The MMP for deformations of Hilb$^n$ $\mathbb{P}^2$

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Abstract

We study the birational geometry of deformations of Hilbert schemes of points on $\mathbb{P}^2$. On one hand, we complete the picture in [1] by giving an explicit correspondence between the stable base locus walls on the Neron-Severi space and the actual walls on the Bridgeland stability space. On the other hand, we show that the birational geometry of a deformed Hilb $\mathbb{P}^2$ is different from that of Hilb $\mathbb{P}^2$.

Introduction

The Hilbert scheme of points on an algebraic variety is the moduli space that parameterizes all of the 0-dimensional subschemes with length $n$ on the variety, where $n$ is a given positive integer. In the case when the variety is a curve, the Hilbert scheme is just the symmetric product of the curve itself. While the variety has dimension greater than 2, the Hilbert scheme has very bad singularities. In the surface case, the Hilbert scheme of points is smooth and connected which becomes a nice geometric object for study. As a moduli space of sheaves on the surface, the Hilbert scheme parameterizes the ideal sheaves with trivial first Chern class and a given second Chern class. The goal of this paper is to understand the birational geometry of the Hilbert scheme and the deformed Hilbert scheme of points on the projective plane. This sits into a huge program which studies the birational geometry of moduli spaces. In the case of the moduli space of curves, this is known as the Hassett-Keel program, and much research has been done. In the case of the moduli space of sheaves, much recent progress is made via Bridgeland stability conditions.

The notation of the stability condition on a triangulated category $T$ has been introduced by Bridgeland in [5]. It is given by abstracting the usual properties of the $\mu$-stability for sheaves on projective varieties. The central charge, which substitutes the slope $\mu$, is a group homomorphism from the numerical Grothendieck group to $\mathbb{C}$, and satisfies some extra conditions including the Harder-Narasimhan filtrations. The Bridgeland stability conditions form a natural topological space $\text{Stab}(T)$, which becomes a manifold of dimension not exceeding that of the numerical Grothendieck group.

Consider the case when $T$ is a bounded derived category of coherent sheaves on a smooth surface $X$. Given a numerical equivalence class $\nu$, and a stability condition $\sigma$ on $D^b(\text{Coh}(X))$, we have the moduli space $\mathcal{M}_\sigma(\nu)$ of $\sigma$-stable complexes of numerical type $\nu$. Two natural questions on $\mathcal{M}_\sigma(\nu)$ are:

1. When is $\mathcal{M}_\sigma(\nu)$ a good geometric object to study?
2. When $\sigma$ changes in $\text{Stab}(D^b(X))$, what is the behavior of $\mathcal{M}_\sigma(\nu)$?
For a general smooth surface, the known answers to both questions are mostly either vague or philosophical. For the first question, there are a few ways to determine when $\mathcal{M}_\sigma(v)$ is projective. For the second question, ideally, the stability space has a well-behaved chamber structure. In each chamber, $\mathcal{M}_\sigma(v) \cong \mathcal{M}_\rho(s)$. Among different chambers, there is a birational map $\mathcal{M}_\sigma(v) \to \mathcal{M}_\rho(s)$. Yet the ideal picture is far from being accomplished. It is only set up or partially/conjecturally set up when $X$ is a K3 surface, the projective plane, a high degree del Pezzo surface, a Hirzebruch surface or an abelian surface.

Let $\mathcal{M}_\sigma(n)$ be the moduli space of complexes on $\mathbb{P}^2$ with numerical type $(r, c_1, \chi) = (1, 0, 1 - n)$, i.e., the numerical type of Hilbert schemes. In [1], the authors studied the two questions in this case. They describe a wall and chamber structure on $\text{Stab}(D^b(\mathbb{P}^2))$ for the invariant $(r, c_1, \chi) = (1, 0, 1 - n)$. On a particular upper half plane slice of $\text{Stab}(D^b(\mathbb{P}^2))$, the walls are a sequence of nested semicircles in each quadrant, plus the vertical axis. For certain $\sigma$ in the second quadrant, they show that $\mathcal{M}_\sigma(n) \cong \text{Hilb}^n \mathbb{P}^2$. By choosing certain representative stability condition in each chamber, the authors prove that there are finitely many chambers for which $\mathcal{M}_\sigma(n)$ is non-empty, and, in this case, projective. Moreover, for small values of $n$, the author also write down an explicit correspondence between the chamber walls of the stability space and the base locus decomposition walls of the effective cone (in the sense of MMP). For general value of $n$, this explicit formula of the correspondence remains conjectural. One difficulty is to get a better answer to the first question above, in other words, to control the behavior of $\mathcal{M}_\sigma(n)$, especially the smoothness and irreducibility. In this paper, we solve these two questions in this case of $\mathbb{P}^2$ with numerical type $(r, c_1, \chi) = (1, 0, 1 - n)$.

**Theorem 0.1** (Corollary [3.6].) Adopting the notations as above, then we have:
1. When $\sigma$ is not on any wall, $\mathcal{M}_\sigma(n)$ is either a smooth, irreducible variety of dimension $2n$ or empty.
2. Given $\sigma$ and $\sigma'$ not on any wall, $\mathcal{M}_\sigma(n)$ and $\mathcal{M}_{\sigma'}(n)$ are birational to each other when both are non-empty.

To prove this theorem we study the GIT construction of $\mathcal{M}_\sigma(n)$ in detail, and control the dimension of the exceptional locus for each birational map associated to wall crossing. Then for each moduli space $\mathcal{M}_\sigma(n)$, we assign an ample line bundle on it. Applying the variation of geometric invariant theory by [9] and [18], we show that a Bridgeland stability wall-crossing of $\mathcal{M}_\sigma(n)$ is the flip with respect to the line bundle. As a result, the nested semicircular walls are one to one correspondence to the stable base locus decomposition walls of the effective divisor cone of $\text{Hilb}^n \mathbb{P}^2$. In addition, given the the location of the destabilizing wall, its corresponding base locus decomposition wall is written out in an explicit way. Notice that in certain cases, this correspondence has been established in a recent paper [2] by Coskun and Huizenga via a different approach.

**Theorem 0.2.** (Theorem [3.17], Proposition [4.2].) For each semicircular actual wall on the second quadrant of Bridgeland stability conditions plane we may assign a divisor $L_{\rho, c_1}$ up to a scalar. $L_{\rho, c_1}$ is on the stable base locus wall of $\text{Hilb}^n S$ and this gives a one to one correspondence between the walls in the stability plane and the stable base locus walls in the effective divisor cone. In particular, the destabilizing semicircular wall on the Bridgeland stability condition space with center $-m - \frac{1}{2}$ corresponds to the base locus wall spanned by divisor $mH - \frac{1}{2}$.

Another important attempt in this paper is to extend this story to the deformations of $\text{Hilb}^n \mathbb{P}^2$ by methods from non-commutative algebraic geometry. Here we use the
notion of Sklyanin algebras $S = \text{Skl}(E, \sigma, \mathcal{L})$, which are non-commutative deformations of the homogeneous coordinate ring of $\mathbb{P}^2$. Such a Sklyanin algebra depends on a cubic curve $E$ on $\mathbb{P}^2$, an automorphism $\sigma$ of $E$ and a degree 3 line bundle. The foundation of such a non-commutative theory has been set up in \cite{2}, \cite{3}, \cite{4}, \cite{16}, \cite{17}. For these non-commutative $\mathbb{P}^2$, we still have $\text{Mss}_{\mathbb{M}}(1, 0, 1 - n)$, which turn out to be smooth varieties (in the ordinary commutative sense!), and are in fact deformations of $\text{Hilb}^n \mathbb{P}^2$ by \cite{16} and \cite{12}. We will call these deformations of $\text{Hilb}^n S$.

In this paper we study the Bridgeland stability conditions of $D^b(\text{Coh}(S))$, which are similar to that of $\mathbb{P}^2$. In particular, we have a similar chamber structure on the upper half plane slice of the Bridgeland space, and the theorem above also holds for $\text{Mss}_{\mathbb{M}}(n)$ associated to non-commutative $\mathbb{P}^2$. However, the behavior of wall-crossing over the vertical wall is different in this case, and this changes the correspondence between the chamber walls of stability space and the base locus decomposition walls of the effective cone. In this case, we have the following theorem:

**Theorem 0.3.** (Theorem 3.17, 3.9) When $n \geq 3$, for each semicircular actual wall on the Bridgeland stability conditions plane we may assign a divisor $L_{\rho,s,t,k}$ up to a scalar. $L_{\rho,s,t,k}$ is on the stable base locus wall of $\text{Hilb}^n S$ and this gives a one to one correspondence between the walls in the stability plane and the stable base locus walls in the effective divisor cone.

In addition this map is ‘monotone’ in the sense that the two most inner walls on the two quadrants correspond to the two edges of the effective cone respectively. When one moves from inner semicircles to the outside, the corresponding stable base locus wall moves in one direction. This reveals a symmetric structure of the Mori decomposition of the effective divisor cone of $\text{Hilb}^n S$. Notice that, given any $n$, the destabilizing walls of $\text{Hilb}^n \mathbb{P}^2$ and $\text{Hilb}^n S$ are computable. Using the location of these destabilizing walls, we can also compute the slopes of base locus decomposition walls in the effective divisor cone. The cartoon of divisor cones of $\text{Hilb}^n \mathbb{P}^2$ and a generic $\text{Hilb}^n S$ are shown below.

In the picture on the left, $\Delta$ is the exceptional divisor of the Hilbert-Chow map to $\text{Sym}^n \mathbb{P}^2$, and $H$ is the pull-back of $O(1)$ on $\text{Sym}^n \mathbb{P}^2$. The picture on the right is for
Hilb^nS. Here \( \Delta \) and \( H \) are the corresponding divisor classes under deformation. It is immediate from the picture that Hilb^nS are Fano and Hilb^nP^2 is log Fano.

After we obtained the results in this paper but before we finished writing it, the paper [7] of Coskun and Huizenga appeared. In [7], the authors obtained the 'correspondence of walls' result for Hilb^nP^2 in certain cases. The paper [7] does not treat the case of Hilb^nS, which is new in this paper. Also, in [7] the author study the zero dimensional monomial subschemes \( Z \) of \( P^2 \), and when \( I_Z \) is destabilized to get their result. our approach is quite different, and the approach in [7] does not apply to the non-commutative case, for example, only \( n \)-dimensional points in Hilb^nS correspond to ideal sheaves. We show the smoothness and irreducibility of each moduli space by showing some Ext^2 vanishing. These good properties allow one to apply the VGIT to get the correspondence.

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1 Background Material

1.1 Review: Sklyanin Algebra and deformed Hilbert scheme of \( P^2 \)

We first recollect some definitions and properties about the Sklyanin algebra from the noncommutative algebraic geometry, further details are referred to [16] by Nevins and Stafford.

Given a smooth elliptic curve \( \iota : E \hookrightarrow P^2 \) with corresponding line bundle \( \mathcal{L} = \iota^*(O_{P^2}(1)) \) of degree 3 and an automorphism \( \sigma \in \text{Aut}(E) \) which is a translation under the group law. Denote the graph of \( \sigma \) by \( \Gamma_\sigma \subset E \times E \). Let \( V := H^0(E, \mathcal{L}) \), then we have a 3-dimensional space \( R(E, \sigma, \mathcal{L}): \)

\[ R = H^0(E \times E, (\mathcal{L} \boxtimes \mathcal{L})(-\Gamma_\sigma)) \subset H^0(E \times E, \mathcal{L} \boxtimes \mathcal{L}) = V \otimes V. \]

Definition 1.1. The 3-dimensional Sklyanin algebra is the algebra

\[ S = \text{Skl}(E, \mathcal{L}, \sigma) = T(V)/(R), \]

where \( T(V) \) denotes the tensor algebra of \( V \).

When \( \sigma \) is the identical morphism, \( \text{Skl}(E, \mathcal{L}, \text{Id}) \) is just the commutative polynomial ring \( \mathbb{C}[x, y, z] \). In general, one may write \( \text{Skl}(E, \mathcal{L}, \sigma) \) as a \( \mathbb{C} \)-algebra with generators \( x_1, x_2, x_3 \) satisfying relations:

\[ ax_i x_{i+1} + bx_{i+1} x_i + cx_i^2 = 0, \quad i = 1, 2, 3 \text{ mod 3,} \quad (\Delta) \]

where \( a, b, c \in \mathbb{C}^* \) are scalars such that \((3abc)^3 \neq (a^3 + b^3 + c^3)^3\).

\[ S = \text{Skl}(E, \mathcal{L}, \sigma) \] is a connected graded algebra with grading induced from \( T(V) \). Write Mod-\( S \) for the category of right \( S \)-modules and Gr-\( S \) for the category of graded right \( S \)-modules, with homomorphisms \( \text{Hom}_S(M, N) \) being graded homomorphisms of degree zero. Given a graded module \( M = \oplus_{n \geq 0} M_n \), the shift \( M(n) \) is the graded module
1.1 Review: Sklyanin Algebra and deformed Hilbert scheme of $\mathbb{P}^2$

with $M(n) = M_{itn}$ for all $i$. $S$ is strongly noetherian (Lemma 5.1 in [16]), we may write $	ext{gr}-S$ for the subcategory of noetherian objects in $\text{Gr}-S$. A module $M \in \text{gr}-S$ is called right bounded if $M_i = 0$ for $i \gg 0$. The full Serre subcategory of $\text{gr}-S$ generated by the right bounded modules is denoted by $\text{rb}-S$ with the quotient category $\text{qgr}-S = \text{gr}-S/\text{rb}-S$. One has an adjoint pair $\pi : \text{gr}-S \cong \text{qgr}-S : \Gamma^*$. Here $\pi$ is the natural projection and $\Gamma^*$ is the ‘global section’ functor. When $S \cong \mathbb{C}[x, y, z]$, $\text{qgr}-S$ is isomorphic to the category of coherent sheaves on $\text{Proj}\mathbb{C}[x, y, z]$. Due to this reason, we call an object $M \in \text{qgr}-S$ as a sheaf on $S$.

A sheaf $M$ on $S$ is called torsion if each element in $\Gamma^* (M)$ is annihilated by a nonzero element of $S$, respectively torsion-free if no element is so. A torsion-free $M$ has rank $r$ if $M$ contains a direct sum of $r$, but not $r + 1$, nonzero submodules. The rank of a general sheaf $M$ is defined to be the rank of its torsion-free quotient part. We write $\text{rk}(M)$ for the rank of $M$.

The first Chern class $c_1(M)$ is defined in [16] Lemma 3.7 as the unique function $c_1 : \text{qgr}-S \rightarrow \mathbb{Z}$ with the following properties: additive on short exact sequences; $c_1(O_S(m)) = m$ for all $m \in \mathbb{Z}$. The Euler character on $\text{qgr}-S$ is defined as usual: $\chi(E, F) := \sum (-1)^i \text{dim} \text{Ext}^i(E, F)$ for $E, F \in \text{qgr}-S$. $\chi(F) := \chi(O, F)$. The Hilbert polynomial of $M$ is $p_M(t) := \chi(M(t))$. The Mumford-Giesker slope of $M$ is defined as $\mu^{\text{MG}}(M) := c_1(M)/\text{rk}(M)$. A torsion-free sheaf $M$ is called Mumford-Giesker stable, if for every non-zero proper submodule $F \subset M$, one has $\text{rk}(M)p_F - \text{rk}(F)p_M < 0$. Given a torsion-free sheaf $M$, it has a Harder-Narasimhan filtration $0 = M_0 \subset M_1 \cdots \subset M_n = M$ such that each quotient $F_i = M_i/M_{i-1}$ is Mumford-Giesker semistable with slope $\mu^{\text{MG}}(F_i) > \mu^{\text{MG}}(F_{i+1})$. We write $\mu^+_e(M)$ for $\mu^{\text{MG}}(F_1)$, and $\mu^-_e(M)$ for $\mu^{\text{MG}}(F_n)$.

Lemma 1.2. Let $M \in \text{qgr}-S$.
1. $c_1(M(s)) = c_1(M) + s \cdot \text{rk}(M)$ for any $s \in \mathbb{Z}$;
2. If $M$ is torsion and non-zero, then $c_1(M) \geq 0$; if in addition $c_1(M) = 0$, then $\chi(M) > 0$.

Proof. Property 1 is the same as the second property of Lemma 3.7 in [16].

For property 2, let $O(j) \rightarrow M$ be a non-zero morphism. By noetherian hypothesis on $M$, the descending chain of the quotient sheaves of $M$ is finite. By the additivity of $c_1$ and $\chi$, we may assume that $O(j) \rightarrow M$ is surjective. To check that $c_1$ is non-negative, by the first property, we may assume $j = 0$. Let $I$ be the kernel of $O \rightarrow M$. Denote $\Gamma^* (I)$ by $I$. Write $c$ for $c_1(I)$. $I(-c)$ is a rank 1, normalized (i.e. $c_1(I(-c)) = 0$), torsion-free sheaf. By Proposition 5.6, Theorem 5.8 and Lemma 6.4 in [16], $I(-c)$ is the homological sheaf $H^0(K)$ of

$$K : O(-1)^{\otimes a} \rightarrow O^{\otimes a+1} \rightarrow O(1)^{\otimes a}$$

at the middle term, where $a$ is $1 - \chi(I(-c))$. Now for $n \gg 0$, recall $I = \oplus_{n \in \mathbb{Z}} I_n$, we have:

$$\dim_{\mathbb{C}} I_n = (2a + 1) \dim_{\mathbb{C}} S(c)_n - a \dim_{\mathbb{C}} S(c - 1)_n - a \dim_{\mathbb{C}} S(c + 1)_n$$

$$= (2a + 1) \binom{n + c + 2}{2} - a \left( \binom{n + c + 1}{2} + \binom{n + c + 3}{2} \right)$$

$$= \binom{n + c + 2}{2} - a.$$

Since $I$ is a subsheaf of $O$, $\dim_{\mathbb{C}} I_n < \dim_{\mathbb{C}} S_n = \binom{n + 2}{2}$ for $n \gg 0$. We get $c \leq 0$, hence $c_1(M) \geq 0$. 

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1.2 Review: Bridgeland Stability Condition on $D^b(qgr-S)$

When $rk(M) = c_1(M) = 0$, by the formula 2 in Lemma 6.1 in [16], the Hilbert polynomial $\chi(M(t)) = p_M(t) = \chi(M)$ is a constant, we may also assume that $j = 0$. Then $I$ is semistable and normalized, by Lemma 6.4 in [16], $\chi(I) \leq 1$ and the equality only holds when $I = 0$. □

Let $D^b(qgr-S)$ be the bounded derived category of $qgr-S$. We rephrase one of the main results in [15].

**Proposition 1.3** (Proposition 6.20 in [16]). $D^b(qgr-S)$ is generated by (i.e. the closure under that extension and the homological shift of) $O(k - 1)$, $O(k)$, $O(k + 1)$ for any $k \in \mathbb{Z}$.

**Proof.** For any integer $k$, by the induction on $k$ and the exact sequence

$$0 \to O(k) \to O(k + 1)^{\oplus 3} \to O(k + 2)^{\oplus 3} \to O(k + 3) \to 0,$$

where $a, b, c$ are coefficients in $\mathbb{A}^3$, $O(k)$ is in the closure. By Proposition 6.20 in [16], all the Mumford-Giesker semi-stable sheaves are in the closure. Since each torsion-free sheaf admits a finite Harder-Narasimhan filtration, and each torsion sheaf is the cokernel of a morphism between two torsion free sheaves, all sheaves are in the closure. □

As a consequence, invariants {rank, first Chern class, Euler character} generate the numerical Grothendieck group of $D^b(qgr-S)$. The importance of Sklyanin algebras is shown in the following theorem in [15]. There the authors prove that deformations of $\text{Hilb}^b \mathbb{P}^2$ can be constructed as the moduli spaces of (semi)stable objects in $qgr-S$ with numerical invariants $(1, 0, 1 - n)$. As pointed out in [12], generically each deformation of $\text{Hilb}^b \mathbb{P}^2$ is constructed in this way.

**Theorem 1.4** (Theorem 8.11, 8.12 in [15]). Let $\mathcal{B}$ be a smooth curve defined over $\mathbb{C}$ and let $S_{\mathcal{B}} (= S_{\mathcal{B}}(E, L, \sigma)) \in \mathbb{A}^3_{\mathcal{B}}$ be a flat family of algebras such that $S_p = \mathbb{C}[x, y, z]$ for some point $p \in \mathcal{B}$. Set $S = S_p$ for any point $b \in \mathcal{B}$. Then $M_{\mathcal{B}}^{(1)}(1, 0, 1 - n)$ is smooth over $\mathcal{B}$, and $M_{\mathcal{B}}^{(1)}(1, 0, 1 - n) \otimes_{\mathcal{B}} \mathbb{C}(b) = M_{\mathcal{B}}^{(1)}(1, 0, 1 - n)$. Each $M_{\mathcal{B}}^{(1)}(1, 0, 1 - n)$ is smooth, projective, fine moduli space for equivalence classes of rank one torsion-free modules $M \in qgr-S$ with $c_1(M) = 0$ and $\chi(M) = 1 - n$. Each $M_{\mathcal{B}}^{(1)}(1, 0, 1 - n)$ is a deformation of $\text{Hilb}^b \mathbb{P}^2$. □

We will write $S^{(n)}$ or $\text{Hilb}^b S$ instead of $M_{\mathcal{B}}^{(1)}(1, 0, 1 - n)$ for short.

**Proposition 1.5.** The Picard number of $\text{Hilb}^b S$ is 2.

**Proof.** By the formula on the second page of [15] by Nakajima, $b_2 (\text{Hilb}^b S) = b_2 (\text{Hilb}^b \mathbb{P}^2) = 2$. Since $\text{Hilb}^b S$ is projective, the Hodge numbers $h^{1,1} \geq 1$ and $h^{0,2} = h^{2,0}$, one must have $h^{1,1} = 2$. □

1.2 Review: Bridgeland Stability Condition on $D^b(qgr-S)$

In this section, we briefly review the stability conditions on derived categories. These notations are introduced in [5] by Bridgeland. Let $N(S)$ be the numerical Grothendieck group of $D^b(qgr-S)$, i.e., the free abelian group generated by $r, c_1,$ and $\chi$. 


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**Definition 1.6.** A numerical stability condition on $D^b(\text{qgr-S})$ is a pair

$$(Z, \mathcal{A}) : \mathcal{A} \subset D^b(\text{qgr-S}),$$

where $Z : \mathcal{N}(\text{S}) \to \mathbb{C}$ is a group homomorphism and $\mathcal{A}$ is the heart of a bounded t-structure, such that the following conditions hold.

1. For any non-zero $E \in \mathcal{A}$, we have

$$Z(E) \in \{re^{i\phi} : r > 0, \phi \in (0, 1]\}.$$

2. Harder-Narasimhan property: for any $E \in \mathcal{A}$, there is a filtration of finite length in $\mathcal{A}$

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

such that each subquotient $F_i = E_i/E_{i-1}$ is $Z$-semistable with $\arg Z(F_i) > \arg Z(F_{i+1})$.

Here an object $E \in \mathcal{A}$ is said to be $Z$-(semi)stable if for any subobject $0 \neq F \subsetneq E$ in $\mathcal{A}$ we have

$$\arg Z(F) < (\leq) \arg Z(E).$$

The group homomorphism $Z$ is called the central charge of the stability condition. The rest of this section is devoted to the construction of numerical stability conditions on $\text{qgr-S}$. First we recall the notion of torsion pairs, which is essential to constructing t-structures. A pair of full subcategories $(\mathcal{F}, \mathcal{T})$ of $\text{qgr-S}$ is called a torsion pair if it satisfies the following two conditions.

1. For all $F \in \text{ob}\mathcal{F}$ and $T \in \text{ob}\mathcal{T}$, we have $\text{Hom}(T, F) = 0$.

2. Each sheaf $E$ in $\text{qgr-S}$ fits in a short exact sequence:

$$0 \to T \to E \to F \to 0,$$

where $T \in \text{ob}\mathcal{T}$, $F \in \text{ob}\mathcal{F}$. In addition, the extension class is uniquely determined up to isomorphism.

A torsion pair defines a t-structure on $D^b(\text{qgr-S})$ by:

$$D^{\geq 0} = \{ C^i | H^i(C^*) \in \mathcal{F} \text{ and } H^i(C^*) = 0 \text{ for } i < -1 \},$$

$$D^{\leq 0} = \{ C^i | H^0(C^*) \in \mathcal{T} \text{ and } H^i(C^*) = 0 \text{ for } i > 0 \}.$$

As in the $\mathbb{P}^2$ case, given $s \in \mathbb{R}$, we can define the full subcategories $\mathcal{T}_s$ and $\mathcal{F}_s$ of $\text{qgr-S}$ as:

$T \in \mathcal{T}_s$ if $T$ is torsion or $\mu^{GM}(T') > s$, where $T'$ is the torsion-free quotient of $T$;

$F \in \mathcal{F}_s$ if $F$ is torsion-free and $\mu_s^{GM}(F) \leq s$.

By Lemma 6.1 in [8], $(\mathcal{F}_s, \mathcal{T}_s)$ is a torsion pair. Let $\mathcal{A}_s$ be the heart of the t-structure determined by the torsion pair $(\mathcal{F}_s, \mathcal{T}_s)$, we may define a central charge $Z_{s,t} = -d_{s,t} + ir_{s,t}$ depending on a parameter $t > 0$ by:

$$r_{s,t}(E) := (c_1 - rs)t,$$

$$d_{s,t}(E) := -rt^2/2 + (s^2/2 - 1)r - (3/2 + s)c_1 + \chi.$$
1.2 Review: Bridgeland Stability Condition on $D^b(qgr-S)$

**Proof.** Any object $E \in \mathcal{A}_t$ fits in an exact sequence:

$$0 \to \mathbb{H}^{-1}(E)[1] \to E \to \mathbb{H}^0(E) \to 0.$$  

in order to check the Property 1 of the central charge in Definition 1.6, we only need to check $\arg(Z(E)) \in (0, \pi]$ for the following cases: 1. $E$ is a torsion sheaf. 2. $E$ is a Mumford-Giesker stable sheaf with $\mu(E) > s$. 3. $E[-1]$ is a Mumford-Giesker stable sheaf with $\mu(E[-1]) \leq s$.

Case 1 is due to Lemma 1.2. Case 2 is clear since $r(E)$ is greater than 0. In case 3, we may assume $c_1(E[-1]) = r(E[-1]), d(E) = rE + 3c_1/2 - \chi + c_1^2/2r \geq rE + r^2/2 + r^2 - 1 > 0$, where $r, c_1, \chi$ stands for $r(E[-1]), c(E[-1]), \chi(E[-1])$ respectively. The first inequality is due to Corollary 6.2 and Proposition 2.4 in [16]: $2\chi - r^2 - 3rc_1 - c_1^2 = \chi(E[-1], E[-1]) \leq 1 + \text{ext}^2(E[-1], E[-1]) = 1 + h^0(E[-1], E[-1])(-3)) = 1$.

When $t$ and $s$ are both rational numbers, the Harder-Narashimhan property is similarly proved as that in Proposition 7.1 in [6] by Bridgeland. In the general case, we need the following lemma to check the descending chain stable property:

**Lemma 1.8.** Given a stability condition $(Z_{s,r}, \mathcal{A}_t)$ and two positive numbers $M_1, M_2$. Then the set

$$((-\infty, M_1] \times [0, M_2]) \cap \{(-d_{s,r}(F), r_{s,r}(F))|F \in \mathcal{A}_t \text{ and is a torsion-free sheaf}\}$$

is finite.

**Proof.** First, we show this holds for all Mumford-Giesker stable sheaves. Write $\chi(F), r(F), c_1(F)$ as $\chi, r, c_1$ for short. By the inequality $2\chi - r^2 - 3rc_1 - c_1^2 \leq 1$, we have

$$\chi \leq \frac{1}{2r}(1 + r^2 + 3rc_1 + c_1^2).$$

Plug this into the formula of $d_{s,r}$, we have

$$-d_{s,r} \geq \frac{rt}{2} - \frac{(c - rs)^2}{2r}.$$  

Since $c_1 - rs \geq 0$ and $M_1 \geq -d_{s,r}, r$ is bounded. As $c_1 - rs \in [0, M_2], c_1$ is bounded. Now by the inequality of $\chi$ and the formula of $d_{s,r}$, we have

$$-M_1 + \frac{rt^2}{2} - \frac{s^2}{2} - 1)r - \frac{3}{2} + s)c_1 \leq \chi \leq \frac{1}{2r}(1 + r^2 + 3rc_1 + c_1^2).$$

Hence $\chi$ is bounded. The set

$$((-\infty, M_1] \times [0, M_2]) \cap \{(-d_{s,r}(F), r_{s,r}(F))|F \in \mathcal{A}_t \text{ and is a torsion-free MG stable sheaf}\}$$

is finite.

Next, we show this holds for torsion free $F \in \mathcal{A}_t$. By the finiteness result above, we may define

$$D := \min(-d_{s,r}(F))|F \in \mathcal{A}_t \text{ and is a torsion-free MG stable sheaf};$$

$$R := \min(r_{s,r}(F))|F \in \mathcal{A}_t \text{ and is a torsion-free MG stable sheaf}.$$
1.2 Review: Bridgeland Stability Condition on $D^b(qgr-S)$

Now given a torsion free sheaf $\mathcal{G} \in \mathcal{A}_r$, if $\mathcal{G}$ has an MG-factor $\mathcal{G}_j$ such that $-d_{a,0}(\mathcal{G}_i) > \frac{D^2}{R} + M_1$, then

$$-d_{a,0}(\mathcal{G}) > \left(\frac{D^2}{R} + M_1\right) + D \cdot \left(\frac{D}{R}\right) = M_1.$$ 

Therefore

$$\{(-d_{a,0}(F), r_{a,0}(F)) | F \in \mathcal{A}_r \text{ and is a torsion-free sheaf} \} \cap \{(-\infty, M_1] \times [0, M_2]\}$$

$$\subset \sum \{(-d_{a,0}(\mathcal{G}), r_{a,0}(\mathcal{G})) | \mathcal{G} \in \mathcal{A}_r \text{ and is a torsion-free MG stable sheaf} \} \cap \{(-\infty, \frac{D^2}{R} + M_1] \times [0, M_2]\},$$

and is finite. \(\square\)

Now we may check the descending chain stable property. Suppose the property does not hold, we have an object $E$ in $\mathcal{A}_{r,t}$ that has an infinite descending chain:

$$\cdots \subset E_{t+1} \subset E_t \cdots \subset E_1 \subset E_0 = E$$

with increasing slopes $\mu_{r,t}(E_{t+1}) > \mu_{r,t}(E_i)$ for all $i$. There are short exact sequences in $\mathcal{A}_{r,t}$: $0 \to E_{i+1} \to E_i \to F_i \to 0$ for $i \geq 0$. By taking the cohomology of sheaves, we have: $\text{H}^{-1}(E_{t+1}) \subset \text{H}^{-1}(E_t)$. We may assume the rank of $\text{H}^{-1}(E_t)$ is constant. Now the cokernel $\text{H}^{-1}(E_t)/\text{H}^{-1}(E_{t+1})$ is torsion, and $\text{H}^{-1}(F_t)$ is torsion-free, we have $\text{H}^{-1}(E_t) \cong \text{H}^{-1}(E_{t+1})$.

Let $T_i$ and $G_i$ be the torsion subsheaf and torsion-free quotient of $H^0(E_t)$ respectively. Since we have the exact sequence

$$0 \to H^{-1}(F_t) \to H^0(E_{t+1}) \to H^0(E_t) \to H^0(F_t) \to 0,$$

and $H^{-1}(F_t)$ is torsion-free, $T_{t+1}$ is a subsheaf of $T_t$. We may assume $c_1(T_t)$ is constant, then $\chi(T_t)$ is non-increasing.

Now we have

$$(-d_{a,0}(H^{-1}(E_t)[-1]), r_{a,0}(H^{-1}(E_t)[-1]) + r_{a,0}(T_t)) \geq (-d_{a,0}(H^{-1}(E_0)[-1]), r_{a,0}(H^{-1}(E_0)[-1]) + r_{a,0}(T_0)).$$

Since the slope is increasing, we also have

$$(-d_{a,0}(E_t), r_{a,0}(E_t)) \leq \max\{-d_{a,0}(E_0), 0\}, r_{a,0}(E_0)).$$

Subtracting the first from the second one, we have

$$(-d_{a,0}(G_t), r_{a,0}(G_t)) \leq \max\{-d_{a,0}(G_0), d_{a,0}(H^{-1}(E_0)[-1]) + d_{a,0}(T_0), 0\}, r_{a,0}(G_0)).$$

Now applying the lemma to $G_i$, combining with the results on $H^{-1}(E_t)[-1]$ and $T_t$, the set of possible values of $(-d_{a,0}(E_t), r(E_t))$ is finite, so we may get the stableness of the descending chain directly as that in $s, t$ rational case.

The ascending chain property can be similarly proved, where one applies the lemma to $H^{-1}(E_t)[-1]$ and the area $(-M_1, +\infty) \times (0, M_2)$ to get the finiteness. Then the argument is the same as that in $s, t$ rational case, and the details are left to the readers. \(\square\)

Let $\mathcal{A}(k)$ be the extension closure of $O(k-1)[2]$, $O(k)[1]$ and $O(k+1)$. Since $O(k+1)$, $O(k), O(k+1)$ is a full strong exceptional collection by Proposition[1,2].
\( \mathcal{A}(k) \) is the heart of a t-structure of \( D^b(\text{qgr-S}) \), see Lemma 3.16 [13]. Objects in \( \mathcal{A}(k) \) are of the form:

\[
O(k - 1) \otimes \mathbb{C}^{n-1} \to O(k) \otimes \mathbb{C}^{n_0} \to O(k + 1) \otimes \mathbb{C}^{n_1},
\]

where \( n_1, n_0, n_1 \) are some non-negative integers. We write \( \vec{n} = (n_1, n_0, n_1) \) and call it the type of the object. One may construct a central charge \( Z \) for \( \mathcal{A}(k) \) by letting \( Z(O(k - 1)[2]) = z_{-1}, Z(O(k)[1]) = z_0 \) and \( Z(O(k + 1)) = z_1 \) for any collection of complex numbers \( z_i \)’s on the upper half plane: \( \{re^{i\phi} : r > 0, \phi \in (0, 1]\} \). \( (Z, \mathcal{A}(k)) \) is a stability condition on \( D^b(\text{qgr-S}) \).

2 Destabilizing Wall

The destabilizing walls on the \((s, t)\)-plane of stability conditions of \( D^b(\text{cohP}^2) \) are discussed in [1] Section 6. In the \( D^b(\text{qgr-S}) \) case, the behavior of the walls is similar to that of the \( \text{P}^2 \) case.

The potential wall associated to a pair of invariants \((r, c, \chi)\) and \((r', c', \chi')\) on \( \text{qgr-S} \) is the following subset of the upper-half \((s, t)\)-plane:

\[
W_{(r, c, \chi), (r', c', \chi')} = \{(s, t) | \mu_{s,t}(r, c, \chi) = \mu_{s,t}(r', c', \chi')\}.
\]

More explicitly, the wall is given by:

\[
W_{(r, c, \chi), (r', c', \chi')} = \{(s, t) \mid \frac{1}{2}[(c_1r'-c_1' r)(r^2+s^2)+(\chi' - \chi) r'+ 3/2 r' c_1 - 3/2 r' c_1'] s + (c_1r'-c_1' r + \chi c_1' - \chi c_1) t = 0\}.
\]

Let \( W_{\text{potential}} := \bigcup_{(r,c,\chi), (r',c',\chi')} W_{(r, c, \chi), (r', c', \chi')} \). In the Hilbert scheme case, where \((r, c_1, \chi) = (1, 0, 1-n) \) (respectively \((-1, 0, n-1) \) when \( s \geq 0 \)), the potential walls form the set \( \{(s, t) - \frac{n}{2} (s^2 + r^2) + (\chi' - (1-n)r') \frac{r}{2 c_1'} s - nc_1' = 0\} \). When \( c_1' = 0 \), the wall is the \( t \)-axis.

When \( c_1' \neq 0 \), these are nested semicircles with center \( x = (\chi' - (1-n)r') / c_1' \) and radius \( \text{Rad} = \sqrt{n^2 - 2n} \). It is not hard to see that \( W_{\text{potential}}^{1,0,1-n} \) on the second quadrant and \( W_{\text{potential}}^{\text{actual}1,0,1-n} \) on the first quadrant are nested semicircles.

We define \( W_{\text{actual}}^{i,0,1-n} \) as

\[
\{(s, t) \mid E \text{ with invariant } (r, c_1, \chi), \text{ which is strictly semistable and locally stable under } (Z_{i,s}, \mathcal{A}_s)\}.
\]

Here by \( "E" \) is locally stable under \((Z_{i,s}, \mathcal{A}_s)\), we mean that for any \( \delta > 0 \), there is \((s', t') \in B_{\delta}(s, t) \) such that \( E \) is stable under \((Z_{s',t'}, \mathcal{A}_s')\). By definition, \( W_{\text{actual}}^{i,0,1-n} \subset W_{\text{potential}}^{i,0,1-n} \). On the second quadrant, \( W_{\text{actual}}^{1,0,1-n} \) is formed by nested semicircular walls. We call such a wall in \( W_{\text{actual}}^{1,0,1-n} \) an actual wall.

**Lemma 2.1.** For any \( k \in \mathbb{Z}, O(k) \) (resp. \( O(k)[1] \)) is a stable object under stability condition \((Z_{i,s}, \mathcal{A}_s)\) for \( s < k \) (resp. \( s \geq k \)).

**Proof.** Since \( Z_{i,s}(E) = Z_{i,s+k}(E(k)) \), we may assume \( k = 0 \). Suppose \( O \) is not stable for some \((Z_{i,s}, \mathcal{A}_s)\), with \( s < 0 \), then there exists \( E \) destabilizing \( O \) under \((Z_{i,s}, \mathcal{A}_s)\). We may assume \((s, t) \) is on a potential wall of \( W_{\text{potential}}^{1,0,1-n} \) which is a semicircle with right corner at the origin. The exact sequence:

\[
0 \to H^{-1}(E) \to H^{-1}(O) \to H^{-1}(O/E) \to H^0(E) \to H^0(O) \to H^0(O/E) \to 0
\]
implies that $H^{-1}(E)$ is 0, we may write $E$ for $H^0(E)$ for short. In addition, $H^0(O/E)$ has rank 0 (otherwise the morphism $H^0(E) \to H^0(O)$ is 0), hence it has non-negative $c_1$. This implies $\mu_{-}^{GM}(E) < 0$.

Let $\mu_{-}^{GM}(E) = s_0 < 0$, we may move $(s, t)$ along the semicircle to the right, when $s < s_0$, $E$ still destabilizes $O$ since $E$ and $O$ are in $\mathcal{A}_i$. Write $E$ as $0 \to E_+ \to E \to E_- \to 0$, where $E_-$ stands for the MG semistable factor with slope $s_0$. When $s$ tends to $s_0$, $r_{s,t}(E_-)$ will tend to 0. As $d_{s,t}(E_-) > l > 0$ for some constant $l$, $E_+$ destabilizes $O$ under $(Z_{s,t}, \mathcal{A}_0)$. By repeating this procedure, we get an $E'$ which destabilize $O$ and has $\mu_{-}(E') \geq 0$, this is a contradiction.

For $O[1]$ and $s > 0$ case, we get $\mu_{-}^{GM}(H^{-1}(O/E)) > 0$, then the same argument also works. When $s = 0$, the only exceptional case is that both $H^{-1}(E)$ and $H^{-1}(O/E)$ has $c_1 = 0$ and $H^0(E)$ is torsion. But that cannot happen since $H^{-1}(E)$ and $H^{-1}(O/E)$ are torsion free and the rank of $O$ is 1.  

The $GL(2, \mathbb{R})^+$ acts on the space of stability condition by the $SL(2, \mathbb{R})$-action on the central charge and the shift on the heart structure. In particular, an element $\phi$ in the subgroup $\mathbb{R}$ acts on $(Z, \mathcal{A})$ as follow: if $\phi$ is an integer, then $\phi \circ (Z, \mathcal{A}) = (Z[\phi], \mathcal{A}[\phi])$ with:

$$\mathcal{A}[\phi] = [A[\phi]] A \text{ is an object of } \mathcal{A} \text{ and } Z[i](A) := (-1)^i Z(A).$$

If $0 < \phi < 1$, then $\mathcal{A}[\phi] := \langle T, \mathcal{F}_\phi(1) \rangle$ and $Z[\phi](E) := e^{-i\phi} Z(E)$ with:

$$\mathcal{T}_\phi = \langle T \in \mathcal{A} \mid T \text{ is stable with } \arg(Z(T)) > \phi \pi \rangle;$$

$$\mathcal{F}_\phi = \langle F \in \mathcal{A} \mid F \text{ is stable with } \arg(Z(F)) \leq \phi \pi \rangle.$$

The moduli spaces of stable objects are unaffected under this $\mathbb{R}$-action.

**Proposition 2.2** (Proposition 7.5 in [1]). *Let $k$ be an integer. If a pair of real numbers $(s, t)$ satisfies

$$(s - k)^2 + t^2 < 1,$$

then there is $\phi_{s,t,k} \in \mathbb{R}$ (not canonically defined), such that under its action $(Z_{s,t}, \mathcal{A}_0[\phi_{s,t,k}])$ is identified with $(Z, \mathcal{A}(k))$ for suitable choice of central charge $(z_{-1}, z_0, z_1)$ for $Z$.

**Proof.** By Lemma 2.1 the rest is the same as Prop 7.5 in [1].  

We call such a semidisic a quiver region.

Consider a central charge $(z_{-1}, z_0, z_1)$ of $\mathcal{A}(k)$:

$$(z_{-1}, z_0, z_1) =: z = \tilde{a} + i \tilde{b},$$

where $\tilde{a}$ and $\tilde{b}$ are real vectors. Fix three non-negative integers $(n_{-1}, n_0, n_1) = \overline{n}$, and let

$$\vec{\rho} = -\tilde{a} + b \left( \frac{\overline{n} \cdot \tilde{a}}{n \cdot \tilde{b}} \right)$$

, then $n \cdot \vec{\rho} = 0$. An object $E$ in $\mathcal{A}(k)$ with type $\overline{n}$ is stable (semistable) with respect to the central charge $\overline{z}$ if and only if for any proper subobject $E'$ with type $\overline{n}'$ one has:

$$\overline{n}' \cdot \vec{\rho} < 0 \ (\leq 0).$$

**Remark 2.3.** $\vec{\rho}$ does not change under the rotation of $\overline{z}$, hence it does not depend on the choice of $\phi_{s,t,k}$ in 2.2. The explicit formula of $\vec{\rho}$ is given at [4.7].
In particular, we will use the following computation in the $k = 0$ case of in the first statement of Proposition 2.5.

Example 2.4. For $\mathcal{A}(0)$ and $s < 0$, let $\vec{m}$ be $(n, 2n + 1, n)$, the $p_{1, s}$ is given as:

$$\frac{ts}{2 + n - \frac{1}{2}} \left( \frac{(1 + s)^2}{2} - \frac{s^2}{2} - \frac{(1 - s)^2}{2} - \frac{r^2}{2} \right) + r(1 + s, -s, s - 1).$$

Consider the space of characters $\vec{p}$, since the subobjects of $E$ have only finitely many possible numerical types, there are finitely many walls (lines in this case) on which an object $E$ with type $\vec{m}$ could be semistable but nonstable with respect to $\vec{r}$. These walls divide the space into chambers. In each chamber, the moduli space of stable objects remains the same, so one may choose an integral vector $\vec{p}$ as a representative in the chamber. By Proposition 3.1 in [13] by King, the moduli space of (semi)stable object with respect to central charge $Z$ consists the $\vec{p}$-(semi)stable points under the $G$-action. As explained in the Chapter 2.2 in [10] by Ginzburg, the moduli space of $Z$-semistable objects is constructed as a GIT quotient:

$$\text{Proj} \left( \bigoplus_{n > 0} C[X]^G \cdot \vec{p}_n \right).$$

Here $X$ is the affine closed subscheme of $\text{Hom}(C^{m-1} \otimes O(k - 1), C^{m} \otimes O(k)) \times \text{Hom}(C^{m} \otimes O(k), C^{m+1} \otimes O(k + 1))$ consists of the complex. $G$ is the reductive group $\text{GL}(n-1, C) \times \text{GL}(n_1) \times \text{GL}(n_2, C)/C^n$ and $\vec{p}$ is the character $(\det^{m-1}, \det^{n_1}, \det^{n_2})$ of $G$. This character is well-defined since $\vec{p} \cdot \vec{m}$ is 0. When $\vec{m}$ is primitive (i.e. gcd$(n-1, n_0, n_1) = 1$), $G$ acts freely on the stable points on $X$. We write $\mathcal{W}_{\vec{p}, \vec{m}}(\vec{m}) := X/_{\vec{p}, G}$ as the moduli space of semistable objects in $\mathcal{A}(k)$ with type $\vec{m}$ and character $\vec{p}$.

Proposition 2.5. 1. Given $n > 0$, for any $s < 0$ and $t > 1$, the moduli space of stable objects with invariants $(r, c_1, \chi) = (1, 0, 1 - n)$ under $(Z_{r,s}, \mathcal{A})$ is the same as that in Mumford-Giesker sense, i.e., the moduli space is the deformed Hilbert scheme $\text{Hilb}^s S$. 2. There are only finitely many actual destabilizing walls for $\text{Hilb}^s S$.

Proof. Let $I$ be a torsion free sheaf with $(r, c_1, \chi) = (1, 0, 1 - n)$. When $k = 0$, $I[1]$ appears in $\mathcal{A}(0)$ with type $\vec{m} = (n, 2n + 1, n)$. By Proposition 7.7 and Proposition 6.20 in [16], let $\vec{p}$ be $((2n + 1)(m - 1), n, -(2n + 1)m)$, then for all $m > 1$, $X^{\vec{p}, \vec{m}}$ consists of complexes which are quasi-isomorphic to $I[1]$ for some torsion-free $I$ with invariants $(1, 0, 1 - n)$. Now by the formula in Example 2.4 there is an open area $A$ in the region $|s|, t|s|^2 + t^2 < 1, s < 0$ with boundary containing $(0, 0 < t < 1)$ such that the stable objects with invariants $(1, 0, 1 - n)$ under $(Z_{r,s}, \mathcal{A})$ are the same as those in the Mumford-Giesker sense.

Note that when $s < 0$, $W_{\text{actual}}$ consists of semicircles with center at $x$ and radius $\sqrt{x^2 - 2n}$. Since $x + \sqrt{x^2 - 2n}$ is decreasing when $x < \sqrt{2n}$, these semicircles are nested with right boundary in the region $(-\sqrt{2n} < s < 0, t = 0)$, hence all the actual destabilizing walls are nested semicircles and each of them corresponds a wall in $\mathcal{A}(k)$ for some $0 < k < \sqrt{2n}$. This tells the finiteness of the actual walls, and in the region outside the first wall, the stable objects with invariants $(1, 0, 1 - n)$ are the same as those in the area $A$.

When $s > 0$, the same argument works for the second statement. □
3 Wall Crossing via GIT

3.1 Properties of stable objects in $\mathcal{A}(k)$

The goal of this section is to show Corollary[3,6] the moduli space $\mathcal{M}^{\beta,n}(n)$ is smooth and irreducible for generic $\beta$. The next two lemmas are about the vanishing property of some Ext$^2$s.

Lemma 3.1. Let $\mathcal{F}$ be a stable object in $(\mathcal{A}_r, Z_{s_{0,t}})$ (for some $s_0 < 0$) with invariant $(r, c_1, \chi) = (1, 0, 1 - n)$ Then we have

$$\text{Hom}(\mathcal{F}, \mathcal{F}[2]) = 0.$$

Proof. Given a point $(\tilde{s}, \tilde{t})$ on the second quadrant, we denote $W_{(1,1)}$ as the unique semicircle with central at $x$ and radius $\sqrt{x^2 - 2n}$ that across $(\tilde{s}, \tilde{t})$, i.e. the potential semicircular wall for the invariant $(1, 0, 1 - n)$ across $(\tilde{s}, \tilde{t})$.

Case I: $W_{(s_0,t_0)}$ has radius greater than $\frac{1}{2}$. The actual destabilizing walls of $\mathcal{F}$ are nested, so $\mathcal{F}$ is a stable object under $(\mathcal{A}_r, Z_{s_{0,t}})$ for all $(s, t) \in W_{(s_0,t_0)}$. $\mathcal{F}(-3)$ is a stable object under $(\mathcal{A}_{t_{0-1}}, Z_{s_{0-2}})$ for any $(s, t) \in W_{(s_0,t_0)}$. These points form the semicircle $W_{(s_0,t_0)} - (3, 0)$. Since the radius of the circle is greater than $\frac{1}{2}$, these two semicircles intersect at $(s_1, t_1)$. In $\mathcal{A}_{t_1}$, under the central charge $Z_{s_1,t_1}$, the slope of $\mathcal{F}$ is $-\frac{n}{3} + \frac{i_1 + 2n}{2s_1}$. Because $s_1$ is less than $-3$, $-\frac{n}{3} + \frac{i_1 + 2n}{2s_1}$ is greater than the slope of $\mathcal{F}(-3)$, which is $-\frac{n}{3} + \frac{i_1 + 2n}{2s_1 + 3}$. Thus $\text{Hom}(\mathcal{F}, \mathcal{F}(-3)) = 0$, and $\text{Hom}(\mathcal{F}, \mathcal{F}[2]) = 0$ by Serre duality.

Case II: $W_{(s_0,t_0)}$ has radius equal to or less than $\frac{1}{2}$. Let $k$ be the positive integer such that

$$(k + 1)(k + 2)/2 \leq n < (k + 2)(k + 3)/2.$$

The semicircle $W_{(k - 1,0)}$ has radius not less than $\frac{1}{2}$, and by Lemma[3,10] after this wall, there is no stable object with invariant $(1, 0, 1 - n)$. The radius of $W_{(k - 1,0)}$ is greater than $\frac{1}{2}$, hence the right edge of $W_{(s_0,t_0)}$ falls into the interval $(-k - 1, -k)$. Therefore $\mathcal{F}[1]$ is an object in $\mathcal{A}(-k)$, and $\mathcal{F}(-3)[1]$ is an object in $\mathcal{A}(-k - 3)$. On the other hand, $W_{(s_0,t_0)}$ is larger than $W_{(k - 1,0)}$, hence its left edge is less than $-k - 2$. Since its radius is not greater than $\frac{1}{2}$, its left edge is greater than $-k - 4$. Combining these two observations, the left edge of $W_{(s_0,t_0)}$ falls into the $\mathcal{A}(-k - 3)$ quiver region. Therefore $\mathcal{F}$ is an object in $\mathcal{A}(-k - 3)$. We have

$$\text{Hom}(\mathcal{F}, \mathcal{F}(-3)) = \text{Hom}(\mathcal{F}, \mathcal{F}(-3)[1][-1]) = 0,$$

where the last equality is because of both $\mathcal{F}$ and $\mathcal{F}(-3)[1]$ are in the same heart $\mathcal{A}(k+3)$, By Serre duality, $\text{Hom}(\mathcal{F}, \mathcal{F}[2]) = 0$. □

On each chamber wall all S-equivalent semistable objects (i.e., their stable factors are the same after rearrangement) are contracted to one point. Let $\mathcal{F}$ be a locally stable object with invariants $(1, 0, 1 - n)$ at $(s_0, t_0)$. Suppose it is destabilized at this point which lies on an semicircular actual wall $W_{(s_0,t_0)}$. Then $\mathcal{F}$ has a filtration in $\mathcal{A}_{s_0}$,

$$\mathcal{F} = \mathcal{F}_m \supset \mathcal{F}_{m-1} \supset \cdots \supset \mathcal{F}_1 \supset \mathcal{F}_0 = 0,$$

such that each factor $\mathcal{E}_1 := \mathcal{F}/\mathcal{F}_1$ is stable under $Z_{s_0,t_0}$. For any point $(s, t)$ on $W$, we have the slope $\mu(s, \mathcal{E}_1) = \mu(s, \mathcal{F})$. Otherwise, $\mathcal{F}$ is always unstable under $Z_{s_0,t_0+k\epsilon}$. Since the actual walls on the second quadrant are nested semicircular walls, this contradicts the fact that $\mathcal{F}$ is locally stable under $Z_{s_0,t_0}$. 

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3.1 Properties of stable objects in $\mathcal{R}(k)$

**Lemma 3.2.** Let $E_1, \ldots, E_m$ be the stable factors of $\mathcal{F}$ as above, then we have:

$$\text{Hom}(E_i, E_j[2]) = 0,$$

for all $1 \leq i, j \leq m$.

**Proof.** In order to apply the same trick as that in Lemma 3.1, we show that each $E_i$ is stable on the whole $W((s, t_0))$. First, we show that $E_i$ is always in $\mathcal{A}_s$. If not so, either $\mu_i^GM(H^{-1}(E_i))$ or $\mu_i^GM(H^0(E_i))$ falls into the open region $W_k := \{ s(s, t) \in W_{(s, t)} \}$ for some $t$. If there exists $\tilde{E}_k$ such that $\mu_i^GM(H^{-1}(\tilde{E}_k)) = \mu_i^GM(H^{-1}(E_k)) \in W_k$, we may assume

$$s_- := \mu_i^GM(H^{-1}(E_k)) \geq \max_i \min_{s \geq s_0} \{ \mu_i^GM(H^{-1}(E_i)) \};
\tilde{s} = \max_i \{ \mu_i^GM(H^{-1}(E_i)) \} = s_-.$$

Then when $s$ tends to $s_-$ from the right along $W_{(s, t_0)}$, $\mu_i^GM(H^{-1}(E_{k}))$ will go to $+\infty$. Let the quotient of $H^{-1}(E_{k})$ in $\mathcal{A}_s$ for all $s_0 < s \leq s_0$ be $E'_k$, then there is a map from $F_k \to E'_k$: $F_k \to E'_k$. By the maximum assumption on $\mu_i^GM(H^{-1}(E_k))$ and $k$, this is surjective for all $s_0 < s \leq s_0$. Let the kernel of $F_k \to E'_k$ be $\mathcal{F}'$. Since $\mu_i^GM(H^{-1}(E_k))$ is bounded on $W_{(s_0, t_0)}$, and $\mu_i^GM(H^{-1}(F_{k}))$ tends to $+\infty$ as $s$ tends to $s_-$, we have $\mu_i^GM(H^{-1}(\tilde{E})) > \mu_i^GM(H^{-1}(E_k))$ when $s$ tends to $s_-$. Since the actual walls of $(1, 0, 1 - n)$ are nested, $\mathcal{F}$ must be locally stable along $W_{(s_0, t_0)}$, which contradicts the inequality $\mu_i^GM(H^{-1}(\tilde{E})) > \mu_i^GM(H^{-1}(E_k))$. In a similar way, we get a contradiction for the case $\mu_i^GM(H^{-1}(E_k)) \in W_k$.

Next, we show the stableness of $E_k$. Suppose $k$ is the maximum number such that $E_k$ is not stable for some $(s, t)$ on $W_{(s_0, t_0)}$. There must be a subobject $E'_k$ of $E_k$ in $\mathcal{A}_s$ for some $(s, t) \in W_{(s_0, t_0)}$ such that $\mu_i^GM(H^{-1}(E'_k)) > \mu_i^GM(H^{-1}(E_k))$. Again let the quotient be $E'_k$ and consider the kernel $F'_k$ of $F_k \to E'_k$. Then $F'_k$ is a subobject of $F_k$ and $\mu_i^GM(H^{-1}(F'_k)) > \mu_i^GM(H^{-1}(F_k))$, which is a contradiction.

Now we repeat the same argument in Lemma 3.1 for $E'_j$ and $E_i$. When the $W_{(s_0, t_0)}$ has radius greater than 3, since on $W_{(s_0, t_0)}$, $\mu_i^GM(H^{-1}(E_{(s_0, t_0)})) = \mu_i^GM(H^{-1}(E_{(s_0, t_0)}))$, and the radius is not greater than 3/2, $E_i$ and $E_i(-3)[1]$ are both in $\mathcal{A}_s(-k - 3)$ (since $F_k(-3)[1]$ is in $\mathcal{A}_s(-k - 3)$ and $E_i(-3)$ has the same slope of $F_k(-3)$ along the wall). In either case, $\text{Hom}(E_j, E_i(-3)) = 0$. We get

$$\text{Hom}(E_j, E_i[2]) \approx \text{Hom}(E_j, E_i(-3))^* = 0.$$

**□**

Based on the previous two lemmas, we will study the phenomenon of wall-crossing in a quiver region. Write $\mathcal{F}[1]$ as an object $K$ in a quiver region $\mathcal{A}(k)$, for some $-\sqrt{2n} < k < 0$. Let $\tilde{p}$ be the character corresponding to $W_{(s_0, t_0)}$. Then the stable factor filtration of $K$ at $\tilde{p}$ in $\mathcal{A}(k)$ is written as:

$$K = K_{s_0} \supset K_{s_0-1} \supset \cdots \supset K_{1} \supset K_{0} = 0,$$

where $K_i$ is $\mathcal{F}[1]$, with $K_i = K_{s_i}/K_{s_i-1} = E_i[1]$. Let $\tilde{p}$ be the character at $(s_0 \pm \epsilon, t_0)$. The point in $\mathcal{F}[s \pm \epsilon]$ that stands for the S-equivalent class with stable factors $(K_1, \ldots, K_m)$, $\mathcal{F}[s \pm \epsilon]$ and $\mathcal{F}[s \pm \epsilon]$ respectively. Let $S^+, S^-$, be the sets defined as follows respectively:
Lemma 3.3. For each $L \in \mathcal{A}(k)$, $L$ has a filtration with stable factors as a strict subset of $[K_1, \ldots, K_m]$ (counting multiples); $L[-1]$ is in $\mathcal{A}_{n+e}$, stable under $Z_{n+e,0}$ and $\overrightarrow{T} \cdot \overrightarrow{\rho}_{+} > 0$ (respectively stable under $Z_{n-e,0}$ and $\overrightarrow{T} \cdot \overrightarrow{\rho}_{-} > 0$),

where $\overrightarrow{T}$ is the type of $L$.

Lemma 3.4. For each $K$ with type $(n - \frac{k(k-1)}{2}, 2n - k^2 + 1, n - \frac{k(k+1)}{2})$ that is stable with respect to the character $\rho_{-}$ and has stable factor filtration with factors $K_1, \ldots, K_m$ (i.e. the point $K$ is in $V^+_{\rho_{-}}(K_1, \ldots, K_m)$), one may write it as an extension of two semistable objects in $\mathcal{A}(k)$:

$$0 \rightarrow K_{-} \rightarrow K \rightarrow K_{+} \rightarrow 0$$

with properties:
1. $\text{Hom}(K_{-}, L_{+}) = \text{Hom}(L_{+}, K_{-}) = 0$, for any $L_{+} \in S_{+}$;
2. $\text{Hom}(K_{+}, L_{-}) = \text{Hom}(L_{-}, K_{+}) = 0$, for any $L_{-} \in S_{-}$.

Proof. Since $K[-1]$ and $L[-1]$ are stable under $Z_{n-e,0}$, and $\overrightarrow{T} \cdot \overrightarrow{\rho}_{+} = \overrightarrow{\rho}_{+} > 0$, $\text{Hom}(L, K) = 0$ for any $L \in S_{+}$. For any sub-object $K_{+}$, we have $\text{Hom}(L_{-}, K_{+}) = 0$. Similarly, we have $\text{Hom}(K_{-}, L_{+}) = 0$ for any $L_{+} \in S_{+}$.

Start from an extension pair $(K_{0}^{0}, K_{1}^{0}) = (K_1, K/K_1)$ for $K$. If $\text{Hom}(L_{+}, K^{0}) \neq 0$ for some $L_{+} \in S_{+}$ (the image is a subobject in $K^{0}$), then we make an adjustment for the pair by moving a subobject $L_{0}^{0}$ in $K^{0}$ to $K_{1}^{0}$. Denote this extension pair by $(K_{+}^{0}, K_{-}^{0})$. Then we move a quotient object $L_{0}^{0}$ in $K_{1}^{0}$ to $K^{0}$ if there is any. Denote the new pair by $(K_{+}^{1}, K_{-}^{1})$. We may repeat this procedure and get pairs $(K_{+}^{i}, K_{-}^{i})_{i \geq 0}$. Denote the type of $K_{+}^{i}$ by $\overrightarrow{k}_{+}^{i}$, it is not hard to see that $\overrightarrow{k}_{+}^{i} \cdot \overrightarrow{\rho}_{+}$ is non-decreasing when $i$ increases. $\overrightarrow{k}_{+}^{i} \cdot \overrightarrow{\rho}_{+}$ stop increasing when there is no adjustment at this step, i.e. $(K_{+}^{i}, K_{-}^{i})$ satisfies the requirements. Since there are only finite possibilities for the value of $\overrightarrow{k}_{+}^{i} \cdot \overrightarrow{\rho}_{+}$, we always get the extension pair.

It is immediate that $\text{Hom}(K_{+}, K_{-}) = 0$. Let $V(K_{+}, K_{-})$ be the sub-variety in $V^+_{\rho_{-}}(K_1, \ldots, K_m)$ consisting of objects that can be written as the extension $0 \rightarrow K_{+} \rightarrow K \rightarrow K_{-} \rightarrow 0$.

Lemma 3.4. Adopting the notation as above, the dimension of $V(K_{+}, K_{-})$ is at most $\dim \text{Ext}^1(K_{+}, K_{-}) - (\dim \text{Aut}(K_{+}) + \dim \text{Aut}(K_{-}) - 1)$.

Proof. The extension is given by an element in $\text{Hom}(K_{+}, K_{-}[1])$. As $\text{Hom}(K_{+}, K_{-}) = 0$, the two extended objects are isomorphic if they are on the same orbit of the $\text{Aut}(K_{+}) \times \text{Aut}(K_{-})$ action. To prove the lemma, we only need to show that if $f \in \text{Hom}(K_{+}, K_{-}[1])$ induces a complex in $V(K_{+}, K_{-})$, then the stabilizers of $f$ in $\text{Aut}(K_{+}) \times \text{Aut}(K_{-})$ are the scalars.

Let $(g^{+}, g^{-})$ be a stabilizer of $f$. Write $K_{+}$ as $O(-k - 1) \otimes H_{-1}^{+} \rightarrow O(-k) \otimes H_{0}^{+} \rightarrow O(-k + 1) \otimes H_{+}^{+}$, then we can represent $g^{+}$ as $(g^{+}_{1}, g^{0}_{0}, g^{+}_{0}) \subset \text{ker}^{0} \subset \text{Hom}(K_{+}^{0}, K_{+}^{0})$, where $g^{+}_{1} \in \text{GL}(H_{-1}^{+})$, $f$ can be written as $(f_{-1}, f_{0}) \in \text{ker}^{0} / \text{im}^{0} \subset \text{Hom}(K_{+}, K_{+}) / \text{im}^{0}$, when $f_{-1} \in \text{Hom}(O(-k - 1) \otimes H_{-1}^{+}, O(-k) \otimes H_{0}^{+})$ and $f_{0} \in \text{Hom}(O(-k) \otimes H_{0}^{+}, O(-k + 1) \otimes H_{+}^{+})$. Then $K_{+}$ is written as:

$$O(-k-1) \otimes (H_{-1}^{+} \otimes H_{-1}^{-}) \rightarrow O(-k) \otimes (H_{0}^{+} \otimes H_{0}^{-}) \rightarrow O(-k+1) \otimes (H_{+}^{+} \otimes H_{+}^{-}).$$
3.1 Properties of stable objects in $\mathcal{A}(k)$

As $(g^*, g^-)$ is a stabilizer, $g^* \circ f \circ (g^-)^{-1} - f$ is an exact cycle in $\text{Hom}^1(K^-, K^*)$ which can be written as $d^0(s)$ for some $s \in \text{Hom}^0(K^-, K^*)$. Write $\tilde{f}$ for $g^* \circ f \circ (g^-)^{-1}$, we have:

$$
\begin{bmatrix}
\text{Id} H_0^+ & s_0 \\
0 & \text{Id} H_0^+
\end{bmatrix}
\begin{bmatrix}
g_0^+ \\
g_0^-
\end{bmatrix}
\begin{bmatrix}
I^* \\
I^-
\end{bmatrix}
= \begin{bmatrix}
\text{Id} H_0^+ & s_0 \\
0 & \text{Id} H_0^+
\end{bmatrix}
\begin{bmatrix}
g_0^+ I^* \\
g_0^- I^-
\end{bmatrix}
= \begin{bmatrix}
I^* g_+^+ \\
I^- g_+^-
\end{bmatrix}
\begin{bmatrix}
\tilde{f}_1 g_+^1 \\
\tilde{g}_1 g_+^1
\end{bmatrix}
= \begin{bmatrix}
I^* \\
I^-
\end{bmatrix}
\begin{bmatrix}
\text{Id} H_0^+ & s_0 \\
0 & \text{Id} H_0^+
\end{bmatrix}
\begin{bmatrix}
g_+^1 \\
g_+^1
\end{bmatrix}.
$$

By changing the labels, we have a similar equality for $J$. $\square$

Let the type of $K_+$ be $\vec{n}_+$, we write $\mathcal{M}_{\vec{p}_0}^{ss}(\vec{n}_+)$ as the moduli space of semistable objects in $\mathcal{A}(k)$ with type $\vec{n}_+$ and character $\vec{p}_0$. It is a projective variety as the case of Hilbert case. For a positive integer $c$, let $\mathcal{M}_{\vec{p}_0}^{ss}(\vec{n}_+)$ be the locus in $\mathcal{M}_{\vec{p}_0}^{ss}(\vec{n}_+)$ consisting of points that $\dim \text{Hom}(K_+, K_-) = c$. As the constrain is algebraic, $\mathcal{M}_{\vec{p}_0}^{ss}(\vec{n}_+)$ is a subvariety in $\mathcal{M}_{\vec{p}_0}^{ss}(\vec{n}_+)$, and we may stratify $\mathcal{M}_{\vec{p}_0}^{ss}(\vec{n}_+)$ as $\cup_{c \in \mathbb{N}} \mathcal{M}_{\vec{p}_0}^{ss}(\vec{n}_+)$). These notations also make sense for $\vec{n}_-$ respectively.

Let $V_{-\vec{p}_0}(\vec{n}_+, \vec{n}_-)$ be the locus of object that can be extended by $K_+$ and $K_-$ which satisfy the properties in Lemma 3.3 with type $\vec{n}_+$ and $\vec{n}_-$. Let $V_{-\vec{p}_0} \subset \mathcal{M}_{\vec{p}_0}^{ss}(\vec{n})$ consisting of objects that are stable with respect to $\vec{p}_0$ but not $\vec{p}_+$, then by Lemma 3.3 $V_{-\vec{p}_0} = \cup_{\vec{n}_+, \vec{n}_-} V_{-\vec{p}_0}(\vec{n}_+, \vec{n}_-)$, we may estimate the dimension of $V_{-\vec{p}_0}$ by studying the dimension of each piece.

**Proposition 3.5.** Adopting the notation as above, when $-\sqrt{2n} < s < 0$, the dimension of $V_{-\vec{p}_0}$ is less than $2n - 2$.

**Proof.** For $c, d \in \mathbb{N}$, let $V_{-\vec{p}_0}(\vec{n}_+, \vec{n}_-)(c, d)$ be the locus where the complex can be extended by objects in $\mathcal{M}_{\vec{p}_0}^{ss}(\vec{n}_+)$ and $\mathcal{M}_{\vec{p}_0}^{ss}(\vec{n}_-)$. Then we have

$$
\dim V_{-\vec{p}_0}(\vec{n}_+, \vec{n}_-)(c, d) 
\leq \dim \mathcal{M}_{\vec{p}_0}^{ss}(\vec{n}+) + \dim \mathcal{M}_{\vec{p}_0}^{ss}(\vec{n}-) + \max_{K_+, K_-} \{ \dim \text{Ext}^1(K_-, K_+) \} - c - d + 1
= \max_{K_+, K_-} \{ \dim \text{Ext}^1(K_-, K_+) \} - c - d + 1
$$

(by Lemma 3.4)

$$
= \dim \text{Ext}^1(K_+, K_+) + \dim \text{Ext}^1(K_-, K_+) + \dim \text{Ext}^1(K_-, K_-) - c - d + 1
$$

(by Lemma 3.2 and 3.3)

$$
\leq 2n + \chi(K_+, K_-).
$$

The remaining task is to estimate $\chi(K_+, K_-)$. Since when $s$ moves from $s_0 + \epsilon$ to $s_0 - \epsilon$, $\vec{p}_+$ will move from $\vec{p}_0 + \epsilon(n_0, -n_{-1})$ to $\vec{p}_0 - \epsilon(n_0, -n_{-1})$, and $K_-$ does not destabilize $K$ on the $s_0 + \epsilon$ side, we have

$$
\vec{n}_+ \cdot (0, n_{-1}, 0) > 0, \quad \vec{n}_+ \cdot (0, n_{1}, -n_{0}) > 0.
$$

(*)
3.2 properties of GIT

Now we have the estimation on $\chi(K_+, K_-)$:

$$\chi(K_+, K_-) \geq \chi(K_+, K_-)$$

$$= (n \cdot n_+ - 3(n_{-1})n_{-1}(0) + n_{-1}(0)n_{+1}) + 6n_{-1}(0)n_{-1}(1) - (n \cdot n_+ - 3(n_{-1})n_{-1}(0)$$

$$+ n_{-1}(0)n_{-1}(1) + 6n_{-1}(1))$$

$$= 3n_+ \cdot (k - 1, -k, k + 1)$$

$$\geq 3.$$

The last inequality is due to $(k - 1, -k, k + 1) = \frac{k+1}{n}(n(0), -n(-1), 0) - \frac{k-1}{n}(0, n(1), -n(0))$ and the formula (7).

$\square$

**Corollary 3.6.** For a generic $\rho$ not on any actual destabilizing wall, $\mathfrak{M}^{\rho,s}(n) = X/\rho G$ is irreducible and smooth.

**Proof.** In the Hilbert scheme chamber, the irreducible components that contain $X^{\rho,s}$ are reduced and irreducible since $\text{Hilb}^S$ is so. Passing to another quiver region does not affect the reduceness property of $X^{\rho,s}$. By Proposition 3.5 while going across one destabilizing wall, the new stable locus $V_{\rho \rho}$ in $X^{\rho,s}/G$ has dimension less than $2n - 2$. Therefore the dimension of $X^{\rho,s} \setminus X^{\rho,s}$ is less than $2n - 2 + \dim G$. On the other hand, the total space $X$ is $\text{Spec} \mathbb{C}[M]/(J \circ I)$, where $M$ is the space $\text{Hom}(H_{-1}, H_0) \otimes \text{Hom}(\mathcal{O}(-k - 1), \mathcal{O}(-k)) \times \text{Hom}(H_0, H_1) \otimes \text{Hom}(\mathcal{O}(-k), \mathcal{O}(-k + 1))$. The dimension of $M$ is $3n - 1 + 3n_0$, and $J \circ I$ has $6n_1 - n_1$ equations. In any quiver region, we have $3n - 1 + 3n_0 - 6n_1 = 2n - 2 + \dim G$. Each irreducible component has dimension at least $2n + \dim G$. Since $X^{\rho,s}$ is open in $X$, and $\dim X^{\rho,s} \setminus X^{\rho,s} < 2n + \dim G$, we get $X^{\rho,s}$ is irreducible.

The dimension of the Zariski tangent space at a point $K = (I_0, J_0)$ is the dimension of $\text{Hom}_C(C[M]/(J \circ I), \mathcal{O}[t]/(t^2))$ at point $(I_0, J_0)$. Each tangent direction is written in a form $(I_0, J_0) + t(I_1, J_1)$. In order to satisfy the equation $J \circ I = 0$, we need

$$J_0 \circ I_1 + J_1 \circ I_0 = 0.$$

Hence the space of $(I_1, J_1)$ is just the kernel of $d^1 : \text{Hom}^1(K, K) \rightarrow \text{Hom}^2(K, K)$. Now by Lemma 3.1, $d^1$ is surjective. The Zariski tangent space has dimension $\dim \text{Hom}^1(K, K)$

$$- \dim \text{Hom}^2(K, K) = 3n_0 + n_1 - 6n_{-1}n_1,$$

which is the dimension of $M$ minus the number of equations. We get the smoothness of $X^{\rho,s}$. Furthermore, since $(n_{-1}, n_0, n_1) = (n - \frac{k+1}{3}, 2n - k^2 + 1, n - \frac{k+1}{3})$ is primitive, $G$ acts freely on $X^{\rho,s}$. By Luna’s étale slice theorem, $X^{\rho,s} \rightarrow X/\rho G$ is a principal bundle. Since $X^{\rho,s}$ is smooth, by Proposition IV.17.7.7 in [11], the base space is also smooth. $\square$

### 3.2 properties of GIT

Birational geometry via GIT has been studied in [9] by Dolgachev and Hu, [18] by Thaddeus. In this section, we recollect some properties in a language of the affine GIT.

Let $X$ be an affine algebraic $G$-variety, where $G$ is a reductive group and acts on $X$ via a linear representation. Given a character $\rho : G \rightarrow \mathbb{C}^*$, the (semi)stable locus is written as $X^{\rho,s}(X^{\rho,s})$. We write $\mathcal{C}[B]^{\rho,s}$ for the $\chi$-semi-invariant functions on $B$. i.e. one has

$$f(g^{-1}(x)) = \chi(g) \cdot f(x), \forall g \in G, x \in B.$$
3.2 properties of GIT

Denote the GIT quotient by $X//_\rho G := \text{Proj} \bigoplus_{n \geq 0} \mathbb{C}[X]^{G, \rho^n}$ and the map from $X^{ss, \rho}$ to $X//_\rho G$ by $F_\rho$.

In additions, we need he following assumptions on $X$ and $G$: 1. there are only finite many walls in the space of characters on which there are semistable but non-stable points, in the chamber we have $X^{ss, \rho} = X^{ss, \rho'}$. 2. $X^{ss, \rho}$ is smooth and the action of $G$ on $X^{ss, \rho}$ is free. 3. $\mathbb{C}[X]^G = \mathbb{C}$, i.e., $X//_\rho G$ is projective and connected. 4. The closure of $X^{ss, \rho}$ (if non-empty) for any $\rho$ is the same irreducible component. 5. Given any point $x \in X$, the set of characters $[\rho] x \in X^{ss, \rho}$ is closed.

Notations and constructions: let $\rho$ be a generic character (i.e. not on any walls) satisfying that $X^{ss, \rho}$ is non-empty, then by assumption we have a $G$-principal bundle $X^{ss, \rho} \to X//_\rho G = X^{ss, \rho}/G$. Giving another character $\rho_0$ of $G$, we denote $L_{\rho, \rho_0}$ to be the line bundle over $X//_\rho G$ by composing the transition functions of the $G$-principal bundles with $\rho_0$. Now we are ready to list some properties from the variation geometric invariant theory (VGIT).

**Proposition 3.7.** Let $X$ be an affine algebraic $G$-variety that satisfies the assumptions 1 to 5 and $L_{\rho, \rho_0}$ be as defined above. We have:

1. $\Gamma(X//_\rho G, L_{\rho, \rho_0}^\oplus) \cong \mathbb{C}[X]^{G, \rho^n}$.

2. If $\rho_+ \text{ and } \rho_-$ are in the same chamber, then $\mathbb{C}[X^{ss, \rho}]^{G, \rho^n} = \mathbb{C}[X]^{G, \rho^n}$ for $n \gg 1$, $L_{\rho_+, \rho_-}$ is ample: if $\rho_0$ is a generic point on the wall of the $\rho$-chamber, then $L_{\rho, \rho_0}$ is nef and semi-ample.

3. Let $\rho_+ \text{ and } \rho_0$ be in the chamber of $\rho$ and on the wall respectively, then there is an inclusion $X^{ss, \rho_+} \subset X^{ss, \rho_0}$ inducing a canonical projective morphism $pr_+: X//_{\rho_+} G \to X//_{\rho_0} G$.

4. A curve $C$ (projective, smooth, connected) in $X//_{\rho_+} G$ is contracted by $pr_+$ if and only if it is contracted by $X//_{\rho_0} G$.

5. Let $\rho_+ \text{ and } \rho_-$ be in two chambers on different sides of the wall, let $\rho_0$ be a generic point on the wall. Assume that $X^{ss, \rho}$ are both non-empty, then the morphisms $X//_{\rho_+} G \to X//_{\rho_0} G$ are proper and birational. If they are both small, then the rational map $X//_{\rho_+} G \to X//_{\rho_0} G$ is a flip with respect to $L_{\rho_+, \rho_-}$.

**Proof.** 1. This is true for general $G$-principal bundle by flat descent theorem, see [8] Exposé I, Théorème 4.5.

2 and 3. By assumption 5, $X^{ss, \rho} \subset X^{ss, \rho_+}$. By assumption 4, the natural maps:

$\mathbb{C}[X]^{G, \rho^n} \to \mathbb{C}[X^{ss, \rho}]^{G, \rho^n} \cong \Gamma(X//_{\rho_0} G, L_{\rho_0, \rho_0}^\oplus)$ in injective for $* = 0, +$ and $n \in \mathbb{Z}_{>0}$. Hence the base locus of $L_{\rho_0, \rho_0}$ is empty. $\mathbb{C}[X]^{G, \rho^n}$ is finitely generated over $\mathbb{C}$.

The canonical morphism $X//_{\rho_0} G \to \text{Proj} \bigoplus_{n \geq 0} \mathbb{C}[X^{ss, \rho}]^{G, \rho^n}$ is birational and projective when $X^{ss, \rho}$ is non-empty. Now we have series of morphisms:

$pr_+: X//_{\rho_+} G \to \text{Proj} \bigoplus_{n \geq 0} \mathbb{C}[X^{ss, \rho}]^{G, \rho^n} \to \text{Proj} \bigoplus_{n \geq 0} \mathbb{C}[X]^{G, \rho^n} = X//_{\rho_0} G$.

The morphism $pr_+$ maps each $[\rho]_S$-equivariant class to itself set-theoretically. When $\rho_+$ is in the same chamber of $\rho$, by the assumption 2, this is an isomorphism, implying that $L_{\rho_+, \rho_0}$ must be ample and $\mathbb{C}[X^{ss, \rho}]^{G, \rho^n} = \mathbb{C}[X]^{G, \rho^n}$ for $n$ large enough. By the definition of $L_{\rho_+, \rho_0}$, it linearly extends to a map from the space of $\mathbb{R}$-characters of $G$ to $\text{NS}_G(X//_{\rho_0} G)$.

Since all elements in the $\rho$ chamber are mapped into the ample cone, $\rho_0$ must be nef.

4. ‘$\to$’ direction: by the assumption 4, $\text{Proj} \bigoplus_{n \geq 0} \mathbb{C}[X^{ss, \rho}]^{G, \rho^n} \to \text{Proj} \bigoplus_{n \geq 0} \mathbb{C}[X]^{G, \rho^n}$ is always surjective. If $C$ is contracted at $\text{Proj} \bigoplus_{n \geq 0} \mathbb{C}[X^{ss, \rho}]^{G, \rho^n}$, then it is also contracted at $\text{Proj} \bigoplus_{n \geq 0} \mathbb{C}[X]^{G, \rho^n}$.
3.3 Walls on the Second Quadrant

‘⇒’: Let \( G’ \) be the kernel of \( \rho_0 \), we show that there is a subvariety \( P \) in \( X^{\sigma^\rho} \) satisfying:

A. \( P \) is a \( G’ \)-principal bundle, and the base space is projective, connected;
B. \( F_{\rho_0}(P) = C \).

Suppose we find such \( P \), then any function \( f \) in \( \mathbb{C}[X^{\sigma^\rho}]^{G>F} \) is constant on each \( G’ \) fiber. Since the base space is projective and connected, it must be constant on the whole space \( P \). Since \( F_{\rho_0}(P) = C \), the value of \( f \) on \( F_{\rho_0}^{-1}(C) \) is determined by this constant. Hence the canonical morphism contracts \( C \) to a point.

To get \( P \), we may assume \( G’ \neq G \), choose \( N \) large enough and finitely many \( f_i’ \)s in \( \mathbb{C}[X^{\sigma^\rho}] \) such that \( \cap_i(V(f_i)) \cap F_{\rho_0}^{-1}(pr_+(C)) \) is empty. Since all points in \( F_{\rho_0}^{-1}(pr_+(C)) \) are \( S \)-equivariant in \( X^{\sigma^\rho} \), each \( Gx \) contains all minimum orbits \( Gy \) in \( F_{\rho_0}^{-1}(pr_+(C)) \).

Choose \( y \) such that \( Gy \) is closed in \( X^{\sigma^\rho} \), let \( P_y \) be

\[
\bigcap_i \{ x \in F_{\rho_0}^{-1}(C) | f_i(x) = f_i(y) \}.
\]

For any \( p \in C \), since the \( G \)-orbit \( F_{\rho_0}^{-1}(p) \) contains \( y \) and \( G \) is reductive, there is a subgroup \( \beta: \mathbb{C}^* \to G \) and \( x_p \in F_{\rho_0}^{-1}(p) \) satisfying that \( y \in \beta(\mathbb{C}) \times x_p \). Since \( y \) belongs to \( X^{\sigma^\rho} \), there is a \( \rho_0 \)-semi-invariant \( f \) such that \( f(y) = 0 \). Therefore \( \rho_0 \circ \beta = 0 \). This implies that for any \( \rho_0 \)-semi-invariant function \( f \) \( f(x_p) = f(y) \). Condition b is checked. Let \( G” \) be the kernel of \( \rho^N \). By the choices of \( f_i’ \)s, another point \( x_q \) on \( Gx_q \) is in \( P_y \) if and only if they are on the same \( G’’ \)-orbit. Since \( G \) acts freely on all stable points, \( P_y \) becomes a \( G” \) principal bundle over base \( C \). As \( [G” \times G’] \) is finite, we may choose a connected component of \( P_y \) and as a \( G’ \)-principal bundle, the induced morphism from base space to \( C \) is finite. Condition A is checked.

5. This is due to Theorem 3.3 in [18].

Remark 3.8. When the difference between \( X^{\sigma^\rho^+} \) and \( X^{\sigma^\rho^-} \) is of codimension two in \( X^{\sigma^\rho^+} \cup X^{\sigma^\rho^-} \), since \( X^{\sigma^\rho^+} \cup X^{\sigma^\rho^-} \) is smooth, irreducible and quasi-affine by the second assumption, we have:

\[
\mathbb{C}[X^{\sigma^\rho^+}]^{G>F} = \mathbb{C}[X^{\sigma^\rho^+} \cup X^{\sigma^\rho^-}]^{G+F} = \mathbb{C}[X^{\sigma^\rho^-}]^{G+F} = \mathbb{C}[X^{\rho^F}] \text{ for } n \gg 0.
\]

In this case, the rational morphism between \( X^{\sigma^\rho^+} \) and \( X^{\sigma^\rho^-} \) identifies \( \text{NS}_{\mathbb{R}}(X/\rho_0 G) \) and \( \text{NS}_{\mathbb{R}}(X/\rho_0 G) \). It maps \( [L_{\rho^+, \rho^0}] \) to \( [L_{\rho^-, \rho^0}] \) for all \( \rho^0 \) in either \( \rho_+ \) chamber.

3.3 Walls on the Second Quadrant

Now the correspondence picture of the stable base locus decomposition of the effective cone and the actual destabilizing walls in the second quadrant is clear:

Theorem 3.9. In the second quadrant of the \((s, t)\)-plane of Bridgeland stability conditions, the semicircular actual walls in \( W_{\text{actual}}(1,0,1-\rho_0) \) is one to one corresponding to stable base locus decomposition walls on one side of the divisor cone of \( \text{Hilb}^\rho S \).

Proof. Each point in \([t,s]) \mid 0 < t < \frac{1}{\sqrt{2}}, -\sqrt{2n} < s < 0 \] falls into some quiver region \( \mathcal{A}(k) \). As explained before Proposition 2.5 the moduli space of \( Z_{\text{steady}} \)-semistable objects with invariants \( (r,c_1,\chi) = (1,0,1-n) \) is parameterized by the quotient space \( X_{1/\rho_{\text{stable}}}^n \). By Proposition 2.5 there are finitely many actual destabilizing walls, and in each chamber the moduli space remains the same. By the formula [4] the character \( \rho_{\text{shift}} = (\rho_{-1},\rho_0,\rho_1) \) always satisfies \( \rho_{-1} > 0 > \rho_1 \).
3.4 The Vertical Wall and the First Quadrant

We first check that the $G_t$-variety $X_k$ satisfies the assumptions of Proposition 3.7 for all $\mathfrak{p}_{t,1,k}$. The assumption 1 ‘finiteness of walls’ is due to the second property in Proposition 2.5. The smoothness and irreducible property is checked in Corollary 3.6. Since $\mathbb{C}[X_k]G_0 = \mathbb{C}$, since Hilb$^aS$ is projective. Since $\mathbb{C}[X_k]G_0 = \mathbb{C}$ if and only if for some $\mathfrak{p}_{t,1,k}, X_k//\mathfrak{p}_{t,1,k}G_k$ is projective or empty, this is checked by induction on $k$. The last assumption 5 holds by King’s criterion \cite{13} for (semi)stable quiver representation.

Now we may assign a divisor $[L_{\mathfrak{p}_{t,1,k}}^-]_k$ to $X_k//\mathfrak{p}_{t,1,k}G_k$, where $\mathfrak{p}_{t,1}$ is the character in the chamber. Starting from a sufficient small $t > 0$ and $-1 < s < 0$, at where $X_0//\mathfrak{p}_{t,1}G_0$ is Hilb$^aS$, let $t$ fix and $s$ decrease. At an actual destabilizing wall, let $pr$ be the morphism from $X_k//\mathfrak{p}_{t,1,k}G_k$ to $X_k//\mathfrak{p}_{t,1,k}G_k$ as that in Proposition 3.7. One of three different cases may happen:
1. $pr$, is a small contraction;
2. $pr$, is birational and has an exceptional divisor;
3. all points in Hilb$^aS$ are destabilized.

Now by Proposition \ref{Semi} in Case 1, we get small morphism on both sides. In addition, $X_k//\mathfrak{p}_{t,1,k}G_k$ has codimension not less than 2, else $X_k//\mathfrak{p}_{t,1,k}G_k$ cannot be projective. By property 5 in Proposition \ref{Semi} this is the flip with respect to the divisor $[L_{\mathfrak{p}_{t,1,k}}^-]_k$. As the different part of $X_k^t\mathfrak{p}_{t,1}$ and $X_k^t\mathfrak{p}_{t,1}$ is of codimension 2, their divisor cones are identified as explained in Remark \ref{Refle}. While $s$ decreases $\rho_1/\rho_{-1}$ is decreasing, so the divisor always jumps to the next chamber and does not go back.

In Case 2, $X_k//\mathfrak{p}_{t,1,k}G_k \rightarrow X_k//\mathfrak{p}_{t,1,k}G_k$ does not have any exceptional divisor by Proposition 3.5 hence the Picard number of $X_k//\mathfrak{p}_{t,1,k}G_k$ is 1. By property 4 in Proposition 3.7, Case 2 only happens when the canonical model associate to $L_{\mathfrak{p}_{t,1,k}}$ contracts a divisor, i.e. the identified divisor of $L_{\mathfrak{p}_{t,1,k}}$ on Hilb$^aS$ is on the boundary of the Movable cone. The next destabilizing wall on the left corresponds to the zero divisor, it must be Case 3. In general, if the boundary of the Movable cone is not the same as that of the Nef cone, then Case 2 happens. Otherwise, case 2 does not happen and the procedure ends up with a Mori fibration of Case 3.

Besides all previous ingredients, we only need to check that Case 3 happens before $s = -\sqrt{2n}$ when $t = 0+$.

**Lemma 3.10.** There is a semicircular wall with radius greater than 1 such that inside the wall, there is no semistable object with invariant $(1, 0, 1 - n)$.

**Proof.** When $(k + 2)(k + 1) > 2n$, $O(-k)[1]$ always has non-zero map to any object $\mathcal{A}(-k)$ with invariant $(n_t, n_0, n_1) = (n - \frac{t(t + 1)}{2}, 2n - k^2 + 1, n - \frac{t(t + 1)}{2})$, since $2n - k^2 + 1 > 3(n - \frac{t(t + 1)}{2})$. $O(-k)[1]$ corresponds to the potential wall across $(-k, 0)$, hence there is no stable object with invariant $(1, 0, 1 - n)$ inside this semicircle. □

By the lemma, Case 3 must happen on this wall or a larger actual wall. □

3.4 The Vertical Wall and the First Quadrant

**Proposition 3.11.** Suppose $\sigma$ of the Sklyanin algebra $\text{Skl}(E, L, \sigma)$ is of infinite order, then no curve is contracted on the vertical wall $s = 0$, i.e., the vertical wall is a faked wall.

20
Lemma 3.13. Let $X$ be the total space of complex $O(-1) \otimes \mathbb{C}^n \to O \otimes \mathbb{C}^{2n+1} \to O(1) \otimes \mathbb{C}^n$, $G_0$ be group $GL_n \times GL_{2n+1} \times GL_m$, $\rho_+$ be the character $(1,0,-1) + \epsilon(n, -2n-1, 0)$ for $\epsilon$ small enough. Then $X_{\rho_+} \cap G_0$ is smooth.
3.4 The Vertical Wall and the First Quadrant

Proof. For a stable complex $K$ with respect of $\rho_+$, we may restricted it to the elliptic curve $E$, since $\text{Hom}(K_0, K_E)$ is $C$, the hypercohomology of $H^2(\text{Hom}^*(K_I, K_{IE}))$ is the same as $\text{Ext}^2(K, K)$. Since $K_{IE}$ is exact at the first term and the homological sheaf at the middle is a line bundle with non-positive degree, it is quasi-isomorphic to $Q \to L^\lambda_{\{0\}}$, where $Q$ is locally free and $\mu_+(Q) \leq 3 = \mu(L)$. Hence $H^2(\text{Hom}^*(K_{IE}, K_{IE})) = 0$. By a similar argument as that in Corollary 3.6, $X_{1/\rho, G}$ is smooth.

By Proposition 3.7 property 5, since no curve is contracted, we have a birational map $T_u : X_0/\rho, G_0 \to X_0/\rho, G_0$, where $X_{1/\rho, G}$ is Hilb$^g S$. As both varieties are smooth and $T_u$ doesn’t have exceptional locus, this is an isomorphism. Under this isomorphism, the line bundle complex remains the same (since they are stable on both sides). Moreover, due to the uniqueness of the $S$-equivariant class, the $T_u$ image of an ideal complex $I_Z$ with $Z$ to be $n$ general distinct points $p_1, \ldots, p_n$ (by the term ‘general’, we mean $\sigma^3(p_i) \neq p_j, p_i \neq p_j$ for any $1 \leq i, j \leq n$) is shown below.

\[
\begin{array}{c}
\xymatrix{ L^* \ar[r]^-{\sigma} & O^2 \ar[r]^-{\rho_+} & \mathcal{T} \\
O \ar[r]^-{\sigma} & O^2 \ar[r]^-{\rho_+} & \mathcal{T} \\
L^* \ar[r]^-{\rho_+} & O^2 \ar[r]^-{\rho_+} & \mathcal{T} \\
0 \ar[r]^-{\rho_+} & O^2 \ar[r]^-{\rho_+} & \mathcal{T} \\
0 \ar[r]^-{\rho_+} & O^2 \ar[r]^-{\rho_+} & \mathcal{T} \\
\end{array}
\]

By writing a complex $K$ in $X_{k^{p_+}}$ as $O(-1) \otimes H_{-1} \xrightarrow{I} O \otimes H_0 \xrightarrow{J} O(1) \otimes H_1$ with $I = xI_1 + yI_2 + zI_3, J = xJ_1 + yJ_2 + zJ_3$. Another morphism $\tilde{T}_I$ from $X_0^{k^{p_+}}$ to $X_0^{k^{p_+}}$ is defined as:

\[(I, J) = (xI_1 + yI_2 + zI_3, xJ_1 + yJ_2 + zJ_3) \mapsto (xI_1^2 + yI_1^2 + zI_1^2, xJ_1^2 + yJ_1^2 + zJ_1^2).
\]

Lemma 3.14. $\tilde{T}_I$ is well-defined and compatible with the $G_0$-action. In addition, it extends to other quiver regions as $\tilde{T}_{I, k} : X_{k^{p_+}}^{\mathbb{P}} \to X_{k^{p_+}}^{\mathbb{P}}$.

Proof. 1. Since $x, y, z$ satisfies the relations (5), the image is really a complex.

2. The stability property is due to the duality. $\tilde{T}_I(K)$ is a complex $O(-1) \otimes H_{-1} \xrightarrow{I} O \otimes H_0 \xrightarrow{J} O(1) \otimes H_1$. A subcomplex in $\tilde{T}_I(K)$ is determined by subspaces $(H_{-1}^{l'}, H_0^{r'}, H_1^{s'})$ in $(H_{-1}^l, H_0^r, H_1^s)$ those are compatible with $\tilde{T}_I(I, J)$. Then $(H_{-1}^{l'}, H_0^{r'}, H_1^{s'})$ in $(H_{-1}, H_0, H_1)$ are compatible with $I$ and $J$, hence they determine a subcomplex of $K$. Since $\rho_+, (h_{-1}^{l'}, h_0^{r'}, h_1^{s'}) > 0$ if and only if $\rho_+ \cdot (n_{-1} - h_{-1}^{l'}, n_0 - h_0^{r'}, n_1 - h_1^{s'}) > 0, \tilde{T}_I(K)$ is $\rho_+$ stable.

As $\tilde{T}_I$ maps a $G_0$-orbit to a $G_0$-orbit, it induces a map from $X_0/\rho, G_0$ to $X_0/\rho, G_0$. We denote this isomorphism between $X_{k^{p_+}}$ to $X_{k^{p_+}}$ by $T_I$. This sets up the symmetry wall crossing picture between the first and second quadrant.

Denote $T := T_I \circ T_u$ by the automorphism of $X_0/\rho, G_0 \simeq \text{Hilb}^n S$. By the definition of $T_u$, we have $T \circ T = \text{Id}$. The following statement shows that when $n \geq 3$, the induced $T$-action on $\text{NS}_g(\text{Hilb}^n S)$ is non-trivial, i.e. the destabilizing wall on the first quadrant destabilizes different points as those on the second quadrant.

Remark 3.15. This involution $T$ is related to the Galois representation of the symplectic resolution.
Proposition 3.16. When \( n \geq 3 \), the automorphism \( T \) induces a non-trivial action on \( H^2(\text{Hilb}^n S, \mathbb{Z}) \).

Proof. When \( n = 3 \), since the \( O(-1) \)-wall (respectively, \( O(1)[1] \)-wall) is the first wall on the left (right) of \( t \)-axis, it is enough to show that these two walls destabilize different points on \( X_0/\{ p \}, G_0 \). We study when an ideal sheave \( I_Z \) can be written as the kernel of \( O \to \oplus O_p \) for 3 general distinct points \( p_1, p_2, p_3 \) on \( E \) is destabilized on the \( O(-1) \)-wall. Let the complex of \( I_Z[1] \) be \( (L^*)^{\oplus 3} \to O_E^{\oplus 3} \to \mathcal{T}^{\oplus 3} \) as the cartoon on the left. Write \( E \) for the kernel of \( O_E^{\oplus 3} \to \mathcal{T}^{\oplus 3} \). As in the cartoon, \( O_E^{\oplus 3} \to \mathcal{T}^{\oplus 3} \) has four parts: \( O \) and three pieces of \( O_E^{\oplus 2} \) to \( \mathcal{T} \). Each \( O_E^{\oplus 2} \to \mathcal{T} \) has kernel \( \mathcal{T}^{-1}(\sigma^3(p_i)) \) and cokernel \( \sigma^{-1}(p_i) \). The map from \( T \) to the direct sum of the three pieces \( O_E^{\oplus 2} \to \mathcal{T} \), has kernel \( O(-\sigma^3(p_1) - \sigma^3(p_2) - \sigma^3(p_3)) \).

Since \( \text{Hom}(O(-1), O(i) \otimes \mathbb{C}^n) \)'s have dimensions 3,21,18, for \( i = -1, 0, 1 \) respectively,

\[
\text{Hom}(O(-1), I_Z) \neq 0
\]
\[
\Rightarrow \text{the map from } \text{Hom}(O(-1), O \otimes \mathbb{C}^3) \text{ to } \text{Hom}(O(-1), O(1) \otimes \mathbb{C}^3) \text{ is not surjective}
\]
\[
\Rightarrow \text{the map from } \text{Hom}(L^*, O_E \otimes \mathbb{C}^3) \text{ to } \text{Hom}(L^*, \mathcal{T} \otimes \mathbb{C}^3) \text{ is not surjective}
\]
\[
\Rightarrow \text{Ext}^1(L^*, E) \neq 0
\]
\[
\Rightarrow \text{Hom}(O(-\sigma^3(p_1) - \sigma^3(p_2) - \sigma^3(p_3)) = L^*.
\]

The last \( \Rightarrow \) is due to the short exact sequence \( 0 \to O(-\sigma^3(p_1) - \sigma^3(p_2) - \sigma^3(p_3)) \to \mathcal{T} \to \oplus \mathcal{T}^{-1}(\sigma^3(p_i)) \to 0 \). A similar argument shows that \( T \) isomorphic to \( O(1)[1] \) if and only if \( O(p_1 + p_2 + p_3) = \mathcal{T} \). Hence \( O(-1) \) has non-zero morphism to \( T(I_Z) \) if and only if \( O(p_1 + p_2 + p_3) = \mathcal{T} \). Since \( \mathcal{T}(p_i) = L^*(\sigma^3(p_i)) \) and \( \sigma \) has infinite order, the locus that is contracted by the \( O(-1) \)-wall and that is contracted by the \( O(1)[1] \)-wall are different.

When \( n \geq 4 \), we do the induction on \( n \). Assume the \( n - 1 \) case is done, then a line bundle \( I \) with \( (r, c, \chi) = (1, 0, 1 - (n - 1)) \) is destabilized by \( O(-1) \), and \( T(I) \) is not destabilized by \( O(-1) \). Consider the morphism \( O(-1) \to I \) restricted on \( E \), the cokernel is a torsion sheaf of length 3, let \( O_p \) be a quotient of the torsion sheaf. Then \( O(-1) \) has a non-zero map to the kernel \( I' \) of \( I \to O_p \). Yet \( T(I') \) is the kernel of \( T(I) \to O_q \) for some \( q \in E \), \( \text{Hom}(O(-1), T(I')) = 0 \).

Since for any destabilize sequence \( O(-1) \to I' \to I'' \). The extension sheaf by \( O(-1) \) and \( I'' \) is a vector bundle if and only if for any non-zero numbers \( l_1, l_2, l_3 \in \mathbb{C}^3 \) on \( E, aI_1 + bI_2 + cI_3 \), is injective i.e \( l_1 I_x^T + l_2 I_y^T + l_3 I_z^T \) is injective. For generic choice of \( \text{Hom}(O(-2) \otimes \mathbb{C}^n, O(-1)) \) and \( I'' \), \( l_1 I_x^T + l_2 I_y^T + l_3 I_z^T \) is injective since for generic \( I'' \) the cokernel of \( xI^{\alpha_1 \beta_1 \gamma_1} + yI^{\alpha_2 \beta_2 \gamma_2} + zI^{\alpha_3 \beta_3 \gamma_3} \) restricts on \( E \) is the direct sum of some skyscraper sheaves of distinct points. Hence on the locus that are destabilized by \( O(-1) \), the set of vector bundles is dense. Therefore there exists a vector bundle that is destabilized by \( O(-1) \) while \( T(-) \) of it is not destabilized by \( O(-1) \). The induction accomplishes. \( \square \)

Combining Theorem 3.9 and Proposition 3.16 we get our main result.

Theorem 3.17. When \( n \geq 3 \), the positivity cone of \( \text{Hilb}^n S \) is symmetric. Each side stable base locus decomposition walls are one to one corresponding to the semicircular actual walls on the first and second quadrant of Bridgeland stability conditions. \( \square \)
4 Examples

Example 4.1. Given n, when $\sqrt{2n} > k \geq 0$, in the quiver region of $\mathcal{A}(-k)$, the character $\overline{\rho}$ is given by:

$$c_k = \frac{s}{\sqrt{2n-s^2}}((s+k+1)^2 - \frac{s}{t} - (s+k)^2 + \frac{s}{t}((s-k+1)^2 - \frac{s}{t}) + t(s+k+1, -s-k, s+k-1).$$

When $t$ tends to 0, the character $\overline{\rho}_{s,k}$ of $G/\mathbb{C}^*$ is up to a scalar given by:

$$(s^2(k+1)+s(2n+(k+1)^2)+2n(k+1), -s^2k-s(2n+k^2)-2nk, (s^2(k-1)+s(2n+(k-1)^2)+2n(k-1)).$$

When $s$ decreases from $-k+1$ to $-k$, the character decreases from $(1, -\frac{m}{n}, 0)$ to $(0, \frac{m}{n}, -1)$. In particular, when $s = -k$ and $t$ tends to 0, up to a scalar $\overline{\rho}_{-k,0+k}$ is $(n_1, 0, -n_{-1})$, it corresponds to the destabilizing walls with type $(0, 1, 0)$, as a sheaf it is just $O(-k)[1]$.

Given an integer $-\sqrt{2n} < k \leq 0$, for $-k+1 < s < -k$, let $A_k$ and $B_k$ be the line bundles (divisors) on $\mathbb{P}_{\mathcal{M}_{\mathbb{C}^*}} G$ that compose with the $G$-principal bundle with characters $(1, *, 0)$ and $(0, *, -1)$ respectively. Then when $s$ is between two integers $-k-1$ and $k$, there are four divisors $A_k, B_k, A_{k+1}$ and $B_{k+1}$. When quiver region only contains flip-type bi-rational morphism, by the Remark \[3\] these divisors satisfy the relation:

$$c_k \begin{bmatrix} A_{k+1} \\ B_{k+1} \end{bmatrix} = \begin{bmatrix} 2n - k(k+1) \\ -2n + (k+1)(k+2) \end{bmatrix} \begin{bmatrix} A_k \\ B_k \end{bmatrix}, \quad \text{(\textcircled{\Delta})}$$

where $c_k$ is a constant only depend on $k$. Furthermore $A_k - B_{k-2}$, where $\sim$ means equal up to a scalar.

Proposition 4.2. Let the notations $A_k, B_k$ be as above. Assume $A_1 \sim H, B_0 \sim A_2 \sim (n-1)H - \frac{1}{n}$, then the divisor at $(s, 0)$ is $(-\frac{2n-s}{n} - k)H - \frac{1}{n}$ up to scalar. In an other word, the destabilizing seminfinite wall on the Bridgeland stability condition space with center $-m - \frac{1}{n}$ corresponds to the divisor $mH - \frac{1}{n}$.

Proof. First of all, we show that $A_k$ and $B_k$ are $(2n + (k+1)(k-4)H - (k-1)\Delta \text{ and } (2n + (k-2)(k+1)H - (k+1)\Delta$ respectively up to a same scalar.

When $k = 1$, we may assume that $A_1 = 2nH, B_1 = b_1((n-1)H - \Delta), A_2 = a_2((n-1)H - \frac{1}{2}H)$. By the equation (\textcircled{\Delta}), we have

$$A_1(2n - 2) + 2nb_1(n - \frac{1}{2}H - \Delta) - (n-1)H - \Delta \frac{1}{2}.$$ 

This implies $b_1 = 2$. By the equation (\textcircled{\Delta}) and induction on $k$, we get $A_k$ and $B_k$.

At a point $(s, 0+)$, the character $\rho_{s,0+k}$ is given in Example 4.1. As

$$\rho_{s,0+k} = -f(n, s, k-1)(0, \frac{n-1}{n_0}, -1) + f(n, s, k+1)(1, -\frac{n-1}{n_0}, 0),$$

where $f(n, s, k) = k(2n + s^2) + s(2n + k^2)$. The divisor at $(s, 0+)$ is up to a scalar given by:

$$-f(n, s, k-1)B_k + f(n, s, k+1)A_k$$

$$\sim f(n, s, k-1)((2n + (k-2)(k+1)H - (k-1)\Delta) + f(n, s, k+1)((2n + (k-4)(k-1)H - (k-1)\Delta)$$

$$= 2(2n - (k-1)(k+1))(2n + s^2 + 3s)H + 2s(2n - (k-1)(k+1)H$$

$$= -2s(2n - (k-1)(k+1))(-\frac{2n + s^2}{2s} - \frac{3}{2}H - \frac{1}{2}\Delta).$$

\[\square\]
4.1 Destabilizing Walls

To compute the ratio of each stable decomposition wall on the Neron-Severi space, we only need compute all the ratio \( \rho_{s,0+r,k} \)'s on the destabilizing chamber walls. We may look at each \( \mathcal{A}(k) \) quiver region to search candidates type of subcomplex that may destabilize a stable complex \( \mathbf{K} \) with type \( (n - \frac{ik+1k}{2}, 2n - k^2 + 1, n - \frac{k(k+1)}{2}) \).

For each quiver region, we only need consider the wall whose right bound is in \((-k - 1, -k)\). Suppose the character \( \rho \) gives an actual wall, then there is a destabilizing sequence: \( \mathbf{K}'' \to \mathbf{K} \to \mathbf{K}' \) with \( \mathbf{K}' \) stable. Let the type of \( \mathbf{K}'' \) be \((a + l, 2a + r + l, a)\), then \( \mathbf{K}' \) has type \((A, A + C - s, C) = (n - \frac{ik+1k}{2}, 2n - k^2 + 1, n - \frac{k(k+1)}{2}) - (a + l, 2a + r + l, a)\).

To achieve an efficient logarithm, we need some restrictions on the candidate type \((a + l, 2a + r + l, a)\).

**Lemma 4.3.** Let \( a, r, l \) be as discussed before, then they satisfy the following inequalities:

\[
\begin{align*}
  a + l & \geq 0; \\
  (A - C)^2 - s(A + C - s) - 2s^2 + 1 & \geq 0; \\
  \frac{k + 1}{n - \frac{nk+1k}{2}}a - r & < l < \frac{k}{n - \frac{nk+1k}{2}}a; \\
  l + r & \leq a; & \text{if } r \geq 2, & \text{then } 2a \geq 3(r + l) \text{ or the type is (0, 3, 1)} & \ldots \\
  k & > \sqrt{(r - 1)(2n + r - 1)r - 1}.\
\end{align*}
\]

**Proof.** Inequality (2) is a consequence of Lemma 3.2. Since \( \text{Ext}^2(\mathbf{K}', \mathbf{K}') = 0 \) and \( \mathbf{K}' \) is stable, we have

\[
\chi(\mathbf{K}', \mathbf{K}') \leq \dim\text{Hom}(\mathbf{K}', \mathbf{K}') = 1.
\]

On the other hand,

\[
\chi(\mathbf{K}', \mathbf{K}') = \dim\text{Hom}^0(\mathbf{K}', \mathbf{K}') - \dim\text{Hom}^1(\mathbf{K}', \mathbf{K}') + \dim\text{Hom}^2(\mathbf{K}', \mathbf{K}')
\]

\[
= (A^2 + (A + C - s)^2 + C^2) - (3A(A + C - s) + 3C(A + C - s)) + 6AC
\]

\[
= -(A - C)^2 + s(A + C - s) + 2s^2.
\]

Inequality (3): by formula (4, 1) the boundary \(-k - 1\) and \(-k\) corresponds to characters \( \rho_{-k-1,0+r,k} \sim (0, n - \frac{nk+1k}{2}, -(2n - k^2 + 1)) \) and \( \rho_{-k,0+r,k} \sim (n - \frac{nk+1k}{2}, 0, -(n - \frac{nk+1k}{2})) \). We have

\[
(a + l, 2a + r + l, a) \cdot \rho_{-k-1,0+r,k} > 0; \\
(a + l, 2a + r + l, a) \cdot \rho_{-k,0+r,k} < 0.
\]

Plug in the values, we get the two boundaries for \( l \).

Inequality (4): if \( 2a + r + l > 3a \), we may consider the intersection of \( \text{ker} J''_y, \text{ker} J''_y \) and \( \text{ker} J''_y \), then \( \mathbf{K}'' \) contains \((0, 1, 0)\) type sub complex, \( \mathbf{K} \) is already destabilized at a previous wall.

The formula is implied by 2 and 3. Write the inequality in terms of \( a, l \) and \( r \): one has

\[
L := (k + l)^2 - (r - 1)(2n - k^2 + 1 - 2a - l - r) - 2(r - 1)^2 + 1 \geq 0.
\]

When \( r \leq 1 \), the inequality holds obviously. We may assume \( r \geq 2 \). When \( l \geq k + 2 - r \), we have \( L \leq (r - 2)^2 - 2(r - 1)^2 + 1 < 0 \), hence \( l \leq k + 1 - r \).
REFERENCES

By the first part of [3] we have:

$$(k - l)^2 - (r - 1)(2n - k^2 + 1 - 2n - \frac{k(k+1)}{k+1}(l + r) - l - r) - 2(2n^2) + 1 > 0.$$ 

Now the left side is a binomial of $l$ with leading coefficient 1. If an $l \in (1 - r, k + 1 - r)$ satisfies the inequality, then either $1 - r$ or $k + 1 - r$ satisfies it. Plug in $l = k + 1 - r$, the inequality always fails. Hence it holds for $l = 1 - r$.

$$(k + r - 1)^2 - (r - 1)(2n - k^2 - 2n - \frac{k(k+1)}{k+1})(l + r) - l - r - 2(r - 1)^2 + 1 \geq 0$$

$\iff rk^2 + (r - 1)k + 1 \geq (r - 1)(r - 1 + \frac{k}{k+1}2n)$$
$$\implies \frac{k+1}{k}rk(k + 1) \geq (r - 1)(r - 1 + 2n)$$
$$\implies k > \sqrt{(r - 1)(2n + r)} / r - 1.$$
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