Abstract. For smooth separated families with enough nice base schemes, we describe the abelian category of coherent sheaves on the generic fiber as a Serre quotient. When the family is a proper effectivization of a formal deformation, the Serre quotient is equivalent to the abelian category of coherent sheaves on the general fiber introduced by Huybrechts–Macrì–Stellari.

Passing to the derived category, we obtain a description of the derived category of the generic fiber as a Verdier quotient. It allows us to induce Fourier–Mukai transforms on the generic and geometric generic fibers from derived-equivalent smooth proper families. As an application, we provide new examples of nonbirational Calabi–Yau threefolds that are derived-equivalent. They consist of geometric generic fibers of smooth projective versal deformations of several known examples, which can be regarded as complex manifolds.

The description as a Verdier quotient also allows us to lift any object along the projection. As another application, given two flat proper families of algebraic varieties over a common base, we prove that if their generic fibers are projective and derived-equivalent then there exists an open subset of the base over which the restrictions become derived-equivalent. This can be regarded as a categorical analogue of the fact that isomorphic generic fibers imply birational families in our setting.

1. Introduction

1.1. The main result. The categorical general fiber was introduced in [HMS11] to study the generic categorical behavior of formal deformations of K3 surfaces. When a formal deformation is effective, one can consult the generic fiber of an effectivization. In analytic setting, Raynaud constructed the generic fiber of formal deformations [Ray74]. Based on his idea, Huybrechts–Macrì–Stellari developed the categorical general fiber, providing a categorical analogue of the generic fiber for non-effective formal deformations of K3 surfaces. As shown in [HMS09], it captures the generic categorical behavior of formal deformations. Below we briefly review their work.

Let $X \to \text{Spf } k[[t]]$ be a formal deformation of a smooth projective $k$-variety $X_0$. Recall that the abelian category of coherent sheaves on the general fiber is the Serre quotient

$$\text{Coh}(X_{k((t))}) := \text{Coh}(X)/\text{Coh}(X)_0,$$

where $\text{Coh}(X)_0 \subset \text{Coh}(X)$ is the full abelian subcategory spanned by coherent $k[[t]]$-torsion $\mathcal{O}_X$-modules. By [Miy91, Theorem 3.2] the derived category $D^b(\text{Coh}(X_{k((t))}))$ of the Serre quotient is equivalent to the Verdier quotient

$$D^b(X)/D^b_0(X),$$

where $D^b_0(X)$ is the full triangulated subcategory spanned by complexes with coherent $k[[t]]$-torsion cohomology. Huybrechts–Macrì–Stellari showed the $k((t))$-linear exact equivalence

$$D^b(\text{Coh}(X_{k((t))})) \cong D^b_{c}(\text{Mod}(\mathcal{O}_X))/D^b_{co} (\text{Mod}(\mathcal{O}_X))$$
when \(X_0\) is a K3 surface [HMST11 Theorem 1.1]. Here \(D^b_c(\text{Mod}(\mathcal{O}_X))\) is the bounded derived category of \(\mathcal{O}_X\)-modules with coherent cohomology and \(D^b_c(\text{Mod}(\mathcal{O}_X))\) is its full triangulated subcategory spanned by complexes with coherent \(k[[t]]\)-torsion cohomology. The latter Verdier quotient is called the derived category of the general fiber.

If \(X\) is effective with a proper effectivization \(X\), i.e., isomorphic to the formal completion of a proper \(k[[t]]\)-scheme \(X\), then by [GD61 Corollary III.1.6] we have
\[
D^b(X) = D^b(X)
\]
and \(D^b(\text{Coh}(X_{k((t))}))\) is equivalent to the Verdier quotient
\[
D^b(X)/D^b_t(X),
\]
which is \(k((t))\)-linear. This can be regarded as an effectivization of \(D^b(\text{Coh}(X_{k((t))}))\). On the other hand, the derived category of the generic fiber \(X_{k((t))}\) of the effectivization \(X\) gives another \(k((t))\)-linear category. By [BFN10, Theorem 1.2] also \(D^b(X_{k((t))})\) is obtained from the \(k[[t]]\)-linear category \(D^b(X)\). As both \(D^b(X)/D^b_t(X)\) and \(D^b(X_{k((t))})\) carry sufficient information on the generic categorical behavior of formal deformations, it is natural to ask whether they are equivalent as a \(k((t))\)-linear triangulated category. Motivated by this question, we prove the following.

**Theorem 1.1** (Theorem 2.5). Let \(X_R\) be a smooth separated family over a noetherian connected regular affine \(k\)-scheme \(\text{Spec} R\) whose closed points are \(k\)-rational. Let \(K\) be the field of fractions of \(R\) and \(X_K\) the generic fiber. Then there exists a \(K\)-linear equivalence
\[
\text{coh}(X_K) \cong \text{coh}(X_R)/\text{coh}(X_R)_0
\]
of abelian categories, where \(\text{coh}(X_R)_0\) is the Serre subcategory spanned by \(R\)-torsion sheaves.

**Corollary 1.2** (Corollary 2.6). Under the same assumption as above, there exists an exact \(K\)-linear equivalence
\[
D^b(X_K) \cong D^b(X_R)/D^b_t(X_R),
\]
where \(D^b_t(X_R)\) is the full triangulated subcategory spanned by complexes with \(R\)-torsion cohomology.

We impose the technical condition on \(\text{Spec} R\) to include smooth proper effectivizations of formal families over formal power series rings, besides smooth proper families over nonsingular affine \(k\)-varieties.

1.2. **The first application.** One advantage to describe the derived category of the generic fiber as a Verdier quotient is that Fourier–Mukai machinery carries over easily. Suppose that
\[
\Phi_E: D^b(X_R) \to D^b(X'_R)
\]
is a relative Fourier–Mukai transform of smooth proper families over \(R\) with kernel \(E \in D^b(X_R \times_R X'_R)\). Then \(\Phi_E\) induces a Fourier–Mukai transform
\[
\Phi_{E_K}: D^b(X_K) \to D^b(X'_K)
\]
of generic fibers, where the kernel \(E_K\) is the pullback along the canonical inclusion \(R \hookrightarrow K\). If, in addition, \(X_K, X'_K\) are projective, then by the standard argument the further base change to the closure defines a Fourier–Mukai transform
\[
\Phi_{E_K}: D^b(X_K) \to D^b(X'_K)
\]
of geometric generic fibers.

Typical examples for our results are given by deformations of higher dimensional Calabi–Yau manifolds. Recently, the author proved the following.
Theorem 1.3 ([Mor, Theorem 1.1]). Let $X_0, X'_0$ be derived-equivalent Calabi–Yau manifolds of dimension more than two. Then there exists a nonsingular affine $k$-variety $\text{Spec} S$ such that smooth projective versal deformations $X_S, X'_S$ over $S$ are derived-equivalent.

Here, the derived equivalence is given by the relative Fourier–Mukai transform with kernel obtained by deformation of the original kernel for central fibers. Similarly, the Fourier–Mukai transform of central fibers extends to proper effectivizations of universal formal families. Combining with our current results, we obtain derived-equivalent geometric generic fibers of the versal deformations and the effectivizations respectively. One can check that they are Calabi–Yau manifolds of dimension $\dim X_0$. When the central fibers are nonbirational, in some special cases one can deduce the nonbirationality of the geometric generic fibers. To summarize, we obtain

Theorem 1.4 (Theorem 5.7). Let $X_0, X'_0$ be derived-equivalent Calabi–Yau manifolds of dimension more than two. Then the geometric generic fibers $X_0, X'_0$ of smooth projective versal deformations and that $X_0, X'_0$ of smooth projective versal deformations are respectively derived-equivalent Calabi–Yau manifolds. If, in addition, we have either $\text{NS}_{\text{tor}} X_0 \neq \text{NS}_{\text{tor}} X'_0$, or $\rho(X_0) = 1$ and $\deg(X_0) \neq \deg(X'_0)$, then they are respectively nonbirational.

Recall that Fourier–Mukai partners are pairs of nonbirational Calabi–Yau threefolds that are derived-equivalent. They are of considerable interest from the view point of string theory and mirror symmetry. For instance, the Gross–Popescu pair [GP01, Sch] and the Pfaffian–Grassmannian pair [BC09, Kuz] satisfy the first and the second conditions in Theorem 1.4 respectively. Thus we obtain new examples of Fourier–Mukai partners over the closure $\overline{K}, \overline{Q}$ of function fields. Note that if $k$ is a universal domain, i.e., an algebraically closed field of infinite transcendence degree of the prime field, then there exists an isomorphism $k \cong \overline{Q}$ [Via13, Lemma 2.1]. In particular, if $k = \mathbb{C}$ then the new examples over $\overline{Q}$ can be regarded as complex manifolds.

When $X_0, X'_0$ are the Pfaffian–Grassmannian pair, we demonstrate the difference between $X_0, X'_0$, and known examples. The geometric generic fibers $X_0, X'_0$ are respectively isomorphic to $X_0, X'_0$ as an abstract scheme, but not as a variety. We will explain why any other known pair $Y_0, Y'_0$ over $k$ cannot be isomorphic to $X_0, X'_0$ even as an abstract scheme. However, we emphasize that one can obtain $X_0, X'_0$ starting from IMOU varieties [IMOU, Kuz18] which are flat degenerations of $X_0, X'_0$. It would be an interesting problem to deduce nonbirationality of the geometric generic fibers for other examples of Fourier–Mukai partners.

1.3. The second application. Another advantage to describe the derived category of the generic fiber as a Verdier quotient is that any object of $D^b(X_S)$ lifts to that of $D^b(X_K)$. The quotient description extends to nonaffine base case for flat proper families of $k$-varieties.

Theorem 1.5 (Theorem 6.1). Let $\pi: X \rightarrow S$ be a flat proper morphism of $k$-varieties. Then there exists a $K$-linear exact equivalence

$$D^b(X)/ \text{Ker}(\iota^*_K) \simeq D^b(X_S),$$

where $K = K(S)$ is the function field and $\iota^*_K: X_K \rightarrow X$ is the canonical morphism.

In particular, there always exists a lift $E \in D^b(X \times_S X')$ of a Fourier–Mukai kernel $E_K \in D^b(X_K \times X'_K)$ whenever $X_K, X'_K$ are projective and derived-equivalent. It allows us to extend the derived equivalence of the generic fibers to that of general fibers.

Theorem 1.6 (Corollary 6.9). Let $\pi: X \rightarrow S, \pi': X' \rightarrow S$ be flat proper morphisms of $k$-varieties. Assume that their generic fibers $X_K, X'_K$ are projective and derived-equivalent. Then
there exists an open subset $U \subset S$ over which the restrictions $X_U, X'_U$ become derived-equivalent. In particular, over $U$ any pair of closed fibers are derived-equivalent.

This can be regarded as a categorical analogue of the fact that isomorphic generic fibers imply birational families in our setting. From [LSTT13 Proposition 1.3] it immediately follows

**Corollary 1.7.** Let $\pi: X \to S, \pi': X' \to S$ be flat proper morphisms of $k$-varieties. Consider a relative integral functor $\Phi_E: D^b(X) \to D^b(X')$ defined by kernel $E \in D^b(X \times_S X')$. Then via restriction $\Phi_E$ induces the derived equivalence of their general fibers if and only if it induces the derived equivalence of their generic fibers.

In classical algebraic geometry, the generic fiber often controls behaviors of general fibers. For instance, if the generic fiber satisfies a certain property which is constructible, then general fibers also satisfy the same property. Such characteristics of the generic fiber must be translated via Gabriel’s theorem [Gab62] into the abelian category of coherent sheaves. Nevertheless, as there exist pairs of nonisomorphic derived-equivalent $K$-varieties, it makes sense to wonder how the derived category of the generic fiber affects that of general fibers.

When $k$ is a universal domain, very general fibers are obtained as base changes of the geometric generic fiber along isomorphisms $k \cong \tilde{K}$ from [Vial13 Lemma 2.1]. Hence in this case the derived equivalence of the projective generic fibers induces that of very general fibers. The proof of [Mor Theorem 1.1] shows that Theorem 1.6 follows as soon as a lift of the Fourier–Mukai kernel induces the derived equivalence of a single pair of closed fibers. However, since the composition of the canonical morphisms $R \to K \to \tilde{K}$ with a fixed $\tilde{K} \equiv k$ does not coincide with the surjection $R \to k$, the induced Fourier Mukai kernel on very general fibers should be different from the restriction of the lift.

Consider another description

$$D^b_{dg}(X_\xi) \cong \text{Perf}_{dg}(X_\xi) \equiv \text{Perf}_{dg}(X) \otimes_{\text{Perf}_{dg}(S)} \text{Perf}_{dg}(K)$$

of the dg enhancement $D^b_{dg}(X_\xi)$ of $D^b(X_\xi)$ obtained from [BFN10] and [Coh]. By [CP21] and [Miy91] the category $\text{Perf}_{dg}(K)$ is a dg enhancement of the Verdier quotient

$$D^b(S)/D^b_{\text{dim}S-1}(S),$$

where $D^b_{\text{dim}S-1}(S)$ is the full triangulated subcategory spanned by objects with cohomology supported on dimension at most $\text{dim}S - 1$. Thus removing the torsion support from $D^b(X)$ is equivalent to removing all closed fibers from the supports of objects of $D^b(X)$. In particular, from a collection of a finite number of objects and Hom-sets between them in $D^b(X)$, one can remove its torsion support by shrinking the base.

In order to show that the restriction of the lift to general fibers define equivalences, we apply the argument in the proof of [Mor Theorem 1.1] to a fixed strong generator $E_U$ of $D^b(X_U)$, which always exists over sufficiently small open subset $U \subset S$. By shrinking $U$ further if necessary, one can remove the torsion parts with respect to the base from $E_U$ and its relevant Hom-sets. Then we invoke some basic categorical results to show that the value of the counit morphism on the trimmed strong generator is an isomorphism, which implies that the restriction of the counit morphism is a natural isomorphism.

Theorem 1.6 tells us that the derived category of the generic fiber determines an $U$-linear triangulated category $D^b(X_U) \cong D^b(X)/D^b_Z(X)$ for some open subset $U \subset S$ and its complement $Z$, where $D^b_Z(X) \subset D^b(X)$ is the full $S$-linear triangulated subcategory with cohomology supported on $X_Z$. In other words, one may say that it is the core of $D^b(X)$. We expect that Theorem 1.5 and Theorem 1.6 provide a way to seek categorical constructible properties and their derived invariance.
Notations and conventions. We work over an algebraically closed field $k$ of characteristic 0 throughout the paper. Every time we apply [Via13, Lemma 2.1] we always assume $k$ to be a universal domain with comments. A $k$-variety is an integral separated $k$-scheme of finite type. A Calabi–Yau manifold $X_0$ is a smooth projective $k$-variety with trivial canonical bundle and $H^i(X_0, \mathcal{O}_{X_0}) = 0$ for $0 < i < \dim X_0$. For a noetherian formal scheme $X$ by $D^b(X)$ we denote the bounded derived category of the abelian category $\text{Coh}(X)$ of coherent sheaves on $X$.

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2. The derived category of the generic fiber

For a scheme $X_R$ over an integral domain $R$, we have the following pullback diagram

$$
\begin{array}{ccc}
X_K & \xrightarrow{i} & X_R \\
\pi_K \downarrow & & \downarrow \pi_R \\
\text{Spec } K & \xrightarrow{j} & \text{Spec } R,
\end{array}
$$

where $K$ is the field of fractions of $R$ and $X_K$ is the fiber over the generic point $\xi \in \text{Spec } R$. In the sequel, we assume that $X_R$ is connected and smooth separated over $R$, and $\text{Spec } R$ is a noetherian connected regular affine $k$-scheme whose closed points are $k$-rational. The assumption guarantees that $X_R$ and $X_K$ are noetherian, separated, and regular. Indeed, $X_{R,x}$ is regular for every closed point $x \in X_R$ by [SP, Tag 031E]. Under the assumption also $X_K$ is connected. An example of $X_R$ we have in mind is a proper efectivization of miniversal formal family of a smooth projective $k$-variety whose deformations are unobstructed. Note that in this case $\text{Spec } R$ is not of finite type over $k$. We impose the technical condition on $\text{Spec } R$ to include this example, besides smooth proper families over nonsingular affine $k$-varieties.

We denote by $\text{coh}(X_R)_0 \subset \text{coh}(X_R)$ the Serre subcategory spanned by $R$-torsion sheaves, i.e., for each $\mathcal{F} \in \text{coh}(X_R)_0$ there is an element $r \in R$ such that $r \mathcal{F} = 0$. We write $C = \text{coh}(X_R)/ \text{coh}(X_R)_0$ for the Serre quotient. The natural projection $p : \text{coh}(X_R) \to C$ which sends $\mathcal{F}$ to $\mathcal{F}_K$ is known to be exact. By universality of Serre quotient, the exact functor

$$(-) \otimes_R K : \text{coh}(X_R) \to \text{coh}(X_K)$$

induces a unique exact functor

$$\Phi : C \to \text{coh}(X_K)$$

such that $(-) \otimes_R K = \Phi \circ p$. Then $\Phi$ defines the derived functor

$$D^b(C) \to D^b(X_K),$$

which induces a functor

$$\Psi : D^b(X_R)/D^b_0(X_R) \to D^b(X_K)$$

via [Miy91, Theorem 3.2]. We show that $\Phi$ and $\Psi$ are equivalences. In particular,

$$D^b(C) = D^b(X_R)/D^b_0(X_R)$$

gives an alternative description of $D^b(X_K)$.

2.1. $K$-linear categorical quotients. As expected from their constructions, both the Serre quotient $C = \text{coh}(X_R)/\text{coh}(X_R)_0$ and the Verdier quotient $D^b(X_R)/D^b_0(X_R)$ carry natural $K$-linear structures. To see this, one can adapt [HMS11, Proposition 2.3, 2.9] to our setting in a straightforward way. We include the proofs for reader’s convenience.

Lemma 2.1 ([HMS11 Proposition 2.3]). The abelian category $C$ is $K$-linear and for all $\mathcal{E}, \mathcal{F} \in \text{coh}(X_R)$ the natural projection $p: \text{coh}(X_R) \rightarrow C$ induces a $K$-linear isomorphism

\[
\text{Hom}_{X_R}(\mathcal{E}, \mathcal{F}) \otimes_R K \cong \text{Hom}_C(\mathcal{E}_K, \mathcal{F}_K).
\]

Proof. As a quotient of the $R$-linear category $\text{coh}(X_R)$, the category $C$ is also $R$-linear. The multiplication with $r^{-1}$ for $r \in R$ is defined as follows. Let $f \in \text{Hom}_C(\mathcal{E}_K, \mathcal{F}_K)$ be a morphism represented by $f: (\mathcal{E} \xrightarrow{s_0} \mathcal{E}_0 \xrightarrow{g} \mathcal{F})$ with $\text{Ker}(s_0), \text{Coker}(s_0) \in \text{coh}(X_R)_0$. Then set $r^{-1}f: (\mathcal{E} \xrightarrow{rs_0} \mathcal{E}_0 \xrightarrow{g} \mathcal{F})$, which is well-defined in $C$, since the objects $\text{Ker}(rs_0)$ and $\text{Coker}(rs_0)$ are in $\text{coh}(X_R)_0$. Moreover, we have $r(r^{-1}f) = f$ due to the commutative diagram

and the $K$-linearity of the composition is obvious. Recall that a morphism in the Serre quotient is an equivalence class of diagrams. In $C$, two morphisms $f: (\mathcal{E} \xleftarrow{s_0} \mathcal{E}_0 \xrightarrow{g} \mathcal{F})$ and $f': (\mathcal{E} \xleftarrow{s_0'} \mathcal{E}_{0}' \xrightarrow{g'} \mathcal{F})$ are equivalent if there is a third diagram $f'': (\mathcal{E} \xleftarrow{s_0''} \mathcal{E}_{0}'' \xrightarrow{g''} \mathcal{F})$ with $\text{Ker}(s_0''), \text{Coker}(s_0'') \in \text{coh}(X_R)_0$ and morphisms $u: \mathcal{E}_{0}'' \rightarrow \mathcal{E}_0$ and $v: \mathcal{E}_{0}'' \rightarrow \mathcal{E}_0'$ in $\text{coh}(X_R)$ which makes the diagram

commute.

Consider the induced $K$-linear map

$\eta_K: \text{Hom}_{X_R}(\mathcal{E}, \mathcal{F}) \otimes_R K \rightarrow \text{Hom}_C(\mathcal{E}_K, \mathcal{F}_K)$.

To prove the injectivity of $\eta_K$, let $f \in \text{Hom}_{X_R}(\mathcal{E}, \mathcal{F})$ be a morphism with $\eta_K(f) = 0$. There exists a commutative diagram

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{f} & \mathcal{F} \\
\downarrow{s} & & \downarrow{f'} \\
\mathcal{E}' & \xrightarrow{0} & \mathcal{F}
\end{array}
\]

with $\text{Ker}(s), \text{Coker}(s) \in \text{coh}(X_R)_0$ and hence $f$ factorizes through

\[
f: \mathcal{E} \xrightarrow{q} \text{Coker}(s) \xrightarrow{f'} \mathcal{F}.
\]

If $r\text{Coker}(s) = 0$, then this yields $rf = f' \circ (rq) = 0$. In particular, $f \otimes 1 \in \text{Hom}_{X_R}(\mathcal{E}, \mathcal{F}) \otimes_R K$ is trivial.

To prove the surjectivity of $\eta_K$, we have to show that for any $f \in \text{Hom}_C(\mathcal{E}_K, \mathcal{F}_K)$ there exists an element $r \in R$ such that $rf$ is induced by a morphism $\mathcal{E} \rightarrow \mathcal{F}$ in $\text{coh}(X_R)$. Write
Proposition 2.3. The functor Canonical functor from the Serre quotient.

2.2. results which rely on it. (results require

\[ E \rightarrow \text{Serre quotient} C \]

\[ \text{2.1} \]

Proof. Any morphism in \( D^b(C) \) is \( K \)-linear and for all \( E, F \in D^b(X_\mathbb{K}) \) the natural projection \( Q : D^b(X_\mathbb{K}) \rightarrow D^b(C) \) induces a \( K \)-linear isomorphism

\[ \text{Hom}_{D^b(X_\mathbb{K})}(E, F) \otimes_K K \cong \text{Hom}_{D^b(C)}(E_\mathbb{K}, F_\mathbb{K}) \]

Proof. Any morphism in \( D^b(C) \) can be represented by a morphism of bounded complexes of objects in \( C \), which is a collection of morphisms in \( C \) compatible with the differentials. Since by Lemma 2.1 both the morphisms and the differentials are \( K \)-linear, the representative is also \( K \)-linear. The \( K \)-linear isomorphism is a direct consequence of Corollary 2.6 whose proof does not rely on it.

Remark 2.1. In [HMST11] only the case where \( R = \mathbb{k}[t] \) was treated. This is because some results require \( R \) to be a DVR. Throughout the paper, we are free from the requirement and results which rely on it.

2.2. Canonical functor from the Serre quotient. Due to the natural \( K \)-linear structure, the Serre quotient \( C \) can be embedded in \( \text{coh}(X_\mathbb{K}) \) via the exact functor \( \Phi \), which is induced by that \((-) \otimes_K K : \text{coh}(X_\mathbb{K}) \rightarrow \text{coh}(X_\mathbb{K}) \).

Proposition 2.3. The functor \( \Phi : C \rightarrow \text{coh}(X_\mathbb{K}) \) is fully faithful.

Proof. The images of \( \mathcal{E}_K, \mathcal{F}_K \in C \) by \( \Phi \) are respectively isomorphic to the pullbacks \( i^* \mathcal{E}, i^* \mathcal{F} \) of some coherent sheaves \( \mathcal{E}, \mathcal{F} \) on \( X_\mathbb{K} \). We have

\[ \text{Hom}_{X_\mathbb{K}}(\Phi(\mathcal{E}_K), \Phi(\mathcal{F}_K)) = \text{Hom}_{X_\mathbb{K}}(i^* \mathcal{E}, i^* \mathcal{F}) \]

\[ \cong \Gamma \circ i^* \text{Hom}_{X_\mathbb{K}}(\mathcal{E}, \mathcal{F}) \]

\[ \cong j^* \circ (\pi_K)_* \text{Hom}_{X_\mathbb{K}}(\mathcal{E}, \mathcal{F}) \]

\[ \cong \text{Hom}_{X_\mathbb{K}}(\mathcal{E}, \mathcal{F}) \otimes_K K \]

\[ \cong \text{Hom}_C(\mathcal{E}_K, \mathcal{F}_K) \].
where the second, the third, and the fourth isomorphisms follow from flat base change, Lemma \ref{flat base change} below, and Lemma \ref{coh is full subcategory} respectively. \hfill \Box

**Lemma 2.4.** For all $\mathcal{E}, \mathcal{F} \in \text{coh}(X_K)$ we have an isomorphism

$$\text{Hom}_{X_K}(\mathcal{E}, \mathcal{F}) \otimes_R K \cong \text{Hom}_{X_K}(\mathcal{E}, \mathcal{F}) \otimes_R K.$$  

**Proof.** We may consider $(\pi_R)_* \text{Hom}_{X_K}(\mathcal{E}, \mathcal{F}) \otimes_R K$ as the stalk of the sheaf $(\pi_R)_* \text{Hom}_{X_K}(\mathcal{E}, \mathcal{F})$ at the generic point $\xi$ of $\text{Spec } R$. For an affine open cover $\text{Spec } R = \{D(f)\}$, $0 \neq f \in R$, take any germ $(D(f), s)$ of $(\pi_R)_* \text{Hom}_{X_K}(\mathcal{E}, \mathcal{F})_{\xi}$. Since $(\pi_R)_* \text{Hom}_{X_K}(\mathcal{E}, \mathcal{F})$ is a quasi-coherent sheaf on an affine scheme as $X_K$ is noetherian, by [Har77] Lemma II15.3 there exists an integer $n \geq 0$ such that $f^n s$ is a global section. Let

$$\phi: (\pi_R)_* \text{Hom}_{X_K}(\mathcal{E}, \mathcal{F})_{\xi} \to \text{Hom}_{X_K}(\mathcal{E}, \mathcal{F}) \otimes_R K$$

be the homomorphism of $R$-algebras which sends $(D(f), s)$ to $f^m s \otimes (1/f^m)$, where $m$ is the minimum integer such that $f^m s$ is a global section. One can check that this is well-defined. The inverse $\phi^{-1}$ is given by the map which sends $v \otimes (g/f) \to (D(f), (gv/f))$ for $v \in \text{Hom}_{X_K}(\mathcal{E}, \mathcal{F})$ and $g \in R$. \hfill \Box

**Theorem 2.5.** The functor $\Phi: C \to \text{coh}(X_K)$ is a $K$-linear equivalence of abelian categories.

**Proof.** It suffices to show that $\Phi$ is essentially surjective. By assumption $X_K$ is connected. Let $\mathcal{F}_{\xi}$ be an object of $\text{coh}(X_K)$. Since $X_K$ is noetherian integral separated regular, by [Har77] Exercise III6.8 any coherent sheaf on $X_K$ can be obtained as the cokernel of a morphism of locally free sheaves of finite rank. The essential image of $\Phi$ is a full abelian subcategory of $\text{coh}(X_K)$. In particular, it is closed under taking cokernels. Hence we may assume that $\mathcal{F}_{\xi}$ is a locally free sheaf of finite rank.

Take an affine open cover $\{U_i\}_{i=1}^m, U_i = \text{Spec } A_i$ of $X_K$ such that the restriction of $\mathcal{F}_{\xi}$ to each affine open subset $V_i = U_i \times_R K$ of $X_K$ is isomorphic to a finite rank free $\tilde{A}_i \otimes_R K$-module

$$F_i = \tilde{A}_i^{\oplus N}.$$  

Let $\phi_{ij} = \phi_i \circ \phi_j^{-1}: F_{ij}|_{V_{ij}} \to F_{ij}|_{V_{ij}}$ be isomorphisms on $V_{ij} = V_i \cap V_j$ where $\phi_i: F_{\xi}|_{V_i} \to F_i$ are trivializations with their inverses $\phi_i^{-1}: F_i \to F_{\xi}|_{V_i}$. In other words, we have the commutative diagrams

$$\begin{array}{c}
F_{\xi}|_{V_{ij}} \xrightarrow{\phi_{ij}} F_{\xi}|_{V_{ij}} \\
\phi_i \downarrow \quad \quad \downarrow \phi_j \\
F_{ij}|_{V_{ij}} \xrightarrow{\phi_{ij}} F_{ij}|_{V_{ij}}.
\end{array}$$

From $F_i$ we obtain a rank $N$ free $\tilde{A}_i$-module

$$E_i = \tilde{A}_i^{\oplus N}$$

with the same generators. By construction tensoring $K$ with $E_i$ recovers $F_i$. Now, up to shrinking the base $\text{Spec } R$, we glue $E_i$ to construct a coherent sheaf $\mathcal{E}$ on $X_K$ such that $\mathcal{E} \otimes_R K \cong \mathcal{F}_{\xi}$. By Lemma \ref{coh is full subcategory} there are lifts $\tilde{\phi}_{ij}: E_{ij}|_{U_{ij}} \to E_{ij}|_{U_{ij}}$ on $U_{ij} = U_i \cap U_j$ of $\phi_{ij}$ along (2.2). Namely, we have

$$\tilde{\phi}_{ij} \otimes_R 1/r_{ij} = \phi_{ij}$$

for some $r_{ij} \in R$.

Consider the affine open subset

$$\text{Spec } T \subset \text{Spec } R$$

8
defined by \( r_{ij} \neq 0, 1 \leq i, j \leq m \). On the base changes \( U_{ij} \otimes_R T \) all \( r_{ij} \) become invertible. Hence \( \phi_{ij} \) canonically lift to isomorphisms

\[
 r_{ij}^{-1} \phi_{ij} : E|_{U_{ij} \times_R T} \to E|_{U_{ij} \times_R T}
\]

along (2.2). Clearly, the lifts satisfy cocycle condition. Thus \( E|_{U_{ij} \times_R T} \) glue to yield a locally free sheaf \( \mathcal{E} \) on \( X_T = X_R \otimes_R T \) such that \( \mathcal{E} \otimes_T K \cong \mathcal{F}_\xi \).

By [Har77, Exercise II5.15] the lift \( \mathcal{E} \) extends to a coherent sheaf \( \mathcal{E} \) on \( X_R \). Since the exact functor \((-) \otimes_R K\) factorizes through \( \text{coh}(X_R) \to \text{coh}(X_T) \to \text{coh}(X_K) \) and it sends \( \mathcal{E} \) to \( \mathcal{F}_\xi \), there is an object \( \mathcal{E} \in C \) which maps to \( \mathcal{F}_\xi \) under \( \Phi \). \( \square \)

2.3. Canonical functor from the Verdier quotient. As the functor \( \Phi : C \to \text{coh}(X_K) \) is exact, termwise application of \( \Phi \) defines the derived functor \( D^b(C) \to D^b(X_K) \). By universality of Verdier quotient, the induced functor

\[
 \Psi : D^b(X_R)/D^b_0(X_R) \to D^b(X_K)
\]

by (2.1) coincides with \( D^b(C) \to D^b(X_K) \). From Theorem 2.5 we obtain

**Corollary 2.6.** The functor \( \Psi : D^b(X_R)/D^b_0(X_R) \to D^b(X_K) \) is a K-linear exact equivalence.

**Remark 2.2.** In Section 6 we will extend Theorem 2.5 and Corollary 2.6 to nonaffine base case for flat proper families of \( k \)-varieties.

3. Comparison with the categorical general fiber

3.1. The abelian category of coherent sheaves on the general fiber. Recall that in [HMS11] for a formal deformation \( X \) of a smooth projective \( k \)-variety over a formal power series ring \( k[[t]] \) the abelian category of coherent sheaves on the general fiber is defined as the Serre quotient

\[
 \text{Coh}(X_{k(t)}) := \text{Coh}(X)/\text{Coh}(X)_0,
\]

where \( \text{Coh}(X) \) is the abelian category of coherent \( \mathcal{O}_X \)-modules and \( \text{Coh}(X)_0 \) is the full abelian subcategory spanned by coherent \( k[[t]] \)-torsion \( \mathcal{O}_X \)-modules. In the case where \( X \) is effective with a proper effectivization, one can obtain \( \text{Coh}(X_{k(t)}) \) via formal completion along the closed fiber in the following sense.

**Corollary 3.1.** Let \( X = \hat{X}_R \to \text{Spf} R \) be an effective formal deformation of a smooth projective variety with a proper effectivization \( X_R \). Then abelian category of coherent sheaves on the general fiber of \( X \) is equivalent to that on the generic fiber \( X_K \) of its effectivization, i.e., there exists a K-linear equivalence

\[
 \text{coh}(X_K) \cong \text{Coh}(X_K)
\]

of abelian categories.

**Proof.** We have the pullback diagram

\[
\begin{array}{ccc}
\hat{X}_R & \xrightarrow{i} & X_R \\
\downarrow \hat{r} & & \downarrow \pi_R \\
\text{Spf} R & \rightarrow & \text{Spec} R
\end{array}
\]
of noetherian formal schemes. Since \( R \) is a complete local noetherian ring, one can apply \([GD61, Corollary III5.1.6]\) to see that the functor
\[
(3.1) \quad \text{coh}(X_R) \to \text{Coh}(X),
\]
which sends each coherent sheaf \( \mathcal{F} \) on \( X_R \) to its formal completion \( \hat{\mathcal{F}} \) along the closed fiber, is an \( R \)-linear equivalence of abelian categories. By universality of Serre quotient, we obtain the induced \( K \)-linear equivalence
\[
\text{coh}(X_R)/\text{coh}(X_R)_0 \to \text{Coh}(X)/\text{Coh}(X)_0.
\]
Then the desired equivalence follows from \([Miy91, Theorem 3.2]\). □

3.2. Serre functor. In the case where \( X \) is effective with a proper effectivization, the Serre functor from \([HMS11, Theorem 1.1]\) constructed when \( X \) is a formal deformation of a \( K3 \) surface, extends to smooth projective varieties and formal power series rings of any finite dimension in a straightforward way.

**Proposition 3.2.** Let \( X = \hat{X}_R \to \text{Spf } R \) be an effective formal deformation of a \( d \)-dimensional smooth projective variety with a proper effectivization \( X_R \). Then a Serre functor on \( D^b(X_k) \) is given by
\[
S(\hat{E}_K) = (E \otimes \omega_{\pi_R} K)[d],
\]
where \( \omega_{\pi_R} \) is the dualizing sheaf for \( \pi_R \).

**Proof.** We have
\[
\text{Hom}_{D^b(\text{Mod}(O_{X_k}))} (\hat{E}_K, \hat{F}_K) \cong \text{Hom}_{D^b(\text{O}_X)} (\hat{E}, \hat{F}) \otimes_R K
\]
\[
\cong \text{Hom}_{D^b(\text{O}_X)} (E, F) \otimes_R K
\]
\[
\cong \text{Hom}_{D^b(\text{O}_X)} (F, E \otimes \omega_{\pi_R}[d])^\vee \otimes_R K
\]
\[
\cong \text{Hom}_{D^b(\text{O}_X)} (\hat{F}, E \otimes \omega_{\pi_R}[d])^\vee \otimes_R K
\]
\[
\cong \text{Hom}_{D^b(\text{Mod}(\text{O}_{X_k}))} (\hat{F}_K, (E \otimes \omega_{\pi_R}) K[d])^\vee,
\]
where the first and the fifth, the second and the fourth, and the third isomorphisms follow from Corollary Lemma 2.2, the equivalence (3.1), and Serre duality for the smooth morphism \( \pi_R \) of relative dimension \( d \) respectively. □

3.3. The derived category of the general fiber. Recall that the derived category of the general fiber is defined as the Verdier quotient
\[
D^b_c (\text{Mod}(\mathcal{O}_X)) / D^b_{c_0} (\text{Mod}(\mathcal{O}_X)),
\]
where \( D^b_c (\text{Mod}(\mathcal{O}_X)) \) is the bounded derived category of \( \mathcal{O}_X \)-modules with coherent cohomology and \( D^b_{c_0} (\text{Mod}(\mathcal{O}_X)) \) is the full triangulated subcategory spanned by complexes with coherent \( \mathbb{k}[t] \)-torsion cohomology. By \([HMS11, Theorem 1.1]\) we have
\[
D^b_c (\text{Mod}(\mathcal{O}_X)) / D^b_{c_0} (\text{Mod}(\mathcal{O}_X)) \cong D^b (\text{Coh}(X_{k((t)))})
\]
when \( X \) is a formal deformation of a \( K3 \) surface. This is deduced from the intermediate \( \mathbb{k}((t)) \)-linear exact equivalence
\[
(3.2) \quad D^b_c (\text{Mod}(\mathcal{O}_X)) / D^b_{c_0} (\text{Mod}(\mathcal{O}_X)) \cong D^b (X) / D^b_0 (X)
\]
established in the proof of \([HMS11, Proposition 3.10]\).

While we have
\[
D^b_{c_0} (\text{Mod}(\mathcal{O}_X)) \cong D^b_0 (X)
\]
by [HMS11, Proposition 2.5], in general the natural inclusion
\[ D^b(X) \hookrightarrow D^b_c \left( \text{Mod} \left( \mathcal{E}_X \right) \right) \]
is not an equivalence. Hence one cannot expect (3.2) to hold for more general \( X \). However, in the case where \( X \) is effective with a proper effectivization, we have
\[ D^b_c \left( \text{Mod} \left( \mathcal{E}_{X_0} \right) \right) / D^b_0 \left( \text{Mod} \left( \mathcal{E}_{X_0} \right) \right) \cong D^b(X_R) / D^b_0(X_R) \cong D^b(X_K) \]
by Corollary 2.6. Note that the first equivalence follows from
\[ D^b_R \left( \text{Mod} \left( \mathcal{E}_{X_0} \right) \right) \cong D^b(X_R), \]
which holds for noetherian schemes. Unless the closed fiber of \( X_R \) is a K3 surface, in general one can only recover the part
\[ D^b(X) / D^b_0(X) \cong D^b(X_K) \]
via formal completion along the closed fiber in the sense of Corollary 3.1.

### 4. Induced Fourier–Mukai transforms

As mentioned in [HMS11], one advantage to describe the derived category of the generic fiber as a Verdier quotient is that the Fourier–Mukai machinery carries over easily. Given a relative integral functor \( \Phi_E : D^b(X_R) \rightarrow D^b(X'_R) \) on smooth proper families \( \pi_R : X_R \rightarrow \text{Spec} \, R \) and \( \pi'_R : X'_R \rightarrow \text{Spec} \, R \), we study the induced derived equivalence on thier generic and geometric generic fibers. We will discuss the opposite direction in Section 6 below.

#### 4.1. Induced Functor from smooth proper families to generic fibers.

**Proposition 4.1.** Let \( X_R, X'_R \) be smooth proper families over \( R \). If \( \Phi_E : D^b(X_R) \rightarrow D^b(X'_R) \) is a relative Fourier–Mukai functor, then the induced integral functor \( \Phi_{E_K} : D^b(X_K) \rightarrow D^b(X'_K) \) is an equivalence. Here, \( E_K \in D^b(X_R \times_R X'_R) / D^b_0(X_R \times_R X'_R) \) is the image of \( E \) by the natural projection.

**Proof.** Since objects of \( D^b(X_R \times_R X'_R) / D^b_0(X_R \times_R X'_R) \) are the same as those of \( D^b(X_R \times_R X'_R) / D^b_0(X_R \times_R X'_R) \) [HMS11, Appendix], the \( R \)-linear functor \( \Phi_E \) induces an integral functor
\[ \Phi_{E_K} : D^b(X_R) / D^b_0(X_R) \rightarrow D^b(X'_R) / D^b_0(X'_R). \]

By Corollary 2.6 we have the commutative diagram
\[
\begin{array}{ccc}
D^b(X_R) & \xrightarrow{\Phi_E} & D^b(X'_R) \\
\downarrow \downarrow & & \downarrow \downarrow \\
D^b(X_K) & \xrightarrow{\Phi_{E_K}} & D^b(X'_K).
\end{array}
\]

The inverse functor \( \Phi_{E_K}^{-1} \) is a left adjoint to \( \Phi_E \) as \( \Phi_E \) is an equivalence. On the other hand, due to the Grothendieck–Verdier duality \( \Phi_E \) has a left adjoint \( \Phi_{E_L} \) with \( \Phi_{E_L} \) a perfect complex on \( X_R \times_R X'_R \). By uniqueness of left adjoint up to isomorphism, it follows \( \Phi_{E_K}^{-1} \cong \Phi_{E_L} \). Then \( \Phi_{E_K}^{-1} \) induces an integral functor \( \Phi_{E_L \times_R X'_R} \) and we obtain natural isomorphisms \( \Phi_{E_L \times_R X'_R} \circ \Phi_{E_K} \cong \text{Id}_{D^b(X_R)} \), \( \Phi_{E_L} \circ \Phi_{E_{L \times_R X'_R}} \cong \text{Id}_{D^b(X'_R)} \). Thus the functor \( \Phi_{E_K} \) is an equivalence. \( \square \)

**Remark 4.1.** By universality of Verdier quotient, Corollary 2.6 induces a mere \( K \)-linear equivalence \( D^b(X_K) \cong D^b(X'_K) \), while Proposition 4.1 preserves Fourier–Mukai kernels.
Corollary 4.2. Let \( X_R, X'_R \) be smooth proper families over \( R \). Assume that \( X_R \) is locally projective. If \( \Phi_{E}: D^b(X_R) \to D^b(X'_R) \) is a relative integral functor whose restrictions to general fibers are equivalences, then the induced integral functor \( \Phi_{E_K}: D^b(X_K) \to D^b(X'_K) \) is an equivalence. Here, \( E_K \in D^b(X_R \times_R X'_R) / D^b(X_R \times_R X'_R) \) is the image of \( E \) by the natural projection.

Proof. We are given a Zariski open subset \( U \subset \text{Spec} R \) such that any pair of closed fibers of \( X_R \) and \( X'_R \) over \( U \) are derived-equivalent. By [LST13, Proposition 1.3] the relative integral functor \( \Phi_{E_K} \) is an equivalence, where \( E_U \) denotes the restriction of \( E \) to \( \text{pr}_1^{-1} \circ \pi_R^{-1}(U) \). Now, the claim follows immediately from Proposition 4.1. \( \square \)

Remark 4.2. The previous corollary provides from general to generic induction of Fourier–Mukai transforms. Conversely, by Corollary 2.6 any integral functor \( \Phi_{E_K}: D^b(X_K) \to D^b(X'_K) \) lifts to a relative integral functor \( \Phi_{E}: D^b(X_K) \to D^b(X'_K) \). In Section 6 we will prove that the induced functor is an equivalence when restricted to general fibers.

4.2. Induced Functor from generic to geometric generic fibers.

Lemma 4.3. ([Huy06, Exercise 5.18]) Let \( X_K, X'_K \) be smooth projective \( K \)-varieties and \( \Phi_{E_K}: D^b(X_K) \to D^b(X'_K) \) an integral functor. Then \( \Phi_{E_K} \) is an equivalence if and only if there are isomorphisms

\[
\mathcal{E}_K \ast (\mathcal{E}_K)_L \cong \mathcal{O}_\Delta, \quad (\mathcal{E}_K)_L \ast \mathcal{E}_K \cong \mathcal{O}_{\Delta'},
\]

where \( \Delta: X_K \hookrightarrow X_K \times X_K, \Delta': X'_K \hookrightarrow X'_K \times X'_K \) are the diagonal embeddings.

Proof. Assume that we are given the isomorphisms (4.1). Regarding the isomorphic objects as kernels, we obtain natural isomorphisms

\[
\Phi_{(\mathcal{E}_K)_L} \circ \Phi_{E_K} \cong \text{Id}_{D^b(X_K)}, \quad \Phi_{E_K} \circ \Phi_{(\mathcal{E}_K)_L} \cong \text{Id}_{D^b(X'_K)}.
\]

Thus the functor \( \Phi_{E_K} \) is an equivalence.

Conversely, assume that \( \Phi_{E_K} \) is an equivalence. Let \( (\Phi_{E_K})^{-1} \) be its inverse. Then we have natural isomorphisms

\[
(\Phi_{E_K})^{-1} \circ \Phi_{E_K} \cong \text{Id}_{D^b(X_K)}, \quad \Phi_{E_K} \circ (\Phi_{E_K})^{-1} \cong \text{Id}_{D^b(X'_K)}.
\]

In particular, it follows that \( (\Phi_{E_K})^{-1} \) is a left adjoint of \( \Phi_{E_K} \). By uniqueness of left adjoint up to isomorphism, we obtain \( (\Phi_{E_K})^{-1} \cong \Phi_{(E_K)_L} \) and (4.2). Thus two pairs of the kernel \( \mathcal{E}_K \ast (\mathcal{E}_K)_L, \mathcal{O}_\Delta \) and \( (\mathcal{E}_K)_L \ast \mathcal{E}_K, \mathcal{O}_{\Delta'} \) respectively define the same derived autoequivalence of \( D^b(X_K) \) and \( D^b(X'_K) \). Since any derived equivalence of smooth projective varieties is defined by a unique Fourier–Mukai kernel up to isomorphism [Orl97, Section 2], we obtain (4.1). \( \square \)

Proposition 4.4. Let \( X_K, X'_K \) be smooth projective \( K \)-varieties. Assume that \( X_K, X'_K \) are derived-equivalent. Then for the canonical embedding \( K \hookrightarrow \tilde{K} \) into the algebraic closure, the base change \( X_{\tilde{K}}, X'_{\tilde{K}} \) are derived-equivalent.

Proof. Let \( E_K \in D^b(X_K \times X'_K) \) be a Fourier–Mukai kernel, which is unique up to isomorphism. By Lemma 4.3 we have isomorphisms

\[
E_K \ast (E_K)_L \cong \mathcal{O}_\Delta, \quad (E_K)_L \ast E_K \cong \mathcal{O}_{\Delta'}.
\]

As \( \tilde{K} \) is a flat \( K \)-module, the pullback by \( \tilde{i}': X_{\tilde{K}} \times X_{\tilde{K}} \to X_K \times X_K \) yields

\[
E_{\tilde{K}} \ast (E_{\tilde{K}})_L \cong \tilde{i}'_* (E_K \ast (E_K)_L) \cong \tilde{i}'_* \mathcal{O}_\Delta \cong \mathcal{O}_{\tilde{\Delta}},
\]

where \( E_{\tilde{K}} = \tilde{i}'_* E_K \) and \( \tilde{\Delta}: X_{\tilde{K}} \hookrightarrow X_K \times X_K \) is the diagonal embedding. Here, we have used \( \Omega_{X_{\tilde{K}} // \tilde{K}} \cong \tilde{i}'_* \Omega_{X_K // K} \) [Har77, Proposition II8.10]. Similarly, we have \( (E_K)_L \ast E_K \cong \mathcal{O}_{\tilde{\Delta'}} \). Again, by Lemma 4.3 we conclude that \( \Phi_{E} \) is an equivalence. \( \square \)
Remark 4.3. When Spec $R$ is a nonsingular affine $k$-variety and $k$ is a universal domain, projective very general fibers of $X_R, X'_R$ are derived-equivalent if and only if so are their geometric generic fibers. Indeed, by [Via13] Lemma 2.1 there is an isomorphism $k \to \bar{K}$ along which the pullback of $X_k \times X'_k$ is isomorphic to $X_R \times X'_R$. Here, $X_k, X'_k$ are very general fibers of $X_R, X'_R$. One can apply the same argument as in the proof of Proposition 4.4. Note that we assume $k$ to be a universal domain to apply [Via13] Lemma 2.1.

4.3. **Induced Functors to effective formal families and their categorical general fibers.** Assume that the families $\pi_R: X_R \to \text{Spec } R$, $\pi'_R: X'_R \to \text{Spec } R$ are effectivizations of formal deformations $X, X'$ over $R$ of smooth projective varieties $X_0, X'_0$, respectively. Assume further that $\pi_R, \pi'_R$ are proper. Here, we will show that the induced Fourier–Mukai functor from smooth proper families to their generic fibers is compatible with formal completion along the closed fibers.

The schemes $X_R, X'_R$, their restrictions to the $n$-th order thickenings, and their formal completions along the closed fibers form the commutative diagram

![Diagram](image)

of noetherian formal schemes. Here, $\hat{q}, \hat{p}$ are canonically determined as the limit by the compatible collections of morphisms $q_n, p_n$ of schemes, and compositions of two sequential vertical arrows give the canonical factorizations of the closed embeddings

$$k_n = t \circ \tau_n, \quad k''_n = t'' \circ \tau''_n, \quad k'_n = t' \circ \tau'_n.$$  

**Proposition 4.5.** Given $\mathcal{E} \in D^b(X_R \times_R X'_R)$, the formal completion $\hat{\mathcal{E}}$ along the closed fiber $X_0 \times X'_0$ defines a relative integral functor

$$\Phi_{\hat{\mathcal{E}}} = R\hat{p}_*(\hat{\mathcal{E}} \otimes^{\mathbb{L}} \hat{q}(-)): D^b(X) \to D^b(X'),$$

which makes the diagram

![Diagram](image)

2-commutative.

**Proof.** Since objects of $D^b(X'), D^b(X \times_R X')$ are quasi-isomorphic to perfect complexes, the functors $\hat{q}: D^b(X) \to D^b(X \times_R X')$ and $\hat{E} \otimes^{\mathbb{L}} (-): D^b(X \times_R X') \to D^b(X \times_R X')$ can be computed by termwise application after replacing objects with perfect complexes. It is known that $R\hat{p}_*$ is well-defined and sends bounded complexes to bounded complexes by the comparison theorem [GD61 Corollary 4.1.7] and Leray spectral sequence. Since by the equivalence (3.1) any object
of $D^b(X)$ can be written as $\hat{E} = G(E) \cong L^*E$ for some object $E \in D^b(X_R)$, we have

$$\Phi_{E^!}(\hat{E}) \otimes_{R}^L R_n = L\tau_n^*R\hat{p}_*(\hat{E} \otimes_{\hat{R}}^L Lq^!\hat{E})$$
$$\cong Rp_nL\tau_n^{**}(E \otimes_{R}^L Lq^*E)$$
$$\cong Rp_n(E_n \otimes_{R_n} q_n^*E)$$
$$\cong \Phi_{E_n}(E_n),$$

where $R_n = R/m_R^{n+1}$, $E_n = \tau_n^*\hat{E}$, and the first isomorphism follows from the comparison theorem. We also have

$$\Phi_{E}(E) \otimes_{R}^L R_n = Lk_n^*Rp_*(E \otimes_{R}^L Lq^*E)$$
$$\cong Rp_n(E_n \otimes_{R_n} q_n^*E)$$
$$\cong \Phi_{E_n}(E_n).$$

Thus we obtain isomorphisms

$$f_n: \Phi_{E^!}(\hat{E}) \otimes_{R}^L R_n \to \Phi_{E}(E) \otimes_{R}^L R_n$$

for any positive integer $n$ satisfying $f_{n+1} \otimes_{R_n}^L id_{R_n} = f_n$. Note that $\Phi_{E^!}(\hat{E})$ is the formal completion of some perfect complex on $X'_R$, as it belongs to $D^b(X') \cong D^b(X'_R)$. Then, by the argument in the proof of [HMST11, Lemma 3.4], taking the limit yields an isomorphism

$$f: \Phi_{E^!}(\hat{E}) \to G(\Phi_{E}(E))$$

which completes the proof.

**Corollary 4.6.** Given a Fourier–Mukai kernel $E \in D^b(X_R \times_R X'_R)$, the functors $\Phi_{E^!}: D^b(X) \to D^b(X')$ and $\Phi_{E_E^!}: D^b(Coh(X_K)) \to D^b(Coh(X'_K))$ are equivalences, where $\hat{E}_K \in D^b(X \times_R X'/K)/D_0^b(X \times_R X')$ is the image of $\hat{E}$ by the natural projection.

5. Fourier–Mukai partners over the closure of function fields

In this section, passing through the deformation theory, we provide new examples of Fourier–Mukai partners, pairs of nonbirational Calabi–Yau threefolds that are derived-equivalent. Our results play a role in deducing the derived equivalences. Let $X_0, X'_0$ be derived-equivalent Calabi–Yau manifolds of dimension more than two. There exists a nonsingular affine $k$-variety $\text{Spec} \ S$ such that smooth projective versal deformations $X_S, X'_S$ over $S$ are derived-equivalent [Mor, Theorem 1.1]. Also effectivizations $X_R, X'_R$ are shown to be derived-equivalent. These equivalences are given by deformed Fourier–Mukai kernels. From our results in Section 4 it immediately follows that the geometric generic fibers of $X_S, X'_S$ and $X_R, X'_R$ are respectively derived-equivalent. One can check that the geometric generic fibers are Calabi–Yau manifolds. If $X_0, X'_0$ satisfy either $\text{NS}_{tor} X_0 \neq \text{NS}_{tor} X'_0$, or $\rho(X_0) = \rho(X'_0) = 1$ and $\deg(X_0) \neq \deg(X'_0)$, then the geometric generic fibers are nonbirational as well as $X_0, X'_0$. Several pairs are known to satisfy one of the two conditions. Thus we obtain new examples of Fourier–Mukai partners over the closure $\bar{K}, \bar{Q}$ of the function fields. Although $\bar{Q}$ is isomorphic to $k$, the Fourier–Mukai partner $X_Q, X'_Q$ are different as a variety from known examples. We demonstrate this difference when $X_0, X'_0$ are the famous Pfaffian–Grassmannian pair.
5.1. **Deformations of Calabi–Yau manifolds.** Let \( X_0 \) be a Calabi–Yau manifold with dimension more than two. The deformation functor

\[
\text{Def}_{X_0} : \text{Art}_k \rightarrow \text{Set}
\]

of \( X_0 \) has a universal formal family \((R, \xi)\), where \( R \) is a formal power series ring with \( \dim_k H^1(X_0, \mathcal{O}_{X_0}) \) valuably, and \( \xi \) belongs to the limit

\[
\text{Def}_{X_0}(R) = \lim_{\leftarrow} \text{Def}_{X_0}(R/m^n_R)
\]

of the inverse system

\[
\cdots \rightarrow \text{Def}_{X_0}(R/m^{n+2}_R) \rightarrow \text{Def}_{X_0}(R/m^{n+1}_R) \rightarrow \text{Def}_{X_0}(R/m^n_R) \rightarrow \cdots
\]

induced by the natural quotient maps \( R/m^{n+1}_R \rightarrow R/m^n_R \). Here, \( m_R \subset R \) is the maximal ideal.

The formal family \( \xi \) corresponds to a natural transformation

\[
h_R = \text{Hom}_{k\text{-alg}}(R, -) \rightarrow \text{Def}_{X_0},
\]

which sends each homomorphism \( f \in h_R(A) \) factorizing through

\[
R \rightarrow R/m^{n+1}_R \xrightarrow{\xi} A
\]

to \( \text{Def}_{X_0}(g)(\xi_a) \). Let \( X_a \) be the schemes defining \( \xi_a \). There is a noetherian formal scheme \( X \) over \( R \) such that \( X_n \cong X \times_R R/m^{n+1}_R \) for each \( n \). Thus for a deformation \((X_A, i_A)\) the scheme \( X_A \) can be obtained as the pullback of \( X \) along some morphism of noetherian formal schemes \( \text{Spec} A \rightarrow \text{Spf} R \).

Now, we briefly recall how to algebrize \( X \). By [GD61, Theorem III5.4.5] there exists a scheme \( X_R \) flat projective over \( R \) whose formal completion along the closed fiber \( X_0 \) is isomorphic to \( X \). Moreover, \( X_R \) is smooth over \( R \) of relative dimension \( \dim X_0 \) [Mor, Lemma 2.4]. We call \( X_R \) an effectivization of \( X \). Consider the extended functor

\[
\text{Def}_{X_0} : k\text{-alg}^{\text{aug}} \rightarrow \text{Set}
\]

from the category of augmented noetherian \( k \)-algebras. Let \( T = k[t_1, \ldots, t_d] \) and \( t \in \text{Spec} T \) be the closed point corresponding to maximal ideal \((t_1, \ldots, t_d)\). There is a filtered inductive system \( \{R_i\}_{i \in I} \) of finitely generated \( T \)-subalgebras of \( R \) whose colimit is \( R \). Since \( \text{Def}_{X_0} \) is locally of finite presentation, \([X_R, i_R] \) is the image of some element \( \zeta_i \in \text{Def}_{X_0}((R_i, m_{R_i})) \) by the canonical map \( \text{Def}_{X_0}((R_i, m_{R_i})) \rightarrow \text{Def}_{X_0}(R) \). By [Art69, Corollary 2.1] there exists an \( \text{étale} \) neighborhood \( \text{Spec} S \) of \( t \) in \( \text{Spec} T \) with first order approximation \( \varphi : R_i \rightarrow S \) of \( R_i \leftarrow R \). Let \( [X_S, i_S] \) be the image of \( \zeta_i \) by the map \( \text{Def}_{X_0}(\varphi) \). The formal completion of \( X_S \) along the closed fiber \( X_0 \) is isomorphic to \( X \).

The triple \((\text{Spec} S, s_0, X_S)\), or sometimes \( X_S \), is called a versal deformation of \( X_0 \). By construction, \( \text{Spec} S \) is an algebraic \( k \)-variety with a distinguished closed point \( s_0 \) mapping to \( t \), and \( X_S \) is flat of finite type over \( S \) whose closed fiber over \( s_0 \) is \( X_0 \). In our setting, one can find a versal deformation \( X_S \) smooth projective over a nonsingular affine \( k \)-variety \( \text{Spec} S \) of relative dimension \( \dim X_0 \) [Mor, Lemma 2.3]. Moreover, given another Calabi–Yau manifold \( X'_0 \) derived-equivalent to \( X_0 \), one can find a smooth projective versal deformation \( X'_S \) over the same base. The construction passes through effectivizations. Namely, there are effectivizations \( X_R, X'_R \) of \( X_0, X'_0 \) over the same regular affine \( k \)-scheme \( \text{Spec} R \).
5.2. Calabi–Yau geometric generic fibers. First, we consider the effectivization $X_R$ of $X$ and its geometric generic fiber. We have the pullback diagram

$$
\begin{array}{ccc}
X_R & \xrightarrow{i} & X_K \\
\pi_K & & \pi_K \\
\Spec \bar{K} & \xrightarrow{\bar{j}} & \Spec \bar{K} \\
& \pi & \bar{\pi} \\
& \Spec K & \xrightarrow{j} \Spec R,
\end{array}
$$

where $K$ is the field of fractions of $R$ and $\bar{K}$ is the closure of $K$.

Lemma 5.1. The geometric generic fiber $X_K$ is a Calabi–Yau manifold over $\bar{K}$.

Proof. Smoothness and projectivity follow from their being stable under base change. One can apply [GD66 Proposition IV15.5.7] to see that $X_K$ is connected. Then $X_K$ must be irreducible, as it is regular. By [Har77 Theorem III12.8] the function $h^0 : \Spec R \rightarrow \mathbb{Z}$ defined as

$$
h^0(r) = \dim_k H^0(X_r, \omega_{X_r/n_1} \mathcal{F}_{X_r} \otimes_R k(r))
$$

for $r \in \Spec R$ is upper semicontinuous, where $\mathcal{F}_{X_r}$ is the relative tangent sheaf. It follows that there is an open neighborhood $U$ of the closed point to which the restriction vanishes. Since $R$ is a domain, $U$ contains the generic point. By flat base change we obtain

$$
H^0(X_K, \omega_{X_K/n_1} \mathcal{F}_{X_K} \otimes_R K) = 0.
$$

Similarly, one can show the vanishing of all the other relevant cohomology.

It remains to show the triviality of the canonical bundle. Consider the formal completion $\hat{\omega}_{X/R}$ of the relative canonical sheaf on $X_R$ along the closed fiber $X_0$. It is given by the limit of inverse system $\{\omega_{X_{n/R}} \otimes R_n \}$. Here, the inverse system consists of the sequence of deformations of $\omega_{X_0}$ along order by order square zero extensions. Since we have $\omega_{X_0} \cong \mathcal{O}_{X_0}$, the inverse system $\{\mathcal{O}_{X_{n/R}} \} \otimes R_n$ also consists of the sequence of deformations of $\omega_{X_0}$. On the other hand, by [Lie06 Theorem 3.1.1] freedom of deformations of $\omega_{X_0}$ as a perfect complex to $X_1$ is given by $\text{Ext}^1_{X_0}(\omega_{X_0}, \omega_{X_1})$, where $l_1$ is the dimension of $k$-vector space $m_1/m_1^2$. This is trivial by the assumption on $X_0$ and there is an isomorphism $\omega_{X_1/R_1} \cong \mathcal{O}_{X_1}$ respecting $\omega_{X_1/R_1} \otimes R_1 \cong \omega_{X_0}$ and $\mathcal{O}_{X_1} \otimes R_1 \cong \mathcal{O}_{X_0}$, $\mathcal{O}_{X_1} \otimes R_1 \cong \mathcal{O}_{X_0}$. Inductively, one finds isomorphisms $\omega_{X_n/R_n} \cong \mathcal{O}_{X_n}$ respecting $\omega_{X_n/R_n} \otimes R_n \cong \omega_{X_0}$ and $\mathcal{O}_{X_n} \otimes R_n \cong \mathcal{O}_{X_0}$. By universality of limit, we obtain $\hat{\omega}_{X/R} \cong \mathcal{O}_{X}$, which in turn induces $\omega_{X/R} \cong \mathcal{O}_{X}$ via the equivalence (3.1).

Next, we consider the versal deformation $X_\mathcal{Q}$ of $X_0$ and its geometric generic fiber. We have the pullback diagram

$$
\begin{array}{ccc}
X_\mathcal{Q} & \xrightarrow{\bar{u}} & X_{\bar{Q}} \\
\pi_0 & & \pi_0 \\
\Spec \bar{Q} & \xrightarrow{\bar{v}} & \Spec Q \\
& \bar{\pi}_0 & \bar{\pi}_0 \\
& \Spec S & \xrightarrow{v} \Spec S,
\end{array}
$$

where $\mathcal{Q}$ is the field of fractions of $S$ and $\bar{Q}$ is the closure of $\mathcal{Q}$.

Lemma 5.2. The geometric generic fiber $X_\mathcal{Q}$ is a Calabi–Yau manifold over $\bar{Q}$.

Proof. Nontrivial part is the vanishing of the canonical bundle. Consider the collection $\{\omega_{X_n/R_n}\}_{n \in I}$ of relative canonical sheaves on $X_R$. It consists of the sequence of deformations of $\omega_{X_0}$. Since we have $\omega_{X_0} \cong \mathcal{O}_{X_0}$, the collection $\{\mathcal{O}_{X_n} \}_{n \in N_0}$ of structure sheaves also consists of the sequence of deformations of $\omega_{X_0}$. We have $\omega_{X_n/R_n} \cong \omega_{X_n/R_n} \otimes R_n$ by [Har77 Proposition III10.10] and the construction of $X_\mathcal{Q}$. One can apply [Lie06 Proposition 2.2.1] to find an isomorphism $\omega_{X_n/R_n} \cong \mathcal{O}_{X_n}$.
for sufficiently large $i$ with respect to the partial order of $I$. We obtain $\omega_{X_S/I} \cong \mathcal{O}_{X_S}$, again by [Har77, Proposition II.8.10] and the construction of $X_S$.

**Remark 5.1.** Assuming $k = \mathbb{C}$, one can show the previous lemma without using the deformation theory of perfect complexes as follows. We include the proof as we could not find any in the literature. Since $\pi_S$ is smooth, we have the short exact sequence
\[
0 \to \pi_S^*\Omega_S \to \Omega_{X_S} \to \Omega_{X_S/I} \to 0.
\]
Recall that Spec $S$ is an étale neighborhood of $T \in$ Spec $T$, which implies $\Omega_S/T \equiv 0$. It follows $\omega_S \equiv \mathcal{S}$, as any line bundle on Spec $T$ is trivial. We obtain
\[
\omega_{X_S} \equiv \omega_{X_S/S} \otimes \pi_S^*\omega_S \equiv \omega_{X_S/S}.
\]

On the other hand, for each closed point $s \in$ Spec $S$, we have the short exact sequence
\[
0 \to \mathcal{I}/(\mathcal{I}^2) \to \Omega_{X_s}|_{X_s} \to \Omega_{X_s} \to 0,
\]
where $\mathcal{I} \subset \mathcal{O}_{X_s}$ is the defining ideal sheaf of the fiber $X_s$ over $s$. Let $m_s \subset S$ be the maximal ideal corresponding to $s$. Then $\mathcal{I}/(\mathcal{I}^2)$ is the pullback of $k$-vector space $m_s/m_s^2$ by $\pi_S$. We obtain
\[
\omega_{X_s}|_{X_s} \equiv \det(\mathcal{I}/(\mathcal{I}^2) \otimes X_s) \equiv \omega_{X_s}.
\]
If $\omega_{X_s}$ is trivial for each closed point $s \in$ Spec $S$, then we have $\omega_{X_s}|_{X_s} \equiv \mathcal{O}_{X_s}$. Note that $X_s$ regarded as an algebraic variety is covered with all closed fibers. This implies that any possible transition function of the line bundle $\omega_{X_s}$ is trivial at each closed point of $X_S$. Thus we conclude $\omega_{X_S} \equiv \mathcal{O}_{X_S}$.

Finally, we show the triviality of the canonical bundle of $X_s$ for each closed point $s \in$ Spec $S$. Let $\omega_s = \omega_{X_s}|_{X_s}$ and $H$ be an ample divisor on $X_s$. Consider the Hilbert polynomial
\[
P_s(m) = \chi(X_s, \omega_s(mH)).
\]
It is independent from $s$, as $X_s$ is flat over a noetherian integral scheme Spec $S$. Now, we will use the assumption $k = \mathbb{C}$. One can apply Riemann–Roch theorem to obtain
\[
\chi(X_s, \omega_s(mH)) = c_1(\omega_s)H^\dim X_s - g(m),
\]
where $g$ is some polynomial. Since we may choose arbitrary $H$, one sees that $c_1(\omega_s)$ must be independent from $s$. Thus $c_1(\omega_s) = 0$ as an element of $H^2(X_s, \mathbb{R})$.

**Remark 5.2.** Assume that $k$ is a universal domain. Let $X_k$ be a very general fiber of $X_S$. Choose a subfield $L \subset k$ over which $X_k$ is defined, i.e., we are given a $L$-variety $X_L$ such that the base change $X_L \times_k k$ is isomorphic to $X_k$. From the proof of [Via13, Lemma 2.1] there exists an isomorphism $k \to \mathcal{Q}$ fixing $L$ such that the base change $X_k \times_k \mathcal{Q}$ is isomorphic to $X_\mathcal{Q}$. Note that $X_k$ and $X_\mathcal{Q}$ are isomorphic as an abstract scheme, but not as a $k$-variety.

### 5.3. The derived equivalence.

Suppose that $X_0$ is derived-equivalent to another Calabi–Yau manifold $X'_0$. Recall that there are effectivizations $X_R, X'_R$ over the same regular affine $k$-scheme Spec $R$. We have the pullback diagram
\[
\begin{array}{ccc}
X_R \times_X X'_R & \xrightarrow{\pi} & X_R \times X'_R \\
\pi_R \times _{\pi'_R} & \downarrow & \downarrow \pi_R \times \pi'_R \\
\text{Spec } \mathcal{K} & \longrightarrow & \text{Spec } K \longrightarrow \text{Spec } R.
\end{array}
\]

Let $\mathcal{E}_0$ be a Fourier–Mukai kernel. By [Mor, Proposition 3.3, Corollary 4.2] one can deform $\mathcal{E}_0$ to a Fourier–Mukai kernel $\mathcal{E}$ on $X_R \times_X X'_R$. Applying Proposition 4.1 and Proposition 4.4, we obtain the derived equivalence of geometric generic fibers
\[
\Phi_{(\pi_R \otimes \pi'_R)^*\mathcal{E}} : D^b(X_R) \to D^b(X'_R).
\]
Remark 5.3. Applying Proposition 4.5 and Corollary 4.6, we obtain the derived equivalence of universal formal families

$$\Phi_E: D^b(X) \to D^b(X')$$

and their categorical general fibers

$$\Phi_{\hat{E}_K}: D^b(\operatorname{Coh}(X_K)) \to D^b(\operatorname{Coh}(X'_K))$$

respectively. Here, $\hat{E}$ is the formal completion of $E$ along the closed fiber and $\hat{E}_K$ is the image of the natural projection. Note that $\hat{E}_K$ is the image of $i''''E$ by (3.1).

Recall that there are smooth projective versal deformations $X_S, X'_S$ of $X_0, X'_0$ over the same nonsingular affine $k$-variety $\Spec S$. We have the following pullback diagram

$$
\begin{array}{ccc}
X_Q \times X'_Q & \xrightarrow{j''} & X_Q \times X'_Q \\
\pi_Q \times \pi'_Q & & \pi_Q \times \pi'_Q \\
\Spec \hat{Q} & \xrightarrow{\phi} & \Spec Q \\
\end{array}
$$

By [Mor, Proposition 3.3] one can deform $E_0$ to a perfect complex $E_S$ on $X_S \times_S X'_S$. After possible shrinking of $\Spec S$, the relative integral functor $\Phi_{E_S}$ is an equivalence [Mor, Theorem 1.1]. Applying Proposition 4.1 and Proposition 4.4, we obtain the derived equivalence of geometric generic fibers

$$\Phi_{(j''''\circ j''')^*E}: D^b(X_Q) \to D^b(X'_Q).$$

5.4. Nonbirationality. Let $X$ be a smooth proper variety over an algebraically closed field.

Recall that the Néron–Severi group $\NS X$ is the quotient of the Picard group $\Pic X$ by the subgroup $\Pic^0 X$ of isomorphism classes of line bundles which are algebraically equivalent to 0. The group $\NS X$ is a finitely generated with rank $\rho(X)$ called the Picard number. We denote by $\NS_{tor} X$ the subgroup of torsion elements, which is known to be a birational invariant.

Lemma 5.3. If $\NS_{tor} X_0, \NS_{tor} X'_0$ are nonisomorphic, then $X_K, X'_K$ are nonbirational.

Proof. By [MP12, Proposition 3.6] there is an injection

$$\sp_{K,k}: \NS X_K \to \NS X_0$$

whose cokernel is torsion free. In particular, $\sp_{K,k}$ is bijective on torsion subgroups. Then $\NS_{tor} X_K, \NS_{tor} X'_K$ cannot be isomorphic. \hfill \Box

The same argument yields

Lemma 5.4. If $\NS_{tor} X_0, \NS_{tor} X'_0$ are nonisomorphic, then $X_Q, X'_Q$ are nonbirational.

If $\rho(X_0) = \rho(X'_0) = 1$, then $X_0, X'_0$ are birational if and only if they are isomorphic [BC09, Section 0.5]. Indeed, birational Calabi–Yau manifolds are connected by a sequence of flops [Kaw08]. However, no such flops are possible on neither $X_0$ nor $X'_0$, since by $\rho(X_0) = \rho(X'_0) = 1$ all their nonzero nef divisors are ample.

Lemma 5.5. Assume that $\rho(X_0) = \rho(X'_0) = 1$ and $\deg(X_0) \neq \deg(X'_0)$. Then $X_K, X'_K$ are nonbirational. Here, the degree is defined with respect to the unique ample generator of the Picard group.
Proof. By \cite[Proposition 3.6]{MP12} we have \( \rho(X_0) \leq \rho(X_0') = 1 \). There is an ample divisor \( H \) on \( X_0 \), as it is a projective variety of positive dimension. It follows that \( H \) is neither torsion nor algebraically equivalent to 0. Indeed, torsion divisors are numerically trivial and one of the two numerically effective divisors is ample if and only if so is the other. Two algebraically equivalent divisors share the degree. Hence we obtain \( \rho(X_0) = 1 \).

Recall that \( \deg(X_0) \) is the highest order coefficient of the Hilbert polynomial of \( X_0 \) multiplied with \((\dim X_0)! \). Since \( \pi_0 \) is flat projective, we have \( \deg(X_0) = \deg(X_0') \). Let \( S(X_0) \) be the homogeneous coordinate ring of \( X_0 \) and \( P_{X_0} \) the Hilbert polynomial of \( X_0 \). By definition \( P_{X_0}(l) \) are given by \( \dim_{K} S(X_0)_l \) for sufficiently large integers \( l \in \mathbb{Z} \). Since \( X_0 \) is irreducible, \( \dim_{K} S(X_0)_l \) is stable under the base change \( X_0 \to X_0' \) along algebraic extension \( K \subset \bar{K} \). Thus we obtain \( \deg(X_0') = \deg(X_0) \) and \( \deg(X_0') \neq \deg(X_0) \).

\( \square \)

**Lemma 5.6.** Assume that \( \rho(X_0) = \rho(X_0') = 1 \) and \( \deg(X_0) \neq \deg(X_0') \). Then \( X_0, X_0' \) are nonbirational. Here, the degree is defined with respect to the unique ample generator of the Picard group.

Proof. The same argument as above works also here. Assuming that \( k = \mathbb{C} \), we show \( \rho(X_0) = 1 \) in another way. By \cite{Ser56} the morphism \( \pi_0 : X_0 \to \text{Spec} S \) corresponds to a proper submersion of complex manifolds \((\pi_0)_* : (X_0)_h \to (\text{Spec} S)_h \). Ehresmann lemma tells us that \((\pi_0)_* \) gives a locally trivial fibration of real manifolds \cite{Ehr52}. In particular, all the fibers of \((\pi_0)_* \) share the differential type and \( H^2((X_0)_h, \mathbb{Z}) \) is independent from closed points \( s \in \text{Spec} S \). On the other hand, we have \( \text{NS}_{X_0} \cong \text{Pic}_{X_0} \cong H^1((X_0)_h, \mathbb{Z}) \), as \( X_0 \) are Calabi–Yau threefolds. By \cite[Theorem 1.1]{MP12}, we obtain \( \rho(X_0) = \rho(X_0') = 1 \).

In summary, we obtain

**Theorem 5.7.** Let \( X_0, X_0' \) be derived-equivalent Calabi–Yau manifolds of dimension more than two. Then the geometric generic fibers \( X_{\bar{k}}, X'_{\bar{k}} \) of proper effectivizations and that \( X_{\bar{k}}, X'_{\bar{k}} \) of smooth projective versal deformations are respectively derived-equivalent Calabi–Yau manifolds. If, in addition, we have either \( \text{NS}_{tor} X_0 \neq \text{NS}_{tor} X_0' \) or \( \rho(X_0) = \rho(X_0') = 1 \) and \( \deg(X_0) \neq \deg(X_0') \), then they are respectively nonbirational.

5.5. Geometric generic Gross–Popescu pair. In this subsection, we temporarily assume that \( k = \mathbb{C} \). Now, we will consider the case of the Gross–Popescu pair

\[ X_0 = V^{1}_{8,3}, \quad X_0' = V^{1}_{8,3}/G \]

where \( V^{1}_{8,3} \) is one of the two small resolutions of \( V_{8,3} \) and \( G = \mathbb{Z}_8 \times \mathbb{Z}_6 \) freely acts on \( V^{1}_{8,3} \). Here, \( V_{8,3} \) is a 3-dimensional complete intersection in \( \mathbb{P}^8 \) of four hypersurfaces parametrized by a general point \( y \in \mathbb{P}^2 \). They are derived-equivalent Calabi–Yau threefolds with \( h^{1,1}(X_0) = h^{2,1}(X_0) = 2 \) \cite{GP01}. Since \( X_0 \) is simply-connected \cite[Theorem 1.4]{GP01}, we have \( H^i(X_0, \mathcal{O}_{X_0}) = 0 \), \( i = 1, 2 \). Although the fundamental group of \( X_0 \) is given by \( G \neq 0 \), we also have \( H^i(X_0', \mathcal{O}_{X_0'}) = 0 \), \( i = 1, 2 \) by \cite[Corollary B]{PS97}. Then \( \text{NS}_{tor} X_0 \neq \text{NS}_{tor} X_0' \), or in this case equivalently \( \text{Pic}_{tor} X_0 \neq \text{Pic}_{tor} X_0' \). Indeed, from the exponential sequence it follows \( \text{Pic} X_0 = H^2((X_0)_h, \mathbb{Z}) \). The torsion part of \( H^2((X_0)_h, \mathbb{Z}) \) is \( \text{Tor}_1(H_1((X_0)_h, \mathbb{Z}), \mathbb{C}) = 0 \) by the universal coefficient theorem and Van Kampen’s theorem. On the other hand, the torsion part of \( H^2((X_0')_h, \mathbb{Z}) \) is \( \text{Tor}_1(H_1((X_0')_h, \mathbb{Z}), \mathbb{C}) = G \). Thus one can apply Theorem 5.7 to see that \( X_{\bar{k}}, X'_{\bar{k}} \) and \( X_{\bar{k}}, X'_{\bar{k}} \) are respectively derived-equivalent but nonbirational Calabi–Yau threefolds.

5.6. Geometric generic Pfaffian–Grassmannian pair. Next, we will consider the case of the Pfaffian-Grassmannian pair

\[ X_0 = \text{Gr}(2, V_7) \cap \mathbb{P}(W), \quad X_0' = \text{Pf}(4, V_7) \cap \mathbb{P}(W^\perp) \]
where $V_7$ is a 7-dimensional $k$-vector space, $W$ is a 14-dimensional general quotient vector space of $\wedge^3 V_7 \rightarrow W$, and $W^\perp = \text{Coker}(W' \hookrightarrow \wedge^2 V_7')$. They are derived-equivalent Calabi–Yau threefolds with $\rho(X_0) = \rho(X_0') = 1$ and $\deg(X_0) \neq \deg(X_0')$. One can apply Theorem 5.7 to see that $X_k, X_k'$ and $X_0, X_0'$ are respectively derived-equivalent but nonbirational Calabi–Yau threefolds. Similarly, one obtains another example from Reye congruence and double quintic symmetroid Calabi–Yau threefolds [HT16].

We will study $X_0, X_0'$ slightly further. Assume that $k$ is a universal domain. Let $L \subset k$ be a subfield over which $X_0, X_0'$ are defined, i.e., we are given $L$-varieties $X_L, X_L'$ such that $X_L \times_L k \cong X_0, X_L' \times_L k \cong X_0'$. From the proof of [Via13, Lemma 2.1] there exists an isomorphism $k \rightarrow \hat{Q}$ fixing $L$ such that the base change of very general fibers $X_s, X_s'$ of $X_L, X_L'$ are respectively isomorphic to $X_\hat{Q}, X_\hat{Q}'$. Note that by [Mor, Lemma 5.1] and [KK16, Corollary 6.3], general fibers of their smooth projective versal deformations $X_s, X_s'$ are isomorphic to $X_0, X_0'$. The isomorphisms $X_s \cong X_0, X_s' \cong X_0'$ of $k$-varieties induces that

$$X_0 \cong X_\hat{Q}, \quad X_0' \cong X_\hat{Q}'$$

of abstract schemes.

There is another Fourier–Mukai partner called IMOU varieties [IMOU, Kuz18], consisting of derived-equivalent Calabi–Yau threefolds $Y_0, Y_0'$ which are deformation equivalent to $X_0, X_0'$ respectively [KK16, IIM19]. Extending $L$ if necessary, we may assume that also $Y_0, Y_0'$ are defined over $L$. Then either of them fails to be isomorphic to $X_\hat{Q}, X_\hat{Q}'$ as an abstract scheme, otherwise we would have $X_L \cong Y_L$ and $X_L' \cong Y_L'$. Hence $Y_0, Y_0'$ cannot be isomorphic $X_\hat{Q}, X_\hat{Q}'$ at the same time even as an abstract scheme. Thus $X_\hat{Q}, X_\hat{Q}'$ provide a new example of Fourier–Mukai partners. They are isomorphic to $X_0, X_0'$ as an abstract scheme, but have different structures from both the Pfaffian–Grassmannian pair and IMOU varieties as a variety. \begin{remark}
General fibers of smooth projective versal deformations of $Y_0, Y_0'$ are isomorphic to $X_0, X_0'$ as a $k$-varieties. This implies that one can obtain $X_\hat{Q}, X_\hat{Q}'$ starting from IMOU varieties. Moreover, one sees that $X_\hat{Q}, X_\hat{Q}'$ are nonbirational, otherwise $X_0, X_0'$ must be birational.
\end{remark}

As in this case, when deformations of Fourier–Mukai partners are well understood, one can deduce nonbirationality of the geometric generic fibers immediately. For instance, $GPK^3$ threefolds [BCP20] are isomorphic to general fibers of smooth projective versal deformations of Kapustka–Rampazzo varieties [KR19] as a $k$-varieties [IIM19, Proposition 4.7]. Then by the same argument the geometric generic fibers are nonbirational and one obtains another Fourier–Mukai partner. Note that for Pfaffian–Grassmannian pair and $GPK^3$ threefolds, the derived equivalences of the geometric generic fibers stems from birationality of noncompact Calabi–Yau manifolds connected by simple $K$-flops [Ued19, Mor21]. \begin{remark}
In this case $X_\hat{Q}, X_\hat{Q}'$ are also $\mathbb{L}$-equivalent. Indeed, the isomorphism $k \rightarrow \hat{Q}$ induces that $K_0(\text{Var}_k) \rightarrow K_0(\text{Var}_\hat{Q})$ of Grothendieck rings. It maps the relation in [Mar16, Theorem 1.1] to

$$([X_\hat{Q}] - [X_\hat{Q}']) \cdot \mathbb{L}_\hat{Q}^6 = 0.$$

\end{remark}

6. Specialization of derived equivalence

As advertised, for flat proper families of $k$-varieties over a common base we study the induced derived equivalence from their generic to general fibers. The key is the ability of Corollary 2.6 to lift Fourier–Mukai kernels along the projection. First, although it is not strictly necessary for our purpose, we extend Corollary 2.6 to nonaffine base case for flat proper families of $k$-varieties. It suffices to show that, when restricted to general fibers, the relative integral functor defined by the lift admits fully faithful left adjoints. As in the proof of [Mor, Theorem
1.1], we show that the associated counit morphism is a natural isomorphism. However, since in general the generic finer is not a subscheme of a family, we have to adapt the proof as follows. Shrinking the base, we remove torsion parts with respect to the base from a fixed strong generator and its relevant Hom-sets. Then we invoke some basic categorical results to show that the value of the counit morphism on the trimmed strong generator is an isomorphism, which implies that the restriction of the counit morphism is a natural isomorphism.

6.1. Lifts of Fourier–Mukai kernels. Let \( \pi: X \to S \) be a flat proper morphism of \( k \)-varieties. Since \( S \) is integral, the function field \( K = K(S) \) is given by local ring \( \mathcal{O}_{X, \xi} \) and it coincides with the field of fractions \( \mathbb{Q}(R) \) for any affine open \( k \)-subvariety \( U = \text{Spec} \ R \) [Har77, Exercise II3.6]. Hence we have the following pullback diagram

\[
\begin{array}{c}
X_\xi \\
\| \downarrow \pi \\
\text{Spec } K \\
\| \downarrow \pi \\
S
\end{array}
\]

where \( \iota_\xi \) is the canonical morphism.

**Definition 6.1.** The **categorical generic fiber** of \( \pi: X \to S \) is the Verdier quotient

\[
\mathcal{D}^b(X)/\text{Ker}(\iota_\xi^*),
\]

where \( \text{Ker}(\iota_\xi^*) \) is the kernel [SP, Tag 05RF] of the exact functor \( \iota_\xi^*: \mathcal{D}^b(X) \to \mathcal{D}^b(X_\xi) \).

Recall that for a smooth separated family \( \pi_R: X_R \to \text{Spec} \ R \) over a nonsingular affine \( k \)-variety, by Corollary 2.6 there exists a \( \mathbb{Q}(R) \)-linear exact equivalence

\[
\mathcal{D}^b(X_R)/\mathcal{D}_0^b(X_R) \simeq \mathcal{D}^b(X_\xi),
\]

where \( \mathcal{D}_0^b(X_R) \) is the full triangulated subcategory spanned by complexes with \( R \)-torsion cohomology. The above definition is an extension of this local description in the following sense.

**Theorem 6.1.** Let \( \pi: X \to S \) be a flat proper morphism of \( k \)-varieties. Then there exists a \( K \)-linear exact equivalence

\[
\mathcal{D}^b(X)/\text{Ker}(\iota_\xi^*) \simeq \mathcal{D}^b(X_\xi).
\]

**Proof.** Let \([\iota_\xi^*]: \mathcal{D}^b(X)/\text{Ker}(\iota_\xi^*) \to \mathcal{D}^b(X_\xi)\) be the unique functor which makes the diagram

\[
\begin{array}{ccc}
\mathcal{D}^b(X) & \xrightarrow{\iota_\xi^*} & \mathcal{D}^b(X_\xi) \\
\downarrow Q & & \downarrow \iota_\xi^* \\
\mathcal{D}^b(X)/\text{Ker}(\iota_\xi^*) & \approx & \mathcal{D}^b(X_\xi)
\end{array}
\]

commute, where \( Q: \mathcal{D}^b(X) \to \mathcal{D}^b(X)/\text{Ker}(\iota_\xi^*) \) is the quotient functor. Take any affine open subset \( U = \text{Spec} \ R \subset S \). Let \( \pi_U: X_U \to U \) and \( \pi_Z: X_Z \to Z \) be the base changes to \( U \) and its complement \( Z = S \setminus U \) respectively. We have

\[
\text{coh}(X_U) \simeq \text{coh}(X)/\text{coh}_Z(X)
\]

where the right hand side is the Serre quotient by the Serre subcategory \( \text{coh}_2(X) \subset \text{coh}(X) \) of sheaves supported on \( X_Z \). Passing to the derived category, via [Miy91, Theorem 3.2] we obtain \( U \)-linear exact equivalence

\[
\mathcal{D}^b(X_U) \simeq \mathcal{D}^b(X)/\mathcal{D}_Z^b(X)
\]
where $D^b_\mathbb{Z}(X) \subset D^b(X)$ is the full $S$-linear triangulated subcategory with cohomology supported on $X_\mathbb{Z}$. Since $D^b_\mathbb{Z}(X)$ is contained in $\text{Ker}(\bar{\iota}^\ast \xi)$, the commutative diagram (6.1) extends to

\[
\begin{array}{ccc}
D^b(X) & \xrightarrow{\bar{\iota}^\ast} & D^b(X_\xi) \\
\downarrow{\bar{\iota}^\ast} & \downarrow{\phi} & \downarrow{[\bar{\iota}^\ast]} \\
D^b(X_U) & \xrightarrow{Q} & D^b(X)/\text{Ker}(\bar{\iota}^\ast).
\end{array}
\]

On the other hand, the inclusion $D^b_\mathbb{Z}(X) \subset \text{Ker}(\bar{\iota}^\ast \xi)$ induces a commutative diagram

\[
\begin{array}{ccc}
D^b(X) & \xrightarrow{\bar{\iota}^\ast} & D^b(X_\xi) \\
\downarrow{\bar{\iota}^\ast} & \downarrow{Q_R} & \downarrow{[Q]} \\
D^b(X_U) & \xrightarrow{Q_R} & D^b(X)/\text{Ker}(\bar{\iota}^\ast).
\end{array}
\]

(6.2)

where $Q_R: D^b(X_R) \to D^b(X_\xi) \cong D^b(X_R)/D^b_0(X_R)$ is the quotient functor. Note that shrinking $U$ if necessary, we may assume that $\pi_R$ is smooth in order to apply Corollary 2.6. Indeed, by [Har77, Theorem 15.3] the singular locus of $X_R$ is a proper closed subset, whose image under the flat proper morphism $\pi_R$ is proper closed subset of $\text{Spec} R$. Changing the base to its complement, we may assume that $X_R$ is nonsingular. Then one can apply [Har77, Corollary III10.7] to find an open subset of $\text{Spec} R$ over which the restriction of $\pi$ becomes smooth. Since $D^b_0(X_R)$ is contained in $\text{Ker}([Q])$, the commutative diagram (6.2) extends to

\[
\begin{array}{ccc}
D^b(X) & \xrightarrow{\bar{\iota}^\ast} & D^b(X_\xi) \\
\downarrow{\bar{\iota}^\ast} & \downarrow{Q_R} & \downarrow{[[Q]]} \\
D^b(X_U) & \xrightarrow{Q_R} & D^b(X)/\text{Ker}(\bar{\iota}^\ast)
\end{array}
\]

with unique $[[Q]]$. Thus we obtain a commutative diagram

\[
\begin{array}{ccc}
D^b(X) & \xrightarrow{Q} & D^b(X)/\text{Ker}(\bar{\iota}^\ast) \\
\downarrow{Q} & \downarrow{[[Q]]=[\bar{\iota}^\ast]} & \downarrow{D^b(X)/\text{Ker}(\bar{\iota}^\ast)}
\end{array}
\]

By universality of Verdier quotient, the composition $[[Q]] \circ [\bar{\iota}^\ast]$ is natural isomorphic to the identity functor. Hence $[\bar{\iota}^\ast]$ is an equivalence. \qed

**Remark 6.1.** The above theorem is a direct consequence of [Miy91, Theorem 3.2] and the $K$-linear equivalence

$$\text{coh}(X)/\text{Ker}(\bar{\iota}^\ast) \cong \text{coh}(X_\xi),$$

which can be deduced by the similar argument. Here, we use the same symbol $\text{Ker}(\bar{\iota}^\ast)$ to denote the kernel [SP Tag 02MR] of the exact functor $\bar{\iota}^\ast$ of abelian categories. In particular, Theorem 2.5 also extends to nonaffine base case for flat proper families of $k$-varieties.

**Corollary 6.2.** Let $\pi: X \to S$ be a flat proper morphism of $k$-varieties. Then any object of $D^b(X_\xi)$ can be lifted to that of $D^b(X)$ along the projection $Q$. 22
6.2. Basic categorical results.

**Lemma 6.3.** Let $\mathcal{C}, \mathcal{D}$ be small categories and $F: \mathcal{C} \to \mathcal{D}, G: \mathcal{D} \to \mathcal{C}$ functors with $F \dashv G$. Assume that there exists an object $D \in \mathcal{D}$ such that the canonical maps

$$\mathrm{Hom}(D, D) \to \mathrm{Hom}(G(D), G(D)), \mathrm{Hom}(D, FG(D)) \to \mathrm{Hom}(G(D), GFG(D))$$

are bijective. Then the counit morphism $\epsilon: FG \Rightarrow 1_D$ induces an isomorphism $\epsilon_D: FG(D) \to D$.

**Proof.** Let $\alpha_D, \alpha_{FG(D)}$ be the compositions

$$\mathrm{Hom}(D, D) \to \mathrm{Hom}(G(D), G(D)) \to \mathrm{Hom}(FG(D), D),$$

$$\mathrm{Hom}(D, FG(D)) \to \mathrm{Hom}(G(D), GFG(D)) \to \mathrm{Hom}(FG(D), FG(D))$$

of the canonical maps. By assumption and definition of adjoint functors $\alpha_D, \alpha_{FG(D)}$ are bijective. We denote $\epsilon_D = \alpha_D(1_D)$ by $f$ and $\alpha_{FG(D)}^{-1}(1_{FG(D)})$ by $g$. Consider the diagrams

$$\begin{array}{ccc}
\mathrm{Hom}(D, D) & \xrightarrow{\alpha_D} & \mathrm{Hom}(FG(D), D) \\
g^o \downarrow & & \downarrow g^o \\
\mathrm{Hom}(D, FG(D)) & \xrightarrow{\alpha_{FG(D)}} & \mathrm{Hom}(FG(D), FG(D)),
\end{array}$$

$$\begin{array}{ccc}
\mathrm{Hom}(D, D) & \xrightarrow{\alpha_D} & \mathrm{Hom}(FG(D), D) \\
f^o \downarrow & & \downarrow f^o \\
\mathrm{Hom}(D, FG(D)) & \xrightarrow{\alpha_{FG(D)}} & \mathrm{Hom}(FG(D), FG(D)),
\end{array}$$

which are commutative by definition of functors and naturality of adjoints. As expressions the images of $1_D, g$ we respectively obtain

$$1_{FG(D)} = \alpha_{FG(D)}(\alpha_{FG(D)}^{-1}(1_{FG(D)})) = \alpha_{FG(D)}(g) = g(\alpha_D(1_D)) = gf,$$

$$\alpha_D(fg) = f(\alpha_{FG(D)}(g)) = f(\alpha_{FG(D)}(\alpha_{FG(D)}^{-1}(1_{FG(D)}))) = f.$$  

From the second line it follows $fg = \alpha_D^{-1}(f) = 1_D$. Hence $f$ is an isomorphism. $\square$

**Remark 6.2.** The above proof is just an adaptation of the proof of the fact that $G$ is fully faithful if and only if $\epsilon$ is a natural isomorphism.

Similarly, one can prove the dual statement.

**Lemma 6.4.** Let $\mathcal{C}, \mathcal{D}$ be small categories and $F: \mathcal{C} \to \mathcal{D}, G: \mathcal{D} \to \mathcal{C}$ functors with $F \dashv G$. Assume that there exists an object $C \in \mathcal{C}$ such that the canonical maps

$$\mathrm{Hom}(C, C) \to \mathrm{Hom}(F(D), F(D)), \mathrm{Hom}(GF(C), C) \to \mathrm{Hom}(FGF(C), F(C))$$

are bijective. Then the unit morphism $\eta: 1_C \Rightarrow GF$ induces an isomorphism $\eta_C: C \to GF(C)$.

**Remark 6.3.** Again, the proof is just an adaptation of the proof of the fact that $F$ is fully faithful if and only if $\eta$ is a natural isomorphism.

6.3. Removal of torsion supports.

**Lemma 6.5.** Let $\pi_R: X \to \mathrm{Spec} R, \pi'_R: X' \to \mathrm{Spec} R$ be a smooth proper morphisms to a nonsingular affine $k$-variety. Assume that their generic fibers $X_K, X'_K$ are derived-equivalent. Let $\Phi_{\mathcal{E}_K}: D^b(X_K) \to D^b(X'_K)$ be the Fourier–Mukai functor giving the equivalence with kernel $\mathcal{E}_K \in D^b(X_K \times X'_K)$. Fix a lift $\mathcal{E} \in D^b(X \times_R X')$ of $\mathcal{E}_K$ along the projection

$$D^b(X \times_R X') \to D^b(X \times_R X')/D^b(\mathcal{E}) \cong D^b(X_K \times X_K')$$

and a strong generator $E$ of $D^b(X)$. Then there exists an affine open subset $U \subset \mathrm{Spec} R$ over which the restriction

$$\Phi_U = \Phi_{\mathcal{E}_U}: D^b(X_U) \to D^b(X'_U)$$
induces bijections

\[
\text{Hom}(E_U, E_U) \to \text{Hom}(\Phi_U(E_U), \Phi_U(E_U)),
\]

\[
\text{Hom}(E_U, \Phi_U^l(E_U)) \to \text{Hom}(\Phi_U(E_U), \Phi_U^l\Phi_U(E_U)),
\]

where \( \Phi_U^l \colon D^b(X_U') \to D^b(X_U) \) is the left adjoint to \( \Phi_U \).

**Proof.** By [BV03, Theorem 2.1.2, Lemma 3.4.1] the base change \( E_K \) of \( E \) along the canonical inclusion \( R \hookrightarrow K \) is a strong generator of \( D^b(X_K) \), which can not be trivial. Consider the support of the \( R \)-torsion part of \( E \), i.e., the union \( \bigcup_i \text{supp} \mathcal{H}^i(E)_{\text{tors}} \) of the supports of the \( R \)-torsion parts \( \mathcal{H}^i(E)_{\text{tors}} \) of \( \mathcal{H}^i(E) \). Each \( \mathcal{H}^i(E)_{\text{tors}} \) is a coherent sheaf on \( X \), as every submodule of a finitely presented module over a noetherian ring is finitely presented. Since the union is finite, \( \bigcup_i \text{supp} \mathcal{H}^i(E)_{\text{tors}} \) is a closed subset of \( X \). Its complement must contain the generic point of \( X \), otherwise \( E_K \) is trivial. Let \( U \subset \text{Spec} \, R \) be the image of the complement under \( \pi_R \), which is a nonempty open subset. By construction over \( U \) the restriction \( E_U = E|_{\Sigma^{-1}(U)} \) is \( \mathcal{O}_U(U) \)-torsion free. Since \( \text{Hom}(E_U, E_U) \) is coherent as an \( \mathcal{O}_U(U) \)-module by Lemma 6.6 below, shrinking \( U \) if necessary, we may assume that it is \( \mathcal{O}_U(U) \)-torsion free. The tensor product \( \Phi_U(E_U) \otimes_{\mathcal{O}_U(U)} K \) can not be trivial, otherwise it does not map to an object quasi-isomorphic to \( E_K \) under \( \Phi_U^l \). Hence by the same argument one finds an affine open subset \( V \subset \text{Spec} \, R \) over which the restriction \( \Phi_V(E_V) \equiv \Phi_E(E)|_{\Sigma^{-1}(V)} \) and \( \text{Hom}(\Phi_V(E_V), \Phi_V(E_V)) \) are \( \mathcal{O}_V(V) \)-torsion free. The intersection \( U \cap V \subset \text{Spec} \, R \) is a nonempty open subset, as \( \text{Spec} \, R \) is integral. Now, we replace \( U \cap V \) with \( U \).

Consider a sequence

\[
0 \to \text{Hom}(E_U, E_U) \xrightarrow{\Phi_U} \text{Hom}(\Phi_U(U), \Phi_U(E_U)) \to 0
\]

of \( \mathcal{O}_U(U) \)-modules. By assumption and Lemma 2.2 tensoring \( K \) yields an exact sequence

\[
0 \to \text{Hom}(E_U, E_U) \otimes_{\mathcal{O}_U(U)} K \xrightarrow{\Phi_U \otimes_{\mathcal{O}_U(U)} K} \text{Hom}(\Phi_U(U), \Phi_U(E_U)) \otimes_{\mathcal{O}_U(U)} K \to 0.
\]

Hence the map \( \Phi_U \) in (6.4) is injective, as \( \text{Hom}(E_U, E_U) \) is \( \mathcal{O}_U(U) \)-torsion free. The associated sheaf with the cokernel

\[
\text{Hom}(\Phi_U(E_U), \Phi_U(E_U))/\text{Hom}(E_U, E_U)
\]

might have nontrivial \( \mathcal{O}_U(U) \)-torsion part. However, since by Lemma 6.6 below it is coherent, one finds an affine open subset \( W \subset U \) to which the restriction of the associated sheaf is \( \mathcal{O}_W(W) \)-torsion free. Now, we replace \( W \) with \( U \). Then the exactness of (6.5) implies that of (6.4). Hence the map \( \Phi_U \) in (6.4) is bijective. Shrinking \( U \) if necessary, by the same argument we conclude that

\[
\text{Hom}(E_U, \Phi_U^l\Phi_U(E_U)) \xrightarrow{\Phi_U} \text{Hom}(\Phi_U(E_U), \Phi_U^l\Phi_U(E_U))
\]

is also bijective. \( \square \)

**Remark 6.4.** For our purpose, we do not need the generator \( E \) to be strong. Nevertheless, we put the adjective “strong” as there always exists a strong generator of \( D^b(X) \) under the assumption.

**Remark 6.5.** Note that the base change

\[
(\text{Hom}(\Phi_U(E_U), \Phi_U(E_U))/\text{Hom}(E_U, E_U)) \otimes_{\mathcal{O}_U(U)} \mathcal{O}_W(W)
\]

is isomorphic to the cokernel of the sequence

\[
0 \to \text{Hom}(E_W, E_W) \xrightarrow{\Phi_W} \text{Hom}(\Phi_W(W), \Phi_W(E_W)) \to 0.
\]
Proof. from [Huy06, Example 2.70, Compatibilities(v)]. Applying the first two, we may assume that variety. Then for any object $E$

**Lemma 6.6.** Let $\Phi(U, E_U) \otimes_{\partial(U)} \mathcal{O}_W(W)$/ \ Hom$(E_U, E_U) \otimes_{\partial(U)} \mathcal{O}_W(W)$ as the pullback by an open immersion is exact. Consider the pullback diagrams

$$
\begin{align*}
X_W' \xrightarrow{i} X_U & \\
\pi_W & \\
W' \xrightarrow{i} U,
\end{align*}
$$

By the derived flat base change we have

$$
\begin{align*}
t'^R \text{Hom}^*(E_U, E_U) & \equiv t' R \pi_U^* R \text{Hom}^*(E_U, E_U) \cong R \text{Hom}^*(E_W, E_W), \\
t'^R \text{Hom}^*(\Phi_U(E_U), \Phi_U(E_U)) & \equiv t'^R R \pi_U^* R \text{Hom}^*(\Phi_U(E_U), \Phi_U(E_U)) \equiv R \text{Hom}^*(\Phi_W(E_W), \Phi_W(E_W)).
\end{align*}
$$

Passing to the 0-th cohomology of complexes, we obtain

$$
\begin{align*}
\text{Hom}(E_U, E_U) \otimes_{\partial(U)} \mathcal{O}_W(W) & \cong \text{Hom}(E_W, E_W), \\
\text{Hom}(\Phi_U(E_U), \Phi_U(E_U)) \otimes_{\partial(U)} \mathcal{O}_W(W) & \cong \text{Hom}(\Phi_W(E_W), \Phi_W(E_W))
\end{align*}
$$

as $t', t''$ are exact.

**Lemma 6.6.** Let $\pi_R: X \to \text{Spec } R$ be a smooth proper morphism to a nonsingular affine $k$-variety. Then for any object $E, F \in D^b(X)$ the $R$-module $\text{Hom}(E, F) = \text{Ext}^0_X(E, F)$ is coherent.

**Proof.** Consider the spectral sequences

$$
\begin{align*}
E_2^{p,q} & = \text{Ext}^p_X(E, \mathcal{H}^q(F)) \Rightarrow \text{Ext}^{p+q}_X(E, F), \\
E_2^{p,q} & = \text{Ext}^p_X(\mathcal{H}^{-q}(E), F) \Rightarrow \text{Ext}^{p+q}_X(E, F), \\
E_2^{p,q} & = H^p(X, \text{Ext}^{q}_X(E, F)) \Rightarrow \text{Ext}^{p+q}_X(E, F),
\end{align*}
$$

from [Huy06 Example 2.70, Compatibilities(v)].Applying the first two, we may assume that $E, F$ are coherent sheaves on $X$. Then $\text{Ext}^0_X(E, F)$ is coherent. Since $\pi_R$ is proper, $R^p \pi_R \text{Ext}^0_X(E, F)$ is also coherent, which is isomorphic to the associated sheaf on $\text{Spec } R$ with $H^p(X, \text{Ext}^0_X(E, F))$ [Har77 Proposition III.8.5]. In the decreasing filtration of $\text{Ext}^0_X(E, F)$, the smallest nontrivial submodule is isomorphic to $E_{0}^{-l}$ for some $l \in \mathbb{Z}$, which is coherent $R$-module. Ascending the filtration, one sees that $\text{Ext}^0_X(E, F)$ is coherent by two out of three principle. \hfill \Box

Similarly, one can prove the dual statement.

**Lemma 6.7.** Let $\pi_R: X \to \text{Spec } R, \pi'_R: X' \to \text{Spec } R$ be a smooth proper morphisms to a nonsingular affine $k$-variety. Assume that their generic fibers $X_K, X'_K$ are derived-equivalent. Let $\Phi_{E_K}: D^b(X_K) \to D^b(X'_K)$ be the Fourier–Mukai functor giving the equivalence with kernel $E_K \in D^b(X_K \times X'_K)$. Fix a lift $E \in D^b(X \times R X')$ of $E_K$ along the projection

$$
D^b(X \times R X') \to D^b(X \times R X')/D^b_0(X \times R X') \cong D^b(X_K \times X'_K)
$$

and a strong generator $E'$ of $D^b(X')$. Then there exists an affine open subset $U \subset \text{Spec } R$ over which the restriction

$$
\Phi^L_U = \Phi^L_{E_U}: D^b(X'_U) \to D^b(X_U)
$$

induces bijections

$$
\begin{align*}
\text{Hom}(E'_U, E'_U) & \to \text{Hom}(\Phi^L_U(E'_U), \Phi^L_U(E'_U)), \\
\text{Hom}(\Phi_U E'_U, \Phi_U E'_U) & \to \text{Hom}(\Phi^L_U \Phi_U E'_U, \Phi^L_U \Phi_U E'_U),
\end{align*}
$$

where $\Phi_U: D^b(X_U) \to D^b(X'_U)$ is the right adjoint to $\Phi^L_U$. 25
6.4. Specialization.

**Theorem 6.8.** Let \( \pi_R : X \to \text{Spec} \, R \), \( \pi'_R : X' \to \text{Spec} \, R \) be smooth proper morphisms to a nonsingular affine \( k \)-variety. Assume that their generic fibers \( X_K, X'_K \) are projective and derived-equivalent. Let \( \Phi_{E_K} : D^b(X_K) \to D^b(X'_K) \) be the Fourier–Mukai functor giving the equivalence with kernel \( E_K \in D^b(X_K \times X'_K) \). Fix a lift \( E \in D^b(X \times_R X') \) of \( E_K \) along the projection

\[
D^b(X \times_R X') \to D^b(X \times_R X')/D^b_0(X \times_R X') = D^b(X_K \times X'_K).
\]

Then there exists an affine open subset \( U \subset \text{Spec} \, R \) over which the restriction

\[
\Phi_U = \Phi_{E|_{\pi'_R^{-1}(U)}} : D^b(U) \to D^b(U')
\]

become an \( \mathcal{O}_U(U) \)-linear exact equivalence. In particular, over \( U \) any pair of closed fibers is derived-equivalent.

**Proof.** The proof is an adaptation of the argument in the proof of [Mor, Theorem 1.1]. Fix a strong generator \( E \) of \( D^b(X) \). The counit morphism \( \epsilon : \Phi_{E} \circ \Phi_{E} \to \text{id}_{D^b(X)} \) gives a distinguished triangle

\[
(6.8) \quad \Phi_{E} \circ \Phi_{E}(E) \xrightarrow{\epsilon} E \to F := \text{Cone} (\epsilon(E)).
\]

Over any open subset \( U \subset \text{Spec} \, R \), (6.8) restricts to a distinguished triangle

\[
\Phi_U \circ \Phi_U(E_U) \xrightarrow{\epsilon_U} E_U \to F_U.
\]

Note that the restriction of the counit morphism is the counit morphism. Choose \( U \) so that we have the bijections (6.3) from Lemma 6.5. Then by Lemma 6.3, the counit morphism \( \epsilon_{E_U} \) on \( E_U \) is an isomorphism. Since \( E_U \) is a strong generator of \( D^b(X_U) \) by [BV03, Theorem 2.1.2, Lemma 3.4.1], this implies that \( \Phi_U \) is fully faithful. Recall that a triangulated category is strongly finitely generated if there exist an object \( E_U \) and nonnegative integer \( k \) such that every object can be obtained from \( E_U \) by taking isomorphisms, finite direct sums, direct summands, shifts, and not more than \( k \) times cones. Now, we may assume that \( E_U \) has no nontrivial direct summands, as \( \Phi_U \) and \( \Phi_U^l \) commute with direct sums on \( D^b(X_U) \) by [BV03, Corollary 3.3.4]. Since \( \epsilon_{E_U} \) is an isomorphism, one inductively sees that over \( U \) the counit morphism on any object is an isomorphism.

Fix a strong generator \( E' \) of \( D^b(X') \). The unit morphism \( \eta : \text{id}_{D^b(X')} \to \Phi_{E} \circ \Phi_{E}^{l} \) gives a distinguished triangle

\[
(6.9) \quad E' \xrightarrow{\eta_{E'}} \Phi_{E} \circ \Phi_{E}^{l}(E') \to F' := \text{Cone}(\eta_{E'}).
\]

Over any open subset \( U \subset \text{Spec} \, R \), (6.9) restricts to a distinguished triangle

\[
E'_U \xrightarrow{\eta_{E'_U}} \Phi_U \circ \Phi_U(E'_U) \to F'_U.
\]

Note that the restriction of the unit morphism is the unit morphism. Choose \( U \) so that we have the bijections (6.7) from Lemma 6.7. Then by Lemma 6.4, the unit morphism \( \eta_{E'_U} \) on \( E'_U \) is an isomorphism. Since \( E'_U \) is a strong generator of \( D^b(X') \) by [BV03, Theorem 2.1.2, Lemma 3.4.1], this implies that \( \Phi_U^l \) is fully faithful. Shrinking \( U \) if necessary, we may assume that over \( U \) both \( \Phi_U \) and \( \Phi_U^l \) are fully faithful. Then \( \Phi_U \) is an equivalence, as a fully faithful functor which admits a fully faithful left adjoint is an equivalence. \( \Box \)

**Remark 6.6.** If \( \Phi_E \) induces the derived equivalence of a single pair of closed fibers, then there exists a Zariski open subset \( U \subset \text{Spec} \, R \) such that the base changes \( X_U, X'_U \) are derived-equivalent. This follows from the proof of [Mor, Theorem 1.1], which exploits the fact that the restriction of the counit morphism \( \epsilon_E : \Phi_E \circ \Phi_E(E) \to E \) to any closed fiber is the counit
morphism for each object $E \in D^b(X)$. However, it does not work for the generic fiber. In general, the generic fiber is not a subscheme of $X_K$, while any closed fiber can naturally be regarded as a subscheme of $X_K$ via the reduced induced structure on the image of the closed immersion.

**Corollary 6.9.** Let $\pi : X \to S, \pi' : X' \to S$ be flat proper morphisms of $k$-varieties. Assume that their generic fibers $X_K, X'_K$ are projective and derived-equivalent. Then there exists an open subset $U \subset S$ to which base changes $X_U, X'_U$ become derived-equivalent. In particular, over $U$ any pair of closed fibers are derived-equivalent.

**Proof.** By [Har77] Theorem I5.3] the singular locus of $X, X'$ are proper closed subsets, whose images under flat proper morphisms $\pi, \pi'$ are proper closed subsets of $S$. Changing the base to the complement of their union, we may assume that $X, X'$ are nonsingular. Then one can apply [Har77] Corollary III10.7] to find an open subset of $S$ over which the restrictions of $\pi, \pi'$ become smooth. Hence we may assume further that $\pi, \pi'$ are smooth.

Let $\Phi_{\mathcal{E}_K} : D^b(X_K) \to D^b(X'_K)$ be the Fourier–Mukai functor giving the equivalence with kernel $\mathcal{E}_K \in D^b(X_K \times X'_K)$. Fix a lift $\mathcal{E} \in D^b(X \times_S X')$ of $\mathcal{E}_K$ along the projection

$$D^b(X \times_S X') \to D^b(X \times_S X)/D^b_0(X \times_S X') = D^b(X_K \times X'_K).$$

Take an affine open cover $\bigcup_{i=1}^N \text{Spec } R_i$ of $S$. One can apply Theorem 6.8 to find open subsets $U_i \subset \text{Spec } R_i$ over which the restrictions

$$\Phi_{U_i} = \Phi_{\mathcal{E}_{U_i}} : D^b(X_{U_i}) \to D^b(X'_{U_i})$$

become $\mathcal{O}_{U_i}(U_i)$-linear exact equivalences. Let $V = \bigcup_{i=1}^N U_i$ be their union, which is an open $k$-subvariety of $S$. Consider the restriction

$$\Phi_V = \Phi_{\mathcal{E}_V} : D^b(X_V) \to D^b(X'_V)$$

over $V$. Since its restriction to any pair of closed fibers over $V$ defines an equivalence, $\Phi_V$ is an equivalence by [LST13] Proposition 1.3].

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