A vanishing theorem for elliptic genera under a Ricci curvature bound

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Abstract

We show that given $n \in \mathbb{N}$ and positive numbers $p, \lambda_1, \lambda_2$ with $p > 2n$, there exists some $\epsilon = \epsilon(p, n, \lambda_1, \lambda_2) > 0$ such that if a compact $4n$-dimensional spin Riemannian manifold $(M, g)$ satisfies $-\lambda_1 \leq \text{Ric}(g) \leq \epsilon, \text{diam}(g) \leq 1, \sqrt{\text{V}(g)} \int_M |\text{R}_g|^p dV \leq \lambda_2$ and has infinite isometry group, then the elliptic genera of $M$ vanish. This extends our simple observation that a compact $4n$-dimensional spin Riemannian manifold with nonpositive Ricci curvature and infinite isometry group has vanishing elliptic genera. To the author’s best knowledge, it is the first time in the literature that the elliptic genera are shown to vanish based on curvature assumptions and analytic methods.

1 Introduction

The purpose of this paper is to prove a vanishing theorem for elliptic genera under certain Ricci curvature bound. We start from the following simple lemma:

Lemma 1.1. Let $(M, g)$ be a compact $4n$-dimensional spin Riemannian manifold with nonpositive Ricci curvature and infinite isometry group, then the elliptic genera of $M$ vanish.

Proof. Since the isometry group of $(M, g)$ is infinite, $M$ admits a nontrivial isometric circle action which gives a nonzero Killing vector field $X$. By a well known theorem of Bochner (page 191 in [53]), any Killing vector field on a compact Riemannian manifold $(M, g)$ with nonpositive Ricci curvature is parallel. It follows that $X$ is nowhere vanishing and so $M$ admits a circle action without fixed points. If $M$ is spin, then by the analytic interpretation of elliptic genera (1.12), (1.13) and the Atiyah-Bott-Segal-Singer Lefchetz fixed point theorem, $M$ has vanishing elliptic genera.

Remark 1. In the above lemma, the spin condition can actually be removed. This can be shown by combining the curvature expression of elliptic genera (1.12) and (1.13), the Bott residue theorem ([8], c.f. page 35 in [65]) and the fixed point free property obtained in the above proof. However in the following theorem and corollaries, which assume a much weaker curvature condition, the spin condition is indispensable to employ the analytic tools as well as the rigidity of elliptic genera.
By a theorem of Lohkamp [50], any compact manifold admits a Riemannian metric with negative Ricci curvature. Hence the assumption in Lemma 1.1 that \((M, g)\) has infinite isometry group is essential. In this paper we show that the same conclusion in Lemma 1.1 still holds for compact spin manifolds under a much weaker curvature assumption.

Let \((M, g)\) be a compact 4n-dimensional spin Riemannian manifold and \(\text{Ric}(g) / \mathcal{R}_g\) be the Ricci curvature and Riemannian curvature tensor of \(g\), respectively. Let \(\text{diam}(g)\) be the diameter of \(g\) and \(V(g)\) be the volume of \(g\). Then we have the following

**Theorem 1.2.** Given \(n \in \mathbb{N}\) and positive numbers \(p, \lambda_1, \lambda_2\) with \(p > 2n\), there exists some \(\epsilon = \epsilon(p, n, \lambda_1, \lambda_2) > 0\) such that if a compact 4n-dimensional spin Riemannian manifold \((M, g)\) satisfies \(-\lambda_1 \leq \text{Ric}(g) \leq \epsilon, \text{diam}(g) \leq 1, \frac{1}{V(g)} \int_M |\mathcal{R}_g|^p dV \leq \lambda_2\) and has infinite isometry group, then the elliptic genera of \(M\) vanish.

A famous theorem of Bochner asserts that the isometry group of a compact manifold with negative Ricci curvature is finite (page 191 in [53]). Theorem 1.2 shows that a compact 4n-dimensional spin Riemannian manifold with nonzero elliptic genera and satisfying the curvature assumptions in Theorem 1.2 must have finite isometry group. Hence Theorem 1.2 can be also viewed as an extension of Bochner’s theorem.

**Example 1.** Let \(M^4 = S^1 \times X^3\), where \(X^3\) is a co-compact quotient of the three dimensional Heisenberg group. \(M^4\) admits a sequence of Riemannian metrics \(g_i\) with infinite isometry group and the absolute value of its sectional curvature \(\leq \frac{1}{4}\) and diameter \(\leq 1\) ([24]). Then for any \(p > 2, \lambda_1, \lambda_2 > 0\), there exists some large \(i\) such that \((M^4, g_i)\) satisfies the curvature assumptions in Theorem 1.2. However, \(g_i\) does not have nonpositive Ricci curvature.

The geometry and topology of Riemannian manifolds with bounded diameter and certain curvature bound (sectional or Ricci curvature) have been studied extensively since 1980’s. See for example [11, 12, 13, 14, 15, 16, 22, 23, 24, 30, 31, 32, 62, 63]. Integral curvature bounds have recently been discovered in various geometric situations, such as the \(L^2\) bound of the curvature tensor for noncollapsed manifolds with bounded Ricci curvature, and the (almost) \(L^4\) bound of the Ricci curvature for the Kähler-Ricci flow as well as the (real) Ricci flow (under certain conditions) [4, 5, 17, 29, 54, 57, 58]. See also [30] for related work on almost Ricci flat manifolds under certain integral bound of the Riemannian curvature tensor. As the Riemannian curvature tensor is determined by sectional curvature, Theorem 1.2 implies the following

**Corollary 1.3.** Given \(n \in \mathbb{N}\) and positive number \(\lambda\), there exists some \(\epsilon = \epsilon(n, \lambda) > 0\) such that if a compact 4n-dimensional spin Riemannian manifold \((M, g)\) satisfies \(\text{Ric}(g) \leq \epsilon, \text{diam}(g) \leq 1, \text{sectional curvature} \geq -\lambda\) and has infinite isometry group, then the elliptic genera of \(M\) vanish.

A similar curvature assumption as in Corollary 1.3 was studied in [22], where the authors showed that given \(n \in \mathbb{N}\) and positive number \(\lambda\), there exists some \(\epsilon = \epsilon(n, \lambda) > 0\) such that if a compact \(n\)-dimensional Riemannian manifold \((M, g)\) has diameter \(\leq 1\) and \(-\lambda \leq \text{sectional curvature} \leq \epsilon\), then the universal covering of \(M\) is diffeomorphic to \(\mathbb{R}^n\). In particular, this gives an affirmative answer to a conjecture of Gromov [25].

As the first elliptic genus \(\text{Ell}_1(M)\) degenerates to the signature of \(M\) (see the following review of elliptic genera for details), Theorem 1.2 implies the following interesting vanishing theorem for signature.

**Corollary 1.4.** Under the assumptions in Theorem 1.2 or Corollary 1.3, the signature of \(M\) vanishes.
Elliptic genera were first constructed by Ochanine [52] and Landweber-Stong in a topological way. Witten gave a geometric interpretation of elliptic genera by showing that formally they are indices of Dirac operators on free loop space [60–61]. The theory of elliptic genera gives a connection among the Atiyah-Singer index theory, Kac-Moody affine Lie algebra, modular forms and quantum field theory. The background and introduction of elliptic genera can be found in [27–34].

We now recall some basic facts about elliptic genera. Let \( M \) be a 4\( n \) dimensional compact oriented manifold and \( \{ \pm 2 \pi \sqrt{-1} z_j, 1 \leq j \leq 2n \} \) denote the formal Chern roots of \( T_C M \), the complexification of the tangent vector bundle \( TM \).

Let
\[
\hat{A}(TM) = \prod_{j=1}^{2n} \frac{\pi \sqrt{-1} z_j}{\sinh(\pi \sqrt{-1} z_j)}, \quad \hat{L}(TM) = \prod_{j=1}^{2n} \frac{2\pi \sqrt{-1} z_j}{\tanh(\pi \sqrt{-1} z_j)}
\]
be the Hirzebruch \( \hat{A} \)-class and \( \hat{L} \)-class of \( M \) respectively.

Let \( E \) be a complex vector bundle. Let \( \text{ch}(E) \) be the Chern character of \( E \). For any complex number \( t \), let
\[
\Lambda_t(E) = |\mathbb{C}|_M + tE + t^2 \Lambda^2(E) + \cdots, \quad S_t(E) = |\mathbb{C}|_M + tE + t^2 S^2(E) + \cdots
\]
denote the total exterior and symmetric powers of \( E \) respectively, which live in \( K(M)[[t]] \) (c.f. page 117-119 in [1]). The following relations on these two operations hold,
\[
S_t(E) = \frac{1}{\Lambda_{-t}(E)}, \quad \Lambda_t(E - F) = \frac{\Lambda_t(E)}{\Lambda_t(F)}. \tag{1.1}
\]

Denote \( \tilde{E} = E - \mathbb{C}^{\text{rank}E} \) in \( K(M) \).

The elliptic genera of \( M \) can be defined as (c.f. chap. 6 in [27] and [40])
\[
\text{Ell}_1(M) = \left\langle \hat{L}(TM) \text{ch} \left( \Theta(T_C M) \otimes \Theta_1(T_C M) \right), [M] \right\rangle \in \mathbb{Q}[[q]],
\]
\[
\text{Ell}_2(M) = \left\langle \hat{A}(TM) \text{ch} \left( \Theta(T_C M) \otimes \Theta_2(T_C M) \right), [M] \right\rangle \in \mathbb{Q}[[q^{\frac{1}{2}}]],
\]
where
\[
\Theta(T_C M) = \bigotimes_{j=1}^{\infty} S_q \left( \tilde{T_C M} \right), \quad \Theta_1(T_C M) = \bigotimes_{j=1}^{\infty} \Lambda_q \left( \tilde{T_C M} \right), \quad \Theta_2(T_C M) = \bigotimes_{j=1}^{\infty} \Lambda_{q^{-}\frac{1}{2}} \left( \tilde{T_C M} \right) \tag{1.2}
\]
are the Witten bundles introduced in [61]. One can expand these elements into Fourier series,
\[
\Theta(T_C M) \otimes \Theta_1(T_C M) = A_0(T_C M) + A_1(T_C M)q + \cdots = \mathbb{C} + (2T_C M - \mathbb{C}^{8n})q + \cdots, \tag{1.3}
\]
\[
\Theta(T_C M) \otimes \Theta_2(T_C M) = B_0(T_C M) + B_1(T_C M)q^{\frac{1}{2}} + \cdots = \mathbb{C} - (T_C M - \mathbb{C}^{4n})q^{\frac{1}{2}} + \cdots. \tag{1.4}
\]

Hence we have
\[
\text{Ell}_1(M) = \left\langle \hat{L}(TM), [M] \right\rangle + 2 \left\langle \hat{L}(TM) \text{ch} \left( T_C M - \mathbb{C}^{4n} \right), [M] \right\rangle q + \cdots, \tag{1.5}
\]
\[
\text{Ell}_2(M) = \left\langle \hat{A}(TM), [M] \right\rangle - \left\langle \hat{A}(TM) \text{ch} (T\mathbb{C}M - \mathbb{C}^{4n}), [M] \right\rangle q^{\frac{1}{2}} + \cdots \tag{1.6}
\]

and see that \(\text{Ell}_1(M)\) is a \(q\)-deformation of \(\sigma(M)\), the signature of \(M\); and \(\text{Ell}_2(M)\) is a \(q\)-deformation of \(\hat{A}(M)\), the \(\hat{A}\) genus of \(M\).

By the Atiyah-Singer index theorem \([3]\), \(\text{Ell}_1(M)\) can be expressed analytically as index of the twisted signature operator

\[
\text{Ell}_1(M) = \text{Ind}(d_s \otimes (\Theta (T\mathbb{C}M) \otimes \Theta_1 (T\mathbb{C}M))) \in \mathbb{Z}[[q]], \tag{1.7}
\]

where \(d_s\) is the signature operator; and furthermore when \(M\) is spin, \(\text{Ell}_2(M)\) can be expressed analytically as index of the twisted Dirac operator,

\[
\text{Ell}_2(M) = \text{Ind}(D \otimes (\Theta (T\mathbb{C}M) \otimes \Theta_2 (T\mathbb{C}M))) \in \mathbb{Z}[[q^{1/2}]], \tag{1.8}
\]

where \(D\) is the Atiyah-Singer spin Dirac operator on \(M\) \([61]\).

Although the definitions of elliptic genera given above depend on the the smooth structure of \(M\), they are in fact homeomorphism invariants. Actually the elliptic genera can be defined via Chern root algorithm as (cf. chap.6 in \([27]\) and \([10]\))

\[
\text{Ell}_1(M) = 2^{2n} \left\langle \prod_{j=1}^{2n} \left( \frac{\theta'(0, \tau) \theta_1(z_j, \tau)}{\theta(z_j, \tau) \theta_1(0, \tau)} \right), [M] \right\rangle, \nonumber
\]

\[
\text{Ell}_2(M) = \left\langle \prod_{j=1}^{2n} \left( \frac{\theta'(0, \tau) \theta_2(z_j, \tau)}{\theta(z_j, \tau) \theta_2(0, \tau)} \right), [M] \right\rangle, \nonumber
\]

with \(\tau \in \mathbb{H}\), the upper half-plane, and \(q = e^{2\pi \sqrt{-1} \tau}\). Here \(\theta(v, \tau), \theta_1(v, \tau)\) and \(\theta_2(v, \tau)\) are the Jacobi theta functions (c.f. \([10]\)):

\[
\theta(v, \tau) = 2q^{1/8} \sin(\pi v) \prod_{j=1}^{\infty} [(1 - q^j)(1 - e^{2\pi \sqrt{-1} v} q^j)(1 - e^{-2\pi \sqrt{-1} v} q^j)], \tag{1.9}
\]

\[
\theta_1(v, \tau) = 2q^{1/8} \cos(\pi v) \prod_{j=1}^{\infty} [(1 - q^j)(1 + e^{2\pi \sqrt{-1} v} q^j)(1 + e^{-2\pi \sqrt{-1} v} q^j)], \tag{1.10}
\]

\[
\theta_2(v, \tau) = \prod_{j=1}^{\infty} [(1 - q^j)(1 - e^{2\pi \sqrt{-1} v} q^{j-1/2})(1 - e^{-2\pi \sqrt{-1} v} q^{j-1/2})] \tag{1.11}
\]

and \(\theta'(0, \tau) = \frac{\partial}{\partial v} \theta(v, \tau)|_{v=0}\). It turns out that the elliptic genera of \(M\) depend only on its rational Pontryagin classes and therefore are homeomorphism invariants of \(M\) by a famous theorem of Novikov. As the elliptic genera only depend on the rational Pontryagin numbers, they are also oriented bordism invariants.

By the Chern Weil theory, elliptic genera can also be expressed in terms of curvature. Suppose \(\nabla^{TM}\) be a connection on \(TM\) and \(R^{TM} = (\nabla^{TM})^2\) be its curvature. One has \((18)\)

\[
\text{Ell}_1(M) = \int_M \det \left( 2\pi^2 \frac{R^{TM}}{\theta'(0, \tau) \theta_1(R^{TM}/2\pi^2, \tau) \theta_1(0, \tau)} \right), \tag{1.12}
\]
One of the important properties of elliptic genera is modularity. More precisely it can be shown that $Ell_1(M)$ is a modular form of weight 2 over $\Gamma_0(2)$ and $Ell_2(M)$ is a modular form of weight 2 over $\Gamma_0(2)$ and they are modularly related (64). See Theorem 2.1 and the proof of Theorem 2.2 for explanations. The theory of elliptic genera triggers the vast development of elliptic (co)homology theory 33, 35.

Another important properties of elliptic genus is rigidity. Let $M$ be a closed smooth manifold and $P$ be an elliptic operator on $M$. We assume that a compact connected Lie group $G$ acts on $M$ nontrivially and that $P$ commutes with the $G$-action. Then the kernel and cokernel of $P$ are finite dimensional representations of $G$. The equivariant index of $P$ is the character of the virtual representation of $G$ defined by

$$\text{Ind}(P, h) = \text{Tr} \left[ h \left| \ker P \right. \right] - \text{Tr} \left[ h \left| \coker P \right. \right], \quad h \in G.$$ (1.14)

$P$ is said to be rigid for the $G$-action if $\text{Ind}(P, h)$ does not depend on $h \in G$. Motivated by physics, Witten conjectured that the operators $d_s \otimes (\Theta(T_C M) \otimes \Theta_1(T_C M))$ and $\Theta_2(T_C M)$ are rigid. The Witten conjecture was first proved by Taubes 56 and Bott-Taubes 9. In 41, 42, using the modular invariance property, Liu presented a simple and unified proof as well as vast generalizations of the Witten conjecture. The rigidity theorems have been generalized in 19, 21, 43, 44, 45, 46, 47, 48, 49 etc. to various situations. We will summarize the basic properties of elliptic genera needed for the proof of Theorem 1.2 in Section 2.

Closely related to elliptic genera is the Witten genus

$$W(M) = \left\langle \hat{A}(TM) \text{ch} (\Theta(T_C M)) , [M] \right\rangle \in \mathbb{Q}[[q]].$$

When $M$ is spin,

$$W(M) = \text{Ind}(D \otimes \Theta(T_C M)) \in \mathbb{Z}[[q]].$$

Using Chern roots algorithm, one has

$$W(M) = \left\langle \prod_{j=1}^{2n} z_j \left( \frac{\theta'(0, \tau)}{\theta(z_j, \tau)} \right), [M] \right\rangle .$$

If $M$ is a string manifold, i.e. $\frac{1}{2}p_1(TM) = 0$, or even weaker, if $M$ is spin and the first rational Pontryagin class of $M$ vanishes, then $W(M)$ is a modular form of weight $2n$ over $SL(2, \mathbb{Z})$ (64). The homotopy theoretical refinement of the Witten genus on string manifolds leads to the theory of topological modular form, the ”universal elliptic cohomology”, developed by Hopkins and Miller 28. The string condition is the orientability condition for this generalized cohomology theory.

The Witten genus is an obstruction to simply connected Lie group actions on string manifolds. Actually it has been shown that a string manifold with a nontrivial $S^3$-action has vanishing Witten genus 41, 20. Recently this vanishing theorem has been generalized to proper actions of non compact Lie groups on non compact manifolds in 26.

The Witten genus is also conjectured to be an obstruction to positive Ricci curvature on string manifolds. More precisely, the famous Stolz conjecture 55 says that if $M$ is a smooth
closed string manifold of dimension $4n$ and admits a Riemannian metric with positive Ricci curvature, then the Witten genus $W(M)$ vanishes. This conjecture can be viewed as the higher version of the classical Lichnerowicz theorem \cite{39}. So far the Stolz conjecture is still open.

Our Theorem 1.2 gives a relationship between Ricci curvature and the elliptic genera. The following example shows that on a closed spin Riemannian manifold, without the curvature assumptions in Theorem 1.2 or Corollary 1.3 even if the isometry group is infinite, the elliptic genera do not necessarily vanish.

**Example 2.** Let $M$ be a $4n$ dimensional smooth closed spin manifold. The famous Atiyah-Hirzebruch vanishing theorem asserts that if $M$ carries a nontrivial $S^1$-action, then $\hat{A}(M) = 0$ \cite{2}. Let $X = X(5,2)$ be a smooth quadric hypersurface in $\mathbb{CP}^5$. This is a 8 dimensional closed spin manifold carrying the linear $SO(6)$ action and therefore a nontrivial $S^1$-action, preserving the Kähler metric on $X$ induced by the embedding $X \subset \mathbb{CP}^5$. Hence $\hat{A}(X) = 0$. We will show that $\int_X \hat{A}(TX) \text{ch}(T_C X) \neq 0$, which implies $\text{Ell}_2(X) \neq 0$ by (1.6). Actually by the 8 dimensional miraculous cancellation formula \cite{40}, one has

$$\sigma(X) = 24 \hat{A}(X) - \int_X \hat{A}(TX) \text{ch}(T_C X),$$

where $\sigma(X)$ is the signature. Since $\hat{A}(X) = 0$, we just need to show that $\sigma(X) \neq 0$. Let $x \in H^2(\mathbb{CP}^5, \mathbb{Z})$ be the generator. Then by the Hirzebruch signature theorem and Poincaré duality, one sees that

$$\sigma(X) = \left\langle \left(\frac{x}{\tan x}\right)^6 \tan(2x), [\mathbb{CP}^5] \right\rangle = \text{Res}_{x=0} \left(\frac{\tan 2x}{(\tan x)^6}\right) = 2.$$

We now discuss the main idea in the proof of Theorem 1.2. We prove it by contradiction. If Theorem 1.2 is not true, given $n \in \mathbb{N}$ and positive numbers $p, \lambda_1, \lambda_2$ with $p > 2n$, then there is a sequence of compact $4n$-dimensional spin Riemannian manifolds $(M_i, g_i)$ with infinite isometry groups and

$$-\lambda_1 \leq \text{Ric}(g_i) \leq \frac{4n - 1}{i}$$

$$\text{diam}(g_i) \leq 1$$

$$\frac{1}{V(g_i)} \int_{M_i} |\mathcal{R}_{g_i}|^p dV \leq \lambda_2$$

$$\text{Ell}_1(M_i) \neq 0 \text{ or } \text{Ell}_2(M_i) \neq 0.$$

By Theorem 2.1, we see $\text{Ell}_2(M_i) = 0$ implies that $\text{Ell}_1(M_i) = 0$. Hence we only deal with the case $\text{Ell}_2(M_i) \neq 0$. By Theorem 2.2 there exists some $k_i$ with $0 \leq k_i \leq \left[\frac{n}{2}\right]$ such that

$$\text{Ind}(D_i \otimes B_{k_i}(T_C M_i)) \neq 0,$$

where $D_i$ is the Atiyah-Singer spin Dirac operator on $M_i$ and $B_{k_i}(T_C M_i)$ is an integral linear combination of bundles of type

$$S^{j_1}(T_C M) \otimes \cdots \otimes S^{j_r}(T_C M) \otimes \Lambda^{j_1}(T_C M) \otimes \cdots \otimes \Lambda^{j_s}(T_C M),$$

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who are subbundles of tensor products of $T_{\mathbb{C}}M$ of power at most $k_i$. Denote the twisted operator $D_i \otimes B_{k_i}(T_{\mathbb{C}}M_i)$ by $P_i$ acting on $\Gamma(S(TM_i) \otimes B_{k_i}(T_{\mathbb{C}}M_i))$, where $S(TM_i)$ is the spinor bundle over $M_i$.

As the isometry group of $(M_i, g_i)$ is infinite, there is a nonzero Killing vector field $X_i$ on $M_i$ generating an isometric $S^1$ action on $M_i$. By passing to a cover of $S^1$, if necessary, we assume that the $S^1$ action preserves the spin structure of $M_i$. By Theorem 2.3, $P_i$ is rigid. So we have

$$\text{Ind } P_i = \text{Ind}(P_i, 1) = \text{Ind}(P_i, \lambda), \ \forall \lambda \in S^1. \quad (1.15)$$

As the equivariant index $\text{Ind}(P_i, \lambda)$ is a Laurent polynomial of $\lambda$ and independent on $\lambda \in S^1$, one must have

$$\text{Ind } P_i = \text{Ind}(P_i, \lambda) = \dim \left( \ker P_i \cap \Gamma(S(TM_i) \otimes B_{k_i}(T_{\mathbb{C}}M_i))^{S^1} \right)$$

$$- \dim \left( \coker P_i \cap \Gamma(S(TM_i) \otimes B_{k_i}(T_{\mathbb{C}}M_i))^{S^1} \right), \quad (1.16)$$

where $\Gamma(S(TM_i) \otimes B_{k_i}(T_{\mathbb{C}}M_i))^{S^1}$ consists of smooth sections of $S(TM_i) \otimes B_{k_i}(T_{\mathbb{C}}M_i)$ invariant under the $S^1$ action.

Consider the following Witten deformation of $P_i$:

$$\tilde{P}_i = P_i + \sqrt{-1}t_i c(X_i), \quad (1.17)$$

where $t_i := \left( \frac{V(g_i)}{\int_{M_i} |X_i|^2 dV_i} \right)^{1/2} > 0$ as $X_i \neq 0$. Clearly, $\tilde{P}_i$ is also $S^1$-invariant, so we have

$$\dim \left( \ker P_i \cap \Gamma(S(TM_i) \otimes B_{k_i}(T_{\mathbb{C}}M_i))^{S^1} \right)$$

$$- \dim \left( \coker P_i \cap \Gamma(S(TM_i) \otimes B_{k_i}(T_{\mathbb{C}}M_i))^{S^1} \right)$$

$$= \dim \left( \ker \tilde{P}_i \cap \Gamma(S(TM_i) \otimes B_{k_i}(T_{\mathbb{C}}M_i))^{S^1} \right)$$

$$- \dim \left( \coker \tilde{P}_i \cap \Gamma(S(TM_i) \otimes B_{k_i}(T_{\mathbb{C}}M_i))^{S^1} \right). \quad (1.18)$$

Since $\text{Ind } P_i \neq 0$ and $\tilde{P}_i$ is self adjoint, we see that there must exist some nonzero $s_i \in \Gamma(S(TM_i) \otimes B_{k_i}(T_{\mathbb{C}}M_i))$ such that

$$\tilde{P}_i s_i = 0,$$

$$L_{X_i} s_i = 0,$$

where $L_{X_i} s_i$ is the Lie derivative of $s_i$ in the direction $X_i$.

By Theorem 3.2, we have the following crucial inequality

$$\int_{M_i} t_i^2 |X_i|^2 |s_i|^2 dV_i \leq C(n) \int_{M_i} t_i |\nabla X_i||s_i|^2 dV_i \quad (1.19)$$

for some constant $C(n)$ depending only on $n$.

As $Ric(g_i) \leq \frac{4n-1}{t_i}$, applying Bochner formula to $X_i$, we get

$$\int_{M_i} |\nabla X_i|^2 dV_i \leq \frac{4n-1}{t_i} \int_{M_i} |X_i|^2 dV_i. \quad (1.20)$$

To prove the desired vanishing theorem for elliptic genera from (1.19) and (1.20), we must get around the difficulty that $X_i$ might have zeros. We will prove a mean value inequality based on
Moser iteration in section 4. Using this mean value inequality and a Poincaré-Sobolev inequality, combined with (1.19) and (1.20), for sufficiently large $i$, we are able to show that

$$\int_{M_i} |s_i|^2 dV_i \leq \frac{1}{2} \int_{M_i} |s_i|^2 dV_i.$$ 

Hence $s_i \equiv 0$ for sufficiently large $i$. Contradiction.

Our method is influenced by the techniques in the classical papers [7, 51, 59] and combines the modularity and rigidity of elliptic genera as well as the mean value inequality obtained in Theorem 4.2.

We emphasize that the rigidity phenomenon of the Witten operators $d_s \otimes (\Theta(T_C M) \otimes \Theta_1(T_C M)), \ D \otimes (\Theta(T_C M) \otimes \Theta_2(T_C M))$ is used in a crucial way in our proof of Theorem 1.2. For this reason, our method can not be used to prove a similar vanishing theorem for Witten genus as the operator $D \otimes \Theta(T_C M)$ is not rigid in general [41].

The paper is organized as follows. In section 2, we recall some basic facts about elliptic genera. In section 3, we prove an integral formula. In section 4, we prove a mean value inequality. Theorem 1.2 will be proved in section 5.

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2 Basic properties of elliptic genera

In this section, we recall some basic properties of elliptic genera. Let $M$ be a $4n$ dimensional compact spin manifold and $Ell_1(M), Ell_2(M)$ be the elliptic genera. The following properties of elliptic genera are essential to us.

**Theorem 2.1.** $Ell_1(M)$ and $Ell_2(M)$ are modularly related as

$$Ell_1(M, -1/\tau) = (2\tau)^{2n}Ell_2(M, \tau).$$

**(2.1)**

*Proof.* See page 119-120 in [27] and [40].

**Theorem 2.2.** (i) $\forall k \geq 0$, the $B_k(T_C M)$ in the expansion (1.4) is a virtual bundle, which is an integral linear combination of bundles of type

$$S^{ij}(T_C M) \otimes \cdots \otimes S^{ir}(T_C M) \otimes \Lambda^{a_1}(T_C M) \otimes \cdots \otimes \Lambda^{a_s}(T_C M),$$

who are subbundles of tensor products of $T_C M$ of power at most $k$;

(ii) $Ell_2(M)$ is determined by $\text{Ind}(D \otimes B_k(T_C M)), 0 \leq k \leq \left[\frac{n}{2}\right]$. 

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Proof. The first statement can be simply observed from (1.1), (1.2) and (1.4).

The proof of second statement can be found in Section 8.2 in [27] and [40]. We repeat here to show how the elliptic genus is determined by $B_k(T_C M)$ more explicitly.

Let

$$\text{SL}_2(\mathbb{Z}) := \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \mid a, b, c, d \in \mathbb{Z}, \ ad - bc = 1 \right\}$$

as usual be the famous modular group. Let

$$S = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right), \quad T = \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right)$$

be the two generators of $\text{SL}_2(\mathbb{Z})$. Their actions on $H$ are given by

$$S : \tau \to -\frac{1}{\tau}, \quad T : \tau \to \tau + 1.$$

Let

$$\Gamma_0(2) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{2} \right\},$$

$$\Gamma^0(2) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{Z}) \mid b \equiv 0 \pmod{2} \right\}$$

be the two modular subgroup of $\text{SL}_2(\mathbb{Z})$. It is known that the generators of $\Gamma_0(2)$ are $T, ST^2ST$, while the generators of $\Gamma^0(2)$ are $STS, T^2STS$ (cf. [10]). It can be shown that $\text{Ell}_1(M)$ is a modular form of weight $2n$ over $\Gamma_0(2)$ and $\text{Ell}_2(M)$ is a modular form of weight $2n$ over $\Gamma^0(2)$ (c.f. [40]).

If $\Gamma$ is a modular subgroup, let $\mathcal{M}_\mathbb{Z}(\Gamma)$ denote the ring of modular forms over $\Gamma$ with real Fourier coefficients. We introduce four explicit modular forms (cf. page 119 in [27]),

$$\delta_1(\tau) = \frac{1}{4} + 6 \sum_{n=1}^{\infty} \sum_{d|n, d \text{ odd}} dq^n, \quad \varepsilon_1(\tau) = \frac{1}{16} + \sum_{n=1}^{\infty} \sum_{d|n} (\varepsilon d^3) q^n,$$

$$\delta_2(\tau) = -\frac{1}{8} - 3 \sum_{n=1}^{\infty} \sum_{d|n, d \text{ odd}} dq^{n/2}, \quad \varepsilon_2(\tau) = \sum_{n=1}^{\infty} \sum_{d|n, n/d \text{ odd}} d^3 q^{n/2}.$$ 

They have the following Fourier expansions in $q^{1/2}$:

$$\delta_1(\tau) = \frac{1}{4} + 6q + 6q^2 + \cdots, \quad \varepsilon_1(\tau) = \frac{1}{16} - q + 7q^2 + \cdots,$$

$$\delta_2(\tau) = -\frac{1}{8} - 3q^{1/2} - 3q + \cdots, \quad \varepsilon_2(\tau) = q^{1/2} + 8q + \cdots.$$ 

where the “$\cdots$” terms are the higher degree terms, all of which have integral coefficients. They also satisfy the transformation laws,

$$\delta_2 \left( -\frac{1}{\tau} \right) = \tau^2 \delta_1(\tau), \quad \varepsilon_2 \left( -\frac{1}{\tau} \right) = \tau^4 \varepsilon_1(\tau). \quad (2.2)$$
One has that $\delta_1(\tau)$ (resp. $\varepsilon_1(\tau)$) is a modular form of weight 2 (resp. 4) over $\Gamma_0(2)$, while $\delta_2(\tau)$ (resp. $\varepsilon_2(\tau)$) is a modular form of weight 2 (resp. 4) over $\Gamma^0(2)$, and moreover $\mathcal{M}_\mathbb{R}(\Gamma^0(2)) = \mathbb{R}[\delta_2(\tau), \varepsilon_2(\tau)]$.

Therefore one can express $\text{Ell}_2(M)$ in terms of $8\delta_2(\tau)$ and $\varepsilon_2(\tau)$ as

$$\text{Ell}_2(M) = h_0(8\delta_2(\tau))^n + h_1(8\delta_2(\tau))^{n-2}\varepsilon_2(\tau) + \cdots + h_{\lfloor \frac{n}{2} \rfloor}(8\delta_2(\tau))^\bar{n}\varepsilon_2(\tau)^{\lfloor \frac{n}{2} \rfloor},$$

where $\bar{n} = 0$ if $n$ is even and $\bar{n} = 1$ if $n$ is odd, and each $h_r$, $0 \leq r \leq \lfloor \frac{n}{2} \rfloor$, is an integer. They are all indices of certain twisted Dirac operators on $M$. Write $\Theta(T_CM) \otimes \Theta_2(T_CM)$ as

$$\Theta(T_CM) \otimes \Theta_2(T_CM) = B_0(T_CM) + B_1(T_CM)q^{\frac{1}{2}} + \cdots.$$  

The $B_i$’s carry canonically induced Hermitian metrics and connections from the Riemannian metric and Levi-Civita connection on $TM$. Then

$$\text{Ell}_2(M) = \text{Ind}(D \otimes B_0(T_CM)) + \text{Ind}(D \otimes B_1(T_CM))q^{\frac{1}{2}} + \cdots.$$  

Comparing the $q$-coefficients in (2.3) and (2.5) and noticing that that $8\delta_2(\tau)$ starts from $-1$, one sees that each $h_r$ is a canonical linear combination of $\text{Ind}(D \otimes B_j(T_CM))$, $0 \leq j \leq r$. So we see that $\text{Ell}_2(M)$ is determined by $B_k(T_CM)$, $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

**Theorem 2.3** (Witten-Bott-Taubes-Liu). The Witten operators

$$d_s \otimes (\Theta(T_CM) \otimes \Theta_1(T_CM)), \ D \otimes (\Theta(T_CM) \otimes \Theta_2(T_CM))$$

are rigid.

**Proof.** See [9, 11, 42, 56].

### 3 Dirac bundles and an integral formula

In this section, we briefly review the Dirac bundles (c.f. page 114 in [37]) and then prove an integral formula as well as an inequality, which will be needed in the proof of Theorem 1.2.

Let $M$ be a compact Riemannian manifold of dimension $m$ and $\nabla^M$ be the Levi-Civita connection. Let $Cl(M)$ be Clifford algebra bundle constructed from the the tangent bundle $TM$ and the Riemannian metric. $\nabla^M$ induces a connection on $Cl(M)$, which we will still denote by $\nabla^M$. Let $E$ be a complex vector bundle of left module over $Cl(M)$ (i.e. a vector bundle over $M$ such that at each point $x \in M$, the fiber $E_x$ is a left module over the algebra $Cl(M)_x$). $E$ together with a Hermitian metric $g^E$ and a compatible connection $\nabla^E$ is called a Dirac bundle if

(i) The Clifford multiplication by unit tangent vectors is unitary, i.e., for each $x \in M$,

$$\langle c(e)s_1, c(e)s_2 \rangle = \langle s_1, s_2 \rangle$$

for all $s_1, s_2 \in E_x$ and unit vectors $e \in T_xM$; this is equivalent to

$$\langle c(e)s_1, s_2 \rangle + \langle s_1, c(e)s_2 \rangle = 0$$

(3.2)
for all \( s_1, s_2 \in E_x \) and unit vectors \( e \in T_x M \);
(ii) The connection \( \nabla^E \) is a module derivation, i.e.,
\[
\nabla^E(\phi \cdot s) = (\nabla^T M \phi) \cdot s + \phi \cdot (\nabla^E s)
\]
for all \( \phi \in \Gamma(Cl(M)) \) and all \( s \in \Gamma(E) \).

The Dirac operator on \( E \) is the first-order differential operator \( D : \Gamma(E) \to \Gamma(E) \) defined by
\[
Ds = \sum_{j=1}^{m} c(e_j)\nabla^E_{e_j} s
\]
at \( x \in M \), where \( e_1, e_2, \ldots, e_m \) is a local orthonormal basis of \( TM \). On \( \Gamma(E) \), there is an inner product induced from the pointwise inner product by setting
\[
(s_1, s_2) = \int_M \langle s_1, s_2 \rangle.
\]
The Dirac operator is formally self-adjoint with respect to this inner product, i.e.,
\[
(Ds_1, s_2) = (s_1, Ds_2)
\]
for any sections \( s_1, s_2 \).

Let \( X \) be a tangent vector field on \( M \). Suppose \( s \in \Gamma(E) \) satisfies
\[
(D + \sqrt{-1}tc(X))s = 0
\]
for some \( t \in \mathbb{R} \).

Then we have the following integral formula.

**Theorem 3.1.**
\[
2\sqrt{-1} \int_M t|c(X)s|^2 = \int_M -2 \langle \nabla^E_X s, s \rangle - \sum_{i=1}^{m} \langle c(\nabla^T M e_i X)s, c(e_i)s \rangle.
\]

**Proof.** Let \( U \) be a vector field on \( M \) defined by
\[
U = \sum_{i=1}^{m} \langle c(X)s, c(e_i)s \rangle e_i,
\]
where \( \{e_i\} \) is a local orthonormal basis. Suppose at point \( p \), we have \( \nabla^T M e_i = 0, \forall i,j \).
Then at point \( p \), we have

\[
\text{div} U = \sum_{j=1}^{m} \left( \langle \nabla_{e_j}^{TM} \left( \sum_{i=1}^{m} \langle c(X)s, c(e_i)s \rangle e_i \right), e_j \rangle \right)
\]

\[
= \sum_{i=1}^{m} \langle \nabla_{e_i}^{E} (c(X)s), c(e_i)s \rangle + \sum_{i=1}^{m} \langle c(X)s, \nabla_{e_i}^{E} (c(e_i)s) \rangle
\]

\[
= \sum_{i=1}^{m} \langle c(\nabla_{e_i}^{TM} X)s + c(X)\nabla_{e_i}^{E} s, c(e_i)s \rangle + \sum_{i=1}^{m} \langle c(X)s, c(e_i)\nabla_{e_i}^{E} s \rangle
\]

\[
= \sum_{i=1}^{m} \langle c(\nabla_{e_i}^{TM} X)s, c(e_i)s \rangle + \sum_{i=1}^{m} \langle c(X)\nabla_{e_i}^{E} s, c(e_i)s \rangle + \langle c(X)s, Ds \rangle \quad (3.8)
\]

\[
= \sum_{i=1}^{m} \langle c(\nabla_{e_i}^{TM} X)s, c(e_i)s \rangle - \sum_{i=1}^{m} \langle c(e_i)c(X)\nabla_{e_i}^{E} s, s \rangle + \langle c(X)s, Ds \rangle
\]

\[
= \sum_{i=1}^{m} \langle (c(X)c(e_i) + 2 \langle c(e_i), X \rangle)\nabla_{e_i}^{E} s, s \rangle + \langle c(X)s, Ds \rangle
\]

\[
= \sum_{i=1}^{m} \langle c(\nabla_{e_i}^{TM} X)s, c(e_i)s \rangle + \langle c(X)Ds, s \rangle + 2 \langle \nabla_{X}^{E} s, s \rangle + \langle c(X)s, Ds \rangle.
\]

But since \( Ds = -\sqrt{-1}tc(X)s \), we have

\[
\langle c(X)s, Ds \rangle = \sqrt{-1}t|c(X)s|^2 = \sqrt{-1}t|X|^2|s|^2; \quad (3.9)
\]

\[
\langle c(X)Ds, s \rangle = -\sqrt{-1} \langle tc(X)c(X)s, s \rangle = \sqrt{-1}t|X|^2|s|^2. \quad (3.10)
\]

The desired formula follows.

Now we apply the integral formula in Theorem 3.1 to the Dirac bundles \( S(TM) \otimes B_k(T\mathbb{C}M) \), \( 0 \leq k \leq \left[ \frac{n}{2} \right] \), where \( S(TM) \) is the spinor bundle over a compact \( 4n \) dimensional spin manifold \( M \) and \( B_k(T\mathbb{C}M) \) involves linear combinations of tensor product of \( T\mathbb{C}M \) at most to power \( k \). We also assume that \( X \) is a Killing vector field generating an isometric \( S^1 \) action on \( M \). By passing to a cover of \( S^1 \), if necessary, we assume that the \( S^1 \) action preserves the spin structure of \( M \). Let \( P = D \otimes B_k(T\mathbb{C}M) \) be the Dirac operator acting on \( \Gamma(S(TM) \otimes B_k(T\mathbb{C}M)) \). Suppose \( s \in \Gamma(S(TM) \otimes B_k(T\mathbb{C}M)) \) satisfies

\[
(P + \sqrt{-1}tc(X))s = 0
\]

\[
L_X s = 0,
\]

where \( t \in \mathbb{R} \) and \( L_X s \) is the Lie derivative of \( s \) in the direction \( X \). Then we have the following crucial inequality

**Theorem 3.2.**

\[
\int_M t^2|X|^2|s|^2dV \leq C(n) \int_M t|\nabla X||s|^2dV, \quad (3.11)
\]

where \( C(n) \) is some constant depending only on \( n \).
Proof. By (1.24) in [59], we get

\[
L_X|_{S(TM)} - \nabla_X^{S(TM)} = - \sum_{j,k=1}^{4n} \frac{1}{4} \left\langle \nabla_{e_j}^{TM} X, e_k \right\rangle c(e_j)c(e_k).
\]

As \(\nabla^{TM}\) is torsion free, we have

\[
L_X - \nabla_X^{TM} = - \nabla^{TM} X.
\]

Since by Theorem 2.2, \(B_k(TC M)\) is an integral linear combination of bundles of type \(S^{i_1}(TC M) \otimes \cdots \otimes S^{i_r}(TC M) \otimes \Lambda^{j_1}(TC M) \otimes \cdots \otimes \Lambda^{j_s}(TC M)\), who are subbundles of tensor products of \(TC M\) of power at most \(k, 0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor\), we see that

\[
\langle \nabla_X s, s \rangle - \langle L_X s, s \rangle \leq C(n) \|\nabla_X s\| s^2
\]

for some constant \(C(n)\) depending only on \(n\). Then Theorem 3.2 is a direct consequence of Theorem 3.1. \(\square\)

4 A mean value inequality

In this section we prove a mean inequality which will be needed in the proof of Theorem 1.2. We firstly recall the following Poincaré-Sobolev inequality, see for example Theorem 2, page 386 and Theorem 3, page 397 in [6].

**Theorem 4.1.** Let \((M, g)\) be a closed \(m\)-dimensional smooth Riemannian manifold such that for some constant \(b > 0\),

\[
r_{\min}(g)(\text{diam}(g))^2 \leq -(m - 1)b^2,
\]

where \(\text{diam}(g)\) is the diameter of \(g\), \(\text{Ric}(g)\) is the Ricci curvature of \(g\) and

\[
r_{\min}(g) = \inf \{\text{Ric}(g)(u, u) : u \in TM, g(u, u) = 1\}.
\]

Let \(R = \frac{\text{diam}(g)}{2C(b)}\), where \(C(b)\) is the unique positive root of the equation

\[
x \int_0^b (cht + xsh)^{m-1} \, dt = \int_0^\pi \sin^{m-1} \theta \, d\theta.
\]

Then for each \(1 \leq l_1 \leq \frac{m l_1}{m - l_2}, l_1 < \infty\) and \(f \in W^{1, l_2}(M)\), we have

\[
\|f - \frac{1}{V(g)} \int_M f \, dV\|_{l_1} \leq S_{l_1, l_2} \|\nabla f\|_{l_2},
\]

\[
\|f\|_{l_1} \leq S_{l_1, l_2} \|\nabla f\|_{l_2} + V(g)^{1/l_1 - 1/l_2} \|f\|_{l_2},
\]

where \(V(g)\) is the volume of \((M, g)\), \(S_{l_1, l_2} = (V(g)/\text{vol}(S^m(1)))^{1/l_1 - 1/l_2} R\Sigma(m, l_1, l_2)\) and \(\Sigma(m, l_1, l_2)\) is the Sobolev constant of the canonical unit sphere \(S^m\) defined by

\[
\Sigma(m, l_1, l_2) = \sup \{\|f\|_{l_1}/\|\nabla f\|_{l_2} : f \in W^{1, l_2}(S^m), f \neq 0, \int_{S^m} f = 0\}.
\]
As an application of Theorem 4.1, we get the following mean value inequality which is a generalization of Theorem 3 in [6], pages 395-396. See also [38] pages 80-84.

**Theorem 4.2.** Let \( m \geq 3 \) and \( (M, g) \) be a closed \( m \)-dimensional smooth Riemannian manifold such that for some constant \( b > 0 \),

\[
 r_{\min}(g)(\text{diam}(g))^2 \geq -(m-1)b^2.
\]

If \( f \in W^{1,2}(M) \) is a nonnegative continuous functions such that \( f \Delta f \geq -h_1 f^2 - \text{div}Y \) (here \( \Delta \) is a negative operator) in the sense of distribution for some nonnegative continuous function \( h_1 \) and \( Y \) is a \( C^1 \) vector field satisfying

\[
 |Y|(x) \leq h_2(x)f^2(x), \forall x \in M
\]

for some nonnegative continuous function \( h_2 \), then

\[
 \max_{x \in M} |f|^2(x) \leq C(m, p, R, \Lambda) \int_M f^2 \, dV.
\]

where \( C(m, p, R, \Lambda) \) is some constant depending only on \( m, p, R = \frac{\text{diam}(g)}{bc(b)} \) and

\[
 \Lambda = \int_M h^p \, dV, \quad p > \frac{m}{2}
\]

\[
 h = h_1 + 2h_2^2.
\]

**Proof.** The proof is a standard application of Moser iteration. For any \( k \geq 1 \), multiply the inequality \( f \Delta f \geq -h_1 f^2 - \text{div}Y \) by \( f^{2k-2} \) and integrate. Then we get

\[
 \int_M f^{2k-1} \Delta f \geq \int_M -h_1 f^{2k} - \text{div}Y f^{2k-2}
\]

\[
 = \int_M -h_1 f^{2k} + \langle Y, \nabla f^{2k-2} \rangle
\]

\[
 = \int_M -h_1 f^{2k} + (2k-2)f^{2k-3} \langle Y, \nabla f \rangle
\]

Hence

\[
 (2k-1) \int_M f^{2k-2} |\nabla f|^2 \leq \int_M h_1 f^{2k} - (2k-2)f^{2k-3} \langle Y, \nabla f \rangle
\]

\[
 \leq \int_M h_1 f^{2k} + (2k-2)f^{2k-1}h_2 |\nabla f|
\]

\[
 \leq \int_M h_1 f^{2k} + (2k-2)h_2^2 f^{2k} + \frac{2k-2}{4} f^{2k-2} |\nabla f|^2.
\]

Then

\[
 \frac{3k-1}{2} \int_M f^{2k-2} |\nabla f|^2 \leq \int_M (h_1 + (2k-2)h_2^2) f^{2k}
\]

And

\[
 \int_M |\nabla f|^2 \leq \frac{2k^2}{3k-1} \int_M (h_1 + (2k-2)h_2^2) f^{2k}
\]
\[ \leq k^2 \int_M \left( h_1 + 2h_2 \right) f^{2k}. \]

So
\[ \| \nabla f^k \|_2 \leq \left( \int_M k^2 h f^{2k} \right)^{\frac{1}{2}}. \]

Applying Theorem 4.1 to \( f^k \), we get
\[ \| f^k \|_{\frac{2m}{m-2}} \leq S_{\frac{2m}{m-2}} \| \nabla f^k \|_2 + V(g)^{\frac{1}{m}} \| f^k \|_2. \quad (4.1) \]

When \( p > \frac{m}{2} \), by the Hölder inequality, we have
\[
k^2 \int_M h f^{2k} \leq k^2 A (\int_M (f^{2k})^{\frac{p}{p-1}})^{\frac{p-1}{p}} \leq k^2 A (\int_M f^{2k})^{\frac{\mu(p-1)-p}{p(\mu-1)}} (\int_M f^{2k\mu})^{\frac{1}{\mu(\mu-1)}}, \quad (4.2)\]

where \( A = (\int_M h^p)^{\frac{1}{p}}, \mu = \frac{m}{m-2}. \)

Define \( \epsilon, \delta, x \) by
\[
\epsilon = \frac{\mu(p-1)-p}{p(\mu-1)} \\
(\delta \epsilon \frac{1}{1-\epsilon} (\frac{1}{\epsilon} - 1))^\frac{1}{2} = \frac{1}{2S_{2\mu,2}} \\
x = (k^2 A)^{\frac{\mu(p-1)}{p(\mu-1)-p}} (\int_M f^{2k}) (\int_M f^{2k\mu})^{\frac{1}{\mu}}. 
\]

Then \( 0 < \epsilon < 1 \) as \( p > \frac{m}{2} \). By Young inequality, we get
\[
x^\epsilon \leq \delta \epsilon \frac{1}{\epsilon} x + \delta \epsilon \frac{1}{1-\epsilon} (\frac{1}{\epsilon} - 1). 
\]

Hence
\[
k^2 A (\int_M f^{2k})^{\frac{\mu(p-1)-p}{p(\mu-1)}} (\int_M f^{2k\mu})^{\frac{1}{\mu(\mu-1)}} \leq \delta \epsilon \frac{1}{\epsilon} (k^2 A)^{\frac{\mu(p-1)}{p(\mu-1)-p}} (\int_M f^{2k}) (\int_M f^{2k\mu})^{\frac{1}{\mu}} + \delta \epsilon \frac{1}{1-\epsilon} (\frac{1}{\epsilon} - 1). 
\]

Multiplying through by \( (\int_M f^{2k\mu})^{\frac{1}{\mu}} \), combined with (4.2), we get
\[
k^2 \int_M h f^{2k} \leq \delta \epsilon \frac{1}{\epsilon} (k^2 A)^{\frac{\mu(p-1)}{p(\mu-1)-p}} \int_M f^{2k} + \delta \epsilon \frac{1}{1-\epsilon} (\frac{1}{\epsilon} - 1)(\int_M f^{2k\mu})^{\frac{1}{\mu}}. 
\]

Then
\[
(k^2 \int_M h f^{2k})^{\frac{1}{2}} \leq \delta \epsilon \frac{1}{2} (k^2 A)^{\frac{\mu(p-1)}{2(p(\mu-1)-p)}} (\int_M f^{2k})^{\frac{1}{2}} + (\delta \epsilon \frac{1}{2} (\frac{1}{\epsilon} - 1))^\frac{1}{2} (\int_M f^{2k\mu})^{\frac{1}{\mu}}. \quad (4.3) 
\]

Combined with (4.1), we get
\[
(\int_M f^{2k\mu})^{\frac{1}{\mu}} \leq S_{2\mu,2} (k^2 \int_M h f^{2k})^{\frac{1}{2}} + V(g)^{-\frac{1}{m}} (\int_M f^{2k})^{\frac{1}{2}} \]

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Let $S_{2\mu, 2} \delta^{\frac{-1}{\epsilon \mu}} (k^2 A)^{\frac{1}{2}} \frac{p(\mu - 1)}{\mu(p - 1) - \mu} (\int_M f^{2k})^{\frac{1}{2}} + S_{2\mu, 2} (\delta \epsilon^{\frac{1}{\epsilon \mu}} (\frac{1}{\epsilon} - 1))^{\frac{1}{2}} (\int_M f^{2k})^{\frac{1}{2}} + V(g)^{-\frac{1}{m}} (\int_M f^{2k})^{\frac{1}{2}}.

As $(\delta \epsilon^{\frac{1}{\epsilon \mu}} (\frac{1}{\epsilon} - 1))^{\frac{1}{2}} = \frac{1}{2S_{2\mu, 2}}$, then $\delta = C(m, p)(\frac{1}{2S_{2\mu, 2}})^2$ for some constant $C(m, p)$ depending only on $m, p$. Moreover, we have

$$ (\int_M f^{2k})^{\frac{1}{2}} \leq 2S_{2\mu, 2} \delta^{\frac{-1}{\epsilon \mu}} (k^2 A)^{\frac{1}{2}} \frac{p(\mu - 1)}{\mu(p - 1) - \mu} (\int_M f^{2k})^{\frac{1}{2}} + 2V(g)^{-\frac{1}{m}} (\int_M f^{2k})^{\frac{1}{2}}. $$

Then

$$ \|f\|_{2k} \leq (2S_{2\mu, 2} \delta^{\frac{-1}{\epsilon \mu}} (k^2 A)^{\frac{1}{2}} \frac{p(\mu - 1)}{\mu(p - 1) - \mu} + 2V(g)^{-\frac{1}{m}}) \|f\|_{2k}. $$

By the choice of $\epsilon$, we have

$$ \frac{\epsilon - 1}{2\epsilon} = \frac{-\mu}{2(\mu(p - 1) - \mu)}. $$

As $S_{2\mu, 2} = C(m)V(g)^{-\frac{1}{m}} R$ for some constant $C(m)$ depending only on $m$, then

$$ \|f\|_{2k} \leq (C(m, p)(V(g)^{-\frac{1}{m}} R) \frac{p(\mu - 1)}{\mu(p - 1) - \mu} (k^2 A)^{\frac{1}{2}} \frac{p(\mu - 1)}{\mu(p - 1) - \mu} + 2V(g)^{-\frac{1}{m}}) \|f\|_{2k} \leq B \frac{1}{k^2} \frac{1}{\mu}(\mu(p - 1) - \mu) \frac{p(\mu - 1)}{\mu(p - 1) - \mu} V(g)^{-\frac{1}{m}} \|f\|_{2k}, $$

where

$$ B = C(m, p)V(g)^{\frac{1}{m}} \frac{p(\mu - 1)}{\mu(p - 1) - \mu} R \frac{p(\mu - 1)}{\mu(p - 1) - \mu} A \frac{1}{\mu(p - 1) - \mu} + 2 = C(m, p)A \frac{1}{\mu(p - 1) - \mu} R \frac{p(\mu - 1)}{\mu(p - 1) - \mu} + 2. $$

Let $k = \mu^i, i = 0, 1, \cdots$. Since $K_1 = \sum i \mu^{-i}$ and $K_2 = \sum \mu^{-i}$ is finite, multiplying (4.4), we get

$$ \max_{x \in M} |f|^2 \leq C(m, p, R, \Lambda) \int_M f^2 dV. $$

$$ C(m, p, R, \Lambda) = \mu^{2K_1} \frac{p(\mu - 1)}{\mu(p - 1) - \mu} B^{2K_2}. $$



\section{Vanishing of elliptic genera}

Now we are ready to prove Theorem 1.2. As discussed in the introduction, we prove it by contradiction. If Theorem 1.2 is not true, given $n \in \mathbb{N}$ and positive numbers $p, \lambda_1, \lambda_2$ with $p > 2n$, then there is a sequence of compact $4n$-dimensional spin Riemannian manifolds $(M_i, g_i)$ with infinite isometry groups and

$$ -\lambda_1 \leq \text{Ric}(g_i) \leq \frac{4n - 1}{i}, $$

$$ \text{diam}(g_i) \leq 1, $$

$$ \frac{1}{V(g_i)} \int_{M_i} |\mathcal{R}_{g_i}|^p dV_i \leq \lambda_2 $$

$$ \text{Ell}_1(M_i) \neq 0 \text{ or } \text{Ell}_2(M_i) \neq 0. $$

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As discussed in the introduction, there must exists some nonzero \( s_i \in \Gamma(S(TM_i) \otimes B_{k_i}(T_C M_i)) \) such that
\[
\tilde{P}_i s_i = 0,
\]
\[
L_X s_i = 0,
\]
where \( X_i \) is a nonzero Killing vector field \( X_i \) on \( M_i \) generating an isometric \( S^1 \) action on \( M_i \) and
\[
\tilde{P}_i = P_i + \sqrt{-1} t_i e(X_i),
\]
\[
t_i = \left( \frac{V(g_i)}{\int_{M_i} |X_i|^2 dV_i} \right)^{1/2} > 0.
\]

By Theorem 3.2, we have the following crucial inequality
\[
\int_{M_i} t_i^2 |X_i|^2 |s_i|^2 dV_i \leq C(n) \int_{M_i} t_i |\nabla X_i||s_i|^2 dV_i
\] (5.1)
for some constant \( C(n) \) depending only on \( n \).

**Lemma 5.1.**
\[
\int_{M_i} t_i^2 |X_i|^2 |s_i|^2 dV_i \leq \frac{C(n)}{\sqrt{i}} |s_i|_{\infty} (\int_{M_i} t_i^2 |X_i|^2 dV_i)^{1/2} (\int_{M_i} |s_i|^2 dV_i)^{1/2}
\]
where \( |s_i|_{\infty} = \max_{x \in M_i} |s_i|(x) \) and \( C(n) \) is some constant depending only on \( n \).

**Proof.** As \( \text{Ric}(g_i) \leq \frac{4n-1}{i} \), applying Bochner formula to \( X_i \), we get
\[
\frac{1}{2} \Delta |X_i|^2 = |\nabla X_i|^2 - \text{Ric}(g_i)(X_i, X_i) \geq |\nabla X_i|^2 - \frac{4n-1}{i} |X_i|^2,
\] (5.2)
where \( \Delta \) is the Laplacian acting on functions which is a negative operator. Then
\[
\int_{M_i} |\nabla X_i|^2 dV_i \leq \frac{4n-1}{i} \int_{M_i} |X_i|^2 dV_i. \] (5.3)

Combining (5.1) and (5.3), we get
\[
\int_{M_i} t_i^2 |X_i|^2 |s_i|^2 dV_i \leq C(n) \int_{M_i} t_i |\nabla X_i||s_i|^2 dV_i
\]
\[
\leq C(n) \left( \int_{M_i} t_i^2 |\nabla X_i|^2 dV_i \right)^{1/2} \left( \int_{M_i} |s_i|^4 dV_i \right)^{1/2}
\]
\[
\leq \frac{C(n)}{\sqrt{i}} |s_i|_{\infty} \left( \int_{M_i} t_i^2 |X_i|^2 dV_i \right)^{1/2} \left( \int_{M_i} |s_i|^2 dV_i \right)^{1/2}, \] (5.4)
where \( |s_i|_{\infty} = \max_{x \in M_i} |s_i|(x) \).

\[\square\]
Lemma 5.2. Under the curvature assumptions in Theorem [4.2], we have

\[ |X_i|^2 = \max_{x \in M} |X_i|^2 (x) \leq C_1(n, R_i) \frac{\int_M |X_i|^2 dV_i}{V(g_i)}, \]  

(5.5)

\[ |s_i|^2 = \max_{x \in M} |s_i|^2 (x) \leq C_2(n, p, R_i, \lambda_2) \frac{\int_M |s_i|^2 dV_i}{V(g_i)}, \]  

(5.6)

where \( C_1(n, R_i), C_2(n, p, R_i, \lambda_2) \) are two constants depending only on \( n, R_i \) and \( n, p, R_i, \lambda_2 \), respectively. Moreover, \( R_i = \frac{diam(g_i)}{\sqrt[4n/(4n-1)]{C(\sqrt{\lambda_1/(4n-1))}}} \) as defined in Theorem [4.2].

Remark 5.3. If we think of \( C_1(n, R_i) \) as a function of \( R_i \), then it is in fact a monotone increasing function of \( R_i \). Similarly \( C_2(n, p, R_i, \lambda_2) \) is a monotone increasing function of \( R_i, \lambda_2 \).

Proof. Since \( X_i \) is a Killing vector field and \( Ric(g_i) \leq \frac{4n-1}{i} \), applying Bochner formula to \( X_i \), we get

\[ \frac{1}{2} \Delta |X_i|^2 = |\nabla X_i|^2 - Ric(g_i)(X_i, X_i) \geq |\nabla X_i|^2 - \frac{4n-1}{i} |X_i|^2, \]  

(5.7)

where \( \Delta \) is the Laplacian acting on functions which is a negative operator. On the other hand, by Kato’s inequality [6], we have \(|\nabla X_i| \geq |\nabla|X_i||. It follows that

\[ |X_i| \Delta |X_i| \geq -\frac{4n-1}{i} |X_i|^2. \]  

(5.8)

Since \( Ric(g_i) \geq -\lambda_1, diam(g_i) \leq 1 \), we have

\[ r_{min}(g_i) diam^2(g_i) \geq -\lambda_1. \]

Applying Theorem [4.2] to \(|X_i|\), we get

\[ |X_i|^2 = \max_{x \in M} |X_i|^2 (x) \leq C_1(n, R_i) \frac{\int_M |X_i|^2 dV_i}{V(g_i)}, \]  

(5.9)

where \( R_i = \frac{diam(g_i)}{\sqrt[4n/(4n-1)]{C(\sqrt{\lambda_1/(4n-1))}}} \) and \( C_1(n, R_i) \) is some constant depending only on \( n, R_i \). If we think of \( C_1(n, R_i) \) as a function of \( R_i \), from the proof of Theorem [4.2] then it is in fact a monotone increasing function of \( R_i \).

Applying the Bochner formula to \( s_i \), we get

\[ \frac{1}{2} \Delta |s_i|^2 = |\nabla s_i|^2 - \langle P_i^2 s_i, s_i \rangle + \langle \Psi_i s_i, s_i \rangle, \]  

(5.10)

where \( \Psi_i \) is a symmetric endomorphism of the bundle \( S(TM_i) \otimes B_k(TC M_i) \) (c.f pages 210-211 in [33] and pages 164-165 in [37]). By Theorem [2.2] \( B_k(TC M_i) \) is an integral linear combination of bundles of type

\[ S^i(TC M) \otimes \cdots \otimes S^r(TC M) \otimes \Lambda^j(TC M) \otimes \cdots \otimes \Lambda^j(TC M), \]

who are subbundles of tensor products of \( TC M \) of at most power \( k_i \) and \( 0 \leq k_i \leq \left[ \frac{R_i}{2} \right] \), then we get

\[ \langle \Psi_i s_i, s_i \rangle \geq -C(n) |R_{g_i}| |s_i|^2 \]
for some constant $C(n)$ depending only on $n$.

Define a vector field $Y_i$ by the condition

$$\langle Y_i, W \rangle = -\langle P_is_i, c(W)s_i \rangle.$$  

Then by the proof of Proposition 5.3 in pages 114-115, \[37\] we get

$$\langle P^2_is_i, s_i \rangle = \langle P_is_i, P_is_i \rangle + \text{div}Y_i.$$  

As $P_is_i + \sqrt{-1}t_ic(X_i)s_i = 0$, then we have

$$\frac{1}{2}\Delta |s_i|^2 \geq |\nabla s_i|^2 - (P_is_i, P_is_i) - \text{div}Y_i - C(n)|\mathcal{R}_{g_i}||s_i|^2$$

$$= |\nabla s_i|^2 - |t_iX_i|^2|s_i|^2 - \text{div}Y_i - C(n)|\mathcal{R}_{g_i}||s_i|^2.$$  

For any $x \in M_i$, by the choice of $t_i$, we have

$$|t_iX_i|^2(x) \leq t_i^2|X_i|_{\infty}^2 \leq t_i^2C_1(n, R_i) \frac{\int_{M_i}|X_i|^2dV_i}{V(g_i)} = C_1(n, R_i).$$

Hence we get

$$\frac{1}{2}\Delta |s_i|^2 \geq |\nabla s_i|^2 - (C_1(n, R_i) + C(n)|\mathcal{R}_{g_i}||s_i|^2 - \text{div}Y_i$$

By Kato’s inequality, we have $|\nabla s_i| \geq |\nabla|s_i||$. It follows that

$$|s_i|\Delta |s_i| \geq -(C_1(n, R_i) + C(n)|\mathcal{R}_{g_i}||s_i|^2 - \text{div}Y_i$$

By the definition of $Y_i$, we get

$$|Y_i| \leq t_i|X_i||s_i|^2 \leq C_1(n, R_i)\frac{1}{2} |s_i|^2.$$  

Applying Theorem 4.2 to $|s_i|$, we get

$$|s_i|_{\infty}^2 := \max_{x \in M_i} |s_i|^2(x) \leq C_2(n, p, R_i, C_2) \frac{\int_{M_i} |s_i|^2dV_i}{V(g_i)}$$

for some constant $C_2(n, p, R_i, C_2)$ depending only on $n, p, R_i, C_2$. If we think of $C_2(n, p, R_i, C_2)$ as a function of $R_i, C_2$, from the proof of Theorem 4.2 then it is a monotone increasing function of $R_i, C_2$. 

\[\square\]

Lemma 5.4.

\[\frac{\int_{M_i}|X_i|^2dV_i}{V(g_i)} \leq \frac{1}{2}\int_{M_i}|s_i|^2dV_i \leq \int_{M_i}|X_i|^2|s_i|^2dV_i + \frac{C(n, R_i)|s_i|_{\infty}^2}{\sqrt{t_i}} \int_{M_i}|X_i|^2dV_i\]

for some constant $C(n, R_i)$ depending only $n, R_i$.  

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Proof. Let \( h_i = |X_i|^2 \) and \( \overline{h_i} = \frac{\int_{M_i} |X_i|^2 dV_i}{V(g_i)} \). By Theorem 4.1, we get

\[
\int_{M_i} |h_i - \overline{h_i}|s_i|^2dV_i \leq |s_i|^2\left(\int_{M_i} |h_i - \overline{h_i}|^2 dV_i\right)^{\frac{1}{2}}(V(g_i))^{\frac{1}{2}} \\
\leq C(n)|s_i|^2R_i\left(\int_{M_i} |\nabla h_i|^2 dV_i\right)^{\frac{1}{2}}(V(g_i))^{\frac{1}{2}} \\
= 2C(n)|s_i|^2R_i\left(\int_{M_i} |X_i|^2|\nabla X_i||^2 dV_i\right)^{\frac{1}{2}}(V(g_i))^{\frac{1}{2}} \\
\leq 2C(n)|s_i|^2R_i\left(\int_{M_i} |X_i|^2|\nabla X_i|^2 dV_i\right)^{\frac{1}{2}}(V(g_i))^{\frac{1}{2}} \\
\leq 2C(n)|s_i|^2R_i|X_i|_\infty(V(g_i))^{\frac{1}{2}}\left(\int_{M_i} |\nabla X_i|^2 dV_i\right)^{\frac{1}{2}} \\
\leq C(n, R_i)|s_i|^2\frac{1}{\sqrt{i}}\int_{M_i} |X_i|^2 dV_i.
\]

It follows that

\[
\frac{\int_{M_i} |X_i|^2 dV_i}{V(g_i)} \int_{M_i} |s_i|^2 dV_i \leq \int_{M_i} |X_i|^2|s_i|^2 dV_i + \frac{C(n, R_i)|s_i|^2}{\sqrt{i}}\int_{M_i} |X_i|^2 dV_i.
\]

Combining Lemma 5.1, 5.2 and 5.4, we get

\[
\frac{\int_{M_i} t_i^2|X_i|^2 dV_i}{V(g_i)} \int_{M_i} |s_i|^2 dV_i \leq \int_{M_i} t_i^2|X_i|^2|s_i|^2 dV_i + \frac{C(n, R_i)|s_i|^2}{\sqrt{i}}\int_{M_i} t_i^2|X_i|^2 dV_i \\
\leq \frac{C(n)}{\sqrt{i}}|s_i|_\infty\left(\int_{M_i} t_i^2|X_i|^2 dV_i\right)^{\frac{1}{2}}\left(\int_{M_i} |s_i|^2 dV_i\right)^{\frac{1}{2}} + \frac{C(n, R_i)|s_i|^2}{\sqrt{i}}\int_{M_i} t_i^2|X_i|^2 dV_i \\
\leq \frac{C(n, p, R_i, \lambda_2)}{\sqrt{i}}\left(\frac{\int_{M_i} t_i^2|X_i|^2 dV_i}{V(g_i)}\right)^{\frac{1}{2}}\int_{M_i} |s_i|^2 dV_i + \frac{C(n, p, R_i, \lambda_2)}{\sqrt{i}}\frac{\int_{M_i} t_i^2|X_i|^2 dV_i}{V(g_i)}\int_{M_i} |s_i|^2 dV_i \\
As \ t_i = \left(\frac{\sqrt{\lambda_1}}{\sqrt{\lambda_1} + \sqrt{\lambda_2}}\right)^{1/2}, \ we \ see \\
\frac{\int_{M_i} t_i^2|X_i|^2 dV_i}{V(g_i)} = 1.
\]

Since \( diam(g_i) \leq 1 \), we see that \( R_i \leq \frac{1}{\sqrt{\lambda_1}/(4n-1)C(\sqrt{\lambda_1}/(4n-1))} \). Then we see that for sufficiently large \( i \),

\[
\int_{M_i} |s_i|^2 dV_i \leq \frac{1}{2}\int_{M_i} |s_i|^2 dV_i.
\]

Hence \( s_i \equiv 0 \) for sufficiently large \( i \), contradiction.

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References

[1] M. F. Atiyah, *K*-theory. Benjamin, New York, 1967.

[2] M. Atiyah and F. Hirzebruch, *Spin-manifolds and group actions*, 1970 Essays on Topology and Related Topics (Memoires dedies Georges de Rham), Springer, New York, 18-28.

[3] M. F. Atiyah and I.M. Singer, *The index of elliptic operators, III*, Ann. Math. 87 (1968), 546-604.

[4] R.H. Bamler, Convergence of Ricci flows with bounded scalar curvature, Ann. of Math. 188 (2018), 753-831.

[5] R.H. Bamler, Q.S. Zhang, Heat kernel and curvature bounds in Ricci flows with bounded scalar curvature, Adv. Math. 319 (2017) 396-450.

[6] P. H. Bérard. From vanishing theorems to estimating theorems: the Bochner technique revisited. Bull. Amer. Math. Soc. 19 (1988), no. 2, 371-406.

[7] J.-M. Bismut, G. Lebeau. *Complex immersions and Quillen metrics*. Publ. Math. IHES. 74, 1-297 (1991).

[8] R. Bott, *Vector fields and characteristic numbers*, Michigan Math. J. 14 (1967), 231-244.

[9] R. Bott, C. Taubes. *On the rigidity theorems of Witten*. J. Amer. Math. Soc. 2 (1989), 137–186.

[10] K. Chandrasekharan, *Elliptic Functions*. Springer-Verlag, 1985.

[11] J. Cheeger and T. Colding. Lower bounds on Ricci curvature and the almost rigidity of warped products. Ann. of Math. (2) 144 (1996), no. 1, 189-237.

[12] J. Cheeger and T. Colding, On the structure of spaces with Ricci curvature bounded below I. J. Diff. Geom. 46 (1997), no. 3, 406-480.

[13] J. Cheeger and T. Colding. On the structure of spaces with Ricci curvature bounded below. II. J. Diff. Geom. 54 (2000), no. 1, 13-35.

[14] J. Cheeger and T. Colding. On the structure of spaces with Ricci curvature bounded below. III. J. Diff. Geom. 54 (2000), no. 1, 37-74.

[15] J. Cheeger, M. Gromov, Collapsing Riemannian manifolds while keeping their curvature bounded. I. J. Diff. Geom. 23 (1986), no. 3, 309-346.

[16] J. Cheeger, M. Gromov, Collapsing Riemannian manifolds while keeping their curvature bounded. II. J. Diff. Geom. 32 (1990), no. 1, 269-298.

[17] J. Cheeger, A. Naber, Regularity of Einstein manifolds and the codimension 4 conjecture, Ann. of Math. 182 (2015) 1093-1165.

[18] Q. Chen and F. Han, *Elliptic Genera, Transgression and Loop Space Chern-Simons Forms*, Comm. Anal. Geom., Vol 17, No.1, Jan 2009, 73–106.
[19] Q. Chen, F. Han and W. Zhang, *Generalized Witten genus and vanishing theorems*, J. Diff. Geom. 88 no. 1 (2011), 1-40.

[20] A. Dessai, *The Witten genus and $S^3$-actions on manifolds*, preprint 1994, Preprint-Reihe des Fachbereichs Mathematik, Univ. Mainz, Nr. 6, February 1994.

[21] A. Dessai, Rigidity for spin$^c$ manifolds, *Topology*, Vol. 39 (2000), 239-258.

[22] K. Fukaya and T. Yamaguchi. Almost nonpositively curved manifolds. J. Diff. Geom. 33 (1991), no. 1, 67-90.

[23] K. Fukaya and T. Yamaguchi. The fundamental groups of almost nonnegatively curved manifolds. Ann. of Math. 136 (1992), no.2, 253-333.

[24] M. Gromov, Almost flat manifolds. J. Diff. Geom. 13 (1978), no. 2, 231-241.

[25] M. Gromov, Synthetic geometry in Riemannian manifolds, Proc. Internat. Congr. Math. (Helsinki, 1978), Acad. Sci. Fennica, Helsinki, 1980, 415-419.

[26] F. Han and V. Mathai, *Witten Genus and Elliptic genera for proper actions*, arXiv: 1807.06863.

[27] F. Hirzebruch, T. Berger and R. Jung, *Manifolds and Modular Forms*, Aspects of Mathematics, vol. E20, Vieweg, Braunschweig 1992.

[28] M. Hopkins, *Algebraic Topology and Modular Forms*, Plenary talk, ICM, Beijing, 2002.

[29] W. Jiang, A. Naber, $L^2$ curvature bounds on manifolds with bounded Ricci curvature, arXiv:1605.05583.

[30] V. Kapovitch and J. Lott. On noncollapsed almost Ricci-flat 4-manifolds, Amer. J. Math. 141 (2019), 737-755.

[31] V. Kapovitch, A. Petrunin and W. Tuschmann. Nilpotency, almost nonnegative curvature, and the gradient flow on Alexandrov spaces. Ann. of Math. 171 (2010), no. 1, 343-373.

[32] V. Kapovitch and B. Wilking. Structure of fundamental groups of manifolds with Ricci curvature bounded below. arXiv:1105.5955v2 [math.DG].

[33] M. Kreck and S. Stolz, $H^2$-bundles and elliptic homology, Acta. Math., 171 (1993), 231-261.

[34] P. S. Landweber, *Elliptic cohomology and modular forms*, in Elliptic Curves and Modular Forms in Algebraic Topology, p. 55-68. Ed. P. S. Landweber. Lecture Notes in Mathematics Vol. 1326, Springer-Verlag (1988).

[35] P. Landweber, D. Ravenel, R. Stong, *Periodic cohomology theories defined by elliptic curves*, in Haynes Miller et al (eds.), The Cech centennial: A conference on homotopy theory, June 1993, AMS (1995).

[36] P.S. Landweber, R.E. Stong, *Circle actions on Spin manifolds and characteristic numbers*, Topology 27(2), 145–161 (1988).
[37] H. B. Lawson and M.-L. Michelsohn, Spin Geometry, Princeton Univ. Press, 1989.

[38] P. Li. Lecture notes on geometric analysis. www.researchgate.net/publication/2634104.

[39] A. Lichnerowicz, *Spinors harmoniques*, C. R. Acad. Sci. Paris, 257 (1963), 7-9.

[40] K. Liu, *Modular invariance and characteristic numbers*, Comm. Math. Phys. 174 (1995), 29-42.

[41] K. Liu, *On modular invariance and rigidity theorems*, J. Diff. Geom. 41, no. 2 (1995), 343-396.

[42] K. Liu, On *elliptic genera and theta-functions*, Topology 35 (1996), 617-640.

[43] K. Liu, *Modular forms and topology*, Moonshine, the Monster, and related topics (South Hadley, MA, 1994), 237-262, Contemp. Math., 193, Amer. Math. Soc., Providence, RI, 1996.

[44] K. Liu and X. Ma, *On family rigidity theorems. I*, Duke Math. J., 102(3):451-474, 2000.

[45] K. Liu and X. Ma, On *family rigidity theorems for Spin^c manifolds*, In Mirror Symmetry, IV, (Montreal, QC, 2000), volume 33 of AMS/IP Stud. Adv. Math., pages 343–360. Amer. Math. Soc., Providence, RI, 2002.

[46] K. Liu, X. Ma and W. Zhang, Spin^c manifolds and *rigidity theorems in K-theory*, Asian J. Math, 4 (2000), 933-959.

[47] K. Liu, X. Ma and W. Zhang, *On elliptic genera and foliations*, Math. Res. Lett. 8 (2001), 361-376.

[48] K. Liu, X. Ma and W. Zhang, *Rigidity and vanishing theorems in K-theory*, Comm. Anal. Geom, 11 (2003), 121-180.

[49] B. Liu and J. Yu, *On the Witten Rigidity Theorem for Stringf Manifolds*, Pacific J. Math., 2013, 266(2): 477-508.

[50] J. Lohkamp, *Metrics of Negative Ricci Curvature*, Ann. Math. 140 (1994), 655-683.

[51] X. Ma and W. Zhang, Geometric quantization for proper moment maps: the Vergne conjecture. Acta Math. 212 (2014), no. 1, 11-57.

[52] Ochanine, S.: *Sur les genres multiplicatifs définis par des intégrales elliptiques*. Topology 26(2), 143–151 (1987).

[53] P. Petersen. Riemannian geometry. Graduate Texts in Mathematics, Vol 171. Springer-Verlag New York, 2006.

[54] M. Simon, Some integral curvature estimates for the Ricci flow in four dimensions, arXiv:1504.02623.

[55] S. Stolz, *A conjecture concerning positive Ricci curvature and the Witten genus*, Math. Ann. 304 (1996), no. 4, 785-800.

[56] C. H. Taubes. *S^1 actions and elliptic genera*. Comm. Math. Phys. 122 (1989), 455-526.
[57] G. Tian, Z.L. Zhang, Regularity of Kähler-Ricci flows on Fano manifolds, Acta Math. 216 (2016), 127-176.

[58] G. Tian, Z.L. Zhang, Convergence of Kähler-Ricci flow on lower dimensional algebraic manifolds of general type, Int. Math. Res. Not. 21 (2016), 6493-6511.

[59] Y. Tian and W. Zhang, An analytic proof of the geometric quantization conjecture of Guillemin-Sternberg, Invent. math. 132 (1998), 229-259.

[60] E. Witten, Elliptic genera and quantum field theory, Comm. Math. Phys. 109 (1987), no. 4, 525-536.

[61] E. Witten, The index of the Dirac operator in loop space, in P.S. Landweber, ed., Elliptic Curves and Modular Forms in Algebraic Topology (Proceedings, Princeton 1986), Lecture Notes in Math., 1326, pp. 161-181, Springer, 1988.

[62] T. Yamaguchi. Manifolds of almost nonnegative Ricci curvature. J. Diff. Geom. 28 (1988) 157-167.

[63] T. Yamaguchi. Collapsing and pinching under a lower curvature bound. Ann. of Math. 133 (1991), no. 2, 317-357.

[64] D. Zagier, Note on the Landweber-Stong elliptic genus, in P.S. Landweber, ed., Elliptic Curves and Modular Forms in Algebraic Topology (Proceedings, Princeton 1986), Lecture Notes in Math., 1326, pp. 216-224, Springer, 1988.

[65] W. Zhang, Lectures on Chern-Weil Theory and Witten Deformations, Nankai Tracts in Mathematics Vol. 4, World Scientific, Singapore, 2001.