MDS Ideal Secret Sharing Scheme
from
AG-codes on Elliptic Curves

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Abstract

For a secret sharing scheme, two parameters $d_{\text{min}}$ and $d_{\text{cheat}}$ are defined in [12] and [13]. These two parameters measure the error-correcting capability and the secret-recovering capability of the secret sharing scheme against cheaters. Some general properties of the parameters have been studied in [12],[9] and [13]. The MDS secret-sharing scheme was defined in [13] and it was proved that MDS perfect secret sharing scheme can be constructed for any monotone access structure. The famous Shamir $(k, n)$ threshold secret sharing scheme is the MDS with $d_{\text{min}} = d_{\text{cheat}} = n - k + 1$. In [3] we proposed the linear secret sharing scheme from algebraic-geometric codes. In this paper the linear secret sharing scheme from AG-codes on elliptic curves is studied and it is shown that many of them are MDS linear secret sharing scheme.
I. Introduction and Preliminaries

In a secret-sharing scheme among the set of players $P = \{P_1, ..., P_n\}$, a dealer $P_0$, not in $P$, has a secret, the dealer distributes the secret among $P$ such that only the qualified subsets of players $P$ can reconstruct the secret from their shares. The access structure, $\Gamma \subset 2^P$, of a secret-sharing scheme is the family of the qualified subsets of $P$. The minimum access structure $\min \Gamma \subset 2^P$ is defined to be the set of minimal elements in $\Gamma$ (here we use the natural order relation $S_1 < S_2$ if and only if $S_1 \subset S_2$ on $2^P$). We call a secret-sharing scheme a $(k, n)$-threshold scheme if the access structure consists of the subset of at least $k$ elements in the set $P$, where the number of elements in the set $P$ is exactly $n$, that is, among the $n$ players any subset of $k$ or more than $k$ players can reconstruct the secret. The first secret-sharing scheme was given independently by Blakley [2] and Shamir [15] in 1979, actually they gave threshold secret-sharing scheme. We call a secret-sharing scheme perfect if the unqualified subsets of players to reconstruct the secret have no information of the secret. The existence of secret-sharing schemes with arbitrary given access structures was proved in [1] and [8]. Let $K$ be a finite field, we refer to [4] for the definition of linear secret sharing scheme (LSSS) over $K$ ($K$-LSSS) and its relation with linear error-correcting codes.

For a secret-sharing scheme, we denote the set of all possible shares $(v_1, ..., v_n)$ (Here $v_i$ is the share of the player $P_i$ for $i = 1, ..., n$) by $V$. Then $V$ is a error-correcting code (not necessarily linear), let $d_{\min}$ be the minimum Hamming distance of this error-correcting code $V$. From the error-correcting capability, it is clear that the cheaters can be identified from any share (presented by the players)$(v_1, ..., v_n)$ if there are at most $\lfloor (d_{\min} - 1)/2 \rfloor$ cheaters. In [12] McEliece and Sarwate proved that $d_{\min} = n - k + 1$ for Shamir’s $(k, n)$-threshold scheme. K. Okada and K. Kurosawa introduced another parameter $d_{\text{cheat}}$ for general secret-sharing scheme, as the the number such that the correct secret value $s$ can be recovered if there are at most $\lfloor (d_{\text{cheat}} - 1)/2 \rfloor$ cheaters (see [13]). It is clear that $d_{\min} \leq d_{\text{cheat}}$. In [13] the authors proved $d_{\text{cheat}} = n - \max_{B \in (2^P - \Gamma)} |B|$, where $|B|$ is the number of the elements in the set $B$. The secret sharing scheme is called MDS if
\[d_{\text{min}} = d_{\text{cheat}} = n - \max_{B \in (2^P - 1)}\] It was also proved in [13] that any monotone access structure can be realized by a perfect MDS secret scheme.

The approach of secret-sharing based on error-correcting codes was studied in [4],[5],[9],[10],[11] and [12]. It is found that actually Shamir’s \((k, n)\)-threshold scheme is just the secret-sharing scheme based on the famous Reed-Solomon (RS) code. The error-correcting code based secret-sharing scheme is defined as follow. Here we suppose \(C\) is a linear error-correcting code over the finite field \(GF(q)\) (where \(q\) is a prime power) with code length \(n + 1\) and dimension \(k\), i.e., \(C\) is a \(k\) dimension subspace of \(GF(q)^{n+1}\). The Hamming distance \(d(C)\) of this error-correcting code \(C\) is defined as follows.

\[d(C) = \min \{\text{wt}(v) : v \in C\}\]

where \(\text{wt}(v)\) is called the Hamming weight of \(v\). Let \(G = (g_{ij})_{1 \leq i \leq k, 0 \leq j \leq n}\) be the generator matrix of \(C\), i.e., \(G\) is a \(k \times (n + 1)\) matrix in which \(k\) rows of \(G\) is a base of the \(k\) dimension subspace \(C\) of \(GF(q)^{n+1}\). Suppose \(s\) is a given secret value of the dealer \(P_0\) and the secret is shared among \(P = \{P_1, ..., P_n\}\), the set of \(n\) players. Let \(g_1 = (g_{11}, ..., g_{k1})^T\) be the 1st column of \(G\). Chosen a random \(u = (u_1, ..., u_k) \in GF(q)^k\) such that \(s = u^Tg_0 = \Sigma u_i g_{i0}\). We have the codeword \(c = (c_0, ..., c_N) = uG\), it is clear that \(c_0 = s\) is the secret, then the dealer \(P_0\) gives the \(i-th\) player \(P_i\) the \(c_i\) as the share of \(P_i\) for \(i = 1, ..., n\). In this secret-sharing scheme the error-correcting code \(C\) is assumed to be known to every player and the dealer. For a secret sharing scheme from error-correcting codes, suppose that \(T_i : GF(q)^k \rightarrow GF(q)\) is defined as \(T_i(x) = x^T g_i\), where \(i = 0, ..., n\) and \(g_i\) is the \(i-th\) column of the generator matrix of the code \(C\). In this form we see that the secret sharing scheme is an ideal linear secret sharing scheme over \(GF(q)\) (\(GF(q)-\text{LSSS}\), see [4]).

We refer the following Lemma to [5],[10] and [11].

**Lemma 1** (see [5], [8] and [11]). Suppose the dual of \(C\), \(C^\perp = \{v = (v_0, ..., v_n) : Gv = 0\}\) has no codeword of Hamming weight 1. In the above secret-sharing scheme based on the error-correcting code \(C\), \((P_{i1}, ..., P_{in})\) can reconstruct the secret if and only if there is a codeword \(v = (1, 0, ..., v_{i1}, ..., v_{im}, ...0)\) in \(C^\perp\) such that \(v_{ij} \neq 0\) for at least one \(j\), where \(1 \leq j \leq m\).
The secret reconstruction is as follows, since $Gv = 0$, $g_i = -\sum_{j=1}^{m} v_{ij} g_{ij}$, where $g_{ih}$ is the $h$–th column of $G$ for $h = 1, \ldots, N$. Then $s = c_0 = ug_1 = -uS_{j=1}^{m} g_{ij} = -\sum_{j=1}^{m} v_{ij} c_{ij}$.

We need recall some basic facts about algebraic-geometric codes. Let $X$ be an absolutely irreducible, projective and smooth curve defined over $GF(q)$ with genus $g$, $D = \{P_0, \ldots, P_n\}$ be a set of $GF(q)$-rational points of $X$ and $G$ be a $GF(q)$-rational divisor satisfying $supp(G) \cap D = \emptyset$. Let $L(G) = \{f : (f) + G \geq 0\}$ is the linear space (over $GF(q)$) of all rational functions with its divisor not smaller than $-G$ and $\Omega(B) = \{\omega : (\omega) \geq B\}$ be the linear space of all differentials with their divisors not smaller than $B$. Then the functional $AG$(algebraic-geometric )code $C_L(D, G) \in GF(q)^{n+1}$ and residual $AG$(algebraic-geometric) code $C_{\Omega}(D, G) \in GF(q)^{n+1}$ are defined. $C_L(D, G)$ is a $[n+1, k = dim(L(G)−dim(L(G−D)), d \geq n+1−deg(G))]$ code over $GF(q)$ and $C_{\Omega}(D, G)$ is a $[n+1, k = dim(\Omega(G−D))−dim(\Omega(G)), d \geq deg(G)−2g+2]$ code over $GF(q)$. We know that the functional code is just the evaluations of functions in $L(G)$ at the set $D$ and the residual code is just the residues of differentials in $\Omega(G_D)$ at the set $D$ (see [16], [17] and [18]).

We also know that $C_L(D, G)$ and $C_{\Omega}(D, G)$ are dual codes. It is known that for a differential $\eta$ that has poles at $P_1, \ldots, P_n$ with residue 1 (there always exists such a $\eta$, see[16]) we have $C_{\Omega}(D, G) = C_L(D, D−G + (\eta))$, the function $f$ corresponds to the differential $f \eta$. This means that functional codes and residue code are essentially same.

II. Main Results

Let $X$ be an absolutely irreducible, projective and smooth curve defined over $GF(q)$ with genus $g$, $D = \{P_0, \ldots, P_n\}$ be a set of $GF(q)$-rational points of $X$ and $G$ be a $GF(q)$-rational divisor with degree $m$ satisfying $supp(G) \cap D = \emptyset$. We can have a LSSS on the $n$ players $P = \{P_1, \ldots, P_n\}$ from the linear code $C_{\Omega}(D, G)$, thus we know that the reconstruction of the secret is based from its dual code $C_L(D, G)$. For the curve of genus 0 over $GF(q)$, we have exactly the same LSSS as Shamir’s $(k, n)$-threshold scheme, since the AG-codes over the curve of genus 0 is just the RS codes (see [16], [17] and [18]).

The following Theorem 4 and Corollary 1 are the main results of this
Theorem 1. For the LSSS over $GF(q)$ from the code $C_{\Omega}(D, G)$ we have $m - 2g + 1 \leq d_{\text{min}} \leq d_{\text{cheat}} \leq m + 1$.

Proof. From the theory of AG-codes ([12-14]), we know $C_{\Omega}(D, G)$ can be identified with $C_L(D, D - G + (\eta))$. Thus $d_{\text{min}}$ is the minimum Hamming weight of $C_L(P, D - G + (\eta))$. We have $d_{\text{min}} \geq m - 2g + 1$.

On the other hand any subset of $P$ less than $n - m$ elements is not qualified from the fact that the minimum Hamming weight of $C_L(D, G)$ is $n + 1 - m$. From the equality $d_{\text{cheat}} = n - \max_{B \in 2^P} |B|$, we have $d_{\text{cheat}} \leq n - (n - m - 1) = m + 1$. The conclusion is proved.

We need to recall the following result in [14].

Theorem 2 (see [14] and [7]). 1) Let $E$ be an elliptic curve over $GF(q)$ with the group of $GF(q)$-rational points $E(GF(q))$. Then $E(GF(q))$ is isomorphic to $Z_{n_1} \oplus Z_{n_2}$, where $n_1$ is a divisor of $q - 1$ and $n_2$.

2) If $E$ is supersingular, then $E(GF(q))$ is either
a) cyclic;
b) or $Z_2 \oplus Z_{2+1}$;
c) or $Z_{\sqrt{q} - 1} \oplus Z_{\sqrt{q} - 1}$;
d) or $Z_{\sqrt{q} + 1} \oplus Z_{\sqrt{q} + 1}$.

For any given elliptic curve $E$ over $GF(q)$, let $D' = \{g_0, g_1, ..., g_H\}$ be a subset of $E(GF(q))$ of $H + 1$ non-zero elements, let $G = mO$ ($O$ is the point of the zero element of $E(GF(q))$). $g_0, ...., g_H$ correspond to the rational points $P_0, P_1, ..., P_H$ of $E(GF(q))$. In the construction, we take $D = D'$ and $P = \{P_1, ..., P_H\}$. We have the following result.

Theorem 3. a) Let $A = \{P_{i_1}, ..., P_{i_t}\}$ be a subset of $P$ with $t$ elements, $B$ is the element in $E(GF(q))$ such that the group sum of $B$ and $g_{i_1}, ..., g_{i_t}$ is zero in the group $E(GF(q))$. Then $A^c$ (Here $A^c$ is the set $P - A$ ) is a qualified subset for the LSSS from $C_{\Omega}(D, G)$ only if $t \leq m$ and

1) When $t = m$, $A^c$ is a minimal qualified subset if and only if $B = O$, the zero element of $E(GF(q))$;
2) When \( t = m - 1 \), \( A^c \) is a minimal qualified subset if and only if \( B \) is not in \( D \) or \( B \) is in the set \( A \).

b) Any subset of \( P \) of more than \( n - m + 2 \) elements is qualified.

**Proof.** From the theory of AG-codes, the minimum Hamming weight of \( C_L(D, G) \) is \( n + 1 - m \), thus \( A^c \) is a qualified subset only if \( t \leq m \).

We know that for any \( t \) points \( W_1, \ldots, W_t \) in \( E(GF(q)) \) the divisor \( W_1 + \ldots + W_t - tO \) is linear equivalent to the divisor \( W - O \), where \( W \) is the group sum of \( W_1, \ldots, W_t \) in the group \( E(GF(q)) \). \( \{P_{t_1}, \ldots, P_{m}\}^c \) is a qualified subset (therefor minimal qualified subset) if there exist a function \( f \in L(G) \) such that \( f(P_{t_1}) = \ldots = f(P_{m}) = 0 \), this means that the divisor \( P_{t_1} + \ldots + P_{m} \) is linearly equivalent to \( G \). The conclusion of a) is proved.

\( \{P_{t_1}, \ldots, P_{m-1}\}^c \) is a qualified subset if there exist a function \( f \in L(G) \) such that \( f(P_{t_1}) = \ldots = f(P_{m-1}) = 0 \), this means that the divisor \( P_{t_1} + \ldots + P_{m-1} + B' \) is linearly equivalent to \( G \) for some effective divisor \( B' \). It is clear that \( deg(B') = 1 \) and \( B' \) is a \( GF(q) \)-rational point in \( E \). Thus \( B' \) is just the \( B \) in the condition. On the other hand we note that \( B \neq P_0 \), so \( B \) has to be in \( A \) or a point not in \( D \). The conclusion of a) is proved.

If \( A \) is a subset of \( P \) such that \( |A| \leq m - 2 \), the divisor \( G - A \) has its degree \( deg(G - A) \geq 2 \). So the corresponding system has no base point. We can find a function in \( L(G - A) \) such that it is not zero at \( P_0 \), thus we have a codeword in \( C_L(D, G) \) which is not zero at \( P_0 \) and zero at all points of \( A \). This implies that \( A^c \) is a qualified subset. The conclusion of b) is proved.

The following Corollary is a direct result of Theorem 3.

**Corollary 1.** If there is a subset of \( P \) of \( H - m + 1 \) elements which is not \( A^c \) of type 2) as in the above Theorem 3 and do not contain any subset of \( H - m \) elements of type a) in Theorem 3, then the LSSS in Theorem 1 is MDS (perfect) ideal secret sharing scheme.

**Theorem 4.** If \( D \cup \{O\} \) is a subgroup of \( E(GF(q)) \), then the ideal LSSS in Theorem 3 is MDS.

**Proof.** We prove that there exist \( m - 1 \) distinct elements \( g_{t_1}, \ldots, g_{m-1} \) in
\( \mathbf{P} \) such that \( g_i + \ldots + g_{n-1} = -g_0 \). First we choose 2 elements \( g_{i1}, g_{i2} \) when \( m - 1 \) is even (or 3 elements \( g_{i1}, g_{i2}, g_{i3} \) when \( m - 1 \) is odd) in the group \( \mathbf{D} \cup \{O\} \) such that \( g_{i1} + g_{i2} = -g_0 \) (\( g_{i1} + g_{i2} + g_{i3} = -g_0 \) when \( m - 1 \) is odd). The other \( m - 3 \) (when \( m - 1 \) is even, or \( m - 4 \) when \( m - 1 \) is odd) elements can be taken to be pairs of elements \((g_{ij}, -g_{ij})\). Since \( \mathbf{D} \cup \{O\} \) is group, thus the desired points can always be found.

For this subset \( A \) of \( m - 1 \) elements in \( \mathbf{P} \), if it is qualified we know that \( B \) in Theorem 3 is \( P_0 \), this is a contradiction to Theorem 3. We have a subset of \( \mathbf{P} \) of \( n - m + 1 \) elements which is not qualified. This implies \( d_{\text{cheat}} \leq m - 1 \). From Theorem 1 \( m - 1 \leq d_{\text{min}} \leq d_{\text{cheat}} \leq m - 1 \), we have \( d_{\text{min}} = d_{\text{cheat}} = m - 1 \). The conclusion is proved.

III. Examples

**Example 1.** Let \( E \) be the elliptic curve \( y^2 = x^3 + 5x + 4 \) defined over \( GF(7) \). Then \( E(GF(7)) \) is a cyclic group of order 10 with \( O \) the point at infinity and \( P_0 = (3, 2), P_1 = (2, 6), P_2 = (4, 2), P_3 = (0, 5), P_4 = (5, 0), P_5 = (0, 2), P_6 = (4, 5), P_7 = (2, 1), P_8 = (3, 5) \). From an easy computation we know that \( P_0 \) is a generator of \( E(GF(7)) \) and \( P_i \) is \((i + 1)P_0 \) (in the group operation of \( E(GF(7)) \)). We take \( G = 3O, \mathbf{D} = \{P_0, P_1, P_3, P_5, P_7\} \), then the access structure of the ideal \( GF(7) \)-LSSS from \( C_{\Omega}(\mathbf{D}, G) \) are the following subsets of \( \mathbf{P} \) = \{\( P_1, P_3, P_5, P_7 \)\}.

1) All subsets of \( \mathbf{P} \) with 3 elements and the set \( \mathbf{P} \);
2) The following 6 subsets of 2 elements \{\( P_1, P_7 \), \{\( P_1, P_3 \), \{\( P_1, P_5 \), \{\( P_3, P_7 \), \{\( P_5, P_7 \)\} are minimal qualified subsets.

We can check that every subset of \( \mathbf{P} \) of 2 elements is qualified so \( d_{\text{cheat}} = 3 \), it is easy to see that \( d_{\text{min}} = 2 \) we conclude that this ideal LSSS is not MDS.

**Example 2.** Let \( E \) be the elliptic curve \( y^2 + y = x^3 \) defined over \( GF(4) \). This is the Hermitian curve over \( GF(4) \), it has 9 rational points and \( E(GF(4)) \) is isomorphic to \( Z_3 \oplus Z_3 \). We take \( G = 3O \), where \( O \) is the zero element in the group \( E(GF(4)) \). Let \( P_{ij} \) be the rational point on \( E \) corresponding to \((i, j)\) in \( Z_3 \oplus Z_3 \). \( \mathbf{D} = \{P_{10}, P_{01}, \ldots, P_{22}\}, \mathbf{P} = \{P_{01}, \ldots, P_{22}\} \). Then the qualified subsets of \( \mathbf{P} \) are as follows.

1) The qualified subsets of 4 elements are \{\( P_{20}, P_{21}, P_{02} \)\}, \{\( P_{01}, P_{20}, P_{22} \)\}, \{\( P_{11}, P_{12}, P_{20} \)\}.
2) The qualified subsets of 5 elements are \{P_{01}, P_{02}\}^c, \{P_{11}, P_{22}\}^c, \{P_{12}, P_{21}\}^c.

3) The subsets of P of 6 elements and the set P are qualified.

The subsets in 1) and 2) are the minimal qualified subsets. It is clear that \(d_{\text{min}} = m - 2g + 1 = 2\) and \(d_{\text{cheat}} = 7 - 5 = 2\). Thus this ideal LSSS is MDS.

**Example 3.** Let E be the elliptic curve \(y^2 + y = x^3\) defined over \(GF(q), q = 2^r\). This is a super-singular elliptic curve, \(E(GF(q))\) has \(2^r + 1\) rational points and is isomorphic to a cyclic group when \(r\) is an odd number; \(E(GF(q))\) has \(2^r + 1 + 2 \cdot 2^{\frac{r}{2}}\) rational points and is isomorphic to the product of two cyclic groups of order \(2^{\frac{r}{2}} + 1\) when \(r\) is an even number. We take \(G = mO\), where \(O\) is the zero element in the group \(E(GF(q))\). Let D be the set of all non-zero rational points and the point \(P_0\) be an arbitrary non-zero point in D. From Theorem 4, the ideal LSSS over \(GF(q)\) is MDS.

For any fixed \(r\), we can calculate the access structure as in Example 2. Now suppose \(r = 3\). Then the access structure can be computed as follows.

In the case over \(GF(8), E(GF(8))\) has 9 rational points and it is a cyclic group of order 9. Let \(P_i\) be the rational point on \(E\) corresponding to \(i\) in \(Z_9 = \{0, 1, 2, ..., 7, 8\}\) for \(i = 1, 2, ..., 8\). Let \(G = 3O\), where \(O\) corresponds to the zero element 0 in the group \(E(GF(8))\), \(D = \{P_1, ..., P_8\}\) and \(P = \{P_2, ..., P_8\}\). Then the access structure of the ideal LSSS from \(C\Omega(D, G)\) is as follows.

1) The minimal qualified subsets of 4 elements are \{\(P_2, P_3, P_4\)^c, \(P_3, P_7, P_8\)^c, \(P_4, P_6, P_8\)^c, \(P_5, P_6, P_7\)^c\}.

2) The minimal qualified subsets of 5 elements are \{\(P_2, P_5\)^c, \(P_2, P_7\)^c, \(P_2, P_8\)^c, \(P_3, P_6\)^c, \(P_4, P_5\)^c, \(P_4, P_7\)^c, \(P_5, P_8\)^c\}.

3) The subsets of P of 6 elements and the set P are qualified.

**IV. Conclusion**

We have proved some sufficient conditions about the MDS ideal linear secret-sharing scheme from the AG-codes on elliptic curves, which can be thought as a natural generalization of Shamir’s \((k, n)\)-threshold scheme(from AG-codes on the genus 0 curve, RS codes). From the main results of this paper many MDS ideal secret sharing schemes can be constructed. This
demonstrates that elliptic curves, perhaps also hyper-elliptic curves, are important resource in the theory and practice of secret-sharing.

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