ON THE GEOMETRY OF \( SL(2) \)-EQUIVARIANT FLIPS

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Dedicated to Ernest Borisovich Vinberg on the occasion of his 70th birthday

Abstract. In this paper, we show that any 3-dimensional normal affine quasihomogeneous \( SL(2) \)-variety can be described as a categorical quotient of a 4-dimensional affine hypersurface. Moreover, we show that the Cox ring of an arbitrary 3-dimensional normal affine quasihomogeneous \( SL(2) \)-variety has a unique defining equation. This allows us to construct \( SL(2) \)-equivariant flips by different GIT-quotients of hypersurfaces. Using the theory of spherical varieties, we describe \( SL(2) \)-flips by means of 2-dimensional colored cones.

Introduction

Let \( X, X^- \) and \( X^+ \) be normal quasiprojective 3-dimensional algebraic varieties over \( \mathbb{C} \). A flip is a diagram

\[
\begin{array}{ccc}
X^- & \xrightarrow{\varphi^-} & X^+ \\
\downarrow & & \downarrow \\
X & & X \\
\uparrow & & \uparrow \\
& \xrightarrow{\varphi^+} & 
\end{array}
\]

in which \( X^- \) and \( X^+ \) are \( \mathbb{Q} \)-factorial and \( \varphi^- : X^- \to X, \varphi^+ : X^+ \to X \) are projective birational morphisms contracting finitely many rational curves to an isolated singular point \( p \in X \). Moreover, the anticanonical divisor \( -K_{X^-} \) and the canonical divisor \( K_{X^+} \) are relatively ample over \( X \). In this case, the singularity at \( p \) is not \( \mathbb{Q} \)-factorial (even not \( \mathbb{Q} \)-Gorenstein). A good understanding of flips is important from the point of view of 3-dimensional birational geometry (see e.g. [CKM88], or [KM98]). We remark that if \( X \) is affine, then the quasiprojective varieties \( X^- \) and \( X^+ \) can be obtained from \( X \) as follows:

\[
X^- := \text{Proj} \bigoplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(-nK_X)), \quad X^+ := \text{Proj} \bigoplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(nK_X)).
\]

Simplest examples of flips can be constructed from 3-dimensional affine toric varieties \( X_\sigma \) which are categorical quotients of \( \mathbb{C}^4 \) modulo \( \mathbb{C}^* \)-actions by diagonal matrices

\[
\text{diag}(t^{-n_1}, t^{-n_2}, t^{n_3}, t^{n_4}), \quad t \in \mathbb{C}^*
\]

where \( n_1, n_2, n_3, n_4 \) are positive integers satisfying the condition \( n_1 + n_2 < n_3 + n_4 \) [Da83, R83]. In this case, the quasiprojective toric varieties \( X^-_\sigma \) and \( X^+_\sigma \) correspond

\[\text{All results of our paper are valid for algebraic varieties defined over an arbitrary algebraically closed field } K \text{ of characteristic 0, but for simplicity we consider only the case } K = \mathbb{C}.\]
to two different simplicial subdivisions of a 3-dimensional cone \( \sigma \) generated by 4 lattice vectors \( v_1, v_2, v_3, v_4 \) satisfying the relation \( n_1 v_1 + n_2 v_2 = n_3 v_3 + n_4 v_4 \).

Another point of view on flips comes from the Geometric Invariant Theory (GIT) which describes a flip diagram as

\[
\begin{array}{ccc}
Y^{ss}(L^-)/G & \overset{\varphi^-}{\longrightarrow} & Y^{ss}(0)/G \\
\downarrow & & \downarrow \\
Y^{ss}(L^+)/G & \overset{\varphi^+}{\longrightarrow} & Y^{ss}(L^+)/G
\end{array}
\]

for some three \( G \)-linearized ample line bundles \( L^-, L^+, L^0 \) on a 4-dimensional variety \( Y \) (see e.g. [Th96]). Here

\[
Y^{ss}(L) := \{ y \in Y : s(y) \neq 0 \text{ for some } s \in \Gamma(Y, L^n)^G \text{ and for some } n > 0 \}
\]
denotes the subset of semistable points in \( Y \) with respect to the \( G \)-linearized ample line bundle \( L \) and \( Y^{ss}(L)/G \) denotes the categorical quotient which can be identified with

\[
\text{Proj} \bigoplus_{n \geq 0} \Gamma(Y, L^n)^G.
\]

In the above toric case, we have \( Y = \mathbb{C}^4, G \cong \mathbb{C}^* \) and \( L^-, L^+, L^0 \) are different \( G \)-linearizations of the trivial line bundle over \( \mathbb{C}^4 \). A classification of 3-dimensional flips in case when \( Y \subset \mathbb{C}^5 \) is hypersurface and both varieties \( Y^{ss}(L^-)/G \) and \( Y^{ss}(L^+)/G \) have at worst terminal singularities was considered by Brown in [Br99].

The purpose of this paper is to investigate another class of quasihomogeneous varieties. We give a geometric description of \( SL(2) \)-equivariant flips in the case when \( X \) is an arbitrary singular normal affine quasihomogeneous \( SL(2) \)-variety.

It follows from a recently result of Gaifullin [Ga08] that a 3-dimensional toric variety \( X_\sigma \) associated with a 3-dimensional cone \( \sigma = \mathbb{R}_{\geq 0} v_1 + \cdots + \mathbb{R}_{\geq 0} v_4 \) is quasi-homogeneous with respect to a \( SL(2) \)-action if and only if \( n_1 = n_2 \) and \( n_3 = n_4 \). One of such toric \( SL(2) \)-equivariant flips \((n_1 = n_2 = 1, n_3 = n_4 = n > 1)\) was described in detail in [KM98, Example 2.7]. However, there exist many normal affine 3-dimensional quasihomogeneous \( SL(2) \)-varieties which are not toric.

According to Popov [P73], every normal affine quasihomogeneous \( SL(2) \)-variety \( E \) is uniquely determined by a pair of numbers \((h,m) \in \{ \mathbb{Q} \cap (0,1) \} \times \mathbb{N} \) (we denote this variety by \( E_{h,m} \)). Let \( h = p/q \leq 1 \) (\( g.c.d. (p,q) = 1 \)). We define \( k := g.c.d. (q - p, m) \),

\[
a := \frac{m}{k}, \quad b := \frac{(q - p)}{k}.
\]

In Section 1 we show that the affine \( SL(2) \)-variety \( E_{h,m} \) is isomorphic to the categorical quotient of the hypersurface \( H_b \subset \mathbb{C}^5 \) defined by the equation

\[
Y^b_0 = X_1X_4 - X_2X_3,
\]

modulo the action of the diagonalizable group \( G \cong \mathbb{C}^* \times \mu_a \), where \( \mathbb{C}^* \) acts by

\[
diag(t^a, t^{-p}, t^{-p}, t^q, t^q), \quad t \in \mathbb{C}^* \]
and \( \mu_a = (\zeta_a) \), \( \zeta_a = e^{2\pi i/a} \) acts by \( \text{diag}(1, \zeta_a^{-1}, \zeta_a^{-1}, \zeta_a, \zeta_a) \). Here we consider the \( SL(2) \)-action on \( H_b \) induced by the trivial action on the coordinate \( Y_0 \) and by left multiplication on the coordinates \( X_1, X_2, X_3, X_4 \):

\[
\left( \begin{array}{c} X \\ Z \\ W \end{array} \right) \cdot \left( \begin{array}{cccc} X_1 & X_3 \\ X_2 & X_4 \end{array} \right) \mapsto \left( \begin{array}{c} X \\ Z \\ W \end{array} \right) \cdot \left( \begin{array}{cccc} X_1 & X_3 \\ X_2 & X_4 \end{array} \right), \quad \left( \begin{array}{c} X \\ Z \\ W \end{array} \right) \in SL(2).
\]

This \( SL(2) \)-action commutes with the \( G \)-action and descends to the categorial quotient \( H_b \!/ G \cong E_{h, m} \). In this way, we obtain a very simple description of the affine \( SL(2) \)-variety \( E_{h, m} \) which seems to be overlooked in the literature.

In Section 2 we consider the notion of the total coordinate ring (or Cox ring) of an algebraic variety \( X \) with a finitely generated divisor class group \( \text{Cl}(X) \) (see e.g. [Ar08, H08]). These rings naturally appear in some questions related to Del Pezzo surfaces and homogeneous spaces of algebraic groups (see [BP04]). Using results from Section 1, we show that the Cox ring of \( E_{h, m} \) is isomorphic to

\[
\mathbb{C}[Y_0, X_1, X_2, X_3, X_4]/(Y_0 - X_1 X_4 + X_2 X_3).
\]

Some similar examples of algebraic varieties whose Cox ring is defined by a unique equation were considered in [BH07]. We remark that our result provides an alternative proof of a criterion of Gaifullin [Ga08]: \( E_{h, m} \) is toric if and only if \( b = 1 \), or if only if \( (q - p) \) divides \( m \). One can use our description of the total coordinate ring of \( E_{h, m} \) as a good illustration of more general resent results of Brion on the total coordinate ring of spherical varieties [B07].

In Section 3 we describe the quasiprojective varieties \( E_{h, m} \), \( E_{h, m}^+ \), \( E_{h, m}^- \) in the \( SL(2) \)-equivariant flip diagram

\[
E_{h, m}^- = H_b^{ss}(L^-) \!/ G \rightarrow E_{h, m}^+ = H_b^{ss}(L^+) \!/ G
\]

by different GIT-quotients of the hypersurface \( H_b \), where \( L^0, L^-, L^+ \) are linearizations of the trivial line bundle over \( H_b \) corresponding to the trivial character \( \chi^0 \) and some nontrivial characters \( \chi^-, \chi^+ \) of \( G \). In fact, we can say something more about \( E_{h, m}^- \) and \( E_{h, m}^+ \): there exist two affine normal toric surfaces \( S^- \subset E_{h, m} \) and \( S^+ \subset E_{h, m} \) which are closures in \( E_{h, m} \) of two orbits of a Borel subgroup \( B \subset SL(2) \) such that

\[
E_{h, m}^- \cong SL(2) \times_B S^-, \quad E_{h, m}^+ \cong SL(2) \times_B S^+.
\]

In particular, both varieties \( E_{h, m}^+ \) and \( E_{h, m}^- \) have at worst log-terminal toroidal singularities. The birational \( SL(2) \)-equivariant morphism \( f^\pm \) is the contraction of the unique 1-dimensional \( SL(2) \)-orbits \( C^\pm \subset E_{h, m}^\pm \) (\( C^+ \cong \mathbb{P}^1 \)) into the unique \( SL(2) \)-fixed singular point \( O \in E_{h, m} \). We remark that the \( \mathbb{Q} \)-factorial \( SL(2) \)-variety \( E_{h, m}^+ \cong SL(2) \times_B S^+ \) was first constructed and investigated by Panyushev in [Pa88, Pa91].
In Section 4 we consider $SL(2)$-equivariant flips from the point of view of the theory of spherical varieties developed by Luna and Vust [LV83] (see also [K91]). It is easy to see that any affine $SL(2)$-variety $E_{h,m}$ admits an additional $\mathbb{C}^*$-action which commutes with the $SL(2)$-action. If $H \subset SL(2) \times \mathbb{C}^*$ is the stabilizer subgroup of a generic point $x \in E_{h,m}$, then $(SL(2) \times \mathbb{C}^*)/H$ is a spherical homogeneous space and $E_{h,m}$ is a spherical embedding. We describe simple spherical varieties $E_{h,m}, E_{-h,m}$, and $E_{h,m}^+$ by colored cones. Thus, any $SL(2)$-equivariant flip provides an illustration of general results on Mori theory for spherical varieties due to Brion [B93], Brion-Knop [BK94]. According to Alexeev and Brion [AB04], any spherical variety $V$ admits a flat degeneration to a toric variety $V'$. We apply this fact to spherical varieties $E_{h,m}, E_{-h,m},$ and $E_{h,m}^+$ and investigate the corresponding degenerations of $SL(2)$-equivariant flips to toric flips. We remark that the idea of toric degenerations appeared already in earlier papers of Popov [P87] and Vinberg [V95]. Toric degenerations of affine spherical varieties (including $SL(2)$-varieties) was considered by Arzhantsev in [Ar99].

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1. Affine $SL(2)$-varieties as categorical quotients

The complete classification of normal affine quasihomogeneous $SL(2)$-varieties has been obtained by Popov [P73]. A shorter modern presentation of this classification is contained in the book of Kraft [Kr84, III.4]:

**Theorem 1.1.** [P73] Every 3-dimensional normal affine quasihomogeneous $SL(2)$-variety containing more than 1 orbit is uniquely determined by a pair of numbers $(h,m) \in \mathbb{Q} \cap (0,1] \times \mathbb{N}$. We denote the corresponding variety by $E_{h,m}$. The number $h$ is called height of $E_{h,m}$. The number $m$ is called degree of $E_{h,m}$ and it equals the order of the stabilizer of a point in the open dense $SL(2)$-orbit $U \subset E_{h,m}$ (this stabilizer is always a cyclic group).

Let $\mu_n = \langle \zeta_n \rangle$ be the cyclic group of $n$-th roots of unity. We denote a cyclic group of order $n$ also by $C_n$. We use the following notations for some closed subgroups in $SL(2)$:

$$T := \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in \mathbb{C}^* \right\}, \quad B := \left\{ \begin{pmatrix} t & u \\ 0 & t^{-1} \end{pmatrix} : t \in \mathbb{C}^*, u \in \mathbb{C} \right\},$$

$$U_n := \left\{ \begin{pmatrix} \xi & u \\ 0 & \xi^{-1} \end{pmatrix} : u \in \mathbb{C}, \xi^n = 1 \right\}, \quad U := \left\{ \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} : u \in \mathbb{C} \right\}.$$

**Remark 1.2.** If $h = 1$, then $E_{1,m}$ is smooth and it contains two $SL(2)$-orbits:

$$U \cong SL(2)/C_m$$

and $D \cong SL(2)/T$.

The geometric description of $E_{1,m}$ is easy and well-known [Kr84, III, 4.5]:

$$E_{1,m} \cong SL(2) \times_{T} \mathbb{C},$$
where the torus $T$ acts on $\mathbb{C}$ by character $\chi_m : t \to t^m$. So $E_{1,m}$ can be considered as a line bundle over $SL(2)/T$.

**Remark 1.3.** If $0 < h < 1$, then $E_{h,m}$ contains a unique $SL(2)$-invariant singular point $O$. We write $h = p/q$ where $g.c.d.(p, q) = 1$ and define $k := g.c.d.(m, q - p)$, $a := m/k$. Then $E_{h,m}$ contains three $SL(2)$-orbits:

$$U \cong SL(2)/C_m, \ D \cong SL(2)/U_{a(p+q)}, \text{ and } \{O\}.$$

The explicit construction of $E_{h,m}$ given in [P73] and [Kr84] involves finding a system of generators of the following semigroup

$$M_{h,m}^+ := \{(i, j) \in \mathbb{Z}_2^+ : j \leq hi, \ m|(i - j)\}.$$

Let $V_n$ be the standard $(n + 1)$-dimensional irreducible representation of $SL(2)$ in the space of binary forms of degree $n$. Denote by $(i_1, j_1), \ldots, (i_r, j_r)$ a system of generators of the semigroup $M_{h,m}^+$. Then $E_{h,m}$ is isomorphic to the closure $SL(2)v$ of the $SL(2)$-orbit of the vector

$$v := (X^{i_1}Y^{j_1}, \ldots, X^{i_r}Y^{j_r}) \in V_{i_1+j_1} \oplus \cdots \oplus V_{i_r+j_r}.$$

For example, if $m = (q-p)a$, $a \in \mathbb{N}$, then the semigroup $M_{h,m}^+$ is minimally generated by $ap + 1$ elements $(m, 0), (m + 1, 1), (m + 2, 2), \ldots, (aq, ap)$ and

$$v := (X^m, X^{m+1}Y, \ldots, X^{aq}Y^{ap}) \in V_m \oplus V_{m+2} \oplus \cdots \oplus V_{aq+ap} \cong V_{aq} \otimes V_{ap}.$$

![Figure 1](image_url)

**Remark 1.4.** It is easy to see that the numbers $h$ and $m$ are uniquely determined by the embedding of the monoid $M_{h,m}^+$ into $\mathbb{Z}_2^+$ (see Figure 1).

There exists another relation between the submonoid $M_{h,m}^+ \subset \mathbb{Z}_2^+$ and $E_{h,m}$:
Theorem 1.5. [Kr84, III,4.3] Let $E$ be a normal affine 3-dimensional quasihomogeneous $SL(2)$-variety with the affine coordinate ring $\mathbb{C}[E]$. Denote by $\mathbb{C}[E]^U$ the $U$-invariant subring. We can consider $\mathbb{C}[E]^U$ as a subring of $\mathbb{C}[SL(2)]^U \cong \mathbb{C}[X,Y]$, where $\mathbb{C}[X,Y]$ is the algebra of regular functions on $SL(2)/U \cong \mathbb{C}^2 \setminus \{(0,0)\}$. Then the monomials $\{X^iY^j : (i,j) \in M^+_{h,m}\}$ form a $\mathbb{C}$-basis of $\mathbb{C}[E]^U$.

Our next purpose is to give a new description of an affine quasihomogeneous $SL(2)$-variety $E_{h,m}$ as a categorical quotient of a 4-dimensional affine hypersurface. Throughout this paper by a categorical quotient $X/G$ of an irreducible affine algebraic variety $X$ over $\mathbb{C}$ by a reductive group $G$ we mean $\text{Spec} \mathbb{C}[X]^G$ [FM82, VP89].

Let $SL(2) \times \mathbb{C}^5 \rightarrow \mathbb{C}^5$ be the $SL(2)$-action on $\mathbb{C}^5$ considered as $V_0 \oplus V_1 \oplus V_1$. We use the coordinates $X_0, X_1, X_2, X_3, X_4$ on $\mathbb{C}^5$ and identify $X_1, X_2, X_3, X_4$ with the coefficients of the $2 \times 2$-matrix

$$\begin{pmatrix} X_1 & X_3 \\ X_2 & X_4 \end{pmatrix}$$

on which $SL(2)$ acts by left multiplication. Denote by $D(5,\mathbb{C})$ the group of diagonal matrices of order 5 acting on $\mathbb{C}^5$.

One of our main results of this section is the following:

Theorem 1.6. Let $E_{h,m}$ be a normal affine $SL(2)$-variety of height $h = p/q \leq 1$ (g.c.d.$(p,q) = 1$) and of degree $m$. Then $E_{h,m}$ is isomorphic to the categorical quotient of the affine hypersurface $H_{q-p} \subset \mathbb{C}^5$ defined by the equation

$$X_0^{q-p} = X_1X_4 - X_2X_3$$

modulo the action of the diagonalizable group $G_0 \times G_m \subset D(5,\mathbb{C})$, where $G_0 \cong \mathbb{C}^*$ consists of diagonal matrices $\{\text{diag}(t,t^{-p},t^{-p},t^q,t^q) : t \in \mathbb{C}^*\}$ and $G_m \cong \mu_m = \langle \zeta_m \rangle$ is generated by $\text{diag}(1,\zeta_m^{-1},\zeta_m^{-1},\zeta_m,\zeta_m)$.

Proof. Case 1: $h = 1$. Then $p = q = 1$,

$$G_0 := \{\text{diag}(t,t^{-1},t^{-1},t,t) : t \in \mathbb{C}^*\}, \quad G_m := \{\text{diag}(1,\zeta_m^{-1},\zeta_m^{-1},\zeta_m,\zeta_m) : \zeta \in \mu_m\},$$

and the hypersurface $H_0$ is defined by the equation $1 = X_1X_4 - X_2X_3$. The algebraic group $G_0 \times G_m$ can be written as a direct product in another way:

$$G_0 \times G_m = G_0 \times G'_m,$$

where $G'_m := \{\text{diag}(1,1,1,1) : \zeta \in \mu_m\}$. We remark that the hypersurface $H_0$ is isomorphic to the product $SL(2) \times \mathbb{C}$. Moreover, the $G_0$-action on the first factor $SL(2)$ is the same as the action of the maximal torus $T$ by right multiplication. On the other hand, $H_0/G'_m$ is again isomorphic to $SL(2) \times \mathbb{C}$, because $G'_m$ acts trivially on $SL(2)$ and $\mathbb{C}/G'_m \cong \mathbb{C}$ (one replaces the coordinate $X_0$ on $\mathbb{C}$ by a new $G'_m$-invariant coordinate $Y_0 = X_0^m$). So the $G_0$-action on the second factor $\mathbb{C}$ in $SL(2) \times \mathbb{C} \cong H_0/G'_m$ is defined by the character $\chi_m : t \rightarrow t^m$. Thus, we come to the already known description of $E_{1,m}$ as a $T$-quotient: $E_{1,m} \cong SL(2) \times T/\mathbb{C}$ (see [1,2]).
CASE 2: \( m = 1, h = p/q < 1 \). The \( SL(2) \)-action on \( \mathbb{C}^5 \) commutes with the \( G_0 \)-action and the hypersurface \( H_{q-p} \) defined by the equation

\[
X_0^{p-q} = X_1 X_4 - X_2 X_3.
\]

is invariant under this \( G_0 \times SL(2) \)-action. Moreover, the point \( x := (1, 1, 0, 0, 1) \in H_{q-p} \) has a trivial stabilizer in \( G_0 \times SL(2) \). Therefore \( H_{q-p} \) is the closure of the \( G_0 \times SL(2) \)-orbit of \( x \) in \( \mathbb{C}^5 \) and \( X_{p,q} := \text{Spec } \mathbb{C}[H_{q-p}]^{G_0} \) is an affine \( SL(2) \)-embedding. One can identify the open dense \( SL(2) \)-orbit \( U \) in \( X_{p,q} \) with the \( G_0 \)-quotient of the open subset in \( H_{q-p} \) defined by the condition \( X_0 \neq 0 \). Moreover, the affine coordinate ring \( \mathbb{C}[U] \) is generated by the \( G_0 \)-invariant monomials

\[
(1) \quad X := X_0^p X_1, \ Y := X_0^{-q} X_3, \ Z := X_0^p X_2, \ W := X_0^{-q} X_4
\]

satisfying the equation

\[
\det \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = X_0^{p-q} X_1 X_4 - X_0^{p-q} X_2 X_3 = 1.
\]

By a theorem of Luna-Vust [Kr84, III,3.3], the normality of \( X_{p,q} := \text{Spec } \mathbb{C}[H_{q-p}]^{G_0} \) follows from the normality of \( \mathbb{C}[X_{p,q}]^U = \mathbb{C}[H_{q-p}]^{G_0 \times U} \). It is easy to see that

\[
\mathbb{C}[H_{q-p}]^U \cong \mathbb{C}[X_0, X_1, X_3].
\]

Since \( U \)-action and \( G_0 \)-action commute, it remains to compute the \( G_0 \)-invariant subring \( \mathbb{C}[X_0, X_1, X_3]^{G_0} \) under the \( \mathbb{C}^* \)-action of \( G_0 \) on \( \mathbb{C}^3 \) defined by \( \text{diag}(t, t^{-p}, t^q) \). Straightforward calculations show that the ring \( \mathbb{C}[X_0, X_1, X_3]^{G_0} \) has a \( \mathbb{C} \)-basis consisting of all monomials \( X^i Y^j = X_0^{pi-qj} X_1^i X_3^j \in \mathbb{C}[U] \) such that \( pi - qj \geq 0, i \geq 0, j \geq 0 \), i.e., \((i, j) \in M_{p,q}^+\). By [L4] and [L5] we obtain simultaneously that \( X_{p,q} \) is normal and that \( X_{p,q} \cong E_{h,1} \).

CASE 3: \( m > 1, h = p/q < 1 \). Let \( X_{p,q}^m \) be the categorical quotient of \( H_{q-p} \) by \( G_0 \times G_m \) where \( G_m \cong \mu_m = \langle \zeta_m \rangle \) acts by \( \text{diag}(1, \zeta_m^{-1}, \zeta_m, \zeta_m) \). By the same arguments as above, one obtains that \( \mathbb{C}[X_{p,q}]^U \cong \mathbb{C}[X_0, X_1, X_3]^{G_0 \times G_m} \) where \( G_m \cong \mu_m \) acts on \( \mathbb{C}^3 \) by \( \text{diag}(1, \zeta_m^{-1}, \zeta_m) \) and \( G_0 \) on \( \mathbb{C}^3 \) by \( \text{diag}(t, t^{-p}, t^q) \). Therefore the ring \( \mathbb{C}[X_{p,q}]^U \subset \mathbb{C}[U]^U = \mathbb{C}[X, Y] \) has a \( \mathbb{C} \)-basis consisting of all monomials \( X^i Y^j = X_0^{pi-qj} X_1^i X_3^j \) such that \( (i, j) \in M_{h,m}^+ \) (the condition \( m \mid (j - i) \) follows from the \( G_m \)-invariance of monomials \( X_0^{pi-qj} X_1^i X_3^j \)). By [L4] and [L5] this shows that \( X_{p,q}^m \cong E_{h,m} \).

\[ \square \]

It will be important to have the following another similar description of an arbitrary affine normal quasihomogeneous \( SL(2) \)-variety \( E_{h,m} \) as a categorical quotient of an affine hypersurface:

Theorem 1.7. Let \( E_{h,m} \) be a normal affine \( SL(2) \)-variety of height \( h = p/q \leq 1 \) \((g.c.d.(p, q) = 1)\) and of degree \( m \). We define \( b := (q-p)/k \). Then \( E_{h,m} \) is isomorphic to the categorical quotient of the affine hypersurface \( H_b \subset \mathbb{C}^5 \) defined by the equation

\[
Y_0^b = X_1 X_4 - X_2 X_3
\]
modulo the action of the diagonalizable group $G := G_0 \times G_a \subset D(5, \mathbb{C})$, where $G_0 \cong \mathbb{C}^*$ consists of diagonal matrices \{diag$(t^k, t^{-p}, t^{-p}, t^q, t^q') : t \in \mathbb{C}^*$\} and $G_a \cong \mu_a = \langle \zeta_a \rangle$ is generated by $\text{diag}(1, \zeta_a^{-1}, \zeta_a^{-1}, \zeta_a, \zeta_a)$.

**Proof.** By [1.6] we have $E_{h,m} = H_{q-p}/(G_0 \times G_m)$. We note that the conditions $k := g.c.d.(p - m)$ and $g.c.d.(q, p) = 1$ imply that $g.c.d.(k, p) = g.c.d.(k, q) = 1$. Since $\zeta_m^k$ is a generator of $\mu_k$ and since the maps $z \to z^p$ and $z \to z^q$ are bijective on $\mu_k$ we can find another generator $\xi \in \mu_k$ such that $\xi^q \zeta_m^k = \xi^q \zeta_m^k = 1$. Therefore, $G_0 \times G_m$ contains the following element

$$g = \text{diag}(\xi, 1, 1, 1, 1) = (\xi, \xi^{-p}, \xi^{-p}, \xi^q, \xi^q) \cdot (1, \zeta_m^{-1}, \zeta_m^{-1}, \zeta_m, \zeta_m).$$

Consider the homomorphism

$$\psi_k : D(5, \mathbb{C}) \to D(5, \mathbb{C}), \ (\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \mapsto (\lambda_0^k, \lambda_1, \lambda_2, \lambda_3, \lambda_4).$$

Then $\psi_k(G_0) = G'_0$ and

$$G'_k := \text{Ker} \psi_k \cap (G_0 \times G_m) = \langle g \rangle = \{\text{diag}(\xi, 1, 1, 1) : \xi \in \mu_k\}.$$

So we obtain a short exact sequence

$$1 \to G'_k \to G_0 \times G_m \to G'_0 \times G_a \to 1,$$

where

$$G_a = \{\text{diag}(1, \zeta^{-1}, \zeta^{-1}, \zeta, \zeta) : \zeta \in \mu_a\}.$$

Therefore the categorical $G$-quotient of $H_{q-p}$ can be divided in two steps. First we divide $H_{q-p}$ by the subgroup $G'_k \subset G_0 \times G_m$ and after that divide by the group $G'_0 \times G_a$. Using a new $G'_k$-invariant coordinate $Y_0 = X_0^k$, we see that $H_{q-p}/G'_k$ is isomorphic to the hypersurface $H_b$ defined by the equation

$$Y_0^b = X_1X_4 - X_2X_3.$$

Since $G_0$ acts on $Y_0$ by character $t \to t^k$, $E_{h,m} \cong X_{p,q}^m = H_{q-p}/(G_0 \times G_m)$ is isomorphic to the categorical quotient of $H_b$ modulo the above $G'_0 \times G_a$-action. \hfill \Box

### 2. The Cox ring of an affine SL(2)-variety

Let us review a definition of the total coordinate ring (or Cox ring) of a normal algebraic variety $X$ with finitely generated divisor class group $\text{Cl}(X)$ (see e.g. [Ar08, H08]).

**Definition 2.1.** Let $X$ be a normal quasiprojective irreducible algebraic variety over $\mathbb{C}$ with the field of rational functions $\mathbb{C}(X)$. We assume that $\text{Cl}(X)$ is a finitely generated abelian group and that every invertible regular function on $X$ is constant. Choose divisors $D_1, \ldots, D_r$ in $X$ whose classes generate $\text{Cl}(X)$ and principal divisors $D'_1, \ldots, D'_s$ which form a $\mathbb{Z}$-basis of the kernel of the surjective homomorphism $\varphi : \mathbb{Z}^r \to \text{Cl}(X)$. Furthermore, we choose rational functions $f_1, \ldots, f_s \in \mathbb{C}(X)$ such
that $D_i^j = (f_i)$ $(i = 1, \ldots, s)$. For any $k = (k_1, \ldots, k_r) \in \mathbb{Z}^r$, we consider a divisor $D(k) := \sum_{j=1}^r k_j D_j$ and put
$$\mathcal{L}(D(k)) := \{ f \in \mathbb{C}(X) : D(k) + (f) \geq 0 \}.$$ Then for every $i \in \{1, \ldots, s\}$, one has an isomorphism
$$\alpha_i : \mathcal{L}(D(k)) \cong \mathcal{L}(D(k) + D_i^1), \quad \alpha_i(f) = \frac{f}{f_i} \quad \forall f \in \mathcal{L}(D(k)).$$
In the $\mathbb{Z}^r$-graded ring
$$\mathcal{R} := \bigoplus_{k \in \mathbb{Z}^r} \mathcal{L}(D(k)),$$
we consider the ideal $\mathcal{I}$ generated by all elements $f - \alpha_i(f) \forall f \in \mathcal{L}(D(k)), \forall k \in \mathbb{Z}^r$, and $\forall i \in \{1, \ldots, s\}$. The ring
$$\text{Cox}(X) := \mathcal{R}/\mathcal{I}$$
is called Cox ring of $X$ associated with divisors $D_1, \ldots, D_r$ and rational functions $f_1, \ldots, f_s$. By [Ar08, Prop. 3.2], Cox$(X)$ is uniquely defined up to isomorphism and does not depend on the choice of generators $D_1, \ldots, D_r$ of Cl$(X)$ and rational functions $f_1, \ldots, f_s$. Moreover, one has a natural Cl$(X)$-grading
$$\text{Cox}(X) = \mathcal{R}/\mathcal{I} \cong \bigoplus_{c \in \text{Cl}(X)} \Gamma(X, \mathcal{O}_X(c)).$$

Let $A$ be a finitely generated abelian group. We shall need the following criterion for a finitely generated factorial $A$-graded $\mathbb{C}$-algebra $R$ with $R^\times = \mathbb{C}^*$ to be a Cox ring of a normal quasiprojective algebraic variety $X$ with $A \cong \text{Cl}(X)$.

**Theorem 2.2.** Let $Y$ be a normal irreducible affine algebraic variety over $\mathbb{C}$ with a factorial coordinate ring $R = \mathbb{C}[Y]$. We assume that $\Gamma(Y, \mathcal{O}_Y^*) = \mathbb{C}^*$ and that $Y$ admits a regular action $G \times Y \to Y$ of a diagonalizable group $G$, or, equivalently, $R$ admits an $A$-grading by the group $A = \text{Hom}_\text{alg}(G, \mathbb{C}^*)$ of algebraic characters of $G$. Then $R$ is a Cox ring of some normal quasiprojective algebraic variety $X$ such that $\text{Cl}(X) \cong A$ and $\Gamma(X, \mathcal{O}_X^*) = \mathbb{C}^*$ if and only the following conditions are satisfied:

(i) there exists an open dense nonsingular $G$-invariant subset $U \subset Y$ such that $\text{codim}_Y(Y \setminus U) \geq 2$ and $G$ acts freely on $U$;

(ii) there exists a character $\chi \in \text{Hom}_\text{alg}(G, \mathbb{C}^*)$ such that $U \subset Y^{ss}(L)$, where $L$ is the $G$-linearization of the trivial line bundle over $Y$ corresponding to $\chi$.

**Proof.** Assume that $Y$ admits a regular $G$-action such that the conditions (i), (ii) are satisfied. We define $X$ to be $Y^{ss}(L)/G$. Then $X$ is a normal irreducible quasiprojective variety and $\Gamma(X, \mathcal{O}_X^*) = \Gamma(Y, \mathcal{O}_Y^*)^G = \mathbb{C}^*$. Moreover, $\overline{U} := U/G$ is a smooth open subset of $X$. Let us show that Cl$(X) \cong A$, where $A = \text{Hom}_\text{alg}(G, \mathbb{C}^*)$.

Since $R$ is factorial and $U$ is a smooth open subset of $Y$, we have Pic$(U) = \text{Cl}(U) = 0$. By a general result in [KKV89, 5.1], the Picard group of $\overline{U}$ is isomorphic to the group of $G$-linearizations of the trivial line bundle over $U$. On the other hand,
since \( \text{codim}_Y(Y \setminus U) \geq 2 \) and \( Y \) is normal, all invertible regular functions on \( U \) extend to invertible regular functions on \( Y \), i.e., they are constant. By [KKV89], the latter implies that the group of \( G \)-linearizations of the trivial line bundle over \( U \) is isomorphic to the group of characters of \( G \), i.e.,

\[
\text{Pic}(\Upsilon) \cong \text{Hom}_{\text{alg}}(G, \mathbb{C}^*) = A.
\]

Since \( \text{Pic}(\Upsilon) = \text{Cl}(\Upsilon) \), it remains to show that \( \text{codim}_X(X \setminus \Upsilon) \geq 2 \). Assume that there exists an irreducible nonempty divisor \( Z \subset X \) such that \( \Upsilon \cap Z = \emptyset \). Since \( X \) is normal, the local ring \( \mathcal{O}_{X,Z} \) is a discrete valuation ring, i.e., there exists an affine open subset \( U' \subset X \) such that \( Z' := U' \cap Z \neq \emptyset \), \( \Upsilon \cap Z' = \emptyset \), and \( Z' \) is a principle divisor in \( U' \) defined by a regular function \( g \in \mathbb{C}[U'] \). Consider the morphism

\[
\pi : Y^{ss}(L) \to X.
\]

Without loss of generality, we can assume \( \tilde{U}' := \pi^{-1}(U') \) is an affine open subset in \( Y^{ss}(L) \) and \( \mathbb{C}[U'] = \mathbb{C}[\tilde{U}']^G \). Then the element \( \tilde{g} := \pi^*(g) \in \mathbb{C}[\tilde{U}'] \) defines a principle divisor \( \tilde{Z}' := (\tilde{g}) \subset \tilde{U}' \) such that \( \tilde{Z}' \cap \Upsilon = \emptyset \) and \( \tilde{Z}' \neq \emptyset \). The latter contradicts to \( \text{codim}_{\tilde{U}'}(\tilde{U}' \setminus (\tilde{U}' \cap \Upsilon)) \geq \text{codim}_Y(Y \setminus U) \geq 2 \), i.e., we must have \( Z' = \emptyset \).

In order to identify \( R = \bigoplus_{a \in A} R_a \) with the Cox ring of \( X \) we consider a finite subset \( \{a_1, \ldots, a_r\} \subset A \) such that the homogeneous components \( R_{a_1}, \ldots, R_{a_r} \) generate the algebra \( R \) and \( R_{a_i} \neq 0 \) for all \( i \in \{1, \ldots, r\} \). Since the class of any effective divisor in \( X \) is a nonnegative integral linear combination of \( a_1, \ldots, a_r \), we obtain that \( a_1, \ldots, a_r \) are generators of \( A \). We choose \( r \) nonzero elements \( g_j \in R_{a_j}, j \in \{1, \ldots, r\} \) which define \( r \) effective principal divisors \( D_j = (g_j) \) in \( Y \) \( (j \in \{1, \ldots, r\}) \). Then we obtain \( r \) effective divisors in \( X \):

\[
D_j := (\overline{D}_j \cap Y^{ss}(L))//G, \quad j \in \{1, \ldots, r\}.
\]

Consider the epimorphism \( \varphi : \mathbb{Z}^r \to A \). For any \( k = (k_1, \ldots, k_r) \in \mathbb{Z}^r \) we define a rational function

\[
g(k) := g_1^{k_1} \cdots g_r^{k_r} \in \mathbb{C}(Y)
\]

and a divisor

\[
D(k) := k_1D_1 + \cdots + k_rD_r \in \text{Div}(X).
\]

If \( a'_1, \ldots, a'_s \) is a \( \mathbb{Z} \)-basis of \( \text{Ker} \varphi \), then \( s \) rational functions \( f_i := g(a'_i) \) \( (i = 1, \ldots, s) \) are \( G \)-invariant, i.e., elements of \( \mathbb{C}(X) \). So we obtain \( s \) principle divisors \( D'_i := D(a'_i) = (f_i) \) in \( X \). On the other hand, for any \( k \in \mathbb{Z}^r \), one has

\[
\mathcal{L}(D(k)) = \left\{ \frac{h}{g(k)} \in \mathbb{C}(X) : h \in R_{\varphi(k)} \right\}.
\]

Consider the \( \mathbb{Z}^r \)-graded ring

\[
\mathcal{R} := \bigoplus_{k \in \mathbb{Z}^r} \mathcal{L}(D(k))
\]

together with the surjective homogeneous homomorphism

\[
\beta : \mathcal{R} \to R = \bigoplus_{a \in A} R_a
\]
whose restriction to $k$-th homogeneous component is an isomorphism
\[
\beta_k : \mathcal{L}(D(k)) \xrightarrow{\cong} R_{\varphi(k)}
\]
defined by multiplication with $g(k)$. Then the elements
\[
\left( \frac{h}{g(k)} - \frac{h}{g(k + a'_i)} \right) = \left( \frac{h}{g(k)} - \frac{h}{g(k)f_i} \right) \in R_k \oplus R_{k + a'_i},
\]
\[
\forall k \in \mathbb{Z}^r, \forall h \in R_{\varphi(k)} = R_{\varphi(k + a'_i)}, \forall i \in \{1, \ldots, s\}
\]
are contained in $\text{Ker} \beta$. Therefore, $\beta$ induces a surjective homogeneous homomorphism of the Cox ring $\mathcal{R}/\mathcal{I}$ to $\mathcal{R}$. By comparing the homogeneous components of $\mathcal{R}/\mathcal{I}$ and $\mathcal{R}$, we obtain an isomorphism $\mathcal{R}/\mathcal{I} \cong \mathcal{R}$.

Now assume that a factorial $A$-graded $\mathbb{C}$-algebra $\mathcal{R}$ is the Cox ring of some normal irreducible quasiprojective variety $X$ with $\text{Cl}(X) \cong A$. Using the same idea as in 2.1 we can define a sheaf-theoretical version of the Cox ring of $X$ (see [H08, Section 2]):
\[
\tilde{\mathcal{R}} = \bigoplus_{a \in A} \mathcal{O}_X(a)
\]
which is a $A$-graded $\mathcal{O}_X$-algebra such that $\Gamma(X, \tilde{\mathcal{R}}) = \mathcal{R}$. Define $Y' := \text{Spec}_X(\tilde{\mathcal{R}})$ as a relative spectrum over $X$. By [H08 Prop.2.2], $Y' \subset Y := \text{Spec}_C(\mathcal{R})$ is an open embedding and the morphism $\pi : Y' \to X$ is a categorical quotient by the action of $G := \text{Spec} \mathbb{C}[A]$. Moreover, $G$ acts freely on the open subset $U := \pi^{-1}(\overline{U})$, where $\overline{U} := X \setminus \text{Sing}(X) \subset X$ the set of all smooth points of $X$ and $\text{codim}_Y(Y \setminus U) \geq 2$. We consider a locally closed embedding $j : X \to \mathbb{P}^n$ and define $\mathcal{L} := j^*\mathcal{O}(1)$. Since $\text{Cl}(Y') = 0$, the pullback $L := \pi^*\mathcal{L}$ is a trivial line bundle over $Y'$ having a $G$-linearization. Since all invertible global regular functions on $Y'$ are constants, this $G$-linearization is determined by a character $\chi \in \text{Hom}_{\text{alg}}(G, \mathbb{C}^*) \cong A$. Since $\pi : Y' \to X$ is a categorical quotient, we have $U \subset Y' \subset Y^{ss}(L)$. Theorem is proved. \hfill \Box

**Remark 2.3.** Methods in [H08] allow to formulate and prove a more general version of 2.2 for algebraic varieties $X$ which are not necessary quasiprojective. Moreover, in Theorem 2.2 it is enough to assume only $A$-graded factoriality of $\mathcal{R}$, i.e., that every $A$-homogeneous divisorial ideal is principal.

Now we begin with the following observation:

**Proposition 2.4.** The affine coordinate ring $\mathbb{C}[H_b]$ of the hypersurface $H_b \subset \mathbb{C}^5$ is factorial. Invertible elements in $\mathbb{C}[H_b]$ are exactly nonzero constants.

**Proof.** Consider the open subset $U_2^+ \subset H_b$ defined by $X_2 \neq 0$. Since $U_2^+$ is isomorphic to a Zariski open subset in $\mathbb{C}^4$, we obtain $\text{Cl}(U_2^+) = 0$. The complement $\widetilde{S}^+ := H_b \setminus U_2^+$ is a principle divisor $(X_2)$. We note that $\widetilde{S}^+$ defined by the binomial equation $Y_0^b = X_1X_4$ which shows that $\widetilde{S}^+$ is isomorphic to the product of $\mathbb{C}$ (with the coordinate $X_3$) and a 2-dimensional affine toric variety with a $A_{b-1}$-singularity defined by the
Proposition 2.5. Consider the following two Zariski open subsets $C$ with nonempty intersection with the smooth $SL(U)$, the statement holds. Let $x \in U$ be a point in $U \cap U'$. Then $g(x)$ is a point in $U \cap U'$. Therefore, $g$ is an element such that $gx = x$. We write $g$ as 

$$g = \text{diag}(t^k, t^{-p}, t^{-p}, t^s, t^q) \cdot \text{diag}(1, \zeta^{-s}, \zeta^{-s}, \zeta^s, \zeta^s), \ t \in \mathbb{C}^*, \zeta \in \mu_a.$$ 

Then $t^{-p} \zeta^{-s} = 1$ (because at least one of $x_1$ and $x_2$ is nonzero), and $t^q \zeta^s = 1$ (because at least one of $x_3$ and $x_4$ is nonzero). Therefore, $t^{-p} \zeta^{-s} t^q \zeta^s = t^{-p} = 1$. Since $q - p$ and $a$ are coprime we obtain that $t^p = \zeta^s = t^q = 1$. Since $g.c.d.(p, q) = 1$ we get $t = 1$. Therefore $g = 1$, i.e., $G$ acts freely on $U \cap U'$.

Now we remark that the open subsets $U^+ \cap U^- \subset H_b$ are $SL(2)$-invariant and have nonempty intersection with the $SL(2)$-invariant divisor $D := \{Y_0 = 0\} \subset H_b$. Therefore, the smooth $SL(2)$-variety $(U^+ \cap U^-)/G$ contains more than one $SL(2)$-orbit. So $(U^+ \cap U^-)/G$ coincides with $E_{h,m} \setminus \text{Sing}(E_{h,m}) = U_{h,m}$ (see 1.2 and 1.3). \hfill $\square$

Corollary 2.6. For any affine $SL(2)$-variety $E_{h,m}$, one has 

$$\text{Cox}(E_{h,m}) \cong \mathbb{C}[H_b] = \mathbb{C}[Y_0, X_1, X_2, X_3, X_4]/(Y_0^b - X_1 X_4 + X_2 X_3).$$

Proof. Let $L_0$ be trivial $G$-linearized line bundle over $H_b$, i.e., $O_{H_b} \cong O_{H_b}(L_0)$ as $G$-bundles. Then $H^0_b(L_0) = H_b$ and $H^0_b(L_0)/G \cong E_{h,m}$. By 2.5 $G$ acts freely on the open subset $U := U^+ \cap U^- \subset H_b$ and $\text{codim}_{H_b}(H_b \setminus U) = 2$. By 2.2 the affine coordinate ring of $H_b$ is isomorphic to the Cox ring of $E_{h,m}$. \hfill $\square$
**Corollary 2.7.** [Ga08] An affine $SL(2)$-variety $E_{h,m}$ is toric if and only if $b = 1$, i.e., $q - p$ divides $m$.

**Proof.** If $b = 0$ (i.e. $h = 1$), then $E_{1,m}$ is smooth and $\text{Cl}(E_{1,m}) \cong \mathbb{Z}$. However, the divisor class group of any smooth affine toric variety is trivial. Hence, $E_{1,m}$ is not toric.

In general, if $X$ is a normal affine toric variety such that all invertible elements in $\mathbb{C}[X]$ are constant, then $\text{Cox}(X)$ is a polynomial ring [Cox95]. In particular, the spectrum of $\text{Cox}(X)$ is nonsingular. On the other hand, if $b > 1$, then the hypersurface $H_b \subset \mathbb{C}^5$ defined by the equation $Y_0^b - X_1X_4 + X_2X_3 = 0$ is singular. Therefore, $E_{h,m}$ is not toric if $b > 1$.

If $b = 1$, then $H_b \cong \mathbb{C}^4$, so $E_{h,m} \cong \mathbb{C}^4/G$ is toric. □

Using (2.7), we obtain a simple interpretation of the following computation of $\text{Cl}(E_{h,m})$ due to Panyushev:

**Proposition 2.8.** [Pa92] Th.2] For any normal affine $SL(2)$-variety $E_{h,m}$, one has

$$\text{Cl}(E_{h,m}) \cong \mathbb{Z} \oplus C_a.$$ 

Let $D \subset E_{h,m}$ be the closure of the unique 2-dimensional $SL(2)$-orbit $\mathcal{D}$. Denote by $S^+ \subset E_{h,m}$ (respectively by $S^- \subset E_{h,m}$) be the closure in $E_{h,m}$ of the $B$-orbit in $U \cong SL(2)/C_m$ defined by the equation $Z^m = 0$ (respectively, by $W^m = 0$). Then $\text{Cl}(E_{h,m})$ is generated by two elements $[D]$ and $[S^+]$, or, respectively, by $[D]$ and $[S^-]$) satisfying the unique relation:

$$aq[D] + m[S^+] = 0,$$

or, respectively,

$$-ap[D] + m[S^-] = 0.$$ 

**Proof.** The isomorphisms

$$\text{Cl}(E_{h,m}) \cong \text{Hom}_{\text{alg}}(G, \mathbb{C}^*) \cong \text{Hom}_{\text{alg}}(G'_0, \mathbb{C}^*) \oplus \text{Hom}_{\text{alg}}(G_a, \mathbb{C}^*) \cong \mathbb{Z} \oplus C_a.$$ 

follow immediately from (2.6). Let $D' \subset E_{h,m}$ be an arbitrary nonzero effective irreducible divisor. Consider the surjective morphism $\pi : U^- \cap U^+ \to (U^- \cap U^+)/G = U_{h,m}$. Then the support of $D'$ has a nonempty intersection with $U_{h,m}$, because $\text{codim}_{E_{h,m}} \text{Sing}(E_{h,m}) \geq 2$. Then the closure $\tilde{D}'$ of $\pi^{-1}(D' \cap U_{h,m}) \subset H_b$ is a $G$-invariant principal irreducible divisor (see 2.4). Therefore, $\tilde{D}'$ is defined by zeros of a polynomial $\tilde{f}(Y_0, X_1, X_2, X_3, X_4)$ such that $\tilde{f}(gX) = \tilde{\chi}(g)\tilde{f}(X)$ and $\tilde{\chi} = \chi_{D'} \in \text{Hom}_{\text{alg}}(G, \mathbb{C}^*)$ is the character representing the class $[D'] \in \text{Cl}(E_{h,m})$.

It is easy to see that the irreducible divisors $\tilde{D}, \tilde{S}^+, \tilde{S}^- \subset H_b$ are defined respectively by polynomials $Y_0, X_2, X_4$. The corresponding characters $\tilde{\chi}$ of $G \cong \mathbb{C}^* \times \mu_a$ are:

$$\chi_D(t, \zeta) = t^k, \quad \chi_{S^+}(t, \zeta) = t^{-p}\zeta^{-1}, \quad \chi_{S^-}(t, \zeta) = t^q\zeta.$$
Since \( g.c.d. (ap, k) = g.c.d. (aq, k) = 1 \) each pair \( \{ \chi_D, \chi_{S^+} \} \) and \( \{ \chi_D, \chi_{S^-} \} \) generate the character group of \( \mathbb{C}^* \times \mu_a \). Moreover, we have
\[
\chi_D^{ap}(t, \zeta)\chi_S^{m}(t, \zeta) = \chi_D^{aq}(t, \zeta)\chi_S^{m}(t, \zeta) = 1 \quad \forall t \in \mathbb{C}^*, \forall \zeta \in \mu_a.
\]
This implies the following two relations in \( \text{Cl}(E_{h,m}) \):
\[
\text{ap}[D] + m[S^+] = -\text{aq}[D] + m[S^-] = 0.
\]
Consider two natural surjective homomorphisms
\[
\psi^+ : \mathbb{Z}^2 \to \text{Cl}(E_{h,m}), (k_1, k_2) \mapsto k_1[D] + k_2[S^+], \\
\psi^- : \mathbb{Z}^2 \to \text{Cl}(E_{h,m}), (k_1, k_2) \mapsto k_1[D] + k_2[S^-].
\]
Then
\[
\text{Ker } \psi^+ = \langle (ap, m) \rangle, \quad \text{Ker } \psi^- = \langle (-aq, m) \rangle,
\]
because each of two elements \( (p, k), (-q, k) \in \mathbb{Z}^2 \) generates a direct summand of \( \mathbb{Z}^2 \), and, by \( ka = m \), we have
\[
\mathbb{Z}^2 / \langle (pa, m) \rangle \cong \mathbb{Z} \oplus C_a \cong \mathbb{Z}^2 / \langle (-qa, m) \rangle.
\]

\[\square\]

3. \( \text{SL}(2) \)-equivariant flips

Let us start with toric \( \text{SL}(2) \)-equivariant flips. It is known that if \( m = a(q - p) \) then the toric variety \( E_{h,m} \) is isomorphic to the closure of the orbit of the highest vector in the irreducible \( \text{SL}(2) \times \text{SL}(2) \)-module \( V_{ap} \otimes V_{aq} \) \cite{Pa92}, Prop.2]. In this case, \( E_{h,m} \) is isomorphic to the affine cone in \( V_{ap} \otimes V_{aq} \cong \mathbb{C}^{(ap+1) \times (aq+1)} \) with vertex 0 over the projective embedding of \( \mathbb{P}^1 \times \mathbb{P}^1 \) into a projective space by the global sections of the ample sheaf \( \mathcal{O}(ap, aq) \). The closure \( D \) of the 2-dimensional \( \text{SL}(2) \)-orbit \( \mathcal{D} \) in \( E_{h,m} \) is isomorphic to the affine cone over \( a(p + q) \)-th Veronese embedding of \( \mathbb{P}^1 \) considered as diagonal in \( \mathbb{P}^1 \times \mathbb{P}^1 \). If \( e_1, e_2, e_3 \) is a standard basis of \( \mathbb{R}^3 \) then the toric variety \( E_{h,m} \) is defined by the cone \( \sigma = \sum_{i=1}^{4} \mathbb{R}_{\geq 0} v_i \) where
\[
v_1 = e_1, \quad v_2 = -e_1 + aqe_3, \quad v_3 = e_2, \quad v_4 = -e_2 + ape_3,
\]
i.e., \( v_1, v_2, v_3, v_4 \) satisfy the equation \( pv_1 + pv_2 = qv_3 + qv_4 \).

Let \( E'_{h,m} \) be the blow up of \( 0 \in E_{h,m} \subset \mathbb{C}^{(ap+1) \times (aq+1)} \). It corresponds to the subdivision of \( \sigma \) into 4 simplicial cones having a new common ray \( \mathbb{R}_{\geq 0} v_5 \) \( (v_5 = e_3) \) and generated by the following 4 sets of lattice vectors
\[
\{ v_1, v_3, v_5 \}, \{ v_2, v_3, v_5 \}, \{ v_2, v_4, v_5 \}, \{ v_1, v_1, v_5 \}.
\]
The exceptional divisor \( D' \) over 0 corresponding to the new lattice vector \( v_5 \) is isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \). Moreover, the whole variety \( E'_{h,m} \) is smooth and can be considered as a line bundle of bidegree \( (-aq, -ap) \) over \( \mathbb{P}^1 \times \mathbb{P}^1 \). Consider two 2-dimensional simplicial cones
\[
\sigma^+ = \mathbb{R}_{\geq 0} v_3 + \mathbb{R}_{\geq 0} v_4, \quad \text{and} \quad \sigma^- = \mathbb{R}_{\geq 0} v_1 + \mathbb{R}_{\geq 0} v_2.
\]
There exist two different subdivisions of $\sigma$ into pairs of simplicial cones

$$\sigma = (\mathbb{R}_{\geq 0}v_1 + \sigma^+) \cup (\mathbb{R}_{\geq 0}v_2 + \sigma^+)^{+}$$

and

$$\sigma = (\mathbb{R}_{\geq 0}v_3 + \sigma^-) \cup (\mathbb{R}_{\geq 0}v_4 + \sigma^-).$$

We denote toric varieties corresponding two these subdivisions by $E_{h,m}^-$ and $E_{h,m}^+$ respectively. Then one obtains the following diagram of toric morphisms:

$$
\begin{array}{ccc}
E_{h,m}' & \gamma^- & \rightarrow & E_{h,m}^{-} \\
\downarrow & & & \downarrow \\
E_{h,m}^{-} & \varphi^- & \rightarrow & E_{h,m} \\
\end{array}
\quad
\begin{array}{ccc}
E_{h,m}^{+} & \gamma^+ & \rightarrow & E_{h,m}^{+} \\
\downarrow & & & \downarrow \\
E_{h,m}^{-} & \varphi^+ & \rightarrow & E_{h,m} \\
\end{array}
$$

The morphisms $\gamma^-$ and $\gamma^+$ restricted to $D'$ are projections of $\mathbb{P}^1 \times \mathbb{P}^1$ onto first and second factors. We denote by $C^-$ (resp. $C^+$) the $\gamma^-$-image (resp. $\gamma^+$-image) of $D'$ in $E_{h,m}^-$ (resp. $E_{h,m}^+$). Then singularities along $C^-$ (resp. along $C^+$) are determined by the 2-dimensional cone $\sigma^-$ (resp. $\sigma^+$). The relations

$$v_3 + v_4 = apv_5, \quad v_1 + v_2 = aqv_5$$

show that the 2-dimensional affine toric variety $X_{\sigma^-}$ (resp. $X_{\sigma^+}$) is an affine cone over $\mathbb{P}^1$ embedded by $\mathcal{O}(ap)$ (resp. by $\mathcal{O}(aq)$) to $\mathbb{P}^{ap}$ (resp. $\mathbb{P}^{aq}$). By $1 \leq p < q$, we obtain that $E_{h,m}^-$ is always singular and $E_{h,m}^+$ is nonsingular if and only if $ap = 1$. Simple calculations in Chow rings of toric varieties $E_{h,m}^-$ and $E_{h,m}^+$ show that

$$C^- \cdot K_{E_{h,m}^-} = \frac{2(p-q)}{aq^2} < 0, \quad C^+ \cdot K_{E_{h,m}^+} = \frac{2(q-p)}{ap^2} > 0.$$ 

So the birational map

$$E_{h,m}^- \rightarrow E_{h,m}^+$$

is a toric flip.

Now we consider a general case for an affine $SL(2)$-variety $E_{h,m}$. Let us begin with the calculation of the canonical class of an arbitrary $SL(2)$-variety $E_{h,m}$ which has been done by Panyushev in [Pa92, Prop.4 and 5]:

**Proposition 3.1.** For any normal affine $SL(2)$-variety $E_{h,m}$, one has

$$K_{E_{h,m}} = -(1+b)[D].$$

**Proof.** Using the description of $E_{h,m}$ as a categorical quotient $H_b\!/\!G$ of the hypersurface $H_b \subset \mathbb{C}^5$, we can consider $E_{h,m}$ as a hypersurface in the 4-dimensional affine toric variety $T_{h,m} := \mathbb{C}^5 \!/\! G$. It is well-known that the canonical divisor of any toric variety consists of irreducible divisors in the complement to the open torus orbit taken with the multiplicity $-1$. If we consider $Y_0, X_1, X_2, X_3, X_4$ as homogeneous coordinates of
the toric variety \( T_{h,m} \), then the canonical class of \( T_{h,m} \) corresponds to the character \\
\[ \chi : G \rightarrow \mathbb{C}^* \]
\[ \chi(t, \zeta) = t^{-k(p\zeta^q)}(t^{-q}\zeta^{-1})^2 = t^{-k+p-2q}. \]

On the other hand, \( G \) acts on the polynomial \( Y_0^b - X_1X_4 + X_2X_3 \) by the character \\
\[ \chi'(t, \zeta) = t^{q-p}. \]

Therefore, by adjunction formula, the canonical class of \( E_{h,m} \) corresponds to the character \( \chi^+ = \chi + \chi' \):
\[ \chi^+(t, \zeta) = t^{-k+p-q}. \]

Since the class \([D] \in \text{Cl}(E_{h,m})\) is defined by the character \( \chi_D(t, \zeta) = t^k \), we obtain that
\[ K_{E_{h,m}} = \frac{-k+p-q}{k}[D] = -(1+b)[D]. \]

\[ \square \]

**Proposition 3.2.** Let \( L^+ \) be the trivial line bundle over \( H_b \) together with the linearization corresponding to the character \( \chi^+ \), then
\[ H_b^{ss}(L^+) = U^+ = H_b \setminus \{X_1 = X_2 = 0\}. \]

**Proof.** The space \( \Gamma(H_b, (L^+)^\otimes n)^G \) consists of all regular functions \( f \) on \( H_b \) such that \( f(gx) = (\chi^+(g))^nf(x) \). It is easy to see that \( \Gamma(H_b, (L^+)^\otimes n)^G \) is generated as a \( \mathbb{C} \)-vector space by restrictions of monomials \( Y_0^{k_0}X_1^{k_1}X_2^{k_2}X_3^{k_3}X_4^{k_4} \) satisfying the above homogeneity condition, i.e.,
\[ t^{k_0k-k_1p-k_2p+k_3q+k_4} \zeta^{-k_1-k_2+k_3+k_4} = t^n(-k+p-q) \forall t \in \mathbb{C}^*, \forall \zeta \in \mu_a. \]

The last condition implies \( a|[(k_3 + k_4 - k_1 - k_2)] \) and
\[ k_0k - k_1p - k_2p + k_3q + k_4q = n(-k + p - q). \]

Since \( n(-k + p - q) < 0 \) and \( k_i \geq 0 \) (0 \leq i \leq 4), we obtain that at least one of the integers \( k_1 \) and \( k_2 \) must be positive, i.e., all monomials \( Y_0^{k_0}X_1^{k_1}X_2^{k_2}X_3^{k_3}X_4^{k_4} \in \Gamma(H_b, (L^+)^\otimes n)^G \) vanish on the subset \( \{X_1 = X_2 = 0\} \cap H_b \). On the other hand, if at least one of two coordinates \( X_1 \) and \( X_2 \) of a point \( x \in H_b \) is not zero, then one of the monomials
\[ X_1^{q-p+k}, \ X_2^{q-p+k} \in \Gamma(H_b, (L^+)^\otimes p)^G \]

does not vanish in \( x \). Hence, \( H_b^{ss}(L^+) = U^+ \). \[ \square \]

**Proposition 3.3.** Let \( L^- \) be the trivial line bundle over \( H_b \) together with the linearization corresponding to the character \( \chi^- = -\chi^+ \), then
\[ H_b^{ss}(L^-) = U^- = H_b \setminus \{X_3 = X_4 = 0\}. \]

**Proof.** The condition \( f(gx) = (\chi^-(g))^nf(x) \) for a monomial
\[ f = Y_0^{k_0}X_1^{k_1}X_2^{k_2}X_3^{k_3}X_4^{k_4} \in \Gamma(H_b, (L^-)^\otimes n)^G \]

implies that
\[ t^{k_0k-k_1p-k_2p+k_3q+k_4q} \zeta^{-k_1-k_2+k_3+k_4} = t^{n(k+q-p)} \forall t \in \mathbb{C}^*, \forall \zeta \in \mu_a. \]
Since \( n(k + q - p) > 0 \), we obtain that at least one of three integers \( k_0, k_3, k_4 \) must be positive. Therefore, all monomials \( Y_0^{k_0} X_1^{k_1} X_2^{k_2} X_3^{k_3} X_4^{k_4} \in \Gamma(H_b, (L^-)^\otimes n)^G \) vanish on the subset \( \{ Y_0 = X_3 = X_4 = 0 \} \cap H_b = \{ X_3 = X_4 = 0 \} \cap H_b \). On the other hand, if at least one of two coordinates \( X_3 \) and \( X_4 \) of a point \( x \in H_b \) is not zero, then one of the monomials
\[
X_3^{q - p + k}, \quad X_4^{q - p + k} \in \Gamma(H_b, (L^-)^\otimes q)^G
\]
does not vanish in \( x \). Hence, \( H_b^{ss}(L^-) = U^- \).

**Theorem 3.4.** Define
\[
E^-_{h,m} := H_b^{ss}(L^-)/G, \quad E^+_{h,m} := H_b^{ss}(L^+)/G.
\]
Then the open embeddings
\[
H_b^{ss}(L^-) = U^- \subset H_b, \quad H_b^{ss}(L^+) = U^+ \subset H_b,
\]
define two natural birational morphisms
\[
\varphi^- : E^-_{h,m} \to E_{h,m}, \quad \varphi^+ : E^+_{h,m} \to E_{h,m},
\]
and the \( SL(2) \)-equivariant flip
\[
E^-_{h,m} \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow E^+_{h,m}
\]
\[
\varphi^- \quad \varphi^+
\]

**Proof.** The statement follows immediately from the isomorphisms
\[
E^-_{h,m} \cong \text{Proj} \bigoplus_{n \geq 0} \Gamma(H_b, (L^-)^\otimes n)^G \cong \text{Proj} \bigoplus_{n \geq 0} \Gamma(E_{h,m}, \mathcal{O}(-nK_{E_{h,m}}))
\]
and
\[
E^+_{h,m} \cong \text{Proj} \bigoplus_{n \geq 0} \Gamma(H_b, (L^+)^\otimes n)^G \cong \text{Proj} \bigoplus_{n \geq 0} \Gamma(E_{h,m}, \mathcal{O}(nK_{E_{h,m}})).
\]

**Corollary 3.5.** One has the following isomorphisms:
\[
E^-_{h,m} \cong \text{Proj} \bigoplus_{n \geq 0} \Gamma(E_{h,m}, \mathcal{O}(-nD)), \quad E^+_{h,m} \cong \text{Proj} \bigoplus_{n \geq 0} \Gamma(E_{h,m}, \mathcal{O}(nD)).
\]

**Proof.** These isomorphisms follow from the equation \( K_{E_{h,m}} = -(1 + b)[D] \) (3.1) and from the isomorphism
\[
\text{Proj} \bigoplus_{n \geq 0} R_n \cong \text{Proj} \bigoplus_{n \geq 0} R_{nl}
\]
for any notherian graded ring \( R = \bigoplus_{n \geq 0} R_n \) and for any positive integer \( l \).

In order to describe the geometry \( E^-_{h,m} \) and \( E^+_{h,m} \) in more detail we need two 2-dimensional affine varieties \( S^+ \) and \( S^- \) having regular \( B \)-actions (see also 2.8).
Proposition 3.6. Let $S^+ \subset E_{h,m}$ be the closure of an $B$-orbit obtained as categorical quotient of $W^+ := H_{g-p} \cap \{X_2 = 0\}$ by $G_0 \times G_m$. Then $S^+$ is isomorphic to the normal affine toric surface $\text{Spec} \mathbb{C}[M^+_{h,m}]$.

Proof. We note that $W^+ = H_{g-p} \cap \{X_2 = 0\} \subset \mathbb{C}^5$ is a 3-dimesional affine toric variety which is a product of $C$ and a 2-dimensional affine toric variety defined by the binomial equation $X_0^{q-p} = X_1X_4$. Let us compute the categorical quotient $W^+ / G_0$. Since $G_0$ acts on $X_0, X_1, X_3, X_4$ by $\text{diag}(t, t^{-p}, t^q, t^q)$ for every nonconstant $G_0$-invariant monomial $X_0^{k_0}X_1^{k_1}X_3^{k_3}X_4^{k_4}$ ($k_i \in \mathbb{Z}_{\geq 0}$) the condition $k_0 - pk_1 + qk_3 + qk_4 = 0$ implies $k_1 > 0$. If at the same time $k_4 > 0$, then

$$X_0^{k_0}X_1^{k_1}X_3^{k_3}X_4^{k_4} - X_0^{k_0+q-p}X_1^{k_1-1}X_3^{k_3}X_4^{k_4-1} \in I(W^-).$$

Using the equation $X_0^{q-p} = X_1X_4$ several times, we can get another monomial $X_0^{k_0'}X_1^{k_1'}X_3^{k_3'}$ such that $X_0^{k_0}X_1^{k_1}X_3^{k_3}X_4^{k_4} - X_0^{k_0'}X_1^{k_1'}X_3^{k_3'} \in I(W^+)$, i.e., vanish on $W^+$. Therefore, the coordinate ring of $W^+ / G_0$ contains a $\mathbb{C}$-basis consisting of all $G_0$-invariant monomials in $X_0, X_1, X_3$. These monomials have form $X_0^{k_0}X_1^{k_1}X_3^{k_3} = X_1X_3^{k_3}$ where $pk_1 - qk_3 \geq 0$ (i.e. $(k_1, k_3) \in M^+_{h,1}$). So the coordinate ring of $S^+ = W^+ / (G_0 \times G_m)$ has a $\mathbb{C}$-basis consisting of $G_m$-invariants monomials $X_1X_3^{k_3} = X_0^{pk_1-qk_3}X_1^{k_1}X_3^{k_3}$ which correspond to lattice points $(k_1,k_3) \in M_{h,m}^+ \cap \{(k_1,k_3) \in \mathbb{Z}^2_{\geq 0} : m|(k_1-k_3)\}$, i.e. $S^+ \cong \text{Spec} \mathbb{C}[M^+_{h,m}]$. \hfill $\Box$

Proposition 3.7. Let $S^- \subset E_{h,m}$ be the closure of an $B$-orbit obtained as categorical quotient of $W^- := H_{g-p} \cap \{X_4 = 0\}$ by $G_0 \times G_m$. Then $S^-$ is isomorphic to the normal affine toric surface $\text{Spec} \mathbb{C}[M^-_{h,m}]$, where the monoid $M^-_{h,m} \subset \mathbb{Z}^2$ (see Figure 2) is defined as follows:

$$M^-_{h,m} := \{(i,j) \in \mathbb{Z}^2 : j \leq hi, \ i \geq 0, \ m|(i-j)\}.$$

Proof. We note that $W^- = H_{g-p} \cap \{X_4 = 0\} \subset \mathbb{C}^5$ is a 3-dimensional toric variety which is a product of $C$ and a 2-dimensional toric variety defined by the equation $X_0^{q-p} = -X_2X_3$. Again the computation of the categorical quotient $W^- / G_0$ reduces to finding all $G_0$-invariant monomials $X_0^{k_0}X_1^{k_1}X_2^{k_2}X_3^{k_3}$. Under the condition $X_0^{q-p} = -X_2X_3$ we can assume that at least one of two variables $X_2$, or $X_3$ does not appear in $X_0^{k_0}X_1^{k_1}X_2^{k_2}X_3^{k_3}$ (i.e., $k_2 = 0$ or $k_3 = 0$). If $k_2 = 0$, then we come to the same situation as in Proposition 3.6 and obtain $G_0$-invariant monomials $X_1X_3^{k_3} = X_0^{pk_1-qk_3}X_1^{k_1}X_3^{k_3}$ $(k_1,k_3) \in M^+_{h,1}$. If $k_3 = 0$, then we obtain $G_0$-invariant monomials $X_0^{pk_1+pk_2}X_1^{k_1}X_2^{k_2} = X_1X_2^{k_2}$, $(k_1,k_2) \in \mathbb{Z}_{\geq 0})$. The equation $X_0^{q-p} = -X_2X_3$ implies that on $W^- / G_0$ we have $YZ = X_0^{-q}X_2X_3 = -1$. So in case $k_2 = 0$ we obtain the monomials in $X_1(Y^{-1})^{k_2}$, $(k_1,k_2) \in \mathbb{Z}_{\geq 0})$. Unifying both cases, we get all $G_0$-invariant monomials $X_1Y^j$, $(i,j) \in M_{h}^1$. The action of the finite group $G_m$ on $X$ and $Y$ gives rise to an additional restriction: $m|(i-j)$. Therefore, $G_0 \times G_m$-invariant monomials can be identified with the set of all lattice points $(i,j) \in M^-_{h,m}$. \hfill $\Box$
Remark 3.8. If \( m = a(q - p) \) (i.e. \( E_{h,m} \) is toric), then \( S^- \cong X_{\sigma^-} \) and \( S^+ \cong X_{\sigma^+} \), where \( \sigma^- \) and \( \sigma^+ \) are 2-dimensional cones as above.

Definition 3.9. Let \( S \) be an algebraic surface with a regular action \( B \times S \to S \) of a Borel subgroup \( B \subset SL(2) \). We denote by \( SL(2) \times_B S \) the \( SL(2) \)-variety \( (SL(2) \times S)/B \), where \( B \) is considered to act on \( SL(2) \) by right multiplication:

\[
\left( \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}, \begin{pmatrix} t & u \\ 0 & t^{-1} \end{pmatrix} \right) \mapsto \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \cdot \begin{pmatrix} t & u \\ 0 & t^{-1} \end{pmatrix}^{-1}, \quad \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \in SL(2).
\]

Theorem 3.10. One has the following isomorphisms

\[
E_{h,m}^- \cong SL(2) \times_B S^-, \quad E_{h,m}^+ \cong SL(2) \times_B S^+.
\]

Proof. Since \( U^+ = H_b \setminus \{X_1 = X_2 = 0\} \) and \( G \) acts on \( (X_1, X_2) \) by scalar matrices, we obtain a natural \( SL(2) \)-equivariant morphism

\[
\alpha^+ : E_{h,m}^+ \cong U^+ / G \to \mathbb{P}^1, \quad (Y_0, X_1, X_2, X_3, X_4) \mapsto (X_1 : X_2)
\]

Analogously, we obtain a natural \( SL(2) \)-equivariant morphism

\[
\alpha^- : E_{h,m}^- \cong U^- / G \to \mathbb{P}^1, \quad (Y_0, X_1, X_2, X_3, X_4) \mapsto (X_3 : X_4)
\]

By \ref{thm:iso1} and \ref{thm:iso2}, we have

\[
S^+ = (\alpha^+)^{-1}(1 : 0), \quad S^- = (\alpha^+)^{-1}(1 : 0).
\]
Since the morphisms $\alpha^+$ and $\alpha^-$ are $SL(2)$-equivariant and $SL(2)$ acts transitively on $\mathbb{P}^1$, we have
\[ S^\pm \cong (\alpha^\pm)^{-1}(z), \quad S^\mp \cong (\alpha^\mp)^{-1}(z) \quad \forall z \in \mathbb{P}^1, \]
i.e., $E_{h,m}^+$ (resp. $E_{h,m}^-$) is a fibration over $\mathbb{P}^1$ with fiber $S^+$ (resp. $S^-$). On the other hand, the projection $SL(2) \times S^\pm \to SL(2)$ defines two natural morphisms $SL(2)$-equivariant morphisms
\begin{align*}
\pi^+ & : SL(2) \times_B S^+ \to SL(2)/B \cong \mathbb{P}^1, \\
\pi^- & : SL(2) \times_B S^- \to SL(2)/B \cong \mathbb{P}^1,
\end{align*}
such that $SL(2) \times_B S^+$ (resp. $SL(2) \times_B S^-$) is a fibration over $\mathbb{P}^1$ with fiber $S^+$ (resp. $S^-$). Consider the morphisms
\begin{align*}
\tilde{\beta}^+ & : SL(2) \times S^+ \to U^+ \! \! /\! / G, \\
\tilde{\beta}^- & : SL(2) \times S^- \to U^- \! \! /\! / G
\end{align*}
defined by
\[ \tilde{\beta}^+(g, x) = gx, \quad \forall g \in SL(2), \quad \forall x \in S^\pm = (\alpha^\pm)^{-1}(1 : 0). \]
Since $\tilde{\beta}^+(gb^{-1}, bx) = \tilde{\beta}^+(g, x) = gx \forall b \in B$, the morphism $\tilde{\beta}^\pm$ descends to a morphism
\[ \beta^\pm : SL(2) \times_B S^\pm \to U^\pm \! \! /\! / G \cong E_{h,m}^\pm. \]
The latter is an isomorphism, because $\beta^\pm$ is $SL(2)$-equivariant and it maps isomorphically the fiber of $\pi^\pm$ over $(1 : 0)$ to the fiber of $\alpha^\pm$ over the $B$-fixed point $[B] \in SL(2)/B$. \hfill \Box

**Remark 3.11.** Since the monoid $M_{h,m}^+$ is submonoid of the monoid $M_{h,m}^-$ we obtain a birational morphism $\psi : S^- \to S^+$ of 2-dimensional normal affine toric varieties $S^-$ and $S^+$. However, $\psi$ is not $B$-equivariant, because an element
\[ \begin{pmatrix} t & u \\ 0 & t^{-1} \end{pmatrix} \in B \]
sends $X^iY^j \in \mathbb{C}[M_h^m]$ to
\[ (tX)^i(tY + uX^{-1})^j \in \mathbb{C}[M_h^m] \]
and sends $X^rY^s \in \mathbb{C}[M_{h,m}^-]$ to
\[ (tX - uY^{-1})^r(tY)^s \in \mathbb{C}[M_{h,m}^-]. \]
This is the reason why there is no any birational $SL(2)$-equivariant morphism from $SL(2) \times_B S^-$ to $SL(2) \times_B S^+$, but only a flip.

**Remark 3.12.** Let $E_{h,m} \hookrightarrow V$ be a closed embedding, where $V$ is an affine space isomorphic to $V_{i_1+j_1} \oplus \cdots \oplus V_{i_r+j_r}$ (see [1,3]). We define a $\mathbb{C}^*$-action on $V$ such that $t \in \mathbb{C}^*$ acts by multiplication with $t^{i-j}$ on $V_{i+j}$. Since this $\mathbb{C}^*$-action commutes with the $SL(2)$-action, the affine variety $E_{h,m} \subset V$ remains invariant under the $\mathbb{C}^*$-action. Consider the weighted blow up $\delta : \tilde{V} \to V$ of $0 \in V$ with respect to weights of this $\mathbb{C}^*$-action. The birational pullback of $E_{h,m}$ under $\delta : \tilde{V} \to V$ is a $SL(2)$-variety.
$E'_{h,m}$ together with surjective morphisms $\gamma^- : E'_{h,m} \to E^-_{h,m}$ and $\gamma^+ : E'_{h,m} \to E^+_{h,m}$ such that the following diagram commutes

\[
\begin{array}{ccc}
E'_{h,m} & \xrightarrow{\gamma^-} & E^-_{h,m} \\
\downarrow \gamma^+ & & \downarrow \gamma^+
\end{array}
\]

The variety $E'_{h,m}$ contains two $SL(2)$-invariant divisors $D' \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\tilde{D} := \delta^*(D)$ whose intersection $C = D' \cap \tilde{D} \cong \mathbb{P}^1$ is the unique 1-dimensional closed $SL(2)$-orbit in $E'_{h,m}$. The morphism $\gamma^+$ contracts $D'$ to $C^+ \subset E^+_{h,m}$. The divisor $D'$ corresponds to the $SL(2)$-invariant discrete valuation of the function field $\mathbb{C}(SL(2))$ defined by above $\mathbb{C}^*$-action on $E_{h,m}$ such that $\mathbb{C}(D')$ is the $\mathbb{C}^*$-invariant subfield $\mathbb{C}(SL(2))^{\mathbb{C}^*} \cong \mathbb{C}(SL(2)/\mathbb{C}^*)$. We note that the $SL(2)$-variety $E'_{h,m}$ has also a toroidal structure, i.e., along the closed 1-dimensional $SL(2)$-orbit $C$, it is locally isomorphic to a product of an affine line $\mathbb{A}^1$ and a 2-dimensional affine toric surface $S'$ which is isomorphic to $\text{Spec} \mathbb{C}[M'_{h,m}]$ where

$$M'_{h,m} := \{(i, j) \in \mathbb{Z}^2 : pj - qi \geq 0, j - i \in m\mathbb{Z}_{\geq 0}\}.$$ 

In particular, $S' \cong \mathbb{A}^2/\mu_b$ and $E'_{h,m}$ is nonsingular along $C$ if and only if $b = 1$, i.e., iff $E_{h,m}$ is toric.

**Proposition 3.13.** The canonical divisor of $E^\pm_{h,m}$ has the following intersection numbers with the 1-dimensional $SL(2)$-orbits $C^\pm \subset E^\pm_{h,m}$:

$$K_{E^-_{h,m}} \cdot C^- = -\frac{(1 + b)k}{aq^2}, \quad K_{E^+_{h,m}} \cdot C^+ = \frac{(1 + b)k}{ap^2}.$$ 

**Proof.** Since $E_{h,m}$, $E^-_{h,m}$ and $E^+_{h,m}$ have the same divisor class group, we can use 2.8 and obtain that

$$ap[D] + m[S^+] = 0 \in \text{Cl}(E^+_{h,m}).$$

The divisor $S^+ \subset E^+_{h,m}$ intersects the curve $C^+$ transversally, but this intersection point is an isolated cyclic quotient singularity of type $A_{ap-1}$ in $S^+$. Therefore, we have $S^+ \cdot C^+ = \frac{1}{ap}$ and

$$D \cdot C^+ = -\left(\frac{m}{ap}S^+\right) \cdot C^+ = -\frac{k}{ap^2}.$$ 

By 3.1 we get

$$K_{E^+_{h,m}} \cdot C^+ = \frac{(1 + b)k}{ap^2}.$$ 

Similarly, the intersection point of $C^-$ and $S^- \subset E^-_{h,m}$ is an isolated cyclic quotient singularity of type $A_{aq-1}$ in $S^-$. Therefore, we have $S^- \cdot C^- = \frac{1}{aq}$ and, by

$$-aq[D] + m[S^-] = 0 \in \text{Cl}(E^-_{h,m}),$$
we obtain

\[ D \cdot C^- = \left( \frac{m}{aq} S^- \right) \cdot C^- = \frac{k}{aq^2}. \]

By 3.1 this implies

\[ K_{E_{h,m}} \cdot C^- = -\frac{(1 + b)k}{aq^2}. \]

\[ \square \]

**Remark 3.14.** In [LV83, Section 9] Luna and Vust gave a description of an arbitrary quasihomogeneous normal $SL(2)$-embedding by a special combinatorial diagram (so called “marked hedgehog”). If we apply this language to the description of quasiprojective varieties $E_{h,1}^-$ and $E_{h,1}^+$ (here we assume $m = 1$), then we find a difference between the using of signs $+$ and $-$ in our paper and in [LV83, Section 9]. For instance, the 1-dimensional orbit $C^+ \subset E_{h,1}^+$ is considered in [LV83] as orbit of type $l_+(D, h)$. Similarly, the 1-dimensional orbit $C^- \subset E_{h,1}^-$ is considered in in [LV83] as orbit of type $l_+(D, h)$.

### 4. Flips and spherical varieties

Let us consider the $\mathbb{C}^*$-action on $H_b$ defined by the diagonal matrices

\[ diag(1, s^{-1}, s^{-1}, s, s), \quad s \in \mathbb{C}^*. \]

We note that this $\mathbb{C}^*$-action commutes with the $SL(2)$-action and with the action of $G = G'_0 \times G_a$. So we obtain a natural $\mathbb{C}^*$-action on the categorical quotient $H_b/\sim \cong E_{h,m}$ which commutes with the $SL(2)$-action. We note that this $\mathbb{C}^*$-action has been already constructed in 3.12 using a closed embedding $E_{h,m} \hookrightarrow V$. This allows to consider $E_{h,m}$ as an affine $SL(2) \times \mathbb{C}^*$-variety.

**Proposition 4.1.** The affine variety $E_{h,m}$ is spherical with respect to the above $SL(2) \times \mathbb{C}^*$-action.

**Proof.** The open subset $U = (H_b \cap \{ Y_0 \neq 0 \})/G \subset E_{h,m}$ is obviously $SL(2) \times \mathbb{C}^*$-invariant. Since $SL(2) \times \mathbb{C}^*$ acts trasitively on $U$, we have $U \cong (SL(2) \times \mathbb{C}^*)/H$ for some closed subgroup $H \subset SL(2) \times \mathbb{C}^*$. It is easy to see that

\[ (H_b \cap \{ Y_0 X_2 X_4 \neq 0 \})/G \subset U \]

is an open dense orbit of the 3-dimensional Borel subgroup $\widetilde{B} := B \times \mathbb{C}^*$ in $SL(2) \times \mathbb{C}^*$. Hence, $E_{h,m}$ is a spherical embedding corresponding to the spherical homogeneous space $(SL(2) \times \mathbb{C}^*)/H$. \[ \square \]

**Remark 4.2.** There exists one more way to define the same $\mathbb{C}^*$ on $E_{h,m}$. We identify $\mathbb{C}^*$ with the maximal torus $T \subset SL(2)$ which acts on $SL(2)$ by right multiplication. Then this action extends to a regular action on $E_{h,m}$ and commutes with the $SL(2)$-action by left multiplication so that we obtain a regular action of $SL(2) \times T$ on $E_{h,m}$. 


By [Kr84], even a more general statement is true: \( E_{h,m} \) admits a regular action of \( SL(2) \times B \).

If we identify \( U \) with \( SL(2)/C_m \) and consider the subgroup \( H \subset SL(2) \times C^* \) as a stabilizer of the class of unit matrix in \( SL(2)/C_m \) then

\[
H = \left\{ (\text{diag}(t, t^{-1}), t^m) : t \in C^* \right\} \subset SL(2) \times C^*.
\]

The lattice \( \Lambda \) of rational \( \tilde{B} \)-eigenfunctions on \( U \) (up to multiplication with a nonzero constant) consists of all Laurent monomials \( Z^iW^j \in \mathbb{C}[SL(2)]^{\mu_m} \) such that \( m|i-j \). Therefore, \( E_{h,m} \) is a spherical embedding of rank 2. This rank equals also the minimal codimension of \( U \)-orbits in \( E_{h,m} \) (we identify \( U \) with the maximal unipotent subgroup in \( SL(2) \times C^* \)).

In order to describe spherical varieties \( E_{h,m}, E^+_{h,m}, \) and \( E^-_{h,m} \) by combinatorial data, we remark that they contain exactly three \( \tilde{B} \)-invariant divisors:

\[
D = H_b \cap \{ Y_0 = 0 \}/G, \quad S^+ = H_b \cap \{ X_2 = 0 \}/G, \quad S^- = H_b \cap \{ X_4 = 0 \}/G.
\]

The restrictions of the corresponding discrete valuations \( \mathbb{C}(U)^* \to \mathbb{Z} \) to the lattice \( \Lambda \) define lattice vectors \( \rho, \rho^+, \rho^- \in \Lambda^* \) in the dual space \( Q := \text{Hom}(\Lambda, \mathbb{Q}) \). We can consider \( \rho^+, \rho^- \) as a \( \mathbb{Q} \)-basis of \( Q \). Then the set of all \( SL(2) \times C^* \)-invariant valuations generate so called valuation cone \( V \subset Q \), \( V = \{ x\rho^+ + y\rho^- \in Q : x + y \leq 0 \} \):

The equations \( Z = X_0^pX_2, W = X_0^{-q}X_4 \) imply

\[
\rho = p\rho^+ - q\rho^- \in V.
\]

It is easy to see that \( E_{h,m}, E^-_{h,m}, \) and \( E^+_{h,m} \) are simple spherical embeddings (i.e., they contain exactly one closed \( SL(2) \times C^* \)-orbit of dimension 1, or 0). Therefore, they can be described by so called colored cones \( (C, F) \), where \( F \) is a subset of \( \{ \rho^+, \rho^- \} \) and \( C \subset Q \) is a strictly convex cone generated by \( F \) and \( \rho \). More precisely we have:

\[
C(E_{h,m}) = \mathbb{Q}_{\geq 0}\rho + \mathbb{Q}_{\geq 0}\rho^- , \quad F(E_{h,m}) = \{ \rho^+, \rho^- \},
\]

\[
C(E^-_{h,m}) = \mathbb{Q}_{\geq 0}\rho + \mathbb{Q}_{\geq 0}\rho^+ , \quad F(E^-_{h,m}) = \{ \rho^+ \},
\]

\[
C(E^+_{h,m}) = \mathbb{Q}_{\geq 0}\rho + \mathbb{Q}_{\geq 0}\rho^- , \quad F(E^+_{h,m}) = \{ \rho^- \}.
\]
Moreover, the spherical variety $E_{h,m}^{'}$ is also simple. However, $E_{h,m}$ contains one more $SL(2) \times \mathbb{C}^*$-invariant divisor $D'$ such that the restrictions of the corresponding discrete valuations to $\Lambda$ defines a lattice vector $\rho' = \rho^+ - \rho^- \in \mathcal{V}$. In this case, we have

$$\mathcal{C}(E_{h,m}^{'}) = \mathbb{Q}_{\geq 0} \rho + \mathbb{Q}_{\geq 0} \rho^-; \quad \mathcal{F}(E_{h,m}^{'}) = \{\rho^-\}.$$ 

**Remark 4.3.** We note that birational morphisms $f : W \to W'$ of simple spherical varieties $W, W'$ where $f \in \{\varphi^-, \varphi^+, \gamma^-, \gamma^+\}$ has an interpretation in terms of colors. In our situation, we see that the set of colors $\mathcal{F}(W')$ is strictly larger than $\mathcal{F}(W')$.

In particular, the birational morphism $\varphi^- : E_{h,m}^{' -} \to E_{h,m}$ combinatorially means that the cone $\mathcal{C}(E_{h,m}^{' -}) = \mathcal{C}(E_{h,m})$ remains unchanged, but it gets an additional color $\rho^+ : \mathcal{F}(E_{h,m}) = \mathcal{F}(E_{h,m}) \cup \{\rho^+\}$. On the other hand, the birational morphism $\varphi^+ : E_{h,m}^+ \to E_{h,m}$ also adds an additional color $\rho^- : \mathcal{F}(E_{h,m}) = \mathcal{F}(E_{h,m}^+) \cup \{\rho^-\}$ such that the color $\rho^+$ becomes an interior point of $\mathcal{C}(E_{h,m})$. This agree with a general description of Mori contractions in [B93, 3.4, 4.4].

**Remark 4.4.** According to Alexeev and Brion [AB04], every spherical $G$-variety $\mathcal{X}$ admits a flat degeneration to a toric variety $\mathcal{X}_0$. In general case, there exist several degenerations depending on different reduced decompositions of the longest element $w_0$ in the Weyl group of the reductive group $G$. However, in the case $G = SL(2) \times \mathbb{C}^*$ the choice of such a decomposition is unique. A simplest example of such a toric degeneration appears in the case $\mathcal{X} := SL(2)$ considered as a spherical homogeneous space of $SL(2) \times \mathbb{C}^*$. Then $\mathcal{X}_0 = \{X_1X_4 - X_2X_3 = 0\}$ is a singular affine 3-dimensional toric quadric. The corresponding deformation is $\mathcal{X}_0 = \lim_{t \to 0} \mathcal{X}_t$ where $\mathcal{X}_t := \{X_1X_4 - X_2X_3 = t\}$.

Let $T_{h,m}$ be the toric degeneration of $E_{h,m}$. Then

$$T_{h,m} := \text{Spec} \mathbb{C}[\tilde{M}_{h,m}]$$

where the semigroup

$$\tilde{M}_{h,m} := \{(i, j, k) \in \mathbb{Z}_{\geq 0}^3 : m|(j - i), jp - qi \geq 0, i + j \geq k\}.$$ 

has surjective homomorphism $\pi : (i, j, k) \mapsto (i, j)$ onto $M_{h,m}^+$ where elements $(i, j)$ can be identified with the highest vector $X_1^iY_j \in V_{i+j}$ and the lattice points $\pi^{-1}(i, j) \subset \tilde{M}_{h,m}$ correspond to the standard basis of $V_{i+j}$. So the toric degeneration $T_{h,m}$ of $E_{h,m}$ is defined by a 3-dimensional cone

$$\sigma_0 = \mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}v_2 + \mathbb{R}_{\geq 0}v_3 + \mathbb{R}_{\geq 0}v_4$$

where $v_1 = (0, 0, 1)$, $v_2 = (1, 1, -1)$, $v_3 = (0, 1, 0)$, $v_4 = (p, -q, 0)$ satisfying the relation

$$pv_1 + pv_2 = (p + q)v_3 + v_4.$$

In the notations of [AB04], the dual 3-dimensional cone $\tilde{\sigma}_0$ has a surjective projection onto 2-dimensional momentum cone $\tilde{\sigma}$ where $\sigma = \mathcal{C}(E_{h,m}) = \mathbb{R}_{\geq 0}v_3 + \mathbb{R}_{\geq 0}v_4$. The
fibers of this projection are 1-dimensional string polytopes. Since \( p + q \neq 1 \), the affine toric variety \( T_{h,m} \) does not admit a quasihomogeneous \( SL(2) \)-action (see also a remark in \cite[Section 8]{Ga08}).

**Remark 4.5.** It is not easy to describe the behavior of toric degenerations under equivariant morphisms of spherical varieties. The simplest example in \cite{LJ} shows that toric degenerations do not preserve equivariant open embeddings: toric generation \( U_0 \) of the open orbit \( U \subset E_{h,m} \) is not an open subset in \( T_{h,m} \), the corresponding birational morphism \( U_0 \to T_{h,m} \) contracts a divisor in \( U_0 \).

We remark that if \( m = 1 \) then \( T^+_{h,m} \) locally isomorphic to product \( \mathbb{A}^2/\mu_p \times \mathbb{A}^1 \).

Therefore, toric degeneration \( T^+_{h,m} \) of \( E^+_{h,m} \) has the same type of toroidal singularity along the curve \( C^+ \subset T^+_{h,m} \) as \( C^+ \subset E^+_{h,m} \). However, the same is not true for the toric degeneration \( T^-_{h,m} \) of \( E^-_{h,m} \). For instance, if \( m = 1 \) then \( T^-_{h,1} \) has only a single isolated singularity, but singular locus of \( E^-_{h,1} \) is the whole curve \( C^- \subset E^-_{h,1} \).

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