Comment on "Solutions of Dirac equation with an improved expression of the Rosen-Morse potential energy model including Coulomb-like tensor interaction Oscillator"

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Abstract

The Nikiforov-Uvarov polynomial method employed by Aguda to solve the Dirac equation with an improved Rosen-Morse potential plus a Coulomb-like tensor potential is shown inappropriate because the conditions of its application are not fulfilled. We clarify the problem and construct the correct solutions in the spin and pseudospin symmetric regimes via the standard method of solving differential equations. For the bound states, we obtain the spinor wave functions in terms of the generalized hypergeometric functions $\text{}_2F_1(a, b; c; z)$ and in each regime we show that the energy levels are determined by the solutions of a transcendental equation which can be solved numerically.

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In a recent paper [1] published in this journal, Aguda claimed to have obtained the approximate analytical solutions of the Dirac equation with an improved Rosen-Morse potential plus a Coulomb-like tensor interaction under the condition of spin and pseudospin symmetry by using the Nikiforov-Uvarov (NU) method. The author of this paper starts with the following Dirac equation for a fermionic particle of mass $M$ in a mixture of scalar and vector potentials and a Coulomb tensor potential:

$$\left\{ \vec{\alpha} \cdot \vec{p} + \beta [M + S(r)] - i \beta \vec{\alpha} \cdot \vec{r} U(r) \right\} \psi_{n_r, \kappa}(\vec{r}) = [E - V(r)] \psi_{n_r, \kappa}(\vec{r}),$$

(1)
and defines the Dirac spinors as

\[ \psi_{n,\kappa}(\mathbf{r}) = \frac{1}{r} \begin{pmatrix} F_{n,\kappa}(r) Y^l_{jm} (\theta, \phi) \\ iG_{n,\kappa}(r) Y^l_{jm} (\theta, \phi) \end{pmatrix}, \]  

where \( F_{n,\kappa}(r) \) and \( G_{n,\kappa}(r) \) are the upper and lower component radial wave functions, respectively. Since \( S(r) \), \( V(r) \) and \( U(r) \) are central potentials, it is clear that, after some simple calculation, the components \( F_{n,\kappa}(r) \) and \( G_{n,\kappa}(r) \) are determined from equation (1) by the system of differential equations

\[ \begin{cases} \left( \frac{d}{dr} + \frac{s}{r} - U(r) \right) F_{n,\kappa}(r) = (M + E_{n,\kappa} - \Delta(r)) G_{n,\kappa}(r), \\ \left( \frac{d}{dr} - \frac{s}{r} + U(r) \right) G_{n,\kappa}(r) = (M - E_{n,\kappa} + \Sigma(r)) F_{n,\kappa}(r), \end{cases} \]  

where \( \Delta(r) = V(r) - S(r) \), \( \Sigma(r) = V(r) + S(r) \) and the Coulomb tensor potential \( U(r) = -\frac{T}{r} \), for \( r \geq R_c \), with \( T = \frac{Z_1 Z_2 e^2}{4 \pi \varepsilon_0} \).

From these equations it follows that

\[ \begin{align*}
& \left[ -\frac{d^2}{dr^2} + \frac{\kappa(\kappa + 1)}{r^2} + (M + E_{n,\kappa} - C_s) \Sigma(r) + \frac{T^2 + T(2\kappa + 1)}{r^2} \right] F_{n,\kappa}(r) \\
& = \left[ E^2_{n,\kappa} - M^2 + C_s (M - E_{n,\kappa}) \right] F_{n,\kappa}(r) 
\end{align*} \]  

in the spin symmetry limit, i.e., when \( d\Delta(r)/dr = 0 \) or \( \Delta(r) = C_s = \text{constant} \). Here \( \kappa = l \) and \( \kappa = -l - 1 \) for \( \kappa < 0 \) and \( \kappa > 0 \), respectively; and \( \Sigma(r) \) is the improved Rosen-Morse potential defined by

\[ \Sigma(r) = D_c \left( 1 - \frac{b}{e^{2\kappa/r} + 1} \right)^2. \]  

Similarly, from (3), one has

\[ \begin{align*}
& \left[ -\frac{d^2}{dr^2} + \frac{\kappa(\kappa - 1)}{r^2} - (M - E_{n,\kappa} + C_{ps}) \Delta(r) + \frac{T^2 + T(2\kappa - 1)}{r^2} \right] G_{n,\kappa}(r) \\
& = \left[ E^2_{n,\kappa} - M^2 - C_{ps} (M + E_{n,\kappa}) \right] G_{n,\kappa}(r) 
\end{align*} \]  

for the pseudospin symmetry limit, i.e., when \( d\Sigma(r)/dr = 0 \) or \( \Sigma(r) = C_{ps} = \text{constant} \).

In this case, \( \kappa = -\tilde{l} \) and \( \kappa = \tilde{l} + 1 \) for \( \kappa < 0 \) and \( \kappa > 0 \), respectively; and it is \( \Delta(r) \) which takes the following form of the improved Rosen-Morse potential:

\[ \Delta(r) = D_c \left( 1 - \frac{b}{e^{2\kappa/r} + 1} \right)^2. \]  

To solve (4) and (6), the author of the Ref. [1] made a homogenous approximation to the improved Rosen-Morse potential to deal with the centrifugal potential term for \( \kappa \neq \pm 1 \) and used the parametric generalization of polynomial NU method [2] without considering of the conditions of its application. By defining a new variable \( s = e^{-2r/d} \) for \( r \in (0, +\infty) \) and \( s \in (0, 1) \) (note that the contradiction in Eq. (A10) where he says \( s \in (0, 1/a_3) \) with \( a_3 = -1 \) in Eq.
(20)) and the polynomial \( \sigma(s) = s(1 - a_3 s) \), Aguda asserts that the solutions of Eqs. (18) and (25) in Ref. [1] can be expressed in terms of the Jacobi polynomials (see (23) and (29) in Ref. [1]) and the relativistic energy equations are given by the Eqs. (22) and (28) in Ref. [1]. However, these solutions cannot be considered as correct. As pointed out in Ref. [2] (see Eq. (17), p. 29), according to the theorem on the orthogonality of hypergeometric-type polynomials, we note that the weight function \( \rho(s) \) does not satisfy the condition

\[
\sigma(s) \rho(s) s^k b^l = 0; \quad (k = 0, 1, 2, \ldots)
\]

Here \((a, b) = (0, 1)\), and the polynomial \( \rho(s) \) is given by (see Appendix A in Ref. [1])

\[
\rho(s) = s^{a_{10}} (1 - a_3 s)^{a_{11}}.
\]

Therefore, we must discard the solutions obtained in Ref. [1] entirely.

In the impossibility of applying the NU method to solve this problem, we can, for example, directly use the standard method of solving differential equations. In the case of the spin symmetry, by ignoring the problem of the validity of the approximation (16) in Ref. [1], we start from equation (18) in Ref. [1] by changing \( s \) to \( (-s) \) and look for solutions of this equation in the form

\[
F_{n, \kappa}(r) = s^\mu (1 - s)^\nu f_{n, \kappa}(s),
\]

in which, on account of boundary conditions, \( \mu \) has to be positive and \( \nu \) may be a real quantity. Substituting (11) into (18) in Ref. [1] and taking

\[
\mu = \frac{d}{2} \sqrt{\beta^2 + \gamma + \delta D_0},
\]

and

\[
\nu_\pm = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 + (\gamma b^2 + \delta D_2) d^2},
\]

we obtain for \( f_{n, \kappa}(s) \) the differential hypergeometric equations

\[
\begin{align*}
\left\{ s (1 - s) \frac{d^2}{ds^2} + [2 \mu + 1 - (2 \mu + 2 \nu_\pm + 1) s] \frac{d}{ds} - (\mu + \nu_\pm)^2 \\
+ [\beta^2 + \gamma b^2 + \gamma - 2 \gamma b + \delta (D_0 - D_1 + D_2)] \frac{d^2}{4} \right\} f_{n, \kappa}(s) = 0,
\end{align*}
\]

with the following notation:

\[
\begin{align*}
\beta &= \sqrt{(M - E_{n, \kappa}) (M + E_{n, \kappa} - C_s)}; \\
\gamma &= (M + E_{n, \kappa} - C_s) D_\epsilon; \\
\delta &= (T + \kappa) (T + \kappa + 1).
\end{align*}
\]

The solutions of these equations are the hypergeometric functions

\[
f_{n, \kappa}(s) = C \, _2F_1 \left( \mu + \nu_\pm + \frac{d}{2} \sqrt{A}, \mu + \nu_\pm - \frac{d}{2} \sqrt{A}, 2 \mu + 1; s \right),
\]
where
\[ A = \beta^2 + \gamma(b - 1)^2 + \delta(D_0 - D_1 + D_2), \quad (15) \]
and \( C \) is a constant factor.

Now, taking into account the formulas (see Ref. [3], Eq. (9.131), p. 1043),
\[
\begin{align*}
2F_1(\alpha, \beta; s) &= (1 - s)^{-\alpha} 2F_1(\gamma - \alpha, \gamma; s), \\
2F_1(\alpha, \beta; s) &= (1 - s)^{-\alpha} 2F_1(\alpha; \gamma - \beta, \gamma; s).
\end{align*}
\quad (16)
\]
we find that the upper spinor component \( F_{n, \kappa}(r) \) is given by
\[
F_{n, \kappa}(r) = N \left( e^{-2r/d} \left( 1 + e^{-2r/d} \right) \right)^{-\mu - \frac{d}{2} \sqrt{A}}
\times 2F_1 \left( \mu + \nu_+ + \frac{d}{2} \sqrt{A}, 1 + \mu - \nu_+ + \frac{d}{2} \sqrt{A}, 2\mu + 1; \frac{1}{e^{2r/d} + 1} \right),
\quad (18)
\]
in which \( N \) is a constant factor. Solution (18) fulfills the boundary condition
\[ F_{n, \kappa}(R_c) = 0, \quad (19) \]
when
\[
2F_1 \left( \mu + \nu_+ + \frac{d}{2} \sqrt{A}, 1 + \mu - \nu_+ + \frac{d}{2} \sqrt{A}, 2\mu + 1; \frac{1}{e^{2r/d} + 1} \right) = 0.
\]
Then, the energy values for the bound states are given by the solution of this transcendental equation (19) which can be solved numerically.

In the case of pseudospin symmetry, to solve equation (25) in Ref. [1], we proceed as before and we obtain the following expression for the lower spinor component \( G_{n, \kappa}(r) \):
\[
G_{n, \kappa}(r) = \overline{N} \left( e^{-2r/d} \left( 1 + e^{-2r/d} \right) \right)^{-\mu - \frac{d}{2} \sqrt{A}}
\times 2F_1 \left( \mu + \nu_+ + \frac{d}{2} \sqrt{A}, 1 + \mu - \nu_+ + \frac{d}{2} \sqrt{A}, 2\mu + 1; \frac{1}{e^{2r/d} + 1} \right),
\quad (20)
\]
where
\[
\begin{align*}
\overline{\mu} &= \frac{d}{2} \sqrt{\beta^2 - \gamma + \delta D_0}, \\
\nu_+ &= \frac{1}{\delta} \left( 1 + \sqrt{1 - (\gamma b^2 + \delta D_2) d^2} \right), \\
A &= \beta^2 - \gamma(b - 1)^2 + \delta(D_0 - D_1 + D_2)
\end{align*}
\quad (21)
\]
and \( \overline{N} \) is a constant factor. The parameters \( \beta, \gamma \) and \( \delta \) involved in (21) have in the present case the following values:
\[ \begin{align*}
\beta &= \sqrt{(M + E_{n_r, \kappa})(M - E_{n_r, \kappa} + C_{ps})}, \\
\gamma &= D \epsilon (M - E_{n_r, \kappa} + C_{ps}), \\
\delta &= (\kappa + T)(\kappa + T - 1).
\end{align*} \] (22)

Then, the energy levels of the physical system can be also found from a numerical solution of the transcendental equation

\[ _2F_1 \left( \frac{\mu + 1}{2}, 1 + \frac{d}{2} \sqrt{A}, 1 + \frac{d}{2} \sqrt{A}, \frac{1}{e^{2R_c/a} + 1} \right) = 0. \] (23)

Therefore, the analytical bound state solutions and the numerical results given by Aguda [1] are not correct because the NU polynomial method does not applicable to this potential type with the Dirichlet boundary conditions. The appropriate solutions of Eqs. (4) and (6) are expressed in terms of hypergeometric series. From these, we have shown by applying the boundary conditions that the energy levels can be found from numerical solution of transcendental equations involving the hypergeometric function.

References

[1] Ekele V. Aguda, Can. J. Phys. 91, 689 (2013).

[2] A. F. Nikiforov and V. B. Uvarov, Special Functions of Mathematical Physics; Birkhäuser: Bassel, 1988.

[3] I. S. Gradshtein and I. M. Ryzhik, Tables of integrals, series and products; Academic Press: New York, 1965.