Effective Quark Interactions from QCD

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Abstract

We propose a new method for an analytical, non-perturbative computation of effective quark interactions from QCD. It is based on an exact flow equation which describes the scale dependence of the effective average action for quarks in presence of gluons.
1 Introduction

Quantum chromodynamics as the theory of strong interactions shows very different facets at short and long distances. The high momentum behaviour is governed by asymptotic freedom \[1\] and the relevant degrees of freedom are quarks and gluons. In contrast, the particles which are observed at large length scales (larger than 1 fm) are mesons and hadrons. The interactions of the pseudoscalar mesons are modeled by chiral perturbation theory \[2\]. The corresponding nonlinear $\sigma$-model shares the flavour symmetries of perturbative QCD. It is believed that the free phenomenological parameters of this model can ultimately be computed from the action of QCD, but this is not a simple task.

Recently, the transition from quark degrees of freedom to meson degrees of freedom has been described by a nonperturbative flow equation \[3\]. It is based on the concept of the effective average action $\Gamma_k$ \[4\] for which only quantum fluctuations with (covariant) momenta $q^2 > k^2$ are integrated out. The average action is the effective action for averages of fields. It acts like a microscope by which we can look at the theory at different length scales, with a “resolution” given by the scale $k^{-1}$. For $k = 0$ one recovers the usual effective action, i.e. the generating functional of the 1PI Green functions, whereas for $k \to \infty$ $\Gamma_k$ equals the classical action. The dependence of $\Gamma_k$ on the scale $k$ is described by an exact nonperturbative flow equation \[5\]. By the introduction of composite fields for the mesons the original equation formulated for quarks can be transmuted into an equivalent exact flow equation involving also the mesons \[3\].

In a first attempt the chiral condensate $< \bar{\psi} \psi >$ and the pion decay constant $f_\pi$ were computed along these lines \[3\]. In this approach the gluons have not been considered explicitly. Their effect was encoded in a phenomenologically motivated four-quark interaction, which may be thought as the result of integrating out the gluons in the defining functional integral. If one aims at a computation of the effective parameters of chiral perturbation theory from the QCD action, this shortcut has to be removed. A full computation should start at short distances with the quark-gluon description of QCD and systematically account for the quantum fluctuations of the gluon field.

Conceptually, one may integrate out the gluons at once and end with an effective
quark theory. In practice, this seems almost impossible: A determination of the resulting complicated nonlocal quark interactions requires more or less a complete solution of QCD. On the other hand, the form of the flow equation at a given scale \( k \) is not sensitive to details of physics at momentum scales much below \( k \). Only the effective propagators and vertices for modes with momenta \( q^2 \approx k^2 \) play a role for this equation. A typical scale where the light mesons form is around 700 MeV and therefore substantially higher than the confinement scale. This situation further improves for problems involving heavier quarks as for example the charmed or beauty mesons. One may hope that very detailed features of confinement are not needed for an understanding of the mesons. An appropriate tool would therefore be a method which only integrates out the gluon fluctuations with momenta \( q^2 > k^2 \) instead of addressing the much more complicated problem of integrating out all gluon fluctuations at once. Formally, it is easy to do this at a given scale \( k_1 \). One needs to compute the effective action for quarks and gluons \( \Gamma_{k_1}[\psi, A] \) at this scale. Solving the classical field equation for the gluon field \( A \) in dependence on \( \psi \) and inserting this classical solution into \( \Gamma_{k_1}[\psi, A] \) yields exactly an effective average action \( \Gamma_{k_1}[\psi] \) involving only the quark fields. The quark-effective action \( \Gamma_{k_1}[\psi] \) may then be used as an initial value for solving a pure fermionic flow equation for \( k < k_1 \). Obviously, the shortcoming of such an approach is the complete omission of the effects of gluon fluctuations with \( q^2 < k_1^2 \). Choosing a different scale \( k_2 \) for eliminating the gluons will lead to a different result. Such a sharp transition between the quark-gluon system and a description involving only quarks necessarily introduces a certain degree of arbitrariness.

We propose here a more refined method which changes the classical solution for the gluon field in the course of the evolution towards lower \( k \), thus reflecting the change in the form of \( \Gamma_{k}[\psi, A] \). The result is a smooth procedure for integrating out the gluons, where at every scale \( k \) all gluon fluctuations with \( q^2 > k^2 \) are included. Nevertheless, we obtain a flow equation for the quark effective average action \( \Gamma_{k}[\psi] \), where gluon fields do not appear explicitly. As a consequence of the inclusion of contributions from additional gluon fluctuations as the scale is lowered, correction terms appear in the flow equation for the quark interactions. In particular, we choose here a formulation where the \( k \)-dependent classical solution for \( A \) as a functional of \( \psi \) includes an effective infrared cutoff \( \sim k \). As a consequence, the only nonlocalities
in $\Gamma_k[\psi]$ concern length scales shorter than $k^{-1}$. For the fermionic low momentum modes $\Gamma_k[\psi]$ is an effectively local action. For example, a derivative expansion is meaningful for $q^2 \ll k^2$. The expected nonlocalities arising from the complete elimination of gluon fields (for example a four-quark interaction $\sim \frac{1}{q^2}$ appearing already in the Born approximation) build up only step by step as $k$ is lowered to zero. As a result of this method we will end with an exact nonperturbative flow equation for the scale dependence of $\Gamma_k[\psi]$. Approximations will be needed to solve this equation but they are not limited to perturbative concepts. The correction terms reflecting the gluon fluctuations require limited knowledge about the effective gluon propagator and vertices. It is hoped that rather crude approximations for the gluonic vertices can already lead to satisfactory results.

In sect. 2 we first demonstrate our formalism for a simple model of two scalar fields. The flow of the effective average action for one of the scalar fields obtains by integrating out the other scalar field at any scale $k$. Subsequently this is generalized to quarks and gluons. We also give a first demonstration how this formalism describes the flow of the two- and four-point function in the effective quark theory. The special case of heavy quarks is addressed in sect. 3. Here we argue that the evolution of the gauge field propagator is needed in this limit. In sect. 4 we compute the corresponding flow equation and discuss the scale dependence of the gluon propagator. In sect. 5 we collect all the ingredients needed for the flow equation in case of light quarks. Finally, our conclusions are contained in sect. 6.

2 Reduction of degrees of freedom

First we consider for simplicity two types of scalar fields, $\varphi$ and $\psi$. We want to develop a formalism how to translate evolution equations for the effective average action for $\varphi$ and $\psi$ into corresponding equations involving only $\psi$. The reader may associate $\varphi$ with the gluon fields and $\psi$ with the quark fields. Our aim is then the construction of the effective average action for quarks out of the coupled quark-gluon system. This amounts to integrating out the gluonic degrees of freedom represented in the simplified model by $\varphi$. We start with the scale-dependent generating func-
tional for the connected Green functions

\[ W_k[J, K] = \ln \int D\varphi' D\psi' \exp \{-S[\varphi', \psi'] + \Delta_k^{(\varphi)} S[\varphi'] + \Delta_k^{(\psi)} S[\psi'] - J^\dagger \varphi' - K^\dagger \psi'\} \]

Here we denote the degrees of freedom contained in \( \varphi' \) (for example the Fourier modes) by \( \varphi'^\alpha \) and similar for \( \psi', J \) and \( K \), with \[ J^\dagger \varphi' = J^\dagger \varphi'^\alpha, \quad K^\dagger \psi' = K^\dagger \psi'^\beta. \]

We have introduced an infrared cutoff quadratic in the fields

\[ \Delta_k^{(\varphi)} S[\varphi'] = \frac{1}{2} \varphi'^\dagger R_k^{(\varphi)} \varphi' \]

and similar for \( \psi \). This suppresses the contribution of fluctuations with small momenta \( q^2 < k^2 \) to the functional integral (2.1). Typically \( R_k^{(\varphi)} \), \( R_k^{(\psi)} \) are functions of \( q^2 \) as, for example,

\[ R_k^{(\varphi)} = \frac{Z_k q^2 \exp \left( -\frac{q^2}{k^2} \right)}{1 - \exp \left( -\frac{q^2}{k^2} \right)} \]

which acts like a mass term \( R_k^{(\varphi)} \sim Z_k k^2 \) for \( q^2 \ll k^2 \). The effective average action \( \Gamma_k[\varphi, \psi] \) is related to the Legendre transform of \( W_k[J, K] \)

\[ \tilde{\Gamma}_k[\varphi, \psi] = -W_k[J, K] + J^\dagger \varphi + K^\dagger \psi \]

by subtracting the infrared cutoff term

\[ \Gamma_k[\varphi, \psi] = \tilde{\Gamma}_k[\varphi, \psi] - \Delta_k^{(\varphi)} S[\varphi] - \Delta_k^{(\psi)} S[\psi] \]

For \( k \to 0 \) the infrared cutoff \( \Delta_k S = \Delta_k^{(\varphi)} S + \Delta_k^{(\psi)} S \) vanishes and \( \Gamma_0 \) is the usual generating function for the 1PI Green functions. Using the quadratic form of \( \Delta_k S \) it is straightforward to derive an exact non-perturbative evolution equation for the dependence of the effective average action on the scale \( k \) \((t = \ln k)\)

\[ \frac{\partial \Gamma_k}{\partial t} = \frac{1}{2} Tr \left\{ (\tilde{\Gamma}_k^{(2)})^{-1} \frac{\partial R_k}{\partial t} \right\} \]

Here \( \tilde{\Gamma}_k^{(2)} = \Gamma_k^{(2)} + R_k \) and the inverse propagator \( \Gamma_k^{(2)} \) is the second functional derivative of \( \Gamma_k \) with respect to the fields. The matrix \( R_k = R_k^{(\varphi)} + R_k^{(\psi)} \) is block diagonal in \( \varphi \) and \( \psi \) spaces. The presence of the infrared cutoff \( R_k \) in \( \tilde{\Gamma}_k^{(2)} \) guarantees infrared finiteness for the momentum integral implied by the trace even in case of

\[ ^1 \text{We use indices } \alpha, \alpha' \text{ etc. for } \varphi \text{ and } \beta, \beta' \text{ etc. for } \psi. \]
massless modes. Ultraviolet finiteness is guaranteed by the exponential decay of \( \partial R_k/\partial t \) (2.3). A solution of the flow equation (2.6) interpolates between the classical action for \( k \to \infty \) (or \( k \) equal to some ultraviolet cutoff \( \Lambda \)) and the effective action for \( k \to 0 \).

The generating functional for the connected Green functions for \( \psi \) obtains from (2.1) for \( J = 0 \)

\[
W_k[K] \equiv W_k[J = 0, K] \quad (2.7)
\]

Correspondingly, we may introduce an effective action expressed only in terms of \( \psi \)

\[
\tilde{\Gamma}_k[\psi] = \tilde{\Gamma}_k[\varphi_k[\psi], \psi] \quad (2.8)
\]

\[
\Gamma_k[\psi] = \tilde{\Gamma}_k[\psi] - \Delta_k^{(\psi)} S[\psi] = \Gamma_k[\varphi_k[\psi], \psi] + \Delta_k^{(\psi)} S[\varphi_k[\psi]] \quad (2.9)
\]

by inserting the \( k \)-dependent solution of the field equation

\[
\frac{\partial \tilde{\Gamma}_k[\varphi; \psi]}{\partial \varphi^\alpha} \bigg|_{\varphi_k[\psi]} = 0 \quad (2.10)
\]

This defines \( \varphi_k \) as a \( k \)-dependent functional of \( \psi \). It is easy to verify that \( \tilde{\Gamma}_k[\psi] \) is the Legendre transform of \( W_k[K] \) (2.7). One concludes for \( k \to 0 \) that \( \Gamma_0[\psi] \) is the generating functional for the 1PI Green functions for \( \psi \).

We want to employ the flow equation (2.6) for finding the \( k \)-dependence of \( \Gamma_k[\psi] \). In addition to the corresponding equation for only one type of fields we have here additional contributions from the \( k \)-dependence of \( \Delta_k^{(\psi)} S \) in (2.7). The evolution equation for \( \Gamma_k[\psi] \) can now be obtained by performing in eq. (2.9) a variable transformation which amounts to a shift of \( \varphi \) around \( \varphi_k[\psi], \varphi^\alpha = \varphi^\alpha - \varphi^\alpha_k[\psi] \). One obtains

\[
\frac{\partial \Gamma_k[\psi]}{\partial t} + \frac{1}{2} \left( \tilde{\Gamma}^{(2)}_k[\varphi_k] + R_k^{(\psi)} \right)^{-1}\left( \frac{\partial R_k^{(\psi)}}{\partial t} \right)^{\beta'}_\beta \left( \frac{\partial \tilde{\Gamma}_k[\varphi_k]}{\partial \tilde{\Gamma}_k[\varphi_k]} \right)^{\alpha'}_\alpha + \frac{1}{2} \varphi^\alpha_{k\alpha'} \left( \frac{\partial R_k^{(\psi)}}{\partial t} \right)^{\alpha'}_\alpha \right)
\]

\[
= \frac{1}{2} \varphi^\alpha_{k\alpha'} \left( \frac{\partial R_k^{(\psi)}}{\partial t} \right)^{\alpha'}_\alpha + \frac{1}{2} \left( \tilde{\Gamma}^{(2)}_k[\varphi = \varphi_k, \psi] \right)^{-1}\left( \frac{\partial \Gamma_k^{(2)}}{\partial t} \right)^{\alpha'}_\alpha \quad (2.11)
\]

It is easy to verify that this equation reduces in the limit \( R_k^{(\psi)} = 0 \) to the equivalent of eq. (2.6) for fields \( \psi \) only. The corrections in the first two terms involve the
explicit form of the "classical solution" $\varphi_k[\psi]$. If one is interested in 1PI Green functions for $\psi$ with a given number of external legs one only needs a polynomial expansion of $\varphi_k[\psi]$ up to a given order. For example, the evolution of the term $\sim \psi^4$ in $\Gamma_k[\psi]$ needs the classical solution up to the order $\psi^4$ if the series $\varphi_k[\psi]$ starts with a term quadratic in $\psi$. Additional knowledge of the form of $\Gamma_k[\hat{\varphi}, \psi]$ beyond its value for $\hat{\varphi} = 0$ is only needed for the last correction term in the form of

$$\left( \Gamma^{(2)}_k[0, \psi] \right)^{\alpha'}_{\alpha} = \frac{\partial^2 \Gamma_k[\varphi, \psi]}{\partial \varphi^*_{\alpha} \partial \varphi^\alpha} |_{\varphi=0} + \left( R^{(\varphi)}_k \right)^{\alpha'}_{\alpha}$$

(2.12)

Only the $\psi$-dependence of the effective $\hat{\varphi}$-propagator plays a role for the study of 1PI functions for $\psi$.

Let us next apply the general flow equation (2.11) explicitly to the quark-gluon system. If $\psi$ is a Grassmann variable as appropriate for fermions the matrix $R_k$ in (2.6) becomes

$$R_k = R_k^{(\varphi)} - R_k^{(\psi)}$$

(2.13)

Also $\psi^*$ should be replaced by $\bar{\psi}$ and the index summation over $\beta$ should involve both $\psi$ and $\bar{\psi}$ separately. For the gauge fields we will choose here a formulation with explicit ghost variables in close analogy, but slightly different from the formulation in ref. [6]. This makes our formulation as close as possible to the language of standard perturbation theory. Details can be found in the appendix.

We start with the action including a gauge-fixing term in the background gauge and a corresponding action for the anticommuting ghost fields $\xi, \bar{\xi}$

$$\tilde{S}[\psi', \xi', a; \bar{A}] = S[\psi', A'] + S_{gf}[a; \bar{A}] + S_{gh}[\xi', a; \bar{A}]$$

(2.14)

Here $S$ is a gauge invariant functional of the fermion fields $\psi, \bar{\psi}$ and the gauge field

$$A'_{\mu} = \bar{A}_{\mu} + a_{\mu}.$$  

(2.15)

The background gauge field $\bar{A}_{\mu}$ appears in the gauge fixing and ghost terms

$$S_{gf} = \frac{1}{2\alpha} \int d^d x G^* G$$

(2.16)

$$G^* = (D^\mu [\bar{A}])^* \bar{a}^\mu$$

(2.17)

$$S_{gh} = \int d^d x \xi^\dagger \xi (D^\mu [\bar{A}] D^\mu [\bar{A} + a])^\dagger \xi^\dagger \xi$$

(2.18)
Here $D_{\mu}[\bar{A}]$ is the covariant derivative in the adjoint representation in presence of the background gauge field $\bar{A}$. The generating functional for the connected Green functions is defined as usual

$$W[\eta, \zeta, K; \bar{A}] = \int \mathcal{D}\psi'\mathcal{D}\bar{\psi}'\mathcal{D}\xi'\mathcal{D}\bar{\xi}' \exp \left\{ \hat{S} - \int d^d x [\bar{\eta}\psi' + \eta\bar{\psi}' + \bar{\zeta}\xi' + \zeta\bar{\xi}' + Ka] \right\}$$

(2.19)

We have introduced here also sources $\zeta$ for the ghost fields and note that the source $K_{\mu}^z$ couples to the gauge field fluctuation $a_{\mu}^z$ and therefore transforms homogeneously under gauge transformations as an adjoint tensor. The $k$-dependent version $W_k$ obtains from $W$ by adding to $\hat{S}$ the infrared cutoff piece

$$\Delta_k S = \Delta_k^{(\psi)} S + \Delta_k^{(A)} S + \Delta_k^{(gh)} S$$

(2.20)

Here the fermionic cutoff reads

$$\Delta_k^{(\psi)} S = \bar{\psi}_\gamma' (R_k^{(\psi)})^{\sigma'}_{\beta'} \psi^\beta = \int d^d x \bar{\psi}' Z_{\psi,k} (i\gamma^\mu D_{\mu}[\bar{A}]) r_k^{(\psi)} (-D^2[\bar{A}]/k^2) \psi'$$

(2.21)

with $D_{\mu}$ the covariant derivative in the appropriate representation ($D^2 = D_{\mu}D^\mu$) and $r_k^{(\psi)}$ a dimensionless function. For the gauge field cutoff we choose

$$\Delta_k^{(A)} S = \frac{1}{2} a^*_\alpha \left( R_k \right)^{\alpha'}_{\alpha} a^\alpha = \frac{1}{2} \int d^d x a^\mu \left[ D[\bar{A}] r_k^{(A)} \left( \frac{Z_{A,k}^{-1} D[\bar{A}]}{k^2} \right) \right]_{\nu\sigma} a^\sigma$$

(2.22)

with $D[\bar{A}]$ an appropriate operator generalizing a covariant Laplacian in the adjoint representation which will be explained below. The matrix $Z_{A,k}$ accounts for an appropriate wave function renormalization. Finally, we take for the ghosts

$$\Delta_k^{(gh)} S = \bar{\xi}_\gamma' (R_k^{(gh)})^{\gamma'}_{\gamma} \xi^\gamma = \int d^d x \bar{\xi}' Z_{gh,k} D_s[\bar{A}] r_k^{(gh)} (-D^2[\bar{A}]/k^2) \xi$$

(2.23)

with $D_s[\bar{A}] = -D^2[\bar{A}]$ in the adjoint representation. A good choice for the dimensionless function $r_k$ is\(^2\)

$$r_k(y) = \frac{e^{-y}}{1 - e^{-y}}$$

(2.24)

\(^2\)The function $r_k^{(\psi)}$ may be chosen differently from (2.24) in order to avoid that $R_k$ diverges for vanishing covariant momenta.
such that
\[ \lim_{D \to 0} R_k = Z_k k^2 \] (2.25)

The \( k \)-dependent functions \( Z_{\psi,k} Z_{A,k} \) and \( Z_{gh,k} \) will be adapted to corresponding wave function renormalization constants in the kinetic terms for the fermions, gauge fields and ghosts. In principle, they can depend on the background field \( \bar{A} \). The infrared cutoff piece \( \Delta_k S \) cuts off all quantum fluctuations with covariant momenta smaller than \( k \) in the functional integral defining \( W_k \). For covariant momenta larger than \( k \) the infrared cutoff is ineffective and its contribution to the propagator is exponentially suppressed.

Performing a Legendre transform and subtracting the IR cutoff piece again (c.f. (2.4), (2.5)) we arrive at the effective average action \( \Gamma_k[\psi, \xi, A, \bar{A}] \), where \( A = \bar{A} + \bar{a} \) and \( \bar{a} \) is conjugate to \( K \). The dependence of \( \Gamma_k \) on the scale \( k \) is described by an exact evolution equation analogous to eq. (2.6), with a negative sign for the contributions \( \sim R_k^{(\psi)} \) and \( R_k^{(gh)} \). It is derived in the appendix (A.12). We note that \( \Gamma_k \) only involves terms with an even number of ghost fields due to the symmetry \( \bar{\xi}' \to -\bar{\xi}', \xi' \to -\xi' \) of the \( S_{gh} \) and \( \Delta_k^{(gh)} S \). In consequence, the ghost field equations
\[ \frac{\delta \Gamma_k}{\delta \xi} = 0, \quad \frac{\delta \Gamma_k}{\delta \bar{\xi}} = 0 \] (2.26)

have always the solution \( \bar{\xi} = \xi = 0 \). We therefore can extract the propagators and vertices for the physical particles from the effective action for \( \bar{\xi} = \xi = 0 \):
\[ \Gamma_k[\psi, A, \bar{A}] = \Gamma_k[\psi, 0, A, \bar{A}] \] (2.27)

Nevertheless, the evolution equation for \( \Gamma_k[\psi, A, \bar{A}] \) obtains a contribution from the variation of the infrared cutoff of the ghost fields as given by
\[ \frac{\partial}{\partial t} \Gamma_k[\psi, A, \bar{A}] = \frac{1}{2} \text{Tr} \left\{ \left( \frac{\partial}{\partial t} R_k^{(A)} \right) \left( \Gamma_k^{(2)} + R_k \right)^{-1} \right\} - \text{Tr} \left\{ \left( \frac{\partial}{\partial t} R_k^{(\psi)} \right) \left( \Gamma_k^{(2)} + R_k \right)^{-1} \right\} - \varepsilon_k \] (2.28)
\[ \varepsilon_k = \text{Tr} \left\{ \left( \frac{\partial}{\partial t} R_k^{(gh)} \right) \left( \Gamma_k^{(gh)(2)} + R_k^{(gh)} \right)^{-1} \right\} \] (2.29)

Here \( \Gamma_k^{(2)} + R_k \) in (2.28) is the matrix of second functional derivatives of \( \Gamma_k + \Delta_k^{(\psi)} S + \Delta_k^{(A)} S \) with respect to \( \psi \) and \( A \) at fixed \( \bar{A} \). As compared to the more symmetric form
of the flow equation (A.12) we have combined here similar pieces in the quark and ghost sector. One should remember, however, that the fermionic part \((\Gamma^k_2 + R_k)^{-1}\) is a submatrix of a larger matrix containing also \(\bar{\psi}\psi\) entries. For the derivation of eq. (2.28) we have exploited that the matrix of second functional derivatives of \((\Gamma_k + \Delta_k S)[\psi, \xi, A, \bar{A}]\) is block diagonal in the \((\psi, A)\) and \(\xi\) components for \(\xi = 0\). The ghost dependence of \(\Gamma_k[\psi, \xi, A, \bar{A}]\) appears in the evolution equation for \(\Gamma_k[\psi, A, \bar{A}]\) only through the term \(\varepsilon_k\) which involves the second functional derivative with respect to the ghost fields \(\Gamma^{(gh)(2)}_k\), which is evaluated at \(\bar{\xi} = \xi = 0\) and may depend on \(\psi, A, \bar{A}\). As a consequence of local gauge invariance the average action \(\Gamma_k\) must obey anomalous Slavnov-Taylor identities which are displayed in the appendix. They constrain, in particular, the ghost dependence of \(\Gamma_k\). We will not pay much attention to the detailed form of \(\Gamma^{(gh)(2)}_k\) in the present paper and approximate it by its “classical” value (cf. (2.18))

\[
\Gamma^{(gh)(2)}_k = -D^\mu [\bar{A}] D_\mu [A],
\]

(2.30)
or a slight generalization thereof (cf. eq. (4.8)). In order to complete the formal setup of our investigation we need to specify the operator \(\mathcal{D}\) in eq. (2.22). A good choice is

\[
\mathcal{D}[\bar{A}] = \Gamma^{(A)(2)}_k[\bar{A}]
\]

(2.31)
where \(\Gamma^{(A)(2)}_k\) is the second functional derivative of \(\Gamma_k[\psi, A, \bar{A}]\) with respect to \(A\) for fixed \(\bar{A}\) and \(\psi = 0\), evaluated at the point \(A = \bar{A}\). As in previous formulations [6], the effective average action \(\Gamma_k[\psi, A, \bar{A}]\) is gauge invariant with respect to simultaneous gauge transformations of \(\psi, A\) and \(\bar{A}\).

We can now apply the formalism of the last section in order to “integrate out” the gluon fields \(A\). The classical field equation, whose solution is \(A_k\), reads

\[
\frac{\delta \tilde{\Gamma}_k[\psi, A, \bar{A}]}{\delta A_\mu^z(x)} \bigg|_{A=A_k} = 0
\]

(2.32)
where the derivative should be taken at fixed \(\bar{A}\). At this point \(A_k\) becomes a functional of \(\psi\) and \(\bar{A}\). For the purpose of the present paper we only consider the special choice \(\bar{A} = 0\) and omit the argument \(\bar{A}\) in the following. In this version \(R^{(A)}_k\) becomes a simple function of momenta. Summarizing our adaptation of eq. (2.11) for
quarks and gluons one obtains

\[
\frac{\partial}{\partial t} \Gamma_k[\psi] = -\left( \Gamma_k^{(2)}[\psi] + R_k^{(\psi)} \right)^{-1} \gamma \psi + \gamma A \bar{\psi} + \gamma_A + \gamma_c - \epsilon
\]  

(2.34)

The flow equation (2.33) is the central equation of this paper. In order to exploit this equation we need \( A_k[\psi], \Gamma_k^{(2)}[\psi, A_k], \) and \( \epsilon_k[\psi, A_k]. \)

Our aim is the solution of the flow equation for \( k \to 0, \) starting at some high scale \( k_0 \) where \( \Gamma_{k_0}[\psi] \) can be reliably computed by solving the field equation for \( A_{k_0} \) in a perturbative context. As \( k \) decreases, we gradually explore the non-perturbative regime and the full quantum-effective action obtains for \( k = 0. \) Obviously, such a program is only feasible with approximations that truncate the most general form of \( \Gamma_k. \) The truncation used in the next two sections concerns the three-gluon and four-gluon vertices which are approximated by a momentum-independent, but \( k \)-dependent running gauge coupling. A similar truncation is used for the ghost contribution. In sect. 5 we truncate, in addition, the most general form of the four- and six-fermion interactions.

We are interested in the evolution of the two- and four-point functions for the quarks. The respective flow equations for these quantities obtain by taking the second and fourth functional derivative of eq. (2.33) at \( \psi = \bar{\psi} = 0. \) We label the different contributions on the r.h.s. of eq. (2.33) by

\[
\frac{\partial}{\partial t} \Gamma_k[\psi] = -\left( \Gamma_k^{(2)}[\psi] + R_k^{(\psi)} \right)^{-1} \gamma \psi + \gamma A \bar{\psi} + \gamma A + \gamma_c - \epsilon
\]  

(2.35)

and discuss them separately. The first term

\[
\gamma_\psi = Tr \left\{ \left( \Gamma_k^{(2)}[\psi] + R_k^{(\psi)} \right)^{-1} \frac{\partial}{\partial t} R_k^{(\psi)} \right\}
\]  

(2.36)

is the standard contribution of a pure fermionic theory. The remaining terms \( \gamma_A, \gamma A \) and \( \gamma_c \) involve \( R_k^{(A)} \) and reflect the contributions from gluons, whereas \( \epsilon \) gives the ghost contribution. The term

\[
\gamma_A = \frac{1}{2} Tr \left\{ \left( \Gamma_k^{(2)} + R_k^{(A)} \right)^{-1} \frac{\partial}{\partial t} R_k^{(A)} \right\}
\]  

(2.37)
involves only a trace over gluonic degrees of freedom and accounts for the contribution of gluon fluctuations around the $\psi$-dependent classical solution. The contribution of $\gamma_A$ to the fermionic two- and four-point functions is given by the dependence of $\Gamma^{(2)}_k[\psi, A_k[\psi], 0]_{\alpha\alpha'}$ on $\psi$. Relevant contributions to $\gamma_A$ therefore arise from terms in $\Gamma_k[\psi, A, 0]$ which are either quadratic in $A$ and also depend on $\psi$ or are cubic or higher-order in $A$.

We use a truncation where the first sort of terms is absent and no relevant contribution to $\gamma_A$ would be present for an abelian gauge theory. For nonabelian gauge theories we get contributions from the three- and four-gluon vertices in $\Gamma_k[\psi, A]$. We approximate here these vertices by the (standard) lowest order expressions which are obtained from functional derivatives of $F_{\mu\nu}F^{\mu\nu}$. More precisely, we use on the r.h.s. of the flow equation

$$\frac{\delta}{\delta A_w(y)} \left( \Gamma^{(2)}_k[\psi, A, \bar{A} = 0] \right)_{\nu z}^{\mu y}(x, x') = \frac{\delta}{\delta A_w(y)} \left( \mathcal{D}[A] \right)_{\nu z}^{\mu y} \delta(x - x')$$

with

$$\tilde{Z}_F^{-1} \left( \mathcal{D}[A] \right)_{\nu z}^{\mu y} = - (\mathcal{D}^2[A])_{\nu z}^{y} \delta_{\nu}^{\mu} + 2i\tilde{g} (T_w)_{\nu z}^{y} F_{\mu}^{w y} + (\tilde{D}_{\nu}[A] \tilde{D}^{\mu}[A])_{\nu z}^{y}$$

(2.38)

Here $\tilde{D}_{\mu}[A] = \partial_{\mu} - i\tilde{g} A_{\mu}^z T_z$ represents the covariant derivative in the adjoint representation with gauge coupling $\tilde{g}$ and $F_{\mu\nu}$ is the nonabelian field strength associated to the gauge field $A_{\mu}$. In this truncation the three- and four-gluon vertices are parametrized by two running parameters $\tilde{g}(k)$ and $\tilde{Z}_F(k)$. The running renormalized gauge coupling $g_k$ is related to them by

$$g_k^2 = \tilde{g}^2 \tilde{Z}_F^{-1}$$

(2.39)

We observe that $\Gamma_k[\psi, A, 0]$ is invariant under global gauge transformations of $\psi$ and $A$. The expression for $\gamma_A$ does not explicitly depend on $\psi$ in our truncation and $\gamma_A[\psi = 0, A_k]$ or $\Gamma_k[0, A_k, 0]$ cannot contain a term linear in $A_k$. There is therefore no contribution from $\gamma_A$ to the flow equation of the fermionic two-point function. An estimate of the contribution to the four-quark interaction from $\gamma_A$ therefore amounts to a computation of the gluon contribution to the evolution of the term quadratic in $A$ in $\Gamma_k[\psi = 0, A, \bar{A} = 0]$.

Similarly the ghost contribution $\epsilon$ is (with the approximation (2.30)) only a functional of $A$, containing terms quadratic in $A$ (and higher orders). We also
observe that $\gamma_A$ and $\epsilon$ only account for the gluon and ghost contributions to the effective gluon propagator, whereas the contribution from quark loops is implicitly contained in $\gamma_\psi$. We note that the latter is not distinguished any more from any other fermionic contributions, as, for example, from an explicit four-quark interaction in $\mathcal{L}_k[\psi]$. The contribution

$$\gamma_c = \frac{1}{2} A_k^{\alpha^\prime} \frac{\partial R^{(A)}}{\partial t} A_k^\alpha$$  \hspace{1cm} (2.40)$$
describes the effect of the “classical” change in the infrared cutoff as $k$ is lowered. It is quadratic in the classical solution $A_k[\psi]$ and therefore gives a contribution to the fermionic four-point function. Finally the piece $\gamma_A \psi$ involves the explicit $\psi$-dependence of the classical solution $\sim \partial A_k/\partial \psi$. It contributes to the running of the fermionic two- and four-point functions.

### 3 Heavy quark approximation

In the limit of infinitely large quark masses our formalism simplifies considerably. For euclidean momenta we can omit in eq. (2.33) the terms involving the inverse fermion propagator $(\Gamma^{(2)}_k[\psi] + R^{(\psi)}_k)^{-1}$ since their contribution is suppressed by inverse powers of the quark masses. In the language of the last section this results in $\gamma_\psi = 0$, $\gamma_A \psi = 0$. The remaining computation amounts to an investigation of the pure gluon theory with static quarks. This is done most easily in the language where the gluon fields are kept explicitly and the relevant effective action is $\Gamma_k[\psi, A, \bar{A} = 0]$. If one wants to extract the effective four-quark interaction, one needs the $k$-dependent effective gluon propagator and the effective vertex $\bar{\psi} \gamma \psi A$. We first describe for arbitrary quark masses the general framework how an effective four-quark interaction obtains from “gluon exchange” in the formulation where both quark and gluon degrees of freedom are kept explicitly. We then specialize to the heavy quark limit.

Let us consider in $\Gamma_k[\psi, A, \bar{A}]$ the term quadratic in $A$

$$\Gamma^{(A)}_{k,2} = \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} A_y^\nu(-q)(\Gamma^{(2)}_k[\psi = 0, A = 0, \bar{A} = 0])^{y\mu}_{\nu z} A_z^\mu(q)$$  \hspace{1cm} (3.1)$$
and parametrize the most general inverse gluon propagator by

$$\left(\Gamma^{(2)}_k[\psi = 0, A = 0, \bar{A} = 0]\right)^{y\mu}_{\nu z}(q) = (G_A(q)\delta_y^\mu + H_A(q)q_\nu q^\mu)\delta_z^y$$  \hspace{1cm} (3.2)$$
For the quark-gluon vertex

$$\Gamma^{(\bar{\psi}\psi A)}_k = \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \bar{\psi}(p)G_\psi(p, q)\gamma^\mu(T^z)\gamma^\nu(p + q)A_\mu^a(q)A_\nu^a(-q)$$  \hspace{1cm} (3.3)

we make the approximation that $G_\psi$ is a simple function not involving Dirac matrices. Knowledge of $G_A, H_A$ and $G_\psi$ permits to compute the classical solution $A_k$ in order $\bar{\psi}\psi$

$$(A_k^{(0)}(q))^\nu_z = -S_\mu^\nu(q) \int \frac{d^4p}{(2\pi)^4} G_\psi(p, q)\bar{\psi}(p)\gamma^\mu(T^z)\gamma^\nu(p + q)$$  \hspace{1cm} (3.4)

Here $S = (\Delta^{(2)}_k)^{-1}$ is the gluon propagator in presence of the infrared cutoff

$$(R_k^{(4)})_{\mu\nu}(q) = (R_k(q))_{\mu\nu} + \tilde{R_k}(q)\{\mu\nu\}$$  \hspace{1cm} (3.5)

and reads

$$S_\mu^\nu(q) = (G_A(q) + R_k(q))^{-1} \left\{ \delta_\mu^\nu - q^\nu q_\mu (H_A(q) + \tilde{R_k}(q)) \right\} \left[ G_A(q) + R_k(q) + q^2 (H_A(q) + \tilde{R_k}(q)) \right]^{-1}$$  \hspace{1cm} (3.6)

Inserting the classical solution into (3.2) and (3.3) and accounting for the term $\Delta^{(4)}_k S$ (2.22) we find the effective quark four point function

$$\Gamma^{(\psi\psi A)}_k = -\frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \frac{d^4p'}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} S_\mu^\nu(q)G_\psi(p, q)G_\psi(p', -q) \left\{ \bar{\psi}^a(p)\gamma^\mu(T^z)\gamma^\nu(p + q) \bar{\psi}^a(p') \right\}$$

$$= -\frac{1}{2} \int \frac{d^4p_1}{(2\pi)^4} \frac{d^4p_2}{(2\pi)^4} \frac{d^4p_3}{(2\pi)^4} \delta(p_1 + p_2 - p_3 - p_4) \left\{ F_1(p_1, p_2, p_3, p_4)\mathcal{M}(p_1, p_2, p_3, p_4) + F_2(p_1, p_2, p_3, p_4)\mathcal{N}(p_1, p_2, p_3, p_4) \right\}$$  \hspace{1cm} (3.7)

with

$$F_1 = G_\psi(-p_1, p_1 - p_3)G_\psi(p_4, p_2 - p_4)(G_A(p_1 - p_3) + R_k(p_1 - p_3))^{-1}$$

$$F_2 = G_\psi(-p_1, p_1 - p_3)G_\psi(p_4, p_2 - p_4)(H_A(p_1 - p_3) + \tilde{R_k}(p_1 - p_3))$$

$$(G_A(p_1 - p_3) + R_k(p_1 - p_3))^{-1}[G_A(p_1 - p_3) + R_k(p_1 - p_3)$$

$${+ (p_1 - p_3)^2 (H_A(p_1 - p_3) + \tilde{R_k}(p_1 - p_3))}^{-1}$$  \hspace{1cm} (3.8)

and

$$\mathcal{N}(p_1, p_2, p_3, p_4) = \left\{ \bar{\psi}^a(p_4)(\bar{\psi}^a - \bar{\psi}^a)(T^z)\bar{\psi}^a(p_2) \right\}$$

$$\left\{ \bar{\psi}^a(p_4)(\bar{\psi}^a - \bar{\psi}^a)(T^z)\bar{\psi}^a(p_2) \right\}$$  \hspace{1cm} (3.9)
\[ \mathcal{M}(p_1, p_2, p_3, p_4) = \left\{ \bar{\psi}_a^i(-p_1)\gamma^\mu(T^z)_k^j p^b_j(-p_3) \right\} \left\{ \bar{\psi}_b^k(p_4)\gamma^\mu(T_z)_k^\ell \psi^b_j(p_2) \right\} \]  

(3.10)

The curled brackets indicate contractions over not explicitly written indices (here spinor indices), \( i, j, k, \ell = 1...N_c \) are the colour indices and \( a, b = 1...N_f \) the flavour indices of the quarks. By an appropriate Fierz transformation and using the identity

\[ (T^z)_k^j (T_z)_k^\ell = \frac{1}{2} \delta^j_\ell \delta^\ell_k - \frac{1}{2N_c} \delta^j_k \delta^\ell_k \]  

(3.11)

we can split \( \mathcal{M} \) into three terms \[3\]

\[
\mathcal{M} = \mathcal{M}_\sigma + \mathcal{M}_\rho + \mathcal{M}_p
\]

(3.12)

\[
\mathcal{M}_\sigma = -\frac{1}{2} \left\{ \bar{\psi}_a^i(-p_1)\psi^b_j(p_2) \right\} \left\{ \bar{\psi}_b^j(p_4)\psi^a_i(-p_3) \right\} 
+ \frac{1}{2} \left\{ \bar{\psi}_a^i(-p_1)\gamma^5\psi^b_j(p_2) \right\} \left\{ \bar{\psi}_b^j(p_4)\gamma^5\psi^a_i(-p_3) \right\}
\]

(3.13)

\[
\mathcal{M}_\rho = \frac{1}{4} \left\{ \bar{\psi}_a^i(-p_1)\gamma^\mu\gamma^5\psi^b_j(p_2) \right\} \left\{ \bar{\psi}_b^j(p_4)\gamma^\mu\gamma^5\psi^a_i(-p_3) \right\} 
+ \frac{1}{4} \left\{ \bar{\psi}_a^i(-p_1)\gamma^\mu\psi^b_j(p_2) \right\} \left\{ \bar{\psi}_b^j(p_4)\gamma^\mu\psi^a_i(-p_3) \right\}
\]

(3.14)

\[
\mathcal{M}_p = -\frac{1}{2N_c} \left\{ \bar{\psi}_a^i(-p_1)\gamma^\mu\psi^a_i(-p_3) \right\} \left\{ \bar{\psi}_b^j(p_4)\gamma^\mu\psi^b_j(p_2) \right\}
\]

(3.15)

In terms of the Lorentz invariants

\[
s = (p_1 + p_2)^2 = (p_3 + p_4)^2
\]

\[
t = (p_1 - p_3)^2 = (p_2 - p_4)^2
\]

(3.16)

we recognize that the quantum numbers of the fermion bilinears in \( \mathcal{M}_\sigma \) correspond to colour singlet, flavour non-singlet scalars in the s-channel and similarly for spin-one mesons for \( \mathcal{M}_\rho \). In analogy to ref. \[3\] we associate these terms with the scalar mesons of the linear \( \sigma \)-model and with the \( \rho \)-mesons. The bilinears in the last term \( \mathcal{M}_p \) correspond to a colour and flavour singlet spin-one boson in the t-channel. These are the quantum numbers of the pomeron. We observe that in the heavy quark approximation where \( \Gamma_{k,4}^{(\psi)} \) arises only from “gluon exchange”, the coefficients of the quark interactions in the \( \sigma, \rho \) and pomeron channel \( (3.12) \) are all given by the same function \( F_1 \).

The general quark bilinear is conveniently parametrized by the real functions \( Z_\psi(q) \) and \( \bar{m}_a(q) \)

\[
\Gamma_{k,2}^{(\psi)} = \sum_a \int \frac{d^4q}{(2\pi)^4} \bar{\psi}_a(q)(Z_\psi(q)\gamma^\mu q_\mu + \bar{m}_a(q)\gamma^5)\psi^a_i(q)
\]

(3.17)
The $k$-dependence of the functions $G_A, H_A, G_\psi, Z_\psi$ and $\bar{m}_a$ relevant for the two- and four-point functions for the quarks can now be studied using the evolution equation (2.28) for $\Gamma_k[\psi, A, \bar{A} = 0]$. In the truncation where only the terms (3.2), (3.3) and (3.17) are kept, it is easy to see that the contributions to the $k$-dependence of $G_\psi, Z_\psi$ and $\bar{m}_a$ all involve quark propagators. In the heavy quark limit they can therefore be neglected for euclidean external momenta. Only the $k$-dependence of $\Gamma^{(A)}_{k,2}$ needs to be considered. For $Z_\psi$ and $G_\psi$ we may take appropriate momentum-independent “short-distance couplings”

\[
\begin{align*}
Z_\psi(q) & = 1 \\
g_\psi(p, q) & = \bar{Z}_F^2(m_\psi) g(m_\psi) = \bar{g}(m_\psi)
\end{align*}
\]

with renormalized gauge coupling $g$ taken at the scale $k = m_\psi$ and $m_\psi$ the heavy quark mass. We also may identify $k = m_\psi$ with the “ultraviolet cutoff” or the scale where the initial values for the flow equation are specified, i.e.

\[
\begin{align*}
\bar{Z}_F(m_\psi) & = 1 \\
G_A(q; k = m_\psi) & = q^2
\end{align*}
\]

Solving the flow equation for $G_A(q)$ for $k \to 0$ yields the effective four-quark interactions for momenta much smaller than the quark mass. For $\alpha_R = 0$ (see next section) the effective four-quark interaction is fully determined by

\[
\Gamma^{(\psi)}_{0,4} = -\frac{1}{2} g^2(m_\psi) \int \frac{d^4p_1}{(2\pi)^4} \cdots \frac{d^4p_4}{(2\pi)^4} (2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4) \delta(p_1 + p_2 - p_3 - p_4)
\]

\[
\lim_{k \to 0} G^{-1}_A(p_1 - p_3) \left\{ \mathcal{M}(p_1, p_2, p_3, p_4) + \frac{1}{(p_1 - p_3)^2} \mathcal{N}(p_1, p_2, p_3, p_4) \right\}
\]

We finally should mention that the heavy quark potential or the scattering amplitude for heavy quarks cannot be extracted directly from the four-point function at small momenta $p^2 \ll m_\psi^2$. For these purposes the momenta appearing in $\mathcal{M}$ should be taken on-shell, i.e. $p_1^2 = p_2^2 = p_3^2 = p_4^2 = -m_\psi^2$. Their size is therefore not small as compared to $m_\psi^2$. For on-shell momenta the vertex function $G_\psi(p, q)$ becomes a function of $q^2$. The flow of $G_\psi(q)$ does not vanish for $k^2 \ll m_\psi^2$ and we need to supplement the computation of the $k$-dependence of $G_A(q)$ by a corresponding one of $G_\psi(q)$. The heavy quark potential can then be extracted as the (three-dimensional) Fourier transform of $F(q) = G^2_\psi(q)G^{-1}_A(q)$.
4 Scale dependence of the gluon propagator

In this section we compute a flow equation for the scale dependence of the effective gluon propagator in the pure Yang-Mills theory. As explained in the last section, this is the central piece which determines the heavy quark interactions. We will derive the flow equation for \( G_A \) in the approximation that the vertices can be extracted from a term \( \frac{1}{4} \tilde{Z}_F F_{\mu\nu}^z F_{\mu\nu}^z \) in the average action, with running wave function renormalization \( \tilde{Z}_F \) and running gauge coupling. The resulting functional form (4.9) of the r.h.s. of the flow equation corresponds to \( \gamma_A - \epsilon \) and is valid beyond the heavy quark approximation. The second part of this section discusses qualitative properties of the solution of the flow equations. This part is valid only for the pure Yang-Mills theory or for the heavy quark theory. For the derivation of the evolution equation we keep the most general gluon propagator. The only truncation concerns the momentum dependence of the effective three-gluon and four-gluon vertices and the ghost sector. One finds that the \( k \)-dependence of the functions \( G_A(q) \) and \( H_A(q) \) is governed by the flow equations

\[
\frac{\partial}{\partial t} G_A(q) = N_c g_k^2 \tilde{Z}_F \int \frac{d^4q'}{(2\pi)^4} \partial_t \left\{ (G_A(q') + R_k(q'))^{-1} \tilde{Z}_F \left[ 3 - \frac{3}{4} b(q') \right] \right. \\
- \frac{1}{2} (G_A(q') + R_k(q'))^{-1} (G_A(q + q') + R_k(q + q'))^{-1} \tilde{Z}_F^2 \\
\left[ 5q'^2 + 2(qq') + \frac{16}{3} q'^2 - \frac{10}{3} \frac{(qq')^2}{q'^2} \right] \\
- \frac{2}{3} b(q') \left( 2q'^2 + 10(qq') + q'^2 + 11 \frac{(qq')^2}{q'^2} + 2 \frac{(qq')^3}{q'^2 q^2} + \frac{(qq')^2}{q^2} \right) \\
+ \frac{1}{3} b(q') b(q + q') \frac{q^2}{(q + q')^2} \left( q'^2 - \frac{(qq')^2}{q'^2} \right) \\
+ \frac{1}{3} P_{gh}^{-1}(q') P_{gh}^{-1}(q + q') \left[ q'^2 - \frac{(qq')^2}{q'^2} \right] \} \tag{4.1}
\]

and

\[
\frac{\partial}{\partial t} H_A(q) = -\frac{1}{2} N_c g_k^2 \tilde{Z}_F \int \frac{d^4q'}{(2\pi)^4} \partial_t \left\{ (G_A(q') + R_k(q'))^{-1} (G_A(q + q') + R_k(q + q'))^{-1} \tilde{Z}_F^2 \\
\left[ \frac{40}{3} \frac{(qq')^2}{(q'^2)^2} - \frac{10 q'^2}{3} + 10 \frac{(qq')}{q'^2} - 2 \right] \right. \\
\left. - \frac{1}{2} (G_A(q') + R_k(q'))^{-1} (G_A(q + q') + R_k(q + q'))^{-1} \tilde{Z}_F^2 \\
\left[ 5q'^2 + 2(qq') + \frac{16}{3} q'^2 - \frac{10}{3} \frac{(qq')^2}{q'^2} \right] \\
- \frac{2}{3} b(q') \left( 2q'^2 + 10(qq') + q'^2 + 11 \frac{(qq')^2}{q'^2} + 2 \frac{(qq')^3}{q'^2 q^2} + \frac{(qq')^2}{q^2} \right) \\
+ \frac{1}{3} b(q') b(q + q') \frac{q^2}{(q + q')^2} \left( q'^2 - \frac{(qq')^2}{q'^2} \right) \\
+ \frac{1}{3} P_{gh}^{-1}(q') P_{gh}^{-1}(q + q') \left[ q'^2 - \frac{(qq')^2}{q'^2} \right] \} \tag{4.2}
\]
\[-2b(q') \left( \frac{2}{3} - \frac{4}{3} \frac{(qq')}{q^2} - \frac{14}{3} \frac{(qq')^2}{q^2 q'^2} + \frac{1}{3} \frac{q'^2}{q^2} - \frac{4}{3} \frac{(qq')^2}{(q^2)^2} - \frac{8}{3} \frac{(qq')^3}{(q^2)^2 q'^2} \right) \]
\[+ \frac{1}{3} b(q') b(q + q') \frac{1}{(q + q')^2} \left( \frac{(qq')^2}{q'^2} - \frac{q'^2}{q^2} \right) \]
\[-\frac{2}{3} P_{gh}^{-1}(q') P_{gh}^{-1} (q + q') \left\{ \frac{3}{q^2} \frac{(qq')^2}{q'^2} + \frac{4}{q^2} \frac{(qq')^2}{(q^2)^2} - \frac{q'^2}{q^2} \right\} \]  \hspace{1cm} (4.2)

Here we use
\[b(q) = \frac{(H_A(q) + \tilde{R}_k(q)) q^2}{G_A(q) + R_k(q) + (H_A(q) + \tilde{R}_k(q)) q^2} \]  \hspace{1cm} (4.3)
and note that the partial derivative \( \tilde{\partial}_t \) acts only on the explicit infrared cutoff terms \( R_k \) and \( \tilde{R}_k \). The parts involving the effective ghost propagator \( P_{gh}^{-1} \) arise from the ghost contribution \( \sim -\epsilon \). After performing the \( q' \) integration the evolution equations (4.1), (4.2) can be interpreted as two coupled nonlinear partial differential equations for the functions \( G_A, H_A \) which depend on two variables \( k \) and \( q^2 \). They describe the scale dependence of the gluon propagator in the approximation where both the three-gluon and the four-gluon vertex are given by a single renormalized gauge coupling \( g_k \) and similarly for the ghost gluon vertex.

Even in this approximation the flow equations are lengthy and difficult to solve. A simplification occurs if we take for the gauge-fixing term
\[S_{gf} = \frac{1}{2\alpha} \int d^4 x (\partial^\mu A_\mu^z)^2 \]  \hspace{1cm} (4.4)
the gauge parameter \( \alpha \to 0 \). In this limit \( H_A \) diverges \( \sim \frac{1}{\alpha} \) and \( b(q) \) approaches one. We may define a \( k \)-dependent renormalized gauge fixing parameter \( \alpha_R \) by
\[H_A(0) = \left( \frac{1}{\alpha_R} - 1 \right) \tilde{Z}_F \]  \hspace{1cm} (4.5)
with \( \lim_{k \to \infty} \tilde{Z}_F = 1 \) and \( \lim_{k \to \infty} \alpha_R = \alpha \). The evolution equation for \( \alpha_R \) follows from (4.2)
\[\frac{\partial}{\partial t} \alpha_R = \frac{\partial \ln \tilde{Z}_F}{\partial t} (\alpha_R - \alpha_R^2) - \alpha_R^2 \frac{\partial}{\partial t} H_A(0) \]  \hspace{1cm} (4.6)
where we note that \( \frac{\partial}{\partial t} H_A(q) \) has a well defined limit for \( q^2 \to 0 \). We conclude that \( \alpha_R = 0 \) is a fixpoint which is infrared stable for \( \partial \ln \tilde{Z}_F / \partial t > 0 \). The approximation \( \alpha_R \to 0 \) remains therefore stable in the course of the evolution. We further observe
\[3\text{We choose } \tilde{R}_k q^2 = \left( \frac{1}{\alpha_R} - 1 \right) R_k \text{ such that } b(q) = 1 + 0(\alpha_R) \text{ for all values of } q^2.\]
that for $\alpha_R \to 0$ the function $F_2$ defined in eq. (3.8) equals $F_1/(p_1 - p_3)^2$ and only the function $G_A(q)$ determines the effective four-quark interaction. The flow equation for $G_A$ can be written in a more compact form using

$$P_A(q) = \tilde{Z}_{F}^{-1}(G_A(q) + R_k(q))$$

(4.7)

We also approximate the ghost part of $\tilde{\Gamma}^{(2)}_k$ by

$$P_{gh}(q) = P_A(q), \quad \partial_t P_{gh}(q) = \partial_t P_A(q)$$

(4.8)

This yields for $\alpha_R \to 0$

$$\frac{\partial}{\partial t} G_A(q) = N_c g_k^2 \tilde{Z}_{F} \int \frac{d^4 q'}{(2\pi)^4} \partial_q \left\{ \frac{9}{4} P_A^{-1}(q') - \frac{1}{6} P_A^{-1}(q') P_A^{-1}(q + q') \left[ 13q^2 - 14(qq') + 10q'^2 \right] - 10 \frac{(qq')^2}{q^2} - 22 \frac{(qq')^2}{q'^2} - 4 \frac{(qq')^3}{q^2 q'^2} + \frac{q^2 q'^2 - (qq')^2}{(q + q')^2} \right\}$$

(4.9)

The evolution equation (4.9) is the central equation of this section. It is a partial nonlinear differential equation for $G_A(q^2; k)$ which can be solved numerically. We observe that the r.h.s. of the flow equation (4.9) involves not only $G_A(q)$ but also the renormalized gauge coupling $g_k$ and the gluon wave function renormalization constant $\tilde{Z}_F$. The precise definition of these $k$-dependent constants is a somewhat subtle issue. We could define $\tilde{g}$ and $\tilde{Z}_F$ (and in consequence $g_k$) in terms of the effective three-gluon vertex $\sim \tilde{g} \tilde{Z}_F A^2 \partial A$ and four-gluon vertex $\sim \tilde{g}^2 \tilde{Z}_F A^4$ which enter the approximation for $\Gamma_k^{(2)}$ used in (2.38). In this way $\tilde{g}$ and $\tilde{Z}_F$ are expressed in terms of third and fourth functional derivatives of $\Gamma_k[\bar{\psi} = 0, A, \bar{A} = 0]$ evaluated at $A = 0$ and projected on the appropriate index structures. One also has to choose appropriate momenta for the external legs for the effective vertices, as for example the limit where all momenta approach zero. Evolution equations for the scale dependence of $\tilde{Z}_F$, $\tilde{g}$ and $g_k$ could then be computed from appropriate functional derivatives of the flow equation (2.28). We will use here a simplification and approximate $\tilde{Z}_F$ by the coefficient of the $q^2$ term in $G_A$. More precisely, we expand for small $q^2$

$$G_A(q) = m_A^2 + G_A^{(1)} q^2 + G_A^{(2)} (q^2)^2 + \ldots$$

(4.10)

and identify $\tilde{Z}_F$ with $G_A^{(1)}$. There is an obvious limitation to this approximation since $G_A^{(1)}$ may turn negative for small $k$ and any reasonable choice of a wave function
renormalization requires positive $\tilde{Z}_F$. For the flow equation for the renormalized
gauge coupling $g_k$ we rely on the fact that the first two coefficients of the $\beta$ function

$$\frac{\partial g_k^2}{\partial t} = \beta g^2 = -c_1 \frac{g_k^4}{16\pi^2} - c_2 \frac{g_k^6}{(16\pi^2)^2} - ... \quad (4.11)$$

are universal

$$c_1 = \frac{22N_c}{3} \quad c_2 = \frac{204}{9} N_c^2 \quad (4.12)$$

In the region of large $g_k$ we may also use nonperturbative estimates of $\beta g^2$ derived by different methods [6]. An ansatz for $\beta g^2$ combined with an estimate of $\tilde{\eta}_F$ fixes also the evolution of $\tilde{g}^2$ and provides all information needed for a numerical solution of the flow equation.

We concentrate first on an analytic discussion of a few prominent features of the solution of equation (4.9). The evolution equations for the mass term $\tilde{m}_A^2$ and for $G_A^{(1)}$ are easily derived by expanding the r.h.s. of eq. (4.9) in powers of $q$. One obtains

$$\frac{\partial}{\partial t} \tilde{m}_A^2 = N_c g_k^2 \tilde{Z}_F \int \frac{d^4q}{(2\pi)^4} \tilde{\eta}_F \left\{ \frac{9}{4} P_A^{-1}(q') - \frac{5}{4} q'^2 P_A^{-2}(q') \right\} \quad (4.13)$$

and the renormalized dimensionless mass term

$$\tilde{m}_A^2 = \tilde{m}_A^2 \tilde{Z}_F^{-1} k^{-2} \quad (4.14)$$

directly obeys

$$\frac{\partial}{\partial t} \tilde{m}_A^2 = (-2 + \tilde{\eta}_F) \tilde{m}_A^2 - \frac{N_c}{8\pi^2} \tilde{g}_k^2 \left( \frac{9}{4} l_{A,1} - \frac{5}{4} l_{A,2}^6 \right) \quad (4.15)$$

Here we have defined the integrals, with $x = q^2$,

$$l_{A,n}^d = -\frac{1}{2} k^{2n-d} \int_0^\infty dx x^{2n-1} \tilde{\eta}_F P_A^{-n}(x) \quad (4.16)$$

The evolution of the ration $\tilde{m}_A^2/g_k^2$

$$\frac{\partial}{\partial t} \left( \frac{\tilde{m}_A^2}{g_k^2} \right) = -\frac{N_c}{8\pi^2} \left( \frac{9}{4} l_{A,1} - \frac{5}{4} l_{A,2}^6 \right) - \left( 2 + \frac{\beta g^2}{g_k^2} - \tilde{\eta}_F \right) \frac{\tilde{m}_A^2}{g_k^2} \quad (4.17)$$

is characterized for small $g_k^2$ by an approximate infrared unstable fixpoint

$$\frac{\tilde{m}_A^2}{g_k^2} = -\frac{3N_c}{128\pi^2} \quad (4.18)$$
Indeed, we can use for small $g_k^2$ the lowest order expressions

$$
G_A(q) = \tilde{Z}_F q^2,
\partial_t P_A(q) = \frac{\partial}{\partial t} P(q)
$$

(4.19)

with

$$
P(q) = q^2 + \tilde{Z}_F^{-1} R_k(q) = \frac{q^2}{1 - \exp \left(-\frac{q^2}{k^2}\right)}
$$

(4.20)

such that $l_{A,1} = l_1 = 1, l_{A,2} = l_2 = \frac{3}{2}$. We also neglect $\beta_g^2 / g^2 - \eta_F$ as compared to two. For small $g_k^2$ the general identities for the dependence of $\Gamma_k[\psi, A, \bar{A}]$ on the background field $\bar{A}$ [6], or, similarly, the generalized Slavnov-Taylor identities [8],[9] imply that the $k$-dependent mass term is indeed described by this fixpoint (cf. Appendix). We conclude that for small $g_k$ the mass term induces only a small correction in the momentum-independent part of $P_A$

$$
P_A \approx P + \tilde{Z}_F^{-1} \tilde{m}_A^2 + O(q^4)
= P - \frac{3N_c}{128\pi^2} g_k^2 k^2 + O(q^4)
= k^2 \left(1 - \frac{3N_c}{128\pi^2} g_k^2\right) + O(q^7)
$$

(4.21)

The behaviour of $\tilde{m}_A^2$ near the confinement scale where $g_k^2$ becomes large is more complicated and best described by evaluating directly the relevant identities [6]. It is conceivable that a negative r.h.s. of eq. (4.13) drives $\tilde{m}_A^2$ to a positive value for $k \to 0$, but we find this scenario not very likely.

The flow equation for $G_A^{(1)}$ can be written in the form

$$
\frac{\partial}{\partial t} G_A^{(1)} = \frac{N_c}{96\pi^2} g_k^2 \tilde{Z}_F (31l_{A,2}^4 - 5m_{A,4}^6)
= \frac{13}{3} N_c c_A \frac{g^2}{16\pi^2}
$$

(4.22)

where we have defined the integrals

$$
m_{A,n}^d = -\frac{1}{2} k^{2n-d-2} \int_0^\infty dx x^d \tilde{\partial}_t \left( \tilde{P}_A^2(x) P_A^{-n}(x) \right)
$$

(4.23)

with $\tilde{P}_A = \frac{d}{dx} P_A$. We note that the r.h.s. of eq. (4.22) is positive for positive $c_A$

$$
c_A = \frac{1}{26} (31l_{A,2}^4 - 5m_{A,4}^6)
= -\frac{31}{52} \int_0^\infty dx x \tilde{\partial}_t \left\{ P_A^{-2} \left(1 - \frac{5}{31} x^2 \tilde{P}_A^2 P_A^{-2} \right) \right\}
$$

(4.24)
Let us first consider small values of the gauge coupling where we can approximate \( \tilde{\partial}_t P_A = \frac{\partial}{\partial t} P \). In this limit we obtain \( c_A = 1 \). For the definition

\[
\tilde{Z}_F = G_A^{(1)}
\]

and therefore finds

\[
\tilde{\eta}_F = - \frac{\partial}{\partial t} \ln \tilde{Z}_F = - \frac{13}{3} N_c c_A \frac{g_k^2}{16\pi^2} \tag{4.25}
\]

For a solution of (4.26) it is convenient to compare \( \tilde{\eta}_F \) with

\[
\eta_F = \frac{\partial}{\partial t} \ln g_k^2 = \frac{\beta g_k}{g_k^2} = - \frac{22}{3} N_c b_A \frac{g_k^2}{16\pi^2} \tag{4.26}
\]

where \( b_A \) may depend on \( g_k \) or \( k \) and reflects the deviation from the one-loop \( \beta \)-function for which \( b_A = 1 \). As long as the \( k \)-dependence of the ratio \( b_A/c_A \) can be neglected, one obtains

\[
\frac{\tilde{Z}_F(k)}{\tilde{Z}_F(k_0)} = \left( \frac{g^2(k_0)}{g^2(k)} \right)^\gamma, \quad \gamma = \frac{13c_A}{22b_A} \tag{4.27}
\]

In the one-loop approximation \( \frac{1}{g^2(k)} \) decreases logarithmically and reaches zero at the confinement scale \( \Lambda_{\text{conf}} \). We conclude that this is exactly the scale where \( \tilde{Z}_F \) would vanish. In this language the divergence of the renormalized gauge coupling is actually entirely due to the vanishing of \( \tilde{Z}_F \): From (4.28) one obtains

\[
\frac{\tilde{g}^2(k)}{g^2(k)} = \left( \frac{g^2(k)}{g^2(k_0)} \right)^{1-\gamma}
\]

\[
\frac{\tilde{Z}_F(k)\tilde{g}^2(k)}{\tilde{Z}_F(k_0)\tilde{g}^2(k_0)} = \left( \frac{g^2(k)}{g^2(k_0)} \right)^{1-2\gamma}
\]

\[
\frac{\tilde{Z}_F(k)\tilde{g}(k)}{\tilde{Z}_F(k_0)\tilde{g}(k_0)} = \left( \frac{g(k)}{g(k_0)} \right)^{1-3\gamma} \tag{4.29}
\]

and for \( \gamma > \frac{1}{2} \) (cf. (4.28)) both the unrenormalized three-point vertex \( \sim \tilde{Z}_F\tilde{g} \) and four-point vertex \( \sim \tilde{Z}_F\tilde{g}^2 \) vanish at the confinement scale.

As mentioned before we should not use (1.22) for \( k \) in the vicinity of the confinement scale. We therefore propose \[1\] to keep \( \tilde{Z}_F \) independent of \( k \) for \( k < k_{np} \) where \( k_{np} \) is defined by

\[
g_{k_{np}}^2 = \frac{4\pi^2}{N_c} \tag{4.30}
\]

\[4\]We have also computed \( \frac{\partial}{\partial t} H_A(0) = - \frac{N_c}{12\pi^2} g^2 \left( 8t_{A,2} + 5m_{A,4}^6 \right) \) and find for small \( g^2 \) that \( \frac{\partial}{\partial t} \left( H_A(0) + G_A^{(1)} \right) \) vanishes as required by the Slavnov-Taylor identity for perturbative transversality.}

21
The definition
\[
\tilde{Z}_F = \begin{cases} 
  G_A^{(1)}(k) & \text{for } g_k^2 < \frac{4\pi^2}{N_c} \\
  \tilde{Z}_F(k_{np}) & \text{for } g_k^2 \geq \frac{4\pi^2}{N_c} 
\end{cases}
\] (4.31)
allows to separate the issue of vanishing \(G_A^{(1)}\) from the choice of the infrared cutoff (i.e. \(\tilde{Z}_F\)).

The vanishing of the term \(G_A^{(1)}q^2\) would have important consequences for the behaviour of the propagator \(\sim G_A^{-1}\). We should therefore investigate if this feature is likely to survive beyond the approximation of small gauge coupling. Turning back to eq. (4.22) we observe that \(G_A^{(1)}\) can only remain positive for \(k \to 0\) if \(\tilde{g}^2\) turns to zero or if \(c_A\) vanishes or becomes negative. In view of eq. (4.29) the first alternative seems not very likely. (This probably generalizes if we go beyond the approximation leading to (4.1) and consider general momentum-dependent three- and four-point functions. Then in eq. (4.22) \(\tilde{g}^2\) has to be replaced by an appropriate momentum-weighted average of these vertices.)

In order to investigate the sign of \(c_A\) we write the integral (4.24) explicitly as
\[
c_A = \frac{31}{26} \int_0^\infty dx x P_A^{-3} S \left\{ 1 - \frac{10}{31} x^2 \left( \frac{\dot{P}_A}{P_A} \right)^2 + \frac{5}{31} x^2 \frac{\dot{P}_A \dot{S}}{P_A S} \right\}
\] (4.32)
with
\[
S(x) = \partial_t P_A(x) = \frac{\partial}{\partial t} P(x) - \tilde{\eta}_F(P(x) - x) = \left( 2 \frac{P(x)}{k^2} - \tilde{\eta}_F \right) (P(x) - x)
\] (4.33)

The integral is dominated by the region \(x \approx k^2\) and can turn negative only if the bracket is negative in this region. As an illustration we take \(P_A = P + m_A^2 + \kappa x^2, m_A^2 = \tilde{Z}_F^{-1} m_A^2\). This implies
\[
\frac{x P_A}{P_A} = \frac{\dot{P} x + 2\kappa x^2}{P + m_A^2 + \kappa x^2}
\] (4.34)

to be compared with
\[
\frac{x \dot{P}}{P} = 1 - \frac{x}{k^2} \exp \left( -\frac{x}{k^2} \right) \left( 1 - \exp \left( -\frac{x}{k^2} \right) \right)^{-1} = 1 - \frac{P - x}{k^2}
\] (4.35)

and
\[
\frac{x \dot{S}}{S} = \frac{x \dot{P}}{P} \left( 1 - \frac{\tilde{\eta}_F k^2}{2 P} \right)^{-1} + 1 - \frac{P}{k^2}
\] (4.36)
We conclude that negative $m_A^2$ (cf. (4.18)) and positive $\kappa$ tend to lower $c_A$. A vanishing $c_A$ for large $\kappa$ cannot be excluded without a more detailed investigation. On the other side, if the integral relation

$$\int_{\Lambda_{\text{conf}}}^{\Lambda} \frac{dk}{k} q^2(k) c_A(k) = \frac{48\pi^2}{13N_c}$$

(4.37)

could be fulfilled for $\Lambda_{\text{conf}} > 0$, the coefficient $G^{(1)}_A$ would vanish at the confinement scale $\Lambda_{\text{conf}}$ and presumably becomes negative for $k < \Lambda_{\text{conf}}$. We emphasize that a negative value of $\bar{m}_A^2$ or of $G^{(1)}_A$ for small $k$ would independently indicate that the groundstate does not correspond to the perturbative ground state $A_\mu = 0 \[12\],[13],[11]\).

Even before reaching the confinement scale, the gluon propagator has to be modified: Whenever the term $G^{(1)}_A q^2$ becomes comparable to $G^{(2)}_A (q^2)^2$ for $q^2 \approx k^2$, i.e. for $G^{(1)}_A \approx G^{(2)}_A k^2$, the gluon propagator cannot be approximated any more by the inverse of $q^2$! As an example for a plausible form one may consider

$$G_A(q) \approx \bar{m}_A^2 + G^{(1)}_A q^2 + \bar{Z}_F \kappa \frac{(q^2)^2}{1 + \delta q^2}$$

(4.38)

where

$$\delta = \frac{\bar{Z}_F \kappa}{1 - G^{(1)}_A}$$

(4.39)

is determined by the requirement that for large $q^2$ one expects $G_A(q) = q^2$ independent of $k$. (This holds up to neglected logarithmic corrections.) Assuming that for $k \approx k_{np}$ the term $\bar{m}_A^2 + G^{(1)}_A q^2$ is small as compared to the $(q^2)^2$ term the approximate form of the propagator

$$G_A^{-1}(q) \approx \frac{1}{\bar{Z}_F \kappa} \frac{1}{(q^2)^2} + \frac{1}{q^2}$$

(4.40)

would be close to the one corresponding to a confining potential.

It is obviously difficult to find an analytical answer to all these questions and it seems preferable to solve the flow equation (4.9) numerically. A numerical investigation has been performed by B. Bergerhoff and the author [14]. We show here only a few first results. In fig. 1 we plot the scale dependence of the wave function renormalization $\bar{Z}_F$ as defined by

$$\bar{Z}_F = \frac{\partial}{\partial q^2} G_A(q)|_{q^2=k^2+2\Lambda_{QCD}^2}$$

(4.41)
with $\Lambda_{QCD}$ the two-loop confinement scale. This definition implements the idea that $\tilde{Z}_F$ should be taken constant for very small $k$ (cf. eq. (4.31)) in a smooth way. One observes the decrease of $\tilde{Z}_F$ as $k$ is lowered corresponding to eq. (4.28). We have started the running in the perturbative region at $k = 40$ GeV with $G_A(q)|_{k=40 \text{ GeV}}$ given by the one-loop perturbative expression (containing the infrared cutoff). The latter was normalized by $\partial G_A/\partial q^2|_{q^2=0} = 1$, which explains the starting value of $\tilde{Z}_F$ (40 GeV) somewhat larger than one for the definition (4.41). In order to concentrate on the deviation of $G_A(q)$ from the linear dependence on $q^2$ we introduce the quantity

$$\chi(q) = \frac{\partial}{\partial \ln q^2} \ln \left( \frac{G_A(q) - G_A(0)}{\tilde{Z}_F q^2} \right)$$

(4.42)

Here the mass term $G_A(0)$ is subtracted from the inverse propagator and the leading perturbative $q^2$-dependence is divided out. In the classical approximation the expression in the bracket equals one, and a nonvanishing value of $\chi(q)$ is entirely due to quantum fluctuations. Since a computation of $\chi(q)$ involves a numerical derivative of a small difference, one needs a numerical solution of the flow equation for $G_A(q)$ with a relatively high precision. We observe that $\chi(q)$ plays the role of a momentum-dependent anomalous dimension. Within renormalization-group improved perturbation theory one expects for $k = 0$ (compare eq. (4.26))

$$\chi(q) = \frac{13}{6} N_c \frac{g^2(q)}{16\pi^2}$$

(4.43)

where $g^2(\mu)$ is the running gauge coupling at the scale $\mu$. In figs. 2 and 3 we compare the numerical determination of $\chi(q)$ with the renormalization-group improved one-loop perturbative result (4.44), with $g$ the two-loop running gauge coupling. Up to a scaling factor of about 10 % the two curves asymptotically coincide for large $q^2$ as $k$ goes to zero. This is not a trivial result since no assumption of this type enters the flow equation (4.9). We note that a propagator $\sim q^{-4}$ for small $q^2$ corresponds to $\lim_{k \to 0} \lim_{q^2 \to 0} \chi(q) = 1$. If a $q^{-4}$ behaviour extends effectively over a certain momentum range $\chi(q)$ should develop a plateau at one in this range. As $k$ is lowered we see in fig. 3 a sizeable increase of $\chi(q)$ for small $q^2$. We also observe a tendency of an extension and flattening of the maximum somewhat below one. This tendency should stabilize as $k$ goes to zero. We have stopped the running at $k = 400$ MeV since for small $k^2$ and $q^2$ the approximations leading to (4.9) become doubtful. At least part of the momentum dependence of the gluon vertices according to (A.55) should presumably be included.
5 Flow equations for light quarks

The formalism for integrating out the gluon degrees of freedom needs not to be restricted to the heavy quark approximation. We want to derive in this section the general flow equation for the quark two-point and four-point function corresponding to (2.33). For simplicity we mainly consider $N_F$ massless quarks – the inclusion of mass terms is straightforward – and we work in the gauge with $\alpha = 0$. At some appropriate short distance scale we start with the “classical action” (cf. (3.7))

$$\Gamma_k[\psi] = Z_\psi \int \frac{d^4q}{(2\pi)^4} \bar{\psi}(q) \gamma_\mu q_\mu \psi(q) - \frac{1}{2} Z_\psi^2 g_k^2.$$

The flow equation then permits to study how $\Gamma_k$ changes its form as $k$ is lowered. In particular one is interested in pole-like structures in the quark four-point function which would indicate the formation of meson bound states [15], [3].

In order to establish the flow equation we have to collect various pieces which have been discussed in the previous sections. We begin with the contribution from the gluon and ghost fluctuations $\gamma_A - \epsilon + \gamma_c$ which only contribute to the four-quark interaction $\Gamma_{k,4}[\psi]$. The direct contributions from gluon and ghost loops read

$$\gamma_A - \epsilon = \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \left( A^{(0)}_k(q) \right)^z_\mu \left( A^{(0)}_k(-q) \right)^\nu_z \left\{ \frac{\partial}{\partial t} \hat{G}_A(q) \delta_\mu^\nu + \frac{\partial}{\partial t} \hat{H}_A(q) q_\mu q_\nu \right\}$$

The functions $\frac{\partial}{\partial t} \hat{G}_A(q)$ and $\frac{\partial}{\partial t} \hat{H}_A(q)$ have been computed in the last section (cf. (4.1), (4.2), or (4.9)), where we have indicated by a hat that only gluon and ghost contributions should be included here, in analogy to the heavy quark approximation. The classical solution $A^{(0)}_k$ is given by eqs. (3.4) and (3.6), where $G_A$ and $H_A$ characterize now the gluon propagator without reference to the heavy quark approximation. We consider again the limit $\alpha \rightarrow 0$ where $S^{\nu}_\mu(q) = (G_A + R_k)^{-1}(\delta^{\nu}_\mu - q^\nu q^\mu/q^2)$. Including also the contribution $\gamma_c$ which is similar in structure we obtain

$$\gamma_A - \epsilon + \gamma_c = -\frac{1}{2} \int \frac{d^4p_1}{(2\pi)^4} \ldots \frac{d^4p_4}{(2\pi)^4} \delta(p_1 + p_2 - p_3 - p_4) \cdot f_1(p_1, p_2, p_3, p_4)(\mathcal{M} + \frac{1}{(p_1 - p_3)^2 N})$$

with

$$f_1(p_1, p_2, p_3, p_4) = -G_\psi(-p_1, p_1 - p_3)G_\psi(p_4, p_2 - p_4).$$
\[ (G_A(p_1 - p_3) + R_k(p_1 - p_3))^{-2} \left( \frac{\partial}{\partial t} \hat{G}_A(p_1 - p_3) + \frac{\partial}{\partial t} R_k(p_1 - p_3) \right) \]  \hspace{1cm} (5.4)

(In the heavy quark approximation there is no difference between \( \hat{G}_A \) and \( G_A \). Also \( G_\psi \) is independent of \( k \) such that \( f_1 = \frac{\partial}{\partial t} F_1 \) \( (3.8) \).) The function \( f_1 \) involves the quark gluon vertex \( G_\psi \) and the gluon propagator \( G_A \) which also enters in the determination of \( \frac{\partial}{\partial t} \hat{G}_A \).

Up to this point the only approximations involve the three- and four-gluon vertices entering \( \frac{\partial}{\partial t} \hat{G}_A \) and the ghost sector. They have been discussed in sect. 4. We may in addition also truncate the quark gluon vertex and use for light quarks the ansatz

\[ G_\psi(p, q) = \tilde{Z}_F^{1/2} Z_\psi g_k \]  \hspace{1cm} (5.5)

The running of the renormalized gauge coupling \( g_k \) is now determined by the \( \beta \)-function including quark contributions and \( \tilde{Z}_F \) may be identified with \( G_A^{(1)} \) for \( k > k_{np} \) (cf. sect. 4). The lowest order truncation for \( G_A(q) \) would be

\[ G_A(q) = \tilde{Z}_F q^2 \]  \hspace{1cm} (5.6)

such that \( P_A(q) \) is replaced by \( P(q) \) in the equation \( (4.9) \) for \( \frac{\partial}{\partial t} \hat{G}_A \). The detailed discussion of the last section shows, however, that this approximation becomes invalid for \( k \) in the vicinity of the confinement scale. There one should rather use a truncation of the form \( (4.38) \) or similar. The corresponding flow equations for \( m_A^2, G_A^{(1)} \) and \( \kappa \) include now additional contributions from quark fluctuations and have to be computed from the evolution equation for \( \frac{\partial}{\partial t} G_A(q) \) in the formulation where both quark and gluon degrees of freedom are present. Fortunately, the formation of meson-bound states occurs typically at a scale substantially higher than the confinement scale. This gives the hope that important features of meson physics can already be extracted using the truncation \( (5.6) \) on the r.h.s. of the flow equation and do not need a very detailed understanding of gluon condensation phenomena.

With the truncation \( (5.5), (5.6) \) the function \( f_1 \) \( (5.4) \) only depends on the Mandelstam variable \( t = (p_1 - p_3)^2 \)

\[ f_1 = -Z_\psi^2 g_k^2 P^{-2}(p_1 - p_3) \left\{ \frac{\partial}{\partial t} P(p_1 - p_3) - \tilde{\eta}_F \left( P(p_1 - p_3) - (p_1 - p_3)^2 \right) + N_c g_k^2 \mathcal{G}(p_1 - p_3) \right\} \]  \hspace{1cm} (5.7)
where
\[
\frac{\partial}{\partial t} \tilde{G}_A(q) = N_c g_k^2 \tilde{Z}_F \mathcal{G}(q)
\]  
(5.8)
is given by the r.h.s. of (4.9) with \( P_A(q) = P(q) \) and \( \tilde{\partial}_t P_A(q) = \frac{\partial}{\partial t} P(q) - \tilde{\eta}_F (P(q) - q^2) \). The first two contributions are proportional \( \tilde{\partial}_t P^{-1} \) and simply reflect the change of the infrared cutoff contained in \( P^{-1} \) in the “classical action” (p.4). Only the last term \( \sim \mathcal{G} \) describes how additional quantum fluctuations are included as \( k \) is lowered - in this case the gluon and ghost contributions to the gluon propagator. If we omit the contribution \( \sim \tilde{\eta}_F \) in \( \tilde{\partial}_t P_A(q) \) the function \( \mathcal{G}(q) \) reads explicitly
\[
\mathcal{G}(q) = \int \frac{d^4 q'}{(2\pi)^4} \frac{\partial}{\partial t} \left\{ \frac{9}{4} P^{-1}(q') - \frac{1}{6} P^{-1}(q') P^{-1}(q + q') \right. \\
\left. \left[ 13 q^2 - 14 (qq') + 10 q'^2 - 10 \frac{(qq')^2}{q'^2} - 22 \frac{(qq')^2}{q'^2} \right. \\
\left. - 4 \frac{(qq')^3}{q'^2 q'^2} + \frac{q^2 q'^2 - (qq')^2}{q'^2 (q + q')^2} \right\} \right.
\]  
(5.9)

We observe that even for our simple truncations the function \( \mathcal{G}(q) \) has a complicated momentum dependence. A solution of the flow equation for the four-quark interaction will go far beyond the effects of a running gauge coupling in the one-loop approximation.

The term \( \gamma_{A\psi} \) contributes to the flow equation for the two-point and the four-point function. The contribution to the two-point function can be extracted from (2.33) using the classical solution (3.4). With \( \alpha = 0 \) and \( G_\psi(p, q) = Z_\psi \tilde{Z}^{1/2}_F g_k \) the lowest order classial solution reads
\[
(A_k^{(0)}(q))^\mu_z = -(G_A(q) + R_k(q))^{-1} Z_\psi \tilde{Z}^{1/2}_F g_k \int \frac{d^4 p}{(2\pi)^4} \bar{\psi}^i(p) \left( \gamma^\mu - \frac{q^\mu q}{q^2} \right) (T_z)_i^j \psi^j(p + q)
\]  
(5.10)

and the next to leading contribution \( A_k^{(1)} \) obtains as
\[
(A_k^{(1)}(q))^\mu_z = -i g_k^2 \tilde{Z}^{1/2} Z^2_f \gamma_5 \bar{p}^1 \cdot \bar{Y} P_A^{-1}(q) \int \frac{d^4 p_1}{(2\pi)^4} \cdots \frac{d^4 p_4}{(2\pi)^4} \delta(q - p_1 - p_2 + p_3 + p_4) P_A^{-1}(p_1 - p_3) P_A^{-1}(p_2 - p_4) \cdot \left\{ \frac{1}{2} (p_1^\mu - p_3^\mu - p_4^\mu + p_2^\mu) \right\} \{ \bar{\psi}(-p_1) \gamma^\nu T_y \psi(-p_3) \{ \bar{\psi}(p_4) \gamma^\nu T_y \psi(p_2) \}
\]
\[
- \frac{(p_1 - p_3)^\nu(p_2 - p_4)^\nu}{(p_1 - p_3)^2(p_2 - p_4)^2} \cdot \{ \bar{\psi}(-p_1)(\bar{\psi}_1 - \bar{\psi}_3)T_\nu \psi(-p_3) \} \{ \bar{\psi}(p_4)(\bar{\psi}_2 - \bar{\psi}_4)T_\nu \psi(p_2) \} \\
- \frac{q^\mu}{(p_2 - p_4)^2} \{ \bar{\psi}(-p_1)(\bar{\psi}_2 - \bar{\psi}_4)T_\nu \psi(-p_3) \} \{ \bar{\psi}(p_4)(\bar{\psi}_2 - \bar{\psi}_4)T_\nu \psi(p_2) \} \\
+ 2 \{ \bar{\psi}(-p_1)(\bar{\psi}_2 - \bar{\psi}_4) - \frac{(p_2 - p_4)^\nu(p_1 - p_3)^\nu}{(p_1 - p_3)^2} (\bar{\psi}_1 - \bar{\psi}_3) \} T_\nu \psi(-p_3) \\
\{ \bar{\psi}(p_4) \gamma^\mu T_\nu \psi(p_2) \} \}
\]  

(5.11)

With the ansatz of a flavour diagonal kinetic term and mass term

\[
\Gamma_k^{(2)}[0] = Z_\psi(c_\psi(q)\bar{\psi} + m_\psi(q)\gamma^5)\delta^i_0\delta^j_1(2\pi)^4 \delta(q - q')
\]  

(5.12)

one finds the following contribution to the flow equation for the two-point function

\[
\gamma_A^{(2)} = \frac{N_c^2 - 1}{2N_c} Z_\psi g_k^2 \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 p}{(2\pi)^4} (q + p)^2 \left( \frac{\partial}{\partial \ell} r_k(q + p) - \tilde{\eta}_\ell r_k(q + p) \right) \\
P_A^{-2}(q + p) \sum_a \{ \bar{\psi}_a(p) [(c_a(q) + r_k^{(a)}(q))^2 q^2 + m_a^2(q)]^{-1} \\
[(c_a(q) + r_k^{(a)}(q))(\bar{\psi} + 2q^2 + (qp)) - 3m_a(q)\gamma^5] \psi_a(p) \}
\]  

(5.13)

The contribution to the four-point function \( \gamma_A^{(4)} \) can be obtained similarly using (5.11).

For a computation of the purely fermionic contribution \( \gamma_\psi \) we will use the following truncation for the term quartic in the fermionic fields

\[
\Gamma_{k,4}^{(\psi)} = -Z_\psi \int \frac{d^4 p_1}{(2\pi)^4} \cdots \frac{d^4 p_4}{(2\pi)^4} (2\pi)^4 \delta(p_1 + p_2 + p_3 + p_4) \\
\{ \lambda_\sigma(p_1, p_2, p_3, p_4)M_\sigma + \lambda_\rho(p_1, p_2, p_3, p_4)M_\rho \\
+ \lambda_p(p_1, p_2, p_3, p_4)M_p + \lambda_n(p_1, p_2, p_3, p_4)N \}
\]  

(5.14)

This yields a contribution to the flow equation for the two-point function

\[
\gamma_\psi^{(2)} = Z_\psi \int \frac{d^4 p}{(2\pi)^4} \sum_a \{ \bar{\psi}_a(p) \int \frac{d^4 q}{(2\pi)^4} \left( \frac{\partial}{\partial \ell} r_k^{(a)}(q) - \eta_\ell r_k^{(a)}(q) \right) \\
\left[ (c_a(q) + r_k^{(a)}(q))^2 q^2 + m_a^2(q) \right]^{-2} \left[ (c_a(q) + r_k^{(a)}(q))^2 q^2 - m_a^2(q) \right] \\
2N_c \lambda_p(-q, -p, p)\bar{\psi} - \frac{2}{N_c} \lambda_p(-q, -p, p)\bar{\psi} \\
- \frac{N_c^2 - 1}{N_c} \lambda_n(-q, -p, p) \left( (p^2 - q^2) \bar{\psi} + 2 (q^2 - (pq)) \bar{\psi} \right) \}
\]

(5.15)
\[+ m_{\alpha}(q) \left( c_{\alpha}(q) + r_k^{(\psi)}(q) \right) q^2 \gamma_5 \left[ 8 N_c \lambda_\sigma(-q, q, -p, p) - \lambda_\rho(-q, q, -p, p) \right] \psi^\alpha(p) \]

\[- \frac{8}{N_c} \lambda_\rho(-q, q, -p, p) - \frac{2(N_c^2 - 1)}{N_c} (q - p)^2 \lambda_\sigma(-q, q, -p, p) \right] \psi^\alpha(p) \}

\[+ Z_\psi \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \left\{ \tilde{g}_\psi(p)q^b(p) \right\} \left( \frac{\partial}{\partial t} r_k^{(\psi)}(q) - \eta_\psi r_k^{(\psi)}(q) \right) \]

\[\sum_a \frac{\left( c_{\alpha}(q) + r_k^{(\psi)}(q) \right)^2 q^2 - m_a^2(q)}{\left( \left( c_{\alpha}(q) + r_k^{(\psi)}(q) \right)^2 + m_a^2(q) \right)^2} \frac{2}{2} \lambda_\sigma(-q, p, -q, p)

\[+ 2 \lambda_\rho(-q, p, -q, p) - 4 \lambda_\rho(-q, p, -q, p) \] . \hspace{1cm} (5.15)

For \( \gamma^{(4)}_\psi \) we present here only the case \( \lambda_\rho = \lambda_\rho = \lambda_n = 0 \) in the chiral limit \( m_\alpha = 0 \).

Omitting all contributions except \( \gamma^{(4)}_\psi \) one finds

\[\frac{\partial}{\partial t} \lambda_\sigma(p_1, p_2, p_3, p_4) = 2 \eta_\psi \lambda_\sigma(p_1, p_2, p_3, p_4) \]

\[+ 8 N_c \int \frac{d^4q}{(2\pi)^4} \frac{q^\alpha(q - p_1 - p_2)}{q^2(q - p_1 - p_2)^2} \left[ c(q) + r_k^{(\psi)}(q) \right]^{-1}

\[\tilde{g}_\psi \left[ c(q - p_1 - p_2) + r_k^{(\psi)}(q - p_1 - p_2) \right]^{-1}

\[\lambda_\sigma(p_1, p_2, q, -q + p_1 + p_2) \lambda_\sigma(q, -q + p_1 + p_2, p_3, p_4) \] . \hspace{1cm} (5.16)

For the special case \( (c(q) + r_k^{(\psi)}(q))^{-1} = \exp(-\frac{q^2}{4}) - \exp(-\frac{q^2}{2}) \), \( \eta_\psi = 0 \) this reproduces the flow equation of ref. [4]. The r.h.s. of this equation should now be supplemented by the contributions from \( \gamma_A - \epsilon + \gamma_c + \gamma_{A\psi} \) which have been discussed before. Furthermore, a better approximation should include the contributions from \( \lambda_\rho \) etc.

Besides this, eq. (5.16) involves the explicit momentum dependence of the fermion kinetic term, i.e. the function \( c(q) \). The scale dependence of the quark propagator can be computed from \( \gamma^{(2)}_\psi \) and \( \gamma^{(2)}_{A\psi} \). Combining eq. (5.15) with (5.13) and (5.12) yields the flow equation for the kinetic term

\[\frac{\partial}{\partial t} (Z_\psi c_{\alpha}(p)) = Z_\psi \int \frac{d^4q}{(2\pi)^4} \left\{ \frac{N_c^2 - 1}{2N_c} g_k \left( \frac{\partial}{\partial t} r_k(p - q) - \eta_F r_k(p - q) \right) \right\}

\[P_A^{-2}(p - q) \left[ \left( c_{\alpha}(q) + r_k^{(\psi)}(q) \right)^2 q^2 + m_a^2(q) \right]^{-1}

\[\left( c_{\alpha}(q) + r_k^{(\psi)}(q) \right) \left( 2q^2 - 3(pq) - 3 \frac{(pq)q^2}{p^2} + 4 \frac{(pq)^2}{p^2} \right) \]

\[- \left( \frac{\partial}{\partial t} r_k^{(\psi)}(q) - \eta_\psi r_k^{(\psi)}(q) \right) \left[ \left( c_{\alpha}(q) + r_k^{(\psi)}(q) \right)^2 q^2 + m_a^2(q) \right]^{-2} \]
\[
\left[ \left(c_a(q) + r_k^{(\psi)}(q) \right)^2 q^2 - m_a^2(q) \right] \left(2N_c \lambda_\rho(-q, q, -p, p) - \frac{2}{N_c} \lambda_\rho(-q, q, -p, p) \right) \frac{(pq)}{p^2} \\
- \frac{N_c^2 - 1}{N_c} \lambda_n(-q, q, -p, p) \left(2q^2 - (pq) - \frac{(pq)q^2}{p^2} \right) \\
+ \frac{(pq)}{p^2} \left( \frac{\partial}{\partial t} r_k^{(\psi)}(q) - \eta_\psi r_k^{(\psi)}(q) \right) \sum_{b=1}^{N_c} \frac{(c_b(q) + r_k^{(\psi)}(q))^2 q^2 - m_b^2(q)}{[(c_b(q) + r_k^{(\psi)}(q))^2 + m_b^2(q)]^2}.
\]

The right-hand side of this equation involves the gauge coupling \( g_k \) and the effective inverse gauge field propagator \( P_A \) as well as the effective fermionic four-point vertices \( \lambda_\sigma, \lambda_\rho, \lambda_\lambda \) and \( \lambda_n \). It is instructive to study this equation in the “classical approximation” for the four-quark vertices \( \lambda_\sigma \), i.e.

\[
\lambda_\sigma(p_1, p_2, p_3, p_4) = \lambda_\rho(p_1, p_2, p_3, p_4) = \lambda_\lambda(p_1, p_2, p_3, p_4) = \frac{1}{4} g_k^2 \left( P_A^{-1}(p_1 - p_3) + P_A^{-1}(p_2 - p_4) \right)
\]

Using

\[
\tilde{\partial}_t P_A^{-1}(q) = -q^2 P_A^{-2}(q) \left( \frac{\partial}{\partial t} r_k(q) - \eta_\psi r_k(q) \right)
\]

\[
\frac{c_a(q) + r_k^{(\psi)}(q)}{(c_a(q) + r_k^{(\psi)}(q))^2 q^2 + m_a^2(q)} = \frac{- (c_a(q) + r_k^{(\psi)}(q))^2 q^2 - m_a^2(q)}{[(c_a(q) + r_k^{(\psi)}(q))^2 + m_a^2(q)]^2} \cdot \left( \frac{\partial}{\partial t} r_k^{(\psi)}(q) - \eta_\psi r_k^{(\psi)}(q) \right)
\]

one obtains

\[
\frac{\partial}{\partial t} (Z_\psi c_a(p)) = Z_\psi \frac{N_c^2 - 1}{2N_c} \frac{g_k^2}{(2\pi)^4} \frac{d^4q}{(pq)^2} \left( \frac{2(pq)}{p^2} - \frac{2q^2p^2 - (pq)(p^2 + q^2)}{p^2(p - q)^2} \right).
\]

We observe that this expression corresponds to the formal \( \tilde{\partial}_t \) derivative of the standard one-loop correction to the fermion kinetic term in presence of an infrared cutoff in the propagator.

30
The fermionic wave function renormalization $Z_{\psi}$ can be defined by $c_a(p_0) = 1$ for suitable $p_0$ and $a$. We will choose here $p_0 = 0$ and use $c_a$ corresponding to a light quark. Defining the anomalous dimension

$$\eta_{\psi} = -\frac{\partial}{\partial t} \ln Z_{\psi}$$  \hspace{1cm} (5.21)$$

one obtains for $m_a = 0$

$$\eta_{\psi} = \frac{3 N_c^2 - 1}{4 N_c} \frac{g_k^2}{q^2} \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2} \tilde{\partial}_t \left\{ (c(q) + r_k(q))^2 P_A^{-1}(q) \left( 1 - q^2 \tilde{P}_A(q) P_A^{-1}(q) \right) \right\}$$  \hspace{1cm} (5.22)$$

where $\tilde{P}_A = \partial P_A/\partial q^2$. In the perturbative limit $c(q) = 1, P_A(q) = P(q), \tilde{\partial}_t P_A(q) = \partial_t P(q)$ this yields a vanishing fermionic dimension

$$\eta_{\psi} = 0$$  \hspace{1cm} (5.23)$$

We also may extract the anomalous mass dimension by expanding eqs. (5.13) and (5.17) in linear order in $m_a$

$$\frac{\partial}{\partial t} m_a(0) = \omega_m m_a(0)$$  \hspace{1cm} (5.24)$$

One obtains

$$\omega_m = \int \frac{d^4q}{(2\pi)^4} \frac{m(q)}{m(0)} \left\{ \frac{3 N_c^2 - 1}{2 N_c} \frac{g_k^2}{(c(q) + r_k(q))^2} \frac{1}{q^2} \tilde{\partial}_t P_A^{-1} \right\}$$  \hspace{1cm} (5.25)$$

which reduces in the “classical approximation” $\lambda_x(-q, q, 0, 0) = \lambda_p(-q, q, 0, 0) = g_k^2 \lambda_n(-q, q, 0, 0) = \frac{1}{2} g^2 P_A^{-1}(q)$ and for $m_a(q) = m_a(0)$ to

$$\omega_m = \frac{3 N_c^2 - 1}{2 N_c} \frac{g_k^2}{q^2} \int \frac{d^4q}{(2\pi)^4} \frac{1}{(c(q) + r_k(q))^2 q^2 P_A(q)}$$  \hspace{1cm} (5.26)$$

In lowest order perturbation theory where $\tilde{\partial}_t [(c(q) + r_k(q))^2 q^2 P_A(q)]^{-1} = \frac{\partial}{\partial t} [(1 + r_k(q))^2 q^2 P(q)]^{-1}$ one recovers the standard perturbative result

$$\omega_m = -\frac{3}{16\pi^2} \frac{N_c^2 - 1}{N_c} g_k^2$$  \hspace{1cm} (5.27)$$
provided \( \lim_{q^2 \to 0} q^2/(1 + r_k^{(\psi)}(q))^2 = 0 \).

To summarize this section, we have explicitly derived a nonperturbative flow equation for the scale dependence of the quark propagator (5.17). We have also given a simplified non-perturbative approximation (5.20) to this evolution equation and checked explicitly the consistency with perturbation theory in the region where the gauge coupling \( g_k \) is small. In the approximation (5.20) the flow equation resembles a differential form of the Schwinger-Dyson equation for the full momentum-dependent quark propagator. In contrast to the usual Schwinger-Dyson equation we notice that only a small range in momenta \( q^2 \approx k^2 \) contributes to the momentum integral on the r.h.s. of eq. (5.20). This is therefore much easier to control. The actual complicated scale and momentum dependence of \( Z_{\psi c}(q) \), which reflects that the physical picture varies from high to low scales, arises then as a property of the solution of the differential equation (5.20) rather than through the form of the equation itself. This solution will provide the anomalous dimension \( \eta_{\psi} \) and the function \( c(q) \) needed for the flow equation of the quark four-point function. We have explicitly computed the contributions \( \gamma_A - \epsilon + \gamma_c \) to the non-perturbative flow equation for the quark four-point function (5.3), (5.7), (5.9). It is straightforward to extract the contributions from \( \gamma_A^{\psi} \) and \( \gamma_{\psi} \) from (5.11) and (5.16). (The latter may be generalized to include \( \lambda_{\rho} \) etc.) The result gives the non-perturbative evolution equation for the momentum-dependent four-quark coupling.

6 Conclusions and discussion

In this paper we have developed a formalism for integrating out the gluon fields in order to obtain an effective action for the quarks. This is not done at once since such an approach would lead to complicated nonlocalities and a reliable direct computation seems almost impossible. Instead, we account for the gluon contributions to an exact nonperturbative flow equation. At every scale \( k \) this needs only information about the “classical solution” for the gauge field in presence of fermions and about quadratic gauge field fluctuations around this solution with momenta \( q^2 \approx k^2 \). The flow equation describes the scale dependence of an effective average action \( \Gamma_k[\psi] \) which only involves the quark fields. In the present work we have concentrated on analytical work whereas the numerical exploitation of our formulae is left to a
Our first main result is the evolution equation (6.9) for the gluon propagator in the heavy quark limit. As described in sect. 3, its quantitative solution is connected with the heavy quark potential. The second result concerns the effective action for light quarks. The evolution equation for the two- and four-point functions can be extracted from sect. 5. The solution for the four-point function is expected to develop pole-like structures connected to mesons. This can be treated with the composite-field methods developed in ref. [3], such that one can finally make a transition to an effective theory for mesons describing the low momentum behaviour of QCD. The main difference of the present formalism as compared to ref. [3] concerns the treatment of the gluons: While ref. [3] accounts for the effects of gluons only by a phenomenologically motivated four-quark interaction, no such term is introduced here by hand. At short distances we simply start with the QCD action for quarks and gluons. The effective four-quark interaction should arise as a property of the solution of the flow equation. For this purpose it is crucial that the gluon contributions to the evolution equation are properly taken into account.

In order to obtain the $k$-dependent classical solution for the gluon $A_\mu$ as a functional of the quark fields $\psi$ we need knowledge about the effective action $\Gamma_k[\psi, A]$ for both quarks and gluons. The same holds true for the quadratic fluctuations of the gluon field around this solution. More concretely, the exact evolution equations for the fermionic two- and four-point functions involve the gluon propagator for $\psi = 0$, the $\bar{\psi}\psi A, (\bar{\psi}\psi)^2 A, (\bar{\psi}\psi) A^2, (\bar{\psi}\psi)^2 A^2$ and $\bar{\psi}\psi A^3$ vertices as well as the gluonic vertices $A^3$ and $A^4$ in $\Gamma_k[\psi, A]$. Obviously, these quantities can be computed only approximately and truncations are needed. We propose to use the nonperturbative evolution equation for $\Gamma_k[\psi, A]$ in order to compute the inverse propagator ($\sim A^2$) and at least one vertex ($\sim A^3$). (Other vertices can then be related to the $A^3$ vertex). Now the reader may ask why we do not work entirely in the framework of the evolution equation for $\Gamma_k[\psi, A]$, extracting $\Gamma[\psi]$ only at the end of the evolution for $k = 0$. Indeed, we have recovered the perturbative $\beta$-functions as limiting cases of our nonperturbative flow equations for small gauge couplings, and these perturbative $\beta$-functions are certainly easier obtained in the framework of the evolution equation for $\Gamma_k[\psi, A]$. Also, the heavy quark potential only involves a computation of the gluon propagator encoded in $\Gamma_k[\psi, A]$. The main virtue of our approach con-
cerns the nonperturbative aspects of the flow equations with light quarks: We want to obtain a quantitatively reliable flow equation for the full momentum dependence of the quark four-point function. This encodes the formation of meson-bound states as pole-like structures in the s-channel. In turn, this requires a control of the momentum dependence on the r.h.s. of the flow equation. Using a flow equation for \( \Gamma_k[\psi, A] \) we would need a computation of the momentum dependence of the effective vertices \( \bar{\psi}\psi A \), \( (\bar{\psi}\psi)^2 A^2 \), \( (\bar{\psi}\psi)^2 A^3 \), \( (\bar{\psi}\psi)^2 A^4 \), \( \bar{\psi}\psi A^3 \) and \( A^4 \) in addition to the momentum dependence of the gluon and quark propagator and the \( (\bar{\psi}\psi)^2 \) vertex in \( \Gamma_k[\psi, A] \). Using the effective average action for quarks \( \Gamma_k[\psi] \) a large part of this momentum dependence is encoded in the two- and four-point functions in \( \Gamma_k[\psi] \). (Note that the effective \( (\bar{\psi}\psi)^2 \) vertex in \( \Gamma_k[\psi] \) is different from the corresponding one in \( \Gamma_k[\psi, A] \) since effects of mixed quark-gluon vertices are included through the classical solution for \( A \).) One may therefore hope that the momentum dependence of the propagator and four-quark interaction in \( \Gamma_k[\psi] \) includes the dominant effects, whereas a less precise estimate is sufficient for the vertices appearing in the contributions from the gluon fluctuations around the classical solution. In the truncation used in this paper we neglect for a computation of the classical solution and the gluon fluctuations the vertices \( \bar{\psi}\psi A^2 \), \( (\bar{\psi}\psi)^2 A^2 \), \( (\bar{\psi}\psi)^2 A^3 \) and \( \bar{\psi}\psi A^3 \) and we describe the three vertices \( \bar{\psi}\psi A \), \( A^3 \) and \( A^4 \) by one common scale-dependent, but momentum-independent, coupling constant \( \bar{g}(k) \). Clearly, establishing the flow equations for the momentum dependence of the \( \bar{\psi}\psi A \), \( A^3 \) and \( A^4 \) vertices and using the appropriate solution on the r.h.s. of the flow equations should be one of the next steps in our approach.

One may suspect that even the non-perturbative treatment of this paper breaks down for scales \( k \) of the order or below the confinement scale. Fortunately the formation of meson-bound states is expected at a scale \( k_\phi \) considerably higher than the confinement scale. For the pseudoscalar mesons a first computation indicates \( k_\phi \approx 650 \text{ MeV} \). In view of this one may hope that an understanding of the formation of mesons does not necessitate a very detailed understanding of the physics near the confinement scale. There are still several “hopes” and “expectations”. A quantitative computation of the quark condensate \( \langle \bar{\psi}\psi \rangle \) and the pion decay constant \( f_\pi \) along similar lines as in ref. [3], but using the flow equations proposed in the present paper should decide whether we are on a reasonable track for an
analytical understanding of QCD.

Note added: Results for the form of the heavy quark potential have recently been obtained using flow equations \[17\] and are closely related to sect. 4 of the present work.

Appendix

For the derivation of general identities it is convenient to work with a generalized field $\chi$

$$\chi = (a^z_\mu, -c^z, \bar{c}^z, -\psi^\prime_m, \bar{\psi}^\prime_m, \varphi'_a, \varphi^a)$$

$$\bar{\chi} = (a^*_\mu, c^z, \bar{c}^z, \psi^\prime_m, \bar{\psi}^\prime_m, \varphi_a, \varphi^a)$$ (A.1)

It is composed of real gauge fields $a^z_\mu = A^z_\mu - \bar{A}^z_\mu$, ghosts $c^z$ and antighosts $\bar{c}^z$ as well as complex spinors $\psi_m$ and complex scalars $\varphi_a$ in some representations of the gauge group. We use here a notation which can be employed both in coordinate space $a(x) = a^*(x)$ and in momentum space where $a^*(p) = a(-p)$ and $\delta \chi^\alpha(p) / \delta \chi^{\bar{\beta}}(p') = (2\pi)^d \delta(p-p') \delta^{\alpha\bar{\beta}}$. For real scalars or Majorana spinors one should omit the doubling of arguments in $\chi$ and impose $\varphi^a(p) = \varphi^a(-p)$ or similar for Majorana spinors. In our notation the quadratic part of the action reads

$$S_2 = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \frac{d^d p'}{(2\pi)^d} S_{\alpha\bar{\beta}}^{(2)}(p,p') \chi^\alpha(p') \bar{\chi}^\beta(p) \equiv \frac{1}{2} S_{\alpha\bar{\beta}}^{(2)} \chi^\alpha \bar{\chi}^\beta$$ (A.2)

$$S_{\alpha\bar{\beta}}^{(2)}(p,p') = \frac{\delta^2 S}{\delta \chi^\alpha(p) \delta \chi^{\bar{\beta}}(p')}$$ (A.3)

In the last part of (A.2) we have combined internal indices $\bar{\alpha}$ and momentum labels $p$ to a collective index $\alpha$. The infrared cutoff is introduced as a quadratic block diagonal piece in the action

$$\Delta_k S = \frac{1}{2} R_{k\alpha\bar{\beta}} \chi^\alpha \bar{\chi}^\beta$$

$$R_k = \text{diag}(R^{(A)}_k, R^{(c)}_k, R^{(c)^*}_k, R^{(\psi)}_k, R^{(\psi^*)}_k, R^{(\varphi)}_k, R^{(\varphi^*)}_k) = R^\dagger_k$$ (A.4)

with $R^{(A)^*}_{yz}(p,p') = R^{(A)}_{yz}(-p,-p')$. We also introduce sources

$$J = (K, \z, \bar{\z}, \eta, \bar{\eta}, j, j^*)$$

$$\bar{J} = (K^*, -\bar{\z}, \z, -\bar{\eta}, \eta, j^*, j)$$ (A.5)
and define
\[
e^{W_k[J]} = \int \mathcal{D}\chi e^{-S}
\]
\[
S = S_0 + S_{gf} + S_{gh} + \Delta_k S + S_{source}
\]
(A.6)

Here \(S_0\) is the classical gauge-invariant action, \(S_{gf}\) and \(S_{gh}\) are gauge-fixing and ghost terms and
\[
S_{source} = -\bar{J}_\alpha \chi_\alpha = -J_\alpha \bar{\chi}_\alpha
\]
(A.7)

For the Legendre transform \(\tilde{\Gamma}_k = \bar{\chi}_\sigma - W_k\) the following identities hold
\[
\sigma_\alpha = \frac{\partial W_k}{\partial J_\alpha}, \quad \bar{\sigma}_\alpha = \frac{\partial W_k}{\partial \bar{J}_\alpha}
\]
(A.8)
\[
\frac{\partial^2 W_k}{\partial J_\alpha \partial J_\beta} \frac{\partial^2 \tilde{\Gamma}_k}{\partial \bar{\sigma}_\beta \partial \sigma_\gamma} = \frac{\partial^2 W_k}{\partial J_\alpha \partial \bar{J}_\beta} \frac{\partial^2 \tilde{\Gamma}_k}{\partial \bar{\sigma}_\beta \partial \bar{\sigma}_\gamma} = M_{\alpha\gamma}
\]
(A.9)
\[
\frac{\partial^2 W_k}{\partial \bar{J}_\alpha \partial \bar{J}_\beta} \frac{\partial^2 \tilde{\Gamma}_k}{\partial \sigma_\beta \partial \sigma_\gamma} = \frac{\partial^2 W_k}{\partial \bar{J}_\alpha \partial J_\beta} \frac{\partial^2 \tilde{\Gamma}_k}{\partial \bar{\sigma}_\beta \partial \sigma_\gamma} = M_{\alpha\gamma}
\]
(A.10)

The appearance of the matrix \(M = \text{diag}(1, -1, -1, -1, -1, 1)\) reflects the anti-commuting properties of the Grassmann variables \(\xi, \bar{\xi}, \psi, \bar{\psi}\) in a notation where
\[
\sigma = (\bar{a}, -\xi, \bar{\xi}, -\psi, \bar{\psi})
\]
\[
\bar{\sigma} = (\bar{a}^*, \xi, \bar{\xi}, \psi^*, \bar{\psi})
\]
(A.11)

and \(\bar{a} = A - \bar{A}\). Taking a derivative of (A.6) with respect to \(t = \ln k\) and noting that only \(\Delta_k S\) depends on \(k\), one finds [5] the flow equation for \(\Gamma_k = \tilde{\Gamma}_k - \frac{1}{2}R_{\alpha\beta} \sigma_\beta \bar{\sigma}_\alpha\)
\[
\frac{\partial \Gamma_k}{\partial t} = \frac{1}{2} S\text{Tr} \left\{ \frac{\partial R_k}{\partial \Gamma_{(2)}} \left( \Gamma_{(2)} + R_k \right)^{-1} \right\}
\]
(A.12)

with \(S\text{Tr} A = \text{Tr} MA, \ S\text{Tr} AB = \text{Tr} BA,\) and \(\text{Tr} = \int \frac{d^d \rho}{(2\pi)^d} \sum \tilde{a}\). The inverse propagator \(\Gamma^{(2)} = \tilde{\Gamma}^{(2)} - R_k\) is given by
\[
\left( \Gamma_{(2)} \right)_{\alpha\beta} = \frac{\partial^2 \Gamma_k}{\partial \bar{\sigma}_\alpha \partial \sigma_\beta}
\]
(A.13)

We next turn to the anomalous Ward-Takahashi or Slavnov-Taylor identities for which we follow closely the treatment of ref. [9]. For a general gauge-fixing \(G^z(a)\) linear in \(a^*_\mu\) (and possibly depending in addition on the background field \(\bar{A}_\mu\))
\[
S_{gf} = \frac{1}{2\alpha} \int d^d x G^z G^z
\]
\[
S_{gh} = - \int d^d x \bar{c}^z \frac{\partial G^z}{\partial a^\mu} (D_\mu (a + \bar{A}))^{\mu w} c_w
\]
(A.14)
the sum $S_0 + S_{gf} + S_{gh}$ is invariant under the BRS variation $\chi \to \chi + \delta_{BRS} \chi, \bar{\chi} \to \bar{\chi} + \delta_{BRS} \bar{\chi}$

$$\delta_{BRS} \chi = \left( \frac{1}{g} (D_\mu (a + \bar{A}) c)^z, -\frac{1}{2} f^{zyw} e^y c^w, -\frac{1}{\alpha g} G^z, ic^z (T_z)_{mn} \psi_n', -ic^z (T_z^\dagger)_{mn} \bar{\psi}_n', ic^z (T_z)_{ab} \varphi_b', -ic^z (T_z^\dagger)_{ab} \bar{\varphi}_b' \right)$$

$$\delta_{BRS} \bar{\chi} = \left( \frac{1}{\bar{g}} (D_\mu (\bar{a} + A) c)^* z, -\frac{1}{2} f^{zyw} e^y c^w, -\frac{1}{\alpha \bar{g}} G^z, -ic^z (T_z^\dagger)_{mn} \bar{\psi}_n', -ic^z (T_z)_{ab} \bar{\varphi}_b', ic^z (T_z)_{ab} \varphi_b' \right)$$

(A.15)

Here $f^{zyw}$ are the structure constants of the gauge group and $T_z$ the hermitean generators in the appropriate representations. It is useful to introduce external sources $(\bar{\beta}_z^\mu(p) = \beta_\mu^z(-p))$

$$\beta = (\beta_\mu^z, \gamma^z, \bar{\gamma}^z, \delta^{(\psi)}_m, \bar{\delta}^{(\psi)}_m, \delta^{(\varphi)}_a, \bar{\delta}^{(\varphi)}_a)$$

$$\bar{\beta} = (\bar{\beta}_\mu^z, -\bar{\gamma}^z, \gamma^z, -\bar{\delta}^{(\psi)}_m, \delta^{(\psi)}_m, -\bar{\delta}^{(\varphi)}_a, \delta^{(\varphi)}_a)$$

(A.16)

for the BRS variations of $\chi$ or $\bar{\chi}$ and to define $W_k[J, \beta]$ similar to (A.6) by adding in (A.7) an additional source term

$$S_{\text{source}}^{(\beta)} = -\bar{\beta}_\alpha (\delta_{BRS} \chi)_\alpha = -\beta_\alpha (\delta_{BRS} \bar{\chi})_\alpha$$

(A.17)

The BRS-invariance of the measure implies

$$0 = <\delta_{BRS} S > = < R_{\alpha \beta} \chi_\beta (\delta_{BRS} \chi)_\alpha >_{|\gamma=0} - < J_\alpha (\delta_{BRS} \bar{\chi})_\alpha >_{|\gamma=0}$$

$$= < R_{\alpha \beta} (\delta_{BRS} \chi)_\beta \bar{\chi}_\gamma M_{\gamma\alpha} >_{|\gamma=0} - < J_\alpha (\delta_{BRS} \chi)_\alpha >_{|\gamma=0}$$

(A.18)

or

$$J_\alpha \frac{\partial W_k}{\partial \beta_\alpha}_{|\gamma=0} = R_{\alpha \beta} \left( \frac{\partial}{\partial J_\beta} + \frac{\partial W_k}{\partial J_\beta} \right) \frac{\partial W_k}{\partial \beta_\alpha}_{|\gamma=0}$$

(A.19)

Using (A.8), (A.9) and the identities

$$\frac{\partial W_k}{\partial \beta_\alpha}_{|J} = -\frac{\partial \tilde{\Gamma}_k}{\partial \beta_\alpha}_{|\bar{\sigma}}$$

(A.20)

$$\frac{\partial^2 W_k}{\partial J_\alpha \partial \beta_\gamma} = -\frac{\partial^2 W_k}{\partial J_\alpha \partial J_\beta} \frac{\partial^2 \tilde{\Gamma}}{\partial \sigma_\beta \partial \beta_\gamma}$$

(A.21)
one arrives at the identity
\[
\frac{\partial \Gamma_k}{\partial \sigma_\alpha} M_{\alpha \beta} \frac{\partial \Gamma_k}{\partial \beta_\beta|_{\gamma=0}} = \frac{\partial \Gamma_k}{\partial \sigma_\alpha} M_{\alpha \beta} \frac{\partial \Gamma_k}{\partial \beta_\beta|_{\gamma=0}} = \text{STr} \left\{ \mathcal{R}(\Gamma^{(2)}_k + \mathcal{R})^{-1} \frac{\partial^2 \Gamma_k}{\partial \sigma \partial \beta} \right\} |_{\gamma=0} = M_{\alpha \beta} \mathcal{R}_{\beta \gamma}(\Gamma^{(2)} + \mathcal{R})^{-1} \frac{\partial^2 \Gamma_k}{\partial \sigma \partial \beta_{\alpha|_{\gamma=0}}} \tag{A.22}
\]

In addition, we note the simple identity
\[
\frac{\partial \Gamma_k}{\partial \gamma} = \frac{1}{\alpha \bar{g}} G^z(\bar{a}) \tag{A.23}
\]
which allows to eliminate the source $\gamma$. Similarly, linearity in $\bar{c}$ yields the field equation for the antighost
\[
\frac{\partial \Gamma_k}{\partial \xi} = \bar{g} \tilde{G}^z \frac{\partial \Gamma_k}{\partial \beta_\mu} \tag{A.24}
\]
where
\[
\tilde{G}^z = \frac{\partial G^z}{\partial \alpha_\mu} = (D_\mu(\bar{A}))^{zy} \tag{A.25}
\]
depends only on the background field $\bar{A}$. For the last identity in (A.25) we have used the particular gauge-fixing $G^z = [D_\mu(\bar{A})]^{zy} a^y_\mu$, which will be assumed in the following. For this gauge we now insert (A.23), (A.24) such that the Ward identity reads in explicit components in momentum space
\[
\int \frac{d^4 p}{(2\pi)^d} \frac{d^4 q}{(2\pi)^d} \left\{ \frac{\partial \Gamma'}{\partial A_\mu^v(p)} \frac{\partial \Gamma'}{\partial \beta_\mu^v(p)} - \frac{\partial \Gamma'}{\partial \xi^z(p)} \frac{\partial \Gamma'}{\partial \gamma^z(p)} \right\} + \frac{\partial \Gamma'}{\partial \psi_\mu(p)} \frac{\partial \Gamma'}{\partial \psi_\mu(p)} + \frac{\partial \Gamma'}{\partial \bar{\varphi}_\mu(p)} \frac{\partial \Gamma'}{\partial \bar{\varphi}_\mu(p)} = \mathcal{A}^{(g)}_{\text{BRS}} + \mathcal{A}^{(m)}_{\text{BRS}} \tag{A.26}
\]
with
\[
\mathcal{A}^{(g)}_{\text{BRS}} = \int \frac{d^4 p}{(2\pi)^d} \frac{d^4 q}{(2\pi)^d} \left\{ \mathcal{R}^{(\alpha\beta\mu\nu)}_{\Gamma^{(2)}_k + \mathcal{R}}(\Gamma^{(2)}_k + \mathcal{R})^{-1} \frac{\partial^2 \Gamma}{\partial \sigma_\alpha(q) \partial \beta_\mu^v(p)} \right\} - \mathcal{R}_{\Gamma^{(2)}_k + \mathcal{R}}(\Gamma^{(2)}_k + \mathcal{R})^{-1} \frac{\partial^2 \Gamma}{\partial \sigma_\alpha(q) \partial \beta_\mu^v(p)} - \frac{1}{\alpha \bar{g}} \int \frac{d^4 p}{(2\pi)^d} \frac{d^4 q}{(2\pi)^d} \left\{ \bar{q} \mathcal{R}^{(c)}_{\Gamma^{(2)}_k + \mathcal{R}}(\Gamma^{(2)}_k + \mathcal{R})^{-1} \frac{\partial^2 \Gamma}{\partial \sigma_\alpha(q) \partial \beta_\mu^v(p)} \right\}
\]
\[
- \bar{g} \int \frac{d^4 p}{(2\pi)^d} \frac{d^4 q}{(2\pi)^d} \left\{ \bar{q} \mathcal{R}^{(c)}_{\Gamma^{(2)}_k + \mathcal{R}}(\Gamma^{(2)}_k + \mathcal{R})^{-1} \frac{\partial^2 \Gamma}{\partial \sigma_\alpha(q) \partial \beta_\mu^v(p)} \right\} \tag{A.27}
\]
and
\[ A_{\text{BRS}}^{(m)} = \int \frac{d^d p}{(2\pi)^d} \frac{d^d p'}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \left\{ -R_{mn}^{(\psi)}(p,p')(\Gamma^{(2)} + \mathcal{R})^{-1}_{\psi_n(p')\sigma_q(q)} \frac{\partial^2 \Gamma}{\partial \sigma_{\tilde{a}}(q) \partial \delta_m^{(\psi)}(p)} - R_{mn}^{(\psi)*}(p,p')(\Gamma^{(2)} + \mathcal{R})^{-1}_{\psi_n(p')\sigma_q(q)} \frac{\partial^2 \Gamma}{\partial \sigma_{\tilde{a}}(q) \partial \delta_m^{(\psi)}(p)} \right. \\
\left. + R_{ab}^{(\phi)}(p,p')(\Gamma^{(2)} + \mathcal{R})^{-1}_{\psi_n(p')\sigma_q(q)} \frac{\partial^2 \Gamma}{\partial \sigma_{\tilde{a}}(q) \partial \delta_m^{(\psi)}(p)} \right\} \right) \] (A.28)

Here we have subtracted the “bare” gauge-fixing term
\[ \Gamma' = \Gamma_{k_{\gamma=0}} - \frac{1}{2\alpha} \int d^d x (D_\mu(A)(A_\mu - \tilde{A}_\mu)^2(D_\nu(A)(A_\nu - \tilde{A}_\nu))^2) \] (A.29)
and we use the index convention (cf. (A.13))
\[ \Gamma_{\phi_n(p)\chi_{\alpha}(q)}^{(2)} = \frac{\partial^2 \Gamma}{\partial \phi_{b}^{*}(p) \partial \chi_{\alpha}(q)} \] (A.30)
\[ \Gamma_{\chi_{\alpha}(q)\chi_{\beta}(q)}^{(2)} = -\frac{\partial^2 \Gamma}{\partial \phi_{b}^{*}(p) \partial \xi_{\epsilon}(q)} \] (A.31)
\[ \Gamma_{\psi_{(p)\psi_{(p)}}^{(2)}}^{(2)} = \Gamma_{\psi_{(p)\psi_{(p)}}^{(2)}}^{(2)} - \frac{\partial^2 \Gamma}{\partial \psi_{(p)} \partial \psi_{(p)}} \] (A.32)

We recover the usual identities in the limit \( k \to 0 \) since \( \mathcal{R} = 0 \) implies a vanishing BRS-anomaly \( A_{\text{BRS}} = 0 \).

The sources \( \bar{\beta} \) appear only linearly in \( S \) and it is straightforward to derive the identity
\[ \frac{\partial \Gamma}{\partial \beta_{\mu}^{z}} = -\frac{\partial W}{\partial \beta_{\mu}^{z}} = -\frac{\partial S}{\partial \beta_{\mu}^{z}} = - \frac{1}{g} D_\mu(A)^{\alpha z} \xi^\alpha - \frac{1}{g} f_{\alpha z}^{\gamma} a_{\mu} w^{\gamma} > c \] (A.31)
where the connected two-point function \( < a c > c \) is related to the appropriate matrix element of the propagator \( (\Gamma^{(2)} + \mathcal{R})^{-1} \) by \( (A.14) \)
\[ < a_{\mu} w^{\gamma} > c = (\Gamma^{(2)} + \mathcal{R})^{-1}_{\alpha \mu w} \xi^\alpha \] (A.32)
This allows one to eliminate the explicit dependence on the source \( \bar{\beta} \) in favour of expressions containing two- and three-point functions and to restrict the discussion

\[ ^{5}\text{Note the minus sign whenever the second index is } \xi \text{ or } \psi. \]
to \( \beta = 0 \) afterwards. Identities similar to (A.31) for the other sources of BRS-variations are easily derived

\[
\frac{\partial \Gamma}{\partial \bar{\gamma}} = \frac{1}{2} f^{zw}(\xi^y \xi^w + \langle c^y c^w \rangle_c) \\
\frac{\partial \Gamma}{\partial \bar{\delta}_a} = -i (T_z)_{ab}(\xi^z \varphi_b^+ + \langle c^z \varphi_b'^* \rangle_c) \\
\frac{\partial \Gamma}{\partial \delta_a} = i (T^*_z)_{ab}(\xi^z \varphi_a^+ + \langle c^z \varphi_a'^* \rangle_c)
\]

(A.33)

Finally, one has the field equation for the ghost field

\[
0 = \langle \frac{\partial S}{\partial \bar{c}_z} \rangle = -\frac{\partial \Gamma}{\partial \bar{\xi}^z} - \frac{1}{g} D_{\mu}(A)^{zy} \bar{\beta}_y - f^{zyw} \xi^w \bar{\gamma}^y \\
+ i \bar{\delta}_m^{(\bar{\psi})}(T_z)_{mn} \psi_n + i \bar{\delta}_m^{(\bar{c}_z)}(T_z)_{mn} \psi_n + i \bar{\delta}_m^{(\bar{\psi})}(T_z)_{ab} \bar{\varphi}_b - i \delta_a^{(\bar{c}_z)}(T_z)_{ab} \bar{\varphi}_b^* \\
+ D_{\mu}(A)^{zy} D_{\mu}(\bar{A})^{yw} \bar{\xi}^w - f^{zyu} \langle c^y c^w \rangle_c
\]

(A.34)

For an abelian gauge theory the vanishing structure constants \( f^{zyw} \) lead to important simplifications: First one can replace (A.31)

\[
\frac{\partial \Gamma}{\partial \bar{\beta}_\mu} = -\frac{1}{g} D_{\mu}(A)^{zy} \bar{\beta}_y = 0
\]

(A.35)

and evaluate everything for \( \bar{\gamma} = \bar{\beta}_\mu = 0 \). The field equations for the ghosts reduce to

\[
\frac{\partial \Gamma}{\partial \bar{\xi}} = -i \bar{\delta} \varphi + i \delta \varphi^* + \partial_{\mu} \partial_{\mu} \bar{\xi} \\
\frac{\partial \Gamma}{\partial \xi} = -\partial_{\mu} \partial_{\mu} \xi
\]

(A.36)

where we have limited the matter content to a complex scalar with \( (T_z)_{ab} = -\delta_{ab} \). The ghost-dependent part of \( \Gamma \) is therefore uniquely determined

\[
\Gamma_{gh} = \int d^d x \left\{ \partial_{\mu} \bar{\xi} \partial_{\mu} \xi - i \bar{\delta} \varphi + i \xi \delta \varphi^* \right\}
\]

(A.37)

It is independent of \( k \) and equals the classical expression. Eq. (A.37) also implies that for \( \delta = \bar{\delta} = 0 \) the connected two-point functions involving \( \xi \) and \( \varphi', \varphi'^* \) vanish

\[
\langle c^y \varphi' \rangle_{c^y, \delta = 0} = \langle c^y \varphi'^* \rangle_{c^y, \delta = 0} = 0
\]

(A.38)

and therefore

\[
\frac{\partial \Gamma}{\partial \delta |_{\delta = 0}} = i \xi \varphi, \quad \frac{\partial \Gamma}{\partial \bar{\delta} |_{\bar{\delta} = 0}} = -i \xi \varphi^*
\]

(A.39)
We can now evaluate the Ward identity (A.20) for \( \bar{\beta}_\mu = 0, \bar{\delta} = \delta = 0 \) where \( A_{BRS}^{(g)} = 0 \):

\[
i \int \frac{d^d q}{(2\pi)^d} \left\{ \frac{1}{g} q_\mu \frac{\partial \Gamma'}{\partial A_\mu(q)} + \int \frac{d^d p}{(2\pi)^d} \left[ \frac{\partial \Gamma'}{\partial \varphi(p)} \varphi(p - q) - \frac{\partial \Gamma'}{\partial \varphi^*(p)} \varphi^*(p + q) \right] \right\} \xi(q) = A_{BRS}^{(m)}
\]

(A.40)

\[
A_{BRS}^{(m)} = i \int \frac{d^d p}{(2\pi)^d} \frac{d^d p'}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \left\{ \mathcal{R}(\varphi)^*(p, p')(\Gamma^{(2)} + \mathcal{R})_{\varphi(p-q)}^{-1} \bar{\phi}(p-q) \bar{\phi}(p) \right\} \xi(q)
\]

(A.41)

Since for \( \bar{\delta} = \delta = 0 \) the ghost sector decouples completely from the \( (\varphi, \varphi^*, A) \) sector we can evaluate \( (\Gamma^{(2)} + \mathcal{R})^{-1} \) for \( \xi = \bar{\xi} = 0 \). Furthermore the definitions (A.10), (A.11) imply

\[
(\Gamma^{(2)} + \mathcal{R})_{\bar{\phi}(p-q)}^{-1} \bar{\phi}(p-q) >_c = (\Gamma^{(2)} + \mathcal{R})_{\bar{\phi}(p-q)}^{-1} \bar{\phi}(p-q)
\]

(A.42)

and \( \mathcal{R}(\varphi)^*(p, p') = \mathcal{R}(\varphi)(p', p) \). Since (A.40) must hold for arbitrary \( \xi(q) \) we finally obtain the modified Ward identity in a form not involving the ghost anymore

\[
\frac{1}{g} q_\mu \frac{\partial \Gamma'}{\partial A_\mu(q)} + \int \frac{d^d p}{(2\pi)^d} \left[ \frac{\partial \Gamma'}{\partial \varphi(p)} \varphi(p - q) - \frac{\partial \Gamma'}{\partial \varphi^*(p)} \varphi^*(p + q) \right] = \int \frac{d^d p}{(2\pi)^d} \frac{d^d p'}{(2\pi)^d} \left[ (\mathcal{R}(\varphi)(p, p' + q) - \mathcal{R}(\varphi)(p - q, p'))(\Gamma^{(2)} + \mathcal{R})_{\bar{\phi}(p-q)}^{-1} \bar{\phi}(p) \right]
\]

(A.43)

The existence of such a form is not surprising since for our choice of gauge fixing the ghost sector is just a free field theory and can be omitted altogether. We observe that the l.h.s. of eq. (A.43) is simply a local gauge variation of \( \Gamma' \) with \( \bar{A} \) kept fixed. The gauge-invariant and \( \bar{A} \)-independent part of \( \Gamma' \) does therefore not contribute. From \( \Gamma_k(\varphi, A, \bar{A}) \equiv \Gamma_k(\varphi, A, \bar{A}, \xi = \bar{\xi} = 0) \) we may subtract a gauge-invariant kernel \( \bar{\Gamma}_k[\varphi, A] = \Gamma_k[\varphi, A, \bar{A} = A] \) and define

\[
\hat{\Gamma}_{gf,k} = \Gamma_k[\varphi, A, \bar{A}] - \bar{\Gamma}_k[\varphi, A] - \Gamma_{gf}[A, \bar{A}]
\]

(A.44)

We can therefore replace \( \Gamma' \) by \( \hat{\Gamma}_{gf} \) on the l.h.s. of (A.43). The Ward identity only constrains the “generalized gauge-fixing term” \( \hat{\Gamma}_{gf,k} \) which contains the \( k \)-dependent counterterms for \( k > 0 \) and vanishes for \( k = 0 \). The invariance of the average action with respect to simultaneous gauge transformations of \( A \) and \( \bar{A} \) implies

\[
\frac{1}{g} q_\mu \left( \frac{\partial \Gamma}{\partial A_\mu(q)} + \frac{\partial \Gamma}{\partial A_\mu(q)} \right) + \int \frac{d^d p}{(2\pi)^d} \left[ \frac{\partial \Gamma}{\partial \varphi(p)} \varphi(p - q) - \frac{\partial \Gamma}{\partial \varphi^*(p)} \varphi^*(p + q) \right] = 0
\]

(A.45)
and similarly for $\Gamma'$ or $\hat{\Gamma}_{gf}$ leading to an alternative form of the identity (A.43)

$$q_\mu \frac{\partial \hat{\Gamma}_{gf}}{\partial A_\mu(q)} = -\tilde{g} \int \frac{d^d p}{(2\pi)^d} \frac{d^d p'}{(2\pi)^d} \left( R^{(\varphi)}(p, p' + q) - R^{(\varphi)}(p - q, p') \right) (\Gamma^{(2)} + R)_{\varphi(p')\varphi(p)}^{-1}$$

(A.46)

It is interesting to compare this equation to an identity for the background field dependence of $\hat{\Gamma}_{gf}$ derived earlier [6]

$$\frac{\partial \hat{\Gamma}_{gf}}{\partial A_\mu(q)} = \int \frac{d^d p}{(2\pi)^d} \frac{d^d p'}{(2\pi)^d} \frac{\partial R^{(\varphi)}(p, p')}{\partial A_\mu(q)} (\Gamma^{(2)} + R)_{\varphi(p')\varphi(p)}^{-1}$$

(A.47)

One can show [16] that the Ward identity (A.46) can be derived from the more general “background field identity” (A.47) by using the invariance of $\Delta^{(\varphi)} S$ under simultaneous gauge transformations of $\varphi, A$ and $\bar{A}$ which yields

$$\frac{1}{\tilde{g}} q_\mu \frac{\partial R^{(\varphi)}(p, p')}{\partial A_\mu(q)} = R^{(\varphi)}(p - q, p') - R^{(\varphi)}(p, p' + q)$$

(A.48)

The background field identity (A.47) has a simple solution in the approximation where the $\bar{A}$ dependence of the propagator $(\Gamma^{(2)} + R)^{-1}$ on the r.h.s. is neglected

$$\hat{\Gamma}_{gf} = \langle \Delta^{(\varphi)} S \rangle - \Delta^{(\varphi)} S[\varphi] = \int \frac{d^d p}{(2\pi)^d} \frac{d^d p'}{(2\pi)^d} R^{(\varphi)}(p, p') \langle \varphi^*(p) \varphi(p') \rangle_c$$

(A.49)

A background field identity for non-abelian gauge theories has also been derived [6], [11]. The precise relation to the Slavnov-Taylor identity has not yet been established. It is clear, however, that the background field identity contains information beyond the Slavnov-Taylor identity.

Let us finally turn to solutions of the Slavnov-Taylor identity for non-abelian gauge theories. We concentrate first on a vanishing infrared cutoff $R_k = 0$. We are interested in solutions for vanishing sources $\beta$ for the BRS-variations. Eq. (A.26) contains then a sum of expressions for which the first factor can be evaluated at $\beta = 0$, whereas the second factor involves the coefficients linear in $\beta$ evaluated at $\beta = 0$. It is straightforward to show that the ansatz

$$\Gamma = \Gamma_{inv}[A, \psi, \varphi] + \Gamma_{gf} + \Gamma_{gh} + \Gamma_s + \Delta \Gamma[\bar{A}]$$

$$\Gamma_{gf} = \frac{1}{2\alpha} \int d^d x \left( D^\mu(A)(A_\mu - \bar{A}_\mu) \right)^2 \left( D^\nu(A)(A_\nu - \bar{A}_\nu) \right)^2$$

$$\Gamma_{gh} = - \int d^d x \xi^2 \left( D^\mu(A) D_\mu(A) \xi \right)^2$$

42
\[
\Gamma_s = - \int d^4x \left\{ \frac{1}{\bar{g}} \bar{\beta}_a (D^\mu (A) \xi)^z + \frac{1}{2} \bar{\gamma}^z f^{zyw} \bar{\xi}^y \xi^w \\
- i \bar{\delta}_m^{(\psi)}(T^z)_{mn} \psi_n + i \delta_m^{(\psi)}(T^z)_{mn} \bar{\psi}_n \\
+ i \bar{\delta}_a^{(\varphi)}(T^z)_{ab} \phi_b - i \delta_a^{(\varphi)}(T^z)_{ab} \bar{\phi}_b \right\}
\] (A.50)

obeys the identity (A.26) with vanishing anomaly on the r.h.s. Here \( \Gamma_{\text{inv}}[A, \psi, \phi] \) is an arbitrary gauge-invariant functional which does not depend on the ghost fields, and \( \Delta \Gamma[\bar{A}] \) is an arbitrary gauge-invariant functional of the background field \( \bar{A} \). In fact, gauge invariance of \( \Gamma_{\text{inv}} \) implies the identity

\[
\int \frac{d^d p}{(2\pi)^d} \left\{ \frac{1}{\bar{g}} \frac{\partial \Gamma_{\text{inv}}}{\partial A^z_\mu (p)} (D^\mu (A) \xi)^z(p) + i \frac{\partial \Gamma_{\text{inv}}}{\partial \psi_m (p)} (\bar{\xi}(T^z)_{mn} \psi_n)(p) \\
- i \frac{\partial \Gamma_{\text{inv}}}{\partial \varphi_a (p)} (\xi(T^z)_{ab} \phi_b)(p) - i \frac{\partial \Gamma_{\text{inv}}}{\partial \bar{\varphi}_a (p)} (\bar{\xi}(T^z)_{ab} \bar{\phi}_b)(p) \right\} = 0
\] (A.51)

This allows to replace \( \Gamma' \) by \( \Gamma_{\text{gh}} \) for the first factors in (A.26). What remains is the relation

\[
\int \frac{d^d p}{(2\pi)^d} \left\{ \frac{\partial \Gamma_{\text{gh}}}{\partial A^z_\mu (p)} (D^\mu (A) \xi)^z(p) - \frac{\bar{g}}{2} f^{zyw} \frac{\partial \Gamma_{\text{gh}}}{\partial \xi^y \xi^w (p)} \right\} = 0
\] (A.52)

which is easily verified using the Jacobi identity for the structure constants \( f^{ztw} f^{tsw} - f^{ztw} f^{tsw} = f^{zty} f^{tsw} \). We observe that the gauge-invariant part \( \Gamma_{\text{inv}} \) remains completely unconstrained by the Slavnov-Taylor identity. We can also verify that the ansatz (A.50) obeys the field equation for the antighost (A.24). In contrast, the source identities (A.31), (A.33) hold only if the pieces involving the connected two-point functions \( < ac >, < cc >, < c\varphi' > \) etc. all vanish. This is generically not the case for the ansatz (A.50), since the cubic vertex \( \sim \bar{g} \bar{\xi} A \) in \( \Gamma_{\text{gh}} \) induces a non-trivial off-diagonal matrix element in the inverse propagator \( \Gamma_k^{(2)} \), mixing the ghost sector to the other fields. This, in turn, is responsible for corresponding off-diagonal elements in \( (\Gamma^{(2)} + \mathcal{R})^{-1} \). We observe that these off-diagonal elements vanish in the limit of small gauge coupling \( \bar{g} \to 0 \) such that (A.31) and (A.33) are obeyed in this approximation. The situation for the ghost field equation (A.34) is completely analogous. We conclude that for small \( \bar{g} \to 0 \) the whole picture becomes formally very similar to the abelian case discussed above, with leading non-abelian structure

6These two-point functions should be evaluated from the ansatz (A.50) for vanishing sources \( \beta \).
given by the ansatz (A.50). We should emphasize that this approximation can be expected to be valid only for the momentum range where the running renormalized gauge coupling remains small enough.

The ansatz (A.50) is, however, not the most general solution. We may try to obtain a more general solution by replacing in (A.50) the ghost field \( \xi^z \) by a functional \( \hat{\xi}^z[A,\xi,\bar{\xi},\psi,\phi] \) and changing the term linear in \( \bar{\gamma}^w \). The functional \( \hat{\xi}^z \) should have ghost number one and obey the same symmetry transformation laws as \( \xi^z \) under gauge transformations acting on \( A \) and \( \bar{A} \) simultaneously. Eq. (A.51) holds also with \( \xi \) replaced by \( \hat{\xi}^z \) such that \( \Gamma_{inv} \) again drops out. The remaining equation relates \( \frac{\partial \Gamma}{\partial \bar{\gamma}^w} \) to the functional form of \( \hat{\xi}^z \):

\[
\int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} (D_\mu(A)D^{\mu}(\bar{A})\hat{\xi})^z(q) \left\{ \frac{\partial \hat{\xi}^z(q)}{\partial \bar{\gamma}^w(p)} \frac{\partial \Gamma}{\partial \bar{\gamma}^w(p)} \right\} 
+ \frac{1}{2} \int \bar{\gamma}^w(\hat{\xi}^w(p))(2\pi)^d \delta(q-p) + \frac{1}{g} \frac{\partial \hat{\xi}^z(q)}{\partial A_\mu^w(p)} (D^{\mu}(A)\hat{\xi})^w(p) 
+ i \frac{\partial \hat{\xi}^z(q)}{\partial \varphi^a(p)} (\hat{\xi}^y(T_y)^{mn}\psi_n(p) - i \frac{\partial \hat{\xi}^z(q)}{\partial \varphi^a(p)} (\hat{\xi}^y(T_y)^{mn}\bar{\psi}_n)(p) 
- i \frac{\partial \hat{\xi}^z(q)}{\partial \bar{\varphi}_a(p)} (\hat{\xi}^y(T_y)^{ab}\varphi_b(p) + i \frac{\partial \hat{\xi}^z(q)}{\partial \bar{\varphi}_a^*(p)} (\hat{\xi}^y(T_y)^{ab}\bar{\varphi}_b^*)(p) \right\} = 0 \tag{A.53}
\]

Given an arbitrary \( \hat{\xi}^z \) one can always solve the equation for \( \frac{\partial \Gamma}{\partial \bar{\gamma}^w} \) provided the relation between \( \hat{\xi}^z \) and \( \xi^z \) remains invertible. At this level we have therefore a general class of solutions for the identity (A.26) since \( \hat{\xi}^z \) remains essentially unconstrained. The constraints arise from the source identities (A.31), (A.33) and the ghost field equation (A.34). (Note that the field equation for the antighost (A.24) is obeyed for arbitrary \( \hat{\xi}^z \).) In fact, the difference between \( \hat{\xi}^z \) and \( \xi^z \) is related to the connected two-point functions involving the ghost field

\[
(D_\mu \hat{\xi})^z - (D_\mu \xi)^z = \bar{g} f^{zwuy} < a^w_{\mu} e^y_c >_c \\
(T_{2a}((\hat{\xi}^z - \xi^z)\varphi_b - c^z \varphi^*_b) < c^z >_c) = 0 \tag{A.54}
\]

Similar relations must hold for \( \varphi^*, \psi \) and \( \bar{\psi} \) and the equation for \( \frac{\partial \Gamma}{\partial \bar{\gamma}^z} \) must be compatible with the solution of (A.53). We conclude that the generalized ansatz with \( \xi \) replaced by \( \hat{\xi} \) can only be used in the approximation where this system of equations for \( \hat{\xi} \) is self-consistent. Beyond this approximation the gauge invariance of the sector with ghost number zero (\( \Gamma_{inv} \)) can probably not be maintained.
In the presence of a nonvanishing infrared cutoff $R_k$ an additional piece $\Gamma_{ct}$ containing counterterms should be added. It vanishes in the limit $k \to 0$. In the perturbative regime $\Gamma_{ct}$ contains a gluon mass term $\sim k^2$ (see sect. 4). There is no reason why $\Gamma_{ct}$ should be gauge-invariant under transformations which leave the background field $\bar{A}$ fixed. We may define $\Gamma_{ct}$ by the requirement that $\Gamma - \Gamma_{ct}$ obeys the anomaly-free Slavnov-Taylor identity. The anomaly $A^{(g)}_{\text{BRS}} + A^{(m)}_{\text{BRS}}$ on the r.h.s. of eq. (A.26) determines then the form of $\Gamma_{ct}$. In a lowest order approximation we may treat $\Gamma_{ct}$ and the anomaly as a small quantity and linearize eq. (A.26) in $\Gamma_{ct}$.

For the computation of the gluon propagator in the present paper we make only a crude approximation for the three- and four-gluon vertices appearing on the r.h.s. of the flow equation. They correspond to the ansatz (A.50) with $\Gamma_{\text{inv}}$ containing only a piece $\sim F_{\mu\nu}F^{\mu\nu}$. An improved treatment of these vertices could generalize $\Gamma_{\text{inv}}$ for a nontrivial momentum dependence of the propagator

$$
\Gamma_{\text{inv}} = \frac{1}{4} \int dx F_{\mu\nu} K(-D^2(A)) F^{\mu\nu}
$$

(A.55)

with $K(x) = (G_A(x) - G_A(0))/x$.

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Figure Captions

Fig. 1: Wave function renormalization $\tilde{Z}_F$ as a function of the average scale $k$.

Fig. 2: Momentum-dependent anomalous dimension $\chi(q)$ in comparison with perturbation theory for various values of $k \geq 1$ GeV.

Fig. 3: The same as fig. 2, for $k \leq 1$ GeV.
Figure 1
Figure 2
Figure 3

\[
\chi(q)
\]

\[
q^2 \text{ [GeV}^2\text{]}
\]

$k = 400$ MeV

$k = 1$ GeV