General soliton solutions to a coupled Fokas-Lenells equation

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In this paper, we firstly establish the multi-Hamiltonian structure and infinite many conservation laws for the vector Kaup-Newell hierarchy of the positive and negative orders. The first nontrivial negative flow corresponds to a coupled Fokas-Lenells equation. By constructing a generalized Darboux transformation and using a limiting process, all kinds of one-soliton solutions are constructed including the bright-dark soliton, the dark-anti-dark soliton and the breather-like solutions. Furthermore, multi-bright and multi-dark soliton solutions are derived and their asymptotic behaviors are investigated.

Keywords: Coupled Fokas-Lenells equation; Tu scheme; Multi-Hamiltonian structure; Generalized Darboux transformation; bright-dark soliton; dark-anti-dark soliton

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I. INTRODUCTION

It is well known that the integrable nonlinear Schrödinger type equations, such as the classical nonlinear Schrödinger (NLS) equation\textsuperscript{1}, derivative-type NLS equation\textsuperscript{2–4}, play an important role in the study of nonlinear wave propagation. Recently, a new integrable model, called the Fokas-Lenells (FL) equation\textsuperscript{5, 6}, was proposed in mono-mode optical fibers when certain higher order nonlinear effects were taken into account. The Hamiltonian structure and inverse scattering transformation for the FL equation were established by Fokas and Lenells in their original paper\textsuperscript{5}. The algebraic geometry solution was constructed\textsuperscript{7}. The rogue wave solutions for this system were constructed by several authors\textsuperscript{8, 9}. It is interesting that, unlike the NLS equation, the FL equation admits both the bright and dark soliton solutions without the sign change of nonlinear term. The bright soliton solutions were constructed by Hirota’s bilinear method\textsuperscript{10}, while the dark soliton solutions were constructed by Hirota’s bilinear method\textsuperscript{11} and by Bäcklund transformation\textsuperscript{12}, respectively.

As pointed out by Lenells, the FL equation is related to the derivative NLS equation. As a matter of fact, there are lots of studies regarding the derivative NLS equation such as the Hamiltonian structure\textsuperscript{17}, the inverse scattering method\textsuperscript{14–16}, the Darboux/Bäcklund transformation\textsuperscript{18–22} and infinite many conservation laws\textsuperscript{23, 24}. Meanwhile, there are quite a few works for the study of multi-component derivative NLS equations such as the Lax pair and its integrable properties\textsuperscript{25–27} and exact solvable methods\textsuperscript{28–30}. Similar to the case of the NLS equation\textsuperscript{31}, it is necessary to consider the two-component or multi-component generalizations of the FL equation for describing the effects of polarization or anisotropy. Most recently, the coupled FL equation has been studied by several authors\textsuperscript{13, 28, 32}.

Darboux transformation (DT) is a useful method to construct exact solutions for the integrable system\textsuperscript{33–35}. Actually, the DT is a method related to the inverse scattering method which can be used to solve the initial value problem of integrable equations. Recently, there are some progresses to use the DT to construct some more general analytical solutions for NLS-type equations\textsuperscript{30, 36–40}. However, in certain physical situations, two or more wave packets of different carrier frequencies appear simultaneously, and their interactions are governed by the coupled equations. Thus, a natural question is how to construct the generalized Darboux transformation and apply it to find exact solutions to the coupled integrable equations.

In this paper, we consider a coupled Fokas-Lenells equation

\begin{align}
  u_{1,xt} + u_1 + i(\sigma|u_1|^2 + \frac{1}{2} \sigma|u_2|^2)u_{1,x} + \frac{i}{2} \sigma u_1 u_2^* u_{2,x} &= 0, \\
  u_{2,xt} + u_2 + i(\sigma|u_2|^2 + \frac{1}{2} |u_1|^2)u_{2,x} + \frac{i}{2} u_2 u_1^* u_{1,x} &= 0\tag{1}
\end{align}

where \( \sigma = \pm 1 \). This equation was first proposed by Guo and Ling\textsuperscript{28} with the matrix generalization of Lax pair. In\textsuperscript{32}, this coupled FL equation was reconstructed by the spectral gradient method. Zhang et. al constructed the bright, breather and first-order rogue wave solutions to a coupled FL equation, which is shown in\textsuperscript{28} to be equivalent to the coupled FL equation\textsuperscript{11} via a gauge transformation. Since the FL equation admits both the bright and dark...
solutions simultaneously, the structure of the soliton solutions to the coupled FL equation \(\text{II}\) is expected to be more complicated. It is the main objective of the present paper to explore the rich structure of soliton solutions to the coupled FL equation.

The Tu scheme \([41-43]\) is an important technique to construct the Hamiltonian structure for the integrable hierarchy. In section II we will apply the Tu scheme to construct the multi-Hamiltonian structure and the infinite many conservation laws for the coupled FL equation. First, Hamiltonian structures for the vector Kaup-Newell spectral problem involving both the positive and negative hierarchy are constructed. The first negative flow of the hierarchy is nothing but the coupled FL equation. Based on the Lax pair, the infinite many conservation laws are constructed in both positive and negative orders. In section III we construct a generalized DT for the coupled FL equation \(\text{II}\). Then the general soliton solution to the coupled FL equation is constructed by the DT method. In section IV based on the general soliton solution, a variety of single soliton solutions from the zero and plane wave seed solutions are constructed and classified. These solutions include the bright soliton, the bright-dark soliton, the dark-dark (D-D) soliton, the dark-anti-dark (D-AD) soliton, the anti-dark-anti-dark (AD-AD) soliton, and the breather-like solution with nonzero boundary conditions. In section VI, we construct the multi-bright and multi-dark/anti-dark soliton solutions and perform its asymptotical analysis. The paper is concluded in Section VII by a brief summary and discussions.

II. HAMILTONIAN STRUCTURE AND CONSERVATION LAWS FOR THE MULTI-COMPONENT KAUP-NEWELL SPECTRAL PROBLEM

It is well known that the important criterions for the integrable hierarchy are multi-Hamiltonian structure and the infinitely many conservation laws. Firstly, we consider the Hamiltonian structure for the multi-component Kaup-Newell hierarchy.

A. Multi-Hamiltonian structure

We consider the following vector Kaup-Newell spectral problem \([2]\)

\[
\Psi_x = (i\lambda^{-2}\sigma_3 + \lambda^{-1}Q) \Psi \equiv U(\lambda)\Psi, \quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -I_N \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & r^T \\ q & 0 \end{pmatrix},
\]

where \(q = (q_1, q_2, \cdots, q_N)^T\), \(r = (r_1, r_2, \cdots, r_N)^T\), \(I_N\) denotes a \(N \times N\) identity matrix, \(\lambda\) is the spectral parameter. To obtain the soliton hierarchy and its Hamiltonian structure, we consider the stationary zero-curvature equation

\[
W_x = [U, W], \quad W = \begin{pmatrix} a & c^T \\ b & d \end{pmatrix},
\]

where \(b\) and \(c\) are \(N \times 1\) matrices, \(d\) is a \(N \times N\) matrix. The equation (3) can be expanded as

\[
a_x = \lambda^{-1}(r^Tb - c^Tq), \quad d_x = \lambda^{-1}(qc^T - br^T),
\]

\[
b_x = -2i\lambda^{-2}b + \lambda^{-1}(aq - dq), \quad c_x = 2i\lambda^{-2}c + \lambda^{-1}(d^Tr - ar).
\]

Firstly, we consider the positive hierarchy. To obtain the recursion operator for the positive hierarchy, we insert equations (3) into equations (4) to obtain

\[
a_x = \frac{i}{2}\lambda(r^Tb_x + c_x^Tq),
\]

\[
d_x = -\frac{i}{2}(qc_x^T + b_xr^T) + \frac{i}{2}((qr^T)d - d(qr^T)).
\]

Then inserting equation (6) into equation (5) yields

\[
i\lambda^{-2}b = -\frac{1}{2}b_x + \frac{1}{2}\lambda^{-1}aq - \frac{1}{2}\lambda^{-1}dq,
\]

\[
i\lambda^{-2}c = \frac{1}{2}c_x - \frac{1}{2}\lambda^{-1}dr + \frac{1}{2}\lambda^{-1}ar.
\]
Finally, supposing $W$ to be of the form
\begin{equation}
\begin{aligned}
a &= \sum_{i=0}^{\infty} a_i \lambda^{2i}, \\
d &= \sum_{i=0}^{\infty} d_i \lambda^{2i}, \\
b &= \sum_{i=0}^{\infty} b_i \lambda^{2i+1}, \\
c &= \sum_{i=0}^{\infty} c_i \lambda^{2i+1},
\end{aligned}
\end{equation}
we can obtain that
\begin{equation}
\begin{aligned}
a_0 &= 2a_0, \\
d_0 &= -2a_0 I_N, \\
b_0 &= q a_0, \\
c_0 &= r a_0,
\end{aligned}
\end{equation}
where $\alpha_0$ is a complex constant, $\text{ad}_{q r T} (\cdot) = [q r T, \cdot]$. The term $-\frac{1}{2} \text{ad}_{q r T} d_i$ does not occur in the scalar equation since it is commutative automatically. Based on the above equations, one can obtain the recursion relation
\begin{equation}
\begin{aligned}
\begin{pmatrix}
b_{i+1} \\
c_{i+1}
\end{pmatrix} = L_1 \partial_x 
\begin{pmatrix}
b_i \\
c_i
\end{pmatrix},
\end{aligned}
\end{equation}
where
\begin{equation}
\begin{aligned}
L_1 &= \frac{i}{2} \begin{pmatrix} I_N & 0 \\ 0 & -I_N \end{pmatrix} + \frac{1}{4} \begin{pmatrix}
q \partial_x^{-1} r T + A & q \partial_x^{-1} q T + C \\
r \partial_x^{-1} r T + D & r \partial_x^{-1} q T + B
\end{pmatrix},
\end{aligned}
\end{equation}
and
\begin{equation}
\begin{aligned}
A &= [(\partial_x - \frac{1}{2} \text{ad}_{q r T})^{-1} (\cdot) r T] q, \\
B &= [(\partial_x + \frac{1}{2} \text{ad}_{r q T})^{-1} (\cdot) q T] r, \\
C &= [(\partial_x - \frac{1}{2} \text{ad}_{r q T})^{-1} (\cdot) q T] q, \\
D &= [(\partial_x + \frac{1}{2} \text{ad}_{r q T})^{-1} (\cdot) r T] r.
\end{aligned}
\end{equation}

**Proposition 1**
\begin{equation}
\begin{aligned}
A^* &= -B, \\
C^* &= -C, \\
D^* &= -D.
\end{aligned}
\end{equation}

**Proof:** By direct calculation, we have
\begin{equation}
\begin{aligned}
\int c^T [(\partial_x - \frac{1}{2} \text{ad}_{q r T}) (b r T)] q dx &= \int c^T (b r T)_x - \frac{1}{2} \text{ad}_{q r T} (b r T) q dx \\
&= \int \{ -c^T (b r T) q - c^T (b r T) q_x - \frac{1}{2} c^T \text{ad}_{q r T} (b r T) q \} dx \\
&= \int b^T [(-\partial_x - \frac{1}{2} \text{ad}_{r q T}) (c q T)] r dx.
\end{aligned}
\end{equation}
Thus, we can obtain that $A^* = -B$ formally. Other two cases can be proved in a similar way. \(\square\)

From the trace identity, one has
\begin{equation}
\begin{aligned}
\frac{\delta}{\delta \omega} \int W \frac{\partial U}{\partial \lambda} dx &= \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \left[ \lambda^\gamma \text{tr} (W \frac{\partial U}{\partial \omega}) \right], \\
\omega &= \begin{pmatrix} q \\ r \end{pmatrix}.
\end{aligned}
\end{equation}

Since
\begin{equation}
\begin{aligned}
\text{tr} \left( W \frac{\partial U}{\partial \lambda} \right) &= -2i \lambda^{-3} [a - \text{tr} (d)] - \lambda^{-2} (c^T q + r^T b) \\
&= -2i [a_0 - \text{tr} (d_0)] - \sum_{i=0}^{\infty} \left[ 2i [a_{i+1} - \text{tr} (d_{i+1})] + (c_i^T q + r^T b_i) \right] \lambda^{2i-1},
\end{aligned}
\end{equation}
and

\[ \text{tr}(W \frac{\partial U}{\partial \omega}) = \lambda^{-1} \left( \begin{array}{c} c \\ b \end{array} \right), \]

it then follows that

\[ \left( \begin{array}{c} c_0 \\ b_0 \end{array} \right) = \frac{\delta}{\delta \omega} H^+_0, \quad H^+_0 = a_0 \int q^T \cdot r \, dx \]

and

\[ \left( \begin{array}{c} c_i \\ b_i \end{array} \right) = \frac{\delta}{\delta \omega} H^+_i, \quad H^+_i = -\frac{1}{2i} \int \left[ 2i(a_{i+1} - \text{tr}(d_{i+1})) + (c_i^T q + r^T b_i) \right] \, dx, \quad i \geq 1. \]

Finally, setting \( V(\lambda) = (\lambda^{2k} W)_+ \), we obtain the multi-Hamiltonian hierarchy

\[ \omega_t = J \frac{\delta}{\delta \omega} H^+_n = JL^n \frac{\delta}{\delta \omega} H^+_0, \]

from the zero curvature condition \( U_t - V_x + [U, V] = 0 \), where

\[ J = \left( \begin{array}{cc} 0 & \partial \\ \partial & 0 \end{array} \right), \quad L = \left( \begin{array}{cc} 0 & I_N \\ I_N & 0 \end{array} \right) L_1 \left( \begin{array}{cc} 0 & I_N \\ I_N & 0 \end{array} \right) \partial. \]

It is readily to see that the operators \( J \) and \( JL^k \) are skew symmetrical.

On the other hand, by assuming

\[ a = \sum_{i=0}^{\infty} a_i \lambda^{-2i}, \quad d = \sum_{i=0}^{\infty} d_i \lambda^{-2i}, \]
\[ b = \sum_{i=0}^{\infty} b_i \lambda^{-2i-1}, \quad c = \sum_{i=0}^{\infty} c_i \lambda^{-2i-1}, \]

one obtains the recursion relation

\[ \left( \begin{array}{c} b_i \\ c_i \end{array} \right)_x = L_2 \left( \begin{array}{c} b_{i-1} \\ c_{i-1} \end{array} \right), \quad i \geq 1, \quad \left( \begin{array}{c} b_0 \\ c_0 \end{array} \right)_x = a_0 \left( \begin{array}{c} q \\ r \end{array} \right), \]

\[ a_0 = \frac{1}{2} a_0, \quad c_0 = -\frac{1}{2} a_0 I_N, \]

\[ a_i = \frac{1}{2} \partial_x^{-1} (r^T b_{i-1} - c_{i-1}^T q), \quad d_i = \partial_x^{-1} (q c_{i-1}^T - b_{i-1} r^T) \]

and

\[ L_2 = -2i \left( \begin{array}{cc} I_N & 0 \\ 0 & -I_N \end{array} \right) + \left( \begin{array}{c} q \partial_x^{-1} r^T + \left( \sum_{k=1}^{N} q_k \partial_x^{-1} r_k \right) I_N \\ -q \partial_x^{-1} q^T - (q \partial_x^{-1} q^T)^T \end{array} \right) \]

\[ -r \partial_x^{-1} r^T - (r \partial_x^{-1} r^T)^T \quad r \partial_x^{-1} q^T + \left( \sum_{k=1}^{N} r_k \partial_x^{-1} q_k \right) I_N. \]

Through the trace identity \( [5] \), we can obtain that

\[ \left( \begin{array}{c} c_i \\ b_i \end{array} \right) = \frac{\delta}{\delta \omega} H^-_i, \quad H^-_i = \int \frac{1}{2(i+1)} \left[ 2i(a_i - \text{tr}(d_i)) + c_i^T q + r^T b_i \right] \, dx. \]

Taking \( V(\lambda) = (\lambda^{2n} W)_+ \), we can obtain the negative hierarchy

\[ \omega_t = -J \frac{\delta}{\delta \omega} H_n = -J K^n \frac{\delta}{\delta \omega} H_0 \]

where

\[ K = \partial_x^{-1} \left( \begin{array}{cc} 0 & I_N \\ I_N & 0 \end{array} \right) L_2 \left( \begin{array}{cc} 0 & I_N \\ I_N & 0 \end{array} \right). \]

It can be readily verified that the operator \( J K^i \) is skew symmetric. Consequently, both the positive and negative flows of the multi-component Kaup-Newell hierarchy are constructed by Tu scheme.
B. Conservation laws

We consider the multi-component Kaup-Newell spectral problem

\[
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}_x = \begin{pmatrix}
i\lambda^{-2} & \lambda^{-1}v^T \\
\lambda^{-1}u_x & -i\lambda^{-2}
\end{pmatrix}
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}
\]  

(10)

and associated evolution equation

\[
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}_t = i\left(\frac{1}{2}\lambda^2 + \frac{1}{2}v^Tu - \lambda^2v_x + \lambda^{-2}uv^T\right)
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}
\]  

(11)

It follows that

\[
v_x^T\begin{pmatrix}
\psi_2 \\
\psi_1
\end{pmatrix}_x = -2i\lambda^{-2}v_x^T\psi_2 - v_x^Tu - \lambda^{-1}\left(v_x^T\psi_2\right)^2
\]  

(12)

and

\[
v^T\begin{pmatrix}
\psi_2 \\
\psi_1
\end{pmatrix}_t = -\frac{i}{2}\lambda^2v^T\psi_2 + \frac{1}{2}\lambda v^Tu - i(v^Tu)v^T\psi_2 + \frac{i\lambda}{2}\left(v^T\psi_2\right)^2
\]  

(13)

To find the conservation laws of negative orders, we substitute an expansion

\[
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix} = \sum_{i=1}^{\infty} P_i \lambda^{2i-1}
\]  

(14)

into equation (12) and obtain

\[
\begin{align*}
P_0 &= u, \\
P_1 &= -\frac{i}{2}u_x, \\
P_{i+1} &= \frac{i}{2} \left[ P_i + \sum_{j=1}^{i} P_j v_x^TP_{i+1-j} \right].
\end{align*}
\]

Then the conservation laws follow

\[
\left(\ln \psi_1\right)_x = -\lambda^{-1} \left[ v_x^T\psi_2 \right]_t = i \left[ \frac{1}{2} (v^Tu)_x - \frac{\lambda}{2} \left(v^T\psi_2\right)_x \right],
\]

i.e.

\[
i \left[ v_x^TP_i \right]_t = \frac{1}{2} \left(v^TP_{i-1}\right)_x, \quad i = 1, 2, \cdots
\]

These conservation laws are all local, among which the first two are listed below

\[
\left[ v_x^Tu_x \right]_t = \frac{1}{2} (v^Tu)_x,
\]

\[
\left[ \frac{i}{4} v^Tu_{xx} + \frac{1}{8} (v^Tu_x)^2 \right]_t = \frac{1}{4} (v^Tu_x)_x.
\]

On the other hand, substituting the following expansion

\[
\begin{pmatrix}
\psi_2 \\
\psi_1
\end{pmatrix} = \sum_{i=1}^{\infty} C_i \lambda^{-2i+1}
\]  

(15)

into equation (13) and equating the coefficients of \(\lambda^{-2i+1}\), one obtains

\[
\lambda : \ C_1 = u,
\]

\[
\lambda^{-2i+1} : \ v^TC_{i+1} = 2iv^TC_{i,t} - 2(v^Tu)(v^TC_i) + \sum_{j=1}^{i} (v^TC_j)(v^TC_{i+1-j}).
\]
which leads to the conservation laws
\[
(\ln \psi_1)_{xt} = \lambda^{-1} \left[ v_x^T \psi_2 \right]_t = i \left[ \frac{1}{2} (v^T u)_x - \frac{\lambda}{2} \left( v^T \psi_2 \right)_x \right],
\]
i.e.
\[
i \left[ v_x^T C_i \right]_t = \left[ \frac{1}{2} v^T C_{i+1} \right]_x, \quad i = 1, 2, \ldots.
\]
The first conversation law is
\[
i \left[ v_x^T u \right]_t = \left[ iv^T u - \frac{1}{2} (v^T u)^2 \right]_x.
\]
In contrast to the conservation laws of negative orders, other conservation laws except the first one of positive orders are nonlocal.

### III. SPECTRAL PROBLEM AND DARBOUX TRANSFORMATION

The coupled FL equation (11) considered in the present paper admits the following Lax pair
\[
\begin{align*}
\Phi_x &= U(x, t; \lambda) \Phi, \quad U(x, t; \lambda) = i\lambda^{-2} \sigma_3 + \lambda^{-1} Q_x, \\
\Phi_t &= V(x, t; \lambda) \Phi, \quad V(x, t; \lambda) = i \left( \frac{1}{4} \lambda^2 \sigma_3 + \frac{1}{2} \sigma_3 (Q^2 - \lambda Q) \right),
\end{align*}
\]
where
\[
\sigma_3 = \text{diag}(1, -1, -1), \quad Q = \begin{pmatrix} 0 & u_1^* & \sigma u_2^* \\ u_1 & 0 & 0 \\ u_2 & 0 & 0 \end{pmatrix}, \quad \sigma = \pm 1.
\]
So the solutions to equation (11) can be solved from its Lax pair (16). To the end, we consider the spectral problem (16)
\[
- i\sigma_3 \left[ \partial_x - \lambda^{-1} Q_x \right] \Phi = \lambda^{-2} \Phi,
\]
which is an energy-dependent spectral problem, here \( Q \in L_{loc}(\mathbb{R}) \). The involution relation for the system (16) can be concluded by the following Lemma:

**Lemma 1** (a) The matrices \( U, V \) are variant under the involution \( \tau_1 : A(\lambda) \mapsto \sigma_3 A(-\lambda) \sigma_3 \) and \( \tau_2 : A(\lambda) \mapsto -J[A(\lambda^*)]^\dagger J \), where \( J = \text{diag}(1, -1, -1, -\sigma) \). (b) If the function \( \Phi(x, t; \lambda) \) satisfies the system (16) with initial data \( \Phi(0, 0; \lambda) = I_{3 \times 3} \), then \( \Phi(x, t; -\lambda) = \sigma_3 \Phi(x, t; \lambda) \sigma_3 \), and \( J[\Phi(x, t; \lambda)]^\dagger J = [\Phi(x, t; \lambda^*)]^{-1} \).

**Proof:** The claim in (a) can be proved by direct calculation, which is omitted here.

b) Since \( \Phi(x, t; \lambda) \) satisfies the system (16) with initial data \( \Phi(0, 0; \lambda) = I_{3 \times 3} \), and \( U, V \) are invariant under the involution \( \tau_1 \), we have
\[
\begin{align*}
\frac{\partial}{\partial x} \Phi(x, t; -\lambda) &= U(x, t; -\lambda) \Phi(x, t; -\lambda), \\
\frac{\partial}{\partial t} \Phi(x, t; -\lambda) &= V(x, t; -\lambda) \Phi(x, t; -\lambda),
\end{align*}
\]
and
\[
\begin{align*}
\frac{\partial}{\partial x} [\sigma_3 \Phi(x, t; \lambda) \sigma_3] &= U(x, t; -\lambda) [\sigma_3 \Phi(x, t; \lambda) \sigma_3], \\
\frac{\partial}{\partial t} [\sigma_3 \Phi(x, t; \lambda) \sigma_3] &= V(x, t; -\lambda) [\sigma_3 \Phi(x, t; \lambda) \sigma_3].
\end{align*}
\]
Based on the existence and uniqueness of ordinary differential equations (ODEs), we have \( \Phi(x, t; -\lambda) = \sigma_3 \Phi(x, t; \lambda) \sigma_3 \).

Similarly, since \( \Phi(x, t; \lambda) \) satisfies the system \( \text{(18)} \) with initial data \( \Phi(0, 0; \lambda) = I_{3 \times 3} \), and \( U, V \) are invariant under the involution \( \tau_2 \), one has

\[
-(J[\Phi(x, t; \lambda)]^\dagger J)_x = (J[\Phi(x, t; \lambda)]^\dagger J)U(x, t; \lambda^*),
-(J[\Phi(x, t; \lambda)]^\dagger J)_t = (J[\Phi(x, t; \lambda)]^\dagger J)V(x, t; \lambda^*)
\]

and

\[
-(\Phi(x, t; \lambda)^{-1})_x = (\Phi(x, t; \lambda)^{-1})U(x, t; \lambda^*),
-(\Phi(x, t; \lambda)^{-1})_t = (\Phi(x, t; \lambda)^{-1})V(x, t; \lambda^*),
\]

we then have \( J[\Phi(x, t; \lambda)]^\dagger J = [\Phi(x, t; \lambda^*)]^{-1} \). □

Therefore, the Darboux matrix \( T(\lambda) \) converts the system \( \text{(18)} \) into a new system of the form

\[
\begin{align*}
\Phi[1]_x &= U[1](x, t; \lambda) \Phi[1], \\
U[1](x, t; \lambda) &= i\lambda^{-2} \sigma_3 + \lambda^{-1} Q[1] x, \\
\Phi[1]_t &= V[1](x, t; \lambda) \Phi[1], \\
V[1](x, t; \lambda) &= i \left( \frac{1}{4} \lambda^2 \sigma_3 + \frac{1}{2} \sigma_3 (Q[1]^2 - \lambda Q[1]) \right).
\end{align*}
\]

(18)

where

\[
\begin{align*}
U[1] &= T_x T^{-1} + T U T^{-1}, \\
V[1] &= T_t T^{-1} + T V T^{-1}.
\end{align*}
\]

(19)

**Lemma 2** Assume that \( T(\lambda) \) satisfies \( T(\lambda) = \sigma_3 T(-\lambda) \sigma_3 \) and \( [T(\lambda)]^{-1} = J[T(\lambda^*)]^\dagger J, \) if the matrices \( U, V \) are invariant under the involution \( \tau_1 \) and \( \tau_2 \), then the new potential functions keep invariant under the involution \( \tau_1 \) and \( \tau_2 \).

**Proof:** From the relation \( \text{(19)} \) and \( T(\lambda) = \sigma_3 T(-\lambda) \sigma_3 \), we have

\[
\begin{align*}
\sigma_3 U[1](-\lambda) \sigma_3 &= \sigma_3 T(-\lambda) x \{ T(-\lambda)^{-1} \sigma_3 + \sigma_3 T(-\lambda) U(-\lambda) [T(-\lambda)^{-1}] \sigma_3 = U[1](\lambda), \\
\sigma_3 V[1](-\lambda) \sigma_3 &= \sigma_3 T(-\lambda) t \{ T(-\lambda)^{-1} \sigma_3 + \sigma_3 T(-\lambda) V(-\lambda) [T(-\lambda)^{-1}] \sigma_3 = V[1](\lambda).
\end{align*}
\]

On the other hand, through the relation \( \text{(19)} \) and \( [T(\lambda)]^{-1} = J[T(\lambda^*)]^\dagger J, \) one obtains

\[
\begin{align*}
J[U[1](\lambda^*)]^\dagger J &= \sigma_3 T(\lambda) x \{ T(\lambda)^{-1} \sigma_3 + \sigma_3 T(\lambda) U(\lambda) [T(\lambda)^{-1}] \sigma_3 = U[1](\lambda), \\
\sigma_3 V[1](\lambda) \sigma_3 &= \sigma_3 T(\lambda) t \{ T(\lambda)^{-1} \sigma_3 + \sigma_3 T(\lambda) V(\lambda) [T(\lambda)^{-1}] \sigma_3 = V[1](\lambda).
\end{align*}
\]

\[ □ \]

Due to the relation \( T(\lambda) = \sigma_3 T(-\lambda) \sigma_3 \), the Darboux matrix can be constructed through the loop group method

\[ T(\lambda) = I + \frac{A_1}{\lambda - \lambda_1} - \frac{\sigma_3 A_1 \sigma_3}{\lambda + \lambda_1}. \]

(20)

Based on the Lemma 2, the inverse of Darboux matrix \( T(\lambda) \) may be chosen as

\[ T(\lambda)^{-1} = J[T(\lambda^*)]^\dagger J = I + \frac{J A_1^\dagger J}{\lambda - \lambda_1} - \frac{\sigma_3 J A_1^\dagger J \sigma_3}{\lambda + \lambda_1}. \]

(21)

The \( L^2(\mathbb{R}) \) eigenfunction can be constructed from the Darboux matrix. Usually, we can construct two wave vector functions which satisfy \( \phi_{\pm}(\lambda_1) \to 0 \) as \( x \to \pm \infty \) with exponential decay, \( \lambda_1 \in \mathbb{C}/\{\mathbb{R} \cup i\mathbb{R}\} \). The Darboux matrix satisfies the following proposition

\[ T(\lambda_1) |y_1\rangle = 0, \quad |y_1\rangle = \phi_{1,+} + \gamma \phi_{1,-}, \]

(22)

where \( \text{rank}(A_1) \) is either 1 or 2. However, since the order for this spectral problem is three, we can assume that \( A_1 = |x_1\rangle \langle z_1| J \), where \( \langle z_1\rangle = |z_1\rangle^\dagger. \) On the other hand, since \( T(\lambda)[T(\lambda)^{-1} = I, \) one can obtain that \( \text{Re}_{\lambda=\lambda_1}(T(\lambda)[T(\lambda)^{-1}) = 0. \) It follows that

\[ I + \frac{A_1}{\lambda_1 - \lambda_1^*} - \frac{\sigma_3 A_1 \sigma_3}{\lambda_1 + \lambda_1^*} \langle z_1\rangle = 0. \]

(23)
Together with equation (22), one arrives at \(|z_1| = c_1|y_1|\). For the sake of convenience, by setting \(c_1 = 1\), equation (23) is rewritten as

\[
|y_1⟩ + \frac{⟨y_1|J|y_1⟩}{\lambda_1 - \lambda_i^*}|x_1⟩ - \frac{⟨y_1|Jσ_3|y_1⟩}{\lambda_1 + \lambda_i}σ_3|x_1⟩ = 0. \tag{24}
\]

Denoting \(|y_1⟩ = (φ_1, ψ_1, χ_1)^T\), one can solve

\[
|x_1⟩ = \begin{pmatrix} \alpha^{-1} & 0 & 0 \\ 0 & β^{-1} & 0 \\ 0 & 0 & β^{-1} \end{pmatrix} |y_1⟩, \tag{25}
\]

where

\[
α = \frac{2[λ_i^*φ_1|^2 - λ_1(|ψ_1|^2 + σ|χ_1|^2)]}{λ_i^2 - λ_i^*}, \quad β = \frac{2[λ_1φ_1|^2 - λ_i^*(|ψ_1|^2 + σ|χ_1|^2)]}{λ_i^2 - λ_i^*}.
\]

Consequently, we obtain the transformation from potential function \(Q\) to \(Q[1]\)

\[
Q[1] = Q + (⟨x_1⟩⟨y_1|J - σ_3|x_1⟩⟨y_1|Jσ_3⟩), \tag{26}
\]

or explicitly

\[
u_1[1] = u_1 + \frac{2}{β}ψ_1φ_1^*, \quad u_2[1] = u_2 + \frac{2}{β}χ_1φ_1^*. \tag{27}
\]

Generally, we could derive the following \(N\)-fold Darboux transformation

\[
T_N = I + \sum_{i=1}^{N} \left[ \frac{A_i}{λ - λ_i^*} - \frac{σ_3A_iσ_3}{λ + λ_i} \right], \tag{28}
\]

where \(A_i = |x_i⟩⟨y_i|J\)

\[
[[|x_{1,1}⟩, |x_{2,1}⟩, \ldots, |x_{N,1}⟩]] = [[|y_{1,1}⟩, |y_{2,1}⟩, \ldots, |y_{N,1}⟩]]B^{-1}, \quad B = (b_{ij})_{N \times N},
\]

\[
[[|x_{1,k}⟩, |x_{2,k}⟩, \ldots, |x_{N,k}⟩]] = [[|y_{1,k}⟩, |y_{2,k}⟩, \ldots, |y_{N,k}⟩]]M^{-1}, \quad M = (m_{ij})_{N \times N}, \quad k = 2, 3,
\]

and

\[
b_{ij} = \frac{⟨y_i|J|y_j⟩}{λ_i^* - λ_j} + \frac{⟨y_i|Jσ_3|y_j⟩}{λ_i^* + λ_j}, \quad m_{ij} = \frac{⟨y_i|J|y_j⟩}{λ_i^* - λ_j} - \frac{⟨y_i|Jσ_3|y_j⟩}{λ_i^* + λ_j}.
\]

The transformation between old and new potential functions is

\[
Q[N] = Q + \sum_{i=1}^{N} (A_i - σ_3A_iσ_3). \tag{29}
\]

Moreover, we have

\[
\begin{pmatrix} u_1[N] \\ u_2[N] \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + 2Y_2M^{-1}Y_1^T \tag{30}
\]

where

\[
Y_1 = (φ_1, φ_2, \ldots, φ_N)^T, \quad Y_2 = (ϕ_1, ϕ_2, \ldots, ϕ_N) \begin{pmatrix} 1 \\ χ_1 \\ χ_2 \ldots \chi_N \end{pmatrix}.
\]
IV. BRIGHT SOLITON SOLUTION WITH VANISHING BOUNDARY CONDITION

A. Single bright soliton solution

Inserting the zero seed solution into Lax pair (10) and introducing \( z = 1/\lambda^2 \), the fundamental solution to the system (10) is \( \Phi_1(\lambda) = \exp[i(zx + x^2/2t)\sigma_3] \). Based on the formula (30), one obtains the single soliton solution with vanishing boundary condition

\[
  u_s[1] = \frac{z_1 - z_1^*}{|z_1|^2} \frac{c_x e^{\omega_t}}{\lambda_1 e^{\omega_t} + \omega_t - \lambda_1^*(|c_1|^2 + \sigma|c_2|^2)}.
\]  

(31)

Let \( \lambda_1 = \lambda_{1R} + i\lambda_{1I} \) and \( \omega_1 = \omega_{1R} + i\omega_{1I} = i(2z_1 + \frac{1}{z_1^2}t) + \delta_1 \), the single soliton solution can be represented as

\[
  u_s[1] = \frac{-2\lambda_{1R}\lambda_{1I} c_x e^{-i\omega_{1I}}}{r_1 [\lambda_{1R} \sinh(\omega_{1R} - \ln(r_1)) + i\lambda_{1I} \cosh(\omega_{1R} - \ln(r_1))]},
\]

(32)

for \( c_1^2 + \sigma c_2^2 > 0 \) and

\[
  u_s[1] = \frac{-2\lambda_{1R}\lambda_{1I} c_x e^{-i\omega_{1I}}}{r_1 [\lambda_{1R} \cosh(\omega_{1R} - \ln(r_1)) + i\lambda_{1I} \sinh(\omega_{1R} - \ln(r_1))]},
\]

(33)

for \( c_1^2 + \sigma c_2^2 < 0 \). Here \( r_1 = (|c_1|^2 + \sigma|c_2|^2)^{1/2} \)

where \( \delta_{1R}, \delta_{1I} \in \mathbb{R} \). Basically, the soliton solution is the bright soliton solution of the bell shape. If \( c_1^2 + \sigma c_2^2 > 0 \), the peak value for \( |u_s[1]|^2 \) is \( 4\lambda_{1R}^2 c_2^2/r_1^2 \) (\( s = 1, 2 \)). Whereas, if \( c_1^2 + \sigma c_2^2 < 0 \), the peak value for \( |u_s[1]|^2 \) is \( 4\lambda_{1R}^2 c_2^2/r_1^2 \) (\( s = 1, 2 \)).

Particularly, if \( \ln(r_1) = \delta_{1R} \), one can obtain a rational soliton solution of the form

\[
  u_s[1] = \frac{2c_x\lambda_{1R} e^{-i(2xz_1^2 + \lambda_{1R}^2 t/2)}}{r_1 [i(4x/\lambda_{1R}^2 - \lambda_{1R}^2 t) - 1]},
\]

as \( \lambda_{1R} \to 0 \). On the other hand, if \( \sigma = -1 \), \( \ln(r_1) = \delta_{1R} \), one has a rational soliton solution

\[
  u_s[1] = \frac{2c_x\lambda_{1I} e^{i(2xz_1^2 + \lambda_{1I}^2 t/2)}}{r_1 [i(4x/\lambda_{1I}^2 - \lambda_{1I}^2 t) + 1]},
\]

(35)

as \( \lambda_{1R} \to 0 \). It can be shown that the rational soliton is also of the bell shape.

B. Multi-bright soliton solutions

Letting \( |y_i| = \Phi_1(\lambda_i)(1, c_{i,1}, c_{i,2})^T \), we can obtain

\[
  m_{i,j} = \frac{2z_i z_j}{z_i - z_j^*} \left[ \lambda_{1} e^{\theta_i + \theta_j} + \lambda_{1}^* \gamma_{i,j} e^{-\theta_i - \theta_j} \right], \quad \gamma_{i,j} = -(c_{i,1}^* c_{j,1} + \sigma c_{i,2}^* c_{j,2})
\]

and

\[
  \varphi_i = e^{\theta_i}, \quad \phi_i = c_{i,1} e^{-\theta_i}, \quad \chi_i = c_{i,2} e^{-\theta_i},
\]

\[
  \theta_i = i \left( z_i x + \frac{1}{4z_i} \right) = -z_{i,R}(x - v_i t) + iz_{i,I}(x + v_i t), \quad v_i = \frac{1}{4|z_i|^2}, \quad z_i = z_{i,R} + iz_{i,I}, \quad z_{i,I} < 0.
\]

Based on the formula (30), we could derive the \( N \)-bright soliton solution formula:

\[
  \det \begin{pmatrix} M & Y_1^T \\ Y_2 & 0 \end{pmatrix}
\]

(36)
FIG. 1: (color online): Breather solution with parameters: \( \lambda_1 = \frac{3}{10} + \frac{3}{5}i \), \( \lambda_2 = \frac{3}{10} + \frac{3}{5}i \), \( c_{1,1} = -2 \), \( c_{2,1} = 1 \), \( c_{1,2} = 1 \), \( c_{2,2} = -2 \).

FIG. 2: (color online): Two-soliton solution with parameters: \( \lambda_1 = \frac{3}{10} + \frac{3}{5}i \), \( \lambda_2 = \frac{3}{10} + \frac{3}{5}i \), \( c_{1,1} = -2 \), \( c_{2,1} = 1 \), \( c_{1,2} = 1 \), \( c_{2,2} = -2 \).

where \( M = (m_{i,j})_{1 \leq i,j \leq N} \) and \( Y^{[s]}_2 \) represents the \( s \)-th row of matrix \( Y_2 \).

To investigate the asymptotic behavior for the \( N \)-bright soliton solution with different velocity, we need to introduce the generalized Cauchy matrix

\[
C(\Delta_k) = \left[ \frac{c^\dagger_A c_j}{z_j - z_i^k} \right]_{i=1,j=1}^{k,k}, \quad C(\Delta_k) = \left[ \frac{c^\dagger_A c_j}{z_j - z_i^k} \right]_{i=k,j=k}^{N,N}, \quad \Lambda = \text{diag}(-1, -\sigma)
\]

\[
\hat{C}_s(\Delta_k) = \left( \hat{C}_{\text{up}} \hat{C}_{\text{lower}} \right), \quad \hat{C}_{\text{up}} = \left[ \frac{c^\dagger_A c_j}{z_j - z_i^k} \right]_{i=1,j=1}^{k-1,k}, \quad \hat{C}_{\text{lower}} = (c_{1,s}, \cdots, c_{k-1,s}, c_{k,s}),
\]

\[
\hat{C}_s(\Delta_k) = \left( \hat{C}_{\text{up}} \hat{C}_{\text{lower}} \right), \quad \hat{C}_{\text{lower}} = \left[ \frac{c^\dagger_A c_j}{z_j - z_i^k} \right]_{i=k+1,j=k}^{N,N}, \quad \hat{C}_{\text{up}} = (c_{k,s}, \cdots, c_{N-1,s}, c_{N,s}),
\]

(37)
where
\[
C(z_k^*, z_j^*) = \begin{bmatrix}
\frac{1}{z_j - z_k^*} & \cdots & \frac{1}{z_j - z_k^*} \\
\vdots & \ddots & \vdots \\
\frac{1}{z_j - z_k^*} & \cdots & \frac{1}{z_j - z_k^*}
\end{bmatrix}_{1 \leq i,j \leq N}^N,
\]
\[
C(z_k, z_j) = \begin{bmatrix}
\frac{1}{z_j - z_k} \\
\vdots \\
\frac{1}{z_j - z_k}
\end{bmatrix}_{1 \leq i,j \leq k}^k,
\]
\[
c_k = (c_i)_{i=1}^k \in \mathbb{C}^k, \quad c_i = z_i \left( \begin{array}{c}
c_{i,1} \\
c_{i,2}
\end{array} \right).
\]

**Proposition 2** When \( t \to \pm \infty \), the asymptotic of the \( N \)-soliton solution is
\[
\lim_{t \to \pm \infty} u_s[N] = \sum_{k=1}^N u_{s,[k]} + O(e^{-c|t|})
\]
where \( c = \min_{1,2,\ldots, N} (|z_i|, |v_j|) \), the expressions of \( u_{s,[k]} \) are given by equations (39) and (40).

**Proof:** Assuming \( v_1 < v_2 < \cdots < v_N \), we have \( \theta_1, \theta_2, \ldots, \theta_{k-1} \to -\infty; \theta_{k+1}, \theta_{k+2}, \ldots, \theta_N \to +\infty \) along the trajectory \( x - v_k t = \text{const} \) as \( t \to -\infty \). On the other hand, \( u_s[N] \) can be rewritten as
\[
u_s[N] = -\frac{\det(\hat{G}_s)}{\det(\hat{M})},
\]
where
\[
\hat{M} = \left( \begin{array}{cc}
\lambda_j e^{2(\theta_1^* + \gamma_1)} & \lambda_j e^{2(\theta_1 + \gamma_1)} \\
\vdots & \ddots \\
\lambda_j e^{2(\theta_1 + \gamma_1)} & \lambda_j e^{2(\theta_1^* + \gamma_1)}
\end{array} \right), \quad \hat{G}_s = \left[ \begin{array}{cc}
\hat{M} & \hat{Y}_1^t \\
\hat{Y}_2 & 0
\end{array} \right],
\]
\[
\hat{Y}_1 = [e^{2\theta_1}, e^{2\theta_2}, \ldots, e^{2\theta_N}], \quad \hat{Y}_2 = \left( \begin{array}{cccc}
c_{1,1} & c_{1,2} & \cdots & c_{1,N,1} \\
c_{2,1} & c_{2,2} & \cdots & c_{N,2}
\end{array} \right).
\]
It then follows
\[
\det(\hat{M}) = e^{4Re(\theta_{k+1} + \theta_{k+2} + \cdots + \theta_N)} \left[ \det(M_k) + O(e^{-c|t|}) \right],
\]
\[
\det(\hat{G}_s) = e^{4Re(\theta_{k+1} + \theta_{k+2} + \cdots + \theta_N)} \left[ \det(G_k^s) + O(e^{-c|t|}) \right],
\]
where
\[
M_k = \left( \begin{array}{cccc}
\frac{z_j^2 \lambda_j^*}{z_j - z_k} & \cdots & \frac{z_j^2 \lambda_j^*}{z_j - z_k} & \frac{z_j^2 \lambda_j^*}{z_j - z_k} \\
\vdots & \ddots & \vdots & \vdots \\
\frac{z_j^2 \lambda_j^*}{z_j - z_k} & \cdots & \frac{z_j^2 \lambda_j^*}{z_j - z_k} & \frac{z_j^2 \lambda_j^*}{z_j - z_k} \\
0 & \cdots & 0 & 0
\end{array} \right),
\]
\[
G_k^s = \left( \begin{array}{cccc}
\frac{z_j^2 \lambda_j}{z_j - z_k} & \cdots & \frac{z_j^2 \lambda_j}{z_j - z_k} & \frac{z_j^2 \lambda_j}{z_j - z_k} \\
\vdots & \ddots & \vdots & \vdots \\
\frac{z_j^2 \lambda_j}{z_j - z_k} & \cdots & \frac{z_j^2 \lambda_j}{z_j - z_k} & \frac{z_j^2 \lambda_j}{z_j - z_k} \\
0 & \cdots & 0 & 0
\end{array} \right).
\]
By direct calculation, we have

\[
\det(G_k^{[n]}) = (-1)^{k+N+1} \prod_{j=1}^{k-1} \lambda_j \det(C_\delta(\Delta_k)) \prod_{j=k+1}^N \left( \lambda_j z_j^2 \frac{z_j - z_k}{z_j - z_k^*} \right) \det(C(z_k^*, z_k^{[l]})) e^{2\theta_k^*},
\]

and

\[
\det(M_k) = \begin{bmatrix}
\frac{|z_1|^2 \lambda_1^*}{z_1 - z_1^*} \gamma_{1,1} & \frac{|z_1|^2 \lambda_1^*}{z_1 - z_1^*} \gamma_{1,k-1} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{|z_1|^2 \lambda_1^*}{z_1 - z_1^*} \gamma_{k-1,1} & \frac{|z_1|^2 \lambda_1^*}{z_1 - z_1^*} \gamma_{k-1,k-1} & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
\frac{|z_1|^2 \lambda_1^*}{z_1 - z_1^*} \gamma_{1,1} & \frac{|z_1|^2 \lambda_1^*}{z_1 - z_1^*} \gamma_{1,k-1} & \frac{|z_1|^2 \lambda_1^*}{z_1 - z_1^*} \gamma_{1,k} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{|z_1|^2 \lambda_1^*}{z_1 - z_1^*} \gamma_{k-1,1} & \frac{|z_1|^2 \lambda_1^*}{z_1 - z_1^*} \gamma_{k-1,k-1} & \frac{|z_1|^2 \lambda_1^*}{z_1 - z_1^*} \gamma_{k-1,k} & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\end{bmatrix}
\]

\[
= \det(C(z_k^*, z_k^{[l]})) \prod_{j=1}^{k-1} \lambda_j \prod_{j=k+1}^N \lambda_j |z_j|^2 \left[ \frac{|z_k|^2 \lambda_k \det(C(\Delta_{k-1}))}{z_k - z_k^*} \prod_{j=k+1}^N \frac{z_k - z_j}{z_k - z_j^*} \right]^2 e^{2(\theta_k^* + \theta_k)} + \lambda_k^* \det(C(\Delta_k))
\]

Thus, along the trajectory \( x - v_k t = \text{const} \), we have

\[
u_{x[N]} = \frac{\det(C_\delta(\Delta_k)) \prod_{j=k+1}^N \left( \frac{|z_j|^2}{z_j^2} \frac{z_j - z_k}{z_j - z_k^*} \right) e^{2\theta_k^*} + O(e^{-c|t|})}{\frac{|z_k|^2 \lambda_k}{z_k - z_k^*} \det(C(\Delta_{k-1})) \prod_{j=k+1}^N \left( \frac{z_k - z_j}{z_k - z_j^*} \right)^2 e^{2(\theta_k^* + \theta_k)} + \lambda_k^* \det(C(\Delta_k))}
\]

\[
u_{x[N]} = u_{x,N} + O(e^{-c|t|})
\]
as \( t \to -\infty \), where

\[
\bar{u}_{s,-}^{[k]} = e^{-\frac{z_k - z_k^*}{|z_k|^2}} \frac{e^{2i\theta_{k,-}}}{\lambda_k e^{2(\theta_{k,-} + \theta_{k,-})} + \lambda_k^* \delta_k}
\]

\[
\hat{\theta}_{k,-} = \theta_k + \frac{1}{2} \sum_{j=k+1}^{N} \ln \left( \frac{z_j - z_k}{z_j - z_k^*} \right),
\]

\[
\hat{c}_{k,s}^{[k]} = \frac{\det(\hat{C}_s(\Delta_k))}{\det(C(\Delta_k - 1))} \prod_{j=k+1}^{N} \frac{z_j^2}{|z_j|^2},
\]

\[
\delta_k^\pm = \frac{z_k - z_k^* \det(C(\Delta_k))}{|z_k|^2 \det(C(\Delta_k - 1))}.
\]

For the general case \( x - vt = \text{const}, \ v \neq v_k (k = 1, 2, \cdots, N) \), we have \( u_s[N] = O(e^{-|v|t}) \). Thus the asymptotic behavior is analyzed as \( t \to -\infty \).

By the same procedure as above, we can obtain the asymptotical behavior as \( t \to +\infty \). To be specific, along the trajectory \( x - v_k t = \text{const} \), one has

\[
u_s[N] = \frac{\det(\hat{C}_s(\Delta^k)) \prod_{j=1}^{k-1} \left( \frac{z_j^2}{|z_j|^2} \right) e^{2i\theta_k} + O(e^{-|v|t})}{\frac{z_k - z_k^*}{z_k - z_k^*} \det(C(\Delta^k + 1)) \prod_{j=1}^{k-1} \left| \frac{z_k - z_j}{z_k - \bar{z}_j} \right| e^{2(\theta_k + \theta_k)} + \lambda_k^* \det(C(\Delta_k))}
\]

as \( t \to +\infty \), where

\[
\bar{u}_{s,+}^{[k]} = e^{-\frac{z_k - z_k^*}{|z_k|^2}} \frac{e^{2i\theta_{k,+}}}{\lambda_k e^{2(\theta_{k,+} + \theta_{k,+})} + \lambda_k^* \delta_k^+}
\]

\[
\hat{\theta}_{k,+} = \theta_k + \frac{1}{2} \sum_{j=1}^{k-1} \ln \left( \frac{z_j - z_k}{z_j - z_k^*} \right),
\]

\[
\hat{c}_{k,s}^{[k]} = \frac{\det(\hat{C}_s(\Delta_k))}{\det(C(\Delta_k + 1))} \prod_{j=1}^{k-1} \frac{z_j^2}{|z_j|^2},
\]

\[
\delta_k^+ = \frac{z_k - z_k^* \det(C(\Delta_k))}{|z_k|^2 \det(C(\Delta_k + 1))}.
\]

Consequently, the asymptotic behaviors of \( N \)-bright soliton’s are analyzed. □

V. THE SOLITON SOLUTIONS WITH NONVANISHING BOUNDARY CONDITION

In this section, we construct all kinds of soliton solutions from the plane wave seed solution. These soliton solutions of coupled FL equation are similar to the ones of coupled nonlinear Schrödinger (CNLS) equation, but they have much richer structures than the ones of the CNLS equation. For instance, the bright-anti-dark soliton and dark-anti-dark soliton solutions, which do not occur in the CNLS equation, exist in the coupled FL equation. We remark here that the soliton is called dark if its amplitude is lower than the background, and the soliton is called anti-dark if its amplitude is higher than the background.

To this end, we start with the plane wave solution for equation (1):

\[
u_i[0] = a_i e^{i\omega_i}, \quad i = 1, 2
\]
Based on the roots of characteristic equation (43) and (44), we can classify the parameters into four cases. Here we assume $a_1 \neq 0$ and introducing $z = 1/\lambda^2$, we have the fundamental solution to the system (42)

$$
\Phi_2(\lambda) = D_1 E_1 \text{diag} \left(e^{\vartheta_1}, e^{\vartheta_2}, e^{\vartheta_3}\right), \quad D_1 = \text{diag} \left(1, e^{i\omega_1}, e^{i\omega_2}\right),
$$

where

$$
\vartheta_i = i(\kappa_i - z) \left(x + \frac{1}{2b_1 b_2} (\kappa_i - z + b_1 + b_2) t\right), \quad i = 1, 2, 3.
$$

Then we have the following conclusions

- If $b_1 = b_2$, then

$$
E_1 = \begin{pmatrix}
\frac{1}{a_1 b_1} & \frac{1}{a_1 b_1} & 0 \\
\frac{1}{\lambda(\kappa_1 + b_1)} & \frac{1}{\lambda(\kappa_2 + b_1)} & -\frac{a_2}{a_1} \\
\frac{1}{\lambda(\kappa_1 + b_1)} & \frac{1}{\lambda(\kappa_2 + b_1)} & 1
\end{pmatrix},
$$

where $\kappa_i, (i = 1, 2)$ satisfies the following equation

$$(\kappa/z - 2)(\kappa + b_1) + (a_1^2 + \sigma a_2^2) b_1^2 = 0, \quad (43)$$

and $\kappa_3 = -b_1$.

- If $b_1 \neq b_2$ and $a_2 \neq 0$, then

$$
E_1 = \begin{pmatrix}
\frac{1}{a_1 b_1} & \frac{1}{a_1 b_1} & \frac{1}{a_1 b_1} \\
\frac{1}{\lambda(\kappa_1 + b_2)} & \frac{1}{\lambda(\kappa_2 + b_2)} & \frac{1}{\lambda(\kappa_3 + b_2)} \\
\frac{1}{\lambda(\kappa_1 + b_2)} & \frac{1}{\lambda(\kappa_2 + b_2)} & \frac{1}{\lambda(\kappa_3 + b_2)}
\end{pmatrix},
$$

where $\kappa_i, (i = 1, 2, 3)$ satisfies the following equation

$$(\kappa/z - 2)(\kappa + b_1)(\kappa + b_2) + a_1^2 b_1^2 (\kappa + b_2) + \sigma a_2^2 b_2^2 (\kappa + b_1) = 0. \quad (44)$$

Based on the roots of characteristic equation (43) and (44), we can classify the parameters into four cases. Here we denote $z = z_1 = \lambda^{-2}$.

**Case (a):** $\Omega_1 \equiv \{a_1, a_2, b_1, b_2|b_1 = b_2, \quad b_1 = \neq 0, \quad 2b_1 - (a_1^2 + \sigma a_2^2) b_1^2 = 0\}$. In this case, the root for the characteristic equation (43) can be solved as

$$
\kappa_1 = 2z_1 - b_1, \quad \kappa_2 = 0, \quad \kappa_3 = -b_1.
$$

**Case (b):** $\Omega_2 \equiv \{a_1, a_2, b_1, b_2|b_1 = b_2, \quad b_1 = \neq 0, \quad 2b_1 - (a_1^2 + \sigma a_2^2) b_1^2 \neq 0\}$. In this case, to avoid the radical expression for the root, we introduce a variable $\xi_1 \in \mathbb{C}$. Denote $\gamma_1 = b_1 - (a_1^2 + \sigma a_2^2) b_1^2$, $\delta_1 = |b_1| \sqrt{|a_1^2 + \sigma a_2^2| |\gamma_1 + b_1|}$, we then have

$$
z_1 = \begin{cases}
\frac{\delta_1}{4} (\xi_1 - \xi_1^{-1}) - \frac{\gamma_1}{2}, & \text{if } (\gamma_1 + b_1)(a_1^2 + \sigma a_2^2) > 0,
\frac{\delta_1}{4} (\xi_1 + \xi_1^{-1}) - \frac{\gamma_1}{2}, & \text{if } (\gamma_1 + b_1)(a_1^2 + \sigma a_2^2) < 0,
\end{cases}
$$

$$
\kappa_1 = -\frac{b_1 + \gamma_1}{2} + \frac{\delta_1}{2} \xi_1, \quad \kappa_2 = -\frac{b_1 + \gamma_1}{2} - \frac{\delta_1}{2} \xi_1^{-1}, \quad \kappa_3 = -b_1. \quad (45)
$$
Remark 1 The relations between the roots and coefficients for the characteristic equation can be represented as

\[ z_1 = \begin{cases} 
\frac{\delta_2}{4} (\xi_2 - \xi_2^{-1}) + \gamma_2, & \text{if } b_1(2/a_1^2 - b_1) > 0, \\
\frac{\delta_2}{4} (\xi_2 + \xi_2^{-1}) + \gamma_2, & \text{if } b_1(2/a_1^2 - b_1) < 0,
\end{cases} \]

(46)

\[ \kappa_1 = -\frac{b_1 + b_2}{2} + \gamma_2 + \frac{\delta_2}{2} \xi_2, \]

\[ \kappa_2 = -\frac{b_1 + b_2}{2} + \gamma_2 + \frac{\delta_2}{2} \xi_2^{-1}, \quad \kappa_3 = 0. \]

Case (d): \( \Omega_4 \equiv \{a_1, a_2, b_1, b_2|a_1, a_2 \neq 0, \sigma b_2 \neq 2a_2^2 - b_1a_1^2a_2^2, b_1 \neq b_2\}. \) In this case, the radical expression for the roots is unavoidable. A direct way of obtaining the roots is to use the Kardan formula. However, one often turns to find numerical solutions for the roots to avoid the complicated formula.

Remark 1 The relations between the roots and coefficients for the characteristic equation are

\[ \sum_{i=1}^{3} \kappa_i + b_1 + b_2 = 2z_1, \]

\[ \prod_{i=1}^{3} (\kappa_i + b_1) = a_1^2 b_1^2 (b_1 - b_2) \left( \sum_{i=1}^{3} \kappa_i + b_1 + b_2 \right), \]

(47)

\[ \prod_{i=1}^{3} (\kappa_i + b_2) = -\sigma a_2^2 b_2^2 (b_1 - b_2) \left( \sum_{i=1}^{3} \kappa_i + b_1 + b_2 \right). \]

In what follows, we will use the formulas (27), (41) and (42) to construct single soliton solutions. Generally, the single soliton solutions can be represented as

\[ u_1[1] = a_1 \left[ \beta + \frac{2 \omega \omega_1^* e^{-i \omega_1}}{a_2} \right] e^{i \omega_1}, \quad u_2[1] = a_2 \left[ \beta + \frac{2 \omega_1 \omega_2^* e^{-i \omega_2}}{a_2} \right] e^{i \omega_2}. \]

(48)

Prior to calculating the explicit form for \( \beta \), we need the following proposition.

Proposition 3 If we choose a special solution \( |y_1\rangle = (\varphi_1, \psi_1, \chi_1)^T \equiv \Phi_2(\lambda_1)(c_1, c_2, c_3)^T, \) then

\[ \beta = 2 \lambda_1^{-1} M_1, \quad M_1 = Z_{l,m} : c_1^* c_m e^{\phi_1^* + \phi_m} \equiv \sum_{l=1}^{3} \sum_{m=1}^{3} Z_{l,m} c_1^* c_m e^{\phi_1^* + \phi_m}, \]

(49)

where \( Z_{l,m} \) are of the following forms depending on four cases mentioned previously.

1. If \( a_1, a_2, b_1, b_2 \in \Omega_1 \), then

\[ Z_{l,m} = \begin{cases} 
\frac{\kappa_1^l}{\kappa_m - \kappa_1^l}, & \text{if } 1 \leq l \leq 2, \ 1 \leq m \leq 2, \ l + m < 4, \\
\frac{|z_1|^2 (\sigma \alpha^2 + \alpha^2 z_1)|z\|_1^{l-1}}{|z_1|^2 - z_1^2}, & \text{if } l = m = 2, \\
\frac{(\sigma \alpha^2 + \alpha^2 z_1)}{z_1^2 - z_1}, & \text{if } l = m = 3, \\
0, & \text{otherwise}.
\end{cases} \]

(50)

2. If \( a_1, a_2, b_1, b_2 \in \Omega_2 \), then

\[ Z_{l,m} = \begin{cases} 
\frac{\kappa_1^l}{\kappa_m - \kappa_1^l}, & \text{if } 1 \leq l \leq 2, \ 1 \leq m \leq 2, \\
\frac{(\sigma \alpha^2 + \alpha^2 z_1)}{z_1^2 - z_1}, & \text{if } l = m = 3, \\
0, & \text{otherwise}.
\end{cases} \]

(51)
• If $a_1, a_2, b_1, b_2 \in \Omega_3$, then

$$Z_{l,m} = \begin{cases} \frac{|z_1|^2(a_1^2 + \sigma a_2^2)z_1 - 1}{|z_1|^2 - z_1^2} & \text{if } l = m = 3, \\ \frac{\kappa_l^*}{\kappa_m - \kappa_l^*} & \text{otherwise}. \end{cases}$$

(52)

• If $a_1, a_2, b_1, b_2 \in \Omega_4$, then

$$Z_{l,m} = \frac{\kappa_l^*}{\kappa_m - \kappa_l^*}.$$  

(53)

**Proof:** Firstly, we have

$$\frac{\kappa_m}{z_1} - 2 + \frac{a_1^2 b_1^2}{\kappa_m b_1} + \frac{\sigma a_2^2 b_2^2}{\kappa_m + b_2} = 0,$$

(54)

and its complex conjugate

$$\frac{\kappa_l^*}{z_1} - 2 + \frac{a_1^2 b_1^2}{\kappa_l^* + b_1} + \frac{\sigma a_2^2 b_2^2}{\kappa_l^* + b_2} = 0.$$  

(55)

Subtracting equation (55) from (54) yields

$$\frac{\lambda_l^2 \kappa_m - \lambda_l^2 \kappa_l^*}{(\kappa_l^* - \kappa_m)} + \frac{a_1^2 b_1^2}{(\kappa_m + b_1)(\kappa_l^* + b_1)} + \frac{\sigma a_2^2 b_2^2}{(\kappa_m + b_2)(\kappa_l^* + b_2)} = 0.$$  

(56)

The coefficient of $e^{\theta_m + \theta_l^*}$ can be simplified as

$$\frac{2c_l c_m}{\lambda_l^2 - \lambda_l^2} \left[ \frac{\lambda_l^2}{\lambda_l^2} \left( \frac{a_1^2 b_1^2}{(\kappa_m + b_1)(\kappa_l^* + b_1)} + \frac{\sigma a_2^2 b_2^2}{(\kappa_m + b_2)(\kappa_l^* + b_2)} \right) \right]$$

$$= \frac{2c_l c_m}{\lambda_l^2 - \lambda_l^2} \left[ \frac{\lambda_l^2}{\lambda_l^2} \left( \frac{\lambda_l^2}{\lambda_l^2} \sum_{l,m=1, l+m<4} \kappa_l^* + b_1 \frac{c_m c_l}{\kappa_m + b_1} \kappa_m \right) \right]$$

$$= \frac{2c_l c_m}{\lambda_l^2 - \lambda_l^2} \left[ \frac{\lambda_l^2}{\lambda_l^2} \right]$$

by referring equation (56). The four cases can be proved by analyzing above relation by tedious work, which is omitted here.

We comment here that if $\kappa_i \in \mathbb{R}$, we can not use the above relation. Instead, we have to use the definition to derive the relation directly. □

A. Bright-dark/Bright-anti-dark solution

First, we consider $a_1, a_2, b_1, b_2 \in \Omega_1$ and $a_2 = 0$. Under this case, one can obtain the following general solution by formula (53)

$$u_1[1] = a_1 \left( \frac{N_1[1]}{M_1} \right) e^{i \omega_1}, \quad u_2[1] = \frac{N_2[1]}{M_1} e^{i \omega_1},$$

where

$$N_1[1] = \sum_{l,m=1, l+m<4} \kappa_l^* + b_1 \frac{c_m c_l}{\kappa_m + b_1} \kappa_m e^{\hat{\theta}_m + \hat{\theta}_l^*} + \frac{|c_2|^2 \sum_{l,m=1} (a_1^2 z_1^* - 1)}{|z_1|^2 - z_1^2} e^{\theta_2 + \theta_2^*} + \frac{\sigma |c_3|^2 |z_1|}{(z_1^* - z_1)} e^{\hat{\theta}_3 + \hat{\theta}_3^*},$$

$$N_2[1] = c_3 \lambda_1 (c_1^* e^{\hat{\theta}_3 + \hat{\theta}_3^*} + c_2^* e^{\hat{\theta}_3 + \hat{\theta}_3^*}).$$

If $c_1 = 0$, $c_2 c_3 \neq 0$, the solution $|u_1[1]|^2 = a_1^2$ and $|u_2[1]|^2$ is a bright soliton along the line $t = \text{const}$. If $c_2 = 0$, $c_1 c_3 \neq 0$, the solution is a bright-dark soliton. If $c_3 = 0$, $c_1 c_2 \neq 0$, then the solution $|u_1[1]|^2$ is a breather solution and
FIG. 3: (color online): Resonant bright-dark-breather sol ution with parameters: $a_1 = 1$, $b_1 = b_2 = 2$, $\lambda_1 = \frac{1}{2}(1 + i)$, $\zeta_1 = -2i$, $\kappa_1 = -2 - 4i$, $\kappa_2 = 0$, $c_1 = c_2 = c_3 = 1$.

If $c_1c_2c_3 \neq 0$, the solution is a resonant bright-dark-breather solution which can be viewed as the nonlinear superposition of above three types of solutions. Then we consider the case $a_1, a_2, b_1, b_2 \in \Omega_2$ and $a_2 = 0$, under which we can obtain the following general solution by formula (48):

\[
\begin{align*}
    u_1[1] &= a_1 \left( \frac{N_1[2]}{M_1} \right) e^{i\omega_1}, \\
    u_2[1] &= \left( \frac{N_2[2]}{M_1} \right) e^{i\omega_1},
\end{align*}
\]

where

\[
\begin{align*}
    N_1[2] &= \sum_{l,m=1}^{2} \frac{\kappa_l^* + b_1 c_m c_l^* \kappa_m \kappa_l^* \kappa_m^* + \omega_m^*}{\kappa_l + b_1 \kappa_m - \omega_l^2} + \sigma |c_3|^2 |z_1| (z^*_1 - z_1) e^{\omega_3 + \omega_3^*}, \\
    N_2[2] &= c_3 \lambda_1 (c_1 e^{\omega_3 + \omega_3^*} + c_2 e^{\omega_3 + \omega_3^*})
\end{align*}
\]

and

\[
\vartheta_1 = i(\kappa_i - z_1) \left[ x + \frac{z_1^*}{2b_1b_2 |z_1|^2} (\kappa_i - z_1 + 2b_1) t \right].
\]

It can be shown that if $c_1 = 0$, $c_2c_3 \neq 0$ ($c_2 = 0$, $c_1c_3 \neq 0$), the solution is either a bright-dark soliton or a bright-anti-dark soliton. Particularly, when $\text{sign}(\text{Re}(\kappa_1)/\text{Re}(z_1)) > 0$, the solution is a bright-dark soliton. The peak for $|u_1[1]|^2$ is along the line $\text{Re}(\vartheta_3 - \vartheta_1) + \ln \left( \frac{|\vartheta_3| |z_1| \text{Im}(\kappa_1)}{|\vartheta_1| \text{Im}(z_1)} \right) = 0$, and the peak values are $a_1^2 \left[ 1 - \frac{2\text{Im}(\kappa_1)\text{Im}(\frac{z_1}{\kappa_1} + \vartheta_1)}{|\kappa_1| \text{Re}(\kappa_1)} \right]$ for $|u_1[1]|^2$ and $\left[ 1 + \frac{2\text{Im}(\kappa_1)\text{Im}(\frac{z_1}{\kappa_1} + \vartheta_1)}{|\kappa_1| \text{Re}(\kappa_1)} \right] |u_1[1]|^2$ for $|u_2[1]|^2$. When $\text{sign}(\text{Re}(\kappa_1)/\text{Re}(z_1)) < 0$, the solution is a bright-anti-dark soliton. The peak values are $a_2^2 \left[ 1 + \frac{2\text{Im}(\kappa_1)\text{Im}(\frac{z_1}{\kappa_1} + \vartheta_1)}{|\kappa_1| \text{Re}(\kappa_1)} \right] |u_1[1]|^2$ and $\left[ 1 + \frac{2\text{Im}(\kappa_1)\text{Im}(\frac{z_1}{\kappa_1} + \vartheta_1)}{|\kappa_1| \text{Re}(\kappa_1)} \right] |u_2[1]|^2$ for $|u_2[1]|^2$.

On the other hand, as $c_1c_2 \neq 0$, $c_3 = 0$, one obtains a breather solution, as $c_1c_2c_3 \neq 0$, one arrives at a resonant bright-dark-breather solution.

B. Dark/anti-dark soliton

In CNLS equation, there only exists dark soliton solution. Whereas there exists dark and anti-dark soliton solutions in the coupled FL equation (1). If we apply the formula (48) for a complex spectral parameter $\lambda_1$, the singularity occurs. With the aid of technique in [37], we can derive the dark/anti-dark soliton to equation (1).
FIG. 4: (color online): Bright-anti-dark soliton solution with parameters: \(a_1 = 2, b_1 = b_2 = 1, z_1 = \frac{3}{2} - \frac{3}{4}\sqrt{2}i, \kappa_1 = 1 - 2\sqrt{2}i, \kappa_2 = 1 + \frac{\sqrt{2}}{2}i, c_1 = 0, c_2 = c_3 = 1\). The peaks value for \(|u_1|^2|\) is 4.399 and for \(|u_2|^2|\) is 0.199.

FIG. 5: (color online): Bright-dark soliton solution with parameters: \(a_1 = 2, b_1 = b_2 = 1, z_1 = \frac{3}{2} - \frac{3}{4}\sqrt{2}i, \kappa_1 = 1 - 2\sqrt{2}i, \kappa_2 = 1 + \frac{\sqrt{2}}{2}i, c_2 = 0, c_1 = c_3 = 1\). The peaks value for \(|u_1|^2|\) is \(\frac{4}{3}\) and for \(|u_2|^2|\) is \(\frac{8}{9}\).

Choosing \(\lambda_1 \in \mathbb{R} \cup i\mathbb{R}\) such that the characteristic equations (43), (44) possess a pair of conjugate complex roots. Further, by taking a special solution

\[|y_1\rangle = D_1 \left[ \left( \begin{array}{c} \frac{1}{\lambda_1(\kappa_1 + b_1)} \alpha_1 b_1 \lambda_1 \kappa_1 \exp(\vartheta_1 + \vartheta_2^*) + \varphi_1 \kappa_1 \exp(\vartheta_1 + \vartheta_2^*) \pm |\kappa_1| \right) e^{i\omega_1} \\ \frac{1}{\lambda_1(\kappa_2 + b_1)} \alpha_1 b_1 \lambda_1 \kappa_2 \exp(\vartheta_1 + \vartheta_2^*) + \varphi_2 \kappa_2 \exp(\vartheta_1 + \vartheta_2^*) \pm |\kappa_2| \right] \]

and combining with the limit \(\lambda_1 \rightarrow \pm \lambda_1^*\), \(\kappa_2 \rightarrow \kappa_2^*\) for an appropriate \(\alpha_1\), we obtain

\[\beta \rightarrow \frac{2\lambda_1^{-1}}{\kappa_1 - \kappa_1^*} \left[ \kappa_1^* e^{\vartheta_1 + \vartheta_2^*} \pm |\kappa_1| \right],\]

\[\psi_1 \varphi_1^\ast \rightarrow \frac{a_1 b_1 \lambda_1^{-1}}{\kappa_1 + b_1} e^{\vartheta_1 + \vartheta_2^*},\]

\[\chi_1 \varphi_1^\ast \rightarrow \frac{a_2 b_2 \lambda_1^{-1}}{\kappa_1 + b_1} e^{\vartheta_1 + \vartheta_2^*}.\]

Finally, we can obtain a soliton solution of either dark or anti-dark type

\[u_s[1] = a_s \left[ \kappa_1 \exp(\vartheta_1 + \vartheta_1^* + i\gamma_s) + c_1 |\kappa_1| \right] e^{i\omega_s},\]

(58)
where \( \exp(i\tau_s) = \frac{\kappa_1 + b_s}{\kappa_1 + b_s} \) and \( \varsigma_1 = \pm 1 \). The peak values of \( |u_s[1]|^2 \) are

\[
a_1^2 = \left[ 1 + \frac{2b_1\kappa_1^2}{(\kappa_1R + \varsigma_1|\kappa_1|)|\kappa_1 + b_1|^2} \right].
\]

It can be shown by direct calculation that if \( \varsigma_1 b_i > 0 \), the solution (58) is an anti-dark soliton; if \( \varsigma_1 b_i > 0 \), it is a dark soliton. The velocity of the dark/anti-dark soliton is

\[
v = \frac{1}{b_1 b_2} (\kappa_1R - z_1 + \frac{b_1 b_2}{2}).
\]

To obtain the stationary solution, we must choose the parameters such that \( b_1 + b_2 = 0, \kappa_1R = z_1 > 0, \kappa_1 = \kappa_1R + i\kappa_1I, \) and

\[
a_1 = \pm \frac{\sqrt{2}}{2|b_1\lambda_1|} \sqrt{(b_1\lambda_1^2 + 1)^2 + \kappa_1^2}\lambda_1^4, \quad a_2 = \pm \frac{\sqrt{2}}{2|b_1\lambda_1|} \sqrt{(b_1\lambda_1^2 - 1)^2 + \kappa_1^2}\lambda_1^4.
\]

If \( \kappa_1 \) is a repeated real root to the characteristic equation (43) or (44), choosing the appropriate parameter \( \varsigma_1 \) in the formula (58), we can obtain the following rational solution through the limit technique

\[
u_s[1] = \sqrt{a_s} \left\{ \frac{2\kappa_1 \left( x + \frac{1}{b_1 b_2} \left( \kappa_1 - z_1 + \frac{b_1 + b_2}{2} \right) t \right) + \frac{1}{\kappa_1 + b_1}}{2\kappa_1 \left( x + \frac{1}{b_1 b_2} \left( \kappa_1 - z_1 + \frac{b_1 + b_2}{2} \right) t \right) + 1} \right\} e^{\omega_s}.
\]

It can be easily shown that the peak values for \( |u_s[1]|^2 \) are \( a_s^2 \left( \frac{\kappa_1 - b_1}{\kappa_1 + b_1} \right)^2 \). What we should point out that the rational solution obtained here is a soliton solution, not a rogue wave solution.

C. Breather-like solution with nonvanishing boundary condition

To gain other types of solutions, it is necessary to provide an expression in the numerator of the formula (58). We conclude it by the following proposition.

**Proposition 4** If we choose a special solution \( |y_1| = (\varphi_1, \psi_1, \chi_1)^T = \Phi_2(\lambda_1)(c_1, c_2, c_3)^T \), then

\[
\beta + \frac{2\varphi_1\chi_1^*}{a_1} e^{-i\omega_1} = 2\lambda_1^{-1} N_1^{[3]}, \quad \beta + \frac{2\varphi_1\chi_1^*}{a_2} e^{-i\omega_2} = 2\lambda_1^{-1} N_2^{[3]}, \quad (59)
\]

where

\[
N_1^{[3]} = K_{1,m}^{[3]} : c_m c_m^* e^{\varphi_m + \chi_m^*} = \sum_{l,m=1}^{3} K_{1,m}^{[3]} c_m c_m^* e^{\varphi_m + \chi_m^*}, \quad s = 1, 2,
\]

and

\[
\vartheta_i = \frac{1}{i}(\kappa_i - z_1) \left[ x + \frac{z_1}{2b_1 b_2|z_1|^2}(\kappa_i - z_1 + b_1 + b_2)t \right], \quad i = 1, 2, 3.
\]
FIG. 7: (color online): Dark-anti-dark soliton solution with parameters: \( a_1 = \frac{19\sqrt{15}}{30}, a_2 = \frac{19\sqrt{10}}{30}, b_1 = 2, b_2 = -3, \lambda_1 = \frac{1}{2}, z_1 = 4, \kappa_1 = 4 \). The peaks value for \( |u_1|^2 \) is \( \frac{1}{30} \) and for \( |u_2|^2 \) is \( \frac{49}{290} \).

- If \( a_1, a_2, b_1, b_2 \in \Omega_1 \) and \( a_2 \neq 0 \), then

\[
K_{l,m}^{[i]} = \begin{cases} 
\frac{\kappa_l^2 + b_i}{\kappa_m + b_1} - \frac{\kappa_m}{\kappa_m - \kappa_l^i} & \text{if } 1 \leq l \leq 2, \ 1 \leq m \leq 2, \ l + m < 4, \\
|z_1|^2 z_1^2 \left( (\sigma + \frac{\kappa_m}{\kappa_1}) z_1 \right) & \text{if } l = m = 2, \\
a_1 \frac{\kappa_m}{z_1 - z_i} & \text{if } l = m = 3, \\
a_2 \lambda_1 & \text{if } l = 1, 2, \ m = 3, \ i = 1, \\
\lambda_1 & \text{if } l = 1, 2, \ m = 3, \ i = 2, \\
0 & \text{otherwise}.
\end{cases}
\]  

(60)

- If \( a_1, a_2, b_1, b_2 \in \Omega_2 \), then

\[
K_{l,m}^{[i]} = \begin{cases} 
\frac{\kappa_l^2 + b_i}{\kappa_m + b_1} - \frac{\kappa_m}{\kappa_m - \kappa_l^i} & \text{if } 1 \leq l \leq 2, \ 1 \leq m \leq 2, \\
|z_1|^2 z_1^2 \left( (\sigma + \frac{\kappa_m}{\kappa_1}) z_1 \right) & \text{if } l = m = 3, \\
a_1 \frac{\kappa_m}{z_1 - z_i} & \text{if } l = m = 3, \ i = 1 \\
a_2 \lambda_1 & \text{if } l = 1, 2, \ m = 3, \ i = 2, \\
0 & \text{otherwise}.
\end{cases}
\]  

(61)

- If \( a_1, a_2, b_1, b_2 \in \Omega_3 \), then

\[
K_{l,m}^{[i]} = \begin{cases} 
\frac{|z_1|^2 z_1^2 \left( (\sigma + \kappa_m) z_1 \right)}{|z_1|^2 - z_i^2} & \text{if } l = m = 3, \\
\frac{\kappa_l^2 + b_i}{\kappa_m + b_1} - \frac{\kappa_m}{\kappa_m - \kappa_l^i} & \text{others},
\end{cases}
\]  

(62)

- If \( a_1, a_2, b_1, b_2 \in \Omega_4 \), then

\[
K_{l,m}^{[i]} = \frac{\kappa_l^2 + b_i}{\kappa_m + b_1} - \frac{\kappa_m}{\kappa_m - \kappa_l^i}.
\]  

(63)

Proposition 4 can be proved in the similar way as Proposition 3, thus we omit the proof here.
In what follows, we present the dynamics for the general solution. When and the center of this soliton turns out to be a special solution CNLS equation. Actually, this type of breather solution also exists in the \( \kappa \) soliton solution. When is given in the formula (46). Based on Propositions 3 and 4, we can obtain the general solution

$\left| \frac{N_1^3}{M_1} \right|^2 = O \left( \frac{(z_1^* - b_1/2) |c_1|^2}{|z_1|(1 + \sigma a_2^*/a_1^*) |c_3|^2} e^{\theta_1 - \theta_3 + \theta_3^* - \theta_3^*} \right),$

and the center of this soliton turns out to be

$\begin{align*}
x &= \frac{1}{4 \text{Im}(z_1)} \ln \frac{(z_1^* - b_1/2) |c_1|^2}{|z_1|(1 + \sigma a_2^*/a_1^*) |c_3|^2},
\end{align*}$

When \( a_1, a_2, b_1, b_2 \in \Omega_3 \) and \( c_1 c_2 \neq 0, c_3 = 0 \), then one has a similar complicated soliton solution, whose center is located at

$\begin{align*}
x &= \frac{1}{\delta_1 \text{Im}(\xi_2 + \xi_2^*)} \ln \frac{|\kappa_1^* (\kappa_2^* - \kappa_2) |c_1|^2}{|\kappa_2^* (\kappa_1^* - \kappa_1) |c_2|^2},
\end{align*}$

here \( \xi_2 \) is given in the formula (16).

![Fig. 8](image)

FIG. 8: (color online): A resonant breather solution with parameters: \( a_1 = \sqrt{2}, a_2 = 1, b_1 = \frac{1}{2}, b_2 = 1, z_1 = \frac{2\imath}{17} + \frac{3\imath}{68}, \kappa_1 = \frac{1}{4} + \frac{1}{4} \imath, \kappa_2 = -\frac{21}{68} + \frac{1}{68} \imath, \kappa_3 = 0, c_1 = 1, c_2 = \frac{1}{6}, c_3 = 0 \).

Other cases lead to the breather-like solution with nonvanishing boundary condition. Particularly, if \( c_1 c_2 c_3 \neq 0 \), a resonant breather solution can be obtained (see Fig. 8). Actually, this type of breather solution also exists in the CNLS equation.

VI. MULTI-DARK/ANTI-DARK SOLITON SOLUTIONS

In this section, we give the multi-dark/anti-dark soliton solution and its asymptotical analysis. To this end, we take a special solution

$\begin{align*}
|y_1| &= D_1 \left[ \left( \frac{1}{\lambda_1 (\kappa_1 + b_1)} \right) e^{\theta_1,1} + \alpha_1 (\lambda_1^2 - \lambda_1^*) \left( \frac{1}{\lambda_1 (\kappa_1 + b_1)} \right) e^{\theta_1,2} \right],
\end{align*}$

$\begin{align*}
\theta_{l,i} &= i(\kappa_{l,i} - z_i) \left( x + \frac{1}{2b_1 b_2 z_i} (\kappa_{l,i} - z_i + b_1 + b_2) t \right), \ l = 1, 2,
\end{align*}$
where $\kappa_{i,l}$ ($l = 1, 2$) satisfies the following equation

$$
(z_{i}^{-1} \kappa_{i} - 2)(\kappa_{i} + b_{1})(\kappa_{i} + b_{2}) + a_{l}^{2}b_{1}^{2}(\kappa_{i} + b_{2}) + \sigma a_{l}^{2}b_{2}^{2}(\kappa_{i} + b_{1}) = 0.
$$

It is noted that as $z_{i} \to z_{i}^{\ast}$, $\kappa_{i,2} \to \kappa_{i,1}^{\ast}$. Based on above equations, one obtains

$$
M = (m_{ij})_{1 \leq i,j \leq N}, \quad m_{ij} = \frac{2\lambda_{j}^{-1}}{\kappa_{j,1} - \kappa_{i,1}^{\ast}} \left[ \kappa_{i,1}^{\ast} e^{\vartheta_{i,1} + \vartheta_{i,1}^{\ast}} + \delta_{i,j} \varsigma_{i} |\kappa_{i,1}| \right],
$$

$$
H_{s} = (h_{ij}^{[s]})_{1 \leq i,j \leq N}, \quad h_{ij}^{[s]} = \frac{2\lambda_{j}^{-1}}{\kappa_{j,1} - \kappa_{i,1}^{\ast}} \left[ \kappa_{i,1}^{\ast} e^{\vartheta_{i,1} + \vartheta_{i,1}^{\ast}} + \delta_{i,j} \varsigma_{i} |\kappa_{i,1}| \right],
$$

where $\delta_{i,j}$ is the Kronecker’s delta, $\varsigma_{i} = \pm 1$. For simplicity, we denote $\kappa_{i,1}$ as $\kappa_{i}$ and assume $\kappa_{i} = \kappa_{i,R} + i\kappa_{i,I}$. By tedious calculations, the multi-dark/anti-dark soliton solution can be represented by

$$
u_{s}[N] = a_{s} \left[ \frac{\det(H_{s})}{\det(M)} \right] e^{i\omega_{i}}.
$$

In what follows, we perform analysis for the asymptotical behavior of the $N$-dark/anti-dark soliton solution. Introducing the determinant of Cauchy matrix

$$
\Delta_{k} \equiv \left| \frac{1}{|\kappa_{j} - \kappa_{i}^{\ast}|} \right|_{1 \leq i,j \leq k},
$$

FIG. 9: (color online): Resonant breather solution with parameters: $a_{1} = a_{2} = 1, b_{1} = -b_{2} = 1, z_{1} = i, \kappa_{1} = 1.359959341 - 0.2573179718i, \kappa_{2} = -0.7352886592 + 2.169427339i, \kappa_{3} = -0.6246706814 + 0.087890633i, c_{1} = 1, c_{2} = 1, c_{3} = 1.$

FIG. 10: (color online): Two-dark soliton solution with parameters: $a_{1} = a_{2} = 1, b_{1} = -b_{2} = 1, \lambda_{1} = 1, \lambda_{2} = 2, \kappa_{1} = 1.347810385 - 1.028852255i, \kappa_{2} = 0.6647417703 - 0.4011272786i, \varsigma_{1} = -1, \varsigma_{2} = 1.$
FIG. 11: (color online): Two-dark soliton solution with parameters: $a_1 = a_2 = 1$, $b_1 = 2$, $b_2 = 1$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\kappa_1 = 0.1766049822 - 1.2028208i$, $\kappa_2 = -0.4486076425 - 0.3327284758i$, $\varsigma_1 = 1$, $\varsigma_2 = 1$.

FIG. 12: (color online): Two-dark soliton solution with parameters: $a_1 = a_2 = 1$, $b_1 = 2$, $b_2 = 1$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\kappa_1 = 0.1766049822 - 1.2028208i$, $\kappa_2 = -0.4486076425 - 0.3327284758i$, $\varsigma_1 = -1$, $\varsigma_2 = -1$.

the asymptotical behavior of $N$-dark soliton can be concluded by

**Proposition 5** As $t \to \pm \infty$, $u_s[N]$ can be expressed as sum of single dark soliton solutions,

$$u_s[N] = a_s \left[ S_1^{[s] \pm} + (S_2^{[s] \pm} - A_1^{[s] \pm}) + \cdots + (S_N^{[s] \pm} - A_N^{[s] \pm} - 1) \right] e^{ix_s} + O(e^{-c|t|})$$

(66)

where $c = \min_{1,2,\ldots,N}(|\kappa_i|) \min_{i \neq j}(|v_i - v_j|)$,

$$S_k^{[s] \pm} = P_k^{[s] \pm} \left[ \kappa_k \exp(2\Re(\vartheta_k) + i\tau_k) \pm |\kappa_k| \right], \quad \exp(i\vartheta_k) = \frac{\kappa_k^* + b_s}{\kappa_k + b_s}$$

and

$$P_k^{[s] \pm} = \prod_{i=1}^{k-1} \left( \frac{\kappa_i^* + b_s}{\kappa_i + b_s} \right), \quad P_k^{[s] +} = \prod_{i=k+1}^{N} \left( \frac{\kappa_i^* + b_s}{\kappa_i + b_s} \right),$$

$$\vartheta_k = \vartheta_k + \sum_{i=1}^{k-1} \ln \left| \frac{\kappa_k - \kappa_i}{\kappa_k^* - \kappa_i} \right|, \quad \vartheta_k^* = \vartheta_k + \sum_{i=k+1}^{N} \ln \left| \frac{\kappa_k - \kappa_i}{\kappa_k^* - \kappa_i} \right|,$$

$$A_k^{[s] \pm} = P_i^{[s] \pm} e^{i\tau_i} \frac{\kappa_i}{\kappa_i^*}, \quad A_i^{[s] +} = P_i^{[s] +}.$$
Proof: The determinant \( \det(H_s) \) and \( \det(M) \) in \( N \)-dark/anti-dark soliton solution can be represented as

\[
\begin{align*}
\det(H_s) &= \frac{1}{\kappa_j - \kappa_i} \left[ \kappa_j^* - \kappa_i^* \right] \left[ \kappa_j^* e^{\theta_j} + \theta_j^* + \delta_{i,j} \xi_i | \kappa_i | \right] \left| \begin{array}{c} \kappa_i^* + b_s \kappa_i^* e^{\theta_j} + \delta_{i,j} \xi_i | \kappa_i | \\ \kappa_j^* + b_s \kappa_j^* e^{\theta_j} + \delta_{i,j} \xi_i | \kappa_i | \end{array} \right|_{1 \leq i,j \leq N}, \\
\det(M) &= \frac{1}{\kappa_j - \kappa_i} \left[ \kappa_j^* e^{\theta_j} + \theta_j^* + \delta_{i,j} \xi_i | \kappa_i | \right] \left| \begin{array}{c} \kappa_i^* + b_s \kappa_i^* e^{\theta_j} + \delta_{i,j} \xi_i | \kappa_i | \\ \kappa_j^* + b_s \kappa_j^* e^{\theta_j} + \delta_{i,j} \xi_i | \kappa_i | \end{array} \right|_{1 \leq i,j \leq N}.
\end{align*}
\]  

(67)

As \( t \to -\infty \), we fix the value of \( \text{Re}(\vartheta_k) \)

\[
\text{Re}(\vartheta_k) = -\kappa_{k-1}(x - v_k t) = \text{const}, \quad v_k = \frac{1}{b_1 b_2 z_k} \left( \kappa_{k-1} - z_k + \frac{b_1 + b_2}{2} \right),
\]

and assume \( v_1 < v_2 < \cdots < v_N \). From \( \text{Re}(\vartheta_i) = -\kappa_{i-1}(x - v_k t + (v_k - v_i)t) \), it is obvious that \( \text{Re}(\vartheta_i) \to +\infty \) for \( 1 \leq i \leq k - 1 \) and \( \text{Re}(\vartheta_i) \to -\infty \) for \( k + 1 \leq i \leq N \). It follows that

\[
\begin{align*}
\det(M) &= e^{2\text{Re}(\vartheta_1 + \vartheta_2 + \cdots + \vartheta_{k-1})} \left[ \det(M_k) + \mathcal{O}(e^{-|\vartheta_i|}) \right], \\
\det(H_s) &= e^{2\text{Re}(\vartheta_1 + \vartheta_2 + \cdots + \vartheta_{k-1})} \left[ \det(H_k^{[s]}) + \mathcal{O}(e^{-|\vartheta_i|}) \right],
\end{align*}
\]

where

\[
\det(M_k) = \begin{vmatrix}
\frac{\kappa_i^*}{\kappa_i - \kappa_j^*} & \cdots & \frac{\kappa_i^*}{\kappa_i - \kappa_k^*} \\
\vdots & \ddots & \vdots \\
\frac{\kappa_i^*}{\kappa_i - \kappa_n^*} & \cdots & \frac{1}{\kappa_n^*} \\
\end{vmatrix}
\]

and

\[
\det(H_k^{[s]}) = \begin{vmatrix}
\frac{\kappa_i^* + b_s \kappa_i^*}{\kappa_i - \kappa_j^*} & \cdots & \frac{\kappa_i^* + b_s \kappa_i^*}{\kappa_i - \kappa_n^*} \\
\vdots & \ddots & \vdots \\
\frac{\kappa_i^* + b_s \kappa_i^*}{\kappa_i - \kappa_n^*} & \cdots & \frac{\kappa_i^* + b_s \kappa_i^*}{\kappa_i} \\
\end{vmatrix}
\]

Consequently, as \( t \to -\infty \), we have the asymptotic behavior along \( \omega_t \) as follows

\[
\begin{align*}
u_k[N] &= a_s \prod_{i=1}^{k-1} \left( \kappa_i^* + b_s \kappa_i^* \right) \left[ \text{Sk} \kappa_k | \Delta_{k-1} + \kappa_k^* + b_s \kappa_k^* e^{\theta_k} + \vartheta_k \right] \left[ \text{Sk} \kappa_k | \Delta_{k-1} + \kappa_k^* e^{\theta_k} + \vartheta_k \right] \left[ \text{Sk} \kappa_k | \Delta_{k-1} + \kappa_k^* + b_s \kappa_k^* e^{\theta_k} + \vartheta_k \right] \left[ \text{Sk} \kappa_k | \Delta_{k-1} + \kappa_k^* e^{\theta_k} + \vartheta_k \right] e^{\omega_t} + \mathcal{O}(e^{-|\vartheta_i|}) \\
&= a_s \prod_{i=1}^{k-1} \left( \kappa_i^* + b_s \kappa_i^* \right) \left[ \text{Sk} \kappa_k | \Delta_{k-1} + \kappa_k^* + b_s \kappa_k^* e^{\theta_k} + \vartheta_k \right] \left[ \text{Sk} \kappa_k | \Delta_{k-1} + \kappa_k^* e^{\theta_k} + \vartheta_k \right] \left[ \text{Sk} \kappa_k | \Delta_{k-1} + \kappa_k^* + b_s \kappa_k^* e^{\theta_k} + \vartheta_k \right] \left[ \text{Sk} \kappa_k | \Delta_{k-1} + \kappa_k^* e^{\theta_k} + \vartheta_k \right] e^{\omega_t} + \mathcal{O}(e^{-|\vartheta_i|}) \\
&= a_s e^{\omega_t} + \mathcal{O}(e^{-|\vartheta_i|}).
\end{align*}
\]
where the relation \( \Delta_{k-1} = \frac{1}{\kappa_k - \kappa_{k-1}} \prod_{i=1}^{k-1} \frac{\kappa_k - \kappa_{k-1}}{\kappa_k^2 - \kappa_{k-1}^2} \) is used.

Similarly, we can prove the asymptotical behavior as \( t \to \infty \), which is omitted here. The proof is complete. □

VII. CONCLUSION AND DISCUSSIONS

In this work, we have constructed the multi-Hamiltonian structure for a multi-component Kaup-Newell hierarchy and the infinite conservation laws for a vector Fokas-Lenells equation. These properties confirm that the vector FL equation is integrable.

Then a generalized Darboux transformation for the coupled FL equation is constructed. By using the DT method, the soliton solutions to the coupled FL equation are thoroughly investigated. Starting from the zero solution, the multi-bright soliton solution is constructed and the analysis of its asymptotic behaviour is performed. On the other hand, starting from a general nonzero seed solution, we have derived a variety of single soliton solutions including the bright-dark soliton, the bright-anti-dark soliton, the dark-dark soliton, the dark-anti-dark soliton and the anti-dark-anti-dark soliton solutions. Particularly, a breather-like solution with nonvanishing boundary condition is also obtained. In the last, multi-dark solution is deduced by a limit technique developed by one of the authors. The asymptotic behaviour is also analyzed. We should point out that, based on the DT and the plane wave seed solution, one can obtain the higher order rogue wave solutions. Since there are several parameters governing the dynamics of rogue wave, the general rogue wave solution for the coupled FL equation need to analyzed carefully. We would like to report the results in a separate work.

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