Generalized semiconfined harmonic oscillator model with a position-dependent effective mass

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Abstract By using a point canonical transformation starting from the constant-mass Schrödinger equation for the isotonic potential, it is shown that a semiconfined harmonic oscillator model with a position-dependent mass in the BenDaniel–Duke setting and the same spectrum as the standard harmonic oscillator can be easily constructed and extended to a semiconfined shifted harmonic oscillator, which could result from the presence of a uniform gravitational field. A further generalization is proposed by considering a $m$-dependent position-dependent mass for $0 < m < 2$ and deriving the associated semiconfined potential. This results in a family of position-dependent mass and potential pairs, to which the original pair belongs as it corresponds to $m = 1$. Finally, the potential that would result from a general von Roos kinetic energy operator is presented and the examples of the Zhu–Kroemer and Mustafa–Mazharimousavi settings are briefly discussed.

1 Introduction

There is much interest in the Schrödinger equation wherein the constant mass is replaced by a position-dependent mass (PDM), because the latter has many applications in problems occurring in several fields of physics [1–12]. As it has been shown [13] that the PDM Schrödinger equation is equivalent to two other unconventional Schrödinger equations, namely the Schrödinger equation resulting from the use of deformed commutation relations [14–16], as well as that in curved space [17–19], this has reinforced the interest in its study.

As a consequence, much attention has been devoted to finding exact solutions of PDM Schrödinger equations because they may provide a conceptual understanding of some physical phenomena, as well as a testing ground for some approximation schemes. The generation of PDM and potential pairs leading to such exact solutions has been achieved by various methods (see, e.g., [20] and references quoted therein). One of the most powerful techniques for such a purpose consists in applying a point canonical transformation (PCT) to an exactly solvable constant-mass Schrödinger equation [21,22]. Recently, such an approach has proved its efficiency again by providing a straightforward generalization [23] of a harmonic oscillator model, wherein both the mass and the angular frequency are dependent on the position [24].

The purpose of the present paper is to re-examine a new model of semiconfined harmonic oscillator with a mass that varies with position, which has the striking property of having the same spectrum as the standard harmonic oscillator model [25]. By using the PCT method, we plan to prove that one can find a family of PDM and semiconfined potential pairs corresponding to such a spectrum and to which the original PDM and semiconfined harmonic oscillator pair belongs.

The paper is organized as follows. In Sect. 2, the model of [25] is reviewed and shown to be derivable by applying the PCT technique to the constant-mass isotonic oscillator model [26,27]. In Sect. 3, an extension of the model is proposed by starting from a more general PDM and determining the associated semiconfined potential. Finally, Sect. 4 contains some comments.

2 Semiconfined harmonic oscillator model and its derivation by the PCT technique

Jafarov and Van der Jeugt recently determined the exact solution of a PDM semiconfined harmonic oscillator model, characterized by the Schrödinger equation [25]

\[
\left(-\frac{d}{dx} \frac{1}{M(x)} \frac{d}{dx} + V_{\text{eff}}(x)\right) \psi_n(x) = E_n \psi_n(x),
\]

(1)

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where the kinetic energy operator has the BenDaniel–Duke form \[28\], and the potential has the harmonic oscillator form

\[
V_{\text{eff}}(x) = \frac{1}{4} M(x) \omega^2 x^2, \tag{2}
\]

except that the mass

\[
M(x) = \begin{cases} 
(1 + \frac{x}{a})^{-1} & \text{if } -a < x < +\infty, \\
+\infty & \text{if } x \leq -a,
\end{cases} \tag{3}
\]

with \(a > 0\), depends on the position in such a way that \(V_{\text{eff}}(-a) = +\infty\) and \(\lim_{x \to +\infty} V_{\text{eff}}(x) = +\infty\).\(^1\)

By directly solving the differential equation (1), they found that the spectrum of this semiconfined model is that of the standard harmonic oscillator,

\[
E_n = \omega \left( n + \frac{1}{2} \right), \quad n = 0, 1, 2, \ldots, \tag{4}
\]

with corresponding wavefunctions

\[
\psi_n(x) = C_n \left( 1 + \frac{x}{a} \right)^{\frac{1}{4} \omega a^2} e^{-\frac{1}{4} \omega a(x + a)} L_n^{(\omega a^2)} \left( \omega a^2 \left( 1 + \frac{x}{a} \right) \right), \quad -a < x < +\infty, \tag{5}
\]

expressed in terms of Laguerre polynomials \(L_n^{(\alpha)}(z)\) and vanishing at \(x = -a\) and \(x \to +\infty\), as it should be. Here, \(C_n\) is a normalization coefficient given by

\[
C_n = (\omega a^2)^{\frac{1}{4}(\omega a^2 + 1)} \frac{n!}{\sqrt{\omega (\omega a^2 + n + 1)}}. \tag{6}
\]

These results may be alternatively derived by applying a PCT to the constant-mass Schrödinger equation for the isotonic oscillator \[26,27\]

\[
\left( -\frac{d^2}{du^2} + U(u) \right) \phi_n(u) = \epsilon_n \phi_n(u), \tag{7}
\]

where

\[
U(u) = \frac{1}{4} \omega^2 u^2 + \frac{g}{u^2}, \quad g > 0, \quad 0 < u < +\infty, \tag{8}
\]

\[
\epsilon_n = \bar{\omega} (2n + \alpha + 1), \quad \alpha = \frac{1}{2} \sqrt{1 + 4g}, \tag{9}
\]

and

\[
\phi_n(u) \propto u^{\alpha + \frac{1}{2}} e^{-\frac{1}{4} \bar{\omega} u^2} L_n^{(\bar{\omega} u^2)} \left( \frac{1}{2} \bar{\omega} u^2 \right). \tag{10}
\]

A PCT transforming an equation such as (7) into a PDM equation of type (1) \[21,22\] consists in making a change of variable

\[
u(x) = \tilde{a} v(x) + \tilde{b}, \quad v(x) = \int^{x} \sqrt{M(x')} \, dx', \tag{11}
\]

and a change of function

\[
\phi_n(u(x)) = [M(x)]^{-1/4} \psi_n(x). \tag{12}
\]

The potential \(V_{\text{eff}}(x)\) and the energy eigenvalues \(E_n\) of the PDM Schrödinger equation are then given in terms of the potential and the energy eigenvalues of the constant-mass one by

\[
V_{\text{eff}}(x) = \tilde{a}^2 U(u(x)) + \frac{M''}{4 M^2} - \frac{7 M^2}{16 M^2} + \tilde{c}, \tag{13}
\]

and

\[
E_n = \tilde{a}^2 \epsilon_n + \tilde{c}, \tag{14}
\]

where a prime denotes derivation with respect to \(x\) and \(\tilde{a}, \tilde{b}, \tilde{c}\) are three real constants.

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\(^1\) Note that we have adopted here units wherein \(\hbar = 2m_0 \) in the original paper.
In the present case, from (3) and (11), we directly obtain
\[ v(x) = 2a \sqrt{1 + \frac{x}{a}}. \]  
(15)

and
\[ \frac{M''}{4M^2} - \frac{7M'^2}{16M^3} = \frac{1}{16a^2} \left( 1 + \frac{x}{a} \right)^{-1} \]  
(16)

for \(-a < x < +\infty\). With the choice \( \bar{a} = \sqrt{\frac{2\omega}{a^2}}, \bar{b} = 0 \), we get for the change of variable (11)
\[ u(x) = a \sqrt{\frac{2\omega}{\bar{a}}} \sqrt{1 + \frac{x}{a}} \]  
(17)

and the change of function (12), together with (10), leads to
\[ \psi_n(x) = C_n \left( 1 + \frac{x}{a} \right)^{\alpha/2} e^{-\frac{1}{2} \omega a(x+a)} L_n^{(\alpha)} \left( \omega a^2 \left( 1 + \frac{x}{a} \right) \right), \]  
(18)

where \( C_n \) turns out to be
\[ C_n = (\omega a^2)^{1/2} \left( \frac{n!}{a \Gamma(\alpha + n + 1)} \right). \]  
(19)

Furthermore, on assuming \( \bar{c} = -\frac{\alpha}{2} \omega \), the transformed potential (13) becomes
\[ V_{\text{eff}}(x) = \frac{a\omega^2}{4(x+a)} \left( x + a - \frac{\alpha}{\omega a} \right)^2, \]  
(20)

with corresponding eigenvalues \( E_n \) given by (4).

If we compare these results with those of [25], we notice that we have obtained the same energy spectrum (4), but with generalized potential and wavefunctions, since the latter depend on an extra parameter \( \alpha \) absent in [25]. By taking \( \alpha = a^2 \omega \), the original results are retrieved, but for other values of \( \alpha \), the potential (20) describes a semiconfined shifted harmonic oscillator. Note that such a potential might be interpreted as a semiconfined harmonic oscillator in a uniform gravitational field as was done for a shifted harmonic oscillator with another type of PDM [29,30].

### 3 Family of generalized semiconfined oscillator models

A further generalization of the model of [25] can be obtained by changing the PDM (3) into a PDM depending on some parameter \( m \) taking values in the interval \( 0 < m < 2 \),
\[ M(x) = \begin{cases} 
(1 + \frac{x}{a})^{-m} & \text{if } -a < x < +\infty, \\
+\infty & \text{if } x \leq -a,
\end{cases} \]  
(21)

and determining the associated potential \( V_{\text{eff}}(x) \) with the assumption that the starting constant-mass Schrödinger equation remains as given in (7) and (8). The results of Sect. 2 will then correspond to the \( m = 1 \) special case.

Equations (15) and (16) are now replaced by
\[ v(x) = \frac{2a}{2-m} \left( 1 + \frac{x}{a} \right)^{1-\frac{m}{2}} \]  
(22)

and
\[ \frac{M''}{4M^2} - \frac{7M'^2}{16M^3} = -\frac{1}{16a^2} \frac{m(3m - 4)}{m - 2} \left( 1 + \frac{x}{a} \right)^{m-2}, \]  
(23)

respectively. On keeping the same values for \( \bar{a}, \bar{b}, \) and \( \bar{c} \) as in Sect. 2, we get a new change of variable
\[ u(x) = \frac{a}{2-m} \sqrt{\frac{2\omega}{\bar{a}}} \left( 1 + \frac{x}{a} \right)^{1-\frac{m}{2}}, \]  
(24)
Fig. 1  Plot of the semiconfined potential (25) in terms of $x$ for $m = 1$ (black line), $m = \frac{1}{2}$ (red line), and $m = \frac{3}{2}$ (green line). The parameter values are $\omega = 1$, $a = 2$, and $\alpha = 4$.

but the resulting energy eigenvalues remain given by (4). From (13), however, the resulting potential turns out to be $m$-dependent and given by

$$V_{\text{eff}}(x) = \frac{a^m \omega^2}{4(2 - m)^2} (x + a)^{2-m} + \frac{[(m - 2)\alpha - (m - 1)][(m - 2)\alpha + m - 1]}{4a^m(x + a)^{2-m}} - \frac{1}{2}a^2 \omega \alpha.$$  (25)

This is also the case for the wavefunctions, which become

$$\psi_n(x) = C_n \left(1 + \frac{x}{a}\right)^{-\frac{\alpha}{2} (2 - m)} e^{-\frac{1}{2} \omega a^2 (1 + \frac{x}{a})^{2-m}} \times L_n^{(\alpha)} \left(\frac{\omega a^2}{(2 - m)^2} \left(1 + \frac{x}{a}\right)^{2-m}\right).$$  (26)

where

$$C_n = \left(\frac{\omega a^2}{(2 - m)^2}\right)^{\frac{\alpha}{2} (2 - m)} \frac{(2 - m)!}{\Gamma(\alpha + n + 1)}.$$  (27)

The new $m$-dependent potential (25) will be a semiconfined potential provided it goes to $+\infty$ for $x \to +\infty$ and $x \to -a$. The former condition is automatically satisfied, but the latter imposes that

$$\alpha > m - \frac{1}{2} - m,$$  (28)

which implies a restriction for $m$ values such that $\frac{m - 1}{2-m} > \frac{1}{2}$, i.e., for those in the interval $\frac{1}{2} < m < 2$. In such a case, the wavefunctions (26) vanish for $x \to +\infty$ and $x \to -a$, as it should be. The minimum of the potential occurs for

$$x_{\min} = -a + \left\{\frac{(2 - m)^2}{a^m \omega} \sqrt{\alpha^2 - \left(\frac{m - 1}{2 - m}\right)^2}\right\}^{1/(2-m)}$$  (29)

and is given by

$$(V_{\text{eff}})_{\min} = \frac{1}{2} \omega \left\{\sqrt{\alpha^2 - \left(\frac{m - 1}{2 - m}\right)^2} - \alpha\right\}.$$  (30)

It is therefore slightly negative, except for $m = 1$ for which it vanishes.

In Fig. 1, we show the dependence of the semiconfined potential (25) on $m$. The black line corresponds to the original semiconfined harmonic oscillator (2).
4 Comments

In the present paper, we have first shown that the PCT method applied to the constant-mass Schrödinger equation for the isotonic oscillator allows us to easily retrieve the results of [25] and to extend them in order to describe a semiconfined shifted harmonic oscillator, which might be interpreted as a semiconfined harmonic oscillator in a uniform gravitational field.

In a second step, we have obtained a further generalization by considering a $m$-dependent PDM for $0 < m < 2$ and by deriving the corresponding semiconfined potential with the same spectrum as the standard harmonic oscillator. We have therefore constructed a family of PDM and potential pairs, to which the original pair belongs as it corresponds to $m = 1$.

In [25], the BenDaniel–Duke ordering [28] was chosen for the momentum and mass operators. One finds, however, in the literature, several other orderings, which are special cases of the von Roos general two-parameter form of the kinetic energy operator [31], for which the Schrödinger operator writes

$$\frac{-1}{2} \left[ M(x)^{\xi} \frac{d}{dx} M(x)^{\eta} \frac{d}{dx} M(x)^{\zeta} + M(x)^{\xi} \frac{d}{dx} M(x)^{\eta} \frac{d}{dx} M(x)^{\zeta} \right] + V_R(x) \right] \psi_n(x) = E_n \psi_n(x),$$

(31)

where $\xi$, $\eta$, $\zeta$ are some real parameters restricted by the condition $\xi + \eta + \zeta = -1$. In particular, the BenDaniel–Duke ordering corresponds to $\xi = \eta = 0$, $\zeta = -1$, and the relation between the potentials in (1) and (31) is given by

$$V_{R}(x) = V_{\text{eff}}(x) - \frac{1}{2} (1 + \eta) \frac{M''}{M} + [\xi (\xi + 1) + 1] \frac{M^2}{M^2}.$$

(32)

For the mass chosen in (21), the latter becomes

$$V_{\text{R}}(x) = V_{\text{eff}}(x) + \left\{ -\frac{1}{2} (1 + \eta)m(m + 1) + [\xi (\xi + 1) + 1]m^2 \right\} \frac{(x + a)^{m-2}}{a^m}.$$

(33)

It is worth noting, in particular, the Zhu–Kroemer [32] and Mustafa–Mazharimousavi [33] orderings, which pass the de Souza Dutra and Almeida test [34] as good orderings. The former corresponds to $\xi = \xi = -\frac{1}{2}$, $\eta = 0$, and leads to replacing (33) by

$$V_{\text{ZK}}(x) = \frac{a^m \omega^2}{4(2 - m)^2 (x + a)^{2m}} + \frac{(m - 2)a - 1}{4a^m (x + a)^{2m}} - \frac{1}{2} \omega a,$$

(34)

while the latter is associated with $\xi = \xi = -\frac{1}{2}$, $\eta = -\frac{1}{2}$, and gives rise to

$$V_{\text{MM}}(x) = \frac{a^m \omega^2}{4(2 - m)^2 (x + a)^{2m}} + \frac{(m - 2)^2(a^2 - \frac{1}{4})}{4a^m (x + a)^{2m}} - \frac{1}{2} \omega a.$$

(35)

These potentials have a behavior very similar to that of $V_{\text{eff}}(x)$, since they are semiconfined for $a$ restricted to $a > 1/(2 - m)$ or for any value of $a$ ($> 1/2$ by definition (9)), respectively. The place of the minimum and its value are given by (29) and (30) provided $\sqrt{\alpha^2 - (m - 1)(2 - m)^2}$ is replaced by $\sqrt{\alpha^2 - 1/(2 - m)^2}$ or $\sqrt{\alpha^2 - 1/4}$.

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