A COMBINATORIAL FRAMEWORK FOR RNA TERTIARY INTERACTION

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ABSTRACT. In this paper we show how to express RNA tertiary interactions via the concepts of tangled diagrams. Tangled diagrams allow to formulate RNA base triples and pseudoknot-interactions and to control the maximum number of mutually crossing arcs. In particular we study two subsets of tangled diagrams: 3-noncrossing tangled-diagrams with \( \ell \) vertices of degree two and 2-regular, 3-noncrossing partitions (i.e., without arcs of the form \((i, i + 1)\)). Our main result is an asymptotic formula for the number of 2-regular, 3-noncrossing partitions, denoted by \( p_{3,2}(n) \), 3-noncrossing partitions over \([n]\). The asymptotic formula is derived by the analytic theory of singular difference equations due to Birkhoff-Trjitzinsky. Explicitly, we prove the formula

\[
p_{3,2}(n + 1) \sim K \frac{n^n}{(1 + c_1/n + c_2/n^2 + c_3/n^3)}
\]

where \( K, c_i, i = 1, 2, 3 \) are constants.

1. Introduction

It is well-known that the functional repertoire of RNA is closely related to the variety of its shapes. Therefore it is of utmost importance to understand the structural “language” of RNA as this will eventually allow for fast folding, identification and discovery of new RNA functionalities. Studies of RNA structural motifs at high resolution by NMR and X-ray crystallographic methods provided insight into the fundamental forces that give rise to the unique structural characteristics of RNA. Non-Watson-Crick purine-pyrimidine, purine-purine, and pyrimidine-pyrimidine base pairing, as well as base-phosphate and base-ribose hydrogen bonding, are known to be important forces for folding and stabilizing RNA structures \[29\]. For RNA pseudoknots (viewed as interactions between unpaired bases) combinatorial abstractions have led to new interpretations, generating functions and enumeration results. Although far from having a complete understanding of RNA pseudoknots

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conceptual progress has been made in identifying the right concepts of, for instance crossing-complexity, which have direct implications for novel RNA pseudoknot folding algorithms. In this paper we build on the concepts derived in the context of RNA pseudoknots.

Before we begin by giving some background on RNA structure, let us remark why “combinatorial frameworks” are of central importance for any prediction algorithm. The above mentioned language of RNA is tantamount to uniquely specifying each element of the variety of shapes. Any prediction involves at some point a search through configurations and has to make sure that shapes are, for instance, not counted multiple times. The enumeration of the combinatorial class and analysis of its mathematical structure are of fundamental importance for designing such a search procedure. The primary sequence of an RNA molecule is its sequence of nucleotides A, G, U and C together with the Watson-Crick (A-U, U-A, G-C, C-G) and (U-G, G-U) base pairings. Single stranded RNA molecules form helical structures whose bonds satisfy the above base pairing rules and which, in many cases, determine their function. Due to the biochemistry of the base pairs stacked base pairs, i.e. arcs of the form \((i, j), (i - 1, j + 1)\) have typically a lower minimum free energy than crossing arcs. Base stacking is as important in determining RNA conformations as hydrogen bonding interactions. With the noncanonical interactions, many single-stranded loop regions such as hairpin loops, bulge loops, and internal loops fold into well-defined secondary structures. The prediction of RNA secondary structure is of complexity \(O(n^3)\) in time and \(O(n^2)\) in space for a sequence of length \(n\) [34, 35] which is result from the fact that no two bonds can cross.

**Figure 1.** The idea behind the notion of 3-noncrossing RNA structures. (a) secondary structure (with isolated labels 3, 7, 8, 10), (b) bi-secondary structure [15], 2, 9 being isolated (c) 3-noncrossing structure, which is not a bi-secondary structure. In fact, this is the smallest 3-noncrossing RNA structure which is not a bi-secondary structure.
While the concept of secondary structure is of fundamental importance, it is well-known that there exist additional types of nucleotide interactions [1]. These bonds are called pseudoknots [26] and occur in functional RNA (RNaseP [24]), ribosomal RNA [23] and are conserved in the catalytic core of group I introns. Stadler et al. [18] suggested a class of RNA pseudoknots called bi-secondary structures which are essentially “superpositions” of the arcs of two “secondary structures” and accordingly generalize from outer-planar to planar graphs, see Figure 1. Prediction algorithms for RNA pseudoknot structures are much harder to derive since there exists no a priori recursion and the subadditivity of local solutions is not guaranteed. The key for enumerating RNA pseudoknot structures is their categorization in terms of the maximal size of sets of mutually crossing bonds [19], i.e. the notion of k-noncrossing structures. To be precise, it is the inherent locality of the property “k-noncrossing” that allows for their enumeration by lattice paths. The diagram representation of a structure illustrates what k-noncrossing means, see Figure 1. In a diagram all nucleotides are drawn horizontally and the backbone bonds are ignored, then all bonds are drawn as arcs in the upper half-plane. The number of 3-noncrossing RNA structures satisfies $S_3(n) \sim 10.4724 \cdot 4^n \cdot \left(\frac{3+\sqrt{21}}{2}\right)^n$ [21], however, it is not the exponential growth rate of $(3+\sqrt{21})$ but the inherent non-recursiveness which makes the prediction difficult.

![Diagram of a tertiary interaction]

**Figure 2.** HIV-2 TAR, [3]. In HIV-2 TAR we have a (C38-G27)·C23+ triple mutant. Improved NMR spectral properties of HIV-2 TAR allowed the observation of the C23 amino and imino protons, providing direct evidence of hydrogen bonding interaction. The tertiary interaction is a tangled-diagram of with one vertex of degree two.

A first step towards RNA-tertiary structures beyond pseudoknot interactions consists in considering single strands interacting with helical regions by forming tertiary contacts with base-paired
nucleotides of the helices. Nucleotide triples occur when single-stranded nucleotides form hydrogen bonds with nucleotides that are already base paired. This hydrogen bonds can involve bases, sugars and phosphates. These interactions function to orient regions of secondary structures in large RNA molecules and to stabilize RNA three-dimensional structures. Base triples are a special case of nucleotide triple interactions in which base-base hydrogen bonding occurs. Single-stranded nucleotides can interact with base paired nucleotides via either the major groove or the minor groove of duplex regions. Nucleotide triples have been shown or proposed to form at junctions of coaxially stacked RNA helices that have adjacent single-stranded regions [29, 10]. Several major groove triples are present in tRNA where they function to stabilize its L-shaped three-dimensional structure. These interactions require to consider tangled diagrams [8], i.e. diagrams with vertices of degree \( \leq 2 \) which exhibit a variety of arc configurations, see Section 2. This variety is motivated from nucleotide interactions observed in RNA structures. In Figure 2 we show the HIV-2 TAR (C38-G27) \( \cdot \) C23\( ^{+} \) triple mutant structure as a tangled-diagram. Let us next have a closer look at the hammerhead structure-motif [10] in Figure 3. Comparing Figure 2 with Figure 3 reveals one feature of the hammerhead motif. It exhibits a lefthand-endpoint of degree 2 (incident to the dashed arc) while all other vertices of degree 2 are left- and righthand-endpoints. These two examples indicate that the majority of the bonds is organized in helical regions, where Watson-Crick and G-U(U-G) base pairs are stacked, additional stacks can be realized forming pseudoknots.

Figure 3. Diagram representation of the hammerhead ribozyme [10], which can be represented as a tangled-diagrams with two vertices of degree two. The gap after C25 indicates that some nucleotides are omitted, which are involved in an unrelated structural motif.
Finally in Figure 4 we display the catalytic core region of the group I self-splicing intron [9]. In order to express tertiary interactions we consider tangled-diagrams introduced in [8], which capture the nucleotide interactions relevant for the tertiary structure of the molecule [10].

![Figure 4. Catalytic core region of the group I self-splicing intron](image)

We will discuss two combinatorial frameworks arising from tangled-diagrams [8], both being suited for expressing RNA tertiary interactions. The first is the set of tangled-diagrams with fixed number of vertices of degree 2 and the second the set of 2-regular $k$-noncrossing partitions. While the former can easily be enumerated the latter requires more work. 2-regular $k$-noncrossing partitions evade lattice path enumeration due to their inherent asymmetry (lacking arcs of length 1). The “straightforward” ansatz via Inclusion-exclusion applied to the set of all $k$-noncrossing partitions revealed a connection between seemingly unrelated combinatorial objects: partitions and enhanced partitions, enumerated by Bousquet-Mélou and Xin [4, 20]. In Lemma 1 [20] we show how this relation can be used to obtain the enumeration. Subsequently, we prove the following a simple formula for the numbers of 2-regular $k$-noncrossing partitions

\[ p_{3,2}(n + 1) \sim K \, 8^n n^{-7} (1 + \frac{c_1}{n} + \frac{c_2}{n^2} + \frac{c_3}{n^3}) , \]
where $K = 6686.408973$, $c_1 = -28$, $c_2 = 455.777778$ and $c_3 = -5651.160494$. As for the quality of approximation we present the sub-exponential factors in the table below, where $g(n) = Kn^{-7}(1 + c_1/n + c_2/n^2 + c_3/n^3)$.

| n   | $\rho_3(n)/8^n$ | $g(n)$     | n   | $\rho_3(n)/8^n$ | $g(n)$     |
|-----|-----------------|------------|-----|-----------------|------------|
| 21  | $1.479 \times 10^{-6}$ | $1.726 \times 10^{-7}$ | 81  | $2.270 \times 10^{-10}$ | $2.264 \times 10^{-10}$ |
| 31  | $1.283 \times 10^{-7}$ | $1.112 \times 10^{-7}$ | 91  | $1.033 \times 10^{-10}$ | $1.031 \times 10^{-10}$ |
| 41  | $2.104 \times 10^{-8}$ | $2.026 \times 10^{-8}$ | 101 | $5.088 \times 10^{-11}$ | $5.081 \times 10^{-11}$ |
| 51  | $5.011 \times 10^{-9}$ | $4.939 \times 10^{-9}$ | 501 | $8.100 \times 10^{-16}$ | $8.095 \times 10^{-16}$ |
| 61  | $1.524 \times 10^{-9}$ | $1.514 \times 10^{-9}$ | 1001| $6.507 \times 10^{-18}$ | $6.502 \times 10^{-18}$ |
| 71  | $5.514 \times 10^{-10}$ | $5.493 \times 10^{-10}$| 10001| $6.672 \times 10^{-25}$ | $6.668 \times 10^{-25}$ |

Our analysis is based on the theory of Birkhoff-Trjitzinsky, which seems to be somewhat overlooked. While the two original papers [5, 6] are hard to read, the paper of Wimp and Zeilberger [33] provides a good introduction and shows via various examples of how to apply the theory. Since the method (if it applies) is quite powerful we give an overview of the analytic theory of singular difference equations in the Appendix.

2. Vacillating tableaux and tangled-diagrams

2.1. Tangled-diagrams. A tangled-diagram over $[n]$ is a triple of sets $(V, E, F)$, where $V$ is a finite non-empty set of $n$ elements called vertices, $E$ is a set of unordered pairs of vertices called arcs and $F$ is the flag set whose elements are the 2-degree points such that they are the ends of two crossing arcs, represented by drawing its vertices in a horizontal line and its arcs $(i, j)$ in the upper halfplane with the following basic configurations and the isolated points.
Composing these motifs we obtain a tangled-diagram, for instance, the tangled-diagram

![Tangled-diagram](image)

has $V = [23]$ and $F = \{1, 18, 23\}$. Let us introduce several important subclasses of 3-noncrossing tangled-diagrams:

1. **3-noncrossing matchings with isolated points** are 3-noncrossing tangled-diagrams in which each vertex has degree at most 1. For instance, RNA pseudoknot structures [19] are 3-noncrossing matchings with isolated points, see Figure 5.

2. **2-regular, 3-noncrossing partitions.** A partition corresponds to a tangled-diagram in which any vertex of degree two, $j$, is incident to the arcs $(i, j)$ and $(j, s)$, where $i < j < s$, for instance, see Figure 4 and Figure 6 (a). Partitions without arcs of the form $(i, i+1)$ are called 2-regular, partitions.

3. **3-noncrossing braids without isolated points** are tangled-diagrams in which all vertices, $j$ of degree two are either incident to loops $(j, j)$ or crossing arcs $(i, j)$ and $(j, h)$, where $i < j < h$, see Figure 6 (b).

4. **3-noncrossing diagrams with $\ell$ vertices of degree 2.** Figure 2, Figure 3 and Figure 4 are 3-noncrossing tangled-diagrams with $\ell = 1, 2, 6$ vertices of degree 2. The following tangled-diagram shows all 4 basic types of degree 2 vertices in tangled diagrams.
In the following, we study the subclasses (2) and (4) since they represent a natural framework for RNA tertiary interactions. It turns out that (3) is of importance since it facilitates the enumeration of (2). To be precise it is shown in [20] that there is a duality between $k$-noncrossing braids without isolated points and 2-regular $k$-noncrossing partitions.

Having introduced the combinatorial framework, one key question is how to enumerate the subclasses (2) and (4). The enumeration is facilitated via a bijection between the tangled-diagrams and certain lattice paths. To derive the latter a bijection between tangled-diagrams and (generalized) vacillating tableaux is constructed. It is then easy to see that vacillating tableaux correspond to lattice paths. In the next Section we provide some background on vacillating tableaux and the bijection.

2.2. Vacillating tableaux. A Young diagram (shape) is a collection of squares arranged in left-justified rows with weakly decreasing number of boxes in each row. A Young tableau is a filling of the squares by numbers which is weakly decreasing in each row and strictly decreasing in each column. A tableau is called standard if each entry occurs exactly once. A tableau-sequence is a sequence $\emptyset = \mu^0, \mu^1, \ldots, \mu^n = \emptyset$ of standard Young diagrams, such that for $1 \leq i \leq n$, $\mu^i$ is obtained from $\mu^{i-1}$ by either adding one square, removing one square or doing nothing.
The RSK-algorithm is a process of row-inserting elements into a tableau. Suppose we want to insert \( k \) into a standard Young tableau \( \lambda \). Let \( \lambda_{i,j} \) denote the element in the \( i \)-th row and \( j \)-th column of the Young tableau. Let \( i \) be the largest integer such that \( \lambda_{1,i-1} \leq k \). (If \( \lambda_{1,1} > k \), then \( i = 1 \).) If \( \lambda_{1,i} \) does not exist, then simply add \( k \) at the end of the first row. Otherwise, if \( \lambda_{1,i} \) exists, then replace \( \lambda_{1,i} \) by \( k \). Next insert \( \lambda_{1,i} \) into the second row following the above procedure and continue until an element is inserted at the end of a row. As a result we obtain a new standard Young tableau with \( k \) included. For instance inserting the number sequence 5, 2, 4, 1, 6, 3 starting with an empty shape yields the following sequence of standard Young tableaux:

A vacillating tableaux \( \mathcal{V}_{\lambda}^{2n} \) of shape \( \lambda \) and length \( 2n \) is a sequence \((\lambda^0, \lambda^1, \ldots, \lambda^{2n})\) of shapes such that (i) \( \lambda^0 = \emptyset \) and \( \lambda^{2n} = \lambda \), and (ii) \( \lambda^{2i-1}, \lambda^{2i} \) is derived from \( \lambda^{2i-2} \), for \( 1 \leq i \leq n \) by either \((\emptyset, \emptyset)\): doing nothing twice; \((-\square, \emptyset)\): first removing a square then doing nothing; \((\emptyset, +\square)\): first doing nothing then adding a square; \((\pm\square, \pm\square)\): adding/removing a square at the odd and even steps, respectively. Let \( \mathcal{V}_{\lambda}^{2n} \) denote the set of vacillating tableaux. For instance, let us consider the following vacillating tableaux:

2.3. A bijection between vacillating tableaux and tangled-diagrams. When constructing the bijection between vacillating tableaux and tangled-diagrams in Theorem 1 below, the notion of the inflation of a tangled-diagram is important. We are now able to discuss the bijection between vacillating tableaux and tangled diagrams.

**Theorem 1.** There exists a bijection between the set of vacillating tableaux of shape \( \emptyset \) and length \( 2n \), \( \mathcal{V}_{\emptyset}^{2n} \) and the set of tangled-diagrams over \( n \) vertices, \( \mathcal{G}_n \).

\[ (2.1) \quad \beta: \mathcal{V}_{\emptyset}^{2n} \rightarrow \mathcal{G}_n. \]
Figure 7. The inflation map: each vertex $i$ of degree 2 is replaced by a pair of vertices, $(i, i')$, each incident to an respective arc.

Figure 8. From tangled-diagrams to lattice paths. First the tangled-diagram (upper left) is resolved into its vacillating tableaux (upper right). Reading the numbers of squares in the corresponding rows (bottom right) induces the $2n$-step lattice path (bottom right), which starts and ends in $(1,0)$. The path has $\emptyset$ (green points), $+\square$ and $-\square$ (red and purple points) induced by the pair steps $(\emptyset, +\square)$, $(-\square, \emptyset)$ and $(-\square, -\square)$. Note that the lattice path does not touch the “wall” $x = y$.

Furthermore a tangled-diagram $G_n$ is $k$-noncrossing if and only if all shapes $\lambda^i$ in its vacillating tableaux have less than $k$ rows. That is $\phi: \mathcal{V}_2^{2n} \rightarrow \mathcal{S}_n$ maps vacillating tableaux having less than $k$ rows into $k$-noncrossing tangled-diagrams.

The proof of Theorem 1 relies on the idea to resolve the vertices of degree 2 via an inflation, i.e. vertex $i$ is resolved by the pair $(i, i')$, where we utilize the linear order $1 < 1' < 2 < 2' < \cdots < (n - 1) < (n - 1)' < n < n'$. The inflation transforms each tangled-diagram into a partial matching with isolated points. For instance, Restricting the steps for vacillating tableaux produces the bijections of Chen et al. Let $\mathcal{M}_k(n)$, $\mathcal{P}_k(n)$ and $\mathcal{B}_k(n)$ denote the set of $k$-noncrossing matchings, partitions and braids without isolated points over $[n]$, respectively. Theorem 1 basically says the tableaux-sequences $\mathcal{M}_k(n)$, $\mathcal{P}_k(n)$ and $\mathcal{B}_k(n)$ are composed by the elements in $S_{\mathcal{M}_k}, S_{\mathcal{P}_k}$ and
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$S_{B_k^+}$, respectively, where

$$S_{M_k} = \{(-\Box_h, \emptyset), (\emptyset, +\Box_h)\}$$

$$S_{P_k} = \{(-\Box_h, \emptyset), (\emptyset, +\Box_h), (\emptyset, (-\Box_h, +\Box_l))\}$$

$$S_{B_k^+} = \{(-\Box_h, \emptyset), (\emptyset, +\Box_h), (+\Box_h, -\Box_l)\} \quad 1 \leq h, l \leq k - 1$$

and $\pm \Box_h$ denote the adding or subtracting of the rightmost square “$\Box_h$” in the $h$th row in a given shape $\lambda$ and let “$\emptyset$” denote doing nothing. To get some intuition above the particular steps and diagram-configurations let us show the key correspondences between tableaux and diagram-motifs

$$3. \text{k-NONCROSSING TANGLED DIAGRAMS AND 2-REGULAR, k-NONCROSSING PARTITIONS}$$

In this section we prove two enumeration results. We give explicit formulas for $k$-noncrossing tangled diagrams with a fixed number of degree 2 vertices and 2-regular $k$-noncrossing partitions. Since the latter formula is quite complicated we provide a simple asymptotic expression in Section 4.

Let $f_k(n)$ denote the number of perfect matching over $[n]$ and $C_m$ be the Catalan number. Our first result reads

**Theorem 2.** The number of the $k$-noncrossing tangled-diagrams over $[n]$ with $\ell$ vertices of degree two, denoted by $d_{\ell,k}(n)$ is given by

$$d_{\ell,k}(n) = \sum_{i=0}^{n} \binom{n}{i} \binom{n-i}{\ell} f_k(n-i+\ell)$$

and in particular for $k = 3$ we have

$$d_{\ell,3}(n) = \sum_{i=0}^{n} \binom{n}{i} \binom{n-i}{\ell} \left( C_{n-i+\ell}^3 C_{n-i+\ell+2}^2 - C_{n-i+\ell+1}^2 \right).$$

**Proof.** Let $D_{i,\ell,k}$ denote the set of tangled-diagrams over $[n]$ with $i$ isolated points and $\ell$ vertices of degree two and $d_{i,\ell,k} = |D_{i,\ell,k}|$. There are $\binom{n}{i} \binom{n-i}{\ell}$ ways to choose the locations of the isolated
points and the vertices of degree two. Furthermore for an arbitrary tangled-diagram over $V = [n]$ with $i$ isolated points $V_1 = \{v_1, v_2, \ldots, v_i\} \subset V$ and $\ell$ vertices of degree two $V_2 = \{v_{i+1}, \ldots, v_{i+\ell}\} \subset V$, let $\tilde{V} = V \setminus (V_1 \cup V_2) = \{v_{i+\ell+1}, \ldots, v_n\}$ be the set of vertices of degree one, via the inflation we will have a perfect matching over $|\{V_2 \cup V_2' \cup \tilde{V}\}| = [2\ell + n - i - \ell] = [n - i + \ell]$, where $V_2' = \{v'_{i+1}, \ldots, v'_{i+\ell}\}$. Since $d_{\ell,k} = \sum_{i=0}^n d_{i,\ell,k}$, the theorem follows.

The first 10 number for $d_{i,\ell,3}$ for $\ell = 1, 2, 3$ and $n = 1 \ldots 10$ are given by

| $\ell,n$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|----------|----|----|----|----|----|----|----|----|----|----|
| 1        | 1  | 2  | 12 | 40 | 165| 606| 2380| 9136| 36099| 142750|
| 2        | 0  | 3  | 9  | 102| 450| 2565| 11823| 57876| 266220| 1243170|
| 3        | 0  | 0  | 14 | 56 | 980| 5320| 38920| 214144| 1251852| 6672120|

We proceed by enumerating 2-regular $k$-noncrossing partitions. A valid approach for this consists in building on the enumeration results of [4] for $k$-noncrossing partitions using the inclusion-exclusion principle. This strategy leads to functional equations which prove that the asymptotic formulas of 2-regular $k$-noncrossing partitions and braids without isolated points coincide. But braids can be enumerated via kernel methods [22, 11, 13] directly, while 2-regular $k$-noncrossing partitions cannot. This suggests an alternative ansatz [20], by directly establishing a relation between partitions and braids and consequently enumerating partitions via braids. In Lemma 1 below we show this correspondence. To this end we replace in a braid without isolated points each loop by an isolated vertex and each pair of crossing arcs at a degree 2 vertex by noncrossing arcs, i.e.
Accordingly, we can identify braids without isolated points with a subset of 3-noncrossing partitions.

Lemma 1. Let \( k \in \mathbb{N} \), \( k \geq 3 \). Then we have the bijection

\[
\vartheta: \mathcal{P}_{k,2}(n) \longrightarrow \mathcal{B}_k^{1}(n-1),
\]

where \( \vartheta \) has the following property: for any \( \pi \in \mathcal{P}_k(n) \) holds: \( (i,j) \) is an arc of \( \pi \) if and only if \( (i,j-1) \) is an arc in \( \vartheta(\pi) \).

Proof. By construction, \( \vartheta \) maps tangled-diagrams over \([n]\) into tangled diagrams over \([n-1]\). Since there exist no arcs of the form \((i,i+1)\), \( \vartheta(\pi) \) is, for any \( \pi \in \mathcal{P}_{k,2}(n) \) loop-free. By construction, \( \vartheta \) preserves the orientation of arcs, whence \( \vartheta(\pi) \) is a partition.

Claim. \( \vartheta: \mathcal{P}_{k,2}(n) \longrightarrow \mathcal{B}_k^{1}(n-1) \) is well-defined.

We first prove that \( \vartheta(\pi) \) is \( k \)-noncrossing. Suppose there exist \( k \) mutually crossing arcs, \((i_s,j_s),\) \( s = 1,\ldots,k \) in \( \vartheta(\pi) \). Since \( \vartheta(\pi) \) is a partition we have \( i_1 < \cdots < i_k < j_1 < \cdots < j_k \). Accordingly, we obtain for the partition \( \pi \in \mathcal{P}_{k,2}(n) \) the \( k \) arcs \((i_s,j_s+1),\) \( s = 1,\ldots,k \) where \( i_1 < \cdots < i_k < j_1 + 1 < \cdots < j_k + 1 \), which is impossible since \( \pi \) is \( k \)-noncrossing. We next show that \( \vartheta(\pi) \) is a \( k \)-noncrossing braid. If \( \vartheta(\pi) \) is not a \( k \)-noncrossing braid, then according to eq. \((3.1)\) \( \vartheta(\pi) \) contains \( k \) arcs of the form \((i_1,j_1),\ldots,(i_k,j_k)\) such that \( i_1 < \cdots < i_k = j_1 < \cdots < j_k \) holds. Then \( \pi \) contains the arcs \((i_1,j_1+1),\) \((i_k,j_k+1)\) where \( i_1 < \cdots < i_k < j_1 + 1 < \cdots < j_k + 1 \), which is impossible since these arcs are a set of \( k \) mutually crossing arcs and the claim follows.

Claim. \( \vartheta \) is bijective.

Clearly \( \vartheta \) is injective and it remains to prove surjectivity. For any \( k \)-noncrossing braid \( \delta \) there exists some \( 2 \)-regular partition \( \pi \) such that \( \vartheta(\pi) = \delta \). We have to show that \( \pi \) is \( k \)-noncrossing. Let \( M' = \{(i_1,j_1),\ldots,(i_k,j_k)\} \) be a set of \( k \) mutually crossing arcs, i.e. \( i_1 < \cdots < i_k < j_1 < \cdots < j_k \). Then we have in \( \vartheta(\pi) \) the arcs \((i_s,j_s-1),\) \( s = 1,\ldots,k \) and \( i_1 < \cdots < i_k < j_1 - 1 < \cdots < j_k - 1 \). If \( M = \{(i_1,j_1-1),\ldots,(i_k,j_k-1)\} \) is \( k \)-noncrossing then we conclude \( i_k = j_1 - 1 \). Therefore \( M \) impossible in \( k \)-noncrossing braids. By transposition we have thus proved that any \( \vartheta \)-preimage is necessarily a
The bijection $\vartheta : \mathcal{P}_{k,2}(n) \rightarrow \mathcal{B}^1_k(n-1)$. Crossings are reduced by contracting the arcs.

$k$-noncrossing partition, whence the claim and the proof of the lemma is complete.

As an illustration of the bijection of Lemma 1 we display Via Lemma 1 we have reduced the enumeration of 2-regular $k$-noncrossing partitions to that of braids without isolated points. Let us discuss how the latter can be enumerated via lattice paths. From Theorem 1, (see Figure 2.3) we know that a 3-noncrossing braid corresponds to a lattice paths in the first quadrant with the following properties:

1. the path starts and ends at $(1,0)$,
2. each step pair $(2i-1, 2i)$, where $1 \leq i \leq n$ is an element of
   \[\{(0,+e_1), (0,+e_2), (-e_1,0), (-e_2,0), (+e_1,-e_1), (+e_2,-e_2), (+e_1,-e_2), (+e_2,-e_1)\} .\]
3. the path never touches the wall $x = y$.

The key result facilitating the enumeration is the reflection principle due D. Andrè in 1887 [2] and subsequently generalized by Gessel and Zeilberger [14]. It is worth mentioning that this strategy is nonconstructive since enumeration is obtained by counting all paths and having paths touching the wall cancel each other.

**Theorem 3. (Reflection-Principle)** [14] Suppose $S \in \{M_3, P_3, B_3\}$ and let $\Omega_S^{(1,0)}(2n)$ denote the number of $S$-walks of length $2n$ from $(1,0)$ to $(1,0)$ that remain in the region $R = \{(x,y) \mid x > y \geq 0, (x,y) \in \mathbb{Z}^2\}$. Let furthermore $f_{(x,y)}^{(x,y)}(2n)$ be the number of $S$-walks from $(x,y)$ to $(x',y')$ of length $2n$ that remain in the first quadrant. Then we have

$$
\Omega_S^{(1,0)}(2n) = f_{(1,0)}^{(1,0)}(2n) - f_{(1,0)}^{(0,1)}(2n).
$$

**Proof.** Suppose $\gamma$ is a $S$-walk starting and ending at $(1,0)$ which remains in the first quadrant and that touches the diagonal $x = y$. Let $(a,a)$ be the first point where $\gamma$ touches the diagonal $y = x$. Reflect all steps of $\gamma$ after $\gamma$ touched the diagonal in $(a,a)$ and denote the resulting walk by $\gamma'$. Then $\gamma'$ is a $S$-walk starting from $(1,0)$ and ending at $(0,1)$. This procedure yields a unique pair $(\gamma, \gamma')$ for each $S$-walk $\gamma$ starting and ending at $(1,0)$ which remains in the first quadrant and that
Figure 10. The reflection principle. The original lattice path (blue) starting and ending at $(1, 0)$ touches the wall $x = y$ at $(3, 3)$ for the first time. The corresponding reflected path (red) starts at $(1, 0)$ and ends at $(0, 1)$ obtained by reflecting all steps after $(3, 3)$ w.r.t. the wall $x = y$.

touches the diagonal $x = y$. According to eq. (3.2) these pairs cancel themselves and only the paths that never touch the diagonal remain, whence the theorem.

Using the reflection principle we can enumerate braids via the kernel method [22, 11, 13]. In fact these computation have been obtained by [4] who enumerated enhanced partitions. Our second result reads

**Theorem 4.** The number of 2-regular, 3-noncrossing partitions is given by

$$p_{3,2}(n+1) = \sum_{s \in \mathbb{Z}} [\beta_n(1,0,s) - \beta_n(1,-1,s) - \beta_n(1,-3,s) + \beta_n(1,-3,s)$$

$$-\beta_n(3,4,s) + \beta_n(3,3,s) + \beta_n(3,0,s) - \beta_n(3,1,s)$$

$$+ \beta_n(2,5,s) - \beta_n(2,4,s) - \beta_n(2,1,s) + \beta_n(2,2,s)] ,$$

where $\beta_n(k, m, s) = \frac{k}{n+1} \binom{n+1}{s} \binom{n+1}{k+s} \binom{n+1}{s+m}$. Furthermore $p_{3,2}(n)$ satisfies the recursion

$$\alpha_1(n) p_{3,2}(n+1) + \alpha_2(n) p_{3,2}(n+2) + \alpha_3(n) p_{3,2}(n+3) - \alpha_4(n) p_{3,2}(n+4) = 0 ,$$
where

\[
\begin{align*}
\alpha_1(n) &= 8(n + 2)(n + 3)(n + 1) \\
\alpha_2(n) &= 3(n + 2)(5n^2 + 47n + 104) \\
\alpha_3(n) &= 3(n + 4)(2n + 11)(n + 7) \\
\alpha_4(n) &= (n + 9)(n + 8)(n + 7).
\end{align*}
\]

For instance, the first 12 numbers of 2-regular, 3-noncrossing partitions are given by

| n  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|
| p_{3,2}(n) | 1  | 1  | 2  | 5  | 15 | 51 | 191 | 772 | 3320 | 15032 | 71084 | 348889 |

We will show in the next section that the formulas given in Theorem 4 have simple asymptotic formulas.

4. Asymptotic Analysis

In this section we employ the particularly elegant theory of singular difference equations due to Birkhoff and Trjitzinsky [6]. The theory of Birkhoff-Trjitzinsky establishes form, existence and properties of such fundamental sets in general, and will be discussed in the Appendix. For our purposes it suffices to identify the unique, monotonously increasing formal series solution (FSS).

**Theorem 5.** There exists some real constants $K > 0$ and $c_1, c_2, c_3$ such that

\[
(4.1) \quad p_{3,2}(n + 1) \sim K \cdot 8^n n^{-7}(1 + c_1/n + c_2/n^2 + c_3/n^3)
\]

holds. Explicitly, we have $K = 6686.408973$, $c_1 = -28$, $c_2 = 455.77778$ and $c_3 = -5651.160494$.

**Proof. Claim.** There exists some $K > 0$ and $c_1, c_2, c_3 \ldots$ such that

\[
(4.2) \quad p_{3,2}(n + 1) \sim K \cdot 8^n n^{-7}(1 + c_1/n + c_2/n^2 + c_3/n^3 \ldots).
\]

Theorem 3 guarantees the existence of 3 linearly independent formal series solutions (FSS) for eq. (3.3). We proceed by constructing these using the following ansatz for $p_{3,2}(n)$:

\[
(4.3) \quad p_{3,2}(n + 1) = E(n) K(n) \quad E(n) = e^{\mu_0 n \ln n + \mu_1 n^0}
\]
(4.4) \[ K(n) = \exp\{\alpha_1 n^{\beta+\alpha_2 n^{\beta-1}/\rho} \}, \quad \alpha_1 \neq 0, \; \beta = j/\rho, \; 0 \leq j < \rho. \]

We immediately derive setting \( \lambda = e^{\mu_0+\mu_1} \)

\[
\frac{p_{3,2}(n+k+1)}{p_{3,2}(n+1)} = n^{\mu_0} \lambda^k \{ 1 + \frac{k\theta + k^2\mu_0/2}{n} + \cdots \} \exp\{ \alpha_1 \beta n^{\beta-1} + \alpha_2 (\beta - 1/\rho) n^{\beta-1/\rho-1} + \cdots \}. 
\]

We arrive at

\[
0 = 1 + \frac{15}{8} \left\{ 1 + \frac{\theta + \mu_0/2 + \frac{27}{2}}{n} + \cdots \right\} \xi \{ 1 + (\alpha_1 \beta n^{\beta-1} + \alpha_2 (\beta - 1/\rho) n^{\beta-1/\rho-1} + \cdots) + \cdots \} \\
+ \frac{3}{4} \left\{ 1 + \frac{2\theta + 2\mu_0 + \frac{21}{2}}{n} + \cdots \right\} \xi^2 \{ 1 + (2\alpha_1 \beta n^{\beta-1} + 2\alpha_2 (\beta - 1/\rho) n^{\beta-1/\rho-1} + \cdots) + \cdots \} \\
- \frac{1}{8} \left\{ 1 + \frac{3\theta + 9\mu_0/2 + 18}{n} + \cdots \right\} \xi^3 \{ 1 + (3\alpha_1 \beta n^{\beta-1} + 3\alpha_2 (\beta - 1/\rho) n^{\beta-1/\rho-1} + \cdots) + \cdots \}. 
\]

First we consider the maximum power of \( n \), which is zero. In view of \( 1 = \frac{3}{8} n^{3\mu_0} \lambda^3 \) we obtain \( \mu_0 = 0 \). This implies \( \rho = 1 \) since \( \rho \geq 1 \) and \( \rho \) should be the smallest integer s.t. \( \rho \mu_0 \in \mathbb{N} \). Equating the constant terms again, we obtain that \( \lambda \) is indeed a root of the cubic polynomial \( P(X) \)

\[
P(X) = 1 + \frac{15}{8} X + \frac{3}{4} X^2 - \frac{1}{8} X^3.
\]

Therefore we have \( \lambda = 8 \) or \(-1\). Notice that \( 0 \leq \beta < 1 \) implies \( \beta = 0 \). Otherwise, equating the coefficient of \( n^{\beta-1} \) implies \( \alpha_1 = 0 \), which is impossible. It remains to compute \( \theta \). For this purpose we equate the coefficient of \( n^{-1} \), i.e. \( 8 \frac{15}{8}(\theta + \frac{27}{8}) + 8^2 \frac{3}{4}(\frac{21}{2} + 2\theta) - 8 \frac{1}{8}(18 + 3\theta) = 0 \) from which we conclude \( \theta = -7 \). Since \( p_{3,2}(n) \) is monotonously increasing \( p_{3,2}(n) \) coincides with the only monotonously increasing FSS, given by

\[
p_{3,2}(n+1) \sim K \cdot 8^n \cdot n^{-7}(1 + c_1/n + c_2/n^2 + c_3/n^3 \cdots)
\]

for some \( K > 0 \) and constants \( c_1, c_2, c_3 \) and the proof of the claim is complete. We compute \( c_1 = -28, \; c_2 = 455.778 \) and \( c_3 = -5651.160494 \) by equating the coefficients of \( n^{-2}, \; n^{-3} \) and \( n^{-4} \),

\[
(2268 + 81c_1 = 0, \; 1683c_1 + 162c_2 - 26712 = 0 \text{ and } -32547c_1 + 729c_2 + 129654 + 243c_3 = 0) \text{ and finally get } K = 6686.408973 \text{ numerically to complete the proof of the theorem.} \]

\( \square \)
5. Appendix

5.1. The Birkhoff-Trjitzinsky theory. Any difference equation with rational coefficients can be written as

\[ \sum_{h=0}^{m} C_m(n) y(n + h) = 0 \quad C_0(n) = 1, \quad C_m(n) \neq 0, \quad n = 0, 1, 2, \ldots \]

where the coefficients possess representations as generalized Poincaré series

\[ C_h(n) \sim n^K \left[ c_{0,h} + c_{1,h} n^{-\frac{1}{\omega}} + c_{2,h} n^{-\frac{2}{\omega}} + \ldots \right], \quad h = 1, 2, \ldots . \]

Here \( K \) is an integer, \( \omega \) is an integer \( \geq 1 \) independent of \( h \) and \( c_{0,h} \neq 0 \) unless \( C_h(n) = 0 \). We shall assume that \( \omega \) is minimal. A set of functions \( z^{(j)}(n) \) is called linearly independent if the determinant

\[ \forall n \in \mathbb{N} \cup \{0\}; \quad \det (z^{(j+1)}(n + i))_{0 \leq i, j \leq h-1} \neq 0. \]

The classical theory of difference equations asserts that eq. (5.1) possesses a set of linearly independent solutions constituting a basis of the solution space. Such a set is called a fundamental set. The Birkhoff-Trjitzinsky theory proves that there exists a fundamental set in which all elements have an asymptotic expansion consisting of an exponential leading term multiplied by a linear combination of descending series of the form eq. (5.2). To provide the notion of formal series solution and Birkhoff series we set

\[
Q(\rho, n) = \rho_0 n \ln(n) + \sum_{j=1}^{\rho} \mu_j n^{\frac{\mu_{j+1} \pm 1}{\omega}}, \quad s(\rho, n) = n^\theta \sum_{j=0}^{t} (\ln(n))^j n^{\frac{r_j}{\omega}} q_j(\rho, n),
\]

\[
q_j(\rho, n) = \sum_{s=0}^{\infty} b_{s,j} n^{-\frac{s}{\rho}}
\]

where \( \rho, r_j, \mu_0 \rho \) are integers, \( \rho \geq 1, \mu_j, \theta, b_{s,j} \in \mathbb{C}, b_{0,j} \neq 0 \), unless \( b_{s,j} = 0 \) for \( s = 0, 1, 2, \ldots, r_0 = 0, -\pi \leq \text{Im}(\mu_1) < \pi \). Then we call

\[ y(\rho, n) = e^{Q(\rho, n)} s(\rho, n) \]

a formal series solution (FSS) of eq. (5.1) if and only if substituted in eq. (5.1) after dividing by \( e^{Q(\rho, n)} \) and corresponding algebraic transformations, the coefficients of

\[ n^{\rho + \frac{t}{\omega}} \ln(n)^j, \quad r, s = 0, 1, \ldots, t \quad r, s = 0, \pm 1, \pm 2, \ldots, \]

are equal to zero. For given sequence \( (f(n))_{n \geq 0} \) we furthermore call

\[ f(n) \sim e^{Q(\rho, n)} s(\rho, n) \]
the Birkhoff series for \( f(n) \) if and only if for every \( k \geq 1 \) there exist bounded functions \( A_{kj}(n) \), \( j = 0, 1, \ldots, t \), such that

\[
e^{-Q(\rho,n)} n^{-\theta} f(n) = \sum_{j=0}^{t} \ln(n)^3 n^{\frac{r_{n+1}}{\theta}} b_{kj} n^{\frac{s}{\theta}} + n^{-\frac{1}{\theta}} \sum_{j=0}^{t} \ln(n)^3 n^{\frac{r_{n+1}}{\theta}} A_{kj}(n).
\]

Following [33] we define

\[
w_k = \det \left( e^{Q_{j+1}(\rho,n+i)} s_{j+1}(\rho,n+i) \right)_{0 \leq i,j \leq k-1}.
\]

The main result of the Birkhoff-Trjitzinsky theory can now be stated as follows

**Theorem 6.** [5, 6] There exist exactly \( m \) FSS of eq. (5.1) of type \( e^{Q(\rho,n)} s(\rho,n) \) where \( \rho = \nu \omega \) for some integer \( \nu \geq 1 \) and each FSS represents asymptotically some solution of the equation. The above FSS are, up to multiplicative constants, unique and the \( m \) solutions so represented constitute a fundamental set for the equation.

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