Abstract. Analogical proportions are expressions of the form “a is to b what c is to d” at the core of analogical reasoning. This paper contributes to the mathematical foundations of analogical proportions in the axiomatic tradition as initiated by Yves Lepage two decades ago. For this we introduce proportoids as sets endowed with a 4-ary analogical proportion relation $a : b :: c : d$ satisfying a suitable set of axioms and study different kinds of proportion-preserving mappings and relations and their properties.

1. Introduction

Analogical proportions are expressions of the form “a is to b what c is to d” at the core of analogical reasoning. The purpose of this paper is to further develop from a mathematical point of view the axiomatic approach to analogical proportions as initiated — in the tradition of the ancient Greeks — by Lepage (2003) two decades ago. We will mostly remain agnostic regarding the concrete set of assumed axioms as we will only assume three evident properties. More formally, we define proportoids as sets endowed with a 4-ary analogical proportion relation $a : b :: c : d$ satisfying the axioms (1–3) (cf. Definition 2), and we define the fundamental concepts of proportional sub proportoids, homomorphisms and congruences. It is important to emphasize that all constructions in this paper are kept very general and do not refer to a concrete model of an analogical proportion relation. However, we do want to point out that the algebro-logical framework of Anti´c (2022, 2023c) provides a canonical analogical proportion relation in any algebraic and logical structure containing functions and relations (see §14).

Functions preserving analogical proportions have already proven to be of practical interest (e.g. Couceiro, Hug, Prade, & Richard, 2017), and studying their mathematical properties is essential for understanding proportions. We therefore define the notion of a proportional homomorphism preserving the analogical proportion relation across different domains by satisfying a stronger version of the proportional inference principle of Couceiro et al. (2017), corresponding to the analogical jump in Davies and Russell (1987) (cf. Couceiro & Lehtonen, 2023) (cf. Remark 10). In §5 we then introduce the fundamental concept of a congruence on a proportoid and show in Theorem 20 that the kernel of every homomorphism is a congruence as expected.

In a similar vein, we then introduce the notion of a proportional analogy preserving the relationship between elements of the source domain. We show in Theorem 35 that analogies satisfy the strong proportion-preserving property (cf. Remark 10) given that the underlying proportoids are transitive. In §8 we provide a procedure for constructing partial proportional analogies using an enumeration of the source structure and a selection function.
In §9 we define the notion of a **proportional identity** and show that it is a congruence relation in a wide class of proportoids (Theorem 60).

Functions between proportoids are fundamental. In §10 we therefore introduce a number of binary relations between functions on proportoids which we believe are appealing from a mathematical point of view since most of them are equivalence relations and some are congruences.

Finally, in §13 we introduce a notion of **similarity** using analogical proportions and show in Theorem 119 that it is a congruence under mild conditions.

In a broader sense, this paper is a further step towards a mathematical theory of analogical proportions.

## 2. Preliminaries

A set $P$ is **denumerable** iff there is a surjective mapping $\mathbb{N} \to P$.

We denote the **identity function** on any set by $I$. A function $f$ on a partially ordered set $(P, \leq)$ is **monotone** iff $a \leq f a$ holds for every $a \in P$. A **partial function** is a function $F : P \to R \cup \{u\}$ possibly yielding the value $u$ ("undefined") for some elements of $P$. We define the **domain** of such a function by

$$
\text{dom } F := \{a \in P \mid F a \neq u\}.
$$

We call $F$ **total** iff $\text{dom } F = P$.

Let $\rho$ be a binary relation on unary mappings on $\mathcal{B}$. An unary mapping $F$ is **idempotent** with respect to $\rho$ iff $F \rho F^2$. As usual, the relation $\rho$ is **left compatible** iff

$$
F \rho G \Rightarrow E F \rho E G, \quad \text{for all } E,
$$

**right compatible** iff

$$
F \rho G \Rightarrow F E \rho E G, \quad \text{for all } E,
$$

and **compatible** iff

$$
E \rho F \text{ and } G \rho H \Rightarrow E G \rho F H.
$$

A left [right] compatible equivalence is called a **left [right] congruence**. A **congruence** is a compatible equivalence relation.

The following characterization of congruences is folklore (see e.g. Howie, 2003, Proposition 1.5.1):

**Proposition 1.** A binary relation on a semigroup is a congruence iff it is both a left and a right congruence.

## 3. Proportoids

In this section, we formally introduce proportoids as sets endowed with a 4-ary analogical proportion relation satisfying a suitable set of axioms which is rooted in but different from the original axiomatization of Lepage (2003) (see Remark 3).

**Definition 2.** A **proportoid** is a pair $\mathcal{B} = (P, ::)$ consisting of a non-empty set $P$ endowed with a 4-ary analogical proportion relation :: on $P$ satisfying the following axioms, for all $a, b, c, d \in P$\footnote{Lepage (2003) used other names for the axioms — we adapt here the terminology in Antić (2022, §4.3).}:

1. $a : b :: a : b$ (reflexivity; r),
2. $a : b :: c : d \Leftrightarrow c : d :: a : b$ (symmetry; s),
3. $a : b :: c : d \Leftrightarrow b : a :: d : c$ (inner symmetry; y).
Beyond that, we will consider the following properties, for \( a, b, c, d, e, f \in P \) and injective function \( g : P \to P \):

1. \( a : a :: c : c \) (inner reflexivity; e)
2. \( a : b :: c : d \iff a : c :: b : d \) (central permutation; c)
3. \( a : a :: a : d \iff d = a \) (determinism; d)
4. \( a : b :: b : a \) (commutativity; o)
5. \( a : b :: c : d \text{ and } c : d :: e : f \iff a : b :: e : f \) (transitivity; t)
6. \( a : b :: c : d \) and \( b : e :: d : f \iff a : e :: c : f \) (inner transitivity; i)
7. \( a : a :: c : d \Rightarrow d = c \) (strong inner reflexivity; v)
8. \( a : b :: a : d \Rightarrow d = b \) (strong reflexivity; x)
9. \( a : ga :: c : gc \) (functionality; f).

The symbol after the semicolon like e.g. “t” will be used in proofs as shortcuts for the respective axiom (in this case “transitivity”) and to denote proportoids satisfying the axiom — for example, a proportoid satisfying transitivity is called a \textit{t-proportoid}.

Moreover, define the \textbf{solution set} of the \textit{proportional equation} (or \textit{p-equation}) \( a : b :: c : x \) by

\[
S_{\mathcal{P}}(a : b :: c : x) := \{ d \in P \mid a : b :: c : d \}.
\]

**Remark 3.** Lepage (2003) uses the axioms (2), (5), (10), and (11) in his original axiomatization in the linguistic context. We agree with (2). Although accepted by many researchers, central permutation (5) is debatable as shown by the simple counterexample:

\[
\begin{array}{ccc}
  a & b & d \\
  \hline
  c & a & \\
\end{array}
\]

That central permutation is problematic has been observed by others as well (see e.g. Murena, Cornuéjols, & Dessalles, 2018; Lim, Prade, & Richard, 2021). The remaining axioms in Lepage’s original list are in general inadequate as well as demonstrated by simple counterexamples in the proof of Antić (2022, Theorem 28).

**Example 4.** Let \( \leq \) be a partial ordering on \( P \). Define

\[
a : b :: c : d \iff (a \leq b \text{ and } c = d) \text{ or } (a < b \text{ and } c < d) \text{ or } (a > b \text{ and } d > c).
\]

Then \( (P, :: \leq) \) is a proportoid.

**Example 5.** Let \( \circ \) be a binary operation on \( P \). Define

\[
a : b :: c : d \iff a \circ b = c \circ d.
\]

Then \( (P, :: \circ) \) is a t-proportoid.

**Example 6.** Let \( \theta \) be a reflexive and symmetric binary relation on \( P \). Define

\[
a : b :: c : d \iff a \theta b \text{ and } c \theta d.
\]

Then \( (P, :: \theta) \) is a proportoid. If \( \theta \) is transitive, then \( (P, :: \theta) \) is transitive.
Example 7. In \(\mathbb{N}\), define the well-known difference proportion relation (and see Antić, 2023a, Difference Proportion Theorem) as follows:
\[
a : b :: c : d :\Leftrightarrow a - b = c - d.
\]
Then \((\mathbb{N}, ::)\) is a proportoid satisfying all properties in Definition 2.

With a notion of structure there is always an associated notion of subproportoid:

Definition 8. A proportoid \(\mathcal{R} = (R, ::)\) is a (proportional) subproportoid of \(\mathcal{B} = (P, ::)\) iff \(R \subseteq P\) and \(a : b :: c : d\) holds in \(\mathcal{R}\) iff it holds in \(\mathcal{B}\), for all \(a, b, c, d \in R\).

Intuitively, a subproportoid preserves the analogical proportion relation between the elements which in general may not be the case. For example, it may be the case that \(a : b :: c : d\) holds in the proportoid \(\mathcal{R}\), whereas if we consider the larger proportoid \(\mathcal{B}\) we can find some \(d'\) such that the relation between \(a\) and \(b\) and between \(c\) and \(d'\) is more similar than between \(c\) and \(d\).

4. Proportional homomorphisms

With any kind of structure there comes a notion of a structure-preserving mapping and proportoids are no exception.

In the rest of the paper, let \(\mathcal{B} = (P, ::)\) and \(\mathcal{R} = (R, ::)\) be proportoids where we often omit the indices from notation in case the underlying sets are clear from the context.

Definition 9. We call a mapping \(H : \mathcal{B} \rightarrow \mathcal{R}\) a (proportional) homomorphism iff for all \(a, b, c, d \in P\),
\[
(13) \quad a : b :: P c : d \quad \Leftrightarrow \quad Ha : Hb :: Hc : Hd.
\]

A (proportional) endomorphism is a homomorphism of the form \(\mathcal{B} \rightarrow \mathcal{B}\), and a (proportional) epimorphism is an onto homomorphism. A (proportional) isomorphism is a bijective homomorphism. We call \(\mathcal{B}\) and \(\mathcal{R}\) are (proportionally) isomorphic — in symbols, \(\mathcal{B} \cong \mathcal{R}\) — iff there is a proportional isomorphism from \(\mathcal{B}\) to \(\mathcal{R}\).

Remark 10. The only if part “\(\Rightarrow\)” of the equivalence in (13) is called the analogical inference principle by Couceiro et al. (2017) (and see Couceiro & Lehtonen, 2023) and it can be viewed as a particular case of the so-called analogical jump by Davies and Russell (1987). We prefer using the term “proportional” instead of “analogical” and proportion-preserving property (or PPP) instead of “analogical inference principle”, and we call the equivalence in (13) the strong proportion-preserving property (or sPPP).

Remark 10 motivates the following definition:

Definition 11. A proportion-preserving mapping (or pp-mapping) is any function \(P : \mathcal{B} \rightarrow \mathcal{R}\) satisfying
\[
(14) \quad a : b :: P c : d \quad \Rightarrow \quad Pa : Pb :: R P c : Pd, \quad \text{for all} \quad a, b, c, d \in P.
\]

Example 12. The iterative successor function given by
\[
S^k a := a + k, \quad \text{for all} \quad a \in \mathbb{N} \text{ and } k \geq 0,
\]
is a homomorphism on \((\mathbb{N}, ::)\) defined as in Example 7.

Example 13. In \(\mathcal{R} := (\{0, 1\}, ::)\), define the boolean proportion relation as follows (Klein, 1982; Antić, 2023b):
\[
a : b :: c : d :\Leftrightarrow (a = b \quad \text{and} \quad c = d) \quad \text{or} \quad (a \neq b \quad \text{and} \quad c \neq d).
\]
The negation function \( \neg \) is an isomorphism since
\[
(a : b :: c : d) \iff (a = b \text{ and } c = d) \text{ or } (a \neq b \text{ and } c \neq d),
\]
\[
\iff (\neg a = \neg b \text{ and } \neg c = \neg d) \text{ or } (\neg a \neq \neg b \text{ and } \neg c \neq \neg d).
\]

**Proposition 14.** The space of all endomorphisms forms a monoid with respect to function composition with the neutral element given by the identity function.

**Proof.** It follows from the definition that homomorphisms are closed under composition, that is, if \( H \) and \( G \) are homomorphisms, then \( H \circ G \) is a homomorphism as well:
\[
(a : b :: c : d) \iff H : Ha : Hb :: Hc : Hd,
\]
\[
\iff G : Ha : Hb :: G : Hc : Hd.
\]
The identity function is clearly a homomorphism. \( \square \)

**Theorem 15** (First Injectivity Theorem). Every homomorphism defined on a d-proportoid is injective.

**Proof.** The following derivation shows that \( Ha = Hb \) implies \( a = b \) for any homomorphism \( H : \mathcal{P} \to \mathcal{R} \) on a d-proportoid \( \mathcal{P} \) and \( a, b \in P \):
\[
\frac{Ha = Hb}{\frac{Ha : Hb :: Ha : Hb}{\frac{a : a :: a : b}{\frac{a = b.}{d}}}}
\]

\( \square \)

**Remark 16.** Theorem 15 motivates the study of non-strong proportion-preserving functions as in Couceiro et al. (2017) (cf. Remark 10).

### 5. Proportional congruences

In universal algebra, congruences provide a mechanism for factorizing proportoids into equivalence classes compatible with the algebraic operations. Here, we require that the equivalence classes preserve the analogical proportion relation giving rise to the notion of a proportional congruence. We will show in Theorem 20 that proportional congruences and homomorphisms via kernels.

**Definition 17.** An equivalence relation \( \theta \) on \( P \) is a (proportional) congruence on \( P = (P, ::) \) iff for all elements \( a, b, c, d, e, f, g, h \in P \),
\[
\frac{a \theta e \quad b \theta f \quad c \theta g \quad d \theta h}{a : b :: c : d \iff e : f :: g : h.}
\]

or, equivalently,
\[
\frac{a \theta e \quad b \theta f \quad c \theta g \quad d \theta h}{a : b :: c : d \iff e : f :: g : h.}
\]

**Proposition 18.** For any congruence \( \theta \) on \( \mathcal{P} \),
\[
\frac{a \theta c}{a : b :: c : d}. 
\]

**Proof.** The following inference rule is an instance of the rule defining a congruence in Definition 17:
\[
\frac{a \theta a \quad b \theta b \quad a \theta e \quad b \theta d}{a : b :: c : d}. 
\]
Now observe that since $\theta$ is reflexive and analogical proportions are reflexive, we can omit $a\theta b$, $b\theta b$, and $a : b :: a : b$ in the first line of the rule which immediately yields the first implication of the proposition.

A standard construction in universal algebra is given by the kernel of a homomorphism (cf. Burris & Sankappanavar, 2000, Definition 6.7) which we directly adapt here:

**Definition 19.** The kernel of a homomorphism $H : \mathfrak{P} \rightarrow \mathfrak{R}$ is given by

$$\ker H := \{ ab \in P^2 \mid Ha = Hb \}.$$ 

We now show that homomorphisms and congruences of proportoids are in the same way related as in universal algebra via kernels:

**Theorem 20.** The kernel of any homomorphism is a congruence.

**Proof.** Let $H : \mathfrak{P} \rightarrow \mathfrak{R}$ be a homomorphism. We show

$$Ha = He \quad Hb = Hf \quad Hc = Hg \quad Hd = Hh \quad a : b :: c : d$$

$e : f :: g : h$ for all $a, b, c, d, e, f, g, h \in P$, by the following derivation:

$$\frac{Ha = He \quad Hb = Hf \quad Hc = Hg \quad Hd = Hh \quad a : b :: c : d}{e : f :: g : h}.$$ 

$$\frac{e : f :: g : h}{Ha : Hb :: Hc : Hd}$$ 

$$\frac{Ha : Hb :: Hc : Hd}{He : Hf :: Hg : Hh}$$ 

$$\frac{He : Hf :: Hg : Hh}{e : f :: g : h}.$$ 

□

6. **Proportional polymorphisms**

The following definition follows the standard definition of a polymorphism on a relational proportoid:

**Definition 21.** A (proportional) polymorphism of $\mathfrak{P} = (P, ::)$ is any operation $f : P^n \rightarrow P$ preserving the analogical proportion relation in the sense that

$$a_1 : b_1 :: c_1 : d_1 \quad \ldots \quad a_n : b_n :: c_n : d_n$$

holds for all $a_i, b_i, c_i, d_i \in P$, $1 \leq i \leq n$.

**Remark 22.** Notice that for any unary function $f : P \rightarrow P$, the above condition amounts to the proportion-preserving property (cf. Remark 10)

$$f a : f b :: f c : f d$$

holds for all $a, b, c, d \in P$. The following definition follows the standard definition of a polymorphism on a relational proportoid:

**Definition 23.** A strong (proportional) polymorphism (or s-polymorphism) is a polymorphism where the implication of the inference rule (PP) is turned into an equivalence.

**Example 24.** The iterated successor function $S^k$ is an s-polymorphism of $(\mathbb{N}, ::)$ defined as in Example 7 thus satisfying

$$a : b :: c : d \quad \Leftrightarrow \quad S^k a : S^k b :: S^k c : S^k d,$$

for all $a, b, c, d \in \mathbb{N}$ and $k \geq 0$. Moreover, addition is a polymorphism of $(\mathbb{N}, ::)$ thus satisfying
Notice that we can use s-polymorphisms to compose and decompose proportions:

**Example 25.** Since \( S \) is an s-polymorphism of \((\mathbb{N}, S)\) as has been observed in Example 24, we immediately obtain the following characterization of the analogical proportion relation in \((\mathbb{N}, ::)\):

\[
a : b :: c : d \iff a - \min(a, b, c, d) : b - \min(a, b, c, d) :: c - \min(a, b, c, d) : d - \min(a, b, c, d).
\]

For example, we can use the fact that \( S \) is an s-polymorphism in the following way:

\[
2 : 3 :: 5 : 7 \iff SS0 :: SSS0 :: SSSSS0 :: SS0 :: SSSSS0 :: 0 : S0 :: SSS0 :: SSSS0 \\
\iff 0 : 1 :: 3 : 5,
\]

which fails in \((\mathbb{N}, ::)\) since \(0 - 1 \neq 3 - 5\). That is, we can decompose each analogical proportion \(a : b :: c : d\) in \((\mathbb{N}, ::)\) in such a way that at least one of \(a, b, c, d\) is equal to 0.

### 7. Proportional analogies

In §4 we defined proportional homomorphisms as mappings satisfying the strong proportion-preserving property (cf. Remark 10). In this section, we are interested in a related but different notion of proportion-preserving mapping defined as follows. First, we need the following construction (compare to Definition 2):

**Definition 26.** Given two proportoids \( \mathfrak{P} = (P, ::_{\mathfrak{P}}) \) and \( \mathfrak{R} = (R, ::_{\mathfrak{R}}) \), we construct the **pair proportoid** \( \mathfrak{P}\mathfrak{R} = (P, R, ::_{\mathfrak{P}\mathfrak{R}}) \), where \( ::_{\mathfrak{P}\mathfrak{R}} \subseteq P^2 \times R^2 \) is a 4-ary analogical proportion relation on \( P \) and \( R \) satisfying, for all \( a, b \in P \cap R \):

\[
(15) \quad a : b ::_{\mathfrak{P}\mathfrak{R}} a : b \quad (\text{reflexivity; } r),
\]

and for all \( a, b \in P \) and \( c, d \in R \):

\[
(16) \quad a : b ::_{\mathfrak{P}\mathfrak{R}} c : d \iff c : d ::_{\mathfrak{R}} a : b \quad (\text{symmetry; } s),
\]

\[
(17) \quad a : b ::_{\mathfrak{P}\mathfrak{R}} c : d \iff b : a ::_{\mathfrak{R} \mathfrak{P}} d : c \quad (\text{inner symmetry; } y).
\]

Notice that every proportoid \((P, ::_{\mathfrak{P}})\) can be turned into a pproportoid \((P, P, ::_{\mathfrak{P}\mathfrak{P}})\) and we will not distinguish between the two.

Moreover, we consider the following properties for all elements from the appropriate sets:

\[
(18) \quad a : b ::_{\mathfrak{P}\mathfrak{R}} c : d \quad \text{and} \quad c : d ::_{\mathfrak{R}\mathfrak{R}} e : f \implies a : b ::_{\mathfrak{P}\mathfrak{R}} e : f \quad (\text{transitivity; } t).
\]

As for proportoids, we add additional symbols to denote pproportoids satisfying specific properties; for example, an **ppt-triple** is a pproportoid satisfying inner transitivity.

A **ppt-triple** is a triple of proportoids

\[
\mathfrak{P}\mathfrak{R}\mathfrak{Q} = (P, R, ::_{\mathfrak{P}\mathfrak{R}})(R, Q, ::_{\mathfrak{R}\mathfrak{Q}})(P, Q, ::_{\mathfrak{P}\mathfrak{Q}})
\]
satisfying transitivity [18].

Finally, we define

\[ S(a : b :: c : x) := \{ d \in R \mid a : b :: c : d \} \]

We will often omit the indices from notation.

**Definition 27.** A (proportional) analogy between the proportoids \( \Psi = (P, ::_\Psi) \) and \( \Re = (R, ::_\Re) \) is any mapping \( A : \Psi \to \Re \) satisfying in the proportoid \( \Psi \Re = (P, R, ::_{\Psi \Re}) \) the properties

\[ a : b ::_\Psi Aa : Ab, \quad \text{for all } a, b \in P. \]

We say that \( \Psi \) and \( \Re \) are (proportionally) analogically isomorphic — in symbols, \( \Psi \cong \Re \) — iff there is a bijective analogy from \( \Psi \) to \( \Re \).

**Fact 28.** The identity function is an analogy in any e-proportoid satisfying inner reflexivity [4].

**Example 29.** The iterative successor function \( S^k \) is an analogy in \( (\mathbb{N}, ::) \) defined as in Example 7.

**Example 30.** The negation operation \( \neg \) is an analogy on \((\{0, 1\}, ::)\) defined as in Example 13.

**Example 31.** Every monotone function is an analogy in \((P, :: \leq)\) defined as in Example 4.

7.1. **Kernels.** The following definition is analogous to the definition of a homomorphism kernel (cf. Definition 19):

**Definition 32.** We define the kernel of an analogy \( A : \Psi \to \Re \) by

\[ \ker A := \{ ab \in P^2 \mid Aa = Ab \}. \]

We have the following implications:

**Proposition 33.** \( ab \in \ker A \quad \frac{a = b}{a = b} \)

**Proof.**

\[ \begin{array}{c}
A \text{ is an analogy} \\
\frac{a : b :: Aa : Ab}{a : b :: Aa : Ab} \\
\frac{ab \in \ker A}{Aa = Ab} \\
\frac{a : b :: Aa : Ab}{a = b.}
\end{array} \]

\[ \square \]

**Proposition 34.** \( ab \in \ker A \quad cd \in \ker A \quad \frac{a : b :: c : d}{t.} \)

**Proof.**

\[ \begin{array}{c}
\frac{ab \in \ker A, cd \in \ker A}{a = b, c = d} \\
\frac{a : b :: c : d}{t.}
\end{array} \]

\[ \square \]
7.2. **Strong proportion-preserving property.** Recall that homomorphisms satisfy the strong proportion-preserving property (cf. Remark 10)

\[ a : b ::_Q c : d \iff Ha : Hb ::_R Hc : Hd \]

for all \( a, b, c, d \in P \). We have the following important result relating homomorphisms and analogies:

**Theorem 35.** Every analogy on a ppt-triple \( \Psi \Re \Psi \) satisfies the strong proportion-preserving property.

**Proof.** Let \( A : \Psi \to \Re \) be an analogy. We only prove the direction from left to right with the other direction being analogous: for any \( a, b, c, d \in P \), we have

\[
\begin{align*}
\frac{a : b ::_Q c : d \quad \text{t}}{Aa : Ad} & \\
\frac{a : b ::_Q Ac : Ad \quad \text{s}}{Ac : Ad} & \\
\frac{Ac : Ad ::_R a : b \quad \text{t}}{Aa : Ab} & \\
\frac{Aa : Ab ::_R Ac : Ad \quad \text{s}}{a : b ::_Q c : d} & \quad \text{t}
\end{align*}
\]

\[ \square \]

7.3. **Second Injectivity theorem.** Interestingly, the next result shows that analogies are injective in the wide range of dt-proportoids as an analogue to the First Injectivity Theorem [15].

**Theorem 36 (Second Injectivity Theorem).** Every analogy on a ppt-triple \( \Psi \Re \Psi \), with \( \Psi \) a d-proportoid, is injective.

**Proof.** Let \( A : \Psi \to \Re \) be an analogy. We show that \( Aa = Ab \) implies \( a = b \), for any \( a, b \in P \), by the following derivation:

\[
\begin{align*}
\frac{a : b ::_Q Aa : Ab \quad \text{t}}{a : b ::_Q Aa : Aa} \quad \text{t} & \\
\frac{AAa : Ab \quad \text{t}}{AAa : Aa ::_R a : a \quad \text{t}} & \\
\frac{a = b \quad \text{d}}{a : a ::_Q a : b \quad \text{s}} & \\
\frac{a = b \quad \text{d}}{a : a ::_Q a : b \quad \text{s}} & \\
\frac{a : a ::_Q a : b \quad \text{s}}{a = b \quad \text{d}} & \\
\frac{a : a ::_Q a : b \quad \text{s}}{a = b \quad \text{d}} & \\
\frac{a = b \quad \text{d}}{a : a ::_Q a : b \quad \text{s}} & \\
\end{align*}
\]

\[ \square \]

7.4. **Closedness under composition.** The composition of two analogies yields another analogy given that the underlying proportoid is transitive which is shown in the next result:

**Theorem 37.** The space of all analogies on a t-proportoid forms a monoid with respect to function composition with the neutral element given by the identity function.

**Proof.** First, it follows from the definition that analogies are closed under composition in case the underlying algebra is transitive, that is, if \( A : \Psi \to \Psi \) and \( B : \Psi \to \Psi \) are analogies, then \( B \circ A \) is an analogy as well by the following derivation:

\[
\begin{align*}
\frac{a : b ::_Q Aa : Ab \quad \text{t}}{Aa : Ab ::_Q BAa : BAb} & \\
\end{align*}
\]

The identity function is an analogy as an immediate consequence of reflexivity. \[ \square \]
7.5. **Proportional idempotency.** Every unary function on a proportoid can be applied repeatedly, which motivates the following definition:

**Definition 38.** We say that \( F : \Psi \to \Psi \) is (proportionally) idempotent iff
\[
F a : F b :: FF a : FF b, \quad \text{holds for all } a, b \in P.
\]

**Fact 39.** Every analogy is idempotent.

**Remark 40.** Notice that by symmetry (2), every idempotent function \( F : \Psi \to \Psi \) on a t-proportoid \( \Psi \) satisfies
\[
F^m a : F^n b :: F^m a : F^n b \quad \text{for all } m, n \geq 0 \text{ and } a, b \in P.
\]

8. **Partial proportional analogies**

In this section, we provide a procedure for constructing partial proportional analogies from a given enumeration of the source proportoid and selection functions on subsets of the target proportoid (which exist by the presumed axiom of choice).

Concretely, let \( \Psi \mathcal{R} \) be a denumerable pair of i-proportoids satisfying inner transitivity (9), let \( \sigma : 2^R \to R \cup \{u\} \) be a selection function (recall that \( u \) stands for “undefined”) such that for every \( S \subseteq R \),
\[
\sigma S = u : \iff S = \emptyset,
\]
and let \( e \) be an enumeration of \( P \). We define
\[
S_{\Psi \mathcal{R}}(a : b :: u : x) := \emptyset,
\]
for all \( a, b \in P \).

We are now ready to introduce the main notion of this section:

**Definition 41.** Define the partial analogy \( A_{\sigma, e} \) inductively, for every \( i \in \mathbb{N} \), by
\[
A_{\sigma, e} e_1 := \sigma S,
\]
\[
A_{\sigma, e} e_{i+1} := \sigma S_{\Psi \mathcal{R}}(e_i : e_{i+1} :: A_{\sigma, e} e_i :: x).
\]
Notice that the domain of \( A_{\sigma, e} \) cannot be empty since we always have \( R \neq \emptyset \) by assumption and thus
\[
e_1 \in \text{dom } A_{\sigma, e}.
\]

The next result shows that the construction of Definition 41 always yields a partial analogy:

**Theorem 42.** On any i-proportoid, the partial function \( A_{\sigma, e} : \Psi \to \mathcal{R} \) is an analogy on its non-empty domain, for every choice of \( e \) and \( \sigma \).

**Proof.** We need to show
\[
a : b :: \Psi \mathcal{R} A_{\sigma, e} a : A_{\sigma, e} b, \quad \text{for all } a, b \in \text{dom } A_{\sigma, e}.
\]

By definition, we have
\[
a = e_{i_a} \quad \text{and} \quad b = e_{i_b}, \quad \text{for some } i_a, i_b \in \mathbb{N}.
\]
Without loss of generality, we can assume \( i_a \leq i_b \) since otherwise we can apply inner symmetry to obtain \( b : a :: A_{\sigma, e} b : A_{\sigma, e} a \). So we have
\[
i_b = i_a + j, \quad \text{for some } j \in \mathbb{N}.
\]
By definition of $A_{e,\sigma}$, we have
\[ e_i : e_{i+1} : : PR A_{\sigma, e} e_i : A_{\sigma, e} e_{i+1} \]
\[ \vdots \]
\[ e_{i+j-1} : e_{i+j} : : PR A_{\sigma, e} e_{i+j-1} : A_{\sigma, e} e_{i+j} \]
and
\[ e_{i+k} \in \text{dom } A_{\sigma, e}, \quad \text{for all } 1 \leq k \leq j. \]
By the assumed inner transitivity (9) axiom, we obtain
\[ e_i : e_{i+j} : : PR A_{\sigma, e} e_i : A_{\sigma, e} e_{i+j} \]
which is equivalent to (20).

Example 43. Let $e$ be the identity on $\mathbb{N}$, let $\sigma : 2^\mathbb{N} \to \mathbb{N} \cup \{u\}$ be a selection function, and let $S$ be the unary successor function. Recall the difference proportion relation in $(\mathbb{N}, ::)$ of Example 7
\[ a : b :: c : d : \Leftrightarrow a - b = c - d. \]
Definition [H1] yields (recall that we have chosen $e$ to be the identity function on $\mathbb{N}$ and it is thus omitted)

\[ A_{\sigma, 1} \sigma \mathbb{N} \]
\[ A_{\sigma, 2} = \sigma S_{(\mathbb{N}, S)}(1 : 2 :: A_{\sigma, 1} : x) \]
\[ = \sigma(2 - 1 + A_{\sigma, 1}) \]
\[ = 1 + A_{\sigma, e} 1 \]
\[ A_{\sigma, 3} = \sigma S_{(\mathbb{N}, S)}(2 : 3 :: A_{\sigma, 2} : x) \]
\[ = \sigma S_{(\mathbb{N}, S)}(2 : 3 :: 1 + A_{\sigma, 1} : x) \]
\[ = \sigma(3 - 2 + 1 + A_{\sigma, 1}) \]
\[ = 2 + A_{\sigma, 1} \]
\[ \vdots \]
\[ A_{\sigma, i} = i + A_{\sigma, 1}, \quad \text{for all } i \in \mathbb{N}. \]
Since $\text{dom } A_{\sigma} = \mathbb{N}$, the function $A_{\sigma} : \mathbb{N} \to \mathbb{N}$ is a total analogy.

Fact 44. Every partial analogy on a ppt-triple $\Psi \Psi \Psi$ satisfies the strong proportion-preserving property on its domain.

Proof. Analogous to the proof of Theorem [55].

Fact 44 tells us that our above procedure for constructing partial analogies automatically yields a procedure for constructing functions satisfying the strong proportion-preserving property and thus for constructing analogy-preserving functions as studied by Couceiro et al. (2017) and Couceiro and Lehtonen (2023).
**Definition 45.** We define the *cardinality* of a partial analogy $A_{\sigma,e}$ by

$$\#A_{\sigma,e} := \begin{cases} \min\{i \in \mathbb{N} \mid A_{\sigma,e}e_i = u\} - 1 & A_{\sigma,e} \text{ is partial,} \\ \infty & A_{\sigma,e} \text{ is total.} \end{cases}$$

We call $A_{\sigma,e}$ *\(\sigma\)-maximal* iff there is no analogy $A_{\sigma',e}$ — with respect to the same enumeration $e$ — such that

$$\text{dom } A_{\sigma,e} \subsetneq \text{dom } A_{\sigma',e}.$$  

**Fact 46.** $A_{\sigma,e}$ is $\sigma$-maximal iff $\#A_{\sigma,e}$ is maximal with respect to $\sigma$. Every total analogy $A_{\sigma,e}$ is $\sigma$-maximal.

**Proof.** An immediate consequence of the fact that by construction, $A_{\sigma,e}e_i = u$ implies $A_{\sigma,e}e_j = u$, for all $j \geq i$. \(\square\)

We shall now show that every analogy is constructed from an enumeration and a selection function as above.

**Theorem 47.** For every partial analogy $A : \mathcal{P} \to \mathcal{R}$ there is an enumeration $e$ of $P$ and a selection function $\sigma : 2^R \to R \cup \{u\}$ such that $A = A_{\sigma,e}$.

**Proof.** Let $e$ be an enumeration of $P$ enumerating first the elements of $\text{dom } A$ and then the rest in arbitrary order. Define, for every $i \geq 1$,

$$A_{\sigma,e}e_i := \sigma R := Ae_1,$$

$$A_{\sigma,e}e_{i+1} := \begin{cases} \sigma S_{\psi R}(e_i : e_{i+1} :: Ae_i : x) := Ae_{i+1} & e_{i+1} \in \text{dom } A, \\ u & \text{otherwise}. \end{cases}$$

We clearly have $A_{\sigma,e} = A$ by construction. \(\square\)

Notice that $\sigma$-maximality is defined with respect to $\sigma$ for some fixed $e$ — this does not guarantee that there is no analogy $A$ constructed by other means with $\text{dom } A_{\sigma,e} \subsetneq \text{dom } A$. We say that $A_{\sigma,e}$ is *maximal* iff there is no such analogy $A$.

**Corollary 48.** A partial analogy $A_{\sigma,e}$ is maximal iff there is no analogy $A_{\sigma',e'}$ such that $\#A_{\sigma,e} < \#A_{\sigma',e'}$.

**Proof.** A direct consequence of Fact 46 and Theorem 47. \(\square\)

Let $A_{\sigma,e}$ be an arbitrary partial analogy and let $n$ be the cardinality of $A_{\sigma,e}$. Suppose there is some $i \in \mathbb{N}$ such that

$$e_i \notin \text{dom } A_{\sigma,e}$$

satisfying

$$S(e_n : e_i :: Ae_n : x) \neq \emptyset. \quad (22)$$

Define the transposition $\tau : \mathbb{N} \to \mathbb{N}$ by

$$\tau n := i$$

$$\tau i := n$$

$$\tau j := j, \text{ for all other } j \in \mathbb{N}.$$
We then have by (22)
\[ A_{\sigma,e_i}e_{tn} = \sigma S(e_i : e_i :: A e_n : x) \neq u, \]
which entails
\[ \#A_{\sigma,e_i} = n + 1 = \#A_{\sigma,e} + 1, \]
and thus
\[ \text{dom } A_{\sigma,e} \not\subseteq \text{dom } A_{\sigma,e_i}. \]
In other words, we have increased the cardinality of the partial analogy \( A_{\sigma,e} \) by one by transforming it into the partial analogy \( A_{\sigma,e_i} \). This procedure can be iterated until a (partial) analogy has been reached whose cardinality can no longer be increased.

9. Proportional identity

In this section, we shall introduce a proportional identity relation (cf. Definition 54). First, we introduce an auxiliary identity relation:

**Definition 49.** Let \( a, b \in P \). We define
\[ a =_c b \iff a : b :: c : c, \quad \text{for some } c \in P. \]

**Fact 50.** \( a =_c b \)

**Proposition 51.** In any cet-proportoid, the relation \( =_c \) is an equivalence relation for any element \( c \).

**Proof.** The reflexivity and symmetry of \( =_c \) follow from inner reflexivity (4) and inner symmetry (3) of the analogical proportion relation, respectively. To prove transitivity, we show the implication
\[ a =_c b \quad b =_c d \]
by the following derivation:
\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
a =_c b \\
a : b :: c : c
\end{array}
\end{array}
\hline
\begin{array}{c}
b =_c d \\
b : d :: c : c
\end{array}
\end{array}
\]
\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
a =_c b \\
a : b :: c : c
\end{array}
\end{array}
\hline
\begin{array}{c}
b =_c d \\
b : d :: c : c
\end{array}
\end{array}
\]
\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
a =_c b \\
a : b :: c : c
\end{array}
\end{array}
\hline
\begin{array}{c}
b =_c d \\
b : d :: c : c
\end{array}
\end{array}
\]
\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
a =_c b \\
a : b :: c : c
\end{array}
\end{array}
\hline
\begin{array}{c}
b =_c d \\
b : d :: c : c
\end{array}
\end{array}
\]
\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
a =_c b \\
a : b :: c : c
\end{array}
\end{array}
\hline
\begin{array}{c}
b =_c d \\
b : d :: c : c
\end{array}
\end{array}
\]
\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
a =_c b \\
a : b :: c : c
\end{array}
\end{array}
\hline
\begin{array}{c}
b =_c d \\
b : d :: c : c
\end{array}
\end{array}
\]
\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
a =_c b \\
a : b :: c : c
\end{array}
\end{array}
\hline
\begin{array}{c}
b =_c d \\
b : d :: c : c
\end{array}
\end{array}
\]
\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
a =_c b \\
a : b :: c : c
\end{array}
\end{array}
\hline
\begin{array}{c}
b =_c d \\
b : d :: c : c
\end{array}
\end{array}
\]
\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
a =_c b \\
a : b :: c : c
\end{array}
\end{array}
\hline
\begin{array}{c}
b =_c d \\
b : d :: c : c
\end{array}
\end{array}
\]
\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
a =_c b \\
a : b :: c : c
\end{array}
\end{array}
\hline
\begin{array}{c}
b =_c d \\
b : d :: c : c
\end{array}
\end{array}
\]
\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
a =_c b \\
a : b :: c : c
\end{array}
\end{array}
\hline
\begin{array}{c}
b =_c d \\
b : d :: c : c
\end{array}
\end{array}
\]
\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
a =_c b \\
\end{array}
\end{array}
\hline
\begin{array}{c}
\end{array}
\end{array}
\]
\]

**Proposition 52.** Every n-ary polymorphism \( f \) satisfies, for any elements \( a_1, b_1, \ldots, a_n, b_n \in P \),
\[ a_1 =_c b_1 \quad \ldots \quad a_n =_c b_n \]
\[ f a_1 \ldots a_n = f c \ldots c \]

**Proof.**
\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
a_1 =_c b_1 \\
a_1 : b_1 :: c : c
\end{array}
\end{array}
\hline
\begin{array}{c}
f a_1 \ldots a_n = f c \ldots c \\
f a_1 \ldots a_n : f b_1 \ldots b_n :: f c \ldots c
\end{array}
\end{array}
\]
\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
a_1 =_c b_1 \\
a_1 : b_1 :: c : c
\end{array}
\end{array}
\hline
\begin{array}{c}
f a_1 \ldots a_n = f c \ldots c \\
f a_1 \ldots a_n : f b_1 \ldots b_n :: f c \ldots c
\end{array}
\end{array}
\]
\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
a_1 =_c b_1 \\
a_1 : b_1 :: c : c
\end{array}
\end{array}
\hline
\begin{array}{c}
f a_1 \ldots a_n = f c \ldots c \\
f a_1 \ldots a_n : f b_1 \ldots b_n :: f c \ldots c
\end{array}
\end{array}
\]
\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
a_1 =_c b_1 \\
a_1 : b_1 :: c : c
\end{array}
\end{array}
\hline
\begin{array}{c}
f a_1 \ldots a_n = f c \ldots c \\
f a_1 \ldots a_n : f b_1 \ldots b_n :: f c \ldots c
\end{array}
\end{array}
\]
Proposition 53. In any ct-proportoid, we have for any elements \(a, b, c, d, e, f, g, h, i \in P\),
\[
\begin{align*}
a &\equiv_i e & b &\equiv_i f & c &\equiv_i g & d &\equiv_i h & a : b :: c : d &\quad e : f :: g : h
\end{align*}
\]
Proof.

We now generalize Definition 49 as follows:

Definition 54. Let \(a, b \in P\). We define the proportional identity relation (or \(p\)-identity) by
\[
a \equiv_p b \iff a =_c b, \quad \text{for some } c \in P,
\]
In case \(a \equiv_p b\), we call \(a\) and \(b\) proportionally identical (or \(p\)-identical).

Proposition 55. In any et-proportoid, we have \(a \equiv_p b\) iff \(a =_c b\) for all \(c\).
Proof. We only have to prove the direction from left to right. By definition of identity, the assumption \(a \equiv_p b\) implies that there is some \(d\) such that \(a : b :: d : d\). By inner reflexivity (4), we have \(d : d :: c : c\). Now apply transitivity to \(a : b :: d : d\) and \(d : d :: c : c\) to obtain \(a : b :: c : c\) which is equivalent to \(a =_c b\). □

Proposition 56. \(a = b\) implies \(a \equiv_p b\) in any proportoid, and \(a \equiv_p b\) implies \(a = b\) in any det-proportoid.

Proof. The first implication is a direct consequence of inner reflexivity (4) which implies \(a : a :: c : c\), for every \(c\). The second implication follows from:
\[
\begin{align*}
a \equiv_p b &\iff a : b :: c : c \quad \text{for some } c \\
&\iff a : b :: c : c \quad \text{for all } c \\
&\iff a : b :: a : a \\
&\iff d \\
&\iff a = b.
\end{align*}
\]
□

Lemma 57. \(a : b :: e : e\) \(c : d :: f : f\) \(a : b :: c : d\) \(t\).

Proof.
Proposition 58. \( a \vdash b \quad c \vdash d \quad t. \)

Proof.

\[
\frac{c \vdash d}{a \vdash b \quad e \vdash e \quad f \vdash f \quad f \vdash f \quad c \vdash d \quad e \vdash e \quad c \vdash d \quad t}
\]

\[
\frac{a \vdash b \quad e \vdash e \quad f \vdash f \quad f \vdash f \quad c \vdash d \quad e \vdash e \quad c \vdash d \quad t}
\]

Proposition 59. Every proportional polymorphism \( f : P^n \to P \) satisfies

\[
\frac{a_1 \vdash b_1 \ldots a_n \vdash b_n}{fa_1 \ldots a_n \vdash fb_1 \ldots b_n.}
\]

Proof.

\[
\frac{a_1 \vdash b_1 \ldots a_n \vdash b_n}{fa_1 \ldots a_n \vdash fb_1 \ldots b_n.}
\]

Theorem 60. The proportional identity relation is a congruence in any cet-proportoid.

Proof. That identity is an equivalence relation is an immediate consequence of Propositions 51 and 55.

It remains to show:

\[
\frac{a \vdash e \quad b \vdash f \quad c \vdash g \quad d \vdash h \quad a : b \vdash c : d \quad e : f \vdash g : h.}{a \vdash e \quad b \vdash f \quad c \vdash g \quad d \vdash h \quad a : b \vdash c : d \quad e : f \vdash g : h.}
\]

To prove this implication, we proceed similar to the proof of Proposition 53:
Theorem 61. For any analogy $A : \Psi \rightarrow \Psi$ on a cft-proportoid $\Psi$ and any injective function $f : P \rightarrow P$,

$$Af a \equiv fA a, \quad \text{for all } a \in P.$$  

Proof. Since $A$ is an analogy by assumption, we have the following derivation, for all $a \in P$:

$$
\begin{array}{c}
\frac{f \text{ is injective}}{Aa : fA a :: a : fa} & \frac{A \text{ is an analogy}}{a : fa :: Aa : Afa} \\
\frac{Aa : fA a :: Aa : Afa}{Aa : Aa :: Afa : fA a} & \frac{Afa : fA a :: Aa : Aa}{Afa = Aa, fA a} \\
\frac{Afa = Aa, fA a}{Afa = fA a} \\
\end{array}
$$

□

Proposition 62. $\frac{a \equiv b}{Pa \equiv Pb}$ holds for any pp-mapping $P$.

Proof.

$$
\begin{array}{c}
a \equiv b \\
\frac{a : b :: c : c}{Pa : Pb :: Pc : Pc} \quad \text{for some } c
\end{array}
$$

(14)

$Pa \equiv Pb.$

□

Proposition 63. For any homomorphism $H : \Psi \rightarrow \Re$,

$$a \equiv b \quad \Leftrightarrow \quad Ha \equiv Hb.$$  

Proof. We have

$$
\begin{align*}
a \equiv b & \quad \Leftrightarrow \quad a : b :: c : c \text{ for some } c \\
& \quad \Leftrightarrow \quad Ha : Hb :: Hc : Hc \text{ for some } c \\
& \quad \Leftrightarrow \quad Ha \equiv Hb.
\end{align*}
$$

□

Definition 64. We call $a \in P$ a $p$-fixed point of $F : P \rightarrow P$ iff $a \equiv Fa$.

Proposition 65. Let $A$ be an analogy with a fixed point on a c-proportoid $(P, ::)$. Then every element of $P$ is a $p$-fixed point of $A$.

Proof. Let $a$ be an arbitrary element of $P$, and let $b \in P$ be a fixed point of $A$, that is, $Ab = b$. We then have

$$
\begin{array}{c}
\frac{A \text{ is an analogy}}{a : b :: Aa : Ab} \\
\frac{a : Aa :: b : Ab}{a \equiv Aa}
\end{array}
$$

$Ab = b$
10. PROPORTIONAL FUNCTION RELATIONS

We now turn our attention to functions on proportoids where we wish to be able to compare two functions with respect to the analogical proportion relation.

10.1. Functional proportionality. Given \( F : \Psi \to \mathbb{R} \) and \( G : \Psi \to \mathcal{Q} \), define their functional proportionality relation by

\[ F :: G \quad \iff \quad F_a : F_b ::_{\mathbb{R} \mathcal{Q}} G_a : G_b, \quad \text{for all } a, b \in P. \]

In case \( F :: G \), we say that \( F \) and \( G \) are proportional. This can be depicted as follows:

\[
\begin{array}{c}
F_a \quad \cdots \quad a \quad \cdots \quad \rightarrow \quad Ga \\
Fb \quad \cdots \quad b \quad \cdots \quad \rightarrow \quad Gb
\end{array}
\]

In case \( A \) and \( B \) are analogies, the figure for \( A :: B \) can be refined to:

\[
\begin{array}{c}
Aa \quad \cdots \quad a \quad \cdots \quad \rightarrow \quad Ba \\
Ab \quad \cdots \quad b \quad \cdots \quad \rightarrow \quad Bb
\end{array}
\]

**Fact 66.** \( F :: I \) iff \( F \) is an analogy.

Before we show that functional proportionality is a congruence relation, we shall first prove an auxiliary lemma which is interesting in its own right as it shows that functional proportionality is in a sense compatible with analogical proportions.

**Lemma 67.** For any analogies \( A : \Psi \to \mathbb{R} \) and \( B : \Psi \to \mathcal{Q} \) on a ppt-triple \( \mathcal{Q} \mathcal{R} \),

\[
\begin{array}{c}
a : b ::_{\Psi} c : d \\
Aa : Ab ::_{\mathbb{R} \mathcal{Q}} Ba : Bb
\end{array}
\]

Proof.

\[
\begin{array}{c}
a : b ::_{\Psi} c : d \\
Ba : Bb ::_{\mathcal{Q}} Bc : Bd \quad \text{sPPP 35} \\
Bs \\
Bc : Bd ::_{\mathbb{Q} \mathbb{R}} Ba : Bb \\
Ba : Bb ::_{\mathbb{R} \mathcal{Q}} Aa : Ab \quad \text{s} \\
Bs \\
Bc : Bd ::_{\mathbb{Q} \mathbb{R}} Aa : Ab \quad \text{s} \\
Aa : Ab ::_{\mathbb{R} \mathcal{Q}} Bc : Bd.
\end{array}
\]

**Theorem 68.** Functional proportionality of analogies is a congruence on any \( t \)-proportoid.

Proof. Reflexivity, symmetry, and transitivity of the proportionality relation follows by reflexivity, symmetry, and the assumed transitivity of analogical proportions. It remains to show that it is compatible with composition:

\[
\begin{array}{c}
A :: B \\
Aa : Ab :: Ba : Bb \\
ACa : ACb :: BDa : BDb \quad \text{t. 67}
\end{array}
\]

\[
\begin{array}{c}
C :: D \\
Ga : Gb :: Da : Db \\
ACa : ACb :: BDa : BDb
\end{array}
\]
Example 69. All iterative successor functions $S^k, S^\ell$ are functionally proportional in $(\mathbb{N}, :)$ (cf. Example [7], that is, $S^k :: S^\ell$ holds for all $k, \ell \geq 0$.

Proposition 70. Functional proportionality is reflexive and symmetric on any proportoid. If the underlying proportoid is transitive, then functional proportionality is transitive and therefore an equivalence relation.

Proof. Reflexivity and symmetry follow from the reflexivity and symmetry of analogical proportions, and the assumed transitivity induces transitivity. □

Theorem 71. Let $\Psi \Re \Re$ be a ppt-triple, let $A : \Psi \to \Re$ be an analogy, and let $F : \Psi \to \Re$ be an arbitrary function. If $A :: F$, then $F$ is an analogy.

Proof. We have the following derivation:

\[
\begin{align*}
A \text{ is an analogy} & \quad A :: F \\
Aa : Ab & \quad Fa : Fb \\
\hline
a : b & \quad Fa : Fb.
\end{align*}
\]

□

Theorem 72. All analogies $A, B : \Psi \to \Re$ on ppt-triples $\Psi \Re \Re$ are functionally proportional.

Proof. We have the following derivation, for any $a, b \in P$:

\[
\begin{align*}
Aa : Ab & \quad a : b ::_{\Psi \Re} a \\
\hline
Aa : Ab & \quad a : b ::_{\Re} Ba : Bb \\
\hline
Aa : Ab & \quad A :: B.
\end{align*}
\]

□

The next result connects analogies with homomorphisms with respect to functional proportionality:

Theorem 73. For any analogy $A$ and homomorphism $H$ on a t-proportoid, we have $HA :: AH$.

Proof. We have the following derivation:

\[
\begin{align*}
A \text{ is an analogy} & \quad H \text{ is a homomorphism} \\
a : b :: Ab & \quad Ha : Hb :: HAa : HAb \\
\hline
Ha : Hb & \quad Ha : Hb :: AHa : AHb \\
\hline
HAa : HAb & \quad HA :: AH
\end{align*}
\]

□
10.2. **Diamond equivalence.** Given \( F, G : \mathfrak{B} \to \mathfrak{R} \), define their (proportional) **diamond equivalence** by

\[
F : \vdash G \iff F_a : G_a :\vdash F_b : G_b, \quad \text{for all } a, b \in P.
\]

This can be depicted as follows:

```
      b
   ___|___
 Fb ----> Gb
       |
      |
    a

Fa ----- Ga
```

Notice the similarity to functional proportionality in §10.1.

**Proposition 74.** All analogies are diamond equivalent on \( i \)-propooids.

**Proof.**

\[
\begin{array}{c}
A \\ a : Aa :: b : Ab
\end{array}

\begin{array}{c}
B \\ a : Ba :: b : Bb
\end{array}

\begin{array}{c}
Aa : Ba :: Ab : Bb
\end{array}

\begin{array}{c}
A : \vdash B
\end{array}
\]

**Fact 75.** \( A : \vdash \) iff \( A \) is an analogy.

**Proposition 76.** Diamond equivalence is an equivalence relation on \( i \)-propooids.

**Proof.** Reflexivity follows from inner reflexivity and symmetry holds trivially. The following derivation proves transitivity:

\[
\begin{array}{c}
F : \vdash G \\
Fa : Ga :: Fb : Gb
\end{array}

\begin{array}{c}
G : \vdash H \\
Ga : Ha :: Gb : Hb
\end{array}

\begin{array}{c}
Fa : Ha :: Fb : Hb
\end{array}

\begin{array}{c}
F : \vdash H
\end{array}
\]

**Fact 77.** In any \( c \)-propooid, we have \( F : \vdash G \) iff \( F :: G \).

**Fact 78.** Diamond equivalence is a congruence on \( ct \)-propooids.

**Proof.** A direct consequence of Theorem 68 and Fact 77.

**Example 79.** In \( (\mathbb{N}, ::) \) defined as in Example 7, we have

\[
F : \vdash G \iff F_a - F_b = G_a - G_b, \quad \text{for all } a, b \in \mathbb{N}.
\]

In particular, we have \( S^k : \vdash S^\ell \), for all \( k, \ell \geq 0 \).
10.3. **Equivalence.** Given mappings \( F, G : \Psi \to \mathbb{P} \), define their *(proportional) equivalence* by

\[
F \equiv G \iff a : a ::_{\Psi \mathbb{P}} Fa : Ga, \quad \text{for all } a \in P.
\]

This situation can be depicted as follows:

\[
\begin{array}{c}
\text{Ga} \\
\bigcirc \\
\downarrow \\
\text{Fa}
\end{array}
\]

The following result provides a simple way to show that two mappings are *not* equivalent given that determinism holds:

**Theorem 80.** Proportionally equivalent mappings have the same fixed points in \( d \)-proportoids.

**Proof.** Let \( a \) be a fixed point of \( F \) thus satisfying \( Fa = a \). Then, by determinism and since \( F \) and \( G \) are equivalent by assumption, we have

\[
a : a :: Fa : Ga \iff Ga = a,
\]

which means that \( a \) is a fixed point of \( G \) as well. By inner reflexivity, an analogous argument shows that every fixed point of \( G \) is a fixed of \( F \). \( \square \)

**Corollary 81.** \( F \equiv I \) iff \( F = I \) holds in \( d \)-proportoids.

**Proof.** A direct consequence of Theorem 80. \( \square \)

**Theorem 82.** Proportional equivalence is an equivalence relation on \( i \)-proportoids, and it is a congruence for all analogies in \( t \)-proportoids.

**Proof.** Reflexivity follows from reflexivity, symmetry follows from inner symmetry, and transitivity follows from inner transitivity.

It remains to show that equivalence is compatible with composition of analogies. In case \( A \) is an analogy, we have

\[
\begin{array}{c}
B \equiv C \\
\hline
a : a :: Ba : Ga \\
\hline
A \text{ is an analogy}
\end{array}
\]

Moreover, we always have (even if \( A \) is not an analogy)

\[
\begin{array}{c}
\text{inner reflexivity} \\
\hline
a : a :: Aa : Aa \\
\hline
B \equiv C
\end{array}
\]

\[
\begin{array}{c}
B \equiv C \\
\hline
Aa : Aa :: BAa : CAa \\
\hline
a : a :: BAa : CAa
\end{array}
\]

This shows that equivalence is left and right compatible, which by Proposition means that it is a congruence. \( \square \)

**Theorem 83.** Proportional equivalence is left cancellative for analogies on any \( t \)-proportoid in the strong sense that for all analogies \( A, B, C \),

\[
AB \equiv AC \iff B \equiv C.
\]
In case \( A \) commutes with \( B \) and \( C \) in the sense that

\[
BAa : CAa :: ABa : ACa, \quad \text{for all } a \in P,
\]

it follows that equivalence is right cancellative in the strong sense as well, that is,

\[
BA \equiv CA \iff A \equiv C.
\]

\textit{Proof.} For the direction from left to right, we have the following derivation:

\[
\begin{array}{c}
AB \equiv AC \\
\hline
a : a :: ABa : ACa \end{array} \quad \begin{array}{c}
A \text{ is an analogy} \\
ABa : ACa :: Ba : Ca \end{array} \quad \begin{array}{c}
A \text{ is an analogy} \\
ABa : ACa :: Ba : Ca \end{array}
\]

The direction from right to left holds trivially:

\[
B \equiv C \quad \begin{array}{c}
A \text{ is an analogy} \\
ABa : ACa :: Ba : Ca \end{array} \quad \begin{array}{c}
B \equiv C.
\end{array}
\]

For the second part, we assume that \( A \) commutes with \( B \) and \( C \) in the sense of (23). The direction from left to right is shown by the following derivation:

\[
\begin{array}{c}
BA \equiv CA \\
\hline
a : a :: BAa : CAa \end{array} \quad \begin{array}{c}
A \text{ commutes with } B \text{ and } C \\
BAa : CAa :: ABa : ACa \end{array} \quad \begin{array}{c}
A \text{ is an analogy} \\
ABa : ACa :: Ba : Ca \end{array} \quad \begin{array}{c}
A \text{ commutes with } B \text{ and } C \end{array} \quad \begin{array}{c}
A \text{ is an analogy} \\
ABa : ACa :: BAa : CAa \end{array}
\]

For the other direction, we compute:

\[
\begin{array}{c}
B \equiv C \\
\hline
a : a :: Ba : Ca \end{array} \quad \begin{array}{c}
A \text{ is an analogy} \\
ABa : ACa :: Ba : Ca \end{array} \quad \begin{array}{c}
B \equiv C.
\end{array}
\]

\textbf{Theorem 84.} \( F \equiv G \) \textit{et.}

\textit{Proof.}

\[
\begin{array}{c}
F \equiv G \\
\hline
a : a :: b : b \end{array} \quad \begin{array}{c}
\text{inner reflexivity} \quad \text{(4)} \quad \text{e} \\
F \equiv G \end{array} \quad \begin{array}{c}
F \equiv G \end{array} \quad \begin{array}{c}
F \equiv G \end{array} \quad \begin{array}{c}
F \equiv G \end{array}
\]

\[
\begin{array}{c}
Fa : Ga :: a : a \\
\hline
F : G.
\end{array}
\]
10.4. **Join equivalence.** Given $F, G : \mathcal{P} \rightarrow \mathcal{P}$, define their (proportional) join equivalence by

$$F \lor G : \iff a : Fa :: a : Ga, \quad \text{for all } a \in P.$$ 

This can be depicted as follows:

$$\begin{array}{c}
F a \\
\downarrow \\
\downarrow \\
G a
\end{array}$$

**Fact 85.** $F \lor G \iff F \equiv G$ in c-proportoids.

**Corollary 86.** Join equivalence is a congruence for analogies on any ct-proportoid.

**Proof.** A direct consequence of Theorem 82 and Fact 85.

Corollary 86 has rather strong assumptions as central permutation often fails in practice and it holds only for analogies on any ct-proportoid. The next result states that for join equivalence to be an equivalence relation (not a congruence), only transitivity is required:

**Fact 87.** Join equivalence is an equivalence relation on any t-proportoid.

The following result gives us a simple method to show that two mappings are not join equivalent in case determinism (6) holds:

**Proposition 88.** Join equivalent mappings have the same fixed points in d-proportoids.

**Proof.** Given join equivalent mappings $F, G$, determinism implies that for any $a \in P$, in case $a$ is a fixed point of $F$ thus satisfying $F a = a$,

$$a : Fa :: a : Ga \iff Ga = a,$$

which means that $a$ is a fixed point of $G$ as well. An analogous argument shows that every fixed point of $G$ is a fixed point of $F$.

**Corollary 89.** $F \lor I$ iff $F = I$ in d-proportoids.

**Proof.** A direct consequence of Proposition 88.

The next observation shows that the converse of Proposition 88 fails in general since $S^k$ and $S^\ell$ have the same fixed points (none) for all $k, \ell \geq 1$:

**Example 90.** In $(N, ::)$ defines as in Example 7, we have $S^k \lor S^\ell$ iff $k = \ell$.

10.5. **Triangular relation.** Given $F, G : \mathcal{P} \rightarrow \mathcal{P}$, define the (proportional) triangular relation by

$$F \bowtie G : \iff F \lor G \quad \text{and} \quad a : Fa :: Fa : Ga \quad \text{and} \quad a : Ga :: Ga : Fa, \quad \text{for all } a \in P.$$ 

This can be depicted as follows (see the similarity to join equivalence in §10.4):

$$\begin{array}{c}
F a \\
\downarrow \\
\downarrow \\
G a
\end{array}$$

The following observation shows that the triangular relation is in general not reflexive:

---

2This is analogous to Theorem 80.
Proposition 91. \( F \triangleleft F \) iff \( F = I \) in d-proportoids.

Proof. We have \( F \triangleleft F \) only if \( a : Fa :: Fa : Fa \), for all \( a \in P \). By determinism, this is equivalent to \( Fa = a \), that is, \( a \) is a fixed point of \( F \) for each \( a \in P \), which is equivalent to \( F = I \). The other direction holds trivially. □

Proposition 92. \( \frac{F \triangleleft G}{F \equiv G} \) c.

Proof. \( F \triangleleft G \) implies \( F \vee G \) which implies \( F \equiv G \) in all c-proportoids by Fact 85. □

Example 93. As a direct consequence of Example 90 in \((N, ::)\) we have \( S^k \triangleleft S^\ell \) iff \( k = \ell \).

10.6. Bowtie relation. Given mappings \( F, G : \mathfrak{B} \to \mathfrak{B} \), define the (proportional) bowtie relation by

\[
F \bowtie G \iff a : GFa :: a : FGa, \quad \text{for all } a \in P,
\]

which can be depicted as follows:

\[
\begin{array}{c}
FGa \\
\downarrow \\
Ga \\
\downarrow \\
Fa \\
\downarrow \\
GFa
\end{array}
\]

Notice that the bowtie relation is connected to the commutation of \( F \) and \( G \) and the next result shows that bowtie equivalent mappings commute with respect to proportional equivalence in c-proportoids satisfying central permutation:

Proposition 94. \( F \bowtie G \) iff \( GF \equiv FG \) in c-proportoids.

Proof. The direction from right to left holds trivially. For the other direction, we have

\[
\frac{F \bowtie G}{a : GFa :: a : FGa} \quad \text{c}
\]

\[
\frac{a : a : GFa :: FGa}{GF \equiv FG}.
\]

Fact 95. \( F \bowtie 1 \) holds for every mapping \( F \).

Proof. An immediate consequence of reflexivity (1). □

Fact 96. The bowtie relation is reflexive and symmetric.

Proof. Follows from the reflexivity (1) and symmetry (2) of the analogical proportion relation. □

Fact 97. For any mapping \( F \), we have \( F^k \bowtie F^\ell \) for all \( k, \ell \geq 0 \). In particular, we have \( F \bowtie F^2 \) which means that \( F \) is idempotent with respect to bowtie equivalence.

Proof. A direct consequence of reflexivity (1). □

Example 98. In \((N, ::)\), we have \( S^k \bowtie S^\ell \) for all \( k, \ell \geq 0 \).
10.7. **Square equivalence.** Given \( F, G : \mathcal{P} \to \mathcal{P} \), define their *(proportional) square equivalence* by

\[
F \Box G \iff a : Fa :: b : Gb \quad \text{and} \quad a : b :: Fa : Gb, \quad \text{for all } a, b \in P,
\]

which can be depicted as follows:

\[
\begin{array}{c c c}
& a & \text{Fa} \\
\downarrow & & \downarrow \\
& b & \text{Gb}
\end{array}
\]

Applying \( F \) and \( G \) iteratively yields:

\[
\begin{array}{c c c c c}
& a & \text{Fa} & \text{FFa} & \cdots \\
\downarrow & & & & \\
& b & \text{Gb} & \text{GGb} & \cdots
\end{array}
\]

Notice the similarity between square equivalence and the definition of an analogy in §7, which immediately yields the following observation:

**Fact 99.** \( F \Box F \) iff \( F \) is an analogy holds in any \( c \)-proportoid.

**Fact 100.** Proportional square equivalence is an equivalence relation for analogies on any \( t \)-proportoid.

**Proof.** Reflexivity follows from the fact that every mapping is an analogy by assumption (and see Fact [99]), symmetry follows from symmetry of analogical proportions, and transitivity follows from transitivity of analogical proportions. \( \square \)

The next result shows that square equivalence is a very strong condition implying equivalence and join equivalence:

**Fact 101.** \( F \Box G \implies F \equiv G \).

**Fact 102.** \( F \Box G \implies F \lor G \).

**Example 103.** As a direct consequence of Example [90] and Fact [102] in \((\mathbb{N},::)\) we have \( S^k \Box S^\ell \) iff \( k = \ell \).

10.8. **Complete square equivalence.** We define the *(proportional) complete square equivalence* of \( F \) and \( G \) by

\[
F \bowtie G \iff a : b :: Fa : Gb \quad \text{and} \quad a : Fa :: b : Gb \quad \text{and} \quad a : Gb :: b : Fa \quad \text{for all } a, b \in P.
\]

This can be depicted as follows:

\[
\begin{array}{c c c}
& a & \text{Fa} \\
\downarrow & & \downarrow \\
& b & \text{Gb}
\end{array}
\]

**Fact 104.** \( F \bowtie G \implies F \Box G \).

**Fact 105.** \( F \bowtie G \implies F \equiv G \).
Proof.

\[
\begin{align*}
F \circ G & \subseteq F \square G \\
F & \equiv G.
\end{align*}
\]

\[\Box\]

11. Proportional circles

Given \(a, b \in P\), the \textbf{(proportional) circle} with center \(a\) and “radius” \(\overline{ab}\) is the solution set

\[c_{ab} := S(a : b :: a : x).\]

The next result shows how to construct further solutions from a given one in any \(t\)-proportoid using circles:

\textbf{Theorem 106.} \(d \in S(a : b :: c : x)\)

\[c_{cd} \subseteq S(a : b :: c : x)\]

\textbf{Proof.} We show the implication

\[e \in c_{cd} \implies e \in S(a : b :: c : x)\]

with the following derivation:

\[
\begin{align*}
& d \in S(a : b :: c : x) \\
\because & e \in c_{cd} \\
\therefore & a : b :: c : d \\
& e \in S(a : b :: c : e) \\
\therefore & e \in S(a : b :: c : x).
\end{align*}
\]

\[\Box\]

12. Function proportions

Every relation on elements of \(P\) can be extended point-wise to a relation of functions of \(P\) of same arity. For the analogical proportion relation extended to functions, we thus obtain the following definition:

\textbf{Definition 107.} Given mappings \(E, F, G, H : P \rightarrow P\), we define the \textbf{function proportion relation} by

\[
E : F :: G : H \iff Ea : Fa :: Ga : Ha, \text{ for all } a \in P.
\]

\textbf{Fact 108.} We have

\[
\begin{align*}
E : F & :: E : F \quad \text{(reflexivity)}, \\
E : F & :: G : H \iff G : H :: E : F \quad \text{(symmetry)}, \\
E : F & :: G : H \iff F : E :: H : G \quad \text{(inner symmetry)}.
\end{align*}
\]

Similarly, all other properties of analogical proportions in Definition 2 transfer to function proportions.

\textbf{Remark 109.} Fact 108 means that from any proportoid \(\mathcal{P} = (P, ::)\), we can construct the proportoid

\[\mathcal{P}^\mathcal{P} := (P^P, ::)\]

of unary functions on \(P\) with :: defined point-wise as in (24).

\textbf{Proposition 110.} In any \(c\)-proportoid, we have

\[E : I :: F : I \iff E \equiv F.\]
Proof. We have

\[ \begin{align*}
E : I &:: F : I \iff Ea : a :: Fa : a, \quad \text{for all } a \in P \\
&\quad \iff Ea : Fa :: a : a, \quad \text{for all } a \in P \\
&\quad \iff a : a :: Ea : Fa, \quad \text{for all } a \in P \\
&\quad \iff E \equiv F.
\end{align*} \]

The next result shows that proportional equivalence is compatible with function proportions:

**Proposition 111.** \( E \equiv F \) \( G \equiv H \) \( E : F :: G : H \) \( t. \)

**Proof.**

\[ \begin{align*}
E \equiv F &\quad \quad G \equiv H \\
\Gamma &\quad \quad \Gamma \\
\end{align*} \]

\[ E : F :: G : H. \]

**Corollary 112.** \( E \Box F \) \( G \Box H \) \( E : F :: G : H \) \( t. \)

**Proof.**

\[ \begin{align*}
E \Box F &\quad G \Box H \\
\Gamma &\quad \quad \Gamma \\
\end{align*} \]

\[ E : F :: G : H. \]

**Corollary 113.** \( E \bowtie F \) \( G \bowtie H \) \( E : F :: G : H \) \( t. \)

**Proof.**

\[ \begin{align*}
E \bowtie F &\quad G \bowtie H \\
\Gamma &\quad \quad \Gamma \\
\end{align*} \]

\[ E : F :: G : H. \]

**Corollary 114.** \( E \triangleright F \) \( G \triangleright H \) \( E : F :: G : H \) \( ct. \)

**Proof.**

\[ \begin{align*}
E \triangleright F &\quad G \triangleright H \\
\Gamma &\quad \quad \Gamma \\
\end{align*} \]

**Definition 115.** We extend proportional identity from elements of \( P \) to unary mappings on \( P \) point-wise by

\[ F \equiv G \iff Fa \equiv Ga, \quad \text{for all } a \in P. \]

The next result is a generalization of Proposition 58 from elements to functions:
Proposition 116. \( \frac{E \equiv F}{E : F :: G : H} \).

Proof.

\[
\begin{array}{c}
E \equiv F \\
Ea \equiv Fa \\
Ga \equiv Ha \\
\hline
Ea : Fa :: Ga : Ha
\end{array}
\]

\( \square \)

13. Proportional similarity

In this section, we shall introduce a notion of similarity in terms of analogical proportions:

Definition 117. Given \( a \in P \) and \( b \in R \) and some set of functions \( \Sigma := \{ \sigma_{ab} : P \to R \mid a, b \in P \} \), define

\[
a \precsim \Sigma b \iff a \vdash c :: b \vdash \sigma_{ab} c,
\]

for every \( c \in P \), and

\[
a \approx \Sigma b \iff a \precsim \Sigma b \quad \text{and} \quad b \precsim \Sigma a.
\]

In case \( a \approx \Sigma b \), we say that \( a \) and \( b \) are \( \Sigma \)-similar.

Notice that

\[
a \precsim \Sigma b \iff \text{for every } c \in P \text{ there is some } d = \sigma_{ab} c \in R \text{ such that } a \vdash c :: b \vdash d,
\]

\[
\implies S(a \vdash c :: b : x) \neq \emptyset, \quad \text{for all } c \in P.
\]

Proposition 118. \( \Sigma \)-similarity is reflexive and symmetric in any proportoid. Moreover, if

\[
\sigma_{ab} \sigma_{cd} = \sigma_{cd} \tag{25}
\]

holds for all \( a, b, c, d \in P \), then \( \Sigma \)-similarity is an equivalence relation in any \( t \)-proportoid.

Proof. Reflexivity follows from the reflexivity of analogical proportions which guarantees that for any \( c \in P \) there is some \( d := c \in R \) such that \( a \vdash c :: b \vdash d \). Symmetry holds trivially. To prove transitivity, we proceed as follows. Suppose \( a \approx \Sigma b \) and \( b \approx \Sigma c \), which means that

\[
a \vdash d :: b \vdash \sigma_{ab} d, \quad \text{for all } d \in P,
\]

\[
b \vdash e :: c \vdash \sigma_{bc} e, \quad \text{for all } e \in P.
\]

Let \( f \in P \) be an arbitrary element. We then have the following derivation:

\[
\begin{array}{c}
a \vdash f :: b \vdash \sigma_{ab} f \\
\hline
\sigma_{ab} f \vdash c : \sigma_{bc} \sigma_{ab} f
\end{array}
\]

\[
\begin{array}{c}
\sigma_{ab} f \vdash c : \sigma_{bc} \sigma_{ab} f \\
\hline
\sigma_{bc} \sigma_{ab} f \vdash a \vdash c : \sigma_{cd} \sigma_{ab} f \tag{25}
\end{array}
\]

\[
\begin{array}{c}
\sigma_{bc} \sigma_{ab} f \vdash a \vdash c : \sigma_{cd} \sigma_{ab} f \\
\hline
a \approx \Sigma c.
\end{array}
\]

\( \square \)

Theorem 119. Let \( \mathcal{P} = (P, ::) \) be an \( it \)-proportoid. If there exists a set of functions \( \Sigma \) satisfying \( \tag{25} \) and the proportions

\[
\sigma_{ae} b : \sigma_{bf} a \vdash \sigma_{cg} d : \sigma_{dh} c \tag{26}
\]

for all \( a, b, c, d, e, f, g, h \in P \), then \( \approx \Sigma \) is a proportional congruence thus satisfying
\[ a \approx \Sigma e \quad b \approx \Sigma f \quad c \approx \Sigma g \quad d \approx \Sigma h \quad a : b :: c : d. \]

**Proof.** We have the derivations

\[
\begin{align*}
 e : f :: g : h & \\
 e : \sigma_{ab}b :: a : b & \\
 a : b :: e : \sigma_{ae}b & \\
 e : \sigma_{ae}b :: \sigma_{ab}b & \\
 e : \sigma_{ab}b :: \sigma_{ae}b & \\
 e : \sigma_{ab}b :: g : \sigma_{cg}d & \\
 c : d :: g : \sigma_{cg}d & \\
 c : d :: e : \sigma_{ae}b & \\
 e : \sigma_{ae}b :: c : d & \\
 a : b :: \sigma_{bf}a & \\
 b : a :: \sigma_{bf}a : f & \\
 \sigma_{bf}a : f :: a : b & \\
 a : b :: \sigma_{bf}a & \\
 e : \sigma_{bf}a :: g : \sigma_{dh}c & \\
 \sigma_{bf}a : f :: \sigma_{dh}c : h & \\
 d : c :: \sigma_{dh}c & \\
 c : d :: \sigma_{dh}c & \\
 e : f :: g : h. & \\
\end{align*}
\]

Now since we assume (26) and inner transitivity, we have

\[
\begin{align*}
 e : \sigma_{ae}b :: g : \sigma_{cg}d & \\
 \sigma_{bf}a : f :: \sigma_{dh}c : h & \\
 \sigma_{bf}a : f :: \sigma_{dh}c : h & \\
 \sigma_{bf}a : f :: \sigma_{dh}c : h. & \\
\end{align*}
\]

**Fact 120.** For any analogy \( A \) and any element \( a \), we have \( a \preceq_A A a. \)

### 14. Proportoids in Universal Algebra and Predicate Logic

This paper is axiomatic in style in the sense that we do not study concrete realizations of the analogical proportion relation. However, in a series of papers the author has shown how a *canonical* notion of an analogical proportion relation can be constructed from any algebra or structure in the sense of universal algebra and first-order logic, respectively (Antić, 2022, 2023c).

More formally, given a first-order language \( L \) consisting of ranked function and relation symbols, an *L-structure* consists of a non-empty set \( P \) together with concrete functions and relations on \( P \) corresponding to the function and relation symbols in \( L \) (see e.g. Hinman, 2005, §2).

Given an *L*-structure \( \mathfrak{A} = (A, \mathfrak{F}, \mathfrak{R}) \) with functions \( \mathfrak{F} \) and relations \( \mathfrak{R} \), we can define the *analogical proportion relation* in \( \mathfrak{A} — in symbols, *
\[
 a : b :\mathfrak{A} c : d
\]

as in Antić (2023c). We do not want to go into technical details here. What is of interest here is that we can *canonical*ly associate with any such *L*-structure \( \mathfrak{A} = (A, \mathfrak{F}, \mathfrak{R}) \) a proportoid \( \mathfrak{P} \) := \( (A, :\mathfrak{A}) \) by defining \( :\mathfrak{A} \) as described in Antić (2023c).

\footnote{Constant symbols are omitted and identified with 0-ary function symbols.}
15. Conclusion

This paper introduced proportoids as sets endowed with a 4-ary analogical proportion relation satisfying a suitable set of axioms which are rooted in but different from Lepage’s initial axiomatization. We then introduced proportional homomorphisms and their congruences and showed that they are related in the usual sense via kernels. Moreover, we introduced proportional analogies and showed that in transitive proportoids, they satisfy the strong proportion-preserving principle and are thus closely related to proportional homomorphisms. We showed how partial proportional analogies can be constructed from an enumeration and a selection function. We introduced a number of binary relations between unary mappings on proportoids. Finally, we introduced a notion of similarity in terms of analogical proportions.

References

Antić, C. (2022). Analogical proportions. *Annals of Mathematics and Artificial Intelligence, 90*(6), 595–644. https://doi.org/10.1007/s10472-022-09798-y

Antić, C. (2023a). Analogical proportions in monounary algebras. *Annals of Mathematics and Artificial Intelligence*, accepted. https://arxiv.org/pdf/2208.06829.pdf

Antić, C. (2023b). Boolean proportions. https://arxiv.org/pdf/2109.00388.pdf

Antić, C. (2023c). Logic-based analogical proportions. https://hal.science/hal-04005139

Burris, S., & Sankappanavar, H. (2000). *A Course in Universal Algebra*. http://www.math.hawaii.edu/~ralph/Classes/619/univ-algebra.pdf

Couceiro, M., Hug, N., Prade, H., & Richard, G. (2017). Analogy-preserving functions: a way to extend boolean samples. In *IJCAI 2017*, pp. 1575–1581.

Couceiro, M., & Lehtonen, E. (2023). Galois theory for analogical classifiers. *Annals of Mathematics and Artificial Intelligence*. https://doi.org/10.1007/s10472-023-09833-6

Davies, T. R., & Russell, S. J. (1987). A logical approach to reasoning by analogy. In McDermott, J. P. (Ed.), *IJCAI 1987*, pp. 264–270. Morgan Kaufmann.

Hinman, P. G. (2005). *Fundamentals of Mathematical Logic*. A K Peters, Wellesley, MA.

Howie, J. M. (2003). *Fundamentals of Semigroup Theory*. London Mathematical Society Monographs New Series. Oxford University Press, Oxford.

Klein, S. (1982). Culture, mysticism and social structure and the calculation of behavior. In *ECAI 1982*, pp. 141–146.

Lepage, Y. (2003). *De L’Analogie. Rendant Compte de la Commutation en Linguistique*. Habilitation à diriger les recherches, Université Joseph Fourier, Grenoble.

Lim, S., Prade, H., & Richard, G. (2021). Classifying and completing word analogies by machine learning. *International Journal of Approximate Reasoning, 132*, 1–25.

Murena, P.-A., Cornuéjols, A., & Dessalles, J.-L. (2018). Opening the parallelogram: Considerations on non-euclidean analogies. In *ICCBR 2018, LNAI 11156*, pp. 597–611. Springer Nature Switzerland AG.