Generalized Proper-Time Approach
For The Case Of Broken Isospin Symmetry

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Abstract

We present a derivation of the low-energy effective meson Lagrangian of the Nambu – Jona-Lasinio (NJL) model on the basis of Schwinger’s proper-time regularization of the one-loop fermion determinant. We consider the case in which the $SU(2) \times SU(2)$ chiral symmetry of the NJL Lagrangian is broken by the current quark mass matrix with $\hat{m}_u \neq \hat{m}_d$. The non-degeneracy of $d$ and $u$ masses destroys one of the most crucial features of the proper-time expansion – the chiral-invariant structure of Seeley – DeWitt coefficients. We show however that systematic resummations inside the proper-time expansion are still possible and derive a result which is in full agreement with the chiral Ward – Takahashi identities.

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1 Introduction

In recent papers [1, 2] we have shown how to determine systematically the low-energy structure of the Nambu – Jona-Lasinio (NJL) model [3] on the basis of Schwinger’s proper time regularization of the one-loop fermion determinant [4, 5]. We have considered the cases with linear and non-linear realizations of explicitly broken $SU(2) \times SU(2)$ chiral symmetry. We have shown that in the presence of the explicit chiral symmetry breaking term in the Lagrangian, the standard definition of $\ln |\det D|$ in terms of a proper-time integral

$$\ln |\det D| = -\frac{1}{2} \int_0^\infty \frac{dT}{T} \rho(T, \Lambda^2) \text{Tr} \left( e^{-T D^\dagger D} \right)$$

(1)

modifies the explicit chiral symmetry breaking pattern of the original quark Lagrangian and needs to be corrected in order to lead to the fermion determinant whose transformation properties exactly comply with the symmetry content of the basic Lagrangian. The reason for the necessary modification of the result obtained on the basis of formula (1) is directly connected with the restrictions imposed by the chiral Ward – Takahashi (WT) identities.

In the present paper we extend this framework to the case in which the $SU(2) \times SU(2)$ chiral symmetry of the NJL Lagrangian is broken by the current quark mass matrix with $\hat{m}_u \neq \hat{m}_d$. The non-degeneracy of $d$ and $u$ masses introduces a rather large violation of isotopic spin conservation and destroys one of the most crucial features of the proper time expansion – the chiral-invariant structure of Seeley – DeWitt coefficients. It means that additionally to the already known problem of correcting the standard definition (1) by the functional $P$ being proportional to the current quark masses [1] one has to find a method which replaces the standard asymptotic expansion of the heat kernel in terms of Seeley – DeWitt coefficients by a new expansion in terms of chiral-invariant structures. As we shall show, systematic resummations inside the proper-time expansion are possible. We describe here how calculations can be organized in this, at the first sight quite fuzzy, situation. The resulting series for the heat kernel is not anymore a proper-time expansion. We calculate its first terms and present the method to determine higher orders.

The spontaneous breakdown of the global $SU(2) \times SU(2)$ chiral symmetry in the NJL model is a consequence of the fact that the Schwinger – Dyson equations for the fermion propagators (“gap”-equations) have nontrivial so-
olutions with \( m_u \neq m_d \neq 0 \). After bosonization the same equations occur as conditions for the minimum of the effective potential. This generates a corresponding redefinition of the scalar fields, from which it follows that the part of the effective Lagrangian which is linear in the scalar fields vanishes (due to the gap-equations), for excitations about the real vacuum state. As we shall show, this immediately affects the isospin symmetry breaking part of the Lagrangian and breaks the chiral WT identities at this level. As a consequence we need to invoke new counterterms which would have not been required if there had not been spontaneous symmetry breakdown. These counterterms have a simple structure and are completely fixed by chiral symmetry.

This article presents a new systematic expansion procedure for the heat kernel of the one loop fermion determinant for the case \( \hat{m}_u \neq \hat{m}_d \). It differs from the methods which have already been used in the literature in the same context, for example in papers [3, 7]. The result is also different. In Sec. II we discuss the Lagrangian of the NJL model and show that chiral \( SU(2) \times SU(2) \) transformations of quark fields dictate the transformation laws of the auxiliary bosonic fields. These collective variables are necessary to rearrange the four-quark Lagrangian of the NJL model in an equivalent Lagrangian which is only quadratic in the quark fields. The chiral symmetry is broken explicitly by the current quark mass matrix with \( \hat{m}_u \neq \hat{m}_d \). We use the classical equations of motion for the pseudoscalar and scalar fields to rewrite the variation of the NJL Lagrangian with respect to the action of the chiral group in terms of meson fields. In Sec. III we discuss in detail the chiral WT identities for the case under consideration. We show how one can integrate these identities to get the symmetry breaking part of the effective Lagrangian without explicit calculations of the quark determinant. In Sec. IV we calculate the fermion determinant. We show how to define it for the case in which explicit symmetry breaking takes place and isospin symmetry is broken. We calculate the first terms of the new expansion of the heat kernel in full detail. We derive the corresponding correcting polynomial from the functional \( P \) and show that it is completely fixed by the symmetry requirements. The effective meson Lagrangian is obtained at the end of this section. In Sec. V we introduce new variables for pseudoscalar and scalar fields in order to describe the physical meson states and to get the meson mass spectrum. The concluding remarks are given in Sec. VI. Finally we show in the Appendix some technical details in our treatment of the heat kernel exponent with different masses for \( u \) and \( d \) quarks.
2 From quarks to mesons: the infinitesimal chiral transformations

Consider the effective quark Lagrangian of strong interactions with only light $u$ and $d$ quark fields which is invariant under a global colour $SU(N_c)$ symmetry

$$\mathcal{L} = \bar{q}(i\gamma^\mu \partial_\mu - \hat{m})q + \frac{G}{2}[(\bar{q}\tau_\alpha q)^2 + (\bar{q}\gamma_5 \tau_\alpha q)^2].$$

Here $q$ is a flavor doublet of Dirac spinors for quark fields $\bar{q} = (\bar{u}, \bar{d})$. The fermion field carries both flavor and color indices. Summation over the color indices is implicit. For the tau $2 \times 2$ matrices, $\tau_a$, where $a = 0, 1, 2, 3,$ and $\text{tr}(\tau_a \tau_b) = 2\delta_{ab}$, we shall use the following notation: $\tau_0 = 1$ and $\tau_i, (i = 1, 2, 3)$ are the standard Pauli matrices. The manifest chiral symmetry breaking occurs via the current quark mass matrix: $\hat{m} = \text{diag}(\hat{m}_u, \hat{m}_d)$ where $\hat{m}_u \neq \hat{m}_d$.

Without this term the Lagrangian (2) would be invariant under global chiral $SU(2) \times SU(2)$ symmetry. We restrict our consideration to the case of only scalar and pseudoscalar four-quark interactions with the coupling constant $G$ being strong enough to create a stable vacuum state with a nontrivial solution of the gap-equations corresponding to spontaneous breakdown of the chiral $SU(2) \times SU(2)$ symmetry.

The transformation law for the quark fields is the following

$$\delta q = i(\alpha + \gamma_5 \beta)q, \quad \delta \bar{q} = -i\bar{q}(\alpha - \gamma_5 \beta),$$

where parameters of global infinitesimal $SU(2) \times SU(2)$ chiral transformations are chosen as $\alpha = \alpha_i \tau_i$, $\beta = \beta_i \tau_i$. Therefore the Lagrangian $\mathcal{L}$ transforms according to the law

$$\delta \mathcal{L} = i\bar{q}([\alpha, \hat{m}] - \gamma_5 \{\beta, \hat{m}\})q.$$
Lagrangian density (2). To integrate over anticommuting c-number quark fields in $Z$ one has to introduce color singlet collective bosonic variables in such a way that the action becomes bilinear in the quark fields and the integration becomes trivial

$$Z = \int \mathcal{D}q\mathcal{D}\bar{q}\mathcal{D}a\mathcal{D}p\mathcal{D}s\mathcal{D}p \exp \left\{ i \int d^4x \left[ \mathcal{L} - \frac{1}{2G} (s_a^2 + p_a^2) \right] \right\}. \quad (5)$$

We suppress external sources in the generating functional $Z$ and assume summation over repeated flavor indices. In order not to destroy the symmetry of the basic quark Lagrangian $\mathcal{L}$ one has to require from the new collective variables that

$$\delta (s_a^2 + p_a^2) = 0 \quad (6)$$

Since there are no kinetic terms for $s_a$ and $p_a$, these fields are auxiliary. Let us replace the variables in $Z$: $s_a, p_a \to \sigma_a, \pi_a$,

$$s_a = \sigma_a - \hat{m} + G(\bar{q}\tau_a q), \quad (7)$$

$$p_a = \pi_a - G(\bar{q}\gamma_5 \tau_a q). \quad (8)$$

Here we put

$$\hat{m}_0 = \frac{\hat{m}_u + \hat{m}_d}{2}, \quad \hat{m}_i = \delta_{i3} \frac{\hat{m}_u - \hat{m}_d}{2}. \quad (9)$$

Requirement (3) together with the chiral transformation laws for the quark fields (3) lead to the transformation laws for the new collective fields:

$$\delta \sigma = i[\alpha, \sigma - \hat{m}] - \{\beta, \pi\}, \quad \delta \pi = i[\alpha, \pi] + \{\beta, \sigma - \hat{m}\}, \quad (10)$$

where $\pi = \pi_a \tau_a$, $\sigma = \sigma_a \tau_a$, and $\hat{m} = \hat{m}_a \tau_a$. In this way the transformation laws of the quark fields with respect to the action of chiral $SU(2) \times SU(2)$ group define the transformation laws of the collective bosonic fields. However Eq.(10) is not yet the final form which we need. In the NJL model the vacuum is not invariant under chiral $SU(2) \times SU(2)$ transformations. To explore the properties of the spontaneously broken theory one has to define new scalar fields with vanishing vacuum expectation values, i.e., one has to rewrite the Lagrangian of the theory in terms of shifted fields:

$$\sigma \to \sigma + m, \quad (11)$$
where
\[ m = m_a \tau_a, \quad m_0 = \frac{m_u + m_d}{2}, \quad m_i = \delta_3 \frac{m_u - m_d}{2}, \] (12)
are the masses of constituent quarks. The chiral transformation properties
of scalar and pseudoscalar fields change correspondingly to the new ones:
\[ \delta \sigma = i[\alpha, \sigma + \Delta] - \{\beta, \pi\}, \quad \delta \pi = i[\alpha, \pi] + \{\beta, \sigma + \Delta\}, \] (13)
where \( \Delta = m - \hat{m} \). This is the final form of infinitesimal chiral transforma-
tions for collective mesonic excitations around the nontrivial vacuum state
in the NJL model.

In accordance with replacements (7) and (8) we obtain the mixed form
for the NJL Lagrangian which includes both quarks and collective de-
grees of freedom
\[ \mathcal{L}(q, \bar{q}, \sigma_a, \pi_a) = \bar{q} D q - \frac{1}{2} G \left[ (\sigma_a + \Delta_a)^2 + \pi_a^2 \right], \] (14)
where
\[ D = i \gamma^\mu \partial_\mu - m - \sigma + i \gamma_5 \pi. \] (15)
The Euler – Lagrange equations for mesonic fields take the form of constraints
\[ \pi_a = G \bar{q} i \gamma_5 \tau_a q, \quad \sigma_a = -\Delta_a - G \bar{q} \tau_a q. \] (16)
By using them one can rewrite Eq.(4) in terms of collective degrees of free-
dom:
\[ \delta \mathcal{L} = \frac{\hat{m}_a}{G} \delta \sigma_a. \] (17)
One concludes that the symmetry breaking pattern of the NJL model is not
affected by the occurrence of spontaneous symmetry breakdown.

The derivation of the action for collective fields \( \pi_a \) and \( \sigma_a \) in the NJL
model is reduced (after integrating out the quark fields) to the calculation of
the functional determinant of the operator \( D \)
\[ S_{\text{coll}} = \int d^4 x \mathcal{L}_{\text{coll}} = -i \ln \det D - \frac{1}{2G} \int d^4 x \left[ (\sigma_a + \Delta_a)^2 + \pi_a^2 \right]. \] (18)
This is a great advantage of the model, for there are methods to study the
determinant of such operators [8, 9]. On the other hand, however, these
methods can not be directly applied in the presence of the manifest chiral
symmetry breaking term in the NJL Lagrangian [1, 2]. We have an even
more complicated case now, because of isospin breaking. We shall consider
this problem in the following sections.
One can ask if it is possible to get the symmetry breaking part of the bosonized NJL Lagrangian without a direct evaluation of the fermion determinant in \((18)\). The answer on this question is positive and hidden in Eq.\((17)\). Indeed, all we need to do is just to solve it. Possible solutions may be constructed by applying a method which is closely related to the one already used in \([10]\). The lack of a direct dependence of chiral transformations on the space-time coordinates simplifies our task. In particular the chiral WT identities can be stated directly by giving the variation of the Lagrangian of collective meson fields, \(\mathcal{L}_{\text{coll}}\), with respect to the infinitesimal chiral \(SU(2) \times SU(2)\) transformations, i.e.,

\[
\delta \mathcal{L}_{\text{coll}} = \frac{\hat{m}_a}{G} \delta \sigma_a. \tag{19}
\]

If chiral symmetry is spontaneously broken, the infinitesimal transformations are given by Eq.\((13)\). It implies that the variation of the functional of the considered collective meson fields is described by the infinitesimal operator

\[
\hat{\delta} = 2(\alpha_i X_i + \beta_i Y_i), \tag{20}
\]

where the generators \(X_i\) and \(Y_i\) are expressed in the space of meson fields as follows

\[
X_i = -\epsilon_{ijk} \left[ \pi_j \frac{\delta}{\delta \pi_k} + (\sigma_j + \Delta_j) \frac{\delta}{\delta \sigma_k} \right], \\
Y_i = (\sigma_i + \Delta_i) \frac{\delta}{\delta \pi_0} + (\sigma_0 + \Delta_0) \frac{\delta}{\delta \sigma_i} + \pi_i \frac{\delta}{\delta \pi_0} - \pi_0 \frac{\delta}{\delta \pi_i}. \tag{21}
\]

The definition of a functional derivative depends on which parameters are considered as fixed in the functions, and which are considered as variable. Our functions \(\pi_a(x)\) and \(\sigma_a(x)\) have fixed arguments with respect to global chiral transformations, i.e., the variational derivative does not produce a delta function of space-time coordinates. In this case the generators satisfy the commutation relations

\[
[X_i, X_j] = \epsilon_{ijk} X_k, \\
[X_i, Y_j] = \epsilon_{ijk} Y_k, \\
[Y_i, Y_j] = \epsilon_{ijk} X_k. \tag{22}
\]
which are the Lie algebra representation of the chiral group. Using the definition (21), we can rewrite Eq.(19) as

\[ X_i L_{\text{coll}} = F_i = -\frac{1}{G} \epsilon_{ijk} \sigma_j \hat{m}_k, \]
\[ Y_i L_{\text{coll}} = G_i = -\frac{1}{G} (\hat{m}_0 \pi_i + \hat{m}_i \pi_0). \] (23)

The fact that the WT identities (23) express the variation of a single function, \( L_{\text{coll}} \), gives immediately strong restrictions on the form of this function. In addition we have integrability conditions:

\[ X_i F_j - X_j F_i = \epsilon_{ijk} F_k, \]
\[ Y_i G_j - Y_j G_i = \epsilon_{ijk} F_k, \]
\[ Y_i F_j - X_j G_i = \epsilon_{ijk} G_k, \] (24)

which tell us if it is possible or not to integrate Eq.(23) with the given functions \( F_i \) and \( G_i \). These conditions are fulfilled in our case. It is clear that from the WT identities one only obtains general symmetry restrictions on the form of the bosonized NJL Lagrangian; in particular, one can use them to fix the symmetry breaking part of \( L_{\text{coll}} \). Indeed, the first equation in the system (23) gives

\[ L_{\text{coll}} = -\hat{m}_3 \sigma^2 + \frac{L_S}{2G \Delta_3} + \Omega, \] (25)

where \( L_S \) is a chiral symmetric part of the Lagrangian, i.e., \( X_i L_S = Y_i L_S = 0 \), and for \( \Omega \) we have

\[ X_i \Omega = 0, \]
\[ Y_i \Omega = -\frac{1}{G} \left( \hat{m}_0 \pi_i + \hat{m}_i \pi_0 + \frac{\hat{m}_3}{\Delta_3} \pi_0 \sigma_i \right). \] (26)

Noting that \( \sigma_0 \) and \( \pi_0 \) are cancelled by the action of the operator \( X_i \) (\( X_i \pi_0 = X_i \sigma_0 = 0 \)), we derive

\[ \Omega = \frac{\hat{m}_0 \sigma_0}{G} - \frac{\hat{m}_3 \pi_0^2}{2G \Delta_3}. \] (27)

Thus

\[ L_{\text{coll}} = -\frac{\hat{m}_3 (\pi_0^2 + \sigma^2)}{2G \Delta_3} + \frac{\hat{m}_0 \sigma_0}{G} + L_S. \] (28)
Since we are looking for solutions corresponding to the real vacuum state, the linear term $\sim \sigma_0$ in the Lagrange density (28) must not occur; it would alter the Schwinger-Dyson equations. Let us use the freedom inherent in the choice of the chiral symmetric piece $L_S$ of $L_{\text{coll}}$ to eliminate the linear scalar field in favor of terms quadratic in the meson fields. There is only one chiral invariant combination which is bilinear in the meson fields and contains the necessary part with $\sigma_0$, it is $a(\sigma_0^2 + \vec{\pi}^2 + 2\Delta_0 \sigma_0)$. The constant $a$ is then fixed by the requirement of cancellation of the linear term in (28): $a = -\hat{m}_0 (2G\Delta_0)^{-1}$. As a result, we obtain the solution of chiral WT identities for the bosonized NJL Lagrangian in the form: $L_{\text{coll}} = L_{SB} + L_S$, where

$$L_{SB} = -\frac{\hat{m}_3 (\pi_0^2 + \vec{\sigma}^2)}{2G\Delta_3} - \frac{\hat{m}_0 (\sigma_0^2 + \vec{\pi}^2)}{2G\Delta_0}. \quad (29)$$

We have achieved our stated aim of integrating Eq.(19). Let us turn now to the direct calculation of the fermion determinant in Eq.(18). Contrary to the chiral symmetric case, for which the proper-time method yields a correct result, we still have to find the way to interpret the fermion determinant for the present problem with explicit chiral and isospin symmetry breaking.

## 4 Dirac fermion determinant with isospin symmetry breaking

To evaluate the real part of the fermion determinant we begin with the standard proper-time representation (1). Although this definition fails when chiral symmetry is explicitly broken, it still can be used as a basis for systematic calculations, if one includes the correcting counterterms in the functional $P$. We have also to note that in the present case the standard proper-time expansion does not work. There are several reasons for that. First, it is the non-commutativity of the quark mass matrix in the heat kernel exponent. This property is reflected in resummations inside the standard proper-time series. We denote them by $T$-resummations and show how to organize them. Second, this $T$-resummations do not lead automatically to chiral invariant groupings at each order of the $P$-exponent. One has to search for the invariants which replace the Seeley-DeWitt coefficients in the asymptotic expansion of the heat kernel. In practice, this is the most difficult task and implies
additional resummations. Third, the gap-equations work against chiral symmetry. One can restore chiral symmetry without afflicting the gap-equations, by adding to the Lagrangian a counterterm local in the collective fields. This is the general philosophy we shall adhere to in the remainder of this paper. As a result, we obtain the collective Lagrangian, $L_{\text{coll}}$, describing the low-energy limit of the four quark NJL dynamics, which is in full agreement with the WT identities.

4.1 Proper-time representation for the heat kernel

The starting point of our analysis in this section is a non-perturbative definition of the fermion path integral, or fermion determinant:

$$\int Dq D\bar{q} \exp \left\{ i \int d^4x \bar{q} Dq \right\} = e^{iW[\pi,\sigma]} \equiv \det D.$$  \hspace{1cm} (30)

The functional $W[\pi, \sigma] = -i \log \det D$ is the fermionic effective action and $D$ is given by Eq.(15). Of course the fermion determinant defined by Eq.(30) is only a formal construction. We shall see below that $W[\pi, \sigma]$ can be defined unambiguously up to a set of local counterterms on the basis of the modified Schwinger proper-time representation. We study here only the real part of the Dirac fermion determinant.

Once a definition for $\det D$ is chosen satisfying some set of consistency requirements, its dependence on the collective meson fields and in particular its behavior under chiral transformations can be analyzed. The naive identification of the real part of this determinant with the proper-time formula Eq.(1) is problematic for a couple of reasons. First, although the Dirac operator $D$ does not include the current quark mass, $\hat{m}$, the transformation law of pseudoscalar and scalar fields does. Thus\(^2\),

$$\delta D_E = i[\alpha, D_E] - i\{\gamma_5\beta, D_E\} - 2i\gamma_5[\hat{m}_0\beta + \hat{m}_3(\beta_3\tau_0 - i\gamma_5\alpha_i\epsilon_{i3k}\tau_k)],$$

where $\gamma\dagger = \gamma$, $\partial = (\nabla, \partial_4)$. So we get $D \rightarrow D_E = i\gamma_r \partial_r - m - \sigma + i\gamma_5\pi$, where $\gamma_5 = \gamma_5$, $\partial_r = (\nabla, \partial_4)$.

\(^2\)For the actual calculations we perform a Wick rotation into Euclidean space-time: $ix_0 = x_4$, $i\gamma_0 = \gamma_4$, $\gamma_5 = \gamma_5$, $\{\gamma_r, \gamma_s\} = -\delta_{rs}$, where $r, s = 1, 2, 3, 4$. We have chosen all $\gamma_r$-matrices anti-Hermitian: $\gamma^\dagger_r = -\gamma_r$. So we get $D \rightarrow D_E = i\gamma_r \partial_r - m - \sigma + i\gamma_5\pi$, where $\gamma_5 = \gamma_5$, $\partial_r = (\nabla, \partial_4)$.\n
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\[ -4\hat{m}_0 \left\{ \vec{\beta} \vec{\pi} + \left[ \gamma_5 \epsilon_{ijk} (\sigma_i + m_i) \beta_j + \beta_k \pi_0 \right] \tau_k \right\} \\
-4\hat{m}_3 \left\{ \gamma_5 (\alpha \pi_3 - \vec{\alpha} \vec{\pi} \tau_3) + \pi_0 \beta_3 \\
+ \epsilon_{3ik} \alpha_i \sigma_k + \left[ \beta_3 \pi_k + (\sigma_0 + m_0) \epsilon_{3ik} \alpha_i \right] \tau_k \right\}. \tag{32} \]

It means that \( \delta \ln |\det D| \) is not equal to zero and has systematic contributions proportional to the current quark mass terms \( \hat{m}_0 \) and \( \hat{m}_3 \). This is similar to the case which has already been considered in [2]. The explicit symmetry breaking terms have to be corrected by the corresponding contributions from the functional \( P \) to lead to a result being in agreement with the WT identities. Second, there is a new type of contributions. These ones emerge after spontaneous symmetry breakdown and are proportional to the isospin symmetry breaking terms \( \sim (m_d - m_u) \). Let us discuss these problems and their solution in full detail.

We start by defining the real part of the fermion determinant by the formula:

\[ \text{Re} (\ln \det D) = \ln |\det D| + P. \tag{33} \]

The first term here is the Schwinger proper-time regularization for the Dirac fermion determinant given by Eq.(1). In the case of nonrenormalizable models like NJL we have to introduce the cutoff \( \Lambda \) to render the integrals over \( T \) convergent. We consider a class of regularization schemes which can be incorporated in the expression (1) through the kernel \( \rho(T, \Lambda^2) \). These regularizations allow to shift in loop momenta. A typical example is the covariant Pauli-Villars cutoff [11]

\[ \rho(T, \Lambda^2) = 1 - (1 + T\Lambda^2)e^{-T\Lambda^2}, \tag{34} \]

which we use in the present calculations. Our strategy is the following: knowing that the first term in Eq.(33) breaks chiral symmetry in a contradictory way, we find the functional \( P[\pi, \sigma] \) by requiring the real part of the fermion determinant to transform in accordance with Eq.(19). The WT identities are used for that. The first step in the realization of this program is the evaluation of the heat kernel \( \text{Tr}(e^{-TR}) \) with the operator \( R = m^2 + B \), where

\[ B = -\partial_r^2 + i\gamma_r (\partial_r \sigma - i\gamma_5 \partial_r \pi) + \sigma^2 + \pi^2 + \{\sigma, m\} + i\gamma_5 [\pi, \sigma + m]. \tag{35} \]

We denote by

\[ m^2 = K\tau_0 + M\tau_3, \quad K = \frac{m_u^2 + m_d^2}{2}, \quad M = \frac{m_u^2 - m_d^2}{2}. \tag{36} \]
the square of the mass matrix. The operator $R$ depends on functions of eu-
clidian coordinates, $x_r$, and derivatives, $\partial_r$. Therefore, following the abstract
formalism developed in [9], we regard $e^{-TR}$ as an operator $e^{-TR}$ acting on a
fictitious Hilbert space, so that the heat kernel $\text{Tr}(e^{-TR})$ reads

$$\text{Tr}(e^{-TR}) = \int d^4x \text{tr} \langle x|e^{-TR}|x \rangle,$$  \hspace{1cm} (37)

where $|x\rangle$ is an eigenvector of a commuting set of Hermitian operators
$\hat{x}_r$ such that $\hat{x}_r|x\rangle = x_r|x\rangle$ and $\langle x|y\rangle = \delta(x - y)$. The Hermitian
operators, $\hat{p}_r = -i\partial_r$, which are conjugate to $\hat{x}_r$, obey canonical commutation
relations: $[\hat{x}_r, \hat{p}_s] = i\delta_{rs}$, $[\hat{x}_r, \hat{x}_s] = [\hat{p}_r, \hat{p}_s] = 0$. Let us take an eigenket $|p\rangle$ of $\hat{p}_r$. Its representative in the Schrödinger representation is $\langle x|p\rangle = (2\pi)^{-2} \exp(-ipx)$. Using this plane wave basis one can evaluate the heat
kernel directly:

$$\langle x|e^{-TR}|p\rangle = \int d^4p \langle x|e^{-TR}|p\rangle = \int \frac{d^4p}{(2\pi)^4} \exp(-ipx) \hat{R} \exp(ipx).$$  \hspace{1cm} (38)

We note then that $e^{-ipx} \hat{R} e^{ipx} = (\hat{R} - 2ip\partial + p^2)1$, where $p_r\partial_r = p\partial$, implies

$$\langle x|e^{-TR}|p\rangle = \int \frac{d^4p}{(2\pi)^4} e^{-T(p^2)} e^{-T(\hat{R} - 2ip\partial)} 1$$

$$= \int \frac{d^4p}{(4\pi^2T)^2} e^{-p^2} \exp \left[ -T \left( \hat{R} - \frac{2ip\partial}{\sqrt{T}} \right) \right] 1.$$  \hspace{1cm} (40)

Finally, combining (37) and (40), results in the desired relation

$$\text{Tr}(e^{-TR}) = \int \frac{d^4p d^4x}{(4\pi^2T)^2} e^{-(p^2+TK)} \text{tr} \{ \exp \left[ -T (M\tau_3 + A) \right] 1 \},$$  \hspace{1cm} (41)

where

$$A = B - \frac{2ip\partial}{\sqrt{T}}.$$  \hspace{1cm} (42)
Similarly as in the case of equal quark masses, one has to separate from the heat kernel exponent the $m^2$ piece. However, in the present case $m^2$ contains apart from the isoscalar, $K$, which we already factorized in Eq. (41), also the non-commuting isovector term proportional to $M$. This non-commutativity is one of the sources of necessary $T$-resummations in the heat kernel expansion, which deviate from the standard case and which we consider next.

4.2 P-exponent and non-commutativity of the mass matrix

We discuss now further the exponent, $\text{tr}[e^{-T(M\tau_3 + A)}]$, contained in the heat kernel. Here we can not use the standard asymptotic expansion in powers of the proper-time $T$, which would lead finally to the Seeley – DeWitt coefficients. One has first to separate the quark mass part $e^{-T M\tau_3}$, otherwise it will generate linear terms in $\sigma$ at any power of $T$. It is not convenient. We need a method which leads to the gap-equations already at the first step of the asymptotic expansion. This separation of the non-commuting part of the squared mass matrix in the heat kernel require the use of the following operator identity:

$$e^{-T(M\tau_3+A)} = e^{-TM\tau_3} \times P \left\{ \exp \left( -\int_0^T ds A(s) \right) \right\}, \quad (43)$$

where

$$A(s) = e^{sM\tau_3} Ae^{-sM\tau_3}. \quad (44)$$

The P-exponent above is defined to mean

$$P \left\{ \exp \left( -\int_0^T ds A(s) \right) \right\} =
1 + \sum_{n=1}^{\infty} (-1)^n \int_0^T ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_{n-1}} ds_n A(s_1)A(s_2)\ldots A(s_n). \quad (45)$$

For our purpose one needs only the first three terms from the right hand side of this expression, i.e.,

$$\text{tr}[e^{-T(M\tau_3+A)}] = \text{tr} \left( e^{-TM\tau_3} \left\{ 1 - TA + \frac{T^2}{4} [A^2 + A\tau_3 A\tau_3 + c(T)(A^2 - A\tau_3 A\tau_3)] + \ldots \right\} \right), \quad (46)$$
\[ c(T) = \frac{1}{2T^2 M^2} \left( e^{2TM\tau_3} - 1 - 2TM\tau_3 \right) \]
\[ = \frac{1}{M^2} e^{TM\tau_3} \int_0^M d\alpha [M \cosh(T\alpha) - \tau_3 \alpha \sinh(T\alpha)]. \]

(47)

The details are given in the Appendix. It is clear now that the asymptotic heat kernel expansion (46) is modified due to systematic resummations generated by the terms \( \sim M \). After evaluating the integrals over 4-momentum \( p_r \) one obtains

\[
\text{Tr}(e^{-T\hat{R}}) = \int \frac{d^4x}{(4\pi T)^2} e^{-TK} \text{tr} \left( e^{-TM\tau_3} \left\{ 1 - TY \right\} \right.
\]
\[ + \left. \frac{T^2}{4} \left[ Y^2 + Y\tau_3 Y\tau_3 + c(T)(Y^2 - Y\tau_3 Y\tau_3) \right] + \ldots \right). \]

(48)

Here we took into account that a Lagrangian density is defined up to total derivatives. The \( Y \) is given by

\[ Y = i\gamma_r (\partial_r \sigma - i\gamma_5 \partial_r \pi) + \sigma^2 + \pi^2 + \{\sigma, m\} + i\gamma_5 [\pi, \sigma + m]. \]

(49)

Finally, the modulus of the fermion determinant may thus be written as

\[
- \ln |\det D_E| = \frac{1}{32\pi^2} \int_0^\infty \frac{dT}{T^{3}} \rho(T, \Lambda^2) e^{-TK} \text{tr} \left( e^{-TM\tau_3} \left\{ 1 - TY \right\} \right.
\]
\[ + \left. \frac{T^2}{4} \left[ Y^2 + Y\tau_3 Y\tau_3 + c(T)(Y^2 - Y\tau_3 Y\tau_3) \right] + \mathcal{O}(Y^3) \right). \]

(50)

### 4.3 Gap-equations against the chiral symmetry

Several points about Eq.(50) should be clarified right away. The first term does not depend on collective fields and has no interest for us. Let us consider the second term which is proportional to \( Y \) and which we denote as \( b_1 \). To simplify this expression one has to take the integral over \( T \) and calculate the traces. The last ones include summations over isotopic, color and euclidean indices. The integrals over \( T \) can be reduced to combinations of some set of elementary integrals \( J_n(m^2) \)

\[
J_n(m^2) = \int_0^\infty \frac{dT}{T^{2-n}} e^{-Tm^2} \rho(T, \Lambda^2), \quad n = 0, 1, 2, \ldots
\]

(51)
These manipulations lead us to the result:

\[
\begin{align*}
    b_1 &= - \frac{N_c}{8\pi^2} \left\{ \left[ J_0(m_u^2) + J_0(m_d^2) \right] \left[ \sigma_a^2 + \pi_a^2 + 2(m_0\sigma_0 + m_3\sigma_3) \right] \right. \\
    &\quad + \left. 2 \left[ J_0(m_u^2) - J_0(m_d^2) \right] (\sigma_0\sigma_3 + \pi_0\pi_3 + m_0\sigma_3 + m_3\sigma_0) \right\} .
\end{align*}
\] (52)

Other terms in the expansion of the integrand in (50) are \( \sim Y^n \), where \( n \geq 2 \), and therefore do not contribute to the term linear in \( \sigma \). We consider small fluctuations of the system about the asymmetric vacuum state and have already shifted the scalar fields correspondingly. It means that linear terms in the scalar fields should not be present in the Lagrangian. This self-consistency requirement can be expressed in terms of the gap-equations:

\[
\begin{align*}
    \frac{m_u - \hat{m}_u}{m_u} &= \frac{N_c G}{2\pi^2} J_0(m_u^2), \\
    \frac{m_d - \hat{m}_d}{m_d} &= \frac{N_c G}{2\pi^2} J_0(m_d^2).
\end{align*}
\] (53) (54)

Once this has been done, one is confronted with a problem related to the fact that the last term in Eq.(52), linear in the scalar fields, i.e.,

\[
- \frac{N_c}{4\pi^2} \left[ J_0(m_u^2) - J_0(m_d^2) \right] (m_0\sigma_3 + m_3\sigma_0)
\] (55)

vanishes. Let us make clear the essence of the problem. The difference of two \( J_0 \) integrals with different arguments can be written as a sum of two \( J_1 \) multiplied by \( M \) and a rest which includes the contributions from \( J_n \) integrals with \( n > 1 \)

\[
J_0(m_u^2) - J_0(m_d^2) = -M [J_1(m_u^2) + J_1(m_d^2)] + O(M^3).
\] (56)

As we soon will see, the first term contributes to the next order of our modified proper-time expansion. This fact is very important, because without this contribution chiral symmetry would be destroyed at the next order. The situation is even more complicated, since the \( O(M^3) \) terms will of course contribute to all remaining orders of the heat kernel expansion. For the same reason relation (56) implies also that the term quadratic in the fields in Eq.(52), multiplying the difference of \( J_0 \) integrals, does not contribute to the leading order. Thus in evaluating the fermion determinant on the basis
of representation \( \mathcal{R} \) one has to perform carefully systematic resummations in the Schwinger's proper-time expansion. This is an embarrassment but not a catastrophe. Indeed, from formula (32) it follows that \( \delta \ln |\det D_E| \sim \hat{m} \). Therefore the resummations are possible in general, since the total expression (1) is already a chiral quasi-invariant, i.e., invariant up to terms \( \sim \hat{m} \). However, the ansatz introduced by the gap-equations destroys this picture, taking away one of the necessary elements. To restore the transformation property of the determinant instead of the removed term (55) one has to add to the functional \( P \) a counterterm which is a quadratic polynomial in the meson fields, and which has the same transformation property as the removed expression (53). The correction predicted by the WT identities thus naturally takes the form

\[
\frac{N_c}{8\pi^2}[J_0(m_u^2) - J_0(m_d^2)]\left\{\frac{1}{\Delta_0\Delta_3}[\Delta_0^2(\sigma^2 + \pi^2_0) + \Delta_3^2(\sigma_0^2 + \pi^2)] \right\}.
\] (57)

4.4 Extracting quasi-invariants

Let us turn to Eq.(52) and consider the first term. This term together with the second one in Eq.(18) gives the part of the meson Lagrangian which is responsible for the explicit symmetry breaking. Using the gap-equations one obtains

\[
\mathcal{L}'_{SB} = -\frac{1}{4G}\left(\frac{m_u}{m_a} + \frac{m_d}{m_d} \right) (\sigma_a^2 + \pi_a^2).
\] (58)

This result is in obvious contradiction with Eq.(29). But this should not surprise us because we already know that the proper-time regularization destroys the explicit symmetry breaking pattern of the theory. Fortunately, this part of the bosonic Lagrangian can be corrected by the appropriate counterterm from the polynomial \( P \) which is unambiguously fixed by the chiral WT identities. The following circumstances do extremely simplify the problem:

1. The symmetry breaking part of the NJL Lagrangian must be proportional to the \( J_0 \) integrals and not to any other \( J_n \). This follows from the form of the gap-equations and from the symmetry breaking pattern of the model. Thus, all steps in the asymptotic expansion of the heat kernel, excluding the first one, must be chiral symmetric.

2. It is possible to reduce the problem of finding the explicit form of these chiral invariants to the problem of deriving quasi-invariants, i.e., groups of
terms which would be chiral invariant if one would add to them the deficient terms depending explicitly on the current quark masses.

The first part of this program requires resummations. The second one can be solved by considering corresponding counterterms in the polynomial $P$.

To see these ideas at work let us restrict ourselves to the easiest case, in which the first quasi-invariants will be obtained. We shall carry out the necessary arguments in detail only up to the second step in the asymptotic expansion; the generalization to the next steps will be more tedious but straightforward. Let us consider the third term in Eq.(50), quadratic in $Y$, which we denote by $b_2$. First, one has to perform the $T$ integration. Using formula (47) for $c(T)$, and noting that the part of the integrand $\sim \sinh(T\alpha)$ in that expression does not contribute to Eq.(50), since the trace over isospin matrices vanishes, one obtains

$$b_2 = \frac{1}{(16\pi)^2} \text{tr} \left\{ \left[ J_1(m_u^2) + J_1(m_d^2) \right] Y^2 + Y\tau_3Y\tau_3 \right\}$$

$$+ \frac{1}{(16\pi)^2} \text{tr} \left\{ \left[ J_1(m_u^2) - J_1(m_d^2) \right] \tau_3Y^2 \right\}$$

$$+ \frac{1}{(16\pi)^2} \text{tr} \left\{ \left[ J_0(m_d^2) - J_0(m_u^2) \right] \left( Y^2 - Y\tau_3Y\tau_3 \right) \right\}.$$ 

We use now Eq.(56) to recast this expression in the form

$$b_2 = \frac{2}{(16\pi)^2} \text{tr} \left\{ \left[ J_1(m_u^2) + J_1(m_d^2) \right] Y^2 + \left[ J_1(m_u^2) - J_1(m_d^2) \right] \tau_3Y^2 \right\}$$

$$+ Q \left( Y^2 - Y\tau_3Y\tau_3 \right)$$.

where

$$Q = \frac{1}{2M} \left\{ K \left[ J_1(m_u^2) - J_1(m_d^2) \right] + \frac{2\Lambda^4}{(\Lambda^2 + m_u^2)(\Lambda^2 + m_d^2)} \right\}$$

$$= -\frac{M^2\Lambda^4}{K(\Lambda^2 + K)^3} + \ldots .$$

The second term in Eq.(60) contributes as $\sim M$ and the third one as $\sim M^2$. It means that these terms contribute to the next and next to the next orders of the asymptotic expansion. At the level of the considered approximation
one has to take into account only the first term in Eq.(60). The trace tr(Y^2) is equal to
\[
\text{tr}(Y^2) = 4N_c \left\{ (\partial_\tau \sigma_a)^2 + (\partial_\tau \pi_a)^2 + \left[ \sigma_a^2 + \pi_a^2 + 2(m_0 \sigma_0 + m_3 \sigma_3) \right]^2 \right. \\
+ 4 \left[ \bar{\pi}^2 (\sigma_i + m_i)^2 - (\bar{\pi} \bar{\sigma} + m_3 \pi_3)^2 + (\pi_i \pi_0 + \sigma_i (\sigma_0 + m_0))^2 \right. \\
+ \left. m_3^2 \sigma_0^2 + 2m_3 \sigma_0 (\pi_0 \pi_3 + \sigma_3 (\sigma_0 + m_0)) \right\}.
\]
(62)

It is not difficult to recognize a quasi-invariant [σ^2 + \bar{\pi}^2 + 2(m_0 \sigma_0 + m_3 \sigma_3)]^2 in the first line. Indeed, the chiral transformations (13) leave invariant the expression [σ^2 + \bar{\pi}^2 + 2(\Delta_0 \sigma_0 + \Delta_3 \sigma_3)]^2 which is obtained from the previous one by adding the terms proportional to the current quark masses \(\hat{m}_0\) and \(\hat{m}_3\). The second and the third lines in Eq.(62) do not have a quasi-invariant form. However at this stage there are contributions from \(b_1\) (see the term \(\sim (\sigma_0 \sigma_3 + \pi_0 \pi_3)\) in Eq.(52)) and from the polynomial \(P\) (see the counterterm (17)), where in the latter one has to take into account only the parts which contribute as \([J_1(m_0^2) + J_1(m_3^2)]\) (see Eq.(56)). As a result we obtain a quasi-invariant structure:
\[
(\pi_0^2 + \bar{\sigma}^2 + 2m_i \sigma_i)(\sigma_0^2 + \bar{\pi}^2 + 2m_0 \sigma_0) - [\pi_i (\sigma_i + m_i) - \pi_0 (\sigma_0 + m_0)]^2.
\]
(63)

We can now adjust this expression to a new one which is invariant under chiral transformations by including a corresponding counterterm to the polynomial \(P\). The possible counterterm which compensates the non-zero variation of (13) is easily constructed through the replacement: \(m_0 \rightarrow \Delta_0\) directly in Eq.(63). In agreement with the symmetry requirement we obtain now
\[
(\pi_0^2 + \bar{\sigma}^2 + 2\Delta_0 \sigma_i)(\sigma_0^2 + \bar{\pi}^2 + 2\Delta_0 \sigma_0) - [\pi_i (\sigma_i + \Delta_i) - \pi_0 (\sigma_0 + \Delta_0)]^2
\]
(64)

instead of (63). Finally, if we collect all terms which contribute to the considered approximation, we obtain a Lagrange density of the bosonized NJL model in the form
\[
\mathcal{L}_{\text{coll}} = -\frac{\hat{m}_3 (\pi_0^2 + \bar{\sigma}^2)}{2G\Delta_3} - \frac{\hat{m}_0 (\sigma_0^2 + \bar{\pi}^2)}{2G\Delta_0} - \frac{N_c}{16\pi^2} \left[ J_1(m_0^2) + J_1(m_3^2) \right] \\
\times \left\{ (\partial_\tau \sigma_a)^2 + (\partial_\tau \pi_a)^2 + \left[ \sigma_a^2 + \pi_a^2 + 2(\Delta_0 \sigma_0 + \Delta_3 \sigma_3) \right]^2 \right. \\
+ 4 \left[ (\pi_0^2 + \bar{\sigma}^2 + 2\Delta_3 \sigma_3)(\sigma_0^2 + \bar{\pi}^2 + 2\Delta_0 \sigma_0) \right. \\
- \left. (\pi_i (\sigma_i + \Delta_i) - \pi_0 (\sigma_0 + \Delta_0))^2 \right\}.
\]
(65)
In closing, we remark that the method developed so far allows for an accurate derivation of the bosonized NJL Lagrangian which describes the low-energy dynamics of Goldstone bosons coupled with scalar fields. This Lagrangian consistently summarizes the effect of the isospin symmetry breaking, $m_u \neq m_d$, and allows for a calculation of its physical consequences. The expression (65) fulfills the chiral symmetry requirements of the fundamental NJL quark Lagrangian permitting us to consider it as an effective low-energy approximation to the chiral quark dynamics described by the Lagrange density (2).

5 Coupling constants and masses

For completeness, we would like to discuss here the coupling constants and masses of the physical collective modes in the Lagrangian (65). We need to bring this expression to the standard form, i.e., to diagonalize its bilinear part and to introduce the physical states by the corresponding field renormalizations. By rewriting the Lagrangian in terms of physical states we complete our work in its analytical part. We are not going to give here the numerical discussion of the resulting mass formulas for composite mesons or to fix the parameters of the model. This numerical part is not relevant for our consideration here and will be done elsewhere together with the discussion of some physical problems.

Let us consider the part of the Lagrange density (65) which is quadratic in the fields

$$\begin{align*}
\mathcal{L}_{\text{free}} &= -\frac{\hat{m}_3(\pi_0^2 + \sigma^2)}{2G\Delta_3} - \frac{\hat{m}_0(\sigma_0^2 + \pi^2)}{2G\Delta_0} - \frac{I}{4} \left[ (\partial_r \sigma_a)^2 + (\partial_r \pi_a)^2 \right] \\
&- I \left[ \Delta_0^2(\sigma_0^2 - \pi_0^2) + \Delta_0^2(\sigma_3^2 - \pi_3^2) + 2\Delta_0\Delta_3(\pi_0\pi_3 + 3\sigma_0\sigma_3) \right],
\end{align*}$$

(66)

where

$$I = \frac{N_c}{4\pi^2} \left[ J_1(m_0^2) + J_1(m_3^2) \right].$$

(67)

There are two kinds of mixing: $\sigma_0-\sigma_3$ and $\pi_0-\pi_3$. Both of them arise as a result of isospin symmetry breaking, and both are diagonalized by the orthogonal rotation:

$$\begin{pmatrix}
\phi_0 \\
\phi_3
\end{pmatrix} = \begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix}
\tilde{\phi}_0 \\
\tilde{\phi}_3
\end{pmatrix},$$

(68)
where we have chosen the general notation $\phi$ which means either $\sigma$ or $\pi$, depending on the case under consideration. The rotation angle specified by the $\theta$'s is equal to $\theta_s$ for scalars and to $\theta_p$ for pseudoscalars. The quadratic form which has to be diagonalized is $-(A\phi_0^2 + C\phi_0\phi_3 + B\phi_3^2)$. Then the angle of the rotation is fixed by the condition: $\tan(2\theta) = C/(B-A)$. From Eq.(66) one obtains

$$\tan 2\theta_s = -\frac{6\Delta_0\Delta_3}{\omega}, \quad \tan 2\theta_p = \frac{2\Delta_0\Delta_3}{\omega},$$

where

$$\omega = \Delta_0^2 - \Delta_3^2 + \frac{1}{2GI} \left( \frac{\tilde{m}_0}{\Delta_0} - \frac{\tilde{m}_3}{\Delta_3} \right).$$

For convenience let us rename $\phi_{1,2} = \tilde{\phi}_{1,2}$. Having introduced the relevant fields, we are now in position to write down the free Lagrangian (66) as

$$\mathcal{L}_{\text{free}} = -\frac{\tilde{m}_3(\tilde{\sigma}_1^2 + \tilde{\sigma}_2^2)}{2G\Delta_3} - \frac{\tilde{m}_0(\tilde{\pi}_1^2 + \tilde{\pi}_2^2)}{2G\Delta_0} - \frac{I}{4} \left[ (\partial_r \tilde{\sigma}_a)^2 + (\partial_r \tilde{\pi}_a)^2 \right] - \left( \tilde{A}_s \tilde{\sigma}_0^2 + \tilde{B}_s \tilde{\sigma}_3^2 + \tilde{A}_p \tilde{\pi}_0^2 + \tilde{B}_p \tilde{\pi}_3^2 \right).$$

The coefficients with tilde are defined as

$$\tilde{A} = A\cos^2 \theta + B\sin^2 \theta - C\sin \theta \cos \theta,$$

$$\tilde{B} = A\sin^2 \theta + B\cos^2 \theta + C\sin \theta \cos \theta,$$

with the following values of coefficients $A$, $B$, and $C$ for the scalar and pseudoscalar cases respectively (see Eq.(66))

$$A_s = \frac{\tilde{m}_0}{2G\Delta_0} + I\Delta_0^2, \quad B_s = \frac{\tilde{m}_3}{2G\Delta_3} + I\Delta_3^2, \quad C_s = 6I\Delta_0\Delta_3,$$

$$A_p = \frac{\tilde{m}_3}{2G\Delta_3} - I\Delta_0^2, \quad B_p = \frac{\tilde{m}_0}{2G\Delta_0} - I\Delta_3^2, \quad C_p = 2I\Delta_0\Delta_3.$$

The physical states are defined by bringing the kinetic terms in the $\mathcal{L}_{\text{free}}$ to the standard form, i.e.,

$$\tilde{\sigma}_a = g\sigma_a^{(ph)}, \quad \tilde{\pi}_a = g\pi_a^{(ph)}, \quad I g^2 = 2.$$

Finally these replacements determine the masses to be

$$m_{\sigma_{1,2}}^2 = \frac{\tilde{m}_3 g^2}{G\Delta_3}, \quad m_{\sigma_3}^2 = 2g^2 \tilde{B}_s, \quad m_{\pi_0}^2 = 2g^2 \tilde{A}_s.$$
\[ m_{\pi_{1,2}}^2 = \frac{\hat{m}_u g^2}{G\Delta_0}, \quad m_{\pi_3}^2 = 2g^2\hat{B}_p, \quad m_{\pi_0}^2 = 2g^2\hat{A}_p, \quad (77) \]

They are different, for example, from the results obtained in \[6\] and \[7, 12\]. They differ also from the ones obtained on the basis of direct calculations with Feynman diagrams \[13, 14\] (and references therein). This can be easily understood, recalling the general symmetry properties of the model discussed in section 2: the chiral transformations of meson fields depend explicitly on the current quark masses, see Eq.(13), but the Dirac operator, (15) does not. Therefore the Feynman amplitudes calculated directly and solely from \( \ln \det D \) will never depend on the current quark masses and consequently not fulfill the symmetry requirements. To have the correct behavior one needs chiral invariant combinations of fields, i.e. the missing current quark mass terms have to be included in the correcting polynomial \( P \), via the WT identities, as extensively discussed in this paper.

6 Concluding remarks

The mesonic degrees of freedom \( \sigma_\alpha \), and \( \pi_\alpha \) are the elementary collective excitations of the non-trivial ground state of the NJL model. The chiral dynamics of this collective modes should be consistent with the standard picture of a spontaneously broken chiral symmetry, which is naturally linked to the explicit symmetry breaking pattern of the underlying four-quark Lagrangian. In this paper we studied the NJL model with the broken chiral and diagonal \( U(2) \) flavour symmetry, choosing the quark mass matrix to be in the non-degenerate form: \( \hat{m}_u \neq \hat{m}_d \). We suggested and discussed in considerable detail a new method to derive the effective bosonic Lagrangian starting from the pure quark action with the chiral four fermion interactions. The collective bosonic modes have been introduced through the generating functional. It is known that quite often the identification of the path integral over quark fields with a determinant is problematic. The case under consideration was not an exclusion. The method we developed deals with the accurate definition and evaluation of the low energy part of the fermion determinant. We studied the real part of the quark effective action and obtained the low energy expansion for it. To get the expansion we have chosen the Schwinger’s proper-time representation of the fermion determinant as a starting point. We have shown that in this case one can not use the standard proper-time
expansion. The mass matrix of quark fields, which does not commute with the mesonic part of the Dirac operator, has first to be factorized from the heat kernel. We used the time-ordered exponent to separate the part of the heat kernel which depends on the meson fields. In this case the expansion of the time-ordered exponent leads directly to the gap-equations.

On the basis of our recent results [1, 2] we knew that a proper-time representation failed in the case of manifestly broken chiral symmetry. Therefore we used the WT identities to correct the \( \hat{m} \)-dependent part of the fermion determinant. In this part of the work we extended the method developed in our previous papers to the isospin asymmetric case. However an additional new problem arose: we found that the naive expansion of the time-ordered exponent has to be modified by additional resummations which touch the terms of the effective action which are proportional to the difference of constituent quark masses \( m_d - m_u \). These resummations are necessary to form groups of terms which are invariant under the chiral transformations.

The gap-equation ansatz made the picture even more intricate. We have found that because of resummations the contributions of linear terms in scalar fields are important for chiral symmetry. As a consequence one has to add the correcting counterterm to the effective action to put the ansatz in agreement with the WT requirements.

The method presented in this work is a necessary mathematical tool which can be applied to the more interesting, from the physical point of view, case of explicitly broken chiral \( SU(3) \times SU(3) \) symmetry. These versions of the NJL model have been already studied but without a careful treatment of the questions indicated in this paper.

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Appendix

Let us obtain here the first four terms in the asymptotic long-wave expansion of the operator

$$\text{tr} \left[ e^{-T(M\tau_3 + A)} \right]. \quad (78)$$

This operator is a part of the heat kernel in the expression (71). The result, as we already know, has been used in Eq.(46). We begin here by making use of the well known formalism of the “time”-ordered exponent, or $P$-exponent:

$$\text{tr} \left[ e^{-T(M\tau_3 + A)} \right] = \text{tr} \left\{ e^{-T(M\tau_3)} P \left[ \exp \left( - \int_0^T ds A(s) \right) \right] \right\} = \sum_{i=0}^{\infty} A_i, \quad (79)$$

where the $P$-exponent is defined by Eq.(45), and $A(s)$, according to the definition (44), is equal to

$$A(s) = e^{sM\tau_3} A e^{-sM\tau_3} = \frac{1}{2} \left[ A + \tau_3 A \tau_3 + e^{2sM\tau_3} (A - \tau_3 A \tau_3) \right]. \quad (80)$$

One can easily verify that the second term of this expression does not contribute to the first nontrivial term of the $P$ exponent, for its isotopic trace vanishes. Thus the first two terms in Eq.(46) are obvious. Let us consider now the integral

$$\int_0^T ds \int_0^s ds_1 A(s) A(s_1)$$

$$= \frac{1}{4M} \int_0^T ds \left[ sM(A^2 + A\tau_3 A \tau_3 + \tau_3 A \tau_3 A + \tau_3 A^2 \tau_3) \right. $$

$$+ sMe^{2sM\tau_3}(A^2 + A\tau_3 A \tau_3 - \tau_3 A \tau_3 A - \tau_3 A^2 \tau_3)$$

$$+ \left. 2 \sinh(sM)e^{sM\tau_3}(A^2 - A \tau_3 A \tau_3) \right]. \quad (81)$$

The explicit form for the third term in the expansion of Eq.(79) is given by

$$A_2 = \text{tr} \left\{ \frac{1}{2M} e^{-TM\tau_3} \int_0^T ds \left[ sM(A^2 + A\tau_3 A \tau_3) \right. $$

$$ + \left. \sinh(sM)e^{sM\tau_3}(A^2 - A \tau_3 A \tau_3) \right] \right\}. \quad (82)$$

Note that the second term in Eq.(81) does not contribute to $A_2$, by the property of the trace, which vanishes. If we integrate over $s$, we find from
the third term in the expansion (46). The calculation of the coefficients \(A_i\) becomes a somewhat more and more difficult exercise at every new step in (79). To finish this Appendix we give the result of such calculations for the coefficient \(A_3\):

\[
A_3 = -\frac{T^3}{8} \text{tr} \left\{ e^{-TM\tau_3} \left[ \frac{A}{3} \{A, \tau_3\}^2 + c_2(T)A \{A, \tau_3\} \tau_3, A \right] + c_3(T)(A^3 - A\tau_3 A\tau_3 A) \right\},
\]

(83)

where the matrices \(c_2(T)\) and \(c_3(T)\) may be written as

\[
c_2(T) = \frac{\tau_3}{2T^3M^3} \left[ 1 + TM\tau_3 + (TM\tau_3 - 1)e^{2TM\tau_3} \right],
\]

(84)

\[
c_3(T) = \frac{\tau_3}{2T^3M^3} \left[ e^{2TM\tau_3} - 1 - 2TM\tau_3 - 2T^2M^2 \right].
\]

(85)

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