QUASICOHERENT SHEAVES ON COMPLEX NONCOMMUTATIVE TWO-TORI

A. POLISHCHUK

ABSTRACT. We introduce the notion of a quasicoherent sheaf on a complex noncommutative two-torus $T$ as an ind-object in the category of holomorphic vector bundles on $T$. Extending the results of [10] and [9] we prove that the derived category of quasicoherent sheaves on $T$ is equivalent to the derived category of usual quasicoherent sheaves on the corresponding elliptic curve. We define the rank of a quasicoherent sheaf on $T$ that can take arbitrary nonnegative real values. We study the category $\mathcal{Qcoh}(\eta_T)$ obtained by taking the quotient of the category of quasicoherent sheaves by the subcategory of objects of rank zero (called torsion sheaves). We show that projective objects of finite rank in $\mathcal{Qcoh}(\eta_T)$ are classified up to an isomorphism by their rank. We also prove that the subcategory of objects of finite rank in $\mathcal{Qcoh}(\eta_T)$ is equivalent to the category of finitely presented modules over a semihereditary algebra.

INTRODUCTION

The goal of this paper is to define and study the category of quasicoherent sheaves on a noncommutative two-torus equipped with a complex structure. Recall that a noncommutative two-torus $T_\theta$ is defined via its algebra of smooth functions $A_\theta$ that is determined by an irrational real number $\theta$. A complex structure on $T_\theta$ is given by a certain derivation $\delta_\tau$ of the algebra $A_\theta$ associated with a complex parameter $\tau$ (see section 1.1). We view this derivation as an analogue of the operator $\partial$. We denote by $T = T_{\theta,\tau}$ the obtained complex noncommutative torus. Holomorphic vector bundles on $T$ are defined as finitely generated projective right $A_\theta$-modules equipped with a lifting of $\delta_\tau$ (see section 1.1).

The category $\text{Vect}(T)$ of holomorphic vector bundles on $T$ was studied in [10], [8] and [9]. In particular, it was proved in [9] that this category is abelian. Furthermore, if one tries to mimick the usual definition of a coherent sheaf in this situation one obtains that every such coherent sheaf is a vector bundle (see Theorem 2.2.2). Thus, there are no analogues of coherent torsion sheaves in our situation. However, as we will show, things become more interesting if we consider quasicoherent sheaves. We define the category $\mathcal{Qcoh}(T)$ of quasicoherent sheaves as the category of ind-objects in $\text{Vect}(T)$. We realize this category explicitly as a full subcategory in the larger category of holomorphic modules on $T$ (these are arbitrary $A_\theta$-modules equipped with a lifting of $\delta_\tau$, see section 2.3).

Our first result is that the category $\mathcal{Qcoh}(T)$ is equivalent to a certain abelian subcategory in the derived category $D^b(\mathcal{Qcoh}(E))$ of usual quasicoherent sheaves on the elliptic curve $E = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$. This extends a similar result for $\text{Vect}(T)$ (see [10], [9]). Furthermore, there is an equivalence of derived categories $D^b(\mathcal{Qcoh}(T)) \simeq D^b(\mathcal{Qcoh}(E))$ (see Theorem 2.4.5).

Supported in part by NSF grant.
Next, we prove that the rank of a vector bundle on $T$ (which is a nonnegative real number of the form $m\theta + n$ with $m, n \in \mathbb{Z}$) extends naturally to an additive function on the category of quasicoherent sheaves taking all possible values in $\mathbb{R}_{\geq 0} \cup \{+\infty\}$ (see Theorem 2.5.1 and Corollary 2.7.4).

We define torsion sheaves as quasicoherent sheaves of rank zero. It turns out that there are many nontrivial torsion sheaves on $T$. In fact, the subcategory $\text{Tors}$ of torsion sheaves is big enough to make the passage from $\text{Qcoh}(T)$ to the quotient-category $\text{Qcoh}(\eta_T) := \text{Qcoh}(T)/\text{Tors}$ in many ways similar to the passage to the general point in commutative algebra geometry.

Note that by definition the rank descends to an additive function on $\text{Qcoh}(\eta_T)$. Let $\text{Qcoh}^f(\eta_T)$ denote the full subcategory of $\text{Qcoh}(\eta_T)$ consisting of quasicoherent sheaves of finite rank.

Our main result is that the category $\text{Qcoh}^f(\eta_T)$ is equivalent to the category of finitely presented modules over a semihereditary ring and that projective objects of $\text{Qcoh}^f(\eta_T)$ are uniquely determined by their rank that can be an arbitrary real number (see Theorem 3.3.3, Proposition 3.3.1, Corollaries 3.2.5 and 3.2.6). We also show that projective objects of $\text{Qcoh}^f(\eta_T)$ correspond to quasi vector bundles, i.e., filtering unions of holomorphic vector bundles (see Theorem 3.1.6). The above equivalence of categories is obtained by taking a projective object $P \in \text{Qcoh}^f(\eta_T)$ and sending $X \in \text{Qcoh}^f(\eta_T)$ to the module $\text{Hom}_{\text{Qcoh}(\eta_T)}(P, X)$ over the ring $R_P = \text{End}_{\text{Qcoh}(\eta_T)}(P)$.

To a large extent the study of the category $\text{Qcoh}(\eta_T)$ reduces to the problem of constructing holomorphic subbundles and quasicoherent subsheaves in a stable holomorphic bundle $V$ on $T$ with given properties (see section 1.1 for the definition of stability for bundles on $T$). One of the constructions we use can be considered as a categorification of a two-sided version of the continuous fraction process (see sections 1.2 and 1.3). Subbundles constructed in this way are numbered by vertices of a binary tree and depend also on some continuous parameters. Theorem 1.3.1 implies that the ranks of the subbundles of $V$ obtained by this construction are constrained only by the requirement that the corresponding slope is smaller than the slope of $V$ (of course, these ranks also should be smaller than $\text{rk} V$).

One may wonder what kind of restrictions one gets for $A_\theta$-modules underlying quasicoherent sheaves on $T$. We consider the subcategory of countably generated quasicoherent sheaves and show that it is a Serre subcategory in $\text{Qcoh}(T)$ and that the underlying $A_\theta$-modules always have projective dimension $\leq 1$ but are not necessarily projective (see Theorem 2.9.4, Corollary 2.7.4). In particular, we derive that every quasicoherent ideal in $A_\theta$ is countably generated. The proof is based on the analogue of the Harder-Narasimhan filtration for quasicoherent subsheaves of holomorphic vector bundles on $T$ (see section 2.8).

One may view the ring $R_P$ appearing above as an algebraic version of the von Neumann factor of type $II_1$. Namely, if $P$ is a vector bundle then such a factor appears as the closure of the endomorphism algebra $\text{End}_{A_\theta}(P)$ in the algebra of bounded operators on the appropriate Hilbert space. We conjecture that algebra $R_P$ contains a “convergent”
subalgebra $R'_P$ with $K_0(R'_P) = K_0(R_P) = \mathbb{R}$ such that $R'_P$ embeds also into the above von Neumann factor.

The plan of the paper is as follows. In section 1 we prove some auxiliary statements about the category $\text{Vect}(T)$ of holomorphic vector bundles. In section 2 we define and study the category $\text{Qcoh}(T)$ of quasicoherent sheaves. This includes realizing $\text{Qcoh}(T)$ explicitly as the category of admissible holomorphic modules in 2.3, establishing the equivalence of derived categories in 2.4, and constructing the extension of the rank function to quasicoherent sheaves in 2.5. We also study the subcategory of countably presented quasicoherent sheaves in 2.9. Finally, in section 3 we prove our results about the category $\text{Qcoh}(\eta_T)$ of “sheaves at the general point of $T$” (and its subcategory $\text{Qcoh}^f(\eta_T)$).

Acknowledgment. I am grateful to Paul Smith for the stimulating question on possible ranks of holomorphic ideals in $A_\theta$. The answer to this question is Theorem 1.3.1. I’m also indebted to Amnon Neeman for a helpful discussion on inductive limits in derived categories.

1. Some facts about the category of holomorphic bundles on $T$

In this section we prove some results about holomorphic bundles on a noncommutative torus $T$ that will be used in our study of quasicoherent sheaves on $T$. After providing some background we describe in 1.3 two constructions of subbundles in holomorphic vector bundles on $T$ that will play a crucial role in section 3. Then in 1.4 we show that every two holomorphic vector bundles of the same rank on $T$ are deformation equivalent.

1.1. Preliminaries. Throughout this paper the number $\theta$ is assumed to be irrational. By a module over a ring we always mean a right module (same convention for ideals).

The algebra $A_\theta$ of smooth functions on the noncommutative torus $T_\theta$ is defined as the algebra of series of the form $\sum_{(m,n) \in \mathbb{Z}^2} a_{m,n} U_1^m U_2^n$, where the generators $U_1$ and $U_2$ satisfy the commutation relation $U_1 U_2 = \exp(2\pi i \theta) U_2 U_1$, and $(a_{m,n})$ is a collection of complex numbers rapidly decreasing as $m^2 + n^2 \to \infty$. We fix $\tau \in \mathbb{C}$ such that $\text{Im} \tau < 0$ and consider the derivation

$$\delta_\tau : A_\theta \to A_\theta : \sum a_{m,n} U_1^m U_2^n \mapsto 2\pi i \sum a_{m,n} (m\tau + n) U_1^m U_2^n$$

as an analogue of the $\overline{\partial}$-operator giving the complex structure on our noncommutative torus. By definition, a vector bundle on $T_\theta$ is a finitely generated projective right $A_\theta$-module. A holomorphic vector bundle on $T_\theta$ is a vector bundle $P$ equipped with an operator $\nabla : P \to P$ such that

$$\nabla(sa) = \nabla(s)a + s\delta_\tau(a) \quad (1.1.1)$$

for all $s \in P$, $a \in A_\theta$.

The category $\text{Vect}(T)$ of holomorphic vector bundles on a noncommutative complex torus $T = T_{\theta,\tau}$ was studied in [10] and [9]. We showed that there is an equivalence of $\text{Vect}(T)$ with a certain abelian subcategory $\mathcal{C}^\theta$ in the derived category $D^b(E)$ of coherent sheaves on the elliptic curve $E = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$. Nonzero objects $F \in \mathcal{C}^\theta$ satisfy

$$\text{rk}_\theta(F) := \deg_{\mathbb{C}}(F)\theta + \text{rk}_E(F) > 0,$$
where $\text{deg}_E$ and $\text{rk}_E$ are the standard degree and rank functions on $D^b(E)$. On the other hand, for a vector bundle on a noncommutative torus there is also a notion of rank defined using the trace $\text{tr} : A_\theta \to \mathbb{R} : \sum_{m,n} a_{m,n} U_1^m U_2^n \mapsto a_{0,0}$. For $V \in \text{Vect}(T)$ we denote this rank by $\text{rk}_V$. If we view $V$ as an object of $\mathcal{C}_\theta$ we obtain the definition of stability

$$
\text{deg}_E(V) = \text{deg}_E(V) \text{ (degree)} \\
\mu(V) = \text{deg}(V)/\text{rk}(V) \text{ (slope)} .
$$

Note that the degree is determined by the rank: $\text{deg}(V) = m\theta + n$ with $m, n \in \mathbb{Z}$.

**Definition.** We say that an object $A$ in $D^b(E)$ is stable (resp., semistable) if $A \simeq F[n]$, where $F$ is either a stable vector bundle on $E$ or the structure sheaf of a point in $E$ (resp., $F$ is either a semistable vector bundle or a torsion sheaf).

Note that every object of $D^b(E)$ can be decomposed into a direct sum of semistable objects. Viewing $\text{Vect}(T)$ as a subcategory in $D^b(E)$ we obtain the definition of stability and semistability for holomorphic vector bundles on $D^b(E)$. It is easy to see that stable holomorphic bundles on $T$ correspond exactly to standard holomorphic structures on basic projective modules over $A_\theta$ (see [10]). In the following lemma we check that the above definition coincides with the notion of stability obtained using slopes of bundles on $T$.

**Lemma 1.1.1.** A holomorphic vector bundle $V$ on $T$ is stable (resp., semistable) iff for every subbundle $W \subset V$ such that $0 < \text{rk} W < \text{rk} V$, one has $\mu(W) < \mu(V)$ (resp., $\mu(W) \leq \mu(V)$).

**Proof.** We will only prove the part concerning semistability and leave the stability part to the reader. Assume first that $V$ is semistable. If $W$ is a semistable bundle on $T$ then $\text{Hom}(W, V) \neq 0$ only if $\mu(W) \leq \mu(V)$ (this follows from Lemma 1.6 of [8] using Serre duality). Since every vector bundle of slope $\mu$ contains a semistable subbundle of slope $\geq \mu$, the “only if” part follows. Conversely, assume that for every $W \subset V$ with $0 < \text{rk} W < \text{rk} V$ one has $\mu(W) \leq \mu(V)$. Let $V_0 \subset V$ be a maximal semistable subbundle of maximal slope (it exists). Then $\mu(V_0) \leq \mu(V)$ hence we should have $V_0 = V$, i.e., $V$ is semistable. \hfill $\square$  

**Lemma 1.1.2.** Let $X \to Y \to Z \to X[1]$ be an exact triangle in $D^b(E)$ with $X, Y \in \mathcal{C}_\theta$. Assume that $Z = \bigoplus_{i=1}^k Z_i$, where each $Z_i$ is semistable with $\text{rk}_\theta(Z_i) > 0$ (resp., $\text{rk}_\theta(Z_i) < 0$). Then $Z \in \mathcal{C}_\theta$ (resp., $Z[-1] \in \mathcal{C}_\theta$).

**Proof.** Let $H^i : D^b(E) \to \mathcal{C}_\theta$ be the cohomology functors associated with the $t$-structure that has $\mathcal{C}_\theta$ as a heart. Then we have $Z \simeq H^0(Z) \oplus H^{-1}(Z)[1]$. Now the condition that $\text{rk}_\theta$ takes positive (resp., negative) values on semistable summands of $Z$ implies that $H^{-1}(Z) = 0$ (resp., $H^0(Z)$). \hfill $\square$

**Remark.** Statements similar to Lemmas 1.1.1 and 1.1.2 hold in the more general framework of stability conditions on derived categories developed in [3]. More precisely, in Lemma 1.1.2 one has to replace the rank with the imaginary part of the central charge function. Also, $Z_i$’s should be replaced by the Harder-Narasimhan constituents of $Z$.  

---

4
Recall that if $\theta$ and $\theta'$ are related by a fractional-linear transformation then the algebras $A_\theta$ and $A_{\theta'}$ are Morita equivalent. The corresponding categories of holomorphic bundles $\text{Vect}(T_\theta,\tau)$ and $\text{Vect}(T_{\theta'},\tau)$ are equivalent. For every such Morita equivalence $\Phi : \text{Vect}(T) \cong \text{Vect}(T')$ we have $\text{rk}(\Phi(V)) = c \cdot \text{rk}(V)$ for some constant $c > 0$. Moreover, for every stable vector bundle $V$ on $T$ there exists a Morita equivalence $\Phi$ such that $\text{rk}(\Phi(V)) = 1$.

1.2. Binary division process associated with an irrational number. Let us denote $L_\theta = \mathbb{Z}\theta + \mathbb{Z}$ and let $\mathcal{P} \subset L_\theta$ be the set of primitive vectors in $L_\theta$. We also denote $\mathcal{P}_{>0} = \mathcal{P} \cap (0, +\infty)$.

We equip $L_\theta = \mathbb{Z}\theta + \mathbb{Z}$ with a $\mathbb{Z}$-valued bilinear form $\chi = \chi_\theta$ by setting

$$\chi(m\theta + n, m'\theta + n') = m'n - mn'.$$

Note that $L_{-\theta} = L_\theta$ but $\chi_{-\theta} = -\chi_\theta$. Recall that $\mathcal{P} \subset L_\theta$ denotes the set of primitive vectors and $\mathcal{P}_{>0} = \mathcal{P} \cap (0, +\infty)$.

**Lemma 1.2.1.** For every $v \in \mathcal{P}_{>0}$ there exists a unique vector $\phi(v) \in \mathcal{P}_{>0}$ such that $\phi(v) < v$ and $\chi(\phi(v), v) = 1$.

**Proof.** Let $v = m\theta + n$. We are looking for a vector $m_1\theta + n_1 \in L_\theta$ such that $0 < m_1\theta + n_1 < m\theta + n$ and $mn_1 - m_1n = 1$. Thus, it suffices to prove the existence and uniqueness of $m_1 \in \mathbb{Z}$ such that $m_1n \equiv -1(m)$ and

$$0 < \frac{m_1(m\theta + n) + 1}{m} < m\theta + n.$$  

The latter condition is equivalent to

$$0 < \text{sign}(m)[m_1 + (m\theta + n)^{-1}] < |m|.$$  

Thus, $m_1$ should be within a given interval of length $|m|$ with irrational ends. Since the residue of $m_1$ modulo $|m|$ is fixed by the condition $m_1n \equiv -1(m)$ we get a unique solution. \hfill $\Box$

Using the map $\phi = \phi_\theta : \mathcal{P}_{>0} \to \mathcal{P}_{>0}$ we can define a canonical way to divide every segment $[a, b]$ such that $b - a \in \mathcal{P}_{>0}$ into two subsegments: $[a, a + \phi(b - a)]$ and $[a + \phi(b - a), b]$. Moreover, each of the two new segments also has the length in $\mathcal{P}_{>0}$. Let us start with the segment $[0, 1]$ and divide it into two subsegments using this recipe. Then divide each of the new segments in two subsegments again, etc. Below we will refer to the subsegments $[a, b] \subset [0, 1]$ that are being divided in this process as division subsegments. Let $\mathcal{B}_\theta \subset (0, 1)$ denote the set of endpoints of all the division subsegments.

**Theorem 1.2.2.** The set $\mathcal{B}_\theta$ coincides with the set of all $m\theta + n \in L_\theta \cap (0, 1)$ such that $m < 0$.

**Proof.** It is easy to see that $\chi_{-\theta} = \chi_\theta$ on $L_{-\theta} = L_\theta$. Hence, $\phi_{-\theta}(v) = v - \phi_\theta(v)$ for $v \in \mathcal{P}_{>0}$. This implies that $\mathcal{B}_{-\theta} = 1 - \mathcal{B}_\theta$. Thus, the assertions of the theorem for $\theta$ and for $-\theta$ are equivalent. Let us first prove that for $m\theta + n \in \mathcal{B}_\theta$ one has $m < 0$. Assume that $v = m\theta + n$ is an element of the $k$-th generation of points obtained by our division process. We use induction in $k$. The first point of the division process is $a - \theta$, where $a$ is the unique integer such that $0 < a - \theta < 1$, so the assertion holds for $k = 1$. Assume
that our claim is true for all $k' < k$ and for all $\theta$. Changing $\theta$ to $-\theta$ if necessary we can assume that $v < a - \theta$. Set $\theta' = 1/(a - \theta)$ and $v' = v/(a - \theta)$. We claim that $v'$ is a $(k - 1)$-th generation point of the division process associated with $\theta'$. Indeed, the map \( \alpha: w \mapsto w/(a - \theta) \) is an order-preserving isomorphism from $L_\theta$ to $L_{\theta'}$. Furthermore, since \( \alpha(1) = \theta' \) and $\alpha(0) = a\theta' - 1$, one can easily check that $\alpha$ is compatible with the forms $\chi_\theta$ and $\chi_{\theta'}$ and hence with maps $\phi_\theta$ and $\phi_{\theta'}$. This implies that $\alpha$ maps the division process of $[0, a - \theta]$ associated with $\theta$ to the division process of $[0, 1]$ associated with $\theta'$ as we claimed. By induction assumption $v' = m'\theta' + n'$ where $m' < 0$. Since $v' > 0$ this implies that $n' < -m\theta' > 0$. Hence,

$$v = (a - \theta)(m'\theta' + n') = m' + (a - \theta)n' = -n'\theta + (m' + an')$$

has negative coefficient with $\theta$.

Now let us prove that conversely every vector $v = m\theta + n \in L_\theta \cap (0, 1)$ with $m < 0$ belongs to $\mathcal{B}_\theta$. We use induction in $|m|$. If $m = -1$ then $v$ coincides with the first division point $a - \theta$, so the base of induction is valid. Assume that the assertion is true for all vectors with smaller $|m|$ (and all $\theta$). Changing $\theta$ to $-\theta$ and $v$ to $1 - v$ if necessary we can assume that $v < a - \theta$ (note that $|m|$ remains invariant under such a change). Now let us consider $v' = v/(a - \theta) \in L_{\theta'}$ where $\theta' = 1/(a - \theta)$. Then $v' = (ma + n)\theta' - m$. We claim that

$$m < ma + n < 0.$$ 

This would finish the proof by applying the induction assumption to $v'$. Since $m < 0$ the inequalities we need are equivalent to

$$1 > a + \frac{n}{m} > 0. \quad (1.2.2)$$

Now the inequalities $0 < m\theta + n < 1, 0 < a - \theta < 1$ imply

$$a + \frac{n}{m} < \theta + 1 + \frac{n}{m} < 1,$$

$$a + \frac{n}{m} > \theta + \frac{n}{m} > \frac{1}{m}.$$ 

It remains to exclude the possibility $a + \frac{n}{m} = 0$. But in this case we would have $m\theta + n = -m(a - \theta)$ which contradicts to $m\theta + n < a - \theta$. \qed

**Corollary 1.2.3.** The set $\mathcal{B}_\theta$ is dense in $(0, 1)$.

1.3. **Construction of subbundles.** We use two methods to construct subbundles in holomorphic vector bundles on $T = T_{\theta, r}$. The first method is based on the binary division process described in the previous section.

**Theorem 1.3.1.** For every stable vector bundle $P$ on $T$ and every $r \in L_\theta$ such that $0 < r < \text{rk } P$ and $\chi(r, \text{rk } P) > 0$, there exists a subbundle $V \subset P$ such that $\text{rk } V = r$.

**Proof.** Using Morita equivalences we can reduce ourselves to the case $\text{rk } P = 1$. Then the condition $\chi(r, \text{rk } P) > 0$ on $r = m\theta + n \in \mathbb{Z}\theta + \mathbb{Z}$ is equivalent to $m < 0$. Thus, we have to show that for every such $r < 1$ there exists a subbundle $V \subset P$ with $\text{rk } V = r$. By Theorem 1.2.2 we have $r \in \mathcal{B}_\theta$. Now we claim that one can associate to every division
subsegment \([a, b]\) a stable vector bundle \(V_{a,b}\) of rank \(b - a\), such that \(V_{0,1} = P\) and for the new division point \(c \in (a, b)\) one has an exact sequence

\[
0 \to V_{a,c} \to V_{a,b} \to V_{b,c} \to 0.
\]

Indeed, assume that \(V_{a,b}\) is already chosen. Then for the new division point \(c \in (a, b)\) we can choose \(V_{a,c}\) to be any stable bundle of rank \(c - a\) and then observe that the condition \(\chi(c-a, b-a) = 1\) implies that \(\text{Hom}(V_{a,c}, V_{a,b})\) is one-dimensional and \(\text{Hom}^1(V_{a,c}, V_{a,b}) = 0\).

We claim that the unique nonzero morphism \(f : V_{a,c} \to V_{a,b}\) is injective. Indeed, identifying \(\text{Vect}(T)\) with the subcategory \(C^\theta \subset D^b(E)\) we can consider the cone of \(f\) as an object in \(D^b(E)\). Since \(\text{Cone}(f)\) is the value on \(V_{a,c}\) of the reflection functor associated with \(V_{a,b}\) (see [11]), it follows that \(\text{Cone}(f)\) is stable. According to Lemma 1.1.2 this implies that \(\text{Cone}(f) \in C^\theta\) which implies our claim. Now we can define inductively a family of subbundles \(V_{0,a} \subset V_{0,1} = P\) for all \(a \in B_\theta\), such that for every division subsegment \([a, b]\) one has \(V_{0,a} \subset V_{0,b}\) and \(V_{0,b}/V_{0,a} \simeq V_{a,b}\). Indeed, assume that \(V_{0,a} \subset V_{0,b}\) for the subsegment \([a, b]\) are already defined. Then for the new division point \(c \in (a, b)\) we define \(V_{0,c} \subset V_{0,b}\) as the preimage of \(V_{a,c} \subset V_{a,b}\) under the projection \(V_{0,b} \to V_{0,b}/V_{0,a} \simeq V_{a,b}\). By the construction the subbundle \(V = V_{0,r}\) has rank \(r\).

Another way to construct subbundles is based on the following lemma.

**Lemma 1.3.2.** Let \(A\) and \(B\) be a pair of stable objects in \(D^b(E)\). Assume that \(\text{Hom}(A, B) \neq 0\) and \(\text{Hom}^i(A, B) = 0\) for \(i \neq 0\). Then for a generic morphism \(f \in \text{Hom}(A, B)\) the object \(\text{Cone}(f)\) is semistable.

**Proof.** Without loss of generality we can assume that \(B = \mathcal{O}_x\) for some \(x \in E\) (one has to use the action of a central extension of \(\text{SL}_2(\mathbb{Z})\) on \(D^b(E)\)). Then \(A\) is a stable vector bundle on \(E\) and we have to prove that for a generic morphism \(f : A \to \mathcal{O}_x\) the kernel of \(f\) is semistable. Let \(r = \text{rk} A\), \(d = \text{deg} A\). For every pair of relative prime numbers \((r', d')\) let \(M_{r', d'}\) be the moduli space of stable bundles of rank \(r'\) and degree \(d'\). Note that if \(\ker(f)\) is unstable then it can be destabilized by some stable bundle \(V \in M_{r', d'}\) such that

\[
\frac{d - 1}{r} < \frac{d'}{r'} < \frac{d}{r}
\]

and \(r' < r\). Note that

\[
\dim \mathbb{P} \text{Hom}(V, A) = dr' - rd' - 1 < r' - 1.
\]

Since \(\dim M_{r', d'} = 1\) we see that the family of possible bundles \(V \subset A\) of this type has dimension \(< r'\). To every such \(V\) there corresponds the \((r - r')\)-dimensional subspace in \(\text{Hom}(A, \mathcal{O}_x)\) consisting of morphisms vanishing on \(V\). Therefore, a generic element \(f \in \text{Hom}(A, \mathcal{O}_x)\) does not belong to any of these subspaces. Hence, for such an element the bundle \(\ker(f)\) will be semistable.

**Remark.** The statement of the above lemma will become false if we assume only that \(A\) and \(B\) are semistable. For example, any morphism from \(A = \mathcal{O}_E\) to \(B = \mathcal{O}_x \oplus \mathcal{O}_x\) has nonzero kernel and cokernel. Hence, its cone is not semistable.

The following result will play a crucial role in the proof of Theorem 3.1.6.
Lemma 1.3.3. For every vector bundle $P$ and for a pair of real numbers $\epsilon > 0$ and $C$ there exists a vector bundle $P' \subset P$ such that $\text{rk} P' > \text{rk} P - \epsilon$ and $P'$ is a direct sum of semistable vector bundles of slopes $< C$.

Proof. First, we can reduce the proof to the case when the vector bundle $P$ is stable. Indeed, it suffices to check that if $P$ fits into the exact sequence

$$0 \to P_1 \to P \to P_2 \to 0$$

and the assertion of the lemma holds for $P_1$ and $P_2$ (and arbitrary $\epsilon$ and $C$), then the assertion holds also for $P$. Let $A$ be the minimum of slopes of semistable bundles in the Harder-Narasimhan filtration of $P_1$. By assumption, we can find subbundles $P'_1 \subset P_1$ and $P'_2 \subset P_2$ such that $\text{rk} P'_1 > \text{rk} P_1 - \epsilon/2$, $\text{rk} P'_2 > \text{rk} P_2 - \epsilon/2$ and such that $P'_1$ and $P'_2$ are direct sums of semistable bundles of slopes $< \min(A, C)$. Then $\text{Ext}^1(P'_2, P'_1) = 0$, so there exists a splitting $P'_2 \to P$. Now we set $P' = P'_1 \oplus P'_2 \subset P$.

Thus, we can assume that $P$ is a stable holomorphic vector bundle. Using principal convergents to $-\theta$ we can choose a sequence of pairs of relatively prime integers $(p_n, q_n)$ such that $p_n + q_n \theta > 0$, $\lim_{n \to \infty} (p_n + q_n \theta) = 0$ and $\lim_{n \to \infty} q_n = +\infty$. Let $V_n$ be a stable holomorphic vector bundle on $T$ with $\text{rk}(V_n) = p_n + q_n \theta$. Then $\lim_{n \to \infty} \mu(V_n) = +\infty$. Hence, for sufficiently large $n$ we have $\mu(V_n) > \mu(P)$ and $\text{rk}(V_n) < \text{rk}(P)$. Now Lemmas 1.3.2 and 1.1.2 imply that a generic morphism $f : P \to V_n$ is surjective and $P' = \ker(f)$ is semistable (provided $n$ is large enough). Note that $\text{deg}(P') = \text{deg}(P) - q_n \to -\infty$ as $n \to \infty$. Hence, for large enough $n$ we will have $\mu(P') < C$ and $\text{rk}(P') > \text{rk} P - \epsilon$. □

1.4. Deformation equivalence. We start with the following combinatorial

Lemma 1.4.1. Let us consider the set of unordered $n$-tuples $(v_1, \ldots, v_n)$ of primitive vectors in $\mathbb{Z}^2$ such that $0 \notin \mathbb{R}_{>0}v_1 + \ldots + \mathbb{R}_{>0}v_n$ (we take the union over all $n \geq 1$). Consider the equivalence relation on this set generated by all relations of the following kind: an $n$-tuple $(v_1, \ldots, v_n)$ is equivalent to $(w_1, \ldots, w_m, v_3, \ldots, v_n)$, where $w$ is repeated $m$ times, if $v_1 + v_2 = mw$. Then $(v_1, \ldots, v_n)$ is equivalent to $(w_1, \ldots, w_m)$ iff $v_1 + \ldots + v_n = w_1 + \ldots + w_m$.

Proof. We are going to show that every unordered $n$-tuple is equivalent to an $n$-tuple of the form $(w, \ldots, w)$. Let us associate to every $n$-tuple $(v_1, \ldots, v_n)$ a nonnegative integer by the following rule:

$$D(v_1, \ldots, v_n) = \sum_{i,j} \delta_{\geq 0}(\det(v_i, v_j)),$$

where $\delta_{\geq 0}(x) = x$ for $x \geq 0$ and $\delta_{\geq 0}(x) = 0$ for $x < 0$. It is clear that $D(v_1, \ldots, v_n) = 0$ iff all $v_i$’s are the same. Thus, it is enough to show that every $n$-tuple as above with not all $v_i$’s equal is equivalent to some $m$-tuple $(w_1, \ldots, w_m)$ with $D(w_1, \ldots, w_m) < D(v_1, \ldots, v_n)$. Without loss of generality we can assume that $v_1 \neq v_2$. Since $v_1 \neq -v_2$ we have $v_1 + v_2 = mw$ for some primitive vector $w$ and some $m > 0$. Consider the corresponding $n - 2 + m$-tuple $(w, \ldots, w, v_3, \ldots, v_n)$ equivalent to $(v_1, \ldots, v_n)$. We claim that

$$D(w, \ldots, w, v_3, \ldots, v_n) < D(v_1, \ldots, v_n).$$
Indeed, since \( D(v_1, v_2) > 0 \) it suffices to show that for every \( i \geq 3 \) one has
\[
m[\delta_{\geq 0}(\det(v_i, w)) + \delta_{\geq 0}(\det(w, v_i))] \leq \\
\delta_{\geq 0}(\det(v_i, v_1)) + \delta_{\geq 0}(\det(v_1, v_i)) + \delta_{\geq 0}(\det(v_2, v_i)).
\]
Assume for example that \( \det(v_i, w) \geq 0 \). Then \( \det(v_i, v_1 + v_2) = m \det(v_i, w) \geq 0 \), so switching \( v_1 \) and \( v_2 \) if necessary we can assume that \( \det(v_i, v_1) \geq 0 \). If \( \det(v_i, v_2) \geq 0 \) then (1.4.1) holds since it fact it becomes an equality. Finally, if \( \det(v_i, v_2) \leq 0 \) then we have
\[
m \det(v_i, w) = \det(v_i, v_1) + \det(v_i, v_2) \leq \det(v_i, v_1)
\]
which implies (1.4.1).

**Definition.** Let us say that two vector bundles on \( T \) are deformation equivalent if they belong to the same class with respect to the minimal equivalence relation on isomorphism classes of vector bundles containing the following relations:

(i) if \( V_1 \) and \( V_2 \) are stable and \( \text{rk} V_1 = \text{rk} V_2 \) then \( V_1 \sim V_2 \);

(ii) if \( 0 \to U \to V \to W \to 0 \) is an exact triple of vector bundles then \( V \sim U \oplus W \);

(iii) if \( V_1 \sim V_2 \) then \( U \oplus V_1 \sim U \oplus V_2 \) for any vector bundle \( U \).

It is clear that deformation equivalent vector bundles have the same rank. The following theorem states that the converse is also true.

**Theorem 1.4.2.** Let \( V_1 \) and \( V_2 \) be vector bundles on \( T \) such that \( \text{rk} V_1 = \text{rk} V_2 \). Then \( V_1 \sim V_2 \).

**Proof.** The idea is to mimick the proof of Lemma 1.4.1. It suffices to show that every vector bundle is deformation equivalent to a bundle of the form \( W_1 \oplus \ldots \oplus W_m \) where \( W_i \)'s are stable and \( \text{rk} W_1 = \ldots = \text{rk} W_m \). Using Harder-Narasimhan filtration and property (ii) of our equivalence we can assume that our vector bundle is a direct sum \( V_1 \oplus \ldots \oplus V_n \), where \( V_i \)'s are stable. Assume that not all of them have the same rank, say, \( \text{rk} V_1 \neq \text{rk} V_2 \). Then we can reorder \( V_1 \) and \( V_2 \) in such a way that \( \text{Hom}(V_1, V_2) = 0 \) and \( \text{Ext}^1(V_1, V_2) \neq 0 \). According to Lemma 1.3.2 for a generic extension
\[
0 \to V_2 \to W \to V_1 \to 0
\]
the bundle \( W \) is semistable. Using property (ii) we see that \( W \sim W_1 \oplus \ldots \oplus W_m \), where \( W_i \)'s are stable and have the same rank \( (\text{rk} V_i + \text{rk} V_2)/m \). Thus, we have an equivalence
\[
V_1 \oplus \ldots \oplus V_n \sim W_1 \oplus \ldots \oplus W_m \oplus V_2 \oplus \ldots \oplus V_n.
\]
As we have seen in the proof of Lemma 1.4.1 repeating this procedure we will eventually arrive at the direct sum of stable bundles of the same rank. \( \square \)

2. Quasicoherent sheaves on \( T \)

2.1. Ind-objects. Let us present some facts about ind-objects following sec. 8 of [6]. Below by an inductive limit we always mean a small filtering inductive limit. By a union of subobjects of an object in an abelian category we always mean a small filtering union.

Recall that with every category \( \mathcal{A} \) one can associate the category \( \text{Ind}(\mathcal{A}) \) of ind-objects of \( \mathcal{A} \) such that in \( \text{Ind}(\mathcal{A}) \) all inductive limits exist (see [6], sec. 8.2 and Prop. 8.5.1). The
category \( \mathcal{A} \) can be identified with a full subcategory of \( \text{Ind}(\mathcal{A}) \). Furthermore, if \( \mathcal{A} \) is abelian then so is \( \text{Ind}(\mathcal{A}) \) ([6], 8.9.9(c)) and the natural embedding functor \( \mathcal{A} \to \text{Ind}(\mathcal{A}) \) is exact ([6], Prop. 8.9.5(a)). Also, Proposition 8.5.1 of [6] implies that if \( (C_i) \) is an inductive system in \( \text{Ind}(\mathcal{A}) \) then for \( A \in \mathcal{A} \) one has

\[
\lim_{\to} \text{Hom}(A, C_i) \simeq \text{Hom}(A, \lim_{\to} C_i) \tag{2.1.1}
\]

Assume that we are given a functor \( f : \mathcal{A} \to \mathcal{C} \), where \( \mathcal{C} \) is a category in which all inductive limits exist. Then by [6], 8.7.2, this functor extends to a functor \( \overline{f} : \text{Ind}(\mathcal{A}) \to \mathcal{C} \) commuting with inductive limits. We will need the following result.

**Proposition 2.1.1.** (a) The functor \( \overline{f} \) is fully faithful iff for every inductive system \( (A_i) \) in \( \mathcal{A} \) the natural map

\[
\lim_{\to} \text{Hom}(A, A_i) \to \text{Hom}(f(A), \lim_{\to} f(A_i))
\]

is bijective.

(b) Assume that \( \mathcal{A} \) is abelian and the functor \( f \) is exact. Then the functor \( \overline{f} \) is also exact.

**Proof.** (a) This is Prop. 8.7.5(a) of [6].

(b) Since the embedding of \( \mathcal{A} \) into \( \text{Ind}(\mathcal{A}) \) (resp., of \( \mathcal{C} \) into \( \text{Ind}(\mathcal{C}) \)) is exact, it is enough to check the exactness of \( \text{Ind}(f) : \text{Ind}(\mathcal{A}) \to \text{Ind}(\mathcal{C}) \). It remains to apply Cor. 8.9.8 of [6]. □

Now let us consider the situation when we have an abelian category \( \mathcal{C} \) in which all small direct sums (and hence, all small filtering inductive limits) exist. Let \( \mathcal{A} \subset \mathcal{C} \) be a full abelian subcategory such that equivalent conditions of Proposition 2.1.1(a) are satisfied. Then we can identify \( \text{Ind}(\mathcal{A}) \) with a full subcategory of \( \mathcal{C} \). In this situation we can give a convenient characterization of objects of \( \mathcal{C} \) that belong to \( \text{Ind}(\mathcal{A}) \) (see Propositions 2.1.4 and 2.1.3 below).

**Lemma 2.1.2.** Let \( A \) be an object of \( \mathcal{A} \), and let \( S \subset A \) be a subobject in \( \mathcal{C} \). Then \( A/S \in \widetilde{\mathcal{A}} \) (equivalently, \( S \in \widetilde{\mathcal{A}} \)) iff \( S = \bigcup_{i \in I} S_i \), where \( (S_i)_{i \in I} \) is a filtering set of subobjects of \( A \) in \( \mathcal{A} \).

**Proof.** First of all, we observe that since \( \widetilde{\mathcal{A}} \subset \mathcal{C} \) is stable under kernels and cokernels, we have \( A/S \in \widetilde{\mathcal{A}} \) iff \( S \in \widetilde{\mathcal{A}} \). If \( S \) is a union of subobjects \( (S_i) \) of \( A \), where \( S_i \in \mathcal{A} \) then \( S \in \widetilde{\mathcal{A}} \) (because such a union can be taken in the category \( \mathcal{A} \) and the embedding \( \widetilde{\mathcal{A}} \to \mathcal{C} \) is exact and commutes with inductive limits). Conversely, assume that \( S \in \widetilde{\mathcal{A}} \). Then \( S = \lim A_i \) for some inductive system \( (A_i) \) in \( \mathcal{A} \). Let \( S_i = \text{im}(A_i \to S) = \text{im}(A_i \to A) \). Then \( S_i \in A \) since \( \mathcal{A} \) is an abelian subcategory. Also, \( S = \bigcup_{i \in I} S_i \) (this union can be taken either in \( \widetilde{\mathcal{A}} \) or in \( \mathcal{C} \)). □

**Definition.** Let us say that an object \( C \in \mathcal{C} \) is \( \mathcal{A} \)-generated if there exists a surjection \( A \to C \) in \( \mathcal{C} \) with \( A \in \mathcal{A} \).

**Proposition 2.1.3.** In the above situation for an \( \mathcal{A} \)-generated object \( C \in \mathcal{C} \) the following conditions are equivalent:
(a) \( C \in \widetilde{\mathcal{A}} \);
(b) \( C \simeq A/S \), where \( A \in \mathcal{A} \) and \( S \) is a union of subobjects \((A_i)\) of \( A \) with \( A_i \in \mathcal{A} \);
(c) For every morphism \( f : A \to C \) where \( A \in \mathcal{A} \), the kernel \( \ker(f) \subset A \) is a union of subobjects \((A_i)\) of \( A \) with \( A_i \in \mathcal{A} \).

Proof. The implication (c) \( \implies \) (b) is clear, while (b) \( \implies \) (a) follows from Lemma 2.1.2. To show that (a) \( \implies \) (c) we note that if \( C \in \widetilde{\mathcal{A}} \) then \( \ker(f) \in \widetilde{\mathcal{A}} \). It remains to apply Lemma 2.1.2 again. \( \square \)

**Proposition 2.1.4.** For an object \( C \in \mathcal{C} \) the following conditions are equivalent:
(a) \( C \in \widetilde{\mathcal{A}} \);
(b) \( C \) is a union \( C = \bigcup_i C_i \), where \( C_i \in \widetilde{\mathcal{A}} \) and \( C_i \) is \( \mathcal{A} \)-generated for all \( i \);
(c) \( C \) is a union of a filtering set of \( \mathcal{A} \)-generated subobjects, and every \( \mathcal{A} \)-generated subobject of \( C \) belongs to \( \widetilde{\mathcal{A}} \);
(c’) \( C \) is a union of a filtering set of \( \mathcal{A} \)-generated subobjects and for every morphism \( f : A \to C \), where \( A \in \mathcal{A} \), the kernel \( \ker(f) \subset A \) is a union of subobjects \((A_i)\) of \( A \) with \( A_i \in \mathcal{A} \).

Proof. The equivalence of (c) and (c’) follows from Proposition 2.1.3. The implications (c) \( \implies \) (b) and (b) \( \implies \) (a) are clear. To prove (a) \( \implies \) (c) we note that if \( A \to C \) is a morphism, where \( A \in \mathcal{A} \) and \( C \in \widetilde{\mathcal{A}} \), then its image also belongs to \( \widetilde{\mathcal{A}} \) (since the subcategory \( \widetilde{\mathcal{A}} \subset \mathcal{C} \) is abelian). Hence, every \( \mathcal{A} \)-generated subobject of \( C \) belongs to \( \widetilde{\mathcal{A}} \). Also, if \( C = \lim_{\longrightarrow} A_i \), where \((A_i)_{i \in I}\) is an inductive system in \( \mathcal{A} \), then \( C = \bigcup_{i \in I} C_i \), where \( C_i \) is the image of the morphism \( A_i \to C \), so that each \( C_i \) is \( \mathcal{A} \)-generated. \( \square \)

**Example.** It is well known (and follows easily from Proposition 2.1.4) that the category of ind-coherent sheaves (ind-objects in \( \text{Coh}(X) \)) on a Noetherian scheme \( X \) is equivalent to the category \( \text{Qcoh}(X) \) of quasicoherent sheaves on \( X \). Note that in this case the only \( \text{Coh}(X) \)-generated objects in \( \text{Qcoh}(X) \) are coherent sheaves.

### 2.2. Holomorphic modules and holomorphic bundles on \( T \)

To realize concretely ind-objects of the category of holomorphic vector bundles on a noncommutative complex torus \( T = T_{\theta, \tau} \) we introduce an auxiliary category \( HM(T) \) of holomorphic modules.

**Definition.** An object of \( HM(T) \) is a right \( A_{\theta} \)-module \( M \) equipped with a holomorphic structure, i.e., with a map \( \nabla : M \to M \) satisfying the Leibnitz rule (1.1.1). The morphisms in \( HM(T) \) are morphisms of \( A_{\theta} \)-modules compatible with holomorphic structures.

It is easy to see that \( HM(T) \) is an abelian category. By definition, a **holomorphic vector bundle** on \( T \) is a holomorphic module \((M, \nabla)\) such that \( M \) is a finitely generated projective (right) \( A_{\theta} \)-module. Thus, \( \text{Vect}(T) \) is a full subcategory in \( HM(T) \). Also, it is easy to see that the natural embedding functor \( \text{Vect}(T) \to HM(T) \) is exact.
Lemma 2.2.1. Let \((M, \nabla)\) be a holomorphic module. For every surjection of \(A_\theta\)-modules \(\phi : A_\theta^{\oplus n} \to M\) there exists a holomorphic structure \(\nabla_\phi\) on \(A_\theta^{\oplus n}\) with respect to which \(\phi\) is holomorphic.

**Proof.** Let \(f_i = \phi(e_i) \in M\), where \(e_1, \ldots, e_n\) is the standard basis of \(A_\theta^{\oplus n}\). Then we have

\[
\nabla(f_j) = \sum_{i=1}^{n} f_i a_{ij}
\]

for some \(A_\theta\)-valued \(n \times n\) matrix \(M = (a_{ij})\). Let us set for \((x_i) \in A_\theta^{\oplus n}\)

\[
\nabla_\phi(x_i) = (\delta(x_i) + \sum_{j=1}^{n} a_{ij} x_j).
\]

One can immediately check that the morphism \(\phi\) becomes holomorphic with respect to \(\nabla_\phi\) and \(\nabla\). \(\square\)

Let us define the category \(\text{Coh}(T)\) of coherent sheaves on \(T\) as the full subcategory in \(HM(T)\) consisting of holomorphic modules \((M, \nabla)\) such that the \(A_\theta\)-module \(M\) is finitely presented.

**Theorem 2.2.2.** One has \(\text{Coh}(T) = \text{Vect}(T)\), i.e., every coherent sheaf on \(T\) is a holomorphic vector bundle.

**Proof.** Let \((M, \nabla)\) be a coherent sheaf. Consider a finite presentation

\[
M = \text{coker}(f : A_\theta^{\oplus m} \to A_\theta^{\oplus n}).
\]

Applying Lemma 2.2.1 to the natural projection \(A_\theta^{\oplus n} \to M\) we find a holomorphic structure \(\nabla_2\) on \(A_\theta^{\oplus n}\) with respect to which this projection is holomorphic. Now the submodule \(\text{im}(f) \subset A_\theta^{\oplus n}\) is preserved by \(\nabla_2\), so we can view \((\text{im}(f), \nabla_2)\) as a holomorphic module. Applying Lemma 2.2.1 to the surjection \(f : A_\theta^{\oplus m} \to \text{im}(f)\) we find a holomorphic structure \(\nabla_1\) on \(A_\theta^{\oplus m}\) such that \(f\) is holomorphic with respect to \(\nabla_1\) and \(\nabla_2\). Hence, \(f\) becomes a morphism in the category \(\text{Vect}(T)\). Since \(\text{Vect}(T)\) is abelian, it follows that \(M = \text{coker}(f)\) is an object of \(\text{Vect}(T)\). \(\square\)

**Corollary 2.2.3.** Let \(P\) be a holomorphic bundle and let \(S \subset P\) be a finitely generated holomorphic submodule. Then \(S\) is a direct summand of \(P\) as an \(A_\theta\)-module. Hence, it is a holomorphic subbundle of \(P\).

**Proof.** The holomorphic module \(P/S\) is finitely presented, hence, it is a vector bundle by the above theorem. \(\square\)

We equip the trivial module \(A_\theta\) with the standard holomorphic structure \(\delta_r\). Thus, a holomorphic ideal in \(A_\theta\) is a right ideal \(I \subset A_\theta\) such that \(\delta_r(I) \subset I\).

**Corollary 2.2.4.** Let \(I \subset A_\theta\) be a finitely generated holomorphic ideal. Then there exists an idempotent \(e \in A_\theta\) such that \(I = eA_\theta\).
Remarks. 1. Let us say that an idempotent \( e \in A_\theta \) is (right) \textit{holomorphic} if
\[
\delta_\tau(e) = \delta_\tau(e),
\]
or equivalently, \( \delta_\tau(e) \in eA_\theta \). The above theorem shows that a map \( e \mapsto eA_\theta \) gives a surjection from the set of holomorphic idempotents to that of finitely generated holomorphic ideals. It is easy to see that the fiber of this map over \( eA_\theta \) coincides with \( e + eA_\theta(1 - e) \).

2. Proposition 2.9.2 below implies that countable filtering unions of finitely generated holomorphic ideals are still projective \( A_\theta \)-modules. However, we will see that such ideals are not necessarily direct summands in \( A_\theta \) (see Theorem 2.7.3).

2.3. Quasicoherent sheaves as holomorphic modules. We define the category of \textit{quasicoherent sheaves} on \( T \) by setting \( \text{Qcoh}(T) = \text{Ind}(\text{Vect}(T)) \). Thus, by definition, quasicoherent sheaves on \( T \) are ind-objects in the category \( \text{Vect}(T) \). They form an abelian category containing \( \text{Vect}(T) \) as a full subcategory. We are going to give a more concrete realization of \( \text{Qcoh}(T) \) using holomorphic modules.

It is easy to see that in \( HM(T) \) all small filtering inductive limits exist. More precisely, the natural embedding of \( HM(T) \) into the category of \( A_\theta \)-modules is exact and commutes with small inductive limits. Therefore, the natural fully faithful exact embedding \( \text{Vect}(T) \hookrightarrow HM(T) \) extends to an exact functor \( \text{Qcoh}(T) \rightarrow HM(T) \) (see Proposition 2.1.1(b)).

**Proposition 2.3.1.** The above functor \( \text{Qcoh}(T) \rightarrow HM(T) \) is fully faithful.

**Proof.** According to Proposition 2.1.1(a) we have to check that for every small filtering inductive system \( (P_i)_{i \in I} \) in \( \text{Vect}(T) \) and every \( P \in \text{Vect}(T) \) the canonical map
\[
\lim_{\rightarrow} \text{Hom}_{\text{Vect}(T)}(P, P_i) \rightarrow \text{Hom}_{\text{HM}(T)}(P, \lim_{\rightarrow} P_i) \tag{2.3.1}
\]
is an isomorphism. It is easy to see that if we replace morphisms in \( HM(T) \) by morphisms of \( A_\theta \)-modules then the similar map is an isomorphism because \( P \) is a finitely generated projective \( A_\theta \)-module. This immediately implies injectivity of (2.3.1). To check surjectivity let us assume that \( f : P \rightarrow \lim_{\rightarrow} P_i \) is any morphism in \( HM(T) \). Then we can lift \( f \) to a morphism of \( A_\theta \)-modules \( f_{i_0} : P \rightarrow P_{i_0} \). Let \( e_1, \ldots, e_n \) be generators of \( P \). Then we have
\[
\nabla(e_i) = \sum_j e_j a_{ij}
\]
for some \( a_{ij} \in A_\theta \). Hence,
\[
\nabla(f(e_i)) = \sum_j f(e_j) a_{ij}.
\]
It follows that for some \( i_1 > i_0 \) we will have
\[
\nabla(f_{i_1}(e_i)) = \sum_j f_{i_1}(e_j) a_{ij},
\]
where \( f_{i_1} : P \rightarrow P_{i_1} \) is the morphism of \( A_\theta \)-modules induced by \( f_{i_0} \). But this means that \( f_{i_1} \) is compatible with holomorphic structures, so it is a morphism in \( HM(T) \). \( \square \)
Definition. Let us call a holomorphic module admissible if it can be represented as an inductive limit of holomorphic bundles.

Corollary 2.3.2. The category $\text{Qcoh}(T)$ is equivalent to the full subcategory of admissible modules in $HM(T)$.

Let us say that a holomorphic module $M$ is finitely generated if it is finitely generated as an $A_\theta$-module. The following lemma shows that this is equivalent to $M$ being $\text{Vect}(T)$-generated.

Lemma 2.3.3. A holomorphic module $M$ is finitely generated iff there exists a holomorphic bundle $P$ and a surjection $P \to M$ in $HM(T)$.

Proof. The “if” part is clear. The “only if” part follows immediately from Lemma 2.2.1. □

Applying Propositions 2.1.3 and 2.1.4 we get the following characterization of admissible holomorphic modules.

Proposition 2.3.4. For a finitely generated holomorphic module $M$ the following conditions are equivalent:
(a) $M$ is admissible;
(b) $M \cong P/S$, where $P$ is a holomorphic vector bundle, and $S$ is a union of holomorphic subbundles in $P$;
(c) For every morphism $f : P \to M$, where $P$ is a holomorphic vector bundle, $\ker(f) \subset P$ is a union of holomorphic subbundles in $P$.

Proposition 2.3.5. For a holomorphic module $M$ the following conditions are equivalent:
(a) $M$ is admissible;
(b) $M$ is the union of its finitely generated admissible holomorphic submodules.
(c) $M$ is the union of its finitely generated holomorphic submodules and every such submodule is admissible;
(c') $M$ is the union of its finitely generated holomorphic submodules and for every morphism $f : P \to M$, where $P$ is a holomorphic vector bundle, $\ker(f)$ is a union of holomorphic subbundles in $P$.

Corollary 2.3.6. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of holomorphic modules such that $M'$ and $M''$ are admissible. Then $M$ is also admissible.

Proof. First, let us show that every element $x \in M$ is contained in a finitely generated holomorphic submodule. Let $N'' \subset M''$ be a finitely generated holomorphic submodule containing the image of $x$ in $M''$. We can lift generators of $N''$ to some elements $e_1, \ldots, e_n \in M$. Then $x = \sum a_i e_i + y$ and $\nabla e_j = \sum a_{ij} e_i + y_j$ for some $a_i, a_{ij} \in A$ and $y, y_j \in M'$. Let $N'$ be a finitely generated holomorphic submodule in $M'$ containing $y$ and $(y_j)$. Then the $A_\theta$-submodule generated by $N'$ and $(e_i)$ is holomorphic and contains $x$. Next, let $f : P \to M$ be any morphism, where $P$ is a holomorphic vector bundle. We have to show that $\ker(f)$ is admissible. Let $f'' : P \to M''$ and $f' : \ker(f'') \to M'$ be the induced morphism. Then $\ker(f'')$ is admissible and hence $\ker(f')$ is admissible. It remains to observe that $\ker(f) = \ker(f')$. □
It is not clear whether there exists a simple characterization of all $A_\theta$-modules underlying quasicoherent sheaves. One obvious condition is flatness (since inductive limits of projective modules are flat). In section 2.9 we will introduce an abelian subcategory of countably presented quasicoherent sheaves and will show that projective dimension of $A_\theta$-modules underlying such sheaves is $\leq 1$ (see Theorem 2.9.4).

2.4. Derived categories equivalence. Recall that the category $\text{Vect}(T)$ is equivalent to the abelian subcategory $C^0$ in the derived category $D^b(E)$ of coherent sheaves on the elliptic curve $E = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$. Moreover, we have an equivalence of derived categories

$$D^b(\text{Vect}(T)) \simeq D^b(E) \quad (2.4.1)$$

(see Proposition 1.4 of [8]). In this section we establish a similar equivalence for the category of quasicoherent sheaves $\text{Qcoh}(T)$.

Let us consider a more general situation. Let $A$ be a Noetherian abelian category and let $\tilde{A} = \text{Ind}(A)$. We view $A$ as a full subcategory in $\tilde{A}$ in a standard way.

**Lemma 2.4.1.** (i) For every surjection $A \to B$ in $\tilde{A}$ with $B \in A$ there exists a subobject $A' \subset A$ such that $A' \in A$ and the induced map $A' \to B$ is surjective.

(ii) The subcategory $A \subset \tilde{A}$ is closed under passing to quotients and subobjects, and under extensions.

**Proof.** (i) Let $A = \lim_i A_i$, where $(A_i)$ is an inductive system in $A$. Let $B_i \subset B$ be the image of the induced map $A_i \to A$. Then $(B_i)$ is a filtering set of subobjects of $B$ such that $\bigcup_i B_i = B$. Hence, $B_i = B$ for some $i$, so we can take as $A'$ the image of the corresponding morphism $A_i \to A$.

(ii) Lemma 2.1.2 easily implies that $A$ is stable under quotients and subobjects in $\tilde{A}$.

Assume that we have an exact sequence

$$0 \to A \to \tilde{B} \to C \to 0$$

in $\tilde{A}$ with $A, C \in A$. By part (i) we can find a subobject $B \subset \tilde{B}$ such that the induced map $B \to C$ is surjective. Then $\tilde{B}$ is isomorphic to a quotient of $A \oplus B$, so $\tilde{B} \in A$.

**Lemma 2.4.2.** Let $(X_i)$ be an inductive system in $\tilde{A}$. Then for every $A \in A$ the natural map

$$\lim_i \text{Ext}^1(A, X_i) \to \text{Ext}^1(A, \lim_i X_i)$$

is an isomorphism.

**Proof.** First, let us check surjectivity. Assume that we have an extension

$$0 \to \lim_i X_i \to \tilde{E} \to A \to 0$$

in $\tilde{A}$. By Lemma 2.4.1(i) there exists a subobject $E \subset \tilde{E}$ with $E \in A$ that still surjects onto $A$. Let $S = E \cap \lim X_i \subset E$. Then the above extension is the push-out of an extension

$$0 \to S \to E \to A \to 0 \quad (2.4.2)$$
by the inclusion map $S \to \lim X_i$. But the latter map factors through a map $S \to X_i$. Taking the push-out of (2.4.2) by this map we get an element of $\text{Ext}^1(A, X_i)$ inducing the original extension.

To check injectivity let us consider an extension

$$0 \to X_{i_0} \to \tilde{E} \to A \to 0 \tag{2.4.3}$$

such that its push-out by $X_{i_0} \to \lim X_i$ becomes trivial. As above we can find an extension (2.4.2) such that (2.4.3) is its push-out by a map $S \to X_{i_0}$. The fact that the push-out by $f : S \to X_{i_0} \to \lim X_i$ becomes trivial means that there is a map $s : E \to \lim X_i$ extending $f : S \to \lim X_i$. But $s$ factors through some map $E \to X_j$. Furthermore, choosing $X_j$ sufficiently far in the inductive system will guarantee that this map restricts to the map $g : S \to X_{i_0} \to X_j$. Hence, the push-out of (2.4.2) by $g$ is trivial. Thus, the induced element of $\text{Ext}^1(A, X_j)$ is trivial. \qed

Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{A}$. Recall (see [7]) that this means that $\mathcal{T}$ and $\mathcal{F}$ are full subcategories in $\mathcal{A}$ such that $\text{Hom}(T, F) = 0$ for $T \in \mathcal{T}$, $F \in \mathcal{F}$, and every object $X \in \mathcal{A}$ fits into an exact triangle

$$0 \to T \to X \to F \to 0$$

with $T \in \mathcal{T}$, $F \in \mathcal{F}$. With a torsion pair $(\mathcal{T}, \mathcal{F})$ one associates the tilted abelian subcategory $\mathcal{A}_t \subset D^b(\mathcal{A})$ by setting

$$\mathcal{A}_t = \{ K \in D^b(\mathcal{A}) \mid H^{-1}K \in \mathcal{F}, H^0K \in \mathcal{T}, H^iK = 0 \text{ for } i \neq -1, 0 \}.$$ 

The pair of subcategories $(\mathcal{F}[1], \mathcal{T})$ is a torsion pair in $\mathcal{A}_t$. A torsion pair $(\mathcal{T}, \mathcal{F})$ is called cotilting (resp., tilting) if every object of $\mathcal{A}$ is a quotient of an object in $\mathcal{F}$ (resp., subobject of an object in $\mathcal{T}$). A torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$ is cotilting iff the pair $(\mathcal{F}[1], \mathcal{T})$ in $\mathcal{A}_t$ is tilting (see [7], Prop. I.3.2).

Set

$$\tilde{\mathcal{T}} = \{ X \in \tilde{\mathcal{A}} \mid \text{there exists a surjection } \oplus_i T_i \to X, \text{ with } T_i \in \mathcal{T} \text{ for all } i \},$$

and let $\tilde{\mathcal{F}} \subset \tilde{\mathcal{A}}$ be the right orthogonal of $\mathcal{T}$ in $\tilde{\mathcal{A}}$.

**Lemma 2.4.3.** (i) The subcategories $\tilde{\mathcal{T}}$ and $\tilde{\mathcal{F}}$ are closed under inductive limits and the natural functors $\text{Ind}(\mathcal{T}) \to \tilde{\mathcal{T}}$, $\text{Ind}(\mathcal{F}) \to \tilde{\mathcal{F}}$ are equivalences.

(ii) $(\tilde{\mathcal{T}}, \tilde{\mathcal{F}})$ is a torsion pair in $\tilde{\mathcal{A}}$. We have $\tilde{\mathcal{T}} \cap \mathcal{A} = \mathcal{T}$, $\tilde{\mathcal{F}} \cap \mathcal{A} = \mathcal{F}$. Furthermore, if $(\mathcal{T}, \mathcal{F})$ is cotilting then so is $(\tilde{\mathcal{T}}, \tilde{\mathcal{F}})$.

(iii) Let $\tilde{\mathcal{A}}_t \subset D^b(\tilde{\mathcal{A}})$ be the tilted abelian category associated with $(\tilde{\mathcal{T}}, \tilde{\mathcal{F}})$. Then in $\tilde{\mathcal{A}}_t$ arbitrary (small) direct sums exist and for any set $I$ the direct sum functor $(\oplus_{i \in I} A_i) \to \oplus_{i \in I} A_i$ is exact. In other words, $\tilde{\mathcal{A}}_t$ satisfies the axiom $AB4$ (see [4], 1.5).

**Proof.** (i) It is clear that $\tilde{\mathcal{T}}$ is closed under direct sums. Since we have a surjection $\oplus_i X_i \to \lim X_i$ for an inductive system $(X_i)$, it follows that $\tilde{\mathcal{T}}$ is closed under inductive limits. The same assertion for $\tilde{\mathcal{F}}$ follows from (2.1.1). The natural functors $\text{Ind}(\mathcal{T}) \to \tilde{\mathcal{A}}$ and $\text{Ind}(\mathcal{F}) \to \tilde{\mathcal{A}}$ are fully faithful by Proposition 2.1.1(a). It remains to show that every
object of $\tilde{T}$ (resp., $\tilde{F}$) can be represented as the limit of an inductive system $(T_i)$ with $T_i \in \mathcal{T}$ (resp., $(F_i)$ with $F_i \in \mathcal{F}$). If $X \in \tilde{T}$ then we have a surjection $\bigoplus_i T_i \rightarrow X$. Let $T'_i$ be the image of the map $T_i \rightarrow X$. Then $X = \lim T'_i$ and $T'_i \in \mathcal{T}$ since $\mathcal{T}$ is closed under quotients. On the other hand, if $X \in \tilde{F}$ then $X = \lim X_i$, where $(X_i)$ is a filtering set of $\mathcal{A}$-generated subobjects in $X$ (see Proposition 2.1.4). By Lemma 2.4.1(ii) we have $X_i \in \mathcal{A}$. Since $X_i$ is a subobject of $X \in \tilde{F}$, it belongs to the right orthogonal of $\mathcal{T}$ in $\mathcal{A}$. Hence, $X_i \in \mathcal{F}$.

(ii) If $X \in \tilde{T}$ and $Y \in \tilde{F}$ then there exists a surjection $\bigoplus_i T_i \rightarrow X$, where $T_i \in \mathcal{T}$. Since $\text{Hom}(T_i, Y) = 0$ this implies that $\text{Hom}(X, Y) = 0$. If $X \in \tilde{A}$ is an arbitrary object then $X = \lim X_i$ for an inductive system $(X_i)$ in $\tilde{A}$. For every $i$ we have an exact sequence

$$0 \rightarrow A_i \rightarrow X_i \rightarrow B_i \rightarrow 0$$

with $A_i \in \mathcal{T}$, $B_i \in \mathcal{F}$. Moreover, these objects assemble into inductive systems $(A_i)$ and $(B_i)$, and the sequence

$$0 \rightarrow \lim A_i \rightarrow \lim X_i \rightarrow \lim B_i \rightarrow 0$$

in $\tilde{A}$ is still exact (the exactness on the left follows from (2.1.1)). Since $\lim A_i \in \tilde{T}$ and $\lim B_i \in \tilde{F}$, it follows that $(\tilde{T}, \tilde{F})$ is a torsion pair in $\tilde{A}$.

It is clear that $\tilde{F} \cap \mathcal{A} = \mathcal{F}$ and that $\mathcal{T} \subset \tilde{T} \cap \mathcal{A}$. On the other hand, $\tilde{T} \cap \mathcal{A}$ is left orthogonal to $\mathcal{F}$, hence it is contained in $\mathcal{T}$.

Finally, assume $(\mathcal{T}, \mathcal{F})$ is cotilting. Then for every $X \in \tilde{A}$ there exists an inductive system $(A_i)$ in $\tilde{A}$ such that $X = \lim A_i$. Pick a surjection $F_i \rightarrow A_i$ with $F_i \in \mathcal{F}$ for every $i$. Then the composed map

$$\bigoplus F_i \rightarrow \bigoplus A_i \rightarrow X$$

is a surjection and $\bigoplus F_i \in \mathcal{F}$.

(iii) Note that the category $\tilde{A} = \text{Ind}(\mathcal{A})$ satisfies the axiom AB4. Indeed, the functor of direct sum is always right exact. The fact that in this case it is left exact follows easily from (2.1.1). Therefore, in the (unbounded) derived category $D(\tilde{A})$ arbitrary (small) direct sums exist and direct sums of triangles are triangles (see [1], Cor. 1.7). Also, by construction these direct sums commute with the cohomology functors. Using (i) we deduce that $\tilde{A}_i \subset D(\tilde{A})$ is stable under direct sums. Now let $X_i \hookrightarrow Y_i$, $i \in I$, be a collection of injective morphisms in $\tilde{A}_i$. Then we have exact triangles of the form

$$X_i \hookrightarrow Y_i \rightarrow Z_i \rightarrow X_i[1]$$

with $Z_i \in \tilde{A}_i$. Taking the direct sums over $i$ we get an exact triangle

$$\bigoplus_i X_i \hookrightarrow \bigoplus_i Y_i \rightarrow \bigoplus_i Z_i \rightarrow \bigoplus_i X_i[1].$$

Since $\bigoplus_i Z_i$ belongs to $\tilde{A}_i$, this implies that $\bigoplus_i X_i \rightarrow \bigoplus_i Y_i$ is injective. \hfill \square

**Theorem 2.4.4.** Let $\mathcal{A}$ be a Noetherian abelian category of homological dimension $\leq 1$, $(\mathcal{T}, \mathcal{F})$ a cotilting torsion pair in $\mathcal{A}$. Set $\tilde{A} = \text{Ind}(\mathcal{A})$ and define the torsion pair $(\tilde{T}, \tilde{F})$
in $\tilde{A}$ as above. Let $\mathcal{A}_t$ (resp., $\tilde{\mathcal{A}}_t$) be the tilted abelian category associated with $(\mathcal{T}, \mathcal{F})$ (resp., $(\tilde{\mathcal{T}}, \tilde{\mathcal{F}})$). Then there is an equivalence of categories

$$\tilde{\mathcal{A}}_t \simeq \text{Ind}(\mathcal{A}_t)$$

and an exact equivalence of triangulated categories

$$D^b(\tilde{A}) \simeq D^b(\text{Ind}(A_t)).$$

**Proof.** We have the natural exact functor $\mathcal{A}_t \to \tilde{\mathcal{A}}_t$. Since $\tilde{\mathcal{A}}_t$ is closed under direct sums, it extends to a functor $\Phi : \text{Ind}(\mathcal{A}_t) \to \tilde{\mathcal{A}}_t$. To check that this functor is fully faithful we have to check that for $X \in \mathcal{A}_t$ and an inductive system $(Y_i)$ in $\mathcal{A}_t$ the natural map

$$\lim \hom_{\tilde{\mathcal{A}}_t}(X, Y_i) \to \hom_{\tilde{\mathcal{A}}_t}(X, \lim Y_i)$$

is an isomorphism (see Proposition 2.1.1). Note that the category $\mathcal{A}_t$ (resp., $\tilde{\mathcal{A}}_t$) is equipped with a torsion pair $((\mathcal{F}, \mathcal{T}), (\mathcal{F}[1], \mathcal{T}))$. Thus, for every $X \in \mathcal{A}_t$ we have an exact sequence

$$0 \to A \to X \to B \to 0$$

in $\mathcal{A}_t$ with $A \in \mathcal{F}[1]$ and $B \in \mathcal{T}$.

**Step 1.** The map (2.4.4) is an isomorphism if all $Y_i$ belong to $\mathcal{F}[1]$. For the proof consider the morphism of the long exact sequences associated with (2.4.5):

$$\begin{array}{ccccccc}
0 & \to & \lim \hom_{\mathcal{A}_t}(B, Y_i) & \to & \lim \hom_{\mathcal{A}_t}(X, Y_i) & \to & \lim \hom_{\mathcal{A}_t}(A, Y_i) & \to & \lim \ext^1_{\mathcal{A}_t}(B, Y_i) \\
\alpha & & \alpha & & \beta & & \beta & & \\
0 & \to & \hom_{\tilde{\mathcal{A}}_t}(B, \lim Y_i) & \to & \hom_{\tilde{\mathcal{A}}_t}(X, \lim Y_i) & \to & \hom_{\tilde{\mathcal{A}}_t}(A, \lim Y_i) & \to & \ext^1_{\tilde{\mathcal{A}}_t}(B, \lim Y_i)
\end{array}$$

By the five-lemma it is enough to check that $\alpha$ and $\beta$ are isomorphisms and that $\gamma$ is injective. The fact that $\beta$ is an isomorphism is the consequence of the equivalence $\tilde{\mathcal{F}} \simeq \text{Ind}(\mathcal{F})$ (see Lemma 2.4.3(i)). The fact that $\alpha$ is an isomorphism follows from Lemma 2.4.2 along with the fact that $\text{Ext}^1_{\tilde{\mathcal{A}}_t}(B, Y_i) \simeq \text{Ext}^1_{\mathcal{A}_t}(B, Y_i)$ since $\mathcal{A}$ is closed under extensions in $\tilde{\mathcal{A}}$ (by Lemma 2.4.1(ii)). Finally,

$$\text{Ext}^1_{\tilde{\mathcal{A}}_t}(B, Y_i) \simeq \text{Ext}^1_{D^b(\mathcal{A})}(B, Y_i),$$

so the vanishing of $\text{Ext}^2$ in $\mathcal{A}$ implies the vanishing of all these groups.

**Step 2.** If $(\tilde{Y}_i)$ is an inductive system in $\tilde{\mathcal{F}}[1]$ and $X \in \mathcal{A}_t$ then the natural map

$$\lim \hom_{\tilde{\mathcal{A}}_t}(X, \tilde{Y}_i) \to \hom_{\tilde{\mathcal{A}}_t}(X, \lim \tilde{Y}_i)$$

is injective. The proof is similar to Step 1 but easier: it is enough to consider only the first three terms in the long exact sequences associated with (2.4.5) and to use Lemma 2.4.2.

**Step 3.** The map (2.4.4) is injective. Indeed, our assumption that $(\mathcal{T}, \mathcal{F})$ is a cotilting torsion pair in $\mathcal{A}$ implies that $(\mathcal{F}[1], \mathcal{T})$ is a tilting torsion pair in $\mathcal{A}_t$, i.e., every object of $\mathcal{A}_t$ can be embedded into an object of $\mathcal{F}[1]$. Let us choose for every $i$ an embedding
We want to replace these embeddings by a new collection of embeddings $Y_i \subset \tilde{Y}_i$ in $\tilde{A}_t$, where $\tilde{Y}_i \in \tilde{F}[1]$, such that $(\tilde{Y}_i)$ form an inductive system and the above embeddings give a morphism of inductive systems. This is achieved by defining $\tilde{Y}_i$ from the following pushout diagram

$$
\begin{array}{ccc}
\oplus_{j \leq i} Y_j & \longrightarrow & Y_i \\
\downarrow & & \downarrow \\
\oplus_{j \leq i} A_j & \longrightarrow & \tilde{Y}_i
\end{array}
$$

Since the horizontal arrows in this diagram are surjective, we have $\tilde{Y}_i \in \tilde{F}[1]$ (because $\tilde{F}[1]$ is the left orthogonal to $\tilde{T}$ in $\tilde{A}_t$). On the other hand, by Lemma 2.4.3(iii) the left vertical arrow is injective, hence the right vertical arrow $Y_i \rightarrow \tilde{Y}_i$ is also injective. It is easy to see that these maps form a morphism of inductive systems. We can include the map (2.4.4) into a commutative square

$$
\begin{array}{ccc}
\lim \longrightarrow \text{Hom}_{A_t}(X, Y_i) & \longrightarrow & \text{Hom}_{\tilde{A}_t}(X, \lim \longrightarrow Y_i) \\
\downarrow & & \downarrow \\
\lim \longrightarrow \text{Hom}_{\tilde{A}_t}(X, \tilde{Y}_i) & \longrightarrow & \text{Hom}_{\tilde{A}_t}(X, \lim \longrightarrow \tilde{Y}_i)
\end{array}
$$

Now the required injectivity follows immediately from the injectivity of the left vertical arrow and from Step 2.

**Step 4.** The map (2.4.4) is an isomorphism. For the proof let us consider for every $i$ the exact sequence

$$0 \rightarrow A_i \rightarrow Y_i \rightarrow B_i \rightarrow 0$$

with $A_i \in F[1]$ and $B_i \in T$. The objects $(A_i)$ (resp., $(B_i)$) form an inductive system and we claim that the limit sequence

$$0 \rightarrow \lim \longrightarrow A_i \xrightarrow{f} \lim \longrightarrow Y_i \rightarrow \lim \longrightarrow B_i \rightarrow 0 \quad (2.4.6)$$

in $\tilde{A}_t$ is exact. Indeed, the only problem is the injectivity of $f$. By Step 1 we obtain that for every $X \in \mathcal{A}_t$ one has

$$\lim \longrightarrow \text{Hom}_{A_t}(X, A_i) \simeq \text{Hom}_{A_t}(X, \lim \longrightarrow A_i).$$

Together with Step 3 this implies that the morphism

$$\text{Hom}_{\tilde{A}_t}(X, \lim \longrightarrow A_i) \rightarrow \text{Hom}_{\tilde{A}_t}(X, \lim \longrightarrow \tilde{Y}_i)$$

induced by $f$ is injective. Therefore, $\text{Hom}_{\tilde{A}_t}(X, \ker(f)) = 0$ for every $X \in \mathcal{A}_t$. Consider the exact sequence

$$0 \rightarrow C \rightarrow \ker(f) \rightarrow D \rightarrow 0$$
in $\tilde{A}_t$ with $C \in \tilde{F}[1]$ and $D \in \tilde{T}$. Since $\tilde{F} \simeq \text{Ind}(\mathcal{F})$ by Lemma 2.4.3(i), it follows that $C = 0$, i.e., $\ker(f) \in \tilde{T}$. Similarly, using the fact that $\tilde{T} \simeq \text{Ind}(\mathcal{T})$ we conclude that $\ker(f) = 0$. Next, let us consider the morphism of the long exact sequences

$$0 \to \lim\limits_{\rightarrow} \text{Hom}_{\tilde{A}_t}(X, A_i) \to \lim\limits_{\rightarrow} \text{Hom}_{\tilde{A}_t}(X, Y_i) \to \lim\limits_{\rightarrow} \text{Hom}_{\tilde{A}_t}(X, B_i) \to \lim\limits_{\rightarrow} \text{Ext}^1_{\tilde{A}_t}(X, A_i)$$

$$\delta$$

$$0 \to \text{Hom}_{\tilde{A}_t}(X, \lim A_i) \to \text{Hom}_{\tilde{A}_t}(X, \lim Y_i) \to \text{Hom}_{\tilde{A}_t}(X, \lim B_i) \to \text{Ext}^1(X, \lim A_i)$$

By the five-lemma and Steps 1 and 3 it remains to check that $\delta$ is an isomorphism. Consider the exact sequence (2.4.5) again. Then we have

$$\lim\limits_{\rightarrow} \text{Hom}_{\tilde{A}_t}(X, B_i) \simeq \lim\limits_{\rightarrow} \text{Hom}_{\tilde{T}}(B, B_i) \simeq \text{Hom}_{\tilde{T}}(B, \lim B_i) \simeq \text{Hom}_{\tilde{A}_t}(X, \lim B_i),$$

where we used again the fact that $\tilde{T} \simeq \text{Ind}(\mathcal{T})$.

**Step 5.** The functor $\Phi : \text{Ind}(\tilde{A}_t) \to \tilde{A}_t$ is essentially surjective on objects. Indeed, since the torsion pair $(\mathcal{T}, \mathcal{F})$ is cotilting the same is true about the torsion pair $(\tilde{T}, \tilde{F})$ in $\tilde{A}_t$ (see Lemma 2.4.3(ii)). Therefore, the torsion pair $(\tilde{F}[1], \tilde{F})$ in $\tilde{A}_t$ is tilting. So for any object $X \in \tilde{A}_t$ we can find an embedding $X \subseteq A$ with $A \in \tilde{F}[1]$. Since $\tilde{F}[1]$ is closed under quotients we obtain that $X$ is the kernel of a morphism $f : A \to A'$ with $A, A' \in \tilde{F}[1]$. Note that $\tilde{F}[1]$ is contained in the essential image of $\Phi$ (since $\tilde{F} \simeq \text{Ind}(\mathcal{F})$ by Lemma 2.4.3(i)). Since by Step 4 the functor $\Phi$ is fully faithful, we derive that the morphism $f$ is contained in the image of $\Phi$. Finally, exactness of $\Phi$ (see Proposition 2.1.1(b)) implies that $X \simeq \ker(f)$ is also contained in the image of $\Phi$.

The statement about the equivalence of derived categories is an immediate consequence of Proposition 5.4.3 of [2].

The category $\text{Vect}(T)$ is equivalent to the tilted abelian category associated with the cotilting torsion pair $(\text{Coh}_{>\theta}(E), \text{Coh}_{<\theta}(E))$ in the category $\text{Coh}(E)$ of the coherent sheaves on the elliptic curve $E$, where $\text{Coh}_{<\theta}(E)$ consists of direct sums of semistable bundles of slopes $\theta$ and $\text{Coh}_{>\theta}(E)$ consists of direct sums of torsion sheaves and semistable bundles of slopes $>\theta$ (see [9] or [8], sec. 1.2). Let $\text{Qcoh}(E) \simeq \text{Ind}(\text{Coh}(E))$ be the category of quasicoherent sheaves on $E$. It is easy to see that the assumptions of Theorem 2.4.4 are satisfied, so we derive the following result.

**Theorem 2.4.5.** The category $\text{Qcoh}(T)$ is equivalent to the tilted abelian category associated with the torsion pair $(\text{Qcoh}_{>\theta}(E), \text{Qcoh}_{<\theta}(E))$ in $\text{Qcoh}(E)$, where $\text{Qcoh}_{<\theta}(E)$ consists of torsion free quasicoherent sheaves $F$, such that any semistable subsheaf of $F$ has slope $\theta$, and $\text{Qcoh}_{>\theta}(E)$ consists of quotients of arbitrary (small) direct sums of torsion sheaves and semistable bundles of slopes $>\theta$. There is also an exact equivalence of derived categories

$$D^b(\text{Qcoh}(T)) \simeq D^b(\text{Qcoh}(E))$$

extending (2.4.1).
2.5. The rank function. The main result of this section is the following

**Theorem 2.5.1.** There exists a unique extension of the function \( \text{rk} \) on holomorphic bundles over \( T \) to a function on quasicoherent sheaves taking values in \( \mathbb{R}_{\geq 0} \cup \{+\infty\} \) and satisfying the following two properties:

(i) \( \text{rk} \) is additive in exact triples;

(ii) If a quasicoherent sheaf is represented as a filtering union of quasicoherent subsheaves, \( M = \cup_{i \in I} M_i \), then

\[
\text{rk} M = \lim_{i \in I} \text{rk} M_i.
\]

Actually, the construction works in the following general framework (we keep the notations and conventions of section 2.1). Let \( \mathcal{A} \) be an abelian category and let \( \text{Ind} \mathcal{A} \) be the corresponding category of ind-objects of \( \mathcal{A} \). Let \( \text{rk} : K_0(\mathcal{A}) \to \mathbb{R} \) be a homomorphism such that \( \text{rk}(A) > 0 \) for any nonzero object \( A \in \mathcal{A} \).

**Theorem 2.5.2.** There exists a unique extension of the function \( \text{rk} \) from objects of \( \mathcal{A} \) to objects of \( \text{Ind} \mathcal{A} \) taking values in \( \mathbb{R}_{\geq 0} \cup \{+\infty\} \) and satisfying the following two properties:

(i) \( \text{rk} \) is additive in exact triples;

(ii) If an ind-object \( X \) is represented as a union of subobjects, \( X = \cup_{i \in I} X_i \), then

\[
\text{rk} X = \lim_{i \in I} \text{rk} X_i.
\]

Note that Theorem 2.5.1 is an immediate consequence of this theorem.

First, we are going to define the rank of \( \mathcal{A} \)-generated ind-objects (see section 2.1). By Proposition 2.1.3 such an ind-object can be presented in the form \( P/S \), where \( P \in \mathcal{A} \), and \( S \) is a union of subobjects of \( P \) in \( \mathcal{A} \). Therefore, it is natural to make the following definition.

**Definition.** Let \( S \in \text{Ind} \mathcal{A} \) be an ind-subobject of \( P \in \mathcal{A} \). Then we set

\[
\text{rk}_P(S) = \sup \{ \text{rk} S' \mid S' \subset S, S' \in \mathcal{A} \},
\]

i.e., we define the rank of \( S \) in \( P \) as the supremum of the ranks of all subobjects of \( P \) in \( \mathcal{A} \) contained in \( S \).

Note that if \( S \) is itself in \( \mathcal{A} \) then \( \text{rk}_P(S) = \text{rk} S \). It is clear from the definition that \( \text{rk}_P(S) \leq \text{rk}(P) \) for every \( S \subset P \) and that \( \text{rk}_P \) is monotone with respect to inclusions. This function also satisfies the following continuity condition.

**Lemma 2.5.3.** Let \( (S_i)_{i \in I} \) be a filtering collection of ind-subobjects of \( P \in \mathcal{A} \). Then for \( S = \cup_{i \in I} S_i \) we have

\[
\text{rk}_P(S) = \lim_{i \in I} \text{rk}_P(S_i).
\]

**Proof.** It is clear that \( \text{rk}_P(S) \geq \sup \{ \text{rk}_P(S_i) \mid i \in I \} \). On the other hand, if \( S' \subset S \) is a subobject such that \( S' \in \mathcal{A} \), then there exists \( i \in I \) such that \( S' \subset S_i \) (by Proposition 2.1.1(a)). This implies that \( \text{rk}_P(S) \leq \sup \{ \text{rk}_P(S_i) \mid i \in I \} \). Hence,

\[
\text{rk}_P(S) = \sup \{ \text{rk}_P(S_i) \mid i \in I \}.
\]

Since the function \( \text{rk}_P \) is monotone, we can replace \( \sup \) with \( \lim \). \( \square \)
The above lemma also implies that in the definition of $\text{rk}_P(S)$ it suffices to take the supremum over any collection of subobjects of $P$ in $\mathcal{A}$ whose union is $S$.

**Proposition 2.5.4.** Let $X = P/S$ be an $\mathcal{A}$-generated ind-object, where $P \in \mathcal{A}$. Then the nonnegative real number

$$\text{rk} \ X := \text{rk} \ P - \text{rk}_P(S)$$

does not depend on a presentation of $X$ in the form $P/S$.

**Proof.** If $P \to X$ and $P' \to X$ are surjections (where $P, P' \in \mathcal{A}$) then they are dominated by the surjection $P \oplus P' \to X$. Hence, it suffices to compare presentations $P/S$ and $P'/S'$ of $X$ in the case $P' \subset P$. In this situation $S' = S \cap P'$ and $P = S + P'$. By assumption we have $S = \bigcup_{i \in I} S_i$, where $(S_i)$ is a filtering collection of subobjects of $S$ contained in $\mathcal{A}$. Then $S' = \bigcup_{i \in I} S_i \cap P'$, so

$$\text{rk}_P(S) = \sup \{\text{rk} \ S_i \mid i \in I\},$$

$$\text{rk}_P(S') = \sup \{\text{rk} \ S_i \cap P' \mid i \in I\}.$$

Now we observe that since $P \in \mathcal{A}$, there exists $i_0 \in I$ such that $P = S_{i_0} + P'$. It follows that for all $S_i \supseteq S_{i_0}$ we have $\text{rk} \ S_i \cap P' = \text{rk} \ S_i + \text{rk} \ P' - \text{rk} \ P$. Hence, $\text{rk}_P(S') = \text{rk}_P(S) + \text{rk} \ P' - \text{rk} \ P$. \hfill \Box

At this point we will only check the following expected property of the rank function on $\mathcal{A}$-generated ind-objects.

**Lemma 2.5.5.** Let $X' \subset X$ be an embedding of $\mathcal{A}$-generated ind-objects. Then $\text{rk} \ X' \leq \text{rk} \ X$.

**Proof.** We can find surjections $P \to X$ and $P' \to X'$ such that $P, P' \in \mathcal{A}$ and $P' \subset P$. Let $S$ and $S'$ be kernels of these maps, so that $X = P/S$, $X' = P'/S'$. Then $S' = S \cap P'$. Let $(S_i)_{i \in I}$ be a filtering collection of subobjects of $S$ contained in $\mathcal{A}$, such that $S = \bigcup_{i \in I} S_i$. As in the proof of Proposition 2.5.4 we see that

$$\text{rk}_P(S') = \sup \{\text{rk} \ S_i \cap P' \mid i \in I\}.$$

Since for every $i \in I$ we have an embedding $S_i/S_i \cap P' \subset P/P'$, it follows that

$$\text{rk} \ S_i - \text{rk} \ S_i \cap P' \leq \text{rk} \ P - \text{rk} \ P'.$$

Passing to the limit in $i \in I$ we derive that

$$\text{rk}_P(S) - \text{rk}_P(S') \leq \text{rk} \ P - \text{rk} \ P'$$

which is equivalent to the desired inequality. \hfill \Box

Since every ind-object $X$ is the union of a filtering collection of its finitely generated ind-subobjects, we can now define the rank of an arbitrary ind-object $X$ by setting

$$\text{rk} \ X = \sup \{\text{rk} \ X^f \mid X^f \subset X, \ X^f \text{ is } \mathcal{A}\text{-generated}\}.$$

By Lemma 2.5.5, if $X$ is itself $\mathcal{A}$-generated then this definition agrees with the old one. We also have the following analogue of Lemma 2.5.3.
Lemma 2.5.6. Let \((X_i)\) be a filtering collection of subobjects in \(X \in \text{Ind} \mathcal{A}\) such that \(X = \bigcup_{i \in I} X_i\). Then
\[
\text{rk } X = \lim_{i \in I} \text{rk } X_i.
\]

Proof. The proof follows the proof of Lemma 2.5.3 step by step (recall that by Lemma 2.5.5 the rank function on \(\mathcal{A}\)-generated ind-objects is monotone).

We need one more lemma for the proof of Theorem 2.5.2.

Lemma 2.5.7. Let \(0 \to A \to B \to C \to 0\) be an exact triple of ind-objects such that \(B\) is \(\mathcal{A}\)-generated. Then \(\text{rk } B = \text{rk } A + \text{rk } C\).

Proof. Let \(B = P/S, C = P/T\), where \(P\) is an object of \(\mathcal{A}\) and \(S \subset T \subset P\) are its ind-subobjects. We have \(\text{rk } B = \text{rk } P - \text{rk } P(S), \text{rk } C = \text{rk } P - \text{rk } P(T)\). Let \((T_i)_{i \in I}\) be a filtering collection of subobjects in \(T\) contained in \(\mathcal{A}\), such that \(T = \bigcup_{i \in I} T_i\). Then \(A = T/S = \bigcup_{i \in I} T_i/T_i \cap S\), so by Lemma 2.5.6 we obtain
\[
\text{rk } A = \lim_{i \in I} \text{rk } T_i/T_i \cap S.
\]

By definition, we have
\[
\text{rk } T_i/T_i \cap S = \text{rk } T_i - \text{rk } T_i(T_i \cap S) = \text{rk } T_i - \text{rk } P(T_i \cap S).
\]

Since \(S = \bigcup_{i \in I} T_i \cap S\), by Lemma 2.5.3 we get
\[
\lim_{i \in I} \text{rk } P(T_i \cap S) = \text{rk } P(S).
\]

Therefore, passing to the limit in (2.5.1) we obtain
\[
\text{rk } A = \text{rk } P(T) - \text{rk } P(S).
\]

Proof of Theorem 2.5.2. It is easy to see that any extension of \(\text{rk}\) satisfying (i) and (ii) should coincide with the rank function constructed above. Note also that our rank function satisfies (ii) by Lemma 2.5.6. It remains to check its additivity in exact triples. Let \(0 \to A \to B \to C \to 0\) be an exact triple in \(\text{Ind} \mathcal{A}\). We have \(B = \bigcup_{i \in I} B_i\), where \(B_i\)’s are \(\mathcal{A}\)-generated subobjects. Let \(C_i\) be the image of \(B_i\) under the map to \(C\) and let \(A_i = A \cap B_i\). Then \(C = \bigcup_{i \in I} C_i\) and \(A = \bigcup_{i \in I} A_i\). By Lemma 2.5.7 for each \(i\) we have
\[
\text{rk } B_i = \text{rk } A_i + \text{rk } C_i.
\]

Passing to the limit and using Lemma 2.5.6 we derive that \(\text{rk } B = \text{rk } A + \text{rk } C\).

2.6. Quasi vector bundles. Recall that these are quasicoherent sheaves that are filtering unions of holomorphic bundles.

Lemma 2.6.1. Every finitely generated holomorphic submodule of a quasi vector bundle is a vector bundle.

Proof. Indeed, let \(M = \bigcup_i M_i\), where \(M_i\) are vector bundles. Then every finitely generated submodule of \(M\) is contained in some \(M_i\), hence itself is a vector bundle by Corollary 2.2.3.
Lemma 2.6.2. Let $M$ be a quasi vector bundle, $N \subset M$ a quasicoherent subsheaf. Then $N$ is a quasi vector bundle.

Proof. Since $N$ is a union of its finitely generated holomorphic submodules, this follows immediately from the previous lemma.

Proposition 2.6.3. Every quasicoherent sheaf on $T$ can be represented in the form $P_1/P_0$, where $P_0 \subset P_1$ are quasi vector bundles.

Proof. Recall that by Lemma 2.6.2 a quasicoherent subsheaf of a quasi vector bundle is itself a quasi vector bundle. Therefore, it is enough to prove that for every quasicoherent sheaf $M$ there exists a quasi vector bundle $P$ and a surjection $P \to M$. By Proposition 2.3.5 we have $M = \bigcup_{i \in I} M_i$ with $M_i = P_i/Q_i$, where $P_i$ are vector bundles. Set $P = \bigoplus_{i \in I} P_i$ and define the morphism $P \to M$ using the natural morphisms $P_i \to M_i \subset M$. It is clear that this morphism is surjective and that $P$ is a quasi vector bundle.

For later use we record here one more simple observation.

Lemma 2.6.4. Let $V \subset M$ be an embedding of a vector bundle into a quasi vector bundle. Then $M/V$ is a quasi vector bundle.

Proof. Note that if $M' \subset M$ is a vector subbundle then $V + M' \subset M$ is still a vector bundle, e.g., by Lemma 2.6.1. It follows that $M$ can be represented as a union of vector bundles $M = \bigcup_i M_i$, where $V \subset M_i$ for all $i$. Hence, $M/V = \bigcup_i M_i/V$.

2.7. Torsion and torsion free sheaves. The following two classes of quasicoherent sheaves will be also important for us.

Definition. Let $M$ be a quasicoherent sheaf on $T$. We say that $M$ is a torsion sheaf if $\text{rk } M = 0$. We say that $M$ is a torsion-free sheaf if for every nonzero submodule $M' \subset M$ one has $\text{rk } M' > 0$.

Proposition 2.7.1. (i) Let $M$ be a torsion sheaf and $N$ be a torsion-free sheaf. Then $\text{Hom}(M,N) = 0$.

(ii) For every quasicoherent sheaf $M$ there exists maximal torsion subsheaf $M_{\text{tors}} \subset M$. The quotient $M/M_{\text{tors}}$ is torsion free.

Proof. (i) If $f : M \to N$ is a morphism then $\text{im}(f)$ has rank 0. But it is a subsheaf of $N$, hence, $\text{im}(f) = 0$.

(ii) If $N_1$ and $N_2$ are torsion subsheaves in $M$. Then there exists a surjection $N_1 \oplus N_2 \to N_1 + N_2 \subset M$. Hence, $\text{rk}(N_1 + N_2) = 0$, so $N_1 + N_2$ is also a torsion sheaf. It follows that the union of all torsion subsheaves in $M$ is itself a subsheaf $M_{\text{tors}} \subset M$. Furthermore, by Lemma 2.5.6 we have $\text{rk } M_{\text{tors}} = 0$. If $N \subset M/M_{\text{tors}}$ is torsion subsheaf then its preimage in $M$ is also a torsion sheaf by additivity of the rank. Hence, $N = 0$.

Proposition 2.7.2. Every quasi vector bundle is torsion free.
Proof. Let $V$ be a quasi vector bundle. It suffices to prove that for every finitely generated quasicoherent subsheaf $W \subset V$ with $\text{rk} W = 0$ one has $W = 0$. But this follows immediately from Lemma 2.6.1. \hfill \square

Modifying slightly the proof of Theorem 1.3.1 we get the following result.

**Theorem 2.7.3.** For every stable bundle $P$ and every real number $r$ such that $0 < r < \text{rk} P$ there exists a countably generated quasicoherent subsheaf $Q \subset P$ such that $\text{rk} Q = r$.

**Proof.** Using Morita equivalences we reduce to the case $\text{rk} P = 1$. Then we can apply the construction of Theorem 1.3.1 to construct a family of subbundles $V_{0,a} \subset P$ for $a \in \mathcal{B}_\theta$, where $\text{rk} V_{0,a} = a$ and $V_{0,a} \subset V_{0,a'}$ for $a < a'$. Let $(a_n)$ be an increasing sequence of numbers in $\mathcal{B}_\theta$ such that $\lim_{n \to \infty} a_n = r$. Then we can take $Q = \bigcup_n V_{0,a_n}$. \hfill \square

Later we will see that in fact all quasicoherent subsheaves of $P$ are countably generated (see Theorem 2.9.4(i)).

**Corollary 2.7.4.** For every real number $r > 0$ there exist a finitely generated torsion free quasicoherent sheaf $M$ of rank $r$ which is not a quasi vector bundle.

**Proof.** Let $P$ be a stable bundle with $\text{rk} P > r$. In the case $r \in \mathbb{Z}\theta + \mathbb{Z}$ we also require that $\chi(\text{rk} P, r) < 0$. Then by the above theorem there exists a quasicoherent subsheaf $S \subset P$ with $\text{rk} S = \text{rk} P - r$. Now we define $M$ to be the quotient of $P/S$ by its torsion part $(P/S)_{\text{tors}}$. Then $M$ is a finitely generated torsion free sheaf of rank $r$. We claim that $M$ is not a vector bundle. Indeed, if $r \not\in \mathbb{Z}\theta + \mathbb{Z}$ then this is clear. Otherwise, using the fact that $M$ is a quotient of a stable bundle $P$ we get a contradiction with the condition $\chi(\text{rk} P, r) < 0$. Finally, Lemma 2.6.1 implies that $M$ is not a quasi vector bundle. \hfill \square

### 2.8. Harder-Narasimhan filtration for quasicoherent subsheaves of vector bundles

In this section we will show that every quasicoherent subsheaf of a vector bundle on $T$ has a canonical exhaustive filtration similar to the Harder-Narasimhan filtration of vector bundles.

**Lemma 2.8.1.** For $N > 0$ consider the subset $\mathcal{M}_N \subset \mathbb{R}$ defined by

$$
\mathcal{M}_N = \left\{ \frac{m}{m\theta + n} \mid (m, n) \in \mathbb{Z}, m \leq 0, 0 < m\theta + n < N \right\}.
$$

Then for every $c > 0$ the set $\mathcal{M}_\theta \cap [-c, 0]$ is finite. In particular, every nonempty subset of $\mathcal{M}_\theta$ has a maximal element.

**Proof.** This follows immediately from the fact that there are at most $N$ numbers of the form $m\theta + n$ in the interval $(0, N)$ with a given value of $m$ and that for such a number we have $|m|/N \leq |m|/(m\theta + n)$. \hfill \square

**Lemma 2.8.2.** Let $P$ be a vector bundle and $\mathcal{F} \subset P$ a quasicoherent subsheaf. Then among subbundles $Q \subset P$ such that $Q \subset \mathcal{F}$ there exists a unique maximal element (by inclusion) of maximal slope. Moreover, such $Q$ is semistable.

**Proof.** It is easy to see that $P$ can be embedded into a vector bundle of the form $P_0^{\oplus N}$, where $P_0$ is a stable vector bundle, so we can replace $P$ with $P_0^{\oplus N}$. Furthermore, applying
Morita equivalence we can assume that \( \text{rk} P_0 = 1 \). In this case the slope of any subbundle of \( P_0^\infty \) belongs to the subset \( \mathcal{M}_N \) considered in the above lemma. Hence, there exists a subbundle \( Q \subset \mathcal{F} \) of a maximal slope \( \mu_{\text{max}} \). Note that such \( Q \) is automatically semisimple. Therefore, among all subbundles of maximal slope there exists a maximal element by inclusion. It remains to prove that it is unique. But if \( Q' \subset \mathcal{F} \) is another subbundle with this property then the exact sequence

\[
0 \to Q \to Q + Q' \to Q'/Q \cap Q' \to 0
\]

shows that the \( \mu(Q + Q') \geq \mu_{\text{max}} \). Indeed, it suffices to see that \( \mu(Q'/Q \cap Q') \geq \mu_{\text{max}} \) but this follows from the semistability of \( Q' \).

\[\square\]

**Proposition 2.8.3.** Let \( P \) be a vector bundle and \( \mathcal{F} \subset P \) a quasicoherent subsheaf. Then there exists sequence of subbundles \( 0 = F_0 \subset F_1 \subset F_2 \subset \ldots \subset P \) such that \( \mathcal{F} = \bigcup F_n \) and \( F_n/F_{n-1} \) is the maximal subbundle of maximal slope in \( \mathcal{F}/F_{n-1} \subset P/F_{n-1} \).

**Proof.** The previous lemma allows to define the sequence of subbundles \( 0 = F_0 \subset F_1 \subset F_2 \subset \ldots \subset P \) such that \( F_n/F_{n-1} \) is the maximal subbundle of maximal slope in \( \mathcal{F}/F_{n-1} \subset P/F_{n-1} \). It remains to show that \( \mathcal{F} = \bigcup F_n \). If \( \mathcal{F} \) is a subbundle then \( (F_n) \) coincides with its usual Harder-Narasimhan filtration (as an object in the derived category \( D^b(E) \)), so in this case our claim holds. Thus, we can assume that \( \mathcal{F} \) is not coherent, so that \( F_n/F_{n-1} \neq 0 \) for all \( n > 0 \). Note that \( \text{rk}(F_n/F_{n-1}) \to 0 \) as \( n \to \infty \). Also, we have \( \mu(F_1/F_0) > \mu(F_2/F_1) > \ldots \). We claim that this implies that \( \mu(F_n/F_{n-1}) \to -\infty \) as \( n \to \infty \). Indeed, if the sequence \( (\mu(F_n/F_{n-1})) \) were bounded then we would have \( \deg(F_n/F_{n-1}) \to 0 \) as \( n \to \infty \). But \( \deg(F_n/F_{n-1}) \) can take value 0 only once, so we get a contradiction that proves our claim. Now let \( Q \subset P \) be an arbitrary subbundle such that \( Q \subset \mathcal{F} \). We have to prove that \( Q \subset F_n \) for \( n \gg 0 \). Since \( Q \) is a direct sum of semistable subbundles, it suffices to consider the case when \( Q \) is semistable. Choose \( n \) such that \( \mu(Q) > \mu(F_n/F_{n-1}) \). Assume that \( Q \not\subset F_n \). Then \( Q' = Q/(Q \cap F_{n-1}) \) is a subbundle of \( P/F_{n-1} \) contained in \( \mathcal{F}/F_{n-1} \). Furthermore, since \( Q \) is semistable, we have \( \mu(Q') \geq \mu(Q) > \mu(F_n/F_{n-1}) \). But this contradicts to the assumption that \( F_n/F_{n-1} \) has maximal slope among all subbundles of \( P/F_{n-1} \) containing \( \mathcal{F}/F_{n-1} \). Therefore, \( Q \) is contained in \( F_{n-1} \). \[\square\]

### 2.9. Countably generated quasicoherent sheaves.

We say that a quasicoherent sheaf \( M \) is *countably generated* if its underlying \( \mathcal{A}_\theta \)-module has a countable set of generators. Equivalently, \( M = \bigcup_{n \geq 1} M_n \) for a chain \( M_1 \subset M_2 \subset \ldots \subset M \) of finitely generated quasicoherent subsheaves. Note that for such \( M \) there exists a surjection \( P \to M \), where \( P \) is a countably generated quasi vector bundle (pick surjections \( P_i \to M_i \) and set \( P = \oplus P_i \)). The importance of countability is due to the following general result.

**Lemma 2.9.1.** Let \( P \) be an object of an abelian category \( \mathcal{A} \) such that \( P = \lim P_i \) for some inductive system \( (P_i) \) with a countable set of indices, such that all objects \( P_i \) are projective and every arrow \( P_i \to P_j \) is an embedding of a direct summand. Then \( P \) itself is projective.

**Proof.** By assumption, for every arrow \( P_i \to P_j \) and every \( A \in \mathcal{A} \) the morphism

\[
\text{Hom}_\mathcal{A}(P_j, A) \to \text{Hom}_\mathcal{A}(P_i, A)
\]

is surjective. Therefore, \( \text{Hom}_\mathcal{A}(P, A) = \bigcup \text{Hom}_\mathcal{A}(P_i, A) \). Since \( P \to \bigcup P_i \) is a surjection, \( P \) is projective. \(\square\)
is surjective. Therefore, the functor
\[ A \rightarrow \text{Hom}_A(P, A) = \text{proj lim}_i \text{Hom}_A(P_i, A) \]
is exact (since the Mittag-Leffler condition is satisfied, see [5], ch. 0, § 13), so \( P \) is projective.

**Proposition 2.9.2.** A countably generated quasi vector bundle is projective as an \( A_\theta \)-module.

**Proof.** This follows immediately from Lemma 2.9.1 since every embedding of holomorphic vector bundles is an embedding of a direct summand on the level of \( A_\theta \)-modules. \( \square \)

**Lemma 2.9.3.** Let \( M \) be a countably generated quasicoherent sheaf, \( N \subset M \) be a quasi-coherent subsheaf. Then \( N \) is countably generated.

**Proof.** Consider a surjection \( P \rightarrow M \), where \( P \) is a countably generated quasi vector bundle. It is enough to prove that every quasicoherent subsheaf \( N \subset P \) is countably generated. Furthermore, if \( P \) is a union of a sequence of vector bundles \( P_1 \subset P_2 \subset \ldots \) then \( N = \cup_i (N \cap P_i) \). Therefore, it suffices to consider the case when \( P \) is a vector bundle. Then the assertion follows from Proposition 2.8.3. \( \square \)

**Theorem 2.9.4.** (i) The full subcategory \( \text{Qcoh}^c(T) \subset \text{Qcoh}(T) \) of countably generated sheaves is closed under passing to subobjects and quotients, and under extensions. In other words, \( \text{Qcoh}^c(T) \) is a Serre subcategory in \( \text{Qcoh}(T) \).

(ii) For any \( M \in \text{Qcoh}^c(T) \) the projective dimension of the underlying \( A_\theta \)-module is at most 1.

(iii) A sheaf \( M \in \text{Qcoh}^c(T) \) is a quasi vector bundle iff the underlying \( A_\theta \)-module is projective.

**Proof.** (i) It is clear that \( \text{Qcoh}^c(T) \) is closed under passing to quotients. The assertion about subobjects is Lemma 2.9.3. The assertion about extensions is clear.

(ii) This follows from Proposition 2.9.2.

(iii) The “only if” part is Proposition 2.9.2. Conversely, assume that \( M \in \text{Qcoh}^c(T) \) is projective as an \( A_\theta \)-module. Let \( M' \subset M \) be a finitely generated quasicoherent subsheaf. Then \( M/M' \) is an object of \( \text{Qcoh}^c(T) \). Hence, by part (ii) \( M' \) is still projective as an \( A_\theta \)-module. Therefore, \( M' \) is a holomorphic vector bundle. Since \( M \) is a union of finitely generated subsheaves, it is a quasi vector bundle. \( \square \)

**Corollary 2.9.5.** Any quasicoherent ideal \( I \subset A_\theta \) is countably generated.

### 3. Sheaves at the general point of a noncommutative torus

#### 3.1. Quasi vector bundles at the general point

Let \( \text{Tors} \subset \text{Qcoh}(T) \) be the full subcategory consisting of torsion sheaves \( M \) (i.e., sheaves with \( \text{rk} M = 0 \)). This is a Serre subcategory of \( \text{Qcoh}(T) \), so we can consider the quotient-category
\[
\text{Qcoh}(\eta_T) = \text{Qcoh}(T)/\text{Tors}
\]
which is a noncommutative analogue of the category of quasicoherent sheaves on a general point of an elliptic curve. Note that $\text{Qcoh}(\eta_T)$ is a $\mathbb{C}$-linear abelian category and there is a canonical exact functor $\text{Qcoh}(T) \rightarrow \text{Qcoh}(\eta_T)$.

**Proposition 3.1.1.** Let $P_1$ and $P_2$ be quasicoherent sheaves on $T$. Assume that $P_2$ is torsion free. Then the natural morphism

$$\text{Hom}_{\text{Qcoh}(T)}(P_1, P_2) \rightarrow \text{Hom}_{\text{Qcoh}(\eta_T)}(P_1, P_2)$$

is injective.

**Proof.** By definition, a morphism from $P_1$ to $P_2$ in $\text{Qcoh}(\eta_T) = \text{Qcoh}(T)/\text{Tors}$ is given by a morphism $P_1' \rightarrow P_2/F$, where $P_1' \subset P$ and $F \subset P_2$ are quasicoherent subsheaves such that $\text{rk} P_1/P_1' = \text{rk} F = 0$. But $P_2$ is torsion free, hence $F = 0$. Thus,

$$\text{Hom}_{\text{Qcoh}(\eta_T)}(P_1, P_2) = \lim_{\rightarrow} \text{Hom}_{\text{Qcoh}(T)}(P_1, P_2).$$

It remains to check that if a morphism $f : P_1 \rightarrow P_2$ vanishes on a subsheaf $P_1' \subset P_1$ such that $\text{rk} P_1/P_1' = 0$ then $f = 0$. But such $f$ factors through a morphism $P_1/P_1' \rightarrow P_2$. Since $\text{rk} P_1/P_1' = 0$ and $P_2$ is torsion free, such a morphism has to be zero. \hfill \Box

**Corollary 3.1.2.** The functor $\text{Vect}(T) \rightarrow \text{Qcoh}(\eta_T)$ is faithful.

**Lemma 3.1.3.** Let $f : M \rightarrow M'$ be a surjection in $\text{Qcoh}(T)$. Then there exist inductive systems $(M_i)$ and $(M'_i)$ in $\text{Vect}(T)$ such that $\lim_{\rightarrow} M_i \simeq M$, $\lim_{\rightarrow} M'_i \simeq M'$, and a morphism of inductive systems $(M_i) \rightarrow (M'_i)$ inducing $f$, such that every morphism $M_i \rightarrow M'_i$ is a surjection.

**Proof.** By Proposition 2.6.3 we can find a quasi vector bundle $P$ and a surjection $P \rightarrow M$. Thus, we can assume that $f$ is a natural morphism $P/S \rightarrow P/S'$, where $S \subset S' \subset P$ are subsheaves. Let $S = \bigcup_{i \in I} S_i$ (resp., $S' = \bigcup_{j \in J} S'_j$), where $S_i \subset P$ (resp., $S'_j \subset P$) are holomorphic vector bundles. We can assume that the sets of indices $I$ and $J$ are the same (e.g., replacing both by $I \times J$). Furthermore, replacing $S'_j$ with $S'_j + S_i$ (which is still a subbundle by Lemma 2.6.1) we can assume that $S_i \subset S'_j$. Then $(P/S_i) \rightarrow (P/S'_j)$ is the required morphism of inductive systems. \hfill \Box

**Lemma 3.1.4.** Assume that we have a commutative diagram in $\text{Qcoh}(T)$ of the form

$$\begin{array}{ccc}
M & \xrightarrow{f} & M' \\
\downarrow g & & \downarrow \\
S & \xrightarrow{i} & P
\end{array}$$

where $S$ is a vector bundle, $P$ is a quasi vector bundle of finite rank, $f$ is a surjection, $i$ is an embedding. Then for every $\epsilon > 0$ there exists a vector bundle $Q \subset P$ such that
$S \subset Q$, $\text{rk} Q > \text{rk} P - \epsilon$ and there exists a morphism $Q \to M$ making the following diagram commutative:

![Diagram](image)

**Proof.** We split the proof in two steps.

**Step 1.** Assume that $M$ and $M'$ are vector bundles. Also, without loss of generality we can assume also that $P$ is a vector bundle. Indeed, otherwise we can replace $\epsilon$ with $\epsilon/2$ and $P$ with some bundle $P' \subset P$ such that $S \subset P'$ and $\text{rk} P' > \text{rk} P - \epsilon/2$. Furthermore, replacing $M$ with the fibered product of $M$ and $P$ over $M'$ we can assume that $P = M'$.

Let $N = \ker(f)$. Then we have an exact sequence

$0 \to N \to M/S \to P/S \to 0.$

Using Lemma 1.3.3 we can find a subbundle $Q' \subset P/S$ such that $\text{rk} Q' > \text{rk} P/S - \epsilon$ and $\text{Ext}^1(Q', N) = 0$. Then the pull-back of the above exact sequence to $Q' \subset P/S$ splits. Let $Q' \to M/S$ be a splitting and let $Q \subset P$ be the preimage of $Q'$ in $P$. Since $M$ is the fibered product of $M/S$ and $P$ over $P/S$ we obtain a morphism $Q \to M$ with required properties.

**Step 2.** Now using Lemma 3.1.3 we can find inductive systems of vector bundles $(M_i)$ and $(M'_i)$ and a system of surjections $f_i : M_i \to M'_i$ inducing $f$. Since the functor $\text{Hom}(S, -)$ on $\text{Qcoh}(T)$ commutes with inductive limits, there exists a commutative diagram of the form

![Diagram](image)

inducing our original diagram. It remains to apply Step 1. $\square$

**Lemma 3.1.5.** Let $M$ be a countably generated quasi coherent sheaf of finite rank. Then there exists a quasi vector bundle $P$ of finite rank and a surjection $P \to M$.

**Proof.** Since $M$ is countably generated, there exists a sequence of finitely generated subsheaves $M_1 \subset M_2 \subset \ldots \subset M$ such that $M = \bigcup_{n \geq 1} M_n$. Furthermore, we can choose a sequence of vector bundles $P_1 \subset P_2 \subset \ldots$ and of compatible surjections $f_n : P_n \to M_n$. Let $K_n = \ker f_n$. We are going to choose recursively a sequence of vector bundles $Q_n \subset K_n$ such that $Q_n = Q_{n+1} \cap P_n$ and $\text{rk} K_n - \text{rk} Q_n < n/(n+1)$. For $n = 1$ we choose $Q_1$ to be any subbundle of $K_1$ such that $\text{rk} K_1 - \text{rk} Q_1 < 1/2$. Assume that $Q_n$ is already constructed and let us set $P'_n = P_n/Q_n$, $P'_{n+1} = P_{n+1}/Q_n$, $K'_n = K_n/Q_n \subset P'_n$ and
$K'_{n+1} = K_{n+1}/Q_n \subset P'_{n+1}$. Note that $K'_{n+1} \cap P'_n = K'_n$ and $\text{rk } K'_n < n/(n+1)$. For every $\epsilon > 0$ we can choose a vector bundle $R \subset K'_{n+1} \subset P'_{n+1}$ such that $\text{rk } R > \text{rk } K'_{n+1} - \epsilon$.

Applying Lemma 3.1.4 to the surjection $R \to R/R \cap P'_n$ we find a subbundle $Q'_{n+1} \subset R \cap P'_n$ such that $Q'_{n+1}$ lifts to a subbundle of $R$ and $\text{rk } Q'_{n+1} > \text{rk } R/R \cap P'_n - \epsilon$. Note that $R \cap P'_n \subset K'_n$. Hence, $\text{rk } R/R \cap P'_n > \text{rk } R - n/(n+1)$, and therefore,

$$\text{rk } Q'_{n+1} > \text{rk } R - n/(n+1) - \epsilon.$$ 

Viewing $Q'_{n+1}$ as a subbundle $Q'_{n+1} \subset R \subset K'_{n+1}$ let us define $Q_{n+1}$ as a preimage of $Q'_{n+1}$ in $K_{n+1}$. Since $Q'_{n+1} \cap P'_n = 0$, we obtain $Q_{n+1} \cap P_n = Q_n$. Also,

$$\text{rk } K_{n+1} - \text{rk } Q_{n+1} = \text{rk } K'_{n+1} - \text{rk } Q'_{n+1} < \text{rk } R + \epsilon - \text{rk } Q'_{n+1} < n/(n+1) + 2\epsilon.$$ 

Thus, if we choose $\epsilon$ sufficiently small we will satisfy the condition $\text{rk } K_{n+1} - \text{rk } Q_{n+1} < (n+1)/(n+2)$.

Now let us consider the sequence of vector bundles $\mathcal{T}_1 \subset \mathcal{T}_2 \subset \ldots$, where $\mathcal{T}_n = P_n/Q_n$. Set $\mathcal{T} = \cup_{n \geq 1} \mathcal{T}_n$. By definition $\mathcal{T}$ is a quasi vector bundle. Furthermore,

$$\text{rk } \mathcal{T}_n = \text{rk } P_n - \text{rk } Q_n < \text{rk } P_n - \text{rk } K_n + n/(n+1) = \text{rk } M_n + n/(n+1).$$ 

Hence, $\text{rk } \mathcal{T} \leq \text{rk } M + 1$.  \qed

**Theorem 3.1.6.** (i) A quasicoherent sheaf of finite rank is a projective object of $\text{Qcoh}(\eta_T)$ (resp., $\text{Qcoh}^f(\eta_T)$) iff it is isomorphic to a quasi vector bundle.

(ii) The categories $\text{Qcoh}(\eta_T)$ and $\text{Qcoh}^f(\eta_T)$ have enough projective objects. The cohomological dimension of $\text{Qcoh}^f(\eta_T)$ is at most 1.

**Proof.** (i) First, let us prove that a vector bundle $P$ considered as an object of $\text{Qcoh}(\eta_T)$ is projective. Every surjection in $\text{Qcoh}(\eta_T)$ can be represented by a morphism $f : M \to M'$ in $\text{Qcoh}(T)$ such that $\text{rk } \text{coker}(f) = 0$. We have to show that every morphism from $P$ to $M'$ in $\text{Qcoh}(\eta_T)$ factors through $f$. By definition, every such morphism is given by a morphism $P' \to M'/F$, where $P' \subset P$ and $F \subset M'$ are such that $\text{rk } P/P' = 0$ and $\text{rk } F = 0$. Replacing $M'$ by $M'/F$ we can assume that $F = 0$. Also by Lemma 2.6.2 we can replace $P$ by $P'$. Thus, it suffices to prove that every morphism $P \to M'$ in $\text{Qcoh}(T)$ factors through $f$ in $\text{Qcoh}(\eta_T)$. Let $P' \subset P$ be the preimage of $\text{im}(f) \subset M'$. Then $\text{rk } P/P' = 0$, so replacing $M'$ by $\text{im}(f)$ and $P$ by $P'$ we can assume that $f$ is surjective.

Iterating Lemma 3.1.4 we can construct a sequence of bundles $S_1 \subset S_2 \subset \ldots \subset P$ such that $\text{rk } S_n > \text{rk } P - 1/n$ equipped with a system of compatible liftings of the induced morphisms $S_n \to M'$ to morphisms $S_n \to M$. Indeed, to construct $S_1$ we apply Lemma 3.1.4 with $S = 0$ and $\epsilon = 1$, and set $S_1 = Q$. If $S_n$ is already constructed then we apply Lemma 3.1.4 with $S = S_n$ and $\epsilon = 1/(n+1)$, and set $S_{n+1} = Q$. Let $P' = \cup_n S_n$. Then the induced morphism $P' \to M'$ factors through $f$ and $\text{rk } P/P' = 0$, so we are done.

Now if $P$ is any quasi vector bundle of finite rank then we can choose a sequence of vector bundles $P_1 \subset P_2 \subset \ldots \subset P$ such that $\lim \text{rk } P_n = \text{rk } P$. Note that $\cup P_n \simeq P$ in $\text{Qcoh}(\eta_T)$, so we can assume that $P = \cup P_n$. As we have seen above, every $P_i$ is a projective object in $\text{Qcoh}(\eta_T)$. Therefore, by Lemma 2.9.1 $P$ is also projective.

Conversely, assume that $M$ is a quasicoherent sheaf of finite rank which is a projective object of $\text{Qcoh}^f(\eta_T)$. Without loss of generality we can assume that $M$ has no torsion. Also, we can choose a sequence of finitely generated subsheaves $M_1 \subset M_2 \subset \ldots \subset M$ such
that \( \lim \operatorname{rk} M_n = \operatorname{rk} M \). Replacing \( M \) by \( \bigcup_{n \geq 1} M_n \) we can assume that \( M \) is countable generated. Then by Lemma 3.1.5 we can find a surjection \( P \to M \) in \( \operatorname{Qcoh}(T) \), where \( P \) is a quasi vector bundle of finite rank. Now our assumption implies that there exists a splitting \( M \to P \) in \( \operatorname{Qcoh}(\eta_T) \). Since \( P \) has no torsion, this splitting is given by a morphism \( f : M' \to P \) in \( \operatorname{Qcoh}(T) \), where \( M' \subset M \) is such that \( \operatorname{rk} M/M' = 0 \). Shrinking \( M' \) if necessary we can assume that the composition \( \pi \circ f : M' \to M \) coincides with the embedding of \( M' \) into \( M \). Hence, \( f \) is an embedding. By Lemma 2.6.2 this implies that \( M' \) is a quasi vector bundle. It remains to observe that \( M' \simeq M \) in \( \operatorname{Qcoh}(\eta_T) \).

(ii) Let \( M \) be any quasicoherent sheaf. Then we can find a collection of vector bundles \( (P_i) \) and a surjection \( P = \bigoplus P_i \to M \). Note that each \( P_i \) is a projective object in \( \operatorname{Qcoh}(\eta_T) \), hence \( P \) is also projective. This shows that \( \operatorname{Qcoh}(\eta_T) \) has enough projective objects. The similar statement for the subcategory \( \operatorname{Qcoh}^f(\eta_T) \) follows from Lemma 3.1.5. Now the fact that the cohomological dimension of \( \operatorname{Qcoh}^f(\eta_T) \) is \( \leq 1 \) follows easily from Lemma 2.6.2 and from part (i).

3.2. Isomorphisms in \( \operatorname{Qcoh}(\eta_T) \).

**Lemma 3.2.1.** Let \( P \) and \( P' \) be holomorphic vector bundles such that \( \operatorname{rk} P = \operatorname{rk} P' \). Then \( P \simeq P' \) in \( \operatorname{Qcoh}(\eta_T) \).

**Proof.** By Theorem 1.4.2 it is enough to prove that deformation equivalent bundles become isomorphic in \( \operatorname{Qcoh}(\eta_T) \). Note that Theorem 3.1.6 implies that every exact triple of vector bundles splits in \( \operatorname{Qcoh}(\eta_T) \). Hence, it is enough to prove the assertion in the case when \( P_1 \) and \( P_2 \) are stable. Using Morita equivalences we can reduce to the case when \( \operatorname{rk} P_1 = \operatorname{rk} P_2 = 1 \). Let us apply the construction of Theorem 1.3.1 to get a family \((V_{a,b})\) (resp., \((V'_{a,b})\)) of stable subquotients of \( P \) (resp., \( P' \)) numbered by the subsegments \([a,b]\) of the division process described in section 1.2. Let also \( V_{0,a} \subset P \) (resp., \( V'_{0,a} \subset P' \)) be the corresponding subbundles numbered by \( a \in \mathcal{B}_\theta \). We can choose these families in such a way that \( V_{a,b} \simeq V'_{a,b} \) for all \([a,b]\) appearing in the division process. Since all exact sequences of vector bundles split in \( \operatorname{Qcoh}(\eta_T) \), it follows that \( V_{0,a} \simeq V'_{0,a} \) in \( \operatorname{Qcoh}(\eta_T) \). Hence, we get an isomorphism

\[
P \simeq \bigcup_{a \in \mathcal{B}_\theta} V_{0,a} \simeq \bigcup_{a \in \mathcal{B}_\theta} V'_{0,a} \simeq P'
\]

in \( \operatorname{Qcoh}(\eta_T) \).

**Lemma 3.2.2.** Let \( V \) be a vector bundle. Then there exists an element \( v \in (\mathbb{Z} + \mathbb{Z}\theta)_{>0} \) such that for every \( r \in \mathbb{Z} + \mathbb{Z}\theta \) such that \( 0 \leq r \leq \operatorname{rk} V \) and \( \chi(r, v) \geq 0 \), there exists a subbundle \( W \subset V \) with \( \operatorname{rk} W = r \). Furthermore, if \( V \) is semistable then we can take \( v = \operatorname{rk} V \).

**Proof.** If \( V \) is stable then the assertion follows from Theorem 1.3.1. In the general case let \( 0 \subset F_1 V \subset F_2 V \subset \ldots \subset F_n V = V \) be a filtration such that the bundles \( F_i V/F_{i-1} V \) are stable and \( \mu(F_1 V) \geq \mu(F_2 V/F_1 V) \geq \ldots \geq \mu(F_n V/F_{n-1} V) \). Set \( v_i = \operatorname{rk}(F_i V/F_{i-1} V) \). We claim that we can take \( v = v_n \). Indeed, for every \( r \) between 0 and \( \operatorname{rk} V \) there exists \( i, 1 \leq i \leq n \), such that \( v_1 + \ldots + v_{i-1} \leq r \leq v_1 + \ldots + v_i \). Set \( r' = r - (v_1 + \ldots + v_{i-1}) \).
Since \( \chi(v_j, v_i) \leq 0 \) for \( j < i \) we have
\[
\chi(r', v_i) = \chi(r, v_i) - \sum_{j=1}^{i-1} \chi(v_j, v_i) \geq 0.
\]
Hence there exists a subbundle \( W' \subset F_1 V/F_{i-1} V \) with \( \text{rk} W' = r' \). It remains to take \( W \) to be the preimage of \( W' \) in \( F_1 V \subset V \).

\[\blacksquare\]

**Lemma 3.2.3.** Let \( V_1 \) and \( V_2 \) be vector bundles. Then for every \( \epsilon > 0 \) there exist subbundles \( W_1 \subset V_1 \) and \( W_2 \subset V_2 \) such that
\[
\text{rk} W_1 = \text{rk} W_2 > \min(\text{rk} V_1, \text{rk} V_2) - \epsilon.
\]

**Proof.** Indeed, let \( v_1 \) and \( v_2 \) be elements of \((\mathbb{Z} + \mathbb{Z} \theta)_{>0}\) chosen as in Lemma 3.2.2 for \( V_1 \) and \( V_2 \), respectively. Without loss of generality we can assume that \( \chi(v_1, v_2) \geq 0 \). Then for every \( r \in \mathbb{Z} + \mathbb{Z} \theta \) such that \( 0 \leq r \leq \min(\text{rk} V_1, \text{rk} V_2) \) and \( \chi(r, v_1) \geq 0 \), there exists subbundles \( W_1 \subset V_1 \) and \( W_2 \subset V_2 \) with \( \text{rk} W_1 = \text{rk} W_2 = r \). Since the set of such \( r \) is dense in the interval \([0, \min(\text{rk} V_1, \text{rk} V_2)]\) the assertion follows.

\[\blacksquare\]

**Theorem 3.2.4.** Let \( P \) and \( P' \) be quasi vector bundles of finite ranks on \( T \) such that \( \text{rk} P = \text{rk} P' \). Then \( P \simeq P' \) in \( \text{Qcoh}(\eta_T) \).

**Proof.** Let \( r = \text{rk} P = \text{rk} P' \). First of all, by definition of the rank and by Lemma 2.6.1, for every \( \epsilon > 0 \) there exist embeddings \( V \subset P \) and \( V' \subset P' \) in \( \text{Qcoh}(T) \), where \( V \) and \( V' \) are vector bundles and of rank \( > r - \epsilon \). Applying Lemma 3.2.3 we find subbundles \( W \subset V \) and \( W' \subset V' \) such that \( \text{rk} W = \text{rk} W' > r - 2\epsilon \). Since \( V/W \) and \( V'/W' \) are again quasi vector bundles by Lemma 2.6.4, we can apply the same procedure to \( V/W \) and \( V'/W' \) again, and so on. Taking \( \epsilon = 1/2n \) at the \( n \)-th step, we will construct in this way a sequence of subbundles \( 0 = W_0 \subset W_1 \subset W_2 \subset \ldots \subset P \) (resp., \( 0 = W'_0 \subset W'_1 \subset W'_2 \subset \ldots \subset P' \) such that \( \text{rk} W_n = \text{rk} W'_n > r - 1/n \). It follows that \( \text{rk} \bigcup_n W_n = \text{rk} \bigcup_n W'_n = r \). Hence, \( P \simeq \bigcup_n W_n \) and \( P' \simeq \bigcup_n W'_n \) in \( \text{Qcoh}(\eta_T) \). Now applying Theorem 3.1.6(i) we derive that \( \bigcup_n W_n \simeq \oplus_{n \geq 1} W_n/W_{n-1} \) (resp., \( \bigcup_n W'_n \simeq \oplus_{n \geq 1} W'_n/W'_n-1 \)) in \( \text{Qcoh}(\eta_T) \). It remains to observe that \( W_n/W_{n-1} \simeq W'_n/W'_n-1 \) in \( \text{Qcoh}(\eta_T) \) for all \( n \geq 1 \) by Lemma 3.2.1.

\[\blacksquare\]

**Corollary 3.2.5.** Projective objects in \( \text{Qcoh}^f(\eta_T) \) are determined up to an isomorphism by their rank. In other words, for a pair of projective objects \( P, P' \in \text{Qcoh}^f(\eta_T) \) one has \( P \simeq P' \) iff \( \text{rk} P = \text{rk} P' \).

**Proof.** Combine Theorem 3.1.6(i) and Theorem 3.2.4.

\[\blacksquare\]

**Corollary 3.2.6.** One has \( K_0(\text{Qcoh}^f(\eta_T)) \simeq \mathbb{R} \) and the effective cone is exactly \( \mathbb{R}_{>0} \subset \mathbb{R} \).

3.3. Equivalences with categories of modules. Let \( P \) be a quasi vector bundle of finite rank and let \( R_P = \text{End}_{\text{Qcoh}(\eta_T)}(P) \). By Theorem 3.1.6 the corresponding functor
\[
\Gamma_P : \text{Qcoh}(\eta_T) \to \text{mod} -R_P : M \mapsto \text{Hom}_{\text{Qcoh}(\eta_T)}(P, M)
\]
is exact. Below we are going to study properties of the ring \( R_P \) and of the functor \( \Gamma_P \).

Recall that a ring \( R \) is called \textit{right semihereditary} if every finitely generated right ideal in \( R \) is projective.
Proposition 3.3.1. For every quasi vector bundle $P$ of finite rank the ring $R_P$ is right semihereditary.

Proof. A finitely generated ideal $I \subset R_P$ is the image of a morphism of $R_P$-modules $R_P^{\oplus n} \to R_P$. Such a morphism is the image under $\Gamma_P$ of a morphism $f : P^{\oplus m} \to P$ in $\text{Qcoh}(\eta_T)$. By Theorem 3.1.6 the projective dimension of $\text{coker}(f)$ is $\leq 1$, hence, $\text{im}(f)$ is projective. It follows that $\text{im}(f)$ is a direct summand of $P^{\oplus n}$. Therefore, $I \simeq \Gamma_P(\text{im}(f))$ is a direct summand of $R_P^{\oplus n}$. □

Lemma 3.3.2. Let $P$ be a projective object in an abelian category $A$, and let $\langle P \rangle \subset A$ denote the full subcategory consisting of objects that can be presented as the cokernel of a morphism of the form $P^{\oplus m} \to P^{\oplus n}$. Then the functor $\Gamma_P : X \mapsto \text{Hom}_A(P, X)$ induces an equivalence of $\langle P \rangle$ with the category $\text{mod}^{fp}_R - R_P$ of finitely presented right modules over $R_P$.

Proof. First, let us construct a functor $F : \text{mod}^{fp}_R - R_P \to \text{Qcoh}(\eta_T)$. For a finitely presented module $M$ we define $F(M)$ as an object representing the functor $X \mapsto \text{Hom}_{R_P}(M, \Gamma_P(X))$. To see that such an object exists we represent $M$ as the cokernel of a morphism of $R_P$-modules $R_P^{\oplus m} \to R_P^{\oplus n}$. Every such a morphism comes from a morphism $f : P^{\oplus m} \to P^{\oplus n}$ and one can easily see that we can take $F(M) = \text{coker}(f)$. It is clear from this construction that the image of $F$ is contained in $\langle P \rangle$ and that $\Gamma_P(F(M)) \simeq M$ for every finitely presented $R_P$-module $M$. This implies that

$$\text{Hom}_{R_P}(M, M') \simeq \text{Hom}_{R_P}(M, \Gamma_P(F(M'))) \simeq \text{Hom}_{\text{Qcoh}(\eta_T)}(F(M), F(M')),$$

so $F$ is an equivalence of $\text{mod}^{fp}_R - R_P$ with the full subcategory of $A$. It is clear that the essential image of $F$ is $\langle P \rangle$. □

Now we can prove our main result about the category $\text{Qcoh}^f(\eta_T)$.

Theorem 3.3.3. For every quasi vector bundle $P$ of finite rank the functor (3.3.1) induces an equivalence of $\text{Qcoh}^f(\eta_T)$ with the category of finitely presented right modules over $R_P$.

This theorem is an immediate consequence of Lemma 3.3.2 and of the following result.

Proposition 3.3.4. For every quasi vector bundle $P$ of finite rank the subcategory $\langle P \rangle \subset \text{Qcoh}(\eta_T)$ coincides with $\text{Qcoh}^f(\eta_T)$.

Proof. It is clear that $\langle P \rangle \subset \text{Qcoh}^f(\eta_T)$. Note also that $\langle P \rangle$ is closed under direct sums and under passing to direct summands. Let $Q$ be any other quasi vector bundle of finite rank. Pick a sufficiently large number $N$ such that $N \text{rk} P > \text{rk} Q$ and a quasi vector bundle $R$ of rank $N \text{rk} P - \text{rk} Q$. By Theorem 3.2.4 there exists an isomorphism

$$P^{\oplus N} \simeq Q \oplus R$$

in $\text{Qcoh}(\eta_T)$. Hence, every quasi vector bundle of finite rank is contained in $\langle P \rangle$. Since $\langle P \rangle$ is closed under taking cokernels, from Theorem 3.1.6 we get that $\langle P \rangle = \text{Qcoh}^f(\eta_T)$. □
Remarks. 1. We do not know whether $R_P$ is actually von Neumann regular, i.e., whether every finitely generated right ideal in it is a direct summand. An equivalent question is whether the category $\text{Qcoh}^f(\eta_T)$ is semisimple. Yet another reformulation of this question is whether every quasicoherent sheaf of finite rank is isomorphic to a quasi vector bundle in $\text{Qcoh}(\eta_T)$ (by Theorem 3.1.6(i)).

2. It is not true that $P$ is a generator of $\text{Qcoh}(\eta_T)$ (even if it is a vector bundle). More precisely, we claim that

$$\text{Hom}_{\text{Qcoh}(\eta_T)}(P, \bigoplus_{n=1}^{\infty} P) \neq \bigoplus_{n=1}^{\infty} \text{Hom}_{\text{Qcoh}(\eta_T)}(P, P).$$

Indeed, using Lemma 1.3.3 it is easy to construct a collection of nonzero subbundles $P_n \subset P$ such that we have an embedding $\bigoplus_{n=1}^{\infty} P_n \subset P$ and $\sum_{n=1}^{\infty} \text{rk} P_n = \text{rk} P$. Therefore, we obtain a direct sum decomposition in $\text{Qcoh}(\eta_T)$

$$P \simeq \bigoplus_{n=1}^{\infty} P_n.$$ 

Hence,

$$\text{Hom}_{\text{Qcoh}(\eta_T)}(P, \bigoplus_{n=1}^{\infty} P) \simeq \text{Hom}_{\text{Qcoh}(\eta_T)}(\bigoplus_{n=1}^{\infty} P_n, \bigoplus_{n=1}^{\infty} P).$$

Taking an element in this space that induces an embedding of $P_n$ into the $n$-th summand $P$, one can easily derive our claim.

References

[1] M. Bökstedt, A. Neeman, Homotopy limits in triangulated categories, Compositio Math. 86 (1993), 209–234.
[2] A. Bondal, M. Van den Bergh, Generators and representability of functors in commutative and noncommutative geometry, Moscow Math. J. 3 (2003), 1–36.
[3] T. Bridgeland, Stability conditions on triangulated categories, preprint math.AG/0212237.
[4] A. Grothendieck, Sur quelques points d’algèbre homologique, Tohoku Math. J. (2) 9 (1957), 119–221.
[5] A. Grothendieck, J. Dieudonné, Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I, Inst. Hautes Études Sci. Publ. Math. 11 (1961), 5–167.
[6] A. Grothendieck, J. L. Verdier, Préfaisceaux, exp. I in Théorie de Topos et Cohomologie Étale des Schémas (SGA4), Lecture Notes in Math. 269, 1–184, Springer-Verlag, 1972.
[7] D. Happel, I. Reiten, S. O. Smalø, Tilting in abelian categories and quasitilted algebras, Memoirs AMS 575, 1996.
[8] A. Polishchuk, Noncommutative two-tori with real multiplication as noncommutative projective varieties, Journal of Geometry and Physics 50 (2004), 162–187.
[9] A. Polishchuk, Classification of holomorphic bundles on noncommutative two-tori, Documenta Math. 9 (2004), 163–181.
[10] A. Polishchuk, A. Schwarz, Categories of holomorphic bundles on noncommutative two-tori, Comm. Math. Phys. 236 (2003), 135–159.
[11] P. Seidel, R. Thomas, Braid group actions on derived categories of coherent sheaves, Duke Math. J. 108 (2001), 37–108.