THETA GROUPS AND PROJECTIVE MODELS OF HYPERKÄHLER VARIETIES

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Abstract. We define the theta group associated to a simple coherent sheaf \( \mathcal{F} \) on a hyperkähler manifold \( X \) of Kummer type or OG6 type, provided \( g^*(\mathcal{F}) \) is isomorphic to \( \mathcal{F} \) for every automorphism \( g \) of \( X \) acting trivially on \( H^2(X) \). Note that this condition is satisfied if \( \mathcal{F} \) is invertible, if \( \mathcal{F} \) is one of the rank 4 stable vector bundles on general polarized HK fourfolds with certain discrete invariants constructed in \([O'G22]\), or if \( \mathcal{F} \) is the tangent bundle. We compute the commutator pairings of theta groups of line bundles and the rank 4 modular vector bundles of \([O'G22]\) (the commutator pairing of the tangent bundle is trivial). We have been motivated by the quest for an explicit description of locally complete families of polarized varieties of Kummer (or OG6) type.

1. Introduction

1.1. Background and motivation. Hyperkähler (HK) manifolds are similar to compact complex tori in many respects. The theta group of a line bundle on an abelian variety plays a key role in the analysis of projective models of such varieties, see \([Mum66]\). In this paper we define and study an analogue of the theta group for HK manifolds of Kummer and OG6 type. Let \( X \) be a HK manifold. The normal subgroup \( \text{Aut}^0(X) \triangleleft \text{Aut}(X) \) of automorphisms acting trivially on \( H^2(X) \) depends only on the deformation class of \( X \), and it has been determined for the known deformation classes. If \( X \) is of type K3\(^n\) or of type OG10 then \( \text{Aut}^0(X) \) is trivial, if \( X \) is of type Kum\(_n\) then \( \text{Aut}^0(X) \) is the semidirect product \( \mathbb{Z}/(2) \rtimes (\mathbb{Z}/(n+1))^4 \) where \( \mathbb{Z}/(2) \) acts on \( \mathbb{Z}/(n+1) \) via multiplication by \(-1\), and if \( X \) is of type OG6 then \( \text{Aut}^0(X) = (\mathbb{Z}/2)^4 \). Our idea is to define the theta group by replacing the group of translations of a complex torus with the largest abelian subgroup \( \mathbb{T}_p(X) \triangleleft \text{Aut}^0(X) \), i.e. (forgetting the deformation classes with trivial \( \text{Aut}^0(X) = (\mathbb{Z}/2)^4 \) if \( X \) is of type Kum\(_n\), and \( (\mathbb{Z}/2)^4 \) if \( X \) is of type OG6. Let \( \mathcal{F} \) be a (coherent) sheaf on \( X \) such that

\[
g^*(\mathcal{F}) \cong \mathcal{F} \quad \forall g \in T(X).
\]

(1.1.1)

Note that (1.1.1) holds if \( \mathcal{F} \) is an invertible sheaf, if \( \mathcal{F} \) is one of the rank 4 stable vector bundles on general polarized HK fourfolds with certain discrete invariants constructed in \([O'G22]\), or if \( \mathcal{F} \) is the tangent bundle. One may mimic Mumford’s definition of theta group \( \mathcal{T}(\mathcal{F}) \) associated to \( \mathcal{F} \). In particular \( \mathcal{T}(\mathcal{F}) \) is a \( \mathbb{C}^* \) extension of \( T(X) \) and there is an associated commutator pairing \( T(X) \times T(X) \rightarrow \mathbb{C}^* \). If the commutator pairing is non degenerate the theta group is isomorphic to a Heisenberg group, and it is well-known that the representations of such groups are severely restricted (Stone - von Neumann). Since \( \mathcal{T}(\mathcal{F}) \) acts naturally on the space...
of sections of $\mathcal{F}$, the upshot is that if the commutator pairing is non degenerate then the representation space $H^0(X, \mathcal{F})$ is fully determined by its dimension.

The present paper has been motivated by the quest for an explicit description of locally complete families of polarized varieties of Kummer (or OG6) type. Because of the analogy with abelian varieties, we believe that an understanding of theta groups will be instrumental in producing such families. More precisely, the philosophy that emerges from the present work is that if the commutator pairing of a sheaf $\mathcal{F}$ on a HK variety $X$ is non degenerate and the space of global sections is non trivial (but not too big), then it should be possible to give an explicit description of the general deformation of the pair $(X, \mathcal{F})$.

The recent paper [Flo22] contains results which are related to the ideas in the present work.

1.2. Main results. For the precise definition of the theta group and related notions we refer to Section 2. Below are simplified versions of our main results on the theta group of a line bundle on a HK manifold of Kummer or OG6 type. More detailed versions are given in Theorems 3.2 and 4.1. Before stating the results we recall that if $X$ is a HK manifold and $\alpha \in H^2(X; \mathbb{Z})$, then the divisibility of $\alpha$ is given by the non negative generator of the ideal $\{q_X(\alpha, \beta) \mid \beta \in H^2(X; \mathbb{Z})\}$ (here $q_X$ is the Beauville-Bogomolov-Fuiki (BBF) symmetric bilinear form of $X$). We denote the divisibility of $\alpha$ by $\text{div}(\alpha)$. If $L$ is a line bundle on $X$ we let $\text{div}(L) = \text{div}(c_1(L))$.

**Theorem 1.1.** Let $X$ be a 2n dimensional HK manifold of Kummer type, and let $L$ be a primitive line bundle on $X$. The theta group of $L$ is a Heisenberg group if and only if the following two conditions are satisfied:

1. $\text{div}(L) = 2$ and $n$ is even, or $\text{div}(L) = 1$ (no restriction on $n$ in this case);
2. $\text{gcd} \left\{ n + 1, \frac{n(2nL)}{q_X(L)} \right\} = 1$ (recall that $q_X$ is even).

**Theorem 1.2.** Let $X$ be a HK manifold of type OG6, and let $L$ be a primitive line bundle on $X$. The theta group of $L$ is a Heisenberg group if and only if $\text{div}(L) = 1$ and $q_X(L)$ is not divisible by 4.

Let $e$ be a positive integer such that $e \equiv -6 \pmod{16}$, and let $(M, h)$ be a general polarized HK fourfold of Kummer type such that $q_M(h) = e$ and the divisibility of $h$ is 2. In [O’G22] we have shown that there exists a slope stable rank 4 vector bundle $\mathcal{F}$ on $M$ such that

$$\det \mathcal{F} \cong O_{\mathcal{F}}(h), \quad \Delta(\mathcal{F}) := 8c_2(\mathcal{F}) - 3c_1(\mathcal{F})^2 = c_2(M).$$

(1.2.1)

We have also proved that $g^*(\mathcal{F}) \cong \mathcal{F}$ for every $g \in \text{Aut}^0(X)$, and hence the theta group $\mathcal{G}(\mathcal{F})$ is defined.

**Theorem 1.3.** Keeping notation as above, the theta group of $\mathcal{F}$ is a Heisenberg group if and only if $q_M(h)$ is not divisible by 3.

Lastly, we remark that the commutator pairing of the tangent bundle of a HK manifold of Kummer type or of type OG6 is trivial, see Example 2.10.

1.3. Outline of the paper. In Section 2 we give the details of the definition of the theta group $\mathcal{G}(\mathcal{F})$, we recall the definition of the Heisenberg representation, and we discuss the representation of $\mathcal{G}(\mathcal{F})$ on the space of global sections of $\mathcal{F}$.

Section 3 is devoted to the computation of the commutator pairing of line bundles on HK manifolds of Kummer type. The main result is Theorem 3.2, which is a more precise version of Theorem 1.1. The proof goes roughly as follows. One may reduce to the case of a generalized Kummer because the commutator pairing is invariant under deformation and under birational maps. For line bundles on a generalized Kummer $K_n(A)$ associated to an abelian surface $A$ one has to treat...
two cases: a line bundle “coming” from $A$, and the square root of the line bundle associated to the divisor $\Delta_n(A)$ parametrizing non reduced subschemes of $A$. If a line bundle “comes” from $A$, i.e. it is equal to $\mu_n(\ell)$ where $\ell$ is a line bundle on $A$ (see Subsection 3.1 for the definition of the map $\mu_n : H^2(A) \to H^2(K_n(A))$), since $K_n(A)$ is regular we denote by the same symbol the isomorphism class of a line bundle and its first Chern class), one may deform $A$ to a product $C_1 \times C_2$ of elliptic curves so that $\ell = \theta_{C_1}(D_1) \boxtimes \theta_{C_2}(D_2)$. It follows that it suffices to compute the commutator pairing of $\mu_n(\theta_{C_i}(D_i))$ for $i \in \{1, 2\}$. There are two lagrangian fibrations $\pi_i : K_n(C_1 \times C_2) \to \mathbb{P}^n$, and $\mu_n(\theta_{C_i}(D_i))$ is a multiple of $\pi_i^*(\theta_{\mathbb{P}^n}(1))$. Thus we are reduced to computing the commutator pairing of $\pi_i^*(\theta_{\mathbb{P}^n}(1))$.

The space of global sections of the latter is identified with the space of global sections of $\mathcal{O}_C((n+1)p_i)$, where $p_i \in C_i$ is the zero of the addition law, and hence is the Heisenberg representation of the theta group of the line bundle $\mathcal{O}_C((n+1)p_i)$ on the elliptic curve $C_i$. From this one gets the commutator pairing of $\pi_i^*(\theta_{\mathbb{P}^n}(1))$.

Lastly, if $L^{\otimes 2} \cong \mathcal{O}_{K_n(A)}(\Delta_n(A))$, one proves that the commutator pairing of $L$ is trivial by lifting the action of $T(K_n(A))$ on $K_n(A)$ to an action on the double cover $\hat{K}_n(A)$ of $K_n(A)$ ramified over $\Delta_n(A)$.

In Section 4 we compute the commutator pairing of line bundles on HK manifolds of type OG6. The main result is Theorem 4.1, which is a more precise version of Theorem 1.2. The proof is a more intricate version of the proof of the main result of Section 3. By deformation it suffices to compute the commutator pairing of line bundles on two models of HK varieties of type OG6 which have been previously studied, namely the symplectic desingularizations $\hat{K}_n(J)$ of an Albanese fiber of the moduli spaces of sheaves on a 2-dimensional Jacobian $J$ with Mukai vectors $v = (0, 2h, -2)$ and $v = (0, 2h, 0)$, where $h$ is the principal polarization of $J$. As in the previous case, the non-trivial contributions to the commutator pairing come from spaces of global sections of “lagrangian line bundles”.

Section 5 contains the proof of Theorem 1.3, which is easy once one has Theorem 1.1.

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2. Theta Groups

2.1. Automorphisms of very general HK manifolds. Let $X$ be a HK manifold. Let $\text{Aut}^0(X) < \text{Aut}(X)$ be the normal subgroup of automorphisms acting trivially on $H^2(X)$. If $Y$ is a HK manifold deformation equivalent to $X$ then $\text{Aut}^0(Y)$ is isomorphic to $\text{Aut}^0(X)$, see Theorem 2.1 in [HT13].

The HK manifold $X$ is of type $K3^{[n]}$ if it is a deformation of the Hilbert scheme (or Douady space) parametrizing length $n$ subschemes of a $K3$ surface, it is of type $Kum_n$ (here $n \geq 2$) if it is a deformation of the generalized Kummer manifold $K_n(A) \subset A^{[n+1]}$, where $A$ is a compact complex torus of dimension 2, see [Bea85]. Lastly $X$ is of type OG10 or of type OG6 if it is a deformation of the symplectic desingularization of the 10 dimensional moduli space of semistable sheaves on a $K3$ surface constructed in [O’G99], respectively the Albanese fiber of the symplectic desingularization of the 10 dimensional moduli space of semistable sheaves on an abelian surface constructed in [O’G03].

Example 2.1. (1) If $X$ is of type $K3^{[n]}$ or OG10, then $\text{Aut}^0(X)$ is trivial, see Proposition 10 in [Bea83] and Theorem 3.1 in [MW17] respectively.

(2) Let $A$ be a compact complex torus of dimension 2, and let $G_A$ be the subgroup of the group of automorphisms of $A$ (as complex manifold, we
forget the group law) generated by multiplication by \((-1)\) and translations \(x \mapsto x + \tau\) where \(\tau \in A[n+1]\). If \(g \in G_A\) and \([Z] \in K_n(A)\) then \(g(Z)\) is a subscheme parametrized by \(K_n(A)\), and hence we get an inclusion \(G_A \leq \text{Aut}(K_n(A))\). In fact \(G_A = \text{Aut}^0(K_n(A))\), see Corollary 5 in [BNWS11]. Note that \(G_A\) is isomorphic to the semidirect product \(\mathbb{Z}/(2) \ltimes \mathbb{Z}/(n+1)^4\) where \(\mathbb{Z}/(2)\) acts on \(\mathbb{Z}/(n+1)^4\) via multiplication by \(-1\).

3) If \(X\) is of type OG6, then \(\text{Aut}^0(X)\) is isomorphic to \((\mathbb{Z}/2)^8\), see Theorem 5.2 in [MW17].

**Proposition 2.2.** Let \(X\) be a HK manifold of type Kum\(_n\) for \(n \geq 2\). There is a unique abelian subgroup of \(\text{Aut}^0(X)\) of index 2, and it is isomorphic to \((\mathbb{Z}/(n+1))^4\).

**Proof.** We may assume that \(X = K_n(A)\). Then \(\text{Aut}^0(K_n(A))\) is isomorphic to the group \(G_A \leq \text{Aut}(A)\) described above. The normal subgroup \(A[n+1] \leq G_A\) is abelian of index 2, and is isomorphic to \((\mathbb{Z}/(n+1))^4\). Suppose that \(H < G_A\) is a different abelian subgroup of index 2. Then \(H \cap A[n+1]\) has index 2 in \(A[n+1]\). Let \(t \in H \cap A[n+1]\) and let \(g \in (H \setminus A[n+1]);\) the equality \(gt = tg\) gives that \(2t = 0\). Hence \(H \cap A[n+1] \leq A[2]\), and this is a contradiction because \(A[n+1] \cap A[2]\) has index greater than 2 in \(A[n+1]\). \(\square\)

**Definition 2.3.** If \(X\) is a HK manifold of type Kum\(_n\) we let \(T(X)\) be the unique abelian subgroup of index 2. If \(X\) is of type OG6, we let \(T(X) := \text{Aut}^0(X)\) the elements of \(T(X)\) are the translations of \(X\).

**Example 2.4.** Let \(A\) be a compact complex torus of dimension 2. By Example 2.1 we have a natural identification \(T(K_n(A)) = A[n+1]\).

**2.2. Theta groups for HK manifolds of Kummer or OG6 type.** In the present subsection \(X\) is a HK manifold of Kummer or OG6 type. Following Mumford [Mum66] we define the theta group of a simple sheaf \(\mathcal{F}\) on \(X\) under the assumption that

\[
g^*(\mathcal{F}) \cong \mathcal{F} \quad \forall g \in T(X).
\]

**Definition 2.5.** Let \(\mathcal{F}\) be a simple sheaf on \(X\) such that (2.2.1) holds. The \textit{theta group} \(\mathcal{G}(\mathcal{F})\) is the set of couples \((g, \phi)\) where \(g \in T(X)\) and \(\phi: \mathcal{F} \to g^*(\mathcal{F})\) is an isomorphism. The product is defined by

\[
(g_1, \phi_1) \cdot (g_2, \phi_2) := (g_1g_2, g_1^*(\phi_1) \circ \phi_2).
\]

**Example 2.6.** If \(L\) is a line bundle, then \(g^*(L) \cong L\) for all \(g \in \text{Aut}^0(X)\). Since \(L\) is simple, the theta group \(\mathcal{G}(L)\) is defined.

**Example 2.7.** In [O’G22] we have constructed rank 4 slope stable vector bundles \(\mathcal{F}\) on a generic polarized HK fourfold \((X, h)\) with \(\text{div}(h) = 2\) and \(q_X(h) \equiv -6\) (mod 16) or \(\text{div}(h) = 6\) and \(q_X(h) \equiv -6\) (mod 144) such that \(c_1(\mathcal{F}) = h\) and \(\Delta(\mathcal{F}) = c_2(X)\), where \(\Delta(\mathcal{F}) = 8c_2(X) - 3c_1(X)^2\) is the discriminant of \(\mathcal{F}\). We have proved that for such vector bundles \(g^*(\mathcal{F}) \cong \mathcal{F}\) for all \(g \in \text{Aut}^0(X)\) (op.cit.). Hence the theta group \(\mathcal{G}(\mathcal{F})\) is defined.

**Example 2.8.** The tangent bundle \(\Theta_X\) is stable with respect to any Kähler metric, in particular it is simple. Since \(\Theta_X^* \cong \Theta_X\) for all \(g \in \text{Aut}^0(X)\), the theta group \(\mathcal{G}(\Theta_X)\) is defined.

The homomorphism \(\mathcal{G}(\mathcal{F}) \to T(X)\) defined by \((g, \phi) \mapsto g\) gives an exact sequence of groups

\[
1 \to \mathbb{C}^* \to \mathcal{G}(\mathcal{F}) \to T(X) \to 1.
\]
The above exact sequence gives rise to the \textit{commutator pairing}

\[
T(X) \times T(X) \xrightarrow{e} \mathbb{C}^* \quad \mapsto \quad \tilde{\alpha} \cdot \tilde{\beta} \cdot \tilde{\alpha}^{-1} \cdot \tilde{\beta}^{-1}
\]  

(2.2.4)

where $\tilde{\alpha}, \tilde{\beta} \in \mathcal{G}(\mathcal{F})$ are lifts of $\alpha, \beta \in T(X)$ respectively. (Note: it is here that we want $T(X)$ to be abelian.) For fixed $\beta \in T(X)$ the maps $T(X) \to \mathbb{C}^*$ defined by $\alpha \mapsto e^\beta(\alpha, \alpha)$ and $\alpha \mapsto e^\beta(\beta, \alpha)$ are characters, and moreover $e^\beta(\alpha, \alpha) = 1$. The commutator pairing defines a homomorphism $E^\beta: T(X) \to \hat{T}(X)$, where $\hat{T}(X)$ is the group of characters of $T(X)$, by setting $E^\beta(\alpha)(\beta) := e^\beta(\alpha, \alpha)$. The commutator pairing is \textit{non degenerate} if $E^\beta$ is an isomorphism.

Below we collect a few observations regarding the commutator pairing.

\textbf{Remark 2.9.} Let $H \subset T(X)$ be a subgroup. The action of $H$ on $X$ lifts to an action on $\mathcal{F}$ if and only if the restriction to $H$ of the commutator pairing is trivial, see p. 293 in [Mum66].

\textbf{Example 2.10.} The commutator pairing of $\mathcal{G}(\Theta_X)$ is trivial because the action of $T(X)$ on $X$ lifts, via the differential, to an action on $\Theta_X$.

\textbf{Remark 2.11.} Let $f: Y \to X$ be a birational (i.e. bimeromorphic) map between HK manifolds. If $\varphi \in \text{Aut}^0(Y)$, then the induced birational map $f^{-1} \circ \varphi \circ f$ is regular, see the proof of Thm 2.1 in [HT13]. Since $f^{-1} \circ \varphi \circ f$ acts trivially on $H^2(Y)$, we get a natural isomorphism

\[
\text{Aut}^0(Y) \xrightarrow{\sim} \text{Aut}^0(Y) \quad f \mapsto f^{-1} \circ \varphi \circ f
\]  

(2.2.5)

\textbf{Remark 2.12.} Let $f: Y \to X$ be a birational map between HK manifolds of Kummer type or of type OG6. The isomorphism in (2.2.5) restricts to an isomorphism $T(X) \to T(Y)$. Since $X, Y$ have trivial canonical line bundles, there exist open subsets $V \subset Y$ and $U \subset X$ with complements of codimension at least 2 such that $f$ is regular on $V$ and it defines an isomorphism $U \to V$. Hence pull-back defines an isomorphism $f^*: \text{Pic}(X) \to \text{Pic}(Y)$. If $L$ is a line bundle on $X$, then the isomorphism $T(X) \to T(Y)$ lifts to an isomorphism of theta groups $\mathcal{G}(L) \to \mathcal{G}(f^*(L))$ because $L|_U$ is identified with $f^*(L)|_V$. In particular the commutator pairings of $\mathcal{G}(L)$ and $\mathcal{G}(f^*(L))$ are isomorphic.

\textbf{Remark 2.13.} Let $f: \mathcal{F} \to T$ be a family of HK manifolds of Kummer or OG6 type over a connected base $T$, and assume that $\mathcal{F}$ is a line bundle on $\mathcal{F}$. If $a, b \in T$ then the commutator pairings $e^{L_a}$ on $T(X_a)$ and $e^{L_b}$ on $T(X_b)$ (here $p_a := f^{-1}(a)$) are isomorphic. More precisely, any arc $\gamma: [0, 1] \to T$ starting at $a$ and ending at $b$ determines an isomorphism $\gamma_*: T(X_a) \to T(X_b)$ (see [HT13]), and we have $e^{L_a}(\alpha, \beta) = e^{L_b}(\gamma_*(\alpha), \gamma_*(\beta))$. In fact the commutator map varies continuously, and since it takes values in a finite subgroup of $\mathbb{C}^*$ (because $T(X)$ is finite) it follows that it is locally constant.

\textbf{Remark 2.14.} Let $L_1, L_2$ be line bundles on $X$, and let $L := L_1 \otimes L_2$. For $(\alpha, \beta) \in T(X) \times T(X)$ we have

\[
e^{L}(\alpha, \beta) = e^{L_1}(\alpha, \beta) \cdot e^{L_2}(\alpha, \beta).
\]  

(2.2.6)

\textbf{2.3. The Heisenberg group.} If the commutator pairing of $\mathcal{G}(\mathcal{F})$ is non degenerate, then $\mathcal{G}(\mathcal{F})$ is isomorphic to a Heisenberg group defined as follows. Let $1 \leq d_1 | d_2 \cdots | d_g$ be natural numbers, let $d := (d_1, \ldots, d_g)$, and let

\[
J(d) := \mathbb{Z}/(d_1) \oplus \cdots \oplus \mathbb{Z}/(d_g).
\]  

(2.3.1)

Let $\hat{J}(d)$ be the group of characters of $J(d)$.
Definition 2.15. The Heisenberg group $\mathcal{H}(d)$ is the set $\mathbb{C}^* \times J(d) \times \tilde{J}(d)$ with the group operation defined by
\[(a, x, f) \cdot (\beta, y, g) := (a \cdot \beta \cdot g(x), x + y, f \cdot g). \tag{2.3.2}\]

The forgetful map $\mathcal{H}(d) \longrightarrow J(d) \times \tilde{J}(d)$ is a homomorphism of groups, and it fits into an exact sequence of groups
\[1 \longrightarrow \mathbb{C}^* \longrightarrow \mathcal{H}(d) \longrightarrow J(d) \times \tilde{J}(d) \longrightarrow 1 \tag{2.3.3}\]
which gives rise to a commutator pairing
\[\left( J(d) \times \tilde{J}(d) \right) \times \left( \tilde{J}(d) \times J(d) \right) \xrightarrow{e_{\mathcal{H}(d)}} \mathbb{C}^*. \tag{2.3.4}\]

The next result follows from Corollary of Th. 1, p. 294 in [Mum66].

Proposition 2.16. Let $X$ be a HK manifold of type Kum$_n$ or of type OG6. Let $\mathcal{F}$ be a simple sheaf on $X$ such that (2.2.1) holds, and such that the commutator pairing $e^\mathcal{F}$ is non degenerate. Then, if $X$ is of type Kum$_n$ there exists an isomorphism $\mathcal{G}(\mathcal{F}) \stackrel{\sim}{\longrightarrow} \mathcal{H}(n + 1, n + 1)$, and if $X$ is of type OG6 there exists an isomorphism $\mathcal{G}(\mathcal{F}) \stackrel{\sim}{\longrightarrow} \mathcal{H}(2, 2, 2)$. In both cases the isomorphism can be chosen so that the exact sequence in (2.2.3) is isomorphic to the exact sequence in (2.3.3).

Next we consider representations of $\mathcal{H}(d)$.

Definition 2.17. The Schrödinger representation of $\mathcal{H}(d)$ is given by
\[\mathcal{H}(d) \times \mathbb{C}^{J(d)} \longrightarrow \mathbb{C}^{J(d)} \quad (\alpha, x, f) \quad \mapsto \quad (y \mapsto \alpha \cdot f(y) \cdot \varphi(x + y)) \tag{2.3.5}\]

The key result about representations is the following.

Proposition 2.18 (Prop. 3, p. 295 in [Mum66]). Let $\rho: \mathcal{H}(d) \rightarrow GL(V)$ be a finite dimensional representation of $\mathcal{H}(d)$ such that $\rho(\alpha, 0, 0) = \alpha \text{Id}_V$ for every $\alpha \in \mathbb{C}^*$. Then $\rho$ is a direct sum of copies of the Schrödinger representation.

2.4. The theta group and global sections. Let $X$ be a HK manifold of type Kum$_n$ or of type OG6, and let $\mathcal{F}$ be a simple sheaf on $X$ such that (2.2.1) holds. The action of $T(X)$ on $\mathbb{P}(H^0(X, \mathcal{F}))$ lifts to an action of $\mathcal{G}(\mathcal{F})$ on $H^0(X, \mathcal{F})$ as follows:
\[\mathcal{G}(\mathcal{F}) \xrightarrow{\Psi} GL(H^0(X, \mathcal{F})) \quad \sigma \mapsto (g \mapsto (g^{-1})^* \varphi(\sigma)). \tag{2.4.1}\]

Note that if $\alpha \in \mathbb{C}^*$ then the element $(\text{Id}_X, \alpha) \in \mathcal{G}(\mathcal{F})$ acts as $\alpha \text{Id}_{H^0(X, \mathcal{F})}$. Now assume that the commutator pairing of $\mathcal{G}(\mathcal{F})$ is non degenerate, and hence $\mathcal{G}(\mathcal{F})$ is isomorphic to a Heisenberg group by Proposition 2.16. Then the $\mathcal{G}(\mathcal{F})$ representation $H^0(X, \mathcal{F})$ is isomorphic to a direct sum of Schrödinger representations by Proposition 2.18.

Remark 2.19. If $H^0(X, \mathcal{F})$ is non zero then $e^\mathcal{F}$ may be read off from the representation of $\mathcal{G}(\mathcal{F})$ on $H^0(X, \mathcal{F})$. More precisely, for $i \in \{1, 2\}$ let $g_i \in T(X)$, and let $\tilde{g}_i = (g_i, \varphi_i)$ be a lift of $g_i$ to $\mathcal{G}(\mathcal{F})$. Then
\[\Psi(\tilde{g}_1) \circ \Psi(\tilde{g}_2) \circ \Psi(\tilde{g}_1)^{-1} \circ \Psi(\tilde{g}_2)^{-1} = (\text{Id}_X, e^\mathcal{F}(g_1, g_2) \cdot \text{Id}_{H^0(X, \mathcal{F})}). \tag{2.4.2}\]

For example, if $h^0(X, \mathcal{F}) = 1$ it follows that the commutator pairing is trivial, i.e. $\mathcal{G}(\mathcal{F})$ is the direct product $\mathbb{C}^* \times T(X)$. More generally, suppose that $H < T(X)$ is a subgroup which acts trivially on $\mathbb{P}(H^0(X, \mathcal{F}))$. If $h \in H$ and $\tilde{h} \in \mathcal{G}(\mathcal{F})$ is a lift of $h$, then $\Psi(\tilde{h})$ is a multiple of $\text{Id}_{H^0(X, \mathcal{F})}$, and hence the equality in (2.4.2) shows that $e^\mathcal{F}(h, g) = 1$ for all $g \in \mathcal{G}(\mathcal{F})$. Thus $H$ is in the kernel of $e^\mathcal{F}$. 
3. The commutator paring for HK manifolds of Kummer type

3.1. Preliminaries on generalized Kummers. Let $A$ be an abelian surface. The generalized Kummer $K_n(A)$ is the fiber over $0$ of the map $A^{[n+1]} \to A$ given by the composition

$$A^{[n+1]} \xrightarrow{h} A^{(n+1)} \xrightarrow{\sigma} A$$

where $h[Z] := \sum_{a \in A} \ell(\mathcal{O}_{Z,a})(a)$ is the Hilbert-Chow map and $\sigma$ is the summation map in the group $A$, i.e. $\sigma((a_1) + \ldots + (a_{n+1})) := a_1 + \ldots + a_{n+1}$. Here and in the rest of the paper we denote by $(a)$ the generator of the group of 0 cycles on $A$ that corresponds to the point $a \in A$. Hence if $k_1, \ldots, k_n+1$ are integers and $a_1, \ldots, a_{n+1} \in A$ then $k_1a_1 + \ldots + k_{n+1}a_{n+1}$ is a 0 cycle while $k_1a_1 + \ldots + k_{n+1}a_{n+1}$ is an element of $A$.

The cohomology group $H^2(K_n(A); \mathbb{Z})$ is described as follows. There is a homomorphism $\mu_n: H^2(A) \to H^2(K_n(A))$ given by the composition

$$H^2(A) \xrightarrow{s_{n+1}} H^2(A^{(n+1)}) \xrightarrow{h^*} H^2(K_n(A))$$

where $s_{n+1}$ is the natural symmetrization map. The map $\mu_n$ is injective but not surjective because $h$ contracts the prime divisor (here we assume that $n \geq 2$)

$$\Delta_n(A) := \{[Z] \in A^{[n+1]} | Z \text{ is not reduced}\}.$$  

The cohomology class of $\Delta_n(A)$ is (uniquely) divisible by 2 in integral cohomology. We let $\delta_n(A) \in H^2(A^{(n+1)}; \mathbb{Z})$ be the class such that

$$2\delta_n(A) = cl(\Delta_n(A)).$$  

(3.1.1)

(Beware: $\delta_n(A)$ is not the class of $\Delta_n(A)$.) One has

$$\mu_n(H^2(A; \mathbb{Z})) \oplus \mathbb{Z}\delta_n(A),$$  

(3.1.2)

where orthogonality is with respect to the BBF quadratic form. Moreover the BBF quadratic form is given by

$$q(\mu_n(\alpha) + x\delta_n(A)) = (\alpha, \alpha)_A - 2(n+1)x^2.$$  

(3.1.3)

(Here $(\alpha, \alpha)_A$ is the self intersection of $\alpha \in H^2(A)$.) We will use the following result of Mongardi-Pacienza.

Proposition 3.1 ([MP18, Theorem 4.2]). Let $X$ be a HK manifold of type $Kum_n$ and let $L$ be a line bundle on $X$. There exist a family $f: \mathcal{X} \to T$ of HK manifolds over a connected base $T$, points $t_0, t_1 \in T$ and a line bundle $\mathcal{L}$ on $\mathcal{X}$ with the following properties:

(a) $X_{t_0} := f^{-1}(t_0)$ is isomorphic to $X$, and $\mathcal{L}|_{X_{t_0}}$ is isomorphic to $L$.

(b) $X_{t_1} := f^{-1}(t_1)$ is isomorphic to a generalized Kummer $K_n(A)$, and

$$c_1(\mathcal{L}|_{X_{t_1}}) = \mu_n(a_1e_1 + a_2e_2) + x\delta_n(A),$$  

(3.1.4)

where $e_1, e_2 \in H^2(A; \mathbb{Z})$ is a standard basis of a hyperbolic sublattice of $H^2(A; \mathbb{Z})$ (i.e. $(e_i, e_j)_A = 0$ and $(e_1, e_2)_A = 1$), $a_1$ divides $a_2$, and of course the class $a_1e_1 + a_2e_2$ belongs to $NS(A)$.

3.2. Main result. Below is the main result of the present section.

Theorem 3.2. Let $X$ be a HK manifold of type $Kum_n$ for $n \geq 2$, and let $L$ be a primitive line bundle on $X$, i.e. such that $c_1(L)$ is primitive. Let $f: \mathcal{X} \to T$ be a family of HK manifolds as in Proposition 3.1, and let $b_i := gcd\{a_i, n+1\}$ where $a_1, a_2$ are as in Item (b) of Proposition 3.1. Then

$$\tilde{T}(X)/\text{Im}(\mathcal{L}) \cong (\mathbb{Z}/(b_1))^2 \oplus (\mathbb{Z}/(b_2))^2$$  

(3.2.1)
Remark 3.3. By Theorem 3.2 $\mathcal{G}(L)$ is a Heisenberg group if and only if $b_1 = b_2 = 1$. Since
\[ \text{div}(L) = \gcd\{a_1, 2(n + 1)\}, \] (3.2.2)
b_1 = 1 if and only if $\text{div}(L) = 1$ or $\text{div}(L) = 2$ and $n$ is even. Since $g(L)/2 = a_1a_2 - n^2(n+1)$, we get that if $b_1 = 1$ then $b_2 = 1$ if and only if $\gcd\{n+1, g(L)/2\} = 1$. This shows that Theorem 1.1 follows from Theorem 3.2.

Remark 3.4. For a prime $p$ and $a \in \mathbb{Q}$, let $\text{ord}_p(a)$ be the integer $m$ such that $a = p^m(b/c)$ where $b$ and $c$ are coprime to $p$ (we let $\text{ord}_p(0) := +\infty$). If $X$ is a HK manifold of type $\text{Kum}^n$ and $L$ is a line bundle on $X$, then
\[ b_1 = \begin{cases} \text{div}(L) & \text{if } \text{ord}_2(\text{div}(L)) \leq \text{ord}_2(n + 1), \\ \frac{\text{div}(L)}{2} & \text{if } \text{ord}_2(\text{div}(L)) < \text{ord}_2(n + 1). \end{cases} \]

Let $p$ be a prime. If $\text{ord}_p(n+1) = 0$ then $\text{ord}_p(b_2) = 0$. Suppose that $\text{ord}_p(n+1) > 0$; if $\text{ord}_p(\text{div}(L)) = \text{ord}_p(n + 1)$ (for $p = 2$ equality is also allowed) then
\[ \text{ord}_p(b_2) = \text{ord}_p\left(\frac{g_X(L)}{2\text{div}(L)}\right). \] (3.2.3)

Remark 3.5. Let $X$ be a HK manifold of type $\text{Kum}^n$ for $n \geq 2$, and let $L$ be an ample primitive line bundle on $X$. Let $g_X(L) = 2\epsilon$. By Kodaira vanishing and Britzke’s formula for Huybrechts’ HRR formula for HK manifolds of Kummer type, we have
\[ h^0(X, L) = (n + 1) \cdot \binom{e + n}{n}. \] (3.2.4)

Now suppose that the commutator pairing of $L$ is non degenerate, i.e. that the hypotheses of Theorem 1.1 hold. Then $H^0(X, L)$ is isomorphic to a direct sum of copies of the Heisenberg representation $\mathcal{H}(n + 1, n + 1)$. Since $\mathcal{H}(n + 1, n + 1)$ has dimension $(n + 1)^2$, it follows that $h^0(X, L)$ must be a multiple of $(n + 1)^2$, and by the equality in (3.2.4) this means that $(n + 1)$ divides $\binom{e + n}{n}$. An elementary argument confirms that this is the case. Moreover we get that $H^0(X, L)$ is the Heisenberg representation if and only if $e = 1$, i.e. $g_X(L) = 2$.

3.3. The commutator pairing for generalized Kummer. Let $A$ be a compact complex torus, and let $\ell \in \text{NS}(A)$. The commutator pairing $e^{(n+1)\ell}$ of a line bundle $L_A$ on $A$ such that $c_1(L_A) = (n+1)\ell$ is defined on $K((n+1)\ell) \times K((n+1)\ell)$, where $K((n+1)\ell)$ is the subgroup of translations $g$ of $A$ such that $g^*g(A) \cong L_A$. Since $A[n+1]$ is a subgroup of $K((n+1)\ell)$, it makes sense to restrict $e^{(n+1)\ell}$ to $A[n+1] \times A[n+1]$. Recall that we have a natural identification $T(K_n(A)) = A[n+1]$, see Example 2.1.

Proposition 3.6. Keep notation as above, and let $L$ be the line bundle on $K_n(A)$ such that $c_1(L) = \mu_n(\ell)$ (notation as in Subsection 3.1). Then $e^\ell$ is equal to the restriction of $e^{(n+1)\ell}$ to $A[n+1] \times A[n + 1]$.

Proof. There exist a family of compact complex tori $f : \mathcal{A} \to T$ over a connected base $T$, points $t_0, t_1 \in T$ and a line bundle $\xi$ on $\mathcal{A}$ with the following properties:

(a) The fiber $A_{t_0} := f^{-1}(t_0)$ is isomorphic to $A$, while the fiber $A_{t_1} := f^{-1}(t_1)$ is isomorphic to the product $C_1 \times C_2$ of elliptic curves.

(b) The line bundle $L_{t_0}$ on $K_n(A_{t_0})$ such that $c_1(L_{t_0}) = \mu_n(c_1(\xi_{t_0}))$ is identified with $L$ via the isomorphism $K_n(A_{t_0}) \cong K_n(A)$ induced by the isomorphism $A_{t_0} \cong A$ of Item (a).

(c) The line bundle $\xi_{t_1}$ is of product type, i.e. $\xi_{t_1} \cong \mathcal{O}_{C_1}(D_1) \boxtimes \mathcal{O}_{C_2}(D_2)$ for divisors $D_1, D_2$ on $C_1, C_2$ respectively.
Considering the relative family of generalized Kummers over $T$ with fiber $K_n(A)$ over $t$ and recalling Remark \ref{rem:connected}, it follows that it suffices to prove the proposition under the additional assumption that $A = C_1 \times C_2$ and $\ell$ is of product type, i.e. $\ell = \rho_1^*(a_1 \eta_1) + \rho_2^*(a_2 \eta_2)$ where $\rho_i: C_1 \times C_2 \to C_i$ is projection, and $\eta_i \in H^2(C_i, \mathbb{Z})$ is the fundamental class. Hence by (2.2.6) we may assume that $\ell = \rho_1^*(\eta_1)$. We are reduced to proving the proposition in this particular case.

Let $A^{(n+1),0} \subset A^{(n+1)}$ and $C^{(n+1),0}_1 \subset C^{(n+1)}_1$ be the subgroups of cycles summing up to 0 in $A$ and $C_1$ respectively. Let $p \in C_1$ be the zero of the addition law. Then $C^{(n+1),0}_1$ is naturally identified with $|\mathcal{O}_{C_1}((n+1)p)| \cong \mathbb{P}^n$. The composition of the Hilbert-Chow map $K_n(A) \to A^{(n+1),0}$ and the projection $A^{(n+1),0} \to C^{(n+1),0}_1$ is a Lagrangian fibration $\pi: K_n(A) \to C^{(n+1),0}_1$. We have $L \cong \pi^*(\mathcal{O}_{\mathbb{P}^n}(1))$. In particular the action of $C_2[n + 1]$ on $|L|$ is trivial and hence $C_2[n + 1]$ is contained in the kernel of $E_L$, see Remark \ref{rem:action}. Hence $e^L$ defines a skew-symmetric pairing on $C_1[n + 1]$ with values in $\mathbb{C}^*$. We claim that this pairing is the same as the commutator pairing of the line bundle $\mathcal{O}_{C_1}((n+1)p)$ on $K(\mathcal{O}_{C_1}((n+1)p)) = C_1[n + 1]$. In fact the latter can be computed by the action of the theta group $\mathcal{G}(\mathcal{O}_{C_1}((n+1)p))$ on the space of sections $H^0(C_1, \mathcal{O}_{C_1}((n+1)p))$, and since the pull-back by $\pi$ defines an isomorphism

$$H^0(C_1, \mathcal{O}_{C_1}((n+1)p)) = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \xrightarrow{\pi^*} H^0(K_n(A), L),$$

our claim follows. Since the first Chern class of $\mathcal{O}_{C_1}((n+1)p) \boxtimes \mathcal{O}_{C_2}$ is equal to $n\ell$ we are done.

\begin{cor}
Keep notation and hypotheses as in Proposition \ref{prop:3.6}, and let $a_1, a_2$ be the elementary divisors of $\ell$, where $a_1 \mid a_2$. Then

\[\hat{T}(K_n(A)) / \text{Im}(E^L) \cong (\mathbb{Z}/(b_1))^2 \oplus (\mathbb{Z}/(b_2))^2\]

(3.3.1)

where $b_i := \gcd(n + 1, a_i)$.

\end{cor}

\begin{proof}
By Proposition \ref{prop:3.6} the commutator pairing $e^L$ is equal to the restriction of $e^{(n+1)\ell}$ to $A[n + 1]$. We have an orthogonal direct sum decomposition

$$K((n + 1)\ell) \cong (\mathbb{Z}/(a_1(n + 1)))^2 \oplus (\mathbb{Z}/(a_2(n + 1)))^2,$$

and an adapted basis $\{\tilde{\alpha}_1, \tilde{\beta}_1, \tilde{\alpha}_2, \tilde{\beta}_2\}$ such that $e^{(n+1)\ell}(\tilde{\alpha}_i, \tilde{\beta}_j) = e_{a_i(n + 1)}$ is a primitive $a_i(n + 1)$-th root of 1. We may choose the basis such that $e^L_{a_1(n + 1)} = e_{a_1(n + 1)}$, call it $\epsilon$. Of course $\epsilon$ is a primitive $(n + 1)$-th root of 1. Let $\alpha_1 := a_1 \tilde{\alpha}_1$, and $\beta_1 := a_1 \tilde{\beta}_1$. Then $\{\alpha_1, \beta_1, \alpha_2, \beta_2\}$ is a basis of $A[n + 1]$, and $e^{(n+1)\ell}(\alpha_1, \beta_1) = e^{a_1}$. The isomorphism in (3.3.1) follows at once.

\end{proof}

\begin{prop}
Let $A$ be a compact complex torus and let $L$ a line bundle on $K_n(A)$ such that $c_1(L)$ is a multiple of $\delta_n$. The commutator pairing of $L$ is trivial.

\end{prop}

\begin{proof}
The linear system $|\Delta_n|$ consists of the single divisor $\Delta_n$ and hence the commutator pairing of $\mathcal{O}_{K_n(A)}(\Delta_n)$ is trivial by Remark \ref{rem:2.19}. This proves that $e^L$ is trivial if $c_1(L) = x \delta_n$ with $x$ even but not for odd $x$ (well, it does if $n$ is even because in that case there are no non trivial 2 torsion elements of $A[n + 1]$). Now suppose that $c_1(L) = \delta_n$. Let $\hat{K}_n(A) \to K_n(A)$ be the double cover ramified over $\Delta_n$. Then

$$\rho_*^{*}(\mathcal{O}_{\hat{K}_n(A)}) = \mathcal{O}_{K_n(A)} \otimes L^{-1}$$

and $L^{-1}$ is the $(-1)$ eigensheaf for the natural action of the covering involution. It follows that in order to prove that $e^L$ is trivial it suffices to lift the action of $T'(K_n(A))$ on $K_n(A)$ to an action on $\hat{K}_n(A)$ (recall Remark \ref{rem:2.9}). This is done by considering the isospectral Hilbert scheme $X_{n+1}(A)$ obtained by blowing up the
big diagonal in $A^{n+1}$ and the finite map $X_{n+1}(A) \to A^{[n+1]}$, see [Hai01]. Let $Y_n(A) \subset X_{n+1}(A)$ be the inverse image of $K_n(A) \subset A^{[n+1]}$. The action of the permutation group $\mathcal{S}_{n+1}$ on $A^{n+1}$ lifts to an action on $X_{n+1}(A)$ and also on $Y_n(A)$. The double cover $\tilde{K}_n(A) \to K_n(A)$ is identified with the double cover $Y_n(A)/\mathcal{S}_{n+1} \to K_n(A)$

where $\mathcal{S}_{n+1} \subset \mathcal{S}_{n+1}$ is the alternating group. The group of translations of $A$ acts on $A^{n+1}$ and it maps the big diagonal to itself, hence the action lifts to an action on $X_{n+1}(A)$. Note that the action commutes with the permutation action. The subgroup of translations in $A[n+1]$ acts on $Y_n(A)$ and hence also on $Y_n(A)/\mathcal{S}_{n+1}$. (Note that we do not need to go through the highly non trivial results of Haiman. In fact it suffices to work away from the codimension 2 subset of $K_n(A)$ parametrizing subschemes $Z$ such that $h(Z)$ has a point of multiplicity greater than 2, where the statement about $X_{n+1}(A)$ being the blow up of the big diagonal is elementary.)

3.4. Proof of Theorem 3.2. By Remark 2.13 and Proposition 3.1 we may assume that $X = K_n(A)$ and $c_1(L) = \mu_n(\ell) + x\delta_n$, where $\ell \in NS(A)$ and $x \in \mathbb{Z}$. In this case the statement of Theorem 3.2 holds by Corollary 3.7 and Proposition 3.8.

4. The commutator pairing for HK manifolds of type OG6

4.1. Main result. Before stating the main result, we note that if $X$ is a HK manifold of type OG6 then a primitive element of $H^2(X;\mathbb{Z})$ has divisibility 1 or 2 (see [Rap08] or (4.2.5)).

Theorem 4.1. Let $X$ be a HK manifold of type OG6, and let $L$ be a primitive line bundle on $X$.

1. If $\text{div}(L) = 1$ and $q_X(L)$ is not divisible by 4 then $e^L$ is non degenerate.
2. If $\text{div}(L) = 1$ and $q_X(L)$ is divisible by 4 then $\tilde{T}(X)/\text{Im}(E_L) \cong (\mathbb{Z}/2)^4$.
3. If $\text{div}(L) = 2$ then $e^L$ is trivial.

The proof for the case $\text{div}(L) = 2$ is in Subsection 4.6, the proof for the case $\text{div}(L) = 1$ is in Subsection 4.10.

Remark 4.2. Theorem 1.2 follows at once from Theorem 4.1.

Remark 4.3. Let $X$ be a HK manifold of type OG6, and let $L$ be an ample primitive line bundle on $X$. Let $q_X(L) = 2e$. By Kodaira vanishing and Huybrechts’ HRR formula for HK manifolds of type OG6, we have

$$h^0(X, L) = 4 \cdot \left( \frac{e + 3}{3} \right).$$

Now suppose that the commutator pairing of $L$ is non degenerate, i.e. that the hypotheses of Theorem 1.2 hold. Then $H^0(X, L)$ is isomorphic to a direct sum of copies of the Heisenberg representation $\mathcal{H}(2, 2, 2, 2)$. Since $\mathcal{H}(2, 2, 2, 2)$ has dimension 16, it follows that $h^0(X, L)$ must be a multiple of 16. An elementary argument confirms that this is the case. Moreover we get that $H^0(X, L)$ is the Heisenberg representation if and only if $e = 1$, i.e. $q_X(L) = 2$.

4.2. Preliminaries on HK manifolds of type OG6. HK manifolds of type OG6 are 6-dimensional and they belong to a single deformation class. The first examples where constructed by the author in [O’G03] as symplectic desingularizations of an Albanese fiber of a suitably chosen singular moduli spaces of semistable sheaves of rank 2 on Jacobians of genus 2 curves. M. Lehn and C. Sorger [LS06] examined in detail the singularities of the relevant moduli spaces and proved that one can construct a similar symplectic desingularization of an Albanese fiber of moduli spaces of semistable sheaves on an abelian surface if the Mukai vector is of the
form \( v = 2w \) where \( w^2 = 2 \) (the square is with respect to the Mukai pairing). 
Rapagnetta and Perego [PR13] proved that all HK varieties obtained this way are deformation equivalent, and Rapagnetta [Rap08] determined their BBF quadratic form. Following is a more detailed exposition. Let \( A \) be an abelian surface, and let 
\[
\tilde{H}(A; \mathbb{Z}) := H^0(A; \mathbb{Z}) \oplus H^2(A; \mathbb{Z}) \oplus H^4(A; \mathbb{Z})
\]  
be its Mukai lattice, where the Mukai quadratic form is defined by \((r, \alpha, s)^2 := \alpha^2 - 2rs\). Let 
\[
v_0 = (r_0, \alpha_0, s_0) \in \tilde{\text{NS}}(A) = H^0(A; \mathbb{Z}) \oplus \text{NS}(A) \oplus H^4(A; \mathbb{Z})
\]  
be a Mukai vector such that \( r_0 \geq 0 \) and moreover \( \alpha \) is effective (non zero) if \( r_0 = 0 \). We assume that \( v_0^2 = 2 \). Let \( v := 2v_0 \) and let \( h \) be a \( v \)-generic polarization of \( A \) (if the Picard number of \( A \) is 1, then any polarization of \( A \) is \( v \)-generic). The moduli space \( M_v(A, h) \) of \( h \) Gieseker-Mukai semistable sheaves \( \mathcal{F} \) on \( A \) with \( \text{ch}(-\mathcal{F}) = v \) is irreducible of dimension 10. There is an embedding of the symmetric square of \( M_v(A, h) \) into \( M_v(A, h) \) 
\[
\begin{align*}
M_v(A, h)^{(2)} & \hookrightarrow M_v(A, h) \\
[\mathcal{F}_1] + [\mathcal{F}_2] & \mapsto [\mathcal{F}_1 \oplus \mathcal{F}_2]
\end{align*}
\]  
whose image (of dimension 8) is the singular locus of \( M_v(A, h) \). Let \( \tilde{M}_v(A, h) \to M_v(A, h) \) be the blow up of the singular locus \( \Sigma_v(A, h) \). Then \( \tilde{M}_v(A, h) \) is smooth with Albanese variety isomorphic to \( A \times A^\vee \). Let \( \tilde{K}_v(A, h) \) be a fiber of (a chosen) Albanese map \( \tilde{M}_v(A, h) \to A \times A^\vee \); then \( \tilde{K}_v(A, h) \) is a HK variety of type OG6. 

Next we describe \( H^2(\tilde{K}_v(A, h); \mathbb{Z}) \) and the BBF quadratic form. Let \( K_v(A, h) \subset M_v(A, h) \) be the image of \( \tilde{K}_v(A, h) \) under the blow up (deingularization) map. First there is the Donaldson-Mukai-Le Potier homomorphism 
\[
\theta_v : v^\perp \longrightarrow H^2(K_v(A, h); \mathbb{Z}),
\]  
where \( v^\perp \subset \tilde{H}(A; \mathbb{Z}) \) is the orthogonal of \( v \) (i.e. of \( v_0 \)) with respect to the Mukai quadratic form. There are some choices to be made in defining \( \theta_v \); we choose to follow the definition in [Yos01], see (1.6) op. cit. The pull back via the blow up map \( \tilde{K}_v(A, h) \to K_v(A, h) \) defines an isometry 
\[
\tilde{\theta}_v : v^\perp \longrightarrow H^2(\tilde{K}_v(A, h); \mathbb{Z}),
\]  
onto a saturated sublattice of \( H^2(\tilde{K}_v(A, h); \mathbb{Z}) \). The orthogonal complement is generated by a class \( \alpha_v \) of BBF square \((-2)\) such that \( 2\alpha_v \) is the Poincaré dual of the exceptional divisor \( \tilde{\Sigma}_v(A, h) \) of the blow up map \( \tilde{K}_v(A, h) \to K_v(A, h) \). Moreover \( H^2(\tilde{K}_v(A, h); \mathbb{Z}) \) is generated over \( \mathbb{Z} \) by \( \text{Im}(\tilde{\theta}_v) \) and \( \alpha_v \). The upshot is that we have an isometry of lattices (see Theorem 3.1 in [PR14]) 
\[
\begin{align*}
(v^\perp \oplus_{-2}) & \longrightarrow H^2(\tilde{K}_v(A, h); \mathbb{Z}) \\
(x, t) & \quad \mapsto \tilde{\theta}_v(x) + t\alpha_v
\end{align*}
\]  
In particular, if \( X \) is HK manifold of type OG6, then \( H^2(X; \mathbb{Z}) \) equipped with the BBF quadratic form is isometric to the lattice 
\[
\Lambda_{\text{OG6}} := U_1 \oplus_{-2} U_2 \oplus_{-2} U_3 \oplus_{-2} \langle g_1 \rangle \oplus_{-2} \langle g_2 \rangle.
\]  
where each \( U_i \) is a hyperbolic plane, and \( \langle g_j, g_j \rangle = -2 \) for \( j \in \{1, 2\} \). We record here the following fact. 

**Proposition 4.4.** If \( a \in \Lambda_{\text{OG6}} \) is primitive (in particular non zero), then one of the following hold:

(I) \( \text{div}(a) = 1 \), 
(II) \( \text{div}(a) = 2 \) and \( (a, a) \equiv -2 \) (mod 8),
4.3. Varieties of type OG6 corresponding to \( v = (0, 2h, -2) \). Let \( J \) be an abelian surface with a principal polarization \( h \). We make the following assumption:

\[
\text{NS}(J) = Zh.
\]  

(4.3.1)

In particular \( J \) is the Jacobian of a (smooth projective) curve \( C \) of genus 2.

Let \( v = (0, 2h, -2) \). We denote the moduli space \( M_v(J, h) \) by \( M_v(J) \) (there is a unique polarization). Points of \( M_v(J) \) parametrize \( h \)-semistable sheaves \( (C_\xi, s(\xi)) \) where \( \iota_C: C \to J \) is the inclusion of a curve with Poincaré dual \( 2h \) and \( \xi \) is a pure sheaf on \( C \) of degree 2.

Let \( \text{Pic}^{2h}(J) \) be the component of the Picard scheme of \( J \) parametrizing line bundles \( \mathcal{L} \) with \( c_1^2(\mathcal{L}) = 2h \), where \( c_1^2(\mathcal{L}) \in H^2(J; \mathbb{Z}) \) is the first Chern class in Betti cohomology. Let \( \sigma: CH_{2}(J) \to J \) be the homomorphism defined (at the level of cycles) by \( \sigma(\sum m_i(x_i)) := \sum m_i x_i \), where the first sum is a formal sum, while the second one is the sum in the group \( J \). Consider the map

\[
M_v(J) \xrightarrow{\text{alb}} \text{Pic}^{2h}(J) \times J, \quad \left( c_1(\mathcal{F}), \sigma(c_2(\mathcal{F})) \right)
\]

(4.3.2)

where Chern classes are taken in the Chow ring of \( J \) and we identify \( \text{CH}^1(J) \) with \( \text{Pic}(J) \). An Albanese fibration of \( M_v(J) \), denote it by \( \text{Alb} \), is provided by the composition \( M_v(J) \to M_v(J) \xrightarrow{\text{alb}} \text{Pic}^{2h}(J) \times J \). Now let \( \Theta_J \) be a symmetric principal polarization of \( J \) with cohomology class \( h \), and let

\[
K_v(J) := \text{alb}^{-1}([2\Theta_J], 0), \quad \tilde{K}_v(J) := \text{Alb}^{-1}([2\Theta_J], 0).
\]

(4.3.3)

The HK variety \( \tilde{K}_v(J) \) is examined in detail in the papers [Rap07] and [MRS18]. To be precise in [MRS18] the Mukai vector is \( (0, 2h, 2) \), but tensorization by \( \mathcal{O}_J(\Theta_J) \) defines an isomorphism between the moduli spaces \( M_{(0,2h,-2)}(J) \) and \( M_{(0,2h)}(J) \).

Here we collect results that will be needed later. Let \( \beta_v, \gamma_v \in \text{NS}(\tilde{K}_v(J)) = \text{NS}(K_v(J)) \) be defined by

\[
\beta_v := \tilde{\theta}_v(2, -h, 1), \quad \gamma_v := \tilde{\theta}_v(2, -h, 0).
\]

(4.3.4)

By the isometry in (4.2.5) and the equality in (4.3.1), we have an orthogonal direct sum decomposition

\[
\text{NS}(\tilde{K}_v(J)) = \mathbb{Z}\alpha_v \oplus \mathbb{Z}\beta_v \oplus \mathbb{Z}\gamma_v.
\]

(4.3.5)

Moreover we have

\[
(\alpha_v, \alpha_v) = -2, \quad (\beta_v, \beta_v) = -2, \quad (\gamma_v, \gamma_v) = 2.
\]

(4.3.6)

The geometric meaning of class the \( \alpha_v \) (or rather \( 2\alpha_v \)) is clear by definition. We give geometric realizations of the classes \( \gamma_v - \beta_v \) and \( \beta_v - \alpha_v \). Since sheaves parametrized by \( K_v(J) \) have rank 0, we have a map

\[
K_v(J) \xrightarrow{\pi_v} [\mathcal{F}] \to \text{Supp}_{\text{det}}(\mathcal{F}), \quad |2\Theta_J| \cong \mathbb{P}^3
\]

(4.3.7)

where \( \text{Supp}_{\text{det}}(\mathcal{F}) \) (the determinantal support of \( \mathcal{F} \)) is the curve defined by the 0-th Fitting ideal of \( \mathcal{F} \) (and is well-defined also for properly semistable sheaves). Let

\[
\tilde{\pi}_v: \tilde{K}_v(J) \to |2\Theta_J| \cong \mathbb{P}^3
\]

(4.3.8)

be the composition of the desingularization map and \( \pi_v \). Then \( \tilde{\pi}_v \) is a Lagrangian fibration. By [LP03] (see Section 2.3) we have that

\[
\gamma_v - \beta_v = \tilde{\theta}_v(0, 0, -1) = c_1(\pi_v^* \mathcal{O}_{\mathbb{P}^3}(1)).
\]

(4.3.9)
The geometric meaning of the class $\beta_v - \alpha_v$ is contained in the proof of the result below.

**Proposition 4.5.** The line bundle $\mathcal{L}_v$ such that $c_1(\mathcal{L}_v) = \beta_v - \alpha_v$ has a unique global section up to rescaling.

**Proof.** Let $w = (2,0,-2)$. Let $\tilde{K}_w(2,0,-2)$ be the Albanese fiber of $\tilde{M}_w(2,0,-2)$ lying over the point $(0,[\sigma_J]) \in \tilde{J}$. In [Rap07] one finds the definition of a birational map

$$
\tau: \tilde{K}_w(2,0,-2) \to \tilde{K}_w(0,2h,-2)
$$

such that

$$
\tau^*(\alpha_v) = \alpha_w.
$$

Let $\tilde{B} \subset \tilde{K}_w(2,0,-2)$ be the prime divisor whose generic point parametrizes a stable non locally free sheaf on $J$. By Theorem 3.5.1 in op.cit. we have

$$
(cl(\tilde{B}), cl(\tilde{B})) = -4, \quad (cl(\tilde{B}), \alpha_w) = 2.
$$

Hence by (4.3.11) we get that

$$
(cl(\tau_*(\tilde{B})), cl(\tau_*(\tilde{B}))) = -4, \quad (cl(\tau_*(\tilde{B})), \alpha_v) = 2.
$$

It follows that

$$
(cl(\tau_*(\tilde{B}))) + \alpha_v, cl(\tau_*(\tilde{B}))) = -2, \quad (cl(\tau_*(\tilde{B}))) + \alpha_v, \alpha_v) = 0.
$$

The orthogonal decomposition in (4.3.5), the “squares” in (4.3.6) and a straightforward computation show that $\pm \beta_v$ are the only classes in $NS(K_v(J))$ orthogonal to $\alpha_v$ and of square $-2$. Hence $cl(\tau_*(\tilde{B}))) + \alpha_v = \pm \beta_v$, i.e. $cl(\tau_*(\tilde{B}))) = \pm \beta_v - \alpha_v$.

Since $\tau_*(\tilde{B})$ is an effective divisor, and $\gamma_v - \beta_v$ is the class of a movable divisor, we have

$$
(cl(\tau_*(\tilde{B}))), \gamma_v - \beta_v) \geq 0,
$$

and hence $cl(\tau_*(\tilde{B})) = \beta_v - \alpha_v$. In order to finish the proof it suffices to show that the divisor $\tilde{B}$ does not move. In fact there is an open dense subset of $\tilde{B}$ which is a (smooth) conic fibration over a 4 dimensional locally closed subset of $J \times M_4(J)$, where $u = (2,0,-1)$ (the “conics” are the generic fibers of the Uhlenbeck map $M_u(J) \to M_w(J)^{Duy}$). By adjunction the restriction to a conic fiber $C$ of the normal bundle of $\tilde{B}$ is the canonical line bundle $\omega_C$, and hence $\tilde{B}$ does not move. \qed

### 4.4. Translations of $\tilde{K}_v(J)$ for $v = (0,2h,-2)$

One defines an action of $J[2] \times \tilde{J}[2]$ on $\tilde{K}_v(J)$ as follows. Let $[\mathcal{F}] \in K_v(J)$. Let $x \in J[2]$ and let $\tau_x: J \to J$ be the translation by $x$. Then $\tau_x(\mathcal{F})$ is parametrized by a point of $K_v(J)$. In fact $\tau_x(\mathcal{F})$ is clearly $h$-semistable, moreover $c_1(\tau_x(\mathcal{F})) = 2\Theta_J$ (here and in what follows Chern classes are taken in the Chow ring) because $J[2]$ is the group of translations sending the rational equivalence class of $2\Theta_J$ to itself, and lastly $\sigma(c_2(\tau_x(\mathcal{F}))) = 0$ because $\deg c_2(\mathcal{F}) = 2$ (all we need is that $\deg c_2(\mathcal{F})$ is even). Similarly, let $y \in \tilde{J}[2]$, and let $\mathcal{L}_y$ be the corresponding line bundle on $J$; then $[\mathcal{F} \otimes \mathcal{L}_y] \in K_v(J)$. In fact $\mathcal{F} \otimes \mathcal{L}_y$ is clearly $h$-semistable, moreover $c_1(\mathcal{F} \otimes \mathcal{L}_y) = c_1(\mathcal{F}) = 2\Theta_J$, and lastly

$$
\sigma(c_2(\mathcal{F} \otimes \mathcal{L}_y)) = \sigma(c_2(\mathcal{F})) + \sigma(c_1(\mathcal{F}) \cdot c_1(\mathcal{L}_y)) = \sigma(2\Theta_J \cdot c_1(\mathcal{L}_y)) = 0.
$$

Thus we have an embedding

$$
J[2] \times \tilde{J}[2] \hookrightarrow \text{Aut}(K_v(J))
$$

$$
(x,y) \mapsto [(\mathcal{F}) \mapsto [\tau_x(\mathcal{F}) \otimes \mathcal{L}_y])
$$

Since the action maps the singular locus to itself, and the map $\pi: \tilde{K}_v(J) \to K_v(J)$ is the blow up of the singular locus, we get an embedding $J[2] \times \tilde{J}[2] \hookrightarrow \text{Aut}(\tilde{K}_v(J))$. 

If \( x \in J[2] \) we let \( \lambda_x : \tilde{K}_v(J) \to \tilde{K}_v(J) \) be the automorphism corresponding to \( \tau_x \), and if \( y \in J[2] \) we let \( \mu_y : \tilde{K}_v(J) \to \tilde{K}_v(J) \) be the automorphism corresponding to tensorization with \( \mathcal{L}_y \). Since one may define similarly an action of \( J \times \tilde{J} \) on \( \tilde{\mathcal{M}}_v(J) \), and each such automorphism of \( \tilde{\mathcal{M}}_v(J) \) acts trivially on cohomology (it is homotopic to the identity), and since the homomorphism \( H^2(\tilde{\mathcal{M}}_v(J)) \to H^2(\tilde{K}_v(J)) \) is surjective, we get an embedding

\[
J[2] \times \tilde{J}[2] \quad \hookrightarrow \quad \text{Aut}^0(\tilde{K}_v(J))
\]

(4.4.1)

By Theorem 5.2 in [MW17], the map in (4.4.1) is an isomorphism.

4.5. The double cover of \( \tilde{K}_v(J) \) for \( v = (0, 2h, -2) \). Since \( 2\alpha_v = \text{cl}(\hat{S}_v(J)) \), there exists a double cover

\[
\tilde{Y}_v(J) \xrightarrow{\tilde{\tau}_v} \tilde{K}_v(J)
\]

ramified over \( \hat{S}_v(J) \), and such that

\[
\tilde{\tau}_v \ast \left( \theta_{\tilde{Y}_v(J)} \right) = \theta_{\tilde{K}_v(J)} \oplus \mathcal{F}_v,
\]

(4.5.2)

where \( \mathcal{F}_v \) is a line bundle, \( c_1(\mathcal{F}_v) = -\alpha_v \), and the addends on the right hand side of (4.5.2) are the \( \pm 1 \) eigenspaces for the natural action on the left hand side of the covering involution of \( \tilde{\tau}_v \).

**Proposition 4.6.** The action of \( \text{Aut}^0(\tilde{K}_v(J)) \) on \( \tilde{K}_v(J) \) lifts to an action on \( \tilde{Y}_v(J) \), i.e. a group homomorphism \( \text{Aut}^0(\tilde{K}_v(J)) \to \text{Aut}(\hat{S}_v(J)) \).

In order to prove the above proposition we recall results of Rapagnetta [Rap07] and Mongardi, Rapagnetta, Saccà [MRS18]. Let

\[
S := \text{Bl}_{J[2]}(J)/\iota
\]

be the quotient of the blow up of \( J \) with center \( J[2] \) by the involution lifting multiplication by \(-1\) on \( J \). Thus \( S \) is the (smooth) Kummer K3 surface associated to \( J \). We have a commutative diagram

\[
\begin{array}{ccc}
\text{Bl}_{J[2]}(J) & \xrightarrow{\mu} & S \\
\downarrow{\nu} \quad \downarrow{\iota} & & \quad \downarrow{\iota} \\
J & & S
\end{array}
\]

(4.5.3)

where \( \mu \) is the blow up map, \( \nu \) is the quotient map, and the (rational) horizontal map identifies \( a \in (J, \mathcal{J}[2]) \) with \((-a) \). Let \( h_S \in \text{NS}(S) \) be the class such that \( \nu^*(h_S) = \mu^*(h) \). Let \( w = (0, h_S, -1) \in H(S; \mathbb{Z}) \), and let \( M_w(S) \) be the moduli space of stable sheaves on \( S \) with Mukai vector \( w \) stable with respect to a \( \nu \)-suitable polarization (note that \( h_S \) is not ample, it contracts the nodal curves of \( S \)). Then \( \tilde{Y}_v(J) \) is birational to \( M_w(S) \). More precisely, we have a rational map

\[
M_w(S) \to \tilde{Y}_v(J)
\]

defined as follows. Let \( [\mathcal{E}] \in M_w(S) \) be a general point, so that \( \mathcal{E} = i_D \ast \xi \) where \( i_D : D \to S \) is the inclusion of a smooth curve with \( \text{cl}(D) = h_S \) and \( \xi \) is a line bundle on \( D \) of degree 1. Let \( C := \mu(\nu^{-1}(D)) \). The restriction to \( C \) of the rational map \( J \to S \) is an étale map \( f_C : C \to D \) of degree 2. Let \( j_C : C \to J \) be the inclusion map. Then \( v(\mu^*(\mathcal{E})) = v \), and since \( C \) is irreducible, \( j_C \ast (f_C^*(\xi)) \) is a stable sheaf. Lastly \( \text{Alb}([j_C \ast (f_C^*(\xi))]) = 0 \). Hence we have rational maps

\[
M_w(S) \quad \to \quad \tilde{K}_v(J)
\]

(4.5.4)
Let $\lambda_D$ be the line bundle on $D$ (with trivial square) determined by the étale double cover $f_C: C \to D$. If $\xi'$ is a line bundle on $D$ then $f_C^*(\xi') \cong f_C^*(\xi)$ if and only if $\xi' \cong \xi$ or $\xi' \cong \xi \otimes \lambda_D$. It follows that $\psi$ has degree 2. In [MRS18], see Lemma 5.2, one finds the proof that $M_w(S)$ is birational to $Y_v(J)$ and that the double cover $\psi$ is identified (birationally) with the double cover $\tilde{e}_v$.

Lemma 4.7. Let $D \subset S$ be a smooth curve with cohomology class $h_S$ (and hence integral), and let $C := \mu(\nu^{-1}(D))$. Let $f_C: C \to D$ be the étale double cover given by the restriction of the horizontal map in (4.5.3), and let $\text{Nm}: \text{Pic}(C) \to \text{Pic}(D)$ be the corresponding norm map. If $[\xi] \in \tilde{J}[2]$, then

$$f_C^*(\text{Nm}(\xi)) = [\xi].$$

Proof. The map $\tilde{J} \to \text{Pic}^0(C)$ defined by restriction identifies $\tilde{J}$ with the Prym variety $\text{Pr}(f_C)$. By irreducibility of the moduli space of étale double covers of curves of a fixed genus (in our case genus 3), it suffices to prove that the equality

$$f_C^*(\text{Nm}(\xi)) = [\xi]$$

holds whenever we have an étale double cover $f_C: C \to D$ of a curve of genus $g$, and $[\xi] \in \text{Pr}(f_C)[2]$. We may also degenerate $C$ to a curve with separating nodes, and it suffices to prove the equality for such double covers. Choose $D = D_1 \cup D_{g-1}$ where $D_1$ has genus 1, $D_{g-1}$ has genus $g-1$, and $q_1 \in D_1$ is glued to $q_{g-1} \in D_{g-1}$. Let $f_1: C_1 \to D_1$ be an étale connected double cover, and let $f_1^{-1}(q_1) = \{p_1', p_1''\}$. Let $C_{g-1}'$ and $C_{g-1}''$ be copies of $D_{g-1}$, with $q_{g-1}' \in C_{g-1}'$ and $q_{g-1}'' \in C_{g-1}''$ corresponding to $q_{g-1}$ respectively. Let $C := C_1 \cup C_{g-1}' \cup C_{g-1}''$ with $p_1'$ glued to $q_{g-1}'$, and $p_1''$ glued to $q_{g-1}''$. We have an obvious étale double cover $f_C: C \to D$. For this double cover $\text{Pr}(f_C) = \text{Pic}^0(D_{g-1})$, and one checks right away that the equality in (4.5.6) holds for all $[\xi] \in \text{Pr}(f_C)[2] = \text{Pic}^0(D_{g-1})[2]$. □

Proof of Proposition 4.6. Let $x \in J[2]$. We define an automorphism $Y_v(J) \to Y_v(J)$ proceeding as follows. The translation $\tau_x: J \to J$ defined by $x$ maps $J[2]$ to itself, hence it lifts to an automorphism of $\text{Bl}_{J[2]}(J)$, and the latter descends to an automorphism $\tau_x: S \to S$ because $\tau_x$ commutes with multiplication by $-1$. Since the generic sheaf parametrized by $M_w(S)$ is the push-forward of a line bundle on an irreducible curve on $S$, we have a birational map

$$M_w(S) \xrightarrow{\lambda_x} M_w(S)$$

$$[\mathcal{E}] \quad \quad \quad \quad \quad \quad \quad \quad \quad [\mathcal{E}]$$

Conjugating with the birational map in (4.5.4), we get a birational map

$$\lambda_x: Y_v(J) \dashrightarrow Y_v(J)$$

such that $\lambda_x \circ \bar{\tau}_v = \bar{\tau}_v \circ \lambda_x$, where $\bar{\tau}_v: Y_v(J) \to K_v(J)$ is the double cover in (4.5.1). Since $\bar{\tau}_v$ is a finite map, it follows that $\lambda_x$ is regular (the graph of $\lambda_x$ has a single point over any point of $Y_v(J)$).

Now let $y \in \tilde{J}[2]$. We define an automorphism $Y_v(J) \to Y_v(J)$ proceeding as follows. Let $\mathcal{L}_y \in \text{Pic}^0(J)$ be the line bundle corresponding to $y$. Let $\mathcal{L}_y := \text{Nm}_\nu(\mu^*(\mathcal{L}_y))$, where $\text{Nm}_\nu: \text{Pic}(\text{Bl}_y(J)) \to \text{Pic}(S)$ is the norm map defined by $\nu$. We claim that we have a birational map

$$M_w(S) \xrightarrow{\phi_y} M_w(S)$$

$$[\mathcal{E}] \quad \quad \quad \quad \quad \quad \quad \quad \quad [\mathcal{E} \otimes \mathcal{L}_y]$$

In fact, let $[\mathcal{E}]$ be a general point of $M_w(S)$. Then $\mathcal{E} = i_{D,w}(\xi)$ where $D \subset S$ is a smooth curve with cohomology class $h_S$ (and hence integral). Let $C := \mu(\nu^*(D))$. Then $\mathcal{E} \otimes \mathcal{L}_y = i_{D,w}(\xi \otimes (\mathcal{L}_y|_D))$, and $\nu(i_{D,w}(\xi \otimes (\mathcal{L}_y|_D))) = w$.
because \( \deg(\mathcal{L}_{|D}) = 0 \) (since \( \deg(\mathcal{L}_{|C}) = 0 \)). Moreover \( \mathcal{E} \otimes \mathcal{L}_{y} \) is stable because it is the push-forward of a line bundle on an integral curve. Conjugating with the birational map in (4.5.4), we get a birational map
\[
\mu_y : Y_v(J) \to Y_v(J)
\] (4.5.10)
such that \( \mu_y \circ \overline{\tau}_v = \overline{\tau}_v \circ \mu_x \). Arguing as in the case of \( \lambda_x \) we get that \( \mu_y \) is regular. We get an embedding
\[
J[2] \times \hat{J}[2] \leftrightarrow \text{Aut}(Y_v(J))
\]
(4.5.11)
The above homomorphism is a lift of the homomorphism in (4.4.1). In fact it is obvious that \( \lambda_x \) lifts \( \lambda_x \), while \( \mu_y \) by Lemma 4.7.

4.6. The commutator pairing for line bundles of divisibility 2. In the present subsection we prove the part of the statement of Theorem 4.1 that refers to line bundles of divisibility 2.

**Proposition 4.8.** Let \( X \) be a HK manifold of type OG6. If \( L \) is a primitive line bundle on \( X \) such that \( \text{div}(L) = 2 \) then \( e^L \) is trivial.

First we prove the above result for two special choices of line bundle \( L \) on \( \hat{K}_v(J) \), where \( v = (0, 2h, -2) \).

**Proposition 4.9.** Let \( J \) and \( v = (0, 2h, -2) \) be as in Subsection 4.3. The commutator pairings of \( L_v \) (notation as in Proposition 4.5) and of \( \mathcal{F}_v \) (see (4.5.2)) are trivial.

**Proof.** The space of global sections of \( L_v \) is one-dimensional by Proposition 4.5, and hence the commutator pairing of \( L_v \) is trivial by Remark 2.19.

By Proposition 4.6 the action of \( \text{Aut}^0(\hat{K}_v(J)) \) on \( \hat{K}_v(J) \) lifts to an action of \( \text{Aut}^0(\hat{K}_v(J)) \) on \( \hat{Y}_v(J) \). Since \( \mathcal{F}_v \) is the \((-1)\) eigenspace for the natural action of the covering involution of \( \hat{\tau}_v : \hat{Y}_v(J) \to \hat{K}_v(J) \) on \( \hat{\tau}_v : \mathcal{G}_{\hat{Y}_v(J)} \), it follows that the action of \( \text{Aut}^0(\hat{K}_v(J)) \) on \( \hat{K}_v(J) \) lifts to an action of \( \text{Aut}^0(\hat{K}_v(J)) \) on \( \mathcal{F}_v \). By Remark 2.9 we get that the commutator pairings of \( \mathcal{F}_v \) is trivial. \( \square \)

Let \( \Lambda_{\text{OG}6} \) be the lattice defined in (4.2.6).

**Proposition 4.10.** Let \( a, b \in \Lambda_{\text{OG}6} \) be primitive elements. Then \( a, b \) belong to the same \( O^+(\Lambda_{\text{OG}6}) \)-orbit if and only if \( (a, a) = (b, b) \) and either Item (I), or Item (II), or Item (III) of Proposition 4.4 holds both for \( a \) and \( b \).

**Proof.** The non trivial implication is “if either Item (I), or Item (II), or Item (III) of Proposition 4.4 holds both for \( a \) and \( b \), then \( a, b \) belong to the same \( O^+(\Lambda_{\text{OG}6}) \)-orbit”. This result holds by Eichler’s Criterion, see Item (i) of Proposition 3.3 in [GHS09]. \( \square \)

**Lemma 4.11.** Let \( X \) be a HK manifold of type OG6 carrying a primitive line bundle \( L \) of divisibility 2, and hence (see Proposition 4.4) either \( q_X(L) \equiv -2 \) (mod 8) or \( q_X(L) \equiv -4 \) (mod 8). Let \( d \in \mathbb{Z} \) be such that
\[
q_X(L) = \begin{cases} 
-2 + 8d & \text{if } q_X(L) \equiv -2 \pmod{8}, \\
-4 + 8d & \text{if } q_X(L) \equiv -4 \pmod{8}.
\end{cases}
\] (4.6.1)

There exist a family \( f : \mathcal{X} \to T \) of HK manifolds over a connected base \( T \), points \( t_0, t_1 \in T \) and a line bundle \( \mathcal{L} \) on \( \mathcal{X} \) with the following properties:

(a) The fiber \( X_{t_0} := f^{-1}(t_0) \) is isomorphic to \( X \), and \( L_{t_0} := \mathcal{L}|_{X_{t_0}} \) is isomorphic to \( L \).
(b) The fiber $X_{t_1} := f^{-1}(t_1)$ is birational to $\overline{K}_v(J)$ where $J$ is a principally polarized abelian surface as in Subsection 4.3, $v = (0, 2h, -2)$ and

$$c_1(L_{t_1}) = \begin{cases} \pm (\alpha_v + 2(-d\alpha_v + d\gamma_v)) & \text{if } q_X(L) \equiv -2 \pmod{8}, \\ \pm (\beta_v - \alpha_v + 2(d\alpha_v + d\gamma_v)) & \text{if } q_X(L) \equiv -4 \pmod{8}. \end{cases}$$ \hspace{1cm} (4.6.2)

(Notation as in Subsection 4.3.)

Proof. First note that

$$\begin{align*}
(\alpha_v + 2(-d\alpha_v + d\gamma_v), \alpha_v + 2(-d\alpha_v + d\gamma_v)) &= -2 + 8d, \\
(\beta_v - \alpha_v + 2(d\alpha_v + d\gamma_v), \beta_v - \alpha_v + 2(d\alpha_v + d\gamma_v)) &= -4 + 8d,
\end{align*}$$

and that both $\alpha_v + 2(-d\alpha_v + d\gamma_v)$ and $\beta_v - \alpha_v + 2(d\alpha_v + d\gamma_v)$ have divisibility 2.

The lemma is a standard consequence of Verbitsky’s global Torelli Theorem, the monodromy computations of Mongardi-Rapagnetta and Proposition 4.10. We quickly go over the argument.

Let $X$ be a HK manifold of type OG6. By Theorem 1.4 in [MR21] the monodromy of $H^2(X; \mathbb{Z})$ is the group $O^+(H^2(X; \mathbb{R}))$ of isometries preserving a (continuous) choice of orientations of the maximal positive definite subspaces of $H^2(X; \mathbb{R})$ (an index 2 subgroup of the orthogonal group $O(H^2(X; \mathbb{Z}))$).

Let $\mathfrak{M}$ be the moduli space of marked HK manifolds of type OG6. By the result on monodromy quoted above, there are exactly 2 connected components of the (non Hausdorff) complex manifold $\mathfrak{M}$, interchanged by mapping $[(X, \varphi)]$ (here $\varphi: H^2(X; \mathbb{Z}) \rightarrow \Lambda_{\text{OG6}}$ is an isometry) to $[(X, -\varphi)]$. Let $\mathfrak{M}_0$ be one of the two connected components of $\mathfrak{M}$.

Let $\mathcal{P} \subset \mathcal{P}(\Lambda_{\text{OG6}} \otimes \mathbb{C})$ be the period domain, and let $\mathcal{P}: \mathfrak{M}_0 \rightarrow \mathcal{P}$ be the period map. Suppose that $[(X_1, \varphi_1)]$ and $[(X_2, \varphi_2)]$ belong to the same fiber of $\mathcal{P}$; then $X_1$ is birational to $X_2$ by Verbitsky’s global Torelli Theorem [Ver13], and $[(X_1, \varphi_1)], [(X_2, \varphi_2)]$ are non separated points in $\mathfrak{M}_0$. There is a Hausdorffization $\mathfrak{M}_0^\prime \rightarrow \mathfrak{M}_0$ with an induced period map $\mathcal{P}: \mathfrak{M}_0^\prime \rightarrow \mathcal{P}$ which is an isomorphism of complex manifolds.

Let $X$ be a HK manifold of type OG6 carrying a line bundle $L$. Let $\varphi: H^2(X; \mathbb{Z}) \rightarrow \Lambda_{\text{OG6}}$ be an isometry, and suppose that $[(X, \varphi)] \in \mathfrak{M}_0$. Then $\mathcal{P}(X, \varphi) = \varphi(c_1(L))^{-1}$. Conversely, if $a \in \Lambda_{\text{OG6}}$, and $[(X, \varphi)] \in \mathfrak{M}_0$ is such that $\mathcal{P}(X, \varphi) = a^\perp$, then $\varphi^{-1}(a)$ is the first Chern class of a line bundle.

Let $a, b \in \Lambda_{\text{OG6}}$ be primitive elements of divisibility 2 such that either $(a, a) = (b, b) = -2 + 8d$ or $(a, a) = (b, b) = -4 + 8d$. By Proposition 4.10 there exists $g \in O^+(\Lambda_{\text{OG6}})$ such that $g(a) = b$ or $g(a) = -b$. (Note that $O(\Lambda_{\text{OG6}})$ is the direct product of $O^+(\Lambda_{\text{OG6}})$ and $(-1)$.) Since the hyperplane $a^\perp \cap \mathcal{P}$ for a fixed $a$ as above is connected, the lemma follows by the monodromy result. (Recall the equalities in (4.6.3) and (4.6.4) and the sentence following those equations.)

Proof of Proposition 4.8. By Lemma 4.11 and by invariance of the commutator pairing under deformation (see Remark 2.13) and birational maps (see Remark 2.12), it suffices to prove that the commutator pairing is trivial for a line bundle $L$ on $\overline{K}_v(J)$ (notation as in Subsection 4.3, in particular $v = (0, 2h, -2)$) such that $c_1(L) = \alpha_v + 2(-d\alpha_v + d\gamma_v)$ or $c_1(L) = \beta_v - \alpha_v + 2(d\alpha_v + d\gamma_v)$. If the former holds there exists a line bundle $\xi$ on $\overline{K}_v(J)$ such that $L = \mathcal{F}_v^{-1} \otimes \xi \otimes \xi$, and if the latter holds there exists a line bundle $\xi$ on $\overline{K}_v(J)$ such that $L = \mathcal{L}_v \otimes \xi \otimes \xi$. (Here $\mathcal{F}_v$ as in (4.5.2) $\mathcal{L}_v$ is as in Proposition 4.5.) It follows that $e^\perp$ is trivial by Proposition 4.9 and because the square of a line bundle on a manifold $X$ of type OG6 has trivial commutator pairing (multiplication by 2 kills every element of $T(X)$).
4.7. Varieties of type OG6 corresponding to \( v = (0, 2h, 0) \). We let \( J \) be as in Subsection 4.3, in particular we assume that (4.3.1) holds. One can repeat all the constructions of that subsection with the Mukai vector \((0, 2h, -2)\) replaced by \( v = (0, 2h, 0) \). Thus we have \( M_v(J), \tilde{M}_v(J) \) and, upon choosing a symmetric principal polarization \( \Theta_J \) of \( J \), we have \( K_v(J) \subset M_v(J) \) and \( K_v(J) \subset \tilde{M}_v(J) \). We analyze \( K_v(J) \) in order to prove the validity of Theorem 4.1 for line bundles of divisibility 1.

By the isomorphism in (4.2.5), the Néron-Severi group of \( \tilde{K}_v(J) \) is freely generated by \( \theta_v(1, 0, 0), \theta_v(0, 0, 1) \) and \( \alpha_v \). We provide geometric descriptions of each of these classes. Proceeding exactly as in Subsection 4.3 one defines a map

\[
K_v(J) \xrightarrow{\pi_v} [\mathcal{F}] \quad \mapsto \quad \text{Supp}_{\text{det}}(\mathcal{F})
\]  

(4.7.1)

Composing with the desingularization map one gets the Lagrangian fibration

\[
\hat{\pi}_v: \tilde{K}_v(J) \rightarrow [2\Theta_J] \cong \mathbb{P}^3
\]  

(4.7.2)

Let \( \Lambda_v := c_1(\pi_v^*\mathcal{O}_{\mathbb{P}^3}(1)) \), and let \( \tilde{\Lambda}_v \) be the pull-back to \( \tilde{K}_v(J) \) of \( \Lambda_v \). Then

\[
\tilde{\theta}_v(0, 0, -1) = \tilde{\Lambda}_v
\]  

(4.7.3)

by the same result quoted in Subsection 4.3. Next, let \( \Theta_v \subset K_v(J) \) be the (reduced) divisor parametrizing sheaves \( \mathcal{F} \) such that \( h^0(\mathcal{F}) > 0 \) (note: this is a divisor because \( \chi(J, \mathcal{F}) = 0 \) for \([\mathcal{F}] \in K_v(J)\)), and let \( \hat{\theta}_v \) be its pull-back to \( \tilde{K}_v(J) \). Since \( \Theta_v \) is the zero locus of the canonical section of the determinant line bundle on \( K_v(J) \), we have \( \theta_v(1, 0, 0) = \text{cl}(\Theta_v) \). Thus

\[
\tilde{\theta}_v(1, 0, 0) = \text{cl}(\hat{\Theta}_v).
\]  

(4.7.4)

Lastly \( 2\alpha_v = \text{cl}(\tilde{\Sigma}_v) \) where \( \tilde{\Sigma}_v(J) \subset \tilde{K}_v(J) \) is the exceptional divisor of the desingularization map \( \tilde{K}_v(J) \rightarrow K_v(J) \), see Subsection 4.2. The conclusion is that we have a direct sum decomposition

\[
\text{NS}(\tilde{K}_v(J)) = \mathbb{Z}\text{cl}(\hat{\Theta}_v) \oplus \mathbb{Z}\tilde{\Lambda}_v \oplus \mathbb{Z}\alpha_v.
\]  

(4.7.5)

We record here the equalities (recall that \( \tilde{\theta}_v \) is an isometry of lattices)

\[
q(\text{cl}(\hat{\Theta}_v), \text{cl}(\hat{\Theta}_v)) = 0, \quad q(\text{cl}(\hat{\Theta}_v), \tilde{\Lambda}_v) = 1, \quad q(\tilde{\Lambda}_v, \tilde{\Lambda}_v) = 0.
\]  

(4.7.6)

4.8. Fourier-Mukai transform. Let \( \hat{J} = \text{Pic}^0(J) \) be the dual of \( J \), where \( J \) is as in Subsection 4.7, in particular (4.3.1) holds. Hence \( \text{NS}(\hat{J}) = \mathbb{Z}\hat{h} \), where \( \hat{h} \) is the unique principal polarization of \( \hat{J} \). Since \( \hat{J} \) is isomorphic to \( J \) all the definitions and considerations of Subsection 4.7 apply to \( \hat{J} \). More precisely, let \( \hat{v} := (0, 2\hat{h}, 0) \).

We have the moduli space \( M_{\hat{J}}(\hat{J}, \hat{h}) \), which we denote by \( M_{\hat{h}}(\hat{J}) \), and, upon choosing a symmetric principal polarization \( \Theta_{\hat{J}} \) of \( \hat{J} \), the singular symplectic variety \( K_{\hat{J}}(\hat{J}) \) and its HK desingularization \( \tilde{K}_{\hat{J}}(\hat{J}) \). On \( \tilde{K}_{\hat{J}}(\hat{J}) \) we have the divisor \( \hat{\Theta}_{\hat{J}} \) and the divisor classes \( \tilde{\Lambda}_{\hat{J}}, \alpha_{\hat{J}} \).

Of course an isomorphism \( J \rightarrow \hat{J} \) mapping \( \Theta_J \) to \( \Theta_{\hat{J}} \) determines an isomorphisms \( \tilde{K}_v(J) \rightarrow \tilde{K}_{\hat{J}}(\hat{J}) \), but we are interested in a different birational mapping \( \tilde{K}_v(J) \rightarrow \tilde{K}_{\hat{J}}(\hat{J}) \), given by a Fourier-Mukai transform. More precisely, let \( \mathcal{S} \) be the Poincaré line bundle on \( J \times \hat{J} \), and let \( \text{FM}: D^b(J) \rightarrow D^b(\hat{J}) \) be the Fourier-Mukai transform with kernel \( \mathcal{S} \).

Let

\[
U_v(J) := \{ [\mathcal{F}] \in K_v(J) \mid \mathcal{F} \text{ polystable, } H^0(J, \mathcal{F} \otimes \mathcal{L}_y) = 0 \text{ for general } y \in \hat{J} \}.
\]  

(4.8.1)
Lemma 4.12. Keeping notation as above, the following properties hold.

(A) Every sheaf $\mathcal{F}$ parametrized by a point of $U_v(J)$ satisfies $\text{WIT}_1$ with respect to the Poincaré line bundle $\mathcal{P}$, and hence $\text{FM}(\mathcal{F})$ is a sheaf shifted by $[-1]$.

(B) The complement of $U_v(J)$ has codimension at least 2 in $K_v(J)$.

(C) The set $\Sigma_v(J)$ of strictly semistable intersects $U_v(J)$.

(D) If $[\mathcal{F}] \in (U_v(J) \setminus \Sigma_v(J))$, the sheaf $\text{FM}(\mathcal{F})[1]$ is $\tilde{h}$-stable.

(E) If $[\mathcal{F}] \in \Sigma_v(J) \cap U_v(J)$, the sheaf $\text{FM}(\mathcal{F})[1]$ is strictly $\tilde{h}$-semistable.

Proof. Item (A) holds for $U_v(J)$ defined as in (4.8.1) because sheaves parametrized by $K_v(J)$ have one dimensional support.

We prove that Item (B) holds. Let $C \in [2\Theta_J]$ be a smooth curve. We claim that the complement of $U_v(J) \cap \pi_v^{-1}(C)$ (see (4.7.1)) in $\pi_v^{-1}(C)$ has codimension at least 2. In fact, let $f_C: C \to D$ be the quotient map for the $\mathbb{Z}/(2)$-action defined by multiplication by $(-1)$ in $J$. Thus $D$ is a smooth plane section of the Kummer surface associated to $J$ (see (4.5.3)). We have $f_C(\Theta_C) = \Theta_D + \lambda_D$ where $\lambda_D$ is a non trivial line bundle whose square is trivial. Let $[\mathcal{F}] \in \pi_v^{-1}(C)$, i.e. $\mathcal{F} = J_{C_\ast}(\xi)$ where $\xi$ is a line bundle of degree 4 on $C$ such that (here Chern classes are in the Chow ring)

$$0 = \sigma(c_2(J_{C_\ast}(\xi))) = \sigma(C \cdot C - J_{C_\ast}(c_1(\xi))) = -\sigma(J_{C_\ast}(c_1(\xi))).$$

(4.8.2)

The equalities in (4.8.2) give that $\xi = f_C^\ast(\zeta)$, where $\zeta$ is a line bundle on $D$ of degree 2. Let $y \in J[2]$, and let $\mathcal{L}_y$ be the corresponding line bundle on $J$; then $\mathcal{L}_y|J_C$ is invariant under multiplication by $(-1)$, and hence there exists a line bundle $\alpha_y$ on $D$ such that $\mathcal{L}_y|C \cong f_C^\ast(\alpha_y)$. Hence

$$H^0(C, \xi \otimes (\mathcal{L}_y|C)) = H^0(C, f_C^\ast(\zeta \otimes \alpha_y)) = H^0(D, (\xi \otimes \alpha_y) \otimes (\zeta \otimes \alpha_y \otimes \lambda_D)).$$

(4.8.3)

Let $\Theta_D \subset \text{Pic}^2(D)$ be the natural theta divisor. The equalities in (4.8.3) give that

$$\pi_v^{-1}(C)|U_v(J) \subset \bigcap_{y \in J[2]} ((\Theta_D + [\alpha_y]) + (\Theta_D + [\alpha_y \otimes \lambda_D])).$$

(4.8.4)

Since $\Theta_D$ is irreducible, it follows that $\pi_v^{-1}(C)|U_v(J)$ has codimension at least 2 in $\pi_v^{-1}(C)$. In order to finish the proof of Item (B) it suffices to prove the complement of $U_v(J)$ does not contain a divisor mapping to one of the two irreducible components of the discriminant hypersurface in $[2\Theta_J]$, i.e. the closure of the locus parametrizing curves $C \in [2\Theta_J]$ with a single node at a point of $J[2]$, and the locus parametrizing curves $C = \tau^+_x(\Theta_J) + \tau^+_{-x}(\Theta_J)$ where $x \in J$. This is easy, we leave details to the reader.

Items (C), (D) and (E) are straightforward.

Let $[\mathcal{F}] \in U_v(J)$. Since $u(\text{FM}(\mathcal{F})[1]) = (0, 2\tilde{h}, 0)$, the Fourier-Mukai transform defines a regular map $U_v(J) \to M_v(J)$. The image lies in an Albanese fiber (we mean the map in (4.3.2) with $J, v$ replaced by $\tilde{J}, \tilde{v}$ respectively) because every map from $K_v(J)$ to an abelian variety is constant. By considering the image of points in $\Sigma_v(J) \cap U_v(J)$ we get that it lands in $K_v(J)$. By Items (C) and (E) in Lemma 4.12 we get a a rational mapping

$$\varphi: \tilde{K}_v(J) \dashrightarrow \tilde{K}_v(J).$$

The map $\varphi$ is birational; the inverse is given by the “reverse” Fourier Mukai transform.

Proposition 4.13. Keeping notation as above, we have

$$\varphi^*(\text{cl}(\tilde{\Theta}_v)) = \tilde{\Lambda}_v, \quad \varphi^*(\tilde{\Lambda}_v) = \text{cl}(\tilde{\Theta}_v), \quad (\varphi^*\sigma_v = \sigma_v).$$

(4.8.5)
Proof. Since the open subset $U_v(J)$ has complement of codimension at least 2 in \( K_v(J) \), one has the equality $\varphi^* (\theta_v(r,xh,s)) = \theta_v(-s,xh,-r)$ by a well-known computation, see Lemma 3.1 in [Yos01]. This gives the first two equalities in (4.8.5). The third equality holds by Items (C) and (E) in Lemma 4.12. \( \square \)

4.9. Translations of $\tilde{K}_v(J)$ for $v = (0,2h,0)$. The definition of the action of $J[2] \times \tilde{J}[2]$ on $\tilde{K}_v(J)$ for $v = (0,2h,-2)$ extends verbatim to give an action of $J[2] \times \tilde{J}[2]$ on $\tilde{K}_v(J)$ for $v = (0,2h,0)$.

The proof that the group of automorphisms of a HK manifold of type OG6 is isomorphic to $\mathbb{Z}/(2)^5$ was achieved by examining $\tilde{K}_v(J)$ for $v = (0,2h,0)$. In fact Mongardi and Wandel proved the following result.

**Theorem 4.14** (Thm. 5.2 in [MW17]). Keep notation as above, in particular $v = (0,2h,0)$. Then the map $J[2] \times \tilde{J}[2] \to \text{Aut}^0(\tilde{K}_v(J))$ is an isomorphism of groups.

Here we recall one element in the proof of Theorem 4.14 because this gives us a chance to correct a statement in [MW17], and also to show that the error does not affect the truth of Theorem 4.14. Lemma 5.4 in op.cit. states that the linear system $|\tilde{\Theta}_v|$ consists of the single divisor $\tilde{\Theta}_v$. This statement is wrong. In fact, by the second equality in (4.8.5), the pull-back $\varphi^*$ gives an isomorphism

$$\varphi^*: H^0(K_v(J), \tilde{\Lambda}_v) \cong H^0(K_v(J), \tilde{\Theta}_v)$$

and $h^0(K_v(J), \tilde{\Lambda}_v) = h^0(K_v(J), \tilde{\Theta}_v) = 4$, see (4.7.1). The reason why the proof of Lemma 5.4 in [MW17] is wrong is that the divisor $\tilde{\Theta}_v$ has an open dense subset which is birational to a $\mathbb{P}^1$-fibration, but it fails to be normal, and hence the hypothesis of Lemma 2.5 in op.cit. is not satisfied by $\tilde{\Theta}_v$.

Lastly, although Lemma 5.4 in [MW17] is wrong, the statement of Theorem 4.14 is valid. In fact Mongardi-Wandel prove that if $g \in \text{Aut}^0(\tilde{K}_v(J))$ then there exists $\lambda_t$ such that $\lambda_t \circ g$ maps each (Lagrangian) fiber of $\tilde{J}$ to itself. Since $\lambda_t \circ g$ maps the line bundle $\mathcal{O}_{\tilde{K}_v(J)}(\tilde{\Theta}_v)$ to itself, the restriction of $\lambda_t \circ g$ to a smooth Lagrangian fiber $A_t$ is equal to the restriction of a translation by an element $y \in \text{Im}(\tilde{J}[2] \to \tilde{\Lambda}_v[2])$ (this holds because the polarization of $\tilde{\Theta}_v$ on $A_t$ has elementary divisors $(1,2,2)$). Clearly $y$ is independent of $t$, i.e. we have $g = \lambda_{-x} \circ \mu_y$.

4.10. The commutator pairing for line bundles of divisibility 1. In the present subsection we prove the part of the statement of Theorem 4.1 that refers to line bundles of divisibility 1.

**Proposition 4.15.** Let $X$ be a HK manifold of type OG6. Suppose that $L$ is a primitive line bundle on $X$ such that $\text{div}(L) = 1$.

1. If $q_X(L)$ is not divisible by 4 then $e^L$ is non degenerate.
2. If $q_X(L)$ is divisible by 4 then $\tilde{T}(X)/\text{Im}(E_L) \cong (\mathbb{Z}/(2))^4$.

Before proving Proposition 4.15 we describe the commutator pairing of the line bundles on $\tilde{K}_v(J)$ given by $\pi^*(\mathcal{O}_{\mathcal{P}}(1))$ and $\mathcal{O}_{\tilde{K}_v(J)}(\tilde{\Theta})$.

**Proposition 4.16.** Keep notation as in Subsections 4.7, 4.8 and 4.9. The commutator pairing of $\pi^*(\mathcal{O}_{\mathcal{P}}(1))$ has kernel $\tilde{J}[2]$ and is non degenerate on $J[2]$. Similarly, the commutator pairing of $\mathcal{O}_{\tilde{K}_v(J)}(\tilde{\Theta}_v)$ has kernel $\tilde{J}[2]$ and is non degenerate on $\tilde{J}[2]$.

**Proof.** Let us prove the first statement of the lemma. By (4.7.1) we have an identification

$$H^0(\tilde{K}_v, \pi^*(\mathcal{O}_{\mathcal{P}}(1)) \cong H^0(J, \mathcal{O}_J(2\Theta_J))^\vee$$

(4.10.1)
Since $\tilde{J}[2]$ acts trivially on $P(\mathcal{H}^0(J, \mathcal{O}_J(2\Theta_J))^{\vee})$, it follows that $\tilde{J}[2]$ is in the kernel of the commutator pairing of $\pi_\ast^\flat(\mathcal{O}_{\mathcal{P}^3}(1))$, see Remark 2.19. On the other hand, let $\mathcal{D}(J[2]) < \mathcal{D}(\pi_\ast^\flat(\mathcal{O}_{\mathcal{P}^3}(1)))$ be the inverse image of $J[2]$ under the natural homomorphism $\mathcal{D}(\pi_\ast^\flat(\mathcal{O}_{\mathcal{P}^3}(1))) \rightarrow \text{Aut}^0(\tilde{K}_v(J))$. Since the action of $\mathcal{D}(J[2])$ on $\mathcal{H}^0(J, \mathcal{O}_J(2\Theta_J))^{\vee}$ is identified with the action of the theta group of $\mathcal{O}_J(2\Theta_J)$ on $\mathcal{H}^0(J, \mathcal{O}_J(2\Theta_J))^{\vee}$, which is the Schrödinger representation, it follows that the commutator pairing of $\pi_\ast^\flat(\mathcal{O}_{\mathcal{P}^3}(1))$ is non degenerate on $J[2]$.

The second statement of the lemma follows from the first one (that we have just proved) because of the Fourier Mukai transform discussed in Subsection 4.8. In fact, the switching in Equation (4.8.5) gives a natural isomorphism

$$\varphi^\ast : H^0(\tilde{K}_v(J), \pi_\ast^\flat(\mathcal{O}_{\mathcal{P}^3}(1))) \cong H^0(\tilde{K}_v(J), \mathcal{O}_{\tilde{K}_v(J)}(\tilde{\Theta}_v))^\vee.$$  

Since the space of sections is not trivial, the commutator pairing can be recovered by the action of the theta group on the space of sections. The action as just been described, provided one keeps in mind that $J = \tilde{J}$.

\[ \Box \]

**Corollary 4.17.** With notation as above, let $L$ be a primitive line bundle on $\tilde{K}_v(J)$ such that

$$c_1(L) = a\hat{\Lambda}_v + b\text{cl}(\tilde{\Theta}_v).$$  

(1) If $a, b$ are both odd, i.e. $q(L)$ is not divisible by 4 (see (4.7.6)), then $e^L$ is non degenerate.

(2) If one among $a, b$ is even, i.e. $q(L)$ is divisible by 4, then $\tilde{T}(\tilde{K}_v(J))/\text{Im}(E_L)$ is isomorphic to $\mathbb{Z}/(2)^4$.

**Proof.** The result follows at once from Proposition 4.16, multiplicativity of the commutator pairing, and the fact that multiplication by 2 kills every element of $T(\tilde{K}_v(J))$. \[ \Box \]

The result below is analogous to Lemma 4.11. We omit the proof because it is analogous to the proof of Lemma 4.11.

**Lemma 4.18.** Let $X$ be a HK manifold of type OG6 carrying a primitive line bundle $L$ of divisibility 1. There exist a family $f : \mathcal{X} \rightarrow T$ of HK manifolds over a connected base $T$, points $t_0, t_1 \in T$ and a line bundle $\mathcal{L}$ on $\mathcal{X}$ with the following properties:

(a) The fiber $X_{t_0} := f^{-1}(t_0)$ is isomorphic to $X$, and $L_{t_0} := \mathcal{L}|_{X_{t_0}}$ is isomorphic to $L$.

(b) The fiber $X_{t_1} := f^{-1}(t_1)$ is birational to $\tilde{K}_v(J)$ where $J$ is a principally polarized abelian surface as in Subsection 4.7, $v = (0, 2h, a)$ and

$$c_1(L_{t_0}) = a\hat{\Lambda}_v + b\text{cl}(\tilde{\Theta}_v)$$  

for suitable $a, b$.

**Proof of Proposition 4.15.** The proposition follows at once from Corollary 4.17, Lemma 4.18, and invariance of the commutator pairing under deformation (see Remark 2.13) and birational maps (see Remark 2.12). \[ \Box \]

5. **The commutator pairing for certain rank 4 vector bundles**

5.1. **The computation.** We recall the setting of Theorem 1.3. Let $e$ be a positive integer such that $e \equiv -6 \pmod{16}$, and let $(M, h)$ be a general polarized HK fourfold of Kummer type with $q_M(h) = e$ and the divisibility of $h$ is 2. In [O’G22]
we have shown that there exists a slope stable rank 4 vector bundle \( \mathcal{F} \) on \( M \) such that
\[
\det \mathcal{F} \cong \mathcal{O}_M(h), \quad \Delta(\mathcal{F}) := 8c_2(\mathcal{F}) - 3c_1(\mathcal{F})^2 = c_2(M). \tag{5.1.1}
\]
We have also proved that \( g^*(\mathcal{F}) \cong \mathcal{F} \) for every \( g \in \text{Aut}^0(M) \), and hence the theta group \( \mathcal{G}(\mathcal{F}) \) is defined.

**Theorem 5.1.** Keeping notation as above, we have
\[
\widehat{T}(M)/\text{Im}(E_{\mathcal{F}}) \cong \begin{cases} 0 & \text{if } q_M(c_1(\mathcal{F})) \text{ is not divisible by 3}, \\ \left(\mathbb{Z}/(3)\right)^2 & \text{if } q_M(c_1(\mathcal{F})) \text{ is divisible by 3}. \end{cases} \tag{5.1.2}
\]

**Proof.** In \cite{OG22} we obtained the sheaves \( \mathcal{F} \) by deformation of certain stable modular sheaves on the generalized Kummer fourfold \( K_2(A) \) associated to an abelian surface \( A \). By invariance under deformation, it suffices to prove that (5.1.2) holds for \( M = K_2(A) \). We recall the definition of the modular sheaves on \( K_2(A) \). Let \( f: B \to A \) be a degree 2 homomorphism of abelian surfaces. By mapping a general \([Z] \in K_3(B)\) to \([f(Z)] \in K_2(A)\) one defines a rational map \( \rho: K_2(B) \dashrightarrow K_2(A) \) whose indeterminacy locus is equal to
\[
V(f) := \{[Z] \in K_2(B) \mid \ell(f(Z)) < \ell(Z) = 2\}. \tag{5.1.3}
\]
The blow up \( \nu: X \to K_2(B) \) of \( V(f) \) resolves the indeterminacies of \( \rho \). Thus we have the commutative diagram
\[
\begin{array}{ccc}
X & \xrightarrow{\nu} & K_2(B) \\
\downarrow & & \downarrow \rho \\
K_2(B) & \dashrightarrow & K_2(A)
\end{array}
\tag{5.1.4}
\]
For a line bundle \( \mathcal{L} \) on \( X \), we let \( \delta(\mathcal{L}) := \rho^*(\mathcal{L}) \). Now assume that \( \mathcal{L} = \nu^*(\overline{\mathcal{L}}) \), where \( \overline{\mathcal{L}} \) is a line bundle on \( K_2(B) \) such that \( c_1(\overline{\mathcal{L}}) = \mu_B(\omega_B) \), where \( \mu_B: H^2(B) \to H^2(K_2(B)) \) is the symmetrization map. We have
\[
c_1(\delta(\mathcal{L})) = 2\mu_A(\omega_A) - \delta_2(A) \tag{5.1.5}
\]
where \( \omega_A := f_*(\omega_B) \). There are assumptions on \( B \) and \( \omega_B \) which guarantee that \( \delta(\mathcal{L}) \) is a stable rank 4 vector bundle, that for a general deformation of \( K_2(A) \) which keeps the class of \( 2\mu_A(\omega_A) - \delta_2(A) \) of type (1, 1), the class is ample of square \( e = 16a - 6 \) and divisibility 2, that \( \delta(\mathcal{L}) \) extends to a (stable) vector bundle on such a general deformation, and that these vector bundles are stabilized by the group of automorphisms acting trivially on \( H^2 \), see Section 6 in \cite{OG22}. In particular, among the relevant assumptions we have that \( \omega_B \) is a primitive class and
\[
\omega_B \cdot \omega_B = 2a, \quad e = 16a - 6. \tag{5.1.6}
\]
The commutative diagram in (5.1.4) gives an identification
\[
H^0(K_2(B), \overline{\mathcal{L}}) \cong H^0(K_2(A), \delta(\mathcal{L})). \tag{5.1.7}
\]
Note that \( \overline{\mathcal{L}} \) is big and nef, hence \( H^0(K_2(B), \overline{\mathcal{L}}) \) is non trivial, in fact the formula in (3.2.4) gives that
\[
3 \cdot \binom{a + 2}{2} = h^0(K_2(B), \overline{\mathcal{L}}) = h^0(K_2(A), \delta(\mathcal{L})). \tag{5.1.8}
\]
The isomorphism \( B[3] \xrightarrow{\sim} A[3] \) induced by the homomorphism \( f: B \to A \) of degree 2 defines an isomorphism
\[
T(K_2(B)) = B[3] \xrightarrow{\sim} A[3] = T(K_2(A)) \tag{5.1.9}
\]
which commutes with the actions of the two groups on the two sides of the isomorphism in (5.1.7). Since we may read off the commutator pairing from the action of
the theta group on $H^0(K_2(A), \mathcal{E}(\mathcal{L}))$ (because it is non trivial), Theorem 3.2 gives that the commutator pairing is non degenerate if $a$ is not divisible by 3, and that $T(M)/\text{Im}(E)$ is isomorphic to $(\mathbb{Z}/(3))^2$ if $a$ is divisible by 3. By the last equality in (5.1.6), this proves the result. □

5.2. An example. We briefly discuss the first instance of non degenerate commutator pairing of Theorem 5.1. Let $(M, h)$ be a general polarized 4 dimensional HK variety of Kummer type with $q(M)(h) = 10$ and divisibility of $h$ equal to 2. Let $\mathcal{F}$ be a stable rank 4 vector bundle on $M$ as in Theorem 5.1. Then the commutator pairing $\varepsilon_{\mathcal{F}}$ is non degenerate, and hence $H^0(M, \mathcal{F})$, which has dimension 9 by (5.1.8), is the Schrödinger representation of $T(M) \cong (\mathbb{Z}/(3))^4$. Let

$$\mathbb{P}(\mathcal{F}^*) \xrightarrow{\phi_{\mathcal{F}}} \mathbb{P}(H^0(M, \mathcal{F})^*)$$

(5.2.1)

be the natural map. If the general fiber of $\phi_{\mathcal{F}}$ is finite, then the image of $\phi_{\mathcal{F}}$ is a hypersurface invariant for the Schrödinger representation.

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