Les Houches Lectures on Strings and Arithmetic*

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These are lecture notes for two lectures delivered at the Les Houches workshop on Number Theory, Physics, and Geometry, March 2003. They review two examples of interesting interactions between number theory and string compactification, and raise some new questions and issues in the context of those examples. The first example concerns the role of the Rademacher expansion of coefficients of modular forms in the AdS/CFT correspondence. The second example concerns the role of the “attractor mechanism” of supergravity in selecting certain arithmetic Calabi-Yau’s as distinguished compactifications.

Jan. 6, 2004

* Summary of lectures delivered at the conference Number Theory, Physics, and Geometry
Les Houches, March, 2003
1. Introduction

Several of the most interesting developments of modern string theory use some of the mathematical tools of modern number theory. One striking example of this is the importance of arithmetic groups in the theory of duality symmetries. Another example, somewhat related, is the occurrence of automorphic forms for arithmetic groups in low energy effective supergravities. These examples are quite well-known.

In the following two lectures we explore two other less-well-known examples of curious roles of number theoretic techniques in string theory. The first concerns a technique of analytic number theory and its role in the AdS/CFT correspondence. The second is related to the “attractor equations.” These are equations on Hodge structures of Calabi-Yau manifolds and have arisen in a number of contexts connected with string compactification. Another topic of possible interest to readers of this volume will appear elsewhere [1].

2. Potential Applications of the AdS/CFT Correspondence to Arithmetic

2.1. Summary

In this talk we are going to indicate how the “AdS/CFT correspondence” of string theory might have some interesting relations to analytic number theory. The main part of the talk reviews work done with R. Dijkgraaf, J. Maldacena, and E. Verlinde which appeared in [2]. Ideas similar in spirit, but, so far as I know, different in detail have appeared in [3].

2.2. Summary of the AdS/CFT correspondence

The standard reviews on the AdS/CFT correspondence are [4,5,6]. In this literature, “anti-deSitter space” comes in two signatures. The Euclidean version is simply hyperbolic space:

\[ AdS_{n+1} = \mathbb{H}^{n+1} = SO(1, n+1)/SO(n+1) \]  \hspace{1cm} (2.1)

while the Lorentzian version is

\[ AdS_{1,n} = SO(2, n)/SO(1,n) \]  \hspace{1cm} (2.2)

where on the right-hand side we should take the universal cover. These spacetimes are nice solutions to Einstein’s equations with negative cosmological constant.

\[ \mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} + \Lambda g_{\mu\nu} = 0 \quad \Lambda = -1 \]  \hspace{1cm} (2.3)
In the context of string theory they arise very naturally in certain solutions to 10- and 11-dimensional supergravity associated with configurations of branes.

Some important examples (by no means all) of such solutions include

1. \( AdS_2 \times S^2 \times M_6 \) where \( M_6 \) is a Calabi-Yau 3-fold. The associated D-brane configurations are discussed in Lecture II below.

2. \( AdS_3 \times S^3 \times M_4 \) where \( M_4 \) is a \( K3 \) surface or a torus \( T^4 \), or \( S^3 \times S^1 \).

3. \( AdS_5 \times S^5 \). This is the geometry associated to a large collection of coincident \( D3 \) branes in 10-dimensional Minkowski space and is the subject of much of the research done in AdS/CFT duality.

At the level of slogans the AdS/CFT conjecture states that **10-dimensional string theory on**

\[
AdS_{n+1} \times K
\]  

is “equivalent” to a super-conformal field theory – i.e., a QFT without gravity – on the conformal boundary

\[
\partial AdS_{n+1}
\]  

The “conformal boundary of AdS” means, operationally,

\[
\partial AdS_{n+1} = S^n \quad \text{or} \quad S^{n-1} \times \mathbb{R}
\]  

More fundamentally it is the conformal boundary in the sense of Penrose.

Of course, the above slogan is extremely vague. One goal of this talk is to give an example where the statement can be made mathematically quite precise. We are explaining this example in the present volume because it involves some interesting analytic number theory. The hope is that a precise version of the AdS/CFT principle can eventually be turned into a useful tool in number theory, and the present example is adduced as evidence for this hope. At the end of the talk we will make some more speculative suggestions along these lines.

**2.2.1. AdS/CFT made a little more precise**

In order to explain our example it is necessary to make the statement of AdS/CFT a little more precise.

Consider 10D string theory on \( X \) which is a noncompact manifold which at infinity looks locally like

\[
X \sim AdS_{n+1} \times K
\]
Let us think of string theory as an infinite-component field theory on this spacetime. In particular the fields include the graviton $g_{\mu\nu}$, as well as (infinitely) many others. Let us denote the generic field by $\phi$. We assume there is a well-defined notion of a partition function of string theory associated to this background. Schematically, it should be some kind of functional integral:

$$Z_{\text{string}} = \int [dg_{\mu\nu}][d\phi] \cdots e^{-\int \sqrt{g}R(g) + (\nabla \phi)^2 + \cdots}$$

(2.8)

Even at this schematic level we can see one crucial aspect of the functional integral: we must specify the boundary conditions of the fields at infinity.

Since spacetime has a factor which is locally $AdS$ at infinity there is a second order pole in the metric at infinity. Let $r$ denote a coordinate so that the conformal boundary is at $r \to \infty$ and such that the metric takes the asymptotic form

$$ds_X^2 \to \frac{dr^2}{r^2} + r^2 g_{ij}(\theta)d\theta^i d\theta^j + ds_K^2$$

(2.9)

where $\theta^i$ denote coordinates on $S^n$. In these coordinates we impose boundary conditions on the remaining fields:

$$\phi(r, \theta) \to r^h \phi_0(\theta)$$

(2.10)

The functional integral (2.8) is thus a function $\hat{\Gamma}$ of the boundary data:

$$Z_{\text{string}}(\hat{g}, \phi_0, \ldots)$$

(2.11)

We can now state slightly more precise versions of AdS/CFT. There is a slightly different formulation for Euclidean and Lorentzian signature.

The Euclidean version of AdS/CFT states that there exists a CFT $\mathcal{C}$ defined on $\partial AdS_{n+1} = S^n$ such that the space $\mathcal{A}$ of local operators in $\mathcal{C}$ is dual to the string theory boundary conditions:

$$\phi_0 \to \Phi_{\phi_0} \in \mathcal{A}$$

(2.12)

such that

$$\left\langle e^{\int_{S^n} \Phi_{\phi_0}(\theta)} \right\rangle_{\text{CFT}} = Z_{\text{string}}(\hat{g}, \phi_0, \ldots)$$

(2.13)

\footnote{In fact, it should be considered as a “wavefunction.” In the closely related Chern-Simons gauge theory/RCFT duality this is literally true.}
This statement of the AdS/CFT correspondence, while conceptually simple, is quite over-simplified. Both sides of the equation are infinite, must be regularized, etc. See the above cited reviews for a somewhat more careful discussion.

The Lorentzian version of AdS/CFT states that there is an isomorphism of Hilbert spaces between the gravity and CFT formulations that preserves certain operator algebras. These are $\mathcal{H}_C$, the Hilbert space of the CFT $\mathcal{C}$ on $S^{n-1} \times \mathbb{R}$, and $\mathcal{H}_{\text{string}}$, the Hilbert space of string (or M) theory on $AdS_{n+1} \times K$. This is already a nontrivial statement when one considers both sides as representations of the superconformal group. An approximation to $\mathcal{H}_{\text{string}}$ is given by particles in the supergravity approximation, and corresponding states in the CFT have been found. See [4]. Whether or not the isomorphism truly holds for the entire Hilbert space is problematic because of multi-particle states and because of the role of black holes. Indeed, it is clear that one must include quantum states in $\mathcal{H}_{\text{string}}$ associated both to black holes and to strings and D-branes in order to avoid contradictions.

2.3. A particular example

In the remainder of this talk we will focus on the example of type $IIB$ string theory on $AdS_3 \times S^3 \times K3$. In this case the dual CFT on $\partial AdS_3$ is a two-dimensional CFT $\mathcal{C}$.

From symmetry considerations it is clear that the dual CFT has $(4, 4)$ supersymmetry. It is thought that $\mathcal{C}$ admits marginal deformations to a supersymmetric $\sigma$-model whose target space $X$ is a hyperkahler resolution

$$X \to (K3)^k/S_k = \text{Sym}^k(K3).$$

In comparing the gravity and CFT side we make the identification

$$k = \ell/4G$$

where $\ell$ is the radius of $S^3$ (which in turn is the curvature radius of $AdS_3$), while $G$ is the Newton constant in 3 dimensions. The quantization of $\ell/4G$ can be seen intrinsically on the gravity side from the existence of certain Chern-Simons couplings for $SU(2)$ gauge fields with coefficient $k$.

The “proof” of the correspondence proceeds by studying the near horizon geometry of solutions of the supergravity equations representing $Q_1$ D1 branes and $Q_5$ D5 branes wrapping $K3 \times S^1$. One studies the low energy excitations of the “string” wrapping the $S^1$ factor. The dynamics of these excitations are are described by a supersymmetric nonlinear
sigma model with target space (2.14) for \( k = Q_1Q_5 + 1 \). The moduli space of supergravity solutions, as well as the moduli space of the supersymmetric sigma model are both the space

\[
\Gamma \backslash SO(4, 21) / SO(4) \times SO(21) \tag{2.16}
\]

where \( \Gamma \) is an arithmetic subgroup of \( SO(4, 21; \mathbb{Z}) \). See [7,8,9,10,11] for some explanation of the details of this.

The correlation function whose equivalence in AdS and CFT formulations we wish to present is a certain partition function which, on the CFT side is the \textit{elliptic genus} of the conformal field theory. The reason we focus on this quantity is that the dual CFT is very subtle. The elliptic genus is a “correlation function” of the CFT \( \mathcal{C} \) which is invariant under many perturbations of the CFT, and is therefore robust and computable. Nevertheless, the resulting function is also still nontrivial and contains much useful information.

Our strategy will be to write the elliptic genus in a form that makes the connection to quantum gravity on \( AdS_3 \) clear. The form in which we can make this connection is a Poincaré series for the elliptic genus.

### 2.4. Review of Elliptic Genera

For some background on the elliptic genus, see [12,13,14,15,16,17,18,19,20,21,22].

Let \( \mathcal{C} \) be a CFT with \( (2, 2) \) supersymmetry. This means the Hilbert space \( \mathcal{H} \) is a representation of superconformal \( \mathcal{V}ir_{\text{left}}^{\mathcal{N}=2} \oplus \mathcal{V}ir_{\text{right}}^{\mathcal{N}=2} \), where the subscript refers to the usual separation of conformal fields into left- and right-moving components.

Let us recall that the \( \mathcal{N} = 2 \) superconformal algebra is generated by Virasoro operators \( L_n \), and \( U(1) \) current algebra \( J_n \), with \( n \in \mathbb{Z} \), and superconformal generators \( G^\pm_r \) with \( r \in \mathbb{Z} + \frac{1}{2} \) for the NS algebra and \( r \in \mathbb{Z} \) for the R algebra. The commutation relations are:

\[
[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \tag{2.17}
\]

\[
[G^+_r, G^+_s] = 2L_{r+s} + (r - s)J_{r+s} + \frac{c}{12}(4r^2 - 1)\delta_{r+s,0} \tag{2.18}
\]

\[
[J_n, J_m] = \frac{c}{3}n\delta_{n+m,0} \tag{2.19}
\]

\[
[L_n, G^+_r] = (\frac{1}{2}n - r)G^+_n \tag{2.20}
\]

\[
[J_n, G^+_r] = \pm G^+_n \tag{2.21}
\]

\[
[L_n, J_m] = -mJ_{n+m} \tag{2.22}
\]
Right-moving generators are denoted $\tilde{L}_n, \tilde{J}_n, \tilde{G}_r^\pm$.

The elliptic genus is

$$\chi(\tau, z) := \text{Tr}_{RR} e^{2\pi i \tau (L_0 - c/24)} e^{2\pi i \tilde{\tau} (\tilde{L}_0 - c/24)} e^{2\pi i z J_0} (-1)^F$$

where the trace is in the Ramond-Ramond sector and $(-1)^F = e^{i\pi (J_0 - \tilde{J}_0)}$. 

In a unitary $(2,2)$ superconformal field theory the operators $L_0, \tilde{L}_0, J_0, \tilde{J}_0$ may be simultaneously diagonalized. In a unitary theory the spectrum satisfies $L_0 - c/24 \geq 0$ in the Ramond sector (and similarly for the right-movers). States with $\tilde{L}_0 = c/24$ are called right-BPS. It follows straightforwardly from the commutation relations (2.18) that only right-BPS states make a nonzero contribution to the trace (2.23) and hence $\chi(\tau, z)$ has Fourier expansion

$$\sum_{n \geq 0, r} c(n, r) q^n y^r$$

where $q = e^{2\pi i \tau}$ and $y = e^{2\pi i z}$.

In this paper we will be considering superconformal theories with $(4,4)$ supersymmetry. These are special cases of the $(2,2)$ theories, but have extra structure: For each chirality, left and right, the $U(1)$ current algebra (2.19) is enhanced to a level $k$ affine $SU(2)$ current algebra $T^a_n$. In addition, for each chirality, there is a global $SU(2)$ symmetry $\hat{T}^a$ and the four supercharges transform in the $(\frac{1}{2}, \frac{1}{2})$ representation of the global $SU(2) \times SU(2)$. The Virasoro central charge is given by $c = 6k$. 

2.4.1. Properties of the Elliptic Genus

The elliptic genus satisfies two key properties: modular invariance and spectral flow invariance. The modular invariance follows from the fact that $\chi(\tau, z)$ can be regarded as a path integral of $\mathcal{C}$ on a two-dimensional torus $S^1 \times S^1$ with odd spin structure for the fermions.

Under modular transformations

$$\chi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = e^{2\pi i k \frac{cz^2}{c\tau + d}} \chi(\tau, z)$$

(2.25)

In order to prove this from the path integral viewpoint note that including the parameter $z$ involves adding a term $\sim \int \tilde{A} \wedge J$ to the worldsheet action. From the singular ope of $J$ with itself one needs to include a subtraction term. After making a modular transformation
this subtraction term must change, the difference is finite and accounts for the exponential prefactor in (2.25).

The $\mathcal{N} = 2$ algebra has a well-known spectral flow isomorphism \[23\]

\[
G^\pm_{n\pm a} \rightarrow G^\pm_{n\pm(a+\theta)}
\]

\[
L_0 \rightarrow L_0 + \theta J_0 + \theta^2 k
\]

\[
J_0 \rightarrow J_0 + 2\theta k
\]

which implies that

\[
\chi(\tau, z + \ell \tau + m) = e^{-2\pi i k (\ell^2 \tau + 2\ell z)} \chi(\tau, z) \quad \ell, m \in \mathbb{Z}
\]

(2.27)

The identities (2.25) and (2.27) above are summarized in the mathematical definition \[24\]:

**Definition** A *weak Jacobi form* $\phi(\tau, z)$ of weight $w$ and index $k$ satisfies the identities:

\[
\phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^w e^{2\pi i k \frac{cs^2}{c\tau + d}} \phi(\tau, z)
\]

(2.28)

\[
\phi(\tau, z + \ell \tau + m) = e^{-2\pi i k (\ell^2 \tau + 2\ell z)} \phi(\tau, z) \quad \ell, m \in \mathbb{Z}
\]

(2.29)

and has a Fourier expansion with $c(n, r) = 0$ unless $n \geq 0$.

Thus, the elliptic genus of a unitary $(4, 4)$ superconformal field theory is a weak Jacobi form of weight 0 and level $k$. Much useful information on Jacobi forms can be found in \[24\].

Using the spectral flow identity (2.23) we find that $c(n, \ell) = c(n + \ell s + ks^2, \ell + 2ks)$, for $s$ an integer, and therefore $c(n, \ell) = c(n - \nu s_0 - ks_0^2, \nu) := c_\nu(4kn - \ell^2)$ if $\ell = \nu + 2ks_0$. Using this it is straightforward to derive

\[
\chi(\tau, z) = \sum_{\mu = -k+1}^k h_\mu(\tau) \Theta_{\mu, k}(z, \tau)
\]

(2.30)

Here $\Theta_{\mu, k}(z, \tau)$ are level $k$ theta functions

\[
\Theta_{\mu, k}(z, \tau) := \sum_{\ell \in \mathbb{Z}, \ell = \mu \text{mod} 2k} q^{\ell^2/(4k)} y^\ell
\]

\[
= \sum_{n \in \mathbb{Z}} q^{k(n+\mu/(2k))^2} y^{(\mu+2kn)}
\]

(2.31)

We denote the combinations even and odd in $z$ by $\Theta_{\mu, k}^\pm$.

Our goal now is to write the elliptic genus for the conformal field theory appearing in the AdS/CFT correspondence in a fashion suitable for interpretation via AdS/CFT. This fashion will simply be a Poincaré series. Before doing this in section 2.5 we make a small digression.
2.4.2. Digression 1: Elliptic Genera for Symmetric Products

If the conformal field theory $C$ is a sigma model with target space $X$, denoted $C = \sigma(X)$, then the elliptic genus of the conformal field theory only depends on the topology of $X$ and hence we can speak of $\chi(\tau, z; X)$. In this case $\chi(\tau, z; X)$ can be interpreted as an equivariant index of the Dirac operator $D$ on the loop space $LX$. The parameter $q$ accounts for rigid rotations of a loop, while $z$ accounts for rotations in the holomorphic tangent space $T^{1,0}X$ of the target.

We will be considering the elliptic genus for the case $X = \text{Sym}^k(K3)$. The elliptic genus for such $X$ is expressed in terms of the elliptic genus of $K3$ itself. For any conformal field theory with Hilbert space $H$ we can consider the symmetric group orbifold of $H^\otimes k$. Denote the Hilbert space of the orbifold theory by $\text{Sym}^k(H)$. This has a decomposition into twisted sectors given by

$$H(\text{Sym}^k(H)) = \oplus \{ k_r \} \otimes_{r > 0} \text{Sym}^{k_r}(H_r)$$

where the sum is over partitions of $k$:

$$\sum r k_r = k$$

The space $H_r$ is isomorphic to the space $H$. It corresponds to “strings of length $2\pi r$” where we scale the usual parameter $\sigma \sim \sigma + 2\pi$ by a factor of $r$. Thus configurations in the symmetric product orbifold theory may be visualized as in Fig. 1.

Fig. 1: A configuration of strings in the symmetric product conformal field theory.
Now, if $\mathcal{H}$ is a conformal field theory based on a sigma model with target space $M$ then (2.32) implies an identity on the orbifold elliptic genus for $\text{Sym}^k(M)$. To be specific, if

$$\chi(\tau, z; M) = \sum c(n, \ell)q^n y^\ell$$

(2.34)

then [25]

$$\sum_{k=0}^{\infty} p^k \chi(\text{Sym}^k M; q, y) = \prod_{n>0, m \geq 0, r} \frac{1}{(1 - p^n q^m y^r)^{c(nm, \tau)}}$$

(2.35)

In the AdS/CFT correspondence we apply this to $M = K3$. The elliptic genus of $K3$ can be computed (say, from orbifold limits or Gepner models) and is

$$\chi(q, y; K3) = 8\left(\left(\frac{\vartheta_2(z|\tau)}{\vartheta_2(0|\tau)}\right)^2 + \left(\frac{\vartheta_3(z|\tau)}{\vartheta_3(0|\tau)}\right)^2 + \left(\frac{\vartheta_4(z|\tau)}{\vartheta_4(0|\tau)}\right)^2\right)$$

(2.36)

and therefore, $\chi(\tau, z; \text{Sym}^k(K3))$ is explicitly known.

The decomposition in terms of theta functions is given by [26]

$$\chi(q, y; K3) = h_0(\tau)\Theta_{0,1} + h_1(\tau)\Theta_{1,1}$$

(2.37)

with

$$h_0(\tau) = \eta(\tau)^{-6}\left(6(\vartheta_2\vartheta_4)^2\vartheta_3(2\tau) - 2(\vartheta_4^4 - \vartheta_2^4)\vartheta_2(2\tau)\right)$$

(2.38)

$$h_1(\tau) = \eta(\tau)^{-6}\left(6(\vartheta_2\vartheta_4)^2\vartheta_2(2\tau) + 2(\vartheta_4^4 - \vartheta_2^4)\vartheta_3(2\tau)\right)$$

(2.39)

with

$$h_0 = 20 + 216q + 1616q^2 + \cdots$$

$$h_1 = q^{-1/4}(2 - 128q + \cdots)$$

(2.40)

Many other interesting aspects of the elliptic genus of $K3$ and its symmetric products, including relations to automorphic infinite products can be found in [27].

2.5. Expressing the elliptic genus as a Poincaré Series

Returning to our main theme, we will explain the basic formula first in a simplified situation. Then we state without proof the analogous result for weak Jacobi forms. The proof may be found in [2].

Let $f \in M_w^*$ be a weak modular form for $SL(2, \mathbb{Z})$ of weight $w \leq 0$. The adjective “weak” means that $f$ is allowed to have a pole of finite order at the cusp at infinity, but
no other singularities in the upper half plane. Thus, the Fourier expansion of $f$ takes the form:

$$f(\tau) = \sum_{n \geq 0} D(n) q^{n+\Delta} \quad (2.41)$$

We refer to the finite sum

$$f^-(\tau) = \sum_{n+\Delta < 0} D(n) q^{n+\Delta} \quad (2.42)$$

as the polar part.

In the physical context, $\Delta = -c/24$, for a unitary CFT, where $c$ is the central charge of the Virasoro algebra. Moreover, $w = -d/2$, where $d$ is the number of noncompact bosons in the CFT. Unfortunately, the letters $c, d$ are quite standard in the theory of modular forms so there is a clash of conventional notations. We will try to avoid the use of $c, d$ for central charge and noncompact dimensions in what follows and use $\Delta, w$ instead.

It turns out to be essential to introduce a map

$$M^*_w \rightarrow M^*_{2-w} \quad (2.43)$$

The explicit map is

$$f(\tau) \rightarrow Z_f(\tau) := (q \frac{\partial}{\partial q})^{1-w} f \quad (2.44)$$

The fact that the right hand side of (2.44) is a modular form is sometimes called Bol’s identity. Note that in terms of the Fourier expansion we have:

$$Z_f = \sum_{n \geq 0} \tilde{D}(n) q^{n+\Delta} \quad (2.45)$$

where

$$\tilde{D}(n) = (n + \Delta)^{1-w} D(n). \quad (2.46)$$

Given a polynomial $\varphi$ in $q^{-1}$ one can construct by hand a modular form of weight $w$ by averaging over the modular group to produce a Poincaré series

$$\sum_{\Gamma \in \Gamma \backslash \Gamma} (c\tau + d)^{-w} \varphi \left( \frac{a\tau + b}{c\tau + d} \right) \quad (2.47)$$

Note that we must sum over cosets of the stabilizer of $i\infty$, that is, we sum over $\Gamma \backslash \Gamma$ where

$$\Gamma \backslash \Gamma := \left\{ \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} | \ell \in \mathbb{Z} \right\} \quad (2.48)$$
The resulting sum is convergent for $w > 2$.

In general, weak modular forms of positive weight $w > 0$ are not uniquely determined by their polar parts. If the space of modular forms $M_w$ is nonzero one can always add an nonzero element to (2.47) to produce another form with the same polar part. However, if a form is in the image of the map (2.44) then it is in fact completely determined by its polar part. To see this, first note that $Z_f$ has no constant term. Next we use a pairing between weak modular forms and cusp forms which was quite useful in [28]. If $f \in M_w^*$ and $g \in S_w$ is a cusp form then we can extend the Petersson inner product by

$$(f, g) := \lim_{\Lambda \to \infty} \int_{F_\Lambda} \frac{dx dy}{y^2} y^w f(x + iy) g(x + iy)$$

(2.49)

Here $F_\Lambda$ is the intersection of the standard fundamental domain of $PSL(2, \mathbb{Z})$ with the set of $\tau = x + iy$ with $y \leq \Lambda$. Using integration by parts we can see that $Z_f$ is orthogonal to the space of cusp forms $S_{2-w}$, and hence it is determined by its polar part.

Let us summarize: We can reconstruct $Z_f$ from the polar part

$$Z_f^- = \sum_{n+\Delta < 0} \tilde{D}(n) q^{n+\Delta}$$

(2.50)

(which is a finite sum) via

$$Z_f(\tau) = \sum_{\Gamma_{\infty} \backslash \Gamma} (c \tau + d)^{w-2} Z_f^- \frac{(a \tau + b)}{(c \tau + d)}$$

(2.51)

This is the kind of formula we are going to interpret in terms of AdS/CFT.

2.5.1. Digression 2: Rademacher’s formula

In the next two subsections we pause to make two more small digressions concerning some related issues: Rademacher’s formula, Cardy’s formula, and the applications to black hole entropy.

The Rademacher formula is a formula for the Fourier coefficients of $f(\tau)$ which is particularly useful for questions about the asymptotic nature of the Fourier coefficients. The formula is easily derived from (2.51) by taking a Fourier transform. On the left hand side we have:

$$\int_{\tau_0}^{\tau_0 + 1} e^{-2\pi i (\ell + \Delta) \tau} Z_f(\tau) d\tau = \tilde{D}(\ell)$$

(2.52)

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on the right hand side, after a little manipulation we have a sum of integrals of the form:

\[
\int (c\tau + d)^{w-2} e^{-2\pi i (\ell + \Delta) \tau} e^{2\pi i (n + \Delta) \frac{\tau + \Delta}{\tau + \Delta}} d\tau
\]  

(2.53)

which can be expressed in terms of Bessel functions. The precise relation we find is

\[
D(\ell) = 2\pi \sum_{n + \Delta < 0} \left(\frac{\ell + \Delta}{|n + \Delta|}\right)^{(w-1)/2} D(n) \\
\cdot \sum_{c=1}^{\infty} \frac{1}{c} Kl(\ell + \Delta, n + \Delta; c) I_{1-w} \left(\frac{4\pi}{c} \sqrt{|n + \Delta| (\ell + \Delta)}\right).
\]

(2.54)

where \(I_\nu(x)\) is the Bessel function growing exponentially at \(\infty\)

\[
I_w(x) \sim \frac{1}{\sqrt{2\pi x}} e^x \quad \Re(x) \to +\infty
\]

(2.55)

while

\[
Kl(n, m; c) := \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^*} \exp \left[2\pi i (\frac{n}{c} d + \frac{m}{c})\right]
\]

(2.56)

is a Kloosterman sum.

While (2.54) is a terribly complicated formula, it is in fact also very useful since it gives the asymptotics of Fourier coefficients of modular forms for large \(\ell\). In fact, it can be a very efficient way to compute the Fourier coefficients exactly if they are known, for example, to be integral.

In the physics literature the leading term,

\[
D(\ell) \sim \frac{D(0)}{\sqrt{2}} \left(\frac{\ell + \Delta}{\Delta}\right)^{\frac{1}{2} w - \frac{3}{4}} \exp \left[4\pi \sqrt{|\Delta| (\ell + \Delta)}\right]
\]

(2.57)

is known as “Cardy’s formula.” It gives the “entropy of states at level \(\ell\)”

The subleading exponential corrections are organized in a beautiful way by Farey sequences. See [29,30,31] or [2], appendix B for details.

2.5.2. Digression 3: Black hole entropy

One very striking application of Cardy’s formula in the string literature is to the statistical accounting for the entropy of certain special black holes. This was first proposed in a famous paper of A. Strominger and C. Vafa [32].
As we have mentioned, the spacetime $\text{AdS}_3 \times \text{S}^3 \times \text{K}^3$ is obtained as a near-horizon geometry from a limit of a system of $Q_1$ D1-branes and $Q_5$ D5-branes wrapping $\text{S}^1 \times \text{K}^3$. The “BPS states” of this system of branes correspond to special black hole solutions of 5-dimensional supergravity. The black hole solution is characterized by three charges $Q_1, Q_5, N$. In the D-brane system, $Q_1, Q_5, N$ specify quantum numbers of BPS states; there is a $\mathbb{Z}_2$-graded vector space of such states: $\mathcal{H}_{BPS}^{(\gamma)}$, with charges $\gamma = (Q_1, Q_5, N)$. The elliptic genus counts the super-dimension of these vector spaces of BPS states:

$$\chi(q, \text{Sym}^k \text{K}^3) = \sum q^N \text{sdim} \mathcal{H}_{BPS}^{(\gamma)} = (Q_1, Q_5, N)$$

The Cardy formula then gives:

$$I \sim \exp \left( 2\pi \sqrt{Q_1 Q_5 N} \right)$$

and confirms the supergravity computation of the Beckenstein-Hawking entropy $^{32}$. $^2$

The Rademacher formula gives an infinite series of subleading corrections

$$\sim \exp \left( \frac{2\pi}{c} \sqrt{Q_1 Q_5 N} \right) \quad c = 2, 3, 4, \ldots$$

organized by terms in the Farey sequences. In section 2.6 we will discuss the physical interpretation of these subleading corrections.

### 2.5.3. Poincaré Series for the Elliptic Genus

Finally, let us return to the main task of this section: Expressing the elliptic genus as a Poincaré series in a form suitable to interpretation within the AdS/CFT correspondence.

The manipulations of section 2.5 above have analogs for Jacobi forms. Let $J_{w,k}$ denote the space of weak Jacobi forms of weight $w$ and index $k$. The analog of the polar part (2.42) is the sum over Fourier coefficients with

$$4kn - \ell^2 < 0.$$  \hfill (2.61)

$^2$ It is important to bear in mind that this is actually counting with signs. It is counting vector-multiplets minus hypermultiplets, and can lead to cancellations, and hence it can underestimate the entropy. In the case examined in $^{32}$ it gives the “right” answer, i.e. the answer that coincides with supergravity.
Applied to the elliptic genus the relevant Poincaré series becomes:

\[
Z\chi(\tau, z) = 2\pi \sum_{(\Gamma_\infty \setminus \Gamma)_0} \sum' \tilde{c}_\mu(4km - \mu^2; \text{Sym}^k(K3)) \\
\exp[-2\pi ik \frac{cz^2}{c\tau + d}] \Theta^+_{\mu,k}(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d}) \\
(c\tau + d)^{-3} \exp\left[2\pi i (m - \frac{\mu^2}{4k}) \frac{a\tau + b}{4k c\tau + d}\right]
\]

where \((\Gamma_\infty \setminus \Gamma)_0\) is the sum over relatively prime pairs \((c, d)\) with \(c \geq 0\), while \(\sum'_m,\mu\) is a finite sum over \((m, \mu)\) with \(4km - \mu^2 < 0\), and \(\Theta^+_{\mu,k}\) was defined in (2.31).

In the next section we are going to sketch how this sum can be interpreted as a sum over solutions to 10D supergravity.

**Note added, Dec. 8, 2007**: Don Zagier pointed out an important error in versions 1-3 of this paper. The map

\[
\phi = \sum c(n, \ell) q^n y^\ell \rightarrow \tilde{\phi} = \sum \tilde{c}(n, \ell) q^n y^\ell
\]

with

\[
\tilde{c}(n, \ell) = |n - \ell^2/4k|^{3/2-w} c(n, \ell)
\]

does not map Jacobi forms \(J_{w,k} \rightarrow J_{3-w,k}\), contrary to what was asserted in versions 1-3. Nevertheless, for \(n - \ell^2/4k > 0\), the \(\tilde{c}(n, \ell)\) can be obtained as Fourier coefficients from the Poincaré series (2.62). For further details see the corrected version 3 of [2], as well as [33], which writes a regularized Poincaré series for the elliptic genus itself, and not its "Fareytail transform."

### 2.6. AdS/CFT Interpretation of the Poincaré Series

In the previous section we wrote down the Poincaré series (2.62) for the elliptic genus. This is a mathematical fact, and we are regarding this exact result as a precious piece of “experimental data” to tell us how to formulate the string theory side of the AdS/CFT correspondence. As we will see, the precise formulation of string theory on \(AdS_3 \times S^3 \times K3\) is full of interesting subtleties. We will now proceed to interpret the various factors in (2.62) in physical terms.
2.6.1. Average over $\Gamma_{\infty}\backslash \Gamma$ and BTZ black holes

We are going to describe the AdS dual to a conformal field theory computation of a partition function. Therefore, the conformal boundary of the $AdS_3$ should be a torus. Therefore, we will be looking at 3-dimensional geometries filling in $S^1_\phi \times S^1_t$. The metric will accordingly have boundary conditions:

$$ds^2 \rightarrow r^2 |d\phi + idt|^2 + \frac{dr^2}{r^2}$$

for $r \rightarrow \infty$. Here $(\phi + it) \sim (\phi + it) + 2\pi(n + m\tau)$, $n, m \in \mathbb{Z}$, and $\tau$ determines the conformal structure of the torus at infinity.

The only smooth complete hyperbolic geometry satisfying these conditions has the topology of a solid torus. One way to realize this geometry is to take a quotient of the upper half plane $\mathbb{H} = \mathbb{C} \times \mathbb{R}^+$ by the group $\mathbb{Z}$ acting as $(z, y) \rightarrow (q^n z, |q^n| y)$. We can compactify the space by adding the boundary at infinity $\hat{\mathbb{C}}$. We must omit $0, \infty \in \hat{\mathbb{C}}$ to get a properly discontinuous group action. Topologically, the resulting space is a solid torus.

While the hyperbolic geometry is unique, in order to do physics we need to make a choice of what is called “space” and what is called “time” in the torus at infinity. This choice will affect computations of action, entropy etc. It is this choice which accounts for the sum over $\Gamma_{\infty}\backslash \Gamma$, that is, over relatively prime integers $(c, d)$ in (2.62). Geometrically, $(c, d)$ describes the unique primitive homology cycle which becomes contractible upon filling in the torus with a solid torus.

For example, let us choose coordinates $(\phi, t)$ on $S^1 \times S^1$. If we choose the term $(c = 0, d = 1)$ then it is the “spatial” $\phi$-circle which is filled in. In this case the geometry has the interpretation of an “AdS gas” – that is, we analytically continue the time in Lorentzian AdS and identify it with $t_E \sim t_E + \beta$.

On the other hand, in the term corresponding to $(c = 1, d = 0)$ it is Euclidean “time” - the $t$-circle - which is filled in. In this case we have the Euclidean “BTZ black hole.” Note that the spatial circle is noncontractible: There is a hole in space, and it is in fact correctly interpreted as a true black hole solution of gravity, as shown in great detail in [34][35].

The general solution is labelled by a point in

$$\Gamma_{\infty}\backslash \Gamma \cong \hat{Q}$$

(2.66)
and is labelled by the homology class of the primitive cycle which is contractible. This family of black holes is the proper interpretation of what Maldacena and Strominger termed an “$SL(2, \mathbb{Z})$ family of black holes” in \[36\]. Thus, the first, and most basic aspect of (2.62) is that it is a sum over this family of black holes (including the AdS gas ($c = 0, d = 1$)).

### 2.6.2. Low energy Chern-Simons theory

Now, we would like to compute the contribution of the string theory path integral to each term in the sum over pairs $(c, d)$ in (2.62). A crucial point is that the elliptic genus is unchanged under deformation of parameters. This allows us to focus on the low energy and long-distance limit of the reduction of 10d supergravity on $AdS_3 \times S^3 \times K3$. In this limit, the dominant term in the supergravity action is that of a Chern-Simons theory. The Chern-Simons supergroup is $SU(2|1, 1) \times SU(2|1, 1)$

$$SU(2|1, 1) \times SU(2|1, 1)$$

and the explicit action is

$$\frac{k}{4\pi} \int \text{Tr}(A dA + \frac{2}{3} A^3) - \text{Tr}(B dB + \frac{2}{3} B^3)$$

The $SU(1, 1) \times SU(1, 1)$ connections are derived from the negative curvature metric via $A_\pm \sim w \pm e$ where $w$ is the spin connection and $e$ is the dreibrein \[38\] \[35\]. The $SU(2) \times SU(2)$ gauge fields arise from Kaluza-Klein reduction on $S^3$. For a detailed derivation of these terms in the action see \[39\] \[40\] \[41\].

We must choose boundary conditions for the Chern-Simons gauge fields. The boundary values of the connections for $SU(2|1, 1)_L$, and $SU(2|1, 1)_R$ couple to CFT left- and right-movers, respectively. The boundary conditions (2.63) determine boundary conditions on the $SU(1, 1)$ gauge fields. In addition: The $SU(2)$ gauge fields become flat at infinity and the proper boundary conditions are:

$$A_u du \rightarrow \frac{\pi}{2\text{Im}\tau} z\sigma^3 du$$

where $u = i(\phi + it)/(c\tau + d)$

Because of our choice of fermion spin structures the boundary conditions of the right-moving $SU(2)$ gauge fields should drop out. This point deserves to be understood more fully.

---

\[3\] An heuristic version of this sum was first written down in \[36\].
2.6.3. Spinning in 6-dimensions

Actually, we have not yet fully enumerated the distinct types of geometry that we must sum over. When we include the $z$-dependence in the elliptic genus it is necessary to consider \emph{six-dimensional geometries}. This leads to an interpretation of the sum on $\mu$ in (2.62).

The BTZ black holes have natural generalizations to quotients of the form

$$\mathbb{Z}\backslash(\mathbb{H}^3 \times S^3)$$

with $\mathbb{Z}$ acting on $S^3 = SU(2)$ by

$$U \rightarrow \tilde{U} = e^{-i\frac{\mu}{2}(t+\phi)\sigma^3}U \quad (2.71)$$

These correspond to solutions spinning in six dimensions with $2j_L = \mu$. Such solutions have been nicely described in detail in [42]. Closely related smooth solutions associated with BPS states have been described in [43].

In the effective $SU(2)$ Chern-Simons theory these solutions correspond to the insertion of a Wilson line in the center of the solid torus as in Fig. 2. Since the $SU(2)$ theory is governed by a Chern-Simons theory we expect to see the wavefunction associated to such theories in the partition function. It is well-known that these wavefunctions are given by the affine Lie algebra characters of $SU(2)$ level $k$ current algebra for spin $j$. Another basis of wavefunctions count states at definite values of $J_3^0$. These are given by level $k$ theta functions:

$$\exp[-2\pi ik\frac{cz^2}{c\tau + d}]\Theta^+_\mu,k\left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right) \quad (2.72)$$

\textbf{Fig. 2:} A black hole spinning in 6 dimensions is effectively equivalent to the partition function on a solid torus with a Wilson line insertion.
To summarize, we can interpret the contribution of \((c, d)\) and \(\mu\) as a BTZ black hole with homology class \((c, d)\) contractible and with Wilson lines inserted so that the Chern-Simons wavefunction has definite values of \(\mu\) modulo \(k\), as in fig. 2.

2.6.4. The light particles of supergravity

Let us now interpret the sum over the polar part in \((2.62)\),

\[
\sum_{m: 4km - \mu^2 < 0} \tilde{c}_\mu \left( 4km - \mu^2; \text{Sym}^k(K3) \right)
\]

(2.73)

In order to do this we must address some aspects of the Lorentzian version of the AdS/CFT correspondence.

In the Lorentzian version, there is an isomorphism of Hilbert spaces between the Hilbert space of the boundary conformal field theory and some much more mysterious Hilbert space of quantum gravity (string theory) on some interior space. The Hilbert space of the conformal field theory is rather well-understood. We will view it as a Hilbert space graded by the values of \((L_0, J_0)\). In the elliptic genus, the left-moving Ramond sector states have quantum numbers \((m, \mu)\) which we identify as the eigenvalues

\[
(m, \mu) = (L_0 - c/24, J_0)
\]

Now, we expect such states to correspond to states in the quantum gravity Hilbert space. Symmetry principles (i.e. matching of superconformal symmetries) show us that we must interpret \(L_0\) as the 2+1 dimensional energy + spin, while \(J_0\) should be viewed as the \(J^3\) eigenvalue for spin in the \(S^3\) directions.

From the point of view of quantum gravity, there is an important distinction between states which are small perturbations on an AdS background - we will refer to these as “particle states” - and states which form black holes. The distinction is governed by the “cosmic censorship bound” \([14] [15] [12]\). Black holes correspond to semiclassical states in \(H_{\text{string}}\). The corresponding states in \(H_{\text{CFT}}\) have \(L_0\) in the Ramond sector related to the mass \(M\) of the black hole by \(M = L_0 - c/24\) \([16]\). On the other hand, the condition for a black hole to have a nonsingular horizon is \(4kM - J_0^2 \geq 0\) \([14] [15] [12]\). Such states therefore have \(4km - J_0^2 \geq 0\). Thus the unitarity region in the \((m, \mu) = (L_0 - k/4, J_0)\) plane is divided into two regions: Supergravity states with \(-k^2 \leq 4km - \mu^2 < 0\) are not sufficiently massive to form black holes, corresponding to the shaded region in Fig. 3, while states with \(4km - \mu^2 \geq 0\) will form black holes. Thus, the states which do not form black holes correspond precisely to the to the polar part of the Jacobi form! Moreover, the degeneracy \(c_\mu(4km - \mu^2; \text{Sym}^k(K3))\) is precisely that of right-BPS supergravity particles from Kaluza-Klein reduction of \((2, 0)\) supergravity on \(AdS_3 \times S^3\) \([37]\).
Fig. 3: The states in the shaded region are not sufficiently energetic to form black holes. These states have quantum numbers corresponding to the polar part of the elliptic genus. Note that quantum numbers not on the $\ell = 2J_0^3$ axis are not BPS states. The discussion above pertains to states which are right-BPS.

2.6.5. Gravitational action and final factor

According to our interpretation, the final factors

\[
(c\tau + d)^{-3} \exp \left[ 2\pi i (m - \frac{\mu^2}{4k}) \frac{a\tau + b}{c\tau + d} \right]
\]

should arise from a careful evaluation of an analytic continuation of $SU(1,1) \times SU(1,1)$ Chern-Simons theory to Euclidean signature.

Thus one is naturally let to attempt a careful evaluation of the gravitational action for the spinning extremal black holes. The Einstein action is

\[
\frac{1}{16\pi G} \int \sqrt{g} (R - \Lambda) + \frac{1}{8\pi G} \int K
\]

where $K$ is the second fundamental form of the boundary. Since the Einstein action on $AdS$ is infinite it must be regularized. The standard way to do this is to introduce a boundary, thus necessitating the second term. The difference of such actions between two geometries in the family (2.66) can be evaluated in a well-defined way and gives:

\[
\pi k \left(\text{Im} \tau - \text{Im} \left(\frac{a\tau + b}{c\tau + d}\right)\right)
\]
Moreover, the computations of [12] produce such an entropy factor weighted by $m - \mu^2/4k$
in the six-dimensional case.

Upon taking a $\bar{\tau} \to \infty$ limit the expression (2.76) closely resembles (2.74), but, so far
as we know, there is no honest and convincing derivation of (2.74) in the literature starting
from the Chern-Simons approach.

Note that (2.74) is odd under $(c,d) \to (-c,-d)$. This is the reason we must put
a restriction $c \geq 0$ on the Poincaré series (2.62). The transformation $(c,d) \to (-c,-d)$
corresponds to the diffeomorphism $-1$ on the boundary torus. The fact that the summand
in (2.62) should be understood better. Perhaps it is due to the fact that only Ramond
groundstates contribute.

2.7. Summary: Lessons & Enigmas

We have presented some evidence to suggest that the full AdS-interpretation of the
elliptic genus of the boundary conformal field theory can be expressed in the form

$$Z_\chi = \sum \Psi_{SU(2|1,1)}^{CS}$$

(2.77)

where $\Psi_{SU(2|1,1)}^{CS}$ is a wavefunction for a Chern-Simons theory and where the sum is over
Euclidean solutions of supergravity of spinning black holes with supergravity particles in
$AdS_3 \times S^3$. It should be clear to the reader that there are gaps and enigmas in this story.

For examples,

1. Why do we need to take the Serre dual to get a reasonable formula?

2. What is the origin of the factor

$$1/(c\tau + d)^3$$

(2.78)

from the string partition function? Note that this factor is crucial for the convergence
of the sum over $(c,d)$. It also has the pleasant property that $Z_\chi dz \wedge d\tau$ is a well-defined
half-density on the universal elliptic curve.

3. Is it sufficient to focus purely on the Chern-Simons sector to evaluate the path integral
or must one take into account the full tower of string fields? (We have been assuming
the latter contribute a trivial factor to $Z_\chi$, because of its topological nature.)

4. Perhaps the most important enigma is the origin of the sum over the polar part
in (2.62). This is probably saying something significant about the Hilbert space of
quantum gravity. It indicates that the nature of the isomorphism between the CFT
Hilbert space and the string theory Hilbert space is qualitatively different for the
infinite set of conformal field theory states above the cosmic censorship bound. What replaces a sum over states in the Euclidean quantum gravity Hilbert space is a sum over a special set of geometries. Note in particular that the \((m = 0, \ell = 0)\) term does not contribute. These are the unique quantum numbers (the so-called “\(M = 0\) BTZ” black hole) of states which are simultaneously topological and black holes. It is possible that this structure is related to the phenomenon of “asymptotic darkness” that has been advocated by T. Banks [47][48].

2.8. Applications

Whether or not one believes the physical interpretation advocated in the previous section, the formula (2.62) is true, and has some some nice applications.

One application is to the thermodynamics of string theory on Euclidean \(AdS_3 \times S^3 \times K3\). One discovers a 3-dimensional version of the deconfining phase transition of large \(N \mathcal{N} = 4\) Yang-Mills theory discussed by Witten [49]. In the \(AdS_3\) case one studies the partition function as a function of

\[
\tau = \Omega + i\beta
\]

where \(\Omega\) is the spin fugacity and \(\beta\) is the inverse temperature. In the large \(k\) limit \(Z_\chi\) becomes a piecewise analytic function of \(\tau\). It is simplest to study the partition function in the \((\text{NS}, R)\) sector (by setting \(z = -\tau/2\)). As \(k \to \infty\) at fixed \(\tau\) the dominant geometry is characterized by the pair \((c, d)\) which maximizes

\[
\frac{\text{Im} \tau}{|c\tau + d|^2}
\]

This geometry contributes a term of order

\[
\frac{1}{|c\tau + d|^3} |\bar{c}(-k^2)| \exp \left[ \frac{\pi k}{2} \frac{\text{Im} \tau}{|c\tau + d|^2} \right]
\]

The standard keyhole region fundamental domain \(\mathcal{F}\) for \(SL(2, \mathbb{Z})\) has the property that the modular image of any point \(\tau \in \mathcal{F}\) has an imaginary part \(\text{Im} \tau' \leq \text{Im} \tau\). Therefore, the phase domains are given by \(\bigcup_{n \in \mathbb{Z}} T^n \cdot \mathcal{F} = \Gamma_\infty \cdot \mathcal{F}\) and its modular images.

As a second application we note that a computation similar in spirit to what we have discussed was performed by Maldacena to resolve a sharp version of the “black hole information paradox” for eternal AdS black holes. See [50].
2.9. Speculations on future applications of AdS/CFT to number theory

In this section we present some speculations on ways in which the AdS/CFT correspondence might have some interesting interactions with number theory. Our speculations are based on ongoing discussions with A. Strominger, and have at times involved B. Mazur, and S. Gukov. For some related ideas see [3]. (Some overlapping remarks were made recently in [51] [52].)

2.9.1. Quotients of AdS/CFT

Suppose string theory on $\text{AdS}_{n+1} \times K$ is dual to a conformal field theory $\mathcal{C}$. Suppose $\Gamma \subset SO(1, n + 1)$ or $\Gamma \subset SO(2, n)$ (2.82) is an infinite discrete group. Since $\Gamma$ acts as a group of isometries in the bulk theory, we can consider string theory on $\Gamma \backslash (\text{AdS} \times K)$ (2.83)

It is natural to ask if string theory on (2.83) makes sense, and if so, whether it is dual to some kind of “quotient” of the conformal field theory $\mathcal{C}$ by $\Gamma$. Note that such a quotient, if it even exists, is very different from an orbifold of a conformal field theory, for $\Gamma$ acts by conformal transformations on the “worldsheet” rather than the “target space” of $\mathcal{C}$.

Such a duality, if it were to make sense, would have very interesting implications in at least two ways. First, there would be important applications to questions of cosmology and time dependence in string theory. Second – and more central to the theme of these lectures – there would be interesting applications to number theory. In the following sections we will sketch some of the possible applications.

The reader should be warned at the outset that there are nontrivial difficulties with the idea that AdS/CFT duality can survive general quotients by such groups $\Gamma$. The difficulties stem from the fact that the “interesting” groups we wish to consider act on the conformal boundary at infinity, $\partial \mathbb{H}^n$, but the action is sometimes ergodic. More precisely, the boundary is divided into a disjoint union of two regions:

$$\partial \mathbb{H}^n = \Omega_\Gamma \cup \Lambda_\Gamma$$ (2.84)

The first region $\Omega_\Gamma$ is the domain of discontinuity. Here the group acts properly discontinuously and the quotient $\Omega_\Gamma / \Gamma$ is, for $n = 2$, a Riemann surface. Note that this Riemann
surface can have cusps and several connected components. The complementary region $\Lambda_{\Gamma}$ is called the limit set. It is the closure of the set of accumulation points of $\Gamma$, and the action on $\Lambda_{\Gamma}$ is ergodic. This means that any "quotient" of the boundary conformal field theory is going to have strange behavior on $\Lambda_{\Gamma}$. To take an extreme example, there are groups $\Gamma$ with no domain of discontinuity. Then the classical quotient $\mathbb{H}^n/\Gamma$ is a compact hyperbolic manifold. So the "boundary theory," if it exists, must surely be something truly unusual.

In fact, the quotient by $\Gamma$ can produce strange causal structure in the Lorentzian case, a fact which probably indicates large backreaction in the context of supergravity. A related point is that the distance between image points $d(x, \gamma \cdot x)$ can get small, again indicating breakdown of the sugra approximation. Indeed, the existence of a boundary theory for groups $\Gamma$ with nontrivial limit set has been argued against by Martinec and McElgin [53][54].

Nevertheless, a successful outcome would undoubtedly lead to many very fascinating things, so let us suppose that a dual boundary theory does exist and briefly ask what it might be good for.

2.9.2. String Cosmology

A few years ago, in [55], interesting cosmologies with singularities were considered based on spacetimes of the form (2.83).

More recently, string theory with time-dependent singularities in "soluble" string models has come under some scrutiny. Amongst the many investigations in this area is the work in [56][57][58][59] which studies the $\mathbb{Z}$-orbifold of $\mathbb{R}^{1,2}$ defined by the action

$$X := \begin{pmatrix} x^+ \\ x \\ x^- \end{pmatrix} \rightarrow g_0 \cdot X = \begin{pmatrix} x^+ \\ x + n v x^+ \\ x^- + n v x + \frac{1}{2} n^2 v^2 x^+ \end{pmatrix}$$

(2.85)

where $(x^+, x, x^-)$ are light-cone coordinates. It turns out that string perturbation theory in such backgrounds is highly problematic. The difficulties are expected to be a generic feature of strings in cosmological singularities. Moreover, nonperturbative effects involving black holes are expected to be important [60]. This is relevant to the present discussion for the following reason. Recall that $AdS_{1,2}$ is the universal covering space $SL(2, \mathbb{R})$. The Lie algebra $sl(2, \mathbb{R}) = \mathbb{R}^{1,2}$ is Minkowski space. Consider the action on $AdS_{1,2}$ by $\mathbb{Z}$ with

$$g \rightarrow g g_0 g_0^{-1},$$

(2.86)
where \( g_0 \) is a parabolic element. In the scaling region of \( g = 1 \) these look like the cosmological models (2.83). On the other hand, since there is a boundary theory summarizing all the nonperturbative physics, it is reasonable to think, \textit{provided the AdS/CFT correspondence survives the quotient construction}, that the boundary theory contains some clue as to the resolution of the cosmological singularity. Some investigations along these lines were carried out in [51], but there is much more to understand.

2.9.3. Potential Applications to Number Theory: Euclidean version

One of the possible applications of these ideas to number theory concerns the theory of modular symbols.

Let us recall (in caricature) the \textit{AdS/CFT} computation of the 2point function of spinless primary fields. In AdS the tree-level 2-point function of scalar fields \( \phi \) is the Green’s function:

\[
(\Delta_1 + m^2)G(P_1, P_2) = \delta(P_1, P_2)
\]  

(2.87)

In \( \mathbb{H}^3 \) we have the simple explicit formula:

\[
G(P_1, P_2) = \frac{1}{2\pi} \frac{e^{-2hd(1,2)}}{1 - e^{-d(1,2)}}
\]  

(2.88)

where

\[
\cosh d(1,2) = 1 + \frac{|z_1 - z_2|^2 + (y_1 - y_2)^2}{2y_1y_2} \quad m^2 = 2h(2h - 2)
\]  

(2.89)

One extracts the 2point correlator from the boundary behavior of the Green’s function:

\[
G(1,2) \rightarrow y_1^{2h}y_2^{2h}\langle \Phi_\phi(z_1)\Phi_\phi(z_2) \rangle
\]  

(2.90)

as \( y_1, y_2 \rightarrow 0 \). This leads to the familiar result:

\[
\langle \Phi_\phi(z_1)\Phi_\phi(z_2) \rangle = \frac{1}{|(z_1 - z_2)^{2h}|^2}
\]  

(2.91)

where \( \Phi_\phi \) is the dual operator of (2.12).

Now, let \( \Gamma \subset PSL(2,\mathbb{C}) \) be discrete and suppose AdS/CFT “commutes with orbifolding.” In the tree-level approximation, the Green’s function on \( \Gamma \backslash \mathbb{H}^3 \) is obtained by the method of images. Therefore, according to (2.90) the boundary CFT correlator should be obtained from the method of images. For a primary field (with spin) of weights \((h,0)\) this would lead to

\[
\langle \Phi(z_1)\Phi(z_2) \rangle_{\Gamma \backslash \Omega_\Gamma} = \sum_{\Gamma} \frac{1}{(z_1 - \gamma \cdot z_2)^{2h}} \frac{1}{(cz_1 + d)^{2h}}.
\]  

(2.92)
We would like to stress that in general in CFT it is not true that the conformal correlators on Riemann surfaces $\Gamma\backslash\Omega_\Gamma$ are obtained by the method of images. While it is true that the Green’s function of a scalar field is obtained by summing over images, in the presence of interactions there are further correlations between a source and its image point. Therefore, at best (2.92) can apply in the large $k$ approximation (which justifies the tree-level supergravity). Even there, AdS/CFT is making a highly nontrivial prediction for the boundary CFT correlators.

Nevertheless, let us accept (2.92). Now suppose there is a flat gauge field in the low energy supergravity coupling to charged scalars $\phi^{\pm}$. Then the boundary correlator becomes:

$$\langle \Phi^+(z_1)\Phi^-(z_2) \rangle_{\Gamma\backslash\Omega_\Gamma} = \sum_{\Gamma} \frac{e^{iq\oint A}}{(z_1 - \gamma \cdot z_2)^{2h}} \frac{1}{(cz_1 + d)^{2h}}$$

(2.93)

For example, we could take $\Gamma = \Gamma_0(N)$ and $A = f(z)dz$, for $f \in S_2(\Gamma_0(N))$, a cusp form of weight 2. In this way we obtain generating functions for modular symbols. Curiously, functions very closely related to (2.93) have recently been studied in attempts to understand the distribution of modular symbols [62]. In view of this, it is interesting to ask if AdS/CFT could give new insights into questions involving modular symbols.

It is also natural to ask about nonabelian generalizations of (2.93). These can be written down. Recalling the relation between boundary CFT and the Chern-Simons-Witten theory, one is lead to a new interpretation of the Verlinde operators of that theory in terms of what might be called “quantum nonabelian modular symbols.” We hope to describe this in detail elsewhere.

2.9.4. Potential Applications to Number Theory: Lorentzian version

As a second illustration of how applications to number theory might arise, let us suppose the Lorentzian AdS/CFT correspondence commutes with orbifolding for $\Gamma \subset SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$. Let us focus on the special case of a Hecke congruence subgroup

$$\Gamma = \Gamma_0(N) \subset SL(2,\mathbb{Z}) \subset SL(2,\mathbb{R})_L$$

(2.94)

so we are considering the spacetime

$$\Gamma\backslash\widetilde{SL(2,\mathbb{R})}$$

(2.95)

---

4 As a simple example, if $\phi$ is a free massless scalar field then $\langle \phi(1)\phi(2) \rangle$ is a sum of images, and therefore $\langle e^{ip\phi}(1)e^{-ip\phi}(2) \rangle$ is a product over images!
which may be pictured as a modular curve, evolving in time. The cusps of the modular
curve trace out null lines at infinity.

Some of the on-shell scalar fields of supergravity are constructed from \( L^2(\Gamma \backslash \widetilde{SL(2, \mathbb{R})}) \). The boundary asymptotics of these forms are, of course, well-studied in number theory, and in this way the the “scattering matrix” for Eisenstein series \([63]\), finds an interpretation in AdS/CFT.

3. Lecture II: Arithmetic and Attractors

3.1. Introduction

Modular forms, congruence subgroups, elliptic curves, are all mathematical objects
of central concern both to number theorists and to some physicists. A nice illustration
of the common interests physicists and mathematicians share in this area is the excellent
predecessor to the present proceedings \([64]\). In this lecture, we will be discussing the
possibility that there are interesting arithmetical issues connected with the theory of string
compactification. We will mostly be reviewing \([65]\), \([66]\), although we will make several new
points along the way.

While there are many common tools and mathematical objects in string compactifi-
cation and in number theory, one often finds that the detailed questions of the number
theorists and the string theorists are quite different. As an illustration of this point, in
string perturbation theory we encounter the elliptic curve

\[
E_\tau := \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})
\]

but in string perturbation theory there isn’t any compelling reason to restrict attention
to elliptic curves defined over \( \mathbb{Q} \) (or any other number field). Moreover, one can argue
that compactification on arithmetic varieties cannot be special. Firstly, physical quanti-
ties such as masses, scattering amplitudes, etc. change continuously with the moduli of
compactification varieties. Secondly, different arithmetic models for the same variety over
\( \mathbb{C} \) have different number-theoretic properties. For example, the elliptic curves \( y^2 = x^3 + n \)
for \( n \in \mathbb{Z} \) are in general inequivalent over \( \mathbb{Q} \), although they are of course equivalent over
\( \mathbb{C} \).

In spite of the above discouraging remarks, in this lecture we’ll present a little evidence
for the contrary viewpoint. We begin by describing the “attractor mechanism.” This is a
mechanism that distinguishes certain complex structure moduli as being special. The point of this talk is that the “attractor mechanism” for susy black holes provides a framework which naturally isolates certain arithmetic varieties. At the level of slogans, one can say that \textit{supersymmetric black holes for IIB string theory on CY 3-folds select arithmetic varieties.} Whether this is really true for arbitrary Calabi-Yau 3-folds, and whether the arithmetic of these varieties has physical significance is still an open problem. We will indicate some ways in which the physics and arithmetic are related.

Some closely related works, which we will not review here, include [67][68][69][70].

### 3.2. The “attractor equations”

The “attractor equations” are conditions on the Hodge structure of Calabi-Yau manifolds. They were introduced in the context of studies of black holes in Calabi-Yau compactification of string theory, for reasons we will explain in the next sections, by S. Ferrara, R. Kallosh, and A. Strominger in [71][72].

Let \( X \) be a compact Calabi-Yau 3-fold, and let \( \tilde{M} \) be the Teichmuller space of complex structures on \( X \). Consider an integral vector \( \gamma \in H^3(X, \mathbb{Z}) \). Given a complex structure \( t \in \tilde{M} \) we have a Hodge decomposition:

\[
\gamma = \gamma^{3,0} + \gamma^{2,1} + \gamma^{1,2} + \gamma^{0,3} \tag{3.2}
\]

**Definition:** The \textit{attractor equations} on the complex structure determined by \( \gamma \) are the equations

\[
\gamma = \gamma^{3,0} + \gamma^{0,3} \tag{3.3}
\]

Equivalently, since \( h^{3,0} = 1 \), we can choose a generator \( \Omega \) for \( H^{3,0}(X) \) and write instead:

\[
2\text{Im}(C\Omega) = \gamma \in H^3(X; \mathbb{Z}) \tag{3.4}
\]

for some constant \( C \). In order to make contact with the literature let us write these equations yet another way. Choose a symplectic basis \( \alpha^I, \beta_I \) for \( H_3 \). Define “flat coordinates”: \( X^I = \int_{\alpha^I} \Omega, F_I = \int_{\beta_I} \Omega \). Then the attractor equations become:

\[
\bar{C}X^I - C\bar{X}^I = ip^I \\
\bar{C}F_I - C\bar{F}_I = iq_I \tag{3.5}
\]

In the remainder of the lectures we will discuss three different ways in which these equations show up in string compactification.
3.3. First avatar: BPS states and black holes in IIB strings on $M_4 \times X$

3.3.1. Compactification of IIB string theory on $M_4 \times X$

In order to set some notation let us consider briefly some aspects of compactification of type IIB string theory on $M_4 \times X$, where $M_4$ is a Lorentzian 4-manifold, such as $\mathbb{R}^{1,3}$, or a spacetime asymptotic to $\mathbb{R}^{1,3}$. If $X$ has generic $SU(3)$ holonomy then there is a unique covariantly constant spinor, up to scale and hence the 32-real dimensional space of supercharges is reduced to an 8-real dimensional space. That is, the low energy supergravity has $\mathcal{N} = 2$ supersymmetry.

$d = 4, \mathcal{N} = 2$ supergravities are highly constrained physical systems [73][74]. For our purposes we only need to know that there are a collection of complex scalar fields in a nonlinear sigma model of maps $t : M_4 \to \tilde{\mathcal{M}}$. (These are the “vectormultiplet scalars.”) In addition there is an abelian gauge theory with gauge algebra $u(1)^{b_3/2}$, where $b_3$ is the Betti number of $X$. These vector fields arise from the self-dual 5-form of IIB supergravity in 10-dimensions and hence the theory is naturally presented without making a choice of electric/magnetic duality frame. The total electric-magnetic fieldstrength:

$$\mathcal{F} \in \Omega^2(M_4; \mathbb{R}) \otimes H^3(X; \mathbb{R})$$

satisfies a self-duality constraint.

$$\mathcal{F} = * \mathcal{F}$$

in ten dimensions. The constraint (3.7) can be usefully expressed in terms of the self-dual and anti-self-dual projections of the two-form on Lorentzian spacetime as:

$$\mathcal{F} = \mathcal{F}^- + \mathcal{F}^+ \quad \mathcal{F}^- \in \Omega^{2,-}(M_4; \mathbb{C}) \otimes \left( H^{3,0}(X) \oplus H^{1,2}(X) \right)$$

Here we have assumed $b_1(X) = 0$ for simplicity. Otherwise we need to decompose the cohomology of $X$ into its primitive parts.

While there are many other fields in the supergravity, for our purposes we need only worry about the fields described above together with the metric $g_{\mu\nu}$ on $M_4$. These three fields are governed by the action

$$I_{\text{boson}} = \int_{M_4} \sqrt{g} R^+ + \| \nabla t \|^2 + \frac{1}{8\pi} \text{Im}(\mathcal{F}^-, \mathcal{F}^-)_{H^3}$$

where we use the natural Weil-Peterson (a.k.a. Zamolodchikov) metric on $\tilde{\mathcal{M}}$ and $(\gamma_1, \gamma_2)_{H^3} = \int_X \gamma_1 \ast \gamma_2$. 

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3.3.2. Superselection sectors

Consider Hamiltonian quantization of the theory described in the previous section, say, on $\mathbb{R}^3 \times \text{time}$. There will be a Hilbert space of states decomposing into superselection sectors described by absolutely conserved charges. The charge group is $K^1(X)$, but for our purposes, we will focus on $H^3(X, \mathbb{Z})$. We will interpret the vector $\gamma \in H^3(X, \mathbb{Z})$ in the attractor equations as specifying a superselection sector. Semiclassically we put a boundary condition at spatial infinity on the electromagnetic flux:

$$\int_{S^2_\infty} F = \gamma \in H^3(X, \mathbb{Z})$$  \hspace{2cm} (3.10)

Thus, we split the Hilbert space into superselection sectors:

$$\mathcal{H} = \bigoplus \gamma \mathcal{H}_\gamma$$  \hspace{2cm} (3.11)

and interpret $\gamma$ as a vector of electric and magnetic charges for the $\frac{1}{2}b_3(X)$ $U(1)$ gauge fields.

The $\mathcal{N} = 2$ supersymmetry algebra acts on the Hilbert spaces $\mathcal{H}_\gamma$ and has a nonzero “central charge” in each of these sectors. That is, the algebra is realized as

$$\{Q_{\alpha i}, Q_{\beta j}\} = \delta_{ij} \gamma_\alpha^\mu P_\mu \quad \{Q_{\alpha i}, Q_{\beta j}\} = \epsilon_{\alpha\beta} \epsilon_{ij} Z$$  \hspace{2cm} (3.12)

where the central charge $Z$ depends on the value of the scalar fields $t(\infty)$ and the charge vector $\gamma$.

**Definition/Proposition:** For $\gamma \in H^3(X; \mathbb{Z}), t(\infty) \in \tilde{\mathcal{M}}$, the central charge is:

$$Z(t; \gamma) := e^{K/2} \int \Omega \wedge \gamma \quad e^{-K} := i \int_X \Omega \wedge \bar{\Omega} > 0$$  \hspace{2cm} (3.13)

This is a result of a direct computation when one expresses the supercharges $Q_{\alpha i}$ in terms of the fields and computes the relevant Poisson brackets. However, for the mathematical reader one can simply take it as a definition of $Z(t; \gamma)$. 

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3.3.3. Attractor points minimize BPS mass

Now we finally meet the attractor equations when we ask about properties of “BPS states.” Let us first explain this term. A simple consequence of the algebra (3.12) is that in the sector \( \mathcal{H}_\gamma \) the Hamiltonian is bounded below

\[
H \geq |Z(t; \gamma)|
\]  

(3.14)

**Definition:** A *BPS state* is a state \( \Psi \in \mathcal{H}_\gamma \) which saturates the bound (3.14).

BPS states have proven to be extremely useful in investigations of nonperturbative physics because the associated representations of the supersymmetry algebra have rigidity properties, and are hence unchanged, under variation of parameters such as coupling constants. Examples of BPS states in the present context are provided by D3 branes wrapped on calibrated 3-cycles in \( X \). The mirror of such states are associated with certain elements of the derived category of coherent sheaves on the mirror of \( X \).

Because of their importance we are interested in the behavior (and existence) of BPS states as a function of moduli. It is here that the attractor equations enter the picture. One useful diagnostic of the existence of such states is associated with the behavior of \( |Z(t; \gamma)|^2 \) as a function on \( \tilde{M} \). The first key result, due to [71][72][75][76] is

**Theorem** If \( |Z(t; \gamma)|^2 \) has a stationary point in \( t \in \tilde{M} \), i.e., \( d|Z(t; \gamma)|^2 = 0 \), then,

a.) If \( Z(t; \gamma) = 0 \), then \( \gamma \in H^{2,1} \oplus H^{1,2} \), \( t \in D_\gamma \in \text{Div}(\tilde{M}) \).

b.) If \( Z(t; \gamma) \neq 0 \), then \( \gamma \in H^{3,0} \oplus H^{0,3} \), \( t = t_* \) is an isolated minimum.

The proof is extremely simple, so let us include it here. Choose \( \Omega(s) \) to vary holomorphically with \( s \in \tilde{M} \) a local holomorphic parameter. Then, if \( \hat{\gamma} \) is Poincaré dual to \( \gamma \),

\[
\partial_s |Z(\gamma)|^2 = \int_{\hat{\gamma}} \left( \partial_s \Omega - \frac{\langle \partial_s \Omega, \bar{\Omega} \rangle}{\langle \Omega, \bar{\Omega} \rangle} \Omega \right) \cdot \frac{\int_{\hat{\gamma}} \bar{\Omega}}{i \int_{\hat{X}} \Omega \wedge \bar{\Omega}}
\]

(3.15)

Now, \( \gamma \) has a Hodge decomposition:

\[
\gamma = \gamma^{3,0} + \gamma^{2,1} + \gamma^{1,2} + \gamma^{0,3}
\]

(3.16)

Stationarity of \( |Z(t; \gamma)|^2 \) implies that \( Z = 0 \) or, \( Z \neq 0 \) and, using \( T^{1,0} \mathcal{M} \cong H^{2,1}(X_3) \), \( \gamma^{2,1} = 0 \). Since \( \gamma \) is real this in turn implies \( \gamma = \gamma^{3,0} + \gamma^{0,3} \).
In case (b) we have a local minimum. To see this we compute

\[ \partial_i \partial_j |Z|^2 = 0 \]
\[ \partial_i \partial_j \log[|Z(\gamma)|^2] = -\partial_i \partial_j \log[i \int \Omega \wedge \bar{\Omega}] = g_{ij} \] (3.17)

so the stationary point is a nondegenerate minimum if the Weil-Peterson metric is nonsingular. That is, if the attractor point is at a regular point in \( \tilde{M} \). (We call such a point a "regular attractor point.")

3.3.4. Attractive fixed points and Black Holes

Let us now consider the relation to black holes. Black holes are certain solutions to (super-)gravity with special causality properties implied by a horizon. The black holes we will consider are "extremal." They have a maximal amount of allowed charge for a given mass, and do not radiate. Semiclassically, they correspond to states in the Hilbert space \( \mathcal{H}_\gamma \) described in section 3.3.2. Semiclassically, we describe these states as field configurations satisfying the equations of motion of supergravity.

We are going to focus on static, spherically symmetric, black holes of charge \( \gamma \).\(^5\) Moreover, we will want to consider "supersymmetric black holes." These conditions force the ansatz for the fields:

\[
\begin{align*}
 ds^2 &= -e^{2U(r)} dt^2 + e^{-2U(r)} (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) \\
 \vec{E} &= e^{2U(r)} \frac{\dot{r}}{r^2} \otimes \text{Im} (\gamma^{2,1} + \gamma^{0,3}) \\
 \vec{B} &= \frac{\dot{r}}{r^2} \otimes \text{Re} (\gamma^{2,1} + \gamma^{0,3}) \\
 t^a &= t^a (r)
\end{align*}
\] (3.18)

Here we have chosen a time direction and \( E_i dx^i = F_{0i} dt dx^i \) while \( B_i dx^i = \frac{1}{2} F_{jk} dx^j dx^k \).

The adjective "supersymmetric black holes" means in this context that the supersymmetric variation of the fermionic fields vanishes. This imposes nontrivial differential equations on the bosonic fields. The supersymmetry variations have the schematic form:

\[
\begin{align*}
 \text{gravitino} \quad &\delta \psi \sim \nabla \epsilon + \Pi^{0,3} (\mathcal{F}^-) \cdot \epsilon \\
 \text{gaugino} \quad &\delta \lambda \sim \partial t \cdot \epsilon + \Pi^{2,1} (\mathcal{F}^-) \cdot \epsilon 
\end{align*}
\] (3.19)

\(^5\) The seemingly innocent restriction to spherical symmetry introduces important limitations, as described briefly in the next subsection.
where $\epsilon$ is a spinor for the supersymmetry variation, $\nabla$ is a spinor covariant derivative, $\phi$ is a Dirac operator, and $\Pi^{0,3}, \Pi^{2,1}$ are the corresponding projection operators to the indicated Hodge type.

Substitution of the ansatz \((3.18)\) into the equations $\delta \psi = \delta \lambda = 0$ yields a system of first order ordinary differential equations in the radial variable $r$. These equations can in turn be interpreted as defining a dynamical system on the Teichmuller space $\tilde{\mathcal{M}}$ as follows. Let $\rho := 1/r$, and define $\mu := e^{-U(r)}$. Then

$$\delta \psi = 0 \quad \rightarrow \quad \frac{d\mu}{d\rho} = \left| Z(t(r); \gamma) \right| \quad (3.20)$$

implies $\mu$ is monotonically increasing as $r \to 0$. We can therefore use it as a flow parameter. Now the equation

$$\delta \lambda = 0 \quad \rightarrow \quad \mu \frac{dt^a}{d\mu} = -g^{a\bar{b}} \partial_{\bar{b}} \log \left| Z \right|^2 \quad (3.21)$$

implies that we have gradient flow in $\tilde{\mathcal{M}}$ to the minimum of $|Z|^2$. The horizon of the black hole appears when there is a zero in the coefficient of $g_{00}$. This happens when $e^{2U(r)} \to 0$, hence at $\mu \to \infty$.

The attractor equations are the fixed point equations for the flow \((3.21)\)

$$t(r) \to t_*(\gamma) \quad \text{such that} \quad \gamma = \gamma^{3,0} + \gamma^{0,3} \quad (3.22)$$

this easily follows since

$$\dot{\gamma}^{2,1} = 0 \Rightarrow t(r) = t_*(\gamma) \quad (3.23)$$

At this fixed point

$$e^{-U_*} = 1 + Z_* / r \quad (3.24)$$

where $Z_* := Z(t_*(\gamma); \gamma)$, and hence the near horizon geometry is $AdS_2 \times S^2$:

$$ds^2 = -\frac{r^2}{Z_*^2} dt^2 + \frac{Z_*^2}{r^2} dr^2 + Z_*^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (3.25)$$

Note that the horizon area is

$$\frac{\text{Horizon Area}}{4\pi} = \left| Z(t_*(\gamma); \gamma) \right|^2 := Z_*^2 \quad (3.26)$$
3.3.5. Summary & Cautionary Remarks

In summary, at the horizon of a susy black hole, the complex structure moduli of the Calabi-Yau $X$ is fixed at an isolated point $t^*\gamma$ such that $\gamma = \gamma^{3,0} + \gamma^{0,3}$. This is also the point at which the mass of states in $H^{BPS}_\gamma$ is minimized.

A remarkable prediction of this picture, in the spirit of the Strominger-Vafa computation is that

$$\log \dim H^{BPS}_\gamma \sim \pi |Z(t^*\gamma;\gamma)|^2$$

(3.27)

for large charges $\gamma$. However, it is important to remark at this point that we have oversimplified things somewhat. In fact, the dynamical system can have several basins of attraction [65]. The multiple-basin phenomenon has been explored in some depth in the papers of F. Denef and collaborators [77,78,79,80]. In particular, Denef et. al.’s investigations have shown that when enumerating BPS states, and accounting for entropy it is quite important not to restrict attention to the spherically symmetric black holes. This leads to the fascinating subject of “split attractor flows,” which clarify considerably the existence of the multiple basins of attraction. Regrettably, all this is outside the scope of these lectures.

3.4. Attractor points for $X = K3 \times T^2$

Now that we have described the significance of the attractor equations for black holes and BPS states let us consider some examples of solutions to these equations. We will focus on the elegant example of the Calabi-Yau $K3 \times T^2$ and comment on other examples in section 3.6 below. Let us choose $a$ and $b$ cycles on $T^2$ so that we have an isomorphism

$$H^3(K3 \times T^2, \mathbb{Z}) \cong H^2(K3; \mathbb{Z}) \oplus H^2(K3; \mathbb{Z})$$

(3.28)

Using (3.28) can take $\gamma = p \oplus q$, with $p, q \in H^2(K3; \mathbb{Z})$: It is easy to solve the equations:

$$2\text{Im} \bar{C} \int_{a \times \gamma} dz \wedge \Omega^{2,0} = p^I$$

(3.29)

$$2\text{Im} \bar{C} \int_{b \times \gamma} dz \wedge \Omega^{2,0} = q_I$$

and the answer is

$$\Omega^{3,0} = dz \wedge (q - \bar{\tau}p)$$

(3.30)

Reference [67] attempts to make this statement a little more precise.
where $dz$ is a holomorphic differential on $T^2$. By the Torelli theorem, the complex structure of the $K3$ surface is determined by $\Omega^{2,0} = (q - \bar{\tau}p)$. Now, note that

$$\int_S \Omega^{0,2} \wedge \Omega^{0,2} = 0 \Rightarrow p^2\tau^2 - 2p \cdot q\tau + q^2 = 0 \Rightarrow (3.31)$$

$$\tau = \tau(p, q) := \frac{p \cdot q + \sqrt{D}}{p^2}$$

$$D = D_{p,q} := (p \cdot q)^2 - p^2q^2$$

Thus, we conclude that a regular attractor point exists for $D_{p,q} < 0$ and, for such charge vectors

$$\frac{A}{4\pi} = |Z_\ast|^2 = \sqrt{-D_{p,q}} = \sqrt{p^2q^2 - (p \cdot q)^2}$$

(3.34)

### 3.4.1. Attractive $K3$ Surfaces

Let us analyze the meaning of the above attractor points more closely. Let $S$ be a $K3$ surface. We may then define its Neron-Severi lattice $NS(S) := \ker\{\sigma \to \int_\sigma \Omega^{2,0}\}$. The rank of the lattice $NS(S)$ is often denoted $\rho(S)$. We define the transcendental lattice $T_S := (NS(S))^\perp$. The generic $K3$ surface is not algebraic and hence $NS(S) = \{0\}$. For the generic algebraic $K3$, $NS(S) = H\mathbb{Z}$, and $\rho(S) = 1$. For the generic elliptically fibered $K3$, $NS(S) = B\mathbb{Z} \oplus F\mathbb{Z}$, and hence $\rho(S) = 2$. For the attractor points, $NS(S) = \langle p, q \rangle^\perp \subset H^2(K3; \mathbb{Z})$ has rank $\rho(S) = 20$ and

$$H^{2,0} \oplus H^{0,2} = T_S \otimes \mathbb{C}$$

(3.35)

These $K$ surfaces are unfortunately called “singular $K3$ surfaces” in the literature, but they are definitely not singular. Sometimes they are called “exceptional $K3$ surfaces.” We will refer to them as “attractive $K3$ surfaces,” because they are rather attractive.

Rather amusingly, from (3.34) we see that the area of a unit cell in $T_S$ is precisely the horizon area $A/(4\pi)$ of the corresponding black hole!

### 3.4.2. Attractive $K3$ Surfaces & Quadratic Forms

There is a beautiful description of the set of attractive $K3$ surfaces in terms of binary quadratic forms. This is summarized by the theorem of Shioda and Inose [81]:

**Theorem** There is a 1-1 correspondence between attractive $K3$ surfaces $S$ and $PSL(2, \mathbb{Z})$ equivalence classes of positive even binary quadratic forms.
In one direction the theorem is easy. Given a surface $S$ we construct the quadratic form:

$$T_S = \langle t_1, t_2 \rangle_{\mathbb{Z}} \leftrightarrow \begin{pmatrix} t_1^2 & t_1 \cdot t_2 \\ t_1 \cdot t_2 & t_2^2 \end{pmatrix}$$

The converse is rather trickier. Given

$$Q = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \quad a, b, c \in \mathbb{Z}$$

we first consider the abelian variety $A_Q = E_{\tau_1} \times E_{\tau_2}$ where

$$\tau_1 = \frac{-b + \sqrt{D}}{2a} \quad \tau_2 = \frac{b + \sqrt{D}}{2} = -c/\tau_1$$

One’s first inclination is to construct the associated Kummer variety, which is the resolution of $A_Q/\mathbb{Z}_2$. Such $K3$ surfaces are indeed attractive $K3$ surfaces, but do not encompass all such surfaces. Shioda and Inose introduce a clever construction involving a pencil of elliptic curves with $E_8$ singularities to construct a branched double cover $Y_Q$ which is itself a $K3$ surface. It is these $Y_Q$ which account for all attractive $K3$ surfaces and are in 1-1 correspondence with the quadratic forms.

Thanks to the Shioda-Inose theorem it is now trivial to describe the attractor points

**Corollary.** Suppose that $\langle p, q \rangle \subset H^2(K3; \mathbb{Z})$ is a primitive sublattice. Then the attractor variety $X_{p, q}$ determined by $\gamma = (p, q)$ is

$$E_{\tau(p, q)} \times Y_{2Q_{p, q}}$$

where $\tau(p, q)$ is given by

$$\tau(p, q) = \frac{p \cdot q + i\sqrt{-D}}{p^2}$$

and $Y_{Q_{p, q}}$ is the Shioda-Inose $K3$ surface associated to the even quadratic form:

$$2Q_{p, q} := \begin{pmatrix} p^2 & -p \cdot q \\ -p \cdot q & q^2 \end{pmatrix}$$

The variety is a double-cover of a Kummer surface constructed from

$$X_{p, q} = Y_{2Q_{p, q}} \times E_{\tau} \rightarrow Km \left( E_{\tau(p, q)} \times E_{\tau'(p, q)} \right) \times E_{\tau(p, q)}$$

with

$$\tau'(p, q) = \frac{-p \cdot q + i\sqrt{-D}}{2}.$$
3.5. U-duality and horizon area

We have now described the attractor varieties. They are beautiful and have the interesting arithmetic property that all their periods are valued in quadratic imaginary fields. We will see in a moment that there is much more nontrivial arithmetic associated to them. However, we would like to know whether this rich arithmetic structure has any physical significance. In this section we attempt to make a connection to physics.

In string theory there are “duality groups.” These are arithmetic groups which map two different charges with “isomorphic physics.” It is thus a natural question to ask how $U$-duality acts on the attractor varieties. For IIB/K3 $\times T^2$ the $U$-duality group is

$$U = SL(2, \mathbb{Z}) \times O(22, 6; \mathbb{Z})$$  \hspace{1cm} (3.44)

The pair of (Electric,Magnetic) charges $(p,q)$, has $p,q \in II^{22,6}$ and forms a doublet under $SL(2, \mathbb{Z})$. In these lectures we are suppressing certain other fields in the supergravity, and hence we are restricting attention to $p,q \in H^2(K3, \mathbb{Z}) \cong II^{19,3} \subset II^{22,6}$, so the duality group should actually be considered to be $SL(2, \mathbb{Z}) \times O(19,3; \mathbb{Z})$.

Now, to a charge $\gamma = (p,q)$ we associate:

$$2Q_{p,q} := \begin{pmatrix} p^2 & -p \cdot q \\ -p \cdot q & q^2 \end{pmatrix}$$  \hspace{1cm} (3.45)

This is manifestly $T$-duality invariant while under $S$-duality

$$Q_{p,q} \rightarrow Q_{p',q'} = mQ_{p,q}m^{tr} \quad m \in SL(2, \mathbb{Z})$$  \hspace{1cm} (3.46)

Note that the near-horizon metric only depends on the discriminant:

$$\frac{A(\gamma)}{4\pi} = \sqrt{-D_{p,q}}$$  \hspace{1cm} (3.47)

Thus, $A(\gamma)$ is invariant under $U(\mathbb{Z})$. Still, it might be that $U$-duality-inequivalent charges $\gamma$ have the same $A(\gamma)$. Asking this question brings us to the topic of class numbers.

3.5.1. Class Numbers

The equivalence of integral binary quadratic forms:

$$m \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} m^{tr} = \begin{pmatrix} a' & b'/2 \\ b'/2 & c' \end{pmatrix} \quad m \in SL(2, \mathbb{Z})$$  \hspace{1cm} (3.48)
is one of the beautiful chapters of number theory. A major result of the efforts of Fermat, Euler, Lagrange, Legendre, and Gauss is a deep understanding of the nature of this equivalence. For a nice discussion of the subject see [64] or [82]. (Reference [55] contains further references.) Let us summarize a few facts here.

Assume, for simplicity, that the quadratic form is primitive, that is, that \( \gcd(a, b, c) = 1 \). There are a finite number of inequivalent classes under \( SL(2, \mathbb{Z}) \). The number of classes is the class number, denoted \( h(D) \), where

\[
D = b^2 - 4ac
\]

is the discriminant. We will be focussing on the case \( D < 0 \). It is a nontrivial fact that one can define the structure of an abelian group on the set of classes \( C(D) \). When \( D \) is a fundamental discriminant then the class group \( C(D) \) is isomorphic to the group of ideal classes of the quadratic imaginary field

\[
K_D := \mathbb{Q}[i\sqrt{|D|}] := \{a + ib\sqrt{|D|} : a, b \in \mathbb{Q}\}
\]

A “fundamental discriminant” is a \( D \) such that it is the field discriminant of a quadratic imaginary field. This turns out to mean that \( D = 1 \mod 4 \) and is squarefree, or, \( D = 0 \mod 4 \), \( D/4 \neq 1 \mod 4 \), and \( D/4 \) is squarefree.

A convenient device for what follows is to associate to a quadratic form

\[
Q = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}
\]

a point \( \tau \in \mathcal{H} \) via:

\[
ax^2 + bxy + cy^2 = a|x - \tau y|^2
\]

that is,

\[
\tau = \frac{-b + \sqrt{D}}{2a}
\]

then \( SL(2, \mathbb{Z}) \) transformations (3.48) act on \( \tau \) by fractional linear transformations, and hence the inequivalent classes may be labelled by points \( \tau_i \in \mathcal{F} \):

**Example:** \( D = -20 \):

\[
\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} x^2 + 5y^2 \quad \tau_1 = i\sqrt{5}
\]

\[
\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} 2x^2 + 2xy + 3y^2 \quad \tau_2 = \frac{-1 + i\sqrt{5}}{2}
\]

The class group is \( \mathbb{Z}_2 \), \( [\tau_1] \) is the identity element, so the class group has multiplication law:

\[
[\tau_2] \ast [\tau_2] = [\tau_1].
\]
3.5.2. U-Duality vs. Area (or Entropy)

It follows immediately from the previous section that there can be U-duality inequivalent BPS black holes with the same horizon area \( A \). More precisely, let \( \mathcal{B}H(D) \) denote the number of U-inequivalent BPS black holes with \( A = 4\pi \sqrt{-D} \). We would like to give a formula for this number. Then, if \( D \) is square-free the associated forms must be primitive and \( \mathcal{B}H(D) = h(D) \). More generally, since \( h(D) \) counts the primitive quadratic forms of discriminant \( D \) we have

\[
\mathcal{B}H(D) = \sum_{m} h(D/m^2) \tag{3.56}
\]

The sum is over \( m \) such that \( D/m^2 = 0 \mod 4 \).

Now, the number of classes grows with \(|D|\). More precisely, it follows from work of Landau, Siegel, and Brauer that \( \forall \epsilon > 0, \exists C(\epsilon) \) with \( h(D) > C(\epsilon)|D|^{1/2-\epsilon} \). Roughly speaking, we can say that at large entropy the number of U-duality inequivalent black holes with fixed area \( A \) grows like \( A \). The U-duality inequivalent black holes are certainly physically inequivalent, nevertheless, the area is a fundamental attribute and the set of black holes with area \( A \) forms a distinguished class of solutions. It is interesting to ask if there is some larger “symmetry” which unifies these. We will give a tentative positive answer to this question in section 3.5.6.

3.5.3. Complex Multiplication

The attractor varieties are closely related to another beautiful mathematical theory, the theory of complex multiplication, which goes back to the 19th century mathematicians Abel, Gauss, Eisenstein, Kronecker, and Weber and continues as an active subject of research to this day. An excellent pedagogical reference for this material is [82]. Further references can be found in [83].

7 The discussion that follows assumes that a primitive lattice \( T \) defined by \((a, b, c)\) has a unique embedding into \( \mathbb{H}^{19,3} \). Indeed, this was blithely asserted in [83], however further reflection shows that the statement is less than obvious. The Nikulin embedding theory characterizes the genus of the complementary lattice \( T^\perp \) in \( \mathbb{H}^{19,3} \), and the embedding is specified by the isomorphism class of the isomorphism of dual quotient groups \( T^*/T \to (T^\perp)^*/T^\perp \). If \( T^*/T \) is \( p \)-elementary then theorem 13, chapter 15 of [83] shows that the class of \( T^\perp \) is unique. When \( T^*/T \) is not \( p \)-elementary there are further subtleties associated with the spinor genus of \( T^\perp \). In addition, there can be distinct isomorphisms between the dual quotient groups. Clearly, this aspect of the counting of \( \mathcal{B}H(D) \) needs further thought.
To introduce complex multiplication let us consider the elliptic curve $E_\tau$. This is an abelian group and we can ask about its group of endomorphisms. Note that there is always a map $z \to nz$, for $n \in \mathbb{Z}$, because

$$n \cdot (\mathbb{Z} + \tau \mathbb{Z}) \subset \mathbb{Z} + \tau \mathbb{Z}.$$ (3.57)

So $\text{End}(E_\tau)$ always trivially contains a copy of $\mathbb{Z}$. However, for special values of $\tau$, namely those for which

$$a\tau^2 + b\tau + c = 0$$ (3.58)

for some integers $a, b, c \in \mathbb{Z}$ the lattice has an extra “symmetry”, that is, $\text{End}(E_\tau)$ is strictly larger than $\mathbb{Z}$, because

$$\omega \cdot (\mathbb{Z} + \tau \mathbb{Z}) \subset \mathbb{Z} + \tau \mathbb{Z} \quad \omega = \frac{D + \sqrt{D}}{2}$$ (3.59)

Here again $D = b^2 - 4ac$. We say that “$E_\tau$ has complex multiplication by $z \to \omega z$”

To see that $E_\tau$ has wonderful properties, we choose a Weierstrass model for $E_\tau$

$$y^2 = 4x^3 - c(x + 1) \quad c = \frac{27j}{j - (12)^3} \quad j \neq 0, 1728$$

$$y^2 = x^3 + 1 \quad j = 0$$

$$y^2 = x^3 + x \quad j = 1728$$ (3.60)

and consider next some remarkable aspects of the $j$-function.

3.5.4. Complex multiplication and special values of $j(\tau)$

The first main theorem of complex multiplication states

**Theorem** Suppose $\tau$ satisfies the quadratic equation $a\tau^2 + b\tau + c = 0$ with $\gcd(a, b, c) = 1$, and $D$ is a fundamental discriminant. Then,

i.) $j(\tau)$ is an algebraic integer of degree $h(D)$.

ii.) If $\tau_i$ correspond to the distinct ideal classes in $\mathcal{O}(K_D)$, the minimal polynomial of $j(\tau_i)$ is

$$p(x) = \prod_{k=1}^{h(D)} (x - j(\tau_k)) \in \mathbb{Z}[x]$$ (3.61)
Moreover: $\hat{K}_D := K_D(j(\tau_i))$ is Galois over $K_D$ and independent of $\tau_i$ (it is a “ring class field”).

Note that $\tau \rightarrow j(\tau)$ is a complicated transcendental function. Thus, the theorem of complex multiplication is truly remarkable.

**Examples:**

\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad j(i) = (12)^3 \quad p(x) = x - 1728
\]
\[
\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad j(i\sqrt{2}) = (20)^3 \quad p(x) = x - 8000
\]
\[
\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \quad j(i\sqrt{5}) = (50 + 26\sqrt{5})^3
\]
\[
\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \quad j\left(\frac{1+i\sqrt{5}}{2}\right) = (50 - 26\sqrt{5})^3
\]
\[
p(x) = x^2 - 1264000 \quad x - 681472000
\]

3.5.5. The Attractor Varieties are Arithmetic

For us, the main consequence of the first main theorem of complex multiplication is that the attractor varieties are *arithmetic varieties*. That is, they are defined by polynomial equations with algebraic numbers as coefficients.

Let us begin with the factor $E_\tau$ in the attractor variety. Here it follows from (3.60) and the above theorem that $E_\tau$ has a model defined over $\hat{K}_D = K_D(j(\tau_i))$.

Now, let us turn to the $K3$ surface factor. The Shioda-Inose construction begins with the abelian surface $E_{\tau_1} \times E_{\tau_2}$ defined by (3.38). Now, $j(\tau_i/c)$ is arithmetic and hence the abelian surface is arithmetic. Moreover, forming the Kummer surface and taking the branched cover can all be done algebraically, but involves the coordinates of the torsion points of $E_\tau$. Now we need the second theorem of complex multiplication:

**Theorem** Let $c = 27j/(j - 1728)$

\[
E_\tau = \{ z : z \sim z + \omega, z \sim z + \omega \tau \} \quad \cong \{ (x, y) : y^2 = 4x^3 - c(x + 1) \}
\]

The torsion points $(x, y)_{a,b,N}$ corresponding to $z = \frac{a+b\tau}{N}\omega$ are arithmetic and generate finite abelian extensions of $\hat{K}_D$. Moreover

\[
\hat{K}_{N,D} = K_D(j, x_{a,b,N})
\]
are “ring class fields.”

Thus, the Shioda-Inose surface is an arithmetic surface and we arrive at the important conclusion: The $K3 \times T^2$ attractor variety, $Y_{p,q} E_{p,q}$ is arithmetic, and is defined over a finite extension of $\hat{K}_D$. It would actually be useful to know more precisely which extensions the variety is defined over. This is an open problem (probably not too difficult).

3.5.6. Gal($\bar{\mathbb{Q}}/\mathbb{Q}$) action on the attractors

In the previous section we have seen that the attractor varieties are defined over finite extensions of $\hat{K}_D$. Therefore, Gal($\bar{\mathbb{Q}}/\mathbb{Q}$) acts on the complex structure moduli of attractors. What can we say about this orbit?

Here again we can use a result of “class field theory”: $\hat{K}_D$ is Galois over $K_D$, and $Gal(\hat{K}_D/K_D)$ is in fact isomorphic to the class group $C(D)$. Indeed, the isomorphism $[\tau] \rightarrow \sigma[\tau] \in Gal(\hat{K}_D/K_D)$ satisfies the beautiful property that

$$[\tau] \rightarrow \sigma[\tau] \in Gal(\hat{K}_D/K_D)$$ (3.65)

is defined by

$$j([\bar{\tau}_i] \ast [\tau_j]) = \sigma[\tau_i](j[\tau_j])$$ (3.66)

Example: Once again, let us examine our simple example of $D = -20$. Here $K_D = \mathbb{Q}(\sqrt{-5})$, and as we have seen

$$D = -20 \quad \hat{K}_{D=-20} = K_{-20}(\sqrt{5}) = \mathbb{Q}(\sqrt{-1}, \sqrt{-5})$$

$$\langle \sigma \rangle = Gal(\hat{K}_D/K_D) \cong \mathbb{Z}/2\mathbb{Z}$$ (3.67)

In this case, (3.65) is verified by:

$$(50 - 26\sqrt{5})^3 = j(\frac{1 + i\sqrt{5}}{2}) = j([\tau_2] \ast [\tau_1])$$

$$= \sigma[\tau_2](j([\tau_1])) = \sigma[\tau_2](j(i\sqrt{5})) = \sigma[\tau_2]((50 + 26\sqrt{5})^3)$$ (3.68)

Now, since Gal($\bar{\mathbb{Q}}/\mathbb{Q}$) permutes the different $j(\tau_i)$ invariants it extends the $U$-duality group and “unifies” the different attractor points at discriminant $D$. In this sense, it answers the question posed at the end of section 3.5.2. Because we have not been very precise about the field of definition of the attractor varieties we cannot be more precise about the full Galois orbit. This, again, is an interesting open problem.
3.5.7. But, the Galois group \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) is not a symmetry of the BPS mass spectrum.

The physical role (if any) of the Galois group action mentioned above remains to be clarified. We would like to stress one important point: The BPS mass spectrum at different attractor points related by the Galois group action are in general different, so the Galois action is not a symmetry in any ordinary sense.

A simple example of this is illustrated by the Calabi-Yau manifold \( X = (S \times E)/\mathbb{Z}_2 \), where \( S \) is the double cover of an Enriques surface. The BPS mass spectrum at an attractor point determined by \( p_0, q_0 \in II^{2,10} \) and turns out to be

\[
|Z(t_s(p_0, q_0); p, q)|^2 = \frac{1}{2|D_{p_0, q_0}|^{3/2}} |A - \tau(p_0, q_0)B|^2
\]  

(3.69)

\( A, B \) are integers depending on \( p, q, p_0, q_0 \). Thus the BPS mass spectrum at the attractor point for \( \gamma = p_0 \oplus q_0 \) is completely determined by the norms of ideals in the ideal class corresponding to \( Q_{p_0, q_0} \). At inequivalent \( \tau_i \) the spectra are in general different.

There have been other attempts at finding a physical role for the Galois group in the present context. Some attempts involve the action on locations of D-branes [65][84], and there are others [68][69]. In a lecture at this workshop A. Connes made a very interesting suggestion of a relation of our discussion to his work with J.-B. Bost on arithmetic spontaneous symmetry breaking [85]. In this view the Galois group is a symmetry, but the symmetry is broken.

3.6. Attractor Points for Other Calabi-Yau Varieties

Let us briefly survey a few known results about attractor points for other Calabi-Yau varieties.

3.6.1. \( T^6 \)

The story here is similar to the case of \( K3 \times T^2 \). For \( IIB/T^6 \) the \( U \)-duality group is \( E_7 \times \mathbb{Z} \) [86]. The charge lattice is a module for \( E_7 \times \mathbb{Z} \) of rank 56. The area of the black hole horizon is \( A/4\pi = \sqrt{-D(\gamma)} \), with \( D(\gamma) = -I_4(\gamma) \), where \( I_4(\gamma) \) is Cartan’s quartic invariant defining \( E_7 \subset Sp(56) \) [87].

If we choose \( \gamma \in H^3(T^6; \mathbb{Z}) \subset \mathbb{Z}^{56} \) (3.70)

then an explicit computation, described in [67], shows that the attractor variety \( \mathbb{C}^3/(\mathbb{Z}^3 + \tau \mathbb{Z}^3) \) is isogenous to \( E_{\tau_0} \times E_{\tau_0} \times E_{\tau_0} \), where \( \tau_0 = i\sqrt{I_4(\gamma)} \), and is therefore defined over a finite extension of \( \mathbb{Q}[i\sqrt{I_4}] \).
3.6.2. Other Exact CY Attractors

Some examples of other exactly known attractors are
1. Orbifolds of $T^6$ and of $K3 \times T^2$.
2. The mirror of the Fermat point $2x_0^3 + x_1^6 + x_2^6 + x_3^6 + x_4^6 = 0$.
3. Consider the Calabi-Yau subvariety in $P^{1,1,2,2}[8]$ defined by

$$x_1^8 + x_2^8 + x_3^4 + x_4^4 + x_5^4 - 8\psi x_1 x_2 x_3 x_4 x_5 - 2\phi x_1^4 x_2^4 = 0$$ (3.71)

From the formulae of Candelas et. al., in ref. [88] we can find exact attractors for $\psi = 0$, via the change of variables:

$$\tilde{\phi}^{-2} = \frac{16z(1-z)}{(1+4z-4z^2)^2}, \quad z = -\frac{\psi^4(\tau)}{\bar{\psi}^4(\tau)}$$ (3.72)

The attractor points correspond to $\tau = a + bi \in \mathbb{Q}[i], -1 < a < 1, b > 0$. In fact, the last two examples are $K3 \times T^2$ orbifolds, as was pointed out to me by E. Diaconescu and B. Florea.

4. Any rigid Calabi-Yau manifold is automatically an attractor variety. We will return to this in remark 5 in the next subsection.

3.6.3. Attractor Conjectures & Remarks

We will now state some conjectures. It is useful to draw the following distinction between attractor points. The attractor equation says that there is an integral vector

$$\gamma \in H^{3,0} \oplus H^{0,3}$$ (3.73)

It can happen that there is a rank 2 submodule $T_X \subset H^3(X; \mathbb{Z})$ with

$$H^{3,0} \oplus H^{0,3} = T_X \otimes \mathbb{C}$$ (3.74)

We call such a point an “attractor of rank 2.” It is simultaneously an attractor point for two charges $\gamma_1, \gamma_2$ with $\langle \gamma_1, \gamma_2 \rangle \neq 0$. If it is not of rank two we call it an “attractor of rank 1.”

Based on the above examples one may jump to a rather optimistic conjecture which we call the Strong Attractor Conjecture: Suppose $\gamma$ determines an attractor point $t_*(\gamma) \in \tilde{M}$. Then the flat coordinates of special geometry are valued in a number field $K_{\gamma}$, and $X_{\gamma}$. 

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is an arithmetic variety over some finite extension of $K_\gamma$. A more modest conjecture, the *Weak Attractor Conjecture* only asserts this for rank 2 attractor points.

Unfortunately, there has been very little progress on these conjectures since they were suggested in [65, 66]. Some salient points are the following:

1. All known exact attractor points are of rank two. Moreover, the evidence is also consistent with the conjecture that all rank 2 attractors are orbifolds of $T^6$ and $K3 \times T^2$. Since rigid Calabi-Yau manifolds are necessarily rank 2 attractors, this suggestion can perhaps be falsified by the interesting examples mentioned in [89].

2. In the course of some discussions with E. Diaconescu and M. Nori, Nori was able to demonstrate that the Hodge conjecture implies that rank 2 attractors are indeed arithmetic. (Thus, one way to falsify the Hodge conjecture is to produce an example of a nonarithmetic rank two attractor.)

3. Attractor points of rank one are expected to be dense. The density can be proved in the limit of large complex structure [65]. On the other hand, attractor points of rank two are expected to be rare. Indeed, this issue can be addressed in a quantitative way using computers. Sadly, a search of some 50,000 attractor points in the moduli space of the mirror of the quintic, performed by F. Denef, revealed no convincing candidates for rank two attractors.

4. On the positive side, we can say that should the attractor conjectures turn out to be true they might imply remarkable identities on trilogarithms and generalized hypergeometric functions. For an explanation of this, see section 9.3 of [65].

5. Finally, we would like to note that there is a notion of “modular Calabi-Yau variety” generalizing the notion of modular elliptic curve. The modular K3-surfaces over $\mathbb{Q}$ turn out to be attractor varieties. For a discussion of this see [89]. The known examples in some unpublished work, R. Bell has checked that some of these examples are indeed arithmetic.

9 Briefly, Denef’s method is the following. Given a complex structure, $Re(\Omega)$ and $Im(\Omega)$ determine a real two-dimensional vector space $V \subset H^4(X, \mathbb{R})$. Given a charge $Q$, Denef computes the attractor point numerically to high precision. Now, $Q$ is an integral vector in $V$. Denef then constructs an orthogonal vector $P$ in $V$ using a Euclidean metric on $H^4(X, \mathbb{Z})$. If the components of $P$ are rational then the complex structure point is a rank 2 attractor. Using the numerical value of the periods he examines the components of $P$ and searches for rational $P$’s using a continued fraction algorithm. (Thus, long continued fractions are considered irrational.) His computer then scans through a list of charges $Q$. 

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of modular Calabi-Yau varieties are rigid, and hence, automatically, are attractors. It would be quite fascinating, to put it mildly, if a relationship between attractors and modular Calabi-Yau varieties persisted in dimension 3.

3.7. Second avatar: RCFT and F-Theory

A second, very different, way attractive K3 surfaces are distinguished in physics is in the context of F-theory. We will now indicate how it is that the compactification of the heterotic string to 8 dimensions on rational conformal field theories (RCFT’s) are dual to the F-theory compactifications on attractive K3-surfaces.

Recall the basic elements of F-theory/Heterotic duality: \( \text{The heterotic string on a torus } T^2 \text{ is dual to a IIB F-theory compactification on a K3 surface } S \). If we fix a hyperbolic plane: \( \langle e, e^* \rangle \subset H^2(S; \mathbb{Z}) \), then \( \langle e, e^* \rangle^\perp \cong II^{2,18} \), and this lattice is identified with the charge lattice in the Narain compactification of F-theory. The moduli space \( \text{Gr}_+(2,II^{2,18} \otimes \mathbb{R}) \) is interpreted in two ways. In IIB theory it is the space of positive definite planes \( \Pi \subset II^{2,18} \otimes \mathbb{R} \), spanned by \( \text{Re}(\Omega) \) and \( \text{Im}(\Omega) \), which defines the complex structure of an elliptically fibered polarized K3-surface. In the heterotic theory is it the moduli space of Narain compactifications.

3.7.1. RCFT’s for the heterotic string

In the heterotic theory, the condition that the right-moving lattice is generated over \( \mathbb{Q} \) (which corresponds to the K3 surface \( S \) being attractive) turns out to be equivalent to the condition that the compactification on \( T^2 \) is along a rational conformal field theory. One can go further, as shown in [65, section 10.3]. Choosing compactifications of the heterotic string to 9 and 10 dimensions is equivalent to choosing a realization of the lattice

\[
\langle w_1, w_1^* \rangle \oplus \langle w_2, w_2^* \rangle \oplus (E_8(-1))^2 \cong II^{2,18}
\] (3.75)

where \( \langle w_i, w_i^* \rangle \) are hyperbolic planes. Using this decomposition the moduli space can be realized as a tube domain in 18-dimensional complex Lorentzian space:

\[
\text{Gr}_+(2,II^{2,18} \otimes \mathbb{R}) \cong \mathbb{R}^{1,17} + iC_+ = \{y = (T,U,\vec{A})\}
\] (3.76)

\[^{10}\text{For more details see [R1, R2].}\]
where $C_+$ is the forward lightcone in $\mathbb{R}^{1,17}$, $U$ is the complex structure of $T^2$, $T$ is the Kahler structure, and $\vec{A}$ encode the holonomy of flat $E_8 \times E_8$ gauge fields. Under the isomorphism (3.76) we identify

$$\Omega = y + w_1 - \frac{1}{2} y^2 w^*_1$$

(3.77)

The conditions for a rational conformal field theory imply that the heterotic theory is compactified on an elliptic curve of CM type with $(T, \vec{A})$ in the quadratic imaginary field defined by $U$. Indeed, the curve has complex multiplication by a rational integral multiple of $\bar{T}$.

There are further interesting relations under this duality, including relations between the Mordell-Weil group of the attractive elliptic K3 surface and the enhanced chiral algebra of the heterotic RCFT. This essentially follows from the fact that the projection of $p \in \Pi^{2,18}$ onto the positive definite space:

$$p_R = e^{K/2} \int_p \Omega^{2,0}$$

(3.78)

in $F$-theory corresponds to “right-moving momentum” in Narain compactification.

The above duality realizes in part an old dream of Friedan & Shenker. Their idea was to approximate superconformal field theories on Calabi-Yau manifolds by rational conformal field theories. Generalizations of the relation between complex multiplication and rational conformal field theories on tori have been studied by K. Wendland in [93][94]. A rather different relation between rational conformal field theories and complex multiplication has been suggested by S. Gukov and C. Vafa [84]. These last authors conjecture that the superconformal field theory with target space given by a K3 surface with complex multiplication will itself be rational.

Finally, we would like to mention the very elegant result of S. Hosono, B. Lian, K. Oguiso, and S.-T. Yau in [95], which may be phrased, roughly, as follows. Consider the map from moduli $(T, U, \vec{A} = 0)$ to the quadratic form characterizing the attractor point. The moduli $T, U$ are valued in $\mathbb{Q}(\sqrt{D})$ and may therefore also be associated to quadratic forms. Reference [95] shows that the three quadratic forms are related by the Gauss product, and uses this to give a classification of $c = 2$ toroidal RCFT’s.

Here is an (over)simplified version of the discussion in [95]. When $\vec{A} = 0$ we have

$$\Omega = w_1 - TU w^*_1 + Tw_2 + U w^*_2$$

(3.79)
A basis (over \( \mathbb{R} \)) for the plane \( \Pi \) is given by
\[
\nu_1 = w_1 + U\bar{T}-\bar{U}T w_1^* + \frac{UT-\bar{U}T}{U-U} w_2
\]
\[
\nu_2 = \frac{T\bar{U}-TU}{U-U} w_1^* + \frac{T-T}{U-U} w_2 + w_2^*
\]
while the orthogonal plane \( \Pi^\perp \) in \( H^{2,2} \otimes \mathbb{R} \) is spanned (over \( \mathbb{R} \)) by
\[
\gamma_1 = w_1 - U\bar{T}-\bar{U}T w_1^* - \frac{T\bar{U}-TU}{U-U} w_2
\]
\[
\gamma_2 = -\frac{U\bar{T}-\bar{U}T}{U-U} w_1^* - \frac{T-T}{U-U} w_2 + w_2^*
\]
Note that these are rational vectors iff \( U,T \in \mathbb{Q}[\sqrt{D}] \). In the latter case, by \( SL(2,\mathbb{Z}) \) transformations we can bring them to the “concordant” form
\[
U = \frac{b+\sqrt{D}}{2a}
\]
\[
T = \frac{b+\sqrt{D}}{2a'} = a'U
\]
in which case the basis vectors simplify to
\[
\nu_1 = w_1 + \frac{c}{a'} w_1^*
\]
\[
\nu_2 = -\frac{b}{a'} w_1^* + \frac{a}{a'} w_2 + w_2^*
\]
\[
\gamma_1 = w_1 - \frac{c}{a'} w_1^* + \frac{b}{a'} w_2
\]
\[
\gamma_2 = -\frac{a}{a'} w_2 + w_2^*
\]
A straightforward computation shows that
\[
(\nu_i \cdot \nu_j) = \frac{1}{a'} \begin{pmatrix} 2c & -b \\ -b & 2a \end{pmatrix}
\]
\[
(\gamma_i \cdot \gamma_j) = -\frac{1}{a'} \begin{pmatrix} 2c & -b \\ -b & 2a \end{pmatrix}
\]
If \( T,U \) are associated with quadratic forms \( (a,b,c) \) and \( (a',b,c') \) then \( t_1 = a'\nu_1, t_2 = a'\nu_2 \) is an integral basis for \( \Pi \), and from (3.84) we see that the quadratic form of this basis is the Gauss product of the quadratic forms associated to \( T,U \).

\[\text{For concordant quadratic forms we further require } a|c, \text{ but we do not use this condition in our discussion in sec. 3.8 below.}\]
3.7.2. Arithmetic properties of the K3 mirror map

The above relation of heterotic RCFT and attractive K3 surfaces raises interesting questions about the arithmetic properties of mirror maps. Recall that the \( j \) function itself can be viewed as a mirror map for 1-dimensional Calabi-Yau manifolds. It is natural to ask if the mirror maps of higher dimensional Calabi-Yau manifolds have arithmetical significance, perhaps playing the role of the transcendental functions sought for in Hilbert’s 12th problem.

The next case to look at is 2-dimensions. In [96] Lian and Yau studied the mirror map for pencils of K3 surfaces and found, remarkably, the occurrence of Thompson series. Hence the mirror map again has arithmetical properties. The perspective on F-theory we have discussed suggests a generalization. We may think of F-theory compactifications in terms of a Weierstrass model:

\[
ZY^2 = 4X^3 - f_8(s,t)XZ^2 - f_{12}(s,t)Z^3
\]

\[
f_8(s,t) = \alpha_{-4}s^8 + \cdots + \alpha_{+4}t^8
\]

\[
f_{12}(s,t) = \beta_{-6}s^{12} + \cdots + \beta_{+6}t^{12}
\]

In this description the moduli space is:

\[
\mathcal{M}_{\text{algebraic}} = \left\{ (\alpha, \beta) \right\} - \mathcal{D} \right/ \text{GL}(2, \mathbb{C})
\]

where \( \mathcal{D} \) is the discriminant variety and the action of \( \text{GL}(2, \mathbb{C}) \) is induced by the action on \( s, t \). The map \( \Phi_F: y \to (\alpha, \beta) \), is a map from flat coordinates to algebraic coordinates and in this sense it can be thought of as the mirror map. From the Shioda-Inose theorem and the theory of complex multiplication it is therefore natural to conjecture that \( \text{The map } \Phi_F \text{ behaves analogously to the elliptic functions in the theory of complex multiplication, i.e., } y^i \in K_D \to \alpha_i, \beta_i \in \hat{K} \text{ for some algebraic number field } \hat{K} \).

In [65] some nontrivial checks on this conjecture were performed. The most comprehensive check is to consider the map \( \Phi_F \) in the limit of stable degenerations (\( T \to \infty \) in terms of the variables defined in (3.76).) In that case, one may use the results of Friedman, Morgan, and Witten [97] [98] to verify the statement.
3.8. Third avatar: Flux compactifications

There is a third manifestation of the attractor varieties. It is related to a topic of current interest in string compactification, namely, compactification with fluxes. The literature on this subject is somewhat vast. See, for examples, [99] [100] [101] for some recent papers with many references to other literature. It turns out that this subject is closely related to the attractor problem for Calabi-Yau four-folds.

We begin by considering compactification of type IIB string theory on a Calabi-Yau manifold $X_3$, now adding “fluxes” instead of wrapped branes, as we have been discussing thus far. In particular, if one considers the RR and NSNS 3-forms $F$ and $H$, then they must be closed, by the Bianchi identity, and they must satisfy a quantization condition on their cohomology classes: $[F], [H] \in H^3(X_3, \mathbb{Z})$. In backgrounds with such fluxes the low energy supergravity develops a superpotential [102], and analysis of this superpotential shows that the supersymmetric minima with zero cosmological constant are characterized by complex structure and complex dilaton such that

$$G_{IIB} := [F] - \phi[H] \in H^{2,1}_{\text{primitive}}$$

for integral vectors $F, H$, where $\phi$ is the axiodil (a.k.a. complex dilaton). (This can also be shown by studying supersymmetry transformations [103] or by using the result of [104] applied to M-theory on $X \times T^2$.) Fluxes with

$$G_{IIB} := [F] - \phi[H] \in H^{2,1}_{\text{primitive}} \oplus H^{0,3}$$

can also in principle be used to obtain supersymmetric AdS compactifications with negative cosmological constant.

Equation (3.88) is usually regarded as an equation on the complex structure of $X_3$ and the complex dilaton $\phi$. For some classes of flux vectors $F$ and $H$ the solutions are isolated points in moduli space. [13] Thus, (3.89) is reminiscent of the attractor equations (as noted in [105] [106]). However, despite its similarity to the attractor equations, the

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12 In our discussion we are suppressing some important physical points. Foremost amongst these is the fact that we need to consider an orientifold of the compactification described above in order to have $d = 4, \mathcal{N} = 1$ supersymmetry. The examples below can be orientifolded.

13 There are also fluxes for which there are no solutions, and fluxes for which there are continuous families of solutions. A general class of examples of the latter type arise by embedding $X$ in some ambient variety $\iota : X \hookrightarrow W$ and choosing $F$ and $H$ to be classes pulled back from $W$. 

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condition (3.88) is in fact a very different kind of constraint on the Hodge structure of the Calabi-Yau manifold, since the left-hand side of (3.88) is complex and nonintegral.

Despite these distinctions the flux compactification problem is in fact related to the attractor problem, but for Calabi-Yau four-folds \( X_4 \). Consider a Calabi-Yau 4-fold with \( \gamma \in H^4(X_4, \mathbb{Z}) \). In analogy to section 3.3.3 above we seek to stationarize the normalized period:

\[
|Z(\gamma)|^2 = \frac{\gamma \cdot \Omega}{\Omega \cdot \bar{\Omega}}.
\]

By exactly the same argument as in section 3.3.3 a stationary point is either a divisor where \( Z(\gamma) = 0 \) or, if \( Z(\gamma) \neq 0 \), a point where \( \gamma^{1,3} = \gamma^{3,1} = 0 \). An important distinction from the 3-fold case is that the Hessian at a critical point is not necessarily positive definite: The first line of (3.17) can be nonzero since \( \gamma \) can have a (2, 2) component which overlaps with the second derivatives of \( \Omega \).

In the physical interpretation of the 4-fold attractor problem we may identify \( \gamma = [G] \) as the cohomology class of the \( G \)-flux of \( M \)-theory. These compactifications can be related to those defined by (3.88) in the case where \( X_4 \) is elliptically fibered, for then we may consider an associated \( F \)-theory compactification. In general, this requires the insertion of 7-branes in the base of the fibration, but when these coincide we can obtain the orientifold compactifications discussed above \([107]\). To specialize further, suppose \( X_4 = X_3 \times T^2 \). Then \( G = H d\sigma^1 + F d\sigma^2 \), with complex structure \( dz = d\sigma_1 + \phi d\sigma_2 \) on \( T^2 \). Then

\[
G = \frac{1}{\phi - \bar{\phi}} ((F - \bar{\phi} H) dz - (F - \bar{\phi} H) d\bar{z}) = \frac{1}{\phi - \bar{\phi}} (G_{IIB}^* dz - G_{IIB} d\bar{z})
\]

so, in particular:

\[
G^{1,3} = \frac{1}{\phi - \bar{\phi}} ((F - \bar{\phi} H)^{0,3} dz - (F - \bar{\phi} H)^{1,2} d\bar{z})
\]

\[
G^{0,4} = -\frac{1}{\phi - \bar{\phi}} (F - \phi H)^{0,3} d\bar{z}
\]

and hence stationary points with \( G^{1,3} = G^{0,4} = 0 \) correspond to supersymmetric Minkowskian compactifications while those with \( G^{0,4} \neq 0 \) are related to more general \( \text{AdS} \) compactifications.

What can we say about exact solutions to the flux compactification problem? One remark is that any attractor point of rank 2 automatically gives a solution to (3.89), for some fluxes. After all, we can choose \([F], [H]\) in the lattice \( T_{X_3} \) in (3.74) and then choose
\( \phi \) so that \([F - \phi H] \in H^{0,3}(X_3)\). Thus, all our rank two attractor examples can be re-interpreted as flux compactifications. For example, using (3.31) (3.32) (3.33) we could take (an orientifold of) \( X_3 = K3 \times T^2 \) and
\[
F = p^2 dx \wedge q + 2p \cdot q dy \wedge q - q^2 dy \wedge p
\]
\[
H = dy \wedge q + dx \wedge p
\]
with \( \phi = p^2 \tau \). Similarly, the example (3.71) (3.72) above provides a simple exact infinite family with \( \phi = i \). For any rational numbers \( a, b, -1 < a < 1, b > 0 \) we have, from section 8.3.2 of [65],
\[
\Omega_{a,b} := \gamma_1 + i \gamma_2
\]
\[
\gamma_1 = 2\alpha_0 - \alpha_1 + (a + 1)\alpha_2 - (a + b - 2)\beta_0 - 2(b + 1)\beta_1 - 4\beta^2
\]
\[
\gamma_2 = \alpha_1 + (b - 1)\alpha_2 - (b - a)\beta_0 - 2(1 - a)\beta_1
\]
Here \( \alpha^i, \beta_i \) is an integral symplectic basis. Thus, suitable integral multiples of \( \gamma_i \) will produce examples. For another recent discussion of exact examples see [108].

In a recent paper, Tripathy and Trivedi analyzed the conditions (3.88) for the case when the Calabi-Yau is \( T^6 \) or \( K3 \times T^2 \) [109]. Their discussion can be interpreted as follows: when the fluxes are such that the solutions admit isolated supersymmetric vacua in complex structure moduli space, those vacua turn out to be precisely attractor points!

With the benefit of hindsight we can easily describe all the solutions in [109] in terms of attractor points on \( S \times T^2 \) with \( S \) a \( K3 \) surface. Choosing a basis \( dx, dy \) of 1-forms on \( T^2 \) we decompose \( F = \alpha_x dx + \alpha_y dy, H = \beta_x dx + \beta_y dy \), where \( \alpha_x, \alpha_y, \beta_x, \beta_y \in \Lambda \cong \mathbb{H}^{3,19} \). The condition (3.88) in this case can be equivalently written in terms of the projection of these vectors into the plane
\[
\Pi = \langle Re\Omega, Im\Omega \rangle \subset \Lambda \otimes \mathbb{R}
\]
and its orthogonal complement \( \Pi^\perp \subset \Lambda \otimes \mathbb{R} \). The condition (3.88) is equivalent to the following six equations for the projection of the vectors into \( \Pi \) and \( \Pi^\perp \):
\[
\beta^\Pi_x = \frac{1}{(\tau - \bar{\tau})(\phi - \bar{\phi})} (\xi \Omega + \bar{\xi} \bar{\Omega})
\]
\[
\alpha^\Pi_x = \frac{1}{(\tau - \bar{\tau})(\phi - \bar{\phi})} (\bar{\phi} \xi \Omega + \phi \bar{\xi} \bar{\Omega})
\]
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\[ \beta_y^\Pi = \frac{1}{(\tau - \bar{\tau})(\phi - \bar{\phi})}(\xi \bar{\tau} \Omega + \xi \tau \bar{\Omega}) \] (3.98)

\[ \alpha_y^\Pi = \frac{1}{(\tau - \bar{\tau})(\phi - \bar{\phi})}(\bar{\phi} \xi \bar{\tau} \Omega + \phi \xi \tau \bar{\Omega}) \] (3.99)

\[ \alpha_x^\perp = \frac{(\bar{\phi} \bar{\tau} - \phi \tau)}{(\phi - \bar{\phi})} \alpha_x^\perp + \frac{\bar{\phi} \phi (\tau - \bar{\tau})}{(\phi - \bar{\phi})} \beta_x^\perp \] (3.100)

\[ \beta_y^\perp = -\frac{(\tau - \bar{\tau})}{(\phi - \bar{\phi})} \alpha_x^\perp + \frac{(\phi \tau - \bar{\phi} \bar{\tau})}{(\phi - \bar{\phi})} \beta_x^\perp \] (3.101)

Here \( \xi \) is a complex number, and \( \tau \) is the period of \( T^2 \). Note that \( \alpha_x^\perp, \beta_x^\perp \) are unconstrained, except that the class \( G \) is primitive iff \( \alpha_x^\perp, \beta_x^\perp \) are orthogonal to the Kahler class \( J \). We will assume the class \( J \) is rational and hence the \( K3 \) surface is algebraic.

When expressed this way it is manifest that for any attractor point there is an infinite set of flux vectors associated to that point. For, if \( Y_Q \) is an attractive \( K3 \) surface associated to \( (a, b, c) \) then \( \Pi \) is rationally generated. Indeed, we may take \( \Omega = t_2 - \omega t_1 \) where \( t_1, t_2 \) is an oriented basis for \( \Pi \) and \( \omega = (b + \sqrt{D})/2a \). If \( \tau, \phi, \xi \in \mathbb{Q}(\sqrt{D}) \), then all the vectors in (3.96),(3.97),(3.98),(3.99),(3.100),(3.101) are rational. The condition that \( \alpha_x^\Pi + \alpha_x^\perp \), etc. lie in \( \Lambda \) reduces to simple Diophantine conditions on \( \xi, \alpha_x^\perp, \beta_x^\perp \) with infinitely many solutions. A similar set of equations can be used to give the general solution to (3.89). In these more general solutions \( \phi, \tau \) and the attractor points can be associated with two distinct quadratic fields.

An even more explicit family of flux vacua can be obtained by combining the 4-fold viewpoint with the formulae (3.80) - (3.83) above. This family can be applied to the 4-folds of the type \( S \times \tilde{S} \) where the surfaces \( S, \tilde{S} \) can be taken to be \( T^4 \) or \( K3 \). Denote by \( T, U \) the moduli for the first factor, and by \( \tilde{T}, \tilde{U} \) the moduli of the second factor. Similarly, a \( \tilde{\ } \) denotes a quantity associated with the second factor. Choose \( 2 \times 2 \) real matrices \( X, Y \) and write

\[ G = (\nu_1 \nu_2) X \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} + (\gamma_1 \gamma_2) Y \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \] (3.102)

This is automatically of type

\[ ((0, 2) + (2, 0)) \otimes ((0, 2) + (2, 0)) + (2, 2) = (4, 0) + (2, 2) + (0, 4). \]
Now, we require $G$ to be an integral vector. Define a $4 \times 4$ matrix of integers so that

$$G = (w_1 \ w_1^* \ w_2 \ w_2^*) N \begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_1^* \\ \tilde{w}_2 \\ \tilde{w}_2^* \end{pmatrix}$$

$$= N_{11} w_1 \otimes \tilde{w}_1 + N_{12} w_1 \otimes \tilde{w}_1^* + N_{13} w_1 \otimes \tilde{w}_2 + N_{14} w_1 \otimes \tilde{w}_2^* + \cdots + N_{44} w_2^* \otimes \tilde{w}_2^* \quad (3.103)$$

Now we have

$$N = M^{tr}(a, a', b, c) \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} M(\tilde{a}, \tilde{a}', \tilde{b}, \tilde{c}) \quad (3.104)$$

where it is useful to define the matrix

$$M(a, a', b, c) = \begin{pmatrix} 1 & c/a' & 0 & 0 \\ 0 & -b/a' & a/a' & 1 \\ 1 & -c/a' & b/a' & 0 \\ 0 & 0 & -a/a' & 1 \end{pmatrix} \quad (3.105)$$

so that

$$\begin{pmatrix} \nu_1 \\ \nu_2 \\ \gamma_1 \\ \gamma_2 \end{pmatrix} = M(a, a', b, c) \begin{pmatrix} w_1 \\ w_1^* \\ w_2 \\ w_2^* \end{pmatrix} \quad (3.106)$$

Now we see that for any pair of attractor points in the complex structure moduli space of $S \times \tilde{S}$, there are infinitely many flux vacua leading to those specified points. To prove this let us choose $T, U \in Q(\sqrt{D})$ and $\tilde{T}, \tilde{U} \in Q(\sqrt{\tilde{D}})$ to be concordant. Then if $X, Y$ are integer matrices divisible by $a'\tilde{a}'$ the resulting matrix $N$ is a matrix of integers. But, by construction, it leads to the specified flux vacuum. For special values of $T, U, \tilde{T}, \tilde{U}$ in fact the vacuum is not an isolated point. However, we expect that for generic $T, U \in Q[\sqrt{D}]$ and $\tilde{T}, \tilde{U} \in Q[\sqrt{\tilde{D}}]$ the vacuum will be isolated. (We did not prove this rigorously.)

It should be stressed that there is no reason in the above construction for the fields $Q[\sqrt{D}]$ and $Q[\sqrt{\tilde{D}}]$ to coincide. As we have mentioned, by including further quantum corrections to the flux potential one can associate an AdS vacuum to stationary points of (3.90) with $G^{4,0} \neq 0$. Moreover the scale of the cosmological constant is, roughly speaking, governed by the value of the normalized period (3.90) with $\gamma = G$. An easy computation shows that for the special vacua under consideration

$$|Z(G)|^2 = \frac{\tilde{a} a}{a'\tilde{a}'} |(x_{11} U - x_{21})\tilde{U} - (x_{12} U - x_{22})|^2 \quad (3.107)$$

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where $x_{ij}$ are the matrix elements of $X$ in (3.102). From this one learn that if one further imposes the condition that $G^{4,0} = 0$ then, for generic $X$, one finds that $U, \bar{U}$ must be in the same quadratic field. Moreover, if $U, \bar{U}$ do not generate the same field then the distribution of values of $|Z(G)|^2$, as $G$ runs over the different fluxes, is dense in $\mathbb{R}$.

In physics, there is another constraint on the fluxes which severely cuts down the above plethora of supersymmetric vacua. In the M-theoretic version the net electric charge for the $C$-field must vanish on a compact space and therefore

$$\int_{X_4} \frac{1}{2} G^2 - \frac{\chi(X_4)}{24} + N_2 = 0 \quad (3.108)$$

where $N_2$ is the number of membranes, and, for supersymmetric vacua, is nonnegative. Thus $[G] \cdot [G]$ is bounded. Equivalently, in the IIB setup the Bianchi identity on the 5-form flux leads to a bound on

$$N_f = \int_{X_3} F \wedge H = \frac{1}{\phi - \bar{\phi}} \int G_{IIB} \wedge G^*_{IIB} \quad (3.109)$$

As pointed out in [101] this leads to an important finiteness property: The number of flux vectors leading to vacua in a compact region of moduli space is finite. Following [101] let us prove this for the more general 4-fold problem. Let $\mathcal{K} \subset \mathcal{M}_{\text{cplx}}(X_4)$ be a compact region in the moduli space of complex structures of the elliptically fibered $X_4$. Consider $\mathcal{K} \times H^4(X_4; \mathbb{R})$. The subbundle of real vectors of type $H^4 \oplus H^2_{\text{primitive}} \oplus H^{0,4}$ has a positive intersection product. With respect to a fixed basis on $H^4(X, \mathbb{R})$ the quadratic form is smoothly varying. Therefore, the set of real vectors satisfying $\frac{1}{2} G^2 \leq B$, for fixed bound $B$, is a compact set in $\mathcal{K} \times H^4(X_4; \mathbb{R})$ and hence projects to a compact set $\mathcal{U}$ in $H^4(X_4; \mathbb{R})$. Therefore, there can be at most a finite number of lattice vectors in $\mathcal{U}$. Note that it is essential to use the primitivity condition.

As an example of how the bound $B$ imposes finiteness, consider the family (3.94), with $F = n\gamma_1$, $H = n\gamma_2$. Then $\int F \wedge H = 4n^2 b$. Since the denominators of $a, b$ must divide $n$, the denominators of $a, b$ are bounded when (3.109) is bounded. Thus, the bound on (3.109) cuts down (3.94) to a finite set of examples. (In fact, we may dispense with the cutoff $\mathcal{K}$ on the region in $\mathcal{M}_{\text{cplx}}$.)

It would be interesting, even in the simple explicit examples above, to give precise bounds for the number for flux vacua associated to a region $\mathcal{K}$ and bound $B$. This should be related to class numbers. For example, the solutions (3.93) have $N_f = 2|D|$, and hence there are $h(D)$ distinct such solutions. Unfortunately, the general relation appears to be complicated. For asymptotic estimates in the case of general CY compactification, under the assumption of uniform distribution, see [101] [11].

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3.9. Conclusions

Complex multiplication is beautiful and profound. Moreover, as we have shown, arithmetic varieties related to number fields do seem to be naturally selected in supersymmetric black holes, F-theory, and flux compactifications. The main open question, as far as the author is concerned, is whether the arithmetic of these varieties has any important physical significance.

Acknowledgements: I would like to thank my collaborators on the work which was reviewed above, R. Dijkgraaf, J. Maldacena, S. Miller, A. Strominger, and E. Verlinde. I also would like to thank F. Denef and E. Diaconescu for numerous detailed discussions on the subject of lecture 2, and N. Yui for some helpful correspondence. In addition I would like to thank B. Acharya, A. Connes, F. Denef, R. Donagi, M. Douglas, S. Kachru, J. Lagarias, J. Marklof, T. Pantev, K. Wendland, and D. Zagier for useful comments and discussions. I would also like to thank the Les Houches École de Physique for hospitality at the wonderful conference and B. Julia and P. van Hove for the invitation to speak at the conference. Finally, this work is supported in part by DOE grant DE-FG02-96ER40949.
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