A Polynomial-Time Algorithm and Applications for Matrix Sampling from Harish-Chandra–Itzykson-Zuber Densities

Jonathan Leake  
TU Berlin

Colin S. McSwiggen  
Brown University

Nisheeth K. Vishnoi  
Yale University

November 12, 2020

Abstract

Given two $n \times n$ Hermitian matrices $Y$ and $\Lambda$, the Harish-Chandra–Itzykson–Zuber (HCIZ) density on the unitary group $U(n)$ is $e^{\text{Tr}(U\Lambda U^* Y)} d\mu(U)$ where $\mu$ is the Haar measure on $U(n)$. Random unitary matrices distributed according to the HCIZ density are important in various settings in physics and random matrix theory. However, the basic question of how to sample efficiently from the HCIZ density has remained open. The main contribution of this paper is an algorithm to sample a matrix from a distribution that is $\varepsilon$-close to the given HCIZ density and whose running time depends polylogarithmically on $1/\varepsilon$. Interestingly, HCIZ densities can also be viewed as exponential densities on $U(n)$-orbits, and in this setting, they have been studied in statistics, machine learning, and theoretical computer science. Thus, our result has several direct applications, including 1) a polynomial-time algorithm to sample from matrix Langevin distributions studied in statistics [9, 8], 2) a polynomial-time algorithm to sample from continuous maximum entropy distributions [25, 24], which in turn implies efficient algorithms to infer quantum states from a given density matrix, and 3) a polynomial-time algorithm for the exponential mechanism for differentially private low-rank approximation [28, 7, 21].

The key obstacle in sampling from an HCIZ density is that the domain is a (non-convex) algebraic manifold, so that the entries of the desired sample matrix are highly correlated. To overcome this, we first consider a mapping that sends each Hermitian matrix $X$ to a natural ordering of the eigenvalues of all leading principal minors of $X$. The image of each $U(n)$-orbit under this map is a convex polytope, and the mapping reveals a recursive structure of $U(n)$-orbits which is hard to see directly in the ambient space of $U(n)$. Subsequently, we develop efficiently computable determinantal expressions for densities that arise in the intermediate steps of sampling from the polytope. Our proof-technique applies to other compact Lie groups and can be viewed as extending the widely studied notion of self-reducibility.
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1 Introduction

Let $U(n)$ denote the group of $n \times n$ unitary matrices and let $\mu$ denote the Haar probability measure on $U(n)$. Given $n \times n$ Hermitian matrices $Y$ and $\Lambda$, consider the following measure on $U(n)$:

$$e^{\text{Tr}(U\Lambda U^*)}d\mu(U).$$

The corresponding density is referred to as the Harish-Chandra–Itzykson–Zuber (HCIZ) density and has been extensively studied, implicitly and explicitly, in physics, random matrix theory, statistics, and theoretical computer science. A major result about the HCIZ density is that its integral over $U(n)$ admits an exact expression as a determinant.

**Theorem 1.1 (HCIZ integral formula)** For $n \times n$ Hermitian matrices $Y$ and $\Lambda$ with distinct eigenvalues $y_1 > \cdots > y_n$ and $\lambda_1 > \cdots > \lambda_n$ respectively, we have the following:

$$\int_{U(n)} e^{\text{Tr}(U\Lambda U^*)}d\mu(U) = \left( \prod_{p=1}^{n-1} p! \right) \frac{\det\left(\prod_{i,j} (y_i-y_j)(\lambda_i-\lambda_j)\right)}{\prod_{i,j} (y_i-y_j)(\lambda_i-\lambda_j)}.$$

Theorem 1.1 was first proved by Harish-Chandra [16] while developing a theory of Fourier analysis on semisimple Lie algebras and, independently, by Itzykson and Zuber [17]. See the blog post by Terry Tao [37] and the expository notes of the second author [29] for more on the HCIZ integral.

**Physics and random matrix theory.** Matrices distributed according to the HCIZ density are important in various settings in physics and random matrix theory. For instance, they appear in multi-matrix models in quantum field theory and string theory [17, 11]. They are related to models of coupled Gaussian matrices [17] as used to solve the Ising model on a planar random lattice [22, 5], and they also arise in certain models of the strong nuclear force [23, 27]. In particular, the moments of HCIZ distributed unitary matrices play a role in computing various physical quantities and have been studied extensively since the 1990’s [30, 35, 13, 14, 34]. In random matrix theory, the HCIZ integral arises in expressions for the joint spectral densities of a number of matrix ensembles, such as Wishart matrices [15]. However, the basic question about sampling from HCIZ density has remained open.

The problem of sampling from the HCIZ density can be equivalently cast as the problem of sampling from an exponential density specified by $Y$ on the $U(n)$-orbit of $\Lambda$. Let $O_\Lambda := \{ U \Lambda U^* \mid U \in U(n) \}$ denote the orbit of $\Lambda$ under the conjugation action of $U(n)$. Note that one can write any $X \in O_\Lambda$ as $X = U \Lambda U^*$ for some $U \in U(n)$, so that the density of the measure (1) can be rewritten as $e^{\langle Y, X \rangle} = e^{\text{Tr}(U\Lambda U^*)}$, where $\langle Y, X \rangle := \text{Tr}(Y^*X)$. Thus, we arrive at the following sampling problem.

**Problem 1.2 (Sampling from unitary orbits)** Given two $n \times n$ Hermitian matrices $\Lambda$ and $Y$, sample an $X \in O_\Lambda$ from the probability distribution

$$d\nu(X) \propto e^{\langle Y, X \rangle}d\mu_\Lambda(X),$$

where $\mu_\Lambda$ is the $U(n)$-invariant probability measure on $O_\Lambda$.

**Statistics.** Distributions of this kind have been studied previously under the name *matrix Langevin* or *matrix Bingham* [9, 8]. These distributions are over manifolds and are a special case of [43], when $\Lambda$ is the diagonal matrix with $k$ ones and $n-k$ zeros. In Section 2.5.2 of [8], the author describes an acceptance-rejection sampling algorithm for these distributions and discusses its time complexity. Their techniques are efficient for small constant values of $n$ and $k$, but beyond this their methods become inefficient. They leave it open as to whether or not their algorithm can be efficiently extended to more general cases.

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1 Although the formula (2) assumes that all $y_i$ and $\lambda_i$ are distinct, when this is not the case we can still use Theorem 1.1 to obtain an exact determinantal formula for the HCIZ integral, simply by applying L'Hôpital's rule to the right-hand side of (2).
Continuous maximum entropy distributions over matrix manifolds. In recent works [25, 24], distributions as in (3) arose as solutions to maximum entropy problems over manifolds, with applications to inferring a quantum state from its density matrix. Concretely, the authors study the following problem: given a matrix $A$ in the convex hull of $O\Lambda$, compute the probability density supported on $O\Lambda$ whose marginal is $A$ and that minimizes the Kullback-Leibler divergence to $\mu\Lambda$. As an example, if we let $\Lambda$ to be the diagonal matrix with exactly one 1 and rest 0s, the convex hull of $O\Lambda$ is exactly the set of PSD matrices with trace one – density matrices. Thus, in this case, the solution to the above entropy problem gives a way to infer a “pure state” from a given density matrix $A$, following the principle of maximum entropy [18, 19, 3, 36]. They show that the solution to the above optimization problem gives rise to the distribution of the form $e^{\langle Y^\star, X \rangle}d\mu\Lambda(X)$ for some $Y^\star$. Their main result is a polynomial-time algorithm to find this optimal solution that runs in time, roughly, the number of bits needed to represent $A$ and the distance of $A$ to the “boundary” of the convex hull of $O\Lambda$. While [25, 24] gave polynomial-time algorithms to compute the optimal value $Y^\star$, designing an algorithm to sample from the corresponding distribution was left as an open problem.

Differentially private algorithms for low-rank approximation. An important technique to obtain differentially private algorithms is the exponential mechanism due to [28]; see also [1, 12] for other approaches. In the context of rank-$k$ approximation, it amounts to sampling from an exponential density of the type (3) on the orbit corresponding to rank-$k$ projections [7, 21]. To the best of our knowledge, the only theoretical result on this problem is an approximate algorithm for the rank-1 case given in [21]. They left it a conceptually open problem to simplify their rank-1 algorithm and also come up with an algorithm for the rank-$k$ case which, in turn, would lead to an algorithm where the “utility” has an improved dependence on $k$.

Our contributions. The main result in this paper is a polynomial-time algorithm to sample from HCIZ density functions or, equivalently, from exponential densities on unitary orbits. This algorithm enables efficient numerical simulation of models in physics, statistics, and random matrix theory where HCIZ densities play a role.

**Theorem 1.3 (Main result)** There is an algorithm that, given an $\varepsilon > 0$ and $n \times n$ Hermitian matrices $\Lambda$ and $Y$, outputs an $L$-bit truncation $\hat{X}$ of a matrix $X$ that is distributed according to a measure that is $\varepsilon$-close in TV-distance to $\nu(X) \propto e^{\langle Y, X \rangle}d\mu\Lambda(X)$. The algorithm evaluates $\text{poly}(n, \|\Lambda\|, \|Y\|, \log 1/\varepsilon) \ n \times n$ determinants and $L$ is $\text{poly}(n, \|\Lambda\|, \|Y\|, \log 1/\varepsilon)$.

We emphasize that the dependence of our algorithm on $\varepsilon$ is polynomial in $\log 1/\varepsilon$. Moreover, the dependence on the norm of $Y$ and $\Lambda$ is expected as they appear in the exponent of the corresponding density. Our algorithm resolves the several open problems mentioned above. In particular, our main result implies 1) a polynomial-time algorithm to sample from matrix Langevin distributions [9, 8], 2) a polynomial-time algorithm to sample from the continuous maximum entropy distributions studied [25, 24], and 3) a polynomial-time algorithm for the exponential mechanism for differentially private low-rank approximation studied in [28, 7, 21].

A simple observation about the exponential density on a unitary orbit is that it is “stable” (see Section D). Thus, our main result might be useful for the problem of robust low-rank approximation introduced recently by [2]: given a matrix $A$ whose entries could be corrupted by an adversary, find a low-rank approximation to $A$. More generally, given the widespread occurrence of HCIZ densities in various disciplines, our result is likely to have further applications.

\footnote{For a real number $x$, its $L$-bit truncation is the largest rational number $\hat{x} \leq x$ whose denominator is $2^L$.}
One of the key difficulties in sampling from an HCIZ density is that its domain, a unitary orbit, is a non-convex algebraic manifold. The individual entries of the desired sample matrix are highly correlated due to the algebraic constraints that define the orbit, which makes it difficult to break the problem into lower-dimensional subproblems.

To overcome this, we use an alternative parameterization of unitary orbits based on the Rayleigh map, which sends a Hermitian matrix $X$ to a natural organization of the eigenvalues of all leading principal submatrices of $X$. The image of each $U(n)$-orbit under the Rayleigh map is a convex polytope called a Gelfand–Tsetlin (GT) polytope, which is cut out by linear inequalities given by the interlacing properties of the eigenvalues. This mapping reveals recursive structure of a given $U(n)$-orbit which is hard to see directly in the ambient space of matrices.

The Rayleigh map from a given $U(n)$-orbit to the corresponding GT polytope is not injective. However, one can show that 1) the HCIZ density on the orbit pushes forward to an exponential density on the polytope, and 2) the density is constant on the fibers of the Rayleigh map. Therefore, to solve the sampling problem on the orbit, it suffices to sample a point from an exponential density on the GT polytope, and then sample a Hermitian matrix uniformly at random from the fiber of the Rayleigh map over that point.

To solve the sampling problem on the GT polytope, we derive efficiently computable determinantal expressions for the density functions of each successive coordinate, conditioned on the preceding coordinates. Interestingly, these expressions can be written exactly as a ratio of determinants. Our result can be viewed as extending the widely studied notion of self-reducibility from the discrete world [20].

We give a detailed overview of the algorithm and the proof of Theorem 1.3 in Section 3. In Section 8, we outline extensions of Theorem 1.3 to the other compact classical Lie groups (the special orthogonal and compact symplectic groups).

## 2 Rayleigh triangles, Gelfand–Tsetlin polytopes, and unitary orbits

In this section we introduce some definitions and facts that we will need in what follows. In particular, we discuss two types of combinatorial objects that are fundamental to the geometry of Hermitian matrices: Rayleigh triangles and Gelfand–Tsetlin polytopes.

**Definition 2.1 (Rayleigh triangle)** For an integer $n \geq 1$, a Rayleigh triangle is a triangular array of real numbers $R = (R_{i,j})_{1 \leq i \leq j \leq n}$ satisfying the interlacing relations

$$R_{i-1,j} \geq R_{i,j-1} \geq R_{i,j} \quad \text{for all } 1 < i \leq j \leq n. \quad (4)$$

The vector $R_{\bullet,j} = (R_{1,j}, \ldots, R_{j,j}) \in \mathbb{R}^j$ is called the $j$th row of $R$, and $R_{\bullet,n} \in \mathbb{R}^n$ is called the top row. If we fix the top row of $R$, we can regard the numbers $R_{i,j}$, $j \leq n - 1$ as coordinates of a point in $\mathbb{R}^{n(n-1)/2}$.

The indexing for the Rayleigh triangle is different from that of matrix notation: the top row is indexed by $n$. For the sake of the reader, we give a picture of a Rayleigh triangle for $n = 4$:

$$
\begin{align*}
R_{1,4} & \quad R_{2,4} & \quad R_{3,4} & \quad R_{4,4} \\
R_{1,3} & \quad R_{2,3} & \quad R_{3,3} \\
R_{1,2} & \quad R_{2,2} \\
R_{1,1} & 
\end{align*}
$$
Definition 2.2 (Gelfand–Tsetlin polytope) Given a vector $\lambda \in \mathbb{R}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n$, the Gelfand–Tsetlin polytope $GT(\lambda)$ is the convex polytope in $\mathbb{R}^{n(n-1)/2}$ consisting of all Rayleigh triangles with top row equal to $\lambda$.

In other words, $GT(\lambda)$ is the polytope cut out by the interlacing inequalities (4), with $R_{\bullet,n} = \lambda$ fixed. The following is a corollary of a classical result of linear algebra known as Rayleigh’s Theorem.

Proposition 2.1 (Hermitian matrices, interlacing, and Rayleigh triangles) Given an $n \times n$ Hermitian matrix $X$, denote by $X[k]$ its $k$th leading submatrix (that is, the $k \times k$ submatrix in the upper left corner of $X$). Let $\lambda_{1,k} \geq \cdots \geq \lambda_{k,k}$ be the eigenvalues (which are real) of $X[k]$. Then the eigenvalues $(\lambda_{j,k})_{1 \leq j \leq n}$ of the leading submatrices of $X$ form a Rayleigh triangle, which we write as $\mathcal{R}(X)$.

Definition 2.3 (Type vector) The type vector of $R$ is defined by

$$\text{type}(R) = (R_{1,1}, R_{1,2} + R_{2,2} - R_{1,1}, \ldots, \sum_{i=1}^n R_{i,n} - \sum_{j=1}^{n-1} R_{j,n-1}).$$

Note that if $R = \mathcal{R}(X)$ for some Hermitian matrix $X$, then $\text{type}(R) = (X_{11}, \ldots, X_{nn})$ is just the diagonal of $X$.

Definition 2.4 (Orbits of $U(n)$) Given a vector $\lambda \in \mathbb{R}^n$ as above, write $\Lambda = \text{diag}(\lambda)$ and let $\mathcal{O}_\Lambda = \{U\Lambda U^* \mid U \in U(n)\}$ be the unitary conjugation orbit of $\Lambda$. Let $\mu_\Lambda$ be the uniform probability measure on $\mathcal{O}_\Lambda$, i.e., the unique probability measure on $\mathcal{O}_\Lambda$ that is invariant under the conjugation action of $U(n)$.

It can be shown that the image $\mathcal{R}(\mathcal{O}_\Lambda)$ is $GT(\lambda)$. In fact, the following stronger result is true: the uniform measure on $\mathcal{O}_\Lambda$ maps to the uniform measure on $GT(\lambda)$; see e.g. [4, 32, 33, 39].

Proposition 2.2 (Pushforward of measures) Under $\mathcal{R}$, the Haar measure $\mu_\Lambda$ on $\mathcal{O}_\Lambda$ pushes forward to the uniform probability measure

$$\mathcal{R}_* \mu_\Lambda = \frac{1}{\text{Vol}(GT(\lambda))} dP$$

on $GT(\lambda)$, where $dP$ is the standard Lebesgue measure.

Note that in the above result, $dP$ indicates the Lebesgue measure on the affine span of $GT(\lambda)$, which is the minimal affine subspace of $\mathbb{R}^{n(n-1)/2}$ containing $GT(\lambda)$. This distinction only matters when not all $\lambda_i$ are distinct, since in this case $GT(\lambda)$ has dimension less than $n(n-1)/2$ so that its volume in the ambient space $\mathbb{R}^{n(n-1)/2}$ is zero.

Proposition 2.2 allows us to prove the following crucial fact that the image of an exponential density on a unitary orbit is an exponential density on the GT polytope.

Theorem 2.3 (Pushforward of the HCIZ density) Let $y \in \mathbb{R}^n$ and $Y = \text{diag}(y)$. Then the pushforward of the measure $e^{\langle Y,X \rangle} d\mu_\Lambda(X)$ by the map $\mathcal{R}$ is

$$\mathcal{R}_*[e^{\langle Y,X \rangle} d\mu_\Lambda(X)] = \frac{1}{\text{Vol}(GT(\lambda))} e^{\langle y, \text{type}(P) \rangle} dP.$$  

Here, $\langle y, \text{type}(P) \rangle$ denotes the inner product between the two vectors.
Proof: First note that \((X_{11}, \ldots, X_{nn}) = \text{type}(R(X))\), because for a Hermitian matrix \(X\),
\[
\sum_{i=1}^{k} \lambda_{i,k} = \text{Tr}(X[k]),
\]
so that the \(k\)th coordinate of \(\text{type}(R(X))\) is just \(\text{Tr}(X[k]) - \text{Tr}(X[k - 1]) = X_{kk}\). Thus, we have
\[
\langle Y, X \rangle = \sum_{i=1}^{n} y_i X_{ii} = \langle y, \text{type}(R(X)) \rangle.
\]
The expression (6) then follows from the formula (5) in Proposition 2.2 for the pushforward of \(\mu_{\Lambda}\) by \(R\).

Finally, to prove the correctness of our sampling algorithm, we will need to describe the set of Hermitian matrices that map to a given Rayleigh triangle under \(R\).

Definition 2.5 (Fiber over a Rayleigh triangle) Given \(R \in GT(\lambda)\), the fiber of the map \(R\) over \(R\) is the set
\[
R^{-1}(R) = \{X \in \mathcal{O}_{\Lambda} \mid R(X) = R\}.
\]
The fiber \(R^{-1}(R)\) is a compact subset of \(\mathcal{O}_{\Lambda}\). The uniform probability measure on \(R^{-1}(R)\) is characterized by the property that if \(X\) is uniformly distributed in \(R^{-1}(R)\), then for \(1 < k \leq n\), \(X[k]\) is uniformly distributed on the compact manifold \(\mathcal{H}(X[k - 1]; R_{\bullet,k})\) of \(k \times k\) Hermitian matrices with eigenvalues \(R_{\bullet,k}\) and leading \((k - 1) \times (k - 1)\) submatrix equal to \(X[k - 1]\). We will show below in Proposition 4.2 that \(\mathcal{H}(X[k - 1]; R_{\bullet,k})\) is a product of spheres.

3 Technical overview

In this section, we give an overview of the algorithm and the proof of our main result (Theorem 1.3), leaving the full details to section 4. We restrict our discussion to the case of unitary orbits, but the cases of other compact groups are similar in spirit (see Section 8 for more discussion). We will also emphasize the important ideas and concepts from the various parts of the proof without going into too much detail. For the interested reader, we will provide links to the relevant sections of the full proof throughout this overview.

Throughout, \(\Lambda\) and \(Y\) will always be \(n \times n\) real diagonal matrices. The unitary orbit \(\mathcal{O}_{\Lambda}\) is defined as the set of all matrices \(U\Lambda U^*\) obtained by conjugating \(\Lambda\) by any unitary matrix. The measure \(d\mu_{\Lambda}(X)\) is the unitarily-invariant probability measure on \(\mathcal{O}_{\Lambda}\), which means that
\[
\int_{\mathcal{O}_{\Lambda}} f(X) \, d\mu_{\Lambda}(X) = \int_{\mathcal{O}_{\Lambda}} f(U\Lambda U^*) \, d\mu_{\Lambda}(X)
\]
for all integrable functions \(f\) and all unitary matrices \(U \in U(n)\). The goal of our algorithm is to return a sample \(X\) from the unitary orbit \(\mathcal{O}_{\Lambda}\) according to the distribution \(e^{\langle Y, X \rangle} d\mu_{\Lambda}(X)\).

3.1 The uniform case

Let us first consider a simple case, when \(Y = 0\). In this case the distribution we want to sample from is precisely the unitarily invariant (uniform) distribution on \(\mathcal{O}_{\Lambda}\). Unitary invariance implies sampling \(X\) from this distribution on \(\mathcal{O}_{\Lambda}\) is equivalent to sampling \(U\) from the Haar probability measure on \(U(n)\) and taking \(U\Lambda U^*\) as our sample in \(\mathcal{O}_{\Lambda}\). Sampling \(U\) from the Haar probability measure on \(U(n)\) then has a classical solution:
1. Sample a standard random Gaussian vector $v_1$ from $\mathbb{C}^n$, and normalize to get $u_1$.

2. Inductively sample a standard random Gaussian vector $v_k$ from the orthogonal complement of $\{u_1, \ldots, u_{k-1}\}$ in $\mathbb{C}^n$, and normalize to get $u_k$.

3. Construct the matrix $U$ with columns given by $u_1, u_2, \ldots, u_n$.

To see why this algorithm works, note that $u_1$ gives a uniform sample from the unit sphere in $\mathbb{C}^n$. Correctness of the algorithm then follows by induction. This shows that sampling from $O_\Lambda$ in the case of $Y = 0$ has a simple, intuitive solution.

**Difficulty in extending the algorithm for the uniform case.** For general $Y$ however, the situation quickly becomes more complicated. The first observation is that unitary invariance is immediately lost, since generically we have

$$e^{\langle Y, X \rangle} \neq e^{\langle Y, UXU^* \rangle}.$$  

This means the method used for $Y = 0$ breaks down, as there is no clear way to generalize the above simple algorithm to exponential weightings of the Haar measure. This is even true in the most basic case when $O_\Lambda$ is the set of rank-one projections (when $\Lambda = \text{diag}(1, 0, \ldots, 0)$), and the difficulty in this case was already realized in previous works \cite{8, 21, 25}. Even though the density is not unitarily invariant, there is still significant symmetry coming from the structure of the orbit $O_\Lambda$. This symmetry leads to the HCIZ integral formula (Theorem 1.1), which gives an efficiently computable formula for the partition function of the HCIZ density. (It should be noted that the proof of this formula is highly non-trivial: it relies on connections between the unitary group and its corresponding Weyl group.) Typically, such an explicit formula for the partition function can be translated into an algorithm for sampling, but it is not clear how to do this for the unitary orbit $O_\Lambda$.

### 3.2 Searching for self-reducibility

In the world of discrete distributions, the seminal work of \cite{20} gives a general way to sample from a distribution using an oracle for the associated partition function. The key property needed to utilize their results is that the distribution needs to be self-reducible. A problem is said to be self-reducible if, roughly speaking, a problem instance with input size $n$ can be reduced in polynomial time to another instance of the same problem with input size $n - 1$. Important examples of self-reducible problems include sampling of matchings and spanning trees of a graph.

As an example of where the ability to compute the partition function can lead to an efficient sampling algorithm, consider the case of sampling matchings from a graph. To uniformly sample a matching, one can first choose an edge $e$ in the graph and then compute the number $k_e$ of matchings that contain $e$, as well as the number $l_e$ of matchings that do not contain $e$. The edge $e$ is then included in the output matching with probability $k_e / l_e$, and the original problem can be reduced to finding a perfect matching in the smaller graph obtained by removing the vertices joined by $e$. To sample non-uniform matchings, the values of $k_e$ and $l_e$ are replaced by evaluations of the partition function.

In our world of continuous distributions on unitary orbits, it is not obvious how to perform a self-reduction similar to that of the discrete world, as described above in the case of sampling matchings, even though we have a formula for the partition function. The obstacle is that self-reducibility depends on preserving the original problem structure: we must reduce to an instance of the same problem, but with smaller input.
One approach towards this is to iteratively sample the individual entries or columns of the matrix, and then to interpret the remaining entries of the matrix as a smaller instance of the original problem. By conditioning on the previously selected entries or columns, one could try to compute the conditioned partition functions using the HCIZ formula. The issue with this approach is that the entries of a matrix $X$ in the unitary orbit $O_\Lambda$ are highly correlated due to the algebraic constraint that $X = U\Lambda U^*$ for some $UU^* = I$. This means that the problem of sampling from a given distribution on the orbit conditional on one or more matrix entries is a priori very different from the original problem, and much more complicated.

There is an alternative way to view a matrix $X \in O_\Lambda$: in terms of its eigenvalues. Of course the problem with viewing $X$ in terms of its eigenvalues is that every $X \in O_\Lambda$ has the same eigenvalues by definition. That is, the map $X \mapsto \text{eig}(X)$ maps $O_\Lambda$ to a single point, and thus the spectrum of $X$ is not enough information to represent $X$ in a useful way.

On the other hand, keeping track of both eigenvalues and eigenvectors is enough information to determine the matrix $X$ completely. Further, a recent result (see [38]) shows how one can (almost) recover the eigenvectors of $X$ using the eigenvalues of the principal submatrices of $X$. When eigenvalues are distinct, they show that

$$|v_{i,j}|^2 = \frac{\prod_{k=1}^{n-1}(\lambda_i(X) - \lambda_k(X_j))}{\prod_{k\neq i}(\lambda_i(X) - \lambda_k(X))},$$

where $\lambda_i(X)$ is the $i$th largest eigenvalue of $X$, $X_k$ is the principal submatrix of $X$ with the $k$th row and column removed, and $v_i$ is the eigenvector corresponding to $\lambda_i(X)$. That is, with the extra information of the eigenvalues of the principal submatrices of $X$, one can determine the eigenvectors of $X$ up to the (complex) sign of the entries.

This is a good sign for us, as it hints at some inductive structure in the eigenvalues of $X$. Can we now understand this relationship between the eigenvectors and eigenvalues of principle submatrices in some recursive manner? As a matter of fact, by considering the matrix $X$ in terms of all of its leading principal submatrices, we are able to prove a similar result (see Section 5). And not only that, but it turns out that this eigenvector information is sufficient for our purposes.

**Self-reducibility in the space of eigenvalues.** This suggests a natural self-reducible structure for the unitary orbit $O_\Lambda$ via the principal submatrices. The Rayleigh map $\mathcal{R}$ (Definition 2.1) maps a matrix $X \in O_\Lambda$ to the length $\binom{n+1}{2}$ vector of the eigenvalues of all the leading principal minors of $X$. These eigenvalues are organized in the form of a triangle called the Rayleigh triangle, denoted $\mathcal{R}(X) = (R_{i,j})_{1 \leq i \leq j \leq n}$ where

$$R_{i,j} := \text{the } i\text{th largest eigenvalue of the top left } j \times j \text{ principal submatrix.}$$

(Note that $R_{\bullet,n}$ are the eigenvalues of $\Lambda$, and counter to matrix indexing intuition, we refer to $R_{\bullet,n}$ is the top row of the Rayleigh triangle. See the picture just following Definition 2.1.) Organizing eigenvalues into a triangle like this then makes the self-reducible structure clear: fixing the top $n - k + 1$ rows of the triangle, $R_{\bullet,n}, \ldots, R_{\bullet,k}$ and leaving the bottom $k$ rows free gives a lower-dimensional Rayleigh triangle and corresponds precisely to the orbit of $U(k)$ given by $O_{R_{\bullet,k}}$.

We now have a self-reducible way to view the elements $X \in O_\Lambda$ in terms of their eigenvalues, but how does this help us to sample from $O_\Lambda$? By Cauchy’s interlacing theorem, for all $X \in O_\Lambda$ the Rayleigh triangle $\mathcal{R}(X)$ is a element of a polytope in $\mathbb{R}^{\binom{n+1}{2}}$ cut out by the inequalities

$$R_{i,j+1} \geq R_{i,j} \geq R_{i+1,j+1} \quad \text{for all valid } i, j.$$
In fact the converse is also true: the image $\mathcal{R}(O_\Lambda)$ is the whole polytope cut out by these inequalities, called the Gelfand–Tsetlin (GT) polytope and denoted $GT(\lambda)$ where $\lambda$ is the vector of eigenvalues of $\Lambda$. What is special about the Rayleigh map $\mathcal{R}$ is then that it projects the uniform measure $d\mu_\Lambda$ on $O_\Lambda$ to the Lebesgue (uniform) measure on the GT polytope.

The questions we now need to answer are as follows.

1. How does the Rayleigh map $\mathcal{R}$ project the exponential HCIZ density from the unitary orbit to the GT polytope?

2. How do we sample from the corresponding distribution on the GT polytope?

3. How do we transfer that sample back to the unitary orbit?

The answers here are reasonable: for (1) the Rayleigh map projects the exponential HCIZ density to an exponential density on the GT polytope (see Theorem 2.3), and for (2) and (3) we have algorithms (see Section 4.1). For the understanding of the reader, we first demonstrate this explicitly in the case of rank-one projections.

### 3.3 The case of rank-one projections

Let us now look at the simplest choice of $\Lambda$: the case where $\Lambda$ is the diagonal matrix with entries $1, 0, 0, \ldots, 0$. This means that $O_\Lambda$ is the set of Hermitian positive semidefinite (PSD) rank-one projections. In this case, each leading principal submatrix of a given $X \in O_\Lambda$ has at most one non-zero eigenvalue. Thus, the entries of the Rayleigh triangle of $X$ are all zero except for $R_{1,j}$, for which we have $1 = R_{1,n} \geq R_{1,n-1} \geq \cdots \geq R_{1,1} \geq 0$. This means that the GT polytope in this case is isomorphic to a simplex by considering the values of $R_{1,n} - R_{1,n-1}, \ldots, R_{1,2} - R_{1,1}$, and $R_{1,1}$, which sum to 1. Since the principal submatrices are all rank at most 1, these differences are the differences of the traces of the submatrices of $X$, which are precisely equal to the diagonal entries of $X$. Hence the map $\text{diag}(X)$, which picks out the diagonal entries of $X$, is equivalent to the Rayleigh map $\mathcal{R}$ in this case, and the image diag($O_\Lambda$) is the standard simplex $\Delta_n$. Therefore diag($X$) maps the uniform measure $d\mu_\Lambda(X)$ to the Lebesgue (uniform) measure on $\Delta_n$, and we can determine the measure on the GT polytope corresponding to the HCIZ density via

$$e^{\langle Y, X \rangle} d\mu_\Lambda(X) = e^{\langle y, \text{diag}(X) \rangle} d\mu_\Lambda(X) \xrightarrow{\text{diag}} e^{\langle y, x \rangle} dx,$$

where $y$ is the vector of diagonal entries of $Y$ (which is itself a diagonal matrix). This suggests an algorithm for sampling from our exponentially weighted distribution $e^{\langle Y, X \rangle} d\mu_\Lambda(X)$ on $O_\Lambda$ when $Y$ is diagonal:

1. Sample $x$ from $\Delta_n$ according to the distribution $e^{\langle y, x \rangle} dx$ where $y = \text{diag}(Y)$.

2. Convert $x$ into a rank-one PSD projection $X \in O_\Lambda$.

We now briefly discuss how to actually implement each of these steps.

First, we want to sample $x$ from $\Delta_n$ according to the distribution $e^{\langle y, x \rangle} dx$. Up to change of variables, the integral on the simplex can be decomposed into a sequence of integrals on its coordinates (up to scalar):

$$\int_{O_\Lambda} e^{\langle Y, X \rangle} d\mu_\Lambda(X) = \int_{\Delta_n} e^{\langle y, x \rangle} dx = \int_0^1 \int_0^{1-x_1} \cdots \int_0^{1-x_1-\cdots-x_{n-2}} e^{\langle y, x \rangle} dx.$$
We can then use this expression to construct a coordinate-by-coordinate sampling algorithm for the simplex. Suppose we have already sampled the first \( k \) coordinates \( x_1, \ldots, x_k \). The cumulative distribution function (CDF) of \( x_{k+1} \), conditioned on those first \( k \) coordinates, can then be written as
\[
F_{k+1}(\beta) \propto \int_0^\beta \int_0^\beta \cdots \int_0^\beta e^{\langle y, x \rangle} dx_{n-1} \cdots dx_{n+1}
\]
for \( \beta \in [0, C] \) where \( C := 1 - x_1 - \cdots - x_k \). The self-reducibility of the simplex now comes into play. Making the change of variables \( x'_j = \frac{x_j}{C} \) for \( j > k \) and \( \beta' = \frac{\beta}{C} \), we have
\[
F_{k+1}(\beta) \propto \int_0^{\beta'} \int_0^{\beta'} \cdots \int_0^{\beta'} e^{C \langle y, x' \rangle} dx'_{n-1} \cdots dx'_{n+1},
\]
which is an integral over a lower-dimensional simplex. That is, the conditioned coordinates of the simplex correspond to coordinates of lower-dimensional simplices, which in turn correspond to lower-dimensional unitary orbits through the diag(\( X \)) map. This means the HCIZ formula gives an effective determinantal expression for the CDF of each conditioned coordinate, and this can be utilized to sample \( x \in \Delta_n \) coordinate-by-coordinate via the inverse CDF method (see Appendix A for more detail).

Note that \( e^{\langle y, x \rangle} \) is a log-concave density function. Thus, instead of using the techniques described above, one can go via the route of sampling from log-concave distributions on polytopes (see [26] for example). This is in contrast to the algorithm used in [21], where the sampling connection between rank-one projections and the simplex had not yet been established. Using these polytope sampling techniques we obtain an analogue to Theorem 1.3, but the dependence on \( \Lambda \) opposed to poly(\( \log(1/\varepsilon) \)) as guaranteed by the theorem using the techniques described above.

Second, we want to convert \( x \) into a rank-one PSD projection \( X \in \mathcal{O}_\Lambda \) in a way which is compatible with our exponentially weighted distribution on \( \mathcal{O}_\Lambda \). Towards this, we first observe that the set of all \( X \) for which diag(\( X \)) = \( x \) is given by
\[
\text{diag}^{-1}(x) = \{ X \in \mathcal{O}_\Lambda : X = vu^* \text{ where } v^* = (e^{-i\theta_1} \sqrt{x_1}, \ldots, e^{-i\theta_n} \sqrt{x_n}) \}.
\]
Secondly, for diagonal \( Y \) the density \( e^{\langle Y, X \rangle} \) does not depend on the choice of \( \theta_1, \ldots, \theta_n \). Therefore if we restrict our distribution on \( \mathcal{O}_\Lambda \) to the subset diag\(^{-1}(x) \), we obtain the uniform distribution. This means we can convert \( x \) into a rank-one PSD projection by uniformly randomly sampling \( e^{i\theta_1}, \ldots, e^{i\theta_n} \) independently from the unit circle and setting \( X := vu^* \) where \( v^* = (e^{-i\theta_1} \sqrt{x_1}, \ldots, e^{-i\theta_n} \sqrt{x_n}) \). Combining these two steps—sampling from the simplex and then transferring that sample to \( \mathcal{O}_\Lambda \)—gives us an algorithm for sampling from \( \mathcal{O}_\Lambda \) according to our exponentially weighted density, completing the proof of the main result in the rank-one case.

**Obstacles towards extending to general \( \Lambda \).** Unfortunately, extending this algorithm beyond the rank-one case immediately runs into issues. Although a Rayleigh triangle has a natural self-reducible structure given by the rows of the triangle, it is not at all clear how to pass this self-reducibility down to the coordinates of each row. That is, for each row we can utilize the same strategy as in the rank-one case above since the bottom \( k \) rows of the triangle correspond to a lower-dimensional unitary orbit. However, this means that we need to be able to sample from the GT polytope row-by-row, which means we then need to find a further self-reducible structure for the entries of each row.

Beyond this, converting a sample from the GT polytope back to the unitary orbit is now also more complicated. In the case of rank-one projections, determining the fiber diag\(^{-1}(x) \cong \mathcal{R}^{-1}(x) \) was straightforward and led to a simple method for sampling an element of the orbit \( \mathcal{O}_\Lambda \). For
general Λ, the fiber $R^{-1}(x)$ does not have such a clear description. We will need to study further the relationship between the unitary orbit $O_Λ$ and the corresponding GT polytope to understand how, if at all, it is possible to adapt the sampling technique used for rank-one projections.

### 3.4 Stronger self-reducibility in the case of general Λ

In the case of general Λ, we can utilize the same over-arching algorithm that was used in the rank-one case: sample from the GT polytope, and then transfer back to the unitary orbit. The first question we need to answer is what the measure on the GT polytope should look like. We know that the Rayleigh map transfers the uniform distribution $d\mu_Λ(X)$ on the unitary orbit to the uniform distribution on the polytope, but what about the distribution $e^{⟨Y,X⟩}d\mu_Λ(X)$?

Since $Y$ is diagonal we can write

$$e^{⟨Y,X⟩}d\mu_Λ(X) = e^{⟨y,\text{diag}(X)⟩}d\mu_Λ(X),$$

where $y = \text{diag}(Y)$. In the rank-one case, the Rayleigh map $R$ was equivalent to the $\text{diag}(X)$ map, and this meant that the projected measure was given by $e^{⟨y,x⟩}dx$. To handle the general case, we need a map which takes a Rayleigh triangle $R ∈ GT(λ)$ to the diagonal vector of the corresponding $X ∈ O_Λ$. This is precisely the $\text{type}(R)$ map (Definition 2.3), which computes differences of the traces of the successive principal submatrices:

$$\text{type}(R) = \left(R_{1,1}, R_{1,2} + R_{2,2} - R_{1,1}, \ldots, \sum_{i=1}^{n} R_{i,n} - \sum_{j=1}^{n-1} R_{j,n-1}\right).$$

This definition implies $\text{type}(R) = \text{diag}(X)$ whenever $R = R(X)$. With this, we can more precisely state our sampling algorithm at a high level.

1. Sample a Rayleigh triangle $R = (R_{i,j})$ from the associated GT polytope according to the distribution $e^{⟨y,\text{type}(R)⟩}dR$, where $y = \text{diag}(Y)$.

2. Convert $R$ into an element $X ∈ O_Λ$ of the unitary orbit.

However, we now still need to answer the questions posed by the previous section of how to generalize the rank-one ideas to the general case. As discussed above, sampling simplex coordinates is analogous to sampling rows of the GT polytope here: the bottom $k$ rows of the Rayleigh triangle correspond to an orbit of $U(k) ⊂ U(n)$.

Note that we can still write the integral over the GT polytope in a coordinate-by-coordinate fashion:

$$\int_{GT(λ)} e^{⟨y,\text{type}(R)⟩}dR = \int_{λ_1}^{λ_2} \ldots \int_{λ_{n-1}}^{λ_n} \int_{R_{1,1}}^{R_{1,n-1}} \ldots \int_{R_{n,1}}^{R_{n,n-1}} \int_{R_{2,2}}^{R_{n-1,n-1}} \int_{R_{n-1,n-1}}^{R_{n,n-1}} \ldots \int_{R_{2,2}}^{R_{n,n-1}} e^{⟨y,\text{type}(R)⟩}dR.$$  \(7\)

The problem remains that if we are halfway through sampling one row of a Rayleigh triangle, there is no change of variables which allows us to view the remaining coordinates as coming from a smaller GT polytope.

**Sampling from the GT polytope.** Our first main technical contribution is then how to sample from the above exponentially weighted density function on the GT polytope using a stronger notion of self-reducibility (see Section 4.1.1). We know by row-by-row self-reducibility discussed above that the first coordinate of each row of a Rayleigh triangle $R$ has a determinantal formula (via the HCIZ
integral formula, Theorem 1.1 for its CDF, conditioned on the higher rows. The question is then: do interior entries of each row of $R$ also have determinantal CDF formulas?

Using the decomposition given in Equation 1 we can obtain CDF formulas for interior entries of a given row of $R$ by applying entry-by-entry integrals to the formula for the first entry, which we know is given by the HCIZ formula. The following simple but powerful lemma (see Lemma 6.2) then helps us to convert these entry-by-entry integrals into derivatives, which are easier to work with:

$$
\int_{t_{k+1}}^{t_k} \cdots \int_{t_{n-1}}^{t_n} \det(e^{\alpha_i x_i})_{i,j=1}^{n-1} \, dx_{n-1} \cdots dx_k \bigg|_{x_1=t_1, \ldots, x_{k-1}=t_{k-1}} = \partial_{i_1} \cdots \partial_{i_{k-1}} \frac{\det(e^{\alpha_i t_i})_{i,j=1}^{n-1}}{\alpha_1 \alpha_2 \cdots \alpha_{n-1}} \bigg|_{\alpha_n=0}.
$$

These derivatives can then be passed through the determinant directly to the matrix itself, since each column of the matrix depends on a different variable $t_j$.

This leads to the following interesting and somewhat surprising fact: the conditioned CDF of every coordinate $R_{i,j}$ of the Rayleigh triangle is given by a determinantal formula. Specifically, we prove (see Proposition 4.1) that the conditioned CDF of $R_{i,j}$ can be expressed as

$$F_{i,j}(x) \propto \det(M_{i,j}[y,R](x)),$$

where for $x \in [R_{i+1,j+1}, R_{i,j+1}]$ and $\alpha_k := y_k - y_{j+1}$ we define

$$M_{i,j}[y,R](x) =
\begin{bmatrix}
\alpha_1 e^{\alpha_1 R_{i,j}} & \cdots & \alpha_1 e^{\alpha_1 R_{i-1,j}} & e^{\alpha_1 x} & e^{\alpha_1 R_{i+1,j+1}} & \cdots & e^{\alpha_1 R_{n,j+1}} \\
\alpha_2 e^{\alpha_2 R_{i,j}} & \cdots & \alpha_2 e^{\alpha_2 R_{i-1,j}} & e^{\alpha_2 x} & e^{\alpha_2 R_{i+1,j+1}} & \cdots & e^{\alpha_2 R_{n,j+1}} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\alpha_j e^{\alpha_j R_{i,j}} & \cdots & \alpha_j e^{\alpha_j R_{i-1,j}} & e^{\alpha_j x} & e^{\alpha_j R_{i+1,j+1}} & \cdots & e^{\alpha_j R_{n,j+1}} \\
0 & \cdots & 0 & 1 & 1 & \cdots & 1
\end{bmatrix},
$$

when the conditioned entries of $R$ are distinct (see Definitions 6.1 and 7.1).

Note that this is stronger than the HCIZ formula, as it holds for all coordinates, including those internal entries which do not correspond to $U(k)$ orbits for $k < n$ via the row-by-row self-reducibility discussed above. Thus we have an efficiently computable expression for the conditioned CDF $F_{i,j}(x)$, and therefore coordinate-by-coordinate application of the inverse CDF method (see Appendix A) yields the desired sample $R$ from the GT polytope.

It is worth mentioning again that the above techniques could be replaced by more general techniques for sampling from log-concave distributions on polytopes, at the same cost of polynomial dependence on $1/\varepsilon$ instead of $\log(1/\varepsilon)$ as discussed above. That said, at the end of the day our techniques rely only on the computing of various determinants, and therefore have the added benefit that one does not need to compute mixing-time bounds needed to utilize the more general techniques.

### 3.5 From the GT polytope back to the unitary orbit

Supposing we have constructed a sample $R$ of the GT polytope using the coordinate-by-coordinate inverse CDF method described above, the final step is to convert $R$ into a sample from the unitary orbit $O_\Lambda$. In the case of the simplex, this was easy because it is easy to describe the fiber $\text{diag}^{-1}(x)$ as well as the restriction of the distribution on the unitary orbit to this fiber. In the case of the GT polytope and the Rayleigh map however, determining $R^{-1}(R)$ and the associated distribution is more complicated.

Fortunately though, we can break the problem down into more manageable pieces corresponding to the row-by-row self-reducible structure discussed above. Observe that for any $k$ we can identify
U(k−1) with the subgroup of U(k) consisting of unitary matrices that have a 1 in the bottom right corner and zeros in all other positions of the last row and column. Inducting on this observation, we obtain inclusions $U(1) \hookrightarrow U(2) \hookrightarrow \cdots \hookrightarrow U(n−1) \hookrightarrow U(n)$, and these inclusions correspond precisely to sub-triangles of our sampled Rayleigh triangle $R$ (the bottom 1, 2, $\ldots$, $n−1$, $n$ rows of the triangle respectively).

This allows us to induct on the size of the unitary group. Assuming that we have an $(n−1) \times (n−1)$ matrix sample $X_0$ from the $U(n−1)$ orbit associated to the bottom $n−1$ rows of $R$, we just need to sample an $X \in \mathcal{O}_\Lambda$ which has $X_0$ as its top-left principal submatrix. That is, given such an $X_0$, we need to sample some

$$X = \begin{bmatrix} X_0 & v \\ v^* & c \end{bmatrix} \in \mathcal{O}_\Lambda.$$ 

To sample such an $X$, first note that the top row of $R$ (that is, the eigenvalues of $\Lambda$) determines the trace of $X$, which specifies deterministically the value of $c$. This leads to a crucial observation: all possible values of the matrix $X$ have the same diagonal entries, and hence our density function $e(Y,X)$ is constant since $Y$ is a diagonal matrix. This means that we may sample $X$ uniformly from the set of all $X$ with the above block form.

With this, we now describe in detail how to sample such a matrix $X$ (see Section 4.1.2).

**The case of distinct eigenvalues.** We first demonstrate how to do this in a simplified case: when $X_0$ is a diagonal matrix with distinct eigenvalues. In this case, we make the following easy observation for $U \in U(n)$:

$$U \begin{bmatrix} X_0 & v \\ v^* & c \end{bmatrix} U^* = \begin{bmatrix} X_0 & w \\ w^* & c \end{bmatrix}$$

for some $v, w \iff U$ is diagonal $\iff U = \text{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n})$.

This immediately gives rise to an algorithm for sampling $X$ of the above block form:

1. Construct any $X \in \mathcal{O}_\Lambda$.

2. Sample $e^{i\theta_1}, \ldots, e^{i\theta_n}$ uniformly and independently from the unit circle.

3. Defining $U := \text{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n})$, our sample is then $UXU^*$.

What remains to be done then is to construct some $X \in \mathcal{O}_\Lambda$, which is equivalent to constructing a valid value of $v$. To this end, we use the special form of $X$ to write down its characteristic polynomial. Letting $X_0 = \text{diag}(R_{1,n−1}, \ldots, R_{n−1,n−1})$, we want to choose $v$ such that

$$\prod_{i=1}^{n} (t - \lambda_i) = \det \left( tI - \begin{bmatrix} X_0 & v \\ v^* & c \end{bmatrix} \right) = (t - c) \prod_{i=1}^{n−1} (t - R_{i,n−1}) + \sum_{i=1}^{n−1} |v_i|^2 \prod_{j \neq i} (t - R_{j,n−1}).$$

Since the values of $R_{\bullet,n−1}$ are distinct, we obtain $n−1$ equations by plugging in $t = R_{k,n−1}$ for each value of $k \in \{1, \ldots, n−1\}$:

$$\prod_{i=1}^{n} (R_{k,n−1} - \lambda_i) = |v_k|^2 \prod_{i \neq k} (R_{k,n−1} - R_{i,n−1}) \implies |v_k|^2 = \frac{\prod_{i=1}^{n} (R_{k,n−1} - \lambda_i)}{\prod_{i \neq k} (R_{k,n−1} - R_{i,n−1})}.$$

This gives us a formula for a choice of $v_k$, so long as the right-hand side is non-negative. In fact, it is always non-negative because the values of $\lambda_\bullet$ and $R_{\bullet,n−1}$ are interlacing by Cauchy’s interlacing theorem for Hermitian matrices. By choosing $v_k \geq 0$ which satisfy the above equalities, we have constructed a valid $X$ from our orbit, and applying the above algorithm gives the desired sample of $\mathcal{O}_\Lambda$. 

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The general case. Handling the cases of non-distinct eigenvalues and non-diagonal $X_0$ is then straightforward. First, if $X_0$ is diagonal with non-distinct ordered eigenvalues, the set of unitary matrices which preserves the block form of $X$ becomes larger:

- **distinct case:** $UXU^* \in O_\Lambda \iff U \in U(1) \times U(1) \times \cdots \times U(1)$
- **non-distinct case:** $UXU^* \in O_\Lambda \iff U \in U(m_1) \times U(m_2) \times \cdots \times U(m_p) \times U(1)$,

where $m_1, m_2, \ldots, m_p$ are the multiplicities of the eigenvalues of $X_0$. That is, we simply need to replace step 2 of the above algorithm by

2. Sample $U_1, \ldots, U_p, U_{p+1}$ uniformly from $U(m_1), \ldots, U(m_p), U(1)$ respectively.

Algorithms to sample uniformly from unitary groups are well-known and were discussed above.

Finally, handling the non-diagonal case is even easier. Letting $U_0 \in U(n-1)$ be such that $U_0X_0U_0^* = D_0$ is diagonal, we reduce to the previous cases by considering

$$
\begin{bmatrix}
U_0 & 0 \\
0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
U_0^* & 0 \\
0 & 1 \\
\end{bmatrix}
= 
\begin{bmatrix}
U_0X_0U_0^* & U_0v \\
(U_0v)^* & c \\
\end{bmatrix}
= 
\begin{bmatrix}
D_0 & U_0v \\
(U_0v)^* & c \\
\end{bmatrix}.
$$

We then first sample a matrix $X'$ by applying the above algorithm to the right-hand side matrix above, since $D_0$ is diagonal. We then obtain our desired sample via the inverse conjugation by $U_0$:

$$
X = \begin{bmatrix} U_0^* & 0 \\ 0 & 1 \end{bmatrix} X' \begin{bmatrix} U_0 & 0 \\ 0 & 1 \end{bmatrix}.
$$

Combining all of this then yields an algorithm which constructs a matrix $X$ in the unitary orbit $O_\Lambda$ from the given Rayleigh triangle $R$ in the GT polytope.

As a final note, while every part of this algorithm looks fairly specific to the unitary group, there are analogous objects to the GT polytope, the HCIZ formula, the Rayleigh map, etc. which would lead to a similar algorithm for other compact Lie groups. See Section 8 for more discussion and details.

4 The sampling algorithm and proof of Theorem 1.3

In this section we present a proof of our main result (Theorem 1.3), a polynomial-time algorithm for sampling from an exponential density on a $U(n)$-conjugation orbit of a Hermitian matrix. We first describe the algorithm. We then prove that it produces the correct output and that it runs in polynomial time. We first remark that there is no loss of generality in Theorem 1.3 if we assume that $Y$ and $\Lambda$ are both diagonal. To see this, note that by the spectral theorem $Y$ and $\Lambda$ are both unitarily diagonalizable. We therefore have $O_\Lambda = O_{\text{diag}(\Lambda)}$, where $\lambda = \text{eig}(\Lambda)$, and we can write $Y = V\cdot\text{diag}(y)\cdot V^*$ for some $V \in U(n)$. Then to sample from $e^{(Y,X)}d\mu_\Lambda(U) = e^{(\text{diag}(y),V^*XY)}d\mu_{\text{diag}(\lambda)}(X)$, we can instead sample from

$$
e^{(\text{diag}(y),X)}d\mu_{\text{diag}(\lambda)}(X)
$$

and then conjugate the result by $V$. This reduces the sampling problem to the case where $Y$ and $\Lambda$ are diagonal. Accordingly, the sampling algorithm only needs to take two vectors $y, \lambda \in \mathbb{R}^n$ as input, corresponding to the diagonal entries of the the diagonal matrices $Y, \Lambda$.

Below, for the ease of readability, we give an “idealized” description of our algorithm where we assume we can sample exactly from conditioned CDFs of the entries of a Rayleigh triangle. In Section 4.3 we sketch the completion of the proof of Theorem 1.3 by describing the additional ideas needed to account for the fact that we do not have exact access to the inverse of the conditional CDFs. The remainder of the paper is devoted to working out these ideas in detail.
4.1 Description of the algorithm

Formally, the input and output of the algorithm are as follows.

- **Input:**
  1. A vector $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, with $\lambda_1 \geq \cdots \geq \lambda_n$.
  2. A vector $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, with $y_1 \geq \cdots \geq y_n$.

  Write $\Lambda = \text{diag}(\lambda)$, $Y = \text{diag}(y)$.

- **Output:** An $n \times n$ Hermitian matrix with eigenvalues $(\lambda_1, \ldots, \lambda_n)$, distributed according to
  the measure $\nu(X) \propto e^{\langle Y, X \rangle} d\mu_\Lambda(X)$ on $\mathcal{O}_\Lambda$.

Recall that Theorem 2.3 implies that the pushforward of the HCIZ density on $\mathcal{O}_\Lambda$ is the exponential
density $\text{Vol}(\mathcal{G}T(\lambda))^{-1} e^{\langle y, \text{type}(P) \rangle} dP$ on the Gelfand-Tsetlin polytope $\mathcal{G}T(\lambda)$. Thus at a high level, the algorithm consists of the following steps.

1. **Step 1: Sample a Rayleigh triangle.** Sample a Rayleigh triangle $R = (R_{j,k})_{1 \leq k \leq j \leq n}$ from
   the density $\text{Vol}(\mathcal{G}T(\lambda))^{-1} e^{\langle y, \text{type}(P) \rangle} dP$ on the Gelfand-Tsetlin polytope $\mathcal{G}T(\lambda)$.

2. **Step 2: Sample from the fiber.** Sample a matrix $S$ uniformly at random from the fiber
   $\mathcal{R}^{-1}(R) = \{ X \in \mathcal{O}_\Lambda | \mathcal{R}(X) = R \}$.

3. **Output** $S$.

We now describe Steps 1 and 2 in detail.

4.1.1 Step 1: Sample a Rayleigh triangle $R$ from $\mathcal{G}T(\lambda)$

To sample a Rayleigh triangle $R \in \mathcal{G}T(\lambda)$, we use the inverse CDF to sample each coordinate $R_{i,j}$.

We move from left to right along each row of $R$, first sampling $R_{1,n-1}$, $R_{2,n-1}$, \ldots, $R_{n-1,n-1}$, then
$R_{1,n-2}$, \ldots, $R_{n-2,n-2}$, etc.. At each step we can explicitly compute the CDF of the next coordinate
to be sampled, conditional on the values of the coordinates that have already been sampled. Here we focus on deriving expressions for the CDF, and in Appendix A we present the (approximate) inverse CDF method. In particular, by Proposition 4.1 below (proved in Sections 6 and 7), the
CDF of $R_{i,j}$ conditional on the given values of $R_{k,l}$ for $l > j$ and $R_{k,j}$ for $k < i$ is equal to the ratio
of determinants

$$F_{i,j}(x) = \frac{\det(M_{i,j}[y, R](x))}{\det(M_{i,j}[y, R](R_{i,j+1}))} \quad (8)$$

for $x \in [R_{i+1,j+1}, R_{i,j+1}]$, where $M_{i,j}[y, R](x)$ is the matrix defined below in Definition 6.1 (if all entries of $y$ and $R$ are distinct) or Definition 7.1 (if $y$ or $R$ has non-distinct entries). The determinantal expression in (8) can be efficiently computed. Concretely then, the steps for sampling $R$ are as follows.

**Sampling procedure for $R \in \mathcal{G}T(\lambda)$:**

1. Set $R_{*,n} = \lambda$.

2. For $j = n-1, \ldots, 1$:
   
   For $i = 1, \ldots, j$:
   
   Sample $R_{i,j}$ using the inverse CDF method and the formula (8) for $F_{i,j}$. 

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3. **Output** \( R = (R_{i,j})_{1 \leq i \leq j \leq n} \).

This sampling procedure is straightforward once we know the expression for \( F_{i,j} \) in (8). Proving that (8) holds is then the main technical contribution of this step of the algorithm.

### 4.1.2 Step 2: Sample a uniform random matrix from the fiber over \( R \)

Once we have sampled \( R \), it remains to sample a matrix \( S \) uniformly at random from the fiber \( R^{-1}(R) \). We construct \( S \) using an inductive procedure, successively sampling the last row and column of each leading submatrix \( S[k] \). We first define \( S[1] \) to be the \( 1 \times 1 \) matrix \( [R_{1,1}] \), and we then sample each submatrix \( S[k] \), for \( 1 < k \leq n \), such that \( S[k] \) is uniformly distributed on the set \( \mathcal{H}(S[k-1]; R_{\bullet,k}) \) of \( k \times k \) Hermitian matrices with eigenvalues \( R_{\bullet,k} \) and \( (k-1) \)th leading submatrix equal to \( S[k-1] \). Explicitly, we sample \( S[k] \) given \( S[k-1] \) as follows.

#### Sampling procedure for \( S[k] \) given \( S[k-1] \) and \( R_{\bullet,k} \):

1. Compute a unitary matrix \( U \in U(k-1) \) such that \( U^* S[k-1] U \) is diagonal.

2. Write
   \[
   S[k] = \begin{bmatrix} S[k-1] & Uv \\ (Uv)^* & c \end{bmatrix},
   \]
   where \( v \in \mathbb{C}^{k-1} \) and \( c \in \mathbb{R} \) are to be determined. Since the diagonal entries of \( S[k] \) are just the type vector of \( \mathcal{R}(S[k]) \), we can compute
   \[
   c = \sum_{i=1}^{k} R_{i,k} - \sum_{j=1}^{k-1} R_{j,k-1}.
   \]

3. It remains to sample \( v \) uniformly at random from the set of vectors in \( \mathbb{C}^{k-1} \) such that the matrix \( S[k] \) in (9) has spectrum \( R_{\bullet,k} \). We prove below in Proposition 4.2 that this can be done using the following procedure. Let \( \delta_1 > \cdots > \delta_m \) be the distinct entries of \( R_{\bullet,k-1} \), where \( \delta_i \) has multiplicity \( n_i \), so that \( n_1 + \cdots + n_m = k - 1 \). The interlacing relations (4) imply that each value \( \delta_i \) occurs in \( R_{\bullet,k} \) with multiplicity at least \( n_i - 1 \). Let \( (\mu_1, \ldots, \mu_{m+1}) \) be the vector obtained by removing \( n_i - 1 \) entries equal to \( \delta_i \) from \( R_{\bullet,k} \), for each \( i \). Then define
   \[
   r_i = \frac{\prod_{j=1}^{m+1} (\delta_i - \mu_j)}{\prod_{j \neq i} (\delta_i - \delta_j)}.
   \]

   The interlacing relations guarantee that \( r_i \geq 0 \). For each \( i = 1, \ldots, m \), we then sample the \( n_i \) coordinates
   \[
   \left( v_1 + \sum_{j=1}^{i-1} n_j, \ldots, v_{n_i} + \sum_{j=1}^{i-1} n_j \right) \in \mathbb{C}^{n_i}
   \]
   uniformly at random from the sphere of radius \( r_i \) in \( \mathbb{C}^{n_i} \). This last step can be accomplished by well-known methods; see e.g. [31].

**Output:** Finally, after iteratively sampling all of the leading submatrices, we output \( S = S[n] \).
4.2 Correctness of the ideal algorithm

To prove the correctness of the algorithm, we first show that if Steps 1 and 2 are each individually correct, then the algorithm produces the desired result. To prove the correctness of Step 1, we need only show that the expression \([5]\) is in fact equal to the desired conditional CDF of the coordinate \(R_{i,j}\). This expression contains certain determinants, and here we explicitly give the simplest case when the entries of \(y\) and \(R\) are all distinct. For \(x \in [R_{i+1,j+1}, R_{i,j+1}]\) and \(\alpha_k \equiv y_k - y_{j+1}\) for all \(k\), we write:

\[
M_{i,j}[y, R](x) = \begin{bmatrix}
\alpha_1 e^{\alpha_1 R_{i,j}} & \ldots & \alpha_1 e^{\alpha_1 R_{i-1,j}} & e^{\alpha_1 x} & e^{\alpha_1 R_{i+1,j+1}} & \ldots & e^{\alpha_1 R_{i+1,j+1}} \\
\alpha_2 e^{\alpha_2 R_{i,j}} & \ldots & \alpha_2 e^{\alpha_2 R_{i-1,j}} & e^{\alpha_2 x} & e^{\alpha_2 R_{i+1,j+1}} & \ldots & e^{\alpha_2 R_{i+1,j+1}} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\alpha_j e^{\alpha_j R_{i,j}} & \ldots & \alpha_j e^{\alpha_j R_{i-1,j}} & e^{\alpha_j x} & e^{\alpha_j R_{i+1,j+1}} & \ldots & e^{\alpha_j R_{i+1,j+1}} \\
0 & \ldots & 0 & 1 & 1 & \ldots & 1
\end{bmatrix}.
\]

Proposition 4.1 (Determinantal formula for CDFs) Fix a vector \(y \in \mathbb{R}^n\) with distinct entries in decreasing order, a Rayleigh triangle \(R\) with \(n\) rows and distinct entries, and an \(x \in [R_{i+1,j+1}, R_{i,j+1}]\). The CDF of the coordinate \(R_{i,j}\), conditional on the values \(R_{k,l}\) for \(l > j\) and \(R_{k,j}\) for \(k < i\) is given by

\[
F_{i,j}(x) = \frac{\det(M_{i,j}[y, R](x))}{\det(M_{i,j}[y, R](R_{i,j+1}))}.
\]

Here \(M_{i,j}[y, R](x)\) is defined as above (distinct entries) and in Definition [7] (non-distinct entries).

The proof of Proposition [1.1], which establishes the correctness of Step 1, is given in Sections [6] and [7]. We prove the correctness of Step 2 in Section [5] but we give the formal statement of the result here.

Proposition 4.2 (Correctness of fiber sampling) Given \(R \in GT(\lambda)\) and a \((k - 1) \times (k - 1)\) Hermitian matrix \(S[k-1]\) with eigenvalues \(R_{\bullet,k-1}\), the procedure described in Step 2 of Section [4.7] samples \(S[k]\) uniformly at random from the space \(H(S[k-1]; R_{\bullet,k})\) of \(k \times k\) Hermitian matrices with eigenvalues \(R_{\bullet,k}\) and leading \((k - 1) \times (k - 1)\) submatrix equal to \(S[k-1]\).

Given correctness of the two main steps, we finally complete the proof of correctness of the ideal algorithm in the following.

Proposition 4.3 (Correctness of the composition of Steps 1 and 2) Let \(R\) be a random Rayleigh triangle distributed according to the measure proportional to \(e^{(y, type(P))} dP\) on \(GT(\lambda)\), and let \(S\) be a uniform random element of \(\mathcal{R}^{-1}(R)\). Then \(S\) is distributed according to the measure \(\nu(X) \propto e^{(Y,X)} d\mu_\Lambda(X)\) on \(\mathcal{O}_\Lambda\).

Proof: From Theorem [2.3] the density \(e^{(Y,X)}\) is constant on the fibers of \(\mathcal{R}\), since \(e^{(Y,X)} = e^{(y, type(R(X))))}\). The statement then follows immediately from the disintegration theorem for probability measures; see Appendix [C] and [6].

4.3 Accounting for approximations and completing the proof

The remaining step in the proof of Theorem [1.3] is to show that we can sample each entry \(R_{i,j}\) efficiently, and we sketch the key ideas here. Since we only have oracle access to the CDF of \(R_{i,j}\) and not its inverse, we must appeal to an approximate inverse CDF method; see Section [A].
Roughly speaking, the approximate CDF method requires (1) access to an oracle for a univariate CDF supported on \([0, 1] \), (2) a bound \( \varepsilon \) on the desired distance of the output to the target density, and (3) a bound \( G \) on the derivative of the associated PDF. It is guaranteed to output a large-enough precision sample from a PDF that is \( \varepsilon \)-close in TV distance to the target PDF, and the algorithm evaluates the CDF oracle \( \log \frac{G}{\varepsilon} \) many times. In our case, each evaluation of the CDF corresponds to evaluating two \( n \times n \) determinants, and we have at most \( n^2 \) different CDFs. So, it remains to bound \( G \).

Considering Proposition 4.1, we note that the CDF, the PDF, and the derivative of the PDF all can be written as a ratio of “Vandermonde-like” determinants. Thus, to bound the derivative of the PDF, we need a lower bound on the denominator and an upper bound on the numerator. The upper bound on the numerator follows from straightforward bounds on the entries of the matrix. However, as the conditioned entries of the Rayleigh triangles become close, the determinant in the denominator tends to zero. Note that if some of these entries become equal, then we can deal with them in an algebraic manner as described in Section 7. Thus, one way to bound the derivative is to restrict how close the conditioned values of \( R_{i,j} \) can get without being equal. This is precisely what we do, and we show that if any two distinct entries of the conditioned part of the Rayleigh triangle are \( 2^{-L} \) apart for some \( L > 0 \), then the derivative is bounded by \( G = 2^{\text{poly}(n, \|y\|, \|\lambda\|, L)} \); see Section 3. Hence, we restrict sampling each \( R_{i,j} \) to \( L \) bits where \( L \approx \log \frac{G}{\varepsilon} \) so that the bound we get on the derivative of the arising PDFs remains controlled.

We now summarize the steps in the proof of Theorem 1.3.

1. Each \( R_{i,j} \) sampled in Step 1 is a rational numbers with denominator bounded by a suitably chosen \( 2^L \) for a positive integer \( L \).
2. In Section 3 (Lemma 3.1), we show that the derivative of any of the arising conditioned PDFs is bounded by \( 2^{\text{poly}(n, \|y\|, \|\lambda\|, L)} \).
3. Thus, using the inverse CDF method from Section A (Lemma A.2), we obtain a guarantee that the Rayleigh triangle \( R \) sampled from Step 1 of our algorithm is an \( L \)-truncation of a distribution \( g \) that is \( \varepsilon \)-close in TV distance to the target density. The number of CDF evaluations is bounded by \( \text{poly}(n, \|y\|, \|\lambda\|, L, \log 1/\varepsilon) \). Each CDF is a determinant of a matrix of size at most \( n \times n \) and \( L = \text{poly}(n, \|y\|, \|\lambda\|, \log 1/\varepsilon) \).
4. Finally, from Lemma C.3 in Section C we obtain that the density on Hermitian matrices induced by pulling back \( g \) to the unitary orbit of \( \lambda \) continues to remain \( \varepsilon \)-close in TV distance to the target density on the unitary orbit.

Combining the steps above, we obtain that the number of determinant evaluations is bounded by \( \text{poly}(n, \|y\|, \|\lambda\|, \log 1/\varepsilon) \), as desired.

5 Correctness of the fiber sampling algorithm

In this section we give a proof of Proposition 4.2. The proof amounts to showing that \( H_k(S^{[k-1]}) \) is a product of spheres as described in Section 4.1. We first recall some notation that we defined in the description of the algorithm for sampling from \( R^{-1}(R) \). Let \( \delta_1 > \ldots > \delta_m \) be the distinct entries of \( R_{i,k-1} \), where \( \delta_i \) has multiplicity \( n_i \), so that \( n_1 + \cdots + n_m = k - 1 \). By the interlacing

---

4 Additionally, if some entries of \( R \) are very close then the support of the PDF of a given \( R_{i,j} \) may not be exactly \([0, 1]\) (as required by the approximate inverse CDF method), but a simple shifting and scaling of the PDF handles this.

---
relations (10), each value \( \delta_i \) must occur in \( R_{i,k} \) with multiplicity at least \( n_i - 1 \). Let \( (\mu_1, \ldots, \mu_{m+1}) \) be the vector obtained by removing \( n_i - 1 \) entries equal to \( \delta_i \) from \( R_{i,k} \), for each \( i \). Then define

\[
r_i = -\prod_{j=1 \atop j \neq i}^{m+1} (\delta_i - \mu_j) / \prod_{j \neq i} (\delta_i - \delta_j).
\]

Since the interlacing relations for \( R_{i,k-1} \) and \( R_{i,k} \) imply the same interlacing relations for \( \delta \) and \( \mu \), we have \( r_i \geq 0 \). Let \( U \in \mathbb{U}(k-1) \) be a unitary matrix diagonalizing \( S[k-1] \), so that \( U^* \cdot S[k-1] \cdot U = \text{diag}(R_{i,k-1}) \). We will show

\[
\mathcal{H}(S[k-1]; R_{i,k}) = \left\{ \left[ \begin{array}{cc} S[k-1] & Uv \\ (Uv)^* & c \end{array} \right] \mid v \in \mathbb{C}^{k-1}, \sum_{l=1}^{n_i} \left| v_i + \sum_{j=1}^{n-1} n_j \right|^2 = r_i \text{ for } i = 1, \ldots, m \right\}, \quad (10)
\]

where we necessarily have

\[
c = \sum_{i=1}^{k} R_{i,k} - \sum_{j=1}^{k-1} R_{j,k-1}
\]
due to the fact that the diagonal of any Hermitian matrix \( X \) is equal to \( \text{type}(\mathcal{R}(X)) \). Write \( D = \text{diag}(R_{i,k-1}) \). To establish (10) we must show that a matrix of the form

\[
S = \begin{bmatrix} U & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} D & v \\ v^* & c \end{bmatrix} \begin{bmatrix} U^* & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} S[k-1] & Uv \\ (Uv)^* & c \end{bmatrix}
\]

has eigenvalues \( R_{i,k} \) if and only if \( \sum_{i=1}^{n_i} \left| v_i + \sum_{j=1}^{n-1} n_j \right|^2 = r_i \) for \( i = 1, \ldots, m \). We prove this by writing the characteristic polynomial of \( S \) in two different ways. First, if \( S \) has eigenvalues \( R_{i,k} \) then

\[
\det(tI - S) = \prod_{i=1}^{k} (t - R_{i,k}). \quad (11)
\]

On the other hand, we must have

\[
\det(tI - S) = \det \left( tI - \begin{bmatrix} D & v \\ v^* & c \end{bmatrix} \right) = \det \begin{bmatrix} tI - D & v \\ v^* & t - c \end{bmatrix},
\]

and expanding along the first row and column we find that this equals

\[
(t-c) \prod_{j=1}^{m} (t - \delta_j)^{n_j} - \sum_{i=1}^{m} \left( |v_{n_1+\ldots+n_{i-1}+1}|^2 + \cdots + |v_{n_1+\ldots+n_i}|^2 \right) (t - \delta_i)^{n_i-1} \prod_{j \neq i} (t - \delta_j)^{n_j}. \quad (12)
\]

We have \( S \in \mathcal{H}(S[k-1]; R_{i,k}) \) exactly when (11) equals (12). Equating these two expressions for the characteristic polynomial and recalling that interlacing of \( R_{i,k} \) and \( R_{i,k-1} \) implies that \( R_{i,k} \) contains the value \( \delta_i \) with multiplicity at least \( n_i - 1 \) for all \( i \), we can divide through both sides by \( (t - \delta_i)^{n_i-1} \) for all \( i \) to obtain

\[
\prod_{i=1}^{m+1} (t - \mu_i) = (t-c) \prod_{j=1}^{m} (t - \delta_j) - \sum_{i=1}^{m} \left( |v_{n_1+\ldots+n_{i-1}+1}|^2 + \cdots + |v_{n_1+\ldots+n_i}|^2 \right) \prod_{j \neq i} (t - \delta_j). \quad (13)
\]

Note that both sides of (13) are monic polynomials of degree \( m+1 \), and

\[
c = \sum_{i=1}^{k} R_{i,k} - \sum_{j=1}^{k-1} R_{j,k-1} = \sum_{i=1}^{m+1} \mu_i - \sum_{j=1}^{m} \delta_j
\]
implies the coefficients of $t^{m+1}$ and $t^m$ on both sides are equal. Therefore the polynomials in (13) are equal if and only if they are equal at $m$ distinct points. Evaluating both sides at $t = \delta_i$ for $i = 1, \ldots, m$, we find that they are equal exactly when

$$\sum_{l=1}^{n_i} |v_{i, l} + \sum_{j=1}^{i-1} n_j|^2 = r_i$$

for $i = 1, \ldots, m$, which is the desired result. This completes the proof of Proposition 4.2.

Remark 5.1 Our proof of Proposition 4.2 is self-contained, and thus our fiber sampling algorithm in the case of the uniform measure (when $Y = 0$) gives an algorithmic proof of the fact that the uniform measure on the unitary orbit $O_{\Lambda}$ maps to the Lebesgue measure on the GT polytope via the Rayleigh map $\mathcal{R}$ (see Proposition 2.2).

6 Computing the CDFs: the distinct case

In this section we present the proof of Proposition 4.1, the purpose of which is to give a formula for the CDF of a single entry $R_{i,j}$ of the Rayleigh triangle $R$, conditioned on previously selected entries $R_{k,j}$ for $k < i$ and $R_{k,l}$ for $l > j$. For ease of reading, we present the definitions and proofs for the case when all the entries of the Rayleigh triangle $R$ and $y$ are distinct. This distinct case already demonstrates the crucial ideas of the proof, and what remains to be shown for the non-distinct case requires more complicated notation and discussion. For that reason we delay the proof in the case of non-distinct entries to Section 7.

The CDF of $R_{i,j}$, denoted $F_{i,j}(x)$, is a ratio of integrals over slices of $GT(\lambda)$ corresponding to the entries which have yet to be selected. The proof is then mainly concerned with giving a nice determinantal formula for these integrals. To obtain this determinantal formula, first note that the bottom $j$ rows of $R$ form a smaller Rayleigh triangle. By induction (or using the HCIZ formula; see Theorem 1.1), we know that the integral over the entries of the bottom $j - 1$ rows of this sub-triangle is given by a determinant of size $j$ (see Lemma 6.4 and Corollary 6.3). We now need to take $j - i + 1$ integrals of this determinant with respect to the last $j - i + 1$ entries of the $j^{th}$ row of $R$ (as in Equation 14). The obvious problem with this is that even though a determinant gives a nice formula, multiple integrals of a determinant can quickly become nasty. The first observation to handle this issue is then Lemma 6.2 which says that integrals of this determinant can be turned into partial derivatives of a determinant of a larger matrix. Of course, taking derivatives of a determinant can be just as problematic as taking integrals. The key observation then is that only one column of the matrix depends on each $R_{k,j}$ for all $k \geq i$. Since partial derivatives are linear operators and the determinant is multilinear in the columns of the input matrix, this means we can pass the partial derivatives directly to the appropriate columns of the matrix. And with this, the partial derivatives of the determinant can actually be expressed as a single determinant of a slightly different matrix. Applying this argument to both the numerator and the denominator then gives an expression for the CDF as a ratio of determinants.

We now describe the matrices which appear in the determinants of the CDF formula. The corresponding definition for the non-distinct case is given in Definition 7.1.

Definition 6.1 (CDF matrix, distinct case) Fix a vector $y \in \mathbb{R}^n$ with distinct entries in decreasing order, a Rayleigh triangle $R$ with $n$ rows and distinct entries, and some non-negative integers $i, j$ such that $1 \leq i \leq j \leq n$. We define the following $(j+1) \times (j+1)$ matrix for
$x \in [R_{i+1,j+1}, R_{i,j+1}]$, where $\alpha_k := y_k - y_{j+1}$ for all $k$:

$$M_{i,j}[y,R](x) = \begin{bmatrix} \alpha_1 e^{\alpha_1 R_{i,j}} & \cdots & \alpha_1 e^{\alpha_1 R_{i-1,j}} & e^{\alpha_1 x} & e^{\alpha_1 R_{i+1,j+1}} & \cdots & e^{\alpha_1 R_{i,j+1}} \\ \alpha_2 e^{\alpha_2 R_{i,j}} & \cdots & \alpha_2 e^{\alpha_2 R_{i-1,j}} & e^{\alpha_2 x} & e^{\alpha_2 R_{i+1,j+1}} & \cdots & e^{\alpha_2 R_{i,j+1}} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_j e^{\alpha_j R_{i,j}} & \cdots & \alpha_j e^{\alpha_j R_{i-1,j}} & e^{\alpha_j x} & e^{\alpha_j R_{i+1,j+1}} & \cdots & e^{\alpha_j R_{i,j+1}} \\ 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \end{bmatrix}.$$  

We now prove Proposition 4.1 formally in the distinct case. Let us first define a specific integral expression which we will use to simplify the proof and the needed lemmas:

$$I_{i,j}[y,R](x) = \int_{R_{i+1,j+1}}^x \cdots \int_{R_{j+1,j+1}}^x e^{-y_{j+1}} \sum_{k=1}^j P_{k,j} \det \begin{bmatrix} e^{y_k P_{j,j}}[j]_{k,l=1}^{j-1} \prod_{k=1}^{j-1} dP_{j-k+1,j} \end{bmatrix}_{P_{m,j}=R_{m,j}, m<i},$$

By Corollary 6.3 and Lemma 6.4 we have

$$\det[e^{y_k P_{j,j}}]_{k,l=1}^{j} = e^{y_j \sum_k P_{k,j} \det[e^{\alpha_k P_{j,j}}]_{k,l=1}^{j}} = e^{y_j \sum_k P_{k,j} \det(M_{1,j-1}[y,P](P_{1,j}))} = I_{1,j-1}[y,P](P_{1,j}) \cdot e^{y_j \sum_k P_{k,j}} \prod_{k=1}^{j-1} (y_k - y_j) = \prod_{1 \leq i < k \leq j} (y_i - y_k) \int_{GT(P_{i,j})} e^{((y_1,\ldots,y_j),\text{type}(P))} dQ.$$  

Combining these, we have

$$\frac{I_{i,j}[y,R](x) \cdot e^{y_{j+1}} \sum_k R_{k,j+1}}{\prod_{1 \leq i < k \leq j} (y_i - y_k)} = \left( \int_{R_{i+1,j+1}}^x \cdots \int_{R_{j+1,j+1}}^x \int_{GT(P_{i,j})} e^{((y_1,\ldots,y_{j+1}),\text{type}(P))} dP \right)_{P_{k,j}=R_{k,j} \text{ for } k<i},$$

where $P$ ranges over all Rayleigh triangles with $j+1$ rows and top row equal to $R_{i,j+1}$. The right-hand side is precisely the formula for the conditioned CDF of $R_{i,j}$ up to normalization factor. Since $x \in [R_{i+1,j+1}, R_{i,j+1}]$, we therefore have

$$F_{i,j}(x) = \frac{I_{i,j}[y,R](x) \cdot e^{y_{j+1}} \sum_k R_{k,j+1}}{I_{i,j}[y,R](R_{i,j+1}) \cdot e^{y_{j+1}} \sum_k R_{k,j+1}} = \frac{I_{i,j}[y,R](x)}{I_{i,j}[y,R](R_{i,j+1})} = \frac{\det(M_{i,j}[y,R](x))}{\det(M_{i,j}[y,R](R_{i,j+1}))},$$

where the last equality follows from Corollary 6.3. This completes the proof of Proposition 4.1 in the distinct case.

**Remark 6.1** Our proof of Proposition 4.1 is self-contained, and thus gives a proof of the HCIZ formula. In fact, Proposition 4.1 is a strengthening of the HCIZ formula: the conditioned density of every entry of the Rayleigh triangle has a determinantal formula, not just those entries that correspond to unitary orbits.
6.1 Technical lemmas and proofs

Lemma 6.2 (Turn integrals into derivatives) For \( \alpha, t \in \mathbb{R}^n \) with distinct entries and \( \alpha_n = 0 \), we have

\[
\int_{t_{k+1}}^{t_k} \cdots \int_{t_1}^{t_{n-1}} \det(e^{\alpha x})_{i,j=1}^{n-1} dx_{n-1} \cdots dx_k \bigg|_{x_1=t_1, \ldots, x_{k-1}=t_{k-1}} = \partial_{t_1} \cdots \partial_{t_{k-1}} \frac{\det(e^{\alpha t_j})_{i,j=1}^{n}}{\alpha_1 \alpha_2 \cdots \alpha_{n-1}}.
\]

Proof: We prove this by induction on \( k \) with the base case being \( k = n \). First for the base case, the derivatives in the expression

\[
\partial_{t_1} \cdots \partial_{t_{n-1}} \det(e^{\alpha t_j})_{i,j=1}^{n}
\]

can be applied column-wise. This is because the determinant is multilinear in the columns, and only one column depends on \( t_j \) for all \( j \in [n] \). The last row of the final matrix after applying all derivatives is \((0, 0, \ldots, 0, 1)\), and the desired result follows.

Now inductively assuming that we know the result for \( k \), we will prove it for \( k - 1 \). That is, we know

\[
\int_{t_{k+1}}^{t_k} \cdots \int_{t_1}^{t_{n-1}} \det(e^{\alpha x})_{i,j=1}^{n-1} dx_{n-1} \cdots dx_k \bigg|_{x_1=t_1, \ldots, x_{k-1}=t_{k-1}} = \partial_{t_1} \cdots \partial_{t_{k-1}} \frac{\det(e^{\alpha t_j})_{i,j=1}^{n}}{\alpha_1 \alpha_2 \cdots \alpha_{n-1}},
\]

and we want to show

\[
\int_{t_k}^{t_{k-1}} \cdots \int_{t_1}^{t_{n-1}} \det(e^{\alpha x})_{i,j=1}^{n-1} dx_{n-1} \cdots dx_{k-1} \bigg|_{x_1=t_1, \ldots, x_{k-2}=t_{k-2}} = \partial_{t_1} \cdots \partial_{t_{k-2}} \frac{\det(e^{\alpha t_j})_{i,j=1}^{n}}{\alpha_1 \alpha_2 \cdots \alpha_{n-1}}.
\]

Equivalently, we want to show

\[
\int_{t_k}^{t_{k-1}} \partial_{t_1} \cdots \partial_{t_{k-1}} \bigg|_{t_{k-1}=x} \det(e^{\alpha t_j})_{i,j=1}^{n} dx = \partial_{t_1} \cdots \partial_{t_{k-2}} \bigg|_{t_{k-2}=x} \det(e^{\alpha t_j})_{i,j=1}^{n}.
\]

By the fundamental theorem of calculus, we have

\[
\int_{t_k}^{t_{k-1}} \partial_{t_1} \cdots \partial_{t_{k-1}} \bigg|_{t_{k-1}=x} \det(e^{\alpha t_j})_{i,j=1}^{n} dx = \left[ \partial_{t_1} \cdots \partial_{t_{k-2}} \bigg|_{t_{k-2}=x} \det(e^{\alpha t_j})_{i,j=1}^{n} \right]_{t_{k-1}=t_k}^{t_{k-1}=x}.
\]

Note that plugging in \( t_k \) for \( x \) on the right-hand side gives the determinant of a matrix which has two identical columns. Therefore that determinant is zero, and we are left with precisely what we wanted to show.

Corollary 6.3 (Determinant expression for \( I_{i,j} \)) Given a vector \( y \in \mathbb{R}^n \) with distinct entries in decreasing order, a Rayleigh triangle \( R \) with \( n \) rows and distinct entries, an \( x \in [R_{i+1,j+1}, R_{i,j+1}] \), and non-negative integers \( i, j \) such that \( 1 \leq i \leq j \leq n \), we have the following expression for the integral given in Equation [14]

\[
I_{i,j}[y, R](x) = \frac{\det(M_{i,j}[y, R](x))}{\prod_{k=1}^{j} (y_k - y_{j+1})}.
\]

Here \( M_{i,j}[y, R](x) \) is defined as in Definition [14].
Proof: Recall from Equation [14] that

\[ I_{i,j}[y,R](x) = \left( \int_{R_{t+1,j+1}}^x \cdots \int_{R_{j+1,j+1}}^{R_{j,j+1}} e^{-y_{j+1} \sum_{k=1}^j P_{k,j}} \det \left[ e^{y_k P_{j,l}} \right]_{k,l=1}^{j-i+1} \prod_{k=1}^{j-1} dP_{j-k+1,j} \right)_{P_{k,j}=R_{k,j} \text{ for } k<i} . \]

Note that we can write

\[ e^{-y_{j+1} \sum_{k=1}^j P_{k,j}} \det \left[ e^{y_k P_{j,l}} \right]_{k,l=1}^{j-i+1} = \det \left[ e^{(y_k-y_{j+1}) P_{j,l}} \right]_{k,l=1}^{j-i+1} . \]

With this, we can apply Lemma 6.2 to get

\[ I_{i,j}[y,R](x) = \left( \int_{R_{t+1,j+1}}^x \cdots \int_{R_{j+1,j+1}}^{R_{j,j+1}} \det \left[ e^{(y_k-y_{j+1}) R_{i,j+i-1}^{j+1}_{l,j>1}} \right]_{k,l=1}^{j-i+1} \prod_{k=1}^{j-1} dP_{j-k+1,j} \right)_{R_{i,j+1}=x} . \]

Here \( \delta_{l>i} \) is 1 if \( l \geq i \) and 0 otherwise. Note that only one column of the matrix \( [e^{(y_k-y_{j+1}) R_{i,j+i-1}^{j+1}_{l,j>1}}]_{k,l=1}^{j-i+1} \) depends on \( R_{k,j+\delta_{l>i}} \) for all \( k \in [j] \). Since the determinant is multilinear in the columns of the input matrix, we can pass the derivatives into the matrix to be applied entry-wise to the relevant columns. This then gives the result.

Lemma 6.4 (Connection between the GT polytope and \( I_{i,j} \)) Given \( y \in \mathbb{R}^n \) and a Rayleigh triangle \( R \) with \( n \) rows and top row equal to \( \lambda \in \mathbb{R}^n \), we have the following expression for the integral given in Equation [14]:

\[ \frac{I_{1,n-1}[y,R](R_{1,n})}{\prod_{1 \leq i < j < n} (y_i - y_j)} = e^{-y_n \sum_{i \lambda_i} \int_{GT(\lambda)} e^{(y,\text{type}(P))} dP} . \]

Note that this expression actually depends only on the top row of \( R \).

Proof: For \( v \in \mathbb{R}^n \), we let \( v' \in \mathbb{R}^{n-1} \) denote the vector of the first \( n-1 \) entries of \( v \). By induction, we have

\[ e^{-y_n \sum_{i \lambda_i} \int_{GT(\lambda)} e^{(y,\text{type}(P))} dP} = \int_{GT(\lambda)} e^{-y_n \sum_{k \lambda_k} P_{k,n-1}} e^{(y',\text{type}(P'))} dP \]

\[ = \int_{R_{1,n}}^{R_{1,n}} \cdots \int_{R_{n,n}}^{R_{n,n}} e^{-y_n \sum_{k \lambda_k} P_{k,n-1}} \int_{GT(P_{n-1,n})} e^{(y',\text{type}(P'))} dP \]

\[ = \int_{R_{2,n}}^{R_{1,n}} \cdots \int_{R_{n,n}}^{R_{n,n}} e^{(y_{n-1} - y_n) \sum_{k \lambda_k} P_{k,n-1}} \frac{I_{1,n-2}[y,P](P_{1,n-1})}{\prod_{1 \leq i < j < n-1} (y_i - y_j)} \prod_{k=1}^{n-1} dP_{n-k,n-1} . \]

By Corollary 6.3 above, we also have

\[ \det[e^{y_k P_{n-1,n-1}}]^{n-1}_{k,l=1} = e^{y_{n-1} \sum_{k \lambda_k} P_{k,n-1}} \det[e^{\alpha_k P_{n-1,n-1}}]^{n-1}_{k,l=1} \]

\[ = e^{y_{n-1} \sum_{k \lambda_k} P_{k,n-1}} \det(M_{n-2}[y,P](P_{1,n-1})) \]

\[ = I_{1,n-2}[y,P](P_{1,n-1}) \cdot e^{y_{n-1} \sum_{k \lambda_k} P_{k,n-1}} \prod_{k=1}^{n-2} (y_k - y_{n-1}) . \]
Combining these gives
\[ e^{-y_n} \sum_i \lambda_i \int_{GT(\lambda)} e^{(y, \text{type}(P))} dP \]
\[ = \int_{R_2,n} \cdots \int_{R_n,n} e^{-y_n} \sum_k \det[e^{y_k P_{i,n-1}}] \prod_{1 \leq i < j \leq n} (y_i - y_j) \prod_{k=1}^{n-1} dP_{n-k,n-1}. \]

By Equation (13) this then implies the result.

The last piece of the proof is the base case, for \( n = 2 \). In this case, the desired expression can give explicitly:
\[ I_{1,1}(y, R)(R_{1,2}) = \int_{R_{1,2}} e^{(y_1 - y_2)P_{1,1}} dP_{1,1} = e^{-y_2(R_{1,2} + R_{2,2})} \int_{GT(\lambda)} e^{(y, \text{type}(P))} dP. \]

Since \( \lambda = R_{\bullet, 2} \) in this case, this is precisely the desired expression.

\[ \square \]

7 Computing the CDFs: the non-distinct case

In Section 6 we proved Proposition 4.1 the determinantal formula for the conditioned CDF, in the case where \( y \) and \( R \) have distinct entries. We now discuss how to prove Proposition 4.1 in the general case. The main difference between these two cases is the definition of the matrix \( M_{i,j} \) which appears in the determinant in the CDF formula. Specifically, the matrix from Definition 6.1 becomes 0 when elements of \( y \) or \( R \) coincide, and the formula for the CDF takes the form \( 0/0 \). To handle this, we know that the conditioned PDF will be continuous by considering it as a marginal density of some slice of the density given in Theorem 2.3. Therefore the CDF will also be continuous, and we can take limits of the ratio of determinants via L’Hospital’s rule. The key observation here is then the same as it was for the distinct case: at most one row or one column depends on each entry of \( y \) or \( R \). Since derivatives are linear and the determinant is multilinear on the rows or columns, we can pass derivatives directly to the relevant entries of the matrices. That is, the only thing that changes in the conditioned CDF formula is the definition of the matrix \( M_{i,j} \).

We give the definition in full generality now.

**Definition 7.1** Fix a vector \( y \in \mathbb{R}^n \) in non-increasing order, a Rayleigh triangle \( R \) with \( n \) rows, and some non-negative integers \( i, j \) such that \( 1 \leq i \leq j \leq n \). Let \( \rho_{1,j+1} > \cdots > \rho_{q,j+1} \) denote the distinct values of \( R_{i+1,j+1}, \ldots, R_{j+1,j+1} \) with multiplicities \( n_{1,j+1}, \ldots, n_{q,j+1} \), let \( \rho_{1,i} > \cdots > \rho_{p,i} \) denote the distinct values of \( R_{1,j}, \ldots, R_{i-1,j} \) with multiplicities \( n_{1,j}, \ldots, n_{p,j} \), and let \( \gamma_1 > \cdots > \gamma_l \) denote the distinct values of \( y_1, \ldots, y_{j+1} \) with multiplicities \( m_1, \ldots, m_l \). We define the following \((j+1) \times (j+1)\) matrix for \( x \in [R_{i+1,j+1}, R_{i,j+1}] \):
\[
M_{i,j}[y, R](x) := \begin{bmatrix}
A_{1,1} & A_{1,2} & \cdots & A_{1,p} & v_1(x) & B_{1,1} & B_{1,2} & \cdots & B_{1,q} \\
A_{2,1} & A_{2,2} & \cdots & A_{2,p} & v_2(x) & B_{2,1} & B_{2,2} & \cdots & B_{2,q} \\
\vvdots & \vvdots & \ddots & \vvdots & \vvdots & \vdots & \vdots & \ddots & \vdots \\
A_{l,1} & A_{l,2} & \cdots & A_{l,p} & v_l(x) & B_{l,1} & B_{l,2} & \cdots & B_{l,q} \\
0 & 0 & \cdots & 0 & 1 & e_1^T & e_1^T & \cdots & e_1^T
\end{bmatrix}.
\]

Here \( e_1^T \) denotes the standard basis row vector with length based on the context. To simplify the description of the entries of \( M_{i,j}[y, R](x) \), we first define a collection of polynomials:
\[
f_{i,j}(t; C) := e^{-tC} \frac{\partial^l}{\partial t^l} (t^i e^{tC}) = \sum_{l=0}^{\min(i,j)} \binom{j}{l} \frac{j!}{(i-l)!} t^{i-l} C^{j-l} = \sum_{l=0}^{\min(i,j)} \binom{i}{l} \frac{j!}{(j-l)!} t^{i-l} C^{j-l}.
\]
For $a < l$, we define $A_{a,b}$ ($m_a \times n_{b,j}$ matrix), $B_{a,b}$ ($m_a \times n_{b,j+1}$ matrix), and $v_a(x)$ (column vector of length $m_a$), via

$$(A_{a,b})_{c,d} := f_{d,c-1}(\gamma a - \gamma b; \rho_{b,j}) \cdot e^{(\gamma a - \gamma b)\rho_{b,j}},$$

$$(B_{a,b})_{c,d} := f_{d-1,c-1}(\gamma a - \gamma b; \rho_{b,j+1}) \cdot e^{(\gamma a - \gamma b)\rho_{b,j+1}},$$

$$(v_a(x))_c := f_{0,c-1}(\gamma a - \gamma b; x) \cdot e^{(\gamma a - \gamma b)x}.$$

For $a = l$, we define $A_{l,b}$ ($(m_l - 1) \times n_{b,j}$ matrix), $B_{l,b}$ ($(m_l - 1) \times n_{b,j+1}$ matrix), and $v_l(x)$ (column vector of length $m_l - 1$) via

$$(A_{l,b})_{c,d} := f_{d,c}(0; \rho_{b,j})$$

$$(B_{l,b})_{c,d} := f_{d-1,c}(0; \rho_{b,j+1}),$$

$$(v_l(x))_c := f_{0,c}(0; x).$$

**Remark 7.1** Whenever all values of $R$ and of $y$ are distinct, then each of $A_{a,b}$, $B_{a,b}$, and $v_a(x)$ in the above definition are $1 \times 1$ matrices for $a < l = j + 1$ and $A_{l,b}$, $B_{l,b}$, and $v_l(x)$ are $0 \times 1$ matrices. In this case, we recover the matrix from Definition 6.1.

**Remark 7.2** Whenever all values of $y$ are equal to 0 and the values of $R$ are distinct, then the matrix in the above definition takes the form

$$M_{i,j}[0,R](x) = \begin{bmatrix} A_{i,1} & \cdots & A_{i,j-1} & v_l(x) & B_{i,1} & \cdots & B_{i,j-i+1} \\ 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \end{bmatrix},$$

where $A_{i,k}$, $B_{i,k}$, and $v_l(x)$ are column vectors of length $j$. In this case, the matrix can be explicitly given as follows:

$$M_{i,j}[0,R](x) = \begin{bmatrix} 1 & \cdots & 1 & x & R_{i+1,j+1} & \cdots & R_{j+1,j+1} \\ 2R_{1,j} & \cdots & 2R_{i-1,j} & x^2 & R^2_{i+1,j+1} & \cdots & R^2_{j+1,j+1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ jR_{1,j}^j & \cdots & jR_{i-1,j}^j & x^j & R_{i+1,j+1}^j & \cdots & R_{j+1,j+1}^j \\ 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \end{bmatrix}.$$
To handle such multiplicities, we compute the limit as elements of $R$ and/or $y$ approach the same value via L’Hopital’s rule (see [25] for similar computations). This computation will involve a number of derivatives with respect to the values of $y$ and $R$ being applied to each of the determinants in the numerator and denominator of the expression given above for $F_{i,j}(x)$, starting with the distinct entries case as the base case. Since only one row or column of the matrix $M_{k,j}[y, R](x)$ depends on each value of $y$ and $R$ in the distinct entries case, these derivatives can be passed directly to the rows/columns of the matrix. In what follows, we describe a bit more precisely this process of applying derivatives to the matrices. Once the reader has a feel for the idea of how this is done, what we discuss below becomes rather tedious. In any case, we have tried to be a bit informal in our discussion for the ease of the reader.

We first take care of multiplicity in the values of $R_{k,j+1}$. As in Definition 7.1, let $\rho_{1,j+1} > \cdots > \rho_{q,j+1}$ denote the distinct values of $R_{i+1,j+1}, \ldots, R_{i+j+1,j+1}$ with multiplicities $n_{1,j+1}, \ldots, n_{q,j+1}$. Let us focus on the one block of the matrix where the values of $R_{i+k,j+1}$ are the same for $1 \leq k \leq n_{1,j+1}$, using $\alpha_k := y_k - y_{j+1}$ and $n := n_{1,j+1}$ for simplicity:

\[
\begin{bmatrix}
\cdots & e^{\alpha_1 R_{i+1,j+1}} & e^{\alpha_1 R_{i+2,j+1}} & e^{\alpha_1 R_{i+3,j+1}} & \cdots & e^{\alpha_1 R_{i+n,j+1}} & \cdots \\
\cdots & e^{\alpha_2 R_{i+1,j+1}} & e^{\alpha_2 R_{i+2,j+1}} & e^{\alpha_2 R_{i+3,j+1}} & \cdots & e^{\alpha_2 R_{i+n,j+1}} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\cdots & e^{\alpha_{j-1} R_{i+1,j+1}} & e^{\alpha_{j-1} R_{i+2,j+1}} & e^{\alpha_{j-1} R_{i+3,j+1}} & \cdots & e^{\alpha_{j-1} R_{i+n,j+1}} & \cdots \\
\cdots & 1 & 1 & 1 & \cdots & 1 & \cdots \\
\end{bmatrix}
\]

We first set $R_{i+1,j+1} = \rho_{1,j+1}$ and limit $R_{i+2,j+1} \to \rho_{1,j+1}$. Using L’Hopital’s rule as discussed above, we need to apply one derivative with respect to $R_{i+2,j+1}$ and then plug in $R_{i+2,j+1} = \rho_{1,j+1}$. Our new matrix then becomes

\[
\begin{bmatrix}
\cdots & e^{\alpha_1 \rho_{1,j+1}} & e^{\alpha_1 \rho_{1,j+1}} & e^{\alpha_1 R_{i+3,j+1}} & \cdots & e^{\alpha_1 R_{i+n,j+1}} & \cdots \\
\cdots & e^{\alpha_2 \rho_{1,j+1}} & e^{\alpha_2 \rho_{1,j+1}} & e^{\alpha_2 R_{i+3,j+1}} & \cdots & e^{\alpha_2 R_{i+n,j+1}} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\cdots & e^{\alpha_{j-1} \rho_{1,j+1}} & e^{\alpha_{j-1} \rho_{1,j+1}} & e^{\alpha_{j-1} R_{i+3,j+1}} & \cdots & e^{\alpha_{j-1} R_{i+n,j+1}} & \cdots \\
\cdots & 1 & 0 & 1 & \cdots & 1 & \cdots \\
\end{bmatrix}
\]

We now limit $R_{i+3,j+1} \to \rho_{1,j+1}$, and so we need to apply a derivative with respect to $R_{i+3,j+1}$ via L’Hopital’s rule. But note that in this case one derivative is not enough; plugging in $R_{i+3,j+1} = \rho_{1,j+1}$ at this point would make the determinant 0 since the second and third columns would be the same. So we must take a second derivative with respect to $R_{i+3,j+1}$ and then plug in $R_{i+3,j+1} = \rho_{1,j+1}$, which gives the matrix

\[
\begin{bmatrix}
\cdots & e^{\alpha_1 \rho_{1,j+1}} & e^{\alpha_1 \rho_{1,j+1}} & e^{\alpha_1 \rho_{1,j+1}} & e^{\alpha_1 R_{i+n,j+1}} & \cdots \\
\cdots & e^{\alpha_2 \rho_{1,j+1}} & e^{\alpha_2 \rho_{1,j+1}} & e^{\alpha_2 \rho_{1,j+1}} & e^{\alpha_2 R_{i+n,j+1}} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\cdots & e^{\alpha_{j-1} \rho_{1,j+1}} & e^{\alpha_{j-1} \rho_{1,j+1}} & e^{\alpha_{j-1} \rho_{1,j+1}} & e^{\alpha_{j-1} R_{i+n,j+1}} & \cdots \\
\cdots & 1 & 0 & 0 & \cdots & 1 \\
\end{bmatrix}
\]

By the same argument, every limit we take will require one more derivative than the previous one. That is, we will in the end apply the differential operator

\[
\partial_{R_{i+n,j+1}}^{n-1} \cdots \partial_{R_{i+3,j+1}}^{2} \partial_{R_{i+2,j+1}}^{1} \partial_{R_{i+1,j+1}}^{0}
\]
and then evaluate all $R_{k,j+1} = \rho_{1,j+1}$ to obtain the matrix

$$
\begin{bmatrix}
\ldots & e^{a_1 \rho_{1,j+1}} & \alpha_1 e^{a_1 \rho_{1,j+1}} & e^{a_2 \rho_{1,j+1}} & \alpha_2 e^{a_2 \rho_{1,j+1}} & \ldots & e^{a_{n-1} \rho_{1,j+1}} & \alpha_{n-1} e^{a_{n-1} \rho_{1,j+1}} & \ldots \\
\ldots & e^{a_2 \rho_{1,j+1}} & \alpha_2 e^{a_2 \rho_{1,j+1}} & e^{a_3 \rho_{1,j+1}} & \alpha_3 e^{a_3 \rho_{1,j+1}} & \ldots & e^{a_{n-2} \rho_{1,j+1}} & \alpha_{n-2} e^{a_{n-2} \rho_{1,j+1}} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
\ldots & e^{a_j \rho_{1,j+1}} & \alpha_j e^{a_j \rho_{1,j+1}} & e^{a_{j+1} \rho_{1,j+1}} & \alpha_{j+1} e^{a_{j+1} \rho_{1,j+1}} & \ldots & e^{a_{n-j} \rho_{1,j+1}} & \alpha_{n-j} e^{a_{n-j} \rho_{1,j+1}} & \ldots \\
1 & 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
\end{bmatrix}
$$

We next apply the exact same logic to handle the multiplicity in all values of $R_{k,j+1}$, as well as to handle the multiplicity in the values of $R_{k,j}$ for valid values of $k$. To this end, let $\rho_{1,j} > \ldots > \rho_{p,j}$ denote the distinct values of $R_{1,j}, \ldots, R_{1,1}$ with multiplicities $n_{1,j}, \ldots, n_{p,j}$. Applying L'Hopital's rule leads to the following form of the matrix, where $A_{a,b}$ and $B_{a,b}$ are defined as in Definition 7.1 in the case that the values of $y_k$ are distinct:

$$
\begin{bmatrix}
A_{1,1} & A_{1,2} & \cdots & A_{1,p} & e^{a_1 x} & B_{1,1} & B_{1,2} & \cdots & B_{1,q} \\
A_{2,1} & A_{2,2} & \cdots & A_{2,p} & e^{a_2 x} & B_{2,1} & B_{2,2} & \cdots & B_{2,q} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\
A_{j,1} & A_{j,2} & \cdots & A_{j,p} & e^{a_j x} & B_{j,1} & B_{j,2} & \cdots & B_{j,q} \\
0 & 0 & 0 & 0 & 1 & e_1 & e_1 & \cdots & e_1 \\
\end{bmatrix}
$$

Let us briefly argue for why the matrix has this form at this point. In this specific case where the values of $y_k$ are distinct, we have that the matrices $A_{a,b}$ and $B_{a,b}$ are actually row vectors. Computing the values of the entries of $A_{a,b}$ and $B_{a,b}$ from Definition 7.1 is then straightforward:

$$(A_{a,b})_{1,d} := f_{d,0}(\alpha_a; \rho_{b,j}) \cdot e^{a_1 \rho_{b,j}} = \alpha_a e^{a_1 \rho_{b,j}},$$

$$(B_{a,b})_{1,d} := f_{d-1,0}(\alpha_a; \rho_{b,j+1}) \cdot e^{a_1 \rho_{b,j+1}} = \alpha_a e^{a_1 \rho_{b,j+1}}.$$  

These expressions correspond precisely to the entries which show up after applying L'Hopital's rule as described above.

Our final task then is to deal with multiplicity in the values of $y$. Let $\gamma_1 > \cdots > \gamma_l$ denote the distinct values of $y_1, \ldots, y_{j+1}$ with multiplicities $m_1, \ldots, m_l$. For $\gamma_1, \ldots, \gamma_{l-1}$, we apply L'Hopital's rule in the same way that we did for multiplicity in the values of $R$. (We will come back to $\gamma_l$.) For example, for $\gamma_1$ we would apply the operator

$$\partial_{y_1}^{m_1-1} \partial_{y_2}^{2} \partial_{y_3}^{1} \partial_{y_4}^{0},$$

followed by setting $y_1 = y_2 = \cdots = y_{m_1} = \gamma_1$. Following the above form of the matrix, we show how this would affect the first few rows:

$$
\begin{bmatrix}
A_{1,1} & A_{1,2} & \cdots & A_{1,p} & e^{a_1 x} & B_{1,1} & B_{1,2} & \cdots & B_{1,q} \\
A_{2,1} & A_{2,2} & \cdots & A_{2,p} & e^{a_2 x} & B_{2,1} & B_{2,2} & \cdots & B_{2,q} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\
A_{m_1,1} & A_{m_1,2} & \cdots & A_{m_1,p} & e^{a_{m_1} x} & B_{m_1,1} & B_{m_1,2} & \cdots & B_{m_1,q} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\
\end{bmatrix}
$$

Applying these derivatives and setting the relevant $y$ values to $\gamma_1$, we have

$$\partial_{y_k}^{k-1} (A_{k,b})_{1,d} = \partial_{y_k}^{k-1} (y_k - y_{j+1})^d e^{(y_k - y_{j+1}) \rho_{b,j}} = f_{d,k-1}(\gamma_1 - \gamma_l; \rho_{b,j}) e^{(\gamma_1 - \gamma_l) \rho_{b,j}},$$

for $d, k, l$. 

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\[
\frac{\partial^{k-1}(B_{k,b})_{1,d}}{y_k} = \frac{\partial^{k-1}(y_k-y_{j+1})}{y_k} d^{-1} e(y_k-y_{j+1}) \rho_{b,j+1} = f_{d-1,k-1}(\gamma_1 - \gamma_l; \rho_{b,j+1}) \rho_{b,j+1}.
\]

The first \( m_l \) rows of the matrix then precisely give the entries of the first row of block matrices in the matrix described in Definition 7.1. Applying such derivatives and plugging in the values of \( \gamma_1, \ldots, \gamma_{l-1} \) for all relevant values of \( y \) further gives the entries of the first \( l - 1 \) rows of block matrices.

The last thing we need to do is then handle multiplicity in the values of \( y \) which equal \( \gamma_l = y_{j+1} \). The key difference here is that the last row of 0s and 1s comes in to play, since setting for example \( y_j = \gamma_l \) will cause the \( j \)th row of the matrix to equal this row of 0s and 1s. Therefore L'Hopital's rule will require that we apply the slightly different differential operator, given by

\[
\frac{\partial^{m_l-1}}{y_{m'+m_l-1}} \cdots \frac{\partial^3}{y_{m'+3}} \frac{\partial^2}{y_{m'+2}} \frac{\partial^1}{y_{m'+1}} \frac{\partial^0}{y_{m'+0}}
\]

where \( m' := m_1 + \cdots + m_{l-1} \). (The difference here is the slight reordering of the \( y \) values corresponding to \( \gamma_l \), as compared to the previous cases.) This means that we will apply one more derivative to each of the relevant rows compared to the previous cases, before plugging in \( \gamma_l \). This is exactly what you see in the definitions of \( A_{l,b}, B_{l,b} \), and \( v_l(x) \) in Definition 7.1, and this completes the proof of Proposition 4.1 in the non-distinct case.

8 Generalization to other Lie groups

The methods developed in this paper are not limited to the unitary group. Here we briefly discuss how similar techniques can be used to sample from the adjoint orbits of the other compact classical groups \( \text{SO}(2n) \), \( \text{SO}(2n+1) \), and \( \text{USp}(n) \). There are only finitely many compact Lie groups that do not fall into the four infinite families \( \text{U}(n) \), \( \text{SO}(2n) \), \( \text{SO}(2n+1) \), and \( \text{USp}(n) \). Therefore these are the only compact Lie groups for which it makes sense to pose questions about the runtime complexity of sampling algorithms as \( n \) grows large. In each case, there is an analogue of the HCIZ integral formula, the Gelfand–Tsetlin polytope, and the theorem stating that the uniform measure on the orbit pushes forward to the uniform measure on the polytope. With these results in hand, the sampling algorithm for each of these other groups is completely analogous to the unitary case. Accordingly we will not repeat the detailed description of the algorithm in each case, and will instead merely sketch the differences between the sampling algorithms for the various classical groups. The results below on generalized GT polytopes are proved in [10]. The generalized HCIZ formulae are special cases of the celebrated formula of Harish-Chandra proved in [16] and are derived explicitly in [29].

8.1 The special orthogonal group \( \text{SO}(N) \)

We first consider the group \( \text{SO}(N) \) of \( N \times N \) orthogonal matrices with determinant 1. Its Lie algebra \( \mathfrak{so}(N) \) consists of \( N \times N \) real antisymmetric matrices. A slight complication that arises when studying the special orthogonal group is that the structure of \( \text{SO}(N) \) differs significantly depending on whether \( N \) is even or odd. Therefore in some formulae we will need to treat these two cases separately, and we will write either \( N = 2n \) or \( N = 2n + 1 \) depending on the case under consideration. By an analogue of the spectral theorem, each element \( X \in \mathfrak{so}(N) \) can be conjugated
The partition function for the density in (16) is given by a determinantal expression analogous to
\[
\nu(X) \propto e^{\text{Tr}(XY)} d\mu_X(X),
\]
where \(\mu_X\) is the SO(N)-invariant probability measure on \(O_X\).

The partition function for the density in (16) is given by a determinantal expression analogous to the HCIZ formula:
\[
\int_{SO(N)} e^{\text{Tr}(YOAT)} dO = \begin{cases} 
\left(\frac{\prod_{p=1}^{n-1} (2p)!}{\prod_{j<k} (y_j - y_k)}\det\left[\frac{\cosh(2y_j \lambda_k)}{y_j y_k (\lambda_j - \lambda_k)(\lambda_j + \lambda_k)}\right]_{j<k=1}^n\right), & N = 2n,
\left(\frac{\prod_{p=1}^{n-1} (2p + 1)!}{\prod_{j<k} (y_j - y_k)}\det\left[\frac{\sinh(2y_j \lambda_k)}{y_j y_k (\lambda_j - \lambda_k)(\lambda_j + \lambda_k)}\right]_{j<k=1}^n\right), & N = 2n + 1,
\end{cases}
\]
where \((\lambda_1, \ldots, \lambda_n)\) and \((y_1, \ldots, y_n)\) are the radial coordinates of \(\Lambda\) and \(Y\), and \(dO\) is the Haar probability measure on \(SO(N)\).

Just as in the unitary case, the sampling algorithm for the special orthogonal group works by reducing the sampling problem on the orbit \(O_X\) to a sampling problem on a generalized Gelfand–Tsetlin polytope. After mapping to the polytope, the problem can be solved by sampling each coordinate according to the inverse CDF method, using the determinantal formula (17) to compute the CDF for each successive coordinate conditional on the coordinates already sampled. We now describe the generalized Gelfand–Tsetlin polytopes for the special orthogonal group, as well as the map from the orbit to the polytope, which sends each \(X \in O_X\) to a type of generalized Rayleigh triangle.

Write \(\lambda_{j,k}, j = 1, \ldots, [k/2]\) for the radial coordinates of the \(k \times k\) leading submatrix \(X[k]\). The array of numbers \(\mathcal{R}(X) = (\lambda_{j,k})_{1 \leq k \leq N, 1 \leq j \leq [k/2]}\) is the analogue of a Rayleigh triangle for \(SO(N)\) and satisfies the following interlacing inequalities:
\[
|\lambda_{1,k}| \geq |\lambda_{1,k-1}| \geq \cdots \geq |\lambda_{1,[k-1]/2},k-1| \geq |\lambda_{1,[k/2],k}|.
\]

The inequalities (18), together with those in (15) and the equalities \(\lambda_{j,n} = \lambda_j\), define the generalized Gelfand–Tsetlin polytope \(GT_{SO(N)}(\lambda)\). Moreover, we have the following uniform pushforward result:

**Lemma 8.2** Under the map \(\mathcal{R} : O_X \rightarrow GT_{SO(N)}(\lambda)\), the uniform probability measure \(\mu_X\) on \(O_X\) pushes forward to the uniform (Lebesgue) probability measure on \(GT_{SO(N)}(\lambda)\).
After sampling a point from \( GT_{\text{SO}(N)}(\lambda) \), it remains to sample from the fiber over that point. This can be accomplished using a procedure exactly analogous to that in Section 5 by writing the characteristic polynomial of each leading submatrix \( X[k] \) in two different ways to obtain a description of the fiber as a product of spheres.

### 8.2 The compact symplectic group \( \text{USp}(n) \)

The compact symplectic group \( \text{USp}(n) \) is a subgroup of \( U(2n) \), and consists of all \( 2n \times 2n \) complex unitary matrices of the form

\[
\begin{bmatrix}
  A & -B \\
  B & A
\end{bmatrix},
\]

where \( A \) and \( B \) are \( n \times n \) blocks and the bar indicates the entrywise complex conjugate. Its Lie algebra \( \text{usp}(n) \) consists of \( 2n \times 2n \) anti-Hermitian matrices of the form

\[
\begin{bmatrix}
  A & B \\
  C & -A^T
\end{bmatrix},
\]

where \( A, B, C \) are \( n \times n \) blocks satisfying \( B = B^T, \ C = C^T \). By an analogue of the spectral theorem, each element \( X \in \text{usp}(n) \) can be conjugated by some element \( S \in \text{USp}(n) \) to obtain a matrix of the form

\[
\Lambda = SXS^* = \text{diag}(i\lambda_1, \ldots, i\lambda_n, -i\lambda_1, \ldots, -i\lambda_n), \quad \lambda_1 \geq \ldots \geq \lambda_{n-1} \geq 0, \quad |\lambda_{n-1}| \geq |\lambda_n|.
\]

(21)

Again we refer to the numbers \( (\lambda_1, \ldots, \lambda_n) \) as the radial coordinates of \( X \) (or of \( \Lambda \)), and the adjoint orbit of \( \Lambda \) is the set

\[
O_\Lambda = \{SAS^* \mid S \in \text{USp}(n)\} \subset \text{usp}(n).
\]

We then have the following sampling problem:

**Problem 8.3 (Sampling from adjoint orbits of \( \text{USp}(n) \))** *Given two \( 2n \times 2n \) matrices \( \Lambda \) and \( Y \) of the form (21), sample an \( X \in O_\Lambda \) from the distribution*

\[
\nu(X) \propto e^{\text{Tr}(YX)}d\mu_\Lambda(X),
\]

(22)

*where \( \mu_\Lambda \) is the \( \text{USp}(n) \)-invariant probability measure on \( O_\Lambda \).*

The partition function for the density in (22) is again given by a determinantal formula:

\[
\int_{\text{USp}(n)} e^{\text{Tr}(YSAS^*)}dS = \left(\prod_{p=1}^{n-1} (2p + 1)!\right) \frac{\det \left[ \sinh(2y_j\lambda_k) \right]_{j,k=1}^n}{\prod_{j=k}^{n} (y_j(y_j - y_k)(\lambda_j - \lambda_k)(\lambda_j + \lambda_k))},
\]

(23)

where \( (\lambda_1, \ldots, \lambda_n) \) and \( (y_1, \ldots, y_n) \) are the radial coordinates of \( \Lambda \) and \( Y \), and \( dS \) is the Haar probability measure on \( \text{USp}(n) \).

Again, the sampling problem can be solved by mapping \( O_\Lambda \) to a generalized Gelfand–Tsetlin polytope and sampling each coordinate according to the inverse CDF method, using the determinantal formula (23) to compute the CDF for each successive coordinate conditional on the coordinates already sampled.

In the case of the compact symplectic group, the pushforward map to the polytope differs slightly from the preceding cases. Write \( \lambda_{1,k} \geq \ldots \geq \lambda_{k,k} \) for the radial coordinates of the leading
submatrix $X[2k]$, and write $\lambda_{1,k-1/2} \geq \ldots \geq \lambda_{k,k-1/2}$ for the ordered absolute values of the $k$ largest

eigenvalues of $iX[2k-1]$, for $k = 1, \ldots, n$.

The array of numbers $\mathcal{R}(X) = (\lambda_{j,k}, \lambda_{j,k-1/2})_{1 \leq j \leq k \leq n}$ is the analogue of a Rayleigh triangle for
$\text{USp}(n)$. It satisfies the interlacing inequalities

$$
\lambda_{1,k} \geq \lambda_{1,k-1/2} \geq \lambda_{2,k} \geq \ldots \geq \lambda_{k,k} \geq \lambda_{k,k-1/2} \geq 0, \quad k = 1, \ldots, n,
$$

$$
\lambda_{1,k-1/2} \geq \lambda_{1,k-1} \geq \ldots \geq \lambda_{k-1,k-1} \geq \lambda_{k,k-1/2} \geq 0, \quad k = 2, \ldots, n.
$$

The generalized Gelfand–Tsetlin polytope $\text{GT}_{\text{USp}(n)}(\lambda) \subset \mathbb{R}^{n^2}$ is defined by the inequalities (24) and by the equalities $\lambda_{j,n} = \lambda_j$, $j = 1, \ldots, n$. Write $\Pi : \mathbb{R}^{n^2} \to \mathbb{R}^{n(n-1)/2}$ for the projection that sends $(\lambda_{j,k}, \lambda_{j,k-1/2})_{1 \leq j \leq k \leq n} \mapsto (\lambda_{j,k})_{1 \leq j \leq k \leq n}$. Then we have the following uniform pushforward result.

**Lemma 8.4** The pushforward of the uniform probability measure $\mu_\Lambda$ on $O_\Lambda$ under the composition of maps $\Pi \circ R$ is equal to the pushforward of the uniform (Lebesgue) probability measure on $\text{GT}_{\text{USp}(n)}(\lambda)$ under the projection $\Pi$.

It can be shown that the density $e^{\text{Tr}(YX)}$, $X \in O_\Lambda$ depends only on $\Omega(\mathcal{R}(X))$, so that we can ignore the pushforward by $\Pi$ in Lemma [8,4] and just sample from an exponential density on $\text{GT}_{\text{USp}(n)}(\lambda)$ as in the preceding cases. Here again, after sampling a point from $\text{GT}_{\text{USp}(n)}(\lambda)$, we can sample from the fiber of $O_\Lambda$ over that point using a procedure analogous to the method presented in Section [3].

**Acknowledgments**

This research was supported in part by NSF CCF-1908347 and NSF DMS-1714187. We would like to thank Ainesh Bakshi, Anay Mehrotra, Kunal Talwar, and Abhradeep Thakurta for useful discussions.

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A Inverse CDF method using binary search

In this section we give the proof of correctness of a method that, given access to a continuous CDF \( F(x) \) (for which the derivative of the PDF \( f(x) \) is bounded by \( G \)), an integer \( L > \log G/\varepsilon \) and an \( \varepsilon > 0 \), outputs a rational number \( s \) from the \( L \)-truncation of a distribution which is \( \varepsilon \)-close to \( f \) in the TV distance. For a density \( g : [0, 1] \rightarrow \mathbb{R}_{\geq 0} \), we define its \( L \)-truncation \( \hat{g}_L \), whose domain is \( \left\{ k/2^L \right\}_{k=0}^{2^L} \), as

\[
\hat{g}_L(k/2^L) = \int_{k/2^L}^{(k+1)/2^L} g(x)dx.
\]

**Assumption A.1** Assume \( f \) is \( G \)-Lipschitz; that is, for all \( x_1, x_2 \in [0, 1] \) we have

\[
|f(x_1) - f(x_2)| \leq G|x_1 - x_2|.
\]

When \( f \) is differentiable, this is the same as \( |f'(x)| \leq G \).

Algorithm to sample from a one dimensional distribution given oracle access to CDF.

1. Set \( \delta := \frac{1}{2^L} \).
2. Let \( z \) be point u.a.r. from \([0, 1]\).
3. Let \( l := 0 \) and \( r := \frac{1}{\delta} \).
4. while \( (l < r - 1) \)
   a. Let \( m := (l + r)/2 \)
   b. if \( F(m \cdot \delta) < z \) then \( l = \lfloor m \rfloor \)
   c. else if \( F(m \cdot \delta) > z \) then \( r = \lceil m \rceil \)
   d. otherwise \( l = \lfloor m \rfloor \) and \( r = \lceil m \rceil \).
5. Output \( \hat{s} = l \cdot \delta \).

Note that in Step 5, if we let \( s \) to be uniformly at random from the interval \([l \cdot \delta, (l + 1) \cdot \delta]\) and \( g \) denote the PDF of \( s \), then \( \hat{s} \) has the PDF \( \hat{g}_L \). The number of iterations of this algorithm is \( O(L) = O(\log G/\varepsilon) \). It remains to show that \( g \) is \( \varepsilon \)-close to \( f \) in TV distance.

**Lemma A.2**

\[
\int_0^1 |g(x) - f(x)|dx \leq \varepsilon.
\]

**Lemma A.3** For all \( x \in [0, 1] \), \( g \) satisfies,

\[
g(x) = \frac{1}{\delta} \cdot \left( F(\lfloor x/\delta \rfloor \cdot \delta) - F(\lceil x/\delta \rceil \cdot \delta) \right).
\]

Pick a small \( \delta > 0 \). Assumption A.1 implies that

\[
F(x + \delta) - F(x) = \int_x^{x+\delta} f(t)dt = \int_x^{x+\delta} \left( \int_t^x f'(z)dz + f(x) \right)dt \leq \frac{G\delta^2}{2} + f(x) \cdot \delta.
\]
Similarly, we have

\[ F(x + \delta) - F(x) = \int_x^{x+\delta} f(t)dt = \int_x^{x+\delta} \left( \int_x^t f'(z)dz + f(x) \right) dt \geq -\frac{G\delta^2}{2} + f(x) \cdot \delta. \]

Combining, we obtain that for all \( x \in [0, 1] \) and small enough \( \delta > 0 \)

\[ |F(x + \delta) - F(x) - f(x) \cdot \delta| \leq \frac{G\delta^2}{2}. \]

Applying the same argument to \( F(x) - F(x - \delta) \) we get

\[ |F(x) - F(x - \delta) - f(x) \cdot \delta| \leq \frac{G\delta^2}{2}. \]

Combining, we obtain that for all \( x \in [0, 1] \) and small enough \( \delta_1, \delta_2 > 0 \)

\[ |F(x + \delta_1) - F(x - \delta_2) - f(x) \cdot (\delta_1 + \delta_2)| \leq |F(x + \delta_1) - f(x) \cdot \delta_1| + |F(x - \delta_2) - f(x) \cdot \delta_2| \leq \frac{G(\delta_1^2 + \delta_2^2)}{2}. \]

Equivalently, we have

\[ \left| \frac{F(x + \delta_1) - F(x - \delta_2)}{\delta_1 + \delta_2} - f(x) \right| \leq \frac{G(\delta_1^2 + \delta_2^2)}{2(\delta_1 + \delta_2)} \leq \frac{1}{4} \cdot G \cdot (\delta_1 + \delta_2) \tag{25} \]

Now, we are ready to prove the theorem.

\[ \int_0^1 |g(x) - f(x)|dx = \int_0^1 \left| \frac{F([x/\delta] \cdot \delta) - F([x/\delta] \cdot \delta)}{\delta} - f(x) \right| dx \]

Let \( \delta_1 := ([x/\delta] - x/\delta) \cdot \delta \) and \( \delta_2 := (x/\delta - [x/\delta]) \). Let \( S \) be the set of values where \( x/\delta \) is integral and \( x \in [0, 1] \), i.e.,

\[ S := \{ \frac{x}{\delta} \in \mathbb{Z} : \ x \in [0, 1] \}. \]

Notice that \( |S| \leq \delta + 1. \) Further, notice that for all \( x \in [0, 1]\setminus S \),

\[ \delta_1 + \delta_2 = \delta \cdot ([x/\delta] - [x/\delta]) = \delta. \tag{26} \]

Using these we get

\[ \int_0^1 |g(x) - f(x)|dx = \int_0^1 \left| \frac{F(x + \delta_1) - F(x - \delta_2)}{\delta} - f(x) \right| dx \]

\[ = \int_{x \in [0,1]\setminus S} \left| \frac{F(x + \delta_1) - F(x - \delta_2)}{\delta} - f(x) \right| dx \]

\[ \leq \int_{x \in [0,1]\setminus S} \left| \frac{F(x + \delta_1) - F(x - \delta_2)}{\delta_1 + \delta_2} - f(x) \right| dx \]

\[ \leq \int_{x \in [0,1]\setminus S} G \cdot (\delta_1 + \delta_2)dx. \]

Notice that for all \( x \in [0, 1], \delta_1 + \delta_2 = \delta \cdot ([x/\delta] - [x/\delta]) \leq \delta \). Using this

\[ \int_0^1 |g(x) - f(x)|dx \leq G \cdot \delta = \frac{G}{2L} \leq \varepsilon. \]
B Bounding the derivative of the PDF

Let $F_{i,j}$ denote the CDF of the coordinate $R_{i,j}$, conditional on the given values $R_{k,l}$, $l > j$ and $R_{k,j}$, $k < i$. That is,

$$F_{i,j}(x) = \frac{\det(M_{i,j}[y,R](x))}{\det(M_{i,j}[y,R](R_{i,j+1}))}$$

for $x \in [R_{i+1,j+1}, R_{i,j+1}]$, where $M_{i,j}[y,R](x)$ is the matrix defined in Definition 7.1. Let $f_{i,j}(x)$ denote the corresponding PDF.

**Lemma B.1 (Bound on the derivative of the PDF)** For all $i, j$ and $x \in (R_{i+1,j+1}, R_{i,j+1})$, we have

$$|f'_{i,j}(x)| \leq 2^{\text{poly}(n \cdot \|y\| \cdot \|\lambda\|, L)}$$

where the entries of $y$ and $R$ given as rational numbers with denominator $2^L$.

**Proof:** Let $S$ denote the $d$-dimensional slice of $GT(\lambda)$ given by fixing $R_{k,l}$ for $k < i$ and $l = j$ and also for all $k$ with $l > j$. Also, let $S_x$ denote the slice of $S$ given by further fixing $R_{i,j} = x \in (R_{i+1,j+1}, R_{i,j+1})$. For any such $x$, we also define a function on all of $S$ via

$$g_x(P) := e^{\langle y, \text{type}(P|R_{i,j}=x) \rangle}.$$ 

For small $\delta$ and any $x$, we use triangle inequality to obtain the following bound:

$$\left| \partial_x \int_{S_x} e^{\langle y, \text{type}(Q) \rangle} dQ \right| \leq \frac{\lim_{\delta \to 0} \left| \int_{S_{x-\delta}} g_{x-\delta}(Q) dQ - \int_{S_x} g_x(Q) dQ \right|}{\delta \cdot \int_S e^{\langle y, \text{type}(P) \rangle} dP}.$$

This idea here is that $C_1$ handles the change in volume, and $C_2$ handles the change in the function.

We first bound $C_1$. Since

$$\|\langle y, \text{type}(P) \rangle\| \leq \|y\|_2 \cdot \|\text{type}(P)\|_2 \leq n \cdot \|y\| \cdot \|\text{type}(P)\| \leq 2n^2 \cdot \|y\| \cdot \|\lambda\|,$$

we have

$$C_1 \leq \lim_{\delta \to 0} \left| \int_{S_{x-\delta}} g_{x-\delta}(Q) dQ - \int_{S_x} g_x(Q) dQ \right| \cdot \frac{\max_{P} e^{\langle y, \text{type}(P) \rangle}}{\min_{P} e^{\langle y, \text{type}(P) \rangle}} = \left| \partial_x \int_{S_x} g_x(Q) dQ \right| \cdot e^{4n^2 \cdot \|y\| \cdot \|\lambda\|}.$$

Now, the expression $\partial_x \int_{S_x} dQ$ is the derivative of the PDF in the case that $y = 0$. In that case, we have from Proposition 4.1 and Remark 7.2 that

$$\partial_x \int_{S_x} dQ = \frac{\partial^2 \det(M_{i,j}[y,R](x))}{\det(M_{i,j}[y,R](R_{i,j+1}))}$$

where

$$M_{i,j}[y,R](x) = \begin{bmatrix}
1 & \cdots & 1 & x & R_{i+1,j+1} & \cdots & R_{i+1,j+1} \\
2R_{1,j} & \cdots & 2R_{i-1,j} & x^2 & R^2_{i+1,j+1} & \cdots & R^2_{i+1,j+1} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
jR^{j-1}_{1,j} & \cdots & jR^{j-1}_{i-1,j} & x^j & R^j_{i+1,j+1} & \cdots & R^j_{i+1,j+1} \\
0 & \cdots & 0 & 1 & 1 & \cdots & 1
\end{bmatrix}.$$
We also obtain the upper bound to the following bound when \( R \) where one of the \( C \) achieve
\[
\int_{\mathbb{R}^2} \left| \partial_y g(x, y) \right| dy \leq \int_{\mathbb{R}^2} e^{\langle y, \text{type}(P) \rangle} dQ = \frac{\max_P \| \nabla e^{\langle y, \text{type}(P) \rangle} \|_2}{\min_P e^{\langle y, \text{type}(P) \rangle}}.
\]
The first term \( \int_{\mathbb{R}^2} dQ \) is the PDF when \( y = 0 \), and we can use the same techniques as above to achieve
\[
\int_{\mathbb{R}^2} dQ \leq \left( n\| \lambda \| 2^L \right)^n.
\]
We now bound the gradient (with respect to the entries of \( P \)) via
\[
\| \nabla e^{\langle y, \text{type}(P) \rangle} \|_2 \leq \sqrt{\frac{n + 1}{2}} \| \nabla e^{\langle y, \text{type}(P) \rangle} \| \leq 2\| y \| \sqrt{\frac{n + 1}{2}} \cdot \max_P e^{\langle y, \text{type}(P) \rangle}.
\]
This gives the bound
\[
\frac{\max_P \| \nabla e^{(y,\text{type}(P))} \|_2}{\min_P e^{(y,\text{type}(P))}} \leq 2\|y\| \sqrt{\binom{n+1}{2}} \cdot \frac{\max_P e^{(y,\text{type}(P))}}{\min_P e^{(y,\text{type}(P))}} \leq 2n\|y\|e^{4\|y\|\|\lambda\|}.
\]
And this finally gives
\[
C_2 \leq 2n\|y\| \cdot \left( n\|\lambda\| \cdot 2^L \cdot e^{4\|y\|\|\lambda\|} \right)^n.
\]
Putting this all together, we obtain
\[
|f_{f,j}'(x)| = \left| \partial_x \int S_x e^{(y,\text{type}(Q))} dQ \right| \leq C_1 + C_2 \leq (2n\|y\| + 1) \left( n\|\lambda\| \cdot 2^L \cdot e^{4\|y\|\|\lambda\|} \right)^n.
\]
This implies the result.

C  Disintegration and pushforwards of probability measures

The disintegration theorem is a kind of a factorization result for probability measures. Given a mapping \( \pi : Y \to X \) between two probability spaces satisfying certain mild assumptions, the theorem describes how to “disintegrate” a probability measure on \( Y \) by decomposing it into a probability measure on \( X \) and a family of probability measures supported on the fibers of \( \pi \). Here we merely state the disintegration theorem and describe how it applies in the special case of the Rayleigh map \( \mathcal{R} : \mathcal{O}_\Lambda \to \text{GT}(\lambda) \). We refer the reader to [6] for further details. We also prove a separate lemma that bounds the total variation distance between two probability measures in terms of the total variation distance between their pushforward measures.

**Theorem C.1 (Disintegration theorem for probability measures)** Let \( X \) and \( Y \) be complete separable metric spaces equipped with their Borel \( \sigma \)-algebras, and let \( \pi : Y \to X \) be a Borel-measurable function. Let \( \mu \) be a probability measure on \( Y \), and write \( \pi_*\mu \) for its pushforward by \( \pi \). Then there exists a family of probability measures \( \{\mu_x\}_{x \in X} \) on \( Y \) such that the following hold:

- For every Borel set \( E \subset Y \), the function \( x \mapsto \mu_x(E) \) is Borel-measurable.
- The measures \( \mu_x \) are supported on the fibers of \( \pi \), i.e. \( \mu_x(\pi^{-1}(x)) = 1 \) for \( \pi_*\mu \)-almost all \( x \in X \).
- For every Borel-measurable function \( f : Y \to [0,\infty] \),
\[
\int_Y f(y) d\mu(y) = \int_X \int_{\pi^{-1}(x)} f(y) d\mu_x(y) d\pi_*\mu(x).
\]
Moreover, the family of measures \( \{\mu_x\}_{x \in X} \) is \( \pi_*\mu \)-almost everywhere uniquely determined.

In the case of the Rayleigh map \( \mathcal{R} : \mathcal{O}_\Lambda \to \text{GT}(\lambda) \) and the \( U(n) \)-invariant probability measure \( \mu_\Lambda \) on \( \mathcal{O}_\Lambda \), Theorem C.1 and Proposition 2.2 together give the following.

**Corollary C.2** For any Borel set \( E \subset \mathcal{O}_\Lambda \) and any Borel-measurable function \( f : \mathcal{O}_\Lambda \to [0,\infty] \),
\[
\int_{\mathcal{O}_\Lambda} f(X) d\mu_\Lambda(X) = \frac{1}{\text{Vol}(\text{GT}(\lambda))} \int_{\text{GT}(\lambda)} \int_{\mathcal{R}^{-1}(P)} f(X) \text{unif}_P(X) dP,
\]
where \( dP \) is Lebesgue measure on \( \text{GT}(\lambda) \) and \( \text{unif}_P \) is the uniform probability measure on the fiber \( \mathcal{R}^{-1}(P) \).
The following lemma bounds the total variation distance between two probability measures $\mu, \nu$ in terms of the total variation distance between their pushforwards $\pi_*\mu$ and $\pi_*\nu$.

**Lemma C.3** Let $X, Y$ and $\pi$ be as in Theorem C.1 and let $\mu, \nu$ be probability measures on $Y$. Then $||\mu - \nu||_{TV} \leq ||\pi_*\mu - \pi_*\nu||_{TV}$.

**Proof:** Recall that $||\mu - \nu||_{TV} = \sup_{E \subseteq Y} |\mu(E) - \nu(E)|$, where the supremum runs over Borel sets $E \subseteq Y$. Then we have:

$$||\mu - \nu||_{TV} = \sup_{E \subseteq Y} |\mu(E) - \nu(E)| \leq \sup_{F \subseteq X} |\mu(\pi^{-1}(F)) - \nu(\pi^{-1}(F))| = \sup_{F \subseteq X} |\pi_*\mu(F) - \pi_*\nu(F)| = ||\pi_*\mu - \pi_*\nu||_{TV},$$

where the inequality comes from restricting the supremum to run only over subsets of $Y$ that are preimages of Borel subsets of $X$.

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**D Stability with respect to perturbations of $Y$**

Here we prove a bound on how much a perturbation of the matrix $Y$ can affect the distribution $e(Y, X) d\mu_A(X)$. Note that the assumption on $Y$ and $Y'$ in the proposition can be achieved by shifting $Y$ and $Y'$ by multiples of the identity, which corresponds to scaling the above integrals. In this case, the functions $e(Y, X)$ and $e(Y', X)$ are in fact probability density functions.

**Proposition D.1** Fix an $n \times n$ diagonal matrix $\Lambda$, and let $Y$ and $Y'$ be $n \times n$ Hermitian matrices such that

$$\int_{\Omega} e(Y, X) d\mu_\Lambda(X) = 1 \quad \text{and} \quad \int_{\Omega} e(Y', X) d\mu_\Lambda(X) = 1.$$

Then for $||Y' - Y||$ small, we have

$$\frac{1}{2} \int_{\Omega} |e(Y, X) - e(Y', X)| d\mu_\Lambda(X) \leq ||Y' - Y|| \cdot ||\Lambda||.$$

**Proof:** When $||Y' - Y|| \cdot ||\Lambda|| < 1$, we can bound the difference between the corresponding probability distributions via

$$\int_{\Omega} |e(Y, X) - e(Y', X)| d\mu_\Lambda(X) \leq \int_{\Omega} e(Y, X) |e(Y' - Y, X) - 1| d\mu_\Lambda(X)$$

$$\leq \sup_{X \in \Omega} |e(Y' - Y, X) - 1|$$

$$\leq \sup_{X \in \Omega} \sum_{m=1}^{\infty} \frac{|(Y' - Y, X)|^m}{m!}$$

$$\leq \sup_{X \in \Omega} \sum_{m=1}^{\infty} \frac{||(Y' - Y) \cdot \|X\||^m}{m!}$$

$$\leq \sum_{m=1}^{\infty} \frac{||Y' - Y|| \cdot ||\Lambda||}{m!}$$

$$\leq (e - 1) \cdot ||Y' - Y|| \cdot ||\Lambda||.$$

Note that replacing $\|X\|$ by $\|\Lambda\|$ while dropping the sup follows from unitary invariance.