Generation of vortices and stabilization of vortex lattices in holographic superfluids

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Abstract

Within the simplest holographic superfluid model and without any ingredients put by hand, it is shown that vortices can be generated when the angular velocity of rotating superfluids exceeds certain critical values, which can be precisely determined by linear perturbation analyses (quasinormal modes of the bulk AdS black brane). These vortices appear at the edge of the superfluid system first, and then automatically move into the bulk of the system, where they are eventually stabilized into certain vortex lattices. For the case of 18 vortices generated, we find (at least) five different patterns of the final lattices formed due to different initial perturbations, which can be compared to the known result for such lattices in weakly coupled Bose-Einstein condensates from free energy analyses.

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I. INTRODUCTION

Vortices play a central role in non-trivial dynamics of superfluids. The study of vortex generation and vortex lattice formation in rapidly rotating superfluids, both experimentally and theoretically, has a long history, from that in the old liquid helium-4 or helium-3 to that in the modern Bose-Einstein condensates (BEC) or superfluid ultracold fermions. Here we will not provide a review of the history of such study, but only refer the readers to a rather incomplete list of references \[1\] on this topic. Most of the related research works are based on the Gross-Pitaevskii (GP) equation \[2,4\] and its generalizations, which is a mean field description of weakly coupled BEC at zero temperature. For example, formation of vortex lattices are studied in \[5,6\]. This GP equation based method is relatively simple, and can reveal some features of vortex generation and vortex lattice formation observed in experiments. However, we can also see some disadvantages of this method. First of all, there is no dissipation in the original GP equation, so in order to yield the dissipation effect at finite temperature a classical thermal cloud has to be introduced. In the numerical simulations by this method, vortices seem to nuclear in the thermal cloud and then enter the BEC to form the lattice. But an explanation how the classical vortices transform to the quantum objects is absent. Second, the GP equation can describe the weakly coupled BEC well, but is not expected to be a proper description of liquid helium systems, which have strong interparticle interactions, as well as general ultracold fermion systems.\[1\]

It is thus valuable to look for alternative methods to cope with the problem of vortex generation and vortex lattice formation. Applied AdS/CFT duality, or holography, just provides one of such alternative methods, which has been proved as a very useful framework to study superfluid physics at finite temperature (see e.g. \[7,8\]). Holography equates certain quantum systems without gravity to gravitational systems in a curved spacetime with one additional spatial dimension \[9,11\]. It describes strongly coupled systems by design and naturally incorporates dissipation by including a black hole on the gravity side. Most interestingly, even the simplest holographic superfluid model \[13\] shows some features of superfluid ultracold fermions \[14,15\]. All the above facts make holography a very promising tool to describe superfluids other than the zero temperature weakly coupled BEC.

In this paper, we use the simplest holographic superfluid model to study the vortex gener-

\[1\] The GP equation can be a good description of superfluid ultracold fermion systems in their BEC limit \[12\].
ation and vortex lattice formation in rapidly rotating superfluids. Without introducing any additional ingredients, we can see vortices generated when the angular velocity of rotating superfluids exceeds certain critical values. Actually, we first investigate the critical angular velocity from the linear instability indicated by critical quasi-normal modes (QNM) of the bulk AdS black brane. Next, in the nonlinear time evolution, it is shown that the vortices appear at the edge of the superfluid system at first, and then automatically move into the bulk of the system, where they are eventually stabilized into certain vortex lattices. In particular, we carefully examine the final configurations when there are in total 18 vortices generated. In this case, we find (at least) five different patterns of the final lattices formed due to different initial perturbations, which can be compared to the known result for such lattices in weakly coupled Bose-Einstein condensates from free energy analyses.

This paper is organized as follows. In the next section, we briefly introduce the holographic superfluid model that will be used. In Sec. III we construct our initial configurations and consider the linear instability from the QNM on this initial background. Then, in Sec. IV the dynamical process of vortex generation and vortex lattice formation is studied numerically by the nonlinear time evolution, and the final vortex lattices with 18 vortices are investigated. Finally, the conclusion and some discussion are provided in Sec. V.

II. HOLOGRAPHIC SETUP

A. Action of the model

The simplest holographic model to describe superfluids is given in [13], which consists of a complex scalar field $\Psi$ coupled to a U(1) gauge field $A_M$ in the (3+1)D gravity with a cosmological constant $\Lambda = -3/L^2$. The action is:

$$S = \int_M \sqrt{-g} d^4x \left[ \frac{1}{2\kappa^2} \left( R + \frac{6}{L^2} \right) - \frac{1}{e^2} \left( \frac{1}{4} F^2 + |D\Psi|^2 + m^2 |\Psi|^2 \right) \right],$$

where $D_M\Psi = (\nabla_M - iA_M)\Psi$, and $\kappa^2 = 8\pi G$ with $G$ Newton’s gravitational constant.

A simple way to solve the model is considering the probe limit, where backreaction of the matter fields is neglected. To do so, we take the limit $e \to \infty$. As a result, the matter fields live in a bulk spacetime that is fixed to be the Schwarzschild-AdS balck brane

$$ds^2 = \frac{L^2}{z^2} \left( -f(z) dt^2 + \frac{1}{f(z)} dz^2 + dx^2 + dy^2 \right),$$
for the finite temperature system, where \( f(z) = 1 - (z/z_h)^3 \) is the blackening factor and \( L \) the AdS radius. The corresponding Hawking temperature is

\[
T = \frac{3}{4\pi z_h},
\]

which is identified with the temperature for the boundary system. The equations of motion are

\[
D^M D_M \Psi - m^2 \Psi = 0,
\]

\[
\nabla_M F^{MN} = j^N,
\]

with \( j^M = i (\Psi^* D^M \Psi - \Psi D^M \Psi^*) \).

Without losing generality, we choose \( m^2 L^2 = -2 \), so the asymptotic behaviors of the matter fields are

\[
\Psi = \frac{z}{L} (\psi_- + z\psi_+ + \cdots),
\]

\[
A_t = \mu - z\rho + \cdots.
\]

In addition, the axial gauge \( A_z = 0 \) is adopted in what follows. According to the dictionary of AdS/CFT duality, \( \mu \) and \( \rho \) are the chemical potential and particle number density of the boundary system, respectively. To describe superfluidity as a spontaneous U(1) symmetry breaking, we will switch off the source \( \psi_- \), and then \( \psi_+ \) corresponds to the condensate (expectation value of the dual operator).

The boundary system is parametrized by \( T/\mu \). When \( T \) is fixed, a phase transition happens at a critical chemical potential \( \mu_c \), above which \( \psi_+ \) does not vanish and the system is in the superfluid phase[13]. We will work in the superfluid phase, so in the following we will choose \( \mu > \mu_c \).

III. INITIAL CONFIGURATIONS AND QUASI-NORMAL MODES

A. Initial configurations

We will work in polar coordinates, so metric of the Schwarzschild-AdS balck brane [2] becomes

\[
ds^2 = \frac{L^2}{z^2} \left( -f(z) dt^2 + \frac{1}{f(z)} dz^2 + dr^2 + r^2 d\theta^2 \right). \tag{8}
\]
For numerical simplicity, we set $z_h = 1$ and $L = 1$ from now on. Hence, $\mu_c = 4.06$ in our numerical computations\cite{13}. We will prepare a rotating BEC as our initial state, which is implemented by taking all fields real and independent of $\theta$:

$$ \Psi (z, r) = \bar{\Psi} (z, r), \quad A_\mu = A_\mu (z, r). \quad (9) $$

With ansatz given above, we can choose $A_r$ to be 0, which actually requires no radial current in the bulk\cite{16}. Then the simplified equations of motion are

\begin{align*}
\partial_z (f \partial_z \psi) + \partial_r^2 \psi + \frac{1}{r} \partial_r \psi + \left( \frac{A_t^2}{f} - \frac{A_\theta^2}{r^2} - z \right) \psi &= 0, \\
 f \partial_z^2 A_t + \partial_r^2 A_t + \frac{1}{r} \partial_r A_t - 2 A_t \psi^2 &= 0, \\
 \partial_z (f \partial_z A_\theta) + \partial_r^2 A_\theta - \frac{1}{r} \partial_r A_\theta - 2 A_\theta \psi^2 &= 0,
\end{align*}

(10) (11) (12)

where we have made the replacement $\Psi = z \psi$.

At the AdS conformal boundary $z = 0$, the boundary conditions

$$ \psi \big|_{z=0} = 0, \quad A_t \big|_{z=0} = \mu, \quad (13) $$

are as usual with $\mu$ the chemical potential. For the superfluid system, a rigid rotation can be introduced by letting

$$ A_\theta \big|_{z=0} = \Omega r^2, \quad (14) $$

where $\Omega$ is the angular velocity. The system should have a finite size in the $r$ direction, i.e. the radius. In what follows, we use $L$ to denote this radius, instead of the AdS radius that has been taken to be 1. The boundary conditions at the edge $r = L$ of the system are taken as

\begin{align*}
\partial_r A_t \big|_{r=L} &= 0, \quad \partial_r \psi \big|_{r=L} = 0, \quad A_\theta \big|_{r=L} = \Omega L^2. \quad (15)
\end{align*}

For detailed discussions of vortices in holographic systems, see \cite{16, 17}. In particular, the differences between superfluid vortices and superconductor vortices in holographic models are discussed in \cite{17}. At the center $r = 0$ of the polar coordinates, we impose the boundary conditions

\begin{align*}
\partial_r A_t \big|_{r=0} &= 0, \quad \partial_r \psi \big|_{r=0} = 0, \quad A_\theta \big|_{r=0} = 0,
\end{align*}

(16)

from the rotation invariance of the initial configuration.

From the above equations of motion and boundary conditions, we can obtain the specific initial configuration numerically, once the chemical potential $\mu$ and the angular velocity $\Omega$
Figure 1: The condensate distribution of our initial configuration with $\Omega = 0.168$, $\mu = 5.5$.

are given. For illustration, we plot our initial state with the angular velocity $\Omega = 0.168$, $\mu = 5.5$, as in Fig. 1. We choose $L = 12$ in all our numerical studies from now on.

B. Quasi-normal modes

Before doing real time evolution, linear analysis of the system is very helpful, which corresponds to quasi-normal modes of the bulk black hole. From now on, we will switch to the ingoing Eddington-Finkelstein coordinates, so that the ingoing boundary conditions at the horizon are easily incorporated by regularity. Under the Eddington-Finkelstein coordinates, the metric (8) becomes

$$ds^2 = \frac{1}{z^2} \left( -f(z) \, dt^2 - 2dtdz + dr^2 + r^2 d\theta^2 \right).$$

(17)

Note that the original axial gauge $A_z = 0$ in the Schwarzschild coordinates (8) is violated after this coordinate transformation. In order to preserve $A_z = 0$ in the new coordinates, a $U(1)$ gauge transformation should be performed after the coordinate transformation:

$$\psi(z, r) \rightarrow e^{i\lambda(z,r)} \psi(z, r), \quad \lambda = \int \frac{A_t}{f} \, dz.$$

(18)

As a result, $A_r = \partial_r \lambda(z, r)$ and so it no longer vanishes, while $\psi$ should not be taken real anymore.

Since the background configurations constructed in the last subsection are time translation invariant and rotation invariant, the linear perturbations can be expanded as

$$\delta\psi = p(z, r) e^{-i\omega t + i m \theta} + q^*(z, r) e^{i\omega t - i m \theta},$$

(19)

$$\delta A_\mu = a_\mu(z, r) e^{-i\omega t + i m \theta} + a_\mu^*(z, r) e^{i\omega t - i m \theta},$$

(20)
where the coefficients $p$, $q$ and $a_\mu$ are all complex functions of $(z, r)$. In particular, $\delta \psi$ is complex, so $q(z, r)$ is independent of $p(z, r)$. Substituting the above expansion of perturbations into the linearized equations of motion, we can get the perturbation equations

$$
\begin{align*}
&\left[-2i(\omega + A_t) \partial_z - i \partial_z A_t - \partial_z (f \partial_z) - (\partial_r - i A_r)^2 - \frac{\partial_r - i A_r}{r} + \frac{(A_\theta - m)^2}{r^2} + z\right] p \\
&\quad - (i \psi \partial_z + 2i \partial_z \psi) a_t + \left(i \psi \partial_r + 2i \partial_r \psi + 2i \psi A_r + \frac{\psi}{r}\right) a_r + \frac{\psi}{r^2} (2A_\theta - m) a_\theta = 0, \\
&\left[-2i(\omega - A_t) \partial_z + i \partial_z A_t - \partial_z (f \partial_z) - (\partial_r + i A_r)^2 - \frac{\partial_r + i A_r}{r} + \frac{(A_\theta - m)^2}{r^2} + z\right] q \\
&\quad + (i \psi^* \partial_z + 2i \partial_z \psi^*) a_t - \left(i \psi^* \partial_r + 2i \partial_r \psi^* - 2i \psi^* A_r + \frac{\psi^*}{r}\right) a_r + \frac{\psi^*}{r^2} (2A_\theta + m) a_\theta = 0, \\
&(-i \psi^* \partial_z + i \partial_z \psi^*) p + (i \psi \partial_z - i \partial_z \psi) q + \partial_z^2 a_t - \left(\frac{\partial_z}{r} + \partial_z \partial_r\right) a_r - \frac{i m}{r^2} \partial_z a_\theta = 0, \\
&\left[i f (\psi^* \partial_z - \partial_z \psi^*) - (\omega + 2A_t) \psi^*\right] p + \left[i f (\partial_z \psi - \psi \partial_z) + (\omega - 2A_t) \psi\right] q \\
&\quad + \left(i \omega \partial_z + \partial_z^2 + \frac{1}{r} \partial_r - 2|\psi|^2 - \frac{m^2}{r^2}\right) a_t + (f \partial_z + i \omega) \left(\partial_r + \frac{1}{r}\right) a_r + \frac{m}{r^2} (i f \partial_z - \omega) a_\theta = 0, \\
&(i \psi^* \partial_r - i \partial_r \psi^* + 2i \psi^* A_r) p + (-i \psi \partial_r + i \partial_r \psi + 2i \psi A_r) q \\
&\quad - \partial_z \partial_r a_t + \left(-2i \omega \partial_z - f \partial_z^2 + f^* \partial_z + \frac{m^2}{r^2} + 2|\psi|^2\right) a_r + \frac{i m}{r^2} \partial_r a_\theta = 0, \\
&(2A_\theta - m) \psi^* p + (2A_\theta + m) \psi q - im \partial_z a_t + im \left(\partial_r - \frac{1}{r}\right) a_r \\
&\quad + \left[-2i \omega \partial_z - \partial_z (f \partial_z) - \partial_r^2 + \frac{\partial_r}{r} + 2|\psi|^2\right] a_\theta = 0.
\end{align*}
$$

Note that the above equations are a set of coupled equations for $(p, q, a_\mu)$, coming from the $e^{-i \omega t + im \theta}$ terms.

At the AdS conformal boundary $z = 0$, Dirichlet boundary conditions should be imposed [18]:

$$
p |_{z=0} = 0, \quad q |_{z=0} = 0, \quad a_t |_{z=0} = 0, \quad a_r |_{z=0} = 0, \quad a_\theta |_{z=0} = 0.
$$

(27)
At $r = 0$, boundary conditions can be obtained by asymptotic analyses of the perturbation equations:

$$\partial_r p |_{r=0} = 0, \quad \partial_r q |_{r=0} = 0, \quad \partial_r a_t |_{r=0} = 0, \quad a_r |_{r=0} = 0, \quad \partial_r a_\theta |_{r=0} = 0. \quad \tag{28}$$

On top of a background configuration constructed in the last subsection and for a given $m$, the quasi-normal frequencies $\omega$ are determined by the condition that the perturbation equations together with the boundary conditions (27) and (28) have nontrivial solutions. Actually, for every $m$, we can obtain a series of $\omega$, among which the one with the maximal imaginary part determines the instability (if there is any $\omega$ with a positive imaginary part) of the $m$ mode: this mode is more unstable if the imaginary part of this $\omega$ is larger. Fig.2a shows our results for the background configuration in Fig.1 where we can see that the system is unstable against the perturbation of the $2 \leq m \leq 48$ modes and the most unstable mode has $m = 26$.

A critical angular velocity $\Omega_m$ can be determined for every $m$, which is just the lowest value of $\Omega$ that makes the $m$ mode unstable. The critical angular velocities for different $m$ are shown in Fig.2b. We can see that the critical angular velocity for the system is $\Omega_c \approx 0.16$, which is the lowest value among all $\Omega_m$. If the angular velocity $\Omega < \Omega_c$, the rotating superfluid system is stable and there will be no vortex generation.
IV. NONLINEAR TIME EVOLUTION

A. Formation of vortex lattices

When the angular velocity $\Omega$ is above the critical value $\Omega_c$, the system is unstable. In real systems, there are random initial perturbations from environment noises, thermal and quantum fluctuations, imperfect preparation of the initial configurations and so on. If the system is unstable, the tiny perturbations of the unstable modes will exponentially grow up at the early stage and then go beyond the linear regime. Will the instability when $\Omega > \Omega_c$ finally result in vortex generation and vortex lattice formation? To answer this question, which is very important but outside the scope of the linear analyses, we will slightly perturb the system from the initial configurations constructed in Sec. [III] and see what will happen by numerically simulating the nonlinear dynamics of this holographic system. To be specific, we will set $\mu = 5.5$ from now on.

Basically, we use the same strategy as in [8, 19, 20] for the numerical evolution here. The only difference and complication is the polar coordinates for this system, which has never been coped with in nonlinear time evolution of holographic superfluids. The full equations of motion in the polar coordinates are as follows:

\[-[\partial_z (f \partial_z \psi) + i (\partial_z A_t) \psi + 2i A_r \partial_z \psi] + \left[-\partial_r^2 \psi + i (\partial_r A_r) \psi + 2i A_r \partial_r \psi - \frac{1}{r} \partial_r \psi \right]
\]
\[+ \frac{1}{r^2} \left[-\partial_\theta^2 \psi + i (\partial_\theta A_\theta) \psi + 2i A_\theta \partial_\theta \psi \right] + \left[A_r^2 + \frac{i A_r}{r} + \frac{A_\theta^2}{r^2} + z \right] \psi + 2i \partial_t \partial_z \psi = 0, \quad (29)\]
\[\partial_r A_t - \partial_t \partial_r A_r - \frac{1}{r} \partial_r A_r - \frac{1}{r^2} \partial_\theta \partial_\theta A_\theta - i (\psi^* \partial_z \psi - \psi \partial_z \psi^*) = 0, \quad (30)\]
\[-\frac{1}{r} \partial_t A_r - \partial_t \partial_z A_t - \partial_t \partial_r A_r - \frac{1}{r^2} \partial_\theta \partial_\theta A_\theta + \frac{f}{r} \partial_z A_r + f \partial_\theta \partial_r A_r + \frac{f}{r^2} \partial_\theta \partial_\theta A_\theta + \partial_r^2 A_t \]
\[+ \frac{1}{r} \partial_r A_t + \frac{1}{r^2} \partial_\theta^2 A_t - i (\psi^* \partial_t \psi - \psi \partial_t \psi^*) - 2A_t \psi^* \psi + i f (\psi^* \partial_z \psi - \psi \partial_z \psi^*) = 0, \quad (31)\]
\[2 \partial_t \partial_z A_r - \partial_z \partial_r A_t - \partial_z (f \partial_z A_r) + \frac{1}{r^2} \left(\partial_r \partial_\theta A_\theta - \partial_\theta^2 A_r \right) \]
\[+ i \left(\psi^* \partial_r \psi - \psi \partial_r \psi^* \right) + 2A_r \psi^* \psi = 0, \quad (32)\]
\[2 \partial_t \partial_z A_\theta - \partial_z (f \partial_\theta A_\theta) - \partial_\theta \partial_r A_t - \partial_r^2 A_\theta + \frac{1}{r} \partial_r A_\theta + \partial_r \partial_\theta A_r - \frac{1}{r} \partial_\theta A_r \]
\[+ i \left(\psi^* \partial_\theta \psi - \psi \partial_\theta \psi^* \right) + 2A_\theta \psi^* \psi = 0. \quad (33)\]

At every step of this evolution, we solve $A_t$ from the restriction equation (30) and use (29, 32, 33) to evolve $(\psi, A_r, A_\theta)$, while equation (31) is used to set a boundary condition for
\( A_t \) (recalling \( \rho = -\partial_z A_t|_{z=0} \)):

\[
\partial_t \rho = - \left( \frac{1}{r} \partial_z A_r + \partial_z \partial_r A_r + \frac{1}{r^2} \partial_z \partial_\theta A_\theta \right) \bigg|_{z=0}.
\]

(34)

We also impose the boundary conditions at \( z = 0 \) as

\[
\partial_t \psi|_{z=0} = 0, \ A_t|_{z=0} = \mu, \ \partial_t A_r|_{z=0} = 0, \ \partial_t A_\theta|_{z=0} = 0.
\]

(35)

At the edge \( r = L \) of the system, we set

\[
\partial_t \partial_r \psi|_{r=L} = 0, \ \partial_t A_r|_{r=L} = 0, \ \partial_t A_\theta|_{r=L} = 0.
\]

(36)

We do spectral expansion in the \((z, r, \theta)\) directions, but skip the coordinate singularity at \( r = 0 \), so no boundary conditions are needed there\[^{25}\]. As well, under the ingoing Eddington-Finkelstein coordinates, no boundary conditions are needed at the horizon \( z = 1 \).

With the setup above, we can do the time evolution by the standard fourth order Runge-Kutta method. Here, we will show our numerical results for a typical process of the vortex lattice formation when \( \Omega = 0.168 \). The results are as follows: When \( t < 50 \), some distortions appear at the edge of the superfluid. These distortions come from instabilities we studied in Sec. III B and will lead to the generation of vortices (see for example at \( t = 30 \) in Fig. 3a). When \( t \approx 60 \), vortices appear at the edge (Fig. 3b). These vortices then move toward the center of the superfluid with the ongoing generation of new vortices at the edge. When \( t \approx 120 \), the generation process ends, while the movement of vortices continues (Fig. 3c). Vortices’ movement basically ends when \( t \approx 360 \), at which the lattice forms (Fig. 3d). We continue the time evolution and find that the structure of the lattice does not change anymore (see Fig. 3e), which means that the lattice gets stabilized.

If changing the angular velocity, we will obtain different lattices. Here, we show another typical lattice that consists 19 vortices (Fig. 4).\[^{2} \] In both cases, vortices are arranged triangularly. A more complex lattice will be obtained if we study the case with more vortices in a larger system. Still, these triangular vortex lattices are good comparisions with results in GP equations.

To make a vortex lattice eventually stabilized, energy dissipation is necessary. In holography, the energy dissipated corresponds to the energy falling into the black hole\[^{7, 21, 22}\].

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\[^{2}\] Movies for the time evolution are available at \[http://people.ucas.edu.cn/~ytian?language=en\#%20687316\].
In this plot, 7 vortices appear, which form the simplest triangular lattice.

To be precise, the energy dissipation in holographic systems is just the energy flux across the bulk black hole horizon:

$$Q = - \int d^3x \sqrt{-g} T^z_t \big|_{z=1},$$

where $T^M_N$ is the energy momentum tensor. Fig. 5 shows the energy dissipation for the whole evolution. We can see that nearly all nonzero values of $Q$ lies in the period $15 < t < 250$, which includes our main process from the initial unstable configurations to the formation of vortex lattices. When $t > 250$, the energy dissipation decreases rapidly and goes to 0 gradually.

**B. Patterns of vortex lattices with 18 vortices**

Vortex lattice has a perfect triangular shape when the number of vortices is infinite, while there are distortions if we consider a finite vortex number case. There are two possibilities. The first case is that the number is so special that interior vortices can form a triangular structure. In this case, there is only one kind of vortex lattice. Fig. 3e for 7 vortices and
Figure 5: Energy dissipation as a function of time. Still we consider the case $\Omega = 0.168$.

Figure 6: Condensates as functions of space coordinates for lattices with 18 vortices at $t = 500$. In these evolutions, we choose $\Omega = 0.22$.

Fig. 4 for 19 vortices are good examples. The second case, in which the number is not so special, is more complicated. Here we will investigate lattices with 18 vortices.

It has been calculated that when the lattice consists of 18 vortices, there are 7 different patterns from free energy analyses based on the GP equation [23]. In our time evolution, we obtain 5 patterns with different initial perturbations (see Fig. 6). Here, we name these different lattices “18-N”, where N is the order of the corresponding pattern in [23].

We have not found the 3rd and 5th patterns from our time evolution. There exist two possible reasons. First, we differentiate lattices by comparing shapes of our lattices with results in [23]. In our model, these 2 absent lattices may not be so distinct that we can merely recognize them by calculating their free energies, which cannot be accomplished because of our current numerical accuracy. The second reason is that these absent lattices are not easy to be realized from time evolution with random initial perturbations, i.e. one may need very special initial perturbations to produce them.
V. CONCLUSION AND DISCUSSION

By holography, we have shown that vortices can be generated when the angular velocity of rotating superfluids exceeds certain critical values. In numerical simulations of the non-linear dynamics, these vortices appear at the edge of the superfluid system first, and then automatically move into the bulk of the system, where they are eventually stabilized into certain vortex lattices. For the case of 18 vortices generated, we have found (at least) five different patterns of the final lattices formed due to different initial perturbations. Actually, these patterns can be recognized as five of seven such patterns obtained from free energy analyses based on the GP equation.

Compared with previous works or traditional methods, our study in this paper has the following advantages. First, in holography, the linear instability that can precisely determine the critical angular velocity is naturally captured by QNM of the bulk black hole. But, as far as we know, such efficient linear analysis in traditional methods is still lacking, though it can be done in principle. More importantly, our study yields all the essential physics of rotating superfluid systems just using the simplest holographic superfluid model without any ingredients put by hand, while the traditional methods cannot incorporate dissipation or finite temperature effects easily and naturally.

For simplicity, we do not consider the backreaction of the matter fields onto the bulk geometry in our holographic superfluid model. But for a full holographic duality, such backreaction should be taken into account\cite{24}. It will be also interesting to investigate the vortex generation and vortex lattice formation in holographic superconductors. These topics are left for future exploration.

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