A MIXED FINITE ELEMENT APPROXIMATION FOR DARCY-FORCHHEIMER FLOWS OF SLIGHTLY COMPRESSIBLE FLUIDS

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Abstract. In this paper, we consider the generalized Forchheimer flows for slightly compressible fluids in porous media. Using Muskat’s and Ward’s general form of Forchheimer equations, we describe the flow of a single-phase fluid in \(\mathbb{R}^d, d \geq 2\) by a nonlinear degenerate system of density and momentum. A mixed finite element method is proposed for the approximation of the solution of the above system. The stability of the approximations are proved; the error estimates are derived for the numerical approximations for both continuous and discrete time procedures. The continuous dependence of numerical solutions on physical parameters are demonstrated. Experimental studies are presented regarding convergence rates and showing the dependence of the solution on the physical parameters.

Key words. Porous media, error analysis, slightly compressible fluid, dependence on parameters, numerical analysis.

AMS subject classifications. 65M12, 65M15, 65M60, 35Q35, 76S05.

1. Introduction. The fluid flow through porous materials, e.g. soil, sand, aquifers, oil reservoir, plants, wood, bones, etc., is a great interest in the research community such as engineering, oil recovery, environmental and groundwater hydrology and medicine. Darcy law, which is the linear relation between the velocity vector and the pressure gradient, is used to describe fluid flow under low velocity and low porosity conditions, see in \([3]\). It has been observed from many experiments that when the fluid’s velocity is high and porosity is non-uniform, the Darcy’s law becomes inadequate. Consequently, the attention has been attracted to the nonlinear equations for describing of this kind of flow. Dupuit and Forchheimer proposed a modified equation, known as Darcy-Forchheimer equation or generalized Forchheimer equation, by adding the nonlinear terms of velocity to Darcy law, see \([16]\). Since then, there have been a growing number of articles studying Darcy-Forchheimer equation in theoretical studies (e.g. \([7, 8, 32, 37, 40, 41]\)) and numerical studies (e.g. \([4, 10, 18, 30, 36, 38]\)).

It is well known that the mixed finite element method is among the popular numerical methods for the modeling flow in porous media because it produces the accurate results for both scalar (density or pressure) and vector (velocity or momentum) functions, see \([36]\). An analysis of mixed finite element method to a Darcy-Forchheimer steady state model was well studied in \([35, 38]\). The mixed methods for a nondegenerate system modeling flows in porous media was studied in \([10, 18, 30, 36]\). The authors in \([1, 14, 15, 43]\) analyzed the mixed finite element approximations of the nonlinear degenerate system modeling water-gas flow in porous media. In their analysis, the Kirchhoff transformation is used to move the nonlinearity from coefficients to the gradient.

The objective of this paper is to analyze mixed finite element approximations to the solutions of the system of equations modeling the flows of a single-phase compressible fluid in porous media subject to the generalized Forchheimer law. This is a nonlinear degenerate system with coefficients depending on the density gradient and degenerating to zero as it approaches to infinity. The Kirchhoff transformation is not applicable for this system. For our degenerate equations, we combine the techniques developed in our previous works in \([19, 24, 28, 29]\) and utilize the special structures of the equations to obtain the stability of the approximated solution and the continuous dependence of the solution on parameters. The error estimates also derives for the density and momentum.

The paper is structured as follows. In section \(\S 2\) we introduce the notations and the relevant results. In section \(\S 3\) we defined a numerical approximation using mixed finite element approximations and the implicit backward difference time discretization to the initial boundary value problem (IVBP) \(\S 1\). In section

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\[ \Omega \subset \mathbb{V} \text{aturally. The fluid flow has velocity } \alpha \text{In particular, when } \]4, we establish many estimates of the energy type norms for the approximate solution \((\rho_h, m_h)\) to the IVBP problem \((3.5)\) in Lebesgue norms in terms of the boundary data and the initial data. In section \(5\) we focus on proving the continuous dependence of the solution on the coefficients of Forchheimer polynomial \(g\). In order to obtain this, we first establish the perturbed monotonicity for our degenerate partial differential equations, see in \((2.23)\). It is then proved in Theorem \((5.1)\) that the difference between the two solutions, which corresponds to two different coefficient vectors \(\vec{a}_1\) and \(\vec{a}_2\) is estimated in terms of \(|\vec{a}_1 - \vec{a}_2|\), see in \((5.2)\).

In section \(6\) we study in Theorem \((6.1)\) the convergence and in Theorem \((6.2)\) the dependence on coefficients of Forchheimer polynomial of the approximated solution to the problem \((3.5)\). Furthermore, we can specify the convergent rate. In section \(7\) we study the fully discrete version of problem \((3.5)\). In Lemma \((7.1)\) the stability of the approximated solution is proved. Theorems \((7.2)\) and \((7.3)\) are for studying the error estimates and the convergent rate. In section \(8\), the numerical experiments in the two-dimensions using the standard finite elements \(P_1\) are presented regarding the convergence rates and the dependence of the solution on the physical parameters.

2. Preliminaries and auxiliaries. We consider a fluid in porous medium occupying a bounded domain \(\Omega \subset \mathbb{R}^d, d \geq 2\) with boundary \(\Gamma\). Let \(x \in \mathbb{R}^d, 0 < T < \infty\) and \(t \in (0, T]\) be the spatial and time variables respectively. The fluid flow has velocity \(v(x, t) \in \mathbb{R}^d\), pressure \(p(x, t) \in \mathbb{R}\) and density \(\rho(x, t) \in \mathbb{R}_+\).

The Darcy–Forchheimer equation is studied in \([2, 19, 20]\) of the form

\[ -\nabla p = \sum_{i=0}^{N} a_i |v|^{\alpha_i} v. \] (2.1)

where \(N \geq 1, a_0 = 0 < a_1 < \ldots < a_N\) are real (not necessarily integral) numbers, and the coefficients satisfy \(a_0, a_N > 0\) and \(a_1, \ldots, a_{N-1} \geq 0\).

In order to take into account the presence of density in the generalized Forchheimer equation, we modify \((2.1)\) using the dimensional analysis by Muskat \([34]\) and Ward \([42]\). They proposed the following equation for both laminar and turbulent flows in porous media:

\[ -\nabla p = F(v^{a_k \frac{\alpha_k}{2} + a_{-1} - 1} \rho^{a_k - a_k}), \text{ where } F \text{ is a function of one variable.} \] (2.2)

In particular, when \(a = 1, 2\), Ward \([42]\) established from experimental data that

\[ -\nabla p = \frac{\mu}{k} v + c_F \frac{\rho}{\sqrt{k}} |v| v, \text{ where } c_F > 0. \] (2.3)

Combining \((2.1)\) with the suggestive form \((2.2)\) for the dependence on \(\rho\) and \(v\), we propose the following equation

\[ -\nabla p = \sum_{i=0}^{N} a_i \rho^{a_i} |v|^{a_i} v. \] (2.4)

Here, the viscosity and permeability are considered constant, and we do not specify the dependence of \(a_i\)'s on them.

Multiplying equation \((2.4)\) to \(\rho\), we obtain

\[ g(|\rho v|) \rho v = -\rho \nabla p, \] (2.5)

where the function \(g\) is a generalized polynomial with non-negative coefficients. More precisely, the function \(g : \mathbb{R}_+ \to \mathbb{R}_+\) is of the form

\[ g(s) = a_0 s^{a_0} + a_1 s^{a_1} + \cdots + a_N s^{a_N}, \quad s \geq 0, \] (2.6)

where \(N \geq 1, a_0 = 0 < a_1 < \ldots < a_N\) are real (not necessarily integral) numbers. The coefficients satisfy \(a_0, a_N > 0\) and \(a_1, \ldots, a_{N-1} \geq 0\). The number \(a_N\) is the degree of \(g\) and is denoted by \(\text{deg}(g)\). Denote the
vectors of powers and coefficients by $\vec{a} = (a_0, \ldots, a_N)$ and $\vec{a} = (a_0, \ldots, a_N)$. The class of functions $g(s)$ as in (2.6) is denoted by $FP(N, \vec{a})$, which is the abbreviation of “Forchheimer polynomials.” When the function $g$ in (2.6) belongs to $FP(N, \vec{a})$, it is referred to as the Forchheimer polynomial.

For slightly compressible fluids, the state equation is

$$\frac{d\rho}{dp} = \frac{\rho}{\kappa}, \quad \kappa = \text{const.}, \kappa \gg 1 \quad (2.7)$$

which yields

$$\rho \nabla p = \kappa \nabla \rho. \quad (2.8)$$

It follows form (2.6) and (2.8) that

$$g(|\rho v|) \rho v = -\kappa \nabla \rho. \quad (2.9)$$

Solving for $\rho v$ from (2.9) gives

$$\rho v = -\kappa K(|\nabla \rho|) \nabla \rho, \quad (2.10)$$

where the function $K : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is defined for $\xi \geq 0$ by

$$K(\xi) = \frac{1}{g(s(\xi))}, \text{ with } s = s(\xi) \text{ being the unique non-negative solution of } sg(s) = \xi. \quad (2.11)$$

The continuity equation is

$$\phi \rho_t + \text{div}(\rho v) = f, \quad (2.12)$$

where $\phi$ is the porosity, and $f$ is the external mass flow rate.

By rescaling the coefficients of the conductivity function $K(\cdot)$, we can assume $\kappa = 1$. We introduce the momentum variables $m = \rho v$.

Combining (2.10) and (2.12), we obtain equations in a density-momentum formulation

$$\begin{cases} m + K(|\nabla \rho|) \nabla \rho = 0, \\ \phi \rho_t + \nabla \cdot m = f. \end{cases} \quad (2.13)$$

We will study the initial boundary value problem (IVBP) associated with the coupled system (2.13). We will derive estimates for the solution, and establish their continuous dependence on the coefficients of the function $g(s)$ in (2.6). As seen below, the generalized permeability tensor $K$ is degenerate to zero when $|\nabla \rho| \rightarrow \infty$. This was not considered in existing papers, e.g. in [13, 30, 35]. Therefore, it creates an additional challenge and requires extra care in the proof and analysis.

Let $g = g(s, \vec{a})$ in $FP(N, \vec{a})$. The following numbers are frequently used in our calculations:

$$\chi(\vec{a}) = \max\{a_0, a_1, \ldots, a_N, \frac{1}{a_0}, \frac{1}{a_N}\} \in [1, \infty), \quad (2.14)$$

$$a = \frac{a_N}{a_N + 1} = \frac{\deg g}{\deg g + 1} \in (0, 1), \quad \beta = 2 - a \in (1, 2), \quad \lambda = \frac{2 - a}{1 - a} = \frac{\beta}{\beta - 1} \in (2, \infty). \quad (2.15)$$

**LEMMA 2.1** (cf. [2][19], Lemma 2.1). Let $g(s, \vec{a})$ be in class $FP(N, \vec{a})$. For any $\xi \geq 0$, One has

(i) $K : [0, \infty) \rightarrow (0, a_0^{-1}]$ and it decreases in $\xi$.

(ii) For any $n \geq 1$, the function $K(\xi)\xi^n$ increases and $K(\xi)\xi^n \geq 0$. 

(iii) Type of degeneracy
\[ c_0(1 + \zeta)^{-a} \leq K(\zeta, \vec{a}) \leq c_1 (1 + \zeta)^{-a}. \tag{2.16} \]

(iv) For all \( n \geq 1, \delta > 0, \)
\[ c_2(\xi^{m-a} - \delta^{m-a}) \leq K(\zeta, \vec{a})\xi^m \leq c_3 \xi^{m-a}, \tag{2.17} \]
where \( c_0, c_1, c_2, c_3 \) depend on \( N \) and \( \chi(\vec{a}), \alpha_N \) only.
In particular, when \( m = 2, \delta = 1, \) one has
\[ c_2(\xi^{2-a} - 1) \leq K(\zeta, \vec{a})\xi^2 \leq c_3 \xi^{2-a}. \tag{2.18} \]

(v) Relation with its derivative
\[-aK(\zeta) \leq K'(\zeta)\zeta \leq 0. \tag{2.19} \]

We define
\[ H(\zeta, \vec{a}) = \int_0^{\xi^2} K(\sqrt{s}, \vec{a})ds \quad \text{for } \zeta \geq 0. \tag{2.20} \]

When vector \( \vec{a} \) is fixed, we denote \( K(\cdot, \vec{a}) \) and \( H(\cdot, \vec{a}) \) by \( K(\cdot) \) and \( H(\cdot) \), respectively. The function \( H(\zeta) \) can be compared with \( \zeta \) and \( K(\zeta) \) by
\[ K(\zeta)\zeta^2 \leq H(\zeta) \leq 2K(\zeta)\zeta^2, \quad c_4(\zeta^{2-a} - 1) \leq H(\zeta) \leq c_5 \zeta^{2-a}, \tag{2.21} \]
where \( c_4, c_5 > 0 \) depend on \( \chi(\vec{a}). \)

For convenience, we use the following notations: let \( \vec{x} = (x_1, x_2, \ldots, x_d) \) and \( \vec{x}' = (x'_1, x'_2, \ldots, x'_d) \) be two arbitrary vectors of the same length, including possible length 1. We denote by \( \vec{x} \vee \vec{x}' \) and \( \vec{x} \wedge \vec{x}' \) their maximum and minimum vectors, respectively, with components \( (\vec{x} \vee \vec{x}')_j = \max(x_j, x'_j) \) and \( (\vec{x} \wedge \vec{x}')_j = \min(x_j, x'_j) \).

**Lemma 2.2.** Let \( S = \{ \vec{a} = (a_0, \ldots, a_N) : a_0, a_N > 0, a_1, \ldots, a_{N-1} \geq 0 \} \) be the set of admissible \( \vec{a} \). For any coefficient vectors \( \vec{a}, \vec{a}' \in S, \) and any \( y, y' \in \mathbb{R}^d, \) one has

(i)
\[
|K(|y|, \vec{a})y - K(|y'|, \vec{a}')y'| \leq (1 + a)|y - y'| \int_0^1 K(|\gamma(t)|, \vec{b}(t))dt \\
+ d_0(|y| \vee |y'|)|\vec{a} - \vec{a}'| \int_0^1 K(|\gamma(t)|, \vec{b}(t))dt.
\tag{2.22} \]

(ii)
\[
(K(|y|, \vec{a})y - K(|y'|, \vec{a}')y') \cdot (y - y') \geq (1 - a)|y - y'|^2 \int_0^1 K(|\gamma(t)|, \vec{b}(t))dt \\
- d_0(|y| \vee |y'|)|\vec{a} - \vec{a}'||y - y'| \int_0^1 K(|\gamma(t)|, \vec{b}(t))dt,
\tag{2.23} \]
where \( a \in (0, 1) \) is defined in \( 2.15, \)
\[ \gamma(t) = ty + (1 - t)y', \quad \vec{b}(t) = \vec{a} + (1 - t)\vec{a}', \quad d_0 = N \left( \min\{a_0, a_0', (1 + a_N)\alpha_N, (1 + a_N)\alpha_N'\} \right)^{-1}. \]

In particular, if \( \vec{a} = \vec{a}' \) then \( 2.22 \) and \( 2.23 \) become
Estimation of $I$

From (i) we find that

\[ |K(|y'|)y' - K(|y|)y| \leq (1 + a)|y' - y| \int_0^1 K(|\gamma(t)|, \widehat{a})\,dt. \tag{2.24} \]

By the Mean Value Theorem, we have

\[ (K(|\gamma'|)\gamma' - K(|\gamma|)\gamma)\cdot (y' - y) \geq (1 - a)|y' - y|^2 \int_0^1 K(|\gamma(t)|, \widehat{a})\,dt. \tag{2.25} \]

Proof. (i). Let $\widehat{a}, \widehat{a}' \in S$ and $y, y', \overline{k} \in \mathbb{R}^d$.

Case 1: The origin does not belong to the segment connecting $y'$ and $y$. Define

\[ z(t) = K(|\gamma(t)|, \widehat{b}(t))\gamma(t) \cdot \overline{k}. \]

By the Mean Value Theorem, we have

\[ I \overset{\text{def}}{=} |K(|y|, \widehat{a})y - K(|y'|, \widehat{a}')y'| \cdot \overline{k} = z(1) - z(0) = \int_0^1 z'(t)\,dt. \]

Elementary calculations give

\[ I = \int_0^1 f_1(t)\,dt + \int_0^1 f_2(t)\,dt \overset{\text{def}}{=} I_1 + I_2, \tag{2.26} \]

where

\[ f_1(t) = K(|\gamma(t)|, \widehat{b}(t))(y - y') \cdot \overline{k} + K_{\xi}(|\gamma(t)|, \widehat{b}(t)) \frac{\gamma(t) \cdot (y - y')}{|\gamma(t)|} \gamma(t) \cdot \overline{k}, \]

\[ f_2(t) = K_{\delta}(|\gamma(t)|, \widehat{b}(t))(\widehat{a} - \widehat{a}')\gamma(t) \cdot \overline{k}. \]

- Estimation of $I_1$. Since

\[ |f_1(t)| \leq \left| K(|\gamma(t)|, \widehat{b}(t))(y - y') + K_{\xi}(|\gamma(t)|, \widehat{b}(t)) \frac{\gamma(t) \cdot (y - y')}{|\gamma(t)|} \gamma(t) \right| |\overline{k}| \leq (1 + a)K(|\gamma(t)|, \widehat{b}(t))|y - y'| |\overline{k}|, \]

we find that

\[ I_1 \leq \int_0^1 |f_1(t)|\,dt \leq (1 + a)|y - y'| |\overline{k}| \int_0^1 K(|\gamma(t)|, \widehat{b}(t))\,dt. \tag{2.27} \]

- Estimation of $I_2$. We find the partial derivative of $K(\xi, \overline{a})$ in $\overline{a}$. For $i = 0, 1, \ldots, N$, taking the partial derivative in $a_i$ of the identity $K(\xi, \overline{a}) = 1/g(s(\xi, \overline{a}), \overline{a})$, we find that

\[ K_{a_i}(\xi, \overline{a}) = -\frac{g_{a_i} + g_s \cdot s_{a_i}}{g^2} = -K(\xi, \overline{a}) \frac{g_{a_i} + g_s \cdot s_{a_i}}{g}. \]

From $sg(s, \overline{a}) = \xi$, we have for $i = 0, 1, \ldots, N$, $s_{a_i} \cdot g + s \cdot (g_{a_i} + g_s \cdot s_{a_i}) = 0$, which implies $s_{a_i} = \frac{-s \cdot g_{a_i}}{g + s \cdot g_s}$.

Hence, we obtain

\[ K_{a_i}(\xi, \overline{a}) = -K(\xi, \overline{a}) \frac{g_{a_i} + g_s \cdot s_{a_i}}{g} = -K(\xi, \overline{a}) \frac{g_{a_i}}{g + s \cdot g_s} = -K(\xi, \overline{a}) \frac{a_{a_i}}{g + s \cdot g_s}. \tag{2.28} \]

This shows that

\[ \sum_{i=0}^N |K_{a_i}(\xi, \overline{a})| \leq K(\xi, \overline{a}) \frac{1 + s^{a_1} + \cdots + s^{a_N}}{a_0 + (1 + a_1)a_1s^{a_1} + \cdots + (1 + a_N)a_Ns^{a_N}}. \]
Using inequality $x^\gamma \leq x^{\gamma_1} + x^{\gamma_2}$ for all $x > 0$, $\gamma_1 \leq \gamma \leq \gamma_2$, we have $s^{\alpha_1}, \ldots, s^{\alpha_N} \leq 1 + s^{\alpha_N}$. Hence,
\[
\sum_{i=0}^{N} |K_{d_i}(\xi, \tilde{a})| \leq K(\xi, \tilde{a}) \frac{N(1 + s^{\alpha_N})}{d_0 + (1 + \alpha_N)d_N s^{\alpha_N}} \leq d(\tilde{a}) K(\xi, \tilde{a}),
\]
where $d(\tilde{a}) = N(\min(d_0, (1 + \alpha_N)d_N))^{-1}$. Thus,
\[
|K_{d_i}(\xi, \tilde{a})| \leq d(\tilde{a}) K(\xi, \tilde{a}). \tag{2.29}
\]
Using the estimate (2.29), we bound
\[
|f_2(t)| \leq |K_{d_i}(\gamma(t), \tilde{b}(t))| |\tilde{a} - \tilde{a}'| |\gamma(t)| |\tilde{k}| \leq d(\tilde{b}(t)) K(|\gamma(t)|, \tilde{b}(t)) |\gamma(t)| |\tilde{a} - \tilde{a}'| |\tilde{k}|. \tag{2.30}
\]
Since $a_i, a'_i$ is positive for $i = 0, N$, the number $d(\tilde{b}(t))$, for all $t \in [0, 1]$, can be bounded by $d(\tilde{b}(t)) \leq d_0$. Using the fact $|\gamma(t)| \leq |\gamma| + |\gamma_\epsilon|$, (2.30) yields
\[
|f_2(t)| \leq d_0 K(|\gamma(t)|, \tilde{b}(t)) (|\gamma| + |\gamma_\epsilon|) |\tilde{a} - \tilde{a}'| |\tilde{k}|, \tag{2.31}
\]
and consequently,
\[
I_2 \leq \int_0^1 |f_2(t)| dt \leq d_0 (|\gamma| + |\gamma_\epsilon|) |\tilde{a} - \tilde{a}'| |\tilde{k}| \int_0^1 K(|\gamma(t)|, \tilde{b}(t)) |\gamma(t)| dt. \tag{2.32}
\]
Thus, we obtain (2.22) by combining (2.26), (2.27) and (2.32).

Case 2: The origin belongs to the segment connect $y'$, $y$. We replace $y'$ by $y' + y_\epsilon$ so that $0 \in [y' + y_\epsilon, y]$ and $y_\epsilon \to 0$ as $\epsilon \to 0$. Apply the above inequality for $y$ and $y' + y_\epsilon$, then let $\epsilon \to 0$.

(ii) If $\tilde{k} = y - y'$ then
\[
K_\xi(|\gamma(t)|, \tilde{b}(t)) \geq -a \frac{K(|\gamma(t)|, \tilde{b}(t))}{|\gamma(t)|}.
\]
This implies
\[
f_1(t) \geq (1 - a)K(|\gamma(t)|, \tilde{b}(t)) |y - y'|^2. \tag{2.33}
\]
It follows (2.33) that
\[
\int_0^1 f_1(t) dt \geq (1 - a) |y' - y|^2 \int_0^1 K(|\gamma(t)|, \tilde{b}(t)) dt, \tag{2.34}
\]
and from (2.31), we see that
\[
I_2 \geq -\int_0^1 |f_2(t)| dt \geq -d_0 (|\gamma| + |\gamma_\epsilon|) |\tilde{a} - \tilde{a}'| |\gamma| |\gamma_\epsilon| \int_0^1 K(|\gamma(t)|, \tilde{b}(t)) |\gamma(t)| dt. \tag{2.35}
\]
Thus, we obtain (2.23) by combining (2.26), (2.34) and (2.35). □

Now we derive the trace estimates suitable for our nonlinear problem.

Lemma 2.3. Assume $v(x)$ is a function defined on $\Omega$.
(i) If $|v| \in W^{1,1}(\Omega)$ then there is a positive constant $C_1$ depending on $\Omega, \beta$ such that for all $\epsilon > 0$,
\[
\int_{\Gamma} |v| d\sigma \leq C_1 \|v\| + \epsilon \|\nabla v\|_{0,\beta}^\beta + C_4 \epsilon^{-\frac{1}{\beta-1}}. \tag{2.36}
\]
(ii) If $u \in L^\infty(\Gamma)$ and $|v| \in W^{1,1}(\Omega)$ then there exists $C_2(\Omega, \beta) > 0$ such that for all $\epsilon > 0$,
\[
|\langle u, v \rangle| \leq \epsilon \left( \|v\|^2 + \|\nabla v\|_{0,\beta}^\beta \right) + C_2 \left( \epsilon^{-1} \|u\|^2_{L^\infty(\Gamma)} + \epsilon^{-\frac{1}{\beta-1}} \|u\|^\beta_{L^\infty(\Gamma)} \right). \tag{2.37}
\]
Consequently,

\[ |\langle u, v \rangle| \leq \frac{1}{4} \left( \| v \|^2 \right. + \| \nabla v \|^2_{0, \beta} \left) + C_3 \left(1 + \| u \|^p_{L^\infty(\tau)} \right) \]  

(2.38)

for a constant \( C_3 > 0 \).

**Proof.** We recall the trace theorem

\[ \int_{\Gamma} |v| d\sigma \leq C_* \int_{\Omega} |v| d\sigma + C_* \int_{\Omega} |\nabla v| d\sigma, \]

for all \( v \in W^{1,1}(\Omega) \), where \( C_* \) is a positive constant depending on \( \Omega \). Using Young's inequality, we obtain (2.36).

(ii) We have

\[ |\langle u, v \rangle| \leq \| u \|_{L^\infty(\Gamma)} \int_{\Gamma} |v| d\sigma. \]  

(2.39)

Using (2.36) and Young's inequality give

\[ |\langle u, v \rangle| \leq \| u \|_{L^\infty(\Gamma)} \int_{\Gamma} |v| d\sigma. \]

Using (2.36) and Young's inequality give

\[ |\langle u, v \rangle| \leq \| u \|_{L^\infty(\Gamma)} \left( \delta \| v \|^2 + \frac{C^2_2|\Omega|}{4}\delta^{-1} + \delta \| \nabla v \|^2_{0, \beta} + C_i \delta^{-1} \right). \]

If \( \| u \|_{L^\infty(\Gamma)} = 0 \) then (2.37) clearly holds true.

Otherwise, selecting \( \delta = \varepsilon \| u \|_{L^\infty(\Gamma)}^{-1} \) and the fact that \( \frac{1}{\delta} + 1 = \lambda \), we obtain (2.37).

Estimate (2.38) follows by choosing \( \varepsilon = 1/4 \) in (2.37) and using Young's inequality.

We recall a discrete version of Gronwall Lemma in backward difference form, which is useful later. It can be proven without much difficulty by following the ideas of the proof in Gronwall Lemma.

**Lemma 2.4.** Assume \( \ell > 0, 1 - \ell \Delta t > 0 \) and the nonnegative sequences \( \{a_n\}_{n=0}^\infty, \{g_n\}_{n=0}^\infty \) satisfying

\[ \frac{a_n - a_{n-1}}{\Delta t} - \ell a_n \leq g_n, \quad n = 1, 2, 3 \ldots \]

then

\[ a_n \leq (1 - \ell \Delta t)^{-n} \left( a_0 + \Delta t \sum_{i=1}^n (1 - \ell \Delta t)^{i-1} g_i \right). \]  

(2.40)

**Proof.** Let \( \tilde{a}_n = (1 - \ell \Delta t)^n a_n \). Simple calculation shows that

\[ \frac{\tilde{a}_n - \tilde{a}_{n-1}}{\Delta t} = (1 - \ell \Delta t)^{n-1} \left( \frac{a_n - a_{n-1}}{\Delta t} - \ell a_n \right) \leq (1 - \ell \Delta t)^{n-1} g_n. \]

Summation over \( n \) leads to

\[ \frac{\tilde{a}_n - \tilde{a}_0}{\Delta t} \leq \sum_{i=1}^n (1 - \ell \Delta t)^{i-1} g_i, \]

and hence (2.40) holds true.

**Notations:** Let \( L^2(\Omega) \) be the set of square integrable functions on \( \Omega \) and \( (L^2(\Omega))^d \) the space of \( d \)-dimensional vectors with all the components in \( L^2(\Omega) \). We denote \( \langle \cdot, \cdot \rangle \) the inner product in either \( L^2(\Omega) \) or \( (L^2(\Omega))^d \). The notation \( \| \cdot \| \) means scalar norm \( \| \cdot \|_{L^2(\Omega)} \) or vector norm \( \| \cdot \|_{(L^2(\Omega))^d} \) and \( \| \cdot \|_{L^p} = \| \cdot \|_{L^p(\Omega)} \) represents the standard Lebesgue norm. Notation \( \| \cdot \|_{L^p(\Omega)} = \| \cdot \|_{L^p(\Omega)} \) means the mixed Lebesgue norm.

For \( 1 \leq q \leq \infty \) and \( m \) any nonnegative integer, let \( W^{m,q}(\Omega) = \{ u \in L^q(\Omega), D^a u \in L^q(\Omega), |a| \leq m \} \) denote a Sobolev space endowed with the norm \( \| u \|_{m,q} = \left( \sum_{|a| \leq m} \| D^a u \|^q_{L^q(\Omega)} \right)^{1/q} \). Define \( H^m(\Omega) = W^{m,2}(\Omega) \) with the norm \( \| \cdot \|_m = \| \cdot \|_{m,2} \).
Our estimates make use of coefficient-weighted norms. For some strictly positive, bounded function, we denote the weighted $L^2$-norm by $\|f\|_\omega$ by

$$\|f\|^2_\omega = \int_{\Omega} \omega |f|^2 \, dx,$$

and if $0 < \omega_* \leq \omega(x) \leq \omega^*$ throughout $\Omega$, we have the equivalent

$$\sqrt{\omega_*} \|f\| \leq \|f\|_\omega \leq \sqrt{\omega^*} \|f\|.$$

We will also use weighted versions of Cauchy-Schwarz. With such a weight function $\omega$, we can bound a standard inner product as

$$(f, g) \leq \|f\|_\omega \|g\|_{\omega^{-1}}.$$

Throughout this paper, we use short hand notations, $\|\rho(t)\| = \|\rho(\cdot, t)\|_{L^2(\Omega)}$, and $\rho^0(\cdot) = \rho(\cdot, 0)$. The letters $C, C_0, C_1, C_2 \ldots$ represent positive generic constants. Their values depend on exponents, coefficients of polynomial $g$, the spatial dimension $d$ and domain $\Omega$, independent of the initial data and boundary data, size of mesh and time step. These constants may be different from place to place.

### 3. A mixed finite element approximation.

In this section, we will present the mixed weak formulation of the Forchheimer equation. We consider the initial boundary value problem (IVBP) associated with (2.13):

\begin{equation}
\begin{aligned}
\begin{cases}
m + K(\|\rho\|) \nabla \rho &= 0 &\text{in } \Omega \times (0, T), \\
\phi \rho_t + \nabla \cdot m &= f &\text{in } \Omega \times (0, T), \\
m \cdot \nu &= \psi(x, t) &\text{in } \Gamma \times (0, T), \\
\rho(x, 0) &= \rho^0(x) &\text{in } \Omega \times (0, T).
\end{cases}
\end{aligned}
\end{equation}

where $\nu$ is an outer normal vector of boundary $\Gamma$, $\rho^0(x)$ and $\psi(x, t)$ are smooth functions.

Assume that $\phi(x) \in C^1(\Omega)$ and $0 < \phi_* < \phi(x) < \phi^*$ for all $x \in \Omega$ then the system (3.1) reduces to the equation form

$$\rho_t - \nabla \cdot (\phi^{-1} K(\|\rho\|) \nabla \rho) + \nabla \phi^{-1} K(\|\rho\|) \nabla \rho - f(x, t) = 0.$$

This equation is a nonlinear degenerate parabolic equation as the density gradient approaches to infinity.

The existence and theory of regularity for degenerate parabolic of this type is studied in [11, 12, 25, 26, 51]. Define $\mathcal{R} = H^1(\Omega)$ and the space

$$\mathcal{M} = H(div, \Omega) = \left\{ m \in (L^2(\Omega))^d, \nabla \cdot m \in L^2(\Omega) \right\}$$

with the norm defined by $\|m\|^2_\mathcal{M} = \|m\|^2 + \|\nabla \cdot m\|^2$.

The variational formulation of (3.1) is defined as follows: Find $(m, \rho) : I = (0, T) \to \mathcal{M} \times \mathcal{R}$ such that

\begin{equation}
\begin{aligned}
(m, z) + \int_{\Omega} K(\|\rho\|) \nabla \rho \cdot z \, dx &= 0, & z \in \mathcal{M}, \\
(\phi \rho_t, r) - (m, \nabla r) &= \langle f, r \rangle - \langle \psi, r \rangle, & r \in \mathcal{R}
\end{aligned}
\end{equation}

with $\rho(x, 0) = \rho^0(x)$.

Let $\{\mathcal{T}_h\}_h$ be a family of quasiuniform triangulations of $\Omega$ with $h$ being the maximum diameter of the mesh elements. Let $\mathcal{M}_h$, $\mathcal{R}_h$ be the space of discontinuous piecewise polynomials of degree $k \geq 0$ over $\mathcal{T}_h$. Let $\mathcal{M}_h \times \mathcal{R}_h$ be the mixed element spaces approximating the space $\mathcal{M} \times \mathcal{R}$.

For density, we use the standard $L^2$-projection operator, see in [9], $\pi : \mathcal{R} \to \mathcal{R}_h$, satisfying

$$(\pi \rho - \rho, r) = 0, \quad \rho \in \mathcal{R}, \forall r \in \mathcal{R}_h.$$

This projection has well-known approximation properties, e.g. [5, 6, 27].
i. For all $\rho \in H^3(\Omega), s \in (0,1)$, there is a positive constant $C_0$ such that
\[
\|\pi \rho\|_s \leq C_0\|\rho\|_s.
\] (3.3)

ii. There exists a positive constant $C_1$ such that for all $\rho \in W^{s,q}(\Omega),$
\[
\|\pi \rho - \rho\|_{0,q} \leq C_1 h^s\|\rho\|_{s,q}, \quad 0 \leq s \leq k+1, 1 \leq q \leq \infty.
\] (3.4)

When $q = 2$, in short hand we write (3.4) as
\[
\|\pi \rho - \rho\| \leq C_1 h^s\|\rho\|_s.
\]

The semidiscrete formulation of (3.2) can read as follows: Find a pair $(m_h, \rho_h) : I \to \mathcal{M}_h \times \mathcal{R}_h$ such that
\[
(m_h, z) + \{ K(|\nabla \rho_h|)\nabla \rho_h, z\} = 0, \quad \forall z \in \mathcal{M}_h,
\]
\[
(\phi \rho_{h,t}, r) - (m_h, \nabla r) = \{ f, r \} - \{ \psi, r \}, \quad \forall r \in \mathcal{R}_h
\] (3.5)

with initial data $\rho_h^0 = \pi \rho^0(x)$.

Let $(t_i)_{i=1}^N$ be the uniform partition of $[0,T]$ with $t_i = i\Delta t$, for time step $\Delta t > 0$. We define $\phi^i = \phi(\cdot, t_i)$.

The discrete time mixed finite element approximation to (3.2) is defined as follows: For given $\{\phi_{i}\}_{i=1}^N \in L^2(\Omega), \{\psi_{i}\}_{i=1}^N \in L^\infty(\Omega)$. Find a pair $(m_h^i, \rho_h^i)$ in $\mathcal{M}_h \times \mathcal{R}_h, i = 1,2,\ldots,N$ such that
\[
\begin{aligned}
(m_h^i, z) + \{ K(|\nabla \rho_h^i|)\nabla \rho_h^i, z\} &= 0, \quad \forall z \in \mathcal{M}_h, \\
(\phi \rho_{h,t}^i - \rho_{h,t}^{i-1} / \Delta t, r) - (m_h^i, \nabla r) &= \{ f^i, r \} - \{ \psi^i, r \}, \quad \forall r \in \mathcal{R}_h.
\end{aligned}
\] (3.6)

4. Stability of semidiscrete approximation. We study the equations (3.2), and (3.5) with fixed functions $g(s)$ in (2.5) and (2.6). Therefore, the exponents $a_i$ and coefficients $a_i$ are all fixed, and so are the functions $K(\xi), H(\xi)$ in (2.13), (2.20).

With the properties (2.16), (2.17), (2.19), the monotonicity (2.25), and by classical theory of monotone operators in (333944), the authors in (1721) proved the global existence and uniqueness of the weak solution of the equation (3.2). For the priori estimates, we assume that the weak solution is a sufficient regularity in both $x$ and $r$ variables.

**Theorem 4.1.** Let $(\rho_h, m_h)$ be a solution to the problem (3.5). Then, there exists a positive constant $C$ such that
\[
\|\rho_h\|_{L^\infty(\Omega)}^2 + \|\nabla \rho_h\|_{L^2(\Omega)} + \|m_h\|_{L^\infty(\Omega)} \leq C\|\rho^0\|^2 + C\mathcal{A}.
\] (4.1)

where
\[
\mathcal{A} = 1 + \|\psi(t)\|_{L^\infty(\Omega)}^2 + \|f(t)\|_{L^2(\Omega)}^2
\] (4.3)

\[
\mathcal{B} = \|\rho^0\|^2 + \|\nabla \rho^0\|_{L^2(\Omega)} + \|\psi(0)\|_{L^2(\Omega)} + \|\rho(t)\|_{L^2(\Omega)} + \|\psi(t)\|_{L^\infty(\Omega)}^2 + C\mathcal{B}.
\] (4.4)

**Proof.** Choosing $z = \nabla \rho_h$ and $r = \rho_h$ in (3.5), and adding the resultants, we find that
\[
(\phi \rho_{h,t} + K(|\nabla \rho_h|)\nabla \rho_h, \rho_h) = \{ f, \rho_h \} - \{ \psi, \rho_h \}.
\] (4.5)

The second term of the LHS in (4.5) is treated by using (2.17) as follows
\[
(K(|\nabla \rho_h|)\nabla \rho_h)dx \geq c_0 \int_{\Omega} (|\nabla \rho_h|^2 - 1)dx = c_0\|\rho_h\|_{0,\beta}^2 - c_0|\Omega|.
\] (4.6)
We use Young’s inequality and (2.36) to obtain

\[
\langle f, \rho_h \rangle - \langle \psi, \rho_h \rangle \leq \frac{1}{2} \| \rho_h \|^2 + \frac{1}{2} f^2 + \| \psi \|_{L^\infty(\Gamma)} \left\{ C_1 \| \rho_h \| + \epsilon \| \nabla \rho_h \|_{0,\beta} + C_1 e^{-\frac{1}{\rho_h}} \right\}. \tag{4.7}
\]

In view of (4.5), (4.6) and (4.7), and selecting \( \varepsilon = \frac{1}{\rho_h} (\| \psi \|_{L^\infty(\Gamma)} + 1)^{-1} \), (4.5) becomes

\[
\frac{d}{dt} \| \rho_h \|_{\phi}^2 + \frac{c_0}{2} \| \nabla \rho_h \|_{0,\beta}^2 \leq C \| \psi \|_{L^\infty(\Gamma)} \left\{ \| \rho_h \| + (\| \psi \|_{L^\infty(\Gamma)} + 1) \frac{1}{\rho_h} \right\} + C \left\{ 1 + \| \rho_h \|^2 + \| f \|^2 \right\} \leq C \| \rho_h \|_{\phi}^2 + C \left\{ 1 + \| \psi \|_{L^\infty(\Gamma)} + \| f \|^2 \right\} . \tag{4.8}
\]

Solving this differential inequality leads to

\[
\| \rho_h \|_{\phi}^2 + \frac{c_0}{2} \int_0^t \| \nabla \rho_h \|_{0,\beta}^2 \, dt \leq \| \rho_h^0 \|_{\phi}^2 + C \int_0^t \left( 1 + \| \psi \|_{L^\infty(\Gamma)} + \| f \|^2 \right) \, dt, \tag{4.9}
\]

which implies

\[
\| \rho_h \|^2 + \frac{c_0}{2} \int_0^t \| \nabla \rho_h \|_{0,\beta}^2 \, dt \leq \| \rho_h^0 \|^2 + C \int_0^t \left( 1 + \| \psi \|_{L^\infty(\Gamma)} + \| f \|^2 \right) \, dt . \tag{4.10}
\]

Note that \( \| \rho_h^0 \| = \| \pi \rho^0 \| \leq \| \rho^0 \| \). Thus inequality (4.11) holds.

(ii) Choosing \( z = \nabla \rho_{h,t} \) and \( r = \rho_{h,t} \) in (4.5), and adding the resulting equations, we obtain

\[
\| \rho_{h,t} \|_{\phi}^2 + \frac{1}{2} d \int_\Omega H(x, t) \, dx = (f, \rho_{h,t}) - \langle \psi, \rho_{h,t} \rangle = (f, \rho_{h,t}) - \frac{d}{dt} \langle \psi, \rho_h \rangle + \langle \psi_t, \rho_h \rangle , \tag{4.10}
\]

where \( H(x, t) = H(\nabla \rho_h(x, t)) \). Let

\[
\mathcal{E}(t) = \int_\Omega H(x, t) \, dx + \| \rho_h \|_{\phi}^2 + 2 \langle \psi, \rho_h \rangle .
\]

Adding (4.10) and (4.8) gives

\[
\| \rho_{h,t} \|_{\phi}^2 + \frac{c_0}{2} \| \nabla \rho_h \|_{0,\beta}^2 + \frac{1}{2} \frac{d}{dt} \mathcal{E}(t) \leq (f, \rho_{h,t}) + \langle \psi_t, \rho_h \rangle + C \| \rho_h \|_{\phi}^2 + C \left\{ 1 + \| \psi \|_{L^\infty(\Gamma)} + \| f \|^2 \right\} .
\]

Using (2.38) and Young’s inequality leads to

\[
\frac{1}{2} \| \rho_{h,t} \|_{\phi}^2 + \frac{c_0}{2} \| \nabla \rho_h \|_{0,\beta}^2 + \frac{1}{2} \frac{d}{dt} \mathcal{E}(t) \leq \frac{1}{4} \left\{ \| \rho_h \|^2 + \| \nabla \rho_h \|_{0,\beta}^2 \right\} + C \left\{ \| \psi_t \|_{L^\infty(\Gamma)} \right\} + \frac{1}{2} \| f \|^2 \right\} + C \left\{ 1 + \| \psi \|_{L^\infty(\Gamma)} + \| f \|^2 \right\} .
\]

Integrating in time gives

\[
\int_0^T \left( \| \rho_{h,t} \|_{\phi}^2 + \frac{c_0}{2} \| \nabla \rho_h \|_{0,\beta}^2 \right) \, dt + \mathcal{E}(t) \leq C \int_0^T \left( \| \rho_h \|^2 + \| \nabla \rho_h \|_{0,\beta}^2 \right) \, dt + C \int_0^T \left\{ 1 + \| \psi \|_{L^\infty(\Gamma)} + \| \psi_t \|_{L^\infty(\Gamma)} + \| f \|^2 \right\} \, dt + \mathcal{E}(0) .
\]

Then using (4.11), we obtain

\[
\int_0^T \left( \| \rho_{h,t} \|_{\phi}^2 + \frac{c_0}{2} \| \nabla \rho_h \|_{0,\beta}^2 \right) \, dt + \int_\Omega H(x, t) \, dx + \| \rho_h \|^2 \leq -2 \langle \psi, \rho_h \rangle + C \left\{ \| \psi_t \|_{L^\infty(\Gamma)} + \| \psi \|_{L^\infty(\Gamma)} + \| f \|^2 \right\} + C \| \rho_0 \|^2 + \mathcal{E}(0) . \tag{4.11}
\]
Applying (2.37) to the first term of the RHS in (4.11) and using the fact that $c_3 (|\nabla \rho_h|^{\beta} - 1) \leq H(x, t) \leq 2c_2 |\nabla \rho_h|^{\beta}$, we have

$$
\int_0^T \|\rho_{h,i}\|_{\phi}^2 dt + c_0 \int_0^T \|\nabla \rho_{h,i}\|_{0,\beta}^2 dt + c_3 \|\nabla \rho_h\|_{0,\beta}^2 + \|\rho_h\|_{\phi}^2 \leq 2\varepsilon \left( \|\rho_h\|^2 + \|\nabla \rho_h\|_{0,\beta}^2 \right) + C \left( \varepsilon^{-1} \|\psi\|_{L^\infty(\Gamma)} + \varepsilon^{1/4} \|\psi\|_{L^4(\Gamma)} \right) + C \left( \|\psi_t\|_{L^2(\Omega)} + \varepsilon \right) + C\|\rho^0\|^2 + \mathcal{E}(0). \tag{4.12}
$$

Then taking $\varepsilon = \min\{c_3, 1/4\}$ and using Young’s inequality, (4.12) leads to

$$
\int_0^T \|\rho_{h,i}\|_{\phi}^2 dt + c_0 \int_0^T \|\nabla \rho_{h,i}\|_{0,\beta}^2 dt + \frac{c_3}{2} \|\nabla \rho_h\|_{0,\beta}^2 + \frac{1}{2} \|\rho_h\|_{\phi}^2 \leq C \left( \|\psi_t\|_{L^2(\Omega)} + \varepsilon \right) + C\|\rho^0\|^2 + \mathcal{E}(0). \tag{4.13}
$$

Note that

$$
\mathcal{E}(0) \leq C \left( \|\rho^0\|^2 + \|\nabla \rho^0\|_{0,\beta}^2 + \|\psi(0)\|_{L^2(\Omega)}^2 \right). \tag{4.14}
$$

Putting estimates (4.13) and (4.14) together, we obtain the first part of (4.2).

Now choosing $z = m_h$ in the first equation of (3.5) gives $\|m_h\|^2 + \langle K(|\nabla \rho_h|) \nabla \rho_h, m_h \rangle = 0$, which leads to

$$
\|m_h\| \leq \|K(|\nabla \rho_h|) \nabla \rho_h\| \leq C \left( \int_\Omega \langle K(|\nabla \rho_h|) \nabla \rho_h \rangle^2 dx \right)^{1/2} \leq C \|\nabla \rho_h\|_{0,\beta}^\beta. \tag{4.15}
$$

This, (4.13) and (4.14) imply the second part of (4.2). The proof is complete.

**Remark 4.2.** The equation (3.5) can be interpreted as the finite system of ordinary differential equations in the coefficients of $(m_h, \rho_h)$ with respect to basis of $\mathcal{M}_h \times \mathcal{R}_h$. The stability estimates (4.11) and (4.12) suffice to establish the local existence of $(m_h(t), \rho_h(t))$ for all $t \in (0, T)$.

The uniqueness of the approximation solution comes from the monotonicity of the operator, see in [19]. In fact, assume that for $i = 1, 2$, $\{m_{h,i}, \rho_{h,i}\}$ are two solutions of (3.5). Let $\mu = m_h, 1 - m_h, \vartheta = \rho_h - \rho_h, 2$. Then

$$
(\mu, z) + \langle K(|\nabla \rho_h|) \nabla \rho_{h,1} - K(|\nabla \rho_h|) \nabla \rho_{h,2}, z \rangle = 0, \quad z \in \mathcal{M}_h,
$$

$$(\varphi_t, r) - \langle \mu, \nabla r \rangle = 0, \quad r \in \mathcal{R}_h. \tag{4.16}
$$

It is easily to see that with $z = \nabla \vartheta$ and $r = \vartheta$ in (4.11)

$$
(\varphi_t, \vartheta) + \langle K(|\nabla \rho_{h,1}|) \nabla \rho_{h,1} - K(|\nabla \rho_{h,2}|) \nabla \rho_{h,2}, \nabla \vartheta \rangle = 0.
$$

Thanks to the monotonicity (2.25), we see that

$$
\frac{1}{2} \frac{d}{dt} \|\vartheta\|_{\phi}^2 + C \int_\Omega \left( \int_0^1 K(|\gamma(s)|) |ds\rangle^2 \right) |\nabla \vartheta|^2 dx \leq 0. \tag{4.17}
$$

Choosing $z = \mu$ in the first equation of (4.16) and using the fact the function $K(\cdot) \leq a_0^{-1}$ lead to

$$
\|\mu\|^2 \leq C \int_\Omega \left( \int_0^1 K(|\gamma(s)|) |ds\rangle^2 \right) |\nabla \vartheta|^2 dx \leq \int_\Omega \left( \int_0^1 K(|\gamma(s)|) |ds\rangle^2 \right) |\nabla \vartheta|^2 dx. \tag{4.18}
$$

Putting (4.16) into (4.17) gives

$$
\frac{1}{2} \frac{d}{dt} \|\vartheta\|_{\phi}^2 + \|\mu\|^2 \leq 0. \tag{4.19}
$$

This implies $\|\vartheta\|_{\phi}^2 + \|\mu\| \leq C \|\vartheta(0)\|_{\phi}^2 = 0$. Hence $\vartheta = 0$ and $\mu = 0$ a.e.
Theorem 4.3. Let $0 < t_0 < T$. Suppose $(m_h, \rho_h)$ be a solution of the problem (3.5). There exists a positive constant $C$ such that for all $t \in [t_0, T]$,

$$
\|\rho_{h,t}(t)\|^2 \leq C (1 + t_0^{-1}) \mathcal{B} + C \int_0^T (1 + \|\psi_t(r)\|_{L^\infty([T])})^{2\lambda} (1 + \|f_t(r)\|)^2 \, dt,
$$

(4.20)

where $\mathcal{B}$ is defined in (4.4).

Proof: Taking time derivative (3.5), choosing $z = \nabla \rho_{h,t}$ and $r = \rho_{h,t}$ in (3.5), we obtain the equations

$$
\begin{align*}
\langle m_{h,t}, \nabla \rho_{h,t} \rangle + \left( K'(\|\nabla \rho_h\|) \frac{\nabla \rho_h \cdot \nabla \rho_{h,t}}{\|\nabla \rho_h\|} \nabla \rho_h + K(\|\nabla \rho_h\|) \nabla \rho_{h,t}, \nabla \rho_{h,t} \right) &= 0, \\
\langle \phi_{h,t}, m_{h,t} \rangle - \langle m_{h,t}, \nabla \rho_{h,t} \rangle &= \langle f_t, \rho_{h,t} \rangle - \langle \psi_t, \rho_{h,t} \rangle.
\end{align*}
$$

(4.21)

Adding these equations yields

$$
\frac{1}{2} \frac{d}{dt} \|\rho_{h,t}\|^2 + \|K^{1/2}(\|\nabla \rho_h\|) \nabla \rho_{h,t}\|^2 = - \left( K'(\|\nabla \rho_h\|) \frac{\nabla \rho_h \cdot \nabla \rho_{h,t}}{\|\nabla \rho_h\|} \nabla \rho_h, \nabla \rho_{h,t} \right) + \langle f_t, \rho_{h,t} \rangle - \langle \psi_t, \rho_{h,t} \rangle.
$$

(4.22)

According to (2.19),

$$
\left| \left( K'(\|\nabla \rho_h\|) \frac{\nabla \rho_h \cdot \nabla \rho_{h,t}}{\|\nabla \rho_h\|} \nabla \rho_h, \nabla \rho_{h,t} \right) \right| \leq a \|K^{1/2}(\|\nabla \rho_h\|) \nabla \rho_{h,t}\|^2.
$$

(4.23)

The inequality (4.22) deduces to

$$
\frac{1}{2} \frac{d}{dt} \|\rho_{h,t}\|^2 + (1 - a) \|K(\|\nabla \rho_h\|) \nabla \rho_{h,t}\|^2 \leq \langle f_t, \rho_{h,t} \rangle - \langle \psi_t, \rho_{h,t} \rangle.
$$

(4.24)

In virtue of Young’s inequality, for all $\varepsilon > 0$

$$
\langle f_t, \rho_{h,t} \rangle \leq \varepsilon \|\rho_{h,t}\|^2 + C\varepsilon^{-1} \|f_t\|^2.
$$

(4.25)

Using Trace Theorem we obtain,

$$
\langle \psi_t, \rho_{h,t} \rangle \leq \|\psi_t\|_{L^\infty([T])} \left( \|\rho_{h,t}\| + \|\nabla \rho_{h,t}\|_{L^\infty([T])} \right).
$$

(4.26)

Again Young’s inequality gives

$$
(\|\rho_{h,t}\|, 1) \leq \varepsilon \|\rho_{h,t}\|^2 + C\varepsilon^{-1}, \quad (\|\nabla \rho_{h,t}\|, 1) \leq \varepsilon_1 (K(\|\nabla \rho_h\|) \nabla \rho_{h,t}, 1) + C\varepsilon_1^{-1} (K^{-1}(\|\nabla \rho_h\|), 1).
$$

(4.27)

By using (2.16) and $(1 + x)^a \leq 1 + x^a, x \geq 0$ imply

$$
(\|\nabla \rho_{h,t}\|, 1) \leq \varepsilon_1 \|K^{1/2}(\|\nabla \rho_h\|) \nabla \rho_{h,t}\|^2 + C\varepsilon_1^{-1} (1 + \|\nabla \rho_h\|_{L^\infty([T])}).
$$

(4.28)

In view of (4.27) and (4.26) becomes

$$
\langle \psi_t, \rho_{h,t} \rangle \leq \|\psi_t\|_{L^\infty([T])} \left( \varepsilon \|\rho_{h,t}\|^2 + C\varepsilon^{-1} + \varepsilon_1 \|K^{1/2}(\|\nabla \rho_h\|) \nabla \rho_{h,t}\|^2 + C\varepsilon_1^{-1} (1 + \|\nabla \rho_h\|_{L^\infty([T])}) \right).
$$

(4.29)

It follows from (4.24), (4.25) and (4.29) that

$$
\frac{1}{2} \frac{d}{dt} \|\rho_{h,t}\|^2 + (1 - a) \|K^{1/2}(\|\nabla \rho_h\|) \nabla \rho_{h,t}\|^2 \leq \|\psi_t\|_{L^\infty([T])} \left( \varepsilon \|\rho_{h,t}\|^2 + C\varepsilon^{-1} + \varepsilon_1 \|K^{1/2}(\|\nabla \rho_h\|) \nabla \rho_{h,t}\|^2 + C\varepsilon_1^{-1} (1 + \|\nabla \rho_h\|_{L^\infty([T])}) \right) + \varepsilon \|\rho_{h,t}\|^2 + C\varepsilon^{-1} \|f_t\|^2.
$$

(4.30)
Selecting $\varepsilon_1 = (1 - a)\varepsilon = \frac{1}{2}\varepsilon_0 (\|\psi_t\|_{L^\infty(\Gamma)} + 1)^{-1}$ yields

$$\frac{d}{dt}\|\rho_{h,t}\|^2 + (1 - a)K\|\nabla \rho_h\|_0\|\rho_{h,t}\|^2 \leq C\|\rho_{h,t}\|^2 + C\|\psi_t\|_{L^\infty(\Gamma)}\|\psi_t\|_{L^\infty(\Gamma)} + 1 + \lambda \|\rho_{h,t}\|_0^\alpha$$

$$+ C\left(\|\psi_t\|_{L^\infty(\Gamma)} + 1\right)\left(\|\psi_t\|_{L^\infty(\Gamma)} + \|f_t\|\right)$$

$$\leq \|\rho_{h,t}\|^2 + \|\nabla \rho_h\|_0^\beta + CZ(t),$$

where $Z(t) = (\|\psi_t\|_{L^\infty(\Gamma)} + 1)^{2\lambda} (1 + \|f_t\|^2) + (1 + \|\psi_t\|_{L^\infty(\Gamma)} + 1)^{2\lambda} \leq 2 (1 + \|\psi_t\|_{L^\infty(\Gamma)} + 1)^{2\lambda} (1 + \|f_t\|^2).$ (4.32)

The inequality (4.20) follows from (4.31) and (4.32). The proof is complete.

In the same manner to the problem (3.2), we have as the following:

**Theorem 4.4.** Let $0 < t_s < T$. Suppose $(\rho, m)$ be a solution of the problem (3.2). Then, there exists a positive constant $C$ such that

$$\|\rho\|^2_{L^2(\Omega)} + \|\nabla \rho\|^2_{L^2(\Omega)} + \|m\|^2_{L^2(\Omega)} \leq C\|\rho^0\|^2 + C\|\mathcal{A}\|,$$

$$\|\rho_t\|^2_{L^2(\Omega)} + \|\nabla \rho\|^2_{L^2(\Omega)} + \|m\|^2_{L^2(\Omega)} \leq C\mathcal{B},$$

$$\|\rho_t\|^2 \leq C(1 + t_s^{-1})\mathcal{B} + C\int_0^T \left(1 + \|\psi_t\|^2\right) \mathcal{B} \quad \forall t \in [t_s, T].$$

where $\mathcal{A}, \mathcal{B}$ are defined in (4.3) and (4.4).

### 5. Dependence of solutions on parameters

In this section, we study the dependence of the solution on the coefficients of Forchheimer polynomial $g(\mu)$ in (2.6). Let $N \geq 1$, the exponent vector $\tilde{a} = (0, a_1, \ldots, a_N)$ and the boundary data $\psi(x, t)$ be fixed. Let $\mathcal{D}$ be a compact subset of $[\tilde{a}] = (a_0, a_1, \ldots, a_N) : a_0, a_N > 0, a_1, \ldots, a_{N-1} \geq 0$. Set $\hat{\chi}(\mathcal{D}) = \max\{\hat{\chi}(\tilde{a}) : \tilde{a} \in \mathcal{D}\}$. Then $\hat{\chi}(\mathcal{D})$ is a number in $[1, \infty]$. Let $g_1(s) = g(s, \tilde{a}_1)$ and $g_2(s) = g(s, \tilde{a}_2)$ be two functions of class $FP(N, \tilde{a}_1)$, where $\tilde{a}_1$ and $\tilde{a}_2$ belong to $\mathcal{D}$.

Let $\rho_1 = \rho_1(x, t; \tilde{a}_1), \rho_2 = \rho_2(x, t; \tilde{a}_2)$ be the two solutions of (3.2) respective to $K(\zeta, \tilde{a}_1), K(\zeta, \tilde{a}_2)$ with the same boundary data $\psi$ and initial data $\rho^0$. We will estimate $\|\rho_1 - \rho_2\|, \|m_1 - m_2\|$ in the term of $|\tilde{a}_1 - \tilde{a}_2|$. Let $\varrho = \rho_1 - \rho_2, \mu = m_1 - m_2$. Then

$$\{\mu, z\} + \{K(\nabla \rho_1, \tilde{a}_1)\nabla \rho_1 - K(\nabla \rho_2, \tilde{a}_2)\nabla \rho_2, z\} = 0, \quad \forall z \in \mathcal{M},$$

$$\{\varphi \psi_t, r\} - \{\mu, \nabla r\} = 0, \quad \forall r \in \mathcal{R}. \quad (5.1)$$
THEOREM 5.1. Given $0 < t_* < T$. Let $(\rho_1, m_1), i = 1, 2$ be two solutions to problem (5.2) corresponding to vector coefficients $\tilde{a}_i$ of Forchheimer polynomial $g(s, \tilde{a}_i)$ in (2.5). There exists a constant positive constant $C$ independent of $|\tilde{a}_1 - \tilde{a}_2|$ such that

$$
\|\rho_1 - \rho_2\|_{L^\infty([t_*;T];L^2(\Omega))} + \|m_1 - m_2\|_{L^\infty([t_*;T];L^2(\Omega))} \leq C|\tilde{a}_1 - \tilde{a}_2|.
$$

(5.2)

Proof. Choosing $z = \nabla \varphi$ and $r = \rho$ in (5.1), and adding the resulting equations, we find that

$$
\frac{1}{2} \frac{d}{dt} \|\rho\|_{\Omega}^2 + (K(|\nabla \rho_1|, \tilde{a}_1) \nabla \rho_1 - K(|\nabla \rho_2|, \tilde{a}_2) \nabla \rho_2, \nabla \varphi) = 0.
$$

(5.3)

According to (2.23),

$$
\frac{1}{2} \frac{d}{dt} \|\rho\|_{\Omega}^2 \leq - (\beta - 1) \int_{\Omega} \left( \int_0^1 K(|\gamma(s)|, \tilde{b}(s)) \varphi \ dx \right) |\nabla \rho|^2 \ dx
$$

$$
+ C|\tilde{a}_1 - \tilde{a}_2| \int_{\Omega} \left( \int_0^1 K(|\gamma(s)|, \tilde{b}(s)) \right) (|\nabla \rho_1| + |\nabla \rho_2|)|\nabla \varphi| \ dx
$$

$$
\leq - \frac{\beta - 1}{2} \int_{\Omega} \left( \int_0^1 K(|\gamma(s)|, \tilde{b}(s)) \right) |\nabla \rho|^2 \ dx
$$

$$
+ C|\tilde{a}_1 - \tilde{a}_2|^2 \int_{\Omega} \left( \int_0^1 K(|\gamma(s)|, \tilde{b}(s)) \right) (|\nabla \rho_1| + |\nabla \rho_2|)^2 \ dx.
$$

(5.4)

Using Poincare's inequality and Hölder's inequality, we obtain

$$
\|\rho\|^2 \leq C_p \|\nabla \rho\|_{0, \beta}^2 \leq C_p \left( \int_{\Omega} \left( \int_0^1 K(|\gamma(s)|, \tilde{b}(s)) \right) |\nabla \rho|^2 \ dx \right) \left( \int_{\Omega} \left( \int_0^1 K(|\gamma(s)|, \tilde{b}(s)) \right) \frac{\rho}{\beta} \ dx \right)^{\frac{\beta}{2}},
$$

(5.5)

which implies that

$$
\int_{\Omega} \left( \int_0^1 K(|\gamma(s)|, \tilde{b}(s)) \right) |\nabla \rho|^2 \ dx \geq C_p^{-1} \|\rho\|^2 \left( \int_{\Omega} \left( \int_0^1 K(|\gamma(s)|, \tilde{b}(s)) \right) \frac{\rho}{\beta} \ dx \right)^{-\frac{\beta}{2}}.
$$

Hence

$$
\frac{1}{2} \frac{d}{dt} \|\rho\|_{\Omega}^2 \leq - C_* \|\rho\|^2 \left( \int_{\Omega} \left( \int_0^1 K(|\gamma(s)|, \tilde{b}(s)) \right) \frac{\rho}{\beta} \ dx \right)^{-\frac{\beta}{2}}
$$

$$
+ C|\tilde{a}_1 - \tilde{a}_2|^2 \int_{\Omega} \left( \int_0^1 K(|\gamma(s)|, \tilde{b}(s)) \right) (|\nabla \rho_1| + |\nabla \rho_2|)^2 \ dx,
$$

(5.6)

where $C_* = \frac{\beta - 1}{2} C_p^{-1} \rho^*$. Define

$$
\Lambda(t) = \int_{\Omega} \left( \int_0^1 K(|\gamma(s)|, \tilde{b}(s)) \right) \frac{\rho}{\beta} \ dx.
$$

Applying Gronwall Lemma to (5.6), and using the fact that $\rho(0) = 0$, we obtain (5.2).

$$
\|\rho\|_{\Omega}^2 \leq C|\tilde{a}_1 - \tilde{a}_2|^2 \int_0^T \left( e^{-\int_0^t \Lambda(\tau) \ dx} \right) \left( \int_0^1 K(|\gamma(s)|, \tilde{b}(s)) \right) (|\nabla \rho_1| + |\nabla \rho_2|)^2 \ dx \ dt.
$$

(5.7)

The last thing is to estimate

$$
\left( \int_0^1 K(|\gamma(t)|, \tilde{b}(t)) \ dx \right) (|\nabla \rho_1| + |\nabla \rho_2|)^2 \leq K(|\nabla \rho_1| + |\nabla \rho_2|, \tilde{a}_1 \wedge \tilde{a}_2) (|\nabla \rho_1| + |\nabla \rho_2|)^2
$$

$$
\leq C(|\nabla \rho_1| + |\nabla \rho_2|)^2 \leq C(|\nabla \rho_1|^\beta + |\nabla \rho_2|^\beta).
$$

(5.8)
Substituting (5.8) to (5.7) leads to
\[ \|\eta\|_{\Phi}^2 \leq C|\bar{a}_1 - \bar{a}_2|^2 \int_0^T \left( \|\nabla \rho_1\|_{0,\beta} + \|\nabla \rho_2\|_{0,\beta} \right) d\tau. \] (5.9)

We estimate the RHS of (5.9) using (4.34) to obtain the estimate for the first term in (5.2).

To the second estimate in (5.2), we rewrite (5.4) as follows
\[ \int_\Omega \left( \int_0^1 K(|\gamma(s)|, \tilde{b}(s)) d\gamma(s) \right) \|\eta\|^2 d\tau \leq C \left[ \|\nabla \rho_1\|_{0,\beta} + \|\nabla \rho_2\|_{0,\beta} \right] \int_0^T (\int_\gamma K(|\gamma(t)|, \tilde{b}(t)) dt)^2 d\tau + |\bar{a}_1 - \bar{a}_2|^2 \int_0^1 K(|\gamma(t)|, \tilde{b}(t)) dt \left( \|\nabla \rho_1\|_{0,\beta} + \|\nabla \rho_2\|_{0,\beta} \right). \] (5.10)

By the mean of the triangle inequality and (4.35) yield
\[ \|\eta\|_{\phi} \leq \|\rho_1, \tau\|_{\phi} + \|\rho_2, \tau\|_{\phi} \leq \Phi^* (\|\rho_1, \tau\| + \|\rho_2, \tau\|) \leq C_{\rho, \phi}. \] (5.11)

Next plugging (5.8), (5.9), (5.11) into (5.10), we obtain
\[ \int_\Omega \left( \int_0^1 K(|\gamma(s)|, \tilde{b}(s)) d\gamma(s) \right) \|\eta\|^2 d\tau \leq C \left[ |\bar{a}_1 - \bar{a}_2| \left( \|\nabla \rho_1\|_{0,\beta} + \|\nabla \rho_2\|_{0,\beta} \right)^{1/2} + |\bar{a}_1 - \bar{a}_2|^2 \left( \|\nabla \rho_1\|_{0,\beta} + \|\nabla \rho_2\|_{0,\beta} \right) \right]. \] (5.12)

Taking \( z = \mu \) in (5.1) gives
\[ \|\mu\|^2 + (K(|\nabla \rho_1|, \bar{a}_1) \nabla \rho_1 - K(|\nabla \rho_2|, \bar{a}_2) \nabla \rho_2, \mu) = 0. \] (5.13)

Applying Hölder's inequality to (5.13) and then (2.22) gives
\[ \|\mu\|^2 \leq \int_\Omega \left( K(|\nabla \rho_1|, \bar{a}_1) \nabla \rho_1 - K(|\nabla \rho_2|, \bar{a}_2) \nabla \rho_2 \right)^2 d\tau \leq 2(1 + a)^2 \int_\Omega \left( \int_0^1 K(|\gamma(t)|, \tilde{b}(t)) dt \right)^2 \|\nabla \rho\|^2 d\tau + C |\bar{a}_1 - \bar{a}_2|^2 \int_\Omega \left( \int_0^1 K(|\gamma(t)|, \tilde{b}(t)) dt \right)^2 \left( \|\nabla \rho_1\|_{0,\beta} + \|\nabla \rho_2\|_{0,\beta} \right). \] (5.14)

Thanks to the upper boundedness of \( K(\cdot) \), (5.12) and (5.8)
\[ \|\mu\|^2 \leq C \left[ |\bar{a}_1 - \bar{a}_2| \left( \|\nabla \rho_1\|_{0,\beta} + \|\nabla \rho_2\|_{0,\beta} \right)^{1/2} + |\bar{a}_1 - \bar{a}_2|^2 \left( \|\nabla \rho_1\|_{0,\beta} + \|\nabla \rho_2\|_{0,\beta} \right) \right]. \]

Then, we use (4.34) to bound \( \|\nabla \rho_i\|_{0,\beta}, i = 1, 2 \) to obtain
\[ \|\mu\|^2 \leq C \left( |\bar{a}_1 - \bar{a}_2| + |\bar{a}_1 - \bar{a}_2|^2 \right). \]

This proves the estimate for the second term in (5.2). The proof is complete.

6. Error estimates for semidiscrete approximation. In this section, we will give the error estimate between the analytical solution and approximate solution. We define the new variables:
\[ m - m_h = m - \pi m - (m_h - \pi m) = \eta - \zeta, \]
\[ \rho - \rho_h = \rho - \pi \rho - (\rho_h - \pi \rho) = \theta - \theta_h. \] (6.1)

**Theorem 6.1.** Given \( 0 < t_* < T \). Let \((\rho, m)\) be the solution of (3.3) and \((\rho_h, m_h)\) be the solution of (3.5). Suppose that \((\rho, m) \in L^\infty(0; H^{k+1}(\Omega)) \times (L^2(0; L^2(\Omega)))^d \) and \( \rho_1 \in L^2(0; H^{k+1}(\Omega)) \). Then there exists a positive constant \( C \) independence of \( h \) such that
\[ \|\rho - \rho_h\|_{L^\infty(0; L^2(\Omega))} + \|m - m_h\|_{L^2(0; L^2(\Omega))} + \|m - m_h\|^2_{L^2(0; T; L^2(\Omega))} \leq C h^{k}. \] (6.2)
Proof. With the error written in (6.1), it suffices, in the view of (3.4), to bound \( \theta_h, \zeta_h \). Subtracting the weak equations and its finite approximation, we obtain the error equations

\[
(m - m_h, z) + (K(\nabla \rho) \nabla \rho - K(\nabla \rho_h) \nabla \rho_h, z) = 0, \quad \forall z \in \mathcal{U}_h,
\]
\[
(\phi(\rho_t - \rho_{h,t}), r) - (m - m_h, \nabla r) = 0, \quad \forall r \in \mathcal{R}_h.
\]  
(6.3)

We rewrite these equations as follows:

\[
(m - m_h, z) + (K(\nabla \rho) \nabla \rho - K(\nabla \rho_h) \nabla \rho_h, z) = 0, \quad \forall z \in \mathcal{U}_h,
\]
\[
(\phi(\theta_{h,t}, r) + (m - m_h, \nabla r) = (\phi t, r), \quad \forall r \in \mathcal{R}_h.
\]  
(6.4)

Selecting \( z = -\nabla \theta_h \) and \( r = \theta_h \), and adding two above equations gives

\[
(\phi(\theta_{h,t}, \theta_h) - (K(\nabla \rho) \nabla \rho - K(\nabla \rho_h) \nabla \rho_h, \nabla \theta_h) = (\phi t, \theta_h),
\]
or

\[
\frac{1}{2} \frac{d}{dt} \| \theta_h \|_{\theta}^2 + (K(\nabla \rho) \nabla \rho - K(\nabla \rho_h) \nabla \rho_h, \nabla \rho - \nabla \rho_h) = (K(\nabla \rho) \nabla \rho - K(\nabla \rho_h) \nabla \rho_h, \nabla \theta) + (\phi t, \theta_h).
\]  
(6.5)

By the monotonicity of \( K(\cdot) \) in (2.25),

\[
(K(\nabla \rho) \nabla \rho - K(\nabla \rho_h) \nabla \rho_h, \nabla \rho - \nabla \rho_h) \leq (\beta - 1) \int_\Omega \left( \int_0^1 K(\gamma(s)) \|ds\right) \|\nabla \rho - \nabla \rho_h\|^2 dx.
\]  
(6.6)

By Young’s inequality, for \( \varepsilon_0 > 0 \)

\[
(\phi t, \theta_h) \leq C\varepsilon_0^{-1} \| \theta_t \|^2_{\theta} + \varepsilon_0 \| \theta_h \|^2_{\theta}.
\]  
(6.7)

Applying (2.24), Young’s inequalities and the upper boundedness of \( K(\cdot) \), we obtain

\[
(K(\nabla \rho) \nabla \rho - K(\nabla \rho_h) \nabla \rho_h, \nabla \theta) \leq (1 + a) \int_\Omega \left( \int_0^1 K(\gamma(s)) \|ds\right) \|\nabla \rho - \nabla \rho_h\|^2 dx \leq \frac{\beta - 1}{2} \int_\Omega \left( \int_0^1 K(\gamma(s)) \|ds\right) \|\nabla \rho - \nabla \rho_h\|^2 dx + C\|\nabla \theta\|^2.
\]  
(6.8)

Combining (6.5), (6.7), (6.8) and (6.5) gives

\[
\frac{1}{2} \frac{d}{dt} \| \theta_h \|_{\theta}^2 + \frac{\beta - 1}{2} \int_\Omega \left( \int_0^1 K(\gamma(s)) \|ds\right) \|\nabla \rho - \nabla \rho_h\|^2 dx \leq C\|\nabla \theta\|^2 + C\varepsilon_0^{-1} \| \theta_t \|^2_{\theta} + \varepsilon_0 \| \theta_h \|^2_{\theta}.
\]

Integrating in time from 0 to \( T \) and then taking sup-norm in time of the resultant shows that

\[
\sup_{t \in [0, T]} \| \theta_h \|^2_{\theta} + \int_0^T \int_\Omega \left( \int_0^1 K(\gamma(s)) \|ds\right) \|\nabla \rho - \nabla \rho_h\|^2 dx dt \leq C\|\nabla \theta\|^2_{L^\infty(L^2)} + C\varepsilon_0^{-1} \| \theta_t \|^2_{L^2(L^2)} + \varepsilon_0 T \sup_{t \in [0, T]} \| \theta_h \|^2_{\theta}.
\]

Selecting \( \varepsilon_0 = 1/(2T) \), we find that

\[
\sup_{t \in [0, T]} \| \theta_h \|^2_{\theta} + \int_0^T \int_\Omega \left( \int_0^1 K(\gamma(s)) \|ds\right) \|\nabla \rho - \nabla \rho_h\|^2 dx dt \leq C\|\nabla \theta\|^2_{L^\infty(L^2)} + CT \| \theta_t \|^2_{L^2(L^2)}.
\]  
(6.9)

Using \( L^2 \)-projection and choose \( z = \zeta_h \) in the first equation in (6.3) yields

\[
\| \zeta_h \|^2 = -(K(\nabla \rho) \nabla \rho - K(\nabla \rho_h) \nabla \rho_h, \zeta_h) \leq C \int_\Omega \left( \int_0^1 K(\gamma(s)) \|ds\right) \|\nabla \rho - \nabla \rho_h\| \| \zeta_h \| dx.
\]
It follows from Cauchy’s inequality and the upper boundedness of the function $K(\cdot)$ that
\[ \|\xi_h\|^2 dt \leq C \int_0^1 \left( \int_0^1 K(|\gamma(s)|) ds \right) \|\nabla \rho - \nabla \rho_h\|^2 dx dt. \] (6.10)

Putting (6.9) and (6.10) together, using equivalent norm, we see that
\[ \|\theta_h\|^2_{L^\infty(I;L^2)} + \|\xi_h\|^2_{L^2(I;L^2)} \leq C \|\nabla \theta\|^2_{L^\infty(I;L^2)} + C T \|\theta_h\|^2_{L^2(I;L^2)}. \] (6.11)

As a consequence, the two first terms in (6.2) follows from (6.11), (6.1) and (3.4).

For the last term in (6.2), we rewrite (6.5) as follows
\[ (K(\nabla \rho)) \nabla \rho - K(\nabla \rho_h) \nabla \rho - \nabla \rho_h = (\phi(\rho_i - \rho_{h,i}), \theta) + (K(\nabla \rho)) \nabla \rho - K(\nabla \rho_h) \nabla \rho_h, \nabla \theta). \] (6.12)

In virtue of the triangle inequality, and Hölder’s inequality, we have
\[ (\rho_i - \rho_{h,i}, \theta) \leq (|\rho_i| + |\rho_{h,i}|, |\theta|) \leq (\|\rho_i\| + \|\rho_{h,i}\|) \|\theta_h\|. \] (6.13)

It follows from (6.12), (6.6), (6.8), and (6.13) that
\[ \int_0^1 \left( \int_0^1 K(|\gamma(s)|) ds \right) \|\nabla \rho - \nabla \rho_h\|^2 dx \leq C \|\rho_h\| + C \|\theta_h\|^2. \] (6.14)

This and (6.10) show that
\[ \|\xi_h\|^2 \leq C \int_0^1 \left( \int_0^1 K(|\gamma(s)|) ds \right) \|\nabla \rho - \nabla \rho_h\|^2 dx \leq C \|\theta_h\| + C \|\theta\|^2. \] (6.15)

Due to (6.11), (6.15) and the fact that \( \|\nabla \theta\| \leq C h^k \|\rho\|_{k+1} \), we obtain (6.2). \( \square \)

**Theorem 6.2.** Given \( 0 < t_s < T \). Let \( (\rho_{h,i}, m_{h,i}), i = 1, 2 \) be two solutions to problems (5.5) corresponding to vector coefficients \( \vec{a}_i \) of Forchheimer polynomial \( g(s, \vec{a}_i) \) in (2.8). Suppose that each \( (\rho_i, m_i) \in L^\infty(I;H^{k+1}(\Omega)) \times (L^2(I;L^2(\Omega)))^d \) and \( \rho_{i,1} \in L^2(I;H^{k+1}(\Omega)) \). Then, there exists a constant positive constant \( C \) independent of \( h \) and \( |\vec{a}_1 - \vec{a}_2| \) such that
\[ \|\rho_{h,1} - \rho_{h,2}\|_{L^\infty(I;L^2(\Omega))} + \|m_{h,1} - m_{h,2}\|^2_{L^2(I;L^2(\Omega))} \leq C (h^k + |\vec{a}_1 - \vec{a}_2|). \] (6.16)

**Proof:** The triangle inequality shows that
\[ \|\rho_{h,1} - \rho_{h,2}\| + \|m_{h,1} - m_{h,2}\|^2 \leq 4 \left( \sum_{i=1,2} (\|\rho_{h,i} - \rho_1\| + \|m_{h,i} - m_i\|^2) + \|\rho_1 - \rho_2\| + \|m_1 - m_2\|^2 \right). \]

Then by using (6.2) to treat the sum-term and (5.2) to the last terms we obtain (6.16). \( \square \)

**7. Error analysis for fully discrete method.** In analyzing this method, we proceed in a similar fashion as for the semidiscrete method. We derive an error estimate for the fully discrete time Galerkin approximation of the differential equation. First, we give some uniform stability results that are crucial in getting the convergence results.

**Lemma 7.1 (Stability).** Let \( (\rho_{h,i}, m_{h,i}) \) solve the fully discrete finite element approximation (3.6) for each time step \( i = 1, 2, \ldots, N \). There exists a positive constant \( C \) independent of \( t, i, \Delta t \) such that for \( \Delta t \) sufficiently small
\[ \|\rho_{h,i}\|^2 + \|m_{h,i}\|^2 \leq C (1 - \Delta t)^{-1} \|\rho_0\|^2 + C \Delta t \sum_{i=1}^N (1 - \Delta t)^{-i+j-1} \left( 1 + \|\psi^j\|_{L^\infty(I)}^4 + \|f^j\|^2 \right). \] (7.1)
Using (2.36) and Hölder’s inequality to the RHS of (7.3) show s that

Putting (7.6) and (7.7) together with note that

ε

According to discrete Gronwall’s inequality in Lemma 2.4,

It follows from (2.18) that

ε

we obtain

ε

Adding the two above equations, and using the identity

ε

we obtain

ε

It follows from (2.18) that

ε

Using (2.36) and Hölder’s inequality to the RHS of (7.3) shows that

ε

Combining (7.3) - (7.5), then selecting ε = \(\frac{C}{2}(\|\psi\|_{L^\infty(\Gamma)} + 1)^{-1}\) yields

ε

We simplify the RHS of the above estimate using the inequality \(\|\psi\|_{L^\infty(\Gamma)}, \|\psi\|_{L^\infty(\Gamma)} \leq C(1 + \|\psi\|_{L^\infty(\Gamma)}^\lambda)\) to obtain

ε

According to discrete Gronwall’s inequality in Lemma 2.4

ε

Now selecting \(z = m^t_h\) in the first equation (3.6) gives \(\|m^t_h\|^2 + (K(\|\nabla \rho^t_h\|_0^\beta)\|\rho^t_h\| + m_t) = 0\), hence

ε

Putting (7.6) and (7.7) together with note that \(\|\rho^t_h\|^2 \leq \|\rho^0\|^2\) implies (7.1). The proof is complete. □

As in the semidiscrete case, we use η = m − \(\pi m\), ξ_i = m_i − \(\pi m\), θ = \(\pi \rho\), \(\theta_h = \rho - \pi \rho\) and \(\eta^t, \theta^t, \xi^t_h, \theta^t_h\) be evaluating η, θ, ξ, \(\theta_h\) at the discrete time levels. We also define

\[\frac{\partial \phi^n - \phi^{n-1}}{\Delta t}\]
where

\begin{equation}
\text{(3.6)}
\end{equation}

Subtracting (3.6) from (7.9), we obtain

\[ \text{We will evaluate (7.12) term by term.} \]

\[ \epsilon_{\mathcal{L}} \]

Then, there exists a positive constant \( C \) independent of \( h \) and \( \Delta t \) such that for \( \Delta t \) sufficiently small

\[ \| \rho^i - \rho_h^i \| + \| m^i - m_h^i \|^2 \leq C \left( h^k + \sqrt{\Delta t} \right). \]

**Proof.** Evaluating equation (3.2) at \( t = t_i \) gives

\[ \left( m^i, z \right) + \left( K(\nabla \rho^i), \nabla \rho^i, z \right) = 0, \]

\[ \left( \phi \rho^i_t, r \right) - \left( m^i, \nabla r \right) = \left( f^i, r \right) - \left( \psi^i, r \right). \]

Subtracting (3.6) from (7.9), we obtain

\[ \left( m^i - m_h^i, z \right) + \left( K(\nabla \rho^i), \nabla \rho^i - K(\nabla \rho_h^i), \nabla \rho_h^i, z \right) = 0, \quad \forall z \in \mathcal{M}_h, \]

\[ \left( \phi \rho^i_t - \vartheta_h^t, r \right) - \left( m^i - m_h^i, \nabla r \right) = 0, \quad \forall r \in \mathcal{R}_h. \]

Choosing \( r = -\vartheta_h^t \), \( z = \nabla r \), and adding the two equations shows that

\[ \left( \phi (\rho^i_t - \vartheta_h^t), \vartheta_h^t \right) + \left( K(\nabla \rho^i), \nabla \rho^i - K(\nabla \rho_h^i), \nabla \rho_h^i, \vartheta_h^t \right) = 0. \]

Since \( \rho^i_t - \vartheta_h^t = \rho^i_t - \rho^i + \vartheta - \vartheta_h^t \), we rewrite (7.11) in the form

\[ \left( \partial \vartheta_h^t, \vartheta_h^t \right) + \left( K(\nabla \rho^i), \nabla \rho^i - K(\nabla \rho_h^i), \nabla \rho_h^i, \vartheta \right) = \left( K(\nabla \rho^i), \nabla \rho^i - K(\nabla \rho_h^i), \nabla \vartheta \right) + \left( \phi (\rho^i_t - \vartheta_h^t), \vartheta_h^t \right) + \left( \phi \vartheta_t, \vartheta_h^t \right). \]

We will evaluate (7.12) term by term.

- For the first term, we use the identity

\[ \left( \partial \vartheta_h^t, \vartheta_h^t \right) = \frac{1}{2\Delta t} \left( \| \vartheta_h^t \|^2 - \| \vartheta_h^{t-1} \|^2 \right) + \frac{\Delta t}{2} \| \vartheta_h^t \|^2. \]

- For the second term, the monotonicity of \( K(\cdot) \) in (2.25) yields

\[ (K(\nabla \rho^i)) \nabla \rho^i - (K(\nabla \rho_h^i)) \nabla \rho_h^i, \nabla \rho^i - \nabla \rho_h^i) \geq (\beta - 1) \int_0^1 \left( \int_0^1 K(\gamma^t(s))) |ds| \right) \nabla \rho^i - \nabla \rho_h^i |^2 dx, \]

where \( \gamma^t(s) = sV \rho^i + (1 - s)V \rho_h^i \).

- For the third term, using (6.18) gives

\[ (K(\nabla \rho^i)) \nabla \rho^i - (K(\nabla \rho_h^i)) \nabla \rho_h^i, \nabla \vartheta \) \leq \frac{\beta - 1}{2} \int_0^1 \left( \int_0^1 K(\gamma^t(s))) |ds| \right) \nabla \rho^i - \nabla \rho_h^i |^2 dx + C \| \vartheta^t \|^2. \]

- For the fourth term, using Taylor expand we see that

\[ \left( \phi (\rho^i_t - \vartheta_h^t), \vartheta_h^t \right) \leq \frac{1}{\Delta t} \int_{t_{i-1}}^{t_i} \rho_{tt}(t) (t - t_{i-1}) dt \| \vartheta_h^t \| \phi \]

\[ \leq \frac{1}{\Delta t} \left( \int_{t_{i-1}}^{t_i} \| \rho_{tt}(t) \|_\phi^2 dt \right)^{1/2} \left( \int_{t_{i-1}}^{t_i} (t - t_{i-1})^2 dt \right)^{1/2} \| \vartheta_h^t \| \phi \leq C \Delta t \int_{t_{i-1}}^{t_i} \| \rho_{tt}(t) \|_\phi^2 dt + \frac{1}{4} \| \vartheta_h^t \| \phi. \]

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• For the last term, using Young’s and Hölder’s inequalities, we find that

\[
\left(\phi \partial \theta^j, \theta^j_h \right) \leq C \| \partial \theta^j \|_\phi^2 + \frac{1}{4} \| \theta^j_h \|_\phi^2 = C \int_{t_{i-1}}^{t_i} \| \theta(t) \|_\phi^2 + \frac{1}{4} \| \theta^j_h \|_\phi^2 \leq C \frac{\int_{t_{i-1}}^{t_i} \| \theta(t) \|_\phi^2 + \frac{1}{4} \| \theta^j_h \|_\phi^2}{\Delta t}.
\]

(7.17)

In view of (7.13)–(7.17), (7.12) yields

\[
\frac{1}{2 \Delta t} \left( \| \theta^j_h \|_\phi^2 - \| \theta^{j-1}_h \|_\phi^2 \right) + \frac{\beta - 1}{2} \int_\Omega \left( \int_0^1 K(\|y^j(s)\|)d s \right) \| \nabla \rho^i - \nabla p^i_h \|_\phi^2 dx \leq C \Delta t \int_{t_{i-1}}^{t_i} \| \rho^i_{tt}(t) \|_\phi^2 dt + C \Delta t \int_{t_{i-1}}^{t_i} \| \theta(t) \|_\phi^2 dt + C \| \nabla \theta^j \|_\phi^2,
\]

(7.18)

which leads to

\[
\frac{\| \theta^j_h \|_\phi^2 - \| \theta^{j-1}_h \|_\phi^2}{\Delta t} - \| \theta^j_h \|_\phi^2 \leq C \Delta t \int_{t_{i-1}}^{t_i} \| \rho^i_{tt}(t) \|_\phi^2 dt + C \int_{t_{i-1}}^{t_i} \| \theta(t) \|_\phi^2 dt + C \| \nabla \theta^j \|_\phi^2.
\]

(7.19)

By mean of discrete Gronwall’s inequality in Lemma 2.4 and the fact \( \theta^j_h = 0 \), we find that

\[
\| \theta^j_h \|^2 \leq C \sum_{j=1}^{l} (1 - \Delta t)^{-l+j-1} \left( \Delta t \int_{t_{i-1}}^{t_i} \| \rho^i_{tt}(t) \|_\phi^2 dt + \Delta t \int_{t_{i-1}}^{t_i} \| \theta(t) \|_\phi^2 dt + \sum_{j=1}^{N} \| \nabla \theta^j \|_\phi^2 \right)
\]

(7.20)

since \( (1 - \Delta t)^{-l+j-1} \leq (1 - \Delta t)^{-N} \leq e^{\frac{N\lambda}{\beta}} \leq e^{C \lambda \Delta t} \).

The first part of (7.8) follows from combining (7.20) and the triangle inequality \( \| \rho^i - p^i_h \| \leq \| \theta^j \| + \| \theta^j_h \| \).

To the other part of (7.8), we first estimate the term \( \int_{\Omega} \left( \int_0^1 K(\|y^j(s)\|)d s \right) \| \nabla \rho^i - \nabla p^i_h \|_\phi^2 dx \) by rewriting (7.11) in the form

\[
\left( K(\|\nabla \rho^i\|)\nabla \rho^i - K(\|\nabla p^i_h\|)\nabla p^i_h, \nabla \rho^i - \nabla p^i_h \right) = (\phi(\rho^i - \partial p^i_h), \theta^j_h) + \left( K(\|\nabla \rho^i\|)\nabla \rho^i - K(\|\nabla p^i_h\|)\nabla p^i_h, \nabla \theta^j \right).
\]

(7.21)

In virtue of the triangle inequality, and Hölder’s inequality

\[
(\phi(\rho^i - \partial p^i_h), \theta^j_h) \leq (\phi(\|\rho^i\| + |\partial p^i_h|), |\theta^j_h|) \leq (\|\rho^i\|_\phi + \|\theta^j_h\|_\phi) \|\theta^j_h\|_\phi.
\]

(7.22)

It follows from (7.21), (6.15), (6.8) and (7.22) that

\[
\int_{\Omega} \left( \int_0^1 K(\|y^j(s)\|)d s \right) \| \nabla \rho^i - \nabla p^i_h \|_\phi^2 dx \leq C(\|\rho^i\|_\phi + \|\theta^j_h\|_\phi) \|\theta^j_h\|_\phi + C \|\nabla \theta^j\|_\phi^2.
\]

(7.23)
Then we chose \( z = \zeta_h^i \) as the test function in the first equation of (7.10) to obtain

\[
\| \zeta_h^i \|^2 = - \left( K(\nabla \rho^j) \nabla \rho^j - K(\nabla \rho_h^j) \nabla \rho_h^j \right) \leq C \int_\Omega \left( \int_0^1 K(|\gamma^j(s)|) ds \right) \nabla \rho^j - \nabla \rho_h^j \| \gamma^j \| dx \\
\leq C \int_\Omega \left( \int_0^1 K(|\gamma^j(s)|) ds \right) \nabla \rho^j - \nabla \rho_h^j dxdt \\
\leq C(\| \theta_h^i \| + \| \nabla \theta^j \|),
\]

(7.24)

The second part of (7.8) follows from the combination of (7.20), (7.24) and the approximation properties. The proof is complete. \( \square \)

The following theorem about an error estimate for \( (\rho^j, m^j) \) is obtained by using the same manner as in the proof of Theorem 6.2.

**Theorem 7.3.** Let \( (\rho_j^i, m_j^i) \), \( j=1,2 \) solve problem (3.2) and \( (\rho_{h,j}^i, m_{h,j}^i) \) solve the fully discrete finite element approximation (5.6) corresponding to vector coefficients \( \tilde{a}_j \) of Forchheimer polynomial \( g(s, \tilde{a}_j) \) in (2.6) for each time step \( i, i=1, \ldots, N \). Suppose that each \( (\rho_j^i, m_j^i) \in L^\infty(\Omega; L^2(\Omega)) \times \{ L^\infty(\Omega; L^2(\Omega)) \}^d \) and \( \rho_{h,j}^i \in L^\infty(\Omega; L^2(\Omega)) \). There exists a positive constant \( C \) independent of \( h, \Delta t \) and \( |\tilde{a}_1 - \tilde{a}_2| \) such that if the \( \Delta t \) is sufficiently small then

\[
\| \rho_{h,1}^i - \rho_{h,2}^i \| + \| m_{h,1}^i - m_{h,2}^i \| \leq C \left( h^k + |\tilde{a}_1 - \tilde{a}_2| + \sqrt{\Delta t} \right).
\]

(7.25)

### 8. Numerical results.

In this section we carry out numerical experiments using mixed finite element approximation to solve problem (3.2) in two dimensional region. For simplicity, the region of examples are unit square \( \Omega = [0,1]^2 \). The triangularization in region \( \Omega \) is uniform subdivision in each dimension. We use the piecewise-linear elements for the both density and momentum variables. Our problem is solved at the first equation generated at each time step.

The numerical examples in this section are constructed in two categories:

- Examples 1A and 2A are used to study the convergence rates of the method proposed in the paper.
  - We test the convergence of our method with the Forchheimer two-term law \( g(s) = 1 + s \). Equation \( (2.11) \) \( g(s) = \zeta, s \geq 0 \) gives \( s = \frac{1 + \sqrt{1 + 4\zeta}}{2} \) and hence \( K(\zeta) = \frac{1}{g(0(\zeta))} = \frac{1}{1 + \sqrt{1 + 4\zeta}} \).
  - Examples 1B and 2B are used to study the dependence of solution on physical parameters. We test the convergence of our method with the Forchheimer two-term law \( g(s) = 1 + 0.9s \). In this case \( K(\zeta) = \frac{10}{5 + \sqrt{25 + 95\zeta}} \).

**Example 1.** To test the convergence rates, we choose the analytical solution

\[
\rho(x, t) = e^{-2t}(x_1 + x_2) \quad \text{and} \quad m(x, t) = \frac{-2e^{-2t}(1; 1)}{1 + \sqrt{1 + 4\sqrt{2e^{-2t}}}} \quad \forall (x, t) \in [0,1]^2 \times (0,1].
\]

For simplicity, we take \( \phi(x) = 1 \) on \( \Omega \). The forcing term \( f \) is determined from equation \( \rho_t + \nabla \cdot m = f \). Explicitly,

\[
f(x, t) = -2e^{-2t}(x_1 + x_2).
\]

The initial condition and boundary condition are determined according to the analytical solution as follows:

\[
\rho^0(x) = x_1 + x_2, \quad \psi(x, f) = \begin{cases} 
\frac{2e^{-2t}}{1 + \sqrt{1 + 4\sqrt{2e^{-2t}}}} & \text{on } [0] \times [0,1] \text{ or } [0,1] \times [0], \\
\frac{-2e^{-2t}}{1 + \sqrt{1 + 4\sqrt{2e^{-2t}}}} & \text{on } [1] \times [0,1] \text{ or } [0,1] \times [1].
\end{cases}
\]

The numerical results are listed in Table 1A.
Table 1A. Convergence study for Darcy–Forchheimer flows using mixed FEM in 2D.

Next, we consider a small change in coefficients of Forchheimer polynomial \( g \), namely \( g(s) = 1 + 0.95s \). In this case, the analytical solution is chosen by

\[
\rho(x, t) = e^{-2t} (x_1 + x_2) \quad \text{and} \quad m(x, t) = -\frac{10e^{-2t}(1; 1)}{5 + \sqrt{25 + 95s^2e^{-2t}}} \forall (x, t) \in [0, 1]^2 \times (0, 1).
\]

The forcing term \( f \), initial condition and boundary condition accordingly are

\[
f(x, t) = -2e^{-2t}(x_1 + x_2), \quad \rho^0(x) = x_1 + x_2, \quad \psi(x, t) = \begin{cases}
\frac{10e^{-2t}}{5 + \sqrt{25 + 95s^2e^{-2t}}} & \text{on } (0) \times [0] \cup [0] \times (0), \\
-\frac{2e^{-2t}}{5 + \sqrt{25 + 95s^2e^{-2t}}} & \text{on } (1) \times [0] \cup [0] \times (1).
\end{cases}
\]

We use \( \|\rho_{1,h} - \rho_{2,h}\| \) and \( \|m_{1,h} - m_{2,h}\| \) as the criterion to measure the dependence of solutions on the coefficients of \( g \). The numerical results are listed in Table 1B.

Table 1B. Study the dependence of solution of Darcy–Forchheimer flows using mixed FEM in 2D.

**Example 2.** In this example, we still take \( \phi(x) = 1 \) on \( \Omega \). The analytical solution is

\[
\rho(x, t) = e^{-t} w^2(x) \quad \text{and} \quad m(x, t) = -\frac{4e^{-t}(x_1; x_2)}{1 + \sqrt{1 + 8e^{-t} w(x)}} \forall (x, t) \in [0, 1]^2 \times (0, 1),
\]

where \( w(x) = \sqrt{x_1^2 + x_2^2} \). The forcing term \( f \), initial condition \( \rho^0(x) \) and boundary condition \( \psi(x, t) \) are as follows

\[
f(x, t) = -e^{-t} w^2(x) + \frac{16e^{-2t} w^2(x)}{w(x) \sqrt{1 + 8e^{-t} w(x)}} \left( 1 + \sqrt{1 + 8e^{-t} w(x)} \right)^2 - \frac{8e^{-t}}{1 + \sqrt{1 + 8e^{-t} w(x)}},
\]

\[
\rho^0(x) = w^2(x), \quad \psi(x, t) = \begin{cases}
0 & \text{on } (0) \times [0] \cup [0] \times (0), \\
\frac{-4e^{-t}}{1 + \sqrt{1 + 8e^{-t} \sqrt{1 + x_1^2}}} & \text{on } (1) \times [0] \cup [0] \times (1), \\
\frac{-4e^{-t}}{1 + \sqrt{1 + 8e^{-t} \sqrt{x_1^2 + 1}}} & \text{on } [0] \times (1).
\end{cases}
\]
The numerical results are listed in Table 2A.

| N  | $\|p - p_h\|$ | Rates | $\|m - m_h\|$ | Rates |
|----|---------------|-------|---------------|-------|
| 4  | $2.331e-01$   | –     | $9.742E-01$   | –     |
| 8  | $1.651E-01$   | 0.489 | $8.926E-01$   | 0.126 |
| 16 | $1.027E-01$   | 0.684 | $7.511E-01$   | 0.249 |
| 32 | $6.249E-02$   | 0.717 | $6.320E-01$   | 0.249 |
| 64 | $3.756E-02$   | 0.734 | $5.318E-01$   | 0.249 |
| 128| $2.245E-02$   | 0.742 | $4.475E-01$   | 0.249 |
| 256| $1.338E-02$   | 0.746 | $3.765E-01$   | 0.249 |
| 512| $7.968E-03$   | 0.748 | $3.168E-01$   | 0.249 |

Table 2A. Convergence study for Darcy–Forchheimer flows using mixed FEM in 2D.

For $g(s) = 1 + 0.95s$, the analytical solution is chosen by

$$
\rho(x, t) = e^{-t} w^2(x) \quad \text{and} \quad m(x, t) = -\frac{20e^{-t}}{5 + \sqrt{25 + 190e^{-t} w(x)}} v(x, t) \in [0, 1]^2 \times (0, 1).
$$

The forcing term $f$, initial condition and boundary condition accordingly are

$$
f(x, t) = -e^{-t} w^2(x) + \frac{1900e^{-2t} u^2(x)}{w(x) \sqrt{25 + 190e^{-t} w(x)}} \left(5 + \sqrt{25 + 190e^{-t} w(x)}\right)^2 - \frac{40e^{-t}}{5 + \sqrt{25 + 190e^{-t} w(x)}}
$$

$$
\rho^0(x) = w^2(x), \quad \psi(x, t) = \begin{cases} 
0 & \text{on } [0] \times [0, 1] \text{ and } [0, 1] \times [0, 1], \\
-20e^{-t} & \text{on } [1] \times [0, 1], \\
5 + \sqrt{25 + 190e^{-t} w^2(x)} & \text{on } [0, 1] \times [1].
\end{cases}
$$

The numerical results are listed in Table 2B.

| N  | $\|p_{1,h} - p_{2,h}\|$ | Rates | $\|m_{1,h} - m_{2,h}\|$ | Rates |
|----|------------------------|-------|------------------------|-------|
| 4  | $1.950E-03$            | –     | $4.430E-03$            | –     |
| 8  | $1.325E-03$            | 0.557 | $4.008E-03$            | 0.115 |
| 16 | $8.858E-04$            | 0.581 | $3.501E-03$            | 0.195 |
| 32 | $5.712E-04$            | 0.633 | $2.975E-03$            | 0.235 |
| 64 | $3.580E-04$            | 0.674 | $2.507E-03$            | 0.247 |
| 128| $2.204E-04$            | 0.700 | $2.108E-03$            | 0.250 |
| 256| $1.341E-04$            | 0.717 | $1.772E-03$            | 0.251 |
| 512| $8.096E-05$            | 0.728 | $1.489E-03$            | 0.251 |

Table 2B. Study the dependence of solution of Darcy–Forchheimer flows using mixed FEM in 2D.

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