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Published in:
Annals of Mathematics

DOI:
10.4007/annals.2020.192.3.5

Publication date:
2020

Document version
Peer reviewed version

Document license:
Other

Citation for published version (APA):
Fournais, S., & Solovej, J. P. (2020). The energy of dilute Bose gases. Annals of Mathematics, 192(3), 893-976. https://doi.org/10.4007/annals.2020.192.3.5
The energy of dilute Bose gases

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May 28, 2019

Abstract

For a dilute system of non-relativistic bosons interacting through a positive $L^1$ potential $v$ with scattering length $a$ we prove that the ground state energy density satisfies the bound $e(\rho) \geq 4\pi a \rho^2 (1 + \frac{128}{15\sqrt{\pi}} \sqrt{\rho a^3} + o(\sqrt{\rho a^3}))$, thereby proving the Lee-Huang-Yang formula for the energy density.

Contents

1 Introduction 2
2 Facts about the scattering solution 4
3 Grand canonical reformulation of the problem 5
4 Strategy of the proof of Theorem 3.1 and the various parameters 6
5 Localization 10
  5.1 Setup and notation ........................................ 10
  5.2 Localization of the kinetic and potential energies ........ 12
  5.3 The localized Hamiltonian .................................. 14
  5.4 Potential energy splitting ................................... 14
6 A priori bounds on particle number and excited particles 18
7 Localization of the number of excited particles $n_+$ 19
8 Localization of the $3Q$-term 21
9 Second quantized operators 22
  9.1 Creation/annihilation operators ............................. 22
  9.2 $c$-number substitution ..................................... 25
10 First energy bounds 27
11 More precise energy estimates 31
  11.1 The Bogolubov Hamiltonian ................................. 32
  11.2 The control of $Q_3(z)$ .................................. 37
Our goal in this paper is to solve the long standing conjecture in mathematical physics to rigorously establish the Lee-Huang-Yang (LHY) formula for the second correction to the thermodynamic (infinite volume) ground state energy per volume of a translation invariant Bose gas in the dilute limit. The formula (i.e., (1.3) below with an equality) is one of the most fundamental results in quantum many-body theory. It appeared for the first time as equation (25) in the seminal 1957 publication [13]. The striking feature of the formula is that the first two terms of the asymptotics of the ground state energy in the dilute limit depend on the interaction potential only through a single parameter, the scattering length $a$. Fairly recently the LHY formula was tested experimentally as reported in [25]. Here the coefficient $\frac{128}{15}\sqrt{\pi} = 4.81$ was measured to be 4.4(5).

The derivation in [13] relies on the pseudo-potential method and offers deep insight into the problem, but nevertheless lacks in mathematical rigor. An alternative, but still non-rigorous, argument was proposed in [15]. We establish the LHY formula rigorously for a large family of two-body potentials (see Assumption 1.1 below), which however does not include the hard core potential.

The importance of the scattering length in understanding the energy and excitation spectrum for interacting many-body gases had already been observed in the celebrated 1947 paper of Bogolubov [5] where he introduced the Bogolubov approximation and laid the foundation for the theory of superfluidity. In this paper Bogolubov studies the excitation spectrum of a Bose gas and finds that it depends on the integral of the potential, not the scattering length. In a famous footnote Bogolubov thanks Landau for making the important remark that this must be wrong and that the correct answer must be to replace the integral of the potential by the scattering length. To establish this rigorously has been a major challenge ever since. The first major rigorous advance was achieved by Dyson in [9] where the leading order asymptotics for the ground state energy was established as an upper bound, but where the lower bound was off by a factor. The correct leading order asymptotics was finally established by Lieb and Yngvason in [21] for all positive interaction potentials with finite scattering length including the hard core potential. This result was extended to the Gross-Pitaevskii limit in the case of trapped gases in [17]. These leading order results are reviewed in the monograph [16] which also contains a non-rigorous derivation of the LHY formula using the Bogolubov approximation. To the best of our knowledge the first works to rigorously establish the validity of the Bogolubov approximation for a many-body problem were [19, 20, 28] which studied the one- and two-component charged Bose gases and established a conjecture of Dyson. Several ideas from [19] are important also in the present work.

The first work to show an upper bound to the LHY order was [10] by Erdős, Schlein, and Yau. This paper makes a very interesting observation about the Bogolubov approximation. The usual approach to the Bogolubov approximation is to approximate the Hamiltonian of the system by what is referred to as a quadratic Hamiltonian. As mentioned above this leads to a wrong approximation for the ground state energy where it will be expressed in terms

1 Introduction

1

12 Proof of the main theorem

A Bogolubov method

B Localization to small boxes

C The explicit localization function
of the integral of the potential rather than the scattering length. Quadratic Hamiltonians have ground states that are quasi-free (or Gaussian) states. In [10] it is observed that if we do not approximate the Hamiltonian by a quadratic Hamiltonian, but instead restrict the evaluation of the full Hamiltonian to quasi-free states then miraculously the scattering length appears in the leading order term, but to LHY order the answer is still wrong. The work in [10] emphasizes that it may often be more fruitful to focus on classes of states rather than to approximate the Hamiltonian. This approach was further pursued in the papers [23, 24] where the positive temperature situation was analyzed for the Hamiltonian restricted to quasi-free states. The leading order correction to the positive temperature free energy for the full many-body problem in the dilute limit was established in [26,30].

For gases confined to a region in the Gross-Pitaevskii regime there is a formula for the second order correction to the ground state energy similar to the LHY formula. This has recently been established in an impressive series of papers by Boccato, Brennecke, Cenatiempo, and Schlein [2–4]. This however does not imply the formula in the original thermodynamic infinite volume setting discussed here. Our proof follows a very different strategy than the one applied in the confined case.

In the confined or trapped case it is also possible to analyze the excitation spectrum of the gas, which is particularly important for understanding superfluidity. The excitation spectrum is also studied in the papers by Boccato et. al. The first result in this direction is, however, due to Seiringer [27] and was also analyzed in [8,12,14,22]. Getting the excitation spectrum in the thermodynamic case seems much more difficult.

The LHY formula in the translation invariant thermodynamic setting was finally rigorously established as an upper bound in the work [29] by Yau and Yin, where they consider smooth fastly decaying interaction potentials. It is this work that we complement by establishing the lower bound in (1.3), in fact, for a much, larger class of interaction potentials. Thus the LHY formula has been proved for all compactly supported potentials satisfying the assumptions in [29]. We shall not discuss the upper bound further in this paper. In Bogolubov theory, the particles not in the condensate constitute pairs of opposite momentum. An important insight, confirmed by the contributions of [29] and the present work, is that in order to get the correct energy to LHY order, one has to go beyond these simple pairs and also consider ‘soft pairs’. This means that not only pairs of particles of exactly opposite momentum contribute. Also pairs of particles with nonzero total momentum - although the individual momenta are much larger than the sum - are important for the energy to this precision.

The LHY formula had previously been established as a lower bound in the restricted case where the interaction potential is allowed to become softer as the gas becomes more dilute. This was first achieved in [11]. In this case, however, the potential still has a range much larger than the inter-particle spacing, which is why the paper has “high density” in the title. Allowing the potential to have range shorter than the inter-particle spacing, but still required to be soft, was recently achieved in [7]. The softness condition was removed in [6], but only to get the ground state energy to the correct LHY order, not with the correct asymptotics. Several of the methods developed in [7] and [6] are crucial to this work.

There has been a large literature also on the dynamics of interacting Bose gases, but we will not review that here.

We now turn to describing the problem in details. We consider $N$ bosons in 3 dimensions described by the Hamiltonian

$$\mathcal{H}_N = \mathcal{H}_N(v) = \sum_{i=1}^{N} -\Delta_i + \sum_{1 \leq i < j \leq N} v(x_i - x_j).$$

(1.1)
Facts about the scattering solution

We will allow interactions described by the following assumptions.

**Assumption 1.1 (Potentials).** The potential \( v \neq 0 \) is non-negative and spherically symmetric, i.e. \( v(x) = v(|x|) \geq 0 \), and of class \( L^1(\mathbb{R}^3) \) with compact support. We fix \( R > 0 \) such that \( \text{supp} \, v \subset B(0, R) \).

We are interested in the thermodynamic limit of the ground state energy density as a function of the particle density \( \rho \).

\[
e(\rho, v) = \lim_{L \to \infty} L^{-3} \inf_{\Psi \in C_0^\infty([0, L]^N) \setminus \{0\}} \frac{\langle \Psi, \mathcal{H}_N(v) \Psi \rangle}{\| \Psi \|^2}.
\] (1.2)

We will omit the dependence on \( v \) from the notation and just write \( e(\rho) \), when the potential is clear from the context. Here the inner product \( \langle \cdot, \cdot \rangle \) and the corresponding norm \( \| \cdot \| \) are in the Hilbert space \( L^2(\Omega_N) \), where we have denoted \( \Omega = [0, L]^3 \). If we talk about bosons the infimum above should be over all symmetric function in \( C_0^\infty(\Omega_N) \). It is however a well-known fact that the infimum over all functions is actually the same as if constrained to symmetric functions. When we restrict to functions with compact support in \( \Omega \) we are effectively using Dirichlet boundary conditions, but it is not difficult to see that the thermodynamic energy is independent of the boundary condition used.

The main result of this work is to establish the celebrated Lee-Huang-Yang formula that gives a two-term asymptotic formula for \( e(\rho) \) in the dilute limit. We express the diluteness in terms of the scattering length \( a \) of the potential \( v \). The definition of the scattering length and its basic properties will be given in Section 2.

**Theorem 1.2 (The Lee-Huang-Yang Formula).** If \( v \) satisfies Assumption 1.1 then in the limit \( \rho a^3 \to 0 \),

\[
e(\rho) \geq 4\pi\rho^2a \left( 1 + \frac{128}{15\sqrt{\pi}} \sqrt{\rho a^3} - C(\rho a^3)^{1/2+\eta} \right).
\] (1.3)

where \( \eta > 0 \) and \( C \) depend on \( R = \int v/(8\pi a) \) and \( R/a \) as given explicitly in Theorem 5.8 below. We have not attempted to optimize this dependence. It follows from Theorem 5.8 that \( R \) and \( R/a \) can be allowed to grow as a negative power of \( \rho a^3 \).

As reviewed above an upper bound consistent with the Lee-Huang-Yang formula was given in [29] under more restrictive assumptions on the potential (see also [1]). Combined with Theorem 1.2 the second term of the energy asymptotics of the dilute Bose gas has therefore been established. It remains an interesting open problem to give upper bounds consistent with (1.3) under less restrictive assumptions on the potential than in [1,29]. It remains, in particular, an open problem to obtain upper and lower bounds for the hard core potential.

**Acknowledgements.** SF was partially supported by a Sapere Aude grant from the Independent Research Fund Denmark, Grant number DFF–4181-00221. JPS was partially supported by the Villum Centre of Excellence for the Mathematics of Quantum Theory (QMATH) and the ERC Advanced grant 321029. Part of this work was carried out while both authors visited the Mittag-Leffler Institute.

### 2 Facts about the scattering solution

In this short section we establish notation and recall results concerning the scattering length and associated quantities.
We suppose that $v$ satisfies Assumption 1.1 and refer to Appendix C of [16] for details and a more general treatment. The scattering equation reads

$$(-\Delta + \frac{1}{2}v(x))(1 - \omega(x)) = 0, \quad \text{with } \omega \to 0, \text{ as } |x| \to \infty. \quad (2.1)$$

The radial solution $\omega$ to this equation satisfies that there exists a constant $a > 0$ such that $\omega(x) = a/|x|$ for $x$ outside supp $v$. This constant $a$ is the scattering length of the potential $v$ and we will refer to $\omega$ as the scattering solution. Furthermore, $\omega$ is radially symmetric and non-increasing with

$$0 \leq \omega(x) \leq 1. \quad (2.2)$$

We introduce the function

$$g := v(1 - \omega). \quad (2.3)$$

The scattering equation can be reformulated as

$$-\Delta \omega = \frac{1}{2} g. \quad (2.4)$$

From this we deduce, using the divergence theorem, that

$$a = (8\pi)^{-1} \int g, \quad (2.5)$$

and that the Fourier transform satisfies

$$\hat{\omega}(k) = \frac{\hat{g}(k)}{2k^2}. \quad (2.6)$$

3 Grand canonical reformulation of the problem

To prove Theorem 1.2 we will reformulate the problem grand canonically on Fock space. Consider, for given $\rho_\mu > 0$, the following operator $\mathcal{H}_{\rho_\mu}$ on the symmetric Fock space $\mathcal{F}_s(L^2(\Omega))$. The operator $\mathcal{H}_{\rho_\mu}$ commutes with particle number and satisfies, with $\mathcal{H}_{\rho_\mu, N}$ denoting the restriction of $\mathcal{H}_{\rho_\mu}$ to the $N$-particle subspace of $\mathcal{F}_s(L^2(\Omega))$,

$$\mathcal{H}_{\rho_\mu, N} = \mathcal{H}_N - 8\pi a \rho_\mu N = \sum_{i=1}^{N} -\Delta_i + \sum_{i<j} v(x_i - x_j) - 8\pi a \rho_\mu N \quad (3.1)$$

$$= \sum_{i=1}^{N} \left(-\Delta_i - \rho_\mu \int_{\mathbb{R}^3} g(x_i - y) \, dy \right) + \sum_{i<j} v(x_i - x_j).$$

Notice that the new term in $\mathcal{H}_{\rho_\mu, N}$ plays the role of a chemical potential justifying the notation.

Define the corresponding ground state energy density,

$$e_0(\rho_\mu) := \lim_{|\Omega| \to \infty} |\Omega|^{-1} \inf_{\Psi \in \mathcal{F}_s \setminus \{0\}} \frac{\langle \Psi, \mathcal{H}_{\rho_\mu} \Psi \rangle}{\|\Psi\|^2}. \quad (3.2)$$

We formulate the following result, which will be a consequence of Theorems 5.7 and 5.8 below.
Theorem 3.1. Suppose that $v$ satisfies Assumption 1.1. Then the thermodynamic ground state energy density of $\mathcal{H}_{\rho\mu}$ satisfies for $\rho_\mu a^3 \to 0$ that
\[
e_0(\rho_\mu) \geq -4\pi \rho_\mu^2 a \left( 1 - \frac{128}{15\sqrt{\pi}} (\rho_\mu a^3)^{1/2} + C(\rho_\mu a^3)^{1/2 + \eta} \right),
\]
where $\eta > 0$ and $C$ depend on $R = \int v/(8\pi a)$ and $R/a$ as given explicitly in Theorem 5.8.

Proof of Theorem 1.2. It is easy to deduce Theorem 1.2 from Theorem 3.1. By inserting the ground state of $\mathcal{H}_N$ as a trial state in $\mathcal{H}_{\rho\mu}$ one gets in the thermodynamic limit for all $\rho, \rho\mu > 0$
\[
e(\rho) \geq e_0(\rho_\mu) + 8\pi a_0 \rho_\mu \geq 8\pi a_0 \rho_\mu - 4\pi a_0^2 \left( 1 - \frac{128}{15\sqrt{\pi}} (\rho_\mu a^3)^{1/2} + C(\rho_\mu a^3)^{1/2 + \eta} \right),
\]
where we have used the lower bound from Theorem 3.1. If we therefore choose $\rho_\mu$ to be equal to $\rho$ we arrive at the LHY formula (1.3).

4 Strategy of the proof of Theorem 3.1 and the various parameters

As already mentioned in the introduction the important parameters given in the problem are
\[a, \int v, R\]
All estimates will in the end depend on these. The most important combination is the diluteness parameter
\[\rho_\mu a^3.\]
The proof introduces a series of additional parameters. There is an integer
\[M \in \mathbb{N}\]
which determines the regularity of the localization function defined in Appendix [C]. It will be chosen explicitly below. The remaining parameters will be chosen to depend on $\rho_\mu a^3$ and $R = \int v/(8\pi a)$. There are dimensionless parameters $0 < s, d, \varepsilon_T$, that will be chosen small, and there are dimensionless parameters $1 < K_\epsilon, K_M, K_H, K_B$ that will be chosen large. The power in the error term will depend on the choice of these 7 parameters in terms of $\rho_\mu a^3$ and $R = \int v/(8\pi a)$.

Let us describe how these parameters enter into the proof and list all the conditions that they must satisfy. Finally we will make choices to show that these conditions can all be satisfied.

The proof will use a double localization approach. First we localize into boxes of length scale
\[\ell := K_\ell(\rho_\mu a)^{-1/2}\]
I.e., boxes that are long on the scale $(\rho_\mu a)^{-1/2}$ which turn out to be the relevant length scale for the Bogolubov calculation and which is often referred to as the healing length in the literature. The length of the box is chosen much longer to get the Bogolubov calculation correct. The kinetic energy localization will be done in such a way that constant functions in the box have zero kinetic energy and such that there is a gap above the zero energy. This gap will allow us to get a priori control on the number of excited particles, i.e., those not in
Strategy of the proof of Theorem 3.1 and the various parameters

the condensate represented by the constant. However, to get this apriori control we need an a priori lower bound on the energy, which is correct to an order which is almost as in LHY. This is achieved by localizing even further to small boxes of length scale

\[ d\ell = dK_\ell (\rho_\mu a)^{-1/2} \ll (\rho_\mu a)^{-1/2} \]  

(4.2)

which gives us our first condition that \( dK_\ell \ll 1 \). Here and below \( f \ll g \) is used in the precise meaning that \( (f/g) \leq (\rho_\mu a^3)^{\varepsilon} \) for some positive \( \varepsilon \) and likewise for \( f \gg g \). In these small boxes we have a much larger energy gap than in the large boxes and this allows us to absorb errors that we cannot estimate in the larger boxes.

The localization of the potential energy is performed by a simple sliding technique described in Lemma 5.6. An important step in controlling the energy in both the small and large boxes is to split the potential energy in terms of writing

\[ 1 = P + Q \]

where \( P \) is the projection onto constant functions. The potential energy can then be written as a sum of 16 terms that contain 0–4 \( Q \)'s. One of the main ideas in this paper is to complete an appropriate square containing the 4\( Q \) term in Lemma 5.9. This will leave renormalized terms with 0–3 \( Q \)s, where the potential has essentially been replaced by the function \( g = v(1 - \omega) \) from (2.3).

The analysis of the small boxes is performed in Appendix B. The parameters \( \varepsilon_T, d, s \) appear in the kinetic energy localization formulas of Section 5.2 and they must satisfy the conditions

\[ d^{-5} s^{M+1} \ll 1, \]  

(4.3)

\[ (dK_\ell)^2 \ll \varepsilon_T K_\ell^{-2} \ll \varepsilon_T \ll sdK_\ell, \]  

(4.4)

\[ sK_\ell \gg 1, \]  

(4.5)

\[ sdK_\ell \gg K_B^{-1}. \]  

(4.6)

Throughout the paper there will also be logarithmic factors. They are ignored here as they are always accomodated by the conditions given. Condition (4.3) is needed to prove the kinetic energy localization into the small boxes (see (B.13)). It relies on a result from [7]. The first condition in (4.4) is needed to have a sufficiently large gap in the small boxes, but in fact, this would only require \( (dK_\ell)^2 \ll \varepsilon_T \). The need for the stronger condition will be explained below. The condition \( dK_\ell \ll 1 \) noted above is contained in (4.4). The last condition in (4.4) is required to finally get the correct LHY constant when the appropriate integral is estimated in Section 11. The condition (4.5) is also needed to control the same integral, in fact, this condition implies that the localized kinetic energy (see (5.20)) in the large boxes is essentially the original kinetic energy at the relevant Bogolubov scales. Finally, (4.6) introduces the parameter \( K_B \) to control that the small boxes are not too small. This is required, in order, to get a good lower bound on the the energy in the small boxes in Appendix B (see Theorem B.5) and hence for the a priori bound on the energy in the large boxes and consequently on the number of particles and excited particles in the large boxes (see Theorem 6.1). The parameter \( K_B \) has to satisfy the additional conditions that

\[ K_B \ll (\rho_\mu a^3)^{-1/6}, \]  

(4.7)

\[ K_B^3 K_\ell^2 \ll (\rho_\mu a^3)^{-1/4}. \]  

(4.8)

Here (4.7) is a very weak condition implying that the a priori lower bound on the energy in Theorem B.6 is at least better than the leading order term. The condition (4.8) ensures that the a priori bounds on the particle number and expected number of excited particles are both correct to leading order (see (6.2)).
Strategy of the proof of Theorem 3.1 and the various parameters

Having established the a priori bound on the energy and the number of excited particles we will be able, using the technique of localizing large matrices from [19], to restrict the analysis to the subspace where the number of excited particles is bounded by a parameter

\[ \mathcal{M} =: K_M(\rho a^3)^{-1/4}. \]  

(4.9)

It must satisfy

\[ K_M^{-2} \int v/a \ll 1, \]  

(4.10)

\[ K_M^3 K_{\ell}^5 \ll \mathcal{M} = K_M(\rho a^3)^{-1/4}, \]  

(4.11)

\[ K_M K_{\ell}^{-3} \ll (\rho a^3)^{-1/4}. \]  

(4.12)

Condition (4.10) is needed to control the error in the energy when restricting to the situation with a bounded number of excited particles. The condition (4.11) says that the upper bound \( \mathcal{M} \) on the number of excited particles must be much bigger than the expected number of these particles, which in Theorem 6.1 is shown to be not much worse than \( K_B^3 K_\ell^2 \rho a^3(\rho a^3)^{1/2} \sim K_B^3 K_\ell^5 \). The condition (4.12) is a very weak condition that ensures

\[ \mathcal{M} \ll \rho a^3, \]  

(4.13)

e.i., that the bound on the number of excited particles is much less than the total number of particles.

When we treat the potential energy a major difficulty will be the terms with 3 \( Q \)'s terms. These terms are responsible for the “soft” pairs that we discussed in the introduction. The main contributions from these terms come when one excited particle has low momentum and the other two have high momenta. This requires introducing an upper cutoff for low momenta, which we choose to be \( K_L(\rho a)^{3/2} \) and a lower cutoff for high momenta which we choose to be (see Section 8)

\[ \bar{K}_H^{-1} (\rho a^3)^{5/12} a^{-1}. \]  

(4.14)

The relevance of the power 5/12 is technical and will appear in the proof of Lemma 9.3. For convenience we also introduce the parameter \( \bar{K}_H = \tilde{K}_H(\rho a^3)^{-5/12}. \)

We will not choose \( K_L \) as a new parameter, but take

\[ K_L =: (K_\ell d^2)^{-1} \gg K_\ell, \]  

(4.15)

where the estimate follows from (4.4).

We get the additional conditions

\[ K_M K_\ell^4 \ll \tilde{K}_H^3 \]  

(4.16)

\[ (K_\ell K_L)^{1-M} = d^{2M-2} \ll (\rho a^3)^{1/2} \]  

(4.17)

\[ K_L \tilde{K}_H = (K_\ell d^2)^{-1} \tilde{K}_H \ll (\rho a^3)^{-1/12}. \]  

(4.18)

The condition (4.18) ensures that the high momenta are disjoint from the low momenta. The condition (4.17) will be ensured by choosing the integer \( M \) that appears in the explicit localization function large enough. The condition is needed to control errors that occur because of the localization function. This error will also appear in the final error on the lower bound on the energy (see (11.41)). The condition (4.16) is needed to control the error (see (9.13)) in cutting off the 3Q terms in momentum by absorbing it into the energy gap. It
is here that the powers in the choice \[4.14\] become important. This step is performed after we have introduced second quantization in Section 9.

After introducing second quantization it turns out to be useful to do \(c\)-number substitution, where the annihilation operator for the constant functions is replaced by a number \(z\) such that \(|z|^2/\ell^3\), which represents the density in the \(c\)-number substituted condensate, is sufficiently close to \(\rho_\mu\). This is done in Section 10. It will require the additional conditions

\[
K^6T_\ell^{-1}M K^2 K^6 = K_M K^4 K^6 d^{-12} (\rho_\mu a^3)^{1/4} \ll 1, \tag{4.19}
\]
\[
K^8T_\ell^{-1}M^3 K^2 K^4 K^6 d^{-12} (\rho_\mu a^3)^{13/12} \ll 1. \tag{4.20}
\]

These conditions ensure that \(\delta\) defined in Lemma 10.1 is small enough to satisfy (11.2). That (11.2) is, indeed, satisfied then follows from (4.6) and (4.8).

Finally, we are then left with (see (11.7))

- Terms with no \(Q\)'s that can be explicitly calculated
- A quadratic Hamiltonian \(K\text{Bog}^3\) including also some linear terms (corresponding to 1\(Q\) terms)
- The 3\(Q\) terms that are left after the momentum cut-offs and additional quadratic and linear terms.

The quadratic Hamiltonian is treated using the simplified Bogolubov method in Appendix A. This together with the no-\(Q\) terms will give the correct energy up to the LHY correction and a positive quadratic operator (the diagonalized Bogolubov Hamiltonian), see (11.8). This requires, however, the condition

\[
K_M K^{-3/2} T_\ell K (\rho_\mu a^3)^{1/12}) M^{-5} \ll (\rho_\mu a^3)^{1/2}. \tag{4.21}
\]

Note that the term taken to the power \(M\) here is small by (4.18) and the estimate in (4.15).

The positive quadratic operator together with the remaining 3\(Q\) and other terms not treated by Bogolubov's method can be shown by a very detailed calculation to be bounded below by a term of lower order than LHY. This last calculation, done in Subsection 11.2, requires the conditions (see Theorem 11.4)

\[
K^2 K_M = (K d)^{-3} K_M \ll (\rho_\mu a^3)^{-1/4}, \tag{4.22}
\]
\[
K^2 K_H^4 d^{-6} \ll (\rho_\mu a^3)^{-1/3}, \tag{4.23}
\]
\[
K_M K^{-3} K_H^{-2} \ll (\rho_\mu a^3)^{-1/12}, \tag{4.24}
\]
\[
K_M K^{-3} d^{-12} (K_H^2 K_H^2 (\rho_\mu a^3)^{1/6})^{M-1} \ll (\rho_\mu a^3)^{3/4}. \tag{4.25}
\]

The conditions (4.24) and (4.25) are needed in order for the errors in Theorem 11.4 to be of lower order than LHY. There are two additional error terms in (11.4) one is, however, already controlled by condition (4.17) and the last term is small. The condition (4.4) above which was not really needed until now will also be needed in Subsection 11.2.

If we choose to let all the parameters depend on a small parameter \(X\) \(< 1\) in the following way

\[
s = X, \ d = X^6, \ \varepsilon_T = X^{23/4}, \ K_\ell = X^{-3/2}, \ K_B = X^{-6}, \ K_M = X^{-1}, \ K_H = X^{-8/3}, \tag{4.26}
\]
then all the conditions (4.4)–(4.6), (4.16) will be satisfied. In order to satisfy (4.7), (4.8), (4.11), (4.12), (4.18)–(4.20), (4.22)–(4.24) of which the most restrictive is (4.19), we can choose

\[ X = (\rho \mu a^3)^{1/323}. \]  

(4.27)

We can choose the integer \( M = 30 \) to ensure that (4.3), (4.17),(4.21), and (4.25) hold. Finally, (4.10) holds if

\[ \int v/a \ll (\rho \mu a^3)^{-2/323}. \]  

(4.28)

To get all the arguments to work we need the assumptions

\[ R \leq K_B (\rho \mu a^3)^{1/4}(\rho \mu a)^{-1/2}, \quad R/\ell \ll (\rho \mu a^3)^{1/4}, \quad R/a \ll (\rho \mu a^3)^{-1/4}. \]  

(4.29)

The third assumption (which could be improved slightly) is the most restrictive and is used in (11.8). The first assumption is used in Appendix B and the second assumption says that the range of the potential should be sufficiently much smaller than the size of the large boxes.

5 Localization

5.1 Setup and notation

The main part of the analysis will be carried out on a box \( \Lambda = [-\ell/2, \ell/2]^3 \) of size \( \ell \) given in (4.1). In this section we will carry out the localization to the box \( \Lambda \). The main result is given at the end of the section as Theorem 5.7 which states that for a lower bound it suffices to consider a ‘box energy’, i.e. the ground state energy of a Hamiltonian localized to a box of size \( \ell \). For convenience, in Theorem 5.8 we state the bound on the box energy that will suffice in order to prove Theorem 3.1.

It will be important to make an explicit choice of a localization function \( \chi \in C_0^{M-1}(\mathbb{R}^3) \), for \( M \in \mathbb{N} \) with support in \([-1/2, 1/2]^3 \). It is given in Appendix C. The function will not be smooth but it will be important in the analysis that we choose \( M \in \mathbb{N} \) finite but sufficiently large. The explicit choice \( M = 30 \) was explained in the previous section. We choose \( \chi \) such that

\[ 0 \leq \chi, \quad \int \chi^2 = 1. \]  

(5.1)

We will also use the notation

\[ \chi_{\Lambda}(x) := \chi(x/\ell). \]  

(5.2)

For given \( u \in \mathbb{R}^3 \), we define

\[ \chi_u(x) = \chi(\frac{x}{\ell} - u) = \chi_{\Lambda}(x - u\ell). \]  

(5.3)

Notice that \( \chi_u \) localizes to the box \( \Lambda(u) := \ell u + [-\ell/2, \ell/2]^3 \).

We will also need the sharp localization function \( \theta_u \) to the box \( \Lambda(u) \), i.e.

\[ \theta_u := 1_{\Lambda(u)}. \]  

(5.4)

Define \( P_u, Q_u \) to be the orthogonal projections in \( L^2(\mathbb{R}^3) \) defined by

\[ P_u\varphi := \ell^{-3}(\theta_u, \varphi)\theta_u, \quad Q_u\varphi := \theta_u\varphi - \ell^{-3}(\theta_u, \varphi)\theta_u. \]  

(5.5)
Localization

In the case $u = 0$ we will use the notations

$$ P_{u=0} = P_\Lambda = P \quad Q_{u=0} = Q_\Lambda = Q $$

(5.6)

Define furthermore

$$ W(x) := \frac{v(x)}{\chi * \chi(x/\ell)}, $$

(5.7)

That $W$ is well-defined for sufficiently small values of $\rho_\mu$, uses that $v$ has finite range. Manifestly $W$ depends on $\ell$ and thus $\rho_\mu$, but we will not reflect this in our notation.

Define the localized potentials

$$ w_u(x, y) := \chi_u(x) W(x - y) \chi_u(y), \quad w(x, y) := w_{u=0}(x, y). $$

(5.8)

Notice the translation invariance,

$$ w_{u+\tau}(x, y) = w_u(x - \ell \tau, y - \ell \tau). $$

(5.9)

For some estimates it is convenient to invoke the scattering solution and thus we introduce the notation, which again is well-defined for $\rho_\mu a^3$ sufficiently small,

$$ W_1(x) := W(x)(1 - \omega(x)) = \frac{g(x)}{\chi * \chi(x/\ell)}, \quad w_1(x, y) := w(x, y)(1 - \omega(x - y)), \quad w_1(x, y) := w(x, y)(1 - \omega^2(x - y)). $$

(5.10)

If we add a subscript $u$ we mean as above the translated versions $w_{1,u}(x, y) = w_1(x-\ell u, y-\ell u)$. For $\rho_\mu a^3$ sufficiently small a simple change of variables yields, for all $u \in \mathbb{R}^3$, the identities

$$ \frac{1}{2} \ell^{-6} \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} \chi_{x/\ell} \chi_{y/\ell} W_1(x - y) dx \, dy = \frac{1}{2} \ell^{-6} \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} w_1(x, y) dx \, dy = \frac{1}{2} \ell^{-3} \int g = 4\pi a \ell^{-3} $$

(5.11)

and likewise

$$ \frac{1}{2} \ell^{-6} \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} w_2(x, y) dx \, dy = \frac{1}{2} \ell^{-3} \int g(1 + \omega). $$

(5.12)

The following simple lemma will often be useful.

**Lemma 5.1.**

$$ g(x) \leq W_1(x) \leq g(x)(1 + C \frac{R^2}{\ell^2}). $$

(5.13)

**Proof.** The proof is an easy estimate of the convolution, noting that its maximum is attained at the origin.

**Lemma 5.2.** Suppose that $f \in L^1(\mathbb{R}^3)$ satisfies $\text{supp} \, f \subset B(0, R)$ and $f(-x) = f(x)$. Then

$$ \left| f * \chi_\Lambda(x) - \chi_\Lambda(x) \int f \right| \leq \max_{i,j} \| \partial_i \partial_j \chi \|_\infty \left( \frac{R}{\ell} \right)^2 \int |f|. $$

(5.14)
Proof. The proof is an easy application of a Taylor expansion and the integral representation
\[ f \ast \chi_\Lambda(x) - \chi_\Lambda(x) \int f = \int f(y) \left[ \chi_\Lambda(x - y) - \chi_\Lambda(x) \right] dy. \]

Lemma 5.3. Suppose that \( R/\ell \leq 1 \). For some universal constant \( C > 0 \) we have
\[ \left| (2\pi)^{-3} \int \frac{\hat{W}_1(k)^2}{2k^2} \, dk - \hat{g} \hat{\omega}(0) \right| \leq C(R/\ell)^2 \hat{g} \hat{\omega}(0). \] (5.15)

We also get
\[ \int \left( \frac{\hat{W}_1(k) - \hat{g}(k)^2}{2k^2} \right) \, dk \leq C \frac{R^4}{\ell^4} \hat{g} \hat{\omega}(0). \] (5.16)

Proof. Recall that \( \hat{\omega}(k) = \frac{\hat{g}(k)}{2k^2} \) by (2.6). Using the Fourier transformation and (5.13) we get
\[ \left| (2\pi)^{-3} \int \frac{\hat{W}_1^2(k) - \hat{g}^2(k)}{2k^2} \, dk \right| = C \int \int \frac{(W_1 - g)(x)(W_1 + g(y))}{|x - y|} \, dx \, dy \]
\[ \leq 3C \frac{R^2}{\ell^2} \int \int \frac{g(x)g(y)}{|x - y|} \, dx \, dy \]
\[ = C \frac{R^2}{\ell^2} \hat{g} \hat{\omega}(0). \] (5.17)

This finishes the proof of (5.15). The proof of (5.16) follows from a similar calculation and is omitted.

5.2 Localization of the kinetic and potential energies

We will use a sliding localization technique developed in the paper [7] where we estimate the kinetic energy \(-\Delta\) in \( \mathbb{R}^3 \) below by an integral over kinetic energy operators in the boxes \( \Lambda(u) \).

The following theorem is essentially Lemma 3.7 in [7].

Lemma 5.4 (Kinetic energy localization). Let \(-\Delta^N_u\) denote the Neumann Laplacian in \( \Lambda(u) \). If the regularity of \( \chi \) has \( M \geq 5 \) (e.g., for our choice 30) and the positive parameters \( \varepsilon_T, d, s, b \) are smaller than some universal constant then for all \( \ell > 0 \) we have
\[ \int_{\mathbb{R}^3} T_u du \leq -\Delta \] (5.18)

where
\[ T_u := \frac{1}{2} \varepsilon_T (d\ell)^{-2} - \frac{-\Delta^N_u}{-\Delta^N_u + (d\ell)^{-2}} + b \ell^{-2} Q_u + b \varepsilon_T (d\ell)^{-2} Q_u \chi_{(d-2\ell, \infty)}(\sqrt{-\Delta}) Q_u + T'_u \] (5.19)

with
\[ T'_u := Q_u \chi_u \left\{ (1 - \varepsilon_T) \left[ \sqrt{-\Delta} - \frac{1}{2}(s\ell)^{-1} \right]^2 + \varepsilon_T \left[ \sqrt{-\Delta} - \frac{1}{2}(d\ell)^{-1} \right]^2 \right\} \chi_u Q_u. \] (5.20)
Localization

Proof. In Lemma 3.7 in [7] we have the same inequality except that the terms above

\[ \frac{1}{2} \varepsilon_T (d \ell)^{-2} \frac{-\Delta u}{-\Delta u} + (d \ell)^{-2} + b \varepsilon_T (d \ell)^{-2} Q_u \chi_{(d^{-2} \ell^{-1}, \infty)} (\sqrt{-\Delta}) Q_u. \]

are replaced by the term \( \varepsilon_T (d \ell)^{-2} \frac{-\Delta u}{-\Delta u} + (d \ell)^{-2} \).

Using scaling it is clear that we may assume \( \ell = 1 \). The proof in Lemma 3.7 in [7] relies on the inequality (see (44) in [7])

\[ d^{-2} \int_{\mathbb{R}^3} \frac{-\Delta u}{-\Delta u} + d^{-2} du \leq d^{-2} \frac{-\Delta}{-\Delta + d^{-2}}. \]

The lemma above will follow in the same way if we can also prove that

\[ bd^{-2} \int_{\mathbb{R}^3} Q_u \chi_{(-d^{-2}, \infty)} (\sqrt{-\Delta}) Q_u du \leq \frac{1}{2} \frac{-\Delta}{-\Delta + d^{-2}}. \]

(5.21)

Using Lemma 3.3 in [7] (with \( \chi_u = \theta_u = \chi^{(u)} \) and \( K(p) = bd^{-2} \chi_{(d^{-2}, \infty)} \)) we can explicitly calculate the operator on the left in (5.21) to be \( H(-i \nabla) \) where

\[ H(p) = (2\pi)^{-3} bd^{-2} \int_{|q| > d^{-2}} (\hat{\theta}(p) \hat{\theta}(q) - \hat{\theta}(q - p))^2 dq \]

\[ \leq (2\pi)^{-3} 2bd^{-2} \left( \hat{\theta}(p) - 1 \right)^2 \int_{|q| > d^{-2}} \hat{\theta}(q)^2 dq + (2\pi)^{-3} 2bd^{-2} \int_{|q| > d^{-2}} \left( \hat{\theta}(q - p) - \hat{\theta}(q) \right)^2 dq. \]

(5.22)

We clearly have \( H(0) = 0 \) and \( 0 \leq H(p) \leq C b d^{-2} \). It is easy to see that

\[ \theta(q)^2 \leq C \frac{1}{(1 + q_1^2)(1 + q_2^2)(1 + q_3^2)}, \quad \left| \hat{\theta}(q - p) - \hat{\theta}(q) \right|^2 \leq C \frac{|p|^2}{(1 + q_1^2)(1 + q_2^2)(1 + q_3^2)} \]

It then follows that \( H(p) \leq C b \min \{|p|^2, d^{-2}\} \). Hence (5.21) holds if \( b \) is smaller than a universal constant.

\[ \square \]

Remark 5.5. The kinetic operator in (5.19) looks complicated. This is partly because we need to localize it even further into smaller boxes in order to get a priori estimates (see Appendix B). The first term in (5.19) will give us a Neumann gap in the small boxes. The second term in (5.19) is a Neumann gap in the large boxes. The third term in (5.19) will control errors coming from excited particles with very large momenta (see Lemma 8.1 and the estimate (11.14) in Lemma 11.5). Finally the term \( T'_u \) is the main kinetic energy term in the large boxes.

The localization of the potential energy is much simpler and relies on the identity in the following lemma which is a straightforward computation similar to Proposition 3.1 in [7].

Lemma 5.6 (Potential energy localization). For points \( x_1, \ldots, x_N \in \mathbb{R}^3 \) we have with the definitions of \( w_{1,u} \) and \( w_u \) in (5.8) and (5.10) that

\[ -\rho \sum_{i=1}^N \int g(x_i - y) dy + \sum_{i<j} v(x_i - x_j) \]

\[ = \int_{\mathbb{R}^3} \left[ -\rho \sum_{i=1}^N \int w_{1,u}(x_i, y) dy + \sum_{i<j} w_u(x_i, x_j) \right] du, \]

(5.23)
5.3 The localized Hamiltonian

The localized Hamiltonian \( H_{\Lambda,u} \) will be an operator on the symmetric Fock space over \( L^2(\mathbb{R}^3) \) preserving particle number. Its action on the \( N \)-particle sector is as

\[
(H_{\Lambda,u}(\rho_\mu))_N := \sum_{i=1}^{N} T_u^{(i)} - \rho_\mu \sum_{i=1}^{N} \int w_{1,u}(x_i, y) dy + \sum_{i<j} w_{u}(x_i, x_j),
\]

where the kinetic energy operator was given in (5.19) above. We abbreviate

\[
T_u := T_{u=0}, \quad H_\Lambda(\rho_\mu) := H_{\Lambda,u=0}(\rho_\mu).
\]

We will also write

\[
\chi_\Lambda := \chi_{u=0} = \chi(\cdot/\ell).
\]

Define the ground state energy and energy density in the box, by

\[
E_\Lambda(\rho_\mu) := \inf \text{Spec } H_\Lambda(\rho_\mu), \quad e_\Lambda(\rho_\mu) := \ell^{-3} \inf \text{Spec } H_\Lambda(\rho_\mu) = \ell^{-3} E_\Lambda(\rho_\mu).
\]

With these conventions, we find

**Theorem 5.7.** We have

\[
e_0(\rho_\mu) \geq e_\Lambda(\rho_\mu).
\]

**Proof.** The proof of this statement follows from the fact that \((H_{\Lambda,u}(\rho_\mu))_N\) and \((H_{\Lambda,u'}(\rho_\mu))_N\) are unitarily equivalent by (5.9). Therefore, using Lemma 5.4 and Lemma 5.6 we find that

\[
H_{\rho_\mu,N}(\rho_\mu) \geq \int_{\Omega + B(0,\ell/2)} (H_{\Lambda,u}(\rho_\mu))_N du \geq \ell^{-3} |\Omega + B(0,\ell/2)| E_\Lambda(\rho_\mu).
\]

Now the desired result follows upon using that \(|\Omega + B(0,\ell/2)|/|\Omega| \to 1\) in the thermodynamic limit. \(\square\)

It is clear, using Theorem 5.7, that Theorem 3.1 is a consequence of the following theorem on the box Hamiltonian. Therefore, the remainder of the paper will be dedicated to the proof of Theorem 5.8 below.

**Theorem 5.8.** Suppose that \( v \) satisfies Assumption 4.28, and the third assumption in (4.29). Then with \( K_\ell, M, R, \) and \( X \) as given in Section 4 we have in the limit \( \rho_\mu a^3 \to 0 \),

\[
e_\Lambda(\rho_\mu) \geq -4\pi \rho_\mu^2 a + 4\pi \rho_\mu^2 a \frac{128}{15\sqrt{\pi}} (\rho_\mu a^3)^{1/2} - C \rho_\mu^2 a (\rho_\mu a^3)^{1/2} \left( X^2 R + \frac{R^2}{a^2} (\rho_\mu a^3)^{1/2} + X^2 \right).
\]

5.4 Potential energy splitting

Using that \( P + Q = 1_\Lambda \) we will in Lemma 5.9 below arrive at a very useful decomposition of the potential.

Define the (commuting) operators

\[
n_0 = \sum_{i=1}^{N} P_i, \quad n_+ = \sum_{i=1}^{N} Q_i, \quad n = \sum_{i=1}^{N} 1_{\Lambda,i} = n_0 + n_+.
\]
We furthermore define
\[
\rho_+ := n_+ \ell^{-3}, \quad \rho_0 := n_0 \ell^{-3}.
\] (5.32)

A crucial idea in this paper is to write the potential energy in the form given in the next lemma, where the important observation is to identify the positive term \(Q^\text{ren}_4\) which we will ignore in our lower bound.

**Lemma 5.9** (Potential energy decomposition). We have
\[
- \rho_\mu \sum_{i=1}^N \int w_1(x_i, y) \, dy + \frac{1}{2} \sum_{i \neq j} w(x_i, x_j) = Q^\text{ren}_0 + Q^\text{ren}_1 + Q^\text{ren}_2 + Q^\text{ren}_3 + Q^\text{ren}_4, \tag{5.33}
\]
where
\[
Q^\text{ren}_4 := \frac{1}{2} \sum_{i \neq j} \left[ Q_i Q_j + (P_i P_j + P_i Q_j + Q_i P_j) \omega(x_i - x_j) \right] w(x_i, x_j)
\]
\[
\times \left[ Q_j Q_i + \omega(x_i - x_j)(P_j P_i + P_j Q_i + Q_j P_i) \right], \tag{5.34}
\]
\[
Q^\text{ren}_3 := \sum_{i \neq j} P_i Q_j w_1(x_i, x_j) Q_j Q_i + h.c. \tag{5.35}
\]
\[
Q^\text{ren}_2 := \sum_{i \neq j} P_i Q_j w_2(x_i, x_j) P_j Q_i + \sum_{i \neq j} P_i Q_j w_2(x_i, x_j) Q_j P_i
\]
\[
- \rho_\mu \sum_{i=1}^N Q_i \int w_1(x_i, y) \, dy + \frac{1}{2} \sum_{i \neq j} (P_i P_j w_1(x_i, x_j) Q_j Q_i + h.c.), \tag{5.36}
\]
\[
Q^\text{ren}_1 := \sum_{i,j} P_j Q_i w_2(x_i, x_j) P_i P_j - \rho_\mu \sum_i Q_i \int w_1(x_i, y) \, dy P_i + h.c. \tag{5.37}
\]
\[
Q^\text{ren}_0 := \frac{1}{2} \sum_{i \neq j} P_i P_j (w_2(x_i, x_j) P_j P_i - \rho_\mu \sum_i P_i \int w_1(x_i, y) \, dy P_i) \tag{5.38}
\]

**Proof.** The identity (5.33) follows using simple algebra and the identitites (5.11) and (5.12). We simply write \(P_i + Q_i = 1_{\Lambda_i}\) for all \(i\). Inserting this identity in both \(i\) and \(j\) on both sides of \(w(x_i, x_j)\) and expanding yields 16 terms, which we have organized in a positive \(Q_4\) term and terms depending on the number of \(Q\)’s occuring. \(\square\)

It will be useful to rewrite and estimate these terms as in the following lemma.

**Lemma 5.10.** If \(v\) and hence \(W_1\) are non-negative we have
\[
Q^\text{ren}_0 = -n_0(n_0 - 1) \frac{1}{2|\Lambda|^2} \iint w_2(x, y) \, dx \, dy - \rho_\mu \frac{n_0}{|\Lambda|} \iint w_1(x, y) \, dx \, dy
\]
\[
= -n_0(n_0 - 1) \frac{1}{2|\Lambda|} \left( g(0) + \tilde{g}(0) \right) - \rho_\mu n_0 g(0), \tag{5.39}
\]
\[
Q^\text{ren}_1 = (n_0|\Lambda|^{-1} - \rho_\mu) \sum_i Q_i \chi_\Lambda(x_i) W_1 * \chi_\Lambda(x_i) P_i + h.c.
\]
\[
+ n_0|\Lambda|^{-1} \sum_i Q_i \chi_\Lambda(x_i)(W_1 \omega) * \chi_\Lambda(x_i) P_i + h.c. \tag{5.40}
\]
\[15\]
and
\[ Q_2^{\text{ren}} \geq \sum_{i \neq j} P_i Q_j w_2(x_i, x_j) P_j Q_i + \frac{1}{2} \sum_{i \neq j} (P_i P_j w_1(x_i, x_j) Q_j Q_i + h.c.) \]
\[ + \left( (\rho_0 - \rho_\mu) \bar{W}_1(0) + \rho_\mu \bar{W}_1(0) \right) \sum_i Q_i \chi_A(x_i)^2 Q_i - C(\rho_\mu + \rho_0) a (R/\ell)^2 n_+ . \] (5.41)

**Proof.** The rewriting of \( Q_0 \) is straightforward. The rewriting of \( Q_1^{\text{ren}} \) follows from
\[ Q_1^{\text{ren}} = \left( n_0 |\Lambda|^{-1} - \rho_\mu \right) \sum_i Q_i \int w_1(x_i, y) dy P_i + h.c. \]
\[ + \left( n_0 |\Lambda|^{-1} \right) \sum_i Q_i \int w_1(x_i, y) \omega(x_i - y) dy P_i + h.c. . \]

We carry out the similar calculation on the part of the 2\( Q \)-term where \( P \) acts in the same variable on both sides of the potential,
\[ Q_2^{\text{ren},1} \geq \sum_{i \neq j} P_i Q_j w_2(x_i, x_j) P_j Q_i + \frac{1}{2} \sum_{i \neq j} (P_i P_j w_1(x_i, x_j) Q_j Q_i + h.c.) \]
\[ + (\rho_0 - \rho_\mu) \sum_i Q_i \chi_A(x_i) W_1 * \chi_A(x_i) Q_i + \rho_\mu \sum_i Q_i \chi_A(x_i) (W_1 \omega) * \chi_A(x_i) Q_i . \]

At this point we invoke Lemma 5.2 to get, for example,
\[ \sum_i Q_i \chi_A(x_i) W_1 * \chi_A(x_i) Q_i \geq (\int W_1) \sum_i Q_i \chi_A(x_i)^2 Q_i \]
\[ - \max_{i,j} \| \partial_i \partial_j \chi \|_\infty (R/\ell)^2 \left( \int W_1 \right) \| \chi \|_\infty n_+ . \] (5.42)

The decomposition in Lemma 5.9 easily implies a simple lower bound on the potential energy.

**Lemma 5.11** (Simple bound on the potential energy). For all \( x_1, \ldots, x_N \in \mathbb{R}^3 \) we have if the 2-body potential \( v \geq 0 \) the following bound on the potential energy
\[ -\rho_\mu \sum_{i=1}^N \int w_1(x_i, y) dy + \frac{1}{2} \sum_{i \neq j} w(x_i, x_j) \geq -C(n^2 \ell^{-3} + \rho_\mu^2 \ell^3) a + \frac{1}{2} Q_4^{\text{ren}} . \] (5.43)

Moreover, we also have the bounds
\[ \pm Q_1^{\text{ren}} \leq C(n^2 \ell^{-3} + \rho_\mu^2 \ell^3) a \] (5.44)
\[ \pm \left( \sum_{i \neq j} Q_j Q_i w_1(x_i, x_j) P_i P_j + h.c. \right) \leq C n^2 \ell^{-3} a + \frac{1}{4} Q_4^{\text{ren}} \] (5.45)
\[ \pm \left( \sum_{i,j} P_j Q_i w_1(x_i, x_j) Q_j Q_i + h.c. \right) \leq C n^2 \ell^{-3} a + \frac{1}{4} Q_4^{\text{ren}} , \] (5.46)

for any (not necessarily selfadjoint) operator \( Q' \) on \( L^2(\mathbb{R}^3) \) with \( QQ' = Q' \) and \( \| Q' \| \leq 1 \).
Proof. Since \( 0 \leq \int W_1 \leq Ca \) we have
\[
0 \leq \rho_\mu \sum_{i=1}^{N} \int w_1(x_i, y) \, dy \leq Ca \| \chi_\Lambda \|_\infty^2 \rho_\mu n \leq Ca \| \chi_\Lambda \|_\infty^2 (\rho_\mu^2 \epsilon^3 + n^2 \epsilon^{-3}). \quad (5.47)
\]
The off-diagonal terms in the one-body potential can be estimated using a Cauchy-Schwarz inequality relying on the positivity of \( w_1 \)
\[
\pm \rho_\mu \left( \sum_{i=1}^{N} P_i \int w_1(x_i, y) \, dy Q_i + h.c. \right) \leq \rho_\mu \sum_{i=1}^{N} P_i \int w_1(x_i, y) \, dy P_i \\
+ \rho_\mu \sum_{i=1}^{N} Q_i \int w_1(x_i, y) \, dy Q_i \\
\leq Ca (1 + \| \chi_\Lambda \|_\infty^2) \rho_\mu n. \quad (5.48)
\]
We also have
\[
0 \leq \sum_{i,j} P_i Q_j w_1(x_i, x_j) P_i Q_j = n_0 |\Lambda|^{-1} \sum_{j} Q_j \chi_\Lambda(x_j) W_1 * \chi_\Lambda(x_j) Q_j \leq C n_0 n_+ \epsilon^{-3} a \| \chi_\Lambda \|_\infty^2
\]
or more generally using again Cauchy-Schwarz inequalities we have for all \( k = 0, 1, \ldots \)
\[
\pm \left( \sum_{i,j} P_i Q'_j (w_1 \omega^k) (x_i, x_j) P_i Q_j + h.c. \right) \leq C n_0 \epsilon^{-3} a \| \chi_\Lambda \|_\infty^2 \left( \varepsilon n_+ + \varepsilon^{-1} \sum_{i} Q'_i Q'^*_i \right), \quad (5.49)
\]
\[
\pm \left( \sum_{i,j} P_i Q'_j (w_1 \omega^k) (x_i, x_j) P_j Q_i + h.c. \right) \leq C n_0 \epsilon^{-3} a \| \chi_\Lambda \|_\infty^2 \left( \varepsilon n_+ + \varepsilon^{-1} \sum_{i} Q'_i Q'^*_i \right), \quad (5.50)
\]
\[
\pm \left( \sum_{i,j} P_j Q'_i (w_1 \omega^k) (x_i, x_j) P_i Q_j + h.c. \right) \leq \sum_{i,j} P_i Q'_i (w_1 \omega^k) (x_i, x_j) Q'^*_i P_j \\
+ \sum_{i,j} P_j P_i (w_1 \omega^k) (x_i, x_j) P_i P_j \\
\leq C n_0 a \epsilon^{-3} \left( \| \chi_\Lambda \|_\infty^2 \sum_{i} Q'_i Q'^*_i + n_0 \right), \quad (5.51)
\]
for all \( \varepsilon > 0 \), where we have abbreviated \( (w_1 \omega^k)(x_1, x_2) = w_1(x_1, x_2) \omega(x_1 - x_2)^k \). In this proof we will choose \( \varepsilon = 1 \) and use \( \sum_{i} Q'_i Q'^*_i \leq n_+ \leq n \). The freedom to choose \( \varepsilon \neq 1 \) will be used in the proof of Corollary 5.12 below. The estimates in \((5.49)-(5.51)\) prove \((5.44)\) if we recall that \( w_2 = w_1 (1 + \omega) \) and choose \( Q' = Q \).

To prove \((5.46)\) we rewrite the terms in \( Q_3^{\text{ren}} \) as follows
\[
\sum_{i,j} P_i Q'_j w_1(x_i, x_j) Q_j Q_i = \sum_{i,j} \left( P_i Q'_j w_1(x_i, x_j) \left[ Q_j Q_i + \omega(x_i - x_j) (P_j P_i + P_j Q_i + Q_j P_i) \right] \right) \\
- \sum_{i,j} \left( P_i Q'_j w_1(x_i, x_j) \omega(x_i - x_j) (P_j P_i + P_j Q_i + Q_j P_i) \right) \quad (5.52)
\]
and likewise for the Hermitian conjugate terms. Thus applying a Cauchy-Schwarz inequality and the estimates \((5.49)-(5.51)\) we arrive at
\[
\pm \left( \sum_{i,j} P_i Q'_j w_1(x_i, x_j) Q_j Q_i + h.c. \right) \leq \frac{1}{2} Q_4^{\text{ren}} + C \sum_{i \neq j} P_i Q'_j w_1(x_i, x_j) (1 - \omega(x_i - x_j)) Q_j^* P_i \\
+ C n^2 a \epsilon^{-3}
\]
17
A priori bounds on particle number and excited particles

which implies (5.46). The estimate (5.45) follows in the same way. Finally, the estimate (5.43) follows from (5.45), (5.46), and (5.47) with $Q' = Q$.

In our more detailed analysis of the $Q_3$ terms in Section 8 we will need the following more refined version of the estimate in (5.46).

**Corollary 5.12.** With the same notation as in Lemma 5.11 we have for all $0 < \varepsilon < 1$

$$\sum_{i,j} \left( P_j Q'_i w_1(x_i, x_j) Q_i Q_j + P_j Q'_i w_1(x_i, x_j) \omega(x_i - x_j) P_i P_j \right) + \text{h.c.} \geq -C n_0 \varepsilon^{-3} a \left( \varepsilon^{-1} \sum_i Q'_i Q'_i^* + \varepsilon n_+ \right) - \frac{1}{4} Q_4^\text{ren}. \quad (5.53)$$

**Proof.** We again use the identity (5.52) and perform the same Cauchy-Schwarz as above, but the term with three $P$ operators now appear on the left and we do not have to estimate it using (5.51). We, however, use (5.49) and (5.50) with $0 < \varepsilon < 1$.

6 A priori bounds on particle number and excited particles

In the section we will give some important a priori bounds on the particle number $n$, the number of excited particles $n_+$ as well as on some of the potential energy terms. The bounds on $n$ and $n_+$ essentially say that for states with sufficiently low energy $n$ is close to what one would expect, i.e., $\rho_n \varepsilon^3$ and the expectation of $n_+$ is smaller with a factor which is not much worse than the relative LHY error. These bounds are difficult to prove and are given in (6.2) below. The proof is in Appendix B. They rely on a very detailed analysis of a further localization into smaller boxes.

**Theorem 6.1 (A priori bounds).** Assume that the conditions (4.3), (4.4), (4.6), (4.7), and (4.29) on $K_B$, $R$, $\varepsilon_T$, $s$, and $d$ are satisfied and that $\rho_n \varepsilon^3$ is small enough. Then there is a universal constant $C > 0$ such that if $\Psi \in F_n(L^2(\Lambda))$ is an $n$-particle normalized state in the bosonic Fock space over $L^2(\Lambda)$ satisfying

$$\langle \Psi, \mathcal{H}_\Lambda(\rho_n) \Psi \rangle \leq -4\pi \rho_n^2 a \varepsilon^3 (1 - J(\rho_n \varepsilon^3)^2) \quad (6.1)$$

for $0 < J \leq K_B^3$ (the freedom to take $J < K_B^3$ will be used Lemma 7.2) then

$$|n \varepsilon^3 - \rho_n| \leq \rho_n C K_B^{3/2} K(\rho_n \varepsilon^3)^{1/4}, \quad \text{and} \quad \langle \Psi, n_+ \Psi \rangle \leq C \rho_n \varepsilon^3 K_B^3 K_2(\rho_n \varepsilon^3)^{1/2}. \quad (6.2)$$

Moreover, we also have

$$0 \leq \langle \Psi, Q_4^\text{ren} \Psi \rangle \leq C \rho_n^2 a \varepsilon^3, \quad (6.3)$$

and

$$\left| \langle \Psi, \rho_n \sum_{i=1}^N (P_i \int w_1(x_i, y) dy Q_i + \text{h.c.}) \Psi \rangle \right| + \left| \langle \Psi, \sum_{i \neq j} (Q_j P_i w(x_i, x_j) P_i P_j + \text{h.c.}) \Psi \rangle \right|$$

$$+ \left| \langle \Psi, \sum_{i,j} (P_j Q_i w(x_i, x_j) Q_i Q_j + \text{h.c.}) \Psi \rangle \right| + \left| \langle \Psi, \sum_{i \neq j} (Q_j Q_i w(x_i, x_j) P_i P_j + \text{h.c.}) \Psi \rangle \right|$$

$$\leq C \rho_n^2 \varepsilon^3 \int v. \quad (6.4)$$

Note that the expressions on the left of (6.4) above contain $w$ instead of $w_1$ which appeared in (5.44)–(5.46). We will need the estimates with $w$ instead of $w_1$ in the next section and this will be the only place where estimates containing $\int v$ will appear.
Localization of the number of excited particles $n_+$

Proof. As explained, the bounds (6.2) are proved in Theorem [3.6]. Due to our assumptions they, in particular, imply that $n \leq C \rho \mu \epsilon^3$.

This a priori bound on $n$, the positivity of the kinetic energy $\mathcal{T}$, and the bound in (5.43) immediately imply

$$\langle \Psi, \mathcal{H}_A(\rho) \Psi \rangle \geq -C \rho^2 \mu \epsilon^3 + \frac{1}{2} \langle \Psi, Q^\text{ren}_4 \Psi \rangle$$

which by the assumption on $\Psi$ gives the bound (6.3).

The bounds on the first two terms in (6.4) follow exactly as the proofs of (5.49)-(5.51) for $k = 0$ and with $w_1$ replaced by $w$ such that $a$ has to be replaced by $\int v \geq 8 \pi a$ in the bounds. The bounds on the last two terms in (6.4) follow the same lines as the proof of (5.45) and (5.46). We sketch it for the last term in (6.4). We rewrite

$$\sum_{i \neq j} P_i P_j w(x_i, x_j) Q_i Q_j = \sum_{i \neq j} P_i P_j w(x_i, x_j) \left( Q_i Q_j + \omega(x_i - x_j) (P_i P_j + Q_i P_j + P_i Q_j) \right)$$

and likewise for the Hermitian conjugate. If we recall that $0 \leq \omega \leq$ the last sum is estimated as in the case of (5.49)-(5.51) again with $a$ replaced by $\int v$. The first term above together with its complex conjugate is after a Cauchy-Schwarz controlled by a similar term and $Q^\text{ren}_4$. I.e., we get

$$\langle \Psi, \left( \sum_{i \neq j} P_i P_j w(x_i, x_j) Q_i Q_j + h.c. \right) \Psi \rangle \leq C \rho^2 \mu \epsilon^3 \int v + C \langle \Psi, Q^\text{ren}_4 \Psi \rangle,$$

which by the bound (6.3) implies what we want. \qed

7 Localization of the number of excited particles $n_+$

As in [7] we shall use the following theorem from [19] to restrict the number of excited particles.

Theorem 7.1 (Localization of large matrices). Suppose that $A$ is an $(N + 1) \times (N + 1)$ Hermitian matrix and let $A^{(k)}$, with $k = 0, 1, \ldots, N$, denote the matrix consisting of the $k$th supra- and infra-diagonal of $A$. Let $\psi \in \mathbb{C}^{N + 1}$ be a normalized vector and set $d_k = \langle \psi, A^{(k)} \psi \rangle$ and $\lambda = \langle \psi, A \psi \rangle = \sum_{k=0}^{N} d_k$ ($\psi$ need not be an eigenvector of $A$). Choose some positive integer $M' \leq N + 1$. Then, with $M'$ fixed, there is some $n' \in [0, N + 1 - M']$ and some normalized vector $\varphi \in \mathbb{C}^{N + 1}$ with the property that $\varphi_j = 0$ unless $n' + 1 \leq j \leq n' + M'$ (i.e., $\varphi$ has localization length $M'$) and such that

$$\langle \varphi, A \varphi \rangle \leq \lambda + \frac{C}{M'^2} \sum_{k=1}^{M'-1} k^2 |d_k| + C \sum_{k=M'}^{N} |d_k|, \quad (7.1)$$

where $C > 0$ is a universal constant. (Note that the first sum starts at $k = 1$.)

This will allow us to prove the following result. We emphasize that this is the only place in this paper where an estimate depends explicitly on $\int v$ and not just on $a$.

Lemma 7.2 (Restriction on $n_+$). Let $M$ be as defined in (4.9) and satisfying (4.10) and (4.11). Assume, moreover, that $p_\mu a^3$ is small enough. There is then a universal $C > 0$ such that if there is a normalized $n$-particle $\Psi \in \mathcal{F}_s(L^2(\Lambda))$ satisfying (6.1) under the assumptions
Localization of the number of excited particles \( n_+ \)

in Theorem 6.1 with \( J = \frac{1}{2} K_B^3 \), then there is also a normalized \( n \)-particle wave function \( \tilde{\Psi} \in \mathcal{F}_n(L^2(\Lambda)) \) with the property that

\[
\tilde{\Psi} = I_{[0,M]}(n_+)^{-1} \tilde{\Psi},
\]

(7.2)
i.e., only values of \( n_+ \) smaller than \( M \) appear in \( \tilde{\Psi} \), and such that

\[
\langle \tilde{\Psi}, \mathcal{H}_\Lambda(\rho_\mu)\tilde{\Psi} \rangle \leq \langle \Psi, \mathcal{H}_\Lambda(\rho_\mu)\Psi \rangle + C K_M^{-2} \rho_\mu^2 \epsilon^3 (\rho_\mu a^3)^{1/2} \int v.
\]

(7.3)

Proof. We may assume from (4.11) that \( M \geq 5 \) and that \( M \leq n \) since otherwise there is nothing to prove.

We shall apply Theorem 7.1 on localization of large matrices to the \( (n+1) \times (n+1) \)-matrix with elements

\[
\mathcal{A}_{i,j} = ||1_{n_+=1}\Psi||^{-1}||1_{n_+=j}\Psi||^{-1}\langle 1_{n_+=1}\Psi, \mathcal{H}_\Lambda(\rho_\mu)1_{n_+=j}\Psi \rangle.
\]

(If any of the norms are zero we set the element to zero.) Then we get a normalized vector \( \psi = (||1_{n_+=0}\Psi||, \ldots, ||1_{n_+=n}\Psi||) \) in \( \mathbb{C}^{n+1} \) and

\[
\langle \psi, \mathcal{A} \psi \rangle = \langle \Psi, \mathcal{H}_\Lambda(\rho_\mu)\Psi \rangle.
\]

Moreover, for the matrix \( \mathcal{A} \), using the notation of Theorem 7.1, only the \( \mathcal{A}^{(k)} \) with \( k = 0, 1, 2 \) are non-vanishing. In fact, we have

\[
d_1 = \langle \psi, \mathcal{A}^{(1)} \psi \rangle = \left\langle \Psi, \left( -\rho_\mu \sum_{i=1}^N (P_i \int w_1(x_i, y)dy x_i + h.c.) + \sum_{i \neq j} (Q_j w(x_i, x_j) P_i P_j + h.c.) \right) \right\rangle
\]

\[
+ \sum_{i \neq j} (P_i Q_j w(x_i, x_j) Q_i + h.c.) \rangle \Psi \right\rangle
\]

and

\[
d_2 = \langle \psi, \mathcal{A}^{(2)} \psi \rangle = \left\langle \Psi, \left( \sum_{i \neq j} (P_i P_j w(x_i, x_j) Q_i + h.c.) \right) \right\rangle \Psi \right\rangle.
\]

(7.4)

It thus follows from (6.4) that \( |d_1|, |d_2| \leq \rho_\mu^2 \epsilon^3 \int v \).

The theorem on localization of large matrices tells us that if we choose \( M' \) equal to the integer part of \( M/2 \) we can find a normalized \( \varphi \in \mathbb{C}^{n+1} \) with localization length \( M' \) such that

\[
\langle \varphi, \mathcal{A} \varphi \rangle \leq \langle \psi, \mathcal{A} \psi \rangle + C M'^{-2} (|d_1| + |d_2|)
\]

\[
\leq \langle \Psi, \mathcal{H}_\Lambda(\rho_\mu)\Psi \rangle + C K_M^{-2} \rho_\mu^2 \epsilon^3 (\rho_\mu a^3)^{1/2} \int v
\]

(7.4)

Let \( \tilde{\varphi} \in \mathbb{C}^{n+1} \) be given by \( \tilde{\varphi}_i = \varphi_i \) if \( ||1_{n_+=i}\Psi|| \neq 0 \) and \( \tilde{\varphi}_i = 0 \) if \( ||1_{n_+=i}\Psi|| = 0 \). Then \( ||\tilde{\varphi}|| \leq 1 \). We then have

\[
\langle \tilde{\varphi}, \mathcal{A} \tilde{\varphi} \rangle = \langle \varphi, \mathcal{A} \varphi \rangle \leq \langle \Psi, \mathcal{H}_\Lambda(\rho_\mu)\Psi \rangle + C K_M^{-2} \rho_\mu^2 \epsilon^3 (\rho_\mu a^3)^{1/2} \int v < 0.
\]

(7.5)

where the negativity follows from \( J = \frac{1}{2} K_B^3 \), (4.7), and (4.10). In particular, \( \tilde{\varphi} \neq 0 \). Define

\[
\tilde{\Psi} = ||\tilde{\varphi}||^{-1} \sum_{i=0}^n \tilde{\varphi}_i ||1_{n_+=i}\Psi||^{-1}1_{n_+=i}\Psi.
\]
Then $\tilde{\Psi}$ is normalized and satisfies
\[
\langle \tilde{\Psi}, H_\Lambda(\rho_\mu)\tilde{\Psi} \rangle = \|\tilde{\varphi}\|^{-2}\langle \tilde{\varphi}, A\tilde{\varphi} \rangle \leq \langle \tilde{\varphi}, A\tilde{\varphi} \rangle
\]
since the term on the right is negative and $\|\tilde{\varphi}\|^{-2} \geq 1$. This proves that $\tilde{\Psi}$ satisfies (7.3).

It remains to prove that $\tilde{\Psi}$ satisfies (7.2). We know from the construction that the possible values of $n_+$ that occur in $\tilde{\Psi}$ lie in an interval of length $\mathcal{M}'$. We need to prove that this interval lies close to zero. This follows from the estimate (7.3), $J = \frac{1}{2}K_B^3$, and (4.10), which imply that we may use the a priori bound (6.2) on the expectation value of $n_+$ in $\tilde{\Psi}$. The consequence is that the interval of $n_+$ values in $\tilde{\Psi}$ must be contained in
\[
[0, \mathcal{M}' + C\rho_\mu\ell^3K_B^3K_1^2(\rho_\mu a^3)^{1/2}] = [0, \mathcal{M}' + CK_B^3K_1^2] \subseteq [0, \mathcal{M}]
\]
by (4.11).

\[
\square
\]

8 Localization of the 3Q-term

In this section we will absorb an unimportant part of the 3Q term in the positive 4Q term.

We first define the ‘low’ and ‘high’ momentum regions as follows.
\[
P_L := \{p| \leq K_L|\mu a^3|\}, \quad P_H := \{p| \geq \tilde{K}_H^{-1}(\rho_\mu a^3)^{1/2} a^{-1}\} = \{p| \geq K_H^{-1}a^{-1}\}
\]
where $K_L, \tilde{K}_H$ were defined in Section 4. The somewhat peculiar definition of $P_H$ is convenient for later estimates (see proof of Lemma 9.3). We will always assume that (4.18) is satisfied. This assures that $P_L$ and $P_H$ are disjoint.

We will define the low momentum localization operator $Q_L$ as follows. Let $f \in C^\infty(\mathbb{R})$ be a monotone non-increasing function satisfying that $f(s) = 1$ for $s \leq 1$ and $f(s) = 0$ for $s \geq 2$. We further define
\[
f_L(s) := f\left(\frac{s}{K_L|\mu a^3|}\right).
\]
I.e. $f_L$ is a smooth localization to the low momenta $P_L$. With this notation, we define
\[
Q_L := Qf_L(\sqrt{-\Delta}), \quad \overline{Q}_L := Q(1 - f_L(\sqrt{-\Delta})).
\]
Notice that $Q_L$ is not self-adjoint.

We will choose $K_L$ such that $K_L|\mu a^3| = d^{-2}\ell^{-1}$—this is equivalent to (4.15)—where $d$ is from the definition of the ‘small boxes’.

We define
\[
n_H^+ := \sum Q_{1(\ell-2\ell-1,\infty)}(\sqrt{-\Delta})Q.
\]
With this definition and the choice of $K_L$ above, we have
\[
\sum \overline{Q}_L(\overline{Q}_L)^* \leq n_H^+.
\]

Lemma 8.1. Define
\[
\tilde{Q}_3^{(1)} := \sum_{i\neq j}(P_iQ_{L,j}w_1(x_i, x_j)Q_jQ_i + h.c.),
\]
We assume (4.4), (4.17) and (6.2) With the notation from (5.34), (5.35), we get,
\[
Q_3^{\text{ren}} + \frac{1}{4} Q_1^{\text{ren}} + \frac{b}{100} (\ell^{-2}n_+ + \varepsilon T(d\ell)^{-2}n_+^H) \geq \tilde{Q}_3^{(1)} - C\rho_\mu^2a\ell^3 \left((K_1L)^{1-M} + \frac{R^2}{\ell^2}\right).
\]
Proof. Using Corollary \(5.12\) with \(Q' = \mathcal{Q}_L\) and \(\varepsilon = cK_\ell^{-2}\) for some sufficiently small constant \(c\), as well as (8.5) we find
\[
\frac{1}{4} Q^{\text{ren}}_L + \frac{b}{100\ell^2} n_+ + Q^{\text{ren}}_3 - \mathcal{Q}_3^{(1)} \geq \sum_{i,j} \left( P_j \mathcal{Q}_L \omega (x_i - x_j) P_i + \text{h.c.} \right) - C \ell^{-2} K_\ell^4 n_+^H.
\] (8.8)
Using (4.4) it is clear that the \(n_+^H\) term is dominated by half of the positive \(n_+^H\) term from (8.7).

To estimate the remaining terms in (8.8) we start by using the estimate on the convolution from Lemma \(5.2\) to get
\[
- \sum_{i \neq j} \left( P_i \mathcal{Q}_L \omega (x_i - x_j) P_j + \text{h.c.} \right)
\geq - I \ell^{-3} \left( \rho_0 \sum_j \mathcal{Q}_L \partial_L \chi^2 (x_j) P_j + \text{h.c.} \right) - C a n_0^2 \ell^{-3} \frac{R^2}{\ell^2},
\] (8.9)
where \(I := \int W_1(y) \omega (y) \leq C a\).

To complete the proof we write, with \(M - 1 \leq 2 \overline{M} \leq M\)
\[
\mathcal{Q}_L \chi^2 P + \text{h.c.} = \mathcal{Q}_L (\ell^{-2} - \Delta)^{-\overline{M}} \left[ (\ell^{-2} - \Delta)^{\overline{M}} \mathcal{Q}_L \right] P + \text{h.c.}
\] (8.10)
and notice that
\[
| (\ell^{-2} - \Delta)^{\overline{M}} \mathcal{Q}_L | \leq C \ell^{-2\overline{M}}.
\] (8.11)
Therefore,
\[
\mathcal{Q}_L \chi^2 P + \text{h.c.} \leq \varepsilon_2 P + \varepsilon_2^{-1} \ell^{2\overline{M}} \mathcal{Q}_L (\ell^{-2} - \Delta)^{-2\overline{M}} (\mathcal{Q}_L)^* \leq \varepsilon_2 P + \varepsilon_2^{-1} (K_\ell K_L)^{-2\overline{M}} \mathcal{Q}_L (\mathcal{Q}_L)^*.
\] (8.12)
Choosing \(\varepsilon_2 = (K_\ell K_L)^{-2\overline{M}}\) and using again (4.4) we get (8.7) upon summing this estimate in the particle indices and absorbing the \(n_+^H\) term as before. \(\square\)

9 Second quantized operators

9.1 Creation/annihilation operators

We will use \(a, a^\dagger\) to denote the standard bosonic annihilation/creation operators on the bosonic Fock space \(\mathcal{F}_s (L^2 (\Lambda))\).

We define \(a_0\) as the annihilation operator associated to the condensate function for the box \(\Lambda\), i.e. \(a_0 = \ell^{-3/2} a (\theta)\), where we recall that \(\theta\) defined in (5.4) is the characteristic function of the box. In more detail, for \(\Psi \in \otimes N L^2 (\Lambda)\) we have
\[
(a_0 \Psi)(x_2, \ldots, x_N) := \sqrt{N} \ell^{3/2} \int \psi (y, x_2, \ldots, x_N) \, dy
\]

Therefore,
\[
\langle \Psi, n_0 \Psi \rangle = \langle \Psi | a_0^\dagger a_0 \Psi \rangle = \frac{N}{\ell^3} \int \int \psi (y, x_2, \ldots, x_N) \, dy \ dy_2 \cdots dy_N.
\] (9.1)
Due to the localization function $\chi_\Lambda$ it is convenient to work with the localized annihilation/creation operators $a_k,a_k^\dagger$ defined in (9.3) below. However, we will also need the non-localized versions $\tilde{a}_k,\tilde{a}_k^\dagger$. Since these are more standard, we give their definition first.

For $k \in \mathbb{R}^3 \setminus \{0\}$ we let

$$
\tilde{a}_k := \ell^{-3/2}a(Q(e^{i k x} \theta)), \quad \tilde{a}_k^\dagger := \ell^{-3/2}a^\dagger(Q(e^{i k x} \theta)) \tag{9.2}
$$

Clearly, for $k,k' \in \mathbb{R}^3 \setminus \{0\},$

$$
[\tilde{a}_k,\tilde{a}_{k'}] = 0, \quad [\tilde{a}_k,\tilde{a}_{k'}^\dagger] = \ell^{-3} (e^{i k x} \theta, Q e^{i k' x} \theta). \tag{9.3}
$$

We also define, for $k \in \mathbb{R}^3 \setminus \{0\},$

$$
a_k := \ell^{-3/2}a(Q(e^{i k x} \chi_\Lambda)) \quad \text{and} \quad a_k^\dagger := \ell^{-3/2}a(Q(e^{i k x} \chi_\Lambda))^*. \tag{9.4}
$$

Then, for all $k, k' \in \mathbb{R}^3 \setminus \{0\},$

$$
[a_k,a_{k'}] = 0, \tag{9.5}
$$

and

$$
[a_k,a_{k'}^\dagger] = \ell^{-3}(Q(e^{i k x} \chi_\Lambda), Q(e^{i k' x} \chi_\Lambda)) = \chi^2((k-k')\ell) - \tilde{\chi}(k\ell)\tilde{\chi}(k'\ell). \tag{9.6}
$$

In particular,

$$
[a_k,a_{k'}^\dagger] \leq 1. \tag{9.7}
$$

Furthermore, we introduce the Fourier multiplier corresponding to the localized kinetic energy (after the separation of the constant term), i.e.

$$
\tau(k) := (1 - \varepsilon_T) \left( ||k| - \frac{1}{2} (s \ell)^{-1} \right)^2 + \varepsilon_T \left( ||k| - \frac{1}{2} (ds \ell)^{-1} \right)^2. \tag{9.8}
$$

We can express the different parts of the Hamiltonian $H_\Lambda(\rho_\mu)$ in second quantized formalism. We give this as the following Lemma [9.1]. The proof is a standard calculation and will be omitted.

**Lemma 9.1.** We have the following expressions for the operators in second quantized formalism (with $\mathcal{T}'$ the part of the kinetic energy operator defined in (5.20))

$$
n_0 = a_0^\dagger a_0, \quad n_0^2 = (a_0^\dagger a_0)^2 = (a_0^\dagger)^2 a_0^2 - a_0^\dagger a_0, \quad n_+ = (2\pi)^{-3} \ell^3 \int_{k \in \mathbb{R}^3} a_k^\dagger a_k dk,
$$

$$
\sum_{j=1}^N \mathcal{T}'_j = (2\pi)^{-3} \ell^3 \int_{k \in \mathbb{R}^3} \tau(k) a_k^\dagger a_k N,
$$

$$
\sum_{i \neq j} P_i P_j w_1(x_i,x_j) Q_j Q_i = (2\pi)^{-3} \int W_1(k) a_i^\dagger a_j a_{-k} a_{-j} dk,
$$

$$
\sum_{j \neq s} P_j Q_s w_2(x_i,x_j) P_s Q_j = (2\pi)^{-3} \int W_2(k) a_{-k}^\dagger a_j a_{0} a_{-s} dk,
$$

$$
\sum_i Q_i f(x_i) \chi_\Lambda(x_i) P_i = (2\pi)^{-3} \int \hat{f}(k) a_k^\dagger a_0 dk,
$$

$$
\sum_{i \neq j} P_i Q_{L,j} w_1(x_i,x_j) Q_j Q_i = \ell^3 (2\pi)^{-6} \int f_L(s) \int W_1(k) a_{0} a_{j}^\dagger a_{s-k} a_k ds. \tag{9.9}
$$
Proposition 9.2. Assume that \( \tilde{\Psi} \) satisfies \((7.2)\) and \((7.3)\) and that the parameters satisfy \((4.16)\), \((4.4)\), and \((4.17)\). Then, in 2nd quantization the operator \( \mathcal{H}_\Lambda(\rho_\mu) \) defined in \((5.24)\) satisfies

\[
\langle \tilde{\Psi}, \mathcal{H}_\Lambda(\rho_\mu)\tilde{\Psi} \rangle \geq \langle \tilde{\Psi}, \mathcal{H}_\Lambda^{2nd}(\rho_\mu)\tilde{\Psi} \rangle - C\rho_\mu^2 \ell^3 \left( (K_L K_M)^{1-M} + \frac{\ell^2}{R^2} \right),
\]

where

\[
\mathcal{H}_\Lambda^{2nd} = (2\pi)^{-3} \ell^3 \int \tau(k) a^\dagger_k a_k \, dk + \frac{b}{2\ell^2} n + \varepsilon_T \frac{b}{2\ell^2} n^H + \frac{1}{2} (\ell^{-3} a^\dagger_0 a_0 \hat{a} \hat{a}^\dagger_0) - \rho_\mu \hat{a} \hat{a}^\dagger_0 a_0 + \int (\chi_\Lambda(k) a^\dagger_k a_k \, dk + h.c.)
\]

\[
+ \frac{1}{2} \frac{\ell^{-3}}{\ell^3} a^\dagger_0 a_0 \hat{W} \hat{\omega}(1)(2\pi)^{-3} \int \chi_\Lambda(k) a^\dagger_k a_0 \, dk + h.c.
\]

\[
+ \frac{2\pi}{\ell^3} \int \left( \hat{W}_1(k) + \hat{W}_1 \hat{\omega}(k) \right) a^\dagger_k a_k a_0 + \frac{1}{2} \hat{W}_1(k) \left( a^\dagger_0 a^\dagger_k a_{-k} + a^\dagger_k a_{-k} a_0 a_0 \right) \, dk
\]

\[
+ \left( \ell^{-3} a^\dagger_0 a_0 - \rho_\mu \hat{W}_1(0) + \ell^{-3} a^\dagger_0 a_0 \hat{W} \hat{\omega}(0) \right) (2\pi)^{-3} \int a^\dagger_k a_k \, dk
\]

\[
+ \bar{Q}_3,
\]

where

\[
\bar{Q}_3 := \ell^3 (2\pi)^{-6} \int_{\{k \in P_H\}} f_L(s) \hat{W}_1(k)(a^\dagger_0 a^\dagger_k a_{-k} a_k + a^\dagger_k a_{-k} a_0 a_0).
\]

Proof. Notice that \((6.2)\) holds, using \((7.3)\) and Theorem 6.1.

We apply Lemma 6.9. For the operators \( Q_0^{\text{ren}} \) and \( Q_1^{\text{ren}} \) we use the simplifications of Lemma 5.10 before making the explicit calculation of their 2nd quantifications. For \( Q_2^{\text{ren}} \) we also use the simplifications of Lemma 5.10. The error term in \((5.41)\) is absorbed in the gap in the kinetic energy. This uses that \( R \ll (\rho_\mu a)^{-1/2} \) and the relation \( n \approx \rho_\mu \ell^3 \) from \((6.2)\).

Finally we consider \( Q_3^{\text{ren}} \) and \( Q_4^{\text{ren}} \). By Lemma 8.1 and the positivity of \( v \) we have the lower bound \((6.5)\). What remains of \( Q_3^{\text{ren}} \) will be discarded for a lower bound. The application of \((8.2)\) also costs a bit of the gap in the kinetic energy. We have left to compare \( \bar{Q}_3^{(1)} \) with \( \bar{Q}_3 \). That is the content of Lemma 9.3 below. Notice that using \((4.16)\) the error term from \((9.13)\) can be absorbed in the gap in the kinetic energy. This finishes the proof of Proposition 9.2. \( \square \)

In the above proof we used the following localization of the 3Q-term.

Lemma 9.3. Assume that \( \tilde{\Psi} \) satisfies \((7.2)\) and \((7.3)\). Let \( \bar{Q}_3^{(1)} \) be as defined in Lemma 8.1 and \( \bar{Q}_3 \) from \((9.12)\).

Then,

\[
\langle \tilde{\Psi}, \bar{Q}_3^{(1)} \tilde{\Psi} \rangle \geq \langle \tilde{\Psi}, \bar{Q}_3 \tilde{\Psi} \rangle - C\rho_\mu^{\frac{n+1}{\ell^3}} K_H^{-3/2} K_M^{1/2}.
\]

Proof. Notice that \((6.2)\) holds, using \((7.3)\) and Theorem 6.1.

In second quantization we have

\[
\bar{Q}_3^{(1)} = \ell^3 (2\pi)^{-6} \int f_L(s) \hat{W}_1(k)(a^\dagger_0 a^\dagger_k a_{-k} a_k + a^\dagger_k a_{-k} \tilde{a}_0 a_0) \, dk \, ds.
\]

24
so we have to estimate the part of the integral where $k \notin P_H$. Let $\varepsilon > 0$. Then,

$$
\ell^3 (2\pi)^{-6} \int_{\{k \leq \frac{M}{3} \varepsilon^3 a^{-1}\}} f_L(s) \tilde{W}_1(k) (a_0^\dagger a_k + a_k^\dagger a_{-k})
$$

$$
\geq -C a \ell^3 (2\pi)^{-6} \int_{\{k \leq \frac{M}{3} \varepsilon^3 a^{-1}\}} f_L(s) \left( \varepsilon a_k^\dagger a_0 a_0^\dagger a_k + \varepsilon^{-1} a_k^\dagger a_{-k} a_{-k} a_k \right)
$$

$$
\geq -C \frac{n_+}{\ell^3} \left( \varepsilon \ell^3 (K_H a)^{-3} + \varepsilon^{-1} \frac{M}{n} \right). \tag{9.15}
$$

Notice that we have not assumed that $\tilde{W}_1(k)$ has a sign and that the Cauchy-Schwarz inequality in \([0.15] \) is valid for $\tilde{W}_1(k)$ of variable sign.

We choose $\varepsilon = \left( \frac{M K_H}{n \ell^3} \right)^{1/2}$. Using the relation $n \approx \rho \ell^3$ from \((6.2) \) the error term in \((\cdot) \) becomes of magnitude $\sqrt{\frac{M}{\rho \ell^3 K_H}} = \frac{K_H^{3/2}}{K_M^{1/2}}$.

It will also be useful to notice the following representation in terms of the operators $\tilde{a}_k$.

**Lemma 9.4.** We have the identities

$$
\left( 2\pi \right)^{-6} \ell^6 \int \tilde{a}_k^\dagger \tilde{a}_{k'}^\dagger ((k - k')\ell) \tilde{a}_{k'} = \sum_{j=1}^N Q_j \chi_j^2 \langle x_j \rangle Q_j, \tag{9.16}
$$

and

$$
\left( 2\pi \right)^{-6} \ell^6 \int f_L(k) f_L(k') \tilde{a}_k^\dagger \tilde{a}_{k'}^\dagger ((k - k')\ell) \tilde{a}_{k'} = \sum_{j=1}^N Q_{L,j} \chi_j^2 \langle x_j \rangle Q_{L,j}. \tag{9.17}
$$

### 9.2 \(c\)-number substitution

It is convenient to apply the technique of $c$-number substitution as described in \([18] \).

Let $\Psi \in \mathcal{F}(L^2(\Lambda))$. We can think of $L^2(\Lambda) = \text{Ran}(P) \oplus \text{Ran}(Q)$, with $\text{Ran}(P)$ being, of course, spanned by the constant vector $\theta$ (defined in \((5.4) \)). This leads to the splitting $\mathcal{F}(L^2(\Lambda)) = \mathcal{F}(\text{Ran}(P)) \otimes \mathcal{F}(\text{Ran}(Q))$. We let $\Omega$ denote the vacuum vector in $\mathcal{F}(L^2(\Lambda))$.

For $z \in \mathbb{C}$ we define

$$
|z\rangle := \exp(-\frac{|z|^2}{2} - z a_{-0}^\dagger) \Omega. \tag{9.18}
$$

Given $z$ and $\Psi$ we can define

$$
\Phi(z) := \langle z| \Psi \rangle \in \mathcal{F}(\text{Ran}(Q)), \tag{9.19}
$$

where the inner product is considered as a partial inner product induced by the representation $\mathcal{F}(L^2(\Lambda)) = \mathcal{F}(\text{Ran}(P)) \otimes \mathcal{F}(\text{Ran}(Q))$.

It is a simple calculation that

$$
1 = \int_{\mathbb{C}} |z\rangle \langle z| d^2 z, \quad \text{and} \quad a_0|z\rangle = z|z\rangle. \tag{9.20}
$$

**Theorem 9.5.** Define

$$
\rho_z := |z|^2 \ell^3, \tag{9.21}
$$
Second quantized operators

\[ K(z) = \frac{b}{2\pi^2} n_+ + \varepsilon \frac{b}{2\pi^2} \rho_z^H + \frac{1}{2} \rho_z^2 \varepsilon^3 \left( g(0) + \tilde{g}(0) \right) - \rho_z \tilde{g}(0) \rho_z \varepsilon^3 \]
\[ + (2\pi)^{-3} \varepsilon^3 \int \left( \tau(k) + \rho_z \tilde{W}_1(k) \right) \frac{a_k^+ a_k}{2} + \frac{1}{2} \rho_z \tilde{W}_1(k) \left( a_k a_{-k} + a_k^+ a_{-k}^\dagger \right) \, dk \]
\[ + (\rho_z - \rho_\mu) \tilde{W}_1(0)(2\pi)^{-3} \varepsilon^3 \int a_k^+ a_k \, dk \]
\[ + Q_1(z) + Q_{1c}(z) + Q_{2c}(z) + Q_3(z), \]  
\tag{9.22}

with

\[ Q_1(z) := \left( (\rho_z - \rho_\mu) \tilde{W}_1(0)(2\pi)^{-3} \int \tilde{\chi}_k(k) a_k^+ a_k \, dk + h.c. \right), \]  
\tag{9.23}

\[ Q_{1c}(z) := \left( \rho_z \tilde{W}_1(k)(2\pi)^{-3} \int \tilde{\chi}_k(k) a_k^+ a_k \, dk + h.c. \right), \]  
\tag{9.24}

\[ Q_3(z) := \varepsilon^3 (2\pi)^{-6} \int \left( f_L(s) \tilde{W}_1(k) \left( a_k^+ a_{s-k} a_k + a_k^+ a_{s-k}^\dagger \tilde{a}_s \right) \right), \]  
\tag{9.25}

and

\[ Q_{2c} = Q_{2c}(z) := (2\pi)^{-3} \rho_z \varepsilon^3 \int \left( \tilde{W}_1(k)(2\pi)^{-3} \int a_k^+ a_k \right). \]  
\tag{9.26}

Assume that \( \tilde{\Psi} \) satisfies [7.2].

Then,

\[ \langle \tilde{\Psi}, H_2^{\text{2nd}}(\rho_\mu) \tilde{\Psi} \rangle \geq \inf_{z \in \mathbb{R}_+} \inf_{\Phi} \langle \Phi, K(z) \Phi \rangle - C \rho_\mu a, \]  
\tag{9.27}

where the second infimum is over all normalized \( \Phi \in \mathcal{F}(\text{Ran}(Q)) \) with

\[ \Phi = 1_{[0, M]}(n_+) \Phi. \]  
\tag{9.28}

Proof. Notice that [6.2] holds, using [7.3] and Theorem 6.1.

We define \( \tilde{K}(z) \) to be the operator \( H_2^{\text{2nd}} \) defined in [6.11] above, but where the following substitutions have been performed:

\[ a_0^+ a_0^+ a_0 a_0 \rightarrow |z|^4 - 4|z|^2 + 2, \]
\[ a_0^+ a_0 a_0 \rightarrow |z|^2 z - 2z, \quad a_0 a_0^+ a_0^+ \rightarrow |z|^2 \tau, \]
\[ a_0 a_0 \rightarrow z^2, \quad a_0^+ a_0^+ \rightarrow \tau^2, \quad a_0^+ a_0 \rightarrow |z|^2 - 1, \]
\[ a_0 \rightarrow z, \quad a_0^+ \rightarrow \tau. \]  
\tag{9.29}

Then, we will prove that

\[ \langle \tilde{\Psi}, H_2^{\text{2nd}} \tilde{\Psi} \rangle = \Re \int \langle \tilde{\Phi}(z), \tilde{K}(z) \tilde{\Phi}(z) \rangle \, d^2 z = \Re \int \langle \tilde{\Phi}(z), \tilde{K}(z) \tilde{\Phi}(z) \rangle n^2(z) \, d^2 z, \]  
\tag{9.30}

where \( n(z) = ||\Phi(z)||_{\mathcal{F}(\text{Ran}(Q))} \) and \( \tilde{\Phi}(z) = \Phi(z)/n(z) \).

To obtain (9.30) we write all polynomials in \( a_0, a_0^+ \) in anti-Wick ordering, for example \( a_0^+ a_0^+ = a_0^+ a_0^+ - 1 \). Therefore,

\[ \langle \Psi, a_0^+ a_0 \Psi \rangle = \int \langle a_0^+ \Psi | z \rangle \langle z | a_0^+ \Psi \rangle \, d^2 z = \int (|z|^2 - 1) \langle \Phi(z) | \Phi(z) \rangle \, d^2 z. \]  
\tag{9.31}
Performing this type of calculation for each term in $H_{\Lambda}^{2nd}$ yields \([9.30]\).

Suppose that $\Psi \in \mathcal{F}_a(L^2(B))$ is such that
\[
\Psi = 1_{[0,M]}(n+)\tilde{\Psi}.
\]
Notice that this relation only involves the part of $\tilde{\Psi} \in \mathcal{F}(\text{Ran}(Q))$. Therefore, we also have for all $z \in \mathbb{C}$,
\[
\tilde{\Phi}(z) = 1_{[0,M]}(n+)\tilde{\Phi}(z).
\]
with $\tilde{\Phi}(z) = \langle z|\tilde{\Psi} \rangle$ as above.

The next step of the proof is to remove the lower order terms coming from the substitutions in \([9.29]\) above.

Let us first consider the negative term $-4|z|^2$ in the substitution of $a_0^\dagger a_0^a a_0$. By undoing the integrations leading to $\tilde{\Psi}$ for this term, we see that it contributes with
\[
\int \langle \Phi(z), -4\frac{1}{2}|z|^2\ell^3(\tilde{g}(0) + \tilde{\omega}(0))\Phi(z) \rangle d^2z \geq -Ca\ell^{-3}(\tilde{\Psi}, a_0^a a_0^\dagger \tilde{\Psi}) \geq -Ca\ell^{-3}(n+1),
\]
in agreement with the error term in \([9.27]\) (using that $n \approx \rho_\mu \ell^3 \gg 1$).

We also estimate the term linear in $z$ coming from the substitution of $a_0^\dagger a_0^a a_0$ in \([9.29]\). This substitution occurs twice, but we will only explicitly treat one of them, namely the term
\[
\Re \int \left\langle \Phi(z), -2\ell^{-3}\tilde{W}_1(0)(2\pi)^{-3} \int \tilde{\chi}_\Lambda(k)a_k^\dagger z \, dk \Phi(z) \right\rangle d^2z \\
= -2\ell^{-3}\tilde{W}_1(0)(2\pi)^{-3} \int \left\langle \Phi(z), \int \tilde{\chi}_\Lambda(k)(a_k^\dagger z + a_k \overline{z}) \, dk \Phi(z) \right\rangle d^2z \\
\geq -Ca\ell^{-3} \int \left\langle \Phi(z), \int |\tilde{\chi}_\Lambda(k)| (\varepsilon a_k^\dagger a_k + \varepsilon^{-1}|z|^2) \, dk \Phi(z) \right\rangle d^2z,
\]
where $\varepsilon > 0$ will be chosen in the end. Notice that $|\tilde{\chi}_\Lambda(k)| = \ell^3|\tilde{\chi}(k\ell^3)|$ and that $\tilde{\chi} \in L^1(\mathbb{R}^3)$ for $M \geq 4$. Redoing the calculation in \([9.34]\) we therefore find with $\varepsilon = \sqrt{\langle \tilde{\Psi}, n_+ \tilde{\Psi} \rangle}/\sqrt{n+1}$,
\[
\Re \int \left\langle \Phi(z), -2\ell^{-3}\tilde{W}_1(0)(2\pi)^{-3} \int \tilde{\chi}_\Lambda(k)a_k^\dagger z \, dk \Phi(z) \right\rangle d^2z \geq -Ca\ell^{-3}\sqrt{n+1}\sqrt{\langle \tilde{\Psi}, n_+ \tilde{\Psi} \rangle}.
\]
This is also easily absorbed in the error term in \([9.27]\).

The other error terms from the substitutions are \([9.29]\) estimated in a similar manner and we will leave out the details.

Finally, we need to restrict to non-negative $z$. Suppose $z = |z|e^{i\varphi}$. In the operator we can replace $a_{\pm k}$ by $e^{i\varphi}a_{\pm k}$. In this way all occurrences of $z$ will be replaced by $|z|$. Notice that this substitution will not affect the commutation relations. This finishes the proof.

\section{First energy bounds}

In this section we will make a rough estimate on the energy. This rough estimate will be used to eliminate the values of $\rho_\mu$ that are far away from $\rho_\mu$.
Lemma 10.1. For any state $\Phi$ satisfying $\langle 9.28 \rangle$ and assuming that $\mathcal{M} \lesssim C^{-1}\rho_\mu\ell^\delta$ for some sufficiently large constant $C$, we have the bound

\[
\langle \Phi, \mathcal{K}(z)\Phi \rangle \geq -\frac{\hat{g}(0)}{2}\rho_\mu^2\ell^\delta + \frac{\hat{g}(0)}{2}(\rho_\mu - \rho_z)^2\ell^\delta - a(\rho_\mu + \rho_z)^{3/2}\rho_\mu^{1/2}\ell^\delta \delta_1 - \rho_z^2a\ell^\delta \delta_2
\]

\[
- C\rho_\mu^2a\ell^\delta \frac{\rho_\mu a^3}{K^6_T(ds)^6},
\]  
with

\[
\delta_1 := C \sqrt{\frac{\mathcal{M}}{\rho_\mu\ell^\delta}} \left( K_3^2 \tilde{K}_2^2 (\rho_\mu a^3)^{2/3} M + K_3^3 K_7^3 \right),
\]

\[
\delta_2 := C \left( \frac{R^2}{\ell^2} + \frac{a}{ds} (1 + \log \left( \frac{ds\ell}{a} \right)) \right).
\]  

Before we give the proof of Lemma 10.1 we will state its main consequence, Proposition 10.2 below.

Notice that by Section 4, our choice of parameters ensures that $\delta_1 + \delta_2 \ll 1$.

Proposition 10.2. Suppose that $\delta_1 + \delta_2 \leq \frac{1}{2}$. Suppose furthermore, for some sufficiently large universal constant $C > 0$, we have

\[
|\rho_z - \rho_\mu| \geq C\rho_\mu \max \left( \delta_1 + \delta_2 + \frac{\rho_\mu a^3}{K^6_T(ds)^6} \right)^{1/2}, (\rho_\mu a^3)^{1/2} \right).
\]  

Then, for any state $\Phi$ satisfying $\langle 9.28 \rangle$, we have

\[
\langle \Phi, \mathcal{K}(z)\Phi \rangle \geq -\frac{\hat{g}(0)}{2}\rho_\mu^2\ell^\delta + 2\rho_\mu^2a\ell^\delta \frac{128}{15\sqrt{\pi}} \sqrt{\rho_\mu a^3}.
\]  

Proof. Using the convexity of $t \mapsto t^\sigma$, for $\sigma \in \{3/2, 2\}$ and Jensen’s inequality, (10.1) implies the bound

\[
\langle \Phi, \mathcal{K}(z)\Phi \rangle \geq -\frac{\hat{g}(0)}{2}\rho_\mu^2\ell^\delta + \frac{\hat{g}(0)}{2}(1 - \delta_1 - \delta_2)(\rho_\mu - \rho_z)^2\ell^\delta - C\rho_\mu^2a\ell^\delta \left( \delta_1 + \delta_2 + \frac{\rho_\mu a^3}{K^6_T(ds)^6} \right)
\]

\[
\geq -\frac{\hat{g}(0)}{2}\rho_\mu^2\ell^\delta + \frac{\hat{g}(0)}{4}(\rho_\mu - \rho_z)^2\ell^\delta - C\rho_\mu^2a\ell^\delta \left( \delta_1 + \delta_2 + \frac{\rho_\mu a^3}{K^6_T(ds)^6} \right).
\]  

If $\langle 10.3 \rangle$ is satisfied, then the term quadratic in $\langle \rho_\mu - \rho_z \rangle$ dominates both the error term above and the LHY correction. This finishes the proof of Proposition 10.2.

Proof of Lemma 10.1. Since, for any $\delta' > 0$, $z\alpha_k^+ + \pi a_k \leq \delta'|z|^2 + (\delta')^{-1}a_k^+a_k$, we find

\[
\int \bar{\chi}_\Lambda(k)(z\alpha_k^+ + \pi a_k) \, dk \leq \delta'|z|^2 \int |\bar{\chi}_\Lambda(k)| \, dk + |\bar{\chi}_\Lambda(0)| (\delta')^{-1} \int a_k^+a_k \, dk \leq C(\delta'|z|^2 + (\delta')^{-1}n_+).
\]  

Therefore, we easily get, setting $\delta' = \sqrt{\mathcal{M}/(\rho_\mu\ell^\delta)}$ and using $\langle 9.28 \rangle$ and the definitions in $\langle 9.23 \rangle$ and $\langle 9.24 \rangle$,

\[
\langle \Phi, (Q_1(z) + Q_1^{ex}(z))\Phi \rangle \geq -C\ell^\delta a \sqrt{\frac{\mathcal{M}|z|^2}{\ell^\delta}} (|\rho_z - \rho_\mu| + \rho_z),
\]  

[28]
in agreement with (10.1) (where we used that $K_L, K_f \geq 1$)

Notice that quadratic terms of the form $\ell^3 \int \hat{W}(k) a_k^\dagger a_k dk$ are easily estimated as

$$\pm \langle \Phi, \ell^3 \int \hat{W}(k) a_k^\dagger a_k dk \Phi \rangle \leq C a^M. \tag{10.8}$$

This allows us to estimate all the quadratic terms in $\hat{K}(z)$ except the kinetic energy and the off-diagonal quadratic terms and to absorb the corresponding terms in the error in (10.1) (using in particular that $M \leq \rho \ell^3$).

Therefore, to establish (10.1) we only have left to estimate the sum of the kinetic energy, $Q_3(z)$ and the ‘off-diagonal’ quadratic terms. This we will do by first adding and subtracting an $n_+$ term, which is easily estimated as above. We will prove the following 3 inequalities, where $\varepsilon < 1/2$ is a (small) parameter that we will optimize in the end (see (10.22)), and where $\Phi$ is a state satisfying (9.28),

$$- \left\langle \Phi, (2\pi)^{-3/2} \rho_\varepsilon \varepsilon^{-1/2} a \int a_k^\dagger a_k dk \Phi \right\rangle \geq -C \varepsilon^{-1/2} \rho \varepsilon a^M, \tag{10.9}$$

$$\left\langle \Phi, \left( (2\pi)^{-3/2} \int \varepsilon \tau(k) a_k^\dagger a_k dk + Q_3(z) \right) \Phi \right\rangle \geq -\varepsilon^{-1} C \rho \varepsilon a^M \ell^3 \left( K_3^\ell K_1^2 M \varepsilon \rho a^3 + K_2^3 \ell^3 \right), \tag{10.10}$$

and

$$(2\pi)^{-3/2} \int \left( A_1(k) a_k^\dagger a_k + \frac{1}{2} B_1(k) \left( a_k^\dagger a_{-k} + a_k a_{-k} \right) \right) dk \geq -\frac{1}{2} \rho_\varepsilon^2 \varepsilon^3 \mu \int (1 + \log \left( \frac{ds\ell}{a} \right)) - C \rho_\varepsilon^2 \varepsilon^3 \mu \frac{\rho a^3}{K_\rho^2 (ds\ell)^6}, \tag{10.11}$$

where we have introduced

$$A_1(k) := (1 - \varepsilon) \tau(k) + \rho \varepsilon^{-1/2} a, \quad B_1(k) := \rho \varepsilon \hat{W}_1(k). \tag{10.12}$$

Notice that (10.9) is easy given the discussion above.

We proceed to prove (10.11). We symmetrize the term in $k$ as

$$(2\pi)^{-3/2} \int \left( A_1(k) a_k^\dagger a_k + \frac{1}{2} B_1(k) a_k^\dagger a_{-k} + \frac{1}{2} B_1(k) a_k a_{-k} \right) dk \tag{10.13}$$

$$= \frac{1}{2} (2\pi)^{-3/2} \int \left( A_1(k) a_k^\dagger a_k + A_1(k) a_{-k}^\dagger a_{-k} + B_1(k) a_k^\dagger a_{-k} + B_1(k) a_{-k} a_k \right) dk. \tag{10.14}$$

At this point we apply the ‘Bogolubov lemma’, Lemma [A.5] to get

$$A_1(k) a_k^\dagger a_k + A_1(k) a_{-k}^\dagger a_{-k} + B_1(k) a_k^\dagger a_{-k} + B_1(k) a_k a_{-k} \geq -\left( A_1(k) - \sqrt{A_1(k)^2 - |B_1(k)|^2} \right), \tag{10.14}$$

where we have also used (9.7).

Notice that (using (5.13)) $|B_1(k)|/|A_1(k)| \leq C \varepsilon^{1/2}$. Therefore, for $\varepsilon$ sufficiently small, a Taylor expansion gives

$$-\left( A_1(k) - \sqrt{A_1(k)^2 - |B_1(k)|^2} \right) \geq -\frac{1}{2} + C \varepsilon \frac{|B_1(k)|^2}{A_1}, \tag{10.15}$$
Below we will need the following estimate of an integral,
\[
\int_{\{k \geq (ds\ell)^{-1}\}} \frac{|B_1|^2}{2\tau(k)} \leq \rho^2 \left( \int \frac{\hat{W}_1(k)^2}{2k^2} + C a^2 \frac{ds\ell}{ds\ell} \int_{\{ds\ell)^{-1} \leq |k| \leq a^{-1}\}} |k|^{-3} + C \frac{a}{ds\ell} \int \frac{\hat{W}_1(k)^2}{2k^2} \right) \\
\leq \rho^2 (1 + \frac{R^2}{\ell^2}) \hat{g_\omega}(0) + C \rho^2 a^2 (ds\ell)^{-1} (1 + \log(ds\ell/a)). \quad (10.16)
\]
where we used that \(0 \leq k^2 - \tau(k) \leq 2|k|(ds\ell)^{-1}\) for \(|k| \geq (ds\ell)^{-1}\) and we also used \((6.15)\).
Inserting these considerations, we find
\[
(2\pi)^{-3} \ell^3 \int A_1(k)a_\epsilon^\dagger a_k + \frac{1}{2} B_1(k) \left( a_k a_{-k}^\dagger + a_k a_{-k} \right) \, dk \\
\geq -\left( \frac{1}{2} + C\varepsilon \right) (2\pi)^{-3} \ell^3 \left( \int_{|k| \leq (ds\ell)^{-1}} \varepsilon^{1/2} \frac{|B_1|^2}{2\rho \varepsilon} + \int_{|k| > (ds\ell)^{-1}} \frac{|B_1|^2}{2(1 - \varepsilon)\tau(k)} \right) \\
\geq -\frac{1}{2} \rho^2 \ell^3 \hat{g_\omega}(0) - C \ell^3 \rho_\omega a \left( \varepsilon^{1/2}(ds\ell)^{-3} + \rho_\omega (\varepsilon + \frac{R^2}{\ell^2}) + C \rho_\omega \frac{a}{ds\ell} (1 + \log(ds\ell/a)) \right) \\
\geq -\frac{1}{2} \rho^2 \ell^3 \hat{g_\omega}(0) - C \ell^3 \rho_\omega a \left( \varepsilon + \frac{R^2}{\ell^2} + \frac{a}{ds\ell} \right) - C a(ds)^{-6} \ell^{-3}. \quad (10.17)
\]
This is easily seen to be consistent with \((10.11)\) and finishes the proof of \((10.11)\).
To prove \((10.10)\) we use a similar approach. Notice that by definition \((9.25)\), \(Q_3(z)\) only lives in the high momentum region \(P_H\). For these momenta we have \(\tau(k) \geq \frac{1}{2} k^2\). Therefore, dropping a part of the kinetic energy, it suffices to bound
\[
\ell^3 (2\pi)^{-3} \int_{\{k \in P_H\}} \left( \frac{\varepsilon}{2} k^2 a_\epsilon^\dagger a_k + (2\pi)^{-3} \int f_L(s) \hat{W}_1(k)(\bar{\omega}_s a_{s-k} a_k + a_k^\dagger a_{s-k}^\dagger \bar{\omega}_s z) \right) \, ds \, dk. 
\]

We estimate, with \(\tilde{b}_k := a_k + 2(2\pi)^{-3} \int f_L(s) \hat{W}_1(k) \bar{\omega}_s a_{s-k}^\dagger \bar{\omega}_s z \, ds\),
\[
\ell^3 (2\pi)^{-3} \int_{\{k \in P_H\}} \left( \frac{\varepsilon}{2} k^2 a_\epsilon^\dagger a_k + (2\pi)^{-3} \int f_L(s) \hat{W}_1(k)(\bar{\omega}_s a_{s-k} a_k + a_k^\dagger \bar{\omega}_s z) \right) \, ds \, dk \\
= \ell^3 (2\pi)^{-3} \int_{\{k \in P_H\}} \left( \frac{\varepsilon}{2} k^2 \tilde{b}_k - 4(2\pi)^{-6} \int f_L(s) f_L(s') \hat{W}_1(k)^2 \frac{|z|^2}{k^2} a_{s-k}^\dagger a_{s-k}^\dagger \bar{\omega}_s z \right) \, ds \, dk \\
\geq -4\varepsilon^{-1} \ell^3 (2\pi)^{-9} \int_{\{k \in P_H\}} \hat{W}_1(k)^2 \frac{|z|^2}{k^2} \int f_L(s) f_L(s') \bar{\omega}_s (a_{s'-k} a_{s'-k}^\dagger + [a_{s'-k}, a_{s'-k}^\dagger]) \, ds \, dk. 
\]

On the term without a commutator, we estimate \(\bar{\omega}_s a_{s-k}^\dagger a_{s'-k} \bar{\omega}_s\) by Cauchy-Schwarz and (since \(k \in P_H\)), \(\hat{W}_1(k)^2 \leq C K_H^2 a^4\). Therefore, for a \(\Phi\) satisfying \((9.28)\), we find
\[
\langle \Phi, \ell^3 \int_{\{k \in P_H\}} \hat{W}_1(k)^2 \frac{|z|^2}{k^2} \int f_L(s) f_L(s') \bar{\omega}_s a_{s-k}^\dagger a_{s'-k}^\dagger \bar{\omega}_s \Phi \rangle \\
\leq C \rho_\omega \left( \int_{\{|s| \leq 2 K_H \sqrt{\tau(s)}\}} ds \right) K_H^2 a^4 \mathcal{M}^2 \\
\leq C \rho_\omega \ell^3 K_H^3 a^3 K_H^4 \mathcal{M}^2. \quad (10.20)
\]
For the commutator term, we estimate (using (9.6) and the Cauchy-Schwarz inequality)

\[ \tilde{a}^\dagger_s[\tilde{a}^\dagger_{s-k}, \tilde{a}^\dagger_{s-k}] \tilde{a}_s \leq 2 \tilde{a}^\dagger_s \tilde{a}_s + 2 \tilde{a}^\dagger_s \tilde{a}_s \]

and \( \int \frac{\tilde{W}(k)^2}{k^2} \leq C a \). This leads to (for a \( \Phi \) satisfying (9.28)),

\[
\langle \Phi, \ell^3 \int_{k \in P_H} \frac{\tilde{W}(k)^2}{k^2} |z|^2 \int \int f_L(s) f_L(s') \tilde{a}^\dagger_{s'}[\tilde{a}^\dagger_{s-k}, \tilde{a}^\dagger_{s-k}] \tilde{a}_s \Phi \rangle 
\leq C M a |z|^2 \int \int_{|s| \leq 2 K \sqrt{\rho \mu}} ds 
\leq C a \rho \frac{M}{\ell^3} K_3^3 K_3^3. 
\] (10.21)

Combining the estimates (10.19), (10.20) and (10.21) proves (10.10).

We choose

\[ \varepsilon = \frac{M^{1/2}}{\sqrt{\rho \mu + \rho \rho}} \] (10.22)

We will add the estimates of (10.9), (10.10) and (10.11) with this choice of \( \varepsilon \). Notice that since \( M \leq \rho \mu \varepsilon^3 \) the contribution from (10.9) will be smaller than the terms appearing in the other estimates. Therefore we get,

\[
\langle \Phi, \left( \frac{1}{2} \rho \varepsilon^3 g \omega(0) + (2\pi)^{-3} \varepsilon^3 \int \tau(k) \alpha^k \alpha^k dk + Q_3(z) \right) \Phi \rangle 
\geq -C \rho \varepsilon a \varepsilon^3 \left( \frac{M(\rho \rho + \rho \varepsilon)}{\varepsilon^3} \left( K_3^3 K_3^3 (\rho \varepsilon^3)^{3/2} M + K_3^3 K_3^3 \right) + \rho \varepsilon R^2 \right) \] (10.23)

This finishes the proof of (10.1).

11 More precise energy estimates

From Proposition (10.2) above, we see that the energy is too high unless \( \rho \rho \approx \rho \mu \). In this section we will give precise energy bounds in the complementary regime. We will always assume that

\[ |\rho \rho - \rho \mu| \leq \rho \mu C \max \left( \frac{\delta + \delta^2}{K_3^3(d s)^{1/2}}, (\rho \mu a^3)^{1/2} \right) \] (11.1)

with the notation from Proposition (10.2)

We will need the condition that

\[ K_3^2 \max \left( \frac{\delta + \delta^2}{K_3^3(d s)^{1/2}}, (\rho \mu a^3)^{1/2} \right) \leq C^{-1}, \] (11.2)

for some sufficiently large universal constant. This condition is satisfied by (4.19), (4.20), (4.6) and (4.8).

Notice, using (11.1) and (11.2), that

\[ |\rho \rho - \rho \mu| \leq C^{-1} K_3^{-2}. \] (11.3)
For convenience of notation, we define the parameter $\delta$ to be the square of the ratio between $\sqrt{\rho_\mu a}$ and the inner radius of $P_H$, i.e.,

$$\delta := \frac{\rho_\mu a}{\sqrt{\rho_\mu a^2} a^{-2}} = (\rho_\mu a^3)^{\frac{1}{2}} K_H^2.$$  \hfill (11.4)

Using \((4.18)\), we see that $\delta \ll 1$.

We define the quadratic Bogolubov Hamiltonian as follows,

$$\mathcal{K}^{\text{Bog}} = \frac{1}{2}(2\pi)^{-3}\ell^3 \int \left( \mathcal{A}(k)(a_k^\dagger a_k + a_{-k}^\dagger a_{-k}) + \mathcal{B}(k)(a_k^\dagger a_{-k} + a_k a_{-k}) + \mathcal{C}(k)(a_k^\dagger + a_{-k} + a_k + a_{-k}) \right) dk,$$  \hfill (11.5)

with

$$\mathcal{A}(k) := \tau(k) + \rho_\mu \tilde{W}_1(k), \quad \mathcal{B}(k) := \rho_\mu \tilde{W}_1(k), \quad \mathcal{C}(k) := \ell^{-3}(\rho_z - \rho_\mu)\tilde{W}_1(0)\chi_\Lambda(z).$$  \hfill (11.6)

With this notation, we can rewrite/estimate $\mathcal{K}(z)$ from \((9.22)\) as follows,

$$\mathcal{K}(z) = \mathcal{K}^{\text{Bog}} + \frac{1}{2} \rho_\mu^2 \ell^3 \left( \hat{g}(0) + \hat{g}\omega(0) \right) - \rho_\mu \hat{g}(0) \rho_\mu \ell^3 + \frac{b}{2\ell^2} \int \left( \frac{b}{2d^2\ell^2} n_+^H + (\rho_z - \rho_\mu)\tilde{W}_1(0)(2\pi)^{-3}\ell^3 \int a_k^\dagger a_k dk \right)
+ \Omega_1^{\text{ex}}(z) + \Omega_2^{\text{ex}}(z) + \Omega_3(z)
\geq -\frac{1}{2} \rho_\mu^2 \ell^3 \hat{g}(0) + \frac{1}{2} \rho_\mu^2 \ell^3 \hat{g}\omega(0) + \frac{1}{2}(\rho_z - \rho_\mu)^2 \ell^3 \hat{g}(0) + \mathcal{K}^{\text{Bog}}
+ \frac{b}{4\ell^2} n_+^H + \frac{b}{2d^2\ell^2} n_+^H + \Omega_1^{\text{ex}}(z) + \Omega_2^{\text{ex}}(z) + \Omega_3(z).$$  \hfill (11.7)

Here we used \((11.2)\) to absorb a quadratic part in the gap.

### 11.1 The Bogolubov Hamiltonian

**Theorem 11.1** (Analysis of Bogolubov Hamiltonian). Assume that $\Phi$ satisfies \((9.28)\) and that $\frac{1}{2}\rho_\mu \leq \rho_z \leq 2\rho_\mu$. Let $\delta$ be the parameter defined in \((11.4)\). Then,

$$\langle \Phi, \mathcal{K}^{\text{Bog}} \Phi \rangle \geq (2\pi)^{-3}\ell^3 \int D_k b_k^\dagger b_k dk \Phi$$

$$- \frac{1}{2}(2\pi)^{-3}\ell^3 \int \left( \mathcal{A}(k) - \sqrt{\mathcal{A}(k)^2 - \mathcal{B}(k)^2} \right) dk$$

$$- (\rho_z - \rho_\mu)^2 \hat{g}(0) \left(1 + C\frac{R^2}{\alpha^2(\rho_\mu a^3)} \right) - C\rho_\mu^2 a^3 K_M K^{3/2}_{\ell}(K^{2}_{\ell}\delta)^{\frac{2\alpha}{2}}. $$  \hfill (11.8)

Here

$$D_k := \frac{1}{2} \left( \mathcal{A}(k) + \sqrt{\mathcal{A}(k)^2 - \mathcal{B}(k)^2} \right),$$  \hfill (11.9)

and

$$b_k := a_k + \alpha_k a_{-k}^\dagger + c_k, $$  \hfill (11.10)
Therefore,

\[ \alpha_k := \mathcal{B}(k)^{-1} \left( \mathcal{A}(k) - \sqrt{\mathcal{A}(k)^2 - \mathcal{B}(k)^2} \right), \quad (11.11) \]

and

\[ c_k := \begin{cases} 
\frac{2C(k)}{\mathcal{A}(k)+\mathcal{B}(k)+\sqrt{\mathcal{A}(k)^2-\mathcal{B}(k)^2}}, & |k| \leq \frac{1}{2} K_H^{-1} a^{-1}, \\
0, & |k| > \frac{1}{2} K_H^{-1} a^{-1}. 
\end{cases} \quad (11.12) \]

**Proof.** To simplify later calculations we start by removing \( C(k) \) for \( |k| > \frac{1}{2} K_H^{-1} a^{-1} \) from \( \mathcal{K}^{\text{Bog}} \), so we aim to prove

\[ \frac{1}{2} (2\pi)^{-3} \ell^3 \int_{\{|k| > \frac{1}{2} K_H^{-1} a^{-1}\}} C(k) (a_k^\dagger + a_{-k}^\dagger + a_k + a_{-k}) \, dk \geq -C \rho_\mu^2 a \ell^3 (\rho_\mu a^3)^\frac{1}{2} + \varepsilon. \quad (11.13) \]

Obviously,

\[ a_k + a_{-k}^\dagger \leq a_k^\dagger a_k + 1. \]

Therefore,

\[ \frac{1}{2} (2\pi)^{-3} \ell^3 \int_{\{|k| > \frac{1}{2} K_H^{-1} a^{-1}\}} C(k) (a_k^\dagger + a_{-k}^\dagger + a_k + a_{-k}) \, dk \geq -(2\pi)^{-3} |\rho_z - \rho_\mu| \tilde{W}_1(0) z | \int_{\{|k| > \frac{1}{2} K_H^{-1} a^{-1}\}} |\bar{\chi}_\lambda(k)| (a_k^\dagger a_k + 1) \, dk \]

\[ \geq -C |\rho_z - \rho_\mu| \tilde{W}_1(0) z |(n_+ + 1) \varepsilon(\chi), \quad (11.14) \]

where

\[ \varepsilon(\chi) := \ell^{-3} \sup_{\{|k| > \frac{1}{2} K_H^{-1} a^{-1}\}} (1 + (k \ell)^2)^2 |\bar{\chi}_\lambda(k)| \leq C(K_\ell^{-2} \delta)^{\tilde{M}^{-2}}, \quad (11.15) \]

where we used Lemma (C.1) to get the last estimate. Estimating \( n_+ \) using (9.28) and using (11.1) to control \( |z| \), it is elementary to conclude (11.13) for this part.

By the estimate above, it suffices to consider

\[ \tilde{\mathcal{K}}^{\text{Bog}} := \frac{1}{2} (2\pi)^{-3} \ell^3 \int (\mathcal{A}(k)(a_k^\dagger a_k + a_{-k}^\dagger a_{-k}) + \mathcal{B}(k)(a_k^\dagger a_{-k}^\dagger + a_k a_{-k})) \]

\[ + \tilde{C}(k)(a_k^\dagger + a_{-k}^\dagger + a_k + a_{-k}) \, dk, \quad (11.16) \]

with \( \mathcal{A}, \mathcal{B} \) from (11.5) and

\[ \tilde{C}(k) := \begin{cases} 
0, & |k| \geq \frac{1}{2} K_H^{-1} a^{-1}, \\
\ell^{-3} (\rho_z - \rho_\mu) \tilde{W}_1(0) \bar{\chi}_\lambda(k) z, & \text{else}. 
\end{cases} \quad (11.17) \]

With the notation from Theorem 11.1 and using Theorem A.1 combined with (9.7) we find

\[ \tilde{\mathcal{K}}^{\text{Bog}} \geq (2\pi)^{-3} \ell^3 \int \mathcal{D}_k b_k^\dagger b_k \, dk \]

\[ - \frac{1}{2} (2\pi)^{-3} \ell^3 \int \left( \mathcal{A}(k) - \sqrt{\mathcal{A}(k)^2 - \mathcal{B}(k)^2} \right) \, dk \]

\[ - (\rho_z - \rho_\mu)^2 \tilde{W}_1(0)^2 z^2 (2\pi)^{-3} \ell^3 \int_{\{|k| \leq \frac{1}{2} K_H^{-1} a^{-1}\}} \frac{|\bar{\chi}_\lambda(k)|^2}{\mathcal{A}(k) + \mathcal{B}(k)}. \quad (11.18) \]
More precise energy estimates

It is elementary, using that $W_1$ is even, that

$$\left| \hat{W}_1(k) - \hat{W}_1(0) \right| \leq C a(k R)^2. \quad (11.19)$$

Therefore we easily get the lower bound

$$\mathcal{A}(k) + \mathcal{B}(k) \geq 2 \rho_2 \hat{W}_1(0)(1 - C (\rho_\mu a^3)^2 R^2 a^{-2}), \quad (11.20)$$

using that the kinetic energy is dominating, unless $|k| \leq C \sqrt{\rho_\mu a}$.

Therefore, the last term in (11.18) becomes controlled as

$$\frac{(\rho_2 - \rho_\mu)^2}{2} \hat{W}_1(0)^2 z^2 (2\pi)^{-3} \ell^{-3} \int_{|k| \leq \frac{1}{4} K H^{-1} a^{-1}} \frac{|\hat{\chi}_A(k)|^2}{\mathcal{A}(k) + \mathcal{B}(k)}$$

$$\leq \frac{(\rho_2 - \rho_\mu)^2}{2} \hat{W}_1(0)^2 \ell^3 (1 + C (\rho_\mu a^3)^2 R^2 a^{-2})$$

$$\leq \frac{(\rho_2 - \rho_\mu)^2}{2} \hat{\vartheta}(0)^2 \ell^3 (1 + C (\rho_\mu a^3)^2 R^2 a^{-2}), \quad (11.21)$$

where we used that $\ell^{-2} \ll \rho_\mu a$ to get the last estimate.

This finishes the proof of Theorem 11.1. \qed

Remark 11.2. We notice that following commutation relations (using the ones for the $a_k$'s (9.6) and that $\hat{\chi}$ is even and real).

$$[b_k, b_{k'}] = (\alpha_k - \alpha_{k'})(\hat{\chi}^2((k + k')\ell) - \hat{\chi}(k\ell)\hat{\chi}(k'\ell)). \quad (11.22)$$

Also,

$$[b_k, b_{k'}] = (1 - \alpha_k \alpha_{k'})(\hat{\chi}^2((k - k')\ell) - \hat{\chi}(k\ell)\hat{\chi}(k'\ell)). \quad (11.23)$$

Lemma 11.3. Assume that (11.1) holds and that $\frac{a^3}{\pi^2} \rho_\mu \leq \rho_2 \leq \frac{1}{\pi} \rho_\mu$. We have the following estimate

$$- \frac{1}{2} (2\pi)^{-3} \ell^3 \int \left( \mathcal{A}(k) - \sqrt{\mathcal{A}(k)^2 - \mathcal{B}(k)^2} \right) dk$$

$$\geq - \frac{\hat{\vartheta}(0)}{2} \rho_2^2 \ell^3 + \frac{4\pi}{15} \rho_2^2 \alpha \sqrt{\rho_\mu a^3} \ell^3 - C \varepsilon(\rho_\mu, \rho_2) \rho_2^2 a \sqrt{\rho_\mu a^3} \ell^3 - C \rho_2^2 \ell^3 \frac{R^2}{\ell^2} \hat{\vartheta}(0), \quad (11.24)$$

with

$$\varepsilon(\rho_\mu, \rho_2) = (\rho_\mu a^3)^{\frac{1}{4}} \sqrt{R} + \varepsilon_T + (K ds)^{-1} \left( 1 + \log(d^{-1}) + \log\left( \frac{K ds}{(\rho_\mu a^3)^{1/2}} \right) \right)$$

$$+ \varepsilon_T (K ds)^{-1} \left( 1 + \log\left( \frac{K ds}{(\rho_\mu a^3)^{1/2}} \right) \right). \quad (11.25)$$

Proof. We regularize the integral as

$$\int \mathcal{A}(k) - \sqrt{\mathcal{A}(k)^2 - \mathcal{B}(k)^2} dk$$

$$= \int \mathcal{A}(k) - \sqrt{\mathcal{A}(k)^2 - \mathcal{B}(k)^2} - \rho_2^2 \frac{\hat{W}_1(k)^2}{2k^2} \, dk + \rho_2^2 \int \frac{\hat{W}_1(k)^2}{2k^2} \, dk. \quad (11.26)$$
The last integral is controlled by (5.15) and contributes with the first and the last term in (11.24).

In the regularized integral in (11.26) we perform the change of variables \(|\sqrt{\rho \zeta} t = k|\). In this way we get

\[
\int A(k) - \sqrt{A(k)^2 - B(k)^2} - \rho \zeta^2 \frac{\tilde{W}_1(k)^2}{2k^2} \, dk = \rho \zeta^2 \sqrt{\rho \zeta} a I_1,
\]

with

\[
I_1 = \int \alpha(t) - \sqrt{\alpha(t)^2 - \beta(t)^2} - \tilde{W}_1(\sqrt{\rho \zeta} at)^2 \, dt,
\]

\[
\alpha(t) = \tilde{T}(t) + a^{-1} \tilde{W}_1(\sqrt{\rho \zeta} at),
\]

\[
\beta(t) = a^{-1} \tilde{W}_1(\sqrt{\rho \zeta} at),
\]

\[
\tilde{T}(t) = (1 - \varepsilon_T) \left[ |t| - \frac{1}{2K_t s} \left( \frac{\rho_t}{\rho_s} \right)^{1/2} \right] + \varepsilon_T \left[ |t| - \frac{1}{2K_t s} \left( \frac{\rho_t}{\rho_s} \right)^{1/2} \right]^2.
\]

We will prove that \(I_1 \approx -64\pi^4 \frac{128}{15\sqrt{\pi}}\). For this we write \(I_1\) as

\[
I_1 = \int \alpha(t) - \sqrt{\alpha(t)^2 - \beta(t)^2} - \frac{\beta^2}{2t^2} \, dt
\]

\[
= \int \alpha(t) - \frac{\beta^2}{2\alpha} - \sqrt{\alpha(t)^2 - \beta(t)^2} + \frac{\beta^2}{2t^2} \, dt
\]

\[
= \int \alpha(t) - \frac{\beta^2}{2\alpha} - \sqrt{\alpha(t)^2 - \beta(t)^2} + \frac{\beta^2 t^2 - \tilde{T} - \beta}{2t^2\alpha} \, dt = I'_1 + I''_1,
\]

with

\[
I'_1 := \int \alpha(t) - \frac{\beta^2}{2\alpha} - \sqrt{\alpha(t)^2 - \beta(t)^2} - \frac{\beta^3}{2t^2\alpha} \, dt
\]

\[
I''_1 := \int \frac{\beta^2 t^2 - \tilde{T}}{2t^2\alpha} \, dt.
\]

It is not difficult to apply dominated convergence to the integral \(I'_1\) to get

\[
I'_1 \approx \int_{\mathbb{R}^3} t^2 + 8\pi - \frac{(8\pi)^2}{2t^2} - \sqrt{(t^2 + 8\pi)^2 - (8\pi)^2} \, dt = -64\pi^4 \frac{128}{15\sqrt{\pi}}.
\]

More precisely, we will prove that

\[
\left| I'_1 - \int_{\mathbb{R}^3} t^2 + 8\pi - \frac{(8\pi)^2}{2t^2} - \sqrt{(t^2 + 8\pi)^2 - (8\pi)^2} \, dt \right| \leq C(\rho_t a)^{\frac{1}{2}} \sqrt{R} + \varepsilon_T + (K_t s)^{-1}.
\]

Notice that this is consistent with (11.25). The part of both integrals where \(|t| \leq 10(K_t s)^{-1}\) is bounded by

\[
C(K_t s)^{-1},
\]

for sufficiently small \(\rho_t\) (using that \(\rho_z \approx \rho_t\)). This is in agreement with (11.32).
More precise energy estimates

For $|t| \geq 10(K_{ls})^{-1}$ we will use

$$|\beta(t) - 8\pi| \leq C \sqrt{\rho_\mu a} R|t|, \quad 0 \leq t^2 - \tilde{\tau}(t) \leq \varepsilon_T t^2 + \frac{1}{K_{ls}} (\frac{\rho_\mu}{\rho_z})^{1/2} |t|. \quad (11.33)$$

Notice that it follows by interpolation that $|\beta(t) - 8\pi| \leq C(\rho_\mu a)^{1/2} R^{3/2} |t|^{3/2}$ and also that $\tilde{\tau} \geq \frac{1}{2} t^2$ when $\varepsilon_T$ is sufficiently small (since $\frac{\rho_\mu}{\rho_z}$ is close to 1).

For $|t| \geq 100$ we use Taylor’s formula with remainder (applied to $\sqrt{1 - x}$) to write

$$\int_{\{|t| \geq 10(K_{ls})^{-1}\}} \alpha(t) - \frac{\beta^2}{2\alpha} - \sqrt{\alpha(t)^2 - \beta(t)^2} - \frac{\beta^3}{2t^2} \, dt$$

$$- \int_{\{|t| \geq 10(K_{ls})^{-1}\}} t^2 + 8\pi - \frac{(8\pi)^2}{2t^2} - \sqrt{(t^2 + 8\pi)^2 - (8\pi)^2} \, dt$$

$$= \int_{\{10(K_{ls})^{-1} \leq |t| \leq 100\}} \left( (\alpha - t^2 - 8\pi) - \frac{\beta^2}{2\alpha} \right) - \frac{(8\pi)^2}{2(t^2 + 8\pi)}$$

$$- \left( \sqrt{\alpha(t)^2 - \beta(t)^2} - \sqrt{(t^2 + 8\pi)^2 - (8\pi)^2} \right) \, dt$$

$$+ \int_{\{|t| \geq 100\}} \int_0^1 f(\tilde{\tau}, \beta, \sigma) - f(t^2, 8\pi, \sigma) \, d\sigma \, dt$$

$$- \int_{\{|t| \geq 10(K_{ls})^{-1}\}} \frac{\beta^3}{2t^2} - \frac{(8\pi)^3}{2t^2(t^2 + 8\pi)} \, dt, \quad (11.34)$$

with

$$f(\tau, \beta, \sigma) := \frac{-\beta^4}{4} \left[ \tau^2 + 2\beta\tau + (1 - \sigma)\beta^2 \right]^{-3/2} (1 - \sigma). \quad (11.35)$$

The last integral in (11.34) is easily estimated, as

$$\left| \int_{\{|t| \geq 10(K_{ls})^{-1}\}} \frac{\beta^3}{2t^2} - \frac{(8\pi)^3}{2t^2(t^2 + 8\pi)} \, dt \right| \leq C \left( (\rho_\mu a)^{1/2} \sqrt{R} + \varepsilon_T + (K_{ls})^{-1} \right), \quad (11.36)$$

in agreement with (11.32).

For the Taylor expansion part in (11.34), we use that $\tilde{\tau}^2 + 2\beta\tilde{\tau} + (1 - \sigma)\beta^2 \geq \frac{1}{4} t^4$, when $|t| \geq 100$. Therefore,

$$\left| f(\tilde{\tau}, \beta, \sigma) - f(t^2, 8\pi, \sigma) \right|$$

$$\leq C|\beta^4 - (8\pi)^4| t^{-6}$$

$$+ C t^2 |(\tilde{\tau}^2 + 2\beta\tilde{\tau} + (1 - \sigma)\beta^2)^{-2} - (t^4 + 16\pi t^2 + (1 - \sigma)(8\pi)^2)^{-2}|$$

$$+ C t^{-8} \sqrt{\tilde{\tau}^4 + 2\beta\tilde{\tau} + (1 - \sigma)\beta^2 + \sqrt{t^4 + 16\pi t^2 + (1 - \sigma)(8\pi)^2}}. \quad (11.37)$$

Now the integrals can easily be estimated to get an error consistent with (11.32).

Finally, we consider the integral over $\{10(K_{ls})^{-1} \leq |t| \leq 100\}$ in (11.34). Here one may estimate term by term and use the finiteness of the domain of integration. Therefore, this part is also consistent with (11.32), which finishes the proof of (11.32).
More precise energy estimates

The integral $I''_1$ from (11.30) is split in 3 parts. For $|t| \leq 10(K_{\ell}s)^{-1}$, we have $0 \leq t^2 - \bar{\tau}(t) \leq t^2$. Therefore,

$$\left| \int_{\{|t| \leq 10(K_{\ell}s)^{-1}\}} \frac{\beta^2 t^2 - \bar{\tau}}{2 t^2 \alpha} \right| \leq C(K_{\ell}s)^{-1}, \quad (11.38)$$

which is consistent with (11.25).

For $10(K_{\ell}s)^{-1} \leq |t| \leq 10(K_{\ell}d{s})^{-1}$, we have (11.33) above. Therefore,

$$\left| \int_{\{10(K_{\ell}s)^{-1} \leq |t| \leq 10(K_{\ell}d{s})^{-1}\}} \frac{\beta^2 t^2 - \bar{\tau}}{2 t^2 \alpha} \right| \leq C \varepsilon_T(K_{\ell}d{s})^{-1} + C(K_{\ell}s)^{-1} \log(d^{-1}), \quad (11.39)$$

in agreement with (11.25).

Finally the case $|t| \geq 10(K_{\ell}d{s})^{-1}$. Here, $0 \leq t^2 - \bar{\tau}(t) \leq C|t|(K_{\ell}s)^{-1} + \varepsilon_T(K_{\ell}d{s})^{-1})$ and $\alpha \geq \frac{1}{2} t^2$. Therefore,

$$\left| \int_{\{|t| \geq 10(K_{\ell}d{s})^{-1}\}} \frac{\beta^2 t^2 - \bar{\tau}}{2 t^2 \alpha} \right|$$

$$\leq C((K_{\ell}s)^{-1} + \varepsilon_T(K_{\ell}d{s})^{-1}) \int_{\{10(K_{\ell}d{s})^{-1} \leq |t| \leq (\rho_2a^3)^{-1/2}\}} |t|^{-3}$$

$$+ C((K_{\ell}s)^{-1} + \varepsilon_T(K_{\ell}d{s})^{-1})(\rho_2a^3)^{1/2}a^{-2} \int \frac{\sqrt{\rho_2at}}{t^2}$$

$$\leq C((K_{\ell}s)^{-1} + \varepsilon_T(K_{\ell}d{s})^{-1}) \left( \log \left( \frac{K_{\ell}d{s}}{\rho_2a^3} \right)^{1/2} \right) + 1. \quad (11.40)$$

Since this estimate is also in agreement with (11.25), this finishes the proof of Lemma 11.3.

### 11.2 The control of $Q_3(z)$

The quadratic Hamiltonian $(2\pi)^{-\frac{3}{2}} \int D_k b_k^1 b_k dk$ from (11.8) turns out to control the 3Q-term $Q_3(z)$ from (9.25). We summarize as follows

**Theorem 11.4.** Assume that $\Phi$ satisfies (9.28). Assume furthermore that (11.1) and (4.29) are satisfied. Let $\delta$ be as defined in (11.4). We will furthermore assume (4.4), (4.11), (4.18), (4.19), (4.22), and (4.23).

Then,

$$\left\langle \Phi, (2\pi)^{-\frac{3}{2}} \int D_k b_k^1 b_k dk + Q_3(z) + \mathcal{Q}^x_2 + \rho_2z \hat{W}_1 \omega(0)(2\pi)^{-\frac{3}{2}} \int \frac{3}{2} \chi^3(s)(\tilde{a}_s^+ + \tilde{a}_s) ds \right\rangle$$

$$+ \frac{b}{50} \left( \frac{1}{\ell^2} n_+ + \frac{\varepsilon_T}{(d_{\ell})^2 n_+} \right) \Phi \right)$$

$$\geq -C\rho_2a^3 \left[ \frac{M}{|z|^2} \left| \tilde{K}_{\ell}^{-1}(\rho_2a^3)^{\frac{5}{12}} + (K_{\ell}K_{\ell})^{-M} + K_{\ell}^3 \delta (K_{\ell}^{-2} \delta)^{M-1} \right]$$

$$+ \delta^3 \left( K_{\ell}^{-3} d_{\ell}^{12} \delta^2 (K_{\ell}^{-2} \delta)^{M-1} \right). \quad (11.41)$$

**Proof of Theorem 11.4** Notice that

$$|B(k)/A(k)| \leq C\delta, \quad \forall |k| \geq \frac{1}{2} \tilde{K}_{\ell}^{-1}(\rho_2a^3)^{5/12}a^{-1}. \quad (11.42)$$
More precise energy estimates

In particular, \(|B(k)/A(k)| \leq \frac{1}{2}\), for \(\rho\) sufficiently small.

This implies, by expansion of the square root that for all \(|k| \geq \frac{1}{2}K_{H}^{-1}(\rho a^{3})^{5/12}a^{-1},\)

\[
|a_{k}| = |B(k)|^{-1} \left( A(k) - \sqrt{A(k)^{2} - B(k)^{2}} \right) \leq C\delta. \quad (11.43)
\]

In particular, \((11.42)\) and \((11.43)\) are valid for \(k = k' - s\), when \(s \in P_{L}\) and \(k' \in P_{H}\).

For later convenience, we reformulate the first-order operator in \((11.41)\) in terms of the \(\tilde{a}_{s}\). We get

\[
-\rho z\bar{W}_{1}(0)(2\pi)^{-3} \int \chi_{A}(s)(\bar{a}_{s} + a_{s}) \, ds
= -\rho z\bar{W}_{1}(0)(2\pi)^{-3} \int \chi_{A}(s)(\bar{a}_{s} + \tilde{a}_{s}) \, ds
= -\rho z\bar{W}_{1}(0)(2\pi)^{-3} \int \chi_{A}(s)(\bar{a}_{s} + \tilde{a}_{s}) \, ds. \quad (11.44)
\]

We start by rewriting \(Q_{\tilde{a}}(z)\) in terms of the \(b_{s}\)'s defined in \((11.10)\). Notice that \(c_{k}, c_{s-k} = 0\) if \(k \in P_{H}\) and \(s \in P_{L}\). We find the basic relation (we will freely use that all involved functions are symmetric; e.g. \(\alpha_{k} = \alpha_{-k}\))

\[
a_{s-k} = \frac{1}{1 - \alpha_{s-k}^{2}} \left( b_{s-k} - \alpha_{s-k}^{} b_{s-k}^{\dagger} \right), \quad a_{k} = \frac{1}{1 - \alpha_{k}^{2}} \left( b_{k} - \alpha_{k}^{} b_{k}^{\dagger} \right). \quad (11.45)
\]

Therefore,

\[
a_{s-k} a_{k} \quad (11.46)
\]

\[
\frac{1}{1 - \alpha_{s-k}^{2}} \frac{1}{1 - \alpha_{k}^{2}} \left( b_{s-k} b_{k} - \alpha_{s-k}^{} b_{s-k}^{\dagger} b_{k} - \alpha_{s-k}^{} b_{k}^{\dagger} b_{s-k} - \alpha_{s-k}^{} \alpha_{k}^{} b_{k} b_{s-k} + \alpha_{k}^{} \alpha_{-k}^{} b_{s-k}^{\dagger} b_{k}^{\dagger} - \alpha_{k}^{} b_{s-k} b_{k}^{\dagger} \right).
\]

We will decompose \(Q_{\tilde{a}}(z)\) according to the different terms in \((11.46)\), i.e.

\[
Q_{\tilde{a}}(z) = Q_{\tilde{a}}^{(1)}(z) + Q_{\tilde{a}}^{(2)}(z) + Q_{\tilde{a}}^{(3)}(z) + Q_{\tilde{a}}^{(4)}(z), \quad (11.47)
\]

where

\[
Q_{\tilde{a}}^{(1)}(z) := z\tilde{\epsilon}^{3}(2\pi)^{-6} \int_{\{k \in P_{H}\}} \frac{f_{L}(s)\bar{W}_{1}(k)}{(1 - \alpha_{k}^{2})(1 - \alpha_{s-k}^{2})} \left( \bar{a}_{s} b_{s-k} b_{k} + \alpha_{s-k}^{} \bar{a}_{s}^{\dagger} b_{s-k}^{\dagger} b_{k}^{\dagger} + h.c. \right),
\]

\[
Q_{\tilde{a}}^{(2)}(z) := -z\tilde{\epsilon}^{3}(2\pi)^{-6} \int_{\{k \in P_{H}\}} \frac{f_{L}(s)\bar{W}_{1}(k)\alpha_{k}}{(1 - \alpha_{k}^{2})(1 - \alpha_{s-k}^{2})} \left( \bar{a}_{s}^{\dagger} b_{s-k} b_{k} + b_{s-k}^{\dagger} b_{k}^{\dagger} \bar{a}_{s} \right),
\]

\[
Q_{\tilde{a}}^{(3)}(z) := -z\tilde{\epsilon}^{3}(2\pi)^{-6} \int_{\{k \in P_{H}\}} \frac{f_{L}(s)\bar{W}_{1}(k)\alpha_{s-k}}{(1 - \alpha_{s-k}^{2})} \left( \bar{a}_{s}^{\dagger} b_{s-k} b_{k} + b_{s-k}^{\dagger} b_{k}^{\dagger} \bar{a}_{s} \right),
\]

and

\[
Q_{\tilde{a}}^{(4)}(z) := (2\pi)^{-6} z\epsilon^{3} \int_{k \in P_{H}} f_{L}(s)\bar{W}_{1}(k) \frac{-\alpha_{k}}{(1 - \alpha_{k}^{2})(1 - \alpha_{s-k}^{2})} [b_{s-k}, b_{k}^{\dagger}](\bar{a}_{s}^{\dagger} + \tilde{a}_{s}). \quad (11.48)
\]

The different \(Q_{\tilde{a}}^{(j)}(z)\)'s will be estimated individually. The result of this is summarized in Lemma \(11.5\). Theorem \(11.4\) follows by adding the estimates of Lemma \(11.5\). We have used that the \(K\)'s are larger than 1 and \((4.11)\) to simplify the total remainder. This finishes the proof.
More precise energy estimates

**Lemma 11.5.** Let $\delta$ be as defined in (11.4). Assume that $\Phi$ satisfies (9.28). Assume furthermore that (11.29), (11.1), (4.18), (4.19), (4.22), (4.4) and (4.23) are satisfied. Then,

$$\langle \Phi, (Q_3^{(1)}(z) + (1 - \delta^2)(2\pi)^{-3}\xi^2) \int_{\{|k| \leq \frac{1}{2} K_H^{-1} a^{-1}\}} D_k b_k^*(b_k + Q_2^\text{ex}) + \frac{b}{100} \left( \frac{1}{\ell^2} n_+ + \varepsilon_T \right) \rangle \Phi \rangle \geq -C \rho^2 a^3 \delta^2 K^{-3/2} (\rho a^3)^{\frac{1}{2}} (M + (K^3 H^2)) (K^{-2} \delta)^{M-1},$$

(11.49)

$$\langle \Phi, (Q_3^{(2)}(z) + Q_3^{(3)}(z) + \delta^2 (2\pi)^{-3} \xi^2 \int_{\{ |k| \geq \frac{1}{2} K_H^{-1} a^{-1}\}} D_k b_k^*(b_k + Q_2^\text{ex}) \rangle \Phi \rangle \geq -C \rho^2 a^3 (K^{-2} \delta)^M K^3 H^2 (\rho a^3)^{\frac{1}{2}},$$

(11.50)

$$\langle \Phi, (Q_3^{(4)}(z) + \rho z \omega(0)(2\pi)^{-3} \int \chi_\Lambda^2(s)(\tilde{a}_s^\dagger + \tilde{a}_s) ds + \frac{b}{100} \left( \frac{1}{\ell^2} n_+ \right) \rangle \Phi \rangle \geq -C \rho^2 a^3 \sqrt{M} \left( K_H^{-1} (\rho a^3)^{\frac{5}{2}} + (K_H K_L)^{-M} + K^2 H^2 (K^{-2} \delta)^{M-1} \right) - C \rho^2 a^3 \delta^4 \frac{q}{\ell}.$$

(11.51)

**Proof of Lemma 11.5.** The proofs of (11.49), (11.50) and (11.51) are each rather lengthy and will be carried out individually.

**Proof of (11.51).** Notice, using Lemma C.1 applied to $\chi^2$ that

$$\left\| \chi_\Lambda^2(s)(1 - f_L(s)) \right\|_\infty \leq C_0 \rho^3 (1 + (K_H K_L)^M)^{-M},$$

(11.52)

with $C_0 = \int \left| (1 - \Delta) M \chi^2 \right|$. Therefore, by a simple application of the Cauchy-Schwarz inequality, we get for any state $\Phi$ satisfying (9.28)

$$\left| \langle \Phi, \int \chi_\Lambda^2(s)(\tilde{a}_s^\dagger + \tilde{a}_s) ds \Phi \rangle \right| \leq C \sqrt{M},$$

(11.53)

and

$$\left| \langle \Phi, \int \chi_\Lambda^2(s)(1 - f_L(s))(\tilde{a}_s^\dagger + \tilde{a}_s) ds \Phi \rangle \right| \leq C \sqrt{M}(K_H K_L)^{-M}.$$  

(11.54)

Therefore, using Lemma 11.6 below to estimate the $k$-integral, we find

$$\left| \langle \Phi, z \left( \rho \tilde{W}_1 \omega(0)(2\pi)^{-3} \int \chi_\Lambda^2(s)(\tilde{a}_s^\dagger + \tilde{a}_s) ds \right) \right| \leq C \rho^2 a^3 \left( K_H^{-1} (\rho a^3)^{\frac{5}{2}} + (K_H K_L)^{-M} \right).$$

(11.55)

The estimate is in agreement with the error term in (11.51).

What remains in order to prove (11.51) is to estimate a difference of two integrals over the same domain. Writing out the commutator using (11.23) we have to estimate

$$z(2\pi)^{-3} \rho^3 \int_{k \in P_H} \tilde{W}_1(k) \alpha_k \chi^2(s \ell) f_L(s) \left( 1 - \frac{1 - \alpha s^{-k} \alpha s^{-k}}{(1 - \alpha_s^{-k})(1 - \alpha_s^{-k})} \right) (\tilde{a}_s^\dagger + \tilde{a}_s),$$

(11.56)
and
\[ z(2\pi)^{-6} \epsilon^3 \int_{k \in P_H} \hat{W}_1(k) \alpha_k f_L(s) \frac{1 - \alpha_{s-k} \alpha_{-k}}{(1 - \alpha_k^2)(1 - \alpha_{s-k}^2)} \hat{\chi}(k \ell) \hat{\chi}((k - s) \ell) (\hat{a}_s + \hat{a}_s^\dagger). \] (11.57)

To estimate (11.56), we use (11.43), (11.62) and Cauchy-Schwarz to get
\[ (11.56) \leq C \rho \alpha^2 \epsilon^3 \int \hat{\chi}^2(s \ell) (\epsilon^{-1} + \epsilon \hat{a}_s^\dagger \hat{a}_s) \leq C \rho \alpha^2 \epsilon^3 (\epsilon^{-1} + \epsilon n_+). \] (11.58)

We choose \( \epsilon^{-1} = D \rho \alpha^2 \epsilon^2 \), for some sufficiently large constant \( D \) to allow the \( n_+ \) term to be absorbed in the kinetic energy gap. Thereby, the magnitude of the error (the \( \epsilon^{-1} \)-term) becomes (using (11.1))
\[ C \rho^2 \alpha^3 \epsilon^4 \frac{a}{\ell}. \] (11.59)

which can clearly be absorbed in the error term in (11.51).

In the second integral (11.57) the terms \( \hat{\chi}(k \ell) \) are very small due to regularity of \( \chi \) and the fact that \( k \in P_H \). Therefore this integral is much smaller. We easily get, for arbitrary \( \epsilon > 0 \),
\[
\nonumber\left(\Phi, z(2\pi)^{-6} \epsilon^3 \int_{k \in P_H} \hat{W}_1(k) \alpha_k f_L(s) \frac{1 - \alpha_{s-k} \alpha_{-k}}{(1 - \alpha_k^2)(1 - \alpha_{s-k}^2)} \hat{\chi}(k \ell) \hat{\chi}((k - s) \ell) (\hat{a}_s + \hat{a}_s^\dagger) \Phi \right) \\
\nonumber \geq -C \rho \alpha^2 \sup_{k \in P_H} |\hat{\chi}(k \ell)| \epsilon^3 \left(\Phi, \int f_L(s)(\epsilon \hat{a}_s^\dagger \hat{a}_s + \epsilon^{-1}) \Phi \right) \\
\geq -C \rho^2 \alpha^3 \epsilon \sqrt{\frac{M}{|z|^2} K_{\ell}^{3/2} K_L^{3/2} (K_\epsilon^{-2} \delta^{-1})} \] (11.60)

where we optimized in \( \epsilon \) and used Lemma 4.1 to get the last estimate. This error term is clearly in agreement with (11.51). This finishes the proof of (11.51). \( \square \)

In the proof of (11.51) we used the following result.

**Lemma 11.6.** Suppose (11.1) and (4.18). We also need the following weaker version of (4.18),
\[
(\rho^3 a^3)^{-\frac{1}{12}} \geq \frac{\tilde{K}_H}{ds K_\ell} (1 + \log(K_H)). \] (11.61)

Then for sufficiently small values of \( \rho \), we have,
\[
\left| \rho z \hat{W}_1 \omega(0) - (2\pi)^{-3} \int_{k \in P_H} \hat{W}_1(k) \alpha_k dk \right| \leq C \rho a (\rho^3 a^3)^{\frac{5}{12}} \tilde{K}_H^{-1}. \] (11.62)

Furthermore,
\[
\left| \hat{W}_1 \omega(0) - (2\pi)^{-3} \int_{k \in P_H} \hat{W}_1(k)^2 \frac{2D_k}{k^2} dk \right| \leq C a (\rho^3 a^3)^{\frac{5}{12}} \tilde{K}_H^{-1}. \] (11.63)

**Proof.** Collecting the estimates below, we really get
\[
\left| \rho z \hat{W}_1 \omega(0) - (2\pi)^{-3} \int_{k \in P_H} \hat{W}_1(k) \alpha_k dk \right| \\
\leq C \rho a \left( K_\ell^{-1} + (\rho^3 a^3) K_H + \frac{R^2}{\ell^2} + (\rho^3 a^3) K_H^3 + \frac{a}{ds} (1 + \log(K_H)) \right). \] (11.64)
More precise energy estimates

From this (11.62) follows upon using (4.18), (11.61) and (4.29) to compare the magnitudes of the different terms.

We calculate,
\[ \rho_z \frac{\tilde{W}_1(0)}{\pi} - (2\pi)^{-3} \int_{k \in P_H} \tilde{W}_1(k) \alpha_k dk \]
\[ = (2\pi)^{-3} \int_{k \in P_H} \tilde{W}_1(k) \left( \rho_z \frac{\tilde{g}(k)}{2k^2} - \alpha_k \right) dk + (2\pi)^{-3} \int_{k \notin P_H} \rho_z \tilde{W}_1(k) \frac{\tilde{g}(k)}{2k^2} dk. \]  
(11.65)

We first estimate the last integral,
\[ \left| \int_{k \notin P_H} \tilde{W}_1(k) \frac{\tilde{g}(k)}{2k^2} dk \right| \leq C a^2 \int_{\{k \leq K_H^{-1} a^{-1}\}} k^{-2} dk = C a K_H^{-1}. \]  
(11.66)

This is consistent with the error term in (11.64).

To continue, we write
\[ \tilde{W}_1(k) \alpha_k = \rho_z^{-1} A(k) \left( 1 - \sqrt{1 - B(k)^2/A(k)^2} \right). \]  
(11.67)

Notice that \(|B(k)/A(k)| \leq \frac{1}{2}\), for \(\rho_\mu\) sufficiently small using (11.42) and (4.18). Therefore,
\[ \left| \tilde{W}_1(k) \alpha_k - \rho_z \tilde{W}_1(k)^2 \right| \leq C \rho_z \tilde{W}_1(k)^2 \leq C \rho_z a^4 k^{-6}, \]  
(11.68)

where we used that \(A(k) \geq \frac{1}{2} k^2\) in \(P_H\). Upon integrating over \(P_H\) we find a term of magnitude
\[ \int_{P_H} \left| \tilde{W}_1(k) \alpha_k - \rho_z \tilde{W}_1(k)^2 \right| \leq C \rho_z a (\rho_z a^3)^2 K_H^3, \]  
(11.69)
in agreement with (11.64).

Finally, we estimate, using \(0 \leq k^2 - \tau(k) \leq 2 |k| (ds_\ell)^{-1}\) in \(P_H\),
\[ \rho_z \left| \int_{k \in P_H} \tilde{W}_1(k) \frac{\tilde{g}(k)}{2k^2} - \tilde{W}_1(k) \right| \]
\[ \leq \rho_z \left| \int_{k \in P_H} \tilde{W}_1(k) \frac{\tilde{g}(k)}{2k^2} - \tilde{W}_1(k) \right| + \rho_z \left| \int_{k \in P_H} \tilde{W}_1(k) \frac{\tilde{g}(k)}{2k^2} \left( 1 - \frac{k^2}{A(k)} \right) \right| \]
\[ \leq \rho_z \left| \int_{k \in P_H} \tilde{W}_1(k) \frac{\tilde{g}(k) - \tilde{W}_1(k)}{2k^2} \right| + \rho_z^2 a^3 \int_{k \in P_H} \left| k^{-4} + C \rho_z (ds_\ell)^{-1} \left( \int_{\{K_H^{-1} \leq k \leq 1\}} a^2 |k|^{-3} + a \int_{2k^2} \tilde{W}_1(k) \right) \right| \]
\[ \leq \rho_z a \frac{R^3}{L^2} + \rho_z a (\rho_z a^3) K_H C \rho_z a (ds_\ell)^{-1} (1 + \log(K_H)), \]  
(11.70)

where the estimate of the first term follows from Cauchy-Schwarz and (5.16). This finishes the proof of (11.62).

The proof of (11.63) is similar. One can for instance use (11.62) and (11.69) and the fact that \(|1 - \frac{A(k)}{2k^2}| \leq C \frac{B(k)^2}{A(k)^2} \leq C \rho_\mu a^2 k^{-4}\) in \(P_H\). Then (11.63) follows. 

41
More precise energy estimates

Proof of (11.50). The two operators $Q^{(2)}_\Delta(z)$ and $Q^{(3)}_\Delta(z)$ are very similar and can be estimated in identical fashion, so we will only explicitly consider the first. We decompose

$$Q^{(2)}_\Delta(z) = I + II,$$

where

$$I := -\varepsilon \ell^{3}(2\pi)^{-6} \int_{(k \in P_H)} \frac{f_L(s) \overline{W}_1(k) \alpha_k}{(1 - \alpha_k^2)(1 - \alpha_{s-k}^2)} \left( b_{s-k}^\dagger \bar{a}_s \bar{a}_s b_{s-k} + b_{s-k}^\dagger \bar{a}_s b_{s-k} + b_{s-k}^\dagger b_{s-k}^\dagger \bar{a}_s \right),$$

$$II := -\varepsilon \ell^{3}(2\pi)^{-6} \int_{(k \in P_H)} \frac{f_L(s) \overline{W}_1(k) \alpha_k}{(1 - \alpha_k^2)(1 - \alpha_{s-k}^2)} \left( \bar{a}_s b_{s-k} + b_{s-k}^\dagger \bar{a}_s \right).$$

The second term $II$ will be very small, due to the smallness of the commutator (notice that $s$ and $k$ are ‘far apart’ since $s \in P_k$ and $k \in P_H$). So the main term is $I$, which we estimate using Cauchy-Schwarz and (11.43) as

$$I \geq -C \ell^{3} z a^\delta \int_{(k \in P_H)} f_L(s) \left( \varepsilon b_{s-k}^\dagger \bar{a}_s \bar{a}_s b_{s-k} + \varepsilon^{-1} b_{s-k}^\dagger b_{s-k}^\dagger b_{s-k} \right).$$

We estimate $\int \bar{a}_s \bar{a}_s \leq \ell^{-3} M$. Upon choosing $\varepsilon = \sqrt{K_L^3 K_L^3 / M}$ and using an easy bound on $D_k$, this leads to the estimate

$$\langle \Phi, I \Phi \rangle \geq -C z a^\delta K_L^{3/2} K_L^{3/2} \int_{(|k| \geq \frac{3}{2} K_H^{1\alpha - 1})} b_k^\dagger b_k \Phi \geq -C \ell^{2} \ell^{3} \left( \frac{K_L^{3} K_L^{3} M}{\rho^3 \ell^{3}} \right)^{1/2} \langle \Phi, \int_{(|k| \geq \frac{3}{2} K_H^{1\alpha - 1})} D_k b_k^\dagger b_k \Phi \rangle$$

Notice that

$$\frac{K_L^{3} K_L^{3} M}{\rho^3 \ell^{3}} = K_L^{3} K_M (\rho \alpha^3)^{3/2} \ll 1,$$

using (4.22). Therefore, $I$ can be absorbed in the $\varepsilon^{2}(2\pi)^{-3} \ell^{3} \int_{(|k| \geq \frac{3}{2} K_H^{1\alpha - 1})} D_k b_k^\dagger b_k$ term in (11.50).

We now return to the term $II$ from (11.72). This is easily estimated as

$$II \geq -2 \ell^{3}(2\pi)^{-6} \sup_{|p| \geq \frac{3}{2} K_H^{1\alpha - 1}} \left| \tilde{\chi}(p) \right| \left( K_L K_\ell \right)^3 \left( \rho \alpha^3 + a \delta \right) \int_{|k| \geq \frac{3}{2} K_H^{1\alpha - 1}} b_k^\dagger b_k$$

The $b_k^\dagger b_k$ is easily absorbed in the $\delta^{2} \ell^{3} \int_{(|k| \geq \frac{3}{2} K_H^{1\alpha - 1})} D_k b_k^\dagger b_k$ term in (11.50). Therefore, using (11.1) and Lemma C.1 $II$ contributes with an error term of order

$$\rho \alpha^3 \ell^{3}(K_L^{3} \delta)^{M} K_L^{3} K_L^{3/2} \rho \alpha^3 \ll 1$$

to (11.50).

This finishes the proof of (11.50).
More precise energy estimates

Proof of (11.49). Finally, we estimate $Q^{(1)}_3(z)$. We rewrite

$$Q^{(1)}_3(z) = z \ell^3 (2\pi)^{-6} \int_{\{k \in P_H\}} \frac{f_L(s)\tilde{W}_1(k)}{(1-\alpha^2_k)(1-\alpha^2_{s-k})} \left( \tilde{a}^\dagger_s b_{s-k} k + \alpha_k \alpha_{s-k} \tilde{a}^\dagger_{-s} b^\dagger_{-s-k} b^\dagger_k + h.c. \right),$$

(11.78)

where we performed a change of variables in the second term to get the equality.

We combine this term with the diagonalized Bogolubov Hamiltonian. We leave a $\delta^2$-part of this operator in order to control error terms appearing below.

Therefore, we consider

$$\begin{align*}
(2\pi)^{-3} \ell^3 &\int_{\{k \in P_H\}} (1-2\delta^2) D_k b^\dagger_k b_k dk \\
&+ z \ell^3 (2\pi)^{-6} \int_{\{k \in P_H\}} \frac{f_L(s)\tilde{W}_1(k)}{(1-\alpha^2_k)(1-\alpha^2_{s-k})} \left( \tilde{a}^\dagger_s b_{s-k} k + \alpha_k \alpha_{s-k} \tilde{a}^\dagger_{-s} b^\dagger_{-s-k} b^\dagger_k + h.c. \right) \\
&= (2\pi)^{-3} \ell^3 \int_{\{k \in P_H\}} (1-2\delta^2) D_k c_k^\dagger c_k + T_1(k) + T_2(k) \\
&\geq (2\pi)^{-3} \ell^3 \int_{\{k \in P_H\}} T_1(k) + T_2(k).
\end{align*}$$

(11.79)

Here we have introduced the operators,

$$c_k := b_k + z(2\pi)^{-3} \int_{\{k \in P_H\}} \frac{f_L(s)\tilde{W}_1(k)}{(1-2\delta^2)D_k(1-\alpha^2_k)(1-\alpha^2_{s-k})} \left( b^\dagger_{s-k} \tilde{a}_s + \alpha_k \alpha_{s-k} \tilde{a}^\dagger_{-s} b^\dagger_{-s-k} \right) ds,$$

(11.80)

$$T_1(k) := -z(2\pi)^{-3} \int_{\{k \in P_H\}} \frac{f_L(s)\tilde{W}_1(k)\alpha_k \alpha_{s-k}}{(1-\alpha^2_k)(1-\alpha^2_{s-k})} \left( b^\dagger_k \tilde{a}_{-s} b^\dagger_{-s-k} + h.c. \right) ds,$$

(11.81)

and

$$T_2(k) := -\frac{|z|^2 \tilde{W}_1(k)^2}{(1-2\delta^2)D_k(1-\alpha^2_k)^2(2\pi)^{-6}} \int_{\{k \in P_H\}} \frac{f_L(s)f_L(s')}{(1-\alpha^2_{s-k})(1-\alpha^2_{s'-k})} \times \left( \tilde{a}^\dagger_{s'} b_{s'-k} + \alpha_k \alpha_{s-k} b_{s'-k} \tilde{a}^\dagger_{-s'} \right) \left( b^\dagger_{s-k} \tilde{a}_s + \alpha_k \alpha_{s-k} \tilde{a}^\dagger_{-s} b^\dagger_{-s-k} \right) ds ds'.
$$

(11.82)

where we used (11.43) to get the estimate on $T_2$. Notice that

$$\tilde{a}^\dagger_{s'} b_{s'-k} + \alpha_k \alpha_{s-k} b_{s'-k} \tilde{a}^\dagger_{-s'} = \left( \tilde{a}^\dagger_{s'} + \alpha_k \alpha_{s-k} \tilde{a}^\dagger_{-s'} \right) b_{s'-k} + \alpha_k \alpha_{s-k} [b_{s'-k}, \tilde{a}^\dagger_{-s'}].$$

(11.83)

The contribution from the commutator term is very small, both due to the factors of $\alpha$ and to the commutator, since $k \in P_H$, $s' \in P_L$. Therefore, we estimate

$$T_2(k) \geq (1+\varepsilon)T'_2(k) + (1+\varepsilon^{-1})T''_2(k),$$

(11.84)
More precise energy estimates

where

\[ T'_2(k) := -(1 + C\delta^2) |z|^2 \frac{\hat{W}_1(k)^2}{D_k} (2\pi)^{-6} \int f_L(s) f_L(s') \]
\[ \times \left( \tilde{a}^\dagger s' + \alpha_k \alpha_{s'-k} \tilde{a}^\dagger_{-s} \right) b_{s'-k} b^\dagger_{s-k} (\tilde{a}_s + \alpha_k \alpha_{s-k} \tilde{a}_{-s}) \right) ds ds' \]
\[ T''_2(k) := -(1 + C\delta^2) |z|^2 \frac{\hat{W}_1(k)^2}{D_k} (2\pi)^{-6} \]
\[ \times \int f_L(s) f_L(s') |\alpha_k|^2 \alpha_{s'-k} \alpha_{s-k} [b_{s'-k}, \tilde{a}^\dagger_{-s}] [\tilde{a}_{-s}, b^\dagger_{s-k}] \]  

(11.85)

For simplicity, we choose \( \varepsilon = \delta^2 \) and can therefore absorb the factor of \( (1 + \varepsilon) \) in \( T'_2(k) \) by simply changing the value of \( C \). With this choice, we estimate using (11.63), (4.18) and (11.43),

\[ (2\pi)^{-3} \varepsilon^3 \int_{k \in P_H} (1 + \varepsilon^{-1}) T'_2(k) dk \geq -C\rho_2 a(K_L)^6 \delta^2 \sup_{k \in P_{H}, s \in P_L} |a_{s-k} b^\dagger_{s-k}|^2 \]
\[ \geq -C\rho_2^2 a^3 (\rho a^3)^{1/2} K_L^6 \delta^2 \sup_{k \in P_{H}, s \in P_L} |a_{s-k} b^\dagger_{s-k}|^2. \]

(11.86)

We continue to estimate the other part of \( T_2(k) \).

\[ T'_2(k) := -(1 + C\delta^2) |z|^2 \frac{\hat{W}_1(k)^2}{D_k} (2\pi)^{-6} \int f_L(s) f_L(s') \]
\[ \times \left( \tilde{a}^\dagger_{s'} + \alpha_k \alpha_{s'-k} \tilde{a}^\dagger_{-s} \right) b_{s'-k} b^\dagger_{s-k} (\tilde{a}_s + \alpha_k \alpha_{s-k} \tilde{a}_{-s}) \right) ds ds' \]
\[ = T'_{2,\text{comm}}(k) + T'_{2,\text{op}}(k), \]  

(11.87)

with

\[ T'_{2,\text{comm}}(k) := -(1 + C\delta^2) |z|^2 \frac{\hat{W}_1(k)^2}{D_k} (2\pi)^{-6} \int f_L(s) f_L(s') \]
\[ \times \left( \tilde{a}^\dagger_{s'} + \alpha_k \alpha_{s'-k} \tilde{a}^\dagger_{-s} \right) [b_{s'-k}, b^\dagger_{s-k}] (\tilde{a}_s + \alpha_k \alpha_{s-k} \tilde{a}_{-s}) \right) ds ds', \]
\[ T'_{2,\text{op}}(k) := -(1 + C\delta^2) |z|^2 \frac{\hat{W}_1(k)^2}{D_k} (2\pi)^{-6} \int f_L(s) f_L(s') \]
\[ \times \left( \tilde{a}^\dagger_{s'} + \alpha_k \alpha_{s'-k} \tilde{a}^\dagger_{-s} \right) b^\dagger_{s-k} b_{s'-k} (\tilde{a}_s + \alpha_k \alpha_{s-k} \tilde{a}_{-s}) \right) ds ds'. \]

(11.88)

We start by estimating the last term in (11.87). We introduce the notation

\[ C := \sup_{s, s' \in P_{H}, k \in P_H} |[\tilde{a}^\dagger_{s'}, \alpha_k \alpha_{s'-k} \tilde{a}^\dagger_{-s}, b^\dagger_{s-k}]| \leq 1, \]

(11.89)

In fact, it follows from (9.6), (11.10), (11.43), and (C.4) that

\[ C \leq C \delta \left(K_L^{-2} \tilde{K}_H^2 (\rho a^3)^{1/2} \right)^{M-1}. \]

(11.90)
More precise energy estimates

To estimate the last term in \([11.87]\) we first apply Cauchy-Schwarz, then commute the \(\tilde{a}'s\) through the \(b's\) and apply Cauchy-Schwarz to the commutator terms. This yields,

\[
\langle \Phi, \int \int f_L(s)f_L(s') \left( \tilde{a}_{s-k}^\dagger + \alpha_k \alpha_{s-k} \tilde{a}_{s-k}^\dagger \right) b_{s-k}^\dagger b_{s-k} \left( \tilde{a}_s + \alpha_k \alpha_{s-k} \tilde{a}_{s-k} \right) \Phi \rangle 
\]

\[
\leq 2 \int \int f_L(s)f_L(s') \langle \Phi, b_{s-k}^\dagger \left( \tilde{a}_{s-k}^\dagger + \alpha_k \alpha_{s-k} \tilde{a}_{s-k}^\dagger \right) \left( \tilde{a}_s + \alpha_k \alpha_{s-k} \tilde{a}_{s-k} \right) b_{s-k} \Phi \rangle 
\]

\[
+ CC \int \int f_L(s)f_L(s') \left( \Phi, \left( \varepsilon b_{s-k}^\dagger b_{s-k} + C \varepsilon^{-1} \tilde{a}_{s-k}^\dagger \tilde{a}_{s-k} + C \right) \Phi \right) 
\]

\[
\leq C(\ell^{-3}M + \varepsilon |P_L|C) \int f_L(s) \langle \Phi, b_{s-k}^\dagger b_{s-k} \Phi \rangle + C\varepsilon^{-1}|P_L|C(\ell^{-3}M + |P_L|) + C|P_L|^2C^2. 
\]  \([11.91]\)

For simplicity, we choose \(\varepsilon = \frac{M}{\ell^3|P_L|C}\) and get

\[
\langle \Phi, \int \int f_L(s)f_L(s') \left( \tilde{a}_{s-k}^\dagger + \alpha_k \alpha_{s-k} \tilde{a}_{s-k}^\dagger \right) b_{s-k}^\dagger b_{s-k} \left( \tilde{a}_s + \alpha_k \alpha_{s-k} \tilde{a}_{s-k} \right) \Phi \rangle 
\]

\[
\leq C\ell^{-3}M \left( \Phi, \int f_L(s)b_{s-k}^\dagger b_{s-k} \Phi \right) + C|P_L|^2C^2 \left( 1 + \frac{\ell^3|P_L|}{M} \right). 
\]  \([11.92]\)

Therefore, using \([11.63]\),

\[
\langle \Phi, (2\pi)^{-3}3^{3\ell^3} \int_{k \in P_H} T_{2, \text{opt}}(k) \Phi \rangle \geq -C\rho_\mu \ell^3 \frac{a^2}{(\min_{k \in \frac{1}{2}P_H} D_k)^2} M|P_L| \langle \Phi, \int \int \int_{\{q \in \frac{1}{2}P_H\}} D_q b_q^\dagger b_q \Phi \rangle 
\]

\[
- \rho_\mu \ell^6 a|P_L|^2C^2 \left( 1 + \frac{\ell^3|P_L|}{M} \right). 
\]  \([11.93]\)

Notice that \(D_k \geq C^{-1}K_H^{-2}(\rho_\mu a^3)^\frac{5}{3}a^{-2}\), for \(k \in \frac{1}{2}P_H\). Therefore, using \([11.4]\) and \([4.22]\),

\[
\rho_\mu \left( \frac{a^2}{(\min_{k \in \frac{1}{2}P_H} D_k)^2} M|P_L| \right) \leq \delta^2 K_L^3 K_M(\rho_\mu a^3)^\frac{5}{3} \ll \delta^2. 
\]  \([11.94]\)

Therefore, the negative \(D_q b_q^\dagger b_q\)-term in \([11.93]\) can be absorbed in a fraction of the similar (positive) term left out in \([11.79]\) exactly for this purpose.

Notice that \(\ell^3|P_L| \leq C(K_L K_H)^3 = d^{-6}\) using \([4.15]\). Therefore, it follows from \([4.19]\) that \(\frac{\ell^3|P_L|}{M} \ll 1\). So, using \([11.90]\) we can estimate the error term in \([11.93]\) as

\[
- \rho_\mu \ell^6 a|P_L|^2C^2 \left( 1 + \frac{\ell^3|P_L|}{M} \right) \geq -C\rho_\mu^2 \ell^6 \sqrt{\rho_\mu a^3} \left( K_\ell^{-3} d^{-12} \delta^2 \left( K_\ell^{-2} K_H(\rho_\mu a^3)^\frac{5}{3} \right)^{M-1} \right), 
\]  \([11.95]\)

This is clearly seen to agree with \([11.49]\).

We next consider the commutator term \(T_{2, \text{comm}}(k)\) from \([11.87]\).

From \([11.23]\) and using Lemma \([C.1]\) we see that

\[
|b_{s-k}^\dagger b_{s-k}^\dagger - \hat{\chi}^2((s-s')\ell)| \leq C\delta^2 \hat{\chi}^2((s-s')\ell) + C(K_\ell^{-2} \delta)^{\frac{M-1}{2}}. 
\]  \([11.96]\)
Notice now, using (4.4), (4.23), (11.4) and (11.1), that

\[ T_{2,\text{comm}}(k) \geq (1 + C\delta^2)|z|^2 \frac{\widehat{W}_1(k)^2}{D_k} (2\pi)^{-6} \int \int f_L(s)f_L(s')\widehat{a}_s^\dagger \widehat{a}_s^\dagger ((s-s')\ell)\widehat{a}_s 
- C|z|^2 \frac{\widehat{W}_1(k)^2}{D_k} \delta^2 |P_L|\ell^{-3} n_+ . \]  

(11.97)

Using (9.17) and (11.63) we see that

\[ -(2\pi)^{-3} \ell^3 \int_{k \in P_H} (1 + C\delta^2)|z|^2 \frac{\widehat{W}_1(k)^2}{D_k} (2\pi)^{-6} \int \int f_L(s)f_L(s')\widehat{a}_s^\dagger \widehat{a}_s^\dagger ((s-s')\ell)\widehat{a}_s 
= -\rho_z(1 + C\delta^2)(2\pi)^{-3} \int_{k \in P_H} \frac{\widehat{W}_1(k)^2}{D_k} \sum_j Q_{L,j}x^2 Q_{L,j} \]

\[ \geq -2\rho_z(1 + C\delta^2)\widehat{W}_1(0)(2\pi)^{-3} \int_{k \in P_H} \frac{\widehat{W}_1(k)^2}{D_k} \sum_j Q_{L,j}x^2 Q_{L,j} . \]  

(11.98)

Here we used (11.63) to control the error from (11.63).

We now notice that, for all \( \varepsilon > 0 \),

\[ \sum_j Q_{L,j}x^2 Q_{L,j} \leq (1 + \varepsilon) \sum_j Q_{L,j}x^2 Q_{L,j} + C\varepsilon^{-1} n_+^H . \]  

(11.99)

We notice that \( \rho_z a = (dK_i)^2 \frac{1}{(d\ell)^2} \). Therefore, choosing \( \varepsilon \) proportional to \( \varepsilon_t^{-1}(dK_i)^2 \), we find, using (11.63),

\[ -(2\pi)^{-3} \ell^3 \int_{k \in P_H} (1 + C\delta^2)|z|^2 \frac{\widehat{W}_1(k)^2}{D_k} (2\pi)^{-6} \int \int f_L(s)f_L(s')\widehat{a}_s^\dagger \widehat{a}_s^\dagger ((s-s')\ell)\widehat{a}_s 
\geq -2\rho_z(1 + C\delta^2 + C\varepsilon_t^{-1}(dK_i)^2)\widehat{W}_1(0)(2\pi)^{-3} \int_{k \in P_H} \frac{\widehat{W}_1(k)^2}{D_k} \sum_j Q_{L,j}x^2 Q_{L,j} - \frac{1}{100} \frac{1}{(d\ell)^2} n_+^H . \]  

(11.100)

Notice now, using (4.4), (4.23), (11.4) and (11.1), that

\[ \rho_z a[\delta^2 + \varepsilon_t^{-1}(dK_i)^2] \ll \ell^{-2} . \]  

(11.101)

Therefore, the above error terms can be absorbed in the energy gap.

To estimate the error term in (11.97) we integrate

\[ -(2\pi)^{-3} \ell^3 \int_{k \in P_H} C|z|^2 \frac{\widehat{W}_1(k)^2}{D_k} \delta^2 |P_L|\ell^{-3} n_+ \geq -C\rho_z a\delta^2(\ell^3|P_L|)n_+ . \]  

(11.102)

Notice that by (4.23) and (4.15) \( \rho_z a\delta^2(\ell^3|P_L|) \ll \ell^{-2} \), so this term can also be absorbed in the energy gap.

We now estimate the other commutator term, namely \( T_1(k) \) from (11.81). We clearly have

\[ T_1(k) \geq -Cz\delta \sup_{k \in P_H, s \in P_L} \left( |[b_{k, s-k}^\dagger, b_{s-k}^\dagger]| \right) |\alpha_k\widehat{W}_1(k)| \int f_L(s)\left( \widehat{a}_{s-k}^\dagger \widehat{a}_{s-k} + 1 \right) ds 
- Cz\delta \sup_{k \in P_H, s \in P_L} \left( |[b_{k, s-k}^\dagger, \widehat{a}_{s-k}^\dagger]| \right) |\alpha_k\widehat{W}_1(k)| \int f_L(s)\left( b_{s-k}^\dagger b_{s-k} + 1 \right) ds . \]  

(11.103)
Proof of the main theorem

Therefore,
\[
\ell^3 (2\pi)^{-3} \int_{k \in F_\ell} T_1(k) \, dk \geq -C \delta \left( \sup_{k \in P_H, s \in P_L} |b_k^\dagger a_{k-s}^\dagger| \right) \rho a(n_+ + (K_\ell K_L)^3) \delta \left( \sup_{k \in P_H, s \in P_L} |b_k^\dagger a_{k-s}^\dagger| \right) \rho a(K_\ell K_L)^3 \delta \left( \sup_{k \in P_H, s \in P_L} |D_{k-s} - k|^{-1} \right) a(K_\ell K_L)^3 \int_{\{|k| \geq \frac{1}{2} K_H^{-1} a^{-1}\}} D_k b_k^\dagger b_k. \tag{11.104}
\]

The last term in this inequality is easily seen to be estimated as
\[
\geq -\delta^2 \left\{ \frac{1}{\rho \ell^3} K_\ell^2 K_L^2 \left( \sup_{k \in P_H, s \in P_L} |b_k^\dagger a_{k-s}^\dagger| \right) \right\} \ell^3 \int_{\{|k| \geq \frac{1}{2} K_H^{-1} a^{-1}\}} D_k b_k^\dagger b_k, \tag{11.106}
\]
and using the properties of the commutator and Lemma \[C.1\] we see that this term can easily be absorbed in the extra $$\delta^2 \ell^3 \int_{\{|k| \geq \frac{1}{2} K_H^{-1} a^{-1}\}} D_k b_k^\dagger b_k$$ omitted in (11.79).

The two remaining terms in (11.104) can be estimated (using, in particular, Lemma \[C.1\] and (11.43)) as
\[
\geq -C \rho_\ell^2 \ell^3 \delta K_\ell^{-3/2} \left( \rho_\ell a^3 \right) \frac{1}{2} \left( M + (K_\ell^3 K_L^3) \right) \left( K_\ell^{-2} \delta \right)^{M-1}. \tag{11.107}
\]

This finishes the proof of (11.49) \qed

Now we have established all three inequalities (11.49), (11.50) and (11.51). This finishes the proof of Lemma 11.5 \qed

12 Proof of the main theorem

In this section we will combine the results of the previous sections in order to prove Theorem 1.2.

Proof of Theorem 1.2: As noted in Section 3 Theorem 1.2 follows from Theorem 3.1 which again—as observed in Section 5.3—follows from Theorem 5.8 We will use the concrete choice of parameters set down in (4.26) and (4.27) in Section 4. Recall in particular the notation X defined in (4.27).

To prove Theorem 5.8 let $$\Psi \in F_s(L^2(\Lambda))$$ be a normalized n-particle trial state satisfying (6.1). Notice that if such a state does not exist, then there is nothing to prove. Using Lemma 7.2 there exists a normalized n-particle wave function $$\tilde{\Psi} \in F_s(L^2(\Lambda))$$ satisfying (7.2) and such that
\[
\langle \Psi, H_A(\rho_\ell) \Psi \rangle \geq \langle \tilde{\Psi}, H_A(\rho_\ell) \tilde{\Psi} \rangle - CX^2 R_{\rho_\ell^2 \ell^3 (\rho_\ell a^3)^{1/2}}. \tag{12.1}
\]
Notice that the error term in (12.1) is consistent with the error term in Theorem 5.8.

Using Proposition 9.2 we find that our localized state $$\Psi$$ satisfies
\[
\langle \tilde{\Psi}, H_A(\rho_\ell) \tilde{\Psi} \rangle \geq \langle \tilde{\Psi}, H_{2nd}(\rho_\ell) \tilde{\Psi} \rangle \geq \langle \tilde{\Psi}, H_{2nd}(\rho_\ell) \tilde{\Psi} \rangle - C \rho_\ell^2 \ell^3 (\rho_\ell a^3)^{1/2} \left( \frac{32}{21} - \frac{1}{2} + X^3 (Ra^{-1})^2 (\rho_\ell a^3)^{1/2} \right), \tag{12.2}
\]
Notice that the error term in (12.1) is consistent with the error term in Theorem 5.8.
where the error is clearly consistent with the error term in Theorem 5.8.

At this point, we can apply Theorem 9.3 to get the lower bound
\[
\langle \tilde{\Psi}, H_{\lambda}^{2nd}(\rho_\mu) \tilde{\Psi} \rangle \geq \inf_{z \in \mathbb{R}^+} \inf_{\Phi} \langle \Phi, K(z) \Phi \rangle - C \rho_\mu a, \tag{12.3}
\]
where the second infimum is over all normalized \( \Phi \in \mathcal{F}(\text{Ran}(Q)) \) satisfying (9.28).

Since
\[
\rho_\mu a = \rho^2_\mu a \ell^3 \sqrt{\rho_\mu a^3 K^{-3}} = \rho^2_\mu a \ell^3 \sqrt{\rho_\mu a^3 X^2}, \tag{12.4}
\]
which is in agreement with the error term in Theorem 5.8, this implies that we need to prove that
\[
\inf_{\Phi} \langle \Phi, K(z) \Phi \rangle \geq -4 \pi \rho^2_\mu a \ell^3 + 4 \pi \rho^2_\mu a \ell^3 \frac{128}{15 \sqrt{\pi}} (\rho_\mu a^3)^{1/2} - C \rho^2_\mu a \ell^3 (\rho_\mu a^3)^{1/2} \left( \frac{R^2}{a^2} (\rho_\mu a^3)^{1/2} + X^{1/2} \right), \tag{12.5}
\]
for all normalized \( \Phi \) satisfying (9.28).

We will use that with our choice of parameters (11.2) is satisfied.

If \( \rho_z = |z|^2 / \ell^3 \) satisfies (10.3), i.e. is ‘far away’ from \( \rho_\mu \), then Proposition 10.2 provides a lower bound on \( \langle \Phi, K(z) \Phi \rangle \) which is larger than needed for (12.5) by a factor of 2 on the LHY-term. Since (11.2) is satisfied the assumptions of Proposition 10.2 are verified.

If \( \rho_z \) satisfies the complementary inequality (11.1) and \( \Phi \) satisfies (9.28), then by (11.7) (using again that (11.2) is satisfied) and Theorem 11.1 combined with Lemma 11.3 we get
\[
\langle \Phi, K(z) \Phi \rangle \geq - \frac{1}{2} \rho^2_\mu a \ell^3 g(0) + 4 \pi \frac{128}{15 \sqrt{\pi}} \rho_\mu a \sqrt{\rho_\mu a^3 \ell^3} + \langle \Phi, \left( \frac{b}{4 \ell^2} n_+ + \varepsilon_T \frac{b}{2 \ell^2} d_+ n_+^H + Q_{1}^{\text{ex}}(z) + Q_{2}^{\text{ex}}(z) + Q_{3}(z) \right) \Phi \rangle + (2 \pi)^{-3} \ell^3 \langle \Phi, \int D_k b_k^\dagger b_k dk \Phi \rangle - \mathcal{E}_1, \tag{12.6}
\]
where the error term \( \mathcal{E}_1 \) satisfies
\[
\mathcal{E}_1 \leq C \rho^2_\mu a \ell^3 (\rho_\mu a^3)^{1/2} \left( \frac{R^2}{a^2} (\rho_\mu a^3)^{1/2} + X^{1/2} + \rho_\mu a^3 \right) (Ra^{-1})^{1/2}. \tag{12.7}
\]

Here the error term in \( X^{1/2} \) comes from the \( \varepsilon(\rho_\mu, \rho_z) \) in Lemma 11.3. Notice that this error is compatible with (12.5) using Young’s inequality.

Now we can apply Theorem 11.4 to obtain the inequality
\[
(2 \pi)^{-3} \ell^3 \langle \Phi, \int D_k b_k^\dagger b_k dk \Phi \rangle + \langle \Phi, \left( \frac{b}{4 \ell^2} n_+ + \varepsilon_T \frac{b}{2 \ell^2} d_+ n_+^H + Q_{1}^{\text{ex}}(z) + Q_{2}^{\text{ex}}(z) + Q_{3}(z) \right) \Phi \rangle \geq -\mathcal{E}_2, \tag{12.8}
\]
with error term
\[
\mathcal{E}_2 \leq C \rho^2_\mu a \ell^3 \sqrt{\rho_\mu a^3 (\rho_\mu a^3)^{1/2} X^{1/2}}. \tag{12.9}
\]
Here the dominant contribution to the error (with our choice of parameters) comes from the $\tilde{K}_H^{-1}(\rho_\mu a^3)^{1/3}$-term. This error is clearly consistent with (12.5).

Combining (12.6) and (12.8), we get

$$\langle \Phi, K(z) \Phi \rangle \geq -\frac{1}{2} \rho_\mu^2 \ell^3 g(0) + 4\pi \frac{128}{15} \rho_\mu a^3 \ell^3$$

$$- (\mathcal{E}_1 + \mathcal{E}_2 + C) \left| \rho_\mu a^3 - a\sqrt{a^3} \right| \ell^3).$$ (12.10)

This establishes (12.5) for $\rho_z$ satisfying (11.1), since by (11.1), (11.2) and (4.26) we have

$$\left| \rho_\mu a^3 - a\sqrt{a^3} \right| \ell^3 \leq C \rho_\mu a^3 \ell^3 \tilde{X}_3.\quad (12.11)$$

This finishes the proof of (12.5) and therefore of Theorem 5.8, which in turn implies Theorem 3.1 and Theorem 1.2.

### A Bogolubov method

In this section we recall a simple consequence of the Bogolubov method (see [19, Theorem 6.3] and [7])

**Theorem A.1** (Simple case of Bogolubov’s method).

Let $a_\pm$ be operators on a Hilbert space satisfying $[a_+, a_-] = 0$ For $A > 0$, $B \in \mathbb{R}$ satisfying either $|B| < A$ or $B = A$ and arbitrary $\kappa \in \mathbb{C}$, we have the operator identity

$$\mathcal{A}(a_+^* a_+ + a_-^* a_-) + \mathcal{B}(a_+^* a_+ + a_+ a_-) + \kappa(a_+^* + a_-) + \overline{\kappa}(a_+ + a_-)$$

$$= \mathcal{D}(b_+^* b_+ + b_-^* b_-) - \frac{1}{2}(A - \sqrt{A^2 - B^2})([a_+, a_+] + [a_-, a_-]) - \frac{2|\kappa|^2}{A + B},$$ (A.1)

where

$$\mathcal{D} := \frac{1}{2} \left( A + \sqrt{A^2 - B^2} \right),$$ (A.2)

and

$$b_+ := a_+ + \alpha a_+^* + c_0, \quad b_- := a_- + \alpha a_-^* + c_0,$$ (A.3)

with

$$\alpha := B^{-1} \left( A - \sqrt{A^2 - B^2} \right), \quad c_0 := \frac{2\pi}{A + B + \sqrt{A^2 - B^2}},$$ (A.4)

In particular,

$$\mathcal{A}(a_+^* a_+ + a_-^* a_-) + \mathcal{B}(a_+^* a_+ + a_+ a_-) + \kappa(a_+^* + a_-) + \overline{\kappa}(a_+ + a_-)$$

$$\geq -\frac{1}{2}(A - \sqrt{A^2 - B^2})([a_+, a_+] + [a_-, a_-]) - \frac{2|\kappa|^2}{A + B}.$$ (A.5)

**Proof.** The identity (A.1) is elementary. From here the inequality (A.5) follows by dropping the positive operator term $\mathcal{D}(b_+^* b_+ + b_-^* b_-)$.
Localization to small boxes

The Hamiltonian $H_B(\rho_\mu)$ defined in (5.24) (with $u = 0$) is localized to the box $\Lambda := \Lambda(0) = [-\ell/2, \ell/2]^3$. In order to arrive at the a priori bounds in Theorem 6.1 we will localize again to boxes with a length scale $\ell d \ll (\rho a)^{-1/2}$. The reason for this second localization is that we need a larger Neumann gap in order to absorb errors. We therefore introduce a new family of boxes (some of which will have a rectangular shape) given by

$$B(u) = [-\ell/2, \ell/2]^3 \cap (\ell du + [-\ell d/2, \ell d/2]^3), \quad u \in \mathbb{R}^3.$$  

The functions that localize to these boxes are

$$\chi_B(u)(x) = \chi\left(\frac{x}{\ell}\right) \chi\left(\frac{x}{d\ell} - u\right), \quad u \in \mathbb{R}^3,$$

where $\chi$ is given in (C.1) in terms of the positive integer $M$. Observe that

$$\int\int \chi_B(u)(x)^2 dx du = \ell^3.$$

As usual we consider the projections

$$P_B(u)\varphi = |B(u)|^{-1}\langle 1_B(u), \varphi \rangle 1_B(u), \quad Q_B(u)\varphi = 1_B(u)\varphi - P_B(u)\varphi.$$

In these small boxes we consider the Hamiltonian

$$H_B(u)(\rho_\mu) = \sum_{i=1}^N \left( T_{B(u),i} - \rho_\mu \int w_{1,B(u)}(x,y) dy \right) + \frac{1}{2} \sum_{i \neq j} w_{B(u)}(x_i, x_j)$$

where (omitting the index $u$)

$$T_B = \frac{1}{2} \varepsilon_T (1 + \pi^{-2})^{-1}(d\ell)^{-2}Q_B + Q_B \chi_B[\sqrt{-\Delta} - (d\ell)^{-1}]^{-2}\chi_B Q_B$$

and

$$w_B(x, y) = \chi_B(x)W^s(x-y)\chi_B(y), \quad w_{1,B}(x, y) = \chi_B(x)W_1^s(x-y)\chi_B(y)$$

with (where the subscript $s$ refers to small)

$$W^s(x) = \frac{W(x)}{\chi \ast \chi(x/(d\ell))}, \quad W_1^s(x) = \frac{W_1(x)}{\chi \ast \chi(x/(d\ell))}.$$}

As in the large boxes we will also need

$$w_{2,B}(x, y) = \chi_B(x)W_2^s(x-y)\chi_B(y), \quad W_2^s(x) = \frac{W_2(x)}{\chi \ast \chi(x/(d\ell))}.$$}

Since $\omega \leq 1$ we have

$$\int W_2^s \leq 2 \int W_1^s(x) \leq Ca$$

We have by a Schwarz inequality that

$$\int\int w_{1,B}(x, y) dx dy \leq \int\int \chi_B(x)^2 W_1^s(x-y) dx dy \leq (\text{cst.})a \int \chi_B^2 \leq Ca|B|. (B.10)$$
Localization to small boxes

Observe also that
\[ \int \int \int w_{1,B(u)}(x,y) dx dy du = \ell^3 \int g = 8 \pi a \ell^3. \]  
(B.11)

It was proved in [7] Theorem 3.10 that the operator \( \mathcal{H}_\Lambda(\rho_\mu) \) defined in (5.24) and (5.25) can be bounded below by (we are for the lower bound ignoring the third term in \( T \) in (5.19))
\[ \mathcal{H}_\Lambda(\rho_\mu) \geq \sum_{i=1}^{N} \frac{b}{2} Q_{\Lambda,i} \ell^{-2} + \int_{\mathbb{R}^3} \mathcal{H}_{B(u)}(\rho_\mu) du, \]  
(B.12)
if
\[ \varepsilon_T, s, ds^{-1}, \text{ and } (s^{-2} + d^{-3})(sd)^{-2}s^M \]  
are smaller than some universal constant. Note that, if \( \rho_\mu a^3 \) is small enough, this is satisfied for our choices in Section 4, in particular, due to (4.3).

In the integral above the operators \( \mathcal{H}_{B(u)}(\rho_\mu) \) are, however, not unitarily equivalent. Depending on \( u \) the boxes \( B(u) \) can be rather small and rectangular. We denote by \( \lambda_1(u) \leq \lambda_2(u) \leq \lambda_3(u) \leq d \ell \) the side lengths of the boxes \( B(u) \). To avoid boxes that are very small, i.e., where \( \lambda_1(u) \leq \rho_\mu^{-1/3} \) we will restrict the integral above to \( u \) such that
\[ \|\ell du\|_\infty \leq \frac{\ell}{2} (1 + d) - \rho_\mu^{-1/3}. \]

Note that since the full integral would be over the set where \( \|\ell du\|_\infty \leq \frac{\ell}{2} (1 + d) \) we see that the restriction implies that all boxes will satisfy \( \lambda_1(u) \geq \rho_\mu^{-1/3} \).

For the kinetic energy and the repulsive potential this restriction will only give a further lower bound. For the chemical potential term we will use the following result.

**Lemma B.1.** For all \( x \in \Lambda \) we have the estimate
\[ -\rho_\mu \int \int w_{1,B(u)}(x,y) dy du \]
\[ \geq -\rho_\mu \int_{\|u\|_{\infty} - \frac{1}{2}(\frac{1}{d}+1)\leq -((\ell \rho_\mu)^{1/3})^{-1}} \int w_{1,B(u)}(x,y) dy du \]
\[ -3 \rho_\mu \int_{-2((\ell \rho_\mu)^{1/3})^{-1} \leq \|u\|_{\infty} - \frac{1}{2}(\frac{1}{d}+1)\leq -((\ell \rho_\mu)^{1/3})^{-1}} \int w_{1,B(u)}(x,y) dy du. \]  
(B.14)

**Proof.** The estimate above follows if we can show that for all \( x, y \in \Lambda \) we have
\[ \chi * \chi \left( \frac{x - y}{\ell d} \right) \leq \int_{\|u\|_{\infty} - \frac{1}{2}(\frac{1}{d}+1)\leq -((\ell \rho_\mu)^{1/3})^{-1}} \chi \left( \frac{x}{\ell d} - u \right) \chi \left( \frac{y}{\ell d} - u \right) du \]
\[ + 3 \int_{-2((\ell \rho_\mu)^{1/3})^{-1} \leq \|u\|_{\infty} - \frac{1}{2}(\frac{1}{d}+1)\leq -((\ell \rho_\mu)^{1/3})^{-1}} \chi \left( \frac{x}{\ell d} - u \right) \chi \left( \frac{y}{\ell d} - u \right) du. \]  
(B.15)
We have
\[ \chi * \chi \left( \frac{x - y}{\ell d} \right) - \int_{|u|_{\infty} - \frac{1}{2}(\frac{1}{d}+1)\leq -((\ell \rho_\mu)^{1/3})^{-1}} \chi \left( \frac{x}{\ell d} - u \right) \chi \left( \frac{y}{\ell d} - u \right) du \]
\[ = \int_{|u|_{\infty} - \frac{1}{2}(\frac{1}{d}+1)\leq -((\ell \rho_\mu)^{1/3})^{-1}} \chi \left( \frac{x}{\ell d} - u \right) \chi \left( \frac{y}{\ell d} - u \right) du. \]  
(B.16)
Localization to small boxes

Since \( x, y \in \Lambda \), the integral on the right is supported on \( ||u||_\infty - \frac{1}{2} \left( \frac{1}{3} + 1 \right) \leq 0 \). Using the fact that \( \rho_\mu^{-1/3} < \ell d/2 \) and that \( \chi \) is a product of symmetric decreasing functions of the coordinates \( u_1, u_2, u_3 \) respectively, we may observe that for fixed \( u_2, u_3 \) we have

\[
\chi \left( \frac{x}{\ell d} - u \right) \chi \left( \frac{y}{\ell d} - u \right) \leq \frac{1}{2} \left( \frac{1}{3} + 1 \right) \int_{||u||_\infty - \frac{1}{2} \left( \frac{1}{3} + 1 \right) \leq ||u||_\infty + \frac{1}{2} \left( \frac{1}{3} + 1 \right) - 1} \chi \left( \frac{x}{\ell d} - u \right) \chi \left( \frac{y}{\ell d} - u \right).
\]

(B.17)

Using this repeatedly (also with \( u_1, u_2 \) and \( u_1, u_3 \) fixed) gives the result in the lemma.

As a consequence of the lemma we find from (B.12), if (B.13) is satisfied, that

\[
\mathcal{H}_\Lambda(\rho_\mu) \geq \frac{b}{2} e^{-2} \sum_{i=1}^{N} Q_{\Lambda,i} + \int_{\ell d u_\infty \leq \frac{1}{2}(1+d) - \rho_\mu^{-1/3}} \mathcal{H}_{B(u)}(m(u)\rho_\mu) du,
\]

where \( m(u) = 1 \) if \( \ell d u_\infty \leq \frac{1}{2}(1+d) - 2\rho_\mu^{-1/3} \) and \( m(u) = 0 \) otherwise, i.e., for \( u \) near the boundary.

The goal in the rest of this section is to give a lower bound on the ground state energy of the operators \( \mathcal{H}_{B(u)}(m(u)\rho_\mu) \) for to conclude an a priori lower bound on the ground state energy of \( \mathcal{H}_\Lambda(\rho_\mu) \). We may now assume that the shortest side length of \( B(u) \) satisfies \( \lambda_1(u) \geq \rho_\mu^{-1/3} \) and we will make use of the fact that the range \( R \) of the potential satisfies \( R \ll \rho_\mu^{-1/3} \). For simplicity we will often omit the parameter \( u \). A main ingredient in getting a lower bound is to get a priori bounds on the operators

\[
n = \sum_{i=1}^{N} 1_{B,i}, \quad n_0 = \sum_{i=1}^{N} P_{B,i}, \quad n_+ = \sum_{i=1}^{N} Q_{B,i}.
\]

(B.19)

Note that the operator \( n \) commutes with \( \mathcal{H}_B \). We will not distinguish the operator \( n \) from its value and talk about \( n \)-particle states.

Applying the decomposition of the potential energy in Subsection 5.4 to the small boxes we arrive at the following lemma.

**Lemma B.2.** There is a constant \( C > 0 \) such that on any small box \( B \) we have

\[
- \rho_\mu \sum_{i=1}^{N} \int w_{1,B}(x, y) dy + \frac{1}{2} \sum_{i \neq j} w_{B}(x_i, x_j) \geq A_0 + A_2 - \frac{C a(\rho_\mu + n_0 |B|^{-1})}{n_+}
\]

where

\[
A_0 = \frac{n_0(n_0 - 1)}{2 |B|^2} \int \int w_{2,B}(x, y) dxdy
\]

\[- \left( \rho_\mu \frac{n_0}{|B|} + \frac{n_0 + 1}{4} \right) \frac{1}{|B|} \int \int w_{1,B}(x, y) dxdy \]

(B.21)

and

\[
A_2 = \frac{1}{2} \sum_{i \neq j} P_{B,i} P_{B,j} w_{1,B}(x_i, x_j) Q_{B,j} Q_{B,i} + h.c.
\]

(B.22)
Localization to small boxes

Proof. We use the identity 5.33 which also holds in the small boxes with $P, Q$ and $w, w_1, w_2$ replaced by $P_B, Q_B$ and $w_B, w_{1,B}, w_{2,B}$ respectively. Let us denote the corresponding terms $Q_{i,B}^\text{ren}, i = 0, \ldots, 4$. Then

$$Q_{0,B}^\text{ren} = \frac{n_0(n_0 - 1)}{2|B|^2} \iint w_{2,B}(x, y) \, dx \, dy - \rho_p \frac{n_0}{|B|} \iint w_{1,B}(x, y) \, dx \, dy.$$

As in the proof of Lemma 5.11 we apply a Cauchy-Schwarz inequality—using the positivity of $w_B$—to absorb $Q_{3,B}^\text{ren}$ in $Q_{4,B}^\text{ren}$. This results in the following inequality,

$$Q_{3,B}^\text{ren} + Q_{4,B}^\text{ren} \geq -C \sum_{i \neq j} P_{B,i} Q_{B,j} w_{1,B}(x_i, x_j) Q_{B,j} P_{B,i} - \sum_{i \neq j} \left( P_{B,i} Q_{B,j} w_{1,B} \omega(x_i, x_j) P_{B,j} P_{B,i} + h.c. \right)$$

$$\geq -C \sum_{i \neq j} P_{B,i} Q_{B,j} w_{1,B}(x_i, x_j) Q_{B,j} P_{B,i} - \sum_{i \neq j} \left( P_{B,i} Q_{B,j} w_{1,B} \omega(x_i, x_j) P_{B,j} P_{B,i} + h.c. \right)$$

$$\geq -C \sum_{i \neq j} \left( P_{B,i} Q_{B,j} w_{1,B} \omega(x_i, x_j) P_{B,j} P_{B,i} + h.c. \right) - C n_0 |B|^{-1} n_+,$$

where we have used the pointwise inequality $0 \leq \omega \leq 1$, and additional Cauchy-Schwarz inequality in the second inequality, and

$$\sum_{i \neq j} P_{B,i} Q_{B,j} w_{1,B}(x_i, x_j) Q_{B,j} P_{B,i} \leq C \|\chi_B\|^2_x n_0 |B|^{-1} n_+ \int W_1^a \leq C n_0 |B|^{-1} n_+,$$

which follows from

$$\int \chi_B(x) W_1^a(x - y) \chi_B(y) \, dy \leq \|\chi_B\|^2_x \int W_1^a.$$

Notice that if we rewrite $Q_{B,1}^\text{ren}$ as in 5.40, the first term on the right side of (B.23) cancels the second line of (5.40). The remaining part of $Q_{B,1}^\text{ren}$ we estimate as follows.

$$|B|^{-1}(n_0 - \rho_p |B|) \sum_i Q_{B,i} \chi_B(x_i) W_1^a * \chi_B(x_i) P_{B,i} + h.c.$$ 

$$= |B|^{-1}(n_0^{1/2} + (\rho_p |B|)^{1/2}) \sum_i Q_{B,i} \chi_B(x_i) W_1^a * \chi_B(x_i) P_{B,i}((n_0 + 1)^{1/2} - (\rho_p |B|)^{1/2}) + h.c.$$ 

$$\geq -4 |B|^{-1} \left( n_0^{1/2} + (\rho_p |B|)^{1/2} \right) \sum_i Q_{B,i} \chi_B(x_i) W_1^a * \chi_B(x_i) Q_{B,i}$$

$$- \frac{1}{4} |B|^{-1} \left( (n_0 + 1)^{1/2} - (\rho_p |B|)^{1/2} \right) \sum_i P_{B,i} \chi_B(x_i) W_1^a * \chi_B(x_i) P_{B,i}.$$

The first term above we estimate similarly to the estimate in (B.24). The last term above is equal to

$$- \frac{1}{4} \frac{n_0}{|B|} \left( (n_0 + 1)^{1/2} - (\rho_p |B|)^{1/2} \right) \iint w_{1,B}(x, y) \, dx \, dy$$

$$\geq - \frac{1}{4} \left( \frac{n_0 + 1}{|B|} - \rho_p \right) \iint w_{1,B}(x, y) \, dx \, dy$$
which together with $Q_{\text{ren}}^{0,B}$ give the $A_0$ term in the lemma.

The first three terms in $Q_{\text{ren}}^{2,B}$ are absorbed into the last term in (B.20) using again the same Cauchy-Schwartz as in the second inequality in (B.23). Finally, the last terms in $Q_{\text{ren}}^{2,B}$ are exactly the terms collected in $A_2$. \(\square\)

We express the term $A_2$ from the lemma in second quantization. Introducing the operators

$$b_p^\dagger = |B|^{-1/2}a^\dagger(Q_B^B e^{-ipx})a_0$$

we can write

$$A_2 = \frac{1}{2}(2\pi)^{-3}\int \tilde{W}_1^B(p)(b_p^\dagger b_{-p}^\dagger + b_{-p} b_p)dp.$$  

We shall control $A_2$ using Bogolubov’s method. In order to do this we will add and subtract a term

$$A_1 = (2\pi)^{-3}K_s a \int (b_p^\dagger b_p + b_{-p}^\dagger b_{-p})dp,$$  

with the constant $K_s > 0$ chosen appropriately. Note that we have

$$A_1 \leq K_s a \frac{n_0 + 1}{|B|} n_+ ||\chi_B||^2_\infty \leq CK_s a \frac{n_0 + 1}{|B|} n_+.$$  

(B.27)

Lemma B.3 (Bogolubov’s method in small boxes). There exists a constant $C > 0$ such that

$$\sum_{i=1}^N Q_{B,i}^B |\sqrt{-\Delta_i} - (ds\ell)^{-1}|_+^2 \chi_{B,i} Q_{B,i} + A_2 \geq$$

$$- \frac{1}{2} \left(1 + C(R/(d\ell)^2)(1 + C(ds\ell)^{-1}a) g_\omega(0) \right) \frac{(n+1)n}{|B|} \int \chi_B^2$$

$$- C \left( a^2(ds\ell)^{-1} \ln(ds\ell a^{-1}) \frac{n+1}{|B|} + a^4(ds\ell)^3 \gomega \frac{n+1}{|B|} + a(ds\ell)^{-3} \right) \frac{n}{|B|} \int \chi_B^2$$

$$- C a \frac{n+1}{|B|} n_+.$$  

(B.28)

Moreover, for all $\varepsilon > 0$ there is a $C_\varepsilon > 0$ such that if

$$(R/d\ell)^2 < C_\varepsilon^{-1}, \quad a(ds\ell)^{-1} \ln(ds\ell a^{-1}) < C_\varepsilon^{-1}$$  

(B.29)

then

$$\sum_{i=1}^N Q_{B,i}^B |\sqrt{-\Delta_i} - (ds\ell)^{-1}|_+^2 \chi_{B,i} Q_{B,i} + A_2 \geq \frac{1}{2} \left(1 + \varepsilon g_\omega(0) + \varepsilon a \right) \frac{(n+1)n}{|B|} \int \chi_B^2$$

$$- C_\varepsilon a(ds\ell)^{-3} \frac{n}{|B|} \int \chi_B^2 - C_\varepsilon a \frac{n+1}{|B|} n_+.$$  

(B.30)

Proof. We add $A_1$ from (B.26) to the term we want to estimate. Using $n_0 \leq n$ we may write

$$\sum_{i=1}^N Q_{B,i}^B |\sqrt{-\Delta_i} - (ds\ell)^{-1}|_+^2 \chi_{B,i} Q_{B,i} + A_1 + A_2 \geq (2\pi)^{-3} \frac{1}{2} \int h(p)dp$$

where $h$ is the operator

$$h(p) = \left( \frac{|B|}{n+1} |p| - (ds\ell)^{-1} \right)_+^2 + 2K_s a \left( b_p^\dagger b_p + b_{-p}^\dagger b_{-p} + \tilde{W}_1^B(p)(b_p^\dagger b_{-p}^\dagger + b_{-p} b_p) \right).$$

54
We observe that
\[ |b_p, b'_p| \leq n_0|B|^{-1} \int \chi_B^2 \leq n|B|^{-1} \int \chi_B^2. \]
We will now apply the simple case of Bogolubov’s method in Theorem \ref{A.1} with
\[ \mathcal{A}(p) = \frac{|B|}{n + 1} |[p] - (\Delta s\ell)^{-1}]_+^2 + 2K_s a, \quad \mathcal{B}(p) = \tilde{W}_1^s(p) \]
We have by \eqref{B.9} that
\[ |\mathcal{B}(p)| = |\tilde{W}_1^s(p)| \leq \int W_1^s \leq C_0 a. \]
If we therefore choose \( K_s \geq C_0 \) we see that \( |\mathcal{B}|/\mathcal{A} \leq 1/2 \) and we get the following lower bound from Theorem \ref{A.1}
\[ h(p) \geq -\frac{1}{2} \left( \mathcal{A}(p) - \sqrt{\mathcal{A}(p)^2 - \mathcal{B}(p)^2} \right) n_0|B|^{-1} \int \chi_B^2. \]
Using that \( |\mathcal{B}|/\mathcal{A} \leq 1/2 \) we have
\[ h(p) \geq -C \frac{\mathcal{B}(p)^2}{\mathcal{A}(p)} n_0|B|^{-1} \int \chi_B^2. \]
We use this for \( |p| < 2(\Delta s\ell)^{-1} \) and we find for the integral over \( |p| < 2(\Delta s\ell)^{-1} \)
\[ \int_{|p|<2(\Delta s\ell)^{-1}} \frac{\mathcal{B}(p)^2}{\mathcal{A}(p)} dp \leq C_0^2 n_0 a \int_{|p|<2(\Delta s\ell)^{-1}} 1 dp \leq \frac{C_0^2}{K_s} (\Delta s\ell)^{-3}. \] (B.31)
For the simple bound \eqref{B.30} we may choose \( K_s \) large depending on \( \epsilon \) to have
\[ h(p) \geq -\frac{1}{2} (1 + \epsilon/2) \frac{\mathcal{B}(p)^2}{\mathcal{A}(p)} n_0|B|^{-1} \int \chi_B^2 \]
and use this in the range \( |p| > 2(\Delta s\ell)^{-1} \). For the more refined bound \eqref{B.28}, in the range \( |p| > 2(\Delta s\ell)^{-1} \), we use
\[ h(p) \geq - \left( \frac{1}{2} \frac{\mathcal{B}(p)^2}{\mathcal{A}(p)} + C \frac{\mathcal{B}(p)^4}{\mathcal{A}(p)^3} \right) n_0|B|^{-1} \int \chi_B^2. \]
For \( |p| > 2(\Delta s\ell)^{-1} \) we have
\[ \frac{\mathcal{B}(p)^2}{\mathcal{A}(p)} \leq \frac{n + 1}{|B|} \frac{\tilde{W}_1^s(p)^2}{(|p| - (\Delta s\ell)^{-1})^2 \leq \frac{\tilde{W}_1^s(p)^2}{p^2} (1 + C(\Delta s\ell)^{-1}|p|^{-1}) n + 1 |B|^{-1}}, \]
and hence by splitting the integral over the error in \( |p| < a^{-1} \) and \( |p| > a^{-1} \) we obtain
\[ \int_{|p|>2(\Delta s\ell)^{-1}} \frac{\mathcal{B}(p)^2}{\mathcal{A}(p)} dp \leq (1 + C(\Delta s\ell)^{-1}) \frac{n + 1}{|B|} \int_{\mathbb{R}^3} \frac{\tilde{W}_1^s(p)^2}{p^2} dp + Ca^2(\Delta s\ell)^{-1} \frac{n + 1}{|B|} \ln(\Delta s\ell a^{-1}). \]
Finally, we use that
\[ \frac{1}{4} (2\pi)^{-3} \int_{\mathbb{R}^3} \frac{\tilde{W}_1^s(p)^2}{p^2} dp = \frac{1}{4} \int_{\mathbb{R}^3} \frac{\tilde{W}_1^s(x)\tilde{W}_1^s(y)}{4\pi|x - y|} dxdy \]
\[ \leq \frac{1}{4} (1 + C(R/(\Delta\ell))^2) \int_{\mathbb{R}^3} \frac{g(x)g(y)}{4\pi|x - y|} dxdy \]
\[ = \frac{1}{4} (1 + C(R/(\Delta\ell))^2) (2\pi)^{-3} \int_{\mathbb{R}^3} \frac{\hat{g}(p)^2}{p^2} dp \]
\[ = \frac{1}{2} (1 + C(R/(\Delta\ell))^2) \hat{g}(0). \] (B.32)
Finally, to get \( (B.28) \) we estimate
\[
\int_{|p| > 2(ds\ell)^{-1}} \frac{B(p)^4}{A(p)^3} \leq (\text{const.}) a^4 \left( \frac{n + 1}{|B|} \right)^3 \int_{|p| > 2(ds\ell)^{-1}} |p|^{-6}dp = C a^4 \left( \frac{n + 1}{|B|} \right)^3 (ds\ell)^3. \tag{B.33}
\]
Using the estimate \( (B.27) \) on \( A_1 \) gives the last term in \( (B.28) \). \( \square \)

In order to use this lemma we will control the negative term quadratic in \( n \) in \( (B.30) \) in terms of the positive term quadratic in \( n_0 \) in \( (B.21) \). The difference between \( n \) and \( n_0 \) will be absorbed in the Neumann gap of \( T_B \). It is, however, important to establish the result in the following lemma

**Lemma B.4.** There is a constant \( C > 0 \) such that if the shortest side length \( \lambda_1 \) of the box \( B \) satisfies \( R \leq \frac{1}{2} C^{-1/2} \min \{ \lambda_1, \ell d \} \) then
\[
\iint w_1, B(x,y) dxdy \geq 8\pi a \left( 1 - C \left( \frac{R}{\lambda_1} \right)^2 \right) \left( 1 - C \left( \frac{R}{\ell d} \right)^2 \right) \int \chi_B^2. \tag{B.34}
\]
\[
\iint w_2, B(x,y) dxdy \geq \iint w_1, B(x,y) dxdy + \left( 1 - C \left( \frac{R}{\lambda_1} \right)^2 \right) \left( 1 - C \left( \frac{R}{\ell d} \right)^2 \right) g\omega(0) \int \chi_B^2. \tag{B.35}
\]
Moreover for any \( 0 < \varepsilon < 1/10 \) we can find a \( C_{\varepsilon} > 0 \) such that if \( R \leq C_{\varepsilon}^{-1} \min \{ \lambda_1, \ell d \} \) then
\[
\iint w_2, B(x,y) dxdy \geq \frac{3}{4} \iint w_1, B(x,y) dxdy + ((1 + \varepsilon) g\omega(0) + \varepsilon a) \int \chi_B^2. \tag{B.36}
\]
**Proof.** The estimate \( (B.35) \) follows from
\[
\iint w_2, B(x,y) dxdy - \iint w_1, B(x,y) dxdy
\]
\[
= \iint \omega(x-y) w_1, B(x,y)
\]
\[
\geq \int \omega(x) W_1^2(x) dx \left( \int \chi_B^2 - CR^2 \| \nabla \chi_B \|_{\infty} \int \chi_B \right)
\]
\[
\geq (1 - C(R\lambda_1^{-1})^2) \int \omega(x) W_1^2(x) dx \int \chi_B^2
\]
\[
\geq (1 - C(R\lambda_1^{-1})^2) (1 - C(R/(\ell d))^2) \left( \int g\omega \right) \int \chi_B^2,
\]
where we have used that \( \omega W \) is spherically symmetric, that \( |B|^{-1} \left( \int \chi_B \right)^2 \leq \int \chi_B^2 \), and that
\[
\| \partial_i \partial_j \chi_B \|_{\infty} \leq C_M \lambda_1^{-2} |B|^{-1} \int \chi_B, \tag{B.37}
\]
which is a simple exercise (see Appendix C). The estimate \( (B.34) \) follows in the same way without \( \omega \) and using \( \int g = 8\pi a \). Finally, \( (B.36) \) follows from \( \omega \leq 1 \). \( \square \)

We are now ready to give the bound on the energy in the small boxes.

**Theorem B.5** (Lower bound on energy in small boxes). Assume \( B \) is a box with shortest side length \( \lambda_1 \geq \rho_\mu^{-1/3} \). There are universal constants \( C, C' > 1 \) and \( 0 < c < 1/2 \) such that
Localization to small boxes

for all $1 \leq K_B \leq C'^{-1}(\rho_\mu a^3)^{-1/6}$ we have for the Hamiltonian defined in (B.4) restricted to $n$-particle states that

$$\mathcal{H}_B(\rho_\mu) \geq \left( c \frac{(n|B|)^{-1} - \rho_\mu}{1 + \frac{n}{|B|\rho_\mu}} - \frac{1}{2} \rho_\mu^2 \right) \int \int w_{1,B}(x,y)dxdy$$

$$- C\rho_\mu^2 a \left( (R\lambda_1^{-1})^2 + K_B^2(\rho_\mu a^3)^{1/2} \right) \int \chi_B^2 - C\rho_\mu a,$$

(B.38)

if

$$C'(\rho_\mu a^3)^{1/2} \leq \varepsilon_T^{1/2} a(d\ell)^{-1} \leq a(d\ell)^{-1} \ln(d\ell a^{-1}) \leq K_B(\rho_\mu a^3)^{1/2},$$

(B.39)

and

$$R \leq K_B^{1/2}(\rho_\mu a^3)^{1/4}(\rho_\mu a)^{-1/2}.$$  

(B.40)

We are assuming that $\varepsilon_T, s, d \leq 1$.

Note that all the assumptions on $K_B$, $R$, $\varepsilon_T$, $s$, and $d$ are satisfied with our choices in Section 4 if $\rho_\mu a^3$ is small enough. In particular, the assumption on $K_B$ is a consequence of (4.7), (B.39) follows from (4.4) and (4.6) and (B.40) was given in (4.29).

Proof. Note that (B.39) implies that

$$(\rho_\mu a)^{-1/2} K_B^{-1} \leq \frac{sd}{\ln(d\ell a^{-1})} \leq d\ell \leq \frac{\varepsilon_T^{1/2}}{C'}(\rho_\mu a)^{-1/2}.$$  

This, in particular, implies that

$$\sum_{i=1}^{N} \frac{1}{2} \varepsilon_T(1 + \pi^{-2})^{-1}(\ell_d)^{-2}Q_{B,i} \geq C''(1 + \pi^{-2})^{-1} \rho_\mu a n_+.$$  

(B.41)

Moreover we see from (B.40) that

$$R\rho_\mu^{1/3} \leq K_B^{1/2}(\rho_\mu a^3)^{1/12}, \quad R/(d\ell) \leq K_B^{3/2}(\rho_\mu a^3)^{1/4}.$$  

We now first choose $\varepsilon$ so small, e.g., to be $1/20$ so that we can apply Lemma B.4. Hence if $C'$ is large enough, we can, since $\lambda_1 > \rho_\mu^{1/3}$, use both (B.35) and (B.36) from Lemma B.4 We choose the same $\varepsilon$ in (B.30) and again by assuming that $C'$ large enough we can ensure that (B.29) is satisfied.

We may of course assume that $n > 0$ as the inequality we want to prove is obviously satisfied if $n = 0$ as the operator is 0 whereas the lower bound is negative in this case. We choose a constant $C_n > 2$ to be determined precisely below to depend only on $C$ and $C_\varepsilon$ in Lemma B.3. Observe that $\rho_\mu |B| \geq 1$. Hence we can choose an integer $n' \geq C_n$ in the interval $[C_n|\rho_\mu|B], (C_n + 1)|\rho_\mu|B])$ and we may write $n = mn' + n''$ with $m, n', n''$ non-negative integers and $n'' < n' < (C_n + 1)|\rho_\mu|B]$. We will get a lower bound on the energy if in the Hamiltonian we think of dividing the particles in $m$ groups of $n'$ particles and one group of $n''$ particles ignoring the positive interaction between the groups. It is not important that the Hamiltonian is no longer symmetric between the particles as we are not considering it as an operator, but only calculating its expectation value in a symmetric state. We arrive at the conclusion that if we denote by $e_B(n, \rho_\mu)$ the ground state energy of $\mathcal{H}_B(\rho_\mu)$ in an $n$-particle state then

$$e_B(n, \rho_\mu) \geq me_B(n', \rho_\mu) + e_B(n'', \rho_\mu).$$  

(B.42)
We have that both $n'$ and $n''$ are less than $(C_n + 1)\rho_n |B| \leq 2C_n\rho_n |B|$. This means that the last terms in (B.20), (B.28), and (B.30) in both cases can be absorbed in the positive term from (B.41) if we choose $C' \geq C_n$. Using (B.10) we see that the same is also true for the errors we get by replacing $n'_0$ and $n''_0$ by $n'$ and $n''$ respectively everywhere in $A_0$ in (B.21).

In the case of the $m$ groups of $n'$ particles we will use Lemma (B.2) and (B.30) to arrive at
\[
e_B(n', \rho_n) \geq \frac{n'^2}{2|B|^2} \int \int w_{2,B}(x, y) \, dxdy - \left( \rho_n \frac{n'}{|B|} + \frac{1}{4} \left( \rho_n - \frac{n'}{|B|} \right)^2 \right) \int \int w_{1,B}(x, y) \, dxdy
\]
\[- \frac{1}{2} \left( (1 + \varepsilon) \tilde{g}(0) + \varepsilon a \right) \frac{n'^2}{|B|^2} \int \chi_B^2 - C\rho_n n' |B|^{-1} \int \chi_B^2.
\]
where we have used that (B.39) implies that $a(df)^{-3} \leq C\rho_n a$. We have also used that the error in replacing $n' + 1$ by $n'$ in several terms can also be absorbed in the last term. Thus applying (B.36) we arrive at
\[
e_B(n', \rho_n) \geq \frac{1}{8} \left( \rho_n - \frac{n'}{|B|} \right)^2 \int \int w_{1,B}(x, y) \, dxdy - C\rho_n n' |B|^{-1} \int \chi_B^2.
\]
It follows, using (B.34) that if we choose the constant $C_n$ large enough then
\[
e_B(n', \rho_n) \geq \frac{1}{9} \left( \rho_n - \frac{n'}{|B|} \right)^2 \int \int w_{1,B}(x, y) \, dxdy \geq \frac{1}{18} \rho_n \frac{n'}{|B|} \int \int w_{1,B}(x, y) \, dxdy \geq 0.
\]
Hence
\[
me_B(n', \rho_n) \geq \frac{1}{18} \rho_n \frac{mn'}{|B|} \int \int w_{1,B}(x, y) \, dxdy = \frac{1}{18} \rho_n \frac{n-n''}{|B|} \int \int w_{1,B}(x, y) \, dxdy
\]
We turn to the group of $n''$ particles. If we apply Lemma (B.2) and (B.28) we see that since $n'' \leq 2C_n\rho_n |B|$ we have
\[
e_B(n'', \rho_n) \geq \frac{n''^2}{2|B|^2} \int \int w_{2,B}(x, y) \, dxdy - \left( \rho_n \frac{n''}{|B|} + \frac{1}{4} \left( \rho_n - \frac{n''}{|B|} \right)^2 \right) \int \int w_{1,B}(x, y) \, dxdy
\]- \frac{1}{2} \tilde{g}(0) \frac{n''^2}{|B|^2} \int \chi_B^2 - CC_n^2 \rho_n^2 aK_B^3(\rho_n a^3)^{1/2} \int \chi_B^2.
\]
The last term comes from repeatedly replacing $n'' + 1$ by $n''$ in the leading terms, which leads to an error $n''a|B|^{-2} \int \chi_B^2 \leq Cn''|B|^{-1} a$. In the error terms we can for the same replacement alternatively use that $1 \leq \rho_n |B|$.

If we now apply the estimate (B.35) in Lemma (B.4) we find that
\[
e_B(n'', \rho_n) \geq \frac{1}{4} \left( \frac{n''}{|B|} - \rho_n \right)^2 \int \int w_{1,B}(x, y) \, dxdy - \frac{1}{2} \rho_n^2 \int \int w_{1,B}(x, y) \, dxdy
\]- C\rho_n a \left( (R\lambda_1^{-1})^2 + K_B^3(\rho_n a^3)^{1/2} \right) \int \chi_B^2 - C\rho_n a.
\]
where we have now ignored the explicit dependence on $C_n$, which is after all now a chosen constant.

We have arrived at the bound that
\[
e_B(n, \rho_n) \geq \left( \frac{1}{4} \frac{n''}{|B|} - \rho_n \right)^2 + \frac{1}{18} \rho_n \frac{n-n''}{|B|} \int \int w_{1,B}(x, y) \, dxdy
\]- \frac{1}{2} \rho_n^2 \int \int w_{1,B}(x, y) \, dxdy - C\rho_n a^2 \left( (R\lambda_1^{-1})^2 + K_B^3(\rho_n a^3)^{1/2} \right) \int \chi_B^2 - C\rho_n a.
\]
This easily implies the result in the theorem. □
Localization to small boxes

We will now apply the small box estimate from the previous theorem to get an a priori bound on the energy and on the number of particles $n$ and excited particles $n_\pm$ in the large box.

**Theorem B.6** (A priori estimates in large box). Assume \([4.1], [B.39], [B.40]\). Then there is a constant $C > 0$ such that if again $1 \leq K_B \leq C^{-1}(\rho_\mu a^\ast)^{1/6}$ and $\rho_\mu a^\ast$ is smaller than some universal constant we have

$$\mathcal{H}_\Lambda(\rho_\mu) \geq -4\pi \rho_\mu^2 a^3 (1 + C K_B^3 (\rho_\mu a^3)^{1/2}).$$  \hspace{1em} (B.45)

Moreover, if there exists a normalized $n$-particle $\Psi \in \mathcal{F}_n(L^2(\Lambda))$ such that \([6.1]\) holds for a $0 < J \leq K_B^3$ then the a priori bounds \([6.2]\) hold.

As explained just after Theorem B.5 the assumptions \([B.39], [B.40]\), and the assumption on $K_B$ are satisfied with our choices in Section 4.

**Proof.** We use \([B.18]\) together with the estimate in Theorem B.5. We will denote by $n(u), n_0(u)$, and $n_\pm(u)$ the operators defined in \([B.19]\). The corresponding operators in the large box $\Lambda$ will be denoted $n, n_0$, and $n_\pm$. On the set

$$\mathcal{I} = \{ u \in \left[ -\frac{1}{2}, \frac{1}{2} \right], \frac{1}{2}, \left[ \frac{1}{2}, \frac{3}{2} \right] \} \cap \left[ \frac{1}{2} \ell (1 + d) - 2 \rho_\mu^{-1/3}, \frac{1}{2} \ell (1 + d) - \rho_\mu^{-1/3} \right],$$

we have that $\rho_\mu$ is replaced by $4\rho_\mu$. On this set we have according to (C.6) that $|\chi_B(u)(x)| \leq C(\rho_\mu^{-1/3}/\ell)^M \leq C(\rho_\mu a^3)^{M/6}$ with $C$ depending on $M$ and therefore

$$\int \int \chi_B(u)(x)^2 dx \, du \leq C(\rho_\mu a^3)^{M/3}(\ell d)^{-3} \leq C(\rho_\mu a^3)^{M/3} \ell^3.$$  \hspace{1em} (B.46)

If we use Theorem B.5 and (B.10) to get the the rough estimate

$$\mathcal{H}_B(4\rho_\mu) \geq -C \rho_\mu^2 a \int \chi_B^2 - C \rho_\mu a$$

we obtain

$$\int \mathcal{H}_B(4\rho_\mu) \geq -C \rho_\mu^2 a (\rho_\mu a^3)^{M/3} \ell^3 - C \rho_\mu a d^{-3}.$$  \hspace{1em} (B.47)

In order to apply the estimate in Theorem B.5 over the remaining $u$ we need to control

$$\int (R \lambda_1(u)^{-1})^2 \int \chi_B^2(x)(x) dx \, du \leq C R^2(\ell d)^{-2} \int (\lambda_1(u)/(\ell d))^{M-2} \int \chi_B(u)(x) dx \, du$$

$$\leq C(\lambda_1/(\ell d)^2) \ell^3 \leq C K_B^3 \ell^3,$$  \hspace{1em} (B.48)

where we have used \([C.5]\) i.e., $\|\chi_B\|_\infty \leq C(\lambda_1/(\ell d))^M$ and $\int \chi_B(u)(x) dx \, du = C \ell^3$. If we combine this with \([B.47]\) (with $M = 8$), \([B.18], [B.11], [B.3]\), and the estimate in Theorem B.5 we arrive at the final a priori lower bound

$$\langle \Psi, \mathcal{H}_\Lambda(\rho_\mu) \Psi \rangle \geq R_\Psi + \langle \Psi, \frac{b}{2\ell^2} n_+ \Psi \rangle - 4 \pi \rho_\mu^2 a^3 + C \rho_\mu a^3 \prec 1/2 \ell^3 - C \rho_\mu a d^{-3}$$

$$\geq R_\Psi + \langle \Psi, \frac{b}{2\ell^2} n_+ \Psi \rangle - 4 \pi \rho_\mu^2 a^3 \left( 1 + C K_B^3 (\rho_\mu a^3)^{1/2} \right),$$

where

$$0 \leq R_\Psi = \langle \Psi, \left( \int \mathcal{I} F \left( \frac{n(u)}{|B(u)|} \right) \int \int w_{1,B(u)}(x,y) dx \, dy \, du \right) \Psi \rangle$$
Localization to small boxes

with \( F(t) = \frac{(t-\rho)^2}{1+4t^2} \) and

\[
\mathcal{L} = \{ u \in \mathbb{R}^3 \mid \| \ell du \|_\infty \leq \frac{1}{2} \ell (1 + d) - 2\rho^{-1/3} \}.
\]

Since \( \mathcal{R}_\Psi \) and \( n_+ \) are non-negative this immediately gives \((B.45)\) and

\[
R_{\Psi} \leq C\rho^3a\ell^3K_B^3(\rho^{-}a^3)^{1/2}, \quad \text{and} \quad \langle \Psi, n_+ \Psi \rangle \leq C\rho^3\ell^3K_B^2(\rho^{-}a^3)^{1/2}
\]

for a normalized \( n \)-particle \( \Psi \) satisfying \((6.1)\). It remains to establish the a priori bound on \( n \) in \((6.2)\).

Using that the function \( F \) is convex and denoting

\[
C = \int_{\mathcal{L}} \int \int w_{1,B(u)}(x,y)dxdydu
\]

we obtain

\[
R_{\Psi} \geq CF \left( C^{-1} \left\langle \Psi, \left( \int_{\mathcal{L}} \frac{n(u)}{|B(u)|} \int \int w_{1,B(u)}(x,y)dxdydu \right) \Psi \right\rangle \right) \quad (B.50)
\]

We have by \((B.11)\) that

\[
8\pi a\ell^3(1 - C(\rho^3a^3)^{M/3}) \leq C \leq 8\pi a\ell^3,
\]

where we used \((B.10)\) and as in \((B.46)\) that \(|\chi_{B(u)}(x)| \leq C(\rho^{1/3}\ell)^M \leq C(\rho^3a^3)^{M/6}\) for \( u \) outside \( \mathcal{L}_- \).

We may write

\[
C^{-1} \int_{\mathcal{L}} \frac{n(u)}{|B(u)|} \int \int w_{1,B(u)}(x,y)dxdydu = \sum_{i=1}^{N} U(x_i)
\]

where

\[
U(z) = C^{-1} \int_{\mathcal{L}} |B(u)|^{-1} \chi_{B(u)}(z) \int \int w_{1,B(u)}(x,y)dxdydu.
\]

Using the form of \( F \) and the a priori bound on \( \mathcal{R}_\Psi \) in \((B.49)\) we see that

\[
\left| \left\langle \Psi, \sum_{i} U(x_i) \Psi \right\rangle - \rho_{\mu} \right| \leq C\rho \mu K_B^{3/2}(\rho^3a^3)^{1/4}
\]

\[
(B.51)
\]

Note that by \((B.10)\) and \( \int \chi_{B(u)}du = \chi_{\Lambda} \) we have that \( U(z) \leq C\ell^{-3} \) and that

\[
P_\Lambda U P_\Lambda = P_\Lambda |\Lambda|^{-1} \int_{\Lambda} U(z)dz = P_\Lambda \ell^{-3}.
\]

Using that for all \( \varepsilon > 0 \)

\[
(1 - \varepsilon) \sum_{i=1}^{N} (P_\Lambda U P_\Lambda)_i - \varepsilon^{-1} \sum_{i}^{N} (Q_\Lambda U Q_\Lambda)_i \leq \sum_{i}^{N} U(x_i) \leq (1 + \varepsilon) \sum_{i=1}^{N} (P_\Lambda U P_\Lambda)_i + (1 + \varepsilon^{-1}) \sum_{i=1}^{N} (Q_\Lambda U Q_\Lambda)_i.
\]

we see that

\[
(1 - \varepsilon)n_0\ell^{-3} - C\varepsilon^{-1}n_+\ell^{-3} \leq \sum_{i=1}^{N} U(x_i) \leq (1 + \varepsilon)n_0\ell^{-3} + (1 + \varepsilon^{-1})Cn_+\ell^{-3}.
\]

Choosing \( \varepsilon = K_B^{3/2}K_{\ell}(\rho^3a^3)^{1/4} \) and using the a priori bounds on the expectation values of \( n_+ \) in \((B.49)\) and \( U \) in \((B.51)\) we conclude the result in the theorem. \( \square \)
The explicit localization function

In this section we discuss the explicit choice of the localization function $\chi$ and its properties. Define

$$\zeta(y) = \begin{cases} \cos(\pi y), & |y| \leq 1/2, \\ 0, & |y| > 1/2, \end{cases}$$

and

$$\chi(x) = C_M (\zeta(x_1) \zeta(x_2) \zeta(x_3))^M. \quad (C.1)$$

Here $M \in \mathbb{N}$ is to be chosen large enough and we explained the need to choose $M = 30$ in Section 4. The constant $C_M$ is chosen such that the normalization $\int \chi^2 = 1$ from (5.1) holds. We have $0 \leq \chi \in C^{M-1}(\mathbb{R}^3)$.

Lemma C.1. Let $\chi$ be the localization function from (C.1). Let $\widetilde{M} = \max\{n \in \mathbb{Z} | 2n \leq M\}$. Then, for all $k \in \mathbb{R}^3$,

$$|\hat{\chi}(k)| \leq C_{\chi}(1 + |k|^2)^{-\widetilde{M}}, \quad (C.2)$$

where

$$C_{\chi} = \int |(1 - \Delta)^{\widetilde{M}} \chi| \quad (C.3)$$

In particular, when $|k| \geq \frac{1}{2} K^{-1}_H (\rho \mu a^3)^{\frac{2}{5}} a^{-1}$, with the notation from (4.14), we have

$$|\hat{\chi}(k)| = |\hat{\chi}(k\ell)| \leq C\ell^2 (K^{-2}_H \widetilde{K}_H^{-2} (\rho \mu a^3)^{\frac{2}{5}})^M. \quad (C.4)$$

The proof of Lemma C.1 is elementary and will be omitted.

The explicit choice of $\chi$ is important when we analyze the behavior of the small box localization function. Recall that according to (B.2) and the explicit choice of $\chi$ we may write $\chi_B(x) = C_M^2 F(x)^M$ where $F(x) = h_{u_1}(x_1) h_{u_2}(x_2) h_{u_3}(x_3)$ and

$$h_v(t) = \zeta \left( \frac{t}{\ell} \right) \zeta \left( \frac{t}{\ell d} - v \right).$$

If we denote by $\lambda_1$ the shortest side length in the box $B$ we see by estimating one of the $\zeta$ factors of scale $d\ell$ and using that it must vanish at one of the sides that

$$\chi_B(x) \leq CC_M^2 (\lambda_1/(d\ell))^M. \quad (C.5)$$

If the shortest side length $\lambda_1$ of the box $B$ satisfies that $\lambda_1 < d\ell$ we can improve this slightly to

$$\chi_B(x) \leq CC_M^2 (\lambda_1/\ell)^M. \quad (C.6)$$

This follows by estimating a $\zeta$ factor of scale $\ell$ and using that it vanishes at one of the sides.

In this rest of this short appendix we will briefly sketch how to get the estimate (B.37) on $\chi_B$. Recall that according to (B.2) and the explicit choice of $\chi$ we may write $\chi_B(x) = C_M^2 F(x)^M$ where $F(x) = h_{u_1}(x_1) h_{u_2}(x_2) h_{u_3}(x_3)$ and

$$h_v(t) = \zeta \left( \frac{t}{\ell} \right) \zeta \left( \frac{t}{\ell d} - v \right).$$
Our first claim is that
\[ \| \chi_B \|_\infty \leq C'_M |B|^{-1} \int \chi_B \]
It is enough to show this for the function \( h_v(t)^M \). Since \( \zeta \) is concave on its support we have that if \( h_v \) is supported on \([a, b]\) and takes its maximum in \( c \) then
\[ h_v(t) \geq \| h_v \|_\infty \min \left\{ \frac{(t-a)^2}{(c-a)^2}, \frac{(t-b)^2}{(c-b)^2} \right\} . \]
In particular, \( h_v \) is bigger than \( \frac{1}{4} \| h_v \|_\infty \) on half the interval. The claim follows from this.

Our second claim is that
\[ \max_i \| \partial_i \chi_B \|_\infty \leq C'_M \lambda_1^{-1} \| \chi_B \|_\infty, \quad \max_{i,j} \| \partial_i \partial_j \chi_B \|_\infty \leq C'_M \lambda_1^{-2} \| \chi_B \|_\infty. \]
It is easy to see that it is enough to show these properties for \( h_v \), i.e., that
\[ \| h_v' \|_\infty \leq C'(b-a)^{-1} \| h_v \|_\infty, \quad \| h_v'' \|_\infty \leq C'(b-a)^{-2} \| h_v \|_\infty. \]
In the case when \((b-a) < \ell d\) we have that one factor in \( h_v \) vanishes at one end point and the other factor vanishes at the other endpoint. It is then easy to see that \( \| h_v' \|_\infty \leq C(b-a)/(d \ell^2) \), \( \| h_v'' \|_\infty \leq C(\ell^{-2} + (\ell d)^{-2}) \| h_v \|_\infty + C(\ell^2 d)^{-1} \), and \( \| h_v \|_\infty \geq c(b-a)^2(d \ell^2)^{-1} \). In case \( b-a = \ell d \), both endpoints occur when the second \( \zeta \) factor in \( h_v \) vanish. Without loss of generality we may consider \( v > 0 \) and let \( D = |\ell(1/2 - dv)| \) denote the distance from the middle of the support of \( h_v \), i.e., \( \ell dv \) to the right endpoint of the support of the first \( \zeta \) factor, i.e., \( \ell/2 \). Then \( \ell d/2 \leq D \leq \ell \) and
\[ \| h_v' \| \leq C'(\ell d)^{-1} D, \quad \| h_v'' \| \leq C'(\ell^{-2} + (\ell d)^{-2}) \| h_v \|_\infty + C'(\ell^2 d)^{-1}, \quad \| h_v \|_\infty \geq cD/\ell. \]

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