A note on Perelman’s no shrinking breather theorem

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As an application of his entropy formula, Perelman [8] proved that every compact shrinking breather is a shrinking gradient Ricci soliton. We give a proof for the complete noncompact case by using Perelman’s \( L \)-geometry. Our proof follows the argument in Lu and Zheng [6] of constructing an ancient solution, and removes a technical assumption made by them. For other proofs with additional assumptions, please refer to Zhang [9].

1 Introduction

After showing that the Ricci flow is the gradient flow of the \( F \) functional

\[
F(g, f) := \int_M (|\nabla f|^2 + R) e^{-f} dg,
\]

Perelman [8] indicated that the Ricci flow on a manifold \( M \) can be regarded as an orbit in the space

\[
\text{Met}(M) / \text{Diff},
\]

where Met(\( M \)) stands for the space of all the Riemannian metrics on \( M \) and Diff represents all the self-diffeomorphisms on \( M \). The breathers are the periodic orbits in this space.

**Definition 1.** A metric \( g(t) \) evolving by the Ricci flow on a Riemannian manifold \( M \) is called a breather, if for some \( t_1 < t_2 \), there exists an \( \alpha > 0 \), a diffeomorphism \( \phi : M \to M \), such that \( \alpha g(t_1) = \phi^* g(t_2) \). If \( \alpha = 1 \), \( \alpha < 1 \), or \( \alpha > 1 \), then the breather is called steady, shrinking, or expanding, respectively.

As a special case of the periodic orbits, the Ricci solitons, moving by diffeomorphisms, are the static orbits in the space Met(\( M \))/Diff.

**Definition 2.** A gradient Ricci soliton is a tuple \((M, g, f)\), where \((M, g)\) is a Riemannian manifold and \( f \) is a smooth function on \( M \) called the potential function, satisfying

\[
\text{Ric} + \nabla^2 f = \frac{\lambda}{2} g,
\]
where \( \lambda = 0, \lambda = 1, \) or \( \lambda = -1 \), corresponding to the cases of steady, shrinking, or expanding solitons, respectively.

It is well understood that when moving by the 1-parameter family of diffeomorphisms generated by the potential function, along with a scaling factor, the pull-back metric on the soliton satisfies the Ricci flow equation, and this Ricci flow is called the canonical form of the Ricci soliton; one may refer to [3] for more details.

Perelman proved that on a closed manifold, any periodic orbit in \( \text{Met}(M)/\text{Diff} \) must be static.

**Theorem 3** (Perelman’s no breather theorem). A steady, shrinking, or expanding breather on a closed manifold is (the canonical form of) a steady, shrinking, or expanding gradient Ricci soliton, respectively. In particular, in the steady or expanding case, the breather is also Einstein.

We extend the no shrinking breather theorem to the complete noncompact case.

**Theorem 4.** Every complete noncompact shrinking breather with bounded curvature is (the canonical form of) a shrinking gradient Ricci soliton.

Our main technique is the \( \mathcal{L} \)-geometry, one of the two monotonicity formulae on the Ricci flow found by Perelman. In section 2 we give a brief introduction to the \( \mathcal{L} \)-functional. In section 3 we prove Theorem 4.

## 2 Perelman’s \( \mathcal{L} \)-geometry

The definitions and results in this section can be found in Perelman [8] and Naber [7]. We consider a backward Ricci flow \( (M, g(\tau)), \tau \in [0, T] \), satisfying

\[
\frac{\partial}{\partial \tau} g(\tau) = 2\text{Ric}(g(\tau)).
\]

(1)

Let \( \gamma(\tau) : [0, \tau_0] \to M \) be a smooth curve, then the \( \mathcal{L} \)-functional of \( \gamma \) is defined by

\[
\mathcal{L}(\gamma) := \int_0^{\tau_0} \sqrt{T} \left( R(\gamma(\tau), \tau) + |\dot{\gamma}(\tau)|^2_{g(\tau)} \right) d\tau.
\]

(2)

The reduced distance between two space-time points \((x_0, 0), (x_1, \tau_1)\), where \( \tau_1 > 0 \), is defined by

\[
l_{(x_0, 0)}(x_1, \tau_1) := \frac{1}{2\sqrt{\tau_1}} \inf_{\gamma} \mathcal{L}(\gamma),
\]

(3)
where the inf is taken among all the (piecewise) smooth curves \( \gamma : [0, \tau_1] \to M \), such that \( \gamma(0) = x_0 \) and \( \gamma(\tau_1) = x_1 \). When regarded as a function of \( (x_1, \tau_1) \), we call \( l_{(x_0,0)}(\cdot, \cdot) \) the reduced distance based at \( (x_0,0) \). When the base point is understood, we also write \( l_{(x_0)}(\cdot, \cdot) \) as \( l \). It is well known that the reduced volume based at \( (x_0,0) \)

\[
V_{(x_0,0)}(\tau) := \int_M (4\pi \tau)^{-\frac{n}{2}} e^{-l_{(x_0,0)}(\cdot, \tau)} dg(\tau)
\]

is monotonically decreasing in \( \tau \). We often write \( V_{(x_0,0)}(\tau) \) as \( V(\tau) \) for simplicity. We also remark here that the integrand \( (4\pi \tau)^{-\frac{n}{2}} e^{-l} \) of the reduced volume is a subsolution to the conjugate heat equation

\[
\frac{\partial}{\partial \tau} u - \Delta u + Ru = 0,
\]

in the barrier sense or in the sense of distribution.

Now we consider an ancient solution \( (M, g(\tau)) \), where \( \tau \in [0, \infty) \) is the backward time. The Type I condition is the following curvature bound.

**Definition 5.** An ancient solution \( (M, g(\tau)) \), where \( \tau \in [0, \infty) \) is the backward time, is called Type I if there exists \( C < \infty \), such that

\[
|Rm|(\tau) \leq \frac{C}{\tau},
\]

for every \( \tau \in (0, \infty) \).

To ensure the existence of a smooth limit, the \( \kappa \)-noncollapsing condition is often required.

**Definition 6.** A backward Ricci flow is called \( \kappa \)-noncollapsed, where \( \kappa > 0 \), if for any space-time point \( (x, \tau) \), any scale \( r > 0 \), whenever \( |Rm| \leq r^{-2} \) on \( B_g(\tau)(x, r) \times [\tau, \tau + r^2] \), it holds that \( \text{Vol}_g(\tau)(B_g(\tau)(x, r)) \geq \kappa r^n \).

We will use the following theorem of Naber [7].

**Theorem 7** (Asymptotic shrinker for Type I ancient solution). Let \( (M, g(\tau)) \), where \( \tau \in [0, \infty) \) is the backward time, be a Type I \( \kappa \)-noncollapsed ancient solution to the Ricci flow. Fix \( x_0 \in M \). Let \( l \) be the reduced distance based at \( (x_0,0) \). Let \( \{(x_i, \tau_i)\}_{i=1}^{\infty} \subset M \times (0, \infty) \) be such that \( \tau_i \nearrow \infty \) and

\[
\sup_{i=1}^{\infty} l(x_i, \tau_i) < \infty.
\]

Then \( \{(M, \tau_i^{-1} g(\tau_i), (x_i, 1))_{\tau \in [1,2]}\}_{i=1}^{\infty} \) converges, after possibly passing to a subsequence, to the canonical form of a shrinking gradient Ricci soliton.
Remark 1: In Naber’s original theorem, he fixes the base points \( x_i \equiv x_0 \). However, it is easy to observe from his proof that so long as (5) holds, all the estimates of \( l \) also hold in the same way as in his case. Hence one may apply the blow-down shrinker part of Theorem 2.1 in [7] to the sequence of space-time base points \((x_i, \tau_i)\) and the scaling factors \( \tau_i^{-1} \).

Remark 2: The estimates for \( l \) and the monotonicity formula for \( V \) in [7] do not depend on the noncollapsing condition. According to Hamilton [4], if the noncollapsing assumption is replaced by

\[
\inf_{i=1}^{\infty} \text{inj}_{\tau_i^{-1}g(\tau)}(x_i) > \delta,
\]

where \( \text{inj}_g(x) \) stands for the injectivity radius of the metric \( g \) at the point \( x \), and \( \delta > 0 \) is a constant, then the conclusion of Theorem 7 still holds.

3 Proof of the main theorem

Following the argument in Lu and Zheng [6], we construct a Type I ancient solution to the Ricci flow starting from a given shrinking breather. After scaling and translating in time, we consider the backward Ricci flow \((M, g_0(\tau))_{\tau \in [0, 1]}\), where \( g_0(\tau) \) satisfies (1), such that there exists \( \alpha \in (0, 1) \) and a diffeomorphism \( \phi : M \to M \), satisfying

\[
\alpha g_0(1) = \phi^* g_0(0). \tag{7}
\]

Furthermore, we let \( C < \infty \) be the curvature bound, that is,

\[
\sup_{M \times [0, 1]} |Rm|(g(\tau)) \leq C. \tag{8}
\]

For notational simplicity, we define

\[
\tau_i = \sum_{j=0}^{i} \alpha^{-j},
\]

where \( i = 0, 1, 2, \ldots \) Apparently, \( \tau_i \to \infty \) since \( \alpha \in (0, 1) \), and we can find a \( C_0 < \infty \) depending only on \( \alpha \) (for instance, one may let \( C_0 = (1 - \alpha)^{-1} \)), such that

\[
\alpha^{-i} \leq \tau_i \leq C_0 \alpha^{-i}, \text{ for every } i \geq 0. \tag{9}
\]

For each \( i \geq 1 \), we define a Ricci flow

\[
g_i(\tau) := \alpha^{-i}(\phi^i)^* g_0(\alpha^i(\tau - \tau_{i-1})), \text{ where } \tau \in [\tau_{i-1}, \tau_i]. \tag{10}
\]
To see all these Ricci flows are well-concatenated, we apply (7) to observe that
\[ g_1(\tau_0) = \alpha^{-1} \phi^* g_0(0) = g_0(1), \]
\[ g_i(\tau_{i-1}) = \alpha^{-i} (\phi^i)^* g_0(0) = \alpha^{-(i-1)} (\phi^i)^* g_0(1) \]
\[ = \alpha^{-(i-1)} (\phi^i)^* g_0 (\alpha^{i-1} (\tau_{i-1} - \tau_{i-2}) ) = g_{i-1} (\tau_{i-1}). \]

Therefore we define an ancient solution
\[ g(\tau) = \begin{cases} 
  g_0(\tau) & \text{for } \tau \in [0, 1] \\
  g_i(\tau) & \text{for } \tau \in [\tau_{i-1}, \tau_i] \text{ and } i \geq 1.
\end{cases} \tag{11} \]

It then follows from the uniqueness theorem of Chen and Zhu [2] that the ancient solution \( g(\tau) \) is smooth.

Now we proceed to show that \((M, g(\tau))_{\tau \in [0, \infty)}\), where \( g(\tau) \) is defined in (11), is Type I. We need only to consider the case when \( \tau \geq 1 \). Let \( i \geq 1 \) be such that \( \tau \in [\tau_{i-1}, \tau_i] \). Then
\[ |Rm(g(\tau))| = |Rm(g_i(\tau))| \leq \alpha^i \sup_{M \times [0, 1]} \left| Rm\left( (\phi^i)^* g_0(\tau) \right) \right| \leq C \alpha^i, \]
where we have used (8), (10), and (11). Then we have
\[ |Rm(g(\tau))| \leq C \alpha^i \leq \frac{C}{\tau} \tau_i \alpha^i \leq \frac{B}{\tau}, \tag{12} \]
where we have used (9), and \( B = CC_0 \) is independent of \( i \).

With all the preparations, we are ready to prove our main theorem.

**Proof of Theorem 4.** Fix an arbitrary point \( y \in M \) as the base point, and for each \( i \geq 0 \) define
\[ x_i = \phi^{-(i+1)}(y). \tag{13} \]

In Lu and Zheng [3], they made an assumption that \( \{x_i\}_{i=1}^\infty \) are not drifted away to space infinity so as to apply Theorem 4.1 in [1] to show that \( \{(M, \tau_i^{-1} g(\tau_i), (x_i, 1))_{\tau_i=[1,2]}\}_{i=1}^\infty \) converges, after passing to a subsequence, to the canonical form of a shrinking gradient Ricci soliton. Instead we will show that \( l(x_i, \tau_i) \), where \( i \geq 0 \) and \( l \) is the reduced distance based at \( (y, 0) \), is a bounded sequence. To see this, we let \( \sigma : [0, 1] \to M \) be a smooth curve such that \( \sigma(0) = y \) and \( \sigma(1) = x_0 \). Let \( A < \infty \) be such that
\[ |\dot{\sigma}(\tau)|_{g_0(\tau)} \leq A, \text{ for all } \tau \in [0, 1]. \tag{14} \]

For each \( i \geq 0 \), we define
\[ \sigma_i(\tau) := \phi^{-(i+1)} \circ \sigma(\alpha_i^{i+1}(\tau - \tau_i)), \text{ where } \tau \in [\tau_i, \tau_{i+1}]. \tag{15} \]
We observe that these \( \sigma_i \)'s and \( \sigma \) altogether define a piecewise smooth curve in \( M \):
\[
\sigma_0(\tau_0) = \phi^{-1} \circ \sigma(0) = \phi^{-1}(y) = x_0 = \sigma(1),
\sigma_i(\tau_i) = \phi^{-(i+1)} \circ \sigma(0) = \phi^{-i} \circ \sigma(1) = \phi^{-i} \circ \sigma(\alpha^i(\tau_i - \tau_{i-1})) = \sigma_{i-1}(\tau_i).
\]
We then define \( \gamma_i : [0, \tau_{i+1}] \to M \), where \( i \geq 0 \), as
\[
\gamma_i(\tau) := \begin{cases} 
\sigma(\tau) & \text{when } \tau \in [0, 1], \\
\sigma_j(\tau) & \text{when } \tau \in [\tau_j, \tau_{j+1}] \text{ and } 0 \leq j \leq i.
\end{cases}
\]
Apparenty \( \gamma_i(\tau) \) is piecewise smooth, and \( \gamma_i(0) = y \), \( \gamma_i(\tau_{i+1}) = \phi^{-(i+2)}(y) = x_{i+1} \). We compute for \( i \geq 0 \)
\[
\mathcal{L}(\gamma_i) = \mathcal{L}(\sigma) + \sum_{j=0}^{i} \int_{\tau_j}^{\tau_{j+1}} \sqrt{\tau \left( R(\sigma_j(\tau), \tau) + |\dot{\sigma}_j(\tau)|^2_{g_{j+1}(\tau)} \right)} d\tau
\leq D + \sum_{j=0}^{i} \int_{\tau_j}^{\tau_{j+1}} \sqrt{\tau \left( \frac{B}{\tau} + A\alpha^{j+1} \right)} d\tau
\]
where in the last inequality we have used \( D \), a constant independent of \( i \), to represent \( \mathcal{L}(\sigma) \), and we have used the Type I condition (12), the definition (15) of \( \sigma_j \), and the assumption (14). Continuing the computation using (9), we have
\[
\mathcal{L}(\gamma_i) \leq D + C_1 \sum_{j=0}^{i} \alpha^{-\frac{j+1}{2}},
\]
where \( C_1 \) is a constant independent of \( i \). It follows from the definition (3) that
\[
l(x_{i+1}, \tau_{i+1}) \leq \frac{1}{2\sqrt{\tau_{i+1}}} \mathcal{L}(\gamma_i)
\leq \frac{1}{2} D\alpha^{-\frac{i+1}{2}} + \frac{1}{2} C_1 \sum_{j=0}^{i} \alpha^{j} \leq C_2 < \infty,
\]
where \( C_2 \) is a constant independent of \( i \), and we have used \( \alpha^{\frac{i}{2}} \in (0, 1) \).

Now we consider the sequence
\[
\{(M, \tau_i^{-1}g(\tau_i), (x_i, 1))_{\tau \in [1, \alpha^{-1}]} \}_{i=1}^{\infty}.
\]
We observe that
\[
\tau_i^{-1}g(\tau_i) = \tau_i^{-1}\alpha^{-(i+1)} \left( \phi^{i+1} \right)^{*} g_0(0),
\]
where $\tau_i^{-1} \alpha^{-(i+1)}$ is bounded from above and below by constants independent of $i$, because of (9). Taking into account the definition (13) of $x_i$, we can use

$$\text{inj}_{g_0(0)}(y) > 0$$

to verify the condition (6). It follows from Theorem 7 that (16) converges smoothly to the canonical form of a shrinking gradient Ricci soliton. Furthermore, since $(M, \tau_i^{-1} g(\tau_i), x_i)$ and $(M, g_0(0), y)$ differ only by a bounded scaling constant and a diffeomorphism that preserves the base points, by the definition of the Cheeger-Gromov convergence, such diffeomorphism does not affect the limit. In other words, there exists a constant $C_3 > 0$, such that

$$(M, \tau_i^{-1} g(\tau_i), x_i) \rightarrow (M, C_3 g_0(0), y)$$

in the pointed smooth Cheeger-Gromov sense. Therefore $(M, g_0(0), y)$ also has a shrinker structure up to scaling. It then follows from the backward uniqueness of Kotschwar [5] that the shrinking breather $(M, g_0(\tau))_{\tau \in [0, 1]}$ is the canonical form of a shrinking gradient Ricci soliton.

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