Verifying Concisely Represented Strategies in One-Counter Markov Decision Processes

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Verifying one-counter Markov decision processes

We study one-counter Markov decision processes (OC-MDPs).

- Markov decision process (MDP): models systems with non-determinism and randomness.
- Counter: can be incremented, decremented, left unchanged.

An OC-MDP induces a countable-state MDP.

Verification problem

Given a strategy, an objective and a threshold, is the probability of the objective being satisfied no less than the threshold?

We focus on a class of memoryless strategies of the infinite MDP that admit a finite representation.

We study variants of reachability objectives.
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Markov decision processes

Markov decision process (MDP) $\mathcal{M}$

- **Finite or countable** state space $S$.
- **Finite** action space $A$.
- **Randomised** transition function $\delta : S \times A \to \mathcal{D}(S)$.

Plays are sequences in $(SA)^\omega$ coherent with transitions.

Example: $s_0 a s_1 b s_1 \ldots$
A strategy is a function $\sigma : (SA)^* S \to D(A)$.

$\sigma$ is memoryless if its choices depend only on the current state.

We view memoryless strategies as functions $S \to D(A)$.

A memoryless strategy $\sigma$ induces a Markov chain over $S$. 

\begin{itemize}
  \item $s_0$ \xrightarrow{a} s_1 \xrightarrow{b} s_0$
  \item $s_1$ \xrightarrow{a} s_3$
  \item $s_2$ \xrightarrow{b} s_2$
  \item $s_3$ \xrightarrow{a} s_3$
\end{itemize}
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One-counter Markov decision processes

One-counter MDP (OC-MDP) \( Q \)

- **Finite** MDP \( (Q, A, \delta) \).
- **Weight function** \( w : Q \times A \rightarrow \{-1, 0, 1\} \).

MDP \( \mathcal{M}^{\leq \infty}(Q) \) induced by \( Q \)

- **Countable** MDP over \( S = Q \times \mathbb{N} \).
- State transitions via \( \delta \).
- Counter updates via \( w \).
Interval strategies

We study two classes of memoryless strategies of $\mathcal{M}^{\leq \infty}(Q)$.

- **Open-ended interval strategies (OEIS):** $\sigma$ is an OEIS if there exists $k_0 \in \mathbb{N}$ such that, for all $q \in Q$ and all $k \geq k_0$, $\sigma(q, k) = \sigma(q, k_0)$.

**Representing an OEIS**

An OEIS is described by a *finite partition* $\mathcal{I}$ of $\mathbb{N}_0$ into intervals and a *function* $Q \times \mathcal{I} \rightarrow \mathcal{D}(A)$.

- **Cyclic interval strategies (CIS):** $\sigma$ is a CIS if there exists $\rho \in \mathbb{N}_0$ such that, for all $q \in Q$ and all $k \in \mathbb{N}_0$, $\sigma(q, k) = \sigma(q, k + \rho)$.

**Representing a CIS**

A CIS is described by a *period* $\rho$, a *partition* $\mathcal{I}$ of $[1, \rho]$ into intervals and a *function* $Q \times \mathcal{I} \rightarrow \mathcal{D}(A)$.
Conciseness of interval strategies

- Let $\sigma$ be an interval strategy of $M^{\leq \infty}(Q)$.
- There exists a strategy of $Q$ that induces the same behaviour as $\sigma$ when an initial counter value is fixed $\leadsto$ memory $=$ counter value.

OEISs may require infinite memory

The OEIS $\sigma$ such that:
- $\sigma(p, 1) = b$
- $\sigma(p, k) = a$ for all $k \geq 2$.
requires infinite memory in $Q$.

- CISs correspond to exponential-size finite-memory strategies of $Q$.
Objectives

Let $Q = (Q, A, \delta, w)$ be an OC-MDP. We consider two objectives for a target $T \subseteq Q$.

- **State reachability**: $\text{Reach}(T)$ is the set of plays visiting $T$.
- **Selective termination**: $\text{Term}(T)$ is the set of plays for which counter value 0 is reached in $T$.

Interval strategy verification problem

Decide whether $\mathbb{P}^{\sigma}_{M \leq \infty} (Q), s_{\text{init}} \geq \alpha$ given an interval strategy $\sigma$, an objective $\Omega \in \{\text{Reach}(T), \text{Term}(T)\}$, a threshold $\alpha \in \mathbb{Q} \cap [0, 1]$ and an initial configuration $s_{\text{init}} \in Q \times \mathbb{N}$.

Goal: explain how to solve the interval strategy verification problem in polynomial space.
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Verification and interval strategies

- We develop techniques to analyse the infinite Markov chain induced by an interval strategy.
- For OEISs, we reduce to the analysis of a finite Markov chain.
- For CISs, we reduce to the analysis of a one-counter Markov chain.

→ We focus on an OEIS $\sigma$ based on a partition $\mathcal{I}$ from here.

Main idea: compressing the configuration space

For each interval $I \in \mathcal{I}$:
- we only keep a subset of the configurations in $Q \times I$ and
- we aggregate several transitions of $M^{\leq \infty}(Q)$ into one.

↝ We define a compressed Markov chain $C_\mathcal{I}^\sigma$. 
Compressing the unbounded interval

- Let $I \in \mathcal{I}$ be the unbounded interval, i.e., $Q \times I$ is infinite.
- We keep only one configuration per state.

Example: $\sigma$ choosing $a$ in all states for the interval $\mathbb{N}_0 = [1, \infty]$.

Transition probabilities can be irrational.

Theorem ([KEM06]$^1$)

The transition probabilities of $C^{\sigma}_{\mathcal{I}}$ with respect to $Q \times I$ are the least non-negative solution of a quadratic system of equations.

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$^1$Kucera et al., “Model Checking Probabilistic Pushdown Automata”, LMCS 2006.
Compressing bounded intervals

Motivation

- Let $I \in \mathcal{I}$ be bounded.
- The set of configurations $Q \times I$ is finite.

Why do we want to compress bounded intervals?

- The bounds of $I$ are encoded in binary.
- Thus $Q \times I$ is of exponential size.
- Goal: polynomial-size compressed Markov chain.
Compressing bounded intervals
State space and transition structure

Main idea: retain configurations by considering counter changes by powers of two.

- The construction requires that $|I| = 2^x - 1$ for some $x \in \mathbb{N}_0$.
- We retain at most $2x - 1$ counter values.
**Compressing bounded intervals**

**Transition probabilities**

**Example:** $\sigma$ playing **uniformly at random** for the interval $[1, 15]$.

Transition probabilities can require **exponential-size** representations.

**Theorem**

The transition probabilities of $C^\sigma_I$ with respect to $Q \times I$ are the least non-negative solution of a quadratic system of equations.
Verification via compressed Markov chains

Summary: compressed Markov chain $C_τ^σ$

- **Polynomial-size** state space.
- Transition probabilities given by *polynomial-size equations systems*.
- Preserves termination probabilities.

For CISs, we can use the same approach to derive a compressed one-counter Markov chain.

| Unbounded counter | Bounded counter |
|-------------------|-----------------|
| **OEIS**          | **OEIS**        |
| co-ETR            | co-ETR          |
| Square-root sum-hard [EWY10]$^2$ | $P^{\text{PosSLP}}$ |

$^2$Etessami et al., “Quasi-Birth-Death Processes, Tree-Like QBDs, Probabilistic 1-Counter Automata, and Pushdown Systems”, Perform. Evaluation 2010.
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