ON THE GEOMETRY OF HERMITIAN ONE-POINT CODES

EDOARDO BALLICO\textsuperscript{1}

Department of Mathematics, University of Trento
Via Sommarive 14, 38123 Povo (TN), Italy

ALBERTO RAVAGNANI\textsuperscript{2,\ast}

Institut de Mathématiques, Université de Neuchâtel
Rue Emile-Argand 11, CH-2000 Neuchâtel, Switzerland

ABSTRACT. Here we describe the Algebraic Geometry of one-point codes arising from the Hermitian curve. In particular, a geometric characterization of the minimum-weight codewords of their dual codes is provided, including explicit closed formulas for their number. We discuss also some natural improvements of the duals of Hermitian one-point codes by means of geometric arguments. Finally, some cohomological tools are developed to characterize the small-weight codewords of such codes.

CONTENTS

0. Introduction \hspace{1cm} 2
  0.1. Main references \hspace{1cm} 2
  0.2. Layout of the paper \hspace{1cm} 3
1. Codes and algebraic curves \hspace{1cm} 4
2. Hermitian one-point codes \hspace{1cm} 5
3. Projective geometry of the Hermitian curve \hspace{1cm} 6
4. The dual minimum distance \hspace{1cm} 8
5. Geometry of minimum-weight codewords \hspace{1cm} 11
6. Improving Hermitian one-point codes \hspace{1cm} 12
7. Geometry of small-weight codewords \hspace{1cm} 14
Appendix A. Proof of Theorem 32 \hspace{1cm} 15
Conclusion \hspace{1cm} 16
References \hspace{1cm} 16

\textsuperscript{1}E-mail addresses: \textsuperscript{1}edoardo.ballico@unitn.it, \textsuperscript{2}alberto.ravagnani@unine.ch.
2010 Mathematics Subject Classification. 94B27; 14C20; 11G20.
Key words and phrases. Hermitian curve; Goppa code; one-point code; minimum-weight codeword.
\textsuperscript{1}Partially supported by MIUR and GNSAGA.
\ast Corresponding author.
0. Introduction

The aim of this paper is to use algebraic-geometric techniques to describe the dual codes of one-point codes on the Hermitian curve. Classical tools of Algebraic Geometry have been recently shown to be of great interest also for coding-theoretic purposes (see, in particular, [3]). The paper by A. Couvreur provides a geometric way to lower-bound the dual minimum distance of a wide class of evaluation codes arising from geometric constructions. Here we restrict to the case of curves in the projective plane, and improve the cited work for the special case of one-point codes arising from the Hermitian curve. These codes are probably the most studied algebraic-geometric codes. Our improvement is essentially due to the introduction of zero-dimensional subschemes of the plane in the framework of [3]. This choice allows us to give a cohomological interpretation to Hermitian one-point codes and study them by means of classical geometric tools.

0.1. Main references. The minimum distance of Hermitian one-point codes was completely determined in [17]. Section 8.3 of [16] is devoted to the study of such codes, and most of their properties can be found there. The interest in the dual minimum-weight code-words of Hermitian one-point codes dates back to [12], whose results have been recently extended in [11]. For codes arising from higher-degree places on the Hermitian curve, see [8]. Improvements of Goppa codes arising from the Hermitian curve have been recently studied in [4]. Efficient decoding (and list-decoding) algorithms for Hermitian one-point codes are known and well-studied in [9], [10] and [14]. A self-contained reference for codes arising from Algebraic Geometry is the book by H. Stichtenoth ([16]), which treats the topic from the point of view of function fields. See [15] for a more geometric approach. Finally, this paper was inspired by the powerful algebraic-geometric techniques introduced by A. Couvreur in [3].

0.2. Layout of the paper. The paper is organized as follows. In Section 1 we recall the basic definitions of Coding Theory. In particular, we introduce Goppa codes on projective curves and briefly describe the properties we are interested in. The Hermitian curve is defined in Section 2, where we summarize some well-known results about one-point codes on it and give them a cohomological interpretation. Section 3 collects some preliminary results about the intersections of the Hermitian curve with lines and conics in the projective plane. In Section 4 we provide a geometric characterization of the dual minimum distance of codes arising from plane smooth curves, and describe in details the case Hermitian codes by using the particular geometry of tangent lines to the Hermitian curve. In Section 5 we study the supports of the minimum-weight codewords of the dual codes of Hermitian one-point codes, providing explicit formulas for their number. The enumeration process combines tools of classical and finite projective geometry. The analysis includes some computational examples. Natural improvements of the dual codes of Hermitian one-point codes are discussed in Section 6, showing that the geometric approach offers a complete control on the parameters of such constructions. Some interesting results on the dual small-weight codewords of Hermitian one-point codes are stated in Section 7. The proofs, rather technical, are given in the algebraic-geometric Appendix A.
1. Codes and algebraic curves

Let $q$ be a prime power and let $\mathbb{F}_q$ denote the finite field with $q$ elements. A $q$-ary linear code of dimension $k$ and length $n$ is simply a $k$-dimensional subspace of $\mathbb{F}_q^n$. We omit the adjective “linear” for the rest of the paper. The elements of a code are called codewords, or simply words. Endow $\mathbb{F}_q^n$ with a metric space structure by defining the Hamming distance $d : \mathbb{F}_q^n \times \mathbb{F}_q^n \to \mathbb{R}$ as $d(v, w) := |\{1 \leq i \leq n : v_i \neq w_i\}|$, for any $v = (v_1, \ldots, v_n), w = (w_1, \ldots, w_n) \in \mathbb{F}_q$. By definition, the weight of a vector $v \in \mathbb{F}_q^n$ is its distance from the 0 vector, i.e., $wt(v) := d(v, 0) = |\{1 \leq i \leq n : v_i \neq 0\}|$. The minimum distance of a code $C \subseteq \mathbb{F}_q^n$ of at least two codewords is defined by

$$d(C) := \min_{v \neq w \in C} d(v, w) = \min_{v \in C \setminus \{0\}} wt(v).$$

The minimum distance of $\{0\}$ is taken to be, by definition, $\infty$. The correction capability $c(C)$ of a code $C \subseteq \mathbb{F}_q^n$ is strictly related to its minimum distance through the formula $c(C) = \lfloor (d(C) - 1)/2 \rfloor$ (for a minimum distance decoder). For this reason, codes whose minimum distance is high are very interesting for applications. The weight distribution of a given code $C \subseteq \mathbb{F}_q^n$ is the collection $(A_i(C))_{i=0}^n$, where $A_i(C) := |\{c \in C : wt(c) = i\}|$. Define the product of $v = (v_1, \ldots, v_n), w = (w_1, \ldots, w_n) \in \mathbb{F}_q^n$ by $v \cdot w := \sum_{i=1}^n v_i \cdot w_i$. The dual code of a code $C \subseteq \mathbb{F}_q^n$ is denoted and defined by $C^\perp := \{v \in \mathbb{F}_q^n : v \cdot c = 0 \text{ for any } c \in C\}$. The set $C^\perp$ is a linear subspace of $\mathbb{F}_q^n$ of dimension $n - \dim C$, i.e., a $q$-ary code of dimension $n - \dim C$.

**Definition 1.** Codes $C, D \subseteq \mathbb{F}_q^n$ are said to be strongly isometric if $C = vD$, where $v \in \mathbb{F}_q^n$ is a vector of non-zero components and $vD := \{(v_1d_1, v_2d_2, \ldots, v_nd_n) : (d_1, d_2, \ldots, d_n) \in D\}$.

**Remark 2.** A strong isometry is an equivalence relation of codes. Strongly isometric codes have the same minimum distance and the same weight distribution. Two codes are strongly isometric if and only if their dual codes are strongly isometric. A strong isometry preserves the supports of the codewords. In Geometric Coding Theory, studying codes up to strong isometries is a well-established praxis (see [13] for details).

Here we recall the definition of Goppa code and state some well-known properties of codes arising from projective curves.

**Definition 3.** Let $q$ be a prime power and let $\mathbb{P}^k$ be the projective space of dimension $k$ over the field $\mathbb{F}_q$. Consider a smooth curve $X \subseteq \mathbb{P}^k$ defined over $\mathbb{F}_q$ and a divisor $D$ on it. Take points $P_1, \ldots, P_n \in X(\mathbb{F}_q)$ not lying in the support of $D$ and set $\overline{D} := \sum_{i=1}^n P_i$. The **Goppa code** $C(\overline{D}, D)$ is defined as the code obtained evaluating the Riemann-Roch space space $L(D)$ at the points $P_1, \ldots, P_n$.

The construction of Definition 3 was proposed in 1981 by the Russian mathematician V. Goppa. For a geometric introduction to Goppa codes see [15] and [16].

**Definition 4.** Take the setup of Definition 3. Choose $s$ distinct $\mathbb{F}_q$-rational points of $X$, say $P_1, \ldots, P_s$, and $s$ integers $a_1, \ldots, a_s$. Set $D := \sum_{i=1}^s P_i$ and $\overline{D} := \sum_{P \in X(\mathbb{F}_q) \setminus \text{Supp}(D)} P$. The code $C(\overline{D}, D)$ is said to be an **$s$-point code** on $X$. 


Goppa codes are known to have good parameters and a designed minimum distance (the so-called Goppa bound, see [15] again). From the true definition we see that curves carrying many rational points may give very interesting codes through Goppa’s construction, because of their length. A very useful fact in geometric Coding Theory is that linear equivalent divisors give rise to strongly isometric Goppa codes.

Remark 5. Take the setup of Definition 3. Let $D$ and $D'$ be divisors on $X$ and take points $P_1, \ldots, P_n \in X(\mathbb{F}_q)$ which do not appear neither in the support of $D$, nor in the support of $D'$. Set $\overline{D} := \sum_{i=0}^{n} P_i$. It is known (see [13], Remark 2.16) that if $D$ and $D'$ are linear equivalent divisors then $C(\overline{D}, D)$ and $C(\overline{D}, D')$ are strongly isometric codes. In particular, $C(\overline{D}, D)\perp$ and $C(\overline{D}, D')\perp$ are strongly isometric codes (see Remark 2).

2. HERMITIAN ONE-POINT CODES

Let $q$ be a prime power and let $\mathbb{P}^2$ denote the projective plane over the field $\mathbb{F}_q$. Let $X \subseteq \mathbb{P}^2$ be the Hermitian curve (see [16], Example VI.3.6) of affine equation $y^q + y = x^{q+1}$.

It is well-known that $X$ is a maximal curve carrying $q^3 + 1$ $\mathbb{F}_q$-rational points (see for instance [15]). Let $P_{\infty}$ be the only point at infinity of $X$, of projective coordinates $(0 : 1 : 0)$.

Notation 6. Let $m > 0$ be an integer. We denote by $C_m$ be code obtained evaluating the Riemann-Roch space $L(mP_{\infty})$ on $B := X(\mathbb{F}_q^3) \setminus \{P_{\infty}\}$. By Definition 4 $C_m$ is a one-point code on the Hermitian curve $X$, i.e., a Hermitian one-point code.

Let $C_m$ be as in Notation 3. It is well-known that $C_m^\perp$ (the dual code of $C_m$), is $C_{m^\perp}$, where $m^\perp$ is defined by $m^\perp := q^3 + q^2 - q - 2 - m$ (see [16], Theorem 2.2.8). The minimum distance of such codes has been completely determined in [17]. Table 1 gives explicit formulas for the minimum distance of any non-trivial code $C_m$.

As pointed out in the Introduction, the results of this paper are based upon a geometric interpretation of Hermitian one-point codes. Our point of view is explained in the following important note.

Remark 7. Let $m > 0$ be an integer. A basis of the Riemann-Roch space $L(mP_{\infty})$ is given by the monomials $\{x^iy^j : 0 \leq j \leq q - 1, \ iq + j(q + 1) \leq m\}$ (see [16], proof of Proposition 8.3.2). Since $X$ is a maximal curve, for any $P \in X(\mathbb{F}_q)$ we have an isomorphism of sheaves $\mathcal{O}_X(1) \cong \mathcal{L}((q + 1)P)$, the latter one being the invertible sheaf associated to the divisor $(q + 1)P$ on $X$. In other words, and more concretely, for any $d > 0$ the Riemann-Roch space $L(d(q + 1)P_{\infty})$ is exactly the vector space $H^0(X, \mathcal{O}_X(d))$ of all the degree $d$ homogeneous forms defined on $X$. Moreover, for any integer $m > 0$ there exists a unique pair of integers $(d, a)$ such that $m = d(q + 1) - a$ and $0 \leq a \leq q$. In this case we clearly have $d > 0$. Write a linear equivalence $mP_{\infty} \sim d(q + 1)P_{\infty} - aP_{\infty}$. Set $E := aP_{\infty}$ and denote by $H^0(X, \mathcal{O}_X(d)(-E))$ the vector space of all the degree $d$ homogeneous forms defined on $X$ and vanishing on the scheme $E$ (this notation is the standard one of algebraic geometers). We see that $C_m$ is strongly isometric to the code obtained evaluating $H^0(X, \mathcal{O}_X(d)(-E))$ on $B$. We denote this code by $C(d, a)$.

The cohomological structure underlined in Remark 7 will be studied in depth in Section 4 after having summarized the main properties of the Hermitian curve.
**Table 1. Minimum distance of any non-trivial code $C_m$.**

| Phase | Values of $m$ | Minimum distance |
|-------|---------------|------------------|
| 1     | $0 < m < q^2 - q$  
$m = aq + \beta$  
$0 \leq \beta < q$ | $q^3 - a(q + 1)$, if $m < q$ or $m \geq q$ and $a \leq \beta$ |
|       | $q^2 - q < m < q^3 - q^2$ | $q^3 - \beta - aq$, if $m \geq q$ and $a > \beta$ |
| 2     | $q^3 - q^2 < m < q^3$  
$m = q^3 - q^2 + aq + b$  
$0 \leq a < b \leq q - 1$ | $q^3 - m$ |
| 3     | $q^3 - q^2 < m < q^3$  
$m = q^3 - q^2 + aq + b$  
$0 \leq a < b \leq q - 1$ | $q^3 - m$ |
| 4     | $q^3 - q^2 < m < q^3$  
$m = q^3 - q^2 + aq + b$  
$0 \leq b \leq a \leq q - 1$ | $q^3 - m + b$ |
| 5     | $q^3 \leq m \leq q^3 + q^2 - q - 2$  
$m_\perp = aq + \beta$  
$0 \leq \beta < q$ | $a + 2$, if $m_\perp < q$ or $m_\perp \geq q$ and $a \leq \beta$ |
|       | $q^3 \leq m \leq q^3 + q^2 - q - 2$  
$m_\perp = aq + \beta$  
$0 \leq \beta < q$ | $a + 1$, if $m_\perp \geq q$ and $a > \beta$ |

### 3. Projective geometry of the Hermitian curve

In this section we collect some results describing the intersections of the Hermitian curve $X \subseteq \mathbb{P}^2$ with lines and conics in the plane.

**Lemma 8.** Let $X$ be the Hermitian curve. Every line $L$ of $\mathbb{P}^2$ either intersects $X$ in $q + 1$ distinct $\mathbb{F}_{q^2}$-rational points, or $L$ is tangent to $X$ at a point $P$ (with contact order $q + 1$). In the latter case $L$ does not intersect $X$ in any other $\mathbb{F}_{q^2}$-rational point different from $P$.

**Proof.** See [7], part (i) of Lemma 7.3.2, at page 247. □

**Lemma 9.** Let $X$ be the Hermitian curve. Fix an integer $e \in \{2, \ldots, q + 1\}$ and a rational point $P \in X(\mathbb{F}_{q^2})$. Let $E \subseteq X$ be the divisor $eP$, seen as a closed degree $e$ subscheme of $\mathbb{P}^2$. Denote by $L_{X,P} \subseteq \mathbb{P}^2$ be the tangent line to $X$ at $P$. It $T \subseteq \mathbb{P}^2$ is an effective divisor (i.e., a plane curve, possibly with multiple components) of degree $\leq e - 1$ and containing $E$, then $L_{X,P} \subseteq T$. In other words, $L_{X,P}$ is one of the components of $T$.

**Proof.** Since $L_{X,P}$ has order of contact $q + 1 \geq e$ with $X$ at $P$, we have $E \subseteq L_{X,P}$. Since $\deg(E) > \deg(T)$ and $E \subseteq T \cap L_{X,P}$, Bezout theorem implies $L_{X,P} \subseteq T$. □

The rational intersections of the Hermitian curve with some parabolas in the affine chart $\{z \neq 0\}$ of the projective plane $\mathbb{P}^2$ have been recently characterized in [11].

**Remark 10.** The authors of [11] take affine parabolas of the form $y = ax^2 + bx + c$, with $a, b, c \in \mathbb{F}_{q^2}$ and $a \neq 0$. A generic conic in $\mathbb{P}^2$ is given by a homogeneous equation of the form

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fyz = 0,$$
with $A, B, C, D, E, F \in \mathbb{F}_q^2$ not all-zero. We see that $P_\infty$ is a rational point of such conic if and only if $B = 0$. Moreover, it is easily checked that the tangent line to $X$ at $P_\infty$ has equation $z = 0$. This line is also the tangent line to the conic at $P_\infty$ if and only if $D = 0$ and $F \neq 0$. It follows that the parabolas studied in [11] are exactly the smooth conics in $\mathbb{P}^2$ passing through $P_\infty$ and tangent to $X$ at $P_\infty$.

Thanks to the previous Remark 10, we can restate part of [11], Theorem 3.1, in the following convenient form.

**Lemma 11.** Let $h > 0$ be an integer and let $\mathcal{T}(h)$ denote the set of the smooth conics $T$ in $\mathbb{P}^2$ passing through $P_\infty$, tangent to the Hermitian curve $X$ at $P_\infty$, and satisfying $\#(T \cap X \cap \{z \neq 0\}) = h$.

1. Assume $q$ odd. Then $|\mathcal{T}(2q)| = q^2(q + 1)(q - 1)/2$. Moreover, if $h > 2q$ then $\mathcal{T}(h) = \emptyset$.
2. Assume $q$ even. If $h > 2q - 1$ then $\mathcal{T}(h) = \emptyset$.

The results of this section will be applied throughout the rest of the paper in order to describe the minimum distance and the minimum-weight codewords of the duals of Hermitian one-point codes.

4. **The dual minimum distance**

The aim of this section is to give a geometric interpretation to the dual minimum distance of certain codes arising from plane smooth curves. The results improve the powerful method by A. Couvreur (see [3]) in the planar case, and explicitly characterize the supports of minimum-weight codewords in terms of cohomological vanishing conditions. The particular case of Hermitian one-point codes is studied in depth.

**Proposition 12.** Let $\mathbb{F}$ be any field and let $\mathbb{P}^2$ denote the projective plane on $\mathbb{F}$. Let $X \subseteq \mathbb{P}^2$ be a smooth plane curve. Fix an integer $d > 0$, a zero-dimensional scheme $E \subseteq X$ and a finite subset $B \subseteq X$ such that $B \cap E_{\text{red}} = \emptyset$. Denote by $C$ the code obtained evaluating the vector space $H^0(C, \mathcal{O}_X(d)(-E))$ at the points of $B$. Set $c := \deg(X)$ and assume $d < c$. The following facts hold.

1. The minimum distance of $C^\perp$ is the minimal cardinality, say $z$, of a subset of $S \subseteq B$ such that $h^1(\mathbb{P}^2, \mathcal{I}_{S \cup E}(d)) > h^1(\mathbb{P}^2, \mathcal{I}_E(d))$.
2. A codeword of $C^\perp$ has weight $z$ if and only if it is supported by a subset $S \subseteq B$ such that $z$.
   (a) $\#S = z$,
   (b) $h^1(\mathbb{P}^2, \mathcal{I}_{E \cup S}(d)) > h^1(\mathbb{P}^2, \mathcal{I}_E(d))$,
   (c) $h^1(\mathbb{P}^2, \mathcal{I}_{E \cup S}(d)) > h^1(\mathbb{P}^2, \mathcal{I}_{E \cup S'}(d))$ for any $S' \subsetneq S$.

**Proof.** Since $X$ is projectively normal (being a smooth plane curve) and we assumed $d < c$, the restriction map $\rho_d : H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \to H^0(X, \mathcal{O}_X(d))$ is bijective. Hence the restriction map $\rho_{d, E} : H^0(\mathbb{P}^2, \mathcal{I}_E(d)) \to H^0(X, \mathcal{O}_X(d)(-E))$ is bijective. It follows that a finite subset $S \subseteq C \setminus E_{\text{red}}$ imposes independent condition to $H^0(X, \mathcal{O}_X(d)(-E))$ if and only if $S$ imposes independent conditions to $H^0(\mathbb{P}^2, \mathcal{I}_E(d))$. Moreover, the set $S$ imposes independent conditions to $H^0(\mathbb{P}^2, \mathcal{I}_E(d))$ if and only if $h^1(\mathbb{P}^2, \mathcal{I}_{E \cup S}(d)) = h^1(\mathbb{P}^2, \mathcal{I}_E(d))$ (here we use again that $S \cap E_{\text{red}} = \emptyset$). To get the existence of a non-zero codeword of $C^\perp$ whose support is $S$ (and not

---

1. Here $E_{\text{red}}$ denotes the reduction of the scheme $E$.
2. We denote the cardinality of a finite set, say $S$, by $\#(S)$. 

only with support contained in $S$) we need that the submatrix $M_S$ of the generator matrix of $C$ obtained by considering the columns associated to the points appearing in $S$ has the property that each of its submatrices obtained deleting one column have the same rank of $M_S$ (each such column is associated to some $P \in S$ and we require that the codeword has support containing $P$). This is equivalent to the last claim in the statement. □

**Remark 13.** From now on, we explicitly focus on the analysis of one-point codes on the Hermitian curve. As pointed out in Remark [7], we need to study codes of type $C(d,a)$ with $d > 0$ and $0 \leq a \leq q$. In the next lemma we show that we may restrict to the analysis of $C(d,a)$ codes such that $d > 0$ and $0 \leq a \leq d$.

**Lemma 14.** Let $X \subseteq \mathbb{P}^2$ be the Hermitian curve. Consider a $C(d,a)$ code, with $d > 1$ and $0 \leq a \leq q$. Set $E := aP_\infty$.

- If $a > d$ then set $d' := d - 1$ and $a' := 0$,
- otherwise set $d' := d$ and $a' := a$.

Then $C(d,a)$ and $C(d',a')$ are strongly isometric codes. In particular, their dual codes are strongly isometric.

**Proof.** First of all, set $E' := a'P_\infty$. The code $C(d,a)$ is obtained evaluating the degree $d$ homogeneous forms vanishing on the scheme $aP_\infty$. If $a > d$ then an $f \in H^0(X, \mathcal{O}_X(d)(-aP_\infty))$ is divided by the equation of the tangent line to $X$ at $P_\infty$, here denoted by $L_{X,P_\infty}$. The division by such equation gives an isomorphism of vector spaces

$$H^0(X, \mathcal{O}_X(d)(-E)) \cong H^0(X, \mathcal{O}_X(d')(E')).$$

Since $L_{X,P_\infty}$ does not intersect $X$ at any rational point different from $P_\infty$ (Lemma [8], we get that $C(d,a)$ and $C(d',a')$ are in fact strongly isometric codes. Their dual codes are also strongly isometric (Remark [2]). □

The following lemma is rather technical and provides some cohomological properties of zero-dimensional subschemes of $\mathbb{P}^2$. The result is taken from [11], Lemma 7, and the proof is omitted here.

**Lemma 15.** Let $\mathbb{F}_{q^2}$ be the finite field with $q^2$ elements ($q$ a prime power) and denote by $\mathbb{P}^2$ the projective plane over the field $\mathbb{F}_{q^2}$. Let $X \subseteq \mathbb{P}^2$ be the Hermitian curve. Choose an integer $d > 0$ and a zero-dimensional scheme $Z \subseteq X(\mathbb{F}_{q^2})$ of degree $z > 0$. The following facts hold.

(a) If $z \leq d + 1$, then $h^1(\mathbb{P}^2, \mathcal{I}_Z(d)) = 0$.

(b) If $d + 2 \leq z \leq 2d + 1$, then $h^1(\mathbb{P}^2, \mathcal{I}_Z(d)) > 0$ if and only if there exists a line $T_1$ such that $\deg(T_1 \cap Z) \geq d + 2$.

(c) If $2d + 2 \leq z \leq 3d - 1$ and $d \geq 2$, then $h^1(\mathbb{P}^2, \mathcal{I}_Z(d)) > 0$ if and only if either there exists a line $T_1$ defined over $\mathbb{F}_{q^2}$ such that $\deg(T_1 \cap Z) \geq d + 2$, or there exists a conic $T_2$ defined over $\mathbb{F}_{q^2}$ such that $\deg(T_2 \cap Z) \geq 2d + 2$.

(d) Assume $z = 3d$ and $d \geq 3$. Then $h^1(\mathbb{P}^2, \mathcal{I}_Z(d)) > 0$ if and only if either there exists a line $T_1$ defined over $\mathbb{F}_{q^2}$ such that $\deg(T_1 \cap Z) \geq d + 2$, or there is a conic $T_2$ defined over $\mathbb{F}_{q^2}$ such that $\deg(T_2 \cap Z) \geq 2d + 2$, or there exists a plane cubic $T_3$ such that $Z$ is the complete intersection of $T_3$ and a plane curve of degree $d$. In the latter case, if $d \geq 4$ then $T_3$ is unique and defined over $\mathbb{F}_{q^2}$ and we may find a plane curve $C_d$ defined over $\mathbb{F}_{q^2}$ and with $Z = T_3 \cap C_d$.\]
(e) Assume $z \leq 4d - 5$ and $d \geq 4$. Then $h^1(\mathbb{P}^2, \mathcal{I}_Z(d)) > 0$ if and only if either there exists a line $T_1$ defined over $\mathbb{F}_{q^2}$ such that $\deg(T_1 \cap Z) \geq d + 2$, or there exists a conic $T_2$ defined over $\mathbb{F}_{q^2}$ such that $\deg(T_2 \cap Z) \geq 2d + 2$, or there exist $W \subseteq Z$ defined over $\mathbb{F}_{q^2}$ with $\deg(W) = 3d$ and plane cubic $T_3$ defined over $\mathbb{F}_{q^2}$ such that $W$ is the complete intersection of $T_3$ and a plane curve of degree $d$, or there is a plane cubic $C_3$ defined over $\mathbb{F}_{q^2}$ such that $\deg(C_3 \cap Z) \geq 3d + 1$.

**Lemma 16.** Let $X$ be the Hermitian curve. Choose integers $d > 0$ and $0 \leq a \leq d$. Set $E := aP_\infty$. Then $h^1(\mathbb{P}^2, \mathcal{I}_E(d)) = 0$.

**Proof.** Assume $h^1(\mathbb{P}^2, \mathcal{I}_E(d)) > 0$. Since $a \leq d$, by Lemma 15 there exists a line $L \subseteq \mathbb{P}^2$ such that $\deg(L \cap E) \geq d + 2$. Since in any case $\deg(L \cap E) \leq d$ we immediately get a contradiction. \* 

5. **Geometry of minimum-weight codewords**

In this section an explicit description of the minimum-weight codewords of $C(d, a) \perp$ codes is provided, for any choice of $d$ with $1 \leq d \leq q$. By Remark 13, we restrict to the case $0 \leq a \leq d$. More precisely, we combine Proposition 12 with the other preliminary results of Section 3 and Section 4 in order to geometrically characterize the supports of the minimum-weight codewords of $C(d, a) \perp$ codes. In particular, here we derive some explicit formulas for their number.

**Theorem 17.** Let $d \leq q - 1$ be a positive integer. Take any integer $0 \leq a \leq d$ and denote by $\delta := \delta(d, a)$ the minimum distance of $C(d, a) \perp$. Let $A_\delta$ be the number of the minimum-weight codewords of $C(d, a) \perp$.

1. If $a = 0$ then $\delta = d + 2$ and a subset $S = \{P_1, ..., P_\delta\} \subseteq X(\mathbb{F}_{q^2}) \setminus \{P_\infty\}$ of cardinality $\delta$ is the support of a minimum-weight codewords of $C(d, a) \perp$ if and only if it consists of $\delta$ collinear points. Moreover,

$$A_\delta \frac{q^2}{q^2 - 1} = \left\{ \begin{array}{ll}
(q^4 - q^3) & \text{if } d = q - 1, \\
q^2 \left(\frac{q}{\delta}\right) + (q^4 - q^3) \left(\frac{q + 1}{\delta}\right) & \text{if } d < q - 1.
\end{array} \right.$$

2. If $a > 0$ then $\delta = d + 1$ and a subset $S = \{P_1, ..., P_\delta\} \subseteq X(\mathbb{F}_{q^2}) \setminus \{P_\infty\}$ of cardinality $\delta$ is the support of a minimum-weight codeword of $C(d, a) \perp$ if and only if $P_\infty, P_1, ..., P_\delta$ are collinear points. Moreover,

$$A_\delta \frac{q^2}{q^2 - 1} = q^2 \left(\frac{q}{\delta}\right).$$

**Proof.** The minimum distance, $\delta(d, a)$, can be easily computed by reversing Table 1. Here we set $E := aP_\infty$. Since $a \leq d$, Lemma 16 gives $h^1(\mathbb{P}^2, \mathcal{I}_E(d)) = 0$. By Proposition 12, a subset $S = \{P_1, ..., P_\delta\} \subseteq X(\mathbb{F}_{q^2}) \setminus \{P_\infty\}$ of cardinality $\delta$ is the support of a minimum-weight codeword of $C(d, a) \perp$ if and only if $h^1(\mathbb{P}^2, \mathcal{I}_{E \cup S}(d)) > 0$. Let $S$ be with this property.

1. Assume $a = 0$. Then we have $E = \emptyset$ and $\deg(E) + \gamma(S) = d + 2 \leq 2d + 1$. Hence, by Lemma 15, $h^1(\mathbb{P}^2, \mathcal{I}_{E \cup S}(d)) > 0$ if and only if $P_1, ..., P_\delta$ are collinear points. By Lemma 15 this condition is also sufficient for $S$ to be the support of a minimum-weight codeword.
(2) If \( a > 0 \) then \( \deg(E) + h(S) \leq 2d + 1 \) and so, again by Lemma \[16\], \( h^1(\mathbb{P}^2, \mathcal{I}_{E} \cup S(d)) > 0 \) if and only if there exists a line \( L \subseteq \mathbb{P}^2 \) such that \( \deg(L \cap (E \cup S)) \geq d + 2 \). Since \( a \leq d \) and Lemma \[8\] holds, \( L \) cannot be the tangent line to the Hermitian curve \( X \) at \( P_\infty \). As a consequence, \( P_\infty \) appears in \( L \) with multiplicity one, and so \( P_\infty, P_1, ..., P_\delta \) are collinear points. By Lemma \[15\] this condition is also sufficient for \( S \) to be the support of a minimum-weight codeword.

To get the formulas for the number of minimum-weight codewords, observe that, in any linear, code two minimum-weight codewords with the same support are (non-zero) multiple one each other. This fact follows from the definitions of linear code and minimum distance. Moreover, any non-zero multiple of a minimum-weight codeword is an other minimum-weight codeword with the same support. Hence we deduce our formulas by using the properties of lines (Lemma \[8\]).

Theorem \[17\] describes the dual code of any one-point code \( C_m^⊥ \) with \( m \leq q^2 - 1 \), providing explicit characterizations of the supports of its minimum-weight codewords. The following result, on the other hand, studies in details \( C(q, a)^⊥ \) codes.

**Theorem 18.** Set \( d := q \) and choose any integer \( 0 \leq a \leq d = q \). Denote by \( \delta := \delta(d, a) \) the minimum distance of \( C(d, a)^⊥ \).

1. If \( a = 0 \) then \( \delta = 2q + 2 \) and a subset \( S = \{P_1, ..., P_\delta\} \subseteq X(\mathbb{F}_{q^2}) \setminus \{P_\infty\} \) of cardinality \( \delta \) is the support of a minimum-weight codeword of \( C(d, a)^⊥ \) if and only if it is contained into a conic of \( \mathbb{P}^2 \).
2. If \( a = 1 \) then \( \delta = 2q + 1 \) and a subset \( S = \{P_1, ..., P_\delta\} \subseteq X(\mathbb{F}_{q^2}) \setminus \{P_\infty\} \) of cardinality \( \delta \) is the support of a minimum-weight codeword of \( C(d, a)^⊥ \) if and only if \( P_\infty, P_1, ..., P_\delta \) lie on a conic of \( \mathbb{P}^2 \).
3. If \( 2 \leq a < q \) then \( \delta = 2q \). The following two facts hold.
   a. Assume \( q \) even. Then a subset \( S = \{P_1, ..., P_\delta\} \subseteq X(\mathbb{F}_{q^2}) \setminus \{P_\infty\} \) of cardinality \( \delta \) is the support of a minimum-weight codeword of \( C(d, a)^⊥ \) if and only if it is contained into two lines meeting at \( P_\infty \). Moreover,
      \[
      A_\delta = (q^2 - 1) \binom{q^2}{2}.\]
   b. Assume \( q \) odd. Then a subset \( S = \{P_1, ..., P_\delta\} \subseteq X(\mathbb{F}_{q^2}) \setminus \{P_\infty\} \) of cardinality \( \delta \) is the support of a minimum-weight codeword of \( C(d, a)^⊥ \) if and only if either it is contained into two lines meeting at \( P_\infty \), or it is contained into a smooth conic of \( \mathbb{P}^2 \) which is tangent to \( X \) at \( P_\infty \). Moreover,
      \[
      A_\delta = (q^2 - 1) \left[ q^2(q + 1)(q - 1)/2 + \binom{q^2}{2} \right].\]

**Proof.** The dual minimum distance, \( \delta = \delta(d, a) \), can be easily computed by reversing Table \[1\] at the beginning of the paper:

\[
\delta(d, a) = \begin{cases} 
2d + 2 - a & \text{if } a \in \{0, 1\}, \\
2d & \text{if } a \geq 2.
\end{cases}
\]

Set \( E := aP_\infty \). By Proposition \[12\] and Lemma \[16\], a subset \( S = \{P_1, ..., P_\delta\} \subseteq X(\mathbb{F}_{q^2}) \setminus \{P_\infty\} \) of cardinality \( \delta \) is the support of a minimum-weight codeword of \( C(d, a)^⊥ \) if and only if \( h^1(\mathbb{P}^2, \mathcal{I}_{E \cup S}(d)) > 0 \). Let \( S \) be with this property.
(1) If \( a = 0 \), then \( \delta = 2d + 2 \) and \( \deg(E) + \sharp(S) = 2d + 2 \). By Lemma \([15]\) there exists either a subscheme \( W \subseteq S \) of degree \( d + 2 = q + 2 \) and contained in a line, or a subscheme \( W \subseteq S \) of degree \( 2d + 2 = 2q + 2 \) and contained in a conic. The former case must be excluded because of Lemma \([8]\). In the latter case we see that \( P_1, \ldots, P_{2q+2} \) lie on a conic. By Lemma \([15]\) this condition is necessary and sufficient for \( S \) to the the support of a minimum-weight codeword.

(2) If \( a = 1 \) then \( \delta = 2d + 1 \) and \( \deg(E) + \sharp(S) = 2d + 2 \). By Lemma \([15]\) there exists either a subscheme \( W \subseteq P_\infty \cup S \) of degree \( d + 2 = q + 2 \) and contained in a line, or a subscheme \( W \subseteq P_\infty \cup S \) of degree \( 2d + 2 = 2q + 2 \) and contained in a conic. The former case must be excluded because of Lemma \([8]\). In the latter case we have that \( P_\infty, P_1, \ldots, P_{2q+2} \) lie on a conic. By Lemma \([15]\) this condition is necessary and sufficient for \( S \) to the the support of a minimum-weight codeword.

(3) Assume \( a \geq 2 \). We have \( \delta = 2d + 2q \) and \( \deg(E) + \sharp(S) = a + 2d \leq 3d - 1 \) (because we assumed \( a < q = d \)). Hence Lemma \([15]\) applies: either there exists a subscheme \( W \subseteq aP_\infty \cup S \) of degree \( d + 2 = q + 2 \) and contained in a line, or there exists a subscheme \( W \subseteq aP_\infty \cup S \) of degree \( 2d + 2 = 2q + 2 \) and contained in a conic. The former case must be excluded, as in the previous cases. If \( W \subseteq aP_\infty \cup \{ P_1, \ldots, P_{2d} \} \), \( \deg(W) = 2d + 2 \) and \( W \) is contained in a conic \( T \) then the multiplicity of \( P_\infty \) in \( W \), say \( e_W(P_\infty) \), must be at least 2. On the other hand, if \( e_W(P_\infty) > 2 \) then (Lemma \([9]\)) the tangent line to \( X \) at \( P_\infty, L_{X,P_\infty} \), turns out to be a component of \( T \). In this case Lemma \([8]\) implies that \( P_1, \ldots, P_{2q} \) lie on the line \( T - L_{X,P_\infty} \), which contradicts Lemma \([8]\) again. As a consequence, \( e_W(P_\infty) = 2 \), and we are done. Indeed, \( L_{X,P_\infty} \) cannot be a component of \( T \) (use Lemma \([8]\) twice) and so \( T \) is either the union of two lines meeting at \( P_\infty \), or a smooth conic which is tangent to \( X \) at \( P_\infty \). By Lemma \([15]\) this condition is also sufficient for \( S \) to the the support of a minimum-weight codeword.

(a) Assume \( q \) even. By Lemma \([11]\) the case of the smooth conic must be excluded. Hence \( S \) is the support of a minimum-weight codeword of \( C(d,a) \) if and only if it is contained in the union of two plane lines meeting at \( P_\infty \). Since (Lemma \([8]\)) any line \( L \subseteq \mathbb{P}^2 \) satisfies \( \deg(L \cap X) = q + 1 \), and \( \delta = 2q \), the supports of the minimum-weight codewords of \( C(d,a) \) are in bijection with the pairs of distinct lines of \( \mathbb{P}^2 \) passing through \( P_\infty \) and not tangent to \( X \) at \( P_\infty \) (use Lemma \([8]\) again). The lines through \( P_\infty \) are \( q^2 + 1 \). One of them is the tangent line to \( X \) at \( P_\infty \). The formula follows.

(b) If \( q \) is odd, then we have to consider also the case of conics. By Lemma \([11]\) a smooth conic \( T \subseteq \mathbb{P}^2 \) which is tangent to \( X \) at \( P_\infty \) cannot intersect \( X \) in more than \( 2q \) affine points. Hence \( S \) must appear exactly as the affine intersection of \( X \) and such a conic. By Lemma \([11]\) there exists \( q^2(q + 1)(q - 1)/2 \) conics with this property. Since \( 2q > 3 \) and a parabola of the form \( y = ax^2 + bx + c \) (with \( a, b, c \in \mathbb{F}_{q^2} \)) is completely determined by three of its points, distinct conics correspond to distinct intersections.

\[ \square \]

**Remark 19.** The formulas given in Theorem \([17]\) and \([18]\) extend those of \([11]\), proved for Hermitian one-point codes of minimum distance smaller or equal than \( q \).

Theorem \([18]\) concludes our analysis of the minimum-weight codewords of the duals of Hermitian one-point codes. Let us examine two explicit examples.
Remark 22. Fix any prime power $q$ and a positive integer $n$. Pick out a non-empty subset $H \subseteq \{1, \ldots, n\}$ and denote by $\pi_H : \mathbb{F}_q^n \rightarrow \mathbb{F}_q^{n-\sharp(H)}$ the projection on the coordinates not appearing in $H$. In other words, given a vector $v = (v_1, \ldots, v_n) \in \mathbb{F}_q^n$, we delete the components associated to any index $i \in H$ by operating $\pi_H(v)$. Let $C \subseteq \mathbb{F}_q^n_⊥$ be a code and let $\delta$ be the minimum distance of $C_⊥$. A subset $H \subseteq \{1, \ldots, n\}$ is said to be an improving subset for $C_⊥$ if the minimum distance of $\pi_H(C_⊥)$ is strictly greater than $\delta$. The map $\pi_H$ will be called an improving projection for the code $C_⊥$.

Definition 23. Let $d > 0$ and $0 \leq a \leq q$ be integers. Set $B := X(\mathbb{F}_q^2) \setminus \{P_\infty\}$ and choose a non-empty subset $H \subseteq B$. We denote by $C(d, a, H)$ the code obtained evaluating the vector space $H^0(X, O_X(d)(-aP_\infty))$ on the set $B \setminus H$ and by $C(d, a, H)_⊥$ its dual code.

Remark 24. By enumerating the points appearing in $B = X(\mathbb{F}_q^2) \setminus \{P_\infty\}$ we can identify $H$ with a subset of $\{1, \ldots, n := q^3\}$ and write $C(d, a, H) = \pi_H(C(d, a))$ in the notations of Definition 22.

Remark 25. The proof of Lemma 14 still works if we replace $B$ and $C(d, a)$ with $B \setminus H$ and $C(d, a, H)$ (respectively). Hence, from now on, we will consider only $C(d, a, H)$ codes with $d > 0$ and $a \leq d$.
Theorem 26. Let $0 < d < q$ and $1 \leq a \leq d$ be integers. Choose a non-empty subset $H \subseteq B = X(\mathbb{F}_{q^2})(\{P_\infty\}$ and let $C(d,a,H)$ be as in Definition\textsuperscript{23}. The minimum distance of $C(d,a,H)$$^\perp$ is at least $d + 1$ and the equality holds if and only if there exist $d + 1$ collinear points in $B \setminus H$ on a line through $P_\infty$.

Proof. Since $1 \leq a \leq d$ we get, by setting $E := aP_\infty$, $h^1(\mathbb{P}^2, \mathcal{I}_E(d)) = 0$ (Lemma\textsuperscript{16}). By Proposition\textsuperscript{12} the minimum distance of $C(d,a,H)$$^\perp$ is the smallest cardinality, say $\delta$, of a subset $S \subseteq B \setminus H$ such that $h^1(\mathbb{P}^2, \mathcal{I}_{E \cup S}(d)) > 0$. Let $S := \{P_1, ..., P_\delta\}$ be the support of a minimum-weight codeword of $C(d,a,H)$$^\perp$. In particular, we have $h^1(\mathbb{P}^2, \mathcal{I}_{E \cup S}(d)) > 0$. If $\delta \leq d$ then $\deg(E \cup S) \leq 2d$ and (Lemma\textsuperscript{15}) there exists a line $L \subseteq \mathbb{P}^2$ such that $\deg(L \cap (E \cup S)) \geq d + 2$. Since $E \cap S = \emptyset$ and $\deg(S) \leq d$ we have that $P_\infty$ appears in $L$ with multiplicity at least two. By Lemma\textsuperscript{9} this means that $L$ is the tangent line to $X$ at $P_\infty$ and (Lemma\textsuperscript{8}) $\deg(L \cap (E \cup S)) = \deg(E) \leq d$, a contradiction. It follows $\delta \geq d + 1$. If $\delta = d + 1$ then $\deg(E \cup S) \leq 2d + 1$ and so (Lemma\textsuperscript{15}) there exists a line $L \subseteq \mathbb{P}^2$ such that $\deg(L \cap (E \cup S)) \geq d + 2$. If $P_\infty$ appears in $L$ with multiplicity greater than one then $L$ is tangent to $X$ at $P_\infty$, contradicting Lemma\textsuperscript{8} ($\deg(E) \leq d$ here). It follows that the points $P_1, ..., P_\delta$ lie on a line through $P_\infty$. Finally, assume that $S = \{P_1, ..., P_\delta\} \subseteq B \setminus H$ is a set of $d + 1$ collinear points on a line through $P_\infty$ (this is possible because we assumed $d < q$, and so $d + 1 < q + 1$). By Lemma\textsuperscript{15} we have $h^1(\mathbb{P}^2, \mathcal{I}_{E \cup S}(d)) > 0$ and hence $S$ contains the support of a minimum-weight codewords of $C(d,a,H)$$^\perp$. Since we proved that the minimum distance of $C(d,a,H)$$^\perp$ is at least $d + 1$ we deduce that $S$ is in fact the support of a minimum-weight codeword of $C(d,a,H)$$^\perp$. This concludes the proof.

Corollary 27. Choose integers $0 < d < q$ and $1 \leq a \leq d$. A non-empty subset $H \subseteq B = X(\mathbb{F}_{q^2}) \setminus \{P_\infty\}$ is an improving subset for $C(d,a)$$^\perp$ if and only if there are no $d + 1$ collinear points in $B \setminus H$ lying on a line through $P_\infty$. In particular, if $H$ is an improving subset for $C(d,a)$$^\perp$ then $\#(H) \geq q^2(q - d)$ and so the length of $C(d,a,H)$$^\perp$ is at most $q^2d$.

Proof. By Theorem\textsuperscript{26} to get an improving subset $H \subseteq B$ we must remove from $B$ any $d + 1$ points lying on a line through $P_\infty$. The lines in $\mathbb{P}^2$ passing through $P_\infty$ and not tangent to $X$ are $q^2$. Every such a line contains $q$ points different from $P_\infty$ (Lemma\textsuperscript{8}). From any line we must remove at least $q - d$ points. This gives the formula.

Remark 28. Corollary\textsuperscript{27} shows that, in order to improve the minimum distance of a non-trivial Hermitian one-point code $C_m^d$ (with $m \leq q^2 - 1$) by evaluating the Riemann-Roch space $L(mP_\infty)$ on a proper subset of rational points of the curve, it is necessary to reduce the length of the code in a signifcative manner.

Example 29. Take $q := 5$. The Hermitian curve $X$ is defined over $\mathbb{F}_{25}$ by the affine equation $y^5 + y = x^6$. Let us consider the Hermitian one-point code $C_{11}$. Its length is $q^3 = 125$. In the notation of Remark\textsuperscript{7} we have $d = 2 < q$ and $a = 1 < d$. Hence the dual minimum distance of $C_4$ is 3 (see Theorem\textsuperscript{17}). As in the statement of Corollary\textsuperscript{27} take $H$ to be a minimal improving subset for $C_{25}^d$. The length of the improved code is 50, while it was 125.

7. Geometry of small-weight codewords

In this Section we state a result which describes the small-weight codewords of certain $C(d,a)$$^\perp$ codes. Our goal is to characterize the supports of such codewords from a geometric point of view.
Remark 30. By Lemma [14] for any $C(d, a)$ code with $d > 1$ and $0 \leq a \leq q$ there exist integers $d' > 0$ and $0 \leq a' \leq d'$ such that $C(d, a) = C(d', a')$. Hence, from now on, we will consider only $C(d, a)$ codes with $d > 0$ and $a \leq d$.

The proof of Theorem 32 is rather technical, and it needs some non-trivial algebra-geometric preliminaries. For these reasons, it is given in Appendix A. Notice that the statement of the theorem can be perfectly understood without any knowledge of Algebraic Geometry.

Notation 31. We denote by $L_{X, P_\infty}$ the tangent line to the Hermitian curve $X$ at $P_\infty$. Moreover, $\mathcal{R}(\infty)$ will be the set of the lines passing through $P_\infty$ which are not tangent to $X$ in any point. $\mathcal{R}$ will denote the set of the lines which do not contain $P_\infty$ and which are not tangent to $X$ at any point.

The following result provides a complete description of the small-weight codewords of any $C(d, a)^\perp$ such that $d \leq q - 1$ and $0 \leq a \leq d$. By Remark 30 here we describe the small-weight codewords of any non-trivial $C_m^\perp$ code such that $m \leq q^2 - 1$.

Theorem 32. Let $0 < d \leq q - 1$ and $0 \leq a \leq d$ be integers. Denote by $S = \{P_1, \ldots, P_w\}$ be the support of a codeword of $C(d, a)^\perp$ of weight $w$.

1. Assume $d + 2 \leq a + w \leq 2d + 1$. Then $S$ must be one of the sets in the following list:
   a) a subset of $w$ elements of $L \cap B$, for an $L \in \mathcal{R}(\infty)$ ($w \geq d + 1$);
   b) a subset of $w$ elements of $L \cap B$, for an $L \in \mathcal{R}$ ($w \geq d + 2$).

Moreover, any such a set appears as the support of a codeword of $C(d, a)^\perp$ of weight exactly $w$.

2. Assume $2d + 2 \leq a + w \leq 3d - 1$. Then either $S$ is one of the sets in cases (a), (b) of the previous list,
   c) or there exist two distinct lines $L, M \subseteq \mathbb{P}^2$ such that
      - $\deg(L \cap (E \cup S)) \geq d + 2$,
      - $\deg(M \cap (E \cup S)) \geq d + 1$,
      - $\deg((L \cup M) \cap E) + w \geq 2d + 2$,
      - either $w \geq 2d + 3$ (if $L, M \in \mathcal{R}$), or $w \geq 2d + 2$ (if $(L, M) \in \mathcal{R} \times \mathcal{R}(\infty)$ or $(M, L) \in \mathcal{R} \times \mathcal{R}(\infty)$), or $w \geq 2d + 1$ (if $L, M \in \mathcal{R}(\infty)$),
   d) or there exists two distinct lines $L, M \subseteq \mathbb{P}^2$ such that
      - $\deg(L \cap (E \cup S)) = \deg(M \cap (E \cup S)) = d + 1$,
      - $\deg((L \cup M) \cap E) + w \geq 2d + 2$,
      - $L \cap M \cap S = \emptyset$,
      - either $w = 2d$ (if and only if $a \geq 2$ and $L \cap M = P_\infty$), or $w = 2d + 1$ (if and only if $a \geq 1$ and $(L, M) \in \mathcal{R} \times \mathcal{R}(\infty)$, or $(M, L) \in \mathcal{R} \times \mathcal{R}(\infty)$), or $w = 2d + 2$ (if and only if $L, M \in \mathcal{R}$),
   e) or there exists a smooth conic $T \subseteq \mathbb{P}^2$ such that
      - $\deg(T \cap E) + w \geq 2d + 2$,
      - $w \geq 2d + 2 - \min\{2, a\}$.

Proof. See Appendix A.

Remark 33. Notice that the number of the small-weight codewords of a $C(d, a)^\perp$ code cannot be derived here from the number of their supports, as in the proofs of Theorem 17 and Theorem 18. Indeed, two small-weight codewords having the same support don’t need to be proportional.
Remark 36. Let $0 < d < q + 1$ and $0 \leq a \leq d$ be integers. Consider the Hermitian one-point code $C(d,a)$. Set $B := X(\mathbb{F}_q) \setminus \{P_\infty\}$, $E := aP_\infty$. Fix a subset $S \subseteq B$ and an integer $e > 0$. There exists a linear subspace of $C(d,a)$ with support contained in $S$ if and only if $h^1(P_2, \mathcal{I}_{E \cup S}(d)) \geq e$.

Proof. Set $V := H^0(X, \mathcal{O}_X(d)(-E))$ and $V(-S) := H^0(X, \mathcal{I}_{S \cup E}(d))$. Write $B = S \sqcup (B \setminus S)$ and $K^S$ as in the proof of Lemma 35 onto its factor $K^S$ and the inclusion $V \hookrightarrow K^B$ induce an inclusion $V/V(-S) \hookrightarrow \mathbb{F}_q^B \setminus S$. Fix $f \in K^B$ with support on $S$. By the latter assumption we have $\sum_{P \in B} f(P)g(P) = \sum_{P \in S} f(P)g(P)$ for all $g \in K^B$. The integer $i(V,S) := \|S\| - h^0(X, \mathcal{O}_X(d)(-E)) + h^0(X, \mathcal{O}_X(d)(-E - S))$ is the number of independent linear relations among the evaluations of $V$ at the points of $S$. Hence $i(V,B)$ is the dimension of the linear subspace of $C^+$ formed by the words with support on $S$. As in the proof of Proposition 12 the restriction map $\rho : H^0(P_2, \mathcal{I}_E(d)) \rightarrow H^0(X, \mathcal{O}_X(-E))$ is bijective. Obviously $\text{Ker}(\rho) = H^0(P_2, \mathcal{I}_E(d))$. Since $i(V,S)$ is the number of conditions that $S$ imposes to $H^0(X, \mathcal{O}_X(-E))$ and $S \subseteq X$, we get $i(V,S) = h^0(P_2, \mathcal{I}_E(d)) - h^0(P_2, \mathcal{I}_{S \cup E}(d))$. Since $S \cap E = \emptyset$ and $h^1(P_2, \mathcal{I}_E(d)) = 0$ (Lemma 16), we have $i(V,S) = h^1(P_2, \mathcal{I}_{S \cup E}(d))$.

Lemma 37. Consider a code $C(d,a)$ with $0 < d < q + 1$ and $0 \leq a \leq d$. Set $E := aP_\infty$. For any integer $h$ such that $1 \leq h \leq (d+2) - \text{deg}(E)$ the smallest minimum distance of a subcode $C^+_h \subseteq C^+$ of dimension $h$ is the minimal cardinality of a set $S \subseteq B$ such that $h^1(P_2, \mathcal{I}_{S \cup E}(d)) \geq h$.

Proof. Apply Lemma 34. \qed

Remark 36. Let $W$ be any projective scheme and $L$ a line bundle on it. Fix any subscheme $E \subseteq Z$. Since $Z$ is zero-dimensional we have $h^1(Z, \mathcal{I}_{E \cap L}) > 0$. Hence the restriction map $H^0(Z, L|Z) \rightarrow H^0(E, L|E)$ is surjective. It follows that if $h^1(W, \mathcal{I}_W \otimes L) > 0$ then $h^1(W, \mathcal{I}_Z \otimes L) > 0$.

Remark 37. For any effective divisor $T \subseteq P_2$ and any zero-dimensional subscheme $Z \subseteq P_2$ let $\text{Res}_T(Z)$ denote the residual scheme of $Z$ with respect to $T$, i.e. the closed subscheme of $P_2$ with $\mathcal{I}_Z : \mathcal{I}_T$ as its ideal sheaf. We have $\text{deg}(Z) = \text{deg}(Z \cap T) + \text{deg}(\text{Res}_T(Z))$. If $Z = Z_1 \cup Z_2$ then $\text{Res}_T(Z) = \text{Res}_T(Z_1) \cup \text{Res}_T(Z_2)$. If $Z$ is reduced (i.e. if $Z$ is a finite set) then $\text{Res}_T(Z) = Z \setminus Z \cap T$. For each $d \in Z$ we have an exact sequence

$$0 \rightarrow \mathcal{I}_{\text{Res}_T(Z)}(d-k) \rightarrow \mathcal{I}_Z(d) \rightarrow \mathcal{I}_{Z \cap T}(d) \rightarrow 0,$$

where $k := \text{deg}(T)$. It follows that, for each integer $i \geq 0$,

$$h^i(P_2, \mathcal{I}_Z(d)) \leq h^i(P_2, \mathcal{I}_{\text{Res}_T(Z)}(d-k)) + h^i(T, \mathcal{I}_{Z \cap T}(d)).$$

Lemma 38. Let $d > 0$ be an integer and $T \subseteq P_2$ be any divisor of degree $k \leq d + 2$. Let $Z \subseteq T$ be any zero-dimensional scheme. Then $h^1(P_2, \mathcal{I}_Z(d)) = h^1(T, \mathcal{I}_Z(d))$.

Proof. Since $Z \subseteq T$, we have $\text{Res}_T(Z) = \emptyset$. Hence the residual exact sequence (1) becomes the exact sequence

$$0 \rightarrow \mathcal{O}_{P_2}(d-t) \rightarrow \mathcal{I}_Z(d) \rightarrow \mathcal{I}_{Z,T}(d) \rightarrow 0.$$  

Use $h^1(P_2, \mathcal{O}_{P_2}(d-k)) = 0$ and deduce (since $d-k \geq -2$) that $h^2(P_2, \mathcal{O}_{P_2}(d-k)) = 0$. \qed
Lemma 39. Let $0 < d < q + 1$ and $0 \leq a \leq d$ be integers. Set $E := aP_\infty$ and fix any line $L \subseteq \mathbb{P}^2$ and a set $S \subseteq L$. If $\sharp(S) - \sharp(L \cap S) + \deg(E) - \deg(E \cap L) \leq d$, then

$$h^1(\mathbb{P}^2, \mathcal{I}_{E \cup S}(d)) = h^1(L, \mathcal{I}_{(E \cup S) \cap L}(d)) = \max(0, \deg(E \cap L) + \sharp(L \cap S) - d - 1).$$

Proof. Since $E \cap S = \emptyset$, we have

(a) $\deg(E \cup S) = \deg(E) + \deg(S)$,
(b) $\deg(\text{Res}_E(E \cup S)) = \deg(\text{Res}_E(E)) + \sharp(S) - \sharp(S \cap L)$,
(c) $\deg(L \cap (E \cup S)) = \deg(E \cap L) + \sharp(S \cap L)$.

The latter equality gives

$$h^1(L, \mathcal{I}_{(E \cup S) \cap L}(d)) = \max(0, \deg(E \cap L) + \sharp(L \cap S) - d - 1),$$

because $L \cong \mathbb{P}^1$. Since $\deg(\text{Res}_E(E \cup S)) \leq d$, we have $h^1(\mathbb{P}^2, \mathcal{I}_{\text{Res}_E(E \cup L)}(d - 1)) = 0$ (2), Lemma 34, or [5], Remarque (i) at p. 116). Hence equation (2) leads to the inequality $h^1(\mathbb{P}^2, \mathcal{I}_{E \cup S}(d)) \leq h^1(L, \mathcal{I}_{(E \cup S) \cap L}(d))$. Since $(E \cup S) \cap L \subseteq E \cup S$, Remark 36 and Lemma 38 imply $h^1(\mathbb{P}^2, \mathcal{I}_{E \cup S}(d)) \geq h^1(L, \mathcal{I}_{(E \cup S) \cap L}(d))$. \hfill \qed

Lemma 40. Let $S \subset B$ be the support of a codeword of a code $C(d, a)^\perp$ with $0 < d < q + 1$ and $0 \leq a \leq d$. Set $E := aP_\infty$ and assume the existence of a plane curve $T$ of degree $k$ such that

$h^1(\mathbb{P}^2, \mathcal{I}_{\text{Res}_T(E \cup S)}(d - k)) = 0$. Then $S \subseteq T$.

Proof. Let $V(S)$ (resp. $V(S \cap T)$) be the subcode of $C(d, a)^\perp$ formed by the codewords whose support is contained in $S$ (resp. in $S \cap T$). We have to prove that $V(S) = V(S \cap T)$. Obviously $V(S \cap T) \subseteq V(S)$. From the sequence (1) we get $h^1(\mathbb{P}^2, \mathcal{I}_{E \cup S}(d)) = h^1(\mathbb{P}^2, \mathcal{I}_{T \cap (E \cup S)}(d))$. Hence Lemma 34 applied to $S \cap T$ and to $S$ gives $V(S) \subseteq V(S \cap T)$. \hfill \qed

Proof of Theorem 32. Let us divide our proof into several steps.

1) Let $S \subseteq B$ be the support of a codeword of weight $w$ of $C(d, a)^\perp$. Observe that $\sharp(S) = w$. By Proposition 12 we have $h^1(\mathbb{P}^2, \mathcal{I}_{E \cup S}(d)) > 0$. Assume $d + 2 \leq a + w \leq 2d + 1$. By Lemma 15 there exists a line $L \subseteq \mathbb{P}^2$ (defined over $\mathbb{F}_{q^2}$) such that $\deg(L \cap (E \cup S)) \geq d + 2$. Since $\deg(\text{Res}_E(E \cup S)) \leq 2d + 1 - d - 2 \leq d$, the case $k = 1$ of Lemma 40 implies $S \subseteq L$. Since $S \neq \emptyset$ and each point of $S$ is defined over $\mathbb{F}_{q^2}$ then also $L$ is defined over $\mathbb{F}_{q^2}$. Set $W := L \cap (E \cup S)$ and note that the multiplicity of $P_\infty$ in $W$, say $e_W(P_\infty)$, must satisfies $e_W(P_\infty) \leq 1$. Indeed, if $e_W(P_\infty) \geq 2$ then Lemma 8 implies $L = L_{X, P_\infty}$, which contradicts $\deg(W) \geq d + 2$ (we assumed $a \leq d$). Hence we have $\sharp(L \cap S) \geq d + 1$ and the support $S$ consists of $w$ points in $L \cap B$ for a certain $L \subseteq \mathcal{R}(\infty) \cup \mathcal{R}$. On the other hand, let $L \in \mathcal{R}(\infty) \cup \mathcal{R}$ and let $S \subseteq L \cap B$ with $\sharp(S) = w$. Assume $a + w \leq 2d + 1$. Observe that $\sharp(S) - \sharp(L \cap S) + \deg(E) - \deg(E \cap L) \leq w - a + d$ and hence by Lemma 39 we have $h^1(\mathbb{P}^2, \mathcal{I}_{E \cup S}(d)) > h^1(\mathbb{P}^2, \mathcal{I}_{S \cup S}(d))$ for any $S' \subset S$. Apply Proposition 12 and deduce that $S$ appears as the support of a codeword of $C(d, a)^\perp$ of weight $w$.

2) Let $S \subseteq B$ be the support of a codeword of weight $w$ of $C(d, a)^\perp$. Observe that $\sharp(S) = w$. By Proposition 12 we have $h^1(\mathbb{P}^2, \mathcal{I}_{E \cup S}(d)) > 0$. Assume $2d + 2 \leq a + w \leq 3d - 1$. By Lemma 15 there exists either a line $L \subseteq \mathbb{P}^2$ (defined over $\mathbb{F}_{q^2}$) such that $\deg(L \cap (E \cup S)) \geq d + 2$, or a plane conic $T$ such that $\deg(T \cap (E \cup S)) \geq 2d + 2$.

2.1) Assume the existence of a line $L \subseteq \mathbb{P}^2$ such that $\deg(L \cap (E \cup S)) \geq d + 2$. If we have $h^1(\mathbb{P}^2, \mathcal{I}_{\text{Res}_L(E \cup S)}(d - 1)) = 0$ then Lemma 40 implies $S \subseteq L$ and we may repeat the proof of case (A). The support $S$ consists of $w$ points in $L \cap B$ for a certain
Every such a line gives a codeword of \( C(d, a) \) of weight \( w \). Now assume \( h^1(\mathbb{P}^2, \mathcal{I}_{\text{Res}_L}(E \cup S))(d - 1) > 0 \). Since \( \deg(\text{Res}_L(E \cup S)) \leq a + w - (d + 2) \leq 2(d - 1) + 1 \), Lemma 15 implies the existence of a line \( M \subseteq \mathbb{P}^2 \) such that \( \deg(M \cap \text{Res}_L(E \cup S)) \geq (d - 1) + 2 = d + 1 \). We easily see that \( M \) is defined over \( \mathbb{F}_{q^2} \) and not tangent to \( X \) in any point (use Lemma 3). Since \( \text{Res}_L(S) = S - (S \cap L) \) we get \( L \neq M \). Observe that \( \deg(L \cap M) \cap (E \cup S) = \deg(L \cap (E \cup S)) + \deg(M \cap \text{Res}_L(E \cup S)) \geq 2d + 3 \). Since neither \( L \) or \( M \) are tangent to \( X \) we have \( \deg(E \cap (L \cup M)) \leq 2 \), with equality if and only if \( L, M \in \mathcal{R}(\infty) \). In this case we have \( w \geq 2d + 1 \) and it will be (Lemma 3) \( d \leq q - 1 \) or \( d = q \) and \( \deg(E \cap (L \cup M)) \leq 1 \). Since \( \deg(\text{Res}_{L \cup M}(E \cup S)) \leq 3d - 1 - (2d + 3) < d - 1 \), we have \( h^1(\mathbb{P}^2, \mathcal{I}_{\text{Res}_{L \cup M}(E \cup S)})(d - 2) = 0 \) and applying Lemma 40 with \( k = 2 \) we deduce \( \mathcal{S} \subseteq L \cup M \).

(2.ii) Assume that there is no line \( L \subseteq \mathbb{P}^2 \) such that \( \deg(L \cap (E \cup S)) \geq d + 2 \). Then there is a plane conic \( T \) (not neccessarily smooth) such that \( \deg(T \cap (E \cup S)) \geq 2d + 2 \). Since \( \deg(\text{Res}_T(E \cup S)) \leq 3d - 1 - (2d + 2) < d - 1 \) we get \( h^1(\mathbb{P}^2, \mathcal{I}_{\text{Res}_T(E \cup S)})(d - 2) = 0 \). Lemma 40 implies \( \mathcal{S} \subseteq T \). Assume that \( T \) is reducible, say \( T = L \cup M \). Since, by assumption, \( \deg(L \cap (E \cup S)) \leq d + 1 \) and \( \deg(M \cap (E \cup S)) \leq d + 1 \) we have \( L \neq M \). Since \( 2d + 2 = \deg(L \cap (E \cup S)) + \deg(M \cap (E \cup S)) \), we get (by assumption) \( \deg(L \cap (E \cup S)) = \deg(M \cap (E \cup S)) = d + 1 \) and \( L \cap M \cap S = \emptyset \). Moreover, if \( P_\infty \) appears in \( L \cap M \) then \( a \geq 2 \). Lemma 3 implies that neither \( L \) or \( M \) can be tangent to \( X \) at any point. Since we assumed \( a < d \) then we are done by Lemma 39. Now assume that \( T \) is smooth. Since we proved that \( S \subseteq T \), Lemma 9 gives \( w = \deg(T \cap S) \geq 2d + 2 - \min(2, a) \).

The proof is concluded.

**Conclusion**

The paper describes Hermitian one-point codes from a purely geometric point of view and provides a geometric interpretation of the dual minimum distance of such codes. The supports of the dual minimum-weight codewords are geometrically characterized, leading to precise formulas for their number. Possible improvements of the dual codes of Hermitian one-point codes are easily controlled by means of the geometric setup here presented. The well-known geometry of tangent lines to the Hermitian curve is applied to study also some small-weight codewords and their supports.

**References**

[1] E. Ballico, A. Ravagnani, A zero-dimensional cohomological approach to Hermitian codes. http://arxiv.org/abs/1202.0894.
[2] A. Bernardi, A. Gimigliano, M. Idà, Computing symmetric rank for symmetric tensors. J. Symbolic. Comput. 46(1), 34–53 (2011).
[3] A. Couvreur, The dual minimum distance of arbitrary dimensional algebraic-geometric codes. J. Algebra 350(1), 84–107 (2012).
[4] I. M. Duursma, R. Kirov, Improved Two-Point Codes on Hermitian Curves. IEEE Trans. Inf. Theory, 57(7), 4469–4476 (2011).
[5] Ph. Ellia, Ch. Peskine, Groupes de points de \( \mathbb{P}^2 \): caractère et position uniforme. Algebraic geometry (L’Aquila, 1988), 111–116, Lecture Notes in Math., 1417, Springer, Berlin, 1990.
[6] G.-L. Feng, T. R. N. Rao, Improved geometric Goppa codes. I. Basic theory. IEEE Trans. Inform. Theory, vol. 41, no. 6, part 1, pp. 1678–1693 (1995). Special issue on algebraic geometry codes.
[7] J. W. P. Hirschfeld, Projective Geometries over Finite Fields. Clarendon Press, Oxford, 1979.
[8] G. Korchmáros, G. P. Nagy, Hermitian codes from higher degree places. http://arxiv.org/abs/1206.4480.

[9] K. Lee, M. E. O'Sullivan, Algebraic Soft-Decision Decoding of Hermitian Codes. IEEE Transaction on Information Theory, 56 (2010), no 6, pp. 2587 – 2600.

[10] K. Lee, M. E. O'Sullivan, List Decoding of Hermitian Codes using Groebner Bases. Journal of Symbolic Computation, 40 (2009), 12, pp. 1662 – 1675.

[11] C. Marcolla, M. Pellegrini, M. Sala On the Hermitian curve, its intersections with some conics and their applications to affine-variety codes and Hermitian codes. http://arxiv.org/abs/1208.1627(2012).

[12] C. Marcolla, M. Pellegrini, M. Sala On the weights of affine-variety codes and some Hermitian codes. WCC 2011, Workshop on coding and cryptography, 273–282 (2011).

[13] C. Munuera, R. Pellikaan, Equality of geometric Goppa codes and equivalence of divisors. Journal of Pure and Applied Algebra, 90, 229–252 (1993).

[14] M. E. O'Sullivan, Decoding of Hermitian Codes: The Key Equation and Efficient Error Evaluation. IEEE Transactions on Information Theory, 46 (2000), no. 2, pp. 512 – 523.

[15] S. A. Stepanov, Codes on Algebraic Curves. Springer, 1999.

[16] Stichtenoth, Algebraic function fields and codes, Second Edition. Springer-Verlag, 2009.

[17] K. Yang, P. V. Kumer, On the True Minimum Distance of Hermitian Codes. Coding Theory and Algebraic Geometry (Stichtenoth and Tsfasman editors), Springer-Verlag, 1992.