INDIVIDUAL ERGODIC THEOREMS IN NONCOMMUTATIVE SYMMETRIC SPACES

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Abstract. It is known that, for a positive Dunford-Schwartz operator in a noncommutative $L^p$-space, $1 \leq p < \infty$ or, more generally, in a noncommutative Orlicz space with order continuous norm, the corresponding ergodic averages converge bilaterally almost uniformly. We show that these averages converge bilaterally almost uniformly in each noncommutative symmetric space $E$ such that $\mu_t(x) \to 0$ as $t \to 0$ for every $x \in E$, where $\mu_t(x)$ is a non-increasing rearrangement of $x$. In particular, these averages converge bilaterally almost uniformly in all noncommutative symmetric spaces with order continuous norm.

1. Preliminaries

Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace $\tau$. Let $\mathcal{P}(\mathcal{M})$ stand for the set of projections in $\mathcal{M}$. If $1$ is the identity of $\mathcal{M}$ and $e \in \mathcal{P}(\mathcal{M})$, we write $e^\perp = 1 - e$. Denote by $L^0 = L^0(\mathcal{M}, \tau)$ the $\tau$-measurable operators affiliated with $\mathcal{M}$. Let $\| \cdot \|_\infty$ be the uniform norm in $\mathcal{M}$. Equipped with the measure topology given by the system

$$V(\epsilon, \delta) = \{ x \in L^0 : \| xe \|_\infty \leq \delta \text{ for some } e \in \mathcal{P}(\mathcal{M}) \text{ with } \tau(e^\perp) \leq \epsilon \},$$

$\epsilon > 0$, $\delta > 0$, $L^0$ is a complete metrizable topological $*$-algebra [14].

Let $x \in L^0$, and let $\{ e_\lambda \}_{\lambda \geq 0}$ be the spectral family of projections for the absolute value $|x|$ of $x$. If $t > 0$, then the $t$-th generalized singular number of $x$ (a non-increasing rearrangement of $x$) (see [9]) is defined as

$$\mu_t(x) = \inf \{ \lambda > 0 : \tau(e_\lambda^\perp) \leq t \}.$$

A Banach space $(E, \| \cdot \|_E) \subset L^0$ is called fully symmetric if the conditions

$$x \in E, \ y \in L^0, \ \int_0^s \mu_t(y) dt \leq \int_0^s \mu_t(x) dt \ \text{for all } s > 0 \ \text{ (writing } y \prec \prec x)$$

imply that $y \in E$ and $\| y \|_E \leq \| x \|_E$. It is known [6] that if $(E, \| \cdot \|_E)$ is a fully symmetric space, $x_n, x \in E$, and $\| x - x_n \|_E \to 0$, then $x_n \to x$ in measure. A fully symmetric space $(E, \| \cdot \|_E)$ is said to possess Fatou property if conditions

$$x_\alpha \in E^+, \ x_\alpha \leq x_\beta \text{ for } \alpha \leq \beta, \ \text{and } \sup_\alpha \| x_\alpha \|_E < \infty$$

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imply that there exists $x = \sup_a x_a \in E$ and $\|x\|_E = \sup_a \|x_a\|_E$. A space $(E, \| \cdot \|_E)$ is said to have order continuous norm if $\|x_a\|_E \downarrow 0$ whenever $x_a \in E$ and $x_a \downarrow 0$.

Let $L^0(0, \infty)$ be the linear space of all (equivalence classes of) almost everywhere finite complex-valued Lebesgue measurable functions on the interval $(0, \infty)$. We identify $L^\infty(0, \infty)$ with the commutative von Neumann algebra acting on the Hilbert space $L^2(0, \infty)$ via multiplication by the elements from $L^\infty(0, \infty)$ with the trace given by the integration with respect to Lebesgue measure. A Banach space $E \subset L^0(0, \infty)$ is called fully symmetric function space on $(0, \infty)$ if the condition above holds with respect to the von Neumann algebra $L^\infty(0, \infty)$.

Let $E = (E(0, \infty), \| \cdot \|_E)$ be a fully symmetric function space. For each $s > 0$ let $D_s : E \to E$ be the bounded linear operator given by $D_s(f)(t) = f(t/s)$, $t > 0$. The Boyd indices $p_E$ and $q_E$ are defined as

$$p_E = \lim_{s \to \infty} \frac{\log s}{\log \|D_s\|_E}, \quad q_E = \lim_{s \to 0^+} \frac{\log s}{\log \|D_s\|_E}.$$

It is known that $1 \leq p_E \leq q_E < \infty$ \[13\] II, Ch.2, Proposition 2.b.2. A fully symmetric function space is said to have non-trivial Boyd indices if $1 < p_E$ and $q_E < \infty$. For example, the space $L^p(0, \infty)$, $1 < p < \infty$, have non-trivial Boyd indices:

$$p_{L^p(0, \infty)} = q_{L^p(0, \infty)} = p$$

\[13\] II, Ch.4, §4, Theorem 4.3. A Banach lattice $(E, \| \cdot \|_E)$ is called q-concave, $1 \leq q < \infty$, if there exists a constant $M > 0$ such that

$$\left( \sum_{i=1}^n |x_i|^q \right)^{\frac{1}{q}} \leq M \left\| \left( \sum_{i=1}^n |x_i|^q \right)^{\frac{1}{q}} \right\|_E$$

for every finite set $\{x_i\}_{i=1}^n \subset E$. Let $E = (E(0, \infty), \| \cdot \|_E)$ be a fully symmetric function space, and let $E$ be q-concave for some $1 \leq q < \infty$. Then there does not exist a sublattice of $E$ isomorphic to $l^\infty$, hence the norm $\| \cdot \|_E$ is order continuous \[16\] Corollary 2.4.3.

If $E(0, \infty)$ is a fully symmetric function space, define

$$E(M) = E(M, \tau) = \{ x \in L^0 : \mu_\tau(x) \in E \}$$

and set

$$\|x\|_{E(M)} = \|\mu_\tau(x)\|_E, \quad x \in E(M).$$

It is shown in \[12\] that $(E(M), \| \cdot \|_{E(M)})$ is a fully symmetric space. If $1 \leq p < \infty$ and $E = L_p(0, \infty)$, the space $(E(M), \| \cdot \|_{E(M)})$ coincides with the noncommutative $L^p$-space $L^p = L^p(M) = (L^p(M, \tau), \| \cdot \|_p)$ because

$$\|x\|_p = \left( \int_0^\infty \mu_\tau^0(x) dt \right)^{1/p} = \|x\|_{L^p(M)}$$

(see \[17\] Proposition 2.4). In addition, $L^\infty(M) = M$ and

$$(L^1 \cap L^\infty)(M) = L^1(M) \cap M, \quad \|x\|_{L^1 \cap L^\infty} = \max\{\|x\|_1, \|x\|_M\},$$

$$(L^1 + L^\infty)(M) = L^1(M) + M,$$
\[ \|x\|_{L^1 + L^\infty} = \inf \{ \|x\|_1 + \|y\|_M : x \in L^1(M), \ y \in M \} = \int_0^1 \mu_t(x) dt \]

(see [6 Proposition 2.5]). (For a comprehensive review of noncommutative \(L^p\)–spaces, see [17, 15].)

Since for a fully symmetric function space \(E(0, \infty), \)
\[ L^1(0, \infty) \cap L^\infty(0, \infty) \subset E \subset L^1(0, \infty) + L^\infty(0, \infty) \]
with continuous embeddings [11, Ch.II, §4, Theorem 4.1], we also have
\[ L^1(M) \cap M \subset E(M, \tau) \subset L^1(M) + M, \]
with continuous embeddings.

Denote
\[ L^0_r = \{ x \in L^0(M) : \mu_t(x) \to 0 \ \text{as} \ t \to 0 \}. \]

Observe that \( L^0_r \) is a linear subspace of \( L^0 \) which is solid in the sense that if \( x \in L^0_r \) and if \( y \in L^0 \) and \( \mu_t(y) \leq \mu_t(x) \), then also \( y \in L^0_r \). It is clear that \( x \in L^0_r \) if and only if \( \tau(\chi_{(\lambda, \infty)}(|x|)) < \infty \) for all \( \lambda > 0 \); moreover, \( L^1 \cap M \subset L^0_r \).

Define
\[ R_\tau = \{ x \in L^1 + M : \mu_t(x) \to 0 \ \text{as} \ t \to 0 \}. \]

It follows from the next proposition that \((R_\tau, \| \cdot \|_{L^1 + M})\) is a Banach space.

**Proposition 1.1.** [6 Proposition 2.7] \( R_\tau \) is the closure of \( L^1 \cap M \) in \( L^1 + M \).

Note that definitions of \( L^0_r \) and \( \| \cdot \|_{L^1 + L^\infty} \) yield that if
\[ x \in R_\tau, \ y \in L^1 + L^\infty, \ \text{and} \ \mu_t(y) \leq \mu_t(x), \]
then \( y \in R_\tau \) and \( \| y \|_{L^1 + M} \leq \| x \|_{L^1 + M}. \) Therefore \((R_\tau, \| \cdot \|_{L^1 + M})\) is a noncommutative symmetric space [6 §2].

2. **Individual Ergodic Theorems in \( R_\tau \)**

A linear operator \( T : L^1 + M \to L^1 + M \) is called a Dunford-Schwartz operator (writing \( T \in DS \)) if
\[ \|T(x)\|_1 \leq \|x\|_1, \ \forall \ x \in L^1 \quad \text{and} \quad \|T(x)\|_\infty \leq \|x\|_\infty, \ \forall \ x \in M. \]

If a Dunford-Schwartz operator \( T \) is positive, that is, \( T(x) \geq 0 \) whenever \( x \geq 0 \), we shall write \( T \in DS^+. \)

If \( T \in DS \), then
\[ (1) \quad \|T\|_{L^1 + M \to L^1 + M} \leq 1 \]
(see, for example, [6 §4]) and, in addition, \( Tx \prec x \) for all \( x \in L^1 + M [6 \text{Theorem } 4.7]. \) Hence
\[ (2) \quad T(R_\tau) \subset R_\tau. \]

Given \( T \in DS \) and \( x \in L^1 + L^\infty \), denote
\[ (3) \quad A_n(x) = A_n(T, x) = \frac{1}{n} \sum_{k=0}^{n-1} T^k(x), \ n = 1, 2, \ldots, \]
the corresponding ergodic averages of the operator \( x \).

Note that, by [11] and [2], for any \( T \in DS \) its restriction on \( R_\tau \) is a linear contraction (also denoted by \( T \)).
Definition 2.1. A sequence $\{x_n\} \subset L^0$ is said to converge to $x \in L^0$ bilaterally almost uniformly (b.a.u.) if for every $\epsilon > 0$ there exists such a projection $e \in \mathcal{P}(\mathcal{M})$ that $\tau(e^+) \leq \epsilon$ and $\|e(x - x_n)e\|_{\infty} \to 0$.

The following groundbreaking result was established in [18] as a corollary of a noncommutative maximal ergodic inequality [18] Theorem 1] (for the assumption $T \in DS^+$, see [4] Remark 1.2] and [10] Lemma 1.1]).

Theorem 2.1. Let $T \in DS^+$ and $x \in L^1$. Then the averages $e^{(k)}$ converge b.a.u. to some $\hat{x} \in L^1$.

The next result is an extension of Theorem 2.1 to $T_{\tau}$.

Theorem 2.2. Let $T \in DS^+$ and $x \in T_{\tau}$. Then the averages $e^{(k)}$ converge b.a.u. to some $\tilde{x} \in L^1 + \mathcal{M}$.

Proof. Without loss of generality assume that $x \geq 0$. Let $\{e_{\lambda}\}_{\lambda \geq 0}$ be the spectral family of $x$. Given $k \in \mathbb{N}$, denote $x_k = \int_{1/k}^{\infty} \lambda d\lambda$ and $y_k = \int_{0}^{1/k} \lambda d\lambda$. Then $0 \leq y_k \leq \frac{1}{k} \cdot 1$, $x_k \in L^1$, and $x = x_k + y_k$ for all $k$.

Fix $\epsilon > 0$. By Theorem 2.1 $A_n(x_k) \to \hat{x}_k \in L^1$ b.a.u. for each $k$. Therefore there exists $e_k \in \mathcal{P}(\mathcal{M})$ such that $\tau(e_k) \leq \frac{\epsilon}{2}$ and $\|e_k(A_n(x_k) - \hat{x}_k)e_k\|_{\infty} \to 0$ as $n \to \infty$.

Then it follows that

$$\|e_k(A_n(x_k) - A_m(x_k))e_k\|_{\infty} < \frac{1}{k} \text{ for all } m, n \geq N(k).$$

Since $\|y_k\|_{\infty} \leq \frac{1}{k}$, we have

$$\|e_k(A_n(x) - A_m(x))e_k\|_{\infty} \leq \|e_k(A_n(x_k) - A_m(x_k))e_k\|_{\infty} + \|e_k(A_n(y_k) - A_m(y_k))e_k\|_{\infty} < \frac{1}{k} + \|e_kA_n(y_k)e_k\|_{\infty} + \|e_kA_m(y_k)e_k\|_{\infty} \leq \frac{3}{k}.$$ 

for each $k$ and all $m, n \geq N(k)$.

Let $e = \bigwedge_{k \geq 1} e_k$. Then $\tau(e^+) \leq \sum_{k=1}^{\infty} \tau(e_k^+) \leq \epsilon$ and

$$\|e(A_n(x) - A_m(x))e\|_{\infty} < \frac{3}{k}$$

for all $m, n \geq N(k)$. This means that $\{A_n(x)\}$ is Cauchy with respect to b.a.u. convergence. Then, by [3] Theorem 2.3], we conclude that the sequence $\{A_n(x)\}$ converges b.a.u. to some $\tilde{x} \in L^0$.

Since $L^1 + \mathcal{M}$ satisfies Fatou property [5] §4], its unit ball is closed in the measure topology [7] Theorem 4.1], and [11] implies that $\tilde{x} \in L^1 + \mathcal{M}$. \qed

Let $C_1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle in $\mathbb{C}$. A function $P : \mathbb{Z} \to \mathbb{C}$ is said to be a trigonometric polynomial if $P(k) = \sum_{j=1}^{N} z_j \lambda_j^k$, $k \in \mathbb{Z}$, for some $s \in \mathbb{N}$, $\{z_j\}_{j=1}^{s} \subset \mathbb{C}$, and $\{\lambda_j\}_{j=1}^{s} \subset C_1$. A sequence $\{\beta_k\}_{k=0}^{\infty} \subset \mathbb{C}$ is called a bounded Besicovitch sequence if

(i) $|\beta_k| \leq C < \infty$ for all $k$;

(ii) for every $\epsilon > 0$ there exists a trigonometric polynomial $P$ such that

$$\limsup_{n} \frac{1}{n+1} \sum_{k=0}^{n} |\beta_k - P(k)| < \epsilon.$$


The following theorem was established in [3].

**Theorem 2.3.** Assume that $\mathcal{M}$ has a separable predual. Let $T \in DS^+$ and let $\{\beta_k\}$ be a Besicovitch sequence. Then for every $x \in L^1(\mathcal{M})$ the averages

$$\frac{1}{n} \sum_{k=0}^{n-1} \beta_k T^k(x)$$

converge b.a.u. to some $\hat{x} \in L^1$.

In view of Theorem 2.3 and since a sequence $\{\beta_k\}$ in question is bounded, the proof of Theorem 2.2 can be carried out for the averages (4), and we obtain the following.

**Theorem 2.4.** Let $\mathcal{M}$, $T$, $\{\beta_k\}$ be as in Theorem 2.3. Then for every $x \in \mathcal{R}_\tau$ the averages (4) converge to some $\hat{x} \in L^1 + \mathcal{M}$.

3. Applications of Theorem 2.2

Let us present some examples of noncommutative symmetric spaces for which Dunford-Schwartz individual ergodic theorem is valid.

1. Let $\Phi$ be an Orlicz function, $L^\Phi = L^\Phi(\mathcal{M}, \tau)$ the corresponding noncommutative Orlicz space, $\| \cdot \|_\Phi$ the Luxemburg norm in $L^\Phi$ (see [5]). Since $(L^\Phi(0, \infty), \| \cdot \|_\Phi)$ has Fatou property, the space $(L^\Phi(\mathcal{M}, \tau), \| \cdot \|_\Phi)$ has it as well [6, §5]. As shown in the proof of [3] Proposition 2.1, $L^\Phi \subset \mathcal{R}_\tau$. It follows then from Theorem 2.2 that we have

**Theorem 3.1.** Let $T \in DS^+$ and $x \in L^\Phi$. Then the averages (3) converge b.a.u. to some $\hat{x} \in L^\Phi$.

2. Let $(E(\mathcal{M}), \| \cdot \|_{E(\mathcal{M})})$ be a noncommutative fully symmetric space with order continuous norm. Then $\tau(\{|x| > \lambda\}) < \infty$ for all $x \in E(\mathcal{M})$ and $\lambda > 0$, so $E(\mathcal{M}) \subset \mathcal{R}_\tau$. Thus we have

**Theorem 3.2.** Let $(E(\mathcal{M}), \| \cdot \|_{E(\mathcal{M})})$ be noncommutative fully symmetric space with order continuous norm. Let $T \in DS^+$ and $x \in E(\mathcal{M})$. Then the averages (3) converge b.a.u. to some $\hat{x} \in E(\mathcal{M})$.

3. Let $\varphi$ be an increasing continuous concave function on $[0, \infty)$ with $\varphi(0) = 0$ and $\varphi(\infty) = \infty$, and let $\Lambda_\varphi(\mathcal{M})$ be the corresponding noncommutative Lorentz space (see, for example, [2]). Since $\Lambda_\varphi(0, \infty)$ is a fully symmetric space with order continuous norm and Fatou property [11, Ch. II, §5], $\Lambda_\varphi(\mathcal{M})$ also has order continuous norm and Fatou property [3, Proposition 3.6 and §5]. Then we have

**Theorem 3.3.** Let $T \in DS^+$ and $x \in \Lambda_\varphi(\mathcal{M})$. Then the averages (3) converge b.a.u. to some $\hat{x} \in \Lambda_\varphi(\mathcal{M})$.

4. Let $E = (E(0, \infty), \| \cdot \|_{E(0, \infty)})$ be a fully symmetric function space, and let $E$ be $q$-concave for some $1 \leq q < \infty$. Then $E$, hence $(E(\mathcal{M}), \| \cdot \|_{E(\mathcal{M})})$, has order continuous norm. Therefore we have the following.

**Theorem 3.4.** Let $E = (E(0, \infty), \| \cdot \|_{E(0, \infty)})$ be a fully symmetric function space. Assume that $E$ is $q$-concave for some $1 \leq q < \infty$. Let $T \in DS^+$ and $x \in E(\mathcal{M})$. Then the averages (3) converge b.a.u. to some $\hat{x} \in L^1 + \mathcal{M}$. In addition, if $E$ has Fatou property, then $\hat{x} \in E(\mathcal{M})$. 
5. Let $E = (E(0, \infty), \| \cdot \|_{E(0, \infty)})$ be a fully symmetric function space. Assume that $E$ has non-trivial Boyd indices. According to [13, II, Ch.2, Proposition 2.b.3], there exist such $1 < p, q < \infty$ that the space $E$ is intermediate for the Banach couple $(L^p(0, \infty), L^q(0, \infty))$. Since $(L^p + L^q)(\mathcal{M}) = L^p(\mathcal{M}) + L^q(\mathcal{M})$ (see [6, Proposition 3.1]), we have $E(\mathcal{M}) \subset L^p(\mathcal{M}) + L^q(\mathcal{M}) \subset \mathcal{R}_\tau$.

Then we also have (cf. [4, Theorem 4.2])

**Theorem 3.5.** Let $E = (E(0, \infty), \| \cdot \|_{E(0, \infty)})$ be a fully symmetric function space, and let $E$ have non-trivial Boyd indices. Given $T \in DS^+$ and $x \in E(\mathcal{M})$, the averages (3) converge b.a.u. to some $\hat{x} \in L^1 + \mathcal{M}$. In addition, if $E$ has Fatou property, then $\hat{x} \in E(\mathcal{M})$.

**Remark 3.1.** It is clear that if one assumes that $\mathcal{M}$ has a separable predual, Theorems 3.1 - 3.5 remain valid for the Besicovitch weighted averages (4).

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