Localization at countable infinitely many prime ideals and applications\footnote{Key words and phrases: Localization; Associated prime ideal. AMS Classification 2010: 13B30; 13D45; 13E99.}

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Abstract

In this paper we present a technique lemma about localization at countable infinitely many prime ideals. We apply this lemma to get many results about the finiteness of associated prime ideals of local cohomology modules.

1 Introduction

In this paper, let $R$ be a commutative Noetherian ring. Localization is one of the most important tools in Commutative algebra. Notice that for any multiplicative subset $S$ of $R$, the canonical extension $R \to R_S$ is flat, and many problems in Commutative algebra have good behavior under flat extensions. Let $p \in \text{Spec}(R)$ be a prime ideal. By localization at $p$ we can reduce a global problem in $R$ to a local problem in $R_p$. We will exhibit many advantages of the local ring $R_p$ such as the Nakayama lemma, the Krull intersection theorem, etc. For a set of finitely many prime ideals $\{p_1, ..., p_k\}$ with no containment relations, set $S = R \setminus \cup_{i=1}^k p_i$, we have $R_S$ is a semilocal ring and $\text{Max}(R_S) = \{p_1 R_S, ..., p_k R_S\}$. This fact follows from the well known prime avoidance lemma. This statement is false for countable infinitely many prime ideals $\{p_i\}_{i \geq 1}$. For example, let $R = \mathbb{Q}[X, Y]$ and $\{p_i\}_{i \in I}$ is the set of prime ideals of height one. Since $R$ is UFD we have a prime ideal of height one is principal. Moreover $R$ is a countable set, so the set $\{p_i\}_{i \in I}$ is countable. On the other hand every non-constant polynomial must be contained in a prime ideal of height one. Thus $S = R \setminus \cup_{i \in I} p_i = \mathbb{Q}$ and so $R_S = R$. This paper is to devote the localization at countable infinitely many prime ideals after passing to a certain flat extension. Concretely, we prove the following result.

Lemma 1.1. Let $R$ be a commutative Noetherian ring and $\{p_i\}_{i \geq 1}$ is a countable set of prime ideals of $R$ with no containment relation. Consider the formal power series rings $R[[X]]$ and set $S = R[[X]] \setminus \cup_{i \geq 1} p_i R[[X]]$ and $T = R[[X]]_S$. Then $R \to T$ is a flat extension and $\text{Max}(T) = \{p_i T\}_{i \geq 1}$.

The above lemma will be proved in the next section. In Section 3 we apply Lemma 1.1 to get many results about the the finiteness of associated prime ideals of local cohomology modules. Among them, is the following:
Corollary 1.2. Let $I$ be an ideal of $R$ and $M$ a finitely generated $R$-module. Then for every $i \geq 0$ the set \( \{ p \in \text{Ass}_R H^i_I(M) : \text{ht}(p/I) \leq 1 \} \) is finite.

Recall that, for any ideal $I$ of $R$ and any $R$-module $M$, the $i^{th}$ local cohomology module of $M$ with respect to $I$ is defined as
\[
H^i_I(M) = \lim_{n \to \infty} \text{Ext}^i_R(R/I^n, M).
\]
We refer the reader to [2] or [3] for more details about local cohomology.

2 Localization at countable infinitely many prime ideals

We start this section by the well known result, countable prime avoidance lemma (see [6, Lemma 13.2]).

Lemma 2.1. Let $A$ be a Noetherian ring satisfying either of these conditions:

(i) $A$ is a complete local ring.

(ii) There is an uncountable set of elements $\{ \mu_\lambda \}_{\lambda \in \Lambda}$ such that $\mu_\lambda - \mu_\gamma$ is a unit of $A$ for every $\lambda \neq \gamma$.

Let $\{ p_i \}_{i \geq 1}$ a countable set of prime ideals of $A$ and $I$ an ideal such that $I \subseteq \bigcup_{i \geq 1} p_i$. Then $I \subseteq p_i$ for some $i$.

The following technique lemma is the main result of this section.

Lemma 2.2. Let $R$ be a commutative Noetherian ring and $\{ p_i \}_{i \geq 1}$ is a countable set of prime ideals of $R$ with no containment relation. Consider the formal power series rings $R[[X]]$ and set $S = R[[X]] \setminus \bigcup_{i \geq 1} p_i R[[X]]$ and $T = R[[X]]_S$. Then $R \to T$ is a flat extension and $\text{Max}(T) = \{ p_i T \}_{i \geq 1}$.

Proof. It is clear that $R \to T$ is flat and $p_i T \in \text{Spec}(T)$ for all $i \geq 1$. We prove that $T_S$ satisfies the condition (ii) of Lemma 2.1. We consider the set of elements in $T$
\[
B := \{ \mu_\lambda = b_0 + b_1 X + \cdots + b_n X^n + \cdots : b_i = 0 \text{ or } 1 \text{ and } \mu_\lambda \neq 0 \}.
\]
It is clear that $B$ is an uncountable set. For every $\mu_\lambda \neq \mu_\gamma$ pair of distinct elements of $B$ we have
\[
\mu_\lambda - \mu_\gamma = a_0 + a_1 X + \cdots + a_n X^n + \cdots
\]
with $a_i = 0, 1$ or $-1$ and at least one $a_i \neq 0$. Let $k$ be the least integer such that $a_k \neq 0$. We have
\[ \mu_{\lambda} - \mu_{\gamma} = X^k(1 + a_{k+1}X + \cdots) \]
or
\[ \mu_{\lambda} - \mu_{\gamma} = X^k(-1 + a_{k+1}X + \cdots). \]
We have both $1 + a_{k+1}X + \cdots$ and $-1 + a_{k+1}X + \cdots$ are units in $R[[X]]$ and so are in $T$. Since $X \notin p_i T$ for all $i \geq 1$ we have $X \in S$. Thus $X$ is a unit in $T$. Therefore $\mu_{\lambda} - \mu_{\gamma}$ is a unit in $T$ for every $\mu_{\lambda} \neq \mu_{\gamma}$. Hence $T$ satisfies the countable prime avoidance lemma. Since $\cup_{i\geq 1} p_i T$ is the set of non-units of $T$, we have $I \subseteq \cup_{i\geq 1} p_i T$ for every proper ideal $I$ of $T$. By the countable prime avoidance lemma we have $I \subseteq p_i T$ for some $i$. Therefore Max($T$) = $\{p_i T\}_{i\geq 1}$. The proof is complete. \qed

\section{Applications}

In this section, let $I$ be an ideal of $R$ and $M$ a finitely generated $R$-module. In general the $i$th local cohomology module $H_i^I(M)$ is not finitely generated. Grothendieck asked the following question: Is Hom($R/I, H_i^I(M)$) finitely generated for all $i \geq 0$? The first counterexample was given by Hartshorne in [1]. In this paper he introduced the notion of $I$-cofinite modules. An $R$-module $L$ is called $I$-cofinite if Supp($L$) $\subseteq V(I)$ and Ext$_R^i(R/I, L)$ if finitely generated for all $i \geq 0$. Hartshorne proved that $H_i^I(M)$ is $I$-cofinite for all $i \geq 0$ if $R$ is a complete regular local ring and $I$ is a prime ideal of dimension one. Hartshorne’s result was extended by many authors. In [1, Theorem 1.1] Bahmanpour and Naghipour proved the following result (see also [8, Theorem 2.10]).

**Lemma 3.1.** Let $I$ be an ideal $R$ of dimension one and $M$ a finitely generated $R$-module. Then $H_i^I(M)$ is $I$-cofinite for all $i \geq 0$.

Now, we are ready to state and prove the fist main result of this section, which is an application of Lemma [2.2]

**Theorem 3.2.** Let $R$ be a Noetherian ring, $I$ an ideal of $R$ and $M$ a finitely generated $R$-module. Then for every $i \geq 0$ and any $j \geq 0$, the set
\[ \mathcal{A} = \{ p \in \text{Ass}_R \text{Ext}_R^i(R/I, H_j^I(M)) : \text{ht}(p/I) \leq 1 \} \]
is finite.

**Proof.** Suppose there are $i$ and $j$ such that the set
\[ \{ p \in \text{Ass}_R \text{Ext}_R^i(R/I, H_j^I(M)) : \text{ht}(p/I) \leq 1 \} \]
is not finite. We can choose an countable set $\{p_k\}_{k \geq 1} \subseteq \text{Ass}_R \text{Ext}^j_R(R/I, H^i_I(M))$ and $\text{ht}(p_k/I) = 1$ for all $k \geq 1$. Let $T$ as Lemma 2.2, we have $R \to T$ is a flat extension and

$$\text{Max}(T) = \{p_kT\}_{k \geq 1}.$$ 

So $p_kT \in \text{Ass}_T \text{Ext}^j_T(T/IT, H^i_{IT}(M \otimes_R T))$ for all $k \geq 1$. On the other hand we have $\dim T/IT = 1$ so $H^i_{IT}(M \otimes_R T)$ is $IT$-cofinite by Lemma 3.1. Thus the $T$-module $\text{Ext}^j_T(T/IT, H^i_{IT}(M \otimes_R T))$ is finitely generated and so the set

$$\text{Ass}_T \text{Ext}^j_T(T/IT, H^i_{IT}(M \otimes_R T))$$

is finite, which is a contradiction. The proof is complete. \hfill \Box

Recall that $\text{Ass}_R H^i_I(M) = \text{Ass}_R \text{Hom}(R/I, H^i_I(M))$ for all $i \geq 0$. So the following result is an immediately consequence of Theorem 3.2.

**Corollary 3.3.** Let $I$ be an ideal of $R$ and $M$ a finitely generated $R$-module. Then for every $i \geq 0$ the set $\{p \in \text{Ass}_R H^i_I(M) \mid \text{ht}(p/I) \leq 1\}$ is finite.

The following results are other applications of Lemma 2.2 to local cohomology modules.

**Corollary 3.4.** Let $R$ be a Noetherian ring, $I$ an ideal of $R$ and $n \geq 1$ be an integer and $M$ be a finitely generated $R$-module such that $\dim(M/IM) = n$. Then for any finitely generated $R$-module $N$ with support in $V(I + \text{Ann}_R(M))$ and each element $L$ of the set

$$\mathcal{J} = \{\text{Ext}^j_R(N, H^i_I(M)) \mid j \geq 0 \text{ and } i \geq 0\},$$

the set

$$\{p \in \text{Ass}_R(L) \mid \dim(R/p) \geq n - 1\}$$

is finite.

**Proof.** Let $J = \text{Ann}(M/IM)$. Then, we have $V(J) = V(I + \text{Ann}_R(M))$. It is not difficult to see that $H^i_I(M) \cong H^i_J(M)$ for all $i \geq 0$. We can assume henceforth that $I = \text{Ann}(M/IM)$ and $\dim R/I = n$. Notice that if $K$ is an $I$-cofinite module, then $\text{Ext}^j_R(N, K)$ is finitely generated for all finitely generated $R$-module $N$ with support $V(I)$ (see [5, Lemma 1]). Now the proof is the same as Theorem 3.2. \hfill \Box

**Corollary 3.5.** Let $R$ be a Noetherian ring, $I$ an ideal of $R$ and $n \geq 1$ be an integer and $M$ be a finitely generated $R$-module such that $\dim(M/IM) = n$. Then for any finitely generated $R$-module $N$ with support in $V(I + \text{Ann}_R(M))$ and each element $L$ of the set

$$\mathcal{J} = \{\text{Tor}^j_R(N, H^i_I(M)) \mid j \geq 0 \text{ and } i \geq 0\},$$

the set

$$\{p \in \text{Ass}_R(L) \mid \dim(R/p) \geq n - 1\}$$

is finite.
Proof. Use [7, Theorem 2.1].

We close the paper with the following result.

**Theorem 3.6.** Let $R$ be a Noetherian ring, $I$ an ideal of $R$ and $M$ an (not necessarily finitely generated) $R$-module. Then for any integer $t \geq 0$, the set

$$\mathcal{S} := \{p \in \text{Ass}_R H^t_I(M) : \text{ht}(p) = t\}$$

is finite. Moreover, we have

$$\mathcal{S} = \{p \in \text{Supp}(H^t_I(M)) : \text{ht}(p) = t\}.$$

**Proof.** It follows from Grothendieck’s Vanishing Theorem, that each element of the set $\{p \in \text{Supp}(H^t_I(M)) : \text{ht}(p) = t\}$ is a minimal element of the set $\text{Supp}(H^t_I(M))$ and so is an associated prime ideal of the $R$-module $H^t_I(M)$. Therefore

$$\{p \in \text{Supp}(H^t_I(M)) : \text{ht}(p) = t\} \subseteq \mathcal{S} \subseteq \{p \in \text{Supp}(H^t_I(M)) : \text{ht}(p) = t\}.$$

Hence

$$\mathcal{S} = \{p \in \text{Supp}(H^t_I(M)) : \text{ht}(p) = t\}.$$

Let $p$ be an arbitrary element of $\{p \in \text{Supp}(H^t_I(M)) : \text{ht}(p) = t\}$ we have $H^t_{1R_p}(M_p) \neq 0$. Notice that $\dim R_p = t$ so by [2, Exercise 6.1.9] we have $H^t_{1R_p}(M_p) = H^t_{1R_p}(R_p) \otimes_{R_p} M_p$. Hence $H^t_{1R_p}(R_p) \neq 0$. Thus for any $R$-module $M$ we have

$$\{p \in \text{Supp}(H^t_I(M)) : \text{ht}(p) = t\} \subseteq \{p \in \text{Supp}(H^t_I(R)) : \text{ht}(p) = t\}.$$

So it is enough to prove the assertion in the case $M = R$. Suppose that $\{p \in \text{Ass}_R H^t_I(R) : \text{ht}(p) = t\}$ is not finite. Then, we can choose a countable infinite subset

$$\{p_i\}_{i \geq 1} \subseteq \{p \in \text{Ass}_R H^t_I(R) : \text{ht}(p) = t\}.$$

Now set $T$ as in Lemma 2.2. Then we have $R \to T$ is a flat extension and $\text{Max}(T) = \{p_i T\}_{i \geq 1}$. In particular, $T$ is a Noetherian ring of dimension $t$ and $p_i T \in \text{Ass}_T H^t_{IT}(T)$ for all $i \geq 1$. But, in view of [7 Proposition 5.1], the $T$-module $H^t_{IT}(T)$ is Artinian and hence has finitely many associated primes, which is a contradiction. The proof is complete.

\[\square\]

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References

[1] K. Bahmanpour and R. Naghipour, Cofiniteness of local cohomology modules for ideals of small dimension, *J. Algebra* 321 (2009), 1997–2011.

[2] M.P. Brodmann and R.Y. Sharp, *Local cohomology; an algebraic introduction with geometric applications*, Cambridge University Press, Cambridge, 1998.

[3] A. Grothendieck, *Local cohomology*, Notes by R. Hartshorne, Lecture Notes in Math., 862 (Springer, New York, 1966).

[4] R. Hartshorne, Affine duality and cofiniteness, *Invent. Math.* 9 (1970), 145–164.

[5] K. -I. Kawasaki, On the finiteness of Bass numbers of local cohomology module, *Proc. Amer. Math. Soc.* 124 (1996), 3275–3279.

[6] G. Leuschke and R. Wiegand, *Cohen-Macaulay representations*, Mathematical Surveys and Monographs, vol. 181, American Mathematical Society, Providence, RI, 2012.

[7] L. Melkersson, Modules cofinite with respect to an ideal, *J. Algebra* 285 (2005), 649–668.

[8] L. Melkersson, Cofiniteness with respect to ideals of dimension one, *J. Algebra* 372 (2012), 459–462.

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