On the Interrelation between Dependence Coefficients of Extreme Value Copulas

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Abstract

For extreme value copulas with a known upper tail dependence coefficient we find pointwise upper and lower bounds, which are used to establish upper and lower bounds of the Spearman and Kendall correlation coefficients. We shown that in all cases the lower bounds are attained on Marshall–Olkin copulas, and the upper ones, on copulas with piecewise linear dependence functions.

Keywords: extreme value copulas, upper tail dependence coefficient, Spearman’s correlation coefficient, Kendall’s correlation coefficient

MSC: 60E15, 60G70, 62G32, 62H20

1 Introduction

For a long time, to describe dependence of random variables, linear Gaussian models were mainly used.

However, by the end of XX century there appeared common understanding that such models are not good enough to well describe many natural, engineering, and social phenomena. Therefore, copulas have become quite popular in the last decades. Their various applications and theoretical studies mutually motivate each other.

A copula $C$ is a multivariate distribution function on $[0,1]^d$, $d \geq 2$, such that all univariate marginal distributions are uniform on $[0,1]$. According to Sklar’s famous theorem, any multivariate function in $\mathbb{R}^d$ can be represented as

$$F(x_1,\ldots,x_d) = C(F_1(x_1),\ldots,F_d(x_d)),$$

where $F_i$, $1 \leq i \leq d$, are marginal distribution functions. Thus, to each multivariate distribution there corresponds its copula. If the marginal distribution functions are continuous, then such a representation is unique.

As an excellent textbook on copulas, we recommend [6].

Below we only consider bivariate copula $C(u,v)$ of random vectors $(X,Y)$ with continuous distribution functions $F_X$ and $F_Y$ of the components, so that for the joint distribution function of $X$ and $Y$ there exists a unique representation

$$F(x,y) = C(F_X(x),F_Y(y)).$$

A survival copula $\hat{C}$ is most simply defined as a copula of the random vector $(-X,-Y)$. It is related with the original copula by

$$\hat{C}(u,v) = u + v - 1 + C(1-u,1-v).$$

The survival copula relates distribution tails instead of the original functions:

$$\hat{F}(x,y) = P(X > x,Y > y) = \hat{C}(\bar{F}_X(x),\bar{F}_Y(y)).$$

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Consider a classical example. Let there be a two-component system with two independent factors that may cause failure of one of the components each. Let there also be a third factor causing failure of both components simultaneously. Assuming that all the factors act with a constant intensity, there occurs a bivariate exponential Marshall–Olkin distribution \[6, Section 3.1.1\]. The survival copula of this distribution is accordingly referred to as a Marshall–Olkin copula and can be represented as
\[
C(u, v) = \min\{u^{1-\alpha}v, uv^{1-\beta}\}, \quad 0 \leq \alpha, \beta \leq 1.
\] (1)

This copula plays a major role in our study, and we will refer to it repeatedly.

There are several dependence coefficients related to copulas. We will need the following ones:

1. **Spearman’s correlation coefficient** $\rho_S$ is defined as standard (Pearson’s) correlation coefficient of the random variables $U = F_X(X)$ and $V = F_Y(Y)$. Taking into account their uniform distribution on $[0,1]$, we have
\[
\rho_S = 12EUV - 3.
\]

2. **Kendall’s correlation coefficient** $\tau_K$ is defined as
\[
\tau_K = \mathbb{E} \text{sign}(X_1 - X_2)(Y_1 - Y_2),
\]
where $(X_1, Y_1)$ and $(X_2, Y_2)$ are independent random vectors distributed as $(X, Y)$.

3. **Upper tail dependence coefficient** $\lambda_U$ is defined as
\[
\lambda_U = \lim_{t \to 1^{-}} \mathcal{P}(X > F_X^{-1}(t) | Y > F_Y^{-1}(t)).
\]

In what follows, instead of $\rho_S$, $\tau_K$, and $\lambda_U$ we will write $\rho$, $\tau$, and $\lambda$.

All these coefficients can be uniquely expressed through copulas and do not depend on marginal distributions of the random variables:
\[
\rho = 12 \int_0^1 \int_0^1 C(u, v) \, du \, dv - 3, \quad \tau = 4 \int_0^1 \int_0^1 C(u, v) \, dC(u, v) - 1,
\]
\[
\lambda = 2 - \lim_{t \to 1^{-}} \frac{1 - C(t, t)}{1 - t}.
\]

In particular, for the Marshall–Olkin copula we have
\[
\rho = \frac{3\alpha\beta}{2\alpha - \alpha\beta + 2\beta}, \quad \tau = \frac{\alpha\beta}{\alpha - \alpha\beta + \beta}, \quad \lambda = \min\{\alpha, \beta\}. \quad (2)
\]

Extreme value copulas are copulas of multivariate extreme value distributions. If we take the componentwise maximum of several i.i.d. random variables with this distribution, its distribution will be of the same type (up to shift-scale transformations of the components). The same distributions appear as limiting ones in the maxima scheme of i.i.d. random vectors (under linear normalization). Respectively, in the minima scheme, survival copulas appear to be extreme value copulas.

A necessary and sufficient condition for a copula to be an extreme value copula is the identity
\[
C(u^s, v^s) = C^s(u, v), \quad \forall s > 0.
\]
Among the copulas mentioned above, the class of extreme value copulas comprises the Marshall–Olkin copula \( (\Pi) \) and the Gumbel copula
\[
C(u, v) = \exp \left\{ - \left( (-\ln u)^\theta + (-\ln v)^\theta \right)^{1/\theta} \right\}, \quad \theta \geq 1. \tag{3}
\]

For the Gumbel copula we have
\[
\tau = 1 - \frac{1}{\theta}, \quad \lambda = 2 - 2^{1/\theta}, \tag{4}
\]
and \( \rho \), unfortunately, cannot be expressed explicitly.

Extreme value copulas have Pickands’ representation:\footnote{Here we follow the definition from \cite[p. 98]{6}, but many other sources, for instance, \cite[p. 312]{4}, use a mirror symmetric representation (for our purposes, this is not significant):}
\[
C(u, v) = \exp \left\{ (\ln u + \ln v)A \left( \frac{\ln v}{\ln u + \ln v} \right) \right\}, \quad A(t) = -\ln C(e^{-(1-t)}, e^{-t}), \tag{5}
\]
where the \textit{dependence function} \( A(t) \) on \([0, 1]\) is convex and satisfies the inequality
\[
\max\{t, 1-t\} \leq A(t) \leq 1.
\]

For example, for the Marshall–Olkin copula, from \((1)\) and \((5)\) we obtain
\[
A(t) = 1 - \min\{\beta t, \alpha(1-t)\}. \tag{6}
\]

In the class of extreme value copulas, we have the following expressions for the dependence coefficients introduced above:
\[
\rho = 12 \int_0^1 \frac{dt}{(A(t) + 1)^2} - 3, \quad \tau = \int_0^1 t(1-t)dA'(t) / A(t), \quad \lambda = 2(1 - A(1/2)). \tag{7}
\]

Furthermore, for an extreme value copula, the upper tail dependence coefficient uniquely determines its behavior on the main diagonal, namely
\[
C(u, u) = u^{2-\lambda}, \quad 0 \leq u \leq 1.
\]

One of the most important and interesting problems in copula theory is establishing interrelations between various dependence coefficients. The classical domain of possible values of \( \rho \) and \( \tau \) is given by
\[
\frac{3\tau - 1}{2} \leq \rho \leq \frac{1 + 2\tau - \tau^2}{2}, \quad \tau \geq 0,
\]
\[
\frac{\tau^2 + 2\tau - 1}{2} \leq \rho \leq \frac{2 + 3\tau}{2}, \quad \tau \leq 0,
\]
whereas for extreme value copulas we have \( \rho, \tau \geq 0 \) and the Hutchinson–Lai inequality
\[
\sqrt{1 + 3\tau} - 1 \leq \rho \leq \min\{(3/2)\tau, 2\tau - \tau^2\}.
\]
holds true, which for long stood as a conjecture and has been proved in [2].

Modern refinements of these results can be found in recent papers [5, 9, 10]. So, in [5] a new sharp inequality for bivariate extreme value copulas is derived:

\[ \rho \geq \frac{3\tau}{2 + \tau}. \]

The coefficient \( \lambda \) was historically paid less attention. In [1, Section 3.2] there were found domains of possible values of \( \lambda \) and \( \tau \) for some families of extreme value copulas (t-EV, BB5, Tawn, Joe). Below we find tight bounds on \( \rho \) and \( \tau \) for a known \( \lambda \) on the whole class of extreme value copulas, but first we pointwise estimate the copulas themselves.

## 2 Main Results and Discussion

**Theorem 1.** For any extreme value copula with a known \( \lambda \in [0, 1] \) we have

\[ C(u, v) \geq \min\{u^{1-\lambda}v, uv^{1-\lambda}\} \] (8)

and there exist \( a, b \geq 0 \), \( a + b = \lambda \), such that

\[ C(u, v) \leq \min\{u, v, u^{1-a}v^{1-b}\}; \] (9)

moreover, these bounds are extreme value copulas with the same \( \lambda \).

**Proof.** By (5), the extreme value copula \( C \) decreases monotonically with \( A \), so it suffices to estimate the function \( A \) from above and below. Consider the plot given in Fig. 1.

The graph of \( A \) (bold line) passes through the points \( K(0; 1) \) and \( M(1; 1) \), and (7) implies that it also passes through \( L(1/2; 1-\lambda/2) \). By the convexity of the curve, no point of the graph lies above the broken line \( KLM \), and the equation of the latter coincides with formula (6) for the Marshall–Olkin copula with \( \alpha = \beta = \lambda \). This yields (8).

Next, by the convexity, the graph of \( A \) has at least one tangent line at \( L \) and does not lie below it at any point. This tangent can be parametrized by the equation

\[ s = (1-a)(1-t) + (1-b)t. \] (10)

By the convexity of \( A \), on the segment \([0, 1]\) this tangent cannot pass above the points \( K \) and \( L \), and therefore \( a, b \geq 0 \). Plugging the coordinates of \( L \) into (10), we obtain \( a + b = \lambda \).

Denote the intersection points of the straight line (10) with \( KN \) and \( OM \) by \( P \) and \( Q \) respectively. The equation of the broken line \( KPQM \), lower bounding the graph of \( A \), is of the form

\[ s = \max\{1-t, t, (1-a)(1-t) + (1-b)t\}, \]

which, taking into account (5), implies (9). \( \Box \)

Let us compare the situation with that studied in [7], see also [6, p. 184, Theorem 5.1.16], where upper and lower pointwise bounds for copulas with known values of \( \rho \) and \( \tau \) were found (without restrictions on a class of copulas). The obtained bounds are found to be copulas but do not have desired values of the coefficients. In our case the
lower pointwise bound belongs to the class of extreme value copulas with a given $\lambda$, but the upper one does not. Nevertheless, instead of the latter there exists a family of copulas

$$C(u, v) = \min\{u, v, u^{1-\nu}v^{1-b}\}, \quad a + b = \lambda, \quad a, b \geq 0,$$

forming a Pareto bound. This copulas are unimprovable in the sense that none of them can be increased at some point without decreasing at another one. This approach could also be useful in other cases where searching for pointwise bounds leads to estimates in the form of quasi-copulas which have no probabilistic sense [3].

**Lemma 1.** For copulas (11) we have

$$\rho = 1 - \frac{16(1 - \lambda)^2}{(4 - \lambda)^2 - 9(a - b)^2}$$

and

$$\tau = \lambda.$$  

**Proof.** Denote for brevity $\nu = a - b$; then

$$a = \frac{\lambda + \nu}{2}, \quad b = \frac{\lambda - \nu}{2}.$$
Again consider the graph plotted in Fig. 1. From the intersection of lines \( s = 1 - t \) and \( s = (1 - a)(1 - t) + (1 - b)t \) we find abscissae of the points \( P \) and \( Q \):

\[
t_P = \frac{a}{1 + \nu}, \quad t_Q = \frac{1 - a}{1 - \nu}.
\]

We have

\[
A(t) = \begin{cases} 
1 - t, & 0 \leq t < t_P, \\
(1 - a)(1 - t) + (1 - b)t, & t_P \leq t < t_Q, \\
t, & t_Q \leq t \leq 1.
\end{cases}
\] (14)

Compute the integral

\[
I = \int_0^1 \frac{dt}{A(t)^2} = \int_0^{t_P} \frac{dt}{(2 - t)^2} + \int_{t_P}^{t_Q} \frac{dt}{((1 - a)(1 - t) + (1 - b)t + 1)^2} + \int_{t_Q}^1 \frac{dt}{(t + 1)^2} = \frac{1 + ab - a^2 - b^2}{(2 + a - 2b)(2 + b - 2a)} = \frac{1}{3} \left(1 - \frac{4(1 - \lambda)^2}{(4 - \lambda)^2 - 9\nu^2}\right);
\]

this, together with \( \rho = 12I - 3 \), yields (12).

To compute \( \tau \), note that the derivative \( A'(t) \) is zero everywhere except for the points \( t_P \) and \( t_Q \), where it has jumps from \(-1\) to \( \nu \) and from \( \nu \) to \( 1 \) respectively. We get

\[
\tau = \int_0^1 \frac{t(1 - t)A'(t)}{A(t)} = \frac{t_P(1 - t_P)(1 + \nu)}{1 - t_P} + \frac{t_Q(1 - t_Q)(1 - \nu)}{t_Q} = a + b = \lambda.
\]

**Theorem 2.** For any extreme value copula with a known \( \lambda \in [0, 1] \), we have

\[
\frac{3\lambda}{4 - \lambda} \leq \rho \leq 1 - 16 \left(\frac{1 - \lambda}{4 - \lambda}\right)^2
\] (15)

and

\[
\frac{\lambda}{2 - \lambda} \leq \tau \leq \lambda
\] (16)

where the lower bounds are attained at Marshall–Olkin copulas with \( \alpha = \beta = \lambda \), and the upper ones, at copulas of the family (11) with \( a = b = \lambda/2 \).

**Proof.** First note that \( \rho \) and \( \tau \) are measures of concordance, which are monotonically nondecreasing with \( C \) \([6, \text{p. 169, Theorem 5.1.9}]\) (though this is not evident from formulas for \( \tau \), in contrast to \( \rho \)).

Thus, the lower bound for \( C \) in Theorem 1 yields lower bounds for \( \rho \) and \( \tau \) in the particular case of a Marshall–Olkin copula with \( \alpha = \beta = \lambda \) according to (2).

The upper bound for \( C \) in Theorem 1, taking into account Lemma 1, immediately gives an upper bound for \( \tau \). One can also observe that, according to (12), the coefficient \( \rho \) decreases in \( |a - b| \) and attains its maximum value when \( a = b = \lambda/2 \). Hence we get the upper bounds. \(\square\)
Figure 2: Bounds on $\rho$ for a known $\lambda$.

Figure 3: Bounds on $\tau$ for a known $\lambda$. 
Bounds of Theorem 2 are presented as solid lines in Figs. 2 and 3. To make the picture complete, by dashed lines we plot the coefficients in the case of the popular Gumbel copula (3), for which (4) gives

\[ \tau = 1 - \log_2(2 - \lambda), \]

and values of \( \rho \) were numerically evaluated by the author and are presented in the following table:

| \( \lambda \) | \( \theta \) | \( \rho \) |
|---|---|---|
| 0 | 1 | 0 |
| 0.1 | 1.080 | 0.110 |
| 0.2 | 1.179 | 0.225 |
| 0.3 | 1.306 | 0.342 |
| 0.4 | 1.475 | 0.461 |
| 0.5 | 1.710 | 0.581 |
| 0.6 | 2.060 | 0.699 |
| 0.7 | 2.641 | 0.808 |
| 0.8 | 3.802 | 0.904 |
| 0.9 | 7.273 | 0.973 |
| 1 | \( \infty \) | 1 |

Let us also mention Blomqvist’s coefficient, which can be defined as

\[ \beta_{X,Y} = \mathbb{E} \text{sign}(X - X_m)(Y - Y_m) \]

where \( X_m \) and \( Y_m \) are medians of \( X \) and \( Y \) respectively, and is expressed through a copula as

\[ \beta_C = 4C(1/2, 1/2) - 1. \]

By (5) and (7), for extreme value copulas this coefficient is uniquely related with the upper tail dependence coefficient:

\[ \beta_C = 2^\lambda - 1, \quad \lambda = \log_2(1 + \beta_C). \]

Thus, the results of Theorem 2 can easily be recalculated to the case of a known Blomqvist’s coefficient instead of the upper tail dependence coefficient.

Main results were briefly presented in [3].

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