W + L)) \leq 3$ and $K_P \neq 0$. Proposition 2 and 3, a) give the following possibilities:

1) $\sum(r_i - 2) = 0$, $K_P = 0$, $B^2 = 4$, $t = K_S^2 - 6$;
2) $\sum(r_i - 2) = 0$, $K_P = 4$, $B^2 = 0$, $t = K_S^2 - 8$;
3) $\sum(r_i - 2) = 2$, $K_P = 2$, $B^2 = 12$, $t = K_S^2 - 8$.

As in the proof of b), the existence of a quadruple or $(3, 3)$-point on $B$ implies $K_S^2 \neq 9$, in case 3).

Consider now cases 1) and 2). From Proposition 5, $P$ has no smooth rational curves. Any singular rational curve $D$ of $B$ satisfies $D^2 \leq 0$, because, since $B$ has only negligible singularities, $D$ is contained in fibres of a fibration $f_A$ of $P$ (see Section 2.3). Therefore $B$ has no $(-2)$-curves, i.e. $t = 0$.

The examples are given in Sections 4.1, 4.2, 4.4, 4.6, 4.7 and 4.8.

3.3 The case $\text{Kod}(S/i) = 2$

Finally we give a list of possibilities for the case $\text{Kod}(S/i) = 2$.

**Theorem 9** Let $S$ and $P$ be as above. If $\text{Kod}(P) = 2$, then $\overline{B}$ has at most negligible singularities and only the following cases can occur:

a) $\chi(O_P) = 2$, $q(P) = 0$ and

- $K_P = 0$, $B^2 = -24$, $t = 12$, $K_S^2 = 2K_P^2$, $K^2 = 2, 3, 4$, or
- $K_P = 2$, $B^2 = -28$, $t = 10 + 2K_P$, $K_S^2 \in \{2K_P^2 + 2, \ldots, 2K_P^2 + 4\}$

b) $\chi(O_P) = 1$, $q(P) = 1$ and

- $K_P = 0$, $B^2 = -8$, $t = 4$, $K_S^2 = 2K_P^2$, $K^2 = 2, 3, 4$, or
- $K_P = 2$, $B^2 = -12$, $K_P^2 = 2$, $t + 2 = K_S^2 \in \{6, 7, 8\}$.
Moreover, there exist examples for

- a) with \( K_S^2 = 4, 6, 7 \) or 8;
- b) with \( K_S^2 = 4 \).

**Proof:**

**Claim:** If \( K_P \mathcal{B} = 0 \), then \( \mathcal{B} \) is a disjoint union of nodal curves.

**Proof:** As \( P \) is of general type, \( \mathcal{B} \) is an union of nodal curves. Suppose that \( \mathcal{B} \) has a singularity. Then it contains two nodal curves \( D_1, D_2 \) such that \( D_1 D_2 \geq 2 \), otherwise \( B \) has a \((-3)\)-curve, contradicting \( B \equiv 0 \) (mod 2). Since \( K_P^2 > 0 \), \( K_P(D_1 + D_2) = 0 \) and \((D_1 + D_2)^2 \neq 0\), the index theorem implies that \( D_1 \) and \( D_2 \) is homologous to zero, a contradiction. \( \Box \)

Since \( p_g(P), q(P) \leq 1 \) and \( p_g(P) \geq q(P) \), we have only the following three cases:

**a)** \( \chi(\mathcal{O}_P) = 2, q(P) = 0. \)

Propositions \( 2 \) a) and \( 4 \) b) give:

\[
h^0(W, \mathcal{O}_W(2K_W + L)) = K_P^2 + K_P \delta + \frac{1}{2} \sum (r_i - 2) - 1,
\]

\[
h^0(W, \mathcal{O}_W(2K_W + L)) \leq \frac{1}{2} K_W^2 + 1 \leq \frac{1}{2} K_P^2 + 1.
\]

From this we get

\[
\frac{1}{2} K_P^2 + K_P \delta + \frac{1}{2} \sum (r_i - 2) \leq 2, \tag{6}
\]

with equality only if \( K_W^2 = K_P^2 \). Since \( K_P^2 > 0 \), \( K_P \delta = 0 \) or 1.

If \( K_P \delta = 1 \), then \( \sum (r_i - 2) = 0 \) and \( K_P^2 = 1 \) or 2.

If \( K_P \delta = 0 \), then \( \sum (r_i - 2) = 0 \), from the Claim above. As \( K_S^2 \leq 9 \), Proposition \( 3 \) implies \( h^0(W, \mathcal{O}_W(2K_W + L)) \leq 3 \).

Now the result follows from Propositions \( 2 \) and \( 3 \) a). Notice that Proposition \( 2 \) gives \( \mathcal{B}^2 \geq -2(12 + 2K_P \delta) \). This implies \( t \leq 12 + 2K_P \delta \), because, since \( q(P) = 0 \) and \( \mathcal{B} \) has only negligible singularities, every component of \( \mathcal{B} \) has non-positive self-intersection.

**b)** \( \chi(\mathcal{O}_P) = 1, q(P) = 1. \)

Equation \( 6 \) is still valid here. As \( p_g(P) = q(P) = 1 \), then \( K_P^2 \geq 2 \), hence

\[
K_P \delta + \frac{1}{2} \sum (r_i - 2) \leq 1.
\]
Using the Claim above and Proposition 2 we have

\[ K_P \delta = 0, \quad \sum (r_i - 2) = 0, \quad K^2_P = 2, 3 \text{ or } 4, \quad B^2 = -8, \quad t = 4, \text{ or} \]

\[ K_P \delta = 1, \quad \sum (r_i - 2) = 0, \quad K^2_P = 2, \quad B^2 = -12. \]

Now the result follows from Proposition 3 a) (notice that \( K^2_P = 2 \) implies \( t \neq 7 \), by Theorem 5).

c) \( \chi(O_P) = 1, \ q(P) = 0 \).

Propositions 2 a), 3 c) and 4 a) imply

\[ h^0(W, O_W(2K_W + L)) = K_P(K_P + \delta) + \frac{1}{2} \sum (r_i - 2) \leq 4 \quad (7) \]

and

\[ h^0(W, O_W(2K_W + L)) \leq 3 + \frac{1}{3} K^2_W, \quad (8) \]

with equality only if \( K^2_S = 9 \). As \( K^2_P \geq 1 \) and \( K^2_W \leq K^2_P \), this implies \( K_P \delta \leq 2 \).

• Suppose \( K_P \delta = 0 \).

We have \( \sum (r_i - 2) = 0 \), by the Claim above. Hence \( h^0(W, O_W(2K_W + L)) = K^2_P \leq 4 \), by (7). Now from Proposition 2 b) and Proposition 3 b), we have \( B^2 = (2\delta)^2 = -8 \) and \( t \geq 4 \). Thus \( t = 4 \) and, using Proposition 3 a), we conclude that

\[ K^2_S = 2K^2_P, \quad 1 \leq K^2_P \leq 4. \]

• Suppose \( K_P \delta = 1 \).

Then \( h^0(W, O_W(2K_W + L)) = 4 \) only if \( K^2_S = 9 \), from (7) and (8). In this case Propositions 3 b) and 2 a) imply \( K^2_P = 3 \) and \( B \) contains an elliptic curve, which contradicts Proposition 0.

If \( K^2_P = 1/2 \sum (r_i - 2) = 1 \), then \( K^2_W = 0 \) and \( K^2_S = 9 \), by (8). The quadruple or (3,3)-point of \( B \) gives rise to an elliptic curve in \( S \), which is impossible from Proposition 0.

Now using (7), Proposition 2 b) and Proposition 3 we obtain

\[ K^2_S = 2, \quad \sum (r_i - 2) = 0, \quad \delta^2 = -3, \quad t = K^2_S - 2 \geq 4 \]

or

\[ K^2_S = 1, \quad \sum (r_i - 2) = 0, \quad \delta^2 = -3, \quad t = K^2_S \geq 4. \]

Hence \( B^2 = -12 \) and then \( t \leq 6 \), because, since \( q(P) = 0 \) and \( B \) has only negligible singularities, every component of \( B \) has non-positive self-intersection.

• Suppose \( K_P \delta = 2 \).

Then \( K^2_P \leq 2 \), from (7), and \( h^0(W, O_W(2K_W + L)) \leq 3 \), from (8). The only possibility allowed by (7), Proposition 2 b) and Proposition 3 is:

\[ K^2_P = 1, \quad \delta^2 = -4, \quad t = K^2_S - 2 \geq 4. \]

It remains to be shown that \( K^2_S \neq 9 \). In this case, the curve \( B \) has at least 8 disjoint components contained in a fibration \( f_A \) of \( P \) (see Section 2.4).
These are independent in Pic($P$) from a general fibre of $\mathcal{F}_k$ and from $K_P$, so Pic($P$) has 10 independent classes. This is a contradiction because the second Betti number of $P$ is

$$b_2(P) = 12\chi(\mathcal{O}_P) - K_P^2 + 4q(P) - 2 = 9.$$ 

The examples can be found in Sections 4.2, 4.3, 4.4, 4.5, 4.6, 4.7 and 4.9.

4 (Bi)double cover examples

The next sections contain constructions of minimal smooth surfaces of general type $S$ with $p_g = q = 1$ which present examples for cases a), b), ... of Theorems 7, 8 and 9. Only the existence of case c) of Theorem 9 remains an open problem.

Each example is obtained as the smooth minimal model of a bidouble cover of a ruled surface (irregular only in Sections 4.8 and 4.9).

A bidouble cover is a finite flat Galois morphism with Galois group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Following [Ca2] or [Pa], to define a bidouble cover $\psi : V \to X$, with $V, X$ smooth surfaces, it suffices to present:

- smooth divisors $D_1, D_2, D_3 \subset X$ with pairwise transverse intersections and no common intersection;
- line bundles $L_1, L_2, L_3$ such that $2L_g \equiv D_j + D_k$ for each permutation $(g, j, k)$ of $(1, 2, 3)$.

If Pic($X$) has no 2-torsion, the $L_i$’s are uniquely determined by the $D_i$’s.

Let $N := 2K_X + \sum_1^3 L_i$. One has:

$$h^0(V, \mathcal{O}_V(2K_V)) \simeq H^0(X, \mathcal{O}_X(N) \oplus \bigoplus_{i=1}^3 H^0(X, \mathcal{O}_X(N - L_i)).$$

The bicanonical map of $V$ is composed with the involution $i_g$, associated to $L_g$, if and only if

$$h^0(X, \mathcal{O}_X(2K_X + L_g + L_j)) = h^0(X, \mathcal{O}_X(2K_X + L_g + L_k)) = 0.$$ 

For more information on bidouble covers see [Ca2] or [Pa].
Denote by $i_1, i_2, i_3$ the involutions of $V$ corresponding to $L_1, L_2, L_3$, respectively. In each example the invariants of $W_j := V/i_j, j = 1, 2, 3$, are given, but the details are not written here. These can be found in the author’s Ph.D. thesis, which can be sent by e-mail under request.

We use the following:

**Notation 10** Let $p_0, \ldots, p_j, \ldots, p_{j+s} \in \mathbb{P}^2$ be distinct points and define $T_i$ as the line through $p_0$ and $p_i, i = 1, \ldots, j$. We say that a plane curve is of type 

$d(m, (n, n)_{T_i}, r^{s})$

if it is of degree $d$ and if it has: an $m$-uple point at $p_0$, an $(n, n)$-point at $p_1, \ldots, p_j$, an $r$-uple point at $p_{j+1}, \ldots, p_{j+s}$ and no other non-negligible singularities. The index $T$ is used if $T_i$ is tangent to the $(n, n)$-point at $p_i$.

An obvious generalization is used if there are other singularities.

Let $p_1', \ldots, p_j'$ be the infinitely near points to $p_1, \ldots, p_j$, respectively. We denote by

$\mu : X \to \mathbb{P}^2$

the blow-up with centers

$p_0, p_1, p_1', \ldots, p_j, p_j', p_{j+1}, \ldots, p_{j+s}$

and by

$E_0, E_1, E_1', \ldots, E_j, E_j', E_{j+1}, \ldots, E_{j+s}$

the corresponding exceptional divisors (with self-intersection $-1$).

The notation $\tilde{\cdot}$ stands for the total transform $\mu^*(\cdot)$ of a curve.

The letter $T$ is reserved for a general line of $\mathbb{P}^2$.

The genus of a general Albanese fibre of $S$ is denoted by $g$.

4.1 $K^2 = 8$, $g = 3$,

$S/i_1$ ruled, $S/i_2$ rational, $\text{Kod}(S/i_1) = 1$

Here we construct a surface of general type $V$ with $p_g = q = 1$ and $g = 3$ such that $K_S^2 = 8$, where $S$ is the minimal model of $V$. The quotients $W_j := V/i_j, j = 1, 2, 3$, satisfy:

- $W_1$ is ruled, $q(W_1) = 1$;
- $W_2$ is rational;
- $\text{Kod}(W_3) = p_g(W_3) = 1, q(W_3) = 0$

This gives an example for case a) of Theorem [8]. The surface $S$ is a Du Val double plane of type III described in [Po3].

- Construction of $S$

Let $Q$ be a reduced curve of type $4(0, (2, 2)^2)$, i.e. $Q$ is the union of two conics tangent to the lines $T_1$ and $T_2$ at $p_1, p_2$. Let $C$ be another non-degenerate conic
tangent to $T_1$, $T_2$ at $p_1$, $p_2$ and let $T_3, \ldots, T_6 \neq T_1, T_2$ be distinct lines through $p_0 \in T_1 \cap T_2$.

Set:

\[ D_1 := \tilde{T}_1 + \cdots + \tilde{T}_6 - \sum_{i=1}^{2}(E_i + E'_i) - 6E_0, \]
\[ D_2 := \tilde{Q} - \sum_{i=1}^{2}(E_i + 3E'_i), \]
\[ D_3 := \tilde{C} - \sum_{i=1}^{2}(E_i + E'_i). \]

We have

\[ L_1 \equiv 3\tilde{T} - \sum_{i=1}^{2}(E_i + 2E'_i), \]
\[ L_2 \equiv 4\tilde{T} - \sum_{i=1}^{2}(E_i + E'_i) - 3E_0, \]
\[ L_3 \equiv 5\tilde{T} - \sum_{i=1}^{2}(E_i + 2E'_i) - 3E_0. \]

and

\[ K_X + L_1 \equiv -E'_1 - E'_2 + E_0, \]
\[ K_X + L_2 \equiv \tilde{T} - 2E_0, \]
\[ K_X + L_3 \equiv 2\tilde{T} - E'_1 - E'_2 - 2E_0. \]

Let $V \rightarrow X$ be the bidouble cover determined by $D_1, D_2, D_3$ and $S$ be the minimal model of $V$. We have

\[ p_g(S) = \sum_{i=1}^{3} h^0(X, \mathcal{O}_X(K_X + L)) = 0 + 0 + 1 = 1, \]
\[ \chi(\mathcal{O}_S) = 4 + \frac{1}{2} \sum_{i=1}^{3} L_i(K_X + L_i) = 4 - 2 - 1 + 0 = 1. \]

One has

\[ h^0(X, \mathcal{O}_X(2K_X + L_1 + L_2)) = 0, \]
\[ h^0(X, \mathcal{O}_X(2K_X + L_1 + L_3)) = 1, \]
\[ h^0(X, \mathcal{O}_X(2K_X + L_2 + L_3)) = 0, \]

thus the bicanonical map of $V$ is composed with the involution $i_2$ and is not composed with the involutions $i_1$ and $i_3$.

The **Albanese fibration** of $S$ is given by the pullback of the pencil of conics tangent to $T_1, T_2$ at $p_1, p_2$.

**· Calculation of $K^2_S$**

We have

\[ N := 2K_X + \sum_{i=1}^{3} L_i = N_1 + N_2, \]

where

\[ N_1 := \tilde{T}_1 + \tilde{T}_2 - 2E_0 - \sum_{i=1}^{2} 2E'_i, \quad N_2 := 4\tilde{T} - 2E_0 - \sum_{i=1}^{2} (E_i + E'_i). \]

Since the support of the pullback of $N_1$ is a disjoint union of $8 = -N^2_1$ (-1)-curves, $|N_2|$ has no fixed component and $N_1N_2 = 0$,

\[ K^2_S = K^2_V + (-N^2_1) = N^2 - N^2_1 = N^2_2 = 8. \]
4.2 $K^2 = 6, \ g = 4,$

\[ \text{Kod}(S/i_1) = 2, \ S/i_2 \text{ rational}, \ \text{Kod}(S/i_3) = 1 \]

This section contains the construction of a bidouble cover $V \rightarrow X$ such that the minimal model $S$ of $V$ is a surface of general type with $K^2 = 6, \ p_g = q = 1, \ g = 4$ and that the quotients $W_j := V/i_j$ satisfy:

- $\text{Kod}(W_1) = 2, \ p_g(W_1) = 1, \ q(W_1) = 0$;
- $W_2$ is rational;
- $\text{Kod}(W_3) = 1, \ p_g(W_3) = 0, \ q(W_3) = 1$.

This is an example for Theorems [8, d] and [9, a].

One can verify that $S$ is a Du Val double plane obtained imposing a 4-uple point to the branch locus of a Du Val’s ancestor of type $D_5$ (cf. [R12]).

Recall Notation [10]

- **Construction of $S$**

From Proposition [12] in Appendix A.1 there is a pencil $\mathcal{P}$, with no base component, of curves of type $7(3, (2, 2)^2_5)$. Let $Q$ be a general element of this pencil and $C$ be a reduced curve of type $4(2, (1, 1)^3_2)$.

Set:

\[
\begin{align*}
D_1 & := \widetilde{T}_1 + \cdots + \widetilde{T}_4 - \sum_4^1 (E_i + E'_i) + (E_5 - E'_5) - 4E_0, \\
D_2 & := \widetilde{T}_5 + \widetilde{Q} - \sum_4^1 (E_i + 3E'_i) - 3E_5 - 3E'_5 - 4E_0, \\
D_3 & := \widetilde{C} - \sum_4^1 (E_i + E'_i) - 2E_0.
\end{align*}
\]

We have

\[
\begin{align*}
L_1 & \equiv 6\widetilde{T} - \sum_4^1 (E_i + 2E'_i) - 2E_5 - 2E'_5 - 3E_0, \\
L_2 & \equiv 4\widetilde{T} - \sum_4^1 (E_i + E'_i) - E'_5 - 3E_0, \\
L_3 & \equiv 6\widetilde{T} - \sum_4^1 (E_i + 2E'_i) - 4E_0.
\end{align*}
\]

and

\[
\begin{align*}
K_X + L_1 & \equiv \left( 2\widetilde{T} - \sum_4^1 E'_i - E_0 \right) + \left( \widetilde{T}_5 - E_5 - E'_5 - E_0 \right), \\
K_X + L_2 & \equiv \widetilde{T} + E_5 - 2E_0, \\
K_X + L_3 & \equiv 3\widetilde{T} - \sum_4^1 E'_i - 3E_0,
\end{align*}
\]

hence

\[ p_g(S) = \sum_1^3 h^0(X, \mathcal{O}_X(K_X + L_i)) = 1 + 0 + 0 = 1 \]

and

\[ \chi(\mathcal{O}_S) = 4 + \frac{1}{2} \sum_1^3 L_i(K_X + L_i) = 4 + 0 - 2 = 1. \]

One has

\[
\begin{align*}
h^0(X, \mathcal{O}_X(2K_X + L_1 + L_2)) & = 0, \\
h^0(X, \mathcal{O}_X(2K_X + L_1 + L_3)) & = 2, \\
h^0(X, \mathcal{O}_X(2K_X + L_2 + L_3)) & = 0,
\end{align*}
\]

thus the bicanonical map of $V$ is composed with the involution $i_2$ and is not composed with the involutions $i_1$ and $i_3$. 
The Albanese fibration of $S$ is given by the pullback of the pencil $l$.

- **Calculation of $K^2_S$**
  We have
  \[ N := 2K_X + \sum_{i=1}^{3} L_i \equiv N_1 + N_2, \]
  where
  \[ N_1 := \tilde{T}_1 + \cdots + \tilde{T}_5 - 5E_0 - \sum_{i=1}^{5} 2E'_i, \quad N_2 := 5\tilde{T} - 3E_0 - \sum_{i=1}^{5} (E_i + E'_i). \]

  As the support of the pullback of $N_1$ is a disjoint union of $20 = -N_1^2$ (-1)-curves, $|N_2|$ has no fixed component and $N_1N_2 = 0$, then
  \[ K^2_S = K^2_V + (-N_1^2) = N^2 - N_1^2 = N_2^2 = 6. \]

4.3 $K^2 = 4$, $g = 3$.
Kod($S/i_1$) = 2, $S/i_2$ rational, Kod($S/i_3$) = 0

In this section a surface of general type $S$ with $p_g = q = 1$, $K^2 = 4$ and $g = 3$ is constructed. It is the minimal model of a double cover of surfaces $W_1, W_2, W_3$ such that:

- Kod($W_1$) = 2, $p_g(W_1) = 1$, $q(W_1) = 0$;
- $W_2$ is rational;
- Kod($W_3$) = 0, $p_g(W_3) = 0$, $q(W_3) = 1$.

This gives an example for Theorems 7, b) and 9, a).

One can verify that $S$ is a Du Val double plane obtained imposing two 4-uple points to the branch locus of a Du Val’s ancestor of type $D_4$ (cf. [R2]).

We keep Notation 10.

- **Construction of $S$**
  From Proposition 12, there is a pencil $l$, with no base component, of curves of type $6(2, (2, 2)^4)$, through points $p_0, \ldots, p_4$ (i.e. of plane curves of degree 6 with a double point at $p_0$ and a tacnode at $p_i$ with tangent line through $p_0, p_i$, $i = 1, \ldots, 4$). Let $Q$ be a general element of this pencil, $C$ be a reduced curve of type $4(2, (1, 1)^4)$ and set:

  \[ D_1 := \tilde{T}_1 + \cdots + \tilde{T}_4 - \sum_{i=1}^{4} (E_i + E'_i) - 4E_0, \]
  \[ D_2 := Q - \sum_{i=1}^{4} (E_i + 3E'_i) - 2E_0, \]
  \[ D_3 := \tilde{C} - \sum_{i=1}^{4} (E_i + E'_i) - 2E_0. \]

  Let $V \to X$ be the bidouble cover determined by $D_1, D_2, D_3$ and $S$ be the minimal model of $V$.

  We have

  \[ L_1 \equiv 5\tilde{T} - \sum_{i=1}^{4} (E_i + 2E'_i) - 2E_0, \]
  \[ L_2 \equiv 4\tilde{T} - \sum_{i=1}^{4} (E_i + E'_i) - 3E_0, \]
  \[ L_3 \equiv 5\tilde{T} - \sum_{i=1}^{4} (E_i + 2E'_i) - 3E_0. \]
and
\[
K_X + L_1 \equiv 2\bar{T} - \sum_1^4 E'_i - E_0, \\
K_X + L_2 \equiv \bar{T} - 2E_0, \\
K_X + L_3 \equiv 2\bar{T} - \sum_1^4 E'_i - 2E_0.
\]

Then
\[
p_g(S) = \sum_1^3 h^0(X, \mathcal{O}_X(K_X + L_i)) = 1 + 0 + 0 = 1
\]
and
\[
\chi(\mathcal{O}_S) = 4 + \frac{1}{2} \sum_1^3 L_i(K_X + L_i) = 4 + 0 - 1 - 2 = 1.
\]

As
\[
h^0(X, \mathcal{O}_X(2K_X + L_1 + L_2)) = 0, \\
h^0(X, \mathcal{O}_X(2K_X + L_1 + L_3)) = 1, \\
h^0(X, \mathcal{O}_X(2K_X + L_2 + L_3)) = 0,
\]
the bicanonical map of \(V\) is composed with the involution \(i_2\), corresponding to \(L_2\), and is not composed with the involutions \(i_1, i_3\), corresponding to \(L_1, L_3\).

The **Albanese fibration** of \(S\) is given by the pullback of the pencil \(l\).

- **Calculation of** \(K_S^2\)
  
  We have
  \[
  N := 2K_X + \sum_1^3 L_i \equiv N_1 + N_2,
  \]
  where
  \[
  N_1 := \tilde{T}_1 + \cdots + \tilde{T}_4 - 4E_0 - \sum_1^4 2E'_i, \\
  N_2 := 4\bar{T} - 2E_0 - \sum_1^4 (E_i + E'_i).
  \]

  As the support of the pullback of \(N_1\) is a disjoint union of \(16 = -N_1^2\) \((-1)\)-curves, \(|N_2|\) has no fixed component and \(N_1N_2 = 0\),
  \[
  K_S^2 = K_V^2 + (-N_1^2) = N^2 - N_1^2 = N_2^2 = 4.
  \]

4.4 \(K^2 = 4, g = 2, S/i_1 \text{ ruled, } \text{Kod}(S/i_2) = 1, \text{Kod}(S/i_3) = 2\)

This section contains the construction of a surface of general type \(S\) with \(p_g = q = 1, K^2 = 4, g = 2\) and \(\deg(\phi_2) = 2\), where \(\phi_2\) is the bicanonical map of \(S\).

The surface \(S\) is the minimal model of a double cover of surfaces \(W_1, W_2, W_3\) such that:

- \(W_1\) is ruled, \(q(W_1) = 1\);
- \(\text{Kod}(W_2) = 1, p_g(W_2) = q(W_2) = 0\);
- \(\text{Kod}(W_3) = 2, p_g(W_3) = 1, q(W_3) = 0\).
This gives an example for Theorems 8 b) and 9 a). Notation 10 is used in this section.

**Construction of \( S \)**

By Proposition 12 in Appendix A.1, there is a pencil \( l \), with no base component, of curves of type \( 6(2, 2; 2) \). Let \( Q_1 \) be a general element of this pencil, \( Q_2 \) be a smooth curve of type \( 3(1, 1; 1) \) and \( Q := Q_1 + Q_2 \). Let \( T_3 \) be a line through \( p_0 \) transverse to \( Q \).

Set:

\[
D_1 := \tilde{T}_1 + \tilde{Q} - 4E_1 - 4E'_1 - \sum_2^4(3E_i + 3E'_i) - 4E_0,
D_2 := \tilde{T}_2 + \cdots + \tilde{T}_5 - \sum_2^4(E_i + E'_i) - 4E_0,
D_3 := \sum_2^4(E_i - E'_i).
\]

We have

\[
L_1 \equiv 2\tilde{T} - \sum_2^4 E'_i - 2E_0, \\
L_2 \equiv 5\tilde{T} - 2E_1 - 2E'_1 - \sum_2^4(E_i + 2E'_i) - 2E_0, \\
L_3 \equiv 7\tilde{T} - \sum_2^4(2E_i + 2E'_i) - 4E_0
\]

and

\[
K_X + L_1 \equiv -\tilde{T} + \sum_2^4 E_i + E'_i - E_0, \\
K_X + L_2 \equiv 2\tilde{T} - \sum_2^4 E'_i - E_1 - E_0, \\
K_X + L_3 \equiv \tilde{T}_1 + \cdots + \tilde{T}_4 - \sum_2^4(E_i + E'_i) - 3E_0.
\]

Let \( \psi : V \to X \) be the bidouble cover determined by \( D_1, D_2, D_3 \) and \( S \) be the minimal model of \( V \). Then

\[
p_g(S) = \sum_1^3 h^0(X, \mathcal{O}_X(K_X + L_i)) = 0 + 0 + 1 = 1
\]

and

\[
\chi(\mathcal{O}_S) = 4 + \frac{1}{2} \sum_1^3 L_i(K_X + L_i) = 4 - 2 - 1 + 0 = 1.
\]

One can verify that

\[
0 = h^0(X, \mathcal{O}_X(2K_X + L_1 + L_2)), \\
0 = h^0(X, \mathcal{O}_X(2K_X + L_1 + L_3)), \\
1 = h^0(X, \mathcal{O}_X(2K_X + L_2 + L_3)).
\]

thus the bicanonical map of \( V \) is composed with the involution \( i_1 \) and is not composed with the involutions \( i_2 \) and \( i_3 \).

The **Albanese fibration** of \( S \) is given by the pullback of the pencil of lines through \( p_0 \).

**Calculation of \( K_S^2 \)**

We have

\[
N := 2K_X + \sum_1^3 L_i \equiv N_1 + N_2,
\]

where

\[
N_1 := \tilde{T}_1 + \cdots + \tilde{T}_4 - 4E_0 - (E_1 + E'_1) - \sum_2^4 2E'_i
\]
and
\[ N_2 := 4\overline{T} - 2E_0 - \sum_{i=1}^{4}(E_i + E'_i). \]

As the support of the pullback of \( N_1 \) is a disjoint union of 14 \(-1\)-
curves, \(|N_2|\) has no fixed component and \( N_1 \cdot N_2 = 0 \), then
\[ K^2_S = N^2 + (-N_1^2) = N_2^2 = 4. \]

\textbf{Degree of} \( \phi_2 \)

The system \( |\psi^*(N)| \) is strictly contained in the bicanonical system of \( V \). Since \( \phi_2 \)
is composed with \( i_1 \) and the map \( \tau : X \rightarrow \mathbb{P}^2 \) induced by \(|N| \) is birational (this
can be verified using the Magma function \texttt{IsInvertible}), one has \( \deg(\phi_2) = 2 \).

\textbf{4.5} \( K^2 = 8 \), \( g = 3 \),
\[
\text{Kod}(S/i_1) = 2, \text{Kod}(S/i_2) = 0, \text{Kod}(S/i_3) = 0
\]

A smooth projective surface \( S \) of general type is said to be a \textit{standard isotrivial fibration} if there exists a finite group \( G \) which acts faithfully on two smooth projective curves \( C \) and \( F \) so that \( S \) is isomorphic to the minimal desingularization of \( T := (C \times F)/G \). The paper \cite{Po1} contains examples of such surfaces with \( K^2 = 8 \).

This section contains the construction of the first surface of general type \( S \) with \( p_g = q = 1 \), \( K^2 = 8 \) and \( g = 3 \) which is not a standard isotrivial fibration.

The surface \( S \) is the minimal model of a double cover of surfaces \( W_1, W_2, W_3 \) such that:
- \( \text{Kod}(W_1) = 2, p_g(W_1) = 1, q(W_1) = 0; \)
- \( \text{Kod}(W_2) = 0, p_g(W_2) = 0, q(W_2) = 1; \)
- \( \text{Kod}(W_3) = 0, p_g(W_3) = 0, q(W_3) = 0. \)

This is an example for Theorems \cite{7a}, b) and \cite{9a}.

We use Notation \cite{10}

\textbf{Construction of} \( S \)

Let \( G \) be a curve of type \( 6(2, (2, 2)_1^2) \) and \( C \) be a curve of type \( 8(4, (2, 2)_1^2, (3, 3)) \)
such that \( G + C \) is reduced and the \( (3, 3) \)-point of \( C \) is tangent to \( G \). The existence of these curves is shown in Appendix \cite{A2}.

Set:
\[
D_1 := \overline{T_1} + \overline{T_2} - \sum_{i=1}^{2}2E'_i + (E_5 - E'_5) - 2E_0,
D_2 := \overline{G} - \sum_{i=1}^{4}(2E_i + 2E'_i) - (E_5 + E'_5) - 2E_0,
D_3 := \overline{T_3} + \overline{T_4} + \overline{C} - \sum_{i=1}^{2}(2E_i + 2E'_i) - \sum_{i=1}^{2}(2E_i + 4E'_i) - (3E_5 + 3E'_5) - 6E_0.
\]

We have
\[
L_1 \equiv 8\overline{T} - \sum_{i=1}^{2}(2E_i + 2E'_i) - \sum_{i=1}^{2}(2E_i + E'_i) - (2E_5 + 2E'_5) - 4E_0,
L_2 \equiv 6\overline{T} - \sum_{i=1}^{2}(E_i + 2E'_i) - 4E_0,
L_3 \equiv 4\overline{T} - \sum_{i=1}^{2}(E_i + E'_i) - \sum_{i=1}^{2}(E_i + E'_i) - E'_5 - 2E_0.
\]
and
\[ K_X + L_1 = 5\tilde{T} - \sum_1^2 (E_i + E_i') - \sum_5^4 (E_4 + 2E_4') - (E_5 + E_5') - 3E_0, \]
\[ K_X + L_2 = 3\tilde{T} - \sum_1^5 E_i' - 3E_0, \]
\[ K_X + L_3 = \tilde{T} - \sum_1^3 E_i' + E_5 - E_0. \]

Let \( V \rightarrow X \) be the bidouble cover determined by \( D_1, D_2, D_3 \) and \( S \) be the minimal model of \( V \). Then
\[ \chi(\mathcal{O}_S) = 4 + \frac{1}{2} \sum_1^3 L_i(K_X + L_i) = 4 + 0 - 1 = 1. \]

The Magma procedure \textit{LinSys} defined in [Ri2] can be used to confirm that
\[ h^0(X, \mathcal{O}_X(K_X + L_1)) = 1 \]
and
\[ h^0(X, \mathcal{O}_X(2K_X + L_1 + L_2)) = 0, \]
\[ h^0(X, \mathcal{O}_X(2K_X + L_1 + L_3)) = 1, \]
\[ h^0(X, \mathcal{O}_X(2K_X + L_2 + L_3)) = 2, \]
therefore
\[ p_g(S) = \sum_1^3 h^0(X, \mathcal{O}_X(K_X + L_i)) = 1 + 0 + 0 = 1 \]
and the bicanonical map of \( V \) is not composed with any of the involutions \( i_1, i_2, i_3 \).

The \textbf{Albanese fibration} of \( S \) is induced by a pencil of curves of type 14(6, (4, 4)^3, (4, 4)), which contains a fibre equal to \( G + C \) (see Appendix A.2). From [Po2 Theorem 3.2], the existence of such reducible fibre implies that \( S \) is not a standard isotrivial fibration, so this is not one of Polizzi’s examples.

\textbf{Calculation of} \( K_S^2 \)

We have
\[ N := 2K_X + \sum_1^3 L_i = N_1 + N_2, \]
where
\[ N_1 := \tilde{T}_1 + \cdots + \tilde{T}_4 - 4E_0 - \sum_1^4 2E_i' + (E_5 - E_5') \]
and
\[ N_2 := 8\tilde{T} - 4E_0 - \sum_1^5 (2E_i + 2E_i'). \]

As the support of the pullback of \( N_1 \) is a disjoint union of 18 = \(-N_1^2\) (-1)-curves, \(|N_2|\) has no fixed component (again this can be confirmed with the Magma procedure \textit{LinSys}) and \( N_1N_2 = 0 \),
\[ K_S^2 = N^2 + (-N_1^2) = N_2^2 = 8. \]
4.6 \( K^2 = 7, \ g = 3 \)

\[ \text{Kod}(S/i_1) = 2, \ \text{Kod}(S/i_2) = 1, \ \text{Kod}(S/i_3) = 0 \]

This section contains the construction of a a bidouble cover \( V \to X \), with \( X \) rational, such that the minimal model \( S \) of \( V \) is a surface of general type with \( K^2 = 7, \ p_g = q = 1 \) and \( g = 3 \) having birational bicanonical map.

Let \( i, j = 1, 2, 3 \), be the involutions associated to the bidouble cover. The quotients \( W_j := V/i_j, \ j = 1, 2, 3, \) satisfy:

- \( \text{Kod}(W_1) = 2, \ p_g(W_1) = 1, \ q(W_1) = 0; \)
- \( \text{Kod}(W_2) = 1, \ p_g(W_2) = 0, \ q(W_2) = 0; \)
- \( \text{Kod}(W_3) = 0, \ p_g(W_3) = 0, \ q(W_3) = 1. \)

This is an example for Theorems 7 b), 8 b) and 9 a).

We keep Notation [10]

- **Construction of \( S \)**

From Appendix A2 there exist a curve \( C \) of type \( 7(3(2, 2)^1, 3) \) (i.e. \( C \) is a plane curve of degree 7 with triple points at \( p_0, p_5 \) and a tacnode at \( p_i \) tangent to the line \( T_i \) through \( p_0, p_1, i = 1, \ldots, 4 \) and a curve \( G \) of type \( 6(2, 2)^1, 1) \), both through points \( p_0, \ldots, p_5 \), such that \( C + G \) is reduced.

Set:

\[
\begin{align*}
D_1 & := \tilde{T}_1 + \tilde{T}_2 + \tilde{T}_3 - \sum_i^3 2E_i' + E_5 - 3E_0, \\
D_2 & := \tilde{T}_4 + \tilde{G} - \sum_i^3 (2E_i + 2E_1') - (2E_4 + 4E_1') - E_5 - 3E_0, \\
D_3 & := C - \sum_i^3 (2E_i + 2E_1') - 3E_5 - 3E_0.
\end{align*}
\]

We have

\[
\begin{align*}
L_1 & := 7\tilde{T} - \sum_i^3 (2E_i + 2E_1') - (2E_4 + 3E_1') - 2E_5 - 3E_0, \\
L_2 & := 5\tilde{T} - \sum_i^3 (E_i + 2E_1') - (E_4 + E_1') - E_5 - 3E_0, \\
L_3 & := 5\tilde{T} - \sum_i^3 (E_i + 2E_1') - 3E_0
\end{align*}
\]

and

\[
\begin{align*}
K_X + L_1 & := 4\tilde{T} - \sum_i^3 (E_i + E_1') - (E_4 + 2E_1') - E_5 - 2E_0, \\
K_X + L_2 & := 2\tilde{T} - \sum_i^3 E_1' - 2E_0, \\
K_X + L_3 & := 2\tilde{T} - \sum_i^3 E_1' + E_5 - 2E_0.
\end{align*}
\]

Let \( \psi : V \to X \) be the bidouble cover determined by \( D_1, D_2, D_3 \) and \( S \) be the minimal model of \( V \). Then

\[
\chi(\mathcal{O}_S) = 4 + \frac{1}{2} \sum_i^3 L_i(K_X + L_i) = 4 + 0 - 1 - 2 = 1.
\]

The Magma procedure LinSys defined in [112] can be used to verify that

\[
h^0(X, \mathcal{O}_X(K_X + L_1)) = 1
\]

and

\[
\begin{align*}
h^0(X, \mathcal{O}_X(2K_X + L_1 + L_2)) & = 1, \\
h^0(X, \mathcal{O}_X(2K_X + L_1 + L_3)) & = 1, \\
h^0(X, \mathcal{O}_X(2K_X + L_2 + L_3)) & = 1,
\end{align*}
\]
therefore
\[ p_g(S) = \sum_1^3 h^0(X, O_X(K_X + L_i)) = 1 + 0 + 0 = 1 \]
and the bicanonical map of \( V \) is not composed with any of the involutions \( i_1, i_2, i_3 \).

The **Albanese fibration** of \( S \) is given by the pullback of the pencil of curves of type \( 6(2, (2, 2)) \).

• **Calculation of** \( K^2_S \)
We have
\[ N := 2K_X + \sum_1^3 L_i = N_1 + N_2, \]
where
\[ N_1 := \tilde{T}_1 + \cdots + \tilde{T}_4 - 4E_0 - \sum_1^4 2E'_i, \quad N_2 := 7\tilde{T} - 3E_0 - \sum_1^4 (2E_i + 2E'_i) - E_5. \]
As the support of the pullback of \( N_1 \) is a disjoint union of \( 16 = -N_1^2 \) \((-1\)-curves, \(|N_2| \) has no fixed component (use the procedure LinSys) and \( N_1N_2 = 0 \),
\[ K^2_S = N^2 + (-N_1^2) = N_2^2 = 7. \]

• **Verification that** \( \phi_2 \) **is birational**
The system \( |\psi^*(N)| \) is strictly contained in the bicanonical system of \( V \). The bicanonical map of \( V \) is not composed with any of the involutions \( i_1, i_2, i_3 \), hence it is birational if the map \( \tau \) given by \( |N| = N_1 + |N_2| \) is birational. This is in fact the case, see Appendix A.2, where Magma is used to show that the image of \( \tau \) is of degree \( 7 = N_2^2 \).

4.7 \( K^2 = 6, g = 3, \)
\[ \text{Kod}(S/i_1) = 2, \text{Kod}(S/i_2) = 1, \text{Kod}(S/i_3) = 0 \]
One can obtain a construction analogous to the one in Section 4.6 but with \( K^2_S = 6 \) instead: replace the triple point of \( C \) by a \((2, 2)\)-point, tangent to \( G \). Such a curve exists, see Appendix A.2. With this change the branch locus in \( W_3 \) has a 4-uple point instead of a \((3, 3)\)-point.

4.8 \( K^2 = 8, g = 3, \)
\[ \text{Kod}(S/i_1) = 1, S/i_2 \text{ ruled}, \text{Kod}(S/i_3) = 1 \]
Here we give the construction of a surface of general type \( S \), with \( K^2 = 8, p_g = q = 1 \) and \( g = 3 \), as a bidouble cover of a ruled surface \( Z \) with \( q(Z) = 1 \).

Let \( i_j, j = 1, 2, 3 \), be the involutions associated to the bidouble cover. The quotients \( W_j := S/i_j, j = 1, 2, 3 \), satisfy:
• \( \text{Kod}(W_1) = 1, p_g(W_1) = 0, q(W_1) = 1; \)
• \( W_2 \) is ruled, \( q(W_2) = 1; \)
\[ K(W_3) = p_g(W_3) = q(W_3) = 1. \]

This is an example for cases c) and d) of Theorem 8.
Notation 10 is used in this section.

**Construction of \( S \)**

Let \( F_1, \ldots, F_4 \) be disjoint fibres of the Hirzebruch surface \( \mathbb{F}_0 \) and \( Z \to \mathbb{F}_0 \) be the double cover with branch locus \( F_1 + \cdots + F_4 \). Clearly \( Z \) is a ruled surface with irregularity 1. Denote by \( \gamma \) the rational fibration of \( Z \).

Let \( G, G_1, \ldots, G_6 \) be distinct smooth elliptic sections of \( \gamma \) and \( \Gamma_1, \ldots, \Gamma_4 \) be distinct fibres of \( \gamma \) such that \( \Gamma_1 + \Gamma_2 \equiv 2\Gamma_3 \equiv 2\Gamma_4 \).

Set
\[
D_1 := \Gamma_1 + \Gamma_2, \\
D_2 := G_1 + \cdots + G_4, \\
D_3 := G_5 + G_6
\]
and
\[
L_1 := 3G + \Gamma_3 - \Gamma_4, \\
L_2 := G + \Gamma_4, \\
L_3 := 2G + \Gamma_3.
\]

We have
\[
K_Z + L_1 \equiv G + \Gamma_3 - \Gamma_4, \\
K_Z + L_2 \equiv -G + \Gamma_4, \\
K_Z + L_3 \equiv \Gamma_3
\]
and then
\[
p_g(S) = \sum_1^3 h^0(Z, \mathcal{O}_Z(K_Z + L_i)) = 0 + 0 + 1 = 1,
\]
\[
\chi(\mathcal{O}_S) = 4 \cdot 0 + \frac{1}{2} \sum_1^3 L_i(K_Z + L_i) = 0 + 0 + 1 = 1.
\]

As
\[
h^0(X, \mathcal{O}_X(2K_X + L_1 + L_2)) = 0, \\
h^0(X, \mathcal{O}_X(2K_X + L_1 + L_3)) = 2, \\
h^0(X, \mathcal{O}_X(2K_X + L_2 + L_3)) = 1,
\]
the bicanonical map of \( V \) is not composed with any of the involutions \( i_1, i_2, i_3 \).

The linear system
\[
|N| := |2K_Z + \sum_1^3 L_i| \equiv |2G + 2\Gamma_3|
\]
has no base component, thus
\[
K_S^2 = N^2 = 8.
\]

4.9 \( K^2 = 4, g = 3, \)

\( S/i_1 \) **ruled**, \( \text{Kod}(S/i_2) = 0, \text{Kod}(S/i_3) = 2 \)

This section contains the construction of a bidouble cover \( V \to Z \), with \( Z \) ruled and \( q(Z) = 1 \), such that the minimal model \( S \) of \( V \) is a surface of general type.
with $K^2 = 4$, $p_g = q = 1$, $g = 3$ and that the bicanonical map $\phi_2$ of $S$ is not composed with any of the involutions $i_1, i_2, i_3$ associated to the bidouble cover.

The quotients $W_j := V/i_j$, $j = 1, 2, 3$, satisfy:

- $W_1$ is ruled, $q(W_1) = 1$;
- $\text{Kod}(W_2) = 0$, $p_g(W_2) = q(W_2) = 1$;
- $\text{Kod}(W_3) = 2$, $p_g(W_3) = 1$, $q(W_3) = 1$; the branch locus of the cover $V \to W_3$ is an union of four $(-2)$-curves.

This is an example for Theorems 7, b) and 9, b).

We use Notation 10.

- **Construction of $S$**

Let $Q_1$ be a general curve of type $5(1, (2, 2)\frac{3}{2})$ (there is a pencil of such curves, see Appendix A.1) and $Q_2$ be a general curve of type $3(1, (1, 1)\frac{3}{2})$, both through points $p_0, \ldots, p_3$.

Let

\[
Q'_1 := \widetilde{Q}_1 - \sum_{1}^{3}(2E_i + 2E'_i) - E_0 \equiv 5\widetilde{T} - \sum_{1}^{3}(2E_i + 2E'_i) - E_0,
\]

\[
Q'_2 := \widetilde{Q}_2 - \sum_{1}^{3}(E_i + E'_i) - E_0 \equiv 3\widetilde{T} - \sum_{1}^{3}(E_i + E'_i) - E_0.
\]

Consider the double cover $\psi : Z \to X$ with branch locus

\[
\widetilde{T}_1 + \cdots + \widetilde{T}_4 - \sum_{1}^{3}2E'_i - 4E_0,
\]

where $T_4$ is a general line through $p_0$.

Let

\[
\Gamma := \frac{1}{2}\psi^*(\widetilde{T}_4 - E_0), \quad \Gamma_i := \frac{1}{2}\psi^*(\widetilde{T}_i - E_0),
\]

\[
C_0 := \psi^*(E_0),
\]

\[
e_i := \frac{1}{2}\psi^*(E_i - E'_i),
\]

\[
e'_i := \psi^*(E'_i) \quad i = 1, 2, 3,
\]

and set

\[
D_1 := \psi^*(Q'_1) \equiv 4C_0 + 10\Gamma - \sum_{1}^{3}(4e_i + 4e'_i),
\]

\[
D_2 := \psi^*(Q'_2) \equiv 2C_0 + 6\Gamma - \sum_{1}^{3}(2e_i + 2e'_i),
\]

\[
D_3 := 0,
\]

\[
L_1 := C_0 + 3\Gamma - \sum_{1}^{3}(e_i + e'_i),
\]

\[
L_2 := 2C_0 + 5\Gamma - \sum_{1}^{3}(2e_i + 2e'_i),
\]

\[
L_3 := 3C_0 + 8\Gamma - \sum_{1}^{3}(3e_i + 3e'_i).
\]
Let $V$ be the bidouble cover of $Z$ determined by the curves $D_i$ and by the divisors $L_i$ and let $S$ be the minimal model of $V$. Since $Z$ is a ruled surface with $q = 1$,

$$K_Z \equiv -2C_0 - 2\Gamma + \sum_{i=1}^{3}[(e_i + e_i') + e_i].$$

Therefore,

$$K_Z + L_1 = -C_0 + \Gamma + \sum_{i=1}^{3} e_i,$$

$$K_Z + L_2 = 3\Gamma - \sum_{i=1}^{3} e_i',$$

$$K_Z + L_3 = C_0 + 6\Gamma - \sum_{i=1}^{3}(e_i + 2e_i').$$

One has

$$\chi(\mathcal{O}_S) = 4 \cdot 0 + \frac{1}{2} \sum_{i=1}^{3} L_i(K_Z + L_i) = 0 + 0 + 1 = 1.$$ 

Let

$$G = \widetilde{T}_1 + \widetilde{T}_2 + \widetilde{T}_3 - \sum_{i=1}^{3}(E_i + E_i') + E_4 - 2E_0$$

and

$$l = 2\widetilde{T} - \sum_{i=1}^{3} E_i' - 2E_0.$$ 

Since the branch locus of $\psi$ is linearly equivalent to $2l$, $K_Z + L_3 \equiv \psi^*(G)$ and $G - l \equiv \widetilde{T} - \sum_{i=1}^{3} E_i,$ then

$$h^0(Z, \mathcal{O}_Z(K_Z + L_3)) = h^0(X, \mathcal{O}_X(G)) + h^0(X, \mathcal{O}_X(G - l)) = 1 + 0 = 1.$$ 

This way,

$$p_g(S) = \sum_{i=1}^{3} h^0(Z, \mathcal{O}_Z(K_Z + L_i)) = 0 + 0 + 1 = 1.$$ 

As

$$h^0(Z, \mathcal{O}_Z(2K_Z + L_1 + L_2)) = 0,$$

$$h^0(Z, \mathcal{O}_Z(2K_Z + L_1 + L_3)) > 0,$$

$$h^0(Z, \mathcal{O}_Z(2K_Z + L_2 + L_3)) > 0,$$

the bicanonical map of $V$ is not composed with any of the involutions $i_1, i_2, i_3$, corresponding to $L_1, L_2, L_3$.

· **Calculation of $K_S^2$**

We have

$$N := 2K_Z + \sum_{i=1}^{3} L_i \equiv N_1 + N_2,$$ 

where

$$N_1 := 2\Gamma_1 + 2\Gamma_2 + 2\Gamma_3 - \sum_{i=1}^{3} 2e_i'$$ 

and

$$N_2 := 2C_0 + 6\Gamma - \sum_{i=1}^{3}(2e_i + 2e_i') \equiv \psi^* \left( 3\widetilde{T} - \sum_{i=1}^{3}(E_i + E_i') - E_0 \right).$$
As the support of the pullback of $N_1$ is a disjoint union of $24 = -N_1^2$ (-1)-
curves, $|N_2|$ has no fixed component and $N_1N_2 = 0$, then

$$K_S^2 = N_2^2 + (-N_1^2) = N_2^2 = 4.$$  

\section{Appendix: Construction of plane curves}

\subsection{Useful pencils}

Here we show the existence of some pencils of plane curves that are useful on
some of the constructions of Section 4. Recall Notation 10.

\begin{lemma}
Let $C \subset \mathbb{P}^2$ be a smooth conic and $p_0 \notin C$, $p_1, \ldots, p_4 \in C$ be
distinct points. Consider the points $p_5, p_6 \in C$ such that the lines through $p_0, p_5$
and $p_0, p_6$ are tangent to $C$.

There exists a smooth curve $Q$ of type $3(1,(1,1)^4,1^2)$, through $p_0, \ldots, p_6$.
\end{lemma}

\begin{proof}
Let $C_x, x \in \mathbb{P}^1$, be a parametrization of the pencil of conics through
$p_1, \ldots, p_4$. Let $p^1_x, p^2_x$ be the points of $C_x$ (not distinct if $C_x$ is singular) such
that the lines through $p_0, p^1_x$ and $p_0, p^2_x$ are tangent to $C_x$. The correspondence

$$x \leftrightarrow \{p^1_x, p^2_x\}$$

gives a plane algebraic curve $Q$, parametrized by $x \in \mathbb{P}^1$, and a double cover $Q \to
\mathbb{P}^1$. This cover is ramified over four points, corresponding to the three degenerate
conics which contain the points $p_1, \ldots, p_4$ plus the conic which contains $p_0$.

Therefore, by the Hurwitz formula, $Q$ is a cubic.

The conic through $p_0, \ldots, p_4$ is not tangent to the line $T_i$ (through $p_0, p_i$) at
$p_0$, thus also $Q$ is not tangent to $T_i$ at $p_0$, $i = 1, \ldots, 4$. Since each conic $C_x$
can be tangent to $T_i$ only at $p_i, i = 1, \ldots, 4$, then $Q$ intersects $T_i$ only at $p_0$ and $p_i,$
$i = 1, \ldots, 4$. This means that $Q$ is tangent to $T_i$ at $p_i, i = 1, \ldots, 4$, and then $Q$
is smooth. 
\end{proof}

\begin{proposition}
In the notation of Notation 10, there exist pencils, without
base components, of plane curves of type:

\begin{enumerate}[(a)]
\item 5(1, (2, 2)^4); 
\item 6(2, (2, 2)^4); 
\item 7(3, (2, 2)^4); 
\item 8(4, (2, 2)^6). 
\end{enumerate}

\begin{proof}
\begin{enumerate}[(a)]
\item This is proved in \cite{Br}. Notice that we are imposing 19 conditions to a linear
system of dimension 20.
\end{enumerate}
\end{proof}

b) Let $A(\mathbb{C})$ be an affine plane and $a, b, c, d \in \mathbb{C} \setminus \{0\}$ be numbers such that $a \neq c$ and $bc \neq \pm ad$. Consider the points of $A$:

$$p_0 := (0, 0), \quad p_1 := (a, b), \quad p_2 := (c, d), \quad p_3 := (c, -d), \quad p_4 := (a, -b)$$

and let $T_i$ be the line through $p_0$ and $p_i$, $i = 1, \ldots, 4$. Let $C_1$ be the conic through $p_1, \ldots, p_4$ tangent to $T_1, T_4$ and $C_2$ be the conic through $p_1, \ldots, p_4$ tangent to $T_2, T_3$.

The curves

$$2C_1 + T_2 + T_3 \quad \text{and} \quad 2C_2 + T_1 + T_4$$

generate a pencil whose general member is a curve of type $6(2, (2, 2))$.

c) Let $C \subset \mathbb{P}^2$ be a non-degenerate conic and $p_0 \notin C, p_1, \ldots, p_5 \in C$ be distinct points such that the lines $T_1, T_5$, defined by $p_0, p_1$ and $p_0, p_5$, are tangent to $C$. From Lemma 11, there exists a curve $Q$ of type $3(1, (1, 1))$, through $p_0, \ldots, p_5$, respectively.

The curves

$$2C + T_2 + T_3 + T_4 \quad \text{and} \quad 2Q + T_5$$

generate a pencil whose general member is a curve of type $7(3, (2, 2))$.

d) This is analogous to the previous case, but now the pencil is generated by

$$2C + T_2 + \cdots + T_5 \quad \text{and} \quad 2Q + T_1 + T_6.$$ 

\[ \square \]

### A.2 Constructions using Magma

In this appendix we construct some curves using the Computational Algebra System Magma ([BCP]). We use the Magma procedure LinSys, defined in [R2]. This procedure calculates the linear system $L$ of plane curves of degree $d$, in an affine plane $A$, having singular points $p_i$ of order $(m_1, m_2)$ with tangent direction given by the slope $td_i$.

Consider, in an affine plane $A$, the points

$$p_0 := (0, 0), \quad p_1 := (2, 2), \quad p_2 := (-2, 2), \quad p_3 := (3, 1), \quad p_4 := (-3, 1).$$

From Section A.1, there exists a pencil of curves of type $6(2, (2, 2))$, with singularities at $p_0, \ldots, p_4$, respectively. Let $G$ be the element of this pencil which contains the point $p_5 := (0, 5)$.

The curve $G$ is reduced and the tangent line to $G$ at $p_5$ is horizontal:

```magma
> A<x,y>:=AffineSpace(Rationals(),2);
> p:=[A!2,2,A!-2,2,A!3,1,A!-3,1,A!0,5,Origin(A)];
> d:=6;m1:=[2,2,2,2,1];m2:=[2,2,2,2,1];
> td:=[p[i][2]/p[i][1]:i in [1..4]] cat [0];
> LinSys(A,d,p,m1,m2,td,"L");
```

The curve $G$ is reduced and the tangent line to $G$ at $p_5$ is horizontal:

```magma
> A<x,y>:=AffineSpace(Rationals(),2);
> p:=[A!2,2,A!-2,2,A!3,1,A!-3,1,A!0,5,Origin(A)];
> d:=6;m1:=[2,2,2,2,1];m2:=[2,2,2,2,1];
> td:=[p[i][2]/p[i][1]:i in [1..4]] cat [0];
> LinSys(A,d,p,m1,m2,td,"L");
```
There exists a reduced curve $C$ of type $8(4, 2, 2), (3, 3)$, singular at $p_0, \ldots, p_5$, such that the $(3, 3)$-point is tangent to $G$. Moreover, $G + C$ is a reduced element of a pencil of curves of type $14(6, 4, 4), (4, 4)$:

\[ d = 8; m_1 = [2, 2, 2, 2, 3, 4]; m_2 = [2, 2, 2, 2, 3]; \]
\[ \text{LinSys}(A, d, p, m_1, m_2, td, ^L); \]
\[ \# \text{Sections}(L); \]
\[ 1 \]
\[ C = \text{Curve}(A, \text{Sections}(L)[1]); \]
\[ \text{IsReduced}(G \text{ join } C); \]
\[ \text{true} \]

Analogously, one can verify that:

- there exists a reduced curve of type $7(3, 2, 2), (2, 2)$, singular at $p_0, \ldots, p_5$, such that the $(2, 2)$-point is tangent to $G$.

Finally we will see that $p_5$ can be chosen such that

- there exist reduced curves $C_1$ of type $7(3, 2, 2), 3$ and $C_2$ of type $6(2, 2, 2), 1$, both through $p_0, \ldots, p_5$, such that $C_1 + C_2$ is reduced and the singularity of $C_1 + C_2$ at $p_5$ is ordinary.
> \( H := h(\text{Sections}(L)) \);
> \( M := [H[i], D(H[i],1), D(H[i],2), D2(H[i],1,1), D2(H[i],1,2),
> D2(H[i],2,2)] : i \in [1..\#H] \);
> \( M_t := \text{Matrix}(M) ; \text{min} := \text{Minors}(M_t, \#H) \);
> \( A := \text{AffineSpace}(R) \);
> \( S := \text{Scheme}(A, \text{Min cat } [x-3, 1+n*(y-x)*(y+x)*(3*y-x)*(3*y+x)]) \);
> //The condition 1+n*(..)=0 implies that
> //the solution is not in \( p \).
> \text{Dimension}(S);
0
> \text{PointsOverSplittingField}(S);

We choose one of the solutions and show that it works:

> \( R<r_1> := \text{PolynomialRing}(\text{Rationals})() \);
> \( K<r_1> := \text{NumberField}(r_1^2 - 1761803/139426560*r_1 + \)
> 1387488001/33730073395200) ;
> \( A<x,y> := \text{AffineSpace}(K, 2) \);
> \( y_1 := -33462374400/102856069*r_1 + 419793163/102856069 ;
> p := [A[2,2], A[-2,2], A[3,1], A[-3,1], A[3,y1], \text{Origin}(A)] ;
> d := 7 ; m_1 := [2,2,2,2,3,3] ; m_2 := [2,2,2,2] ;
> t_d := [p[i][2]/p[i][1] : i \in [1..\#m_2]] ;
> \text{LinSys}(A, d, p, m_1, m_2, t_d, "L") ; \#\text{Sections}(L) ;
1
> \text{C}_1 := \text{Curve}(A, \text{Sections}(L)[1]) ;
> \text{IsOrdinarySingularity}(\text{C}_1, p[5]) ;
true
> d := 6 ; m_1 := [2,2,2,1,2] ; m_2 := [2,2,2,2] ;
> \text{LinSys}(A, d, p, m_1, m_2, t_d, "L") ; \#\text{Sections}(L) ;
1
> \text{C}_2 := \text{Curve}(A, \text{Sections}(L)[1]) ;
> \text{IsReduced}(\text{C}_1 \text{ join } \text{C}_2) ;
true
> \text{IsSingular}(\text{C}_2, p[5]) ;
false
> \text{IsOrdinarySingularity}(\text{C}_1 \text{ join } \text{C}_2, p[5]) ;
true

The verification that the singularities are no worst than stated is left to the
reader (use the Magma functions \text{ProjectiveClosure}, \text{SingularPoints}, \text{HasSingularPointsOverExtension} and \text{ResolutionGraph}).

The calculations for Section 4.6 (verification that \( \phi_2 \) is birational) are as
follows:

> d := 7 ; m_1 := [2,2,2,1,3] ; m_2 := [2,2,2,2] ;
> \text{LinSys}(A, d, p, m_1, m_2, t_d, "L") ;
> \#\text{Sections}(L) ; \text{BaseComponent}(L) ;
5
Scheme over K defined by
1
> P4:=ProjectiveSpace(K,4);
> tau:=map<A->P4|Sections(L)>;
> Degree(tau(Scheme(A,Sections(L)[3])));
7
thus an hyperplane section of the image of \( \tau \) is of degree 7.

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