Signal Reconstruction from Frame and Sampling Erasures

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Received: 1 September 2014 / Revised: 16 March 2015 / Published online: 22 April 2015
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Abstract We give some new methods for perfect reconstruction from frame and sampling erasures in a small number of steps. By bridging an erasure set we mean replacing the erased Fourier coefficients of a function with respect to a frame by appropriate linear combinations of the non-erased coefficients. We prove that if a minimal redundancy condition is satisfied bridging can always be done to make the reduced error operator nilpotent of index 2 using a bridge set of indices no larger than the cardinality of the erasure set. This results in perfect reconstruction of the erased coefficients. We also obtain a new formula for the inverse of an invertible partial reconstruction operator. This leads to a second method of perfect reconstruction from frame and sampling erasures in a small number of steps. This gives an alternative to the bridging method for many (but not all) cases. The methods we use employ matrix techniques only of the order of the cardinality of the erasure set, and are applicable to rather large finite erasure sets for infinite frames and sampling schemes as well as for finite frame theory. These methods are usually more efficient than inverting the frame operator for the remaining coefficients because the size of the erasure set is usually much smaller than the dimension of the underlying Hilbert space. Some new classification theorems for frames are obtained and some new methods of measuring redundancy are introduced based on our bridging theory.

Keywords Finite frame · Omission · Erasure · Reconstruction · Bridging
Mathematics Subject Classification 46G10 · 46L07 · 46L10 · 46L51 · 47A20 · 42C15 · 46B15 · 46B25 · 47B48

1 Introduction

Frame and sampling techniques are often used to analyze and digitize signals and images when they are represented as vectors or functions in a Hilbert space. There is a large literature on the pure and applied mathematics of this subject (c.f. [3,10, 11,15,17,21]). A number of articles have been written on problems and methods for reconstruction from erasures (c.f. [5,7,9,12,18,23]). We give some new methods for perfect reconstruction from frame and sampling erasures in a small number of steps.

Let \( \{f_j\} \) be a Parseval frame for a Hilbert space \( \mathcal{H} \), or more generally let \( \{f_j, g_j\} \) be a dual frame pair. (See definitions below.) Let \( f \) be a vector in \( \mathcal{H} \), and let \( \Lambda \) be a finite subset of the index set. If \( f \) is analyzed with \( \{g_j\} \) and if the frame coefficients for \( \Lambda \) are erased, then by bridging the erasures we mean replacing the erased coefficients with appropriate linear combinations of the non-erased coefficients. We show that bridging can always be done to make the resulting reduced error operator nilpotent of index 2 using a bridge set no larger than the cardinality of the erasure set. From this, an algorithm for perfect reconstruction from erasures follows easily. The resulting algorithms use only finite matrix methods of order the cardinality of the erasure set and are thus more efficient than inverting the frame operator for a reduced frame. Frames can be infinite, such as Gabor and wavelet frames. The only delimiter in a computational sense seems to be the size of the erasure set, which we take to be finite in this article. This method adapts equally well to sampling theory, such as Shannon–Whittaker sampling theory ([4,19,29]). Our bridging results suggest some new classification techniques and new measures of redundancy for finite and infinite frames and sampling schemes.

We conclude with an apparently new “direct” formula for inverting the partial reconstruction operator. If we are dealing with a frame of length \( N \) in an \( n \)-dimensional Hilbert space, then a common method of inverting the frame operator of the remaining coefficients requires an \( n \times n \) matrix inversion. However with the techniques in Sect. 6 of this paper, if we are dealing with \( L \) erasures, perfect reconstruction is possible by inverting an \( L \times L \) matrix. This is particularly useful when \( L \) is small compared to \( n \).

We would like to thank Deguang Han for useful discussions on this work, and for piquing our interest in frame erasure problems in the recent interesting article [27]. We thank Stephen Rowe for useful Matlab and programming advice in the experimental phases of this work. Many of our mathematical results were obtained after numerous computer experiments. We thank the referees for several helpful suggestions.

2 Preliminaries

A frame \( F \) for a Hilbert space \( \mathcal{H} \) is a sequence of vectors \( \{f_j\} \subset \mathcal{H} \) indexed by a finite or countable index set \( J \) for which there exist constants \( 0 < A \leq B < \infty \) such that, for every \( f \in \mathcal{H} \),
\[ A \|f\|_2^2 \leq \sum_{j \in J} |\langle f, f_j \rangle|^2 \leq B \|f\|_2^2. \]  

(2.1)

The constants \( A \) and \( B \) are called lower and upper frame bounds, respectively. The optimal lower frame bound is the supremum over all lower frame bounds, and the optimal upper frame bound is the infimum over all upper frame bounds. If \( \{f_j\}_{j \in J} \) has optimal frame bounds \( A_0 \) and \( B_0 \), and \( A_0 = B_0 \), then \( \{f_j\} \) is called a tight frame. If \( A_0 = B_0 = 1 \), \( \{f_j\} \) is called a Parseval frame. If we only require that a sequence \( \{f_j\} \) satisfies the upper bound condition in (2.1), then \( \{f_j\} \) is called a Bessel sequence. A frame which is a Schauder basis is called a Riesz basis. Orthonormal bases are special cases of Parseval frames. A Parseval frame \( \{f_j\} \) for a Hilbert space \( \mathcal{H} \) is an orthonormal basis if and only if each \( f_j \) is a unit vector.

The analysis operator \( \Theta \) for a Bessel sequence \( \{f_j\} \) is a bounded linear operator from \( \mathcal{H} \) to \( \ell^2(\mathbb{J}) \) defined by

\[ \Theta f = \sum_{j \in \mathbb{J}} \langle f, f_j \rangle e_j, \]  

(2.2)

where \( \{e_j\} \) is the standard orthonormal basis for \( \ell^2(\mathbb{J}) \). It is easily verified that

\[ \Theta^* e_j = f_j, \quad \forall j \in \mathbb{J}. \]

The Hilbert space adjoint \( \Theta^* \) is called the synthesis operator for \( \{f_j\} \). The positive operator \( S := \Theta^* \Theta : \mathcal{H} \rightarrow \mathcal{H} \) is called the frame operator, or sometimes the Bessel operator if the Bessel sequence is not a frame, and we have

\[ Sf = \sum_{j \in \mathbb{J}} \langle f, f_j \rangle f_j, \quad \forall f \in \mathcal{H}. \]  

(2.3)

We can also use rank one operator notation \((x \otimes y)(z) = \langle z, y \rangle x\) to write (2.3) as

\[ S = \sum_{j \in \mathbb{J}} f_j \otimes f_j. \]

Similarly, \( \Theta = \sum_{j \in \mathbb{J}} e_j \otimes f_j \) and \( \Theta^* = \sum_{j \in \mathbb{J}} f_j \otimes e_j \). The operator \( \Theta \Theta^* : \ell^2(\mathbb{J}) \rightarrow \ell^2(\mathbb{J}) \) is called the Gramian operator (or Gram Matrix) and is denoted \( Gr(F) \). Then

\[ Gr(F) = \sum_{j,k \in \mathbb{J}} \langle f_k, f_j \rangle e_j \otimes e_k = \left( \langle f_k, f_j \rangle \right)_{j,k}. \]

In outer product notation which is often used in frame theory, these formulas become \( S = \sum_{j \in \mathbb{J}} f_j f_j^*, \quad \Theta = \sum_{j \in \mathbb{J}} e_j f_j^*, \quad \Theta^* = \sum_{j \in \mathbb{J}} f_j e_j^*, \) and \( Gr(F) = \sum_{j,k \in \mathbb{J}} (f_k, f_j) e_j e_k^* \).

Given a frame \( \{f_j\}_{j \in \mathbb{J}} \) for a Hilbert space \( \mathcal{H} \), from (2.3) we obtain the reconstruction formula (or frame decomposition)
\[ f = \sum_{j \in J} \langle f, S^{-1} f_j \rangle f_j = \sum_{j \in J} \langle f, f_j \rangle S^{-1} f_j \quad \forall f \in \mathcal{H}, \]

where the convergence is in the norm of \( \mathcal{H} \). The frame \( \{ S^{-1} f_j \} \) is called the canonical or standard dual of \( \{ f_j \} \). In the case that \( \{ f_j \} \) is a Parseval frame for \( \mathcal{H} \), we have \( S = I \) and hence \( f = \sum_{j \in J} \langle f, f_j \rangle f_j, \quad \forall f \in \mathcal{H} \). More generally, if a Bessel sequence \( \{ g_j \} \) satisfies a reconstruction formula

\[ f = \sum_{j \in J} \langle f, g_j \rangle f_j \quad \forall f \in \mathcal{H} \]

then \( \{ g_j \} \) is called an alternate dual of \( \{ f_j \} \). (Hence \( \{ g_j \} \) is also necessarily a frame.) The canonical and alternate duals are often simply referred to as duals, and \( (F, G) := \{ f_j, g_j \}_{j \in J} \) is called a dual frame pair. The second frame \( G \) in the ordered pair will be called the analysis frame and the first frame \( F \) will be called the synthesis frame.

It will be convenient to define a frame pair which is not necessarily a dual frame pair to be simply a pair of frames \( F = \{ f_j \} \) and \( G = \{ g_j \} \) indexed by the same set \( J \) for which the operator \( \tilde{S} f = \sum_{j \in J} \langle f, g_j \rangle f_j \) is invertible. We will call the operator \( \tilde{S} \) the cross-frame operator for \( F \) and \( G \), and the operator \( \text{Gr}(F, G) = \sum \langle f_k, g_j \rangle e_j \otimes e_k \) the cross-Gramian. If \( \{ f_1, \ldots, f_L \} \) and \( \{ g_1, \ldots, g_L \} \) are finite sets of vectors, we will write \( \text{Gr}(\{ f_1, \ldots, f_L \}, \{ g_1, \ldots, g_L \}) \) for the cross-Gram matrix,

\begin{equation}
\text{Gr}(\{ f_1, \ldots, f_L \}, \{ g_1, \ldots, g_L \}) = \left( \langle f_k, g_j \rangle \right)_{j,k} := \begin{pmatrix}
    \langle f_1, g_1 \rangle & \langle f_2, g_1 \rangle & \cdots & \langle f_L, g_1 \rangle \\
    \langle f_1, g_2 \rangle & \langle f_2, g_2 \rangle & \cdots & \langle f_L, g_2 \rangle \\
    \vdots & \vdots & \ddots & \vdots \\
    \langle f_1, g_L \rangle & \langle f_2, g_L \rangle & \cdots & \langle f_L, g_L \rangle 
\end{pmatrix}.
\end{equation}

We will use this notation mainly when \( \{ f_j \} \) and \( \{ g_j \} \) are frames or subsets of frames. It is useful to note that if \( \{ f_1, \ldots, f_L \} \) and \( \{ g_1, \ldots, g_L \} \) are two bases for the same Hilbert space \( \mathcal{H} \), then \( \text{Gr}(\{ f_1, \ldots, f_L \}, \{ g_1, \ldots, g_L \}) \) is invertible. Indeed, if \( \{ e_j \} \) is an orthonormal basis for \( \mathcal{H} \), and \( A \) and \( B \) are invertible matrices with \( Ae_j = f_j \) and \( Be_j = g_j \), then \( \text{Gr}(\{ f_1, \ldots, f_L \}, \{ g_1, \ldots, g_L \}) \) is just the matrix of \( B^*A \) with respect to \( \{ e_j \} \).

### 3 Nilpotent Bridging

Let \( F = \{ f_j \}_{j \in J} \) be a frame. An erasure set for \( F \) is defined to be simply a finite subset of \( J \). We say that an erasure set \( \Lambda \) for a frame \( F \) satisfies the minimal redundancy condition if \( \text{span}\{ f_j : j \notin \Lambda \} = \mathcal{H} \). Theorem 5.4.7 in [15] shows that \( \Lambda \) satisfies the minimal redundancy condition if and only if \( \{ f_j : j \in \Lambda^c \} \) is also a frame for \( \mathcal{H} \). For a Parseval frame \( F \) and an erasure set \( \Lambda \) satisfying the minimal redundancy condition, \( \{ f_j \}_{j \in \Lambda^c} \) still forms a frame, and the operator \( R_\Lambda = \sum_{j \in \Lambda^c} f_j \otimes f_j \) is the frame operator for the reduced frame \( \{ f_j \}_{j \in \Lambda^c} \), hence it is invertible. Let \( f_R = R_\Lambda f \)
be the partial reconstruction of the vector \( f \). It is possible to reconstruct \( f \) from the “good” Fourier coefficients by \( f = R^{-1}_A f_R \).

Let \( (F, G) = \{ f_j, g_j \}_{j \in \mathbb{J}} \) be a dual frame pair. As above, an erasure set for \((F, G)\) is simply a finite subset of \( \mathbb{J} \). We say that an erasure set \( \Lambda \) satisfies the minimal redundancy condition for the dual frame pair \((F, G)\) if \( \operatorname{span}\{g_j : j \notin \Lambda \} = \mathcal{H} \). We point out that the minimal redundancy condition for a dual frame pair \((F, G)\) as we have defined it is a condition on only the analysis frame \( G \). The redundancy properties of the synthesis frame \( F \) play a role here only in that it is required to be a dual frame to \( G \). For the special case where \( G \) is the standard dual of \( F \), \( F \) and \( G \) have the same linear redundancy properties. The Parseval frame case, where \( F = G \), is a special case of this. For a dual frame pair \((F, G)\), if \( \Lambda \) satisfies the minimal redundancy condition then since \( \{g_j : j \in \Lambda^c\} \) is a frame for \( \mathcal{H} \) it has some frame dual (in general many duals) that will yield the reconstruction of \( f \) from the coefficients over \( \Lambda^c \), so there is enough information in \( \{\langle f, g_j \rangle : j \in \Lambda^c\} \) to reconstruct \( f \). On the other hand if \( \Lambda \) fails the minimal redundancy condition then some nonzero vector \( f \) will be orthogonal to \( g_j \) for all \( j \in \Lambda^c \), and hence no reconstruction of \( f \) is possible using only the coefficients \( \{\langle f, g_j \rangle : j \in \Lambda^c\} \). This justifies the use of the word “minimal” in the description of the minimal redundancy condition.

Given a dual frame pair \((F, G)\) indexed by \( \mathbb{J} = \{1, 2, \ldots, N\} \) and an erasure set \( \Lambda \) satisfying the minimal redundancy condition, the partial reconstruction operator \( R_\Lambda := \sum_{j \in \Lambda} f_j \otimes g_j \) need not be invertible. In fact invertibility of \( R_\Lambda \) can fail even if both \( F \) and \( G \) separately satisfy the minimal redundancy condition for \( \Lambda \). The following simple example shows that this can happen and \( R_\Lambda \) can even be the zero operator.

**Example 3.1** Let \( \{f_j, g_j\}_{j=1}^N \) be a dual frame pair. Suppose

\[
\begin{align*}
f_j &= f_{j+N} = f_{j+2N} \quad 1 \leq j \leq N \\
g_{j+N} &= -g_j \quad 1 \leq j \leq N \\
g_{j+2N} &= g_j \quad 1 \leq j \leq N.
\end{align*}
\]

Then, it is easily verified that \( \{f_j, g_j\}_{j=1}^3 \) is a dual frame pair, and \( \Lambda = \{1, 2, \ldots, N\} \) satisfies the minimal redundancy condition with respect to both frames. However,

\[
R_\Lambda = \sum_{j=N+1}^{3N} f_j \otimes g_j
= \sum_{j=N+1}^{2N} f_j \otimes g_j + \sum_{j=2N+1}^{3N} f_j \otimes g_j
= \sum_{j=1}^{3N} f_j \otimes (-g_j)
+ \sum_{j=1}^{N} f_j \otimes g_j = 0.
\]

\( \Box \)

Even when \( R_\Lambda \) is invertible, computing \( R^{-1}_\Lambda \) can be a computationally costly process. We denote by \( f_R \) the partial reconstruction of a vector \( f \) given by \( f_R = R_\Lambda f = \sum_{j \in \Lambda^c} \langle f, g_j \rangle f_j \). The error for the partial reconstruction is \( f_E = f - f_R \),
and the associated error operator for the partial reconstruction is \( E_\Lambda = I - R_\Lambda = \sum_{j \in \Lambda} f_j \otimes g_j \). Then \( R_\Lambda^{-1} = (I - E_\Lambda)^{-1} \), and if the norm, or more generally the spectral radius of \( E_\Lambda \) is strictly less than 1 then \( R_\Lambda^{-1} \) can be computed using the Neumann series expansion \( R_\Lambda^{-1} = I + E + E^2 + \cdots = \sum_{j=0}^\infty E^j \).

For certain very special cases \((F, G)\), with corresponding erasure set \( \Lambda \), the error operator, \( E_\Lambda \) will be nilpotent of index 2, (i.e. \( E_\Lambda^2 = 0 \)) such as the example below. In this case, \( R_\Lambda^{-1} = I + E_\Lambda \), and moreover, the error \( f_E \) of \( f \), can be obtained by applying the error operator to the partial reconstruction \( f_R \) instead of \( f \). (That is, \( f_E = E_\Lambda f = E_\Lambda(f_E + f_R) = E_\Lambda^2 f + E_\Lambda f_R = E_\Lambda f_R \).)

**Example 3.2** Let \( \{e_1, e_2\} \) be the standard orthonormal basis for \( \mathbb{C}^2 \). Let \( F = \{e_1, -e_1, e_1, e_2\} \) and \( G = \{e_2, e_2, e_1, e_2\} \). Let \( \Lambda = \{1\} \). Then \( E_\Lambda = e_1 \otimes e_2 \). So

\[
E_\Lambda^2 = (e_1 \otimes e_2)(e_1 \otimes e_2) = \langle e_1, e_2 \rangle (e_1 \otimes e_2) = 0.
\]

Therefore \( R_\Lambda^{-1} = I + E_\Lambda \). \( \square \)

Let \((F, G)\) be a dual frame pair for a Hilbert space \( \mathcal{H} \). Let \( \Lambda \) be an erasure set, \( \Omega \subset \Lambda^c \), and \( f \in \mathcal{H} \). The main idea behind bridging is to replace each erased coefficient \( \langle f, g_j \rangle \) for \( j \in \Lambda \) with \( \langle f, g_j' \rangle \) for \( g_j' \in \text{span}\{g_k : k \in \Omega\} \) for some \( \Omega \subset \Lambda^c \). (i.e. \( \langle f, g_j' \rangle \) is a weighted average of the \( \langle f, g_k \rangle \) for \( k \in \Omega \).) Any set \( \Omega \subset \Lambda^c \) will be called a bridge set. The point of this preconditioning is to make the inverse problem of recovering \( f \) more efficient than simply inverting \( R_\Lambda \).

The partial reconstruction with bridging is

\[
\tilde{f} = f_R + f_B
\]

where \( f_B = \sum_{j \in \Lambda} \langle f, g_j' \rangle f_j \). We call \( f_B \) the bridging supplement and \( B_\Lambda := \sum_{j \in \Lambda} f_j \otimes g_j' \) the bridging supplement operator. The reduced error is \( f_{\tilde{E}} := f - \tilde{f} \), and the associated reduced error operator is \( \tilde{E}_\Lambda = I - R_\Lambda - B_\Lambda \). We have

\[
\tilde{E}_\Lambda f = f_{\tilde{E}} = \sum_{j \in \Lambda} \left( \langle f, g_j - g_j' \rangle f_j \right).
\]

There are various ways to choose the \( g_j' \in \text{span}\{g_j : j \in \Omega\} \), but in this paper, we choose \( g_j' \) so that the reduced error operator is nilpotent of index 2. Then the logic in the sentence just above Example 3.2 will apply, leading to perfect reconstruction. It is straightforward to verify that the condition

\[
f_j \perp (g_k - g_k') \quad \forall j, k \in \Lambda \quad (3.1)
\]

forces the reduced error operator to be nilpotent of index 2. So, writing

\[
g_k' = \sum_{\ell \in \Omega} c^{(k)}_\ell g_\ell \quad (3.2)
\]
we seek coefficients $c^{(k)}_{\ell}$ so that (3.1) is satisfied. We have

$$0 = \langle f_j, g_k - \sum_{\ell \in \Omega} c^{(k)}_{\ell} g_{\ell} \rangle = \langle f_j, g_k \rangle - \sum_{\ell \in \Omega} c^{(k)}_{\ell} \langle f_j, g_{\ell} \rangle.$$  

For each $k \in \Lambda$, we obtain a system of $|\Lambda|$ equations with $|\Omega|$ unknowns:

$$\langle f_j, g_k \rangle = \sum_{\ell \in \Omega} C^{(k)}_{\ell} \langle f_j, g_{\ell} \rangle.$$  

If we enumerate $\Lambda = \{\lambda_j\}_{j=1}^L$ and $\Omega = \{\omega_j\}_{j=1}^M$ we obtain the matrix equation

$$(\begin{pmatrix} \langle f_{\lambda_1}, g_{\omega_1} \rangle \\
\langle f_{\lambda_2}, g_{\omega_1} \rangle \\
\vdots \\
\langle f_{\lambda_L}, g_{\omega_1} \rangle 
\end{pmatrix} \begin{pmatrix} \langle f_{\lambda_1}, g_{\omega_2} \rangle \\
\langle f_{\lambda_2}, g_{\omega_2} \rangle \\
\vdots \\
\langle f_{\lambda_L}, g_{\omega_2} \rangle 
\end{pmatrix} \cdots \begin{pmatrix} \langle f_{\lambda_1}, g_{\omega_M} \rangle \\
\langle f_{\lambda_2}, g_{\omega_M} \rangle \\
\vdots \\
\langle f_{\lambda_L}, g_{\omega_M} \rangle 
\end{pmatrix} \begin{pmatrix} C^{(k)}_{\omega_1} \\
C^{(k)}_{\omega_2} \\
\vdots \\
C^{(k)}_{\omega_M} 
\end{pmatrix} = \begin{pmatrix} \langle f_{\lambda_1}, g_k \rangle \\
\langle f_{\lambda_2}, g_k \rangle \\
\vdots \\
\langle f_{\lambda_L}, g_k \rangle 
\end{pmatrix})$$  

(3.3)

for all $k \in \Lambda$. We call the matrix in (3.3) the bridge matrix and denote it $B(F, G, \Lambda, \Omega)$. Since the bridge matrix is independent of $k$, we can solve for all of the coefficients simultaneously with the equation

$$(\begin{pmatrix} \langle f_{\lambda_j}, g_{\omega_k} \rangle 
\end{pmatrix} C^{(k)}_{\omega_j})_{j,k} = \begin{pmatrix} \langle f_{\lambda_j}, g_{\lambda_k} \rangle 
\end{pmatrix}.$$  

(3.4)

We can rewrite this equation as

$$B(F, G, \Lambda, \Omega)C = B(F, G, \Lambda, \Lambda),$$  

(3.5)

where $C$ denotes our coefficient matrix (actually, $C$ is the matrix of complex conjugates of the coefficients $c^{(k)}_{\omega_j}$ in 3.4).

**Remark 3.3**

1. The transpose of the bridge matrix $B(F, G, \Lambda, \Omega)$ is a skew (i.e. diagonal-disjoint) minor of the cross-Gram matrix $Gr(F, G)$ of the frames $F$ and $G$, and the transpose of $B(F, G, \Lambda, \Lambda)$ is a principle minor of $Gr(F, G)$.

2. The form of the bridge matrix in Eq. (3.3) depends on the particular enumerations one takes of $\Lambda$ and $\Omega$. However, for two different enumerations one bridge matrix will transform into the other by interchanging appropriate rows and columns, and so the norm and the rank of the matrices will be the same. In particular, one will be invertible if and only if the other is.

Given a dual frame pair $(F, G)$, and an erasure set $\Lambda$, a bridge set $\Omega$ is said to satisfy the robust bridging condition (or $\Omega$ is a robust bridge set) if Eq. (3.5) has a solution.

Now, given $f \in \mathcal{H}$,

$$f = f_{\bar{E}} + \tilde{f}.$$
However, $\tilde{E}_\Lambda(f - \tilde{f}) = \tilde{E}_\Lambda^2 f = 0$. Thus, $f_E = \tilde{E}_\Lambda \tilde{f}$, and we can reconstruct $f$ from the good Fourier coefficients by

$$f = \tilde{f} + \tilde{E}_\Lambda \tilde{f}. \quad (3.6)$$

Furthermore, $f_B \in \text{span}\{f_j : j \in \Lambda\}$, so by (3.1), $\tilde{E}_\Lambda f_B = 0$. Therefore, to reconstruct $f$, we have

$$f = \tilde{f} + \tilde{E}_\Lambda f_R. \quad (3.7)$$

Let $\alpha_j = \langle f, g_j \rangle$ and $\beta_j = \langle f_R, g_j \rangle$. Then, $\alpha_j$ for $j \in \Omega$ are known coefficients, and the $\beta_j$ are computable. The theorem below gives a direct algorithm for the reconstruction that involves nilpotent bridging and then applying the error operator.

**Theorem 3.4** Let $(F, G)$ be a dual frame pair with erasure set $\Lambda$ satisfying the minimal redundancy condition, and $\Omega$ be a robust bridge set. Assume $C = (c^{(j)}_{kj})_{j \in \Omega, k \in \Lambda}$ solves the matrix equation $B(F, G, \Lambda, \Omega)C = B(F, G, \Lambda, \Lambda)$. Then,

$$\left(\langle f, g_j \rangle\right)_{j \in \Lambda} = C^T \left( (\alpha_j)_{j \in \Omega} - (\beta_j)_{j \in \Omega} \right) + (\beta_j)_{j \in \Lambda},$$

where $C^T$ denotes the transpose of $C$.

**Proof** Let $\{f_j, g_j\}_{j \in J}$ be a dual frame pair, $\Lambda$ be an erasure set, and $\Omega$ be a corresponding robust bridge set. For $j \in \Lambda$ and $f \in \mathcal{H}$

$$\langle f, g_j \rangle = \langle f, g'_j \rangle + \langle f, g_j - g'_j \rangle$$

$$= \langle f, g'_j \rangle + \langle f - f_R, g_j - g'_j \rangle + \langle f_R, g_j - g'_j \rangle.$$

Since $f - f_R \in \text{span}\{f_j : j \in \Lambda\}$, Eq. (3.1) says that $f - f_R \perp g_j - g'_j$. So,

$$\langle f, g_j \rangle = \langle f, g'_j \rangle + \langle f_R, g_j - g'_j \rangle$$

$$= \langle f - f_R, g'_j \rangle + \langle f_R, g_j \rangle$$

$$= \sum_{k \in \Omega} c^{(j)}_k \langle f - f_R, g_k \rangle + \langle f_R, g_j \rangle.$$

Therefore, we can recover the erased coefficients with the following equation:

$$\left(\langle f, g_j \rangle\right)_{j \in \Lambda} = C^T \left( (f - f_R)_{k \in \Omega} \right) + \left(\langle f_R, g_j \rangle\right)_{j \in \Lambda}. $$

That is,

$$\left(\langle f, g_j \rangle\right)_{j \in \Lambda} = C^T \left( (\alpha_j)_{j \in \Omega} - (\beta_j)_{j \in \Omega} \right) + (\beta_j)_{j \in \Lambda}.$$
Remark 3.5 Notice that we need only solve a $|\Lambda| \times |\Omega|$ matrix equation. Theorem 3.7 discusses how if $\Lambda$ satisfies the minimal redundancy condition, we can find a robust bridge set $\Omega$ satisfying $|\Omega| \leq |\Lambda|$. In Sect. 5 we show that for “most” dual frame pairs, and for $|\Lambda| \leq \min \{n, N - n, \frac{N}{2}\}$, any $\Omega \subset \Lambda^c$ satisfying $|\Omega| = |\Lambda|$ is a robust bridge set. (See Sect. 5 for a more precise explanation of “most.”) Thus, in “most” cases the above algorithm shows that to recover the lost frame coefficients, we need only invert a $|\Lambda| \times |\Lambda|$ matrix and perform a few basic matrix multiplications and additions. This is more efficient than inverting the $n \times n$ matrix $R_\Lambda$ when $|\Lambda|$ is small compared to $n$.

Example 3.6 Consider the case where $\Lambda = \{k\}$, and choose a set $\Omega = \{\ell\}$. Then, $g'_k = c g_\ell$. For Nilpotent bridging, we require that $\langle f_k, g_k - g'_k \rangle = 0$. In solving for $c$, we get

$$0 = \langle f_k, g_k - g'_k \rangle = \langle f_k, g_k \rangle - \overline{c} \langle f_k, g_\ell \rangle.$$ 

So, if $\langle f_k, g_\ell \rangle \neq 0$, then $\Omega$ is a robust bridge set for $\Lambda$ and

$$g'_k = \frac{\langle g_k, f_k \rangle}{\langle g_\ell, f_k \rangle} g_\ell.$$ 

In particular any singleton set $\{\ell\}$ is a robust bridge set for $\Lambda$ provided $\langle f_k, g_\ell \rangle \neq 0$. So, in a suitably random frame, any singleton set disjoint from $\Lambda$ will be a robust bridge set. \qed

The following result provides a necessary and sufficient condition for the existence of a robust bridge set for a given erasure set.

**Theorem 3.7** Let $(F, G)$ be a dual frame pair for a Hilbert space $\mathcal{H}$, and let $\Lambda$ be an erasure set. Then there is a robust bridge set $\Omega$ for $\Lambda$ if and only if $\Lambda$ satisfies the minimal redundancy condition for $G$. In this case we can take $|\Omega| = \dim(\mathcal{F})$, where $\mathcal{F} = \text{span}\{f_j : j \in \Lambda\}$.

**Proof** Assume that $\Lambda$ satisfies the minimal redundancy condition. Let $\mathcal{F} = \text{span}\{f_j : j \in \Lambda\}$. Let $q = \dim(\mathcal{F})$. Let $\{h_j\}_{j \in \mathbb{N}}$ be a Schauder basis for $\mathcal{F}^\perp$. Since $\mathcal{F}^\perp$ has codimension $q$, we can complete this set to a Schauder basis $\{h_j\}_{j \in \mathbb{N}} \cup \{g_j\}_{k=1}^q$, where each $j_k \in \Lambda^c$. Let $\Omega = \{j_k\}_{k=1}^q$. Then $|\Omega| = q$ and $\Lambda \cap \Omega = \emptyset$. For each $\ell \in \Lambda$, write

$$g_\ell = \sum_{k=1}^q c_{jk}^{(\ell)} g_{j_k} + \sum_{j \in \mathcal{J}} b_j^{(\ell)} h_j.$$ 

Let

$$g'_\ell = \sum_{k=1}^q c_{jk}^{(\ell)} g_{j_k}.$$
Then $g_\ell - g'_\ell \in \mathcal{F}^\perp$. Therefore, by (3.1), the $c^{(\ell)}_j$ solve the bridge equation (3.5) and $\Omega$ is a robust bridge set.

To prove the converse, assume that $\Omega$ is a robust bridge set. Assume that $f \perp \text{span}\{g_j : j \in \Lambda^c\}$. Then,

$$f = \sum_{j \in \Lambda} \langle f, g_j \rangle f_j = \sum_{j \in \Lambda} \langle f, g_j \rangle f_j.$$ 

So, $f \in \text{span}\{f_j : j \in \Lambda\}$. We have

$$f = \sum_{j \in \Lambda} \langle f, g_j - g'_j \rangle f_j + \sum_{j \in \Lambda} \langle f, g'_j \rangle f_j.$$ 

However, since $f \in \text{span}\{f_j : j \in \Lambda\}$, Eq. (3.1) says that $\langle f, g_j - g'_j \rangle = 0$ for all $j \in \Lambda$. Since $g'_j \in \text{span}\{g_j : j \in \Lambda^c\}$, $\langle f, g'_j \rangle = 0$ for all $j \in \Lambda$. Hence, $f = 0$. Therefore, $\mathcal{H} = \text{span}\{g_j : j \in \Lambda^c\}$ and $\Lambda$ satisfies the minimal redundancy condition with respect to $G$. \hfill \Box

The following is a useful criterion for sufficiency of robustness of a bridge set.

**Theorem 3.8** Let $(F, G)$ be a dual frame pair, and $\Lambda$ be an erasure set. If $\Omega \subset \Lambda^c$ is a bridge set for which

$$\text{rank}(B(F, G, \Lambda, \Omega)) = \dim(\mathcal{F})$$

where $\mathcal{F} = \text{span}\{f_j : j \in \Lambda\}$, then $\Omega$ is a robust bridge set. In particular if $|\Lambda| = |\Omega|$ and $B(F, G, \Lambda, \Omega)$ is invertible, then $\Omega$ is a robust bridge set.

**Proof** First consider the special case where $\{f_j : j \in \Lambda\}$ is a linearly independent set and $|\Lambda| = |\Omega|$. The rank condition (3.8) is then just the condition that $B(F, G, \Lambda, \Omega)$ is invertible. Then the system (3.5) has a unique solution $C = B(F, G, \Lambda, \Omega)^{-1}B(F, G, \Lambda, \Lambda)$. So, $\Omega$ is robust.

Now consider the general case. Let $\kappa = \text{rank}(B(F, G, \Lambda, \Omega)) = \dim(\mathcal{F})$. Let $\Lambda_0 \subset \Lambda$ be such that $\{f_j : j \in \Lambda_0\}$ is a basis for $\mathcal{F}$. The rows of $B(F, G, \Lambda, \Omega)$ are linear combinations of the rows of $B(F, G, \Lambda_0, \Omega)$. Thus, $\text{rank}(B(F, G, \Lambda_0, \Omega)) = \kappa$. So, $|\Lambda_0| = \kappa$. Then, there is a subset $\Omega_0 \subset \Omega$ with $|\Omega_0| = \kappa$. It follows that $\text{rank}(B(F, G, \Lambda, \Omega_0)) = \kappa$. By the first paragraph of this proof, $\Omega_0$ is a robust bridge set for $\Lambda_0$. Since the rows of $B(F, G, \Lambda, \Omega)$ are linear combinations of the rows of $B(F, G, \Lambda_0, \Omega)$, it follows that the rows of $B(F, G, \Lambda, \Omega)$ are the same linear combinations of the rows of $B(F, G, \Lambda_0, \Omega_0)$. Thus $\Omega_0$ is a robust bridge set for $\Lambda$. So since $\Omega$ contains $\Omega_0$, $\Omega$ is a robust bridge set for $\Lambda$. \hfill \Box

**Remark 3.9** Theorem 3.8 says that the rank condition (3.8) on the bridge matrix is sufficient for robustness of $\Omega$. In the general case it is not necessary, as shown by Example 3.2. In that case, the unreduced error operator is already nilpotent of index 2, so any bridge set is robust for it. From experiments, it appears that the minimal rank possible of the bridge matrix for a robust bridge set and the minimal size of $\Omega$ is linked to the number of nonzero eigenvalues of the unreduced error operator. (See
Theorem 3.13 for a result relating to this.) However, for Parseval frames, the converse of Theorem 3.8 holds.

**Corollary 3.10** Let $F$ be a Parseval frame. If $\Lambda$ is an erasure set for $F$, and $\Omega \subset \Lambda^c$, then $\Omega$ is a robust bridge set for $\Lambda$ if and only if $\text{rank}(B(F, G, \Lambda, \Omega)) = \dim(\mathcal{F})$, where $\mathcal{F} = \text{span}\{f_j : j \in \Lambda\}$. In particular, if $\{f_j : j \in \Lambda\}$ is linearly independent and $|\Omega| = |\Lambda|$, then $\Omega$ is a robust bridge set for $\Lambda$ if and only if $B(F, G, \Lambda, \Omega)$ is invertible.

**Proof** The “only if” part holds by Theorem 3.8 for the dual frame pair $(F, G)$ with $G = F$. For the “if” part, suppose $F$ is a Parseval frame, $\Lambda$ is an erasure set, and $\Omega$ is a robust bridge set for $\Lambda$. By definition, for each $j \in \Lambda$ there exists $f_j' \in \text{span}\{f_k : k \in \Omega\}$ such that $f_j - f_j' \in \mathcal{F}^\perp$, where $\mathcal{F} = \text{span}\{f_j : j \in \Lambda\}$. Let $\kappa = \dim(\mathcal{F})$. Let $P$ be the orthogonal projection onto $\mathcal{F}$. Then for $j \in \Lambda$, $f_j = Pf_j = Pf_j'$. So $\mathcal{F} = P\text{span}\{f_j' : j \in \Omega\}$. It follows that $|\Omega| \geq \kappa$. Since $\text{span}\{Pf_j : j \in \Omega\} = \mathcal{F}$, there is a subset $\Omega_0 \subset \Omega$ such that $\{Pf_j : j \in \Omega_0\}$ is a basis for $\mathcal{F}$. Similarly there is a subset $\Lambda_0 \subset \Lambda$ such that $\{f_j : j \in \Lambda_0\}$ is a basis for $\mathcal{F}$. Then $\{f_j : j \in \Lambda_0\}$ and $\{Pf_j : j \in \Omega_0\}$ are two bases for the same Hilbert space, so $|\Lambda_0| = |\Omega_0|$, and the cross-Gramian $\langle Pf_j, f_j \rangle_{j \in \Lambda_0, k \in \Omega_0}$ is invertible (see the preliminaries), so it has rank $\kappa$. But for each $j \in \Lambda_0$ and $k \in \Omega_0$,

$$\langle Pf_k, f_j \rangle = \langle f_k, Pf_j \rangle = \langle f_k, f_j \rangle,$$

so $\{(Pf_k, f_j)\}_{j \in \Lambda_0, k \in \Omega_0}$ is just the bridge matrix $B(F, G, \Lambda_0, \Omega_0)$. Since it has rank $\kappa$, and it is a minor of $B(F, G, \Lambda, \Omega)$, $\text{rank}(B(F, G, \Lambda, \Omega)) \geq \kappa = \dim(\mathcal{F})$. But $\dim(\mathcal{F}) = \kappa$ implies that $B(F, G, \Lambda, \Omega)$ can not have more than $\kappa$ linearly independent rows, and hence $\text{rank}(B(F, G, \Lambda, \Omega)) \leq \kappa$. Thus the rank of the bridge matrix must be $\kappa$. It follows that if $\{f_j : j \in \Lambda\}$ are linearly independent then $B(F, G, \Lambda, \Omega)$ is invertible as claimed.\]

The next two examples illustrate the relationship between the minimal redundancy condition and the invertability of $R_\Lambda$. For the examples, we consider the dual frame pair

$$F = \left\{(1, 1)^T, (-1, 1)^T, (-1, -1)^T, (1, -1)^T\right\}$$

and

$$G = \left\{(1, 0)^T, \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}^T, \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}^T, (1, 0)^T\right\}.$$ 

Our first example is an example where the 2-nilpotent bridging algorithm works, but $R_\Lambda$ is not invertible.
Example 3.11 Let \( \Lambda = \{1\} \),
\[
R_{\Lambda} = \sum_{j=2}^{4} f_j \otimes g_j = I - f_1 \otimes g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}
\]
is not invertible. Therefore, methods that require the inversion of \( R_{\Lambda} \) won’t work. Furthermore,
\[
E_{\Lambda} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}
\]
is idempotent, so Neumann series approximations also fail. However, since \( \langle f_1, g_2 \rangle \neq 0 \) and \( \langle f_1, g_4 \rangle \neq 0 \), Example 3.6 shows that nilpotent bridging works with \( \Omega = \{2\} \) or \( \Omega = \{4\} \). Note that \( \Omega = \{3\} \) won’t work for Nilpotent bridging since \( \langle f_1, g_3 \rangle = 0 \).

While for robustness \( \Lambda \) needs to satisfy the minimal redundancy condition with respect to \( G \), the second example shows that \( \Lambda \) need not satisfy the minimal redundancy condition with respect to \( F \).

Example 3.12 Let \( \Lambda = \{2, 4\} \), and \( \Omega = \{1, 3\} \). Then \( \Lambda \) does not satisfy the minimal redundancy condition for \( F \). But, we have
\[
f_2, f_4 \perp g_2 - 0g_1 - 0g_3 \quad \text{and} \quad f_2, f_4 \perp g_4 - g_1 - 0g_3.
\]
Letting \( f = (4, 2)^T \), we get
\[
f_R = R_{\Lambda} f = (f_1 \otimes g_1)(f) + (f_3 \otimes g_3)(f) = (3, 3)^T
\]
and
\[
f_B = B_{\Lambda} f = (f_2 \otimes 0)(f) + (f_4 \otimes g_1)(f) = (4, -4)^T.
\]
So,
\[\tilde{f} = f_R + f_B = (7, -1)^T.\]
We have
\[
f_{\tilde{E}} = \tilde{E}_{\Lambda} f_R = (f_2 \otimes (g_2 - 0g_1 - 0g_3))(f_R) + (f_4 \otimes (g_4 - g_1 - 0g_3))(f_R)
= (-3, 3)^T.
\]
Therefore we recover our original vector as
\[\tilde{f} + f_{\tilde{E}} = (4, 2)^T.\]
Consider a dual frame pair \((F, G)\) with erasure set \(\Lambda\) and bridge set \(\Omega\). Computer experiments indicated that if \(|\Omega| < |\Lambda|\), then \(\sigma(\tilde{E}_{\Lambda}) \setminus \{0\} = |\Lambda| - |\Omega|\). So, if one chooses a bridge set that is too small, \(\tilde{E}_{\Lambda}\) will have nonzero eigenvalues, but may have fewer nonzero eigenvalues than \(E_{\Lambda}\) (the error operator without bridging). The following gives a mathematical proof of this fact.

**Theorem 3.13** Let \((F, G)\) be a dual frame pair. Assume \(\Lambda\) satisfies the minimal redundancy condition with respect to \(G\), and \(|\Lambda| = L\). Then, there is a bridge set \(\Omega\) of any size \(M \leq L\) so that \(|\sigma(\tilde{E}_{\Lambda}) \setminus \{0\}| \leq L - M\).

**Proof** By Theorem 3.7, we can find a robust bridge set \(\Omega' \subset \Lambda_c\) satisfying \(|\Omega'| < L\). That is, for each \(k \in \Lambda\) we can find \(g'_k = \sum_{j \in \Omega'} c_j(k) g_j\) so that \(g'_k \perp \text{span}\{f_j: j \in \Lambda\}\). Assume that \(\Omega' = \{\omega_1, \ldots, \omega_{|\Omega'|}\}\). Let \(\Omega = \{\omega_1, \ldots, \omega_M\}\) and

\[
g''_k = \sum_{j \in \Omega} c_j(k) g_j.
\]

Then,

\[
\tilde{E}_{\Lambda} = \sum_{k \in \Lambda} f_k \otimes (g_k - g'_k) = \sum_{k \in \Lambda} f_k \otimes (g_k - g'_k) + \sum_{k \in \Lambda} f_k \otimes (g'_k - g''_k).
\]

Let \(N = \tilde{E}_{\Lambda} = \sum_{k \in \Lambda} f_k \otimes (g_k - g'_k)\), and \(A = \sum_{k \in \Lambda} f_k \otimes (g'_k - g''_k)\). Then, it is easily verified that \(N\) is nilpotent of index 2, and \(NA = 0\). Since \(\text{range}(A^*) \subset \{g'_k - g''_k: k \in \Lambda\} \subset \{g_{\omega_k}: k = M + 1, \ldots, |\Omega'|\}\), the rank of \(A\) is at most \(L - M\).

Let \(\lambda \in \sigma(N + A) \setminus \{0\}\). Both \(N\) and \(A\) are finite rank operators, so \(\lambda\) must be an eigenvalue of \(N + A\). Thus, there exists \(x \in \mathcal{H}\) so that

\[
(N + A)x = \lambda x.
\]

Multiplying by \(N\) on the left on both sides yields

\[
0 = \lambda Nx.
\]

Since \(\lambda \neq 0\), we have \(Nx = 0\). Thus, \(Ax = \lambda x\) and \(\lambda \in \sigma(A)\). Since \(A\) can have at most \(L - M\) distinct eigenvalues, it follows that \(\tilde{E}_{\Lambda}\) has at most \(L - M\) nonzero eigenvalues. \(\square\)
4 Applications to Sampling Theory

There are well-known deep established connections between frame theory and modern sampling theory. We cite for instance the excellent references ([4, 19, 29]). We note that a good account of sampling theory for our purposes is contained in [22, Chapter 9]. Let \( X \) be a metric space and let \( \mu \) be a Borel measure on \( X \). Let \( \mathcal{H} \) be a closed subspace of \( L^2(X, \mu) \) consisting of continuous functions. Let \( T = \{ t_j \}_{j \in \mathbb{J}} \subset X \) and define the sampling transform \( \Theta \) mapping \( \mathcal{H} \) into the complex sequences by \( \Theta(f) = (f(t_j))_{j \in \mathbb{J}} \). If \( \Theta : \mathcal{H} \to \ell^2(\mathbb{J}) \) and is bounded, then the point evaluation functionals \( \gamma_j : \mathcal{H} \to \mathbb{C} \) defined by \( \gamma_j(f) = f(t_j) \) are bounded, and hence by the Riesz Representation Theorem, \( \gamma_j(f) = \langle f, g_j \rangle \) for some \( g_j \in \mathcal{H} \). If the sampling transform is also bounded below, then \( \{ g_j \}_{j \in \mathbb{J}} \) forms a frame for \( \mathcal{H} \), and thus we can find some dual \( F := \{ f_j \}_{j \in \mathcal{H}} \). We then have the identity

\[
 f = \sum_{j \in \mathbb{J}} \langle f, g_j \rangle f_j = \sum_{j \in \mathbb{J}} f(t_j) f_j \quad \forall f \in \mathcal{H}. \tag{4.1}
\]

We will refer to \((X, F, T)\) as a sampling scheme for \( \mathcal{H} \). The most well-known sampling scheme comes from the Shannon–Whittaker Sampling Theorem. For this scheme, \( \mathcal{H} = \mathcal{P}W[-\pi, \pi], T = p\mathbb{Z} (p \in (0, 1]), f_j = \text{sinc}(\pi t - (j \pi)) \). Then \( g_j = p \text{sinc}(\pi t - (j \pi)) \), where \( \text{sinc}(x) = \frac{\sin x}{x} \).

Let \( \Lambda \) be an erasure set for a sampling scheme \((X, F, T)\), with corresponding bridge set \( \Omega \). We can think of the erased coefficients as either \( \{ f, g \} \) or as \( f(t_j) \) for \( j \in \Lambda \). For this case, the bridge matrix is

\[
 B(F, G, \Lambda, \Omega) = \left( \langle f_j, g_k \rangle \right)_{j \in \Lambda, k \in \Omega} = \left( f_j(t_k) \right)_{j \in \Lambda, k \in \Omega}. \tag{4.2}
\]

Similarly, \( B(F, G, \Lambda, \Omega) = (f_j(t_k))_{j, k \in \Lambda} \). Note that these matrices only involve the sampled values of the \( \{ f_j \} \) over the points \( \{ t_k \} \) and do not explicitly involve the \( \{ g_k \} \). Let us simply write \( B(\Lambda, \Omega) \) and \( B(\Lambda, \Lambda) \) for these two matrices. Then Theorem 3.4 becomes the following Theorem:

**Theorem 4.1** Let \((X, F, T)\) be a sampling scheme with erasure set \( \Lambda \) satisfying the minimal redundancy condition, and \( \Omega \) be a robust bridge set for \( \Lambda \). Suppose \( C = \left( \frac{t_k}{c_j(k)} \right)_{j \in \Omega, k \in \Lambda} \) solves the bridging equation

\[
 B(\Lambda, \Omega) C = B(\Lambda, \Lambda),
\]

where \( B(\Lambda, \Omega) = (f_j(t_k))_{j \in \Lambda, k \in \Omega} \) and \( B(\Lambda, \Lambda) = (f_j(t_k))_{j, k \in \Lambda} \). Then

\[
 \left( f(t_j) \right)_{j \in \Lambda} = C^T \left( (f(t_j))_{j \in \Omega} - (f_R(t_j))_{j \in \Omega} \right) + \left( f_R(t_j) \right)_{j \in \Lambda}. 
\]
5 Generic Duals

In this section we deal only with finite frames in finite dimensional Hilbert spaces. Assume that $\mathcal{H}$ is an $n$-dimensional Hilbert space. We denote the set of $N$-tuples of vectors in $\mathcal{H}$ by $\mathcal{H}^N$. The space $\mathcal{H}^N$ can be equipped with many equivalent norms, but the one we will use is defined by $\| F \| := \max_{1 \leq j \leq N} \| f_j \|$ for $F = \{ f_j \}_{j=1}^N \in \mathcal{H}^N$.

Let $F = \{ f_j \}_{j=1}^N$ be a frame in $\mathcal{H}^N$. For a frame, $F$, we define $D(F) = \{ G \in \mathcal{H}^N : (F, G) \text{ is a dual frame pair} \}$ and call it the dual set of $F$.

In the frame literature, a class of frames is sometimes called generic if it is open and dense in the set of all frames (c.f. [1,2,26]). We will say that a class of duals to a given frame $F$ is generic if it is open and dense in the relative topology on $D(F)$ inherited as a subspace of $\mathcal{H}^N$.

Proposition 5.1 $D(F)$ is a closed, convex subset of $\mathcal{H}^N$.

Proof Let $G, G' \in D(F)$. Then, for any $t \in [0, 1]$, we see that

$$\sum_{j=1}^N f_j \otimes \left( (1-t)g_j + tg'_j \right) = (1-t) \sum_{j=1}^N f_j \otimes g_j + t \sum_{j=1}^N f_j \otimes g'_j$$

$$= (1-t)I + tI = I.$$

Hence, $(1-t)G + tG' \in D(F)$, so $F$ is convex. The proof that $D(F)$ is closed is elementary. $\Box$

Since $D(F)$ is a closed subset of $\mathcal{H}$, $D(F)$ is a complete metric space with the norm topology inherited from $\mathcal{H}^N$.

Theorem 5.2 Let $\Lambda$ be an erasure set for a frame $F$ with the minimal redundancy condition and let $\{ g_j \}_{j \in \Lambda}$ be assigned arbitrarily. Then, $\{ g_j \}_{j \in \Lambda}$ can be extended to a dual frame $\{ g_j \}_{j=1}^N \in D(F)$.

Proof We first show that under the same conditions on $F$, the set $\{ h_j \}_{j \in \Lambda}$ can be extended to $\{ h_j \}_{j=1}^N$ so that $\sum_{j=1}^N f_j \otimes h_j = 0$. Let $A = \sum_{j \in \Lambda} f_j \otimes h_j$. Let $\{ k_j \}_{j \in \Lambda^c}$ be a dual to the reduced frame $\{ f_j \}_{j \in \Lambda^c}$. Then, $I = \sum_{j \in \Lambda^c} f_j \otimes k_j$. So,

$$A = \left( \sum_{j \in \Lambda^c} f_j \otimes k_j \right) A = \sum_{j \in \Lambda^c} f_j \otimes (A^*k_j).$$

For each $j \in \Lambda^c$, let $h_j = -A^*k_j$. Then,

$$\sum_{j=1}^N f_j \otimes h_j = \sum_{j \in \Lambda^c} f_j \otimes h_j + \sum_{j \in \Lambda} f_j \otimes h_j$$

$$= -\sum_{j \in \Lambda^c} f_j \otimes A^*k_j + A = A - \left( \sum_{j \in \Lambda^c} f_j \otimes k_j \right) A = A - IA = 0.$$
Now, let \( \{g'_j\}_{j=1}^N \in \mathcal{D}(F) \). Let \( h_j = g_j - g'_j \) for \( j \in \Lambda \). Then, as above, we can extend \( \{h_j\}_{j \in \Lambda} \) to \( \{h_j\}_{j=1}^N \) so that \( \sum_{j=1}^N f_j \otimes h_j = 0 \). For all \( j \), let \( \tilde{g}_j = g'_j + h_j \). So,

\[
\sum_{j=1}^N f_j \otimes \tilde{g}_j = \sum_{j=1}^N f_j \otimes g'_j + \sum_{j=1}^N f_j \otimes h_j = I + 0 = I.
\]

Thus, \( \{\tilde{g}_j\}_{j=1}^N \in \mathcal{D}(F) \). Furthermore, for \( j \in \Lambda \),

\[
\tilde{g}_j = g'_j + h_j = g'_j + g_j - g'_j = g_j.
\]

Therefore, \( \{\tilde{g}_j\}_{j=1}^N \) is the desired extension of \( \{g_j\}_{j \in \Lambda} \).

\[\square\]

**Remark 5.3** The above theorem shows that in the presence of the minimal redundancy condition, one can pick “designer duals” that satisfy certain conditions with respect to \( \Lambda \). Theorem 5.7 (below) is our main result in this direction.

In the frame literature (c.f. [1,14]) a frame \( F \) is said to have *spark* \( k \) if every collection of \( k - 1 \) vectors in \( F \) is linearly independent and there is a collection of \( k \) vectors in \( F \) which are linearly dependent. \( F \) is said to have *full spark* if it has spark \( n + 1 \) (\( n \) is the dimension of \( \mathcal{H} \)). It is known that the set of full spark frames is an open dense set in \( \mathcal{H}^N \) (c.f. [1,26]). Moreover, it is known that the set of all full spark frames is dense in the set of all frames, and that the set of equal norm, full spark Parseval frames is dense in the set of equal norm Parseval frames (c.f. [8]).

**Lemma 5.4** Let \( (F, G) \) be a dual frame pair of length \( N \) in an \( n \)-dimensional Hilbert space, \( \mathcal{H} \). Let \( \Lambda \) be an erasure set, and \( \Omega \) be a bridge set satisfying \( |\Lambda| = |\Omega| \). A necessary (but not sufficient) condition for \( B(F, G, \Lambda, \Omega) \) to be an invertible matrix is

\[
|\Lambda| \leq \min \left\{ n, N - n, \frac{N}{2} \right\}
\]

(5.1)

**Proof** If \( |\Lambda| > n \), then the rows of the bridge matrix \( B(F, G, \Lambda, \Omega) \) will be linearly dependent (since \( \mathcal{H} \) is an \( n \)-dimensional space). Thus, \( B(F, G, \Lambda, \Omega) \) will fail to be invertible.

Assume that \( B(F, G, \Lambda, \Omega) \) is invertible, and \( |\Lambda| > N - n \). Then, since the bridge equation \( B(F, G, \Lambda, \Omega)C = B(F, G, \Lambda, \Lambda) \) has a solution \( C = B(F, G, \Lambda, \Omega)^{-1} B(F, G, \Lambda, \Lambda) \), Theorem 3.7 asserts that \( \Lambda \) satisfies the minimal redundancy condition with respect to \( G \). Therefore, \( |\Lambda^c| \geq n \). So, \( N = |\Lambda| + |\Lambda^c| > N - n + n > N \). This is a contradiction, and therefore, if \( B(F, G, \Lambda, \Omega) \) is invertible, then \( |\Lambda| \leq N - n \).

If \( |\Lambda| > \frac{N}{2} \), then \( |\Lambda| + |\Omega| > N \). This is a contradiction since \( \Lambda \) and \( \Omega \) are disjoint subsets of \( \{1, \ldots, N\} \).

\[\square\]

**Corollary 5.5** Assume that \( F \in \mathcal{H}^N \) has full spark. Let \( \Lambda \) be an erasure set satisfying \( |\Lambda| \leq \min\{n, N - n, \frac{N}{2}\} \), and \( \Omega \) be a bridge set satisfying \( |\Lambda| = |\Omega| \) and \( \Lambda \cap \Omega = \emptyset \). Then there exists a dual frame \( G \) to \( F \) so that \( B(F, G, \Lambda, \Omega) \) is invertible.
Proof Define a bijection \( \varphi : \Omega \rightarrow \Lambda \). Let \( \{g_j\}_{j \in \Omega} = \{f_{\varphi(j)}\}_{j \in \Omega} \). By Theorem 5.2, we can extend \( \{g_j\}_{j \in \Omega} \) to a dual frame \( G \) for \( F \). Then \( B(F, G, \Lambda, \Omega) \) is the Gram matrix of a permutation of the finite sequence \( \{f_j : j \in \Lambda\} \), which is invertible since \( \{f_j : j \in \Lambda\} \) is linearly independent. \( \square \)

We say that a dual frame pair \((F, G)\) has skew-spark \( k \) if for every erasure set \( \Lambda \) with \( |\Lambda| < k \), and any bridge set \( \Omega \subset \Lambda^c \) satisfying \( |\Lambda| = |\Omega| \), \( B(F, G, \Lambda, \Omega) \) is invertible. If \((F, G)\) has skew-spark \( \min\{N/2, n, N-n\} + 1 \), then \((F, G)\) is said to have full skew-spark.

**Proposition 5.6** If the dual frame pair \((F, G)\) for \( \mathcal{H} \) has skew-spark \( k \), then \( F \) and \( G \) each have spark at least \( k \).

**Proof** Let \( \Lambda \) be an erasure set of cardinality \( k \). Let \( \Omega \) be any subset of \( \Lambda^c \) of cardinality \( k \). By hypothesis the matrix \( B(F, G, \Lambda, \Omega) \) is invertible, so its rows and columns are linearly independent. This implies that \( \{f_j : j \in \Lambda\} \) is linearly independent. Since \( \Lambda \) was arbitrary, this shows that \( F \) has spark \( k \). The proof for \( G \) is analogous. \( \square \)

This also shows that if \( n \leq \min\{N/2, N-n\} \), then the full skew-spark property implies the full spark property.

Let \( G = \{G \in D(F) : (F, G) \) has full skew-spark\}.

**Theorem 5.7** Assume that \( F \) has full spark. Then, \( G := \{G \in D(F) : (F, G) \) has full skew-spark\} is generic in \( D(F) \).

**Proof** Let \( \Gamma = \{\Lambda \subset \{1, \ldots, N\} : |\Lambda| \leq \min\{N/2, N-n, n\}\} \). For a given \( \Lambda \in \Gamma \), let \( \Phi_\Lambda = \{\Omega \subset \{1, \ldots, N\} : |\Omega| = |\Lambda|, \Omega \cap \Lambda = \emptyset\} \). Then, \( G = \bigcap_{\Lambda \in \Gamma} \bigcap_{\Omega \in \Phi_\Lambda} G_{\Lambda, \Omega}, \) where \( G_{\Lambda, \Omega} = \{G \in D(F) : \det(B(F, G, \Lambda, \Omega)) \neq 0\} \). Since we are intersecting over all possible erasure sets and all corresponding bridge sets, the above intersection is finite. So by the Baire category theorem, if we show that each \( G_{\Lambda, \Omega} \) is open and dense, then \( G \) will also be open and dense.

Fix an erasure set \( \Lambda \), and a corresponding bridge set \( \Omega \). It is easily verified that the maps \( G \mapsto \lambda \mapsto B(F, G, \Lambda, \Omega) \) and \( B(F, G, \Lambda, \Omega) \mapsto \det(B(F, G, \Lambda, \Omega)) \) are continuous. So, \( G_{\Lambda, \Omega} = (\det \omega)^{-1}(\mathbb{C} \setminus \{0\}) \) is an open set.

To show density of \( G_{\Lambda, \Omega}, \) let \( \epsilon > 0 \), and assume that \( G_0 \in D(F) \setminus G_{\Lambda, \Omega} \). Since \( F \) satisfies the full spark property, \( \Lambda \) satisfies the minimal redundancy condition with respect to \( F \). Thus, by Corollary 5.5, there is a \( G_1 \in D(F) \) so that \( \det(B(F, G_1, \Lambda, \Omega)) \neq 0 \). Let \( G_t = (1-t)G_0 + tG_1 \). By Proposition 6.1, \( G_t \in D(F) \). Furthermore, \( \det(B(F, G_t, \Lambda, \Omega)) \) is a polynomial in \( t \) satisfying \( \det(B(F, G_t, \Lambda, \Omega))(0) = 0 \) and \( \det(B(F, G_t, \Lambda, \Omega))(1) \neq 0 \). Thus, \( \det(B(F, G_t, \Lambda, \Omega)) \) has only finitely many zeros. So, we can find \( 0 < t_0 < \frac{\epsilon}{\|G_1 - G_0\|} \) so that \( G_{t_0} \in G_{\Lambda, \Omega} \). Furthermore, \( \|G_{t_0} - G_0\| = \|(1-t_0)G_0 + t_0G_1 - G_0\| = \|t_0(G_1 - G_0)\| \leq |t_0| \|G_1 - G_0\| < \epsilon \).

Hence, \( G_{\Lambda, \Omega} \) is dense in \( D(F) \).

Therefore, by the Baire-Category theorem, \( G \) is generic in \( D(F) \). \( \square \)

In short, what we have proven in this section is that for most frames \( F \in \mathcal{H}^N \), and most duals \( G \) to \( F \), the pair \((F, G)\) has full skew-spark.
Remark 5.8 We found it convenient to present and prove the topological results of this section for the metric topology. A similar argument can be used to obtain these for the Zariski topology.

6 Computing an Inverse for $R_A$

In this section, we obtain a basis-free closed-form formula for the inverse of the partial reconstruction operator $R_A$ for a finite erasure set. (The underlying Hilbert space may be either finite or infinite: only the erasure set need be finite.) By basis-free we mean that the computations do not depend on any preassigned orthonormal or Schauder basis for the space, and by closed-form we mean that it is of the same general form as $R_A$ is given in and does not require an iterative process such as the Neuman series formula. This gives a second method of perfect reconstruction from frame and sampling erasures in a small number of steps that applies when $R_A^{-1}$ exists. Furthermore, it requires only a $|\Lambda| \times |\Lambda|$ matrix inversion which is usually “much” smaller than the dimension of the space. Thus, this method is more efficient than if we were to invert $R_A$ without using the tools from this section.

Let $(F, G)$ be a dual frame pair indexed by $J$, and $\Lambda$ be an erasure set. Recall that

$$R_A = \sum_{j \in J \setminus \Lambda} f_j \otimes g_j = I - \sum_{j \in \Lambda} f_j \otimes g_j.$$ 

We derive a simple method for computing inverses of operators of the form

$$R = I - \sum_{j=1}^{L} f_j \otimes g_j$$

that was motivated by our work on bridging.

Proposition 6.1 Assume that $R = I - \sum_{j=1}^{L} f_j \otimes g_j$ is invertible. Then, $R^{-1}$ has the form $I + \sum_{j,k=1}^{L} c_{jk} f_j \otimes g_k$ for some $c_{jk} \in \mathbb{C}$.

Proof We must show that $R^{-1} - I$ is a linear combination of the elementary tensors $\{f_j \otimes g_k\}_{j,k=1}^{L}$.

Let $E = \sum_{j=1}^{L} f_j \otimes g_j$. Note that $(I - R^{-1}) R = R - I = -E$. Since $R$ is invertible, this shows that the range of $I - R^{-1}$ is contained in span${f_j}_{j=1}^{L}$.

We have,

$$-E = R - I = (I - R^{-1}) R = (I - R^{-1})(I - E) = I - R^{-1} - (I - R^{-1}) E.$$ 

Therefore,

$$R^{-1} = I + E - (I - R^{-1}) E.$$
From above, we know that \((I - R^{-1})f_k = \sum_{j=1}^{L} b_{jk} f_j\) for some \(b_{jk} \in \mathbb{C}\). So,

\[
(I - R^{-1})E = \sum_{k=1}^{L} (I - R^{-1}) f_k \otimes g_k = \sum_{k=1}^{L} \sum_{j=1}^{L} b_{jk} f_j \otimes g_k
\]

a linear combination of the \(\{f_j \otimes g_k\}_{j,k=1}^{L}\).

Since \(E\) is also a linear combination of the \(f_j \otimes g_k\), \(\sum_{j,k=1}^{L} c_{jk} f_j \otimes g_k\) for appropriate constants \(c_{jk} \in \mathbb{C}\).

So \(R^{-1} = I + \sum_{j,k=1}^{L} c_{jk} f_j \otimes g_k\).

\[
\square
\]

Although the elementary tensors \(f_j \otimes g_k\) in the representation of \(R^{-1}\) in Proposition 6.1 are generally not linearly independent and hence the coefficients \(\{c_{jk}\}_{j,k=1}^{L}\) are not unique, we can derive a simple matricial formula that gives a valid choice of the \(c_{jk}\).

**Theorem 6.2** Let \(R = I - \sum_{j=1}^{L} f_j \otimes g_j\), where \(\{f_j\}_{j=1}^{L}, \{g_j\}_{j=1}^{L}\) are finite sequences and \(\{f_j\}_{j=1}^{L}\) is linearly independent. If \(R\) is invertible, then a formula for the inverse is

\[
R^{-1} = I + \sum_{j,k=1}^{L} c_{jk} f_j \otimes g_k
\]

where the coefficient matrix \(C := (c_{jk})_{j,k=1}^{L}\) is given by

\[
C = (I - M)^{-1}
\]

where \(I\) is the \(L \times L\) identity matrix and

\[
M = Gr(\{f_1, \ldots, f_L\}, \{g_1, \ldots, g_L\}) := \begin{bmatrix}
\langle f_1, g_1 \rangle & \langle f_2, g_1 \rangle & \cdots & \langle f_L, g_1 \rangle \\
\langle f_1, g_2 \rangle & \langle f_2, g_2 \rangle & \cdots & \langle f_L, g_2 \rangle \\
& \vdots & \ddots & \vdots \\
\langle f_1, g_L \rangle & \langle f_2, g_L \rangle & \cdots & \langle f_L, g_L \rangle
\end{bmatrix}
\]

\[
(6.3)
\]

**Proof** By Proposition 6.1 we can write

\[
R^{-1} = I + \sum_{j=1}^{L} \sum_{k=1}^{L} c_{jk} f_j \otimes g_k \text{ for some } c_{jk} \in \mathbb{C}. \text{ Compute:}
\]

\[
\square
\]

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So there would exist a nonzero vector $x$ where

$$I = R^{-1}R$$

$$= \left( I + \sum_{j=1}^{L} \sum_{k=1}^{L} c_{jk} f_j \otimes g_k \right) \left( I - \sum_{j=1}^{L} f_j \otimes g_j \right)$$

$$= I + \sum_{j=1}^{L} \sum_{k=1}^{L} c_{jk} f_j \otimes g_k - \sum_{j=1}^{L} f_j \otimes g_j - \sum_{j=1}^{L} \sum_{k=1}^{L} c_{jk} (f_j \otimes g_k)(f_\ell \otimes g_\ell)$$

$$= I + \sum_{j=1}^{L} \sum_{k=1}^{L} c_{jk} f_j \otimes g_k - \sum_{j=1}^{L} f_j \otimes g_j - \sum_{j=1}^{L} \sum_{\ell=1}^{L} c_{j\ell} (f_\ell \otimes g_k)(f_j \otimes g_k)$$

In the last sum, we switched indices $k$ and $\ell$. Thus,

$$\sum_{j=1}^{L} f_j \otimes g_j = \sum_{j=1}^{L} \sum_{k=1}^{L} c_{jk} f_j \otimes g_k - \sum_{\ell=1}^{L} \sum_{j=1}^{L} c_{j\ell} (f_\ell \otimes g_k)(f_j \otimes g_k).$$

By simply setting the coefficients of the $f_j \otimes g_k$ to $\delta_{j,k}$, we obtain the following system of equations:

$$c_{jk} = \sum_{\ell=1}^{L} c_{j\ell} (f_\ell \otimes g_k) = \delta_{jk}. \quad (6.4)$$

For a fixed value of $j$, we have the system

$$\left( \delta_{jk} \right)_{k=1, \ldots, L}^T = \begin{pmatrix}
1 - \langle f_1, g_1 \rangle & -\langle f_1, g_2 \rangle & \cdots & -\langle f_1, g_L \rangle \\
-\langle f_2, g_1 \rangle & 1 - \langle f_2, g_2 \rangle & \cdots & -\langle f_2, g_L \rangle \\
\vdots & \vdots & \ddots & \vdots \\
-\langle f_L, g_1 \rangle & -\langle f_L, g_2 \rangle & \cdots & 1 - \langle f_L, g_L \rangle
\end{pmatrix} \left( c_{jk} \right)_{k=1, \ldots, L}^T.$$

Let $C = \left( c_{jk} \right)_{j,k}$. Combining the equations for all $j$ gives

$$I = (I - M^T)C^T$$

where $M = Gr(\{f_1, \ldots, f_L\}, \{g_1, \ldots, g_L\})$. So $C(I - M) = I$.

We will show that under our hypothesis that $\{f_1, \ldots, f_L\}$ is linearly independent the matrix $I - M$ is invertible, so this system has a unique solution. This will yield a valid choice of the $c_{jk}$. If $I - M$ were singular then 1 would be an eigenvalue of $M$. So there would exist a nonzero vector $x = (x_k)_{k=1}^{L} \in \mathbb{C}^n$ so that $Mx = x$. Computing gives $\sum_{j=1}^{L} (f_j, g_k)x_j = x_k$ for each $k$. Let $z = \sum_{j=1}^{L} x_j f_j$. Since $x$ is nonzero not all of the $x_j$ are zero. By hypothesis $\{f_1, \ldots, f_L\}$ is linearly independent, so $z$ cannot
be the zero vector. Compute:

\[ Rz = z - \sum_{k=1}^{L} \langle z, g_k \rangle f_k = z - \sum_{k=1}^{L} \sum_{j=1}^{L} x_j \langle f_j, g_k \rangle f_k = z - \sum_{k=1}^{L} x_k f_k = z - z = 0. \]

So \( z \) is in the kernel of \( R \) contradicting our hypothesis that \( R \) is invertible. Thus \( I - M \) is a nonsingular matrix, and the system has the unique solution \( C = (I - M)^{-1} \) as claimed.

Remark 6.3 In order to apply the above theorem to inverting a frame partial reconstruction operator, if \( \{f_j : j \in \Lambda \} \) is not linearly independent one must first use linearity of the elementary tensors \( f \otimes g \) in the first component and conjugate linearity in the second component to precondition \( R \) to the form \( I - \sum_{j=1}^{L} f'_j \otimes g'_j \) with the first component set \( \{f'_j : j \in \Lambda \} \) linearly independent. In many cases this will be simple and even automatic, but in other cases this may be computationally expensive. The main point is that if \( R \) is invertible, the computation above, perhaps with preconditioning, always yields a formula for the inverse. Furthermore, it may be useful to note that since we are solving a matrix equation, it follows that the coefficients \( c_{jk} \) are given by rational functions of the \( \langle f_j, g_k \rangle \). In this sense the formula is indeed basis-free.

7 Concluding Remarks

The main results in this article are presented for reconstruction from finite erasure subsets of frames, so much of our theory is finite dimensional. However, the reconstruction results can be applied to finite subsets of infinite frames, including the well known classes of Gabor (Weyl–Heisenberg) frames, Laurent frames, infinite group frames, and wavelet frames, as well as abstract sampling theory. There may be applications to the pure and applied aspects of these classes, including classification results. In fact, our initial computer experiments suggest to us that many of these natural classes of infinite frames may be full skew-spark in the sense that they have skew-spark \( k \) for all finite \( k \). But mathematical proofs of general theorems on this have eluded us so far. In addition, there may be applications to the three closely related topics that deal with frames in blocks: operator-valued frames, fusion frames, and G-frames (c.f. [11,25,28]). Finally, we should mention that we expect that there will be applications to the more abstract theories: frames for Banach spaces and related topics of Banach frames, atomic decompositions, and framings (c.f [13,16,20]), the theory of frames for Hilbert \( C^* \)-modules, and in the purely algebraic direction: frames for other fields such as \( p \)-adic frames and binary frames (c.f. [6,24]).

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