Inherently Global Nature of Topological Charge Fluctuations in QCD

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Abstract

We have recently presented evidence that in configurations dominating the regularized pure-glue QCD path integral, the topological charge density constructed from the overlap Dirac operator organizes into an ordered space-time structure. It was pointed out that, among other properties, this structure exhibits two important features: it is low-dimensional and geometrically global, i.e. consisting of connected sign-coherent regions with local dimensions $1 \leq d < 4$, and spreading over arbitrarily large space–time distances. Here we show that the space-time structure responsible for the origin of topological susceptibility indeed exhibits global behavior. In particular, we show numerically that topological fluctuations are not saturated by localized concentrations of most intense topological charge density. To the contrary, the susceptibility saturates only after the space-time regions with most intense fields are included, such that geometrically global structure is already formed. We demonstrate this result both at the fundamental level (full topological density) and at low energy (effective density). The drastic mismatch between the point of fluctuation saturation ($\approx 50\%$ of space-time at low energy) and that of global structure formation ($< 4\%$ of space-time at low energy) indicates that the ordered space-time structure in topological charge is inherently global and that topological charge fluctuations in QCD cannot be understood in terms of individual localized pieces. Description in terms of global brane-like objects should be sought instead.

1. Introduction. Understanding the nature of topological charge fluctuations is an essential step toward clarifying the structure of the QCD vacuum. Indeed, not only are there important physical phenomena associated with topological charge (such as $\eta'$ mass, $\theta$–dependence and possibly spontaneous chiral symmetry breaking (SChSB)), but there are also intriguing theoretical reasons to believe that topological charge structure in QCD may provide a particularly clear window on the gauge theory/string theory connection, since topological charge
is dual to Ramond-Ramond charge in type IIA string theory [1]. Nevertheless, it is only recently that a first-principles direct study of space-time topological structure in a well-defined lattice-regularized setting was carried out [2, 3]. This development was made possible by important advances in understanding lattice chiral symmetry [4, 5, 6] which, among other things, led to the construction of novel lattice topological charge density operators [7, 5]. These operators are not just convenient in that the corresponding topological susceptibility can be computed without any subjective manipulations of the gauge field and that theoretical properties of the lattice topological field match those in the continuum [8]. In addition, they exhibit features that are inherent to smooth backgrounds on continuum manifolds, but quite unexpected for general backgrounds contributing to the regularized QCD path integral. In particular, the index theorem is exactly satisfied relative to the lattice Dirac operator used to define the density [7], and the global charge of a generic configuration is strictly stable with respect to small local changes of the gauge field.

In previous work we have studied the space-time behavior of such topological density based on the overlap Dirac operator in pure-glue QCD [2]. We found a clear excess of ordered structure compared to random distributions of charge. In fact, topological charge organizes into two oppositely-charged low-dimensional “sheets” that fill a macroscopic fraction of space-time, and are built around a connected “skeleton” of points with high intensity 1. This structure is embedded in space-time in a non-trivial geometric manner and it should be regarded as fundamental since it includes fluctuations at all scales up to the lattice cutoff. 2 We note that a viable candidate for the fundamental structure in typical configurations contributing to the QCD path integral must reproduce the correct behavior of the topological density correlator [3]. A particularly notable requirement is that it has to conform to the fact that \(\langle q(0)q(x) \rangle \leq 0\) at arbitrary non-zero distance [10]. Indeed, for the above two-sheet/skeleton structure the non-positivity of the correlator at non-zero distances arises in an ordered manner due to the fact that layers of low-dimensional structure with alternating sign permeate the space-time.

Two basic aspects of the topological charge structure described in [2] are particularly intriguing and possibly far-reaching. First is the strictly low-dimensional nature of the structure 3. By that we mean that the structure only contains parts with local dimensions strictly less than four 4. This suggests some interesting possibilities for the way in which gauge fields guide the propagation of pions, and will hopefully shed new insight into the relation between topological charge fluctuations and SChSB. The second aspect is the geometrically global nature of the structure. This expresses a geometric fact that the subset of space-time occupied by the structure (sign-coherent sheet or skeleton) is a connected set spreading over maximal available distances. In this paper we focus on the distribution of charge within the geometric structure and show that the global behavior is necessarily present in the physically

1From the geometric point of view, both the sheets and the skeleton behave somewhat similarly to Peano’s space-filling curve [2].

2Analogous structure was recently reported to be seen also in the 2-d CP(N-1) models [9].

3The low-dimensional nature of fundamental topological field is reflected to some degree also in low-lying Dirac modes [11, 12]. More precise form of this correspondence is yet to be understood. The notion of strictly low-dimensional structure is also emerging using indirect projection techniques [13]. Its relation (if any) to the structure in topological field is yet to be understood.

4The dimension is at least one since the sign-coherent sheet/skeleton is a connected global structure.
relevant substructure responsible for the origin of topological susceptibility. In particular, we demonstrate that the bulk of topological susceptibility in QCD cannot be carried by localized concentrations of most intense topological charge. Just to the contrary, we will argue quantitatively that the space-time structure carrying topological charge fluctuations is inherently global (super-long-distance), in the sense discussed in Ref. [14]. In other words, our results indicate that this structure (both at the fundamental level and at low energy) cannot be broken into localized objects that could be used as a basis for valid physical analysis of the origin of topological susceptibility in QCD. Analysis in terms of global low-dimensional brane-like objects should be sought instead.

2. The Argument. The methodology for analysis of global behavior in space-time structure of local observables has been developed in Ref. [14]. Let us first review the basic ingredients of this approach in case of topological charge density. In what follows, we will assume that \( q(x) \) is a well-defined lattice topological density in the sense that the topological susceptibility can be directly computed via the associated correlator or, equivalently, via \( \chi = \frac{\langle Q^2 \rangle - \langle Q \rangle^2}{V} \), where \( Q = \sum_x q(x) \). If \( D \) is the overlap Dirac operator [6] based on the Wilson-Dirac kernel with mass \( -\rho \), then the density \[ q(x) = \frac{1}{2\rho} \text{tr} \gamma_5 D_{x,x} \equiv -\text{tr} \gamma_5 (1 - \frac{1}{2\rho} D_{x,x}) \] (1) satisfies this requirement.

The basic idea of Ref. [14] is to simultaneously monitor the physics (in this case topological susceptibility) and the geometry (in this case global behavior) as the space-time structure in \( q(x) \) is built starting from the most intense point and then gradually lowering the intensity threshold for points included in the structure. Formally, we define the sets \( S^q(f) \) (fraction supports) containing a fraction \( f = N(S^q(f))/N(\Omega) \) of the most intense points as ranked by \( |q(x)| \). Here \( N(\Gamma) \) denotes the number of points contained in the subset \( \Gamma \) of discretized torus \( \Omega \). The fraction \( f \) thus plays the role of the monitoring variable. We emphasize that what we mean by “structure” here is the partition of \( S^q(f) \) into maximal path-connected subsets \( S^q_i(f) \) containing the points with topological density of the same sign \([14, 2]\). This definition of geometric structure emphasizes connectedness because extended connected concentrations of strong fields can naturally facilitate the long-distance physics (e.g. lead to inherently global Dirac eigenmodes [14] and Goldstone boson propagation). Considering sign coherence is physically motivated by the fact that quarks of preferred chirality are attracted to regions of definite sign.

To monitor the saturation of topological fluctuations, we compute the cumulative function \( C(f) \) of topological susceptibility [14], which represents the fraction of the total susceptibility carried by a fraction \( f \) of the most intense points, i.e.

\[
C(f) \equiv \frac{\chi(f)}{\chi(1)} \quad \chi(f) \equiv \frac{\langle Q^2(f) \rangle - \langle Q(f) \rangle^2}{V} \tag{2}
\]

where

\[
Q(f) \equiv \sum_{x \in S^q(f)} q(x) \quad V \equiv a^4 N(\Omega) \tag{3}
\]

"By path-connected lattice set we mean a set of points any pair of which can be connected by a nearest-neighbor path passing only through points within the set."
are the charge associated with the fraction support and the total volume respectively. To monitor the global behavior in the structure, we compute two associated characteristic functions. The first is the maximal linear size \( \ell_{\text{max}}(f) \) of the connected component, i.e.

\[
\ell_{\text{max}}(f) \equiv \max_i \{ \ell(S^q_i(f)) \}
\]  

(4)

Here the linear size \( \ell(\Gamma) \) for arbitrary \( \Gamma \subset \Omega \) is the maximal distance between two points of the set, i.e. \( \ell(\Gamma) \equiv \max \{ |x-y| : x,y \in \Gamma \} \). The second characteristic is the point-wise average of the linear size over the points of the structure. More precisely, to \( x \in S^q_i(f) \) we assign the linear size \( \ell_f(x) \equiv \ell(S^q_i(f)) \) of the corresponding connected component, and take the average over \( S^q(f) \)

\[
\ell_{\text{pta}}(f) \equiv \langle \ell_f(x) \rangle_{S^q(f)}
\]

(5)

The motivation for introducing \( \ell_{\text{max}}(f) \) and \( \ell_{\text{pta}}(f) \) is that their values signal different regimes of global behavior in the structure. In particular, when \( \ell_{\text{max}}(f) \) reaches values \( \ell_{\text{max}}(f) \approx \ell(\Omega) \) (close to the “size of the space-time”) at some typical fraction \( f_{\text{max}} \), then \( f_{\text{max}} \) characterizes the onset of global behavior. In other words, an intensity threshold that corresponds to \( f \geq f_{\text{max}} \) is low enough to include not only the most intense local concentrations of the charge distribution, but also the connecting ridges which join them into a global structure. In addition, for fractions \( f \geq f_{\text{max}} \), where the largest connected piece is global, we would also like to know at what value of \( f \) most of the points in the structure are part of the global component. The function \( \ell_{\text{pta}}(f) \) characterizes this property and its saturation at some value \( f_{\text{pta}} \geq f_{\text{max}} \) indicates that global behavior is prevalent throughout the structure.

We computed the above characteristics for the ensemble (51 independent configurations) of Iwasaki gauge action [15] on an \( 8^4 \) lattice with scale \( a = 0.165 \text{ fm} \) determined from string tension. The physical volume of the system is thus \( V = 3 \text{ fm}^4 \). The topological charge density operator (1) with \( \rho = 1.368 \ (\kappa = 0.19) \) has been used to compute the site-by-site values of full topological charge density. The details of numerical implementation for overlap matrix–vector operation can be found in Ref. [16]. The results are shown in Fig. 1.

The data indicates that \( \ell_{\text{max}}(f) \) has a strict (full) saturation point at \( f_{\text{max}} \approx 0.14 \). Also, \( \ell_{\text{pta}}(f) \) exhibits a clear saturation “knee” at \( f_{\text{pta}} \approx 0.20 \). At the same time, the cumulative function of susceptibility saturates more gradually and reaches a full saturation point (within the statistics) at \( f_C \approx 0.80 \). The central message of Fig. 1 is that at fractions where the geometric structure consists of localized pieces (i.e. \( \ell_{\text{pta}}(f) \leq \ell_{\text{max}}(f) \ll 1 \)) only a small percentage of the total topological susceptibility is accounted for. In other words, topological susceptibility is not dominated by localized carriers of topological charge. Moreover, the susceptibility saturates only at fractions obviously larger than those at which the structure becomes prevalently global (i.e. \( f > f_{\text{pta}} \)). This result quantitatively justifies the proposition put forward in Ref. [2], that the physically relevant vacuum structure in topological charge density is inherently global (super-long-distance). In other words, our data indicate that it is not correct to view vacuum fluctuations of topological charge in terms of localized lumps of topological charge, while it strongly supports the idea that global extended objects should be used as a basis for valid physical analysis.

It should be noted here that our arguments started with the assumption that space-time regions containing the most intense fields (in this case topological charge density) are most

\[\text{In what follows we will always quote distances in units of } \ell(\Omega), \text{ the maximal possible distance.}\]
relevant for the physics in question (in this case topological susceptibility). This assumption is a posteriori justified by the fact that the computed $C(f)$ is a concave function on its domain within the errorbars. In other words, the ordering of space-time points by intensity results in monotonically decreasing contribution to susceptibility as the fraction is increased.

3. Low Energy. Our analysis in the previous section was carried out for full topological charge density, i.e. with fluctuations at all scales up to the lattice cutoff included. An attractive feature of topological charge density operators based on Ginsparg-Wilson fermionic actions is the possibility of defining an effective topological field \cite{3} via the eigenmode expansion of Eq. (1), i.e.

$$ q^{\Lambda_F}(x) = - \sum_{\lambda \leq \Lambda_F a} (1 - \frac{\lambda}{2\rho}) c^\lambda(x) $$

where $c^\lambda(x) = \psi^\dagger(x) \gamma_5 \psi^\lambda(x)$ is the local chirality of the mode $\psi^\lambda$ with eigenvalue $\lambda$. The effective field $q^{\Lambda_F}(x)$ describes topological fluctuations up to energy scale $\Lambda_F$ defined by the probing chiral fermion. Since zero modes are included in the expansion for arbitrary $\Lambda_F$, we have $\sum_x q(x) = Q = \sum_x q^{\Lambda_F}(x) = n_- - n_+$. Here $n_{-(+)}$ is the number of zero modes with...
negative (positive) chirality. This has three noteworthy implications [3]. (a) The topological susceptibility is independent of $\Lambda_F$. (b) Similarly to $q(x)$, the effective density is a perfect topological field in the sense that (for generic backgrounds) its global sum is strictly stable with respect to small local changes of the gauge field. (c) $q^{\Lambda_F}(x)$ satisfies the index theorem on the lattice in the same sense that $q(x)$ does [7].

The introduction of the effective topological density opens the possibility of describing the space-time structure of topological fluctuations in a scale-dependent manner as is appropriate in field theory [3]. The emerging precise relation between the structure at low energy and the fundamental structure will be described elsewhere. Here we focus on the issue of global behavior at low energy and how it compares to the behavior at the fundamental level. To do so, we have constructed effective densities at $\Lambda_F = 600$ MeV for three ensembles of Wilson gauge action with parameters described in Table 1. Lattices at different cutoffs have the same physical volumes $V \approx 3 \text{ fm}^4$. The quoted values of lattice spacing were obtained via the Sommer parameter using the interpolation formula given in Ref. [17]. For each configuration in a given ensemble we have calculated the zero modes and a fixed number of complex-conjugate eigenmode pairs specified in Table 1. The Ritz variational method [18] was used for this calculation. A typical accuracy of calculated eigenvalues (as measured by differences of complex-conjugate pairs) was one part in $10^5$ or better. The largest cutoff at which the effective density could be constructed for $E_3$ was just above $\Lambda_F = 600$ MeV (larger for $E_1$ and $E_2$).

We have calculated the cumulative function of topological susceptibility $C(f)$ and the global characteristics $\ell_{\text{max}}(f)$ and $\ell_{\text{pta}}(f)$ for the three ensembles with the results shown in Fig. 2 (left column). One can see that the main change at low energy compared to the fundamental structure is that the functions $\ell_{\text{max}}(f)$ and $\ell_{\text{pta}}(f)$ saturate even faster. For example, $\ell_{\text{max}}(f)$ in $E_3 (a = 0.055 \text{ fm})$ reaches full saturation point at $f_{\text{max}} < 0.06$. We thus conclude that the inherently global nature of the topological structure in the QCD vacuum is even more apparent at low energy. Moreover, the wide range of cutoffs spanned by our data shows convincingly that this conclusion will not change in the continuum limit.

To see the relation between the degree of saturation in susceptibility and in global behavior more directly, we follow Ref. [14] and eliminate the “monitoring variable” $f$ in favor of $C$. This defines the functions $\ell_{\text{max}}(C)$ and $\ell_{\text{pta}}(C)$ that quantify the degree of global behavior for any given value of susceptibility saturation. These functions are plotted in the right column of Fig. 2 and exhibit the strongly concave behavior characteristic of inherently global structure. From the data one can see that the structure becomes prevalently global when it carries only $C \approx 0.2$ of total susceptibility.

To quantify the vast discrepancy between the fractions of space-time volume characteriz-

| ensemble | $\beta$ | $a$ [fm] | $V_{\text{lat}}$ | configs | eigen-pairs |
|----------|---------|----------|-----------------|---------|-------------|
| $E_1$    | 5.91    | 0.110    | $12^4$          | 20      | 12          |
| $E_2$    | 6.07    | 0.082    | $16^4$          | 20      | 16          |
| $E_3$    | 6.37    | 0.055    | $24^4$          | 20      | 12          |

Table 1: Ensembles of Wilson gauge configurations. Zero modes and fixed number of lowest complex eigen-pairs were computed for each configuration.
Figure 2: The cumulative function of topological susceptibility and the global characteristics of the geometric structure at low energy. See the discussion in the text. Distances are measured in units of $\ell(\Omega)$. 
Figure 3: The continuum extrapolation of $f^{0.9}$ associated with $\ell_{\text{max}}(f)$, $\ell_{\text{pta}}(f)$ and $C(f)$. The linearly extrapolated values are quoted in the plot.

The goal of the study described here was to justify the proposition of Ref. [2] that the ordered vacuum structure in topological charge density, first observed there, is based on underlying global connected object(s) present in the vacuum. The original suggestion of Ref. [2] was based on the purely geometrical observation that the regions of space-time containing the most intense topological density have a strong preference to organize into a global long-range structure via the formation of a skeleton $^7$, i.e. the minimal skeleton.

$^7$In the language developed in Ref. [14] the formation of a skeleton corresponds to saturation of $\ell_{\text{pta}}$. This definition is simpler than the one used in Ref. [2] and leads to similar quantitative results.
hard-core substructure exhibiting the prevalently global behavior. In order to ascertain that a description of physics entails the existence of global geometric objects rather than some hypothetical localized objects forming the skeleton, it is necessary to evaluate the contribution of such objects to the relevant physical observable. Ref. [14] developed a formalism for performing such a calculation in the case of topological susceptibility. We adopted this formalism and the results presented here demonstrate that the global behavior dominates at decisively smaller fractions of space-time than those that are necessary to saturate the topological susceptibility. This is true both at the fundamental level and at low energy where the difference of relevant fractions is strikingly large. From these observations we conclude that the vacuum topological structure behaves as inherently global (super-long-distance) in QCD. 8 The relevance of this result resides in the fact that it directly indicates that a valid theoretical description of topological charge fluctuations in QCD vacuum needs to be based on global brane-like objects.

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