Suppose that two parties, Alice and Bob, communicate over a noisy classical channel. While there are many examples of how Alice and Bob may benefit when they upgrade to a quantum channel, examples in which shared entanglement improves communication over a classical channel have only recently been discovered [1,2]. That these examples exist at all is somewhat surprising, as neither shared entanglement nor the assistance of non-signaling correlations is available. In this section, we prove the limit of such protocols. As an example, we show that the Prevedel et al. protocol is optimal for two-qubit entanglement. We also prove some simple upper bounds on the improvement that can be obtained from quantum and non-signaling correlations.

We use two distinct approaches to quantify the extent to which entanglement can help Alice and Bob. In our first approach, we derive a simple formula for Succ_{Q}(N) in terms of the dimension of the entanglement. This formula, which is given by maximizing a quantity over a family of positive semidefinite operators, is easy to work with, and as an example of its applicability, we show that the protocol from [2] is in fact optimal for their channel and for 2-dimensional entanglement assistance.

While our first approach is quite general, it does not give a closed form for the success probability. Our second approach obtains explicit closed-form upper bounds for the success probability. As a first step, we prove the following general bound on non-signaling assistance. Let r be the number of elements in the input alphabet of N. Then,

\[
\frac{\text{Succ}\text{NS}(N) - \frac{1}{2}}{\text{Succ}(N) - \frac{1}{2}} \leq 2 - \frac{2}{r}.
\]

The quantity (Succ(N) − ½) measures the advantage that Alice and Bob have over a random strategy; thus, 1 measures the additional advantage gained by non-signaling correlations. Our proof of 1 uses the linear program characterization of Succ_{NS}(N) from 2. From this, we derive an upper bound on the amount of assistance from a binary quantum device; we use the fact that any quantum correlation can be decomposed into a probabilistic mixture of a local correlation and a non-signaling correlation (the concept of local fraction). We show that both of these bounds are the best possible, in the sense that there are channels for which equality is achieved.

A common thread in both approaches above is the use of the radius of a subset of a normed vector space. Our formula for Succ_{Q}(N) depends on maximizing the radius of a family of Hermitian operators. In the second approach we use a formula for Succ_{NS}(N) (an alternate formulation of Proposition 14 from 2) which is expressed in terms of the radius of a particular set of vectors.
Notation and terminology. Throughout this paper, we assume that Alice is trying to transmit a single bit to Bob across a classical channel. Alice and Bob will have access to a two-part input-output device $D$ (Figure 1), which may be classical, quantum, or implement an arbitrary non-signaling correlation. Each two-part input output device $D$ gives rise to a correlation between Alice and Bob, given by

$$\{D(pq|rs) \mid p \in \mathcal{P}, q \in \mathcal{Q}, r \in \mathcal{R}, s \in \mathcal{S}\}$$

so that $D(pq|rs)$ is the probability of outputs $p$ and $q$ given inputs $r$ and $s$. We will abuse notation by identifying the device $D$ with the correlation it induces.

We say a device is non-signaling if the partial sums $\sum_{p \in \mathcal{P}} D(pq|rs)$ do not depend on $r$, and the partial sums $\sum_{q \in \mathcal{Q}} D(pq|rs)$ do not depend on $s$. We say a non-signaling device $D$ is quantum if there exist Hilbert spaces $V_A$ and $V_B$, families of POVMs $\{\{A^i_x\}_x\}_r$, $\{\{B^i_y\}_y\}_s$, and a density operator $\Lambda$ on $V_A \otimes V_B$ such that $D(rs|pq) = Tr((A^i_x \otimes B^i_y)\Lambda)$. The device is quantum with dimension $n$ if both $V_A$ and $V_B$ are $n$-dimensional, and binary if the input and output alphabets have size 2.

A classical channel $N$ is given by a matrix of conditional probabilities $\{N(y|x) \mid x \in \mathcal{X}, y \in \mathcal{Y}\}$, where $N(y|x)$ is the probability of seeing an output $y \in \mathcal{Y}$ given the input $x \in \mathcal{X}$. For any channel $N$, let $\text{Succ}(N)$ denote the maximum probability with which a single bit can be sent across $N$ (without assistance). Let $\text{Succ}(N, D)$ denote the maximum probability for a single-bit transmission across $N$ with the assistance of $D$. If $S$ is a set of two-part devices, write $\text{Succ}_S(N) := \sup_{S \in S} \text{Succ}(N, S)$. We will be concerned with three choices of $S$. We consider the set $N_S$ of non-signaling devices; the sets $Q$ and $Q(n)$ of quantum and $n$-dimensional quantum devices; and the set $Q_b$ of binary quantum devices.

General quantum devices. In this section, we derive a formula for $\text{Succ}_{Q(n)}(N)$, and give an example of how to use our formula. We will use the radius of a finite set $\{H_i\}_i \subseteq \mathcal{H}$ of Hermitian operators on a finite-dimensional Hilbert space $V$, defined by $\text{Rad}(\{H_i\}_i) := \min_{C} \max_i \|H_i - C\|$ where the minimum is taken over all Hermitian operators $C$ on $V$. The following lemma, which is proved in section 1 of the supplementary material, gives an alternative expression for the radius.

**Lemma 1.** For any finite set $\{H_i\}_i \subseteq \mathcal{H}$ of Hermitian operators on a finite-dimensional Hilbert space $V$, the radius of $\{H_i\}$ is equal to

$$\max_{\lambda_i \geq 0, \lambda_i' \geq 0} \frac{1}{2} \left( \sum_{i \in \mathcal{I}} \text{Tr}((\lambda_i - \lambda_i')H_i) \right).$$

Here, the maximization is over all Hermitian operators $\{\lambda_i\}_i \subseteq \mathcal{H}$ and $\{\lambda_i'\}_i \subseteq \mathcal{H}$ on $V$ satisfying the given constraints.

Using this lemma, we will prove the following theorem, which characterizes $\text{Succ}_{Q(n)}(N)$.

**Theorem 2.** For any channel $N$, and any integer $n \geq 2$,

$$\text{Succ}_{Q(n)}(N) = \frac{1}{2} + \max_{\{B_y\}_y \subseteq \mathcal{Y}} \left( \text{Rad} \left\{ \sum_{y \in \mathcal{Y}} N(y|x)B_y \right\} \right)_{x \in \mathcal{X}},$$

where the maximization is over all families $\{B_y\}_y \subseteq \mathcal{Y}$ of Hermitian operators on $\mathbb{C}^n$ satisfying $0 \leq B_y \leq I$.

**Proof.** Consider the following quantum-assisted protocol for transmitting a single bit across $N$. Alice and Bob possess a bipartite quantum system represented by a density matrix $\Lambda$ on a Hilbert space $V_A \otimes V_B$. Alice wishes to transmit a message $a \in \{0, 1\}$. Depending the value of $a$, she applies one of two possible POVMs $\{A^0_x\}_x \subseteq \mathcal{X}$ or $\{A^1_x\}_x \subseteq \mathcal{X}$ to $V_A$ and sends the result of the measurement to the channel $N$. Bob receives the output $y$ of the channel, and according to this output, applies one of a family of binary POVMs $\{\{B^0_y, B^1_y\}_y\}_y \subseteq \mathcal{Y}$ to $V_B$. The result of this output is Bob’s guess at Alice’s original message.

In order to compute the success probability for this protocol, it is not necessary to know the state $\Lambda$ or the operators $\{A^0_x\}_x, a$: it is only necessary to know the operators $\rho^0_x := \text{Tr}_A((A^0_x \otimes I)\Lambda)$, which represent the state of Bob’s quantum system when the outcome of Alice’s measurement is $x$. These operators satisfy $\sum_x \rho^0_x = \sum_x \rho^1_x$ and $\text{Tr}(\rho^0_x) = 1$, and, in fact, any family of operators satisfying those two constraints can be induced by an appropriately chosen state $\Lambda$ and appropriately chosen POVMs $\{A^0_x\}_x \subseteq \mathcal{X}$ or $\{A^1_x\}_x \subseteq \mathcal{X}$. Thus, for our purposes, to specify an $(n, n)$-dimensional entanglement-assisted strategy for communicating a single bit across $N$, it suffices to specify a collection of binary POVMs $\{\{B^0_y, B^1_y\}_y\}_y \subseteq \mathcal{Y}$ and a collection of positive semidefinite operators $\{\rho^0_x \mid a \in \{0, 1\}, x \in \mathcal{X}\}$ on $\mathbb{C}^n$ satisfying

$$\sum_x \rho^0_x = \sum_x \rho^1_x \quad \text{and} \quad \text{Tr}(\sum_x \rho^0_x) = 1. \quad (2)$$

The success probability of the protocol is given by

$$\frac{1}{2} \left( \sum_{y \in \mathcal{Y}} N(y|x) \sum_{x \in \mathcal{X}} \text{Tr}(\rho^0_x B^y_y) \right) + \frac{1}{2} \left( \sum_{y \in \mathcal{Y}} N(y|x) \sum_{x \in \mathcal{X}} \text{Tr}(\rho^1_x B^1_y) \right).$$
Let \( B_y = B_y^0 \). Since \( B_y^1 = I - B_y \), the expression above simplifies to

\[
\frac{1}{2} + \frac{1}{2} \cdot \text{Tr} \left[ \sum_{x \in \mathcal{X}} (\rho^x_0 - \rho^x_1) \sum_{y \in \mathcal{Y}} N(y|x) B_y \right].
\] (3)

The quantity \( \text{Succ}_{Q(n)}(N) \) is the maximum of this expression over all \( n \times n \) Hermitian operators \( \{B_y\}_{y \in \mathcal{Y}} \) satisfying \( 0 \leq B_y \leq I \) and all \( n \times n \) positive semidefinite operators \( \{\rho^x_a\}_{x \in \mathcal{X}, a \in \{0, 1\}} \) satisfying (2) above. Applying Lemma 1 yields the desired formula. □

A convexity argument (see section 2 of the supplementary information) proves the following stronger version of Theorem 2

**Corollary 3.** The formula in Theorem 2 holds also when the maximum is taken only over families \( \{B_y\} \) that consist of projections on \( \mathbb{C}^n \).

As an example of the utility of Corollary 3 consider the channel \( M \) in Figure 2 which is defined in 2. The input alphabet for \( M \) is \{1, 2, 3, 4\}, and the output alphabet is \{1, 2, 3, 4, 5, 6\}. In section 3 of the supplementary information, we prove that for any \( 2 \times 2 \) projection operators \( P_1, \ldots, P_6 \), the radius of the set \( \{\sum_{y=1}^{6} M(y|x)P_y \mid x = 1, 2, 3, 4\} \) is no more than \( \frac{1}{3} + \frac{1}{3\sqrt{2}} \). This maximum is achieved when \( P_1 = 0, P_2 = I \), and \( \{P_3, P_4\} \) and \( \{P_5, P_6\} \) are two different Pauli measurements. Therefore,

\[
\text{Succ}_{Q(2)}(M) = \frac{2}{3} + \frac{1}{3\sqrt{2}},
\]

and the protocol from 2 is optimal for 2-dimensional entanglement assistance. (We note that this generalizes the paper 3, which showed the optimality of 2 within a more restricted class of protocols.)

| 1  | 2  | 3  | 4  | 5  | 6  |
|----|----|----|----|----|----|
| 1/3| 0  | 0  | 1/3| 0  | 1/3|
| 2/3| 0  | 0  | 1/3| 0  | 1/3|
| 3  | 0  | 1/3| 1/3| 0  | 1/3|
| 4  | 0  | 1/3| 0  | 1/3| 1/3|

**FIG. 2.** The channel \( M \), from 2.

**Non-signaling devices.** In order to prove more explicit bounds on the limits of quantum assistance, we first turn our attention to assistance by a non-signaling correlation. The next proposition asserts a formula for the optimal non-signaling assisted success probability of a channel. For any finite set of vectors \( S \subseteq \mathbb{R}^k \), let \( \text{Rad}_1(S) \) denote the radius of \( S \) under the 1-norm.

**Proposition 4.** Let \( N \) be a classical channel, and for each \( x \in \mathcal{X} \), let \( \{N(y \mid x)\}_{y \in \mathcal{Y}} \in \mathbb{R}^\mathcal{Y} \). Then,

\[
\text{Succ}_{\text{NS}}(N) = \frac{1}{2} + \frac{1}{2} \cdot \text{Rad}_1 \{n_x \mid x \in \mathcal{X}\}.
\] (4)

Note that in the above formula, we take the radius of \( \{n_x\} \) as a subset of \( \mathbb{R}^3 \), not as a subset of the set of probability distributions on \( \mathcal{Y} \).

**Proof of Proposition 4.** By Proposition 14 from 4,

\[
\text{Succ}_{\text{NS}}(N) = 1 - \max_{\rho \in \mathbb{R}^{\mathcal{Y}}} \sum_{y \in \mathcal{Y}} (\min_{c_y, N(y \mid x)} - c_y/2) = \min_{\rho \in \mathbb{R}^{\mathcal{Y}}} \sum_{y \in \mathcal{Y}} (1 - \min_{c_y, N(y \mid x)} + c_y/2).
\]

Using the easily proved fact that \( \|u - v\|_1 = \sum_i u_i + \sum_i v_i - 2 \sum_i \min\{u_i, v_i\} \), the above formula simplifies to

\[
\text{Succ}_{\text{NS}}(N) = \min_{\rho \in \mathbb{R}^{\mathcal{Y}}} \left( \frac{1}{2} + \frac{1}{2} \cdot \|c - n_x\|_1 \right),
\]

which implies (4) above. □

Formula (4) allows us to relate the quantity \( \text{Succ}_{\text{NS}}(N) \) to the quantity \( \text{Succ}(N) \).

**Theorem 5.** Let \( N \) be a classical channel, and let \( r = |\mathcal{X}| \) denote the size of the input alphabet of \( N \). Then,

\[
\text{Succ}_{\text{NS}}(N) - \frac{1}{2} \leq \left( 2 - \frac{2}{r} \right) \left[ \text{Succ}(N) - \frac{1}{2} \right].
\] (5)

**Proof.** Let \( \{n_x\} \) be the vectors defined in Proposition 4. The unassisted one-shot success probability can be expressed in terms of these vectors like so:

\[
\text{Succ}(N) = \frac{1}{2} + \frac{1}{4} \cdot \text{Diam}_1 \{n_x\},
\]

where \( \text{Diam}_1 \) denotes diameter under the 1-norm. A triangle-inequality argument shows that the distance from the mean vector \( (\sum_x n_x) / r \) to the set \( \{n_x\} \) cannot exceed \( (1 - \frac{1}{r}) \text{Diam}_1 \{n_x\} \). Therefore, \( \text{Rad}_1 \{n_x\} \leq (1 - \frac{1}{r}) \text{Diam}_1 \{n_x\} \), which implies the desired result. □

Theorem 5 is the best possible in the sense that there are channels where equality is achieved in 5. Consider the following example, which is a generalization of the channel \( M \) from Figure 2. Let \( s \) be a positive integer. For any \( i \in \{0, 1, 2, \ldots, 2^s - 1\} \), let \( b_i \in \mathbb{F}_2^s \) denote the binary representation of \( i \). Define a channel \( T \) as follows. The input alphabet of \( T \) is \{0, 1, 2, \ldots, 2^s - 1\}, and the output alphabet is \{1, 2, \ldots, 2^s - 1\} \times \{0, 1\}. On given input \( i \), the channel chooses an element \( j \in \{1, \ldots, 2^s - 1\} \) uniformly at random and outputs the pair \((j, b_i \cdot b_j)\) (where \( b_i \cdot b_j \) denotes the inner product of \( b_i \) and \( b_j \) mod 2).

For any \( i \in \{0, 1, 2, \ldots, 2^s - 1\} \), let \( \epsilon_i \in \mathbb{R}^{2^{(2^s - 1)}} \) denote the probability vector which expresses the output of \( T \) on input \( i \). It is easy to see that the diameter of \( \{\epsilon_i\} \) is \( 2^s /(2^s - 1) \), and thus \( \text{Succ}(T) = \frac{1}{2} + 2^{s-2} /\left(2^s - 1\right) \). On
the other hand, the radius of \( \{ \ell_i \} \) is 1, as can be seen from the following calculation. For any \( c \in \mathbb{R}^{2s} \),
\[
\max_{0 \leq i \leq 2s-1} \| \ell_i - c \|_1 \geq 2^{-s} \sum_{i=0}^{2s-1} \| \ell_i - c \|_1
\]
\[
= 2^{-s} \sum_{0 \leq i \leq 2s-1} [2^{s-1} |c_{jt} - (2^{s-1} - 1)| + 2^{s-1} |c_{jt} - 0|]
\]
\[
\geq 2^{-s} \sum_{0 \leq i \leq 2s-1} [2^{s-1}(2^{s-1} - 1)] = 1.
\]

Thus \( \text{Succ}_{NS}(T) = 1 \). (And, indeed, a perfect communication protocol for \( T \) exists—see section 4 of the supplementary information.) The channel \( T \) achieves equality in (5).

The following modified version of Theorem 5 will be useful in our analysis of entanglement assistance.

**Theorem 6.** Let \( N \) be a classical channel, and let \( D \) be a non-signaling correlation arising from a two-part device \( (D_A, D_B) \). Let \( m \) denote the size of the output alphabet of \( D_A \). Then,
\[
\text{Succ}(N, D) - \frac{1}{2} \leq \left( 2 - \frac{1}{m} \right) \left[ \text{Succ}(N) - \frac{1}{2} \right]. \tag{7}
\]

**Proof.** A protocol for communicating a single bit \( a \) using \( N \) and \( D \) proceeds as follows. Alice uses \( a \) to choose an input to \( D_A \), and then uses \( a \) and the output of \( D_A \) to choose an input to \( N \). Bob uses the output of \( N \) to choose an input to \( D_B \), and then uses the outputs of \( N \) and \( D_B \) together to guess the bit \( a \).

The optimal success probability \( \text{Succ}(N, D) \) can be achieved by a deterministic protocol (i.e., a protocol in which Alice and Bob make their choices according to deterministic functions). As there are only \( 2m \) possible inputs that Alice could make to \( N \) in a deterministic protocol, the success probability of such a protocol is bounded by \( (2 - 2/(2m))\text{Succ}(N) \) by Theorem 5.

**Binary quantum devices.** Finally, we will use our bounds for non-signaling devices to obtain bounds for assistance by binary quantum devices.

A two-part device \( D \) is local-deterministic if the output of each part is a deterministic function of its input. A non-signaling correlation is local if it is a convex combination of local-deterministic correlations. We define the local fraction of a non-signaling correlation, a concept which is used in [4, 10].

**Definition 7.** Let \( D \) be a non-signaling correlation. The local fraction of \( D \), denoted \( \text{loc}(D) \), is the largest real number \( \alpha \in [0, 1] \) such that there exists a decomposition
\[
D = \alpha L + (1 - \alpha) F,
\tag{8}
\]
where \( L \) is a local correlation and \( F \) is a non-signaling correlation.

For any classical channel \( N \), it is easy to see that when a decomposition such as (8) exists with \( L \) local and \( F \) non-signaling,
\[
\text{Succ}(N, D) \leq \alpha \text{Succ}(N, L) + (1 - \alpha) \text{Succ}(N, F)
\]
\[
\leq \alpha \text{Succ}(N) + (1 - \alpha) \text{Succ}_{NS}(N).
\]

This implies the following stronger version of Theorem 8.

**Theorem 8.** Let \( N \) be a channel, and let \( D \) be a non-signaling correlation arising from a two-part device \( (D_A, D_B) \). Let \( m \) denote the size of the output alphabet of \( D_A \). Then
\[
\frac{\text{Succ}(N, D) - \frac{1}{2}}{\text{Succ}(N) - \frac{1}{2}} \leq 1 + \left( 1 - \frac{1}{m} \right) (1 - \text{loc}(D)). \tag{9}
\]

Thus, to obtain improved upper bounds on \( \text{Succ}(N, D) \) for quantum correlations \( D \), it suffices to find lower bounds on the local fractions of quantum correlations. In section 5 of the supplementary material, we use facts about the geometry of quantum and non-signaling correlations [11] to prove the following bound for binary quantum correlations.

**Proposition 9.** Let \( D \) be a binary quantum correlation. Then \( \text{loc}(D) \geq 2 - \sqrt{2} \).

Combining Theorem 8 and Proposition 9 yields the following.

**Corollary 10.** For any classical channel \( N \),
\[
\frac{\text{Succ}_{Q} (N) - \frac{1}{2}}{\text{Succ}(N) - \frac{1}{2}} \leq 1 + \frac{1}{\sqrt{2}}.
\]

Note that equality occurs in Corollary 10 for the case discussed in [2].

**Conclusion.** We have given a formula for the \( n \)-dimensional entanglement-assisted one-shot success probability of a classical channel, and have shown its utility by using it to show that the protocol in [2] is optimal. We derived a more explicit bound on the advantage gained by binary quantum correlations (which is an equality in the case of [2]). Along the way, we established a bound on the advantage gained by non-signaling assistance and provided an example where equality is achieved.

Future research could explore methods for evaluating the formula from Theorem 2 (Section 3 of the supplementary information provides methods which might generalize.) Also, it would be interesting to try to prove stronger bounds on the increase in \( \text{Succ}(N) \) that is provided by entanglement. (This might involve generalizations of Proposition 9.) Another natural next step would be to consider the one-shot success probability for non-binary messages.

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SUPPLEMENTARY INFORMATION

1. The radius of a set of Hermitian operators

Lemma 1. For any finite set \( \{ H_i \}_{i \in I} \) of Hermitian operators on a finite-dimensional Hilbert space \( V \), the radius of \( \{ H_i \} \) is equal to

\[
\max_{\lambda_i \geq 0, \lambda_i' \geq 0 \atop \sum \lambda_i = \sum \lambda_i'} \left[ \frac{1}{2} \sum_{i \in I} \text{Tr} \left( (\lambda_i - \lambda_i') H_i \right) \right].
\]

Proof. Any family of Hermitian operators \( \{ H_i \} \) may be translated to a family \( \{ H_i + W \} \) which contains the operator 0. This translation does not affect the radius nor the expression from the statement of the lemma. Therefore, we may assume that \( \{ H_i \} \) contains 0. By definition,

\[
\text{Rad} \{ H_i \} = \min_{C, r} \left( r \right),
\]

where the maximization is over Hermitian operators \( C \) and real numbers \( r \). Since 0 \( \in \{ H_i \} \), whenever the constraints in this maximization are satisfied we have in particular that \( C \geq -rI \). Letting \( Z = C + rI \), we obtain the following alternate expression:

\[
\text{Rad} \{ H_i \} = \min_{Z, r} \left( r \right).
\]

By semidefinite programming duality, this is equivalent to

\[
\text{Rad} \{ H_i \} = \max_{\lambda_i \geq 0, \lambda_i' \geq 0 \atop \sum \lambda_i - \sum \lambda_i' \leq 0 \atop 2 \text{Tr}(\sum \lambda_i') \leq 1} \left[ \left( \sum \lambda_i \text{Tr}(H_i) - \sum \lambda_i' \text{Tr}(H_i) \right) \right].
\]

It is easy to see that this maximum is achieved by a pair of families \( \{ \lambda_i \}, \{ \lambda_i' \} \) satisfying \( \sum \lambda_i = \sum \lambda_i' \) and \( 2 \text{Tr}(\sum \lambda_i') = 1 \). \( \square \)

2. The proof of Corollary 3 in the main text

The radius function is convex in the following sense: for any families of operators \( \{ J_y \}_{y \in Y} \) and \( \{ K_y \}_{y \in Y} \), and real number \( \alpha \in [0, 1] \),

\[
\text{Rad} \{ \alpha J_y + (1 - \alpha) K_y \} \leq \alpha \text{Rad} \{ J_y \} + (1 - \alpha) \text{Rad} \{ K_y \}.
\]

(For, if we let \( J' \) be such that the distance from \( J' \) to \( \{ J_y \} \) is equal to \( r := \text{Rad} \{ J_y \} \), and we let \( K' \) be such that the distance from \( K' \) to \( \{ K_y \} \) is equal to \( r' := \text{Rad} \{ K_y \} \), then the distance from \( \alpha J' + (1 - \alpha) K' \) to \( \{ \alpha J_y + (1 - \alpha) K_y \} \) is no more than \( \alpha r + (1 - \alpha) r' \) by the triangle inequality.) In particular, this
convexity property implies that the radius of \( \{\alpha J_y + (1 - \alpha) K_y\}_y \) is no more than the maximum of \( \text{Rad}\{J_y\}_y \) and \( \text{Rad}\{K_y\}_y \).

Since any Hermitian operator \( B \) satisfying \( 0 \leq B \leq \mathbb{I} \) is a convex combination of projection operators, Corollary 3 follows from Theorem 2.

3. AN EXAMPLE CALCULATION

Let \( M \) be the channel defined in figure 2 in the main text. In this section we will use Theorem 2 from the main text to calculate the quantity \( \text{Succ}_{Q(2)}(M) \).

First, we will prove the following lemma which provides a simplified formula for \( \text{Succ}_{Q(2)} \). Additionally, we will use Theorem 2 from the main text to calculate the quantity \( \text{Succ}_{Q(2)}(M) \).

Let \( B_1, B_2, B_3, B_4, B_5, B_6 \) on \( \mathbb{C}^n \), let \( P \) be projection onto the orthogonal complement of \( P \).

**Lemma 2.** For any \( n \geq 1 \), the quantity \( \text{Succ}_{Q(n)}(M) \) is equal to

\[
\frac{1}{2} + \left( \frac{1}{3} \right) \max_{X,Y,Z} \{ \text{Rad} \{X + Y + Z, X + Y^\perp + Z^\perp, X^\perp + Y + Z^\perp, X^\perp + Y^\perp + Z\} \},
\]

where the maximum is taken over all projection operators \( X, Y, Z \) on \( \mathbb{C}^n \).

**Proof.** For any Hermitian operators \( B_1, B_2, B_3, B_4, B_5, B_6 \) on \( \mathbb{C}^n \), let

\[
F(B_1, B_2, B_3, B_4, B_5, B_6)
\]

be equal to the quantity

\[
\text{Rad} \{B_1 + B_3 + B_5, B_1 + B_4 + B_6, B_2 + B_3 + B_6, B_2 + B_4 + B_5\}.
\]

By the formula from Theorem 2 in the main text,

\[
(3) \quad \text{Succ}_{Q(n)}(M) = \frac{1}{2} + \left( \frac{1}{3} \right) \max_{0 \leq B_i \leq \mathbb{I}} F(B_1, B_2, B_3, B_4, B_5, B_6).
\]

Let

\[
(4) \quad m = \max_{0 \leq B_i \leq \mathbb{I}} F(B_1, B_2, B_3, B_4, B_5, B_6).
\]

It suffices to prove that this maximum is achieved by some 6-tuple of the form \( (X, X^\perp, Y, Y^\perp, Z, Z^\perp) \), where \( X, Y, \) and \( Z \) are projections.

As noted in section 2 of the supplementary information, the radius function is convex in the sense that if \( (H_1, H_2, H_3, H_4) \) and \( (H'_1, H'_2, H'_3, H'_4) \) are Hermitian operators and \( \alpha \in [0, 1] \) is a real number,

\[
(5) \quad \text{Rad}\{\alpha H_i + (1 - \alpha) H'_i\}_i \leq \alpha \text{Rad}\{H_i\}_i + (1 - \alpha) \text{Rad}\{H'_i\}_i.
\]

It follows easily by linearity that a similar convexity property holds for \( F \): for any Hermitian operators \( B_1, \ldots, B_6 \) and \( B'_1, \ldots, B'_6 \), and any \( \alpha \in [0, 1] \),

\[
F(\alpha B_1 + (1 - \alpha) B'_1, \ldots, \alpha B_6 + (1 - \alpha) B'_6) \leq \alpha F(B_1, \ldots, B_6) + (1 - \alpha) F(B'_1, \ldots, B'_6).
\]

In particular,

\[
(6) \quad F(\alpha B_1 + (1 - \alpha) B'_1, \ldots, \alpha B_6 + (1 - \alpha) B'_6) \leq \max \{ F(B_1, \ldots, B_6), F(B'_1, \ldots, B'_6) \}.
\]

Additionally, \( F \) is translation-invariant in the following sense: for any Hermitian operators \( B_1, \ldots, B_6 \), and any Hermitian operators \( K, L, \) and \( M \),

\[
(7) \quad F(B_1 + K, B_2 + K, B_3 + L, B_4 + L, B_5 + M, B_6 + M) = F(B_1, \ldots, B_6).
\]
Let $X_1, X_2, Y_1, Y_2, Z_1, Z_2$ be Hermitian operators satisfying $0 \leq X_i, Y_i, Z_i \leq I$ such that $F(X_1, X_2, Y_1, Y_2, Z_1, Z_2) = m$. Let $X_+$ and $X_-$ be a pair of positive semidefinite operators having mutual orthogonal supports which are such that

\begin{equation}
X_1 - X_2 = X_+ - X_-.
\end{equation}

Define $Y_+, Y_-, Z_+, Z_-$ similarly. By property (7) above,

\begin{equation}
F(X_+, X_-, Y_+, Y_-, Z_+, Z_-) = F(X_1, X_2, Y_1, Y_2, Z_1, Z_2) = m.
\end{equation}

The pair $(X_+, X_-)$ can be expressed as a convex combination of pairs of projections $(P_1^{(i)}, P_2^{(i)})$ where for each $i$, the support of $P_1^{(i)}$ is orthogonal to $P_2^{(i)}$. A similar decomposition exists for $(Y_+, Y_-)$ and $(Z_+, Z_-)$. Therefore by property (10) above, there exist pairs of projections $(P_1, P_2), (Q_1, Q_2), (R_1, R_2)$, with each pair having mutually orthogonal supports, such that

\begin{equation}
F(P_1, P_2, Q_1, Q_2, R_1, R_2) = m.
\end{equation}

Let $P_3 = I - P_1 - P_2$, and define $Q_3$ and $R_3$ similarly. By (11),

\begin{equation}
F\left(\frac{P_1 + P_3}{2}, P_2 + \frac{P_3}{2}, Q_1 + \frac{Q_3}{2}, Q_2 + \frac{Q_3}{2}, R_1 + \frac{R_3}{2}, R_2 + \frac{R_3}{2}\right) = m.
\end{equation}

The 6-tuple on the left hand side of the equation above is a convex combination of the 6-tuples

\begin{align*}
(P_1 + P_3, P_2, Q_1 + Q_3, Q_2, R_1 + R_3, R_2) & \\
\text{and} (P_1, P_2 + P_3, Q_1, Q_2 + Q_3, R_1, R_3 + R_3).
\end{align*}

By (9), at least one of these 6-tuples must achieve the maximum $m$. This completes the proof. \qed

**Lemma 3.** For any projection operators $X, Y, Z$ on the two-dimensional vector space $\mathbb{C}^2$, the radius of the set

\begin{equation}
\{X + Y + Z, X + Y^\perp + Z^\perp, X^\perp + Y + Z^\perp, X^\perp + Y^\perp + Z\}.
\end{equation}

is less than or equal to $\frac{1}{2} + \frac{1}{\sqrt{2}}$.

**Proof.** Case 1: The matrices $X, Y, Z$ are all scalar matrices. In this case, each of $X, Y, Z$ is equal to either $0$ or $I$. This case is trivial, since the radius of the set $\{3I, I\}$ is $1$, and the radius of the set $\{2I, 0\}$ is $1$.

Case 2: Two of the matrices $X,Y,Z$ are scalar matrices and one is a nonscalar. We may assume without loss of generality that $X$ is the nonscalar matrix. Then the set $\{\}^{12}$ is equal to either

\begin{equation}
\{0, X + I, 2I\}
\end{equation}

or

\begin{equation}
\{X, X + 2I, I\}.
\end{equation}

In the former case, the operator-norm distance from the operator $I$ to the set $\{0, X + I, 2I\}$ is $1$. In the latter case, the operator-norm distance from the operator $X + I$ to the set $\{X, X + 2I, I\}$ is $1$. The desired result follows.
Case 3: Exactly one of the matrices $X, Y, Z$ is a scalar matrix. We may assume that $X$ and $Y$ are nonscalar matrices and $Z$ is scalar. Also, by replacing $(X,Y,Z)$ with $(X^\perp,Y,Z^\perp)$ if necessary, we may assume that $Z = I$.

Let $X = |x\rangle \langle x|$ and $Y = |y\rangle \langle y|$ where $x, y \in \mathbb{C}^2$ are unit vectors, and let $\theta = \arccos(|x \cdot y|)$. Both of the operators
\[
X + Y + I, X^\perp + Y^\perp + I
\]
have eigenvalues $\{2 + \cos \theta, 2 - \cos \theta\}$, and both of the operators
\[
X + Y^\perp, X^\perp + Y
\]
have eigenvalues $\{1 + \sin \theta, 1 - \sin \theta\}$. If we let
\[
C = \left(\frac{3}{2} + \frac{\cos \theta - \sin \theta}{2}\right) I,
\]
then the operator norm distance from $C$ to each of the elements of (12) is
\[
\frac{1}{2} + \frac{\cos \theta + \sin \theta}{2} \leq \frac{1}{2} + \frac{1}{\sqrt{2}}.
\]

Case 4: Each of $X, Y, Z$ is a nonscalar matrix. As in case 3, let $X = |x\rangle \langle x|$ and $Y = |y\rangle \langle y|$ and let $\theta = \arccos(|x \cdot y|)$.

Let
\[
C = I + \left(\frac{1}{2} + \frac{\cos \theta - \sin \theta}{2}\right) Z + \left(\frac{1}{2} - \frac{\cos \theta - \sin \theta}{2}\right) Z^\perp.
\]
Then, the operator norm of the difference
\[
(X + Y + Z) - C = (X + Y) - \left(\frac{3}{2} + \frac{\cos \theta - \sin \theta}{2}\right) I
\]
is $\frac{1}{2} + \frac{\cos \theta + \sin \theta}{2}$, which is less than or equal to $\frac{1}{2} + \frac{1}{\sqrt{2}}$. A similar calculation shows that the distance from $C$ to each of the other three elements of set (12) is equal to $\frac{1}{2} + \frac{\cos \theta + \sin \theta}{2}$. This completes the proof. \qed

For any angle $\theta \in \mathbb{R}$, let $P_\theta : \mathbb{C}^2 \to \mathbb{C}^2$ denote projection onto the unit vector $\cos(\theta)|0\rangle + \sin(\theta)|1\rangle$. Consider the set
\[
\{P_0 + P_{\pi/4} + I, P_0 + P_{3\pi/4}, P_{\pi/2}, P_{3\pi/4}, P_{\pi/2} + P_{3\pi/4} + I\}
\]
A direct calculation shows that the distance from the operator $(\frac{3}{2}) I$ to set (20) is $\frac{1}{2} + \frac{1}{\sqrt{2}}$. The next lemma asserts that this quantity is in fact the radius of (20).

Lemma 4. The radius of the set
\[
\{P_0 + P_{\pi/4} + I, P_0 + P_{3\pi/4}, P_{\pi/2} + P_{3\pi/4}, P_{\pi/2} + P_{3\pi/4} + I\}
\]
is is equal to $\frac{1}{2} + \frac{1}{\sqrt{2}}$.

Proof. For any Hermitian operator $H : \mathbb{C}^2 \to \mathbb{C}^2$, let us write $\overline{H}$ to denote the trace-zero operator $H - (\text{Tr}(H))I/2$. In the proof that follows, we will make use of the following fact: for any two Hermitian operators $Q, R : \mathbb{C}^2 \to \mathbb{C}^2$,
\[
||Q - R|| = |\text{Tr}(Q) - \text{Tr}(R)| + ||\overline{Q} - \overline{R}||
\]
Suppose, for the sake of contradiction, that there exists a Hermitian operator $Z$ whose distance from each of the elements of set (20) is strictly less than $\frac{1}{2} + \frac{1}{\sqrt{2}}$. Then,

$$\begin{align*}
2 \left( \frac{1}{2} + \frac{1}{\sqrt{2}} \right) &> \| (P_0 + P_{3\pi/4} - Z \| + \| (P_{\pi/2} + P_{\pi/4} - Z \|
= \| (P_0 + P_{3\pi/4} - Z \| + \| (P_{\pi/2} + P_{\pi/4} - Z \| + 2 \cdot |2 - \text{Tr}(Z)|
\geq \| (P_0 + P_{3\pi/4} - (P_{\pi/2} + P_{\pi/4}) \| + 2 \cdot |2 - \text{Tr}(Z)|
= \sqrt{2} + 2 \cdot |2 - \text{Tr}(Z)|.
\end{align*}$$

Therefore, $\text{Tr}(Z) < \frac{3}{2}$. Similarly,

$$\begin{align*}
2 \left( \frac{1}{2} + \frac{1}{\sqrt{2}} \right) &> \| (P_0 + P_{\pi/4} + Z \| + \| (P_{\pi/2} + P_{3\pi/4} - I \|
= \| (P_0 + P_{\pi/4} - Z \| + \| (P_{\pi/2} + P_{3\pi/4} - Z \| + 2 \cdot |3 - \text{Tr}(Z)|
\geq \| (P_0 + P_{\pi/4} - (P_{\pi/2} + P_{3\pi/4}) \| + 2 \cdot |3 - \text{Tr}(Z)|
= \sqrt{2} + 2 \cdot |3 - \text{Tr}(Z)|,
\end{align*}$$

which implies $\text{Tr}(Z) > \frac{5}{2}$. This is a contradiction. \(\square\)

Combining Lemmas [2][3] we have the following proposition.

**Proposition 5.** The quantity $\text{Succ}_{Q(2)}(M)$ is equal to $\frac{2}{3} + \frac{1}{3\sqrt{2}}$. \(\square\)

4. **An example of optimal non-signaling assistance**

In this section we discuss an example in which equality occurs in Theorem 5 from the main text. This example is a generalization of the protocol from [2].

Let $m$ be a positive integer. Let

$$\begin{align*}
Z &= \mathbb{F}_2^m, \\
W &= (\mathbb{F}_2^m \setminus \{0\}) \times \mathbb{F}_2.
\end{align*}$$

Let $K$ be a channel defined as follows:

1. The input alphabet of $K$ is $Z$, and the output alphabet of $K$ is $W$.
2. For any given input $v \in \mathbb{F}_2^m$, the output of $K$ is uniformly distributed over the set

$$\{ (w, w \cdot v) \mid w \in \mathbb{F}_2^m \setminus \{0\} \}.$$  

(Here, $w \cdot v \in \mathbb{F}_2$ denotes the inner product of $w$ and $v$.)

Let $(E_1, E_2)$ be a two part input-output device defined as follows. (See Figure 1)

1. The input alphabet for $E_1$ is $\mathbb{F}_2$, and the output alphabet for $E_1$ is $Z$.
2. The input alphabet for $E_2$ is $W$, and the output alphabet for $E_2$ is $\mathbb{F}_2$.
3. If the inputs to $E_1$ and $E_2$ are $a \in \{0, 1\}$ and $(w, r) \in (\mathbb{F}_2^m \setminus \{0\}) \times \mathbb{F}_2$, then the output of $E_1$ is uniformly distributed over all vectors $a = (a_1, a_2, \ldots, a_m)$ that satisfy $a_1 = a$, and the output of $E_2$ is $a \oplus r \oplus (w \cdot a)$.

It can be checked that the correlation $E$ arising from $(E_1, E_2)$ is non-signaling. Additionally, one can see (by substitution) that using $E$ to assist $K$ yields a perfect transmission of a single bit. (See figure [2])

Now, let us calculate the quantity $\text{Succ}(K)$. For any two distinct vectors $x_0, x_1 \in \mathbb{F}_2^m$, the probability that a randomly chosen vector $w \in \mathbb{F}_2^m \setminus \{0\}$ will satisfy
w \cdot x_0 \neq w \cdot x_1 is equal to \( \frac{2^{m-1}}{2^m - 1} \). This fact has the following consequence: if Alice employs the deterministic encoding strategy \([0 \mapsto x_0, 1 \mapsto x_1]\) to send a single bit, then the optimal probability with which Bob can decode is

\[
\frac{2^{m-1}}{2^{m-1}} (1) + \left[ \frac{2^{m-1} - 1}{2^m - 1} \right] \left( \frac{1}{2} \right)
\]

(26)

\[
= \frac{2^m + 2^{m-1} - 1}{2^{m+1} - 2}.
\]

Therefore, \( \text{Succ}(K) \) is equal to quantity (27), while \( \text{Succ}_{\text{NS}}(K) \) is equal to 1. Theorem 5 from the main text asserts the following bound on \( \text{Succ}_{\text{NS}}(K) \):

\[
\text{Succ}_{\text{NS}}(K) \leq \frac{1}{2} + \left( 2 - \frac{2}{2^m} \right) \left[ \text{Succ}(K) - \frac{1}{2} \right]
\]

\[
= \frac{1}{2} + 2 \left( \frac{2^m - 1}{2^m} \right) \left( \frac{2^{m-1}}{2^{m+1} - 2} \right)
\]

(27)

\[
= 1.
\]

Therefore, equality is achieved in Theorem 5 from the main text when \( N = K \).

5. The Local Fraction of a Binary Quantum Correlation

In this section, we prove the following proposition from the main text.

**Proposition 6.** Let \( D \) be a binary quantum correlation. Then \( \text{loc}(D) \geq 2 - \sqrt{2} \).

**Proof.** For any binary non-signaling correlation \( G \), let

\[
f_1(G) = \sum_{a,x,b,y \in \{0,1\}} (-1)^{x \oplus b \oplus (a \wedge y)} G(xb|ay).
\]

This is the function which defines the CHSH inequality [1]. let \( f_2 \), \( f_3 \), and \( f_4 \) be the functions defined by the same expression with \( a \wedge y \) replaced by \( \neg a \wedge y \), \( a \wedge \neg y \), and \( \neg a \wedge \neg y \), respectively.
We note the following facts. (See [3].)

(1) A non-signaling correlation $G$ is local if and only if $-2 \leq f_i(G) \leq 2$ for $i = 1, 2, 3, 4$.
(2) If $G$ is a quantum correlation, then for $i = 1, 2, 3, 4$,
$$-2\sqrt{2} \leq f_i(G) \leq 2\sqrt{2}.$$
(3) There are eight non-signaling correlations $\{P^+_i\}_{i=1}^4$ and $\{P^-_i\}_{i=1}^4$, satisfying
$$f_j(P^+_i) = \begin{cases} 
\pm 4 & \text{if } j = i \\
0 & \text{otherwise}
\end{cases}$$

These are the Popescu-Rohrlich (PR) boxes.
(4) Every non-signaling correlation is a convex combination of local correlations and the eight PR boxes. Further, for any two distinct PR boxes $P$ and $P'$, the correlation $(P + P')/2$ is local.

From the second part of item [3] it follows that any convex combination of local boxes and PR boxes can be simplified into an expression of the form $\alpha L + (1 - \alpha)Q$, where $L$ is local, $Q$ is a PR box, and $\alpha \in [0, 1]$. Any non-signaling correlation can thus be expressed as a convex combination of a local correlation and a single PR box.

Let $D = \alpha L + (1 - \alpha)Q$, where $L$ is local and $Q$ is a PR box. First suppose that $Q = P^+_j$. Let $L_\beta = (\alpha L + (\beta - \alpha)P^+_j)/\beta$ for any $\beta \in [\alpha, 1]$. Then $L_\beta$ is local whenever $f_j(L_\beta) \leq 2$. If $f_j(L_1) < 2$, then $L_1(= D)$ is local, and the proposition follows easily. Otherwise, there is a value $\beta \in [\alpha, 1]$ such that $f_j(L_\beta) = 2$. We have $D = \beta \cdot L_\beta + (1 - \beta)P^+_j$. The quantity $\beta$ must be at least $2 - \sqrt{2}$, since otherwise (2) would be violated. Therefore $\text{loc}(D) \geq 2 - \sqrt{2}$.

A similar argument completes the proof in the case where $Q = P^-_j$. \qed

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