The \( k \)-Compound of a Difference-Algebraic System

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Abstract

The multiplicative and additive compounds of a matrix have important applications in geometry, linear algebra, and the analysis of dynamical systems. In particular, the \( k \)-compounds allow to build a \( k \)-compound dynamical system that tracks the evolution of \( k \)-dimensional parallelotopes along the original dynamics. This has recently found many applications in the analysis of non-linear systems described by ODEs and difference equations. Here, we introduce the \( k \)-compound system corresponding to a differential-algebraic system, and describe several applications to the analysis of discrete-time dynamical systems described by difference-algebraic equations.

Key words: Multiplicative compounds, Drazin inverse, evolution of volumes, wedge product.

1 Introduction

There is a growing interest in the applications of compound matrices to systems and control theory (see, e.g. [Margaliot and Sontag 2019], [Bar-Shalom et al. 2023], [Katz et al. 2020], [Alseidi et al. 2019], [Ben-Avraham et al. 2020], [Wu et al. 2022], [Alseidi et al. 2021], [Grussler et al. 2022], [Grussler and Sepulchre 2020], [Grussler et al. 2022], [Grussler and Sepulchre 2022]). In particular, \( k \)-compound matrices have been used to generalize important classes of dynamical systems leading to linear \( k \)-positive and non-linear \( k \)-cooperative systems [Weiss and Margaliot 2021], \( k \)-contracting systems [Murdock 1990], [Wu et al. 2022], [Miron et al. 2022], and \( k \)-diagonally stable systems [Wu and Margaliot 2022].

Here, we introduce the \( k \)-compound system of the difference-algebraic equation (DAE):

\[
Bx(j + 1) = Ax(j), \quad j = 0, 1, \ldots
\]

and also the \( k \)-compound of the corresponding matrix pencil.

Given matrices \( A, B \in \mathbb{C}^{n \times n} \), the corresponding matrix pencil is the matrix polynomial

\[
(A, B) := A - \lambda B, \quad \lambda \in \mathbb{C}.
\]

Matrix pencils have numerous applications in linear algebra and in systems and control theory (see e.g., the monographs [Kunkel and Mehrmann 2006], [Milano et al. 2021], [Benner et al. 2015]). In particular, matrix pencils and their generalized eigenvalues and eigenvectors play an important role in the analysis of DAEs (see [Kunkel and Mehrmann 2006], [Ben-Israel and Greville 2003]).

We show that the \( k \)-compound system of a DAE describes the evolution of \( k \)-parallelotopes under the DAE. We analyze properties such as consistency of initial conditions, tractability, and stability of the \( k \)-compound system and relate them to the analogue properties in the original DAE. We also present a result that relates the stability of the \( k \)-compound system to the existence of a stable subspace of the original DAE. The dimension of this stable subspace decreases as \( k \) increases.

Several papers considered matrix pencils and used matrix compounds in their analysis [Karcianis and Mitrouli 1994], [Mitrouli et al. 1996], [Kalogeropoulos et al. 1989], [Iwata 2003]. This is closely related to the Smith form of a matrix polynomial (see, e.g., [Gohberg et al. 2009]), and also to the GCD of a given set of polynomials (see, e.g., [Karcianis 1989]). However, to the best of our knowledge the \( k \)-compound system of a DAE that we introduce here, and its applications, are novel.

The remainder of this note is organized as follows. The next section briefly reviews some known results that are used later on. Section 3 introduces the \( k \)-multiplicative compound system associated with a DAE. This is also a DAE, and its analysis, described in Section 5, is based on the so called \( k \)-multiplicative compound matrix pencil introduced in Sec-
tion 4. Section 6 shows an application of these theoretical results by linking the stability of the \( k \)-multiplicative compound system with that of the original DAE. Our results use the Drazin inverse of the \( k \)-multiplicative compound of a matrix. For the sake of completeness, it is shown in the Appendix that this is equal to the \( k \)-multiplicative of the Drazin inverse of the original matrix.

We use standard notation. Vectors [matrices] are denoted by small [capital] letters. A square matrix \( A \) is called regular [singular] if \( \det(A) \neq 0 [\det(A) = 0] \). The complex conjugate transpose of \( A \) is denoted by \( A^* \). If \( A \) is real then this is just the transpose of \( A \) denoted \( A^{T} \). Given an integer \( n \geq 1 \) and \( k \in \{1, \ldots, n\} \), let \( Q(k, n) \) denote the list of all \( k \)-tuples: \( \alpha_1 < \cdots < \alpha_k \), with \( \alpha_i \in \{1, \ldots, n\} \), ordered lexicographically. For example,

\[
Q(3, 4) = \{(1, 2, 3), (1, 2, 4), (1, 3, 4), (2, 3, 4)\}. \tag{3}
\]

We refer to the \( i \)-th element of \( Q(k, n) \) as \( \alpha^i \) and use subscripts to refer to an entry, e.g., if \( k = 3 \) and \( n = 4 \), then \( \alpha_1^4 = 2 \). Given \( \alpha, \beta \in Q(k, n) \), let \( A[\alpha|\beta] \) denote the submatrix of \( A \) obtained by taking the rows [columns] with indices in \( \alpha [\beta] \). For example, \( A[(1, 2)][(1, 3)] = \begin{pmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{pmatrix} \).

Let \( A[\alpha|\beta] := \det(A[\alpha|\beta]) \) denote the corresponding minor. For square matrices \( A_i \in \mathbb{C}^{n_i \times n_i} \), \( i = 1, \ldots, \ell \), let \( \text{diag}_{i \in \{1, \ldots, \ell\}}(A_i) \) denote the \( (\sum_{i=1}^{\ell} n_i) \times (\sum_{i=1}^{\ell} n_i) \) diagonal block matrix with blocks \( A_1, \ldots, A_\ell \). The \( n \times n \) identity matrix is denoted by \( I_n \).

2 Preliminaries

To make this paper more self-contained, we first review the multiplicative compound of a matrix (for more details see, e.g., [Muldowney 1990], [Fiedler 2008]), and provide a very brief overview of matrix pencils and their applications in DAEs.

2.1 Multiplicative compound of a matrix

Let \( A \in \mathbb{C}^{n \times m} \), and fix \( k \in \{1, \ldots, \min\{n, m\}\} \). A \( k \)-minor of \( A \) is the determinant of a \( k \times k \) submatrix of \( A \). The \( k \)-

\[ A^{(k)} = \begin{bmatrix}
A((1,2)[(1,2)] & A((1,2)[(1,3)] & A((1,2)[(2,3)]) \\
A((1,3)[(1,2)] & A((1,3)[(1,3)] & A((1,3)[(2,3)]) \\
A((2,3)[(1,2)] & A((2,3)[(1,3)] & A((2,3)[(2,3)])
\end{bmatrix}. \]

In particular, \( A^{(1)} = A \), and if \( m = n \) then \( A^{(n)} = \det(A) \).

The Cauchy-Binet theorem asserts that for any \( A \in \mathbb{C}^{n \times m} \), \( B \in \mathbb{C}^{m \times \ell} \), and \( k \in \{1, \ldots, \min\{n, m, \ell\}\} \), we have

\[
(AB)^{(k)} = A^{(k)}B^{(k)}. \tag{4}
\]

This justifies the term multiplicative compound. When \( k = n = m = \ell \), \( (4) \) reduces to \( \det(AB) = \det(A)\det(B) \).

The definition of the multiplicative compound implies that \( (A^*)^{(k)} = (A^{(k)})^* \), and \( I_n^{(k)} = I_r \), with \( r := \binom{n}{k} \). If \( A \) is square and regular then \( (4) \) gives

\[
I_n^{(k)} = (A^{-1})^{(k)} = (AA^{-1})^{(k)} = (A^{-1})^{(k)}(A)^{(k)} = (A^{-1})^{(k)}(A^{(k)}). \tag{5}
\]

so \( A^{(k)} \) is also regular and its inverse is \( (A^{-1})^{(k)} \). A similar argument shows that if \( U \in \mathbb{C}^{n \times n} \) is unitary, that is, \( U^*U = UU^* = I_n \), then \( (U^{(k)})^*U^{(k)} = U^{(k)}(U^{(k)})^* = I_r \), so \( U^{(k)} \) is also unitary.

If \( A \) is upper triangular (lower triangular, diagonal) then \( A^{(k)} \) is upper triangular (lower triangular, diagonal), and the diagonal entries of \( A^{(k)} \) are

\[
A^{(k)}_{i,i} = \prod_{j=1}^{k} a^{i,j}_i. \tag{6}
\]

where \( \alpha^i \) is the \( i \)-th sequence in \( Q(k, n) \). For example, for \( n = 4 \) and \( k = 3 \), \( \alpha^2 = (1, 2, 4) \), so \( (5) \) becomes \( (A^{(3)})_{2,2} = a_{11}a_{22}a_{44} \).

If \( \lambda_i, i = 1, \ldots, n \), are the eigenvalues of \( A \in \mathbb{C}^{n \times n} \), then the eigenvalues of \( A^{(k)} \) are \( \binom{n}{k} \)-products: \( \prod_{i=1}^{k} \lambda_i \), \( \alpha \in Q(k, n) \). For \( k = n \) this reduces to \( \det(A) = \prod_{i=1}^{n} \lambda_i \).

If \( A \) is rectangular, the singular values of \( A \) and of \( A^{(k)} \) satisfy the same multiplicative property as the eigenvalues. Then using the singular value decomposition and the Cauchy-Binet theorem yields

\[
\text{rank}(A^{(k)}) = \binom{\overline{\ell}}{k}, \tag{6}
\]

where \( \ell := \text{rank}(A) \), and \( \binom{\overline{\ell}}{k} \) is defined as zero when \( k > \ell \).

It follows from \( (6) \) that \( A^{(k)} = 0 \) if and only if \( (\text{iff}) \) \( k > \text{rank}(A) \), and that \( A^{(k)} \) has non-full rank if \( A \) has non-full rank.

One reason for the usefulness of the \( k \)-compounds in systems and control theory is that they have an important geometric application.

2.1.1 Geometric interpretation of the multiplicative compound

Fix \( k \) vectors \( x^1, \ldots, x^k \in \mathbb{R}^n \), and let \( P(x^1, \ldots, x^k) := \{ \sum_{i=1}^{k} s_i x^i | s_i \in [0, 1] \} \) denote the parallelotope with vertices \( x^1, \ldots, x^k \) and 0 (see Fig. 1). Let \( \text{vol}(P) \) denote the volume of \( P \). Define \( \text{vol}(X) \) for any \( n \times k \) matrix \( X := [x^1 \ldots x^k] \), and the \( k \) \( k \)-non-negative definite matrix \( G(x^1, \ldots, x^k) := X^T X \). Then \( \text{vol}(P) = \sqrt{\det(G)} \) [Gantmacher 1960, Ch. IX]. To express this using the multiplicative compound, note that \( \det(G) = G^{(k)} \). so
vol(P) = \sqrt{\text{vol}(X^T X)^{\text{min}}}, \quad \text{vol}(P) = |X(k)|_2, \quad \text{where} \quad |\cdot|_2 \text{ denotes the } L_2 \text{ norm. In the special case } k = n, \text{ this reduces to the well-known formula}

\text{vol}(P(x^1, \ldots, x^n)) = |\det\left[ x^1 \ldots x^n \right] |.

2.2 Matrix pencils

Given \( A, B \in \mathbb{C}^{n \times n} \), the associated matrix pencil is the matrix polynomial (2). The matrix pencil is called regular if there exists a \( \lambda \in \mathbb{C} \) such that \( \det(A - \lambda B) \neq 0 \). Otherwise, it is called singular. The normal rank of \( (A, B) \) is

\[ \text{nrank}(A, B) := \max_{\lambda \in \mathbb{C}} \text{rank}(A - \lambda B). \]

Any \( \lambda_0 \in \mathbb{C} \) for which

\[ \text{rank}(A - \lambda_0 B) < \text{nrank}(A, B) \tag{8} \]

is called a finite (generalized) eigenvalue of \( (A, B) \). For any such \( \lambda_0 \) there exists a vector \( v \in \mathbb{C}^n \setminus \{0\} \) such that

\[ Av = \lambda_0 Bv. \tag{9} \]

If \( (A, B) \) is regular then (9) implies that \( \lambda_0 \) is a finite eigenvalue of \( (A, B) \). If \( \det(B) = 0 \) then \( (A, B) \) also has an eigenvalue at infinity, which corresponds to the zero eigenvalue of the matrix pencil \( (B, A) \).

Any \( A, B \in \mathbb{C}^{n \times n} \) may be jointly triangularized using the generalized Schur decomposition (GSD) \[\text{ [Colub and Van Loan, 2013, Thm. 7.7.1] }\], that is, there exist unitary matrices \( U, V \in \mathbb{C}^{n \times n} \) such that

\[ UAV = T, \quad UBV = S, \tag{10} \]

where \( T \) and \( S \) are upper triangular. The GSD is particularly useful when studying the spectrum of a matrix pencil, as

\[ \det(A - \lambda B) = \det(U) \det(T - \lambda S) \det(V) = \det(U) \prod_{i=1}^n (T_{ii} - \lambda S_{ii}). \]

Thus, \( (A, B) \) is singular iff there exists \( i \in \{1, \ldots, n\} \) such that \( T_{ii} = S_{ii} = 0 \). If \( (A, B) \) is regular, then its eigenvalues may be read from the diagonal entries of \( T \) and \( S \) for every \( i \in \{1, \ldots, n\} \) such that \( S_{ii} \neq 0 \), \( T_{ii}/S_{ii} \) is a finite eigenvalue of \( (A, B) \), and \( (A, B) \) has an eigenvalue at infinity iff there exists \( i \in \{1, \ldots, n\} \) such that \( S_{ii} = 0 \).

2.3 Difference-algebraic equations

Consider the DAE (1) with \( x : \{0, 1, \ldots\} \to \mathbb{R}^n \), and \( B, A \in \mathbb{R}^{n \times n} \). If \( B \) is regular then this is equivalent to the discrete-time LTI system \( x(j + 1) = B^{-1} A x(j) \), but we will assume that \( B \) is singular. Then (1) may be interpreted as a discrete-time dynamical system with algebraic constraints.

An initial condition \( x(0) \) is called consistent if (1) admits a corresponding solution \( x(j) \) for all \( j \geq 0 \). For example, \( x(0) = 0 \) is always consistent. The system (1) is called tractable (some authors use instead the term solvable) if for any consistent initial condition \( x(0) \) the system (1) admits a unique solution \( x(j), j = 0, 1, \ldots \).

The next two results relate the system-theoretic properties of (1) to the matrix pencil \( (A, B) \). To state them, we recall the notions of the Drazin index and the Drazin inverse.

**Definition 1.** [Drazin, 1958] The Drazin index of a square matrix \( A \), denoted \( \text{index}(A) \), is the minimal integer \( k \geq 0 \) such that \( \text{rank}(A^k) = \text{rank}(A^{k+1}) \).

For example, if \( A \) is regular then \( \text{rank}(A^0) = \text{rank}(A^1) \), so \( \text{index}(A) = 0 \). If \( N \) is nilpotent, i.e. there exists a minimal integer \( k \) such that \( N^k = 0 \), then \( \text{index}(N) = k \).

**Definition 2.** [Drazin, 1958] Let \( A \) be a square matrix. The Drazin inverse of \( A \) is a matrix \( X \) such that

\[ (1) \quad A^{\text{index}(A)+1} X = A^{\text{index}(A)}, \]

\[ (2) \quad AX = XA, \]

\[ (3) \quad XAX = X. \]

It is known that the Drazin inverse, denoted \( A^D \), always exists and is unique. If \( A \) is regular then \( A^D = A^{-1} \), and if \( N \) is nilpotent then \( N^D = 0 \). If the Jordan decomposition of \( A \) is

\[ A = T^{-1} \begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix} T, \tag{11} \]

with \( C \) regular and \( N \) nilpotent, then \( A^D = T^{-1} \begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix} T \).

**Proposition 1.** [Ben-Israel and Greville, 2003] The DAE (1) is tractable iff there exists \( \lambda \in \mathbb{C} \) such that \( \det(A - \lambda B) \neq 0 \), that is, iff \( (A, B) \) is regular.

**Proposition 2.** [Ben-Israel and Greville, 2003, Belov et al., 2018] Assume that (1) is tractable. Fix \( \lambda \in \mathbb{C} \) such that \( \det(A - \lambda B) \neq 0 \), and let

\[ \tilde{B}_\lambda := (A - \lambda B)^{-1} B, \quad \hat{A}_\lambda := (A - \lambda B)^{-1} A. \tag{12} \]

Let \( i := \text{index}(\tilde{B}_\lambda) \). An initial condition \( x(0) \) is consistent iff \( x(0) \) is in the range of \( (\tilde{B}_\lambda)^i \), and for such an initial
condition the unique solution of (1) is

\[ x(j) = \left( (\hat{B}_\lambda)^D \hat{A}_\lambda \right)^j x(0), \quad j = 0, 1, \ldots \quad (13) \]

Furthermore, \( \lim_{t \to \infty} x(j) = 0 \) for any consistent initial condition \( x(0) \) iff all the finite eigenvalues of \((A, B)\) lie in the open unit disk.

The next sections describe our main results.

3 The \( k \)-multiplicative compound DAE

We begin by generalizing a DAE to a corresponding \( k \)-compound DAE. We consider the general case of a time-varying DAE:

\[ B(j + 1)x(j + 1) = A(j)x(j), \quad j = 0, 1, \ldots \quad (14) \]

with \( x \in \mathbb{R}^n \), and \( A, B \in \mathbb{R}^{n \times n} \).

We introduce a new definition.

**Definition 3.** Fix \( k \in \{1, \ldots, n\} \), and let \( r := \binom{n}{k} \). The \( k \)-multiplicative compound DAE corresponding to (14) is

\[ B(k)(j + 1)y(j + 1) = A(k)(j)y(j), \quad j = 0, 1, \ldots \quad (15) \]

with \( y \in \mathbb{R}^r \).

Note that for \( k = 1 \), Eq. (15) is the original DAE (14), whereas for \( k = n \), Eq. (15) becomes the scalar equation: \( \det(B(j + 1))y(j + 1) = \det(A(j))y(j) \).

The next result shows that the \( k \)-compound DAE tracks the evolution of volumes of \( k \)-parallelopetes under the DAE (14).

**Proposition 3.** Fix \( k \in \{1, \ldots, n\} \), and pick \( k \) consistent initial conditions \( a^1, \ldots, a^k \in \mathbb{R}^n \) of (14). Let \( x^i(\ell) := x(\ell, a^i) \) denote a solution at time \( \ell \) of (14) emanating from \( x(0) = a^i \). Define the \( n \times k \) matrix

\[ X(j) := \begin{bmatrix} x^1(j) & \ldots & x^k(j) \end{bmatrix}, \]

and the \( \binom{n}{k} \)-dimensional column vector

\[ y(j) := (X(j))^{(k)}. \quad (16) \]

Then \( y \) is a solution of the \( k \)-compound DAE (15).

The proof is straightforward. By (14), \( B(j + 1)X(j + 1) = A(j)X(j) \). Taking the \( k \)th multiplicative compound on both sides, and using the Cauchy-Binet Theorem completes the proof.

4 The \( k \)-multiplicative compound matrix pencil

In the time-invariant case, the \( k \)-compound DAE is

\[ B(k)y(j + 1) = A(k)y(j). \quad (17) \]

Our next goal is to extend known analysis results for time-invariant DAEs to (17). Since the known analysis results are closely related to the pencil \((A, B)\), we begin by introducing the \( k \)-pencil associated to the \( k \)-multiplicative compound DAE.

**Definition 4.** Given \( A, B \in \mathbb{C}^{n \times n} \) and \( k \in \{1, \ldots, n\} \), the \( k \)-multiplicative compound pencil of \((A, B)\) is the matrix pencil

\[ (A, B)^{(k)} := A^{(k)} - \lambda B^{(k)}, \quad \lambda \in \mathbb{C}. \quad (18) \]

Note that \((A, B)^{(1)}\) is just \((A, B)\), and \((A, B)^{(n)} = \det(A) - \lambda \det(B)\). Also, \((A, 0)^{(k)}\) is just \( A^{(k)}\).

**Remark 1.** Several authors associate with \((A, B)\) the matrix polynomial

\[ (A - \lambda B)^{(k)}, \quad \lambda \in \mathbb{C}. \quad (19) \]

This algebraic construction is particularly useful in studying the Smith normal form of \((A, B)\). It should be noted that (18) and (19) are quite different. For example, let \( n = 2 \), \( B = I_2 \), and \( A = \text{diag}(\mu_1, \mu_2) \). Then

\[ (A - \lambda B)^{(2)} = (\mu_1 - \lambda)(\mu_2 - \lambda), \]

whereas (18) gives

\[ (A, B)^{(2)} = A^{(2)} - \lambda B^{(2)} = \mu_1 \mu_2 - \lambda. \]

In particular, here the generalized eigenvalues of \((A - \lambda B)^{(2)}\) are simply the generalized eigenvalues of \((A - \lambda B)\), whereas \((A, B)^{(2)}\) admits a single generalized eigenvalue \( \mu_1 \mu_2 \). In this sense, neither \((A, B)^{(k)}\) nor \((A - \lambda B)^{(k)}\) is a generalization or a special case of the other.

Propositions 1 and 2 imply that the matrix pencil \((A, B)\) determines important system-theoretic properties of (14), and thus the matrix pencil \((A, B)^{(k)}\) determines the same system-theoretic properties for (17). In particular, it follows from Prop. 1 that the \( k \)-compound system (17) is tractable iff the matrix pencil \((A, B)^{(k)}\) is regular.

4.1 Regularity of \((A, B)^{(k)}\)

The next result provides a necessary and sufficient condition for regularity of the pencil \((A, B)^{(k)}\), with \( k \geq 2 \).

**Proposition 4.** Let \( A, B \in \mathbb{C}^{n \times n} \). Fix \( k \in \{2, \ldots, n\} \). The following four conditions are equivalent:

1. The pencil \((A, B)^{(k)}\) is singular.
2. For any GSD (10) there exists \( \alpha \in Q(k, n) \) such that \( \prod_{i=1}^k T_{\alpha_i, \alpha_i} = \prod_{i=1}^k S_{\alpha_i, \alpha_i} = 0 \).
3. \( \det(A) = \det(B) = 0 \).
4. \( \ker(A^{(k)}) \cap \ker(B^{(k)}) \neq \{0\} \).

**Proof.** We begin by showing that (1) and (2) are equivalent. Applying the Cauchy-Binet Theorem to (10) gives

\[ U^{(k)} A^{(k)} V^{(k)} = T^{(k)}, \quad U^{(k)} B^{(k)} V^{(k)} = S^{(k)}. \quad (20) \]
Furthermore, either \( \lambda \) and this completes the proof of Prop. 4.

Suppose that \( (A, B)^{(k)} \) is singular. Then at least one diagonal entry of \( T \) and at least one diagonal entry of \( S \) are zero, so

\[
det(A) = det(U) det(V) det(T) = 0, \\
det(B) = det(U) det(V) det(S) = 0.
\]

This proves that (1) implies (3). We now show that (3) implies (4) for \( k = 2 \). Assume that (3) holds. Then there exist \( x, y \in \mathbb{C}^n \setminus \{0\} \) such that \( Ax = 0, By = 0 \). We consider two cases. If \( x, y \) are linearly dependent then \( z = sy \) for some scalar \( s \neq 0 \). Pick \( z \in \mathbb{R}^n \) such that \( x, z \) are linearly independent. Then

\[
A^{(2)} [\begin{array}{c} x \\ z \end{array}]^{(2)} = [Ax \ Axz]^{(2)} = [0 \ Axz]^{(2)} = 0,
\]

and similarly \( B^{(2)} [\begin{array}{c} x \\ z \end{array}]^{(2)} = B^{(2)} [sy \ z]^{(2)} = 0 \), so (4) holds. Now assume that \( x, y \) are linearly independent. Let \( z := [x \ y]^{(2)} \). Then

\[
A^{(2)} z = [Ax \ Ay]^{(2)} = [0 \ Ay]^{(2)} = 0,
\]

and similarly \( B^{(2)} z = 0 \), so \( z \in \ker(A^{(2)}) \cap \ker(B^{(2)}) \). Since \( x, y \) are linearly independent, \( z \neq 0 \). A similar argument shows that for any \( j \in \{2, \ldots, n\} \), we have \( \ker(A^{(j)}) \cap \ker(B^{(j)}) \neq \{0\} \), so (3) implies (4).

Now suppose that (4) holds. Let \( x \neq 0 \) be a vector such that \( x \in \ker(A^{(k)}) \cap \ker(B^{(k)}) \). Then \( (A^{(k)} - \lambda B^{(k)}) z = 0 \) for any \( \lambda \in \mathbb{C} \), so (1) holds. We conclude that (4) implies (1), and this completes the proof of Prop. 4.

Remark 2. Prop. 4 implies in particular that if \( (A, B)^{(k)} \) is singular then \( (A, B)^{(k)} \) is singular for any \( k \in \{1, \ldots, n\} \). Furthermore, either \( (A, B)^{(k)} \) is regular for all \( k > 1 \), or it is singular for all \( k > 1 \).

Prop. 4 demonstrates a perhaps surprising property of the \( k \)-multiplicative compound of a pencil. A sufficient, but not necessary, condition for a pencil \( (A, B) \) to be singular is that \( \ker(A) \cap \ker(B) \neq \{0\} \). However, this condition is both sufficient and necessary for the singularity of for \( (A, B)^{(k)} \), with \( k > 1 \).

The next example illustrates Prop. 4.

Example 1. Suppose that \( A = \text{diag}(0,1,2) \) and \( B = \text{diag}(1,2,0) \). Note that \( \det(A) = \det(B) = 0 \). Then \( \det(A - \lambda B) = 2(2\lambda - 1) \), so \((A, B)\) is regular. Also, \( A^{(2)} = \text{diag}(0,0,2) \) and \( B^{(2)} = \text{diag}(2,0,0) \), so \( \det(A^{(2)} - \lambda B^{(2)}) = 0 \) for any \( \lambda \in \mathbb{C} \) and thus \((A, B)^{(2)} \) is singular. Consider the GSD in (10). Then there exists exactly one \( i \in \{1,2,3\} \) such that \( T_{ii} = 0 \), and exactly one \( j \in \{1,2,3\} \) such that \( S_{ij} = 0 \). Also, \( i \neq j \), as otherwise \((T, S)\) is singular and this is impossible as \((A, B)\) is regular. Let \( \alpha \) be the sequence in \((2,3)\) that includes \( i \) and \( j \). Then \( \prod_{r=1}^{2} T_{rr,\alpha r} = \prod_{i=1}^{2} S_{ir,\alpha r} = 0 \). Note also that \( [0 \ 1 \ 0]^T \in \ker(A^{(2)}) \cap \ker(B^{(2)}) \).

4.2 Spectral properties of \((A, B)^{(k)}\)

Recall that if \( A \in \mathbb{C}^{n \times n} \) and \( k \in \{1, \ldots, n\} \) then any eigenvalue of \( A^{(k)} \) is the product of \( k \) eigenvalues of \( A \). The next result demonstrates that the matrix pencil \((A, B)^{(k)}\) satisfies a similar property.

Proposition 5. Let \( A, B \in \mathbb{C}^{n \times n} \). Fix \( k \in \{1, \ldots, n\} \). Suppose that \((A, B)^{(1)}\) is regular. Then every eigenvalue of \((A, B)^{(k)}\) is the product of \( k \) eigenvalues of \((A, B)\), where we define the product of infinity with any value as infinity.

Proof. Let \( r := \binom{k}{1} \). By Eq. (21), \( \det((A, B)^{(k)}) = \det((T, S)^{(k)}) = \prod_{i=1}^{r} (T^{(k)})_{ii} - \lambda(S^{(k)})_{ii} \), so the eigenvalues of \((A, B)^{(k)}\) are \((T^{(k)})_{ii}(S^{(k)})_{ii}, i \in \{1, \ldots, r\} \), where we define \( c/0 \) with \( c \neq 0 \), as infinity (note that the assumption that the pencil is regular guarantees that the case \((T^{(k)})_{ii} = (S^{(k)})_{ii} = 0 \) is not possible). In particular, the eigenvalues of \((A, B)\) are \( T_{ii}/S_{ii}, i \in \{1, \ldots, n\} \). Since \( T \) is upper triangular, Eq. (5) implies that entry \((i, i)\) of \( T^{(k)} \) is \( \prod_{r=1}^{n} T_{ir,i} \alpha^r \), where \( \alpha^r \) is the \( r \)-th sequence in \((Q, n)\), and similarly for \( S^{(k)} \). This completes the proof.

Given \( k \) finite eigenvalues and the corresponding \( k \) eigenvectors of \((A, B)\), the following result gives an explicit formula for the corresponding eigenvalue and eigenvector of \((A, B)^{(k)}\).

Proposition 6. Let \( A, B \in \mathbb{C}^{n \times n} \), and pick \( k \in \{1, \ldots, n\} \). Suppose that \( \lambda_1, \ldots, \lambda_k \in \mathbb{C} \) and \( v^1, \ldots, v^k \in \mathbb{C}^n \setminus \{0\} \) satisfy

\[
Av^i = \lambda_i Bv^i, \quad i = 1, \ldots, k.
\]

Define \( \tilde{v} := [v^1 \ldots v^k]^{(k)} \) and \( \tilde{\lambda} := \binom{k}{1} \lambda_i \). Then

\[
A^{(k)} \tilde{v} = \tilde{\lambda} B^{(k)} \tilde{v}.
\]

This implies in particular that if \( v^1, \ldots, v^k \) are linearly independent eigenvectors of \((A, B)\) with corresponding eigenvalues \( \lambda_1, \ldots, \lambda_k \), then \( \tilde{v} \in \mathbb{R}^n \setminus \{0\} \) is an eigenvector of \((A, B)^{(k)}\) corresponding to the eigenvalue \( \tilde{\lambda} \).
Proof. Applying the Cauchy-Binet Theorem yields

\[
A^{(k)} \left[ v^1 \ldots v^k \right]^{(k)} = \left[ Av^1 \ldots Av^k \right]^{(k)} = \left[ \lambda_1 Bv^1 \ldots \lambda_k Bv^k \right]^{(k)} = \hat{\lambda} \left[ Bv^1 \ldots Bv^k \right]^{(k)} = \lambda B^{(k)} \left[ v^1 \ldots v^k \right]^{(k)},
\]

and this completes the proof.

Remark 3. The multiplicative compound of a matrix pencil has a geometric interpretation similar to that of the multiplicative compound of a matrix. Let \( r := \binom{n}{k} \). The dimensions of \( \hat{v} \) are \( r \times \binom{n}{k} \), i.e., it is an \( r \)-dimensional column vector, and \( |\hat{v}|_2 \) is the volume of the parallelotope with vertices \( 0, v^1, \ldots, v^k \). Eq. (25) thus implies that the volume of the parallelotope generated by \( 0, Av^1, \ldots, Av^k \) is equal to \( \prod_{i=1}^{k} \lambda_i \) times the volume of the parallelotope generated by \( 0, Bv^1, \ldots, Bv^k \). Indeed, this follows from (23).

5 Analysis of the \( k \)-multiplicative compound DAE

It is clear that the algebraic properties of the pencil \((A,B)^{(k)}\) are closely related to the dynamical properties of (17). The following example demonstrates this relation from the point of view of the spectral properties studied in Prop. 5.

Example 2. Consider (1) with \( n = 3 \),

\[
A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.
\]

Here \( \det(A) \neq 0 \), so \( A \) is regular. As discussed in Subsection 5.1 below, this implies that the \( 2 \)-multiplicative compound system is tractable. Furthermore, the eigenvalues of \((A,B)\) are \( 1, -1, \infty \), so by Prop. 5 \((A,B)^{(2)}\) has eigenvalues \( 1, \infty, \infty \). Therefore, given any two consistent initial conditions of the DAE, the corresponding solution of the \( 2 \)-compound DAE is constant in time. It follows from Prop. 3 and the geometric properties of multiplicative compounds that the corresponding solution of the \( 2 \)-compound DAE equals the area of the parallelotope defined by the solutions of the DAE. We would therefore expect this parallelotope to have a constant area. This can be seen in Fig. 2, that depicts two trajectories of the DAE with initial conditions \( \begin{bmatrix} 1 & 1 \end{bmatrix}^T \) and \( \begin{bmatrix} 1.5 & 0.75 \end{bmatrix}^T \), and the triangles they define (with area which is half of the area of the corresponding parallelotopes). The corresponding solution of the \( 2 \)-compound DAE is \( y(j) = 0.75 \), and this agrees with the fact that the area of the triangles is constant in time.

5.1 Tractability and asymptotic stability of the \( k \)-multiplicative compound DAE

We begin by studying the tractability of the \( k \)-multiplicative compound DAE (17). Combining Prop. 1 and Prop. 4, we have that (17) is tractable if and only if the matrix pencil \((A,B)^{(k)}\) is regular, or equivalently, if and only if at least one of the matrices \( A, B \) is regular.

As a specific example, consider \( A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \), and \( B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \).

Then \( A^{(2)} = \det(A) = 0 \) and \( B^{(2)} = \det(B) = 0 \), so \((A,B)^{(2)} = A^{(2)} - \lambda B^{(2)} = 0\). Note that in this case \((A,B)\) is regular, and \((A,B)^{(2)}\) is singular.

Important dynamical properties of (17) follow from combining Prop. 2 and Prop. 5. Suppose that the \( k \)-multiplicative compound DAE is tractable. Then there exists \( \lambda \in \mathbb{C} \) such that \( \det(A^{(k)} - \lambda B^{(k)}) \neq 0 \). Let

\[
\hat{B}_{k,\lambda} := (A^{(k)} - \lambda B^{(k)})^{-1} B^{(k)}, \quad \hat{A}_{k,\lambda} := (A^{(k)} - \lambda B^{(k)})^{-1} A^{(k)},
\]

and let \( i := \text{index}(\hat{B}_{k,\lambda}) \). An initial condition \( y(0) \in \mathbb{R}^{\binom{n}{k}} \) is consistent iff \( y(0) \) is in the range of \((\hat{B}_{k,\lambda})^i\), and for such an initial condition the solution of (17) is

\[
y(j) = \left((\hat{B}_{k,\lambda})^D \hat{A}_{k,\lambda}\right)^j y(0), \quad j = 0, 1, \ldots
\]

Furthermore, if \((A,B)\) has \( s \geq k \) finite eigenvalues, denoted \( \lambda_i, \quad i = 1, \ldots, s \), then (17) is asymptotically stable iff \( \prod_{i=1}^{k} |\lambda_{\alpha,i}| < 1 \) for all \( \alpha \in Q(k,s) \).

5.2 Consistent and non-consistent initial conditions of the \( k \)-multiplicative compound DAE

Consider the LTI DAE (1). We already know that given \( k \) \( \in \{1, \ldots, n\} \) solutions to (1), the vector \( y(j) \) in (16) is a solution to the \( k \)-multiplicative compound DAE (17). However, Prop. 4 implies that under certain conditions a tractable DAE...
will induce a non-tractable \( k \)-multiplicative compound DAE for any \( k > 1 \).

On the other hand, when \( (A,B)^{(k)} \) is regular for all \( k \in \{1, \ldots, n\} \) it is possible that for large values of \( k \) the only consistent initial condition of the \( k \)-multiplicative compound system is zero. Indeed, for large \( k \), Eq. (1) may not have \( k \) linearly-independent consistent initial conditions. The following results analyze these issues. We begin with the case where \( (A,B)^{(\ell)} \) is regular for all \( \ell \in \{1, \ldots, n\} \).

**Proposition 7.** Suppose that \( (A,B)^{(\ell)} \) is regular for any \( \ell \geq 1 \). Fix \( k \in \{1, \ldots, n\} \). Let \( \mathcal{V}^{1} \) denote the subspace of consistent initial conditions of (1), and let \( \mathcal{V}^{k} \) denote the subspace of consistent initial conditions of the \( k \)-multiplicative compound DAE (17). Then

\[
\dim(\mathcal{V}^{k}) = \left( \dim(\mathcal{V}^{1}) \right)^{k}, \tag{27}
\]

where \( \left( \dim(\mathcal{V}^{1}) \right) \) is defined to be zero for \( k > \dim(\mathcal{V}^{1}) \).

Prop. 7 implies in particular that (17) will have zero as its only consistent initial condition for any \( k > \dim(\mathcal{V}^{1}) \).

**Proof.** Let \( s \) denote the number of finite eigenvalues of \( (A,B) \), counting multiplicities. Recall that \( \dim(\mathcal{V}^{1}) = s \). It follows from Prop. 5 that \( (A,B)^{(k)} \) has \( \binom{k}{s} \) finite eigenvalues, and this completes the proof of Prop. 7.

Note that \( \dim(\mathcal{V}^{1}) = \text{rank}\left( (\hat{B}_{k})^{\text{index}(\hat{B}_{k})} \right) \leq \text{rank}(B) \). Suppose that \( \text{rank}(B) < n \) and fix \( k > \text{rank}(B) \). Then \( \hat{B}_{k,\lambda} = 0 \), so \( \dim(\mathcal{V}^{k}) = 0 \). However, often \( \dim(\mathcal{V}^{1}) \) is strictly smaller than \( \text{rank}(B) \), and then there exists \( k \leq \text{rank}(B) \) such that \( \dim(\mathcal{V}^{k}) = 0 \) but \( \hat{B}_{k,\lambda} \neq 0 \). In this case \( \hat{B}_{k,\lambda} \) will be nilpotent. The next example illustrates this.

**Example 3.** Consider (17) with

\[
A = \begin{bmatrix} -2 & -3 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

Let \( \lambda = 1 \), and note that \( \det(A - B) \neq 0 \). It may be verified that \( \text{index}(\hat{B}_{1}) = 2 \) and \( \text{rank}(\hat{B}_{1}^{\text{index}(\hat{B}_{1})}) = 1 \), so \( \dim(\mathcal{V}^{1}) = 1 \), and Prop. 7 implies that \( \dim(\mathcal{V}^{k}) = 0 \) for \( k = 2, 3 \).

We now show directly that \( \dim(\mathcal{V}^{2}) = 0 \). The 2-compound system is

\[
B^{(2)}y(j+1) = A^{(2)}y(j). \tag{28}
\]

The matrix \( A^{(2)} - B^{(2)} \) is regular and multiplying (28) on the left by \( (A^{(2)} - B^{(2)})^{-1} \) gives \( \hat{B}_{2,1}y(j+1) = \hat{A}_{2,1}y(j) \), that is,

\[
\begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} y(j+1) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} y(j).
\]

The first equation here gives \( y_{1}(j) = 0 \). Using this in the second equation gives \( y_{2}(j) = 0 \), and now the third equation gives \( y_{3}(j) = 0 \), so indeed the only consistent initial condition is \( y(0) = 0 \). Note that the matrix \( \hat{B}_{2,1} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \) is nilpotent, as expected.

We now turn to consider the case where \( (A,B) \) is regular, but \( (A,B)^{(k)} \) is singular for any \( k > 1 \). The following result shows that in this case the \( k \)-compound system will have a consistent non-zero initial condition for any \( k \).

**Proposition 8.** Let \( A, B \in \mathbb{R}^{n \times n} \) be such that \( (A,B) \) is regular and \( \det(A) = \det(B) = 0 \). Fix \( k > 1 \). Then there exists a vector \( z \in \mathbb{R}^{(n)} \setminus \{0\} \) such that:

(1) \( z \) is a consistent initial condition for the \( k \)-multiplicative compound DAE (17);

(2) if \( y(j), j = 0, 1, \ldots \) is a solution of (17) then \( y(j) + z, j = 0, 1, \ldots \) is another solution of (17) for the initial condition \( y(0) \).

**Proof.** Since \( \det(A) = \det(B) = 0 \), Prop. 4 implies that there exists \( z \in \ker(A^{(k)}) \cap \ker(B^{(k)}) \) with \( z \neq 0 \). Consider the sequence \( y(j) \equiv z \) for all \( j \geq 0 \). Since \( B^{(k)}y(j+1) = A^{(k)}y(j) = 0 \), \( y(j) \) is a solution of the \( k \)-compound system and, in particular, the vector \( z \) is indeed a consistent initial condition of (17). This proves (1). The proof of (2) follows similarly, and this completes the proof of Prop. 8.

The singularity of \( (A,B)^{(k)} \) implies that the \( k \)-compound system may have consistent initial conditions and solutions that do not correspond to \( k \)-compounds of consistent initial conditions and solutions of the original system. The next example illustrates this.

**Example 4.** Consider (17) with \( A = \text{diag}(0,1/2,1) \) and \( B = \text{diag}(1,1,0) \), that is,

\[
x_{1}(j+1) = 0, \\
x_{2}(j+1) = x_{2}(j)/2, \\
0 = x_{3}(j).
\]

Note that the pencil \( (A,B) \) is regular, but since \( \det(A) = \det(B) = 0 \), the pencil \( (A,B)^{(k)} \) is singular for any \( k > 1 \). The subspace of consistent initial conditions of (29) is

\[
\mathcal{V}^{1} = \text{span}(e_{1}, e_{3}), \tag{30}
\]

where \( e_{i} \) is the \( i \)th canonical vector in \( \mathbb{R}^{3} \). Furthermore, given a consistent initial condition \( x \in \mathcal{V}^{1} \), the corresponding solution is \( x(0) = a \) and \( x(j) = 2^{-j}a_{2}e_{2}, j = 1, 2, \ldots \).

Consider now the 2-compound system. Since \( A^{(2)} = \text{diag}(0,0,1/2) \) and \( B^{(2)} = \text{diag}(1,0,0) \), the 2-compound
This implies that $Y^2 = Y^1$. For any $a, b \in Y^1$, we have
\[ \begin{bmatrix} a \\ b \end{bmatrix}^{(2)} = \begin{bmatrix} a_1 b_2 - b_1 a_2 & 0 \\ 0 & 0 \end{bmatrix}^T. \]
Thus, the 2-compound system has consistent initial conditions that are not 2-compounds of consistent initial conditions of the original system. Furthermore, it is easy to see that the 2-compound system has solutions that are not 2-compounds of solutions of the original system.

6 An Application: the set of stable initial conditions of a DAE

We now use our results to analyse stable subspaces of a DAE by studying the stability of the $k$-compound DAE. This is inspired by [Muldowney 1990], which related the subspace of initial conditions of the linear time-varying ODE $\dot{x}(t) = A(t)x(t)$ which lead to an asymptotically stable solution to the $k$-compound ODE. Here we rephrase this result in terms of discrete-time systems and generalize it to linear time-varying DAEs.

We require the following definition. The linear time-varying DAE (1) is called uniformly stable if for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $x(j_0) \in \mathbb{R}^n$, if $x(j_0)$ is a consistent initial condition and $|x(j_0)| < \delta$, then $|x(j)| < \epsilon$ for all $j \geq j_0$.

**Theorem 9.** Suppose that the time-varying DAE (14) is tractable and uniformly stable. Fix an initial time $j_0$, and let $V^1(j_0)$ denote the set of consistent initial conditions. Fix $k \in \{1, \ldots, \dim(V^1(j_0))\}$. Consider the following assertions:

(a) Every solution of the $k$-multiplicative compound DAE (15) satisfies $\lim_{j \to \infty} y(j) = 0$;

(b) The DAE (14) admits a subspace $\mathcal{X}(j_0) \subseteq V^1(j_0)$, with $\dim(\mathcal{X}(j_0)) = \dim(V^1(j_0)) - k + 1$, such that
\[ \lim_{j \to \infty} x(j) = 0 \quad \text{for any} \quad x(j_0) \in \mathcal{X}(j_0). \]

Then (a) implies (b).

**Proof:** Suppose that every solution of (15) satisfies $\lim_{j \to \infty} y(j) = 0$. Pick $k$ vectors $a^1, \ldots, a^k \in \mathbb{R}^n$. Define $X(j) := \begin{bmatrix} x(j, a^1) & \cdots & x(j, a^k) \end{bmatrix}$. Since uniform stability implies that all trajectories are bounded (by a constant which depends on the initial condition), there exists an increasing sequence of times $j_i$ such that $\lim_{j \to \infty} j_i = \infty$ and $P := \lim_{i \to \infty} X(j_i)$ exists. By Prop. 3,
\[ B^{(k)}(j+1)X^{(k)}(j+1) = A^{(k)}(j)X^{(k)}(j), \]
so $P^{(k)} = 0$, and by (6) $P$ has non-full rank, so there exists $c \in \mathbb{R}^k \setminus \{0\}$ such that $Pc = 0$, that is,
\[ 0 = \lim_{i \to \infty} \sum_{\ell=1}^k c_\ell x(j, a^\ell) = \lim_{j \to \infty} x(j, \sum_{\ell=1}^k c_\ell a^\ell), \]
where the last step follows from uniform stability.

Note that not the case for ODEs, the existence of a stable set of initial conditions with dimension $\dim(V^1) - k + 1$ does not imply that the $k$-compound system is asymptotically stable. This is due to the fact that the $k$-compound system might have solutions which do not correspond to compounds of solutions of the original system. **Example 5.** Consider again the system from Example 4. It is easy to verify that all solutions converge to the origin asymptotically, so the system is uniformly stable, and we may take $X = V^1$. However, Prop. 8 implies that the $k$-compound system will have a constant non-zero solution, so it is not asymptotically stable.

6.1 Application to a 3D Leslie model

We describe an application of Thm. 9 to the Leslie model from mathematical demography (see, e.g., [Kot 2001 Ch. 22]). Consider the system:
\[ z(j + 1) = Lz(j), \]
with $z \in \mathbb{R}^3$ and $L = \begin{bmatrix} b_1 & b_2 & 0 \\ p_1 & 0 & 0 \\ 0 & p_2 & 0 \end{bmatrix}$. The parameters $b_i > 0 \quad [p_i > 0]$ represent age-class fertilities [age-class survival probabilities]. Suppose that we can measure the population at the current time, denoted $\ell$, and we are interested in projecting the population dynamics backwards in time [Campbell and Meyer 2009 Ch. 9]. Letting $x(0) := z(\ell)$, $x(1) := z(\ell - 1)$, and so on, gives $Lx(1) = Lz(\ell - 1) = z(\ell) = x(0)$, so we are led to consider the DAE
\[ Lx(j + 1) = x(j). \]
and for any \( x(0) \in \mathcal{V}^1 \) the unique solution of (35) is

\[
x(j) = (L^D)^j x(0), \quad j = 0, 1, \ldots
\]

(36)

A calculation gives

\[
L^D = \begin{bmatrix}
0 & p_1^{-1} & 0 \\
-b_2^{-1} & -b_1 b_2^{-1} p_1^{-1} & 0 \\
-b_1 p_2 b_2^{-2} p_1^{-1} & c & 0
\end{bmatrix},
\]

with \( c := \frac{b_2 p_2}{b_1 p_1^2} + \frac{p_2}{b_2 p_1} \).

The 2-compound system (17) is \( L^{(2)} y(j + 1) = y(j) \), with

\[
L^{(2)} = \begin{bmatrix}
-b_2 p_1 & 0 & 0 \\
0 & b_1 p_2 & 0 \\
0 & 0 & p_1 p_2
\end{bmatrix}. \]

Note in particular that the 2-compound system has a simpler structure than the original system, and is therefore easier to analyse. Note also that this simple structure would still hold even if the system were time-varying, i.e. if \( b_1, p_1 \) vary with the time \( j \).

Assume that \( b_2 p_1 > 1 \). Then condition (a) holds, so Thm. 9 implies that the DAE (35) admits a one-dimensional set of initial conditions \( X \) such that (32) holds. Indeed, it can be shown that one of the finite eigenvalues of \( (I_3, L) \) (or equivalently of \( L^D \)) is \( 2/(b_1 + \sqrt{b_1^2 + 4b_2 p_1}) \), which is smaller than one when \( b_2 p_1 > 1 \). Therefore, (35) has a one-dimensional subspace of initial conditions for which the corresponding solutions converge to the origin. It can also be shown that this stable eigenvalue has a corresponding eigenvector with positive entries, consistent with the fact that we expect the state-variables (that represent populations) to be non-negative. To explain this result, note that \( b_2 p_1 \) is the ratio of current children to future offspring: \( p_1 \) describes the proportion of children which survive to the second age group, and \( b_2 \) describes the fertility of the surviving individuals. Intuitively, the condition \( b_2 p_1 > 1 \) implies that the population in (34) grows with (forward) time, so the population in (35) decreases.

Fig. 3 shows two trajectories of (35) with the parameters \( p_1 = 0.9, p_2 = 0.7, b_1 = 1.1 \) and \( b_2 = 2.3 \) (so \( b_2 p_1 > 1 \)), projected for simplicity onto the 2-dimensional space \( \mathcal{V}^2 \) using the orthogonal projection matrix derived by applying the Gram-Schmidt process. The figure shows convergence along one direction (clearly demonstrated by the trajectory with circle markers), and diverging oscillations along a different direction. Fig. 3 also shows the corresponding solution to the 2-compound system, which converges to the origin asymptotically.

7 Conclusion

Given square matrices \( A, B \), we defined the \( k \)-multiplicative compound of the matrix pencil \( (A, B) \). This is a matrix pencil, denoted \( (A, B)^{(k)} \), that for \( k = 1 \) reduces to \( (A, B) \).

We studied the relation between \( (A, B) \) and \( (A, B)^{(k)} \) and illustrated several applications to DAEs. In particular, we showed that the DAE corresponding to \( (A, B)^{(k)} \) describes the evolution of \( k \)-parallelotopes in the DAE corresponding to \( (A, B) \).

An interesting line of research is defining also the \( k \)-additive compound of a matrix pencil, and using it to analyze differential-algebraic equations.

The \( k \)-compounds of a matrix have been recently used to define non-trivial generalizations of several classes of both continuous-time and discrete-time dynamical systems including \( k \)-positive linear systems and \( k \)-cooperative nonlinear systems [Weiss and Margaliot, 2021], \( k \)-contracting systems [Wu et al, 2022], \( k \)-diagonally stable systems [Wu and Margaliot, 2022], and more. Another possible research direction is to use the compounds of matrix pencils to define such generalizations for difference-algebraic and differential-algebraic systems.

Appendix: Drazin inverse of the \( k \)-multiplicative compound

Eq. (26) includes the Drazin inverse of the \( k \)-multiplicative compound of a matrix. The next result shows that this is equal to the \( k \)-multiplicative of the Drazin inverse of the original matrix.

**Proposition 10.** Let \( A \in \mathbb{C}^{n \times n} \), and fix \( k \in \{1, \ldots, n\} \). Then \( (A^{(k)})^D = (A^D)^{(k)} \).

**Proof.** Denote \( i := \text{index}(A) \), \( j := \text{index}(A^{(k)}) \), and \( E := A^{(k)} \). We need to show that \( E^D = (A^D)^{(k)} \). Since \( A^D \) is the Drazin inverse of \( A \), we have

\[
A^{i+1} A^D = A^i, \quad AA^D = A^D A, \quad A^D AA^D = A^D. \quad (37)
\]

Taking the \( k \)-multiplicative compounds of these equations and using the Cauchy-Binet Theorem gives

\[
E^{i+1} (A^D)^{(k)} = E^i,
\]

\[
E (A^D)^{(k)} = (A^D)^{(k)} E,
\]

\[
(A^D)^{(k)} E (A^D)^{(k)} = (A^D)^{(k)}. \quad (38)
\]

Thus, \( (A^D)^{(k)} \) satisfies two of the requirements for the Drazin inverse of \( E \), and we only need to show that

\[
E^{i+1} (A^D)^{(k)} = E^j. \quad (39)
\]
Let $A$ have the Jordan decomposition in (11). Then the index of nilpotency of $N$ is also $i$, that is, $i$ is the minimal integer such that $N^i = 0$. We prove the proposition when $T = I$, so $A = \text{diag}(C, N)$, $A^s = \text{diag}(C^s, 0)$, and $A^D = \text{diag}(C^{-1}, 0)$. The proof in the more general case is very similar. Denote the dimension of $C$ by $s$. Then $\text{rank}(A^i) = s$. We consider two cases.

Case 1: Assume that $k > s$. Then every $(k \times k)$-submatrix of $A^D$ includes either a column of zeros or a row of zeros, so $(A^D)^{(k)} = 0$. Also, since every eigenvalue of $A^{(k)}$ is the product of $k$ eigenvalues of $A$, every eigenvalue of $E$ is zero. Thus, $E$ is nilpotent, so $E^j = 0$. We conclude that (39) holds.

Case 2: Assume that $k \leq s$. We will show that in this case $j = i$. Fix an integer $\ell \geq 0$. Then
\[
\text{rank}(E^{\ell+1}) = \text{rank}((A^{(k)})^{\ell+1}) = \text{rank}(A^{(k)}),
\]
where the last equation follows from (6). Combining this with the definition of $i$ gives
\[
\text{rank}(E^{i+1}) = \text{rank}(A^{(k)}),
\]
and thus (38) implies (39). This completes the proof of Prop. 10.

Example 6. Let $A = \text{diag}(a_1, \ldots, a_s, 0, \ldots, 0) \in \mathbb{R}^{n \times n}$, with $a_i \neq 0$. Fix $k \leq s$. Then on the one-hand $A^{(k)} = \text{diag}(\prod_{i=1}^k a_i, \ldots, \prod_{i=s-k+1}^s a_i, 0, \ldots, 0)$, so $(A^{(k)})^D = \text{diag}(\prod_{i=1}^k a_i^{-1}, \ldots, \prod_{i=s-k+1}^s a_i^{-1}, 0, \ldots, 0)$. On the other-hand, $A^D = \text{diag}(a_1^{-1}, \ldots, a_s^{-1}, 0, \ldots, 0)$ and thus
\[
(A^{(k)})^D = \text{diag} \left( \prod_{i=1}^k a_i^{-1}, \ldots, \prod_{i=s-k+1}^s a_i^{-1}, 0, \ldots, 0 \right),
\]
so $(A^{(k)})^D = (A^{(k)})^D$.

Note that if $A$ is regular then $A^D = A^{-1}$, so Prop. 10 reduces to the well-known relation $(A^{(k)})^{-1} = (A^{-1})^{(k)}$.

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