A Stochastic Line Search Method with Convergence Rate Analysis

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Abstract

For deterministic optimization, line-search methods augment algorithms by providing stability and improved efficiency. We adapt a classical backtracking Armijo line-search to the stochastic optimization setting. While traditional line-search relies on exact computations of the gradient and values of the objective function, our method assumes that these values are available up to some dynamically adjusted accuracy which holds with some sufficiently large, but fixed, probability. We show the expected number of iterations to reach a near stationary point matches the worst-case efficiency of typical first-order methods, while for convex and strongly convex objective, it achieves rates of deterministic gradient descent in function values.

1 Introduction

In this paper we consider the classical stochastic optimization problem

$$
\min_{x \in \mathbb{R}^n} \left\{ f(x) = \mathbb{E}[\tilde{f}(x;\xi)] \right\},
$$

where $\xi$ is a random variable obeying some distribution. In the case of empirical risk minimization with a finite training set, $\xi_i$ is a random variable that is defined by a single random sample drawn uniformly from the training set. More generally $\xi$ may represents a sample or a set of samples drawn from the data distribution.

The most widely used method to solve (1.1) is the stochastic gradient descent (SGD) [16]. Due to its low iteration cost, SGD is often preferred to the standard gradient descent (GD) method for empirical risk minimization. Despite the prevalent use of SGD, it has known challenges and inefficiencies. First, the direction may not represent a descent direction, and second, the method is sensitive to the step-size (learning rate) which is often poorly overestimated. Various authors have attempted to address this last issue, see [8, 10, 12, 13]. Motivated by these facts, we turn to the deterministic optimization approach for adaptively selecting step sizes - GD with Armijo back-tracking line-search.

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Related work. In general, GD with back-tracking requires computing a full gradient and function evaluation - too expensive of an operation for the general problem (1.1). On the other hand, the per-iteration convergence rate for GD is superior to SGD making it an attractive alternative. Several works have attempted to transfer ideas from deterministic GD to the stochastic setting with the intent of diminishing the gradient computation, by using dynamic gradient sampling, e.g. [5, 9, 11]. However, these works address only convex setting. Moreover for them to obtain convergence rates matching those of GD in expectation, a small constant step-size must be known in advance and the sample size needs to be increased at a pre-described rate thus decreasing the variance of gradient estimates. Recently, in [4] an adaptive sample size selection strategy was proposed where sample size is selected based on the reduction of the gradient (and not pre-described). For convergence rates to be derived, however, an assumption has to be made that these sample sizes can be selected based on the size of the true gradient, which is, of course, unknown. In [18] a second-order method that subsamples gradient and Hessian is proposed, however, the sample size is simply assumed to be sufficiently large, so that essentially, the method behaves as a deterministic inexact method with high probability.

In [4] and [9] a practical back-tracking line search is proposed, combined with the their sample size selection. In both cases the backtracking is based on Armijo line search condition applied to function estimates that are computed on the same batch as the gradient estimates and is essentially a heuristic. A very different type of line-search based on probabilistic Wolfe condition is proposed in [14], however, it aims at improving step size selection for SGD and has no theoretical guarantees.

Our contribution. In this work we propose an adaptive backtracking line-search method, where the sample sizes for gradient and function estimates are chosen adaptively using knowable quantities along with the step-size. We show that this method converges to the optimal solution with probability one and derive strong convergence rates that match those of the deterministic gradient descent methods in the nonconvex $O(\varepsilon^{-2})$, convex $O(\varepsilon^{-1})$, and strongly convex $O(\log(\varepsilon^{-1}))$ cases. This paper offers the first stochastic line search method with convergence rates analysis, and is the first to provide convergence rates analysis for adaptive sample size selection based on knowable quantities.

Background. There are many types of (deterministic) line-search methods, see [15, Chapter 3], but all share a common philosophy. First, at each iteration, the method computes a search direction $d_k$ by e.g. the gradient or (quasi) Newton directions. Next, they determine how far to move in the direction through the univariate function, $\phi(\alpha) = f(x_k + \alpha d_k)$, to find the stepsize $\alpha_k$. Typical line-searches try out a sequences of potential values for the stepsize, accepting $\alpha$ once some verifiable criteria becomes satisfied. One popular line-search criteria specifies an acceptable step length should give sufficient decrease in the objective function $f$:

\begin{equation}
\text{(Armijo condition \[1\])} \quad f(x_k + \alpha d_k) \leq f(x_k) - \theta \alpha \|\nabla f(x_k)\|^2, \tag{1.2}
\end{equation}

where the constant $\theta \in (0, 1)$ is chosen by the user and $d_k = -\nabla f(x_k)$. Larger step sizes imply larger gains towards optimality and lead to fewer overall iterations. When step sizes get too small or worse 0, no progress is made and the algorithm stagnates. A popular way to systematically search the domain of $\alpha$ while simultaneously preventing small step sizes is backtracking. Backtracking starts with an overestimate of $\alpha$ and decreases it until (1.2) becomes true. Our exposition is on a stochastic version of backtracking using the stochastic gradient estimate as a search direction and stochastic function estimates in (1.2). In the remainder of the paper, all random quantities will be denoted by capitalized letters and their respective realizations by corresponding lower case letters.
**Algorithm 1: Line search method**

**Initialization:** Choose constants $\gamma > 1$, $\theta \in (0, 1)$ and $\alpha_{\text{max}}$. Pick initial point $x_0$, $\alpha_0 = \gamma^{j_0} \alpha_{\text{max}}$ for some $j_0 \leq 0$, and $\delta_0$.

Repeat for $k = 0, 1, \ldots$

1. **Compute a gradient estimate** Based on $\alpha_k$ compute a gradient estimate $g_k$. Set the step $s_k = -\alpha_k g_k$

2. **Compute function estimates** Based on $\delta_k$, $g_k$ and $\alpha_k$ obtain estimates of $f_k^0$ and $f_k^s$ of $f(x_k)$ and $f(x_k + s_k)$ respectively.

3. **Check sufficient decrease**
   Check if
   \[ f_k^s \leq f_k^0 - \alpha_k \theta \|g_k\|^2. \] (2.2)

4. **Successful step**
   If (2.2) set $x_{k+1} = x_k - \alpha_k g_k$ and $\alpha_{k+1} = \min\{\alpha_{\text{max}}, \gamma \alpha_k\}$.
   - **Reliable step:** If $\alpha_k \|g_k\|^2 \geq \delta_k^2$, then increase $\delta_{k+1}^2 = \gamma \delta_k^2$.
   - **Unreliable step:** If $\alpha_k \|g_k\|^2 < \delta_k^2$, then decrease $\delta_{k+1}^2 = \gamma^{-1} \delta_k^2$.

5. **Unsuccessful step**
   Otherwise, set $x_{k+1} = x_k$, $\alpha_{k+1} = \gamma^{-1} \alpha_k$, and $\delta_{k+1}^2 = \gamma^{-1} \delta_k^2$.

Let $k = k + 1$.

2 **Stochastic back-tracking line search method**

We present here our main algorithm for GD with back-tracking line search. We impose the standard assumption on the objective function.

**Assumption 2.1.** We assume that all iterates $x_k$ of Algorithm 1 satisfy $x_k \in \Omega$ where $\Omega$ is a set in $\mathbb{R}^n$. Moreover, the gradient of $f$ is $L$-Lipschitz continuous for all $x \in \Omega$ and that $f_{\text{min}} \leq f(x)$, for all $x \in \Omega$.

2.1 **Outline of method**

At each iteration, our scheme computes a random direction $g_k$ via e.g. a minibatch stochastic gradient estimate or sampling the function $f(x)$ itself and using finite differences. Then, we compute stochastic function estimates at the current iterate and prospective new iterate, resp. $f_k^0$ and $f_k^s$. We check the Armijo condition [1] using the stochastic estimates

\[ (\text{Stochastic) Armijo} \quad f_k^s \leq f_k^0 - \theta \alpha_k \|g_k\|^2. \] (2.1)

If (2.1) holds, the next iterate becomes $x_{k+1} = x_k - \alpha_k g_k$ and stepsize $\alpha_k$ increases; otherwise $x_{k+1} = x_k$ and $\alpha_k$ decreases, as is typical in (deterministic) back-tracking line searches.

Algorithm [1] describes our method [1]. Unlike classical back-tracking line search, there is an additional control, $\delta_k^2$, which serves as a guess of the true function decrease and controls the accuracy of the function estimates. We discuss this further next.

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1We state the algorithm using the lower case notation to represent a realization of the algorithm.
Challenges with randomized line-search. Due to the stochasticity of the gradient and/or function values, two major challenges result:

- a series of erroneous unsuccessful steps cause $A_k$ to become arbitrarily small;
- steps may falsely satisfy (2.1) leading to objective value at the next iteration arbitrarily larger than the current iterate.

Convergence proofs for deterministic line searches rely on the fact that neither of the above problems arise. Our approach controls the probability with which the random gradients and function values are representative of their true counterparts. When this probability is large enough, the method tends to make successful steps when $A_k$ is sufficiently small, hence $A_k$ behaves like a random walk with an upward drift thus staying away from 0.

Yet, even when the probability of good gradients/function estimates is near 1, it is not guaranteed that $\mathbb{E}(f(X_{k+1})|X_k) < f(X_k)$ holds at each iteration due to the second issue - possible arbitrary increase of the objective. Since random gradient may not be representative of the true gradient the function estimate accuracy and thus the expected improvement needs to be controlled by a different quantity, $\Delta_k^2$. When the predicted decrease in the true function matches the expected function estimate accuracy ($\Delta_k^2 \leq A_k \|G_k\|^2$), we call the step reliable and increase the parameter $\Delta_k^2$ for the next iteration; otherwise our prediction does not match the expectation and we decrease $\Delta_k^2$.

Moreover, unlike the typical stochastic convergence rate analysis, which bounds expected improvement in either $\mathbb{E}(\|\nabla f(x)\|)$ or $\mathbb{E}(f(x) - f_{\max})$ after a given number of iteration, our convergence rate analysis bounds the total expected number of steps that the algorithm takes before either $\|\nabla f(x)\| \leq \varepsilon$ or $f(x) - f_{\max} \leq \varepsilon$ is reached. Our results rely on a stochastic process framework introduced and analyzed in [3] to provide convergence rates for stochastic trust region method.

2.2 Random gradient and function estimates

Overview. At each iteration, we compute a stochastic gradient and stochastic function values. With probability $p_g$, the random direction $G_k$ is close to the true gradient. We measure closeness or accuracy of the random direction using the current step length, which is a known quantity. This procedure naturally adapts the required accuracy as the algorithm progresses. As the steps get shorter (i.e. either the gradient gets smaller or the step-size parameter does), we require the accuracy to increase, but the probability $p_g$ of encountering a good gradient $G_k$ at any iteration is the same.

A similar procedure applies to function estimates, $F_k^0$ and $F_k^s$. The accuracy of the function estimates to the true function values at the points $x_k$ and $x_{k+1}$ are tied to the size of the step, $A_k \|G_k\|$. At each iteration, there is a probability $p_f$ of obtaining good function estimates. By choosing the probabilities of good gradient and estimates, we show Algorithm 1 converges. To formalize this procedure, we introduce the following.

Notation and definitions. Algorithm 1 generates a random process $\{G_k, X_k, A_k, \Delta_k, S_k, F_k^0, F_k^s\}$, in what follows we will denote all random quantities by capital letters and their realization by small letters. Hence random gradient estimate is denoted by $G_k$ and its realizations - by $g_k = G_k(\omega)$. Similarly, let the random quantities $x_k = X_k(\omega)$ (iterates), $\alpha_k = A_k(\omega)$ (stepsize), control size $\Delta_k(\omega) = \delta_k$, and $s_k = S_k(\omega)$ (step) denote their respective realizations. Similarly, we let $\{F_k^0, F_k^s\}$ denote estimates of $f(X_k)$ and $f(X_k + S_k)$, with their realizations denoted by $f_k^0 = F_k^0(\omega)$ and $f_k^s = F_k^s(\omega)$. Our goal is to show that under some assumptions on $G_k$ and $\{F_k^0, F_k^s\}$ the resulting
satisfy the condition with respect to the corresponding sequence Assumption 2.4.

To formalize the conditioning on the past, let \( \mathcal{F}^{G,F}_{k-1} \) denote the \( \sigma \)-algebra generated by the random variables \( G_0, G_1, \ldots, G_{k-1} \) and \( F^0_k, F^0_{k-1}, F^1_k, F^1_{k-1}, \ldots, F^s_k, F^s_{k-1} \) and let \( \mathcal{F}^{G,F}_{k-1/2} \) denote the \( \sigma \)-algebra generated by the random variables \( G_0, G_1, \ldots, G_k \) and \( F^0_k, F^0_0, F^1_k, F^1_0, \ldots, F^s_k, F^s_{k-1} \). For completeness, we set \( \mathcal{F}^{G,F}_0 = \sigma(x_0) \). As a result, we have that \( \mathcal{F}^{G,F}_k \) for \( k \geq -1 \) is a filtration. By construction of the random variables \( X_k \) and \( \mathcal{A}_k \) in Algorithm 1 we see \( \mathbb{E}[X_k|\mathcal{F}^{G,F}_{k-1}] = X_k \) and \( \mathbb{E}[A_k|\mathcal{F}^{G,F}_{k-1}] = A_k \) for all \( k \geq 0 \).

We measure accuracy of the gradient estimates \( G_k \) and function estimates \( F^0_k \) and \( F^s_k \) using the following definitions.

**Definition 2.2.** We say that a sequence of random directions \( \{G_k\} \) is \( (p_g) \)-probabilistically \( \kappa_g \)-sufficiently accurate for Algorithm 1 for the corresponding sequence \( \{\mathcal{A}_k, X_k\} \), if there exists a constant \( \kappa_g > 0 \), such that the events

\[
I_k = \{ ||G_k - \nabla f(X_k)|| \leq \kappa_g A_k ||G_k|| \}
\]

satisfy the condition

\[
\mathbb{P}(I_k|\mathcal{F}^{G,F}_{k-1}) = \mathbb{E}[1_{I_k}|\mathcal{F}^{G,F}_{k-1}] \geq p_g
\]

In addition to sufficiently accurate gradients, we require estimates on the function values \( f(x_k) \) and \( f(x_k + s_k) \) to also be sufficiently accurate.

**Definition 2.3.** A sequence of random estimates \( \{F^0_k, F^s_k\} \) is said to be \( (p_f) \)-probabilistically \( \varepsilon_f \)-accurate with respect to the corresponding sequence \( \{X_k, \mathcal{A}_k, S_k\} \) if the events

\[
J_k = \{ |F^0_k - f(x_k)| \leq \varepsilon_f A^2_k \|G_k\|^2 \text{ and } |F^s_k - f(x_k + s_k)| \leq \varepsilon_f A^2_k \|G_k\|^2 \}
\]

satisfy the condition

\[
\mathbb{P}(J_k|\mathcal{F}^{G,F}_{k-1/2}) = \mathbb{E}[1_{J_k}|\mathcal{F}^{G,F}_{k-1/2}] \geq p_f
\]

We note here that the filtration \( \mathcal{F}^{G,F}_{k-1/2} \) includes \( \mathcal{A}_k \) and \( G_k \); hence the accuracy of the estimates is measured with respect to fixed quantities. Next, we state the key assumption on the nature of the stochastic information in Algorithm 1.

**Assumption 2.4.** The following hold for the quantities in the algorithm:

(i) The sequence of random gradients \( G_k \) generated by Algorithm 1 is \( (p_g) \)-probabilistically \( \kappa_g \)-sufficiently accurate for some sufficiently large \( p_g \in (0, 1) \).

(ii) The sequence of estimates \( \{F^0_k, F^s_k\} \) generated by Algorithm 1 is \( (p_f) \)-probabilistically \( \varepsilon_f \)-accurate estimates for some \( \varepsilon_f \leq \frac{\kappa_f \theta}{\theta_\max} \) and sufficiently large \( p_f \in (0, 1) \).

(iii) The sequence of estimates \( \{F^0_k, F^s_k\} \) generated by Algorithm 1 satisfies a \( \kappa_f \)-variance condition for all \( k \geq 0 \):

\[
\mathbb{E}[\|F^s_k - f(X_k + S_k)\|^2|\mathcal{F}^{G,F}_{k-1/2}] \leq \max\{\kappa^2_f A^2_k \|\nabla f(X_k)\|^4, \theta^2 \Delta^2_k\}
\]

and

\[
\mathbb{E}[\|F^0_k - f(X_k)\|^2|\mathcal{F}^{G,F}_{k-1/2}] \leq \max\{\kappa^2_f A^2_k \|\nabla f(X_k)\|^4, \theta^2 \Delta^2_k\}.
\]
A simple calculation shows that under Assumption 2.4 the following hold

$$\mathbb{E}[1_{I_k \cap J_k} | F_{k-1}^G] \geq p_g p_f, \quad \mathbb{E}[1_{I_k \cap J_k} | F_{k-1}^G] \leq 1 - p_g, \quad \text{and} \quad \mathbb{E}[1_{I_k} | F_{k-1}^G] \leq 1 - p_f.$$  

**Remark 1.** We are interested in deriving convergence results for the case when $\kappa_g$ may be large. For the rest of the exposition, without loss of generality $\kappa_g \geq 2$. It is clear if $\kappa_g$ happens to be smaller, somewhat better bounds that the ones we derive here will result since the gradients give tighter approximations of the true gradient. We are interested in deriving bound for the case when $\kappa_g$ is large. Equation (2.3) includes the maximum of two terms - one of the terms $\|\nabla f(X_k)\|$ is unknown. When one possesses external knowledge of $\|\nabla f(X_k)\|$, one could use this value. This is particularly useful when $\|\nabla f(X_k)\|$ is big since it allows large variance in the function estimates, for example assumption that $\|\nabla f(X_k)\| \geq \varepsilon$ implies that this variance does not have to be driven to zero, before the algorithm reaches a desired accuracy. Yet, for convergence and since a useful lower bound on $\|\nabla f(X_k)\|$ may be unknown, we include the parameter $\Delta_k$ as a way to adaptively control the variance. As such $\kappa_f$ should be small, in fact, can be set equal to 0. The analysis can be performed for any other values of the above constants - the choices here are for simplicity and convenience.

This assumption on the accuracy of the gradient and function estimates is key in our convergence rate analysis. We derive specific bounds on $p_g$ and $p_f$ under which these rates would hold. We note here that if $p_f = 1$ then Assumption 2.4(iii) is not needed and condition $p_g > 1/2$ is sufficient for the convergence results. This case can be considered as an extension of results in [6]. Before concluding this section, we state a result showing the relationship between the variance assumption on the function values and the probability of inaccurate estimates.

**Lemma 2.5.** Let Assumption 2.4 hold. Suppose $\{X_k, G_k, F_k^0, F_k^s, A_k, \Delta_k\}$ is a random process generated by Algorithm 1 and $\{F_k^0, F_k^s\}$ are $p_f$-probabilistically accurate estimates. Then for every $k \geq 0$ we have

$$\mathbb{E}[1_{I_k} | F_k^s - f(X_k) | F_{k-1/2}^G] \leq (1 - p_f)^{1/2} \max\{\kappa_f, A_k \|\nabla f(X_k)\|^2, \theta \Delta_k^2\}$$

and

$$\mathbb{E}[1_{I_k} | F_k^0 - f(X_k) | F_{k-1/2}^G] \leq (1 - p_f)^{1/2} \max\{\kappa_f, A_k \|\nabla f(X_k)\|^2, \theta \Delta_k^2\}.$$  

**Proof.** We show the result for $F_k^0 - f(X_k)$, but the proof for $F_k^s - f(X_k + S_k)$ is the same. Using Holder’s inequality for conditional expectations, we deduce

$$\mathbb{E}\left[\frac{|F_k^0 - f(X_k)|}{\max\{\kappa_f A_k \|\nabla f(X_k)\|^2, \theta \Delta_k^2\}} | F_{k-1/2}^G\right] \leq \left(\mathbb{E}\left[|F_k^0 - f(X_k)|^2 / \max\{\kappa_f A_k \|\nabla f(X_k)\|^4, \theta \Delta_k^4\}\right] / F_{k-1/2}^G\right)^{1/2} \left(\mathbb{E}\left[|F_k^0 - f(X_k)|^2 / \max\{\kappa_f A_k \|\nabla f(X_k)\|^4, \theta \Delta_k^4\}\right] / F_{k-1/2}^G\right)^{1/2}.$$  

The result follows after noting by (2.3)

$$\left(\mathbb{E}\left[|F_k^0 - f(X_k)|^2 / \max\{\kappa_f A_k \|\nabla f(X_k)\|^4, \theta \Delta_k^4\}\right] F_{k-1/2}^G\right)^{1/2} \leq 1.$$  

\[\square\]

### 2.3 Computing $G_k$, $F_k^0$, and $F_k^s$ to satisfy Assumption 2.4

Assuming that the variance of random function and gradient realizations is bounded as

$$\mathbb{E}(\|\nabla \tilde{f}(x, \xi_i) - \nabla f(x)\|^2) \leq V_g \quad \text{and} \quad \mathbb{E}(|\tilde{f}(x, \xi_i) - f(x)|^2) \leq V_f,$$
Assumption 2.4 can be made to hold if $G_k$, $F_0^k$ and $F_0^k$ are computed using a sufficient number of samples. In particular, let $S_k$ be a sample of realizations $\nabla f(x, \xi_i)$, $i \in S_k$ and $G_k := \frac{1}{|S_k|} \sum_{i \in S_k} \nabla \hat{f}(X_k, \xi_i)$. By using results e.g. in [18, 19] we can show that if $|S_k| \geq \tilde{O}(V_g \kappa_g^2 A_k^2 \|G_k\|^2) \ (2.4)$ (where $\tilde{O}$ hides the log factor of $1/(1 - p_g)$), then Assumption 2.4(i) is satisfied. While $G_k$ is not known when $|S_k|$ is chosen, one can design a simple loop by guessing the value of $\|G_k\|$ and increasing the number of samples until (2.4) is satisfied, this procedure is discussed in [6]. Similarly to satisfy Assumption 2.4(ii), it is sufficient to compute $F_0^k = \frac{1}{|S_0^k|} \sum_{i \in S_0^k} \tilde{f}(X_k, \xi_i)$ with $|S_0^k| \geq \tilde{O}(V_f \kappa_f^2 \Delta_k \|G_k\|^4)$ (where $\tilde{O}$ hides the log factor of $1/(1 - p_f)$) and to obtain $F_s^k$ analogously. Finally, it is easy to see that Assumption 2.4(iii) is simply satisfied if $|S_0^k| \geq \frac{|V_f|}{\theta^2 \Delta_k}$ by standard properties of variance.

We observe that:

- unlike [5,9], the number of samples for gradient and function estimation does not increase at any pre-defined rate, but is closely related to the progress of the algorithm. In particular if $A_k\|G_k\|$ and $\Delta_k$ increase then the sample sets sizes can decrease.

- Also, unlike [18] where the number of samples is simply chosen large enough a priori for all $k$ so that the right hand side in Assumption 2.4(i) is bounded by a predefined accuracy $O(\varepsilon)$, our algorithm can be applied without knowledge of $\varepsilon$.

- Finally, unlike [4] where theoretical results require that $|S_k|$ depends on $\|\nabla f(X_k)\||, which is unknown, our bounds on the sample set sizes all use knowable quantities, such as bound on the variance and quantities computed by the algorithm.

We also point out $\kappa_g$ can be arbitrarily big and $p_g$ depends only on the backtracking factor $\gamma$ and is not close to 1; hence the number of samples to satisfy Assumption 2.4(i) is moderate. On the other hand, $p_f$ will have to depend on $\kappa_g$; hence a looser control of the gradient estimates results in tighter control, i.e. larger sample sets, for function estimates.

Our last comment is that $G_k$ does not have to be an unbiased estimate of $\nabla f(X_k)$ and does not need to be computed via gradient samples. Instead it can be computed via stochastic finite differences, as is discussed for example in [7].

3 Renewal-Reward Process

In this section, we define a general random process introduced in [3] and its stopping time $T$ which serve as a general framework for analyzing behavior of stochastic trust region method in [3] and stochastic line search in this paper. We state the relevant definitions, assumptions, and theorems and refer the reader to the proofs in [3].

**Definition 3.1.** Given a discrete time stochastic process $\{X_k\}$, a random variable $T$ is a stopping time for $\{X_k\}$ if the event $\{T = k\} \in \sigma(X_0, \ldots, X_k)$. 

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Let \( \{ \Phi_k, A_k \} \) be a random process such that \( \Phi_k \in [0, \infty) \) and \( A_k \in [0, \infty) \) for \( k \geq 0 \). Let us also define a biased random walk process, \( \{ W_k \}_{k=1}^{\infty} \), defined on the same probability space as \( \{ \Phi_k, A_k \} \). We denote \( \mathcal{F}_k \) the \( \sigma \)-algebra generated by \( \{ \Phi_0, A_0, W_0, \ldots, \Phi_k, A_k, W_k \} \), where \( W_0 = 1 \). In addition, \( W_k \) obeys the following dynamics

\[
\Pr(W_{k+1} = 1|\mathcal{F}_k) = p \quad \text{and} \quad \Pr(W_{k+1} = -1|\mathcal{F}_k) = (1 - p) \tag{3.1}
\]

We define \( T_\varepsilon \) to be a family of stopping times parameterized by \( \varepsilon \). In \[3\] a bound on \( E(T_\varepsilon) \) is derived under the following assumption on \( \{ \Phi_k, A_k \} \).

**Assumption 3.2.** The following hold for the process \( \{ \Phi_k, A_k, W_k \} \).

(i) \( A_0 \) is a constant. There exists a constant \( \lambda \in (0, \infty) \) and \( \alpha_{\text{max}} = A_0 e^{\lambda j_{\text{max}}} \) (for some \( j_{\text{max}} \in \mathbb{Z} \)) such that \( A_k \leq \alpha_{\text{max}} \) for all \( k \geq 0 \).

(ii) There exists a constant \( A = A_0 e^{\lambda \bar{j}} \) for some \( \bar{j} \in \mathbb{Z} \) and \( \bar{j} < 0 \), such that, the following holds for all \( k \geq 0 \),

\[
1_{\{ T_\varepsilon > k \}} A_{k+1} \geq 1_{\{ T_\varepsilon > k \}} \min \left\{ A_k e^{\lambda W_{k+1}}, \bar{A} \right\}
\]

where \( W_{k+1} \) satisfies (3.1) with \( p > \frac{1}{2} \).

(iii) There exists a nondecreasing function \( h : [0, \infty) \to (0, \infty) \) and a constant \( \Theta > 0 \) such that

\[
1_{\{ T_\varepsilon > k \}} E[\Phi_{k+1}|\mathcal{F}_k] \leq 1_{\{ T_\varepsilon > k \}} (\Phi_k - \Theta h(A_k)).
\]

Assumption 3.2 (iii) states that conditioned on the event \( T_\varepsilon > k \) and the past, the random variable \( \Phi_k \) decreases by \( \Theta h(A_k) \) at each iteration. Whereas Assumption 3.2 (ii) says that once \( A_k \) falls below the fixed constant \( \bar{A} \), the sequence has a tendency to increase. Assumptions 3.2 (i) and (ii) together also ensures that \( \bar{A} \) belongs to the sequence of values taken by the sequence \( A_k \). As we will see this is a simple technical assumption that can be satisfied w.l.o.g.

**Remark 2.** Computational complexity (in deterministic methods) measures the number of iterations until an event such as \( \| \nabla f(x) \| \) is small or \( f(x_k) - f^* \) is small, or equivalently, the rate at which the gradient/function values decreases as a function of the iteration counter \( k \). For randomized or stochastic methods, previous works tended to focus on the second definition, i.e. showing the expected size of the gradient or function values decreases like \( 1/k \). Instead, here we bound the expected number of iterations until the size of the gradient or function values are small, which is the same as bounding the stopping times \( T_\varepsilon = \inf \{ k \geq 0 : \| \nabla f(X_k) \| < \varepsilon \} \) and \( T_\varepsilon = \inf \{ k \geq 0 : f(X_k) - f^* \leq \varepsilon \} \), for a fixed \( \varepsilon > 0 \).

**Remark 3.** In the context of deterministic line search, when the stepsize \( \alpha_k \) falls below the constant \( 1/L \), where \( L \) is the Lipschitz constant of \( \nabla f(x) \), the iterate \( x_k + s_k \) always satisfies the sufficient decrease condition, namely \( f(x_k + s_k) \leq f(x_k) - \theta \alpha_k \| \nabla f(x_k) \|^2 \). Thus \( \alpha_k \) never falls much below \( 1/L \). To match the dynamics behind deterministic line search, we expect \( \Phi_{k+1} - \Phi_k \approx f(X_{k+1}) - f(X_k) \) with \( \Theta h(A_k) \approx A_k \| \nabla f(X_k) \|^2 \) and the constant \( \bar{A} \approx 1/L \). However, in the stochastic setting there is a positive probability of \( A_k \) being arbitrarily small. Theorem 3.3, below, is derived by observing that on average \( A_k \geq \bar{A} \) occurs frequently due to the upward drift in the random walk process. Consequently, \( E[\Phi_{k+1} - \Phi_k] \) can be bounded by a negative fixed value (dependent on \( \varepsilon \)) frequently; thus we can derive a bound on \( E[T_\varepsilon] \).

The following theorem (Theorem 2.2 in [3]) bounds \( E[T_\varepsilon] \) in terms of \( h(\bar{A}) \) and \( \Phi_0 \).

**Theorem 3.3.** Under Assumption 3.2

\[
E[T_\varepsilon] \leq \frac{p}{2p-1} \cdot \frac{\Phi_0}{\Theta h(\bar{A})} + 1.
\]
4 Convergence of Stochastic Line Search

Our primary goal is to prove convergence of Algorithm 1 by showing a \( \liminf \) convergence result, \( \liminf_{k \to \infty} \| \nabla f(X_k) \| = 0 \) a.s. We that typical convergence results for stochastic algorithms prove either high probability results or that the expected gradient at an averaged point converges. Our result is slightly stronger than these results since we show a subsequence of the \( \| \nabla f(X_k) \| \) converges a.s. With this convergence result, stopping times based on either \( \| \nabla f(x) \| < \varepsilon \) and/or \( f(x) - f_{\text{min}} < \varepsilon \) are finite almost surely. Our approach for the \( \liminf \) proof is twofold: (1) construct a function \( \Phi (\approx f) \) whose expected progress decreases proportionally to \( \| \nabla f(x) \|^2 \) and (2) the \( \limsup \) of the step sizes is strictly larger than 0 a.s.

4.1 Useful results

Before delving into the convergence statement and proof, we state some lemmas similar to those derived in [2, 6, 7].

**Lemma 4.1** (Accurate gradients \( \Rightarrow \) lower bound on \( \| g_k \| \)). Suppose \( g_k \) is \( \kappa_g \)-sufficiently accurate. Then

\[
\frac{\| \nabla f(x_k) \|}{(\kappa_g \alpha_{\text{max}} + 1)} \leq \| g_k \|.
\]

**Proof.** Because \( g_k \) is \( \kappa_g \)-sufficiently accurate together with the triangle inequality implies

\[
\| \nabla f(x_k) \| \leq (\kappa_g \alpha_k + 1) \| g_k \| \leq (\kappa_g \alpha_{\text{max}} + 1) \| g_k \|.
\]

\[\Box\]

**Lemma 4.2** (Accurate gradients and estimates \( \Rightarrow \) successful iteration). Suppose \( g_k \) is \( \kappa_g \)-sufficiently accurate and \( \{ f_k^0, f_k^s \} \) are \( \varepsilon_f \)-accurate estimates. If

\[
\alpha_k \leq \frac{1 - \theta}{\kappa_g + \frac{L}{2} + 2\varepsilon_f}
\]

then the trial step \( x_k + s_k \) is successful. In particular, this means \( f_k^s \leq f_k^0 - \theta \alpha_k \| g_k \|^2 \).

**Proof.** The \( L \)-smoothness of \( f \) and the \( \kappa_g \)-sufficiently accurate gradient immediately yield

\[
f(x_k + s_k) \leq f(x_k) - \alpha_k (\nabla f(x_k) - g_k)^T g_k - \alpha_k \| g_k \|^2 + \frac{L\alpha_k^2}{2} \| g_k \|^2
\leq f(x_k) + \kappa_g \alpha_k^2 \| g_k \|^2 - \alpha_k \| g_k \|^2 + \frac{L\alpha_k^2}{2} \| g_k \|^2.
\]

Since the estimates are \( \varepsilon_f \)-accurate, we obtain

\[
f_k^0 - \varepsilon f \alpha_k^2 \| g_k \|^2 \leq f(x_k + s_k) - f_k^s + f_k^s
\leq f(x_k) - f_k^0 + f_k^0 + \kappa_g \alpha_k^2 \| g_k \|^2 - \alpha_k \| g_k \|^2 + \frac{L\alpha_k^2}{2} \| g_k \|^2
\leq f_k^0 + \varepsilon f \alpha_k^2 \| g_k \|^2 + \kappa_g \alpha_k^2 \| g_k \|^2 - \alpha_k \| g_k \|^2 + \frac{L\alpha_k^2}{2} \| g_k \|^2.
\]

The result follows by noting \( f_k^0 \leq f_k^0 - \alpha_k \| g_k \|^2 (1 - \alpha_k (\kappa_g + \frac{L}{2} + 2\varepsilon_f)) \).

\[\Box\]
Lemma 4.3 (Good estimates ⇒ decrease in function). Suppose \( \varepsilon_f < \frac{\theta}{4\alpha_{\max}} \) and \( \{f^0_k, f^1_k\} \) are \( \varepsilon_f \)-accurate estimates. If the trial step is successful, then the improvement in function value is

\[
f(x_{k+1}) \leq f(x_{k}) - \frac{\theta \alpha_k}{2} \|g_k\|^2. \tag{4.1}
\]

If, in addition, the step is reliable, then the improvement in function value is

\[
f(x_{k+1}) \leq f(x_{k}) - \frac{\theta \alpha_k}{4} \|g_k\|^2 - \frac{\theta}{4} \sigma^2_k. \tag{4.2}
\]

Proof. The iterate \( x_k + s_k \) is successful and the estimates are \( \varepsilon_f \)-accurate so we conclude

\[
f(x_k + s_k) \leq f(x_k + s_k) - f_k^s + f_k^0 - f(x_k) + f(x_k) - \alpha_k \theta \|g_k\|^2 \leq f(x_k) + 2\varepsilon_f \alpha^2_k \|g_k\|^2 - \alpha_k \theta \|g_k\|^2 \leq f(x_k) - \alpha_k \|g_k\|^2 (\theta - 2\varepsilon_f \alpha_{\max}),
\]

where the last inequality follows because \( \alpha_k \leq \alpha_{\max} \). The condition \( \varepsilon_f < \frac{\theta}{4\alpha_{\max}} \) immediately implies (4.1). By noticing \( \frac{\sigma_k}{2} \|g_k\|^2 \geq \frac{\theta \alpha_k}{4} \|g_k\|^2 + \frac{\theta \sigma^2_k}{4} \) holds for reliable steps, we deduce (4.2). \( \square \)

Lemma 4.4. Suppose the iterate is successful. Then

\[
\|\nabla f(x_{k+1})\|^2 \leq 2(L^2\alpha^2_k \|g_k\|^2 + \|\nabla f(x_{k})\|^2).
\]

In particular, the inequality holds

\[
\frac{1}{L^2} \left( \alpha_{k+1} \|\nabla f(x_{k+1})\|^2 - \alpha_k \|\nabla f(x_{k})\|^2 \right) \leq 2\gamma \alpha_k (\alpha^2_{\max} \|g_k\|^2 + \frac{1}{L^2} \|\nabla f(x_{k})\|^2).
\]

Proof. An immediate consequence of \( L \)-smoothness of \( f \) is \( \|\nabla f(x_{k+1})\| \leq \alpha_k \|g_k\| + \|\nabla f(x_{k})\| \). The result follows from squaring both sides and applying the bound, \( (a + b)^2 \leq 2(a^2 + b^2) \). To obtain the second inequality, we note that in the case \( x_k + s_k \) is successful, \( \alpha_{k+1} = \gamma \alpha_k \). \( \square \)

Lemma 4.5 (Accurate gradients and estimates ⇒ decrease in function). Suppose \( g_k \) is \( \kappa_g \)-sufficiently accurate and \( \{f^0_k, f^1_k\} \) are \( \varepsilon_f \)-accurate estimates where \( \varepsilon_f \leq \frac{\theta}{4\alpha_{\max}} \). If the trial step is successful, then

\[
f(x_{k+1}) - f(x_{k}) \leq -\frac{\theta \alpha_k}{4} \|g_k\|^2 - \frac{\theta \alpha_k}{4(\kappa_g \alpha_{\max} + 1)^2} \|\nabla f(x_{k})\|^2. \tag{4.3}
\]

In addition, if the trial step is reliable, then

\[
f(x_{k+1}) - f(x_{k}) \leq -\frac{\theta \alpha_k}{8} \|g_k\|^2 - \frac{\theta \sigma^2_k}{8} - \frac{\theta \alpha_k}{4(\kappa_g \alpha_{\max} + 1)^2} \|\nabla f(x_{k})\|^2. \tag{4.4}
\]

Proof. Lemma 4.4 implies

\[
-\frac{\theta}{2} \alpha_k \|g_k\|^2 \leq -\frac{\theta}{4} \alpha_k \|g_k\|^2 - \frac{\theta}{4(\kappa_g \alpha_{\max} + 1)^2} \alpha_k \|\nabla f(x_{k})\|^2. \tag{4.5}
\]

We combine this result with Lemma 4.3 to conclude the first result. For the second result, since the step is reliable, equation (4.5) improves to

\[
-\frac{\theta}{2} \alpha_k \|g_k\|^2 \leq -\frac{\theta}{8} \alpha_k \|g_k\|^2 - \frac{\theta}{8} \sigma^2_k - \frac{\theta}{4(\kappa_g \alpha_{\max} + 1)^2} \alpha_k \|\nabla f(x_{k})\|^2,
\]

and again the result follows from Lemma 4.3. \( \square \)
4.2 Definition and analysis of \{\Phi_k, A_k, W_k\} process for Algorithm 1

We base our proof of convergence on properties of the random function

\[ \Phi_k = \nu (f(X_k) - f_{\min}) + (1 - \nu) \frac{1}{L^2} A_k \|\nabla f(X_k)\|^2 + (1 - \nu) \theta \Delta_k^2. \]  

(4.6)

for some (deterministic) \( \nu \in (0, 1) \) and \( f_{\min} \leq f(x) \) for all \( x \). The goal is to show that \{\Phi_k, A_k\} satisfies Assumption 3.2, in particular, that \( \Phi_k \) is expected to decrease on each iteration. Due to inaccuracy in function estimates and gradients, the algorithm may take a step that increases the objective and thus \( \Phi_k \). We will show that such increase if bounded by a value proportional to \( \|\nabla f(x)\|^2 \). On the other hand, as we will show, on successful iteration with accurate function estimates, the objective decreases proportionally \( \|\nabla f(x)\|^2 \), while on unsuccessful steps, equation (4.6) is always negative because both \( A_k \) and \( \Delta_k \) are decreased. The function \( \Phi \) is chosen to balance the potential increases and decreases in the objective with changes inflicted by unsuccessful steps.

**Theorem 4.6.** Let Assumptions 2.1 and 2.4 hold. Suppose \{\( X_k, G_k, F_k^0, F_k^g, A_k, \Delta_k \)\} is the random process generated by Algorithm 1. Then there exist probabilities \( p_g, p_f > 1/2 \) and a constant \( \nu \in (0, 1) \) such that the expected decrease in \( \Phi_k \) is

\[ \mathbb{E}[\Phi_{k+1} - \Phi_k | F_k] \leq -p_g p_f (1 - \nu)(1 - \gamma^{-1}) \left( \frac{A_k}{L^2} \|\nabla f(X_k)\|^2 + \theta \Delta_k^2 \right). \]  

(4.7)

In particular, the constant \( \nu \) and probabilities \( p_g, p_f > 1/2 \) satisfy

\[ \frac{\nu}{1 - \nu} \geq \max \left\{ \frac{32 \gamma \alpha_{\max}^2}{\theta}, 16(1 - 1), \frac{16 \gamma (\kappa_0 \alpha_{\max} + 1)^2}{\theta} \right\}, \]  

(4.8)

and

\[ \frac{p_g p_f}{1 - p_f} \geq \max \left\{ \frac{8 L^2 \nu \kappa_f + 16 \gamma (1 - \nu)}{(1 - \nu)(1 - \gamma^{-1})}, \frac{8 \nu}{(1 - \nu)(1 - \gamma^{-1})} \right\}. \]  

(4.10)

| \multicolumn{4}{c|}{Upper bound on \( \mathbb{E}[\Phi_{k+1} - \Phi_k] \)} |
|-----------------|-----------------|-----------------|
| \begin{tabular}{l} \text{Accurate gradients} \\
\text{Accurate functions} \\
\text{w/ prob.} \( p_g p_f \) \end{tabular} | \begin{tabular}{l} \text{Bad gradients} \\
\text{Accurate functions} \\
\text{w/ prob.} \( (1 - p_g)p_f \) \end{tabular} | \begin{tabular}{l} \text{Bad functions} \\
w/ prob. \( 1 - p_f \) \end{tabular} |
| \begin{tabular}{l}
\text{Success} \\
\text{decrease} \end{tabular} | \begin{tabular}{l}
\text{Unsuccess} \\
\text{decrease} \end{tabular} | \begin{tabular}{l}
\text{Overall} \\
\text{worst case } \\
\text{improv.} \end{tabular} |
| \begin{tabular}{l}
\( -\frac{A^2}{L^2} \|\nabla f(X_k)\|^2 - \Delta_k^2 \) \\
\text{decrease} \end{tabular} | \begin{tabular}{l}
\( \frac{A^2}{L^2} \|\nabla f(X_k)\|^2 \) \\
\text{increase} \end{tabular} | \begin{tabular}{l}
\( \frac{A^2}{L^2} \|\nabla f(X_k)\|^2 + \Delta_k^2 \) \\
\text{increase} \end{tabular} |

Table 1: We summarize the proof of Theorem 4.6 by displaying the values of \( \Phi_{k+1} - \Phi_k \). The proof considers cases: accurate grad./functions estimates, bad grad./accurate functions estimates, and bad function estimates. Each of these is further broken into whether the step was successful/unsucessful. We summarize the expected upper bounds on \( \Phi_{k+1} - \Phi_k \) up to constants.
Proof of Theorem 4.6. Our proof considers three separate cases: good gradients/good estimates, bad gradients/good estimates, and lastly bad estimates. Each of these cases will be broken down into whether a successful/unsuccesful step is reliable/unreliable. To simplify notation, we introduce three sets

\[ \text{Succ} := \{ X_k + S_k \text{ is successful, namely sufficient decrease occurs} \}, \]
\[ R := \{ X_k + S_k \text{ is reliable, i.e. } A_k \parallel G_k \parallel^2 \geq \Delta_k^2 \}, \]
and \[ U := \{ X_k + S_k \text{ is unreliable, i.e. } A_k \parallel G_k \parallel^2 < \Delta_k^2 \}. \]

Using this notation we can write

\[ \mathbb{E}[\Phi_{k+1} - \Phi_k | F_{k-1}^{G,F}] = \mathbb{E}[(1_{I_k \cap J_k} + 1_{I_k^c \cap J_k} + 1_{J_k^c})(\Phi_{k+1} - \Phi_k) | F_{k-1}^{G,F}]. \]

For each case we will derive a bound on the expected decrease (increase) in \( \Phi_k \). In particular, we will show that, under an appropriate choice of \( \nu \), when the model and the estimates are good all three types of steps result in a decrease of \( \Phi_k \) proportional to \( A_k \parallel \nabla f(X_k) \parallel^2 \) and \( \Delta_k^2 \), while when model is bad, but the estimates are good, \( \Phi_k \) may increase by an amount proportional to \( A_k \parallel \nabla f(X_k) \parallel^2 \). Finally, when both the model and estimates are both bad, the expected increase in \( \Phi_k \) is bounded by an amount proportional to \( A_k \parallel \nabla f(X_k) \parallel^2 \) and \( \Delta_k^2 \). Thus, by choosing the right probability values for these events, we can ensure overall expected decrease. These bounds are derived in the proof below and are summarized in Table 1.

Case 1 (Accurate gradients and estimates, \( 1_{I_k \cap J_k} = 1 \)). We will show that the \( \Phi_k \) decreases no matter what type of step occurs and that the smallest decrease happens on the unsuccessful step. Thus this case dominates the other two and overall we conclude that

\[ \mathbb{E}[1_{I_k \cap J_k}(\Phi_{k+1} - \Phi_k) | F_{k-1}^{G,F}] \leq -p_f p_f (1 - \nu)(1 - \gamma^{-1}) \left( \frac{A_k}{L^2} \parallel \nabla f(X_k) \parallel^2 + \theta \Delta_k^2 \right). \]  \hfill (4.11)

(i). Successful and reliable step (\( 1_{\text{Succ}1_R} = 1 \)). The iterate is successful and both the gradient and function estimates are accurate so a decrease in the true objective occurs, specifically, from Lemma 4.4, \[ 1_{I_k \cap J_k} 1_{\text{Succ}1_R} \nu(f(X_{k+1}) - f(X_k)) \]

\[ \leq -\nu 1_{I_k \cap J_k} 1_{\text{Succ}1_R} \left( \frac{\theta A_k}{8} \parallel G_k \parallel^2 + \frac{\theta}{8} \Delta_k^2 + \frac{\theta A_k}{4(\kappa g \alpha_{\max} + 1)^2} \parallel \nabla f(X_k) \parallel^2 \right). \] \hfill (4.12)

As the iterate is successful, the term \( A_k \parallel \nabla f(X_k) \parallel^2 \) may increase, but its change is bounded due to Lemma 4.4

\[ 1_{I_k \cap J_k} 1_{\text{Succ}1_R}(1 - \nu) \frac{1}{L^2} \left( A_{k+1} \parallel \nabla f(X_{k+1}) \parallel^2 - A_k \parallel \nabla f(X_k) \parallel^2 \right) \]

\[ \leq 1_{I_k \cap J_k} 1_{\text{Succ}1_R}(1 - \nu) 2 \gamma A_k \left( \alpha_{\max}^2 \parallel G_k \parallel^2 + \frac{1}{L^2} \parallel \nabla f(X_k) \parallel^2 \right). \] \hfill (4.13)

Lastly because we have a reliable step, \( \Delta_k^2 \) increases by \( \gamma \). Consequently, we deduce that

\[ 1_{I_k \cap J_k} 1_{\text{Succ}1_R}(1 - \nu) \theta(\Delta_{k+1}^2 - \Delta_k^2) = 1_{I_k \cap J_k} 1_{\text{Succ}1_R}(1 - \nu) \theta(\gamma - 1) \Delta_k^2. \] \hfill (4.14)
We take conditional expectations with respect to $F$. We chose $\nu$ sufficiently large so that the term on the right hand side of (4.12) dominates the right hand sides of (4.13), and (4.14), specifically, $\nu \theta A_k^2 \|G_k\|^2 + (1 - \nu)2 \gamma A_k \alpha_{\max}^2 \|G_k\|^2 \leq -\frac{\nu \theta A_k}{8 \|G_k\|^2}$, $-\frac{\nu \theta A_k}{4L^2(\kappa_g \alpha_{\max} + 1)^2} \|\nabla f(X_k)\|^2 + (1 - \nu)2 \gamma A_k \frac{L^2}{\kappa_g \alpha_{\max} + 1} \|\nabla f(X_k)\|^2 \leq -\frac{\nu \theta A_k}{8L^2(\kappa_g \alpha_{\max} + 1)^2} \|\nabla f(X_k)\|^2$, and $-\frac{\nu \theta A_k}{8 \|G_k\|^2} \|\nabla f(X_k)\|^2 + (1 - \nu)(\gamma - 1)\theta \Delta_k^2 \leq -\frac{\nu \theta A_k}{16 \Delta_k^2}$. (4.15)

We combine Equations (4.12), (4.13), and (4.14) to conclude

$$1_{I_k \cap J_k} \mathbf{1}_{\text{Succ}}(\Phi_{k+1} - \Phi_k) \leq -1_{I_k \cap J_k} \mathbf{1}_{\text{Succ}}(\Phi_{k+1} - \Phi_k) \leq \frac{\nu \theta A_k}{8L^2(\kappa_g \alpha_{\max} + 1)^2} \|\nabla f(X_k)\|^2 + (1 - \nu)(1 - \gamma^{-1})\theta \Delta_k^2). \quad (4.16)$$

(ii). Successful and unreliable step ($1_{\text{Succ}}1_U = 1$). Because the iterate is successful and our gradient/estimates are accurate, we again apply Lemma 4.5 to bound $f(X_{k+1}) - f(X_k)$ but this time using (4.3) which holds for unreliable steps. The possible increase from the $(1 - \nu)\frac{1}{L^2} \|\nabla f(X_k)\|^2$ term is the same as (4.13) where we replace $1_R$ with $1_U$ since Lemma 4.4 still applies. Lastly with an unreliable step, the change in $\Delta_k^2$ is $1_{I_k \cap J_k} \mathbf{1}_{\text{Succ}}1_U(1 - \nu)\theta(\Delta_{k+1}^2 - \Delta_k^2) \leq -1_{I_k \cap J_k} \mathbf{1}_{\text{Succ}}1_U(1 - \nu)(1 - \gamma^{-1})\theta \Delta_k^2. \quad (4.17)$

Therefore by choosing $\nu$ such that (4.42) holds, we have that

$$1_{I_k \cap J_k} \mathbf{1}_{\text{Succ}}1_U(\Phi_{k+1} - \Phi_k) \leq -1_{I_k \cap J_k} \mathbf{1}_{\text{Succ}}1_U \left(\frac{\nu \theta A_k}{8L^2(\kappa_g \alpha_{\max} + 1)^2} \|\nabla f(X_k)\|^2 + (1 - \nu)(1 - \gamma^{-1})\theta \Delta_k^2 \right). \quad (4.18)$$

(iii). Unsuccessful iterate ($1_{\text{Succ}} = 1$). Because the iterate is unsuccessful, the change in the function values is 0 and the constants $A_k$ and $\Delta_k^2$ decrease. Consequently, we deduce that

$$1_{I_k \cap J_k} \mathbf{1}_{\text{Succ}}(\Phi_{k+1} - \Phi_k) \leq -1_{I_k \cap J_k} \mathbf{1}_{\text{Succ}}(1 - \nu)(1 - \gamma^{-1}) \left(\frac{A_k}{L^2} \|\nabla f(X_k)\|^2 + \theta \Delta_k^2 \right). \quad (4.19)$$

We chose $\nu$ sufficiently large to ensure that the third case (iii), unsuccessful iterate (4.19), provides the worst case decrease when compared to (4.16) and (4.18). Specifically $\nu$ is chosen so that

$$-\frac{\nu \theta A_k}{8L^2(\kappa_g \alpha_{\max} + 1)^2} \|\nabla f(X_k)\|^2 \leq -(1 - \nu)(1 - \gamma^{-1}) \frac{A_k}{L^2} \|\nabla f(X_k)\|^2 \quad (4.20)$$

and $-\frac{\nu \theta A_k}{16} \|\nabla f(X_k)\|^2 \leq -(1 - \nu)(1 - \gamma^{-1})\theta \Delta_k^2.$

As such, we bounded the change in $\Phi_k$ in the case of accurate gradients and estimates by

$$1_{I_k \cap J_k}(\Phi_{k+1} - \Phi_k) \leq -1_{I_k \cap J_k}(1 - \nu)(1 - \gamma^{-1}) \left(\frac{A_k}{L^2} \|\nabla f(X_k)\|^2 + \theta \Delta_k^2 \right). \quad (4.21)$$

We take conditional expectations with respect to $F^{G,F}_{k-1}$ and using Assumption 2.4, equation (4.11) holds.
As before, we consider three separate cases.

\( k \)

The right hand sides of (4.23) and (4.24) are trivially upper bounded by the positive term \( A_k \Phi_k \).

Inequality (4.22) follows by taking expectations with respect to \( f \), which can lead to an increase in \( \Phi_k \). By adjusting the probability of outcome (Case (3)) to be sufficiently small, we can ensure that in expectation \( \Phi_k \) is upper bounded by the positive term \( A_k \Phi_k \).

Case 2 (Bad gradients and accurate estimates, \( 1_{I_k^c \cap J_k} = 1 \)) Unlike the previous case, \( \Phi_k \) may increase, since the step along an inaccurate probabilistic gradients may not provide enough decrease to cancel the increase from the \( \| \nabla f(X_k) \|^2 \). Precisely, the successful and unreliable case dominates the worst case increase in \( \Phi_k \):

\[
E[1_{I_k^c \cap J_k}(\Phi_{k+1} - \Phi_k)|F_{k-1}^G] \leq (1 - p_y)(1 - \nu)\frac{2\gamma A_k}{L^2} \| \nabla f(X_k) \|^2. \tag{4.22}
\]

As before, we consider three separate cases.

(i) **Successful and reliable step** \( (1_{\text{Succ}}_{1_R} = 1) \). A successful, reliable step with accurate function estimates but bad gradients has functional improvement (Lemma 4.3, equation (4.2)):

\[
1_{I_k^c \cap J_k} 1_{\text{Succ}}_{1_R}(f(X_{k+1}) - f(X_k)) \leq -1_{I_k^c \cap J_k} 1_{\text{Succ}}_{1_R} \nu \left( \frac{A_k \theta \| G_k \|^2}{4} + \frac{\theta \Delta_k^2}{4} \right).
\]

In contrast to (4.12), we lose the \( \| \nabla f(X_k) \|^2 \) term. A reliable, successful step increases both constants \( A_{k+1} \) and \( \Delta_{k+1}^2 \), leading to (4.13) and (4.14) with \( 1_{I_k \cap J_k} \) replaced by \( 1_{I_k^c \cap J_k} \). Hence by choosing \( \nu \) to satisfy (4.42), the dominant term in \( \Phi_k \) is

\[
1_{I_k^c \cap J_k} 1_{\text{Succ}}_{1_R}(\Phi_{k+1} - \Phi_k)
\]

\[
\leq 1_{I_k^c \cap J_k} 1_{\text{Succ}}_{1_R} \left( -\frac{\nu \theta A_k}{16} \| G_k \|^2 - \frac{\nu \theta}{16} \Delta_k^2 + \frac{2\gamma(1 - \nu)}{L^2} A_k \| \nabla f(X_k) \|^2 \right). \tag{4.23}
\]

(ii) **Successful and unreliable step** \( (1_{\text{Succ}}_{1_U} = 1) \). Lemma 4.3 holds, but this time equation (4.1) for unreliable steps applies. Moreover, (4.13) and (4.14) that bound the change in the last two terms of \( \Phi_k \) also apply. Again by choosing \( \nu \) to satisfy (4.42), we deduce

\[
1_{I_k^c \cap J_k} 1_{\text{Succ}}_{1_U}(\Phi_{k+1} - \Phi_k)
\]

\[
\leq 1_{I_k^c \cap J_k} 1_{\text{Succ}}_{1_U} \left( -\frac{\nu \theta A_k}{16} \| G_k \|^2 - (1 - \nu)(1 - \gamma^{-1})\theta \Delta_k^2 + \frac{2\gamma(1 - \nu)}{L^2} A_k \| \nabla f(X_k) \|^2 \right). \tag{4.24}
\]

(iii) **Unsuccessful** \( (1_{\text{Succ}} = 1) \). As in the previous case, equation (4.19) holds.

The right hand sides of (4.23) and (4.24) and (4.19) are trivially upper bounded by the positive term \( A_k \| \nabla f(X_k) \|^2 \). Hence, we conclude that

\[
1_{I_k^c \cap J_k}(\Phi_{k+1} - \Phi_k) \leq 1_{I_k^c \cap J_k} \frac{2\gamma(1 - \nu)}{L^2} A_k \| \nabla f(X_k) \|^2. \tag{4.25}
\]

Inequality (4.22) follows by taking expectations with respect to \( F_{k-1}^G \) and noting that \( E[1_{I_k^c \cap J_k} | F_{k-1}^M] \leq 1 - p_y \) as in Assumption 2.4.

Case 3 (Bad estimates, \( 1_{I_k} = 1 \)). Inaccurate estimates can cause the algorithm to accept a step which can lead to an increase in \( f \), \( A \), and \( \Delta \) and hence in \( \Phi_k \). We control this increase in \( \Phi_k \) by bounding the variance in the function estimates, as in (2.23), which is the key reason for Assumption 2.4(iii). By adjusting the probability of outcome (Case (3)) to be sufficiently small, we can ensure that in expectation \( \Phi_k \) is sufficiently reduced. Precisely, we will show

\[
E[1_{I_k}(\Phi_{k+1} - \Phi_k)|F_{k-1}^G] \leq 2\nu(\sqrt{1 - p_f}) \max \{ \kappa_f A_k \| \nabla f(X_k) \|^2, \theta \Delta_k^2 \}
\]

\[
+ (1 - p_f)(1 - \nu)\frac{2\gamma}{L^2} A_k \| \nabla f(X_k) \|^2. \tag{4.26}
\]

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A successful step leads to the following bound
\[
1_{J_k^1}1_{\text{Succ}}\nu(f(X_{k+1}) - f(X_k)) \leq 1_{J_k^1}1_{\text{Succ}}\nu((F_k^s - F_k^0) + |f(X_{k+1}) - F_k^s| + |F_k^0 - f(X_k)|)
\]
\[
\leq 1_{J_k^1}1_{\text{Succ}}\nu \left( -\theta A_k \|G_k\|^2 + |f(X_{k+1}) - F_k^s| + |F_k^0 - f(X_k)| \right),
\]
(4.27)

where the last inequality is due to the sufficient decrease condition. As before, we consider three separate cases.

(i). **Successful and reliable step** \((1_{\text{Succ}}1_R = 1)\). With a reliable step we have \(-A_k\|G_k\|^2 \leq -\Delta_k^2\), thus (4.27) implies
\[
1_{J_k^1}1_{\text{Succ}}1_R\nu(f(X_{k+1}) - f(X_k)) \leq 1_{J_k^1}1_{\text{Succ}}1_R\nu \left( -\frac{\theta}{2}A_k \|G_k\|^2 - \frac{\theta}{2}\Delta_k^2 + |f(X_{k+1}) - F_k^s| + |F_k^0 - f(X_k)| \right).
\]

We note that \(\Phi_{k+1} - \Phi_k\) is upper bounded by the right hand side of the above inequality and the right hand sides of (4.13) and (4.14). As before, by choosing \(\nu\) as in (4.42) we ensure \(-\nu\theta A_k \|G_k\|^2 + (1 - \nu)2\gamma A_k\alpha_{\text{max}} \|G_k\|^2 \leq 0\) and \(-\nu\theta \Delta_k^2 + (1 - \nu)(\gamma - 1)\theta \Delta_k^2 \leq 0\). It follows that
\[
1_{J_k^1}1_{\text{Succ}}1_R(\Phi_{k+1} - \Phi_k) \leq 1_{J_k^1}(\nu|f(X_{k+1}) - F_k^s| + \nu|F_k^0 - f(X_k)| + (1 - \nu)\frac{\nu\theta}{2}\Delta_k^2 A_k \|\nabla f(X_k)\|^2).
\]
(4.28)

(ii). **Successful and unreliable step** \((1_{\text{Succ}}1_U = 1)\). Since on unreliable steps, \(\Delta_{k+1}^2\) is decreased, then the increase in \(\Phi_k\) is always smaller than the worst-case increase we just derived for the successful and reliable step. Thus (4.28) holds with \(1_R\) replaced by \(1_U\).

(iii). **Unsuccessful** \((1_{\text{Succ}}^c = 1)\) As we decrease both \(\Delta\) and \(A\), and \(X_{k+1} = X_k\), we conclude that (4.19) hold.

The equation (4.28) dominates (4.19); thus in all three cases (4.28) holds. We take expectations of (4.28) and apply Lemma 2.5 to conclude that
\[
\mathbb{E}[1_{J_k^1}(\Phi_{k+1} - \Phi_k)\mathcal{F}_{k-1}^{G,F}] \leq 2\nu(1 - p_f)^{1/2} \max\{\kappa_f A_k \|\nabla f(X_k)\|^2, \theta \Delta_k^2\}
\]
\[
\leq (1 - p_f)(1 - \nu)\frac{\nu\theta}{2}\|\nabla f(X_k)\|^2.
\]
(4.29)

Now we combine the expectations (4.11, 4.22), and (4.26) to obtain
\[
\mathbb{E}[\Phi_{k+1} - \Phi_k|\mathcal{F}_{k-1}^{G,F}] = \mathbb{E}[(1_{I_k^1}1_{J_k^1} + 1_{I_k^1\cap J_k^1} + 1_{J_k^1})(\Phi_{k+1} - \Phi_k)|\mathcal{F}_{k-1}^{G,F}]
\]
\[
\leq -p_g p_f(1 - \nu)(1 - \nu)(1 - \gamma^{-1})(\frac{\Delta_k}{L^2} \|\nabla f(X_k)\|^2 + \theta \Delta_k^2) + p_f(1 - p_g)^{2\nu(1 - \nu)A_k}{\|\nabla f(X_k)\|^2}
\]
\[
+ 2\nu(1 - p_f)^{1/2} \left( \kappa_f A_k \|\nabla f(X_k)\|^2 + \theta \Delta_k^2 \right) + (1 - p_f)^{1/2} \frac{2\nu(1 - \nu)A_k}{L^2} \|\nabla f(X_k)\|^2
\]

where the inequality follows from \(1 - p_f \leq (1 - p_f)^{1/2}\) and \(1 - p_g = p_f(1 - p_g) + (1 - p_f)(1 - p_g) \leq p_f(1 - p_g) + (1 - p_f)^{1/2}\). Let us choose \(p_g \in (0, 1]\) so that (4.40) holds which implies
\[
\left( -p_g p_f \frac{(1 - \nu)(1 - \gamma^{-1})A_k}{L^2} + p_f(1 - p_g)^{2\nu(1 - \nu)A_k}{\|\nabla f(X_k)\|^2} \right) \|\nabla f(X_k)\|^2 \leq -p_g p_f \frac{(1 - \nu)(1 - \gamma^{-1})A_k}{2L^2} \|\nabla f(X_k)\|^2.
\]
We have now reduced the number of terms in the conditional expectation

\[ E[\Phi_{k+1} - \Phi_k | F_{k-1}^G] \leq -p_g p_f \frac{1}{2}(1 - \nu)(1 - \gamma^{-1}) \left( \frac{A_k}{L^2} \| \nabla f(X_k) \|^2 + \theta \Delta_k^2 \right) \]

\[ + 2\nu(1 - p_f)^{1/2} \left( \kappa_f A_k \| \nabla f(X_k) \|^2 + \theta \Delta_k^2 \right) + (1 - p_f)^{1/2} \frac{A_k}{L^2} \| \nabla f(X_k) \|^2. \]

We choose \( p_f \in (0, 1) \) large enough, so that \( \frac{p_g p_f}{\sqrt{1 - p_f}} \) satisfies (4.41) which implies

\[ \left( -\frac{p_g p_f (1 - \nu)(1 - \gamma^{-1})}{2L^2} + (1 - p_f)^{1/2} \left( 2\nu \kappa_f + \frac{4\gamma(1 - \nu)}{L^2} \right) \right) A_k \| \nabla f(X_k) \|^2 \leq -\frac{p_g p_f (1 - \nu)(1 - \gamma^{-1})}{4L^2} A_k \| \nabla f(X_k) \|^2 \]

\[ -p_g p_f \left( 1 - \nu \right) \left( 1 - \gamma^{-1} \right) \theta \Delta_k^2 + 2\nu(1 - p_f)^{1/2} \theta \Delta_k^2 \leq -p_g p_f \frac{1}{2}(1 - \nu)(1 - \gamma^{-1}) \theta \Delta_k^2. \]

The proof is complete.

\[ \square \]

**Remark 4.** To simplify the expression for the constants we will assume that \( \theta = 1/2 \) and \( \gamma = 2 \) which are typical values for these constants. We also assume that without loss of generality \( \kappa_g \geq 2 \) and \( \nu \geq 1/2 \). The analysis can be performed for any other values of the above constants - the choices here are for simplicity and illustration. The conditions on \( p_g \) and \( p_f \) under the above choice of constants will be shown in our results.

**Theorem 4.7.** Let Assumptions 2.1 and 2.4 hold and chose constants as in Remark 4. Suppose \( \{X_k, G_k, F_k^0, F_k^s, A_k, \Delta_k\} \) is the random process generated by Algorithm 2. Then there exists probabilities \( p_g, p_f > 1/2 \) and a constant \( \nu \geq 1/2 \) such that the expected decrease in \( \Phi_k \) is

\[ E[\Phi_{k+1} - \Phi_k | F_{k-1}^G] \leq -\frac{1}{8192 (\kappa_g \alpha_{max} + 1)^2} \left( \frac{A_k}{L^2} \| \nabla f(X_k) \|^2 + \frac{1}{2} \Delta_k^2 \right). \]

(4.30)

In particular, the constant \( \nu \) and probabilities \( p_g, p_f > 1/2 \) must satisfy

\[ \frac{\nu}{1 - \nu} = 64(\kappa_g \alpha_{max} + 1)^2, \]

(4.31)

\[ p_g \geq \frac{16}{17}, \]

(4.32)

and

\[ \frac{p_g p_f}{\sqrt{1 - p_f}} \geq \max \left\{ 1024 \kappa_f L^2 (\kappa_g \alpha_{max} + 1)^2 + 64, 1024 (\kappa_g \alpha_{max} + 1)^2 \right\}. \]

(4.33)

**Proof.** We plug in the values for \( \gamma \) and \( \theta \) and use the fact that \( \kappa_g \geq 2 \) to obtain the expression for \( \nu/(1 - \nu) \) and \( p_g \). In order to deduce the expression for \( p_g p_f/(1 - p_f)^{1/2} \), we assume that \( \nu/(1 - \nu) = 64(\kappa_g \alpha_{max} + 1)^2 \). Lastly, we suppose \( \nu > 1/2 \) and \( p_g p_f \geq 1/2 \) and \( \frac{p_g p_f}{\sqrt{1 - p_f}} \geq \max \{ 1024 \kappa_f L^2 (\kappa_g \alpha_{max} + 1)^2 + 64, 1024 (\kappa_g \alpha_{max} + 1)^2 \} \). Therefore, we have

\[ \frac{-p_g p_f (1 - \nu)(1 - \gamma^{-1})}{4} \leq \frac{-(1 - \nu)}{64} \leq \frac{-\nu}{4096 (\kappa_g \alpha_{max} + 1)^2} \leq \frac{1}{8192 (\kappa_g \alpha_{max} + 1)^2}. \]

The result is shown.

\[ \square \]

**Corollary 4.8.** Let the same assumptions as Theorem 4.6 hold. Suppose \( \{X_k, G_k, F_k^0, F_k^s, A_k, \Delta_k\} \) is the random process generated by Algorithm 2. Then we have

\[ \sum_{k=0}^{\infty} E[A_k \| \nabla f(X_k) \|^2] < \infty. \]

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We define two new random variables $R_b$ and $Z_b$ that are measurable with respect to the same $\sigma$-algebra namely $\mathcal{F}_{k-1}$ for $k \geq 0$. We establish a contradiction by proving that $R_b$ and $Z_b$ are null sets. By construction, the random variables $R_b$ and $Z_b$ are measurable with respect to the same $\sigma$-algebra namely $\mathcal{F}_{k-1}$ for $k \geq 0$. We next show that $R_b \geq Z_b$.

The following proceeds by induction whereas the base case is given by definition. Without loss

**Proof of Corollary 4.8.** By taking expectations of (4.38) and summing up, we deduce

$$\tilde{C} \sum_{k=0}^{\infty} \mathbb{E}[\mathcal{A}_k \| \nabla f(X_k) \|^2] \leq \sum_{k=0}^{\infty} \mathbb{E}[\Phi_k] - \mathbb{E}[\Phi_{k+1}] \leq \Phi_0 < \infty,$$

where $\tilde{C}$ is the constant in front of the $\mathcal{A}_k \| \nabla f(X_k) \|^2$ in (4.38).

**4.3 The liminf convergence**

We are ready to prove the liminf-type of convergence result, i.e. a subsequence of the iterates drive the gradient of the objective function to zero. The proof closely follows [2,7] for trust regions; we adapt their proofs to handle line search.

We set

$$\bar{A} = \xi^{-1} \text{ where } \xi \geq \max \left\{ \frac{\kappa_0 + L/2 + 2\varepsilon f}{1 - \theta}, \frac{1}{\alpha_{\max}} \right\}.$$

For simplicity and without loss of generality, we assume that $\mathcal{A}_0 = \gamma^i \bar{A}$ and $\alpha_{\max} = \gamma^j \bar{A}$ for integers $i, j > 0$. In this case, for any $k$, $\mathcal{A}_k = \gamma^j \bar{A}$ for some integer $i$.

**Theorem 4.9.** Let the assumptions of Theorem 4.6 hold. Then the sequence of random iterates generated by Algorithm 1, $\{X_k\}$, almost surely satisfy

$$\liminf_{k \to \infty} \| \nabla f(X_k) \| = 0.$$

**Proof.** We prove this result by contradiction. With positive probability, there exists constants $\mathcal{E}(\omega) > 0$ and $K_0(\omega)$ such that $\| \nabla f(X_k) \| > \mathcal{E}$ for all $k \geq K_0$. Because of Corollary 4.8 following Theorem 4.6 we have that $\mathcal{A}_k \| \nabla f(X_k) \|^2 \to 0$ a.e. Hence, we have

$$\mathbb{P}(\{\omega : \| \nabla f(X_k) \| \geq \mathcal{E} \text{ for all } k \geq K_0 \text{ and } \lim_{k \to \infty} \mathcal{A}_k \| \nabla f(X_k) \|^2 = 0\}) > 0.$$

Let $\{x_k\}, \{\alpha_k\}, \varepsilon$, and $k_0$ be the realizations of $\{X_k\}, \{\mathcal{A}_k\}, \mathcal{E}$, and $K_0$, respectively, for which $\| \nabla f(x_k) \| > \varepsilon$ for all $k \geq k_0$ and $\lim_{k \to \infty} \alpha_k \| \nabla f(x_k) \|^2 = 0$. An immediate consequence is that $\alpha_k \to 0$. Consequently, we deduce that

$$0 < \mathbb{P}(\{\omega : \| \nabla f(X_k) \| \geq \mathcal{E} \text{ for all } k \geq K_0 \text{ and } \lim_{k \to \infty} \mathcal{A}_k \| \nabla f(X_k) \|^2 = 0\}) \leq \mathbb{P}(\{\omega : \lim_{k \to \infty} \mathcal{A}_k = 0\}).$$

We define two new random variables $R_k = \log(\mathcal{A}_k)$ and $Z_k$ defined by the recursion

$$Z_{k+1} = \min \left\{ \log(\bar{A}), 1_{I_k} 1_{J_k} (\log(\gamma) + Z_k) + (Z_k - \log(\gamma))(1 - 1_{I_k} J_k) \right\} \quad \text{and} \quad Z_0 = R_0 = \log(\alpha_0).$$

We observe that $Z_k$ is bounded from below and that $R_k$, by our assumption has a positive probability of diverging to $-\infty$. We establish a contradiction by proving that $R_k \geq Z_k$, which is what we do below.

The sequence of random variables increase by $\log(\gamma)$, unless it hits the maximum, with probability $p_f p_g$ and otherwise decreases by $\log(\gamma)$. Our main argument is to show that $\{\omega : \lim_{k \to \infty} \mathcal{A}_k = 0\} = \{\omega : \lim_{k \to \infty} R_k = \infty\}$ are null sets. By construction, the random variables $R_k$ and $Z_k$ are measurable with respect to the same $\sigma$-algebra namely $\mathcal{F}_{k-1}$ for $k \geq 0$. We next show that $R_k \geq Z_k$.
of generality assume there exists a \( j \in \mathbb{Z} \) such that \( \gamma^j \alpha_0 = \bar{A} \). Assume the induction hypothesis, namely, \( R_k \geq Z_k \). If \( R_k > \log(\bar{A}) \), then

\[
R_{k+1} \geq \log(\gamma^{-1} A_k) = R_k - \log(\gamma) \geq \log(\bar{A}) \geq Z_k,
\]

where the third inequality follows because the assumption \( R_{k+1} \) strictly larger than \( \log(\bar{A}) \) implies that \( R_{k+1} \geq \log(\bar{A}) + \log(\gamma) \). Now suppose \( R_{k+1} \leq \log(\bar{A}) \) and we consider some cases. If \( 1_{I_k} \neq \emptyset \), then by Lemma 4.2 we know \( R_{k+1} = \log(\alpha_{\text{max}}) \). Suppose \( R_{k+1} = \log(\alpha_{\text{max}}) \). Then by definition of \( \bar{A} \) and \( Z_{k+1} \), \( R_k \geq Z_{k+1} \). On the other hand, suppose \( R_{k+1} = R_k + \log(\gamma) \). Then by the induction hypothesis, we have \( R_{k+1} \geq Z_k + \log(\gamma) \geq \min\{\bar{A}, Z_k + \log(\gamma)\} = Z_{k+1} \). Next, suppose \( 1_{I_k} = \emptyset \). It follows that \( Z_{k+1} = Z_k - \log(\gamma) \geq R_k - \log(\gamma) = \log(A_k \gamma^{-1}) \leq R_{k+1} \). Therefore, we showed that \( R_k \geq Z_k \) for all \( k \geq 0 \). Moreover, we see that \( \{Z_k\} \) is a random walk with a maximum and a drift upward. Therefore,

\[
1 = \mathbb{P}(\limsup_k Z_k \geq \log(\bar{A})) = \mathbb{P}(\limsup_k R_k \geq \log(\bar{A})).
\]

However, this contradicts the fact that \( \mathbb{P}(\omega : \limsup_k R_k = -\infty) > 0 \). \( \square \)

### 4.4 Convergence rates for the nonconvex case

Our primary goal in this paper is to bound the expected number of steps that the algorithm takes until \( \|\nabla f(X_k)\| \leq \varepsilon \). Define the stopping time

\[
T_\varepsilon = \inf\{k \geq 0 : \|\nabla f(X_k)\| < \varepsilon\}.
\]

We show in this section, under the simplified assumptions on the constant from Theorem 4.7,

\[
\mathbb{E}[T_\varepsilon] \leq O(1) \cdot \frac{p_0 p_f}{2 p_0 p_f - 1} \cdot L^3 (\kappa_g \alpha_{\text{max}} + 1)^2 \Phi_0 + 1.
\]

Here \( O(1) \) hides universal constants and dependencies on \( \theta, \gamma, \alpha_{\text{max}} \). We derive this result from Theorem 3.3, therefore, the remainder of this section is devoted to showing Assumption 3.2 holds. Given Theorem 4.6, it is immediate the random variable \( \Phi_k \) defined, as in equation (4.6), satisfies Assumption 3.2 (iii) by multiplying both side by the indicator, \( 1_{\{T_\varepsilon > k\}} \). In particular, we define the function \( h(A_k) = A_k \varepsilon^2 \) (i.e. \( \approx A_k \|\nabla f(X_k)\|^2 \)) to obtain from Theorem 4.6

\[
\mathbb{E}[1_{\{T_\varepsilon > k\}} (\Phi_{k+1} - \Phi_{k}) | J_{k-1}] \leq -\Theta h(A_k) 1_{\{T_\varepsilon > k\}},
\]

where \( \Theta = \frac{1}{2 \varepsilon^2 L^2 (\alpha_{\text{max}} + 1)^2} \). It remains to show Assumption 3.2 (ii) holds.

**Lemma 4.10.** Let \( p_g \) and \( p_f \) be such that \( p_g p_f \geq 1/2 \) then Assumption 3.2 (ii) is satisfied for \( W_k = 2(1_{I_k \cap J_k} - 1/2) \), \( \lambda = \log(\gamma) \), and \( p = p_g p_f \).

**Proof.** By the choice of \( \bar{A} \) we have that \( \bar{A} = A_0 e^{\lambda j} \) for some \( j \in \mathbb{Z} \) and \( j \leq 0 \). It remains to show that

\[
1_{\{T_\varepsilon > k\}} A_{k+1} \geq 1_{\{T_\varepsilon > k\}} \min\{\bar{A}, \min\{\alpha_{\text{max}}, \gamma A_k\} I_k J_k + \gamma^{-1} A_k (1 - 1_{I_k \cap J_k})\}.
\]

Suppose \( A_k > \bar{A} \). Then \( A_k \geq \gamma \bar{A} \) and hence \( A_{k+1} \geq \bar{A} \). Now, assume that \( A_k \leq \bar{A} \). By definition of \( \xi \), we have that

\[
A_k \leq \frac{1 - \theta}{\kappa_g + L/2 + 2 \varepsilon_f}.
\]

Assume that \( I_k = 1 \) and \( J_k = 1 \). It follows from Lemma 4.2 that the iteration \( k \) is successful, i.e., \( x_{k+1} = x_k + s_k \) and \( \alpha_{k+1} = \max\{\alpha_{\text{max}}, \gamma \alpha_k\} \). If \( I_k J_k = 0 \), then \( \alpha_{k+1} \geq \gamma^{-1} \alpha_k \). \( \square \)
Finally substituting the expressions for $h$, $\bar{A}$, and $\Phi_k$ into the bound on $\mathbf{E}[T_\varepsilon]$ from Theorem 3.3, we obtain the following complexity result.

**Theorem 4.11.** Under the assumptions in Theorem 4.7, suppose the probabilities $p_g, p_f > 1/2$ satisfy

$$p_g \geq \frac{16}{17} \quad \text{and} \quad \frac{p_g p_f}{\sqrt{1 - p_f}} \geq \max \left\{ 1024 \kappa_f L^2 (\kappa_g \alpha_{\max} + 1)^2 + 64, 1024 (\kappa_g \alpha_{\max} + 1)^2 \right\},$$

with $\frac{\nu}{1 - \nu} = 64 (\kappa_g \alpha_{\max} + 1)^2$. Then the expected number of iterations that Algorithm 1 takes until $\|\nabla f(X_k)\|^2 \leq \varepsilon$ occurs is bounded as follows

$$\mathbf{E}[T_\varepsilon] \leq \frac{p_g p_f}{2 p_g p_f - 1} \frac{L^2 (\kappa_g + L/2 + 2 \varepsilon_f) (\kappa_g \alpha_{\max} + 1)^2}{\Theta \xi^2} \Phi_0 + 1,$$

where $\Theta = 1/16384$ and $\Phi_0 = \nu (f(X_0) - f_{\min}) + (1 - \nu) (1/L A_0 \|\nabla f(X_0)\|^2 + 1/2 \Delta_0^2)$.

**4.5 Convex Case**

We now analyze line search (Algorithm 1) under the setting that the objective function is convex.

**Assumption 4.12.** Suppose in addition to Assumption 2.1, $f$ is convex. Let $x^*$ denote the global minimizer of $f$ and $f^* = f(x^*)$. We assume there exists a constant $D$ such that

$$\|x - x^*\| \leq D \quad \text{for all} \ x \in \Omega,$$

where $\Omega$ is the set that contains all iteration realizations as stated in Assumption 2.1. Moreover, we assume there exists a $L_f > 0$ such that $\|\nabla f(x)\| \leq L_f$ for all $x \in \Omega$.

**Remark 5.** In deterministic optimization it is common to assume that function $f$ has bounded level sets and that all iterates remain within the bounded set defined by $f(x) \leq f(x_0)$. For the stochastic case, it is not guaranteed that all iterates remain in the bounded level set because it is possible to take steps that increase the function value. Clearly iterates remain in a (large enough) bounded set with high probability. Alternatively, if it is known that the optimal solution lies within some bounded set, Algorithm 1 can be simply modified to project iterates onto that set. This modified version for the convex case can be analyzed in almost identical way as is done in Theorem 4.6. However, for simplicity of the presentation, for the convex case we simply impose Assumption 4.12.

In convex setting, the goal is to bound the expected number of iterations $T_\varepsilon$ of Algorithm 1 until

$$f(x_k) - f^* < \varepsilon.$$

In deterministic case, the complexity bound is derived by showing that $1/(f(x_k) - f^*)$ has a constant decrease, until the $\varepsilon$-accuracy is reached. For the randomized line search we follow the same idea, replacing $f(x_k) - f^*$ with $\Phi_k$ (modified by substituting $f_{\min}$ in (4.6) by $f^*$) and defining the function

$$\Psi_k = \frac{1}{\nu \varepsilon} - \frac{1}{\Phi_{k \land T_\varepsilon}} \quad \text{(4.34)}$$

We show the random process $\{\Psi_k, A_k\}$ satisfies Assumption 3.2 for all $k$. To simplify the argument, we impose an upper bound on $\Delta_k$.

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4We use $a \wedge b = \min\{a, b\}$. 

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Assumption 4.13. Suppose there exists a constant $\delta_{\max}$ such that the random variable $\Delta_k \leq \delta_{\max}$. First, with a simple modification to Algorithm 1 we can impose this assumption. Second, the dynamics of the algorithm suggest $\Delta_k$ eventually decreases until it is smaller than any $\varepsilon > 0$.

The random variables $A_k$ behaves the same as in the nonconvex setting. We ensure the positivity of the random process $\{\Psi_k\}$ by incorporating the stopping time $T_k$ directly into the function $\Psi$; hence the dependency on $\varepsilon$ for convergence rates is built directly into the function $\Psi$. The main component of this section is proving Assumption 3.2 (ii) holds for this $\Psi$; hence the dependency on $\varepsilon$.

Theorem 4.14. Let Assumptions 2.1, 2.4, 4.12, and 4.13 hold. Suppose $\{X_k, G_k, F^0_k, F^e_k, A_k, \Delta_k\}$ is the random process generated by Algorithm 1 Then there exists probabilities $p_g$ and $p_f$ and a constant $\nu \in (0, 1)$ such that

$$1_{\{T_k > k\}} E[\Psi_{k+1} - \Psi_k|F^G_{k-1}] \leq -\frac{p_g p_f (1 - \nu)(1 - \gamma^{-1})}{8(\nu DL + (1 - \nu)\alpha_{\max} L_f) + (1 - \nu)\sqrt{\theta} \delta_{\max}} A_k 1_{\{T_k > k\}},$$

where $\Psi_k$ is defined in (4.34). In particular, the probabilities $p_g$ and $p_f$ and constant $\nu$ satisfy (4.39), (4.40), and (4.41) from Theorem 4.6.

Proof. First, by convexity, we have that

$$\Phi_k = \nu(f(X_k) - f^*) + (1 - \nu)A_k \frac{\|\nabla f(X_k)\|^2}{L^2} + (1 - \nu)\theta \Delta_k$$

$$\leq \nu\langle \nabla f(X_k), X_k - x^* \rangle + (1 - \nu)\alpha_{\max} \frac{\|\nabla f(X_k)\|^2}{L^2} + (1 - \nu)\theta \delta_{\max} \Delta_k$$

$$\leq \left(\nu DL + (1 - \nu)\alpha_{\max} L_f + (1 - \nu)\sqrt{\theta} \delta_{\max}\right) \left(\|\nabla f(X_k)\| + \sqrt{\theta} \Delta_k\right),$$

where we used $\|\nabla f(X_k)\| < L_f$. Without loss of generality, we assume $\alpha_{\max} \leq 1$; one may prove the same result with any stepsize, but for the sake simplicity we will defer to the standard case when $\alpha_{\max} \leq 1$. By squaring both sides, we conclude

$$\frac{A_k \Phi_k^2}{C} := \frac{A_k \Phi_k^2}{2(\nu DL + (1 - \nu)\alpha_{\max} L_f + (1 - \nu)\sqrt{\theta} \delta_{\max})^2} \leq \frac{A_k \|\nabla f(X_k)\|^2}{L^2} + \theta \Delta_k^2,$$

where we used the inequality $(a + b)^2 \leq 2(a^2 + b^2)$. From the above inequality combined with (4.38) we have

$$E[1_{\{T_k > k\}}(\Phi_{k+1} - \Phi_k)|F^G_{k-1}] \leq -\frac{p_g p_f (1 - \nu)(1 - \gamma^{-1})A_k \Phi_k^2}{4C} 1_{\{T_k > k\}}.$$

Now using the simple fact that $1_{\{T_k > k\}}(\Phi_{k+1} - \Phi_k) = \Phi_{(k+1)\wedge T_k} - \Phi_{k\wedge T_k}$ we can write

$$E[\Phi_{(k+1)\wedge T_k} - \Phi_{k\wedge T_k}|F^G_{k-1}] \leq -\frac{p_g p_f (1 - \nu)(1 - \gamma^{-1})A_k \Phi_k^2}{4C} 1_{\{T_k > k\}}.$$

We can then use Jensen’s inequality to derive

$$\mathbb{E} \left[ \frac{1}{\Phi_{k\wedge T_k}} - \frac{1}{\Phi_{(k+1)\wedge T_k}} \mid F^G_{k-1} \right] \leq \frac{1}{\Phi_{k\wedge T_k}} - \mathbb{E}[\Phi_{(k+1)\wedge T_k} |F^G_{k-1}] = \frac{\mathbb{E}[\Phi_{(k+1)\wedge T_k} - \Phi_{k\wedge T_k} |F^G_{k-1}]}{\Phi_{k\wedge T_k} \mathbb{E}[\Phi_{(k+1)\wedge T_k} |F^G_{k-1}]} \cdot 1_{\{T_k > k\}}$$

$$\leq -\frac{p_g p_f (1 - \nu)(1 - \gamma^{-1})A_k \Phi_k^2}{4C} \cdot 1_{\{T_k > k\}}.$$
where the last inequality follows from $E[\Phi_{(k+1)\wedge T_e} | \mathcal{F}^G_{k-1}] \leq \Phi_{k\wedge T_e}$. The result follows after noting that $1_{\{T_e > k\}}(\Psi_{k+1} - \Psi_k) = \Phi^{-1}_{k\wedge T_e} - \Phi^{-1}_{(k+1)\wedge T_e}$.

The expected improvement in $\Psi_k$ allows us to use Theorem 3.3 which directly gives us the convergence rate.

**Theorem 4.15.** Let the assumptions of Theorem 4.14 hold with constant $\nu$ and probabilities $p_gp_f$ as in Theorem 4.14. Then the expected number of iterations that Algorithm 1 takes until $f(X_k) - f^* < \varepsilon$ is bounded as follows

$$E[T\varepsilon] \leq O(1) \cdot \frac{p_g p_f}{2p_g p_f - 1} \cdot \left( \frac{(\kappa_g \alpha_{\max} + 1)^2 (\kappa_g + L + \varepsilon_f)(\nu DL + \frac{(1 - \nu)\alpha_{\max} L_f}{L} + (1 - \nu)\sqrt{d_{\max}})^2}{\varepsilon} + 1 \right).$$

The bound in Theorem 4.15 can be further simplified as follows

$$E[T\varepsilon] \leq O(1) \cdot \frac{p_g p_f}{2p_g p_f - 1} \left( \frac{L^3 \kappa_g^3 (D^2 + L_f^2 + d_{\max}^2)}{\varepsilon} \right).$$

### 4.6 Strongly convex case

Lastly, we analyze the stochastic line search (Algorithm 1) under the setting that the objective function is strongly convex. As such, we assume the following is now true of the objective function while dropping Assumption 4.12 and the bound on $\Delta_k$.

**Assumption 4.16.** Suppose that in addition to Assumption 2.1, $f$ is $\mu$-strongly convex, namely for all $x, y \in \mathbb{R}^n$ the following inequality holds

$$f(x) \geq f(y) + \nabla f(y)^T (x - y) + \frac{\mu}{2} \|x - y\|^2.$$

Our goal, like the convex setting, is to bound the expected number of iterations $T\varepsilon$ until $f(x) - f^* < \varepsilon$. We show that this bound is of the order of $\log(1/\varepsilon)$, as in the deterministic case. Our proof follows the same technique used in deterministic which relies on showing that $\log(f(x_k) - f^*)$ decreases by a constant at each iteration. Here, instead of tracking the decrease in $\log(f(x_k) - f^*)$, we define the function

$$\Psi_k = \log(\Phi_{k\wedge T_e}) + \log \left( \frac{1}{\nu \varepsilon} \right).$$

(4.35)

We show the random process $\{\Psi_k, A_k\}$ satisfies Assumption 3.2. Again, the dynamics of $A_k$ do not change and $\Psi \geq 0$ since we incorporated the stopping time directly into the definition of $\Psi$.

**Theorem 4.17.** Let Assumptions 2.1, 2.4, and 4.16 hold. Suppose $\{X_k, G_k, F^0_k, F^s_k, A_k\}$ is the random process generated by Algorithm 1. The expected improvement is

$$1_{\{T\varepsilon > k\}} E[\Psi_{k+1} - \Psi_k | \mathcal{F}^G_{k-1}] \leq -\frac{p_g p_f (1 - \nu)(1 - \gamma^{-1})}{4(\frac{L^2}{4\mu} + (1 - \nu)\alpha_{\max} + (1 - \nu))} A_k \cdot 1_{\{T\varepsilon > k\}},$$

where $\Psi_k$ is defined in (4.35) and the probabilities $p_g$ and $p_f$ and constant $\nu$ are defined in Theorem 4.6.
Proof. By strong convexity, for all \( x \), we have \( f(x) - f^* \leq \frac{1}{2\mu} \| \nabla f(x) \|^2 \); hence we obtain
\[
\Phi_k = \nu (f(X_k) - f^*) + (1 - \nu) \left( A_k \frac{\| \nabla f(X_k) \|^2}{L^2} + \theta \Delta_k^2 \right) \leq \left( \frac{\nu L^2}{2\mu} + (1 - \nu) \alpha_{\text{max}} \right) \frac{\| \nabla f(X_k) \|^2}{L^2} + (1 - \nu) \theta \Delta_k^2
\]
\[
\leq \left( \frac{\nu L^2}{2\mu} + (1 - \nu) \alpha_{\text{max}} + (1 - \nu) \right) \left( \frac{\| \nabla f(X_k) \|^2}{L^2} + \theta \Delta_k^2 \right).
\]

For simplicity of notation we define \( \tilde{C} = \left( \frac{\nu L^2}{2\mu} + (1 - \nu) \alpha_{\text{max}} + (1 - \nu) \right) \). Also for simplicity and without loss of generality, we assume \( \alpha_{\text{max}} \leq 1 \); hence, we conclude
\[
1_{\{T > k\}} \Phi_k \leq 1_{\{T > k\}} \tilde{C} \left( \frac{A_k \| \nabla f(X_k) \|^2}{L^2} + \theta \Delta_k^2 \right).
\]

Theorem 4.6 and the equality \( 1_{\{T > k\}} (\Phi_{k+1} - \Phi_k) = \Phi_{(k+1)\wedge T_k} - \Phi_{k\wedge T_k} \) give
\[
E[\Phi_{(k+1)\wedge T_k} - \Phi_{k\wedge T_k} | F_{k-1}^{G,F}] \leq - \frac{p_g p_f (1 - \nu)(1 - \gamma^{-1})}{4} \left( A_k \frac{\| \nabla f(X_k) \|^2}{L^2} + \theta \Delta_k^2 \right) 1_{\{T > k\}}
\]
\[
\leq - \frac{p_g p_f (1 - \nu)(1 - \gamma^{-1}) A_k}{4C} \Phi_k \cdot 1_{\{T > k\}}
\]
\[
\Rightarrow E[\Phi_{(k+1)\wedge T_k} | F_{k-1}^{G,F}] \leq \left( 1 - \frac{p_g p_f (1 - \nu)(1 - \gamma^{-1}) A_k}{4C} \cdot 1_{\{T > k\}} \right) \Phi_{k\wedge T_k}.
\]

Consequently, using Jensen’s inequality, we have the following
\[
E[\log(\Phi_{(k+1)\wedge T_k}) - \log(\Phi_{k\wedge T_k}) | F_{k-1}^{G,F}] \leq \log \left( \frac{E[\Phi_{(k+1)\wedge T_k} | F_{k-1}^{G,F}]}{\Phi_{k\wedge T_k}} \right)
\]
\[
= \log \left( \frac{1 - \frac{p_g p_f (1 - \nu)(1 - \gamma^{-1}) A_k}{4C} \cdot 1_{\{T > k\}}}{\Phi_{k\wedge T_k}} \right),
\]
where the last inequality follows by (4.36). Because \( \log(1 - x) \leq -x \) for \( x < 1 \), we deduce our result.

Using the above theorem allows us to use Theorem 3.3 and after simplifying some constants, we have the following complexity bound.

**Theorem 4.18.** Let the assumptions of Theorem 4.14 hold with constant \( \nu \) and probabilities \( p_f p_g \) as in Theorem 4.14. Then the expected number of iterations that Algorithm 4 takes until \( f(X_k) - f^* < \varepsilon \) is bounded as follows
\[
E[T_\varepsilon] \leq O(1) \cdot \frac{p_g p_f}{2p_f p_f} \left( \kappa_g \alpha_{\text{max}}^2 (\kappa_g + L + \varepsilon f) \left( \frac{L^2}{2\mu} + \alpha_{\text{max}} \right) \right) \left( \log(\Psi_0) + \log \left( \frac{1}{\varepsilon} \right) \right) + 1
\]

Simplifying the bound further gives us
\[
E[T_\varepsilon] \leq O(1) \cdot \frac{p_g p_f}{2p_f p_f} \left( \frac{L^3 (\kappa_g \alpha_{\text{max}})^3}{\mu} \right) \left( \log(\Psi_0) + \log \left( \frac{1}{\varepsilon} \right) \right).
\]
4.7 General descent, nonconvex case

In this subsection, we extend the analysis of our line search method to the general setting where
the search direction is any descent direction \( d_k \), which is meant to be a decent direction, but may
not be due to stochasticity. For example \( d_k \) can be a direction computed by applying subsampled
Newton method [17]. Algorithm 1 is then modified as follows

- a step is reliable when \(- \alpha_k g^T_k d_k \geq \delta_k^2\) instead of \( \alpha_k \|g_k\|^2 \geq \delta_k^2\);
- the stepsize \( s_k = \alpha_k d_k \) (instead of \(- \alpha_k g_k\)).

The sufficient decrease (2.2) is replaced with

\[
f(x_k + \alpha_k d_k) \leq f(x_k) + \alpha_k \theta d^T_k g_k. \tag{4.37}
\]

- \( d_k \) satisfies the following standard conditions.

**Assumption 4.19.** Given a gradient estimate \( g_k \) we assume the following hold for the descent
direction \( d_k \)

1. There exists a constant \( \beta > 0 \), such that
   \[
   \frac{d^T_k g_k}{\|d_k\| \|g_k\|} \leq -\beta, \quad \text{for all } k.
   \]

2. There exist constants \( \kappa_1, \kappa_2 > 0 \) such that
   \[
   \kappa_1 \|g_k\| \leq \|d_k\| \leq \kappa_2 \|g_k\|, \quad \text{for all } k.
   \]

We now provide simple variants of lemmas derived in Section 4.1.

**Lemma 4.20 (Variant of Lemma 4.4).** Suppose the iterate is successful and the descent direction
\( d_k \) satisfies Assumption 4.19. Then

\[
\|\nabla f(x_{k+1})\|^2 \leq 2(L^2 \alpha_k^2 \kappa_2^2 \|g_k\|^2 + \|\nabla f(x_k)\|^2).
\]

In particular, the inequality holds

\[
\frac{1}{\beta} \left( \alpha_{k+1} \|\nabla f(x_{k+1})\|^2 - \alpha_k \|\nabla f(x_k)\|^2 \right) \leq 2 \gamma \alpha_k^2 \left( \alpha_{\text{max}}^2 \kappa_2^2 \|g_k\|^2 + \frac{1}{\beta} \|\nabla f(x_k)\|^2 \right)
\]

Proof. An immediate consequence of \( L \)-smoothness of \( f \) is \( \|\nabla f(x_{k+1})\| \leq L \alpha_k \|d_k\| + \|\nabla f(x_k)\| \).

The result follows from Assumption 4.19 (ii) then squaring both sides and applying the bound,
\((a+b)^2 \leq 2(a^2 + b^2)\). To obtain the second inequality, we note that in the case \( x_k + s_k \) is successful,
so \( \alpha_{k+1} = \gamma \alpha_k \). \( \square \)

The analysis for the steepest descent relies on successful iterations occurring whenever the
stepsize is sufficiently small. We provide a similar result for general descent case.

**Lemma 4.21 (Accurate gradients/estimates ⇒ successful iteration, variant of Lemma 4.2).** Suppose \( g_k \) is \( \kappa_g \)-sufficiently accurate, the descent direction \( d_k \) satisfies Assumption 4.19 and \( \{f_k, \delta_k\} \)
are \( \varepsilon_f \)-accurate estimates. If

\[
\alpha_k \leq \frac{\beta(1-\theta)}{\kappa_g + \frac{L \kappa_2^2}{\kappa_1}}
\]

then the trial step \( x_k + s_k \) is successful. In particular, this means \( f_k^s \leq f_k^0 + \theta \alpha_k g^T_k d_k \).
Since the estimates are \( \varepsilon_f \)-accurate, we obtain
\[
f^s_k - \varepsilon f \alpha^2_k \| g_k \|^2 \leq f(x_k + s_k) - f^s_k + f^s_k
\leq f(x_k) - f^0_k + \kappa^2 g^2_k \| g_k \| \| d_k \| + \alpha_k g^T_k d_k + \frac{L^2 \alpha^2_k}{2} \| d_k \|^2
\leq f^0_k + \varepsilon f \alpha^2_k \| g_k \|^2 + \kappa^2 \alpha^2_k \| g_k \| \| d_k \| + \alpha_k g^T_k d_k + \frac{L^2 \alpha^2_k}{2} \| d_k \|^2.
\]
The above inequality with Assumption \[4.19\] implies
\[
f^s_k - f^0_k \leq \alpha^2_k \left( \frac{2 \varepsilon f}{\kappa_1} + \kappa^2 + \frac{L^2 \alpha^2_k}{2} \right) \| g_k \| \| d_k \| + \alpha_k g^T_k d_k
\leq -\frac{\alpha^2_k}{\beta} \left( \frac{2 \varepsilon f}{\kappa_1} + \kappa^2 + \frac{L^2 \alpha^2_k}{2} \right) \| g_k \| \| d_k \| + \alpha_k g^T_k d_k.
\]
The result follows by noting \( f^s_k \leq f^0_k + \alpha_k g^T_k d_k \left( 1 - \frac{\alpha}{\beta} \left( \kappa_1 + \frac{L^2 \alpha^2_k}{2} + \frac{2 \varepsilon f}{\kappa_1} \right) \right). \]

As in the steepest descent case, we can use the same function \( \Phi_k \) as defined in \[4.10\]. Using the sufficient decrease condition \[4.37\] and Assumption \[4.19\] on \( d_k \), a successful iterate yields a decrease of
\[
f(x_k + \alpha_k d_k) \leq -\theta \alpha_k \kappa_1 \beta \| g_k \|^2.
\]
Hence, we can derive, as in the steepest descent scenario, an expected decrease in \( \Phi_k \).

**Theorem 4.22.** Let Assumptions \[2.7\] \[2.4\] and \[4.19\] hold. Suppose \( \{X_k, D_k, G_k, F^0_k, F^s_k, A_k, \Delta_k\} \) is the random process generated by Algorithm \[7\]. Then there exist probabilities \( p_g, p_f \) \( > 1/2 \) and a constant \( \nu \in (0, 1) \) such that the expected decrease in \( \Phi_k \) is
\[
E[\Phi_{k+1} - \Phi_k | F^k] \leq -p_g p_f (1 - \nu)(1 - \gamma^{-1}) \left( \frac{A_k}{L^2} \| \nabla f(X_k) \|^2 + \theta \Delta^2_k \right).
\]

In particular, the constant \( \nu \) and probabilities \( p_g, p_f \) \( > 1/2 \) satisfy
\[
\frac{\nu}{1 - \nu} \geq \max \left\{ \frac{32 \gamma \alpha^2_{\max} \kappa^2}{\theta \kappa_1 \beta} , 16(\gamma - 1), \frac{16 \gamma (\kappa_1 \alpha^2_{\max} + 1)^2}{\theta \kappa_1 \beta} \right\},
\]
\[
p_g \geq \frac{2 \gamma}{1/2(1 - \gamma^{-1}) + 2 \gamma}
\]
and
\[
\frac{p_g p_f}{\sqrt{1 - p_f}} \geq \max \left\{ \frac{8 L^2 \nu \kappa_f + 16 \gamma (1 - \nu)}{(1 - \nu)(1 - \gamma^{-1})}, \frac{8 \nu}{(1 - \nu)(1 - \gamma^{-1})} \right\}
\]

**Proof.** Using Assumption \[4.19\] on the descent direction \( d_k \) when an iterate is successful, we see
\[
1_{\text{Succ}}(f(X_k + A_k D_k) - f(X_k)) \leq -1_{\text{Succ}} \theta A_k \kappa_1 \beta \| G_k \|^2.
\]
Hence, we may replace $\theta$ in the proof of Theorem 4.6 with $\theta \kappa_1 \beta$. The only other change to the proof and the resulting constants lies in the replacement of $\nu$ of Lemma 4.4 by Lemma 4.20. This implies a change in the choice of $\nu$ in equations (4.42). In particular, we choose $\nu$ to now satisfy

\[-\frac{\nu \theta \kappa_1 \beta A_k}{8} \|G_k\|^2 + (1 - \nu)2\gamma A_k \alpha_{\max}^2 \|G_k\|^2 \leq -\frac{\nu \theta \kappa_1 \beta A_k}{16} \|G_k\|^2 , \]

\[-\frac{\nu \theta \kappa_1 \beta A_k}{4L^2(\kappa_g \alpha_{\max} + 1)^2} \|\nabla f(X_k)\|^2 + (1 - \nu)\frac{2\gamma A_k}{L^2} \|\nabla f(X_k)\|^2 \leq -\frac{\nu \theta \kappa_1 \beta A_k}{8L^2(\kappa_g \alpha_{\max} + 1)^2} \|\nabla f(X_k)\|^2 , \]

and

\[-\frac{\nu \theta}{8} \Delta_k^2 + (1 - \nu)(\gamma - 1) \theta \Delta_k^2 \leq -\frac{\nu \theta}{16} \Delta_k^2 . \quad (4.42)\]

Using Lemma 4.21, we can set

$$\bar{A} = \frac{\beta(1 - \theta)}{\kappa_g + \frac{L A_k}{2} + \frac{2 \nu L^2}{\kappa_1}}$$

and apply Theorem 3.3.

**Theorem 4.23.** Under the assumptions in Theorem 4.22 and constants chosen in Remark 4, suppose the probabilities $p_g, p_f > 1/2$ satisfy

$$p_g p_f \geq \frac{16}{17} \text{ and } \frac{p_g p_f}{\sqrt{1 - p_f}} \geq \max \left\{ \frac{1024 \kappa_1 \beta L^2(\max\{\kappa_g, 2\kappa_2\} \alpha_{\max} + 1)^2}{\kappa_1 \beta} + 64, \frac{1024(\max\{\kappa_g, 2\kappa_2\} \alpha_{\max} + 1)^2}{\kappa_1 \beta} \right\}$$

with $\frac{\nu}{1 - \nu} = \frac{64(\max\{\kappa_g, 2\kappa_2\} \alpha_{\max} + 1)^2}{\kappa_1 \beta}$. Then the expected number of iterations that Algorithm 7 takes until $\|\nabla f(X_k)\|^2 \leq \varepsilon$ occurs is bounded as follows

$$E[T_\varepsilon] \leq \frac{p_g p_f}{2p_0 p_f - 1} \cdot \frac{L^3 \kappa_2^2 \kappa_2^2 \Phi_0}{\kappa_1 \beta^2} \cdot \frac{1}{\varepsilon^2} + 1,$$

where $\Phi_0 = \nu(f(X_0) - f_{\min}) + (1 - \nu)(1/L^2 A_0 \|\nabla f(X_0)\|^2 + 1/2 \Delta_0^2)$.

**5 Conclusions**

We have used a general framework based on analysis of stochastic processes proposed in [3] with the purpose of analyzing convergence rates of stochastic optimization methods. In [3] the framework is used to analyze stochastic trust region method, while in this paper we were able to use the same framework to develop and analyze stochastic back-tracking line search method. Our method is the first implementable stochastic line-search method that has theoretical convergence rate guarantees. In particular, the the accuracy of gradient and function estimates is chosen dynamically and the requirements of this accuracy are all stated in terms of knowable quantities. We establish complexity results for convex, strongly convex and general nonconvex, smooth stochastic functions.

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