Improved Menter’s one-equation turbulence model

Md Mizanur Rahman¹,³, Yang Zhang² and Zefei Zhu¹
¹Hangzhou Dianzi University, School of Mechanical Engineering, 310018 Hangzhou, China
²Xi’an Jiaotong University, School of Aerospace, China
³Email: mizanur.rahman@hdu.edu.cn

Abstract. Compatible modifications to original Menter model (OMM) are incorporated using an elliptic relaxation approach to accurately account non-local characteristics accompanied by near-wall turbulence. Coefficients/functions of improved Menter model (IMM) are parameterized with the elliptic relaxation function to imitate the combined effects of near-wall turbulence and non-equilibrium. The characteristic length scale involved in the elliptic relaxation scheme is constructed in terms of viscous and turbulent length scales in conjunction with a non-singular strain-rate invariant. Non-local effects are consequently reinforced by the mean flow and turbulent variables. A non-linear eddy-viscosity damping function is invoked to relax the viscous length-scale coefficient pertaining to the elliptic relaxation model. Comparisons indicate that the IMM substantially improves the accuracy of flow simulations compared to the OMM and widely used Spalart-Allmaras (SA) model. A good consensus is obtained between IMM and DNS (direct numerical simulation)/measured data.

Nomenclature

| Symbol | Definition |
|--------|------------|
| $C_f$  | skin-friction coefficient |
| $C_{1,2}, C_{1}, C_{k1}, C_{k2}$ | model closure coefficients |
| $C_\mu$ | eddy-viscosity coefficient |
| $C_l$  | viscous length scale coefficient |
| $D_{1,2}$ | damping functions |
| $f_\mu$ | eddy-viscosity damping function |
| $f_R$  | elliptic-relaxation parameter |
| $h, k$ | channel or inlet height, turbulent kinetic energy |
| $L_R$  | elliptic-relaxation length scale |
| $P_k$  | turbulent production |
| $R$    | pseudo-eddy viscosity |
| $Re$   | Reynolds number |
| $S$    | invariant of mean strain-rate tensor |
| $S_\alpha$ | mean strain-rate correction |
| $t$    | time |
| $u, v$ | velocities in $x$- and $y$-directions |
| $u_\tau$ | wall-friction velocity |
| $u^+$  | non-dimensional flow velocity |
| $y^+$  | non-dimensional wall distance |
| $\epsilon$ | dissipation-rate |
| $\kappa$ | von Karman constant |
| $\nu, \nu_T$ | kinematic laminar and turbulent eddy viscosities |
| $\rho$ | density |
| $\sigma$ | turbulent Schmidt/Prandtl number |
| $\chi$ | viscosity ratio, $R/\nu$ |

Subscript

| Symbol | Definition |
|--------|------------|
| ref    | reference condition |
1. Introduction

Innovative researches have been conducted to improve the competency of one-equation models for both simple and anisotropic turbulent flows [1–10], due to their simplicity, low computational cost and anticipated level of accuracy when compared with higher-order models. The Baldwin–Barth (BB) model [1] is perhaps one of the first one-equation models in which algebraic length scales are not explicitly used; these features make it very attractive from a computational point of view. Nevertheless, it employs a number of additional simplifying assumptions which enfeeble its link with the $k-\epsilon$ closure. Empiricism and dimensional arguments are used to formulate the well-known one-equation Spalart and Allmaras (SA) model [2]. The SA model avoids its link with the $k-\epsilon$ model and replicates more/less physically accepted results for both internal and external flows. It is worth noting that the SA model preserves the origin of BB model. In principle, the status of one-equation models lies between the algebraic and two-equation models since they retain transport effects.

Recently, Rahman et al. [4–7] have consistently promoted the BB model, recovering the link to the $k-\epsilon$ model. The modified version accounts for non-equilibrium effects with non-isotropic model coefficients, providing better results for complex flows. While transferring the $k-\epsilon$ closure to a one-equation model, Menter [3] also revives a close resemblance to the $k-\epsilon$ model than that of BB model applying the Bradshaw-relation [11], which relates the shear stress with the turbulent kinetic energy in the boundary layer. Menter additionally claims that the Bradshaw-relation is good at modeling complex turbulent flows. Using the Bradshaw-relation, several modifications have been introduced with one-equation models [4–10] in order to account for near-wall turbulence; these modifications essentially provide good results for flows with separation and reattachment. The Bradshaw-relation is also efficient in devising anisotropic source and sink term coefficients with one-equation models, yielding a natural decay of turbulence in the free-stream regime.

Using the artifacts of BB model and Bradshaw-relation, a wall-distance-free one-equation turbulence model has been proposed by Menter [3] based on the $k-\epsilon$ closure. However, the model explicitly contains several constant coefficients that may provoke deficiencies in predicting wall-bounded flows even with mild adverse pressure gradients [12]. This inaccuracy can be largely ascribed to the over-estimation of eddy-viscosity by the model. Inclusion of an elliptic blending function in the eddy-viscosity transport equation may enhance the predictive capability of one-equation model. The elliptic-relaxation method is an excellent way to integrate near-wall anisotropy in an eddy-viscosity turbulence model. The wall-blocking is influenced by an elliptic Helmholtz-type partial differential equation, invoking beneficial non-local near-wall effects. Rahman et al. [4, 5], Durbin et al. [13] and Elkhoury [14] have applied an elliptic relaxation equation to a one-equation model for accounting the wall-blocking phenomenon.

The current research includes low-Reynolds number (LRN) and near-wall modifications with the OMM, deserving relatively several desirable attributes: (a) the elliptic blending suppresses the production term in the wall-vicinity, preventing the over-estimation of eddy-viscosity; (b) a modified strain-rate is used to avoid the singularity problem with the inverse of von Karman and elliptic-relaxation length scales; (c) the viscous length-scale coefficient $C_l$ associated with the elliptic-relaxation approach is relaxed by introducing an eddy-viscosity damping function $f_{\mu}$ and (d) coefficients of both source and sink terms imitate the non-linear characteristic of the elliptic relaxation parameter. Ostensibly, the suggested modifications to the OMM augment its capability to account for near-wall turbulence and non-equilibrium effects.

The performance of turbulence models is validated against well-documented experimental and DNS data; selected test cases consist of a fully developed channel flow, a flat plate boundary layer flow with zero pressure-gradient and an asymmetric plane diffuser flow. Chosen test cases presumably justify the competency of improved Menter model (IMM) in replicating the characteristics of LRN and near-wall anisotropy, pertaining to non-equilibrium turbulence.
Conventionally, these cases are probably the most popular numerical experiments to investigate the predictive performance of a turbulence model.

2. Original Menter model (OMM)

The derivation of Menter one-equation turbulence model commences with the transformation of standard high-Reynolds number k-\( \epsilon \) equations, given as follows:

\[
\frac{Dk}{Dt} = P_k - \epsilon + \nabla \cdot \left( \frac{R}{\sigma_k} \nabla k \right) \tag{1}
\]

\[
\frac{De}{Dt} = C_{\epsilon 1} \frac{\epsilon}{k} P_k - C_{\epsilon 2} \frac{\epsilon^2}{k} + \nabla \cdot \left( \frac{R}{\sigma_\epsilon} \nabla \epsilon \right) \tag{2}
\]

where \( D/Dt \) implies a substantial derivative. \( \sigma_k, \sigma_\epsilon, C_{\epsilon 1} \) and \( C_{\epsilon 2} \) are model constants; \( P_k \) is the production term.

The eddy-viscosity \( R = C_{\mu} k^2/\epsilon \) transport equation can be constructed from the relation:

\[
\frac{DR}{Dt} = \frac{D(C_{\mu} k^2/\epsilon)}{Dt} = C_{\mu} \left( \frac{2k Dk}{\epsilon} \frac{Dt}{k} - \frac{k^2 D\epsilon}{\epsilon^2 Dt} \right) = \frac{2R Dk}{k} \frac{Dt}{\epsilon} - \frac{R D\epsilon}{\epsilon Dt} \tag{3}
\]

After combining Eqs. (1)-(3), physically two assumptions are prosecuted to avoid the dependency of Eq. (3) on \( k \) and \( \epsilon \). Firstly, the Bradshaw relation \([11]\), entailing the proportionality of turbulent shear stress to the turbulence kinetic energy, is valid:

\[
|\nu u| = \sqrt{C_{\mu} k} = R \frac{\partial u}{\partial y} \approx RS = R\sqrt{2S_{ij}S_{ij}}; \quad S_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tag{4}
\]

where \( S \) is the invariant of mean strain-rate \( S_{ij} \) tensor; secondly, diffusion coefficients in the \( k \) and \( \epsilon \) equations are identical, i.e., \( \sigma_k = \sigma_\epsilon = \sigma \). A detailed derivation of the single equation model can be found in Reference \([3]\); the LRN form yields:

\[
\frac{DR}{Dt} = \frac{\partial}{\partial x_j} \left[ \left( \nu + \frac{\nu_T}{\sigma} \right) \frac{\partial R}{\partial x_j} \right] + C_1 D_1 RS - C_2 \min \left[ \left( \frac{R}{S} \right)^2 \frac{\partial S}{\partial x_j} \frac{\partial R}{\partial x_j}; \frac{C_3}{\partial x_j} \frac{\partial R}{\partial x_j} \right] \tag{5}
\]

with the damping function \( D_1 = \left(1 + \nu_T/\nu \right)/(1 + \chi) \) in the production term where \( \chi = R/\nu \).

The eddy-viscosity is computed from:

\[
\nu_T = D_2 R; \quad D_2 = 1 - \exp \left[ -\left( \frac{\chi}{\kappa A^+} \right)^2 \right] \tag{6}
\]

The standard \( k-\epsilon \) model constants are used in Eq. (5): \( C_{\epsilon 1} = 1.44, C_{\epsilon 2} = 1.92, C_{\mu} = 0.09 \) and \( \sigma = 1.0; \) the von Karman constant \( \kappa = 0.41, C_1 = \sqrt{C_{\mu} (C_{\epsilon 2} - C_{\epsilon 1})}, C_2 = (1/\sigma + C_1/k^2), C_3 = 7.0 \) and \( A^+ = 13.0 \). Note that the above-mentioned equations are strictly local, independent of the wall-distance. Damping functions \( D_{1,2} \) are displayed in Figure 1, exhibiting \( D_{1,2} \sim 1 \) after \( y^+ \approx 30 \) (e.g., known as viscous affected regions).

3. Improved Menter model (IMM)

Combined impacts of viscous and wall-blocking effects, comprising the LRN and near-wall turbulence can be reinforced by parameterizing the coefficients of eddy-viscosity model with the elliptic relaxation function. The construction of IMM is subjected to enhanced accuracy and greater flexibility compared to the OMM. Coefficients of the OMM are modified such as to
encounter the dominance of elliptic relaxation scheme encompassing near-wall effects and flow inhomogeneity:

\[
\frac{D R}{D t} = \frac{\partial}{\partial x_j} \left[ \left( \nu + \nu_T \right) \frac{\partial R}{\partial x_j} \right] + C_{1ke} R S - C_{2ke} \left( \frac{R}{S} \right)^2 \frac{\partial S}{\partial x_j} \frac{\partial S}{\partial x_j}
\]

where the modified strain-rate \( \tilde{S} \) is defined subsequently. The damping functions \( D_{1,2} \) in the OMM are replaced by an elliptic-relaxation function \( f_R \) and an eddy-viscosity function \( f_\mu \), respectively. The eddy viscosity \( \nu_T = f_\mu R \) and the significance of elliptic-relaxation blending is to entangle wall-effect substantially, thereby repressing the eddy-viscosity production near a solid surface. To invoke this phenomenon in the IMM, the elliptic relaxation equation is devised as follows:

\[-L_R^2 \nabla^2 f_R + f_R = 1.0 \]

Note-worthily, the entire active region of elliptic Helmholtz Eq. (8) is affected by the solid wall and its solution approaches the homogeneous one beyond the wall-proximity. Since turbulence fluctuations at near-wall regions rely on \( \nu \) (e.g., laminar viscosity), the viscous scaling \( \nu/S \) may dominate the lower bound of turbulent length scale. Matching the computational data of simple flow or complex flow with separation and reattachment leads the length scale as:

\[L_R^2 = \max \left( \frac{C_1 R}{9 C_1}; C_1 \nu \right) / \tilde{S} \approx C_1 \nu \sqrt{1.0 + \left( \frac{\chi}{9 C_1} \right)^2} / \tilde{S}
\]

where \( C_1 = 4 + \sqrt{\chi} \) and \( C_1 = 0.144 \). Equation (8) is solved collaterally with the \( R \)-equation; the solution can be initiated with a guess \( 0 \leq f_R \leq 1.0 \) in the flow domain, however, on wall boundaries \( f_R = 0 \). The eddy-viscosity damping function \( f_\mu \) partially shares the wall-blocking effect and in coexistence, reduces the burden on the viscous length-scale coefficient \( C_1 \) to suppress near-wall turbulence. Compatible parametric relations are constructed for the IMM as follows:

\[f_\mu = f_R \times \frac{\chi^3}{\chi^3 + C_\omega^w}; \quad C_{1ke} = f_R - 1.0 + C_1 \]

\[C_{2ke} = \min \left[ C_2; 1.0 - f_R + 8.1 C_1 \right], \quad C_\omega = C_1 \]

where \( C_2 = (1/\sigma + C_1/\kappa^2) \approx 1.86 \) with \( \sigma = 1.0 \). It is worth pointing out that compared with the OMM model, the IMM retains non-linear model coefficients (anisotropic in nature) which are parameterized with the elliptic relaxation function \( f_R \). Remarkably, anisotropic coefficients in Eqs. (9) and (10) are recommended in order to replicate near-wall turbulence inhomogeneity, to reduce the sensitivity of elliptic-relaxation length-scale to a large constant and to enhance the solver stability.

It is worth mentioning that a modified strain-rate \( \tilde{S} \) is used in Eqs. (7) and (9) to avoid the singularity problem with the inverse of von Karman length scale

\[
\frac{1}{S^2} \frac{\partial S}{\partial x_j} \frac{\partial S}{\partial x_j}
\]

and the elliptic-relaxation length scale when the mean strain-rate \( S \) reaches very small values; it is modified as:

\[\tilde{S} = \sqrt{S^2 + S_\alpha^2}; \quad S_\alpha = \frac{f_\mu}{f_R} \frac{\partial \sqrt{R}}{\partial x_i} \frac{\partial \sqrt{R}}{\partial x_i}
\]

where \( S_\alpha \) is the mean strain-rate correction. Evidently, \( S >> S_\alpha \) in the proximity of wall (viscous sublayer and boundary layer), however, away from the wall (defect layer and free-stream region), Eq. (11) ensures eventually a non-zero strain-rate.
Plots of eddy-damping function $f_{\mu}$ and elliptic-relaxation parameter $f_R$ are shown in Figure 1. As can be seen, the relaxation function $f_R \sim 1.0$ after the viscous sub-layer. Therefore, with the elliptic blending contrivance, the net production $(f_R - 1 + C_{1k}) RS$ is negative at near-wall regions (i.e., production is partly dissipated); however, quickly becomes positive away from the solid wall as is obvious from Figure 1. This tendency is due to a relevant choice of length scale in the relaxation equation. As mentioned previously, the wall-blocking effect is partially shared by $f_{\mu}$ implies that a further reduction in the production term by the elliptic blending is exterminated. This is an advantage of integrating both viscous (near-wall turbulence) and blocking (non-local) effects in a turbulence model, having a positive significance. In all computations, $R$ has been non-negative and the devised elliptic blending can essentially be identified as an interpolation method, remaining inside the wall-vicinity with the interpolation coefficient supplied by a non-local model.

4. Computations
Turbulence model equations are solved in conjunction with Reynolds-averaged Navier-Stokes (RANS) equations; turbulent flows, comprising a developed turbulent channel flow, a zero pressure-gradient flat-plate boundary-layer flow and an asymmetric plane diffuser flow are considered to evaluate the turbulence model performance. Numerical computations from the Spalart-Allmaras (SA) model [2] are included for comparisons. Note that compared with the OMM and SA model, the IMM has been made sensitive to non-equilibrium flows by including anisotropic model coefficients which depend non-linearly on the elliptic-relaxation parameter $f_R$. An artificial compressibility (AC) method with a cell-centered finite-volume scheme is applied to solve governing equations [15–18]. A fully second-order upwind scheme is used to evaluate the
convective terms. Roe’s [19] damping term is utilized to compute the cell-face fluxes. A DDADI (diagonally dominant alternating direction implicit time integration method) is employed to solve discretized equations iteratively [20]. The convergence-acceleration is initiated by a multigrid method [21]. Fundamental implementations of an AC approach and associated aspects may be available in References [15–18].

4.1. Turbulent channel flow
A fully developed channel flow at \( Re_{\tau} = 640 \) is computed to show the generality and accuracy of turbulence models; for this test-case DNS data are available in Reference [22]. A one-dimensional RANS solver is used to conduct simulations in the half-width of a channel. Computations involve a \( 1 \times 128 \) non-uniform grid for \( Re_{\tau} = 640 \); grid-independent studies (not shown) justify that the considered grid resolution is sufficiently accurate. The first near-wall grid node is placed at \( y^+ \approx 0.3 \) to resolve the viscous sub-layer. Results are plotted using \( u^+ = u/u_{\tau} \) and \( uv^+ = \overline{uv}/u_{\tau}^2 \), versus \( y^+ \) for comparisons.

Figure 2 depicts the comparisons of velocity and shear stress profiles for different turbulence models. As can be noticed, predicted velocity profiles of both IMM and SA models fairly agree with DNS data. The OMM model slightly under-estimates the mean velocity profile in the logarithmic and defect layers. However, turbulent shear-stress profiles, adhering to all turbulence models maintain good correspondence with DNS data.

4.2. Flat-plate boundary layer flow
Near-wall behaviors of selected turbulence models are further investigated by computing the flow over a flat-plate having a high free-stream intensity of turbulence. This computational geometry is identified as a well-known T3B case for bypass transition, extracted from "ERCOFTAC" Fluid Dynamics Database WWW Services (http://fluindigo.mech.surrey.ac.uk/) conserved by P. Voke. J. Coupland at Rolls-Royce conducted measurements down to \( x = 1.495 \) \( m \) corresponding to a local Reynolds number \( Re_x \approx 94000 \). The inlet reference velocity \( U_r = 9.4 \) \( m/s \) with a zero pressure-gradient; the upstream turbulence intensity \( T_u = 6.0\% \) which is defined as \( T_u = \sqrt{2k}/U_r \). The free-stream turbulent to laminar viscosity ratio is set to \( \nu_T/\nu = 1.0 \).

Computations start with 16 \( cm \) forwarded leading edge, employing symmetric/mirror boundary conditions. The dimension of computational domain is \( 1.6 \times 0.3 \) \( m \); the first
near-wall grid node is placed at $y^+ < 1.0$, however, with the leading edge-point it is at $y^+ = 2.1$. A non-uniform grid resolution $96 \times 64$ is applied; the grid is compressed heavily at the near-wall region to resolve sharp gradients and a grid-independent solution is ensured (not shown).

Figure 3 compares predicted skin-friction coefficients ($C_f = 2u_r^2/U_r^2$) with measured data. Evidently, the SA model has the best agreement with data, preserving a compelling phenomenon that a strong transition appears at the right location. On the contrary, both OMM and IMM convolve earlier transition compared to the experiment; Savill experienced the similar characteristic in his observation [23]. Apparently, congruence between numerical and experimental results is obviously striking after the end of transition (e.g., beyond $x = 0.195 \text{ m}$). Nonetheless, both OMM and IMM are incompetent in recovering measured profiles; their predictions stay somewhat on a lower level than experiments show. Profiles of mean velocity are illustrated at the fully turbulent region, corresponding to $x = 1.495 \text{ m}$ in Figure 4. Both IMM and SA model fairly match the data but the OMM shows the similar behavior to that of a fully developed channel flow as mentioned earlier.

![Figure 3. Streamwise skin-friction coefficient of boundary layer flow.](image)

![Figure 4. Mean velocity profiles of boundary layer flow.](image)

### 4.3. Asymmetric plane diffuser flow

To validate performances against complex flows with separation and reattachment, turbulence models are further involved in computing an asymmetric plane diffuser flow, experimental data of which can be found from Reference [24]. The opening angle of diffuser is $10^\circ$ with an expansion ratio of 4.7, due to which a separation bubble is generated on the deflected wall, yielding a test-case for adverse pressure-driven separation. A plane channel at the diffuser-entrance is used to produce a fully developed flow at $Re = 2.0 \times 10^4$; $Re$ is evaluated based on the center-line velocity $U_r$ and inlet channel height $h$. The computational domain length is set to $76h$ and the first cell-thickness $y^+ \leq 1$ on both deflected and flat walls. The applied non-uniform grid size is $120 \times 72$ and heavily clustered at near-wall regimes, ensuring a grid-independent numerical solution (not shown).

Figure 5 exhibits skin-friction coefficients predicted by three independent models along bottom and straight top walls. Results of IMM and SA reasonably match experiments at
the bottom wall, although the IMM produces a slight amplification in the $C_f$-profile. It can be observed that the OMM predicts the $C_f$-profiles, worse than those of IMM and SA. Nevertheless, along the straight top wall both OMM and IMM show comparable agreement with experiment. Remarkably, the IMM shows the best performance; it seems likely that the elliptic-relaxation length-scale $L$ has a significant impact on the prediction of flow recirculation and reattachment.

Figure 6 shows the mean velocity profiles at four representative locations in the diffuser. The competency of IMM in reproducing the velocity profiles is remarkable. Unlike the OMM and SA model, the IMM employs an elliptic blending approach together with a non-linear damping function to repress the near-wall eddy-viscosity, and hence predicted results are in better conformity with experimental data. However, the peak of $u$-profile is over-predicted by the IMM in the outlet-vicinity of diffuser, for instance, $x/h = 30$. Apparently, both OMM and SA model replicate identical results.

Comparisons of Reynolds shear stresses at different $x/h$ positions are plotted in Figure 7. Results demonstrate that all turbulence models have reasonable consistency with measured data. The IMM acceptably replicates the experimental data for $u$ (except after $x/h = 25$) and $uv$ profiles, since it entangles the solution to the transport equations with the elliptic blending.

5. Conclusions
The OMM is modified to replace the damping functions $D_{1,2}$ by elliptic-relaxation and non-linear eddy-viscosity damping functions, accounting for near-wall turbulence and wall-blocking
effects. The parameterized coefficients with the elliptic function are validated against simple and complex flows: the elliptic blending function $f_R$ enhances the IMM accuracy in predicting wall-bounded flows involving separation and reattachment. Comparing numerical predictions with DNS and measured data demonstrates that the IMM offers significant improvements over the OMM and SA model. The potential importance of elliptic-relaxation scheme is conspicuous.

References
[1] Baldwin BS, Barth TJ 1990: A one-equation turbulence transport model for high-Reynolds number wall-bounded flows. NASA TM-102847.
[2] Spalart PR, Allmaras SR 1992: A one-equation turbulence model for aerodynamic flows. AIAA Paper No. 92-0439.
[3] Menter FR 1997: Eddy viscosity transport equations and their relation to the $k$–$\varepsilon$ model. ASME Journal of Fluids Engineering, 119:876-884.
[4] Rahman MM, Siikonen T, Agarwal RK 2011: Improved low-Reynolds-number one-equation turbulence model. AIAA Journal, 49(4):735-747.
[5] Rahman MM, Wallin S, Siikonen T 2012: Exploring $k$ and $\varepsilon$ with $R$-equation model using elliptic relaxation function. Flow, Combustion and Turbulence, 89:121-148.
[6] Rahman MM, Agarwal RK, Siikonen T 2016: RAS one-equation turbulence model with non-singular eddy-viscosity coefficient. Int. J. Comput. Fluid Dynamics, 30(2):89-106.
[7] Rahman MM, Keskinen K, Vuorinen V, Larmi M, Siikonen T 2019: Consistently formulated eddy-viscosity coefficient for $k$-equation model. J. Turbulence, 19(11-12):959-994.
[8] Elkhoury M 2007: Assessment and modification of one-equation models of turbulence for wall-bounded flows. ASME Journal of Fluids Engineering, 129:921-928.
[9] Fares E, Schröder W 2004: A general one-equation turbulence model for free shear and wall-bounded flows. Flow, Turbulence and Combustion, 73:187-215.
[10] Nagano Y, Pei CQ, Hattori H 1999: A new low-Reynolds-number one-equation model of turbulence. Flow, Turbulence and Combustion, 63:135-151.
[11] Bradshaw P, Ferriss DH, Atwell NP 1967: Calculation of boundary layer development using the turbulent energy equations. Journal of Fluid Mechanics, 23:31-64.
[12] Elkhoury M 2008: Effect of $C_{t,1}$ on the performance of the Menter one-equation model of turbulence. Journal of Aircraft, 45(2):733-736.
[13] Durbin PA, Mansour NN, Yang Z 1994: Eddy viscosity transport model for turbulent flow. Physics of Fluids, 6(2):1007-1015.
[14] Elkhoury M 2017: On the eddy viscosity transport models with elliptic relaxation. *Journal of Turbulence*, 18(3):240-259.

[15] Rahman MM, Rautaheimo P, Siikonen T 1997: Numerical study of turbulent heat transfer from a confined impinging jet using a pseudo-compressibility method, *Second Int. Symposium on Turbulence, Heat and Mass transfer*, Delft, The Netherlands, Hanjalic K and Peeters TWJ, (eds), Delft University Press: Delft, 511-520.

[16] Rahman MM, Siikonen T 2001: An artificial compressibility method for incompressible flows. *Numer. Heat Transfer, Part B*, 40:391-409.

[17] Rahman MM, Siikonen T 2002: A dual-dissipation scheme for pressure-velocity coupling. *Numerical Heat Transfer, Part B*, 42, 231-242.

[18] Rahman MM, Siikonen T 2008: An artificial compressibility method for viscous incompressible and low Mach number flows. *Int. J. Numer. Meth. Engg.*, 75:1320-1340.

[19] Roe PL 1981: Approximate Riemann solvers, parameter vectors, and difference schemes. *J. Comput. Physics*, 43:357-372.

[20] Lombard C, Bardina J, Venkatapathy E, Oliger J 1983: Multi-dimensional formulation of CSMC - an upwind flux difference eigenvector split method for the compressible Navier-Stokes equations. *Sixth AIAA Comput. Fluid Dynamics Conference*, AIAA Paper 83-1895-CP, 649-664.

[21] Jameson A, Yoon S 1986: Multigrid solution of the Euler equations using implicit schemes. *AIAA Journal*, 24:1737-1743.

[22] Kawamura H, Abe H, Matsuo Y 1999: DNS of turbulent heat transfer in channel flow with respect to Reynolds and Prandtl number effect. *International Journal of Heat and Fluid Flow*, 20(3):196-207. doi:10.1016/S0142-727X(99)00014-4.

[23] Savill AM 1993: Some recent progress in the turbulence modeling of by-pass transition. *In Near-Wall Turbulent Flows*, So RMC, Speziale CG, and Launder BE, (eds.), Elsevier, 829-848.

[24] Buice C, Eaton JK 1997: Experimental investigation of flow through an asymmetric plane diffuser. *Department of Mechanical Engineering, Thermoscience Div.*, Rept. TSD-107, Stanford University, California, CA.