A New Class of LRS Bianchi Type VI\(_0\) Universes with Free Gravitational Field and Decaying Vacuum Energy Density

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Abstract

A new class of LRS Bianchi type VI\(_0\) cosmological models with free gravitational fields and a variable cosmological term is investigated in presence of perfect fluid as well as bulk viscous fluid. To get the deterministic solution we have imposed the two different conditions over the free gravitational fields. In first case we consider the free gravitational field as magnetic type whereas in second case ‘gravitational wrench’ of unit ‘pitch” is supposed to be present in free gravitational field. The viscosity coefficient of bulk viscous fluid is assumed to be a power function of mass density. The cosmological constant \(\Lambda\) is found to be a decreasing function of time and positive which is corroborated by results from recent supernovae Ia observations. The physical and geometric aspects of the models are discussed.

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1 Introduction and Motivations

The problem of the cosmological constant is salient yet unsettled in cosmology. The smallness of the effective cosmological constant recently observed \((\Lambda_0 \leq 10^{-56}\text{cm}^{-2})\) poses the most difficult problems involving cosmology and elementary particle physics theory. To explain the striking cancellation between
the “bare” cosmological constant and the ordinary vacuum energy contributions of the quantum fields, many mechanisms have been proposed [1]. The “cosmological constant problem” can be expressed as the discrepancy between the negligible value of \( \Lambda \) for the present universe as seen by the successes of Newton’s theory of gravitation [2] whereas the values \( 10^{50} \) larger is expected by the Glashow-Salam-Weinberg model [3] and by grand unified theory (GUT) it should be \( 10^{107} \) larger [4]. The cosmological term \( \Lambda \) is then small at the present epoch simply because the universe is too old. The problem in this approach is to determine the right dependence of \( \Lambda \) upon \( R \) or \( t \).

Models with a relic cosmological constant \( \Lambda \) have received ample attention among researchers recently for various reasons (see Refs. [5] - [10] and references therein). Some of the recent discussions on the cosmological constant “problem” and on cosmology with a time-varying cosmological constant by Ratra and Peebles [11], Dolgov et al. [12] [13], Dolgov [14], and Sahni and Starobinsky [15] point out that in the absence of any interaction with matter or radiation, the cosmological constant remains a “constant”, however, in the presence of interactions with matter or radiation, a solution of Einstein equations and the assumed equation of covariant conservation of stress-energy with a time-varying \( \Lambda \) can be found. For these solutions, conservation of energy requires decrease in the energy density of the vacuum component to be compensated by a corresponding increase in the energy density of matter or radiation. Earlier researches on this topic, are contained in Zeldovich [16], Weinberg [1] and Carroll, Press and Turner [17]. Recent observations by Perlmutter et al. [18] and Riess et al. [19] strongly favour a significant and positive value of \( \Lambda \). Their findings arising from the study of more than 50 type Ia supernovae with redshifts in the range \( 0.10 \leq z \leq 0.83 \) suggest Friedmann models with negative pressure matter such as a cosmological constant (\( \Lambda \)), domain walls or cosmic strings (Vilenkin [20], Garnavich et al. [21]). Recently, Carmeli and Kuzmenko [22], Behar and Carmeli [23] have shown that the cosmological relativistic theory predicts the value for cosmological constant \( \Lambda = 1.934 \times 10^{-35}\text{s}^{-2} \). This value of “\( \Lambda \)” is in excellent agreement with the measurements recently obtained by the High-Z Supernova Team and Supernova Cosmological Project (Garnavich et al. [21], Perlmutter et al. [18], Riess et al. [19], Schmidt et al. [24]) The main conclusion of these observations is that the expansion of the universe is accelerating.

Several ansätze have been proposed in which the \( \Lambda \) term decays with time (see Refs. Gasperini [25], Berman [26], Berman et al. [27], Özer and Taha [7], Freese et al. [8], Peebles and Ratra [28], Chen and Wu [29], Abdussattar and Viswakarma [30], Gariel and Le Denmat [31], Pradhan [32], Pradhan et al. [33]). Of the special interest is the ansatz \( \Lambda \propto S^{-2} \) (where \( S \) is the scale factor of the Robertson-Walker metric) by Chen and Wu [29], which has been considered or modified by several authors (Abdel-Rahaman [34], Carvalho et al. [9], Silveira and Waga [10], Vishwakarma [35]).

In classical electromagnetic theory, the electromagnetic field has two inde-
The classification of the field is characterized by the property of the scalar $k^2 = (F_{ij}F^{ij})^2 + (\ast F_{ij}F^{ij})^2$. When $k = 0$ the field is said to be null and for any observer $|E| = |H|$ and $E.H = 0$ where $E$ and $H$ are the electric and magnetic vectors respectively. When $k \neq 0$, the field is non-null and there exists an observer for which $mE = nH$, $m$ and $n$ being scalars. It is easy to see that if $\ast F_{ij}F^{ij} = 0$, $F_{ij}F^{ij} \neq 0$ then either $E = 0$ or $H = 0$. These we call the magnetic and electric fields, respectively. If $F_{ij}F^{ij} = 0$, $\ast F_{ij}F^{ij} \neq 0$ then $E = \pm H$. This we call the ‘electromagnetic wrench’ with unit ‘pitch’. In this case Maxwell’s equations lead to an empty electromagnetic field with constant electric and magnetic intensities. In the case of gravitational field, the number of independent scalar invariants of the second order is fourteen. The independent scalar invariants formed from the conformal curvature tensor are four in number. In the case of Petrov type D space-times, the number of independent scalar invariants are only two, viz. $C_{hijl}C_{hijl}$ and $\ast C_{hijl}C_{hijl}$. Analogous to the electromagnetic case, the electric and magnetic parts of free gravitational field for an observer with velocity $v^i$ are defined as $E_{\alpha\beta} = C_{\alpha j\beta i}v^jv^l$ and $H_{\alpha\beta} = \ast C_{\alpha j\beta i}v^jv^l$ [36]. It is clear from the canonical form of the conformal form of the conformal curvature for a general type D space-time that there exists an observer for which $E_{\alpha\beta} = (\frac{n}{m})H_{\alpha\beta}$, where $m, n$ being integers and $m \neq 0$. The field is said to be purely magnetic type for $n = 0$, $m \neq 0$. In this case we have $E_{\alpha\beta} = 0$ and $H_{\alpha\beta} \neq 0$. The physical significance for the gravitational field of being magnetic type is that the matter particles do not experience the tidal force. When $m \neq 0$ and also $n \neq 0$, we call that there is a ‘gravitational wrench’ of unit ‘pitch’ $\frac{|mn|}{|mn|}$ in the free gravitational field [37]. If ‘pitch’ is unity then we have $E_{\alpha\beta} = \pm H_{\alpha\beta}$.

The space-time having a symmetry property is invariant under a continuous group of transformations. The transformation equations for such a group of order $r$ is given by

$$X^i = f^i(x^1, ..., x^r, a^1, ..., a^r)$$

which satisfy the differential equations

$$\frac{\partial X^i}{\partial x^\alpha} = \xi_{(\beta)}^i(X)A_\beta^\alpha(a), \quad (\alpha, \beta = 1, ..., r)$$

where $a^1, ..., a^r$ are $r$ essential parameters. The vectors $\xi^\alpha_i$ are the Killing vectors for the group $G_r$ of isometry satisfying the Killing’s equation

$$\xi_{(\alpha)i;j} + \xi_{(\alpha)j;i} = 0$$

A subspace of space-time is said to be the surface of transitivity of the group if any point of this space can be transformed to another point of it by the action of this group. A space-time is said to be spatially homogeneous if it admit a group $G_r$ of isometry which is transitive on three dimensional space-like hyper-surfaces. The group $G_3$ of isometry was first considered by [38] who obtained nine different types of isometry group known as the Bianchi types. The
space-time which admits $G_4$ group of isometry is known as locally rotationally symmetric (LRS) which always has a $G_3$ as its subgroup belonging to one of the Bianchi type provided this $G_3$ is simply transitive on the three dimensional hyper-surface $t =$ constant.

Considerable work has been done in obtaining various Bianchi type cosmological models and their inhomogeneous generalization. Barrow [39] pointed out that Bianchi VI$_0$ models of the universe give a better explanation of some of the cosmological problems like primordial helium abundance and they also isotropize in a special sense. Looking to the importance of Bianchi type VI$_0$ universes, many authors [40]−[44] have studied it in different context. Recently Bali et al. [45] have obtained some LRS Bianchi type VI$_0$ cosmological models imposing two types of conditions over the free gravitational fields.

In this paper we have revisited and generalized the solutions obtained by Bali et al. [45]. We have considered an LRS Bianchi type VI$_0$ space-time and obtained models with free gravitational field of purely ‘magnetic type’ and also in the presence of ‘gravitational wrench’ of unit ‘pitch’ in the free gravitational field. It is found that the ‘magnetic’ part of the free gravitational field induces shear in the fluid flow, which is zero in the case of a ‘electric’ type free gravitational field representing an unrealistic distribution in this case. This paper is organized as follows. The introduction and motivation are laid down in Sec. 1. The metric and the field equations are given in Sec. 2. In Sec. 3, solutions representing LRS Bianchi type VI$_0$ cosmological models with perfect fluid and bulk viscous fluid are obtained imposing the condition when the free gravitational field is purely magnetic type ($m \neq 0, n = 0$). In Sec. 4, we obtain the solution in presence of perfect fluid imposing the condition when there is a ‘gravitational wrench’ of unit ‘pitch’ in the free gravitational field i.e. $E_{\alpha\beta} = \pm H_{\alpha\beta}$. Discussion and concluding remarks are given in the last Sec 5.

2 The Metric and Field Equations

We consider an LRS Bianchi type VI$_0$ universe for which

$$ds^2 = \eta_{ab}\theta^a\theta^b,$$  \hspace{1cm} (4)

where $\theta^1 = A(t)dx$, $\theta^2 = B(t)\exp(x)dy$, $\theta^3 = B(t)\exp(-x)dz$, $\theta^4 = dt$.

The energy-momentum tensor for a perfect fluid distribution with comoving flow vector $v^i$ is given by

$$T^i_j = (\rho + p)v_iv^j + p\delta^i_j,$$  \hspace{1cm} (5)

where $v^i = \delta^i_4$, $\rho$ and $p$ being respectively, energy density and thermodynamic pressure of the fluid. Here we obtain

$$T^1_1 = T^2_2 = T^3_3 = p, T^4_4 = -\rho.$$  \hspace{1cm} (6)
The Einstein’s field equations (in gravitational units \( c = 1, \ G = 1 \)) read as

\[
R^i_j - \frac{1}{2} R \delta^i_j + \Lambda \delta^i_j = -8\pi T^i_j, \tag{7}
\]

for the line element (4) has been set up as

\[
\frac{2\dot{B}}{B} + \frac{\dot{B}^2}{B^2} + \frac{1}{A^2} + \Lambda = -8\pi p, \tag{8}
\]

\[
\frac{\ddot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{1}{A^2} + \Lambda = -8\pi p, \tag{9}
\]

\[
2\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} - \frac{1}{A^2} + \Lambda = 8\pi \rho. \tag{10}
\]

Here, and also in the following expressions a dot indicates ordinary differentiation with respect to \( t \).

The average scale factor \( S \) for LRS Bianchi type VI\(_0\) model is defined by

\[
S = (AB^2)^{\frac{2}{3}}. \tag{11}
\]

A volume scale factor is given by

\[
V = S^3 = (AB^2). \tag{12}
\]

The generalized mean Hubble parameter \( H \) is given by

\[
H = \frac{1}{3}(H_1 + H_2 + H_3), \tag{13}
\]

where \( H_1 = \frac{\dot{A}}{A}, \ H_2 = H_3 = \frac{\dot{B}}{B} \).

The expansion scalar \( \theta \) and shear scalar \( \sigma \) are obtained as

\[
\theta = v^i_i = \frac{\dot{A}}{A} + \frac{2\dot{B}}{B}, \tag{14}
\]

and

\[
\sigma = \frac{1}{\sqrt{3}} \left( \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right), \tag{15}
\]

respectively. The average anisotropy parameter is given by

\[
A_p = \frac{1}{3} \sum_{i=1}^{3} \left( \frac{\Delta H_i}{H} \right)^2, \tag{16}
\]

where \( \Delta H_i = H_i - H (i = 1, 2, 3) \).

The deceleration parameter \( (q) \) is defined as

\[
q = -\frac{\ddot{S}}{\dot{S}^2} \tag{17}
\]
The non-vanishing physical components of $C_{ijkl}$ for the line (18) are given by

$$C_{2323} = -\frac{1}{2} C_{3131} = C_{1212} = -C_{1414} = \frac{1}{2} C_{2424} = -C_{3434} =$$

$$= \frac{1}{6} \left[ \frac{2\ddot{A}}{A} - \frac{2\ddot{B}}{B} - \frac{2\dot{A}\dot{B}}{AB} + \frac{2\dot{B}^2}{B^2} + \frac{4}{A^2} \right],$$

(18)

$$C_{2314} = -\frac{1}{2} C_{3124} = C_{1234} = -\frac{1}{A} \left[ \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right].$$

(19)

Equations (8)-(10) are three relations in five unknowns $A$, $B$, $p$, $\rho$ and $\Lambda$. For complete solutions of equations (8)-(10), we need two extra conditions. The research on exact solutions is based on some physically reasonable restrictions used to simplify the Einstein equations. Although the distribution of matter at each point determines the nature of expansion in the model, the latter is also affected by the free gravitational field through its effect on the expansion, vorticity and shear in the fluid flow. A prescription of such a field may therefore be made on an a priori basis. The cosmological models of Friedman Robertson Walker, as well as the universe of Einstein - de Sitter, have vanishing free gravitational fields. In the following two cases we impose different conditions over the free gravitational field to find the deterministic solutions.

3 First Case: Free Gravitational Field is Purely Magnetic Type

3.1 Solution in Presence of Perfect Fluid

In this section we have extended the solution obtained by Bali et al. [45] by revisiting their solution. When free gravitational field is purely magnetic type ($m \neq 0, n = 0$), we have $H_{\alpha\beta} \neq 0$ and $E_{\alpha\beta} = 0$. From (18), we obtain

$$\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{\dot{B}^2}{B^2} + \frac{2}{A^2} = 0.$$  

(20)

Equation (8) together with (9) reduce to

$$\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{\dot{B}^2}{B^2} - \frac{2}{A^2} = 0.$$  

(21)

From Eqs. (20) and (21), we obtain two independent equations

$$\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} = 0,$$  

(22)

$$\frac{\dot{A}\dot{B}}{AB} - \frac{\dot{B}^2}{B^2} - \frac{2}{A^2} = 0.$$  

(23)
Let $\frac{\dot{A}}{A} = U$. Then Eqs. (22) and (23) take the form

$$UU_{\xi \xi} + UU_{\xi} \dot{A} - 2U_{\xi}^2 = 0,$$

(24)

and

$$U_{\xi}^2 - UU_{\xi} \dot{A} + 2U^2 = 0,$$

(25)

respectively, where $\xi$ is defined by

$$\frac{d}{dt} = \frac{1}{A} \frac{d}{d\xi}.$$

(26)

Eqs. (24) and (25) lead to

$$UU_{\xi \xi} - U_{\xi}^2 + 2U^2 = 0.$$  

(27)

On substituting $U = \exp (\mu)$, the above second order differential equation reduces to the form

$$\mu_{\xi \xi} + 2 = 0,$$

(28)

which gives

$$\mu = k_1 \xi - \xi^2 + k_2,$$

(29)

where $k_1$ and $k_2$ are arbitrary constants. Hence, we obtain

$$U = k_3 \exp [-T(T - k_1)],$$

(30)

where $T$ stands for $\xi$. Therefore, from Eqs. (23), (25) and (30), we obtain

$$A = \frac{k_4 \exp [-T(T - k_1)]}{(k_1 - 2T)},$$

(31)

$$B = \frac{k_5}{(k_1 - 2T)},$$

(32)

where $T$ is given by

$$\frac{dT}{dt} = \frac{1}{k_4} (k_1 - 2T) \exp [T(T - k_1)],$$

(33)

$k_3$, $k_4$ and $k_5 (= \frac{k_4}{k_3})$ being arbitrary constants. Therefore the geometry of the universe (4) reduces to the form

$$ds^2 = \frac{k_3^2 \exp [-2T(T - k_1)]}{(k_1 - 2T)^2} \left[ -dT^2 + dx^2 + \frac{\exp [2T(T - k_1)]}{k_3^2} \right]

\{ \exp (2x)dy^2 + \exp (-2x)dz^2 \}.$$

(34)

The expressions for pressure $p$ and density $\rho$ for the model (34) are given by

$$8\pi p = -\frac{3}{k_3^2} [4 - (k_1 - 2T)^2] \exp [2T(T - k_1)] - \Lambda(T),$$

(35)
\[ 8\pi \rho = \frac{3}{k_4^2} \left[ 4 + (k_1 - 2T)^2 \right] \exp \left[ 2T(T - k_1) \right] + \Lambda(T), \quad (36) \]

For the specification of \( \Lambda(T) \), we assume that the fluid obeys an equation of state of the form

\[ p = \gamma \rho, \quad (37) \]

where \( \gamma (0 \leq \gamma \leq 1) \) is a constant.

Using (37) in (35) and (36), we obtain

\[ 8\pi (1 + \gamma) \rho = \frac{6}{k_4^2} (k_1 - 2T)^2 \exp \left[ 2T(T - k_1) \right] \quad (38) \]

Eliminating \( \rho \) between Eqs. (36) and (38), we obtain

\[ (1 + \gamma) \Lambda = -\frac{12(1 + \gamma)}{k_4^2} \exp \left[ 2T(T - k_1) \right] + \frac{3(1 - \gamma)}{k_4^2} (k_1 - 2T)^2 \exp \left[ 2T(T - k_1) \right] \quad (39) \]

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{energy_density.png}
\caption{The plot of energy density \( \rho(t) \) Vs. time}
\end{figure}

From Eq. (33), we observe that \( \rho(t) \) is a decreasing function of time and \( \rho > 0 \) for all times. Figure 1 shows this behaviour of energy density. From Eq. (39), we note the cosmological term \( \Lambda \) is a decreasing function of time and it approaches a small positive value with increase in time. From Figure 2 (\( \rho \) and \( \lambda \) are in geometrical units in entire paper), we note the same character of \( \Lambda \). This
Figure 2: The plot of cosmological term $\Lambda(t)$ Vs. time

is to be taken as a representative case of physical viability of the model.

The behaviour of the universe in this model will be determined by the cosmological term $\Lambda$, this term has the same effect as a uniform mass density $\rho_{\text{eff}} = -\Lambda/4\pi$ which is constant in space and time. A positive value of $\Lambda$ corresponds to a negative effective mass density (repulsion). Hence, we expect that in the universe with a positive value of $\Lambda$ the expansion will tend to accelerate whereas in the universe with negative value of $\Lambda$ the expansion will slow down, stop and reverse. In a universe with both matter and vacuum energy, there is a competition between the tendency of $\Lambda$ to cause acceleration and the tendency of matter to cause deceleration with the ultimate fate of the universe depending on the precise amounts of each component. This continues to be true in the presence of spatial curvature, and with a nonzero cosmological constant it is no longer true that the negatively curved (“open”) universes expand indefinitely while positively curved (“closed”) universes will necessarily recollapse - each of the four combinations of negative or positive curvature and eternal expansion or eventual recollapse become possible for appropriate values of the parameters. There may even be a delicate balance, in which the competition between matter and vacuum energy is needed drawn and the universe is static (non expanding). The search for such a solution was Einstein’s original motivation for introducing the cosmological constant. Recent cosmological observations (Garnavich et al. [21], Perlmutter et al. [18], Riess et al. [19], Schmidt et al. [24]) suggest the existence of a positive cosmological constant $\Lambda$ with the magnitude
Λ(G \bar{h}/c^3) \approx 10^{-123}. These observations on magnitude and red-shift of type Ia supernova suggest that our universe may be an accelerating one with induced cosmological density through the cosmological Λ-term. Thus the nature of Λ in our derived model of the universe is consistent with recent observations (Garnavich et al. [21], Perlmutter et al. [18], Riess et al. [19], Schmidt et al. [24]).

Some Physical and Geometric Features of the Model

The expressions for kinematical parameters i.e. the scalar of expansion θ, shear scalar σ, average scale factor S, proper volume V^3 and average anisotropy parameter A_p for the model (36) are given by

\[ θ = \frac{1}{k_4^4} \left[ 6 + (k_1 - 2T)^2 \right] \exp [T(T - k_1)], \] (40)

\[ σ = \frac{1}{k_3} \sqrt{3} \left( k_1 - 2T \right)^2 \exp [T(T - k_1)], \] (41)

\[ S = \frac{k_1^4 k_3^2}{(k_1 - 2T) \exp \frac{T(T - k_1)}{3}}, \] (42)

\[ V^3 = \sqrt{-g} = \frac{k_1^4 \exp \left[ -2T(T - k_1) \right]}{k_3^2 (k_1 - 2T)^4}, \] (43)

\[ A_p = \frac{4}{3}. \] (44)

The directional Hubble’s parameters H_x, H_y and H_z are given by

\[ H_x = \frac{1}{k_4^4} \left[ (k_1 - 2T)^2 + 2 \right] \exp [T(T - k_1)], \] (45)

\[ H_y = H_z = \frac{2}{k_4} \exp [T(T - k_1)], \] (46)

where the mean Hubble’s parameter is given by

\[ H = \frac{1}{3k_4^4} \left( 6 + (k_1 - 2T)^2 \right) \exp [T(T - k_1)] \] (47)

The model (36) starts expansion with a big-bang singularity from T = −∞ and it goes on expanding till \( T = \frac{2\sqrt{3}}{k_3} \) respectively correspond to the cosmic time \( t = 0 \) and \( t = \infty \). The is found to be realistic everywhere in this time interval for \( \Lambda > -\frac{12}{k_3^2} \exp \left( -\frac{k_3^2}{2} \right) \). The model behaves like a steady-state de-Sitter type universe at late times where the physical and kinematic parameters ρ, p, θ tend to a finite value, however shear vanishes there. The model has a point type singularity at time \( T = k_1 \). The singular behaviour may be close to cosmic origin or outside the evolution. The average anisotropy parameter \( A_p \) remains uniform and isotropic through out the evolution of the universe. This would

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depend on physical properties of matter and radiation. This may need detailed study to make better quantifiable view. From figure 3, it can be seen that in the early stages of the universe, i.e., \( t = 0 \), the scale factor of the universe had been approximately constant and had increased very slowly. At specific time the universe had exploded suddenly and expanded to large scale. This is good matching with big bang scenario. This is indicated in first part (top) of figure 3. Later singular behaviour depends on \((k_1, T)\).

### 3.2 Solutions For Bulk Viscous Fluid

Astronomical observations of large-scale distribution of galaxies of our universe show that the distribution of matter can be satisfactorily described by a perfect fluid. But large entropy per baryon and the remarkable degree of isotropy of the cosmic microwave background radiation, suggest that we should analyze dissipative effects in cosmology. Further, there are several processes which are expected to give rise to viscous effect. These are the decoupling of neutrinos during the radiation era and the recombination era [46], decay of massive super string modes into massless modes [47], gravitational string production [48, 49] and particle creation effect in grand unification era [50]. It is known that the introduction of bulk viscosity can avoid the big bang singularity. Thus, we should consider the presence of a material distribution other than a perfect fluid to have realistic cosmological models (see Grøn [51] for a review on cosmological models with bulk viscosity). A uniform cosmological model filled with fluid
which possesses pressure and second (bulk) viscosity was developed by Murphy [52]. The solutions that he found exhibit an interesting feature that the big bang type singularity appears in the infinite past.

In presence of bulk viscous fluid distribution, we replace isotropic pressure $p$ by effective pressure $\bar{p}$ in Eq. (35) where

$$\bar{p} = p - \xi v^i_i,$$  \hspace{1cm} (48)

where $\xi$ is the coefficient of bulk viscosity.

The expression for effective pressure $\bar{p}$ for the model Eq. (34) is given by

$$8\pi\bar{p} = 8\pi(p - \xi v^i_i) = -\frac{3}{k^4} [4 - (k_1 - 2T)^2] \exp [2T(T - k_1)] - \Lambda(T).$$  \hspace{1cm} (49)

Thus, for given $\xi(t)$ we can solve for the cosmological parameters. In most of the investigation involving bulk viscosity is assumed to be a simple power function of the energy density (Pavon [53], Maartens [54], Zimdahl [55], Santos [56])

$$\xi(t) = \xi_0 \rho^n,$$  \hspace{1cm} (50)

where $\xi_0$ and $n$ are constants. For small density, $n$ may even be equal to unity as used in Murphy’s work [52] for simplicity. If $n = 1$, Eq. (24) may correspond to a radiative fluid (Weinberg [2]). Near the big bang, $0 \leq n \leq \frac{1}{2}$ is a more appropriate assumption (Belinskii and Khalatnikov [57]) to obtain realistic models.

For simplicity sake and for realistic models of physical importance, we consider the following two cases ($n = 0, 1$):

### 3.2.1 Model I: When $n = 0$

When $n = 0$, Eq. (50) reduces to $\xi = \xi_0 = \text{constant}$. With the use of Eqs. (36), (37) and (40), Eq. (49) reduces to

$$8\pi\rho = 8\pi(1 + \gamma)\rho = \frac{6}{k^4} (k_1 - 2T)^2 \exp [2T(T - k_1)] +$$

$$\frac{8\pi\xi_0}{k^4} \{ 6 + (k_1 - 2T)^2 \} \exp [2T(T - k_1)].$$  \hspace{1cm} (51)

Eliminating $\rho(t)$ between Eqs. (36) and (51), we obtain

$$(1 + \gamma)\Lambda = -\frac{12(1 + \gamma)}{k^4} \exp [2T(T - k_1)] + \frac{3(1 - \gamma)}{k^4} (k_1 - 2T)^2 \exp [2T(T - k_1)]$$

$$+ \frac{8\pi\xi_0}{k^4} \{ 6 + (k_1 - 2T)^2 \} \exp [T(T - k_1)].$$  \hspace{1cm} (52)
Figure 4: The plot of energy density $\rho(t)$ Vs. time

Figure 5: The plot of cosmological term $\Lambda(t)$ Vs. time
Figure 6: The plot of energy density $\rho(t)$ Vs. time

Figure 7: The plot of cosmological term $\Lambda(t)$ Vs. time
3.2.2 Model II: When \( n = 1 \)

When \( n = 1 \), Eq. (50) reduces to \( \xi = \xi_0 \rho \). With the use of Eqs. (36), (37) and (40), Eq. (49) reduces to

\[
\rho = \frac{3(k_1 - 2T)^2 \exp [2T(T - k_1)]}{4 \pi k_4 [k_4 (1 + \gamma) - \xi_0 \{6 + (k_1 - 2T)^2 \} \exp [T(T - k_1)]]}
\] (53)

Eliminating \( \rho(t) \) between Eqs. (36) and (53), we obtain

\[
\Lambda = \frac{12}{k_4^2} \exp [2T(T - k_1)] - \frac{3(k_1 - 2T)^2 \exp [2T(T - k_1)]}{k_4^2 [k_4 (\gamma + 1) - \xi_0 \{6 + (k_1 - 2T)^2 \} \exp [T(T - k_1)]]} \times \\
\left[ k_4 (\gamma - 1) - \xi_0 \{6 + (k_1 - 2T)^2 \} \exp [T(T - k_1)] \right]
\] (54)

where \( k_1, k_4, \xi_0, \) and \( \gamma (0 \leq \gamma \leq 1) \) are constants.

From Eqs. (51) and (53), we observe that \( \rho(t) \) in both models are a decreasing function of time and \( \rho > 0 \) for all times. Figures 4 and 6 show this behaviour of energy density. From Eqs. (52) and (54), we see that the cosmological terms \( \Lambda \) in both models are a decreasing function of time and they approach a small positive value at late time. From Figures 5 and 7, we note this behaviour of cosmological term \( \Lambda \) in both models I and II. Thus, our models are consistent with the results of recent observations (Garnavich et al. [21], Perlmutter et al. [18], Riess et al. [19], Schmidt et al. [24]).

The effect of bulk viscosity is to produce a change in perfect fluid and therefore exhibits essential influence on the character of the solution. A comparative inspection of Figures 1, 4, 6 show apparent evolution of time due to perfect fluid and bulk viscous fluid. It is apparent that the vacuum energy density \( \rho \) decays much fast in later case. Also shows effect of uniform viscosity model and linear viscosity model. Even in these case decay of vacuum energy density is much faster than uniform. So the coupling parameter \( \xi_0 \) would be related with physical structure of the matter and provides mechanism to incorporate relevant property. In order to say more specific, detailed study would be needed which would be reported in future. Similar behaviour is observed for the cosmological constant \( \Lambda \). We also observe here that Murphy’s [52] conclusion about the absence of a big bang type singularity in the infinite past in models with bulk viscous fluid in general, is not true. The results obtained by Myung and Cho [58] also show that, it is not generally valid since for some cases big bang singularity occurs in finite past. For both models, it is observed that the effect of viscosity prevents the shear and the free gravitational field from withering away.
4 Solution of Field Equations for Second Case

In this case there is a ‘gravitational wrench’ of unit ‘pitch’ in the free gravitational field i.e. $E_{\alpha\beta} = \pm H_{\alpha\beta}$. Therefore we have

$$E_{\alpha\beta} = \kappa H_{\alpha\beta}, \quad \kappa^2 = 1.$$  \hfill (55)

In this case, Bali et al. [45] have investigated the solution given by

$$ds^2 = \left(\frac{2\tau^2 + 3\kappa\tau + 2}{\tau^2}\right)^\frac{\nu}{2} \exp \left(\frac{3\kappa}{\sqrt{7}} \tan^{-1} \left(\frac{4\tau + 3\kappa}{\sqrt{7}}\right)\right) \left[-\frac{d\tau^2}{(2\tau^2 + 3\kappa\tau + 2)^{\frac{3}{2}}} + dx^2 + (2\tau^2 + 3\kappa\tau + 2)^{\frac{1}{2}} \exp \left(-3\kappa \sqrt{7} \tan^{-1} \left(\frac{4\tau + 3\kappa}{\sqrt{7}}\right)\right) \{e^{2r} dy^2 + e^{-2r} dz^2\}\right].$$  \hfill (56)

The expressions for pressure $p$ and energy density $\rho$ for the model (56) are obtained as

$$8\pi p = \frac{12\kappa^3 + 29\tau^2 + 72\kappa\tau - 48}{4(2\tau^2 + 3\kappa\tau + 2)^{\frac{3}{2}}} \exp \left(-3\kappa \sqrt{7} \tan^{-1} \left(\frac{4\tau + 3\kappa}{\sqrt{7}}\right)\right) - \Lambda(\tau)$$  \hfill (57)

$$8\pi \rho = \frac{12\kappa^3 + 39\tau^2 + 72\kappa\tau + 48}{4(2\tau^2 + 3\kappa\tau + 2)^{\frac{3}{2}}} \exp \left(-3\kappa \sqrt{7} \tan^{-1} \left(\frac{4\tau + 3\kappa}{\sqrt{7}}\right)\right) + \Lambda(\tau)$$  \hfill (58)

For the specification of $\Lambda(\tau)$, we assume that the fluid obeys an equation of state of the form [37]. Using Eqs. (37) in (57) and (58), we obtain

$$8\pi(1 + \gamma)\rho = \frac{6\kappa^3 + 17\tau^2 + 36\kappa\tau}{(2\tau^2 + 3\kappa\tau + 2)^{\frac{3}{2}}} \exp \left(-3\kappa \sqrt{7} \tan^{-1} \left(\frac{4\tau + 3\kappa}{\sqrt{7}}\right)\right).$$  \hfill (59)

Eliminating $\rho$ between (57) and (59), we obtain

$$\Lambda = -\left[\frac{12(\gamma - 1)\kappa(\tau^2 + 6)\tau + T^2(39\gamma - 29) + 48(\gamma + 1)}{4(\gamma + 1)(2T^2 + 3\kappa T + 2)^{\frac{3}{2}}} \exp \left(-3\kappa \sqrt{7} \tan^{-1} \left(\frac{4T + 3\kappa}{\sqrt{7}}\right)\right)\right].$$  \hfill (60)

From preliminary study, we find that both energy density and cosmological constant are negative. Hence it will not be studied. It also shows singular behaviour in energy density at later stage of the evolution. Hence, the model is unphysical for further study.

5 Discussion and Concluding Remarks

In this paper, we have studied properties of the free gravitational field and their invariant characterizations and obtained LRS Bianchi type VI$_0$ cosmological models imposing different conditions on the free gravitational field. In first case,
where the gravitational field is purely magnetic type, we observe that the energy density $\rho$ and cosmological constant $\Lambda$ are well behaved. Also the effect of bulk viscous fluid distribution in the universe is compared with perfect fluid model. We observe that due to presence of bulk viscous fluid, the rate of decrease in energy density is faster compared to perfect fluid model. The linear relation of the coefficient of bulk viscosity with mass density provides further enhanced decrease rate (see figures 1, 4, 6). Since we had considered extension of the various models, detailed physical parameter study would be reported in future. Important incorporation is time-dependence of cosmological constant $\Lambda$. The similar enhanced decrease rate is also observed for $\Lambda$ (see figures 2, 5, 7). The scale factor dependence is also reported. The nature of $\Lambda$ in our derived models of the universe is found to be consistent with recent observations (Garnavich et al. [21], Perlmutter et al. [18], Riess et al. [19], Schmidt et al. [24]).

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