Witten Effect and Fractional Electric Charge on the Domain Wall between Topological Insulators and Spin Ice Compounds

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Abstract. We propose that there might be emergent quasiparticles with fractional electronic charge such dyons on the domain wall between topological insulators and spin ice compounds through the Witten effect and interaction between the Dirac fermions and excited magnetic monopoles.

1. Introduction
The Quantum Spin Hall state has been theoretically predicted [1] and experimentally observed in HgTe quantum wells [2]. Time-Reversal-Invariant topological insulators have also been classified in (3+1) dimension [3, 4]. These three-dimensional states carry spin Hall current in the insulating state [5]. Recently the three-dimensional topological insulator Bi$_{1-x}$Sb$_x$ in a certain range of compositions [6] has been predicted. In addition, Dirac fermions on the surface of three-dimensional topological insulator have been observed by angle-resolved photoemission spectroscopy [7]. Also Castelnovo et al. [8] have proposed that magnetic monopoles emerge in a class of exotic magnets known collectively as spin ice compounds Dy$_2$Ti$_2$O$_7$ and Ho$_2$Ti$_2$O$_7$. Spin ice is identified as a very unusual magnet when it is noted that it does not order to the lowest temperatures $T$ even though it appeared to have ferromagnetic interaction. Indeed, spin ice is found to have a residual entropy at low temperature, which is well-approximated by the Pauling entropy for water ice. Recently, one of us [9] has proposed exotic quasiparticles with fractional charges in semiconductor-dot from collectively induced-charge effect on a domain wall shell. In this study, we propose that there might be emergent quasiparticles with fractionalized electric charge on the domain wall between topological insulator and spin ice compounds, through the Witten effect [10] and interaction between the Dirac fermions and excited magnetic monopoles.

2. Exotic particle on the domain wall between the topological insulator and spin ice
To study the response properties of the (3+1) dimension system, the effective action $S_{3D}(A, \theta)$ with an adiabatic parameter $\theta$ can be defined as

$$e^{iS_{3D}(A,\theta)} = \int D\psi D\bar{\psi} \exp \left[ i \int dt \sum_{x} \bar{\psi}_x (i\partial_t - A_{x0}) \psi_x - \mathcal{H}[A,\theta] \right].$$
Figure 1. The Feynman diagram that contributes to topological term. The loop is a fermion propagator and the dashed lines are external legs corresponding to the gauge field.

$\mathcal{H}(A, \theta)$ is the $(3+1)$ dimensional Hamiltonian of Dirac model coupled to an external $U(1)$ gauge field $A$ [11]. $\psi$ is the wavefunction of the Dirac electron. Where $\vec{x}$ stands for the three-dimensional coordinates. A Taylor expansion of $S_{3D}$ can be carried out around the gauge field configuration $A_s(\vec{x}, t)$, whose $s = 1, 2, 3$ stands for the $x, y, z$ directions, $\theta(\vec{x}, t) \equiv \theta_0$, which contains a nonlinear-response term derived from the $(4+1)$ dimensional Chern-Simon action, $S_{3D}(A, \theta)$.

$$S_{3D}(A, \theta) = \frac{G_3(\theta_0)}{4\pi} \int d^3x \, dt \, \varepsilon_{\mu\nu\sigma\tau} \delta \theta \partial_\mu A_\nu \partial_\sigma A_\tau .$$

The field $\delta \theta(\vec{x}, t) = \theta(\vec{x}, t) - \theta_0$ plays the role of $A_4$ in the $(4+1)$ dimension, and the coefficient $G_3(\theta_0)$ is determined by Goldstone and Wilczek-type Feynman diagram [12] in Fig. 1. Consequently, $G_3(\theta_0)$ can be calculated as follows.

$$G_3(\theta_0) = \frac{\pi}{6} \int \frac{d^3 k \, d\omega}{(2\pi)^3} \varepsilon_{\mu\nu\sigma\tau} \left[ \left( G^{-1}_\mu \right) \partial_\nu G^{-1}_\sigma \partial_\tau G^{-1}_\theta \right]$$

where $q^\mu = (\omega, k_x, k_y, k_z)$ and $G(q_{\mu}) = [\omega + i\delta - v(k_i)]^{-1}$ is the single-particle Green’s function. In addition, $G_3(\theta_0)$ is determined from the Berry phase curvature as

$$G_3(\theta_0) = \frac{1}{8\pi^2} \int d^3 k \varepsilon_{ij} k Tr[f_{\theta_i} f_{\theta_j}]$$

in which the Berry phase gauge field is defined in the four-dimensional space $(k_x, k_y, k_z, \theta_0)$, i.e.,

$$a_i^{\alpha\beta} = -i \left\langle \tilde{k}, \theta_0; \alpha \left| \frac{\partial}{\partial \theta_i} \right| \tilde{k}, \theta_0; \beta \right\rangle$$

and

$$a_\theta^{\alpha\beta} = -i \left\langle \tilde{k}, \theta_0; \alpha \left| \frac{\partial}{\partial \theta_0} \right| \tilde{k}, \theta_0; \beta \right\rangle .$$

A generalized polarization $P_3(\theta_0)$ can also be defined in $(3+1)$ dimensions so that $G_3(\theta_0) = \frac{\partial P_3(\theta_0)}{\partial \theta_0} = \int d^3 k \partial_\theta K^\theta$, with

$$P_3(\theta_0) = \frac{1}{16 \pi^2} \int d^3 k \varepsilon_{ij} k Tr \left\{ \left( f_{ij} - \frac{1}{3} [a_i, a_j] \right) \cdot a_k \right\} .$$

The effective action for the $(3+1)$-dimensional system is finally written as

$$S_{3D}(A, \theta) = \frac{1}{16 \pi^2} \int d^3 x \, dt \, \theta(\vec{x}, t) \varepsilon_{\mu\nu\sigma\tau} \partial_\mu A_\nu \partial_\sigma A_\tau .$$
In the case of the (3+1) dimensional topological insulator, \( \theta_0(\vec{x}, t) \) correspond to \( \pi \). Here we shall consider the domain wall between the topological insulator and spin ice as shown in Fig. 2. We must consider the (3+1) dimensional Dirac fermion with a domain wall configuration of the \( \theta_0(z) \) field given by

\[
\theta_0(z) = \frac{\pi}{2} \left[ 1 - \tanh \left( \frac{z}{4\xi} \right) \right]
\]

which has the asymptotic behavior \( \theta_0(z \to -\infty) = \pi, \theta_0(z \to \infty) = 0 \). The domain wall width is \( \xi \).

For the spin ice compound Dy\(_2\)Ti\(_2\)O\(_7\), the magnitude \( \mu \) of the magnetic moments equals approximately ten Bohr magnetrons \( \mu \approx 10 \mu_B \). The thermodynamic properties of this compound are known to be described by an energy term that accounts for the nearest-neighbor exchange and the long-range dipolar interactions [13, 14],

\[
\mathcal{H} = \frac{J}{3} \sum_{<i,j>} S_i \cdot S_j + D a^3 \sum_{<i,j>} \left[ \frac{\hat{e}_i \hat{e}_j}{|r_{ij}|^3} - \frac{3(\hat{e}_i r_{ij})(\hat{e}_j r_{ij})}{|r_{ij}|^5} \right] S_i S_j.
\]

The distance between spins is \( r_{ij} \) and \( a \approx 3.54 \) Å is the pyrochlore nearest-neighbor distance,

\[
D = \frac{\mu_0 \mu^2}{4\pi a^3} = 1.41 \text{ K}
\]

is the coupling constant of the dipolar interaction. Here we shall consider the excited states. The excited inverted-dumbbell corresponds to two adjacent sites with net magnetic charge \( \pm q_m = \pm 2 \mu \frac{a}{a_d} - a \) nearest neighbor monopole-antimonopole pair. Where \( a_d = \sqrt{\frac{3}{2}} a \) is the diamond lattice bond length. The monopoles can be separated from one another without further violation of local neutrality by flipping a chain of adjacent dumbbells. As shown in Fig. 2, when excited magnetic monopoles in the spin ice inject into the domain wall with the \( \theta_0(z) \) as follows,

\[
\theta_0(z) \sim \frac{\pi}{2} \left[ 1 - \tanh \left( \frac{z}{4\xi} \right) \right].
\]

The magnetic monopole transfers to the dyon with the fractional electric charge

\[
Q \sim \frac{1}{2\pi} \theta_0(z) q_m = \frac{\theta_0(z)}{2\pi} \frac{2\mu}{a_d}
\]
due to the Witten effect [10]. Now we shall consider the relaxation of the fermion on the domain wall due to the dyons, which are derived from magnetic monopoles in the spin ice. The Green-function of fermion is represented approximately as follow,

$$G(k, \varepsilon + i\delta) = \frac{1}{\varepsilon + i\delta - \varepsilon_k - \sum(k, \varepsilon + i\delta)}$$

$$\sum(k, \varepsilon + i\delta) \propto i\pi N_{\text{dyon}} \left( \frac{e}{4\pi} \frac{2\mu}{a_d} \right)^2 n \equiv \frac{i}{2\tau}$$

where $N_{\text{dyon}}$ is the density of dyons and $n$ is the fermion density per spin on the domain wall. $\tau$ is relaxation time. $N_{\text{dyon}}$ is proportional to the magnetic monopole density $\rho_{\text{mono}}$. Taking into account the monopole density in the spin ice [15], it has been deduced that $N_{\text{dyon}}$ at 10 K is approximately 10 times of one at 1 K in the case of the domain wall between the spin ice $\text{Dy}_2\text{Ti}_2\text{O}_7$ and the topological insulator $\text{Bi}_{1-x}\text{Sb}_x$. That is, it is expected that the relaxation time $\tau$ at 10 K is reduced 1/10 in comparison with the $\tau$ at 1 K.

3. Conclusion

It has been proposed that there might be emergent quasiparticle with fractional electronic charge such as dyons on the domain wall between topological insulator and spin ice components, through the Witten-effect and interaction between the Dirac fermions and excited magnetic monopoles.

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