Generalised velocity-dependent one-scale model for current-carrying strings

C. J. A. P. Martins, † Patrick Peter, ‡ I. Yu. Rybak, † and E. P. S. Shellard

1 Centro de Astrofísica da Universidade do Porto, Rua das Estrelas, 4150-762 Porto, Portugal
2 Instituto de Astrofísica e Ciências do Espaço, CAUP, Rua das Estrelas, 4150-762 Porto, Portugal
3 GRyCO – Institut d’Astrophysique de Paris, CNRS & Sorbonne Université, UMR 7095 98 bis boulevard Arago, 75014 Paris, France
4 Centre for Theoretical Cosmology, Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, United Kingdom

(Dated: November 20, 2020)

We develop an analytic model to quantitatively describe the evolution of superconducting cosmic string networks. Specifically, we extend the velocity-dependent one-scale (VOS) model to incorporate arbitrary currents and charges on cosmic string worldsheets under two main assumptions, the validity of which we also discuss. We derive equations that describe the string network evolution in terms of four macroscopic parameters: the mean string separation (or alternatively the string correlation length) and the root mean square (RMS) velocity which are the cornerstones of the VOS model, together with parameters describing the averaged timelike and spacelike current contributions. We show that our extended description reproduces the particular cases of wiggly and chiral cosmic strings, previously studied in the literature. This VOS model enables investigation of the evolution and possible observational signatures of superconducting cosmic string networks for more general equations of state, and these opportunities will be exploited in a companion paper.

I. INTRODUCTION

The early stage of the Universe’s evolution is thought to have included phase transitions that may have led to the production of cosmic strings, as originally suggested by Tom Kibble [1] (for exhaustive introductions see [2, 3]). Scenarios including cosmic strings are ubiquitous in grand unified theories (GUT) [4–6] and models of inflation [7–9]. A prominent feature of these one-dimensional topological defects is their stability, i.e. once such a network is produced in the early universe, it will generically survive until the present day. These networks lead to many potentially detectable observational signatures, including anisotropies of the Cosmic Microwave Background (CMB) [10, 11], gravitational waves [12, 13] and lensing [14, 15]. Thus, astrophysical searches for cosmic string networks also probe the high energy physics of the early universe.

To obtain a precise connection between the high energy stage of the universe and observational constraints on cosmic strings, one needs to have an accurate model for string network evolution. There are two approaches to tackling this problem: on the one hand, high-resolution numerical simulations of the detailed network evolution and, on the other, thermodynamical evolution models for the averaged network properties. These prove to be complementary because analytic models must be calibrated with numerical simulations, whose restricted dynamic range can then be extrapolated to cosmological scales. The simulations themselves can be performed using two alternative methods. First, by modelling the full field theory equations, in principle, all the key physical properties of cosmic strings are reproduced, but at huge computational cost [16–18], although this can be mitigated with highly efficient accelerated codes [19, 20] or by using adaptive mesh refinement [21]. Secondly, by relying on the effective Nambu-Goto action, cosmic strings are approximated to be infinitely thin, thus considerably increasing the dynamic range of simulations [22–24].

While most numerical simulations to date have been performed for the simplest Abelian-Higgs (or Nambu-Goto) model, it is expected that realistic cosmic strings have non-trivial internal structure. A first example of such strings was discussed in [25], where the string superconductivity is due to an additional charged scalar boson or fermion that is trapped in the string core. Current carrying cosmic strings were also found to be typical outcomes of supersymmetric theories [26–28], some non-Abelian models [29–31] and other possible scenarios [32–35]. Thus, analytic and numerical studies of these extended models are highly desired.

To study the evolution of superconducting strings it is convenient to use an infinitely thin effective model, which describes the original field theory in the same way as the Nambu-Goto action describes strings from the Abelian-Higgs model. A first effective model for superconducting cosmic strings was given in [25], while a more realistic description was developed in [36, 37]. The properties of the current for such strings were also studied in various works [38, 39]; a detailed review can be found in [40].

It is expected that the existence of a current flowing along cosmic strings impacts the evolution of the string network and thus its observational predictions. A relevant example is a superconducting loop that, unlike a standard currentless loop, can have an equilibrium configuration, known as a vorton [41, 42]. A more detailed
treatment of vortons was carried out through an effective model [43, 44] and also by numerically studying its field theory [45, 46] leading to additional observational constraints [47, 48].

While it is possible to numerically investigate particular configurations of superconducting strings, it is more challenging to perform full simulations of a superconducting cosmic string network. What is currently lacking is the extension of the thermodynamical approach, where the evolution of the string network is described by macroscopic parameters. There are several approaches for an analytic description of a string network evolution [49, 50].

We will follow the quantitative velocity-dependent one-scale (VOS) model [51–53], which has already been shown to be extendable to include the effects of non-trivial internal structure. Specifically, such extensions have already been reported for elastic strings [54, 55], also known as wiggly strings, to treat the small-scale structure on strings, and for chiral superconducting strings [56].

The purpose of this work is to fill the gap in the literature, by introducing and starting the exploration of a further extension of the VOS model that applies to generic current-carrying string networks, using the previously known results for the specific cases of wiggly and chiral strings as validation of our methodology. The plan for this paper is as follows. We start by reviewing the dynamical effects of currents on cosmic strings in Section 2, and by outlining our key assumptions in Section 3. Both of these are then used to systematically derive the generalised VOS model for strings with currents, which we do in Section 4. We then discuss the stability of general scaling behaviours and some validating special cases in Sections 5 and 6 respectively, and finally present our conclusions in section 7.

II. DYNAMICS OF CURRENT-CARRYING COSMIC STRINGS IN EXPANDING UNIVERSES

We start by providing the microscopic equations of motions driving the cosmological dynamics of generic current-carrying cosmic strings in the infinitely thin approximation. These will be the basis for the subsequent development of the VOS model.

A. Embedding in background spacetime

It is well known [36, 40, 43, 57] that the influence of the current on the motion of the string worldsheet can be described as in the usual Lorentz-invariant (Nambu-Goto) case by integrating the transverse degrees of freedom, except that one is now left with a more general action [58]

\[ S = \int L(\kappa) \sqrt{-\gamma} \, d^2 \sigma = -\mu_0 \int f(\kappa) \sqrt{-\gamma} \, d\sigma^0 \, d\sigma^1, \]  

where \( \gamma \) is the determinant of the induced metric

\[ \gamma_{ab} \equiv g_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^a} \frac{\partial X^\nu}{\partial \sigma^b} = g_{\mu\nu} X^\mu_a X^\nu_b, \]  

given in terms of the internal string coordinates \( \sigma^a \), \( a \in [0, 1] \), with \( \sigma^0 \) timelike and \( \sigma^1 \) spacelike. The string spans the two-dimensional worldsheet defined by \( X^\mu(\sigma^a) = \{ X^0(\sigma^a), \sigma^a \} \).

We assume that the background metric \( g \) has signature \(-2\): in the cosmological setup which will be the focus of our analysis, the relevant line element is given by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric,

\[ ds^2_{\text{FLRW}} = a^2(\tau) \left( d\tau^2 - dx^2 \right) = dt^2 - a^2(t) dx^2, \]

in terms of the cosmic \((t)\) or conformal \((\tau)\) times. We shall work below in terms of the conformal time \( \tau \) only.

In Eq. (1), the surface lagrangian \( L = -\mu_0 f(\kappa) \) is normalized by means of a parameter \( \mu_0 \), a constant with units of mass squared (or mass per unit length, in natural units with \( \hbar = c = 1 \)), and depends on a so-called state parameter \( \kappa \), defined through a scalar field \( \varphi(\sigma^a) \) living on the string worldsheet; in practice, this is identified with the phase of a condensate [25, 39]. Specifically, this scalar quantity of the worldsheet reads

\[ \kappa = \gamma^{ab} \frac{\partial \varphi}{\partial \sigma^a} \frac{\partial \varphi}{\partial \sigma^b} = \gamma^{ab} \varphi_{,a} \varphi_{,b}. \]  

To complete the macroscopic description, one needs the stress-energy tensor \( T_{\text{string}}^{\mu\nu} \), which can be locally diagonalized

\[ T_{\text{string}}^{\mu\nu} = U u^\mu u^\nu - \delta T_{\text{string}}^{\mu\nu}, \]

where the normalized eigenvectors \( u \) and \( v \) are respectively timelike and spacelike; the slightly different case of a chiral current, usually treated in a completely different manner [59], appears in our framework as the limit when the current becomes lightlike, i.e. \( \kappa \to 0 \).

It can be shown that the eigenvalues of \( T_{\text{string}}^{\mu\nu} \) appearing in Eq. (5), namely the energy per unit length \( U \) and tension \( T \), can be expressed through the state parameter \( \kappa \) and the dimensionless lagrangian function \( f(\kappa) \) as [36, 58]

\[ U = \mu_0 \left[ f - (1 - s) \kappa \frac{df}{d\kappa} \right], \]

\[ T = \mu_0 \left[ f - (1 + s) \kappa \frac{df}{d\kappa} \right], \]

where \( s = +1 \) when \( \kappa f_k > 0 \) and \( s = -1 \) when \( \kappa f_k < 0 \), i.e. \( s = \kappa f_k / |\kappa f_k| = \text{sign} (\kappa f_k) \). This ensures that \( U - T = 2\mu_0 |\kappa f_k| > 0 \), in agreement with the definition and meaning of \( U \) and \( T \). Here and in what follows, we denote \( f_k \equiv df(\kappa)/d\kappa \); we use this unusual notation instead of the traditional \( f' \) in order to avoid any confusion with derivatives with respect to the spacelike coordinate \( \sigma^1 \).
As a side remark, we shall also make use of another useful notation, namely the Legendre transform \( \tilde{f} \) of the generating function \( f \). It is defined through
\[
\tilde{f} \equiv f - 2\kappa f_\kappa. \quad (7)
\]

It can be seen from Eqs. (6) that \( f \) and \( \tilde{f} \) yield in turn the energy \( U \) and tension \( T \) depending on whether the current is, respectively, timelike (electric regime, \( \kappa \geq 0 \)) or spacelike (magnetic regime, \( \kappa \leq 0 \)).

With the energy per unit length \( U \) and tension \( T \) defined above, one derives \[36\] the velocities of propagation for transverse \( (c_t) \) and longitudinal \( (c_l) \) perturbations. They are given by
\[
c_t^2 = \frac{T}{U} = \frac{f - (1 + s) \kappa f_\kappa}{f - (1 - s) \kappa f_\kappa},
\]
\[
c_l^2 = -\frac{dT}{dU} = \frac{s f_\kappa + \kappa f_{\kappa \kappa} (s + 1)}{s f_\kappa + \kappa f_{\kappa \kappa} (s - 1)}.
\]

In order for any model to make sense, it should be causal, and therefore these velocities must be less than unity. From the definition of \( s \) as the sign of \( \kappa f_\kappa \), it is clear that the first of these condition, \( c_t \leq 1 \), is trivially satisfied, while the limit of the longitudinal perturbation velocity sets bounds on the equation of state, specifically
\[
\kappa f_{\kappa \kappa} \leq 0. \quad (10)
\]

Moreover, string stability also demands both \( c_t^2 \geq 0 \) and \( c_l^2 \geq 0 \). The first of these constraints in turn imposes that both \( U \) and \( T \) are positive \[60\]. This yields
\[
f > 2\kappa f_\kappa. \quad (11)
\]

Given the definitions (6), this justifies the global sign in (1) in order to include the Nambu-Goto and chiral limiting cases for which \( \kappa \to 0 \): the dimensionless function \( f \) must be assumed positive definite, \( f > 0 \). Similarly, the other stability criterion, with respect to longitudinal perturbations \( c_l^2 \geq 0 \), provides
\[
f_\kappa > 2\kappa f_{\kappa \kappa} \quad \text{if} \quad f_\kappa > 0, \quad f_\kappa < 2\kappa f_{\kappa \kappa} \quad \text{if} \quad f_\kappa < 0. \quad (12)
\]

The models are discussed below.

**B. Equations of motion**

In what follows, we use the gauge in which the string worldsheet timelike coordinate \( \sigma^0 \) coincides with the conformal time, i.e., \( \sigma^0 = \tau \) and \( \sigma^1 \equiv \sigma \), and restrict attention to the choice
\[
X^0 = \tau \quad \text{and} \quad \frac{\partial X}{\partial \tau} \cdot \frac{\partial X}{\partial \sigma} \equiv \dot{X} \cdot X' = 0, \quad (13)
\]

thereby defining the microscopic notation for derivatives of a quantity \( A \) by \( A' \equiv \partial A/\partial \sigma \) and \( A \equiv \partial A/\partial \tau \). With this choice and within the framework of the metric (3), the induced metric (2) becomes
\[
\gamma_{ab} = \text{diag} \left[ a^2 \left( 1 - X^2 \right), -a^2 X'^2 \right], \quad (14)
\]
leading to the determinant
\[
\sqrt{-\gamma} = a^2 \sqrt{X'^2 \left( 1 - X^2 \right)}, \quad (15)
\]
and the inverse metric
\[
\gamma_{ab} = \text{diag} \left[ \frac{1}{a^2 \left( 1 - X^2 \right)}, -\frac{1}{a^2 X'^2} \right], \quad (16)
\]
from which one obtains the state parameter as
\[
\kappa = \frac{\dot{\varphi}^2}{a^2 \left( 1 - X^2 \right)} - \frac{\varphi'^2}{a^2 X'^2}. \quad (17)
\]

In what follows, as in most previous literature on this topic, we will make use of the quantity \( \epsilon \) defined by
\[
\epsilon^2 = \frac{X'^2}{1 - X^2}. \quad (18)
\]

It turns out that the equations of motion derived from the action (1) with equation of state (6) are also more tractable if one uses the dimensionless variables \( \bar{U}, \bar{T} \) and \( \Phi \) defined through the relations, derived in Ref. \[58\]
\[
\bar{U} \equiv f - 2\gamma_{00} \frac{df}{d\kappa} \dot{\varphi}^2 = f - 2q^2 f_\kappa,
\]
\[
\bar{T} \equiv f - 2\gamma_{11} \frac{df}{d\kappa} \varphi'^2 = f + 2j^2 f_\kappa, 
\]
\[
\Phi \equiv -\frac{2}{\sqrt{-\gamma}} \frac{df}{d\kappa} \varphi' \dot{\varphi} = -2qj f_\kappa,
\]
where for convenience we have defined
\[
q^2 \equiv \gamma_{00} \dot{\varphi}^2 = \frac{\dot{\varphi}^2}{a^2 \left( 1 - X^2 \right)}, 
\]
\[
j^2 \equiv -\gamma_{11} \varphi'^2 = \frac{\varphi'^2}{a^2 X'^2} = q^2 - \kappa. \quad (20)
\]

With our metric convention (3), \( q^2 \) and \( j^2 \) are positive definite. In terms of these variables, the state parameter (17) has the simple form
\[
\kappa = q^2 - j^2, \quad (21)
\]
and is thus easily seen to be a worldsheet Lorentz scalar.

We are now in position to write down the equations of motion, again assuming the gauge choice (13). As in Ref. \[58\], we find the following explicit form...
\[ \frac{\partial}{\partial \tau} \left( \epsilon \bar{U} \right) + \frac{\dot{\epsilon}}{\alpha} \left( \bar{U} + \dot{T} \right) \bar{X}^2 + \bar{U} - \dot{T} = \partial_{\sigma} \Phi, \]  
(22a)

\[ \dot{X} \epsilon \bar{U} + \frac{\dot{\epsilon}}{\alpha} \left( \bar{U} + \dot{T} \right) \left( 1 - \dot{X}^2 \right) \bar{X} = \partial_{\sigma} \left( \frac{\bar{T}}{\epsilon} \bar{X}' \right) + 2 \Phi \bar{X}' + \bar{X}' \left( \dot{\Phi} + 2 \frac{\dot{\epsilon}}{\alpha} \Phi \right), \]  
(22b)

\[ \frac{\partial}{\partial \tau} \left( f_a \alpha \sqrt{q^2 \bar{X}^2} \right) = \partial_{\sigma} \left[ f_a \alpha \sqrt{j^2 (1 - \dot{X}^2)} \right]. \]  
(22c)

The dynamical equation (22b) must be solved under the constraint (22a), while Eq. (22c) completes the system (22) with the dynamics of the current.

It is worth mentioning that Eqs. (22) were obtained without any assumption. They describe the dynamics of individual strings. Below, we discuss an approach, based on Refs. [51, 52, 56], thanks to which one can average equations Eqs. (22) and obtain a thermodynamical description which extends the currently available VOS model. In order to implement this scheme however, we will need two specific assumptions, pertaining to boundary conditions and uncorrelated variables, which we now discuss.

### III. KEY MODELLING ASSUMPTIONS

In order to obtain the macroscopic variables for the VOS network evolution model, one needs to integrate over the spacelike variable \( \sigma \) along all the strings in the network [51, 52]. In practice, this means that we average over boxes of strings, since this also assumes a similar summation, as illustrated in Fig. 1.

![Figure 1](image)

Figure 1. A schematic illustration of our averaging process, which is calculated by integration over \( \sigma \) and represents a summation over segments that are uncorrelated on distances larger than \( \xi \).

We start by introducing two macroscopic parameters, namely the total energy \( E \) and bare (currentless) energy \( E_0 \)

\[ E = a \mu_0 \int \bar{U} \epsilon \, d\sigma \quad \text{and} \quad E_0 = a \mu_0 \int \epsilon \, d\sigma \]  
(23)

and define the average value of any generic function \( \mathcal{O} \) of the worldsheet coordinates,\(^1\) denoted by \( \langle \mathcal{O} \rangle \), through

\[ \langle \mathcal{O} \rangle \equiv \frac{\int \mathcal{O} \epsilon \, d\sigma}{\int \epsilon \, d\sigma}. \]  
(24)

In particular, we define the total charge \( Q \) and current \( J \) energy densities through the relations

\[ Q^2 \equiv \langle q^2 \rangle, \quad J^2 \equiv \langle j^2 \rangle, \]  
(25)

and the RMS velocity

\[ v \equiv \sqrt{\langle \dot{X}^2 \rangle}. \]  
(26)

Since over distances larger than the correlation length the string parameters become, by definition, uncorrelated, we can understand the above averaging process as a sum of uncorrelated segments with length \( \approx \xi \) [23, 61]. In the current-carrying case, we expect the current correlation length to be comparable to that of the uncharged strings, although in full generality, one could consider a different correlation length for the current. Indeed, in the case of the wiggly model, which has been previously studied in [54, 55], the VOS model effectively describes the small-scale structure on the strings through an average at a mesoscopic scale, which is intermediate between the microscopic scale (where the RMS velocity is defined) and the correlation length scale. A similar situation could occur for the charge and current densities introduced above.

#### A. Assumption 1: Uncorrelated variables

Our first assumption is that for microscopic variables with a definite sign (generally positive definite), we can

\(^1\) In Refs. [54, 55], the notation \( \langle \mathcal{O} \rangle \) refers to the weighted average involving the measure \( U \epsilon \, d\sigma \) instead of our choice \( \epsilon \, d\sigma \) only. These choices of weighting averages are of course not equivalent unless one assumes uncorrelated variables as will be discussed presently.
use the approximation
\[ \langle \mathcal{O}^2 \rangle \approx \langle \mathcal{O} \rangle^2, \] (27)
or even, as we shall do later, \( \langle F(\mathcal{O}) \rangle \approx F(\langle \mathcal{O} \rangle) \), for any given arbitrary function \( F \) of the function \( \mathcal{O} \).

We can discuss how fair this assumption actually is, by considering the specific example \( \langle \dot{X}^4 \rangle = \langle (\dot{X}^2)^2 \rangle \), which we therefore claim can be approximated by \( v^4 = (v^2)^2 \). We note that this is already the case in the standard VOS model and was discussed extensively in [51]. Thus averaging Eq. (22b) (with zero current \( q^2 = 0, j^2 = 0 \)) yields terms of the form
\[ \langle \dot{X}^4 \rangle = \langle \dot{X}^2 \rangle^2 + \text{cov}(\dot{X}^2, \dot{X}^2) = v^4 + \text{var}(\dot{X}^2). \] (28)

Lacking other theoretical input, we resort to numerical simulations results to check how small the variance of the RMS velocity is. Using the results from the simulations reported in [23], we fitted normal distributions to the RMS velocity for both small scales (\( \ell = 0.005\xi \)) and correlation length scales (\( \ell = \xi \)); this provides the results shown in Fig. 2. The distributions are clearly close to normal and the relative corrections due to the variance are consistent across the two different lengthscales, implying that this may be a good approximation provided one is interested in the scaling or proportionality of the quantities being averaged (rather than their absolute values). The main conclusion is that this appears to be reasonable approximation on correlation length scales in light of these caveats.

Given that the one scale modeling assumption underlies the successful VOS approach, we therefore adopt this assumption in what follows, bearing in mind its potential weakness. This can in principle be mitigated by incorporating the effect of the actual velocity distributions into the VOS model [51], specifically by allowing for differences between \( \langle \dot{X}^4 \rangle \) and \( \langle \dot{X}^2 \rangle^2 \) in the standard VOS model [51]. One possible such treatment is outlined in Appendix A, where we show that this cannot be done in closed form without resorting to some additional assumption. This is not surprising given the underlying one-scale context (which would clearly have to be extended to allow for velocity distributions), but in any case this as-

Figure 2. Normal density probability function \( N \) for string velocities averaged over length scales \( \ell = 0.005\xi \) (solid lines) and \( \ell = \xi \) (dashed lines), fitted to data from Nambu-Goto simulations [23] of radiation, matter and \( \Lambda \)-dominated universes, and as well as the averaged values for all three epochs. The derived best-fit parameters are shown in the accompanying table 1.
assumption should not be a strong limiting factor in our analysis.

B. Assumption 2: Vanishing boundary terms

The second assumption which is required in order to integrate the macroscopic equations of motion concerns the terms expressed as a total derivative and averaged over the string network: endowing the entire network with periodic boundary conditions, such integrals automatically become closed loop integrals and thus vanish identically. In practice, we will set

$$\int \partial_{\tau} \left\{ \mathcal{F} (X (\sigma, \tau)) \right\} d\sigma \rightarrow \oint \partial_{\tau} \left\{ \mathcal{F} (X (\sigma, \tau)) \right\} d\sigma \approx 0.$$  \hspace{1cm} (29)

Condition (29) should be valid at all times provided we assume that all strings, even the so-called infinite ones, are actually loops on larger scales. This assumption also holds when a phenomenological loop chopping parameter is introduced.

It should be mentioned that one can imagine a situation in which the integral (29) does not vanish when it is calculated for one string. This happens for instance when the function $\mathcal{F}$ is discontinuous, i.e., when the string has kink or cusp like structures. However, when we integrate and sum over the entire string network, it is reasonable to assume that the sum (29) will contain both positive and negative contributions that one can expect, on average, to compensate, yielding an overall vanishing result.

IV. VOS MODEL FOR STRINGS WITH CURRENTS

Under the assumptions discussed in the previous section and after some algebraic manipulations, it is possible to obtain a system of differential equations for the relevant macroscopic variables describing the string network. It rests on the following two assumptions, both of which are central to the VOS model:

- the cosmic string network is Brownian, i.e.,

$$E = \frac{\mu_0 V}{L_c^2 a^2} \quad \text{and} \quad E_0 = \frac{\mu_0 V}{\xi^2 a^2}$$  \hspace{1cm} (30)

inside the volume $V$; this defines the current-carrying characteristic length $L_c$ and the bare (or Nambu-Goto) correlation length $\xi$. Given the definition (23) of the energies and that of $\bar{U}$ in Eq. (19), one finds that

$$E = E_0 \langle f - 2q^2 f_\kappa \rangle \implies \frac{E}{E_0} = F - 2Q^2 F',$$  \hspace{1cm} (31)

where we have introduced the macroscopic notation for $\langle f \rangle$ and its derivatives through the following

$$F \equiv \langle f \rangle, \quad F' \equiv \langle f_\kappa \rangle, \quad F'' \equiv \langle f_{\kappa\kappa} \rangle; \hspace{1cm} (32)$$

we also assume that since the function $f$ depends on $\kappa = q^2 - J^2$, its averaged counterpart similarly satisfies $F = F(Q^2 - J^2)$.

With (30), in turn, we obtain the required connection between the two characteristic lengths, namely

$$L_c \sqrt{F - 2Q^2 F'} = \xi_c,$$  \hspace{1cm} (33)

which are clearly not independent. In what follows we will mostly work with $L_c$, but will occasionally also discuss the behaviour of $\xi_c$.

- the average comoving radius of curvature for the string network coincides with its physical characteristic length $R_c = \xi_c$ (see [51, 52] for further details).

One ends up with the following system of evolution equations, the derivation of which is outlined in B,

\[
\begin{align}
\frac{dL_c}{d\tau} &= \frac{\dot{a}}{a} \frac{L_c}{F - 2Q^2 F'} \left\{ v^2 \left[ F - (Q^2 - J^2) F' \right] - (Q^2 + J^2) F' \right\}, \hspace{1cm} (34a) \\
\frac{dv}{d\tau} &= \frac{(1 - v^2)}{F - 2Q^2 F'} \left\{ \frac{k(v)}{L_c \sqrt{F - 2Q^2 F'}} \left[ F + 2J^2 F' \right] - 2v \frac{\dot{a}}{a} \left[ F - (Q^2 - J^2) F' \right] \right\}, \hspace{1cm} (34b) \\
\frac{dJ^2}{d\tau} &= 2J^2 \left[ \frac{vk(v)}{L_c \sqrt{F - 2Q^2 F'}} - \frac{\dot{a}}{a} \right], \hspace{1cm} (34c) \\
\frac{dQ^2}{d\tau} &= 2Q^2 \frac{F' + 2J^2 F''}{F' + 2Q^2 F''} \left[ \frac{vk(v)}{L_c \sqrt{F - 2Q^2 F'}} - \frac{\dot{a}}{a} \right], \hspace{1cm} (34d)
\end{align}
\]

where we have used notation for $F'$, $F''$ defined in (32). In Eqs. (34b), (34c) and (34d), the momentum parameter
where the parameters $k_0$, $\alpha$ and $\beta$ have been obtained from a robust statistical analysis and have the values $k_0 \approx 1.3$, $\alpha \approx 2.3$, $\beta \approx 1.5$; different models are expected to yield a similar functional dependence in the velocity, with potentially different values of the macroscopic parameters $k_0$, $\alpha$ and $\beta$. For comparison, the relevant momentum parameter functions (36) and (37) are depicted in Fig. 3.

Eqs. (34) do not include energy loss terms, which should thus be added phenomenologically. String intercommutings can lead to the production of loops, which eventually shrink and radiate their energy away, thereby reducing the energy contained in the string network, hence increasing the correlation length. To incorporate the energy loss terms into Eqs. (34), we follow the arguments of [56]. We first write down the energy loss for $\xi_c$ in the conventional form [51]

$$\frac{\dot{\xi}_c}{\xi_c} = \cdots + \frac{\tilde{c}}{2} \frac{v}{\xi_c},$$

where the chopping efficiency function $\tilde{c}$ is a constant for the simplest string models [18] but could conceivably depend on the charge and current carried by the network [54, 55].

The corresponding energy loss term for the overall characteristic length $L_c$ has an analogous form, but should generically include a bias function $g(J,Q)$,

$$\frac{\dot{L}_c}{L_c} = \cdots + g(J,Q) \frac{\tilde{c}}{2} \frac{v}{\xi_c}.$$  

This bias function phenomenologically allows for the possibility that regions with different amounts of charge or current may be subject to intercommutings and be incorporated into loops with different probabilities (bearing in mind, for example, that they are likely to have different velocities).

Upon using Eq. (33) between the characteristic length $\xi_c$ associated with the bare energy $E_0$ and that associated with the total $E$, namely $L_c$, we obtain

$$2 \frac{\dot{L}_c}{L_c} + \cdots + g \tilde{c} \frac{v}{\xi_c} = 2 \frac{\tilde{\xi}_c}{\xi_c} + \cdots + \tilde{c} \frac{v}{\xi_c} + \frac{[(Q^2)^* + (J^2)] F' + 2Q^2 [(Q^2)^* - (J^2)] F''}{F - 2Q^2 F'}$$

and

$$\frac{(Q^2)^* F' + 2Q^2 F''}{F' - 2Q^2 F'} + \frac{(J^2)^*}{L_c} = \cdots + \frac{\tilde{c} v}{L_c} \frac{\sqrt{F - 2Q^2 F'}}{F'} (g - 1).$$

It might be the case that the momentum parameter also depends on the current, or alternatively that it has a different velocity dependence, hereinafter we will assume that momentum parameter has standard velocity-dependent form and leave the question of charge dependence for future studies.
To be consistent with Eq. (41) — in other words, to ensure energy conservation — one must also add analogous phenomenological terms for the charge and current loss through chopping in the following way

\[
(J^2)' = \cdots + \rho \xi v \frac{1}{L_c} \sqrt{F - 2Q^2F} (g - 1),
\]

\[
(Q^2)' = \cdots + (1 - \rho) \xi v \frac{1}{L_c} \sqrt{F - 2Q^2F} (g - 1),
\]

where \( \rho \) is an arbitrary constant. We emphasize that \( g \) and \( \rho \) are phenomenological bias parameters, whose fiducial values in the unbiased Nambu-Goto case are respectively \( g = 1 \) and \( \rho = 1/2 \). In particular, note that if \( \rho \) is biased then \( g \) must also be biased, but the opposite need not be true.

As a result, one obtains the generalized evolution equation for the correlation length

\[
\dot{\xi}_c = \frac{1}{W^2} \left\{ \frac{\dot{a}}{a} \xi v^2 \left[ W + (Q^2 + J^2) F' \right] - v k(v) (J^2 + Q^2)^2 \right\} + \frac{v}{2},
\]

given here for comparison convenience, together with the full system of our generalized VOS model including arbitrary charges and currents

\[
\dot{L}_c = \frac{\dot{a}}{a^2} \left\{ v^2 \left[ W^2 + (Q^2 + J^2) F' \right] - (Q^2 + J^2) F' \right\} + \frac{g}{W^2} \frac{\dot{c}}{c},
\]

\[
\dot{v} = \frac{1 - v^2}{W^2} \left\{ \frac{k(v)}{L_c W} \left[ W^2 + 2 (Q^2 + J^2) F' \right] - 2 \frac{\dot{a}}{a} \left[ W^2 + (Q^2 + J^2) F' \right] \right\},
\]

\[
(J^2)' = 2J^2 \left\{ \frac{vk(v)}{L_c W} - \frac{\dot{a}}{a} \right\} + \frac{\rho}{c} \frac{v}{W} \frac{1}{F} \frac{1}{2} \frac{v}{F^2} (g - 1),
\]

\[
(Q^2)' = 2Q^2 \left\{ \frac{2J^2 F'}{F} + \frac{2Q^2 F'}{F} - \frac{vk(v)}{L_c W} - \frac{\dot{a}}{a} \right\} + \frac{(1 - \rho) v}{L_c W} \frac{1}{F} \frac{1}{2} \frac{v}{F^2} (g - 1),
\]

where for simplicity we have used the notation \( W = \sqrt{F - 2Q^2F} \), assuming it to be positive-definite.

Having introduced the general evolution equations for the VOS model, we now discuss specific examples of models already considered in the literature, both at the microscopic levels, which our formalism accommodated.

**V. MICROSCOPIC MODELS AND THEIR STABILITY**

There are many ways to introduce a current along a cosmic string and to describe it in terms of the timelike or spacelike state parameter \( \kappa \); their general properties are discussed in Ref. [36], which we summarize and adapt to our notations below.

The simplest option, which we shall refer to as the *linear* model, consists in assuming the current to be small and the correction it induces on the equation of state to represent a perturbation, that is, we only allow for a first order term and write

\[
f_{\text{lin}} = 1 - \frac{\kappa}{2\mu_0} = 1 - \frac{\kappa}{2} \quad \Rightarrow \quad \tilde{f}_{\text{lin}} = 1 + \frac{\kappa}{2},
\]

in which we factored out the string scale \( \mu_0 \) and introduced the dimensionless degree of freedom \( \tilde{\kappa} \) through \( \kappa = \mu_0 \tilde{\kappa} \). This possibly over-simplified equation of state also has the advantage of being self-dual [36, 62].

One can extend this model to include higher order terms at the expense of introducing another mass parameter \( m_\sigma \), as we shall see below, but before doing so, one can consider another self-dual model, based on the generating function

\[
f_{\text{KK}} = \sqrt{1 - \kappa} \quad \Rightarrow \quad \tilde{f}_{\text{KK}} = \frac{1}{\sqrt{1 - \kappa}},
\]

and resulting from the motion of a Nambu-Goto string in a 5 dimensional Kaluza-Klein spacetime with the extra-dimension curled into a circle [63, 64]. This model also describes the so-called wiggly case for which one integrates over the small-scale structure of the string, which becomes effectively current-carrying [65, 66].

A field theoretical model for superconducting strings was proposed by Witten in Ref. [25] and explored numerically in full detail in Refs. [38, 39, 60]. Analytic approximations were proposed, based on asymptotic properties of the classical fields making up the internal string structure, which yield two distinct phenomenological models. This “realistic” current-carrying string model is based on a string-forming Higgs field, breaking a U(1) symmetry, and a bosonic or fermionic charge-carrier that condenses in the vortex core at an energy scale \( m_\sigma \). This condensate may or may not [38, 67] be coupled to a long-range gauge field, whose influence on the worldsheet dynamics
is mostly negligible [39, 60]; it is such a coupling with a long-range field, when the latter is identified to electromagnetism, that led to the name “superconducting” for such strings. Figure 4 illustrates the kind of equation of state, i.e. the energy per unit length and tension as functions of the state parameter $\kappa$ that can be obtained in a realistic, though simplified, Witten neutral model.

Once the current-carrier degree of freedom is integrated over a cross-section of the string, the latter becomes effectively two-dimensional, and for a spacelike current [68], one finds that increasing the equation of state parameter leads first to an increase of the current, followed by a saturation effect. After that limit, any further increase of the state parameter (phase gradient of the current carrier) leads to a decrease of the corresponding current. While the energy per unit length always increases, the tension decreases for increasing current until saturation is reached, and then increases. This implies an instability with respect to longitudinal perturbations ($c^2 < 0$).

A generic feature derivable from the Witten field theory model for superconducting strings is that the relevant macroscopic model is supersonic, in other words that it satisfies $c_T > c_L$. Assuming this result to hold, the constraint
\[ \kappa f_{kk} \leq 0 \] (46)
should also apply.

The Witten model provides two separate (although numerically very close) equations of state. The first provides the best approximation in the magnetic regime for which the current is spacelike. It is derived from the function
\[ f_{\text{mag}} = 1 - \frac{1}{\kappa} \frac{\bar{\kappa}}{1 - \alpha \bar{\kappa}} \implies \tilde{f}_{\text{mag}} = 1 + \frac{(1 + \alpha \bar{\kappa})}{2(1 - \alpha \bar{\kappa})}. \] (47)
The electric regime for which the current is timelike, on the other hand, is better described by
\[ f_{\text{elec}} = 1 + \frac{\ln(1 - \alpha \bar{\kappa})}{2\alpha} \implies \tilde{f}_{\text{elec}} = 1 + \frac{\bar{\kappa}}{1 - \alpha \bar{\kappa}} + \frac{\ln(1 - \alpha \bar{\kappa})}{2\alpha}. \] (48)

In Eqs. (47) and (48), we introduced the dimensionless parameter $\alpha$
\[ \alpha = \frac{\mu_0}{m_{\sigma}^2} = \left( \frac{m_{\text{Higgs}}}{m_{\sigma}} \right)^2, \] (49)
given by the ratio of the string-forming Higgs field mass $m_{\text{Higgs}}$ to that of the current-carrier $m_{\sigma}$. In order for the current to condense in the string, this ratio must be less than unity [25]; it is however hardly bounded from below [42].

Applying the constraints derived earlier to the specific models discussed above, we find the following limits on either the state parameter $\kappa$ or the extra parameter $\alpha$:

- **Linear equation of state**
  The constraints on (44) reduce to
  \[ f_{\text{lin}} \geq 0 \implies \kappa < 2, \] (50)
  \[ \tilde{f}_{\text{lin}} \geq 0 \implies \kappa > -2, \] (51)
  with the derivatives being constant, namely $f_{\text{lin}} = -\frac{1}{2}$ and $\tilde{f}_{\text{lin}} = \frac{1}{2}$.

- **Kaluza-Klein equation of state**
  For this other self-dual model, Eq. (45) yields
  \[ f_{\text{KK}} \geq 0 \text{ and } \tilde{f}_{\text{KK}} \geq 0 \implies \kappa < 1, \] (52)
  and this is the only requirement since
  \[ f_{\text{KK}} = -\frac{1}{2\sqrt{1 - \kappa}} \leq 0 \text{ } \forall \kappa < 1, \] (53)
  \[ \tilde{f}_{\text{KK}} = \frac{1}{2(1 - \kappa)^{3/2}} \geq 0 \text{ } \forall \kappa < 1, \] (54)

- **Witten model magnetic equation of state**
  This behavior is encoded in the phenomenological function (47). For the tension and the energy per unit length to be positive, one first needs to enforce
  \[ f_{\text{mag}} \geq 0 \implies \kappa \leq \frac{2}{1 + 2\alpha}. \] (55)
For the dual function, one finds that for $\alpha > 1/16$, $f^\text{mag}(\kappa) > 0$ provided $\kappa$ satisfies the constraint above. Indeed, $f^\text{mag} = 0$ for $\kappa = -1/(3\alpha)$, and $f^\text{mag}[-1/(3\alpha)] = 1 - 1/(16\alpha)$. If $\alpha \leq 1/16$, the equation $f^\text{mag} = 0$ has two solutions, both negative (corresponding to a spacelike current), namely

$$\kappa_{\pm} = \frac{4}{4\alpha - 1 \pm \sqrt{1 - 16\alpha}}$$

with $|\kappa_{+}| \geq |\kappa_{-}|$. (56)

In actual model calculations, however, one finds that the current-carrier mass $m_\sigma$ must be sufficiently smaller than the Higgs mass in order for the condensate to occur, and this means one can safely assume $\alpha > 1/16$ in what follows.

- **Witten model electric equation of state**

The magnetic equation of state provides an accurate description for a spacelike current-carrying cosmic string and for the most part of the timelike current case, but numerical simulations and asymptotic expansions also showed another effect in the electric regime, namely that of phase frequency threshold [38]. This amounts to a simple pole in the classical calculation of the condensed charge (as opposed to a second order pole arising from $f^\text{mag}$), and thus to a logarithmic generating function (48). The situation is roughly the same as for the magnetic Witten case, but the logarithm prevents analytic calculations to be made throughout. Again, the positiveness of $f^\text{elec}$ yields the phase frequency threshold

$$f^\text{elec} \geq 0 \quad \Rightarrow \quad \kappa \leq \frac{1 - e^{-2\alpha}}{\alpha}$$

and that of its dual $f^\text{elec}$ leads to two negative solutions $\kappa_{\pm}$ which must be calculated numerically. As above, $f^\text{elec}$ is however positive definite provided $\alpha$ exceeds a limiting value, numerically estimated to $\alpha_{\text{num}} \approx 0.156$. We shall restrict attention to this physically motivated situation.

Since, as with the previous models, one finds that $f^\text{elec}_{\kappa} - [2(1 - \alpha\kappa)]^{-1} < 0$, the stability is then established provided $f^\text{elec}_{\kappa} > 0$, which leads to $\kappa > -1/\alpha$.

These equations of state are shown graphically in Fig. 5.

**VI. MACROSCOPIC MODELS**

Here we first show how the system consisting of Eqs. (43) can be reduced to already known wiggly [54, 55, 65, 66] and chiral [56, 59] models, and then introduce a macroscopic version of the linear model which will be the starting point for our exploitation of the new VOS model in the companion paper.

![Figure 5. Analytic equations of state constructed from the generating functions $f^\text{lin}$ [Eq. (44)], $f^\text{sea}$ [Eq. (45)], $f^\text{mag}$ [Eq. (47)] and $f^\text{elec}$ [Eq. (48)]. For the magnetic and electric Witten models, we set $\alpha = 0.6$ for representation purposes.](image_url)

**A. The wiggly case**

For the wiggly model, we define

$$F = \sqrt{1 - Q^2} \equiv \mu^{-1},$$

where $J = 0$. Additionally, one can manage to be consistent with the notations in Refs. [54, 55] by introducing the parameters $\eta$ and $D$ such that

$$\begin{align*}
\tilde{c} &= [1 + D (1 - \mu^{-2})] c, \\
g &= \frac{1 + \eta (1 - \mu^{-1/2})}{1 + D (1 - \mu^{-2})}, \\
\rho &= 0,
\end{align*}$$

where $c$ is a constant loop chopping efficiency [51], while $\eta$ quantifies how much energy the long string network loses to small-scale loops, which are produced due to the presence of wiggles when a large (typically correlation length-sized) loop is produced; the parameter $D$ quantifies the energy transferred from the bare string into wiggles as a result of any intercommuting (whether or not loops are produced). In other words, it quantifies the energy transfer from large to small scales.

Plugging Eqs. (58) and (59) into Eqs. (43), one recovers the wiggly model described in Refs. [54, 55], namely...
\[ \dot{L}_c = \frac{\dot{\alpha}}{2\alpha} L_c \left( 1 + v^2 - \frac{1 - v^2}{\mu^2} \right) + \frac{[1 + \eta (1 - \mu^{-1/2})] c}{2\sqrt{\mu}}, \]  
(60a)
\[ \dot{v} = (1 - v^2) \left[ \frac{k(v)}{L_c \mu^{5/2}} - \frac{\dot{\alpha}}{a} \left( 1 + \frac{1}{\mu^2} \right) \right], \]  
(60b)
\[ \frac{\dot{\mu}}{\mu} = \frac{v}{L_c \sqrt{\mu}} \left\{ k(v) \left( 1 - \frac{1}{\mu^2} \right) + c \left[ \eta \left( 1 - \mu^{-1/2} \right) - D \left( 1 - \frac{1}{\mu^2} \right) \right] \right\} - \frac{\dot{\alpha}}{a} \left( 1 - \frac{1}{\mu^2} \right), \]  
(60c)

which are in agreement with Eqs. (78–80) of Ref. [58].

Note that since \( \rho = 0 \), this is a maximally biased model (in the previously discussed sense), while the rest of Eqs (59) can be written

\[ \dot{c} = (1 + DQ^2) c, \]
\[ g \sim 1 + \left( \frac{1}{4} \eta - D \right) Q^2; \]  
(61)
the fact the small-scale structure is seen to grow in Nambu-Goto simulations indicates that \( D > \eta/4 \) and therefore \( g < 1 \) in this case.

**B. The chiral case**

For the chiral model, we set

\[ F = 1, \quad F' = -\frac{1}{2}, \quad F'' = 0 \quad \text{and} \quad J^2 = Q^2, \]  
(62)
so as to ensure that the current is everywhere lightlike, i.e., \( \kappa \to 0 \). The evolution equations for \( Q^2 \) and \( J^2 \) are then identical provided we choose

\[ \rho = \frac{1}{2}, \]  
(63)
and it therefore suffices to fix \( J^2(\tau_{ini}) = Q^2(\tau_{ini}) \) for some initial time \( \tau_{ini} \) to ensure the current remains chiral at all times. The only physically relevant variable to describe the effect of the current is thus its amplitude, namely

\[ Y \equiv \frac{1}{2} (Q^2 + J^2). \]  
(64)

The other macroscopic parameter, \( g \), can be chosen in many different ways, depending on the model that we wish to reproduce in Ref. [56]. It should be also pointed out that we reproduce the model of Ref. [56] with vanishing \( s \) (\( s = 0 \)), due to our previously discussed assumption of vanishing boundary terms.

The set of equations to describe the chiral model is then obtained in a straightforward manner from Eqs. (43). They read

\[ \dot{L}_c = \frac{\dot{\alpha}}{a} \frac{L_c}{1 + Y} (v^2 + Y) + \frac{g \dot{c}}{2\sqrt{1 + Y}} v, \]  
(65a)
\[ \dot{v} = \frac{1 - v^2}{1 + Y} \left[ \frac{k(v)}{L_c \sqrt{1 + Y}} \right] - 2 \frac{\dot{\alpha}}{a}, \]  
(65b)
\[ \dot{Y} = 2Y \left[ \frac{v k(v)}{L_c \sqrt{1 + Y}} - \frac{\dot{\alpha}}{a} \right] - \frac{\dot{v}}{L_c} \dot{c} (g - 1) \sqrt{1 + Y}, \]  
(65c)

which correspond to Eqs. (105-107) of Ref. [58] provided one sets \( s \to 0 \) and \( \dot{c} \to c \), and if one assumes the bias \( g \) in (39) compensates exactly the difference between the total and bare correlation lengths (33), in other words if \( g = \sqrt{1 + Y} \). This can therefore be seen as the minimally biased model.

**C. The linear case**

The linear case is the natural macroscopic version of the microscopic linear model and is therefore obtained by setting

\[ F(K) = 1 - \frac{K}{2} \implies F' = -\frac{1}{2} \quad \text{and} \quad F'' = 0, \]  
(66)
with the macroscopic state parameter

\[ K \equiv Q^2 - J^2, \]  
(67)
effectively measuring the distance to chirality and leading to \( W = \sqrt{1 + Y} \), where the average current amplitude \( Y \) is defined by Eq. (64). This transforms Eqs. (43) into (65) for the variables \( L_c, v \) and \( Y \), together with

\[ \dot{K} = 2K \left[ \frac{v k(v)}{L_c \sqrt{1 + Y}} - \frac{\dot{\alpha}}{a} \right] \]
\[ - 2 \frac{\dot{v}}{L_c} \dot{c} (g - 1) (1 - 2\rho) \sqrt{1 + Y}, \]  
(68)
for the chirality parameter \( K \).

For an initially very small current with \( K \ll 1 \) and \( Y \ll 1 \), the linear model applies whatever the true model, and it becomes possible to figure out the conditions under which the current might grow, at least in the case when the source term in Eq. (68) vanishes (\( \rho = 1/2 \)). In this special case, supposing the other quantities reach a scaling solution in which the linear regime is still a valid approximation, the averaged 4-current magnitude \( K(\tau) \) behaves as \( K_{\tau=1/2} = \tau^{2\alpha} \), with

\[ \alpha = \frac{v_{sc} k(v_{sc})}{\zeta_{sc} \sqrt{1 + Y_{sc}}} - n \approx \frac{v_{sc} k(v_{sc})}{\zeta_{sc}} - n, \]  
(69)
to zeroth order in \( Y_{sc} \ll 1 \), where \( v_{sc}, \zeta_{sc} \) and \( Y_{sc} \) are the scaling values of the relevant functions.
If $\alpha > 0$, the average 4-current $K(\tau)$ grows so the “distance” to chirality increases and the non-linear regime may be reached to yield another, non-trivial and current-carrying, scaling solution. When $\alpha < 0$ on the other hand, the string network is dragged back to its original condition, approaching the chiral conditions $K(\tau) \to 0$ at late times, although perhaps with a non-vanishing current amplitude $Y_{sc} \neq 0$. It is interesting to note that such a non-linear current is more probable to build during the radiation dominated era ($n = 1$) than during the subsequent matter dominated era ($n = 2$).

**VII. CONCLUSION**

We have proposed a natural extension of the velocity one-scale VOS model, originally aimed at describing Nambu-Goto cosmic string networks through the evolution of their most salient statistical properties, namely a characteristic length scale or correlation length and a root mean square velocity, to include superconducting current properties that are, in principle, expected in many particle physics scenarios.

An arbitrary equation of state supposedly derivable from the microscopic structure of the string yields a non-linear $\sigma$-model description, enabling the identification of a single Lagrangian function of a state parameter, itself leading unambiguously to dynamical charge and current densities along the string network. Averaging, one obtains a generalization of the VOS model, namely Eqs. (43) which, among others, applies to the wiggly and chiral cases examined in earlier studies. As in these two specific cases, previously discussed in some detail, such extended models include two different length scales, denoted $L_c$ and $\xi_c$, that are related through the remaining degrees of freedom. Broadly speaking, the former encodes the total energy while the latter (which retains the physical interpretation of a correlation length) encodes the energy in the bare string. This is to be contrasted with the original one-scale model, where the string correlation length, inter-string separation and string curvature radius are all assumed to coincide.

We have also started the exploration of our new formalism by briefly considering the simplest non-trivial case—the small-current limit described by Eqs. (68). This very preliminary analysis confirms expectations, informed by previous work on the wiggly and chiral cases, that the behaviour of the additional degrees of freedom (in our case, the charges and currents) will depend on a competition between the cosmological expansion rate and available physical mechanisms determining how these charges and currents are produced (through reconnection or say primordial magnetic fields) and removed from the network (through reconnection and loop production). Naturally, such physical mechanisms are expected to be different for different models.

What our preliminary analysis already suggests is that a faster expansion rate (say the matter era, as opposed to the radiation era) facilitates the evolution towards the chiral limit, with equal amounts of charge and current. One can therefore envisage significantly different properties of superconducting string networks in the radiation and matter eras, leading to correspondingly different observational signatures That said, one must also bear in mind that the Nambu-Goto limit, with zero charge and current, is a (trivial) case of this chiral limit. While it is natural that such a Nambu-Goto limit exists for some parameter range within these models (indeed, all the more so in the linear model), the interesting question is whether or not there is also a parameter range for which a chiral solution with a non-trivial charge and current also exists.

**ACKNOWLEDGMENTS**

This work was financed by FEDER—Fundão Europeu de Desenvolvimento Regional funds through the COMPETE 2020—Operational Programme for Competitiveness and Internationalisation (POCI), through grants POCI-01-0145-FEDER-028987 and POCI-01-0145-FEDER-031938 and by Portuguese funds through FCT - Fundação para a Ciência e a Tecnologia in the framework of the projects PTDC/FIS-AST/28987/2017 and PTDC/FIS-PAR/31938/2017.

PP wishes to thank Churchill College, Cambridge, where he was partially supported by a fellowship funded by the Higher Education, Research and Innovation Dpt of the French Embassy to the United-Kingdom during this research. PS acknowledges funding from STFC Consolidated Grant ST/P000673/1.
Appendix A: $\langle \dot{X}^4 \rangle$ and $\langle \dot{X}^2 \rangle^2$ differences for standard VOS model

The difference between $\langle \dot{X}^4 \rangle$ and $\langle \dot{X}^2 \rangle^2$ can be studied in the standard VOS model perturbatively. We first set all charge and current terms to zero and study equation (22b), which can be rewritten as

$$\frac{1}{2} \frac{d \dot{X}^2}{d \tau} + 2 \dot{X}^2 \frac{\dot{a}}{a} \left( 1 - \dot{X}^2 \right) = \frac{X'' \cdot \dot{X}}{\epsilon},$$

$$\frac{1}{4} \frac{d \dot{X}^4}{d \tau} + 2 \dot{X}^4 \frac{\dot{a}}{a} \left( 1 - \dot{X}^2 \right) = \dot{X}^2 \frac{X'' \cdot \dot{X}}{\epsilon},$$

and similar expressions for the time derivative of higher powers of $\dot{X}^2$.

Averaging (A1), one obtains an infinite series of equations

$$\frac{1}{2} \frac{d v^2_{(2)}}{d \tau} + \frac{\dot{a}}{a} \left[ v^4_{(2)} - v^4_{(4)} \right] + 2 v^2_{(2)} \frac{\dot{a}}{a} - 2 v^4_{(4)} \frac{\dot{a}}{a} = \frac{k[v(2)]}{R_c} \left[ 1 - v^2_{(2)} \right],$$

$$\frac{1}{4} \frac{d v^4_{(4)}}{d \tau} + \frac{\dot{a}}{a} \left[ v^6_{(6)} - v^6_{(4)} v^2_{(2)} \right] + 2 v^4_{(4)} \frac{\dot{a}}{a} - 2 v^6_{(6)} \frac{\dot{a}}{a} = \frac{v(4) k[v(4)]}{R_c} \left[ v^2_{(2)} - v^4_{(4)} \right];$$

$$\frac{1}{6} \frac{d v^6_{(6)}}{d \tau} + \frac{\dot{a}}{a} \left[ v^8_{(8)} - v^6_{(6)} v^2_{(2)} \right] + 2 v^6_{(6)} \frac{\dot{a}}{a} - 2 v^8_{(8)} \frac{\dot{a}}{a} = \frac{v(6) k[v(6)]}{R_c} \left[ v^4_{(4)} - v^6_{(6)} \right];$$

...,

where the vector $u$ is a unit normal vector oriented towards the radius of curvature and $R_c$ is the averaged comoving radius of curvature, $v_{(p)} = \langle \dot{X}^2 \rangle^{1/2} = v$ as defined in the main text, Eq. (26), and similarly, $v_{(2p)} = \langle \dot{X}^{2p} \rangle^{1/(2p)}$ for any $p \in \mathbb{N}$, and we need to distinguish products of the form $\langle \dot{X}^2 \rangle^2 \neq \langle \dot{X}^4 \rangle$, which merely mean that $v_{(2p)} \neq v_{(2q)}$ for $p \neq q$.

The set of equations (A2) for the averages of various powers of $\dot{X}^2$ is, in theory, an infinite hierarchy. In practice however, since the velocities are less than unity, the difference $v^2_{(n^2)} - v^2_{(n-2)} v^2_{(2)}$ is getting increasingly smaller with larger values of $n$ so that it is reasonable to end the series for some finite value of $n$: truncating these equations at a given point merely amounts to choosing the required precision. Let us in what follows truncate the chain of equations (A2) on the third step, and therefore assume $v^8_{(8)} \approx v^6_{(6)} v^2_{(2)} + O\left[ v^8_{(8)} - v^6_{(6)} v^2_{(2)} \right]$.

We wish to compare the standard VOS model [51, 52] and the model with distinguished $v_{(2)}$, $v_{(4)}$ and $v_{(6)}$. Introducing the chopping efficiency $c$ and combining Eq. (A2) with the average energy relation (22b) we get

$$\frac{1}{2} \frac{d v^2_{(2)}}{d \tau} = \sqrt{v^2} \frac{k[v(2)]}{L_c} \left[ 1 - v^2_{(2)} \right] + \frac{\dot{a}}{a} \left[ v^4_{(4)} - v^4_{(2)} \right] - 2 \frac{\dot{a}}{a} \left[ v^2_{(2)} - v^4_{(4)} \right],$$

$$\frac{1}{4} \frac{d v^4_{(4)}}{d \tau} = v(4) \frac{k[v(4)]}{L_c} \left[ v^2_{(2)} - v^4_{(4)} \right] + \frac{\dot{a}}{a} \left[ v^6_{(6)} - v^4_{(4)} v^2_{(2)} \right] - 2 \frac{\dot{a}}{a} \left[ v^4_{(4)} - v^6_{(6)} \right];$$

$$\frac{1}{6} \frac{d v^6_{(6)}}{d \tau} = v(6) \frac{k[v(6)]}{L_c} \left[ v^4_{(4)} - v^6_{(6)} \right] + \frac{\dot{a}}{a} \left[ v^8_{(8)} - v^6_{(6)} v^2_{(2)} \right] - 2 \frac{\dot{a}}{a} \left[ v^6_{(6)} - v^8_{(8)} \right];$$

$$\frac{dL_c}{d\tau} = \frac{\dot{a}}{a} \frac{L_c v^2_{(2)} + c v^2_{(2)}}{2},$$

where we assumed that the network is Brownian, so that $E = \mu_0 V/(a L_c)^2$ in the volume $V$ [Eq. (30)], we also approximated that $R_c = L_c$, and defined the truncation at $v^8_{(8)} \approx v^6_{(6)} v^2_{(2)}$. The initial conditions are chosen to be different for $v_{(2)}$, $v_{(4)}$ and $v_{(6)}$, with differences of order $\approx 0.1$. The numerical solution of the system (A3) is compared to the standard VOS model, and the result is shown on figure 6.

It is seen that even if we try to impose the difference between $\langle \dot{X}^4 \rangle$ and $\langle \dot{X}^2 \rangle^2$ in the VOS model, by introducing the new variables $v_{(2)}$, $v_{(4)}$, and so forth, the variance eventually goes to zero as the system (A3) evolves. It might be an illustration of the fact that VOS model works on large scale (distances larger than $\xi$) and cannot properly grasp small-scale structure dynamics. This issue should be addressed to the studies of the models that are legitimate on different scales.

Similar assumption for terms as $\langle \dot{X}q \rangle$ should be valid even with higher accuracy, due to smaller correlations between the current and the velocity. For any variable $O$ satisfying $\langle O^2 \rangle \neq \langle O \rangle^2$ and leading to an expansion in the average, we anticipate a behavior similar to that obtained for the velocities.
Figure 6. Numerical solution of (A.3) for $a \propto \tau$, with initial conditions $v^2_{(2)0} = 0.4^2 + 0.1$, $v^4_{(4)0} = 0.4^4 + 0.1$, $v^6_{(6)0} = 0.4^6 + 0.1$, $L_{c0} = 0.001$, $c = 0.23$, and $v_*$ is the velocity for the standard VOS model with the assumption 1.

Appendix B: Derivation of the macroscopic equations

The system of Eqs. (22) can be rewritten with a supplementary equation, which arises from the combinations of Eqs. (22b) and (22c), in the following form

\[ \partial_t \left[ \epsilon \left( f - 2q^2 f_k \right) \right] + 2 \frac{\dot{a}}{a} \left\{ X^2 \left[ f - f_k \left( q^2 - j^2 \right) \right] - f_k \left( j^2 + q^2 \right) \right\} = -2 \partial_\sigma \left( f_k q j \right), \quad (B1a) \]

\[ \dot{X} \epsilon \left( f - 2q^2 f_k \right) + 2 \dot{X} \epsilon \left( 1 - X^2 \right) \left[ f - f_k \left( q^2 - j^2 \right) \right] = \partial_\sigma \left( \frac{f + 2f_k p}{\epsilon} X' \right) - 4 X' f_k q j \]

\[ \quad - 2 X' \left[ 2 \frac{\dot{a}}{a} f_k q j + \partial_\tau \left( f_k q j \right) \right], \quad (B1b) \]

\[ f_k \epsilon q \sqrt{1 - X^2} \left[ \frac{\dot{a}}{a} - \frac{X''}{X'^2} + \frac{X'}{X'^2} \left( q^2 - j^2 \right) \right] = \partial_\sigma \left( f_k j \sqrt{1 - X^2} \right), \quad (B1c) \]

\[ \epsilon \left( 1 - \frac{j^2}{q^2} \right) f_k \left\{ \frac{\dot{a}}{a} - \frac{X''}{X'^2} + \frac{2q^2 f_k q j + f_k}{2f_k \left( q^2 - j^2 \right)} \left[ (q^2 - j^2) \right] \right\} = \partial_\sigma \left( f k j q \right), \quad (B1d) \]

where we used relations from the gauge condition (13)

\[ X' \cdot X = -\dot{X} \cdot X', \quad X' \cdot X' = -\dot{X} \cdot X'', \quad (B2) \]

and

\[ \frac{\epsilon'}{\epsilon} = \frac{X' \cdot X''}{1 - X'^2}, \quad (B3) \]

\[ \frac{\dot{\epsilon}}{\epsilon} = \frac{\dot{X} \cdot \dot{X}}{1 - X'^2} - \frac{\dot{X} \cdot X''}{1 - X'^2}. \quad (B4) \]

Let us use the macroscopic variables defined in (23)–(26) to obtain a thermodynamical description of the system (B1). This provides a connection between the bare and total energies of the string network, namely

\[ E = a \mu_0 \int \epsilon \left( f - 2q^2 f_k \right) d\sigma \]

\[ = a \mu_0 \int \epsilon d\sigma \left( \int \frac{\epsilon f d\sigma}{\epsilon d\sigma} - 2 \int \frac{\epsilon q^2 f_k d\sigma}{\epsilon d\sigma} \right) \]

\[ = E_0 \left[ (Q^2 - J^2) F - 2Q^2 \left( Q^2 - J^2 \right) F' \right], \quad (B5) \]
where we used the notations (32) together with assumption 1. This leads to
\[
\dot{L}_c = \frac{\dot{a}}{a} \frac{L_c}{F - 2Q^2F'} \left\{ v^2 \left[ F - (Q^2 - J^2)F' \right] - F'(Q + J) \right\},
\] (B6)

once the Brownian assumption (30) is also used. Similarly, the average equation of motion for the velocity (B1b) can be written as
\[
\dot{\nu} = \frac{(1 - v^2)}{F - 2Q^2F'} \left\{ \frac{k(v)}{L_c \sqrt{F - 2Q^2F'}} \left( F + 2J^2F' \right) - 2v \frac{\dot{a}}{a} \left[ F - (Q^2 - J^2)F' \right] \right\},
\] (B7)

where we assumed \( R_c = \xi_c \) (\( R_c = L_c \sqrt{F - 2Q^2F'} \)) and used both assumptions 1 and 2.

Finally, the average Eqs. (B1c) and (B1d) for the charge and current take the form
\[
\langle J^2 \rangle^* = 2J^2 \left[ v \frac{k(v)}{L_c \sqrt{F - 2Q^2F'}} - \frac{\dot{a}}{a} \right],
\]
\[
\langle Q^2 \rangle^* = 2Q^2 \frac{2J^2F'' + F'}{2Q^2F'' + F'} \left[ v \frac{k(v)}{L_c \sqrt{F - 2Q^2F'}} - \frac{\dot{a}}{a} \right].
\] (B8)

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