The Bivariate Normal Copula

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We collect well-known and less-known facts about the bivariate normal distribution and translate them into copula language. In addition, we provide various (equivalent) expressions for the bivariate normal copula, we compute its Gini’s gamma, and we derive improved bounds and approximations on its diagonal.

Keywords Bivariate normal distribution; Copula; Inequalities; Measures of concordance.

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1. Introduction

When it comes to modeling dependent random variables, in practice one often resorts to the multivariate normal distribution, because it is easy to parameterize and to deal with, and because in some settings its use can be motivated by (a multivariate version of) the central limit theorem. In recent years, in particular in the quantitative finance community, we have also witnessed a trend to separate marginal distributions and dependence structure, using the copula concept. The normal (or Gauss or Gaussian) copula has even come to the attention of the general public due to its use (and misuse) in the valuation of structured products and the decline of these products during the financial crisis of 2007 and 2008.

The multivariate normal distribution has been studied since the 19th century. Many important results have been published in the 1950s and 1960s. Applications in quantitative finance include pricing of options and estimation of asset correlations; the impressive list by Balakrishnan and Lai (2009) mentions applications in agriculture, biology, engineering, economics and finance, the environment, genetics, medicine, psychology, quality control, reliability and survival analysis, sociology, physical sciences, and technology.

In this article, we will concentrate on the bivariate case. We will give an extensive review of the properties of the bivariate normal distribution, formulated in terms of the associated copula. We will also provide new results, including a general expression for the bivariate normal copula implying other well known expressions,
a derivation of its Gini’s gamma, and improved and simple bounds for the diagonal of the copula.

For collections of facts on the bivariate (or multivariate) normal distribution we refer to the books of Balakrishnan and Lai (2009), Kotz et al. (2000), Patel and Read (1996), and Fang et al. (1990), and to the survey article of Gupta (1963a) with its extensive bibliography (Gupta, 1963b). For theory on copulas we refer to the book of Nelsen (2006).

We will use the symbols \( P, E, \) and \( V \) for probabilities, expectations, and variances.

2. Definitions and Basic Properties

Let \( x, y, h, k \in \mathbb{R} \). Denote by

\[
\varphi(x) := \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x^2}{2} \right), \quad \Phi(h) := \int_{-\infty}^{h} \varphi(x) \, dx,
\]

the density and distribution function of the univariate standard normal distribution, and by

\[
\varphi_2(x, y; \varrho) := \frac{1}{2\pi \sqrt{1 - \varrho^2}} \exp \left( -\frac{x^2 - 2\varrho xy + y^2}{2(1 - \varrho^2)} \right),
\]

\[
\Phi_2(h, k; \varrho) := \int_{-\infty}^{h} \int_{-\infty}^{k} \varphi_2(x, y; \varrho) \, dy \, dx,
\]

the density and distribution function of the bivariate standard normal distribution with correlation parameter \( \varrho \in (-1, 1) \).

Since the margins \( \Phi(\cdot) \) of \( \Phi_2(\cdot, \cdot; \varrho) \) are continuous, by Sklar’s theorem (cf. Sec. 2.3 of Nelsen, 2006) there is a unique copula \( C(\cdot, \cdot; \varrho) \) with

\[
\Phi_2(h, k; \varrho) = C(\Phi(h), \Phi(k); \varrho), \quad (h, k) \in \mathbb{R}^2.
\]

Consequently, we can define the bivariate normal (or Gauss or Gaussian) copula with parameter \( \varrho \) via

\[
C(u, v; \varrho) := \Phi_2 \left( \Phi^{-1}(u), \Phi^{-1}(v); \varrho \right), \quad (u, v) \in [0, 1]^2. \tag{2.1}
\]

For \( \varrho \in \{-1, 1\} \), the correlation matrix of the bivariate standard normal distribution becomes singular. Nevertheless, the distribution, and hence the normal copula, can be extended continuously. We may define

\[
C(u, v, -1) := \lim_{\varrho \to -1} C(u, v; \varrho) = \max(u + v - 1, 0), \tag{2.2}
\]

\[
C(u, v, +1) := \lim_{\varrho \to +1} C(u, v; \varrho) = \min(u, v). \tag{2.3}
\]

Hence, \( C(\cdot, \cdot; \varrho) \), for \( \varrho \to -1 \), approaches the lower Fréchet-Hoeffding bound,

\[
W(u, v) := \max(u + v - 1, 0),
\]
and, for $\varrho \to 1$, approaches the upper Fréchet-Hoeffding bound,

$$M(u, v) := \min(u, v).$$

For $\varrho = 0$ we have

$$C(u, v; 0) = uv =: \Pi(u, v), \tag{2.4}$$

the independence copula. Furthermore, the following differential equation derived by Plackett (1954) holds:

$$\frac{d}{d\varrho} \Phi_2(x, y; \varrho) = \varphi(x, y; \varrho) = \frac{d^2}{dx \, dy} \Phi_2(x, y; \varrho). \tag{2.5}$$

We find

$$C(u, v; \varrho) - C(u, v; \sigma) = \int_{\varrho}^{\sigma} \varphi_2(\Phi^{-1}(u), \Phi^{-1}(v); r) \, dr, \tag{2.6}$$

and, in particular,

$$C(u, v; \varrho) = W(u, v) + l(u, v; -1, \varrho), \tag{2.7}$$

$$= \Pi(u, v) + l(u, v; 0, \varrho), \tag{2.8}$$

$$= M(u, v) - l(u, v; \varrho, 1). \tag{2.9}$$

In other words, the bivariate normal copula allows comprehensive total concordance ordering with respect to $\varrho$, cf. Sec. 2.8 of Nelsen (2006):

$$W(\cdot, \cdot) = C(\cdot, \cdot; -1) \prec C(\cdot, \cdot; \varrho) \prec C(\cdot, \cdot; \sigma) \prec C(\cdot, \cdot; 1) = M(\cdot, \cdot)$$

for $-1 \leq \varrho \leq \sigma \leq 1$, i.e., for all $(u, v) \in [0, 1]^2$.

$$W(u, v) = C(u, v; -1) \leq C(u, v; \varrho) \leq C(u, v; \sigma) \leq C(u, v, 1) = M(u, v).$$

Figure 1 illustrates the bivariate normal copula, for selected values of $\varrho$, on the diagonal $u = v$ (we will see in the following sections that the bivariate normal copula can always be reduced to certain diagonals). For further illustration (e.g., perspective plots, contour plots, and density plots), cf. Sec. 5.1.2 of Embrechts et al. (2005) and Sec. 3.2.1 of Cherubini et al. (2004).

By substituting $u = v = \frac{1}{2}$ into (2.8) we obtain

$$C\left(\frac{1}{2}, \frac{1}{2}; \varrho\right) = \frac{1}{4} + \int_{\varrho}^{1} \frac{1}{2\pi \sqrt{1 - r^2}} \, dr = \frac{1}{4} + \frac{1}{2\pi} \arcsin(\varrho), \tag{2.10}$$

a result already known to Stieltjes (1889).

Mehler (1866) and Pearson (1901a), among other authors, obtained the tetrachoric expansion in $\varrho$:

$$C(u, v; \varrho) = uv + \varphi(\Phi^{-1}(u))\varphi(\Phi^{-1}(v)) \sum_{k=0}^{\infty} \frac{\text{He}_k(\Phi^{-1}(u))\text{He}_k(\Phi^{-1}(v))}{(k+1)!} \varrho^{k+1} \tag{2.11}$$
where

$$H_{\ell}(x) = \sum_{i=0}^{\lfloor k/2 \rfloor} \frac{k!}{i!(k-2i)!} \left( -\frac{1}{2} \right)^i x^{k-2i}$$

are the Hermite polynomials.

The bivariate normal copula inherits the symmetries of the bivariate normal distribution:

$$C(u, v; \varrho) = C(v, u; \varrho)$$  (2.12)

$$= u - C(u, 1 - v; -\varrho)$$  (2.13)

$$= v - C(1 - u, v; -\varrho)$$  (2.14)

$$= u + v - 1 + C(1 - u, 1 - v; \varrho).$$  (2.15)

Here, (2.12) is clear from the definition, and (2.13) follows easily from (3.5) by considering the symmetry $\Phi(-x) = 1 - \Phi(x)$ of the univariate normal distribution. Combining (2.12) and (2.13) gives (2.14) and (2.15). In copula language, (2.12) is a consequence of exchangeability, and (2.15) is a consequence of radial symmetry, cf. Sec. 2.7 of Nelsen (2006).

In the following sections we will discuss expressions for the bivariate normal copula, its numerical evaluation, bounds and approximations, measures of concordance, and univariate distributions related to the bivariate normal copula. It will be convenient to assume $\varrho \notin \{-1, 0, 1\}$ unless explicitly stated otherwise (cf. (2.2), (2.4), (2.3) for the simple formulation in the missing cases).

3. Expressions for the Bivariate Normal Copula

We will start with a general construction based on writing the components of a bivariate normally distributed vector as linear combinations of certain univariate standard normal variables. Details and a proof of the resulting expression (3.1) will be provided in Sec. A.1. The construction has occurred in credit risk modelling but without the link to the bivariate normal copula, cf. Bluhm and Overbeck (2003).
Let \( \alpha, \beta, \gamma \in (-1, 1) \) with \( \alpha \beta \gamma = q \). Then the following holds:

\[
C(u, v; q) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi \left( \frac{\Phi^{-1}(u) - \alpha x}{\sqrt{1 - x^2}} \right) \Phi \left( \frac{\Phi^{-1}(v) - \beta y}{\sqrt{1 - y^2}} \right) \varphi(x, y; \gamma) dy \, dx \tag{3.1}
\]

where

\[
c(u, v; q) := \frac{\partial^2}{\partial u \partial v} C(u, v; q) = \frac{\varphi_2(\Phi^{-1}(u), \Phi^{-1}(v); q)}{\varphi(\Phi^{-1}(u)) \varphi(\Phi^{-1}(v))} \tag{3.2}
\]

\[
= \frac{1}{\sqrt{1 - q^2}} \exp \left( \frac{2q\Phi^{-1}(u)\Phi^{-1}(v) - q^2 (\Phi^{-1}(u)^2 + \Phi^{-1}(v)^2)}{2(1 - q^2)} \right)
\]

is the copula density.

Other expressions for \( C(u, v; q) \) are obtained by carefully studying the limit as some of the variables \( \alpha, \beta, \gamma \) approach 1. The interesting cases (modulo the symmetry in \( \alpha \) and \( \beta \)) are listed in Table 1.

By letting \( \alpha \) tend to 1 in (3.1) we obtain:

\[
C(u, v; q) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi \left( \frac{\Phi^{-1}(v) - \beta y}{\sqrt{1 - y^2}} \right) \varphi(x, y; \gamma) dy \, dx \tag{3.3}
\]

\[
= \int_{0}^{1} \int_{0}^{1} \Phi \left( \frac{\Phi^{-1}(v) - \beta \Phi^{-1}(t)}{\sqrt{1 - (\beta t)^2}} \right) c(s, t; \gamma) dt \, ds.
\]

However, expression (3.3) may be considered rather unattractive. By letting \( \gamma \) tend to 1 in (3.1) instead we obtain a more interesting expression:

\[
C(u, v; q) = \int_{-\infty}^{\infty} \Phi \left( \frac{\Phi^{-1}(u) - \alpha z}{\sqrt{1 - z^2}} \right) \Phi \left( \frac{\Phi^{-1}(v) - \beta z}{\sqrt{1 - z^2}} \right) \varphi(z) dz \tag{3.4}
\]

\[
= \int_{0}^{1} \Phi \left( \frac{\Phi^{-1}(u) - \alpha \Phi^{-1}(t)}{\sqrt{1 - (\alpha t)^2}} \right) \Phi \left( \frac{\Phi^{-1}(v) - \beta \Phi^{-1}(t)}{\sqrt{1 - (\beta t)^2}} \right) dt.
\]
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Expression (3.4) seems to have been discovered repeatedly, sometimes in more general context (e.g., multivariate, cf., Steck and Owen, 1962), sometimes for special cases. Gupta (1963a) gives credit to Dunnett and Sobel (1955).

By letting both $\alpha$ and $\gamma$ tend to 1 in (3.1) we obtain:

$$C(u, v; \varrho) = \int_0^u \Phi \left( \frac{\Phi^{-1}(v) - \varrho \Phi^{-1}(t)}{\sqrt{1 - \varrho^2}} \right) dt = \int_0^u \frac{\partial}{\partial t} C(t, v; \varrho) dt. \quad (3.5)$$

This expression reflects the fact that if $(X, Y)$ are bivariate standard normally distributed with correlation $\varrho$ then $Y$ conditional on $X = x$ is normally distributed with expectation $\mathcal{N}(\varrho x)$ and variance $1 - \varrho^2$.

Finally, by letting both $\alpha$ and $\beta$ tend to 1 in (3.1) we rediscover the trivial expression

$$C(u, v; \varrho) = \int_0^u \int_0^v c(s, t; \varrho) dt ds. \quad (3.6)$$

In order to evaluate the bivariate normal distribution function numerically, Owen (1956) defined the following useful function, to which we will refer as Owen’s $T$-function:

$$T(h, a) = \frac{1}{2\pi} \int_0^a \exp \left( -\frac{1}{2} h^2 (1 + x^2) \right) \frac{dx}{1 + x^2} = \varphi(h) \int_0^a \varphi(hx) \frac{dx}{1 + x^2}. \quad (3.7)$$

He proved that

$$C(u, v; \varrho) = \frac{u + v}{2} - T(\Phi^{-1}(u), u) - T(\Phi^{-1}(v), v) - \delta(u, v) \quad (3.8)$$

where

$$\delta(u, v) := \begin{cases} \frac{1}{2}, & \text{if } u < \frac{1}{2}, \quad v \geq \frac{1}{2} \text{ or } u \geq \frac{1}{2}, \quad v < \frac{1}{2} \\ 0, & \text{else} \end{cases} \quad (3.9)$$

and

$$z_u = \frac{1}{\sqrt{1 - \varrho^2}} \left( \Phi^{-1}(u) - \varrho \right), \quad z_v = \frac{1}{\sqrt{1 - \varrho^2}} \left( \Phi^{-1}(v) - \varrho \right). \quad (3.10)$$

In particular, on the lines defined by $v = \frac{1}{2}$ and by $u = v$, the following expressions hold:

$$C \left( u, \frac{1}{2}; \varrho \right) = \frac{u}{2} - T \left( \Phi^{-1}(u), -\frac{\varrho}{\sqrt{1 - \varrho^2}} \right), \quad (3.11)$$

$$C(u, u; \varrho) = u - 2T \left( \Phi^{-1}(u), \frac{1 - \varrho}{\sqrt{1 + \varrho}} \right). \quad (3.12)$$

From (3.8) and (3.11) we can derive the useful expression

$$C(u, v; \varrho) = C \left( u, \frac{1}{2}; \varrho_u \right) + C \left( v, \frac{1}{2}; \varrho_v \right) - \delta(u, v), \quad (3.13)$$
where
\[ \varphi_u = -\frac{x_u}{\sqrt{1 + x_u^2}} = \sin(\arctan(-x_u)), \quad \varphi_v = -\frac{x_v}{\sqrt{1 + x_v^2}} = \sin(\arctan(-x_v)). \]

On the diagonal \( u = v \), (3.13) reads:
\[ C(u, u; \varrho) = 2C\left(u, \frac{1}{2}; -\sqrt{\frac{1-\varrho}{2}}\right). \]  \hfill (3.14)

Inversion of (3.14) using (2.13) gives:
\[ C\left(u, \frac{1}{2}; \varrho\right) = \begin{cases} \frac{1}{2} C(u, u; 1 - 2\varrho^2), & \varrho < 0, \\ u - \frac{1}{2} C(u, u; 1 - 2\varrho^2), & \varrho > 0. \end{cases} \]  \hfill (3.15)

Applying (3.5) to (3.14) we obtain, cf. also Steck and Owen (1962),
\[ C(u, u; \varrho) = 2 \int_0^u g(t; \varrho) dt \]  \hfill (3.16)

with
\[ g(u; \varrho) := \Phi\left(\sqrt{\frac{1-\varrho}{1+\varrho}} \Phi^{-1}(u)\right). \]  \hfill (3.17)

We find
\[ \frac{d}{du} C(u, u; \varrho) = 2g(u; \varrho). \]  \hfill (3.18)

The function \( g \) will become important in Sec. 5. Below we list some properties of \( g \):
\[ \lim_{u \to 0^+} g(u; \varrho) = 0, \quad \lim_{u \to 1^-} g(u; \varrho) = 1, \]  \hfill (3.19)
\[ g(1-u; \varrho) = 1 - g(u; \varrho), \]  \hfill (3.20)
\[ g(g(u; \varrho); -\varrho) = u. \]  \hfill (3.21)

In particular, (3.18) and (3.19) show that the bivariate normal copula does not exhibit lower tail dependence (cf. Secs. 5.2.3 and 5.3.1 of Embrechts et al., 2005):
\[ \lim_{u \to 0^+} \frac{C(u, u; \varrho)}{u} = \lim_{u \to 0^+} \frac{d}{du} C(u, u; \varrho) = \lim_{u \to 0^+} 2g(u; \varrho) = 0. \]

By radial symmetry (cf. (2.15)), the bivariate normal copula does not exhibit upper tail dependence as well. That is, random variables linked by a bivariate normal copula will be asymptotically independent in both tails, regardless of the correlation. Therefore, one should be careful when assuming multivariate normality in applications using tail-related quantities, e.g., very high quantiles in financial risk management.

Substitution of \( t = g(s; \varrho) \) in (3.16) and application of (3.21) lead to the identity, cf. also Steck and Owen (1962):
\[ C(u, u; \varrho) = 2ug(u; \varrho) - C(g(u; \varrho), g(u; \varrho); -\varrho). \]  \hfill (3.22)
4. Numerical Evaluation

The bivariate normal copula has to be evaluated numerically. Since there are excellent algorithms available for evaluation of \( \Phi^{-1} \), cf. Acklam (2004), the main problem is evaluation of the bivariate normal distribution function \( \Phi_2 \).

In the literature on evaluation of \( \Phi_2 \) there are basically two approaches: application of a multivariate method to the bivariate case, and explicit consideration of the bivariate case. For background on multivariate methods we refer to the recent book by Bretz and Genz (2009). In most cases, bivariate methods will be able to obtain the desired accuracy in less time. In the following we will provide an overview on the literature. We will concentrate on methods and omit references dealing with implementation only. Comparisons of different approaches in terms of accuracy and running time have been provided by numerous authors, e.g., Aćga and Chance (2003), Terza and Welland (1991), and Wang and Kennedy (1990).

Before the advent of widely available computer power, extensive tables of the bivariate normal distribution function had to be created. Using (3.13) or similar approaches, the three-dimensional problem (two variables and the correlation parameter) was reduced to a two-dimensional one.

Pearson (1901a) used the tetrachoric expansion (2.11) for small \(|\varrho|\), and quadrature for large \(|\varrho|\). Nicholson (1943), building on ideas of Sheppard (1900), worked with a two-parameter function, denoted \( V \)-function. Owen (1956) introduced the \( T \)-function (3.7) which is closely related to Nicholson’s \( V \)-function. For many years, quadrature of the \( T \)-function was the method of choice for evaluation of the bivariate normal distribution. Numerous authors, e.g., Borth (1973), Daley (1974), Young and Minder (1974), and Patefield and Tandy (2000), have been working on improvements, e.g., by dividing the plane into many regions and choosing specific quadrature methods in each region.

Sowden and Ashford (1969) applied Gauss-Hermite quadrature to (3.4) and Simpson’s rule to (3.5). Drezner (1978) used (3.13) and Gauss quadrature. Divgi (1979) relied on polar coordinates and an approximation to the univariate Mills’ ratio. Vasicek (1998) proposed an expansion which is more suitable for large \(|\varrho|\) than the tetrachoric expansion (2.11).

Drezner and Wesolowsky (1990) applied Gauss-Legendre quadrature to (2.8) for \(|\varrho| \leq 0.8\), and to (2.9) for \(|\varrho| > 0.8\). Improvements of their method in terms of accuracy and robustness have been provided by Genz (2004) and West (2005).

Most implementations today will rely on variants of the approaches of Divgi (1979) or of Drezner and Wesolowsky (1990). The method of Drezner (1978), although less reliable, is also very common, mainly because it is proposed in Hull (2008) and other prevalent books.

5. Bounds and Approximations

Nowadays, high-precision numerical evaluation of the bivariate normal copula is usually available. Nevertheless, bounds and approximations may still be of interest. Bounds, for example, if they are not too weak, can be used for checking numerical algorithms. Moreover, derivation of bounds or approximations often provides valuable insight into the mathematics behind the function to be bounded or approximated.

In the following, we will concentrate on bounds and approximations explicitly derived for the bivariate normal copula (multivariate approximations applied to
the bivariate case are usually rather weak, cf., Lu and Li, 2009, and the references therein). Throughout this section we will only consider the case \( \rho > 0, \quad 0 < u = v < 1/2 \). Note that by successively applying, if required, (3.13), (3.15), (3.22) and (2.15), we can always reduce \( C(u, v; \rho) \) to a sum of two terms of that form. Any approximation or bound given for the special case can be translated to an approximation or bound for the general case, with at most twice the absolute error. Note also that for many existing approximations and bounds the diagonal \( u = v \) may be considered a worst case, cf. Willink (2004).

Mee and Owen (1983) elaborated on the conditional approach proposed by Pearson (1901b). If \((X, Y)\) are bivariate standard normally distributed with correlation \( \rho \) then we can write

\[
\Phi_2(h, k; \rho) = \Phi(h) \mathbb{P}(Y \leq k \mid X \leq h). \tag{5.1}
\]

The distribution of \( Y \) conditional on \( X = h \) is normal but the distribution of \( Y \) conditional on \( X \leq h \) is not. Nevertheless, it can be approximated by a normal distribution with the same mean and variance. In terms of the bivariate normal copula, the resulting approximation is

\[
C(u, u; \rho) \approx u \Phi \left( \frac{u \Phi^{-1}(u) + \rho \varphi \left( \Phi^{-1}(u) \right)}{\sqrt{u^2 - \rho^2 \varphi \left( \Phi^{-1}(u) \right) (u \Phi^{-1}(u) + \varphi \left( \Phi^{-1}(u) \right))}} \right). \tag{5.2}
\]

The approximation works well for \( |\rho| \) not too large. For \( |\rho| \) large there are alternative approximations, e.g., Albers and Kallenberg (1994).

The simpler of the two approximations proposed by Cox and Wermuth (1991) replaces the second factor in (5.1) by the mean of the conditional distribution (the more complicated approximation adds a term of second order). In terms of the bivariate normal copula, the resulting approximation is

\[
C(u, u; \rho) \approx u \Phi \left( \frac{1 - \rho}{1 + \rho} \cdot \frac{u \Phi^{-1}(u) + \rho \varphi \left( \Phi^{-1}(u) \right)}{(1 + \rho) u} \right).
\]

Mallows (1959) gave two approximations to Owen’s \( T \)-function (3.7). In terms of the bivariate normal copula, the simpler one reads

\[
C(u, u; \rho) \approx 2u \Phi \left( \sqrt{\frac{1 - \rho}{1 + \rho}} \cdot \left( \Phi^{-1} \left( \frac{u}{2} + \frac{1}{4} \right) - \Phi^{-1} \left( \frac{3}{4} \right) \right) \right).
\]

Further approximations were derived by Cadwell (1951) and Pólya (1949).

There are not too many bounds available in the literature. For \( \rho > 0 \) there are, of course, the trivial bounds (2.2) and (2.3):

\[
w^2 \leq C(u, u; \rho) \leq u.
\]

The upper bound given by Pólya (1949) is just the one above. His lower bound is too weak (even negative) on the diagonal. A recent overview on known bounds, and derivation of some new ones, is provided by Willink (2004). We will present some of his bounds below in more general context.
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Theorem 5.1. Let \( q \geq 0 \) and \( 0 \leq u \leq 1/2 \). Then \( C(u, u; q) \) is bounded as follows, where \( g(u; q) \) is defined as in (3.17):

\[
C(u, u; q) \geq ug(u; q), \tag{5.3}
\]

\[
C(u, u; q) \leq ug(u; q) \cdot 2. \tag{5.4}
\]

The lower bound (5.3) is tight for \( q = 0 \) or \( u = 0 \). The maximum error of (5.3) equals \( 1/4 \) and is attained for \( (u, q) = (1/2, 1) \). The upper bound (5.4) is tight for \( q = 1 \) or \( u = 0 \). The maximum error of (5.4) equals \( 1/4 \) and is attained for \( (u, q) = (1/2, 0) \).

A proof of Theorem 5.1 is provided in Sec. A.2, together with a proof of the following refinement.

**Theorem 5.2.** Let \( q > 0 \) and \( 0 \leq u \leq 1/2 \). Then \( C(u, u; q) \) is bounded as follows, where \( g(u; q) \) is defined as in (3.17):

\[
C(u, u; q) \geq ug(u; q) \cdot \left(1 + \frac{2}{\pi} \arcsin(q)\right), \tag{5.5}
\]

\[
C(u, u; q) \leq ug(u; q) \cdot (1 + q). \tag{5.6}
\]

These bounds are the optimal ones of the form \( ug(u) \cdot a(q) \). They are tight for \( q = 0 \), \( q = 1 \), or \( u = 0 \). The maximum error of (5.6) is attained for \( (u, q) = (1/2, \sqrt{1 - 4/\pi^2}) \approx (0.5, 0.7712) \), the value being

\[
\frac{1}{4} \left(\sqrt{1 - \frac{4}{\pi^2}} - \frac{2}{\pi} \arcsin\left(\sqrt{1 - \frac{4}{\pi^2}}\right)\right) \approx 0.05263.
\]

The lower bound (5.5) is tight for \( u = 1/2 \).

The bounds (5.3) and (5.6) have been discussed, without explicit computation of the maximum error, by Willink (2004). The maximum error of (5.5) is difficult to grasp analytically. Figure 2 displays the maximum error for \( q \in [0, 1] \). Numerically, the error always stays below 0.006.

An alternative upper bound is given by the following theorem, the proof of which is provided in Sec. A.3.

![Figure 2. Maximum error of (5.5) for \( q \in [0, 1] \).](image)
Theorem 5.3. Let $\varrho > 0$ and $0 \leq u \leq 1/2$. Then,

$$C(u, u; \varrho) \leq 2ug\left(\frac{u}{2}; \varrho\right).$$

The bound is tight for $\varrho \in \{0, 1\}$ or $u = 0$. The maximum error is attained for

$$u = 1/2, \quad -\frac{1 + \varrho}{2\pi\Phi^{-1}(\frac{1}{2})} = \varphi\left(\sqrt{\frac{1 - \varrho}{1 + \varrho}} \cdot \Phi^{-1}\left(\frac{1}{4}\right)\right).$$

i.e., $\varrho \approx 0.5961$, the value being approx. 0.015.

It is also possible to derive good approximations to $C(u, u; \varrho)$ by considering the family $C(u, u; \varrho) \approx ug(u; \varrho) \cdot (a(\varrho) + b(\varrho)u)$. In particular, the choice

$$a(\varrho) := 1 + \varrho, \quad b(\varrho) := 2\left(1 + \frac{2}{\pi}\arcsin(\varrho) - (1 + \varrho)\right) = \frac{4}{\pi}\arcsin(\varrho) - 2\varrho$$

is attractive because the resulting approximation

$$C(u, u; \varrho) \approx ug(u; \varrho) \cdot \left(1 + \varrho + \left(\frac{4}{\pi}\arcsin(\varrho) - 2\varrho\right)u\right) \quad (5.7)$$

is tight for $\varrho \in \{0, 1\}$ or $u \in \{0, 1/2\}$, and for $u \to 0^+$ it has the same asymptotic behaviour as (5.6). By visual inspection we may conjecture that it is even an upper bound, with an error almost cancelling the error of the lower bound (5.5) most of the time. Consequently, an even better approximation (but not a lower bound, for $\varrho$ large) is given by

$$C(u, u; \varrho) \approx ug(u; \varrho) \cdot \left(1 + \varrho + \left(\frac{2}{\pi}\arcsin(\varrho) - \varrho\right)u\right). \quad (5.8)$$

Again, (5.8) is tight for $\varrho \in \{0, 1\}$ or $u \in \{0, 1/2\}$. Numerically, the absolute error always stays below 0.0006. Hence, (5.8) is comparable in performance with (5.2), and much better for $\varrho$ large.

6. Measures of Concordance

In the study of dependence between (two) random variables, properties and measures that are scale-invariant, i.e., invariant under strictly increasing transformations of the random variables, can be expressed in terms of the (bivariate) copula of the random variables. Among these are the measures of concordance, in particular, Kendall’s tau, Spearman’s rho, Blomqvist’s beta, and Gini’s gamma. For background and general definitions and properties we refer to Sec. 5 of Nelsen (2006). In this section we will provide expressions for measures of concordance for the bivariate normal copula, depending on the correlation parameter $\varrho$.

Blomqvist’s beta follows immediately from (2.10):

$$\beta(\varrho) := 4C\left(\frac{1}{2}, \frac{1}{2}; \varrho\right) - 1 = \frac{2}{\pi}\arcsin(\varrho). \quad (6.1)$$
For the bivariate normal copula, Kendall’s tau equals Blomqvist’s beta:

\[ \tau(q) := 4 \int_0^1 \int_0^1 C(u, v; q) dC(u, v; q) - 1 = \frac{2}{\pi} \arcsin(q). \]  
(6.2)

For a proof of (6.1) and (6.2) cf. Sec. 5.3.2 of Embrechts et al. (2005). Both Blomqvist’s beta and Kendall’s tau can be generalized to (copulas of) elliptical distributions, cf. Lindskog et al. (2003). This is not the case for Spearman’s rho, cf. Hult and Lindskog (2002), which is given by:

\[ q_s(q) := 12 \int_0^1 \int_0^1 C(u, v; q) du dv - 3 = \frac{6}{\pi} \arcsin\left(\frac{q}{2}\right). \]  
(6.3)

For proofs of (6.3), cf. Kruskal (1958) or Sec. 5.3.2 of Embrechts et al. (2005).

Gini’s gamma for the bivariate normal copula is given as follows:

\[ \gamma(q) := 4 \left( \int_0^1 C(u, u; q) du + \int_0^1 C(u, 1 \pm u; q) du - \frac{1}{2} \right) \]
\[ = 4 \left( \int_0^1 C(u, u; q) du + \int_0^1 u - C(u, u; -q) du - \frac{1}{2} \right) \]
\[ = \frac{2}{\pi} \left( \arcsin\left(\frac{1 + q}{2}\right) - \arcsin\left(\frac{1 - q}{2}\right) \right) \]  
(6.4)
\[ = \frac{4}{\pi} \left( \arcsin\left(\frac{\sqrt{1 + q}}{2}\right) - \arcsin\left(\frac{\sqrt{1 - q}}{2}\right) \right) \]  
(6.5)
\[ = \frac{4}{\pi} \arcsin\left(\frac{1}{4} \left( \sqrt{(1 + q)(3 + q)} - \sqrt{(1 - q)(3 - q)} \right) \right). \]
(6.6)

Proofs will be provided in Sec. A.4.

Expression (6.6) can be inverted which may be useful for estimation of \( q \) from an estimate for \( \gamma(q) \):

\[ q = \sin\left(\gamma(q) \frac{\pi}{4}\right) \sqrt{3 - \tan\left(\gamma(q) \frac{\pi}{4}\right)}. \]  
(6.7)

Expressions (6.1), (6.2), and (6.3) can be inverted for \( q \) as well. That is, if two marginals are linked by the bivariate normal copula, each of the measures \( q, \beta, \tau, \varrho_s, \) and \( \gamma \) fully characterizes the dependence structure. Therefore, estimating just one of these measures is a valid procedure to determine the level of dependence. However, such reasoning is not appropriate in non-normal settings. For example, Kaas et al. (2009) show that relevant quantities, such as quantiles (often termed Value-at-Risk in finance applications), may vary widely even when the marginal distributions are known and the value of a measure of concordance is given.

7. Univariate Distributions

In this section we will discuss two univariate distributions being closely related to the bivariate normal copula (or distribution).
7.1. The skew-Normal Distribution

A random variable $X$ on $\mathbb{R}$ is skew-normally distributed with skewness parameter $\lambda \in \mathbb{R}$ if it has a density function of the form

$$f_{\lambda}(x) = 2\varphi(x)\Phi(\lambda x).$$  \hspace{1cm} (7.1)

The skew-normal distribution was introduced by O’Hagan and Leonard (1976) and studied and made popular by Azzalini (1985, 1986). Its cumulative distribution function is given by

$$\mathbb{P}(X \leq x) = \int_{-\infty}^{x} 2\varphi(t)\Phi(\lambda t)dt = \Phi(\lambda^{-1}(t))dt.$$  \hspace{1cm} (7.2)

In the light of (3.16) and (3.14), cf. also Azzalini and Capitanio (2003), we find

$$\mathbb{P}(X \leq x) = 2\Phi_{2}\left(x, 0; \frac{\lambda}{\sqrt{1 + \lambda^2}}\right)$$  \hspace{1cm} (7.3)

$$= \begin{cases} \Phi_{2}\left(x, x; \frac{1 - \lambda^2}{1 + \lambda^2}\right), & \lambda \geq 0, \\ 1 - \Phi_{2}\left(-x, -x; \frac{1 - \lambda^2}{1 + \lambda^2}\right), & \lambda \leq 0. \end{cases}$$  \hspace{1cm} (7.4)

In particular, the bounds given in Theorem 5.2 can be applied.

7.2. The Vasicek Distribution

A random variable $P$ on the interval $[0, 1]$ is Vasicek distributed with parameters $p \in (0, 1)$ and $\varrho \in (0, 1)$ if $\Phi^{-1}(P)$ is normally distributed with mean

$$\mathbb{E}(\Phi^{-1}(P)) = \frac{\Phi^{-1}(p)}{\sqrt{1 - \varrho}}$$  \hspace{1cm} (7.5)

and variance

$$\mathbb{V}(\Phi^{-1}(P)) = \frac{\varrho}{1 - \varrho}.$$  \hspace{1cm} (7.6)

In Sec. A.1, it is proved implicitly that

$$\mathbb{E}(P) = p, \quad \mathbb{E}(P^2) = C(p, p; \varrho),$$

so that

$$\mathbb{V}(P) = \mathbb{E}(P^2) - \mathbb{E}(P)^2 = C(p, p; \varrho) - p^2.$$  

Furthermore, we have

$$\mathbb{P}(P \leq q) = \mathbb{P}(\Phi^{-1}(P) \leq \Phi^{-1}(q)) = \Phi\left(\frac{\sqrt{1 - \varrho}\Phi^{-1}(q) - \Phi^{-1}(p)}{\sqrt{\varrho}}\right).$$
The (one-sided) \( \alpha \)-Quantile \( q_\alpha \) of \( P \), with \( \alpha \in (0, 1) \), is therefore given by

\[
q_\alpha = \Phi \left( \frac{\sqrt{\alpha} \Phi^{-1}(x) + \Phi^{-1}(p)}{\sqrt{1 - q}} \right).
\]  

In particular, the median of \( P \) is simply

\[
q_{0.5} = \Phi \left( \frac{\Phi^{-1}(p)}{\sqrt{1 - q}} \right) = \Phi(\Phi^{-1}(P)).
\]  

The density of \( P \) is

\[
\frac{d}{dq} \mathbb{P}(P \leq q) = \sqrt{1 - q} \varphi \left( \frac{\sqrt{1 - \varrho \Phi^{-1}(q) - \Phi^{-1}(p)}}{\sqrt{1 - q}} \right) \frac{1}{\varphi(\Phi^{-1}(q))}.
\]

The distribution is unimodal with the mode at

\[
\Phi \left( \frac{\sqrt{1 - q} \Phi^{-1}(p)}{1 - 2q} \right)
\]

for \( q < 0.5 \), monotone for \( q = 0.5 \), and U-shaped for \( q > 0.5 \).

Let \( \tilde{P} \) be Vasicek distributed with parameters \( \tilde{p}, \tilde{q} \), and let

\[
\text{corr}(\Phi^{-1}(P), \Phi^{-1}(\tilde{P})) = \gamma.
\]

Then

\[
\text{cov}(\Phi^{-1}(P), \Phi^{-1}(\tilde{P})) = \gamma \sqrt{\frac{q}{1 - q \sqrt{1 - \tilde{q}}}},
\]

\[
\mathbb{E}(p\tilde{P}) = C(p, \tilde{p}; \gamma \sqrt{q\tilde{q}}),
\]

and

\[
\text{cov}(P, \tilde{P}) = C(p, \tilde{p}; \gamma \sqrt{q\tilde{q}} - p\tilde{p}).
\]

The Vasicek distribution does not offer immediate advantages over other two-parametric continuous distributions on \( (0, 1) \), such as the beta distribution. Its importance stems from its occurrence as mixing distribution in linear factor models set up as in Sec. A.1. It is a special case of a probit-normal distribution; it is named after Vasicek who introduced it into credit risk modeling.

For (different) details on the material in this section we refer to Vasicek (1987, 2002) and Tasche (2008). Estimation of the parameters \( p \) and \( q \) is also discussed in Meyer (2009).

A Proofs

A.1 Proof of (3.1)

Let \( X = \alpha Y + \sqrt{1 - \alpha^2} \epsilon, \tilde{X} = \beta \tilde{Y} + \sqrt{1 - \beta^2} \tilde{\epsilon} \), where \( \alpha, \beta \in (-1, 1) \setminus \{0\} \) are parameters and where \( Y, \tilde{Y}, \epsilon, \tilde{\epsilon} \) are all standard normal and pairwise independent, except
\(\gamma := \text{corr}(Y, \widetilde{Y}) = \text{cov}(Y, \widetilde{Y})\). By construction, \(X\) and \(\widetilde{X}\) are standard normal again with
\[
\text{corr}(X, \widetilde{X}) = \text{cov}(X, \widetilde{X}) = z \beta \gamma.
\]

We define indicator variables \(Z = Z(X) \in \{0, 1\}\), \(\widetilde{Z} = \widetilde{Z}(\widetilde{X}) \in \{0, 1\}\) calibrated to expectations \(u, v\):

\[
Z = 1 \iff X \leq \Phi^{-1}(u), \quad \widetilde{Z} = 1 \iff \widetilde{X} \leq \Phi^{-1}(v).
\]

Conditional on \((Y = y, \widetilde{Y} = \widetilde{y})\), \(Z\) and \(\widetilde{Z}\) are independent. We find
\[
\mathbb{P}(Z = 1 | Y = y) = \mathbb{P}(X \leq \Phi^{-1}(u) | Y = y) = \mathbb{P}
\left(y + \sqrt{1 - x^2} \epsilon \leq \Phi^{-1}(u)\right) = \Phi\left(\frac{\Phi^{-1}(u) - y}{\sqrt{1 - x^2}}\right).
\]

Now we define the random variables
\[
P := P(Y) := \Phi\left(\frac{\Phi^{-1}(u) - zY}{\sqrt{1 - x^2}}\right), \quad \tilde{P} := \tilde{P}(\tilde{Y}) := \Phi\left(\frac{\Phi^{-1}(v) - \beta Y}{\sqrt{1 - \beta^2}}\right).
\]

We find
\[
u = \mathbb{E}(Z) = \mathbb{E}(\mathbb{E}(Z | Y)) = \mathbb{E}(\mathbb{P}(Z = 1 | Y)) = \mathbb{E}(P), \quad v = \mathbb{E}(\tilde{P})
\]
and
\[
\mathbb{P}\left(Z = 1, \widetilde{Z} = 1\right) = \mathbb{P}\left(X \leq \Phi^{-1}(u), \widetilde{X} \leq \Phi^{-1}(v)\right)
= \Phi_2\left(\Phi^{-1}(u), \Phi^{-1}(v), \text{cov}(X, \widetilde{X})\right) = \Phi_2\left(\Phi^{-1}(u), \Phi^{-1}(v), z \beta \gamma\right).
\]

On the other hand,
\[
\mathbb{P}\left(Z = 1, \widetilde{Z} = 1\right) = \mathbb{E}(ZZ) = \mathbb{E}\left(\mathbb{E}(Z \tilde{Z} | Y, \widetilde{Y})\right)
= \mathbb{E}\left(\mathbb{E}(Z \tilde{Z} | Y, \widetilde{Y}) \mathbb{E}(Z | Y, \widetilde{Y})\right)
= \mathbb{E}\left(\mathbb{E}(Z | Y) \mathbb{E}(\tilde{Z} | \widetilde{Y})\right) = \mathbb{E}\left(PP\right)
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(x) \tilde{P}(y) \phi_2(x, y, \gamma) dx dy
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi\left(\frac{\Phi^{-1}(u) - x}{\sqrt{1 - x^2}}\right) \Phi\left(\frac{\Phi^{-1}(v) - \beta y}{\sqrt{1 - \beta^2}}\right) \phi_2(x, y, \gamma) dx dy.
\]
A.2 Proof of Theorems 5.1 and 5.2

We will assume \( g \) as fixed and write \( C(u) := C(u, u; g) \), \( g(u) := g(u; g) \). The upper bound (5.4) follows from (3.22). Regarding the lower bound (5.3) we note that

\[
g'(u) = \sqrt{\frac{1 - \varrho}{1 + \varrho}} \exp \left( \frac{\varrho}{1 + \varrho} \Phi^{-1}(u)^2 \right) > 0,
\]

\[
g''(u) = g'(u) \frac{2 \varrho}{1 + \varrho} \frac{\Phi^{-1}(u)}{\varphi(\Phi^{-1}(u))} < 0.
\]

Hence, \( g \) is increasing and concave on \( (0, 1/2) \) and we conclude

\[
C(u) = 2 \int_0^u g(t) dt \geq 2 \frac{1}{2} u g(u) = u g(u).
\]

Now we define

\[
D_u(u) := u g(u) \cdot a
\]

with \( a \in [1, 2] \), \( u \in [0, 1/2] \). We start by noting that

\[
C'(u) = 2 g(u) > 0, \quad C''(u) = 2 g'(u) > 0,
\]

hence \( C \) is increasing and convex on \( (0, 1/2) \). Furthermore,

\[
D'_u(u) = a \left( g(u) + u g'(u) \right) > 0,
\]

\[
D''_u(u) = a \left( 2 g'(u) + u g''(u) \right) = 2 a g'(u) \left( 1 + \frac{1}{1 + \varrho} \frac{u \Phi^{-1}(u)}{1 + \varrho \varphi(\Phi^{-1}(u))} \right),
\]

\[
= 2 a g'(u) \left( 1 - \frac{1}{1 + \varrho} \frac{H(-\Phi^{-1}(u))}{\varphi(\Phi^{-1}(u))} \right)
\]

with \( H(x) = x R(x) \), where

\[
R(x) = \frac{1 - \Phi(x)}{\varphi(x)}
\]

is Mills' ratio. Pinelis (2002) has shown that \( H'(x) > 0 \) for \( x > 0 \), \( H(0) = 0 \), and \( \lim_{x \to \infty} H(x) = 1 \). Hence \( D'_u(u) \) is increasing and convex on \( (0, 1/2) \) as well. For \( u \in (0, 1/2) \), \( C'(u) = D'_u(u) \) is equivalent with

\[
f(u) := \frac{u g'(u)}{g(u)} = \frac{2 - a}{a}, \quad \text{or} \quad a = \frac{2}{1 + f(u)}.
\]

We will show that \( f \) is strictly increasing on \( (0, 1/2) \). We have

\[
f(u) = \frac{\Phi(\Phi^{-1}(u)) \varphi(\lambda \Phi^{-1}(u))}{\varphi(\Phi^{-1}(u)) \Phi(\lambda \Phi^{-1}(u))} = \lambda F_\lambda(-\Phi^{-1}(u))
\]

with \( \lambda = \sqrt{1 - \varrho} \in [0, 1] \) and

\[
F_\lambda(x) := \frac{R(x)}{R(\lambda x)}, \quad x \geq 0.
\]
We find
\[ F'_\lambda(x) = \frac{R'(x)R(\lambda x) - R(x)R'\lambda(x)}{R(\lambda x)^2} = F_\lambda(x) \left( \frac{R'(x)}{R(x)} - \lambda \frac{R'\lambda(x)}{R(\lambda x)} \right) < 0 \]
for \( \lambda < 1 \). Here, we have used that \( F_\lambda(x) > 0 \) and that the function
\[ y \mapsto \frac{R'(y)}{R(y)} \]
is strictly decreasing on \((0, \infty)\), cf. Baricz (2008). We conclude
\[ f'(u) = -\lambda \frac{F'(\Phi^{-1}(u))}{\Phi(\Phi^{-1}(u))} > 0. \]
Furthermore, we find
\[ f \left( \frac{1}{2} \right) = \lambda \frac{R(0)}{R(0)} = \sqrt{\frac{1 - q}{1 + q}} \]
and
\[ \lim_{u \to 0^+} f(u) = \lim_{u \to 0^+} \frac{g'(u) + ug''(u)}{g'(u)} = \lim_{u \to 0^+} 1 + \frac{2q}{1 + q} \frac{u\Phi^{-1}(u)}{\Phi(\Phi^{-1}(u))} = \lim_{u \to 0^+} 1 - \frac{2q}{1 + q} H(\Phi^{-1}(u)) = 1 - \frac{2q}{1 + q} = \frac{1 - q}{1 + q}. \]
We have \( C(0) = D_a(0) = 0, C'(0) = D'_a(0) = 0 \), and
\[ \lim_{u \to 0^+} \frac{D_a(u)}{C(u)} = \lim_{u \to 0^+} \frac{D'_a(u)}{C'(u)} = \lim_{u \to 0^+} \frac{a}{2} (1 + f(u)) = \frac{a}{1 + q}. \]
By standard calculus we conclude that:
- for \( a \geq 1 + q \) we have \( D'_a(u) \geq C'(u) \), and hence \( D_a(u) \geq C(u) \), for all \( u \in [0, 1/2] \);
- for \( a \leq 2 \left( 1 + \sqrt{1 + q} \right)^{-1} \) we have \( D'_a(u) \leq C'(u) \), and hence \( D_a(u) \leq C(u) \), for all \( u \in [0, 1/2] \);
- for
\[ a \in \left( 2 \left( 1 + \sqrt{1 + q} \right)^{-1}, 1 + q \right) \]
there exists \( u_0 \in (0, 1/2) \) with \( D'_a(u) < C'(u) \) for \( u \in (0, u_0) \), and \( D'_a(u) > C'(u) \) for \( u \in (u_0, 1/2) \). Consequently, the best lower bound for \( C \) of the form \( D_a \) is obtained if \( C(1/2) = D_a(1/2) \), i.e., \( a = 1 + \frac{4}{\pi} \arcsin(q) \). Moreover, the upper bound \( D_a \) with \( a = 1 + q \) can not be improved.

The maximum error of \( D_a \) with \( a = 1 + q \) is attained if
\[ \frac{d}{dq} [D_a(1/2) - C(1/2)] = \frac{1}{4} - \frac{1}{2\pi\sqrt{1 - q^2}} = 0, \]
which is equivalent with \( q = \sqrt{1 - \frac{4}{\pi}} \).
A.3 Proof of Theorem 5.3

We will assume $\varrho$ as fixed and write $C(u) := C(u, u; \varrho)$. We have

$$C(u) = 2 \int_0^u g(t) dt = 2ug(v(u))$$

with $v(u) := v(u; \varrho) \leq u$. Since $g$ is concave and increasing, we even know that $v(u) \leq u/2$, and hence

$$C(u) = 2ug(v(u)) \leq 2ug\left(\frac{u}{2}\right).$$

Moreover, for the same reason we have

$$\frac{d}{du} \left(2ug\left(\frac{u}{2}\right) - C(u)\right) = 2\left(\frac{u}{2}g'\left(\frac{u}{2}\right) - \left(g\left(\frac{u}{2}\right) - g\left(\frac{u}{2}\right)\right)\right) \geq 0,$$

and hence, for $\varrho$ fixed, the maximum error is attained for $u = 1/2$, the value being

$$\Phi\left(\sqrt{\frac{1 - \varrho}{1 + \varrho}} \Phi^{-1}\left(\frac{1}{4}\right)\right) - \frac{1}{4} - \frac{1}{2\pi} \arcsin(\varrho).$$

Derivation of the above expression with respect to $\varrho$ gives the result.

Note that by (3.21) we can write $v(u; \varrho) = g(C(u)/2u; -\varrho)$. That is, the function $2ug(u/2)C(1/2)/g(1/4)$ is another good approximation for $C(u)$. Unfortunately, for large $\varrho$, the function $v$ is not convex, and the approximation is not an upper bound.

A.4 Proof of (6.4), (6.5), (6.6)

In a first step, using (2.8) we find:

$$\int_0^1 C(u, u; \varrho) du = \int_0^1 u^2 + \frac{1}{2\pi} \int_0^1 \int_0^\infty \frac{1}{\sqrt{1 + r^2}} \exp\left(-\Phi^{-1}(u)^2\right) \frac{d\varphi}{1 + r} dr du$$

$$= \frac{1}{3} + \frac{1}{2\pi} \int_0^\infty \int_{-\infty}^\infty \varphi(v) \exp\left(-\frac{v^2}{1 + r}\right) dv dr$$

$$= \frac{1}{3} + \frac{1}{2\pi} \int_0^\infty \int_{-\infty}^\infty \varphi(s) \frac{1 + r}{3 + r} ds dr$$

$$= \frac{1}{3} + \frac{1}{2\pi} \int_0^\infty \frac{1}{\sqrt{1 + r^2}} dr$$

$$= \frac{1}{3} + \frac{1}{2\pi} \arcsin\left(\frac{1 + \varrho}{2}\right) - \arcsin\left(\frac{1}{2}\right)$$

$$= \frac{1}{4} + \frac{1}{2\pi} \arcsin\left(\frac{1 + \varrho}{2}\right).$$
We conclude that
\[
\gamma(q) = 4 \left( \int_0^1 C(u, u; q) \, du + \int_0^1 u - C(u, u; -q) \, du - \frac{1}{2} \right)
\]
\[
= 4 \left( \frac{1}{4} + \frac{1}{2\pi} \arcsin \left( \frac{1 + q}{2} \right) + \frac{1}{2} - \frac{1}{4} - \frac{1}{2\pi} \arcsin \left( \frac{1 - q}{2} \right) - \frac{1}{2} \right)
\]
\[
= \frac{2}{\pi} \left( \arcsin \left( \frac{1 + q}{2} \right) - \arcsin \left( \frac{1 - q}{2} \right) \right).
\]

In a similar way, using again (2.8), we can compute
\[
\int_0^1 C \left( u, \frac{1}{2}; q \right) \, du = \int_0^1 u + \frac{1}{2\pi} \int_0^\infty \frac{1}{\sqrt{1 - r^2}} \exp \left( - \frac{\Phi^{-1}(u)^2}{2(1 - r^2)} \right) \, dr \, du
\]
\[
= \frac{1}{4} + \frac{1}{2\pi} \arcsin \left( \frac{q}{\sqrt{2}} \right),
\]

which, using (3.14), leads to
\[
\int_0^1 C(u, u; q) \, du = 2 \int_0^1 C \left( u, \frac{1}{2}; -\sqrt{1 - q} \right) \, du = \frac{1}{2} - \frac{1}{\pi} \arcsin \left( \frac{\sqrt{1 - q}}{2} \right).
\]

We obtain alternative expressions for Gini’s gamma, the second one using the addition theorem for the arcsin function:
\[
\gamma(q) = \frac{4}{\pi} \left( \arcsin \left( \frac{\sqrt{1 + q}}{2} \right) - \arcsin \left( \frac{\sqrt{1 - q}}{2} \right) \right)
\]
\[
= \frac{4}{\pi} \arcsin \left( \frac{1}{4} \left( \sqrt{(1 + q)(3 + q)} - \sqrt{(1 - q)(3 - q)} \right) \right).
\]

References

Acklam, P. (2004). An algorithm for computing the inverse normal cumulative distribution function. http://home.online.no/~pjacklam/notes/invnorm/index.html

Albers, W., Kallenberg, W. C. M. (1994). A simple approximation to the bivariate normal distribution with large correlation coefficient. J. Multivariate Anal. 49(1):87–96.

 Ağca, Ş., Chance, D. M. (2003). Speed and accuracy comparison of bivariate normal distribution approximations for option pricing. J. Computat. Fin. 6(4):61–96.

Azzalini, A. (1985). A class of distributions which includes the normal ones. Scand. J. Statist. 12(2):171–178.

Azzalini, A. (1986). Further results on a class of distributions which includes the normal ones. Statistica 46(2):199–208.

Azzalini, A., Capitanio, A. (2003). Distributions generated by perturbation of symmetry with emphasis on a multivariate skew t distribution. J. Roy. Statist. Soc. Ser. B 65:367–389.

Balakrishnan, N., Lai, C. D. (2009). Continuous Bivariate Distributions. 2nd ed. New York: Springer.

Baricz, Á. (2008). Mills’ ratio: Monotonicity patterns and functional inequalities. J. Mathemat. Anal. Applic. 340(2):1362–1370.
The Bivariate Normal Copula

Bluhm, C., Overbeck, L. (2003). Estimating systematic risk in uniform credit portfolios. In: Bol, G., Nakhaeizadeh, G., Rachev, S. T., Ridder, T., Vollmer, K.-H., eds. Credit Risk: Measurement, Evaluation and Management. Contributions to Economics. Heidelberg: Physica-Verlag.

Borth, D. M. (1973). A modification of Owen’s method for computing the bi-variate normal integral. J. Roy. Statist. Soc. Ser. C (Appl. Statist.) 22(1):82–85.

Bretz, F., Genz, A. (2009). Computation of Multivariate Normal and t Probabilities. Lecture Notes in Statistics 195. Heidelberg: Springer.

Bretz, F., Genz, A. (2009). Computation of Multivariate Normal and t Probabilities. Lecture Notes in Statistics 195. Heidelberg: Springer.

Cadwell, J. H. (1951). The bivariate normal integral. Biometrika 38(3–4):475–479.

Cherubini, U., Luciano, E., Vecchiato, W. (2004). Copula Methods in Finance. Wiley Finance Series. Chichester, UK: John Wiley & Sons.

Cox, D. R., Wermuth, N. (1991). A simple approximation for bivariate and trivariate normal integrals. Int. Statist. Rev. 59(2):263–269.

Daley, D. J. (1974). Computation of bi- and tri-variate normal integrals. J. Roy. Statist. Soc. Ser. C (Appl. Statist.) 23(3):433–438.

Divgi, D. R. (1979). Calculation of univariate and bivariate normal probability functions. Ann. Statist. 7(4):903–910.

Drezner, Z. (1978). Computation of the bivariate normal integral. Math. Computat. 32(141):277–279.

Drezner, Z., Wesolowsky, G. O. (1990). On the computation of the bivariate normal integral. J. Statist. Computat. Simul. 35:101–107.

Dunnett, C. W., Sobel, M. (1955). Approximations to the probability integral and certain percentage points of a multivariate analogue of Student’s t-distribution. Biometrika 42(1–2):258–260.

Dunnett, C. W., Sobel, M. (1955). Approximations to the probability integral and certain percentage points of a multivariate analogue of Student’s t-distribution. Biometrika 42(1–2):258–260.

Embretchts, P., Frey, R., McNeil, A. J. (2005). Quantitative Risk Management: Concepts, Techniques, Tools. Princeton, NJ: Princeton University Press.

Fang, K.-T., Kotz, S., Ng, K. W. (1990). Symmetric Multivariate and Related Distributions. London: Chapman & Hall.

Genz, A. (2004). Numerical computation of rectangular bivariate and trivariate normal and t probabilities. Statist. Comput. 14(3):151–160.

Gupta, S. (1963a). Probability integrals of multivariate normal and multivariate t. Ann. Mathemat. Statist. 34(3):792–828.

Gupta, S. (1963b). Bibliography on the multivariate normal integrals and related topics. Ann. Mathemat. Statist. 34(3):829–838.

Hull, J. (2008). Futures, Options, and Other Derivatives. 7th ed. Englewood Cliffs, NJ: Prentice Hall.

Hult, H., Lindskog, F. (2002). Multivariate extremes, aggregation and dependence in elliptical distributions. Adv. Appl. Probab. 34(3):587–608.

Kaas, R., Laeven, R. J. A. Nelsen, R. B. (2009). Worst VaR scenarios with given marginals and measures of association. Insur. Math. Econ. 44(2):146–158.

Kotz, S., Balakrishnan, N., Johnson, N. L. (2000). Continuous Multivariate Distributions, Volume 1: Models and Applications. 2nd ed. Wiley Series in Probability and Statistics. New York: John Wiley & Sons.

Kruskal, W. H. (1958). Ordinal measures of association. J. Amer. Statist. Assoc. 53(284):814–861.

Lindskog, F., McNeil, A. J., Schmock, U. (2003). Kendall’s tau for elliptical distributions. In: Bol, G., Nakhaeizadeh, G., Rachev, S. T., Ridder, T., Vollmer, K.-H., eds. Credit Risk: Measurement, Evaluation and Management. Contributions to Economics, Heidelberg: Physica-Verlag.

Lu, D., Li, W. V. (2009). A note on multivariate Gaussian estimates. J. Mathemat. Anal. Applic. 354:704–707.

Mallows, C. L. (1959). An approximate formula for bivariate normal probabilities. Technical Report No. 30. Statistical Techniques Research Group, Princeton University.
Mee, R. W., Owen, D. B. (1983). A simple approximation for bivariate normal probability. *J. Qual. Technol.* 15:72–75.

Mehler, G. (1866). Reihenentwicklungen nach Laplaceschen Functionen höherer Ordnung. *J. Reine Angewandte Mathematik* 66:161–176.

Meyer, C. (2009). Estimation of intra-sector asset correlations. *J. Risk Model Valid.* 3(4): 47–79.

Nelsen, R. B. (2006). *An Introduction to Copulas.* 2nd ed. New York: Springer.

Nicholson, C. (1943). The probability integral for two variables. *Biometrika* 33(1):59–72.

O’Hagan, A., Leonard, T. (1976). Bayes estimation subject to uncertainty about parameter constraints. *Biometrika* 63(1):201–203.

Owen, D. B. (1956). Tables for computing bivariate normal probability. *Ann. Mathemat. Statist.* 27:1075–1090.

Patefield, M., Tandy, D. (2000). Fast and accurate computation of Owen’s $T$-function. *J. Statist. Software* 5(5).

Patel, J. K., Read, C. B. (1996). *Handbook of the Normal Distribution.* New York: Dekker.

Pearson, K. (1901a). Mathematical contributions to the theory of evolution. VII. On the correlation of characters not quantitatively measurable. *Philosoph. Trans. Roy. Soc. London Ser. A* 195:1–47.

Pearson, K. (1901b). Mathematical contributions to the theory of evolution. XI. On the influence of natural selection on the variability and correlation of organs. *Philosoph. Trans. Roy. Soc. London Ser. A* 200:1–66.

Plackett, R. L. (1954). A reduction formula for normal multivariate integrals. *Biometrika* 41(3):351–360.

Pólya, G. (1949). Remarks on computing the probability integral in one and two dimensions. *Proc. of the First Berkeley Symposium on Mathematical Statistics and Probability.* University of California Press, Berkeley.

Sheppard, W. F. (1900). On the calculation of the double integral expressing normal correlation. *Trans. Cambridge Philosop. Soc.* 19:23–69.

Sowden, R. R., Ashford, J. R. (1969). Computation of the bivariate normal integral. *J. Roy. Statist. Soc. Series C (Appl. Statist.)* 18(2):169–180.

Steck, G. P., Owen, D. B. (1962). A note on the equicorrelated multivariate normal distribution. *Biometrika* 49(1–2):269–271.

Stieltjes, T. S. (1889). Extrait d’une lettre adressée à M. Hermite. *Bull. Sci. Math. Ser.* 2 13:170.

Tasche, D. (2008). The Vasicek distribution. http://www-m4.ma.tum.de/pers/tasche/

Terza, J. V., Welland, U. (1991). A comparison of bivariate normal algorithms. *J. Statist. Computat. Simul.* 39(1–2):115–127.

Vasicek, O. (1987). Probability of loss on loan portfolio. http://www.moodyskmv.com/research/portfolioCreditRisk_wp.html

Vasicek, O. (1998). A series expansion for the bivariate normal integral. *J. Computat. Fin.* 1(4):5–10.

Vasicek, O. (2002). The distribution of loan portfolio value. *Risk* 15(12):160–162.

Wang, M., Kennedy, W. J. (1990). Comparison of algorithms for bivariate normal probability over a rectangle based on self-validated results from interval analysis. *J. Statist. Computat. Simul.* 37(1–2):13–25.

West, G. (2005). Better approximations to cumulative normal functions. *Wilmott Mag.* May:70–76.

Willink, R. (2004). Bounds on the bivariate normal distribution function. *Commun. Statist. Theor. Meth.* 33(10):2281–2297.

Young, J. C., Minder, Ch.E. (1974). An integral useful in calculating non-central $t$ and bivariate normal probabilities. *J. Roy. Statist. Soc. Ser. C (Appl. Statist.)* 23(3):455–457.