Forced-self-excited system of iced transmission lines under planar harmonic excitations

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Abstract This work involved analyzing the self-excited and forced vibrations of iced transmission lines. By introducing an external excitation load, the effect of dynamic wind on nonlinear vibration equations was reflected by the vertical aerodynamic force. The approximate analytic solution of the non-resonance of the forced-self-excited system was obtained using the multiple scale method. With an increase in excitation amplitude, the nonlinearity of the system was enhanced, and the forced-self-excited system experienced three vibration stages—namely, self-excited vibration, the superposition forms of self-excited and forced vibrations, and forced vibration controlled by nonlinear damping. Among these, the accuracy of the approximate analytic solution decreased with increase in nonlinear strength variations. When the excitation amplitude was greater than the critical value, the quenching phenomenon appeared in the forced-self-excited system, and the discriminant formula was derived in this work. In addition, the third-order Galerkin method, which considered the small sag effect, was used to discretize the nonlinear galloping governing equation. The response (principal resonance, harmonic resonance) of the forced-self-excited system was analyzed by time history displacement curves and phase diagrams. The conclusions of this work may contribute to the practical engineering of iced transmission lines. More importantly, as a combination of the Duffing equation and Rayleigh equation, the forced-self-excited system may have high theoretical research value.

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1 Introduction

The galloping of iced transmission lines consists of typical self-excited vibrations [1]. Under the effect of wind load, an iced transmission line will be prone to galloping for a long time [2]. As a result, the iced transmission line structure may short-circuit, cause frequent tripping, cause broken strands or wires, or cause other accidents [3]. Thus, to some extent, studying the nonlinear characteristics of transmission line galloping can have great engineering application value as well as significance for theoretical research.

Iced transmission lines consist of suspended flexible cables, and the stay cables of cable-stayed bridges fall under these types of cables as well. Thus, transmission lines and stay cables both feature cable structures. Under the action of external excitation, rain and wind-induced galloping [4], dry galloping [5], and ice galloping will occur in the stay cables of cable-stayed bridges.

Over the past few years, cable structures have been studied by various scholars, with some investigating suspended elastic cables and stay cables. Arafat and Nayfeh [6] investigated the nonlinear forced responses of shallow suspended cables, mainly including principal resonance and auto-parametric resonance. According to the different excitation forms of cable structures, the dynamic response (internal resonance response, principal resonance response, and harmonic resonance response) of a system may be excited [7–9]. For example, under the condition of external excitation, Nayfeh et al. [10] studied the nonplanar nonlinear response of a suspended cable by considering different symmetrical modes and different internal resonance conditions. Under the condition of harmonic excitation, Rega and Benedettini [11, 12] studied the plane nonlinear dynamic response (sub-harmonic resonance and super-harmonic resonance) of an elastic cable. Under parametric and external excitation conditions, Chen et al. [13] and Zhang et al. [14] studied the bifurcation and chaotic dynamics of suspension cables. In addition, cable structures with different boundary motion conditions have been found to generate unique nonlinear dynamic characteristics. For example, Perkins et al. [15] established a theoretical model of a suspension cable excited by support vibrations; experiments and theoretical analysis demonstrated that internal resonance tended to weaken the plane stability of the system. Sun et al. [16] studied the influence of support excitation with different phases on the responses of suspension cables. Meanwhile, Berlioz and Lamarque [17] carried out theoretical and experimental research on stay cables, proposing a model to predict nonlinear behavior. Based on the modified Irvine formula, Wu et al. [18] derived a formula for inclined cables with small sag, which could be utilized to calculate in-plane natural frequencies and the shapes of the modes. References [6–18] mainly studied the nonlinear vibration response characteristics of cable structures, such as principal resonance, internal resonance, and harmonic resonance, through experimental verification and theoretical modeling analysis. However, the nonlinear behavior of cable structures in dynamic-wind loads has rarely been studied. Zulli and Luongo [19, 20] considered the dynamics of two towers exposed to turbulent wind flow, linked by a nonlinear viscous device. They noted that the steady component of the wind was responsible for self-excitation, while the turbulent part caused both parametric and external excitations, supposed under a specific resonance condition.

Based on the analyses in references [6–20], this work investigated the galloping characteristics of iced transmission lines under dynamic-wind loads. The hysteretic nonlinear restoring force term was considered, which included the quadratic and cubic nonlinear terms, as well as the Rayleigh damping term. In reference [21], the effect of dynamic wind on the structure of iced transmission lines was studied by analyzing the forced-self-excited system, while the nonlinear response characteristics under different harmonic excitations were discussed. Based on reference [21], the present work examined the influence of varying excitation amplitudes on a forced-self-excited system, providing a systematic analysis of the harmonic resonance of the forced-self-excited system using numerical techniques. With variations in excitation amplitude, the vibrations of the forced-self-excited system presented three stages: (1) pure self-excited vibrations (excitation amplitude was equal to
0), (2) the superposition form of self-excited and forced vibrations, and (3) forced vibrations affected by Rayleigh damping, known as the quenching phenomenon, where the excitation amplitude was greater than the critical value. In addition, we identified the damage levels of different resonance forms for the iced transmission line system. The damage of principal resonance to the transmission line was greater than that of super-harmonic resonance, while the damage of super-harmonic resonance to the transmission line was greater than that of sub-harmonic resonance. The findings of this work may be helpful for the galloping super-harmonic resonance to the transmission line was greater than the critical value. In addition, we identified the damage levels of different resonance forms for the iced transmission line system. The damage of principal resonance to the transmission line was greater than that of super-harmonic resonance, while the damage of super-harmonic resonance to the transmission line was greater than that of sub-harmonic resonance. The findings of this work may be helpful for the galloping analysis of iced transmission lines. This work could also serve as a reference for practical engineering.

The rest of this paper is structured as follows. In Sect. 2, under the condition of dynamic-wind loads, the mathematical formulation of an iced transmission line system (forced-self-excited system) is presented, where the galloping governing equation of the system was obtained using the Galerkin method. In Sect. 3, the approximate analytic solution of the forced-self-excited system was obtained using the multiple scale method, and the three vibration stages of the system response were analyzed. In Sect. 4, the nonlinear response of the forced-self-excited system under harmonic excitation with different frequencies was obtained using the Runge–Kutta function, where the system dynamic responses under different excitation amplitudes were analyzed systematically. A detailed summary of the results is presented in Sect. 5.

2 Dynamic model of iced transmission lines

In this work, we used the same physical model as reference [21]. Figure 1 shows the model of single-span transmission lines with equal height, where \( u(s,t) \) and \( v(s,t) \) are the displacements measured from the dynamic equilibrium configuration in the \( x \)-axis and \( y \)-axis directions, respectively; \( p(s)\cos(\Omega t) \) is the external excitation. Because the iced transmission lines have a small initial sag-to-span ratio (less than 1:8) and low static strain, the associated static equilibrium configuration of iced transmission lines can be described through the parabola \( y = 4d[s/L-(s/L)^2] \) (\( s \) is the curvilinear abscissa; \( L \) is the span length; and \( d \) is the sag) [22]. In addition, the initial static equilibrium configuration was \( \zeta_1 \), with a dynamic configuration of \( \zeta_2 \). We considered that the wind was along the \( z \)-axis, as shown in Fig. 2.

Figure 2 shows the model of a cross section for the mid-span micro-segment \( ds \) of the iced transmission line model shown in Fig. 1. The micro-segment \( ds \) was taken as the research object in this work, and Fig. 2a plots the physical model of the iced transmission line cross section. Figure 2b depicts the aerodynamic forces on the cross section of the iced transmission line, where the \( z_2 \)-axis is the axis of symmetry of the cross section, \( O_1z_1 \) is the direction in which the horizontal wind acts on the iced transmission line during vibrations, the \( z \)-axis is the horizontal axis, \( z \) is the wind attack angle, \( \alpha_0 \) is the initial wind attack angle, \( \alpha_t \) is the relative wind attack angle, \( U \) is the steady wind velocity, \( U_0 \) is the relative wind velocity, and \( \dot{v} \) is the vertical vibration velocity. Aerodynamic drag \( F_D \) and lift \( F_L \) will be caused by the relative wind velocity \( U_0 \).

According to Fig. 2, we could obtain:

\[
\tan(\alpha) = \frac{\dot{v}}{U} \approx \alpha, \tag{1}
\]

where Eq. (1) is based on the quasi-static assumption, ignoring the influence of conductor motion on the wind field [23, 24]. Relative wind will act on the iced transmission lines, resulting in aerodynamic drag \( F_D \) along the relative wind direction and an aerodynamic lift \( F_L \) perpendicular to the relative wind direction. According to fluid-induced vibration theory, the expressions of \( F_L \) and \( F_D \) can be listed as [25]:

\[
\begin{align*}
F_L &= \rho U^2 C_L / 2; \\
F_D &= \rho U^2 C_D / 2; \\
F_\gamma &= F_L \cos(\alpha) - F_D \sin(\alpha) = \rho U^2 C_\gamma / 2, \tag{2a, b}
\end{align*}
\]

where \( C_L \) is the aerodynamic lift coefficient, \( C_D \) is the aerodynamic drag coefficient, \( C_\gamma \) is the vertical coefficient of aerodynamics, \( F_\gamma \) is the vertical aerodynamic effect of the static wind, \( \rho \) is the air density, and \( D \) is the diameter of the iced transmission lines.

The relationship of the vertical aerodynamic coefficient in Eq. (2) is obtained by

\[ ...\]
\[ C_y = C_L \cos(x) - C_D \sin(x). \]  

In Eq. (3), the aerodynamic coefficients \( C_L \) and \( C_D \) can be fitted into a quintic nonlinear curve related to \( x \) by mathematical software, and considering small deformation, where \( \sin(x) \approx x, \cos(x) \approx 1 \). Based on the Taylor’s law, the aerodynamic coefficients \( (C_y) \) can also be written as

\[
C_y = \frac{1}{0!} C_L + \frac{1}{1!} \left( \frac{\partial C_L}{\partial x} - C_D \right) x
+ \frac{1}{2!} \left( \frac{\partial^2 C_L}{\partial x^2} - C_L - 2 \frac{\partial C_D}{\partial x} \right) x^2
+ \frac{1}{3!} \left( \frac{\partial^3 C_L}{\partial x^3} - 3 \frac{\partial C_L}{\partial x} - 3 \frac{\partial^2 C_D}{\partial x^2} + C_D \right) x^3. \tag{4}
\]

The aerodynamic coefficient \( (C_L, C_D) \) of the iced transmission lines were measured experimentally. According to Eq. (4), the coefficient \( C_L / 0! \) will be irrelevant to the vertical vibration velocity \( \dot{v} \), where the coefficients of \( x, x^2 \), and \( x^3 \) denote the undetermined coefficients related to the aerodynamic loads, and the coefficient of \( x^2 \) is unrelated to the wind velocity; thus, in this work, only the aerodynamic coefficients of \( x \) and \( x^3 \) are considered.

Under the effect of wind load, iced transmission lines will contain self-excited vibration with a constant amplitude. However, in practice, the effect of dynamic-wind load will be unstable. Based on this concept, combined Eqs. (1–4) along with reference [21], the vertical aerodynamic effect of the dynamic wind can be expressed by

\[
F_{y1} = \ddot{a} \dot{v} + \ddot{b} \dot{v}^3 - p* \cos(\Omega t). \tag{5}
\]

The parameters in Eq. (5) are as follows:

\[
\ddot{a} = \rho UD \left( \frac{\partial C_L}{\partial x} - C_D \right) / 2,
\]

\[
\ddot{b} = \rho D \left( \frac{\partial^3 C_L}{\partial x^3} - 3 \frac{\partial C_L}{\partial x} - 3 \frac{\partial^2 C_D}{\partial x^2} + C_D \right) / 12U. \tag{6}
\]

According to nonlinear vibration theory, when the response contains sub-harmonic and super-harmonic components, the relationship between the restoring force term and displacement of the system is a nonlinear closed curve, it will manifest nonlinear hysteretic characteristics. The simplified vertical aerodynamic force \( F_{y1} = \ddot{a} \dot{v} + \ddot{b} \dot{v}^3 - p* \cos(\Omega t) \) of dynamic wind will include the external excitation term \(-p* \cos(\Omega t)\) and the Rayleigh damping term \( \ddot{a} \dot{v} + \ddot{b} \dot{v}^3 \). In addition, the nonlinear hysteretic force of the system will consist of the Rayleigh damping, quadratic, and cubic nonlinear restoring force terms. The stable part of dynamic wind was injected into the iced transmission line to generate self-excited vibration, as the unstable part would cause external excitation [19, 20]. As indicated in Eq. (5), the average wind (stable part) in the natural wind is expressed as the Rayleigh damping term \( F_{y} = \ddot{a} \dot{v} + \ddot{b} \dot{v}^3 \), where the unstable part of the dynamic wind was simplified as the effect of the external excitation term on the iced transmission line system. According to the vertical galloping mechanism proposed by Den Hartog in reference [26, 33], and combined with Eq. (5), we could easily obtain the governing equation of the vertical movement of iced transmission lines under the effect of dynamic wind:

\[
\{H\ddot{v} + (ES/L)(\dot{y}' + v') \int \frac{L}{0} [\dot{y}' v' + v'^2 / 2] dx \}'
- F_{y1} + p* \cos(\Omega t) - \mu \ddot{v} = m \ddot{v}, \tag{7}
\]

where \( H \) is the tension of iced transmission lines, \( E \) is the Young’s Modulus of iced transmission lines, and \( L \) and \( S \) denote the spans and cross-sectional area of iced transmission lines, respectively. \( v' \) is the first derivative of the vertical motion function with respect to \( x \), \( y' \) is the first derivative of the parabolic equation with respect to \( x \), \( \dot{v} \) and \( \ddot{v} \) denote the first derivative and
second derivative of the vertical motion function with respect to time $t$, respectively, $\mu$ is the structural damping, and $m$ is the self-weight per unit unstretched length.

Because the contribution of the fourth-order modal response to the system response could be ignored \cite{27}, we only considered the contribution of the first three-order modes to the vibration of iced transmission lines. The displacement $v(x, t)$ and the excitation term $p_\ast$ in Eq. (7) can be written as

$$p_\ast = f_0(x)p; v(x, t) = f_1(x)q_1(t) + f_2(x)q_2(t) + f_3(x)q_3(t)$$ \hfill (8)

where

$$f_2(x) = \zeta_2 \sin(2\pi x/L),$$ \hfill (9a)

$$f_n(x) = (32\zeta_n ESd^2/m_{0n}^2 L^5)[1 - \tan(\omega_0 \sqrt{m/H/2}) \sin(\omega_0 \sqrt{m/Hx}) - \cos(\omega_0 \sqrt{m/Hx})](n = 1, 3),$$ \hfill (9b)

where $f_n(x)$ denote the $n$th modal functions of iced transmission lines, considering small sag effect \cite{28}. When $n$ is odd, the system mode is symmetrical, as $n$ is even, and the system mode was antisymmetric. In Eq. (9b), the coefficient $\zeta_n$ could be obtained using the modal normalization method, with $\omega_0$ is the frequency of transmission line system ignoring damping and nonlinear restoring force. The frequency $\omega_0$ is related to the span, static tension, sag, other factors, which could be obtained by the transcendental equation

$$\tan(\omega_0 L \sqrt{m/H/2}) = \omega_0 L \sqrt{m/H/2} - \omega_0 L^3 \sqrt{m/H/2(2\lambda^2 H)}.$$ \hfill (10)

The Irvine parameter is $\lambda = 512 E A d^3/(mgL^4)$. In addition, the uniform distributed load ($p_\ast$) in Eq. (7) was considered, where $f_0(x) = 1$ is the modal function of external excitation, $q(t)$ is the displacement function, and $q_n(t)(n = 1, 2, 3)$ denote the displacement functions of the modal functions $f_n(x)(n = 1, 2, 3)$.

To obtain the analytical solution of non-resonant response, the single-degree-of-freedom model was given first. Based on the Galerkin method, ignoring the mode-coupling effects, the nonlinear partial differential equation could be obtained by substituting Eqs. (8) and (9b) into Eq. (7), according to

$$\ddot{q} + \omega^2 q + c_1 q^2 + c_2 q^3 + (\mu - c_3) \ddot{q} + c_4 q^3 = p \cos(\Omega t),$$ \hfill (11a)

where the coefficient $\omega$ is the natural frequency of transmission lines, $c_1$ is the coefficient of the quadratic restoring force ($q^2$), $c_2$ is the coefficient of the cubic restoring force ($q^3$), where $c_1$, $c_2$ are related to structural physical parameters. $c_3$ is the coefficient of the linear aerodynamic damping term ($\ddot{q}$), $c_4$ is the coefficient of the cubic aerodynamic damping term ($q^3$), where $c_3$ and $c_4$ are related to environmental parameters and aerodynamic shape of the structure, and $\mu$ is the coefficient of the structural damping term. Equation (10) was used to obtain the transmission line vibration response dominated by a single mode.

The quadratic nonlinearities, due to geometry (or initial sag) and cubic nonlinearities, were due to stretching of the centerline \cite{28, 29}. Then, Eq. (10) could be regarded as the combined form of the Duffing equation and the Rayleigh damping term \cite{21}.

A three-degree-of-freedom model was proposed for analyzing the modal interactions in coupled response of first three-order modes. Similarly, based on the Galerkin method, $n = 3$, where the three-degree-of-freedom model in Eq. (8) was obtained by substituting Eqs. (8–9) into Eq. (7):

$$\ddot{q}_1 + \omega^2 \dot{q}_1 + c_{5a} \dot{q}_2 + c_{6a} \dot{q}_3 + c_{7a} q_1^2 + c_{8a} q_2^2 + c_{9a} q_3^2 + c_{10} \dot{q}_1 q_2 + c_{11} \dot{q}_1 q_3 + c_{12} \dot{q}_2 q_3 + c_{13} q_1^3 + c_{14} q_1 q_2^2 + c_{15} q_1 q_3^2 + c_{16} q_2^2 q_2 + c_{17} q_1^2 q_3 + c_{18} \dot{q}_1 q_2 q_3 + c_{19} \dot{q}_1 q_3^3 + c_{20} \dot{q}_2 q_3^2 + c_{21} \dot{q}_1 q_2 q_3^3 + c_{22} \dot{q}_2 \dot{q}_1 q_3^2 + c_{23} \dot{q}_2 \dot{q}_3 q_1 + c_{24} \dot{q}_1^3 q_2 + c_{25} \dot{q}_1^2 q_2 q_3 + c_{26} \dot{q}_1 \dot{q}_2 q_3^2 + c_{27} \dot{q}_2 q_1^3 + c_{28} \dot{q}_1^2 q_2^2 + c_{29} \dot{q}_1 q_2^3 + c_{30} \dot{q}_2 q_1 q_3^2 + c_{31} \dot{q}_2 q_3 q_1^2 + c_{32} \dot{q}_1 q_2 q_3^3 + c_{33} \dot{q}_1 \dot{q}_2 \dot{q}_3 + c_{34} \dot{q}_1 \dot{q}_2 \dot{q}_3^2 + c_{35} \dot{q}_1 \dot{q}_2 \dot{q}_3^3 q_1 + c_{36} \dot{q}_2 q_1^3 q_3 + c_{37} \dot{q}_2 q_1^2 q_3^2 + c_{38} \dot{q}_2 q_1 q_3^3 + c_{39} \dot{q}_2 q_3 q_1^2 + c_{40} \dot{q}_2 q_3 q_1 q_3 + c_{41} \dot{q}_2 q_3 q_1^2 q_3 + c_{42} \dot{q}_2 q_3^2 q_1 + c_{43} \dot{q}_2 q_3 q_1 q_3 + c_{44} \dot{q}_2 q_3 q_1^2 + c_{45} \dot{q}_2 q_3 q_1^2 q_3 + c_{46} \dot{q}_2 q_3 q_1 q_3 + c_{47} \dot{q}_2 q_3^2 q_1 + c_{48} \dot{q}_2 q_3 q_1^2 + c_{49} \dot{q}_2 q_3 q_1^2 q_3 + c_{50} \dot{q}_2 q_3^2 q_1 + c_{51} \dot{q}_2 q_3 q_1 q_3 + c_{52} \dot{q}_2 q_3 q_1^2 + p_1 \cos(\Omega t) = 0,$$ \hfill (11a)
\[ q_2 + \omega_2^2 q_2 + c_{20}q_1 + c_{26}q_3 + c_{27}q_1^2 + c_{28}q_2 + c_{29}q_3^2 + c_{30}q_1 q_2 + c_{32}q_2 q_3 + c_{33}q_3^3 + c_{34}q_1^2 q_2 + c_{35}q_1 q_3 + c_{36}q_2 q_3 + c_{37}q_3^2 q_2 + c_{38}q_1 q_2 q_3 + c_{39}q_3^3 + c_{40}q_3^3 + c_{41}q_3^3 + c_{42}q_1^2 q_2 + c_{43}q_1 q_3 + c_{46}q_2 q_3 + c_{47}q_3^2 q_2 + c_{48}q_1 q_2 q_3 + c_{49}q_3^3 + p_2 \cos(\Omega t) = 0, \] (11b)

\[ q_3 + \omega_3^2 q_3 + c_{30}q_1 + c_{36}q_2 + c_{47}q_1^2 + c_{48}q_2^2 + c_{49}q_3^3 + c_{31}q_1 q_3 + c_{32}q_2 q_3 + c_{33}q_3^3 + c_{34}q_1^2 q_3 + c_{35}q_1 q_3 + c_{36}q_2 q_3 + c_{37}q_3^2 q_3 + c_{38}q_1 q_2 q_3 + c_{39}q_3^3 + c_{40}q_3^3 + c_{41}q_3^3 + c_{42}q_1^2 q_3 + c_{43}q_1 q_3 + c_{46}q_2 q_3 + c_{47}q_3^2 q_3 + c_{48}q_1 q_2 q_3 + c_{49}q_3^3 + q_3 + p_3 \cos(\Omega t) = 0. \] (11c)

Equation (11) presents the third-order modal coupling equation obtained using the third-mode Galerkin method, including nonlinear term and modal coupling term. It is worth noting that Eq. (11) contains nonlinear damping coupling terms of different-order modal responses, which will be generated in the discrete process due to the cubic nonlinear damping term; therefore, the influence of wind velocity on the coupling effect of nonlinear damping terms could not be ignored. In addition, in Eq. (11), the coupling effect of nonlinear restoring force terms generated by quadratic and cubic nonlinear restoring force terms was also considered. The parameters in Eqs. (10–11) are presented in Appendix A.

Crescent-shaped icing is a common ice type observed with galloping, according to on-site observations. The parameters of iced transmission lines can be obtained through testing, field measurements, and in situ observations. To facilitate analysis and comparison, the geometrical parameters, material parameters, and related aerodynamic parameters were cited in the reference, as shown in Table 1 [30].

### 3 Approximate analytic solution of amplitude for non-resonance

Equation (10) includes the forced excitation and Rayleigh damping terms. With an excitation amplitude of \( p = 0 \), the system will belong to pure self-excited vibration, and when \( p > 0 \), the system will belong to the forced-self-excited system. To study the non-resonance response characteristics of the forced-self-excited system, the second-order multiple scale method was employed in this section.

\( D_1 \) represents the partial derivative of \( T_k \), where \( T_k \) represents the \( k \)-order time scale \( (k = 0, 1, 2) \), \( \varepsilon \) represents the perturbation parameters set for solving the nonlinear system, \( A \) represents the amplitude of the system response under the action of self-excited force, and \( q \) represents the displacement function, according to

\[
T_0 = t; T_1 = \varepsilon t; T_2 = \varepsilon^2 t
\]

\[
\frac{d}{dt} = \frac{\partial}{\partial T_0} + \frac{\partial}{\partial T_1} \frac{\partial T_2}{\partial T_2} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2
\]

\[
\frac{d^2}{dt^2} = \left( D_0 + \varepsilon D_1 + \varepsilon^2 D_2 \right)^2 = D_0^2 + 2\varepsilon D_0 D_1 + 2\varepsilon^2 D_0 D_2 + 2\varepsilon D_1 D_1 + \varepsilon^2 D_2 D_2
\]

\[
DA = D_0 A + \varepsilon D_1 A + \varepsilon^2 D_2 A
\]

\[
q = q_0 + \varepsilon q_1 + \varepsilon^2 q_2.
\] (12a-e)

If Eq. (10) is rewritten as a forced vibration far away from resonance, Eq. (10) can be set as follows using the second-order multiple scale method [11]:

\[
\ddot{q} + \omega^2 q + \varepsilon [c_1 q^2 + (\mu - c_3) \dot{q}] + \varepsilon^2 (c_2 q^3 + c_4 q^3) = p \cos(\Omega t).
\] (13)

We can substitute Eq. (12) into Eq. (13), and equating coefficients of like powers of \( \varepsilon^k \) \((k = 0, 1, 2)\) will lead to the following linear ordinary equations, respectively [31]:

\[
D_0^2 q_0 + \omega^2 q_0 = p \cos(\Omega t_1)
\]

\[
D_0^2 q_1 + \omega^2 q_1 = -[2D_0 D_1 q_0 + c_1 q_0^2 + (\mu + c_3) D_1 q_0]
\]

\[
D_0^2 q_2 + \omega^2 q_2 = -[2D_0 D_1 q_1 + 2D_0 D_2 q_0 + D_1^2 q_0 + c_2 q_0^3 + (\mu + c_3) (D_1 q_0 + D_0 q_1) + c_4 D_0^3 q_0^3].
\] (14a-c)

The solution to Eq. (14a) can be written in the following form:

\[
D_0 A = 0
\]

\[
q_0 = A(T_1, T_2) e^{i\omega T_0} + \bar{A}(T_1, T_2) e^{-i\omega T_0} + Be^{i\Omega T_0} + B e^{-i\Omega T_0}.
\] (15a,b)
In Eq. (15), $A$ represents the amplitude of the system response under the action of self-excited force, while $B$ represents the amplitude of the system response under the action of external excitation; $\overline{A}$ and $\overline{B}$ represent the conjugate terms of $A$ and $B$, respectively. Thus, the $A$ and $B$ can be set as

$$A(T_1, T_2) = a(T_1, T_2) \exp(i\beta(T_1, T_2))/2$$

$$B = P/[2|\omega^2 - \Omega^2]|.$$  \hspace{1cm} (16a, b)

If the external excitation frequency($\Omega$) does not meet following conditions, $\Omega \neq \omega + \omega \alpha$, $\Omega \neq 2\omega + \omega \alpha$, $\Omega \neq 3\omega + \omega \alpha$, $\Omega \neq \omega/2 + \omega \alpha$, and $\Omega \neq \omega/3 + \omega \alpha$, then the harmonic resonance of the system will be easy to excite [21]. To avoid the generation of super-harmonic resonance and sub-harmonic resonance, we could set excite [21]. To avoid the generation of super-harmonic resonance and sub-harmonic resonance, we could set

$$\Omega = \omega/4.$$  \hspace{1cm} (16b)

By substituting Eqs. (15) and (17) into Eq. (14c), we could obtain:

$$D_1A = -(\mu + c_3)A/2$$

$$q_1 = -4iB(\mu + c_3)e^{0.25i\omega t}/15w + c_1A^2e^{2i\omega t}/(3\omega^2)$$

$$-3c_2\overline{A}\overline{B}e^{0.75i\omega t}/(7\omega^2) + 32c_1ABe^{1.25i\omega t}/(9\omega^2)$$

$$-4c_1B^2e^{0.5i\omega t}/(3\omega^2) - c_1(\overline{A}\overline{A} + \overline{B}\overline{B})/\omega^2.$$  \hspace{1cm} (17a, b)

By substituting Eqs. (15) and (17) into Eq. (14c), we could also obtain:

$$D_2A = -3c_4\omega^2\overline{A}^2/2 - 3c_4\omega^2A\overline{AB}/16$$

$$+3ic_2\omega^2\overline{A}/2\omega + 3ic_2AB\overline{B}/\omega + i(\mu^2 + c_3^2)A/8\omega$$

$$+i(\mu c_3 - \mu)A/4\omega + ic_3/4\omega$$

$$q_2 = 3ic_2\omega^2B^2e^{0.75i\omega t}/28 - [16c_2B^2B/(5\omega^2)$$

$$+ic_2B^2/20 + 32c_2A\overline{A}/5\omega^2] + 4c_1B^2/(15\omega^2)$$

$$+8ic_2\omega^2B/15\omega^2 + 8ic_2\omega^2\overline{A}/S)e^{0.25i\omega t}$$

$$-16c_2B^2e^{0.75i\omega t}/(7\omega^2) - 16ic_2\overline{B}^2e^{-1.75i\omega t}/11$$

$$-3ic_2\omega^2\overline{B}^2e^{1.5i\omega t}/20 + ic_4\omega^2B^2e^{0.75i\omega t}/28$$

$$+12c_2AB^2e^{0.5i\omega t}/(5\omega^2) + 16ic_2AB^2e^{0.75i\omega t}/7$$

$$-4c_1\overline{AB}e^{-0.5i\omega t}/\omega^2 + c_2A^3e^{1.75i\omega t}/(8\omega^2)$$

$$+16c_2A^2\overline{B}e^{1.75i\omega t}/(11\omega^2) + 48c_2A^2B^2e^{2.25i\omega t}/(65\omega^2).$$  \hspace{1cm} (18a, b)

The averaging equation of amplitude and phase could be obtained by combining Eqs. (12d), (15a), (16), (17a), and (18a) [32]:

$$\dot{a} = -v(\mu + c_3)a/2 - v^2(3\omega^2c_4A^2/8 + 3c_4\omega^2B^2a/16)$$

$$\dot{b} = v^2(3c_2a^2/8\omega + 3c_2B^2/\omega$$

$$+ (\mu^2 + 2c_3\mu + c_3^2 - 2\mu)/8\omega + c_3/2\omega)$$

$$+(\mu^2 + 2c_3\mu + c_3^2 - 2\mu)/8\omega + c_3/2\omega)$$

$$\dot{\alpha} = (\mu^2 + 2c_3\mu + c_3^2 - 2\mu)/8\omega + c_3/2\omega.\hspace{1cm} (19a, b)$$

After integrating Eq. (19) over time ($t$), we could obtain:

$$a = \sqrt{1/[(1/a_0^2 + n/m)e^{-2mt} - n/m]}$$

$$\beta_1 = \beta_0 + v^2(3c_2a^2/8\omega + 3c_2B^2/\omega$$

$$+ (\mu^2 + 2c_3\mu + c_3^2 - 2\mu)/8\omega + c_3/2\omega)$$

$$\dot{\alpha}$$

$$\dot{\beta}$$

where $a(t)$ is the amplitude function, and where $m$ and $n$ depend on the environmental parameters and aerodynamic shape of iced transmission lines, and $m$ determines the amplitude variation trend.

With an excitation amplitude($p$) less than the critical value, self-excited vibration will occur in the forced-self-excited system. Therefore, the necessary and sufficient conditions for the existence of self-excited vibration will follow:

$$a_0 < \lim_{t \to \infty} \sqrt{1/[(1/a_0^2 + n/m)e^{-2mt} - n/m]} = C.$$  \hspace{1cm} (20a, b)

If the parameter $(m)$ satisfies the condition $(m>0)$, Eq. (20a) will be convergent, and the excitation amplitude($p$) will satisfy the condition of Eq. (21). Then, the system will still form the self-excited vibration without destroying the condition of self-excited vibration, and the steady-state motion of the system will be composed of the self-excited and forced vibrations.

If the excitation amplitude($p$) does not meet the condition of Eq. (21), parameter $(m)$ will satisfy the condition $(m<0)$. The destroying conditions of self-excited vibration will be as follows:

$$\lim_{t \to \infty} \sqrt{1/[(1/a_0^2 + n/m)e^{-2mt} - n/m]} = 0.  \hspace{1cm} (22)$$

If the excitation amplitude meets the condition of Eq. (22), the amplitude of self-excited vibration will tend to zero with time. Then, self-excited vibration will not be able to form, resulting in no self-excited vibration in the forced-self-excited system. The parameters in Eq. (20a), Eq. (21), and Eq. (22) are given as follows:
$$m = -\varepsilon (\mu + c_3)/2 - 3\varepsilon^2 c_4 \omega^2 B^2 /16,$$

$$n = -3\varepsilon^2 \omega^2 c_4 /8.$$  

The approximate analytic amplitude expression can be obtained by combining Eqs. (12e), (15–18), and (20):

$$q = a \cos \psi + 2B \cos (\psi/4) + 4c_1 a^2 \cos (2\psi)/(6\omega^2) + 8B(\mu + c_3) \sin (\psi/4)/(15\omega^2) - 32c_1 a B \cos (3\psi/4)/(7\omega^2) + 32c_2 a B \cos (5\psi/4)/(9\omega^2) - 8c_1 B^2 \cos (\psi/2)/(3\omega^2) - c_1 (a^2 + 4BB)/(2\omega^2)
+ \varepsilon^2 \{ -3oc_4 a^2 B \sin (3\psi/4)/56 - [32c_2 B^2/(5\omega^2)] + 16c_2 a^2 B/(5\omega^2) + 8c_3 B^2/(15\omega^2)
+ 16\omega c_1 B/(15\omega^2) \sin (\psi/4) - (\varepsilon B)^2 B/(10) + 4oc_4 a^2 B/(\psi^2) - 32c_2 B^2 \cos (3\psi/4)/(7\omega^2)
+ 8c_3 B a^2 \sin (-\psi/4)/11 + 3oc_4 a B^2 \sin (3\psi/2)/(20 - 4oc_4 a^3 \sin (3\psi/4)/14 + 12c_2 a^2 B \cos (3\psi/2)/(5\omega^2)
- 4oc_4 a^3 \sin (3\psi/4)/7 - 4c_3 a B^2 \cos (\psi/2)/(\omega^2) + c_2 a^3 \cos (3\psi/(32\omega^2) + 8c_3 a^2 B \cos (7\psi/4)/(11\omega^2)
+ 24c_2 a^2 B \cos (9\psi/4)/(65\omega^2)\},$$  

(24)

where the $\psi$ is:

$$\psi = \omega t + \beta_1$$

In the following section, small sag single mode was adopted, as shown in Fig. 3, where the time history displacement curve in Fig. 3 could not reflect the coupling effect of different-order modes. Figure 3 is used for the non-resonant response analysis of forced-self-excited system, and in Figs. 4, 5, 6, 7, 8 and 9, the different-order modal coupling effects of the forced-self-excited system were considered, where the harmonic resonance responses of the forced-self-excited system were analyzed under different excitation frequencies and amplitudes.

Figure 3a–b shows the time history displacement curves of the forced-self-excited system, where the partial curve of the time history displacement curve was 20 s. The blue spherical solid line in the figure represents the numerical solution, which was obtained by solving Eq. (10) with the Runge–Kutta function in MATLAB, while the red square solid line represents the approximate analytic solution of the multiple scale method, corresponding to Eq. (24). When the excitation amplitude($p$) in Fig. 3a was equal to 0, the system belonged to self-excited vibrations, and when the excitation amplitude($p$) in Fig. 3b was greater than 0, the system belonged to forced-self-excited vibrations.

Figure 3a shows the excitation amplitude $p = 0$ in Eq. (10), where the feature of time history displacement curve belongs to pure self-excited vibration. The initial disturbance of the system was $q_0 = 0.01$ m, $\dot{q}_0 = 0$, and the other parameters are shown in Table 1. Under the conditions of the negative damping and cubic damping of aerodynamic force, the galloping amplitude of the system increased from 0.010 m to 0.211 m gradually. The numerical solution was in good agreement with the approximate analytic solution of the multiple scale method shown in Fig. 3a, where the average error between them was only 0.50%. As shown in Fig. 3b, $p = 0.5$ N/m, and the self-excited vibration characteristics of the time history displacement curve changed. Under the action of external excitation, the initial vibration amplitude of the system increased from 0.010 m to 0.054 m. The galloping amplitude of the system increased from 0.211 m to 0.253 m, while the galloping time was shortened from 1,750 s to 1,100 s. The galloping time was the time required for the system response to reach stability, known as stable time. As indicated in Fig. 3b, the comparison time range of the time history displacement curve used the stable numerical solution of 1,355–1,375 s and the stable analytic solution of 1,925–1,945 s, respectively. Moreover, the form of time history displacement curve also changed from simple harmonic vibration to irregular vibration.

As indicated in Eq. (24), when the excitation amplitude was $p = 0$, the governing Eq. (10) was consistent with that in reference [31]. However, with an increase in the excitation amplitude($p$), the error between the approximate analytic solution and numerical solution increased gradually. When $p = 0.5$ N/m, as shown in Fig. 3b, the galloping times of the numerical and the analytical solution were 1,100 s and 1,750 s, respectively. With the enhancement of external excitation ($p$), the nonlinear characteristics of system response became increasingly prominent, and the error of the approximate analytic solution of the multiple scale method increased as well [21]. Therefore, the strongly nonlinear and multimodal coupled govern equations in the following paper were analyzed by numerical methods.
Figure 3c–d is calculated by Maple software, which correspond to the amplitude analytic solution from Eq. (20a), which only represent the stable amplitude of self-excited vibration in the forced-self-excited system. Curves b and c in Fig. 3c were obtained by substituting excitation amplitudes \( p = 0.5 \) and 8.0 N/m into Eq. (20a). In addition, curve b satisfied the discriminant formula (21), while curve c satisfied the discriminant formula (22). When \( p = 0.5 \) N/m, the response amplitude of self-excited vibration corresponded to point \( a_b \), and when \( p = 8.0 \) N/m, the response amplitude of self-excited vibration tended to be close to 0, which also verified the accuracy of discriminant Eqs. (21) and (22). Figure 3d depicts the relationship between time \( t \), excitation amplitude \( p \), and the response amplitude \( a \), and the curve in Fig. 3d corresponded to curve b in Fig. 3c when \( p = 0.5 \) N/m. With an increase in excitation amplitude, we also verified that the self-excited vibration in the forced-self-excited vibration system decreased gradually. Hence, with self-excited force in the system, the nonlinear system would reach a stable state under the action of self-excitation. However, the system contained self-excited force and external forced excitations, and the contribution of the self-excited force to the system was much greater than forced excitation, while the system response showed the vibration characteristics of self-excited vibrations. When the contribution of self-excitation to the system response was far less than forced excitation, the system response showed the vibration characteristics of forced vibration, and when the contribution of self-

Fig. 3 Non-resonance time history displacement curve of forced-self-excited system
(a) The time history displacement curve and phase diagram considering different-order modal coupling

(b) The time history displacement curve of different-order modes

(c) The time history displacement curve and phase diagram of the first-order mode

(d) The time history displacement curve and phase diagram of the second-order mode

(e) The time history displacement curve and phase diagram of the third-order mode
excitation was consistent with forced excitation, it belonged to the forced-self-excited system mentioned above.

After the above analysis of Fig. 3, we could easily determine that with variations in excitation amplitude \( p \), the vibration form of the forced-self-excited system went through three stages.

1. In Fig. 3a, the excitation amplitude \( p \) was equal to 0, and the vibration form of the system response was the pure self-excited vibration.
2. In Fig. 3b, the excitation amplitude \( p \) satisfied the discriminant formula (21); the vibration form of the system response was the superposition form of self-excited and forced vibrations.
3. The curves \( c \) in Fig. 3c satisfied the discriminant formula (22), while the vibration form of the system response consisted of forced vibration regulated by Rayleigh damping.

In a forced-self-excited system, the self-excited vibration condition of iced transmission lines under wind load excitation will be destroyed by forced excitation, known as the quenching phenomenon.

4 Numerical solution of forced-self-excited system for resonance response

With an increase in the wind velocity and excitation amplitude, the nonlinearity of the system will be enhanced, and the error of the approximate analytic solution will increase [21]. For the forced-self-excited system, to study the influence of excitation amplitude and frequency on the dynamic response, the Runge–Kutta function was used to solve Eq. (11) directly. The time history displacement curves and phase diagrams in Figs. 4, 5, 6, 7, 8 and 9 were obtained using MATLAB, where the initial disturbances of time history displacement curves were 0.01 m (\( q_1, q_2, q_3 \)) and 0 m/s (\( \dot{q}_1, \dot{q}_2, \dot{q}_3 \)). The nonlinear dynamic behaviors of the principal resonance, super-harmonic resonance, and sub-harmonic resonance were also analyzed. Of note, Eq. (11) consists of a nonlinear galloping governing equation with the coupling effect of multimodal, and Figs. 4, 5, 6, 7, 8 and 9 represent the time history displacement curves considering the coupling effect of different-order modes [34].

With an excitation amplitude of \( p = 0 \), in Fig. 4, the time history displacement curves and phase diagrams corresponded to the small sag multi-mode. In Fig. 4a, the time history displacement curve was obtained by superimposing the first-order, second-order, and three-order modal response amplitudes according to the displacement superposition principle, where the velocity of the phase diagram was obtained by the vector superposition of the first-order, second-order, and third-order modal response velocities. The time history displacement curves and phase diagrams in Figs. 5, 6, 7, 8, and 9 were also obtained using the method in Fig. 4a and b depicts the time history displacement curve corresponding to the different-order modes in Fig. 4a. Figure 4c–e plots the time history displacement curves and phase diagrams of the first-order, second-order, and third-order modal response, respectively, and Fig. 4c–e corresponds to \( q_1, q_2, q_3 \) in Eq. (11), respectively.

As shown in Fig. 4a, when the excitation amplitude was \( p = 0 \), this indicates that the galloping time \( t \) of the time history displacement curve was 10,800 s, while the peak values of galloping amplitude and galloping velocity were \(-0.132 \text{ m/}-0.734 \text{ m/s}, \) and \(-0.131 \text{ m}/0.733 \text{ m/s}, \) and the limit cycle of the stable system was a closed elliptical cycle. When the excitation amplitude was \( p = 0 \), we only observed self-excited vibration in the system, namely pure self-excited vibration. By comparing the time history displacement curves of Figs. 3a and 4a, we determined that Fig. 3a could only analyzed the response of the first-order mode, while Fig. 4a reflects the coupling effect of different-order modes. Figure 3a shows that the galloping time was 1,750 s, where the system response reached a stable state and the response amplitude of the system remained unchanged. As shown in Fig. 4a, from 1,500–6,000 s, the response amplitude of the system was maintained at around \( \pm 0.173 \text{ m} \). From 6,000–12,000 s, the response amplitude of the system would decrease to \( \pm 0.132 \text{ m} \) and remain stable. As indicated in Fig. 4b, it was not difficult to observe the second-order mode reaches a steady state, making an indispensable contribution to the system response. As shown in Fig. 4c and d, this belonged to periodic motion, where the phase trajectory in Fig. 4e was an irregular periodic vibration. In Fig. 4c and Fig. 4e, when the system response was
stable, little contributions were applied to the system response by the first-order modal and the third-order modal responses. The second-order modal responses in Fig. 4d contributed the most when the system response was stable. The main reasons for this phenomenon were as follows. In Eq. (11), the first-order \((q_1)\), second-order \((q_2)\), and third-order \((q_3)\) modal responses were coupled with each other. Under the effect of nonlinearity and multimodal coupling, the natural frequency was close to the second-order modal frequency, causing the second-order modal response to dominate the system response when the system was stable.

4.1 Principal resonance

Under the condition of external excitation, when the excitation frequency \(\Omega\) was equivalent to the natural frequency \(\omega_1\) of the system, the system response would be prone to resonance, known as the principal resonance. As shown in Fig. 5a, when the excitation amplitude was \(p = 0.1\) N/m, the galloping time \(t\) of the time history displacement curve was 150 s, and the classical galloping phenomenon formed. Therefore, under the condition of external excitation, the galloping time \(t\) decreased from 10,800 s of \(p = 0\) to 150 s of \(p = 0.1\) N/m. In addition, the peak values of galloping amplitude and galloping velocity increased to \(-0.531\) m/\(-1.833\) m/s, and \(0.528\) m/\(1.833\) m/s. As shown in Fig. 5b, when \(p = 3.0\) N/m, the galloping time \(t\) of the time history displacement curve was 16 s, and the peak values for galloping amplitude and galloping velocity increased to \(-1.609\) m/\(-5.378\) m/s, and \(1.517\) m/\(5.449\) m/s. By analyzing Fig. 5, the galloping time of the system shortened sharply, while the galloping amplitude and galloping velocity increased rapidly. Another problem of note was that when the system consisted of a pure self-excited system, the time history displacement curve and phase diagram in Fig. 4a were irregular due to the interaction of different-order modal responses. However, when
Fig. 6 Time history displacement curve and phase diagram of 2-order super-harmonic resonance
Fig. 7  The time history displacement curve and phase diagram of 3-order super-harmonic resonance
Fig. 8 The time history displacement curve and phase diagram of 1/2-order sub-harmonic resonance
Fig. 9  Time history displacement curve and phase diagram of 1/3-order sub-harmonic resonance
the excitation amplitude was $p > 0$, the system changed into forced-self-excited system, and the time history displacement curves and phase diagrams in Fig. 5 were regular. Hence, when external excitation was close to the natural frequency of the self-excited system, the system response would resonate, resulting in an irregular response changed into the stable response.

4.2 2-order super-harmonic resonance

When the excitation frequency($\Omega$) was equivalent to half of the natural frequency($\omega_1/2$) of the system, the resonance of the system was known as 2-order super-harmonic resonance.

As shown in Fig. 6, with variations in excitation amplitude, the phase diagram first changed from a closed elliptical ring to the period-two periodic motion, then changed to the period-one periodic motion with a concave heart shape. In addition, the phenomenon in Fig. 6 was consistent with that in Fig. 5, where the galloping time of the system shortened, and the galloping amplitude and galloping velocity increased. However, the variation trends of galloping time, amplitude, and velocity in Fig. 6 were slower than that in Fig. 5. When the excitation amplitude was $p = 0.1$ N/m, as shown in Fig. 6a, the galloping time($t$) of the time history displacement curve was 8,000 s, where the peak values of galloping amplitude and galloping velocity were $-0.133$ m/ $-0.736$ m/s, and $0.133$ m/ $0.732$ m/s. As shown in Fig. 6b, when $p = 1.0$ N/m, the peak value of galloping amplitude and galloping velocity also increased to $-0.253$ m/$-0.735$ m/s, and $0.239$ m/$0.825$ m/s, where the phase diagram consisted of a closed elliptical ring with period-two periodic motion.

As shown in Fig. 6c, when the excitation amplitude was $p = 10.0$ N/m, the galloping time was $t = 150$ s, and the peak values of galloping amplitude and galloping velocity increased to $-0.755$ m/$-1.415$ m/s, and $0.843$ m/$2.154$ m/s. In addition, the internal limit cycle of the phase diagram changed into an intersection point. When $p = 20.0$ N/m, as shown in Fig. 6d, the galloping time was $t = 60$ s, and the peak values of galloping amplitude and galloping velocity also increased to $-1.403$ m/$-1.906$ m/s, and $1.402$ m/$3.476$ m/s; however, the phase diagram consisted of a concave limit cycle, which changed from period-two periodic motion to period-one periodic motion.

4.3 3-order super-harmonic resonance

When the excitation frequency($\Omega$) was equal to one-third of the natural frequency($\omega_1/3$) of a system, the resonance response of the system was known as 3-order super-harmonic resonance. As shown in Fig. 7, the vibration center of the inner limit cycle shifted from the range near the equilibrium position to the range above (below) the equilibrium position.

When the excitation amplitude was $p = 1.0$ N/m, as shown in Fig. 7a, the phase diagram added a period of vibration to the upper and lower sides of the equilibrium position, respectively, and the system response behaved as a closed elliptical ring with period-three periodic motion. As indicated in Fig. 7b, when $p = 5.0$ N/m, the system response still maintained the form of period-three periodic motion, which different from Fig. 7a. As shown in Fig. 7c, when $p = 10.0$ N/m, the phase diagram consisted of a concave shape limit cycle with two intersection points. With variations in the excitation amplitude, the phase diagram changed from the period-three periodic motion to the period-one periodic motion, as shown in Fig. 7d.

4.4 1/2-order sub-harmonic resonance

When the excitation frequency($\Omega$) was twice the natural frequency($2\omega_1$) of the system, it was known as 1/2-order sub-harmonic resonance.

Under the condition of 1/2-order sub-harmonic resonance, the excitation amplitude($p$) was equal to 0, which was the time history displacement curve of Fig. 4a. Compared to the galloping amplitude and galloping velocity in Figs. 4a and 8a–d, this indicated that with the variation in excitation amplitude, the galloping amplitude and galloping velocity of the forced-self-excited system with 1/2-order sub-harmonic resonance had four stages (first decreased, then increased, decreased, and finally increased).

When the excitation amplitude was $p = 0$ N/m, the peak values of galloping amplitude and galloping velocity were $-0.132$ m/$-0.734$ m/s, and $0.131$ m/$0.733$ m/s, as shown in Fig. 4a. When $p = 0.1$ N/m, as shown in Fig. 8a, the peak values of galloping amplitude and galloping velocity decreased...
to \(-0.131\) m/\(-0.728\) m/s, and \(0.130\) m/\(0.725\) m/s. When \(p = 1.0\) N/m, the phase diagram showed a concave limit cycle in Fig. 8b, and the peak values of galloping amplitude and galloping velocity also increased to \(-0.195\) m/\(-0.641\) m/s, and \(0.198\) m/\(0.720\) m/s. When \(p = 8.0\) N/m, as shown in Fig. 8c, the peak values of galloping amplitude and galloping velocity decreased to \(-0.050\) m/\(-0.342\) m/s, and \(0.049\) m/\(0.340\) m/s. When \(p = 55.0\) N/m, as shown in Fig. 8d, the peak values of galloping amplitude and galloping velocity increased to \(-0.403\) m/\(-2.688\) m/s, and \(0.358\) m/\(2.601\) m/s.

### 4.5 1/3-order sub-harmonic resonance

When the excitation frequency(\(\Omega\)) was triple the natural frequency(3\(\omega_1\)) of the system, it was called the 1/3-order sub-harmonic resonance. As shown in Fig. 9, with variations in the excitation amplitude, the phenomenon showed that the galloping amplitude of the system with 1/3-order sub-harmonic resonance first decreased, then increased, decreased, and finally increased. The phase diagram changed from a closed elliptical ring with period-three periodic motion to a closed elliptical ring with period-one periodic motion.

To obtain the response characteristics of the system response amplitude with the excitation amplitude(\(\rho\)), which increased from \(p = 0\) gradually, the time history displacement curves in Fig. 4a and Fig. 9a were compared. As shown in Fig. 4a, the excitation amplitude was \(p = 0\) N/m, and the peak values of galloping amplitude and galloping velocity were \(-0.132\) m/\(-0.734\) m/s, and \(0.131\) m/\(0.733\) m/s. When \(p = 0.5\) N/m, the limit cycle of the forced-self-excited system excited by the 1/3-order sub-harmonic was also a closed elliptical cycle, and the galloping time(\(t\)) of the time history displacement curve was shortened to 10,000 s, as shown in Fig. 9a, where the galloping amplitude and galloping velocity decreased to \(-0.092\) m/\(-0.761\) m/s, and 0.086 m/\(0.755\) m/s. As shown in Fig. 9b, when \(p = 5.5\) N/m, the galloping time(\(t\)) of the time history displacement curve was 3,500 s, and the galloping amplitude and galloping velocity increased to \(-0.163\) m/\(-1.185\) m/s, and \(0.163\) m/\(1.181\) m/s. This showed that the phase diagram increased from the period-one periodic motion to period-three periodic motion. As shown in Fig. 9c, when \(p = 10.0\) N/m, the galloping amplitude and galloping velocity decreased to \(-0.118\) m/\(-1.187\) m/s, and \(0.118\) m/\(1.192\) m/s, and the phase diagram gradually changed from the shape of period-three periodic motion to the closed elliptical ring with period-one periodic motion. As shown in Fig. 9d, when \(p = 20.0\) N/m, the galloping amplitude and galloping velocity increased to \(-0.217\) m/\(-2.238\) m/s, and 0.215 m/\(2.245\) m/s, and the phase diagram changed from the shape of period-three periodic motion to the closed elliptical ring with period-one periodic motion.

Figures 5, 6, 7, 8 and 9 correspond to the principal resonance, 2-order super-harmonic resonance, 3-order super-harmonic resonance, 1/2-order sub-harmonic resonance, and 1/3-order sub-harmonic resonance of the forced-self-excited system, respectively. Analyzing of the galloping time, galloping amplitude, and galloping velocity of the time history displacement curves in Sects. 4.1–4.5, showed that for the principal resonance and super-harmonic resonance, with the variation in the excitation amplitude, the galloping amplitude and galloping velocity of the forced-self-excited system continued to increase, and the galloping time continued to decrease. For the sub-harmonic resonance, with an increase in excitation amplitude, the galloping amplitude and galloping velocity of the forced-self-excited system continued to increase, and the galloping time continued to decrease. For the sub-harmonic resonance, with an increase in excitation amplitude, the galloping amplitude and galloping velocity of the forced-self-excited system went through four stages: first decreased, then increased, decreased, and finally increased.

Compared to harmonic resonance and principal resonance, the principal resonance response increased

| parameter | Symbol | Units | Value |
|-----------|--------|-------|-------|
| Tension   | \(H\)  | N     | 30 000 |
| Span      | \(L\)  | m     | 125.88 |
| Young’s modulus | \(E\)   | N/mm² | 47 803.3 |
| Diameter  | \(D\)  | m     | 0.0286 |
| Mass per unit length | \(m\)  | Kg/m  | 2.379 |
| Traverse area | \(S\) | mm²   | 423.24 |
| Air mass density | \(\rho\) | Kg/m³ | 1.2929 |
| Wind velocity | \(U\) | m/s   | 4.0 |
| Sag       | \(d\)  | m     | 1.5432 |
| Vertical dumping | \(\mu\) |       | 0.0005 |
| Aerodynamic parameters | \(A\) |       | -0.1667 |
| Aerodynamic parameters | \(B\) |       | 8.3581 |
the most, and the increase in sub-harmonic resonance response was the smallest. By analyzing the different response characteristics, we found that under the influence of the principal resonance response, the galloping phenomenon of transmission line appeared in the shortest time, and the galloping amplitude and velocity were also the largest. However, the sub-harmonic resonance possibly weakened the system response as the excitation amplitude was in a specific range. Therefore, for the transmission line structure, the damage of principal resonance was greater than super-harmonic resonance, and the damage of super-harmonic resonance was greater than sub-harmonic resonance.

5 Conclusion

A new forced-self-excited system was obtained by establishing a physical model of iced transmission lines under the condition of dynamic wind. The approximate analytic periodic solution (non-resonance) of the forced-self-excited system was obtained using the multiple scale method. Then, the third-order Galerkin method, which considered the small sag effect, was used to discretize nonlinear partial differential equations, where the three-degree-of-freedom galloping governing equations considering geometric nonlinearity and multimode coupling were obtained [25]. The time history displacement curves and phase diagrams of the different excitation frequencies and different excitation amplitudes were obtained using numerical methods. The results showed that the forced-self-excited system had different harmonic resonance forms due to the different excitation frequencies, as follows:

(1) For the forced-self-excited system, when the excitation amplitude was \( p = 0 \), the vibration form of the system was pure self-excited vibration, and when \( p > 0 \), the vibration form of the system consisted of the superposition form of self-excited and forced vibrations. When the excitation amplitude \( (p) \) was greater than the critical value, the response characteristics of self-excited vibration disappeared, resulting in the quenching phenomenon.

(2) Under the condition of frequency-doubling excitation, the periodic vibration of corresponding multiples appeared in the forced-self-excited system. The 2-order super-harmonic resonance corresponded to the vibration form of period-two periodic motion, while the 3-order super-harmonic and 1/3-order sub-harmonic resonances corresponded to the vibration form of period-three periodic motion. When the amplitude of external excitation was relatively limited, the system response was complex with irregular periodic vibration due to the coupling effect of the different-order modes and the nonlinearity of the system. As the external excitation amplitude increased gradually, the phenomenon of the vibration form with period-two and period-three periodic motion appeared. When the excitation amplitude exceeded the critical value, the system response would turn into the vibration form with period-one periodic motion.

(3) For the forced-self-excited system of iced transmission lines, the damage of the principal resonance was greater than that of super-harmonic resonance, and the damage of the super-harmonic resonance was greater than that of sub-harmonic resonance. In general, principal resonance and super-harmonic excitation promoted the system response, and the sub-harmonic excitation blocked the system response in the variation interval of the independent variable part. Therefore, in practical engineering applications, the natural frequency of transmission lines should be far away from the frequency region of principal resonance and super-harmonic resonance.

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Data availability The authors declare that the data are available regarding the publication of this paper.
Declarations

Conflict of interest The authors declare that there are no conflicts of interest regarding the publication of this paper.

Appendix  
\[ \omega_2^2 = \frac{1}{m \frac{I_a}{I_i}}; c_1 = 12 \frac{dE}{m^2 I_i^2}; c_2 = \frac{E}{2mL \frac{I_a}{I_i}}; \]
\[ c_3 = \frac{a}{m}; c_4 = \frac{b}{m \frac{I_a}{I_i}}; p = \frac{p^*}{m \frac{I_a}{I_i}}; \]
\[ \omega_1^2 = \frac{1}{m \frac{I_a}{I_i}}; c_5a = 64 \frac{E}{m^2 I_i^2}; c_5a = 64 \frac{E}{m^2 I_i^2}; \]
\[ c_{7a} = \frac{4E}{m^2 I_i^2}; c_{7a} = \frac{4E}{m^2 I_i^2}; \]
\[ c_{10} = \frac{8E}{m^2 I_i^2}; c_{12} = \frac{8E}{m^2 I_i^2}; \]
\[ c_{13} = \frac{8E}{m^2 I_i^2}; c_{15} = \frac{8E}{m^2 I_i^2}; \]
\[ c_{16} = \frac{16E}{m^2 I_i^2}; c_{17} = \frac{16E}{m^2 I_i^2}; c_{18} = \frac{16E}{m^2 I_i^2}; \]
\[ c_{19} = \frac{b}{m \frac{I_a}{I_i}}; c_{20} = \frac{b}{m \frac{I_a}{I_i}}; c_{21} = \frac{b}{m \frac{I_a}{I_i}}; \]
\[ c_{22} = \frac{3}{m \frac{I_a}{I_i}}; c_{23} = \frac{3}{m \frac{I_a}{I_i}}; c_{24} = \frac{3}{m \frac{I_a}{I_i}}; \]
\[ c_{25} = \frac{3}{m \frac{I_a}{I_i}}; c_{26} = \frac{3}{m \frac{I_a}{I_i}}; c_{27} = \frac{3}{m \frac{I_a}{I_i}}; \]
\[ c_{28} = \frac{6}{m \frac{I_a}{I_i}}; c_{29} = \frac{6}{m \frac{I_a}{I_i}}; c_{30} = \frac{6}{m \frac{I_a}{I_i}}; \]
\[ c_{31} = \frac{1}{m \frac{I_a}{I_i}}; c_{32} = \frac{1}{m \frac{I_a}{I_i}}; c_{33} = \frac{1}{m \frac{I_a}{I_i}}; \]
\[ c_{34} = \frac{1}{m \frac{I_a}{I_i}}; c_{35} = \frac{1}{m \frac{I_a}{I_i}}; c_{36} = \frac{1}{m \frac{I_a}{I_i}}; \]
\[ \omega_2^2 = \frac{1}{m \frac{I_a}{I_i}}; c_{5b} = 64 \frac{E}{m^2 I_i^2}; c_{6b} = 64 \frac{E}{m^2 I_i^2}; \]
\[ c_{20} = \frac{8E}{m^2 I_i^2}; c_{21} = \frac{8E}{m^2 I_i^2}; c_{22} = \frac{8E}{m^2 I_i^2}; \]
\[ c_{23} = \frac{8E}{m^2 I_i^2}; c_{24} = \frac{8E}{m^2 I_i^2}; c_{25} = \frac{8E}{m^2 I_i^2}; \]
\[ c_{26} = \frac{16E}{m^2 I_i^2}; c_{27} = \frac{16E}{m^2 I_i^2}; \]
\[ c_{28} = \frac{16E}{m^2 I_i^2}; c_{29} = \frac{16E}{m^2 I_i^2}; c_{30} = \frac{16E}{m^2 I_i^2}; \]
\[ c_{31} = \frac{64}{m^2 I_i^2}; c_{32} = \frac{64}{m^2 I_i^2}; c_{33} = \frac{64}{m^2 I_i^2}; \]
\[ c_{34} = \frac{64}{m^2 I_i^2}; c_{35} = \frac{64}{m^2 I_i^2}; c_{36} = \frac{64}{m^2 I_i^2}; \]
\[ I_0 = H \int \frac{l}{(f')}^2 \, dx + 64\frac{d^2ES}{L^3} \int \frac{l}{f'} \, dx; \]
\[ I_1 = \int \frac{l}{(f')}^2 \, dx + \int \frac{l}{f'} \, dx, I_2 = (\int l \frac{l}{f'}^2 \, dx); \]
\[ I_m = \int \frac{l}{(f')}^2 \, dx; I_b = \int \frac{l}{f'}^2 \, dx; I_p = \int \frac{l}{f'} \, dx; \]
\[ I_a = 64ESd^2(\int \frac{l}{f'} dx)^2/L^5 + H \int \frac{l}{(f')}^2 \, dx; \]
\[ I_{5a} = \int \frac{l}{f'} dx \int \frac{l}{f_2} dx; I_{6a} = \int \frac{l}{f'} dx \int \frac{l}{f_2} dx; \]
\[ I_{7a} = \int \frac{l}{f'} x \int \frac{l}{(f')}^2 \, dx + 2 \int \frac{l}{(f')}^2 \, dx \]
\[ \int \frac{l}{(f')}^3 \, dx; I_{8a} = \int \frac{l}{f'} dx \int \frac{l}{(f_2)^2} \, dx; \]
\[ I_{9a} = \int \frac{l}{f'} dx \int \frac{l}{(f_2)^2} \, dx; I_{10} = \int \frac{l}{f'} dx \]
\[ \int \frac{l}{f'}^2 f_2 \, dx + \int \frac{l}{(f_2)^2} \, dx \int \frac{l}{f'} f_3 dx; \]
\[ I_{11} = \int \frac{l}{f'} dx \int \frac{l}{f'} f_2 \, dx + \int \frac{l}{(f_2)^2} \, dx \int \frac{l}{f'} f_4 dx; \]
\[ I_{12} = \int \frac{l}{f'} dx \int \frac{l}{f'_2 f'_2} \, dx; I_{13} = (\int \frac{l}{(f')}^2 \, dx); \]
\[ I_{14} = \int \frac{l}{(f')^2} \, dx \int \frac{l}{(f_2)^2} \, dx; \]
\[ I_{15} = \int \frac{l}{(f')^2} \, dx \int \frac{l}{(f_3)^2} \, dx; \]
\[ I_{16} = \int \frac{l}{(f')^2} \, dx \int \frac{l}{(f_2)^2} \, dx; \]
\[ I_{17} = \int \frac{l}{(f')^2} \, dx \int \frac{l}{(f_2)^2} \, dx; \]
\[ I_{18} = \int \frac{l}{(f')^2} \, dx \int \frac{l}{(f_3)^2} \, dx; \]
\[ I_{19} = \int \frac{l}{(f')^2} \, dx \int \frac{l}{(f_3)^2} \, dx; \]
\[ I_{20} = \int \frac{l}{(f')^2} \, dx \int \frac{l}{(f_3)^2} \, dx; \]
\[ I_{21} = \int \frac{l}{(f')^2} \, dx \int \frac{l}{(f_3)^2} \, dx; \]
\[ I_{22} = \int \frac{l}{(f')^2} \, dx \int \frac{l}{(f_3)^2} \, dx ; I_{23} = (\int \frac{l}{(f')^2} \, dx); \]
\[ I_{24} = \int \frac{l}{(f')^2} \, dx \int \frac{l}{(f_4)^2} \, dx; \]
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\[ I_{49} = \int \frac{l}{(f')^2} \, dx \int \frac{l}{(f_5)^2} \, dx; \]
\[ I_{50} = \int \frac{l}{(f')^2} \, dx \int \frac{l}{(f_5)^2} \, dx; \]
\[ I_{51} = \int \frac{l}{(f')^2} \, dx \int \frac{l}{(f_5)^2} \, dx; \]
\[ I_{52} = \int \frac{l}{(f')^2} \, dx \int \frac{l}{(f_5)^2} \, dx; \]
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\[ I_{56} = \int \frac{l}{(f')^2} \, dx \int \frac{l}{(f_5)^2} \, dx; \]
\[ I_{57} = \int \frac{l}{(f')^2} \, dx \int \frac{l}{(f_5)^2} \, dx; \]
\[ I_{58} = \int \frac{l}{(f')^2} \, dx \int \frac{l}{(f_5)^2} \, dx; \]
\[ I_c = 64Esd^2 \left( \int_0^L f_0' dx \right)^2 \frac{L^5}{2} + H \int_0^L f_0'' dx; \]

\[ I_{sc} = \int_0^L f_0' dx \int_0^L f_0'' dx; I_{sc} = \int_0^L f_0 dx \int_0^L f_0 dx; \]

\[ I_{s0} = \int_0^L f_0' f_0'' dx \int_0^L f_0 dx; I_{s0} = \int_0^L f_0' dx \int_0^L f_0'' dx; \]

\[ I_{sc} = \int_0^L f_0' dx \int_0^L f_0'' dx; I_{s0} = \int_0^L f_0 dx \int_0^L f_0 dx; \]

\[ I_{2c} = \left( \int_0^L f_0' dx \right)^2; I_{3c} = \int_0^L f_0' dx \int_0^L f_0 dx; \]

\[ I_{10} = \int_0^L f_0' dx; I_{20} = \int_0^L f_0 dx; \]

\[ I_{12} = \int_0^L f_0' dx; I_{22} = \int_0^L f_0 dx; \]

\[ I_{13} = \int_0^L f_0' dx; I_{23} = \int_0^L f_0 dx; \]

\[ I_{14} = \int_0^L f_0' dx; I_{24} = \int_0^L f_0 dx; \]

\[ I_{34} = \int_0^L f_0' dx; I_{34} = \int_0^L f_0 dx; \]

\[ I_{50} = \int_0^L f_0' dx; I_{50} = \int_0^L f_0 dx; \]

\[ I_{60} = \int_0^L f_0' dx; I_{60} = \int_0^L f_0 dx; \]

\[ I_{70} = \int_0^L f_0' dx; I_{70} = \int_0^L f_0 dx; \]

\[ I_{80} = \int_0^L f_0' dx; I_{80} = \int_0^L f_0 dx; \]

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