Air expansion in the water rocket

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We study the thermodynamics of the water rocket in the thrust phase, taking into account the expansion of the air with water vapor, vapor condensation and the energy taken from the environment. We set up a simple experimental device with a stationary bottle and verified that the gas expansion in the bottle is well approximated by a polytropic process $PV^\beta = \text{constant}$, where the parameter $\beta$ depends on the initial conditions. We find an analytical expression for $\beta$ that only depends on the thermodynamic initial conditions and is in good agreement with the experimental results.

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I. INTRODUCTION

The water rocket is a popular toy that is already several decades old [1]. Its inexpensive version can be built with a plastic bottle filled with water and pressurized air obtained with a bicycle pump. When the bottle is opened, the internal air pressure pushes the water out and this causes an increase in the momentum of the bottle. The water ejection gives the bottle the thrust that allows it to move for several meters, see Ref. [2].

The launch of water rockets is used in our undergraduate laboratory to illustrate several physics concepts, such as: (a) Newton laws, (b) conservation of momentum, (c) work-energy theorem, (d) Bernoulli equation, (e) mass conservation equation and (f) ideal gas expansion. On this last point it is necessary to clarify whether the expansion of the gas in the rocket is an adiabatic or an isothermal process.

The theoretical description of the water rocket is not trivial and in general it is necessary to adopt certain premises to obtain a friendly caricature of reality. In particular, the rocket’s only source of energy, the air expansion, is in general modeled [2–5] as an isothermal or an adiabatic process involving dry air. The qualitative predictions of these theoretical approaches give reasonable results, however their quantitative description is incorrect, showing that some hypothesis of the model is inadequate. In particular observing the launch of the water rocket, we see that when the ejection of the water concludes, an additional ejection of fog follows. Therefore, it is clear that the expansion occurs for a mixture of dry air, water vapor and condensed water (fog). In reference [2] the author deepens in this point and he builds a pressure-volume relation assuming that the total entropy of the mixture is conserved during the expansion; he additionally shows that the solution of the pressure-volume equation can be approximated by a polytropic process.

II. POLYTROPIC PROCESS AND EXCHANGE OF ENERGY

We present here a brief revision of the polytropic process and the calculation of some quantities of heat involved. A polytropic process is a quasistatic process carried out in such a way that the specific heat remains constant during the entire process [7–9]. Therefore the heat exchange when the temperature changes is

$$\frac{dQ}{dT} = Nc,$$

where $Q$ is the heat absorbed by the system, $T$ is the temperature, $c$ is a constant specific heat and $N$ is the number of moles. Pressure and volume can change arbitrarily in this process, therefore it may be thought as a polytropic process with infinite specific heat.

Using the first law of thermodynamics together with the internal energy of an ideal gas $U = c_vNT$ and Eq. (1),
we have for an infinitesimal polytropic process
\[(e_v - c)NdT + PdV = 0,\] (2)
where \(P\) is the pressure and \(V\) the volume. From Eq. (2) and using the ideal gas equation of state, it is easy to prove that the polytropic process satisfies
\[PV^\beta = constant,\] (3)
where
\[\beta = \frac{c_p - c}{c_v - c}.\] (4)

Usually thermodynamic textbooks define a polytropic process through Eq. (3) (for example \[11\]). However we have started with Eq. (1) which gives a direct link between heat and temperature changes. In particular for an adiabatic process \(c = 0\) and \(\beta = \gamma \equiv c_p/c_v\) and for an isothermal process, \(c = \infty\) and \(\beta = 1\). For the process treated in this paper we find that \(1 < \beta < \gamma\) where \(\gamma = 1.4\) for dry air at room temperature. The specific heat as a function of \(\beta\) is
\[c = \frac{c_p - \beta c_v}{1 - \beta}.\] (5)
The total heat \(Q\) exchanged between the gas and the environment can be calculated integrating Eq. (1)
\[Q = cN(T - T_0),\] (6)
where \(T\) is the gas temperature and \(T_0\) is the environment temperature, which is also the water temperature and the initial temperature of the gas in the bottle. An experimental measurement of the time dependence of \(T\) determines the time dependence of \(Q\) through Eq. (6).

As the water is expelled from the bottle the gas expands and cools. In this process the relative humidity increases up to saturation and some of the vaporcondenses. The quantity of condensed water vapor at the end of the experiment will depend on the initial relative humidity in the gas and the initial and final temperatures. We want to estimate the mass and the heat involved in this condensation. The mass of water vapor \(m_v\) can be calculated using \[12\]
\[m_v = \frac{P_vV}{R_vT},\] (7)
where \(P_v\) is the partial pressure of the vapor at \(T\) and \(R_v = 0.4615 \text{ kJ/(kg K)}\) is the ideal gas constant for the vapor. The vapor partial pressure is determined by de equation \(P_v = \varepsilon w P_0\) where \(\varepsilon \in [0,1]\) is the fraction of relative humidity and \(w\) is the saturation pressure which depends only on the temperature and may be obtained from a vapor table. The pressurized air is generated by the compressor at a higher temperature than the environment and its cooling results in the saturation of the air with vapor inside the system. Thus we assume that we are working with \(w = 1\) during all the course of the experiments, then \(P_v = w w P_0\). This assumption implies the maximization of the energy delivered by the vapor-liquid phase transition. In our case the initial temperature is \(T_0 = 292 \text{ K}\) and partial pressure is \(P_w = 2.3 \text{ kPa}\) (this last value was taken from a vapor table).

In order to get a fuller understanding about the heat exchange due to vapor condensation we start with the Clapeyron-Clausius equation \[11\] for the vapor-liquid phase transition
\[\frac{d\varepsilon w}{dT} = \frac{L_v \varepsilon w}{R_v T^2},\] (8)
where \(L_v\) is the latent heat, which for our purposes may be considered constant, \(L_v = 2500 \text{ kJ/kg}\). The solution of Eq. (8) gives the partial pressure \(P_v = \varepsilon w\) as a function of the temperature \(T\)
\[P_v = P_{w_0} e^{\frac{L_v}{R_v T_0}}(\frac{T}{T_0} - \frac{1}{\gamma - 1}),\] (9)
where the constant was adjusted so that \(P_v(T_0) = P_{w_0}\).

The condensation produces a change in the mass of vapor in the gas. The mass of condensed vapor can be calculated using Eq. (7) twice, that is
\[M = \frac{P_{v_0} V_0}{R_v T_0} - \frac{P_v V}{R_v T}.\] (10)

It is easy to prove that the polytropic process is also characterized by the equation \(PT^\beta = constant\). Then,
\[\frac{V}{T} = \frac{NR}{P} = \frac{NR}{P_0} \left(\frac{T_0}{T}\right)^\beta,\] (11)
where \(R\) is the ideal gas constant. To obtain the latent heat of condensation \(Q_v = ML_v\) we substitute Eqs. (9,11) into Eq. (10) that is
\[Q_v = \frac{NR P_{v_0} L_v}{R_v P_0} \left[1 - e^{\varepsilon w P_0 (1 - \frac{T_0}{T}) \left(\frac{T}{T_0}\right)^\beta}\right].\] (12)

One may wonder if there are other types of heat exchange between the system and the environment. As will be seen our experiment lasts for less than one second, then it is very reasonable to assume that the heat exchanged through the walls is not important, the energy delivered by the vapor-liquid phase transition remains as the sole source of heat for the system.

It is important to point out that, due to the water-vapor condensation, the number of moles \(N\) in the gas is not constant. However, Eqs. (6,12) were obtained using the ideal gas law for air and water vapor, which assume that \(N\) and \(N_v\) (the number of moles of water vapor) are constants. Therefore, Eqs. (6,12) may be used only if the variation of moles is negligible during condensation. The relative number of moles condensed \(\delta N_v/N_v\) can be estimated using Eqs. (7,10)
\[\frac{\delta N_v}{N_v} = \frac{M}{m_{v_0}} = 1 - \left(\frac{T_0}{T}\right)^\beta e^{\varepsilon w P_0 (1 - \frac{T_0}{T}) \left(\frac{T}{T_0}\right)^\beta}.\] (13)
where \( m_{w_0} \) is the initial water vapor mass. If we estimated the final temperature as \( T = T_0 - 10 \) K then Eq. (13) gives us the value \( \delta N_v/N_v \sim 0.6 \). The total relative variation of moles in the gas \( \delta N_v/N \) is

\[
\frac{\delta N_v}{N} = \frac{\delta N_v}{N_v} = \frac{P_{v_0} \delta N_v}{P_0 N_v} \sim 10^{-3},
\]

where we have used the ideal equation gas both for the gas as for the water vapor. Therefore we have the following situation, \( \delta N_v/N \) is negligible but \( \delta N_v/N_v \) is not. In the deduction of Eq. (12), we have used Eq. (7) which is correct only if \( \delta N_v/N_v \sim 0 \). Then, it is clear that Eq. (12) can only be correct in the limit \( T \to T_0 \). Therefore, we must substitute Eq. (12) by the following exact differential equation

\[
dQ_v = \frac{L_v}{T_0 R_v} \frac{P_{v_0} N R}{P_0} \left( \frac{\beta}{1 - \beta} + \frac{L_v}{R_v T_0} \right) dT,
\]

that it is obtained replacing \( T = T_0 + dT \) in Eq. (12) and taking the terms to first order in \( dT \). Now it is possible to equate \( dQ_v \) with \( dQ \) taken from Eq. (4)

\[
dQ = dQ_v,
\]

and using Eqs. (5,15) in Eq. (10) lead to

\[
\frac{c_p/R - \beta c_v/R}{1 - \beta} = \frac{L_v}{T_0 R_v} \frac{P_{v_0} N R}{P_0} \left( \frac{\beta}{1 - \beta} + \frac{L_v}{R_v T_0} \right).
\]

This last equation gives a theoretical expression to obtain the constant parameter \( \beta \),

\[
\beta = \frac{c_p/R + (L_v/R_v T_0)^2 (P_{v_0}/P_0)}{c_v/R + (L_v/R_v T_0)^2 (P_{v_0}/P_0) - (L_v/R_v T_0) (P_{v_0}/P_0)}.
\]

This is the main theoretical result in this paper and from its analysis we conclude: (a) If \( P_{v_0} \sim 0 \) or \( L_v \sim 0 \) then \( \beta \sim \gamma = \frac{m_{v_0}}{m_{v_0} + \rho_{w_0}} \), this means that the process is adiabatic for dry air. (b) \( \beta \) depends only on the thermodynamical initial conditions of the gas, this is unexpected since other parameters like the water ejection time do not appear. Table I shows the theoretical values of \( \beta \) Eq. (18), which we call \( \beta \), computed for our experimental initial conditions.

It is also interesting to point out that the model developed in Ref. [8] also predicts that the exponent \( \beta \) depends mostly on the initial relative humidity and proposes the empirical function

\[
\beta = 1.15 + (1.4 - 1.15) \exp(-36P_{v_0}/P_0).
\]

Fig. (1) shows \( \beta \) (Eqs. (18,19)) as a function of the initial ratio \( P_{v_0}/P_0 \) (that is the molar fraction), the figure also shows the experimental values obtained in the present paper.

### III. EXPERIMENTAL RESULTS

In order to elucidate the nature of the expansion of the air in the water rocket we designed an experimental device to measure the heat exchange between the compressed gas of the water rocket and the environment. The water rocket is fixed to the laboratory frame and then the time dependence of the pressure and volume of the expanding gas in the bottle is measured. The stationary bottle is depicted in Fig. (2). The bottle is a transparent
acrylic cylinder of inner diameter 5.40 cm, wall thickness 0.30 cm and length 1 m, that is maintained in vertical position. The upper and lower covers are both of aluminum. The upper one has a tire valve and a manometer (WIKA EN 837 – 1) working from 0 to 4 bar. The lower cover has a conical shape with a slope of 30° with a 0.5” faucet that allows to fill and evacuate the water in the bottle. The device is quite simple and the experiment is not dangerous for the range of pressures used in this paper, so it is appropriate for undergraduate students. In a typical trial the bottle is filled with about 1400 cm$^3$ of water and the rest with air. It is placed in vertical position with the faucet closed. The compressed air is introduced through the valve and then the faucet is opened. The level and pressure of the gas is filmed with a PIXELINK PL-B776F camera. The video was obtained at 50 frames per second, with a exposure time of 10 ms per frame. The ejection of water from the bottle lasted between 0.64 s to 0.88 s, depending of the initial pressure, see Tab. I. Finally the filming was processed manually to obtain the pressure and volume of the gas as functions of time.

Fig. 3 shows the results for five sets of experimental data of pressure and volume. Each set corresponds to different initial pressures with the same initial volume; for each initial pressure the experiment was repeated five times and all the data are incorporated in the same figure. The data are adjusted with a power law $PV^\beta =$constant. The values of $\beta$, the slope of the adjustment in the figure, are given in Tab. I. The experimental values of the polytropic exponents $\beta$ are in very good agreement with their theoretical values $\beta^*$ given by Eq. (18).

\[
\begin{array}{ccccc}
P_0/P_A & \beta & \beta^* & c/R & t(s) & Q (J) \\
4.8 & 1.22 & 1.27 & -2.0 & 0.64 & 161.7 \\
4.4 & 1.24 & 1.26 & -2.2 & 0.66 & 158.3 \\
4.0 & 1.20 & 1.25 & -2.6 & 0.75 & 154.7 \\
3.6 & 1.19 & 1.24 & -2.7 & 0.75 & 144.8 \\
3.2 & 1.17 & 1.23 & -3.5 & 0.88 & 144.9 \\
\end{array}
\]

TABLE I:

The fifth column of Tab.I shows the ejection time at which end the water is exhausted from the bottle. The heat $Q$ at the end of the process shown in the last column of Tab.I it is calculated using Eq.(6) the experimental data and the ideal gas equation.

IV. DISCUSSION AND CONCLUSION

The typical ejection time of the water in our experiments is 0.7 s and the typical ejection time in the usual water rocket is 0.1 s [2]. The difference in the ejection rates must be analyzed: (i) The main geometrical difference between the bottles is related to the nozzles. The nozzle section of the stationary bottle $a_1$ and the standard soda bottle $a_2 = 3.46 \times 10^{-4} \text{m}^2$ are linked by $a_1 = 0.4a_2$. A smaller nozzle was necessary in order to accommodate the capability of our video camera. (ii) Another difference between the experiments is the apparent gravity. In the stationary bottle the water is only subjected to the weight. In the water rocket, the water mass is subjected to a force which is the sum of its weight plus a fictitious force due to the rocket acceleration. The apparent gravitational field is several times larger than the Earth’s gravitational field. (iii) Finally, our theoretical calculation shows that $\beta^*$ does not depend on the water output rate, it only depends on the thermodynamic initial conditions of the gas. Additionally $\beta^*$ agrees with our experimental values. Then we conclude that same process should occur in the water rocket with the same thermodynamic initial conditions.

In summary, we have presented an experimental study of the air expansion in the water rocket for the first time, using a stationary bottle. It is shown that the air expansion follows a polytropic process of the type $PV^\beta =$constant where $\beta$ is an average value with slight dependence on the initial conditions. Furthermore we obtained an analytical expression for the polytropic exponent which agrees reasonably with the experimental values. This theoretical exponent only depends on the thermodynamic initial conditions of the gas and it is also suitable to study the air expansion in the usual water rocket. Finally we conclude that the vapor condensation is the main energy source for the gas expansion process in the water rocket.

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