CHUDNOVSKY-RAMANUJAN TYPE FORMULAE FOR THE LEGENDRE FAMILY

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Abstract. We apply the method established in our previous work to derive a Chudnovsky-Ramanujan type formula for the Legendre family of elliptic curves. As a result, we prove two identities for $1/\pi$ in terms of hypergeometric functions.

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1. Introduction

In [8], Ramanujan derived a number of rapidly converging series for $1/\pi$. Later, Chudnovsky and Chudnovsky [5], [6] derived an additional such series based on the modular $j$-function, which is often used in practice for the computation of the digits of $\pi$. In [4], we generalized the method of [5], [6] to derive a complete list of Chudnovsky-Ramanujan type formulae for the modular $j$-function. The method is systematic, and in principle applicable to any family of elliptic curves parametrized by a triangular Fuchsian group of genus zero.

The purpose of this paper is to illustrate the method of [4] in another case. Consider the Legendre family of elliptic curves given by

$$C_{\lambda} : y^2 = x(x - 1)(x - \lambda),$$

which is parametrized by the modular $\lambda$ function. The Picard-Fuchs differential equation for this family is well known and given by

$$\lambda(1 - \lambda) \frac{d^2P}{d\lambda^2} + (1 - 2\lambda) \frac{dP}{d\lambda} - \frac{P}{4} = 0,$$

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which is a hypergeometric differential equation with parameters \( a = 1/2, b = 1/2, c = 1 \), and it has three regular singular points: 0, 1, \( \infty \).

Let \( 2F_1(a, b; c; z) \) denote the hypergeometric function. Kummer’s method yields six distinct hypergeometric solutions of the form

\[
\lambda^\alpha (1 - \lambda)^\beta 2F_1(a, b; c; \nu(\lambda)),
\]

where \( \nu(\lambda) \) is one of

\[
\lambda, \quad 1 - \lambda, \quad \frac{1}{\lambda}, \quad \frac{1}{1 - \lambda}, \quad \frac{\lambda}{\lambda - 1}, \quad \frac{\lambda - 1}{\lambda}.
\]

In particular, we have the solutions:

Around \( \lambda = 0 \):

\[
2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \lambda\right),
\]

\[
(1 - \lambda)^{-1/2} 2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{\lambda}{\lambda - 1}\right).
\]

Around \( \lambda = 1 \):

\[
2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \lambda\right),
\]

\[
\lambda^{-1/2} 2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{\lambda - 1}{\lambda}\right).
\]

Around \( \lambda = \infty \):

\[
\lambda^{-1/2} 2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{1}{\lambda}\right),
\]

\[
(1 - \lambda)^{-1/2} 2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{1}{1 - \lambda}\right).
\]

We can restrict our attention to only the first solution of each pair above because the second solution of each pair is obtained from the first by a Pfaff transformation.

Each of these solutions will be valid near one of the singular points \( \lambda = 0, 1, \infty \) and will give rise to an expression of the period \( P_1(\lambda) \) in terms of a hypergeometric function and \( \lambda \).

Using the method of [4], we turn one of these period expressions into a Chudnovsky-Ramanujan type formulae valid near \( \lambda = 0 \). As a result, we derive the following identities for \( 1/\pi \) in terms of hypergeometric functions:

\[
\frac{8}{\pi} = 2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{1}{2}\right) 2F_1\left(\frac{3}{2}, \frac{3}{2}; 2; \frac{1}{2}\right),
\]

\[
\frac{1}{\pi} = 2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; -1\right)^2 - 2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; -1\right) 2F_1\left(\frac{3}{2}, \frac{3}{2}; 2; -1\right).
\]

**Remark.** Although the above identities could be verified by other means, the purpose of this work is to illustrate how systematic the method in [4] is to discover and derive such formulae.
2. Periods and families of elliptic curves

Let \( C_\lambda : y^2 = x(x-1)(x-\lambda) \) be the Legendre family. We first explain how to specify periods and quasi-periods for \( C_\lambda \). Suppose

1. \( \alpha_1(\lambda) \) is a simple loop which encircles 0 and 1, and does not pass through \( \lambda \);
2. \( \alpha_2(\lambda) \) is a simple loop which encircles 1 and \( \lambda \), and does not pass through 0.

Then \( \{\alpha_1(\lambda), \alpha_2(\lambda)\} \) forms a \( \mathbb{Z} \)-basis for \( H_1(E_\lambda, \mathbb{Z}) \) as depicted in [9, Chapter VI, Section 1, Figure 6.5], and it is possible to choose \( \alpha_i(\lambda), i = 1, 2 \), so they vary continuously for \( \lambda \) in a simply connected open subset of \( \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\} \). See [4] for details.

We then define the periods \( P_i \) and quasi-periods \( Q_i \) for \( C_\lambda \) as:

\[
P_i = \int_{\alpha_i} \frac{dx}{y},
Q_i = -\int_{\alpha_i} \frac{x}{y}.
\]

Up to a modification of the \( \mathbb{Z} \)-basis \( \{\alpha_1(\lambda), \alpha_2(\lambda)\} \) for \( H_1(C_\lambda, \mathbb{Z}) \), that is, a transformation in \( \text{SL}_2(\mathbb{Z}) \), we obtain another \( \mathbb{Z} \)-basis \( \{\gamma_1(\lambda), \gamma_2(\lambda)\} \) such that \( \tau = P_2/P_1 \) is in the fundamental domain for \( \Gamma(2) \), and is near the points 0, 1, \( \infty \) in the sense that \( \lambda(\tau), 1-\lambda(\tau), 1/(1-\lambda(\tau)) \) have absolute value less than unity, respectively.

Under the change of variables

\[
x \mapsto x + \frac{\lambda + 1}{3},
\]
\[
y \mapsto \frac{y}{2},
\]

the Legendre family is put into the Weierstrass form:

\[
E_\lambda : y^2 = 4x^3 - \frac{4}{3}(\lambda^2 - \lambda + 1)x - \frac{4}{27}(\lambda + 1)(2\lambda - 1)(\lambda - 2) = 4(x - e_0)(x - e_1)(x - e_\lambda),
\]

where

\[
e_0 = -\frac{1 + \lambda}{3}, \quad e_1 = \frac{2 - \lambda}{3}, \quad e_\lambda = \frac{2\lambda - 1}{3}.
\]

This defines an isomorphism \( \pi : E_\lambda \rightarrow C_\lambda \).

The normalized \( j \)-invariant \( J = j/12^3 \) of \( C_\lambda \) is given by

\[
J = \frac{4}{27} \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(1-\lambda)^2}.
\]

The map \( \lambda \mapsto J(\lambda) \) gives the covering map \( X(2) \rightarrow X(1) \), where \( X(n) \) denotes the modular curve with full level \( n \)-structure.

The choice of \( P_i \) and \( Q_i \) for \( C_\lambda \) above fixes the periods and quasi-periods of \( E_\lambda \), which we denote by \( \Omega_i \) and \( H_i \), respectively. In fact, we have that

\[
P_i = 2\Omega_i,
Q_i = 2H_i - \frac{1 + \lambda}{3}P_i.
\]

Fix an elliptic curve \( E : y^2 = 4x^3 - g_2x - g_3 \). For \( u \neq 0 \), the map \( \varphi_u : (x, y) \mapsto (u^2x, u^3y) \) gives an isomorphism \( \varphi_u : E \xrightarrow{\cong} E' \) from an elliptic \( E \) over \( \mathbb{C} \) to the
elliptic curve $E'$ over $\mathbb{C}$, where
\[ E : y^2 = 4x^3 - g_2x - g_3 \quad \text{and} \quad E' : y^2 = 4x^3 - g_2'x - g_3', \]
and
\[ \begin{cases} g_2' = u^4 g_2, \\
g_3' = u^6 g_3, \\
\Delta(E') = u^{12} \Delta(E). \end{cases} \tag{4} \]

By the uniformization theorem, $E(\mathbb{C}) = E_\Lambda(\mathbb{C})$ for some lattice $\Lambda \subset \mathbb{C}$. It then follows that $E'(\mathbb{C}) = E_{u^{-1}\Lambda}(\mathbb{C})$, and we have the following commutative diagram:
\[
\begin{array}{ccc}
\mathbb{C}/\Lambda & \xrightarrow{\iota_\Lambda} & E_\Lambda(\mathbb{C}) = E(\mathbb{C}) \\
\phi \downarrow & & \downarrow \phi_u \\
\mathbb{C}/u^{-1}\Lambda & \xrightarrow{\iota_{u^{-1}\Lambda}} & E_{u^{-1}\Lambda}(\mathbb{C}) = E'(\mathbb{C})
\end{array}
\]
so that the isomorphism $\phi_u$ corresponds to scaling $\Lambda$ by $u^{-1}$.

We now introduce three families of elliptic curves: $E$, $E_\tau$, $E_\lambda$, and compare their discriminants, associated lattices, and periods.

Consider the elliptic curve $E$ over $\mathbb{C}$ given by
\[ E : y^2 = 4x^3 - g_2x - g_3, \quad \Delta(E) = \Delta = g_2^3 - 27g_3^2, \quad \Lambda(E) = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2. \]

Taking $u = \omega_1$, and using (4) we see that $E$ is isomorphic to $E_\tau : y^2 = 4x^3 - g_2(\tau)x - g_3(\tau)$, $\Delta(E_\tau) = \Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2$, $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$, where $\tau = \omega_2/\omega_1$.

To find a $u$ so that $E$ is isomorphic to $E_\lambda$, we claim that it is necessary to set
\[
\begin{cases} u^4 g_2 = \frac{4}{3}(\lambda^2 - \lambda + 1), \\
u^6 g_3 = \frac{4}{27}(\lambda + 1)(2\lambda - 1)(\lambda - 2). \end{cases}
\]
Dividing the second equation by the first, we obtain
\[
u^2 \frac{g_3}{g_2} = \frac{(\lambda + 1)(2\lambda - 1)(\lambda - 2)}{9(\lambda^2 - \lambda + 1)},
\]
or
\[
u = \sqrt{\frac{(\lambda + 1)(2\lambda - 1)(\lambda - 2) g_2}{9(\lambda^2 - \lambda + 1) g_3}}.
\]
The above is a necessary condition for $u$. To show sufficiency, observe that
\[
u^4 g_2 = \frac{g_2}{g_3} \left\{ \frac{(\lambda + 1)(2\lambda - 1)(\lambda - 2)}{9(\lambda^2 - \lambda + 1)} \right\}^2
= \frac{9g_2}{4} \frac{12}{J - 1} \frac{J - 1}{J} \left\{ \frac{(\lambda + 1)(2\lambda - 1)(\lambda - 2)}{9(\lambda^2 - \lambda + 1)} \right\}^2
= \frac{27J}{J - 1} \left\{ \frac{(\lambda + 1)(2\lambda - 1)(\lambda - 2)}{9(\lambda^2 - \lambda + 1)} \right\}^2
= \frac{4}{3}(\lambda^2 - \lambda + 1),
\]
we see that

and using (4), we see that

and with \( \Im(\tau) \rightarrow \infty \))

Therefore, we have

Thus, taking

Thus, taking

Thus, taking

and using (4), we see that \( E \) is isomorphic to

\[
E_\lambda : y^2 = 4x^3 - \frac{4}{3}(\lambda^2 - \lambda + 1)x - \frac{4}{27}(\lambda + 1)(2\lambda - 1)(\lambda - 2),
\]

\[
\Delta(E_\lambda) = 16\lambda^2(1 - \lambda)^2, \quad \Lambda(E_\lambda) = \mathbb{Z}\Omega_1 + \mathbb{Z}\Omega_2.
\]

Taking

Thus, taking

we see that \( E_\tau \) is isomorphic to \( E_\lambda \). Hence, we have that \( \Lambda(E_\lambda) = \mu(\tau)\Lambda_\tau \), where

\[
\mu(\tau) = \sqrt{\frac{9(\lambda^2 - \lambda + 1)}{(\lambda + 1)(2\lambda - 1)(\lambda - 2)}} \frac{g_2(\tau)}{g_3(\tau)},
\]

\[
\left\{
\begin{aligned}
\mu(\tau) &= \sqrt{\frac{9(\lambda^2 - \lambda + 1)}{(\lambda + 1)(2\lambda - 1)(\lambda - 2)}} \frac{g_2(\tau)}{g_3(\tau)}, \\
\Delta(\tau)^{1/12} \frac{J_1^{1/6}}{J_1^{1/6}} &= \sqrt{\frac{9(\lambda^2 - \lambda + 1)}{(\lambda + 1)(2\lambda - 1)(\lambda - 2)}} \\
\end{aligned}
\right.
\]

\[
\begin{aligned}
\mu(\tau) &= \sqrt{\frac{9(\lambda^2 - \lambda + 1)}{(\lambda + 1)(2\lambda - 1)(\lambda - 2)}} \frac{g_2(\tau)}{g_3(\tau)}, \\
\Delta(\tau)^{1/12} \frac{J_1^{1/6}}{J_1^{1/6}} &= \sqrt{\frac{9(\lambda^2 - \lambda + 1)}{(\lambda + 1)(2\lambda - 1)(\lambda - 2)}} \\
\end{aligned}
\]

\[
\begin{aligned}
\Delta(\tau)^{1/12} \frac{J_1^{1/6}}{J_1^{1/6}} &= \sqrt{\frac{9(\lambda^2 - \lambda + 1)}{(\lambda + 1)(2\lambda - 1)(\lambda - 2)}} \\
\end{aligned}
\]

3. LIMIT VALUES OF THE MODULAR \( \lambda \) FUNCTION

It is well known that for \( \tau \) in the upper half-plane we have

\[
\lambda(\tau) = 16q^{1/2} - 2q + 704q^{3/2} - O(q^2), \quad q = e^{2\pi i \tau},
\]

or with \( x = \sqrt{q} \),

\[
\lambda(\tau) = 16x - 128x^2 + 704x^3 - O(x^4), \quad x = \sqrt{q}.
\]

In what follows and throughout the paper, the notation \( \tau \rightarrow \infty \) will be synonymous with \( \Im(\tau) \rightarrow \infty \). Therefore, we have

\[
\lim_{\tau \rightarrow \infty} \lambda(\tau) = \lim_{x \rightarrow 0} (16x - 128x^2 + 704x^3 - O(x^4)) = 0.
\]
Moreover, in view of \( \lambda(-1/\tau) = 1 - \lambda(\tau) \), we see that
\[
\lim_{\tau \to 0} \lambda(\tau) = \lim_{\tau \to \infty} \lambda(-1/\tau) = \lim_{\tau \to \infty} (1 - \lambda(\tau)) = 1.
\]
Furthermore, using \( \lambda(\tau + 1) = \lambda(\tau)/(\lambda(\tau) - 1) \) together with \( \lambda(-1/\tau) = 1 - \lambda(\tau) \), we find that
\[
\lim_{\tau \to 1} \lambda(\tau) = \lim_{\tau \to \infty} \lambda(1 - 1/\tau) = \lim_{\tau \to \infty} \frac{\lambda(-1/\tau)}{\lambda(-1/\tau) - 1} = \lim_{\tau \to \infty} \frac{\lambda(\tau) - 1}{\lambda(\tau)} = \infty.
\]
Below is the table summarizing these limit values.

| \( \tau \) | \( \lambda(\tau) \) |
|---|---|
| \( \infty \) | 0 |
| 0 | 1 |
| 1 | \( \infty \) |

Table 1: Limit values of \( \lambda(\tau) \)

### 4. Hypergeometric representations of \( \Omega_1 \)

Recall that the first solution of (1) around \( \lambda = 0 \) is
\[\text{2F}_1 \left( \frac{1}{2}, \frac{1}{2}; 1; \lambda \right).\]

Following the method in [1] with the homothety factor \( \mu(\tau) \) in (5), we have that
\[\text{2F}_1 \left( \frac{1}{2}, \frac{1}{2}; 1; \lambda \right) = \frac{(A + B \tau) \eta(\tau)^2}{(\lambda(1 - \lambda))^{1/6}}.\]

Replacing \( \tau \) by \( \tau + 2 \) leaves \( \lambda(\tau) \) invariant but changes \( \eta(\tau)^2 \) by \( \zeta \eta(\tau)^2 \), where \( \zeta = e^{\pi i/3} \). Hence, we get the identity
\[\frac{(A + B \tau) \eta(\tau)^2}{(\lambda(1 - \lambda))^{1/6}} = (A + B(\tau + 2)) \frac{\zeta \eta(\tau)^2}{(\lambda(1 - \lambda))^{1/6}}\]
valid for \( \tau \) around \( \infty \). Since both sides are nonzero for \( \tau \) around \( \infty \), we obtain
\[A + B \tau = (A + B(\tau + 2))\zeta\]
for \( \tau \) around \( \infty \). Upon solving for \( B/A \) and taking the limit as \( \tau \to \infty \), shows that \( B = 0 \). Recalling that
\[\lambda(\tau) = 16q^{1/2} - 128q + 704q^{3/2} - O(q^2), \quad q = e^{2\pi i \tau},\]
we find that \( A = 16^{1/6} = 4^{1/3} \), and thus we have the identity
\[\text{2F}_1 \left( \frac{1}{2}, \frac{1}{2}; 1; \lambda \right) = 4^{1/3} \frac{\eta(\tau)^2}{(\lambda(1 - \lambda))^{1/6}}.\]

We thus obtain the following period expression.

**Theorem 4.1.** For \( \tau \) around \( \infty \), we have
\[\omega_1 = 2^{1/3} \pi (\lambda(1 - \lambda))^{1/6} \Delta(E)^{-1/12} \text{2F}_1 \left( \frac{1}{2}, \frac{1}{2}; 1; \lambda \right).\]

**Proof.** Since \( \Delta(\tau) = (2\pi)^{12} \eta(\tau)^{24} \) and \( \Delta(\tau) = \omega_1^{12} \Delta \), the result following from the identity above. \qed
We state the above period expression for $E$, but if $E = E_\lambda$, then the period expression becomes the classical relation:

\[(6)\quad P_1 = 2\pi_2 F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \lambda\right),\]

where $P_1 = 2\Omega_1$, because $\Delta(E_\lambda) = 16\lambda^2(1 - \lambda)^2$.

Now, we use the fact that the Galois covering $X(2) \to X(1)$ has Galois group $\text{GL}_2(\mathbb{F}_2)$, which is isomorphic to $S_3$. Explicitly, we have generators:

\[(7)\quad \begin{cases} 
\lambda(\tau \pm 1) = \frac{\lambda(\tau)}{\lambda(\tau) - 1}, \\
\lambda(-1/\tau) = 1 - \lambda(\tau).
\end{cases}\]

Applying these transformations to the above period expression around $\lambda = 0$ gives all period expressions around $\lambda = 0, 1, \infty$. We also require the following functional equations of the Dedekind eta function:

\[(8)\quad \begin{cases} 
\eta(\tau \pm 1) = e^{\pm \pi i / 12}\eta(\tau), \\
\eta(-1/\tau) = \eta(\tau)\sqrt{-i\tau}.
\end{cases}\]

**Theorem 4.2.** For $\tau$ around 0, we have

\[\omega_1 = 2^{1/3}\pi i \frac{\pi i}{\tau}(\lambda(1 - \lambda))^{1/6}\Delta(E)^{-1/12}F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \lambda\right).\]

**Proof.** This is proven by applying the transformation $\tau \to -1/\tau$ in Theorem 4.1 and using (7) and (8). \hfill \square

**Theorem 4.3.** For $\tau$ around 1, we have

\[\omega_1 = 2^{1/3}\pi i \frac{\tau}{(\tau + 1)\sqrt{1 - \lambda}}(\lambda(1 - \lambda))^{1/6}\Delta(E)^{-1/12}F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{1}{1 - \lambda}\right).\]

**Proof.** This is proven by applying the transformation $\tau \to \tau + 1$ in Theorem 4.2 and using (7) and (8). \hfill \square

### 5. Complex multiplication and quasi-period relations

Suppose $\tau$ satisfies $a\tau^2 + b\tau + c = 0$ for mutually coprime integers $a$, $b$, $c$ such that $a > 0$ and $-d = b^2 - 4ac$, that is,

\[(9)\quad \tau = \frac{-b + \sqrt{-d}}{2a}.
\]

Let $q = e^{2\pi i \tau}$ with $\text{Im}(\tau) > 0$, and define

\[s_2(\tau) = \frac{E_4(\tau)}{E_6(\tau)}\left(E_2(\tau) - \frac{3}{\pi \text{Im}(\tau)}\right),\]
where
\[
E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n},
\]
\[
E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3q^n}{1-q^n},
\]
\[
E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5q^n}{1-q^n}.
\]

Then we have from [4] that
\[
(10) \quad \Omega_1 H_1 \text{Im}(\tau) - \Omega_1^2 \text{Im}(\tau) \frac{3g_3}{2g_2} s_2(\tau) = \pi \]

by taking \( E = E_\lambda \) so that \( \omega_i = \Omega_i \) and \( \eta_i = H_i \).

6. A Chudnovsky-Ramanujan Type Formula

Using the differential relations derived by Bruns [3] pp. 237-238] with
\[
g_2 = \frac{4}{3}(\lambda^2 - \lambda + 1),
\]
\[
g_3 = \frac{4}{27}(\lambda + 1)(2\lambda - 1)(\lambda - 2),
\]
we obtain
\[
\frac{d\Omega_1}{d\lambda} = -\frac{H_1}{2\lambda(\lambda - 1)} - \frac{2\lambda - 1}{6\lambda(\lambda - 1)} \Omega_1,
\]
\[
\frac{dH_1}{d\lambda} = \frac{\lambda^2 - \lambda + 1}{18\lambda(\lambda - 1)} \Omega_1 + \frac{2\lambda - 1}{6\lambda(\lambda - 1)} H_1.
\]

Remark. These relations can also be obtained by using Fricke’s and Klein’s method of deriving the Picard-Fuchs differential [7] pp. 33-34] if applied to \( E_\lambda \).

Using the first differential relation above, together with a period expression, yields a Chudnovsky-Ramanujan type formula.

Theorem 6.1. If \( \tau \) satisfies (9) and is around \( \infty \), and
\[
g_2 = \frac{4}{3}(\lambda^2 - \lambda + 1),
\]
\[
g_3 = \frac{4}{27}(\lambda + 1)(2\lambda - 1)(\lambda - 2),
\]
then we have that
\[
-F^2 \left[ \frac{2\lambda - 1}{3} + \frac{3g_3}{2g_2} s_2(\tau) \right] + \lambda(1 - \lambda) \frac{dF^2}{d\lambda} = \frac{2a}{\pi \sqrt{d}},
\]
where \( \lambda = \lambda(\tau) \) and \( F = \text{}_2F_1(1/2, 1/2; 1; \lambda) \).

Proof. The first differential relation is equivalent to
\[
(11) \quad H_1 = -2\lambda(\lambda - 1) \frac{d\Omega_1}{d\lambda} - \frac{2\lambda - 1}{3} \Omega_1.
\]
Substitute (11) into (10) to get
\[
\frac{2\pi a}{\sqrt{d}} = -\Omega_1 \left[ 2\lambda(\lambda - 1) \frac{d\Omega_1}{d\lambda} + \frac{2\lambda - 1}{3} \Omega_1 + \Omega_1 \frac{3g_3}{2g_2} s_2(\tau) \right]
\]
\[
= -\Omega_1^2 \left[ \frac{2\lambda - 1}{3} + \frac{3g_3}{2g_2} s_2(\tau) \right] + 2\lambda(1 - \lambda) \frac{d\Omega_1}{d\lambda}.
\]

Then, substituting \( \Omega_1 = \pi F \) from (6), where \( F = 2F_1(1/2, 1/2; 1; \lambda) \), into the above, we obtain
\[
\frac{2a}{\pi \sqrt{d}} = -F^2 \left[ \frac{2\lambda - 1}{3} + \frac{3g_3}{2g_2} s_2(\tau) \right] + \lambda(1 - \lambda) \frac{dF^2}{d\lambda}.
\]

Under the covering \( X(2) \to X(1) \) given by
\[
J = \frac{4}{27} \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(1 - \lambda)^2},
\]
the only values for \( \tau \) for which \( \lambda(\tau) \in \mathbb{Q} \) correspond to \( J = 1 \) and \( \lambda = 2, 1/2, -1 \), each with ramification index 2. In the fundamental domain for \( \Gamma(2) \), we have
\[
\lambda\left(\frac{i \pm 1}{2}\right) = 2,
\]
\[
\lambda(i) = \frac{1}{2},
\]
\[
\lambda(i + 1) = -1.
\]

Using
\[
\frac{3g_3}{2g_2} = \Omega_1^2 \frac{\pi^2}{3} E_6(\tau)
\]
in Theorem 5.1 yields
\[
\frac{2a}{\pi \sqrt{d}} = -F^2 \left[ \frac{2\lambda - 1}{3} + \frac{\pi^4}{3} \left\{ E_2(\tau) - \frac{3}{\pi \text{Im}(\tau)} \right\} F^2 \right] + \lambda(1 - \lambda) \frac{dF^2}{d\lambda},
\]

which can be evaluated at \( \tau = i \). Recall that \( E_2(\tau) \) is a quasi-modular form, and \( E_2(-1/\tau) = \pi^2 E_2(\tau) - 6i \tau / \pi \) for \( \tau \) in the upper half-plane. Setting \( \tau = i \), shows that \( E_2(i) = 3/\pi \), and we obtain (2). Furthermore, it is evident that \( E_2(\tau + 1) = E_2(\tau) \) because \( g(\tau + 1) = g(\tau) \). Thus, setting \( \tau = i + 1 \), we obtain (3).

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