Diffusive and ballistic motion in superconducting hybrid structures.

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We examine transport properties of superconducting hybrid mesoscopic structures, in both the diffusive and ballistic regimes. For diffusive structures, analytic results from quasi-classical theory are compared with predictions from numerical, multiple-scattering calculations performed on small structures. For all structures, the two methods yield comparable results and in some cases, quantitative agreement is obtained. These results not only demonstrate that quasi-classical theory can yield the ensemble averaged conductance \( <G> \) of small structures of dimensions of order 10-20 Fermi wavelengths, but also establish that numerical scattering calculations on such small structures can yield results for \( <G> \) which are characteristic of much larger systems. Having compared the two approaches, we extend the multiple-scattering analysis to the ballistic limit, where the sample dimensions become smaller than the elastic mean free path and demonstrate that the properties of certain Andreev interferometers are unchanged in the clean limit.

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I. INTRODUCTION

When a normal metal (N) makes contact with a superconductor (S), classical tunnelling theory predicts that as a consequence of the existence of a superconducting energy gap, the low-temperature sub-gap conductance will vanish. Andreev scattering \[1\] provides an alternative mechanism for charge transport through such an N-S junction and for an ideal interface \[2\], leads to a zero-voltage conductance which is almost twice that of the normal state. In the presence of an insulating layer (I), Andreev scattering becomes less effective and in the absence of disorder, the sub-gap conductance of an N-I-S junction is predicted to be depressed compared with that of the normal state. In contrast, at low enough temperatures, experiments on N-I-S junctions \[3\] reveal the existence of a zero-voltage conductance peak, with a value comparable to the conductance in the normal state. This effect is due to the interplay between disorder-induced scattering and tunnelling and has been observed in experiments involving quantum wells with superconducting electrodes \[4\], superconductor-normal metal micro-junctions \[5\] and superconductor-2DEG-superconductor structures \[6\]. These have been interpreted using a number of theoretical approaches, including quasi-classical Green function techniques \[7\]-\[12\], tunnelling Hamiltonian methods \[13\],\[14\], multiple-scattering techniques \[15\]-\[17\], and random matrix theory \[18\],\[19\].

Recently, following a number of theoretical papers on disordered transport in the presence of two superconducting contacts \[20\]-\[22\], several new experiments aimed at probing the phase-coherent nature of Andreev scattering have been carried out. These involve a normal metal in contact with superconductors \( S_1 \) and \( S_2 \), with order parameter phases \( \phi_1, \phi_2 \), whose order parameter phase difference \( \phi = \phi_1 - \phi_2 \) can be controlled by external means. For a diffusive system of size \( L \), with diffusion coefficient \( D \), early theoretical work \[21\],\[21\] predicted that in the high temperature limit \( T > T^* \), where \( k_B T^* = \hbar D/L^2 \), the ensemble averaged conductance \( <G> \) should be a periodic function of \( \phi \), with fundamental period \( \pi \), whereas in the low temperature regime \( T < T^* \), it was predicted \[22\],\[23\] that the fundamental periodicity of \( <G> \) would be \( 2\pi \). This prediction of a \( 2\pi \) periodicity is a common feature of all recent theories of \( <G> \) \[3\],\[4\],\[24\],\[25\] and of recent experiments \[27\]-\[31\].

Despite much progress, many details of such Andreev interferometers remain to be understood. In particular, there exists no general theory of the amplitude of oscillation, the nature of the zero-phase extremum and the harmonic content of the conductance-phase characteristic. Experiments on various geometries, in different transport regimes have yielded amplitudes of oscillation which differ by many orders of magnitude. It has also been questioned whether analytical theories that mainly apply in the diffusive limit can be used in the quasi-ballistic case \[32\].

One difficulty in establishing a general theory is that most theoretical papers are based on a single technique, with little detailed comparison with the results of other approaches. For example, distinct analytical theories exist for the ballistic, diffusive and strongly disordered regimes, but there is no analytical theory capable of describing the cross-over between them. In contrast, exact numerical solutions of the Bogoliubov - de Gennes equation \[32\] can easily cross from one regime to another, but are limited to system sizes of order a hundred Fermi wavelengths.

In this paper, we undertake a detailed comparison between an exact numerical, multiple-scattering technique and quasi-classical theory. The former was first used to solve the Bogoliubov - de Gennes equation for disordered, one-dimensional systems \[33\] and soon after generalized to higher dimensions \[34\], while the latter, developed in the context of nonequilibrium superconductivity \[35\]-\[38\], when supplemented by the appropriate boundary
conditions [11], [12], has recently yielded a variety of results for ensemble averaged conductances in the diffusive limit [3]. Once agreement between the two approaches is obtained in the diffusive limit, we then use the numerics to follow the crossover to the ballistic regime, where the electron mean free path becomes comparable with the system size.

The comparison will be carried out by examining two canonical examples of mesoscopic superconducting hybrid structures, namely an N-I-S junction and an Andreev interferometer. In section II we briefly recall some results of the quasi-classical Green function approach for these structures and in section III highlight the main features of the numerical scattering approach. Whereas a quasi-classical approach yields a one-dimensional theory whose results depend only on the topology of the structure, the numerical calculations, in common with real experiments, require the specification of a suitable geometry. In section IV we present a detailed discussion of the geometry and numerical parameters needed to reproduce the results of quasi-classical theory. Having established agreement between the two approaches we then depart from the results of quasi-classical theory. Having established agreement between the two approaches we then depart from the diffusive limit and investigate the cross-over to ballistic transport, which is particularly relevant to the experiments of reference [30].

II. RESULTS FROM QUASI-CLASSICAL THEORY.

In this section, we highlight some predictions of the quasi-classical approach of references [6], [13]. These theories focus on the ensemble averaged conductance \( \langle G \rangle \) and ignore the weak localisation contribution discussed in [24], [25]. The latter does not scale with the system size and in systems with a conductance much larger than \( 2e^2/h \), can be neglected. The results of references [6]-[12] are obtained by solving the following equation for the quasi-classical Green function \( \hat{g} \) in the diffusive limit \( D\partial_t (\hat{g}\partial_t \hat{g}) + iE[\hat{r}_z, \hat{g}] = 0 \), where \( E \) is the quasi-particle excitation energy [11]. In what follows, we shall consider only the \( E = 0 \) limit of this equation, which applies to a diffusive system, whose length \( L \) is assumed to satisfy the inequalities \( L \ll \xi \), where \( \xi^2 = D/\Delta_0 \) and \( \xi^2 = D/E \), in the superconducting and normal region respectively. Here \( D \) is the diffusion coefficient, \( \Delta_0 \) the energy gap and \( E = eV \), where \( V \) is the applied voltage. Furthermore all temperatures and voltages are assumed to be much smaller than \( \Delta_0 \). For convenience in what follows, we shall also restrict the analysis to zero temperature.

In units of \( 2e^2/h \) theory predicts [11], [13] that the total conductance \( G \) of the structure shown in figure 2(a) may be computed from the equation

\[
\frac{1}{G} = \frac{1}{G_{\text{diff}}} + \frac{1}{G_{\text{tun}} \sin \theta}, \tag{1}
\]

where \( \theta \) is a solution of the transcendental equation

\[
G_{\text{diff}} \theta = G_{\text{tun}} \cos \theta \tag{2}
\]

and \( G_{\text{tun}}, G_{\text{diff}} \) are the respective conductances of the tunnel junction and diffusive region in isolation.

There are two obvious limits to take. The first is where \( G_{\text{tun}}/G_{\text{diff}} \rightarrow 0 \) for which \( \theta \approx G_{\text{tun}}/G_{\text{diff}} \approx \sin \theta \) and hence

\[
\frac{1}{G} \approx \frac{G_{\text{diff}}}{G_{\text{tun}}}.
\tag{3}
\]

The second limit is \( G_{\text{tun}}/G_{\text{diff}} \rightarrow \infty \) in which \( \theta \rightarrow \pi/2 \). This yields

\[
\frac{1}{G} \approx \frac{1}{G_{\text{diff}}} + \frac{1}{G_{\text{tun}}}.
\tag{4}
\]

Eq.(3) has been also directly obtained in the tunneling Hamiltonian limit [13]. Furthermore, as emphasized in Ref. [10], the change of the power in the dependence on the tunnel junction conductance \( G_{\text{tun}} \), as described by the equations (3) and (4), reflects the combined effect of the Andreev scattering at the N-S interface and the interference effects in the mesoscopic phase coherent disordered normal region. In the regime described by eq.(4) and when \( G_{\text{tun}} \ll 1 \), the conductance of the N-S junction appears to be the same as in the normal state and the total resistance is obtained by simply adding up in series the resistances of the diffusive region and tunnel junction. Equations (3) and (4) may be solved numerically to yield the resistance \( R_{\text{tot}} = 1/G \) of the system as a function of \( G_{\text{diff}} \) and \( G_{\text{tun}} \). The result of such an exercise for a fixed value \( G_{\text{diff}} = 1.6 \) and a variable \( G_{\text{tun}} \) is shown by the curve (a) of figure 2. Also plotted are the limits given by equations (3) and (4) represented by the dashed lines (e) and (d). The curve (b) shows the results of the numerical simulation described in section IV.

As a second example, quasi-classical theory predicts [11] that the conductance \( G \) of the structure depicted in figure 3(a) may be computed from the equation

\[
G = \frac{4 \cos^2(\phi/2)}{\{G_1^2 + 4G_2^2 \cos^2(\phi/2)\}^{3/2}},
\tag{5}
\]

where \( G_1 \) is the conductance of the tunnel junction (1), \( G_2 \) is the conductance of the tunnel junction (2) and \( \phi \) is the phase difference between the two superconductors. In the limit \( G_1 \gg G_2 \), this simplifies to the expression

\[
G = \frac{4G_2^2}{G_1} \cos^2(\phi/2),
\tag{6}
\]

whereas if \( G_2 \gg G_1 \) then

\[
G = \frac{1}{2} \frac{G_2^2}{G_1} \left| \frac{1}{\cos^2(\phi/2)} \right|.
\tag{7}
\]
III. A SCATTERING APPROACH TO TRANSPORT IN MESOSCOPIC SUPERCONDUCTORS.

During the past six years [32, 33] we have developed numerical codes capable of yielding exact solutions of the Bogoliubov - de Gennes equation for disordered structures in arbitrary dimensions. Currently there are two independent sets of codes available at Lancaster; one of these is based on a transfer matrix approach and the other is based on a recursive Green’s function method. Typically these are used as independent cross-checks and therefore there can be no doubt about the accuracy of the results obtained for a given structure. For a two-dimensional system of width less than a few hundred Fermi wavelengths, or for a three-dimensional system of width less than a few tens of Fermi wavelengths, the problem of computing dc transport properties of a phase-coherent system described by mean-field BCS theory is therefore no longer an issue. Just as the appearance of pocket calculators rendered approximate methods for computing elementary functions redundant, the existence of these codes has, for several years, allowed transport properties of small systems to be calculated without further approximation. For larger systems, the key issue is how to extrapolate the results of such calculations to larger numbers of channels. By making contact with quasi-classical theory, the results which follow establish that for many systems, the ensemble averaged conductance obtained from small systems exhibits the essential features of much larger structures.

The numerical codes evaluate multi-channel scattering formulæ for the dc electrical conductance [15, 32] and more recently thermoelectric properties [16]. In what follows, we focus on hybrid structures connected to normal external reservoirs only [16]. For a structure connected to two such reservoirs, the zero-temperature, zero-bias electrical conductance can be written [17, 32] (in units of $2e^2/h$),

$$G = T_0 + T_a + \frac{2(R_a R'_a - T_a T'_a)}{R_a + R'_a + T_a + T'_a} \quad (8)$$

In this expression, $R_0, T_0$ ($R_a, T_a$) are the coefficients for normal (Andreev) reflection and transmission for zero-energy quasi-particles from reservoir 1, while $R'_0, T'_0$ ($R'_a, T'_a$) are the corresponding coefficients for quasi-particles from reservoir 2. If each of the external leads connecting the reservoirs to the scatterer contains $N$ open channels, these satisfy $R_0 + T_0 + R_a + T_a = R'_0 + T'_0 + R'_a + T'_a = N$ and $T_0 + T_a = T'_0 + T'_a$. Furthermore, in the absence of a magnetic field, all reflection coefficients are even functions of $\phi$, while the transmission coefficients satisfy $T'_0(\phi) = T_0(-\phi)$, $T'_a(\phi) = T_a(-\phi)$. Consequently on quite general grounds, in the absence of a magnetic field, $G$ is predicted to be an even function of $\phi$.

In the absence of quasi-particle transmission between the two external probes, equation (8) reduces to

$$G^{-1} = (2R_a)^{-1} + (2R'_a)^{-1}, \quad (9)$$

where $2R_a$ (2$R'_a$) are left (right) boundary conductances, introduced by Blonder, Tinkham and Klapwijk [2]. The Lancaster codes yield all possible scattering coefficients, but in what follows, to compare results with the theory leading to eqs. (8) and (9), we shall analyze structures with no transmission and restrict attention to $R_a$ only.

The numerical codes compute the scattering coefficients of a tight-binding lattice, described by a Bogoliubov - de Gennes (BdG) Hamiltonian of the form

$$H = \left( \begin{array}{cc} H_0 & \Delta \\ \Delta^* & -H_0^* \end{array} \right). \quad (10)$$

If an index $n$ is used to label sites on the lattice and any internal spin-degrees of freedom, then $H_0$ is of the form

$$H_0 = \sum_n \epsilon_n |n\rangle\langle n| + \sum_{(n,m)} V_{n,m} |n\rangle\langle m|. \quad (11)$$

To model a given physical structure, it is necessary to specify certain phenomenological parameters which capture the essential physics. As an example, in the absence of spin-orbit scattering, spin degrees of freedom can be ignored and in the absence of a magnetic field, one chooses $V_{n,m} = -\gamma$ for nearest neighbour pairs $(n,m)$. If $(n,m)$ are not nearest neighbours, then $V_{n,m} = 0$. In a region free from disorder, the diagonal elements $\epsilon_n$ are set equal to a constant $\epsilon_0$, whereas in a disordered region, $\epsilon_n$ is a random number uniformly distributed between $\epsilon_0 - W$ and $\epsilon_0 + W$. In the presence of spin singlet, local s-wave pairing, $\Delta$ is a diagonal order parameter matrix with elements $\Delta_n$. In a normal region, $\Delta_n = 0$, whereas in a clean superconducting region, $|\Delta_n|$ is set to a constant value $\Delta_0$. The phase of $\Delta_n$ is chosen to equal a value assigned to the superconducting region to which site $n$ belongs. In what follows, the energy scale will be fixed by making the choice $\gamma = 1$.

Of course, the above parameters are not directly accessible experimentally and are not an explicit feature of quasi-classical theory. Therefore when making comparisons, some effort is needed to map one analysis onto
another. In $d$-dimensions, for a clean system on a square or cubic lattice, the chemical potential relative to the band bottom is $\mu = \epsilon_0 + 2d\gamma$, the band width is $4d\gamma$ and the effective mass for excitations near the band bottom is $m^* = h^2/(2\gamma a^2)$, where $a$ is the lattice constant. A key parameter in the problem is the dimensionless ratio $\Delta = \Delta_0/\mu$, which takes a value $10^{-3}$ for conventional low $T_c$ superconductors such as Niobium, but can be as large as 0.1 for high-temperature superconductors, or for a 2DEG in contact with a conventional superconductor. Andreev’s approximation, which underpins many analytic theories, including quasi-classical and random matrix descriptions, is valid only when this parameter is much less than unity.

Other parameters which are needed when making comparisons are the Thouless energy $E^*$, which for a diffusive structure of width $M$, length $L_{\text{diff}}$ and normal-state conductance $G$, is given by $E^* = hfL_{\text{diff}}^2 = (h/2e^2)G/(n(0)L_{\text{diff}}M^{d-1})$, where $n(0)$ is the density of states per site. A second parameter is the normal-state, elastic mean-free path $l$, which for a diffusive sample connected to external lead with $N$ open channels, is given by $G = (2e^2/h)n(0)L_{\text{diff}}$. Within a numerical simulation on a given geometry, once the model parameters $W$, $\gamma$, $\epsilon_0$ and $\Delta_0$ are chosen, the parameters $G$, $l$, $n(0)$ and $E^*$ are computed explicitly.

### IV. NUMERICAL RESULTS FOR A N-I-S STRUCTURE.

In this section, we present a comparison between the predictions of quasi-classical theory and the above numerical scattering approach, for an N-I-S structure. Our aim is to highlight the steps required to obtain a suitable choice of parameters, from which a meaningful comparison can be made. In the literature, numerical results in two-dimensions have been obtained by first solving for the scattering matrix of a normal diffusive structure, with or without a tunnel junction and then employing Andreev’s approximation to model the Andreev scattering induced by a nearby superconductor. As noted above, this approximation requires that $\Delta_0$ be small compared with the Fermi energy and that there be no disorder in the superconductor. Furthermore for a clean system, Andreev’s approximation can yield incorrect results, because even at a clean N-S interface, the approximation breaks down when scattering channels are almost closed. For these reasons a comparison with an exact solution of the Bogoliubov - de Gennes equation allows one to examine changes occurring away from the Andreev limit, in a region of parameter space which is more relevant to high-temperature superconductors.

The system to be examined is shown in figure 1(b) and consists of a disordered region in contact with a tunnel junction, which is in turn adjacent to a superconducting probe. The simulated structure consists of a two-dimensional tight-binding lattice of width $M$ sites. The disordered region is of length $L_{\text{diff}}$ sites, the tunnel junction is $L_{\text{tun}}$ sites long and the superconductor has a length $L_{\text{sup}}$. In units of $2e^2/h$, the conductance of a particular realisation of the structure will be denoted $G$ and the ensemble-averaged conductance will be written $\langle G \rangle$. In order to make a comparison with quasi-classical theory, it is necessary that the properties of the two resistive components and the superconducting probe be compatible with the assumptions made by the theory.

To identify a suitable choice of parameters, consider first a normal diffusive portion of length $L_{\text{diff}}$ and width $M$, connected to crystalline leads. In units of $2e^2/h$, the conductance of a particular realisation of this structure will be denoted $G_{\text{diff}}$ and the ensemble-averaged conductance will be written $\langle G_{\text{diff}} \rangle$. The conductance of a diffusive material is inversely proportional to its length and therefore a plot of $\langle G_{\text{diff}}\rangle/L_{\text{diff}}$ as a function of $L_{\text{diff}}$ will exhibit a plateau in the diffusive regime, with a mean free path given by $l = \langle G_{\text{diff}}\rangle/L_{\text{diff}}/N$.

For a sample of width $M = 10$, figure 3 shows a plot of $\langle G_{\text{diff}}\rangle/L_{\text{diff}}/N$ versus $L_{\text{diff}}$. This structure has periodic boundary conditions in the direction transverse to the current flow and the choice $\epsilon_0 = 0.2$ was made, which yields $N = 9$. Results are shown for a disorder of $W = 1$. The length $L_{\text{diff}}$ of the disordered region was incremented in steps of two sites from $L_{\text{diff}} = 2$ to 40. For each value of $L_{\text{diff}}$, 2000 realizations of disorder were chosen and the conductance $G_{\text{diff}}$ computed for each. Then the ensemble average $\langle G_{\text{diff}} \rangle$ and the standard deviation $\delta G_{\text{diff}}$ were calculated.

Figure 3 shows that in the interval $20 < L_{\text{diff}} < 40$ the system exhibits diffusive behaviour, with a mean free path of $l \approx 4.6$. For smaller values of $L_{\text{diff}}$, the system is in the ballistic regime and for larger values, the onset of localisation causes the curve to fall. A diffusive system is one for which $l \ll L_{\text{diff}}$ and $l \ll M$. Furthermore if weak localization corrections are to be neglected, we require $NL_{\text{diff}}$. For these reasons a judicious choice of length yielding a diffusive system whilst minimizing CPU time is $L_{\text{diff}} = 25$. Such a system has an average conductance $\langle G_{\text{diff}} \rangle = 1.6$.

To compare with quasi-classical theory, a knowledge of the conductance of the isolated tunnel junction $G_{\text{tun}}$ as a function of the barrier height $\epsilon_b$ is also required. In what follows, we consider a clean tunnel junction of dimensions $M = 10$ and $L_{\text{tun}} = 1$, obtained by setting the diagonal elements $\epsilon_n$ of all barrier-sites $n$ equal to $\epsilon_b + \epsilon_0$. For an isolated barrier connected to crystalline leads of width 10, figure 4 shows a plot of $G_{\text{tun}}$ as a function of $\epsilon_b$. To obtain this plot, the conductance $G_{\text{tun}}$ is computed for 1000 successive values of the barrier height $\epsilon_b$ in the range $0.0 < \epsilon_b < 10.0$. This choice of barrier heights yields a spread of barrier conductances in the convenient range
Finally, before a comparison with theory can be made, the properties of the superconductor must be chosen such that there be no transmission through the superconducting region and that Andreev’s approximation of neglecting normal reflection at the N-S interface is valid. To avoid quasi-particle transmission, it is necessary to choose \( L_{\text{sup}} > \xi \), where \( \xi = \mu/\Delta_0 \) and to minimise normal reflection it is necessary that \( \xi >> 1 \). For \( \Delta_0 = 0.05 \) and \( \epsilon_0 = 0.2 \) the superconducting coherence length is \( \xi = 76 \) and it is found that transmission is negligible for \( L_{\text{sup}} > 100 \). For the above choice of parameters, one finds for the normal and Andreev reflection and transmission coefficients: \( R_0 = 0.06485, R_a = 8.84762, T_0 = 0.08748, \) and \( T_a = 0.00005 \).

It should be noted that the condition \( \xi >> 1 \) is not sufficient to completely exclude normal reflection at an N-S interface. It is also necessary that \( \epsilon_0 \) be chosen such that the number of open channels in the external leads is not sensitive to small changes in \( \epsilon_0 \). This feature is illustrated in figure 3 which shows as a function of \( \epsilon_0 \), the conductance of a clean superconducting region of width \( M = 50 \) and length \( L_{\text{sup}} = 5 \), attached to crystalline normal leads. Results are shown for 5 values of \( \Delta_0 \): \( \Delta_0 = 0, 10^{-2}, 10^{-1}, 0.3 \) and 0.5. For \( \Delta_0 = 0 \), the conductance is equal to the number of open channels and changes by unity whenever an external quasi-particle channel closes. At these values of \( \epsilon_0 \), switching on an infinitesimal \( \Delta_0 \) causes the conductance to decrease by unity. As shown in the figure, finite values of \( \Delta_0 \) smear the conductance steps and suppress the conductance. Both of these features lie outside Andreev’s approximation. To achieve compatibility with the assumptions of circuit theory, the choice \( \epsilon_0 = 0.2 \) was made, which places a system of width 10 between two conductance steps and avoids the above sensitivity to changes in \( \Delta_0 \).

Having identified a choice of parameters which is compatible with quasi-classical theory, numerical results for the combined structure of figure 2b can now be obtained. To summarize, this structure has the following properties: width \( M = 10 \), number of open channels \( N = 9 \), band filling factor \( \epsilon_0 = 0.2 \) leading to a chenial potential \( \mu = 3.8 \), length of tunnel junction \( L_{\text{tun}} = 1 \), barrier heights \( 0 < \epsilon_b < 10.0 \), barrier conductances \( 0 < G_{\text{tun}} < 9.0 \), length of diffusive region \( L_{\text{diff}} = 25 \), diffusive disorder width \( W = 1 \), conductance of diffusive region \( \langle G_{\text{diff}} \rangle = 1.6 \), length of superconductor \( L_{\text{sup}} = 100 \), superconducting coherence length \( \xi = 76 \), superconducting order parameter \( \Delta_0 = 0.05 \), elastic mean free path \( l = 4.5 \).

First consider the case of no barrier, where \( \epsilon_b = 0 \). In this case, quasi-classical theory insists that the conductance of a diffusive region in contact with a superconductor should be identical with the normal-state conductance of the diffusive region. Figure 3 shows plots of the mean conductance \( \langle G \rangle = N - R_0(\Delta_0) + R_a(\Delta_0) \) as a function of \( \Delta_0 \), for disordered regions of four different lengths. In the normal state \( \langle \Delta_0 = 0 \rangle \) \( \langle G \rangle \) reduces to \( T_0(0) - R_0(0) \) and in the presence of a sufficiently-long superconductor, to the BTK conductance \( 2R_a(\Delta_0) \). The left inset shows the quantity \( < g > = (N - R_0(\Delta_0) + R_a(\Delta_0))/T_0(0) \) (ie the conductance divide by the normal state conductance). The right inset shows the root mean square deviation \( \sigma = (\langle G(\Delta_0) \rangle - \langle G(\Delta_0) \rangle)^2)^{1/2} \). These show that in the ballistic limit \( L_{\text{diff}} = 0 \), the conductance rises to a value almost double that of the normal state, before decreasing with increasing \( \Delta_0 \). In contrast, the mean conductance of a diffusive normal region is relatively insensitive to the onset of superconductivity, with the largest relative change corresponding to the largest value of \( L_{\text{diff}} \), ( ie the smallest value of the normal state conductance). It should be noted however that even though the mean is insensitive to \( \Delta_0 \), the fact that the rms deviation \( \sigma \) is non-zero reveals that for individual samples, large changes of arbitrary sign can occur.

Having examined a diffusive conductor with no barrier, we now turn to the case of finite \( \epsilon_b \). Curve (b) of figure 3 shows numerical results in the presence of a tunnel barrier. For 50 equally-spaced barrier heights in the range \( 0 < \epsilon_b < 10.0 \), 500 realizations of disorder in the diffusive region were selected and the total conductance \( G \) computed for each realisation. The ensemble-averaged conductance \( \langle G \rangle \) was then calculated and finally the total resistance \( R_{ \text{tot} } = 1/\langle G \rangle \) plotted against the computed conductance \( G_{\text{tun}} \) of the isolated tunnel junction. Since the average conductance of the diffusive region \( \langle G_{\text{diff}} \rangle \) is also known, equation (1) can be evaluated to yield the corresponding analytical result, curve (a) of figure 3.

This demonstrates that in the range of validity of quasi-classical theory, quantitative agreement with the numerical scattering approach is obtained.

V. NUMERICAL RESULTS FOR ANDREEV INTERFEROMETERS.

Having examined a simple N-I-S structure, we now compare numerical results for the tight-binding structure of figure 3b, with the predictions of quasi-classical theory for the one-dimensional system of figure 3(a). The latter comprises a tunnel junction connected by diffusive 1-D wires to a fork. Each of the two arms of the fork is a diffusive wire, connected via tunnel junctions to infinitely long superconductors. The conductance of the diffusive wires is assumed to be much greater than that of the tunnel junctions.

The two-dimensional tight-binding realisation of this structure is shown in figure 3(b), which consists of a tunnel junction (1) lying next to a diffusive region which is
in turn adjacent to two superconductors. The superconductors are separated from each other by an insulating layer and from the diffusive region by two identical tunnel junctions (2). The superconductors \( i = 1, 2 \) have order parameter phases \( \phi_i \), but are identical in every other respect. In order that they may successfully represent superconducting probes of infinite length, they are chosen in such a way that quasiparticle transmission through them is negligible.

The diffusive region is of length \( L_{\text{diff}} \), sites, each tunnel junction is \( L_{\text{tun}} \), sites long and the superconductors have a length \( L_{\text{sup}} \). To model a superconducting reservoir, \( L_{\text{sup}} \) is again chosen sufficiently large so that there is negligible transmission through the superconductor. The system width and the width of both the diffusive region and the tunnel junction (1) is \( M \) sites. On the right of the diffusive region, the three insulating layers are each one site thick and therefore the superconductors are each of width \( M' \) (where \( 2M' = M - 3 \)).

In the simulation, the conductances \( G_1 \) and \( G_2 \) of the tunnel junctions are fixed at values which replicate the three situations of figure 3, namely \( G_1 = G_2 \), \( G_1 \ll G_2 \) and \( G_1 \gg G_2 \) to enable comparisons to be made with the analytic results. In each case, the phase difference between the two superconductors is varied and the total conductance \( G \) plotted as a function of phase for a particular realisation of disorder in the diffusive region. Ensemble-averaging over many disorder realisations yields the conductance \( \langle G \rangle \) which is independent of the microscopic configuration of the system and may be usefully compared with the results of equation (3) (see below) and figure 4.

In what follows, we examine a sample of width \( M = 15 \), with a diagonal matrix element \( \epsilon_0 = 0.2 \) and periodic boundary conditions, for which the number of open channels is \( N = 13 \). The disorder is chosen to be \( W = 1 \) and again from a graph of the form of figure 5, we obtain a mean free path of \( l \approx 4.9 \). In most cases of experimental interest, the conductance of the diffusive ‘wires’ may be considered to be much greater than that of the tunnel junctions. In order to take into account this situation in our numerical simulations, one has to put some restrictions on the length of the diffusive region, since the conductance decreases with length. A compromise must therefore be found between the desire to increase the length into the diffusive regime and the wish to decrease it in order to maintain a high conductance. In what follows a choice \( L_{\text{diff}} = 10 \) is made, for which \( \langle G_{\text{diff}} \rangle \approx 5.1 \).

To create a tunnel barrier of length \( L_{\text{tun}} = 1 \) and width \( M \), all diagonal elements \( \epsilon_i \) of sites within the barrier were set to a value \( \epsilon_b = \epsilon_0 + \epsilon_b \). For an isolated tunnel junction (1) of width \( M = 15 \), the values \( \epsilon_b = 3.52, 12.27 \) yield respectively the conductances \( G_1 = 2.0, 0.2 \) and for an isolated tunnel junction (2) of width \( M' = 6 \), the values \( \epsilon_b = 1.87, 7.75 \) yield conductances \( G_2 = 2.0, 0.2 \). These values of \( \epsilon_b \) were used in the simulations of figure 9. As in the previous section, the choice \( \epsilon_0 = 0.2, \Delta_0 = 0.05, L_{\text{sup}} = 100 \) was made. For the superconductor alone, connected to crystalline normal leads of width \( M' = 6 \), the values of the normal and Andreev reflection and transmission coefficients were found to be: \( R_0 = 0.02155, R_s = 4.92905, T_0 = 0.04937, T_s = 0.00003 \). For such a structure, there are 5 open channels and as a consequence, the sum of these four coefficients is 5. Finally, in order to model an insulating barrier, the diagonal matrix elements \( \epsilon_i \) referring to a site \( i \) on the barrier between the superconductors, were each set to the large number \( \epsilon_i = \epsilon_0 + 50 \).

By combining the above components to yield the complete structure of figure 3(b), one obtains the structure to be analysed numerically, whose parameters are as follows: total width \( M = 15 \), superconductor width \( M' = 6 \), number of open channels in normal lead \( N = 13 \), band filling \( \epsilon_0 = 0.2 \), chemical potential \( \mu = 3.8 \), length of tunnel junctions \( L_{\text{tun}} = 1 \), length of diffusive region \( L_{\text{diff}} = 10 \), disorder width \( W = 1.0 \), conductance of diffusive region \( \langle G_{\text{diff}} \rangle = 5.1 \), length of superconductor \( L_{\text{sup}} = 100 \), superconducting coherence length \( \xi = 76 \), superconducting order parameter \( \Delta_0 = 0.05 \).

To carry out the simulation, the conductances \( G_1 \) of tunnel junction (1) and \( G_2 \) of tunnel junction (2) were fixed and a particular realisation \{\( \epsilon_i \)\} of disorder in the diffusive region was selected. Then the phase difference \( \phi = \phi_1 - \phi_2 \) between superconductors 1 and 2 was varied from zero to \( 2\pi \). This was done by fixing \( \phi_1 = 0 \) and choosing 50 evenly spaced values of \( \phi_2 \). For each value of \( \phi \), 200 different diffusive regions were obtained and the conductance \( G \) of the whole system computed for each. The ensemble-averaged conductance \( \langle G \rangle \) was then calculated. The graphs of figure 9 show plots of \( \langle G \rangle \) as a function of the phase difference \( \phi \) for the following four combinations of \( G_1 \) and \( G_2 \). They are (a) \( G_1 = 0.2, G_2 = 0.2 \), (b) \( G_1 = 2.0, G_2 = 0.2 \), (c) \( G_1 = 0.2, G_2 = 2.0 \), (d) \( G_1 = 2.0, G_2 = 2.0 \).

Apart from the different vertical scales, the numerical results of figure 9(a) and the analytic results of figure 4 share many qualitative features and also exhibit some interesting differences. Figure 9(d) is comparable with 4(b); each exhibits a zero-phase minimum and a further minimum at \( \phi = \pi \). Similarly 9(b) is comparable with 4(b); each exhibits a zero-phase maximum, with a minimum at \( \phi = \pi \). The remaining curves compare less favourably. Whereas the analytic results of figures 4(a) and 4(d) are necessarily identical, there is no such restriction on the numerics and as shown in figure 9(a), decreasing the conductances \( G_1 \) and \( G_2 \) can produce qualitative changes. As a consequence, figure 9(a) possesses a zero-phase maximum, whereas figure 9(b) possesses a zero-phase minimum.

Figures 9(c) and 9(c) also reveal some differences. Each possesses a zero-phase minimum, but at \( \phi = \pi \), where the analytic result vanishes, the numerical result is
almost maximal. The inset in figure 9(c) shows a ‘blowing up’ of the region $\pi - 0.15 \leq \phi \leq \pi + 0.15$, with $G_1 = 0.2$ and $G_2 = 2.0$. The inset shows three curves, obtained by averaging over different numbers of disordered samples, namely 200, 1000 and 2000 realisations of the disorder. These demonstrate that in contrast with eq. (3), the numerical results 9(c) possess a shallow, local minimum at $\phi = \pi$.

Finally, we end this discussion by noting that for systems with a small number of open channels, the behaviour of an individual sample can be very different from that of the mean. For each of the four cases (a) to (d), figure 10 shows each of the 200 plots of conductance $G$ from which the ensemble averages of figure 9 were calculated. Apart from the case (b), where individual members of the ensemble behave in the same manner as the ensemble average, the nature of the extrema at $\phi = 0, \pi$ depends on the microscopic realisation of the disorder. We also note that by changing the dimensions of the sample, one can change the details of figure 9, but not the qualitative shape. For example by increasing the length $L_{diff}$ from 10 to 15, the local minimum at $\phi = \pi$ in figure 9c becomes more pronounced, but further increasing $L_{diff}$ to 20 causes the minimum to become more shallow.

Having compared the quasi-classical theory of references [7]-[12] with the numerical scattering approach in the diffusive limit, we now examine the cross-over to the ballistic regime, where the former is inapplicable. We focus attention on the interferometer of figure 3(b) and examine the change in behaviour as the length $L_{diff}$ of the diffusive region becomes smaller than the elastic scattering length $l$. Apart from the change in $L_{diff}$ all other parameters are fixed to the values used in figure 9. Figure 11 shows results for a diffusive region of length $L_{diff} = 5$ and figure 12 for a length $L_{diff} = 1$. Remarkably, apart from the overall increase in the conductance, the qualitative shape of the curves is unchanged, despite the fact that the restrictions on quasi-classical theory are violated by these structures.

VI. DISCUSSION

In this paper, for the first time, a detailed comparison between quasi-classical theory and numerical multiple-scattering calculations has been carried out. To ensure that the simulated structures fall within the parameter range where the approximations of quasi-classical Green function methods hold, we have painstakingly examined each component of a given structure. For the N-I-S structures of figure 1, figure 2 shows that there is quantitative agreement between the two methods. For the interferometers of figure 3, there is broad qualitative agreement, although as shown in figures 3 and 4, some interesting differences are present. The theory of references [7]-[12] is a quasi-one-dimensional theory and for the symmetric structures of figure 3, necessarily predicts a vanishing conductance at $\phi = \pi$. This symmetry is not present at a microscopic level and therefore there is no such restriction on the results from an exact solution of the Bogoliubov - de Gennes equation. Figures 4(a,b,d) suggest that for certain structures, this microscopic symmetry-breaking may be unimportant, but for other strengths of the tunnel barriers, figure 4(c) suggests that this artifact of quasi-classical theory will not be observed experimentally. In the clean limit, figures 11 and 12 show that although the overall conductance is increased, the qualitative shape of the conductance-phase curves is unchanged.

The above results demonstrate that quasi-classical theory yields the correct shape for the ensemble-averaged conductance even down to extremely small system sizes and that results obtained for dirty systems can be applicable, in some cases, even in the clean limit. They also demonstrate that even without attempting a systematic extrapolation to a large number of channels, numerical multiple-scattering calculations on small structures, can yield results for ensemble-averaged properties of much larger systems.

The results of figure 12 demonstrate that the essential properties of these interferometers are unchanged in the clean limit and therefore as already noted in 4 disorder is not a necessary feature of large amplitude Andreev interferometers.

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FIG. 1. (a) Schematic picture of an N-I-S junction for analysis using circuit theory. (b) Picture of the tight-binding lattice used to model a two-dimensional N-I-S structure.

FIG. 2. Conductance of the N-I-S structures of figure 1, as a function of the conductance $G_{tun}$ of the tunnel junction. The various curves refer to: a) analytic theory; b) numerical simulation; c) resistance of the diffusive region $R_{diff} = 0.63 = \langle G_{diff} \rangle^{-1}$; d) asymptotics at high transparency $G_{tun} \gg \langle G_{diff} \rangle$; e) asymptotics at low transparency $G_{tun} \ll \langle G_{diff} \rangle$.

FIG. 3. (a) Schematic picture of an interferometer analyzed using circuit theory. (b) Picture an interferometer formed from a two-dimensional tight-binding lattice.

FIG. 4. Analytic results for the conductance versus phase in the diffusive limit: a) $G_1 = 0.2$ and $G_2 = 0.2$. b) $G_1 = 2.0$ and $G_2 = 0.2$. c) $G_1 = 0.2$ and $G_2 = 2.0$. d) $G_1 = 2.0$ and $G_2 = 2.0$. 

FIG. 5. $G_{diff} > L_{diff}/N$ versus $L_{diff}$. The plateau region signifies the diffusive regime. Here the number of open channels is $N = 9$. In the inset standard deviation is also shown.

FIG. 6. Conductance $G_{tun}$ of the tunnel junction as a function of the barrier height $\epsilon_0$. 

FIG. 7. The conductance $G(\Delta_0)$ of a clean superconducting region of length $L_{sup} = 5$, width $M = 50$, plotted as a function of the site energy $\epsilon_0$, for 5 different values of $\Delta_0$.

FIG. 8. Plots of the mean conductance $\langle G(\Delta_0) \rangle = N - R_0 + R_a$ as a function of $\Delta_0$, for a diffusive conductor of width $M = 10$ and four different lengths. The left inset shows the conductance $\langle g \rangle = \langle G(\Delta_0) \rangle / \langle G(0) \rangle$, scaled by the normal state conductance. The right inset shows the rms deviation $\sigma = \langle (G(\Delta_0) - \langle G(\Delta_0) \rangle)^2 \rangle^{1/2}$. 

FIG. 9. Numerical results for the conductance versus phase for $L_{diff} = 10$: a) $G_1 = 0.2$ and $G_2 = 0.2$. b) $G_1 = 2.0$ and $G_2 = 0.2$. c) $G_1 = 0.2$ and $G_2 = 2.0$. d) $G_1 = 2.0$ and $G_2 = 2.0$. 

FIG. 10. (a) Schematic picture of an N-I-S junction for analysis using circuit theory. (b) Picture of the tight-binding lattice used to model a two-dimensional N-I-S structure.
FIG. 10. Numerical results for the conductance versus phase for individual realizations of the disorder: a) \( G_1 = 0.2 \) and \( G_2 = 0.2 \). b) \( G_1 = 2.0 \) and \( G_2 = 0.2 \). c) \( G_1 = 0.2 \) and \( G_2 = 2.0 \). d) \( G_1 = 2.0 \) and \( G_2 = 2.0 \).

FIG. 11. Numerical results for the conductance versus phase for \( L_{diff} = 5 \): a) \( G_1 = 0.2 \) and \( G_2 = 0.2 \). b) \( G_1 = 2.0 \) and \( G_2 = 0.2 \). c) \( G_1 = 0.2 \) and \( G_2 = 2.0 \). d) \( G_1 = 2.0 \) and \( G_2 = 2.0 \).

FIG. 12. Numerical results for the conductance versus phase for \( L_{diff} = 1 \): a) \( G_1 = 0.2 \) and \( G_2 = 0.2 \). b) \( G_1 = 2.0 \) and \( G_2 = 0.2 \). c) \( G_1 = 0.2 \) and \( G_2 = 2.0 \). d) \( G_1 = 2.0 \) and \( G_2 = 2.0 \).