We compute the three-graviton tree amplitude in Type IIB superstring theory compactified to six dimensions using the manifestly (6d) supersymmetric Berkovits-Vafa-Witten worldsheet variables. We consider two cases of background geometry: the flat space example $\mathbb{R}^6 \times K3$, and the curved example $AdS_3 \times S^3 \times K3$ with Ramond flux, and compute the correlation functions in the bulk.
1. Introduction

We compute string tree correlation functions using the manifestly supersymmetric covariant formulation of Berkovits-Vafa-Witten (BVW) for Type IIB superstrings compactified to six dimensions\[1-6\]. Unlike the ten-dimensional covariant pure spinor quantization\[7-13\], the six-dimensional version\[1-3\] we use here incorporates an $N = 4$ topological string formulation to compute tree level scattering amplitudes.

For IIB superstrings either on flat six-dimensional space times $K3$, or on $AdS_3 \times S^3 \times K3$, the massless degrees of freedom correspond to a $D = 6$, $N = (2, 0)$ supergravity multiplet and 21 tensor multiplets. In this paper, we consider the string theory tree level scattering of three gravitons both in the flat case and the $AdS_3 \times S^3$ case, where the latter has background Ramond flux. For these amplitudes, the relevant massless compactification independent vertex operator, in the BVW worldsheet formalism, contains the graviton, dilaton and two-form field which contribute to the supergravity and one of the tensor multiplets.

In this formalism\[1-3\], the Type IIB superstring compactified on $K3$, which has 16 supercharges corresponding to 16 unbroken supersymmetries, has 8 which are manifest in that they act geometrically on the target space. These are given by $F_a, \bar{F}_a$ described below and are related to the presence of 8 theta world sheet variables $\theta^a, \bar{\theta}^a$. The other 8 supersymmetries $E_a, \bar{E}_a$ are not manifest, but can still be expressed in terms of ordinary world sheet fields, i.e. not spin operators. Therefore, in addition to making some of the supersymmetry geometric (either in $R^6$ or $AdS_3 \times S^3$), this formalism is also advantageous to describing background fields belonging to the Ramond-Ramond (RR) sector, since the worldsheet fields which couple to the RR background fields are not spin fields. Thus for strings on AdS that require RR backgrounds, the purpose of using BVW variables is that RR background fields can be added to the worldsheet action without adding spin fields to the worldsheet action\[3\].

On flat space ($R^6$), the BVW variables describe a free worldsheet conformal field theory. Their operator products (OPE’s) are reviewed in sect. 2, together with the $N = 4$ topological method for computing string correlation functions. In sect. 3 we use these OPE’s to evaluate the flat space three graviton tree amplitude in position space, and show that it reduces to the conventional answer.

On curved space, the BVW variables do not satisfy free operator product relations. Nonetheless, we proceed to evaluate the three graviton correlation function on $AdS_3 \times S^3$
by assuming a form for the OPE’s. It is motivated by [4], where the vertex operator
constraint equations were derived for $AdS_3 \times S^3$ by requiring them to be invariant under
the $AdS$ supersymmetry transformations. It can be shown that our assumed OPE’s result
in the same constraint equations. In sect. 4, we compute the curved space three graviton
tree amplitude using these OPE’s.

2. Review of Components

The $N = 4$ topological prescription [1-3] for calculating superstring tree-level ampli-
tudes is

$$< V_1(z_1)(G^+_0V_2(z_2))(G^+_0V_3(z_3)) \prod_{r=4}^{n} \int dz_r G^-_r G^+_0 V(z_r) > . \quad (2.1)$$

where $G^\pm_n$ are elements of the topological $N = 4$ super Virasoro algebra, and the notation
$O_n V(z)$ denotes the pole of order $d + n$ in the OPE of $O(\zeta)$ with $V(z)$, when $O$ is an
operator of conformal dimension $d$.

2.1. $N = 4$ Superconformal Algebra in Flat Space

For the IIB superstring there is both a holomorphic $N = 4$ superconformal algebra
and another anti-holomorphic one. The holomorphic generators, specialized for the IIB
string compactified to six dimensions, and in terms of BVW worldsheet variables, are [3]

$$\tilde{T}(z) = -\frac{1}{2} \partial x^m \partial x_m - p_a \partial \theta^a - \frac{1}{2} \partial \rho \partial \sigma - \frac{1}{2} \partial \sigma \partial \rho + \frac{3}{2} \partial^2 (\rho + i \sigma) + \tilde{T}_C$$

$$G^+(z) = -e^{-2\rho - i\sigma} \partial_x^4 + \frac{1}{2} e^{-\rho} p_a p_b \partial x^{ab}$$

$$+ e^{\frac{i}{2} \rho} \left( -\frac{1}{2} \partial x^m \partial x_m - p_a \partial \theta^a - \frac{1}{2} \partial (\rho + i \sigma) \partial (\rho + i \sigma) + \frac{1}{2} \partial^2 (\rho + i \sigma) \right) + G^+_C$$

$$G^-(z) = e^{-i\sigma} + G^-_C$$

$$J(z) = \partial (\rho + i \sigma) + J_C$$

$$J^+(z) = e^{\rho + i\sigma} J_C^+ = -e^{\rho + i\sigma + iH_C}$$

$$J^-(z) = -e^{-\rho - i\sigma} J_C^- = -e^{-\rho - i\sigma - iH_C}$$

$$\tilde{G}^+(z) = e^{iH_C + \rho} + e^{\rho + i\sigma} \tilde{G}^+_C$$

$$\tilde{G}^-(z) = e^{-iH_C} \left[ -\frac{1}{2} \partial x^m \partial x_m - p_a \partial \theta^a - \frac{1}{2} \partial (\rho + i \sigma) \partial (\rho + i \sigma) + \frac{1}{2} \partial^2 (\rho + i \sigma) \right]$$

$$- e^{-\rho - i\sigma} \tilde{G}^-_C .$$
These currents are given in terms of the left-moving bosons $\partial x^m$, $\rho$, $\sigma$, and the left-moving fermionic worldsheet fields $p_a, \theta^a$, where $0 \leq m \leq 5; 1 \leq a \leq 4$. The conformal weights of $p_a, \theta^a$ are 1 and 0, respectively. The BVW variables no longer exhibit the matter times ghost sector structure familiar from the conventional Ramond-Neveu-Schwarz formalism, with cancelling contributions of $\pm 15$ to the central charge. In BVW, the residual ghost fields are $\rho, \sigma$. (In the ten-dimensional version, these are promoted to complex worldsheet boson fields $\lambda^a$ with a spacetime Majorana spinor index $\alpha$, which are parameterized by eleven complex fields $\bar{\lambda}^{\alpha} \lambda^a$.) We define $p^4 \equiv \frac{1}{4!} \epsilon^{abcd} p_a p_b p_c p_d = p_1 p_2 p_3 p_4$; and $\partial x^{ab} = \partial x^m \sigma^a \sigma^b_m$ where the sigma matrices in flat space satisfy $\sigma^a \sigma^b + \sigma^b \sigma^a = \eta_{mn} \delta^a \delta^b$. Here lowered indices mean $\sigma_m \sigma^m = \frac{1}{2} \epsilon_{abcd} \sigma^c \sigma^d_m$. Note that $e^\rho$ and $e^{i\sigma}$ are worldsheet fermions. Also $e^{\rho+i\sigma} \equiv e^\rho e^{i\sigma} = -e^{i\sigma} e^\rho$. Here $J_C \equiv i \partial H_C, J^+_C \equiv -e^{i H_C}, J^-_C \equiv e^{-i H_C}$. Both $\tilde{T}, G^\pm, J, J^\pm, \tilde{G}^\pm$ and the generators describing the $K3$ compactification $\tilde{T}_C, G^\pm_C, J_C, J^\pm_C, \tilde{G}^\pm_C$ satisfy the twisted $N = 4, c = 6$, superconformal algebra, i.e. both $\tilde{T}$ and $\tilde{T}_C$ have zero central charge. However, $c$ still appears in the twisted $N = 4$ and $N = 2$ algebras in the products involving the supercurrents and the SU(2) currents; and the N=2 generators in (2.2) $\tilde{T}, G^\pm, J$ decompose into a $c = 0$ six-dimensional part and a $c = 6$ compactification-dependent piece.

The other non-vanishing OPE’s are $x^m(z, \bar{z}) x^n(\zeta, \bar{\zeta}) = -\eta_{mn} \ln |z - \zeta|$; for the left-moving worldsheet fermion fields $p_a(z) \theta^a(\zeta) = (z - \zeta)^{-1} \delta^a_b$; and for the left-moving worldsheet bosons $\rho(z) \rho(\zeta) = -\ln(z - \zeta)$; $\sigma(z) \sigma(\zeta) = -\ln(z - \zeta)$. Right-movers are denoted by barred notation and have similar OPE’s.

2.2. $N = 4$ Superconformal Algebra for $AdS_3 \times S^3$

We also recall the expression for the twisted holomorphic $N = 4$ generators when $\mathbf{R}^6$ is replaced by $AdS_3 \times S^3$ in the presence of Ramond flux:

$$\bar{T}(z) = \frac{1}{8} \epsilon^{abcd} K_{ab} K_{cd} - F^a E^a - \frac{1}{2} \partial \rho \partial \rho - \frac{1}{2} \partial \sigma \partial \sigma + \frac{3}{2} \partial^2 (\rho + i \sigma) + \bar{T}_C$$

$$G^+(z) = -\frac{1}{6} e^{-2 \rho - i \sigma} \epsilon^{abcd} U_{ab} U_{cd} + i e^{-\rho} K_{ab} U_{ab}$$

$$+ e^{i \sigma} \left(-\frac{1}{8} \epsilon^{abcd} K_{ab} K_{cd} - F^a E^a - \frac{1}{2} \partial (\rho + i \sigma) \partial (\rho + i \sigma) + \frac{1}{2} \partial^2 (\rho + i \sigma)\right) + G^+_C$$

$$G^-(z) = e^{-i \sigma} + G^-_C$$

$$J(z) = \partial (\rho + i \sigma) + J_C$$

$$\tilde{G}^+(z) = e^{i H_C + \rho} + e^{\rho + i \sigma} \tilde{G}^+_C,$$

(2.3)
where
\[
U_{ab} = (1 - \frac{1}{4}e^{\varphi+\bar{\varphi}})^{-2} \left[ \frac{1}{2} F_a F_b - \frac{1}{8} e^{2\varphi} E_a E_b - \frac{1}{4} e^{\varphi} (E_a E_b + F_a F_b) \right],
\] (2.4)
and the remaining generators \( J^\pm, \tilde{G}^- \) can be constructed from \( T, G^\pm, J \) [3]. Here \( e^\varphi \equiv e^{-\rho - i\sigma} \), and \( F_a, E_a, K_{ab} \) are the fermionic and bosonic z-components of the right invariant \( PSU(2|2) \) currents. There is a corresponding anti-holomorphic \( N = 4 \) algebra in terms of barred worldsheet fields. Although the \( N = 4 \) generators have definite holomorphicity, the worldsheet fields \( F_a, E_a, K_{ab}, \bar{F}_a, \bar{E}_a, \bar{K}_{ab} \) do not. They are each functions of \( z \) and \( \bar{z} \) and have non-free operator products, which are not known in closed form. They reduce to the free conformal fields \( p_a, \partial \theta_a, \partial x_{ab}, \bar{p}_a, \bar{\partial} \bar{\theta}_a, \bar{\partial} x_{ab} \) only in the flat limit.

3. Three-graviton tree amplitude in flat space

We first compute the six-dimensional three-graviton tree level amplitude in (6d) flat space, for Type IIB superstrings on \( R^6 \times K3 \) in the BVW formalism. It is contained in the closed string three-point function
\[
< V(z_1, \bar{z}_1) \left( G_0^+ \bar{G}_0^+ V(z_2, \bar{z}_2) \right) \left( \bar{G}_0^+ \bar{G}_0^+ V(z_3, \bar{z}_3) \right) > \] (3.1)
where the \( N = 4 \) supercurrents are found in (2.2), and the vertex operators are given by
\[
V(z, \bar{z}) = e^{i\sigma(z) + \rho(z)} e^{i\bar{\sigma}(\bar{z}) + \bar{\rho}(\bar{z})} \theta^a(z) \theta^b(\bar{z}) \bar{\theta}^\bar{a}(\bar{z}) \bar{\theta}^\bar{b}(z) \sigma_{ab}^m \sigma_{\bar{a}\bar{b}}^n \phi_{mn}(X(z, \bar{z})),
\] (3.2)
when the field
\[
\phi_{mn} = g_{mn} + b_{mn} + \bar{g}_{mn} \phi
\]
satisfies the constraint equations \( \square \phi_{mn} = 0 \), and \( \Box \phi_{mn} = 0 \). These constraints imply the gauge conditions \( \partial^m b_{mn} = 0 \) for the two-form, and \( \partial^m g_{mn} = -\partial_n \phi \) for the traceless graviton \( g_{mn} \) and dilaton \( \phi \). The constraints follow from the physical state conditions which in this formalism are implemented by the \( N = 4 \) generators, as shown in [3]. (Since the \( N = 4 \) algebra is twisted, \( i.e. \) topological, the nilpotent generators \( G^+, \tilde{G}^+, \tilde{G}^+, \bar{G}^+ \) are dimension one, and their cohomology essentially determines the physical states.) There is a residual gauge symmetry
\[
g_{mn} \rightarrow g_{mn} + \partial_m \xi_n + \partial_n \xi_m, \quad \phi \rightarrow \phi, \quad b_{mn} \rightarrow b_{mn}
\] (3.3)
with \( \Box \eta = 0 \), \( \partial \cdot \xi = 0 \). To evaluate (3.4), we first extract the simple poles

\[
G_0^+ \tilde{G}_0^+ V(z, \tilde{z}) = e^{i\sigma} e^{i\bar{\sigma}} (-4) \left[ \phi_{mn}(X) \partial X^m \partial X^n - p_a \theta^b \sigma^m_{cb} \sigma^{nca} \partial X^n \partial p \phi_{mn}(X) - \bar{p}_a \bar{\theta}^\dagger \sigma^m_{cb} \sigma^{nca} \partial X^m \partial p \phi_{mn}(X) + p_a \theta^b \bar{p}_a \bar{\theta}^\dagger \sigma^m_{cb} \sigma^{nca} \sigma^{\bar{q}ca} \partial \bar{p} \partial h \phi_{mn}(X) \right].
\]

(3.4)

Then, using the OPE's for the ghost fields and \( H_C \), we partially compute (3.4) by exhibiting their contribution as [14]-[17]

\[
\langle V_1(z_1, \tilde{z}_1) (G_0^+ \tilde{G}_0^+ V_2(z_2, \tilde{z}_2))(\tilde{G}_0^+ \tilde{G}_0^+ V_3(z_3, \tilde{z}_3)) \rangle = (z_1 - z_2)(z_2 - z_3)(z_1 - z_3)^{-1}(\bar{z}_1 - \bar{z}_2)(\bar{z}_2 - \bar{z}_3)(\bar{z}_1 - \bar{z}_3)^{-1} \cdot 4 \langle e^{iH_C(z_3)} e^{\rho(z_3) + 2}\bar{\sigma}(z_3) e^{iH_C(\bar{z}_3)} e^{\bar{\rho}(\bar{z}_3) + 2}\sigma(\bar{z}_3) e^{3i\bar{\sigma}(z_3)} \rangle \cdot \theta^a(z_1) \theta^b(\bar{z}_1) \theta^\dagger(\bar{z}_1) \sigma^m_{ab} \sigma^n_{ab} \phi_{mn}(X(z_1, \bar{z}_1)) \cdot \phi_{jk}(X(z_2, \bar{z}_2)) \partial X^j(z_2) \partial X^k(\bar{z}_2) - p_e(z_2) \theta^f(z_2) \sigma^i_{ef} \sigma^{\mu\nu e} \partial X^i(\bar{z}_2) \partial p \phi_{jk}(X(z_2, \bar{z}_2)) - \bar{p}_e(\bar{z}_2) \bar{\theta}^\dagger(\bar{z}_2) \sigma^i_{ef} \sigma^{\bar{\mu}\bar{\nu} \bar{e}} \partial X^i(\bar{z}_2) \partial \bar{p} \phi_{jk}(X(z_2, \bar{z}_2)) + p_e(z_2) \theta^f(z_2) \bar{p}_e(\bar{z}_2) \bar{\theta}^\dagger(\bar{z}_2) \sigma^i_{ef} \sigma^{\mu\nu e} \partial \bar{p} \partial h \phi_{jk}(X(z_2, \bar{z}_2)) \rangle \cdot \theta^c(z_3) \theta^d(z_3) \theta^\dagger(\bar{z}_3) \theta^\dagger(\bar{z}_3) \sigma^q_{cd} \sigma^h_{cd} \phi_{gh}(X(z_3, \bar{z}_3)) \rangle
\]

Computing the remaining \( z_2, z_3 \) operators products, and using the \( SL(2, C) \) invariance of the amplitude to take the three points to constants \( z_1 \to \infty, \bar{z}_1 \to \infty, z_2 \to 1, \bar{z}_2 \to 1, z_3 \to 0, \bar{z}_3 \to 0 \), we have

\[
\langle V_1(z_1, \tilde{z}_1) (G_0^+ \tilde{G}_0^+ V_2(z_2, \tilde{z}_2))(\tilde{G}_0^+ \tilde{G}_0^+ V_3(z_3, \tilde{z}_3)) \rangle = (z_2 - z_3)(\bar{z}_2 - \bar{z}_3)(\bar{z}_2 - \bar{z}_3)^{-1} \cdot 4 \cdot \langle e^{iH_C(0) + 3i\sigma(0)} e^{i\bar{H}_C(0) + 3i\bar{\sigma}(0)} \theta_0^a \theta_0^b \theta_0^c \theta_0^d \bar{\theta}_0^a \bar{\theta}_0^b \bar{\theta}_0^c \bar{\theta}_0^d \rangle \cdot [\sigma^m_{ab} \sigma^n_{cd} \sigma^h_{cd} \sigma^\dagger_{cd} < \phi_{mn}(X(\infty)) \phi_{jk}(X(1)) \partial^j \partial^k \phi_{gh}(X(0)) > + 2\sigma^m_{ab} \sigma^n_{cd} \sigma^h_{cd} \sigma^\dagger_{cd} < \phi_{mn}(X(\infty)) \partial_p \phi_{jk}(X(1)) \partial^k \phi_{gh}(X(0)) > + 2\sigma^m_{ab} \sigma^n_{cd} \sigma^h_{cd} \sigma^\dagger_{cd} < \phi_{mn}(X(\infty)) \partial_p \phi_{jk}(X(1)) \partial^j \phi_{gh}(X(0)) > + 4\sigma^m_{ab} \sigma^n_{cd} \sigma^h_{cd} \sigma^\dagger_{cd} < \phi_{mn}(X(\infty)) \partial_\bar{p} \partial h \phi_{jk}(X(1)) \phi_{gh}(X(0)) >]
\]

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where the second equality follows from the vacuum expectation value of the ghost fields, $H_C$ and eight fermion zero modes $< e^{iH_C(0)+3\rho(0)+3i\sigma(0)}e^{i\bar{H}_C(0)+3\bar{\rho}(0)+3i\bar{\sigma}(0)}\partial_0^a\partial_0^b\partial_0^c\partial_0^d\partial_{\alpha}^a\partial_{\beta}^b\partial_{\gamma}^c\partial_{\delta}^d > = \frac{1}{16} \epsilon^{abcd} \epsilon^{\bar{a}\bar{b}\bar{c}\bar{d}}$, and various sigma matrix identities [3-4]. Further using $(\sigma^m \sigma^n \sigma^p \sigma^q)^d = \bar{g}^{mn} \bar{g}^{pq} + \bar{g}^{mq} \bar{g}^{np} - \bar{g}^{mp} \bar{g}^{nq}$ where in flat space $\bar{g}_{mn} = \eta_{mn}$, we have

\[
\begin{align*}
\langle V_1(z_1, \bar{z}_1) (G_0^+ \bar{G}_0^+ V_2(z_2, \bar{z}_2))(\bar{G}_0^+ \bar{G}_0^+ V_3(z_3, \bar{z}_3)) \rangle \\
= 4 \left[ \bar{g}^{mn} \bar{g}^{nh} < \phi_{mn}(x_0) \partial^i \partial^k \phi_{gh}(x_0) > \\
- \bar{g}^{nh} (\bar{g}^{mj} \bar{g}^{pg} - \bar{g}^{mp} \bar{g}^{jg}) < \phi_{mn}(x_0) \partial_p \phi_{jk}(x_0) \partial^k \phi_{gh}(x_0) > \\
- \bar{g}^{mq} (\bar{g}^{nk} \bar{g}^{ph} - \bar{g}^{np} \bar{g}^{kh}) < \phi_{mn}(x_0) \partial_p \phi_{jk}(x_0) \partial^j \phi_{gh}(x_0) > \\
+ (\bar{g}^{mj} \bar{g}^{pq} + \bar{g}^{mq} \bar{g}^{jp} - \bar{g}^{mp} \bar{g}^{jq}) (\bar{g}^{nk} \bar{g}^{qh} + \bar{g}^{nh} \bar{g}^{kq} - \bar{g}^{na} \bar{g}^{kh}) \right] \\
\cdot < \phi_{mn}(x_0) \partial_p \partial_q \phi_{jk}(x_0) \phi_{gh}(x_0) > 
\end{align*}
\]

Using the gauge condition $\partial^m \phi_{mn} = 0$ again, then finally

\[
\begin{align*}
\langle V_1(z_1, \bar{z}_1) (G_0^+ \bar{G}_0^+ V_2(z_2, \bar{z}_2))(\bar{G}_0^+ \bar{G}_0^+ V_3(z_3, \bar{z}_3)) \rangle \\
= 4 \left[ \bar{g}^{mn} \bar{g}^{nh} < \phi_{mn}(x_0) \partial^i \partial^k \phi_{gh}(x_0) > \\
- \bar{g}^{nh} (\bar{g}^{mj} \bar{g}^{pg} - \bar{g}^{mp} \bar{g}^{jg}) < \phi_{mn}(x_0) \partial_p \phi_{jk}(x_0) \partial^k \phi_{gh}(x_0) > \\
- \bar{g}^{mq} (\bar{g}^{nk} \bar{g}^{ph} - \bar{g}^{np} \bar{g}^{kh}) < \phi_{mn}(x_0) \partial_p \phi_{jk}(x_0) \partial^j \phi_{gh}(x_0) > \\
+ (\bar{g}^{mj} \bar{g}^{pq} + \bar{g}^{mq} \bar{g}^{jp} - \bar{g}^{mp} \bar{g}^{jq}) (\bar{g}^{nk} \bar{g}^{qh} + \bar{g}^{nh} \bar{g}^{kq} - \bar{g}^{na} \bar{g}^{kh}) \right] \\
\cdot < \phi_{mn}(x_0) \partial_p \partial_q \phi_{jk}(x_0) \phi_{gh}(x_0) > 
\end{align*}
\]

\[
= 12 \left[ < \phi_{mn}^0(x_0) \phi_{jk}^0(x_0) \partial_m \partial_n \phi_{jk}(x_0) > + 2 < \phi_{mn}^0(x_0) \partial_m \phi_{jk}^0(x_0) \partial_j \phi_{nk}(x_0) > \right].
\]

(3.6)

To make contact with the supergravity field theory expression, we can evaluate this dual model amplitude in either momentum space or in position space. For this flat space case, momentum space is often used, since momentum is conserved, i.e. $k_1 + k_2 + k_3 = 0$. For AdS, momentum is not conserved, so we will just work in the position space representation in both cases.
The Einstein-Hilbert action is \( I = \int d^d x \sqrt{|g|} \left\{ -\frac{R}{2K} \right\} \). Expanding to third order in \( K \) using \( g_{\mu \nu} = \eta_{\mu \nu} + 2K h_{\mu \nu} \), we find the three-point interaction \( I_3 \). In harmonic gauge, \( i.e. \) when \( \partial^\mu h_{\mu \nu} - \frac{1}{2} \partial_\nu h_\rho^\rho = 0 \), and on shell \( \Box h_{\mu \nu} = 0 \), it is given by

\[
I_3 = -K \int d^d x \left[ h^{\mu \nu} h^{\rho \sigma} \partial_\mu \partial_\nu h_{\rho \sigma} + 2h^{\mu \nu} \partial_\mu h^\rho_\nu \partial_\rho h_{\nu \sigma} \right].
\]  

(3.7)

The gauge transformations

\[
h_{\mu \nu} \rightarrow h_{\mu \nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \tag{3.8}
\]

leave invariant the harmonic gauge condition and \( I_3 \), given in (3.7), when \( \Box \xi = 0 \). Using this gauge symmetry, we could further choose \( h_\rho^\rho \neq 0 \), \( \partial^\mu h_{\mu \nu} = 0 \). Then \( I_3 \) represents the three-graviton amplitude, and is invariant under residual gauge transformations that have \( \partial \cdot \xi = 0 \).

To extract the string theory three-graviton amplitude from (3.6), we set \( b_{mn} \) to zero, and use the field identifications\[^4\] that relate the string fields \( g_{mn}, \phi \) to the supergravity field \( h_{mn} \) via \( \phi \equiv -\frac{1}{3} h_\rho^\rho \) and \( g_{mn} \equiv h_{mn} - \frac{1}{6} \bar{g}_{mn} h_\rho^\rho \), where \( h_{mn} \) is in harmonic gauge. Then \( \phi_{mn} = h_{mn} - \frac{1}{2} \bar{g}_{mn} h_\rho^\rho \), and from (3.6) the on shell string tree amplitude is

\[
-\frac{K}{12} < V_1(z_1, \bar{z}_1) (G_0^+ \bar{G}_0^+ V_2(z_2, \bar{z}_2)) (\bar{G}_0^+ G_0^+ V_3(z_3, \bar{z}_3)) > \\
= -K \int d^d x \left[ \phi^{mn} (x) \phi^{jk} (x) \partial_m \partial_n \phi_{jk} (x) + 2 \phi^{mn} (x) \partial_m \phi^{jk} (x) \partial_j \phi_{nk} (x) \right] \\
= -K \int d^d x \left[ h^{mn} h^{jk} \partial_m \partial_n h_{jk} + 2 h^{mn} \partial_m h^{jk} \partial_j h_{nk} \right] + K \int d^d x \ h^{mn} \partial_m h_k^k \partial_n h_p^p \\
= I_3 + I'_3
\]

(3.9)

where \( I'_3 \) is the one graviton - two dilaton amplitude, \( I_3 \) is the three graviton interaction in harmonic gauge, and \( d = 6 \). We note that \( I_3 \) and \( I'_3 \) separately are invariant under the gauge transformation (3.8) with \( \Box \xi = 0 \) and \( \partial \cdot \xi = 0 \), which corresponds to the gauge symmetry of the string field \( \phi_{mn} \rightarrow \phi_{mn} + \partial_m \xi_n + \partial_n \xi_m \). Furthermore, \( I_3 \) is also invariant under gauge transformations for which \( \partial \cdot \xi \neq 0 \), and these can be used to eliminate the trace of \( h_{mn} \) in \( I_3 \). In the string gauge, the trace of \( \phi_{mn} \) is related to the dilaton \( \phi_m^m = 6 \phi \), so even when \( b_{mn} = 0 \), (3.6) contains both the three graviton amplitude and the one graviton - two dilaton interaction. From (3.9) we see that we could have extracted \( I_3 \) from (3.6) merely by setting both \( b_{mn} = 0 \) and \( \phi = 0 \), since then \( \phi_{mn} = g_{mn} \) and \( \partial^m g_{mn} = 0 \).
4. Three-graviton tree amplitude in $AdS_3 \times S^3$

In this section, we compute the six-dimensional three-graviton amplitude in the Type IIB superstring on $AdS_3 \times S^3 \times K3$ with background Ramond flux. Consider

$$< V_1(z_1, \bar{z}_1) (G_0^+ \tilde{G}_0^+ V_2(z_2, \bar{z}_2))(\tilde{G}_0^+ \tilde{G}_0^+ V_3(z_3, \bar{z}_3) >$$

(4.1)

where the $N = 4$ supercurrents are reviewed in (2.3),(2.4). They are expressed in terms of the left action generators defined in [4] as

$$F_a = \frac{d}{d\theta^a}, \quad K_{ab} = -\theta_a \frac{d}{d\theta^b} + \theta_b \frac{d}{d\theta^a} + t_{Lab}$$

$$E_a = \frac{i}{2} \epsilon_{abcd} \theta^b (t'^c_L - \theta^c \frac{d}{d\theta_d}) + h_{a\bar{b}} \frac{d}{d\theta_{ar{b}}},$$

(4.2)

and corresponding right action generators $F_a, E_a, K_{ab}$. The vertex operators are

$$V(z, \bar{z}) = e^{i\sigma(z)+\rho(z)} e^{i\bar{\sigma}(\bar{z})+\bar{\rho}(\bar{z})} \theta^a(z, \bar{z}) \theta^b(\bar{z}, z) \bar{\theta}^a(z, \bar{z}) \bar{\theta}^b(\bar{z}, z) \sigma_{ab}^{m} \sigma_{\bar{a}\bar{b}}^{n} \phi_{mn}(X(z, \bar{z})) .$$

On $AdS$, in addition to the equation of motion, the string field $\phi_{mn} = g_{mn} + b_{mn} + g_{mn} \phi$ satisfies constraints given by $t'^{ab}_L h^a_L h^b_L \sigma^{mn}_{ab} \sigma^n_{gh} \phi_{mn} = 0, t'^{ab}_R h^a_R h^b_R \sigma^{mn}_{ab} \sigma^n_{gh} \phi_{mn} = 0$, which were derived in [4] where $t'^{ab}_L, t'^{ab}_R$ describe invariant derivatives on the $SO(4)$ group manifold. These can be related to covariant derivatives $T'^{cd}_L \equiv -\sigma^{p cd} D_p, T'^{cd}_R \equiv \sigma^{p c d} D_p$, where for example, acting on a function, $T_L = t_L$ and $T_R = t_R$. But when acting acting on fields that carry vector or spinor indices, they differ so that for example on spinor indices $t'^{ab}_L V_e = T'^{ab}_L V_e + \frac{1}{2} \delta^{ab}_e \delta^m V_e - \frac{1}{2} \delta^{ac}_e \delta^{bc} V_e$. In terms of covariant derivatives, the constraints are $D^m g_{mn} = D^m g_{nm} = -D^m \phi + \bar{H}_{nrs} b^p s$, and $D^m b_{nm} = 0 = D^m b_{nm}$. These are the $AdS_3 \times S^3$ analog of the flat space constraints $\partial^m \phi_{mn} = 0$. On $AdS_3 \times S^3$,

$$G_0^+ \tilde{G}_0^+ V(z, \bar{z}) = e^{iH_c + 2\rho + i\sigma} e^{i\bar{H}_c + 2\bar{\rho} + i\bar{\sigma}} \theta^a \theta^b \bar{\theta}^a \bar{\theta}^b \sigma_{ab}^{m} \sigma_{\bar{a}\bar{b}}^{n} \phi_{mn}(X).$$

(4.4)
In deriving (4.3), we keep only the contribution $ie^{-\rho}K^{ab}U_{ab}$ to $G^+$ with $U_{ab} \sim \frac{1}{2} F_aF_b$ since other terms in $G^+$ do not survive the ghost measure in the vacuum expectation value. We have also assumed that the OPE of $F_a(z, \bar{z})$ with $\theta^{\rho}(\zeta, \bar{\zeta})$ can be replaced with $F_a(z, \bar{z})\theta^{\rho}(\zeta, \bar{\zeta}) \sim (z - \zeta)^{-1}\delta^\rho_a$, in accordance with (4.2). This is motivated by the observation that evaluating the OPE’s in this manner leads to the constraint equations found in [4], where those equations were derived solely by requiring supersymmetric invariance (and not from the action of the $N = 4$ generators).

$$< V_1(z_1, \bar{z}_1)(\bar{G}^+ \bar{G}_0^+ V_2(z_2, \bar{z}_2))(\bar{G}^+ \bar{G}_0^+ V_3(z_3, \bar{z}_3)) >$$

$$= (z_1 - z_2)(z_2 - z_3)(z_1 - z_3)^{-1}(\bar{z}_1 - \bar{z}_2)(\bar{z}_2 - \bar{z}_3)(\bar{z}_1 - \bar{z}_3)^{-1}$$

$$\cdot 4 < e^{i\hat{H}_C(z_3)}e^{\rho(z_3) + 2\rho(z_3)}e^{3i\sigma(z_3)}e^{i\hat{H}_C(z_3)}e^{\rho(z_3) + 2\rho(z_3)}e^{3i\sigma(z_3)}$$

$$\cdot \theta^a(z_1)\theta^b(z_1)\bar{\theta}^a(z_1)\bar{\theta}^b(z_1)\sigma_{ab}^m\sigma_{ab}^n\phi_{mn}(X(z_1, \bar{z}_1))$$

$$\cdot [\frac{1}{4} \phi_{jk}(X(z_2, \bar{z}_2))\sigma^j_{ef}\sigma^k_{ef}K^{ef}(z_2, \bar{z}_2)\bar{K}^{\ell f}(z_2, \bar{z}_2)$$

$$- \frac{1}{2} F_e(z_2, \bar{z}_2)\theta^f(z_2, \bar{z}_2)\bar{K}^{\ell f}(z_2, \bar{z}_2)(t^\ell_L - \delta^\ell f)\sigma^j_{ef}\sigma^k_{ef}\phi_{jk}(X)$$

$$- \frac{1}{2} F_e(z_2, \bar{z}_2)\bar{\theta}^f(z_2, \bar{z}_2)K^{ef}(z_2, \bar{z}_2)(t^\ell_R - \delta^\ell f)\sigma^j_{ef}\sigma^k_{ef}\phi_{jk}(X)$$

$$+ F_e(z_2, \bar{z}_2)\theta^f(z_2, \bar{z}_2)\bar{F}_e(z_2, \bar{z}_2)\bar{\theta}^f(z_2, \bar{z}_2)(t^\ell_R - \delta^\ell f)(t^\ell_L - \delta^\ell f)\sigma^j_{ef}\sigma^k_{ef}\phi_{jk}(X)]$$

$$\cdot \theta^d(z_3)\theta^d(z_3)\bar{\theta}^d(z_3)\bar{\theta}^d(z_3)\sigma_{cd}^q\sigma_{cd}^h\phi_{gh}(X(z_3, \bar{z}_3)) > .$$

(4.5)

Evaluating at $z_1 \to \infty$, $\bar{z}_1 \to \infty$, and restricting $\phi_{mn}, \phi_{jk}, \phi_{gh}$ to be symmetric, we have

$$< V_1(z_1, \bar{z}_1)(\bar{G}^+ \bar{G}_0^+ V_2(z_2, \bar{z}_2))(\bar{G}^+ \bar{G}_0^+ V_3(z_3, \bar{z}_3)) >$$

$$= (z_2 - z_3)(\bar{z}_2 - \bar{z}_3)(z_2 - z_3)^{-1}(\bar{z}_2 - \bar{z}_3)^{-1}$$

$$\cdot < e^{i\hat{H}_C(0)+\rho(0)+3i\sigma(0)}e^{i\hat{H}_C(0)+\rho(0)+3i\sigma(0)}\theta^a_0\theta^b_0\theta^d_0\theta^d_0\phi_{gh}(X(0)) >$$

$$\cdot [\frac{1}{16} \sigma^m_{ab}\sigma^n_{ab}\sigma^j_{ef}\sigma^k_{ef} < \phi_{mn}(X(\infty))\phi_{jk}(X(1)) ] t^\ell_L t^\ell_R \sigma^q_{cd} \sigma^h_{cd} \phi_{gh}(X(0)) ] >$$

(4.6)

$$+ \sigma^m_{ab}\sigma^n_{ab} < \phi_{mn}(X(\infty)) [ t^\ell_L t^\ell_R \sigma^j_{ef} \sigma^k_{ef} \phi_{jk}(X(1)) ] \sigma^q_{cd} \sigma^h_{cd} \phi_{gh}(X(0)) ] >$$

$$+ \sigma^m_{ab}\sigma^n_{ab} < \phi_{mn}(X(\infty)) [ t^\ell_R t^\ell_L \sigma^j_{ef} \sigma^k_{ef} \phi_{jk}(X(1)) ] \sigma^q_{cd} \sigma^h_{cd} \phi_{gh}(X(0)) ] >$$

$$+ 4\sigma^m_{ab}\sigma^n_{ab}\sigma^h_{cd} < \phi_{mn}(X(\infty)) [ t^\ell_R t^\ell_L \sigma^j_{ef} \sigma^k_{ef} \phi_{jk}(X(1)) ] \phi_{gh}(X(0)) ] >$$

where the $z_2, z_3$ OPE’s have been evaluated in similar fashion, the $SL(2, C)$ invariance sets $z_2 \to 1, z_3 \to 0$, and cancellations occur among the contributions to the OPE’s from the four terms in the sum in (4.5).
Then

\[
< V_1(z_1, \tilde{z}_1) (G_0^+ \tilde{G}_0^+ V_2(z_2, \tilde{z}_2))(\tilde{G}_0^+ \tilde{G}_0^+ V_3(z_3, \tilde{z}_3)) > \\
= 4 \cdot \left[ \frac{1}{16} g^{mcd} \sigma^{\ast cd} \sigma^{j}_{ef} \sigma^{k}_{ef} < \phi_{mn}(X(\infty)) \phi_{jk}(X(1)) \left[ t^{ef}_L t^{ef}_R \sigma^{g}_{cd} \sigma^{h}_{cd} \phi_{gh}(X(0)) \right] > \\
+ \frac{1}{4} g^{mcd} \sigma^{\ast cd} < \phi_{mn}(X(\infty)) \left[ t^{ef}_L \sigma^{k}_{ef} \phi_{jk}(X(1)) \left[ t^{ef}_R \sigma^{g}_{cd} \phi_{gh}(X(0)) \right] > \\
+ \frac{1}{4} g^{mcd} \sigma^{\ast cd} < \phi_{mn}(X(\infty)) \left[ t^{ef}_R \sigma^{j}_{ef} \phi_{jk}(X(1)) \left[ t^{ef}_L \sigma^{g}_{cd} \phi_{gh}(X(0)) \right] > \\
+ g^{mcd} \sigma^{\ast cd} < \phi_{mn}(X(\infty)) \left[ t^{ef}_R \sigma^{j}_{ef} \phi_{jk}(X(1)) \left[ t^{ef}_L \sigma^{g}_{cd} \phi_{gh}(X(0)) \right] > \right] \\
\right]
\]

where the sigma matrices are expressed in terms of the \( AdS_3 \times S^3 \) background fields

\[
g_{mn} = \frac{1}{2} \sigma^{ab}_m \sigma_{nab}, \quad R_{mn} = -\frac{1}{2} \sigma^{ab}_m \sigma^{cd}_n \delta_{ac} \delta_{bd}, \quad H_{mpq} = \frac{1}{2} (\sigma_m \sigma_p \sigma_q)_{ab} \delta^{ab}, \quad (4.8)
\]

and the Ricci tensor is related to the self-dual three-form flux as \( \tilde{R}_{mn} = -\tilde{H}_{mpq} \tilde{H}_{n}^{pq} \).

Dropping terms proportional to total divergences, we have finally from (4.7)

\[
< V_1(z_1, \tilde{z}_1) (G_0^+ \tilde{G}_0^+ V_2(z_2, \tilde{z}_2))(\tilde{G}_0^+ \tilde{G}_0^+ V_3(z_3, \tilde{z}_3)) > \\
= -12 \left[ \phi^{mn}(x_0) \phi^{jk}(x_0) D_m D_n \phi_{jk}(x_0) > + 2 \phi^{mn}(x_0) D_m \phi^{jk}(x_0) D_j \phi_{nk}(x_0) > \\
- 4 \tilde{H}^{mjg} \tilde{H}^{nh} < \phi_{mn}(x_0) \phi_{jk}(x_0) \phi_{gh}(x_0) > \\
- 8 \tilde{g}^{mjg} \tilde{H}_{p}^{knh} < \phi_{mn}(x_0) \phi_{jk}(x_0) \phi_{gh}(x_0) > \\
- \frac{2}{3} < \phi_{mn}(x_0) D_j \phi^{mj}(x_0) D_k \phi^{nk}(x_0) > \\
+ \frac{2}{3} < \phi^{mn}(x_0) \phi_{mn}(x_0) D_j D_k \phi^{jk}(x_0) > \right].
\]

(4.9)

In this derivation, we have evaluated terms with invariant derivatives such as

\[
\frac{1}{16} g^{mcd} \sigma^{\ast cd} \sigma^{j}_{ef} \sigma^{k}_{ef} < \phi_{mn}(X(\infty)) \phi_{jk}(X(1)) \left[ t^{ef}_L t^{ef}_R \sigma^{g}_{cd} \sigma^{h}_{cd} \phi_{gh}(X(0)) \right] > \\
= -g^{mg} \tilde{g}^{nh} < \phi_{mn}(x_0) \phi_{jk}(x_0) D^j D^k \phi_{gh}(x_0) > \\
+ 2 \tilde{H}^{mjg} \tilde{H}^{nh} < \phi_{mn}(x_0) \phi_{jk}(x_0) \phi_{gh}(x_0) > \\
+ 2 \tilde{H}^{pjg} \tilde{H}^{kh} \tilde{g}^{mg} < \phi_{mn}(x_0) \phi_{jk}(x_0) \phi_{gh}(x_0) > \right.
\]

(4.10)
To interpret (4.9) on shell, we recall [4] that the first order linearized duality equation of motion for one of the supergravity fields related to \( b_{mn} \) is \( g_{1n}^{\mu} - \frac{1}{6} \varepsilon_{n}^{\rho\sigma} \epsilon_{\mu}^{\rho\sigma} g_{1}^{\mu} = -\tilde{H}_{ng} g_{j}^{\mu} + \tilde{H}_{jqr} g_{n}^{\mu} + \tilde{H}_{n}^{q} g_{r}^{\mu} \). Part of (4.9) is then identified as
\[
-4 \tilde{H}^{mg} \tilde{h}^{nkh} g_{mn} g_{jk} g_{gh} - 8 \bar{g}^{mg} \tilde{h}^{npj} \tilde{h}_{p}^{kh} g_{mn} g_{jk} g_{gh} = -4 \tilde{H}^{nkh} g_{jk} g_{gh} \left( g_{1}^{\mu} \bar{e}_{n}^{\rho\sigma} \right) (4.11).
\]
Assuming that the only non-vanishing string field fluctuation is \( g_{mn} \), we have \( \phi_{mn} = g_{mn} \) and the gauge condition becomes \( D^{m}g_{mn} = 0 \).

Then on \( AdS_{3} \times S^{3} \), the string theory three graviton amplitude is
\[
< V_{1}(z_{1}, \bar{z}_{1}) (G_{0}^{+} \bar{G}_{0}^{+} V_{2}(z_{2}, \bar{z}_{2})) (\bar{G}_{0}^{+} \bar{G}_{0}^{+} V_{3}(z_{3}, \bar{z}_{3})) > \\
= -12 \left[ < g_{mn}^{mn} g_{jk} D_{m} D_{n} g_{jk} > +2 < g_{mn}^{mn} g_{jk} D_{j} g_{nk} > \right. \\
-4 < \tilde{H}^{mg} g_{mn} g_{jk} g_{gh} > \\
-8 < \bar{g}^{mg} \tilde{h}^{npj} g_{mn} g_{jk} g_{gh} > \\
= -12 \int d^{5}x \sqrt{g} \left[ g_{mn}^{mn} g_{jk} D_{m} D_{n} g_{jk} + 2 g_{mn}^{mn} D_{m} g_{jk} D_{j} g_{nk} \right].
\]
(4.12) is the curved space analog of (3.7). In three dimensions, the graviton has no propagating degrees of freedom, which means the graviton field can be gauged to zero. Nonetheless, expanding (4.12) in spherical harmonics on \( S^{3} \), and noting that \( \bar{g}_{mn} = \bar{g}_{\mu\nu}, \bar{g}_{\alpha\beta} \), the background metric of \( AdS_{3} \times S^{3} \) for \( 1 \leq \mu, \nu \leq 3 \) and \( 1 \leq \alpha, \beta \leq 3 \), we find that the covariant derivatives factorize and the string theory calculation retains the familiar structure \( \sim \int d^{3}x \sqrt{g_{3}} \left[ g^{\mu\nu} g^{\rho\sigma} D_{\mu} D_{\nu} g_{\rho\sigma} + 2 g^{\mu\rho} D_{\mu} g^{\rho\sigma} D_{\rho} g_{\nu\sigma} \right] \).

In general, \( \alpha' \) corrections are expected to occur in four or higher \( n \)-point string tree amplitudes, but will be calculable only as an expansion in \( \alpha' \) since the worldsheet theory is not free. To study the AdS/CFT correspondence, the bulk correlations functions on shell can be related to correlations on the boundary. Since \( \alpha' \) is related the coupling constant of the spacetime conformal field theory, to investigate this correspondence systematically it would be of interest to attain tree-level expressions that are exact in \( \alpha' \), perhaps by adapting integrable methods for sigma models which have a supergroup manifold target space [15] such as this \( AdS_{3} \times S^{3} \) theory.
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