THE CONNECTEDNESS OF THE MODULI SPACE OF MAPS TO HOMOGENEOUS SPACES

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0. Introduction

Let $X$ be a compact algebraic homogeneous space: $X = G/P$ where $G$ is a connected complex semisimple algebraic group and $P$ is a parabolic subgroup. Let $\beta \in H_2(X, \mathbb{Z})$. The (coarse) moduli space $\overline{M}_{g,n}(X, \beta)$ of $n$-pointed genus $g$ stable maps parameterizes the data 

$$[\mu : C \to X, p_1, \ldots, p_n]$$

satisfying:

(i) $C$ is a complex, projective, connected, reduced, (at worst) nodal curve of arithmetic genus $g$.

(ii) The points $p_i \in C$ are distinct and lie in the nonsingular locus.

(iii) $\mu_*[C] = \beta$.

(iv) The pointed map $\mu$ has no infinitesimal automorphisms.

Since $X$ is convex, the genus 0 moduli space $\overline{M}_{0,n}(X, \beta)$ is of pure dimension

$$\dim(X) + \int_\beta c_1(T_X) + n - 3.$$ 

Moreover, $\overline{M}_{0,n}(X, \beta)$ is locally the quotient of a nonsingular variety by a finite group. For general $g$, the space $\overline{M}_{g,n}(X, \beta)$ may have singular components of different dimensions. Stable maps in algebraic geometry were first defined in [Ko]. Basic properties of the moduli space $\overline{M}_{g,n}(X, \beta)$ can be found in [BM], [FP], and [KoM]. The following connectedness result is proven here.

Theorem 1. $\overline{M}_{g,n}(G/P, \beta)$ is a connected variety.

This result may be viewed as analogous to the connectedness of the Hilbert scheme of projective space proven by Hartshorne. As in [Har], connectedness is obtained via maximal degenerations.

Since $\overline{M}_{0,n}(X, \beta)$ has quotient singularities, connectedness is equivalent to irreducibility.

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Corollary 1. \( \overline{M}_{0,n}(G/P, \beta) \) is an irreducible variety.

Corollary 1 is easy to verify in case \( X \) is a projective space. When \( X \) is a Grassmannian, the irreducibility follows from Strømme’s Quot scheme analysis [S]. A proof of Corollary 1 can be found in case \( G = \text{SL} \) in [MM]. For the variety of partial flags in \( \mathbb{C}^n \), a proof of irreducibility using flag-Quot schemes is established in [Ki]. Results of Harder closely related to Corollary 1 appear in [Ha]. There is an independent proof by J. Thomsen for the irreducibility of \( \overline{M}_{0,n}(G/P, \beta) \) in [T].

The moduli space \( \overline{M}_{g,n}(X, \beta) \) has a natural locally closed decomposition indexed by stable, pointed, modular graphs \( \tau \) (see [BM]). The strata correspond to maps with domain curves of a fixed topological type and a fixed distribution \( \beta_\tau \) of \( \beta \). The graph \( \tau \) determines a complete moduli space of stable maps \( \overline{M}_{\tau,n}(X, \beta_\tau) \) together with a canonical morphism:

\[
\pi_\tau : \overline{M}_{\tau,n}(X, \beta_\tau) \to \overline{M}_{g,n}(X, \beta).
\]

A closed decomposition is determined by the images of these morphisms. Theorem 1 is a special case of the following result.

Theorem 2. \( \overline{M}_{\tau,n}(G/P, \beta_\tau) \) is a connected variety.

Since \( \overline{M}_{\tau,n}(X, \beta_\tau) \) is normal in the genus 0 case, we obtain the corresponding corollary.

Corollary 2. Let \( g = 0 \). \( \overline{M}_{\tau,n}(G/P, \beta_\tau) \) is an irreducible variety.

In particular, all the boundary divisors of \( \overline{M}_{0,n}(X, \beta) \) are irreducible.

Theorem 2 is proven by studying the maximal torus action on \( X \). The method is to degenerate a general \( G \)-translate of a map \( \mu : C \to X \) onto a canonical 1-dimensional configuration of \( \mathbb{P}^1 \)’s in \( X \) determined by the maximal torus and the Bialynicki-Birula stratification of \( X \).

In the genus 0 case, we study the Bialynicki-Birula stratification of \( \overline{M}_{0,n}(X, \beta) \). The following result is then deduced from the rationality of torus fixed components.

Theorem 3. \( \overline{M}_{0,n}(G/P, \beta) \) is rational.

The fixed component rationality is equivalent to a rationality result for certain quotients of \( \text{SL}_2 \)-representations proven by Katsylo and Bogomolov [Ka], [Bog]. It should be noted that the fixed components will in general be contained in the boundary of the moduli space of maps –
the compactification by stable maps therefore plays an important role in
the proof.

The rationality of the Hilbert schemes of rational curves in projective
space (birational to \( \overline{M}_{0,0}(\mathbb{P}^r, d) \)) is a consequence of Katsylo’s results
[Ka] and was also studied by Hirschowitz in [Hi].

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1. The torus action on \( G/P \)

Let \( G \) be a connected complex semisimple algebraic group. Let \( P \)
be a parabolic subgroup. Select a maximal algebraic torus \( T \) and Borel
subgroup \( B \) of \( G \) satisfying:

\[
T \subset B \subset P \subset G.
\]

Let \((G/P)^T\) denote the fixed point set of the left \( T \)-action on \( G/P \).
Three special properties of this \( T \)-action will be needed:

(i) The \( T \)-action has isolated fixed points.

(ii) For every point \( p \in (G/P)^T \), there exits a \( T \)-invariant open set \( U_p \)
containing \( p \) which is \( T \)-equivalent to a vector space representation
of \( T \).

(iii) Let \( C^\ast \subset T \) correspond to an interior point of a Weyl chamber.
Then, \((G/P)^{C^\ast} = (G/P)^T \), and the Bialynicki-Birula decomposi-
tion obtained from the \( C^\ast \)-action is an affine stratification of \( G/P \).

A stratification is a decomposition such that the closures of the strata
are unions of strata. In general, the Bialynicki-Birula decomposition
obtained from a \( C^\ast \)-action on a nonsingular variety need not be a strat-
ification.

The claims (i)-(iii) are well known. Only a brief summary of the
arguments will be presented here. Let \( W \) be the Weyl group of \( G \)
relative to \( T \).

**Lemma 1.** \(|(G/B)^T| = |W|\), and \( W \) acts transitively on \((G/B)^T\).

**Proof.** See, for example, [Bor].

In particular, \((G/B)^T\) is a finite set.
Lemma 2. The natural map \((G/B)^T \to (G/P)^T\) is surjective.

Proof. Let \(p \in (G/P)^T\). The invariant fiber (isomorphic to \(P/B\)) over the fixed point \(p\) is a nonsingular projective variety, and hence contains a \(T\)-fixed point by the Borel fixed point theorem (or, alternatively, this is a Hamiltonian action on a compact manifold).

Therefore, \(W\) acts transitively on the finite set \((G/P)^T\).

A representation \(\psi : T \to GL(V)\) is fully definite if there exists a \(\mathbb{C}^*\)-basis of \(T\) for which all the weights of the representation are positive integers. Equivalently, a fully definite representation can be written

\[
\psi(t_1, \ldots, t_r)v_j = \prod_{i=1}^{r} t_i^{\lambda_{ij}} \cdot v_j
\]

where \(\lambda_{ij} > 0\) for some choice of \(\mathbb{C}^*\)-basis of \(T\) and \(\mathbb{C}\)-basis \(\{v_j\}\) of \(V\).

The point \(1 \in G/B\) corresponding to the identity element of \(G\) is a \(T\)-fixed point. The \(T\)-action induces a representation

\[
\phi : T \to GL(Tan_1 G/B).
\]

Lemma 3. The representation \(\phi\) is fully definite.

Proof. The natural quotient map \(q : G \to G/B\) is \(T\)-equivariant for the conjugation action on \(G\) and the left action on \(G/B\). The differential of \(q\) yields an isomorphism from the Adjoint representation of \(T\) on \(\text{Lie}(G)/\text{Lie}(B)\) to \(\phi\). \(\text{Lie}(G)/\text{Lie}(B)\) is the space of positive roots. This \(T\)-representation space has \(n\) simple roots (where \(n\) is the rank of \(G\)). All the 1-dimensional representations in \(\text{Lie}(G)/\text{Lie}(B)\) are non-negative tensor products of these simple roots. Moreover, the \(n\) weight vectors of these simple roots are independent in the lattice of 1-dimensional representations of the torus \(T\). Lemma 3 now follows from Lemma 4 below.

Lemma 4. Let \(\psi : T \to GL(\mathbb{C}^n)\) be an \(n\) dimensional representation of a rank \(n\) torus \(T\). If the \(n \times n\) matrix of weights is nonsingular, then the representation is fully definite.

Proof. See [Bi].

Lemma 5. The \(T\)-representation \(Tan_1 G/P\) is fully definite.

Proof. There is a surjection of \(T\)-modules given by the differential \(Tan_1 G/B \to Tan_1 G/P\).
Proposition 1. For every \( p \in (G/P)^T \), there exists a \( T \)-invariant Zariski open set \( U_p \subset G/P \) of \( p \) which is \( T \)-equivalent to a vector space representation of \( T \).

Proof. By a theorem of Bialynicki-Birula [Bi], it suffices to show the tangent representation of \( T \) is fully definite at \( p \). This is a consequence of Lemma 5 and the transitivity of the \( W \)-action on \((G/P)^T\). (In fact, only definiteness of the tangent representation is needed in [Bi].) \( \square \)

Let \( C^* \subset T \) correspond to an interior point of a Weyl chamber. By the analysis of the tangent representation \( \phi \), every point of \((G/B)^T\) is an isolated fixed point of \( C^* \). The equality \((G/B)^{C^*} = (G/B)^T\) follows. Since the map \((G/B)^C \rightarrow (G/P)^{C^*}\) is surjective, \((G/P)^{C^*} = (G/P)^T \).

For each \( p \in (G/P)^T \), let \( A_p \) be the set of points \( x \in G/P \) such that \( \lim_{t \to 0} tx = p \).

By Proposition 1, \( A_p \) is isomorphic to the affine space \( C^{r_p} \) where \( r_p \) is the number of positive weights in the \( C^*\)-representation \( \text{Tan}_{p} G/P \). The set \( \{A_p\} \) is the Bialynicki-Birula affine decomposition of \( G/P \). In fact, \( \{A_p\} \) coincides (up to the Weyl group action) with the (open) Schubert cell stratification of \( G/P \). This is essentially proven in [Bor] for the case \( G/B \). The general case \( G/P \) is proven in [A]. Therefore, \( \{A_p\} \) is a stratification.

2. The \( C^* \)-flow

Let \( C^* \subset T \) correspond to an interior point of a Weyl chamber. Let \( s, x_1, \ldots, x_l \in (G/P)^T \) be the fixed points corresponding to the unique maximal dimensional stratum \( A_s \) and the complete set of codimension 1 strata, \( A_1, \ldots, A_l \), respectively. The points of \( A_s \) flow \( (t \to 0) \) to \( s \), and the points of \( A_i \) flow \( (t \to 0) \) to \( x_i \). Let \( U = A_s \cup A_1 \cup \ldots \cup A_l \). Since the Bialynicki-Birula decomposition \( \{A_p\} \) is a stratification, \( U \) is a Zariski open set with complement of codimension at least 2.

The inverse action of \( C^* \) on \( G/P \) is also a torus action on \( G/P \) with the same fixed point set. Let \( A'_s, A'_1, \ldots, A'_l \) be the affine strata for the inverse action corresponding to the fixed points \( s, x_1, \ldots, x_l \). Let \( \dim(G/P) = m \). Since,

\[
\dim(A_p) + \dim(A'_p) = m,
\]

\( A'_1, \ldots, A'_l \) are the complete set of 1-dimensional strata for the inverse action. Moreover, the closure \( P_i = \overline{A_i} \) can contain only the unique 0-dimensional stratum \( A'_s = s \). We have shown the closures \( P_i \) are contained in \( U \). Each \( P_i \) is isomorphic to \( \mathbb{P}^1 \) (Chevelley [C] proves the
closed Schubert cells have singularities in codimension at least 2). The intersection pairing
\[ P_i \cap A_j = \delta(i - j) \]
follows from the above analysis. Since the closed strata of the inverse action freely generate the integral homology, the classes
\[ [P_1], \ldots, [P_l] \in H_2(G/P, \mathbb{Z}) \]
span an integral basis of \( H_2(G/P, \mathbb{Z}) \).

Let \( f : C \to G/P \) be a non-constant stable map satisfying the following properties:

(i) The image \( f(C) \) lies in \( U \).
(ii) \( C \) intersects (via \( f \)) the divisors \( A_i \) transversely at nonsingular points of \( C \).
(iii) All the markings of \( C \) have image in \( A_s \).

If \([f]\) represents the class \( \beta = \sum_{i=1}^l a_i [P_i] \in H_2(G/P, \mathbb{Z}) \)
then let \( C \) meet \( A_i \) at the \( a_i \) distinct points \( \{x_{i,1}, \ldots, x_{i,a_i}\} \).

We will study the induced \( \mathbb{C}^* \)-action on \( \overline{M}_{g,n}(G/P, \beta) \) by translation of maps. Let \( F : C_0 \to G/P \) be the limit in the space of stable maps,
\[ F = \lim_{t \to 0} tf \]
where \( t \in \mathbb{C}^* \).

Define a map \( \tilde{F} : \tilde{C} \to G/P \) as follows. Let the domain \( \tilde{C} \) be:
\[ \tilde{C} = C \cup \bigcup_{i=1}^l (\bigcup_{j=1}^{a_i} \mathbb{P}_1^{i,j}) \]
where \( \mathbb{P}_1^{i,j} \) is a projective line attached to \( C \) at the point \( x_{i,j} \). Let the markings of \( \tilde{C} \) coincide with the markings of \( C \) (note the markings of \( C \) are disjoint from the set \( \{x_{i,j}\} \) by condition (ii)). Define \( \tilde{F} \) by \( \tilde{F}(C \subset \tilde{C}) = s \) and
\[ \tilde{F}|_{\mathbb{P}^{i,j}_{i,j}} : \mathbb{P}^{i,j}_{i,j} \simeq P_i \]
for each \( i \) and \( j \).

**Proposition 2.** If \( f \) satisfies conditions (i-iii), then the \( t \to 0 \) limit \( F \) equals the stabilization of \( \tilde{F} \).
Proof. Let \( \triangle^\circ \subset \triangle \) be the punctured holomorphic disk at the origin. Let
\[
h : C \times \triangle^\circ \to \mathbb{G}/\mathbb{P}
\]
be the map defined by \( h(c, t) = tf(c) \). The \( \mathbb{C}^* \)-action on \( A_s \) extends to a map
\[
C \times A_s \to A_s
\]
since the \( \mathbb{C}^* \)-action on \( A_s \) is a vector space representation with positive weights. The map \( h \) thus extends to a map
\[
h : C \times \triangle \setminus \{x_{i,j} \times 0\} \to \mathbb{G}/\mathbb{P}
\]
since the \( f \)-image of \( C \setminus \{x_{i,j}\} \) lies in \( A_s \). Note,
\[
(2) \quad h(C \setminus \{x_{i,j}\}, 0) = s.
\]
After a suitable blow-up
\[
\gamma : S \to C \times \triangle
\]
supported along the isolated nonsingular points \( \{x_{i,j} \times 0\} \) of \( C \times \triangle \), there is a morphism \( h' : S \to \mathbb{G}/\mathbb{P} \).

The limit as \( t \to 0 \) of \( tf(x_{i,j}) \) equals \( x_i \). Hence, the exceptional divisor \( C_{i,j} \) of \( \gamma \) over \( x_{i,j} \) connects the points \( x_i \) to \( s \) under the map \( h' \). The image \( h'(C_{i,j}) \) thus represents an effective curve class containing the class \([P_i]\). By degree considerations over all the exceptional divisors \( C_{i,j} \), we conclude \( h'(C_{i,j}) \) is of curve class exactly \([P_i]\). As \( P_i \) is the unique \( \mathbb{C}^* \)-fixed curve of class \([P_i]\) connecting the points \( x_i \) and \( s \),
\[
h'(C_{i,j}) = P_i.
\]

We may assume \( S \) to be nonsingular (away from the original nodes of \( C \)) and each \( C_{i,j} \) to be a normal crossings divisor – possibly after further blow-ups and base changes altering only the special fiber over \( 0 \in \triangle \). We then conclude each \( C_{i,j} \) has a single component which is mapped to \( P_i \) isomorphically (and the other components of \( C_{i,j} \) are contracted).

After blowing-down the \( h' \)-contracted components of each \( C_{i,j} \), we obtain a map \( h'' : S'' \to \mathbb{G}/\mathbb{P} \) which is a family of nodal maps over \( \triangle \). The fiber of \( S'' \) over \( t = 0 \) is isomorphic to \( \tilde{C} \). Moreover, the condition \( \tilde{F}(C \subset \tilde{C}) = s \) follows directly from (3).

The limit stable map \( F \) is then simply obtained by stabilizing the map \( \tilde{F} \). We have carried out the stable reduction of the family of maps \( tf \) (see [FP]). \( \square \)
3. Connectedness

Let $[\mu]$ denote the point $[\mu : C \to X, p_1, \ldots, p_n] \in \overline{M}_{g,n}(X, \beta)$. The stable, pointed, modular graph $\tau$ with $H_2(X, \mathbb{Z})$-structure canonically associated to $[\mu]$ consists of the following data:

(i) The pointed dual graph of $C$:
   (a) The vertices $V_\tau$ correspond to the irreducible components of the curve $C$.
   (b) The edges correspond to the nodes.
   (c) The markings correspond to the marked points $p_i$.

(ii) The genus function, $g_\tau : V_\tau \to \mathbb{Z} \geq 0$, where $g_\tau(v)$ is the geometric genus of the corresponding component of $C$.

(iii) The $H_2(X, \mathbb{Z})$-structure, $\beta_\tau : V_\tau \to H_2(X, \beta)$, where $\beta_\tau(v)$ equals the $\mu$ push-forward of the fundamental class of the corresponding component of $C$.

Following [BM], define $M_{\tau,n}(X, \beta_\tau)$ to be the moduli space of maps $\mu$ together with an isomorphism of $\tau_\mu$ with a fixed stable graph $\tau$. The space $\overline{M}_{\tau,n}(X, \beta_\tau)$ is the compactification via stable maps where the vertices of $V_\tau$ may correspond to nodal curves. Note $M_{\tau,n}(X, \beta_\tau)$ may not be dense in $\overline{M}_{\tau,n}(X, \beta_\tau)$.

There is a canonical morphism

$$\pi_\tau : \overline{M}_{\tau,n}(X, \beta_\tau) \to \overline{M}_{g,n}(X, \beta).$$

As $\tau$ varies over possible graphs, the images of $\pi_\tau$ determine a (closed) decomposition of the moduli space of maps.

Let $\tau$ be a stable, pointed, modular graph with $H_2(G/P, \mathbb{Z})$-structure. The connectedness of $\overline{M}_{\tau,n}(G/P, \beta_\tau)$ will now be established.

Proof of Theorem 2. If $\beta_\tau = 0$, the irreducibility of $\overline{M}_{\tau,n}(G/P, \beta_\tau)$ is a direct consequence of the irreducibility of the corresponding stratum in $\overline{M}_{g,n}$ and the irreducibility of $G/P$. We may thus assume $\beta_\tau \neq 0$.

Fix the $\mathbb{C}^*$-action on $G/P$ as studied in Section 2. Consider an arbitrary point

$$[\mu] \in \overline{M}_{\tau,n}(G/P, \beta_\tau).$$

By the Kleiman-Bertini Theorem, a general $G$-translate $f$ of $\mu$ satisfies conditions (i-iii) of Section 3. As $G$ is connected, $[\mu]$ is connected to its general $G$-translate $[f]$.

The point $[f]$ is connected to the limit:

$$[F] = \lim_{t \to 0} [tf].$$
To prove the connectedness of $\overline{M}_{\tau,n}(G/P, \beta_\tau)$, it suffices to prove the set of limits $F$ lies in a connected locus of the moduli space. We will first construct the required connected locus of $\overline{M}_{\tau,n}(G/P, \beta_\tau)$.

The pair $(\tau, \beta_\tau)$ canonically determines a family of maps $\gamma_b$ with nodal domains over a base $b \in B$. For $v \in V_\tau$, let $\beta_\tau(v) = \sum_i a_i^v [P_i]$. Define the base space $B$ as follows:

$$B = \prod_{v \in V_\tau} \overline{M}_{g(v), \mathit{val}(v)+\sum_i a_i^v}.$$ 

where $\mathit{val}(v)$ is the valence of $v$ in $\tau$ (including nodes and markings). The extra $\sum_i a_i^v$ markings each correspond to a basis homology element -- with $a_j^v$ of these markings corresponding to $[P_j]$. The degenerate cases $\overline{M}_{0,1}$ and $\overline{M}_{0,2}$ in the product $B$ are taken to be points. $B$ is irreducible and hence connected.

For $b = \prod_v [b_v] \in B$, let

$$\gamma_b : D_b \to G/P$$

be defined as follows:

(i) $D_b$ is obtained by attaching the curves $b_v$ by connecting nodes as specified by $\tau$ and further attaching $\mathbb{P}^1$'s to each of the extra points $\sum_i a_i^v$.

(ii) For each subcurve $b_v \subset D_b$, $\gamma_b(b_v) = s$.

(iii) For each $\mathbb{P}^1$ corresponding to $[P_j]$, $\gamma_b(\mathbb{P}^1) \cong P_j$.

The family of maps $\gamma_b$ over $B$ then defines a morphism (via stabilization):

$$\epsilon : B \to \overline{M}_{\tau,n}(G/P, \beta_\tau).$$

Certainly the image variety $\epsilon(B)$ is connected.

By Proposition 2, the limit $F$ is simply the stabilization of $[\tilde{F}]$. Since $\tilde{F} = \gamma_b$ for some $b$, the set of limits $F$ lies in a connected locus of $\overline{M}_{\tau,n}(G/P, \beta_\tau)$. This concludes the proof of Theorem 2.

Theorem 1 is a special case of Theorem 2 (where $\tau$ has a single vertex). Corollary 2 is a simple consequences of Theorem 2.

**Proof of Corollary 2.** In the genus 0 case, $\tau$ is a tree with genus function identically zero. The moduli stack

$$\overline{M}_{\tau,n}(G/P, \beta_\tau)$$

is constructed as a fiber product over the evaluation maps obtained from the edges of $\tau$. We will prove $\overline{M}_{\tau,n}(G/P, \beta_\tau)$ is a nonsingular Deligne-Mumford stack by induction on the number of vertices of $\tau$. 


First, suppose $\tau$ has only 1 vertex $v$. Then, the moduli stack (3) is $\mathcal{M}_{0,\text{val}(v)}(G/P, \beta_\tau(v))$ – a nonsingular moduli stack by the convexity of $G/P$.

Next, let $\tau$ have $m$ vertices and let $v$ be an extremal vertex ($v$ is incident to exactly 1 edge). Let $p \in G/P$ be a point. By the Kleiman-Bertini Theorem,

\[(4) \quad ev_1^{-1}(p) \subset \mathcal{M}_{0,\text{val}(v)}(G/P, \beta_\tau(v))\]

is a nonsingular Deligne-Mumford stack for the general point $p$ (and hence every point $p$). Let $\tau'$ be the graph obtained by removing $v$ from $\tau$ and adding an extra marking corresponding to the broken node. The moduli stack (3) is fibered over

\[(5) \quad \mathcal{M}_{\tau',n'+1}(G/P, \beta'_{\tau'})\]

with fiber (4). As (3) is nonsingular by induction, the stack (3) is thus nonsingular. This completes the induction step.

Finally, since $\mathcal{M}_{\tau,n}(G/P, \beta_\tau)$ is a nonsingular and connected Deligne-Mumford stack, it is irreducible.

**4. Rationality**

We first review a basic rationality result proven in a sequence papers by Katsylo and Bogomolov [Ka], [Bog]. Let $V = \mathbb{C}^2$ be a vector space. Let $a_1, a_2, \ldots, a_n$ be a sequence of positive integers with $\sum a_i \geq 3$. Then, the quotient

\[(6) \quad \mathbb{P}(\text{Sym}^{a_1}V^*) \times \cdots \times \mathbb{P}(\text{Sym}^{a_n}V^*) \// \mathbb{PGL}(V)\]

is a rational variety – we may take any non-empty invariant theory quotient. Geometrically, the quotient (4) is birational to the moduli space quotient

\[(7) \quad M_{\tau,n,\sum a_i} / \Sigma a_1 \times \Sigma a_2 \times \cdots \times \Sigma a_n\]

where $\Sigma$ is the symmetric group. Essentially, the rationality of (4) is deduced from rationality in case $n = 1$ [Ka]. Proofs in the $n = 1$ case may be found in [Ka], [Bog].

We will also need the following simple Lemma.

**Lemma 6.** Let $W$ be any finite dimensional linear representation of $A$ where $A = \Sigma_2$ or $A = \Sigma_3$. Then, $W/A$ is rational.

**Proof.** By the complete reducibility of representations and the fact that a $\text{GL}$-bundle is locally trivial in the Zariski topology, it suffices to prove the Lemma in case $W$ is an irreducible representation. It is then
easily checked by hand the two irreducible representation of $\Sigma_2$ and the three irreducible representations of $\Sigma_3$ have rational quotients. □

**Proof of Theorem 3.** Fix the $\mathbb{C}^*$-action on $G/P$ as studied in Section 2. We first consider the moduli space

$$M = M_{0,n}(G/P, \beta = \sum_i a_i [P_i])$$

where the property

$$n + \sum_i a_i \geq 4$$

is satisfied.

Let $\tau$ be the graph with a single vertex $v$ with $n$ markings, and let $\beta_\tau(v) = \sum_i a_i [P_i]$. Let $\gamma_b$ over $B$ be the family of maps constructed canonically from $(\tau, \beta_\tau)$ in the proof of Theorem 2. The base $B$ is simply:

$$B = M_{0,n+\sum_i a_i}.$$  \hspace{1cm} (9)

The map $\gamma_b$ over a general point $b \in B$ has no map automorphisms (as $n + \sum_i a_i \geq 4$). Hence, the image $\epsilon(B)$ in $\overline{M}$ intersects the nonsingular (automorphism-free) locus of the moduli space $\overline{M}_0 \subset \overline{M}$. Let

$$\epsilon(B)^0 = \epsilon(B) \cap \overline{M}_0,$$

and let $B^0 = \epsilon^{-1}(\epsilon(B)^0)$. The map

$$B^0 \rightarrow \epsilon(B)^0$$

is simply a quotient of $B^0$ by the natural $\Sigma_{a_1} \times \cdots \times \Sigma_{a_n}$ action on (3). By the rationality result (5), $\epsilon(B)^0$ is rational.

Consider now the $\mathbb{C}^*$-action on $\overline{M}^0$ by translation. As $\overline{M}^0$ is a nonsingular, irreducible, quasi-projective variety, we may study the Bialynicki-Birula stratification of $\overline{M}^0$. By the proof of Theorem 2, $\epsilon(B)^0$ is a $\mathbb{C}^*$-fixed locus which contains the limit,

$$\lim_{t \to 0} t[f],$$

of the general point $[f] \in \overline{M}^0$. By [Bi], $\overline{M}^0$ is birational to an affine bundle over $\epsilon(B)^0$. Therefore, $\overline{M}$ is rational. The proof of Theorem 3 is complete in case (8) is satisfied.

Next, we will consider the case where the sum (8) is at most 3. In this case, the base $B$ is a point. If $\epsilon(B)$ lies in the automorphism-free locus, the previous argument proving the rationality of $M_{0,n}(G/P, \beta)$
is still valid. There are exactly four cases in which the point \( \epsilon(B) \) corresponds to a map with nontrivial automorphisms:

(i) \( n = 0, \beta = 3[P_i] \).
(ii) \( n = 0, \beta = 2[P_i] + [P_j], i \neq j \).
(iii) \( n = 0, \beta = 2[P_i] \).
(iv) \( n = 1, \beta = 2[P_i] \).

Here, the Deligne-Mumford stack structure of these moduli spaces is important. The automorphism group in case (i) is \( \Sigma_3 \) and in cases (ii-iv) is \( \Sigma_2 \). In each case, we will show the coarse moduli space \( \overline{M}_{0,n}(G/P, \beta) \) is birational to a quotient of a linear representation of the corresponding automorphism group.

Consider first the case (i): \( n = 0, \beta = 3[P_i] \). Let \( \epsilon(B) = [\gamma] \). Let \( \mu \) denote the unique 3-pointed stable map obtained from \( \gamma \) by marking each \( \mathbb{P}^1 \cong P_i \) by a point lying over \( x_i \). Certainly, \( [\mu] \in \overline{M}_{0,3}^0(G/P, \beta) \).

We will study:

\[
N \subset \overline{M}_{0,3}^0(G/P, \beta)
\]

where \( N \) is the component of the locus of transverse intersection of the three divisors \( \text{ev}_1^{-1}(A_i) \), \( \text{ev}_2^{-1}(A_i) \), and \( \text{ev}_3^{-1}(A_i) \) containing \( [\mu] \). The torus \( \mathbb{C}^* \) acts on \( N \) by translation. By an argument exactly parallel to the flow result of Proposition 2, we deduce

\[
\lim_{t \to 0} t[f] = [\mu]
\]

for a general element \( [f] \in N \). As \( N \) is a nonsingular, quasi-projective scheme, Theorem 2.5 of [Bi] implies that \( N \) is \( \mathbb{C}^* \)-equivariantly birational to the tangent \( \mathbb{C}^* \)-representation at \( [\mu] \).

There is a \( \Sigma_3 \)-action on \( N \) by permutation of the markings. The \( \mathbb{C}^* \) and \( \Sigma_3 \) actions commute. A slightly refined version of Theorem 2.5 of [Bi] shows \( N \) is \( \mathbb{C}^* \times \Sigma_3 \)-equivariantly birational to the tangent \( \mathbb{C}^* \times \Sigma_3 \)-representation at \( [\mu] \). Lemma \( \mathbb{L} \) below explains the refinements of the results of [Bi] needed here. \( N/\Sigma_3 \) is birational to \( \overline{M}_{0,3}^0(G/P, \beta) \).

Hence, by Lemma \( \mathbb{L} \), Theorem 3 is proven in case (i).

A similar strategy is used in cases (ii-iv). In each of these cases, let \( \epsilon(B) = [\gamma] \) and let \( [\mu] \) denote the rigidification by adding 2 new markings \( \bullet, \bullet' \) which lie over \( x_i \). The locus \( N \) is chosen as the corresponding transverse intersection locus of \( \text{ev}_1^{-1}(A_i) \) and \( \text{ev}_2^{-1}(A_i) \) in the maps space with the new markings. \( N \) is then \( \mathbb{C}^* \times \Sigma_2 \)-equivariantly birational to the tangent \( \mathbb{C}^* \times \Sigma_2 \)-representation of \( N \) at \( [\mu] \) by the refined Lemma \( \mathbb{L} \). Theorem 3 is then a consequence of Lemma \( \mathbb{L} \) since \( N/\Sigma_2 \) is birational to the moduli space of maps considered in the case. \( \square \)
Lemma 7. Let $A$ be a finite group. Let $S$ be a nonsingular, irreducible, quasi-projective scheme with a $\mathbb{C}^* \times A$-action and a $\mathbb{C}^* \times A$-fixed point $s \in S$. Let $T_s$ denote the $\mathbb{C}^* \times A$-representation on the tangent space at $s$. Suppose the $\mathbb{C}^*$-action is fully definite at $s$. Then, there is $\mathbb{C}^* \times A$-equivariant isomorphism between an open set of $(S, s)$ and $(T_s, 0)$.

Proof. We note $\mathbb{C}^* \times A$ is a linearly reductive group. By Theorem 2.4 of [Bi] for linearly reductive group actions, we may find a third nonsingular irreducible pointed space $(Z, z)$ with a $\mathbb{C}^* \times A$-action and equivariant, étale, morphisms:

$$\pi_1 : (Z, z) \to (S, s),$$

$$\pi_2 : (Z, z) \to (T_s, s).$$

In the proof of Theorem 2.5 of [Bi], such morphisms $\pi_1$ and $\pi_2$ are proven to be open immersions by a study of only the $\mathbb{C}^*$-action. Hence, the morphisms $\pi_1$ and $\pi_2$ are open immersions in our case. By the full definiteness of the $\mathbb{C}^*$-representation on $T_s$, the morphism $\pi_2$ is then an isomorphism. \qed

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