INDEX 1 COVERS OF LOG
TERMINAL SURFACE SINGULARITIES

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ABSTRACT. We shall investigate index 1 covers of 2-dimensional log terminal singularities. The main result is that the index 1 cover is canonical if the characteristic of the base field is different from 2 or 3. We also give some counterexamples in the case of characteristic 2 or 3. By using this result, we correct an error in [K2].

1. Introduction

We fix an algebraically closed field \( k \) and let \( p \) be its characteristic. Let \( S \) be a normal surface over \( k \), let \( P \) be a closed point of \( S \), and let \( D \) be an effective and reduced Weil divisor on \( S \) through \( P \). We consider the germ of the pair \((S, D)\) at \( P \). Let \( \mu : S' \to S \) be an embedded resolution of the singularity for the pair \((S, D)\). The numerical pull-back \( \mu^*(K_S + D) \) is defined as a \( \mathbb{Q} \)-divisor on \( S' \) such that the equality \( (\mu^*(K_S + D)) \cdot C = (K_S + D) \cdot \mu^*C \) holds for any curve \( C \) on \( S' \). We write \( \mu^*(K_S + D) = K_{S'} + D' + E \) for a \( \mathbb{Q} \)-divisor \( E \) on \( S' \), where \( D' = \mu_1^{-1}D \) is the strict transform of \( D \). The pair \((S, D)\) is said to be log terminal at \( P \) if the coefficients of \( E \) are strictly less than 1. It is called canonical if the coefficients of \( E \) are non-positive. \( S \) is said to be canonical or log terminal if \((S, 0)\) is so (cf. [KMM]).

The index \( r \) of the pair \((S, D)\) is the smallest positive integer such that \( r(K_S + D) \) is a Cartier divisor. Let \( \theta_0 \) and \( \theta \) be non-zero sections of \( \mathcal{O}_S(K_S + D) \) and \( \mathcal{O}_S(r(K_S + D)) \), respectively. Assume that \( \theta \) generates \( \mathcal{O}_S(r(K_S + D)) \). Let \( L \) be the rational function field of \( S \), and write \( \theta = \alpha \theta_0^\beta \) for \( \alpha \in L \). The normalization \( \pi : T \to S \) of \( S \) in the field extension \( L(\alpha^{1/r}) \) is called the index 1 cover of \( S \) associated to the section \( \theta \). We note that the index 1 cover depends on the choice of \( \theta \). The index 1 cover is called the log canonical cover in [KMM], but its construction is not canonical at all, and in order to avoid a confusion, we use instead this terminology.

If \( p \) does not divide the degree \( r \) of the morphism \( \pi \), then \( \pi \) is etale over \( S \setminus \{P\} \), and the pair \((T, D_T)\) for \( D_T = \pi^*D \) is known to be canonical (cf. [KMM]). But if \( p \) divides \( r \), then \( \pi \) is inseparable, and the situation is totally different. The following is the main result of this paper.

Theorem 1. Let \( S \) be a normal surface over an algebraically closed field \( k \) of characteristic \( p \neq 2, 3 \), \( D \) a reduced curve, and \( P \) a closed point such that the pair \((S, D)\) is log terminal of index \( r \) at \( P \). Let \( \theta \) be a nowhere vanishing section of \( \mathcal{O}_S(r(K_S + D)) \), and \( \pi : T \to S \) the index 1 cover associated to \( \theta \). Set \( D_T = \pi^*D \)

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and $Q = \pi^{-1}(P)$. If $\theta$ is chosen to be general enough, then $(T, D_T)$ is canonical at $Q$.

We have counterexamples in the case of characteristic 2 or 3 (Example 5). By using the above result, we shall correct an error in [K2] in §3. We would like to thank Professor K. Matsuki for pointing out this error.

2. Proof of Theorem 1

We keep the notation of the introduction.

Lemma 2. Assume that a pair $(S, D)$ is log terminal at a point $P$. Then $(S, P)$ is a rational singularity.

Proof. Since $S$ is also log terminal, we may assume that $D = 0$. Let $\mu : S' \to S$ be the minimal resolution and write $\mu^*K_S = K_{S'} + E$ for a $\mathbb{Q}$-divisor $E$. Let $Z$ be the fundamental cycle. Since the coefficients of $E$ are non-negative and less than 1, the divisor $Z - E$ is effective and its support is the whole exceptional locus of $\mu$. Hence $(Z^2) + (Z \cdot K_{S'}) = (Z \cdot (Z - E)) < 0$.

It follows that the index $r$ of the pair $(S, D)$ is equal to the smallest positive integer such that $rE$ becomes a divisor.

Let us consider the divisor on $S'$ which is the sum of the exceptional locus of $\mu$ and the strict transform of $D$. Then we can classify the dual graphs of these divisors ([K1, TM]). We note that this classification is purely numerical and characteristic free. For example, the dual graphs for canonical singularities are Dynkin diagrams of type $A, D$ or $E$. The dual graphs of log terminal singularities are the same as those of quotient singularities in characteristic 0, but they are not necessarily quotient singularities in general.

We assume that $p | r$ from now on. We start the proof of Theorem 1 with the calculation of the log canonical divisor on an index $r/p$ cover of a log terminal singularity of index $r$. For a 1-form $\omega$ on a normal variety $S$, we denote by $\text{div}_S(\omega)$ the divisorial part of its zero or pole.

Lemma 3. Let $S$ be a normal affine surface, $D$ a reduced curve, $P$ a closed point such that the pair $(S, D)$ is log terminal of index $r$ at $P$. Assume that $p \neq 2, 3$ and $p$ divides $r$. Let $\theta_0$ and $\theta$ be non-zero sections of $O_S(K_S + D)$ and $O_S(r(K_S + D))$, respectively. Assume that $\theta$ is nowhere vanishing, and write $\theta = \alpha \theta_0^l$ for $\alpha \in L$, the rational function field of $S$. Assume that $\text{div}_S(\alpha) = \text{div}_S(\alpha^l)$ as Weil divisors on $S$. Let $\pi : \tilde{S} \to S$ be the normalization of $S$ in the field $L = L(\alpha^{1/p})$. Set $\tilde{D} = \pi^*D$ and $\tilde{P} = \pi^{-1}(P)$. Then $(\tilde{S}, \tilde{D})$ is again log terminal at $\tilde{P}$ of index $r/p$. Moreover, if $p^2 | r$, then $\text{div}_{\tilde{S}}(\alpha^{1/p}) = \text{div}_{\tilde{S}}(\alpha^{1/p})$ as Weil divisors on $\tilde{S}$.

Proof. Since $O_S(\pi^*(K_S + D))$ is not invertible, we have $\alpha \notin L^p$, and $\tilde{L}/L$ is a purely inseparable extension of degree $p$. By [RS, Proposition 2], we have

$$K_{\tilde{S}} = \pi^*(K_S - (1 - 1/p)\text{div}_S(\alpha)).$$

Since $\text{div}_S(\alpha) = \text{div}_S(\alpha) \sim 0$, we have $K_{\tilde{S}} = \pi^* K_S$ by a different choice of the identification. Therefore, $\tilde{\theta} = \alpha^{1/p} \pi^* \theta_0^{l/p}$ is a nowhere vanishing section of $O_{\tilde{S}}(\pi^*(K_{\tilde{S}} + \tilde{D}))$, and the index of the pair $(\tilde{S}, \tilde{D})$ is $r/p$. 


Let \( \mu : S' \to S \) be a projective birational morphism from a smooth surface, and \( D' = \mu_*(-1)D \) the strict transform of \( D \). Since \((S, D)\) is log terminal, we can write

\[
\mu^*(K_S + D) = K_{S'} + D' + \sum_j a_j C_j
\]

with \( a_j < 1 \) for each irreducible component \( C_j \) of the exceptional locus \( C \) of \( \mu \). By the adjunction, the \( C_j \) are isomorphic to \( \mathbb{P}^1 \), and intersect transversally.

Let \( \pi' : \tilde{S}' \to S' \) be the normalization in \( L \), \( \tilde{\mu} : \tilde{S}' \to \tilde{S} \) the induced birational morphism, and \( \tilde{D}' = \tilde{\mu}_*^{-1}\tilde{D} = \pi'^*D' \). We can write

\[
\tilde{\mu}^*(K_{\tilde{S}} + \tilde{D}) = K_{\tilde{S}'} + \tilde{D}' + \sum_j \tilde{a}_j \tilde{C}_j,
\]

where the \( \tilde{C}_j \) are prime divisors such that \( \pi'(\tilde{C}_j) = C_j \). We know that \( \pi'^*C_0 = \tilde{C}_0 \) or \( p\tilde{C}_0 \). We shall prove that \( \tilde{a}_j < 1 \) for all \( j \) and for any \( \mu \).

By [RS, Proposition 2] again, we have

\[
K_{\tilde{S}'} = \pi'^*(K_{S'} - (1 - 1/p)\text{div}_{S'}(d\alpha)).
\]

Since \( \mu^*\text{div}_S(\alpha) = \text{div}_{S'}(\alpha) \), we have

\[
\sum_j \tilde{a}_j \tilde{C}_j = \pi'^*(\sum_j a_j C_j - (1 - 1/p)(\text{div}_{S'}(\alpha) - \text{div}_{S'}(d\alpha))).
\]

Let \( G = \text{div}_S(\theta_0) \), and \( G' = \mu_*^{-1}G \) its strict transform. Then we can write

\[
\text{div}_{S'}(\alpha) + rG' = \sum_j m_j C_j
\]

\[
\text{div}_{S'}(d\alpha) + rG' = \sum_j m'_j C_j
\]

for some \( m_j, m'_j \in \mathbb{Z} \). Thus

\[
\tilde{a}_j \tilde{C}_j = (a_j - (1 - 1/p)(m_j - m'_j))\pi'^*C_j.
\]

Since \( G \sim K_{\tilde{S}} + D \), there exists a divisor \( F \) supported on \( C \) such that \( F + G' \sim K_{\tilde{S}'} + D' \). Then \( \text{div}_{S'}(\alpha) + rF + rG' + \sum_j a_j rC_j \) is numerically trivial, hence \( m_j + a_j r \equiv 0 \pmod{r} \).

Let us fix an irreducible component of \( C \), say \( C_0 \). We consider 2 cases (we shall prove later that these are the only cases provided that \( p \neq 2, 3 \):

Case 1. We assume that \( p \) does not divide \( m_0 \).

We take a general closed point \( P' \) on \( C_0 \). Let \((x, y)\) be local coordinates such that \( C_0 = \text{div}(x) \) near \( P' \). We can write \( \alpha = u x^{m_0} \) near \( P' \) such that \( u(P') \neq 0 \). Then we have \( d\alpha = x^{m_0 - 1}(m_0 u dx + x du) \), hence \( m'_0 = m_0 - 1 \). Since \( p \) does not divide \( m_0 \), there are integers \( s, t \) such that \( ps + m_0 t = 1 \). Then \( \text{div}_{S'}(x^{ps} \alpha^t) = C_0 \) near \( P' \), hence \( \pi'^*C_0 = p\tilde{C}_0 \) with \( \text{div}_{\tilde{S}'}(x^{s} \alpha^{t/p}) = \tilde{C}_0 \) near \( \tilde{P}' = \pi'^{-1}(P') \). Therefore,

\[
\tilde{a}_j \tilde{C}_j = (a_j - (1 - 1/p)(m_j - m'_j))\pi'^*C_j.
\]
Case 2. We assume that \( p|m_0 \). In this case, we assume in addition that \( C_0 \) intersects at most 2 other components of \( C \), say \( C_1 \) and \( C_2 \) (\( C_2 \) may not exist). Moreover, we assume that \( p \) does not divide \( m_1 \).

Since \( m_1 + m_2 \equiv ra_1 + ra_2 \equiv 0 \pmod{p} \), \( C_2 \) necessarily exists and \( p \) does not divide \( m_2 \). Let \( P' \) be an arbitrary closed point on \( C_0 \) except \( P'_i = C_0 \cap C_i \) for \( i = 1, 2 \), and \((x, y)\) local coordinates such that \( C_0 = \text{div}(x) \) near \( P' \). We can write \( \alpha = uv^px^mdu \) near \( P' \) in such a way that \( u_0 = u|_{C_0} \) is a rational function on \( C_0 \) such that \( \text{div}_{C_0}(u_0) = m_1P'_1 + m_2P'_2 + pQ' \) for some divisor \( Q' \) on \( C_0 \) whose support does not contain \( P' \). Since the \( m_i \) are not divisible by \( p \), we have \( du_0 \neq 0 \). Thus \( \deg(du_0) = -2 \), and we have \( \text{div}_{C_0}(du_0) = (m_1 - 1)P'_1 + (m_2 - 1)P'_2 + pQ' \). Therefore, \( du_0 \) does not vanish at \( P' \). Since \( d\alpha = v^p x^m du \) near \( P' \), we have \( m'_0 = m_0 \). Moreover, \( u_0 - u_0(P') \) gives a local coordinate of \( C_0 \) at \( P' \). Hence \( (\pi^*x, \pi^*(u - u(P'))^{1/p}) \) give local coordinates at \( \tilde{P}' = \pi^{-1}(P') \). Thus \( \pi^*C_0 \) is reduced, and \( \tilde{S}' \) is smooth at \( \tilde{P}' \). In particular, \( \pi^*C_0 = \tilde{C}_0 \) and \( \tilde{a}_0 = a_0 < 1 \).

We shall prove that any irreducible component \( C_0 \) of \( C \) satisfies the assumptions of one of the above two cases. First, we consider the case in which \( \mu = \mu_0 : S' = S'_0 \to S \) coincides with the minimal resolution.

Assuming that \( C_0 \) intersects 3 other components, say \( C_1, C_2, C_3 \), we shall prove that we have Case 1 for \( C_0 \). Assume the contrary that \( p|m_0 \). Then \( p|m_0 \). In the case in which the dual graph for \( S' \) is of type \( D \), we have \( (C^2_1) = (C^2_2) = -2 \) after the permutation of the indices. Then we calculate that \( a_1 = a_2 = a_0/2 \). Since \( m_1 + m_2 \equiv 0 \pmod{r} \) and \( p \neq 2 \), we have \( p|m_1 \) and \( p|m_2 \). Since \( p|m_1 + m_2 + m_3 - (C^2_0)m_0 \), we have \( p|m_3 \). Then we have \( p|m_4 \) for an irreducible component \( C_4 \) which intersects \( C_3 \). In this way, we conclude that \( p|m_4 \) for all \( j \). It follows that \( \frac{1}{p}(K_{S'} + D' + \sum_j a_j C_j) \) is a divisor on \( S' \). Since this divisor is numerically trivial and \( S \) is a rational singularity, it is a pull back of a divisor on \( S \), a contradiction with the assumption that \( r \) is the index.

In the case in which the dual graph for \( S' \) is of type \( E \), we have two cases after the permutation of the indices: (i) \( (C^2_1) = (C^2_2) = -2 \) and \( C_2 \) intersects another irreducible component \( C_4 \) such that \( (C^2_4) = -2 \) while \( C_1 \) does not intersect other components, or (ii) \( (C^2_1) = -2 \), \( (C^2_2) = -3 \), and \( C_1 \) and \( C_2 \) intersect no other irreducible components. We have \( a_1 = a_0/2 \) and \( a_2 = 2a_0/3 \) in the former case, and \( a_1 = a_0/2 \) and \( a_2 = (a_0 + 1)/3 \) in the latter. Since \( p \neq 2, 3 \), we have \( p|m_1 \) and \( p|m_2 \), and obtain a contradiction as before.

If we assume that \( p|m_0 \), then by the above argument, \( C_0 \) intersects at most 2 other components of \( C \), say \( C_1 \) and \( C_2 \) (\( C_2 \) may not exist). Suppose that \( p|m_1 \). Then \( C_1 \) intersects at most 1 other component, say \( C_3 \), and that \( p|m_3 \). Moreover, if \( C_2 \) exists, then we have also \( p|m_2 \). Then we have \( p|m_j \) for all \( j \) as before, a contradiction. Therefore, we have Case 2 for \( C_0 \).

Next, we consider the general case. Let \( \mu \) and \( j \) be arbitrary. If the center of \( C_j \) on the minimal resolution \( S'_0 \) is a curve, then the above argument showed our assertion. Assume that the center \( Q \) of \( C_j \) on \( S'_0 \) is a point. We have 3 cases: (a) \( Q \) is contained in only one irreducible component \( C_0 \) of \( C \) such that \( p|m_0 \), (b) \( Q \) is contained in only one irreducible component \( C_0 \) of \( C \) such that \( p \nmid m_0 \), (c) \( Q \) is contained in two irreducible components \( C_0 \) and \( C_1 \) of \( C \) such that \( p \nmid m_0 \).

In the case (a), the covering \( S'_0 \) is smooth at the point \( \tilde{Q} \) above \( Q \) by the argument of Case 2, hence we obtain \( \tilde{a}_j < 1 \) after any sequence of blow-ups of \( S'_0 \) above \( Q \).

In the case (b) or (c), we replace \( S' \) by its blow-up at \( Q \), and we obtain again (a).
the exceptional divisors. We have constructed by blowing up suitably a smooth surface and then contracting some of and other intersection numbers are 0. We note that a surface \( S \)

\[ \text{dual graph of the exceptional divisors} \]

singularity over an algebraically closed field of characteristic \( C \).

**Example 5.**

(1) Let \( \mu : S' \to S \) be the minimal resolution of \( S \) as above can be constructed by blowing up suitably a smooth surface and then contracting some of the exceptional divisors. We have

\[ \mu^* K_S = K_{S'} + \frac{1}{2} C_1 + \frac{1}{4} C_2 + \frac{1}{4} C_3 + \frac{1}{2} C_4. \]

\( S \) is a rational triple point and the index \( r = 4 \). Let \( \pi : T \to S' \) and \( \pi_1 : S \to S' \) be the index 1 cover associated to a general section \( \theta \) of \( O_S(4K_S) \), where \( \pi_1 \) and \( \pi_2 \) are purely inseparable morphisms of degree 2. We claim that \( T \) is not log terminal.

Finally, in order to prove the last statement, we claim that

\[ p\text{div}_{S}(d\alpha^{1/p}) = \pi^*(\text{div}_S(d\alpha)). \]

We shall check this equality at all but finitely many points on \( S \). As in the proof of [RS, Proposition 2], we may assume that there exist local coordinates \( (x, y) \) of the completion of \( S \) at the point \( Q \) such that \( (\tilde{x}, \tilde{y}) \) with \( \tilde{x} = \pi^* x \) and \( \tilde{y} = \pi^* y^{1/p} \) give local coordinates of the completion of \( \tilde{S} \) at \( \tilde{Q} = \pi^{-1}(Q) \). We can write \( \alpha = u^r \sum_{i=0}^{p-1} c_i y^i \) for \( u \in L \) and \( c_i \in \hat{L} \), where \( \hat{L} \) is the fraction field of the completed local ring, such that \( \text{div}_{\tilde{S}}(\alpha) = r\text{div}_S(u) \). Since \( \text{div}_S(d\alpha) = \text{div}_S(\alpha) \), we may assume that \( c_1(Q) \neq 0 \). Since

\[ d\alpha^{1/p} = u^{r/p} \sum_{i=0}^{p-1} (ic_i \tilde{y}^{i-1}d\tilde{y} + \tilde{y}^idc_i), \]

we have \( p\text{div}_{\tilde{S}}(d\alpha^{1/p}) = r\text{div}_S(u) \).

**Proof of Theorem 1.** Since \( \theta \) is chosen to be general enough, we deduce that \( \text{div}(d\alpha) = \text{div}(\alpha) \) if we replace \( S \) by a suitable neighborhood of \( P \) by the dimension count argument as in p. 472 of [K2]. We apply Lemma 3 until the index becomes coprime to \( p \), then apply the usual argument to obtain our assertion (cf. [KMM]).

**Remark 4.** (1) The formula for \( K_{\tilde{S}} \) depends on the choice of \( \alpha^{1/p} \) which generates the field extension \( \tilde{L}/L \). This choice is equivalent to the splitting of a free \( L \)-module \( \tilde{L} \) as

\[ \tilde{L} = \bigoplus_{m=0}^{p-1} L\alpha^{m/p}. \]

The construction of index 1 cover as in [K2] uses this kind of splitting explicitly and thus there is a canonical divisor formula.

(2) Lemma 3 is still true in the case of characteristic 2 or 3 if the minimal resolution diagram of \( S \) is of type \( A \).

**Example 5.** (1) Let \( \mu : S' \to S \) be the minimal resolution of a log terminal singularity over an algebraically closed field of characteristic \( p = 2 \). Assume that the dual graph of the exceptional divisors \( C \) is of type \( D \) as follows: \( C = C_1 + C_2 + C_3 + C_4 \) with \( (C_1^2) = (C_2^2) = (C_3^2) = -2, (C_4^2) = -3, (C_1 \cdot C_2) = (C_1 \cdot C_3) = (C_1 \cdot C_4) = 1, \) and other intersection numbers are 0. We note that a surface \( S \) as above can be constructed by blowing up suitably a smooth surface and then contracting some of the exceptional divisors. We have

\[ \mu^* K_S = K_{S'} + \frac{1}{2} C_1 + \frac{1}{4} C_2 + \frac{1}{4} C_3 + \frac{1}{2} C_4. \]
Indeed, as in Case 2 in the proof of Lemma 3, since $2|m_1$, we can write $\alpha = uv^2x^m$ near a general closed point $P'$ of $C_4$ in such a way that $u_4 = u|_{C_4}$ is a rational function on $C_4$ such that $\text{div}_{C_4}(u_4) = 2Q'$ for some divisor $Q'$ on $C_4$. Thus $u_4 = v_4^2$ for some rational function $v_4$ on $C_4$. It follows that the natural morphism $\tilde{C}_4 \to C_4$ is birational, hence $\pi'^*C_4 = 2\tilde{C}_4$. Since $m_4' \geq m_4$, we have $\tilde{a}_4 \geq 1$. If we denote by $b_j$ the coefficients for $K_T$ in a suitable way, then we deduce that $b_4 \geq 1$ by the same argument as in the proof of Lemma 3.

(2) Let $\mu : S' \to S$ be the minimal resolution of a log terminal singularity over an algebraically closed field of characteristic $p = 3$. Assume that the dual graph of $\mu$ is of type $E_6$ as follows: $C = C_1 + C_2 + C_3 + C_4 + C_5$ with $(C_2^3) = (C_3^2) = (C_4^2) = (C_5^2) = -2$, $C_1 \cdot C_2 = (C_1 \cdot C_3) = (C_3 \cdot C_4) = (C_1 \cdot C_5) = 1$, and other intersection numbers are 0. Then we have

$$\mu'^*K_S = K_{S'} + \frac{2}{3}C_1 + \frac{1}{3}C_2 + \frac{4}{9}C_3 + \frac{2}{9}C_4 + \frac{5}{9}C_5.$$ 

$S$ is a rational quintuple point and $r = 9$. Let $\pi : T \to \tilde{S} \to S$ be the index 1 cover associated to a general section $\theta$ of $O_S(9K_S)$. We claim that $T$ is not log terminal. Indeed, we have $b_2 \geq 1$ as in (1).

3. Correction to [K2]

Kenji Matsuki pointed out that the proof of Theorem 3.1 of [K2] is insufficient because the calculation in the middle of p. 473 is wrong. We shall replace the proof of Theorems 3.1 and 4.1 of [K2] by a different argument and prove them under the additional assumption that the residue characteristic is different from 2 or 3. We note that it is still an open question in the case of characteristic 2 or 3.

Proof of Theorems 3.1 and 4.1 of [K2] in the case $p \neq 2, 3$. We prove Theorem 4.1 by a slightly modified argument. Theorem 3.1 follows a posteriori from Theorem 4.1. We use the notation in Theorem 3.1; let $f : X \to \Delta = \text{Spec} \ A$ be a family satisfying Assumption 1.1. Let $p$ be the characteristic of the residue field at the closed point of $\Delta$. We assume that $p \neq 2, 3$. We take a closed point $P \in X$ of index $r$. The index 1 cover $\pi : Y \to X$ is constructed by using a general section $\theta$ of $O_X(rK_X/\Delta)$ as

$$\pi_*O_Y \cong \bigoplus_{m=0}^{r-1} O_X(-mK_X/\Delta)t^m, \, t^r = \theta.$$ 

We shall prove that the closed fiber $Y_s$ is canonical or normal crossing, but we do not prove that the singularity of $Y$ is isolated at this point.

First, assume that the closed fiber $X_s$ of $X$ is irreducible. Since $O_X(-mK_X/\Delta) \otimes\otimes O_{X_s} \cong O_{X_s}(-mK_X/s)$ by Assumption 1.1 (6), $Y_s$ is isomorphic to the index 1 cover of $X_s$ constructed by using the restriction of $\theta$ to $X_s$. By Theorem 1, $Y_s$ is canonical. We can prove that its completed local ring at $Q$ is isomorphic to the completion of $A[x_1, x_2, x_3]/(F)$ with $\text{ord}(F_s) \leq 2$ as in the original proof of Theorem 4.1, where the action of $\mu_r$ on the coordinates $(x_1, x_2, x_3)$ is given by $x_i \mapsto \zeta^{a_i} \otimes x_i$, $(i = 1, 2, 3)$. Since $O_X(-K_X/\Delta)$ is not invertible, there exists at least 2 coordinates whose weights $a_i$ are coprime to $r$. Let $x_1, \ldots, x_c$ ($c = 2$ or 3) be such coordinates. Since $\theta = t^r$ never vanishes and the natural homomorphism $O_X(-K_X/\Delta) \otimes\otimes O_{X_s} \to \otimes O_{X_s}$, we have

$$\pi_*O_Y \cong \bigoplus_{m=0}^{r-1} O_X(-mK_X/\Delta)t^m, \, t^r = \theta.$$ 

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$$\pi_*O_Y \cong \bigoplus_{m=0}^{r-1} O_X(-mK_X/\Delta)t^m, \, t^r = \theta.$$ 

We shall prove that the closed fiber $Y_s$ is canonical or normal crossing, but we do not prove that the singularity of $Y$ is isolated at this point.

First, assume that the closed fiber $X_s$ of $X$ is irreducible. Since $O_X(-mK_X/\Delta) \otimes\otimes O_{X_s} \cong O_{X_s}(-mK_X/s)$ by Assumption 1.1 (6), $Y_s$ is isomorphic to the index 1 cover of $X_s$ constructed by using the restriction of $\theta$ to $X_s$. By Theorem 1, $Y_s$ is canonical. We can prove that its completed local ring at $Q$ is isomorphic to the completion of $A[x_1, x_2, x_3]/(F)$ with $\text{ord}(F_s) \leq 2$ as in the original proof of Theorem 4.1, where the action of $\mu_r$ on the coordinates $(x_1, x_2, x_3)$ is given by $x_i \mapsto \zeta^{a_i} \otimes x_i$, $(i = 1, 2, 3)$. Since $O_X(-K_X/\Delta)$ is not invertible, there exists at least 2 coordinates whose weights $a_i$ are coprime to $r$. Let $x_1, \ldots, x_c$ ($c = 2$ or 3) be such coordinates. Since $\theta = t^r$ never vanishes and the natural homomorphism $O_X(-K_X/\Delta) \otimes\otimes O_{X_s} \to \otimes O_{X_s}$, we have

$$\pi_*O_Y \cong \bigoplus_{m=0}^{r-1} O_X(-mK_X/\Delta)t^m, \, t^r = \theta.$$
\( \mathcal{O}_X(-rK_{X/\Delta}) \) is surjective outside \( \{ P \} \), we have \( \{ x_1 = \cdots = x_c = F = 0 \} = \{ Q \} \).

It follows that all the \( a_i \) are coprime to \( r \) and \( F \) contains a term in \( A \). Thus \( F_s \) is \( \mu_r \)-invariant, and \( \text{ord}(F_s) = 2 \). If \( F_s \) contains a term of the form \( x_1 x_2 \), then we are done. If it contains \( x_i^2 \) and there are no other terms of degree 2, then \( r = 2 \). But there is a term of degree 3 in \( F_s \), a contradiction.

Next, assume that \( X_s \) is reducible. Let \( X_{s,i} (1 \leq i \leq d) \) be its irreducible components. Since the \( X_{s,i} \) are \( \mathbb{Q} \)-Cartier divisors and the pairs \((X_{s,i}, D_i)\) for \( D_i = \sum_{j \neq i} X_{s,j} \cap X_{s,i} \) are log terminal, we have \( d = 2 \) or 3. Let \( r_i \) be the indices of the \((X_{s,i}, D_i)\). If \( d = 3 \), then \( r_i = 1 \) for all \( i \), and there are nowhere vanishing sections \( \theta_i \) of \( \mathcal{O}_{X_{s,i}}(K_{X_{s,i}} + D_i) \) which coincide each other on the double locus of \( X_s \) to give a nowhere vanishing section \( \theta_{X_s} \) of \( \mathcal{O}_{X_s}(K_{X_s}) \). Here we used the assumption that \( \mathcal{O}_{X_s}(K_{X_s}) \) has depth 2 at \( P \). Therefore, \( r = 1 \), a contradiction.

We consider the case \( d = 2 \). \( \theta \) induces a section \( \theta_{X_s} \) of \( \mathcal{O}_{X_s}(rK_{X_s}) \) and the sections \( \theta_i \) of \( \mathcal{O}_{X_{s,i}}(r(K_{X_{s,i}} + D_i)) \). Thus \( r_i | r \). We write \( r = r' p^f \) and \( r_i = r'_i p^f \) with \( (r', p) = 1 \) and \( (r'_i, p) = 1 \). We can construct a covering \( \pi' : X' \to X \) of degree \( r' \) by

\[
\pi'_* \mathcal{O}_{X'} \cong \bigoplus_{m=0}^{r'-1} \mathcal{O}_X(-mp^f K_{X/\Delta}) t^m, \quad t' = \theta.
\]

Then \( X'_{s,i} = \pi'^{-1} X_{s,i} \) is a union of \( r'/r'_i \) prime divisors which intersect only at a point \( \pi'^{-1}(P) \). Since \( X'_{s,i} \) supports a Cartier divisor on \( X' \), it follows that \( r' = r'_i \) for \( i = 1, 2 \).

Let \( \theta_0 \) be a section of \( \mathcal{O}_{X_s}(K_{X_s}) \) which does not vanish identically along the double locus \( D \) of \( X_s \). We write \( \theta_{X_s} = \alpha \theta_0^a \) as in Lemma 3. Since \( \theta \) is general, we may assume that \( \alpha_D = \alpha | D \not\subseteq L^p_D \), where \( L_D \) is the rational funtctin field of \( D \). Let \( Y_{s,i} = \pi^{-1}(X_{s,i}) \). We can extend Lemma 3 and apply it to the induced morphism \( \pi_i : Y_{s,i} \to X_{s,i} \) even if \( r_i \) might be smaller than \( r \), because \( L_D(\alpha_D^{1/p^f})/L_D \) is a purely inseparable field extension. Since \( \mathcal{O}_X(-mK_{X/\Delta}) \otimes \mathcal{O}_{Y_{s,i}} \cong \mathcal{O}_X(-m(K_{X_{s,i}} + D_i)) \) on \( Y_{s,i} \setminus \{ P \} \), \( Y_{s,i} \) is smooth possibly except at \( Q \), and \( \pi_i^* D_i \) is a reduced smooth divisor on \( Y_{s,i} \setminus \{ Q \} \). Thus \( Y_{s,i} \setminus \{ Q \} \) is a normal crossing divisor on \( Y \setminus \{ Q \} \). Since \( Y_s \) has depth 2, we conclude that the completed local ring of \( Y \) is isomorphic to the completion of \( A[x_1, x_2, x_3]/(F) \) with \( F_s = x_1 x_2 \) as in the original proof of Theorem 4.1. We may assume that the action of \( \mu_r \) on the coordinates \( (x_1, x_2, x_3) \) is given by \( x_i \mapsto \zeta^{a_i} \otimes x_i (i = 1, 2, 3) \), because the ideal \( (F_s) \) is preserved by this action. By the same reason as in the case where \( X_s \) is irreducible, all the \( a_i \) are coprime to \( r \) and \( F \) contains a term in \( A \) so that \( F = x_1 x_2 + \tau \). Since the \( X_{s,i} \) are Cartier divisors on \( X \setminus \{ P \} \), so are the \( Y_{s,i} \) on \( Y \setminus \{ Q \} \). Therefore, \( Y \setminus \{ Q \} \) is regular, and \( \tau \) is a generator of the maximal ideal of \( A \).

\[ \square \]

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