On the Davey-Stewartson hierarchy: construction by two scalar pseudo-differential operators and compatibility for infinite many flows

G.Yi * and X. Liao

School of Mathematics, Hefei University of Technology, Hefei 230601, China

* Corresponding author: ge.yi@hfut.edu.cn

May 17, 2022

Abstract

The infinite many symmetries of Davey-Stewartson (DS) system are closely connected to the integrable deformations of surfaces in a four-dimensional space. In this paper, we give a direct algorithm to construct the expression of the DS hierarchy by two scalar pseudo-differential operators involving partial derivatives.

1 Introduction

The KP (Kadomtsev-Petviashvili) equation \[1, 2\]

\[(u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0,\] (1)

and the DS (Davey-Stewartson) system \[2, 3\]

\[iq_t + \frac{1}{2}(q_{xx} + \sigma^2 q_{yy}) + \delta q\phi = 0,\] (2a)

\[\sigma^2 \phi_{yy} - \phi_{xx} + (|q|^2)_{xx} + \sigma^2 (|q|^2)_{yy} = 0,\] (2b)

as the most important classical integrable models in (2+1) dimensions, have been extensively studied with many important results obtained \[4, 26\]. An integrable system is usually associated with a hierarchy of nonlinear partial differential equations defining infinitely many symmetries. This is one of the most important and valuable properties of integrable systems.
The KP hierarchy plays a fundamental role in the theory of integrable systems, a key reason is the clear and explicit definition via Lax equations of pseudo-differential operators [29–36]. Let

\[ L = \partial + u_1\partial^{-1} + u_2\partial^{-2} + \cdots, \quad \partial = \frac{\partial}{\partial x}, \]  

be a pseudo-differential operator whose coefficients \( u_i \) depending on the spatial coordinate \( x \). The KP hierarchy is defined as the following set of equations

\[ \frac{\partial L}{\partial t_n} = [(L^n)_+, L], \quad n = 1, 2, 3, \cdots. \]  

Here the subscript ” + ” means to take the purely differential part (nonnegative part) of the pseudo-differential operator, while the subscript ” − ” means to take the negative part. The well-known GD (Gelfand-Dickey) hierarchy is a reduction of the KP hierarchy with the constraint \( (L^m)_− = 0 \) for some natural number \( m \).

The KP hierarchy has been generalized to multicomponent cases with scalar pseudo-differential operators replaced by matrix-value ones [37–40]. Wu, Zhou, and Lu studied an extension of the KP hierarchy by considering two particular pseudo-differential operators [41, 42]. The infinite many symmetries of DS system are closely connected to the 2-component KP hierarchy [2, 43]. Konopelchenko, Landolfi and Taimanov studied the infinite many symmetries of DS system and pointed out that any symmetry induces an infinite family of geometrically different deformations of tori in \( \mathbb{R}^4 \) preserving the Willmore functional. They defined the DS hierarchy by considering the compatibility of undetermined differential operators in terms of \( \partial_z \) and \( \hat{\partial}_z \) [44–46] and gave examples of \( t_2 \) and \( t_3 \) flows. But how to characterize the compatibility for the infinite many equations in the whole hierarchy?

In this paper, we give a direct algorithm to construct the expression of the flows of the DS hierarchy by two scalar pseudo-differential operators involved with \( \partial, \hat{\partial} \) and proof the compatibility for these infinite flows. The \( (1+1) \) dimensional reduction and some examples are discussed in the final section.
2 The Davey-Stewartson hierarchy

Firstly we introduce two scalar pseudo-differential operators

\[ L = \partial + \sum_{j=1}^{\infty} u_j \partial^{-j}, \quad (5) \]

\[ \hat{L} = \hat{\partial} + \sum_{j=1}^{\infty} \hat{u}_j \hat{\partial}^{-j}, \quad (6) \]

in which the coefficients \( u_j = u_j(t_{mn}) \) and \( \hat{u}_j = \hat{u}_j(t_{mn}) \) depend on complex variables \( t_{mn} \) (\( m, n \) are nonnegative integers and \( m + n \geq 1 \)). In particular, \( t_{10} \equiv z, t_{01} \equiv \hat{z} \). Here and hereafter we denote \( \partial = \frac{\partial}{\partial z}, \hat{\partial} = \frac{\partial}{\partial \hat{z}} \) in this paper.

Similar to the definition of the KP hierarchy, we denote

\[ A_m = (L^m)_+, \quad (7a) \]

\[ B_n = (\hat{L}^n)_+, \quad (7b) \]

and we introduce

\[ \tilde{A}_m = (\partial^{-1} \circ q \circ A_m)_+, \quad (8a) \]

\[ \tilde{B}_n = (\hat{\partial}^{-1} \circ p \circ B_n)_+, \quad (8b) \]

here and hereafter the subscript “+” means to take the purely differential part (nonnegative part) of the pseudo-differential operators both in \( \partial \) and \( \hat{\partial} \), while the subscript “−” means to take the negative part, \( q = q(t_{mn}) \) and \( p = p(t_{mn}) \) depend on complex variables \( t_{mn} \) are unknown complex-valued functions in this DS hierarchy. The main result of this paper is the following theorem.

**Theorem.** The compatibility condition of the following linear system

\[ L \varphi = \sigma_1 \varphi, \quad (9a) \]

\[ \hat{L} \psi = \sigma_2 \psi, \quad (9b) \]
\[ \varphi_{t_{mn}} = A_m \varphi + \tilde{B}_n \psi, \]  
(9c)

\[ \psi_{t_{mn}} = \tilde{A}_m \varphi + B_n \psi, \]  
(9d)

is equivalent to the equations about the complex-valued functions \( q \) and \( p \):

\[ q_{t_{mn}} = B_n (q) - A^*_m (q), \]  
(10a)

\[ p_{t_{mn}} = A_m (p) - B^*_n (p), \]  
(10b)

in which \( \sigma_1 \) and \( \sigma_2 \) are two parameters, \( A_m, B_n, \tilde{A}_m, \tilde{B}_n \) are defined by (7)(8), \( F^* \) means the adjoint operator to \( F \). This compatible system (9) is defined as DS (Davey-Stewartson) hierarchy, and the infinite number of (2+1) dimensional nonlinear partial differential equations (10) are corresponding flow equations of the DS hierarchy.

**Remark 1.** Without the effect of \( \tilde{A}_m, \tilde{B}_n \), the above system is nothing but two separated KP hierarchies. In fact, \( \tilde{A}_m, \tilde{B}_n \) connect the two KP hierarchies and keep the compatibility. This is the key point for constructing this DS hierarchy.

**Remark 2.** Notice that \( \varphi_{t_{10}} = \varphi_z, \psi_{t_{01}} = \psi_{\hat{z}} \). This implies \( \frac{\partial}{\partial t_{10}} = \partial, \frac{\partial}{\partial t_{01}} = \hat{\partial} \). This is the reason we identify \( t_{10} \) with \( z \) and \( t_{01} \) with \( \hat{z} \) in this paper. Therefore, one obtains the following Dirac system

\[ \varphi_{\hat{z}} = \varphi_{t_{01}} = \tilde{B}_1 \psi = p \psi, \]  
(11a)

\[ \psi_{\hat{z}} = \psi_{t_{10}} = \tilde{A}_1 \varphi = q \varphi. \]  
(11b)

Then the compatibility condition of linear system (9) reads as follows

\[ \varphi_{\hat{z}} = p \psi, \]  
(12a)

\[ \psi_{z} = q \varphi, \]  
(12b)

\[ \varphi_{t_{mn}} = A_m \varphi + \tilde{B}_n \psi, \]  
(12c)
\[ \psi_{t_{mn}} = \tilde{A}_m \varphi + B_n \psi. \] (12d)

The above hierarchy is called the DS hierarchy for the reason that its $t_{22}$ flow is the well-known DS system \[2\].

To prove this main theorem, we need the following two lemmas.

**Lemma 1.** The pseudo-differential operators $L$ and $\hat{L}$ satisfy the following structure equations respectively

\[
\frac{\partial L^m}{\partial \hat{z}} + [L^m, R] = 0, \tag{13a}
\]

\[
\frac{\partial \hat{L}^n}{\partial z} + [\hat{L}^n, \hat{R}] = 0, \tag{13b}
\]

in which

\[ R = p \circ \partial^{-1} \circ q, \quad \hat{R} = q \circ \hat{\partial}^{-1} \circ p. \] (14)

Correspondingly, $A_m$ and $B_n$ satisfy

\[
\frac{\partial A_m}{\partial \hat{z}} + [A_m, R]_+ = 0, \tag{15a}
\]

\[
\frac{\partial B_n}{\partial z} + [B_n, \hat{R}]_+ = 0. \tag{15b}
\]

**Proof.** In fact, the equation \[9a\] yields $L^m \varphi = \sigma_1^m \varphi$. By taking the derivative of this equation with respect to $\hat{z}$, one obtains

\[
\frac{\partial L^m}{\partial \hat{z}} \varphi + L^m \varphi_{\hat{z}} - \sigma_1^m \varphi_{\hat{z}} = 0,
\]

which leads to

\[
\left( \frac{\partial L^m}{\partial \hat{z}} + L^m \circ p \circ \partial^{-1} \circ q - p \circ \partial^{-1} \circ q \circ L^m \right) \varphi = 0.
\]

Hence, (13a) is true. The discussion for (13b) is similar. By taking the differential part (nonnegative part) of (13a) and (13b), one obtains (15a) and (15b).

\(\square\)
Remark 3. From the structure equations (13), we can deduce that all the coefficients \( u_j \) and \( \hat{u}_j \) of the pseudo-differential operators \( L \) and \( \hat{L} \) depend on the two complex-valued functions \( p, q \) and their derivatives or integrals with respect to the independent variables \( z, \hat{z} \). The leading terms of \( L \) and \( \hat{L} \) can be obtained directly by (13) as follows

\[
\begin{align*}
    u_1 &= -\frac{\partial}{\partial \hat{z}}^1 ((pq)_z), \\
    u_2 &= -\frac{\partial}{\partial \hat{z}}^1 ((q_z p)_\hat{z}), \\
    u_3 &= -\frac{\partial}{\partial \hat{z}}^1 ((pqzz)_z) - \frac{\partial}{\partial \hat{z}}^1 ((pq)_z \frac{\partial}{\partial \hat{z}}^1 ((pq)_z)) + \frac{\partial}{\partial \hat{z}}^1 (pq \frac{\partial}{\partial \hat{z}}^1 ((pq)_zz)), \\
\end{align*}
\]

\[
\cdot \cdot \cdot \tag{16a}
\]

\[
\begin{align*}
    \hat{u}_1 &= -\frac{\partial}{\partial z}^1 ((pq)_\hat{z}), \\
    \hat{u}_2 &= -\frac{\partial}{\partial z}^1 ((p_z q)_\hat{z}), \\
    \hat{u}_3 &= -\frac{\partial}{\partial z}^1 ((pz q)_z) - \frac{\partial}{\partial z}^1 ((pq)_z \frac{\partial}{\partial z}^1 ((pq)_\hat{z})) + \frac{\partial}{\partial z}^1 (pq \frac{\partial}{\partial z}^1 ((pq)_zz)), \\
\end{align*}
\]

\[
\cdot \cdot \cdot \tag{16b}
\]

Lemma 2. The two pairs of differential operators \( A_m, \tilde{A}_m \) and \( B_n, \tilde{B}_n \) satisfy

\[
\begin{align*}
    \partial \circ \tilde{A}_m - q \circ A_m &= -A_m^*(q), \quad \text{(17a)} \\
    \hat{\partial} \circ \tilde{B}_n - p \circ B_n &= -B_n^*(p). \quad \text{(17b)}
\end{align*}
\]

Proof. In fact, \( A_m \) defined in (7) reads as

\[
A_m = \sum_{k=0}^{m} a_k \partial^k,
\]

where \( a_m = 1, a_{m-1} = 0 \) and \( a_k (k = 0, 1 \cdots, m - 2) \) are differential polynomials in \( u_j (j = 1, 2, \cdots, m - 1) \). Therefore,

\[
\begin{align*}
    \partial \circ \tilde{A}_m - q \circ A_m &= \partial \circ (\partial^{-1} \circ q \circ A_m)_+ - \partial \circ \partial^{-1} \circ q \circ A_m \\
    &= -\partial \circ (\partial^{-1} \circ q \circ A_m)_- \\
    &= -\partial \circ \left( \partial^{-1} \circ \left( \sum_{k=0}^{m} (-1)^k (qa_k)^{(k)} \right) \right) \\
    &= -A_m^*(q),
\end{align*}
\]

where \( (qa_k)^{(k)} = \frac{\partial^k(qa_k)}{\partial z^k} \). The proof of (17b) is similar. \( \square \)
With the help of the two lemmas, we come to the proof of the main theorem.

**Proof of the theorem.** In fact, the compatibility condition of (12a) and (12c) reads as

\[
\frac{\partial^2 \psi}{\partial z \partial t_{mn}} = \frac{\partial^2 \psi}{\partial t_{mn} \partial z}.
\]

By direct calculation, one obtains

\[
\frac{\partial^2 \psi}{\partial z \partial t_{mn}} = q_{t_{mn}} \varphi + qA_m \varphi + q\tilde{B}_n \psi,
\]

\[
\frac{\partial^2 \psi}{\partial t_{mn} \partial z} = \partial(\tilde{A}_m \varphi) + \partial(B_n \psi).
\]

Therefore,

\[
\frac{\partial^2 \psi}{\partial z \partial t_{mn}} - \frac{\partial^2 \psi}{\partial t_{mn} \partial z} = q_{t_{mn}} \varphi + \left(qA_m \varphi - \partial(\tilde{A}_m \varphi)\right) + \left(q\tilde{B}_n \psi - \partial(B_n \psi)\right) = 0.
\]

(18)

By the virtue of (17a) in Lemma 2, one obtains

\[
qA_m \varphi - \partial(\tilde{A}_m \varphi) = \left(q \circ A_m - \partial \circ \tilde{A}_m\right) (\varphi) = A^*_m(q) \varphi.
\]

(19)

The other part in (18) can be simplified by direct calculation as follows

\[
q\tilde{B}_n \psi - \partial(B_n \psi) = q\tilde{B}_n \psi - \frac{\partial B_n}{\partial z} \psi - B_n(q \varphi)
\]

\[
= q\tilde{B}_n \psi - \frac{\partial B_n}{\partial z} \psi - B_n \circ q \circ \hat{\partial}^{-1} \circ p(\psi)
\]

\[
= \left(q \circ \tilde{B}_n - \left(B_n \circ q \circ \hat{\partial}^{-1} \circ p\right)_+ - \frac{\partial B_n}{\partial z}\right)(\psi) - \left(\partial \circ B_n \circ \hat{\partial}^{-1} \circ p\right)_-(\psi)
\]

\[
= \left(\frac{\partial B_n}{\partial z} + [B_n, \tilde{R}]_+\right)(\psi) - \left(B_n(q) \circ \hat{\partial}^{-1} \circ p\right)(\psi)
\]

\[
= \left(\frac{\partial B_n}{\partial z} + [B_n, \tilde{R}]_+\right)(\psi) - B_n(q) \varphi.
\]

(20)
Then the structure equation (15b) in Lemma 1 gives
\[ q\tilde{B}_n\psi - \partial(B_n\psi) = -B_n(q)\varphi. \] (21)

Therefore the compatibility equation (18) can be simplified as
\[ \frac{\partial^2 \psi}{\partial z \partial t_{mn}} - \frac{\partial^2 \psi}{\partial t_{mn} \partial \hat{z}} = (q_{tmn} + A_m^*(q) - B_n(q)) \varphi = 0, \] (22)

which leads to the flow equation (10a).

Similarly, by considering the compatibility condition
\[ \frac{\partial^2 \varphi}{\partial \hat{z} \partial t_{mn}} = \frac{\partial^2 \varphi}{\partial t_{mn} \partial \hat{z}}, \]

one obtains the flow equation (10b).

\[ \square \]

3 Examples and (1+1) dimensional reduction

Some examples from the DS hierarchy and the (1+1) dimensional reduction are presented below.

**Example 1:**
By taking \( m = n = 2 \), then
\[ A_2 = \partial^2 + 2u_1 = \partial^2 - 2\partial^{-1}_z ((pq)_z), \] (23a)
\[ B_2 = \hat{\partial}^2 + 2\hat{u}_1 = \hat{\partial}^2 - 2\partial^{-1}_z ((pq)_{\hat{z}}), \] (23b)
\[ \tilde{A}_2 = q\partial - q_z, \] (23c)
\[ \tilde{B}_2 = p\hat{\partial} - p_{\hat{z}}. \] (23d)

Therefore, linear system (12) reads as
\[ \varphi_{\hat{z}} = p\psi, \] (24a)
ψ = qφ, \quad (24b)

ϕ_t = (2\partial^2 - 2\partial^{-1}_z((pq)_z)) \varphi + \left( p\hat{\partial} - p\hat{z} \right) \psi, \quad (24c)

ψ_t = (q\varphi - q\hat{z}) \varphi + \left( \hat{\partial}^2 - 2\partial^{-1}_z((pq)_z) \right) \psi. \quad (24d)

Then the following system

\begin{align*}
q_{t_{22}} &= B_2(q) - A_2(q) = q_{\hat{z}\hat{z}} - q_{zz} - \phi q, \quad (25a) \\
p_{t_{22}} &= A_2(p) - B_2^*(p) = p_{zz} - p_{\hat{z}\hat{z}} + \phi p, \quad (25b) \\
\phi_{z\hat{z}} &= 2(pq)_{\hat{z}\hat{z}} - 2(pq)_{zz}, \quad (25c)
\end{align*}

in the form (10) arising from (24) is nothing but the well-known DS system with \( p = \tilde{q} \).

**Example 2:**

By taking \( m = n = 3 \), then

\begin{align*}
A_3 &= \partial^3 + 3u_1 \partial + 3u_2 + 3u_1z \\
&= \partial^3 - 3\partial^{-1}_z((pq)_z) \partial - 3\partial^{-1}_z((q_zp)_z) - 3\partial^{-1}_z((pq)_{zz}), \quad (26a)
\end{align*}

\begin{align*}
B_3 &= \hat{\partial}_3 + 3\hat{u}_1 \hat{\partial} + 3\hat{u}_2 + 3\hat{u}_1\hat{z} \\
&= \hat{\partial}^3 - 3\partial^{-1}_z((pq)_z) \partial - 3\partial^{-1}_z((p_zq)_z) - 3\partial^{-1}_z((pq)_{\hat{z}\hat{z}}), \quad (26b)
\end{align*}

\begin{align*}
\tilde{A}_3 &= q\partial^2 - q_z \partial + q_{zz} + 3qu_1, \quad (26c) \\
\tilde{B}_3 &= p\hat{\partial}^2 - p_z \hat{\partial} + p_{\hat{z}\hat{z}} + 3pu_1. \quad (26d)
\end{align*}

Correspondingly, the adjoint operators read as

\begin{align*}
A_3^* &= -\partial^3 + 3u_1 \partial + 3u_2 + 3u_1z \\
&= -\partial^3 + 3\partial^{-1}_z((pq)_z) \partial - 3\partial^{-1}_z((q_zp)_z) - 3\partial^{-1}_z((pq)_{zz}), \quad (27a)
\end{align*}
\[ B_3^* = -\hat{\partial}^3 - 3\hat{u}_1\hat{\partial} + 3\hat{u}_2 + 3\hat{u}_1\hat{\partial} \]
\[ = -\hat{\partial}^3 + 3\partial_z^{-1} ((pq)\hat{z}) \hat{\partial} - 3\partial_z^{-1} ((pzq)\hat{z}) - 3\partial_z^{-1} ((pq)\hat{z}) \]  \hspace{1cm} (27b)

Then the communication of linear system (12) is equivalent to the following integrable system

\[ q_{t_{43}} = B_3(q) - A_3^*(q), \]  \hspace{1cm} (28a)
\[ p_{t_{43}} = A_3(p) - B_3^*(p). \]  \hspace{1cm} (28b)

**Example 3:**

By taking \( m = 2, n = 3 \), then the communication condition of the linear system (12) reads as

\[ q_{t_{23}} = B_3(q) - A_2^*(q), \]  \hspace{1cm} (29a)
\[ p_{t_{23}} = A_2(p) - B_3^*(p). \]  \hspace{1cm} (29b)

Now, we consider the (1+1) dimensional reduction of the DS hierarchy which includes the important integrable model NLS (nonlinear Schrödinger) equation. By considering the conjugate independent variables \( z = x + iy, \) \( \hat{z} = x - iy \) and the reduced condition \( \partial = i\hat{\partial}, \) i.e., \( x = -y, \) one obtains the following (1+1) reduction of the linear system (12) under the transform \( p \rightarrow \frac{1-\i}{2} p, q \rightarrow \frac{1+\i}{2} q \)

\[ \varphi_x = p\psi, \]  \hspace{1cm} (30a)
\[ \psi_x = q\varphi, \]  \hspace{1cm} (30b)
\[ \varphi_{t_{mn}} = \alpha_m \varphi + \beta_n \psi, \]  \hspace{1cm} (30c)
\[ \psi_{t_{mn}} = \alpha_m \varphi + \beta_n \psi, \]  \hspace{1cm} (30d)
where \(\alpha_m, \beta_n, \tilde{\alpha}_m\) and \(\tilde{\beta}_n\) are differential operators in terms of \(\partial_x\). The following nontrivial example of this reduced hierarchy is the NLS equation.

**Example 4:**

By taking \(m = n = 2\), the linear system (30) reads as

\[
\begin{align*}
\varphi_x &= p\psi, \\
\psi_x &= q\varphi, \\
\varphi_{t_2} &= \left(\frac{i}{2} \partial_x^2 - ipq\right)\varphi + \left(-\frac{i}{2} p \partial_x + \frac{i}{2} p x\right) \psi, \\
\psi_{t_2} &= \left(\frac{i}{2} q \partial_x - \frac{i}{2} q x\right)\varphi + \left(-\frac{i}{2} q \partial_x^2 + ipq\right) \psi.
\end{align*}
\]

The compatibility condition of (31) is equivalent to

\[
\begin{align*}
iq_t - q_{xx} + 2pq^2 &= 0, \\
ipt + p_{xx} - 2pq^2 &= 0,
\end{align*}
\]

which is exactly the classical NLS equation with \(p = \pm \bar{q}\).

### 4 Outlook

Research on the dispersionless integrable systems (integrable systems of hydrodynamic type) which arise from the commutation condition of vector fields Lax pairs, is an important subject. One important kind of dispersionless integrable systems comes from the semiclassical limit (dispersionless limit) of the classical integrable systems. In [47], we discussed the semiclassical limit of the DS system (2) and the relevant nonlinear Riemann-Hilbert problem. In [48], we defined a new class of dispersionless integrable systems called dDS.
(dispersionless Davey-Stewartson) hierarchy. In fact, the semiclassical limit (dispersionless limit) of the DS hierarchy (9)(12) is closely connected to the dDS hierarchy. We will show the details in the next separated paper.

Acknowledgements: This work has been supported by the Key Laboratory Foundation (No.6142209180306) and National Natural Science Foundation of China (No. 11501222).

References

[1] B. Kadomtsev and V. Petviashvili, The stability of solitary waves in weakly dispersive media, Doklady Akademii Nauk SSSR 192(6), 532-541 (1970).

[2] M. Ablowitz and P. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, London Math. Society, Lecture Notes (1991).

[3] A. Davey and K. Stewartson, On three dimensional packets of surface waves, Proc. R. Soc. A 338(1613), 101-110 (1974).

[4] M. Ablowitz and H. Segur, On the evolution of packets of water waves, J. Fluid Mech. 92(04), 691-715 (1979).

[5] M. Mulase, Solvability of the super KP equation and a generalization of the Birkhoff decomposition, Invent. Math. 92(1), 1-46 (1988).

[6] A. Bobenko and L. Bordag, Periodic multiphase solutions of the Kadomtsev-Petviashvili equation, J. Phys. A: Gen. 22(9), 1259 (1989).

[7] A. Fokas, Symmetries and Integrability, Stud. Appl. Math. 77(3), 253-299 (1987).

[8] J. Satsuma, N-Soliton Solution of the Two-Dimensional Korteweg-de Vries Equation, J. Phys. Soc. Jpn. 40(1), 286-290 (1976).

[9] S. Chakravarty and Y. Kodama, Soliton solutions of the KP equation and application to shallow water waves, Stud. Appl. Math. 123(1), 83-151 (2010).
[10] Y. Kodama, Young diagrams and N-soliton solutions of the KP equation, J. Phys. A: Gen. 37(46), 11169-11190 (2004).

[11] S. Deng, D. Chen, and D. Zhang, The Multisoliton Solutions of the KP Equation with Self-consistent Sources, J. Phys. Soc. Jpn. 72(9), 2184-2192 (2014).

[12] E. Date, M. Jimbo, M. Kashiwara, and T. Miwa, Quasi-Periodic Solutions of the Orthogonal KP Equation-Transformation Groups for Soliton Equations V, Publications of the Research Institute for Mathematical Sciences 18(3), 409-23 (2009).

[13] A. Wazwaz, Multiple-soliton solutions for the KP equation by Hirota’s bilinear method and by the tanh-`coth method, Appl. Math. Comput. 190(1), 633-640 (2007).

[14] D. Anker and N. C. Freeman, On the soliton solutions of Davey-Stewartson equation for long waves, Proc. R. Soc. London A 360, 529-540 (1978).

[15] V. A. Arkadiev, A. K. Pogrebkov, and M. C. Polivanov, Inverse scattering transform method and soliton solutions for Davey-Stewartson II equation, Physica D 36, 189-197 (1989).

[16] A. Nakamura, Explode-decay mode lump solitons of a two-dimensional nonlinear Schrödinger equation, Phys. Lett. A 88(2), 55-56 (1982).

[17] A. Nakamura, Exact explode-decay soliton solutions of a 2-dimensional nonlinear Schrödinger equation, J. Phys. Soc. Japan 51, 19-20 (1983).

[18] M. Boiti, J. Leon, L. Martina, and F. Pempinelli, Scattering of localized solitons in the plane, Phys. Lett. A 132, 432-439 (1988).

[19] B. Champagne and P. Winternitz, On the infinite-dimensional symmetry group of the Davey-Stewartson equations, J. Math. Phys. 29, 1-8 (1988).

[20] M. Omote, Infinite-dimensional symmetry algebras and an infinite number of conserved quantities of the (2+1)-dimensional Davey-Stewartson equation, J. Math. Phys. 29(12), 2599-2603 (1988).

[21] M. Tajiri, Similarity reductions of the one and two dimensional nonlinear schrödinger equations, J. Phys. Soc. Jpn. 52, 1908-1917 (1983).
[22] S. Ganesan and M. Lakshmanan, Singularity-structure analysis and Hirota’s bilinearisation of the Davey-Stewartson equation, J. Phys. A: Math. Gen. 103(20), L1143-L1147 (1987).

[23] A. S. Fokas and P. M. Santini, Recursion operators and bi-Hamiltonian structures in multidimensions. II, Commun. Math. Phys. 116(3), 449-474 (1988).

[24] P. M. Santini and A. S. Fokas, Recursion operators and bi-Hamiltonian structures in multidimensions. I, Commun. Math. Phys. 115, 375-419 (1988).

[25] A. Fokas, The Davey-Stewartson Equation on the Half-Plane, Commun. Math. Phys. 289(3), 957-993 (2009).

[26] O. Assainova, C. Klein, R. McLaughlin, and P. Miller, A study of the direct spectral transform for the defocusing Davey-Stewartson II equation the semiclassical limit, Commun. Pur. Appl. Math. 72(7), 1-74 (2019).

[27] J. Liang and C. Ruan, Two-dimensional doubly localized rogue waves in the Davey-Stewartson III equation, Rom. Rep. Phys. 73, 126 (2021).

[28] K. Tian, J. He, and A. Foerster, Negative generators of the Virasoro constraints for BKP hierarchy, Rom. Rep. Phys. 72, 101 (2020).

[29] I. Gel’Fand and L. Dikii, The resolvent and Hamiltonian systems, Func. Anal. Appl. 11(2), 93-105 (1977).

[30] M. Sato, E. Jimbo, Nonlinear integrable systems–classical theory and quantum theory, World Scientific, Singapore (1983).

[31] L. Dickey, Soliton equations and Hamiltonian systems, World Scientific, Singapore (1994).

[32] C. Terng and U. Karen, Bäcklund transformations and loop group actions, Commun. Pur. Appl. Math. 53(1), 1-75 (2012).

[33] Y. Watanabe, Hamiltonian structure of Sato’s hierarchy of KP equations and a coadjoint orbit of a certain formal Lie group, Lett. Math. Phys. 7(2), 99-106 (1983).
[34] A. Orlov and E. Shulman, Additional symmetries for integrable and conformal algebra representation, Lett. Math. Phys. 12(3), 171-179 (1986).

[35] M. Adler, T. Shiota, and P. Moerbeke, From the \( w_\infty \)-algebra to its central extension: a \( \tau \)-function approach, Phys. Lett. A 194, 33-43 (1994).

[36] M. Adler, T. Shiota, and P. Moerbeke, A Lax representation for the vertex operator and the central extension, Commun. Math. Phys. 171(3), 547-588 (1995).

[37] Y. Zhang, On a reduction of the multi-component KP hierarchy, J. Phys. A: Math. Gen. 32(36), 6461 (1999).

[38] M. Sato, Soliton Equations as Dynamical Systems on a Infinite Dimensional Grassmann Manifolds (Random Systems and Dynamical Systems), North Holl. Math. Stud. 81(1), 259-271 (1983).

[39] L. Dickey, On tau-functions of Zakharov-Shabat and other matrix hierarchies of integrable equations, Algebraic Aspects of Integrable Systems, 26, 49-74, Birkhäuser, Boston (1997).

[40] J. van de Leur, KdV type hierarchies, the string equation and \( W_{1+\infty} \) constraints, J. Geom. Phys. 17(2), 95-124 (1995).

[41] C. Wu and X. Zhou, An extension of the Kadomtsev-Petviashvili hierarchy and its hamiltonian structures, J. Geom. Phys. 106, 327-341 (2016).

[42] J. Lu and C. Wu, Bilinear equation and additional symmetries for an extension of the Kadomtsev-Petviashvili hierarchy, arXiv:1911.12727 (2019).

[43] B. Konopelchenko, Introduction to Multidimensional Integrable Equations, Plenum Press, New York (1992).

[44] B. Konopelchenko, Weierstrass Representations for Surfaces in 4D Spaces and Their Integrable Deformations via DS Hierarchy, Ann. Glob. Anal. Geom. 18(1), 61-74 (2000).

[45] B. Konopelchenko and G. Landolfi, Induced surfaces and their integrable dynamics. II. Generalized Weierstrass representations in 4D spaces and
deformations via DS hierarchy, Stud. Appl. Math. 104(2), 129-169 (2000).

[46] I. Taimanov, Surfaces in the four-space and the Davey-Stewartson equations, J. Geom. Phys. 56(8), 1235-1256 (2006).

[47] G. Yi, On the dispersionless Davey-Stewartson system: Hamiltonian vector field Lax pair and relevant nonlinear Riemann-Hilbert problem for dDS-II system, Lett. Math. Phys. 110, 445-463 (2020).

[48] G. Yi, On the dispersionless Davey-Stewartson hierarchy: Zakharov-Shabat equations, twistor structure and Lax-Sato formalism, arXiv:1812.10220v2 (2018).