Appropriateness of Performance Indices for Imbalanced Data Classification: An Analysis

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Abstract

Indices quantifying the performance of classifiers under class-imbalance, often suffer from distortions depending on the constitution of the test set or the class-specific classification accuracy, creating difficulties in assessing the merit of the classifier. We identify two fundamental conditions that a performance index must satisfy to be respectively resilient to altering number of testing instances from each class and the number of classes in the test set. In light of these conditions, under the effect of class imbalance, we theoretically analyze four indices commonly used for evaluating binary classifiers and five popular indices for multi-class classifiers. For indices violating any of the conditions, we also suggest remedial modification and normalization. We further investigate the capability of the indices to retain information about the classification performance over all the classes, even when the classifier exhibits extreme performance on some classes. Simulation studies are performed on high dimensional deep representations of subset of the ImageNet dataset using four state-of-the-art classifiers tailored for handling class imbalance. Finally, based on our theoretical findings and empirical evidence, we recommend the appropriate indices that should be used to evaluate the performance of classifiers in presence of class-imbalance.

Keywords: Imbalanced classification, Performance evaluation indices, Precision, Recall, GMean, Area under the curve

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1. Introduction

1.1. Overview

Classification is a fundamental supervised learning problem where the task is to develop classifiers (for example, $k$-Nearest Neighbor ($k$NN) [1], Multi-Layer Perceptron (MLP) [2], Support Vector Machine (SVM) [11] etc.) which can approximate a many-to-one mapping from a set $X$ of $d$-dimensional data points to a set $C = \{1, 2, \cdots, C\}$ of class labels. An allied challenge comes in the form of designing indices [3] which can accurately evaluate the performance of a classifier, considering the particular nature of the classification problem as well as the pertinent data irregularities [4].

Class imbalance [5, 6] is a form of data irregularity which is fairly common in many real-world classification problems [6, 7, 4] such as like medical diagnosis, fraud detection, etc. A training set $P \subseteq X$ is considered as class imbalanced when it does not contain equal number of training instances from all the classes (especially those corresponding to the rare and therefore important events). This leads the classifier to be biased in favor of the majority classes and consequently suffers from higher misclassification on the minority classes. Evidently, such bias should be properly compensated during performance evaluation. This indicates the need for special indices, unlike the widely used Accuracy measure which lays more stress on the performance over the majority classes, being unsuitable in presence of class imbalance [8].

Over the years, for a binary imbalanced classification problem indices like Recall, Specificity, and Precision [9] were considered to be the basic measures of performance. However, by design, Recall (or Sensitivity) measures the accuracy over the minority (positive) class, Specificity does the same for the majority (negative) class, and Precision [9] considers the fraction of positives which are accurately classified (true positive) to the number of instances predicted as positives. In other words, these three measures offer different criteria of evaluation by respectively focusing on true positive, true negative (analogous to true positive for the negative class), and false positive (negative instances wrongly classified as positive) counts, and a good classifier is expected to optimize all of them. However, optimizing multiple indices simultaneously is difficult in practice, especially if a trade-off is required. Therefore, attempts were made to combine two or more of these basic indices together to form new measures which can consider multiple distinct aspects during evaluation as well as provide easy interpretability. For example, the GMean [10] index is calculated by taking the geometric mean of Sensitivity and Specificity, while Area Under Receiver Operating Characteristics (AUROC) [11] measure is found by plotting Recall
against False Positive Rate (FPR). Similarly, the Precision and Recall can be combined to form the Area Under Recall Precision Curve (AURPC) \([12]\) index.

In case of the multi-class classification, a direct extension of the GMean index is available \([7]\). The multi-class analog of Recall is the Average Class-Specific Accuracy (ACSA) \([13]\). AUROC can be extended for multi-class classification problems by either the One Versus One (OVO) strategy to calculate AUROC-OVO \([11]\), or by the One Versus All (OVA) strategy to find AUROC-OVA \([14]\). Similarly, the multi-class version of AURPC is called AURPC-OVA \([14]\), as the extension warrants use of the OVA strategy. In the following Table 1 we briefly describe the indices which are analysed in detail in the subsequent sections of this article.

| Index         | Brief description                                                                 |
|---------------|-----------------------------------------------------------------------------------|
| GMean \([10]\) | Geometric mean of all the class-specific accuracies. Applicable to two-class as well as multi-class classification problems. |
| AUROC \([11]\) | Can be reduced to the arithmetic mean of the class-specific accuracies in a two-class classification problem.           |
| Precision \([9]\) | In a two-class classification problem it is defined as the fraction of true positives to the total number of instances which are classified as positives. |
| AURPC \([12]\) | Reduces to the arithmetic mean of accuracy over the positive class and Precision in a two-class classification problem. |
| ACSA \([13]\) | Arithmetic mean of the class-specific accuracies in a multi-class classification problem.                                |
| AUROC-OVA \([14]\) | Direct extension of AUROC for multi-class classification using OVA strategy.                                           |
| AUROC-OVO \([11]\) | Direct extension of AUROC for multi-class classification using OVO strategy.                                             |
| AURPC-OVA \([14]\) | Direct extension of AURPC for multi-class classification using OVA strategy.                                             |

1.2. Background

The growing number of classification performance measures inspired the research community to investigate their uniqueness, compare their applicability to class imbalanced problems in general, and evaluate their suitability for specific applications. Studies like \([15, 16, 17]\) attempted to empirically find the inter-relation between indices, while observing their behaviour under different scenarios. However, these empirical analyses are dependent on the choice of classifiers as well as the datasets, therefore failing to provide general conclusions. In contrast, a theoretical approach is taken in \([18, 19]\) to model the change of an index value with respect to varying class imbalance. However, these preliminary approaches, in addition to being complicated did not consider the possible disparity between the training and test sets. A simpler, more structured framework was proposed by Sokolava and Lapalme \([20, 21]\) which was later extended by Brzezinski et al. \([22]\). They formalized a set of transformation conditions on the confusion
matrix \[10\] to imitate changes in classification performance as well as alterations in the test set. An index is called invariant (or considered unaffected) by a certain transformation if its value does not change despite the transformation. Luque \textit{et al.} \[23\] took a different direction by building upon the measure of class imbalance proposed in \[24\] and defining a set of indicators to theoretically analyze the bias of various indices in binary classification problems. Recently, the work done by Brzezinski \textit{et al.} \[22\] was further extended in \[25\] for imbalanced streaming data classification \[26, 27\]. The authors attempted to properly interpret the value returned by an index especially under the effect of the dynamically changing class priors between the training and test sets, which is fairly common for streaming data. However, these works mostly focused on the application-specific suitability of an index. This limits them from discussing on a set of necessary conditions, violation of which may deem the index as undesirable for general use, along with offering any remedial modifications to impose invariance. Moreover, they considered the key dataset properties such as the number of classes as constant. This restricts them from addressing the pivotal role that the altering number of classes may play in distorting an index, a situation common to open set classification problems \[28\].

1.3. Motivation

In an attempt to rectify the shortcomings of the existing literature (as discussed in Section 1.2), we carry out a systematic theoretical study on the desirable properties of the indices. The presented properties are fundamental in the sense that they ensure the invariance of an index against the following
two types of undesirable distortions:

**Type 1 distortion:** The fraction of representatives from a class in the test set $Q \subseteq X$ (containing $n$ data instances) may not always be similar to that of $P$. However, under such conditions, the value returned by some performance indices (such as Precision [29]) tends to vary with changes in the number of test points from the different classes. An example of this type of distortion is illustrated in Figure 1a where the mapping within the range of the index (which remains unchanged) becomes increasingly warped. The change in the fraction of representatives between training and test (or validation) sets may happen in a real-world classification problem due to a couple of reasons. Firstly, concept drift [26, 27] can result in continuous alterations of class priors (and consequently the degree of imbalance) over time. Such drifting is fairly common in imbalanced streaming data classification problems [25], resulting in different extents of class imbalance in training, validation, and test sets. Secondly, prior probability shift between training, validation and test sets also occurs in current large scale benchmark datasets such as LSUN [30] and ImageNet [31], where the the class priors in the training set are not retained in the predefined validation and test sets. It is important to note that remedial measures like stratified cross-validation [32] cannot be efficiently applied in both of these situations.

If an index suffering from the above-mentioned distortion is used during validation, then the classifier will be improperly evaluated and consequently miscalibrated. Further, in streaming data classification, an application may require the classifier to be tested at regular intervals, so that the classifier parameters can be periodically fine-tuned according the latest performance. Here also, use of an index which is susceptible to this first type of distortion may lead to inappropriate judgment about the quality of the classifier and mislead the periodic retraining procedure.

**Type 2 distortion:** The range of possible values to be returned by some performance indices (for example, AUROC-OVO as shown in Theorem 3) gets diminished with increase in the number of classes in the data. Thus, an index affected by such a distortion may fail to identify the better classifier with decreasing confidence as the number of classes increases, even when the contenders are of diverse quality. In the worst case, on a very large number of classes, due to rounding error a set of classifiers may end up being evaluated as similar, all providing commendable performance instead of reflecting their actual quality. An example of this distortion is also illustrated in Figure 1b where the lower bound of the index gradually increases.
Example 1. To better illustrate the two types of distortions we present an example in Figure 2. For Type 1 distortion, we take a two class dataset where each class is drawn from a normal distribution. To quantize the level of class imbalance in the test set, we use a measure called Ratio of Representatives in the Test set (RRT), which for a two-class classification problem can be expressed as \( \frac{n_{maj}}{n_{min}} \), where \( n_{maj} \) and \( n_{min} \) are respectively the number of test points from the majority and the minority class (see Definition 2). A well behaved test set as shown in Figure 2a, is created by sampling 5000 points from each class. Progressively worse behaved test sets are formed by varying the RRT between 2, 4, 6, 8, and 10. Now as shown in Figures 2a and 2b, let us shift a linear classifier from the best possible position (which accurately separates the two classes) to the worst (which only perfectly classifies the majority). We measure the Precision of the different classifiers on the varying test sets and plot them in Figure 2c. We can observe that even though the classifier maintains its quality by remaining at a fixed position, the Precision decreases with increasing RRT (for example, at the class boundary \( x = 4 \) the Precision decreases).

Footnote 1: The construction of all the datasets used in this example is detailed in the supplementary document.
deteriorates from 0.93 to 0.59 when RRT is altered from 1 to 10). The change in Precision is high when the classifier is of poor quality, while the gap progressively closes down with improving performance.

In case of Type 2 distortion we start with a three class dataset as in Figure 2d where the classes are sampled from normal distributions centered at the three adjacent vertices of a regular hexagon. We gradually increase the number of classes to six in Figure 2e by similarly sampling on the rest of the three vertices in an anticlockwise order. In each case we start with an OVA ensemble of linear classifiers which performs as worse as an uniformly random assignment and gradually move towards the best which achieves perfect accuracy. As previous we calculate the AUROC-OVA of the different classifiers on the various datasets and plot them in Figure 2f. We can observe that even when a classifier is accurately classifying the same fraction of points from each class the AUROC-OVA index gradually returns a higher value with the increasing number of classes (for example, when the classifier correctly predicts 60% points from each class on average, the AUROC-OVA reaches from 0.70 to 0.77 with C altering from 3 to 6).

Evidently, an evaluation index may suffer from either or both of these distortions with the change in the properties of the dataset (such as number of classes, size of the test set, and extent of class imbalance) even if the classifier retains a consistent performance. Consequently, such types of distortions primarily complicate the interpretation of a value returned by an index. For example, the classifier with moderate performance in Figure 1 can either be assigned a higher index value or a lower index value (compared to the ideal) depending on the nature of the distortion suffered by the performance indices. Hence, the actual index values cannot be used to properly assess the merit of a classifier. Another issue arises when the difference between the values yielded by a bad classifier and a good one gets diminished to the extent of being ignored due to the rounding of values in practical experiments.

Even when an index is found to be unaffected by the two types of distortions it may still provide a value from which adequate information about the performance of a classifier over all the classes is difficult to extract. This usually occurs in multi-class imbalanced classification problems, where high misclassification in a single class (irrespective of the classifier’s performance over the other classes) results in a severely deteriorated index value.

1.4. Contribution

In this study, we identify two necessary conditions (described in Section 2) that an index must satisfy to be considered ideal for evaluating the performance of classifiers on imbalanced datasets. Contrary
Table 2: Summary of contributions made in this article in comparison to existing literature (no references are provided for original contributions).

| Topics | Reference | Our contribution |
|--------|-----------|------------------|
| Condition 1 | [21] | Established as a necessary safeguard against Type 1 distortion. |
| Invariance of Recall, Precision, and AUROC to Condition 1 | [21] | Re-validated using a mathematical framework. |
| Invariance of GMean, ACSA, AUROC-OVA, AUROC-OVO, and AURPC-OVA to Condition 1 | - | Validated using a mathematical framework. |
| Remedial modification of Precision, AURPC, and AURPC-OVA to satisfy Condition 1 | [29] | Validated using a mathematical framework. Established as an effective replacement to the original ones, which satisfy Condition 1. |
| Remedial normalization of AUROC-OVA to satisfy Condition 1 | - | Proposed and validated using a mathematical framework. |
| Condition 2 | - | Proposed as a necessary safeguard against Type 2 distortion. |
| Invariance of Recall, Precision, AUROC, GMean, ACSA, AUROC-OVO, AUROC-OVA, and AURPC-OVA to Condition 2 | - | Validated using a mathematical framework. |
| Remedial normalization of AUROC-OVO, and AUROC-OVA to satisfy Condition 2 | - | Proposed by us and validated using a mathematical framework. |
| Condition 3 | - | Proposed to validate the quality of the information returned by an index which satisfies Condition 1 and 2. |
| Invariance of GMean, ACSA, and modified AURPC-OVA to Condition 3 | - | Validated using a mathematical framework. |

to the prior work, we do not focus on application specific suitability of any index, and aim to propose a set of constraints which will evaluate an index from a more generalized perspective. To elaborate we look at the nature of changes in the data itself which can affect an index. Evidently, it is expected that an index should remain invariant to any changes in the training/test set if the classifier performs uniformly. Therefore, ensuring such can be considered as a fundamental requirement over all other types of secondary consistency checks. In essence, under the assumption that the classifier sustains its performance over each of the classes, we formulate two transformation conditions on the confusion matrix. Invariance to both of these transformations will ensure immunity of an index against the two types of distortions. In Table 2 we put our contributions in proper context with the existing works. We further summarize the contributions as follows:

1. The first condition guards against the Type 1 distortion by ensuring invariance of an index with alterations of the size and sample distributions among the different classes in the test test. This condition was first introduced by Sokolova and Lapalme as properties $I_6$ and $I_8$ in [21] (where the former is a special case of the latter). However, they were not motivated to evaluate the effects of distortions over the indices. Therefore, their analysis did not elaborate on the implication of
the properties or identify them as fundamental constraints. In this article we bridge this gap by establishing this condition as a necessary measure against the Type 1 distortion. Moreover, in Theorems 1 and 3 under the light of the first condition we analytically discuss the properties of GMean, ACSA, AUROC-OVO, AUROCC-OVA, and AURPC-OVA, none of which were covered in the previous studies.

2. We propose the second condition which deals with the Type 2 distortion assuring invariance to varying number of classes in the test set, as shown in Theorem 3.

3. We show in Theorem 2 that contrary to the regular Precision and AURPC indices, the modifications proposed in [29] are indeed capable of inducing invariance under the first condition. We further propose the normalized variants of AUROC-OVO (which essentially reduces to ACSA) and AUROC-OVA, which offer immunity against the effects of the two types of distortions.

4. We also propose the third condition to ensure that in a multi-class classification problem, an index which fulfills the two fundamental desirable properties are also capable to provide sufficient information about the classifier’s performance over all the classes, even when a single class suffers extremely high misclassification. We show in Theorem 5 that except GMean, both ACSA and AURPC-OVA offer invariance under the third condition.

In this paper, we also present an empirical analysis on some selected subsets of ImageNet [31] in Section 5 to experimentally validate our theoretical findings and effectiveness of the prescribed remedies. Finally, in Section 6 we present a discussion on the applicability of different indices in an imbalanced classification tasks, and make recommendations as per situation, and subsequently conclude in Section 7.

2. Desirable Properties for Performance Indices

Various performance evaluating indices depend on the diverse properties (such as imbalance, number of classes etc.) of the training and/or test set to different extents, resulting in improper/ambiguous evaluation of a classifier. This issue can be resolved by defining a set of necessary but not sufficient conditions to ensure the quality of the evaluation by an index. We start with the definition of measures to quantify the extent of class imbalance in the training and test sets.

Definition 1 (Datta and Das [33]). For a 2-class classification problem the Imbalance Ratio (IR) is defined as the ratio of the number of points in the majority class to that of the minority class in the
training set. Analogously, for a \( C \)-class classification problem \( IR \) is calculated as the maximum among all the pairwise \( IRs \) (represented by the set \( I = \{ \frac{p_i}{p_j} | i, j \in C; i \neq j \} \), where \( p_i \) is the number of training points from the \( i^{th} \) class) among the \( C \) classes (i.e. \( IR = \max I \)). Therefore, \( P \) can be considered as imbalanced if \( IR > 1 \).

However, a classifier is trained on a single training set \( P \) with a fixed predefined \( I \), whereas all possible test sets might not follow a distribution of the representatives among the classes (i.e. \( I \)) similar to \( P \). Therefore, analogous to \( IR \) we define \( RRT \) for quantifying the ratio of representatives among the classes in the test set.

**Definition 2.** For a 2-class classification problem, the \( RRT \) is defined as the proportion of the number of points from the majority class to that of the minority class, where the majority and the minority classes are named according to the training set. This definition can be extended for the \( C \)-class classification case in a manner similar to \( IR \), where the set of a pairwise ratio of the number of the data instances among the \( C \) classes is denoted by \( T \), i.e. \( RRT = \max T \).

Performance of the classifier on a \( C \)-class classification problem can be expressed in the form of a matrix called the *confusion matrix*, which is defined as follows:

**Definition 3 (Kubat et al. [10]).** A confusion matrix over a test set \( Q \) for a \( C \)-class classification problem can be defined as \( M_C = [m_{ij}]_{C \times C} \), where \( m_{ij} \) represents the number of points which actually belongs to \( i^{th} \) class but are predicted as a member of class \( j \), for all \( i, j \in C \). Thus, the diagonal elements i.e. \( m_{ii} \) are those instances of class \( i \) which are correctly classified while the rest are different misclassifications. Evidently, each entry in the confusion matrix must be a non-negative integer i.e. \( m_{ij} \in \mathbb{Z}^+ \cup \{0\}; \forall i, j \in C \)

There are some important properties of the confusion matrix which we detail in the following discussion.

**Property 1:** The sum of entries in the \( i^{th} \) row of the confusion matrix is denoted by \( n_i \) (i.e. \( \sum_{j=1}^{C} m_{ij} = n_i; \forall i \in C \)), which is the number of test points belonging to the \( i^{th} \) class. We assume that \( n_i > 0 \), as there should be at least one point from each class in the test set.

**Property 2:** The sum of entries in the \( i^{th} \) column of the confusion matrix is denoted by \( k_i \) (i.e. \( \sum_{j=1}^{C} m_{ji} = k_i; \forall i \in C \)), which is the number of test points predicted as \( i^{th} \) class.

**Property 3:** The total number of test points \( n = \sum_{i=1}^{C} \sum_{j=1}^{C} m_{ij}; \forall i, j \in C \).
**Property 4**: In case of two-class classification, the entries of $M_2$ are specially named, as True Positive (TP), False Positive (FP), True Negative (TN), and False Negative (FN), when the test instances from the majority and the minority classes are respectively labelled as -1 (negative) and +1 (positive). Therefore, $M_2$ can be formally represented as:

$$M_2 = \begin{bmatrix} TP & FN \\ FP & TN \end{bmatrix}.$$  \hfill (1)

Before proceeding further we need to describe our primary assumption based on which the following theory will be built.

**Assumption 1.** The class-specific performance (the fraction of correct classification as well as the proportion of misclassification to each of the other classes) of a classifier remains the same over any random subset of the dataset.

The class-specific performance of a classifier can be considered as an equivalence relation, which can partition the set $M_C$ containing all possible $C$-class confusion matrices into some equivalence classes. In any of these equivalence classes, the class-specific performance of a classifier remains constant over all the classes. To elaborate, given two confusion matrices say $M_C$ and $M'_C$ if the equivalence relation $m_{ij}/n_i = m'_{ij}/n'_i$, satisfies for all $i, j \in C$, then they can be considered as members of the same equivalence class, i.e. $M_C \sim M'_C$. In other words the equivalence property essentially corresponds to constant performance by a classifier or formally represents Assumption 1. Using the notion of the confusion matrix, we can now formally define a performance evaluation index as a function $f$ mapping from the set of all possible confusion matrices $M_C$ to a real scalar quantity. Such a representation is important as it helps us define some functionals to formally describe our proposed conditions, in the following manner.

**Condition 1.** The value of an index should not be dependent on RRT, if the classifier performs equivalently, i.e.

$$V_{M_C}(f) = V_{M'_C}(f); \forall M_C \sim M'_C,$$

where $V_M$ is a functional evaluating the index $f$ on the confusion matrix $M$.

\[2\] Evidently, $M'_C = [m'_{ij}]_{C \times C} \colon \sum_{i=1}^{C} \sum_{j=1}^{C} m'_{ij} = n'; \sum_{j=1}^{C} m'_{ij} = n'_i; \text{ and } \sum_{j=1}^{C} m'_{ji} = k'_i$ for all $i \in C$. 

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As an extension of the work by Sokolova and Lapalme [21], in this article we propose Condition 1 as a necessary measure against the Type 1 distortion, violation of which may alter the value of an index with the changes of RRT in the test set even when the classifier remains the same.

**Condition 2.** The lower and the upper bounds of an index $f$ should not be dependent on the number of classes, i.e.

\[
\mathcal{L}_{M_C}(f) = \mathcal{L}_{M_{C+1}}(f) \forall C \in \mathbb{Z}^+ \setminus \{1\},
\]

and

\[
\mathcal{U}_{M_C}(f) = \mathcal{U}_{M_{C+1}}(f) \forall C \in \mathbb{Z}^+ \setminus \{1\},
\]

where, $M_C$ and $M_{C+1}$ are respectively the sets of all possible $C$-class and $(C+1)$-class confusion matrices. Moreover, $\mathcal{L}$ and $\mathcal{U}$, are two functionals of $f$, respectively calculating the minimum and maximum value of $f$ over all $M_C$ and $M_{C+1}$.

Condition 2 ensures that under the assumption of a consistent performance by a classifier, the value of an index should not be biased to differing number of classes in the test set.

If we consider a $C$-class confusion matrix, where $m_{ii}/n_i = \epsilon$, only for the $i^{th}$ class ($i \in C$) while $m_{jj}/n_j \geq (1 - \epsilon)$ for all the other classes ($j \in C \setminus \{i\}$), and $\epsilon = \frac{1}{C}$, then all such matrices form a set $\mathcal{M}_C(i) \subset M_C$. In other words, $\mathcal{M}_C(i)$, is the set of all such $C$-class confusion matrices where the classifier performed extremely poor only on the $i^{th}$ class.

**Condition 3.** An index $f$ while evaluating a multi-class classifier should not be biased towards the misclassification of a single class, i.e. $\mathcal{W}_{\mathcal{M}_C(i)}(f) > \mathcal{L}_{M_C}(f)$, where $\mathcal{W}$ is a functional which calculates the limit of $f$, as $\epsilon \to 0^+$. In other words, Condition 3 ensures that the value returned by the index will not excessively degrade if a single class is almost entirely misclassified in a multi-class classification problem. Violating Condition 3 will lead the index to lose information about the classifier’s performance on all the other classes. Evidently, index failing to satisfy Condition 3 will be unable to distinguish between two classifiers, one of which achieves good class-specific accuracies on all but one class, while the other achieves high misclassification on all.

Before proceeding further, we list down the various notations which will be used throughout the rest of this article in Table 3.
3. Analysis of the two-class performance evaluation indices

In this section, we analyze the characteristics of four indices, namely GMean, AUROC, Precision, and AURPC which are used to evaluate the performance of a classifier in presence of class imbalance for a two-class classification problem. We only require to validate if the indices satisfy Condition 1 as the other is only applicable for multi-class classification.

**Definition 4.** For a two-class classification problem, given a confusion matrix $M_2$ as in (1),

1. The GMean index, denoted by $\gamma_2$ is defined as:

$$\gamma_2(M_2) = \left( \frac{TP}{TP + FN} \right) \left( \frac{TN}{FP + TN} \right)^{\frac{1}{2}}. \quad (2)$$

2. The AUROC index, denoted by $\rho$ is defined as:

$$\rho(M_2) = \frac{1}{2} \left( \frac{TP}{TP + FN} + \frac{TN}{FP + TN} \right). \quad (3)$$

3. The Precision index, denoted by $\zeta$ is defined as:

$$\zeta(M_2) = \frac{TP}{TP + FP}. \quad (4)$$

4. The AURPC index, denoted by $\kappa$ is defined as:

$$\kappa(M_2) = \frac{\beta(M_2) + \zeta(M_2)}{2}, \quad (5)$$

For mathematical simplicity we have restricted ourselves to the discrete version of the index, which is popularly used in practice.
where, \( \beta(M_2) = \frac{TP}{TP + FN} \) is the Recall.

Evidently, the formal definitions of the indices do correspond to their pedagogical description in Table 1. We may now proceed to analyzing the behavior of the indices under the light of Condition 1 in the following theorem.

**Theorem 1.** For a two-class classification problem, given two confusion matrices \( M_2 \) and \( M'_2 \), the following statements can shown to be true if \( M_2 \sim M'_2 \),

1. The index \( \gamma_2 \) satisfies Condition 1.
2. The index \( \rho \) satisfies Condition 1.
3. The index \( \zeta \) does not satisfy Condition 1.
4. The index \( \kappa \) does not satisfy Condition 1.

**Proof.** Let us define \( M_2 \), as in (1), while \( M'_2 \) can be constructed as,

\[
M'_2 = \begin{bmatrix}
b_1TP & b_1FN \\
b_2FP & b_2TN
\end{bmatrix},
\]

where \( b_1, b_2 \in \mathbb{R}^+, b_1 \neq b_2 \) and \( m'_{ij} \in \mathbb{Z}^+, \forall i, j \in \{1, 2\} \). Such a form of \( M'_2 \), will ensure that \( m_{ij}/n_i = m'_{ij}/n'_i \), is satisfied for all \( i, j \in \{1, 2\} \), or \( M_2 \sim M'_2 \). With this initial setup we start the proof of the first statement by finding the value of \( V_{M'_2}(\gamma_2) \), following (2):

\[
V_{M'_2}(\gamma_2) = \left( \frac{b_1TP}{b_1TP + b_1FN} \right)^{\frac{1}{2}} \left( \frac{b_2TN}{b_2FP + b_2TN} \right)^{\frac{1}{2}},
\]

\[
= \left( \frac{TP}{TP + FN} \right)^{\frac{1}{2}} \left( \frac{TN}{FP + TN} \right)^{\frac{1}{2}} = V_{M_2}(\gamma_2).
\]

Therefore, the \( \gamma_2 \) index satisfies Condition 1.

The second statement can be proved in a similar manner by starting from \( V_{M'_2}(\rho) \) using (3),

\[
V_{M'_2}(\rho) = \frac{1}{2} \left( \frac{b_1TP}{b_1(TP + FN)} + \frac{b_2TN}{b_2(FP + TN)} \right),
\]

\[
= \frac{1}{2} \left( \frac{TP}{TP + FN} + \frac{TN}{FP + TN} \right) = V_{M_2}(\rho).
\]

This proves the second statement.
Similarly, we show the third statement to be true by calculating $V_{M_2}(\zeta)$ as per (4),

$$V_{M_2}(\zeta) = \frac{b_1TP}{b_1TP + b_2FP}.$$  

(6)

From (6) it is evident that $V_{M_2}(\zeta)$ can be equal to $V_{M_2}(\zeta)$, only when $b_1 = b_2$, which implies that $\zeta$ violates Condition 1.

Finally, we prove the fourth statement by finding the value of $V_{M_2}(\kappa)$ according to (5),

$$V_{M_2}(\kappa) = \frac{TP}{TP + FN} + \frac{b_1TP}{b_1(TP) + b_2(FP)}.$$  

(7)

Therefore, from (7), we can conclude in a manner similar to (6) that $V_{M_2} = V_{M_2}$ only holds when $b_1 = b_2$, thus completing the proof.

From Theorem 1 we can see that while GMean and AUROC indices satisfy Condition 1, the Precision and AURPC indices do not, thus being susceptible to RRT. In other words, both of Precision and AURPC may evaluate a good classifier as a poor choice, with the increase in RRT, even when the class-specific performances are retained. This is due to the increasing number of test points from the majority class which considerably increases FP. This was first observed by Bradley et al. [29], who proposed a solution by incorporating the class priors in the definition of Precision. In their modified definition of Precision (and consequently AURPC) the direct use of FP is replaced with the ratio of false positives to the number of majority instances. The modified Precision and AURPC, respectively called mPrecision and mAURPC, are described in the following definition.

**Definition 5 (Bradley et al. [29]).** For a two-class classification problem, given a confusion matrix $M_2$ as in (1), where $TP + FN = n_1$ and $FP + TN = n_2$,

1. The mPrecision index, denoted by $\hat{\zeta}$ is defined as:

$$\hat{\zeta}(M_2) = \frac{TP/n_1}{(TP/n_1) + (FP/n_2)}.$$  

(8)
2. The mAURPC index, denoted by $\hat{\kappa}$ is defined as:

$$\hat{\kappa}(M_2) = \frac{1}{2}(\beta(M_2) + \hat{\zeta}(M_2)).$$  \hfill (9)

**Theorem 2.** For a two-class classification problem, given two confusion matrices $M_2$, and $M'_2$, the following statements can shown to be true, if $M_2 \sim M'_2$,

1. The index $\hat{\zeta}$ satisfies Condition 1.
2. The index $\hat{\kappa}$ satisfies Condition 1.

**Proof.** Given $M_2$, as in (1), we construct $M'_2$, such that $M_2 \sim M'_2$, in a manner similar to Theorem 1. Then to prove the first statement we proceed by calculating $\mathcal{V}_{M'_2}(\hat{\zeta})$ using (8).

$$\mathcal{V}_{M'_2}(\hat{\zeta}) = \frac{b_1TP/b_1n_1}{b_1TP/b_1n_1 + b_2FP/b_2n_2},$$

$$= \frac{TP/n_1}{(TP/n_1) + (FP/n_2)} = \mathcal{V}_{M_2}(\hat{\zeta}).$$

This completes the proof of first statement.

We prove the second statement by finding $\mathcal{V}_{M'_2}(\hat{\kappa})$, which from (8), and (9) can also be written as

$$\mathcal{V}_{M'_2}(\hat{\kappa}) = \frac{b_1TP}{2b_1n_1} + \frac{b_1TP/b_1n_1}{2b_1TP/b_1n_1 + 2b_2FP/b_2n_2},$$

$$= \frac{TP}{2n_1} + \frac{TP/n_1}{(TP/2n_1) + (FP/2n_2)} = \mathcal{V}_{M_2}(\hat{\kappa}).$$

Thus, the second statement is proved, completing the proof of this Theorem. \qed

From Theorem 2 we can conclude that the proposed modification of Precision and AURPC can improve their immunity over RRT by satisfying Condition 1 and in the process will be able to better evaluate a classifier.

4. Analysis of the multi-class performance evaluation indices

In this section, we will define five multi-class evaluation indices, namely GMean, ACSA, AUROC-OVO, AUROC-OVA, and AURPC-OVA. Similar to the two-class indices we present an analysis in the perspective of the first two conditions and prescribe modification/normalization as per requirement.
Definition 6. Given a \( C \)-class confusion matrix \( M_C \) as defined in Definition 3, \( C \in \mathbb{N} \):

1. The GMean index, denoted by \( \gamma_C \) is defined as:
\[
\gamma_C(M_C) = \left( \prod_{i=1}^{C} \frac{m_{ii}}{n_i} \right) ^{\frac{1}{C}}.
\] (10)

2. The ACSA index, denoted by \( \alpha \) is defined as:
\[
\alpha(M_C) = \frac{1}{C} \sum_{i=1}^{C} \frac{m_{ii}}{n_i}.
\] (11)

3. The AUROC-OVO index, denoted by \( \rho_o \) is defined as:
\[
\rho_o(M_C) = \frac{1}{2C} \sum_{i=1}^{C} \left( 1 + \frac{m_{ii}}{n_i} - \frac{m_{ii}}{(C-1)n_j} \right).
\] (12)

4. The AUROC-OVA index, denoted by \( \rho_a \) is defined as:
\[
\rho_a(M_C) = \frac{1}{2C} \sum_{i=1}^{C} \left( 1 + \frac{m_{ii}}{n_i} - \frac{k_i - m_{ii}}{n - n_i} \right).
\] (13)

5. The AURPC-OVA index, denoted by \( \kappa_a \) is defined as:
\[
\kappa_a(M_C) = \frac{1}{2C} \sum_{i=1}^{C} \left( \frac{m_{ii}}{k_i} + \frac{m_{ii}}{n_i} \right).
\] (14)

Similar to the two-class case, here also the indices reflect their informal description from Table 1 to their mathematical definition. Before proceeding further, we need to first prove three supporting lemmas, which respectively comment on the range of ACSA index, and highlights the key properties of AUROC-OVO and AUROC-OVA.

Lemma 1. In a \( C \)-class classification problem the value of index \( \alpha \) lies between 0, and 1.

Proof. According to Definition 3, in a \( C \)-class confusion matrix \( 0 \leq m_{ii}/n_i \leq 1 \), for all \( i \in \mathcal{C} \). Using this and the definition of \( \alpha \) in (11), we can conclude that \( 0 \leq \alpha \leq 1 \). Specifically, \( \alpha = 0 \), when the classifier wrongly classified every test point i.e. \( m_{ii} = 0 \), for all \( i \in \mathcal{C} \), and \( \alpha = 1 \), if the classifier correctly predicts the class label for each member of the test set. 

\[\square\]
Lemma 2. The value $\rho_o(M_C)$ can be expressed as a linear function of $\alpha(M_C)$, with a constant coefficient and bias both of which are dependent on $C$, as follows:

$$\rho_o(M_C) = \frac{C}{2(C-1)} \alpha(M_C) + \frac{C - 2}{2(C - 1)}. \quad (15)$$

Moreover, the lower bound of $\rho_o(M_C)$ can be expressed as $L_{M_C}^{\rho_o}(\rho_o) = \frac{C - 2}{2(C - 1)}$, while the upper bound $U_{M_C}^{\rho_o}(\rho_o) = 1$.

Proof. We first start with a $M_C \in M_C$, then following from (12) after some algebraic manipulation we express $\rho_o(M_C)$ as:

$$\rho_o(M_C) = \frac{1}{2} + \frac{1}{2C} \sum_{i=1}^{C} \frac{m_{ii}}{n_i} - \frac{1}{2C(C - 1)} \sum_{i=1}^{C} \sum_{j \in C \setminus \{i\}} \frac{m_{ji}}{n_j},$$

$$= \frac{1}{2} + \frac{1}{2C} \sum_{i=1}^{C} \frac{m_{ii}}{n_i} - \frac{1}{2C(C - 1)} \sum_{i=1}^{C} \frac{n_i - m_{ii}}{n_i},$$

$$= \frac{1}{2} + \frac{1}{2C} \sum_{i=1}^{C} \frac{m_{ii}}{n_i} - \frac{1}{2C(C - 1)} \left(1 - \frac{1}{C} \sum_{i=1}^{C} \frac{m_{ii}}{n_i}\right),$$

$$= \frac{1}{2} + \frac{\alpha(M_C)}{2} - \frac{1}{2C(C - 1)} \left(1 - \alpha(M_C)\right),$$

$$= \frac{C}{2(C - 1)} \alpha(M_C) + \frac{C - 2}{2(C - 1)}. \quad (16)$$

Interestingly, from (16) we can conclude that for a given $C$, the index $\rho_o(M_C)$ can be expressed as a linear function of $\alpha(M_C)$, with a constant coefficient and a bias. Now $M_C$ is an arbitrary matrix belonging to the set $M_C$. Therefore, we can safely extend (16) to:

$$L_{M_C}^{\rho_o}(\rho_o) = \frac{C}{2(C - 1)} L_{M_C}^{\rho_o}(\alpha) + \frac{C - 2}{2(C - 1)}, \quad (17)$$

and,

$$U_{M_C}^{\rho_o}(\rho_o) = \frac{C}{2(C - 1)} U_{M_C}^{\rho_o}(\alpha) + \frac{C - 2}{2(C - 1)}. \quad (18)$$

Plugging the values of $L_{M_C}^{\rho_o}(\alpha), U_{M_C}^{\rho_o}(\alpha)$ from Lemma 1 respectively in (17), and (18) we obtain.

$$L_{M_C}^{\rho_o}(\rho_o) = \frac{C - 2}{2(C - 1)} \text{ and } U_{M_C}^{\rho_o}(\rho_o) = 1,$$

which completes the proof.
Lemma 3. If we assume for simplicity, without loss of generality that $n_1 \leq n_2 \cdots \leq n_C$, then the lower bound of $\rho_a(M_C)$ can be expressed as

$$L_{M_C}(\rho_a) = \frac{1}{2C} \left( C - 1 - \frac{n_C}{n - n_{C-1}} \right)$$

while the upper bound $U_{M_C}(\rho_a) = 1$.

Proof. If the classifier misclassifies all of the test points then $m_{ii} = 0, \forall i \in C$. However, as evident from (13) the value of $L_{M_C}(\rho_a)$ is also dependent on the actual predictions as the cost of misclassification to all the classes are not equal. To elaborate, we take a $C$-class confusion matrix $M_C$, and construct $M'_C$, such that $M_C + \Delta = M'_C$, where, $\Delta = [\delta_{ij}]_{C \times C}$, $\sum_{i=1}^{C} \delta_{ji} = 0$, $m'_{ij} = m_{ij} + \delta_{ij} \geq 0$, and $\sum_{i=1}^{C} \delta_{ij} = \bar{k}_i \forall i, j \in C$, while conserving $n_i, \forall i \in C$, and $n$. Now, $\rho_a(M_C) - \rho_a(M'_C)$ can be calculated as

$$\rho_a(M'_C) - \rho_a(M_C) = \frac{1}{2C} \sum_{i=1}^{C} \delta_{ii} - \frac{1}{2C} \sum_{i=1}^{C} \bar{k}_i - \delta_{ii}.$$  

(20)

Now for simplicity without loss of generality if we assume that $n_1 \leq n_2 \cdots \leq n_C$, then for any $i > j, \forall i, j \in C$, from (20) we can conclude that, increase in $\bar{k}_i - \delta_{ii}$ (i.e. the misclassification to other classes) will have larger effect on the value of $\rho_a(M'_C)$ than $\bar{k}_j - \delta_{jj}$. In other words, the cost of misclassifications is higher for the majority class. Hence the value of $\rho_a(M_C)$ will be minimum when all the points from classes other than $C$ are wrongly predicted as class $C$, while the points from class $C$ are misclassified as $C - 1$. Hence, from (13) we get,

$$L_{M_C}(\rho_a) = \frac{1}{2C} \left( C - 1 - \frac{n_C}{n - n_{C-1}} \right).$$

(21)

If the classifier correctly classifies all the test points then $k_i - m_{ii} = 0$, while $m_{ii}/n_i = 1, \forall i \in C$. Plugging these values in (13) gives $U_{M_C}(\rho_a) = 1$, thus finishing the proof.

We can now state the following theorem which investigates the behavior of different indices under the effect of varying RRT and number of classes.

Theorem 3. Given a $C$-class classification problem:

1. The index $\gamma_C$ satisfies both of Condition 4 and 5.
2. The index $\alpha$ satisfies both of Condition 1 and 3.
3. The index $\rho_a$ satisfies Condition [2] but not Condition [4].

4. The index $\rho_a$ fails to satisfy both of Condition [2] and [4].

5. The index $\kappa_a$ satisfies Condition [2] but not Condition [4].

Proof. Let us consider two $C$-class confusion matrices $M_C$ and $M'_C$. If we define $M_C$ as per Definition [3], then we can construct a new confusion matrix $M'_C$ by multiplying all the elements in the $i^{th}$ row by a $b_i$ where, $b_i \in \mathbb{R}^+, b_im_{ij} \in \mathbb{Z}^+$, and $b_i \neq b_j; \forall i, j \in C$, such that $M_C \sim M'_C$.

1) Using (10) we find the value of $\mathcal{V}_{M'_C}(\gamma_C)$ as follows:

$$\mathcal{V}_{M'_C}(\gamma_C) = \left( \prod_{i=1}^{C} \frac{b_im_{ii}}{b_in_i} \right)^{\frac{1}{b_i}} = \left( \prod_{i=1}^{C} \frac{m_{ii}}{n_i} \right)^{\frac{1}{b_i}} = \mathcal{V}_{M_C}(\gamma_C).$$

Hence, it is proved that $\gamma_C$ satisfies Condition [1].

Given a $C$-class confusion matrix $M_C$, the $\gamma_C(M_C)$ is only dependent on the values of $m_{ii}$, and $n_i$, as can be inferred from its definition in (10). Now, from Definition [3] we know that the values of $n_i > 0$, and $m_{ii} \geq 0$ (non-zero positive when at least one point from the class is correctly classified, 0 otherwise) for all $i \in C$. Thus, from (10), it is evident that $\gamma_C(M_C) \geq 0$ (non-zero only when $m_{ii} > 0; \forall i \in C$), which implies that $L_{M_C}(\gamma_C) = 0$. Similarly, from the fact that $m_{ii} \leq n_i; \forall i \in C$, as $n_i = \sum_{j=1}^{C} m_{ij}$, we can conclude that $0 \leq m_{ii}/n_i \leq 1$. Therefore, from (10) the value of $\mathcal{U}_{M_C}(\gamma_C)$ can found to be 1. Now, given the family of $C + 1$-class confusion matrices, by the similar argument it can be shown that $L_{M_C+1}(\gamma_C) = 0$, and $\mathcal{U}_{M_C+1}(\gamma_C) = 1$, which satisfies the Condition [2]. This completes the first part of the theorem.

2) We take a $C$-class confusion matrix $M_C$, and construct $M'_C$, such that they belong to the same equivalence class. We then find the value of $\mathcal{V}_{M'_C}(\alpha)$ as per (10):

$$\mathcal{V}_{M'_C}(\alpha) = \frac{1}{C} \sum_{i=1}^{C} \frac{b_im_{ii}}{b_in_i} = \frac{1}{C} \sum_{i=1}^{C} \frac{m_{ii}}{n_i} = \mathcal{V}_{M_C}(\alpha).$$

Therefore, $\alpha$ satisfies Condition [1].

It is evident from Lemma [1] that $L_{M_C}(\alpha) = 0$, $U_{M_C}(\alpha) = 1$. By the same logic it can be claimed that $L_{M_C+1}(\alpha) = 0$, $U_{M_C+1}(\alpha) = 1$. Therefore, the lower and upper bound of $\alpha$ does not change with the increase in the number of classes, proving the second part of the theorem.

3) We start by a $C$-class confusion matrix $M_C$, and construct $M'_C$, ensuring that $M_C \sim M'_C$. Now to
confirm if \( \rho_o \) satisfies Condition 1, we find \( V_{MC}^\prime (\rho_o) \), using (12) as follows:

\[
V_{MC}^\prime (\rho_o) = \frac{1}{2C} \sum_{i=1}^{C} \left( 1 + \frac{b_im_{ii}}{b_in_i} - \sum_{j=1}^{C} \frac{b_jm_{ji}}{(C-1)b_jn_j} \right),
\]

\[
= \frac{1}{2C} \sum_{i=1}^{C} \left( 1 + \frac{m_{ii}}{n_i} - \sum_{j \neq i}^{C} \frac{m_{ji}}{(C-1)n_j} \right) = V_{MC} (\rho_o).
\]

Thus we show the invariance of \( \rho_o \) under Condition 1.

We first start with a \( MC \in MC \), then following from Lemma 2, we get:

\[
L_{MC} (\rho_o) = \frac{C - 2}{2(C-1)} \quad \text{and} \quad U_{MC} (\rho_o) = 1.
\]

Approaching similarly for a \( C+1 \)-class confusion matrix, we see that:

\[
L_{MC+1} (\rho_o) = \frac{C - 1}{2C} \neq L_{MC} (\rho_o), \quad (22)
\]

\[
U_{MC+1} (\rho_o) = 1 = U_{MC} (\rho_o). \quad (23)
\]

Therefore, from (22), and (23), we conclude that the lower bound of \( \rho_o \) is dependent on the number of classes while the upper bound is remained at 1, violating Condition 2 and completing the third part of the theorem.

4) Similar to the previous approaches given a \( C \)-class confusion matrix \( MC \), we construct \( MC' \), and express \( V_{MC}^\prime (\rho_o) \) as follows:

\[
V_{MC} (\rho_o) = \frac{1}{2C} \sum_{i=1}^{C} \left( 1 + \frac{b_im_{ii}}{b_in_i} - \sum_{j=1}^{C} \frac{b_jm_{ji} - b_im_{ii}}{\sum_{j=1}^{C} b_jn_j - b_in_i} \right),
\]

\[
= \frac{1}{2C} \sum_{i=1}^{C} \left( 1 + \frac{m_{ii}}{n_i} - \sum_{j=1}^{C} \frac{b_jm_{jj} - b_im_{ii}}{\sum_{j=1}^{C} b_jn_j - b_in_i} \right) \neq V_{MC} (\rho_o).
\]

Hence, \( \rho_o \) do not satisfy Condition 1.

It is evident from Lemma 3 that for a \( C+1 \)-class problem \( U_{MC+1} (\rho_o) = U_{MC} (\rho_o) = 1 \). Moreover,
similar to (21), we can calculate:

$$L_{MC+1}(\rho_a) = \frac{1}{2C+2} \left( C - \frac{n_{C+1}}{n-n_{C}} \right).$$  \tag{24}$$

From (21) and (24) we can show $$L_{MC}(\rho_a) \neq L_{MC+1}(\rho_a)$$, indicating that $$\rho_a$$ does not satisfy Condition 2, which completes the fourth part of the theorem.

5) As previous we take a $$C$$-class confusion matrix $$M_C$$, and construct $$M'_C$$ satisfying the equivalence relation. Let us now find $$V_{M'_C}(\kappa_a)$$ by (14),

$$V_{M'_C}(\kappa_a) = \frac{1}{2C} \sum_{i=1}^{C} \left( \frac{b_i m_{ii}}{\sum_{j=1}^{C} b_j m_{ji}} + \frac{m_{ii}}{b_i} \right),
= \frac{1}{2C} \sum_{i=1}^{C} \left( \frac{b_i m_{ii}}{\sum_{j=1}^{C} b_j m_{ji}} + \frac{m_{ii}}{n_i} \right) \neq V_{M_C}(\kappa_a).$$

Therefore, we conclude that $$\kappa_a$$ violates Condition 1.

We know from Definition 3, if $$m_{ii}$$ becomes $$n_i$$ for all $$i \in C$$, i.e. when all the points in the test set are correctly classified in their respective classes, then $$k_i = m_{ii}; \forall i \in C$$. On the other hand if all the test points are misclassified then $$m_{ii} = 0$$ for all $$i \in C$$. Consequently, $$0 \leq m_{ii}/k_i \leq 1$$, and $$0 \leq m_{ii}/n_i \leq 1$$, where both reaches the lower bound of 0 when $$m_{ii} = 0$$ (at the worst performance of the classifier), and the upper bound 1 when $$m_{ii} = n_i$$ (i.e. the classifier has achieved the best performance). Following this observation we can calculate $$L_{MC}(\kappa_a) = \frac{1}{2}(0+0) = 0$$ and $$U_{MC}(\kappa_a) = \frac{1}{2}(1+1) = 1$$. We can similarly find $$L_{MC+1}(\kappa_a)$$ and $$U_{MC+1}(\kappa_a)$$, which will be equal to their respective values for the set of $$C$$-class confusion matrices. Hence, $$\kappa_a$$ satisfies Condition 2.

If we consider the case of AUROC-OVO then it is evident from Lemma 2, that the lower limit of $$\rho_o$$ gradually increases with the number of classes thus becomes affected by the Type 2 distortion. A solution to mitigate this problem is to apply a normalization to $$\rho_o$$, such that its lower bound can be made independent of $$C$$. This can be done by first subtracting the bias from $$\rho_o$$ and then dividing the result by the coefficient (both terms are dependent on the choice of $$C$$) found in (15), which necessarily reduces the index to ACSA.

In a similar fashion we can discuss the nature of AUROC-OVA as well. As per Lemma 3 the lower bound of the $$\rho_a$$ index is dependent on the number of classes as well as on the number of test points
from the top two majority classes. Therefore, normalizing such an index will require to make certain assumptions on the representatives of the majority classes in the test set. In a special situation where the test set only contains \( \frac{n}{2} \) points each from the top two majority classes, then \( \mathcal{L}_{\mathcal{M}_C}(\rho_a) \) reduces down to \( \lambda_C = \frac{C - 2}{4C^2} \), which is a weak lower limit for \( \rho_a \). We call the normalized AUROC-OVA, as nAUROC-OVA, and calculate it by first subtracting the reduced lower limit and then dividing the result by the difference between the reduced lower limit and unity. In other words nAUROC-OVA can be expressed as \( \frac{\rho_a(M_C) - \lambda_C}{1 - \lambda_C} \).

The reason for violating Condition \( \Pi \) by AURPC-OVA is the direct consideration of \( k_i \) (which involve the true as well as false predictions in the \( i^{th} \) class) in the precision counterpart. Therefore, we propose a modified AURPC-OVA such that while calculating the precision the \( m_{ji} \) values are properly scaled by their corresponding \( n_j \)'s, for all \( j, i \in C \). We describe the modified AURPC-OVA called as mAURPC-OVA, in the following Definition 7.

**Definition 7.** For a \( C \)-class confusion matrix \( M_C \) the mAURPC-OVA index, denoted by \( \hat{\kappa}_a \) is defined as:

\[
\hat{\kappa}_a(M_C) = \frac{1}{2C} \sum_{i=1}^{C} \left( \frac{m_{ii}/n_i}{\sum_{j=1}^{C} m_{ji}/n_j} + \frac{m_{ii}}{n_i} \right).
\]  

(25)

We now proceed to confirm Condition \( \Pi \) and \( \Pi \overline{2} \) for mAURPC-OVA, in the following theorem.

**Theorem 4.** The index \( \hat{\kappa}_a \) satisfies both of the Conditions.

**Proof.** Proceeding in a manner similar to the one taken for \( \kappa_a \) in Theorem \( \Pi \) if we consider the two \( C \)-class confusion matrices \( M_C \) and \( M_C' \), then \( \mathcal{V}_{M_C'}(\hat{\kappa}_a) \) can be expressed as follows:

\[
\mathcal{V}_{M_C'}(\hat{\kappa}_a) = \frac{1}{2C} \sum_{i=1}^{C} \left( \frac{b_i m_{ii}/b_i n_i}{\sum_{j=1}^{C} b_j m_{ji}/b_j n_j} + \frac{b_i m_{ii}}{b_i n_i} \right),
\]

\[
= \frac{1}{2C} \sum_{i=1}^{C} \left( \frac{m_{ii}/n_i}{\sum_{j=1}^{C} m_{ji}/n_j} + \frac{m_{ii}}{n_i} \right) = \mathcal{V}_{M_C}(\hat{\kappa}_a),
\]

which indicates that \( \hat{\kappa}_a \), satisfies Condition \( \Pi \).

Similar to Theorem \( \Pi \) we can see that \( 0 \leq m_{ii}/n_i \leq 1 \), and \( 0 \leq \sum_{j=1}^{C} m_{ji}/n_j \leq 1 \), for all \( i, j \in C \). Both of these terms reach their corresponding lower bound when classifiers performs the worst and upper bound at the accurate classification as previously described in the fifth part of Theorem \( \Pi \). Therefore, following Theorem \( \Pi \) we can conclude that both of \( \mathcal{L}_{\mathcal{M}_C}(\hat{\kappa}_a) = 0 = \mathcal{L}_{\mathcal{M}_C+1}(\hat{\kappa}_a) \) and \( \mathcal{U}_{\mathcal{M}_C}(\hat{\kappa}_a) = \mathcal{U}_{\mathcal{M}_C+1}(\hat{\kappa}_a) \),
hold implying that Condition 2 is satisfied by $\kappa_a$. 

From Theorem 3 and 4 we can conclude that among all only GMean, ACSA, and AURPC-OVA satisfy both of Condition 1 and 2 and thus can be applicable to evaluate multi-class imbalanced classifiers in presence of varying RRT or number of classes. However, the question about the quality of information provided by these indices under extremely poor classification performance on a single class is still required to be answered. Therefore, we proceed to the following Theorem 5 which evaluates the indices under the light of Condition 3.

Theorem 5. Among the three indices which are immune to the two types of distortions, except $\gamma_C$ both of $\alpha$ and $\kappa_a$ also satisfy Condition 3.

Proof. To prove that $\gamma_C$ fails to satisfy the third condition we start by finding $W_{M_C(i)}(\gamma_C)$, which by (10) can be expressed as follows:

$$W_{M_C(i)}(\gamma_C) = \lim_{\epsilon \to 0^+} \left( \epsilon \prod_{j=1, j \neq i}^C (1 - \epsilon) \right)^{\frac{1}{\epsilon}} = \lim_{\epsilon \to 0^+} \epsilon^\frac{1}{\epsilon} (1 - \epsilon) \frac{\epsilon^{\frac{\epsilon^2}{1-\epsilon}}}{C} = 0. \quad (26)$$

From (26) and Theorem 3 we can see that for $\gamma_C$ index $W_{M_C(i)}(\gamma_C) = L_{\beta_C}(\gamma_C)$, thus violating Condition 3.

We begin by calculating $W_{M_C(i)}(\alpha)$, which according to (10) is as follows:

$$W_{M_C(i)}(\alpha) = \lim_{\epsilon \to 0^+} \frac{1}{C} \left( \epsilon + \sum_{j=1, j \neq i}^C (1 - \epsilon) \right),$$

$$= \lim_{\epsilon \to 0^+} \frac{\epsilon + (C - 1)(1 - \epsilon)}{C} = \frac{C - 1}{C}. \quad (27)$$

From (27) and Theorem 3 it is evident that $\alpha$ index $W_{M_C(i)}(\alpha) > L_{\beta_C}(\alpha)$, therefore, $\alpha$ satisfies Condition 3.
As per (7) the value of $W_{MC(i)}(\hat{\kappa}_a)$ can be calculated as:

$$W_{MC(i)}(\hat{\kappa}_a) = \lim_{\epsilon \to 0^+} \frac{1}{2C} \sum_{j=1}^{C} \left( 1 - \epsilon + \frac{1 - \epsilon}{1 - \epsilon + \frac{k_i - m_j}{n - n_j}} \right)$$

$$+ \lim_{\epsilon \to 0^+} \frac{1}{2C} \left( \epsilon + \frac{\epsilon}{\epsilon + \frac{k_i - m_{11}}{n - n_{11}}} \right),$$

$$\Rightarrow W_{MC(i)}(\hat{\kappa}_a) > \lim_{\epsilon \to 0^+} \frac{1}{2C} \sum_{j=1}^{C} \left( 1 - \epsilon + \frac{1 - \epsilon}{1 - \epsilon + \frac{n - n_j + n_j \epsilon}{n - n_j}} \right)$$

$$+ \lim_{\epsilon \to 0^+} \frac{1}{2C} \left( \epsilon + \frac{\epsilon}{\epsilon + \frac{n - n_i \epsilon}{n - n_i}} \right),$$

$$\Rightarrow W_{MC(i)}(\hat{\kappa}_a) > \frac{C - 1}{2C} \left( 1 + \frac{1}{2} \right) + 0 = \frac{3(C - 1)}{4C}.$$  \hfill (28)

From (28) and Theorem 4, it is evident that $W_{MC(i)}(\hat{\kappa}_a) > L_{MC}(\hat{\kappa}_a)$, confirming that the index $\hat{\kappa}_a$ satisfies Condition 3, which completes the proof.

5. Experiments and results

This section first provides a brief description of the used dataset, followed by details of the experiment protocol and finally illustrates the different results alongside appropriate discussion.

5.1. Description of datasets

We have used a subset of the widely popular ImageNet [31] classification dataset for all our experiments. The ImageNet dataset provides a vast collection of natural images categorized into a large number of structured classes (1000 leaf classes alongside 860 higher level concepts following a predefined tree). For our experiments we have taken a subset of the ImageNet training set by sampling images from 12 higher level classes (formed by combining 1-5 leaf concepts and containing a total of 1300-6500 data instances). Since our chosen classifiers are only applicable to real valued data, given the images, we need to extract quality features. Therefore, for the purpose of feature extraction we have used the state-of-the-art Inception V3 [34], an end-to-end deep neural network, which learns the map from the image space to a set of classes through a 2048-dimensional real values distributed representation space. The Inception V3 used by us is a standard implementation pre-trained on the complete ImageNet training set, publicly available from Keras deep learning API at [https://keras.io/applications](https://keras.io/applications). Thus, in our case the
selected subset of images can be mapped to useful feature vectors by a simple forward pass through the 
pre-trained network. We have then created 12 two-class classification problems, having \( IR \) between 5 
and 40 for experimentally validating the effect of Type 1 distortion over the various indices. Moreover, 
we have formed a total of 10 multi-class imbalanced datasets (4 sets for 3-class, while 3 sets each for 
the 5-class and 10-class classification problems) having \( IR \) between 20-30 for the purpose of empirically 
evaluating the effect of Type 2 distortion. A detailed description of the datasets used in our experiments 
can be found in Section 2 of the supplementary document.

5.2. Experiment protocol

We perform two sets of experiments respectively over two-class and multi-class datasets to inspect the 
behavior of different indices in light of Condition 1 and Condition 2. We conduct our experiments using 
four state-of-the-art, classifiers of diverse nature, all specifically tailored for handling class-imbalance, 
namely Dual-LexiBoost with \( k \)-Nearest Neighbor as the base classifier [35], Near Bayesian SVM (NBSVM) 
[36], RUSBoost [37, 38] with decision trees [39] as the base classifier, and MLP [2] combined with SMOTE 
[40]. The parameter settings of these methods can be found in Section 3 of the supplementary material.

For both experiments, the classifier is first trained with the training set and then tested by multiple 
test sets having different RRT values. This is done in an attempt to mimic the two primary causes of 
the first type of distortion as described in Section 1.3. In case of two-class datasets the RRT is varied 
between 1 (balanced), 0.5 (more number of minority points are taken compared to majority), half of the 
original IR (reduced effect of imbalance), the original IR of the training set, and twice of the original IR. 
Similarly in case of multi-class datasets, we have used 5 different test sets with varying RRT (The first is 
balanced, in the second the IR between the classes are reversed, in the third the original IR between the 
minority class and all others are halved, the fourth maintains the original IR, while the last doubles the 
test points from all classes except the minority). Such an experimental setting helps us to understand the 
effect of the varying number of test points from different classes on the values of the indices. Additionally, 
the experiments on multi-class datasets provide us with a way to inspect the effect of increasing \( C \) on 
the indices. Average results over 5 independent runs of 5-fold stratified cross validation (which also aids 
in parameter tuning for the classifiers) are reported to ensure the reliability of our findings.

5.3. Validating the two-class classification performance evaluation indices in light of Condition 1

For each of the dataset, we have found the standard deviation of the mean performance in terms of 
an index over the five test sets and four classifiers. We plot the findings in Figure 3 which shows that
the Precision index achieves the highest variability over the test sets for a given training set. However, the low standard deviation of GMean and AUROC suggests that the classifiers retain an almost similar performance over the various test sets. Therefore, the high standard deviation of Precision must be due to the changes in the actual numbers of the respective test points from the two classes, which vary significantly due to the diverse choice of RRT. These observations reflect the theoretical analysis which shows Precision to be sensitive over RRT even when the class-specific classification performance is retained, thus failing to satisfy Condition 1. Due to having Precision as a component AURPC also suffers from the same issue, though the additional consideration of Recall helps to mitigate the effect of altering RRT to some extent. Interestingly, mPrecision and mAURPC closely follow the GMean and AUROC indices indicating their immunity against the effect of RRT.

5.4. Validating the multi-class classification performance evaluation indices in light of Condition 1

We use an approach similar to the two-class case for validating Condition 1 for the multi-class performance evaluation indices. However, the indices which are susceptible to Condition 2 are expected
to have a smaller range with increasing value of \( C \), and may result into a lower standard deviation over the test sets for high number of classes. Thus, comparing these indices with those indices satisfying Condition 2 may lead to a bias against the later and will not help to reach a conclusive remark. Hence, in Figure 4a we only compare the standard deviations of indices satisfying Condition 2 viz. GMean, ACSA, nAUROC-OVA, AURPC-OVA, and mAURPC-OVA, over the various test sets for each of the datasets. A close inspection reveals that the minimum variability (especially improving from AURPC-OVA) is achieved by mAURPC-OVA establishing it as the better choice among the five contenders. Interestingly, ACSA has shown slightly higher variability compared to nAUROC-OVA, which is unlikely as the later violates Condition 1. This leads us to investigate further, by normalizing the standard deviation of the ACSA and nAUROC-OVA indices for each of the datasets by the respective minimum standard deviation achieved over all the multi-class datasets. This kind of normalized standard deviation can be considered as a measure of stability as it quantifies the variability of an index from its best stable performance (a lower value signifies that the index can equivalently evaluate similar performing classifiers). We plot the results in Figure 4b which shows the normalized standard deviation to be slightly greater for nAUROC-OVA than that of ACSA. This indicates nAUROC-OVA to be less stable compared to ACSA, and the lower standard deviation of the former in Figure 4a may be due to the fact that AUROC-OVA is normalized using a weak lower bound.

5.5. The effect of the number of classes (Condition 2) over the different indices

We consider AUROC-OVA, and AUROC-OVO for this experiment as they are seen to have a higher lower bound with increasing number of classes. Their respective normalized version, i.e. nAUROC-OVA and ACSA are also considered to establish the improvement achieved through normalization, alongside GMean as a reference. We plot the minimum value achieved by these indices for each of the datasets in Figure 5. The results shows that on three and five class datasets the AUROC-OVA, and AUROC-OVO performs almost equivalently to their normalized counterparts. However, on the ten class datasets the minimum index value achieved by AUROC-OVA and AUROC-OVO are significantly higher than ACSA, and nAUROC-OVA. This validates the bias of AUROC-OVA and AUROC-OVO towards a higher value with increasing number of classes, and also demonstrated the ability of the respective normalized versions to counter this bias.
5.6. The effect of Condition 3 on multi-class indices

From Figure 5, the minimum values of GMean are consistent with the other indices over the three, and five class datasets. However, for the ten-class datasets the index produced significantly lower values compared to the others. Moreover, on a ten-class dataset GMean produced its lowest possible value of 0, indicating the worst possible classification performance. However, the values of the other indices over the same dataset clearly indicate that the classifier managed to successfully classify many of the test points. Therefore, despite satisfying Conditions 1 and 2, GMean fails to do the same for Condition 3 as poor performance on a single class results in the loss of all information about the performance on every other class.

6. Discussion on the applicability of indices

Based on the satisfaction of the two fundamental conditions the indices can be grouped as shown in Figure 6a. Moreover, the multi-class indices which satisfy Condition 1 and 2 are further classified by...
Condition 3 in Figure 6b. Therefore, using Figure 6 we can proceed to recommend an appropriate choice of indices for different applications.

In case of two-class classification, all four of GMean, AUROC, mPrecision, and mAURPC satisfy Condition 1, thus any one of these can be a good index of choice. However, GMean is biased towards the accuracy of that class which is poorly classified compared to the other. In a two-class scenario, this property of GMean may prove useful as it will identify the high bias of a classifier towards a particular class. AUROC, on the other hand, accords equal weight to the performance in both classes. Therefore, we recommend GMean for general evaluation of the performance of a two-class classifier.

Recall and Precision (consequently AURPC, mPrecision, and mAURPC) both depend on the choice of the positive class. Recall is focused on the classification performance over the minority class, thus can be used in applications where false positives do not lead to severe consequences. For example, we may consider the case of benign and malignant tumor classification in medical diagnostic systems, where wrongly classifying a sample from the minority class of malignant tumors may result in fatal outcome. On the other hand, Precision (and AURPC) can be effectively used when the application attempts to limit the number of false positives while the class priors do not significantly vary over time. One can think of the spam filtering problem where even though the non-spam mails are considerably high in numbers, labelling one of them as spam may lead to loss of important information. Evidently, mPrecision and mAURPC indices can act as the respective replacement of Precision and AURPC if the application under concern can cause Type 1 distortion.

In case of multi-class classification, even though GMean satisfies both of Condition 1 and 2 it may still be biased in case of extremely poor performance over a single class, as indicated by its violation of Condition 3. GMeans can however still prove beneficial if the target is to achieve non-zero classification accuracy on each class. On the contrary, ACSA and mAURPC satisfy all three of the conditions, and thus any of the two can be an appropriate choice of index. Finally, despite their violation of Condition 1, nAUROC-OVA and AURPC-OVA can be used in those applications where the misclassification from different classes are associated with different costs. For example, in a multi-class medical diagnostic application a somewhat similar set of syndromes may correspond to different diseases of varying severity and rarity.
7. Conclusion and future works

In this article, we show that the common indices used for evaluating the performance of a classifier in presence of class imbalance may suffer from different forms of distortions, depending on the character of the data, especially the test set. We formally define two conditions that an index needs to satisfy to be resilient to such distortions. We present theoretical analyses detailing the traits of the indices in light of these conditions and propose necessary remedies as per need. We further define a third condition to evaluate the quality of the information provided by an index, especially under adverse conditions such as exceptionally poor accuracy over a single class. We also undertake empirical analysis to support our theoretical findings. Finally, we discuss on the applicability of different indices and make recommendations.

A natural future extension of this work would be to investigate the behavior of indices which are used in imbalanced multi-label [41] and multi-instance [42] classification problems. One may also consider validating the efficacy of the modified/normalised indices on class imbalanced problems where the two types of distortions are naturally occurring. For example, Type 1 distortion is quite inherent in image foreground and background classification [43]. This is because foreground usually spreads over less number of pixels compared to background resulting in class imbalance. Moreover, the the fraction of background to foreground in an image i.e. the RRT may significantly vary between images consequently causing distortion in performance indices which fail to satisfy Condition 1. Type 1 distortion is also possible during sentiment analysis from tweets, where not all sentiments may occur with equal frequency [44], while the prior probability of different sentiments may notably change over time. Thus, if a sentiment analyzing classifier is periodically tested after deployment for potential fine tuning and an index susceptible to Type 1 distortion is utilized for quantifying its performance, then the result may mislead the quality assessment. On the other hand, Type 2 distortion can occur in open set recognition or incremental learning problems where the number of classes may increase over time [45], thus use of an index which satisfies Condition 2 may be beneficial.

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