On Walkup’s class $\mathcal{K}(d)$ and a minimal triangulation of a 4-manifold

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Abstract

For $d \geq 2$, Walkup’s class $\mathcal{K}(d)$ consists of the $d$-dimensional simplicial complexes all whose vertex-links are stacked $(d-1)$-spheres. Kalai showed that for $d \geq 4$, all connected members of $\mathcal{K}(d)$ are obtained from stacked $d$-spheres by finitely many elementary handle additions. According to a result of Walkup, the face-vector of any triangulated 4-manifold $X$ with Euler characteristic $\chi$ satisfies $f_1 \geq 5f_0 - \frac{15}{2}\chi$, with equality only for $X \in \mathcal{K}(4)$. Kühnel observed that this implies $f_0(f_0 - 11) \geq -15\chi$, with equality only for 2-neighbourly members of $\mathcal{K}(4)$. Kühnel also asked if there is a triangulated 4-manifold with $f_0 = 15$, $\chi = -4$ (attaining equality in his lower bound). In this paper, guided by Kalai’s theorem, we show that indeed there is such a triangulation. It triangulates a non-orientable closed 4-manifold with first Betti number $\beta_1 = 3$. Because of Kühnel’s inequality, the given triangulation of this manifold is a vertex-minimal triangulation. We also present a self-complete proof of Kalai’s result.

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1 Walkup’s class $\mathcal{K}(d)$

A weak pseudomanifold without (respectively, with) boundary is a pure simplicial complex in which each face of co-dimension one is in exactly (respectively, at most) two facets (face of maximum dimension). The dual graph $\Lambda(X)$ of a weak pseudomanifold $X$ is the graph (simplicial complex of dimension $\leq 1$) whose vertices are the facets of $X$, two such vertices being adjacent in $\Lambda(X)$ if the corresponding facets of $X$ meet in a co-dimension one face.

We say that $X$ is a pseudomanifold if $\Lambda(X)$ is connected. Any triangulation of a closed and connected manifold is automatically a pseudomanifold without boundary.

A stacked ball of dimension $d$ (in short, a stacked $d$-ball) may be defined as a $d$-dimensional pseudomanifold $X$ with boundary such that $\Lambda(X)$ is a tree. (We recall that a tree is a minimally connected graph, i.e., a connected graph which is disconnected by the removal of any of its edges.) A stacked $d$-sphere may be defined as the boundary of a stacked $(d + 1)$-ball. Since a tree on at least two vertices has (at least two) end vertices, a

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trivial induction shows that a stacked $d$-ball actually triangulates a topological $d$-ball, and hence a stacked $d$-sphere triangulates a topological $d$-sphere.

For a simplicial complex $X$ of dimension $d$, $f_j = f_j(X)$ denotes the number of $j$-dimensional faces of $X$ ($0 \leq j \leq d$), and the vector $f(X) = (f_0, \ldots, f_d)$ is called the face vector of $X$. From our definitions, it is easy to see that the face vector of a stacked $d$-sphere $X$ is determined by its dimension $d$ and the number of vertices $f_0$, as follows.

**Lemma 1.** The face-vector of any $d$-dimensional stacked sphere satisfies

$$f_j = \begin{cases} \binom{d+1}{j} f_0 - \frac{j}{d+1+1} \binom{d+2}{j+1}, & \text{if } 1 \leq j < d \\ df_0 - (d+2)(d-1), & \text{if } j = d. \end{cases}$$

In [6], Walkup defined the class $K(d)$ as the family of all $d$-dimensional simplicial complexes all whose vertex-links are stacked $(d-1)$-spheres. Clearly, all the members of $K(d)$ are triangulated manifolds and, indeed, for $d \leq 2$, $K(d)$ consists of all triangulated $d$-manifolds.

**Proposition 1.** Let $d$ be an even number and let $X$ be a connected member of $K(d)$ with Euler characteristic $\chi$. Then the face-vector of $X$ is given by

$$f_j = \begin{cases} \binom{d+1}{j} f_0 - \frac{j}{d+1+1} \binom{d+2}{j+1} \chi, & \text{if } 1 \leq j < d \\ df_0 - \frac{1}{2}(d+2)(d-1) \chi, & \text{if } j = d. \end{cases}$$

**Proof.** Let’s count in two ways the number of ordered pairs $(x, \tau)$, where $\tau$ is a $j$-face of $X$ and $x \in \tau$ is a vertex. This yields the formula

$$f_j = \frac{1}{j+1} \sum_{x \in V(X)} f_{j-1}(\text{lk}(x)).$$

Let, as usual, $\deg(x)$ denote the degree of $x$ in $X$ (i.e., the number of vertices in $\text{lk}(x)$). Since all the vertex-links $\text{lk}(x)$ of $X$ are stacked $(d-1)$-spheres, Lemma 1 applied to these links shows that

$$f_j = \begin{cases} \frac{1}{j+1} \sum_{x \in V(X)} \left( \binom{d}{j-1} \deg(x) - (j-1) \binom{d+1}{j} \right), & 1 \leq j < d \\ \frac{1}{j+1} \sum_{x \in V(X)} ((d-1) \deg(x) - (d-2)(d+1)), & j = d. \end{cases}$$

But $\sum_{x \in V(X)} \deg(x) = 2f_1$. Therefore, we obtain

$$f_j = \begin{cases} \frac{2}{j+1} \binom{d}{j} f_1 - \frac{j-1}{j+1} \binom{d+1}{j} f_0, & 1 \leq j < d \\ \frac{2d-2}{d+1} f_1 - (d-2) f_0, & j = d. \end{cases} \tag{1}$$

Substituting (1) into $\chi = \sum_{j=0}^{d} (-1)^j f_j$, and remembering that $d$ is even, we get $\chi = 2a f_1 - bf_0$, where $a := \frac{d+1}{d+1} + \sum_{j=1}^{d} \frac{1}{j+1} \binom{d+1}{j-1}$ and $b := d - 2 + \sum_{j=0}^{d-1} (-1)^j \frac{1}{j+1} \binom{d+1}{j}$. But the binomial theorem together with Euler’s formula, relating his Beta and Gamma integrals, yields:

$$\sum_{j=1}^{d+1} (-1)^j \frac{1}{j+1} \binom{d}{j} = \int_0^1 (1-x)^d dx = \beta(2, d+1) = -\frac{1}{(d+1)(d+2)}.$$
Proof. This is immediate on substituting \( f_1 = (d + 1)f_0 - \frac{1}{2}(d+2)\chi \) in the trivial inequality \( f_1 \leq \binom{d}{2} \). □

**Proposition 2** (Walkup [6], Kühnel [4]). Let \( X \) be a connected triangulated 4-manifold with Euler characteristic \( \chi \). Then the face-vector of \( X \) satisfies the following.

(a) \[
\begin{align*}
\sum_{j} (-1)^j \frac{1}{j+1} \binom{d+1}{j} \chi & = \int_{0}^{1} \left(1 - x^{d+1}\right) dx = \frac{1}{d+2}.
\end{align*}
\]

Hence (still remembering that \( d \) is even), we get \( a = 2/(d + 2) \) and \( b = 4/(d + 1)(d + 2) \). Thus, \( \chi = 4f_0/(d + 2) - 4f_1/(d + 1)(d + 2) \). In other words, \( f_1 = (d + 1)f_0 - \frac{1}{2}(d+2)\chi \).

Substituting this value of \( f_1 \) in (1), we get the expression for \( f_j \) in terms of \( f_0 \) and \( \chi \), as claimed.

Notice that, till the proof of (1), we have not used the assumption that \( d \) is even. Thus (1) is valid for all dimensions \( d \). However, there is no further simplification when \( d \) is odd.

A simplicial complex is said to be 2-neighbourly if any two vertices are joined by an edge, i.e., \( f_1 = \binom{d}{2} \). Thus Proposition 1 has the following immediate consequence:

**Corollary 1.** Let \( d \) be an even number and \( X \) be a connected member of \( \mathcal{K}(d) \) with Euler characteristic \( \chi \). Then the face-vector of \( X \) satisfies \( f_0(f_0 - 2d - 3) \geq \binom{d+2}{2}\chi \), and equality holds if and only if \( X \) is 2-neighbourly.

**Proof.** This is immediate on substituting \( f_1 = (d + 1)f_0 - \frac{1}{2}(d+2)\chi \) in the trivial inequality \( f_1 \leq \binom{d}{2} \). □

Further, equality holds here for some \( j \geq 1 \) if and only if \( X \in \mathcal{K}(4) \).

(b) \( f_0(f_0 - 11) \geq -\binom{15}{2}\chi \), and equality holds here if and only if \( X \) is a 2-neighbourly member of \( \mathcal{K}(4) \).

**Proof.** As a well-known consequence of the Dehn-Sommerville equations, the face-vector of \( X \) satisfies (cf. [4]) \( f_2 = 4f_1 - 10(f_0 - \chi) \), \( f_3 = 5f_1 - 15(f_0 - \chi) \) and \( f_4 = 2f_1 - 6(f_0 - \chi) \). Therefore, to prove Part (a), it suffices to do the case \( j = 1 \): \( f_1 \geq 5f_0 - 15\chi/2 \), with equality only for \( X \in \mathcal{K}(4) \). But, applying the lower bound theorem (LBT) for normal pseudomanifolds (cf. [2]) to the vertex links of \( X \), we get \( f_2 = \frac{1}{3} \sum_{x \in V(X)} f_1(\text{lk}(x)) \geq \frac{1}{3} \sum_{x \in V(X)} (4\deg(x) - 10) = \frac{8}{3}f_1 - \frac{10}{3}f_0 \). On substituting \( f_2 = 4f_1 - 10(f_0 - \chi) \), this simplifies to \( f_1 \geq 5f_0 - 15\chi/2 \). Since equality in the LBT holds only for stacked spheres, equality holds only for \( X \in \mathcal{K}(4) \). This proves (a).

In conjunction with the trivial inequality \( f_1 \leq \binom{d}{2} \), Part (a) implies Part (b). □

Clearly, any \( d \)-dimensional weak pseudomanifold without boundary has at least \( d + 2 \) vertices, with equality only for the standard \( d \)-sphere \( S_{d+2}^d \) (whose faces are all the proper subsets of a set of \( d + 2 \) vertices). \( S_{d+2}^d \) is a stacked \( d \)-sphere: it is the boundary of the standard \( (d + 1) \)-ball \( B_{d+2}^{d+1} \) (with only one facet). Since any tree on at least two vertices has at least two end vertices (i.e., vertices of degree one), the following lemma is immediate from the definitions of stacked balls and stacked spheres. (See [2] for a proof.)

**Lemma 2.** Let \( X \) be a stacked \( d \)-sphere.
(a) Then $X$ has at least two vertices of (minimum) degree $d+1$.

(b) Let $X$ have $f_0 > d + 2$ vertices. Suppose $x$ is a vertex of degree $d + 1$. Let $\sigma$ denote the set of neighbours of $x$ in $X$. Let $X_0$ be the pure simplicial complex whose facets are $\sigma$ together with the facets of $X$ not containing $x$. Then $X_0$ is a stacked $d$-sphere.

**Lemma 3.** Let $X$ be a stacked sphere of dimension $d \geq 2$ with edge graph (1-skeleton) $G$. Let $\bar{X}$ be the simplicial complex whose faces are all the cliques (sets of mutually adjacent vertices) of $G$. Then $\bar{X}$ is a stacked $(d + 1)$-ball whose boundary is $X$.

**Proof.** Let $X$ have $n$ vertices. If $n = d + 2$ then $X = S^d_{d+2}$ and $\bar{X} = B^d_{d+1}$, and there is nothing to prove. So assume that $n > d + 2$ and we have the result for all stacked $d$-spheres with fewer vertices. Let $x, \sigma, X_0$ be as in Lemma 2 (b). Notice that (as $d \geq 2$), the edge graph $G_0$ of the $(n - 1)$-vertex stacked $d$-sphere $X_0$ is obtained from $G$ by deleting all the edges through $x$ (and the vertex $x$ itself). Therefore, the cliques of $G$ are $\alpha \cup \{x\}$, where $\alpha \subseteq \sigma$; and the cliques of $G_0$. Hence the facets of $\bar{X}$ are $\bar{\sigma} := \sigma \cup \{x\}$ and the facets of the stacked $(d + 1)$-ball $\bar{X}_0$. Thus the dual graph $\Lambda(\bar{X})$ is obtained from the tree $\Lambda(X_0)$ by adding an end vertex ($\bar{\sigma}$). So, $\Lambda(\bar{X})$ is a tree, i.e., $\bar{X}$ is a stacked $(d + 1)$-ball. Since $X_0$ is the boundary of $\bar{X}_0$, it is immediate that $X$ is the boundary of $\bar{X}$. 

Notice that Lemma 3 shows that any stacked sphere is uniquely determined by its 1-skeleton. (This is, of course, trivial for $d = 1$.)

Now, let $X$ be a member of $\mathcal{K}(d)$, $d \geq 3$. Let $S$ be a set of $d + 1$ vertices of $X$ such that the induced subcomplex $X[S]$ of $X$ on the vertex set $S$ is isomorphic to the standard $(d - 1)$-sphere $S^d_{d+1}$. Then, with notations as in Lemma 3, it is clear that for any $x \in S$, $\sigma := S \setminus \{x\}$ is a $(d - 1)$-face in the interior of the stacked $d$-ball $\overline{\operatorname{lk}}(x)$. The proof of the following lemma shows that when $d \geq 4$, the converse is also true: if $\sigma$ is an interior $(d - 1)$-face of $\overline{\operatorname{lk}}(x)$ for some vertex $x$, then $X \in \mathcal{K}(d)$ induces an $S^d_{d+1}$ on the vertex set $\sigma \cup \{x\}$.

**Lemma 4** For $d \geq 4$, every member of $\mathcal{K}(d)$, excepting $S^d_{d+2}$, has an $S^d_{d+1}$ as an induced subcomplex.

**Proof.** Let $X \in \mathcal{K}(d)$, $X \neq S^d_{d+2}$. Then $X$ has a vertex of degree $\geq d + 2$. Fix such a vertex $x$. Then the stacked $d$-ball $\overline{\operatorname{lk}}(x)$ given by Lemma 3 has an interior $(d - 1)$-face $\sigma$. (If there was no such $(d - 1)$-face, then we would have $\overline{\operatorname{lk}}(x) = B^d_{d+1}$, and hence $\deg(x) = d + 1$, contrary to the choice of $x$.) We claim that $X$ induces an $S^d_{d+1}$ on $S := \sigma \cup \{x\}$. Clearly, every proper subset of $S$, with the possible exception of $\sigma$, is a face of $X$, while $S$ itself is not a face of $X$ since $\sigma$ is not a boundary face of $\overline{\operatorname{lk}}(x)$. Therefore, to prove the claim, we need to show that $\sigma \in X$. Notice that $\operatorname{lk}(x)$ and $\overline{\operatorname{lk}}(x)$ have the same $(d - 2)$-skeleton. In particular, as $d - 2 \geq 2$ and $\sigma \in \overline{\operatorname{lk}}(x)$, it follows that each 3-subset of $\sigma$ is in $\operatorname{lk}(x)$. Therefore, for any vertex $y \in \sigma$, each 3-subset of $\sigma \cup \{x\} \setminus \{y\}$ containing $x$ is in $\operatorname{lk}(y)$. Hence each 2-subset of $\sigma \setminus \{y\}$ is in $\operatorname{lk}(y)$, i.e., $\sigma \setminus \{y\}$ is a clique in the edge graph of $\operatorname{lk}(y)$. Hence $\sigma \setminus \{y\} \subseteq \overline{\operatorname{lk}}(y)$. Since $\sigma \setminus \{y\}$ is a $(d - 2)$-face of $\overline{\operatorname{lk}}(y)$, and $\operatorname{lk}(y)$ has the same $(d - 2)$-skeleton as $\overline{\operatorname{lk}}(y)$, it follows that $\sigma \setminus \{y\} \subseteq \operatorname{lk}(y)$, i.e., $\sigma \in X$, as was to be shown. 

Now, let $X$ be a triangulated closed $d$-manifold and $\sigma_1, \sigma_2$ be two facets of $X$. A bijection $\psi : \sigma_1 \to \sigma_2$ is said to be admissible if, for each vertex $x \in \sigma_1$, $x$ and $\psi(x)$ are at distance at least three in the edge graph of $X$ (i.e., there is no path of length at most two
joining $x$ to $\psi(x))$. In this case, the triangulated $d$-manifold $X^\psi$, obtained from $X \setminus \{\sigma_1, \sigma_2\}$ by identifying $x$ with $\psi(x)$ for each $x \in \sigma_1$, is said to be obtained from $X$ by an elementary handle addition. Notice that the induced subcomplex of $X^\psi$ on the vertex set $\sigma_1 \simeq \sigma_2$ is an $S^d_{d+1}$. In case $X = X_1 \sqcup X_2$, for vertex-disjoint subcomplexes $X_1, X_2$ of $X$, and $\sigma_1 \in X_1$, $\sigma_2 \in X_2$, any bijection $\psi: \sigma_1 \rightarrow \sigma_2$ is admissible. In this situation, we write $X_1 \# X_2$ for $X^\psi$, and $X_1 \# X_2$ is called a (combinatorial) connected sum of $X_1$ and $X_2$.

In Lemma 1.3 of [1], we have shown (in particular) that if $Y$ is a connected triangulated closed manifold of dimension $d \geq 3$, with a vertex set $S$ on which $Y$ induces an $S^d_{d+1}$, the above construction can be reversed. Namely, then there exists a unique triangulated closed $d$-manifold $\tilde{Y}$, together with an admissible map $\psi: \sigma_1 \rightarrow \sigma_2$, such that $Y = (\tilde{Y})^\psi$, and $\beta = \beta_1(\tilde{Y}) + 1$, or else $\tilde{Y}$ has exactly two connected components, say $Y_1$ and $Y_2$, and we have $Y = Y_1 \# Y_2$, in the latter case. It is also easy to see that $Y \in \mathcal{K}(d)$ if and only if $\tilde{Y} \in \mathcal{K}(d)$ (cf. Lemma 2.6 in [1]). We use these results in the following proof.

**Proposition 3 (Kalai [3]).** Let $d \geq 4$. Then $X$ is a connected member of $\mathcal{K}(d)$ if and only if $X$ is obtained from a stacked $d$-sphere by $\beta_1(X)$ combinatorial handle additions.

**Proof.** Clearly, stacked $d$-spheres are in $\mathcal{K}(d)$. Hence so are simplicial complexes obtained from stacked $d$-spheres by finitely many elementary handle additions. This proves the “if” part. We prove the “only if” part by induction on the integral Betti number $\beta_1(X)$. To start the induction, we need:

**Claim:** For $d \geq 4$, $X \in \mathcal{K}(d)$ and $\beta_1(X) = 0$ implies $X$ is a stacked sphere.

We prove the claim by induction on the number $n$ of vertices in $X$. If $n = d + 2$ then $X = S^d_{d+2}$, and the result is obvious. So, assume $n > d + 2$ and we have the result for members of $\mathcal{K}(d)$ with fewer vertices and vanishing first Betti number. By Lemma 4, $X$ has an induced subcomplex isomorphic to $S^d_{d+1}$. Therefore, we may obtained $\tilde{X} \in \mathcal{K}(d)$ by a handle deletion. Then $\tilde{X}$ must be disconnected since otherwise we get the contradiction $\beta_1(\tilde{X}) > \beta_1(\tilde{X}) \geq 0$. Therefore $X = X_1 \# X_2$, where $X_1, X_2 \in \mathcal{K}(d)$ are the connected components of $\tilde{X}$. Since $\beta_1(X_1) = 0 = \beta_1(X_2)$, induction hypothesis yields that $X_1, X_2$ are both stacked spheres. But the combinatorial connected sum of stacked spheres is easily seen to be a stacked sphere (cf. Lemma 2.5 in [1]). So, $X$ is a stacked sphere. This completes the induction, proving the claim.

Thus, we have the “only if” part when the Betti number is 0. So, assume that the Betti number $\beta_1 > 0$ and we have the result for members of $\mathcal{K}(d)$ with smaller first Betti number.

If possible, assume that the result is not true, i.e., there exists a member of $\mathcal{K}(d)$ with Betti number $\beta_1 > 0$ which can’t be obtained from a stacked $d$-sphere by $\beta_1$ combinatorial handle additions. Choose one such member, say $X$, of $\mathcal{K}(d)$ with the smallest number of vertices. As before, obtain $\tilde{X}$ from $X$ by an combinatorial handle deletion. If $\tilde{X}$ is connected then $\beta_1(\tilde{X}) = \beta_1 - 1$. So, by induction hypothesis, $\tilde{X}$ is obtained from a stacked sphere by $\beta_1(\tilde{X})$ combinatorial handle additions. Then $X$ is obtained from the same stacked sphere by $\beta_1 = \beta_1(\tilde{X}) + 1$ combinatorial handle additions. Therefore, from our hypothesis, $\tilde{X}$ is not connected. So, $\tilde{X} = X_1 \sqcup X_2$ and $X = X_1 \# X_2$, for some $X_1, X_2 \in \mathcal{K}(d)$. Then $\beta_1 = \beta_1(X_1) + \beta_1(X_2)$ and $\beta_1(X_1), \beta_1(X_2) \geq 0$. If $\beta_1(X_1), \beta_1(X_2) < \beta_1$, then, by induction hypothesis, $X_i$ is obtained from a stacked sphere $S_i$ by $\beta_1(X_i)$ combinatorial handle additions, for $1 \leq i \leq 2$ and hence $X$ is obtained from the stacked sphere $S_1 \# S_2$ by $\beta_1 = \beta_1(X_1) + \beta_1(X_2)$ combinatorial handle additions. By our assumption, this is not
possible. So, one of $\beta_1(X_1), \beta_1(X_2)$ is equal to $\beta_1$ and the other is 0. Assume, without loss, that $\beta_1(X_1) = \beta_1$. This is a contradiction to our choice of $X$, since $f_0(X_1) \leq f_0(X) - 1$. Thus, the result is true for Betti number $\beta_1$. The result now follows by induction. \hfill \square

2 Example

By Proposition 2, any $n$-vertex triangulated connected 4-manifold $X$, with Euler characteristic $\chi$, satisfies $n(n - 11) \geq -15\chi$. Thus, when $n(n - 11) = -15\chi$, $X$ must be a minimal triangulation of its geometric carrier (requiring the fewest possible vertices). The smallest values of $n$ for which equality may hold is $n = 11$. Indeed, there is a unique 11-vertex 4-manifold with $\chi = 0$ (cf [1]): it triangulates $S^1 \times S^3$. In [4], Kühnel asked if the next feasible case $n = 15$, $\chi = -4$ can be realized. Notice that by Proposition 2, any 15-vertex triangulated 4-manifold with $\chi = -4$ must be a (2-neighbourly) member of $\mathcal{K}(4)$. By Proposition 3, it must arise from a 30-vertex stacked 4-sphere by three elementary handle additions (since it must have $\beta_1 = 3$). Now, three such operations require three pairs of facets (each containing five vertices) in the original stacked sphere, with admissible bijection within each pair. As $30 = 5 \times 6$, it seems reasonable to demand that these six facets in the sought after 30-vertex stacked 4-sphere be pairwise disjoint, covering the vertex set (though we are unable to prove that this must be the case). This strategy works!

The Construction: Let $B_{30}^5$ denote the pure 5-dimensional simplicial complex with 30 vertices $a_i, a'_i, b_i, b'_i, c_i, c'_i, 1 \leq i \leq 5$, and 25 facets $\delta, \alpha_j, \lambda_j, \gamma_j, 1 \leq j \leq 8$ given as follows:

\[
\begin{align*}
\delta &= a_1a_2b_1b_2c_2c_1, \\
\alpha_1 &= a_1a_2a_3a_4b_1b_2, \\
\alpha_2 &= a_1a_2a_3a_4b_1b_2, \\
\alpha_3 &= a_1a_2a_3a_4a_5b_1, \\
\alpha_4 &= a_2a_3a_4a_5b_1c'_5, \\
\alpha_5 &= a_3a_4a_5b_1c'_5, \\
\alpha_6 &= a_3a_4a_5c'_5c'_2c'_1, \\
\alpha_7 &= a_3a_5c'_2c'_3c'_4c'_5, \\
\alpha_8 &= c'_1c'_2c'_3c'_4c'_5a_3, \\
\lambda_1 &= a_1a_2b_1c_2c_4c_2, \\
\lambda_2 &= a_1a_2b_1c_2c_4c_2, \\
\lambda_3 &= a_1c_2c_3c_5c_4, \\
\lambda_4 &= a_1c_2c_3c_5b'_5, \\
\lambda_5 &= a_2c_1c_3c_4c'_5b'_5, \\
\gamma_1 &= a_2b_1b_2b_4c_2c_1, \\
\gamma_2 &= b_1b_2b_3b_4c_1c_2, \\
\gamma_3 &= b_1b_2b_3b_4b_5c_1, \\
\gamma_4 &= b_1b_2b_3b_4b_5c_1, \\
\gamma_5 &= a'_4a'_3b_3b_4b_5c_1, \\
\gamma_6 &= a'_3a'_4b_3b_4b_5c_1, \\
\gamma_7 &= a'_4a'_3a'_4b_3b_4, \\
\gamma_8 &= a'_1a'_2a'_3a'_4b_3b_5.
\end{align*}
\]

The dual graph $\Lambda(B_{30}^5)$ is the following tree.

Thus, $B_{30}^5$ is a 30-vertex stacked 5-ball, and its boundary $S_{30}^4$ is a 30-vertex stacked 4-sphere. Let $M_{15}^4$ be the simplicial complex obtained from $S_{30}^4 \setminus \{a_1a_2a_3a_4a_5, b_1b_2b_3b_4b_5, \ldots$
\[c_1 c_2 c_3 c_4 c_5, a'_1 a'_2 a'_3 a'_4 a'_5, b'_1 b'_2 b'_3 b'_4 b'_5, c'_1 c'_2 c'_3 c'_4 c'_5\] by the identifications \(a'_i \equiv a_i, b'_i \equiv b_i, c'_i \equiv c_i,\ 1 \leq i \leq 5\). It is easy to see that each of these three identifications is admissible for \(S^3_{20}\), and remains so when the other two identifications are already made. (Just verify that there is exactly one edge among the four vertices \(a_i, a'_j, b_j, b'_j\) for each \(i, j\), and similarly for \(b_i\)’s and \(c_i\)’s or \(c_i\)’s and \(a_i\)’s.) Therefore, \(M^4_{15}\) is indeed a 2-neighbourly 4-manifold in the class \(\mathcal{K}(4)\), with \(\beta_1 = 3\) and hence \(\chi = -4\). (If \(N^5_{15}\) is the simplicial complex obtained from \(B^5_{30}\) by the above identification, then \(M^4_{15}\) is the boundary of \(N^5_{15}\). Then \(\Lambda(N^5_{15})\) can be obtained from \(\Lambda(B^5_{30})\) by adding three more edges \(a_8 \lambda_3, \lambda_8 \gamma_3\) and \(\gamma_8 \alpha_3\).

Notice that the permutation \(\prod_{i=1}^5 (a_i, b_i, c_i)(a'_i, b'_i, c'_i)\) is an automorphism of order 3 in \(B^5_{30}\) which induces the automorphism \(\prod_{i=1}^5 (a_i, b_i, c_i)\) of \(M^4_{15}\). Observe that the degrees of the edges in \(M^4_{15}\) are given by the following two tables (and the above automorphism):

| Edges within a’s         | Edges between a’s and b’s |
|--------------------------|--------------------------|
| \(a_1\) | 10 6 7 5 | \(b_1\) | 7 9 7 8 6 |
| \(a_2\) | 6 8 9 | \(b_2\) | 7 8 4 5 4 |
| \(a_3\) | 7 9 11 11 | \(b_3\) | 4 5 6 7 8 |
| \(a_4\) | 5 7 11 11 | \(b_4\) | 4 4 4 5 6 |
| \(a_5\) | 5 7 11 11 | \(b_5\) | 5 4 5 6 7 |

These tables clearly shows that the full automorphism group of \(M^4_{15}\) is of order 3. By Proposition 1, the face-vector of \(M^4_{15}\) is \((15,105,1230,240,96)\). The following is an explicit list of the 96 facets of \(M^4_{15}\):

\[
\begin{align*}
& a_1 a_2 b_1 b_2 c_1, a_1 b_1 b_2 c_1, a_2 a_3 a_1 b_1, a_2 a_3 b_1 c_2, a_1 b_2 b_1 b_4 c_1, a_1 b_2 c_1 c_2 c_4, \\
& a_1 a_2 a_3 b_2, a_2 b_1 b_2 c_2, a_2 b_1 b_2 c_2 c_4, a_1 a_4 b_1 b_2 c_2, a_2 b_1 b_4 c_2, a_1 a_2 b_2 c_2 c_4, \\
& a_1 a_2 b_2 c_4, a_2 a_1 b_1 b_2 c_2, a_2 b_2 b_1 c_2 c_4, a_1 a_2 a_3 a_4 b_2, b_1 b_2 b_4 c_2, a_2 b_1 c_2 c_3 c_4, \\
& a_1 a_2 a_3 b_1, b_1 b_2 b_3 c_2 c_3, a_1 a_2 c_1 c_2 c_3, a_1 a_3 a_4 b_1 b_2, b_1 b_3 b_1 c_1 c_2, \\
& a_1 a_2 c_2 c_4, a_2 a_3 a_4 b_1 b_2, b_2 b_4 c_1 c_2, a_1 a_2 a_3 a_5 b_1, b_1 b_2 b_5 c_1, a_1 c_1 c_2 c_3 c_5, \\
& a_1 a_2 a_4 a_5 b_1, b_1 b_4 b_5 c_1, a_1 c_1 c_2 c_4 c_5, a_1 a_3 a_4 a_5 b_1, b_1 b_3 b_4 c_1, a_1 c_1 c_3 c_4 c_5, \\
& a_2 a_3 a_4 a_5 c_5, a_5 b_3 b_4 b_5, b_5 c_2 c_3 c_4 c_5, a_2 a_3 a_4 b_1 c_5, a_5 b_2 b_4 c_1, a_1 b_5 c_2 c_3 c_4, \\
& a_2 a_3 a_5 b_1 c_5, a_5 b_2 b_3 c_5, a_1 b_2 b_3 c_3 c_5, a_2 a_4 a_5 b_1 c_5, a_5 b_2 b_4 c_1, a_1 b_5 c_2 c_4 c_5, \\
& a_2 a_3 a_4 a_5 c_4, a_5 b_3 b_4 b_5, b_5 c_3 c_4 c_5, a_2 a_3 a_4 b_1 c_5, a_5 b_2 b_4 c_1, a_1 b_5 c_2 c_3 c_4, \\
& a_2 a_3 a_5 b_1 c_4, a_5 b_3 b_4 b_5, a_1 b_4 c_3 c_4 c_5, a_2 a_4 a_5 b_1 c_4, a_5 b_3 b_4 c_1, a_1 b_4 b_3 c_4 c_5, \\
& a_2 a_3 a_4 a_5 c_4, a_5 b_3 b_4 b_5, b_5 c_3 c_4 c_5, a_2 a_3 a_4 b_1 c_5, a_5 b_2 b_4 c_1, a_1 b_5 c_2 c_3 c_4, \\
& a_2 a_3 a_5 b_1 c_4, a_5 b_3 b_4 b_5, b_5 c_3 c_4 c_5, a_2 a_4 a_5 b_1 c_4, a_5 b_3 b_4 c_1, a_1 b_4 b_3 c_4 c_5, \\
& a_2 a_3 a_4 a_5 c_4, a_5 b_3 b_4 b_5, b_5 c_3 c_4 c_5, a_2 a_3 a_4 b_1 c_5, a_5 b_2 b_4 c_1, a_1 b_5 c_2 c_3 c_4, \\
& a_2 a_3 a_5 b_1 c_4, a_5 b_3 b_4 b_5, a_1 b_4 c_3 c_4 c_5, a_2 a_4 a_5 b_1 c_4, a_5 b_3 b_4 c_1, a_1 b_4 b_3 c_4 c_5,
\end{align*}
\]

If we take the simplices \(\delta, a_1, \ldots, a_8, \lambda_1, \ldots, \lambda_8, \gamma_1, \ldots, \gamma_8\) given above as positively oriented oriented simplices then that gives a coherent orientation on \(B^5_{30}\). This orientation gives a co-
herent orientation on $S^4_{30}$ in which $b'_2c_2c_3c_4c_5$, $a_1c_3c_4c_5c_1$, $a_1c_4c_5c_1c_2$, $a_1c_5c_1c_2c_3$, $a_2c_1c_2c_3c_4$, $a_3c_2c_3c_4c_5$, $a_3c_3c_4c_5c_1$, $a_3c_4c_5c_1c_2$, $a_3c_5c_1c_2c_3$, $a_3c_1c_2c_3c_4$, $a_5c_1c_2c_3c_4$, $a_5c_2c_3c_4c_5$, $a_5c_3c_4c_5c_1$, $a_5c_4c_5c_1c_2$, $a_5c_5c_1c_2c_3$, $a_5c_1c_2c_3c_4$c\, are positively oriented.

Let $X = S^4_{30} \setminus \{c_1c_2c_3c_4c_5, c'_1c'_2c'_3c'_4c'_5\}$. Let $Y$ be obtained from $X$ by the identifications $c'_i \equiv c_i$, $1 \leq i \leq 5$. Then $X$ triangulates $S^3 \times [0, 1]$ and the above orientation on $S^4_{30}$ induces a coherent orientation on $X$ with positively oriented simplices (on the boundary of $X$) $c_2c_3c_4c_5$, $c_3c_4c_5c_1$, $c_4c_5c_1c_2$, $c_5c_1c_2c_3$, $c_1c_2c_3c_4$, $c'_2c'_3c'_4c'_5$, $c'_3c'_4c'_5c'_1$, $c'_4c'_5c'_1c'_2$, $c'_5c'_1c'_2c'_3$, $c'_1c'_2c'_3c'_4$. This implies that the geometric carrier of $Y$ is the twisted product $S^1 \times S^3$ (cf. [5, pages 134–135]). So, $Y$ is non-orientable. Since $M^4_{15}$ is obtained from $Y$ by attaching two more handles, it follows that $M^4_{15}$ is non-orientable.

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