How to Catch $L_2$-Heavy-Hitters on Sliding Windows

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Abstract

Finding heavy-elements (heavy-hitters) in streaming data is one of the central, and well-understood tasks. Despite the importance of this problem, when considering the sliding windows model of streaming (where elements eventually expire) the problem of finding $L_2$-heavy elements has remained completely open despite multiple papers and considerable success in finding $L_1$-heavy elements. In this paper, we develop the first poly-logarithmic-memory algorithm for finding $L_2$-heavy elements in sliding window model. (That is, finding elements that appear more times than a specified threshold of the $L_2$ frequency norm.) This is especially significant since finding $L_2$-heavy elements has far greater applicability in many practical scenarios and is considered a more central question. What is especially interesting regarding our solutions is that finding $L_2$-heavy elements is not a “smooth” function, and thus previous methods of working with sliding windows via smooth histograms (that in turn generalizes exponential histograms) are not applicable.

To obtain our results on $L_2$ heavy elements, we “merely” combine (but in a non-trivial and novel way) existing techniques from the literature. In fact, variants of techniques sufficient to derive similar results were known since 2002, yet, no algorithm for $L_2$ heavy elements was reported. Since $L_2$ heavy elements play a central role for many fundamental streaming problems (such as frequency moments), we believe our method would be extremely useful for many sliding-windows algorithms and applications. For example, our technique allows us not only to find $L_2$-heavy elements, but also heavy elements with respect to any $L_p$ for $0 < p < 2$ on sliding windows. Thus, our paper completely resolves the question of finding $L_p$-heavy elements for sliding windows with poly-logarithmic memory for all values of $p$ since it is well known that for $p > 2$ this task is impossible.

Our method may have other applications as well. We demonstrate a broader applicability of our novel yet simple method on two additional examples: we show how to obtain a sliding window approximation of other properties such as the similarity of two streams, or the fraction of elements that appear exactly a specified number of times within the window (the $\alpha$-rarity problem). In these two illustrative examples of our method, we replace the current expected memory bounds with worst case bounds.

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1 Introduction

A data stream $S$ is an ordered multiset of elements $\{a_0, a_1, a_2, \ldots\}$ where each element $a_t \in \{1, \ldots, u\}$ arrives at time $t$. In the sliding window model we consider at each time $t \geq N$ the last $N$ elements of the stream, i.e. the window $W = \{a_{t-(N-1)}, \ldots, a_t\}$. Those elements are called active elements, whereas elements that arrived prior to the current window $\{a_i \mid 0 \leq i < t-(N-1)\}$ are expired. For $t < N$, the window consists of all the elements received so far, $\{a_0, \ldots, a_t\}$.

Usually, both $u$ and $N$ are considered to be extremely large so it is not applicable to save the entire stream (or even one entire window) in memory. The problem is to be able to calculate various characteristics about the window’s elements using a small amount of memory (usually, poly-logarithmic in $N$ and $u$). We refer the reader to the books of Muthukrishnan [39] and Aggarwal (ed.) [1] for extensive surveys on data stream models and algorithms.

One of the main open problems in data streams deals with the relations between the different streaming models [36], specifically between the unbounded stream model and the sliding window model. In this paper we provide another important step in clarifying the connection between these two models by showing that finding $L_p$-heavy hitters is just as doable on the sliding windows model as on the entire stream.

Motivation. We focus on approximation-algorithms for certain statistical characteristics of the data streams, specifically, finding frequent elements. The problem of finding frequent elements in a stream is very interesting for many applications, such as network monitoring [12] and DoS prevention [23, 18, 4], and was extensively explored over the last decade (see [39, 17] for a definition of the problem and a survey of existing solutions, as well as [14, 55, 26, 32, 19, 3, 20, 43, 27]).

Determining which element is considered heavy can be interpreted in several ways. The most common way is to define a fraction of the stream size, and say that every element that appears more times than this threshold is considered heavy. A more powerful notion is to define a heavy element as such that appears more times than a specified fraction of the second frequency norm of the stream (or window). Recall that the $L_p$ norm of the frequency vector \(^1\) is defined by $L_p = (\sum_i n_i^p)^{1/p}$, where $n_i$ is the frequency of element $i \in [u]$, i.e., the number of times $i$ appears in the window. Thus, the $L_1$ norm is fixed to the size of the stream (or window), while the $L_2$ norm is affected by the specific frequencies of the elements in the stream.

Finding frequent elements with respect to the $L_2$ norm is a more difficult task than the equivalent $L_1$ problem. To demonstrate this let us regard the following example: let $S$ be a stream of size $N$, in which the element $a_1$ appears $\sqrt{N}$ times, while the rest of the elements $a_2, \ldots, a_t$ appear exactly once in $S$. It follows that $n_1 = \frac{1}{\sqrt{N}}L_1$ while $n_1 = cL_2$, where $c$ is a constant, lower bounded by $\frac{1}{\sqrt{2}}$. Any algorithm that finds all the elements which are heavier then $\gamma L_1$ with memory $poly(\gamma^{-1}, \log N, \log u)$, can not identify $a_1$ as an heavy element\(^2\).

Generally speaking, identifying frequent elements (heavy-hitters) with respect to $L_p$ is better for larger values of $p$ [30]. We focus on solving the following $L_2$-heaviness problem:

**Definition 1** (($\gamma, \epsilon$)-approximation of $L_2$-frequent elements, [30, 38]). For $0 < \epsilon, \gamma < 1$, output any element $i \in [u]$ such that $n_i > \gamma L_2$ and no element such that $n_i < (1-\epsilon)\gamma L_2$.

The $L_2$ norm is the most powerful norm for which we can expect a poly-logarithmic solution, for the frequent-elements problem. This is due to the known lower bound of $\Omega(u^{1-2/p})$ for calculating

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1 Throughout the paper we use the term “$L_p$ norm” to indicate the $L_p$ norm of the frequency vector, i.e., the $p$th root of the $p$th frequency moment $F_p = \sum_i n_i^p$ [2], rather than the norm of the data itself.

2 In order for the algorithm to be poly-logarithmic, it must be the case that $\gamma^{-1} = O(\log^c N)$ for some constant $c$. 


\(L_p\) over a stream, which results from observing the amount of memory required to identify an element which is a constant fraction of the \(L_p\) norm.

There has been a lot of progress on the question of finding \(L_1\)-frequent elements, in the sliding window model \([3, 43, 27]\), however those algorithms can not be used to find \(L_2\)-frequent elements with an efficient memory. In 2002, Charikar, Chen and Farach-Colton \([14]\) showed an algorithm that can approximate the “top \(k\)” frequent-elements in an unbounded stream, where \(k\) is given as an input. Formally, their algorithm outputs only elements with frequency larger than \((1 - \epsilon)\phi_k\), where \(\phi_k\) is the frequency of the \(k^{th}\) most frequent element in the stream, using memory proportional to \(L_2^2/(\epsilon \phi_k)^2\). Since the “heaviness” in this case is relative to \(\phi_k\), and the memory is bounded by the fraction \(L_2^2/(\epsilon \phi_k)^2\), Charikar et al.’s algorithm finds in fact heaviness in terms of the \(L_2\) norm. A natural question is whether one can develop an algorithm for finding frequent-elements that appear at least \(\gamma L_2\) times in the \textit{sliding window model}, using \(\text{poly}(\gamma^{-1}, \log N, \log u)\) memory.

We give the first solution for the problem of finding an \(\epsilon\)-approximation of the \(L_2\)-frequent elements in the sliding window model. Our algorithm is able to identify elements that appear within the window a number of times which is at least a \(\gamma\)-fraction of the \(L_2\) norm of the window, up to a multiplicative factor of \((1 - \epsilon)\). In addition, the algorithm guarantees to output all the elements with frequency at least \((1 + \epsilon)\gamma L_2\). Our solution gives another step in the direction of making a connection between the unbounded and sliding window models, as it provides an answer for the very important question of heavy hitters in the sliding window model. The result joins the various solutions of finding \(L_1\)-heavy hitters in sliding windows \([26, 3, 40, 4, 43, 27, 28]\), and can be used in various algorithms that require identifying \(L_2\) heavy hitters, such as \([31, 8]\) and others. We note that problem being considered here is the most powerful \(L_p\) which can have a poly-logarithmic solution, specifically, the \(L_2\) norm \([30]\). More generally, our paper resolves the question of finding \(L_p\)-heavy elements on sliding windows for all values of \(p\) that allows small memory one-pass solutions (i.e. all \(0 < p \leq 2\)).

To achieve our result on \(L_2\) heavy hitters, we combine (in a non-trivial way) existing techniques. Variants of these techniques sufficient to derive similar results were known since 2002. Surprisingly, no algorithm for \(L_2\) heavy hitters was reported despite several papers on \(L_1\) heavy hitters. Since \(L_2\) heavy hitters play a central role for many fundamental streaming problems (such as frequency moments and many others), we suspect that our \(L_2\) heavy hitters algorithm will significantly increase our understanding of sliding windows.

A Broader Perspective. In fact, one can consider the tools we develop for the frequent elements problem as a general method that allows obtaining a sliding window solution out of an algorithm for the unbounded model, for a wide range of functions. We explain this concept in this section.

Many statistical properties were aggregated into families, and efficient algorithms were designed for those families. For instance, Datar, Gionis, Indyk and Motwani, in their seminal paper \([21]\) showed that a sliding window estimation is easy to achieve for any function which is \textit{weakly-additive} by using a data structure named \textit{exponential histograms} \([21]\); for certain functions that decay with time, one can maintain time-decaying aggregates \([16]\); another data structure, named \textit{smooth-histogram} \([9]\) can be used in order to approximate an even larger set of functions, known as \textit{smooth functions}. See \([1]\) for a survey of synopsis construction.

In this paper we introduce a new concept which uses a smooth-histogram in order to perform sliding window approximation of \textit{non-smooth} properties. Informally speaking, the main idea is to relate the

\[\text{Indeed, we use the algorithm of Charikar et al. \([14]\) that is known since 2002. Also, it is possible to replace (with some non-trivial effort) our smooth histogram method for } L_2 \text{ computation with the algorithm of Datar, Gionis, Indyk and Motwani \([21]\) for } L_2 \text{ approximation.}\]
non-smooth property $f$ with some other, smooth property $g$, such that changes in $f$ are bounded by the changes in $g$. By maintaining a smooth-histogram for the smooth function $g$, we partition the stream into sets of sub-streams (buckets). Due to the properties of the smooth-histogram we can bound the error (of approximating $g$) for every sub-stream, and thus get an approximation of $f$. We use the term semi-smooth to describe these kinds of algorithms.

We demonstrate the above idea by showing a concrete efficient sliding window algorithm for the properties of rarity and similarity [22], we stress that neither is smooth. In addition to those properties, we believe that the tools we develop here can be used to build efficient sliding window approximations for many other (non-smooth) properties and provides a general new method for computing on sliding windows. It is important to note that trying to build a smooth-histogram (or any other known sketch) directly to $f$ will not preserve the required invariants, and the memory consumption might be not be efficient.

1.1 Previous works

**Frequent elements problem.** Finding elements that appear many times in the stream (“heavy hitters”) is a very central question and thus has been extensively studied both for the unbounded model [19, 37] and for the sliding window model [3, 40, 43, 27] as well as other variants such as the offline stream model [35], insertion and deletion model [20, 32], finding heavy-distinct-hitter [4], etc. Reducing the processing time was done by [34] into $O(\frac{1}{\epsilon})$ and by [28] into $O(1)$.

Another problem which is related to finding the heavy hitters, is the top-$k$ problem, namely, finding the $k$ most frequent elements. As mentioned above, Charikar, Chen and Farach-Colton [14] provide an algorithm that finds the $k$ most frequent elements in the unbounded model (up to a precision of $1 \pm \epsilon$). Golab, DeHaan, Demaine, López-Ortiz and Munro [26] solve this problem for the jumping window model.

**Similarity and $\alpha$-rarity problem.** The similarity problem was defined in order to give a rough estimation of closeness between files over the web [13] (and independently in [15]). Later, it was shown how to use min-hash functions [29] in order to sample from the stream, and estimate the similarity of two streams.

The notion of $\alpha$-rarity, introduced by Datar and Muthukrishnan [22], is that of finding the fraction of elements that appear $\alpha$ times within the stream. This quantity can be seen as finding the fraction of elements with frequency within certain bounds.

The questions of rarity and similarity were analyzed, both for the unbounded stream and the sliding window models, by Datar and Muthukrishnan [22], achieving an expected memory bound of $O(\log N + \log u)$ words of space for a constant $\epsilon, \alpha, \delta$. At the bit level, their algorithm yields $O(\alpha \cdot \epsilon^{-3} \log \delta^{-1} \log N (\log N + \log u))$ bits for $\alpha$-rarity and $O(\epsilon^{-3} \log \delta^{-1} \log N (\log N + \log u))$ bits for similarity, with $1 - \delta$ being the probability of success.

1.2 Our results

We build the first efficient sliding window approximation for the $L_2$-frequent-element problem, prove its correctness and bound its memory consumption by $O(\frac{1}{\gamma^4 \epsilon^4} \log N \log N + \frac{1}{\epsilon^4} \log N \log \frac{1}{\epsilon})$ words.

**Theorem 1.** There exists a sliding window algorithm that obtains an $\epsilon$-approximation for the $L_2$ frequent-elements problem, in space $\text{poly}(\epsilon^{-1}, \gamma^{-1}, \log N, \log \delta^{-1})$.

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4 Of course, other kinds of aggregations can be used, however our focus is on smooth histograms.

5 These bounds are not explicitly stated in the paper [22], but follow from the analysis (see Lemma 1 and Lemma 2 in [22]).
Other results reported in this paper are efficient algorithms for the rarity and similarity problems (see Sections 4.2 and 4.3 for a detailed definition of the problems). Although for these problems there already exist algorithms with efficient yet expected memory bounds, our techniques achieve a worst case memory bounds of essentially the same magnitude (up to a factor of log log N).

As mentioned above, the tools we develop here can be seen as a general method of building a sliding window approximation for a non-smooth property \( f \), using a related smooth property \( g \). For the problem of finding frequent elements, we use the \( L_2 \) norm as the smooth property \( g \), while for estimating rarity and similarity we set \( g \) to be the number of distinct elements in the stream.

1.3 High-Level Ideas

The main idea behind our approach is quite simple: partitioning the streams into a poly-logarithmic number of partitions (buckets) and estimating the frequencies in each. Although this approach is not new, most of the previous algorithms partition the stream in a correlative way to the estimated function (e.g., exponential and smooth histogram). There have been several works that partitioned the stream arbitrarily, e.g., by splitting the stream into Basic Windows [44] which provides a solution for a slightly different model [26].

The novel technique, and the main conceptual contribution of our paper is to partition the stream according to some other function \( g \), rather than the function we wish to estimate. As long as \( g \) has structurally nice properties (e.g., is a smooth function, weakly additive, etc.) and there can be a special type correlation (as we illustrate in this paper on a number of examples) between \( g \) and the function we wish to estimate — the partition according to \( g \) then gives useful way of obtaining estimation of \( f \) with efficient memory.

It is important to note that the method presented here is different from the concept of Purifier [10], recently presented by Braverman and Ostrovsky, despite superficial similarities. More specifically, in [10] a description of Purifier is given: an informal method of reductions that is applied to solve a wide class of functions. Both methods use an “easier” function \( g \) as a building block for computing a “difficult” function \( f \). However, the ways of applying the reductions are completely different. To begin with, the purifier is applied on unbounded streams while our method is applied on sliding windows. Most importantly, in [10] the “easier” functions \( g \) is used as a substitute for \( f \) under certain conditions. In this paper we use “easier” functions to “format” the stream into right buckets. We explicitly state that \( g \) cannot be used as an approximation of \( f \) since \( g \) is smooth and \( f \) is not. Applying \( g \) as a right formatting tool is exactly the novelty of our method.

Our scheme begins with building a smooth-histogram [9] for approximating the \( L_2 \) norm of the frequency vector of the window. By doing so, we partition the stream into \( O(\frac{1}{\epsilon^2} \log N) \) buckets, so that the current window is always a suffix of the first bucket \( A_1 \), and their frequency \( L_2 \) norm is “close”. Moreover, the smooth-histogram provides us with an \( \epsilon \)-estimation of the frequency \( L_2 \) norm of the window, \( \hat{L}_2 \). This value is also a good estimation of the frequency \( L_2 \) norm of the bucket \( A_1 \) (with a different parameter \( \epsilon' \)), since the window and the bucket have a close frequency \( L_2 \) norm.

Next we show that every element that is “heavy” in the window, is also heavy in the bucket \( A_1 \). This allows us to use the method of Charikar et al. in order to obtain a list of the most frequent elements in the bucket, along with their approximated frequencies, \( \hat{n}_i \). Setting the memory to \( O(\epsilon^{-2} \gamma^{-2} \log N) \), Charikar et al.’s algorithm is guaranteed to output any element with frequency higher than a specified threshold, specifically \( (1 + \epsilon) \gamma L_2 \). However, it might output other elements which are less frequent.

Finally, we use the approximated frequencies \( \hat{n}_i \) of the elements in the bucket \( A_1 \), as outputted by Charikar et al.’s algorithm, along with the \( L_2 \) approximation of the window, \( \hat{L}_2 \), as given by the smooth-histogram, in order to output only those elements which are heavy enough in the window.
This method is used again for estimating the $\alpha$-Rarity and the Similarity of two streams. Instead of partitioning the stream according to the estimated value of the rarity (or similarity), we partition the stream according to the number of distinct elements. Namely, we build a smooth-histogram so that the bucket $A_1$ and the window would have approximately the same number of distinct elements. We then use approximated min-wise hash functions [29] to sample elements from the bucket. Using this sampling, following the techniques of Broder [11] and of Datar and Muthukrishnan [22] for unbounded streams, we are able to obtain the specific estimation (i.e., of similarity or $\alpha$-rarity) on the bucket. Finally, we show that for this specific partition, an estimation on the bucket is a good estimation on the current window.

1.4 Organization of the paper

Section 2 describes our notations and several of the tools we use. In Section 3 we define and discuss the frequent-elements problem, and provide an algorithm which solves the approximation version of the frequent-elements problem in the sliding window model. Section 4 describes and analyzes a semi-smooth algorithm for the $\alpha$-rarity problem and a semi-smooth algorithm for approximating the similarity of two streams, in the sliding window model. Conclusions are given in Section 5.

2 Preliminaries

2.1 Notations

We say that an algorithm $A$ is an $(\epsilon, \delta)$-approximation of a function $f$, if $(1 - \epsilon)f \leq A \leq (1 + \epsilon)f$, except for probability $\delta$ over $A$’s coin tosses. We denote this relation as $A \in (1 \pm \epsilon)f$ for short. We denote an output of an approximation algorithm with a hat symbol, e.g., the estimator of $f$ is denoted $\hat{f}$.

The set $\{1, 2, \ldots, n\}$ is usually denoted as $[n]$. If the stream $B$ is a suffix of $A$, we denote $B \subseteq_r A$. For instance, let $A = \{q_1, q_2, \ldots, q_n\}$ then $B = \{q_{n_1}, q_{n_1+1}, \ldots, q_n\} \subseteq_r A$, if $1 \leq n_1 \leq n$. The notation $A \cup C$ denotes the concatenation of the stream $C = \{c_1, c_2, \ldots, c_m\}$ to the end of stream $A$, i.e., $A \cup C = \{q_1, q_2, \ldots, q_n, c_1, c_2, \ldots c_m\}$. The notation $|A|$ denotes the number of different elements in the stream $A$, that is the cardinality of the set imposed by the multiset $A$. The size of the stream (i.e. of the multiset) $A$ will be denoted as $|A|$, e.g., for the example above $|A| = n$.

We use the notation $\tilde{O}(\cdot)$ to indicate an asymptotic bound which suppresses any term of magnitude $poly(log \frac{1}{\epsilon}, \log \log \frac{1}{\delta}, \log \log N, \log \log u)$.

2.2 Smooth Histograms

Recently, Braverman and Ostrovsky [9] showed that any function $f$ that can be calculated (or approximated) in the unbounded stream model, and belongs to a family of functions named smooth-functions, can also be $\epsilon$-approximated in the sliding window model. Formally,

**Definition 2** (Smooth-Functions [9]). A polynomial function $f$ is $(\alpha, \beta)$-smooth if it satisfies the following properties: (i) $f(A) \geq 0$; (ii) $f(A) \geq f(B)$ for $B \subseteq_r A$; and (iii) there exist $0 < \beta \leq \alpha < 1$ such that if $(1 - \beta)f(A) \leq f(B)$ for $B \subseteq_r A$, then $(1 - \alpha)f(A \cup C) \leq f(B \cup C)$ for any adjacent $C$.

If an $(\alpha, \beta)$-smooth $f$ can be calculated (or $(\epsilon, \delta)$-approximated) on an unbounded stream with memory $g(\epsilon, \delta)$, then there exists an $(\alpha + \epsilon, \delta)$-estimation of $f$ in the sliding window model using $O(\frac{1}{\beta} \log N(g(\epsilon, \frac{\delta \beta}{\log N}) + \log N))$ bits [9].
The key idea is to save a “smooth-histogram” (SH) [9], a structure that contains estimations on \(O(\frac{1}{\beta} \log N)\)-suffixes of the stream, \(A_1 \supseteq A_2 \supseteq \ldots \supseteq A_r \supseteq n_{c^2 \log(n)}\). Each suffix \(A_i\) is called a Bucket. Each new element in the stream initiates a new bucket, however adjacent buckets with a close estimation of \(f\) are removed (keeping only one representative). Since the function is “smooth”, i.e., monotonic and slowly-changing, it is enough to save \(O(\frac{1}{\beta} \log N)\) buckets in order to maintain a reasonable approximation of the window. At any given time, the current window \(W\) is between buckets \(A_1\) and \(A_2\), i.e. \(A_1 \supseteq W \supseteq A_2\). Once the window “slides” and the first element of \(A_2\) expires, we delete the bucket \(A_1\) and renumber the indices so that \(A_2\) becomes the new \(A_1\), \(A_3\) becomes the new \(A_2\), etc. We will use the value of the smooth-function on bucket \(A_1\) to estimate the value of the function on the current window. The relation between the window and the bucket as derived in [9] is given by
\[
(1 - \alpha)f(A_1) \leq f(A_2) \leq f(W) \leq f(A_1).
\]

### 3 A Semi-Smooth Estimation of Frequent Elements

In this section we develop an efficient semi-smooth algorithm for finding the elements which occur frequently within the window. Let \(n_i\) be the frequency of an element \(i \in \{1, \ldots, u\}\), i.e., the number of times \(i\) appear in the window. The first frequency norm and the second frequency norm of the window are defined by \(L_1 = \sum_{i=1}^{u} n_i = N\) and \(L_2 = (\sum_{i=1}^{u} n_i^2)^{\frac{1}{2}}\). Our notion of approximating frequent elements is given in Definition 1. An equivalent definition to Definition 1 which we use in our proof is the following:

**Definition 3.** For \(0 < \epsilon, \gamma < 1\), output any element \(i \in [u]\) with frequency higher than \((1 + c\epsilon)\gamma L_2\), and do not output any element with frequency lower than \((1 - c\epsilon)\gamma L_2\), for a constant \(c\).

Many papers [19, 3, 39, 43, 27] make use of the following (weaker) definition

**Definition 4** ((\(\gamma, \epsilon\))-approximation of \(L_1\)-heavy hitters). Output any element \(i \in [u]\) such that \(n_i \geq \gamma L_1\) and no element such that \(n_i \leq (1 - \epsilon)\gamma L_1\).

An \(L_2\) approximation is stronger than the above \(L_1\) definition [30], since when a certain element is heavy in the means of the \(L_1\) norm, it is also heavy in the means of the \(L_2\) norm,
\[
n_i \geq \gamma L_1 = \gamma \sum_j n_j \implies n_i^2 \geq \gamma^2 \left( \sum_j n_j \right)^2 \geq \gamma^2 \sum_j n_j^2 = (\gamma L_2)^2,
\]
while the opposite direction is not true for the general case.

In order to identify the frequent elements in the current window, we use an algorithm by Charikar et al. [14], which provides an \(\epsilon\)-approximation (for the unbounded stream model) of the following problem

**Definition 5** ((\(k, \epsilon\))-top frequent approximation [14]). Output a list of \(k\) elements such that every outputted element \(i\) has a frequency \(n_i > (1 - \epsilon)\phi_k\), where \(\phi_k\) is the frequency of the \(k\)th most frequent element in the stream.

The algorithm of Charikar et al. guarantees that any element that satisfies \(n_i > (1 + \epsilon)\phi_k\), appears in the output. This algorithm runs on a stream of size \(n\) and succeeds with probability at least \(1 - \delta\) with memory complexity of
\[
O\left( \left( k + \frac{1}{(\epsilon \gamma)^2} \right) \log \frac{n}{\delta} \right).
\]
for every $\delta > 0$, given that $\phi_k \geq \gamma L_2$.

Definition 5 and Definition 1 do not describe the same problem, yet they are strongly connected. In fact, our method allows solving the frequent elements problem under both definitions, however in this paper we focus on solving the $L_2$-frequent-elements problem, as defined by Definition 3. In order to do so, we use the algorithm COUNTSKETCH$_b$ given by Charikar et al., with specific parameters tailored for our problem. The algorithm outputs a list of elements, and is guaranteed to output every element with frequency at least $(1 + \epsilon')\gamma L_2$ and no element of frequency less than $(1 - \epsilon')\gamma L_2$, for an input parameter $\epsilon'$. See Appendix A for a full definition of COUNTSKETCH$_b$ and a proof of its properties.

Note that a solution of the $k$-top frequent problem (Definition 5) in the sliding window model can be done by using the same technique using the original algorithm of Charikar et al.

### 3.1 Semi-smooth algorithm for frequent elements approximation

Our algorithm builds a smooth-histogram for the $L_2$ norm, and partitions the stream into buckets accordingly. It is known that the $L_2$ property is a $(\epsilon, \epsilon^2)$-smooth function [9]. Using the method of Charikar et al. [14], separately on each bucket, with a careful choice of parameters, we are able to approximate the $(\gamma, \epsilon)$-frequent elements problem on a sliding window (Figure 1).

**ApproxFreqElements**($\gamma, \epsilon, \delta$)

1. Maintain an $(\epsilon/2, \delta/2)$-estimation of the $L_2$ norm of the window, using a smooth-histogram.
2. For each bucket of the smooth-histogram, $A_1, A_2, \ldots$ maintain an approximated list of the $k = \frac{1}{\epsilon^2} + 1$ most frequent elements, by running $(\gamma, \frac{\epsilon}{2}, \frac{\delta}{2})$-COUNTSKETCH$_b$.
   (see COUNTSKETCH$_b$’s description in Appendix A).
3. Let $\hat{L}_2$ be the approximated value of the $L_2$ norm of the current window $W$, as given by the the smooth-histogram. Let $q_1, \ldots, q_k \in \{1, \ldots, u\}$ be the list of the $k$ most heavy elements in $A_1$, along with $\hat{n}_1, \ldots, \hat{n}_k$ the estimation of their frequencies, as outputted by COUNTSKETCH$_b$.
4. Output any element $q_i$ that satisfies $\hat{n}_i > \frac{1}{1+\epsilon}\hat{L}_2$.

**Figure 1:** A semi-smooth algorithm for the frequent elements problem

**Theorem 2.** The semi-smooth algorithm ApproxFreqElements (Fig. 1) is a $(\gamma, \epsilon)$-approximation of the $L_2$-frequent elements problem, with success probability at least $1 - \delta$.

I.e., there exists a constant $c$ such that for a small enough $\epsilon$, the algorithm returns a list which includes all the elements with frequency at least $(1 + ce)\gamma L_2(W)$ and no element with frequency lower than $(1 - ce)\gamma L_2(W)$.

See proof in Appendix B

**Memory usage.** The memory usage of the protocol is composed of two parts: maintaining a $(\epsilon/2, \delta/2)$-smooth-histogram of $L_2$, and running COUNTSKETCH$_b$ on each of the buckets. According to [9] (corollary 5), maintaining a smooth-histogram for $L_2$ can be done with memory

$$O\left(\frac{1}{\epsilon^2} \log^2 N + \frac{1}{\epsilon^2} \log N \log \frac{\log N}{\delta \epsilon}\right)$$

for a relative error of $\epsilon/2 + \epsilon^2/8$, with success probability at least $1 - \delta/2$. For a small enough $\epsilon$, $\epsilon/2 + \epsilon^2/8 < \epsilon$, as required.
As for the second part, recall that one instance of CountSketch requires a memory of $O\left(\frac{1}{\varepsilon^2} \log \frac{n}{\delta}\right)$ (see Appendix A), where $n$ is the size of the input. In our case the maximal size of the input is the size of the first bucket, $\|A_1\|$. Note that $\log \|A_1\| = O(\log N)$ since $(1 - \alpha) L_2(A_1) \leq L_2(W) \leq N$. The number of CountSketch instances is bounded by the number of buckets, $O\left(\frac{1}{\varepsilon^2} \log N\right)$ [9], which leads to a total memory bound of $O\left(\frac{1}{\gamma^2 \varepsilon^4} \log N \log \frac{N}{\delta} + \frac{1}{\varepsilon^4} \log N \log \frac{1}{\varepsilon}\right)$.

**Extensions to different $L_p$.** It is easy to see that the same method can be used in order to approximate $L_p$-heavy elements for any $0 < p < 2$, up to a $1 \pm \varepsilon$ precision. The algorithms and analysis remains the same, except for using a smooth-histogram for the $L_p$ norm, and changing the parameters by constants.

**Theorem 3.** There exists a sliding window algorithm that outputs all elements with frequency at least $(1 + \varepsilon) \gamma L_p$, and no element with frequency less then $(1 - \varepsilon) \gamma L_p$. The algorithm succeeds with probability at least $1 - \delta$ and takes memory of $\text{poly}(\varepsilon^{-1}, \gamma^{-1}, \log N, \log \delta^{-1})$.

## 4 Estimation of Non-Smooth Properties Relativized to the Number of Distinct Elements

In this section we extend the method shown above and apply it on other non-smooth functions. While above we relate the frequent-elements problem to the smooth $L_2$ problem, in this section we use a different smooth function to partition the stream, namely the distinct elements count problem. This allows us to provide efficient semi-smooth approximations for the Similarity and $\alpha$-Rarity (non-smooth) properties.

### 4.1 Preliminaries

We now show that counting the number of distinct elements in a stream is smooth. This allows us to partition the stream into a smooth-histogram structure, where any two adjacent buckets have approximately the same number of distinct elements.

**Proposition 4.** Define $\text{DEC}(A)$ as the number of distinct elements in the stream $A$, i.e., $\text{DEC}(A) = |A|$. The function $\text{DEC}$ is an $(\varepsilon, \varepsilon)$-smooth-function, for every $0 \leq \varepsilon \leq 1$.

**Proof.** Properties (i) and (ii) of Definition 2 follow directly from $\text{DEC}$’s definition. As for property (iii), assume that $B \subseteq \tau A$ and $(1 - \varepsilon) \text{DEC}(A) \leq \text{DEC}(B)$, then

\[
(1 - \varepsilon) \text{DEC}(A \cup C) = (1 - \varepsilon) [\text{DEC}(A) + \text{DEC}(C \setminus A)] \\
\leq \text{DEC}(B) + (1 - \varepsilon) \text{DEC}(C \setminus A) \\
\leq \text{DEC}(B) + \text{DEC}(C \setminus B) \\
= \text{DEC}(B \cup C),
\]

where “$A \setminus B$” represents the set consisting of the elements in $A$ but not in $B$. \qed

There have been many works on counting distinct elements in streams, initiated by Flajolet and Martin [24], and later improved by [2, 25, 7, 5]. Very recently Kane, Nelson and Woodruff provided an optimal algorithm for $(\varepsilon, \delta)$-approximating the number of distinct elements [33], with memory $O((\frac{1}{\varepsilon^2} + \log u) \log \frac{1}{\delta})$ bits and time $O(1)$. We use the method of Kane et al. in order to build a
smooth-histogram for the distinct elements count with memory $O\left(\left(\log u + \frac{1}{\epsilon^2}\right)\log N \log \frac{1}{\delta} + \frac{1}{\epsilon} \log^2 N\right)$, suppressing $\log \log N$ and $\log \frac{1}{\epsilon}$ terms.

Another tool we use is min-wise hash functions [11, 12], used in various algorithms in order to estimate different characteristics of data streams, especially the similarity of two streams [11]. Informally speaking, these functions have a meaning of sampling an element uniformly from the stream, and thus they are a very useful tool.

**Definition 6 (min-hash [12]).** Let $\Pi = \{\pi_i\}$ be a family of permutations over $[u] = \{1, \ldots, u\}$. For a subset $A \subseteq [u]$ define $h_i$ to be the minimal permuted value of $\pi_i$ over $A$,

$$h_i = \min_{a \in A} \pi_i(a).$$

A family $\{h_i\}$ of such functions is called exact min-wise independent hash functions (or min-hash) if for any subset $A \subseteq [u]$ and $a \in A$

$$\Pr_i(h_i(A) = \pi_i(a)) = \frac{1}{|A|}.$$

The family $\{h_i\}$ is called $\epsilon$-approximated min-wise independent hash functions (or $\epsilon$-min-hash) if for any subset $A \subseteq [u]$ and $a \in A$,

$$\Pr_i(h_i(A) = \pi_i(a)) \in \left[\frac{1}{|A|}, 1\right] (1 \pm \epsilon).$$

A specific construction of $\epsilon$-min-hash functions was presented by Indyk [29], using only $O(\log \frac{1}{\epsilon} \log u)$ bits. The time per hash calculation is bounded by $O(\log \frac{1}{\epsilon})$.

Min-hash functions can be used in order to estimate the similarity of two sets, by using the following lemma,

**Lemma 5.** ([12]. See also [22].) For any two sets $A$ and $W$ and an $\epsilon'$-min-hash function $h_i$,

$$\Pr_i [h_i(A) = h_i(W)] = \frac{|A \cap W|}{|A \cup W|} \pm \epsilon'.$$

### 4.2 A Semi-Smooth Estimation for the $\alpha$-Rarity Problem

In the following section we present an algorithm that estimates the $\alpha$-rarity of a stream (in the sliding window model), i.e., the ratio of elements that appear exactly $\alpha$ times in the window. The rarity property is known not to be smooth, yet by using a smooth-histogram for the distinct elements count, we are able to partition the stream into $O\left(\frac{1}{\epsilon^2} \log N\right)$ buckets, and estimate the $\alpha$-rarity in each bucket.

**Definition 7 ($\alpha$-rarity, [22]).** An element $x$ is $\alpha$-rare if it appears exactly $\alpha$ times in the stream. The $\alpha$-rarity measure, $\rho_\alpha$, denotes the ratio of $\alpha$-rare elements in the entire stream $S$, i.e.,

$$\rho_\alpha = \frac{|\{x \mid x \text{ is } \alpha\text{-rare in } S\}|}{\text{DEC}(S)}.$$

Our algorithm follows the method used in [22] to estimate $\alpha$-rarity in the unbounded model. The estimation is based on the fact that the $\alpha$-rarity is equal to the portion of min-hash functions that their min-value appears exactly $\alpha$ times in the stream.

However, in order to estimate rarity over sliding windows, one needs to estimate the ratio of min-hash functions of which the min-value appears exactly $\alpha$ times within the window. Our algorithm builds a smooth-histogram for DEC in order to split the stream into buckets, such that each two consecutive buckets would have approximately the same number of distinct elements. In addition,
we sample the bucket using a min-wise hash, and count the $\alpha + 1$ last occurrences of the sampled element $x_i$ in the bucket. We estimate the $\alpha$-rarity of the window by calculating the fraction of the min-hash functions of which the appropriate min-value $x_i$ appears exactly $\alpha$ times \textit{within the window}. Due to feasibility reasons we use approximated min-wise hashes, and prove that this estimation is an $\varepsilon$-approximation of the $\alpha$-rarity of the current window (up to a pre-specified additive precision). The semi-smooth algorithm \textbf{ApproxRarity} for $\alpha$-rarity is defined in Figure 2.

\begin{table}[h]
\centering
\begin{tabular}{|c|}
\hline
\textbf{ApproxRarity$(\varepsilon, \delta)$} \\
\hline
1. Randomly choose $k = \frac{\varepsilon}{\delta}$-min-hash functions $h_1, h_2, \ldots, h_k$. \\
2. Maintain an $(\varepsilon, \delta)$-estimation of the number of distinct elements by building a smooth histogram. \\
3. For every bucket instance $A_j$ of the smooth-histogram and for each one of the hash functions $h_i, i \in [k]$ \\
   (a) Maintain the value of the min-hash function $h_i$ over the bucket, $h_i(A_j)$. \\
   (b) Maintain a list $L_i(A_j)$ of the most recent $\alpha + 1$ occurrences of $h_i(A_j)$ in $A_j$. \\
   (c) Whenever the value $h_i(A_j)$ changes, re-initialize the list $L_i(A_j)$, and continue maintaining the occurrences of the new value $h_i(A_j)$. \\
4. Output $\hat{\rho}_\alpha$, the ratio of the min-hash functions $h_i$, which has exactly $\alpha$ active elements in $L_i(A_1)$, i.e. the ratio $\hat{\rho}_\alpha = \left|\{i \text{ s.t. } L_i(A_1) \text{ consists exactly } \alpha \text{ active elements}\}\right|/k$. \\
\hline
\end{tabular}
\caption{Semi-smooth algorithm for $\alpha$-rarity}
\end{table}

The \textbf{ApproxRarity} algorithm provides an $(\varepsilon, \delta)$-approximation for the $\alpha$-rarity problem, up to an additive error of $\varepsilon$. As proven by Datar et al. [22], the ratio of min-hash functions that have exactly $\alpha$ active elements in the window is an estimation of $\rho_\alpha$. This is true even if we use the min-value of the inclusive bucket $A_1$ rather than the min-value of the current windows $W$.

\textbf{Theorem 6.} The semi-smooth algorithm (Fig. 2) is an $(\varepsilon, \delta)$-approximation for the $\alpha$-rarity problem, up to an additive precision.

See proof in Appendix C

\section*{Memory Usage.}

The memory consumption of the \textbf{ApproxRarity} algorithm is as follows. Maintaining a smooth histogram for DEC is done using the method of Kane et al. [33] as the underlying algorithm for DEC in the unbounded model, with memory $O((\log u + \frac{1}{\varepsilon})\frac{1}{\varepsilon}\log N \log \frac{1}{\varepsilon} + \frac{1}{\varepsilon} \log^2 N)$; $k$ seeds for the $\frac{1}{2}$-min-hash functions: $O(k \log \frac{1}{\varepsilon} \log u)$; Saving a list $L_i$ and a value $h_i$ for each bucket $A_j$ and for $i \in [k]$: $O([\log u + \alpha \log N]) + \log N)$. We note that this improves the \textit{expected} memory bound of Datar et al. [22] into a \textit{worst case} bound of the same magnitude (up to a $\log \log N$ term). In most of the practical cases $\log u$ and $\log N$ are very close, and we can assume that $\log u = O(\log N)$. In that case, the space complexity is $O \left( \frac{k\alpha}{\varepsilon} \log^2 N \right)$ bits, with $k = \Omega(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon})$, and the time complexity is $O \left( \frac{k\alpha}{\varepsilon} \log N \right)$ calculations per element, suppressing $\text{poly}(\log \frac{1}{\varepsilon}, \log \log N)$ terms.
4.3 A Semi-Smooth Estimation of Streams Similarity

In this section we present an algorithm for calculating the similarity of two streams $X$ and $Y$. As in the case of the rarity, the similarity property is known not to be smooth, however we are able to design a semi-smooth algorithm that estimates it. We maintain a smooth-histogram of the distinct elements count in order to partition each of the streams, and sample each bucket of this partition using a min-hash function. We compare the ratio of sample agreements in order to estimate the similarity of the two streams.

**Definition 8 (similarity).** The (Jaccard) similarity of two streams, $X$ and $Y$ is given by

$$S(X, Y) = \frac{|X \cap Y|}{|X \cup Y|}.$$  

Recall that for two streams $X$ and $Y$, a reasonable estimation of $S(X, Y)$ is given by the number of min-hash values they agree on [22]. In other words, let $h_1, h_2, \ldots, h_k$ be a family of $\epsilon$-min hash functions and let

$$\hat{S}(X, Y) = \frac{|\{i \in [k] \text{ s.t. } h_i(X) = h_i(Y)\}|}{k},$$

then $\hat{S}(X, Y) \in (1 \pm \epsilon)S(X, Y) + \epsilon(1 + p)$, with success probability at least $1 - \delta$, where $p$ and $\delta$ are determined by $k$. Based on this fact, Datar et al. [22] showed an algorithm for estimating similarity in the sliding window model, that uses expected memory of $O(k(\log \frac{1}{\epsilon} + \log N))$ words with $k = \Omega(\frac{1}{\epsilon^3} \log \frac{1}{\delta})$. Using smooth-histograms, our algorithm reduces the expected memory bound into a worst-case bound. The semi-smooth algorithm **ApproxSimilarity** is rather straightforward and is given in Figure 3.

**ApproxSimilarity**($\epsilon, \delta$)

1. Randomly choose $k$ $\epsilon'$-min-hash functions, $h_1, \ldots, h_k$. The constant $\epsilon'$ will be specified later, as a function of the desired precision $\epsilon$.

2. For each stream ($X$ and $Y$) maintain an ($\epsilon', \frac{\delta}{2}$)-estimation of the number of distinct elements by building a smooth histogram.

3. For each stream and for each bucket instance $A_1, A_2, \ldots$, separately calculate the values of each of the min-hash functions $h_i$, $i = 1 \ldots k$.

4. Let $A_X$ ($A_Y$) be the first smooth-histogram bucket that includes the current window $W_X$ ($W_Y$) of the stream $X$ ($Y$). Output the ratio of hash-functions $h_i$ which agree on the minimal value, i.e.,

$$\hat{\sigma}(W_X, W_Y) = \frac{|\{i \in [k] \text{ s.t. } h_i(A_X) = h_i(A_Y)\}|}{k}.$$  

**Figure 3:** A semi-smooth algorithm for estimating similarity

**Theorem 7.** The semi-smooth algorithm for estimating similarity (Fig. 3), is an ($\epsilon, \delta$)-approximation for the similarity problem, up to an additive precision.

See proof in Appendix D
Memory Usage. Let us summarize the memory consumption of the ApproxSimilarity algorithm. Maintaining a smooth histogram for DEC: \( \tilde{O}(\log u + \frac{1}{\epsilon}) \log N \log \frac{1}{\delta} + \frac{1}{\epsilon} \log^2 N) \); \( k \) seeds for \( \epsilon/2 \)-min-hash functions: \( O(k \log \frac{1}{\epsilon} \log u) \); Keeping the hash value for each \( h_i \): \( O(k \log N \log u) \).

Our algorithm improves the currently known expected bound \( \tilde{O}(k) \) into a worst case bound of the same magnitude (up to a \( \log \log N \) term). Taking \( k = \Omega(\frac{1}{\epsilon} \log \frac{1}{\epsilon}) \) and assuming \( \log u = O(\log N) \), we achieve a memory bound of \( \tilde{O}(k \log^2 N) \), with \( \tilde{O}(k \log N) \) calculations per element, suppressing poly\( (\log \frac{1}{\epsilon}, \log \log N) \) elements.

5 Conclusions

To conclude, we have shown the first poly-logarithmic algorithm for identifying \( L_2 \) heavy-hitters up to a \( 1 \pm \epsilon \) precision, over sliding windows. Our result supplies another insight about the relations of the unbounded and sliding window model, for the central question of heavy-hitters. As the \( L_p \)-heavy-hitters problem is more difficult for larger \( p \), and for \( p > 2 \) there can not exist a poly-logarithmic solution, our algorithm provides a solution to the strongest \( L_p \) norm with small memory.

Although our main concern was the \( L_2 \) norm, the algorithm can easily be extended for any \( L_p \) with \( 0 < p < 2 \). Moreover, a poly-logarithmic approximation of the top-\( k \) problem in sliding window is immediate using our methods.

We have demonstrated that the tools shown in this paper can be applied to many other properties, if there exists a smooth function which is correlated to our target function. We have shown how to employ the same techniques in order to obtain a sliding window algorithm for the similarity and \( \alpha \)-rarity problems, which yields essentially the same memory bound as the current state of the art, yet the bound is for the worst case scenario. We expect that our method can improve the memory efficiency of many sliding-window algorithms when applied to other non-smooth properties.

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Appendix

A  The COUNTSKETCH$_b$ algorithm

In this section we describe the COUNTSKETCH$_b$ algorithm and prove several of its properties. Define \((\gamma, \epsilon', \delta)\)–COUNTSKETCH$_b$ for \(0 < \epsilon', \gamma, \delta \leq 1\) as the algorithm COUNTSKETCH defined in [14], setting \(k = \frac{1}{\gamma^2} + 1\) and limiting the memory usage by using the parameter \(b = \frac{256}{\gamma^2 \epsilon^2}\). The choice of \(k\) follows from the following known fact

**Lemma 8.** There are at most \(\frac{1}{\gamma^2}\) elements with frequency higher than \(\gamma L_2\).

**Proof.** Assume that there are \(m\) elements with frequency higher than \(\gamma L_2\). It follows that \(L_2 = (\sum_{j=1}^{u} n_j^2)^{1/2} \geq \sqrt{m} \cdot \gamma L_2\). Clearly, \(m \leq \frac{1}{\gamma^2}\). \(\square\)

Setting \(k = \frac{1}{\gamma^2} + 1\) ensures that the outputted list is large enough to contain all the elements with frequency \(\gamma L_2\) or more.

However, COUNTSKETCH$_b$ does not guarantee anymore to output all the elements with frequency higher than \((1 + \epsilon') \phi_k\) and no element of frequency less than \((1 - \epsilon') \phi_k\) (Lemma 5 of [14]), since the value of \(b\) might not satisfy the conditions of that lemma.

We can still follow the analysis of [14] and claim that the frequency approximation of each element is still bounded (Lemma 4 of [14]),

**Lemma 9.** with probability \(1 - \delta\), for all elements \(i \in [u]\) in the stream \(S\),

\[
|\hat{n}_i - n_i| < 8 \frac{L_2(S)}{\sqrt{b}} < \frac{1}{2} \gamma' \epsilon' L_2(S)
\]

where \(\hat{n}_i\) is the approximated frequency of \(i\) calculated by COUNTSKETCH$_b$, and \(n_i\) is the real frequency of the element \(i\).

The above lemma allows us to bound the frequencies of the outputted elements.
Proposition 10. The $(\gamma, \epsilon', \delta)$–CountSketch$_b$ algorithm does not output any element with frequency less than $(1 - \epsilon')\gamma L_2(S)$

Proof. There exist $k$ elements which satisfy $n_i \geq \phi_k$. The estimated frequency of those elements satisfies $\hat{n}_i > n_i - \frac{1}{k} \epsilon' \gamma L_2(S) \geq n_k - \frac{1}{k} \epsilon' \gamma L_2(s)$. Every element with frequency less than $\phi_k - \epsilon' \gamma L_2(s)$ would have a lower frequency estimation than the above $k$ elements, and thus could not be in the outputted list. For $k = \frac{1}{\gamma^2} + 1$, Lemma 8 suggests that $\phi_k < \gamma L_2(S)$ and thus, $\phi_k - \epsilon' \gamma L_2(S) < (1 - \epsilon') \gamma L_2(S)$.

Proposition 11. The $(\gamma, \epsilon', \delta)$–CountSketch$_b$ algorithm outputs all elements with frequency at least $(1 + \epsilon') \gamma L_2(S)$

Proof. An element is not in the outputted list only if there are (at least) $k$ elements with higher approximated frequency. Due to Lemma 9 any element $i$ with frequency $n_i > (1 + \epsilon') \gamma L_2(S)$ has an estimated frequency of at least $\hat{n}_i \geq (1 + \frac{1}{2} \epsilon') \gamma L_2(S)$, so it can be replaced only by an element with frequency higher than $\gamma L_2(s)$, however, there are at most $k$ elements with $n_i \geq \gamma L_2(S)$, specifically, at most $k - 1$ elements other than $i$ itself, which completes the proof.

The memory consumption of CountSketch$_b$ is bounded by $O((k + b) \log \frac{|S|}{\epsilon})$ [14], which in our case gives $O(\frac{1}{\gamma^2 \epsilon^2} \log \frac{|S|}{\delta})$.

B Proof of Theorem 2

Theorem 2 The semi-smooth algorithm ApproxFreqElements (Fig. 1) is a $(\gamma, \epsilon)$-approximation of the $L_2$-frequent elements problem, with success probability at least $1 - \delta$.

i.e., there exists a constant $c$ such that for a small enough $\epsilon$, the algorithm returns a list which includes all the elements with frequency at least $(1 + c\epsilon)\gamma L_2(W)$ and no element with frequency lower than $(1 - c\epsilon)\gamma L_2(W)$.

Proof. Recall that the smooth-histograms data structure for $L_2$ guarantees us an estimation $\hat{L}_2$ which is $(1 \pm \epsilon)L_2(W)$; in addition there exists some $\alpha$ such that $(1 - \alpha)L_2(A_1) \leq L_2(W) \leq L_2(A_1)$. In our case the inequality is satisfied for $\alpha = \epsilon/2$ (see Theorem 3 and Definition 3 in [9]). Any element $j$ with frequency $n_j(W) > (1 + \epsilon)\gamma L_2(W)$ satisfies

$$n_j(A_1) \geq n_j(W) \geq (1 + \epsilon)\gamma L_2(W) \geq (1 + \epsilon)(1 - \epsilon/2)\gamma L_2(A_1),$$

and will be outputted in Step 2, since Proposition 11 guarantees that any element $i$ such that $n_i(A_1) > (1 + \epsilon/4)\gamma L_2(A_1)$ is outputted by CountSketch$_b$ (assuming $\epsilon < \frac{1}{2}$).

In order to show that all of the required elements survive Step 4, we use Lemma 9 to bound the estimated frequency $\hat{n}_i$ reported by CountSketch$_b$, and show it is above the required threshold. If $n_i(W) > (1 + \epsilon)\gamma L_2(W)$ then

$$\hat{n}_i(A) > n_i(A) - \frac{\epsilon}{8} \gamma L_2(A) > n_i(W) - \frac{\epsilon}{2 - \epsilon} L_2(W) > \left[1 + \epsilon - \frac{\epsilon}{2 - \epsilon}\right] \gamma L_2(W),$$

recalling that $\hat{L}_2 < (1 + \epsilon)L_2(W)$ implies that the element survives Step 4.

While we are guaranteed that all the $(1 + \epsilon)\gamma L_2(W)$-frequent elements appear in the outputted list, it might contain as well many other elements, which are not heavy enough. We now prove that Step 4 eliminates any element of frequency less than $(1 - c\epsilon)\gamma L_2(W)$, for a constant $c$. 

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Lemma 12. If for an element $i$ there exists some $\zeta > \sqrt{\epsilon}$ such that $n_i(A_1) > \zeta L_2(A_1)$, then there exist a constant $\xi > 0$ such that $n_i(W) > \xi L_2(W)$.

Proof. By the properties of the smooth-histogram,

$$L_2(W)^2 > (1 - \epsilon/2)L_2(A_1)^2 > (1 - \epsilon)L_2(A_1)^2$$

$$n_i(W)^2 + \sum_{j \neq i} n_j(W)^2 > n_i(A_1)^2 + \sum_{j \neq i} n_j(A_1) - \epsilon L_2(A_1)^2$$

$$n_i(W)^2 > n_i(A_1)^2 - \epsilon L_2(A_1)^2 > (\zeta^2 - \epsilon)L_2(A_1)^2$$

and $n_i(W) > \xi L_2(W)$ for $\xi \leq \sqrt{\zeta^2 - \epsilon}$.

Assume that an element $i$ survived Step 4 thus $\hat{n}_i(A_1) > \frac{1}{1 + \epsilon} \gamma \hat{L}_2 > \frac{1 - \epsilon}{1 + \epsilon} \gamma L_2(W)$. By Lemma 9,

$$n_i(A_1) \geq \hat{n}_i(A_1) - \frac{\epsilon}{8} \gamma L_2(A_1) \geq \left(\frac{(1 - \epsilon)(1 - \frac{\epsilon}{4})}{1 + \epsilon} - \frac{\epsilon}{8}\right) \gamma L_2(A_1) > (1 - 3\epsilon) \gamma L_2(A_1),$$

and by Lemma 12 $n_i(W) \geq \sqrt{1 - 7\epsilon} \cdot \gamma L_2(W)$. This proves that for small enough $\epsilon$ there exist some constant $c$ such that the algorithm doesn’t output any element with frequency lower than $(1 - c \epsilon) \gamma L_2(W)$.

To conclude, except for probability $\delta/2$ we are able to partition the stream into $L_2$-smooth buckets, and except for probability $\delta/2$, the COUNTSKETCH algorithm outputs a list which can be used to identify the frequent elements of the window. Using the union bound we conclude that the entire algorithm succeeds except for probability $\delta$. This completes the proof of the theorem.

C Proof of Theorem 6

Theorem 6 The semi-smooth algorithm (Fig. 2) is an $(\epsilon, \delta)$-approximation for the $\alpha$-rarity problem, up to an additive precision.

Proof. For the sake of simplicity we treat the multisets $A_1, W$, etc., as sets. Let $R_\alpha$ be the set of elements which are $\alpha$-rare in the window $W$. Following Lemma 5 with $R_\alpha \subseteq A_1$,

$$\Pr[h_i(R_\alpha) = h_i(A_1)] = \frac{|R_\alpha \cap A_1|}{|R_\alpha \cup A_1|} \pm \frac{\epsilon}{2} = \frac{|R_\alpha|}{|A_1|} \pm \frac{\epsilon}{2}.$$ 

The algorithm outputs an approximation of $\Pr[L_i(A_1) \text{ consists of exactly } \alpha \text{ active elements}]$, which equals to $\Pr[h_i(A_1) = h_i(R_\alpha)]$, since $h_i(A_1) = h_i(R_\alpha)$ if and only if $L_i(A_1)$ consists of $\alpha$ active elements. Let $x_i$ be the element which minimizes $h_i$ on $A_1$, $h(x_i) = h(A_1)$. If the number of active elements in $L_i(A_1)$ is not $\alpha$, then $x_i \notin R_\alpha$, thus $h(A_1) \neq h(R_\alpha)$. For the other direction, if $h_i(A_1) = h_i(R_\alpha)$ then $L_i(A_1)$ counts the number of occurrences of $x_i$ in the bucket, and since $x_i \in R_\alpha$, it appears exactly $\alpha$ times within the window.

We build a smooth-histogram for DEC by using the algorithm of Kane et al. [33] as an approximation of DEC for the unbounded model (see Theorem 3 in [9]). The smooth-histogram guarantees that $(1 - \epsilon)|A_1| \leq |W| \leq |A_1|$, thus

$$\frac{|R_\alpha|}{|A_1|} \leq \frac{|R_\alpha|}{|W|} = \rho_\alpha,$$

$$\frac{|R_\alpha|}{|A_1|} \geq (1 - \epsilon)\frac{|R_\alpha|}{|W|} \geq (1 - \epsilon)\rho_\alpha.$$ 

\footnote{Actually, it guarantees even a better bound, specifically, $(1 - \frac{\epsilon}{2})|A_1| \leq |W| \leq |A_1|$.}
Therefore, estimating the ratio $\rho_\alpha$ using $k$ hash functions results with a value $(1 \pm \epsilon)\rho_\alpha \pm \frac{\delta}{2}$ up to some additive error $\epsilon$ determined by $k$. Finally, using Chernoff’s inequality we can bound the additive error so that $\epsilon < \frac{\delta}{2}$, except for probability $\frac{\delta}{2}$. In order to achieve the desired precision we require $k = \Omega(\frac{1}{\epsilon^2} \log \frac{1}{\delta})$, and the estimation satisfies

$$\hat{\rho}_\alpha \in (1 \pm \epsilon)\rho_\alpha \pm \epsilon,$$

except for probability at most $\delta$. This concludes the correctness of the algorithm. \hfill \Box

### D Proof of Theorem 7

**Theorem 7** The semi-smooth algorithm for estimating similarity (Fig. 3), is an $(\epsilon, \delta)$-approximation for the similarity problem, up to an additive precision.

**Proof.** Following Lemma 5

$$Pr[h_i(A_X) = h_i(A_Y)] = \frac{|A_X \cap A_Y|}{|A_X \cup A_Y|} \pm \epsilon'.$$

For convenience, we once again treat the buckets $A_X, A_Y, W_X, W_Y$ as sets. Notice that we can write $A_X = W_X \cup (A_X \setminus W_X)$ and that $0 \leq |A_X \setminus W_X| \leq \frac{\epsilon'}{1 - \epsilon'} |W_X|$, which follows from the guarantee of the smooth-histogram that $(1 - \epsilon')|A_X| \leq |W_X| \leq |A_X|$ (and same for $A_Y$ and $W_Y$). Using elementary set operations, we can estimate $|W_X \cup W_Y|$ using $|A_X \cup A_Y|$,

$$|W_X \cup W_Y| \leq |A_X \cup A_Y| \leq |W_X \cup W_Y| + \frac{\epsilon'}{1 - \epsilon'} |W_X| + \frac{\epsilon'}{1 - \epsilon'} |W_Y| \leq |W_X \cup W_Y| + 2\frac{\epsilon'}{1 - \epsilon'} |W_X \cup W_Y| = \frac{1 + \epsilon'}{1 - \epsilon'} |W_X \cup W_Y|.$$

In addition, any two sets $S, Q$ satisfy $\frac{|S \cap Q|}{|S \cup Q|} = \frac{|S| + |Q|}{|S \cup Q|} - 1$, thus the similarity estimation satisfies

$$\frac{|A_X \cap A_Y|}{|A_X \cup A_Y|} = \frac{|A_X| + |A_Y|}{|A_X \cup A_Y|} - 1 \leq \frac{\frac{1}{1 - \epsilon'} |W_X| + \frac{1}{1 - \epsilon'} |W_Y|}{|W_X \cup W_Y|} - 1 = \frac{1}{1 - \epsilon'} \frac{|W_X \cap W_Y|}{|W_X \cup W_Y|} + \frac{\epsilon'}{1 - \epsilon'},$$

and

$$\frac{|A_X \cap A_Y|}{|A_X \cup A_Y|} \geq \frac{|W_X \cap W_Y|}{1 + \epsilon' |W_X \cup W_Y|}.$$

Finally, setting $\epsilon' < \epsilon/2$ gives an estimation $\hat{\rho}(W_X, W_Y) \in (1 \pm \epsilon)S(W_X, W_Y) \pm 3\epsilon/2$, up to an additional additive error, which can be arbitrarily decreased using Chernoff’s bound, by increasing $k$. Specifically, this additional error is bounded by $O(\epsilon)$ when $k = \Omega(\frac{1}{\epsilon^2} \log \frac{1}{\delta})$, with success probability at least $1 - O(\delta)$. \hfill \Box