On the True Nature of Turbulence

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Abstract. In this article, I would like to express some of my views on the nature of turbulence. These views are mainly drawn from the author’s recent results on chaos in partial differential equations [10].

Fluid dynamicists believe that Navier-Stokes equations accurately describe turbulence. A mathematical proof on the global regularity of the solutions to the Navier-Stokes equations is a very challenging problem. Such a proof or disproof does not solve the problem of turbulence. It may help understanding turbulence. Turbulence is more of a dynamical system problem. Studies on chaos in partial differential equations indicate that turbulence can have Bernoulli shift dynamics which results in the wandering of a turbulent solution in a fat domain in the phase space. Thus, turbulence can not be averaged. The hope is that turbulence can be controlled.

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1. The Governing Equations of Turbulence

It has been overwhelmingly accepted by fluid dynamicists that the Navier-Stokes equations are accurate governing equations of turbulence. Their delicate experimental measurements on turbulence have led them to such a conclusion. A simple form of the Navier-Stokes equations, describing viscous incompressible fluids, can be written as

\[ u_{i,t} + u_j u_{i,j} = -p_i + \text{Re}^{-1} u_{i,jj} + f_i , \quad u_{i,i} = 0 ; \]

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where $u_i$'s are the velocity components, $p$ is the pressure, $f_i$'s are the external force components, and Re is the Reynolds number. There are two ways of deriving the Navier-Stokes equations: (1). The fluid dynamicist’s way of using the concept of fluid particle and material derivative, (2). The theoretical physicist’s way of starting from Boltzmann equation. According to either approach, one can replace the viscous term $Re^{-1} u_{i,jj}$ by for example

\[ Re^{-1} u_{i,jj} + \alpha u_{i,jj,kk} + \cdots. \]

Here the only principle one can employ is the Einstein covariance principle which eliminates the possibility of third derivatives for example. According to the fluid dynamicist’s way, the viscous term $Re^{-1} u_{i,jj}$ was derived from a principle proposed by Newton that the stress is proportional to the velocity’s derivatives (strain, not velocity). Such fluids are called Newtonian fluids. Of course, there exist non-Newtonian fluids like volcanic lava for which the viscous term is more complicated and can be nonlinear. According to the theoretical physicist’s way, the viscous term was obtained from an expansion which has no reason to stop at its leading order term $Re^{-1} u_{i,jj}$.

2. Global Well-Posedness of the Navier-Stokes Equations

It is well known that the global well-posedness of the Navier-Stokes equations (1.1) has been selected by the Clay Mathematics Institute as one of its seven million dollars problems. Specifically, the difficulty lies at the global regularity \[6\]. More precisely, the fact that

\[ \int \int u_{i,j} u_{i,j} \, dx \, dt \]

being bounded only implies

\[ \int u_{i,j} u_{i,j} \, dx \]

being bounded for almost all $t$, is the key of the difficulty. In fact, Leray was able to show that the possible exceptional set of $t$ is actually a compact set of measure zero. There have been a lot of more recent works on describing this exceptional compact set \[2\]. The claim that this possible exceptional compact set is actually empty, will imply the global regularity and the solution of the problem. The hope for such a claim seems slim.

Even for ordinary differential equations, often one can not prove their global well-posedness, but their solutions on computers look perfectly globally regular and sometimes chaotic. Chaos and global regularity are compatible. The fact that fluid experimentalists quickly discovered shocks in compressible fluids and never found any finite time blow up in incompressible fluids, indicates that there might be no finite time blow up in Navier-Stokes equations (even Euler equations). On the other hand, the solutions of Navier-Stokes equations can definitely be turbulent (chaotic).

Replacing the viscous term $Re^{-1} u_{i,jj}$ by higher order derivatives (1.2), one can prove the global regularity \[4\]. This leaves the global regularity of (1.1) a more challenging and interesting mathematical problem. Assume that the unthinkable event happens, that is, someone proves the existence of a meaningful finite time blow up in (1.1), then fluid experimentalists need to identify such a finite time blow up in the experiments. If they fail, then the choice will be whether or not to replace
the viscous term $\text{Re}^{-1} \ u_{1,22}$ in the Navier-Stokes equations \ref{1.1} by higher order derivatives like \ref{1.2} to better model the fluid motion.

Even after the global regularity of \ref{1.1} is proved or disproved, the problem of turbulence is not solved although the global regularity information will help understanding turbulence. Turbulence is more of a dynamical system problem. Often a dynamical system study does not depend on global well-posedness. Local well-posedness is often enough. In fact, this is the case in my proof on the existence of chaos in partial differential equations \ref{10}.

3. Chaos in Partial Differential Equations

Ever since the discovery of chaos in low dimensional systems, people have been trying to use the concept of chaos to understand turbulence \ref{17}. There are two types of fluid motions: Laminar flows and turbulent flows. Laminar flows look regular, and turbulent flows are non-laminar and look irregular. Chaos is more precise, for example, in terms of Bernoulli shift dynamics. On the other hand, even in low dimensional systems, there are solutions which look irregular for a while, and then look regular again. Such a dynamics is often called a transient chaos.

Everyone knows that the signature of chaos is sensitive dependence on initial data. Often the word “sensitive” is over-imagined. For any fixed large time, the chaotic solution still depends on its initial condition continuously. It is the infinite time that leads to sensitive dependence.

Low dimensional chaos is the starting point of a long journey toward understanding turbulence. To have a better connection between chaos and turbulence, one has to study chaos in partial differential equations \ref{10}. Take the simple perturbed sine-Gordon equation for example \ref{9} \ref{15} (3.1)

$$u_{tt} = c^2 u_{xx} + \sin u + \epsilon \left[ -au_t + \cos t \sin^3 u \right],$$

which is subject to periodic boundary condition

$$u(t, x + 2\pi) = u(t, x),$$

and even or odd constraint

$$u(t, -x) = u(t, x) \quad \text{or} \quad u(t, -x) = -u(t, x),$$

where $u$ is a real-valued function of two real variables $(t, x)$, $c$ is a real constant, $\epsilon \geq 0$ is a small perturbation parameter, and $a > 0$ is an external parameter. One can view (3.1) as a flow defined in the phase space

$$(u, u_t) \in H^1 \times L^2$$

where $H^1$ and $L^2$ are the Sobolev spaces on $[0, 2\pi]$. A point in the phase space corresponds to two profiles

$$(u(x), u_t(x)) \ .$$

One can prove that there exists a homoclinic orbit $(u, u_t) = h(t, x)$ asymptotic to $(u, u_t) = (0, 0)$ \ref{9} \ref{15}. Let us define two orbits segments

$$\eta_0 : (u, u_t) = (0, 0), \quad t \in [-T, T], \quad \eta_1 : (u, u_t) = h(t, x), \quad t \in [-T, T] .$$

When $T$ is large enough, $\eta_1$ is almost the entire homoclinic orbit (chopped off in a small neighborhood of $(u, u_t) = (0, 0)$). To any binary sequence

(3.2)

$$a = \{\cdots a_{-2}a_{-1}a_0, a_1 a_2 \cdots \}, \quad a_k \in \{0, 1\} ;$$
one can associate a pseudo-orbit
\[ \eta_a = \{ \cdots \eta_{a-2} \eta_{a-1} \eta_{a0} , \eta_{a1} \eta_{a2} \cdots \} . \]
The pseudo-orbit \( \eta_a \) is not an orbit but almost an orbit. One can prove that for any such pseudo-orbit \( \eta_a \), there is a unique true orbit in its neighborhood \[9\] \[15\]. Therefore, each binary sequence labels a true orbit. All these true orbits together form a chaos. In order to talk about sensitive dependence on initial data, one can introduce the product topology by defining the neighborhood basis of a binary sequence
\[ a^* = \{ \cdots a_{-2}^* a_{-1}^* a_0^* , a_1^* a_2^* \cdots \} \]
as
\[ \Omega_N = \{ a : a_n = a_n^* , |n| \leq N \} . \]
The Bernoulli shift on the binary sequence \[3.2\] moves the comma one step to the right. Two binary sequences in the neighborhood \( \Omega_N \) will be of order \( \Omega_1 \) away after \( N \) iterations of the Bernoulli shift. Since the binary sequences label the orbits, the orbits will exhibit the same feature. In fact, the Bernoulli shift is topologically conjugate to the perturbed sine-Gordon flow.

Replacing a homoclinic orbit by its fattened version – a homoclinic tube, or by a heteroclinic cycle, or by a heteroclinically tubular cycle; one can still obtain the same Bernoulli shift dynamics \[7\] \[8\] \[9\] \[15\].

Adding diffusive perturbation \( \epsilon b u_{txx} \) to \[3.1\], one can still prove the existence of homoclinics or heteroclinics, but the Bernoulli shift result has not been established \[9\] \[15\].

Another system studied is the complex Ginzburg-Landau equation \[11\] \[12\],
\[ i q_t = q_{xx} + 2 \left[ |q|^2 - \omega^2 \right] + i \epsilon \left[ q_{xx} - \alpha q + \beta \right] , \]
which is subject to periodic boundary condition and even constraint
\[ q(t, x + 2\pi) = q(t, x) , \quad q(t, -x) = q(t, x) , \]
where \( q \) is a complex-valued function of two real variables \( (t, x) \), \( (\omega, \alpha, \beta) \) are positive constants, and \( \epsilon \geq 0 \) is a small perturbation parameter. In this case, one can prove the existence of homoclinic orbits \[11\]. But the Bernoulli shift dynamics was established under generic assumptions \[12\].

A real fluid example is the amplitude equation of Faraday water wave, which is also a complex Ginzburg-Landau equation \[13\],
\[ i q_t = q_{xx} + 2 \left[ |q|^2 - \omega^2 \right] + i \epsilon \left[ q_{xx} - \alpha q + \beta \bar{q} \right] , \]
subject to the same boundary condition as \[3.3\]. For the first time, one can prove the existence of homoclinic orbits for a water wave equation \[3.4\] \[13\]. The Bernoulli shift dynamics was also established under generic assumptions \[13\]. That is, for the first time, one can prove the existence of chaos in water waves under generic assumptions.

The nature of the complex Ginzburg-Landau equation is a parabolic equation which is near a hyperbolic equation. The same is true for the perturbed sine-Gordon equation with the diffusive term \( \epsilon b u_{txx} \) added. They contain effects of diffusion, dispersion, and nonlinearity. The Navier-Stokes equations are diffusion-advection equations. The advective term is missing from the perturbed sine-Gordon equation and the complex Ginzburg-Landau equation. But the modified KdV equation does contain an advective term. In principle, perturbed modified KdV equation
should have the same feature as the perturbed sine-Gordon equation. Turbulence happens when the diffusion is weak, i.e. in the near hyperbolic regime. One should hope that turbulence should share some of the features of chaos in the perturbed sine-Gordon equation. There is a popular myth that turbulence is fundamentally different from chaos because turbulence contains many unstable modes. In both the perturbed sine-Gordon equation and the complex Ginzburg-Landau equation, one can incorporate as many unstable modes as one likes, the resulting Bernoulli shift dynamics is still the same. On a computer, the solution with more unstable modes may look rougher, but it is still chaos. So I think the issue of number of unstable modes between turbulence and chaos is an illusion.

Turbulence is any flow that is non-laminar. Sometimes, turbulence can happen in a localized spot of a fluid domain, or during a finite period of time. These are not chaos. I have a favorite simile of the situation: One can think turbulence as marbles; and those flows for which the existence of chaos can be rigorously proved, as diamonds. Marbles are everywhere, while diamonds are rare. Understanding diamonds can help understanding marbles. Diamonds are precious, while marbles are realistically useful in engineering.

A simple setup for studying the chaotic nature of turbulence is posing the Navier-Stokes equation \( \text{(1.1)} \) on a spatially periodic domain, with a temporally and spatially periodic external force. In this case, one can take the advantage of Fourier series. One can show that there are well-defined invariant manifolds \( 14 \). A thorough numerical investigation of this dynamical system should be significant for a better understanding of turbulence.

4. Control of Turbulence

When dealing with random solutions to a stochastic equation, researchers are not content with the random solutions as they are. Various averagings will be conducted to gain more certain quantifications of the random solutions, since uncertainty is never the favorite to researchers in contrast to certainty. Fundamentally encouraging to such thoughts is that these averagings are very successful in describing the random solutions.

When dealing with Navier-Stokes equations which are nonlinear deterministic equations, fluid engineers are very happy with laminar solutions as they are, but not turbulent solutions. They have been trying hard to quantify turbulent solutions with averaging techniques. Reynolds envisioned a relatively long time averaging to the turbulent solutions. Such an averaging failed miserably.

From what we learn about chaos in partial differential equations, turbulent solutions not only have sensitive dependences on initial conditions, but also are densely packed inside a domain in the phase space. They are far away from the feature of fluctuations around a mean. In fact, they wander around in a fat domain rather than a thin domain in the phase space. Therefore, averaging makes no sense at all. One has to be content with turbulent solutions as they are.

In real life, turbulence often represents unpleasant or disastrous events. When an airplane meets turbulence, the passengers do not feel comfortable and the airplane can be damaged. The fundamental question here is whether or not turbulence can be controlled. Here the word “control” represents a wide spectrum of actions: Taming turbulent states into laminar states \( 1 \), reducing turbulent drag \( 16 \ 5 \).
enhancing turbulent mixing \cite{16, 5}, and gearing a turbulent orbit to a specific target \cite{3} etc. The final motto that I am aiming at is:

- Turbulence can not be averaged, but can be controlled.

Specific control tools have been developed. These are sensors and actuators placed in flow fields. These sensors and actuators hopefully can be placed by MEMS (Micro-Electro-Mechanical-System) technology in the future to obtain a more effective control.

One can re-interpret the Reynolds averaging as a control of taming turbulence into a laminar flow. According to Reynolds, one splits the variables in (1.1) into two parts:

\[ u_i = U_i + \tilde{u}_i, \quad p = P + \tilde{p} \]

where the capital letters represent relatively long time averages which are still a function of time and space, and the tilde-variables represent mean zero fluctuations,\[ U_i = \langle u_i \rangle, \quad \langle \tilde{u}_i \rangle = 0, \quad P = \langle p \rangle , \quad \langle \tilde{p} \rangle = 0 . \]

A better interpretation is by using ensemble average of repeated experiments. One can derive the Reynolds equations for the averages,

\begin{equation}
U_{i,t} + U_j U_{i,j} = -P_{i,i} + Re^{-1} U_{i,ji} - \langle \tilde{u}_i \tilde{u}_j \rangle , j + f_i , \quad U_{i,i} = 0 .
\end{equation}

The term \( \langle \tilde{u}_i \tilde{u}_j \rangle \) is completely unknown. Fluid engineers call it Reynolds stress. The Reynolds model is given by

\begin{equation}
\langle \tilde{u}_i \tilde{u}_j \rangle = -R^{-1} U_{i,j} ,
\end{equation}

where \( R \) is a constant. There are many more models on the term \( \langle \tilde{u}_i \tilde{u}_j \rangle \). But no one leads to a satisfactory result. One can re-interpret the Reynolds equations \cite{4.1} as control equations of the original Navier-Stokes equations \cite{1.1}, with the term \( \langle \tilde{u}_i \tilde{u}_j \rangle , j \) being the control of taming a turbulent solution to a laminar solution (hopefully nearby). The Reynolds model \cite{1.2} amounts to changing the fluid viscosity which can bring a turbulent flow to a laminar flow. This laminar flow may not be anywhere near the turbulent flow though. Thus, the Reynolds model may not produce satisfactory result in comparison with the experiments. Fluid engineers gradually gave up all these Reynolds’ type models and started directly computing the original Navier-Stokes equations \cite{1.1}

An advantage of the control theory is that it can be conducted in a trial-correction manner without a detailed knowledge of turbulence. Of course, better knowledge of turbulence will help the control. In a sense, locating chaos and controlling chaos are intertwined. The Melnikov integral can predict the existence of chaos \cite{10}, at the same time, it also predicts the non-existence of chaos when parameters are changed.

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