A sampling-based approximation of the complex error function and its implementation without poles

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Abstract

Recently we developed a new sampling methodology based on incomplete cosine expansion of the sinc function and applied it in numerical integration in order to obtain a rational approximation for the complex error function \( w(z) = e^{-z^2} \left( 1 + \frac{2i}{\sqrt{\pi}} \int_0^z e^{t^2} dt \right) \), where \( z = x + iy \). As a further development, in this work we show how this sampling-based rational approximation can be transformed into alternative form for efficient computation of the complex error function \( w(z) \) at smaller values of the imaginary argument \( y = \text{Im}[z] \). Such an approach enables us to avoid poles in implementation and to cover the entire complex plain with high accuracy in a rapid algorithm. An optimized Matlab code utilizing only three rapid approximations is presented.

Keywords: complex error function; rational approximation; sampling; sinc function

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1 Introduction

The complex error function, also known as the Faddeeva function, can be defined as \[1, 2\]

\[
w(z) = e^{-z^2} \left( 1 + \frac{2i}{\sqrt{\pi}} \int_{0}^{z} e^{t^2} dt \right),
\]

where \( z = x + iy \). Despite simple representation, the integral in equation \[1\] cannot be taken analytically. Therefore, the integral equation \[1\] for the complex error function must be computed numerically.

This function \( w(z) \) is the most important in family of the Faddeeva functions. In particular, comparing this equation with the error function of complex argument \[2\]

\[
\text{erf}(z) = 2 \frac{2}{\sqrt{\pi}} \int_{0}^{z} e^{-t^2} dt
\]

one can see that

\[
\text{erf}(z) = 1 - e^{-z^2} w(iz).
\]

Consequently, we can conclude that the error function \( \text{erf}(z) \) is just a reformulation of the complex error function \( w(z) \). Some other functions of practical importance that can be reformulated in terms of the complex error function are the Dawson’s integral \[3, 4, 5, 6, 7\]

\[
\text{daw}(z) = e^{-z^2} \int_{0}^{z} e^{t^2} dt = \frac{\sqrt{\pi}}{2i} \left( w(z) - e^{-z^2} \right),
\]

the Fresnel integral \[2\]

\[
F(z) = \int_{0}^{z} e^{i(\pi/2)t^2} dt = (1 + i) \left[ 1 - e^{i(\pi/2)z^2} w(\sqrt{\pi} (1 + i) z/2) \right]/2
\]

and the plasma dispersion function \[8\]

\[
Z(z) = PV \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-t^2}}{t - z} dt = i \sqrt{\pi} w(z),
\]
where the principal value signifies that it has no discontinuity at \( y = \text{Im}[z] = 0 \).

The equation (1) is absolutely identical to the complex probability function on the upper half of the complex plain \(^9\) (see also \(^2\))

\[
W(z) = \text{PV} \frac{i}{\pi} \int_{-\infty}^{\infty} e^{-t^2} \frac{1}{z-t} dt = w(z), \quad \text{Im}[z] \geq 0,
\]

where the principal value implies again that this function has no discontinuity at \( y = \text{Im}[z] = 0 \). The real part of the complex probability function is known as the Voigt function \(^10\), \(^11\), \(^12\), \(^13\), \(^14\)

\[
K(x,y) = \text{Re}[W(z)] = \text{PV} \frac{y}{\pi} \int_{-\infty}^{\infty} e^{-t^2} \frac{1}{y^2 + (x-t)^2} dt,
\]

widely used in atmospheric science to describe the spectral line broadening effects in photon absorption and emission by various gas molecular species in a planetary atmosphere.

There is a remarkable property of the complex error function \(^2\), \(^15\), \(^16\)

\[
w(z) = 2e^{-z^2} - w(-z).
\]

From equation (2) it immediately follows that

\[
w(x, -|y|) = 2e^{-(x+iy)^2} - w(-x, |y|)
\]

This signifies that it is sufficient to consider only \(^1\text{st}\) and \(^2\text{nd}\) quadrants in order to cover the entire complex plane. Thus, in order to simplify algorithmic implementation we will imply further that \( y \geq 0 \).

The identity (2) can be used not only to avoid a direct computation of the complex error function at negative \( y < 0 \). In this work we show how this identity can also be generalized to derive a sampling-based approximation of the complex error function that excludes all its poles in algorithmic implementation. This approach sustains high accuracy in computation at smaller values of the parameter \( y \) that is commonly considered difficult for computation of the Voigt/complex error function \(^10\), \(^17\), \(^18\).
2 Methodology and derivation

In our earlier publication [19] we have developed a new methodology of sampling based on incomplete cosine expansion of the sinc function. This technique of sampling is especially efficient in numerical integration. As an example, we have shown that applying the incomplete cosine expansion of the sinc function to equation (1), the following sampling-based rational approximation of the complex error function

\[ w(z) \approx \sum_{m=1}^{2K-1} A_m B_m (z + i\varsigma/2) C_m^2 - (z + i\varsigma/2)^2, \]  

where the expansion coefficients are

\[ A_m = \frac{\sqrt{\pi} (2m - 1)}{2^{2K} h} \sum_{n=-N}^{N} e^{c^2/4-n^2h^2} \sin \left( \frac{\pi (2m - 1) (nh + \varsigma/2)}{2^{K} h} \right), \]

\[ B_m = -\frac{i}{2^{K-1} \sqrt{\pi}} \sum_{n=-N}^{N} e^{c^2/4-n^2h^2} \cos \left( \frac{\pi (2m - 1) (nh + \varsigma/2)}{2^{K} h} \right), \]

and

\[ C_m = \frac{\pi (2m - 1)}{2^{K+1} h}, \]

can be obtained. Specifically, by taking \( h = 0.25, \varsigma = 2.75, K = 5 \) and \( N = 23 \) the approximation [3] alone covers with average accuracy \( \sim 10^{-14} \) [19] the entire domain \( 0 \leq x \leq 40,000 \) and \( 10^{-4} \leq y \leq 10^2 \) of the Voigt line-shapes that arise from the collection of HITRAN molecular transitions [20] at low terrestrial altitudes and for spectral displacements less than 25 \( cm^{-1} \).

Later we have shown that the truncation integer in approximation [3] may not be necessarily equal to \( 2^{K-1} \). In particular, this restriction can be avoided by replacing \( 2^{K-1} \) with an arbitrary integer \( M \) (see [21] for more details). Consequently, the approximation (3) can be rewritten in form

\[ w(z) \approx \sum_{m=1}^{M} a_m + b_m (z + i\varsigma/2) C_m^2 - (z + i\varsigma/2)^2, \]  

where the expansion coefficient are modified correspondingly as

\[ a_m = \frac{\sqrt{\pi} (m - 1/2)}{2M^2 h} \sum_{n=-N}^{N} e^{c^2/4-n^2h^2} \sin \left( \frac{\pi (m - 1/2) (nh + \varsigma/2)}{Mh} \right), \]
\[ b_m = -\frac{i}{M\sqrt{\pi}} \sum_{n=-N}^{N} e^{\xi^2/4-n^2h^2} \cos \left( \frac{\pi (m - 1/2)(nh + \varsigma/2)}{Mh} \right) \]

and
\[ c_m = \frac{\pi (m - 1/2)}{2Mh}. \]

Although the approximation (4) can cover the entire HITRAN domain, its accuracy deteriorates as the parameter \( y \) decreases. In order to resolve this problem we rewrite equation (2) as follows
\[ w(z) = e^{-z^2} + \frac{w(z) - w(-z)}{2}. \] (5)

Substituting approximation (4) into right side of the equation (5), after some trivial rearrangements we can transform it into alternative form as follows
\[ w(z) \approx e^{-z^2} + z \sum_{m=1}^{M} \alpha_m - \beta_m z^2 + \gamma_m \]
\[ w(z) \approx e^{-z^2} + z \sum_{m=1}^{M} \alpha_m - \beta_m z^2 + \gamma_m z^4 \] (6)

where
\[ \alpha_m = c_m \left[ c_m^2 - \left( \frac{\varsigma^2}{2} \right)^2 \right] + i a_m \varsigma = b_m \left[ \left( \frac{\pi (m - 1/2)}{2Mh} \right)^2 - \left( \frac{\varsigma}{2} \right)^2 \right] + i a_m \varsigma, \]
\[ \beta_m = b_m, \]
\[ \gamma_m = c_m^4 + \frac{c_m^2 \varsigma^2}{2} + \frac{\varsigma^4}{16} = \left[ \left( \frac{\pi (m - 1/2)}{2Mh} \right)^2 + \left( \frac{\varsigma}{2} \right)^2 \right]^2 \]
and
\[ \theta_m = 2c_m^2 - \frac{\varsigma^2}{2} = 2 \left( \frac{\pi (m - 1/2)}{2Mh} \right)^2 - \frac{\varsigma^2}{2}. \]

Since the new equation (6) is derived by transformation from the sampling-based rational approximation (4), it also represents a sampling-based approximation.

It should be noted that application of the identity (5) in derivation of approximation of the complex error function in alternative form has been proposed already in our recent publication [22] (see also the corresponding Matlab code [23]). However, in this work we show its generalization leading to approximation (6) that can be used without poles in a rapid algorithm. Furthermore, we also suggest that equation (5) may be applied to other approximations in order to compute more accurately the Voigt/complex error function at smaller values of the parameter \( y \).
3 Implementation

3.1 Approximations and boundaries

Similar to our previous work [7], we applied only three approximations bounded inside domains as shown in Fig. 1; in fact, due to symmetric properties of the complex error function

\[
\text{Re} \left[ w(x,y) \right] = \text{Re} \left[ w(-x,y) \right]
\]

and

\[
\text{Im} \left[ w(x,y) \right] = - \text{Im} \left[ w(-x,y) \right],
\]

it is sufficient to consider only the I\textsuperscript{st} quadrant of the complex plain. The complex plain is divided into external and internal domains. Taking as an objective a worst relative accuracy of \(10^{-13}\), we separated these domains by boundaries accordingly for the best optimization.

Internal domain is situated inside a circle \(|x + iy| = 8\) and consists of two subdomains, the primary subdomain and secondary subdomain. The secondary subdomain is bounded by a straight line \(y = 0.05|x|\).

It is very convenient for algorithmic implementation to rewrite the approximation (4) as follows

\[
w(z) \approx \Omega \left( z + i\frac{c}{2} \right)
\]

\[
\Rightarrow \Omega(z) \triangleq \sum_{m=1}^{M} \frac{a_m + b_m z}{c_m - z^2}.
\]  

(7)

At \(M = 23\) this approximation meets the requirement for accuracy exceeding \(10^{-13}\) within primary subdomain.

For secondary subdomain we may apply the approximation [6] without any modification. However, at \(M = 23\) its accuracy becomes \(\sim 10^{-12}\) in the area near the origin. In order to resolve this problem we should increase the number of summation terms by two as given by [4]

\[
w(z) \approx e^{-z^2} + z \sum_{m=1}^{M+2} \frac{\alpha_m - \beta_m z^2}{\gamma_m - \theta_m z^2 + z^4}.
\]  

(8)

\(^1\)These two additional terms may be optional if the requirement for accuracy \(\sim 10^{-12}\) is sufficient for users.
Inclusion of these two terms almost does not decelerate the computation and sustains high accuracy exceeding $10^{-13}$ everywhere within the secondary domain.

External domain utilizes the following Laplace continued fraction [2, 15, 24] given by

$$w(z) \approx \frac{(i/\sqrt{\pi})}{z^{1/2}}.$$  \hspace{1cm} (9)

An optimized Matlab code, implemented according to this scheme, is shown in Appendix A.

![Diagram](image)

Fig. 1. Boundaries and location of 25 poles in I$^\text{st}$ quadrant of the complex plain.

3.2 Location of poles

Approximation (7) contains poles that can be readily found by solving the following quadratic equation

$$c_m^2 - (z + i\varsigma/2)^2 = 0.$$
It is not difficult to see that the solution for the equation above results to two poles at each index \( m \). Particularly, we can find that

\[
z_{1,2} = \pm c_m - i\varsigma/2.
\]

Fortunately, all these poles are located in the III\textsuperscript{rd} and IV\textsuperscript{th} quadrants only since \( \text{Im}[z_{1,2}] = -\varsigma/2 = -1.375 \). Therefore, these poles do not affect the computation.

The new sampling-based approximation \( (8) \) also contains poles. In particular, solving the following quartic equation

\[
\gamma_m - \theta_m z^2 + z^4 = 0 \tag{10}
\]

one can find four poles associated with each index \( m \). These poles are

\[
z_{1,2} = \pm \frac{\sqrt{\theta_m + \sqrt{\theta_m^2 - 4\gamma_m}} + \sqrt{\theta_m^2 - 4\gamma_m}}{\sqrt{2}}
\]

and

\[
z_{3,4} = \pm \frac{\sqrt{\theta_m - \sqrt{\theta_m^2 - 4\gamma_m}} + \sqrt{\theta_m^2 - 4\gamma_m}}{\sqrt{2}}.
\]

All these poles are located along two horizontal lines, since (see Appendix B)

\[
\text{Im}[z_{1,4}] = \varsigma/2 = 1.375
\]

and

\[
\text{Im}[z_{2,3}] = -\varsigma/2 = -1.375.
\]

The location of \( M + 2 = 25 \) poles on the I\textsuperscript{st} quadrant is shown in Fig. 1 by open circles with dots inside. Since these poles are situated far beyond the secondary subdomain, they also do not affect the computation.

### 3.3 Error analysis

The error analysis is performed by using relative errors defined as

\[
\Delta_{\text{Re}} = \left| \frac{\text{Re}[w_{\text{ref}}(x, y)] - \text{Re}[w(x, y)]}{\text{Re}[w_{\text{ref}}(x, y)]} \right|
\]
\[ \Delta_{\text{Im}} = \left| \frac{\text{Im}[w_{\text{ref}}(x, y)] - \text{Im}[w(x, y)]}{\text{Im}[w_{\text{ref}}(x, y)]} \right|, \]

where \( w_{\text{ref}}(x, y) \) is the reference, for the real and imaginary parts, respectively.

Fig. 2. Logarithm of the relative errors for (a) real and (b) imaginary parts over the domain \( 0 \leq x \leq 15 \) and \( 0 \leq y \leq 15 \).

Fig. 3. Logarithm of the relative errors over the domains (a) \( 4 \leq x \leq 10 \) and \( 0 \leq y \leq 0.1 \), (b) \( 0 \leq x \leq 1 \) and \( 0 \leq y \leq 0.5 \) for the real and imaginary parts, respectively.

Figures 2a and 2b show the logarithm of relative errors for the real and imaginary parts, respectively, over the area \( 0 \leq x \leq 15 \) and \( 0 \leq y \leq 15 \). Figures 3a and 3b depict the logarithm of relative errors with worst accuracies.
As we can see, over the real part the worst accuracy is about $2 \times 10^{-14}$ while over the imaginary part the worst accuracy is about $8 \times 10^{-14}$.

Thus, the provided accuracy at double precision computation of the complex error function $w(z)$ is absolutely consistent with CERNLIB, libcerf and RooFit packages (see [25] for detailed information regarding accuracy of these packages).

4 Run-time test

The Matlab is an array programing language. Consequently, the number of applied equations should be minimized in order to reduce computational flow consisting of large size arrays [7]. The algorithm we develop employs only three rapid approximations (7), (8) and (9). Although equation (8) involves quartic polynomial in its denominator, this practically does not decelerate the computation since the arrays $z^2$, $z^4$ can be predefined prior to the nested sum (see the function \texttt{SD2 = subdom2(z)} inside body of the Matlab code in Appendix A). Furthermore, the array of the exponential function $e^{-z^2}$ is computed just once outside the nested loop. Consequently, the approximation (8) is almost as rapid as the rational approximation (7).

The run-time test has been performed by comparing our Matlab code, shown in Appendix A, with Ab-Initio group of MIT implementation written in C/C++ programing language by Steven Johnson [26, 27]. The corresponding algorithm in this C/C++ implementation is based on some modification of the Algorithm 680 [24, 28] with additional inclusion of the Salzer’s approximation for smaller values of the imaginary part $y = \text{Im} [z]$ (see our recent paper [7] for detailed information, see also [29, 30]).

Although Ab-Initio group of MIT implementation utilizes only earliest equations published for the complex error function, it can provide, nevertheless, a rapid computation with relative errors smaller than $10^{-13}$ [26].

Despite that Matlab programs are usually slower than their C/C++ analogs, the run-time test shows that with 10 million random numbers, generated within the domain $0 < x < 6$ and $0 < y < 0.1$, our Matlab code is faster by a factor 1.68 (11.6 and 6.9 seconds, respectively). This can be explained from the fact that the Salzer’s equation covering this domain in C/C++ implementation is not simple and, therefore, relatively slow due to requirement to compute multiple times the hyperbolic sine and hyperbolic cosine functions in a nested recurrence procedure (see our recent work [7].
for C/C++ and Matlab command lines and technical aspects describing why this domain is especially important for consideration).

Computational test also reveals that with 10 million random numbers, generated within the domain $|x + iy| < 15$, the Matlab code is faster than C/C++ code by a factor 1.51 (5.9 and 3.9 seconds, respectively). Within the domain $|x + iy| < 10,000$ the C/C++ implementation is faster than the Matlab code by a factor 1.33 (2.1 and 2.8 seconds, respectively, for 10 million random numbers). However, since a high spectral resolution in computational spectroscopy is required only for the sharp lines and absolutely unnecessary for the flat curve regions, this advantage of the C/C++ implementation disappears when we perform a computation with non-equidistantly distributed grid-points. For example, if we take, say, 9 million grid points over the domain $|x + iy| < 15$ and the remaining 1 million grid points beyond this domain, then the Matlab code remains faster than the C/C++ implementation. It should be noted that application of the non-equidistantly distributed grid-points is very common in radiative transfer applications since this approach significantly accelerates the computation. In atmospheric radiative transfer applications the Lorentzian distribution can be used to select non-equidistantly spaced grid-points along $x$-axis at some fixed value $y$.

The present Matlab code is as fast as that of reported in our recent publication with some very minor differences over specific domains. To the best of our knowledge this and our recent Matlab codes are the most rapid in computation of the complex error function as compared to any other Matlab codes with comparable accuracy ever reported in scientific literature or elsewhere online (see for example this Matlab codes). Therefore, the proposed approximations may be also be useful in radiative transfer models like MODRTAN and bytran, where our previously reported approximations are currently used for rapid and accurate computation of the Voigt/complex error function.

The run-time test has been performed on a typical desktop computer (Intel(R), CPU at 2.6 GHz, RAM 8 GB, Windows 10) by using the Intel compiler and Matlab 2009b distributive.
5 Conclusion

In this work we show how the sampling-based approximation can be obtained and applied for efficient computation of the complex error function $w(z)$ at smaller values of the imaginary argument $y = \text{Im}[z]$. This approach results in coverage of the entire complex plain. An optimized Matlab code providing rapid computation with high accuracy is presented.

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Appendix A

function FF = fadsamp(z)

% This function file computes the complex error function (also known as
% the Faddeeva function) by using a new method of sampling based on
% incomplete cosine expansion of the sinc function [1, 2]. External domain
% is computed by the Laplace continued fraction [3]. The description of the
% algorithm is presented in the work [4].
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The code is written by Sanjar M. Abrarov, Brendan M. Quine and Rajinder K. Jagpal, York University, Canada, February 2018.

All variables in this section are global within the function file.

\[ h = 0.25; \]
\[ \text{step} \]
\[ \text{stigma} = 2.75; \]
\[ \text{shift constant} \]
\[ \text{m} = 23; \]
\[ \text{truncating integer for index m} \]
\[ \text{n} = 23; \]
\[ \text{truncating integer for index n} \]

\[ \text{n} = \text{-n} \text{max:n} \text{max}; \]
\[ \text{array for index n} \]

\[ \text{tab} = \text{ones}(6, \text{m} \text{max}); \]
\[ \text{initiate the table} \]
\[ \text{m} = 1; \]
\[ \text{counter} \]

while \( \text{m} \leq \text{m} \text{max} + 2 \)

\[ \text{tab}(1, \text{m}) = (\text{sqrt} \text{pi} \times (\text{m} - 1/2))/(2 \times (\text{m} \text{max}^2 \times h) \times \text{sum} (\text{exp} \text{stigma}^2/4 - \text{n}^2 \times h^2) \times \text{sin} ((\text{pi} \times (\text{m} - 1/2) \times (\text{n} \times h + \text{stigma}/2))/(\text{m} \text{max} \times h))); \]

\[ \text{tab}(2, \text{m}) = -\text{i}/(\text{m} \text{max} \times \text{sqrt} \text{pi}) \times \text{sum} (\text{exp} \text{stigma}^2/4 - \text{n}^2 \times h^2) \times \text{cos} ((\text{pi} \times (\text{m} - 1/2) \times (\text{n} \times h + \text{stigma}/2))/(\text{m} \text{max} \times h)); \]

\[ \text{tab}(3, \text{m}) = \text{pi} \times (\text{m} - 1/2)/(2 \times (\text{m} \text{max} \times h)); \]

\[ \text{tab}(4, \text{m}) = \text{tab}(2, \text{m}) \times ((\text{pi} \times (\text{m} - 1/2))/(2 \times (\text{m} \text{max} \times h)))^2 \times (\text{stigma}/2)^2 + \text{i} \times \text{tab}(1, \text{m}) \times \text{stigma}; \]

\[ \text{tab}(5, \text{m}) = ((\text{pi} \times (\text{m} - 1/2))/(2 \times (\text{m} \text{max} \times h)))^2 + (\text{stigma}/2)^2; \]

\[ \text{tab}(6, \text{m}) = 2 \times ((\text{pi} \times (\text{m} - 1/2))/(2 \times (\text{m} \text{max} \times h)))^2 - \text{stigma}^2/2; \]

\[ \text{m} = \text{m} + 1; \]
\[ \text{increment the counter} \]

The expansion coefficients are:
\[ \text{a} = \text{tab}(1,:); \]
\[ \text{b} = \text{tab}(2,:); \]
\[ \text{c} = \text{tab}(3,:); \]
\[ \text{alpha} = \text{tab}(4,:); \]
\[ \text{beta} = \text{tab}(2,:); \]
\[ \text{gamma} = \text{tab}(5,:); \]
\[ \text{theta} = \text{tab}(6,:); \]

end
ind_neg = imag(z)<0; % if some imag(z) values are negative, then ...
    z(ind_neg) = conj(z(ind_neg)); % ... bring them to the upper-half plane

FF = zeros(size(z));

ind_ext = abs(z)>8; % external indices

FF(~ind_ext) = intf(z(~ind_ext));
    FF(ind_ext) = contfr(z(ind_ext)); % continued fraction (external region)

function CF = contfr(z) % the Laplace continued fraction approximation
    bN = 11; % initial integer
    bN = 1:bN;
    bN = bN/2;
    CF = bN(end)./z; % start computing from the last bN
    for k = 1:length(bN) - 1
        CF = bN(end-k)./(z - CF);
    end
    CF = 1i/sqrt(pi)./(z - CF);
end

function IF = intf(z) % internal function
    IF = zeros(size(z));

    ind_pr = imag(z)>0.05*abs(real(z));

    IF(ind_pr) = Omega(z(ind_pr) + 1i*stigma/2);
    IF(~ind_pr) = subdom2(z(~ind_pr)); % secondary subdomain

function OF = Omega(z) % Omega function for primary subdomain
    zP2 = z.^2; % define repeating array

    OF = 0;
    for k = 1:n_max
        OF = OF + (tab(1,k) + tab(2,k)*z)./(tab(3,k).^2 - zP2);
    end
end
function SD2 = subdom2(z)  \% secondary subdomain
    zP2 = z.^2;  \% first repeating array
    zP4 = zP2.^2;  \% second repeating array
    SD2 = 0;
    for k = 1:n_max + 2  \% increased by 2 terms!
        SD2 = SD2 + (tab(4,k) - tab(2,k)*zP2)./(tab(5,k) - tab(6,k)* ... 
                    zP2 + zP4);
    end
    SD2 = exp(-z.^2) + z.*SD2;
end

% Convert for negative imag(z) values
FF(ind_neg) = conj(2*exp(-z(ind_neg).^2) - FF(ind_neg));
end

Appendix B

There are four solutions of the quartic equation (10). In order to find them it is convenient to represent equation (10) in a biquadratic form as follows

\[
\gamma_m - \theta_m Z + Z^2 = 0, \tag{B.1}
\]

where

\[
Z = z^2. \tag{B.2}
\]

Two solutions of the biquadratic equation (B.1) are given by

\[
Z_{1,2} = \frac{\theta_m \pm \sqrt{\theta_m^2 - 4\gamma_m}}{2}. \tag{B.3}
\]

Since

\[
\theta_m = 2c_m^2 - \frac{s^2}{2}
\]

and

\[
\gamma_m = c_m^4 + \frac{c_m^2 s^2}{2} + \frac{s^4}{16}
\]

it follows that

\[
\theta_m^2 - 4\gamma_m = -4c_m^2 s^2.
\]
Consequently, from equation (B.3) we have
\[ Z_{1,2} = \frac{(2c_m^2 - \varsigma^2/2) \pm \sqrt{-4c_m^2\varsigma^2}}{2} = \frac{(2c_m^2 - \varsigma^2/2) \pm 2ic_m\varsigma}{2} = \frac{1}{4}(2c_m \pm i\varsigma)^2. \]
(B.4)
Lastly, taking into account the relation (B.2) from equation (B.4) we obtain four solutions for the equation (10)
\[ z_1 = \frac{1}{2} (2c_m + i\varsigma) = c_m + i\frac{\varsigma}{2}, \]
\[ z_2 = -\frac{1}{2} (2c_m + i\varsigma) = -c_m - i\frac{\varsigma}{2}, \]
\[ z_3 = \frac{1}{2} (2c_m - i\varsigma) = c_m - i\frac{\varsigma}{2}, \]
and
\[ z_4 = -\frac{1}{2} (2c_m - i\varsigma) = -c_m + i\frac{\varsigma}{2}. \]

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