General form of the solutions of some difference equations via Lie symmetry analysis

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Abstract

In this paper, we obtain exact solutions of the following rational difference equation

\[ x_{n+1} = \frac{x_n x_{n-2} x_{n-4}}{x_{n-1} x_{n-3}(a_n + b_n x_n x_{n-2} x_{n-4})}, \]

where \( a_n \) and \( b_n \) are random real sequences, by using the technique of Lie symmetry analysis. Moreover, we discuss the periodic nature and behavior of solutions for some special cases. This work is a generalization of some works by Elsayed and Ibrahim in [E.M.Elsayed, T. F. Ibrahim, Solutions and periodicity of a rational recursive sequences of order five, Bulletin of the Malaysian Mathematical Sciences Society 38:1 (2015), 95-112].

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1 Introduction

Recently, there has been a considerable interest in studying the dynamics of rational difference equations. One of the reasons is that difference equations have many applications in several mathematical models in biology, ecology, physics, economics, genetics, population dynamics, medicine, physiology and so forth, see for example [1, 6, 15, 17]. Furthermore, symmetry methods for differential equations especially Lie symmetry analysis method have been investigated by several researchers. The method of Lie symmetry has been applied to difference equations in the past years and interesting results has been made by the authors (see [7, 10, 16]). The area of Lie symmetry analysis is a very rich research field.
The Norwegian mathematician Sophus Lie studied the group of mappings which leaves the differential equations invariant \([11]\). With this approach, one can solve difference equations using the group of transformations that leaves the equations invariant, similarly. After the inspiring works of Lie, the invariance properties of these equations under groups of point transformations attracted great interest. The symmetry method has been used to obtain the form of the solutions of difference equations, see for example \([3, 4, 5, 14]\). Moreover, Maeda \([12]\) showed how to use symmetry methods and get the solutions of the system of first-order ordinary difference equations.

In \([8]\), the author investigated the solutions and properties of the difference equation

\[
x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(a + bx_n x_{n-2})}
\]

where initial values are nonnegative real numbers.

E.M. Elsayed and T. F. Ibrahim \([2]\) obtained the solutions of the following difference equations of order five

\[
x_{n+1} = \frac{x_n x_{n-2} x_{n-4}}{x_{n-1} x_{n-3} (\pm 1 \pm x_n x_{n-2} x_{n-4})}, \quad n = 0, 1, 2, \ldots
\]

where the initial conditions \(x_{-4}, x_{-3}, x_{-2}, x_{-1}\) and \(x_0\) are arbitrary real numbers.

Our aim in this paper is to give the form of the solutions of the following difference equation

\[
x_{n+1} = \frac{x_n x_{n-2} x_{n-4}}{x_{n-1} x_{n-3} (a_n + b_n x_n x_{n-2} x_{n-4})}
\]  \((1)\)

in closed form, where \((a_n)_{n \in \mathbb{N}_0}, (b_n)_{n \in \mathbb{N}_0}\) are non-zero real sequences, via the technique of Lie group analysis. In this work, we are motivated by the results of E.M. Elsayed and T. F. Ibrahim \([2]\).

For the sake of convenience, we instead investigate the Kovalevskaya form of \((1)\):

\[
u_{n+5} = \frac{u_n u_{n+2} u_{n+4}}{u_{n+1} u_{n+3} (A_n + B_n u_n u_{n+2} u_{n+4})}.
\]  \((2)\)

2 Preliminaries

We begin by introducing some basic definitions and theorems needed in the sequel. For details, see \([7]\).

**Definition 1** Let \(G\) be a local group of transformations acting on a manifold \(M\). A subset \(S \subset M\) is called \(G\)-invariant, and \(G\) is called symmetry group of \(S\), if whenever \(x \in S\), and \(g \in G\) is such that \(g.x\) is defined, then \(g.x \in S\).

**Definition 2** Let \(G\) be a connected group of transformations acting on a manifold \(M\). A smooth real-valued function \(V : M \to \mathbb{R}\) is an invariant function for \(G\) if and only if

\[
X(V) = 0 \quad \text{for all} \quad x \in M,
\]  \((3)\)
and every infinitesimal generator $X$ of $G$.

**Definition 3** A parameterized set of point transformations,

$$
\Gamma_\varepsilon : x \rightarrow \hat{x}(x; \varepsilon),
$$

where $x = x_i$, $i = 1, \ldots, p$ are continuous variables, is a one-parameter local Lie group of transformations as long as the following conditions are met:

a. $\Gamma_0$ is the identity map if $\hat{x} = x$ when $\varepsilon = 0$.

b. $\Gamma_a \Gamma_b = \Gamma_{a+b}$ for every $a$ and $b$ sufficiently close to 0.

c. Each $\hat{x}_i$ can be represented as a Taylor series (in a neighborhood of $\varepsilon = 0$ that is determined by $x$), and therefore

$$
\hat{x}_i(x; \varepsilon) = x_i + \varepsilon \xi_i(x) + O(\varepsilon^2), i : 1, \ldots, p.
$$

Consider the $p$th-order difference equation

$$
u_{n+p} = \Omega (u_n, \ldots, u_{n+p-1}),
$$

for some smooth function $\Omega$. Assume the point transformations are of the form

$$
\hat{n} = n; \quad \hat{u}_n = u_n + \varepsilon Q(n, u_n) O(\varepsilon^2)
$$

with the corresponding infinitesimal symmetry generator

$$
X = Q(n, u_n) \frac{\partial}{\partial u_n} + S^{(1)} Q(n, u_n) \frac{\partial}{\partial u_{n+1}} + \ldots + S^{(p-1)} Q(n, u_n) \frac{\partial}{\partial u_{n+p-1}},
$$

where $S^{(j)}$ is the shift operator, i.e., $S^{(j)} : n \rightarrow n + j$. The symmetry condition is defined as

$$
\hat{u}_{n+p} = \Omega (n, \hat{u}_n, \hat{u}_{n+1}, \ldots, \hat{u}_{n+p-1}),
$$

whenever (8) is true. Substituting the Lie point symmetries (6) into the symmetry condition (7) leads to the linearized symmetry condition

$$
S^{(p)} Q - X \Omega = 0|_{u_{n+p}=\Omega(u_n,\ldots,u_{n+p-1})}. \tag{8}
$$

### 3 Reduction and solutions

Consider difference equations of the form (2), i.e.,

$$
u_{n+5} = \Omega = \frac{u_n u_{n+2} u_{n+4}}{u_{n+1} u_{n+3} (A_n + B_n u_n u_{n+2} u_{n+4})}. \tag{9}$$
To get the symmetry algebra of equation (9), we apply the invariance criterion (8) to (9) to get

\[ Q(n + 5, \Omega) - \frac{u_{n+2}u_{n+4}A_n}{u_{n+1}u_{n+3}(A_n + B_nu_nu_{n+2}u_{n+4})^2} Q(n, u_n) \]

\[ + \frac{u_{n+3}u_{n+1}(A_n + B_nu_nu_{n+2}u_{n+4})}{u_{n+4}u_{n+2}u_n} Q(n + 1, u_{n+1}) \]

\[ - \frac{u_{n+1}u_{n+3}(A_n + B_nu_nu_{n+2}u_{n+4})}{u_{n+1}u_{n+3}(A_n + B_nu_nu_{n+2}u_{n+4})^2} Q(n + 2, u_{n+2}) \]

\[ + \frac{u_{n+1}u_{n+3}(A_n + B_nu_nu_{n+2}u_{n+4})}{u_{n+1}u_{n+3}(A_n + B_nu_nu_{n+2}u_{n+4})^2} Q(n + 3, u_{n+3}) \]

\[ - \frac{u_{n+2}u_nA_n}{u_{n+1}u_{n+3}(A_n + B_nu_nu_{n+2}u_{n+4})^2} Q(n + 4, u_{n+4}) = 0. \] (10)

By acting the partial differential operator

\[ L = \frac{\partial}{\partial u_n} - \frac{\partial u_{n+3}}{\partial u_n} \frac{\partial}{\partial u_{n+3}} = \frac{\partial}{\partial u_n} - \left( \frac{\partial \Omega / \partial u_n}{\partial \Omega / \partial u_{n+3}} \right) \frac{\partial}{\partial u_{n+3}}, \]

on (10), we have

\[ \left( \frac{2u_{n+2}u_nB_n + A_n}{u_{n+3}} \right) Q - (A_n + B_nu_nu_{n+2}u_{n+4})Q' + (B_nu_nu_{n+4})S^{(2)}Q \]

\[ - \frac{(A_n + B_nu_nu_{n+2}u_{n+4})}{u_{n+3}} S^{(3)} Q' + (B_nu_nu_{n+4}) S^{(4)} Q = 0. \] (11)

The notation \( \prime \) stands for the derivative with respect to the continuous variable. Differentiation of (11) with respect to \( u_n \) twice, while \( u_{n+2}, u_{n+3} \) and \( u_{n+4} \) are fixed, yields the equation

\[ \frac{2A_n}{u_n^3} Q - \frac{2A_n}{u_n^2} Q' + \frac{A_n}{u_n} Q'' - A_n Q''' - (B_nu_nu_{n+2}u_{n+4})Q''' = 0. \] (12)

The equation above is solved by separation of variables in powers of shifts of \( u_n \). Thus, we have the system

\[ 1 : Q'''(n, u_n) - \frac{1}{u_n} Q''(n, u_n) + \frac{2}{u_n^2} Q'(n, u_n) - \frac{2}{u_n^3} Q(n, u_n) = 0 \]

\[ u_{n+2}u_{n+4} : Q''(n, u_n) = 0 \]

whose solution is as follows:

\[ Q(n, u_n) = \alpha_n u_n^2 + \beta_n u_n, \] (13)

for some arbitrary functions \( \alpha_n \) and \( \beta_n \) of \( n \) to be found. Substituting (13) and its first, second and third shifts in (10), and thereafter, replacing the expression
of $u_{n+5}$ given in (9) in the obtained equation, it follows that
\[
\begin{align*}
&u_n^2 u_{n+2}^2 u_{n+4}^2 u_{n+5} - A_n \alpha_{n+2} u_n^2 u_{n+1} u_{n+2} u_{n+3} u_{n+4} + A_n \alpha_{n+1} u_{n+1} u_{n+2} u_{n+3} u_{n+4} \\
&- A_n \alpha_{n+2} u_{n+1} u_{n+2} u_{n+3} u_{n+4} + A_n \alpha_{n+3} u_{n+1} u_{n+2} u_{n+3} u_{n+4} \\
&- A_n \alpha_{n+4} u_{n+1} u_{n+2} u_{n+3} u_{n+4} + \alpha_{n+1} B_n u_n^2 u_{n+2} u_{n+3} u_{n+4} \\
&+ \alpha_{n+3} B_n u_{n+2} u_{n+3} u_{n+4} u_{n+5} + A_n u_n u_{n+1} u_{n+2} u_{n+3} u_{n+4} (\beta_{n+5} - \beta_n + \beta_{n+1} \\
&- \beta_{n+2} + \beta_{n+3} - \beta_{n+4}) + B_n u_n^2 u_{n+2} u_{n+3} u_{n+4} (\beta_{n+5} + \beta_{n+1} + \beta_{n+3}) = 0.
\end{align*}
\]
(14)

Equating coefficients of all powers of shifts of $u_n$ to zero and simplifying the resulting system, we get its reduced form
\[
\alpha_n = 0, \quad \beta_n + \beta_{n+2} + \beta_{n+4} = 0
\]
(15)
and its solutions are
\[
\alpha_n = 0 \quad \text{and} \quad \beta_n = (-1)^n \beta^n c_1 + (-1)^n \beta^n c_2 + \beta^n c_3 + \beta^n c_4
\]
for some arbitrary constants $c_i, i = 1, \ldots, 4$, and where $\beta = \exp (\pi i / 3)$. So, we get four characteristics given by
\[
\begin{align*}
Q_1(n, u_n) &= (-1)^n \beta^n u_n, \\
Q_2(n, u_n) &= (-1)^n \beta^n u_n, \\
Q_3(n, u_n) &= \beta^n u_n, \\
Q_4(n, u_n) &= \beta^n u_n.
\end{align*}
\]
(16)
The four corresponding symmetry generators $X_1, X_2, X_3$ and $X_4$ are as follows:
\[
\begin{align*}
X_1 &= \sum_{j=0}^{4} (-\beta)^{n+j} u_{n+j} \partial u_{n+j}, \quad X_2 = \sum_{j=0}^{4} (-\beta)^{n+j} u_{n+j} \partial u_{n+j}, \\
X_3 &= \sum_{j=0}^{4} (\beta)^{n+j} u_{n+j} \partial u_{n+j}, \quad X_4 = \sum_{j=0}^{4} (\beta)^{n+j} u_{n+j} \partial u_{n+j}.
\end{align*}
\]
(17)
Here, using $Q_4$, we introduce the canonical coordinate $[9]
\[
S_n = \int \frac{du_n}{Q_4(n, u_n)} = \int \frac{du_n}{\beta^n u_n} = \frac{1}{\beta^n} \ln |u_n|.
\]
(18)
Using relation (19), we derive the function
\[
\tilde{V}_n = S_n \beta^n + S_{n+2} \beta^{n+2} + S_{n+4} \beta^{n+4}
\]
(19a)
and let
\[
|V_n| = \exp \left\{-\tilde{V}_n\right\}.
\]
(19b)
In other words, $V_n = \pm 1 / (u_n u_{n+2} u_{n+4})$. One can show by using (9) and (19b) that
\[
V_{n+1} = A_n V_n + B_n,
\]
(20)
that is,
\[ V_n = V_0 \prod_{k_1=0}^{n-1} A_{k_1} + \sum_{l=0}^{n-1} B_l \prod_{l+1=0}^{n-1} A_{k_2}. \]  \tag{21}

Here, we first use (18) to have
\[ |u_n| = \exp(S_n \beta^n). \]

Then, invoking (19) yields
\[ |u_n| = |H_n| \exp \left[ \frac{2\sqrt{3}^{n-1}}{3} \sum_{k=0}^{n-1} \left( \cos \left( \frac{(n-k)\pi}{2} \cos \frac{(n-k+1)\pi}{6} \right) \ln|V_k| \right) \right], \]  \tag{22}

where \( V_k \) is given in (21) and note that the \( H_n \)'s satisfy
\[ H_0 = x_0, \quad H_1 = x_1, \quad H_2 = x_2, \quad H_3 = x_3, \quad H_4 = \frac{1}{x_0x_2}, \quad H_5 = \frac{1}{x_1x_3}, \quad H_{6n+j} = H_j. \]

We can simplify the solution (22) further by splitting it into six categories. We have
\[ |u_{6n}| = H_{6n} \exp \left[ \frac{2\sqrt{3}^{6n-1}}{3} \sum_{k=0}^{6n-1} \left( \cos \left( \frac{(6n-k)\pi}{2} \cos \frac{(6n-k+1)\pi}{6} \right) \ln|V_k| \right) \right] \]
\[ = H_0 \exp \left( \ln V_0 - \ln V_2 + \ln V_6 - \ln V_8 + ... + \ln V_{6n-6} - \ln V_{6n-4} \right) \]
\[ = x_0 \prod_{k=0}^{n-1} \frac{V_{6k}}{V_{6k+2}}. \]  \tag{23}

Thus
\[ u_{6n} = u_0 \prod_{k=0}^{n-1} \frac{V_{6k}}{V_{6k+2}}. \]  \tag{24}

It can be verified, using \( V_n = 1/(u_n u_{n+2} u_{n+4}) \), that there is no need for absolute value function in (24). Similarly, for any \( j = 0, \ldots, 5 \), we obtain the following:
\[ u_{6n+j} = u_j \prod_{k=0}^{n-1} \frac{V_{6k+j}}{V_{6k+j+2}}. \]  \tag{25}

For \( j = 0 \), we get
\[ u_{6n} = u_0 \prod_{k=0}^{n-1} \frac{V_{6k}}{V_{6k+2}} \]
\[ = u_0 \prod_{k=0}^{n-1} \left( \prod_{k_1=0}^{6k-1} A_{k_1} \right) + u_0 u_2 u_4 \sum_{l=0}^{6k-1} \left( B_l \prod_{k_2=l+1}^{6k-1} A_{k_2} \right). \]
For $j = 1$, we have

$$u_{6n+1} = u_1 \prod_{k=0}^{n-1} \frac{V_{6k+1}}{V_{6k+3}}$$

$$= u_1 \prod_{k=0}^{n-1} \frac{6k}{k_1=0} A_{k_1} + u_0 u_2 u_4 \sum_{l=0}^{6k} \left( B_l \prod_{k_2=l+1}^{6k} A_{k_2} \right)$$

For $j = 2$, we have

$$u_{6n+2} = u_2 \prod_{k=0}^{n-1} \frac{V_{6k+2}}{V_{6k+4}}$$

$$= u_2 \prod_{k=0}^{n-1} \frac{6k+1}{k_1=0} A_{k_1} + u_0 u_2 u_4 \sum_{l=0}^{6k+1} \left( B_l \prod_{k_2=l+1}^{6k+1} A_{k_2} \right)$$

For $j = 3$, we have

$$u_{6n+3} = u_3 \prod_{k=0}^{n-1} \frac{V_{6k+3}}{V_{6k+5}}$$

$$= u_3 \prod_{k=0}^{n-1} \frac{6k+2}{k_1=0} A_{k_1} + u_0 u_2 u_4 \sum_{l=0}^{6k+2} \left( B_l \prod_{k_2=l+1}^{6k+2} A_{k_2} \right)$$

For $j = 4$, we have

$$u_{6n+4} = u_4 \prod_{k=0}^{n-1} \frac{V_{6k+4}}{V_{6k+6}}$$

$$= u_4 \prod_{k=0}^{n-1} \frac{6k+3}{k_1=0} A_{k_1} + u_0 u_2 u_4 \sum_{l=0}^{6k+3} \left( B_l \prod_{k_2=l+1}^{6k+3} A_{k_2} \right)$$
Finally, for \( j = 5 \), we have

\[
\begin{align*}
    u_{6n+5} &= u_5 \prod_{k=0}^{n-1} V_{6k+5} + u_0 u_2 u_4 \sum_{l=0}^{6k+4} \left( B_l \prod_{k_2=l+1}^{6k+4} A_{k_2} \right) \\
    &= u_5 \prod_{k=0}^{n-1} \left( \prod_{k_1=0}^{6k+4} A_{k_1} \right) + u_0 u_2 u_4 \sum_{l=0}^{6k+4} \left( B_l \prod_{k_2=l+1}^{6k+4} A_{k_2} \right).
\end{align*}
\]

More compactly, we have the solutions of (9) as follows:

\[
\begin{align*}
    u_{6n+j} &= u_j \prod_{k=0}^{n-1} \left( \prod_{k_1=0}^{6k+j-1} A_{k_1} \right) + u_0 u_2 u_4 \sum_{l=0}^{6k+j-1} \left( B_l \prod_{k_2=l+1}^{6k+j-1} A_{k_2} \right),
\end{align*}
\]

\( j = 0, 1, 2, 3, 4, 5 \). Therefore the solutions of (11) become

\[
\begin{align*}
    x_{6n+j-4} &= x_{j-4} \prod_{k=0}^{n-1} \left( \prod_{k_1=0}^{6k+j-1} a_{k_1} \right) + x_{-4} x_{-2} x_0 \sum_{l=0}^{6k+j-1} \left( b_l \prod_{k_2=l+1}^{6k+j-1} a_{k_2} \right), 0 \leq j \leq 5.
\end{align*}
\]

(26)

### 3.1 The case \( a_n \) and \( b_n \) are 1-periodic sequences.

In this case, \( a_n = a \) and \( b_n = b \) where \( a, b \in \mathbb{R} \). Then, equations in (26) simplify to

\[
\begin{align*}
    x_{6n+j-4} &= x_{j-4} \prod_{k=0}^{n-1} \left( a^{6k+j} + x_{-4} x_{-2} x_0 b \sum_{l=0}^{6k+j-1} a^l \right), 0 \leq j \leq 5.
\end{align*}
\]

(27)

**Theorem 4** The difference equation

\[
    x_{n+1} = \frac{x_n x_{n-2} x_{n-4}}{x_{n-1} x_{n-3} (a + b x_n x_{n-2} x_{n-4})}
\]

has a periodic solution of period six if and only if \( x_{-4} x_{-2} x_0 = \frac{1-a}{b}, \ a \neq 1 \).
Proof. Here, \( x_\cdot x_\cdot_2x_0 = \frac{1}{b} \) and then (27) simplifies to

\[
x_{6n+j-4} = \prod_{k=0}^{n-1} \frac{a^{6k+j} + (1-a) \left( \sum_{l=0}^{6k+j-1} a^l \right)}{a^{6k+j+2} + (1-a) \left( \sum_{l=0}^{6k+j+1} a^l \right)}, 0 \leq j \leq 5
\]

for \( i = 0, \ldots, 5 \). □

Example 5 Consider the Eq. (28) where \( a = -1, b = 1 \) and the initial conditions \( x_\cdot -4 = 0.2, x_\cdot -3 = 9, x_\cdot -2 = 5, x_\cdot -1 = 7, x_0 = 2 \) to verify our theoretical results.

![Figure 1: Plot of \( x_{n+1} = \frac{x_n x_{n-2} x_{n-4}}{x_{n-1} x_{n-3} (1 + bx_n x_{n-2} x_{n-4})} \)](image)

Theorem 6 Let \( \{x_n\} \) be a well-defined solution to the sequence

\[
x_{n+1} = \frac{x_n x_{n-2} x_{n-4}}{x_{n-1} x_{n-3} (1 + bx_n x_{n-2} x_{n-4})}
\]

with \( b \neq 0 \). Then

\[
\lim_{n \to \infty} x_n = 0.
\]

Proof. Using (27), with \( a = 1 \), we have

\[
x_{6n+j-4} = \prod_{k=0}^{n-1} \frac{1 + (6k + j)bx_\cdot -4x_\cdot -2x_0}{1 + (6k + j + 2)bx_\cdot -4x_\cdot -2x_0} = \prod_{k=0}^{n-1} \left[ 1 - \frac{2bx_\cdot -4x_\cdot -2x_0}{1 + (6k + j + 2)bx_\cdot -4x_\cdot -2x_0} \right].
\]
Clearly, $1 + (6k + j + 2)bx_{-4}x_{-2}x_0 \to \infty$ as $k \to \infty$. Therefore, we can always find a sufficiently large $n_0 \in \mathbb{N}$ such that for all $k > n_0$, we have

$$1 + (6k + j + 2)bx_{-4}x_{-2}x_0 \simeq (6k + j + 2)bx_{-4}x_{-2}x_0.$$ 

So

$$x_{6n+4} = x_{j-4} \Gamma(n_0) \prod_{k=n_0+1}^{n-1} \left(1 - \frac{2}{6k + j + 2}\right)$$

$$= x_{j-4} \Gamma(n_0) \prod_{k=n_0+1}^{n-1} \exp \left[ \ln \left(1 - \frac{2}{6k + j + 2}\right) \right].$$

Note that

$$\Gamma(n_0) = \prod_{k=0}^{n_0} \left(1 - \frac{2}{6k + j + 2}\right). \quad (30)$$

Now, using the fact that $\ln(1 + x) = x + O(x^2)$ for small $x$, we obtain

$$x_{6n+4} = x_{j-4} \Gamma(n_0) \prod_{k=n_0+1}^{n-1} \exp \left[ - \frac{2}{6k + j + 2} + O \left( \frac{1}{(6k + j + 2)^2} \right) \right]$$

$$= x_{j-4} \Gamma(n_0) \exp \left[ - \sum_{k=n_0+1}^{n-1} \left( \frac{2}{6k + j + 2} \right) \right] \prod_{k=n_0+1}^{n-1} \exp \left[ O \left( \frac{1}{(6k + j + 2)^2} \right) \right],$$

$j = 0, \ldots, 5$, and hence $x_n \to 0$ as $n \to \infty$. $lacksquare$

Next, we deal with the cases $a = \pm 1$ and $b = \pm 1$.

3.1.1 Case : $a = 1$ and $b = \pm 1$.

We can verify our results from [2] (see Theorems 1 and 6) that

$$x_{6n+4} = x_{j-4} \prod_{k=0}^{n-1} \frac{1 + b(6k + j)x_{-4}x_{-2}x_0}{1 + b(6k + j + 2)x_{-4}x_{-2}x_0}.$$ 

Clearly, in terms of the initial values $x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0$, we have

$$x_{6n-4} = x_{-4} \prod_{k=0}^{n-1} \frac{1 + 6bkx_{-4}x_{-2}x_0}{1 + (6k + 2)bx_{-4}x_{-2}x_0},$$

$$x_{6n-3} = x_{-3} \prod_{k=0}^{n-1} \frac{1 + (6k + 1)bx_{-4}x_{-2}x_0}{1 + (6k + 3)bx_{-4}x_{-2}x_0}.$$
\begin{align*}
x_{6n-2} &= x_{-2} \prod_{k=0}^{n-1} \frac{1 + (6k + 2)bx_{-4}x_{-2}x_0}{1 + (6k + 4)bx_{-4}x_{-2}x_0}, \\
x_{6n-1} &= x_{-2} \prod_{k=0}^{n-1} \frac{1 + (6k + 3)bx_{-4}x_{-2}x_0}{1 + (6k + 5)bx_{-4}x_{-2}x_0}, \\
x_{6n} &= x_0 \prod_{k=0}^{n-1} \frac{1 + (6k + 4)bx_{-4}x_{-2}x_0}{1 + (6k + 6)bx_{-4}x_{-2}x_0}
\end{align*}

and
\begin{equation*}
x_{6n+1} = \frac{x_{-4}x_{-2}x_0}{x_{-2}x_{-3}(1 + bx_{-4}x_{-2}x_0)} \prod_{k=0}^{n-1} \frac{1 + (6k + 5)bx_{-4}x_{-2}x_0}{1 + (6k + 7)bx_{-4}x_{-2}x_0}.
\end{equation*}

**Example 7** Consider the Eq. (28) where \(a, b = 1\) and the initial conditions \(x_{-4} = -0.2, x_{-3} = 3, x_{-2} = 1.3, x_{-1} = 0.7, x_0 = -2\) to verify our theoretical results.

![Figure 2: Plot of \(x_{n+1} = \frac{x_{n}x_{n-2}x_{n-4}}{x_{n-1}x_{n-3}(1 + x_{n-2}x_{n-4})}\)](image)

3.1.2 **Case :** \(a = -1\) and \(b = \pm 1\).

In this case, the solution becomes
\begin{equation*}
x_{6n+j-4} = x_{j-4} \prod_{k=0}^{n-1} \frac{(-1)^j + bx_{-4}x_{-2}x_0}{(-1)^j + bx_{-4}x_{-2}x_0} \left( \frac{1 - (-1)^j}{2} \right)
\end{equation*}

\begin{equation*}
= x_{j-4}
\end{equation*}

where \(j = 0, 1, 2, 3, 4, 5\).

Therefore, in this case it is easy to see that the solutions are periodic with period six. It is clear that, here, the extra condition \(x_{-4}x_{-2}x_0 = \frac{1-a}{b}\) in the above theorem is not needed. We can check our results from [2] (see Theorems 3 and 8).
4 Conclusion

Lie symmetry generators of difference equations of the form (2) were obtained, which in turn were useful in finding the solutions to the difference equation (1). Closed form formulas for the solutions were given and specific cases exhibited. Furthermore, periodic nature and behavior of solutions for some special cases were discussed.

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