Covering with Chang models over derived models*†‡

Grigor Sargsyan

October 13, 2021

Abstract

We present a covering conjecture that we expect to be true below super-
strong cardinals. We then show that the conjecture is true in hod mice. This
work is a continuation of the work that started in [4], and the main conjecture
of the current paper is a revision of the UB-Covering Conjecture of [4].

One of the main projects of inner model theory is to identify canonical structures
like Gödel’s $L$ that are close to the universe of sets. While what exactly closeness
should mean is open to interpretation, perhaps the simplest way of interpreting it
is by demanding that the successor of some singular cardinal is computed correctly.
Perhaps the most well-known result of this kind is Jensen’s covering lemma. A weaker
version of it says that if $0^\#$ doesn’t exist then $L$ computes the successor of any singular
cardinal correctly, i.e., if $\kappa$ is singular then $(\kappa^+)^L = \kappa^+$. In this paper, our goal is to
introduce a covering principle that is based on the idea that the information coded
into the universally Baire sets is enough to describe the successor cardinals. To state
our conjecture we need to introduce universally Baire sets and extenders.

Recall from [1] that a set of reals is universally Baire if all of its preimages in
Hausdorff spaces have the property of Baire. As shown in [1], $A \subseteq \mathbb{R}$ is universally
Baire if and only if for each cardinal $\kappa$ there is a pair of trees $(T, S)^1$ on $\kappa$ such that
$p[T] = A$ and for all $< \kappa$-generics $g^2$, $V[g] \vDash p[T] = p[S]^c$. If $A$ has the property on
the right side of the equivalence then we say that $A$ is $\kappa$-universally Baire. Clearly,
$A$ is universally Baire if and only if it is $\kappa$-universally Baire for all $\kappa$.

---

*2000 Mathematics Subject Classifications: 03E15, 03E45, 03E60.
†Keywords: Mouse, inner model theory, descriptive set theory, hod mouse.
‡The author’s research was partially supported by the NSF Career Award DMS-1352034.
¹A tree on $\kappa$ is a downward closed subset of $\bigcup_{n<\omega} \omega^n \times \kappa^n$. If $T$ is a tree then $[T]$ are the set of
branches of $T$ and $p[T]$ is the projection of $[T]$ on the first coordinate.
²This means that the poset in question has size $< \kappa$. 

---

1
Following Woodin (see [17]), we let $\Gamma^\infty$ be the set of universally Baire sets. Given a $V$-generic $g$ we set $\Gamma^\infty_g = (\Gamma^\infty)^{V[g]}$ and $\mathbb{R}_g = \mathbb{R}^{V[g]}$. Given a universally Baire set $A$ and a generic $g$, we let $A^g$ be the canonical interpretation of $A$ in $V[g]$. More precisely, fixing $\kappa$ larger than the size of the poset and a pair of trees $(T, S)$ witnessing that $A$ is $\kappa$-universally Baire, $A^g = (p[T])^{V[g]}$.

An extender is a system of ultrafilters. More precisely, we say $E$ is a $(\kappa, \lambda)$-extender over $M$ if there is an embedding $j : M \rightarrow N$ such that

1. $\text{crit}(j) = \kappa$,
2. $j(\kappa) \geq \lambda$,
3. $E = \{(a, A) : a \in \lambda^{<\omega}, A \in \mathcal{V}(\kappa^{[a]}) \cap M$ and $a \in j(A)\}$,
4. $N = \{j(f)(a) : f \in M, f : \kappa^{[a]} \rightarrow M$ and $a \in \lambda^{<\omega}\}$.

Notice that for each $a \in \lambda^{<\omega}$, $E_a = \{A : (a, A) \in E\}$ is an ultrafilter measuring $\mathcal{V}(\kappa^{[a]}) \cap M$. The extenders we have defined are usually called short, where shortness refers to the fact that $j(\kappa) \geq \lambda$. The fact that $E_a$ is an ultrafilter on $\kappa$ is a consequence of shortness. Long extenders have measures concentrating on more than one cardinal. Just like ultrafilters, extenders can be defined without a reference to the embedding $j$. In our set up, $N$ is the ultrapower of $M$ by $E$, and it can be shown to be completely determined by the pair $(M, E)$.

From now on, when we say we work in the short extender region we mean that we are in the region of large cardinals that can be defined using only short extenders. Large cardinal notions such as strong cardinals, Woodin cardinals, Shelah cardinals, superstrong cardinals and subcompact cardinals are all in the short extender region, while supercompactness is not. We will drop short from now on, and to state our results, will use Steel’s NLE (see [16]), which stands for “no long extender”.

**Conjecture 0.1 (Covering with Chang Models)** Assume NLE and suppose there are unboundedly many Woodin cardinals and strong cardinals. Let $\kappa$ be a limit of Woodin cardinals and strong cardinals and such that either $\kappa$ is a measurable cardinal or $\text{cf}(\kappa) = \omega$. Then there is a transitive model $M$ of $\text{ZFC} - \text{Powerset}$ such that

1. $\text{Ord} \cap M = \kappa^+$,
2. $M$ has a largest cardinal $\nu$,

\[3\text{Absoluteness implies that } A^g \text{ is independent of } \kappa \text{ and } (T, S). \text{ See [1].}\]
3. for any \( g \subseteq \text{Coll}(\omega, < \kappa) \), letting \( R^* = \bigcup_{\alpha < \kappa} R^* \mid g \cap \text{Coll}(\omega, \alpha) \) and \( \Gamma^* = \{ A^g \cap R^* : \exists \alpha < \kappa (A \in \Gamma^g \cap \text{Coll}(\omega, \alpha)) \} \), in \( V(R^*) \),

\[
L(M, \bigcup_{\alpha < \nu} \alpha^\omega, \Gamma^*, R^*) \models \text{AD}.
\]

4. If in addition there is no inner model with a subcompact cardinal then

\[
L(M) \models \text{ZFC + } \varphi(\nu) = \varphi(\nu)^M + \square_\nu.
\]

We state the following two immediate corollaries of \textbf{Covering with Chang Models}.

\textbf{Corollary 0.2} Assume \textbf{Covering with Chang Models} holds and there is no inner model with a subcompact cardinal. Assume further that there are unboundedly many Woodin cardinals and strong cardinals. Let \( \kappa \) be a limit of Woodin cardinals and strong cardinals such that either \( \kappa \) is a measurable cardinal or \( \text{cf}(\kappa) = \omega \). Then \( \square(\kappa^+) \) holds.

\textit{Proof.} Let \( M \) be as in \textbf{Covering with Chang Models} applied to \( \kappa \) and let \( \nu \) be its largest cardinal. Then as \( L(M) \models \square_\nu \), we have a sequence \( \vec{C} = (C_\alpha : \alpha < \kappa^+) \in L(M) \) such that

1. each \( C_\alpha \) is a closed cofinal subset of \( \alpha \);
2. for each limit point \( \beta \) of \( C_\alpha \), \( C_\beta = C_\alpha \cap \beta \);
3. the order type of each \( C_\alpha \) is at most \( \nu \).

Condition 3 above implies that in \( V \) there is no thread \( C \subseteq \kappa^+ \) of \( \vec{C} \). Thus, \( \vec{C} \) is a \( \square(\kappa^+) \) sequence in \( V \).

\textbf{Corollary 0.3} Assume \textbf{PFA} and suppose \textbf{Covering with Chang Models} holds. Then there is an inner model with a subcompact cardinal.

Much has been done towards establishing covering results using canonical inner models resembling \( L \). For example, the reader can consult [2], [3], [9], [4], [10], [14] and [18]. All of these papers except the last one deal with the \textit{short-extender-region}. The last one has conjectures in the region of supercompact cardinals. The introduction of [4] contains a lengthier discussion of Conjecture 0.1 and its role in inner model theory.

Conjecture 0.1 connects determinacy theories with other natural frameworks. Perhaps our approach to the program of finding canonical covering structures is
somewhat unconventional as the covering structures that we propose are not constructed by conventional backgrounded constructions such as \( L[\vec{E}] \) constructions or \( K^c \) constructions. The reader can learn more about the conventional approach from [2], which is based on the existence of \( K^c \).

Our main motivation comes from the desire to isolate large models of determinacy in natural extensions of \( \text{ZFC} \). We strongly believe that the information coded into canonical sets of reals describes the core of the universe. Thus, our approach is closer to the Ultimate \( L \) framework proposed by Woodin than the conventional approach. Our ideas are also heavily influenced by Steel’s inter-translatability ideas that appear in [15].

Besides stating Conjecture 0.1, our goal here is to verify that it holds in \( \text{hod mice} \), which are canonical structures that appear in the analysis of HOD of models of determinacy. The reader can learn more about them by consulting [5], [6], [7] and [16]. The following is the main theorem of the paper. We heavily rely on the terminology of [16].

Suppose \( M \) is a transitive model of some fragment of set theory and \( \kappa \) is a cardinal of \( M \). Then \( \Gamma^\kappa \) is the set of all \( \kappa \)-universally Baire set of reals of \( M \), and if \( g \) is \( M \)-generic then \( \Gamma^\kappa_g \) is the set of \( \kappa \)-universally Baire sets of \( M[g] \).

Suppose \( V \models \text{ZFC} \) is a hod premouse and \( \kappa \) is a limit of Woodin cardinals of \( V \). Let \( g \subseteq \text{Coll}(\omega, \kappa) \) be \( V \)-generic. Set \( g_\alpha = g \cap \text{Coll}(\omega, \alpha) \), \( R^*_g = \bigcup_{\alpha < \kappa} R^{V[g_\alpha]} \) and

\[
\Gamma^*_g = \{ A^g \cap R^*: \exists \alpha < \kappa (A \in \Gamma^\kappa_{g_\alpha}) \}. 
\]

The following is our main theorem.

**Theorem 0.4** Suppose \( (V, \Omega) \) is an excellent pair (see Definition 2.1) and \( V \models \text{ZFC} \). Suppose further that \( \kappa \) is a limit of Woodin cardinals of \( V \) such that either \( \kappa \) is regular or its cofinality is not a measurable cardinal. Then the following stronger version of Conjecture 0.1 is true in \( V \) at \( \kappa \).

There is a transitive \( \text{lbr hod premouse} \) \( M \) such that whenever \( g \subseteq \text{Coll}(\omega, \kappa) \) is generic.

1. \( \text{Ord} \cap M = \kappa^+ \),
2. \( M \) has a largest cardinal \( \nu \),
3. for any generic \( g \subseteq \text{Coll}(\omega, \kappa) \),

\[
L(M, \bigcup_{\alpha < \nu} (M|\alpha)^\omega, \Gamma^*_g, R^*_g) \models \text{AD}.
\]
4. If in addition \( \kappa \) is not subcompact then \( M \models \Box_{\nu} \).
In fact we will give a precise definition of $M$ as above (see Theorem 4.1).

The reader may wonder if there are hod mice in which there are Woodin cardinals and strong cardinals below them. Recently, the author constructed, assuming large cardinals, hod mice that do have Woodin cardinals that are limits of Woodin cardinals. At the time of writing the current paper, this is the best known existence result on hod mice. In particular, since we present Theorem 0.4 as an evidence for Conjecture 0.1, perhaps it is wiser to strengthen the minimality assumption used in Conjecture 0.1 to the non-existence of inner models with a Woodin cardinal that is a limit of Woodin cardinals. We confess that at the time of writing the current paper, the universe beyond a Woodin cardinal that is a limit of Woodin cardinals is a somewhat darker region.

Another mystery may be the requirement in Conjecture 0.1 that $\kappa$ be a limit of strong cardinals. After all no such requirement was made in Theorem 0.4. One reason for this requirement is rooted in technical arguments that have been used in our attempts to prove instances of Conjecture 0.1. For example, the reader may consult the introduction of [9] where proving Conjecture 0.1 is presented as the main goal of the core model induction.

Another reason is that recent unpublished results of the author and Paul Larson seem to suggest that the failure of square at $\omega_3$ along with $2^{\omega_3} = \omega_2$ and $2^{\omega_2} = \omega_3$ is weaker than a Woodin cardinal that is a limit of Woodin cardinals. This result seems to suggest that Conjecture 0.1 for successor cardinals or cardinals $\kappa$ for which $V_\kappa$ is not rich maybe false in that the model $M$ may not have a largest cardinal. This is exactly what seems to happen in the model mentioned above in which $\neg \square_{\omega_3}$ holds.

Another issue with stating such covering conjectures for non-measurable regular cardinals is that for a regular cardinal $\kappa$ one can collapse $\kappa^+$ to $\kappa$ without adding bounded subsets of $\kappa$. This suggests that one cannot hope to obtain $M$ as in Conjecture 0.1, as such an $M$ is most likely absolute between $V$ and $V[g]$ where $g$ collapses $\kappa^+$ to $\kappa$ (the problematic condition seems to be the requirement that $M$ has height $\kappa^+$).

Inner model theory is perhaps closer to physics than other branches of mathematics. The fundamental goal of the subject seems to be finding truths about the universe of sets rather than solving particular precisely stated problems about concrete mathematical structures or discovering truths about concrete mathematical structures such as $\mathbb{N}$ or $L$. Just like many theories describing the nature of the physical universe, inner model theory has gone through many stages of revising truths believed by the community. We certainly do not think that Conjecture 0.1 will be the last word one will say about the short extender region of the universe of sets. There seem to be a lot of mysteries that yet to be understood, and so the intention
of Conjecture 0.1 is to suggest a concrete direction for practitioners of inner model theory. It is very likely that as we know more, just like with many theories put forward by physicists, Conjecture 0.1 will be revisited and modified.

Acknowledgments. The author wishes to thank the referee for several useful comments, especially for not trusting an earlier version of this paper. The author’s work was partially supported by the NSF Career Award DMS-1352034.

1 Terminological Interlude

In this paper, our goal is to give a proof of Theorem 0.4 without getting into nuts and bolts of inner model theory. We will thus isolate some properties of iteration strategies that Steel’s mouse pairs have and work with strategies that have these properties. We, however, will not explain how one obtains such strategies as this is simply beyond the scope of this paper (appropriate references will be given).

In this paper, we will use some aspects of Jensen’s terminology, though not so much his indexing scheme. In this paper, the indexing issues are irrelevant. However, we warn the readers that in this paper, following Jensen, we simply use “iterations” as opposed to “iteration trees”.

A notational digression. Suppose $\mathcal{M}$ is an lbr hod premouse and $\Lambda$ is an iteration strategy for $\mathcal{M}$.

1. For any transitive set $o(\mathcal{M}) = M \cap \text{Ord}$.

2. Given $\xi \leq \text{Ord} \cap M$, we let $\Lambda_{\mathcal{M}^{|\xi}}$ be the strategy of $\mathcal{M}^{|\xi}$ induced by $\Lambda$.

3. If $\mathcal{T}$ is an iteration of $\mathcal{M}$ according to $\Lambda$ with last model $Q$, we let $\Lambda_{\mathcal{T},Q}$ be the strategy of $Q$ induced by $\Lambda$. More precisely, $\Lambda_{\mathcal{T},Q}(U) = \Lambda(\mathcal{T}^\mathcal{T}U)$.

4. Suppose $\mathcal{T}$ is an iteration of some $\mathcal{M}^{|\kappa}$ where $\kappa$ is an inaccessible cardinal of $\mathcal{M}$ with the property that $\rho_\omega(\mathcal{M}) > \kappa$. We then let $\mathcal{T}^\mathcal{M}$ be the id-copy of $\mathcal{T}$ on $\mathcal{M}$. Because of our choice of $\kappa$, $\mathcal{T}^\mathcal{M}$ and $\mathcal{T}$ have the same extenders and the same tree structure.

5. Similarly, if $\mathcal{T}$ is an iteration of $\mathcal{M}$ and is based on $\mathcal{M}^{|\kappa}$ where $\kappa$ is an inaccessible cardinal of $\mathcal{M}$ with the property that $\rho_\omega(\mathcal{M}) > \kappa$, we let $\mathcal{T} \upharpoonright \mathcal{M}^{|\kappa}$ be

---

4One can think of $\Lambda_{\mathcal{M}^{|\xi}}$ as id-pullback of $\Lambda$. 
6. Given an interval \((\alpha, \beta)\) with \(\beta \leq o(\mathcal{M})\), we say that iteration \(\mathcal{T}\) is based on \(\mathcal{M}|(\alpha, \beta)\) or just \((\alpha, \beta)\) if all extenders used in \(\mathcal{T}\) have critical points \(> \alpha\) and for each \(\gamma < \text{lh}(\mathcal{T})\), if \(\pi^\mathcal{T}_{0, \gamma}\) exists then \(\text{lh}(E_\gamma) < \pi^\mathcal{T}_{0, \gamma}(\beta)\). We say \(\mathcal{T}\) is based on \(\mathcal{M}|\beta\) if it is based on \((0, \beta)\).

7. We say \(\mathcal{T}\) is above \(\alpha\) if \(\mathcal{T}\) is based on \((\alpha, o(\mathcal{M}))\).

8. If \(\mathcal{T}\) is an iteration of \(\mathcal{M}\) and \(\alpha < \text{lh}(\mathcal{T})\) then \(\mathcal{T}_{\leq \alpha} = \mathcal{T} \upharpoonright \alpha + 1\) and \(\mathcal{T}_{\geq \alpha}\) is \(\mathcal{T}\) after stage \(\alpha\).

9. Suppose \(\mathcal{Q}\) is a \(\Lambda\)-iterate of \(\mathcal{M}\) and \(\mathcal{R}\) is a \(\Lambda_{\mathcal{Q}}\)-iterate of \(\mathcal{Q}\). Suppose \(\Lambda\) has full normalization (see clause 1 of Definition 2.1). Then we let \(\mathcal{T}_{\mathcal{Q}, \mathcal{R}}^\Lambda\) be the unique normal \(\Lambda_{\mathcal{Q}}\)-iteration of \(\mathcal{Q}\) with last model \(\mathcal{R}\). If \(\pi^\mathcal{T}_{\mathcal{Q}, \mathcal{R}}\) exists then we let \(\pi^\Lambda_{\mathcal{Q}, \mathcal{R}}\) be this embedding. When \(\Lambda\) is clear from context, we will drop it from our notation.

10. We say that \((\mathcal{N}, \Phi)\) is an iterate of \((\mathcal{M}, \Lambda)\) if \(\mathcal{N}\) is a \(\Lambda\)-iterate of \(\mathcal{M}\) via \(\mathcal{T}\) and \(\Phi = \Lambda_{\mathcal{T}, \mathcal{N}}\).

11. We say \(\mathcal{T}\) is strongly non-dropping if for each \(\alpha < \text{lh}(\mathcal{T})\), \(\nu^\mathcal{T}_\alpha\), the supremum of the generators of \(E^\mathcal{T}_\alpha\), is an inaccessible cardinal in \(\mathcal{M}^\mathcal{T}_\alpha\).

12. We let \(S^\mathcal{M}\) is the strategy predicate of \(\mathcal{M}\).

13. In this paper, to keep the matters simple, when we say that some cardinal has large cardinal properties in a fine structural model we always mean that the large cardinal properties in question are witnessed by the extenders on the sequence. Schlutzenberg has done substantial work showing that many such large cardinals are indeed witnessed by the extenders on the sequence of the fine structural model (see for example [11]).

14. We say \(w = (\eta^w, \delta^w)\) is a window of \(\mathcal{M}\) if \(\mathcal{M} \models \text{“there are no Woodin cardinals in the interval } (\eta^w, \delta^w)\text{ and } \delta^w \text{ is a Woodin cardinal”}\).

15. We say \(w\) is a maximal window if in addition \(\eta^w\) is the least \(\mathcal{M}\)-inaccessible \(> \sup\{\xi : \mathcal{M}|\delta^w \models \text{“} \xi \text{ is a Woodin cardinal”}\}\).

\(5\) The fact that \(\kappa\) is inaccessible or that \(\rho_\omega(\mathcal{M}) > \kappa\) are largely irrelevant for these two notions, provided \(\mathcal{M}\) has an iteration strategy with strong hull condensation. However defining the meaning of \(\mathcal{T} \upharpoonright \mathcal{M}|\kappa\) and \(\mathcal{T}^\mathcal{M}\) when \(\kappa\) is arbitrary is not as straightforward. We do not need them in this paper, so we will not be dealing with these notions.
16. Given two windows $w$ and $v$, we write $w <_W v$ if $\delta^w < \delta^v$ (notice that this implies that $\delta^w < \eta^w$).

17. We say that an iteration $\mathcal{T}$ of $\mathcal{M}$ is based on $w$ if $\mathcal{T}$ is based on $(\eta^w, \delta^w)$.

18. We let $\mathsf{EA}^w_M$ be the extender algebra of $\mathcal{M}$ associated with $\delta^w$ that only uses extenders $E$ such that $\text{crit}(E) > \eta^w$ and $\nu_E$, the sup of the generators of $E$, is an inaccessible cardinal of $\mathcal{M}$.

2 Excellent hod pairs

In this section, we isolate some abstract properties of Steel’s notion of mouse pairs (see [12, Chapter 0.1]). These abstract properties are the properties that we will need in our proof of Theorem 0.4.

**Definition 2.1** Suppose $\Sigma$ is a strategy for an lbr hod mouse $\mathcal{P}$. We say that $\Sigma$ is almost excellent if

1. $\Sigma$ has full normalization, i.e., whenever $\mathcal{T}$ is an iteration of $\mathcal{P}$ according to $\Sigma$ with last model $Q$, there is a normal iteration $U$ of $\mathcal{P}$ according to $\Sigma$ with last model $Q$ and such that
   
   (a) $\pi^\mathcal{T}$ exists if and only if $\pi^U$ exists,
   
   (b) if $\pi^\mathcal{T}$ exists then $\pi^\mathcal{T} = \pi^U$, and
   
   (c) if $\mathcal{T}$ is strongly non-dropping then $U$ is strongly non-dropping.

2. $\Sigma$ is positional, i.e., if $Q$ is a $\Sigma$-iterate of $\mathcal{P}$ via both $\mathcal{T}$ and $U$ then $\Sigma_{\mathcal{T},Q} = \Sigma_{\mathcal{U},Q}$,

3. $\Sigma$ is stable, i.e., if $Q$ and $R$ are $\Sigma$-iterates of $\mathcal{P}$ then $Q \preceq R$ implies that $Q = R$,

4. $\Sigma$ is segmentally normal, i.e., whenever $\eta$ is an inaccessible cardinal of $\mathcal{P}$ such that $\rho_\omega(\mathcal{P}) > \eta$, $\mathcal{T}$ is a strongly non-dropping $\Sigma$-iteration of $\mathcal{P}$ that is above $\eta$, $Q$ is the last model of $\mathcal{T}$, $U$ is a $\Sigma_Q$-iteration of $Q$ that is based on $Q|\eta$ such that $\pi^U$ exists and $R$ is the last model of $U$ then

---

6We will not specify which generator version of the extender algebra we use, though we will never use $\delta^w$-generator version.

7We will thus drop $\mathcal{T}$ from $\Sigma_{\mathcal{T},Q}$.
We say that \( \Sigma \) premouse and \( \Sigma \) P cardinal of \( g \) an easy consequence of genericity iterations. First, we will let \( \Lambda \) in clause 6, and \( \Lambda \) g Q P and \( P \) Let

**Proof.**

Lemma 2.2 Suppose \((P, \Sigma)\) is an excellent pair, \( \kappa \) is a limit of Woodin cardinals of \( P \) and \( \eta < \kappa \). Let \( g \subseteq \text{Coll}(\omega, \eta) \) be \( P \)-generic and \( h \subseteq \text{Coll}(\omega, < \kappa) \) be generic over \( P[g] \). Then \( \Lambda^{g,h} \upharpoonright \text{HC}^P(R_{g,h}) \) is directed.

**Proof.** Let \( Q \) and \( R \) be \( \Lambda^{g,h} \)-iterates of \( P|\eta \) in \( P(R_{g,h}) \). Let \( \xi \in (\eta, \kappa) \) be such that \( Q, R \in \mathcal{P}|\xi[g * h_\xi] \), and let \( \delta \in (\xi, \kappa) \) be a Woodin cardinal of \( \mathcal{P}[g * h_\xi] \) and \( S \) be a

---

(a) \( \Sigma_{P|\eta} = (\Sigma_Q)_{P|\eta} \) and

(b) let \( W = (U \upharpoonright (Q|\eta))^P \) and \( S \) be the last model of \( W \), \( R \) is a normal \( \Sigma_S \)-iterate of \( S \) that is strongly non-dropping and is above \( \pi_{P,S}(\eta) \).

5. \( \Sigma \) is **directed**, i.e., if \( Q \) and \( R \) are \( \Sigma \)-iterates of \( P \) then there is \( S \) that is a \( \Sigma_Q \)-iterate of \( Q \) and a \( \Sigma_R \)-iterate of \( R \), and \( |S| = \max(|Q|, |R|) \).

6. \((P, \Sigma)\) satisfies **generic interpretability**, i.e., for any \( \kappa \) that is a limit of Woodin cardinals of \( P \) and for any \( \eta < \kappa \) and any \( P \)-generic \( g \subseteq \text{Coll}(\omega, \eta) \), setting \( \Lambda = \Sigma \upharpoonright (\text{HC}^P[g]) \),

(a) \( \Lambda \in \mathcal{P}[g] \),

(b) \( \Lambda \upharpoonright (\text{HC}^P[g]) \) is \( \kappa \)-universally Baire in \( \mathcal{P}[g] \) as witnessed by trees \( T, S \) that are definable over \( \mathcal{P}[g] \) from the pair \((g, \Sigma_{\mathcal{P}[g]})\),

(c) if \((T, S) \in \mathcal{P}[g] \) is any pair of trees witnessing clause 2 above, for any \( \xi, \kappa \)-generic \( h \) over \( \mathcal{P}[g] \), \((p[T])^P[g][h] = \Sigma \upharpoonright (\text{HC}^P[g][h]) \),

(d) in \( \mathcal{P}[g] \), \( \Lambda \) is the unique strategy \( \Phi \) of \( \mathcal{P}|\eta \) that extends \( \Sigma_{\mathcal{P}[g]|\eta} \upharpoonright (\text{HC}^P[g]) \) and satisfies clauses (b).

7. \( \Sigma \) is **pullback consistent**, i.e., if \( Q \) is a \( \Sigma \)-iterate of \( P \) such that \( \pi_{P,Q} \) is defined and \( \eta < \text{Ord} \cap \mathcal{P} \) then \( \Sigma_{\mathcal{P}|\eta} \) is the \( \pi \)-pullback of \( \Sigma_{\mathcal{Q}|\pi_{P,Q}(\eta)} \).

We say that \( \Sigma \) is **excellent** if whenever \( \mathcal{M} \leq \mathcal{P} \) is such that \( o(\mathcal{M}) \) is an inaccessible cardinal of \( \mathcal{P} \) and \( \rho_\omega(\mathcal{P}) > o(\mathcal{M}) \), \( \Sigma_{\mathcal{M}} \), the id-pullback of \( \Sigma \), is almost excellent.

In this paper, we say \((P, \Sigma)\) is an **excellent pair** if \( P \) is a countable lbr hod premouse and \( \Sigma \) is an excellent \((\omega_1, \omega_1 + 1)\)-iteration strategy for \( P \).

The reader might be wondering if \( \Lambda \) of clause 6 above is directed in \( \mathcal{P}[g] \). This is an easy consequence of genericity iterations. First, we will let \( \Lambda^g \) be the unique \( \Lambda \) as in clause 6, and \( \Lambda^{g,h} \) be its canonical extension in \( \mathcal{P}[g][h] \).

---

8See clause 4 and 5 of Section 1.
common iterate of \( Q \) and \( R \) via \( \Sigma_Q \) and \( \Sigma_R \). Let \( P_0 \) be a non-dropping iteration of \( P \) based on \( (\xi, \delta) \) such that \( S \) is generic over \( P_0 \). Then \( P_0[g \ast h_\xi][S] \) thinks that \( Q \) and \( R \) have a common \( \Lambda^{g \ast h_\xi}_Q \) and \( \Lambda^{g \ast h_\xi}_R \) iterate. This fact pulls back to \( P[g \ast h_\xi] \), and as \( P|\delta \) is countable in \( P[g \ast h] \), we can find \( S \in HC_{P}(R^{g \ast h}) \) that is a common \( \Lambda^{g \ast h}_Q \) and \( \Lambda^{g \ast h}_R \) iterate of \( Q \) and \( R \).

\[ \square \]

**Remark 2.3**  
1. The assumptions made in Definition 2.1, while legitimate, are not part of the defining conditions of a hod premouse; they are only consequences of Steel’s notion of mouse pair which, for example, appears in [12, Chapter 0.1]. We advice the reader unfamiliar with the nuts and bolts of normalization to glance over [12, Chapter 0.1], and consider the references mentioned in that chapter. However, exactly how various normalization procedures work are irrelevant for the current paper.

2. The following are the relevant definitions and theorems of [12] that the reader is encouraged to consult: Definition 0.1, Theorem 0.3 and the entire Chapter 1. These definitions and results have their origins in [16]. The reader can easily locate them in [16] by following the references listed in [12].

3. Notice that clause 1a, 1b and 2 of Definition 2.1 are simply Theorem 1.1 and Corollary 1.2 of [12]. Clause 3 is a consequence of the Dodd-Jensen property (see Theorem 0.3 of [12]). Clause 1c is a consequence of embedding normalization (see the next comment).

4. Clause 4 is an immediate consequence of embedding normalization as spelled out in [16, Chapter 3]. For example, consider [16, Claim 3.2] and [16, Proposition 3.18].

5. Clause 5 is a consequence of the comparison theorem (see [16, Theorem 1.3]).

6. Clause 6 is verified in [16, Chapter 8.1, Theorem 8.1].

7. Clause 7 is a standard consequence of hull condensation and can be verified by consulting [16, Lemma 4.9].

8. The requirement that \( \Sigma_M \) is almost excellent is a consequence of the fact that initial segments of mouse pairs are mouse pairs (see [16]). However, notice that this requirement is made only for those \( M \) for which \( o(M) \) is an inaccessible cardinal of \( P \) and \( \rho_\omega(P) > o(M) \). Given such an \( M \) and an iteration \( T \) of it, there is essentially no difference between \( T^P \) and \( T \). Thus, for such \( M \), \( \Sigma_M \) straightforwardly inherits all of the condensation properties that \( \Sigma \) might have.
9. It is straightforward to verify that if $\Sigma$ is excellent and $Q$ is a $\Sigma$-iterate of $P$ then $\Sigma Q$ is also excellent.

Thus, the following theorem, due to Steel, summarizes the above remark.

**Theorem 2.4** Assume $AD^+$ and suppose $(P, \Sigma)$ is a mouse pair such that $P$ is an lbr hod premouse. Then $(P, \Sigma)$ is an excellent pair.

If $\Sigma$ is an excellent strategy and $Q$ is a $\Sigma$-iterate of $P$ via $T$ then $\Sigma T Q$ is independent of $T$, and so we will drop $T$ from our notation (see [12, Corollary 1.2]).

The following is a simple consequence of excellence that we will need in this paper.

**Lemma 2.5** Suppose $(P, \Sigma)$ is an excellent pair and

1. $\lambda$ is an inaccessible cardinal of $P$ such that $\rho_\omega(P) > \lambda$, and

2. $Q_0$ and $Q_1$ are $\Sigma$-iterates of $P$ via normal iterations $U_0$ and $U_1$ respectively such that both $U_0$ and $U_1$ are based on $P|\lambda$ and do not have drops on their main branch.

Let for $k \in 2$, $\lambda_k = \pi^{U_k}(\lambda)$. Suppose next that $(S, \Phi)$ is an iterate of both $(Q_0|\lambda_0, \Sigma Q_0|\lambda_0)$ and $(Q_1|\lambda_1, \Sigma Q_1|\lambda_1)$. Let $^9$

$$U_2 = (T_{Q_0|\lambda_0,\Phi})^{Q_0} \text{ and } U_3 = (T_{Q_1|\lambda_1,\Phi})^{Q_3}.$$ 

Finally, let $Q_2$ be the last model of $U_2$ and $Q_3$ be the last model of $U_3$. Then $Q_2 = Q_3$ (and clearly, $\Sigma Q_2 = \Sigma Q_3$).

**Proof.** First note that both $\pi^{U_2}$ and $\pi^{U_3}$ are defined and setting $\nu = S \cap \text{Ord}$,

$$\pi^{U_2}(\lambda_0) = \nu = \pi^{U_3}(\lambda_1).$$

The above equality is a consequence of the full normalization of $\Sigma P|\lambda$. Indeed, full normalization of $\Sigma P|\lambda$ implies that

$$\pi^{U_2}|Q_0|\lambda_0 \circ \pi^{U_0} = \pi^{U_3}|Q_1|\lambda_1 \circ \pi^{U_1}. $$

$^9$See the above notational digression.
Because $\lambda$ is inaccessible, we have that both $\pi^{U_2} \circ \pi^{U_0}$ and $\pi^{U_3} \circ \pi^{U_1}$ are continuous at $\lambda$.

Let now $T$ be the normal iteration of $P|\lambda$ according to $\Sigma_{P|\lambda}$ with last model $S$. Thus, $T$ is the full normalization of both $(U_0 \upharpoonright P|\lambda) \lessdot (U_2 \upharpoonright Q_0|\lambda_0)$ and $(U_1 \upharpoonright P|\lambda) \lessdot (U_3 \upharpoonright Q_1|\lambda_1)$, and it is obtained via the least extender comparison process between $P|\lambda$ and $S$. Because $\Sigma$ is excellent, the last branch of $T$ doesn’t drop. An argument like the one given above implies that $\pi^T(\lambda) = \nu$. Let $R$ be the last model of $T^P$. We have that $S \sqsubseteq R$.

We now claim that $R = Q_2$. Suppose not. Clearly, $R|\nu = Q_2|\nu = S$. Let $W$ be the normal iteration of $P$ according to $\Sigma$ with last model $Q_2$. Thus, $W$ is the normalization of $U_0 \lessdot U_2$, and we must have that $\pi^W = \pi^{U_2} \circ \pi^{U_0}$. Because we are assuming $R \neq Q_2$, it follows that $T \lessdot W$ (recall that $W$ is just the iteration produced by the least extender comparison between $P$ and $Q_2$ and $P$-to-$R$ iteration is part of it). Let $\alpha$ be such that $M^W_\alpha = R$. We then must have that $lh(E^W_\alpha) > \nu$.

Because $U_0 \lessdot U_2$ is based on $P|\lambda$ we have that the generators of $U_0 \lessdot U_2$ are contained in $\nu$. Thus, the generators of $W$ are contained in $\nu$. Notice that if $\text{crit}(E^W_\alpha) < \nu$ then we cannot have that $\nu \in \text{rng}(\pi^W)$ which contradicts full normalization as $\nu \in \text{rng}(\pi^{U_2} \circ \pi^{U_0})$. If $\text{crit}(E^W_\alpha) > \nu$ then $W$ has generators above $\nu$ implying that $U_0 \lessdot U_2$ must also have such a generator.

By a symmetric argument, we also have that $R = Q_3$. Hence, $Q_2 = Q_3$. 

\section{Internal direct limit constructions}

Suppose $(\mathcal{V}, \Omega)$ is an excellent pair and $\kappa$ is a limit of Woodin cardinals of $\mathcal{V}$ such that if $\text{cf}^\mathcal{V}(\kappa) < \kappa$ then $\text{cf}^\mathcal{V}(\kappa)$ is not a measurable cardinal of $\mathcal{V}$. We will drop $\Omega$ from now on. In fact, its only use is to guarantee that the internal strategy of $\mathcal{V}$ has the properties stated in Definition 2.1.

Set $\mathcal{P} = \mathcal{V}|(\kappa^+)^\mathcal{V}$. We let $\Sigma$ be the $(\kappa, \kappa + 1)$-fragment\footnote{Equality cannot hold because $\nu$ is inaccessible in $R$ and $< \text{ cannot hold because otherwise we will have } R|\nu \neq S$.} of $S^\mathcal{V}_\mathcal{P}$ that acts on iterations that are based on $\mathcal{P}|\kappa$. Given any $\mathcal{V}$-generic $h$, we let $\Sigma^h$ be the extension

\[ \text{\footnote{Equality cannot hold because } \nu \text{ is inaccessible in } R \text{ and } < \text{ cannot hold because otherwise we will have } R|\nu \neq S.} \]

\[ \text{\footnote{In this case if } \nu \in \text{rng}(\pi^W) \text{ then we must have that } \alpha \text{ is on the main branch of } W \text{ and the extender used at } \alpha \text{ on the main branch has a critical point in the interval } (\nu, lh(E^W_\alpha)). \text{ This easily implies that } W \text{ must have generators above } \nu.} \]

\[ \text{\footnote{I.e., } \kappa \text{-rounds each of which can be } \kappa + 1 \text{ length. Player I can start a new round only if the previous round has length strictly shorter than } \kappa.} \]

\[ \text{\footnote{Equality cannot hold because } \nu \text{ is inaccessible in } R \text{ and } < \text{ cannot hold because otherwise we will have } R|\nu \neq S.} \]

\[ \text{\footnote{In this case if } \nu \in \text{rng}(\pi^W) \text{ then we must have that } \alpha \text{ is on the main branch of } W \text{ and the extender used at } \alpha \text{ on the main branch has a critical point in the interval } (\nu, lh(E^W_\alpha)). \text{ This easily implies that } W \text{ must have generators above } \nu.} \]

\[ \text{\footnote{I.e., } \kappa \text{-rounds each of which can be } \kappa + 1 \text{ length. Player I can start a new round only if the previous round has length strictly shorter than } \kappa.} \]
of \( \Sigma \) in \( V[h] \) (see Definition 2.1)\(^{13}\). Suppose \( g \subseteq \text{Coll}(\omega, < \kappa) \) is \( V \)-generic. The following is a notation that we will use in this paper.

1. For \( \alpha < \kappa \), let \( g_\alpha = g \cap \text{Coll}(\omega, < \alpha) \).

2. Let \( \mathcal{I}^{g_\alpha}(\mathcal{P}) \) be the set of \( \Sigma^{g_\alpha} \)-iterates of \( \mathcal{P} \) that are obtained via iterations \( \mathcal{T} \in V[g_\alpha] \) such that \( lh(\mathcal{T}) \leq \kappa + 1 \), \( \mathcal{T} \) is based on \( \mathcal{P}|\kappa \), \( \pi^{\mathcal{T}} \) is defined and \( \pi^{\mathcal{T}}(\kappa) = \kappa \).

3. Set \( \mathcal{I}^g(\mathcal{P}) = \bigcup_{\alpha < \kappa} \mathcal{I}^{g_\alpha}(\mathcal{P}) \).

4. Given \( Q \in \mathcal{I}^{g_\alpha}(\mathcal{P}) \) and \( \beta \in (\alpha, \kappa) \), we let \( F^{g_\alpha}_{\mathcal{Q}} \) be the set of \( \Sigma^{g_\alpha} \)-iterates of \( Q \) of \( \mathcal{P} \) such that \( lh(T_{Q,R}) < \kappa \) and \( \pi^{T_{Q,R}} \) is defined.

5. Set \( F^{g}_{\mathcal{Q}} = \bigcup_{\beta \in (\alpha, \kappa)} F^{g_{\beta}}_{\mathcal{Q}} \). Clearly, \( F^{g}_{\mathcal{Q}} = F^{g_{\beta}}_{\mathcal{Q}} \), and so we drop \( \alpha \) from our notation and just write \( F^{g}_{\mathcal{Q}} \).

Below we introduce window-based iterations and genericity-iterations.

**Definition 3.1** Suppose \( \mathcal{R} \in \mathcal{I}^{g}(\mathcal{P}) \). We say \( Q \) is a window-based iterate of \( R \) if there is \( \iota < \kappa \) such that \( R \in V[g_\iota] \), an \( \langle \mathcal{W} \rangle \)-increasing sequence of windows \( \mathcal{W}_\alpha = (\eta_\alpha, \delta_\alpha) : \alpha < cf(\kappa) \) of \( R \) and a sequence \( (Q_\alpha : \alpha < cf(\kappa)) \subseteq F^{g_{\iota}}_{\mathcal{R}} \) (in \( V[g_\iota] \)) such that

1. for all \( \alpha < cf(\kappa) \), \( \delta_\alpha < \kappa \),
2. \( \sup_{\alpha < cf(\kappa)} \delta_\alpha = \kappa \),
3. \( Q_0 \in F^{g_{\iota}}(\mathcal{R}) \) and \( \mathcal{T}_{\mathcal{R},Q_0} \) is based on \( \mathcal{R}|\eta_0 \),
4. \( Q_{\alpha+1} \in F^{g_{\iota}}_{Q_\alpha} \),
5. \( Q_{\alpha+1} \) is obtained from \( Q_\alpha \) via an iteration according to \( \Sigma^{g_{\iota}}_{Q_\alpha} \) that is based on \( \pi^{\mathcal{T}_{\mathcal{R},Q_\alpha}}(w_\alpha) \),
6. for limit ordinals \( \lambda \), \( Q_\lambda \) is the direct limit of \( (Q_\alpha, \pi^{Q_\alpha}_{Q_\alpha}: \alpha < \beta < \lambda) \),
7. \( Q \) is the direct limit of \( (Q_\alpha, \pi^{Q_\alpha}_{Q_\alpha}: \alpha < \beta < cf(\kappa)) \).

We set \( \eta_{\alpha,\beta} = \pi^{\mathcal{R},Q_\alpha}(\eta_\beta), \eta^\alpha = \eta_{\alpha,\alpha}, \delta_{\alpha,\beta} = \pi^{\mathcal{R},Q_\alpha}(\delta_\beta) \) and \( \delta^\alpha = \delta_{\alpha,\alpha} \).

It should be clear to the reader that there are unique sequences \( (w_\alpha = (\eta_\alpha, \delta_\alpha) : \alpha < cf(\kappa)) \) and \( (Q_\alpha : \alpha < cf(\kappa)) \) witnessing that \( Q \) is a window-based iterate of \( \mathcal{R} \).

\(^{13}\)We will only use this notation for posets of size strictly smaller than \( \kappa \) or for \( \text{Coll}(\omega, < \kappa) \).
**Definition 3.2** Suppose \( R \in T^9(P) \). We say that \( Q \) is a genericity iterate of \( R \) if it is a window-based iterate of \( R \) as witnessed by \( (w_\alpha : \alpha < cf(\kappa)) \) and \( (Q_\alpha : \alpha < cf(\kappa)) \) such that

1. if \( x \in R_g \) then for some \( \alpha < \kappa \), \( x \) is generic for the extender algebra \( EA^{Q}_{\pi R,Q(x_\alpha)} \);
2. for each \( \alpha < cf(\kappa) \), \( w_\alpha \in \text{rng}(\pi P, R) \);
3. for each \( \alpha \), \( T_{Q_\alpha, Q_{\alpha+1}} \) is a strongly non-dropping iteration of \( Q_\alpha \).

Clause 1 of Definition 3.2 may seem odd to the reader as \( x \in R_g \) while \( Q \) is a \( \Sigma^g_{\kappa_i} \)-iterate of \( R \) for some \( i < \kappa \). Fix an inaccessible cardinal \( \nu \in (i, \kappa) \) such that \( x \in P[g_\nu] \). In \( V[g_\nu] \), given a window \( w \) of \( R \), we can iterate \( R \) in the window \( w \) to make \( a = \text{def} P|^{\nu} [g_\nu] \) generic. If \( S \) is the result of this iteration then \( x \) is generic over \( S^{14} \). If \( Q \) now is an iterate of \( R \) such that for each \( \nu < \kappa \), \( P|^{\nu^+} \) is generic over some window of \( Q \) then clause 1 of Definition 3.2 will be satisfied. Notice that because we are using \( EA \) as our extender algebra, the genericity iterations described above are all strongly non-dropping. The following, then, follows from the above discussion.

**Proposition 3.3** Suppose \( R \) is a genericity iterate of \( P \) and \( S \in F^a_{R} \). Then \( T_{R,S} \) can be normally continued to a genericity iterate of \( R \). More precisely, there is \( Q \) that is a genericity iterate of \( S \) such that \( (T_{R,S})^{-1} T_{S,Q} \) is normal and is a genericity iterate of \( R \).

The following is a consequence of excellence.

**Proposition 3.4** Suppose \( R \) is a genericity iterate of \( P \), \( Q \) is a genericity iterate of \( R \) and \( S \) is a genericity iterate of \( Q \). Then \( S \) is a genericity iterate of \( R \).

**Proof.** Let \( (w_\alpha : a < cf(\kappa)) \) be the windows used in \( T_{R,Q} \) and \( (u_\alpha : a < cf(\kappa)) \) be the windows used in \( T_{Q,S} \). Notice that the full normalization of \( (T_{R,Q})^{-1} T_{Q,S} \) is strongly non-dropping. For each \( \alpha < cf(\kappa) \), set \( u'_\alpha = \pi^{-1}_{R,Q}(u_\alpha) \). Notice that for each \( \alpha < cf(\kappa) \), \( u'_\alpha \in \text{rng}(\pi P, R) \). Let \( (v_\alpha : a < cf(\kappa)) \) be the enumeration of \( \{w_\alpha : a < cf(\kappa)\} \cup \{u'_\alpha : a < cf(\kappa)\} \) in increasing order. It then follows from excellence that \( S \) is a window iterate of \( R \) as witnessed by \( (v_\alpha : a < cf(\kappa)) \). It remains to see that for each \( x \) there is \( \alpha < cf(\kappa) \) such that \( x \) is generic for \( EA^S_{\pi R,S(v_\alpha)} \). Let \( \alpha \) be such that \( x \) is generic for \( EA^S_{\pi R,S(v_\alpha)} \) and let \( \beta \) be such that \( u'_\alpha = v_\beta \). Then \( x \) is generic for \( EA^S_{\pi R,S(v_\beta)} \). \( \square \)

The following is a straightforward consequence of excellence and Lemma 2.5.

---

\(^{14}\text{Notice that } T_{R,S} \in P[g_\nu]. \text{ Thus, } H^{S[a]}_{\nu^+} = P|^{\nu^+}[g_\nu]. \text{ Since } x \text{ is generic over } P|^{\nu^+}[g_\nu], x \text{ is generic over } S[a], \text{ and hence, over } S.\)
Lemma 3.5 Suppose $Q$ is a window based iterate of $P$ and $\mathcal{R}_0, \mathcal{R}_1 \in \mathcal{F}^g_Q$. There is then an $\mathcal{R} \in \mathcal{F}^g_Q$ such that $(\mathcal{R}, \Sigma^g_\mathcal{R})$ is a common iterate of $(\mathcal{R}_0, \Sigma^g_{\mathcal{R}_0})$ and $(\mathcal{R}_1, \Sigma^g_{\mathcal{R}_1})$.

Proof. Let $\mathcal{T} = \mathcal{T}_{P,Q}$, $U_0 = \mathcal{T}_{\mathcal{R}_0, \mathcal{R}_0}$ and $U_1 = \mathcal{T}_{\mathcal{R}_1, \mathcal{R}_1}$. Notice that because for $i \in 2$, $lh(U_i) < \kappa$, we must have that there is some $\eta' < \kappa$ such that $\eta'$ is an inaccessible cardinal of $Q$ and for $i \in 2$, $U_i$ is a normal iteration of $Q$ based on $Q|\eta'$. Let $\eta$ be the least such $\eta'$ and let $\xi < lh(\mathcal{T})$ be such that $M^\xi_\mathcal{T}|\eta = Q|\eta$. Let $\lambda_0$ be the least inaccessible of $\mathcal{P}$ such that $\mathcal{T} \upharpoonright \xi$ is based on $\mathcal{P}|\lambda_0$. Let $(w_\alpha = (\eta_\alpha, \delta_\alpha) : \alpha < cf(\kappa))$ be the windows used in $\mathcal{T}$. Let $\lambda = \eta_\alpha$ where $\alpha$ is the least such that $\lambda_0 < \eta_\alpha$. Because $\mathcal{T}$ is a window-based iteration, it follows that if $\xi < \kappa$ is the least such that $M^\xi_\mathcal{T}|\pi^\xi(\lambda) = Q|\pi^\xi(\lambda)$ then (notice that $\pi^\xi(\lambda) = \eta^\alpha$)

(a) $\mathcal{T}_{\geq \xi}$ is a normal window-based iteration of $M^\xi_\mathcal{T}$ that is above $\pi^\xi(\lambda)$, and
(b) $\mathcal{T} \upharpoonright \xi + 1$ is based on $\mathcal{P}|\lambda$.

Set $\nu = \pi^\xi(\lambda)$. Let now for $k \in 2$, $S_k$ be the last model of $U_k \upharpoonright Q|\nu$. Using the fact that $\Sigma^g_{\mathcal{P}|\lambda}$ is directed (see Definition 2.1) we can find a common iterate $(S, \Phi)$ of $(S_0, \Sigma^g_{S_0})$ and $(S_1, \Sigma^g_{S_1})$. Let $\mathcal{R}$ be the last model of $(\mathcal{T}_{S_0, S})^{R_0}$. It follows from Lemma 2.5 that $\mathcal{R}$ is the last model of $(\mathcal{T}_{S_1, S})^{R_1}$. □

Suppose now that $Q$ is a window based $\Sigma^g$-iterate of $\mathcal{P}$. For $\mathcal{R}, S \in \mathcal{F}^g_Q$, we let $\mathcal{R} \leq_Q S$ if $S \in \mathcal{F}^g_{\mathcal{R}}$. It follows from Lemma 3.5 that $\leq_Q$ is directed. Moreover, notice that if $\mathcal{R}, S \in \mathcal{F}^g_Q$ and $W \in \mathcal{F}^g_{\mathcal{R}} \cap \mathcal{F}^g_S$ then $W$ is a normal $\Sigma^g_Q$-iterate of $Q$ and

$$\pi_{\mathcal{R}, W} \circ \pi_{Q, \mathcal{R}} = \pi_{Q, W} = \pi_{S, W} \circ \pi_{Q, S}.$$ 

We can then let $M^\infty_Q(\mathcal{Q})$ be the direct limited of the directed system $(\mathcal{F}^g_Q, \leq_Q)$ under the iteration embedding. Given $\mathcal{R} \in \mathcal{F}^g_Q$ we let $\pi^Q_{\mathcal{R}, \infty} : \mathcal{R} \to M^\infty_Q(\mathcal{Q})$ be the direct limit embedding. We also set $\kappa^Q = \pi^Q_{\mathcal{R}, \infty}(\kappa)$.

Suppose $Q$ is a genericity iterate of $\mathcal{R}$. There is then $h \subseteq Coll(\omega, < \kappa)$ with $h \in V[g]$ such that $h$ is $\mathcal{Q}$-generic and $(\mathcal{R}_h)_{Q[h]} = \mathbb{R}^*_g$. We call such an $h$ a maximal generic.

Lemma 3.6 Suppose $Q$ is a genericity iterate of $P$. Let $h \subseteq Coll(\omega, < \kappa)$ be a maximal $Q$-generic. Then $M^\infty_Q(\mathcal{Q}) = (M^\infty_Q(\mathcal{Q}))_{Q[h]}$. □

Proof. This is simply because $\mathcal{F}^g_Q = (\mathcal{F}^h_Q)^{Q[h]}$.

Suppose $\iota < \kappa$ and $Q \in \mathcal{I}^g_P$ is a window-based iterate of $P$ as witnessed by $(w_\alpha = (\eta_\alpha, \delta_\alpha) : \alpha < cf(\kappa))$ and $(Q_\alpha : \alpha < cf(\kappa))$. We set up the following notation.
1. Suppose $R \in F_Q^g$ and $y \in R|\kappa$. Let $\alpha_{R,y}$ be the least $\alpha$ such that $T_{Q,R}$ is based on $Q|\eta^\alpha$ and $y \in R|\pi_{Q,R}(\eta^\alpha)$.

2. Given $\beta \geq \alpha_{R,y}$, let $W(R,y,\beta)$ be the last model of $(T_{Q,R} \upharpoonright (Q|\eta^\alpha))^Q_\beta$.

The following lemma is an easy consequence of excellence (see Definition 2.1); in particular, of segmental normality.

**Lemma 3.7** Suppose $\iota < \kappa$ and $Q \in I^\iota(\mathcal{P})$ is a genericity iterate of $\mathcal{P}$ as witnessed by $(w_\alpha = (\eta_\alpha, \delta_\alpha) : \alpha < \text{cf}(\kappa))$ and $(Q_\alpha : \alpha < \text{cf}(\kappa))$. Let $\xi \in [\iota, \kappa)$, $R$ and $y$ be such that $R \in F^g_Q$ and $y \in R|\kappa$. Let $\beta \leq \gamma < \text{cf}(\kappa)$ be such that $\alpha_{R,y} \leq \beta$. Then $W(R,y,\gamma)$ is a normal $\Sigma^g_{W(R,y,\beta)}$-iterate of $W(R,y,\beta)$ via a normal strongly non-dropping iteration $U$ such that $U$ is above $\pi_{Q_\beta, W(R,y,\beta)}(\eta^\beta)$ and $U$ is based on $W(R,y,\beta)|\pi_{Q_\beta, W(R,y,\beta)}(\delta_{\beta,\gamma})$.

**Proof.** First apply segmental normality to $Q_\beta|\delta_{\beta,\gamma}$. Then apply Lemma 2.5. \qed

The following is the main theorem of this section.

**Theorem 3.8** Suppose $Q$ is a genericity iterate of $\mathcal{P}$ as witnessed by $(w_\alpha = (\eta_\alpha, \delta_\alpha) : \alpha < \text{cf}(\kappa))$ and $(Q_\alpha : \alpha < \text{cf}(\kappa))$. Then

$$M_\infty(Q) = M_\infty(\mathcal{P}).$$

**Proof.** We define $j : M_\infty(Q)|\kappa_\infty^Q \rightarrow M_\infty(\mathcal{P})|\kappa_\infty^P$ as follows. Given $x \in M_\infty(Q)|\kappa_\infty^Q$, let $R \in F^g_Q$ be such that $\pi^Q_{R,\kappa}(y) = x$ for some $y \in R$. Notice that for every $\beta \in [\alpha_{R,y}, \text{cf}(\kappa))$, $W(R,y,\beta) \in F^g_P$. Set

$$j(x) = \pi^P_{W(R,y,\alpha_{R,y}),\infty}(y).$$

**Claim 1.** For $\beta \geq \alpha_{R,y}$, $j(x) = \pi^P_{W(R,y,\beta),\infty}(y)$.

**Proof.** Set $\alpha = \alpha_{R,y}$. Because $\text{crit}(\pi^P_{W(R,y,\alpha),W(R,y,\beta)}) > \pi_{Q_\alpha, W(R,y,\alpha)}(\eta^\alpha)$, we have the following equalities.

$$\pi^P_{W(R,y,\alpha),\infty}(y) = \pi^P_{W(R,y,\beta),\infty}(\pi_{W(R,y,\alpha),W(R,y,\beta)}(y)) = \pi^P_{W(R,y,\beta),\infty}(y)$$

\qed

**Claim 2.** $j$ is well-defined.

**Proof.** Fix $\xi \in [\iota, \kappa)$, $R, R' \in F^g_Q$, $y \in R$ and $y' \in R'$ such that $\pi^Q_{R,\infty}(y) = \pi^Q_{R',\infty}(y') = x$. Let $\beta > \max(\alpha_{R,y}, \alpha_{R',y'})$. Because of Claim 1 above, it is enough to show that
\[ \pi_{W(\mathcal{R}, y, \beta), \infty}^P(y) = \pi_{W'(\mathcal{R}', y', \beta), \infty}^P(y'). \]

Set \( \mathcal{W} = \mathcal{W}(\mathcal{R}, y, \beta) \) and \( \mathcal{W}' = \mathcal{W}(\mathcal{R}', y', \beta) \). Let \( \eta = \pi_{Q_{\beta}, \mathcal{W}(\eta^\beta)} \) and \( \eta' = \pi_{Q_{\beta}, \mathcal{W}'(\eta^\beta)} \). Notice that \( \Sigma_{Q_{\beta}|\eta^\beta}^g \) is excellent.

Using the fact that \( \Sigma_{Q_{\beta}|\eta^\beta}^g \) is directed we can find a common \( \Sigma_{\mathcal{W}|\eta}^g \) and \( \Sigma_{\mathcal{W}'|\eta'}^g \) iterate \( \mathcal{M} \) of \( \mathcal{W}|\eta \) and \( \mathcal{W}'|\eta' \). Set \( \mathcal{U} = \mathcal{T}_{\mathcal{W}|\eta,M} \) and \( \mathcal{U}' = \mathcal{T}_{\mathcal{W}'|\eta',M} \). It follows from excellence that

\[ \pi_{\mathcal{W}|\eta,M} \circ \pi_{Q_{\beta}|\eta^\beta, \mathcal{W}|\eta} = \pi_{\mathcal{W}'|\eta',M} \circ \pi_{Q_{\beta}|\eta^\beta, \mathcal{W}'|\eta'}. \]

Set

1. \( \mathcal{Y} = \mathcal{T}_{Q_{\beta}|\eta^\beta, \mathcal{W}|\eta} \) and \( \mathcal{Y}' = \mathcal{T}_{Q_{\beta}|\eta^\beta, \mathcal{W}'|\eta'}; \)
2. \( \mathcal{X} = \mathcal{T}_{Q_{\beta}|\eta^\beta, \mathcal{M}} \), \( \mathcal{X}_0 = \mathcal{X}^Q_{\beta} \) and \( \mathcal{X}_1 = \mathcal{X}^Q \),
3. \( \mathcal{U}_0 = \mathcal{U}^\mathcal{Y} \) and \( \mathcal{U}'_0 = \mathcal{U}'^\mathcal{Y'}; \)
4. \( \mathcal{U}_1 = \mathcal{U}^\mathcal{R} \) and \( \mathcal{U}'_1 = \mathcal{U}'^\mathcal{R'} \).

Let \( S \) be the last model of \( \mathcal{X}_0 \). Then \( S \) is also the last model of \( \mathcal{U}_0 \) and \( \mathcal{U}'_0 \) (see Lemma 2.5).

Notice that \( \mathcal{X}_1 \) is the full normalization of \( (\mathcal{T}_{Q, \mathcal{R}})^{-1} \mathcal{U}_1 \) and \( (\mathcal{T}_{Q, \mathcal{R}'})^{-1} \mathcal{U}'_1 \). It follows that \( \mathcal{X}_1, \mathcal{U}_1 \) and \( \mathcal{U}'_1 \) have the same last model. Call it \( \mathcal{N} \). It follows from full normalization that \( \mathcal{N} \) is a \( \Sigma^g \)-iterate of \( S \) via a normal iteration that is above \( \pi_{Q_{\beta}, S}(\eta^\beta) \).

Because \( \pi_{\mathcal{R}, \infty}^Q(y) = \pi_{\mathcal{R}', \infty}^Q(y') \), we have that \( \pi_{\mathcal{R}, \mathcal{N}}(y) = \pi_{\mathcal{R}', \mathcal{N}}(y') \). Hence, \( \pi_{\mathcal{W}, S}(y) = \pi_{\mathcal{W}', S}(y') \) (this uses the fact that \( \mathcal{T}_{S, \mathcal{N}} \) is above \( \pi_{\mathcal{X}_0(\eta^\beta)} \)). We now have the following equalities.

\[ \pi_{\mathcal{W}, \infty}(y) = \pi_{S, \infty}(\pi_{\mathcal{W}, S}(y)) \]
\[ = \pi_{S, \infty}(\pi_{\mathcal{W}', S}(y')) \]
\[ = \pi_{\mathcal{W}', \infty}(y'). \]

To finish the proof of the lemma, we show that \( j \) is onto and \( \Sigma_1 \)-elementary. Fix \( u \in \mathcal{M}_\infty(\mathcal{P})|_{\kappa_{\mathcal{R}}^P} \). We want to see that there is \( x \in \mathcal{M}_\infty(\mathcal{Q})|_{\kappa_{\mathcal{Q}}^Q} \) such that \( j(x) = u \). To start with, let \( \mathcal{R} \in \mathcal{F}_{\mathcal{P}}^\beta \) be such that for some \( v \in \mathcal{R}, \pi_{\mathcal{R}, \infty}(v) = u \). Let \( \alpha \) be such that \( \mathcal{T}_{\mathcal{P}, \mathcal{R}} \) is based on \( \mathcal{P}|_{\eta_\alpha} \) and \( v \in \mathcal{R}|_{\pi_{\mathcal{P}, \mathcal{R}}(\eta_\alpha)} \). Using the fact that \( \Sigma_{\mathcal{P}|_{\eta_\alpha}}^g \)
is directed, we can find an $S$ that is a common $\Sigma^g_R$ and $\Sigma^g_{Q_\alpha}$ iterate of $R$ and $Q_\alpha$ respectively\(^{15}\). Notice that $T_{Q_\alpha,S}$ is based on $Q_\alpha|\eta^\alpha$ and $T_{R,S}$ is based on $R|\pi_P R(\eta^\alpha)$. Set $X = (T_{Q_\alpha,S} \upharpoonright (Q_\alpha|\eta^\alpha))^Q$. Letting $N$ be the last model of $X$, we have that $N$ is a normal strongly non-dropping $\Sigma^g_S$-iterate of $S$ via an iteration that is above $\pi_{Q_\alpha,S}(\eta^\alpha)$. Let $y = \pi_{R,S}(v)$. Notice that $\pi_{S,N}(y) = y$.

We now easily have that $j(\pi_{N,\infty}(y)) = u$. Indeed,

$$j(\pi_{N,\infty}(y)) = \pi_{P,\infty}(y) = \pi_{S,\infty}(\pi_{R,S}(v)) = \pi_{R,\infty}(v) = u.$$  

To see that $j$ is $\Sigma_2$-elementary, it is enough to notice that given a $\Sigma_1$-formula $\phi(...)$, $M_\infty(Q)|\kappa_\infty^Q \models \phi(\bar{x})$ if and only if there is $R \in F^g_Q$ such that for some $\vec{y} \in R|\kappa$ and some $\alpha > \alpha_{R,\vec{y}}, \pi_{Q_\alpha,\infty}(\vec{y}) = \bar{x}$ and

$$W(R, \vec{y}, \alpha)|\pi_{Q_\alpha, W(R, \vec{y}, \alpha)}(\eta^\alpha) \models \phi(\vec{y}).$$

As $j(\bar{x}) = \pi_{W(R, \vec{y}, \alpha), \infty}(\vec{y})$, the $\Sigma_1$-elementarity follows.

Since $j$ is onto and $\Sigma_1$-elementary, it follows that

$$M_\infty(Q)|\kappa_\infty^Q = M_\infty(P)|\kappa_\infty^P.$$  

The fact that $M_\infty(Q) = M_\infty(P)$ follows from full normalization. \(\square\)

Set $\kappa_\infty = \kappa_\infty^P$ and $M_\infty = M_\infty(P)$. The following useful corollary is an immediate consequence of excellence and Theorem 3.8.

**Corollary 3.9** Suppose $R$ is a genericity iterate of $P$ and $Q$ is a genericity iterate of $R$. Then $Q$ is a genericity iterate of $P$ (see Proposition 3.4) and

$$\pi_{R,\infty}^R = \pi_{Q,\infty}^Q \circ \pi_{R, Q}.$$  

The following proposition will be used in the next section.

**Proposition 3.10** Suppose $Q$ is a genericity iterate of $P$ and $h$ is a maximal $Q$-generic. Suppose $X \in V[y]$ is a countable subset of $M_\infty(P)|\xi$ for some $\xi < \kappa_\infty^P$. Then $X \in Q(R_h^*)$.

\(^{15}\)A similar argument was used in the proof of Claim 2 to conclude that $X_0$, $U_0$ and $U'_0$ have the same last model.
Proof. Let \((w_\alpha = (\eta_\alpha, \delta_\alpha) : \alpha < \text{cf}(\kappa))\) and \((Q_\alpha : \alpha < \text{cf}(\kappa))\) witness that \(Q\) is a genericity iterate of \(P\). We can fix \(R \in F^g_P, \alpha < \text{cf}(\kappa)\), a real \(z \in R^g\) that codes a bijection \(f : \omega \to R|_{\pi_P, R(\eta_\alpha)}\) and a \(y \subseteq \omega\) such that

\[
\beta \in X \text{ if and only if there is } i \in y \text{ such that } \pi^P_{R, \infty}(f(i)) = \beta.
\]

Let \(S\) be a common iterate of \(Q_\alpha\) and \(R\). We have that \(S \in F^g_{Q_\alpha}\). Let \(N\) be the last model of \((T_{Q_\alpha, S} | Q_\alpha|^{\eta_\alpha})^Q\). Let \(\sigma : R|_{\pi_P, R(\eta_\alpha)} \to N|_{\pi_{Q, N}(\eta_\alpha)}\) be the iteration embedding. We now have that

\[
\beta \in X \text{ if and only if there is } i \in y \text{ such that } \pi^Q_{N, \infty}(\sigma(f(i))) = \beta.
\]

It follows from Lemma 3.6 that \(X \in Q[z]\). □

4 A Chang model over the derived model

Suppose \(V \models \text{ZFC}\) is a hod premouse and \(\kappa\) is a limit of Woodin cardinals of \(V\) such that if \(\text{cf}(\kappa) < \kappa\) then \(\text{cf}(\kappa)\) is not a measurable cardinal. Let \(g \subseteq \text{Coll}(\omega, < \kappa)\) be \(V\)-generic and let \(P = V|(\kappa^+)^V\). Recall that for \(\beta < \kappa, g_\beta = g \cap \text{Coll}(\omega, < \beta)\). Working in \(V(R^*_g)\), set

\[
\begin{align*}
\mathbb{R}^*_g &= \bigcup_{\beta < \kappa} \mathbb{R}^{|g[\beta]}|, \\
\Gamma^*_g &= \{ A^g \cap \mathbb{R}^* : \exists \beta < \kappa (A \in \Gamma^*_g) \}, \\
C(g) &= L(M_{\infty}, \cup_{\xi < \kappa} \cup_{\omega_1} (M_{\infty}|\xi), \Gamma^*_g, \mathbb{R}^*_g), \\
C_\alpha(g) &= L_\alpha(M_{\infty}, \cup_{\xi < \kappa} \cup_{\omega_1} (M_{\infty}|\xi), \Gamma^*_g, \mathbb{R}^*_g)
\end{align*}
\]

If \(Q \in T^g(P)\) then we let \(V_Q\) be the last model of \((T_{P, Q})^V\).

**Theorem 4.1** Suppose \((V, \Omega)\) is an excellent pair, \(\kappa\) is a limit of Woodin cardinals of \(V\) such that either \(\kappa\) is regular or its cofinality is not a measurable cardinal, and \(g \subseteq \text{Coll}(\omega, < \kappa)\) is \(V\)-generic. Then \(C(g) \models \text{AD}^+\).

As with Section 3, the use of \(\Omega\) is to make sure that \(S^V\) has the desired properties. We will drop \(\Omega\) from now on.

**Proposition 4.2** Suppose \(Q\) is a genericity iterate of \(P\) and \(h\) is a maximal \(Q\)-generic. Then \(C(g) = (C(h))^{|Q[\Omega]\}^h}\).
Proof. Because of Theorem 3.8 and Proposition 3.10, it is enough to show that \( \Gamma^*_g = (\Gamma^*_h)^{V_\delta} \). This is a standard fact about hod mice, and reduces to the fact that \( \Sigma^g \) is pullback consistent. Indeed, suppose \( B \in \Gamma^*_g \). Let \( \eta < \kappa \) be an inaccessible cardinal such that the if \( \delta \) is the least Woodin cardinal of \( V \) above \( \eta \) then there is a pair of trees \((T, S) \in V[g_\eta]\) witnessing that \( B \) is \( \kappa \)-uB. Using genericity iterations, it can be shown that \( B \) is projective in the code of \( \Sigma^g_{\mathcal{P}|\delta} \). Given this information, notice next that \( \Sigma^g_{\mathcal{P}|\delta} \in (\Gamma^*_h)^{V_\delta} \) as it can be computed from \( \Sigma^g_{\mathcal{Q}|\pi_{\mathcal{P},\mathcal{Q}(\delta)}} \) (using pullback consistency). This shows that \( \Gamma^*_g \subseteq (\Gamma^*_h)^{V_\delta} \). That \( (\Gamma^*_h)^{V_\delta} \subseteq \Gamma^*_g \) can be easily verified using the fact that \( \mathcal{Q} \) is an iterate of \( \mathcal{P} \). We now show that \( B \) is indeed projective in \( \Sigma^g_{\mathcal{P}|\delta} \). Set \( w = (\eta, \delta) \).

Working in \( \mathcal{P}[g_\eta] \), let \( \tau \in \mathcal{P}[g_\eta]^{Coll(\omega, \delta)} \) be the standard term relation that is always realized as \( p[T] \) in \( \mathcal{P}[g_\eta]^{Coll(\omega, \delta)} \).

We now have that \( x \in B \) if and only if whenever \( S \) is a \( \Sigma \)-iterate of \( \mathcal{P} \) based on \( w \) such that \( \pi_{\mathcal{P},S} \) is defined and \( h \subseteq Coll(\omega, \pi_{\mathcal{P},S}(\delta)) \) is \( S[g_\eta] \)-generic such that \( x \in S[g_\eta]|h \), \( x \in (\pi_{\mathcal{P},S}(\tau))_h \).

Indeed, let \( x \in B \) and fix \( (S, h) \) as in the right side of the equivalence. Then since \( x \in p[T] \), \( x \in p[\pi_{\mathcal{P},S}(T)] \) and therefore, \( x \in (\pi_{\mathcal{P},S}(\tau))_h \). Conversely, if \( (S, h) \) is as in the right side of the equivalence and \( x \in (\pi_{\mathcal{P},S}(\tau))_h \), then \( x \in p[\pi_{\mathcal{P},S}(T)] \). But then \( x \in B \) as otherwise \( x \in p[S] \), implying that \( x \in p[\pi_{\mathcal{P},S}(S)] \), contradiction. \( \square \)

We spend the rest of this section proving Theorem 4.1. To start with, fix \( A \subseteq R^*_g \) and let \( \alpha \) be the least such that \( A \in C_\alpha(g) \). A Skolem hull argument shows that \( \alpha < \kappa^{++} \). Let \( (\gamma, Y, B, x, s, \phi) \) be such that

1. \( \gamma < \kappa_\infty \),
2. \( Y \in \wp_\omega(M_\infty|\gamma) \),
3. \( B \in \Gamma^*_g \),
4. \( x \in R^*_g \),
5. \( s \in Ord^{< \omega} \),
6. for all \( u \in R^*_g \),

\[ u \in A \iff C_\alpha(g) \models \phi[Y, B, x, s, u] \].

\( \text{16} \)That this is indeed the case is a standard fact and appears in many places in literature.

\( \text{17} \)Indeed, working in \( V \), we can find \( \beta \geq \kappa^{++} \) such that \( A \in C_\beta(g) \), and \( \pi : M \rightarrow V|\beta \) such that \( M^{\omega} \subseteq M, |M| = \kappa^+ \) and \( A \in M(R^*_g) \). But \( C(g)^M = C_\alpha(g) \) where \( \alpha = Ord \cap M \).
It is enough to show that $A \in \Gamma^*_{g}$, and for this, it is enough to show that for some $\lambda < \kappa$, $A$ is projective in $\Sigma^g_{\mathcal{P}|\lambda}$. Notice that there is some $\lambda_0$ such that $B$ is projective in $\Sigma^g_{\mathcal{P}|\lambda_0}$ (for instance, see the proof of Proposition 4.2). Thus, without loss of generality we can assume that $B$ is the code of $\Sigma^g_{\mathcal{P}|\lambda_0}$ (denoted by Code($\Sigma^g_{\mathcal{P}|\lambda_0}$)), and the real involved in the definition of our old $B$ is now part of $x$. We thus have that for every $u \in \mathbb{R}^*_g$,

(1) $u \in A$ if and only if $C_\alpha(g) \Vdash \phi[Y, \text{Code}(\Sigma^g_{\mathcal{P}|\lambda_0}), x, s, u]$.

Suppose $\mathcal{R}$ is a genericity iterate of $\mathcal{P}$. We say $\mathcal{R}$ is A-stable if whenever $Q$ is a genericity iterate of $\mathcal{R}$,

$$\pi_{\mathcal{R},Q}[Y] = Y \text{ and } \pi_{\mathcal{R},Q}(\alpha, s) = (\alpha, s).$$

Lemma 4.3 There is an A-stable $\mathcal{R}$.

Proof. First find a genericity iterate $\mathcal{R}_0$ of $\mathcal{P}$ such that $Y \subseteq \text{rng}(\pi^\mathcal{R}_{\mathcal{R}_0,\infty}[\kappa])$. Next find a genericity iterate $\mathcal{R}$ of $\mathcal{R}_0$ such that for any genericity iterate $Q$ of $\mathcal{R}$, $\pi_{\mathcal{R},Q}(\alpha, s) = (\alpha, s)$. Notice that it follows from Corollary 3.9 that we still have that $Y \subseteq \text{rng}(\pi^\mathcal{R}_{\mathcal{R},\infty}[\kappa])$.

We have that $\mathcal{R}$ is a genericity iterate of $\mathcal{P}$. Moreover, whenever $Q$ is a genericity iterate of $\mathcal{R}$, $\pi_{\mathcal{R},Q}[Y] = Y$. Indeed, fix such a $Q$ and let $\xi \in \pi_{\mathcal{R},Q}[Y]$. Fix $\beta \in Y$ such that $\xi = \pi_{\mathcal{R},Q}(\beta)$ and let $\theta$ be such that $\pi^\mathcal{R}_{\mathcal{R},\infty}(\theta) = \beta$. But then

$$\pi_{\mathcal{R},Q}(\xi) = \pi_{\mathcal{R},Q}(\pi^\mathcal{R}_{\mathcal{R},\infty}(\theta))$$

$$= \pi^Q_{\mathcal{R},\infty}(\pi_{\mathcal{R},Q}(\theta)) \text{ (uses the fact that } \pi^\mathcal{R}_{\mathcal{R},\infty} \in \mathcal{V}_{\mathcal{R}})$$

$$= \pi^\mathcal{R}_{\mathcal{R},\infty}(\theta) \text{ (see Corollary 3.9)}$$

$$= \beta.$$

Hence, $\xi = \beta$ implying that $\xi \in Y$. The proof that $Y \subseteq \pi_{\mathcal{R},Q}[Y]$ is similar. □

We now fix some objects.

1. Let $\mathcal{R}$ be A-stable, and let $\tau < \kappa$ be such that $Y \subseteq \pi^\mathcal{R}_{\mathcal{R},\infty}[\mathcal{P}|\tau]$.

2. Let $u \in \mathbb{R}^*_g$ be a real that codes $(\mathcal{P}|\lambda_0, \pi_{\mathcal{P},\mathcal{R}} \upharpoonright (\mathcal{P}|\lambda_0), \mathcal{R}|\pi_{\mathcal{P},\mathcal{R}}(\lambda_0))$. Notice that $\Sigma^g_{\mathcal{P}|\lambda_0}$ can be easily defined over $\mathcal{R}(\mathbb{R}^*_g)$ from $u$ and $\Sigma^g_{\mathcal{R}|\pi_{\mathcal{P},\mathcal{R}}(\lambda_0)}$ as the $\pi_{\mathcal{P},\mathcal{R}} \upharpoonright (\mathcal{P}|\lambda_0)$-pullback of $\Sigma^g_{\mathcal{R}|\pi_{\mathcal{P},\mathcal{R}}(\lambda_0)}$.

---

18 This is one of the places where we use the fact that if $\kappa$ is singular then its cofinality is not a measurable cardinal. Otherwise there may not be such an $\mathcal{R}_0$ as $\pi_{\mathcal{P},\infty}(\kappa) > \sup(\pi_{\mathcal{P},\infty}[\kappa])$. 

21
3. Next, let \( v \in \mathbb{R}^*_y \) be a real coding a pair of reals \( (v_0, v_1) \) such that \( v_1 \) codes a bijection \( k : \omega \to \mathcal{R}|\tau \) and \( Y = \{ \pi^\mathcal{R}_{R,\infty}(k(n)) : n \in v_0 \} \).

Notice that \( Y \in \mathcal{R}[u, v] \) and \( \mathcal{R}[u, v] \models \text{"} Y \text{ is countable"} \).

We can now get a formula \( \psi \) such that for every \( z \in \mathbb{R}^*_y \), \( z \in A \) if and only if
\[
\mathcal{V}_{\mathcal{R}}[x, u, v, z] \models \psi[\mathcal{R}|_{\kappa^{++}}, \kappa, \alpha, s, x, u, v, z].
\]

The formula \( \psi(W, p, q, r, a, b, c, d) \) essentially says the following (in the language of lbr hod mice):

1. \( W \) is an lbr hod mouse,
2. \( p \) is a limit of Woodin cardinals and \( p^{++} = \text{Ord} \cap W \),
3. \( q < p^{++} \) is an ordinal and \( r \) is a finite sequence of ordinals,
4. \( (a, b, c, d) \in \mathbb{R}^4 \),
5. \( H_{p^{++}} \) is the universe of \( W \),
6. \( b \) codes a triple \( (M, i, N) \) such that \( i : M \to N \) is an elementary embedding and \( N \subseteq W|p \),
7. \( c \) codes a pair of reals \( (c_0, c_1) \) such that for some \( \mathcal{K} \subseteq W|p \), \( c_0 \) codes an enumeration of \( k : \omega \to \mathcal{K} \),
8. setting \( Z = \{ \pi^W_{W[p^{++}]}(k(n)) : n \in c_1 \} \) and letting \( \Lambda \) be the \( i \)-pullback of \( S^W_N \), whenever \( h \subseteq \text{Coll}(\omega, p) \) is \( W \)-generic \( (C_q(h))^{W[\mathbb{R}^*_y]} \models \phi[Z, \text{Code}(\Lambda^h), a, r, d] \).

Let now \( w \) be a window of \( \mathcal{R} \) such that \( \max(\pi_{\mathcal{P}, \mathcal{R}}(\lambda_0), \tau) < \eta^w \) and \( (x, u, v) \) is generic for a poset in \( \mathcal{R}|\eta^w \). We claim that \( \lambda = \delta^w \) is as desired, i.e., \( A \) is projective in \( \Sigma_{\mathcal{R}|\lambda} \). The following lemma shows exactly this.

Suppose \( S \) is an iterate of \( \mathcal{R} \) via an iteration based on \( w \) such that \( \pi_{\mathcal{R}, S} \) is defined. Let \( \tau_S \in S[x, u, v]^\text{Coll}(\omega, \lambda) \) be the standard term that is always realized as the set of reals \( z \) such that
\[
\mathcal{V}_S[x, u, v, z] \models \psi[S|_{\kappa^{++}}, \kappa, \alpha, s, x, u, v, z].
\]

We have that \( \tau_S \in S[x, u, v]|(\lambda^{++})^S \).

**Lemma 4.4** Suppose \( z \in \mathbb{R}^*_y \). Then \( z \in A \) if and only if for any \( S \in \mathcal{F}_\mathcal{R}^0 \) such that \( \tau_{\mathcal{R}, S} \) is based on \( w \) and \( z \) is generic over \( S \) for \( \mathcal{E}_A^S_{\pi_{\mathcal{R}, S}(w)} \), and for any \( S[x, u, v] \)-generic \( h \subseteq \text{Coll}(\omega, \pi_{\mathcal{R}, S}(\lambda)) \) with the property that \( z \in S[x, u, v][h] \), \( z \in (\tau_S)_h \).
Proof. Assume $z \in A$ and let $(S, h)$ be such that

1. $S \in \mathcal{F}^g_R$, $\mathcal{T}_{R,S}$ is based on $w$ and $z$ is generic over $S$ for $\text{EA}^S_{\pi_{R,S}(w)}$, and

2. $h \subseteq \text{Coll}(\omega, \pi_{R,S}(\lambda))$ is $S[x, u, v]$-generic with the property that $z \in S[x, u, v][h]$.

We need to see that $z \in (\tau_S)_h$. Continue $\mathcal{T}_{R,S}$ normally to get a $Q$ that is a genericity iterate of $\mathcal{R}$ (see Proposition 3.3). Because $\text{crit}(\pi_{R,S}) > \eta^w$, we have that

1. $u$ codes $(P|\lambda_0, \pi_{P,Q} | P|\lambda_0, Q|\pi_{P,Q}(\lambda_0))$ and

2. $v$ codes a pair of reals $(v_0, v_1)$ such that $v_1$ codes a bijection $k : \omega \to Q|\tau$ and $Y = \{\pi_{Q,\omega}(k(n)) : n \in v_0\}$.

Notice now that because $z \in A$, we have that $C_\alpha(g) \models \phi[\text{Code}(\Sigma^g_{P|\lambda_0}), x, s, z]$. Tracing definitions, we see that $V_Q[x, u, v, z] \models \psi[Q|\kappa^+, \kappa, \alpha, s, x, u, v, z]$. It then follows that $z \in (\tau_Q)_h$. Because $\mathcal{R}$ is $A$-stable, we have that $\pi_{R,Q}(\alpha, s) = (\alpha, s)$ and $\pi_{R,Q}(Y) = \pi_{R,Q}[Y] = Y^{19}$, implying that $\pi_{S,Q}(\tau_S) = \tau_Q$. Because $\text{crit}(\pi_{S,Q}) > \pi_{R,S}(\lambda)$, it follows that $\tau_S = \pi_{S,Q}(\tau_S)$. Hence, $\tau_S = \tau_Q$ and therefore, $x \in (\tau_S)_h$.

Assume now that $z$ has the property that $S \in \mathcal{F}^g_R$ such that $\mathcal{T}_{R,S}$ is based on $w$ and $z$ is generic over $S$ for $\text{EA}^S_{\pi_{R,S}(w)}$, and for any $S[x, u, v]$-generic $h \subseteq \text{Coll}(\omega, \pi_{R,S}(\lambda))$ with the property that $z \in S[x, u, v][h]$, $z \in (\tau_S)_h$. We now want to see that $z \in A$.

Once again continue $\mathcal{T}_{R,S}$ normally to obtain a $Q$ that is a genericity iterate of $\mathcal{R}$. We now have that $z \in (\tau_Q)_h$ (because $\tau_S = \pi_{S,Q}(\tau_S) = \tau_Q$). It then follows that $Q[x, u, v, z] \models \psi[Q|\kappa^+, \kappa, \alpha, s, x, u, v, z]$.

Reversing the above implications, we get that $C_\alpha(g) \models \phi[Y, \text{Code}(\Sigma^g_{P|\lambda_0}), x, s, z]$. Thus, $z \in A$. □

---

19 $\pi_{R,Q}(Y)$ makes sense as $Y \in \mathcal{R}[x, u, v]$ and $\text{crit}(\pi_{R,Q}) > \eta^w$. 

23
5 The proof of Theorem 0.4

Let $\mathcal{M} = \mathcal{M}_\infty$. We then have that $\nu = \kappa_\infty$. Clauses 1-3 of Theorem 0.4 follow from Theorem 4.1. Clause 4 follows from the results of [13].

Conjecture 0.1 will eventually be settled by a robust core model induction technique which at the moment of writing this paper humanity does not posses. However, it is an interesting question whether one can prove more instances of it in nice structures. We suspect that it can be derived from just assuming that the universe has a unique universally Baire iteration strategy. For more on this concept see [8].

References

[1] Qi Feng, Menachem Magidor, and Hugh Woodin. Universally Baire sets of reals. In Set theory of the continuum (Berkeley, CA, 1989), volume 26 of Math. Sci. Res. Inst. Publ., pages 203–242. Springer, New York, 1992.

[2] Ronald Jensen, Ernest Schimmerling, Ralf Schindler, and John Steel. Stacking mice. The Journal of Symbolic Logic, 74(01):315–335, 2009.

[3] W. J. Mitchell, E. Schimmerling, and J. R. Steel. The covering lemma up to a Woodin cardinal. Ann. Pure Appl. Logic, 84(2):219–255, 1997.

[4] Grigor Sargsyan. Covering with universally Baire operators. Advances in Mathematics, 268:603–665, 2015.

[5] Grigor Sargsyan. Hod Mice and the Mouse Set Conjecture, volume 236 of Memoirs of the American Mathematical Society. American Mathematical Society, 2015.

[6] Grigor Sargsyan. Descriptive inner model theory. Bull. Symbolic Logic, 19(1):1–55, 2013.

[7] Grigor Sargsyan and Nam Trang. The largest Suslin axiom. Submitted. Available at math.rutgers.edu/~gs481/lsa.pdf.

[8] Grigor Sargsyan and Nam Trang. Sealing from iterability. Available at http://www.grigorsargs.net/papers.html.

[9] Grigor Sargsyan and Nam Trang. The exact consistency strength of generic absoluteness for universally Baire sets. 2019. Available at math.rutgers.edu/~gs481/lsa.pdf.
[10] E. Schimmerling and W. H. Woodin. The Jensen covering property. *J. Symbolic Logic*, 66(4):1505–1523, 2001.

[11] Farmer Schlutzenberg. The definability of $\vec{E}$ in self-iterable mice. Submitted. Available at https://sites.google.com/site/schlutzenberg/home-1/research/papers-and-preprints.

[12] John Steel. Mouse pairs and Suslin cardinals. Available at https://math.berkeley.edu/~steel/papers/mousepairs.suslin.pdf.

[13] John Steel and Nam Trang. Condensation for mouse pairs. Available at https://math.berkeley.edu/~steel/papers/.

[14] John R. Steel. *The core model iterability problem*, volume 8 of *Lecture Notes in Logic*. Springer-Verlag, Berlin, 1996.

[15] John R. Steel. Gödel’s program. In *Interpreting Gödel*, pages 153–179. Cambridge Univ. Press, Cambridge, 2014.

[16] John R. Steel. Normalizing iteration trees and comparing iteration strategies. 2016. Available at math.berkeley.edu/~steel/papers/Publications.html.

[17] Hugh Woodin. Iteration hypotheses and the strong sealing of universally baire sets. *Slides from Reflections on Set Theoretic Reflection, available at http://www.ub.edu/RSTR2018/slides.htm*, 2018.

[18] W. Hugh Woodin. In search of Ultimate-$L$: the 19th Midrasha Mathematicae Lectures. *Bull. Symb. Log.*, 23(1):1–109, 2017.