Velocity and Heat Flow in a Composite Two Fluid System

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We describe the stress energy of a fluid with two unequal stresses and heat flow in terms of two perfect fluid components. The description is in terms of the fluid velocity overlap of the components, and makes no assumptions about the equations of state of the perfect fluids. The description is applied to the metrics of a conformally flat system and a black string.

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I. INTRODUCTION

Descriptions of fluid systems used in general relativity have become more complex as we acquire new data about our universe. Simple perfect fluids are still useful models in many cases [1], but, increasingly, extensions of the perfect fluid stress-energy to include anisotropy and fluid interactions are necessary considerations. The stress-energy for an anisotropic fluid can easily be written in terms of metric based tetrads $[U_a, R_a, \Theta_a, \Phi_a]$

$$T_{ab} = (\varepsilon) \hat{U}_a \hat{U}_b + (P_r) \hat{R}_a \hat{R}_b + (P_\theta) \hat{\Theta}_a \hat{\Theta}_b + (P_\phi) \hat{\Phi}_a \hat{\Phi}_b.$$  \hspace{1cm} (1)

without addressing the physical origins of the anisotropy. Fluid anisotropy has many physical causes [2, 3]. For a low density fluid, anisotropy can be modeled by multi-perfect fluid descriptions with differences in the component fluid velocities [4] or equations of state [5] generating the anisotropy. One method of incorporating these differences is through stress-energy equivalence [6, 7] where a composite system can be written as the sum of the two component perfect fluids:

$$(\varepsilon) \hat{U}_a \hat{U}_b + (P_r) \hat{R}_a \hat{R}_b + (P_\theta) \hat{\Theta}_a \hat{\Theta}_b + (P_\phi) \hat{\Phi}_a \hat{\Phi}_b = (\varepsilon_1 + P_1) \hat{U}_a^{(1)} \hat{U}_b^{(1)} + (\varepsilon_2 + P_2) \hat{U}_a^{(2)} \hat{U}_b^{(2)} + (P_1 + P_2) g_{ab}$$ \hspace{1cm} (2)

This form allows the component fluid motions and individual equations of state to be built into the composite fluid description and has parameter freedom for modeling assumptions.
A related method is due to Letelier [8], where he explicitly rotates the component fluid velocities to create a new composite tetrad. Using the new tetrad, the composite stress-energy tensor describes a fluid with one anisotropic stress, $\sigma$, associated with a direction of fluid anisotropy $\Upsilon_a$.

$$T_{ab} = (\varepsilon + \Pi) \hat{U}_a \hat{U}_b + (\sigma - \Pi) \hat{\Upsilon}_a \hat{\Upsilon}_b + (\Pi) g_{ab}$$  \hspace{1cm} (3)

Letelier’s stress-energy form follows from an assumption of zero heat flow in the composite fluid and implies a relation between the stress-energies of the component fluids, $\varepsilon_1 + P_1 = const \times (\varepsilon_2 + P_2)$. Not all physical examples will obey this stress-energy relation or will have zero heat flow, and Eq.(3) can follow from other physical assumptions. Multi-fluid models of complex fluids are increasingly being used, not only in formal general relativity [9] but also in other physics sub areas. Two and three fluid descriptions cover phenomena like multi charge species in magnetized plasmas [10], superfluids [11–13], cosmological models [14–17], particles in heavy ion collisions [18], Fermi-Bose fluids [19] and hydrodynamic nuclear models [20]. A need has emerged for a range of two fluid descriptions to use as a basis for n-fluid generalizations [21–25]. In this paper we suggest an anisotropic two fluid stress form based on the overlap of the component fluid velocity vectors rather than a stress-energy assumption. The description allows heat flow along the direction of anisotropy in the composite stress-energy and has no initial stress-energy assumptions. In the next section we set up the stress-energy description. The relation between the 4-velocity overlap and the stress-energy is discussed in the third section, and some examples are discussed in the fourth part of the paper.

**II. COMBINING COMPONENT FLUIDS**

The stress-energy form

Consider a manifold which contains two perfect fluid flows and metric $g_{ab}$. A composite stress-energy for two perfect fluids is

$$T_{ab} = (\varepsilon_1 + P_1) \hat{U}_a^{(1)} \hat{U}_b^{(1)} + (\varepsilon_2 + P_2) \hat{U}_a^{(2)} \hat{U}_b^{(2)} + g_{ab}(P_1 + P_2).$$  \hspace{1cm} (4)

In order to express the composite stress-energy in a simpler form, following Letelier [8], we map the timelike unit vectors $[\hat{U}_a^{(1)}, \hat{U}_a^{(2)}]$ to an un-normed pair $[U_a^*, \Upsilon_a^*]$ where $U_a^*$ is timelike
and \( \Upsilon_a^* \) is spacelike. A general unimodular transformation between these two sets of vectors can be written as

\[
\begin{bmatrix}
\hat{U}_a^{(1)} \\
\hat{U}_a^{(2)}
\end{bmatrix} =
\begin{bmatrix}
A \cos \alpha & -B \sin \alpha \\
D \sin \alpha & C \cos \alpha
\end{bmatrix}
\begin{bmatrix}
\Upsilon_a^* \\
\end{bmatrix}
\]

(5)

with unit transformation determinant \( AC \cos^2 \alpha + BD \sin^2 \alpha = 1 \) and transformation inverse

\[
\begin{bmatrix}
\hat{U}_a^{(1)} \\
\hat{U}_a^{(2)}
\end{bmatrix} =
\begin{bmatrix}
C \cos \alpha & B \sin \alpha \\
-D \sin \alpha & A \cos \alpha
\end{bmatrix}
\begin{bmatrix}
\hat{U}_a^{(1)} \\
\hat{U}_a^{(2)}
\end{bmatrix}
\]

(6)

Since \( U^{*a} \) and \( \Upsilon^{*a} \) are not normalized, two of the constants can be absorbed into their definitions. Choosing \( A = C = 1 \), the unit determinant condition is

\[
\cos^2 \alpha + BD \sin^2 \alpha = 1
\]

(7)

The rotation angle \( \alpha \) is fixed by requiring \( g^{ab} \Upsilon_a^* U_b^* = 0 \), with \( U_a^* \) timelike, \( \Upsilon_a^* \) spacelike and \( g^{ab} \hat{U}_a^{(2)\beta} \hat{U}_b^{(2)\beta} = g^{a\beta} \hat{U}_a^{(1)\beta} \hat{U}_b^{(1)\beta} = -1 \). The orthogonality condition implies

\[
-(B - D) \cos \alpha \sin \alpha + \hat{U}^{(2)\beta} \hat{U}_a^{(1)\beta} (\cos^2 \alpha - BD \sin^2 \alpha) = 0,
\]

(8)

where \( \hat{U}^{(2)\beta} \hat{U}_a^{(1)\beta} \) is the velocity overlap. There are several special values of \( B \) which can be eliminated from the parameter range by the requirement that \( \Upsilon_a^* \) be spacelike. Condition \( B = 0 \) in determinant Eq.(7) implies \( \alpha = 0, \pi \). From the transformation equation Eq.(5), this choice also identifies both \( U_a^* \) and \( \Upsilon_a^* \) as timelike vectors so that overlap equation (8) is not valid for values \( B = 0, \alpha = 0, \pi \). The choice \( B = \pm 1 \) requires \( D = \pm 1 \). The overlap equation thus becomes \( \hat{U}^{(2)\beta} \hat{U}_a^{(1)\beta} \cos 2\alpha = 0 \), with \( \alpha = \pi/4, 3\pi/4 \). These values of \( \alpha \) in \( \cos 2\alpha = 0 \) are also excluded by the spacelike condition on \( \Upsilon_a^* \). From Eq.(6) with \( B = \pm 1 \), we have

\[
\Upsilon_a^* = \mp \sin \alpha \hat{U}_a^{(1)} + \cos \alpha \hat{U}_a^{(2)}
\]

(9)

\[
U^{(1,2)} := \hat{U}_a^{(1)} \hat{U}_a^{(2)a}
\]

(10)

\[
\Upsilon_a^* \Upsilon^{*a} = -1 \pm \sin 2\alpha [-U^{(12)}]
\]

(11)

The velocity overlap \( U^{(1,2)} \) must be negative, since it is the product of two future pointing timelike vectors. For \( \Upsilon_a^* \) to be spacelike requires \( \pm \sin 2\alpha [-U^{(12)}] > 1 \), or \( \alpha < \pi/4 \) for \( B > 0 \). The ranges we will consider are \( B \neq 0,1 \) and \( 0 < \alpha < \pi/4 \). These ranges and the unit determinant condition require \( BD = 1 \). The two normalized unit vectors are

\[
\hat{U}^b = \frac{U^{*b}}{\sqrt{-U^{*a}U_a^*}}, \quad \hat{T}^b = \frac{\Upsilon^{*b}}{\sqrt{\Upsilon^{*a}\Upsilon_a^*}}.
\]

(12)
with norms

\[
\begin{align*}
\bar{N}^2 &= \Upsilon_a^* \Upsilon^a = \frac{1}{2} \left[ \frac{1 - B^2}{B^2 \cos 2\alpha} - \left( \frac{1 + B^2}{B^2} \right) \right] \\
N^2 &= -U^a U_a^* = \frac{1}{2} \left[ \frac{1 - B^2}{\cos 2\alpha} + (1 + B^2) \right]
\end{align*}
\]

(13)

III. OVERLAP AND STRESS-ENERGY

Stress-Energy

The rotation angle can be expressed in terms of the 4-velocity overlap.

\[
\tan 2\alpha = -\frac{2B}{1 - B^2} U^{(1,2)}
\]

(14)

\[
B \neq 0, 1 \quad 0 < \alpha < \pi/4
\]

The composite stress-energy tensor in Eq.(11) can now be written as

\[
T_{ab} = (\varepsilon + \Pi) \hat{U}_a \hat{U}_b + (\sigma - \Pi) \hat{\Upsilon}_a \hat{\Upsilon}_b + (\Pi) g_{ab} + Q_a \hat{U}_b + Q_b \hat{U}_a
\]

(15)

with fluid parameters

\[
\Pi = p_1 + p_2
\]

(16a)

\[
Q_a = \hat{\Upsilon}_a \bar{N}\bar{N}[B^{-1}(\varepsilon_2 + P_2) - B(\varepsilon_1 + P_1)] \sin \alpha \cos \alpha
\]

(16b)

\[
\varepsilon + \Pi = \bar{N}^{-2}[(\varepsilon_1 + P_1) \cos^2 \alpha + B^{-2}(\varepsilon_2 + P_2) \sin^2 \alpha]
\]

(16c)

\[
\sigma - \Pi = \bar{N}^{-2}[B^2(\varepsilon_1 + P_1) \sin^2 \alpha + (\varepsilon_2 + P_2) \cos^2 \alpha]
\]

(16d)

and \(B\) an unspecified constant except for \(B \neq 0, 1\).

Heat Flow and \(B\)

Letelier’s [8] choice, \(B^2 = (\varepsilon_2 + p_2)/(\varepsilon_1 + p_1)\) reproduces the original two fluid stress-energy with \(Q_a = 0\). However, the underlying physics of the description can depend on both the velocity overlap and general equations of state in the component fluids. For example, if the two fluids move together with the same 4-velocity, the overlap is \(-1\) and we have \(\tan(2\alpha_0) = 2B/(1 - B^2)\). Another way of choosing \(B\) that allows non zero heat flow is to use the \(\hat{U}_a^{(1)} = \hat{U}_a^{(2)}\) condition to define \(B\) as

\[
B = \tan \alpha_0
\]

(17)
with $\alpha = \alpha_0$ producing the 'aligned' fluid. Substituting the alignment condition in Eq. (9) requires the aligned fluid to have zero anisotropy vector, $\Upsilon^*_a = 0$ [recall, $U^*_a = \cos \alpha \hat{U}^{(1)}_a + B \sin \alpha \hat{U}^{(2)}_a$, $\Upsilon^*_a = (-1/B) \sin \alpha \hat{U}^{(1)}_a + \cos \alpha \hat{U}^{(2)}_a$]. The conditions on $B$ set the positive range $0 < \alpha_0 < \pi/4$, such that $\alpha_0 \leq \alpha < \pi/4$. The velocity overlap is

$$\tan 2\alpha = -U^{(1,2)} \tan 2\alpha_0$$  \hspace{1cm} (18)

The general normalizations are

$$\tilde{N}^2 = \Upsilon^*_a \Upsilon^* a = \frac{1}{2 \sin^2 \alpha_0} \left[ \frac{\cos 2\alpha_0}{\cos 2\alpha} - 1 \right]$$  \hspace{1cm} (19)

$$N^2 = -U^{*a} U^*_a = \frac{1}{2 \cos^2 \alpha_0} \left[ \frac{\cos 2\alpha_0}{\cos 2\alpha} + 1 \right]$$  \hspace{1cm} (20)

Note that both $\tilde{N}$ and $\Upsilon^*_a$ have zero values when the fluid is aligned with $\alpha = \alpha_0$. The fluid parameters are

$$\Pi = P_1 + P_2$$  \hspace{1cm} (21a)

$$\varepsilon + \Pi = \frac{2}{\sin^2 \alpha_0} \left[ \frac{\cos 2\alpha_0}{\cos 2\alpha} + 1 \right] \left[ (\varepsilon_1 + P_1) \sin^2 \alpha_0 \cos^2 \alpha + (\varepsilon_2 + P_2) \cos^2 \alpha_0 \sin^2 \alpha \right]$$  \hspace{1cm} (21b)

$$\sigma - \Pi = \frac{2}{\sin^2 \alpha_0} \left[ \frac{\cos 2\alpha_0}{\cos 2\alpha} - 1 \right] \left[ (\varepsilon_1 + P_1) \sin^2 \alpha_0 \sin^2 \alpha + (\varepsilon_2 + P_2) \cos^2 \alpha_0 \cos^2 \alpha \right]$$  \hspace{1cm} (21c)

The heat flow vector is

$$Q_a = \left( \frac{\sin \alpha \cos \alpha}{2 \sin^2 \alpha_0} \sqrt{\left( \frac{\cos 2\alpha_0}{\cos 2\alpha} \right)^2 - 1} \left[ (\varepsilon_2 + P_2) - \tan^2 \alpha_0 (\varepsilon_1 + P_1) \right] \right) \hat{\Upsilon}_a$$  \hspace{1cm} (22)

and is zero for $\alpha = \alpha_0$.

**The $\alpha, \alpha_0$ boundary**

The $\hat{U}^{(1)}_a = \hat{U}^{(2)}_a$ alignment condition has stress-energy relations $Q = 0$ and $\varepsilon + \Pi = \varepsilon_1 + P_1 + \varepsilon_2 + P_2$. When $\alpha$ is close to $\alpha_0$, the 4-velocities should be only slightly different and the fluids should show small deviations from the aligned relations. There are three parameter boundary cases: (1) both $\alpha$ and $\alpha_0$ near zero, (2) both $\alpha$ and $\alpha_0$ near $\pi/4$, and (3) $\alpha_0$ near zero and $\alpha$ just under $\pi/4$.

One expects the first two cases to describe a composite fluid only slightly different from the aligned composite. For the first case, $\alpha_0 = \delta_0$, $\alpha = \delta$, choose $\alpha_0$ and $\alpha$ small but of the
same order with $\delta_0 < \delta$. We have $R \approx 1 + 2\delta^2 - 2\delta_0^2$ and $\tan 2\alpha / \sqrt{R^2 - 1} \approx \delta / \sqrt{\delta^2 - \delta_0^2} \gg 1$. The fluid parameters are

\[-U^{(12)} \sim \delta / \delta_0 \quad (23a)\]
\[\varepsilon + \Pi \sim (\varepsilon_1 + P_1) + (\varepsilon_2 + P_2)(\delta / \delta_0)^2 \quad (23b)\]
\[\sigma - \Pi \sim [(\delta / \delta_0)^2 - 1](\varepsilon_2 + P_2) \quad (23c)\]

with heat flow

\[Q \sim (\delta / \delta_0)\sqrt{(\delta / \delta_0)^2 - 1}(\varepsilon_2 + P_2)\]

The second case, $\alpha = \pi/4 - \delta$, $\alpha_0 = \pi/4 - \delta_0$, is very similar to the first with $\delta_0 > \delta$. The velocity overlap is again, almost aligned, $-U^{(1,2)} \sim \delta_0 / \delta$, but the composite fluid relations to the component fluids are multiples of the aligned fluid description.

\[Q \sim \frac{1}{2} \sqrt{(\delta_0 / \delta)^2 - 1}[(\varepsilon_2 + P_2 - (\varepsilon_1 + P_1))] \quad (24a)\]
\[\varepsilon + \Pi \sim \frac{1}{2} [\delta_0 / \delta + 1][\varepsilon_1 + P_1 + \varepsilon_2 + P_2] \quad (24b)\]
\[\sigma - \Pi \sim \frac{1}{2} [\delta_0 / \delta - 1][\varepsilon_1 + P_1 + \varepsilon_2 + P_2] \quad (24c)\]

The stress-energy in both these cases obeys the $U_a^{(1)} = U_a^{(2)}$ condition to lowest order, $\varepsilon - \sigma + 2\Pi \sim \varepsilon_1 + P_1 + \varepsilon_2 + P_2$. The physical difference between the two cases can be explained by the composite * vector relation to the component fluid velocities, Eq.(6). For parameter values near zero, the component fluid velocity is dominated by the first fluid, $U_a^* \sim \hat{U}_a^{(1)}$ with direction of anisotropy $\Upsilon_a^* \sim - (\delta / \delta_0)\hat{U}_a^{(1)} + \hat{U}_a^{(2)}$. For parameter values near $\pi/4$, $U_a^* \sim (\hat{U}_a^{(1)} + \hat{U}_a^{(2)}) / \sqrt{2}$ and $\Upsilon_a^* \sim (-\hat{U}_a^{(1)} + \hat{U}_a^{(2)}) / \sqrt{2}$. The straight combination of velocities in the second case explaining its close similarity to the aligned fluid case. The third case is an example of strong non-alignment. When $\alpha$ and $\alpha_0$ are at opposite ends of the parameter range, $\alpha_0 = \delta_0$, $\alpha = \pi/4 - \delta$, we have $R = \cos 2\delta_0 / \sin 2\delta \gg 1$ and $\tan(\pi/2 - 2\delta) / \sqrt{R^2 - 1} \approx 1$. The fluid parameters obey a very different relation than the aligned fluid with $\varepsilon + 2\Pi - \sigma \sim 0$, rather than the sum of the component stress-energy. We also have

\[\varepsilon + \Pi \sim \frac{(\varepsilon_2 + P_2)}{8\delta_0^2} \quad (25a)\]
\[\sigma - \Pi \sim \frac{(\varepsilon_2 + P_2)}{8\delta_0^2} \quad (25b)\]
\[Q \sim \frac{(\varepsilon_2 + P_2)}{8\delta_0^2} \quad (25c)\]
For this case the $\ast$ vectors are related to the component velocities by $U_{a}^{\ast} \sim (\hat{U}_{a}^{(1)} + \delta \hat{U}_{a}^{(2)})/\sqrt{2}$ and $\Upsilon_{a}^{\ast} \sim (-\hat{U}_{a}^{(1)}/\delta + \hat{U}_{a}^{(2)})/\sqrt{2}$. In the next section we give some metric examples.

IV. APPLICATIONS

As an application of the two perfect fluid description we consider three different examples. The first two are metric based with an anisotropic stress-energy following from the field equations. The inverse of Eqs. (21b, 21c) give the component stress-energies in terms of the composite descriptions:

\begin{align*}
\varepsilon_{1} + P_{1} &= \cos^{2} \alpha_{0} \left[ (\varepsilon + \Pi) \frac{\cos 2\alpha + 1}{\cos 2\alpha_{0} + \cos 2\alpha} + (\sigma - \Pi) \frac{\cos 2\alpha - 1}{\cos 2\alpha_{0} - \cos 2\alpha} \right] \quad (26) \\
\varepsilon_{2} + P_{2} &= \sin^{2} \alpha_{0} \left[ (\varepsilon + \Pi) \frac{\cos 2\alpha - 1}{\cos 2\alpha_{0} + \cos 2\alpha} + (\sigma - \Pi) \frac{\cos 2\alpha + 1}{\cos 2\alpha_{0} - \cos 2\alpha} \right] \quad (27)
\end{align*}

The two metric examples are a conformally flat spacetime and a black string. The stress-energy from the field equations for both spacetimes has $\varepsilon + \Pi = 0$, and we can describe the related component perfect fluids. The third example uses two dusty component perfect fluids and a single anisotropic stress with no heat flow. The component fluid density relations are examined along with the equation of state in the composite. For this example, the Letelier description and the description in this paper coincide.

Example: A conformally flat spacetime

A simple conformally flat spacetime has metric and field generated fluid parameters as seen by a comoving observer $\hat{U}_{a}^{\ast} = (e^{-2az}, 0, 0, 0)$.

\begin{align*}
\left.ds^{2}\right|_{z} &= e^{2az}(-dt^{2} + dr^{2} + r^{2}d\phi^{2} + dz^{2}), \quad (28a) \\
8\pi\varepsilon &= -a^{2}e^{-2az}, \quad (28b) \\
8\pi\Pi &= a^{2}e^{-2az}, \quad (28c) \\
\Pi &= P_{r} = P_{\phi}, \quad (28d) \\
8\pi\sigma &= 8\pi P_{z} = 3a^{2}e^{-2az}, \quad (28e)
\end{align*}

with the anisotropy in the $z$-direction, $\Upsilon_{a}^{\ast} = (0, 0, 0, e^{-2az})$. The $z$-dependent negative density in this solution does not lend itself to a physical description outside of a cosmological constant.
or Casimir effects. The two fluid description of the composite fluid model has the advantage of explaining the negative composite density in terms of tension in the first fluid. For this fluid $\varepsilon + \Pi = 0$ and $8\pi(\sigma - \Pi) = 2a^2e^{-2az}$. From Eq. (21), the component fluid parameters that could create a composite fluid are

$$
\varepsilon_1 + P_1 = \frac{2a^2e^{-2az}(\cos 2\alpha - 1)}{8\pi} \frac{\cos^2 \alpha_0}{(\cos 2\alpha_0 - \cos 2\alpha)}
$$

$$
\varepsilon_2 + P_2 = \frac{2a^2e^{-2az}(\cos 2\alpha + 1)}{8\pi} \frac{\sin^2 \alpha_0}{(\cos 2\alpha_0 - \cos 2\alpha)}
$$

The second fluid could have both positive stress and density. The first fluid, if it has positive density, must describe a fluid with tension rather than pressure. The heat flow is axial

$$
Q_a = \left( \frac{a^2e^{-2az}\sin 2\alpha}{8\pi} \right) \sqrt{\frac{\cos 2\alpha_0 + \cos 2\alpha}{\cos 2\alpha_0 - \cos 2\alpha}} \hat{\Upsilon}_a
$$

The 4-velocities of the component fluid come directly from the transformation equations and involve, unsurprisingly, a coordinate velocity in the z-direction.

$$
U^{(1)a} = [e^{-az}N \cos \alpha, 0, 0, -e^{-az}\tilde{N} \tan \alpha_0 \sin \alpha]
$$

$$
U^{(2)a} = [e^{-az}N \frac{\sin \alpha}{\tan \alpha_0}, 0, 0, e^{-az}\tilde{N} \cos \alpha]
$$

$\hat{U}^{(i)a}\hat{U}^{(i)a} = -1$ was imposed in Eq. (12). Both $U^{(1)a}$ and $U^{(2)a}$ are expanding, accelerating and shear-free. Requiring the velocity overlap, Eq. (10) to be constant, $U^{(12)} = \text{const}$, can be restrictive for shear-free fluids, requiring acceleration and expansion (see Appendix A). That restriction is met for this example.

**Example: A black string**

A similar example to the conformal metric is the black string of Lemos and Zanchin [26], with metric

$$
ds^2 = -(\Lambda r^2 - m)dt^2 + \frac{dr^2}{(\Lambda r^2 - m)} + r^2d\phi^2 + dz^2
$$

and comoving stress-energy following from the field equations

$$
8\pi\varepsilon = -\Lambda,
$$

$$
8\pi\Pi = \Lambda,
$$

$$
8\pi\sigma = 8\pi P_z = 3\Lambda.
$$
From the first example, the replacement $a^2 e^{-2az} \to \Lambda$, gives the component stress-energy and heat flow for this case. The direction of anisotropy is along the string axis. Here, since the cosmological constant can be negative and does not depend on position, a negative energy density is possible. The fluid matter obeys $\varepsilon + \Pi = 0$ and $\sigma = 3\Pi$. The 4-velocities for the black string components are

\[ U^{(1)a} = \left[ \frac{N}{\sqrt{\Lambda r^2 - m}} \cos \alpha, \ 0, \ 0, \ -\tilde{N} \tan \alpha_0 \sin \alpha \right] \quad (36) \]

\[ U^{(2)a} = \left[ \frac{N}{\sqrt{\Lambda r^2 - m}} \sin \alpha, \ 0, \ 0, \ \tilde{N} \cos \alpha \right] \quad (37) \]

These component velocities are expansion-free but have acceleration, shear, and vorticity.

**Example: Linear composite equation of state**

For the third example consider a composite fluid with $\Pi = 0$, and two component dust fluids, $P_1 = P_2 = 0$. The composite fluid has a linear equation of state related to $\alpha$ and $\alpha_0$:

\[ \frac{\varepsilon}{\cos 2\alpha_0 / \cos 2\alpha} + 1 = \frac{\sigma}{\cos 2\alpha_0 / \cos 2\alpha} - 1 \quad (38) \]

With Eq.(21) we have for the composite fluid

\[ \varepsilon_1 \sin^2 \alpha_0 = \varepsilon_2 \cos^2 \alpha_0 \quad (39) \]

and this example has the Letelier 

stress-energy relation with zero heat flow.

\[ \varepsilon = \frac{\varepsilon_1}{2 \cos^2 \alpha_0} \left[ \frac{\cos 2\alpha_0}{\cos 2\alpha} + 1 \right] \quad (40) \]

\[ \sigma = \frac{\varepsilon_1}{2 \cos^2 \alpha_0} \left[ \frac{\cos 2\alpha_0}{\cos 2\alpha} - 1 \right] \quad (41) \]

The composite fluid has a dusty equation of state for $\alpha$ close to $\alpha_0$, $R = (\cos 2\alpha_0 / \cos 2\alpha) \sim 1$, changing to a stiff equation of state for large $R$.

**V. DISCUSSION**

In conclusion, we have presented an anisotropic fluid recipe based on component velocity overlap. It is useful in describing anisotropic fluids in terms of possible perfect fluid components, and may also be of use in developing broader descriptions of multi-fluid models. The
description includes heat flow driven by fluid velocity non-alignment. The heat flow is along
the direction of anisotropy in the stress-energy form. Heat flows with several spatial vector
components are possible \[9\], especially if spatial tetrads are chosen to describe acceleration,
or a null-vector construction, or other non-heat related parameters. Including more general
heat descriptions is an idea for future work. The fluids considered here are non-interacting;
another possible extension is to component fluids with non zero, but balancing stress-energy
divergence \[32–35\], and to acoustic phenomena related to the heat flow \[36–38\].

Appendix A: Restrictions on the Component Fluids

The behavior of the composite fluid is determined by the component velocity overlap and
equation of state. The velocity overlap equation following from tetrad orthogonality can
be restrictive since it requires the velocity overlap to be constant, \( U^{(1,2)} = \text{const} \). This
will impose restrictions on the component fluids. Consider the covariant derivative of the
constant velocity overlap \( \nabla_b U^{(1,2)} = 0 \)

\[
\dot{U}^{(1)a} U^{(2)}_{a:b} = -U^{(1)a}_b \dot{U}^{(2)}
\]  

(A1)

Each of the velocity derivatives can be expanded in terms of its acceleration, expansion,
shear and vorticity \( \kappa^i, \Theta^i, \sigma^i_{ab}, \omega^i_{ab} \) and projection operator \( h^{(i)}_{ab} = g_{ab} + \hat{U}^i_a \hat{U}^i_b \).

\[
[-\hat{U}^{(1)}_a \hat{U}^{(1)}_b + \sigma_{1ab} + \omega_{1ab} + \frac{\theta_1}{3} h^{(1)}_{ab}] \dot{U}^{(2)} = -\hat{U}^{(1)a} [-\hat{U}^{(2)}_b \dot{U}^{(2)}_b + \sigma_{2ab} + \omega_{2ab} + \frac{\theta_2}{3} h^{(2)}_{ab}] \]  

(A2)

The fluid acceleration can be parameterized with the Frenet tetrad associated with each
velocity vector \( \hat{U}^{(i)}_a, \hat{A}^{(i)}_a, \hat{B}^{(i)}_a, \hat{C}^{(i)}_a \). The acceleration lies along the vector \( \hat{A}^{(i)}_a \) and can be
written as \( \dot{U}^{(i)}_a = U^{(i)b} \dot{U}^{(i)b} = \kappa A^{(i)}_a \). The acceleration for fluid one can be isolated with \( U^{(1)b} \)
multiplication and similarly, for fluid two with \( U^{(2)b} \) multiplication:

\[
k_1 \hat{A}^{(1)}_a \dot{U}^{(2)a} = k_2 \hat{A}^{(2)}_a \dot{U}^{(1)}_a U^{(1,2)} - \sigma_{2ab} \dot{U}^{(1)a} \dot{U}^{(1)b} - \frac{\theta_2}{3} [-1 + (U^{(1,2)})^2]
\]

\[
k_2 \hat{A}^{(2)}_a \dot{U}^{(1)}_a = k_1 \hat{A}^{(1)}_a \dot{U}^{(2)}_a U^{(1,2)} - \sigma_{1ab} \dot{U}^{(2)a} \dot{U}^{(2)b} - \frac{\theta_1}{3} [-1 + (U^{(1,2)})^2]
\]

If the component fluids are shear-free we have

\[
k_1 \hat{A}^{(1)}_a \dot{U}^{(2)a} = k_2 \hat{A}^{(2)}_a \dot{U}^{(1)}_a U^{(1,2)} - \frac{\theta_2}{3} [-1 + (U^{(1,2)})^2]
\]  

(A3)

\[
k_2 \hat{A}^{(2)}_a \dot{U}^{(1)}_a = k_1 \hat{A}^{(1)}_a \dot{U}^{(2)}_a U^{(1,2)} - \frac{\theta_1}{3} [-1 + (U^{(1,2)})^2]
\]  

(A4)
If the two component fluids are shear free and have no expansion then they satisfy

\[
\kappa_1 A_1^{(1)} U_1^{(2)a} = \kappa_1 A_1^{(1)} U_1^{(2)a} (U_1^{(1,2)})^2 \\
\kappa_2 A_2^{(2)} U_2^{(1)a} = \kappa_2 A_2^{(2)} U_2^{(1)a} (U_1^{(1,2)})^2
\]

If the expansions are zero, this can only be satisfied by identical component fluid velocities, \(U_1^{(1,2)} = -1\) or by zero accelerations, \(\kappa_1, \kappa_2 = 0\). The component fluids must be unaccelerated or aligned. If the fluids are not aligned or if there is acceleration, one (or both) must have expansion if they are shear-free. In the first metric example, the component velocities Eqs.(32,33) are shear and vorticity free, and have both expansion and acceleration. The black string component fluids have zero expansion but non-zero radial acceleration, with shear components \(\sigma_{0r}\) and \(\sigma_{zr}\), and non-zero vorticity. Collins [27] has conjectured that, if shear-free perfect fluids obey a barotopic equation of state, then either the expansion or vorticity must be zero. If the conjecture is true [28–31], the shear-free, accelerated, unaligned component fluids must have zero vorticity or a non-barotopic equation of state. The conjecture holds for the conformal metric and does not apply to the black string.

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