STANDING WAVES OF THE COMPLEX GINZBURG-LANDAU EQUATION

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Abstract. We prove the existence of nontrivial standing wave solutions of the complex Ginzburg-Landau equation $\phi_t = e^{i\theta} \Delta \phi + e^{i\gamma} |\phi|^\alpha \phi$ with periodic boundary conditions. Our result includes all values of $\theta$ and $\gamma$ for which $\cos \theta \cos \gamma > 0$, but requires that $\alpha > 0$ be sufficiently small.

1. Introduction

We consider the complex Ginzburg-Landau equation

$$\phi_t = e^{i\theta} \Delta \phi + e^{i\gamma} |\phi|^\alpha \phi,$$

where $\alpha > 0$, both on the whole space $\mathbb{R}^N$, with periodic boundary conditions, and on a bounded domain $\Omega$ of $\mathbb{R}^N$ with Dirichlet boundary conditions. We look for standing wave solutions of the form

$$\phi(t, x) = e^{i\omega t} u(x)$$

with $\omega \in \mathbb{R}$. The resulting equation for $u$ is then

$$e^{i\theta} \Delta u + e^{i\gamma} |u|^\alpha u = i\omega u.$$

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Equation (1.1) is used to model such phenomena as superconductivity, chemical turbulence and various types of fluid flows. See [6] and the references cited therein. Local and global well-posedness of (1.1), on both \( \mathbb{R}^N \) and a domain \( \Omega \subset \mathbb{R}^N \), are known under various boundary conditions and assumptions on the parameters, see e.g. [7, 9, 10, 13, 14, 16, 18, 19, 20, 21]. Concerning standing wave solutions, the particular case of the nonlinear Schrödinger equation (i.e. \( \theta = \pm \gamma = \pm \frac{\pi}{2} \)) leads to the elliptic equation \( \Delta u \pm |u|^{\alpha}u \pm \omega u = 0 \). This equation is the object of a literature too vast to be cited here. Another well-known case is \( \omega = 0 \), i.e. stationary solutions. Then necessarily \( \gamma = \theta \) modulo \( 2\pi \) (see Remark 1.4 (ii) below) and so equation (1.3) reduces to \( \Delta u + |u|^{\alpha}u = 0 \), which is a special case of the previous equation. In the other cases, we are not aware of mathematical results concerning the existence of standing wave solutions. Numerous papers discuss the existence of special solutions (holes, fronts, pulses, sources, sinks, etc), see e.g. [2, 3, 4, 5, 11, 12, 15, 17, 23, 24, 25].

Throughout this paper, all the function spaces are made up of complex-valued functions, but are considered as real Hilbert or Banach spaces. For example, \( L^2(\Omega) \) is the real Hilbert space of all complex-valued square integrable functions on \( \Omega \) with the (real) inner product
\[
(u, v)_{L^2} = \Re \int_{\Omega} u \overline{v}.
\] (1.4)
In addition, we consider the \( N \) dimensional torus \( \mathbb{T}^N = (\mathbb{R}/2\pi\mathbb{Z})^N \) and the space
\[
H^2(\mathbb{T}^N) = \{ u \in H^2_{\text{loc}}(\mathbb{R}^N); u \text{ is } 2\pi\text{-periodic in all variables} \},
\] (1.5)
equipped with the norm of \( H^2(\Omega) \) with \( \Omega = (0, 2\pi)^N \).

Our first result is the existence of spatially periodic standing wave solutions of (1.1) for small \( \alpha \).

**Theorem 1.1.** Suppose \( \gamma, \theta \) satisfy
\[
-\frac{\pi}{2} < \theta, \gamma < \frac{\pi}{2}.
\] (1.6)
It follows that there exist \( \alpha_0 > 0 \) and continuous maps \( u : (0, \alpha_0) \rightarrow H^2(\mathbb{T}^N) \) and \( \omega : [0, \alpha_0] \rightarrow \mathbb{R} \) such that for every \( \alpha \in (0, \alpha_0) \), \( u = u(\alpha) \) is a nontrivial solution of (1.3) with \( \omega = \omega(\alpha) \). In particular, the resulting function \( \phi \) given by (1.2) is a standing wave solution of (1.1).

Note that it is part of the statement of Theorem 1.1 that \( |u|^\alpha u \in L^2_{\text{loc}}(\mathbb{R}^N) \), since both \( \Delta u \) and \( u \) belong to \( L^2_{\text{loc}}(\mathbb{R}^N) \). Our proof of Theorem 1.1 proceeds by first constructing solutions of the equation (1.3) on the set \( \Omega = (0, \pi)^N \) which vanish on the boundary \( \partial \Omega \), and then extending these solutions to \( \mathbb{R}^N \) by reflection. Thus, to prove Theorem 1.1, we need first to prove a similar result, but on a bounded domain of \( \mathbb{R}^N \), which we now describe.

We consider a bounded, connected open subset \( \Omega \) of \( \mathbb{R}^N \) and we set
\[
H = \{ u \in H^1_0(\Omega); \Delta u \in L^2(\Omega) \},
\] (1.7)
so that \( H \) equipped with the scalar product
\[
(u, v)_H = \Re \int_{\Omega} u \overline{v} + \Re \int_{\Omega} \Delta u \Delta \overline{v},
\] (1.8)
is a real Hilbert space and \( H \hookrightarrow H^1_0(\Omega) \). We show the following result.
Theorem 1.2. Suppose $\Omega$ is a bounded, connected, open subset of $\mathbb{R}^N$. Let $\gamma, \theta$ satisfy (1.6) and let $H$ be defined by (1.7)-(1.8). It follows that there exist $\alpha_0 > 0$ and continuous maps $u : (0, \alpha_0) \to H$ and $\omega : [0, \alpha_0] \to \mathbb{R}$ such that for every $\alpha \in (0, \alpha_0)$, $u(\alpha)$ is a nontrivial solution of (1.3) with $\omega(\alpha) = \omega(\alpha)$. In particular, the resulting function $\phi$ given by (1.2) is a standing wave solution of (1.1).

We prove Theorem 1.2 by a perturbation argument from the case $\alpha = 0$, using the implicit function theorem. Indeed, equation (1.3) with $\alpha = 0$ reduces to the eigenvalue problem $-\Delta u = (e^{i(\gamma - \theta)} - i\omega e^{-i\theta})u$. As is well known, all eigenvalues of $-\Delta$ with Dirichlet boundary conditions are positive real numbers, while $\lambda^* =: e^{i(\gamma - \theta)} - i\omega e^{-i\theta}$ is not in general real. It turns out that $\lambda^*$ is real precisely when $\omega = \frac{\cos \theta}{\cos \gamma}$, in which case $\lambda^* = \frac{\cos \theta}{\cos \gamma}$. Unfortunately, this value of $\lambda^*$ is not always an eigenvalue of $-\Delta$. To overcome this problem, we introduce another parameter $\mu > 0$ and consider the equation

$$\Delta v + \mu e^{i(\gamma - \theta)}|v|^\alpha v - i\omega e^{-i\theta}v = 0. \quad (1.9)$$

(See also Remark 2.1.) Note that if $\alpha > 0$, then a solution of (1.9) can be turned into a solution of (1.3) by a simple multiplicative factor.

Equation (1.9) in the case $\alpha = 0$ now becomes $-\Delta v = (\mu e^{i(\gamma - \theta)} - i\omega e^{-i\theta})v$. Given any $\lambda > 0$, one sees that $\mu_0 e^{i(\gamma - \theta)} - i\omega_0 e^{-i\theta} = \lambda$ if and only if

$$\omega_0 = \lambda \frac{\sin(\gamma - \theta)}{\cos \theta}, \quad (1.10)$$

and

$$\mu_0 = \lambda \frac{\cos \theta}{\cos \gamma}. \quad (1.11)$$

The implicit function theorem now yields the following result.

Theorem 1.3. Suppose $\Omega$ is a bounded, connected, open subset of $\mathbb{R}^N$ and let $H$ be defined by (1.7)-(1.8). Let

$$\lambda = \lambda(-\Delta), \quad (1.12)$$

be an eigenvalue of $-\Delta$ in $L^2(\Omega)$ with domain $H$ and $\varphi$ a corresponding eigenvector such that

$$\int_{\Omega} |\varphi|^2 = 1. \quad (1.13)$$

Assume that $\lambda$ is a simple eigenvalue, in the sense that the corresponding eigenspace is $\mathbb{C}\varphi$. (For instance, $\lambda$ can be the first eigenvalue of $-\Delta$.) Let $\gamma, \theta$ satisfy (1.6) and let $\omega_0$ and $\mu_0$ be defined by (1.10)-(1.11). It follows that there exist $\alpha_0 > 0$ and continuous maps $v : [0, \alpha_0] \to H$, $\mu : [0, \alpha_0] \to \mathbb{R}$ and $\omega : [0, \alpha_0] \to \mathbb{R}$ such that $v(0) = \varphi$, $\mu(0) = \mu_0$, $\omega(0) = \omega_0$ and such that (1.9) holds for all $0 \leq \alpha \leq \alpha_0$.

The above results call for several remarks. Since our proof of Theorem 1.3 is based on a perturbation argument, we have no information on the size of $\alpha_0$. In addition, based on what is known about standing waves of the nonlinear Schrödinger equation and stationary solutions of the nonlinear heat equation, one would expect that, at least in space dimension $N \geq 2$, there would exist an infinite family of standing wave solutions (all with the same $\omega$). Our results do not address this question at all. Another important issue is the stability (both linear and dynamical) of the standing waves.

We next make a few remarks concerning the conditions on $\theta$ and $\gamma$. 

Remark 1.4. Suppose for example equation (1.3) is set on a bounded, connected subset $Ω$ of $\mathbb{R}^N$ with Dirichlet boundary conditions. (Similar calculations can be made in the case of periodic boundary conditions.) Let $u ∈ H^1_0(Ω) \cap L^{α+2}(Ω)$ with $Δu ∈ L^2(Ω)$, $u ≠ 0$ be a solution of (1.3). Since
\[ \int_Ω πΔu = - \int_Ω |∇u|^2, \quad (1.14) \]
we deduce from (1.3) that
\[ e^{θ} \int_Ω |∇u|^2 = e^{γ} \int_Ω |u|^α + 2 - iω \int_Ω |u|^2. \quad (1.15) \]
We now can draw the following consequences.

(i) Considering the real part of (1.15), we obtain
\[ \cos θ \int_Ω |∇u|^2 = \cos γ \int_Ω |u|^α + 2. \quad (1.16) \]
Thus we see that either $\cos θ = \cos γ = 0$ or else $\cos θ \cos γ > 0$. If $\cos θ = \cos γ = 0$, then the equation (1.1) becomes the nonlinear Schrödinger equation
\[ iφ_t = ±Δφ ± |φ|^α φ, \]
whose standing wave solutions have been extensively studied. Assume now $\cos θ \cos γ > 0$. Changing $(γ, θ, ω)$ to $(γ + π, θ + π, −ω)$ leaves the equation invariant, so we may assume that $\cos γ > 0$ and $\cos θ > 0$. Therefore, we may assume without loss of generality that (1.6) holds.

(ii) It follows easily from (1.15) and (1.6) that $ω = 0$ (i.e. $u$ is a stationary solution of (1.1)) if and only if $γ = θ$ modulo $2π$.

(iii) Finally, observe that changing $(u, γ, θ, ω)$ to $(π, −γ, −θ, −ω)$ leaves the equation invariant.

The next remark gives some variants of the main results.

Remark 1.5. (i) If $u$ is a periodic solution of (1.3), as in Theorem 1.1, then for every positive integer $n$, $u_n(x) = nπ u(nx)$ is also a solution of (1.3) in $H^2(\mathbb{T}^N)$, but with $ω$ replaced by $n^2ω$. In this way, we obtain infinitely many solutions on $\mathbb{T}^N$ (starting from one given solution), but for values of $ω$ that change with the solution.

(ii) In Theorem 1.1, we may replace the torus $(\mathbb{R}/2π\mathbb{Z})^N$ by $(\mathbb{R}/2ℓ_1\mathbb{Z}) × \cdots × (\mathbb{R}/2ℓ_N\mathbb{Z})$, where $ℓ_1, \ldots, ℓ_N > 0$. It suffices to apply Theorem 1.2 with $Ω = (0, ℓ_1) × \cdots × (0, ℓ_N)$ instead of $Ω = (0, π)^N$. The above remark about rescaling applies in this situation as well.

(iii) In the case where $Ω$ is the unit ball of $\mathbb{R}^N$, there is version of Theorem 1.3 in the space $L^2_{rad}(Ω)$ of radially symmetric functions. Note that all the eigenvalues of $−Δ$ in $L^2_{rad}(Ω)$ with Dirichlet boundary conditions are simple. Thus for every integer $n$, there exists $α_0 > 0$ such that for $0 < α < α_0$ there exist $n$ different standing wave solutions of (1.1) (assuming $γ ≠ θ$). Indeed, the solutions are different because for sufficiently small $α$, the corresponding value of $ω$ is close to $ω_0$ given by (1.10); and the values of $ω_0$ corresponding to different eigenvalues are all different. It would be interesting to know if these solutions are related by dilation, as is true for the eigenfunctions of $−Δ$ in $L^2_{rad}(Ω)$.

Remark 1.6. The method we use to prove Theorem 1.2 is not valid in the case $Ω = \mathbb{R}^N$. In fact, the conclusion of Theorem 1.2 is false if $Ω = \mathbb{R}^N$. Indeed, suppose $θ = γ$, in which case $ω = 0$ by the same argument as in Remark 1.4 (ii). In this case,
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Equation (1.3) becomes \(-\Delta u = |u|^\alpha u\), which has no nontrivial solutions in \(H^1(\mathbb{R}^N)\) if \(\alpha\) is Sobolev subcritical (i.e. \((N - 2)\alpha < 4\)). This fact is a consequence of the Pohožaev identity [22]. For the precise formulation needed here, see [1, Corollary 1, p.321].

2. Proof of Theorem 1.3

In this section, we prove Theorem 1.3.

Proof of Theorem 1.3. It follows from (1.10)-(1.11) that

\[ \mu_0 e^{i(\gamma - \theta)} - i\omega_0 e^{-i\theta} = \lambda. \]  

(2.1)

Therefore, \(\varphi\) is a solution of (1.9) with \(\alpha = 0, \mu = \mu_0\) and \(\omega = \omega_0\). For \(\alpha > 0\) small, \(\mu\) and \(\omega\) close to \(\mu_0\) and \(\omega_0\), we seek a solution \(v\) of (1.9) of the form \(v = \varphi + \zeta\) with \(\zeta \in H_1\), where \(H_1\) is the orthogonal complement of \(\mathbb{C}\varphi\) in \(H\). The main tool we use is the implicit function theorem. The first order of business is to define an appropriate mapping \(F\). We fix

\[ 0 < \tilde{\alpha} < \infty \text{ such that } (N - 2)\tilde{\alpha} \leq 2, \]  

(2.2)

so that \(H \hookrightarrow L^{2(\tilde{\alpha}+1)}(\Omega)\) by Sobolev’s embedding. We set

\[ X = \mathbb{R}^2 \times H_1, \]  

(2.3)

and we define the map \(F : (-\infty, \tilde{\alpha}] \times X \to L^2(\Omega)\) by

\[ F(\alpha, \mu, \omega, \zeta) = \Delta v + \mu e^{i(\gamma - \theta)} g(\alpha, v) - i\omega e^{-i\theta} v, \]  

(2.4)

\[ v = \varphi + \zeta, \]  

(2.5)

where the function \(g : \mathbb{R} \times \mathbb{C} \to \mathbb{C}\) is given by

\[ g(\alpha, v) = \begin{cases} |v|^\alpha v, & \alpha > 0, \\ v, & \alpha \leq 0. \end{cases} \]  

(2.6)

It follows that if \(F(\alpha, \mu, \omega, \zeta) = 0\) and \(\alpha > 0\), then \(v\) is a solution of equation (1.9). Since

\[ \Delta \varphi + \lambda \varphi = 0, \]  

(2.7)

we deduce from (2.1) that

\[ F(0, \mu_0, \omega_0, 0) = 0. \]  

(2.8)

It follows (for instance from Proposition A.1 below) that the map \((\alpha, v) \mapsto g(\alpha, v(\cdot))\) is continuous \((-\infty, \tilde{\alpha}] \times H \to L^2(\Omega)\), from which we deduce that \(F\) is continuous \((-\infty, \tilde{\alpha}] \times X \to L^2(\Omega)\). Furthermore, if \(\alpha \leq \tilde{\alpha}\), then by Proposition A.1 the map \(v \mapsto g(\alpha, v(\cdot))\) is differentiable everywhere on \(H\), so that the map \((\mu, \omega, \zeta) \mapsto F(\alpha, \mu, \omega, \zeta)\) is differentiable everywhere. In addition, using (A.5), we have

\[ \frac{\partial F(\alpha, \mu, \omega, \zeta)}{\partial \mu} = e^{i(\gamma - \theta)} g(\alpha, v(\cdot)), \]  

(2.9)

\[ \frac{\partial F(\alpha, \mu, \omega, \zeta)}{\partial \omega} = -ie^{-i\theta} v, \]  

(2.10)
and
\[ \frac{\partial F}{\partial \zeta}(\alpha, \mu, \omega, \zeta) w = \begin{cases} 
\Delta w + \mu e^{i(\gamma - \theta)\alpha} |v|^\alpha w + e^{i(\gamma - \theta)\alpha} |v|^{\alpha - 2}v \mathbb{R}(v w) - i\omega e^{-i\theta} w & \alpha > 0, \\
\Delta w + [\mu e^{i(\gamma - \theta)} - i\omega e^{-i\theta}] w & \alpha \leq 0.
\end{cases} \] (2.11)

We now show that the derivative
\[ \frac{\partial F}{\partial (\mu, \omega, \zeta)}(0, \mu_0, \omega_0, 0) : X \rightarrow L^2(\Omega) \]
is a bijection. Indeed, we deduce from (2.9)–(2.11) that
\[ \begin{align*}
\frac{\partial F}{\partial \mu}(0, \mu_0, \omega_0, 0) &= e^{i(\gamma - \theta)} \varphi, \\
\frac{\partial F}{\partial \omega}(0, \mu_0, \omega_0, 0) &= -ie^{-i\theta} \varphi, \\
\frac{\partial F}{\partial \zeta}(0, \mu_0, \omega_0, 0) w &= \Delta w + \lambda w.
\end{align*} \] (2.12–2.14)

where we used (2.1) in the last identity. Therefore,
\[ \begin{bmatrix} \frac{\partial F}{\partial (\mu, \omega, \zeta)}(0, \mu_0, \omega_0, 0) \end{bmatrix} (a, b, w) = A(a, b, w), \] (2.15)

where
\[ A(a, b, w) = ae^{i(\gamma - \theta)} \varphi - ibe^{-i\theta} \varphi + \Delta w + \lambda w. \] (2.16)

We first claim that the kernel of \( A \) is trivial. Indeed, suppose
\[ A(a, b, w) = 0. \] (2.17)

Multiplying the equation (2.17) by \( \overline{\varphi} \), integrating by parts on \( \Omega \) and using (1.13) and (2.7), we obtain \( 0 = ae^{i(\gamma - \theta)} - ibe^{-i\theta} \), i.e. \( ae^{i\gamma} = ib \). Using (1.6), we conclude that \( a = b = 0 \). It then follows from (2.17) that \( \Delta w + \lambda w = 0 \), so that \( w \in \mathbb{C} \varphi \). Since \( \mathbb{C} \varphi \cap H_1 = \{0\} \), this proves the claim.

We next claim that \( A \) is surjective. Let \( M(z) = ae^{i(\gamma - \theta)} - ibe^{-i\theta} \), for \( z = a + bi \). Considering \( \mathbb{C} \) as a real linear space, we see that \( M \) is a linear operator \( \mathbb{C} \rightarrow \mathbb{C} \). As shown above, \( \ker M = \{0\} \), and so \( M \) is a bijection. Thus, given \( f \in L^2(\Omega) \) there exist \( a, b \in \mathbb{R} \) such that
\[ ae^{i(\gamma - \theta)} - ibe^{-i\theta} = \int_{\Omega} f \overline{\varphi}. \] (2.18)

It follows from (1.13) and (2.18) that \( f - ae^{i(\gamma - \theta)} \varphi + ibe^{-i\theta} \varphi \) belongs to the orthogonal of \( \mathbb{C} \varphi \). Therefore, there exists a unique \( w \in H_1 \) such that
\[ \Delta w + \lambda w = f - ae^{i(\gamma - \theta)} \varphi + ibe^{-i\theta} \varphi, \]
i.e. \( A(a, b, w) = f \). This proves surjectivity.

At this point, we wish to apply the implicit function theorem [26, Theorem 4.B, p.150]. The only condition that we have not yet verified is that the map \( \partial_{(\mu, \omega, \zeta)} F \) given by
\[ \begin{cases} 
(-\infty, \overline{\alpha}] \times X \rightarrow L(X, L^2(\mathbb{R}^N)) \\
(\alpha, \mu, \omega, \zeta) \mapsto \frac{\partial F}{\partial (\mu, \omega, \zeta)}(\alpha, \mu, \omega, \zeta)
\end{cases} \] (2.19)
is continuous at the point $(0, \mu_0, \omega_0, 0)$. This is an immediate consequence of Proposition A.2, since $\varphi \neq 0$ a.e. in $\Omega$. To see this last property, we note that $\varphi$ is analytic in the connected, open set $\Omega$, see e.g. [8], so that it cannot vanish on a set of positive measure.

By the above cited the implicit function theorem, there exist $\alpha_0 > 0$ and continuous maps $\zeta : [0, \alpha_0] \to H_1$, $\mu : [0, \alpha_0] \to \mathbb{R}$ and $\omega : [0, \alpha_0] \to \mathbb{R}$ such that $\zeta(0) = 0$, $\mu(0) = \mu_0$, $\omega(0) = \omega_0$ and such that $F(\alpha, \mu(\alpha), \omega(\alpha), \zeta(\alpha)) = 0$ for $0 \leq \alpha \leq \alpha_0$. This completes the proof (with $v(\alpha) = \varphi + \zeta(\alpha)$).

\[ \square \]

**Remark 2.1.** The parameter $\mu$ is not only useful to ensure that the equation (1.9) has the form $\Delta v + \lambda v = 0$ when $\alpha = 0$, where $\lambda$ is an eigenvalue of $-\Delta$. It also provides a second parameter in the implicit function theorem so that the linearized operator is bijective.

## 3. Proof of Theorems 1.2 and 1.1

**Proof of Theorem 1.2.** We apply Theorem 1.3. (Note that there always exists $\lambda$ as in the statement, for example $\lambda$ can be the first eigenvalue of $-\Delta$.) Observe that

\[ \cos \gamma > 0, \quad \cos \theta > 0, \quad (3.1) \]

by (1.6), so that $\mu_0 > 0$ by (1.11). Therefore, we may choose $\alpha_0$ small enough so that $\mu(\alpha) > 0$ for $\alpha \in [0, \alpha_0)$. Thus $u = \mu \hat{\varphi} v$ is well defined and satisfies (1.3) (since $v$ satisfies (1.9)). Since $v(0) = \varphi$, we have $v(\alpha) \neq 0$, hence $u(\alpha) \neq 0$, for $\alpha > 0$ sufficiently small. Thus, by choosing $\alpha_0 > 0$ possibly smaller, we have $u(\alpha) \neq 0$ for $0 < \alpha < \alpha_0$.

\[ \square \]

**Proof of Theorem 1.1.** We apply Theorem 1.2 with $\Omega = (0, \pi)^N$, and we obtain $0 < \alpha_0 < \frac{2}{(N-2)_+}$ and continuous maps $\ti u : (0, \alpha_0) \to H$ (defined by (1.5)) and $\omega : [0, \alpha_0] \to \mathbb{R}$ such that for every $\alpha \in (0, \alpha_0)$, $\ti u = \ti u(\alpha)$ is a solution of (1.3) on $\Omega$. We now extend $\ti u$ to $\mathbb{R}^N$ by symmetry. More precisely, given $x \in \mathbb{R}^N$, there exists a unique family of integers $(k_j)_{1 \leq j \leq N}$ such that $k_j \pi \leq x_j < (k_j + 1)\pi$, and we set

\[ u(x) = (-1)^j \sum_{k=1}^N k \ti u(\ti x), \quad (3.2) \]

where $\ti x_j = x_j - k_j \pi$. It follows that $u \in H^1_{\text{loc}}(\mathbb{R}^N) \cap L^{2(\alpha+1)}_{\text{loc}}(\mathbb{R}^N)$ is a solution of (1.3) on $\mathbb{R}^N$, and by standard elliptic regularity, $u \in H^2_{\text{loc}}(\mathbb{R}^N)$. Since $u$ is clearly $2\pi$-periodic in all variables, we see that $u \in H^2(T^N)$.

\[ \square \]

**Remark 3.1.**

(i) Recall that $u(\alpha)$ constructed in the proof of Theorem 1.2 is given by $u(\alpha) = \mu(\alpha) \hat{\varphi} v(\alpha)$, where the functions $v(\alpha)$ and $\mu(\alpha)$ are given by Theorem 1.3. Furthermore, $\mu(\alpha) \to \mu_0$ given by (1.11) as $\alpha \to 0$. Clearly, if $\mu_0 > 1$ (i.e. $\lambda > \frac{\cos \gamma}{\cos \theta}$), then $\|u(\alpha)\|_{L^2} \to \infty$ as $\alpha \to 0$, while if $\mu_0 < 1$ (i.e. $\lambda < \frac{\cos \gamma}{\cos \theta}$), then $\|u(\alpha)\|_{L^2} \to 0$ as $\alpha \to 0$. In the latter case, the curve $u(\alpha)$ bifurcates from the trivial branch of solutions of (1.3) at $\alpha = 0$. The same conclusions hold for the solutions constructed in Theorem 1.1.

(ii) In the context of Theorem 1.3, suppose $\lambda$ and $\tilde{\lambda}$ are two different simple eigenvalues of $-\Delta$ with corresponding eigenvectors $\varphi$ and $\tilde{\varphi}$. Let $v, \mu, \omega$ and $\ti v, \ti \mu, \ti \omega$ be the resulting continuous maps constructed by Theorem 1.3. Since $v(\alpha) \to \varphi$ and $\ti v(\alpha) \to \tilde{\varphi}$ as $\alpha \to 0$, it is clear that $v(\alpha) \neq \ti v(\alpha)$ for $\alpha > 0$ small. Similarly, it is clear (see (1.10)) that $\omega(\alpha) \neq \ti \omega(\alpha)$ for $\alpha > 0$ small. The functions $u(\alpha) = \mu(\alpha) \hat{\varphi} v(\alpha)$ and $\ti u(\alpha) = \ti \mu(\alpha) \hat{\varphi} \ti v(\alpha)$ are solutions, respectively, of
equation (1.3) and of equation (1.3) with $\bar{\omega}$ instead of $\omega$. Thus it is also clear that $u(\alpha) \neq \bar{u}(\alpha)$ for $\alpha > 0$ small.

(iii) Suppose $\Omega$ is the unit ball of $\mathbb{R}^N$ and consider radially symmetric standing waves. As noted above (see Remark 1.5 (iii)) all the eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots$ of $-\Delta$ in $L^2_{\text{rad}}(\Omega)$ with Dirichlet boundary conditions are simple. And so, Theorem 1.3 can be applied with each $\lambda = \lambda_k$ for each $k \geq 1$, thus producing an infinite family of curves $u_k(\alpha)$ of standing waves. If $\lambda_k < \frac{\cos \gamma}{\cos \theta}$ (which can happen only for finitely many $k$), then $\|u_k(\alpha)\|_{L^2} \to 0$ as $\alpha \to 0$. On the other hand, if $\lambda_k > \frac{\cos \gamma}{\cos \theta}$ (which necessarily happens infinitely many $k$), then $\|u_k(\alpha)\|_{L^2} \to \infty$ as $\alpha \to 0$. Observe that the number of eigenvalues such that $\lambda_k < \frac{\cos \gamma}{\cos \theta}$ obviously depends on $\theta$ and $\gamma$.

**Appendix A. An Extension of the Map $(\alpha, v) \mapsto |v|^\alpha v$**

In this section we construct an explicit extension of the map $(\alpha, v) \mapsto |v|^\alpha v$ to include negative values of $\alpha$ and we study its differentiability with respect to $v$.

We consider the function $g : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ defined by (2.6) and we define $H : \mathbb{R} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ by

$$H(\alpha, v, u) = \begin{cases} |v|^\alpha u + \alpha|v|^{\alpha-2}v\Re(\overline{v}u) & \alpha > 0, v \neq 0, \\ 0 & \alpha > 0, v = 0, \\ u & \alpha \leq 0. \end{cases}$$

(A.1)

It follows easily that $g \in C(\mathbb{R} \times \mathbb{C}, \mathbb{C})$ and $H$ is continuous, except at the points $(0, 0, u)$ with $u \neq 0$ (where it is discontinuous). Moreover, $g$ is differentiable with respect to $v$ at every point $(\alpha, v) \in \mathbb{R} \times \mathbb{C}$ (where $\mathbb{C}$ is considered as a real Hilbert space), and

$$\partial_v g(\alpha, v)u = H(\alpha, v, u),$$

(A.2)

for all $\alpha \in \mathbb{R}$ and $u, v \in \mathbb{C}$.

Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$ and, given $1 \leq r \leq \infty$, let $L^r(\Omega)$ be the usual Lebesgue space of complex valued functions, equipped with its standard norm $\| \cdot \|_{L^r}$, considered as a real Banach space. We fix $a > 0$ and set $p = 2(a + 1)$. Given $v \in L^p(\Omega)$, we define

$$G(\alpha, v)(\cdot) = g(\alpha, v(\cdot)), \quad (A.3)$$

where $g$ is given by (2.6).

**Proposition A.1.** The map $G$ is continuous $(-\infty, a] \times L^p(\Omega) \to L^2(\Omega)$. Moreover, $G$ is Fréchet differentiable with respect to $v$ everywhere on $(-\infty, a] \times L^p(\Omega)$ and

$$\partial_v G(\alpha, v) = L_{\alpha, v},$$

(A.4)

with

$$L_{\alpha, v}u = H(\alpha, v(\cdot), u(\cdot)), \quad (A.5)$$

for all $\alpha \in (-\infty, a)$, $v \in L^p(\Omega)$ and $u \in L^2(\Omega)$, where $H$ is defined by (A.1).

**Proof.** Since $\Omega$ is bounded, it follows from the estimate

$$|g(\alpha, v)| \leq \begin{cases} |v|^{|\alpha|+1} & \alpha \geq 0, \\ |v| & \alpha < 0, \end{cases}$$

(A.6)
that $G(\alpha, v) \in L^2(\Omega)$ for all $\alpha < a$. Moreover, it follows easily from (A.6) and the dominated convergence theorem that $G \in C((-\infty, a) \times L^p(\Omega), L^2(\Omega))$. Next, we deduce from the estimate
\[
|H(\alpha, v, u)| \leq \begin{cases} 
(\alpha + 1)|v|^\alpha |u| & \alpha > 0, \\
|u| & \alpha \leq 0,
\end{cases}
\] (A.7)
that
\[
\|L_{\alpha, v}u\|_{L^2} \leq \begin{cases} 
(\alpha + 1)\|v\|_{L^p(\Omega)}^\alpha \|u\|_{L^p} & 0 < \alpha < a, \\
\|u\|_{L^p} & \alpha \leq 0.
\end{cases}
\]
Therefore, given any $\alpha < a$ and $v \in L^p(\Omega)$, we see that $L_{\alpha, v}$ is linear and continuous $L^p(\Omega) \to L^2(\Omega)$. It is not difficult to verify, using (A.7) and the dominated convergence theorem that
\[
(\alpha, v) \mapsto L_{\alpha, v} \text{ is continuous } ((-\infty, a) \setminus \{0\}) \times L^p(\Omega) \to L(L^p(\Omega), L^2(\Omega)).
\] (A.8)
Given $0 < \alpha < a$, $v, u \in L^p(\Omega)$ and $0 < t \leq 1$, it follows from (A.2) that
\[
G(\alpha, v + u) - G(\alpha, v) - L_{\alpha, v}u = \int_0^1 H(\alpha, v + \sigma u, u) \, d\sigma - L_{\alpha, v}u = \int_0^1 [L_{\alpha, v+\sigma u}u - L_{\alpha, v}u] \, d\sigma.
\] (A.9)
Applying (A.8), we deduce that
\[
\left\| \int_0^1 [L_{\alpha, v+\sigma u}u - L_{\alpha, v}u] \, d\sigma \right\|_{L^2} = o(\|u\|_{L^p}) \text{ as } \|u\|_{L^p} \to 0.
\] (A.10)
Thus we see that $G$ is Fréchet differentiable with respect to $v$ at $(\alpha, v)$ with derivative $L_{\alpha, v}$, provided $0 < \alpha < a$. Since $G(\alpha, v) = v$ if $\alpha \leq 0$, we see that $G(\alpha, v)$ is differentiable at $(\alpha, v)$ for any $L^p(\Omega)$, with derivative $\partial_v G(0, v) = I$. This completes the proof. \(\square\)

**Proposition A.2.** Let $v \in L^p(\Omega)$ satisfy $v(x) \neq 0$ for a.a. $x \in \Omega$. It follows that $\partial_v G(\alpha, v)$ is continuous at $(0, v)$.

**Proof.** Consider a sequence $(\alpha_n, v_n)_{n \geq 1} \subset \mathbb{R} \times L^p(\Omega)$ such that $\alpha_n \to 0$ in $\mathbb{R}$ and $v_n \to v$ in $L^p(\Omega)$. By (A.4), we need to show that $L_{\alpha_n, v_n} \to L_{0, v} = I$ in $L(L^p(\Omega), L^2(\Omega))$ as $n \to \infty$. Since $L_{\alpha, v} = I$ if $\alpha \leq 0$, we may assume that $\alpha_n > 0$. We write
\[
L_{\alpha_n, v_n} = L_{\alpha_n, v_n}^1 + L_{\alpha_n, v_n}^2,
\] (A.11)
where
\[
L_{\alpha_n, v_n}^1 u = |v_n|^{\alpha_n} u, \quad L_{\alpha_n, v_n}^2 u = \alpha_n|v_n|^{\alpha_n-2} v_n \Re(v_n u).
\]
We first note that $\alpha_n|v_n|^{\alpha_n} \leq \alpha_n(1 + |v_n|^2)$, so that
\[
\|L_{\alpha_n, v_n}^2\|_{L(L^p, L^2)} \leq \alpha_n(\|\alpha_n\|_1 + \|v_n\|_{L^p}^2) \to 0.
\] (A.12)
Furthermore, since $|v| > 0$ a.e. in $\Omega$, we see that
\[
|v_n|^{\alpha_n} \to 1 \text{ a.e. in } \Omega,
\]
a.e. in $\Omega$, and it follows by dominated convergence that
\[
L_{\alpha_n, v_n}^1 \to I.
\] (A.13)

---

1. Recall that we consider the space $L^2(\Omega)$ of complex-valued functions as a **real** Banach space.
in $L^p(\Omega), L^2(\Omega)$). The result is now a consequence of (A.11)–(A.13). □

References

[1] Berestycki H. and Lions P.-L. Nonlinear scalar field equations, I. Existence of a ground state, Arch. Ration. Mech. Anal. 82 (1983), no. 4, 313–345. (MR0695535) (doi: 10.1007/BF00250555)

[2] Chung K.W. and Cao Y.Y. Exact front, soliton and hole solutions for a modified complex Ginzburg-Landau equation from the harmonic balance method. Appl. Math. Comput. 218 (2012), no. 9, 5140–5145. (MR2870036) (doi: 10.1016/j.amc.2011.10.080)

[3] Cruz-Pacheco G., Levermore C.D. and Luce B.P. Complex Ginzburg-Landau equations as perturbations of nonlinear Schrödinger equations. Phys. D 197 (2004), no. 3-4, 269–285. (MR2093575) (doi: 10.1016/j.physd.2004.07.012)

[4] Descalzi O., Argentina M. and Tirapegui E. Stationary localized solutions in the subcritical complex Ginzburg-Landau equation. Spatio-temporal complexity. Internat. J. Bifur. Chaos Appl. Sci. Engrg. 12 (2002), no. 11, 2459–2465. (MR1956001) (doi: 10.1142/S0218127402005960)

[5] Doelman A. Traveling waves in the complex Ginzburg-Landau equation. J. Nonlinear Sci. 3 (1993), no. 2, 225–266. (MR1220175)

[6] Doering C.R., Gibbon J.D., Holm, D.D. and Nicolaenko B. Low-dimensional behaviour in the complex Ginzburg-Landau equation. Nonlinearity 1 (1988), no. 2, 279–309. (MR0937004) (doi: 10.1088/0951-7715/1/2/001)

[7] Doering C.R., Gibbon J.D. and Levermore C.D. Weak and strong solutions of the complex Ginzburg-Landau equation. Phys. D 71 (1994), 285–318. (MR1264120) (doi: 10.1016/0167-2789(94)90150-3)

[8] Friedman A. On the regularity of the solutions of nonlinear elliptic and parabolic systems of partial differential equations. J. Math. Mech. 7 (1958), no. 1, 43–59. (MR0118970) (doi: 10.1512/iumj.1958.7.57004)

[9] Ginibre J. and Velo G. The Cauchy problem in local spaces for the complex Ginzburg-Landau equation I: compactness methods, Phys. D 95 (1996), no. 3-4, 191–228. (MR1406282) (doi: 10.1016/0167-2789(96)00055-3)

[10] Ginibre J. and Velo G. The Cauchy problem in local spaces for the complex Ginzburg-Landau equation II: contraction methods, Comm. Math. Phys. 187 (1997), no. 1, 45–79. (MR1463822) (doi: 10.1007/s002200050129)

[11] Lan Y., Garnier N. and Cvitanović P. Stationary modulated-amplitude waves in the 1D complex Ginzburg-Landau equation. Phys. D 188 (2004), no. 3-4, 193–212. (MR2043730) (doi: 10.1016/S0167-2789(03)00289-6)

[12] Lega J. and Fauve S. Traveling hole solutions to the complex Ginzburg-Landau equation as perturbations of nonlinear Schrödinger dark solitons. Phys. D 102 (1997), no. 3-4, 234–252. (MR1439689) (doi: 10.1016/S0167-2789(96)00218-7)

[13] Levermore C.D. and Oliver M. The complex Ginzburg-Landau equation as a model problem. In Dynamical systems and probabilistic methods in partial differential equations (Berkeley, CA, 1994), 141–190, Lectures in Appl. Math., 31 , Amer. Math. Soc., Providence, RI, 1996. (MR1363028)

[14] Levermore C.D. and Oliver M. Distribution-valued initial data for the complex Ginzburg-Landau equation. Comm. Partial Differential Equations 22 (1997), no. 1-2, 39–48. (MR1434137) (doi: 10.1080/03605309708821254)

[15] Mancas S.C. and Choudhury S.R. The complex cubic-quintic Ginzburg-Landau equation: Hopf bifurcations yielding traveling waves. Math. Comput. Simulation 74 (2007), no. 4-5, 281–291. (MR2323319) (doi: 10.1016/j.matcom.2006.10.022)

[16] Mischakow K. and Morita Y. Dynamics on the global attractor of a gradient flow arising from the Ginzburg-Landau equation. Japan J. Indust. Appl. Math. 11 (1994), no. 2, 185–202. (MR1286431) (doi: 10.1007/BF03167221)

[17] Mohamadou A., Ndzana F.II and Kofané T.C. Pulse solutions of the modified cubic complex Ginzburg-Landau equation. Phys. Scr. 73 (2006), no. 6, 596–600. (MR2247673) (doi: 10.1088/0031-8949/73/6/011)

[18] Okazawa N. and Yokota T. Monotonicity method for the complex Ginzburg-Landau equation, including smoothing effect. Proceedings of the Third World Congress of Nonlinear
Analysts, Part 1 (Catania, 2000). Nonlinear Anal. 47 (2001), no. 1, 79–88. (MR1970632) (doi: 10.1016/S0362-546X(01)00158-4)

[19] Okazawa N. and Yokota T. Global existence and smoothing effect for the complex Ginzburg-Landau equation with $p$-Laplacian, J. Differential Equations 182 (2002), 541–576. (MR1900334) (doi: 10.1006/jdeq.2001.4097)

[20] Okazawa N. and Yokota T. Monotonicity method applied to the complex Ginzburg-Landau and related equations, J. Math. Anal. Appl. 267 (2002), 247–263. (MR1888827) (doi: 10.1006/jmaa.2001.7770)

[21] Okazawa N. and Yokota T. Perturbation theory for $m$-accretive operators and generalized complex Ginzburg-Landau equations, J. Math. Soc. Japan 54 (2002), 1–19. (MR1864925) (doi: 10.2969/jmsj/1191593952)

[22] Pohožaev S.I. Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, Soviet Math. Dokl. 6 (1965), 1408–1411.

[23] Popp S., Stiller O., Aranson I. and Kramer L. Hole solutions in the 1D complex Ginzburg-Landau equation. Phys. D 84 (1995), no. 3–4, 398–423. (MR1336543) (doi: 10.1016/0167-2789(95)00070-K)

[24] Popp S., Stiller O., and Kramer L. From dark solitons in the defocusing nonlinear Schrödinger to holes in the complex Ginzburg-Landau equation. Phys. D 84 (1995), no. 3-4, 424–436. (MR1336544) (doi: 10.1016/0167-2789(95)00071-B)

[25] van Saarloos W. and Hohenberg P.C. Fronts, pulses, sources and sinks in generalized complex Ginzburg-Landau equations. Phys. D 56 (1992), no. 4, 303–367. (MR1169610) (doi: 10.1016/0167-2789(92)90175-M)

[26] Zeidler E. Nonlinear functional analysis and its applications. I. Fixed-point theorems. Translated from the German by Peter R. Wadsack. Springer-Verlag, New York, 1986. (MR0816732)