Conserved supercurrents and Fayet-Iliopoulos terms in supergravity

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Abstract

Recently there has appeared in the literature a sequence of papers questioning the consistency of supergravity coupled to Fayet-Iliopoulos terms. A key feature of these arguments is a demonstration that the conventional superspace stress tensor fails to be gauge invariant. We briefly show here how this can be understood as defining the stress tensor in a non-covariant Brans-Dicke frame in an underlying superconformal theory. When converted to the Einstein frame, the inconsistency vanishes, which is consistent with the emergence of a global symmetry discussed in these papers.
1 Introduction

Globally supersymmetric theories admit a peculiar type of term, known as a Fayet-Iliopoulos D-term, which in superspace language may be written

$$S_{FI} = 2\xi \int d^8z V = -\xi \int d^4x D$$  \hspace{1cm} (1.1)

up to a total derivative (which we shall always discard). Here $V$ is the gauge prepotential of some $U(1)$ (denote it $U(1)_\xi$), $D = -D^\alpha \bar{D}^\alpha V/8$ is its highest component, and $\xi$ is a parameter of dimension two. The gauge invariance of this expression follows since under a gauge transformation

$$V \rightarrow V + \Lambda + \bar{\Lambda}$$

for chiral $\Lambda$,

$$\int d^8z \Lambda = 0.$$  \hspace{1cm} (1.2)

The coupling of such terms to conventional (old minimal) supergravity is not entirely straightforward since the direct analogue of (1.2) fails to be true there, but a construction exists which we will review shortly.

Recently it has been argued by Komargodski and Seiberg \cite{1} that the inclusion of such a term in supergravity is problematic: specifically, its contribution to the superspace stress tensor of supergravity fails to be gauge invariant at the classical level without requiring a global symmetry. This has been expounded upon in detail by Dienes and Thomas \cite{3}, who examined the situation in further detail in both old and new minimal supergravity, and by Kuzenko \cite{4}, who clarified several issues relating to the existence of the supercurrents.

We would like to comment on some of the issues involved in this argument by considering the full coupling of the theory to supergravity in superspace, specifically, the super-Weyl redefinitions which must accompany any coupling of a D-term in conventional (old minimal) supergravity. The easiest way to see the issue is to consider not an actual Fayet-Iliopoulos term, but another term rather closely related and even more mundane: the Kähler potential. Recall that a supersymmetric non-linear sigma model can be constructed from the D-term

$$\int d^8z K = \int d^4x \left(-K_{\bar{\phi}^i \phi^j} \partial^m \phi^i \partial^m \bar{\phi}^j + \ldots\right)$$  \hspace{1cm} (1.3)

On the right hand side of this formula only the Kähler metric $K_{\bar{\phi}^i \phi^j}$ (and its derivatives) appears; the reason is that the left hand side is invariant under $K \rightarrow K + F + \bar{F}$ for chiral $F$ and so must be the right side. (In fact, the Kähler potential term may be understood as a composite Fayet-Iliopoulos term for some $U(1)_K$.)

Naively coupling such a term to supergravity is a bit problematic. Students of conventional (old minimal) supergravity are well-aware of a certain curious feature: the integral of the supervolume is proportional to the supersymmetric Einstein-Hilbert term. Specifically

$$-\frac{3}{\kappa^2} \int d^8z E = S_{EH} = -\frac{1}{2\kappa^2} \int d^4x \sqrt{g} R + \ldots$$  \hspace{1cm} (1.4)

\footnote{1}{See the extensive citations in \cite{1} and \cite{3} on the subject.}

\footnote{2}{Within the last two weeks, Komargodski and Seiberg have released another paper \cite{2} constructing an alternative supercurrent which appears to correspond to a non-minimal 16+16 supergravity in which the FI term is more easily accounted for. This seems to correspond to adding both a chiral and linear superfield to conformal supergravity to yield the non-minimal Poincaré sector.}
where $\kappa^2$ is the reduced Planck length. This particularly causes problems when coupling a Kähler potential to supergravity. The coupling one would naively propose is $\int d^4 \theta E K$; however, it is clear that such a term must yield a non-canonical Einstein-Hilbert term

$$\int d^8 z E K = \int d^4 x \sqrt{g} \left( \frac{1}{6} K R - K_{ij} \nabla^i \phi^j \nabla^m \bar{\phi}^j + \ldots \right)$$

as the previous formula is simply a special case of constant $K$. This Brans-Dicke like interaction can be cured by performing a super-Weyl transformation on the supergravity sector, and one finds that the proper way to couple supergravity to chiral matter is via an exponential and a super-Weyl redefinition \[5\]

$$-\frac{3}{\kappa^2} \int d^8 z E e^{-\kappa^2 K/3} = \int d^4 x \sqrt{g} e^{-\kappa^2 K/3} \left( -\frac{1}{2\kappa^2} \mathcal{R} - K_{ij} \nabla^m \phi^i \nabla^m \bar{\phi}^j + \ldots \right)$$

The first equality, (1.6), is supergravity coupled to chiral matter in what we shall call the Brans-Dicke frame; the second equality, (1.7), is in the Einstein frame. An important feature of the right-hand side of either of these formulae is that there is no clear distinction between a pure supergravity term and a chiral matter term; rather, they are blended together. This is obvious for (1.6) but it is also true for (1.7) since after the super-Weyl transformations, the supersymmetry transformation rule for the gravitino includes details of the matter sector (albeit suppressed by factors of $\kappa^2$). In fact, the entire artifice of Kähler superspace has been worked out to explain the details of this intertwining \[6\]. One important detail that we should keep in mind is that (1.6) is not Kähler invariant without performing additional super-Weyl transformations whereas (1.7) is. Classically then we should regard (1.7) as the proper frame in which to perform our calculations if we would like Kähler invariance to be preserved. Indeed, this is what one normally does \[5\].

Moreover, and this is the critical part, the two equations (1.6) and (1.7) differ in their form even in the small $\kappa^2$ limit. Even though we have suppressed all but two terms, these alone clearly differ in their form by $K \mathcal{R}/6$, which survives in the small $\kappa^2$ limit. Although this term certainly vanishes when one turns off supergravity, we find the stress tensor by varying the metric and then setting it to zero; thus terms linear in the curvature can indeed alter the stress tensor. This one, for example, contributes to the stress tensor a term $\partial_m \partial_n K$ which is clearly not Kähler invariant. Thus, the canonical definition of the stress tensor can (and does!) differ between the Brans-Dicke and the Einstein frames.

The coupling of an FI term to supergravity is equivalent to that of $K$. One can simply make the replacement of $K \rightarrow 2\xi V$ in the above argument, and quite analogously we expect that the definition of the stress tensor should differ between the two frames. Our contention is that the specific calculation recounted in \[1, 3\] while performed in the rigid limit is ultimately equivalent to a calculation performed in the superspace analogue of the Brans-Dicke frame; since that frame fails to be gauge invariant it is unsurprising that its supercurrent should have the same problem. We will further show that the superspace Einstein frame possesses a gauge invariant stress tensor, but implies the additional symmetry these authors discussed, reaffirming their main point.

Our paper is structured as follows. In section 2, we review the calculation of the gauge supercurrent in a globally supersymmetric theory both as a warmup and to set our conventions. In section 3, we consider the supergravity supercurrent, first in the superspace
context where we discuss the difference between the Brans-Dicke and Einstein frames, and
then in the component context where we demonstrate that the Einstein frame currents do
give the standard globally supersymmetric currents when supergravity is decoupled. An
appendix is attached which briefly summarizes the calculation of the Noether currents.

2 Review: The gauge supercurrent and some conventions

Before diving into the details of supergravity, we will briefly discuss gauge supercurrents
both to standardize our gauge superfield notation and to review how superspace currents
yield the more familiar component currents.

The action of a $U(1)$ gauge sector with a Fayet-Iliopoulos term coupled to a single chiral
superfield of charge $g$ in global supersymmetry may be written in superspace as

$$S = \int d^8z \left( \bar{\phi}e^{2gV} \phi + 2\xi V \right) + \frac{1}{4} \int d^6z W^\alpha W_\alpha + \frac{1}{4} \int d^6\bar{z} \bar{W}^\dot{\alpha} \bar{W}^{\dot{\alpha}}$$  \hspace{1cm} (2.1)

The chiral superfield $\phi$ is (conventionally) chiral, obeying $D^\dot{\alpha} \phi = 0$, as is the $U(1)$ gaugino field strength. The gauge invariance of the action follows from the transformations

$$\phi \to e^{-2g\Lambda} \phi, \quad \bar{\phi} \to \bar{\phi} e^{-2g\bar{\Lambda}}, \quad V \to V + \Lambda + \bar{\Lambda}$$  \hspace{1cm} (2.2)

where $\Lambda$ is also chiral. (Normal gauge transformations correspond to $\lambda = i(\Lambda - \bar{\Lambda})$.) In
the conventions we will use here, the gaugino field strength is defined in terms of the $U(1)$
prepotential $V$ as

$$W^\alpha = \frac{1}{4} D^2 D^\alpha V, \quad \bar{W}^{\dot{\alpha}} = \frac{1}{4} D^2 \bar{D}^\dot{\alpha} V$$  \hspace{1cm} (2.3)

The fundamental dynamical variables of this theory are $\phi, \bar{\phi}$, and $V$. If we vary the
action under small deformations of each of these parameters,

$$\delta V \equiv \Sigma, \quad \delta \phi \equiv \eta, \quad \delta \bar{\phi} \equiv \bar{\eta}$$  \hspace{1cm} (2.4)

we find a general first order structure

$$S^{(1)} = \int d^8z \Sigma J_V + \int d^6z \eta J_\phi + \int d^6\bar{z} \bar{\eta} \bar{J}_\phi$$  \hspace{1cm} (2.5)

where for our simple model

$$J_V = 2\xi + D^\alpha W_\alpha + 2g \bar{\phi} e^{2gV} \phi$$  \hspace{1cm} (2.6)

$$J_\phi = -\frac{1}{4} D^2 (\bar{\phi} e^{2gV})$$  \hspace{1cm} (2.7)

$$\bar{J}_\phi = -\frac{1}{4} D^2 (e^{2gV} \phi)$$  \hspace{1cm} (2.8)

Note that the currents have properties similar to the first order variations to which they
couple: $J_V$ is real, $J_\phi$ is chiral, and $\bar{J}_\phi$ is antichiral.

If the original action is gauge invariant, then $S^{(1)}$ should vanish when we choose the
deformations of the fields to be gauge transformations. Thus for infinitesimal $\Lambda$ we find the
general structure

$$0 = \delta_\Lambda S = \int d^6z \Lambda \left( -\frac{1}{4} D^2 J_V - 2g\phi J_\phi \right) + \text{h.c.}$$  \hspace{1cm} (2.9)
implying the classical conservation equations
\[-\frac{1}{4} \bar{D}^2 J_{\phi} = 2g \phi J_{\phi}, \quad -\frac{1}{4} D^2 J_V = 2g \bar{\phi} J_{\bar{\phi}} \] (2.10)

When the matter fields are placed on shell, \( J_{\phi} \) and \( \bar{J}_{\phi} \) vanish and this becomes the usual superfield version of current conservation.

To better see this, we can turn to a component formulation of the same theory. The original way of doing this is to go to Wess-Zumino gauge for the prepotential \( V \), but there is a more elegant geometric approach. One promotes the conventionally chiral superfield \( \phi \) to a covariantly chiral \( \Phi \) by endowing the superspace derivatives with a gauge connection. In this language
\[ \bar{\phi} e^{2gV} \phi \rightarrow \bar{\Phi} \Phi, \quad D^2 (e^{2gV} \phi) \rightarrow D^2 \Phi, \quad D^2 (\bar{\phi} e^{2gV}) \rightarrow D^2 \bar{\Phi} \]

and except for the explicit FI term, the prepotential \( V \) need not be mentioned explicitly\(^3\). We then define the covariant combinations
\[ \chi_{\alpha} = \frac{1}{\sqrt{2}} D_{\alpha} \Phi, \quad F = -\frac{1}{4} D^2 \Phi \]

for the matter sector. For the gauge sector, we have
\[ A_m = -\frac{1}{4} \bar{\sigma}_{\alpha} [D_{\alpha}, \bar{D}_{\alpha}] V \equiv -\Delta_m V \] (2.11)
\[ \lambda_{\alpha} = W_{\alpha}, \quad D = -\frac{1}{2} D^a W_a \] (2.12)

We have introduced the notation \( \Delta_{\alpha\dot{\alpha}} \equiv -[D_{\alpha}, \bar{D}_{\dot{\alpha}}]/2 \) for selecting out the vector component (i.e. the component of \( \theta \sigma^m \bar{\theta} \) in the superfield expansion). Writing (2.11) in component notation gives
\[ \mathcal{L} = -D^m \bar{\phi} D_m \phi - i \bar{\chi} \sigma^m D_m \chi + \bar{F} F + \sqrt{2} g (\lambda \bar{\chi}) \phi + \sqrt{2} g (\bar{\lambda} \chi) \phi - g D \phi \phi \]
\[ -\frac{1}{4} F^{mn} F_{mn} - i \bar{\lambda} \sigma^m D_m \lambda + \frac{1}{2} D^2 - \xi D \] (2.13)

Its first order variation has the general form
\[ S^{(1)} = \int d^4 x \left( \delta \phi J_{\phi}^{(\phi)} + \delta \chi^\alpha J_{\phi\alpha} + \delta F J_{\phi} + \text{h.c.} \right. \]
\[ \left. + \delta A_m J_{V}^m + \delta \lambda^\alpha J_{V\alpha} + \delta \bar{\lambda}_{\dot{\alpha}} J_{\bar{V}}^\dot{\alpha} + \delta D J_{V}^{(D)} \right) \] (2.14)

For our specific case, it is easy to work out the various currents \( J \) in the above expression, but a more profitable approach is to identify them from the component version of (2.5). Doing so gives
\[ J_{\phi}^{(\phi)} = \frac{1}{4} D^2 J_{\phi} \] (2.15)
\[ J_{\phi\alpha} = -\frac{1}{\sqrt{2}} D_{\alpha} J_{\phi} \] (2.16)

\(^3\)See the classic textbooks [7, 8] for a discussion of this.
for the matter supermultiplet currents, and
\[
J_{V\alpha} = \frac{1}{2} D_\alpha J_V, \quad J_{\bar{V}}^\alpha = \frac{1}{2} \bar{D}^\alpha J_V
\] (2.17)
\[
J_V^m = \frac{1}{2} \Delta^m J_V
\] (2.18)
\[
J_{V}^{(D)} = -\frac{1}{2} J_V
\] (2.19)
for the gauge supermultiplet currents.

Invariance of the component action under gauge transformations implies
\[
\partial_m J_{V}^m = ig\phi J_{\phi}^{(\phi)} + ig\chi^\alpha J_{\phi\alpha} + igFJ_{\phi} + \text{h.c.}
\] (2.20)
It is easy to see that (2.20) is a consequence of (2.10): the latter implies
\[
\frac{i}{32}[D^2, \bar{D}^2]J_V = -\frac{i}{8} D^2(2g\phi J_{\phi}) + \text{h.c.}
\] (2.21)
which reduces to the former after some algebra.

Turning off the gauge sector amounts to setting \(V = 0\), which yields in our simple case the current superfield
\[
J_V = 2\xi + 2g\bar{\phi}\phi
\]
whose vector component
\[
J_V^m = ig\partial^m \phi - ig\bar{\phi}\partial^m \phi - g(\bar{\chi}\sigma^m \chi)
\]
is precisely what would have been constructed by the Noether procedure. This is an elegant (if historically backward) approach to constructing Noether currents associated with gauge fields: vary the action with respect to the gauge field and then set it to zero. It is this approach which is most easily replicated in superspace to construct the supergravity supercurrents.

### 3 The supergravity supercurrent

We turn now to our actual object of interest, the supergravity supercurrent. In order to understand the superfield form of the current equations we will derive, it helps to recast the original theory in a superconformal form\[^3\] We will briefly review why.

### 3.1 The relevance of superconformal concerns

Recall that when the bosonic conformal algebra is placed in the structure group of a regular Poincaré theory, the covariant d’Alembertian receives a correction proportional to the Ricci scalar when acting on a scalar fields \(\phi\) of conformal dimension one:
\[
\nabla^a \nabla_a \phi = D^a D_a \phi + \frac{1}{6} \mathcal{R} \phi
\]
\[^4\]The importance of superconformal methods in understanding the occasionally bizarre structure of conventional supergravity cannot be overstated. For the construction of superspace with the full superconformal algebra as the structure group, see the discussion in [9].
In the above, $\nabla$ denotes the covariant derivative with the conformal algebra in the structure group while $D$ denotes the normal Poincaré derivative. The conformally invariant kinetic term for a scalar, strangely normalized with a factor of $-3$ contains a Brans-Dicke like term:

$$-3 \int d^4x \sqrt{g} \phi \nabla^a \nabla_a \phi = \int d^4x \sqrt{g} \left( -3 \phi D^a D_a \phi - \frac{1}{2} R \phi^2 \right)$$

The field $\phi$ is known as the “conformal compensator,” and taking the conformal gauge $\phi = 1$ then yields the conventional Einstein-Hilbert Lagrangian.

A similar approach “explains” the bizarre features of conventional supergravity. Introducing a chiral superfield $\phi_0$ with conformal dimension one, the kinetic term (again with an additional factor of $-3$) is

$$-3 \kappa^2 \int d^4\theta \bar{\phi}_0 \phi_0 e^{-\kappa^2 K/3} = \int d^4x \sqrt{g} \left( -3 \kappa^2 e^{-\kappa^2 K/3} \bar{\phi}_0 \nabla^a \nabla_a \phi_0 + \frac{1}{16} |\phi_0|^2 \nabla^a \nabla^b \nabla_a K + \ldots \right)$$

Indeed, the part of the action corresponding to the conventional Kähler action is simply the $D$-term of the composite $U(1)$ factor:

$$-3 \kappa^2 \int d^4\theta \bar{\phi}_0 \phi_0 e^{-\kappa^2 K/3} = \int d^4x \sqrt{g} \left( -3 \kappa^2 e^{-\kappa^2 K/3} \bar{\phi}_0 D^a D_a \phi_0 - \frac{1}{2} e^{-\kappa^2 K/3} R |\phi_0|^2 + \ldots \right)$$

Indeed, the part of the action corresponding to the conventional Kähler action is simply the $D$-term of the composite $U(1)_K$.

It is clear that the a certain superconformal gauge choice ($\phi = 1$) must correspond to the Brans-Dicke frame (1.6) while another ($\phi = e^{\kappa^2 K/6}$) must correspond to the Einstein frame (1.7). The advantage of working in a superconformal framework is that we can impose these gauge choices at the superfield level without first going to components. This will be critical in finding the Einstein frame superspace supercurrents.

It was shown in [9] that one can reduce conformal superspace to Poincaré superspace with a residual $U(1)$ structure on which the conformal transformations are realized non-linearly. The resulting Poincaré $U(1)$ superspace (which we will refer to simply as “Poincaré” from now on) is that of [10]. It is not necessary for the $U(1)$ to actually contain degrees of freedom; one can, for example, rewrite the original supergravity of [5] in this structure, so it is quite generic. In [6], it was shown how to use the $U(1)$ structure to encode Kähler transformations, but the discussion there can easily be generalized to include a $U(1)_\xi$ rather than a Kähler potential. It is this structure we eventually expect to recover in the superspace currents.

### 3.2 The superspace conformal and Poincaré stress tensors

We will use the convenient shorthands

$$[Z]_D \equiv \int d^4x d^4\theta E Z, \quad [W]_F \equiv \int d^4x d^2\theta E W$$

(3.3)

to denote integrations over the full superspace with integrand $Z$ and over the chiral superspace with integrand $W$, respectively. For now we will maintain manifest superconformal
invariance at all times; thus the above terms are invariant only if \(Z\) is conformally primary with conformal dimension two and vanishing \(U(1)_R\) weight and if \(W\) is conformally primary and chiral with conformal dimension three and \(U(1)_R\) weight two. (If a superfield has conformal dimension \(\Delta\) and \(U(1)_R\) weight \(w\), we will refer to its weight as \((\Delta, w)\); thus \(Z\) is weight \((2, 0)\) and \(W\) is weight \((3, 2)\).) \(E\) denotes the superdeterminant of the supervierbein, a suitable volume measure for the full superspace, while \(E\) is the chiral volume measure \([9]\).

We are interested ultimately in examining the classical supercurrent for the D-term action

\[
S = -\frac{3}{\kappa^2} \left[ \phi_0 \overline{\phi}_0 e^{-2\kappa^2 \xi V/3} \right]_D + \left[ \frac{1}{4} W^\alpha W_\alpha \right]_F + \left[ \frac{1}{4} \bar{W}_{\dot{\alpha}} \bar{W}_{\dot{\alpha}} \right]_F
\]

(3.4)

where \(W_\alpha\) is the gaugino superfield and \(V\) the prepotential associated with the \(U(1)_\xi\). It is convenient to switch from conventional chirality to covariant chirality for the purposes of making contact with the notation used in \([11]\):

\[
S = -\frac{3}{\kappa^2} \left[ \Phi_0 \bar{\Phi}_0 \right]_D + \left[ \frac{1}{4} W^\alpha W_\alpha \right]_F + \left[ \frac{1}{4} \bar{W}_{\dot{\alpha}} \bar{W}_{\dot{\alpha}} \right]_F
\]

(3.5)

where \(\Phi_0\) is covariantly chiral with a \(U(1)_\xi\) charge of \(-\kappa^2 \xi/3\). The first order variation of this model has the form\([7]\):

\[
S^{(1)} = \left[ H^\dot{\alpha} J_{\alpha \dot{\alpha}} + \Sigma J_\xi \right]_D + \left[ \eta_0 J_0 \right]_F + \left[ \bar{\eta}_0 \bar{J}_0 \right]_F
\]

(3.6)

where \(H_a\) is an Hermitian superfield whose components encode the variation of the conformal supergravity multiplet, \(\Sigma\) is a Hermitian superfield encoding the variation of the \(U(1)_\xi\) gauge multiplet, and \(\eta_0\) is a chiral superfield encoding the variation of the chiral compensator. More precise definitions of the variational superfields are given in \([11]\)\([9]\).

Of immediate concern are the superconformal properties of the various objects. \(H_a\) is weight \((-1, 0)\) and so \(J_a\) must be weight \((3, 0)\). \(\Sigma\) is weight \((0, 0)\) and thus \(J_\xi\) is weight \((2, 0)\). Note that \(\eta_0\) is necessarily of weight \((1, 2/3)\) and so \(J_0\) has weight \((2, 4/3)\). Also of importance is that \(J_0\) has \(U(1)_\xi\) charge of \(+\kappa^2 \xi/3\), opposite that of \(\Phi_0\).

Under a (quantum) gauge transformation

\[
\delta \Sigma = \Lambda + \bar{\Lambda}, \quad \delta \eta_0 = \frac{2}{3} \kappa^2 \xi \Lambda \Phi_0, \quad \delta \bar{\eta}_0 = \frac{2}{3} \kappa^2 \xi \bar{\Lambda} \bar{\Phi}_0
\]

Gauge invariance of the first-order action implies that\([9]\)

\[
\mathcal{P} J_\xi = -\frac{2}{3} \kappa^2 \xi \Phi_0 J_0, \quad \mathcal{P} J_0 = -\frac{2}{3} \kappa^2 \xi \bar{\Phi}_0 J_0
\]

(3.7)

where \(\mathcal{P} \equiv -\nabla^2/4\) and \(\mathcal{P} \equiv -\nabla^2/4\). This is the superfield version of gauge current conservation which we have discussed previously in the globally supersymmetric case.

\[\text{in [11], we used } V_o \text{ to denote } H_o. \text{ Here we use the latter notation to agree with the general convention.}\]

\[\text{In particular, the definition of } \Sigma \text{ differs from the conventional definition by terms involving } H_a \text{ which render its transformation rule covariant. The question of which (if either) definition is more correct is moot since we will soon place the gauge superfield on shell.}\]

\[\text{Recall that when a theory with a gauge invariance is expanded in terms of first order quantum variations about a background, there exist two different notions of gauge transformation: background transformations under which the quantum variations transform homogeneously and quantum transformations under which the background is invariant. The latter are important for figuring out the currents.}\]

\[\text{In conformal superspace, these are chiral projection operators, just as they are in global supersymmetry, and so } \mathcal{P} J_\xi \text{ is a chiral superfield.}\]
A similar structure exists for conformal supergravity transformations. Under these,

$$\delta H_{\alpha\hat{\alpha}} = \nabla_\alpha L_{\hat{\alpha}} - \nabla_{\hat{\alpha}} L_\alpha, \quad \delta \Sigma = L^\alpha W_\alpha + L_{\hat{\alpha}} W^{\hat{\alpha}}$$

$$\delta \eta_0 = \mathcal{P}(L^\alpha \nabla_\alpha \Phi_0) + \frac{1}{3} \Phi_0 \mathcal{P} \nabla^\alpha L_\alpha, \quad \delta \bar{\eta}_0 = \bar{\mathcal{P}}(L_{\hat{\alpha}} \nabla^{\hat{\alpha}} \bar{\Phi}_0) + \frac{1}{3} \bar{\Phi}_0 \bar{\mathcal{P}} \nabla_{\hat{\alpha}} L^{\hat{\alpha}}$$

(3.8)

which imply the superfield version of energy-momentum conservation

$$\nabla_{\hat{\alpha}} J_{\alpha\hat{\alpha}} = \nabla_{\hat{\alpha}} W_\alpha J_\alpha + \nabla_\alpha \Phi_0 J_0 - \frac{1}{3} \nabla_\alpha (\Phi_0 J_0)$$

$$\nabla^\alpha J_{\alpha\hat{\alpha}} = \nabla^\alpha W_{\hat{\alpha}} J_{\hat{\alpha}} + \nabla^{\hat{\alpha}} \bar{\Phi}_0 \bar{J}_0 - \frac{1}{3} \nabla^{\hat{\alpha}} (\bar{\Phi}_0 \bar{J}_0)$$

(3.9)

Using the techniques developed in [11], one can show that for the model under consideration here,

$$J_{\alpha\hat{\alpha}} = \frac{2}{\kappa^2} X \hat{G}_{\alpha\hat{\alpha}} + W_\alpha W_{\hat{\alpha}}$$

(3.10)

$$J_0 = -\frac{6}{\kappa^2 \Phi_0} X \hat{R}$$

(3.11)

$$J_\xi = 2\xi X + \nabla^\alpha W_\alpha$$

(3.12)

where we have defined

$$X \equiv \Phi_0 \bar{\Phi}_0$$

(3.13)

$$\hat{G}_{\alpha\hat{\alpha}} \equiv -X^{1/2} \Delta_{\alpha\hat{\alpha}} X^{-1/2}$$

(3.14)

$$\hat{R} \equiv \frac{1}{8} X^{-1} \hat{\nabla}^2 X$$

(3.15)

The superfields $\hat{G}_{\alpha\hat{\alpha}}$ and $\hat{R}$ are superconformally primary when $X$ is of dimension two and reduce to the Poincaré superfields of the same name when the gauge choice $X = 1$ is made.

We now have the formulae governing the conservation of energy and momentum in superspace. In order to find the Poincaré stress energy relation, we first put the gauge sector on shell (i.e. we set $J_\xi$ to zero) and then choose the conformal gauge. There are essentially two options available to us.

### 3.2.1 The Brans-Dicke frame

The Brans-Dicke frame corresponds to choosing the conventionally chiral superfield $\phi_0$ to be unity. This is equivalent to choosing

$$X = \frac{\bar{\phi}_0 \phi_0 e^{-2\xi \kappa^2 V/3}}{e^{-2\xi \kappa^2 V/3}}$$

in the original action and thus corresponds (in the small $\kappa^2$ limit) to the choice made in [13]. The easiest way to see this is to rewrite the superfields in conventionally chiral notation (that is, remove $U(1)_\xi$ from the structure group) and then to set $\phi_0 = 1$, fixing the $U(1)_R$ gauge along with the conformal symmetry. The superfields $\hat{G}_{\alpha\hat{\alpha}}$ and $\hat{R}$ that we have previously defined become

$$\hat{G}_{\alpha\hat{\alpha}} = G - \frac{1}{3} \xi \kappa^2 \Delta_{\alpha\hat{\alpha}} V + \mathcal{O}(\kappa^4)$$

(3.16)

$$\hat{R} = R + \frac{1}{12} \xi \kappa^2 \hat{D}^2 V + \mathcal{O}(\kappa^4)$$

(3.17)

$^9$Here differs by a factor of $i$ from that defined in [13].
where the Brans-Dicke frame superfields $G$ and $R$ obey the constraint
\[ D^\dot{\alpha}G_{\alpha\dot{\alpha}} = D_\alpha R \] (3.18)
We can rewrite the conservation equation as
\[ D^\dot{\alpha}J_{\alpha\dot{\alpha}} = -\frac{2}{3}\kappa^2\xi D_\alpha V(\Phi_0J_0) - \frac{1}{3}D_\alpha(\Phi_0J_0) \] (3.19)
where we have left the gauge invariant combination $\Phi_0J_0$ in covariant form. (Note that since $\Phi_0J_0$ is gauge invariant, it is both covariantly and conventionally chiral.) It expands out as
\[ \Phi_0J_0 = -\frac{6}{\kappa^2}e^{-2\kappa^2\xi V/3}\dot{R} = -\frac{6}{\kappa^2}R + 4\xi VR - \frac{1}{2}\xi\bar{D}^2V + O(\kappa^2) \] (3.20)
The supercurrent is
\[ J_{\alpha\dot{\alpha}} = \frac{2}{\kappa^2}G_{\alpha\dot{\alpha}} - \frac{4}{3}\xi VG_{\alpha\dot{\alpha}} - \frac{2}{3}\xi\Delta_{\alpha\dot{\alpha}}V + W_\alpha W_{\dot{\alpha}} + O(\kappa^2) \] (3.21)
and it is a straightforward exercise to verify that (3.21) and (3.20) do indeed satisfy the rather strange-looking conservation equation (3.19).

Within the Brans-Dicke frame, we may freely set the entire superfields $G$ and $R$ to zero, and then send $\kappa^2$ to zero. Doing so, we find the supercurrent
\[ J_{\alpha\dot{\alpha}} = -\frac{2}{3}\xi\Delta_{\alpha\dot{\alpha}}V + W_\alpha W_{\dot{\alpha}} \]
obeying the conservation equation
\[ D^\dot{\alpha}J_{\alpha\dot{\alpha}} = \frac{1}{6}\xi D_\alpha D^2V \] (3.22)
This is (up to normalizations) the non-covariant conservation equation found in [1]. It is clear that these non-covariant supercurrents will yield non-covariant component currents, so we do not bother calculating those here. They will invariably correspond to the component currents one would calculate in the Brans-Dicke frame and have no chance of being gauge invariant.

### 3.2.2 The Einstein frame

The choice corresponding to the Einstein frame is to choose the covariantly chiral superfield $\Phi_0$ to be set to unity. This gauge choice fixes the dilatation symmetry, while the $U(1)_R$ symmetry is identified with $U(1)_\xi$. This is easy to prove by examining the covariant chirality condition:
\[ 0 = \nabla^\dot{\alpha}\Phi_0 = D^\dot{\alpha}\Phi_0 - \frac{2i}{3}A^\dot{\alpha}\Phi_0 + \frac{i}{3}\kappa^2\xi A^\dot{\alpha}_\xi \Phi_0 \] (3.23)
In the gauge $\Phi_0 = 1$, this implies $A^{\dot{\alpha}} = \frac{1}{2}\kappa^2\xi A^\dot{\alpha}_{\dot{\xi}}$. A similar argument for the conjugate superfield then necessarily implies that for the full superfield connections
\[ A_\alpha = \frac{1}{2}\kappa^2\xi A^\xi_\alpha, \quad A^{\dot{\alpha}} = \frac{1}{2}\kappa^2\xi A^\dot{\alpha}_{\dot{\xi}} \]
\[ A_{\alpha\dot{\alpha}} = -\frac{3}{2}G_{\alpha\dot{\alpha}} + \frac{1}{2}\kappa^2\xi A^\xi_{\alpha\dot{\alpha}} \] (3.24)
in this conformal gauge. The superfields $\hat{G}_{a\dot{a}}$ and $\hat{R}$ become the superfields with the same names of Poincaré supergravity. They obey the constraint

$$\mathcal{D}_\alpha R - \mathcal{D}^\dot{\alpha} G_{a\dot{a}} = X_\alpha = -\xi\kappa^2 W_\alpha$$  \hspace{1cm} (3.25)

where $\mathcal{D}$ is the Poincaré derivative and $X_\alpha$ is the gaugino superfield associated with the $U(1)^R$ structure, now identified with $U(1)_\xi$. The conservation equation becomes

$$\mathcal{D}^\dot{\alpha} J_{a\dot{a}} = -\frac{1}{3} \mathcal{D}_\alpha J_0$$  \hspace{1cm} (3.26)

where

$$J_{a\dot{a}} = \frac{2}{\kappa^2} G_{a\dot{a}} + W_\alpha W_{\dot{\alpha}}, \quad J_0 = -\frac{6}{\kappa^2} R$$  \hspace{1cm} (3.27)

and follows trivially from (3.25) and the $U(1)_\xi$ equation of motion.

Observe that all these superfields are manifestly gauge covariant under the $U(1)_\xi$, which has been absorbed into the structure group of superspace via its identification with the $U(1)^R$. It necessarily follows that any components of these superfields, such as the component stress tensor or supersymmetry current, ought to share this manifest covariance.

The curious feature of the Einstein frame is that it necessarily intertwines the $U(1)_\xi$ with supergravity. The instinct that global supersymmetry should be restored by sending $G_{a\dot{a}}$ and $R$ to zero as superfields is incorrect in this frame by virtue of the constraint (3.25). One instead suspects that one must turn off the supergravity multiplet by setting its component fields to zero and then sending $\kappa^2$ to zero. This rather intricate structure is, unfortunately, the only way to maintain $U(1)_\xi$ covariance. Proving that this yields the correct globally supersymmetric currents is the only task left.

### 3.3 The component currents

The previous subsection has relied rather heavily on superfield current arguments. While elegant, this is a somewhat unsatisfying line of attack since superspace should tell us nothing that we couldn’t have already deduced by more difficult means in component language. We therefore turn to a calculation of the component currents associated with the superconformal first order action. In analogy to the gauge supercurrent calculation in section 2, we anticipate that the component first order action should have the following form:

$$e^{-1} \mathcal{L}^{(1)} = \delta \phi_0 J_0^{(0)} + \delta \chi^0_J J_{0\alpha} + \delta F_0 J_0 + \text{h.c.}$$

$$+ \delta A_m J^{(1)} + \delta \lambda^{\alpha} J_{\xi} + \delta \bar{\lambda}^{\dot{\alpha}} J_{\bar{\xi}} + \delta D^{\xi} J_{(D)}$$

$$+ \delta e_m^a J_{a \dot{m}} + \delta \psi_m^\alpha J_{a \alpha} + \delta \tilde{\psi}_m^{\dot{\alpha}} J_{a \dot{\alpha}} + \delta A_m J^{(5)}$$  \hspace{1cm} (3.28)

The components of the chiral compensator multiplet are $(\phi_0, \chi_\alpha, F_0)$; those of the $U(1)_\xi$ gauge multiplet are $(A_\alpha^\xi, \lambda^\alpha, \bar{\lambda}^{\dot{\alpha}}, D^{\xi})$; those of the conformal supergravity multiplet are $(e_m^a, \psi^\alpha_m, \tilde{\psi}^{\dot{\alpha}}_m, A_m)$. Recall that the last of these, $A_m$, is the gauge field associated with the $U(1)^R$ of the superconformal algebra.\(\text{[11]}\)

---

\text{[10]} The appearance of $G_{a\dot{a}}$ in the bosonic connection has to do with the convention of defining $F_{a\dot{a}} = -3G_{a\dot{a}}$ for the $U(1)$ curvature. Choosing $F_{a\dot{a}} = 0$ removes $G_{a\dot{a}}$ from the expression for $A_{a\dot{a}}$.

\text{[11]} There also exists a gauge field associated with dilatations; however, its coefficient in the above expansion must vanish (provided the original action was conformally invariant) since it is the only field which transforms under special conformal transformations. \([9]\)
Clearly the various component $J$'s described above must be defined in terms of the superfield expressions considered in the previous section. We have worked out these relations \[12\] and some are rather complicated. For our purposes, we will simplify them by ignoring all terms involving the background gravitino. For the chiral and gauge sectors, the results are exactly those of global supersymmetry:

\[
J_i^{(\phi)} \sim -\frac{1}{4} \nabla^2 J_i , \quad J_{i\alpha} \sim -\frac{1}{\sqrt{2}} \nabla_\alpha J_i
\]

(3.29)

and for the gauge sector,

\[
J_{\xi\alpha} \sim \frac{1}{2} \nabla_\alpha J_{\xi} , \quad J^{\alpha}_{\xi} \sim \frac{1}{2} \nabla^\alpha J_{\xi}
\]

(3.30)

\[
J_{\xi} \sim \frac{1}{2} \Delta^a J_{\xi} , \quad J^{(D)}_{\xi} \sim -\frac{1}{2} J_{\xi}
\]

(3.31)

(3.32)

For the $U(1)_R$ and supersymmetry currents, we have

\[
J^{a(5)} \sim -2J^a
\]

(3.33)

\[
J_\alpha^b \sim -i \frac{1}{2} \sigma^{b\beta\dot{\beta}} J_{\alpha\beta\dot{\beta}} + i \frac{1}{4} \phi_0 \sigma^{b\alpha\dot{\alpha}} \nabla_\alpha J_{\dot{\alpha}}
\]

(3.34)

where we have defined

\[
J_{\alpha\beta\dot{\beta}} \equiv \frac{1}{2} \nabla_\alpha J_{\beta\dot{\beta}} + \frac{1}{2} \nabla_\beta J_{\alpha\dot{\beta}}
\]

(3.35)

For the stress tensor,

\[
J_{ab} \sim -\frac{1}{2} \Delta_a J_b - \frac{1}{2} \Delta_b J_a + \frac{1}{4} \eta_{ab} \Delta_c J^c
\]

\[
- \frac{1}{2} (\chi^0 \sigma_{ab})^\alpha J_{\alpha 0} - \frac{1}{2} (\lambda^\xi \sigma_{ab})^\alpha J_{\alpha \xi} + h.c.
\]

\[
+ \frac{1}{4} \eta_{ab} \left( 2 D^\xi J_{\xi} + \frac{3}{2} \lambda^\alpha \xi J_{\alpha \xi} + \frac{3}{2} \lambda^\xi \dot{\alpha} J_{\dot{\alpha}} \right)
\]

\[
+ \frac{1}{4} \eta_{ab} \left( 2 F_0 J_0 + \frac{3}{2} \lambda^\alpha \xi J_{\alpha 0} + \phi_0 J^{(\phi)}_0 + h.c. \right)
\]

(3.36)

These formulae are rather complicated (even after neglecting the gravitino!) and so it is useful to have a number of independent checks. The third and fourth lines of $J_{ab}$ correspond to the trace, which is dictated by scale invariance of the component action. Similarly, the second line of $J_{ab}$ is antisymmetric and dictated by Lorentz invariance. The first line of $J_{ab}$ and the entirely of $J_{\alpha}^b$ can be checked using the fermionic special superconformal symmetry.

As these superconformal currents are gauge invariant, fixing the conformal gauge should yield gauge invariant Einstein frame Poincaré currents\[12\]

\[12\]Note that Dienes and Thomas have also explored the components of the superstress tensor in \[13\].
3.3.1 The Einstein frame currents

We begin by putting the entire gauge sector on shell, eliminating the $J_\xi$ multiplet. Then going to the conformal gauge $\Phi_0 = 1$, we find that the fermion $\chi^\alpha_0$ vanishes and $F_0 = 2\bar{R}$ becomes the scalar auxiliary field of Poincaré supergravity. Our goal is to calculate the $U(1)_R$ supersymmetry, and energy-momentum currents associated with the super-Maxwell system with an FI term. A Noether current calculation on the globally supersymmetric system (see Appendix A for a brief discussion) gives

$$J_m^{(5)} = \lambda \sigma_m \bar{\lambda}$$

$$J_\alpha^m = \frac{1}{2} F^{mn}(\sigma_n \lambda)_\alpha - \frac{i}{4} \epsilon^{mdcb}(\sigma_b \bar{\lambda})_\alpha F_{cd} - \frac{i}{2} \xi (\sigma^m \bar{\lambda})_\alpha$$

$$J_{nm} = F_{n}^{\rho} F_{mp} - \frac{1}{4} \eta_{nm} F^{ab} F_{ab} + \frac{i}{4} (\lambda \sigma_{(m} D_{n)} \bar{\lambda}) + \frac{i}{4} (\bar{\lambda} \sigma_{(m} D_{n)} \lambda) - \frac{1}{2} \eta_{nm} \xi^2$$

We begin with the $U(1)_R$ current:

$$J_a^{(5)} \sim -2J_a \sim -\frac{4}{\kappa^2} G^a + \lambda \sigma^a \bar{\lambda}$$

The lowest component of $G^a$ belongs to the supergravity multiplet and so we will set it to zero when decoupling supergravity, giving identically what the Noether procedure dictates.

For the supersymmetry current, we have

$$J_\alpha^b \sim -\frac{i}{2} \bar{\sigma}^b \beta \beta J_{\alpha \beta \bar{\beta}} + \frac{i}{4} \bar{\phi}_0 \sigma_{\dot{\alpha} \dot{\beta}} \nabla_{\dot{\alpha}} J_0$$

and we first need to calculate

$$J_{\alpha \beta \bar{\beta}} = \frac{1}{2} \mathcal{D}_{(\alpha} J_{\beta)\bar{\beta}} = \frac{1}{\kappa^2} \mathcal{D}_{(\alpha} G_{\beta)\bar{\beta}} + \frac{1}{2} \mathcal{D}_{(\alpha} W_{\beta)} W_{\bar{\beta}}$$

To evaluate the spinor derivative of $G$ requires certain supergravity relations, given for example in [6], but all of the terms contained within involve the gravitino and so vanish when the supergravity sector is turned off. The second term involves the product of a gaugino and a field strength; when contracted with a sigma matrix it gives

$$-\frac{i}{2} \bar{\sigma}^b \beta \beta J_{\alpha \beta \bar{\beta}} = -\frac{1}{2} (\sigma_c \bar{\lambda})_\alpha F^{cb} - \frac{i}{4} \epsilon^{decb} (\sigma_a \bar{\lambda})_\alpha F_{cd}$$

Next we need to calculate the additional spin-1/2 term involving $\mathcal{D}_a \bar{J}_0$. Recall that $\bar{J}_0 = -6\bar{R}/\kappa^2$ and so, consulting the structure of Poincaré superspace [6], one finds that

$$\mathcal{D}_a \bar{J}_0 = -\frac{6}{\kappa^2} \mathcal{D}_a \bar{R} \sim \frac{2}{\kappa^2} X_{\dot{\alpha}} = -2\xi W_{\dot{\alpha}}$$

Thus we find that the supersymmetry current is

$$J_\alpha^b = -\frac{1}{2} (\sigma_c \bar{\lambda})_\alpha F^{cb} - \frac{i}{4} \epsilon^{decb} (\sigma_a \bar{\lambda})_\alpha F_{cd} - \frac{i}{2} \xi (\sigma^b \bar{\lambda})_\alpha$$

which is indeed the same (up to index reshufflings) as that produced by the Noether procedure.
The stress tensor is a significantly more complicated beast. Assuming the gauge sector is on shell and that the chiral compensator is in the appropriate gauge, we find the terms

$$J_{ab} \sim -\frac{1}{2} \Delta_a J_b - \frac{1}{2} \Delta_b J_a + \frac{1}{4} \eta_{ab} \Delta_c J^c + \frac{1}{4} \eta_{ab} \left( 2F_0 J_0 + J_{(\phi)}^0 + \text{h.c.} \right)$$

(3.42)

We begin with the calculation of the symmetric traceless part of

$$\Delta_a J_b = \frac{2}{\kappa^2} \Delta_a G_b - \frac{1}{2} \Delta_a (W_{\sigma} b \bar{W})$$

The first term yields the symmetric traceless part of the Ricci tensor and so we can discard it in the limit where we turn off supergravity. The second term is more complicated and yields

$$-\frac{i}{4} (\lambda_{\sigma} (b \nabla a) \bar{\lambda}) - \frac{i}{4} (\bar{\lambda} \bar{\sigma} (b \nabla a) \lambda) - F_a^m F_{bm} + \ldots$$

where $\ldots$ denotes terms which are either antisymmetric or proportional to $\eta_{ab}$. Next we calculate the trace of the current. Dropping all terms which vanish when supergravity is turned off gives

$$\frac{1}{4} J_{(\phi)}^0 + \text{h.c.} \geq \frac{3}{8 \kappa^2} (D^2 R + \bar{D}^2 \bar{R}) \geq -\frac{1}{4 \kappa^2} D^a X_a = + \frac{\xi}{4} D^a W_\alpha = -\frac{1}{2} \xi^2$$

(3.43)

Putting everything together, we find the stress energy tensor

$$J_{ab} = F_a^m F_{bm} - \frac{1}{4} \eta_{ab} F^{mn} F_{mn} + \frac{i}{4} (\lambda_{\sigma} (b \partial a) \bar{\lambda}) + \frac{i}{4} (\bar{\lambda} \bar{\sigma} (b \partial a) \lambda) - \frac{1}{2} \xi^2 \eta_{ab}$$

(3.44)

This, too, is as expected from the Noether procedure.

### 4 Conclusion

We have attempted to demonstrate that the curious features of conventional (old minimal) supergravity make defining the supercurrent superfield somewhat subtle. The most straightforward way of going about it involves a Brans-Dicke frame which for the case of an FI term yields an inconsistent energy momentum tensor, while the current analogous to the Einstein frame involves a subtle intertwining of gravity and gauge fields prior to the decoupling of supergravity. In particular, the supersymmetry generators $Q$ now carry $U(1)_\xi$ charge, and thus so does the gravitino and any superpotential we turn on. The difficulties with the latter have been discussed particularly in [1][3] and references therein and remains a major objection to any FI term in conventional supergravity. Moreover, that the gravitino should carry some $U(1)_\xi$ charge implies that maintaining gauge invariance at the quantum level might necessitate some nontrivial anomaly cancellation.

Other flavors of supergravity can accommodate FI terms more easily. New minimal supergravity, for example, involves a different supercurrent conservation equation as it involves a linear superfield $L$ as a conformal compensator [8]. In the superconformal framework, FI terms in such a theory take the simple form

$$2\xi \int d^8 z E L V$$
where $L$ is conformal dimension two. Gauge invariance holds since the integral of $L\Lambda$ for chiral $\Lambda$ vanishes. The Poincaré theory is recovered by taking $L$ to unity, and the FI term has exactly the same form in the supergravity theory as it does in the globally supersymmetric one. The difficulty with new minimal supergravity is that it does not fix the $U(1)_R$ invariance, which remains a symmetry of the Poincaré theory. This severely constrains the type of matter which can be coupled even at the classical level, while at the quantum level it is generically anomalous [7,13].

It is a possibility that one could combine features of both theories by using both chiral and linear compensators, which would yield a non-minimal supergravity with sixteen off-shell bosonic and sixteen off-shell fermionic degrees of freedom, but could more straightforwardly accommodate FI terms (and Kähler potentials). Indeed, this seems like it could lie at the root of the $S$-multiplet recently constructed by Komargodski and Seiberg [2] and warrants further investigation.
A Noether currents for super-Maxwell theory

The component super-Maxwell theory (with FI term) has the Lagrangian

\[ \mathcal{L} = -\frac{1}{4} F^{mn} F_{mn} - i \sigma^{m} \partial_{m} \lambda - i \lambda \sigma^{m} \partial_{m} \lambda + \frac{1}{2} D^2 - \xi D \quad (A.1) \]

and is invariant (up to a total derivative) under the global supersymmetry transformations

\[ \delta \xi D = i (\xi \sigma^{m} \partial_{m} \lambda) + h.c. \quad (A.2) \]
\[ \delta \xi \lambda^{\alpha} = - i (\xi \sigma^{ba})^{\alpha} F_{ab} + \xi^{\alpha} D \quad (A.3) \]
\[ \delta \xi A_{m} = \xi \sigma_{m} \lambda + h.c. \quad (A.4) \]

where the spinor parameter \( \xi^{\alpha} \) is not to be confused with the FI parameter \( \xi \).

A comment is necessary about this definition for the supersymmetry transformations. While perfectly sensible in the component formulation, these transformations do not correspond to the most straightforward definition of “supersymmetry” as supertranslation in superspace. Rather, if one works with conventionally chiral superfields and Wess-Zumino gauge for the \( U(1) \) prepotential, one must augment the supertranslation by a further gauge transformation to restore Wess-Zumino gauge, which yields these rules. If instead one works with covariantly chiral superfields, the supersymmetry transformation described above is exactly that of a covariant super Lie derivative in superspace \(^ {13} \) This latter interpretation is clearly more elegant.

We define the supersymmetry current \( J_{\alpha}^{m} \) by calculating the shift in the action for local \( \xi^{\alpha} \) when the gauge sector is on shell. One finds

\[ \delta \xi S = -2 \int d^{4} x \partial_{m} \xi^{\alpha} J_{\alpha}^{m} \quad (A.5) \]

where

\[ J_{\alpha}^{m} = \frac{1}{2} F^{mn} (\sigma_{n} \lambda)_{\alpha} - \frac{i}{4} \epsilon^{mdcb} (\sigma_{b} \lambda)_{\alpha} F_{cd} - \frac{i}{2} \xi (\sigma^{m} \lambda)_{\alpha} \quad (A.6) \]

which is manifestly gauge invariant. The factor of two in the definition is conventional since under the normalization of supersymmetry used here, the gravitino transforms as \( 2 \partial_{m} \xi^{\alpha} \).

A caveat is also necessary regarding the construction of the current associated with translations. As is well known, the energy-momentum tensor for physical reasons ought to be both symmetric and gauge invariant, but the straightforward Noether current constructed from translations is neither, and must be augmented by a certain “improvement term,” whose addition changes neither the conservation of the current nor the definition of the conserved charge. The result, known as the Belinfante tensor, is conserved, symmetric, and gauge invariant (see for example the discussion in \(^ {14} \)). For the super-Maxwell Lagrangian here, the Belinfante tensor is

\[ J_{nm} = F_{n}^{p} F_{mp} - \frac{1}{4} \eta_{nm} F^{ab} F_{ab} + \frac{i}{4} (\lambda \sigma_{\{m} D_{n\} \lambda}) + \frac{i}{4} (\lambda \tilde{\sigma}_{\{m} D_{n\} \lambda}) - \frac{1}{2} \eta_{nm} \xi^{2} \quad (A.7) \]

Note that its trace is proportional to \( \xi^{2} \) (since the gauginos obey their equation of motion \( \tilde{\sigma}^{m} \partial_{m} \lambda \)), and so it is only the FI term which spoils the classical scale invariance of the theory.

\(^ {13}\)See the discussion in \(^ {6} \).
In addition to global supersymmetry and translations, the theory described above admits a global $U(1)_R$ rotation under which the supersymmetry generator $Q_{\alpha}$ has charge -1 and $Q^{\dot{\alpha}}$ has charge +1. This leads to $\lambda_{\alpha}$ carrying charge +1, with the other fields neutral, and so the $U(1)_R$ current is simply the gaugino vector bilinear

$$J^{(5)}_m = \lambda \sigma_m \bar{\lambda}$$

where we have defined

$$\delta S = - \int d^4x \partial_m \alpha_{(5)} J^{m(5)}$$

for local $U(1)_R$ rotation parameter $\alpha_{(5)}$.

It is a straightforward exercise to check that each of these Noether currents is conserved using their equations of motion, and they agree with the currents constructed in [3] (up to improvement terms).

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