SOME GLOBAL DYNAMICS OF A LOTKA-VOLterra
COMPETITION-DIFFUSION-ADVECTION SYSTEM

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Abstract. This paper studies some population dynamics of a diffusive Lotka-Volterra competition advection model under no-flux boundary condition. We establish the main results that determine the stability of semi-trivial steady states.

1. Introduction and statement of the main results. The classical Lotka-Volterra competition-diffusion system

\[
\begin{aligned}
  u_t &= \mu \Delta u + u(m(x) - u - bv) \quad \text{in } \Omega \times (0, +\infty), \\
  v_t &= \nu \Delta v + v(m(x) - cu - v) \quad \text{in } \Omega \times (0, +\infty), \\
  \frac{\partial u}{\partial n} &= \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial \Omega \times (0, +\infty), \\
  u(x, 0) &= u_0(x) \geq \neq 0 \quad \text{in } \Omega, \\
  v(x, 0) &= v_0(x) \geq \neq 0 \quad \text{in } \Omega,
\end{aligned}
\]  

(1.1)

models two competing species. Here \( u(x, t) \) and \( v(x, t) \) denote respectively the population densities of two competing species at location \( x \in \Omega \) and time \( t > 0 \), and \( \mu, \nu > 0 \) are random diffusion rates of species \( u \) and \( v \) respectively. The habitat \( \Omega \) is a bounded domain in \( \mathbb{R}^N \), with smooth boundary \( \partial \Omega \), \( n \) denotes the unit outer normal vector on \( \partial \Omega \), and the no flux boundary condition means that no individuals cross over the boundary. The function \( m(x) \in C^2(\Omega) \) represents their common intrinsic growth rate or local carrying capacity, which is non-constant and \( m(x) > 0 \) in \( \Omega \). \( b > 0 \) and \( c > 0 \) are interspecific competition coefficients. Then the maximum principle [18] yields that \( u(x, t) > 0, v(x, t) > 0 \) for every \( x \in \Omega \) and every \( t > 0 \). By both mathematicians and ecologists, particular interests in two-species Lotka-Volterra competition models with spatially homogeneous or heterogeneous interactions are the dynamics of (1.1). See [2, 7, 8, 9, 10, 11, 14] and the references therein. We say that a steady state \( (u_s, v_s) \) of (1.1) is a coexistence state if both components are positive, and it is a semi-trivial state if one component is positive and the other is identically zero. We make the following basic assumption on \( b, c \).

Assumption 1. \( 0 < b, c \leq 1 \).

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It is well-known in [2, 17] that the equation
\[
\begin{align*}
ll\mu\Delta\Theta + \Theta(m(x) - \Theta) &= 0 \quad \text{in } \Omega, \\
\frac{\partial\Theta}{\partial n} &= 0 \quad \text{on } \partial\Omega,
\end{align*}
\]  
(1.2)
has a unique positive solution, denoted by \(\Theta_\mu\), which indicates that (1.1) has two semi-trivial steady states, denoted by \((\Theta_\mu, 0)\) and \((0, \Theta_\nu)\), for every \(\mu > 0\) and \(\nu > 0\).

For the weak competition case \(0 < b, c < 1\), it is shown in [14] that if \(\mu\) and \(\nu\) are sufficiently small, then (1.1) has a unique, globally asymptotically stable positive steady state \((u^*, v^*)\). Moreover, \((u^*, v^*)\) converges to \(\left(\frac{1-b}{1-bc}m(x), \frac{1-c}{1-bc}m(x)\right)\) in \(L^\infty(\Omega)\) as \(\mu \to 0\) and \(\nu \to 0\). When both \(\mu\) and \(\nu\) are sufficiently large, it is not difficult to see that (1.1) again has a unique, globally asymptotically stable positive steady state \((u^{**}, v^{**})\), which converges to \(\left(\frac{1-b}{1-bc}\int_{\Omega} m(x)dx, \frac{1-c}{1-bc}\int_{\Omega} m(x)dx\right)\) as \(\mu \to +\infty\) and \(\nu \to +\infty\).

Under above conditions, Lou [17] verified that there exists some constant \(c_* \in (0, 1)\) such that (i) for any \(c \in (0, c_*)\), the steady state \((\Theta_\mu, 0)\) is unstable for all \(\mu, \nu > 0\); (ii) for any \(c \in (c_*, 1)\), there exist \(\bar{\mu} > 0\) and \(\bar{\nu} > 0\) such that \((\Theta_{\bar{\mu}}, 0)\) is unstable. Meanwhile, when \(c \in (c_*, 1)\), there exists \(b_* \in (0, 1)\) such that for some \((\mu, \nu) \in (0, +\infty) \times (0, +\infty)\), \((\Theta_\mu, 0)\) is globally asymptotically stable provided that \(b \in (0, b_*]\). Furthermore He and Ni [9] provided a complete classification on the global dynamics of system (1.1), which says that either one of the two semi-trivial steady states is globally asymptotically stable, or there is a unique co-existence steady state which is globally asymptotically stable, or the system is degenerate in the sense that there is a compact global attractor consisting of a continuum of steady states which connect the two semi-trivial steady states (see [9], Theorems 1.3 and 3.4). We refer the interested readers to [7, 8, 10, 11] for some investigations on system (1.1).

Besides random dispersal, it seems reasonable to suppose that species could move upward along the resource gradient (see e.g. [2, 19] and the references therein). In this paper we deal with a diffusive Lotka-Volterra competition-advection model in spatially heterogeneous environment
\[
\begin{align*}
\frac{\partial u}{\partial t} &= \mu\Delta u - \alpha\nabla \cdot (u\nabla m_1(x)) + u(m_1(x) - u - bv) \quad \text{in } \Omega \times (0, +\infty), \\
\frac{\partial v}{\partial t} &= \nu\Delta v - \beta\nabla \cdot (v\nabla m_2(x)) + v(m_2(x) - cu - v) \quad \text{in } \Omega \times (0, +\infty), \\
\frac{\partial u}{\partial n} - \alpha u \frac{\partial m_1}{\partial n} &= \nu \frac{\partial v}{\partial n} - \beta v \frac{\partial m_2}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0, +\infty), \\
u(x, 0) &= u_0(x) \geq 0 \quad \text{in } \Omega, \\
v(x, 0) &= v_0(x) \geq 0 \quad \text{in } \Omega.
\end{align*}
\]  
(1.3)
The functions \(m_1(x)\) and \(m_2(x)\) stand for the local carrying capacity or intrinsic growth rates of species \(u\) and \(v\), respectively. \(b, c > 0\) are inter-species competition coefficients. The advection rates of two species are denoted by \(\alpha, \beta \geq 0\), respectively. Here the movement strategies, growth rates and competition abilities of two species are taken into account and allowed to be different.

Throughout this paper, besides Assumption 1, we define
\[
\eta := \frac{\alpha}{\mu} \geq 0, \quad \xi := \frac{\beta}{\nu} \geq 0,
\]  
(1.4)
and also give the following assumption.
Assumption 2. \( m_i(x) \in C^2(\Omega) (i = 1, 2) \) is non-constant, and \( m_i(x) \geq 0 \) in \( \Omega \).

This paper is devoted to some global dynamics of (1.3).

1.1. Motivation and related work. System (1.3) has important applications in biological scenarios. For example, by letting \( m_1(x) = m_2(x) = m(x), b = c = 1 \), one obtains the following competition model

\[
\begin{aligned}
    u_t &= \mu \Delta u - \alpha \nabla \cdot (u \nabla m(x)) + u(m(x) - u - v) \quad \text{in } \Omega \times (0, +\infty), \\
v_t &= \nu \Delta v - \beta \nabla \cdot (v \nabla m(x)) + v(m(x) - u - v) \quad \text{in } \Omega \times (0, +\infty), \\
    \mu \frac{\partial u}{\partial n} - \alpha n \frac{\partial m}{\partial n} = \nu \frac{\partial v}{\partial n} - \beta v \frac{\partial m}{\partial n} &= 0 \quad \text{on } \partial \Omega \times (0, +\infty), \\
    u(x, 0) &= u_0(x) \geq 0 \quad \text{in } \Omega, \\
v(x, 0) &= v_0(x) \geq 0 \quad \text{in } \Omega.
\end{aligned}
\]

Recently (1.5) has been frequently used as a standard model to study the evolution of conditional dispersal, see, e.g., [1, 4, 6] for \( \alpha, \beta > 0 \). Basically speaking, system (1.5) models the competition between two species with the same population dynamics but different movement strategies as reflected by their diffusion and/or advection rates.

Motivated by the above work, we shall consider more global dynamics of (1.3).

The rest of this paper is organized as follows. In Section 1.2, we present some preliminary results, which are helpful to verify our results. In Section 1.3 we establish our main results (Theorem 1.6, Theorem 1.7). The proof will be given in Section 2.

1.2. Preliminaries. Under Assumption 2, (1.3) has two semi-trivial steady states for all \( \mu, \nu > 0 \) and \( \alpha, \beta \geq 0 \) ([2]), denoted by \((\theta_{\mu,\eta,m_1,0}, 0), (0, \theta_{\nu,\xi,m_2})\) respectively, where \( \theta_{\mu,\eta,m_1} \) is the unique positive solution of

\[
\begin{aligned}
    \mu \nabla \cdot (\nabla \theta_{\mu,\eta,m_1} - \eta \theta_{\mu,\eta,m_1} \nabla m_1) + \theta_{\mu,\eta,m_1}(m_1(x) - \theta_{\mu,\eta,m_1}) &= 0 \quad \text{in } \Omega, \\
    \frac{\partial \theta_{\mu,\eta,m_1}}{\partial n} - \eta \frac{\partial m_1}{\partial n} &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

and \( \theta_{\nu,\xi,m_2} \) is the unique positive solution of

\[
\begin{aligned}
    \nu \nabla \cdot (\nabla \theta_{\nu,\xi,m_2} - \xi \theta_{\nu,\xi,m_2} \nabla m_2) + \theta_{\nu,\xi,m_2}(m_2(x) - \theta_{\nu,\xi,m_2}) &= 0 \quad \text{in } \Omega, \\
    \frac{\partial \theta_{\nu,\xi,m_2}}{\partial n} - \xi \frac{\partial m_2}{\partial n} &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

To study the dynamics of system (1.3), we should study the stability of semi-trivial steady states \((\theta_{\mu,\eta,m_1,0}, 0), (0, \theta_{\nu,\xi,m_2})\). Mathematically, the stability of \((\theta_{\mu,\eta,m_1,0})\) is determined by the principal eigenvalue \( \sigma_1(\nu, \xi, m_2 - c \theta_{\mu,\eta,m_1}) \) of the elliptic eigenvalue problem

\[
\begin{aligned}
    \nu \nabla \cdot (\nabla \psi - \xi \psi \nabla m_2) + (m_2 - c \theta_{\mu,\eta,m_1}) \psi &= \sigma \psi \quad \text{in } \Omega, \\
    \frac{\partial \psi}{\partial n} - \xi \psi \frac{\partial m_2}{\partial n} &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

Similarly, the stability of \((0, \theta_{\nu,\xi,m_2})\) is determined by the principal eigenvalue \( \sigma_1(\mu, \eta, m_1 - b \theta_{\nu,\xi,m_2}) \) of the linear problem as follows:

\[
\begin{aligned}
    \mu \nabla \cdot (\nabla \varphi - \eta \varphi \nabla m_1) + (m_1 - b \theta_{\nu,\xi,m_2}) \varphi &= \sigma \varphi \quad \text{in } \Omega, \\
    \frac{\partial \varphi}{\partial n} - \eta \varphi \frac{\partial m_1}{\partial n} &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

More precisely, the following criterion is well-known.
Lemma 1.1 ([2]). \((\theta_{\mu,\eta, m_1}, 0)\) is linearly stable if \(\sigma_1(\nu, \xi, m_2 - c\theta_{\mu, \eta, m_1}) < 0\) and is linearly unstable if \(\sigma_1(\nu, \xi, m_2 - c\theta_{\mu, \eta, m_1}) > 0\). Similarly, \((0, \theta_{\nu, \xi, m_2})\) is linearly stable if \(\sigma_1(\mu, \eta, m_1 - b\theta_{\nu, \xi, m_2}) < 0\) and is linearly unstable if \(\sigma_1(\mu, \eta, m_1 - b\theta_{\nu, \xi, m_2}) > 0\).

The following result is due to [1].

Lemma 1.2. Assume that \(m(x) \in C^2(\bar{\Omega})\), \(m \neq \) constant, and \(m(x) \geq 0\) in \(\bar{\Omega}\). Fixed \(\eta > 0\). Then the steady state problem

\[
\begin{cases}
  \mu \nabla \cdot (\nabla \theta - \eta \theta \nabla m) + \theta (m(x) - \theta) = 0 & \text{in } \Omega, \\
  \frac{\partial \theta}{\partial n} - \eta \theta \frac{\partial m}{\partial n} = 0 & \text{on } \partial \Omega,
\end{cases}
\]

has a unique positive solution \(\theta_{\mu, \eta, m}\). Furthermore the following statements hold:

(i) \(\int_\Omega \theta_{\mu, \eta, m} e^{\eta m} dx > \int_\Omega m e^{\eta m} dx\).

(ii) \(\lim_{\mu \to 0} \int_\Omega \theta_{\mu, \eta, m} e^{\eta m} dx = \lim_{\mu \to +\infty} \int_\Omega \theta_{\mu, \eta, m} e^{\eta m} dx = \int_\Omega m e^{\eta m} dx\).

Let \(\lambda_1(\xi, h, m)\) denote the unique nonzero principal eigenvalue of

\[
\begin{cases}
  \nabla \cdot (\nabla \phi - \xi \phi \nabla m) + \lambda_1 h(x) \phi = 0 & \text{in } \Omega, \\
  \frac{\partial \phi}{\partial n} - \xi \phi \frac{\partial m}{\partial n} = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \(h \neq \) constant, could change sign. In fact \(\lambda_1(\xi, h, m)\) is also the principal eigenvalue of

\[
\begin{cases}
  \Delta \zeta + \xi \nabla m \nabla \zeta + \lambda_1 h(x) \zeta = 0 & \text{in } \Omega, \\
  \frac{\partial \zeta}{\partial n} = 0 & \text{on } \partial \Omega.
\end{cases}
\]

The following result can be obtained in [2, Lemma 2.16].

Lemma 1.3. The problem (1.11) has a nonzero principal eigenvalue \(\lambda_1 = \lambda_1(\xi, h, m)\) if and only if \(h\) changes sign and \(\int_\Omega h(x)e^{\xi m} dx \neq 0\). More precisely, if \(h\) changes sign, then \(\int_\Omega h e^{\xi m} dx < 0 \Rightarrow \lambda_1(\xi, h, m) > 0\).

In order to analyze the principal eigenvalue of problem (1.8) and (1.9), it is more convenient to consider the following more general form of eigenvalue problem:

\[
\begin{cases}
  \mu \nabla \cdot (\nabla \phi - \eta \phi \nabla m) + h(x) \phi = \sigma \phi & \text{in } \Omega, \\
  \frac{\partial \phi}{\partial n} - \eta \phi \frac{\partial m}{\partial n} = 0 & \text{on } \partial \Omega,
\end{cases}
\]

which is equivalent to

\[
\begin{cases}
  \mu \nabla \cdot (e^{\eta m} \nabla \psi) + e^{\eta m} h(x) \psi = \sigma e^{\eta m} \psi & \text{in } \Omega, \\
  \frac{\partial \psi}{\partial n} = 0 & \text{on } \partial \Omega.
\end{cases}
\]

The principal eigenvalue of problem (1.13), denoted by \(\sigma_1(\mu, \eta, h)\), is expressed by the following variational formula (see, e.g. [2])

\[
\sigma_1(\mu, \eta, h) = \max_{\psi \in W^{1,2}(\Omega), \psi \neq 0} \frac{-\mu \int_\Omega e^{\eta m} |\nabla \psi|^2 dx + \int_\Omega e^{2\eta m} h(x) \psi^2 dx}{\int_\Omega e^{\eta m} \psi^2 dx}.
\]

The following lemmas collects some useful properties of \(\sigma_1(\mu, \eta, h)\).
Lemma 1.4 ([2]). The first eigenvalue $\sigma_1(\mu, \eta, h)$ of (1.13) has the following property: if $\lambda_1(\eta, h, m) > 0$, then $\sigma_1(\mu, \eta, h) < 0 \Leftrightarrow \mu > \frac{1}{\lambda_1(\eta, h, m)}$.

Lemma 1.5. Assume that $m(x) > 0$ and $h(x)$ is non-constant in $\overline{\Omega}$. Then the first eigenvalue $\sigma_1(\mu, \eta, h)$ of (1.13) has the following property: $\sigma_1(\mu, \eta, h)$ is strictly decreasing in $\mu$, provided that either $\alpha > 0$ is fixed and $\alpha \|m\|_\infty \leq \mu$ or $\eta > 0$ is fixed.

Proof. For the case where $\alpha > 0$ is fixed and $\alpha \|m\|_\infty \leq \mu_1 < \mu_2$, one can see that

$$
\mu_2 \frac{\int_\Omega e^{\frac{\alpha}{\mu_2}} |\nabla \psi|^2 dx - \int_\Omega e^{\frac{\alpha}{\mu_2}} m h(x) \psi^2 dx}{\int_\Omega e^{\frac{\alpha}{\mu_2}} \psi^2 dx} > \mu_1 \frac{\int_\Omega e^{\frac{\alpha}{\mu_1}} |\nabla \psi|^2 dx - \int_\Omega e^{\frac{\alpha}{\mu_1}} m h(x) \psi^2 dx}{\int_\Omega e^{\frac{\alpha}{\mu_1}} \psi^2 dx},
$$

for any $\psi \in W^{1,2}(\Omega), \psi \not\equiv 0$. This implies

$$
-\mu_2 \frac{\int_\Omega e^{\frac{\alpha}{\mu_2}} |\nabla \psi|^2 dx + \int_\Omega e^{\frac{\alpha}{\mu_2}} m h(x) \psi^2 dx}{\int_\Omega e^{\frac{\alpha}{\mu_2}} \psi^2 dx} < -\mu_1 \frac{\int_\Omega e^{\frac{\alpha}{\mu_1}} |\nabla \psi|^2 dx + \int_\Omega e^{\frac{\alpha}{\mu_1}} m h(x) \psi^2 dx}{\int_\Omega e^{\frac{\alpha}{\mu_1}} \psi^2 dx}.
$$

Therefore there exists $\psi_0 \in W^{1,2}(\Omega), \psi_0 \not\equiv 0$ such that

$$
\sigma_1(\mu_2, \frac{\alpha}{\mu_2}, h) = \max_{\psi \in W^{1,2}(\Omega), \psi \not\equiv 0} -\mu_2 \frac{\int_\Omega e^{\frac{\alpha}{\mu_2}} |\nabla \psi|^2 dx + \int_\Omega e^{\frac{\alpha}{\mu_2}} m h(x) \psi^2 dx}{\int_\Omega e^{\frac{\alpha}{\mu_2}} \psi^2 dx} = \mu_2 \frac{\int_\Omega e^{\frac{\alpha}{\mu_2}} |\nabla \psi_0|^2 dx + \int_\Omega e^{\frac{\alpha}{\mu_2}} m h(x) \psi_0^2 dx}{\int_\Omega e^{\frac{\alpha}{\mu_2}} \psi_0^2 dx}.
$$

\[ \leq \max_{\psi \in W^{1,2}(\Omega), \psi \not\equiv 0} -\mu_1 \frac{\int_\Omega e^{\frac{\alpha}{\mu_1}} |\nabla \psi|^2 dx + \int_\Omega e^{\frac{\alpha}{\mu_1}} m h(x) \psi^2 dx}{\int_\Omega e^{\frac{\alpha}{\mu_1}} \psi^2 dx} = \sigma_1(\mu_1, \frac{\alpha}{\mu_1}, h). \] (1.16)
On the other hand, when \( \eta \) is fixed, and when \( \psi \) is non-constant, \( \psi \in W^{1,2}(\Omega) \),

\[
-\mu_2 \int_\Omega e^{\eta m} |\nabla \psi|^2 \, dx + \int_\Omega e^{2\eta m} h(x) \psi^2 \, dx \\
\frac{1}{\int_\Omega e^{\eta m} \psi^2 \, dx}
\]

follows immediately for any \( 0 < \mu_1 < \mu_2 \). Similar to (1.16), we have that \( \sigma_1(\mu_2, \eta, h) < \sigma_1(\mu_1, \eta, h) \), since in this case the maximum \( \sigma_1(\mu_2, \eta, h) \) can be obtained at some \( \nu \) which is non-constant.

To describe our first result, according to [9], for any \( \mu > 0, \nu > 0 \), we define

\[
\begin{align*}
\Gamma & := \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+; \\
\Sigma_u & := \{(\mu, \eta, \nu, \xi) \in \Gamma : (\theta_{\mu,\eta,m_1},0) \text{ is linearly stable}\}; \\
\Sigma_v & := \{(\mu, \nu, \xi) \in \Gamma : (0, \theta_{\nu,\xi,m_2}) \text{ is linearly stable}\}; \\
L_u & := \inf_{\mu>0,0<\eta \leq 1} \frac{m_2}{\theta_{\mu,\eta,m_1}} \in [0, +\infty); \\
L_v & := \inf_{\nu>0,0<\xi \leq 1} \frac{m_2}{\theta_{\nu,\xi,m_2}} \in [0, +\infty); \\
S_u & := \sup_{\mu>0,0<\eta \leq 1} \sup_{1/\Pi} \frac{m_1}{\theta_{\mu,\eta,m_1}} \in (0, +\infty]; \\
S_v & := \sup_{\nu>0,0<\xi \leq 1} \sup_{1/\Pi} \frac{m_1}{\theta_{\nu,\xi,m_2}} \in (0, +\infty],
\end{align*}
\]

and

\[
\begin{align*}
I_c & := \{(\mu, \eta) : \int_\Omega (m_2 - c\theta_{\mu,\eta,m_1}) \, dx < 0\} := I_c^0 \cup I_c^1, \\
I_c^0 & := \{(\mu, \eta) : m_2 - c\theta_{\mu,\eta,m_1} \leq (\neq) 0 \text{ on } 1/\Pi\}, \\
I_c^1 & := \{(\mu, \eta) \in I_c : \sup_{1/\Pi} (m_2 - c\theta_{\mu,\eta,m_1}) > 0\}, \\
I_b & := \{(\nu, \xi) : \int_\Omega (m_1 - b\theta_{\nu,\xi,m_2}) \, dx < 0\} := I_b^0 \cup I_b^1, \\
I_b^0 & := \{(\nu, \xi) : m_1 - b\theta_{\nu,\xi,m_2} \leq (\neq) 0 \text{ on } 1/\Pi\}, \\
I_b^1 & := \{(\nu, \xi) \in I_b : \sup_{1/\Pi} (m_1 - b\theta_{\nu,\xi,m_2}) > 0\}.
\end{align*}
\]

In fact, by Lemma 1.1 we have the following equivalent descriptions.

\[
\begin{align*}
\Sigma_u & := \{(\mu, \eta, \nu, \xi) \in \Gamma : \sigma_1(\nu, \xi, m_2 - c\theta_{\mu,\eta,m_1}) < 0\}; \\
\Sigma_v & := \{(\mu, \eta, \nu, \xi) \in \Gamma : \sigma_1(\mu, \eta, m_1 - b\theta_{\nu,\xi,m_2}) < 0\}.
\end{align*}
\]

1.3. **Main results.** Based on the above preparations, we are now ready to state our main results concerning the steady state of (1.3).

**Theorem 1.6.** Assume that \( m_i(x)(i = 1, 2) \) satisfies Assumption 2 and assume \( \max_{x \in \Omega} m_1(x) = \max_{x \in \Omega} m_2(x) \). Suppose that Assumption 1 holds. Let \( L_u, S_u, L_v, \) and \( S_v \) be defined as in (1.17). Then the following results hold for (1.3):
Assume that Theorem 1.7. and I
have $\geq I$ and $b$ that one of the following conditions holds: verified similarly. The proof is split into four steps. Proof of Theorem 1.6. 2. Proof of the main results. (i) For any $\mu > 0$, there exists $\Lambda(\mu) > 0$ such that when $\xi > \Lambda(\mu)$ and $0 < \eta \leq \frac{1}{\max \sigma} m_1$,
\[
\Sigma_u = \begin{cases} 
\emptyset & (\mu, \eta, \xi) \in I_c, \nu > \nu^*(\mu, \eta, \xi) > 0 \\
\{(\mu, \eta, \nu, \xi) : (\mu, \eta) \in I_c, \nu > \nu^*(\mu, \eta, \xi) > 0, \xi > \Lambda(\mu)\} & 0 < c \leq L_u, \\
\{(\mu, \eta, \nu, \xi) : c \geq S_u, \xi > \Lambda(\mu)\} & L_u < c < S_u,
\end{cases}
\] (1.20)
where $\nu^*(\mu, \eta, \xi)$ is defined as follows:
\[
\nu^*(\mu, \eta, \xi) = \begin{cases} 
0 & (\mu, \eta) \in I_0^c, \\
\frac{1}{\lambda_1(\xi, m_2 - c\theta_{\mu, \eta, m_1}, m_2)} & (\mu, \eta) \in I_1^c,
\end{cases}
\] (1.21)
and $I_c, I_0^c, I_1^c$ are defined in (1.18).
(ii) For any $\nu > 0$, there exists $\Pi(\nu) > 0$ such that when $\eta > \Pi(\nu)$ and $0 \leq \xi \leq \frac{1}{\max \sigma} m_2$,
\[
\Sigma_v = \begin{cases} 
\emptyset & (\mu, \eta, \nu, \xi) \in I_b, \mu > \mu^*(\nu, \xi, \eta) > 0 \\
\{(\mu, \eta, \nu, \xi) : (\mu, \eta) \in I_b, \mu > \mu^*(\nu, \xi, \eta) > 0, \xi > \Lambda(\mu)\} & 0 < b \leq L_v, \\
\{(\mu, \eta, \nu, \xi) : b \geq S_v, \xi > \Lambda(\mu)\} & L_v < b < S_v,
\end{cases}
\] (1.22)
where $\mu^*(\nu, \xi, \eta)$ is defined as follows:
\[
\mu^*(\nu, \xi, \eta) = \begin{cases} 
0 & (\nu, \xi) \in I_0^b, \\
\frac{1}{\lambda_1(\eta, m_1 - b\theta_{\nu, \xi}, m_1)} & (\nu, \xi) \in I_1^b,
\end{cases}
\] (1.23)
and $I_b, I_0^b, I_1^b$ are defined in (1.18).

**Theorem 1.7.** Assume that $m_1(x) = m_2(x) = m(x)$ satisfies Assumption 2 and $b = c = 1$. Then the steady state $(\theta_{\mu, \eta, m_1}, 0)$ is globally asymptotically stable, provided that one of the following conditions holds:
(i). $\mu < \nu$ and $\xi = \eta$;
(ii). $\alpha = \beta$, $0 < \alpha\|m\|_{\infty} \leq \mu < \nu$ and $\{x \in \Omega : m(x) = 0\} \neq \emptyset$.

2. **Proof of the main results.** In this section, we verify our main results. 

**Proof of Theorem 1.6.** In fact it is sufficient to prove case (i), and case (ii) can be verified similarly. The proof is split into four steps.

Step 1. $(\mu, \eta, \nu, \xi) \in \Sigma_u$ indicates that $(\mu, \eta) \in I_c$.

Suppose that $(\mu, \eta) \notin I_c$, where $I_c$ is defined in (1.18). Then $\int_{\Omega}(m_2 - c\theta_{\mu, \eta, m_1}) dx \geq 0$, which implies that either $m_2 \geq c\theta_{\mu, \eta, m_1}$ or $m_2 - c\theta_{\mu, \eta, m_1}$ changes sign in $\Pi$.

If $m_2 \geq c\theta_{\mu, \eta, m_1}$, then we get that
\[
\int_{\Omega} e^{\xi m_2}(m_2 - c\theta_{\mu, \eta, m_1}) dx \geq 0
\]
directly, and hence by (1.15) we reach that $\sigma(\nu, \xi, m_2 - c\theta_{\mu, \eta, m_1}) \geq 0$, i.e., $(\mu, \eta, \nu, \xi) \notin \Sigma_u$.

If $m_2 - c\theta_{\mu, \eta, m_1}$ changes sign, we claim that in this case there exists a positive constant $\Lambda(\mu)$ (independent of $\xi$) such that if $0 \leq \eta \leq \frac{1}{\max \sigma} m_1$ and $\xi \geq \Lambda(\mu)$, we have
\[
\int_{\Omega} e^{\xi m_2}(m_2 - c\theta_{\mu, \eta, m_1}) dx > 0.
\]
Since by $0 \leq \eta \leq \frac{1}{\max \mu m_1}$ and $\max_{x \in \Omega} m_1(x) = \max_{x \in \Omega} m_2(x)$, we have that
\[ \|\theta_{\mu,\eta,m_1}\|_\infty \leq \max_{x \in \Omega} m_2(x). \] (2.1)

It then follows that
\[ \left| \int_{\{x \in \Omega : m_2(x) \leq c\|\theta_{\mu,\eta,m_1}\|_\infty\}} e^{\xi(m_2 - c\|\theta_{\mu,\eta,m_1}\|_\infty)} (m_2 - c\theta_{\mu,\eta,m_1}) dx \right| \]
\[ \leq \int_{\{x \in \Omega : m_2(x) \leq c\|\theta_{\mu,\eta,m_1}\|_\infty\}} e^{\xi(m_2 - c\|\theta_{\mu,\eta,m_1}\|_\infty)} (m_2 - c\theta_{\mu,\eta,m_1}) dx \]
\[ \leq 2\|m_2\|_\infty |\Omega|. \] (2.2)

Set
\[ \varepsilon = \frac{1}{2} \min_{0 \leq \eta \leq \frac{1}{\max \mu m_1}} (\max \mu m_2 - c\|\theta_{\mu,\eta,m_1}\|_\infty) > 0. \] (2.3)

Let $x_0$ be a point such that $m_2(x_0) = \max_{\Omega} m_2 = \max_{\Omega} m_1$. Due to (2.1), it follows that
\[ x_0 \in \{x \in \Omega : m_2(x) > c\|\theta_{\mu,\eta,m_1}\|_\infty\}. \] (2.4)

Hence, the continuity of $\theta_{\mu,\eta,m_1}$ with respect to $\eta$ yields that we can choose $\delta > 0$ independent of $\eta$, such that for every $\eta \in [0, \frac{1}{\max \mu m_1}]$,
\[ m_2(x) - c\|\theta_{\mu,\eta,m_1}\|_\infty \geq \frac{1}{2} (\max \mu m_2 - c\|\theta_{\mu,\eta,m_1}\|_\infty) \geq \varepsilon > 0, \] (2.5)

if $x \in \Omega$ and $|x - x_0| \leq \delta$. Then
\[ \int_{\{x \in \Omega : m_2(x) > c\|\theta_{\mu,\eta,m_1}\|_\infty\}} e^{\xi(m_2 - c\|\theta_{\mu,\eta,m_1}\|_\infty)} (m_2 - c\theta_{\mu,\eta,m_1}) dx \]
\[ \geq \int_{\{x \in \Omega : |x - x_0| \leq \delta\}} e^{\xi(m_2 - c\|\theta_{\mu,\eta,m_1}\|_\infty)} (m_2 - c\theta_{\mu,\eta,m_1}) dx \]
\[ \geq e^{\xi\varepsilon} \varepsilon dx \to +\infty \] (2.6)
as $\xi \to +\infty$. Therefore,
\[ \int_\Omega e^{\xi(m_2 - c\|\theta_{\mu,\eta,m_1}\|_\infty)} (m_2 - c\theta_{\mu,\eta,m_1}) dx > 0, \] (2.7)

provided that $\xi > \Lambda(\mu)$ (independent of $\xi$). Hence, our claim is correct. This also indicates that $\sigma_1(\nu, \xi, m_2 - c\theta_{\mu,\eta,m_1}) > 0$, i.e., $(\mu, \eta, \nu, \xi) \notin \Sigma_u$. Hence $(\mu, \eta, \nu, \xi) \in \Sigma_u$ implies that $(\mu, \eta) \in I_u$.

We next characterize the set $I_u$ for all $c > 0$ in detail.

**Step 2.** If $c \leq L_u$, then the definition of $L_u$ in (1.17) leads to
\[ \int_\Omega (m_2 - c\theta_{\mu,\eta,m_1}) dx \geq 0. \] (2.8)

Hence $I_u = \emptyset$. Similar to **Step 1**, we get that $\Sigma_u = \emptyset$.

**Step 3.** If $I_u^1 \neq \emptyset$, then $L_u < c < S_u$. If $(\mu, \eta) \in I_u^1 \neq \emptyset$, then $L_u < c$ and for some $x_0 \in \Omega$, there holds
\[ e^{\xi m_2(x_0)} (m_2 - c\theta_{\mu,\eta,m_1})(x_0) > 0. \] (2.9)

Therefore
\[ c < \frac{m_2(x_0)}{\theta_{\mu,\eta,m_1}(x_0)} \leq S_u. \] (2.10)
Hence $I^1_c \neq \emptyset$ gives rise to $L_u < c < S_u$. On the other hand, if $L_u < c < S_u$, then there exists some $\mu' > 0$, $y_0 \in \Omega$, $\eta' \geq 0$ such that
\[
\int_\Omega (m_2 - c\theta_{\mu',\eta',m_1})dx < 0 \quad \text{and} \quad (m_2 - c\theta_{\mu',\eta',m_1})(y_0) > 0. \tag{2.11}
\]
That is, $(\mu', \eta') \in I^1_c \neq \emptyset$, which finishes the proof of Step 3.

**Step 4.** $I_c$ admits the following decomposition:
\[
I_c = \begin{cases} 
0 & c \leq L_u, \\
I^0_c \cup I^1_c \subset \mathbb{R}^+ \times \left[0, \frac{1}{\max \{m_1\}}\right] & L_u < c < S_u, \\
I_c = I^0_c = \mathbb{R}^+ \times \left[0, \frac{1}{\max \{m_1\}}\right] & S_u \leq c.
\end{cases} \tag{2.12}
\]
Indeed, it suffices to show that if $S_u < +\infty$ and $c \geq S_u$, then $I_c = I^0_c = \mathbb{R}^+ \times \left[0, \frac{1}{\max \{m_1\}}\right]$. By the definition of $S_u$ in (1.17), we deduce that $m_2 - c\theta_{\mu,\eta,m_1} \leq 0$ in $\Omega$. Hence to show that $I_c = I^0_c = \mathbb{R}^+ \times \left[0, \frac{1}{\max \{m_1\}}\right]$, it suffices to show that $m_2 - c\theta_{\mu,\eta,m_1} \neq 0$ for $c \geq S_u$. This is obviously true if $c > S_u$. Hence it suffices to prove that $m_2 - S_u\theta_{\mu,\eta,m_1} \neq 0$. Assume for contradiction that $m_2 - S_u\theta_{\mu,\eta,m_1} \equiv 0$ for some $\mu > 0$, $\eta \in [0, 1]$. Then the definition of $S_u$ yields that $\theta_{\mu,\eta,m_1} = S_u \geq \sup_\Omega \frac{m_2}{\theta_{\mu,\eta,m_1}} \geq \frac{m_2}{\theta_{\mu,\eta,m_1}}$, which implies that $\theta_{\mu,\eta,m_1} \leq (\neq) \theta_{\mu,\eta,m_1}$ on $\Omega$ for all $(\mu, \eta) \neq (\bar{\mu}, \bar{\eta})$. However, this is impossible since $\int_\Omega \theta_{\mu,\eta,m_1}e^{\bar{m}_1}dx$ cannot attain its global minimum at some $(\bar{\mu}, \bar{\eta})$ by Lemma 1.2. This finishes the proof of (2.12).

Hence
\[
c \geq S_u \Rightarrow I_c = I^0_c \Rightarrow m_2 - c\theta_{\mu,\eta,m_1} \leq \neq 0 \Rightarrow \sigma_1(\nu, \xi, m_2 - c\theta_{\mu,\eta,m_1}) < 0, \tag{2.13}
\]
i.e.
\[
\Sigma_u = \{(\mu, \eta, \nu, \xi) : \nu > 0, \xi > \Lambda(\mu)\}. \tag{2.14}
\]
Now assume that $L_u < c < S_u$ and $(\mu, \eta) \in I_c$. If $(\mu, \eta) \in I^0_c$, then $\sigma_1(\nu, \xi, m_2 - c\theta_{\mu,\eta,m_1}) < 0$ by (1.15). If $(\mu, \eta) \in I^1_c$, then
\[
\sigma_1(\nu, \xi, m_2 - c\theta_{\mu,\eta,m_1}) < 0 \iff \nu > \frac{1}{\lambda_1(\xi, m_2 - c\theta_{\mu,\eta,m_1})} > 0 \tag{2.15}
\]
by Lemma 1.3 and 1.4. Hence after defining
\[
\nu^* = \begin{cases} 
0 & (\mu, \eta) \in I^0_c, \\
\frac{1}{\lambda_1(\xi, m_2 - c\theta_{\mu,\eta,m_1})} & (\mu, \eta) \in I^1_c,
\end{cases} \tag{2.16}
\]
we obtain that $(\mu, \eta, \nu, \xi) \in \Sigma_u$ if and only if $(\mu, \eta) \in I_c$ and $\nu > \nu^*$. This finishes the proof of Theorem 1.6.

**Proof of Theorem 1.7.** Step 1. $(\theta_{\mu,\eta,m}, 0)$ is locally asymptotically stable.

We linearize the corresponding elliptic system of (1.3) at $(\theta_{\mu,\eta,m}, 0)$:
\[
\begin{cases} 
\nu \nabla \cdot (\nabla \psi - \xi \psi \nabla m) + (m - \theta_{\mu,\eta,m})\psi = \sigma \psi & \text{in } \Omega, \\
\partial \psi = -\xi \psi \frac{\partial m}{\partial n} = 0 & \text{on } \partial \Omega,
\end{cases} \tag{2.17}
\]
i.e.
\[
\begin{cases} 
\nabla \cdot (\nu e^{\xi m} \nabla \psi) + (m - \theta_{\mu,\eta,m})e^{\xi m} \psi = \sigma e^{\xi m} \psi & \text{in } \Omega, \\
\frac{\partial \psi}{\partial n} = 0 & \text{on } \partial \Omega,
\end{cases} \tag{2.18}
\]
where \( \tilde{\psi} = \psi e^{-\xi m} \). Since the condition (i) or (ii) holds, it then follows from Lemma 1.5 that
\[
\sigma_1(\nu, \xi, m - \theta_{\mu, \eta, m}) < \sigma_1(\mu, \eta, m - \theta_{\mu, \eta, m}) = 0, \tag{2.19}
\]
and \((\theta_{\mu, \eta, m}, 0)\) is locally asymptotically stable.

**Step 2.** \((0, \theta_{\nu, \xi, m})\) is unstable.

Similarly, we linearize the corresponding elliptic system of (1.3) at \((0, \theta_{\nu, \xi, m})\):
\[
\begin{aligned}
\mu \nabla \cdot (\nabla \varphi - \eta \varphi \nabla m) + (m - \theta_{\nu, \xi, m}) \varphi &= \varphi \quad \text{in } \Omega, \\
\frac{\partial \varphi}{\partial n} - \eta m \frac{\partial m}{\partial n} &= 0 \quad \text{on } \partial \Omega,
\end{aligned} \tag{2.20}
\]
i.e.
\[
\begin{aligned}
\nabla \cdot (\mu e^{\eta m} \nabla \phi) + (m - \theta_{\nu, \xi, m}) e^{\eta m} \phi &= \sigma e^{\eta m} \phi \quad \text{in } \Omega, \\
\frac{\partial \phi}{\partial n} &= 0 \quad \text{on } \partial \Omega,
\end{aligned} \tag{2.21}
\]
where \( \tilde{\phi} = \phi e^{-\eta m} \). Also, the condition (i) or (ii), and Lemma 1.5 indicate that
\[
\sigma_1(\mu, \eta, m - \theta_{\nu, \xi, m}) > \sigma_1(\nu, \xi, m - \theta_{\nu, \xi, m}) = 0. \tag{2.22}
\]
Thus \((0, \theta_{\nu, \xi, m})\) is unstable.

**Step 3.** (1.3) does not have any coexistence steady state.

Suppose \((U, V)\) is a coexistence steady state of (1.3), where \(U > 0, V > 0\) and
\[
\begin{aligned}
\mu \nabla \cdot (\nabla U - \eta \nabla (U \nabla m)) + U(m - U - V) &= 0 \quad \text{in } \Omega, \\
\nu \nabla \cdot (\nabla V - \xi \nabla (V \nabla m)) + V(m - U - V) &= 0 \quad \text{in } \Omega, \\
\frac{\partial U}{\partial n} - \eta U \frac{\partial m}{\partial n} &= \frac{\partial V}{\partial n} - \xi V \frac{\partial m}{\partial n} = 0 \quad \text{on } \partial \Omega.
\end{aligned} \tag{2.23}
\]
Setting \((\tilde{U}, \tilde{V}) = (U e^{-\eta m}, V e^{-\xi m})\), we arrive at
\[
\begin{aligned}
\mu \Delta \tilde{U} + \alpha \nabla m \nabla \tilde{U} + \tilde{U}(m - U - V) &= 0 \quad \text{in } \Omega, \\
\nu \Delta \tilde{V} + \beta \nabla m \nabla \tilde{V} + \tilde{V}(m - U - V) &= 0 \quad \text{in } \Omega, \\
\frac{\partial \tilde{U}}{\partial n} &= \frac{\partial \tilde{V}}{\partial n} = 0 \quad \text{on } \partial \Omega.
\end{aligned} \tag{2.24}
\]
Then, if \(U + V \neq m(x)\), we have that
\[
\sigma_1(\mu, \eta, m - U - V) = 0 = \sigma_1(\nu, \xi, m - U - V), \tag{2.25}
\]
a contradiction to Lemma 1.5. If \(U + V \equiv m(x)\), then we have
\[
\mu \Delta \tilde{U} + \alpha \nabla m \nabla \tilde{U} \equiv 0 \equiv \nu \Delta \tilde{V} + \beta \nabla m \nabla \tilde{V}, \tag{2.26}
\]
which implies that \(\tilde{U}\) is a constant, so is \(\tilde{V}\). Consequently, we reach that \(U \equiv c_1 e^{\eta m}, V \equiv c_2 e^{\xi m}\) and therefore \(m(x) \equiv c_1 e^{\eta m} + c_2 e^{\xi m}\), where \(c_1, c_2\) are two positive constants. If \(\eta = \xi\), then \(m(x) \equiv (c_1 + c_2)e^{\xi m}\). That is, \(mc^{-\xi m} \equiv \text{constant}\), and then \(m\) is a constant, a contradiction. If \(\beta = \alpha \leq \mu < \nu\), then \(\eta > \xi\). So,
\[
(c_1 + c_2)e^{\xi m} \leq c_1 e^{\eta m} + c_2 e^{\xi m} = m(x) \leq (c_1 + c_2)e^{\eta m}. \tag{2.27}
\]
That is,
\[
0 < c_1 + c_2 \leq me^{-\xi m}, \tag{2.28}
\]
again a contradiction to the condition (ii) of Theorem 1.7.

**Step 4.** Finally, the theory of monotone flow [12, 13] guarantees that there is a connecting orbit from \((\theta_{\mu, \eta, m}, 0)\) to \((0, \theta_{\nu, \xi, m})\). Moreover, \((\theta_{\mu, \eta, m}, 0)\) is globally asymptotically stable, and we complete the proof of Theorem 1.7. \(\square\)
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