CATALAN TRIANGLES AND TIED ARC DIAGRAMS

F. AICARDI

Abstract. The Catalan triangle, as well as a Fuss-Catalan triangle, enter a problem of counting particular tied arc diagrams. This setting allows us to prove some combinatorial properties of these triangles.

1. Introduction

The Catalan numbers $C_n := \frac{1}{n+1} \binom{2n}{n}$ can be decomposed into $n$ integers defined by a recurrence: they form the so-called Catalan triangle (see [6, 7]). Other decompositions exist for the Fuss-Catalan numbers $A_n(p, q) := \frac{q}{pq+q} \binom{pn+q}{n}$ (see [4]).

Since the seminal paper [1], several tied knot algebras of different kinds were introduced and studied. In particular, for some families of such algebras, a diagrammatic interpretation of the generators leads to the definition of tied arc diagrams.

The tied arc diagrams can be obtained recursively. It turns out that their number, in the case of the so called tb-diagrams, is calculated by means of the Catalan triangle. In the case of the so called ta-diagrams the recurrence yields a Fuss-Catalan triangle, whose rows sum to the Fuss-Catalan numbers $A_n(4, 1)$. We use the tb-diagrams to prove a combinatorial result (Theorem 1), that is used in [2] to calculate the dimension of a tied-tangle monoid.

The combinatorial results of Theorems 2, proved by means of the ta-diagrams, are used in [3] to get the dimensions of tied Temperley-Lieb algebras.

2. Results

2.1. The Catalan Triangle.

Definition 1. [6] The integers $T(n, k)$ are defined for $n \geq 0$, $0 \leq k \leq n$ by the initial conditions,

\begin{equation}
T(0, 0) = 1, \quad T(n, 0) = 1, \quad T(n, n) = 0, \quad n > 0;
\end{equation}

and the recurrence

\begin{equation}
T(n, k) = T(n, k - 1) + T(n - 1, k), \quad n > 1, \quad 0 < k < n.
\end{equation}
Observe that the above equations imply

\[ T(n, k) = \sum_{j=0}^{k} T(n-1, j) \quad n > 1, \quad 0 < k < n. \]

Here the Catalan triangle \( T(n, k) \), for \( n \leq 8, 0 \leq k \leq n \)

\[
\begin{array}{cccccccc}
0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 \\
2 & 1 & 2 & 2 & 0 \\
3 & 1 & 3 & 5 & 5 & 0 \\
4 & 1 & 4 & 9 & 14 & 14 & 0 \\
5 & 1 & 5 & 14 & 28 & 42 & 42 & 0 \\
6 & 1 & 6 & 20 & 48 & 90 & 132 & 132 & 0 \\
7 & 1 & 7 & 27 & 75 & 165 & 297 & 429 & 429 & 0 \\
8 & 1 & 8 & 35 & 105 & 231 & 429 & 714 & 1140 & 1140 & 0
\end{array}
\]

**Proposition 1.** \([6]\) The Catalan triangle satisfies, for every \( n > 0 \):

\[ T(n, n-1) = C_{n-1}, \quad \sum_{k=0}^{n} T(n, k) = C_n. \]

**Theorem 1.** The Catalan triangle satisfies, for every \( n > 0 \):

\[ \sum_{k=0}^{n} T(n, k) 2^{n-1-k} = \binom{2n-1}{n}. \]

2.2. A Fuss-Catalan triangle.

The triangle here defined by recurrence was already obtained in \([4]\) by another procedure.

**Definition 2.** The integers \( F(n, k) \) are defined for \( n \geq 0, 0 \leq k \leq n \) by the initial conditions

\[ F(0, 0) = 1; \quad F(n, n) = 0, \quad \text{for } n > 0; \]

and the recurrence for \( n > 0 \) and \( 0 \leq k < n \):

\[ F(n, k) = \sum_{j=0}^{k} \binom{k-j+2}{2} F(n-1, j). \]

Here the triangle \( F(n, k) \) for \( n \leq 8, 0 \leq k \leq n \)

\[
\begin{array}{cccccccc}
0 & 1 & 1 & 0 \\
1 & 1 & 2 & 2 & 0 \\
2 & 1 & 3 & 5 & 5 & 0 \\
3 & 1 & 4 & 9 & 14 & 14 & 0 \\
4 & 1 & 5 & 14 & 28 & 42 & 42 & 0 \\
5 & 1 & 6 & 20 & 48 & 90 & 132 & 132 & 0 \\
6 & 1 & 7 & 27 & 75 & 165 & 297 & 429 & 429 & 0 \\
7 & 1 & 8 & 35 & 105 & 231 & 429 & 714 & 1140 & 1140 & 0 \\
8 & 1 & 9 & 42 & 132 & 297 & 612 & 1140 & 2079 & 32890 & 0
\end{array}
\]
Theorem 2. The integers \( F(n, k) \) satisfy, for every \( n > 0 \)

\[
\sum_{k=0}^{n} F(n, k) = A_n(4, 1); 
\]

\[
\sum_{k=0}^{n} F(n, k) \binom{n - k + 3}{3} = A_n(4, 4); 
\]

and

\[
F(n, n - 1) = A_{n-1}(4, 3) 
\]

where \( A_n(4, 1), A_n(4, 4) \) and \( A_n(4, 3) \) are respectively the Fuss-Catalan numbers \( \frac{\binom{4n+1}{n}}{4n+1}, \frac{\binom{4n+4}{n}}{4n+4} \) and \( \frac{\binom{4n+3}{n}}{4n+3} \).

Remark 1. Statements \([6]\) and \([8]\) can be obtained also from \([4]\). However, we give here a proof in terms of ta-diagrams for completeness.

3. Proof of Proposition 1 in terms of arc diagrams

Proposition \([1]\) says that the numbers \( T(n, k) \) form a particular partition of \( C_n \). We will consider one of the definitions of \( C_n \), namely: \( C_n \) is the number of semicircle diagrams with \( n \) semicircles. More precisely:

Definition 3. An \( n \)-arcdiagram is a planar diagram of \( n \) non intersecting semicircles or arcs with end points on a straight line. All arcs lie on one of the closed half–planes defined by the line.

Lemma 1. The number of \( n \)-arcdiagrams is the Catalan number \( C_n \)

Proof. Recall that the Catalan number \( C_n \) is, among others, the number of expressions containing \( n \) pairs of parentheses which are correctly matched. E.g., for \( n = 3 \), \( C_n = 5 \):

\[
( ) ( ) ( ), ( ( ) ) ( ), ( ) ( ( ) ), ( ( ) ( ) ), ( ( ) ( ) ).
\]

To obtain the corresponding five arcdiagrams, it is sufficient to substitute every pair of open-closed parentheses with a semicircle in this way:

\[
( ) \rightarrow ( \bigcirc ) \rightarrow \bigcirc \bigcirc
\]

This defines evidently a bijection, since from an \( n \)-arcdiagram we get viceversa a unique set of \( n \) pairs of parentheses.

Observe that an \( n \)-arcdiagram divides the upper half plane in \( n + 1 \) regions, of which only one, said \( U \), is unbounded.

Definition 4. A block of an arcdiagram, is a semidisc defined by an arc at the boundary of \( U \). A block may contain other arcs.
Let us denote $D^n_k$ the number of the $n$-arcdiagrams having $k$ blocks.

**Lemma 2.** $D^n_k = T(n, n-k)$.

*Proof.* Evidently, $D^n_n = 1$, i.e., there is one diagram with $n$ blocks, each one with a sole semicircle, and $D^n_0 = 0$, since a diagram with no blocks has no arcs; so, in particular, $D^n_0 = 1$. The initial conditions (1) are fulfilled. We will verify the recurrence (3) that reads

\begin{equation}
D^n_k = \sum_{j=k-1}^{n-1} D^n_{j-1}.
\end{equation}

Consider the set of arcdiagrams with $n-1$ arcs. It is partitioned into $n-1$ parts corresponding to the number $j$ of blocks, so that $C_{n-1} = \sum_{j=1}^{n-1} D^n_{j-1}$. Suppose that we want to get, starting from this set, the set of arcdiagrams with $n$ arcs that have exactly $k$ blocks, by inserting in a suitable way the $n$-th arc. Let $d$ be one of such diagrams. If the number $j$ of blocks of $d$ is less than $k-1$, the new diagram cannot have $k$ blocks by adding a new arc. Therefore we start with diagrams with $j \geq k-1$. We put the left endpoint of the $n-th$ arc at left of the diagram $d$. If $d$ has $k-1$ blocks, then the right point of the new arc will be at left of the first block of $d$. If $d$ has $j > k-1$ arcs, the end point will be after the $j-k+1$th block, till $j = n-1$. To finish, we have to prove that in this way we get all diagrams with $n$ arcs and $k$ blocks, and that all diagrams so obtained are all different. Firstly, for every $j$ we consider a set of diagrams with $j$ blocks and $n-1$ arcs, that are all different, and hence the procedure we use produces different diagrams. Two diagrams obtained starting from two sets with different values of $j$ cannot coincide since the first blocks of them contain different quantities of blocks of the original diagrams. Suppose now that a diagram $d'$ with $k$ blocks and $n$ arcs is not reached by the above procedure. The diagram $d'$ contains one first block. Removing the first arc, we get a sequence (possibly empty) of $h$ blocks $b_1, \ldots, b_h$ $0 \leq h \leq n-k$, plus $k-1$ blocks $b_{h+1}, \ldots, b_{h+k}$. The sequence of blocks $b_1, \ldots, b_{h+k}$ form a diagram with $n-1$ arcs and with a number of blocks at least $k-1$. Then it is impossible that it has been missed by the procedure. \hfill \square

*Proof of Proposition 1.* By Lemma 2, we have to verify that $D^n_1 = C_{n-1}$. Indeed, this is what says Eq. (9) with $k = 1$, by using Lemma 1. As for the second equation, we have by Lemma 1, $\sum_{k=1}^{n} D^n_k = C_n$, that we can rewrite as

$$\sum_{k=0}^{n-1} D^n_{n-k} = C_n.$$  

Then Proposition 1 follows from Lemma 2, since $D^n_0 = 0$. \hfill \square

4. **TB-diagrams and proof of Theorem 1**

A *tied* arcdiagram is an arcdiagram with *ties* that may connect each other two arcs, avoiding selfintersections and intersections with the arcs. See Figure 1.
We firstly consider particular tied arcdiagrams named tb-diagrams.

**Definition 5.** We call *top arc* of an arcdiagram $d$ a semicircle bounding a block of $d$. A diagram has at least one top arc. The top arcs are naturally ordered from left to right.

**Definition 6.** A *tb-diagram* is a tied arcdiagram whose ties may exist only in the unbounded region of the complement to the diagram and can connect only successive top arcs.

**Lemma 3.** The number of tb-diagrams with $n$ arcs and $k$ blocks is $2^{k-1}D_n^k$.

**Proof.** It is an immediate consequence of Definition 6, since there are $k - 1$ pairs of successive top arcs admitting a tie in between. \qed

We will define a bijection between the tb-diagrams with $n$ arcs, and the $n$-combinations of $2n - 1$ objects. Then Theorem 1 will follow from Lemma 2 and Lemma 3.

Let $d$ be a tb-diagram with $n$ arcs. Label by $0, 1, \ldots, 2n - 1$ the endpoints of its arcs. We associate to $d$ one $n$-combination of $\{1, 2, \ldots, 2n - 1\}$ as follows.

**Procedure 1.**
- If $d$ has no ties, then take the $n$ labels of the right endpoints of the $n$ arcs.
- If $d$ has ties, then take the labels of the the right endpoints of the arcs of the first block, and the left ones of the arcs inside every block connected to the preceding block by a tie.

**Example.** In Figure 2 see two tb-diagrams with 6 arcs, and the corresponding combinations of the integers from 1 to 11.

**Lemma 4.** Every $n$-combination of $\{1, 2, \ldots, 2n - 1\}$ is obtained by Procedure 1 starting from one and only one tb-diagram.
Proof. Consider firstly a tb-diagram without ties and a function \( f \) defined on the points \( i \), taking value \(-1\) on the left endpoint and \(+1\) on the right endpoint of each arc. Define the function \( F \) as

\[
F(p) := \sum_{j=0}^{p} f(j)
\]

Let \( \{2k_i - 1\}_{i=1}^{m} \) be the zeroes of \( F \). Now we observe the trivial facts:

1. \( f \) has at least one zero for \( k_m = n \);
2. the points \( 2k_i - 1 \) are the right endpoints of the top arcs of \( d \), so \( d \) has \( m \) blocks;
3. the point \( p_0 \) and and the points \( p_{2k_i} \) are the left endpoints of the top arcs;

Consider the arc diagrams inside any block of \( d \), obtained by removing the top arc. The function \( f \) takes value \(-1\) at the first points \( p_{2k_i+1} \) and \(+1\) at the last points \( 2k_i+1 \). Then, by relabeling the indices \( (2k_i + 1, \ldots, 2k_{i+1}) \) by \( (0, \ldots, 2n-1) \), we can repeat all the preceding observations, and so on for every subdiagram inside the blocks of the arc diagrams just considered.

It is therefore evident that \( f \) defines uniquely the arc diagram, and that the integers \( i \) such that \( f(i) = +1 \) define one \( n \)-combination of \( \{1, 2, \ldots, 2n-1\} \).

It is also clear that the \( n \)-combinations obtained this way are not all: for instance, all such combinations contain the integer \( 2n-1 \), since for every arc diagram \( f(2n-1) = +1 \).

We observe now that the function \(-f\) defines the same arc diagram as \( f \), simply exchanging left endpoints with right endpoints. Moreover, every function \( f' \) obtained from \( f \) by reversing the values inside one or more blocks, still defines the same arc diagram as \( f \): the value \(+1\) will be assigned to the left endpoints inside the blocks where \( f' = -f \).

![Figure 3](image-url)

**Figure 3.** The functions \( f \) and \( F \) of the left tb-diagram of Figure 2

Thus Procedure 1 associates different \( n \)-combinations to all \( 2^{k-1} \) tb-diagrams obtained by adding ties to an arc diagram \( d \) with \( k \) blocks. By the preceding observations, the functions \( f \) associated to these tb-diagrams differ each other only by a reversion of sign inside some blocks of \( d \).

Now we assign uniquely one tb-diagram to a chosen \( n \)-combination.

Procedure 2.
Let us start with a \( n \)-combination \( C \) of \( \{1, 2, \ldots, 2n - 1\} \). Consider \( 2n \) points on a line labeled by \( \{0, 1, \ldots, 2n - 1\} \) and a function \( f \) defined on these points:

\[
f(i) = +1 \quad \text{if} \ i \in C, \quad f(i) = -1 \quad \text{otherwise}
\]

Observe that \( f(0) = -1 \). Define \( F(p) = \sum_{i=0}^{p} f(i) \). Since the quantity of zeroes and ones is the same, surely \( F(p) = 0 \) at least at \( p = 2n - 1 \). Suppose there are no other zeroes. It means that \( F \) is negative till \( i = 2n - 2 \), and that \( f(2n - 1) = +1 \). I.e., \( 2n - 1 \in C \). So, \( d \) has only one block, and \( (0, 2n - 1) \) are the endpoints of the unique top arc. Remove this top arc and define \( F^1(k) = \sum_{i=1}^{k} f(i) \). Note that \( f(1) = -1 \), otherwise \( 1 \) should be a zero of \( F \). The zeroes of \( F^1 \) defines the right endpoints of the top arcs inside the block of \( d \). Observe that \( F^1(2n - 2) = 0 \), and if \( k \) is a zero of \( F^1 \), then \( f(k + 1) = -1 \), otherwise \( k + 1 \) should be a zero of \( F \). We proceed this way inside the blocks defined by the top arcs, determining all new top arcs, and so on inside all corresponding blocks, concluding that at the left endpoint \( l \) of every arc, \( f(l) = -1 \), since every arc is the top arc of one block of some subdiagram of \( d \). The diagram \( d \) is thus completely defined. Observe that in this case \( d \) and has no ties.

Suppose now that \( F(i) = 0 \) for \( i = 2m - 1, \ m < n \). The points \( (0, 2m - 1) \) define the top arc of the first block of \( d \), and inside it we proceed as in the preceding case, putting all arcs, that result to have left endpoints where \( f = -1 \). Observe that this follows only from the fact that \( f(0) = -1 \) by hypothesis.

We proceed now to define the second block of \( d \). If \( f(2m) = -1 \), and \( F(2r - 1) = 0 \) for \( m < r \leq n \), then \( f(2r - 1) = +1 \), and we proceed as previously inside the second block. But if \( f(2m) = +1 \), and \( F(2r - 1) = 0 \) for \( m < r \leq n \), then \( f(2r - 1) = -1 \), so that on the endpoints of the second top arc \( (2m, 2r - 1) \), \( f \) takes value +1 on the left, and -1 on the right. Then we proceed as in the preceding case but exchanging +1 with -1, so that the second block is uniquely defined, and the left endpoints of all arcs inside it have \( f = +1 \). In this case we put a tie between the first and the second block of \( d \).

We proceed the same way to define the successive blocks of \( d \), tied or not with the preceding one. The \( \text{tb-diagram} \) \( d \) is uniquely defined by the \( n \)-combination.

It is clear that, having obtained by Procedure 2 a \( \text{tb-diagram} \) \( d \) from a chosen \( n \)-combination, the same combination is obtained by applying to \( d \) Procedure 1. So we have proved that by Procedure 1 all \( n \)-combinations are attained.

5. Ta-diagrams

In this section we introduce another class of tied arc diagrams, called ta-diagrams.

In fact, ta-diagrams are equivalence classes of tied arc diagrams, the equivalence relation being given below.

We order the arcs of a \( n \)-arc diagram according to the order of their right endpoints on the line from left to right, see Figure 4.

In a ta-diagram with arcs \( a_1, \ldots, a_n \), the ties define a partition of the set of arcs in this way: two arcs connected by a tie belong to the same part of the partition. We shall
denote this partition by ta-partition, and for short we write only the indices 1, \ldots, n of the arcs.

**Definition 7.** Two ta-diagrams are equivalent if they coincide forgetting the ties, and the ta-partition defined by the ties of one diagram coincides with that of the other.

**Example.** See Figure 4.

![Figure 4. Two equivalent ta-diagrams with ta-partition \{\{1\}, \{2, 3, 4, 6\}, \{5\}\}](image)

We say that $m > 2$ ties form a cycle, if there is a sequence of $m$ arcs $a_{i_1}, \ldots, a_{i_m}$ such that there is a tie between $a_{i_k}$ and $a_{i_{k+1}}$, for $k < m$, and there is a tie between $a_{i_m}$ and $a_{i_1}$.

**Definition 8.** A ta-diagram is said irreducible if between two arcs there is at most one tie, and the ties do not form cycles. Otherwise, the ta-diagram is said reducible.

**Proposition 2.** Every ta-diagram is equivalent to an irreducible one.

**Proof.** If a ta-diagram is reducible, then it becomes irreducible by canceling all ties between two arcs but one, and by canceling just one tie, for every cycle of $m$ arcs connected by $m$ ties. By these canceling operations the ta-partition defined by the ties is preserved.  

**Example.** The left ta-diagram of Figure 4 is reducible, the right diagram is irreducible.

Observe that an arc $a$ divides the half-plane in two parts that we call the interior and the exterior of $a$.

**Definition 9.** An arc $a$ is inside an arc $b$ if it lies in the interior of $b$, otherwise it is outside $b$.

**Definition 10.** A ta-diagram is standard if it is irreducible and in each part consisting of $m$ arcs with ordered indices $i_1 < i_2 < \cdots < i_m$, every arc with index $i_k$ is connected by a tie at most to one arc of index $i_r > i_k$, and at most to two arcs of indices $i_p < i_q < i_k$, with $a_{i_p}$ outside $a_{i_k}$, and $a_{i_q}$ inside $a_{i_k}$.

**Example.** The irreducible ta-diagram in Figure 4 is not standard, since the arc $a_6$ is connected by ties to $a_3$ and $a_4$, both outside $a_6$. The corresponding standard diagram is in Figure 5, at left.
Proposition 3. Every ta-diagram is equivalent to a standard ta-diagram.

Proof. Consider an irreducible ta-diagram. Suppose that the diagram is not standard. Then, let the arc $a_h$ be the arc with the minimum index for which the conditions of being standard are not fulfilled. There are three possibilities:

1) $a_h$ is connected by ties to two or more arcs inside it;
2) $a_h$ is connected by ties to two or more arcs with higher indices;
3) $a_h$ is connected by ties to two or more arcs with lower indices outside it.

We see how these situations can be corrected to obtain a standard diagram, without affecting the ta-partition.

Observe that in the case (1), there are $m \geq 2$ arcs $a_{i_1}, \ldots, a_{i_m}$ inside $a_h$ tied with $a_h$. The arcs are ordered by their increasing indices $i_1 < \cdots < i_m$. (in Figure 6 $m = 2$). We cancel $m - 1$ ties between $a_{i_j}$ and $a_h$ for $j < m$ and we put a tie from $a_{i_j}$ and $a_{i_{j+1}}$ for $i = 1, \ldots, m - 1$. In the case (2), we order as before the $m \geq 2$ arcs connected to $a_h$, we cancel the ties but that from the first arc to $a_h$ and we put a tie from the other successive arcs, so that it remains only the tie from $a_h$ to the arc with minimum index.

In the case (3) we do the same as in case (2).

Now, observe that the part of the ta-partition containing $a_h$ remains unchanged by the above operations and the ta-diagram obtained is irreducible. It remains to prove that we can always put the new ties without crossing other arcs or other ties.

Consider the case (1). The arcs $a_{i_j}$ and $a_{i_{j+1}}$ were initially connected by a tie to the arc $a_h$. See Figure 6. This means that both these arcs were not inside other arcs between them and $a_h$. Then a tie between $a_{i_j}$ and $a_{i_{j+1}}$ does not cross any arc. This tie could cross another tie between an arc $a_k$ with $i_j < k < i_{j+1}$ and another arc with index $l$ with $l < i_j$ or $i_{j+1} < l < h$, belonging to another part of the ta-partition. (In Figure 6 hypothetical ties of such kind are shown as dotted lines.) But such a tie does not exist since it should cross the tie connecting $a_h$ either to $a_{i_j}$ or to $a_{i_{j+1}}$, see Figure 6.
Figure 6. Creating a new tie between the arcs $a_{i_1}$ and $a_{i_2}$

The cases (2) and (3) are similar.

**Definition 11.** A block of a ta-diagram, is a subdiagram, having either only one top arc, or a number $m > 1$ of top arcs such that the first and the the $m$-th belong to the same part of the partition.

**Example.** In Figure 5 the ta-diagram at left has a sole block, the ta-diagram at right has two blocks. In Figure 7 the ta-diagram at left has two blocks.

In the next section we will build the standard ta-diagrams with $n$ arcs from those with $(n - 1)$ arcs. Taking a ta-diagram $d$ with $n - 1$ arcs, we will add a new arc $a_n$ and a possible tie from $a_n$ to some arc of $d$. Observe that this arc is the top arc with maximum index of a bloc of $d$. We will say for short that the tie connects $a_n$ to that block. See Figure 7.

![Figure 7. Left. A standard ta-diagram $d$ with 6 arcs and two blocks, $b$ and $b'$. Right. A tie connects the arc $a_7$ to the block $b$.](image)

6. **Proof of Theorem 2**

We denote by $A_n$ the set of classes of ta-diagrams with $n$ arcs and by $A_n$ the number of standard ta-diagrams. By Proposition 3, $A_n = |A_n|$.

We denote by $B_n^j$ the set of classes of ta-diagrams having $n$ arcs and $j$ blocks, and $B_n^j$ its cardinality. Then

\[ A_n = \sum_{j=1}^{n} B_n^j. \]
Remark 2. Observe that $A_0$ contains the diagram without arcs, and $B_0^0$ the diagram with no arcs and no blocks, whereas $B_0^n$ is empty, for $n > 1$. So

$$A_0 = 1, \quad B_0^0 = 1, \quad B_0^n = 0 \quad \text{for} \quad n > 0$$

The proof of Theorem 2 is subdivided in two parts. In the first part we prove that

$$A_n = A_{n-1}(4, 1), \quad B_1^1 = A_{n-1-1}(4, 3).$$

In the second part we prove that

$$F(n, k) = B_{n-k}^n.$$

6.1. Part 1. To prove Equations (12), we firstly observe that $A_n$ can be written through the $A_k$ and the $B_{n-k}^1$, for $0 \leq k < n$.

**Lemma 5.** For every $n > 0$, the numbers $A_k$ and $B_{n-k}^1$, for $k = 0, \ldots, n - 1$, satisfy the equation:

$$A_n = \sum_{k=0}^{n-1} B_{n-k}^1 A_k.$$

**Proof.** Consider any standard ta-diagram $d$ in $A_n$, and decompose it in two ta-diagrams $D'$ and $D''$: $D''$ consists of the last block of $d$, $D'$ is the remaining ta-diagram, which is void when $d$ has only one block, i.e. when $D'' = d$. Consider now the ta-diagrams for which $D'$ and $D''$ contain respectively $k$ and $n - k$ arcs. Their number is equal to the number of ta-diagrams with $k$ arcs, $A_k$, multiplied by the number of ta-diagrams with one block and $n - k$ arcs, i.e. $B_{n-k}^1$. Since for every standard ta-diagram $d$ in $A_n$ this decomposition is unique, we get

$$A_n = A_0 B_1^n + A_1 B_{n-1}^1 + A_{n-2} B_2^1 + \cdots + A_{n-1} B_1^1$$

i.e., eq. (14). \qed

In order to use Equation (14) as a recurrence to calculate $A_n$, we need the following lemma.

**Lemma 6.** The number $B_{n}^1$ of standard ta-diagrams with $n$ arcs and only one block satisfies

$$B_{n}^1 = \sum_{k=0}^{n-1} A_k \sum_{j=0}^{n-1-k} A_j A_{n-1-k-j}.$$ 

**Proof.** Observe that formula (15) says that $B_{n}^1$ equals the sum of all products of three ordered factors $A_i A_j A_k$ such that $i + j + k = n - 1$. So, we prove that, for any ordered triple of non negative integers $(i, j, k)$ such that $i + j + k = n - 1$, we can associate to every triple of standard ta-diagrams in $A_i \times A_j \times A_k$, one and only one standard diagram in $B_{n}^1$, and that each ta-diagram in $B_{n}^1$ can be uniquely decomposed in one triple of ta-diagrams in $A_i \times A_j \times A_k$ for some ordered triple $(i, j, k)$ such that $i + j + k = n - 1$. 

Let us take three standard ta-diagrams, \(D_1 \in \mathcal{A}_i, D_2 \in \mathcal{A}_j, \) and \(D_3 \in \mathcal{A}_k, \) with \(i + j + k = n - 1.\) The diagram in \(\mathcal{A}_0\) is a diagram without arcs. We define a standard ta-diagram in \(B^{1}_n\) the following way: we put \(D_2\) at right of \(D_1,\) and \(D_3\) at right of \(D_2.\) We get a diagram in \(\mathcal{A}_{n-1}\) which is standard. We add one \(n\)-th arc, \(a_n,\) with right endpoint at right of \(D_3,\) and left endpoint between \(D_1\) and \(D_2.\) Then we put one tie from \(a_n\) to the first block of \(D_1,\) if \(D_1\) is not void, and one tie from \(a_n\) to the last block of \(D_2,\) if \(D_2\) is not void. Such operations guarantee that the new diagram is standard, is uniquely defined starting by \(D_i, D_j, D_k\) and has one block by construction. See Figure 8.

\[\text{Figure 8. A standard ta-diagram with one block built from three standard diagrams } D_1, D_2, D_3 \in \mathcal{A}_2. \text{ Ties connect the arc } a_7 \text{ to the first block of } D_1 \text{ and to the last block of } D_2.\]

Now, consider a standard diagram \(d \in B^{1}_n.\) Take the last top arc of the unique block of \(d,\) i.e. \(a_n.\) If the top arc is unique, then \(i = 0.\) Otherwise, \(a_n\) is connected by a tie to another top arc. By canceling this tie, we get a ta-diagram with \(i < n\) arcs at left of \(a_n.\) We call this diagram \(D_1.\) Since \(d\) is standard, there is at most one tie from \(a_n\) to an arc \(a_m\) inside \(a_n.\) If there is no such a tie, then \(j = 0.\) Otherwise, consider the ta-diagram \(D_2\) consisting of all arcs with indices from \(i + 1\) to \(m,\) i.e. \(j = m - i.\) Observe that all these arcs are between the left endpoint of \(a_n\) and the right endpoint of \(a_m,\) since \(a_m\) is connected by a tie to \(a_n.\) The arcs from \(a_m\) to \(a_{n-1}\) form the ta-diagram \(D_3 \in \mathcal{A}_k,\) \(k = n - 1 - m.\) If \(m = n - 1,\) then \(k = 0.\) See Figure 9. Observe that the diagrams \(D_1, D_2, D_3\) are in this way uniquely defined.

\[\text{Figure 9. The standard ta-diagram at left is decomposed in } D_1 \in \mathcal{A}_4, D_2 \in \mathcal{A}_2, D_3 \in \mathcal{A}_0.\]

Now, we recall in the next two lemmas [5] two known properties of the Fuss-Catalan numbers.
Lemma 7. \( A_n(p, q) = A_n(p, q - 1) + A_{n-1}(p, p + q - 1) \).

Lemma 8. \( A_n(p, r + s) = \sum_{i=0}^{n} A_i(p, r)A_{n-i}(p, s) \).

By Lemma 7 for \( p = 4, q = 1 \) we have
\[
A_n(4, 1) = A_n(4, 0) + A_{n-1}(4, 4) = A_{n-1}(4, 4),
\]
since \( A_n(p, 0) = 0 \) for every \( p \).

By Lemma 8 we obtain, for \( p = 4, r = 1 \) and \( s = 3 \)
\[
A_{n-1}(4, 4) = \sum_{i=0}^{n-1} A_i(4, 1)A_{n-1-i}(4, 3).
\]

Then, by Eq. (16)
\[
A_n(4, 1) = \sum_{i=0}^{n-1} A_i(4, 1)A_{n-1-i}(4, 3).
\]

Using again Lemma 8 for \( p = 4, r = 1 \) and \( s = 2 \) we get
\[
A_m(4, 3) = \sum_{k=0}^{m} A_k(4, 1)A_{m-k}(4, 2);
\]
and again for for \( p = 4, r = 1 \) and \( s = 1 \),
\[
A_{m-k}(4, 2) = \sum_{j=1}^{m-k} A_j(4, 1)A_{m-k-j}(4, 1).
\]

Therefore,
\[
A_m(4, 3) = \sum_{k=0}^{m} A_k(4, 1) \sum_{j=1}^{m-k} A_j(4, 1)A_{m-k-j}(4, 1).
\]

Observe now that Equations (14) and (15), with initial condition (11), define by recursion the values \( B_n^1 \) and \( A_n \), for every integer \( n > 0 \).

Notice, moreover, that \( A_0(4, 1) = 1 \), and \( A_0(4, 3) = 1 \). Comparing now Equations (14) and (15) respectively with Equations (17) and (18), we get Equations (12). So, part 1 is proved.

6.2. Part 2. To prove Equation (13), we prove that the integers \( B_{n-k}^m \) satisfy the same recurrence of \( F(n, k) \).

I.e., for a given \( n \), we write the integers \( B_n^k \), for \( 0 < k \leq n \) in term of the integers \( B_{n-1}^j \), for \( k - 1 \leq j \leq n - 1 \).

Firstly, we prove this lemma
Lemma 9. We have

\[ A_n = \sum_{k=1}^{n-1} \binom{k+3}{3} B_{n-1}^k. \]

Proof. When the ta-diagram has only one arc, \(a_1\), the ta-partition is \(\{\{1\}\}\), and there is only one block, with top arc \(a_1\). So,

\[ B_1^1 = 1, \quad A_1 = 1. \]

We define a procedure that generates \(\binom{k+3}{3}\) standard ta-diagrams with \(n\) arcs, starting by a standard ta-diagram in \(B_{n-1}^k\). We prove that every standard ta-diagram in \(A_n\) is uniquely generated by this procedure from one standard diagram in \(B_n^k\), for some \(k\).

Let \(d\) be a standard ta-diagram in \(B_{n-1}^k\) with \(k\) blocks. We represent a block by a black half disc. We put at right of \(d\) the right endpoint of the \(n\)-th arc \(a_n\). The left end point of this arc can be put in \((k+1)\) places, numbered by \(i = 0, 1, \ldots, k\); namely: the first at left of \(d\), then in \((k-1)\) places between successive blocks, and finally at right of \(d\). See Figure 10.

For a given position \(i\) of the left end point of \(a_n\), we consider now the possible ties form \(a_n\) to the blocks. When we say that a tie connects \(a_n\) to a block, we mean that the tie connects \(a_n\) to the top arc with maximum index of the bloc. So, we can add at most one tie form \(a_n\) to one of the blocks at the exterior of \(a_n\) (there are \(i\) of such blocks); and, contemporarily, we can add at most one tie form \(a_n\) to one of the blocks in the interior of \(a_n\) (they are \(k - i\)). Indeed, if we should add two ties from \(a_n\) to two blocks \(b_r\) and \(b_s\) both at the exterior or at the interior of \(a_n\), the resulting ta-diagram, having two ties connecting a top arc of \(b_r\) and a top arc of \(b_s\) to \(a_n\), should be non standard, see Definition 10.

So, the total number of standard ta-diagrams obtained by adding an arc \(a_n\) connected by ties to a diagram \(d\) with \(n-1\) arcs and \(k\) blocks, counting the possibilities of zero ties at the exterior and in the interior of \(a_n\), is

\[ \sum_{i=0}^{k} (i+1)(k-i+1) = \binom{k+3}{3}. \]

Now, consider any standard ta-diagram \(d\) in \(A_n\). Since \(d\) is standard, the arc \(a_n\) is connected at most by one tie to an arc at its interior and at most by one tie to an arc in its exterior. Let’s erase these ties and the arc \(a_n\). By erasing \(a_n\) and these ties, we get a diagram in \(A_{n-1}\), for which the sc-partition is obtained from that of \(d\) by erasing \(n\) from the part of the sc-partition of \(d\) containing \(n\). We thus get a unique standard ta-diagram in \(B_{n-1}^j\) for some \(j\), from which \(d\) is uniquely obtained by the procedure.
Remark 3. The number of standard ta-diagrams with $n$ arcs generated by a standard ta-diagram $d$ with $n - 1$ arcs is independent from $n$ and depends only on the number of blocks of $d$.

We obtain now a stronger result. In what follows we shall refer to the procedure above simply by the procedure.

Lemma 10. For $j = 1, \ldots, n$,

$$(20) \quad B_n^j = \sum_{k=j-1}^{n-1} B_n^k \binom{k - j + 3}{2}$$

Proof. We denote $N_j(d^k)$ the number of standard ta-diagrams with $n$ arcs and $j$ blocks generated by the procedure from one standard ta-diagram $d^k$ with $n - 1$ arcs and $k$ blocks. Observe that $j$ satisfy $1 \leq j \leq k + 1$. If $j = k + 1$, there is only one diagram, consisting of $d^k$ with at right the arc $a_n$, without ties, so $N_{k+1}(d^k) = 1$. If $j \leq k$, in the new standard diagram with $j$ blocks, say $d^j$, there are the first $j - 1$ blocks of $d^k$, unaltered, while the other $k - j + 1$ blocks of $d_k$ will form with $a_n$ a sole block. If $k - j = r$, there are $r + 2$ possibilities, since a block $b$ of $d^k$ belongs to a sole block of $d^j$ containing $a_n$ either if it is inside the new arc $a_n$, or if it is connected by a tie to $a_n$, or if it is below the tie from $a_n$ to the $j$-th block of $d^k$. See Figure 11.

$\Box$

Figure 10. Left: Possible places for the left endpoints of the arc $a_n$. Right: Dashed lines represent the possibilities for placing one tie.

Figure 11. Diagrams with 3 blocks generated by a diagram with 5 blocks.
We must now take into account the multiplicities of the above $r + 2$ diagrams, by counting the possibilities for adding one allowed tie from $a_n$ to a block inside $a_n$. In fact, each diagram of the above list has multiplicity $1 + h$ where $h$ is the number of blocks inside $a_n$, $h = 0, \ldots, r + 1$. So,

$$N_j(d^k) = \sum_{h=0}^{k-j+1} (1 + h) = \binom{k-j+3}{2}.$$ 

Therefore Lemma 10 follows.

Comparing now Equations (11) with the initial conditions (4) and Equation (20) with the recurrence (5) we conclude that the integers $B_{n-k}^n$ fulfill the same recurrence as $F(n,k)$, as states Equation (13). Part 2 is proven.

**Proof of Theorem 2.** Because of Equation (13), observe that the three statements of the theorem follows from the proved lemmas concerning $ta$-diagrams. More precisely: statement (6) follows from Equation (10) together with Equation (12). Statement (7) follows from Lemma 9, using (12) and (16). Statement (8) follows from Equation (12).

References

[1] F.Aicardi, J.Juyumaya, Tied Links, J. Knot Theory Ramifications, 25 (2016), no. 9, DOI: 10.1142/S02182165164100171.

[2] F.Aicardi, D.Arcis, J.Juyumaya Brauer and Jones tied monoids, in preparation.

[3] F.Aicardi, J.Juyumaya, P.Papi Tied Temperley-Lieb algebras, in preparation.

[4] H.M. Finucan, Some decomposition of generalized Catalan numbers, Proceedings of the Ninth Australian Conference of Combinatorial Mathematics, University of Queensland, Brisbane, Australia, 1981, 275-293.

[5] J. Riordan, Combinatorial identities(1968). Wiley. ISBN 978-0471722755.

[6] D.G. Rogers, Pascal Triangles, Catalan Numbers and Renewal Arrays Discrete Mathematics, Vol. 22, 1978, 301–310.

[7] A.G. Shannon, Notes on Number Theory and Discrete Mathematics Print ISSN 1310-5132, Online ISSN 2367–8275 Vol. 22, 2016, No. 2, 10–16.