FUNCTIONAL INEQUALITIES ON MANIFOLDS WITH NON-CONVEX BOUNDARY

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ABSTRACT. In this article, new curvature conditions are introduced to establish functional inequalities including gradient estimates, Harnack inequalities and transportation-cost inequalities on manifolds with non-convex boundary.

1. INTRODUCTION

Let $(M,g)$ be a $d$-dimensional complete and connected Riemannian manifold with Riemannian distance $\rho$, boundary $\partial M$ and inward pointing unit normal vector $N$. Define the second fundamental form of the boundary by

$$II(X,Y) = -\langle \nabla_X N, Y \rangle, \quad X,Y \in T_x \partial M, \quad x \in \partial M$$

where $T \partial M$ denotes the tangent bundle of $\partial M$. In order to study non-convex boundaries, we will perform a conformal change of metric such that the boundary is convex under the new metric. In particular, we will use the fact that if $\phi \in C^2(M)$ with $\phi \geq -N \log \phi$ (see [15, Theorem 1.2.5]).

Given a $C^1$-vector field $Z$ on $M$, consider the elliptic operator $L := \Delta + Z$ and let $X^t$ be a reflecting $L$-diffusion process starting from $X^0 = x$. Then $X^t$ solves the Stratonovich equation

$$dX^t = \sqrt{2} u^t \circ dB_t + Z(X^t) \, dt + N(X^t) \, dl^t, \quad X^0 = x$$

where $u^t$ is the horizontal lift of $X^t$ to the orthonormal frame bundle $O(M)$ with $\pi(u^t) = x$, $B_t$ is a standard $\mathbb{R}^d$-valued Brownian motion defined on a complete naturally filtered probability space $(\Omega, \mathcal{F}_{t \geq 0}, \mathbb{P})$ and $l^t$ is a continuous adapted nondecreasing and nonnegative process which increases only on $\{ t \geq 0 : X^t \in \partial M \}$. The process $l^t$ is the local time of $X^t$ on $\partial M$.

We assume that $X^t$ is non-explosive for each $x \in M$. Then the diffusion process $X^t$ gives rise to the Neumann semigroup $P_t$ which solves the diffusion equation $(\partial_t - L) P_t = 0$ with Neumann boundary condition $NP_t = 0$. Furthermore $P_t f(x) = \mathbb{E}[f(X^t)]$ for each $f \in C_b(M)$.

In [7], Hsu found a probabilistic formula for $\nabla P_t f$ for compact manifolds with boundary, which he used to derive a gradient estimate. Feng-Yu Wang extended it to the non-compact case [15, Theorem 3.2.1] under the assumption that $|\nabla P_t f|$ is uniformly bounded on $[0,t] \times M$. Wang’s formula is given

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In particular, we prove (see Theorem 2.2) that if \( \mathbf{ Ric}^{Z} := \mathbf{ Ric} - \nabla Z \geq K \) for some \( K \in C(\mathcal{M}) \) and there exists \( \phi \in \mathcal{D} \) such that
\[
\mathbf{ K}_{\phi} := \inf_{\mathcal{M}} \left\{ \phi^{2} K + \frac{1}{2} \mathbf{ L} \phi^{2} - | \nabla \phi^{2} | | Z | - (d - 2) | \nabla \phi |^{2} \right\} > -\infty
\]
then \( | \nabla P \phi | \) is uniformly bounded on \([0, t] \times \mathcal{M} \).

In this article, we revisit this problem by using coupling methods to weaken this curvature condition. In particular, we prove (see Theorem 2.2) that if there exists \( \phi \in \mathcal{D} \) and a constant \( K_{\phi} \) such that
\[
\mathbf{ Ric}^{Z} + L \log \phi - \nabla \log | \phi |^{2} \geq K_{\phi}
\]
then \( | \nabla P \phi | \) is uniformly bounded on \([0, t] \times \mathcal{M} \). The upper bound we obtain improves that of [15, Proposition 3.2.7]. We construct such a function \( \phi \) in Proposition 3.3 under the assumption that there exist non-negative constants \( \sigma \) and \( \theta \) such that \( -\sigma \leq \mathbf{ I} \leq \theta \) and a positive constant \( r_{0} \) such that on \( \partial_{\mathcal{M}} := \{ x \in \mathcal{M} : \rho_{\mathcal{M}}(x) \leq r_{0} \} \) the function \( \rho_{\mathcal{M}} \) is smooth, the norm of \( Z \) is bounded and \( \text{Sect} \leq k \) for some positive constant \( k \).

F.-Y. Wang also used coupling methods to consider Harnack and transportation-cost inequalities on manifolds with boundary [15]. We reconsider these problems too and find that the curvature conditions used to establish these inequalities can also be weakened and simplified. It is worth mentioning that we find a transportation-cost inequality on the path space of the reflecting diffusion process which (see Theorem 2.8) recovers the results for the convex boundary case, making the theory of functional inequalities on path space complete.

This article is organized as follows. In Section 2, we prove the gradient estimates, Harnack inequalities and transportation-cost inequalities for the Neumann semigroup via coupling methods. In Section 3, we construct a function \( \phi \) which satisfies the new curvature conditions.

## 2. Functional Inequalities

### 2.1. Gradient estimates.

The derivative formula for \( \mathbf{ P}_{\phi} \) is known as Bismut formula (see [4, 5]). The formula we introduce is of a more general type due to Thalmaier [9]. As mentioned in the introduction, Hsu [7] found this type of formula for compact manifolds with boundary. The following formula for manifolds with boundary, due to F.-Y. Wang [15, Theorem 3.2.1], does not require compactness. See also [11] for recent work on probabilistic representations of the derivative of Neumann semigroups.

**Theorem 2.1.** Let \( t > 0 \) and \( u_{0} \in \mathcal{O}_{\mathcal{A}}(\mathcal{M}) \) be fixed. Let \( K \in C(\mathcal{M}) \) and \( \sigma \in C(\partial \mathcal{M}) \) be such that \( \mathbf{ Ric}^{Z} \geq K \) and \( \mathbf{ I} \geq \sigma \). Assume that
\[
\sup_{x \in [0, t]} \mathbb{ E}^{x} \left[ \exp \left( - \int_{0}^{s} K(X_{r}) \, dr - \int_{0}^{s} \sigma(X_{r}) \, dl_{r} \right) \right] < \infty.
\]
Then there exists a progressively measurable process \( \{ Q_{s} \}_{s \in [0, t]} \) on \( \mathbb{ R}^{d} \otimes \mathbb{ R}^{d} \) such that
\[
Q_{0} = I, \quad \| Q_{s} \| \leq \exp \left( - \int_{0}^{s} K(X_{r}) \, dr - \int_{0}^{s} \sigma(X_{r}) \, dl_{r} \right), \quad s \in [0, t]
\]
and for any \( f \in C_{c}^{1}(\mathcal{M}) \) such that \( \nabla P_{\phi} f \) is bounded on \([0, t] \times \mathcal{M} \), for any \( h \in C^{1}([0, t]) \) with \( h(0) = 0 \) and \( h(t) = 1 \), we have
\[
u_{0}^{-1} \nabla P_{\phi} f(x) = \mathbb{ E}^{x} \left[ Q_{s} u_{t}^{-1} \nabla f(X_{s}) \right] = \frac{1}{\sqrt{2}} \mathbb{ E}^{x} \left[ f(X_{s}) \int_{0}^{s} h(s) Q_{s} \, dB_{s} \right].
\]

In order to use this formula it is necessary to check the uniform boundedness of \( \nabla P_{\phi} f \) on \([0, t] \times \mathcal{M} \). In [15] Proposition 3.2.7, F.-Y. Wang did so using a conformal change of metric such that under the new metric the boundary is convex, and by then making a time change of the \( L \)-diffusion process \( X_{t} \). Here, we use coupling methods to study this problem again and obtain improved upper bounds.
Theorem 2.2. If there exist \( \phi \in \mathcal{D} \) and a constant \( K_\phi \) such that
\[
\text{Ric}^Z + L \log \phi - |\nabla \log \phi|^2 \geq K_\phi
\]
(2.1)
then for \( f \in C^1(M) \) such that \( f \) is constant outside a compact set,
\[
|\nabla P_t f| \leq \|\phi\|_{\infty} \|\nabla f\|_{\infty} e^{-K_\phi t}, \quad t > 0.
\]

Proof. We use a coupling method to prove the gradient inequality. To this end, we need to conformally change the metric \( g \). Since \( \phi \in \mathcal{D} \), the boundary \( \partial M \) is convex under new metric \( g' := \phi^{-2}g \). Let \( \Delta' \) and \( \nabla' \) be the Laplacian and gradient operator associated with the metric \( g' \). Then
\[
L = \phi^{-2} (\Delta' + \phi^2 (Z + (d - 2)\nabla \log \phi))
\]
\[
= \phi^{-2} \left( \sum_{i=1}^d V_i^2 + \phi^2 (Z + (d - 2)\nabla \log \phi) \right)
\]
\[
= \sum_{i=1}^d (\phi^{-1} V_i)^2 + \phi^{-2} Z',
\]
where \( \{V_i\}_{i=1}^d \) is a \( g' \)-orthonormal basis of \( T_xM \) and \( Z' := \phi^2 (Z + (d - 1)\nabla \log \phi) \) (the factor \( d - 1 \) corrects the factor \( d - 2 \) appearing in the proof of [15 Proposition 3.2.7]). Now consider the process \( X_t \) generated by \( L \) on the manifold \( (M, g') \). Denoting by \( d_t \) the Itô differential on \( M \), it is easy to see that \( X_t \) is a solution to the equation
\[
d_t X_t = \sqrt{2} \phi^{-1}(X_t) u_t \, dB_t + \phi^{-2}(X_t) Z'(X_t) \, dt + N'(X_t) \, dl_t, \quad X_0 = x
\]
(2.3)
where the horizontal lift \( u_t \) and boundary local time \( l_t \) are now defined with respect to the metric \( g' \).

Recall that in local coordinates, the Itô differential of a continuous semimartingale \( X_t \) on \( M \) is given (see [6] or [2]) by
\[
(d_t X_t)^k = dX_t^k + \frac{1}{2} \sum_{i,j=1}^d \Gamma_{ij}^k(X_t) \, d(X^i, X^j)_t, \quad 1 \leq k \leq d
\]
where \( \Gamma_{ij}^k \) are the Christoffel symbols of \( g' \). Similarly, let \( Y_t \) solve
\[
d_t Y_t = \sqrt{2} \phi^{-1}(Y_t) \tilde{u}_t \, d\tilde{B}_t + \phi^{-2}(Y_t) Z'(Y_t) \, dt + N'(Y_t) \, d\tilde{l}_t, \quad Y_0 = y
\]
(2.4)
with horizontal process \( \tilde{u}_t \) and boundary local time \( \tilde{l}_t \) and where \( \tilde{B}_t \) is a new Brownian motion satisfying
\[
1_{\{r,(r,Y) \notin \text{cut}\}} \tilde{u}_t \, d\tilde{B}_t = 1_{\{(r,Y) \notin \text{cut}\}} P^t_{X_t,Y_t} u_t \, dB_t
\]
with cut \( \subset M \times M \) denoting the set of cut points. For the sake of conciseness, we may assume without loss of generality that the cut locus of \( M \) is empty. Denote by \( \rho' \) the distance function for the metric \( g' \).

Since the boundary \( \partial M \) is convex under \( g' \), by the Itô formula, we have
\[
d \rho'(X_t, Y_t) \leq \sqrt{2} \left( \phi^{-1}(X_t) - \phi^{-1}(Y_t) \right) \, dB_t + \left\{ \sum_{i=1}^d (U_i)^2 \rho'(X_t, Y_t) \right\} \, dt
\]
\[
+ \left\{ \phi^{-2}(Y_t) Z'(Y_t), \nabla' \rho'(X_t, \cdot)(Y_t) \right\}' + \left\{ \phi^{-2}(X_t) Z'(X_t), \nabla' \rho'(\cdot, Y_t)(X_t) \right\}' \right\} \, dt
\]
where \( b_t \) is a one-dimensional Brownian motion, \( \{U_i\}_{i=1}^d \) are vector fields on \( M \times M \) such that \( \nabla' U_i(X_t, Y_t) = 0 \) and
\[
U_i(X_t, Y_t) = (\phi^{-1}(X_t) V_i, \phi^{-1}(Y_t) P^t_{X_t,Y_t} V_i), \quad 1 \leq i \leq d
\]
for \( \{V_i\}_{i=1}^d \) a \( g' \)-orthonormal basis of \( T_XM \). Here \( P'_{\gamma} \) denotes parallel displacement from \( x \) to \( y \) with respect to the metric \( g' \). Write \( \rho' = \rho'(X_i, Y_i) \) and for a minimizing \( g' \)-geodesic \( \gamma \) with \( \gamma(0) = X_i \) and \( \gamma'(p') = Y_i \) let

\[
J_i(s) = \phi^{-1}(\gamma(s)) P'_{\gamma(0), \gamma(s)} V_i, \quad 1 \leq i \leq d
\]

where \( J_i(0) = \phi^{-1}(X_i) V_i \) and \( J_i(\rho') = \phi^{-1}(Y_i) P'_{\gamma(0), \gamma(s)} V_i \). Since \( P'_{\gamma(0), \gamma(s)} V_i \) are parallel vector fields along \( \gamma \) with respect to the metric \( g' \), we have

\[
\sum_{i=1}^d (U_i)^2 \rho'(X_i, Y_i)
\]

\[
\leq \sum_{i=1}^d \int_0^{\rho'} \left\{ \|\nabla_{\gamma} J_i\|^2 - \left\langle R'(\gamma, J_i) J_i, J_i \right\rangle \right\} ds
\]

\[
= d \int_0^{\rho'} \phi^{-2}(\gamma(s)) \langle \nabla \log \phi(\gamma(s)), \gamma(s) \rangle^2 ds - \int_0^{\rho'} \phi^{-2}(\gamma(s)) \text{Ric}'(\gamma(s), \gamma(s)) ds. \tag{2.5}
\]

On the other hand

\[
\phi^{-2}(X_i) \langle Z'(X_i), \nabla' \rho'(\cdot, Y_i)(X_i) \rangle' + \phi^{-2}(Y_i) \langle Z'(Y_i), \nabla' \rho'(X_i, \cdot)(Y_i) \rangle'
\]

\[
= \int_0^{\rho'} \frac{d}{ds} \left\{ \phi^{-2}(\gamma(s)) \langle Z'(\gamma(s)), \gamma(s) \rangle' \right\} ds
\]

\[
= \int_0^{\rho'} \phi^{-2}(\gamma(s)) \left\langle (\nabla_{\gamma} Z) \circ \gamma, \gamma \right\rangle' (s) ds
\]

\[
- 2 \int_0^{\rho'} \phi^{-2}(\gamma(s)) \langle \nabla \log \phi(\gamma(s)), \gamma(s) \rangle \langle Z'(\gamma(s)), \gamma(s) \rangle' ds. \tag{2.6}
\]

Moreover

\[
\langle Z'(\gamma(s)), \gamma(s) \rangle' = \langle Z, \gamma(s) \rangle + (d - 1) \langle \nabla \log \phi, \gamma(s) \rangle.
\]

Combining this with (2.5) and (2.6), we have

\[
d\rho'(X_i, Y_i) \leq \sqrt{2} (\phi^{-1}(X_i) - \phi^{-1}(Y_i)) dB_i - \int_0^{\rho'} \phi^{-2} \left[ (\text{Ric}^Z)'(\gamma(s), \gamma(s)) + (d - 3) \langle \nabla \log \phi, \gamma(s) \rangle^2 + 2 \langle \nabla \log \phi, \gamma(s) \rangle \langle Z, \gamma(s) \rangle \right] ds. \tag{2.7}
\]

By [3] Theorem 1.159 we know that

\[
(\text{Ric}^Z)'(\gamma, \gamma) = \text{Ric}'(\gamma, \gamma) - \langle \nabla_{\gamma} Z', \gamma \rangle'
\]

\[
= \text{Ric}^Z(\gamma, \gamma) + \frac{1}{2} L \phi^2 - 2 \langle \nabla \log \phi, \gamma \rangle \langle Z, \gamma \rangle - (d - 2) \langle \gamma, \nabla \log \phi \rangle^2 - 2 |\nabla \phi|^2
\]

and, noting that \( |\gamma| = \phi \), we thus have

\[
(\text{Ric}^Z)'(\gamma(s), \gamma(s)) + (d - 3) \langle \nabla \log \phi, \gamma(s) \rangle^2 + 2 \langle \nabla \log \phi, \gamma(s) \rangle \langle Z, \gamma(s) \rangle
\]

\[
= \text{Ric}^Z(\gamma(s), \gamma(s)) + \frac{1}{2} L \phi^2 - \langle \gamma, \nabla \log \phi \rangle^2 - 2 |\nabla \phi|^2
\]

\[
\geq \text{Ric}^Z(\gamma(s), \gamma(s)) + \frac{1}{2} L \phi^2 - 3 |\nabla \phi|^2
\]

\[
= \text{Ric}^Z(\gamma(s), \gamma(s)) + \phi^2 L \log \phi - |\nabla \phi|^2. \tag{2.8}
\]

Consequently, if

\[
\text{Ric}^Z + (L \log \phi - |\nabla \log \phi|^2) \langle \cdot, \cdot \rangle \geq K_{\phi} \langle \cdot, \cdot \rangle
\]

then, combining (2.7) with (2.8), we arrive at

\[
d\rho'(X_i, Y_i) \leq \sqrt{2} (\phi^{-1}(X_i) - \phi^{-1}(Y_i)) dB_i - K_{\phi} \rho'(X_i, Y_i) dt.
\]
Then, observing that $\rho' \leq \rho \leq \|\phi\|_\infty \rho'$, we have
\[
|\nabla P_t f|(x) = \lim_{y \to x} \left| \frac{P_t f(x) - P_t f(y)}{\rho(x, y)} \right|
= \lim_{y \to x} \left| \mathbb{E}(x, y) \left[ \frac{f(X_t) - f(Y_t)}{\rho(X_t, Y_t)} \frac{\rho'(X_t, Y_t) \rho'(x, y)}{\rho(X_t, Y_t) \rho'(x, y)} \right] \right|
\leq \|\phi\|_\infty \|\nabla f\|_\infty e^{-K_{\phi} t}
\]
which completes the proof.

\[\square\]

**Remark 2.3.**

(i) Since $(U_d)^2 \rho' \neq 0$, it is indeed necessary to account for this quantity in inequality (2.5), correcting the proof of [15] Theorem 3.4.6.

(ii) Compared with the proof of [15] Theorem 3.4.6, our choice of vector field $J_i$ yields a simpler result.

(iii) In [16], a certain technical assumption which was used to ensure the uniformly boundedness of $|\nabla P_t f|$ on $[0, t] \times M$ is no longer needed in the results.

The following results remove the additional condition in [15] Corollary 3.6.5 (1) or [16] Corollary 1.2 (1) to ensure the uniform boundedness of $|\nabla P_t f|$ on $[0, t] \times M$ and give a another proof to extend these inequalities to $L^p$ forms for $p > 1$:

**Theorem 2.4.** If there exists $\phi \in \mathcal{D}$ such that for $p > 1$ the inequality
\[
\text{Ric}^Z + L \log \phi - p|\nabla \log \phi|^2 \geq K_{\phi, p}
\]
holds, then for $t > 0$ and $f \in C^1_t(M)$,
\[
|\nabla P_t f| \leq \frac{1}{\phi} e^{-K_{\phi, p}} \left( P_t(\phi|\nabla f|)^{p/(p-1)} \right)^{(p-1)/p}.
\]

**Proof.** The lower bound (2.9) implies $\text{Ric}^Z + L \log \phi - |\nabla \log \phi|^2$ is bounded below. By Theorem 2.2, it follows that $|\nabla P_t f|$ is bounded on $[0, t] \times M$. Furthermore
\[
\text{Ric}^Z \geq K_{\phi, p} - L \log \phi + p|\nabla \log \phi|^2 = K_{\phi, p} + \frac{1}{p} \phi^p \phi^{-p} \text{ and } \text{II} \geq -N \log \phi
\]
and so, by Theorem 2.1 there exists $\{Q_s\}_{s \in [0, t]}$ such that
\[
|Q_t| \leq \exp \left( -K_{\phi, p} t - \frac{1}{p} \int_0^t \phi^p \phi^{-p} (X_s) ds + \int_0^t N \log \phi (X_s) dl_s \right)
\]
with
\[
|\nabla P_t f|^p \leq \left( P_t(\phi|\nabla f|)^{p/(p-1)} \right)^{p-1} \mathbb{E} \left[ \phi^{-p}(X_t)|Q_t|^p \right].
\]
It therefore suffices to give the upper bound estimate of the following term:
\[
\mathbb{E} \left[ \phi^{-p}(X_t) \exp \left( -\int_0^t \phi^p \phi^{-p} (X_s) ds + p \int_0^t N \log \phi (X_s) dl_s \right) \right].
\]
To this end, by the Itô formula, it is easy to see that
\[
d\phi^{-p}(X_t) = (\nabla \phi^{-p}(X_t), u_t dB_t) + L \phi^{-p}(X_t) dt + N \phi^{-p}(X_t) dl_t
\]
\[
= (\nabla \phi^{-p}(X_t), u_t dB_t) - p \phi^{-p}(X_t) \left( -\frac{1}{p} \phi^p L \phi^{-p}(X_t) dt + N \log \phi (X_t) dl_t \right).
\]
So
\[
M_t = \phi^{-p}(X_t) \exp \left( -\int_0^t \phi^p (X_s) L \phi^{-p} (X_s) ds + p \int_0^t N \log \phi (X_s) dl_s \right)
\]
is a local martingale. Thus,
\[ \mathbb{E} \left[ \phi^{-p}(X_t) \exp \left( -\int_0^t \phi^p(X_s)L\phi^{-p}(X_s) \, ds + p \int_0^t N \log \phi(X_s) \, dl_s \right) \right] \leq \phi^{-p}(x). \]
Combining this and (2.11) and (2.10) completes the proof. \( \square \)

**Corollary 2.5.** If there exists \( \phi \in \mathcal{D} \) such that for \( p > 1 \) the inequality
\[ \text{Ric}^Z + L \log \phi - p |\nabla \log \phi|^2 \geq K_{\phi, p} \]
holds, then for \( t > 0 \) and \( f \in C^1_b(M) \),
\[ |\nabla P_t f| \leq ||\phi||_\infty e^{-K_{\phi, p}(P_t|f|^{p/(p-1)})(p-1)/p}; \]
and for \( f \in \mathcal{B}_b(M) \) and \( t > 0 \),
\[ |\nabla P_t f|^2 \leq ||\phi||_\infty^2 \frac{K_{\phi, 2}}{e^{2K_{\phi, 2} t - 1}} P_t f^2. \] (2.12)

**Proof.** The first assertion follows from Theorem 2.4 by observing \( \phi \geq 1 \). As Theorem 2.1 can be used under our condition directly, the main idea of the proof of (2.12) is similar to that of [15 Corollary 3.2.8], we skip it here. \( \square \)

Note that taking the limit \( p \downarrow 1 \) in Corollary 2.5 recovers Theorem 2.2.

### 2.2. Harnack inequalities

In [15 Theorem 3.4.7] or [13 Theorem 3.1], F.-Y. Wang used a coupling method to obtain dimension free Harnack inequalities and a log-Harnack inequality on manifolds with boundary assuming \( \text{Ric}^Z \geq K \) for some \( K \in C(M) \) with \( \phi \in \mathcal{D} \) such that \( K_{\phi} \) is finite (where the quantity \( K_{\phi} \) is defined as in (1.1)). The coefficient involved in these inequalities is:
\[ K_{\phi} = 2K_{\phi, p} + 4||\phi Z + (d-2)\nabla \phi||_\infty ||\nabla \log \phi||_\infty + 2d ||\nabla \log \phi||_\infty^2. \]
However, in [15 Corollary 3.6.5 (2)] or [16 Corollary 1.2 (2)], F.-Y. Wang used a modified curvature condition to obtain a log-Harnack inequality with coefficient \( K_{\phi, 2} \). A natural question raised is how to get dimension-free Harnack inequalities with coefficient \( K_{\phi, 2} \) which are consistent with the log-Harnack inequality [15 Corollary 3.6.5 (2)] or [16 Corollary 1.2 (2)]. We give the answer as follows, also by using coupling methods:

**Theorem 2.6.** Assume there exists \( \phi \in \mathcal{D} \) such that
\[ \text{Ric}^Z + L \log \phi - 2 |\nabla \log \phi|^2 \geq K_{\phi, 2} \] (2.13)
for some constant \( K_{\phi, 2} \). Then
(i) for \( T > 0 \), \( x, y \in M \) and a measurable function \( f \geq 1 \), we have
\[ P_T \log f(y) \leq \log P_T f(x) + \frac{K_{\phi, 2} ||\phi||_\infty^2 \rho^2(x, y)}{2(e^{2K_{\phi, 2} T} - 1)}; \]
(ii) for \( T > 0 \), \( x, y \in M \), \( p > ||\phi||_\infty^2 \) and function \( f \in C^1_b(M) \), we have
\[ (P_T f(y))^p \leq P_T f^p(x) \exp \left( \frac{\sqrt{p}(\sqrt{p} - 1)K_{\phi, 2} ||\phi||_\infty^2 \rho^2(x, y)}{8K_{\phi, 2}(\sqrt{p} - 1 - \delta_p)(e^{2K_{\phi, 2} T} - 1)} \right), \]
where \( \delta_p = \max \left\{ \frac{p}{2}, \frac{\sqrt{p} - 1}{2} \right\} \).

**Proof.** Fix \( x, y \in M \) and \( T > 0 \). As in the proof of Theorem 2.2, we consider the process \( X_t \) generated by \( L = \Delta + Z \) under the metric \( g' := \phi^{-2} g \), for which the boundary \( (\partial M, g') \) is convex. Let \( X_t \) solve equation (2.3) with \( X_0 = x \). For a strictly positive function \( \xi \in C([0, T]) \), to be later determined, let \( Y_t \) solve
\[ d_t Y_t = \sqrt{2} \phi^{-1}(Y_t) P_t \phi_{X_t, Y_t} u_t \, dB_t + \phi_t^{-2} Z(Y_t) \, dt - \frac{\phi^{-1}(Y_t) p'(X_t, Y_t)}{\phi^{-1}(X_t) \xi(t)} \nabla' p'(X_t, \cdot)(Y_t) \, dt + N'(Y_t) \, d\tilde{t}, \]
for \( t \in [0,T) \) with \( Y_0 = y \), where \( \tilde{t} \) is the local time of \( Y \) on \( \partial M \). Now consider the process \((X_t, Y_t)\) starting from \((x, y)\), which is a well defined continuous process for \( t \leq T \wedge \zeta \) where \( \zeta \) is the explosion time of \( Y \); that is \( \zeta := \lim_{n \to \infty} \zeta_n \) for \( \zeta_n := \inf\{ t > 0 : \rho'(y, Y_t) \geq n \} \). As in the proof of Theorem 2.2, we can assume that the cut-locus of \((M, g')\) is empty, so that parallel displacement is smooth. Let

\[
d\tilde{B}_t = dB_t + \frac{\rho'(X_t, Y_t)}{\sqrt{2\xi(t)}} u_t^{-1} \nabla \rho'(\cdot, Y_t)(X_t) \, dt, \quad 0 \leq t < T \wedge \zeta. \tag{2.14}
\]

Since

\[
\text{Ric}^Z + L \log \phi - |\nabla \log \phi|^2 \geq K_{\phi, 2} + |\nabla \log \phi|^2, \quad t \in [0, T],
\]

by a similar calculation as for (2.7) we find

\[
d\rho'(X_t, Y_t) \leq \sqrt{2} (\phi^{-1}(X_t) - \phi^{-1}(Y_t)) \langle \nabla \rho'(\cdot, Y_t)(X_t), u_t dB_t \rangle' \\
- \left( \int_0^t \rho'(X_s, Y_s) \left( \frac{d}{ds}(\phi^{-1}(X_s) - \phi^{-1}(Y_s)) \right) ds \right) \frac{\rho'(X_t, Y_t)^2}{\xi(t)} \, dt, \quad 0 \leq t < T \wedge \zeta \tag{2.15}
\]

which implies

\[
\frac{d\rho'(X_t, Y_t)^2}{\xi(t)} \leq \frac{2\sqrt{2}}{\xi(t)} \rho'(X_t, Y_t) \langle \phi^{-1}(X_t) - \phi^{-1}(Y_t) \rangle \langle \nabla \rho'(\cdot, Y_t)(X_t), u_t dB_t \rangle' \\
- \frac{\rho'(X_t, Y_t)^2}{\xi(t)} \left( \frac{\xi(t)}{2K_{\phi, 2}} + 2 \right) \, dt, \quad 0 \leq t < T. \tag{2.16}
\]

Now for \( \theta \in (0, 2) \) let

\[
\frac{\xi(t)}{2K_{\phi, 2}} + 2 = \theta, \quad t \in [0, T)
\]

so that \( \xi \) solves the equation

\[
\frac{\xi(t)}{2K_{\phi, 2}} + 2 = \theta, \quad t \in [0, T).
\]

Combining this with (2.16), we find

\[
\frac{d\rho'(X_t, Y_t)^2}{\xi(t)} \leq \frac{2\sqrt{2}}{\xi(t)} \rho'(X_t, Y_t) \langle \phi^{-1}(X_t) - \phi^{-1}(Y_t) \rangle \langle \nabla \rho'(\cdot, Y_t)(X_t), u_t dB_t \rangle' \\
- \frac{\rho'(X_t, Y_t)^2}{\xi(t)} \theta \, dt.
\]

The remainder of argument is given by the proof of [15] Theorem 3.4.7. \( \square \)

2.3. **Transportation-cost inequalities.** Consider \( \mu, \nu \in \mathcal{P}(M) \) where \( \mathcal{P}(M) \) denotes the space of all probability measures on \( M \). Recall the \( L^p \)-Wasserstein distance between \( \mu \) and \( \nu \) is

\[
W_p(\mu, \nu) = \inf_{\eta \in \mathcal{C}(\mu, \nu)} \left\{ \int_M \rho(x, y)^p \, d\eta(x, y) \right\}^{1/p}
\]

where \( \mathcal{C}(\mu, \nu) \) is the set for couplings of \( \mu \) and \( \nu \). When the manifold has no boundary, it is well known that the curvature condition,

\[
\text{Ric}^Z \geq K \quad \text{for some constant } K
\]

is equivalent to

\[
W_p(\mu P_t, \nu P_t) \leq W_p(\mu, \nu) e^{-Kt}, \quad \mu, \nu \in \mathcal{P}(M),
\]

where \( \mu P_t \in \mathcal{P}(M) \) is defined by \((\mu P_t)(A) = \mu(P_t 1_A)\) for measurable set \( A \). This equivalence is due to [10] which is extended to the manifolds with convex boundary [14]. Using a coupling method, we obtain the following transportation-cost inequality.

**Theorem 2.7.** If there exists \( \phi \in \mathcal{D} \) and a constant \( K_{\phi, 2} \) satisfying

\[
\text{Ric}^Z + L \log \phi - 2|\nabla \log \phi|^2 \geq K_{\phi, 2}
\]

then

\[
W_2(\mu P_t, \nu P_t) \leq \|\phi\|_{\infty} e^{-K_{\phi, 2} t} W_2(\mu, \nu).
\]
Proof. By \[15\] Theorem 4.4.2, it suffices to only consider \( \mu = \delta_x \) and \( \nu = \delta_y \). Let \( \phi \) be a smooth function in \( \mathcal{D} \) and recall that \( L = \sum_{i=1}^d (\phi^{-1} V_i)^2 + \phi^{-2}Z' \) for the manifold \((M, g')\) and \( \{V_i\}_{i=1}^d \) the \( g'\)-orthonormal basis of \( T_M \), where \( g' = \phi^{-2}g \). Let \( X_t \) and \( Y_t \) solve the following SDEs respectively:

\[
dt X_t = \sqrt{2}\phi^{-1}(X_t)u_t \, dB_t + \phi^{-2}(X_t)Z'(X_t) \, dt + N'(X_t) \, dt, \quad X_0 = x;
\]

\[
dt Y_t = \sqrt{2}\phi^{-1}(Y_t)P_{X_t,Y_t}u_t \, dB_t + \phi^{-2}(Y_t)Z'(Y_t) \, dt + N'(Y_t) \, dt, \quad Y_0 = y.
\]

Then, as explained in the proof of Theorem \([22]\) we have

\[
d\rho'(X_t,Y_t) \leq \sqrt{2}(\phi^{-1}(X_t) - \phi^{-1}(Y_t)) \left\langle \nabla' \rho'(\cdot,Y_t)(X_t), u_t \, dB_t \right\rangle' - \left( \int_0^t \rho'(X_s,Y_s) \left( \phi^{-2} \text{Ric}^Z(\gamma(s),\gamma(s)) + L \log \phi - |\nabla \log \phi|^2 \right)(\gamma(s)) \, ds \right) \, dt.
\]

Therefore

\[
d\rho'(X_t,Y_t)^2 = 2d\rho'(X_t,Y_t) \, d\rho'(X_t,Y_t) + (\phi^{-1}(X_t) - \phi^{-1}(Y_t))^2 \, dt
\]

\[
\leq d\tilde{M}_t + 2 \left[ \int_0^t \rho'(X_s,Y_s) \left\langle \nabla' \phi^{-1}(\gamma(s)), \gamma(s) \right\rangle' \, ds \right]^2 \, dt
\]

\[
- 2 \rho'(X_t,Y_t) \left[ \int_0^t \rho'(X_s,Y_s) \left( \phi^{-2} \text{Ric}^Z(\gamma(s),\gamma(s)) + L \log \phi - |\nabla \log \phi|^2 \right)(\gamma(s)) \, ds \right] \, dt
\]

\[
\leq d\tilde{M}_t - 2\rho'(X_t,Y_t) \left[ \int_0^t \rho'(X_s,Y_s) \left\{ \phi^{-2}(\gamma(s)) \text{Ric}^Z(\gamma(s),\gamma(s)) \right. \right. \\
\]

\[
\left. \left. + L \log \phi(\gamma(s)) \right\} \left| \gamma(s) \right|^2 \, ds \right] \, dt
\]

\[
\leq d\tilde{M}_t - 2K_{\phi,2} \rho'(X_t,Y_t)^2 \, dt,
\]

where

\[
d\tilde{M}_t = 2\sqrt{2} \rho'(X_t,Y_t)(\phi^{-1}(X_t) - \phi^{-1}(Y_t)) \left\langle \nabla' \rho'(\cdot,Y_t)(X_t), u_t \, dB_t \right\rangle'.
\]

It follows that

\[
W_2(\delta_{\tilde{M}_1}, \delta_{\tilde{M}_2})^2 \leq \mathbb{E}[\phi(\gamma)^2] \leq \mathbb{E} \left[ |\rho'(X_t,Y_t)^2| \right] \leq \mathbb{E} \left[ |\rho'(X_t,Y_t)^2| \right] \leq \mathbb{E} \left[ |\rho'(x,y)^2| \right] \leq \mathbb{E} \left[ |\rho'(x,y)^2| \right]
\]

which completes the proof. \(\square\)

We now investigate Talagrand-type inequalities with respect to the uniform distance on the path space \( W_T := C([0,T];M) \) of the (reflecting) diffusion process, for a given positive constant \( T \). Let \( X^\mu_{[0,T]} \) be the (reflecting if \( \partial M \neq \emptyset \)) diffusion process generated by \( L \) with initial distribution \( \mu \in \mathcal{P}(M) \). Let \( \Pi^\mu_T \) be the distribution of

\[
X^\mu_{[0,T]} := \{ X^\mu_t : t \in [0,T] \},
\]

which is a probability measure on the (free) path space \( W_T \). When \( \mu = \delta_x \) we denote \( \Pi_T = \Pi_T^\mu \) and \( X^\delta_{[0,T]} = X_{[0,T]} \). For any non-negative measurable function \( F \) on \( W_T \) such that \( \Pi_T^\mu(F) = 1 \), one has

\[
\mu^T_F(dx) := \Pi_T^\mu(F) \mu(dx) \in \mathcal{P}(M).
\]  

(2.17)

The the uniform distance on \( W_T \) is given by

\[
\rho_\infty(\gamma, \eta) := \sup_{t \in [0,T]} \rho(\gamma_t, \eta_t), \quad \gamma, \eta \in W_T.
\]
Let $W^p_2$ be the $L^2$-Wasserstein distance (or $L^2$-transportation cost) induced by $\rho_\infty$. In general, for any $p \in [1, \infty)$ and two probability measures $\Pi_1, \Pi_2$ on $W^T$,

$$W^p_2(\Pi_1, \Pi_2) := \inf_{\phi \in \mathcal{G}(\Pi_1, \Pi_2)} \frac{1}{p} \left\{ \int_{W^T \times W^T} \rho_\infty(\gamma, \eta)^p \pi(d\gamma, d\eta) \right\}^{1/p}$$

is the $L^p$-Wasserstein distance (or $L^p$-transportation cost) of $\Pi_1$ and $\Pi_2$, induced by the uniform norm, where $\mathcal{G}(\Pi_1, \Pi_2)$ is the set of all couplings for $\Pi_1$ and $\Pi_2$. Moreover, for $F \geq 0$ with $\Pi^T_\mu(F) = 1$, let

$$\mu^T_\mu(dx) = \Pi^T_\mu(F) \mu(dx).$$

The following result improves [14, Theorems 4.1 and 4.2] or [15, Theorems 4.5.3 and 4.5.4]:

**Theorem 2.8.** If there exists $\phi \in \mathcal{D}$ and a constant $K_\phi$ satisfying

$$\operatorname{Ric}^\mathcal{L} + L \log \phi - |\nabla \log \phi|^2 \geq K_\phi$$

then

(i) For $F \geq 0$, $\Pi^T_\mu(F) = 1$ and $\mu \in \mathcal{P}(\mathcal{M})$,

$$W^2_2(F\Pi^T_\mu, \Pi^T_\nu)^2 \leq \frac{2 \|\phi\|_2^2}{K_\phi} \left( e^{2K_\phi T} - e^{2K_\phi T} \right) \inf_{R > 0} \left\{ (1 + R^{-1}) \exp \left( 8(1 + R) \|\nabla \log \phi\|_\infty^2 \right) \right\} \Pi^T_\mu(F \log F);$$

(ii') For $F \geq 0$, $\Pi^T_\mu(F) = 1$ and $\mu \in \mathcal{P}(\mathcal{M})$,

$$W^2_2(F\Pi^T_\mu, \Pi^T_\nu)^2 \leq \frac{2 \|\phi\|_2^2}{K_\phi} (1 - e^{-2K_\phi T}) \inf_{R > 0} \left\{ (1 + R^{-1}) \exp \left( 8(1 + R) \|\nabla \log \phi\|_\infty^2 e^{2K_\phi T} \right) \right\} \Pi^T_\mu(F \log F);$$

(ii) For any $\mu, \nu \in \mathcal{P}(\mathcal{M})$,

$$W^2_2(\Pi^T_\mu, \Pi^T_\nu) \leq 2 \|\phi\|_\infty e^{(K_\phi + \|\nabla \log \phi\|_\infty) T} W_2(\mu, \nu).$$

**Remark 2.9.**

(a) When $\|\nabla \log \phi\|_\infty > 0$ and $K_\phi > 0$ the upper bound in (i) is better than that in (i').

(b) When the boundary is convex we can choose $\phi \equiv 1$. In this case $\nabla \log \phi = 0$ and the estimate in (i') is consistent with [15, Theorem 4.4.2 (2)] for the convex case.

(c) We note that [15, Theorem 4.4.2 (6)] needs to be corrected as follows:

$$W^2_2(\Pi^T_\mu, \Pi^T_\nu) \leq e^{K T} W_2(\mu, \nu),$$

where $K$ is the lower bound of Ricci curvature. It is then consistent with Theorem 2.8 (ii) when $\phi \equiv 1$ and the boundary is convex.

**Proof of Theorem 2.8.**

(i) Simply denote $X_{[0,T]}^1 = X_{[0,T]}$. Let $F$ be a positive bounded measurable function on $W^T$ such that $\inf F > 0$ and $\Pi^T_\mu(F) = 1$. Let

$$dQ = F(X_{[0,T]}) d\mathbb{P}.$$

Since $E F(X_{[0,T]}) = \Pi^T_\mu(F) = 1$, $Q$ is a probability measure on $\Omega$. Then, we conclude that there exists a unique $\mathcal{F}_t$-predictable process $\beta_t$ on $\mathbb{R}^d$ such that

$$F(X_{[0,T]}) = \exp \left( \int_0^T \langle \beta_s, dB_s \rangle - \frac{1}{2} \int_0^T \|\beta_s\|^2 ds \right)$$

and

$$\int_0^T E_Q \|\beta_s\|^2 ds = 2E \left[ F(X_{[0,T]}) \log F(X_{[0,T]}) \right]. \quad (2.18)$$

Then, by the Girsanov theorem, $\tilde{B}_t := B_t - \int_0^t \beta_s \, ds$, $t \in [0, T]$ is a $d$-dimensional Brownian motion under the probability measure $Q$. 

As explained in the proof of [15, Theorem 4.5.3], it suffices to assume \( \mu = \delta_x, x \in M \). In this case, the desired inequality involves
\[
\mu^T = \delta_x \quad \text{and} \quad \Pi^T_{\mu}(F \log F) = \Pi^T_x(F \log F).
\]
Since the diffusion coefficients are non-constant, it is convenient to adopt the Itô differential \( d \) for the Girsanov transformation. So the reflecting \( L \) diffusion process \( X \) can be constructed by solving the Itô SDE
\[
d_t X_t = \sqrt{2} \phi^{-1}(X_t) u_t \, dB_t + \phi^{-2}(X_t) Z(X_t) \, dt + \Phi(X_t) \, d\ell_t,
\]
where \( B_t \) is the \( d \)-dimensional Brownian motion with natural filtration \( \mathcal{F}_t \). Then
\[
d_t Y_t = \sqrt{2} \phi^{-1}(Y_t) P_{X_t,Y_t} \, u_t \, d\tilde{B}_t + \phi^{-2}(Y_t) Z(Y_t) \, dt + \Phi(Y_t) \, d\tilde{\ell}_t,
\]
and let \( l_t \) and \( \tilde{l}_t \) are the local times of \( X_t \) and \( Y_t \) on \( \partial M \), respectively. Moreover, for any bounded measurable function \( G \) on \( W^T \),
\[
\mathbf{E}_Q G(X_{[0,T]}) := \mathbf{E}(FG)(X_{[0,T]}) = \Pi^T_t(FG). \]
We conclude that the distribution of \( X_{[0,T]} \) under \( Q \) coincides with \( F \Pi^T_t \). Therefore,
\[
W^Q_t(F \Pi^T_t, \Pi^T_t)^2 \leq \mathbf{E}_Q \left[ \rho_\infty(X_{[0,T]}, Y_{[0,T]\}}^2) \right] = \mathbf{E}_Q \left[ \max_{r \in [0,T]} \rho(X_t, Y_t)^2 \right]
\]
\[
\leq \langle \phi \rangle^2 \mathbf{E}_Q \left[ \max_{r \in [0,T]} \rho(X_t, Y_t)^2 \right]. \quad (2.21)
\]
Note that due to the convexity of the boundary,
\[
\langle N'(x), \nabla' \rho(\cdot, y)(x) \rangle \leq 0, \quad x \in \partial M.
\]
From this and equations (2.19) and (2.20), it follows that
\[
d \rho(X_t, Y_t) \leq \sqrt{2} (\phi^{-1}(X_t) - \phi^{-1}(Y_t)) \langle \nabla' \rho(\cdot, y)(X_t), u_t, d\tilde{B}_t \rangle' - K \rho(X_t, Y_t) \, dt + \sqrt{2} \| \beta_t \| \, dr.
\]
Since
\[
M_t := \sqrt{2} \int_0^t e^{k_s} \phi^{-1}(X_t) - \phi^{-1}(Y_t) \langle \nabla' \rho(\cdot, y)(X_t), u_s, \tilde{d}B_s \rangle' \quad t \in [0, T]
\]
\[
is a \mathcal{F}_t\text{-martingale, we have}
\[
\rho(X_t, Y_t) \leq e^{-k_t} \left( M_t + \sqrt{2} \int_0^t e^{k_s} \| \beta_s \| \, ds \right), \quad t \in [0, T].
\]
So to prove (i), we will estimate the function
\[
h_t = \mathbf{E}_Q \max_{s \in [0,t]} e^{2k_s} \rho(X_s, Y_s)^2.
\]
By the Doob inequality, for any \( R > 0 \), we have
\[
h_t \geq \mathbf{E}_Q \left[ \max_{s \in [0,t]} e^{2k_s} \rho(X_s, Y_s)^2 \right]
\]
\[
\leq (1 + R) \mathbf{E}_Q \max_{s \in [0,t]} M^2_s + 2(1 + R^{-1}) \max_{s \in [0,t]} \mathbf{E}_Q \left[ \left( \int_0^s e^{k_s} \| \beta_s \| \, ds \right)^2 \right]
\]
\[
\leq 4(1 + R) \mathbf{E}_Q M^2_t + 2(1 + R^{-1}) \int_0^t e^{2k_s} \, ds \int_0^t \mathbf{E}_Q \| \beta_s \|^2 \, ds
\]
\[
\leq 8(1 + R) \| \nabla \log \rho \|_{\infty} \int_0^t h_s \, ds + 2(1 + R^{-1}) \int_0^T e^{2k_s} \, ds \int_0^T \mathbf{E}_Q \| \beta_s \|^2 \, ds, \quad t \in [0, T]. \quad (2.22)
\]
Since $h_0 = 0$, by using the Gronwall inequality, this inequality further implies
\[ h_T \leq 2(1 + R^{-1}) \exp \left( (8(1 + R)\|\nabla \log \phi\|_\infty^2) \right) \int_0^T \mathbb{E}_Q |\beta_s|^2 \, ds \] 
(2.23)
By (2.18) and (2.23) we thus have
\[ \mathbb{E}_Q \left[ \max_{s \in [0,T]} \rho'(X_s, Y_s)^2 \right] \leq 4(1 + R^{-1}) \exp \left( 8(1 + R)\|\nabla \log \phi\|_\infty^2 \right) \frac{e^{2K_{sT}} - e^{2K_{sT}}}{2K_\phi} \Pi_x^T (F \log F). \]

(i') For this we use the function
\[ \tilde{h}_t = e^{2K_{sT}} \mathbb{E}_Q \left[ \max_{s \in [0,t]} \rho'(X_s, Y_s)^2 \right]. \]

The inequality (2.22) should then be modified as follows:
\[ \tilde{h}_t := e^{2K_{sT}} \mathbb{E}_Q \left[ \max_{s \in [0,t]} \rho'(X_s, Y_s)^2 \right] \]
\[ \leq e^{2K_{sT}} \left( 1 + R \right) \mathbb{E}_Q \left[ \max_{s \in [0,t]} e^{-2K_{sT}} M_s^2 \right] + 2(1 + R^{-1}) \max_{s \in [0,t]} \mathbb{E}_Q \left[ \left( \int_s^t e^{-K_{x(s-r)}} |\beta_r| \, dr \right)^2 \right] \]
\[ \leq 4(1 + R) e^{2K_{sT}} \mathbb{E}_Q M_t^2 + 2(1 + R^{-1}) \int_0^t e^{2K_{sT}} \, ds \]
\[ \leq 8(1 + R) \|\nabla \log \phi\|_\infty^2 e^{2K_{sT}} \int_0^t \tilde{h}_s \, ds + 2(1 + R^{-1}) \int_0^T e^{2K_{sT}} \, ds. \]

Since $\tilde{h}_0 = 0$, this inequality implies
\[ \tilde{h}_T \leq 2(1 + R^{-1}) \exp \left( 8(1 + R)\|\nabla \log \phi\|_\infty^2 e^{2K_{sT}} \right) \int_0^T e^{2K_{sT}} \, ds. \]

We then conclude that
\[ \mathbb{E}_Q \max_{s \in [0,T]} \rho'(X_s, Y_s)^2 \leq 4(1 + R^{-1}) \exp \left( 8(1 + R)\|\nabla \log \phi\|_\infty^2 e^{2K_{sT}} \right) \frac{1 - e^{-2K_{sT}}}{2K_\phi} \Pi_x^T (F \log F). \]

(ii) Without loss of generality, we consider $\mu = \delta_x$, and $\nu = \delta_y$. Let $X_t$ and $Y_t$ solve the following SDEs, respectively:
\[ \begin{align*}
\dot{X}_t &= \sqrt{2} \phi^{-1}(X_t) u_t \, dB_t + \phi^{-2}(X_t) Z'(X_t) \, dt + N'(X_t) \, dt, \quad X_0 = x; \\
\dot{Y}_t &= \sqrt{2} \phi^{-1}(Y_t) P_{X,Y}^{*} u_t \, dB_t + \phi^{-2}(Y_t) Z'(Y_t) \, dt + N'(Y_t) \, dt, \quad Y_0 = y.
\end{align*} \]

Then, as explained in the proof of Theorem 2.2, we have
\[ \begin{align*}
d\rho'(X_t, Y_t) &= \sqrt{2} \left( \phi^{-1}(X_t) - \phi^{-1}(Y_t) \right) \left( \nabla \rho'(\cdot, Y_t)(X_t), u_t \right) \, dB_t \\
&\quad - \left( \int_0^t \rho'(X_s, Y_s) \left( \phi^{-2} \text{Ric}^Z(\gamma(s), \gamma(s)) + L \log \phi - \|\nabla \log \phi\|_\infty^2 \right) \gamma(s) \, ds \right) \, ds.
\end{align*} \]
(2.24)

Therefore,
\[ \rho'(X_t, Y_t) \leq e^{-K_{sT}} (\tilde{M}_t + \rho'(x,y)), \quad t \geq 0 \] 
(2.25)
for
\[ \tilde{M}_t := \sqrt{2} \int_0^t e^{K_{sT}} \phi^{-1}(X_s) - \phi^{-1}(Y_s) \left( \nabla \rho_s(\cdot, Y_t)(X_s), u_s \right) \, dB_s \] 
(2.26)
Again using the Itô formula, we have
\[ d\rho'(X_t, Y_t)^2 \leq d\tilde{M}_t - 2(K_\phi - \|\nabla \log \phi\|_\infty^2) \rho'(X_t, Y_t)^2 \, dt \]
where
\[ d\tilde{M}_t = 2\rho'(X_t, Y_t) (\phi^{-1}(X_t) - \phi^{-1}(Y_t)) \left( \nabla \rho'(\cdot, Y_t)(X_t), u_t \right) \, dB_t \]
which implies
\[ E\rho'(X_t, Y_t) \leq e^{-2(K_{\phi} - \|\nabla \log \phi\|_\infty)} \rho'(x, y)^2. \]

Combining this with (2.25) we arrive at
\[ W_2^p(\Pi_x^T, \Pi_y^T)^2 \leq \|\phi\|_\infty^2 E \max_{t \in [0, T]} \rho'(X_t, Y_t)^2 \]
\[ \leq \|\phi\|_\infty^2 e^{2K_{\phi} T} E \max_{t \in [0, T]} (\hat{M}_t + \rho'(x, y))^2 \]
\[ \leq 4\|\phi\|_\infty^2 e^{2K_{\phi} T} \left( 2 \int_0^T e^{2K_{\phi} t} \|\nabla \log \phi\|_\infty^2 E\rho'(X_t, Y_t)^2 dt + \rho'(x, y)^2 \right) \]
\[ \leq 4\|\phi\|_\infty^2 e^{2K_{\phi} + \|\nabla \log \phi\|_\infty^2 T} \rho'(x, y)^2 \]
\[ \leq 4\|\phi\|_\infty^2 e^{2K_{\phi} + \|\nabla \log \phi\|_\infty^2 T} \rho(x, y)^2 \]

where the second inequality is due to the Doob inequality. This implies the desired inequality for \( \mu = \delta_x \) and \( \nu = \delta_y \).

\[ \square \]

**Corollary 2.10.** If there exists \( \phi \in \mathcal{D} \) and a constant \( K_{\phi} \) satisfying
\[ \text{Ric}^\mathcal{L} + L \log \phi - |\nabla \log \phi|^2 \geq K_{\phi} \]
then
(i) for \( F \geq 0 \), \( \Pi_{\mu}^T(F) = 1 \) and \( \mu \in \mathcal{P}(M) \),
\[ W_2^p(\Pi_{\mu}^T, \Pi_{\mu}^T_F)^2 \leq \frac{2\|\phi\|_\infty^2}{K_{\phi}} \left( e^{2K_{\phi} T} - e^{-2K_{\phi} T} \right) \exp \left( 8\|\nabla \log \phi\|_\infty^2 + 4\sqrt{2}\|\nabla \log \phi\|_\infty \right) \Pi_{\mu}^T(F \log F); \]

(i') for \( F \geq 0 \), \( \Pi_{\mu}^T(F) = 1 \) and \( \mu \in \mathcal{P}(M) \),
\[ W_2^p(\Pi_{\mu}^T, \Pi_{\mu}^T_F)^2 \leq \frac{2\|\phi\|_\infty^2}{K_{\phi}} \left( 1 - e^{-2K_{\phi} T} \right) \exp \left( 8\|\nabla \log \phi\|_\infty^2 e^{2K_{\phi} T} + 4\sqrt{2}\|\nabla \log \phi\|_\infty e^{K_{\phi} T} \right) \Pi_{\mu}^T(F \log F). \]

**Proof.** It is easily observed that
\[ (1 + R^{-1}) \exp \left( 8(1 + R)^2 \|\nabla \log \phi\|_\infty^2 \right) \leq \exp \left( R^{-1} + 8(1 + R)\|\nabla \log \phi\|_\infty \right). \]
Taking the infimum about \( R \) on the right side above, we arrive at
\[ \exp \left( R^{-1} + 8(1 + R)\|\nabla \log \phi\|_\infty^2 \right) \geq \exp \left( 8\|\nabla \log \phi\|_\infty^2 + 4\sqrt{2}\|\nabla \log \phi\|_\infty \right) \]
which allows to prove (i). The inequality (i') can be checked in the same way. \[ \square \]

### 3. New construction of function \( \log \phi \)

In this section, we give a new construction of a function \( \phi \) which satisfies the conditions of the previous section. To do so, we let \( \rho_\partial \) be the Riemannian distance to the boundary \( \partial M \) and use a comparison theorem for \( \Delta \rho_\partial \) near the boundary, essentially due to [8]. Note that, by using local charts, it is clear that \( \rho_\partial \) is smooth in a neighbourhood of \( \partial M \). We call
\[ i_\partial := \sup \{ r > 0 : \rho_\partial \text{ is smooth on } \{ \rho_\partial < r \} \} \]
the injectivity radius of \( \partial M \). Obviously, \( i_\partial > 0 \) if \( M \) is compact, but it could be zero in the non-compact case (sup \( \emptyset = 0 \) by convention). As [15] Theorem 1.2.3 we have:
Lemma 3.1. Let $\theta, k$ be constants such that $\frac{1}{2} \leq \theta$ and $\operatorname{Sect} \leq k$. Let

$$h(t) := \begin{cases} \cos \sqrt{kt} - \frac{\theta}{k} \sin \sqrt{kt}, & k \geq 0, \\ \cosh \sqrt{-kt} - \frac{\theta}{\sqrt{-k}} \sinh \sqrt{-kt}, & k < 0 \end{cases}$$

(3.1)

for $t \geq 0$. Let $h^{-1}(0)$ be the first zero of $h$ (with $h^{-1}(0) := \infty$ if $h(t) > 0$ for all $t \geq 0$). Then for any $x \in M$ such that $\rho_\theta(x) \leq i_\theta \wedge h^{-1}(0)$ we have

$$\Delta \rho_\theta(x) \geq (d - 1) \frac{h'(\rho_\theta(x))}{h}.$$

(3.2)

Note that if $k$ is positive then

$$h^{-1}(0) = \frac{1}{\sqrt{k}} \arcsin \left( \sqrt{\frac{k}{k + \theta^2}} \right).$$

We now work under the following assumption:

**Assumption (A)** There exist non-negative constants $\sigma$ and $\theta$ such that $-\sigma \leq \frac{1}{2} \leq \theta$ and a positive constant $r_0$ such that on $\partial \rho_\theta M := \{ x \in M : \rho_\theta(x) \leq r_0 \}$ the function $\rho_\theta$ is smooth, the norm of $Z$ is bounded and $\operatorname{Sect} \leq k$ for some positive constant $k$.

Using this assumption, F.-Y. Wang constructed a function $\phi$ satisfying $\phi \in \mathcal{D}$ (see [12, p.1436] or [15, Theorem 3.2.9]). Following his construction, we define

$$\log \phi(x) = \frac{\sigma}{\alpha} \int_0^{\rho_\theta(x)} [h(s) - h(r_1)]^{1-d} ds \int_{h(r_1)}^{r_1} [h(u) - h(r_1)]^d du,$$

where $r_1 := r_0 \wedge h^{-1}(0)$ and

$$\alpha := (1 - h(r_1))^{1-d} \int_0^{r_1} [h(s) - h(r_1)]^d ds.$$

Then from the proof of [11, Theorem 1.1], we know:

**Theorem 3.2.** Suppose that Assumption (A) holds and $\operatorname{Ric}^Z \geq K$. Define

$$K_p = K - \sigma \left( \delta_{r_1}(Z) + \frac{d}{r_1} \right) - p \sigma^2,$$

where

$$\delta_{r_1}(Z) := \sup \{ |Z(x)| : x \in \partial r_1 M \}.$$  

(3.3)

Then all results in Section 2 hold by replacing

$$K_\phi, K_{\phi,p}, \|\phi\|_\infty \text{ and } \|\nabla \log \phi\|_\infty$$

with

$$K_1, K_p, e^{\frac{1}{2} \sigma d r_1} \text{ and } \sigma$$

respectively.

In the following we give a new construction of function $\phi$ by using the modifier

$$\ell(r) = \begin{cases} e^{-2} - e^{-2(1-2r)^{-1}}, & 0 \leq r < \frac{1}{2}, \\ e^{-2}, & r \geq \frac{1}{2} \end{cases}.$$

**Proposition 3.3.** Suppose that Assumption (A) holds. Let

$$H(r) := \frac{\sqrt{k + \theta^2}}{k} \cos \left( \arcsin \left( \sqrt{\frac{k}{k + \theta^2}} \right) - \sqrt{k} (r \wedge r_1) \right) - \frac{\theta}{k}.$$
Then the function
\[ \log \phi(x) := \frac{1}{2} \sigma \epsilon^2 \ell \left( \frac{H(\rho_\delta(x))}{2H(r_1)} \right) H(r_1) \] (3.4)
satisfies
\[ N \log \phi |_{\partial M} = \sigma \geq - \Pi. \]
Moreover,
\[ \| \phi \|_{\infty} \leq e^{\frac{1}{2} \sigma H(r_1)}, \quad |\nabla \log \phi| \leq \sigma \]
and
\[ L \log \phi(x) \geq -\sigma \left( d \sqrt{\theta^2 + k} + \delta_1(Z) + \frac{5}{2H(r_1)} \right). \]

**Proof of Proposition 3.3** First it is easy to see that the modifier \( \ell \) satisfies \( \ell \leq e^{-2} \). Differentiating \( \ell \) we obtain
\[ \ell'(r) = \begin{cases} \left( \frac{1}{2} - r \right)^{-2} e^{-\left( \frac{1}{2} - r \right)^{-1}}, & 0 \leq r < \frac{1}{2}; \\ 0, & r \geq \frac{1}{2} \end{cases} \]
and
\[ \ell''(r) = \begin{cases} -2r \left( \frac{1}{2} - r \right)^{-4} e^{-\left( \frac{1}{2} - r \right)^{-1}}, & 0 \leq r < \frac{1}{2}; \\ 0, & r \geq \frac{1}{2}. \end{cases} \]
As \( \ell'' < 0 \) on \([0, 1/2]\), the function \( \ell' \) is at its maximal point when \( r = 0 \), which implies \( 0 \leq \ell' \leq 4e^{-2} \).
Using the same method, when \( r = \sqrt{3}/6 \) the function \( \ell'' \) reaches the minimal value, which implies
\[ \ell'' \geq - 3^{-1/2}(3 + \sqrt{3})^4 e^{-(3+\sqrt{3})} > -20 e^{-2}. \]
Using these results, we have
\[ N \log \phi |_{\partial M} = \frac{1}{4} \epsilon^2 \sigma \ell'(0) \rho_\delta = \sigma, \]
and
\[ |\nabla \log \phi| = \frac{1}{4} \epsilon^2 \sigma \ell' \left( \frac{H(\rho_\delta)}{2H(r_0)} \right) H' \left( \rho_\delta \right) \leq \sigma. \]
Moreover, by Lemma 3.1 we have
\[ L \log \phi = \frac{1}{4} \epsilon^2 \sigma \left( \ell' \left( \frac{H(\rho_\delta)}{2H(r_0)} \right) h(\rho_\delta) L \rho_\delta + \ell'' \left( \frac{H(\rho_\delta)}{2H(r_0)} \right) \frac{h(\rho_\delta)^2}{2H(r_0)} + \ell' \left( \frac{H(\rho_\delta)}{2H(r_0)} \right) h'(\rho_\delta(x)) \right) \]
\[ \geq \frac{1}{4} \epsilon^2 \sigma \left( \ell' \left( \frac{H(\rho_\delta)}{2H(r_0)} \right) \left( d h'(\rho_\delta) - \sup_{\partial_0 M} |Z| \right) + \frac{h(\rho_\delta)^2}{2H(r_0)} \ell'' \left( \frac{H(\rho_\delta)}{2H(r_0)} \right) \right), \]
where \( h \) is defined as in (3.1) for \( k \geq 0 \). It is easy to calculate that
\[ h'(r) \geq - \sqrt{\theta^2 + k}. \]
Combining this with properties of \( \ell \), we conclude that
\[ L \log \phi \geq - \sigma \left( d \sqrt{\theta^2 + k} + \sup \left\{ |Z(x)|: x \in \partial_0^{\lambda h^{-1}(0)M} \right\} + \frac{5}{2H(r_0)} \right) \]
which completes the proof. \( \square \)
Corollary 3.4. Suppose that Assumption (A) holds and $\text{Ric}^Z \geq K$. Define

$$\tilde{K} = K - \sigma \left( d\sqrt{\theta^2 + k + \delta_1(Z) + \frac{5}{2H(r_1)}} \right),$$

and $\tilde{K}_p = \tilde{K} - p\sigma^2$ with $\delta_1(Z)$ as defined in (3.3). Then all results in Section 2 hold by replacing

$$K_{\phi, p}, \|\phi\|_\infty \text{ and } \|\nabla \log \phi\|_\infty$$

with

$$\tilde{K}_p, e^{\frac{1}{2}\sigma H(r_1)} \text{ and } \sigma,$$

respectively.

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