Generalized Hopf Bifurcation for planar vector fields via the inverse integrating factor

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Abstract

In this paper we study the maximum number of limit cycles that can bifurcate from a focus singular point $p_0$ of an analytic, autonomous differential system in the real plane under an analytic perturbation. We consider $p_0$ being a focus singular point of the following three types: non-degenerate, degenerate without characteristic directions and nilpotent. In a neighborhood of $p_0$ the differential system can always be brought, by means of a change to (generalized) polar coordinates $(r, \theta)$, to an equation over a cylinder in which the singular point $p_0$ corresponds to a limit cycle $\gamma_0$. This equation over the cylinder always has an inverse integrating factor which is smooth and non-flat in $r$ in a neighborhood of $\gamma_0$. We define the notion of vanishing multiplicity of the inverse integrating factor over $\gamma_0$. This vanishing multiplicity determines the maximum number of limit cycles that bifurcate from the singular point $p_0$ in the non-degenerate case and a lower bound for the cyclicity otherwise.

Moreover, we prove the existence of an inverse integrating factor in a neighborhood of many types of singular points, namely for the three types of focus considered in the previous paragraph and for any isolated singular point with at least one non-zero eigenvalue.

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1 Introduction and statement of the results

Let us consider a planar, real, analytic, autonomous differential system with a singular point which we assume to be at the origin, that is, we consider a differential system of the form:

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y),$$

(1)

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where $P(x, y)$ and $Q(x, y)$ are real analytic functions in a neighborhood $\mathcal{U}$ of the origin such that $P(0, 0) = Q(0, 0) = 0$. As usual, we associate to system (1) the vector field $\mathcal{X}_0 = P(x, y)\partial_x + Q(x, y)\partial_y$. We assume that the origin $p_0 = (0, 0)$ is an isolated singular point, that is, there exists a neighborhood of it without any other singular point, and we assume that it is a monodromic singular point. Therefore, it is either a center (i.e. it has a neighborhood filled with periodic orbits) or a focus (i.e. it has a neighborhood where all the orbits spiral in forward or in backward time to the origin).

We consider an analytic perturbation of system (1) of the form:

$$\dot{x} = P(x, y) + \bar{P}(x, y, \varepsilon), \quad \dot{y} = Q(x, y) + \bar{Q}(x, y, \varepsilon),$$

where $\varepsilon \in \mathbb{R}^p$ is the perturbation parameter, $0 < \|\varepsilon\| << 1$ and the functions $\bar{P}(x, y, \varepsilon)$ and $\bar{Q}(x, y, \varepsilon)$ are analytic for $(x, y) \in \mathcal{U}$, analytic in a neighborhood of $\varepsilon = 0$ and $\bar{P}(x, y, 0) = \bar{Q}(x, y, 0) \equiv 0$. We associate to this perturbed system the vector field $\mathcal{X}_\varepsilon = (P(x, y) + \bar{P}(x, y, \varepsilon))\partial_x + (Q(x, y) + \bar{Q}(x, y, \varepsilon))\partial_y$.

We say that a limit cycle $\gamma_\varepsilon$ of system (2) *bifurcates from the origin* if it tends to the origin (in the Hausdorff distance) as $\varepsilon \to 0$. We are interested in giving a sharp upper bound for the number of limit cycles which can bifurcate from the origin $p_0$ of system (1) under any analytic perturbation with a finite number of parameters. The word sharp means that there exists a system of the form (2) with exactly that number of limit cycles bifurcating from the origin, that is, the upper bound is realizable. This sharp upper bound is called the *cyclicity* of the origin $p_0$ of system (1) and will be denoted by $\text{Cycl}(\mathcal{X}_\varepsilon, p_0)$ all along this paper.

As defined in [4, 21, 29], a *Hopf bifurcation*, also denoted by Poincaré-Andronov-Hopf bifurcation, is a bifurcation in a neighborhood of a singular point like the origin of system (1). If the stability type of this point changes when subjected to perturbations, then this change is usually accompanied with either the appearance or disappearance of a small amplitude periodic orbit encircling the equilibrium point.

We remark that in all the examples considered in this paper we take a perturbed system (2) with the origin as singular point, that is, $\bar{P}(0, 0, \varepsilon) = \bar{Q}(0, 0, \varepsilon) \equiv 0$. Moreover, all the limit cycles $\gamma_\varepsilon$ that bifurcate from the origin in our examples surround the origin.

We consider systems of the form (1) where the origin $p_0$ is a focus singular point of the following three types: non-degenerate, degenerate without characteristic directions and nilpotent (the definitions are stated below). The study of the cyclicity of a degenerate focus has been tackled in very few sources; we mention the papers [1, 3, 10, 12, 18, 22, 24] where some (partial) results can be found. In relation with normal forms and integrability of degenerate singular points, we cite [8, 22, 25]. Our results do not establish that the cyclicity of this type of singular points is finite.
but give an effective procedure to study it. In the three mentioned types of focus points, we will consider a change to (generalized) polar coordinates which embed the neighborhood $\mathcal{U}$ of the origin into a cylinder $C = \{(r, \theta) \in \mathbb{R} \times S^1 : |r| < \delta \}$ for a certain sufficiently small value of $\delta > 0$. This change to polar coordinates is a diffeomorphism in $\mathcal{U} - \{(0, 0)\}$ and transforms the origin of coordinates to the circle of equation $r = 0$. In fact, the neighborhood $\mathcal{U}$ is transformed into the half-cylinder in which $r \geq 0$, but we can consider the extension to the values in which $r < 0$ by using several symmetries of the considered (generalized) polar coordinates. In these new coordinates, system (1) can be seen as a differential equation over the cylinder $C$ of the form:

$$\frac{dr}{d\theta} = \mathcal{F}(r, \theta),$$

where $\mathcal{F}(r, \theta)$ is an analytic function in $C$. The circle $r = 0$ needs to be a particular periodic solution of the equation (3) and, therefore, $\mathcal{F}(0, \theta) \equiv 0$ for all $\theta \in S^1$.

Throughout the rest of the paper, we consider an inverse integrating factor $V(r, \theta)$ of equation (3). We recall that an inverse integrating factor of equation (3) is a function $V : C \to \mathbb{R}$ of class $C^1(C)$, which is non locally null and which satisfies the following partial differential equation:

$$\frac{\partial V(r, \theta)}{\partial \theta} + \frac{\partial V(r, \theta)}{\partial r} \mathcal{F}(r, \theta) = \frac{\partial \mathcal{F}(r, \theta)}{\partial r} V(r, \theta).$$

We remark that since $V(r, \theta)$ is a function defined over the cylinder $C$ it needs to be $T$–periodic in $\theta$, where $T$ is the minimal positive period of the variable $\theta$, that is, we consider the circle $S^1 = \mathbb{R}/[0, T]$. The function $V(r, \theta)$ is smooth ($C^\infty$) and non–flat in $r$ in a neighborhood of $r = 0$. The existence of an inverse integrating factor $V(r, \theta)$ with this regularity is proved in [13] using the result of [30], see also Lemma 21 of the present paper. A characterization of when $V(r, \theta)$ is analytic in a neighborhood of $r = 0$ is given in [13].

Let us consider the Taylor expansion of the function $V(r, \theta)$ around $r = 0$: $V(r, \theta) = v_m(\theta) r^m + \mathcal{O}(r^{m+1})$, where $v_m(\theta) \neq 0$ for $\theta \in S^1$ and $m$ is an integer number with $m \geq 0$. As we will see in the following section, in fact, $v_m(\theta) \neq 0$ for all $\theta \in S^1$, cf. Lemma 20. We say that $m$ is the vanishing multiplicity of $V(r, \theta)$ on $r = 0$. The aim of this paper is to show the correspondence between this vanishing multiplicity $m$ and the cyclicity $\text{Cycl}(\mathcal{X}_\epsilon, p_0)$ of the origin $p_0$ of system (1).

One of our hypothesis is that the origin of system (1) is a focus, and thus, we obtain that the circle $r = 0$ is an isolated periodic orbit (i.e. a limit cycle) of equation (3). This hypothesis implies that there can only exist one inverse integrating factor $V(r, \theta)$ smooth and non–flat in $r$ in a neighborhood of $r = 0$, up to a nonzero multiplicative constant, see Lemma 21. The uniqueness of $V(r, \theta)$ implies that the number $m$ corresponding to the vanishing multiplicity of $V(r, \theta)$ on $r = 0$ is well–defined.
The existence of $V(r, \theta)$ gives the existence of an inverse integrating factor $V_0(x, y)$ for system (11) by undoing the change to (generalized) polar coordinates. We recall that $V_0 : \mathcal{U} \to \mathbb{R}$ is said to be an inverse integrating factor of system (11) if it is of class $C^1(\mathcal{U})$, it is not locally null and it satisfies the following partial differential equation:

$$P(x, y) \frac{\partial V_0(x, y)}{\partial x} + Q(x, y) \frac{\partial V_0(x, y)}{\partial y} = \left( \frac{\partial P(x, y)}{\partial x} + \frac{\partial Q(x, y)}{\partial y} \right) V_0(x, y).$$

The function $V_0(x, y)$, obtained from $V(r, \theta)$ by undoing the change to polar coordinates, does not need to be smooth at the origin $p_0$ since the change to polar coordinates is a diffeomorphism except in the origin. Besides, we are also interested in the problem of the regularity of an inverse integrating factor $V_0(x, y)$ in a neighborhood of the origin of system (11), whenever it exists, and we also analyze this question, see Theorem 16 and Corollaries 12 and 19.

The zero-set of $V_0(x, y)$, which we denote by $V_0^{-1}(0) := \{ p \in \mathcal{U} : V_0(p) = 0 \}$, is formed by orbits of system (11) and usually contains those orbits which determine the dynamics of the system: singular points, limit cycles and graphics, see [7, 15, 17, 19]. If there exists an inverse integrating factor in a neighborhood of a limit cycle, then it is contained in $V_0^{-1}(0)$, as it has been proved in [19]. Since the origin $p_0 = (0, 0)$ of (11) is a focus, we have that $V_0(0, 0) = 0$, as proved in [7, 17]. Moreover, the set $V_0^{-1}(0)$ contains all the singular points of (11) having at least one parabolic or elliptic sector in its domain of definition, as it is also proved in [7, 17].

In [15], we proved that the vanishing multiplicity of an analytic $V(r, \theta)$ on the limit cycle $r = 0$ coincides with the multiplicity (and, thus, the cyclicity) of $r = 0$ as an orbit of equation (3). The statements and the proofs given in [15] can be repeated verbatim with the weaker assumption that $V(r, \theta)$ is smooth and non–flat in $r$ in a neighborhood of $r = 0$, see Corollary 24 in the following section. The multiplicity of the limit cycle $r = 0$ is related to the cyclicity of the origin of system (11) by the change from polar coordinates. The aim of this work is to study the cyclicity of the origin of system (11) through the vanishing multiplicity of $V(r, \theta)$ on $r = 0$. We remark that the multiplicity $m$ of $r = 0$ as a limit cycle of equation (3) can also be established by successively solving the variational equations. However, this method implies the computation of iterated integrals of non–elementary periodic functions. If we explicitly know an inverse integrating factor $V_0(x, y)$ for system (11), then we have an inverse integrating factor $V(r, \theta)$ of (3) and we can immediately know the value of $m$ through the vanishing multiplicity of $V(r, \theta)$ in $r = 0$. On the other hand, since the existence of an inverse integrating factor $V(r, \theta)$, which is smooth and non–flat in $r$ in a neighborhood of $r = 0$, for equation (3) is ensured (see Lemma 21), we have an alternative method to the variational equations to determine the value of $m$. 

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We are going to study the cyclicity of the origin of system (1) in the following three cases: the origin is a non-degenerate focus, the origin is a degenerate focus without characteristic directions and the origin is a nilpotent focus. The two first cases are transformed to an equation of the form (3) using polar coordinates and the latter case using the so-called generalized polar coordinates, see the definition below. We state our results for each type of coordinates, separately. The following Subsection 1.1 contains the results related to the non-degenerate and degenerate without characteristic directions case. Subsection 1.2 is devoted to the nilpotent focus. In Subsection 1.3 we analyze the existence (and regularity conditions) of an inverse integrating factor in a neighborhood of an isolated singular point of system (1). We give some general results on an equation (3) over a cylinder in Section 2. Finally, Section 3 contains the proofs of our main results.

1.1 A focus without characteristic directions

We say that a focus at the origin of system (1) is non-degenerate if the linear part of system (1) has complex eigenvalues of the form \( \alpha \pm \beta i \) with \( \alpha, \beta \in \mathbb{R} \) and \( \beta \neq 0 \).

After a linear change of coordinates and a rescaling of time, if necessary, system (1) can be written in the form:

\[
\begin{align*}
\dot{x} &= \zeta x - y + P_2(x, y), \\
\dot{y} &= x + \zeta y + Q_2(x, y),
\end{align*}
\]

where \( \zeta \in \mathbb{R} \) and \( P_2(x, y) \) and \( Q_2(x, y) \) are analytic functions in a neighborhood of the origin without constant nor linear terms.

We say that the origin of system (1) is a degenerate singular point if the determinant associated to the linear part of (1) is zero. We consider a system (1) of the form:

\[
\begin{align*}
\dot{x} &= p_d(x, y) + P_{d+1}(x, y), \\
\dot{y} &= q_d(x, y) + Q_{d+1}(x, y),
\end{align*}
\]

where \( d \geq 1 \) is an odd number, \( p_d(x, y) \) and \( q_d(x, y) \) are homogeneous polynomials of degree \( d \) and \( P_{d+1}(x, y), Q_{d+1}(x, y) \in \mathcal{O}(\| (x, y) \|^{d+1}) \), that is, they are analytic functions in a neighborhood of the origin with order at least \( d + 1 \). We assume that \( p_d^2(x, y) + q_d^2(x, y) \neq 0 \). When \( d > 1 \), the origin of system (5) is a degenerate singular point.

A characteristic direction for the origin of system (5) is a linear factor in \( \mathbb{R}[x, y] \) of the homogeneous polynomial \( xq_d(x, y) - yp_d(x, y) \). It is obvious that, unless \( xq_d(x, y) - yp_d(x, y) \equiv 0 \), the number of characteristic directions for the origin of system (5) is less than or equal to \( d + 1 \). If there are no characteristic directions, then the origin is a monodromic singular point of system (5). We observe that the reciprocal is not true. A singular point with characteristic directions can be
monodromic. The origin of system (1) is a monodromic singular point if it is either a center or a focus, see for instance [4, 16, 24] for further information about monodromic singular points and characteristic directions. We assume that the origin of system (5) is a focus without characteristic directions.

In the non-degenerate case \((d = 1)\), that the cyclicity of a focus point is finite is well-known as well as several methods to determine it. The most usual method is to compute the first non-vanishing Liapunov constant and its order gives the cyclicity of the focus. Indeed, the same method allows to study the limit cycles which bifurcate from the origin in this case. When an inverse integrating factor is known, we give a shortcut in the study of the cyclicity as it can be given in terms of the vanishing multiplicity of the inverse integrating factor at the origin.

As far as the authors know, the cyclicity of a degenerate focus \((d > 1)\) of system (5) is not proved to be bounded. In fact, very few techniques appear in the literature to tackle the cyclicity of this type of singular points. Usually, polar coordinates are taken and the corresponding equation over the cylinder is studied. Our approach is to take profit from the knowledge of an inverse integrating factor to avoid the study of the differential equation over the cylinder.

We remark that if \(d = 1\) and the origin of system (5) is a focus point (without characteristic directions), then it can be written in the form (4).

We use polar coordinates, \((x, y) = (r \cos \theta, r \sin \theta)\), in order to transform a neighborhood of the origin into the cylinder with period \(T = 2\pi\), and system (5) into an ordinary differential equation of the form (3).

In relation with system (2), an analytic perturbation field \((\bar{P}(x, y, \varepsilon), \bar{Q}(x, y, \varepsilon))\) is said to have subdegree \(s\) if \((\bar{P}(x, y, \varepsilon), \bar{Q}(x, y, \varepsilon)) = O(||(x, y)||^s)\). In this case, we denote by \(X_{\varepsilon}^{[s]} = (P(x, y) + \bar{P}(x, y, \varepsilon)) \partial_x + (Q(x, y) + \bar{Q}(x, y, \varepsilon)) \partial_y\) the vector field associated to such a perturbation.

**Theorem 1** We assume that the origin \(p_0\) of system (5) is monodromic and without characteristic directions. Let \(V(r, \theta)\) be an inverse integrating factor of the corresponding equation (3) which has a Laurent expansion in a neighborhood of \(r = 0\) of the form \(V(r, \theta) = v_m(\theta) r^m + O(r^{m+1})\), with \(v_m(\theta) \neq 0\) and \(m \in \mathbb{Z}\).

(i) If \(m \leq 0\) or \(m\) is even, then the origin of system (5) is a center.

(ii) If the origin of system (5) is a focus, then \(m \geq 1\), \(m\) is an odd number and the cyclicity \(\text{Cycl}(X_{\varepsilon}, p_0)\) of the origin of system (5) satisfies \(\text{Cycl}(X_{\varepsilon}, p_0) \geq (m + d)/2 - 1\). In this case \(m\) is the vanishing multiplicity of \(V(r, \theta)\) on \(r = 0\).

(iii.1) If, moreover, the focus is non–degenerate \((d = 1)\), then the aforementioned lower bound is sharp, that is, \(\text{Cycl}(X_{\varepsilon}, p_0) = (m - 1)/2\).
If only perturbations whose subdegree is greater than or equal to \( d \) are considered, then the maximum number of limit cycles which bifurcate from the origin is \((m - 1)/2\), that is, \( \text{Cycl}(\mathcal{X}_e^{[d]}, p_0) = (m - 1)/2 \).

The proof of this theorem is given in Section 3.

**Remark 2** As we will see in the proof of this theorem, if there exists an inverse integrating factor \( V_0(x, y) \) of system (5) such that \( V_0(r \cos \theta, r \sin \theta)/r^d \) has a Laurent expansion in a neighborhood of \( r = 0 \), then the exponents of the leading terms of \( V_0(r \cos \theta, r \sin \theta)/r^d \) and \( V(r, \theta) \) coincide. Thus, the vanishing multiplicity \( m \) can be computed without passing the system to polar coordinates.

We provide several examples of application of Theorem 1.

**Example 3** The following system

\[
\begin{align*}
\dot{x} &= -y ((2\mu + 1)x^2 + y^2) + x^3 (\lambda_1 x^2 + \lambda_2 (x^2 + y^2)), \\
\dot{y} &= x (x^2 + (1 - 2\mu) y^2) + x^2 y (\lambda_1 x^2 + \lambda_2 (x^2 + y^2)),
\end{align*}
\]

where \( \mu, \lambda_1 \) and \( \lambda_2 \) are real parameters, appears in [31], where it is shown that the origin is a focus for a non semi-algebraic set of values of \((\mu, \lambda_1, \lambda_2)\) and it is a center otherwise.

We have that this system is written in the form (5) with \( d = 3 \) and it has no characteristic directions as \( xq_3(x, y) - yp_3(x, y) = (x^2 + y^2)^2 \). Easy computations show that the function

\[ V_0(x, y) = e^{\frac{-2\mu x^2}{x^2 + y^2}} (x^2 + y^2)^3 \]

is an inverse integrating factor of system (6) which satisfies the hypothesis of our Theorem 1 with \( V(r, \theta) = e^{-2\mu \cos^2 \theta} r^3 \). We remark that \( V_0(x, y) \) is not an analytic function in a neighborhood of \( r = 0 \) unless \( \mu = 0 \). As a consequence of Lemma 21 which we prove in Section 3, we deduce that when the origin of system (5) is a focus, there are no analytic inverse integrating factors \( \bar{V}_0(x, y) \) defined in a neighborhood of the origin. If it existed, its transformation to polar coordinates would produce an analytic inverse integrating factor \( \bar{V}(r, \theta) \) defined in a neighborhood of \( r = 0 \) and different from \( V(r, \theta) \), up to a multiplicative nonzero constant. In [31] it is shown that there exist values of \((\mu, \lambda_1, \lambda_2)\), either with \( \mu = 0 \) or with \( \mu \neq 0 \), for which the origin of system (5) is a focus. Applying Theorem 1 we deduce that whenever the origin of system (5) is a focus, then its cyclicity is greater than or equal to 2. In the particular case of a perturbation of subdegree greater or equal than 3, the maximum number of limit cycles that bifurcate from the origin is 1.
Example 4 Let \( k, s \) be integers such that \( s \geq 2k \geq 0 \) and consider the following differential system:

\[
\begin{align*}
\dot{x} &= -y(x^2 + y^2)^k + xR_s(x,y), \\
\dot{y} &= x(x^2 + y^2)^k + yR_s(x,y), \\
\end{align*}
\]

where \( R_s(x,y) \) is a homogeneous polynomial of degree \( s \). The origin of this system is a monodromic singular point since there are no characteristic directions. We take polar coordinates \( x = r \cos \theta, y = r \sin \theta \) and system (7) reads for

\[
\begin{align*}
\dot{r} &= r^{s+1}R_s(\cos \theta, \sin \theta), \\
\dot{\theta} &= r^{2k}, \\
\end{align*}
\]

where we have used that \( R_s(r \cos \theta, r \sin \theta) = r^sR_s(\cos \theta, \sin \theta) \) since \( R_s(x,y) \) is a homogeneous polynomial of degree \( s \). Hence, we see that the origin of system (7) is a focus if, and only if, the following integral is different from zero:

\[
\mathcal{L} = \int_0^{2\pi} R_s(\cos \theta, \sin \theta) \, d\theta \neq 0.
\]

We remark that if \( s \) is an odd number, then the origin of system (7) is a center. Easy computations show that

\[
V_0(x,y) = (x^2 + y^2)^{s/2+1}
\]

is an inverse integrating factor for system (7) and that \( V(r, \theta) = r^{s+1-2k} \) is an inverse integrating factor of the ordinary differential equation on a cylinder corresponding to system (7). Thus, applying Theorem 1 we deduce that the cyclicity \( \text{Cycl}(\mathcal{X}_\varepsilon, p_0) \) of the origin of system (7), with \( \mathcal{L} \neq 0 \), is \( \text{Cycl}(\mathcal{X}_\varepsilon, p_0) \geq s/2 \).

We observe that when \( k = 0 \), we have that the origin of system (7) is a non–degenerate monodromic singular point and \( \text{Cycl}(\mathcal{X}_\varepsilon, p_0) = s/2 \). The center problem for any even value of \( s \) is determined only by one Lyapunov constant, namely \( \mathcal{L} \), whereas the cyclicity of the origin (in case it is a focus) is given by \( s/2 \). We see, in this way, that the center problem and the determination of the cyclicity for a non–degenerate monodromic singular point are strongly related but are not equivalent problems.

Example 5 In this example we show that no limit cycles bifurcate from a focus of a homogeneous system of degree \( d \), under perturbations of subdegree \( \geq d \). We consider a homogeneous system:

\[
\begin{align*}
\dot{x} &= p_d(x,y), \\
\dot{y} &= q_d(x,y), \\
\end{align*}
\]

where \( p_d(x,y) \) and \( q_d(x,y) \) are homogeneous polynomials of degree \( d \) with \( d \geq 1 \). A focus of system (8) has no characteristic directions because any linear real factor of \( yp_d(x,y) - xq_d(x,y) \) gives an invariant straight line of system (8) through the
origin. Since \( yp_d(x, y) - xq_d(x, y) \) has no linear real factors, we have that \( d \) is odd. In polar coordinates system (8) becomes

\[
\dot{r} = r^d R_d(\theta), \quad \dot{\theta} = r^{d-1} F_d(\theta),
\]

where

\[
R_d(\theta) = p_d(\cos \theta, \sin \theta) \cos \theta + q_d(\cos \theta, \sin \theta) \sin \theta, \\
F_d(\theta) = q_d(\cos \theta, \sin \theta) \cos \theta - p_d(\cos \theta, \sin \theta) \sin \theta.
\]

The hypothesis that the origin is a focus implies that there are no characteristic directions. The non-existence of characteristic directions is equivalent to \( F_d(\theta) \neq 0 \) for \( \theta \in [0, 2\pi) \). We see that the origin of system (8) is a focus if, and only if,

\[
\mathcal{L} = \int_0^{2\pi} \frac{R_d(\theta)}{F_d(\theta)} \, d\theta \neq 0.
\]

On the other hand, easy computations show that \( V_0(x, y) = yp_d(x, y) - xq_d(x, y) \) is an inverse integrating factor for system (8), where we have used Euler’s Theorem for homogeneous polynomials. Indeed \( V(r, \theta) = r \) is an inverse integrating factor for the corresponding differential equation on the cylinder. By Theorem [\ref{thm:unstable-focus} we have that, when the origin of system (8) is a focus (i.e. \( \mathcal{L} \neq 0 \)), no limit cycle can bifurcate from it under perturbations of subdegree \( \geq d \).

However, if we take perturbations of lower degree, there can appear limit cycles which bifurcate from the origin. By Theorem [\ref{thm:limit-cycles} we have that at least \((d - 1)/2\) limit cycles bifurcate from the origin of system (8).

For instance, let us consider the following system:

\[
\dot{x} = p_3(x, y) = (x - y)(x^2 + y^2), \quad \dot{y} = q_3(x, y) = (x + y)(x^2 + y^2), \tag{9}
\]

which has an unstable degenerate focus at the origin. The function \( V_0(x, y) = (x^2 + y^2)^2 \) is an inverse integrating factor for this system and, as we are under the hypothesis of Theorem [\ref{thm:limit-cycles}, we deduce that \( m = 1 \) and that no limit cycle can bifurcate from system (8) under perturbations of subdegree \( \geq 3 \). Let us consider the following perturbed system:

\[
\dot{x} = (x - y)(x^2 + y^2) - \varepsilon(x + y), \quad \dot{y} = (x + y)(x^2 + y^2) + \varepsilon(x - y),
\]

which has the invariant algebraic curve \( x^2 + y^2 = \varepsilon \). For \( \varepsilon > 0 \), this algebraic curve is a hyperbolic limit cycle of the system which bifurcates from the origin. The function \( V_\varepsilon(x, y) = (x^2 + y^2)(x^2 + y^2 - \varepsilon) \) is an inverse integrating factor of this system.
1.2 A nilpotent focus

We say that the origin of system (1) is a nilpotent singular point if it is a degenerate singularity that can be written as:

\[
\begin{align*}
\dot{x} &= y + P_2(x, y), \\
\dot{y} &= Q_2(x, y),
\end{align*}
\]

with \(P_2(x, y)\) and \(Q_2(x, y)\) analytic functions near the origin without constant nor linear terms. The following theorem is due to Andreev [2] and it solves the monodromy problem for the origin of system (10), that is, it determines when the origin is a monodromic singular point.

**Theorem 6** [2] Let \(y = F(x)\) be the solution of \(y + P_2(x, y) = 0\) passing through \((0, 0)\). Define the functions \(f(x) = Q_2(x, F(x)) = ax^\alpha + \cdots\) with \(a \neq 0\) and \(\alpha \geq 2\) and \(\phi(x) = (\partial P_2/\partial x + \partial Q_2/\partial y)(x, F(x))\). We have that either \(\phi(x) = bx^\beta + \cdots\) with \(b \neq 0\) and \(\beta \geq 1\) or \(\phi(x) \equiv 0\). Then, the origin of (10) is monodromic if, and only if, \(a < 0\), \(\alpha = 2n - 1\) is an odd integer and one of the following conditions holds:

(i) \(\beta > n - 1\).

(ii) \(\beta = n - 1\) and \(b^2 + 4an < 0\).

(iii) \(\phi(x) \equiv 0\).

**Definition 7** We consider a system of the form (10) with the origin as a monodromic singular point. We define its Andreev number \(n \geq 2\) as the corresponding integer value given in Theorem 6.

We consider system (10) and we assume that the origin is a nilpotent monodromic singular point with Andreev number \(n\). Then, the change of variables

\[
(x, y) \mapsto (x, y - F(x)),
\]

where \(F(x)\) is defined in Theorem 6 and the scaling

\[
(x, y) \mapsto (\xi x, -\xi y),
\]

with \(\xi = (-1/a)^{1/(2-2n)}\), brings system (10) into the following analytic form for monodromic nilpotent singularities

\[
\begin{align*}
\dot{x} &= y (-1 + X_1(x, y)), \\
\dot{y} &= f(x) + y \phi(x) + y^2 Y_0(x, y),
\end{align*}
\]

where \(X_1(0, 0) = 0\), \(f(x) = x^{2n-1} + \cdots\) with \(n \geq 2\) and either \(\phi(x) \equiv 0\) or \(\phi(x) = bx^\beta + \cdots\) with \(\beta \geq n - 1\). We remark that we have relabelled the functions...
\( f(x), \phi(x) \) and the constant \( b \) with respect to the ones corresponding to system \((10)\). We recall, cf. Theorem 6, that when \( \beta = n - 1 \) we also have that \( b^2 - 4n < 0 \).

We are going to transform system \((13)\) to an equation over a cylinder of the form \((3)\). The transformation depends on the Andreev number \( n \) and it is given through the generalized trigonometric functions defined by Lyapunov \([23]\) as the unique solution \( x(\theta) = \text{Cs} \theta \) and \( y(\theta) = \text{Sn} \theta \) of the following Cauchy problem

\[
\frac{dx}{d\theta} = -y, \quad \frac{dy}{d\theta} = x^{2n-1}, \quad x(0) = 1, \quad y(0) = 0. \tag{14}
\]

We observe that, in the particular case \( n = 1 \), the previous definition gives the classical trigonometric functions.

We introduce in \( \mathbb{R}^2 \setminus \{(0, 0)\} \) the change to generalized polar coordinates, \((x, y) \mapsto (r, \theta)\), defined by

\[
x = r \text{Cs} \theta, \quad y = r^n \text{Sn} \theta. \tag{15}
\]

In relation with this change, we say that a polynomial \( R(x, y) \in \mathbb{C}[x, y] \) is a \((1, n)\)-quasihomogeneous polynomial of weighted degree \( w \) if the following identity is satisfied:

\[
R(\lambda x, \lambda^n y) = \lambda^w R(x, y),
\]

for all \((x, y, \lambda) \in \mathbb{R}^3\). We observe that a homogeneous polynomial of degree \( w \) is, with this definition, a \((1, 1)\)-quasihomogeneous polynomial of weighted degree \( w \).

Since we are going to use some properties of the generalized trigonometric functions and the relations satisfied by \((1, n)\)-quasihomogeneous polynomials of weighted degree \( w \), we summarize them up in the following proposition. The proof of each of its statements can be found in \([23]\).

**Proposition 8** \([23]\) We fix an integer \( n \geq 1 \) and we consider \((\text{Cs} \theta, \text{Sn} \theta)\) the solution of the Cauchy problem \((14)\). The following statements hold.

(a) The functions \( \text{Cs} \theta \) and \( \text{Sn} \theta \) are \( T_n \)-periodic with \( T_n = 2\sqrt{\frac{\pi}{n}} \frac{\Gamma\left(\frac{1}{2n}\right)}{\Gamma\left(\frac{n+1}{2n}\right)} \), where \( \Gamma(\cdot) \) denotes the Euler Gamma function.

(b) \( \text{Cs}^{2n} \theta + n \text{Sn}^2 \theta = 1 \) (the fundamental relation).

(c) \( \text{Cs}(-\theta) = \text{Cs} \theta, \quad \text{Sn}(-\theta) = -\text{Sn} \theta, \quad \text{Cs}(\theta + T_n/2) = -\text{Cs} \theta, \quad \text{Sn}(\theta + T_n/2) = -\text{Sn} \theta. \)

(d) Euler Theorem for quasihomogeneous polynomials: if \( R(x, y) \) is a \((1, n)\)-quasihomogeneous polynomial of weighted degree \( w \), then

\[
x \frac{\partial R(x, y)}{\partial x} + n y \frac{\partial R(x, y)}{\partial y} = w R(x, y).
\]
(e) \( \text{Cs} \varphi = - \text{Cs} \theta, \text{Sn} \varphi = (-1)^n \text{Sn} \theta, \) where \( \varphi = (-1)^{n+1} (\theta + T_n/2). \) If \( R(x, y) \) is a \((1,n)\)-quasihomogeneous polynomial of weighted degree \( w, \) then
\[
R(\text{Cs} \varphi, \text{Sn} \varphi) = (-1)^w R(\text{Cs} \theta, \text{Sn} \theta).
\]

In particular,
\[
R(-1, 0) = R(\text{Cs} (T_n/2), \text{Sn} (T_n/2)) = (-1)^w R(\text{Cs} 0, \text{Sn} 0) = (-1)^w R(1, 0).
\]

Analogously to the case of a degenerate focus without characteristic directions, we can also provide the maximum number of limit cycles which can bifurcate from a nilpotent focus when only certain perturbations are taken into account. In this sense, and in relation with system (2), we consider the following definition, which will be used in the following Theorem (10).

**Definition 9** An analytic perturbation vector field \( (\bar{P}(x, y, \varepsilon), \bar{Q}(x, y, \varepsilon)) \) is said to be \((1,n)\)-quasihomogeneous of weighted subdegrees \((w_x, w_y)\) if \( \bar{P}(\lambda x, \lambda^n y, \varepsilon) = \mathcal{O}(\lambda^{w_x}) \) and \( \bar{Q}(\lambda x, \lambda^n y, \varepsilon) = \mathcal{O}(\lambda^{w_y}). \) In this case, we denote by \( \mathcal{X}_{\varepsilon}^{[w_x, w_y]} \) \( = (P(x, y) + \bar{P}(x, y, \varepsilon))\partial_x + (Q(x, y) + \bar{Q}(x, y, \varepsilon))\partial_y \) the vector field associated to such a perturbation.

We remark that the perturbative functions \( \bar{P}(x, y, \varepsilon), \bar{Q}(x, y, \varepsilon) \) do not need to be \((1,n)\)-quasihomogeneous of a certain degree, they just need to have a \((1,n)\)-quasihomogeneous subdegree high enough.

The following theorem is one of the main results of this work. The symbol \( [x] \) denotes the integer part of \( x. \)

**Theorem 10** We assume that the origin of system (10) is monodromic with Andreev number \( n. \) Let \( V(r, \theta) \) be an inverse integrating factor of the corresponding equation (3) which has a Laurent expansion in a neighborhood of \( r = 0 \) of the form
\[ V(r, \theta) = v_m(\theta) r^m + \mathcal{O}(r^{m+1}), \]
with \( v_m(\theta) \neq 0 \) and \( m \in \mathbb{Z}. \)

(i) If \( m \leq 0 \) or \( m + n \) is odd, then the origin of system (10) is a center.

(ii) If the origin of system (10) is a focus, then \( m \geq 1, \) \( m + n \) is even and its cyclicity \( \text{Cycl}(\mathcal{X}_\varepsilon, p_0) \) satisfies \( \text{Cycl}(\mathcal{X}_\varepsilon, p_0) \geq (m + n)/2 - 1. \) In this case, \( m \) is the vanishing multiplicity of \( V(r, \theta) \) on \( r = 0. \)

(iii) If the origin of system (13) is a focus and if only analytic perturbations of \((1,n)\)-quasihomogeneous weighted subdegrees \((w_x, w_y)\) with \( w_x \geq n \) and \( w_y \geq 2n - 1 \) are taken into account, then the maximum number of limit cycles which bifurcate from the origin is \( [(m - 1)/2], \) that is, \( \text{Cycl}(\mathcal{X}_\varepsilon^{[n, 2n-1]}, p_0) = [(m - 1)/2]. \)
The proof of this theorem is given in Section 3.

Remark 11 The proof of this theorem shows that if there exists an inverse integrating factor $V_0^*(x, y)$ of system \([13]\) such that $V_0^*(r \text{Cs} \theta, r^n \text{Sn} \theta)/r^{2n-1}$ has a Laurent expansion in a neighborhood of $r = 0$, then the exponents of the leading terms of $V_0^*(r \text{Cs} \theta, r^n \text{Sn} \theta)/r^{2n-1}$ and $V(r, \theta)$ coincide. Therefore, the value of $m$ can be determined without performing the transformation of the system to generalized polar coordinates.

The following corollary establishes a necessary condition for system \([10]\) to have an analytic inverse integrating factor $V_0(x, y)$ defined in a neighborhood of the origin.

Corollary 12 We assume that the origin of system \([10]\) is a nilpotent focus with Andreev number $n$, and that there exists an inverse integrating factor $V_0(x, y)$ of \([10]\) which is analytic in a neighborhood of the origin. Then, $n$ is odd.

The proofs of Theorem \([10]\) and Corollary \([12]\) are given in Section 3. Before the proofs of the main results of this paper we provide several examples of application of Theorem \([10]\).

We would like to remark that the change to system \([13]\) is not always necessary to arrive at an equation over a cylinder, using generalized polar coordinates. The following proposition establishes sufficient conditions for an analytic system in cartesian coordinates (nilpotent or not) to be transformed to an equation over a cylinder by the change to generalized polar coordinates. We remark that given any analytic function $P(x, y)$ in a neighborhood of the origin and any positive integer number $\tilde{n}$, we can always develop $P(x, y)$ as a series of $(1, \tilde{n})$–quasihomogeneous polynomials. That is, we can always define $(1, \tilde{n})$–quasihomogeneous polynomials $p_i(x, y)$ of weighted degree $i$ such that the following identity is satisfied $P(x, y) = \sum_{i \geq 0} p_i(x, y)$.

Proposition 13 Let $\tilde{n} \geq 2$ be an integer number and consider an analytic system of the form:

$$
\dot{x} = \sum_{i \geq a} p_i(x, y), \quad \dot{y} = \sum_{i \geq b} q_i(x, y),
$$

where $p_i(x, y)$ and $q_i(x, y)$ are $(1, \tilde{n})$–quasihomogeneous polynomials of weighted degree $i$ and $p_a(x, y), q_b(x, y)$ are not identically null. If $a - 1 = b - \tilde{n} \geq 0$ and

$$
\text{Cs} \theta q_b (\text{Cs} \theta, \text{Sn} \theta) - \tilde{n} \text{Sn} \theta p_a (\text{Cs} \theta, \text{Sn} \theta) \neq 0
$$

for all $\theta \in [0, T_\tilde{n}]$, then the origin of system \([16]\) is monodromic. Moreover, the change of coordinates $x = r \text{Cs} \theta$, $y = r^{\tilde{n}} \text{Sn} \theta$ brings the system to an equation of the form \([3]\) which is analytic in a neighborhood of its periodic solution $r = 0$. 

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Proof of Proposition 13. The change of variables \((x, y) \mapsto (r, \theta)\) gives
\[
\begin{align*}
\dot{r} &= \frac{x^{2n-1} P(x, y) + y Q(x, y)}{r^{2n-1}} = \frac{r^{2n-1}Cs^{2n-1} \theta \sum_{i \geq a} r^i \tilde{p}_i(\theta) + r^n \sum_{i \geq b} r^i \tilde{q}_i(\theta)}{r^{2n-1}} \\
\dot{\theta} &= \frac{x Q(x, y) - \tilde{n} y P(x, y)}{r^{n+1}} = \frac{rCs \theta \sum_{i \geq b} r^i \tilde{q}_i(\theta) - \tilde{n} r^n \sum_{i \geq a} r^i \tilde{p}_i(\theta)}{r^{n+1}},
\end{align*}
\]
where \(\tilde{p}_i(\theta) := p_i(Cs \theta, Sn \theta), \tilde{q}_i(\theta) := q_i(Cs \theta, Sn \theta)\). When \(a \geq 1\) and \(b \geq \tilde{n}\), this system is analytic at \(r = 0\). In this case, we have that
\[
\begin{align*}
\dot{r} &= Cs^{2n-1} \theta \sum_{i \geq a} r^i \tilde{p}_i(\theta) + r \sum_{i \geq b} r^i \tilde{q}_i(\theta) \\
\dot{\theta} &= Cs \theta \sum_{i \geq b} r^{i-\tilde{n}} \tilde{q}_i(\theta) - \tilde{n} r^n \sum_{i \geq a} r^{i-1} \tilde{p}_i(\theta).
\end{align*}
\]
Moreover, we get that
\[
\dot{\theta} = \begin{cases} 
-\tilde{n} r^n \sum_{i \geq a} r^i \tilde{p}_a(\theta) + \ldots & \text{if } a - 1 > b - \tilde{n} \\
Cs \theta \sum_{i \geq b} r^{i-\tilde{n}} \tilde{q}_i(\theta) + \ldots & \text{if } a - 1 < b - \tilde{n} \\
(Cs \theta \sum_{i \geq b} r^{i-\tilde{n}} \tilde{q}_i(\theta) - \tilde{n} r^n \sum_{i \geq a} r^{i-1} \tilde{p}_a(\theta)) r^{a-1} + \ldots & \text{if } a - 1 = b - \tilde{n},
\end{cases}
\]
where the dots denote terms of higher order in \(r\). In this way, when \(a - 1 = b - \tilde{n}\) and \(Cs \theta \sum_{i \geq b} r^{i-\tilde{n}} \tilde{q}_i(\theta) - \tilde{n} r^n \sum_{i \geq a} r^{i-1} \tilde{p}_a(\theta) \neq 0\) for all \(\theta \in [0, T_{\tilde{n}}]\), we have that \(\dot{\theta}\) is different from zero for all \(|r|\) small enough.

We want to remark that not every system with a nilpotent singularity at the origin satisfies the hypothesis of Proposition 13. Moreover, the value of the Andreev number \(n\) of a nilpotent system satisfying Proposition 13 does not need to coincide with the value of \(\tilde{n}\) which appears in the statement of this proposition.

Example 14 Let \(m\) and \(n\) be two positive integers such that \(n \geq 2\), \(m \geq 1\) and \(m + n\) is even. We consider the following system:
\[
\begin{align*}
\dot{x} &= y + x R(x, y), \quad \dot{y} = -x^{2n-1} + n y R(x, y),
\end{align*}
\]
where \(R(x, y)\) is a \((1, n)\)-quasihomogeneous polynomial of weighted degree \(m + n - 2\). The origin of system (17) is a nilpotent singularity and it is a monodromic point applying Theorem 6. We observe that the Andreev number of system (17) is \(n\). The change to generalized polar coordinates brings the system to the form:
\[
\begin{align*}
\dot{r} &= r^{m+n-1} R(Cs \theta, Sn \theta), \\
\dot{\theta} &= r^{n-1}.
\end{align*}
\]
where we have used the fundamental relation $C_s^{2n} \theta + n S_n^2 \theta = 1$. Hence, the condition for the origin of system (17) to be a focus is that

$$\mathcal{L} = \int_0^{T_n} R(C_s \theta, S_n \theta) \, d\theta \neq 0,$$

where $T_n$ is the period defined in Proposition 8. For instance, if we take $R(x, y) = x^{m+n-2}$ we always have that $\mathcal{L} \neq 0$ because the integrand is positive. The symmetry properties of the generalized trigonometric functions $C_s \theta, S_n \theta$ stated in Proposition 8 imply that in case $m + n$ is odd, the value of $\mathcal{L}$ is zero.

Easy computations show that

$$V_0(x, y) = \left(x^{2n} + n y^2\right)^{\frac{m-1}{2n+1}}$$

is an inverse integrating factor for system (17), where we have used Euler Theorem for $(1, n)$-quasihomogeneous polynomials, cf. Proposition 8. It is clear that since $m$ and $n$ are arbitrary positive integers with the restriction of being of the same parity, the function $V_0(x, y)$ does not need to be analytic in a neighborhood of the origin. We observe that $V_0(x, y)$ is analytic in a neighborhood of the origin if, and only if, there exists an integer $k \geq 0$ such that $m = 2kn + 1$, which is an odd integer. We observe that:

- When the origin of system (17) is a focus, then $m$ and $n$ have the same parity.
- When system (17) has an analytic inverse integrating factor $V_0(x, y)$ defined in a neighborhood of the origin, then $m = 2kn + 1$, which is an odd integer. Therefore, accordingly to Corollary 12, we deduce that in this case and if the origin is a focus, then $n$ needs to be an odd integer.

The function $V(r, \theta)$ defined in the statement of Theorem 10 is $V(r, \theta) = r^m$ which implies that the cyclicity $\operatorname{Cycl}(\mathcal{X}_r, p_0)$ of the origin of system (17), when it is a focus, satisfies $\operatorname{Cycl}(\mathcal{X}_r, p_0) \geq (m + n)/2 - 1$.

Another way to see that the origin of system (17) is always a center in case that $m + n$ is odd, is by using Theorem 10.

**Example 15** We fix an integer $n \geq 2$ and we consider the following planar differential system:

$$\dot{x} = y - \nu_1 x^n, \quad \dot{y} = -x^{2n-1} + \nu_2 x^{n-1} y,$$

(18)

where $\nu_1$ and $\nu_2$ are real parameters such that $\nu_1 \nu_2 - 1 < 0$ and $(\nu_2 + n \nu_1)^2 - 4n < 0$. We remark that the Andreev number of the nilpotent monodromic singular point at the origin of system (18) is $n$. System (18) is the most general $(1, n)$-quasihomogeneous planar polynomial differential system of weighted degree $n$ with
a nilpotent monodromic singularity at the origin (we have used Andreev's Theorem \[6\]). The change to generalized polar coordinates \[15\] transforms system \[18\] in:

\[
\begin{align*}
\dot{r} &= r^n \cos^{n-1} \theta (-\nu_1 \cos^{2n} \theta + \nu_2 \sin^{2} \theta), \\
\dot{\theta} &= -r^{n-1} (1 - (\nu_2 + \nu_1 n) \cos^n \theta \sin \theta).
\end{align*}
\]

Hence, we have that the origin of system \[18\] is a focus if, and only if, the following integral is different from zero:

\[
L = \int_{0}^{T_n} \frac{r^n \cos^{n-1} \theta (-\nu_1 \cos^{2n} \theta + \nu_2 \sin^{2} \theta)}{1 - (\nu_2 + \nu_1 n) \cos^n \theta \sin \theta} \, d\theta \neq 0,
\]

where \(T_n\) is the minimal positive period of these generalized trigonometric functions, see Proposition \[8\]. We define \(z(\theta) := 1 - (\nu_2 + \nu_1 n) \cos^n \theta \sin \theta\) and we have that \(z(\theta) > 0\) for all \(\theta \in \mathbb{R}\) because

\[
z(\theta) = \left( \cos^n \theta - \frac{\nu_2 + \nu_1 n}{2} \sin \theta \right)^2 + \frac{1}{4} \left( 4n - (\nu_2 + n \nu_1)^2 \right) \sin^2 \theta,
\]

where we have used the fundamental relation \(\cos^{2n} \theta + n \sin^2 \theta = 1\) and that, under our hypothesis, \((\nu_2 + n \nu_1)^2 - 4n < 0\).

We remark that, using the symmetries of the generalized trigonometric functions stated in Proposition \[8\], we deduce that if \(n\) is even, then \(L\) vanishes. Therefore, a necessary condition for the origin of system \[18\] to be a focus is that \(n\) must be odd. Indeed, easy computations using the properties of the generalized trigonometric functions, show that

\[
\frac{r^n \cos^{n-1} \theta (-\nu_1 \cos^{2n} \theta + \nu_2 \sin^{2} \theta)}{1 - (\nu_2 + \nu_1 n) \cos^n \theta \sin \theta} = \frac{\nu_2 - n \nu_1}{2n} \frac{r^n \cos^{n-1} \theta}{z(\theta)} + \frac{z'(\theta)}{2nz(\theta)}.
\]

Hence, we deduce that

\[
L = \frac{\nu_2 - n \nu_1}{2n} \int_{0}^{T_n} \frac{r^n \cos^{n-1} \theta}{z(\theta)} \, d\theta.
\]

Under the assumptions \(n > 2\) and odd, \(\nu_1 \nu_2 - 1 < 0\) and \((\nu_2 + n \nu_1)^2 - 4n < 0\), we have that the integrand in the previous expression \(r^n \cos^{n-1} \theta/z(\theta) \geq 0\) for any value of \(\theta \in [0, T_n]\). Therefore, we have that, under these assumptions, the origin of system \[18\] is a focus if, and only if, \(\nu_2 \neq n \nu_1\).

The function \(V_0(x, y) = ny \dot{x} - x \dot{y} = x^{2n} - (\nu_2 + \nu_1 n) x^n y + ny^2\) is an inverse integrating factor of system \[18\], which is a polynomial and, thus, it is analytic in the whole plane. We observe that the parity of \(n\) agrees with Corollary \[12\]. From the given expression of \(V_0(x, y)\), we deduce that \(V(r, \theta) = r\) is an inverse
integrating factor of the equation (3) corresponding to system (18). Applying Theorem 10, we get that $m = 1$. Thus, if we consider an analytic perturbation which is $(1, n)$-quasihomogeneous of weighted subdegrees $(w_x, w_y)$, with $w_x \geq n$ and $w_y \geq 2n - 1$, then no limit cycles can bifurcate from the origin. Indeed, we have that the cyclicity of the origin of system (18) is at least $(n - 1)/2$.

1.3 On the existence of an inverse integrating factor

The following result is a summary and a generalization of several results on the existence of a smooth and non–flat inverse integrating factor $V_0(x, y)$ in a neighborhood of an isolated singular point, see [9, 13, 20].

**Theorem 16** Let the origin be an isolated singular point of (1) and let $\lambda, \mu \in \mathbb{C}$ be the eigenvalues associated to the linear part of (1). If $\lambda \neq 0$, then there exists a smooth and non–flat inverse integrating factor $V_0(x, y)$ in a neighborhood of the origin.

We can ensure the existence of an inverse integrating factor with stronger regularity in some particular cases. We recall that a singular point is said to be strong if the function $\partial P/\partial x + \partial Q/\partial y$ is not zero on it and it is said to be weak otherwise.

By using a translation, we can assume that the origin is the singular point under consideration. We say that the origin is analytically linearizable if there exists an analytic, near-identity change of variables such that the transformed vector field is linear. We say that the origin is orbitally analytically linearizable if there exists an analytic, near-identity change of variables such that the transformed vector field is a linear multiplied by a scalar unit function.

**Corollary 17** Let the origin be an isolated singular point of (1) and let $\lambda, \mu \in \mathbb{C}$ be the eigenvalues associated to the linear part of (1). Then, the following statements hold.

(i) (Strong focus) If $\lambda = \alpha + i\beta$ and $\mu = \alpha - i\beta$ with $\alpha, \beta \in \mathbb{R}\{0\}$, then $V_0(x, y)$ is analytic and it is unique up to a multiplicative constant.

(ii) (Center) If $\lambda = i\beta$ and $\mu = -i\beta$ with $\beta \in \mathbb{R}\{0\}$ and the origin is a center, then $V_0(x, y)$ is analytic.

(iii) (Linearizable point) If the origin is (orbitally) analytically linearizable and $\lambda \neq 0$, then $V_0(x, y)$ is analytic.

(iv) (Node) If $\lambda, \mu \in \mathbb{R}$ and $\lambda \mu > 0$, then $V_0(x, y)$ is analytic and it is unique up to a multiplicative constant.
We include here the references where the previous statements have been proved. All the proofs are based upon the same idea: take the transformation to normal form, which is smooth in the considered cases, see [11] for instance. There is an analytic inverse integrating factor for the vector field in normal form (usually polynomial) and, thus, undoing the transformation, the obtained inverse integrating factor is smooth. In this way, if the singular point is isolated and with \( \lambda \neq 0 \), then its (orbitally) linearizability implies the existence of an analytic inverse integrating factor as it has been shown in [9]. If \( \lambda \neq 0 \), we have that the origin is either:

- A strong focus, and the existence of an analytic inverse integrating factor is given in [20, 9, 13]. A strong focus is a particular case of a linearizable point.

- A weak focus, that is, \( \lambda = i\beta \) and \( \mu = -i\beta \) with \( \beta \in \mathbb{R}\{0\} \) and the origin is not a center. The existence of a smooth inverse integrating factor is given in [13], where a characterization of the existence of an analytic inverse integrating factor and an example of a weak focus without an analytic inverse integrating factor defined in a neighborhood of it are also given. In the following paragraph we include a sketch of the proof of this fact for the sake of completeness.

- A non-degenerate center, and the existence of an analytic inverse integrating factor is given by the normal form of a center given by Poincaré. We recall that this normal form implies that a non-degenerate center is orbitally analytically linearizable.

- A (hyperbolic) node, that is \( \lambda, \mu \in \mathbb{R} \) and \( \lambda \mu > 0 \), and the fact that the transformation to normal form is analytic is proved in [5]. Thus, there is an analytic inverse integrating factor defined in a neighborhood of it. Indeed, in a neighborhood of a node there cannot exist a first integral, see [9], so that the analytic inverse integrating factor is unique up to a multiplicative constant.

- A (hyperbolic) saddle, that is \( \lambda, \mu \in \mathbb{R} \) and \( \lambda \mu < 0 \), and the existence of a smooth inverse integrating factor is shown in [13].

- A semi–hyperbolic point, that is \( \mu = 0 \) but \( \lambda \neq 0 \). When it is isolated, the existence of a smooth inverse integrating factor is given in [13].

For the sake of completeness and in relation with the topic of this paper, we give the normal form in the case of a non–degenerate weak focus, that is, we consider the analytic system \( \{ x \} \{ \zeta = 0 \} \) and we suppose that its origin is a focus. In [6], it
is shown the existence of a smooth and non-flat transformation that brings the considered system to the Birkhoff normal form:

\[ \begin{align*}
\dot{x} &= -y + S_1(r^2)x - S_2(r^2)y, \\
\dot{y} &= x + S_2(r^2)x + S_1(r^2)y,
\end{align*} \tag{19} \]

where \( S_1 \) and \( S_2 \) are formal series on \( r^2 = x^2 + y^2 \). We use Borel’s Theorem, see for instance [27], to ensure the existence of smooth functions representing \( S_1 \) and \( S_2 \).

**Theorem 18** [Borel’s Theorem] For every point \( p \in \mathbb{R}^n \) and for every formal series in \( n \) variables, there exists a \( C^\infty \) function \( f \) defined in a neighborhood of \( p \) whose Taylor series at \( p \) is equal to the given formal series.

Therefore, we have a smooth change of coordinates which brings system (4) \( \{ \zeta = 0 \} \) to system (19). Smooth changes of coordinates in normal form theory usually come from an order-to-order change and may induce flat terms in the normal form. However, since system (4) \( \{ \zeta = 0 \} \) is analytic in a neighborhood of the origin, all of these flat terms can be removed by a suitable smooth change of coordinates. On the contrary, if system (4) \( \{ \zeta = 0 \} \) were only smooth and with an infinite codimension focus at the origin (that is a focus with all its Liapunov constants equal to zero), then the normal form (19) would be only formal because we cannot ensure the removal of all the flat terms. For example, the Birkhoff normal form of system

\[ \begin{align*}
\dot{x} &= -y + x \exp \left( -\frac{1}{x^2 + y^2} \right), \\
\dot{y} &= x + y \exp \left( -\frac{1}{x^2 + y^2} \right)
\end{align*} \]

is \( \dot{x} = -y, \dot{y} = x \) but its origin is a focus. Therefore, the flat terms cannot be removed by any smooth change. This is an example of a smooth system with a focus of infinite codimension. It is known from Poincaré that an analytic system (4) has no foci of infinite codimension. In this work we only consider analytic differential systems.

We take the analytic system (4) \( \{ \zeta = 0 \} \) and we perform the smooth change of coordinates which brings it to the Birkhoff normal form (19). It is well known, that the origin of system (4) \( \{ \zeta = 0 \} \) is a center if, and only if, \( S_1 \equiv 0 \). Since we are in the case that the origin is a focus, we have that \( S_1 \) is not identically null and easy computations show that \( V_0(x, y) = (x^2 + y^2) S_1(x^2 + y^2) \) is an inverse integrating factor for the Birkhoff normal form. By undoing the change to the original system, we obtain a smooth and non-flat inverse integrating factor.

For a strong focus, there is a unique analytic inverse integrating factor, as it has been shown in [9]. For a weak focus, when there is an analytic inverse integrating factor, then it is unique up to a multiplicative constant. However, there may exist other inverse integrating factors with lower regularity, as it is shown, for instance, in the forthcoming example with system (22).
The following result is a consequence of the results given in Theorems 1 and 10 and it ensures the existence of an inverse integrating factor, of class at least $C^1$, in a neighborhood of certain degenerate focus points, namely for the origin of system (5) without characteristic directions and the origin of system (10).

**Corollary 19** There exists an inverse integrating factor $V_0(x, y)$, of class at least $C^1$, in a neighborhood of the following two types of singular points: a degenerate focus without characteristic directions and a nilpotent focus.

The proof of this corollary is given in Section 3.

### 2 Ordinary differential equations over a cylinder

This section is devoted to several results related with ordinary differential equations of the form (3) defined over a cylinder $C = \{(r, \theta) \in \mathbb{R} \times S^1 : |r| < \delta\}$ for a certain $\delta > 0$ sufficiently small. We denote by $T$ the minimal positive period of the variable $\theta$, that is, we consider the circle $S^1 = \mathbb{R}/[0, T]$. Thus, we consider an ordinary differential equation of the form (3):

$$\frac{dr}{d\theta} = \mathcal{F}(r, \theta),$$

where $\mathcal{F}(r, \theta)$ is an analytic function on the cylinder $C$ and $\mathcal{F}(0, \theta) \equiv 0$. We have that, by assumption, the circle $r = 0$ is a periodic orbit of equation (3).

We assume that equation (3) has an inverse integrating factor $V(r, \theta)$ which is analytic in a neighborhood of $r = 0$. All the results remain true if we assume that $V(r, \theta)$ is a smooth function in $r$ in a neighborhood of $r = 0$. We remark that $V(r, \theta)$ is a function over the cylinder $C$ and, thus, $V(r, \theta)$ is $T$–periodic in $\theta$. We recall that we have defined the vanishing multiplicity $m$ of $V(r, \theta)$ on $r = 0$ as the value such that

$$V(r, \theta) = v_m(\theta) r^m + \mathcal{O}(r^{m+1}), \quad (20)$$

with $v_m(\theta) \neq 0$. The following lemma is already stated and proved in [15]. We include here a proof for the sake of completeness.

**Lemma 20** [15] If $V(r, \theta)$ is an inverse integrating factor of (3) which is smooth and non–flat in $r$ in a neighborhood of $r = 0$ and with vanishing multiplicity $m$ over $r = 0$, then the function $v_m(\theta)$ defined in (20) satisfies that $v_m(\theta) \neq 0$ for $\theta \in [0, T)$. 

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Proof. By hypothesis $F(0, \theta) \equiv 0$ and we define the function $F_1(\theta)$ as the one which satisfies
\[
F(r, \theta) = F_1(\theta) r + O(r^2).
\]
We note that $F_1(\theta)$ may be identically null. We have that $V(r, \theta)$ satisfies the following partial differential equation
\[
\frac{\partial V(r, \theta)}{\partial \theta} + \frac{\partial V(r, \theta)}{\partial r} F(r, \theta) = \frac{\partial F}{\partial r} V(r, \theta),
\]
and equating the coefficients of the order $r^m$ in the previous identity, we get that $v'_m(\theta) = (1 - m) F_1(\theta) v_m(\theta)$. Since $v_m(\theta) \neq 0$, let $\theta_0 \in [0, T)$ be such that $v_m(\theta_0) \neq 0$. We deduce that
\[
v_m(\theta) = v_m(\theta_0) \exp \left\{ (1 - m) \int_{\theta_0}^{\theta} F_1(\sigma) d\sigma \right\}.
\]
Therefore, we conclude that $v_m(\theta) \neq 0$ for $\theta \in [0, T)$. \qed

As we will see below, $m$ coincides with the multiplicity of the limit cycle $r = 0$ of equation (3). We remark that the integral $\int_0^T F_1(\sigma) d\sigma$ is the characteristic exponent of the periodic orbit $r = 0$ in equation (3). In particular, either $m = 1$ or $\int_0^T F_1(\sigma) d\sigma = 0$. Thus, from the formula (21), we confirm that $v_m(\theta)$ is always a $T$-periodic function.

The following lemma establishes the existence and uniqueness of $V(r, \theta)$ when the periodic orbit $r = 0$ is isolated, that is, when $r = 0$ is a limit cycle of (3). The existence is proved in [13] and the uniqueness is also stated and proved in [15].

Lemma 21 [13, 15] If the circle $r = 0$ is a limit cycle of (3), then there exists an inverse integrating factor $V(r, \theta)$ of (3) which is smooth and non-flat in $r$ in a neighborhood of $r = 0$. Indeed, $V(r, \theta)$ is unique, up to a nonzero multiplicative constant.

Proof. We recall here the main ideas to prove this statement, for the sake of completeness.

Let us assume that $r = 0$ is a limit cycle of multiplicity $m$ of equation (3). When $m = 1$, we say that $r = 0$ is hyperbolic. Following the ideas given in [13], and by a result in [30], in a neighborhood of $r = 0$, we can consider a smooth, and non-flat in $\rho$ in a neighborhood of $\rho = 0$, change of coordinates $(r, \theta) \to (\rho, \tau)$ which takes equation (3) to
\[
\frac{d\rho}{d\tau} = \lambda \rho \quad \text{with } \lambda \neq 0, \quad \text{if } m = 1;
\]
\[
\frac{d\rho}{d\tau} = \rho^m + a \rho^{2m-1} \quad \text{with } a \in \mathbb{R}, \quad \text{if } m > 1.
\]
In the case that \( r = 0 \) is hyperbolic \( (m = 1) \), we have that the change of coordinates is, indeed, analytic in a neighborhood of \( r = 0 \). We remark that the function \( \bar{V}(\rho, \tau) \) defined by:

\[
\bar{V}(\rho, \tau) := \begin{cases} 
\rho & \text{if } m = 1; \\
\rho^m + a \rho^{2m-1} & \text{if } m > 1;
\end{cases}
\]

is an inverse integrating factor of the latter equation. By undoing the change of coordinates, we have a smooth, and non-flat in \( r \) in a neighborhood of \( r = 0 \), inverse integrating factor \( V(r, \theta) \) for equation (3). In the case that \( r = 0 \) is hyperbolic, we have an analytic inverse integrating factor in a neighborhood of \( r = 0 \).

To show the uniqueness of \( V(r, \theta) \), let us assume that there exist two linearly independent inverse integrating factors \( V(r, \theta) \) and \( \tilde{V}(r, \theta) \) of equation (3) which are both smooth and non-flat in \( r \) in a neighborhood of \( r = 0 \). We assume that \( V(r, \theta) = v_m(\theta) r^m + O(r^{m+1}) \) and \( \tilde{V}(r, \theta) = \tilde{v}_{\tilde{m}}(\theta) r^{\tilde{m}} + O(r^{\tilde{m}+1}) \) and that \( m \geq \tilde{m} \). We have that the function on the cylinder \( C \) defined by

\[
H(r, \theta) := V(r, \theta)/\tilde{V}(r, \theta)
\]

is not locally constant, smooth in \( r \) and of class \( C^1 \) in \( \theta \), by using the Lemma 20.

If \( m > \tilde{m} \), we have that \( H(r, \theta) \) is constant equal to 0 all over the circle \( r = 0 \) and if \( m = \tilde{m} \), using the proof of Lemma 20 we have that \( H(0, \theta) = v_m(0)/\tilde{v}_{\tilde{m}}(0) \), from which we deduce that \( H(r, \theta) \) takes a constant value all over the circle \( r = 0 \). Moreover, this function \( H(r, \theta) \) is a first integral of equation (3), since it satisfies

\[
\frac{\partial H(r, \theta)}{\partial \theta} + \frac{\partial H(r, \theta)}{\partial r} F(r, \theta) = 0.
\]

Thus, \( H(r, \theta) \) is constant on each orbit of equation (3). If \( r = 0 \) is a limit cycle, then the orbits in a neighborhood of this circle accumulate on it. By continuity of \( H(r, \theta) \), this fact implies that \( H(r, \theta) \) needs to take the same value on any point in a neighborhood of \( r = 0 \) in contradiction with the fact that \( H(r, \theta) \) is not locally constant.

The following example, which is described in page 219 of [7], shows that the conditions for \( V(r, \theta) \) to be smooth and non-flat in \( r \) in a neighborhood of \( r = 0 \) and \( 2\pi \)-periodic in \( \theta \) are essential to have a unique inverse integrating factor. We consider the planar differential system

\[
\dot{x} = -y + x(x^2 + y^2), \quad \dot{y} = x + y(x^2 + y^2),
\]

which has a non-degenerate (unstable) focus at the origin. The following functions are two inverse integrating factors of the system which are of class \( C^1 \) in a neighborhood of the origin

\[
V_0(x, y) = (x^2 + y^2)^2 \quad \text{and} \quad \tilde{V}_0(x, y) = (x^2 + y^2)^2 \sin \left( 2 \arctan \left( \frac{y}{x} \right) + \frac{1}{x^2 + y^2} \right).
\]
In polar coordinates, this system reads for $\dot{r} = r^3$, $\dot{\theta} = 1$, which has the following inverse integrating factor $V_1(r, \theta) = V_0(r \cos \theta, r \sin \theta)/r = r^3$ analytic in a neighborhood of $r = 0$. Moreover, the function $V_2(r, \theta) = \bar{V}_0(r \cos \theta, r \sin \theta)/r = r^3 \sin(2\theta + r - 2)$ is an inverse integrating factor of class $C^1$ in $r \neq 0$. Indeed, $V_3(r, \theta) = r + 2r^3\theta$ is another inverse integrating factor of the system in polar coordinates, which is analytic in $r$ but not $2\pi$–periodic in $\theta$.

We will show that when the periodic orbit $r = 0$ is a limit cycle of equation (3), the vanishing multiplicity of an inverse integrating factor $V(r, \theta)$ is strictly positive.

**Lemma 22** Let us consider a differential equation of the form (3) over a cylinder $C = \{(r, \theta) \in \mathbb{R} \times S^1 : |r| < \delta\}$ for a certain $\delta > 0$ and let us assume that $r = 0$ is a periodic orbit of the equation. We assume that $V(r, \theta)$ is an inverse integrating factor for equation (3) defined in $C \setminus \{r = 0\}$ and which has a Laurent series of the form:

$$V(r, \theta) = v_m(\theta) r^m + O(r^{m+1}),$$

with $v_m(\theta) \neq 0$ and $m \in \mathbb{Z}$. If $m \leq 0$, the periodic orbit $r = 0$ has a neighborhood filled with periodic orbits, that is, it is not a limit cycle.

**Proof.** Let $\omega = dr - F(r, \theta) d\theta$ be the Pfaffian 1–form associated to equation (3). Since $V(r, \theta)$ is an inverse integrating factor of the equation (3), we have that $\omega/V$ is a closed 1–form. We observe that the proof of Lemma 20 also applies and, therefore, we have that $v_m(\theta) \neq 0$ for any $\theta \in [0, T)$. In case that $m \leq 0$, we have that $\omega/V$ is well-defined in the whole cylinder $C$. Let us consider any non–contractible cycle in this cylinder, for instance the cycle $r = 0$. By virtue of De Rham’s Theorem, see [14], we have that the closed 1–form $\omega/V$ is exact if, and only if, the value of the line integral $\int_{r=0} \omega/V$ is zero. This value is zero since the oval $r = 0$ is an orbit of the 1-form $\omega$ and, thus, $\omega|_{r=0} \equiv 0$. Hence, we have that $\omega/V = dH$ for a certain $C^2$ function $H(r, \theta)$, which turns out to be a first integral of equation (3). The existence of this first integral implies that the cycle $r = 0$ is surrounded by periodic orbits, formed by the level curves of $H$. 

In order to relate the vanishing multiplicity $m$ of $V(r, \theta)$ on $r = 0$, which we have proved to be positive, and the cyclicity of the focus at the origin of system (1), we use a previous result which is already stated and proved in [15]. Our result gives an ordinary differential equation for the Poincaré map associated to equation (3) in terms of the inverse integrating factor $V(r, \theta)$. If the minimal positive period of $\theta$ in (3) is denoted by $T$ and $\Psi(\theta; r_0)$ is the flow of (3) with initial condition $\Psi(0; r_0) = r_0$, we recall that the Poincaré map $\Pi$ : $\Sigma \subseteq \mathbb{R} \to \mathbb{R}$ is defined as $\Pi(r_0) = \Psi(T; r_0)$. We have that $\Pi$ is an analytic diffeomorphism defined in a neighborhood $\Sigma$ of $r_0 = 0$ whose fixed points correspond to periodic orbits of the
Since we have that $r = 0$ is a limit cycle of equation (3), we deduce that the Poincaré map is not the identity map. We recall that the limit cycle $r = 0$ is said to have multiplicity $k$ if the expansion of the analytic Poincaré map in a neighborhood of $r_0 = 0$ is of the form
\[
\Phi(r_0) = r_0 + c_k r_0^k + O(r_0^{k+1}),
\]
where $c_k \neq 0$. The multiplicity of the limit cycle $r = 0$ in equation (3) allows us to determine the cyclicity of the focus at the origin of system (1), as we will see below.

We state the result proved in [15] in the terms used in the present paper.

**Theorem 23** [15] Let us assume that $V(r, \theta)$ is an inverse integrating factor of equation (3) defined in a neighborhood of the periodic orbit $r = 0$, whose minimal positive period is denoted by $T$. We consider $\Phi(r_0)$ the Poincaré map associated to the periodic orbit $r = 0$ of equation (3). Then, the following identity holds:
\[
V(\Phi(r_0), T) = V(r_0, 0) \Phi'(r_0).
\] (23)

As a consequence of equation (23), we can prove the following result.

**Corollary 24** [15] Let us assume that $V(r, \theta)$ is an inverse integrating factor of equation (3) which is smooth and non-flat in $r$ in a neighborhood of the limit cycle $r = 0$ and whose vanishing multiplicity over it is $m$. Then, $r = 0$ is a limit cycle of multiplicity $m$.

**Proof.** Since $V(r, \theta)$ is assumed to be a function on the cylinder $C$, we have that it needs to be $T$ periodic in $\theta$. We have that $V(r_0, T) = V(r_0, 0)$ and we consider its development in a neighborhood of $r_0 = 0$:
\[
V(r_0, 0) = \nu_m r_0^m + O(r_0^{m+1}),
\]
where $\nu_m \neq 0$. We observe that the index $m$ appearing in this decomposition coincides with the vanishing multiplicity of $V(r, \theta)$ in $r = 0$ by Lemma 20. Recalling that $\Phi(r_0) = r_0 + c_k r_0^k + O(r_0^{k+1})$, where $c_k \neq 0$, we consider equation (23) and we subtract $V(r_0, 0)$ from both members to obtain that:
\[
\nu_m (\Phi(r_0)^m - r_0^m) + O(r_0^{m+1}) = (\nu_m r_0^m + O(r_0^{m+1})) (k c_k r_0^{k-1} + O(r_0^k)).
\]
The lowest order terms in both sides of the previous identity correspond to $r_0^{m+k-1}$ and the equation of their coefficients is $m c_k \nu_m = k c_k \nu_m$, which implies that $k = m$.

### 3 Proofs of the results

**Proof of Theorem 1**

The following lemma establishes the first step in the proof of Theorem 1. We show that the transformation to polar coordinates $x = r \cos \theta$, $y = r \sin \theta$, of system (1) gives an equation over a cylinder of the form (3).
Lemma 25 We consider system (5) with \( d \geq 1 \) and odd integer and we assume there are no characteristic directions. Then the transformation to polar coordinates brings system (5) to an ordinary differential equation over a cylinder.

Proof. In polar coordinates system (5) becomes
\[
\begin{align*}
\dot{r} &= r^d R(r, \theta) = r^d (R_d(\theta) + O(r)), \\
\dot{\theta} &= r^{d-1} F(r, \theta) = r^{d-1} (F_d(\theta) + O(r)),
\end{align*}
\]
where
\[
\begin{align*}
R_d(\theta) &= p_d(\cos \theta, \sin \theta) \cos \theta + q_d(\cos \theta, \sin \theta) \sin \theta, \\
F_d(\theta) &= q_d(\cos \theta, \sin \theta) \cos \theta - p_d(\cos \theta, \sin \theta) \sin \theta.
\end{align*}
\]
The hypothesis that there are no characteristic directions is equivalent to say that \( F_d(\theta) \neq 0 \) for \( \theta \in [0, 2\pi) \). We can, therefore, consider the ordinary differential equation associated to the orbits of system (24) which takes the form (3):
\[
\frac{dr}{d\theta} = \frac{r R(r, \theta)}{F(r, \theta)}. \tag{25}
\]

The following lemma establishes that the center problem for the origin of system (5) is equivalent to the determine when the circle \( r = 0 \) is contained in a period annulus for equation (25).

Lemma 26 We consider system (5) with \( d \geq 1 \) and odd integer and we assume there are no characteristic directions. The circle \( r = 0 \) is a limit cycle of equation (25) if, and only if, the origin of system (5) is a focus.

Proof. The origin of system (5) is transformed to the periodic orbit \( r = 0 \) of equation (25) by the transformation to polar coordinates. This transformation gives a one-to-one correspondence between any point in a punctured neighborhood of the origin in the plane \((x, y)\) and a cylinder \( \{(r, \theta) : 0 < r < \delta, \theta \in S^1\} \) for \( \delta > 0 \) sufficiently small. Thus, any orbit spiraling from or towards the origin of system (5) is transformed to an orbit spiraling (from or towards) the circle \( r = 0 \) in equation (25).

We have a symmetry for equation (25) which is inherited by the symmetries of the polar coordinates.

Lemma 27 Let us consider a planar \( C^1 \) differential system \( \dot{x} = P(x, y), \dot{y} = Q(x, y) \), and perform the change to polar coordinates \( x = r \cos \theta, y = r \sin \theta \). The resulting system \( \dot{r} = \Xi(r, \theta), \dot{\theta} = \Theta(r, \theta) \) is invariant under the change of variables \((r, \theta) \mapsto (-r, \theta + \pi)\).
Proof. We observe that the monomials $r \cos \theta$ and $r \sin \theta$ are invariant by the change of variables. Since

$$\dot{r} = \frac{r \cos \theta P(r \cos \theta, r \sin \theta) + r \sin \theta Q(r \cos \theta, r \sin \theta)}{r},$$

$$\dot{\theta} = \frac{r \cos \theta Q(r \cos \theta, r \sin \theta) - r \sin \theta P(r \cos \theta, r \sin \theta)}{r^2},$$

the result follows.

Lemma 28 We assume that the origin of system (5) is a focus without characteristic direction and we consider the corresponding equation (25) with associated Poincaré map $\Pi(r_0) = r_0 + c_m r_0^m + O(r_0^{m+1})$, with $c_m \neq 0$. Then, $m$ is odd.

Proof. By Lemma 26 we have that the origin of system (5) is a focus if, and only if, the circle $r = 0$ is a limit cycle of equation (25). We assume that the circle $r = 0$ is a limit cycle with multiplicity $m$ and we consider its associated Poincaré map $\Pi(r_0)$. By Lemma 27 we have that equation (25) has the discrete symmetry $(r, \theta) \mapsto (-r, \theta + \pi)$ because it comes from a system in cartesian coordinates (5). This symmetry implies that $r = 0$ is either a stable or an unstable limit cycle. Thus, $m$ needs to be odd.

The next lemma states that, under our hypothesis, we have an inverse integrating factor $V(r, \theta)$ for equation (25) which is smooth and non-flat in $r$ in a neighborhood of $r = 0$ and such that $V(0, \theta) \equiv 0$.

Lemma 29 We consider system (5) with $d \geq 1$ and odd integer and we assume there are no characteristic directions.

(i) If system (5) has an inverse integrating factor $V_0(x, y)$ defined in a neighborhood of the origin, then the function defined by

$$V(r, \theta) := \frac{V_0(r \cos \theta, r \sin \theta)}{r^d F(r, \theta)}$$

is an inverse integrating factor for equation (25) in $r \neq 0$.

(ii) Let $V(r, \theta)$ be an inverse integrating factor of equation (25) which has a Laurent expansion in a neighborhood of $r = 0$ of the form $V(r, \theta) = v_m(\theta) r^m + O(r^{m+1})$, with $v_m(\theta) \neq 0$ and $m \in \mathbb{Z}$. Then, if $m \leq 0$ the origin of system (5) is a center.
Proof. (i) Since the jacobian to polar coordinates is $r$, we have that the function $V_0(r \cos \theta, r \sin \theta)/r$ is an inverse integrating factor for system (24). We see that equation (25) is the equation of the orbits associated to system (24) and, therefore, the function $V(r, \theta) := V_0(r \cos \theta, r \sin \theta)/(r^d F(r, \theta))$ is an inverse integrating factor of (25). We remark that $V(r, \theta)$ does not need to be well-defined in a neighborhood of $r = 0$. Thus, we only have, at the moment, that it is an inverse integrating factor for equation (25) in $r \neq 0$.

(ii) If $m \leq 0$, we are under the hypothesis of Lemma 22 and we conclude that the cycle $r = 0$ is surrounded by periodic orbits, which give rise to a neighborhood of the origin of system (5) filled with periodic orbits. Therefore, the origin of system (5) is a center.

The previous Lemma 29 together with the statements given in Lemmas 25 and 26 establishes that under the hypothesis of Theorem 1, we have that $m \geq 1$. Moreover, by Lemma 28 and Corollary 24 we have that $m$ needs to be odd. We have proved statement (i) in Theorem 1.

Assuming that the origin of system (5) is a focus, the following step of the proof is to relate the multiplicity of the limit cycle $r = 0$ of equation (25) and the cyclicity of the origin of system (5).

Lemma 30 We consider system (5) with $d \geq 1$ and odd integer, we assume there are no characteristic directions and that the origin $p_0$ is a focus with cyclicity $\text{Cycl}(X_\varepsilon, p_0)$. We consider the corresponding equation (25) and we assume that $r = 0$ is a limit cycle of multiplicity $m = 2k+1$. Then, $\text{Cycl}(X_\varepsilon, p_0) \geq (m+d)/2 - 1$. Moreover:

1. When $d = 1$, we have $\text{Cycl}(X_\varepsilon, p_0) = k$.

2. If we only consider perturbations of system (5) whose subdegree is $\geq d$, then at most $k$ limit cycles can bifurcate from the origin of system (5), that is, $\text{Cycl}(X_\varepsilon^{[d]}, p_0) = k$.

Proof. The particular case $d = 1$ of statement 1. and system (5) with a focus at the origin is proved in Theorem 40, Ch. IX in [4] (page 254), see also [29].

We first provide an example of a perturbation of equation (25), with $m$ limit cycles bifurcating from $r = 0$, whose transformation to cartesian coordinates gives a perturbation of system (5) of the form (2) with exactly $(m-1)/2 = k$ limit cycles bifurcating from the origin. This example shows that $\text{Cycl}(X_\varepsilon, p_0) \geq k$.

We take the corresponding equation (25) and we assume that $r = 0$ is a limit cycle of multiplicity $m$, which is an odd integer with $m \geq 1$. We consider the
associated system (24) from which (25) comes from, we define $k = (m - 1)/2$ and we perturb system (24) in the following way:

$$
\dot{r} = r^d R(r, \theta) + \sum_{i=0}^{k-1} \varepsilon^{k-i} a_i r^{2i+d}, \quad \dot{\theta} = r^{d-1} F(r, \theta),
$$

(26)

with the convention that if $k = 0$ no perturbation term is taken. The real constant $\varepsilon$ is the perturbation parameter $0 < |\varepsilon| << 1$ and the $a_i, i = 0, 1, 2, \ldots, k - 1,$ are real constants to be chosen in such a way that the Poincaré map $\Pi(r_0; \varepsilon)$ associated to the ordinary differential equation

$$
\frac{dr}{d\theta} = \frac{r^d R(r, \theta) + \sum_{i=0}^{k-1} \varepsilon^{k-i} a_i r^{2i+d}}{r^{d-1} F(r, \theta)} = \frac{r R(r, \theta) + r \sum_{i=0}^{k-1} \varepsilon^{k-i} a_i r^{2i}}{F(r, \theta)}
$$

has $2k + 1$ real zeroes; $k$ of them positive. We recall that $F(0, \theta) \neq 0$ for all $\theta \in [0, 2\pi)$ and, thus, the perturbative terms are an analytic perturbation in a neighborhood of $r = 0$ and $\varepsilon = 0$ of equation (25). The proof of the fact that this choice of $a_i$ can be done is analogous to the one described in [4], pp 254–259. More concretely, the exponent of the leading term of the displacement function $d(r_0; 0)$ of system (25) is $m$ and the considered perturbation (26) produces that $d(r_0; \varepsilon)$ has all the monomials of odd powers of $r_0$ up to order $m$. The coefficient of each monomial, for $\varepsilon$ sufficiently small, is dominated by one of the constants $a_i$.

Undoing the change to polar coordinates, system (26) gives rise to an analytic system in a neighborhood of the origin which is a perturbation of system (5) of the form (2) and with $k = (m - 1)/2$ limit cycles bifurcating from the origin. If system (5) is written as $\dot{x} = P(x, y)$ and $\dot{y} = Q(x, y)$, then the change to cartesian coordinates from (26) reads for:

$$
\dot{x} = P(x, y) + x K(x, y, \varepsilon), \quad \dot{y} = Q(x, y) + y K(x, y, \varepsilon),
$$

(27)

where $K(x, y, \varepsilon) = \sum_{i=0}^{k-1} \varepsilon^{k-i} a_i (x^2 + y^2)^i + \frac{d-1}{2}$. We recall that $d$ is odd and $d \geq 1$. In this way, we have that $\text{Cycl}(\mathcal{X}_\varepsilon, p_0) \geq k$.

We provide now an example of an analytic perturbation of system (5) with at least $(m + d)/2 - 1$ limit cycles bifurcating from the origin. We take system (27) and we perturb it in order to produce $\ell = (d - 1)/2$ additional limit cycles bifurcating from the origin when $\varepsilon \to 0$. Let us consider the smallest limit cycle $\gamma$ surrounding the origin of system (27) and let us assume that it is an attractor. We have that the $(m - 1)/2$ limit cycles of system (27) which bifurcate from the origin are hyperbolic, by choosing the parameters $a_i$ conveniently. Since $\gamma$ is the smallest limit cycle, we have that the origin needs to be a repeller. Let us take a
convenient real value \(b_{l-1}\) such that the system

\[
\begin{align*}
\dot{x} &= P(x, y) + x K(x, y, \varepsilon) + \varepsilon^{k+1} x b_{l-1} (x^2 + y^2)^{l-1}, \\
\dot{y} &= Q(x, y) + y K(x, y, \varepsilon) + \varepsilon^{k+1} y b_{l-1} (x^2 + y^2)^{l-1}
\end{align*}
\]

still has the previous limit cycle \(\gamma\) as an attractor (thus, \(|b_{l-1}|\) needs to be small enough) and the origin also becomes an attractor (this implies that \(b_{l-1} < 0\)). Therefore, there is a limit cycle bifurcating from the origin, surrounding it and smaller than \(\gamma\). If \(\gamma\) was a repeller and the origin an attractor in system (27), we take \(b_{l-1} > 0\) in order to make the new limit cycle to bifurcate. This bifurcated limit cycle is hyperbolic by conveniently choosing the value \(b_{l-1}\). The previous system maintains the \((m-1)/2\) limit cycles of (27) because they are all hyperbolic. Thus, the previous system has at least \((m-1)/2 + 1\) limit cycles bifurcating from the origin when \(\varepsilon \to 0\).

By induction, and a relabelling of the parameters \(a_i := b_{i+(d-1)/2}\), we deduce that the following system

\[
\begin{align*}
\dot{x} &= P(x, y) + x \bar{K}(x, y, \varepsilon), \\
\dot{y} &= Q(x, y) + y \bar{K}(x, y, \varepsilon),
\end{align*}
\]

(28)

where \(\bar{K}(x, y, \varepsilon) = \sum_{i=0}^{L-1} \varepsilon^{L-i} b_i (x^2 + y^2)^i\), \(L := (m + d)/2 - 1\), has at least \((m + d)/2 - 1\) limit cycles bifurcating from the origin. We recall that both \(m\) and \(d\) are odd and \(d \geq 1\), \(m \geq 1\). In this way, we have that \(\text{Cycl}(\mathcal{X}_\varepsilon, p_0) \geq (m + d)/2 - 1\).

Finally, we will prove statement 2. If we only consider perturbations of system (5) whose subdegree is \(\geq d\), that is vector fields of the type \(\mathcal{X}_\varepsilon^{[d]}\), then, the transformation of these perturbative terms to polar coordinates gives rise to a perturbation of the corresponding equation (25) which is analytic in a neighborhood of \(r = 0\) and \(\varepsilon = 0\). Let us assume that the circle \(r = 0\) is a limit cycle with multiplicity \(m\) and we consider the Poincaré map \(\Pi(r_0)\) defined in the previous section, which satisfies:

\[
\Pi(r_0) = r_0 + c_m r_0^m + \mathcal{O}(r_0^{m+1}),
\]

with \(c_m \neq 0\). We recall that equation (25) has the discrete symmetry \((r, \theta) \mapsto (-r, \theta + \pi)\) because it comes from a system in cartesian coordinates (5). This symmetry implies that \(r = 0\) is a limit cycle which cannot be semistable and therefore \(m\) is odd.

The key point of the proof is that any perturbation of (25) analytic near \((r, \varepsilon) = (0, 0)\), with \(\varepsilon \in \mathbb{R}^p\) small, has a displacement function \(d(r_0; \varepsilon) = \Pi(r_0; \varepsilon) - r_0\) which is analytic near \((r_0, \varepsilon) = (0, 0)\) and when \(\varepsilon = 0\) coincides with the displacement function \(d(r_0; 0) = \Pi(r_0) - r_0\) of the unperturbed equation (25). By using standard arguments (counting zeroes with Weierstrass Preparation Theorem of \(d(r_0; \varepsilon)\) near
$(r_0, \varepsilon) = (0, 0)$, the cyclicity of the circle $r = 0$ under analytic perturbations of equation (25) is $m$. We recall that the cyclicity and the multiplicity of a limit cycle are equal, see [4]. However, since the displacement function $d(r_0; 0) = \Pi(r_0) - r_0$ of the unperturbed equation (25) is of odd order $m$ at $r_0 = 0$, and taking into account the above mentioned discrete symmetry, we have that only $(m - 1)/2$ zeroes of $d(r_0; \varepsilon)$ can appear for $r_0 > 0$ and $\|\varepsilon\|$ small enough. This fact gives that at most $(m - 1)/2$ limit cycles bifurcate from the origin of system (5) when only this kind of perturbative terms are taken into account. Therefore, we have proved that $\text{Cycl}(X^{[\varepsilon]}_2, p_0) = (m - 1)/2$. Indeed, the example given in (27) shows that this upper bound is sharp.

Proof of Theorem 10.

The proof of this theorem goes analogously to the proof of Theorem 1, only with some technical differences.

We first show that the transformation to generalized polar coordinates $x = r \cos \theta$, $y = r^n \sin \theta$ transforms system (13) to an equation over the cylinder of the form (3).

Lemma 31 We assume that the origin of system (13) is a nilpotent monodromic singular point. Then the transformation to generalized polar coordinates $(x, y) = (r \cos \theta, r^n \sin \theta)$ brings system (13) to an ordinary differential equation (3) over a cylinder.

Proof. Taking into account that $\cos^{2n} \theta + n \sin^2 \theta = 1$, we get that the Jacobian determinant of the former change is

$$J(r, \theta) = \frac{\partial(x, y)}{\partial(r, \theta)} = r^n.$$ 

Since $x^{2n} + ny^2 = r^{2n}$, we deduce that

$$\dot{r} = \frac{x^{2n-1} \dot{x} + y \dot{y}}{r^{2n-1}}, \quad \dot{\theta} = \frac{x \dot{y} - ny \dot{x}}{r^{n+1}}.$$ 

In particular, system (13) adopts the form

$$\dot{r} = \tilde{p}(\theta) r^{n+1} + O(r^{n+2}), \quad \dot{\theta} = r^{n-1} + O(r^n), \quad \text{(29)}$$

when $\beta > n - 1$ or $\phi(x) \equiv 0$, and

$$\dot{r} = b \cos^{n-1} \theta \sin^2 \theta r^n + O(r^{n+1}), \quad \dot{\theta} = (1 + b \cos^n \theta \sin \theta) r^{n-1} + O(r^n), \quad \text{(30)}$$

when $\beta = n - 1$. 

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We also observe that in the latter case \((\beta = n - 1)\) we have the following decomposition

\[
1 + b \, C_{s}^{n} \theta \, S_{n} \theta = \left( \frac{C_{s} \theta + \frac{b}{2} S_{n} \theta}{2} \right)^{2} + \frac{1}{4} \left( 4n - b^{2} \right) S_{n}^{2} \theta,
\]

where we have used that \(C_{s}^{2n} \theta + n S_{n}^{2} \theta = 1\). Since \(4n - b^{2} > 0\) in this case, due to Andreev’s conditions for monodromy, we deduce that \(1 + b \, C_{s} \theta \, S_{n} \theta > 0\) for any \(\theta \in \mathbb{R}\).

We denote by \(\Xi(r, \theta)\) the function defined by \(\dot{r}\) and by \(\Theta(r, \theta)\) the function defined by \(\dot{\theta}\) in both cases, and we have that,

\[
\dot{r} = \Xi(r, \theta), \quad \dot{\theta} = \Theta(r, \theta) = \Theta_{n-1}(\theta) r^{n-1} + \mathcal{O}(r^{n}),
\]

where

\[
\Theta_{n-1}(\theta) = \begin{cases} 
1 & \text{if } \beta > n - 1 \text{ or } \phi(x) \equiv 0, \\
1 + b \, C_{s} \theta \, S_{n} \theta & \text{if } \beta = n - 1.
\end{cases}
\]

Hence, the equation of the orbits corresponding to system \((29)\) or \((30)\) writes as

\[
\frac{d r}{d \theta} = \begin{cases} 
\frac{O(r^{2})}{1 + O(r)} & \text{if } \beta > n - 1 \text{ or } \phi(x) \equiv 0, \\
\frac{O(r)}{1 + b \, C_{s} \theta \, S_{n} \theta + O(r)} & \text{if } \beta = n - 1.
\end{cases}
\]

We observe that \(\Theta_{n-1}(\theta) \neq 0\) for any \(\theta \in [0, T_{n})\). In short, we have proved that in a neighborhood of any monodromic singular point of the form \((10)\), we can perform a transformation, which is the composition of the changes \((11)\) and \((12)\) and the transformation to generalized polar coordinates, which brings the system to an equation over a cylinder \(C\):

\[
\frac{d r}{d \theta} = \mathcal{F}(r, \theta),
\]

where \(\mathcal{F}(r, \theta)\) is \(T_{n}\)-periodic in \(\theta\) and \(\mathcal{F}(0, \theta) \equiv 0\).

The center problem for the origin of system \((10)\) is equivalent to determine when the circle \(r = 0\) is contained in a period annulus for equation \((32)\).

**Lemma 32** We assume that the origin of system \((10)\) is a monodromic singular point with Andreev number \(n\). The circle \(r = 0\) is a limit cycle of equation \((32)\) if, and only if, the origin of system \((10)\) is a focus.
Proof. The proof is analogous to the proof of Lemma 26. The transformation from system (10) to equation (32) gives a one-to-one correspondence between each point in a punctured neighborhood of the origin in the plane \((x, y)\) and each point on a cylinder \(\{(r, \theta) : 0 < r < \delta, \theta \in S^1\}\) for \(\delta > 0\) sufficiently small.

The following proposition establishes a symmetry for equation (32) which is inherited by the symmetries of the generalized trigonometric functions.

**Proposition 33** Let us consider a planar \(C^1\) differential system
\[
\dot{x} = P(x, y), \quad \dot{y} = Q(x, y),
\]
we take any positive integer \(n\) and perform the change to generalized polar coordinates
\[
x = r \cos \theta, \quad y = r^n \sin \theta.
\]
The resulting system \(\dot{r} = \Xi(r, \theta), \quad \dot{\theta} = \Theta(r, \theta)\) is invariant under the change of variables \((r, \theta) \mapsto (-r, (-1)^{n+1} [\theta + T_n/2])\).

**Proof.** We observe that, due to Proposition 8, the following composition is the identity \((X = x, Y = y)\):
\[
(x, y) \mapsto (r, \theta) \mapsto (R, \varphi) \mapsto (X, Y),
\]
where \(R = -r, \varphi = (-1)^{n+1}[\theta + T_n/2]\) and \(X = R \cos \varphi, Y = R^n \sin \varphi\). Since
\[
\dot{r} = \frac{x^{2n-1}P(x, y) + yQ(x, y)}{r^{2n-1}}, \quad \dot{\theta} = \frac{xQ(x, y) - nyP(x, y)}{r^{n+1}},
\]
the proposition follows.

The previous symmetry of equation (32) imposes a condition on the circle \(r = 0\) to be a limit cycle.

**Lemma 34** We assume that the origin of system (10) is a focus with Andreev number \(n\). We consider the corresponding equation (32) and its Poincaré map \(\Pi(r_0) = r_0 + c_m r_0^m + \mathcal{O}(r_0^{m+1})\), with \(c_m \neq 0\).

(i) If \(n\) is odd, then \(r = 0\) cannot be a semistable limit cycle of equation (32), that is, \(m\) is odd.

(ii) If \(n\) is even, then \(r = 0\) is a semistable limit cycle of equation (32), that is, \(m\) is even.

**Proof.** The change to generalized polar coordinates ensures that the origin of system (10) is a focus if, and only if, the circle \(r = 0\) is a limit cycle of equation (32). We assume that the circle \(r = 0\) is a limit cycle with multiplicity \(m\) and we consider its associated Poincaré map \(\Pi(r_0)\). We recall that equation (32) has the discrete symmetry \((r, \theta) \mapsto (-r, (-1)^{n+1} [\theta + T_n/2])\) because it comes from a system in cartesian coordinates (10), see Proposition 33. This symmetry implies that \(r = 0\) is a semistable limit cycle if, and only if, \(n\) is even.
Corollary 35  If equation (32) has a periodic orbit different from $r = 0$, then it has two periodic orbits (one in the upper half cylinder and one in the lower half cylinder).

Proof. The properties of the periodic orbits of (32) stated in this corollary are straightforward consequences of the discrete symmetry given in Proposition 33.

Let $V_0(x, y)$ be an inverse integrating factor defined in a neighborhood of the origin for system (10). Since the changes of variables (11) and (12) have constant non–vanishing Jacobian, it follows that the transformed system (13) has the following inverse integrating factor defined in a neighborhood of the origin:

$$V_0^* (x, y) = V_0(\xi^{-1}x, -\xi^{-1}y + F(\xi^{-1}x)).$$

Next lemma is the analogous to Lemma 29 and states that, under our hypothesis, we have an inverse integrating factor $V(r, \theta)$ for equation (32) which is analytic in $r$ in a neighborhood of $r = 0$ and such that $V(0, \theta) \equiv 0$.

Lemma 36  We assume that the origin of system (13) is a nilpotent monodromic singularity.

(i) If system (13) has an inverse integrating factor $V_0^*(x, y)$ defined in a neighborhood of the origin, then the function defined by

$$V(r, \theta) := \frac{V_0^*(r \cos \theta, r^n \sin \theta)}{r^n \Theta(r, \theta)},$$

where $\Theta(r, \theta)$ is the function defined in (31), is an inverse integrating factor for equation (32) in $r \neq 0$.

(ii) Let $V(r, \theta)$ be an inverse integrating factor of equation (32). We assume that $V(r, \theta)$ has a Laurent expansion in a neighborhood of $r = 0$ of the form

$$V(r, \theta) = v_m(\theta) r^m + O(r^{m+1}),$$

with $v_m(\theta) \neq 0$ and $m \in \mathbb{Z}$. Then, if $m \leq 0$ the origin of system (13) is a center.

Proof. (i) Taking into account the Jacobian $r^n$ of the change to generalized polar coordinates $(x, y) = (r \cos \theta, r^n \sin \theta)$, we see that the differential equation (32) has the inverse integrating factor $V(r, \theta)$ described in the statement which is a $T_n$–periodic function of $\theta$. We observe that $V(r, \theta)$ may not be well-defined on $r = 0$.

(ii) The assumption $m \leq 0$ establishes that we are under the hypothesis of Lemma 22 and we conclude that the cycle $r = 0$ is surrounded by periodic orbits, which give rise to a neighborhood of the origin of system (13) filled with periodic
orbits.

The previous Lemmas 34 and 36 together with Corollary 24 ensure that, under the hypothesis of Theorem 10 and if the origin of system (10) is a focus, then \( m \geq 1 \) and \( m + n \) needs to be even. Thus, we have proved statement (i) in Theorem 10.

To end with, we relate the multiplicity of the limit cycle \( r = 0 \) of equation (32) and the cyclicity of the origin of system (10).

**Lemma 37** We assume that the origin of system (10) is a nilpotent focus with Andreev number \( n \). We consider the corresponding equation (32) for which we assume the circle \( r = 0 \) to be a limit cycle with multiplicity \( m \).

1. The cyclicity \( \text{Cycl}(\mathcal{X}_\varepsilon, p_0) \) of the origin of system (10) satisfies \( \text{Cycl}(\mathcal{X}_\varepsilon, p_0) \geq (m + n)/2 - 1 \).

2. If only analytic perturbations of system (13) with \((1, n)\)-quasihomogeneous weighted subdegrees \((w_x, w_y)\) with \( w_x \geq n \) and \( w_y \geq 2n - 1 \) are taken into account, then the maximum number of limit cycles which bifurcate from the origin of system (13) (and, thus, of system (10)) is \( [(m - 1)/2] \), that is, \( \text{Cycl}(\mathcal{X}_\varepsilon[n, 2n-1], p_0) = [(m - 1)/2] \).

**Proof.** The fact that \( m \) and \( n \) have the same parity is a consequence of Lemma 34. If \( n \) is odd, the symmetry given in Proposition 33 implies that at most \( (m - 1)/2 \) limit cycles can bifurcate from the limit cycle \( r = 0 \) of equation (32) in the region \( r > 0 \). If \( n \) is even, since equation (32) comes from cartesian coordinates, we also need to take into account that \( r = 0 \) is always a solution. Therefore, by using the symmetry again, at most \( (m - 2)/2 \) limit cycles can bifurcate from \( r = 0 \) in the region \( r > 0 \).

We provide an example of a perturbation of equation (32), with \( m \) limit cycles bifurcating from \( r = 0 \) (counting multiplicities), whose transformation to cartesian coordinates gives a perturbation of system (10) with exactly \( [(m - 1)/2] \) limit cycles bifurcating from the origin. This example proves that \( \text{Cycl}(\mathcal{X}_\varepsilon, p_0) \geq k \). We take equation (32) and we assume that \( r = 0 \) is a limit cycle of multiplicity \( m \), which is an integer with the same parity as \( n \) and such that \( m \geq 1 \). We consider the associated system (31) from which (32) comes from, we define \( k = [(m - 1)/2] \) and we perturb system (31) in the following way:

\[
\begin{align*}
\dot{r} &= \Xi(r, \theta) + \sum_{i=0}^{k-1} \varepsilon^{k-i} a_i r^{n+2i} (C_\varepsilon \theta)^{n-1+2i} , \quad \dot{\theta} = \Theta(r, \theta) , \quad \text{if } n \text{ is odd,} \\
\dot{r} &= \Xi(r, \theta) + \sum_{i=0}^{k-1} \varepsilon^{k-i} a_i r^{n+1+2i} (C_\varepsilon \theta)^{n+2i} , \quad \dot{\theta} = \Theta(r, \theta) , \quad \text{if } n \text{ is even,}
\end{align*}
\]
with the convention that if \( k = 0 \) no perturbation terms are taken. The real value \( \varepsilon \) is the perturbation parameter \( 0 < |\varepsilon| \ll 1 \) and the \( a_i \), \( i = 0, 1, 2, \ldots, k - 1 \), are real constants to be chosen so that the Poincaré map \( \Pi(r_0; \varepsilon) \) associated to the ordinary differential equation

\[
\frac{dr}{d\theta} = \left\{ \begin{array}{ll}
\Xi(r, \theta) + \sum_{i=0}^{k-1} \varepsilon^{-i} a_i r^{n+2i} (C_s \theta)^{n+2i}, & \text{if } n \text{ is odd,} \\
\Theta(r, \theta) + \sum_{i=0}^{k-1} \varepsilon^{-i} a_i r^{n+1+2i} (C_s \theta)^{n+2i}, & \text{if } n \text{ is even,}
\end{array} \right.
\]

has \( 2k + 1 \) real zeroes; \( k \) of them positive. The proof of this fact is analogous to the proof given in [4], pp. 254–259.

Undoing the change to generalized polar coordinates system (33) gives rise to an analytic system in a neighborhood of the origin which is a perturbed system from (13) of the form (2) and with \( k = \lfloor (m - 1)/2 \rfloor \) limit cycles bifurcating from the origin. All these limit cycles are hyperbolic by taking convenient values of the parameters \( a_i \). If system (13) is written as \( \dot{x} = P(x, y) \) and \( \dot{y} = Q(x, y) \), then the change to cartesian coordinates from (33) reads for:

\[
\dot{x} = P(x, y) + x K(x, y, \varepsilon), \quad \dot{y} = Q(x, y) + ny K(x, y, \varepsilon), \quad (34)
\]

where

\[
K(x, y, \varepsilon) = \left\{ \begin{array}{ll}
\sum_{i=0}^{k-1} \varepsilon^{-i} a_i x^{n-1+2i} & \text{if } n \text{ is odd,} \\
\sum_{i=0}^{k-1} \varepsilon^{-i} a_i x^{n+2i} & \text{if } n \text{ is even.}
\end{array} \right.
\]

Then, undoing the changes (11) and (12), we obtain a perturbation of system (10) with \( k = \lfloor (m - 1)/2 \rfloor \) limit cycles bifurcating from the origin. We observe that these perturbative terms satisfy the \( (1, n) \)-quasihomogeneous subdegree conditions established in the statement 2 of the lemma.

In order to generate the \( \ell := \lfloor n/2 \rfloor \) limit cycles that we lack to prove the first statement of this lemma, we will consider perturbations with lower subdegree. We observe that the stability of the origin in system (34) is given by the sign of \( a_0 \). Thus, if we choose a convenient value \( b_{\ell-1} \) such that the following system

\[
\dot{x} = P(x, y) + x K(x, y, \varepsilon) + \varepsilon^{k+1} x b_{\ell-1} x^{2(\ell-1)}, \\
\dot{y} = Q(x, y) + ny K(x, y, \varepsilon) + \varepsilon^{k+1} y b_{\ell-1} x^{2(\ell-1)}
\]

satisfies that the stability of the smallest limit cycle in (34) does not change (thus \( |b_{\ell-1}| \) needs to be small enough) and with the origin of the contrary stability
(nämlich, \(b_{\ell-1}\) of contrary sign to \(a_0\)), then there is a new limit cycle bifurcating from the origin. This limit cycle is hyperbolic by choosing \(b_{\ell-1}\) conveniently. By induction, and relabelling \(a_i := b_i + \ell\), we have that the system

\[
\dot{x} = P(x, y) + x\overline{K}(x, y, \varepsilon), \quad \dot{y} = Q(x, y) + ny\overline{K}(x, y, \varepsilon),
\]

(35)

where

\[
\overline{K}(x, y, \varepsilon) = \sum_{i=0}^{L-1} \varepsilon^{L-i} b_i x^{2i}
\]

and \(L = (m + n)/2 - 1\), has at least \((m + n)/2 - 1\) limit cycles bifurcating from the origin. By undoing the changes (11) and (12), we have thus proved that the cyclicity of the origin of system (10) is at least \((m + n)/2 - 1\).

We are going to prove statement 2 of this lemma. By the results established on Section 2 we can control the maximum number of limit cycles which bifurcate from \(r = 0\) in equation (32) under analytic perturbations of this equation. The hypothesis that we only take analytic perturbations of system (13) with \((1, n)\)-quasihomogeneous weighted subdegrees \((w_x, w_y)\) with \(w_x \geq n\) and \(w_y \geq 2n - 1\) is equivalent to say that we take a perturbation of equation (32) which is analytic in a neighborhood of both \(r = 0\) and \(\varepsilon = 0\). System (34) provides an example where the upper bound of \(k = \lfloor (m - 1)/2 \rfloor\) limit cycles bifurcating from the origin \(p_0\) of system (10), when only these perturbations are considered, is attained. In this way we have that \(\text{Cycl}(X_{\varepsilon^{[n,2n-1]}}, p_0) = \lfloor (m - 1)/2 \rfloor\).

**Proof of Corollary 12**

Let us consider system (10) with a nilpotent focus at the origin and the corresponding Andreev number \(n\). Let \(V_0(x, y)\) be an inverse integrating factor of this system which is analytic in a neighborhood of the origin. Therefore, we have an inverse integrating factor \(V^*_0(x, y)\) of the corresponding system (13) which is analytic in a neighborhood of the origin. We consider the Taylor development around \(r = 0\) of the following function:

\[
V^*_0(r \cos \theta, r^n \sin \theta) = v^*_m(\theta) r^M + \mathcal{O}(r^{M+1}),
\]

where \(v^*_m(\theta) \neq 0\).

Let us consider the transformation of system (13) to an equation over a cylinder by generalized polar coordinates, see Lemma 31 and the corresponding inverse integrating factor \(V(r, \theta)\) which is smooth and non-flat in \(r\) in a neighborhood of \(r = 0\). We take the Taylor development around \(r = 0\) of the following functions:

\[
V(r, \theta) = v_m(\theta) r^m + \mathcal{O}(r^{m+1}), \quad \Theta(r, \theta) = \Theta_{n-1}(\theta) r^{n-1} + \mathcal{O}(r^n),
\]

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where $\Theta(r, \theta)$ is the function defined in (31). We recall, see again the proof of
Lemma 31, that $\Theta_{n-1}(\theta) \neq 0$ for any $\theta \in [0, T_n)$. Indeed, by Lemma 20 we have
that $v_m(\theta) \neq 0$ for any value of $\theta \in [0, T_n)$.

By statement (i) in Lemma 36 we deduce that $v_M^*(\theta) = v_m(\theta) \Theta_{n-1}(\theta)$. There-
fore, we have that $v_M^*(\theta) \neq 0$ for any value of $\theta \in [0, T_n)$. On the other hand,
since $V_0^*(x, y)$ is analytic in a neighborhood of the origin, we deduce that $v_M^*(\theta)$ is a $(1, n)$–quasihomogeneous trigonometric polynomial of weighted degree $M$. Using
the symmetries of the generalized trigonometric functions described in statement (e) of Proposition 8, we see that

$$v_M^* \left( \frac{T_n}{2} \right) = (-1)^M v_M^*(0).$$

Therefore, we conclude that $M$ needs to be an even number.

We consider the value of $m$ defined in Theorem 10 and Remark 11 and we have
that $M = 2n - 1 + m$. Since $M$ is even, we deduce that $m$ is odd. Moreover, since
the origin of system (10) is assumed to be a focus, we have that $m$ is an integer
number with $m \geq 1$ and that $m$ and $n$ need to have the same parity, see statement
(ii) of Theorem 10. Thus, $n$ is odd.

**Proof of Corollary 19**

As we have already stated, see Lemmas 25 and 31 in a neighborhood of these
singular points, we can transform the system to an equation over a cylinder by
means of (generalized) polar coordinates. Since the origin is a focus, we have that
the periodic orbit $r = 0$ is a limit cycle for the equation on the cylinder. Lemma 21
ensures the existence of a smooth, and non–flat in $r$ in a neighborhood of $r = 0$,
inverse integrating factor $V(r, \theta)$ for the equation over the cylinder. Indeed, we have
that its vanishing multiplicity $m$ at the origin is at least 1, that is, there exists a
smooth function $f(r, \theta)$ defined on the cylinder such that $V(r, \theta) = r^m f(r, \theta)$, with
$m \geq 1$. The inverse integrating factor $V(r, \theta)$ gives rise to an inverse integrating
factor $V_0(x, y)$ in cartesian coordinates.

In the case of polar coordinates, we have that $r^2 = x^2 + y^2$ and $\theta = \arctan(y/x)$. We observe that the functions $\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, r \frac{\partial \theta}{\partial x}$ and $r \frac{\partial \theta}{\partial y}$ are bounded functions in a neigh-
borhood of the origin. By Remark 2 and if we consider system (5) with $d \geq 1$, we
have that $V_0(x, y) = r^{m+d} \tilde{f}(r, \theta)$ with $r$ and $\theta$ expressed in cartesian coordinates and $\tilde{f}(r, \theta)$ a bounded function, with bounded derivatives, in a neighborhood of the origin.

In the case of generalized polar coordinates, we have that $r = \sqrt{x^{2n} + ny^2}$
and we observe that the functions $\frac{\partial r}{\partial x}$ and $r^{n-1} \frac{\partial r}{\partial y}$ are bounded in a neighborhood
of the origin because

$$\frac{\partial r}{\partial x} = \frac{x^{2n-1}}{(x^{2n} + ny^2)^{\frac{2n-1}{2n}}} \quad \text{and} \quad \frac{\partial r}{\partial y} = \frac{y}{(x^{2n} + ny^2)^{\frac{2n-1}{2n}}}.$$  

If, in these expressions, we consider again the change of coordinates (15) we have that $\frac{\partial r}{\partial x} = Cs^{2n-1}\theta$ and $r^{n-1}\frac{\partial r}{\partial y} = Sn\theta$. From the change of coordinates (15) and using the definition of the generalized trigonometric functions, see (14), it can be shown that

$$\frac{\partial \theta}{\partial x} = -\frac{ny}{(x^{2n} + ny^2)^{\frac{n+1}{2n}}} \quad \text{and} \quad \frac{\partial \theta}{\partial y} = \frac{x}{(x^{2n} + ny^2)^{\frac{n+1}{2n}}}.$$  

Thus, we have that the functions $r\frac{\partial \theta}{\partial x}$ and $r^n\frac{\partial \theta}{\partial y}$ are bounded in a neighborhood of the origin by an analogous argument as before. By Remark 11, and if we consider system (10) with $n > 1$, we have that $V_0(x, y) = r^{m+2n-1}\tilde{f}(r, \theta)$ with $r$ and $\theta$ expressed in cartesian coordinates and $\tilde{f}(r, \theta)$ a bounded function, with bounded derivatives, in a neighborhood of the origin.

We have, in both cases, that $V_0(x, y) = r^a\tilde{f}(r, \theta)$, with $r$ and $\theta$ expressed in cartesian coordinates and $\tilde{f}(r, \theta)$ a bounded function, with bounded derivatives, in a neighborhood of the origin. Since $m \geq 1$, we have that the exponent $a > 1$ in the case of polar coordinates and $a > n$ in the case of generalized polar coordinates. Thus, the limit of this product when $(x, y)$ tends to the origin exists and it is equal to zero. Therefore, $V_0(x, y)$ is continuous in a neighborhood of the origin and $V_0(0, 0) = 0$.

By the chain rule, we have that

$$\frac{\partial V_0}{\partial x} = \frac{\partial V_0}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial V_0}{\partial \theta} \frac{\partial \theta}{\partial x} = \left(a r^{a-1} \tilde{f}(r, \theta) + r^a \frac{\partial \tilde{f}}{\partial r}\right) \frac{\partial r}{\partial x} + r^a \frac{\partial \tilde{f}}{\partial \theta} \frac{\partial \theta}{\partial x}.$$  

This expression can also be written as the product of a function tending to zero (because $a > 1$ or $a > n$, respectively, in each case) and a bounded function. Thus, the function $\frac{\partial V_0}{\partial x}$ is continuous in a neighborhood of the origin and it is zero on the point $(0, 0)$. An analogous argument holds for $\frac{\partial V_0}{\partial y}$. We have that the function $V_0(x, y)$ and its first derivatives are continuous and vanish at the origin. Therefore, $V_0(x, y)$ is at least of class $C^1$ in a neighborhood of the origin.

We remark that in Example 14 we have given an inverse integrating factor $V_0(x, y)$ for system (17) which might be only of class $C^1$ in a neighborhood of the origin, depending on the values of $m$ and $n$.  

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