TIE-POINTS AND FIXED-POINTS IN $\mathbb{N}^*$

ALAN DOW AND SAHARON SHELAH

Abstract. A point $x$ is a (bow) tie-point of a space $X$ if $X \setminus \{x\}$ can be partitioned into (relatively) clopen sets each with $x$ in its closure. Tie-points have appeared in the construction of non-trivial autohomeomorphisms of $\beta\mathbb{N} \setminus \mathbb{N}$ (e.g. [10]) and in the recent study of (precisely) 2-to-1 maps on $\beta\mathbb{N} \setminus \mathbb{N}$. In these cases the tie-points have been the unique fixed point of an involution on $\beta\mathbb{N} \setminus \mathbb{N}$. This paper is motivated by the search for 2-to-1 maps and obtaining tie-points of strikingly differing characteristics.

1. Introduction

A point $x$ is a tie-point of a space $X$ if there are closed sets $A, B$ of $X$ such that $\{x\} = A \cap B$ and $x$ is an adherent point of each of $A$ and $B$. We picture (and denote) this as $X = A \bowtie_B x$ where $A, B$ are the closed sets which have a unique common accumulation point $x$ and say that $x$ is a tie-point as witnessed by $A, B$. Let $A \equiv_x B$ mean that there is a homeomorphism from $A$ to $B$ with $x$ as a fixed point. If $X = A \bowtie_B x$ and $A \equiv_x B$, then there is an involution $F$ of $X$ (i.e. $F^2 = F$) such that $\{x\} = \text{fix}(F)$. In this case we will say that $x$ is a symmetric tie-point of $X$.

An autohomeomorphism $F$ of $\beta\mathbb{N} \setminus \mathbb{N}$ (or $\mathbb{N}^*$) is said to be trivial if there is a bijection $f$ between cofinite subsets of $\mathbb{N}$ such that $F = \beta f \upharpoonright \beta\mathbb{N} \setminus \mathbb{N}$. If $F$ is a trivial autohomeomorphism, then $\text{fix}(F)$ is clopen; so of course $\beta\mathbb{N} \setminus \mathbb{N}$ will have no symmetric tie-points in this case if all autohomeomorphisms are trivial.

If $A$ and $B$ are arbitrary compact spaces, and if $x \in A$ and $y \in B$ are accumulation points, then let $A \bowtie_{x\to y} B$ denote the quotient space of

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A ⊕ B obtained by identifying x and y and let xy denote the collapsed point. Clearly the point xy is a tie-point of this space.

We came to the study of tie-points via the following observation.

**Proposition 1.1.** If x, y are symmetric tie-points of βN \ N as witnessed by A, B and A’, B’ respectively, then there is a 2-to-1 mapping from βN \ N onto the space A ⊖ xy ⊖ B'.

The proposition holds more generally if x and y are fixed points of involutions F, F' respectively. That is, replace A by the quotient space of βN \ N obtained by collapsing all sets \{z, F(z)\} to single points and similarly replace B' by the quotient space induced by F'. It is an open problem to determine if 2-to-1 continuous images of βN \ N are homeomorphic to βN \ N [5]. It is known to be true if CH [3] or PFA [2] holds.

There are many interesting questions that arise naturally when considering the concept of tie-points in βN \ N. Given a closed set A ⊂ βN \ N, let \mathcal{I}_A = \{a ⊂ N : a^* ⊂ A\}. Given an ideal \mathcal{I} of subsets of N, let \mathcal{I} = \{b ⊂ N : (\forall a ∈ \mathcal{I}) a ∩ b = ∅\} and \mathcal{I} = \{d ⊂ N : (\forall a ∈ I) d \setminus a ∈ \mathcal{I}\}. If \mathcal{J} ⊂ [N]^ω, let J = \bigcup_{J \in \mathcal{J}} P(J). Say that \mathcal{J} ⊂ \mathcal{I} is unbounded in \mathcal{I} if for each a ∈ \mathcal{I}, there is a b ∈ \mathcal{J} such that b \setminus a is infinite.

**Definition 1.1.** If \mathcal{I} is an ideal of subsets of N, set cf(\mathcal{I}) to be the cofinality of \mathcal{I}; b(\mathcal{I}) is the minimum cardinality of an unbounded family in \mathcal{I}; δ(\mathcal{I}) is the minimum cardinality of a subset \mathcal{J} of \mathcal{I} such that \mathcal{J} is dense in \mathcal{I}.

If βN \ N = A ⊖ B, then \mathcal{I}_B = \mathcal{I}_A and x is the unique ultrafilter on N extending \mathcal{I}_A ⊕ \mathcal{I}_B. The character of x in βN \ N is equal to the maximum of cf(\mathcal{I}_A) and cf(\mathcal{I}_B).

**Definition 1.2.** Say that a tie-point x has (i) b-type; (ii) δ-type; respectively (iii) bδ-type, (κ, λ) if βN \ N = A ⊖ B and (κ, λ) equals: (i) (b(\mathcal{I}_A), b(\mathcal{I}_B)) (ii) (δ(\mathcal{I}_A), δ(\mathcal{I}_B)); and (iii) each of (b(\mathcal{I}_A), b(\mathcal{I}_B)) and (δ(\mathcal{I}_A), δ(\mathcal{I}_B)). We will adopt the convention to put the smaller of the pair (κ, λ) in the first coordinate.

Again, it is interesting to note that if x is a tie-point of b-type (κ, λ), then it is uniquely determined (in βN \ N) by λ many subsets of N since x will be the unique point extending the family ((\mathcal{J}_A)^+ \cap ((\mathcal{J}_B)^+)^+ where \mathcal{J}_A and \mathcal{J}_B are unbounded subfamilies of \mathcal{I}_A and \mathcal{I}_B.

**Question 1.1.** Can there be a tie-point in βN \ N with δ-type (κ, λ) with κ ≤ λ less than the character of the point?
**Question 1.2.** Can \( \beta \mathbb{N} \setminus \mathbb{N} \) have tie-points of \( \delta \)-type \((\omega_1, \omega_1)\) and \((\omega_2, \omega_2)\)?

**Proposition 1.2.** If \( \beta \mathbb{N} \setminus \mathbb{N} \) has symmetric tie-points of \( \delta \)-type \((\kappa, \kappa)\) and \((\lambda, \lambda)\), but no tie-points of \( \delta \)-type \((\kappa, \lambda)\), then \( \beta \mathbb{N} \setminus \mathbb{N} \) has a 2-to-1 image which is not homeomorphic to \( \beta \mathbb{N} \setminus \mathbb{N} \).

One could say that a tie-point \( x \) was radioactive in \( X \) (i.e. \( \triangleright \)). If \( X \setminus \{ x \} \) can be similarly split into 3 (or more) relatively clopen sets accumulating to \( x \). This is equivalent to \( X = A \triangleright \diamondsuit B \) such that \( x \) is a tie-point in either \( A \) or \( B \).

Each point of character \( \omega_1 \) in \( \beta \mathbb{N} \setminus \mathbb{N} \) is a radioactive point (in particular is a tie-point). P-points of character \( \omega_1 \) are symmetric tie-points of \( \theta \delta \)-type \((\omega_1, \omega_1)\), while points of character \( \omega_1 \) which are not P-points will have \( \theta \)-type \((\omega, \omega_1)\) and \( \delta \)-type \((\omega_1, \omega_1)\). If there is a tie-point of \( \delta \)-type \((\kappa, \lambda)\), then of course there are \((\kappa, \lambda)\)-gaps. If there is a tie-point of \( \delta \)-type \((\kappa, \lambda)\), then \( p \leq \kappa \).

**Proposition 1.3.** If \( \beta \mathbb{N} \setminus \mathbb{N} = A \triangleright \diamondsuit B \), then \( p \leq \delta(I_A) \).

*Proof.* If \( J \subset I_A \) has cardinality less than \( p \), there is, by Solovay’s Lemma (and Bell’s Theorem) an infinite set \( C \subset \mathbb{N} \) such that \( C \) and \( \mathbb{N} \setminus C \) each meet every infinite set of the form \( J \setminus (\bigcup J') \) where \( \{ J \} \cup J' \in [J]^\omega \). We may assume that \( C \notin x \), hence there are \( a \in I_A \) and \( b \in I_B \) such that \( C \subset a \cup b \). However no finite union from \( J \) covers \( a \) showing that \( J \) can not be dense in \( I_A \). \( \square \)

Although it does not seem to be completely trivial, it can be shown that PFA implies there are no tie-points (the hardest case to eliminate is those of \( \theta \delta \)-type \((\omega_1, \omega_1)\)).

**Question 1.3.** Does \( p > \omega_1 \) imply there are no tie-points of \( \theta \)-type \((\omega_1, \omega_1)\)?

Analogous to tie-points, we also define a tie-set: say that \( K \subset \beta \mathbb{N} \setminus \mathbb{N} \) is a tie-set if \( \beta \mathbb{N} \setminus \mathbb{N} = A \triangleright \diamondsuit B \) and \( K = A \cap B \), \( A = A \setminus K \), and \( B = B \setminus K \). Say that \( K \) is a symmetric tie-set if there is an involution \( F \) such that \( K = \text{fix}(F) \) and \( F[A] = B \).

**Question 1.4.** If \( F \) is an involution on \( \beta \mathbb{N} \setminus \mathbb{N} \) such that \( K = \text{fix}(F) \) has empty interior, is \( K \) a (symmetric) tie-set?

**Question 1.5.** Is there some natural restriction on which compact spaces can (or can not) be homeomorphic to the fixed point set of some involution of \( \beta \mathbb{N} \setminus \mathbb{N} \)?

Again, we note a possible application to 2-to-1 maps.
Proposition 1.4. Assume that $F$ is an involution of $\beta N \setminus N$ with $K = \text{fix}(F) \neq \emptyset$. Further assume that $K$ has a symmetric tie-point $x$ (i.e. $K = A \sqsupset x B$), then $\beta N \setminus N$ has a 2-to-1 continuous image which has a symmetric tie-point (and possibly $\beta N \setminus N$ does not have such a tie-point).

Question 1.6. If $F$ is an involution of $N^*$, is the quotient space $N^*/F$ (in which each $\{x, F(x)\}$ is collapsed to a single point) a homeomorphic copy of $\beta N \setminus N$?

Proposition 1.5 (CH). If $F$ is an involution of $\beta N \setminus N$, then the quotient space $N^*/F$ is homeomorphic to $\beta N \setminus N$.

Proof. If $\text{fix}(F)$ is empty, then $N^*/F$ is a 2-to-1 image of $\beta N \setminus N$, and so is a copy of $\beta N \setminus N$. If $\text{fix}(F)$ is not empty, then consider two copies, $(N_1^*, F_1)$ and $(N_2^*, F_2)$, of $(N^*, F)$. The quotient space of $N_1^*/F_1 \oplus N_2^*/F_2$ obtained by identifying the two homeomorphic sets $\text{fix}(F_1)$ and $\text{fix}(F_2)$ will be a 2-to-1-image of $N^*$, hence again a copy of $N^*$. Since $N_1^* \setminus \text{fix}(F_1)$ and $N_2^* \setminus \text{fix}(F_2)$ are disjoint and homeomorphic, it follows easily that $\text{fix}(F)$ must be a P-set in $N^*$. It is trivial to verify that a regular closed set of $N^*$ with a P-set boundary will be (in a model of CH) a copy of $N^*$. Therefore the copy of $N_1^*/F_1$ in this final quotient space is a copy of $N^*$.

2. A SPECTRUM OF TIE-SETS

We adapt a method from [1] to produce a model in which there are tie-sets of specified $b\delta$-types. We further arrange that these tie-sets will themselves have tie-points but unfortunately we are not able to make the tie-sets symmetric. In the next section we make some progress in involving involutions.

Theorem 2.1. Assume GCH and that $\Lambda$ is a set of regular uncountable cardinals such that for each $\lambda \in \Lambda$, $T_\lambda$ is a $<\lambda$-closed $\lambda^+$-Souslin tree. There is a forcing extension in which there is a tie-set $K$ (of $b\delta$-type $(c, c)$) and for each $\lambda \in \Lambda$, there is a tie-set $K_\lambda$ of $b\delta$-type $(\lambda^+, \lambda^+)$ such that $K \cap K_\lambda$ is a single point which is a tie-point of $K_\lambda$. Furthermore, for $\mu \leq \lambda < c$, if $\mu \neq \lambda$ or $\lambda \notin \Lambda$, then there is no tie-set of $b\delta$-type $(\mu, \lambda)$.

We will assume that our Souslin trees are well-pruned and are ever $\omega$-ary branching. That is, if $T_\lambda$ is a $\lambda^+$-Souslin tree, we assume that for each $t \in T$, $t$ has exactly $\omega$ immediate successors denoted $\{t^\ell : \ell \in \omega\}$ and that $\{s \in T_\lambda : t < s\}$ has cardinality $\lambda^+$ (and so has successors on every level). A poset is $<\kappa$-closed if every directed subset
of cardinality less than $\kappa$ has a lower bound. A poset is $\prec$-distributive if the intersection of any family of fewer than $\kappa$ dense open subsets is again dense. For a cardinal $\mu$, let $\mu^-$ be the minimum cardinal such that $$(\mu^-)^+ \geq \mu \text{ (i.e. the predecessor if } \mu \text{ is a successor)}.$$ The main idea of the construction is nicely illustrated by the following.

**Proposition 2.2.** Assume that $\beta\mathbb{N}\setminus\mathbb{N}$ has no tie-sets of $b\delta$-type $(\kappa_1, \kappa_2)$ for some $\kappa_1 \leq \kappa_2 < c$. Also assume that $\lambda^+ < c$ is such that $\lambda^+$ is distinct from one of $\kappa_1, \kappa_2$ and that $T_\lambda$ is a $\lambda^+$-Souslin tree and $$\{(a_t, x_t, b_t) : t \in T_\lambda\} \subset (\mathbb{N}^\omega)^3$$ satisfy that, for $t < s \in T_\lambda$:

1. $\{a_t, x_t, b_t\}$ is a partition of $\mathbb{N}$,
2. $x_{t-j} \cap x_{t-j} = \emptyset$ for $j < \ell$,
3. $x_s \subset^* x_t, a_t \subset^* a_s$, and $b_t \subset^* b_s$,
4. for each $\ell \in \omega, x_{t+\ell} \subset^* a_{t+\ell}$ and $x_{t+\ell} \subset^* b_{t+\ell}$,
then if $\rho \in [T_\lambda]^{\lambda^+}$ is a generic branch (i.e. $\rho(\alpha)$ is an element of the $\alpha$-th level of $T_\lambda$ for each $\alpha \in \lambda^+$), then $K_\rho = \bigcap_{\alpha \in \lambda^+} x^*_\rho(\alpha)$ is a tie-set of $\beta\mathbb{N}\setminus\mathbb{N}$ of $b\delta$-type $(\lambda^+, \lambda^+)$, and there is no tie-set of $b\delta$-type $(\kappa_1, \kappa_2)$.

5. Assume further that $\{(a_\xi, x_\xi, b_\xi) : \xi \in c\}$ is a family of partitions of $\mathbb{N}$ such that $\{x_\xi : \xi \in c\}$ is a mod finite descending family of subsets of $\mathbb{N}$ such that for each $Y \subset \mathbb{N}$, there is a maximal antichain $A_Y \subset T_\lambda$ and some $\xi \in c$ such that for each $t \in A_Y$, $x_t \cap x_\xi$ is a proper subset of either $Y$ or $\mathbb{N} \setminus Y$, then $K = \bigcap_{\xi \in c} x^*_\xi$ meets $K_\rho$ in a single point $z_\lambda$.

6. If we assume further that for each $\xi < \eta < c, a_\xi \subset^* a_\eta$ and $b_\xi \subset^* b_\eta$, and for each $t \in T_\lambda, \eta$ may be chosen so that $x_t$ meets each of $(a_\eta \setminus a_\xi)$ and $(b_\eta \setminus b_\xi)$, then $z_\lambda$ is a tie-point of $K_\rho$.

**Proof.** To show that $K_\rho$ is a tie-set it is sufficient to show that $K_\rho \subset \bigcup_{\alpha \in \lambda^+} a_\alpha^* \cap \bigcup_{\alpha \in \lambda^+} b_\alpha^*$. Since $T_\lambda$ is a $\lambda^+$-Souslin tree, no new subset of $\lambda$ is added when forcing with $T_\lambda$. Of course we use that $\rho$ is $T_\lambda$ generic, so assume that $Y \subset \mathbb{N}$ and that some $t \in T_\lambda$ forces that $Y^* \cap K_\rho$ is not empty. We must show that there is some $t < s$ such that $s$ forces that $a_s \cap Y$ and $b_s \cap Y$ are both infinite. However, we know that $x_{t-\ell} \cap Y$ is infinite for each $\ell \in \omega$ since $t-\ell \Vdash_{T_\lambda} "K_\rho \subset x^*_t"$. Therefore, by condition 4, for each $\ell \in \omega, Y \cap a_{t-\ell}$ and $Y \cap b_{t-\ell}$ are both infinite.

Now let $\kappa_1, \kappa_2$ be regular cardinals at least one of which is distinct from $\lambda^+$. Recall that forcing with $T_\lambda$ preserves cardinals. Assume that in $V[\rho], K \subset \mathbb{N}^* \text{ and } \mathbb{N}^* = C \bowtie^K D$ with $b(I_C) = \delta(I_C) = \kappa_1$ and $b(I_D) = \delta(I_D) = \kappa_2$. In $V$, let $\{c_\gamma : \gamma \in \kappa_1\}$ be $T_\lambda$-names for the increasing cofinal sequence in $I_C$ and let $\{d_\xi : \xi \in \kappa_2\}$ be $T_\lambda$-names for the increasing cofinal sequence in $I_D$. Again using the fact that $T_\lambda$ adds
no new subsets of \( N \) and the fact that every dense open subset of \( T_\lambda \) will contain an entire level of \( T_\lambda \), we may choose ordinals \( \{\alpha_\gamma : \gamma \in \kappa_1\} \) and \( \{\beta_\xi : \xi \in \kappa_2\} \) such that each \( t \in T_\lambda \), if \( t \) is on level \( \alpha_\gamma \) it will force a value on \( c_\gamma \) and if \( t \) is on level \( \beta_\xi \) it will force a value on \( d_\xi \). If \( \kappa_1 < \lambda^+ \), then \( \sup \{\alpha_\gamma : \gamma \in \kappa_1\} < \lambda^+ \), hence there are \( t \in T_\lambda \) which force a value on each \( c_\gamma \). If \( \lambda^+ < \kappa_2 \), then there is some \( \beta < \lambda^+ \), such that \( \{\xi \in \kappa_2 : \beta_\xi \leq \beta\} \) has cardinality \( \kappa_2 \). Therefore there is some \( t \in T_\lambda \) such that \( t \) forces a value on \( d_\xi \) for a cofinal set of \( \xi \in \kappa_2 \). Of course, if neither \( \kappa_1 \) nor \( \kappa_2 \) is equal to \( \lambda^+ \), then we have a condition that decided cofinal families of each of \( \mathcal{I}_C \) and \( \mathcal{I}_D \). This implies that \( N^* \) already has tie-sets of \( \mathfrak{b}\delta \)-type \( (\kappa_1, \kappa_2) \).

If \( \kappa_1 < \kappa_2 = \lambda^+ \), then fix \( t \in T_\lambda \) deciding \( \mathcal{C} = \{c_\gamma : \gamma \in \kappa_1\} \), and let \( \mathcal{D} = \{d \subseteq \mathbb{N} : (\exists s > t)s \Vdash_{T_\lambda} "d^* \subseteq D\"\}. \) It follows easily that \( \mathcal{D} = \mathcal{C}^\perp \). But also, since forcing with \( T_\lambda \) can not raise \( \mathfrak{b}(\mathcal{D}) \) and can not lower \( \delta(\mathcal{D}) \), we again have that there are tie-sets of \( \mathfrak{b}\delta \)-type in \( V \).

The case \( \kappa_1 = \lambda^+ < \kappa_2 \) is similar.

Now assume we have the family \( \{(a_\xi, x_\xi, b_\xi) : \xi \in c\} \) as in (5) and (6) and set \( K = \bigcap_\xi x_\xi^* \), \( A = \{K\} \cup \bigcup\{a_\xi^* : \xi \in c\}, \) and \( B = \{K\} \cup \bigcup\{b_\xi^* : \xi \in c\}. \) It is routine to see that (5) ensures that the family \( \{x_\xi \cap x_{\rho(\alpha)} : \xi \in c \) and \( \alpha \in \lambda^+\} \) generates an ultrafilter when \( \rho \) meets each maximal antichain \( A_Y \) (\( Y \subseteq \mathbb{N} \)). Condition (6) clearly ensures that \( A \setminus K \) and \( B \setminus K \) each meet \( (x_\xi \cap x_{\rho(\alpha)})^* \) for each \( \xi \in c \) and \( \alpha \in \lambda^+ \). Thus \( A \cap K_\rho \) and \( B \cap K_\rho \) witness that \( z_\lambda \) is a tie-point of \( K_\rho \).

Let \( \theta \) be a regular cardinal greater than \( \lambda^+ \) for all \( \lambda \in \Lambda \). We will need the following well-known Easton lemma (see [4, p234]).

**Lemma 2.3.** Let \( \mu \) be a regular cardinal and assume that \( P_1 \) is a poset satisfying the \( \mu \)-cc. Then any \( < \mu \)-closed poset \( P_2 \) remains \( < \mu \)-distributive after forcing with \( P_1 \). Furthermore any \( < \mu \)-distributive poset remains \( < \mu \)-distributive after forcing with a poset of cardinality less than \( \mu \).

**Proof.** Recall that a poset \( P \) is \( < \mu \)-distributive if forcing with it does not add, for any \( \gamma < \mu \), any new \( \gamma \)-sequences of ordinals. Since \( P_2 \) is \( < \mu \)-closed, forcing with \( P_2 \) does not add any new antichains to \( P_1 \). Therefore it follows that forcing with \( P_2 \) preserves that \( P_1 \) has the \( \mu \)-cc and that for every \( \gamma < \mu \), each \( \gamma \)-sequence of ordinals in the forcing extension by \( P_2 \times P_1 \) is really just a \( P_1 \)-name. Since forcing with \( P_1 \times P_2 \) is the same as \( P_2 \times P_1 \), this shows that in the extension by \( P_1 \), there are no new \( P_2 \)-names of \( \gamma \)-sequences of ordinals.
functions of posets, then we will use \( \Pi \).

Definition 2.1. If \( \Lambda \) is a set of cardinals and \( \{ P_\lambda : \lambda \in \Lambda \} \) is a set of posets, then we will use \( \Pi_{\lambda \in \Lambda} P_\lambda \) to denote the collection of partial functions \( p \) such that

1. \( \text{dom}(p) \subseteq \Lambda \),
2. \( |\text{dom}(p) \cap \mu| < \mu \) for all regular cardinals \( \mu \),
3. \( p(\lambda) \in P_\lambda \) for all \( \lambda \in \text{dom}(p) \).

This collection is a poset when ordered by \( q < p \) if \( \text{dom}(q) \supseteq \text{dom}(p) \) and \( q(\lambda) \leq p(\lambda) \) for all \( \lambda \in \text{dom}(p) \).

Lemma 2.4. For each cardinal \( \mu \), \( \Pi_{\lambda \in \Lambda \cap \mu} T_\lambda \) is \( \mu^+ \)-closed and, if \( \mu \) is regular, \( \Pi_{\lambda \in \Lambda \cap \mu} T_\lambda \) has cardinality at most \( 2^{<\mu} \leq \min(\Lambda \setminus \mu) \).

Lemma 2.5. If \( P \) is ccc and \( G \subseteq P \times \Pi_{\lambda \in \Lambda} T_\lambda \) is generic, then in \( V[G] \), for any \( \mu \) and any family \( \mathcal{A} \subseteq [\mathbb{N}]^\omega \) with \( |\mathcal{A}| = \mu \):

1. if \( \mu \leq \omega \), then \( \mathcal{A} \) is a member of \( V[G \cap P] \);
2. if \( \mu = \lambda^+ \), \( \lambda \in \Lambda \), then there is an \( \mathcal{A}' \subseteq \mathcal{A} \) of cardinality \( \lambda^+ \) such that \( \mathcal{A}' \) is a member of \( V[G \cap (P \times T_\lambda)] \);
3. if \( \mu^- \notin \Lambda \), then there is an \( \mathcal{A}' \subseteq \mathcal{A} \) of cardinality \( \mu \) which is a member of \( V[G \cap P] \).

Corollary 2.6. If \( P \) is ccc and \( G \subseteq P \times \Pi_{\lambda \in \Lambda} T_\lambda \) is generic, then for any \( \kappa \leq \mu < \mathfrak{c} \) such that either \( \kappa \neq \mu \) or \( \kappa \notin \{ \lambda^+ : \lambda \in \Lambda \} \), if there is a tie-set of \( \mathfrak{d}\)-type \( (\kappa, \mu) \) in \( V[G] \), then there is such a tie-set in \( V[G \cap P] \).

Proof. Assume that \( \beta \mathbb{N} \setminus \mathbb{N} = A \not\leq_{k} B \) in \( V[G] \) with \( \mu = b(A) \) and \( \lambda = b(B) \). Let \( \mathcal{J}_A \subseteq \mathcal{I}_A \) be an increasing mod finite chain, of order type \( \mu \), which is dense in \( \mathcal{I}_A \). Similarly let \( \mathcal{J}_B \subseteq \mathcal{I}_B \) be such a chain of order type \( \lambda \). By Lemma 2.5, \( \mathcal{J}_A \) and \( \mathcal{J}_B \) are subsets of \( [\mathbb{N}]^\omega \cap V[G \cap P] = [\mathbb{N}]^\omega \).

Choose, if possible \( \mu_1 \in \Lambda \) such that \( \mu_1^+ = \mu \) and \( \lambda_1 \in \Lambda \) such that \( \lambda_1^+ = \lambda \). Also by Lemma 2.5, we can, by passing to a subcollection, assume that \( \mathcal{J}_A \subseteq V[G \cap (P \times T_{\mu_1})] \) (if there is no \( \mu_1 \), then let \( T_{\mu_1} \) denote...
the trivial order). Similarly, we may assume that $\mathcal{J}_B \in V[G \cap (P \times T_\lambda)]$. Fix a condition $q \in G \subset (P \times \Pi_{\lambda \in A} T_\lambda)$ which forces that $(\mathcal{J}_A)^1$ is a $\subset$-dense subset of $\mathcal{I}_A$, that $(\mathcal{J}_B)^1$ is a $\subset$-dense subset of $\mathcal{I}_B$, and that $(\mathcal{I}_A)^2 = \mathcal{I}_B$.

Working in the model $V[G \cap P]$ then, there is a family $\{\dot{a}_\alpha : \alpha \in \mu\}$ of $T_{\mu_1}$-names for the members of $\mathcal{J}_A$; and a family $\{\dot{b}_\beta : \beta \in \lambda_1\}$ of $T_{\lambda_1}$-names for the members of $\mathcal{J}_B$. Of course if $\mu = \lambda$ and $T_{\mu_1}$ is the trivial order, then $\mathcal{J}_A$ and $\mathcal{J}_B$ are already in $V[G \cap P]$ and we have our tie-set in $V[G \cap P]$.

Otherwise, we assume that $\mu_1 < \lambda_1$. Set $\mathcal{A}$ to be the set of all $a \subset \mathbb{N}$ such that there is some $q(\mu_1) \leq t \in T_{\mu_1}$ and $\alpha \in \mu$ such that $t \forces_{T_{\mu_1}} "a = \dot{a}_\alpha"$. Similarly let $\mathcal{B}$ be the set of all $b \subset \mathbb{N}$ such that there is some $q(\lambda_1) \leq s \in T_{\lambda_1}$ and $\beta \in \lambda$ such that $s \forces_{T_{\lambda_1}} "b = \dot{b}_\beta"$. It follows from the construction that, in $V[G]$, for any $(a', b') \in \mathcal{A} \times \mathcal{J}_B$, there is an $(a, b) \in \mathcal{A} \times \mathcal{B}$ such that $a' \subset^* a$ and $b' \subset^* b$. Therefore the ideal generated by $\mathcal{A} \cup \mathcal{B}$ is certainly dense. It remains only to show that $\mathcal{B} \subset (\mathcal{A})^2$. Consider any $(a, b) \in \mathcal{A} \times \mathcal{B}$, and choose $(q(\mu_1), q(\lambda_1)) \leq (t, s) \in T_{\mu_1} \times T_{\lambda_1}$ such that $t \forces_{T_{\mu_1}} "a \in \mathcal{J}_A"$ and $s \forces_{T_{\lambda_1}} "b \in \mathcal{J}_B"$. It follows that for any condition $\bar{q} \leq q$ with $\bar{q} \in (P \times \Pi_{\lambda \in A} T_\lambda)$, $\bar{q}(\mu_1) = t$, $\bar{q}(\lambda_1) = s$, we have that

$$\bar{q} \forces_{(P \times \Pi_{\lambda \in A} T_\lambda)} "a \in \mathcal{J}_A \text{ and } b \in \mathcal{J}_B".$$ 

It is routine now to check that, in $V[G \cap P]$, $\mathcal{A}$ and $\mathcal{B}$ generate ideals that witness that $\bigcap\{(\mathbb{N} \setminus (a \cup b))^* : (a, b) \in \mathcal{A} \times \mathcal{B}\}$ is a tie-set of $b\delta$-type $(\mu, \lambda)$. \hfill \qed

Let $T$ be the rooted tree $\{\emptyset\} \cup \bigcup_{\lambda \in A} T_\lambda$ and we will force an embedding of $T$ into $\mathcal{P}(\mathbb{N})$ mod finite. In fact, we force a structure $\{(a_t, x_t, b_t) : t \in T\}$ satisfying the conditions (1)-(4) of Proposition 2.2.

**Definition 2.2.** The poset $Q_0$ is defined as the set of elements $q = (n^q, T^q, f^q)$ where $n^q \in \mathbb{N}$, $T^q \in [T]^{<\omega}$, and $f^q : n^q \times T^q \to \{0, 1, 2\}$. The idea is that $x_t$ will be $\bigcup_{q \in G} \{j \in n^q : f^q(j, t) = 0\}$, $a_t$ will be $\bigcup_{q \in G} \{j \in n^q : f^q(j, t) = 1\}$ and $b_t = \mathbb{N} \setminus (a_t \cup x_t)$. We set $q < p$ if $n^q \geq n^p$, $T^q \supseteq T^p$, $f^q \supseteq f^p$ and for $t, s \in T^p$ and $i \in [n^p, n^q]$

1. if $t < s$ and $f^q(i, t) \in \{1, 2\}$, then $f^q(i, s) = f^q(i, t)$;
2. if $t < s$ and $f^q(i, s) = 0$, then $f^q(i, t) = 0$;
3. if $t \perp s$, then $f^q(i, t) + f^q(i, s) > 0$.
4. if $j \in \{1, 2\}$ and $\{t-\ell, t-(\ell+j)\} \subset T^p$ and $f^q(i, t-(\ell+j)) = 0$, then $f^q(i, t-\ell) = j$.

The next lemma is very routine but we record it for reference.
Lemma 2.7. The poset $Q_0$ is ccc and if $G \subset Q_0$ is generic, the family $X_T = \{(a_t, x_t, b_t) : t \in T\}$ satisfies the conditions of Proposition 2.2.

We will need some other combinatorial properties of the family $X_T$.

Definition 2.3. For any $\tilde{T} \in [T]^{<\omega}$, we define the following ($Q_0$-names).

1. for $i \in \mathbb{N}$, $[i]_{\tilde{T}} = \{j \in \mathbb{N} : (\forall t \in \tilde{T}) i \in x_t \iff j \in x_t\}$,
2. the collection fin($\tilde{T}$) is the set of $[i]_{\tilde{T}}$ which are finite.

We abuse notation and let fin($\tilde{T}$) $\subset n$ abbreviate fin($\tilde{T}$) $\subset P(n)$.

Lemma 2.8. For each $q \in Q_0$ and each $\tilde{T} \subset T^q$, fin($\tilde{T}$) $\subset n^q$ and for $i \geq n_q$, $[i]_{\tilde{T}}$ is infinite.

Definition 2.4. A sequence $S_W = \{(a_\xi, x_\xi, b_\xi) : \xi \in W\}$ is a tower of $T$-splitters if for $\xi < \eta \in W$ and $t \in \tilde{T}$:

1. $\{a_\xi, x_\xi, b_\xi\}$ is a partition of $\mathbb{N}$,
2. $a_\xi \subset^* a_\eta$, $b_\xi \subset^* b_\eta$,
3. $x_t \cap x_\xi$ is finite.

Definition 2.5. If $S_W$ is a tower of $T$-splitters and $Y$ is a subset $\mathbb{N}$, then the poset $Q(S_W,Y)$ is defined as follows. Let $E_Y$ be the (possibly empty) set of minimal elements of $T$ such that there is some finite $H \subset W$ such that $x_t \cap Y \cap \bigcup_{\xi \in H} x_\xi$ is finite. Let $D_Y = E_Y^\perp = \{t \in T : (\forall s \in E_Y) t \perp s\}$. A condition $q \in Q(S_W,Y)$ is a tuple $(n^q, a^q, x^q, b^q, T^q, H^q)$ where

1. $n^q \in \mathbb{N}$ and $\{a^q, x^q, b^q\}$ is a partition of $n^q$,
2. $T^q \in [T]^{<\omega}$ and $H^q \in [W]^{<\omega}$,
3. $(a_\xi \setminus a_\eta)$, $(b_\xi \setminus b_\eta)$, and $(x_\eta \setminus x_\xi)$ are all contained in $n^q$ for $\xi < \eta \in H^q$.

We define $q < p$ to mean $n^p \leq n^q$, $T^p \subset T^q$, $H^p \subset H^q$, and

4. for $t \in T^p \cap D_Y$, $x_t \cap (x^q \setminus x^p) \subset Y$,
5. $x^q \setminus x^p \subset \bigcap_{\xi \in H^p} x_\xi$,
6. $a^q \setminus a^p$ is disjoint from $b_{\max(H^p)}$,
7. $b^q \setminus b^p$ is disjoint from $a_{\max(H^p)}$.

Lemma 2.9. If $W \subset \gamma$, $S_W$ is a tower of $T$-splitters, and if $G$ is $Q(S_W,Y)$-generic, then $S_W \cup \{(a_\gamma, x_\gamma, b_\gamma)\}$ is also a tower of $T$-splitters where $a_\gamma = \bigcup\{a_q : q \in G\}$, $x_\gamma = \bigcup\{x_q : q \in G\}$, and $b_\gamma = \bigcup\{b_q : q \in G\}$. In addition, for each $t \in D_Y$, $x_t \cap x_\xi \subset^* Y$ (and $x_t \cap x_\xi \subset^* \mathbb{N} \setminus Y$ for $t \in E_Y$).

Lemma 2.10. If $W$ does not have cofinality $\omega_1$, then $Q(S_W,Y)$ is $\sigma$-centered.
As usual with \((\omega_1,\omega_1)\)-gaps, \(Q(S_W,Y)\) may not (in general) be ccc if \(W\) has a cofinal \(\omega_1\) sequence.

Let \(0 \notin C \subset \theta\) be cofinal and assume that if \(C \cap \gamma\) is cofinal in \(\gamma\) and \(\text{cf}(\gamma) = \omega_1\), then \(\gamma \in C\).

**Definition 2.6.** Fix any well-ordering \(\prec\) of \(H(\theta)\). We define a finite support iteration sequence \(\{P_\gamma, \dot{Q}_\gamma : \gamma \in \theta\} \subset H(\theta)\). We abuse notation and use \(Q_0\) rather than \(\dot{Q}_0\) from definition 2.2. If \(\gamma \notin C\), then let \(\dot{Q}_\gamma\) be the \(\prec\)-least among the list of \(P_\gamma\)-names of ccc posets in \(\dot{Q}_\gamma \in Q(\theta) \setminus \{Q_\xi : \xi \in \gamma\}\). If \(\gamma \in C\), then let \(\dot{Y}_\gamma\) be the \(\prec\)-least \(P_\gamma\)-name of a subset \(\mathbb{N}\) which is in \(H(\theta) \setminus \{Y_\xi : \xi \in C \cap \gamma\}\). Set \(\dot{Q}_\gamma\) to be the \(P_\gamma\) name of \(Q(S_{C \cap \gamma}, \dot{Y}_\gamma)\) adding the partition \(\{\dot{a}_\gamma, \dot{x}_\gamma, \dot{b}_\gamma\}\) and, where \(S_{C \cap \gamma}\) is the \(P_\gamma\)-name of the \(T\)-splitting tower \(\{(a_\xi, x_\xi, b_\xi) : \xi \in C \cap \gamma\}\).

We view the members of \(P_\theta\) as functions \(p\) with finite domain (or support) denoted \(\text{dom}(p)\).

The main difficulty to the proof of Theorem 2.1 is to prove that the iteration \(P_\theta\) is ccc. Of course, since it is a finite support iteration, this can be proven by induction at successor ordinals.

**Lemma 2.11.** For each \(\gamma \in C\) such that \(C \cap \gamma\) has cofinality \(\omega_1\), \(P_{\gamma+1}\) is ccc.

**Proof.** We proceed by induction. For each \(\alpha\), define \(p \in P_\alpha^*\) if \(p \in P_\alpha\) and there is an \(n \in \mathbb{N}\) such that

\[
(1) \text{ for each } \beta \in \text{dom}(p) \cap C, \text{ with } H^\beta = \text{dom}(p) \cap C \cap \beta, \text{ there are subsets } a^\beta, x^\beta, b^\beta \text{ of } n \text{ and } T^\beta \in [T]^{<\omega} \text{ such that } p \upharpoonright \beta \models_{P_\beta} \text{“}p(\beta) = (n, a^\beta, x^\beta, b^\beta, T^\beta, H^\beta)\text{”}
\]

Assume that \(P_\beta^*\) is dense in \(P_\beta\) and let \(p \in P_{\beta+1}\). To show that \(P_{\beta+1}^*\) is dense in \(P_{\beta+1}\) we must find some \(p^* \leq p\) in \(P_{\beta+1}^*\). If \(\beta \notin C\) and \(p^* \in P_\beta^*\) is below \(p \upharpoonright \beta\), then \(p^* \cup \{(\beta, p(\beta))\}\) is the desired element of \(P_{\beta+1}^*\). Now assume that \(\beta \in C\) and assume that \(p \upharpoonright \beta \in P_\beta^*\) and that \(p \upharpoonright \beta\) forces that \(p(\beta)\) is the tuple \((n_0, a, x, b, \bar{T}, \bar{H})\). By an easy density argument, we may assume that \(\bar{H} \subset \text{dom}(p)\). Let \(n^*\) be the integer witnessing that \(p \upharpoonright \beta \in P_\beta^*\). Let \(\zeta\) be the maximum element of \(\text{dom}(p) \cap C \cap \beta\) and let \(p \upharpoonright \zeta \models_{P_\beta^*} \text{“}p(\zeta) = (n^*, a^\zeta, x^\zeta, b^\zeta, T^\zeta, H^\zeta, \zeta)\text{”}\) as per the definition of \(P_{\zeta+1}^*\). Notice that since \(\bar{H} \subset H^{\zeta}\) we have that

\[
p \upharpoonright \beta \models_{P_\beta^*} \text{“(}n^*, a^*, x, b^*, T^\zeta \cup \bar{T}, H^\zeta \cup \{\zeta\}\text{) ≤ }p(\beta)\text{”}
\]

where \(a^* = a \cup ([n_0, n^*) \setminus b^\zeta]\) and \(b^* = b \cup ([n_0, n^*) \cap b^\zeta]\). Defining \(p^* \in P_{\beta+1}\) by \(p^* \upharpoonright \beta = p \upharpoonright \beta\) and \(p^*(\beta) = (n^*, a^*, x, b^*, T^\zeta \cup \bar{T}, H^\zeta \cup \{\zeta\})\)
completes the proof that $P^*_\beta$ is dense in $P^*_{\beta+1}$, and by induction, that this holds for $\beta = \gamma$.

Now assume that $\{p_\alpha : \alpha \in \omega_1\} \subset P^*_{\gamma+1}$. By passing to a subcollection, we may assume that

1. the collection \(\{T^{p_\alpha(\gamma)} : \alpha \in \omega_1\}\) forms a $\Delta$-system with root $T^*$;
2. the collection \(\{\text{dom}(p_\alpha) : \alpha \in \omega_1\}\) also forms a $\Delta$-system with root $R$;
3. there is a tuple \((n^*, a^*, x^*, b^*)\) so that for all $\alpha \in \omega_1$, \(a^{p_\alpha(\gamma)} = a^*, x^{p_\alpha(\gamma)} = x^*, \) and \(b^{p_\alpha(\gamma)} = b^*\).

Since $C \cap \gamma$ has a cofinal sequence of order type $\omega_1$, there is a $\delta \in \gamma$ such that $R \subset \delta$ and, we may assume, \(\text{dom}(p_\alpha) \setminus \delta \subset \min(\text{dom}(p_\beta) \setminus \delta)\) for $\alpha < \beta < \omega_1$. Since $P_\delta$ is ccc, there is a pair $\alpha < \beta < \omega_1$ such that $p_\alpha \upharpoonright \delta$ is compatible with $p_\beta \upharpoonright \delta$. Define $q \in P_{\gamma+1}$ by

1. $q \upharpoonright \delta$ is any element of $P_\delta$ which is below each of $p_\alpha \upharpoonright \delta$ and $p_\beta \upharpoonright \delta$.
2. if $\delta \leq \xi \in \gamma \cap \text{dom}(p_\alpha)$, then $q(\xi) = p_\alpha(\xi)$,
3. if $\delta \leq \xi \in \text{dom}(p_\beta) \setminus C$, then $q(\xi) = p_\beta(\xi)$,
4. if $\delta \leq \xi \in \text{dom}(p_\beta) \cap C$, then $q(\xi) = (n^*, a^{p_\beta(\xi)}, x^{p_\beta(\xi)}, b^{p_\beta(\xi)}, T^{p_\beta(\xi)}, H^{p_\beta(\xi)} \cup H^{p_\alpha(\gamma)})$.

The main non-trivial fact about $q$ is that it is in $P_{\gamma+1}$ which depends on the fact that, by induction on $\eta \in C \cap \gamma$, $q \upharpoonright \eta$ forces that

\[(a_\eta \setminus a_\xi) \cup (b_\eta \setminus b_\xi) \cup (x_\xi \setminus x_\eta) \subset n^* \text{ for } \xi \in C \cap \eta.\]

It now follows trivially that $q$ is below each of $p_\alpha$ and $p_\beta$. \qed

**Proof of Theorem 2.7** This completes the construction of the ccc poset $\Pi$ (or $P_\theta$ as above). Let $G \subset (\Pi \times \Pi_{\lambda \in \Lambda} T_\lambda)$ be generic. It follows that $V[G \cap P]$ is a model of Martin’s Axiom and $\mathfrak{c} = \theta$. Furthermore, by applying Lemma 2.3 with $\mu = \omega$ and Lemma 2.3 we have that $P_2 = \Pi_{\lambda \in \Lambda} T_\lambda$ is $\omega_1$-distributive in the model $V[G \cap P]$. Therefore all subsets of $\mathbb{N}$ in the model $V[G]$ are also in the model $V[G \cap P]$.

Fix any $\lambda \in \Lambda$ and let $p_\lambda$ denote the generic branch in $T_\lambda$ given by $G$. Let $G^\lambda$ denote the generic filter on $\Pi \times \Pi_{\mu \neq \lambda} T_\mu$ and work in the model $V[G^\lambda]$. It follows easily by Lemma 2.3 and Lemma 2.3 that $T_\lambda$ is a $\lambda^+$-Souslin tree in this model. Therefore by Proposition 2.2, $K_\lambda = \bigcap_{\alpha < \lambda} x^*_\alpha$ is a tie-set of $\mathfrak{b}$-type $(\lambda^+, \lambda^+)$ in $V[G]$. By the definition of the iteration in $P$, it follows that condition (4) of Lemma 2.2 is also satisfied, hence the tie-set $K = \bigcap_{\xi \in \mathcal{C}} x^*_\xi$ meets $K_\lambda$ in a single point $z_\lambda$. A simple genericity argument confirms that conditions (5) and (6) of Proposition 2.2 also holds, hence $z_\lambda$ is a tie-point of $K_\lambda$. 
It follows from Corollary 2.6 that there are no unwanted tie-sets in \( \beta \mathbb{N} \setminus \mathbb{N} \) in \( V[G] \), at least if there are none in \( V[G \cap P] \). Since \( p = c \) in \( V[G \cap P] \), it follows from Proposition 1.3 that indeed there are no such tie-sets in \( V[G \cap P] \).

Unfortunately the next result shows that the construction does not provide us with our desired variety of tie-points (even with variations in the definition of the iteration). We do not know if \( b \delta \)-type can be improved to \( \delta \)-type (or simply exclude tie-points altogether).

**Proposition 2.12.** In the model constructed in Theorem 2.1, there are no tie-points with \( b \delta \)-type \((\kappa_1, \kappa_2)\) for any \( \kappa_1 \leq \kappa_2 < c \).

**Proof.** Assume that \( \beta \mathbb{N} \setminus \mathbb{N} = A \upharpoonright \mathbb{N} \) and that \( \delta(A) = \kappa_1 \) and \( \delta(I) = \kappa_2 \). It follows from Corollary 2.6 that we can assume that \( \kappa_1 = \kappa_2 = \lambda^+ \) for some \( \lambda \in \Lambda \). Also, following the proof of Corollary 2.6, there are \( P \times T_\lambda \)-names \( J_A = \{ \tilde{a}_\alpha : \alpha \in \lambda^+ \} \) and \( P \times T_\lambda^+ \)-names \( J_B = \{ \tilde{b}_\beta : \beta \in \lambda^+ \} \) such that the valuation of these names by \( G \) result in increasing (mod finite) chains in \( I_A \) and \( I_B \) respectively whose downward closures are dense. Passing to \( V[G \cap P] \), since \( T_\lambda \) has the \( \theta \)-cc, there is a Boolean subalgebra \( B \in [\mathcal{P}(\mathbb{N})]^{<\theta} \) such that each \( \tilde{a}_\alpha \) and \( \tilde{b}_\beta \) is a name of a member of \( B \). Furthermore, there is an infinite \( C \subseteq \mathbb{N} \) such that \( C \notin x \) and each of \( b \cap C \) and \( b \setminus C \) are infinite for all \( b \in B \). Since \( C \notin x \), there is a \( Y \subseteq \mathbb{N} \) (in \( V[G] \)) such that \( C \cap Y \in I_A \) and \( C \setminus Y \in I_B \). Now choose \( t_0 \in T_\lambda \) which forces this about \( C \) and \( Y \). Back in \( V[G \cap P] \), set

\[
\mathcal{A} = \{ b \in B : (\exists t_1 \leq t_0) \ t_1 \vdash_{T_\lambda} "b \in J_A \cup J_B" \}.
\]

Since \( V[G \cap P] \) satisfies \( p = \theta \) and \( A^1 \) is forced by \( t_0 \) to be dense in \( [\mathbb{N}]^{\omega} \), there must be a finite subset \( \mathcal{A}' \) of \( A \) which covers \( C \). It also follows easily then that there must be some \( a, b \in \mathcal{A}' \) and \( t_1, t_2 \) each below \( t_0 \) such that \( t_1 \vdash_{T_\lambda} "a \in J_A" \), \( t_2 \vdash_{T_\lambda} "b \in J_B" \), and \( a \cap b \) is infinite. The final contradiction is that we will now have that \( t_0 \) fails to force that \( C \cap a \subseteq^* Y \) and \( C \cap b \subseteq^* (\mathbb{N} \setminus Y) \).

3. \( T \)-INVLATIONS

In this section we strengthen the result in Theorem 2.1 by making each \( K \cap K_\lambda \) a symmetric tie-point in \( K_\lambda \) (at the expense of weakening Martin’s Axiom in \( V[G \cap P] \)). This is progress in producing involutions with some control over the fixed point set but we are still not able to make \( K \) the fixed point set of an involution. A poset is said to be \( \sigma \)-linked if there is a countable collection of linked (elements are pairwise compatible) which union to the poset. The statement MA(\( \sigma \) – linked)
Let $D$ be the (possibly empty) set of minimal elements of $\lambda$. Thus, for each $\lambda \in \Lambda$, there is a $\lambda$-closed $\lambda^+$-Souslin tree.

Our approach is to replace $T$-splitting towers by the following notion. If $f$ is a (partial) involution on $\mathbb{N}$, let $\text{min}(f) = \{ n \in \mathbb{N} : n < f(n) \}$ and $\text{max}(f) = \{ n \in \mathbb{N} : f(n) < n \}$ (hence $\text{dom}(f)$ is partitioned into $\text{min}(f) \cup \text{fix}(f) \cup \text{max}(f)$).

**Definition 3.1.** A sequence $\mathcal{T} = \{ (A_\xi, f_\xi) : \xi \in W \}$ is a tower of $T$-involutions if $W$ is a set of ordinals and for $\xi < \nu \in W$ and $t \in T$

1. $A_\nu \subset^* A_\xi$;
2. $f_\xi^2 = f_\xi$ and $f_\xi \upharpoonright (\mathbb{N} \setminus \text{fix}(f_\xi)) \subset^* f_\eta$;
3. $f_\xi[x_t] = x_t$ and $\text{fix}(f_\xi) \cap x_t$ is infinite;
4. $f_\xi([n, m)) = [n, m)$ for $n < m$ both in $A_\xi$.

Say that $\mathcal{T}$, a tower of $T$-involutions, is **full** if $K = K_\mathcal{T} = \bigcap \{ \text{fix}(f_\xi)^* : \xi \in W \}$ is a tie-set with $\beta \mathbb{N} \setminus \mathbb{N} = A \upharpoonright^K B$ where $A = K \cup \bigcup \{ \text{min}(f_\xi)^* : \xi \in W \}$ and $B = K \cup \bigcup \{ \text{max}(f_\xi)^* : \xi \in W \}$.

If $\mathcal{T}$ is a tower of $T$-involutions, then there is a natural involution $F_\mathcal{T}$ on $\bigcup_{\xi \in W} (\mathbb{N} \setminus \text{fix}(f_\xi))^*$, but this $F_\mathcal{T}$ need not extend to an involution on the closure of the union - even if the tower is full.

In this section we prove the following theorem.

**Theorem 3.1.** Assume $\text{GCH}$ and that $\Lambda$ is a set of regular uncountable cardinals such that for each $\lambda \in \Lambda$, $T_\lambda$ is a $<\lambda$-closed $\lambda^+$-Souslin tree. Let $T$ denote the tree sum of $\{ T_\lambda : \lambda \in \Lambda \}$. There is forcing extension in which there is $\mathcal{T}$, a full tower of $T$-involutions, such that the associated tie-set $K$ has $\mathfrak{b}\delta$-type $(c, c)$ and such that for each $\lambda \in \Lambda$, there is a tie-set $K_\lambda$ of $\mathfrak{b}\delta$-type $(\lambda^+, \lambda^+)$ such that $F_\mathcal{T}$ does induce an involution on $K_\lambda$ with a singleton fixed point set $\{ z_\lambda \} = K \cap K_\lambda$. Furthermore, for $\mu \leq \lambda < c$, if $\mu \neq \lambda$ or $\lambda \notin \Lambda$, then there is no tie-set of $\mathfrak{b}\delta$-type $(\mu, \lambda)$.

**Question 3.1.** Can the tower $\mathcal{T}$ in Theorem 3.1 be constructed so that $F_\mathcal{T}$ extends to an involution of $\beta \mathbb{N} \setminus \mathbb{N}$ with $\text{fix}(F) = K_\mathcal{T}$?

We introduce $T$-tower extending forcing.

**Definition 3.2.** If $\mathcal{T} = \{ (A_\xi, f_\xi) : \xi \in W \}$ is a tower of $T$-involutions and $Y$ is a subset of $\mathbb{N}$, we define the poset $Q = Q(\mathcal{T}, Y)$ as follows. Let $E_Y$ be the (possibly empty) set of minimal elements of $T$ such that there is some finite $H \subset W$ such that $x_t \cap Y \cap \bigcap_{\xi \in H} \text{fix}(f_\xi)$ is finite. Let $D_Y = E_Y^\perp = \{ t \in T : (\forall s \in E_Y) t \perp s \}$. A tuple $q \in Q$ if $q = (a^q, f^q, T^q, H^q)$ where:
We define $p < q$ if $n^p \leq n^q$, and for $t \in T^p$ and $i \in [n^p, n^q)$:

1. $a^p = a^q \cap n^p$, $T^p \subset T^q$, and $H^p \subset H^q$,
2. $a^q \setminus a^p \subset A_{a^p}$,
3. $f^q(i) \neq i$ implies $f^q(i) = f_{a^p}(i)$,
4. $f^q((n, m)) = [n, m)$ for $n < m$ both in $a^q \setminus a^p$,
5. $f^q(x_t \cap [n^p, n^q)) = x_t \cap [n^p, n^q)$,
6. if $t \in D^p$ and $i \in x_t \cap \text{fix}(f^q)$, then $i \in Y$

It should be clear that the involution $f$ introduced by $Q(\xi, Y)$ satisfies that for each $t \in D_Y$, $\text{fix}(f) \cap x_t \subset^* Y$, and, with the help of the following density argument, that $\xi \cup \{(\gamma, A, f)\}$ is again a tower of $T$-involutions where $A$ is the infinite set introduced by the first coordinates of the conditions in the generic filter.

**Lemma 3.2.** If $W \subset \gamma$, $Y \subset \mathbb{N}$, and $\xi = \{(A_\xi, f_\xi) : \xi \in W\}$ is a tower of $T$-involutions and $p \in Q(\xi, Y)$, then for any $\tilde{T} \in [T]^\omega$, $\zeta \in W$, and any $m \in \mathbb{N}$, there is a $q < p$ such that $n^q \geq m$, $\zeta \in H^q$, $T^q \supset \tilde{T}$, and $\text{fix}(f^q) \cap (x_t \setminus n^q)$ is not empty for each $t \in T^p$.

**Proof.** Let $\beta$ denote the maximum $\alpha^p$ and $\zeta$ and let $\eta$ denote the minimum. Choose any $n^\eta \in A_{a^q} \setminus m$ large enough so that

1. $f_{a^p}[x_t \setminus n^q] = x_t \setminus n^q$ for $t \in \tilde{T}$,
2. $f_{\eta}[\mathbb{N} \setminus (n^q \cup \text{fix}(f_\eta))] \subset f_\beta$,
3. $A_\beta \setminus A_\eta$ is contained in $n^q$,
4. $n^q \cap [i]_{T^p} \cap \text{fix}(f_{a^p})$ is non-empty for each $i \in \mathbb{N}$ such that $[i]_{T^p}$ is in the finite set $\{[i]_{T^p} : i \in \mathbb{N}\} \setminus \text{fin}(T^p)$,
5. if $i \in x_t \cap n^q \setminus n^p$ for some $t \in D_Y \cap T^p$, then $Y$ meets $[i]_{T^p} \cap n^q \setminus n^p$ in at least two points.

Naturally we also set $H^q = H^p \cup \{\zeta\}$ and $T^q = T^p \cup \tilde{T}$. The choice of $n^q$ is large enough to satisfy (3), (4), (5) and (6) of Definition 3.2. We will set $a^q = a^p \cup \{n^q\}$ ensuring (1) of Definition 3.2. Therefore for any $f^q \supset f^p$ which is an involution on $n^q$, we will have that $q = (a^q, f^q, T^q, H^q)$ is in the poset. We have to choose $f^q$ more carefully to ensure that $q \leq p$. Let $S = [n^p, n^q) \cap \text{fix}(f_{a^p})$, and $S' = [n^p, n^q) \setminus S$. We choose $\tilde{f}$ an involution on $S$ and set $f^q = f^p \cup (f_{a^p} \upharpoonright S') \cup \tilde{f}$. We leave
it to the reader to check that it suffices to ensure that \( \bar{f} \) sends \([i]_{Tp} \cap S\) to itself for each \( t \in T^p \) and that \( \text{fix}(\bar{f}) \cap x_t \subset Y \) for each \( t \in T^p \cap D_Y \).

Since the members of \( \{[i]_{Tp} \cap S : i \in \mathbb{N}\} \) are pairwise disjoint we can define \( \bar{f} \) on each separately.

For each \([i]_{Tp} \cap S\) which has even cardinality, choose two points \( y_i, z_i \) from it so that if there is a \( p \in D_Y \cap T^p \) such that \([i]_{Tp} \subset x_t\), then \( \{y_i, z_i\} \subset Y \). Let \( \bar{f} \) be any involution on \([i]_{Tp} \cap S\) so that \( y_i, z_i \) are the only fixed points. If \([i]_{Tp} \cap S\) has odd cardinality then choose a point \( y_i \) from it so that if \([i]_{Tp}^p\) is contained in \( x_t \) for some \( t \in D_y \cap T^p \), then \( y_i \in Y \cap [i]_{Tp} \cap S \). Set \( \bar{f}(y_i) = y_i \) and choose \( \bar{f} \) to be any fixed-point free involution on \([i]_{Tp} \cap S \setminus \{y_i\}\).

Let \( P_\gamma \) now be the finite support iteration defined as in Definition 2.6 except for two important changes. For \( \gamma \in C \), we replace \( T\)-splitting towers by the obvious inductive definition of towers of \( T \)-involutions when we replace the posets \( \dot{Q}(S_{\mathcal{C}\gamma}, \dot{Y}_\gamma) \) by \( \dot{Q}(\Sigma_{\mathcal{C}\gamma}, \dot{Y}_\gamma) \). For \( \gamma \notin C \) we require that \( \Vdash_{P_\gamma} \dot{Q}_\gamma \) is \( \sigma \)-linked.

Special (parity) properties of the family \( \{x_t : t \in T\} \) are needed to ensure that \( \Vdash_{P_\gamma} \dot{Q}(S_{\mathcal{C}\gamma}, \dot{Y}_\gamma) \) is ccc even for cases when \( \text{cf}(\gamma) \) is not \( \omega_1 \).

The proof of Theorem 3.1 is virtually the same as the proof of Theorem 2.11 (so we skip) once we have established that the iteration is ccc.

**Lemma 3.3.** For each \( \gamma \in C \), \( P_{\gamma+1} \) is ccc.

**Proof.** We again define \( P_\alpha^* \) to be those \( p \in P_\alpha \) for which there is an \( n \in \mathbb{N} \) such that for each \( \beta \in \text{dom}(p) \cap C \), there are \( n \in a^\beta \subset n+1 \), \( f^\beta \in n^n \), \( T^\beta \in [T]^{<\omega} \), and \( H^\beta = \text{dom}(p) \cap C \cap \beta \) such that \( p \upharpoonright \beta \Vdash_{P_\beta} p(\beta) = (a^\beta, f^\beta, T^\beta, H^\beta) \). However, in this proof we must also make some special assumptions in coordinates other than those in \( C \). For each \( \xi \in \gamma \setminus C \), we fix a collection \( \{\dot{Q}(\xi, n) : n \in \omega\} \) of \( P_\xi \)-names so that

\[
1 \Vdash_{P_\xi} \dot{Q}_\xi = \bigcup_n \dot{Q}(\xi, n) \text{ and } (\forall n) \dot{Q}(\xi, n) \text{ is linked.}
\]

The final restriction on \( p \in P_\alpha^* \) is that for each \( \xi \in \alpha \setminus C \), there is a \( k_\xi \in \omega \) such that \( p \upharpoonright \xi \Vdash_{P_\xi} \text{"} p(\xi) \in \dot{Q}(\xi, k_\xi) \".

Just as in Lemma 2.11, Lemma 3.2 can be used to show by induction that \( P_\alpha^* \) is a dense subset of \( P_\alpha \). This time though, we also demand that \( \text{dom}(f^{p(0)}) = n \times T^{p(0)} \) is such that \( T^\beta \subset T^{p(0)} \) for all \( \beta \in \text{dom}(p) \cap C \) and some extra argument is needed because of needing to decide values in the name \( \dot{Y}_\gamma \) as in the proof of Lemma 3.2. Let \( p \in P_{\beta+1} \) and assume that \( P_\beta^* \) is dense in \( P_\beta \). By density, we may assume that \( p \upharpoonright
$\beta \in P^*_\beta$, $H^{p(\beta)} \subseteq \dom(p)$, $T^{p(\beta)} \subseteq T^{p(0)}$, and that $p \upharpoonright \beta$ has decided the members of the set $D_\beta \cap T^{p(\beta)}$. We can assume further that for each $t \in D_\beta \cap T^{p(\beta)}$, $p \upharpoonright \beta$ has forced a value $y_t \in \hat{Y}_\beta \cap x_t \setminus \bigcup \{ x_s : s \in T^p \text{ and } s \not\subseteq t \}$ such that $y_t > n^{p(\beta)}$. We are using that $T$ is not finitely branching to deduce that if $t \in D_\beta$, then $p \upharpoonright \beta \Vdash_{P_\beta}$ "$\hat{Y}_\beta \cap x_t \setminus \bigcup \{ x_s : s \in T^p \text{ and } s \not\subseteq t \}$ is non-empty" (which follows since $\hat{Y}_\beta$ must meet $x_s$ for each immediate successor $s$ of $t$). Choose any $m$ larger than $y_t$ for each $t \in T^{p(\beta)}$. Without loss of generality, we may assume that the integer $n^*$ witnessing that $p \upharpoonright \beta \in P^*_\beta$ is at least as large as $m$ and that $n^* \in \bigcap_{\xi \in H^{p(\beta)} A_\xi}$. Construct $\bar{f}$ just as in Lemma 3.2 except that this time there is no requirement to actually have fixed points so one member of $\hat{Y}_\beta$ in each appropriate $[i]_{T^{p(\beta)}}$ is all that is required. Let $\zeta = \max(\dom(p) \cap \beta)$. No new forcing decisions are required of $p \upharpoonright \beta$ in order to construct a suitable $\bar{f}$, hence this shows that $p \upharpoonright \beta \cup \\{ (\beta, q) \}$ (where $q$ is constructed below $p(\beta)$ as in Corollary 3.2) in which $H^{p(\zeta)} \cup \{ \zeta \}$ is add to $H^q$ is the desired extension of $p$ which is a member of $P^*_{\beta+1}$.

Now to show that $P_{\gamma+1}$ is ccc, let $\{ p_\alpha : \alpha \in \omega_1 \} \subseteq P^*_{\gamma+1}$. Clearly we may assume that the family $\{ p_\alpha(0) : \alpha \in \omega_1 \}$ are pairwise compatible and that there is a single integer $n$ such that, for each $\alpha \in \omega_1$, $\dom(p_\alpha(0)) = n \times T^\alpha$ for some $T^\alpha \subseteq \gamma$. Also, we may assume that there is some $(a, h)$ such that, for each $\alpha$,

$$p_\alpha \upharpoonright \gamma \Vdash_{P_\gamma} " p(\gamma) = (a, h, T^\alpha, H^\alpha)"$$

where $H^\alpha = \dom(p_\alpha) \cap C \cap \gamma$.

The family $\{ \dom(p_\alpha) \cap \gamma : \alpha \in \omega_1 \}$ may be assumed to form a $\Delta$-system with root $R$. For each $\xi \in R$, we may assume that, if $\xi \notin C$, there is a single $k_\xi \in \omega$ such that, for all $\alpha$, $p_\alpha \upharpoonright \xi \Vdash_{P_\xi} " p_\alpha(\xi) \in \hat{Q}(\xi, k_\xi)"$, and if $\xi \in C$, then there is a single $(a_\xi, h_\xi)$ such that $p_\alpha \upharpoonright \xi \Vdash_{P_\xi} " p_\alpha(\xi) = (a_\xi, h_\xi, T^\alpha, H^\alpha \cap \xi)"$. For convenience, for each $\xi \notin C$ let $\tp_\xi$ be a $P_\xi$-name of a function from $\omega \times \hat{Q}_\xi^2$ such that, for each $k \in \omega$,

$$1 \Vdash_{P_\xi} " \tp_\xi(k, q, q') \leq q, q' \ (\forall q, q' \in \hat{Q}(\xi, k))".$$

Fix any $\alpha < \beta < \omega_1$ and let $H = H^\alpha \cup H^\beta$. Recall that $p_\alpha(0)$ and $p_\beta(0)$ are compatible. Recursively define a $P_\xi$-name $q(\xi)$ for $\xi \in \omega_1$
Now we check that \( C \) in this show that condition (4) of Definition 3.2 will hold in all coordinates \( q \) that \( H \) in and \( A \max(\ q \inductive\ hypothesis\ above). \) It follows then that \( "q(\xi) + 1. \)

By the definition of the ordering on \( \dot{q} \), \( \xi \in dom(\ p_\alpha) \ \setminus\ (R \cup C) " \).

Remark 2. We also prove, by induction on \( \xi \), that \( q \upharpoonright \xi \) forces that for \( \eta < \delta \) both in \( H \cap \xi \) and \( t \in T^\alpha \cup T^\beta \), \( f_\delta[x_t \setminus n] = x_t \setminus n, f_\eta \upharpoonright (N \setminus (fix(f_\eta) \cup n)) \subset f_\delta \) and \( A_\delta \setminus n \subset A_\eta. \)

Given \( \xi \in H \) and the assumption that \( q \upharpoonright \xi \in P_\xi \), and \( \alpha = \alpha^{\xi(q)} = \max(H \cap \xi), \) condition (3), (5), and (6) of Definition 3.2 hold by the inductive hypothesis above. It follows then that \( q \upharpoonright \xi \models P_\xi "q(\xi) \in \dot{Q}_\xi " \).

By the definition of the ordering on \( \dot{Q}_\xi \), given that \( H \cap \xi = H^{\alpha(\xi)} \) and \( T^\alpha \cup T^\beta = T^{\alpha(\xi)} \), it follows that the inductive hypothesis then holds for \( \xi + 1. \)

It is trivial for \( \xi \in dom(q) \setminus C, \) that \( q \upharpoonright \xi \in P_\xi \) implies that \( q \upharpoonright \xi \models P_\xi "q(\xi) \in \dot{Q}_\xi " \). This completes the proof that \( q \in P_\gamma + 1, \) and it is trivial that \( q \) is below each of \( p_\alpha \) and \( p_\beta. \)

Remark 1. If we add a trivial tree \( T_1 \) to the collection \( \{T_\lambda : \lambda \in \Lambda\} \) (i.e. \( T_1 \) has only a root), then the root of \( T \) has a single extension which is a maximal node \( t, \) and with no change to the proof of Theorem 3.1 one obtains that \( F \) induces an automorphism on \( x^*_t \) with a single fixed point. Therefore, it is consistent (and likely as constructed) that \( \beta \in \setminus \) \( N \) will have symmetric tie-points of type \( (\xi, \xi) \) in the model \( V[G \cap P] \) and \( V[G]. \)

Remark 2. In the proof of Theorem 2.1 it is easy to arrange that each \( K_\lambda (\lambda \in \Lambda) \) is also \( K_{\xi_\lambda} \) for a \( (T_\lambda\text{-generic}) \) full tower, \( \xi_\lambda, \) of \( \in\text{-}\)involutions. However the generic sets added by the forcing \( P \) will prevent this tower of involutions from extending to a full involution.

4. Questions

In this section we list all the questions with their original numbering.
Question 1.1. Can there be a tie-point in $\beta\mathbb{N} \setminus \mathbb{N}$ with $\delta$-type $(\kappa, \lambda)$ with $\kappa \leq \lambda$ less than the character of the point?

Question 1.2. Can $\beta\mathbb{N} \setminus \mathbb{N}$ have tie-points of $\delta$-type $(\omega_1, \omega_1)$ and $(\omega_2, \omega_2)$?

Question 1.3. Does $p > \omega_1$ imply there are no tie-points of $b$-type $(\omega_1, \omega_1)$?

Question 1.4. If $F$ is an involution on $\beta\mathbb{N} \setminus \mathbb{N}$ such that $K = \text{fix}(F)$ has empty interior, is $K$ a (symmetric) tie-set?

Question 1.5. Is there some natural restriction on which compact spaces can (or can not) be homeomorphic to the fixed point set of some involution of $\beta\mathbb{N} \setminus \mathbb{N}$?

Question 1.6. If $F$ is an involution of $\mathbb{N}^*$, is the quotient space $\mathbb{N}^*/F$ (in which each $\{x, F(x)\}$ is collapsed to a single point) a homeomorphic copy of $\beta\mathbb{N} \setminus \mathbb{N}$?

Question 3.1. Can the tower $\mathcal{T}$ in Theorem 3.1 be constructed so that $F_{\mathcal{T}}$ extends to an involution of $\beta\mathbb{N} \setminus \mathbb{N}$ with $\text{fix}(F) = K_{\mathcal{T}}$?

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__Department of Mathematics, Rutgers University, Hill Center, Piscataway, New Jersey, U.S.A. 08854-8019__

__Current address__: Institute of Mathematics, Hebrew University, Givat Ram, Jerusalem 91904, Israel

__E-mail address__: shelah@math.rutgers.edu