W-Strings 93†

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ABSTRACT

We present a review of the status of W string theories, their physical spectra, and their interactions.

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1. Introduction

$W$ algebras have received a considerable amount of attention since their discovery [1] in the middle of the 1980’s. These efforts have broadly concentrated in two directions, namely the construction and classification of $W$ algebras on the one hand, and their application to $W$ gravity and $W$ strings on the other. Since $W$ algebras are higher-spin extensions of the Virasoro algebra, it is natural to expect that they could be associated with some kind of extensions of the usual bosonic string. The idea of building $W$-string theories was first developed in [2]. One of the hopes was that they might have massless physical states with spins greater than 2 [2], but so far all the explicit examples that have been constructed do not give rise to such higher-spin massless states. Nonetheless, the structures that emerge in $W$-string theories are not without interest, revealing intriguing connections with Virasoro and $W$ minimal models.

Most of the efforts so far in constructing $W$-string theories have been concentrated on the case of the $W_3$ string, since $W_3$ is the simplest non-trivial $W$ algebra. It has a primary current of spin 3 in addition to the energy-momentum tensor. Like all $W$ algebras, it is non-linear. The starting point for building a $W$ string is to construct an anomaly-free quantum theory of the associated $W$ gravity. By far the simplest way to do this is by BRST methods [3]. The essential requirement, therefore, is to find the BRST operator $Q_B$ for the $W$ algebra in question. From this, one can write down the full Lagrangian, with ghosts and gauge-fixing terms. The requirement of anomaly-freedom translates into the requirement that the BRST operator $Q_B$ be nilpotent. This is achieved provided that the matter currents satisfy a closed (albeit non-linear) algebra at the full quantum level, with the correct value of central charge to cancel the anomaly from the ghosts for the gauge fields.

Having arrived at an anomaly-free quantum $W$ gravity, the next step in building a $W$ string is to find an appropriate $W$ matter system, with the critical value of the central charge, that admits some sort of a spacetime interpretation. Thus some of the matter fields, which we may suggestively call $X^\mu$, should be worldsheet scalars that enter the matter currents in a Lorentz-invariant way. It is non-trivial that this can be done, since the $W$ algebra is non-linear and one cannot, by contrast with ordinary string theory, simply take tensor products of elementary realisations to obtain new ones with larger central charge.

The basic elementary realisations of $W$ algebras are those that come from Hamiltonian reduction and the Miura transformation. In the case of $W_3$, for example, this is a two-scalar realisation [1]. The central charge can be adjusted to its critical value ($c = 100$ for $W_3$) by an appropriate choice of a free background-charge parameter. This realisation gives rise to what one may call a theory of pure $W_3$ gravity, and can be thought of as a generalisation of a one-scalar realisation of the Virasoro algebra, or pure Liouville gravity.

In [4], the observation was made that one of the two-scalars of the Miura realisation of $W_3$ appears in the spin-2 and spin-3 currents only through its energy-momentum tensor. Thus one can obtain more general realisations of $W_3$ by replacing this energy-momentum
tensor by that for an arbitrary matter system $T^{\text{eff}}$ with the same central charge $c^{\text{eff}}$, the only other requirement being that it should commute with the other original scalar field, which we shall call $\phi$. The value of $c^{\text{eff}}$, which is dictated by the details of the Miura realisation and the required critical value ($c = 100$) for the total matter central charge, turns out to be $c^{\text{eff}} = \frac{51}{2}$. It is no coincidence, as we shall see later, that this is 26 minus the central charge of the Ising model. By realising $T^{\text{eff}}$ in terms of worldsheet scalars $X^\mu$, the desired goal of obtaining a $W_3$ realisation with a string-like spacetime interpretation is achieved. The first investigations of the $W_3$ string based on this realisation were carried out in [5].

Although these multi-scalar realisations enable one to construct $W_3$ strings that admit a sensible-looking multi-dimensional spacetime interpretation, in some sense the way in which the $W_3$ symmetry is realised on the fields $X^\mu$ is a little trivial, and this ultimately shows up when one calculates the spectrum and scattering of physical states in the theory. As we shall see later, the multi-scalar $W_3$ string behaves, at least at tree level, precisely like a bosonic string in which criticality is achieved by tensoring a $c^{\text{eff}} = \frac{51}{2}$ energy-momentum tensor for $X^\mu$ (with a background charge) with the $c = \frac{1}{2}$ Ising model. While it is not without interest that such a system displays a “hidden” $W_3$ symmetry, the original hopes that $W$ strings might be new kinds of string theories, maybe even with higher-spin massless states, seem to have evaporated.

In fact, although it is possibly physically less realistic, the two-dimensional $W_3$ string based on the original two-scalar Miura realisation is mathematically-speaking a much more interesting object. It turns out that the physical states of the two-scalar $W_3$ string divide into two categories; there are some which generalise to the multi-scalar case if one makes the replacement described above, and then there are other physical states that have no analogues in the multi-scalar case. The first category is in some sense “uninteresting,” in that again the physical states are tensor products of $c^{\text{eff}} = \frac{51}{2}$ Virasoro states with Ising-model primary fields. The second category comprises physical states that are not simply tensor products of Virasoro-string states with Ising model states, and as such they could be regarded as states on which the $W_3$ symmetry is realised in a more non-trivial way.

One could, of course, build a $W$ string theory based on more or less any $W$ algebra, such as $W_N$. However, since each higher-spin current in a $W$ algebra represents a constraint on the physical states of the associated $W$ string, there is a sense in which one obtains richer $W$-string theories by considering algebras with fewer higher-spin currents. To this end, recently some examples of other kinds of $W$ strings have been constructed, in which there is just a single current of spin $s > 2$ in addition to the energy-momentum tensor [6]. Curiously, it has been found that one does not necessarily even need an underlying $W$ algebra in order to construct a nilpotent BRST operator. We shall describe some recent work on these spin-2 plus spin-$s$ strings later.
2. $W_3$ strings

2.1 The physical spectrum

We shall begin by describing the multi-scalar $W_3$ string. As was discussed in the introduction, the key ingredient for the construction of the $W_3$ string is its BRST operator. This was found in [7] (see also [8]). It takes the form

$$Q_B = \oint dz \left[ c(T + \frac{1}{2}T_{gh}) + \gamma (W + \frac{1}{2}W_{gh}) \right],$$

(2.1)

and is nilpotent provided that the matter currents $T$ and $W$ generate the $W_3$ algebra with central charge $c = 100$, and that the ghost currents are chosen to be

$$T_{gh} = -2b \partial c - \partial b e^{-3\beta} \partial \gamma - 2\partial \beta \gamma,$$

(2.2)

$$W_{gh} = -\partial b e^{-3\beta} \partial c - \frac{8}{261} \left[ \partial (b \gamma T) + b \partial \gamma T \right] - \frac{25}{1155} \left( 2\gamma \partial^3 b + 9\partial \gamma \partial^2 b + 15\partial^2 \gamma \partial b + 10\partial^3 \gamma b \right),$$

(2.3)

where the ghost-antighost pairs $(c, b)$ and $(\gamma, \beta)$ correspond respectively to the $T$ and $W$ generators. A matter realisation of $W_3$ with central charge 100 can be given in terms of $n \geq 2$ scalar fields, as follows [4]:

$$T = -\frac{1}{2} (\partial \varphi)^2 - \alpha \partial^2 \varphi + T^{\text{eff}},$$

$$W = -\frac{2i}{\sqrt{261}} \left[ \frac{1}{3} (\partial \varphi)^3 + \alpha \partial \varphi \partial^2 \varphi + \frac{1}{3} \alpha^2 \partial^3 \varphi + 2 \partial \varphi T^{\text{eff}} + \alpha \partial T^{\text{eff}} \right],$$

(2.4)

where $\alpha^2 = \frac{49}{8}$ and $T^{\text{eff}}$ is an energy-momentum tensor with central charge $\frac{51}{2}$ that commutes with $\varphi$. Since $T^{\text{eff}}$ has a fractional central charge, a background-charge vector $a_\mu$ is needed in order to realise it with $d$ scalar fields $X^\mu$:

$$T^{\text{eff}} = -\frac{1}{2} \partial X_\mu \partial X^\mu - ia_\mu \partial^2 X^\mu,$$

(2.5)

with $a_\mu$ chosen so that $\frac{51}{2} = d - 12a_\mu a^\mu$.

It is now in principle a straightforward matter to construct physical states $|\chi\rangle$, by demanding that they be annihilated by the BRST operator and that they be BRST non-trivial:

$$Q_B |\chi\rangle = 0, \quad |\chi\rangle \neq Q_B |\psi\rangle.$$  

(2.6)

Rather than follow the historical course of events here, we shall jump ahead to a discovery made in [9], which leads to a considerable simplification of the BRST operator and the form of the physical states. In [9], the the following non-linear redefinition, under which the ghosts and the $\varphi$ matter field become mixed, was introduced:

$$c \rightarrow c + \frac{21}{8\sqrt{2}} \partial \gamma + \frac{9}{8} b \partial \gamma \gamma - \frac{3}{2} \partial \varphi \gamma,$$

$$b \rightarrow b,$$

$$\gamma \rightarrow \frac{9\sqrt{29}}{8} \gamma,$$

$$\beta \rightarrow -\frac{8i}{9\sqrt{29}} \left\{ \beta + \frac{21}{8\sqrt{2}} \partial b + \frac{9}{8} \partial b \beta + \frac{3}{2} \partial \varphi b \right\},$$

$$\varphi \rightarrow \varphi - \frac{3}{2} b \gamma.$$  

(2.7)
These transformations are canonical, in the sense that the redefined fields satisfy the same set of OPEs as the original ones. We have taken the opportunity to rescale away some tiresome numerical coefficients at the same time. In terms of the new fields, which we shall use exclusively from now on, the BRST operator becomes [9]

\[ Q_B = Q_0 + Q_1, \quad (2.8) \]

where

\[ Q_0 = \oint dz \left( T^{\text{eff}} + T_\varphi + T_{\gamma,\beta} + \frac{1}{2} T_{c,b} \right), \quad (2.9) \]
\[ Q_1 = \oint dz \gamma \left( (\partial \varphi)^3 + 3\alpha \partial^2 \varphi \partial \varphi + \frac{19}{8} \partial^3 \varphi + \frac{9}{2} \partial \varphi \beta \partial \gamma + \frac{3}{2} \alpha \partial \beta \partial \gamma \right), \quad (2.10) \]

with the energy-momentum tensors given by

\[ T_\varphi \equiv -\frac{1}{2} (\partial \varphi)^2 - \alpha \partial^2 \varphi, \quad (2.11) \]
\[ T_{\gamma,\beta} \equiv -3 \beta \partial \gamma - 2 \partial \beta \gamma, \quad (2.12) \]
\[ T_{c,b} \equiv -2 b \partial c - \partial b c, \quad (2.13) \]
\[ T^{\text{eff}} \equiv -\frac{1}{2} \partial X^\mu \partial X^\nu \eta_{\mu\nu} - ia_\mu \partial^2 X^\mu. \quad (2.14) \]

The BRST operator is graded, with \( Q_0^2 = Q_1^2 = \{Q_0, Q_1\} = 0 \).

The physical states of the \( W_3 \) string have been analysed in considerable detail. The early discussions [5,3,10,11] concentrated on physical states having what one might call “standard” ghost structure. In ordinary bosonic string theory, standard ghost structure means that the physical operators \( V(z) \) that create physical states \( |\text{phys}\rangle = V(0) |0\rangle \) are of the form \( V = c Y(X) \), where \( c \) is the ghost field and \( Y(X) \) is independent of ghosts. (\( Y = e^{i p \cdot X} \) for tachyons, \( \xi \cdot \partial X^{ip \cdot X} \) at level 1, etc. Note that here, and always in this paper, \( |0\rangle \) denotes the \( SL(2,C) \)-invariant vacuum.) For the \( W_3 \) string, the analogous notion of states with “standard” ghost structure is when the physical operator \( V \) takes the form \( V = c \partial \gamma \gamma Y(\varphi, X) \). The physical-state conditions (2.6) imply that such operators are physical if

\[ V = c \partial \gamma \gamma e^{\mu \varphi} Y^{\text{eff}}_\Delta (X), \quad (2.15) \]

where \( Y^{\text{eff}}_\Delta (X) \) is a highest-weight operator under \( T^{\text{eff}} \) with weight \( \Delta \), and

\[ \mu = -\frac{8}{7} \alpha, \quad \Delta = 1; \quad \text{or} \quad \mu = -\frac{6}{7} \alpha, \quad \Delta = 1; \quad \text{or} \quad \mu = -\alpha, \quad \Delta = \frac{15}{16}. \quad (2.16) \]

Thus we see that the momentum in the \( \varphi \) direction is frozen to three discrete values. From the point of view of the effective spacetime described by the \( X^\mu \) coordinates, these physical states look like two sectors of ordinary bosonic strings, one having intercept 1, and the other having intercept \( \frac{15}{16} \) [5,10].
It turns out that this is by no means the end of the story as far as physical states are concerned. There are also physical states with “non-standard” ghost structure. The first example of such a state was found in [12], in the case of the two-scalar $W_3$ string. Subsequently, more physical states with non-standard ghost structure were found in [13]. The main focus of [13] was to examine analogues of the “ground-ring” of ghost-number zero physical operators for the two-scalar $W_3$ string. However, it was also shown that some (but not all) of the non-standard ghost structure physical states of the two-scalar $W_3$ string generalise to physical states of the multi-scalar $W_3$ string [13]. For example, there is a physical state described by the operator

$$V = c \gamma e^{\mu \varphi} Y^\text{eff}_\Delta (X)$$

(2.17)

with $\mu = -\frac{4}{7} \alpha$ and $\Delta = \frac{1}{2}$ [13,14,9]. From the effective spacetime point of view, this corresponds to a third sector of effective bosonic string states, with intercept $\frac{1}{2}$.

We shall not present a detailed discussion of all the physical states of the $W_3$ string here, but instead just give a summary of the final conclusions. Further details may be found in [14,9]. Let us first consider what have been called “prime states.” These are the physical states of lowest possible ghost number at a given momentum and level. From the prime states, one can build quartets of physical states by normal ordering with either or both of the “ghost boosters” $a_\varphi \equiv [Q_B, \varphi]$ and $a_{X\mu} \equiv [Q_B, X^\mu]$ [13,14,9]. In addition there are conjugate quartets [13]. Thus it suffices to describe the prime states in order to characterise the entire physical spectrum.

The physical prime states divide into four sectors. The first three of these sectors comprise physical states with continuous on-shell spacetime momentum, and are described by operators of the form [9]

$$V = c U(\varphi, \beta, \gamma) Y^\text{eff}_\Delta (X).$$

(2.18)

The operator $Y^\text{eff}_\Delta (X)$ is highest-weight under $T^\text{eff}$, with conformal weight $\Delta$ taking one of the three values $1, \frac{15}{16}, \frac{1}{2}$. $U(\varphi, \beta, \gamma)$ has conformal weight $h = 1 - \Delta$ under $T_\varphi + T_{\gamma, \beta}$, and in addition satisfies $\{Q_1, U\} = 0$. For simplicity, we may take the operator $Y^\text{eff}_\Delta (X)$ to be tachyonic for now, since the process of constructing excited spacetime operators is identical to that in ordinary bosonic string theory. Thus we may classify the physical states by their level number $\ell$, which represents the level of ghost and $\varphi$ excitations only. One finds that as one goes to higher and higher levels, the same set of three $\Delta$ values $1, \frac{15}{16}, \frac{1}{2}$ recur repeatedly, with the ghost numbers of the associated operators $U(\varphi, \beta, \gamma)$ becoming more and more negative, and the $\varphi$ momentum frozen to more and more positive values. The lowest-level examples for $\Delta = 1$ and $\Delta = \frac{15}{16}$ are the standard ghost structure $\ell = 0$ operators (2.16), and the lowest-level example for $\Delta = \frac{1}{2}$ is the level $\ell = 1$ operator (2.17). Many more examples may be found in [14,9].
The fourth sector of physical prime states in the multi-scalar $W_3$ string comprises states with discrete momentum (in fact $p_\mu = 0$ or $p_\mu = -2a_\mu$) in the effective spacetime described by $X^\mu$. Those with $p_\mu = 0$ take the form

$$V = c U_1(\varphi, \beta, \gamma) + U_2(\varphi, \beta, \gamma),$$

(2.19)

where the operators $U_i(\varphi, \beta, \gamma)$ satisfy the conditions consequent upon (2.6). The lowest-level examples occur at $\ell = 1$ [13], but a more interesting discrete state is the one at level $\ell = 6$, which is described by the operator [13,9]

$$D = \left[ c \beta + \frac{1}{4} \left( 6\sqrt{2} \partial \beta \gamma - 3\sqrt{2} \beta \partial \gamma + 12\partial \varphi \beta \gamma + 4\sqrt{2} \partial \varphi \partial \varphi + 2\partial^2 \varphi \right) \right] e^{\frac{2}{7} \alpha \varphi}.$$  

(2.20)

From the pattern of physical states, their frozen $\varphi$-momenta and ghost numbers, obtained in [14,9], one can see that the level-6 discrete operator $D$, and its associated screening current $\oint b D$, might be used to build up each entire tower of physical states in each sector from its lowest-level member. Thus the $\Delta = 1$ and $\frac{15}{16}$ level $\ell = 0$ states (2.15) [5,10], and the $\Delta = \frac{1}{2}$ level $\ell = 1$ state (2.17) [13] could be viewed as the basic building blocks for all the continuous-momentum physical states of the multi-scalar $W_3$ string. This idea was first proposed in [13], where the discrete state $D$ (2.20) was derived as one of the grounding generators of the $W_3$ string. The fermionic screening current $\oint b D$ was first explicitly presented in [15]; in the simplified formalism introduced in [9] that we are using in this paper, it takes the form $\beta e^{\frac{2}{7} Q \varphi}$ [9] (clearly in general, every discrete operator (2.19) has an associated screening current $U_1(\varphi, \beta, \gamma)$). Dealing with the multiple contour integrals that arise is very cumbersome, since only certain powers of $D$ give well-defined products. In practice, it seems that the easiest way to obtain explicit expressions for higher-level physical states is by directly solving the physical-state conditions (2.6). The utility of the screening current $\beta e^{\frac{2}{7} Q \varphi}$ is largely restricted to a descriptive role in organising the higher-level physical spectrum once it is already known by other means, rather than as a tool for deriving the full cohomology of the BRST operator. Some further details of its action were presented in [16], where its use in obtaining physical states already found in [14,9] was exhibited. It has recently been shown that there are discrete physical operators at level $\ell = 15$ which are invertible, and are thus guaranteed to give BRST-non-trivial physical states when normal ordered with any physical operators [17]. These, by contrast, can be used to derive the complete cohomology of the $W_3$ string, both in the two-scalar and the multi-scalar cases [17].

2.2 Interactions

Constructing BRST-invariant scattering amplitudes for the $W_3$ string remained an outstanding problem for quite a long time. The difficulty was that with the states of standard ghost structure known at that time, it was simply not possible to build multi-point correlation functions in which the total ghost number of the operators took the correct value.
(G_{b,c} = 3 and G_{\beta,\gamma} = 5) and the total \( \varphi \) momentum took the correct value \(-2\alpha\), where \( \alpha \) is the background charge). One can easily see this by looking at the form of the standard ghost structure states (2.16).

The key to building \( W_3 \) scattering amplitudes was the discovery [12,13] of physical states in the multi-scalar \( W_3 \) string with non-standard ghost structure. Now, it turns out that both of the above difficulties are eliminated at a stroke [14,9]. In fact, one builds BRST-invariant \( W_3 \)-string scattering amplitudes by precisely the same techniques as one uses for the ordinary bosonic string [14,9]. In other words, one simply builds all possible correlation functions of the four sectors of physical operators of the theory. Those which achieve the correct balance of \( \varphi \) momentum and the correct overall ghost structure can be non-zero. (The use of the ghost boosters \( a_\varphi \) and \( a_{X\mu} \) is often important in order to achieve the correct overall ghost structure.) For three-point functions, this is the end of the story; since the physical operators have conformal weight zero under the total energy-momentum tensor, the results are simply constants, independent of the locations of the three operators on the worldsheet.

For higher \( N \)-point functions, it is necessary, as in ordinary bosonic string theory, to make the replacement \( V(z_i) \rightarrow \oint dw \, b(w) V(z_i) \) for \( (N - 3) \) of the physical operators, so as to make spin-1 currents (screening operators) that can be integrated over the positions \( z_i \) to give an invariant amplitude [14,9].†

We shall not give any detailed scattering-amplitude calculations here; many examples can be found in [14,9]. The upshot is that the “tower” of physical operators in any one of the four sectors described above all behave equivalently from the effective spacetime point of view, and essentially can be thought of as equivalent representatives of the same effective-spacetime physical state. Thus one effectively has three Virasoro-like string sectors, but with intercepts \( \Delta = 1, \frac{15}{16} \) and \( \frac{1}{2} \), and a fourth sector of discrete operators which, having \( \Delta = 0 \) in the effective spacetime, all behave like representatives of the identity.

To uncover the pattern of non-vanishing correlation functions, one must choose representatives appropriately from the four sectors (and use ghost-boosters if necessary) so as to achieve the correct \( \varphi \)-momentum balance and overall ghost structure. Note that for some amplitudes the fourth sector, of discrete physical states, can play an essential rôle in achieving the \( \varphi \)-momentum balance. This is the case for example for the scattering of four physical

† After this procedure for computing scattering amplitudes for the \( W_3 \) string was obtained in [14,9], papers appeared [16] making the incorrect claim that the prescription in [14,9] was incomplete. Specifically, the authors of [16] claimed that unless one augmented the prescription of [14,9] by introducing a new kind of amplitude in which a certain screening charge was included, then one could not construct all the non-zero amplitudes of the \( W_3 \) string. The claim in [16] is obviously wrong, since the authors simply rediscovered the screening current \( \oint dw \, b(w) D(z) \) where \( D(z) \) is given by (2.20). (As is well known, acting on any physical operator with \( \oint dw \, b(w) \) gives a screening current.) Thus inclusion of the screening charge is just a particular case of the general procedure of computing \( N \)-point correlation functions of physical operators that we described above, where one of the \( N > 3 \) operators happens to be the discrete physical operator \( D \) of (2.20). Apparently the authors of [16] were unaware that the screening current was simply \( \oint dw \, b(w) D \), and therefore was already included in the procedure for calculating scattering amplitudes given in [14,9].
states from the $\Delta = \frac{15}{16}$ sector. In an obvious notation $\langle \frac{15}{16} \mid b \frac{15}{16} \mid \frac{15}{16} \rangle$ itself is zero, but $\langle \frac{15}{16} \mid b \frac{15}{16} \mid bD \frac{15}{16} \rangle$ is non-zero, where $D$ is the discrete physical operator (2.20). From the effective-spacetime point of view, the latter is also a four-point function, since $D$ has $p_\mu = 0$. (It happens that this four-point function was not originally evaluated in [14,9]. Possibly the authors of [16] mistakenly thought that this meant that it could not be calculated, giving rise to the incorrect claims in [16] discussed in the footnote on the previous page.)

The full set of tree-level scattering amplitudes for the multi-scalar $W_3$ string turns out to admit a very simple interpretation: The scattering amplitudes are exactly what one would get for a critical Virasoro string in which one tensors the $c = \frac{51}{2}$ energy-momentum tensor (2.5) with the $c = \frac{1}{2}$ energy-momentum tensor for the Ising model. In fact, if one associates the $\Delta = 1, \frac{15}{16}$ and $\frac{1}{2}$ sectors of the theory with the identity operator 1, the spin field $\sigma$, and the energy operator $\epsilon$ of the Ising model, then the $W_3$ scattering amplitudes reproduce the fusion rules of the Ising model [14,9].

In a sense, this conclusion is a somewhat unexciting one. What one has learned is that the operators $U(\varphi, \beta, \gamma)$ in (2.18) provide a representation of the primary fields of the Ising model, realised on the $(\varphi, \beta, \gamma)$ system. (From (2.11) and (2.12) one easily sees that $T_\varphi$ has central charge $\frac{149}{2}$, and $T_{\gamma, \beta}$ has central charge $-74$, so their total is indeed the central charge $\frac{1}{2}$ of the Ising model.) Some further aspects were discussed in [18].

As mentioned in the introduction, if one solves the physical-state conditions (2.6) for the case of the two-scalar $W_3$ string (i.e. just one field $X$ in addition to the field $\varphi$), the spectrum turns out to have many more states than simply those corresponding to specialising the physical states discussed in section 2.1 to the case of a single $X$ coordinate [12,13,14,9,19,17]. To put it another way, the two-scalar $W_3$ string has additional states in its physical spectrum that do not generalise to the multi-scalar case. These states have the property that they cannot be written in a factorised form such as (2.18) or (2.19). For example, at level $\ell = 3$ there is a physical operator $[21b c \gamma - 28\partial \gamma - \frac{16}{3}a c + 12(\alpha \partial \varphi + a \partial X) \gamma]e^{-\frac{2}{7}a \varphi + \frac{3}{7}a X}$, where $a = \alpha/\sqrt{3}$ is the background charge for the $X$ coordinate. For such states, there is no effective-spacetime interpretation, and the $W_3$ symmetry evidently acts in a more non-trivial way than it does on the states of the multi-scalar $W_3$ string. It may well be, therefore, that the two-scalar $W_3$ string will provide more insights into the meaning of $W_3$ geometry.

3. Higher-spin generalisations

A natural generalisation of the above discussion would be to consider a string theory based on a different $W$ algebra, such as the $W_N$ algebra. It is believed that the multi-scalar $W_N$ string would admit an effective spacetime interpretation as a $c = 26 - \frac{6}{N(N+1)}$ Virasoro-type string tensored with the $N$'th unitary Virasoro minimal model [5,20]. More interesting possibilities emerge if one considers a $W$ string with fewer higher-spin currents, since there will be correspondingly fewer constraints and therefore a richer physical spectrum. A case that has been considered recently is where one has just two currents, namely the
energy-momentum tensor and a primary current of spin $s$ [6]. The BRST operator for such a string can be obtained by writing an ansatz that generalises (2.8)–(2.14). In fact the only changes are to replace (2.10) by

$$Q_1 = \oint dz \gamma F(\varphi, \beta, \gamma), \quad (3.1)$$

where $F(\varphi, \beta, \gamma)$ is a spin-$s$ operator to be solved for, and $(\beta, \gamma)$ is now a ghost-system for spin $s$, so $\beta$ has spin $s$, $\gamma$ has spin $(1 - s)$, and (2.12) is replaced by

$$T_{\gamma, \beta} = -s \beta \partial \gamma - (s - 1) \partial \beta \gamma. \quad (3.2)$$

One can now attempt to solve for operators $F(\varphi, \beta, \gamma)$ which give a nilpotent BRST charge. Solutions for $F(\varphi, \beta, \gamma)$ were found in [6] for the cases of $s = 4, 5$ and 6. It seems likely that solutions exist for arbitrary $s$, but their complexity grows rapidly with increasing $s$. In fact for $s = 4$ there are two distinct solutions, for $s = 5$ there is one, and for $s = 6$ there four. Just one solution in each case seems to be associated with a unitary string theory; it corresponds to $T_{\text{eff}}$ having the central charge $c_{\text{eff}} = 26 - \frac{2(s-2)}{(s+1)}$ [6].

We shall discuss only the solutions for $Q_B$ that appear to be associated with unitary string theories here. It seems that in the multi-scalar case the physical states can again be interpreted as coming from a Virasoro-like string theory, in this case tensored with a $c = \frac{2(s-2)}{(s+1)}$ minimal model. This is the central charge of the lowest unitary minimal model of the $W_{s-1}$ algebra. The physical states for the models with $s = 4, 5$ and 6 were analysed in some detail in [6], and evidence was found supporting the conjecture that the physical states correspond to those for an effective Virasoro string tensored with the lowest unitary $W_{s-1}$ minimal model. In particular, it was found in [6] that some of the continuous-momentum physical operators, which take the general form (2.18), correspond to effective Virasoro operators $Y_{\text{eff}}(X)$ with intercepts $\Delta$ that are conjugate to the conformal weights $h$ of the primary fields of the relevant $W_{s-1}$ minimal model, in the sense that $\Delta = 1 - h$. The remaining continuous-momentum physical operators correspond to intercepts that are conjugate to weights having the form of a positive integer plus a weight of a primary field of the minimal model. This can be easily understood, and was investigated in detail in [21]:

The fields of the $W_{s-1}$ minimal model can be viewed as fields of a Virasoro model with the same central charge. Primary fields of the $W_{s-1}$ model will also be primary under Virasoro. In addition, fields that are $W$ secondaries in the $W_{s-1}$ model (i.e. fields that are built by acting with the negative modes of the higher-spin currents of the $W_{s-1}$ algebra) will also be Virasoro primaries, with weights that are increased by some integers relative to those of the $W_{s-1}$ primaries. Thus what we are seeing explicitly in these examples is the way in which the set of primary fields of a given Virasoro model can be more economically described in terms of a smaller set of primaries of a $W$ minimal model of the same central charge. When $s = 4$, the $W_3$ minimal model has central charge $c = \frac{4}{3}$, so there is just a finite
set of Virasoro primaries in this case. However, for $s = 5$ and $s = 6$ the corresponding $W_4$ and $W_5$ minimal models have central charges $c = 1$ and $c = \frac{8}{7}$ respectively. In these cases, an infinite number of Virasoro primaries are obtained from a finite number of $W$ primaries and their $W$ descendants.

We have seen for the examples of $s = 4$, 5 and 6 that the lowest unitary $W_{s-1}$ minimal model, with central charge $c = \frac{2(s-2)}{(s+1)}$, can be realised in terms of the $(\varphi, \beta, \gamma)$ system, with the primary and $W$-secondary fields of the model described by the set of operators $U(\varphi, \beta, \gamma)$ appearing in the physical operators (2.18) of the theory. Included amongst the $W_{s-1}$ primary operators are the primary currents of the $W_{s-1}$ algebra themselves, with spins $3, 4, \ldots (s-1)$. In fact they arise as physical operators with zero $\varphi$ momenta, at levels that can easily be seen to be given by all integers $\ell$ in the interval $\frac{1}{2}s(s-1) + 3 \leq \ell \leq \frac{1}{2}s(s+1) - 1$. Thus we obtain an explicit realisation of $W_{s-1}$ at central charge $c = \frac{2(s-2)}{(s+1)}$ in terms of a scalar field $\varphi$ and the $(\beta, \gamma)$ ghost system with spins $(s, 1-s)$. Of course one can bosonise the $(\beta, \gamma)$ ghosts, thereby obtaining a two-scalar realisation of $W_{s-1}$ at the specific value of central charge given above. In interesting feature of these rather unusual realisations is that the OPEs of these primary currents close on the relevant $W_{s-1}$ algebra modulo the appearance of certain additional null fields on the right-hand side [21]. For example, the case $s = 4$ gives a two-scalar realisation at $c = \frac{4}{5}$ which closes on $W_3$ modulo the appearance of an additional spin-4 primary null field in the OPE of the spin-3 generator with itself. Such realisations were studied previously in [22].

4. Conclusions

In this paper we have presented a brief review of the current status of some of the developments in $W$-string theory. We have concentrated mostly on the $W_3$ string, since this is the simplest non-trivial example and it already illustrates many of the new features that result from generalising ordinary bosonic string theory. We have also concentrated mostly on the multi-scalar case, since this enables one to have sufficiently many dimensions that one can give the theory a reasonably sensible spacetime interpretation. As we have seen, the spacetime theory that emerges is essentially that of a $c = \frac{51}{2}$ Virasoro string tensored with the Ising model.

Theories that are perhaps more interesting arise if we consider a $W$ string in which the higher-spin local worldsheet symmetries are relatively sparse. In particular, if one considers the case of a $W$ string with just two local symmetries, generated by currents of spins 2 and $s$, then as we saw in section 3 the multi-scalar theory is essentially equivalent to that of a $c = 26 - \frac{2(s-2)}{(s+1)}$ Virasoro string tensored with the lowest unitary $W_{s-1}$ minimal model. The way in which this $W_{s-1}$ symmetry is realised on the physical states, and the manner in which it enables one to classify the physical states of the theory, is quite intriguing.

In some sense the more subtle aspects of the $W$ symmetry are lost in the multi-scalar $W$ string. This reflects itself in the way in which the physical states factorise, as in (2.18), into
the product of Virasoro states times minimal-model fields. If, however, one works instead with the basic “Miura”-type realisations of the $W$ algebra, one finds that there are many more physical states that cannot factorise in this way, suggesting that they correspond to a more non-trivial realisation of the $W$ symmetry. For the $W_3$ string, and the spin-2 plus spin-$s$ strings discussed in section 3, these basic realisations are in terms of just two scalar fields, $\varphi$ and $X$. If one is looking for a non-trivial realisation of the $W$ symmetry, as opposed to a “realistic” spacetime theory, these basic “pure $W$-gravity” theories are probably of greater interest.

We have not touched on several important topics, notably the construction of a non-critical BRST operator for the $W_3$ string [15,23], which has two independent realisations of the $W_3$ symmetry, one for the “Liouville sector” and the other for the “matter sector.” This is an extremely interesting area deserving of further study. Another topic that we have not covered is the attempt to give a full and rigorous derivation of the cohomology of the BRST operator. Progress in this direction has recently been achieved in [19,17].

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