Closability of Quadratic Forms Associated to Invariant Probability Measures of SPDEs *

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Abstract

By using the integration by parts formula of a Markov operator, the closability of quadratic forms associated to the corresponding invariant probability measure is proved. The general result is applied to the study of semilinear SPDEs, infinite-dimensional stochastic Hamiltonian systems, and semilinear SPDEs with delay.

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1 Introduction

Let \(\mathbb{B}\) be a separable Banach space and \(\mu\) a reference probability measure on \(\mathbb{B}\). For any \(k \in \mathbb{B}\), let \(\partial_k\) denote the directional derivative along \(k\). According to [8], the form

\[
\mathcal{E}_k(f, g) := \mu((\partial_k f)(\partial_k g)) := \int_{\mathbb{B}} (\partial_k f)(\partial_k g) d\mu, \quad f, g \in C^2_b(\mathbb{B}),
\]

is closable on \(L^2(\mu)\) if \(\rho_s := \frac{d\mu(sk+)}{d\mu}\) exists for any \(s\) such that \(s \mapsto \rho_s\) is lower semi-continuous \(\mu\)-a.e.; i.e. for some fixed \(\mu\)-versions of \(\rho_s, s \in \mathbb{R}\),

\[
\liminf_{s \to t} \rho_s(x) \geq \rho_t(x), \quad \mu-a.e. \, x, \, t \in \mathbb{R}.
\]

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In this paper, we aim to investigate the closability of $\mathcal{E}_k$ for $\mu$ being the invariant probability measure of a (degenerate/delay) semilinear SPDE. Since in this case the above lower semi-continuity condition is hard to check, in this paper we make use of the integration by parts formula for the associated Markov semigroup in the line of \cite{10} using coupling arguments.

The main motivation to study the closability of $\mathcal{E}_k$ (respectively of $\partial_k$) on $L^2(\mu)$ is that it leads to a concept of weak differentiability on $\mathcal{B}$ with respect to $\mu$ and one can define the corresponding Sobolev space on $\mathcal{B}$ in $L^p(\mu)$, $p \in [1, \infty)$. In particular, one can analyze the generator of a Markov process (e.g. arising from a solution of an SPDE) on these Sobolev spaces when $\mu$ is its (infinitesimally) invariant measure, see e.g. \cite{7} for details.

Before considering specific models of SPDEs, we first introduce a general result on the closability of $\mathcal{E}_k$ using the integration by parts formula. To this end, we consider a family of $\mathcal{B}$-valued random variables $\{X^x\}_{x \in \mathcal{B}}$ measurable in $x$, and let $P(x, dy)$ be the distribution of $X^x$ for $x \in \mathcal{B}$. Then we have the following Markov operator on $\mathcal{B}_b(\mathcal{B})$:

$$Pf(x) := \int_{\mathcal{B}} f(y)P(x, dy) = \mathbb{E}f(X^x), \quad x \in \mathcal{B}, f \in \mathcal{B}_b(\mathcal{B}).$$

A probability measure $\mu$ on $\mathcal{B}$ is called an invariant measure of $P$ if $\mu(Pf) = \mu(f)$ for all $f \in \mathcal{B}_b(\mathcal{B})$.

**Proposition 1.1.** Assume that the Markov operator $P$ has an invariant probability measure $\mu$. Let $k \in \mathcal{B}$. If there exists a family of real random variables $\{M_x\}_{x \in \mathcal{B}}$ measurable in $x$ such that $M_x \in L^2(\mathbb{P} \times \mu)$, i.e.

$$\mathbb{P} \times \mu(|M_x|^2) := \int_{\mathcal{B}} \mathbb{E}|M_x|^2 \mu(dx) < \infty;$$

and the integration by parts formula

$$P(\partial_k f)(x) = \mathbb{E}\{f(X^x)M_x\}, \quad f \in C^2_b(\mathcal{B}), \mu\text{-a.e. } x \in \mathcal{B}$$

holds, then $(\mathcal{E}_k, C^2_b(\mathcal{B}))$ is closable in $L^2(\mu)$.

**Proof.** Since $\mu$ is $P$-invariant, by (1.1) and (1.2) we have

$$\mu(\partial_k f) = \int_{\mathcal{B}} P(\partial_k f)(x)\mu(dx) = (\mathbb{P} \times \mu)(f(X^x)M_x), \quad f \in C^2_b(\mathcal{B}).$$

So,

$$\mathcal{E}_k(f, g) := \mu((\partial_k f)(\partial_k g)) = \mu(\partial_k \{f\partial_k g\}) - \mu(f \partial_k^2 g) = (\mathbb{P} \times \mu)(\{f\partial_k g\}(X^x)M_x) - \mu(f \partial_k^2 g), \quad f, g \in C^2_b(\mathcal{B}).$$

It is standard that this implies the closability of the form $(\mathcal{E}_k, C^2_b(\mathcal{B}))$ in $L^2(\mu)$. Indeed, for $\{f_n\}_{n \geq 1} \subset C^2_b(\mathcal{B})$ with $f_n \to 0$ and $\partial_k f_n \to Z$ in $L^2(\mu)$, it suffices to prove that $Z = 0$. 

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Since $\mu(f_n^2) \to 0$ and $(\mathbb{P} \times \mu)((f_n \partial_k g)^2(X^t))) = \mu((f_n \partial_k g)^2)$ as $\mu$ is $P$-invariant, the above formula yields

$$|\mu(Zf)| = \lim_{n \to \infty} |\mu(g \partial_k f_n)|$$

$$= \lim_{n \to \infty} \left| (\mathbb{P} \times \mu)(\{f_n \partial_k g\}(X^t)M) - \mu(f_n \partial_k^2 g) \right|$$

$$\leq \liminf_{n \to \infty} \left\{ \sqrt{(\mathbb{P} \times \mu)((f_n \partial_k g)^2(X^t)))} \cdot (\mathbb{P} \times \mu)(|M|^2) + \sqrt{\mu(f_n^2)\mu(|\partial_k^2 g|^2)} \right\}$$

$$\leq \liminf_{n \to \infty} \left\{ \|\partial_k g\|\sqrt{\mu(f_n^2)} \cdot (\mathbb{P} \times \mu)(|M|^2) + \|\partial_k^2 g\|\sqrt{\mu(f_n^2)} \right\} = 0, \quad g \in C^2_b(\mathbb{R}).$$

Therefore, $Z = 0$.  

\[ \square \]

Remark 1.1. The integration by parts formula $(1.2)$ implies the estimate $(1.3)$

$$|\mu(\partial_k f)|^2 \leq (\mathbb{P} \times \mu)(|M|^2)\mu(f^2).$$

As the main result in [3] (Theorem 10), this type of estimate, called Fomin derivative estimate of the invariant measure, was derived as the main result for the following semilinear SPDE on $\mathbb{H} := L^2(\Omega)$ for any bounded open domain $\Omega \subset \mathbb{R}^n$ for $1 \leq n \leq 3$:

$$dX(t) = [\Delta X(t) + p(X(t))]dt + (-\Delta)^{-\gamma/2}dW(t),$$

where $\Delta$ is the Dirichlet Laplacian on $\Omega$, $p$ is a decreasing polynomial with odd degree, $\gamma \in (\frac{3}{2} - 1, 1)$, and $W(t)$ is the cylindrical Brownian motion on $\mathbb{H}$. The main point of the study is to apply the Bismut-Elworthy-Li derivative formula and the following formula for the semigroup $P_t^\alpha$ for the Yoshida approximation of this SPDE (see [3] Proposition 7) :

$$P_t^\alpha \partial_k f = \partial_k P_t^\alpha - \int_0^t P_{t-s}(\partial_{Ak} + \partial_{kP} P_s^\alpha f) ds.$$  

In this paper we will establish the integration by parts formula of type $(1.2)$ for the associated semigroup which implies the estimate $(1.3)$. Our results apply to a general framework where the operator $(-\Delta)^{-\gamma/2}$ is replaced by a suitable linear operator $\sigma$ (see Section 2) which can be degenerate (see Section 3), and the drift $p(x)$ is replaced by a general map $b$ which may include a time delay (see Section 4). However, the price we have to pay for the generalization is that the drift $b$ should be regular enough.

2 Semilinear SPDEs

Let $(\mathbb{H}, \langle \cdot, \cdot \rangle, | \cdot |)$ be a real separable Hilbert space, and $(W(t))_{t \geq 0}$ a cylindrical Wiener process on $\mathbb{H}$ with respect to a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$. Let $\mathcal{L}(\mathbb{H})$ and $\mathcal{L}_{HS}(\mathbb{H})$ be the spaces of all linear bounded operators and Hilbert-Schmidt operators on $\mathbb{H}$ respectively. Let $\| \cdot \|$ and $\| \cdot \|_{HS}$ denote the operator norm and the Hilbert-Schmidt norm respectively.
Consider the following semilinear SPDE
\begin{equation}
\label{eq:2.1}
dX(t) = \{AX(t) + b(X(t))\}dt + \sigma dW(t),
\end{equation}
where

\textbf{(A1)} \ (A, \mathcal{D}(A)) \text{ is a negatively definite self-adjoint linear operator on } \mathbb{H} \text{ with compact resolvent.}

\textbf{(A2)} \ Let \ \mathbb{H}^{-2} \text{ be the completion of } \mathbb{H} \text{ under the inner product}
\begin{equation}
\langle x,y \rangle_{\mathbb{H}^{-2}} := \langle A^{-1}x,A^{-1}y \rangle.
\end{equation}
Let \( b : \mathbb{H} \rightarrow \mathbb{H}^{-2} \) be such that
\begin{equation}
\int_{0}^{1} |e^{tA}b(0)|dt < \infty, \ |e^{tA}(b(x) - b(y))| \leq \gamma(t) |x - y|, \ x, y \in \mathbb{H}, t > 0
\end{equation}
holds for some positive \( \gamma \in C((0, \infty)) \) with \( \int_{0}^{1} \gamma(t)dt < \infty \).

\textbf{(A3)} \ \sigma \in \mathcal{L}(\mathbb{H}) \text{ with } \text{Ker}(\sigma \sigma^*) = \{0\} \text{ and } \int_{0}^{1} \|e^{tA} \sigma\|_{HS}^2 dt < \infty.

According to \textbf{(A1)}, the spectrum of \( A \) is discrete with negative eigenvalues. Let \( 0 < \lambda_0 \leq \cdots \leq \lambda_n \cdots \) be all eigenvalues of \(-A\) counting the multiplicities, and let \( \{e_i\}_{i \geq 1} \) be the corresponding unit eigen-basis. Denote \( \mathbb{H}_{A,n} = \text{span}\{e_i : 1 \leq i \leq n\}, n \geq 1 \). Then \( \mathbb{H}_A := \cup_{n=1}^{\infty} \mathbb{H}_{A,n} \) is a dense subspace of \( \mathbb{H} \). In assumption \textbf{(A2)} we have used the fact that for any \( t > 0 \), the operator \( e^{tA} \) extends uniquely to a bounded linear operator from \( \mathbb{H}^{-2} \) to \( \mathbb{H} \), which is again denoted by \( e^{tA} \).

Due to assumptions \textbf{(A1), (A2) and (A3)}, by a standard iteration argument we conclude that for any \( x \in \mathbb{H} \) the equation \eqref{eq:2.1} has a unique mild solution \( X^x(t) \) such that \( X^x(0) = x \) (see [4]). Let
\begin{equation}
P_tf(x) = \mathbb{E}f(X^x(t)), \ f \in \mathcal{B}(\mathbb{H}), x \in \mathbb{H}
\end{equation}
be the associated Markov semigroup.

Let
\begin{equation}
\|x\|_{\sigma} = \inf \{ |y| : y \in \mathbb{H}, \sqrt{\sigma \sigma^*}y = x \}, \ x \in \mathbb{H},
\end{equation}
where \( \inf \emptyset := \infty \) by convention. Then \( \|x\|_{\sigma} < \infty \) if and only if \( x \in \text{Im}(\sigma) \).

**Theorem 2.1.** Assume that \( P_t \) has an invariant probability measure \( \mu \) and \( \mathbb{H}_A \subset \text{Im}(\sqrt{\sigma \sigma^*}) \).

1. For any \( k \in \mathbb{H}_A \) such that
\begin{equation}
\sup_{x \in \mathbb{H}} \|\partial_k b(x)\|_{\sigma} := \sup_{x \in \mathbb{H}} \limsup_{\varepsilon \downarrow 0} \frac{\|b(x + \varepsilon k) - b(x)\|_{\sigma}}{\varepsilon} < \infty,
\end{equation}
the form \( (\mathcal{E}_k, C^2_b(\mathbb{H})) \) is closable in \( L^2(\mu) \).
(2) If $\sigma \sigma^*$ is invertible and $b : \mathbb{H} \to \mathbb{H}$ is Lipschitz continuous, then $(\mathcal{E}_k, C_b^2(\mathbb{H}))$ is closable in $L^2(\mu)$ for any $k \in \mathcal{D}(A)$.

Proof. Since $d \tilde{W}_t := (\sigma \sigma^*)^{-1/2} \sigma dW_t$ is also a cylindrical Brownian motion and $\sigma dW_t = \sqrt{\sigma \sigma^*} d\tilde{W}_t$, we may and do assume that $\sigma$ is non-negatively definite.

(1) Without loss of generality, we may and do assume that $k$ is an eigenvector of $A$, i.e. $Ak = \lambda k$ for some $\lambda \in \mathbb{R}$. We first prove the case where $b$ is Fréchet differentiable along the direction $k$. By $Ak = \lambda k$ we have

$$k(t) := \int_0^t e^{sA}k ds = \frac{e^{\lambda t} - 1}{\lambda} k, \quad t \geq 0,$$

where for $\lambda = 0$ we set $\frac{e^{sA} - 1}{A} = t$. Due to $\|k\|_\sigma < \infty$ and (2.2), the proof of [10, Theorem 5.1(1)] leads to the integration by parts formula

$$(2.3) \quad P_T(\partial_k f)(x) = \mathbb{E}\{ f(X^x(T)) M_{x,T} \}, \quad f \in C_b^1(\mathbb{H}), x \in \mathbb{H}, T > 0,$$

where

$$M_{x,T} := \frac{\lambda}{e^{\lambda T} - 1} \int_0^T \left\langle \sigma^{-1} \left( k - \frac{e^{\lambda t} - 1}{\lambda} (\partial_k b)(X^x(t)) \right), dW(t) \right\rangle.$$

Since (2.2) implies

$$(2.4) \quad \int_{\mathbb{H}} \mathbb{E}|M_{x,T}|^2 \mu(dx) \leq \frac{\lambda^2}{(e^{\lambda T} - 1)^2} \int_0^T \left\| \sigma^{-1} \left( k - \frac{e^{\lambda t} - 1}{\lambda} \partial_k b \right) \right\|_\infty^2 dt < \infty,$$

$(\mathcal{E}_k, C_b^2(\mathbb{H}))$ is closable in $L^2(\mu)$ according to Proposition 5.1.

In general, for any $\varepsilon > 0$ let

$$b_\varepsilon(x) = \frac{1}{\sqrt{2\pi \varepsilon}} \int_{\mathbb{R}} b(x + rk) \exp \left( -\frac{r^2}{2\varepsilon} \right) dr, \quad x \in \mathbb{H}.$$

Then for any $\varepsilon > 0$, $b_\varepsilon$ is Fréchet differentiable along $k$ and (2.2) holds uniformly in $\varepsilon$ with $b_\varepsilon$ replacing $b$. Let $P^\varepsilon_T$ be the semigroup for the solution $X_\varepsilon(t)$ associated to equation (2.1) with $b_\varepsilon$ replacing $b$. By simple calculations we have:

(i) $\lim_{\varepsilon \to 0} \mathbb{E}|X^x_\varepsilon(t) - X^x(t)|^2 = 0, \quad t \geq 0, x \in \mathbb{H}$.

(ii) For any $T > 0$, the family

$$M_{x,T}^\varepsilon := \frac{\lambda}{e^{\lambda T} - 1} \int_0^T \left\langle \sigma^{-1} \left( k - \frac{e^{\lambda t} - 1}{\lambda} (\partial_k b_\varepsilon)(X^x_\varepsilon(t)) \right), dW(t) \right\rangle, \quad \varepsilon > 0$$

is bounded in $L^2(\mathbb{P} \times \mu)$; i.e. $\sup_{\varepsilon > 0} \int_{\mathbb{H}} \mathbb{E}|M_{x,T}^\varepsilon|^2 \mu(dx) < \infty$.

(iii) $P^\varepsilon_T(\partial_k f)(x) = \mathbb{E}\{ f(X^x_\varepsilon(T)M_{x,T}^\varepsilon) \}, \quad f \in C_b^1(\mathbb{H}), \varepsilon > 0$. 


So, there exist $M, T \in L^2(\mathbb{P} \times \mu)$ and a sequence $\varepsilon_n \downarrow 0$ such that $M^{x_n}_t \rightarrow M, T$ weakly in $L^2(\mathbb{P} \times \mu)$. Thus, by taking $n \rightarrow \infty$ in (iii) and using (i), we prove (2.3) for $\mu$-a.e. $x \in \mathbb{B}$. Then the proof of the first assertion is completed as in the first case.

(2) Since $\sigma$ is invertible, (A3) implies $\alpha := \sum_{i=1}^{\infty} \frac{1}{\lambda_i} < \infty$. Next, since the Lipschitz constant $\|\partial_b\|_\infty$ of $b$ is finite, the integration by parts formula (2.3) also implies explicit Fomin derivative estimates on the invariant probability measure, which were investigated recently in [3]. Indeed, it follows from (2.3) and (2.4) that

$$
|\mu(\partial_k f)| = \inf_{T > 0} |\mu(P_T(\partial_k f))| \leq \inf_{T > 0} \sqrt{\mu(P_T f^2)} \left( \int_\mathbb{R} |E|M_{x,T}|^2 \mu(dx) \right)^{\frac{1}{2}}
$$

$$
\leq |k| \cdot \|f\|_{L^2(\mu)} \inf_{T > 0} \frac{\lambda}{e^{\lambda T} - 1} \left( \int_0^T \|\sigma^{-1}(I - e^{\lambda T} - \lambda^{-1}\partial_b)\|_\infty^2 dt \right)^{\frac{1}{2}}, \quad Ak = \lambda k.
$$

By taking $k = e_i, T = \lambda_i^{-1}$ and $\lambda = -\lambda_i$ in the above estimate, for any $k \in \mathcal{D}(A)$ we have

$$
|\mu(\partial_k f)| \leq \sum_{i=1}^{\infty} |\langle k, e_i \rangle \mu(\partial_{e_i} f)| \leq \left( \sum_{i=1}^{\infty} \frac{\lambda_i^2 |\langle k, e_i \rangle|^2}{\lambda_i} \right)^{\frac{1}{2}} \left( \sum_{i=1}^{\infty} \frac{1}{\lambda_i^2} \mu(\partial_{e_i} f)^2 \right)^{\frac{1}{2}}
$$

$$
\leq |Ak| \left( \sum_{i=1}^{\infty} \frac{\lambda_i}{e - 1} \right)^{\frac{1}{2}} \left( 1 + \lambda_i^{-1} \|\partial_b\|_\infty^2 \right)^{\frac{1}{2}} \|f\|_{L^2(\mu)}
$$

$$
\leq C|Ak| \cdot \|f\|_{L^2(\mu)},
$$

where $C := \|\sigma^{-1}\|_{\infty} \sqrt{\alpha} \left( 1 + \frac{e - 1}{\lambda_1} \|\partial_b\|_\infty \right)^{\frac{1}{2}}$. This implies the closability of $(\partial_k, C^2_b(\mathbb{H}))$ as explained in the proof of Proposition 1.1. Indeed, if $\{f_n\}_{n \geq 1} \subset C^2_b(\mathbb{B})$ satisfies $f_n \rightarrow 0$ and $\partial_k f_n \rightarrow Z$ in $L^2(\mu)$, then (2.3) implies

$$
|\mu(gZ)| = \lim_{n \rightarrow \infty} |\mu(g \partial_k f_n)| = \lim_{n \rightarrow \infty} |\mu(\partial_k(f_n g) - \mu(f_n \partial_k g)|
$$

$$
\leq C|Ak| \lim_{n \rightarrow \infty} \sqrt{\mu((f_n g)^2)} = 0, \quad g \in C^2_b(\mathbb{B}),
$$

so that $Z = 0$. \hfill \square

To conclude this section, let us recall a result concerning existence and stability of the invariant probability measure. Let $W_A(t) = \int_0^t e^{A(t-s)} \sigma dW(s), t \geq 0$. Assume that $b$ is Lipschitz continuous and $\int_0^\infty \|e^{tA}\|^2_{HS} dt < \infty$. We have

$$
\sup_{t \geq 0} \mathbb{E}(\|W_A(t)\|^2 + |b(W_A(t))|^2) < \infty.
$$

Therefore, by [5] Theorem 2.3, if there exist $c_1 > 0, c_2 \in \mathbb{R}$ with $c_1 + c_2 > 0$ such that

$$
\langle A(x - y), x - y \rangle \leq -c_1 |x - y|^2, \quad \langle b(x) - b(y), x - y \rangle \leq -c_2 |x - y|^2, \quad x, y \in \mathbb{H},
$$

then $P_t$ has a unique invariant probability measure such that $\lim_{t \rightarrow \infty} P_t f = \mu(f)$ holds for $f \in C_b(\mathbb{H})$.  

6
3 Stochastic Hamiltonian systems on Hilbert spaces

Let \( \mathbb{H} \) and \( \mathbb{H} \) be two separable Hilbert spaces. Consider the following stochastic differential equation for \( Z(t) := (X(t),Y(t)) \) on \( \mathbb{H} \times \mathbb{H} \):

\[
\begin{aligned}
    \begin{cases}
        dX(t) = BY(t)dt, \\
        dY(t) = \{AY(t) + b(t,X(t),Y(t))\}dt + \sigma dW(t),
    \end{cases}
\end{aligned}
\]

where \( B \in \mathcal{L}(\mathbb{H} \rightarrow \mathbb{H}) \), \( (A, \mathcal{D}(A)) \) satisfies (A1), \( \sigma \) satisfies (A3), \( W(t) \) is the cylindrical Brownian motion on \( \mathbb{H} \), and \( b : [0,\infty) \times \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H}^{-2} \) satisfies: for any \( T > 0 \) there exists \( \gamma \in C((0,T]) \) with \( \int_0^T \gamma(t)dt < \infty \) such that

\[
\begin{aligned}
    \sup_{s \in [0,T]} \int_0^T |e^{tA}b(s,0)|dt &< 1, \\
    \sup_{s \in [0,T]} |e^{tA}(b(s,z) - b(s,z'))| &\leq \gamma(t)|z - z'|, \quad t \in [0,T], z, z' \in \mathbb{H} \times \mathbb{H}.
\end{aligned}
\]

Obviously, for any initial data \( z := (x,y) \in \mathbb{H} \), the equation has a unique mild solution \( Z^z(t) \). Let \( P_t \) be the associated Markov semigroup.

When \( \mathbb{H} \) and \( \mathbb{H} \) are finite-dimensional, the integration by parts formula of \( P_t \) has been established in [10, Theorem 3.1]. Here, we extend this result to the present infinite-dimensional setting.

**Proposition 3.1.** Assume that \( BB^* \in \mathcal{L}(\mathbb{H}) \) with \( \text{Ker}(BB^*) = \{0\} \). Let \( T > 0 \) and \( k := (k_1,k_2) \in \text{Im}(BB^*) \times \mathbb{H} \) be such that

\[
Ak_2 = \theta_2k_2, \quad AB^*(BB^*)^{-1}k_1 = \theta_1B^*(BB^*)^{-1}k_1
\]

for some constants \( \theta_1, \theta_2 \in \mathbb{R} \). For any \( \phi, \psi \in C^1([0,T]) \) such that

\[
\phi(0) = \phi(T) = \psi(0) = \psi(T) = 1 = \int_0^T e^{\theta_2t}\psi(t)dt = 0, \quad \int_0^T \phi(t)e^{\theta_1t}dt = e^{\theta_1T},
\]

let

\[
\begin{aligned}
    h(t) &= B^*(BB^*)^{-1}k_1 \int_0^t \phi'(s)e^{\theta_1(s-T)}ds + k_2 \int_0^t \psi'(s)e^{\theta_2(s-T)}ds, \\
    \bar{h}(t) &= \phi(t)e^{\theta_1(t-T)}B^*(BB^*)^{-1}k_1 + \psi(t)e^{\theta_2(t-T)}k_2, \\
    \Theta(t) &= \left( \int_0^t B\bar{h}(s)ds, \bar{h}(t) \right), \quad t \in [0,T].
\end{aligned}
\]

If for any \( t \in [0,T] \), \( b(s,\cdot) \) is Fréchet differentiable along \( \Theta(t) \) such that

\[
\int_0^T \sup_{z \in \mathbb{H} \times \mathbb{H}} \|h'(t) - (\partial_{\Theta(t)}b(t,\cdot))(z)\|_\sigma^2 dt < \infty,
\]

then for any \( f \in C^1_b(\mathbb{H} \times \mathbb{H}) \),

\[
P_T(\partial_k f) = \mathbb{E}\left\{ f(Z(T)) \int_0^T \left( (\sigma \sigma^*)^{-1/2} \{h'(t) - (\partial_{\Theta(t)}b(t,\cdot))(Z(t))\}, dW(t) \right) \right\}.
\]
Proof. As explained in the proof of Theorem 2.1, we simply assume that \( \sigma = \sqrt{\sigma^*} \). Let \((X^0(t), Y^0(t)) = (X(t), Y(t))\) solve (3.1) with initial data \((x, y)\), and for \( \varepsilon \in (0, 1] \) let \((X^\varepsilon(t), Y^\varepsilon(t))\) solve the equation

\[
\begin{cases}
    dX^\varepsilon(t) = BY^\varepsilon(t)dt, & X^\varepsilon(0) = x, \\
    dY^\varepsilon(t) = \sigma dW(t) + \left\{b(t, X(t), Y(t)) + AY^\varepsilon(t) + \varepsilon h'(t)\right\}dt, & Y^\varepsilon(0) = y.
\end{cases}
\]

Then it is easy to see from (3.3) and (3.4) that

\[
Y^\varepsilon(t) - Y(t) = \varepsilon \int_0^t e^{(t-s)A}h'(s)ds + \varepsilon B^* \int_0^t \phi'(s)e^{\theta_1(s-T)}e^{\theta_2(t-s)}ds + \varepsilon k_2 \int_0^t \psi'(s)e^{\theta_2(1-T)}ds
\]

and hence,

\[
X^\varepsilon(t) - X(t) = \varepsilon \int_0^t B\tilde{h}(s)ds + \varepsilon k_1 \int_0^t \phi(r)e^{\theta_1(r-T)}dr + (Bk_2) \int_0^t \psi(r)e^{\theta_2(r-T)}dr.
\]

So,

\[
X^\varepsilon(t) - X(t) = \varepsilon \Theta(t), \quad t \in [0, T],
\]

and in particular

\[
(X^\varepsilon(T), Y^\varepsilon(T)) = (X(T), Y(T)) + \varepsilon k
\]

due to (3.4). Next,

\[
\xi^\varepsilon(s) = \varepsilon h'(s) + b(s, X(s), Y(s)) - b(s, X^\varepsilon(s), Y^\varepsilon(s))
\]

and

\[
R^\varepsilon = \exp \left[ -\int_0^T \langle \sigma^{-1}\xi^\varepsilon(s), dW(s) \rangle - \frac{1}{2} \int_0^T |\sigma^{-1}\xi^\varepsilon(s)|^2ds \right].
\]

We reformulate (3.6) as

\[
\begin{cases}
    dX^\varepsilon(t) = BY^\varepsilon(t)dt, & X^\varepsilon(0) = x, \\
    dY^\varepsilon(t) = \sigma dW(t) + \left\{b(t, X^\varepsilon(t), Y^\varepsilon(t)) + AY^\varepsilon(t)\right\}dt, & Y^\varepsilon(0) = y,
\end{cases}
\]

where by (3.5) and (3.7),

\[
W^\varepsilon(t) := W(t) + \int_0^t \sigma^{-1}\xi^\varepsilon(s)ds, \quad t \in [0, T]
\]
is a cylindrical Brownian motion under the weighted probability measure $Q_{\varepsilon} := R_{\varepsilon}P$. Since $|\xi_{\varepsilon}|$ is uniformly bounded on $[0, T]$, by the dominated convergence theorem and (3.7), for any $f \in C^1_b(\mathbb{H} \times \mathbb{H})$ we obtain

$$
P_T(\partial_k f) = \lim_{\varepsilon \to 0} \mathbb{E} f((X(T), Y(T)) + \varepsilon k) - f((X(t), Y(t)))
= \lim_{\varepsilon \to 0} \mathbb{E} \frac{f((X^\varepsilon(T), Y^\varepsilon(T))) - R_{\varepsilon} f((X^\varepsilon(T), Y^\varepsilon(T)))}{\varepsilon}
= \mathbb{E} \left( f(Z(T)) \lim_{\varepsilon \to 0} \frac{1 - R_{\varepsilon}}{\varepsilon} \right)
= \mathbb{E} \left( f(Z(T)) \int_0^T \sigma^{-1}(h'(t) - (\partial_{\Theta(t)}b)(Z(t)), dW(t)) \right).
$$

}\hspace{1cm} \Box

To apply this result, we present here a specific choice of $(\phi, \psi)$ such that (3.4) holds:

$$
\phi(t) = \frac{e^{\theta_1 t} (T-t)}{\int_0^T s(T-s)e^{\theta_1 s} ds}, \quad \psi(t) = \frac{e^{\theta_2 (T-t)}}{T} \left( \frac{3t^2}{T} - 2t \right), \quad t \in [0, T].
$$

**Theorem 3.2.** Let $\mathbb{H} = \mathbb{H} = \mathbb{H}$ and $\text{Ker}(B) = \{0\}$. Let $b(t, \cdot) = b$ do not dependent on $t$ such that $P_t$ has an invariant probability measure $\mu$. If

$$
(3.11) \quad \sup_{(x,y) \in \mathbb{H} \times \mathbb{H}} \lim_{r \downarrow 0} \frac{\|b(x + rB^{-1}\tilde{k}, y + rk) - b(x, y)\|_{\sigma}}{r} < \infty, \quad (\tilde{k}, k) \in (B\mathbb{H}_A) \times \mathbb{H}_A.
$$

Then for any $(k_1, k_2) \in (B\mathbb{H}_A) \times \mathbb{H}_A$, the form $(\partial_{k_1} C^2_b(\mathbb{H} \times \mathbb{H}))$ is closable in $L^2(\mu)$.

**Proof.** It suffices to prove for $k = (k_1, k_2)$ such that $B^{-1}k_1$ and $k_2$ are eigenvectors of $A$, i.e. $AB^{-1}k_1 = \theta_1 B^{-1}k_1$ and $AK_2 = \theta_2 k_2$ hold for some $\theta_1, \theta_2 \in \mathbb{R}$. As explained above there exists $T > 0$ such that (3.4) holds for some $\phi, \psi \in C^\infty([0, T])$. Moreover, as explained in the proof of Theorem 2.1 by taking

$$
b_{\varepsilon}(s, x, y) = \frac{1}{\sqrt{2\pi\varepsilon}} \int_{\mathbb{R}} b((x, y) + r \Theta(s)) \exp \left[ - \frac{r^2}{2\varepsilon} \right] dr, \quad s \in [0, T], (x, y) \in \mathbb{H} \times \mathbb{H}
$$

for $\varepsilon > 0$, such that (3.11) holds uniformly in $\varepsilon > 0$ and $s \in [0, T]$ with $b_{\varepsilon}(s, \cdot)$ replacing $b$, we may and do assume that $b(s, \cdot)$ is Fréchet differentiable along $\Theta(s)$. Then the integration by parts formula in Proposition 3.1 holds, and due to (3.11) we have

$$
M_{s,t} := \int_0^T \left\{ (\sigma \sigma^*)^{-1/2} h'(t) - (\partial_{\Theta(t)} b(t, \cdot))(Z(t)) \right\} dW(t) \in L^2(\mathbb{P} \times \mu).
$$

Therefore, by Proposition 3.1 the form $(\partial_{k_1} C^2_b(\mathbb{H} \times \mathbb{H}))$ is closable on $L^2(\mu)$.

Below are typical examples of the stochastic Hamiltonian system with invariant probability measure such that Theorem 3.2 applies.
Example 3.1. Let $\mathbb{H} = \mathbb{H} = \mathbb{H}$.

(1) Let $\mathbb{H} = \mathbb{R}^d$ for some $d \geq 1$. When $\sigma = B = I$, $A \leq -\lambda I$ for some $\lambda > 0$ is a negatively definite $d \times d$-matrix, and $b(x, y) = A^{-1}\nabla V(x)$ for some $V \in C^2(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} e^{-V(x)}dx < \infty$. Then the unique invariant probability measure of $P_t$ is

$$
\mu(dx, dy) = C e^{-V(x) + \frac{1}{2}(Ay, y)} dx dy,
$$

where $C > 0$ is the normalization. See [2, 6, 9] for the study of hypercoercivity of the associated semigroup $P_t$ with respect to $\mu$, as well as [12] for the stronger property of hypercontractivity.

(2) In the infinite-dimensional setting, let $\sigma = B = I$ and $A$ be negatively definite such that $A^{-1}$ is of trace class. Take $b(x, y) = A^{-1}Qx$ for some positively definite self-adjoint operator $Q$ on $\mathbb{H}$ such that $Q^{-1}$ is of trace class and

$$
\int_0^1 \|e^{tA}A^{-1}Q\|dt < 1.
$$

Then it is easy to see that

$$
\mu(dx, dy) = N_{Q^{-1}}(dx)N_{-A^{-1}}(dy)
$$

is an invariant probability measure.

(3) More generally, let $\sigma = B = I$ and

$$
b(x, y) = \tilde{b}(x) := A^{-1}\nabla V(x), \quad (x, y) \in \mathbb{H} \times \mathbb{H}_A
$$

for some Fréchet differentiable $V : \mathbb{H}_A \to \mathbb{R}$ such that (3.11) holds. For any $n \geq 1$, let

$$
V_n(r) = V \circ \varphi_n(r), \quad \varphi_n(r) = \sum_{i=1}^n r_i e_i, \quad r = (r_1, \cdots, r_n) \in \mathbb{R}^n.
$$

If $\int_{\mathbb{R}^n} e^{-V_n(r)}dr < \infty$ and when $n \to \infty$ the probability measure

$$
\nu_n(D) := \frac{1}{\int_{\mathbb{R}^n} e^{-V_n(r)}dr} \int_{\varphi_n^{-1}(D)} e^{-V_n(r)}dr, \quad D \in \mathcal{B}(\mathbb{H})
$$

converges weakly to some probability measure $\nu$, then $\mu := \nu \times N_{-A^{-1}}$ is an invariant probability measure of $P_t$. This can be confirmed by (1) and a finite-dimensional approximation argument. Indeed, let $\pi_n : \mathbb{H} \to \mathbb{H}_{A,n}$ be the orthogonal projection, and let $A_n = \pi_n A, W_n = \pi_n W$ and $b_n(x, y) = \pi_n \nabla V(x)$. Let $X_n(t)$ solve the finite-dimensional equation

$$
\begin{align*}
\begin{cases}
dX_n(t) = Y_n(t)dt, \\
dY_n(t) = \{A_nY_n(t) + b_n(X_n(t))\}dt + dW_n(t)
\end{cases}
\end{align*}
$$

with $(X_n(0), Y_n(0)) = (\pi_n X(0), \pi_n Y(0))$. Then the proof of [11, Theorem 2.1] yields that for every $t \geq 0$,

$$
\lim_{n \to \infty} \mathbb{E}(|X_n(t) - X(t)|^2 + |Y_n(t) - Y(t)|^2) = 0
$$
uniformly in the initial data \((X(0), Y(0)) \in \mathbb{H} \times \mathbb{H}\). Thus, letting \(P_{t}^{(n)}\) be the semigroup for \((X_n(t), Y_n(t))\), we have
\[
\lim_{n \to \infty} \sup_{(x,y) \in \mathbb{H} \times \mathbb{H}} |P_{t}^{(n)} f(\pi_n x, \pi_n y) - P_t f(x, y)| = 0, \quad f \in C^1_0(\mathbb{H} \times \mathbb{H}).
\]

Combining this with the assertion in (1) and noting that \(\nu_n \times (N_{A-1} \circ \pi_n^{-1}) \to \mu\) weakly as \(n \to \infty\), we conclude that \(\mu\) is an invariant probability measure of \(P_t\).

4 Semilinear SPDEs with delay

For fixed \(\tau > 0\), let \(\mathcal{C}_\tau = C([-\tau, 0]; \mathbb{H})\) be equipped with the uniform norm \(\|\eta\|_\infty := \sup_{\theta \in [-\tau, 0]} |\eta(\theta)|\). For any \(\xi \in C([-\tau, \infty); \mathbb{H})\), we define \(\xi \in C([0, \infty); \mathcal{C}_\tau)\) by letting
\[
\xi_t(\theta) = \xi(t + \theta), \quad \theta \in [-\tau, 0], t \geq 0.
\]

Consider the following stochastic differential equation with delay:
\[
(4.1) \quad dX(t) = \{AX(t) + b(X_t)\} dt + \sigma dW(t), \quad X_0 \in \mathcal{C}_\tau,
\]
where \((A, \mathcal{D}(A))\) satisfies (A1), \(\sigma\) satisfies (A3), and \(b : \mathcal{C}_\tau \to \mathbb{H}\) satisfies: for any \(T > 0\) there exists \(\gamma \in C((0, T])\) with \(\int_0^T \gamma(t) dt < \infty\) such that
\[
(4.2) \quad \int_0^T \sup_{s \in [0, T]} |e^{tA}b(s, 0)|^2 dt < \infty, \quad |e^{tA}(b(s, \xi) - b(s, \eta))|^2 \leq \gamma(t) \|\xi - \eta\|_\infty^2, \quad t, s \in [0, T].
\]

Then for any initial datum \(\xi \in \mathcal{C}_\tau\), the equation has a unique mild solution \(X^\xi(t)\) with \(X_0 = \xi\). Let \(P_t\) be the Markov semigroup for the segment solution \(X_t\).

Let
\[
\mathcal{C}_1 = \left\{ \eta \in \mathcal{C}_\tau : \eta(\theta) \in \mathcal{D}(A) \text{ for } \theta \in [-\tau, 0], \int_{-\tau}^0 (|A\eta(\theta)|^2 + |\eta'(\theta)|^2) d\theta < \infty \right\}.
\]

The following result is an extension of [10, Theorem 4.1(1)] to the infinite-dimensional setting.

**Proposition 4.1.** For any \(\eta \in \mathcal{C}_1\) and \(T > \tau\), let
\[
\Gamma(t) := \begin{cases} \frac{e^{(s+\tau-T)A}A(\eta(-\tau), \eta'(-\tau))}{T-\tau} & \text{if } s \in [0, T - \tau], \\ \eta'(s - T) - A\eta(s - T) & \text{if } s \in (T - \tau, T], \end{cases}
\]
and
\[
\Theta(t) := \int_0^t \Gamma(s) ds, \quad t \in [-\tau, T].
\]

If \(b(t, \cdot)\) is Fréchet differentiable along \(\Theta_t\) for \(t \in [0, T]\) such that
\[
(4.3) \quad \sup_{\xi \in \mathcal{C}_\tau} \int_0^T \|\Gamma(t) - (\nabla_{\Theta_t} b(T, \cdot))(\xi)\|_\sigma^2 dt < \infty,
\]


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then
\begin{equation}
(4.4)
P_T(\partial_{\eta}f) = \mathbb{E}\left( f(X_T) \int_0^T \langle (\sigma\sigma^*)^{-1/2}(\Gamma(t) - (\nabla_{\Phi}b(t, \cdot))(X_t)), dW(t) \rangle \right), \quad f \in C_b^{1}(\mathbb{C}_r).
\end{equation}

**Proof.** Simply let \( \sigma = \sqrt{\sigma\sigma^*} \) as in the proof of Theorem 2.1. For any \( \varepsilon \in (0, 1) \), let \( X^\varepsilon(t) \) solve the equation
\begin{equation}
(4.5)
dX^\varepsilon(t) = \{ AX^\varepsilon(t) + b(t, X_t) + \varepsilon(\Gamma(t)) dt + \sigma dW(t) \}, \quad X^\varepsilon_0 = X_0.
\end{equation}
We have
\begin{align}
X^\varepsilon(t) - X(t) &= \varepsilon \int_0^{t+} e^{t-s}A\Gamma(s)ds \\
&= \frac{\varepsilon^T}{T - \tau} e^{(\tau - T)A}(\eta(-\tau)1_{[T - \tau, T]}(t) + \varepsilon\eta(t - T)1_{[T - \tau, T]}(t), \quad t \in [-\tau, T].
\end{align}
In particular, we have \( X^\varepsilon_T - X_T = \varepsilon\eta \). To formulate \( P_T \) using \( X^\varepsilon_T \), rewrite \( (4.5) \) by
\begin{equation}
(4.6)
dX^\varepsilon(t) = \{ AX^\varepsilon(t) + b(t, X^\varepsilon_t) \} dt + \sigma dW^\varepsilon(t), \quad X^\varepsilon_0 = X_0,
\end{equation}
where
\begin{equation}
W^\varepsilon(t) := W(t) + \int_0^t \xi^\varepsilon(s)ds, \quad \xi^\varepsilon(s) := b(s, X_s) - b(s, X^\varepsilon_s) + \varepsilon\Gamma(s).
\end{equation}
By \([4.3]\) and the Girsanov theorem, we see that \( \{W^\varepsilon(t)\}_{t \in [0, T]} \) is a cylindrical Brownian motion on \( \mathbb{H} \) under the probability measure \( dQ^\varepsilon := R^\varepsilon dP \), where
\begin{equation}
R^\varepsilon := \exp\left[ \int_0^T \langle \sigma^{-1}(b(t, X^\varepsilon_t) - b(t, X_t) - \varepsilon\Gamma(t)), dW(t) \rangle \right].
\end{equation}
Then
\begin{equation}
\mathbb{E}(f(X_T)) = P_T f = \mathbb{E}(R^\varepsilon f(X^\varepsilon_T)).
\end{equation}
Combining this with \( X^\varepsilon_T = X_T + \varepsilon\eta \) and using \( (4.6) \), we arrive at
\begin{align}
P_T(\partial_{\eta}f) &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mathbb{E}\{f(X_T + \varepsilon\eta) - f(X_T)\} = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mathbb{E}\{f(X^\varepsilon_T) - R^\varepsilon f(X^\varepsilon_T)\} \\
&= \mathbb{E}\left( f(X_T) \lim_{\varepsilon \downarrow 0} \frac{1 - R^\varepsilon}{\varepsilon} \right) = \mathbb{E}\left\{ f(X_T) \int_0^T \langle \sigma^{-1}(\Gamma(t) - (\nabla_{\Phi}b(t, \cdot))(X_t)), dW(t) \rangle \right\}.
\end{align}

**Theorem 4.2.** Let \( b(t, \cdot) = b \) be independent of \( t \) such that \( P_t \) has an invariant probability measure \( \mu \). If \( \text{Im}(\sigma) \supset \mathbb{H}_A \) and
\begin{equation}
(4.7)
\sup_{\xi \in \mathbb{C}_r} \sup_{\varepsilon \downarrow 0} \frac{||b(\xi + \varepsilon\eta) - b(\xi)||_\sigma}{\varepsilon} < \infty, \quad \eta \in \mathbb{C}_r \cap \left( \bigcup_{n \geq 1} C([-\tau, 0]; \mathbb{H}_{A,n}) \right),
\end{equation}
then
then for any \( \eta \in \mathcal{C}_1 \cap \left( \cup_{n \geq 1} C([-\tau,0]; \mathbb{H}_{A,n}) \right) \), which is dense in \( \mathcal{C}_\tau \), the form

\[
\mathcal{E}_\eta(f,g) := \int_{\mathcal{C}_\tau} (\partial_{\eta} f)(\partial_{\eta} g) d\mu, \quad f, g \in C^2_b(\mathcal{C}_\tau)
\]

is closable in \( L^2(\mu) \).

**Proof.** For any \( \varepsilon \in (0,1) \) let

\[
b_{\varepsilon}(t,\xi) = \frac{1}{\sqrt{2\pi\varepsilon}} \int_{\mathbb{R}} b(\xi + r\Theta_t) \exp \left[ -\frac{r^2}{2\varepsilon} \right] dr, \quad \xi \in \mathcal{C}_\tau.
\]

Then \( b_{\varepsilon}(t,\cdot) \) is Fréchet differentiable along \( \Theta_t \) and \( (4.7) \) holds uniformly in \( \varepsilon \) with \( b_{\varepsilon}(t,\cdot) \) replacing \( b \). Moreover, \( \eta \in \mathcal{C}_1 \cap \left( \cup_{n \geq 1} C([-\tau,0]; \mathbb{H}_n) \right) \) implies that \( \Theta_t \in \mathcal{C}_1 \cap \left( \cup_{n \geq 1} C([-\tau,0]; \mathbb{H}_n) \right) \) and \( (4.7) \) holds uniformly in \( t \in [0,T] \) and \( \varepsilon \in (0,1) \) with \( \Theta_t \) and \( b_{\varepsilon}(t,\cdot) \) replacing \( \eta \) and \( b \) respectively. Combining this with \( \text{Im}(\sigma) \supset \mathbb{H}_A \), we conclude that \( (4.3) \) holds uniformly in \( \varepsilon \) with \( b_{\varepsilon}(\cdot) \) replacing \( b \). Therefore, as explained in the proof of Theorem 2.1, we may assume that \( b \) is Fréchet differentiable along \( \Theta_t, t \in [0,T] \), and by Proposition 4.1 the integration by parts formula \( (4.4) \) holds. Moreover, \( (4.7) \) implies

\[
M_{t,T} := \int_0^T \left\langle (\sigma\sigma^*)^{-1/2} (\Gamma(t) - (\nabla_{\Theta_t} b(t,\cdot))(X_t)), dW(t) \right\rangle \in L^2(\mathbb{P} \times \mu).
\]

Then the proof is finished by Proposition 4.1.

Finally, we introduce the following example to illustrate Theorem 4.2.

**Example 4.1.** Let \( b(\xi) = F(\xi(-\tau)), \xi \in \mathcal{C}_\tau \), for some \( F \in C^1_b(\mathbb{H}) \). If \( \sigma \) is Hilbert-Schmidt and

\[
\langle x, Ax + F(y) - F(y') \rangle \leq -\lambda_1 |x|^2 + \lambda_2 |y - y'|^2, \quad x, y, y' \in \mathbb{H},
\]

for some constants \( \lambda_1 > \lambda_2 \geq 0 \), then according to [1, Theorem 4.9] \( P_t \) has a unique invariant probability measure \( \mu \). If moreover \( \text{Im}(\sigma) \supset \mathbb{H}_A \) and for any \( y \in \mathbb{H}_A \) there exists a constant

\[
\lim_{\varepsilon \downarrow 0} \sup_{x \in \mathbb{H}} \frac{\|F(x + \varepsilon y) - F(x)\|_\sigma}{\varepsilon} < \infty,
\]

then by Theorem 4.2 for any \( \eta \in \mathcal{C}_1 \cap \left( \cup_{n \geq 1} C([-\tau,0]; \mathbb{H}_{A,n}) \right) \) the form \( (\mathcal{E}_\eta, C^2_b(\mathcal{C}_\tau)) \) is closable on \( L^2(\mu) \).

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