On \((k, l)\)-stable vector bundles over algebraic curves.

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Abstract
In this paper, we study the \((k, l)\)-stable vector bundles over non-singular projective curve \(X\) of genus \(g \geq 2\), its relation with stability and Segre invariants. For rank 2 and 3, we give an explicit description and relation of \((k, l)\)-stability and Brill-Noether loci.

1 Introduction.
In \([8, 9]\) Narasimhan and Ramanan introduced the notion of \((k, l)\)-stability for vector bundles over a non-singular projective curve \(X\) of genus \(g \geq 2\). A vector bundle \(E\) is \((k, l)\)-stable if for all proper subbundle \(F \subset E\) the differences of the slopes \(\mu(E) - \mu(F)\) is greater than a rational number that involves \(k\) and \(l\) (see Definition 2.1). Mainly they use \((0, 1), (1, 0)\) and \((1, 1)\)-stability to define an open set in the moduli space \(M(n, L)\) of stable bundles over \(X\) with fix determinant \(L\) that allows them to define the Hecke cycles and Hecke curves. Also compute some cohomology groups of \(M(n, L)\).

In this paper we study the \((k, l)\)-stability for all \(k, l \in \mathbb{Z}\). Denote by \(A_{(k,l)}(n, d)\) the set of isomorphic classes of \((k, l)\)-stable vector bundles of rank \(n\) and degree \(d\) over \(X\). The non-emptiness conditions of \(A_{(k,l)}(n, d)\) for any pair \((k, l)\) of integers are given as follows: if \(k(n - 1) + l < (n - 1)(g - 1)\) and \(k((n - 1) + l < (n - 1)(g - 1)\) then \(A_{(k,l)}(n, d) \neq \emptyset\) (Proposition 2.4). This bound could be improve for fix values of \(d\) and \(g\) (Theorem 2.7). Moreover, we obtain that whenever \(A_{(k,l)}(n, d) \neq \emptyset\), there exist a stable vector bundle \(E \in A_{(k,l)}(n, d)\).

If \(k, l\) satisfies \(0 < k(n - 1) + l\) and \(0 < k + (n - 1)l\), then every \((k, l)\)-stable vector bundle is stable, i.e. \(A_{(k,l)}(n, d) \subset M(n, d)\). In this case we compute the

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dimension and codimension of the complement of $A_{(k,l)}(n,d)$ in $M(n,d)$ (see, Theorem 2.12).

For others values of $k, l$ is possible to have semistable vector bundles which are $(k,l)$-stables. For example, if $k(n - 1) + l < 0$ and $k + (n - 1)l < 0$, then every semistable vector bundle is $(k,l)$-stable (Proposition 2.9), in particular this holds for $k,l$ negatives.

Let $E$ and $F$ be $S$-equivalent vector bundles and suppose that $E$ is $(k,l)$-stable. Then it does not implies that $F$ is $(k,l)$-stable (see Remark 2.13). In general, the $(k,l)$-stability splits the elements in the $S$-equivalence class of strict semistable vector bundles.

If $(k,l)$ is such that $k(n - 1) + l < 0$ and $k + (n - 1)l < 0$, then there are unstable vector bundles (actually with automorphisms) that are $(k,l)$-stable. However, there exist decomposable unstable vector bundles which are not $(k,l)$-stable. If we consider indecomposable vector bundles only, then is possible to obtain conditions over $k,l$ for which every indecomposable vector bundle is $(k,l)$-stable (Theorem 3.4).

Using the above results we make explicit computations for rank 2 and 3 cases. We give the necessary and sufficient conditions for non-emptiness of $A_{(k,l)}(n,d)$ (see, Theorem 3.2 and Table 1). Especially, in rank 2 case, we prove that every indecomposable vector bundle is $(k,l)$-stable if $k + l < 2 - 2g$ (see Theorem 3.4), hence we obtain a complete classification of $(k,l)$-stable vector bundle of rank 2.

Moreover, for rank 3 we study the relation between semistability and $(k,l)$ stability and we give the splitting of $S$-equivalence classes using $(k,l)$-stability (see, Theorem 4.1). Finally we apply of $(k,l)$-stability on Brill-Noether theory. We study the relation between $A_{(k,l)}(n,d)$ and $B(n,d,r)$ (see, Theorem 5.2).

This paper is as follows: Section 2 presents some basic properties and known results. We give necessary and sufficient conditions for non-emptiness of $A_{(k,l)}(n,d)$ and we compute of codimension for $(k,l)$-stable vector bundles of general rank. Section 3 establish the results for rank 2 case and Section 4 the results for rank 3 case. In Section 5 we relate the $(k,l)$-stability and the Brill-Noether loci.

2. $(k,l)$-stability.

From now on, $X$ denotes a non-singular projective curve of genus $g \geq 2$ over $\mathbb{C}$. In this section we recall basic properties of $(k,l)$-stability, the proofs of some of the results can be found in [9].

For any integer $k$, the $k$-slope of a vector bundle $E$ on $X$ is the quotient

$$
\mu_k(E) := \frac{\deg E + k}{rk E}.
$$
Definition 2.1. Let $E$ be a vector bundle over $X$ and $k, l \in \mathbb{Z}$. Then $E$ is a $(k, l)$-stable vector bundle if for all proper subbundle $F \subset E$ we have $\mu_k(F) < \mu_{k-l}(E)$, i.e.

$$\frac{\deg F + k}{\text{rk} F} < \frac{\deg E + k - l}{\text{rk} E}.$$ 

If the inequality is not strict then $E$ is $(k, l)$-semistable.

Let us denote by $A_{(k,l)}(n,d)$ the set of isomorphic classes of $(k, l)$-stable vector bundles of rank $n$ and degree $d$ over $X$.

The inequality in Definition 2.1 is equivalent to

$$(k(n - m) + ml)/nm < \mu(E) - \mu(F). \quad (2.1)$$

Remark 2.2. An easy computation shows the following statements:

1. If $(k, l) = (0, 0)$, then $A_{(k,l)}(n,d) = M(n,d)$.
2. If $E \in A_{(k,l)}(n,d)$, then $E^* \in A_{(l,k)}(n,-d)$.
3. If $E \in A_{(k,l)}(n,d)$ and $L$ a line bundle of degree $d_L$, then $E \otimes L \in A_{(k,l)}(n,d+nd_L)$.
4. If $k, l \geq 0$, then $A_{(k,l)}(n,d) \subseteq M(n,d)$.
5. If $k, l \leq 0$, then $A_{(k,l)}(n,d) \supseteq M(n,d)$.

It is known that the $(k, l)$-stability is an open property (see [9, Proposition 5.3]). Hence by Remark 2.2 (4), if $k, l \geq 0$, then $A_{(k,l)}(n,d)$ is an open variety of the moduli space $M(n,d)$.

Another important property of $(k, l)$-stability is its behavior under elementary transformations. In this sense [9] Lemma 5.5] state that if $E \in A_{(k,l)}(n,d)$, $x \in X$ and $0 \to E' \to E \to \mathcal{O}_x \to 0$ is an exact sequence of sheaves with $E', E$ locally free. Then $E' \in A_{(k,l-1)}(n,d-1)$. We reproduce the proof of this for the convenance of the reader.

Proof. Let $F' \subset E'$, and $F \subset E$ the generated bundle by the map $F \to E$. Then $F' \to F$ is of maximal rank and hence $\deg F' \leq \deg F$. Now $\mu_k(F') \leq \mu_k(F) < \mu_{k-l}(E) = \mu_{k-l+1}(E')$.

$\Box$
Remark 2.3. From definition of \((k, l)\)-stability and using the inequality (2.1), we can observe that if \(E\) is a \((k, l)\)-stable vector bundle, then \(E\) is \((k, l-1)\)-stable and \((k-1, l)\)-stable. Thus we have the following filtration

\[
A_{(k,l)}(n,d) \subseteq A_{(k,l-1)}(n,d) \subseteq A_{(k,l-2)}(n,d) \subseteq \cdots
\]

\[
A_{(k,l)}(n,d) \subseteq A_{(k-1,l)}(n,d) \subseteq A_{(k-2,l)}(n,d) \subseteq \cdots
\]

The conditions for non-emptiness of \(A_{(0,1)}(n,d)\), \(A_{(1,0)}(n,d)\), and \(A_{(1,1)}(n,d)\) are given by Narasimhan and Ramanan in [9, Proposition 5.4], this is

1. Except when \(g = 2\), \(n = 2\) and \(d\) odd, \(A_{(0,1)}(n,d) \neq \emptyset\).

2. \(A_{(1,1)}(n,d) \neq \emptyset\) except in the following cases:

   (a) \(g = 3\), and \(d\) both even.
   (b) \(g = 2\), \(d \equiv 0, \pm 1 \pmod{n}\).
   (c) \(g = 2\), \(n = 4\), \(d \equiv 2 \pmod{4}\).

We study the non-emptiness conditions for \(A_{(k,l)}(n,d)\) with any value of \(k\) and \(l\). Observe that if \(A_{(k,l)}(n,d)\) is non empty, then \(A_{(k,l-1)}(n,d)\) and \(A_{(k-1,l)}(n,d)\) are non empty (Remark 2.3). By this reason we will prove the non-emptiness for \(k\) and \(l\) bigger enough. Following the idea of Narasimhan and Ramanan we obtain the following result, which implies [9, Proposition 5.4].

Proposition 2.4. If \(k, l \in \mathbb{Z}\) are such that

\[
k(n-1) + l < (n-1)(g-1) \tag{2.2}
\]

and

\[
k + l(n-1) < (n-1)(g-1) \tag{2.3}
\]

then \(A_{(k,l)}(n,d) \neq \emptyset\).

Proof. If \((k, l)\) satisfies (2.2) and (2.3), we will prove that there exist stable vector bundles that are \((k,l)\)-stable. Let \(E \in M(n,d)\) be a not \((k,l)\)-stable vector bundle on \(X\) of rank \(n\) degree \(d\). Hence by definition there is a proper subbundle \(F \subset E\) such that

\[
\mu_{k-l}(E) \leq \mu_k(F), \tag{2.4}
\]

and \(F\) determine the following extension \(0 \to F \to E \to E/F \to 0\). If \(m\) and \(\delta\) denote the rank and degree of \(F\), then the number of such extensions is bounded
by \(m^2(g-1) + 1 + (n-m)^2(g-1) + 1 + h^1((E/F)^* \otimes F) - 1\). Now, using (2.2), (2.3) and (2.4) we obtain
\[
m^2(g-1) + 1 + (n-m)^2(g-1) + 1 + h^1((E/F)^* \otimes F) - 1 < n^2(g-1) + 1.
\]
This implies that the dimension of vector bundles which satisfies (2.4) is less than \(\dim M(n,d)\). Therefore, the dimension of no \((k,l)\)-stable vector bundles is less than \(\dim M(n,d)\), this is for any \(m = 1, \ldots, n-1\). As \(m\) take a finite values then does not cover the moduli space \(M(n,d)\). An this proves the proposition.

Hence, we can rephrase [9, Proposition 5.3] as follows:

**Corollary 2.5.** Under the hypotheses of Proposition 2.4, the very general vector bundle in \(M(n,d)\) is \((k,l)\)-stable.

To obtain the pairs \((k,l)\) such that \(A_{(k,l)}(n,d) = \emptyset\), we will give a brief discussion about Segre invariants. The classical work of Segre invariant in rank 2 case is [6]. For a general treatment we refer the reader to [5]. A more complete theory in general rank may be obtained in [2, 10].

Let \(E\) be vector bundle on \(X\) of rank \(n\), degree \(d\) and \(m \in \mathbb{Z}\) such that \(1 \leq m \leq n-1\). Recall that the \(m\)-Segre invariant for a vector bundle \(E\) is denoted by \(s_m(E)\) and defined as
\[
s_m(E) = md - n \cdot \deg F_{\text{max}},
\]
where \(F_{\text{max}} \subset E\) is a proper subbundle of rank \(m\) and maximal degree. Clearly, \(s_m(E) \equiv md \mod n\).

Hirschowitz proved in [4] that,
\[
s_m(E) \leq m(n-m)(g-1) + (n-1). \tag{2.5}
\]
Moreover, let \(X\) be a curve of genus \(g\) and let \(E\) be a vector bundle of rank \(n\) and degree \(d\). There is an unique integer \(\delta_m\) with \(0 \leq \delta_m \leq n-1\) and \(m(n-m)(g-1) + \delta_m \equiv md \mod n\), such that
\[
s_m(E) \leq m(n-m)(g-1) + \delta_m. \tag{2.6}
\]
The equality holds if \(E\) is general.

Denote by \(M(n,d,m,s)\) the set of stable vector bundles of rank \(n\) and degree \(d\) such that the \(m\)-Segre invariant is \(s\), that is
\[
M(n,d,m,s) := \{ E \in M(n,d) | s_m(E) = s \}. \tag{2.7}
\]
From [10, Theorem 0.1] (see, [2, Theorem 4.2]) we have that if \( s \) is an integer such that, \( 0 < s \leq m(n - m)(g - 1) \) and \( s \equiv md \mod n \) and \( g \geq 2 \), then \( M(n, d, m, s) \) is non-empty irreducible and
\[
\dim M(n, d, m, s) = n^2(g - 1) + 1 + s - m(n - m)(g - 1).
\]

Using above results we will prove the emptiness conditions of \( A_{(k,l)}(n, d) \).

**Proposition 2.6.** Let \( X \) be a non-singular projective algebraic curve of genus \( g \). If \( k, l \in \mathbb{Z} \) are such that
\[
k(n - 1) + l \geq (n - 1)g,
\]
(2.8)
or
\[
k + l(n - 1) \geq (n - 1)g,
\]
(2.9)
then \( A_{(k,l)}(n, d) = \emptyset \).

**Proof.** Let \((k_0, l_0)\) be such that satisfies (2.8), we will prove that there does not exist any vector bundle which is \((k_0, l_0)\)-stable.

Let \( E \) be a vector bundle of rank \( n \) and degree \( d \) and \( L_0 \subset E \) a line subbundle of maximal degree, then by (2.5) and (2.8) respectively we obtain,
\[
d - n \cdot \deg L_0 = s_1(E) \leq (n - 1)g \leq k_0(n - 1) + l_0.
\]

This implies \( \mu_{k_0}(L_0) \geq \mu_{k_0-l_0}(E) \), thus \( E \) is not \((k_0, l_0)\)-stable vector bundle. Similarly if \((k_0, l_0)\) satisfies (2.9) consider \( F \subset E \) a subbundle of rank \( n - 1 \) and maximal degree. This complete the proof.

The set of pairs \((k, l)\) defined in Proposition 2.6 will be denoted by \( R_0 \), i.e.
\[
R_0 := \left\{ (k, l) \in \mathbb{Z} \times \mathbb{Z} \mid k(n - 1) + l \geq (n - 1)g \quad \text{or} \quad k + l(n - 1) \geq (n - 1)g \right\}.
\]
(2.10)

Hence if \((k, l) \in R_0\) then \( A_{(k,l)}(n, d) = \emptyset \).

However the bound given in Proposition 2.4 and Proposition 2.6 can be improved if we consider the degree, Segre invariants and the genus of the curve.

**Theorem 2.7.** The set \( A_{(k,l)}(n, d) \neq \emptyset \) if and only if the pair \((k, l)\) is such that for all \( m, 1 \leq m \leq n - 1 \) the following inequality holds:
\[
k(n - m) + ml < m(n - m)(g - 1) + \delta_m,
\]
(2.11)
where \( \delta_m \) is the unique integer (which depends of \( m \)) with \( 1 \leq \delta_m \leq n - 1 \) and \( m(n - m)(g - 1) + \delta_m \equiv md \mod n \).
Proof. $(\Rightarrow)$ Let $E \in A_{(k,l)}(n,d) \neq \emptyset$, then combining equation (2.1), and (2.6) we have $k(n-m) + ml < s_m(E) \leq m(n-m)(g-1) + \delta_m$ for all $m$, this implies (2.11).

$(\Leftarrow)$ If $(k,l)$ is such that for all $m$ satisfies (2.11), then by (2.6) the generic vector bundle $E$ is such that $m(n-m)(g-1) + \delta_m = s_m(E)$, for all $m$. Hence using equation (2.1), we conclude that $E \in A_{(k,l)}(n,d)$, and this complete the proof.

Remark 2.8. Observe that by inequality (2.1) the following statements holds:

1. If $(k,l)$ is such that $k(n-m) + ml \geq 0$ for all $m$, $1 \leq m \leq n-1$. Then every $(k,l)$-stable vector bundle is stable.

2. If $(k,l)$ be such that $0 \leq k(n-1) + l \leq (n-1)(g-1)$ and $0 \leq k + l(n-1) \leq (n-1)(g-1)$. Then $(k,l)$-stability implies stability.

The set of pairs that satisfies Remark 2.8 (2), will be denoted by $R_1$, that is

$$R_1 := \left\{(k,l) \in \mathbb{Z} \times \mathbb{Z} \mid 0 \leq k(n-1) + l \leq (n-1)(g-1) \text{ and } 0 \leq k + l(n-1) \leq (n-1)(g-1)\right\}. \quad (2.12)$$

Hence, if $(k,l) \in R_1$, then $A_{(k,l)}(n,d) \neq \emptyset$.

Now we want to know when the stability implies $(k,l)$-stability. In this way it is easily seen that if for all $m$ we have $k(n-m) + ml \leq 0$, then every stable vector bundle $E$ is $(k,l)$-stable. For this, note that for all subbundle $F$ of rank $m$ we have, $\mu(E) - \mu(F) > 0 \geq (k(n-m) + ml)/nm$.

Proposition 2.9. If $(k,l)$ is such that $(n-1)k + l \leq 0$ and $k + (n-1)l \leq 0$. Then every stable vector bundle is $(k,l)$-stable.

Proof. Let $(k,l)$ be a pair of integer such that satisfies the conditions of proposition. We have divided the proof in three cases: First $k,l \leq 0$, second $k \leq 0$, $l > 0$ and third $k > 0$, $l \leq 0$.

If $k,l \leq 0$, it follows easily that every stable vector bundle is $(k,l)$-stable. Now, if $k \leq 0$ and $l > 0$, using the fact that $k + (n-1)l \leq 0$, we obtain

$$k(n-m) + ml \leq (l(1-n))(n-m) + ml = ml(m+1-n) \leq 0,$$

for all $m$. Hence, the assertion follows from Remark 2.8 (1). Similarly, if we suppose that $k > 0$, $l \leq 0$, then taking in count that $k(n-1) + ml \leq 0$ we have

$$k(n-m) + ml \leq k(n-m) + m(k(1-n)) = nk(1-m) \leq 0,$$

for all $m$. Thus for any pair $(k,l)$ which satisfies the hypothesis, is such that $k(n-m) + ml \leq 0$, for all $m$. Now, combining this inequalities with Remark 2.1 (3), the proposition follows.
As above we define the following region.

\[ R_2 := \left\{ (k, l) \in \mathbb{Z} \times \mathbb{Z} \mid (n-1)k + l \leq 0 \quad \text{and} \quad k + (n-1)l \leq 0 \right\}. \quad (2.13) \]

The relation between \((k, l)\)-stability and stability in the different regions described in the above propositions is rewrite as:

1. If \((k, l) \in R_0\), then \(A_{(k, l)}(n, d) = \emptyset\).
2. If \((k, l) \in R_1\), then \(A_{(k, l)}(n, d) \subset M(n, d)\).
3. If \((k, l) \in R_2\), then \(M(n, d) \subset A_{(k, l)}(n, d)\).

We mentioned above that \((k, l)\)-stability is an open property. Thus, if \(A_{(k, l)}(n, d) \subset M(n, d)\) then \(A_{(k, l)}(n, d)\) has dimension \(n^2(g-1) + 1\). The relation between \(A_{(k, l)}(n, d)\) and \(M(n, d)\) is as follows.

**Proposition 2.10.** Over an algebraic curve \(X\) of genus \(g \geq 2\), if \((k, l) \in R_1\) then

\[
A_{(k, l)}(n, d) = \bigcap_{m=1}^{n-1} \left( \bigcup_{s > k(n-m)+ml} M(n, d, m, s) \right). 
\]

**Proof.** It is easily seen that if \(E \in A_{(k, l)}(n, d)\), then \(s_m(E) > k(n-m) + ml\) for all \(1 \leq m \leq n - 1\), which implies the contention (\(\subseteq\)).

Now suppose that

\[
E \in \bigcap_{m=1}^{n-1} \left( \bigcup_{s > k(n-m)+ml} M(n, d, m, s) \right), 
\]

thus for all \(m\), \(s_m(E) > k(n-m) + ml\). This implies that \(E \in A_{(k, l)}(n, d)\) and the proof is complete.

\[ \square \]

Now, if we denote by \(A_{(k, l)}^c(n, d)\) the complement of \(A_{(k, l)}(n, d)\) in \(M(n, d)\) then we compute its codimension.

**Proposition 2.11.** If \((k, l) \in R_1\), then

\[
\text{codim} A_{(k, l)}^c(n, d) \geq \min \left\{ \frac{(n-1)(g-1) - k(n-1) - l}{(n-1)(g-1) - k - l(n-1)} \right\}.
\]
Proof. Let \( E \in A_{(k,l)}^c(n,d) \) and \( F \subset E \) a subbundle of rank \( m \) and degree \( \delta \). As in proof of Proposition 2.4, the dimension of stable vector bundles which have a subbundle \( F \) of rank \( m \) and degree \( \delta \) such that \( \mu_{k-l}(E) > \mu_k(F) \) is
\[(n^2 - nm + m^2)(g - 1) + 1 + dm - n\delta.\]
Moreover, this number is upper bounded by \((n^2 - nm + m^2)(g - 1) + 1 + (n - m)k + ml\). Then
\[
\dim M_X(n,d) - \dim(A_{(k,l)}^c(n,d)) \geq (nm - m^2)(g - 1) - (n - m)k - ml.
\]
Considering \( m \) as variable, we can see that the maximum of \((nm - m^2)(g - 1) - (n - m)k - ml\) is obtained when \( m = 1 \) or \( n - 1 \). Consequently, the codimension of \( A_{(k,l)}^c(n,d) \) is lower bounded by
\[
\min\{(n - 1)(g - 1) - k(n - 1) - l, (n - 1)(g - 1) - k - l(n - 1)\}.
\]
This is the desired conclusion.

To compute explicitly the dimension and codimension of \( A_{(k,l)}^c(n,d) \), we define the following variables:

Let
\[
\tilde{s}_m := \max\{s \mid s \leq k(n - m) + ml, s \equiv md \mod n\}. \tag{2.14}
\]
\[
s_\Delta := \min_m\{m(n - m)(g - 1) - \tilde{s}_m\}. \tag{2.15}
\]

**Theorem 2.12.** If \((k,l) \in R_1\), then
\begin{enumerate}
\item \( \dim A_{(k,l)}^c(n,d) = n^2(g - 1) + 1 - s_\Delta. \)
\item \( \text{codim} A_{(k,l)}^c(n,d) = s_\Delta. \)
\end{enumerate}
Proof. (1) By Proposition 2.10, we have the following

\[
\dim \left( A_{(k,l)}(n,d) \right)^c = \dim \left[ \bigcap_{m=1}^{n-1} \left( \bigcup_{s > k(n-m)+ml} M(n,d,m,s) \right)^c \right],
\]

\[
= \dim \left[ \bigcup_{m=1}^{n-1} \left( \bigcup_{s > k(n-m)+ml} M(n,d,m,s) \right)^c \right],
\]

\[
= \dim \left[ \bigcup_{m=1}^{n-1} \left( \bigcup_{s \leq k(n-m)+ml} M(n,d,m,s) \right) \right],
\]

\[
= \max_m \left\{ \max_s \left\{ \dim \left( M(n,d,m,s) \right) \right\} \right\},
\]

\[
= \max_m \left\{ \max_s \left\{ n^2(g-1) + 1 + s - m(n-m)(g-1) \right\} \right\},
\]

\[
= \max_m \left\{ n^2(g-1) + 1 + s_m - m(n-m)(g-1) \right\},
\]

\[
= n^2(g-1) + 1 - s_\Delta.
\]

and this proves (1).

(2) As \( \mathcal{A}_{(k,l)}(n,d) \) is closed, the proof is straightforward from the difference

\[
\dim M(n,d) - \dim \mathcal{A}_{(k,l)}(n,d) = \frac{n^2(g-1) + 1 - (n^2(g-1) + 1 - s_\Delta)}{s_\Delta}.
\]

\[
\square
\]

2.1 Semistability and \((k,l)\)-stability.

It is well known that, when degree and rank are coprime, semistability and stability coincide. Moreover, for semistable vector bundles there is an equivalence relation called \(S\)-equivalence. This equivalence relation is defined via the graduation of vector bundles which is obtained with the Jordan-Hölder filtration. Thus two vector bundles are \(S\)-equivalent if their graduations are isomorphic. However, it is also possible that two vector bundles with different Jordan-Hölder filtration can be \(S\)-equivalents \([11]\). Therefore we want to use the \((k,l)\)-stability in order to distinguish the strict semistable vector bundles, i.e. to determine the Jordan-Hölder filtration for each semistable vector bundle. We present this phenomena in the following example.

Example 2.13. Consider \(E\) and \(E'\) two \(S\)-equivalent vector bundles of rank \(n\) and degree \(d\). Suppose that \(\text{gr}(E) = \text{gr}(E') = F_1 \oplus F_2\) with \(0 \subset F_1 \subset E'\) of rank \(n_1\) and \(0 \subset F_2 \subset E\) of rank \(n_2\), with \(n_1 < n_2\).
If $k \geq -l > 0$, then $E'$ is not $(k, l)$-stable because $\mu_k(F_1) < \mu_{k-l}(E')$ implies $k < (n_1/n_2)(-l)$ which is a contradiction. Therefore if $E \in A_{(k,l)}(n,d)$, then $(k,l)$-stability split the class of $S$-equivalence of $E$.

More generally, the $(k, l)$-stability split the $S$-equivalence classes in the different types of Jordan-Hölder filtration. Each type of Jordan-Hölder filtration will correspond to a region in $(k, l)$-plane. Now, as a first step, we describe the following two regions named $R_{3k}$ and $R_{3l}$.

\[
R_{3k} := \left\{ (k, l) \middle| 0 < k(n-1) + l < (n-1)(g-1) \text{ and } k + l(n-1) < 0 \right\}. \tag{2.16}
\]

\[
R_{3l} := \left\{ (k, l) \middle| 0 < k + (n-1)l < (n-1)(g-1) \text{ and } k(n-1) + l < 0 \right\}. \tag{2.17}
\]

**Remark 2.14.** The regions are defined considering the values $(k, l)$. By definition, if $(k, l) \in R_{3k}$, then $k > 0, l < 0$ and if $(k, l) \in R_{3l}$ then $l > 0, k < 0$.

In both cases the first inequality in (2.16) (respectively (2.17)), is to consider the non-emptiness given by Proposition 2.4. Using both regions we define $R_3$ as the union, i.e.

\[
R_3 := R_{3k} \cup R_{3l}. \tag{2.18}
\]

Hence $R_3$ is the region that determine the relation between $(k, l)$-stability and the Jordan-Hölder filtration (see, Figure 1).

For $n = 3$, the regions $R_{3k}$ and $R_{3l}$ will split the graduation in Jordan-Hölder filtration for strict semistable vector bundle of rank 3 (see, Theorem 4.1). This is because the graduation of a strict semistable vector bundle is $L_1 \oplus L_2$ or $L \oplus F$. Now, if $gr(E) = L_1 \oplus L_2$, then the Jordan-Hölder filtration is $0 \subset L_i \subset F \subset E$ for $i = 1$ or 2 and $E$ is not $(k, l)$-stable if $(k, l) \in R_3$. Moreover, if $gr(E) = L \oplus F$, then the Jordan-Hölder filtration is $0 \subset L \subset E$ or $0 \subset F \subset E$. The first one implies that $E \notin A_{(k,l)}(n,d)$ if $(k, l) \in R_{3k}$. The second one implies that $E \notin A_{(k,l)}(n,d)$ if $(k, l) \in R_{3l}$.

For $n \geq 4$, we need subdivide $R_{3k}$ and $R_{3l}$ in more regions in order to classify the different types. Such subdivision is given by the lines $k(n - m) + ml = 0$, with $1 \leq m \leq n - 1$. In Section 4 we describe the rank 3 case and the ideas that we use there, can be easily generalized for $n \geq 4$.

### 3 Rank 2 case.

In this section we describe the above results for rank 2. By inequality (2.1) for rank 2 case of $(k, l)$-stability depends of the sum $k + l$ only. That is, a vector
bundle $E$ of rank 2 is $(k, l)$-stable if for any line subbundle $L \subset E$ satisfies

$$\mu(E) - \mu(L) > \frac{k + l}{2}.$$ 

Remark 3.1. In order to simplify notation we will write $A_t(2, d) := A_{(k,l)}(2, d)$, when $k + l = t$, and will be called $t$-stable instead $(k,l)$-stable. Using this notation (by Theorem 2.7), we have that:

1. If $t = 0$, then $A_t(2, d) = M(2, d)$
2. If $t > 0$ then $A_t(2, d) \subseteq M(2, d)$.
3. If $t = t'$, then $A_t(2, d) = A_{t'}(2, d)$.
4. If $t > t'$, then $A_t(2, d) \subseteq A_{t'}(2, d)$.

Moreover, in this case the Proposition 2.4, Proposition 2.6 and Theorem 2.7 are combined to obtain:

**Theorem 3.2.** If $X$ be a non-singular projective curve of genus $g$. Then we have the following statements.

1. For $g \not\equiv d \mod 2$, $A_t(2, d) \neq \emptyset$ if and only if $t < g - 1$.
2. For $g \equiv d \mod 2$, $A_t(2, d) \neq \emptyset$ if and only if $t \leq g - 1$. 

Figure 1: Regions defined by $(k, l)$-stability.
Proof. \((1,\Rightarrow)\) Let \(E \in A_r(2, d)\) and let \(L \subset E\) be a line subbundle of maximal degree, hence \(t < d - 2\deg L\) \(\equiv s_1(E)\). Moreover, if \(g \neq d \mod 2\), it follows that \(s_1(E) \neq g\). Thus, using (2.3) we have that \(t < g - 1\) as required.

\((1,\Leftarrow)\) This implication is a consequence of Proposition 2.4 taking \(n = 2\).

\((2,\Rightarrow)\) The proof is similar to \((1,\Rightarrow)\) considering \(g \equiv d \mod 2\).

\((2,\Leftarrow)\) By Proposition 2.4, if \(t < g - 1\) then \(A_r(2, d) \neq \emptyset\). Hence it is enough to show the implication when \(t = g - 1\).

Let \(E \in A_{g-2}(2, d)\) and \(L \subset E\) be a line subbundle, then \(d - 2\deg L > g - 2\).

By hypothesis \(g - 1 \neq d \mod 2\), which implies \(d - 2\deg L \neq g - 1\). Hence \(d - 2\deg L > g - 1\) and therefore \(E \in A_{g-1}(2, d)\) which proves that \(A_{g-1}(2, d)\) is non-empty.

Thus, the proof of \((2,\Leftarrow)\) gives more, namely \(A_{g-2}(2, d) = A_{g-1}(2, d)\) if \(g \equiv d \mod 2\). The following result makes explicit this relation.

**Proposition 3.3.**

1. If \(d\) is even, \(r \in \mathbb{Z}\) and \(2r \leq g - 1\), then \(A_{2r}(2, d) = A_{2r+1}(2, d)\).

2. If \(d\) is odd, \(r \in \mathbb{Z}\) and \(2r + 1 \leq g - 1\), then \(A_{2r+1}(2, d) = A_{2r+2}(2, d)\).

**Proof.** By (4) in Remark 3.1 we only need to prove the contentions \(\subseteq\).

\((1,\subseteq)\) Let \(E \in A_{2r}(2, d)\), then \(d - 2\deg L \geq 2r + 1\) for any line subbundle \(L \subset E\). But \(d - 2\deg L\) is even which implies \(d - 2\deg L > 2r + 1\). Therefore \(E \in A_{2r+1}(2, d)\).

Similar arguments apply to prove \((2)\).

Clearly, if \(t < 0\) then every semistable vector bundle of rank 2 is \(t\)-stable. However this does not apply to unstable case, because it is possible construct a unstable vector bundle which is not \(t\)-stable. For this choose any \(t_0 < 0\) and consider two line bundles \(L_1\) and \(L_2\) such that \(\deg L - \deg L_1 \leq t_0\). Now define \(E\) as \(E = L_1 \oplus L_2\). Hence \(E\) is unstable vector bundle such that is not \(t_0\)-stable as we desire.

Thus, if we consider only the indecomposable case, then is possible establish a lower bound such that every indecomposable vector bundle of rank 2 is \(t\)-stable.

**Theorem 3.4.** If \(E\) is an indecomposable vector bundle of rank 2 and degree \(d\) then \(E \in A_{1-2g}(2, d)\).

**Proof.** If \(E\) is semistable then the proof is clear. Suppose that \(E\) is an indecomposable and unstable vector bundle such that \(E \not\in A_{1-2g}(2, d)\). Hence if \(L_0\) is the line subbundle of maximal degree of \(E\) then \(d - 2\deg L_0 \leq 1 - 2g < 2 - 2g\), and we have the following extension

\[0 \to L_0 \to E \to L_1 \to 0.\]
Thus, $\deg L_1 - \deg L_0 = d - 2\deg L_0 < 2 - 2g$, in consequence $d(L_1^* \otimes L_0) > 2g - 2$ which implies $H^1(L_1^* \otimes L_0) = 0$, i.e. the extension splits. This contradicts the fact that $E$ is indecomposable and proves the theorem. 

\[ \square \]

**Remark 3.5.** The theorem asserts that if $t$ is such that $t \leq 1 - 2g$, then all rank 2 degree $d$ indecomposable vector bundle is $t$-stable. Moreover, if $r \in \mathbb{N}$, then we have the following contentions

1. If $d$ is even then

   \[ \emptyset = A_g(2, d) \subset \cdots \subset A_{2r+1}(2, d) = A_{2r}(2, d) \subset \cdots \]

   \[ \cdots \subset A_1(2, d) = M(2, d) = A_0(2, d) \subset A_{-1}(2, d) \subset \cdots \subset A_{1-2g}(2, d). \]

2. If $d$ is odd then

   \[ \emptyset = A_g(2, d) \subset \cdots \subset A_{2r+2}(2, d) = A_{2r+1}(2, d) \subset \cdots \]

   \[ \cdots = A_1(2, d) \subset A_0(2, d) = M(2, d) = A_{-1}(2, d) \subset \cdots \subset A_{1-2g}(2, d). \]

Now, we will describe in terms of $t$ the different regions given in Section 2.1 for rank 2 case. By Theorem 3.2 we have two cases:

**Case** $g \equiv d \mod 2$.

1. $R_0 = \{t \mid t \geq g\}$ and $A_t(2, d) = \emptyset$ if and only if $t \in R_0$.
2. $R = \{t \mid t \leq g - 1\}$ and $A_t(2, d) \neq \emptyset$ if and only if $t \in R$.
3. $R_1 = \{t \mid 0 \leq t \leq g - 1\}$ and $A_t(2, d) \subset M(2, d)$ if $t \in R_1$.
4. $R_2 = \{t \mid t \leq 0\}$ and $A_t(2, d) \supset M(2, d)$ if $t \in R_2$.

**Case** $g \not\equiv d \mod 2$.

1. $R_0 = \{t \mid t \geq g - 1\}$ and $A_t(2, d) = \emptyset$ if and only if $t \in R_0$.
2. $R = \{t \mid t \leq g - 2\}$ and $A_t(2, d) \neq \emptyset$ if and only if $t \in R$.
3. $R_1 = \{t \mid 0 \leq t \leq g - 2\}$ and $A_t(2, d) \subset M(2, d)$ if $t \in R_1$.
4. $R_2 = \{t \mid t \leq 0\}$ and $A_t(2, d) \supset M(2, d)$ if $t \in R_2$.

**Remark 3.6.** In this case $R_3$ defined in (2.18) is empty when $n = 2$. The reason is that we have the line $k + l = 0$ only.
Moreover, it is possible compute explicitly the dimension and codimension of \( A_2^c(t, d) \). In order to do this, combining (2.14) and (2.15) we have that
\[
\tilde{s}_m := \begin{cases} 
  t, & \text{if } t \equiv d \mod 2, \\
  t - 1, & \text{if } t \not\equiv d \mod 2,
\end{cases}
\]
and
\[
s_\Delta = \begin{cases} 
  g - t - 1, & \text{if } t \equiv d \mod 2, \\
  g - t - 2, & \text{if } t \not\equiv d \mod 2.
\end{cases}
\]
Now using Theorem 2.12 we obtain the following result.

**Theorem 3.7.** If \( t \) be such that \( t \in R_1 \) then:
\[
\dim A_2^c(t, d) = \begin{cases} 
  3g + t - 2, & \text{if } t \equiv d \mod 2, \\
  3g + t - 3, & \text{if } t \not\equiv d \mod 2.
\end{cases}
\]
\[
codim A_2^c(t, d) = \begin{cases} 
  3g + t - 1, & \text{if } t \equiv d \mod 2, \\
  3g + t - 2, & \text{if } t \not\equiv d \mod 2.
\end{cases}
\]

## 4 Rank 3 case.

Let \( E_1 \) and \( E_2 \) be two vector bundles of rank \( n \) and degree \( d \), with \( n \) and \( d \) not coprime. If \( E_1 \) and \( E_2 \) are strictly semistable and \( S \)-equivalents, is false that the \((k, l)\)-stability of \( E_1 \) implies the \((k, l)\)-stability of \( E_2 \) (see, Example 2.13). Hence, in order to establish the behavior of \((k, l)\)-stability in semistable vector bundles we study explicitly the relation of the \((k, l)\)-stability in rank 3 and the Jordan-Hölder filtration.

Remember that in Section 2 we define the region \( R_3 \) as the union of \( R_{3l} \) and \( R_{3k} \) which are defined using the lines \( k(n - 1) + l = 0 \) and \( k + l(n - 1) = 0 \) (see, (2.16), (2.17) and (2.18)).

**Theorem 4.1.** Let \( E \) be a vector bundle on \( X \) strictly semistable of rank 3. Then the Jordan-Hölder filtration of \( E \) is of one of these types:

1. \( 0 \subset L \subset E \), for some line bundle \( L \) if and only if there exist a pair \((k, l) \in R_{3l} \) such that \( E \) is \((k, l)\)-stable.
2. \( 0 \subset F \subset E \), for some vector bundle \( F \) of rank 2 if and only if there exist a pair \((k, l) \in R_{3k} \) such that \( E \) is \((k, l)\)-stable.
3. \( 0 \subset L \subset F \subset E \), for some line bundle \( L \) ans some vector bundle \( F \) of rank 2 if and only if \( E \) is not \((k, l)\)-stable for all \((k, l) \in R_3 \). Moreover, \( E \) is \((k, l)\)-stable for \((k, l) \in R_2 \).
Proof. For (1) the proof is based on the following observation. If the Jordan-Hölder filtration of $E$ is of type $0 \subset L \subset E$, then every subbundle $G \subset E$ of rank 2 satisfies $0 < \mu(E) - \mu(G)$. Moreover, $\deg E$ is multiple of 3, for otherwise $E$ will be stable.

$(1, \Rightarrow)$ Suppose the implication is false. Thus, $E$ is not $(k, l)$-stable for all $(k, l) \in R_{3l}$. Taking an arbitrary point $(k_0, l_0) \in R_{3l}$, we have $0 < k_0 + 2l_0$, $2k_0 + l_0 < 0$ by definition of $R_{3l}$. As $E$ is not $(k_0, l_0)$-stable, there is a subbundle $G_0 \subset E$ of rank $m$ such that

$$\mu(E) - \mu(G_0) \leq (k_0(3-m) + ml_0)/3m, \text{ with } m = 1 \text{ or } 2.$$ 

If $m = 1$ and $0 < \mu(E) - \mu(G_0) \leq ((2k_0 + l_0)/3)$ which is a contradiction to $2k_0 + l_0 < 0$.

It follows that $m = 2$, thus $0 < \mu(E) - \mu(G_0) \leq ((k_0 + 2l_0)/6)$, which implies $2 \deg E - 3 \deg G_0 \leq k_0 + 2l_0$. Now, as $(k_0, l_0)$ is arbitrary we can choose it such that $k_0 + 2l_0 = 1$, but this implies that $\deg E$ and 3 are coprime and this is a contradiction (remember that 3 divides to $\deg E$). This conclude that $E$ is $(k, l)$-stable for some $(k, l) \in R_{3l}$.

$(1 \Leftarrow)$ Note that is sufficient to prove that, if $G \subset E$ is a subbundle of rank 2 then $\mu(G) < \mu(E)$. However, as $E \in A_{(k_0, l_0)}(3, d)$ for some $(k_0, l_0) \in R_{3l}$ and $k_0 + 2l_0 > 0$ then,

$$\mu(E) - \mu(G) > (k_0 + 2l_0)/6,$$

establishes the desired conclusion. For (2) and (3) the proofs are similar.

To complete the description of $A_{(k,l)}(3, d)$ we need consider degree and genus module 3 (see, Theorem 2.7). To get a better idea we will fix in $g = 2$ and consider all the possible cases, that is: $d \equiv i \mod 3$, for $i = 0, 1, 2$ (compare with [9, Proposition 5.4]).

$d \equiv 0 \mod 3$. We will make the following computations. By (2.0) we know that if $E$ is a rank 3, degree $d$ vector bundle then $s(E) \leq 3$ and $s_2(E) \leq 3$. Hence $A_{(k,l)}(3, d) \neq \emptyset$ if and only if $2k + l < 3$ and $k + 2l < 3$. Hence $A_{(1,1)}(3, d) = \emptyset$, $A_{(0,1)}(3, d)$ and $A_{(1,0)}(3, d)$ are non-empty. Moreover, $A_{(0,1)}(3, 0) = A_{(1,0)}(3, 0) = M(3, 0)$

$d \equiv 1 \mod 3$. As above using (2.0), we compute the bound for Segre invariants of $E$, $s_1(E) \leq 4$ and $s_2(E) \leq 2$. Hence $A_{(k,l)}(3, d) \neq \emptyset$ if and only if $2k + l < 4$ and $k + 2l < 2$. Hence $A_{(1,0)}(3, d) \neq \emptyset$ and $A_{(1,1)}(3, d)$, $A_{(0,1)}(3, d)$ are empty.

$d \equiv 2 \mod 3$. Similarly, $s_1(E) \leq 2$ and $s_2(E) \leq 4$. Hence $A_{(k,l)}(3, d) \neq \emptyset$ if and only if $2k + l < 2$ and $k + 2l < 4$. Therefore $A_{(0,1)}(3, d) \neq \emptyset$ and
Remark 4.2. If we allow to vary the genus and following similar arguments we can obtain the necessary an sufficient conditions for \( A(\cdot, \cdot)(3, d) \neq \emptyset \). Table 1 consider the nine cases for rank 3 (\( g, d \equiv 0, 1, 2 \mod 3 \)) and this complete the information about \( A(\cdot, \cdot)(3, d) \).

More general if \( n \geq 4 \), then it is necessary consider the \( n^2 \) possible cases of \( d, g \equiv 0, 1, \ldots, n - 2 \mod n \).

| \( d \equiv \mod 3 \) | \( g \equiv 0 \mod 3 \) | \( g \equiv 1 \mod 3 \) | \( g \equiv 2 \mod 3 \) |
|---|---|---|---|
| \( d \equiv 0 \mod 3 \) | \( A(\cdot, \cdot)(3, d) \neq \emptyset \) iff \( 2k + l < 2g \) | \( A(\cdot, \cdot)(3, d) \neq \emptyset \) iff \( 2k + l < 2g - 2 \) | \( A(\cdot, \cdot)(3, d) \neq \emptyset \) iff \( 2k + l < 2g - 1 \) |
| \( d \equiv 1 \mod 3 \) | \( A(\cdot, \cdot)(3, d) \neq \emptyset \) iff \( 2k + l < 2g - 2 \), \( k + 2l < 2g - 1 \) | \( A(\cdot, \cdot)(3, d) \neq \emptyset \) iff \( 2k + l < 2g - 1 \), \( k + 2l < 2g - 2 \) | \( A(\cdot, \cdot)(3, d) \neq \emptyset \) iff \( 2k + l < 2g \), \( k + 2l < 2g - 1 \) |
| \( d \equiv 2 \mod 3 \) | \( A(\cdot, \cdot)(3, d) \neq \emptyset \) iff \( 2k + l < 2g - 1 \), \( k + 2l < 2g - 2 \) | \( A(\cdot, \cdot)(3, d) \neq \emptyset \) iff \( 2k + l < 2g \), \( k + 2l < 2g - 1 \) | \( A(\cdot, \cdot)(3, d) \neq \emptyset \) iff \( 2k + l < 2g - 2 \), \( k + 2l < 2g \) |

Table 1: Non-emptiness for \( A(\cdot, \cdot)(3, d) \).

5 Application to Brill-Noether theory.

It is well known that there is a filtration of the moduli space \( M(n, d) \) given by the Brill-Noether loci \( B(n, d, r) \).

\[
B(n, d, r) := \{ E \in M(n, d) | h^0(E) \geq r \}
\]

We refer the reader to [1, 3, 7] for a general reference of the Brill-Noether theory.
Let $\mu$ and $\lambda$ denote the quotients $\mu = d/n$ and $\lambda = r/n$. In this section we study the relation between $B(n, d, r)$ and $A_{(k,l)}(n, d)$. The interest in this relation is given by Remark 4.5.

Let $E \in B(n, d, r)$ be a $(k,l)$-stable vector bundle, i.e. $E \in B(n, d, r) \cap A_{(k,l)}(n, d)$. If we suppose that $\mathcal{O} \subset E$, then has sections and we have that

$$\frac{k(n-1)+l}{n} < \mu(E) - \mu(\mathcal{O}),$$

which implies $k(n-1)+l < d$. Then $E \not\in A_{(k,l)}(n, d)$ if $l \geq d - k(n - 1)$. Now, by Remark 2.3, $E \not\in A_{(k,l)}(n, d)$ if $k \geq 0$, $l \geq d$. This prove the following

**Proposition 5.1.** If $k \geq 0$, $l \geq d$ and $E \in B(n, d, r)$ is such that $\mathcal{O} \subset E$, then $E \not\in A_{(k,l)}(n, d)$.

The above result implies that if $\mathcal{O} \subset E$ and $E \in B(n, d, r)$, then we have $E \not\in M(n, d, 1, s_1)$ for $s_1 > d$. To have a better description of the relation between $B(n, d, r)$ and $A_{(k,l)}(n, d)$ we consider a different regions defined by the Brill-Noether theory.

**(BGN).** For $0 < \mu \leq 1$, then $B(n, d, r) \neq \emptyset$ if and only if

$$(\mu, \lambda) \in \{(\mu, \lambda) | 0 < \mu \leq 1, 1 \leq \mu + (1 - \lambda)g, (\mu, \lambda) \neq (1, 1)\}.$$

**(M).** If $1 < \mu < 2$, then $B(n, d, r) \neq \emptyset$ if and only if

$$(\mu, \lambda) \in \{(\mu, \lambda) | 1 \leq \mu + (1 - \lambda)g\}.$$

**(BMNO).** If $d = nd' + d''$, $0 < d'' < 2n$, $0 \leq d'$ and $(d'', r) \neq (n, n)$, then $B(n, d, r) \neq \emptyset$. Moreover, if exists a line bundle $L$ such that $h^0(L) \geq u$ with $1 \leq u \leq g$, then $B(n, d, ur) \neq \emptyset$.

Considering the information in BGN and M regions we relate the Brill-Noether loci with $(k,l)$-stability.

**Theorem 5.2.** Let $L$ a line bundle of degree $d_L$ such that $h^0(L) = s$. Let $n \geq 2$, $E \in B(n, d, r)$ be a vector bundle and let $\mu = d/n$, $\lambda = r/n$. Then we have the following statements

1. If $(\mu, \lambda) \in BGN$, $k \geq 1$ and $l \geq 0$, then $E \not\in A_{(k,l)}(n, d)$. Moreover, $E \otimes L \in B(n, d + nd_L, rs)$ and $E \otimes L \not\in A_{(k,l)}(n, d + nd_L)$.

2. If $(\mu, \lambda) \in M$, $k \geq 2$, $l \geq 0$ and $d \neq 2n - 1$, then $E \not\in A_{(k,l)}(n, d)$. Moreover, $E \otimes L \in B(n, d + nd_L, rs)$ and $E \otimes L \not\in A_{(1,0)}(n, d + nd_L)$.
Proof. (1) Suppose that $E \in A_{(1,0)}(n,d)$, with $(\mu, \lambda) \in BGN$ hence $\mu(E) < 1$ and we have the following exact sequence

$$0 \to \mathcal{O} \to E \to Q \to 0.$$  

(5.1)

Thus $(n-1)/n < \mu(E) - \mu(\mathcal{O}) = \mu(E) < 1$ which is impossible, consequently $E$ is not $(1,0)$-stable. Moreover, if $k \geq 1$ and $l \geq 0$, then $A_{(k,l)}(n,d) \subset A_{(1,0)}(n,d)$ and therefore $E$ is not $(k,l)$-stable (see, Remark 2.3). Using Remark 2.2 (3), it follows that $E \otimes L \notin A_{(k,l)}(n,d + nd_L)$. This prove the first statement.

(2) Suppose that $E \in A_{(2,0)}(n,d)$, then there is a line subbundle $L \subset E$ with sections such that $0 \leq \text{deg}(L) \leq 1$. Combining the $(2,0)$-stability of $E$ with the hypothesis over $\mu$ we obtain,

$$\frac{2(n-1)}{n} < \mu(E) - \mu(L) = \mu(E) < 2.$$  

However this implies $2n - 2 < d(E) < 2n$, which is a contradiction. In consequence, $E \notin A_{(2,0)}(n,d)$ and by Remark 2.3 we have $E \notin A_{(k,l)}(n,d)$ with $k \geq 2$ and $l \geq 0$. Using Remark 2.2 (3), it follows that $E \otimes L \notin A_{(k,l)}(n,d + nd_L)$. This prove the second statement.

Now, we can rewrite the above result for rank 2 case. For this remember that that the $(k,l)$-stability depends of the sum $k + l$. Hence we using the notation of $t$-stability given in Section 3, it follows easily that if $t \geq d$, then $E \notin A_t(2,d)$. Moreover, when $s_1 > d$ we have $E \notin M(2,d,1,s_1)$. Hence, from Theorem 5.2 we obtain the following result.

**Corollary 5.3.** Let $E \in B(2,d,r)$ be a vector bundle. If $E \in BGN$, then $E \notin A_t(2,d)$ for all $t \geq 1$.

**Proposition 5.4.** Let $E \in B(2,d,1)$ such that $1 < \mu(E) < 2$. If $E$ is 1-stable, then $E \in M(2,3,1,3)$.

Proof. By hypothesis, $d = 3$. There is a line subbundle with sections $L \subset E$, then $0 \leq d(L) < 3/2$. By 1-stability we have $1/2 < \mu(E) - d(L) = (3/2) - d(L)$. Thus, $d(L) = 0$ and $s_1(E) = d$, which is the desire conclusion.

In the same sense, we have the same result for rank 3. This is, if $E \in B(3,d,r)$ determine a point in BGN, then $E \notin A_{(1,0)}(3,d)$. Moreover, if $E \in B(3,d,r)$ in $M$, then for $d = 4, 5$. Thus for $d = 4, E \notin A_{(2,0)}(3,4)$ but for $d = 5$ there are many possible results. Therefore we have the following existence for 1-stable vector bundles.

Let $E \in B(3,d,1)$ with $3 \leq d \leq 5$ and $E \in A_{(1,0)}(3,d)$. If $L \subset E$ is a maximal line subbundle, then $0 \leq d(L)$. Using $(1,0)$-stability of $E$, we have
that \(1 \leq d(L) + 1 < (d+1)/3 \leq 2\), which implies \(d(L) = 0\). Therefore \(s_1(E) = d\) and using notation of Segre invariants (see, (2.7)) we can see that

\[ E \in M(3, 3, 1, 3) \cup M(3, 4, 1, 4) \cup M(3, 5, 1, 5). \]

Now, as \(E\) is \((1, 0)\)-stable, then \(s_2(E) \geq 2\). We thus get the following result.

**Proposition 5.5.** Let \(E \in B(3, d, 1)\) be a vector bundle such that \(1 \leq \mu(E) < 2\). If \(E\) is \((1, 0)\)-stable, when one of the cases holds:

1. \(E \in M(3, 3, 1, 3) \cap M(3, 3, 2, s)\), for some \(s \geq 2\),
2. \(E \in M(3, 4, 1, 4) \cap M(3, 4, 2, s)\), for some \(s \geq 2\),
3. \(E \in M(3, 5, 1, 5) \cap M(3, 5, 2, s)\), for some \(s \geq 2\).

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