REMARKS ON GENERATING SERIES FOR SPECIAL CYCLES

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Abstract. In this note, we consider special algebraic cycles on the Shimura variety $S$ associated to a quadratic space $V$ over a totally real field $F$, $|F : \mathbb{Q}| = d$, of signature

$((m, 2)^{d_+}, (m + 2, 0)^{d-d_+}), \quad 1 \leq d_+ < d.$

For each $n$, $1 \leq n \leq m$, there are special cycles $Z(T)$ in $S$, of codimension $nd_+$, indexed by totally positive semi-definite matrices with coefficients in the ring of integers $O_F$. The generating series for the classes of these cycles in the cohomology group $H^{2nd_+}(S)$ are Hilbert-Siegel modular forms of parallel weight $\frac{m}{2} + 1$. One can form analogous generating series for the classes of the special cycles in the Chow group $\text{CH}^{nd_+}(S)$. For $d_+ = 1$ and $n = 1$, the modularity of these series was proved by Yuan-Zhang-Zhang. In this note we prove the following: Assume the Bloch-Beilinson conjecture on the injectivity of Abel-Jacobi maps. Then the Chow group valued generating series for special cycles of codimension $nd_+$ on $S$ is modular for all $n$ with $1 \leq n \leq m$.

1. Introduction

The goal of the present note is to probe the limits of what we know about certain special cycle generating series. Suppose that $V$ is a quadratic space over a totally real field $F$ of degree $d$ such that the signature of $V$ is

$((m, 2), \ldots, (m, 2), (m + 2, 0), \ldots, (m + 2, 0)) = ((m, 2)^{d_+}, (m + 2, 0)^{d-d_+}), \quad d_+ > 0.$

We also suppose that $m > 0$. The special cycles in the associated orthogonal Shimura variety $S$ have codimensions $nd_+$ for $1 \leq n \leq m$. Thus there is a significant difference between the case $d_+ = 1$, where there are special cycles of every codimension, and the case $d_+ > 1$, where there are not. The modularity of Chow group valued generating series in the case $d_+ = 1$ is established in many cases; we will review what is known in a moment. However, when $d_+ > 1$ the modularity of such series is more problematic, due to a lack of any systematic source of relations.

As a concrete example, suppose that $B$ is a totally indefinite division quaternion algebra over a real quadratic field $F$ which is not a base change from $\mathbb{Q}$. The space $V$ of trace zero elements in $B$ with quadratic form given by the reduced norm has signature $((1, 2), (1, 2))$. 

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We work with Chow groups with rational coefficients and write $\text{CH}(X)$ rather than $\text{CH}(X)_{\mathbb{Q}}$.

This example arose in discussions with Luis Garcia and Jan Bruinier and was the initial motivation for this paper.
The special cycles on the associated Shimura surface $S$ are 0-cycles indexed by totally positive elements of the ring of integers $O_F$, and the generating series for their degrees is a Hilbert modular form of weight $(3/2, 3/2)$. The modularity of the generating series for their classes in $\text{CH}^2(S)$ is not known however and would depend on the existence of many relations among these 0-cycles. Recall that relations arise from collections $\{(C_i, f_i)\}$ where $C_i$ is a curve on $S$, $f_i$ is a meromorphic function on $C_i$, and the 0-cycle on $S$ given by $\sum_i \text{div}_{C_i}(f_i)$ is zero in $\text{CH}^2(S)$. But there are no evident curves on $S$ and hence any relations among the special 0-cycles have no evident modular construction and would have to arise in some other way. Of course, the situation is the same whenever $d_+ > 1$, since there are no special cycles of codimension $nd_+ - 1$ available to generate relations.

The modularity of generating series for certain divisor classes on the orthogonal Shimura variety $S$ associated to a quadratic space $V$ of signature $(m, 2)$ over $\mathbb{Q}$ was proved by Borcherds, [5]. His proof depends on the existence of a sufficient supply of meromorphic functions on $S$ with explicitly known divisors, constructed by means of his regularized theta lift. They provide the relations among the special divisors in $\text{CH}^1(S)$ and these relations among the coefficients of the generating series imply modularity. The problem of showing modularity of analogous generating series for special cycles of higher codimension, series with coefficients in $\text{CH}^n(S)$, was suggested in [13]. In his thesis [25], Wei Zhang showed that such series are indeed the $q$-expansions of Siegel modular forms of genus $n$ under the assumption that the series are convergent. His proof is based on an induction, beginning with the result of Borcherds for divisors, and a calculation of the Fourier-Jacobi coefficients of the generating series. Subsequently, Bruinier and Westerholt-Raum [7] established the required convergence by an argument based on an analysis of the dimensions of the spaces of Jacobi forms that arise as Fourier-Jacobi coefficients. Such an argument has its roots in the work of [1] and [12].

Over a totally real field $F$ and in the case $d_+ = 1$, the generating series for special cycles of codimension $n$ was considered in [14], where the modularity of its image under the cycle class map to the (Betti) cohomology group $H^{2n}(S)$ is shown to be a consequence of the results of [16], [17], [18]. Using the vanishing of the first Betti number of such varieties, it is shown in [24] that modularity of the $\text{CH}^1(S)$-valued generating series for special divisors follows from the modularity of the $H^2(S)$-valued series. Moreover, it is shown in [24] that the inductive argument of [25] can be carried over to the $d_+ = 1$ case and yields modularity of the $\text{CH}^n(S)$-valued generating series, again assuming the convergence of the series. At present, no analogue of the Bruinier-Westerholdt-Raum result is available for totally real fields of degree $d > 1$, and so modularity of the $\text{CH}^n(S)$-valued generating series in remains open.

In the present paper we consider the case in which $d_+$ is arbitrary. Since we want to avoid a discussion of compactifications, we will assume that $V$ is anisotropic and hence, when $m \geq 3$, the low dimensional exceptions are handled by the embedding trick which we explain in Section 7.
that \( d_+ < d \). The definition of both the connected and weighted special cycles given in [14] for \( d_+ = 1 \) goes over to the general case with almost no change. One important difference, however, is that the role played by the hyperplane section line bundle in Section 6 of [14] is now played by a class \( c_S \in \text{CH}^{d_+}(S) \) constructed as a product of the Chern classes of inverses of tautological bundles. The weighted special cycles \([Z(T, \varphi)] \in \text{CH}^{nd_+}(S)\) are indexed by pairs \((T, \varphi)\) where \( T \in \text{Sym}_n(F) \) is positive semi-definite at each archimedean place of \( F \) and \( \varphi \in S(V(A_f)^n) \) is a Schwartz function on the finite adeles of \( V \). We establish the analogues for general \( d_+ \) of various properties of these cycles proved, for \( d_+ = 1 \), in [14] and in [24]. For example, there is a product formula in the Chow ring \( \text{CH}^\bullet(S) \), Proposition 5.2,

\[
(1.2) \quad Z(T_1, \varphi_1) \cdot Z(T_2, \varphi_2) = \sum_{T \in \text{Sym}_{n_1+n_2}(F) \geq 0} Z(T, \varphi_1 \otimes \varphi_2) \in \text{CH}^{(n_1+n_2)d_+}(S).
\]

In the case \( d_+ = 1 \), this is proved in [24], while the analogous cup product formula for images in cohomology is proved in [14]. The proof we give in Section 5 for general \( d_+ \) makes use of the intersection theory from Fulton [11], a computation of excess bundles, and some Jaffee Lemma arguments, cf. Lemma 4.3 and Proposition 4.13, which allow us to pass to suitable covers to achieve regular embeddings. Also, there is a formula for the pullback of special cycles to Shimura subvarieties associated to totally positive definite subspaces \( U \) of \( V \). This formula plays a key role in the embedding trick.

The generating series for special cycles of codimension \( nd_+ \) is the formal \( q \)-series

\[
(1.3) \quad \phi_n(\tau, \varphi, S) = \sum_{T \in \text{Sym}_n(F) \geq 0} [Z(T, \varphi)] q^T \in \text{CH}^{nd_+}(S)[[q]],
\]

where \( \tau = (\tau_1, \ldots, \tau_d) \in \mathcal{H}_n^d, \varphi \in S(V(A_f)^n), \) and

\[
(1.4) \quad q^T = e(\sum_{j=1}^d \text{tr}(\sigma_j(T)\tau_j)).
\]

Here \( \mathcal{H}_n \) is the Siegel space of genus \( n \) and \( \Sigma = \{\sigma_j\}_{1 \leq j \leq d} \) is the set of archimedean embeddings of \( F \).

The product formula implies the following identity for the formal generating series:

\[
(1.5) \quad \phi_n(\tau_1, \varphi_1 \otimes \varphi_2, S) = \phi_{n_1}(\tau_1, \varphi_1, S) \cdot \phi_{n_2}(\tau_2, \varphi_2, S),
\]

whose analogue for generating series for cohomology classes was proved in [14], for \( d_+ = 1 \), using the theta series. Here the product on the right side is take in the Chow ring of \( S \).

\(^4\)We write \( \text{Sym}_n(F) \geq 0 \) for the space of such totally positive semi-definite matrices.
The series (1.3) is said to be modular if, for every complex valued linear functional on \( \text{CH}^{nd+}(S) \) the formal Fourier series

\[
\phi_n(\tau, \varphi, S, \lambda) = \sum_{T \in \text{Sym}_n(F) \geq 0} \lambda( [Z(T, \varphi)]) q^T \in \mathbb{C}[[q]],
\]

is absolutely convergent and the resulting holomorphic function on \( S^d_n \) is a Hilbert-Siegel modular form.

For example, the image

\[
\phi_n(\tau, \varphi, S, \text{cl}) = \sum_{T \in \text{Sym}_n(F) \geq 0} \text{cl}[Z(T, \varphi)] q^T \in H^{2nd+}(S)[[q]]
\]

of this series under the cycle class map \( \text{cl} = \text{cl}_{nd+} : \text{CH}^{nd+}(S) \to H^{2nd+}(S) \) is the \( q \)-expansion of a Hilbert-Siegel modular form of parallel weight \((m^2+1, \ldots, m^2+1)\), again by the results of [16], [17], [18]. Of course, if the cycle class map happens to be injective, then the modularity of (1.3) follows from this immediately. As observed in [24], such injectivity would result from a combination of the Bloch-Beilinson conjecture, which predicts that the kernel of \( \text{cl}_{nd+} \) maps injectively to the intermediate Jacobian \( J^{nd+}(S) \) under the Abel-Jacobi map, and the vanishing of \( H^{2nd+1}(S) \), which implies that \( J^{nd+}(S) = 0 \).

A main result of this paper is that we can use a variant of this observation to obtain the following.

**Theorem 1.1.** Assume the Bloch-Beilinson conjecture. Then the \( \text{CH}^{nd+}(S) \)-valued generating series (1.3) is modular for all \( n \).

The idea is to combine the embedding trick with a peculiar property of the Hodge diamond for orthogonal Shimura varieties. If \( U_0 \) is a totally positive quadratic space of dimension \( 4\ell \) over \( F \), the orthogonal sum \( \tilde{V} = U_0 + V \) has signature \( ((m+4\ell, 2)^{d+}, (m+2+4\ell, 0)^{d-d+}) \), and there is a corresponding Shimura variety \( \tilde{S} \) with an embedding

\[
\rho : S \to \tilde{S}
\]

of Shimura varieties. The image of the (formal) generating series \( \phi_n(\tau, \varphi, \tilde{S}) \) under the pullback

\[
\rho^* : \text{CH}^{nd+}(\tilde{S}) \to \text{CH}^{nd+}(S)
\]

is a finite linear combination of products

\[
\theta(\tau, \varphi^0) \phi_n(\tau, \varphi^1, S)
\]

where \( \theta(\tau, \varphi^0) \) is a theta series for \( \varphi^0 \in S(U_0(\mathbb{A}_f)^n) \) and \( \phi_n(\tau, \varphi^1, S) \) is a (formal) \( \text{CH}^{nd+}(S) \)-valued generating series for \( \varphi^1 \in S(V(\mathbb{A}_f)^n) \). On the other hand, the results of Vogan and

\[\text{7}^5\text{This is made more precise in Section 7.}\]

\[\text{6}^6\text{Recall that our Chow groups are taken with rational coefficients and all of our varieties and special cycles are defined over number fields.}\]
Zuckermann, explained in detail in Section 9, imply that
\begin{equation}
H^{2nd_+ - 1}(\tilde{S}) = 0, \quad \text{for } \ell > nd_+.
\end{equation}

Therefore, if we assume the Bloch-Beilinson conjecture, the series $\phi_n(\tau, \varphi, \tilde{S})$ is modular, and Theorem 1.1 follows from the pullback relations (1.7). This pullback argument is analogous to the argument in [24], p1159. In fact, in our case, the proof of this consequence given in Section 7 is quite non-trivial and was provided by Jan Bruinier. It depends on the normality of the Baily-Borel compactification of the Hilbert-Siegel modular variety and some results of Knöller, [13].

Remark 1.2. (i) Theorem 1.1 provides support for the conjectured modularity of the Chow group valued generating series, even in the ‘problematic’ $d_+ > 1$ cases. Note that the Bloch-Beilinson conjecture serves as an existence result for the required (but non-evident) relations.

(ii) One can obviously consider analogous unitary Shimura varieties with respect to a CM field over $F$ associated to a Hermitian space of signature $((m, 1)^{d_+}, (m + 1, 0)^{d-d_+})$. The special cycles occur in codimensions $nd_+$ so that, when $d_+ > 1$, the modularity of the Chow group valued generating series for such cycles case is again problematic. Unfortunately, there is no evident Hodge diamond argument in this case.

We now give a brief summary of the contents of this paper. In Section 2, we define the special cycles and the generating series for them in classical language. We also explain how the modularity of the Chow group valued generating series follows from the Bloch-Beilinson conjecture together with a vanishing theorem for low odd degree cohomology of orthogonal Shimura varieties. In Section 3, we point out a couple of natural questions/problems that arise when $d_+ > 1$. Section 4 is the core of paper. Here, working in classical language, we give formulas for the intersection products on special cycles using the machinery of Fulton [11]. There are several basic ingredients. First, using the Jaffee Lemma, Lemma 4.3 and its variant, Proposition 4.13 we pass to covers so that the embedding of the special cycles and the components of their intersections are regular embeddings. In this situation, the intersection product can be expressed in terms of Chern classes and Segre classes of normal cones, Proposition 4.7. These, in turn, can be computed in terms of an excess bundle which is finally related, cf. Proposition 4.11 to the ‘co-tautological’ bundle $C$, defined in (2.6). Thus we obtain a nice formula for the intersection product, in classical language, Theorem 4.15. In Section 5, we give the definition of weighted special cycles in adelic language. These cycles are compatible with pullbacks and hence define classes in the Chow group $\text{CH}^{nd_+} (S) := \lim_K \text{CH}^{nd_+} (S_K)$ as $K$ runs over compact open subgroups of $G(\mathbb{A}_f)$, where $G = R_{F/\mathbb{Q}} \text{GSpin}(V)$. The product formula (1.2) and Proposition 5.2 for weighted special cycles then follows from the classical version. A formula for pullbacks of weighted special cycles to Shimura subvarieties is proved in Section 6, Proposition 6.2. This provides the basis for the first step in the embedding trick discussed in Section 7, an identity, (7.1), expressing the pullback of the formal generating series for an ambient orthogonal Shimura variety as a product of the formal generating series.
for $S$ and a Hilbert-Siegel theta function. The fact that the modularity of the ambient generating series for a family of such identities implies the modularity of the series for $S$ is proved in Section 8. The vanishing of the low odd degree cohomology of an orthogonal Shimura variety as a consequence of the results of Vogan and Zuckerman, is explained in Section 9. Finally, in Section 10 we work out in detail the relation between the weighted special cycles as defined in Section 5 and an alternative definition analogous to that used in [14]. This relation, which involves a careful discussion of the connected component and the structure of the special 0-cycles arising when $n = m$, will be useful in certain applications.

2. Generating series for special cycles: classical version

In this section, we set up the generating series for algebraic cycles on our orthogonal Shimura variety over a totally real field. Here we formulate things in classical language so that the geometric aspects are clearer. An ad`elic version is described in Section 4.

Let $F$ be a totally real field of degree $d = |F : \mathbb{Q}|$ and let $\Sigma = \{ \sigma_j \}$ be the set of archimedean embeddings of $F$. Let $V, (\cdot, \cdot)$ be a quadratic space over $F$ with $\text{sig}(V_j) = \begin{cases} (m, 2) & \text{for } 1 \leq j \leq d_+ \\ (m + 2, 0) & \text{for } d_+ < j \leq d, \end{cases}$ where $V_j = V \otimes_{F, \sigma_j} \mathbb{R}$. We will write $\Sigma_+ = \{ \sigma_j \mid j \leq d_+ \}$. Let

$$D^+ = \prod_{1 \leq j \leq d_+} D^{(j),+},$$

where $D^{(j),+}$ is one component of the space $D^{(j)}$ of oriented negative 2-planes in $V_j$. Thus $D^+$ is connected and $\dim_{\mathbb{C}} D^+ = md_+$. The space

$$D = \prod_j D^{(j)}$$

has $2^{d_+}$ connected components and will be used in the ad`elic version in Section 4.

Let $L \subset V$ be an $O_F$-lattice on which $Q(x) = \frac{1}{2}(x, x)$ is $O_F$-valued and let

$$L^\vee = \{ x \in V(F) \mid (x, L) \subset \partial_F^{-1} \} \supset L$$

be the dual lattice, where $\partial_F^{-1}$ is the inverse different of $F/\mathbb{Q}$. Let

$$\Gamma_L = \{ \gamma \in \text{SO}(V) \mid \gamma L = L, \gamma | L^\vee / L = \text{id} \},$$

and let $\Gamma \subset \Gamma_L$ be a neat subgroup of finite index which stabilizes the component $D^+$. In particular, $\Gamma$ is torsion free. The quotient

$$S = S_\Gamma = \Gamma \backslash D^+ \overset{\pi}{\leftarrow} D^+, \quad \pi = \pi_\Gamma,$$

Later, when we consider the Weil representation, this definition of $L^\vee$ will be appropriate when we use the ‘standard’ additive character $\psi_0$ of $\mathbb{Q}_A/\mathbb{Q}$ and the character $\psi = \psi_0 \circ \text{tr}_{F/\mathbb{Q}}$ for $F_A/F$. 
is then (isomorphic to the set of complex points of) a smooth quasi-projective variety over \( \mathbb{C} \) and is projective if \( d_+ < d \). It is a connected Shimura variety with a canonical model over a number field, but we will not need this for the moment. Let \( \text{CH}^i(S) \) be the Chow group of algebraic cycles of codimension \( i \) on \( S \) modulo rational equivalence and let
\[
\text{CH}^\bullet(S) = \bigoplus_{i=0}^{md_+} \text{CH}^i(S)
\]
be the Chow ring of \( S \). We frequently make the identification \( \text{Pic}(S) = \text{CH}^1(S), \mathcal{L} \mapsto c_1(\mathcal{L}) \).

Special cycles are defined as follows. For a subspace \( W \subset V \) which is totally positive definite for \( Q \), let
\[
D^+_W = \prod_j D^{(j), +}_W,
\]
where
\[
D^{(j), +}_W = \{ z_j \in D^{(j), +} \mid z_j \subset W^\perp \otimes \mathbb{R} \}.
\]
In particular, the codimension of \( D^+_W \) in \( D^+ \) is \( r(W)d_+ \) where \( r(W) = \dim_F W \), and
\[
Z(W) = Z(W)_{\Gamma} = \pi_{\Gamma}(D^+_W)
\]
is an algebraic cycle of codimension \( r(W)d_+ \) in \( S \). The corresponding class in \( \text{CH}^{r(W)d_+}(S) \) will be denoted by \([Z(W)]\).

On the quadric model
\[
D^{(j)} \simeq \{ w_j \in (V_j)_\mathbb{C} \mid (w_j, w_j) = 0, \ (w_j, \bar{w}_j) < 0 \} / \mathbb{C}^\times \subset \mathbb{P}((V_j)_\mathbb{C}),
\]
let \( \mathcal{L}_j \) be the restriction of the tautological line bundle on \( \mathbb{P}((V_j)_\mathbb{C}) \). Let \( \mathcal{L}_j = \text{pr}_j^* \mathcal{L}^5_j \) be the pullback of \( \mathcal{L}^5_j \) to \( D \), where \( \text{pr}_j \) is the projection onto the \( j \)th factor. The restriction of this line bundle to \( D^+ \) descends to \( S \), where we denote it by the same symbol, and we obtain a class \( c_1(\mathcal{L}_j) \in \text{CH}^1(S) \). Let
\[
c_{S} = \prod_{j=1}^{d_+} c_1(\mathcal{L}^\vee_j) \in \text{CH}^{d_+}(S).
\]
We will also need the vector bundle, the co-tautological bundle,
\[
\mathcal{C}_S = \bigoplus_j \mathcal{L}^\vee_j
\]
of rank \( d_+ \). The fibers of this bundle are naturally \( F \)-vector spaces and
\[
c_S = c_{d_+}(\mathcal{C}_S) \cap [S]
\]
where \( c_{d_+}(\mathcal{C}_S) \) is the top Chern class of \( \mathcal{C}_S \). Here we are using the conventions of Chapter 3 of [11]. Later, when we vary \( \Gamma \), we will write \( c_{\Gamma} \) and \( \mathcal{C}_{\Gamma} \) to indicate the dependence on \( \Gamma \).

For \( x \in V^n \), let \( W(x) \) be the subspace of \( V \) spanned by the components of \( x \) and let \( r(x) = \dim W(x) \). Let
\[
[Z(x)] = \begin{cases} [Z(W(x))] \cdot c_S^{n-r(x)} & \text{if } W(x) \text{ is positive definite,} \\ 0 & \text{otherwise.} \end{cases}
\]
Thus, \([Z(x)] \in \text{CH}^{nd+}(S)\). For example, \([Z(0)] = c^n\). When we vary \(\Gamma\), we will write \([Z(x)_{\Gamma}]\).

The following equivariance property will be useful later. If \(\eta \in \text{SO}(V)(F)\), under the natural isomorphism,

\[
\eta : S_{\Gamma} \xrightarrow{\sim} S_{\eta \Gamma \eta^{-1}}, \quad z \mapsto \eta z, \tag{2.8}
\]

\([\eta \ast Z(W)_{\Gamma}] = Z(\eta W)_{\eta \Gamma \eta^{-1}}\) and \([\eta \ast [Z(x)]_{\Gamma}] = [Z(\eta x)]_{\eta \Gamma \eta^{-1}}\).

For the lattice \(L\), let \(S(L) = \mathbb{C}[(L^\vee/L)^n]\) be the group algebra of \((L^\vee/L)^n\). Define the generating series

\[
\phi_n(\tau, S) = \sum_{\mu \in (L^\vee/L)^n} \sum_{x \in \mu + L^n \mod \Gamma} [Z(x)] q^{Q(x)} \cdot e_\mu \in \text{CH}^{nd+}(S) \otimes S(L)[[q]],
\]

where \(\{e_\mu\}\) is the coset basis for \(S(L)\) and, for \(T \in \text{Sym}_n(F)\) and \(\tau \in (\mathfrak{H}_n)^d\), \(q^T\) is given by (1.4).

There is a unitary representation \(\rho_L\) of \(\Gamma'\) on the space \(S(L)\) where \(\Gamma' = \text{Sp}_m(O_F)\), if \(m\) is even, \(s\) and is a 2-fold central extension of this group, if \(m\) is odd.

The expectation is that \(\phi_n(\tau, S)\) is the \(q\)-expansion of a Hilbert-Siegel modular form of genus \(n\) and parallel weight \(\kappa = \frac{m}{2} + 1\). This means that, for any linear functional \(\lambda : \text{CH}^n(S) \longrightarrow \mathbb{C}\), the series

\[
\phi_n(\tau, S, \lambda) = \sum_{\mu \in (L^\vee/L)^n} \sum_{x \in \mu + L^n \mod \Gamma} \lambda([Z(x)]) q^{Q(x)} \cdot e_\mu,
\]

with coefficients in \(S(L)\), is termwise absolutely convergent and that the resulting analytic function on \(\mathfrak{H}_n\) satisfies

\[
\phi_n(\gamma(\tau), S, \lambda) = \prod_j \det(c \tau_j + d)^n \rho_L(\gamma) \phi_n(\tau, S, \lambda),
\]

for all \(\gamma \in \Gamma'\).

As motivation, one has the fact that the image of \(\phi_n(\tau, S)\) under the cycle class map

\[
\text{cl}^{nd+} : \text{CH}^{nd+}(S) \longrightarrow H^{2nd+}(S) \tag{2.10}
\]

is the \(q\)-expansion of a Hilbert-Siegel modular form,

\[
\phi_n(\tau, S, \text{cl}) = \sum_{\mu \in (L^\vee/L)^n} \sum_{x \in \mu + L^n \mod \Gamma} \text{cl}([Z(x)]) q^{Q(x)} \cdot e_\mu \in H^{2nd+}(S) \otimes S(L),
\]

by the results of [16], [17] and [18].

\(^8\)As usual, \(S(L)\) can be identified with a subspace of the Schwartz space \(S(V(\mathbb{A}_f)^n)\) of finite adeles over \(F\) of \(V^n\) and the representation \(\rho_L\) has a natural construction in this language.
Of course, as observed in [24], when the cycle class map [2.10] is injective, the modularity of $\phi_n(\tau, S, \text{cl})$ implies that of $\phi_n(\tau, S)$. Let $\text{CH}^N(S)^0 = \ker(\text{cl}^N)$ be the subgroup of $\text{CH}^N(S)$ of cohomologically trivial cycles and let

$$AJ_N : \text{CH}^N(S)^0 \to J^N(S)$$

be the Abel-Jacobi map to the $N$th intermediate Jacobian of $S$. The Bloch-Beilinson Conjecture asserts that the map $AJ_N$ is injective up to torsion. Here recall that $S$ is defined over a number field. On the other hand, we will show in Section 9 that $J^N(S) = 0$ for $2N - 1 < m - \lceil \frac{m}{2} \rceil$. This proves the following.

**Proposition 2.1.** Assume the Bloch-Beilinson conjecture. Then the $\text{CH}^{nd_+}(S)$-valued series $\phi_n(\tau, S)$ is modular for

$$nd_+ < \begin{cases} \frac{m+2}{4} & \text{for } m \text{ even,} \\ \frac{m+3}{4} & \text{for } m \text{ odd.} \end{cases}$$

Now we apply the ‘embedding trick’, as described in [21], p.1159. Our description here is imprecise; a precise version using the ad` elic formul ation of the generating series of Section 4 will be given in Section 7. Let $U_0$ be totally positive quadratic space over $F$ of dimension $4\ell$, and let $\tilde{V} = U_0 \oplus V$. Suppose that $L_{U_0}$ is an even integral lattice in $U_0$ and let $\tilde{L} = L_{U_0} \oplus L$. Note that

$$\text{sig}(\tilde{V}) = ((m + 4\ell, 2)^{d_+}, (m + 2 + 4\ell, 0)^{d-d_+})$$

in the obvious notation. Let $\tilde{D}^+ = \prod_{j=1}^{d_+} \tilde{D}^{(j)}$ be the associated symmetric space and take a neat subgroup $\tilde{\Gamma}$ of finite index in the group $\Gamma_{\tilde{L}}$. We suppose that $\tilde{\Gamma} \cap \text{SO}(V) = \Gamma$ and thus have an embedding

$$j : S \longrightarrow \tilde{S} = \tilde{\Gamma} \backslash \tilde{D}.$$ 

If $\ell$ is sufficiently large, e.g., $\ell > nd_+$ will always work, and assuming the Bloch-Beilinson conjecture, the series $\phi_n(\tau, \tilde{S})$ is modular of weight $\kappa + 2\ell$, valued in $\text{CH}^{nd_+}((\tilde{S}) \otimes S(\tilde{L}))$. On the other hand, the pullback of this series under $j$, should be expressible as a finite linear combinations of products of theta series associated to $L_{U_0}$, of weight $2\ell$, and components of generating series $\phi_n(\tau, S)$. Using a suitable cancellation property, the modularity of $\phi_n(\tau, S)$ will follow for all $n$! Thus, up to several compatibilities and the pullback formula and cancellation properties which will be carefully formulated in Section 9 we have the following.

**Theorem 2.2.** Assume the Bloch-Beilinson conjecture. Then the $\text{CH}^{nd_+}(S)$-valued series $\phi_n(\tau, S)$ is modular for all $n$.

Note that the range of vanishing of odd Betti numbers given by Corollary [9.4] plays a crucial role here.

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9This will be handled in a better way in Section 9
3. Problematic examples

In this section, we make some observations about relations among special cycles. The key point is that the codimensions of special cycles are multiples of \(d_+\). Thus, when \(d_+ = 1\), there are special cycles defined in each codimension and it is reasonable to imagine that relations among cycles of codimension \(n\) arise from meromorphic functions on special cycles of codimension \(n-1\). When \(d_+ > 1\), this is no longer the case and there is no evident source of such relations, whereas the modularity of the generating series for such cycles implies that such relations must exist in abundance. Thus there is an essential difference between the cases \(d_+ = 1\) and \(d_+ > 1\).

In the case \(d_+ = 1\) and \(F = \mathbb{Q}\), one might imagine that the meromorphic functions on special cycles of codimension \(n-1\) giving rise to relations among special cycles of codimension \(n\) are those constructed by Borcherds on such \(m-n+1\)-dimensional orthogonal Shimura subvarieties. In fact, the Zhang, Bruinier-Westerholt-Raum proof of modularity does not proceed in this way and this suggests the following problem.

**Problem 1.** In the case \(F = \mathbb{Q}\), what are the relations among the special cycles

\[
Z(T) = \sum_{\begin{subarray}{c} x \in (L')^n \\ Q(x) = T \mod \Gamma \end{subarray}} Z(x)
\]

implied by the modularity of the generating series. Can these be described in terms of Borcherds forms on \(Z(y)\)'s where \(y \in (L')^{n-1}\)?

Now suppose that \(d_+ > 1\).

**Example.** Suppose that \(F\) is a real quadratic field and let \(B\) be a division quaternion algebra over \(F\) that is split at the archimedean places \(\sigma_1\) and \(\sigma_2\). We also suppose that \(B\) is not a base change of an indefinite quaternion algebra over \(\mathbb{Q}\), e.g., that \(B_p\) is a division algebra for some non-archimedean place over a rational prime \(p\) that is not split in \(F\). Let \(V\) be the subspace of elements \(x \in B\) with \(\text{tr}(x) = 0\), where \(\text{tr} : B \to F\) is the reduced trace, and let \(Q(x) = \nu(x)\) be the reduced norm of \(x\). Then \(\text{sig}(V) = ((1,2),(1,2))\) so that \(m = 1\) and \(d_+ = 2\) in our notation above. Choosing \(L\) and \(\Gamma\), we obtain a smooth projective surface \(S\) with a large supply of 0-cycles \(Z(x)\) defined by vectors \(x \in V\) with \(Q(x) \gg 0\). The associated generating series is

\[
\phi_1(\tau, S) = \sum_{\mu \in (L'/L)^n} \sum_{x \in \mu + L \mod \Gamma} [Z(x)] \cdot q^{Q(x)} \cdot e_{\mu} \in \text{CH}^2(S) \otimes S(L)[[q]].
\]

The image of this series under the cycle class map

\[
\text{cl}^2 : \text{CH}^2(S) \to H^4(S, \mathbb{C}) = \mathbb{C}
\]
is a Hilbert modular form
\[ \phi_1(\tau, S, \text{cl}) = \sum_{\mu \in (L^\vee/L)^n} \sum_{x \in \mu + L \mod \Gamma} \deg Z(x) \cdot q^{Q(x)} \cdot e_\mu \]
of parallel weight $\frac{3}{2}$ valued in $\mathbb{S}(L)$. The modularity of $\phi_1(\tau, S)$ must entail a large number of relations among the 0-cycles $Z(x)$, but such relations would arise from collections $\sum_j (C_j, f_j)$ where $C_j$ is a curve on $S$ and $f_j$ is a meromorphic function on $C_j$. But there are no evident curves on $S$! Nonetheless, Proposition 2.2 asserts the modularity of $\phi_1(\tau, S)$ assuming the Bloch-Beilinson conjecture.

**Problem 2.** Find explicit relations among the 0-cycles $Z(x)$ on $S$.

**Problem 3.** Use them to prove modularity of (3.1).

### 4. Some intersection theory

In this section, we record some results about the geometry and intersections of the special cycles. We begin with the classical version of Section 2 and will pass back and forth, via GAGA, between topological and algebraic geometric arguments. In particular, since we will be working with projective varieties over $\mathbb{C}$, we follow the treatment of intersection theory given in Fulton, [11]. Thus we sometimes write $A_k(X)$ for the group of $k$-cycles modulo rational equivalence and note that $A_k(X) = \text{CH}^{n-k}(X)$ if $X$ is smooth of dimension $n$.

**Remark 4.1.** The intersection of (weighted adelic) special cycles was considered in [24] in the case $d_+ = 1$. It may be that their formulation can be extended to the case $d_+ > 1$, but we felt that the more classical approach given here with complete proofs provides a better insight into the geometry.

As in Section 2, we suppose that $\Gamma$ is a neat subgroup of $\Gamma_L$ preserving the component $D^+$ and is, in particular, torsion free.

#### 4.1. Some preliminary results.

If $\Gamma' \subset \Gamma$ is a subgroup of finite index, the map $\text{pr} : S_{\Gamma'} \to S_\Gamma$ is, topologically, a covering map and hence, algebraically, is finite étale of degree $|\Gamma : \Gamma'|$. The map
\[ \text{pr}^* : \text{CH}^\bullet(S_\Gamma) \longrightarrow \text{CH}^\bullet(S_{\Gamma'}) \]
is a ring homomorphism. Since, for $\alpha \in A_k(S_\Gamma)$,
\[ \text{pr}_* \text{pr}^*(\alpha) = |\Gamma : \Gamma'| \alpha, \]
$\text{pr}^*$ is injective. In particular,
\[ \text{pr}_*(\text{pr}^*(\alpha) \cdot \text{pr}^*(\beta)) = \text{pr}_*(\text{pr}^*(\alpha \cdot \beta)) = |\Gamma : \Gamma'| \alpha \cdot \beta, \]
so that identities involving products of elements of $\text{CH}^*(S_\Gamma)$ can be checked on their pullbacks. Also note that

$$\text{pr}_* (c_{\Gamma'} ) = c_{\Gamma}.$$ 

**Remark 4.2.** For a totally positive subspace $U$ in $V$, the cycle $D_U$ in $D$ is a holomorphic and totally geodesic submanifold and $\Gamma$ acts on $D$ by holomorphic isometries. Thus, if the restriction of the (topological covering) map $\pi _\Gamma: D^+ \to \Gamma \backslash D^+$ to $D^+_U$ is injective, the image is a totally geodesic holomorphic submanifold and the inclusion of this image in $S_\Gamma$ is (algebraically) a regular embedding. Similarly, for totally positive subspaces $W \subset U \subset V$, if the restriction of $\pi _\Gamma$ to $D^+_W$ is injective, then the image of $D^+_W$ is a totally geodesic submanifold of the image of $D^+_U$ and the inclusion is a regular embedding.

The following result and its variants will be useful. It holds in a much more general context, c.f., [22]. For convenience, we include the proof.

**Lemma 4.3.** [21, 22]. Let $U$ be a subspace of $V$ which is totally positive definite for $Q$, and let $\sigma _U$ be the isometry of $V$ with $\sigma _U |_U = -1$ and $\sigma _U |_{U^\perp} = +1$.

(i) Let $\tilde{\Gamma}_U$ be the centralizer of $\sigma _U$ in $\Gamma$, i.e., the stabilizer in $\Gamma$ of the subspace $U$. Let $\Gamma_U$ be the subgroup of $\tilde{\Gamma}_U$ whose elements act trivially on $U$. Then, since $\Gamma$ is neat, $\tilde{\Gamma}_U = \Gamma_U$.

(ii) (Jaffee Lemma) Suppose that $\sigma _U \Gamma \sigma _U = \Gamma$. Then the map

$$\tilde{\Gamma}_U \backslash D^+_U \longrightarrow \Gamma \backslash D^+$$

is injective.

In particular, this implies that $Z(U)_{\Gamma} = \Gamma_U \backslash D^+_U$ is a submanifold of $S_\Gamma$ and the map

$$f: Z(U)_{\Gamma} \longrightarrow S_\Gamma$$

is a regular embedding of codimension $r(U)d_+$. 

**Proof.** To prove (i), note that, since $\Gamma$ is neat, so are $\tilde{\Gamma}_U$ and its image in $O(U)$. Since $U$ is totally positive definite and the image of $\tilde{\Gamma}_U$ in $O(U)(\mathbb{R})$ is discrete, this image must be torsion and hence trivial. Thus $\gamma \in \Gamma_U$, as required. To prove (ii), suppose that $z$ and $z' \in D^+_U$ and that $\gamma z' = z$ for some $\gamma \in \Gamma$. Then, since $\sigma _U$ fixes $D^+_U$ pointwise, $\sigma _U \gamma \sigma _U z' = z$ as well. Since $\Gamma$ is torsion free and hence acts without fixed points on $D^+$, we must have $\gamma^{-1} \sigma _U \gamma \sigma _U = 1$ and so $\sigma _U \gamma \sigma _U = \gamma \in \tilde{\Gamma}_U$, as required. Combining this with (i) gives the last statement. 

**Remark 4.4.** (i) If a lattice $L$ satisfies $L = U \cap L + U^\perp \cap L$, then $\sigma _U \Gamma L \sigma _U = \Gamma_L$.

(ii) The condition of the lemma will always hold after passing to a subgroup of finite index. For example, for a totally positive subspace $U$ of $V$, let

$$\Gamma' = \Gamma \cap \sigma _U \Gamma \sigma _U.$$
Then $\sigma_U \Gamma' \sigma_U = \Gamma'$ and we have

$$Z(U)_{\Gamma'} = \Gamma' \setminus D_U^+ \xrightarrow{f'} \Gamma' \setminus D^+$$

$$Z(U)_{\Gamma} = \Gamma \setminus D_U^+ \xrightarrow{f} \Gamma \setminus D^+,$$

where $f'$ is a regular embedding.

(iii) If $f : Z(U)_{\Gamma} \to S_{\Gamma}$ is a regular embedding and $\Gamma' \subset \Gamma$ has finite index, then $f' : Z(U)_{\Gamma'} \to S_{\Gamma'}$ is also a regular embedding.

4.2. Intersections. Suppose that $U_1$ and $U_2$ are totally positive subspaces of $V$ with associated classes

$$[Z(U_i)_{\Gamma}] \in \text{CH}^{r_i, d_i}(S_{\Gamma}), \quad r_i = r(U_i).$$

We want to compute the product

$$[Z(U_1)_{\Gamma}] \cdot [Z(U_2)_{\Gamma}] \in \text{CH}^{(r_1 + r_2), d_i}(S_{\Gamma}).$$

The following is the analogue of Proposition 2.2 in [24].

**Proposition 4.5.** (i) As a set, the fiber product

$$|I(U_1, U_2)_{\Gamma}| \xrightarrow{g} |Z(U_2)_{\Gamma}|$$

$$|Z(U_1)_{\Gamma}| \xrightarrow{g} |S_{\Gamma}|$$

is given by

$$|I(U_1, U_2)_{\Gamma}| = \bigcup_{W} |Z(W)_{\Gamma}|,$$

where $W$ runs over the set

$$\Gamma \setminus \{ W = \gamma_1 U_1 + \gamma_2 U_2 \mid \gamma_1, \gamma_2 \in \Gamma \}.$$

(ii) As a scheme, the fiber product

$$I(U_1, U_2)_{\Gamma} \xrightarrow{g} Z(U_2)_{\Gamma}$$

$$Z(U_1)_{\Gamma} \xrightarrow{g} S_{\Gamma}$$

is given by

$$I(U_1, U_2)_{\Gamma} = \bigcup_{\gamma} Z(W_{\gamma})_{\Gamma},$$

where the subspace $W_{\gamma}$ is given by $\gamma_1 U_1 + \gamma_2 U_2$ as the pair $\gamma = (\gamma_1, \gamma_2)$ runs over representatives for the $\Gamma$-orbits in the set

$$\text{Inc}(U_1, U_2)_{\Gamma} := \Gamma/\Gamma_{U_1} \times \Gamma/\Gamma_{U_2}.$$
Remark 4.6. (i) Note that the map

\[ \Gamma \setminus \frac{\Gamma}{\Gamma U_1 \times \Gamma U_2} \to \Gamma \setminus \{ W = \gamma_1 U_1 + \gamma_2 U_2 \mid \gamma_1, \gamma_2 \in \Gamma \} \]

has finite fibers which give rise to multiplicities in the fiber product. For example, suppose that \( \dim_F U_2 < \dim_F U_1 \) and that \( \gamma_0 U_2 \subset U_1 \) and \( \gamma'_0 U_2 \subset U_1 \), for some \( \gamma_0 \) and \( \gamma'_0 \in \Gamma \). Then the pairs \( (1, \gamma_0) \) and \( (1, \gamma'_0) \) both map to the \( \Gamma \)-orbit of \( U_1 \) in (4.3). But the subspaces \( \gamma_0 U_2 \) and \( \gamma'_0 U_2 \) of \( U_1 \) can be distinct and hence, since elements of \( \Gamma U_1 \) act trivially on \( U_1 \), the double cosets \( \Gamma U_1 \gamma_0 \Gamma U_2 \) and \( \Gamma U_1 \gamma'_0 \Gamma U_2 \) can be distinct as well.

(ii) We will often use representatives of the form \((1, \gamma)\) for orbits in (4.6), with a slight abuse of notation.

By Proposition 6.1 (a) of [11] we have the following.

**Proposition 4.7.** Suppose that \( i_1 : Z(U_1)_{\Gamma} \to S \) is a regular embedding. Let \( N = g^*(N_{Z(U_1)_{\Gamma}}(S_{\Gamma})) \) be the pullback of the normal bundle, where \( g \) is as in (4.4), and let \( I = I(U_1, U_2)_{\Gamma} \).

(i) Then

\[ Z(U_1)_{\Gamma} \cdot Z(U_2)_{\Gamma} = \{ c(N) \cap s(I, Z(U_2)_{\Gamma}) \}_\kappa, \]

where

\[ \kappa = \dim Z(U_1) + \dim Z(U_2) - \dim S, \]

\( c(N) \) is the total Chern class of \( N \), and

\[ s(I, Z(U_2)_{\Gamma}) = s(C) \]

is the Segre class of the normal cone

\[ C = C_I(Z(U_2)_{\Gamma}) \]

of \( I \) in \( Z(U_2)_{\Gamma} \).

We will see below that this expression can be written as a sum of contributions from the various \( Z(W_{\gamma})_{\Gamma} \). To evaluate these contributions we will use the next result, whose proof we omit, compare Section 6.1 of Fulton [11], in particular, Example 6.1.7.

**Proposition 4.8.** For totally positive subspaces \( U_1 \) and \( U_2 \) of \( V \) and \( W = U_1 + U_2 \), suppose that all of the morphisms in the diagram

\[ (4.7) \]

are regular embeddings. Let

\[ N = q^* N_{Z(U_1)_{\Gamma}} S_{\Gamma} \]

\[ \text{This need not be the fiber product.} \]
be the pullback to $Z(W)\Gamma$ of the normal bundle to $Z(U_1)\Gamma$ in $S\Gamma$ and let
\[ N' = N_{Z(W)} Z(U_2)\Gamma \]
be the normal bundle to $Z(W)\Gamma$ in $Z(U_2)\Gamma$. Note that $N'$ is a sub-bundle of $N$ and that these bundles have ranks $(r(W) - r(U_2))d_+$ and $r(U_1)d_+$ respectively. The excess bundle $E = N/N'$ has rank
\[ e = (r(U_1) + r(U_2) - r(W))d_+ \]  
Then
\[ \{ c(N) \cap s(Z(W)\Gamma, Z(U_2)\Gamma) \} \kappa = c_e(E) \cap [Z(W)\Gamma] \in A_{k-e}(Z(W)\Gamma), \]
where $c_e(E)$ is the top Chern class of $E$ and $k = \dim D_W^+ = (m - r(W))d_+$. Pushing this forward to $S\Gamma$ yields a class
\[ (4.8) \quad c_e(E) \cap [Z(W)\Gamma] \in A_{k-e}(S\Gamma) = \text{CH}^{(r_1+r_2)d_+}(S\Gamma). \]

4.3. Excess bundles. In this section we compute the excess bundle in the situation of Proposition 4.8 working with complex manifolds.

For a point $z_j \in D^{(j)+}$, we have a canonical identification of the tangent space\footnote{Here we note that if $e_1$ and $e_2$ is a properly oriented orthogonal basis for $z_j$ with $(e_1, e_1) = (e_2, e_2) = -1$, then the complex structure $J_{z_j}$ on $z_j$ given by $J_{z_j}e_1 = -e_2$, $J_{z_j}e_2 = e_1$ induces a complex structure on $T_{z_j}(D^{(j)+})$; it depends only on the orientation of $z_j$. The map (2.4) sending $z_j$ to the $+i$-eigenspace of $J_{z_j}$ in $(z_j)_c$ is holomorphic.}
\[ T_{z_j}(D^{(j)+}) = \text{Hom}(z_j, z_j^\perp), \]
and hence
\[ T_z(D^+) = \bigoplus_{j=1}^{d_+} \text{Hom}(z_j, z_j^\perp) = \text{Hom}_F(z, z^\perp). \]
Here we are working with subspaces of $V \otimes_{\mathbb{Q}} \mathbb{R}$, and the idempotents in $F \otimes_{\mathbb{Q}} \mathbb{R}$ give the direct sum decomposition of the space of $F$-linear maps. As a slight abuse of notation, we are writing $\text{Hom}_F$ for the space of $F_\mathbb{R}$-linear maps. Similarly, if $z \in D_W^+$, we have tangent spaces
\[ T_z(D_U^+) = \text{Hom}_F(z, z^\perp \cap U_i^\perp), \quad i = 1, 2, \]
and
\[ T_z(D_W^+) = \text{Hom}_F(z, z^\perp \cap W^\perp). \]
Since $z \in D_U^+$, we have an orthogonal decomposition
\[ V_\mathbb{R} = U_{1,\mathbb{R}} + z + (z^\perp \cap U_{1,\mathbb{R}}^\perp). \]
Thus, the fiber at $z$ of the normal bundle to $D_U^+$ in $D^+$ is given by
\[ (4.9) \quad N_{D_U^+}(D^+_z) \simeq \text{Hom}_F(z, z^\perp/(z^\perp \cap U_i^\perp)) \simeq \text{Hom}_F(z, U_{1,\mathbb{R}}). \]
Lemma 4.9. Suppose that the map \( Z(U_1)_\Gamma \rightarrow S_\Gamma \) is a regular embedding. Then
\[
N_{Z(U_1)_\Gamma}(S_\Gamma) \simeq (\mathcal{C}_\Gamma \otimes_F U_1)|_{Z(U_1)_\Gamma}.
\]

Next consider \( z \in D^+_W \subset D^+_U \), so that the fiber of the normal bundle to \( D^+_W = D^+_{U_1+U_2} \) in \( D^+_U \) at \( z \) is
\[
(4.10) \quad \text{Hom}_F(z, (z^\perp \cap U_2^\perp)/(z^\perp \cap W^\perp)) = \text{Hom}_F(z, (z^\perp \cap U_2^\perp)/(z^\perp \cap U_1^\perp \cap U_2^\perp)),
\]
where \( W^\perp = U_1^\perp \cap U_2^\perp \). We have the diagram
\[
\begin{array}{ccc}
   z^\perp & \overset{j}{\rightarrow} & z^\perp \\
   \downarrow & & \downarrow \\
   (z^\perp \cap U_2^\perp)/(z^\perp \cap W^\perp) & \rightarrow & z^\perp/(z^\perp \cap U_1^\perp)
\end{array}
\]
where \( j \) is injective. This exhibits \( N_{D^+_W}(D^+_U) \) as a sub-bundle of \( N_{D^+_U}(D^+) \), since
\[
N_{D^+_W}(D^+_U)_z = \text{Hom}_F(z, (z^\perp \cap U_2^\perp)/(z^\perp \cap W^\perp)) \hookrightarrow \text{Hom}_F(z, z^\perp/(z^\perp \cap U_1^\perp)) = N_{D^+_U}(D^+)_z.
\]
The fiber of the excess bundle at \( z \) is given by \( F \)-linear homomorphisms from \( z \) into the cokernel of \( j \),
\[
E_z = \text{Hom}_F(z, z^\perp/(z^\perp \cap U_1^\perp + z^\perp \cap U_2^\perp)).
\]
But, in fact, we have a nicer expression.

Lemma 4.10.
\[
E_z = \text{Hom}_F(z, (U_1 \cap U_2)_R).
\]

Proof. Recall that \( \text{Hom}_F \) is the space of \( F_R \)-linear maps. Since \( z \in D_W = D_{U_1} \cap D_{U_2} \), we have an inclusion \((U_1 \cap U_2)_R \rightarrow z^\perp \). If
\[
x \in (U_1 \cap U_2)_R \cap (z^\perp \cap U_1^\perp + z^\perp \cap U_2^\perp),
\]
write \( x = w_1 + w_2 \) with \( w_i \in U_i^\perp \cap z^\perp \). Then \((x, x) = (x, w_1) + (x, w_2) = 0\), so that \( x = 0 \). But
\[
\dim_F U_1 \cap U_2 = e = \dim_R z^\perp/(z^\perp \cap U_1^\perp + z^\perp \cap U_2^\perp),
\]
so the inclusion gives an isomorphism
\[
(U_1 \cap U_2)_R \overset{\sim}{\rightarrow} z^\perp/(z^\perp \cap U_1^\perp + z^\perp \cap U_2^\perp).
\]
Thus we have the following nice expression for the excess bundle.
Proposition 4.11. In the situation of Proposition 4.8, the excess bundle is given by
\[ E \simeq (C_{\Gamma} \otimes_{F} (U_1 \cap U_2)) \big|_{Z(W)_{\Gamma}}. \]

Corollary 4.12. In the situation of Proposition 4.8,
\[ c_e(E) \cap [Z(W)_{\Gamma}] = c^{r_1+r_2-r(W)}_{\Gamma} \cap [Z(W)_{\Gamma}] \in \text{CH}^{(r_1+r_2)+d_1}(S_{\Gamma}). \]

4.4. Passing to covers. To compute \( Z(U_1 \cap U_2)_{\Gamma} \), we pass to a cover where the geometry becomes nice so that Corollary 4.12 can be applied.

If \( \Gamma' \) has finite index in \( \Gamma \) and \( \text{pr}_{\Gamma'}: S_{\Gamma'} \to S_{\Gamma} \) is the projection, then
\[ (\text{pr}_{\Gamma'})_*([Z(U)_{\Gamma}]) = [Z(U)_{\Gamma}], \]
and
\[ \text{pr}_{\Gamma'}^*([Z(U)_{\Gamma}]) = \sum_{\gamma \in \Gamma' \setminus \Gamma / \Gamma} [Z(\gamma U)]_{\Gamma'}. \]

Moreover,
\[ \text{pr}_{\Gamma'}^*(I(U_1, U_2)) = \bigcup_{\gamma} Z(W_{\gamma})_{\Gamma'}, \]
where \( W_{\gamma} = \gamma_1 U_1 + \gamma_2 U_2 \) as \( \gamma = (\gamma_1, \gamma_2) \) runs over a set of orbit representatives for \( W_{\gamma_{\Gamma'}} \in \Gamma' \setminus (\Gamma / \Gamma_{U_1} \times \Gamma / \Gamma_{U_2}). \)

Here note that we are simply passing from the \( \Gamma \)-orbits in Proposition 4.10 to \( \Gamma' \)-orbits.

Proposition 4.13. For totally positive subspaces \( U_1 \) and \( U_2 \) of \( V \), choose a set of representatives \( \{ \gamma_j \} \) for the \( \Gamma \)-orbits in (4.6) and let \( W_j = W_{\gamma_j} \). Then there exists a subgroup \( \Gamma' \) of finite index in \( \Gamma \) such that, for all \( \gamma \in \Gamma \), all of the morphisms
\[ Z(\gamma U_1)_{\Gamma'} \rightarrow S_{\Gamma'}, \quad Z(\gamma U_2)_{\Gamma'} \rightarrow S_{\Gamma'}, \quad \text{and} \quad Z(\gamma W_j)_{\Gamma'} \rightarrow S_{\Gamma'}, \]
are injective and hence are regular embeddings. Moreover, for \( W_j = \gamma_1 U_1 + \gamma_2 U_2 \) and for all \( \gamma \in \Gamma \), the morphism
\[ Z(\gamma W_j)_{\Gamma'} \rightarrow Z(\gamma \gamma_{\Gamma'} U_{\Gamma'}) \]
is injective and hence is a regular embedding as well.

Proof. Suppose that \( \Gamma' \) is a normal subgroup of finite index in \( \Gamma \). For any \( \gamma \in \Gamma \) and totally positive subspace \( U \) of \( V \), \( \sigma_{\gamma U} = \gamma \sigma_U \gamma^{-1} \). Thus, if \( \Gamma' \subset \Gamma \cap \sigma_U \Gamma \sigma_U \), we have
\[ \Gamma' \subset \Gamma \cap \sigma_U \Gamma \sigma_U = \gamma (\Gamma \cap \sigma_U \Gamma \sigma_U) \gamma^{-1} \]
as well. By Lemma 4.3 and (iii) of Remark 4.4, it follows that \( Z(\gamma U)_{\Gamma'} \rightarrow S_{\Gamma'} \) is a regular embedding. Thus any normal subgroup \( \Gamma' \) of \( \Gamma \) such that
\[ \Gamma' \subset \Gamma \cap (\sigma_{U_1} \Gamma \sigma_{U_1}) \cap (\sigma_{U_2} \Gamma \sigma_{U_2}) \cap \bigcap_{j} (\sigma_{W_j} \Gamma \sigma_{W_j}) \]
has the required properties. Note that, for $W_j = \gamma_{1,j} U_1 + \gamma_{2,j} U_2$, there is a factorization

$$Z(\gamma W_j)_{\Gamma'} \to Z(\gamma \gamma_{2,j} U_2)_{\Gamma'} \to S_{\Gamma'},$$

so that the map

$$Z(\gamma W_j)_{\Gamma'} \to Z(\gamma \gamma_{2,j} U_2)_{\Gamma'}$$

is injective and hence is a regular embedding. □

For totally positive subspaces $U_1$ and $U_2$ of $V$, take a subgroup $\Gamma' \subset \Gamma$ of finite index as in Proposition 4.13 so that we have the the diagram

$$I(U_1, U_2)_{\Gamma'} \xrightarrow{j} Z(U_2)_{\Gamma'}$$

$$\xrightarrow{g} Z(U_1)_{\Gamma'} \xrightarrow{i_1} S_{\Gamma'}$$

is given by

$$I(U_1, U_2)_{\Gamma'} = \bigcup_{\gamma} Z(W_{\gamma})_{\Gamma'},$$

where $\gamma$ runs over a set of orbit representatives for

$$\Gamma' \setminus (\Gamma / \Gamma U_1 \times \Gamma / \Gamma U_2).$$

Moreover, $i_1$ and $i_2$ are regular embeddings and $I(U_1, U_2)_{\Gamma'}$ is a union of smooth subvarieties $Z(W_{\gamma})_{\Gamma'}$ of $S_{\Gamma'}$ intersecting cleanly. In this situation, we have the following result, whose proof we omit.

**Lemma 4.14.** Let $C' = C_{I(U_1, U_2)_{\Gamma'}}(Z(U_2)_{\Gamma'})$ be the normal cone of $I(U_1, U_2)_{\Gamma'}$ in $Z(U_2)_{\Gamma'}$. Then the decomposition of $C'$ into irreducible components is given by

$$C' = \bigcup_{\gamma} C'_{\gamma}, \quad C'_{\gamma} = N_{Z(W_{\gamma})_{\Gamma'}}(Z(U_2)_{\Gamma'}).$$

Combining Proposition 4.13 and Corollary 4.12, we obtain our main formula for the intersection of special cycles.

**Theorem 4.15.** For totally positive subspaces $U_1$ and $U_2$ of $V$ with $\dim F U_i = r_i$,

$$[Z(U_1)_{\Gamma}] \cdot [Z(U_2)_{\Gamma}] = \sum_{\gamma} c_{\Gamma}^{r_1 + r_2 - r(W_{\gamma})} \cap [Z(W_{\gamma})_{\Gamma}] \in CH^{(r_1 + r_2)d_+}(S_{\Gamma}),$$

where $\gamma$ runs over the index set $\Gamma \setminus (\Gamma / \Gamma U_1 \times \Gamma / \Gamma U_2)$.

**Proof.** For totally positive subspaces $U_1$ and $U_2$ of $V$, there is a subgroup $\Gamma'$ of finite index in $\Gamma$ such that

$$pr^*(Z(U_1)_{\Gamma} \cdot Z(U_2)_{\Gamma}) = \sum_{\gamma} c_{\Gamma'}^{r_1 + r_2 - r(W_{\gamma})} \cap [Z(W_{\gamma})_{\Gamma}] \in CH^{(r_1 + r_2)d_+}(S_{\Gamma}).$$
where \( \gamma \) runs over (4.15). This follows from the discussion following Proposition 6.1 in Fulton where the irreducible components of the normal cone are described by Lemma 4.14 and their contributions, according to Example 6.1.1 of Fulton, are given by Proposition 4.13 and Corollary 4.12.

We will need the following fact.

**Lemma 4.16.** Let \( \Gamma_\gamma \) be stabilizer in \( \Gamma \) of the coset \( \gamma \in \Gamma / \Gamma U_1 \times \Gamma / \Gamma U_2 \). Then \( \Gamma_\gamma = \Gamma_{W_\gamma} \).

**Proof.** Elements of \( \Gamma_\gamma \) preserve the subspace \( W_\gamma \) and hence lie in \( \tilde{\Gamma}_{W_\gamma} = \Gamma_{W_\gamma} \), in the notation of (i) of Lemma 4.3. Conversely, if \( \gamma_0 \in \Gamma_{W_\gamma} \), then \( \gamma_0 \) acts trivially on \( W_\gamma \) and hence it preserves the subspaces \( \gamma_1 U_1 \) and \( \gamma_2 U_2 \). It then lies in both \( \Gamma_{\gamma_1 U_1} = \gamma_1 \Gamma U_1 \gamma_1^{-1} \) and \( \Gamma_{\gamma_2 U_2} = \gamma_2 \Gamma U_2 \gamma_2^{-1} \) and hence in \( \Gamma_\gamma \). \( \square \)

Now we pass back to our original \( \Gamma \). For \( \gamma \in \Gamma / \Gamma U_1 \times \Gamma / \Gamma U_2 \), let \( r(\gamma) = \dim_F W_\gamma \). With this notation and using Lemma 4.16, we have

\[
\sum_{\gamma \in \Gamma \setminus \Gamma / \Gamma_{W_\gamma}} [Z(W_\gamma)_{\Gamma'}] = \sum_{\gamma \in \Gamma \setminus \Gamma / \Gamma_{W_\gamma}} [Z(\gamma W_\gamma)_{\Gamma'}] = \sum_{r(\gamma) = r \mod \Gamma'} \text{pr}^*([Z(W_\gamma)_{\Gamma'}]).
\]

Thus

\[
\text{pr}^*(Z(U_1)_{\Gamma} \cdot Z(U_2)_{\Gamma'}) = \sum_r c_{r_1 + r_2 - r} \cap \left( \sum_{r(\gamma) = r \mod \Gamma'} \text{pr}^*([Z(W_\gamma)_{\Gamma'}]) \right),
\]

and hence

\[
\text{pr}_* \text{pr}^*(Z(U_1)_{\Gamma} \cdot Z(U_2)_{\Gamma'}) = \sum_r \text{pr}_* \left( c_{r_1 + r_2 - r} \cap \left( \sum_{r(\gamma) = r \mod \Gamma'} \text{pr}^*([Z(W_\gamma)_{\Gamma'}]) \right) \right)
\]

\[
= \sum_r \text{pr}_* \left( c_{r_1 + r_2 - r} \right) \cap \left( \sum_{r(\gamma) = r \mod \Gamma'} [Z(W_\gamma)_{\Gamma'}] \right).
\]

But by definition \( c_{r'} = \text{pr}^*(c_{r}) \), and so, canceling the index \( \lvert \Gamma : \Gamma' \rvert \) from both sides, we have

\[
Z(U_1)_{\Gamma} \cdot Z(U_2)_{\Gamma'} = \sum_{\gamma \mod \Gamma} c_{r_1 + r_2 - r(\gamma)} \cap [Z(W_\gamma)_{\Gamma'}]
\]

as claimed. \( \square \)
5. Generating series for special cycles: adèlic version

In this section, we formulate a version of the generating series for special cycles in adèlic language using cycles weighted by Schwartz functions on the finite adèles of \( V \). This is a slight variation\(^{12}\) of the setup of [14] also used in [24]. It is far more convenient than the classical setup of Section 2 for pullback arguments and intersection relations.

Most of the definitions and results of [14], where \( d_+ = 1 \), remain unchanged when \( d_+ \) is arbitrary. We will continue to assume that \( d_+ < d \), however, so that our varieties will be proper over \( \mathbb{C} \). We will not repeat all of the discussion of [14], but will simply recall the main results and note where minor modifications are needed. There will be a slight shift in notation.

5.1. Weighted cycles. Let \( G = R_{F/\mathbb{Q}} \text{GSpin}(V) \) and for a compact open subgroup \( K \subset G(\mathbb{A}_f) \), let

\[
S_K = G(\mathbb{Q}) \backslash D \times G(\mathbb{A}_f) / K.
\]

If \( K \) is neat, \( S_K \) is a union of smooth projective varieties as discussed in Section 2. More precisely, we write

\[
G(\mathbb{A}_f) = \bigsqcup_j G(\mathbb{Q})_+ g_j K,
\]

where \( G(\mathbb{Q})_+ \) is the subgroup of element with totally positive spinor norm, and, for \( g \in G(\mathbb{A}_f) \), we let \( \Gamma_g = G(\mathbb{Q})_+ \cap gKg^{-1} \). Then, we have the decomposition, [14] (1.5),

\[
S_K \simeq \bigsqcup_j \Gamma_{g_j} \backslash D^+ = \bigsqcup_j S_{\Gamma_{g_j}}.
\]

If \( U \) is a totally positive definite subspace of \( V \), \( Z(U)_{\Gamma_{g_j}} \) is a connected cycle of codimension \( r(U)d_+ \) in \( S_{\Gamma_{g_j}} \). In the notation of [14], (3.3), \( Z(U)_{\Gamma_{g_j}} = c(U, g, K) \).

The weighted cycles are defined as follows. For \( \varphi \in S(V(\mathbb{A}_f)^n) \) and \( T \in \text{Sym}_n(F)_{\geq 0} \) totally positive semi-definite, let

\[
Z(T, \varphi, K) := \sum_j \sum_{x \in \Omega_T(F) \mod \Gamma_{g_j}} \varphi(g_j^{-1}x) [Z(U(x))_{\Gamma_{g_j}}] \cap c_{\Gamma_{g_j}}^{n-r(x)},
\]

where

\[
\Omega_T = \{ x \in V^n \mid Q(x) = T \}.
\]

Then \( Z(T, \varphi, K) \) is an element of \( \text{CH}^{ad_+}(S_K) \otimes \mathbb{Q} R \), where \( R \) is the subfield of \( \mathbb{C} \) where \( \varphi \) takes values. We will take \( R = \mathbb{C} \) from now on. Note that

\[
Z(0, \varphi, K) = c_S^\mathbb{A} \cdot \varphi(0),
\]

where the restriction of \( c_S \) to \( S_{\Gamma_{g_j}} \) is \( c_{\Gamma_{g_j}} \).

\(^{12}\)In previous cases the parameter \( d_+ = 1 \).
Remark 5.1. Here we have taken as our definition the analogue of the expression given in Proposition 5.4 of [14] in the case \( d_+ = 1 \). The alternative definition in terms of ‘natural’ cycles and the the proof of the coincidence of the two definitions is given in Section 10.

The equivariance and pullback properties of Propositions 5.9 and 5.10 of [14] go over without change\(^{13}\).

For any \( g \in G(\mathbb{A}_f) \),

\[
Z(T, \omega(g) \varphi, gKg^{-1}) = Z(T, \varphi, K) \cdot g^{-1}.
\]

If \( K \) is neat and \( K' \subset K \) is another compact open subgroup of \( G(\mathbb{A}_f) \), then

\[
pr^*(Z(T, \varphi, K)) = Z(T, \varphi, K'),
\]

where \( pr: S_{K'} \to S_K \) is the natural projection. As a result, we have well defined classes

\[
Z(T, \varphi) \in CH^{nd_+}(S) := \lim_{\rightarrow K} CH^{nd_+}(S_K).
\]

In the case \( d_+ = 1 \), the following result was suggested in [14], Remark 6.3, and proved in [24], Proposition 2.6.

Proposition 5.2. For \( T_i \in \text{Sym}_{n_i}(F)_{\geq 0} \) and \( \varphi_i \in S(V(\mathbb{A}_f)^{n_i})^K \),

\[
Z(T_1, \varphi_1, K) \cdot Z(T_2, \varphi_2, K) = \sum_{T \in \text{Sym}_{n_1+n_2}(F)_{\geq 0}} Z(T, \varphi_1 \otimes \varphi_2, K) \in CH^{(n_1+n_2)d_+}(S_K).
\]

Remark 5.3. By invariance under pullback, for classes in the direct limit \((5.7)\), we have

\[
Z(T_1, \varphi_1) \cdot Z(T_2, \varphi_2) = \sum_{T \in \text{Sym}_{n_1+n_2}(F)_{\geq 0}} Z(T, \varphi_1 \otimes \varphi_2) \in CH^{(n_1+n_2)d_+}(S).
\]

Proof. Choose \( K \) neat such that \( \varphi_1 \) and \( \varphi_2 \) are \( K \)-invariant and compute

\[
Z(T_1, \varphi_1, K) \cdot Z(T_2, \varphi_2, K)
\]

\[
= \sum_j \sum_{x_1 \in \Omega_{T_1}(F) \mod \Gamma_{g_j}} \varphi_1(g_j^{-1}x_1) \varphi_2(g_j^{-1}x_2) \sum_{x_2 \in \Omega_{T_2}(F) \mod \Gamma_{g_j}} \times [Z(U(x_1))_{\Gamma_{g_j}}] \cdot [Z(U(x_2))_{\Gamma_{g_j}}] \cap \mathbb{R}^{n_1+n_2-r(x_1)-r(x_2)}.
\]

\(^{13}\)It should be noted however, that the proofs depend on the relations between the connected cycles and the ‘natural’ cycles of Section 2 of [13] and Section 10 below. These relations depend, in turn, on Lemma 5.7 of [13], which is due to Weil.
By Theorem 4.15, the intersection number is given by
\[
[Z(U(x_1))_{T_g}] \cdot [Z(U(x_2))_{T_g}] = \sum_r c_r^{g_1+g_2} \cap \left( \sum_{\gamma} [Z(W_{\gamma})_{T_g}] \right)
\]
where \( \gamma \) runs over the \( \Gamma_g \)-orbits in the set
\[
\{ \gamma \in \Gamma_g / \Gamma_g, U(x_1) \times \Gamma_g, U(x_2) \mid r(\gamma) = r \}.
\]
The whole intersection number then unwinds to
\[
Z(T_1, \varphi_1, K) \cdot Z(T_2, \varphi_2, K)
= \sum_j \sum_{T \in \text{Sym}_{n_1+n_2}(F) \geq 0} \sum_{x \in \Omega_T(F)} (\varphi_1 \otimes \varphi_2)(g_j^{-1} x) [Z(U(x))_{T_g}] \cap c_r^{g_1+g_2} \cap [\gamma]\,
\]
and this gives the claimed expression. \( \square \)

5.2. The generating series. Here we use the notation of Sections 7 and 8 of [14].

For \( g \in G(A_f), \tau \in S_{g_1}^d \) and \( \varphi \in S(V(A_f)^n)_K \), define the formal generating series
\[
(5.9) \quad \phi_n(\tau; \varphi, K) = \sum_{T \in \text{Sym}_{n_1+n_2}(F) \geq 0} [Z(T, \varphi, K)] q^T \in \text{CH}^{nd}(S_K)_C[[q]].
\]
Note that for \( g \in G(A_f) \), by (5.5), we have
\[
(5.10) \quad \phi_n(\tau; \varphi) \cdot g^{-1} = \sum_{T \in \text{Sym}_{n_1+n_2}(F) \geq 0} [Z(T, \omega(g)\varphi, gKg^{-1})] q^T \in \text{CH}^{nd}(S_{gKg^{-1}})_C[[q]].
\]
Passing to the limit over \( K \), noting (5.6), and writing
\[
(5.11) \quad \phi_n(\tau; \varphi) = \sum_{T \in \text{Sym}_{n_1+n_2}(F) \geq 0} [Z(T, \varphi)] q^T \in \text{CH}^{nd}(S)_C[[q]],
\]
we see that the formal series is equivariant with respect to the actions of \( G(A_f) \) on \( S(V(A_f)^n) \) and on \( \text{CH}^{nd}(S) \).

As a consequence of the product formula (5.8), we have a product formula for the formal generating series. For \( n = n_1 + n_2 \) with \( n_i \geq 1 \), let
\[
J : S_{g_1}^d \times S_{g_2}^d \rightarrow S_{g_1}^d, \quad (\tau_1, \tau_2) \mapsto \left( \begin{array}{c} \tau_1 \\ \tau_2 \end{array} \right).
\]

Proposition 5.4. For \( n = n_1 + n_2 \) with \( n_i \geq 1 \) and for \( \varphi_1 \in S(V(A_f)^{n_1}) \) and \( \varphi_2 \in S(V(A_f)^{n_2}) \),
\[
(5.12) \quad \phi_n(\left( \begin{array}{c} \tau_1 \\ \tau_2 \end{array} \right); \varphi_1 \otimes \varphi_2) = \phi_{n_1}(\tau_1; \varphi_1) \cdot \phi_{n_2}(\tau_2; \varphi_2).
\]
In particular, the \( T_2 \)-coefficient with respect to \( q_2 \) of the pullback is given by
\[
(5.13) \quad \phi_n(\left( \begin{array}{c} \tau_1 \\ \tau_2 \end{array} \right); \varphi_1 \otimes \varphi_2)_{T_2} = \phi_{n_1}(\tau_1; \varphi_1) \cdot Z(T_2; \varphi_2).
\]
and the $q_2$-constant term of the pullback

\[(5.12) \quad \phi_n\left(\begin{array}{c} T_1 \\ T_2 \end{array}; \varphi_1 \otimes \varphi_2\right)_0 = \varphi_2(0) \sum_{T_1} Z(T_1, \varphi_1) \cdot c_{S-n}^{n-1} q_1^{T_1} \]

**Proof.**

\[
\phi_n(\tau; \varphi_1) \cdot \phi_n(\tau; \varphi_2) = \sum_{T_1, T_2} Z(T_1, \varphi_1) \cdot Z(T_2, \varphi_2) q_1^{T_1} q_2^{T_2} = \sum_{T_1, T_2} T = \left(\begin{array}{c} T_1 \\ \ast \end{array} \right) Z(T, \varphi_1 \otimes \varphi_2) J^* q^T. \]

\[\square\]

**Remark 5.5.** In [14], such a product formula for generating series valued in cohomology groups was a consequence of the product formula for the theta forms (5.19), and was used to prove the cup product version of (5.8). Here we have given a direct geometric proof of (5.8) and obtain (5.10) from it.

Using the definition (5.2) of the weighted cycles, we can also write our formal series as

\[(5.13) \quad \phi_n(\tau; \varphi, K) = \sum_j \sum_{x \in V(F)^n} \varphi(g_j^{-1} x) \left[ Z(U(x))_{\Gamma_{g_j}} \right] \cap C_{\Gamma_{g_j}}^{n-r(x)} \cdot q^{Q(x)}, \]

a kind of formal theta-function.

For any complex valued linear functional $\lambda$ on $\text{CH}^{nd_+}(S_K)$, let

\[(5.14) \quad \phi_n(\tau; \varphi, K, \lambda) = \sum_{T \in \text{Sym}_n(F)_{\geq 0}} \lambda(\left[ Z(T, \varphi, K) \right]) q^T \in \mathbb{C}[[q]]. \]

Recall that the generating series $\phi_n(\tau; \varphi, K)$ is said to be modular if, for every linear functional $\lambda$, the formal power series (5.14) is absolutely convergent and the resulting holomorphic function on $\mathcal{H}_n$ is a Hilbert-Siegel modular form.

### 5.3. Classes in cohomology

Taking $\lambda$ to be the cycle class map $\text{cl} : \text{CH}^{nd_+}(S_K) \to H^{2nd_+}(S_K)$, we have

\[(5.15) \quad \phi_n(\tau; \varphi, K, \text{cl}) = \sum_{T \in \text{Sym}_n(F)_{\geq 0}} \text{cl}(\left[ Z(T, \varphi, K) \right]) q^T \in H^{2nd_+}(S_K). \]

The modularity of this series is the analogue with Theorem 8.1 of [14] for $d_+$ arbitrary. We briefly sketch the argument, referring to Section 7 of [14] for details.

Let $G' = R_{F/Q} \text{Sp}(n)$ and let $\tilde{G}'(\mathbb{A})$ be the metaplectic cover of $G'(\mathbb{A})$. The metaplectic group $\tilde{G}'(\mathbb{A})$ acts on $S(V(\mathbb{A})^n)$ via the Weil representation $\omega = \omega_{\psi}$ determined by the additive
character $\psi$ of $F_\kappa/F$. This action commutes with the linear action of $g \in G(A)$, $\omega(g)\varphi(x) = \varphi(g^{-1}x)$.

Let

$$\varphi_\infty = \frac{\varphi^{(n)} \otimes \ldots \otimes \varphi^{(n)} \otimes \varphi_+^0 \otimes \ldots \otimes \varphi_+^0}{d_+^{d-d_+}} \in [S(V^n_R) \otimes A^{(nd_+,nd_+)}(D)]^{G(\mathbb{R})},$$

(5.16)

where, for $1 \leq j \leq D_+$, $\varphi^{(n)} \in S(V^n_j) \otimes A^{(n,n)}(D_j)$ is the Schwartz form for $V_j$, and, for $j > d_+$, $\varphi_+^0 \in S(V^n_j)$ is the Gaussian for $V_j$, cf. Sections 7 and 8 of [14].

For use later, we define a $(1,1)$-form on $D^{(j)}$ by

$$\Omega_j = \varphi^{(1)}(0),$$

(5.17)

and note that $\varphi^{(n)}(0) = \Omega_j^n$. By Corollary 4.12 of [14], the form $\Omega_j$ on the factor $D^{(j)}$ is the first Chern form of the inverse of the tautological bundle on the space of oriented negative 2-planes in $V_j$. Thus

$$\Omega_S^n = \varphi_\infty(0)$$

(5.18)

is an $(nd_+,nd_+)$-form representing the class $c^n_S$, the $n$-th power of the top Chern class of the vector bundle $C_S$ defined in (2.6).

For $\varphi \in S(V(A_j)^n)\mathbb{K}$, let $\widetilde{\varphi} = \varphi_\infty \otimes \varphi \in S(V(A)^n)$. Then the theta series

$$\theta(g', g; \varphi) = \sum_{x \in \langle V(F)^n \rangle} \omega(g') \widetilde{\varphi}(g^{-1}x), \quad g \in G(A_f), \quad g' \in \widetilde{G'}(A),$$

(5.19)

defines a closed $(nd_+,nd_+)$-form on $S_K$ and is left invariant for the (canonical) image of $G'(\mathbb{Q})$ in $\widetilde{G'}(A)$. When $g = 1$, we will write simply $\theta(g'; \varphi)$ for this series.

Let $Mp(n,R)$ be the metaplectic cover of $Sp(n,R)$, and, for $g'_0 \in Mp(n,R)$, write

$$g'_0 = \begin{pmatrix} u & 1 \\ 1 & v^{-\frac{1}{2}} \\ & t_v^{-\frac{1}{2}} \end{pmatrix} k', \quad \tau = u + iv \in \mathfrak{h}_n,$$

as in (7.21) of [14], where $k' \in K'$, the 2-fold cover of $U(n)$. Then for $T \in \mathrm{Sym}_n(\mathbb{R})$, define the Whittaker function

$$W_T(g'_0) = \det(v)^{m+\frac{1}{2}} e(\mathrm{tr}(T\tau)) \det(k')^{m+\frac{1}{2}}.$$

When $k' = 1$, we write $g'_0 = g'_+$ and note that

$$\det(v)^{-m-\frac{1}{2}} W_T(g'_+) = e(\mathrm{tr}(T\tau)).$$

There is a surjection $Mp(n,R)^d \to \widetilde{G'}(\mathbb{R})$ and an inclusion $\widetilde{G'}(\mathbb{R}) \hookrightarrow \widetilde{G'}(A)$. For $T \in \mathrm{Sym}_n(F)$ and $g' \in \widetilde{G'}(\mathbb{R})$, let

$$W_T(g') = W_{T_1}(g'_1) \ldots W_{T_d}(g'_d),$$

where $T_j = \sigma_j(T)$ and $(g_1, \ldots, g_d) \in Mp(n,R)^d$ is a preimage of $g'$. Note that

$$q^T = N(\det(v))^{-m+\frac{1}{2}} W_T(g'_+),$$
where $N(\det(v)) = \prod_j \det(v_j)$.

The analogue of Theorem 8.1 of [14] is that, as a consequence of [16], [17] and [18], the cohomology class\(^{14}\) of $\theta(g'; \varphi)$ is given by

$$[[\theta(g'; \varphi)]] = N(\det(v))^{\frac{m+2}{4}} \sum_{T \in \text{Sym}_n(F) \geq 0} \text{cl}([Z(T, \varphi)]) q^T \in H^{2n+1}(S).$$

The invariance of $\theta(g'; \varphi)$ under $G'(Q)$ implies that the series (5.15) is the $q$-expansion of a Hilbert-Siegel modular form of weight $(\frac{m}{2} + 1, \ldots, \frac{m}{2} + 1)$.

6. A PULLBACK FORMULA

6.1. The classical version. In this section we suppose that $U_0$ is a totally positive subspace of $V$ and let

$$\rho : D_0^+ = D_{U_0}^+ \to D^+.$$ Let $\Gamma_0 = \Gamma_{U_0}$ and suppose that $\sigma_0 \Gamma_0 = \Gamma$, where $\sigma_0 = \sigma_{U_0}$, so that we have a regular embedding

$$\rho : S^0_\Gamma = Z(U_0)_{\Gamma} = \Gamma_0 \backslash D_0^+ \to \Gamma \backslash D^+ = S_\Gamma$$
of non-singular varieties. There is then a pullback homomorphism

$$\rho^* : \text{CH}^\bullet(S_{\Gamma}) \to \text{CH}^\bullet(S^0_\Gamma)$$of Chow rings. Note that the tautological bundles $L_j$ on $S_{\Gamma}$ pull back to the corresponding tautological bundles on $S^0_\Gamma$ and hence $\rho^* c_S = c_{S^0}$. The proof of the following result is analogous to that of Theorem 4.15 and will be omitted.

**Proposition 6.1.** Suppose that $U_0$ is a totally positive definite subspace of $V$, as above. Then for any totally positive subspace $U$ in $V$,

$$\rho^*([Z(U)]_{\Gamma}) = \sum_{\gamma \in \Gamma_0 \backslash \Gamma_{U_0}} [Z(\text{pr}_0(\gamma U))]_{\Gamma_0} \cap c^{(r(U) - r(\text{pr}_0(\gamma U)))}_{S^0},$$
where $\text{pr}_0 : V \to U_{U_0}^+$ is the orthogonal projection.

6.2. The ad`elic version. For a totally positive subspace $U_0$ in $V$, let $G^0 = R_{F/\mathbb{Q}} \text{GSpin}(U_{U_0}^+)$ so that $G^0 \subset G = R_{F/\mathbb{Q}} \text{GSpin}(V)$. For a compact open subgroup $K \subset G(\mathbb{A}_f)$ and $g \in G(\mathbb{A}_f)$, let $K_g = G^0(\mathbb{A}_f) \cap g K g^{-1}$ and let

$$\rho_{g,K} : S_{K_g}^0 = G^0(\mathbb{Q}) \backslash D_0 \times G^0(\mathbb{A}_f)/K^0 \to S_K = G(\mathbb{Q}) \backslash D \times G(\mathbb{A}_f)/K, \quad [z, h] \mapsto [z, hg],$$
be the natural morphism. Suppose that $K$ is neat. The corresponding pullback homomorphisms

$$\rho^*_{g,K} : \text{CH}^\bullet(S_K) \to \text{CH}^\bullet(S_{K_g}^0)$$
are compatible with the systems of projections (5.7) and define a homomorphism

$$\rho^*_g : \text{CH}^\bullet(S) \to \text{CH}^\bullet(S^0)$$

\(^{14}\)Which we denote by $[[\theta(g'; \varphi)]]$.\)
on the direct limits.

**Proposition 6.2.** For a Schwartz function $\varphi \in S(V(\mathbb{A}_f)^n)^K$ and $T \in \text{Sym}_n(F)$ totally positive semi-definite,

$$
\rho_{g,K}(Z(T, \varphi, K)) = \sum_r \sum_{x_0 \in U_0(F)^n} \varphi_r^0(x_0) Z(T - Q(x_0), \varphi_r^1, K_g^0).
$$

where

$$
(6.1) \quad \omega(g)\varphi = \sum_r \varphi_r^0 \otimes \varphi_r^1 \in S(U_0(\mathbb{A}_f)) \otimes S(U_0^+(\mathbb{A}_f)).
$$

**Proof.** The description of $\rho_{g,K}$ on connected components is given in Section 4 of [14]. Write

$$
G(\mathbb{A}_f) = \coprod_j G(\mathbb{Q})_+g_jK,
$$

and

$$
G^0(\mathbb{A}_f) = \coprod_i G^0(\mathbb{Q})_+h_iK_g^0,
$$

and, for each $i$, write

$$
G(\mathbb{Q})_+h_iG = G(\mathbb{Q})_+g_jK, \quad j = j(i), \quad h_ig = \gamma_i^{-1}g_jk_i.
$$

Here the index $j = j(i)$ depends on $i$. Let

$$
(6.2) \quad \Gamma_i^0 = G^0(\mathbb{Q})_+ \cap h_iK_g^0h_i^{-1} = G^0(\mathbb{Q})_+ \cap h_igKg^{-1}h_i^{-1} \subset G^0(\mathbb{Q})_+ \cap g_jKg_j^{-1}\gamma_i,
$$

and $\Gamma_j = \Gamma_j^0 = G(\mathbb{Q})_+ \cap g_jKg_j^{-1}$. Let

$$
\pi_i^0 : D^{0,+} \to \Gamma_i^0\backslash D^{0,+} \quad \text{and} \quad \pi_j : D^+ \to \Gamma_j\backslash D^+
$$

be the projections. Here note that $\Gamma_i^0$ is a subgroup of $\gamma_i^{-1}\Gamma_j\gamma_i$. Then

$$
\rho_{g,K} : \coprod_i \Gamma_i^0\backslash D^{0,+} \simeq S_{K_g^0} \to S_K \simeq \coprod_j \Gamma_j\backslash D^+, \quad \rho_i : \pi_i^0(z) \mapsto \pi_j(\gamma_i z),
$$

where $j = j(i)$ as in (6.2) and $\rho_i : \Gamma_i^0\backslash D^{0,+} \to \Gamma_j\backslash D^+$ is the restriction of $\rho_{g,K}$ to the component $\Gamma_i^0\backslash D^{0,+}$. For $\varphi \in S(V(\mathbb{A}_f)^n)^K$, the part of the special cycle $Z(T, \varphi, K)$ lying in the connected component $S_{\Gamma_j} = \Gamma_j\backslash D^+$ is given by

$$
(6.3) \quad \sum_{x \in O_{\mathbb{A}_f}(F) \mod \Gamma_j} \varphi(g_j^{-1}x) [Z(U(x))_{\Gamma_j}] \cap c_{\Gamma_j, g_j}^{n-r(x)}.
$$

For a fixed $i$ and taking $j = j(i)$, write $\rho_i = [\gamma_i] \circ \rho_i^5$ where $[\gamma_i] : S_{\gamma_i^{-1}\Gamma_j\gamma_i} \simeq S_{\Gamma_j}$. For convenience, we write $\Gamma_j' = \gamma_i^{-1}\Gamma_j\gamma_i$. 


By Proposition 6.1, the pullback of (6.3) under $\rho_i$ is
\[
\sum_{x \in \Omega_f(F) \mod \Gamma_j} \varphi(g_j^{-1}x) \rho_i^* \left( [Z(U(x))_{\Gamma_j}] \cap c_{T,j}^{n-r(x)} \right)
\]
\[
= \sum_{x \in \Omega_f(F) \mod \Gamma_j} \varphi(g_j^{-1}x) (\rho_i^*)^* \left( [Z(\gamma_i^{-1}U(x))_{\Gamma_j}] \cap c_{T,j}^{n-r(x)} \right)
\]
\[
= \sum_{x \in \Omega_f(F) \mod \Gamma_j} \varphi(g_j^{-1}\gamma_i x) (\rho_i^*)^* \left( [Z(U(x))_{\Gamma_j}] \cap c_{T,j}^{n-r(x)} \right)
\]
\[
= \sum_{x \in \Omega_f(F) \mod \Gamma_j} \varphi(g_j^{-1}\gamma_i x) \sum_{\gamma \in \Gamma^0 \cap \Gamma_j \cap U(U(x))} [Z(pr_0(\gamma U(x)))_{\Gamma^0_0}] \cap c_{T,0}^{r(U(x)) - r(pr_0(\gamma U(x)))} \cap c_{T,0}^{n-r(x)}
\]
\[
= \sum_{x \in \Omega_f(F) \mod \Gamma_j} \varphi(g_j^{-1}\gamma_i x) [Z(pr_0(U(x)))_{\Gamma^0_0}] \cap c_{T,0}^{n-r(pr_0(U(x)))}
\]
\[
= \sum_{x_0 \in U_0(F)^n, \ x_1 \in U_1(F)^n \mod \Gamma^0_0} \varphi(g^{-1}(x_0 + h_i^{-1}x_1)) [Z(U(x_1))_{\Gamma^0_0}] \cap c_{T,0}^{n-r(x_1)}.
\]

Here in the last line we note that $g_j\gamma_i = k_i g_j^{-1} h_i^{-1}$ and that $h_i$ acts trivially on $U_0$. Using (6.1), the sum on $i$ of the last expression is
\[
\sum_i \sum_{x_0 \in U_0(F)^n, x_1 \in U_1(F)^n \mod \Gamma^0_0} \varphi_r(x_0) \varphi_r^1(h_i^{-1}x_1) [Z(U(x_1))_{\Gamma^0_0}] \cap c_{T,0}^{n-r(x_1)}
\]
\[
= \sum_r \sum_{x_0 \in U_0(F)^n} \varphi_r^0(x_0) \sum_i \sum_{x_1 \in U_1(F)^n \mod \Gamma^0_0} \varphi_r^1(h_i^{-1}x_1) [Z(U(x_1))_{\Gamma^0_0}] \cap c_{T,0}^{n-r(x_1)}
\]
\[
= \sum_r \sum_{x_0 \in U_0(F)^n} \varphi_r^0(x_0) [Z(T - Q(x_0), \varphi_r^1, K_g^0)].
\]

\[\square\]

6.3. The pullback of the generating series. Applying Proposition 6.2, we obtain a formula for the pullback of the generating series.

**Proposition 6.3.** With the notation of the previous section, suppose that $\varphi \in S(V(A_f)^n)^K$ satisfies (6.7). Then
\[
\rho_{g,K}^* \left( \phi_n(\tau; \varphi, K) \right) = \sum_r \theta(\tau, \varphi_r^0) \cdot \phi_n(\tau, \varphi_r^1, K_g^0)
\]
where
\[
\theta(\tau, \varphi^0_\tau) = \sum_{x_0 \in U_0(F)^n} \varphi^0_\tau(x_0) q^{Q(x_0)}
\]
and
\[
\varphi_n(\tau, \varphi^1_\tau, K_0^0) \in \text{CH}^{nd+}(S_{K_0^0}^0)[[q]]
\]
is the formal generating series for \(S_{K_0^0}^0\).

Note that the decomposition (6.1) depends on \(g\).

7. The embedding trick

We now slightly vary the situation of Section 6. Let \(U_0\) be a totally positive definite space over \(F\) of dimension \(4\ell\) and let \(\tilde{V} = U_0 \oplus V\) be the orthogonal sum. The signature of \(\tilde{V}\) is given by (2.11). Let \(\tilde{G} = \text{R}_{F/Q} \text{GSpin}(\tilde{V})\) so that there is a natural homomorphism \(G \to \tilde{G}\). For \(\tilde{K}\) compact open in \(\tilde{G}(F)\) and \(K = G(F) \cap \tilde{K}\), we obtain a morphism
\[
\rho_{\tilde{K}} : S_K \to \tilde{S}_{\tilde{K}},
\]
and, assuming that \(\tilde{K}\) is neat so that these are smooth varieties, a ring homomorphism
\[
\rho^*_{\tilde{K}} : \text{CH}^*(\tilde{S}_{\tilde{K}}) \to \text{CH}^*(S_K).
\]
Passing to the limit over \(\tilde{K}\), we also have
\[
\rho^* : \text{CH}^*(\tilde{S}) \to \text{CH}^*(S).
\]

For \(\varphi \in S(V(F)^n)\) and \(\varphi^0 \in S(U_0(F)^n)\), Proposition 5.3 yields
\[
(7.1)
\rho^*\left( \phi_n(\tau; \varphi^0 \otimes \varphi) \right) = \theta(\tau, \varphi^0) \cdot \phi_n(\tau, \varphi).
\]

Thus the two formal generating series are related by multiplication by a holomorphic Hilbert-Siegel theta series. The following result will be proved in the next section.

**Proposition 7.1.** Suppose that for all choices of \(\varphi^0 \in S(U_0(F)^n)\) the series \(\phi_n(\tau; \varphi^0 \otimes \varphi)\) is modular. Then the series \(\phi_n(\tau, \varphi)\) is modular.

As explained in [24], we have the following non-vanishing result.

**Lemma 7.2.** For any point \(\tau_0 \in \mathcal{H}_d\), there exists a function \(\varphi^0 \in S(U_0(F)^n)\) such that \(\theta(\tau_0, \varphi^0) \neq 0\).

**Proof.** Suppose that the linear functional
\[
\varphi^0 \mapsto \theta(\tau_0, \varphi^0) = N(\det v_0)^{-\ell} \theta(g'_0, \varphi^0), \quad N(\det v_0) = \prod_{j=1}^d \det(v_{0,j}),
\]
on $S(U^0(A_f))^n$ is zero. Then, for all $\varphi^0$, the function $\theta(g',\varphi^0)$ on $G'(A)$ vanishes on the subset $G'(\mathbb{Q})g'_0G'(A_f)$. But this set is dense in $G'(A)$ and the functions $\theta(g',\varphi^0)$ are not all zero. □

8. Formal Fourier series

In this section, we prove Proposition 7.1. The argument, using formal Fourier series and a special case of a result of [13], was provided by Jan Bruinier.

Let $S_F = \text{Sym}_n(O_F)$ and let

$$S_F^\vee = \{ x \in \text{Sym}_n(F) \mid \text{tr}_{F/Q} \text{tr}(xy) \in \mathbb{Z}, \forall y \in S_F \} = \text{Sym}_n(\mathbb{Z})^\vee \otimes \mathbb{Z} \partial_F^{-1}. $$

Let $\mathcal{C}$ be the cone of totally positive definite elements in $\text{Sym}_n(\mathbb{R})$ so that

$$S_F^\vee \cap \mathcal{C} = \{ x \in S_F^\vee \mid \sigma_j(x) \geq 0, \text{for all } j \}. $$

For a subgroup $\Gamma \subset \text{Sp}_n(F)$ commensurable with $\text{Sp}_n(O_F)$, there is a positive integer $\nu \in \mathbb{Z}_{>0}$ such that $\Gamma$ contains the principal congruence subgroup $\Gamma(\nu)$ of $\text{Sp}_n(O_F)$. It will be sufficient to consider the case $\Gamma = \Gamma(\nu)$. We will assume that $\nu \geq 3$ to eliminate sign issues.

Let $N = N_P$ (resp. $M = M_P$) be the unipotent radical (resp. the standard Levi factor) of the Siegel parabolic $P$ of $\text{Sp}_n/F$ and let

$$\Gamma_N = \Gamma \cap N(F) = \{ n(\beta) \mid \beta \in \nu \cdot S_F \}, \quad n(\beta) = \left( \begin{array}{cc} 1_n & \beta \\ \beta & 1_n \end{array} \right),$$

and

$$\Gamma_M = \Gamma \cap M(F) = \{ m(\epsilon) \mid \epsilon \in \text{GL}_n(O_F), \epsilon \equiv 1_n \text{ mod } \nu O_F \}, \quad m(\epsilon) = \left( \begin{array}{c} \epsilon \\ \epsilon^{-1} \end{array} \right).$$

We write $\Lambda = \{ \epsilon \in \text{GL}_n(O_F), \epsilon \equiv 1_n \text{ mod } \nu O_F \}$.

For $k \in \mathbb{Z}_{\geq 0}$, let $M_k(\Gamma)$ be the vector space of Hilbert-Siegel modular forms of parallel weight $k$ with respect to $\Gamma$. The graded ring $M_*(\Gamma) = \oplus_{k \geq 0} M_k(\Gamma)$ is an integral domain. We write $Q(M_*(\Gamma))$ for its quotient field. It can be viewed as a subfield of the field of meromorphic functions on $\Gamma_N \backslash \mathfrak{H}^d$. A function $f \in M_k(\Gamma)$ has a Fourier expansion of the form

$$f(\tau) = a_f(0) + \sum_{T \in S_F} a_f(T) q_T^T,$$

where $q_T^T$ is given by (1.4) and where, to lighten notation, we write

$$S_F^* = (\nu^{-1} \cdot S_F^\vee \setminus \{0\}) \cap \mathcal{C}. $$

The case of half-integral weight can be formulated in exactly the same way using the metaplectic group. We leave this to the reader.
This set, which is denoted by $L'$ in [3], depends on $\nu$, although we omit this dependence from the notation. Note that the Fourier series is ‘symmetric’ with respect to $\Lambda$, i.e., for all $\epsilon \in \Lambda$,

\[ a_f(\epsilon \cdot T) = a_f(T), \quad \epsilon \cdot T = \epsilon T^t \epsilon. \]  

A formal Fourier series over $F$ of genus $n$ is a function $a : \text{Sym}_n(F) \to \mathbb{C}$. It can be viewed as a formal Laurent series

\[ \sum_{T \in \text{Sym}_n(F)} a(T) q^T. \]

Define the support of such a series as

\[ \text{supp}(a) = \{ T \in \text{Sym}_n(F) \mid a(T) \neq 0 \}. \]

Let $\text{FFS}$ be the complex vector space of all such formal series and let

\[ \text{FFS}' = \{ a \in \text{FFS} \mid \text{supp}(a) \subset S_F' \cup \{0\} \}. \]

Note that $\text{FFS}'$ is a ring for the product defined by

\[ a \cdot b = c, \quad c(T) = \sum_{R \in S_F' \cup \{0\}} a(R) b(T - R). \]

The sum is finite since, for $v \in \text{Sym}_n(\mathbb{R})_{\geq 0}$, the set $\{ w \in \text{Sym}_n(\mathbb{R}) \mid w \geq 0, v - w \geq 0 \}$ is compact and the image of $S_F' \cup \{0\}$ is discrete in $\text{Sym}_n(\mathbb{R})^d$.

An element $a \in \text{FFS}$ is symmetric with respect to $\Lambda$ if it satisfies (8.3). We denote by $\text{FFS}'_\Lambda$ the subring of symmetric elements in $\text{FFS}'$. We postpone the proof of the following crucial fact.

**Proposition 8.1.** $\text{FFS}'_\Lambda$ is an integral domain.

There is a natural injective ring homomorphism

\[ \varphi : M_*(\Gamma) \to \text{FFS}'_\Lambda, \quad f \mapsto a_f(0) + \sum_{T \in S_F'} a_f(T) \cdot q^T, \]

taking a holomorphic modular form to its Fourier series. It induces an inclusion of quotient fields

\[ Q(\varphi) : Q(M_*(\Gamma)) \to Q(\text{FFS}'_\Lambda), \]

such that the diagram

\[ \begin{array}{ccc}
M_*(\Gamma) & \xrightarrow{\varphi} & \text{FFS}'_\Lambda \\
\downarrow & & \downarrow \\
Q(M_*(\Gamma)) & \xrightarrow{Q(\varphi)} & Q(\text{FFS}'_\Lambda)
\end{array} \]

commutes.
Proposition 8.2. Let \( c \in \text{FFS}_\Lambda^* \). Suppose the following conditions are satisfied.

(i) There are modular forms \( f \in M_{k+l}(\Gamma) \) and \( g \in M_l(\Gamma) \) such that \( \varphi(f) = \varphi(g) \cdot c \in \text{FFS}_\Lambda^* \).

(ii) For any \( z \in \mathcal{S}^d_n \) there exist holomorphic modular forms \( f_z \in M_{k+l'}(\Gamma') \) and \( g_z \in M_l(\Gamma') \), where the weight \( l' \) and \( \Gamma' \), a congruence subgroup of \( \Gamma \), may depend on \( z \), such that
   
   (a) \( \varphi(f_z) = \varphi(g_z) \cdot c \in \text{FFS}_{\Lambda'}^* \),
   
   (b) \( g_z(z) \neq 0 \).

Then there exists an \( h \in M_k(\Gamma) \) such that \( c = \varphi(h) \), that is, \( c \) is the Fourier expansion of the holomorphic Hilbert-Siegel modular form \( h \). In particular, the series \( c \) is absolutely convergent on \( \mathcal{S}^d_n \).

Proof. By (i) and the diagram (5.5), we have

\[
Q(\varphi) \left( \frac{f}{g} \right) = \frac{\varphi(f)}{\varphi(g)} = c \in Q(\text{FFS}_\Lambda^*).
\]

Let \( h = g^{-1}f \) so that \( h \) is a meromorphic modular form for \( \Gamma \) of weight \( k \). Let \( z \in \mathcal{S}^d_n \) and let \( \Gamma' \) be as in (ii)(a). The inclusion \( M_e(\Gamma) \rightarrow M_e(\Gamma') \) induces an inclusion of the diagram (8.5) for \( \Gamma \) into the corresponding diagram for \( \Gamma' \), and we have

\[
Q(\varphi) \left( \frac{f_z}{g_z} \right) = c.
\]

Since \( Q(\varphi) \) is injective, we obtain

\[
h = \frac{f}{g} = \frac{f_z}{g_z}.
\]

Thus, by (ii)(b), \( h \) is holomorphic in neighborhood of \( z \) and hence is holomorphic on all of \( \mathcal{S}^d_n \). But this implies that \( h \in M_k(\Gamma) \) and \( c = \varphi(h) \).

Proof of Proposition 8.7. The proposition follows from a special case of the results of Knöller, [13]. We sketch the idea. Let \( M_\Gamma = \Gamma \backslash \mathcal{S}^d_n \) the Hilbert-Siegel modular variety with respect to \( \Gamma \) and let \( \mathcal{M}_\Gamma^{BB} \) be its Baily-Borel compactification.\(^{16}\) Let \( \xi_P \) be the point boundary stratum of \( \mathcal{M}_\Gamma^{BB} \) associated to the Siegel parabolic \( P \) and let \( R = \mathcal{O}_{\xi_P} \) be the corresponding local ring. By the normality of the Baily-Borel compactification, \([4]\), \( R \) is a normal noetherian local ring with maximal ideal \( m = m_{\xi_P} \) and its \( m \)-adic completion

\[
\hat{R} = \lim_{\leftarrow} R/m^r
\]

is also normal and, in particular, an integral domain.

Let

\[
A = \{ v \in \mathcal{C} \mid \text{tr} \frac{F}{Q} \text{tr}(xv) \geq 1, \forall x \in \mathcal{S}^d_P \}.
\]

\(^{16}\)This case was proved earlier by Baily, [3].
This is a ‘standard kernel’, [13], p.19, [2], Theorem 5.2 (d). Note that \( R \) is isomorphic to the ring of symmetric Fourier expansions \((8.2)\) that are termwise absolutely convergent in sets of the form
\[
(8.6) \quad \{ \tau = u + iv \in \mathcal{H}^d_n \mid v \in t \cdot A \},
\]
for some \( t > 0 \). Indeed, if \( f \) is a holomorphic function on some open neighborhood of \( \xi_\Gamma \) in \( \mathcal{M}^{BB}_\Gamma \), the pullback to \( \mathcal{H}^d_n \) of its restriction to \( \mathcal{M}_\Gamma \) is holomorphic in some open set of the form \((8.6)\), cf. Section 6.1 of [2], and has a symmetric Fourier expansion there. Conversely, the restriction of such a convergent series to a sufficiently small neighborhood \((8.6)\) extends to a holomorphic function on an open neighborhood of \( \xi_\Gamma \) in \( \mathcal{M}^{BB}_\Gamma \) since the boundary has codimension \( \geq 2 \). Note that \( m \) is the ideal of such expansions where \( a_f(0) = 0 \).

Following Section 2 of [13], define \( \lambda : S_F^r \to \mathbb{Z}_{\geq 1} \) as
\[
\lambda(x) = \max\{ k \mid x = x_1 + \cdots + x_k, \ x_i \in S_F \}.
\]
This function satisfies \( \lambda(x+y) \geq \lambda(x) + \lambda(y) \) and \( \lambda(\epsilon \cdot x) = \lambda(x) \), for all \( \epsilon \in \Lambda \). As in Section 3 of [13], let \( I_0 = R \) and, for \( k \geq 1 \), let
\[
I_k = \{ f \in R \mid a_f(0) = 0, \ a_f(x) = 0, \forall x \in S_F^r \text{ with } \lambda(x) < k \}.
\]
Then \( I_k \) is an ideal in \( R \) and these ideals satisfy \( I_1 = m, I_k \subset I_{k-1} \), and \( I_k \cdot I_{k'} \subset I_{k+k'} \).

The following result is the analogue of the Hilfsatz on p.127 of [10] and is proved using standard facts about Poincaré series with respect to \( \Lambda \).

**Lemma 8.3.**
\[
\mathfrak{FF}_{\Lambda} = \lim_{\leftarrow k} R/I_k.
\]

Now, as a special case of Satz 3.1.3 of [13], the filtrations \((I_k)\) and \((m^k)\) define the same topology on \( R \) and hence
\[
\mathfrak{FF}_{\Lambda} = \hat{R}
\]
is an integral domain, as claimed. \qed

**9. What Vogan-Zuckerman says about Hodge numbers**

Assume that \( 1 \leq d_+ < d \), and that \( \Gamma \) is neat so that \( S = S_\Gamma \) as in \((2.1)\) is a smooth compact complex manifold. The Betti cohomology and the Hodge numbers of \( S \) can be described in term of the \((\mathfrak{g}, K)\)-cohomology of the space of smooth \( K\)-finite functions on \( \Gamma \backslash G(\mathbb{R})^+ \), where \( G = \text{SO}(V) \). Here \( G(\mathbb{R})^+ \) is the identity component of \( G(\mathbb{R}) \),
\[
G(\mathbb{R})^+ = \text{SO}_0(m,2)^{d_+} \times \text{SO}(m)^{d-d_+},
\]
and
\[
K = (\text{SO}(m) \times \text{SO}(2))^{d_+} \times \text{SO}(m)^{d-d_+},
\]
is a maximal compact subgroup. Vogan and Zuckerman, [23], describe all irreducible Harish-Chandra modules that can contribute to this cohomology and the degrees in which these contribution occurs. In the case at hand, their results imply the vanishing of certain Hodge numbers.

9.1. Harish-Chandra modules with non-trivial \((g, K)\)-cohomology for \(SO_0(m, 2)\). In this section, we work out the results of [23] very explicitly (and naively) in this special case. This information is also to be found in the literature, cf. [20], for example, and probably elsewhere, but we feel it might be useful to provide more details. In particular, we do not know of a reference for the picture of Hodge diamond given below.

We slightly shift notation and let \(G = SO_0(m, 2)\) and \(K = SO(m) \times SO(2)\), with real Lie algebras \(g_0\) and \(k_0\). Let \(g = (g_0)_C\) and \(k = (k_0)_C\) be their complexifications, let \(g_0 = k_0 + p_0\) be the Cartan decomposition, and let \(\theta\) be the corresponding Cartan involution, \(\theta|_{k_0} = +1\), \(\theta|_{p_0} = -1\).

We write elements of \(g_0\) as

\[
X = \begin{pmatrix} X_1 & X_2 \\ tX_2 & X_4 \end{pmatrix}, \quad X_1 \in \text{Skew}_m(\mathbb{R}), X_4 \in \text{Skew}_2(\mathbb{R}), X_2 \in M_{m, 2}(\mathbb{R}).
\]

Here \(k_0\) is the subalgebra where \(X_2 = 0\) and \(p_0\) is the subspace where \(X_1 = 0\) and \(X_4 = 0\). The element

\[
h = \begin{pmatrix} 1_m & J \\ J & -1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]

lies in the center of \(K\) and the Harish-Chandra decomposition is

\[
g = k + p_+ + p_-,
\]

where \(p_{\pm}\) are the \(\pm i\)-eigenspaces of \(\text{Ad}(h)|_p\). Concretely, these subspaces are given by

\[
p_{\pm} = \{ X \in p \mid X_2 = (x_2, \pm ix_2), \ x_2 \in \mathbb{C}^m \},
\]

where we note that \((x_2, \pm ix_2)iJ = \pm (x_2, \pm ix_2)\). A Cartan subalgebra \(t_0\) of \(g_0\) is given by

\[
t(a) = t(a_0, a_1, \ldots, a_{m'}) = \begin{pmatrix} \text{diag}(a_1J, \ldots, a_{m'}J, 0_s) \\ a_0J \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]

where \(a_j \in \mathbb{R}\), \(m' = \left\lceil \frac{m}{2} \right\rceil\), and \(0_s\) indicates an extra 0 when \(m\) is odd.

The representations \(A_q\) are irreducible Harish-Chandra modules associated to \(\theta\)-stable parabolic subalgebras \(q\) of \(g\). These subalgebras are constructed as follows, [23], Section 2. For integers \(r, s\) with \(0 \leq r \leq m'\), \(0 \leq s \leq 1\), \(r + s < \frac{1}{2}m\), let

\[
x_{r,s}(a) = i t(a_0, a_1, \ldots, a_{m'}) \in i t_0,
\]

where \(a_j \neq 0\) for \(1 \leq j \leq r\), \(a_j = 0\) for \(j > r\), \(a_0 = 0\) for \(s = 0\), and \(a_0 \neq 0\) for \(s = 1\). The endomorphism \(\text{ad}(x_{r,s})\) of \(g\) is diagonalizable and the subalgebra

\[
q = \text{sum of the ad}(x_{r,s})\text{-eigenspaces with eigenvalue } \mu \geq 0.
\]
is a \( \theta \)-stable parabolic subalgebra with decomposition \( q = I + u \) where \( I \) is the sum of the eigenspaces with \( \mu = 0 \) and \( u \) is the sum of the eigenspaces with \( \mu > 0 \).

The representation \( A_q \) associated to \( q \) is characterized by Theorem 2.5 of [23]. As explained in Section 6 of [23], there is a Hodge type decomposition

\[
H^i(g, K; A_q) = \bigoplus_{p+q=i} H^{p,q}(g, K; A_q)
\]

of the \( (g, K) \)-cohomology. The main fact that we need is the following result, extracted from Proposition 6.19 of [23].

**Proposition 9.1.** For a \( \theta \)-stable parabolic \( q = I + u \), let \( R_{\pm} = \dim(u \cap p_{\pm}) \). Then

\[
H^{p,q}(g, K; A_q) = 0
\]

unless \( p - q = R_+ - R_- \).

The next result records the possible values for \( (R_+, R_-) \).

**Proposition 9.2.** Suppose that \( q \) is is constructed, as above, from \( x_{r,s}(a) \).

(i) If \( s = 0 \), then \( (R_+, R_-) = (r, r) \).

(ii) If \( s = 1 \), let \( \delta_+ \) (resp. \( \delta_- \)) be the number of \( j \)'s for which \( a_j + a_0 = 0 \) (resp. \( a_j - a_0 = 0 \)). Then

\[
(R_+, R_-) = \begin{cases} 
(r - \delta_+, m - r - \delta_-) & \text{if } a_0 > 0 \\
(m - r - \delta_+, r - \delta_-) & \text{if } a_0 < 0.
\end{cases}
\]

In particular, \( 0 \leq \delta_+ + \delta_- \leq r \leq \left[ \frac{m}{2} \right] \) and

\[
R_+ + R_- = m - \delta_+ - \delta_- \geq m - \left[ \frac{m}{2} \right].
\]

**Proof.** We want to calculate \( \dim(u \cap p_{\pm}) \) and so we consider the action of \( \text{ad}(x_{r,s}(a)) \) and \( \text{Ad}(J) \) on an element of \( p \). Write

\[
\text{ad}(x_{r,s}(a)) : X_2 = \begin{pmatrix} A_1 \\ \vdots \\ A_r \\ \alpha_1 \\ \vdots \\ \alpha_{m-2r} \end{pmatrix} \mapsto \begin{pmatrix} a_1 i A_1 - a_0 A_1 i J \\ \vdots \\ a_r i A_r - a_0 A_r i J \\ -a_0 \alpha_1 i J \\ \vdots \\ -a_0 \alpha_{m-2r} i J \end{pmatrix}.
\]

for \( 2 \times 2 \) blocks \( A_j \) and row vectors \( \alpha_k \). The eigenvalues of the transformation

\[
A \mapsto a i J A - a_0 A i J, \quad A \in M_2(\mathbb{C}), \ a, a_0 \in \mathbb{C}
\]

are \( \pm(a - a_0) \) and \( \pm(a + a_0) \). Corresponding eigenvectors are

\[
\begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -i \\ -i & 1 \end{pmatrix},
\]

where we note that the \( \pm(a - a_0) \)-eigenvectors satisfy \( A i J = -A \) and the \( \pm(a + a_0) \)-eigenvectors satisfy \( A i J = A \). For a row vector \( \alpha \), the eigenvalues of the transformation \( \alpha \mapsto -a_0 \alpha i J \) are
\[ \pm a_0 \] with corresponding eigenvectors \( \pm (1, i) \). Thus the \( a_0 \)-eigenvector satisfies \( \alpha i J = -\alpha \) and the \( -a_0 \)-eigenvector satisfies \( \alpha i J = \alpha \).

Suppose that \( s = 1 \) so that \( a_0 \neq 0 \) and that \( x_{r,1}(a) \) is regular, i.e., that \( a_j \pm a_0 \neq 0 \) for \( 1 \leq j \leq r \). Then in \( j \)th block, precisely two eigenspaces have positive eigenvalue and one of them lies in \( p_+ \) and the other in \( p_- \). Each such block contributes 1 to both \( R_+ \) and \( R_- \). The eigenvalues of the transformation \( \alpha \mapsto -a_0 \alpha i J \) are \( \pm a_0 \). For \( a_0 > 0 \) (resp. \( a_0 < 0 \)), such a row contributes 1 to \( R_- \) (resp. \( R_+ \)). This proves (ii) in the regular case.

Now suppose that \( s = 1 \) and let \( \delta_+ \) (resp. \( \delta_- \)) be the number of \( j \)'s for which \( a_j + a_0 = 0 \) (resp. \( a_j - a_0 = 0 \)). Now the \( A_j \) blocks with \( a_j + a_0 = 0 \) do not make a contribution to \( R_+ \), since the corresponding eigenvalue \( \mu = 0 \), while the \( A_j \) blocks with \( a_j - a_0 = 0 \) do not make a contribution to \( R_- \). This proves (ii) in general. We omit the easier case (i). \( \square \)

**Remark.** Note that when \( s = 1 \) and \( x_{r,1}(a) \) is regular, the whole space \( p \) is spanned by non-zero eigenspaces. Thus we have \( l \subset \mathfrak{k} \) so that the \( A_q \)'s are discrete series representations, \( [23] \), p. 58, and contribute only to the cohomology in the middle dimension, with Hodge numbers \( (r, m - r) \), for \( a_0 > 0 \) and \( (m - r, r) \), for \( a_0 < 0 \).

We thus have the following vanishing result for Hodge numbers, which plays an essential role in our argument.

**Corollary 9.3.** If \( H^{p,q}(g, K; A_q) \neq 0 \) for some \( (p, q) \) with \( p \neq q \), then \( p + q \geq m - \left\lceil \frac{m}{2} \right\rceil \).

### 9.2. Global consequences for Hodge numbers.

Now return to the global situation where \( S = \Gamma \backslash D^+ = \Gamma \backslash G(\mathbb{R})^+/K \), for \( \Gamma \) neat and \( 1 \leq d_+ < d \), or \( d_+ = d \) and \( V \) is anisotropic. Following the discussion in the introduction of \( [23] \), write

\[ L^2(\Gamma \backslash G) \cong \bigoplus \limits_{\pi \in \hat{G}} m_\pi \mathcal{H}_\pi, \]

a Hilbert space direct sum of irreducible unitary representations of \( G \) with finite multiplicities. The cohomology of \( S \) is then described as

\[ H^*(S, \mathbb{C}) \cong \bigoplus \limits_{\pi \in \hat{G}} m_\pi H^*(g, K; \mathcal{H}_\pi^K), \]

where \( \mathcal{H}_\pi^K \) is the Harish-Chandra module of \( \pi \). Vogan and Zuckerman, \( [23] \), Theorem 4.1, show that \( H^*(g, K; \mathcal{H}_\pi^K) \neq 0 \) only if \( \mathcal{H}_\pi^K \cong A_q \) for some \( q \). Thus, via the Kunneth formula for relative Lie algebra cohomology and (9.1), we have the following.

**Corollary 9.4.** Suppose that \( H^{(p,q)}(S) \neq 0 \) for some \( (p, q) \) with \( p \neq q \). Then \( p + q \geq m - \left\lceil \frac{m}{2} \right\rceil \). In particular \( H^{2k-1}(S, \mathbb{C}) = 0 \) if \( 2k - 1 < m - \left\lceil \frac{m}{2} \right\rceil \) and the intermediate Jacobian \( J^k(S) = 0 \) in this range.

\( \text{17This require that } V \text{ be anisotropic at some finite place of } F \text{ and hence can only occur when } \dim_{F} V \leq 4. \)
Note that one should be able to slightly extend this vanishing range using relations between archimedean components of automorphic representations coming from Arthur’s classification, but we will not need such an improvement.

10. Weighted cycles and ‘natural’ cycles

In this section, we extend the discussion of Sections 2–5 of [14] to the case of $d_+ > 1$. In [14], Definition 5.2, the weighted cycles were defined in terms of the ‘natural’ cycles given by sub-Shimura varieties. An expression for them in terms of connected or classical cycles was then proved in Proposition 5.5 of loc. cit. Here in our definition (5.2) of weighted cycles in Section 5, we have taken the expression in terms of connected/classical cycles as the starting point, since this expression is what is needed for the intersection theory calculations of Section 4. In the present section, we prove that the weighted cycles defined by the analogue of Definition 5.2 of [14] coincide with the ones defined in Section 5 above. Especially in the case $n = m$, this requires some care about connected components and so we give a detailed discussion.

Having fixed a connected component $D^+$ in Section 2 above, we can index the components $D^+$ of $D$ by collections $\epsilon = (\epsilon_1, \ldots, \epsilon_{d_+})$, where $\epsilon_j = \pm 1$.

For a totally positive subspace $W \subset V$ with $\dim_F W = n \leq m$, we let $H = G_W$ be the pointwise stabilizer of $W$ in $G$, so that $H \simeq R_{F/\mathbb{Q}} \text{GSpin}(W^\perp)$. The space $W^\perp_j = W^\perp \otimes_{F, \sigma_j} \mathbb{R}$, for $1 \leq j \leq d_+$, has signature $(m-n, 2)$. We let $D^{(j)}_W$ be the space of oriented negative 2-planes in $W^\perp_j$, let $D^\epsilon_W = \prod_j D^{(j)}_W$, and let $D^\epsilon_W = D_W \cap D^\epsilon$. Thus, $D^\epsilon_W = D_W \cap D^+$, as in (2.2), and $D_W$ has $2^{d_+}$ components.

Note that, if $n < m$, the group $H(\mathbb{R})$ also has $2^{d_+}$ components and acts transitively on $D_W$, whereas, if $n = m$, the group

$$H(\mathbb{R}) \simeq \text{GSpin}(2)^d$$

is connected and acts trivially on the finite set of points $D_W$. This distinction will lead to slight differences in the treatment of the two cases.

For $g \in G(\mathbb{A}_f)$, the codimension $d_+ n$ cycle

$$Z(W, g, K)^\mathbb{C} := H(\mathbb{Q}) \backslash D_W \times H(\mathbb{A}_f)/K_{H,g} \rightarrow G(\mathbb{Q}) \backslash D \times G(\mathbb{A}_f)/K, \quad [z, h] \mapsto [z, hg]$$

is called a ‘natural’ cycle in [14]. Here $K_{H,g} = H(\mathbb{A}_f) \cap gKg^{-1}$. On the other hand, for any $g_0 \in G(\mathbb{A}_f)$, if we let $\Gamma'_{g_0} = G(\mathbb{Q})_+ \cap g_0Kg_0^{-1}$ and let $\Gamma_{g_0}$ be its image in $\text{SO}(V)$, then the connected cycle defined in (2.3) is

$$Z(W)_{\Gamma_{g_0}} = \pi_{\Gamma_{g_0}}(D^+_W) \subset \Gamma_{g_0} \backslash D^+.$$
We next describe the relation between these cycles.

For each $\epsilon$, choose $\gamma_\epsilon \in G(\mathbb{Q})$ such that
\[ \gamma_\epsilon D^\epsilon = D^+ . \]

Writing
\[ G(\mathbb{A}_f) = \bigsqcup_j G(\mathbb{Q})_+ g_j K , \]
we have an isomorphism
\[ (10.1) \quad G(\mathbb{Q}) \backslash D \times G(\mathbb{A}_f) / K \xrightarrow{\sim} \bigsqcup_j \Gamma_{g_j} \backslash D^+ , \]
where, if $z \in D^\epsilon$ and $\gamma_\epsilon g = \gamma^{-1} g_j k$ with $\gamma \in G(\mathbb{Q})_+$, then
\[ [z, g] \mapsto \pi_{\Gamma_{g_j}} (\gamma_\epsilon z) . \]

As in Lemma 4.1 of [14], write
\[ (10.2) \quad H(\mathbb{A}_f) = \bigsqcup_i H(\mathbb{Q})_+ h_i K_{H,g} . \]
If $n < m$, then we can choose $\gamma_{i,H} \in H(\mathbb{Q})$ with $\gamma_{i,H} D^\epsilon_W = D^+_W$ and obtain
\[ (10.3) \quad H(\mathbb{Q}) \backslash D_W \times H(\mathbb{A}_f) / K_{H,g} \xrightarrow{\sim} \bigsqcup_i \Gamma_{H,i} \backslash D^+_W , \]
where $\Gamma_{H,i} = H(\mathbb{Q})_+ \cap h_i K_{H,g} h_i^{-1} = H(\mathbb{Q})_+ \cap h_i g K g^{-1} h_i^{-1}$. For each $i$, write $h_i g = \gamma_i^{-1} g_j k_i$ where $\gamma_i \in G(\mathbb{Q})_+$ and $j = j(i)$ depend on $h_i$. Then the point $\pi_{\Gamma_{H,i}} (z)$, $z \in D^+_W$ on the right side of (10.3), with preimage $[z, h_i]$ on the left side, maps to the point $\pi_j (\gamma_i z)$ on the right side of (10.1). Note that $\gamma_i z \in D^+_W$. This proves the following.

**Lemma 10.1.** If $n < m$, then, under the isomorphism (10.1),
\[ (10.4) \quad Z(W, g, K)^\natural = \sum_i Z(\gamma_i W)^\Gamma_j , \]
where, for coset representatives $h_i$ as in (10.2), $h_i g = \gamma_i^{-1} g_j k_i$, with $\gamma_i \in G(\mathbb{Q})_+$, $k_i \in K$, and $j = j(i)$ depending on $h_i$.

Now suppose that $n = m$ so that $H(\mathbb{Q}) = H(\mathbb{Q})_+$ and
\[ (10.5) \quad H(\mathbb{Q}) \backslash D_W \times H(\mathbb{A}_f) / K_{H,g} = D_W \times H(\mathbb{Q})_+ \backslash H(\mathbb{A}_f) / K_{H,g} \xrightarrow{\sim} \bigsqcup_i D_W . \]
For each $i$ and $\epsilon$, write
\[ (10.6) \quad \gamma_\epsilon h_i g = \gamma_{i,\epsilon}^{-1} g_j k , \]
where $j, \gamma_{i,\epsilon} \in G(\mathbb{Q})_+$, and $k$ depend on $h_i$ and $\gamma_\epsilon$. The point $z \in D^+_W$ in the $i$-th part of the right side of (10.5), with preimage $[z, h_i]$ on the left side, maps to the point $\pi_{\Gamma_j} (\gamma_{i,\epsilon} \gamma_\epsilon z)$ on the right side of (10.1), via
\[ [z, h_i] \mapsto [z, h_i g] = [\gamma_\epsilon z, \gamma_\epsilon h_i g] = [\gamma_{i,\epsilon} \gamma_\epsilon z, g_j] \mapsto \pi_{\Gamma_j} (\gamma_{i,\epsilon} \gamma_\epsilon z) . \]
Lemma 10.2. If \( n = m \), then, under the isomorphism (10.1),
\[
Z(W, g, K)^2 = \sum_{\epsilon} \sum_{i} Z(\gamma_i, \epsilon W)_{\Gamma_{ij}}.
\]

These expressions for the natural cycles in terms of connected cycles are the analogue of parts (i) and (ii) of Lemma 4.1 of [13].

Next we consider the weighted cycles. The discussion of Section 5 of [14] carries over with minor changes. First suppose that \( T \in \text{Sym}_n(F) \) is totally positive definite and that \( \Omega_T(F) \neq \emptyset \), where \( \Omega_T \) is the hyperboloid in \( V^n \), as in (5.3). Fix \( x_0 \in \Omega_T(F) \). Then \( \Omega_T(A_f) = G(A_f) \cdot x_0 \) and, for a \( K \)-invariant Schwartz function \( \varphi \in S(V(A_f)^n) \), write
\[
(\text{supp} \varphi) \cap \Omega_T(A_f) = \bigcup_r K \cdot \xi_r^{-1} \cdot x_0,
\]
with \( \xi_r \in G(A_f) \) as in (5.4) of [14]. Consider the weighted sum of natural cycles
\[
Z(T, \varphi, K)^2 := \sum_r \varphi(\xi_r^{-1} x_0) Z(W(x_0), \xi_r, K)^2,
\]
as in Definition 5.2 of [14].

Proposition 10.3. The two definitions of weighted special cycles (5.2) and (10.8) coincide,
\[
Z(T, \varphi, K) = Z(T, \varphi, K)^2.
\]

Proof. We only give the proof in the case \( n = m \) and \( T \in \text{Sym}_m(F)_{>0} \) totally positive definite. The remaining cases follow by similar arguments. Let \( \nu : G \to R_{F/Q}S_m \) denote the spinor norm and note that
\[
\nu(G(Q)) = \{ \alpha \in F^\times \mid \sigma_j(\alpha) > 0, \forall j > d_+ \}.
\]
This implies that \( G(Q)_+ \) has \( 2^{d_+} \)-orbits in \( \Omega_T(F) \). These can be indexed as follows. Fix an (ordered) \( F \)-basis for \( V \). This basis determines an orientation for \( V_j \) for all \( j \). An \( n \)-frame \( x \in \Omega_T(F) \) determines an orientation of \( W(x) \otimes F_{\sigma_j} \mathbb{R} \) for all \( j \) and hence, since an orientation of \( V_j \) has been fixed, an orientation for \( W(x)_{1} \otimes F_{\sigma_j} \mathbb{R} \). Thus we obtain a point in \( D^e = D^{e(x)} \). This defines a function \( \epsilon : \Omega_T(F) \rightarrow (\pm)^{2^{d_+}} \) which distinguishes the \( G(Q)_+ \)-orbits.

We can write
\[
Z(T, \varphi, K) = \sum_{\epsilon} \sum_j \sum_{x \in \Omega_T(F)^e \mod \Gamma_{sj}} \varphi(g_j^{-1} x) Z(W(x))_{\Gamma_{sj}},
\]
where we note that the only \( x \)'s that contribute to the \( j \)-th summand are those in the set
\[
\Omega_T(F)^e \cap ( g_j \text{supp}(\varphi) \cap \Omega_T(A_f) ) = \bigcup_r G(Q)_+ \gamma_c x_0 \cap g_j K \xi_r^{-1} x_0.
\]
Here we suppose that $\epsilon(x_0) = (+1, \ldots, +1)$. Thus we have

$$Z(T, \varphi, K) = \sum_{\epsilon} \sum_{j} \sum_{r} \sum_{x \in \Omega_T(F) \cap g_j K \xi_r^{-1} x_0} \varphi(g_j^{-1} x) Z(W(x)) \Gamma_{g_j}.$$ 

Now, by Weil’s Lemma, [14], Lemma 5.7 (ii), there is a bijection

$$\Gamma_{g_j} \text{-orbits in } G(\mathbb{Q}) \gamma_\epsilon x_0 \cap g_j K \xi_r^{-1} x_0,$$

(10.10) $$\downarrow$$

$$H(\mathbb{Q}) \backslash \left( H(\mathbb{A}_f) \cap \gamma_\epsilon^{-1} G(\mathbb{Q}) \gamma_\epsilon g_j K \xi_r^{-1} \right) / K H \xi_r,$$

given by

$$\Gamma_{g_j} \cdot x \mapsto H(\mathbb{Q}) \gamma_\epsilon^{-1} g_j k \xi_r^{-1} K H \xi_r, \quad x = \gamma_\epsilon x_0 = g_j k \xi_r^{-1} x_0.$$

Here note that the two expressions for $x$ insure that $\gamma_\epsilon^{-1} g_j k \xi_r^{-1} \in H(\mathbb{A}_f)$ and that $\varphi(g_j^{-1} x) = \varphi(\xi_r^{-1} x_0)$.

We can then rearrange the sum as

$$Z(T, \varphi, K) = \sum_{r} \varphi(\xi_r^{-1} x_0) \sum_{\epsilon} \sum_{j} \sum_{x \in \Omega_T(F) \cap g_j K \xi_r^{-1} x_0} Z(W(x)) \Gamma_{g_j}.$$

Now, for each $\epsilon$ and $r$,

$$H(\mathbb{A}_f) = \bigsqcup_j H(\mathbb{A}_f) \cap \gamma_\epsilon^{-1} G(\mathbb{Q}) \gamma_\epsilon g_j K \xi_r^{-1},$$

where, we see that subsets on the right side give the elements of $h \in H(\mathbb{A}_f)$ such that $\gamma_\epsilon h \xi_r$ lies in $G(\mathbb{Q}) \gamma_\epsilon g_j K$. Returning to (10.2) with $g = \xi_r$, and writing a double coset representative as $h_i = \gamma_\epsilon^{-1} g_j k \xi_r^{-1}$, the corresponding $x$ is $\gamma_\epsilon \gamma_i x_0$. Indexing the sum by the double cosets, we have

$$Z(T, \varphi, K) = \sum_{r} \varphi(\xi_r^{-1} x_0) \sum_{\epsilon} \sum_{i} Z(W(\gamma_\epsilon \gamma_i x_0)) \Gamma_{g_j}$$

$$= \sum_{r} \varphi(\xi_r^{-1} x_0) Z(W(x_0), \xi_r, K)^i$$

$$= Z(T, \varphi, K)^i,$$

by Lemma [10.2] \hfill \Box

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