Chain inflation revisited

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Abstract. This paper represents an in-depth treatment of the chain inflation scenario. We fully determine the evolution of the universe in the model, the conditions necessary in order to have a successful inflationary period, and the matching with the observational results regarding the cosmological perturbations. We study in great detail, and in general, the dynamics of the background, as well as the mechanism of generation of the perturbations. We also find an explicit formula for the spectrum of adiabatic perturbations. Our results prove that chain inflation is a viable model for solving the horizon, entropy and flatness problems of standard cosmology and for generating the right amount of adiabatic cosmological perturbations. The results are radically different from those found in previous works on the subject. Finally, we argue that there is a natural way to embed chain inflation into flux compactified string theory. We discuss the details of the implementation and how to fit observations.

Keywords: cosmological perturbation theory, cosmological phase transitions, string theory and cosmology, inflation

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1. Introduction

String theory is our best developed theory of quantum gravity and should describe the evolution and birth of our universe. In particular, it should provide a mechanism for inflation. Unfortunately, all models constructed so far suffer from various deficiencies. A significant difficulty is that potentials appropriate for slow roll seem to be uncommon in string theory. On the other hand, the theory is in many cases poorly known, and therefore so are the corrections to the potentials, which can be important for large values of the fields.

Most successful models of inflation are based on second-order phase transitions with a field that rolls slowly through field configuration space. In the simplest models the field travels long distances of the order of the Planck mass, which makes a low energy approximation questionable. There is also the issue of fine-tuning of the models if one wants to obtain enough inflation.

In this paper we will explore the possibility of using a series of first-order phase transitions to generate inflation. While it is well known that a single first-order phase transition, as in the case of old inflation, cannot do the job, the situation could be dramatically different with a large number of transitions. The idea is called chain
The basic idea is that inflation need not proceed in one step. Instead there is a long series of metastable minima where the inflaton field or fields spend just a short time allowing for a fraction of an e-folding. The field then tunnels to the next metastable minima in the series. Only after adding up several such steps will there be a sufficient number of e-foldings.

It is widely believed that string theory has a rich structure of many metastable vacua separated by potential barriers related to domain walls (D- and M-branes). Such a setup favors first-order transitions, and should provide suitable conditions for chain inflation. The basic idea is that the early universe searched its way through the landscape experiencing a large number of tunneling events before settling down in the present state. During this process the universe should have gone through periods of inflation.

Our motivation for studying chain inflation is twofold. On the one hand, this scenario has not been studied in sufficient detail to make it robust. Moreover, results in previous works are not fully consistent with a complete treatment of the subject [1]–[7]. On the other hand, our interest is directly related to a previous study of the topography of the string landscape. In [8,9] it was discovered that series of simply related vacua are a generic feature of the string landscape in the context of flux compactifications (for a review see for example [10,11] and references therein). Contrary to previous studies of chain inflation, this provides us with a reasonably constrained framework for implementing chain inflation. Furthermore, in this case we are able to propose an inflationary model where the agents are complex structure moduli, in contrast to the models so far described in the literature, which involve Kähler moduli or open string moduli.

The paper is divided into four main sections. In section 2 we discuss the basic problems of inflation with first-order phase transitions, and discuss the various constraints that need to be met for successfully solving the horizon, entropy and flatness problems and achieving percolation and large scale thermalization. In section 3 we discuss the generation of cosmological perturbations in chain inflation, proposing a specific set of equations for determining them and computing the power spectrum.

In the first sections our discussion is at a general level, while a detailed model based on string theory is specified in section 4. There, we review the construction of a series of minima using flux compactification and apply it to chain inflation. We compare our results to the experimental data in section 5. We then end with some conclusions, speculations and outlook.

2. Successful chain inflation

The key point of chain inflation is that the slow rolling of the inflaton is substituted by a series of subsequent tunneling processes. Chain inflation must still fulfill the same demands as ordinary inflation with a slow evolution of the energy density of the vacuum allowing for an accelerated expansion of the universe. At the same time, in order to obtain a homogeneous and isotropic universe, we need the tunneling events to proceed in such a way as to guarantee percolation and homogenization at every step of the ladder of transitions.

While these requirements are enough\(^1\) to solve the horizon, entropy and flatness problems of standard cosmology (we can call them background requirements), the model

\(^1\) A viable model for inflation must also provide a suitable mechanism for reheating, but we will not address this question in this paper.
must also provide the right amount of density perturbations and the correct features of the spectrum.

We will deal with this last aspect later in the paper (section 3); for the moment let us proceed by establishing the background requirements for the tunneling processes.

2.1. Basic setup

Consider a theory with a potential with many metastable minima at different energies\(^2\). The number of fields could in principle be very large, as it is supposed to be for string moduli, and their dynamics as they pass through the various minima could be complex, with rolling, tunneling, and jumping phases. In the following we will simplify the description by modeling the dynamics purely through the stress energy tensors of the fields and the gravity field.

The phase transitions among the various vacua proceed by nucleation of bubbles. The bubbles are filled by the new vacuum at a lower energy, and all the latent heat of the transition is stored within the walls of the bubbles. When bubbles collide the energy of the walls is converted in radiation (massless string states). The process of bubble collision and the transfer of the energy to radiation are complex phenomena, that can be studied numerically. We are not going to do so, and will instead stick to a simplified model based on a number of assumptions:

- we consider values for the decay rate per unit physical volume \(\Gamma_n\) between vacua \(n\) and \(n-1\) such that percolation and large scale thermalization are achieved at every phase transition (we will investigate which values allow this in the following);
- decays between two distant vacua are strongly suppressed;
- the bubbles are produced with negligible volume and expand at the speed of light;
- the energy of the walls is optimally transferred to radiation in the collision and the thermalization time is very short.

In this way, we can assume that a homogeneous and isotropic space–time is obtained in a very short time in every patch and phase, and we can therefore describe the metric using the Friedmann–Robertson formula

\[
ds^2 = -dt^2 + a(t)^2 \, d\mathbf{x}^2.\tag{1}
\]

Schematically we have the following picture. As time passes the universe is divided into regions of different phases (new vacua) created through bubble nucleation and rapid percolation. The collisions among bubble walls produce an amount of radiation which remains trapped within every phase. Our actual universe is dominated by one of these phases. From our point of view, therefore, we observe the universe as filled with the radiation produced from the collisions of the walls of the bubbles of our phase and small patches, distributed in an homogeneous way, composed of the previous phases (which are contained within one another, together with the radiation produced at every step).

It is convenient to write the equations involving the energy tensors of the vacuum and the radiation by formally distinguishing every phase with an index \(n\) and associating

\(^2\) If we are allowed to consider the string landscape as governed by a very complicated effective action with a potential depending on many fields, this description applies to that case as well.
with each one a vacuum and a radiation stress–energy tensor, distinguished by $\ell = V, r$:

$$T_{\mu\nu}^{\ell,m} = (\rho_{\ell,m} + p_{\ell,m})u_\mu^{\ell,m}u_\nu^{\ell,m} + p_{\ell,m}g_{\mu\nu},$$

(2)

where $u_\mu^{\ell,m}, \rho_{\ell,m}, p_{\ell,m}$, for $\ell = \{r, V\}$, are the proper 4-velocity, energy and pressure densities. We write the energy density, pressure and stress–energy tensors for radiation and vacuum as

$$\rho^{\ell} \equiv \sum_m \rho_{\ell,m}^{\ell,m}, \quad P^{\ell} \equiv \sum_m P_{\ell,m}^{\ell,m}, \quad T_{\mu\nu}^{\ell} \equiv \sum_m T_{\mu\nu}^{\ell,m},$$

(3)

and the total energy density, pressure and stress–energy tensor as

$$\rho \equiv \sum_{\ell} \rho^{\ell}, \quad P \equiv \sum_{\ell} P^{\ell}, \quad T_{\mu\nu} \equiv \sum_{\ell} T_{\mu\nu}^{\ell}.$$

(4)

We furthermore write

$$\rho^{V} \equiv \rho^{V} + \rho^{W},$$

(5)

expressing that the ‘vacuum energy’ is actually the sum of the energy density in the interior of the bubbles, $\rho^{V}$, plus the energy stored in the bubbles walls, $\rho^{W}$.

The energy and pressure densities of our generalized fluids depend on space and time, but, due to the hypothesis of homogeneity and isotropy within every phase, their average over the volume occupied by the phases depends only on time:

$$\langle \rho^{V,r}(\mathbf{x},t) \rangle = \rho^{V,r}(t).$$

(6)

Written in terms of $\rho^{V}$ and $\rho^{r}$, the Friedmann and Chaudhuri equations are

$$H^2 = \frac{8\pi G}{3}(\rho^{V} + \rho^{r}),$$

(7)

$$\dot{H} = -4\pi G \sum_{\ell=V,r} \left( \rho^{\ell} + P^{\ell} \right).$$

(8)

### 2.2. A slow roll through the landscape

In order to have a successful chain inflation mechanism, the system must undergo a series of successive tunnelings such that the decay of the vacuum energy is effectively slowed down. In this way inflation can effectively cure the problems related to entropy, flatness and horizon size, by lasting for a sufficient number of e-foldings ($N_e \sim 60$).

To avoid a too rapid decay of the vacuum energy it is necessary that each tunneling step occurs only after a suitable and sufficient fraction of the universe volume has completed the preceding tunneling event. We will show how this condition arises through multiple tunnelings, and thereby establish that chain inflation is a viable model within string theory or other gravity theories with multiple metastable vacua. Although our results are in agreement with the chain inflation model expectations, our formulas for the

3 As is well known, the hydrodynamical fluid description does not apply for scalar fields and dark energy, and therefore we cannot use it for modeling the dynamics of our vacuum fields here. Nevertheless, it is possible and well defined to speak of energy density and radiation with reference to the components of the energy tensor; that is why we speak of generalized fluids.
energy density of the vacua, their lifetime and persistence are in fact very different from those usually found in the literature (see for instance [1]–[7]). In many cases the results have not been derived, but instead were inferred from calculations based on old inflation, not acknowledging that in that case only one tunneling step is considered.

To proceed we write the various contributions to the energy density as

\[ \rho_V(t) = \sum_{m=0}^{\infty} \epsilon_m p_m(t), \quad (9) \]

\[ \rho_W(t) = \sum_{m=1}^{\infty} \Delta \epsilon_m \sum_{n=0}^{m-1} p_n(t) F_{m,m-1}(t), \quad (10) \]

\[ \dot{\rho}_r(t) = -4H \rho_r + \sum_{m=1}^{\infty} \Delta \epsilon_m \partial_t \left( 1 - \sum_{n=0}^{m-1} p_n(t) F_{m,m-1}(t) \right), \quad (11) \]

where \( \epsilon_m \) is the value of the potential energy in the vacuum \( m \), \( \Delta \epsilon_m \equiv (\epsilon_m - \epsilon_{m-1}) \), \( p_m(t) \) is the fraction of volume still occupied by the vacuum \( m \) at time \( t \) and, finally, \( F_{m,m-1}(t) \) is the energy weighted fraction of uncollided walls at time \( t \). \( m = 0 \) corresponds to the vacuum that we live in at the present time\(^4\). The energy density of the \( m \)th vacuum phase is therefore given by

\[ \rho_{V,m} = \epsilon_m p_m + \Delta \epsilon_{m+1} \sum_{n=0}^{m-1} p_n(t) F_{m+1,m}(t). \quad (12) \]

In chain inflation the \( p_n(t) \) are obtained from a coupled set of differential equations:

\[ \dot{p}_N = -\tilde{\Gamma}_N p_N \]

\[ \dot{p}_{N-1} = -\tilde{\Gamma}_{N-1} p_{N-1} + \tilde{\Gamma}_N p_N \]

\[ \ldots \]

\[ \dot{p}_0 = \tilde{\Gamma}_1 p_1, \quad (13) \]

with

\[ \sum_m p_m(t) = 1. \]

We have

\[ \tilde{\Gamma}_m(t) = \Gamma_m V^{\text{physical}}(t_0, t), \quad (14) \]

where \( \Gamma_m \) is the decay rate per unit time and (physical) volume. As we will see later, the amount of radiation is negligible and therefore the decay rate will essentially be given by the Coleman–Callan tunneling process, and not by thermal nucleation of bubbles.

The uncollided fraction of walls of bubbles of the phase \( m-1 \) is given by (see also \([12]\))

\[ F_{m,m-1}(t) = \frac{\int_0^t d\tau' \Gamma_m V^{\text{physical}}(t, \tau') p_m(\tau') p_{m,\text{un}}(t, \tau')} {\int_0^t d\tau' \Gamma_m V^{\text{physical}}(t, \tau') p_m(\tau')}, \quad (15) \]

\(^4\) Note that in general \( \epsilon_0 \neq 0 \).
where \( p_{m,\text{un}}(t, t') \) is the fraction of a (reference) wall generated at time \( t' \) which is still not collided at time \( t \).

This fraction represents the probability that a point of the (reference) wall has not collided at time \( t \), and is equal to the probability that a point just outside the wall is still in the phase \( m \), so

\[
p_{m,\text{un}}(t, t') = e^{-\int_0^t \tilde{\Gamma}_m dt'}.
\]

(16)

Note that \( p_{m,\text{un}}(t', t') = 1 \), which simply means that at the time of nucleation the wall is completely uncollided.

The solution of the system (13) differs depending on whether the nucleation rates per time \( \tilde{\Gamma}_m \) are

(a) equal for every \( m \) (that is \( \tilde{\Gamma}_m = \tilde{\Gamma} \)),

(b) different for every \( m \).

We will show how the correct requirement for having a successful inflationary periods can arise, so that chain inflation is a viable model within string theory or other gravity theories with multiple metastable vacua. Scenario (b) will be dealt with in detail in section 2.4, and we concentrate on case (a) in the following section 2.3.

### 2.2.1. Percolation and large scale thermalization

We must now discuss for which values of the parameters in our model (essentially the decay rates) we are able to achieve percolation (and large scale thermalization) at every phase transition. There are three time scales that can be used to study the final stage of an inflationary expansion [12, 13].

- **\( t_{m}^{p} \)**: the time when the volume of the patch starts to contract, that is the time when

\[
V_{\text{phys},m}^{-1} \frac{dV_{\text{phys},m}}{dt} < 0.
\]

(17)

This can be found in the following way [13]. The bubble nucleation rate per unit time and coordinate volume is

\[
\frac{d\bar{\Gamma}}{dt \, d^3x} = a^3(t) \Gamma_{m+1} \theta(t - t_B),
\]

(20)
where \( t_B \) represents the time when the nucleation of bubbles starts. Since the bubbles expand at the speed of light (in our model) we can relate their radius to the time using the formula \( dr = a(t)^{-1} dt \). We choose a reference radius \( r_1 \), corresponding to a time \( t_{r_1} \). Bubbles with radius \( r > r_1 \) would nucleate at times \( t < t_{r_1} \). Therefore the nucleation rate per unit coordinate volume for a sphere having \( r > r_1 \) is given by

\[
\frac{d\Upsilon}{d^3x} \bigg|_{r>r_1} = \Gamma_{m+1} \int_{t_B}^{t_{r_1}} a^3(t) \, dt. \tag{21}
\]

Successful percolation (and, as can be shown, also large scale thermalization; see [13]) at time \( t_m \), for \( r_1 \) small, can then be achieved if

\[
\frac{\text{total volume of bubbles with } r > r_1}{\text{total volume}} \bigg|_{t_m} > \frac{4\pi}{3} r_1^3 \frac{d\Upsilon(t_m)}{d^3x} \bigg|_{r>r_1} \gtrsim 0.34, \tag{22}
\]

where \( (4\pi/3)r_1^3 \) is the volume of bubbles of radius \( r_1 \).

- \( t_{m_{\text{sf}}} \): the time by which the patch of phase \( m \) has become a small fraction of the total volume, which for example can be chosen to correspond to the moment when \( p_m(t_{m_{\text{sf}}}) = 0.01 \).

We will analyze these requirements in more detail in the following sections, when we will have the explicit form of \( p_m(t) \).

### 2.3. Equal tunneling rates at each of the steps of the ladder

We will now assume that the tunneling rates are constant and equal at every step of the ladder of decay events. That is

\[
\tilde{\Gamma}_m(t) \sim \tilde{\Gamma}. \tag{23}
\]

In this case the solution to (13) is easily seen to be

\[
p_{N-k}(t) = \frac{\left( \tilde{\Gamma}t \right)^k}{k!} e^{-\tilde{\Gamma}t}. \tag{24}
\]

Furthermore, the uncollided fraction of walls of bubbles of the phase \( N - k - 1 \) is given by

\[
\mathcal{F}_{N-k,N-k-1}(t) = \frac{p_{N-k-1}(t) k!}{k! - \Gamma(k+1,t)}, \tag{25}
\]

where

\[
\Gamma(k+1,t) = k! e^{-y} \sum_{s=0}^{k} \frac{y^s}{s!} \tag{26}
\]

is the incomplete Gamma function.

It is now possible to determine for which values of the decay rate \( \tilde{\Gamma} \) successful percolation and large scale thermalization are achieved. Formula (18) for \( m = N - k \)
now reads
\[ t^N_c - k \geq \frac{k}{\Gamma - 3H}. \tag{27} \]
and if we take $H$ to be constant during inflation\textsuperscript{5}, we see from formula (22) that
\[ \frac{\tilde{\Gamma}}{3H}(1 - e^{3H(t^m_r - t^m_p)}) > 0.34 \Rightarrow t^m_p > -\frac{\log(1 - 0.34(3H/\tilde{\Gamma}))}{3H}, \tag{28} \]
where we have used (14) to rewrite the formula in terms of $\tilde{\Gamma}$ and we have sent $t_{r_1} \to 0$. We can see that if $\tilde{\Gamma}$ is sufficiently larger than $3H$, then the contraction of volume, percolation and large scale thermalization are achieved in very short times.

We can now calculate the vacuum energy, which is given by (see (12))
\[ \rho^V(t) = \rho^V(t) + \rho^W(t) \]
\[ = e^{-y} \sum_{k=1}^{N} \epsilon_{N-k}^N \frac{y^k}{k!} \left( 1 + \frac{\Delta \epsilon_{N-k+1} \Gamma(N+1,y)(k-1)! - \Gamma(k,y)N!}{N!(k-1)! - \Gamma(k,y)} \right) + \epsilon_N e^{-y}, \tag{29} \]
where we have used the variable $y = \tilde{\Gamma} t \equiv \frac{t}{\tau}$.

Apart from the last steps of tunneling, it is possible to see that we have
\[ \frac{\rho^V_{N-k}}{\epsilon_{N-k}} \approx \frac{\rho^W_{N-k}}{\Delta \epsilon_{N-k+1}}. \tag{30} \]
Note, then, that the transfer of energy from the vacuum to radiation is quite low during the main part of the tunneling chain. We can then write
\[ \rho^N(t) \sim e^{-y} \sum_{k=1}^{N} \epsilon_{N-k+1} \frac{y^k}{k!} + \epsilon_N e^{-y}. \tag{31} \]

Motivated by the specific model that we will describe in section 4, we consider two possible functional forms for the energy density at every stage:
\[ \epsilon_n = \begin{cases} m_4^4 n & \text{case I} \\ \frac{m_4^4}{2} n^2 & \text{case II}, \end{cases} \tag{32} \]
where $m_4$ is a suitable energy scale. In this way we can easily compute the vacuum energy, with the result
\[ \rho^V(t) \sim \begin{cases} \frac{m_4^4}{N!} \left( \left( N + 1 - \frac{t}{\tau} \right) \Gamma \left( N + 1, \frac{t}{\tau} \right) + \left( \frac{t}{\tau} \right)^{N+1} e^{-t/\tau} \right) & \text{case I} \\ \frac{m_4^4}{2N!} \left( \left( \left( N + 1 - \frac{t}{\tau} \right)^2 + \frac{t}{\tau} \right) \Gamma \left( N + 1, \frac{t}{\tau} \right) \right. \\ \left. + \left( N - \frac{t}{\tau} + 1 \right) \left( \frac{t}{\tau} \right)^{N+1} e^{-t/\tau} \right) & \text{case II}. \end{cases} \tag{33} \]

\textsuperscript{5} The accuracy of this approximation can be verified by solving the Friedmann and Chaudhuri equations using the formula for the energy density of the vacuum that we will provide in (34).
If $N$ is large, the incomplete gamma function is to a very high degree of approximation constant, $\Gamma(N + 1, t/\tau) \sim N!$, for values of $t$ almost all the way up to $N\tau$. We also have that $(t/\tau)^N e^{-t/\tau} \ll \Gamma(N + 1, t/\tau)$. As a consequence we find, to a high degree of accuracy,

$$\rho^V(t) \sim \begin{cases} m_i^4 \left( N + 1 - \frac{t}{\tau} \right) & \text{case I} \\ \frac{m_i^4}{2} \left( N + 1 - \frac{t}{\tau} \right)^2 & \text{case II} \end{cases}$$

which does indeed lead to the expected slow roll.

It is convenient to define an effective slow roll parameter

$$\varepsilon = -\frac{\dot{H}}{H^2}. \quad (35)$$

Using the excellent approximation

$$n = N + 1 - \frac{t}{\tau}, \quad (36)$$

it can be written as

$$\varepsilon \equiv -\frac{\dot{H}}{H^2} = \begin{cases} \frac{1}{2nH\tau} & \text{case I} \\ \frac{1}{nH\tau} & \text{case II} \end{cases}$$

As regards the number of e-foldings ($N_e$), a straightforward computation gives

$$N_e = \int H \, dt \sim \begin{cases} \frac{1}{3\varepsilon} \frac{n_i^{3/2} - n_f^{3/2}}{n_i^{3/2}} & \text{case I} \\ \frac{1}{2\varepsilon} \frac{n_i^2 - n_f^2}{n_i^2} & \text{case II} \end{cases} \quad (38)$$

where we have used $H^2 \sim (8\pi G/3)\rho^V$ and formula (34) for the time dependence of the energy density. The subscript i/f indicates quantities at the beginning/end of the inflationary period (in particular $n_i = N$ using our previous notation). It is easy to accommodate the desired value for $N_e$ by choosing $n_f$ not too close to $n_i$ (but not necessarily as low as 1 or 0) and an appropriate value for $\varepsilon$. The specific details regarding the end of the inflationary period (final metastable vacuum, reheating, etc) will, as we have already stressed, not be treated in this work.

### 2.4. Different tunneling rates at each of the steps of the ladder

In the case when $\tilde{\Gamma}_i \neq \tilde{\Gamma}_j$ in (13), the solution of the coupled set of differential equations is given by

$$p_{N-k} = c_0^{N-k} e^{-\tilde{\Gamma}_N t} \left( 1 + \sum_{i=N-k+1}^{N} c_i^{N-k} e^{-\tilde{\Gamma}_i-N-k} \right), \quad (39)$$
where the coefficients $c_{i}^{N-k}$ are given by the recursive equations

$$
c^{N-k+1}_0 = c^{N-k}_0 N^{N-k+1} \frac{\Gamma_{N-k} - \Gamma_{N-k+1}}{\Gamma_{N-k+1}}$$

$$
c^{N-k+1}_i = c^{N-k}_i N^{N-k+1} \frac{\Gamma_{N-k} - \Gamma_{N-i}}{\Gamma_{N-k+1}}$$

for $N-k+2 \leq i \leq N$ (40)

$$
\sum_{i=N-k+1}^{N} c_i^{N-k} = -1.
$$

As in the previous case we are interested in investigating the conditions for which there is a slow roll regime, and in studying the values for the tunneling rates $\tilde{\Gamma}_m$ leading to percolation and large scale thermalization.

In general it is possible to understand how a slow roll behavior can arise, by observing that every $p_{N-k}$ for $k \neq 0$ is a function peaked (how much depending on the $\tilde{\Gamma}_m$s) at a certain time $t_{\max}$. Therefore the tunneling chain is effectively slowed down at every step, until the fraction of volume of the new phase reaches its maximum. It is however not possible to write a formula for the total vacuum energy such as (33) unless a model is specified, so that the precise dependence of the $\tilde{\Gamma}_m$ on $m$ is known.

Nevertheless it is possible to estimate the slow roll parameter as follows. During slow roll we naturally expect

$$
H^2 \sim \frac{8\pi G}{3} \rho^V
$$

and 6

$$
\rho^V \sim \sum_{m} \epsilon_m p_m.
$$

From (13) we find

$$
\dot{\rho}^V = - \sum_{m} \Delta \epsilon_m \tilde{\Gamma}_m p_m,
$$

and from (32) we see that

$$
\Delta \epsilon_m = \frac{\epsilon_m}{m} \quad \text{case I}
$$

$$
\Delta \epsilon_m \sim \frac{2 \epsilon_m}{m} \quad \text{case II}.
$$

Then, using this and (41), we find for the slow roll parameter

$$
\varepsilon = \frac{1}{2H} \times \left\{ \begin{array}{ll}
\sum_{m} \epsilon_m \tilde{\Gamma}_m p_m m^{-1} & \text{case I} \\
2 \sum_{m} \epsilon_m \tilde{\Gamma}_m p_m m^{-1} & \text{case II}
\end{array} \right.
$$

6 We in general expect this formula because the energy density of uncollided walls is proportional to the energy difference between two consecutive vacua, while the one for the interior of bubbles is proportional to the energy level, which is greater than the difference. See also the discussion in the previous section.
We now define an average \( \langle \tilde{\Gamma}_n \rangle \) as
\[
\langle \tilde{\Gamma}_n \rangle = \frac{\sum_m \epsilon_m \tilde{\Gamma}_m p_m m^{-1}}{\sum_n \epsilon_n p_n}.
\] (46)

Then the slow roll parameter is given by
\[
\varepsilon \equiv -\frac{\dot{H}}{H^2} \approx \begin{cases} 
\frac{1}{2H} \langle \tilde{\Gamma} \rangle & \text{case I} \\
\frac{1}{H\tau} \langle \tilde{\Gamma} \rangle & \text{case II.}
\end{cases}
\] (47)

It is interesting (since it will be needed later on, when we will deal with the spectral index of perturbations) to compute the time derivative of the slow roll parameter. From (45) and using (44),
\[
\dot{\varepsilon} = \begin{cases} 
\frac{\varepsilon}{2} \langle \tilde{\Gamma} \rangle + \frac{1}{2H} \left( \frac{\sum_m \epsilon_m \tilde{\Gamma}_m p_m m^{-1}}{\sum_n \epsilon_n p_n} + \frac{1}{(\sum_n \epsilon_n p_n)^2} \left( \sum_m \epsilon_m \tilde{\Gamma}_m p_m m^{-1} \right)^2 \right) & \text{case I} \\
\varepsilon \langle \tilde{\Gamma} \rangle + \frac{1}{H} \left( \frac{\sum_m \epsilon_m \tilde{\Gamma}_m p_m m^{-1}}{\sum_n \epsilon_n p_n} + 2 \frac{\sum_m \epsilon_m \tilde{\Gamma}_m p_m m^{-1}}{(\sum_n \epsilon_n p_n)^2} \left( \sum_m \epsilon_m \tilde{\Gamma}_m p_m m^{-1} \right)^2 \right) & \text{case II.}
\end{cases}
\] (48)

Let us focus on the two terms in each of the brackets. The second one is simply
\[
\frac{\left( \sum_m \epsilon_m \tilde{\Gamma}_m p_m m^{-1} \right)^2}{(\sum_n \epsilon_n p_n)^2} = \langle \tilde{\Gamma} \rangle^2.
\] (49)

It is easy to see that using (13), the numerator in the first term reads
\[
\sum_m \epsilon_m \tilde{\Gamma}_m p_m m^{-1} = \sum_m \tilde{\Gamma}_{m+1} p_{m+1} \left( \frac{\epsilon_m}{m} \tilde{\Gamma}_m - \frac{\epsilon_{m+1}}{m+1} \tilde{\Gamma}_{m+1} \right) = -\sum_m \tilde{\Gamma}_{m+1} p_{m+1} \left( \Delta \left( \frac{\epsilon_{m+1}}{m+1} \tilde{\Gamma}_{m+1} + \frac{\epsilon_m}{m} \tilde{\Gamma}_m \right) \right),
\] (50)

where we have defined, for any quantity \( f_m \),
\[
\Delta (f_m) \equiv f_m - f_{m-1}.
\] (51)

We find that
\[
\Delta \left( \frac{\epsilon_{m+1}}{m+1} \right) = \begin{cases} 
0 & \text{case I} \\
\frac{m^2}{2} = \frac{\epsilon_{m+1}}{(m+1)^2} & \text{case II.}
\end{cases}
\] (52)

We now define
\[
\sigma^2(\tilde{\Gamma}/n) = \left\langle \left( \frac{\tilde{\Gamma}}{n} \right)^2 \right\rangle - \left\langle \frac{\tilde{\Gamma}}{n} \right\rangle^2.
\] (53)
Then, from (48) and (47), we have

\[
\dot{\epsilon} \sim \begin{cases} 
3H\epsilon^2 - \epsilon \left( \frac{n}{\Delta \Gamma} \right) \left( \frac{\Gamma}{n} \right)^{-1} & \text{case I} \\
2H\epsilon^2 - \epsilon \sigma^2 \frac{\Gamma}{n} \left( \frac{\Gamma}{n} \right)^{-1} - \epsilon \left( \frac{n}{\Delta \Gamma} \right) \left( \frac{\Gamma}{n} \right)^{-1} & \text{case II},
\end{cases}
\] (54)

where all the averaging has been done using the distribution \( \rho_m = \epsilon_m p_m \). We note that in case I, the derivative of the slow roll parameter contains a term which will also be found in the case of equal tunneling rates at every step (since it does not depend on the difference of these decay rates) and a term that is depending on the differences themselves. We call them the "universal" term and the "difference" term.

In case II, instead, we find, beyond the universal and the difference terms, also a contribution deriving from the spread of the distribution \( \rho_m \).

3. Cosmological perturbations in chain inflation

3.1. General considerations

Inflationary models must be able to fulfill a series of requirements such as providing a solution to the flatness, entropy, horizon and monopole problems of the standard big bang. But at the same time, they must also meet experimental constraints coming from measurements of the spectrum of cosmological perturbations.

This last point represents a serious issue for chain inflation, as well as for any inflationary model not depending on a single field. Indeed, the CMB analysis strongly constrains the amount of entropy perturbations and the form of the spectrum (and higher order non-Gaussianities) of the adiabatic density perturbations.

In this section we will derive a set of equations that enable us to discuss the cosmological perturbations and compute their adiabatic spectrum.

3.2. Scalar-type perturbations

3.2.1. The nature of perturbations. Within a chain inflation model, density perturbations are generated and evolved in a complicated way. Contrary to the usual slow roll, chaotic inflation model (or any model not containing first-order phase transitions), different kinds of perturbations are being produced. In a slow roll model the seeds of cosmological perturbations are provided only by the quantum fluctuations that accompany the classical evolutions of the fields, metric and fluids. Their dynamics can be described by a set of differential equations that relate the perturbations of the different entities (metric, fields, fluids).

Within a first-order inflationary scenario, we have instead two kinds of perturbations being generated: on the one hand the quantum fluctuations as in the case of slow roll, chaotic inflation, and on the other hand the statistical fluctuations of bubble nucleation and collision. The presence of these two seeds of cosmological perturbations makes it interesting and important to study and analyze this aspect of the model.
It is clear that we cannot straightaway say which of the two mechanisms of generation is the dominating one. Indeed, the basic quantity that we use to describe the energy density for vacuum and radiation is represented by the fraction of universe occupied by the $m$th vacuum:

$$
\dot{p}_m(t) = -\Gamma_m V^{\text{physical}}(t, t_f)p_m(t) + \Gamma_{m+1} V^{\text{physical}}(t, t_f)p_{m+1}(t)
\equiv -\widetilde{\Gamma}_m(t)p_m(t) + \widetilde{\Gamma}_{m+1}(t)p_{m+1}(t).
$$

(55)

The quantity $\widetilde{\Gamma}_m(t)$ represents the average number of bubbles of the $m$th vacuum generated at a time $t$, irrespective of the bubbles generated at previous times. The actual number of such bubbles is therefore distributed according to a Poisson distribution with mean value $\widetilde{\Gamma}_m(t)$. Therefore, a first kind of fluctuation in $p_m(t)$ is due to the statistical fluctuations in this quantity, being equal to $\sqrt{\widetilde{\Gamma}_m(t)}$. In general, solving for $p_m$, we can determine the perturbation in the energy density due to these statistical fluctuations.

But this is not the only seed of fluctuations in $p_m(t)$. Also, the volume $V^{\text{physical}}(t, t')$ depends on the metric which it is measured by. And therefore the quantum fluctuations in the metric field generate and are sourced by perturbations in the energy density of the vacuum and of the radiation as well, due to the coupling between these two.

Our goal will therefore be to obtain a set of equations for the perturbations (relating those of the metric and the various energy and pressure densities), to solve them and to obtain the spectrum of the fluctuations.

We will use the longitudinal gauge, where the most general metric with scalar perturbations can be written as

$$
\text{d}s^2 = -(1 + 2\phi)\,\text{d}t^2 + 2a\partial_i B\,\text{d}t\,\text{d}x^i + ((1 - 2\psi)\delta_{ij} + D_{ij}E)\,\text{d}x^i\,\text{d}x^j,
$$

(56)

with $B = E = 0$. Note that here $D_{ij} \equiv \partial_i \partial_j - \frac{1}{2} \delta_{ij} \nabla^2$.

Let us now pause for a moment to discuss the general approach to cosmological perturbations. The distinction that is made in cosmology between perturbed and unperturbed universes is a fictitious one. The actual universe has a metric of the form (56), and we compare it with a fictitious universe where the metric is a perfect Robertson–Walker one. In our case, $p_m$ has a well-defined geometrical meaning independent of the metric used to compute $\widetilde{\Gamma}_m(t)$, and has to obey the equation

$$
\dot{p}_m = -\Gamma_m p_m + \Gamma_{m+1} p_{m+1},
$$

(57)

in both the perturbed and the unperturbed cases. It is therefore possible to determine the perturbations in $p_m$ just by writing $\Gamma$ in terms of the perturbed fields and find the quantum part (for example in $V^{\text{physical}}(t, t')$) by adding the relevant statistical perturbation. Similarly, we find the form of $\mathcal{F}^{m+1, m}$ to be determined by the geometry also in the presence of perturbations.

3.2.2. Notation and basic quantities. As before it is convenient to formally distinguish every phase with an index $m$, and associate with each one a vacuum and radiation stress–energy tensor, distinguished by $\ell = V, r$ as in formulas (3). We will also distinguish the quantities

---

We consider only scalar perturbations for the moment.
in the perturbed (real) universe form those in the ‘background’ one by writing the first one with a bar, the second one without. Hence we write

\[ \bar{A} \text{ is the perturbed value} \]
\[ A \text{ is the background}, \]

(58)

with

\[ \bar{A} = A + \delta A + O((\delta A)^2). \]

In this formalism the fundamental equations governing the dynamics of the various generalized fluids are

\[ \nabla_\mu \bar{T}_{\ell,m}^{\mu \nu} = \bar{Q}_{\nu}^{\ell,m}, \]

(59)

with, in order for the total stress–energy tensor to be conserved,

\[ \sum_{\ell,m} \bar{Q}_{\nu}^{\ell,m} = 0 \Rightarrow \sum_{\ell,m} Q_{\nu}^{\ell,m} = \sum_{\ell,m} \delta Q_{\nu}^{\ell,m} = 0. \]

(60)

Let us focus, since it will be especially important in the following, on the vacuum sector. First of all, the energy–momentum tensor is given by

\[ \bar{T}_V^{\mu \nu} = \sum_m (\bar{\rho}_V^{m} + \bar{P}_V^{m}) \bar{u}_V^{\mu} \bar{u}_V^{\nu} + \bar{P}_V^{m} \bar{g}^{\mu \nu}, \]

(61)

where \( \bar{u}_V^\alpha \) is the proper velocity for the vacuum, and therefore we have

\[ \bar{u}_V^{\alpha} \bar{T}_V^{\alpha \beta} = \bar{\rho}_V^{m} \bar{u}_V^\beta. \]

(62)

Second, recall that the formula for the vacuum energy at every phase is given by

\[ \rho_V^{m} = \rho_V^{m} + \rho_W^{m}, \]

(63)

where \( \rho_V^{m} \) is the energy in the interior of the bubbles of the \( m \)th phase and \( \rho_W^{m} \) is the energy stored within the walls between phase \( m+1 \) and phase \( m \). We also know that the dynamical evolution of the energy density is

\[ \ddot{\bar{\rho}}_V^{m} = -\bar{\Gamma}_m \bar{\rho}_V^{m} + \bar{\Gamma}_{m+1} \bar{\rho}_V^{m+1} - \bar{\gamma}_m \bar{\rho}_W^{m}, \]

(64)

where \( \bar{\gamma}_m \) represents the conversion of the energy of the walls into radiation\(^8\).

By comparing with (59), it is then straightforward to find that the covariant form of \( \bar{Q}_V^{\nu} \) is

\[ \bar{Q}_V^{\nu} = -\bar{\Gamma}_m \bar{u}_V^{\alpha} \bar{T}_V^{\alpha \nu} + \bar{\Gamma}_{m+1} \bar{u}_V^{\alpha} \bar{T}_V^{\alpha \nu} + \bar{\gamma}_m \bar{u}_V^{\alpha} \bar{T}_W^{\alpha \nu}, \]

(65)

where in the last line we have used (62). Note also that \( u_V^{\alpha} \) is the same for all components with different \( m \), since it is just the proper velocity of the total vacuum, whose energy–momentum tensor is (61).

\(^8\) At the microscopical level this would be a series of complicated processes of collisions of walls and decays, but we will instead use a well-defined coarse-grained description.
Chain inflation revisited

The perturbations in the sources $Q^{\ell,m}$ can be written as [14]–[17]

\begin{align}
\delta Q^{0,m}_0 &= -Q^{\ell,m}_0 \phi - \delta Q^{V,m}_0, \\
\delta Q^{\ell,m}_\ell &= \partial_\ell \left( Q^{\ell,m}_\ell v + f^{\ell,m}_\ell \right),
\end{align}

where $v$ is the velocity potential, obtained by expanding the spatial components of the average proper velocity $u^i$ in harmonic functions as in [15, 16]. In our case, for $\ell = V, r$, we find

\begin{equation}
v = \sum_{\ell,m} \frac{\rho^{\ell,m} + P^{\ell,m}}{\rho + P} v^{\ell,m} = \sum_m v^{r,m} = v^r.
\end{equation}

For ease of notation, we will also use

\begin{equation}
\delta q \equiv (p + P) v,
\end{equation}

where

\begin{align}
\delta q &\equiv \sum_\ell \delta q^\ell, \\
\delta q^{\ell,m}_\ell &\equiv (\rho^{\ell,m} + P^{\ell,m}) v^{\ell,m}.
\end{align}

By expanding (65) in background plus perturbations, writing the proper velocity in terms of the velocity potential, comparing with (67) and using (68) we find for the vacuum

\begin{equation}
Q^{V,m} v^V = Q^{V,m} v^V + f^{V,m}_V = Q^{V,m} v^r + f^{V,m}_r.
\end{equation}

In order to study entropy (isocurvature) perturbations, it is convenient to define the gauge invariant relative entropy perturbation between fluids $\ell$ and $\ell'$,

\begin{equation}
S_{\ell,\ell'} = -3 \left( \mathcal{R}_\ell - \mathcal{R}_{\ell'} \right).
\end{equation}

In the usual definition, for hydrodynamical matter, $S_{\ell,\ell'}$ is defined in terms of the curvature perturbation on uniform density hypersurfaces. Since we are now studying a system where many scalar fields play a role, the correct definition (see [18]) is in terms of the comoving curvature perturbation\textsuperscript{9}

\begin{equation}
\mathcal{R}_\ell = \phi - H v_\ell.
\end{equation}

The total density perturbation is given by [14]

\begin{equation}
\mathcal{R} = \phi - H v.
\end{equation}

By using these definitions and (68), we find

\begin{equation}
f^V = \sum_m f^{V,m}_V = Q^V (v^V - v^r) = \frac{Q^V}{3H} S_{V,r}.
\end{equation}

\textsuperscript{9} In [18] this is indicated as $\zeta$, which is unfortunately also used for the curvature perturbation on uniform energy density hypersurfaces.
3.2.3. *Equations of motion and spectrum of perturbations.* The most general set of equations for a set of generalized fluids (including scalar fields) for the case of pure Einstein–Hilbert theory has been written in [17] (for generalized theories of gravity, the enlarged set of equations is in [16]). In particular we will use (see also [14])

\[ \psi = \phi, \] (77)

\[ \frac{k^2}{a^2} \phi - 3H(\dot{H} + \dot{\phi}) = 4\pi G \delta \rho, \] (78)

\[ (H\phi + \dot{\phi}) = -4\pi G (\rho + p)v = -4\pi G \delta q, \] (79)

\[ \dot{\delta \rho}^{\ell,m} + 3H(\delta \rho^{\ell,m} + \delta P^{\ell,m}) = 3(\rho^{\ell,m} + P^{\ell,m})\left(\dot{\phi} + \frac{k^{\ell,m}}{a}\right) + Q^{\ell,m} \phi + \delta Q^{\ell,m}, \] (80)

\[ \dot{\delta q}^{\ell,m} + 3H\delta q^{\ell,m} + (\rho^{\ell,m} + P^{\ell,m})\phi + \delta P^{\ell,m} = Q^{\ell,m} \frac{\delta q}{\rho + p} + f^{\ell,m}, \] (81)

\[ \dot{\delta q} + 3H\delta q + (\rho + P)\phi + \delta P = 0. \] (82)

Note that \( \delta \rho^{\ell,m}, \delta P^{\ell,m} \), etc, include the contributions from quantum as well as statistical fluctuations. In order to rewrite this system of equations in a useful way, we need to establish the relation between the total pressure and energy densities. We already know that

\[ \delta P^{\ell,m} = \frac{1}{3} \delta \rho^{\ell,m}, \] (83)

but we also need to find the relation between the corresponding quantities for the vacuum energy. This can be done by inspecting the relevant equation of motion. From (59), we have

\[ \dot{\rho}^{V,m} = Q^{V,m}. \] (84)

Since, according to our discussion above, this relation is valid both for the unperturbed and for the perturbed universes, we immediately obtain

\[ \dot{\delta \rho}^{V,m} = \delta Q^{V,m}. \] (85)

By comparing this result with the general equation (80) for \( \ell = V \), we find

\[ \delta P^{V,m} = -\delta \rho^{V,m} + \frac{Q^{V,m}}{3H} \phi, \] (86)

and therefore

\[ \delta P^V = \sum_m \delta P^{V,m} = -\delta \rho^V + \sum_m \frac{Q^{V,m}}{3H} \phi \]

\[ = -\delta \rho + \delta \rho^\phi + \sum_m \frac{Q^{V,m}}{3H} \phi. \] (87)

If we compare this with the result that we would obtain in a standard single-field slow roll or chaotic scenario, we see the importance of the presence of perturbations of...
the radiation at every step of the tunneling chain. In fact, $\delta \rho_r$ plays here a role similar to that of the perturbation of the scalar field kinetic energy in standard slow roll. It is therefore important to keep track of it, as it will contribute substantially to the dynamical equation of the perturbations.

On the other hand, from the definition of the momentum perturbation in (71), we see that

$$\delta q^{V,m} = 0.$$  

(88)

Therefore, using (81) for $\ell = V$, we find

$$\delta P^{V,m} = \frac{Q^{V,m}}{3H(\rho + P)} \left( \delta \rho + \frac{k^2}{4\pi G a^2} \phi \right) + f^{V,m},$$  

(89)

and so, summing over all $m$s,

$$\delta P^V = \frac{Q^V}{3H(\rho + P)} \left( \delta \rho + \frac{k^2}{4\pi G a^2} \phi \right) + f^V.$$  

(90)

Then, from (87), (90), (76) and neglecting a sub-leading term $\propto \dot{\rho}$, we get

$$\delta \rho^r = - \sum_m Q^{V,m} \phi - \frac{k^2}{4\pi G a^2} \phi + \sum_m \frac{Q^{V,m}}{H} S^{V,r},$$  

(91)

where we have used

$$\frac{4\pi G}{3H} \sum_m Q^{V,m} = 4\pi G \frac{\dot{\rho}^V}{3H} = 4\pi G \frac{d}{dt} \left( \frac{3(2 - \epsilon)}{16\pi G} H^2 \right) \sim \dot{H}.$$  

(92)

As a consequence

$$\delta P = \delta P^V + \delta P^r$$

$$= - \delta \rho - \frac{4}{3} \frac{k^2}{4\pi G a^2} \phi - \frac{1}{3} \sum_m \frac{Q^{V,m}}{3H} \phi + 4 \sum_m \frac{Q^{V,m}}{3H} S^{V,r}.$$  

(93)

Using (92) together with (93), we finally get the system of equations (from (79) and (82))

$$(a\phi)' = -4\pi G a \delta q$$

$$\frac{\delta q}{\rho + p} = - \left( 1 - \frac{1}{3} \frac{k^2}{4\pi G a^2(\rho + P)} - \frac{1}{3} \frac{\dot{H}}{3\pi G(\rho + P)} \right) \phi - \frac{4}{3} \frac{\dot{H}}{4\pi G(\rho + P)} S^{V,r}.$$  

(94)

It is convenient to introduce the new variables $R, \xi$ defined by

$$a \phi = 4\pi G H \xi$$

$$\frac{\delta q}{\rho + p} = - \frac{\dot{R}}{H} + \frac{4\pi G}{a} \xi,$$  

(95)

(96)
Chain inflation revisited

where we easily recognize in $\mathcal{R}$ the *comoving curvature perturbation*\(^{10}\). In terms of these new variables, the system becomes

$$\dot{\xi} = \frac{a(\rho + p)}{H^2} \mathcal{R}$$

$$\mathcal{R} = \frac{1}{3 a^3 (\rho + p)} \left( - k^2 + a^2 H^2 \varepsilon \right) \xi - \frac{4}{3} H S_{V,r}. \quad (97)$$

For the moment we concentrate on the adiabatic perturbations, neglecting the term proportional to $S_{V,r}$. Non-adiabatic isocurvature perturbations (as well as non-Gaussianities\(^{11}\)) are expected to be present in chain inflation models because of the presence of various fields that can follow trajectories in field space different from the inflaton one, but this depends strongly on the specific features of the various models, which our very general picture does not accommodate.

General arguments justifying the negligibility of the isocurvature perturbations are often found in the literature, but they do not apply in our case, or in any other case with coexisting fields and conventional hydrodynamical fluids that interact. Lacking a general argument regarding the amount of the isocurvature perturbations, we would need to rely on specific analysis of the detailed models. This we do not attempt to do, and our conclusions in what follows are valid only in those cases where the detailed model does indeed contain a negligible contribution of isocurvature perturbations.

As a word of comment, note that the difference between our final result and the usual one (for non-interacting fields and fluids) is, apart from the entropy term, proportional to $(Q^{V,m}/3H) \sim \varepsilon$ (see (93)). It is therefore, as one could expect, due to the interactions.

In order to normalize the amplitude of quantum fluctuations, we need the action governing their dynamics. Our setup, being so general, lacks this information. Nevertheless, following [19], we can infer the action from the equations of motion (97) themselves, up to a time independent factor. Following the same steps as in [19], we introduce the canonical quantization variable $\varsigma = z \mathcal{R}$ with

$$z \equiv a(\rho + p)^{1/2} \left( \frac{\dot{O}}{(-k^2 + H^2 \varepsilon)} \right)^{1/2}, \quad (98)$$

where we have used the conformal time defined as $\eta = \int a^{-1} \, dt$. The time independent operator $\dot{O}$ can be determined by comparing our result with well-known ones (for example with the pure radiation case). We find then that $\dot{O} = -k^2$. The action for the variable $\varsigma$ is in our case found to be (up to a total derivative term)

$$S = \frac{1}{2} \int \left[ \varsigma'' + \frac{1}{3} \varsigma (\Delta + H^2 \varepsilon) + \frac{z''}{z} \varsigma^2 \right] \, d\eta \, d^3x, \quad (99)$$

and the equation of motion is

$$\varsigma'' + \left( \frac{1}{3} k^2 - \frac{1}{3} H^2 \varepsilon - \frac{z''}{z} \right) \varsigma = 0, \quad (100)$$

with the obvious long wavelength solution $\varsigma \sim z \rightarrow \mathcal{R} \sim \text{const.}.$

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\(^{10}\) In [19] this is indicated as $\zeta$.

\(^{11}\) See for example [20].
Since \( z''/z \sim a''/a \sim 2(Ha)^2 \) we find\(^\text{12}\) following [19]

\[
P^R_k = \frac{H^2}{8\pi^2 M_{\text{Planck}}^2 \sqrt{\frac{1}{3} \epsilon}}.
\]

(101)

for the spectrum of perturbations, where quantities have to be evaluated at a time given by \( aH = k/\sqrt{3} \).

### 4. A potential for chain inflation from flux compactification

In order for chain inflation to be a serious alternative to chaotic inflation, one needs a framework where the required potential arises in a natural way. As we will see, such a framework is provided by flux compactified string theory.

#### 4.1. Series of minima from type IIB flux compactifications

A particularly interesting compactification of string theory is type IIB with fluxes. It serves as a useful playground for constructing models of the early universe, and exhibits in a rather explicit way the problems and promises of the string landscape.

The potential for a flux compactified type IIB string is given by

\[
V(z, \tau) = e^K g_{i\bar{i}} D_i W D_{\bar{i}} W + e^K g_{\tau\bar{\tau}} D_\tau W D_{\bar{\tau}} W - 3 e^K |W|^2,
\]

(102)

where \( W \) is the superpotential and \( K \) is the Kähler potential. We will focus on the dependence on the complex structure moduli and are therefore mostly interested in the structure of the superpotential. It is obtained from the fluxes as

\[
W = F \cdot \Pi - \tau H \cdot \Pi,
\]

(103)

where the period vectors are defined to be \( \Pi = (\int \Omega_1, \int \Omega_2, \ldots)^T \), with \( \Omega \) as the holomorphic 3-form.

As shown in [8, 9] it is rather easy to find fluxes where the potential has a minimum somewhere in moduli space. Of particular interest are minima where one of the components of the flux vector, say \( F_2 \), is much smaller than the corresponding conjugated component, i.e. \( F_1 \). We can then make use of conifold monodromies to generate new minima. Specifically, we need to move in moduli space around the conifold point where the cycle \( A_1 \) vanishes. When we do that we go through a cut and end up on a new sheet. Furthermore, the cycles transform as \( B_1 \rightarrow B_1 + n A_1 \), where \( n \) is the number of times we go around the conifold. The potential after the transformation can equivalently be evaluated using the old periods but with

\[
F_{(3)} \rightarrow F_{(3)} + n (F_2, 0, 0, \ldots, 0),
\]

(104)

\[
H_{(3)} \rightarrow H_{(3)} + n (H_2, 0, 0, \ldots, 0).
\]

(105)

If \( F_2 < F_1 \) we can be reasonably sure that we will find a new minimum near the old one on the next sheet. In this way we can generate an infinite series of minima if the \( F_1 \rightarrow \infty \) limit of \( F_{(3)} \) corresponds to a potential with a minimum. The minima will be positioned along a spiral staircase turning around the conifold point.

\(^\text{12}\) We use the reduced Planck mass \( M_{\text{Pl}} = (8\pi G)^{-1} \).
In [8] it was discussed in detail how to obtain such series. In [9] it was suggested that one could make use of geometric transitions between different Calabi–Yau manifolds to help generate the series.

4.2. A potential for chain inflation

Instead of working with explicit flux compactifications for specific Calabi–Yau manifolds, we will capture the essential features of the potential with a simple toy model. The main results will depend on just a limited number of parameters that in turn will depend on the complex geometry of the actual compactification.

The value of the potential at the minima can effectively be described using the results of [8] and general properties of the potential used in type IIB flux compactifications. There are two cases that are of particular interest. In [8] it was found that when the axiodilaton modulus is fixed by the fluxes themselves, the potential at the minima has a form such that

$$V_n = m_f^4 n = m^3 \varphi_n \quad \text{case I.}$$

If the axiodilaton is fixed by other mechanisms, we instead expect

$$V_n = \frac{1}{2} m_f^4 n^2 = \frac{1}{2} m^2 \varphi_n^2 \quad \text{case II.}$$

$$n$$ is in both cases given by the flux $$F_1$$, and we have assumed that we are in a limit where the flux is large but the minima still remain. In general the potential $$V$$ is the sum of an underlying $$m^3 \varphi$$ (case I) or $$\frac{1}{2} m^2 \varphi^2$$ (case II) and superimposed barriers. What we have found in this way is a realization of the chain inflation potential studied in the previous section.

It immediately follows that the energy difference between two adjacent minima is given by

$$\Delta V = \begin{cases} m_f^4 & \text{case I} \\ m_f^4 n & \text{case II.} \end{cases}$$

We can also calculate the field values at the minima with the result

$$\varphi = \begin{cases} m_f^4 n & \text{case I} \\ \frac{m_f^4}{m^3} n & \text{case II}, \end{cases}$$

and the distance in field space between two adjacent minima is given by

$$\Delta \varphi = \begin{cases} m_f^4 & \text{case I} \\ \frac{m_f^4}{m^3} & \text{case II}, \end{cases}$$

As we see, there are two energy scales involved, $$m$$ and $$m_f$$, that ultimately depend on the details of the compactification. $$m_f$$ is given by parameters involving coupling strengths and the size of the extra dimensions, while $$m$$ crucially depends on details of the complex geometry of the Calabi–Yau manifolds.
geometry determining the distance in field space of the minima from a conifold point. In the following we will treat $m_f$ and $m$ as free parameters.

Let us now turn to the tunneling. In our model the Coleman–Callan decay is dominant; hence we have that the decay rate per unit time and physical volume is

$$\Gamma_n \sim e^{-B} = e^{-(27\pi^2/2)(S^4/(\Delta V)^3)}.$$  \hfill (111)

$$S = \int d\varphi \sqrt{V}$$  \hfill (112)

is taken over the path that minimizes the integral. In our discussion of chain inflation we have assumed a potential depending on a single (real) scalar field. The potentials generated from flux compactifications are far more complicated. Even if we restrict our attention to just one complex structure moduli, the problem is essentially two dimensional with a series of minima winding around the conifold point. We will not attempt to perform a complete analysis of the problem; instead we will focus on a simple order of magnitude estimate of the ingoing quantities.

We assume that the least action will essentially be given by a path encircling the conifold point with a length given by the circumference. Since, for large $n$, the potential $V$ scales like $n$ or $n^2$, it follows that the barriers between the two minima have a height of the order of

$$\lambda \approx \begin{cases} m_f^4 n & \text{case I} \\ m_f^4 n^2 & \text{case II}. \end{cases}$$  \hfill (113)

As a consequence we find

$$S \sim \Delta \varphi \lambda^{1/2} \sim \begin{cases} m_f^6 \sqrt{n} & \text{case I} \\ m_f^4 n & \text{case II}, \end{cases}$$  \hfill (114)

and the exponent becomes

$$B = \begin{cases} \frac{27\pi^2}{2} \left( \frac{m_f}{m} \right)^{12} n^2 & \text{case I} \\ \frac{27\pi^2}{2} \left( \frac{m_f}{m} \right)^4 n & \text{case II}. \end{cases}$$  \hfill (115)

In order for the tunneling not to be too suppressed, and $\tilde{\Gamma}_n = \Gamma_n \times$ physical volume to be larger than 1, we need that $B$ is of order 1.

5. Matching cosmological data

We have found that the primordial spectrum is of the form

$$\Delta_{R}^2 = \frac{H^2}{8\pi^2 M_{Pl}^2} \frac{1}{\sqrt{3}} \varepsilon.$$  \hfill (116)

13 Recall that the decay rate per unit time is given by $\tilde{\Gamma}_n = \Gamma_n \times$ physical volume.

14 This is valid for models where the amount of isocurvature perturbations is negligible compared to that of the adiabatic ones. As we have pointed out, there is no general argument to establish when this is the case.
This leads to the spectral index

\[ n_s = 1 + \frac{1}{H} \left( \frac{\Delta_R^2}{\Delta_R^2} \right) = 1 - 2\epsilon - \frac{\dot{\epsilon}}{H\epsilon} \]

\[ = \begin{cases} 
1 - 5\epsilon & \text{case I} \\
1 - 4\epsilon + \frac{1}{H} \sigma_{\tilde{\Gamma}/n}^2 \left( \frac{\tilde{\Gamma}}{n} \right)^{-1} + \frac{1}{H} \left( \frac{\tilde{\Gamma} \Delta \tilde{\Gamma}}{n} \right) \left( \frac{\tilde{\Gamma}}{n} \right)^{-1} & \text{case II,} 
\end{cases} \tag{117} \]

where we have used formula (54).

Note that in the case of equal and constant tunneling rate for all tunneling events, we get the result

\[ n_s = \begin{cases} 
1 - 5\epsilon & \text{case I} \\
1 - 4\epsilon & \text{case II.} \tag{118} 
\end{cases} \]

For case II this is identically the same result as for the case of slow roll with a quadratic potential. The reason for this agreement is that the quadratic potential in slow roll models is special in the sense that it gives rise to dynamics where the scalar field rolls with a constant speed. This corresponds perfectly to our case II with constant tunneling time \( \tau \). For other slow roll models with different potentials, such as the linear potential of case I, the results based on equal tunneling time will differ from ordinary slow roll based on Hubble friction.

In general, in order to satisfy the requirements on the spectral index, we expect the contributions in (117) due to \( \sigma_{\tilde{\Gamma}/n}^2 \) and to the difference in tunneling rates to be of order \( \epsilon \) at most. As a confirmation, it is possible to see that using formula (111), we would obtain a contribution \( B\epsilon \) to the spectral index from the dependence on \( n \) of the exponent. There is also a further dependence on \( n \) in the prefactor of the potential that is more tricky to derive. We will not attempt to do this. We just note that given the slow roll, the corrections to the spectral index have to be fractions of \( \epsilon \). Let us use this for a rough estimate of the ingoing parameters.

Matching the observations requires\(^{15}\)

\[ \Delta_R^2 = \eta, \tag{119} \]

where \( \eta \sim 2.5 \times 10^{-9} \). The Hubble constant is then given by

\[ H = \sqrt{\frac{8\pi^2 \eta \epsilon}{\sqrt{3}}} M_{\text{Pl}}, \tag{120} \]

which is way below the Planck mass. Writing the Hubble constant in terms of the energy density leads to

\[ m_f = \begin{cases} 
(8\sqrt{3}\pi^2 \eta \epsilon)^{1/4} n^{-1/4} M_{\text{Pl}} & \text{case I} \\
(8\sqrt{3}\pi^2 \eta \epsilon)^{1/4} 2^{1/4} n^{-1/2} M_{\text{Pl}} & \text{case II,} \tag{121} 
\end{cases} \]

and therefore \( m_f \) is also below the Planck scale.

\(^{15}\) We take the data from the five-year WMAP survey [21].
To estimate the parameter $m$ we must use the expression for the tunneling amplitude, $(115)$. We find that

$$m = \begin{cases} \left(\frac{27\pi^2}{2B}\right)^{1/12} n^{1/6} m_f & \text{case I} \\ \left(\frac{27\pi^2}{2B}\right)^{1/4} n^{1/4} m_f & \text{case II.} \end{cases}$$

(122)

Since measurements indicate $n_s \sim 0.97$, we must have $\varepsilon \sim 10^{-2}$. With $\varepsilon = 10^{-2}$, the normalization of the spectrum requires $H/M_{\text{pl}} \sim 2.6 \times 10^{-5}$. With $n \sim 10^4$ we find

$$m_f \sim \begin{cases} 7.6 \times 10^{-4} M_{\text{pl}} & \text{case I} \\ 9 \times 10^{-5} M_{\text{pl}} & \text{case II}, \end{cases}$$

and with $B \sim 1$ we find

$$m \sim \begin{cases} 5.3 \times 10^{-3} M_{\text{pl}} & \text{case I} \\ 3.1 \times 10^{-3} M_{\text{pl}} & \text{case II.} \end{cases}$$

(123)

(124)

These values for $m$ and $m_f$ are quite reasonable and can easily be lowered by increasing $n$ further\(^\text{16}\).

It is also of interest to note that the field values at the highest minima are quite small, from (109):

$$\phi_n \sim \begin{cases} 2.2 \times 10^{-2} M_{\text{pl}} & \text{case I} \\ 2.7 \times 10^{-2} M_{\text{pl}} & \text{case II.} \end{cases}$$

(125)

Unlike in the case of slow roll in chaotic inflation, we see that we need not move over large regions in field space to achieve inflation. Comparing with chaotic inflation one finds that the average motion of the field is in fact much slower than would be the case in a potential of the form $\frac{1}{2} m^2 \varphi^2$ for the above field values. The reason for the slow roll is the presence of the barriers that make the field temporarily stuck before it tunnels. The barriers take the role of the Hubble friction term, which otherwise is too weak for slow roll. To be precise, the average motion of the field $\varphi$ in chain inflation corresponds to $\dot{\varphi}^2 \sim \varepsilon^2$, rather than a single $\varepsilon$ as in the slow roll regime of chaotic inflation. In chain inflation we instead have a contribution from radiation, $\rho_r \sim \varepsilon$, that takes over the role of the kinetic term. We should also add that it is straightforward to verify that we obtain the correct number of e-foldings with the parameters we are using.

It is interesting to note that the different speed of sound appearing in the expression for $\Delta_k^2$ effectively increases the strength of the scalar modes compared with the tensor modes with a factor three. On the other hand, the chain inflation scenario sources additional gravitational waves through the collisions of the bubble walls, beside those due to quantum fluctuations in the metric field (which are the only ones present in the usual slow roll chaotic scenario). These effects constitute an observational signature of our model. However, to really verify them, one must analyze in detail the physics of bubble collisions and derive a precise relation between the spectral index and $\varepsilon$. We expect this to be extremely model dependent and defer such an analysis to future work.

\(^{16}\) It is possible then to check that with these values, in order to provide the correct number of e-foldings and satisfy the requirements of percolation, large scale thermalization and contraction of volume at every phase, the inflationary period ends before the field tunnels to the very last minima (with vacuum numbers 1 or 0).
Chain inflation revisited

6. Conclusion

In this paper we have found that chain inflation can be a viable model of inflation\(^\text{17}\). We have established that a series of tunneling events can give rise to a slowly changing vacuum energy in much the same way as Hubble friction gives rise to slow roll in chaotic inflation. We have calculated the spectrum of adiabatic perturbations and found it to be very similar to that of chaotic inflation up to a difference in the effective speed of sound. This is due to the fact that radiation plays a similar role in chain inflation to that of the kinetic term for the inflaton in chaotic inflation.

We have also found that general properties of the type IIB string landscape can be used to generate chain inflation. The key is special properties of flux compactifications that naturally give rise to series of minima useful for chain inflation. The relevant potentials are not flat or fine-tuned to be flat. Instead, one carefully needs to choose the values of other parameters that determine the amplitude for tunneling between different vacua. Nevertheless it is interesting to see that one need not have particularly extreme values to achieve inflation.

In order to estimate the likelihood of finding an appropriate potential one needs to examine the statistics of the topography of the landscape in a way that has not been done so far. We expect this to be a quite challenging task.

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\(^{17}\) In order to establish this in a complete and rigorous way, it is still necessary to study the reheating stage and the actual contribution and presence of isocurvature perturbations, which we leave for future research, when more detailed models will be available.
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