Computational Improvements to Matrix Operations

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Abstract

An alternative to the matrix inverse procedure is presented. Given a bit register which is arbitrarily large, the matrix inverse to an arbitrarily large matrix can be performed in $O(N^2)$ operations, and to matrix multiplication on a vector in $O(N)$. This is in contrast to the usual $O(N^3)$ and $O(N^2)$. A finite size bit register can lead to speeds up of an order of magnitude in large matrices such as $500 \times 500$. The FFT can be improved from $O(N \ln N)$ to $O(N)$ steps, or even fewer steps in a modified butterfly configuration.
The matrix inverse is the backbone to the modern STAP process; this has to be performed every time a set of data is trained in the field of view. The complexity is one of the primary origins of the cost to the large computing required to process the data. Any improvement in the complexity to performing the matrix inverse is a desirable.

Typically the matrix inverse to $M_{N \times N}$ is performed in $O(N^3)$ operations. There are several variants, including the LU and QR variants. The LU diagonalization is twice as fast as the (Guass) QR form [1]. The LU factorization requires splitting the matrix into the product of upper and lower diagonal matrices $M = LU$. The inverse is performed by inverting the respective components. The QR form requires splitting the matrix $M$ into an orthogonal component $Q$ times its projection $R$.

There is a simplification of the matrix inverse by grouping the entries of the matrix $M_{ij}$ into larger numbers. For example, the first row of the matrix has elements $N_1, N_2, \ldots$. A larger number can be built of these entries by placing the digits into one number $N_1N_2\ldots$. For the computing purposes a zero number $N_0$ with as many digits as the entries $M_{ij}$ is required, and the rows are grouped into the number $N_1N_0N_2N_0\ldots$. The rows of the matrix are now used as a single number in the diagonalization procedure in the LU factorization.

For example, the zeroing out of the matrix’s first column requires using the first row; a number $b_{i1}$ is used to multiply $M_{1j}$ so that $M_{i1} = -b_{i1}M_{i1}/M_{11}$. This number multiplies the entire row of the matrix $M_{ij}$ and is added to the $i$th row. In doing so, a set of zeroes is produced in the first column of the matrix $M$; the numbers $b_{i1}$ are placed in the lower diagonal factor matrix of $L$. The procedure is iterated using the diagonal elements $M_{jj}$ to construct upper diagonal and lower matrices $L$ and $U$. The computational cost of one of the multiplications is $N^2$ due to the $N$ elements in the row and the multiplications and additions to the $N$ elements in the column.

In using the larger number the $N^2$ operations to create a zero column can be reduced to $N$ operations. This requires the zero number $N_0$ to have a sufficient number of digits so that

$$b(N_1N_0N_2N_0\ldots) = (bN_1)N_0(bN_2)N_0\ldots$$  \hspace{1cm} (1)

$$(N_1N_0N_2N_0\ldots) + (M_1M_0M_2M_0\ldots) = (N_1 + M_1)N_0(N_2 + M_2)N_0\ldots$$  \hspace{1cm} (2)

as one number. The bit register in the processor has to be able to handle these two operations, multiplication by a scalar and addition. The $N$ operations in the bit
register to treat the multiplication and addition of the original row has been reduced
to one multiplication and one addition. The numbers $b$ which multiply the larger
numbers $N_1N_0N_2N_0\ldots$ are collected into the lower diagonal matrix $L$.

A separate matrix is required to discern if the subtraction of a positive number
to this number is negative or positive. For example,

$$(N_1N_0N_2N_0\ldots) - (M_1M_0M_2M_0\ldots)$$

could have negative entries $N_i - M_i$ but the absolute value is used in the composition
of the number $N - M$. The subtraction process does not work well in the procedure,
and the numbers are separated into $N_i - M_i$ independently.

The processing of using the larger numbers instead of the smaller numbers is
that typical processes such as the matrix inverse and the FFT can be reduced in
complexity from $\mathcal{O}(N^3)$ and $N \ln N$ to $\mathcal{O}(N^2)$ and $N$.

**STAP Example**

The use of spacetime adaptive processing requires the training of data using a
covariance matrix. This matrix is canonically symmetric in the acquisition of data,
satisfying the multiplicative product $X = x_i x_j$. The inverse of the covariance matrix
is unwieldy, being performed in $\mathcal{O}(N^3)$ steps, but must be performed in conventional
STAP processes and signal location.

The product $x_i x_j$ represents a probability distribution, with positive entries. An
alternative matrix satisfies $X_{ij} = -X_{ij}$ with positive entries along the diagonal, is
far more convenient in the matrix inverse procedure. The inverse of the latter matrix
can be performed theoretically in $\mathcal{O}(N^2)$ steps. The limitation is set by the number
of bits in the bit register; the $\mathcal{O}(N^2)$ arises from an arbitrarily large bit register.

Consider the LU reduction of the alternative covariance matrix $\tilde{X}$. The process
of adding the rows to null the lower left triangular portion of $U$ requires only adding
the numbers $M_1/N_1 \ast (N_1N_0N_2N_0\ldots)$ to $(M_1M_0M_2M_0\ldots)$. For example the second
row modeled by the single number $(M_1M_0M_2M_0\ldots)$ has a negative entry for $M_1$ and
positive entries $M_j$. The addition of $M_1/N_1 \ast N_j$ to the $M_j$ occurs in one operation
theoretically due to the size of the individual numbers. The $M_1/N_1$ is stored in the
lower left triangular matrix $L$ in the process of the LU factorization. The operation
is repeated to first nullify the left column of $U$ (except the diagonal component), and
then the process is repeated for the other columns. As there are $N^2$ components in
the matrix the LU factorization of \( \tilde{X} \) requires \( N^2 \) steps; this is considerably faster than \( N^3 \) when \( N \) is of the order of a thousand or more.

The question is whether data can be trained with the alternate to the covariance matrix. The \( \tilde{X} \) contains the same information but with minus signs placed to partially antisymmetrize. It appears clear by the conventional use of stap in locating signals, and in eliminating noise, that this should be possible.

**Matrix Multiplication**

The same use of the bit register and organizing the rows of the matrix in terms of whole numbers can be used to simplify matrix multiplication. Usual multiplication of a matrix by a vector requires \( \mathcal{O}(N^2) \) steps. This can be reduced to \( \mathcal{O}(N) \) with a reordering of the matrix and vector information.

Consider all positive entries in the matrix \( M \) and all positive entries in the vector \( v \). The vector consists of one number \( (v_1v_0v_2v_0\ldots) \), and the columns of the matrix consist of individual numbers \( \tilde{M}_j = (M_1M_0M_2M_0\ldots) \). The multiplication of one column by the vector element is accomplished in one step: \( v_i \cdot \tilde{M}_j \), with the element of the vector used. This results in \( v_i\tilde{M}_j = (v_1M_1M_0v_iM_2M_0\ldots) \). The total matrix multiplication is then accomplished by adding the previous multiplications: \( \sum v_i\tilde{M}_j \). The vector resultant from the matrix multiplications are stored in the decomposition of \( \sum v_i\tilde{M}_j \). The total operations to matrix multiply is \( 2N \) steps, and not \( \mathcal{O}(N^2) \). This reduction can be substantial for numbers \( N \) of the order of a thousand.

The previous example pertains to the matrix \( M \) and vector consisting of positive values. Minus signs in the matrix can be incorporated very simply by separating the elements \( \tilde{M}_j = (M_1M_0M_2M_0\ldots) \) into the respective positive entries and negative entries: \( \tilde{M}_j^+ \) and \( \tilde{M}_j^- \). A vector with positive entries can be used to multiply both the \( v_i\tilde{M}_j = v_i\tilde{M}_j^+ + v_i\tilde{M}_j^- \) entries. The addition of the column vectors is achieved by adding separately the positive entries and the negative entries: \( \sum v_i\tilde{M}_j^+ \) and \( \sum v_i\tilde{M}_j^- \). Then the individual entries of the two terms are required to be subtracted. The net total number of steps is \( \mathcal{O}(N) \).

A simple application of the of the matrix multiplication of a vector is the fast fourier transform. The butterfly reduction of the usual multiplication of the vector lowers the \( \mathcal{O}(N^2) \) to \( \mathcal{O}(N \ln N) \) steps. Theoretically, by separating the matrix into real and complex parts, with the minus signs handled separately, can achieve a theoretical FFT in \( \mathcal{O}(N) \) steps. This exponentially faster than the butterfly configuration.

The butterfly configuration can also be analyzed with subtle memory allocation
of the data transfer to an approximate \( \ln N \) operations. The data has to be reordered in traversing the butterfly.

**Conclusions**

The theoretical improvements in the matrix inverse from \( \mathcal{O}(N^3) \) steps to \( \mathcal{O}(N^2) \) steps, and matrix times vector from \( \mathcal{O}(N^2) \) to \( \mathcal{O}(N) \) steps has profound impact in computational science. Unfortunately, a bit register of a large size is required. In conventional computing registers, there is a waste depending on the data size. For example, a 256 bit register handling 32 bit data can be optimized by a factor of 8, which is still substantial.

Ideally, the theoretical drop of the matrix inverse and the matrix multiplication by a factor of \( N \) is suitable for more advanced computing apparatus. The theoretical bounds in the optimization are achieved with large bit registers, which can be designed in several contexts.
References

[1] Gene H. Golub and Charles F. Van Loan, *Matrix Computations*, Johns Hopkins Studies in Mathematical Sciences, 1996.