The extended mapping class group is generated by 3 symmetries

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Abstract

We prove that for \( g \geq 1 \) the extended mapping class group is generated by three orientation reversing involutions. To cite this article: M. Stukow, C. R. Acad. Sci. Paris, Ser. I 338 (2004).

Résumé

Le groupe modulaire étendu est engendré par 3 symétries. Nous prouvons que pour chaque \( g \geq 1 \) le groupe moduloire étendu est engendré par trois inversions qui inversent l’orientation. Pour citer cet article : M. Stukow, C. R. Acad. Sci. Paris, Ser. I 338 (2004).

1. Introduction

Let \( S_g \) be a closed orientable surface of genus \( g \). Denote by \( \mathcal{M}_g^{\pm} \) the extended mapping class group, i.e., the group of isotopy classes of homeomorphisms of \( S_g \). By \( \mathcal{M}_g \) we denote the mapping class group, i.e., the subgroup of \( \mathcal{M}_g^{\pm} \) consisting of orientation preserving maps. We will make no distinction between a map and its isotopy class, so in particular by the order of a homeomorphism \( h : S_g \to S_g \) we mean the order of its class in \( \mathcal{M}_g^{\pm} \).

By \( C_i, U_i, Z_i \) we denote the right Dehn twists along the curves \( c_i, u_i, z_i \) indicated in Fig. 1. It is known that this set of generators of \( \mathcal{M}_g \) is not minimal, and a great deal of attention has been paid to the problem of finding a minimal (or at least small) set of generators or a set of generators with some additional property. For different approaches to this problem see [3,5,7,8,10,11] and references there. The main purpose of this Note is to prove that for \( g \geq 1 \) the extended mapping class group \( \mathcal{M}_g^{\pm} \) is generated by three symmetries, i.e. orientation reversing involutions. This generalises a well known fact for \( \mathcal{M}_1^{\pm} \equiv \text{GL}(2, \mathbb{Z}) \).

As was observed in [4], the fact that \( \mathcal{M}_g^{\pm} \) is generated by symmetries is rather simple. Namely, suppose that \( S_g \) is embedded in \( \mathbb{R}^3 \) as shown in Fig. 1. Define the sandwich symmetry \( \tau : S_g \to S_g \) as a reflection across the \( yz \)-plane. Now if \( u \) is any of the curves indicated in Fig. 1, then the twist \( U \) along this curve satisfies the relation:
\[ \tau U \tau = U^{-1}, \] i.e. the element \( \tau U \) is a symmetry. This proves that each of generating twists is a product of two symmetries. Note that for the composition of mappings we use the following convention: \( fg \) means that \( g \) is applied first.

### 2. Preliminaries

Suppose that \( S_g \), for \( g \geq 2 \), is embedded in \( \mathbb{R}^3 \) as shown in Fig. 1. Let \( \rho : S_g \to S_g \) be a hyperelliptic involution, i.e., the half turn about \( y \)-axis.

The hyperelliptic mapping class group \( \mathcal{M}_g^h \) is defined to be the centraliser of \( \rho \) in \( \mathcal{M}_g \). By [2] the quotient \( \mathcal{M}_g^h / \langle \rho \rangle \) is isomorphic to the mapping class group \( \mathcal{M}_{0,2g+2} \) of a sphere \( S_{0,2g+2} \) with \( 2g+2 \) marked points \( P_1, \ldots, P_{2g+2} \). This set of marked points corresponds (under the canonical projection) to fixed points of \( \rho \) (Fig. 1). In a similar way, we define the extended hyperelliptic mapping class group \( \mathcal{M}_g^{h\pm} \) which projects onto the extended mapping class group \( \mathcal{M}_{0,2g+2}^{\pm} \) of \( S_{0,2g+2} \). Denote this projection by \( \pi : \mathcal{M}_g^h \to \mathcal{M}_g^{h\pm} \). In case \( g = 2 \) it is known that \( \mathcal{M}_2 = \mathcal{M}_2^h \) and \( \mathcal{M}_2^{h\pm} = \mathcal{M}_2^{h\pm}_2 \).

Denote by \( \sigma_1, \sigma_2, \ldots, \sigma_{2g+1} \) the images under \( \pi \) of twist generators \( C_1, U_1, Z_1, U_2, Z_2, \ldots, U_g, Z_g \) respectively. These generators of \( \mathcal{M}_{0,2g+2} \) are closely related to Artin braids, cf. [2].

Let \( \tilde{M} : S_{0,2g+2} \to \tilde{S}_{0,2g+2} \) be a rotation of order \( 2g+1 \) with a fixed point \( P_1 \) such that: \( \tilde{M}(P_i) = P_{i+1}, \) for \( i = 2, \ldots, 2g+1 \) and \( \tilde{M}(P_{2g+2}) = P_2 \) (Fig. 2). In terms of the generators \( \sigma_1, \ldots, \sigma_{2g+1} \) we have:

\[
\tilde{M} = \sigma_2 \sigma_3 \cdots \sigma_{2g+1}. \tag{1}
\]

In particular \( M \) has order \( 4g+2 \). Using the technique described in [10] it is easy to write \( M \) as a product of twists: \( M = \tilde{U}_1 \tilde{Z}_1 \tilde{U}_2 \tilde{Z}_2 \cdots \tilde{U}_g \tilde{Z}_g \).

Since every finite subgroup of \( \mathcal{M}_g \) can be realised as the group of automorphisms of a Riemann surface [6], \( M \) has maximal order among torsion elements of \( \mathcal{M}_g \) [12]. Geometric properties of \( M \) played a crucial role in the problem of finding particular sets of generators for \( \mathcal{M}_g \) and \( \mathcal{M}_g^{h\pm} \), cf. [3, 7, 8, 11].

Following [1], let \( t_1, s_1, \ldots, t_g, s_g \) be generators of the fundamental group \( \pi_1(S_g) \) as in Fig. 3. In terms of these generators, \( \pi_1(S_g) \) has the single defining relation: \( R = t_g^b s_g^{e-1} \cdots s_1^b s_1^{e-1} s_2^{e-1} \cdots s_g^{e-1} \) where by \( a^b \) we denote the conjugation \( b a b^{-1} \).

It is well known [9] that the mapping class group \( \mathcal{M}_g^{\pm} \) is isomorphic to the group \( \text{Out}(\pi_1(S_g)) \) of outer automorphisms of \( \pi_1(S_g) \). In terms of this isomorphism, elements of \( \mathcal{M}_g^{\pm} \) correspond to the elements of \( \text{Out}(\pi_1(S_g)) \) which map the relation \( R \) to its conjugate, and elements of \( \mathcal{M}_g^{\pm} \setminus \mathcal{M}_g \) to those elements of \( \text{Out}(\pi_1(S_g)) \) which map \( R \) to a conjugate of \( R^{-1} \).
Using representations of twist generators as automorphisms of $\pi_1(S_g)$ [1] we could derive the following representation for the rotation $M$:

$$M : t_i \mapsto s_i^{t_1} \cdots s_i^{t_1} t_{i+1}$$

for $i = 1, \ldots, g$,

$$s_i \mapsto t_i^{-1} s_i^{t_1} \cdots s_i^{t_1} t_i$$

for $i = 1, \ldots, g - 1$,

$$s_g \mapsto t_i^{-1} s_i^{t_1} \cdots s_g^{t_1} t_i.$$

As in the case of maps and their isotopy classes, we abuse terminology by identifying an element of $\text{Out}(\pi_1(S_g))$ with its representative in $\text{Aut}(\pi_1(S_g))$.

3. $M_g^\pm$ is generated by 3 symmetries

If we represent the action of the rotation $\tilde{M}$ as the orthogonal action on the unit sphere, it becomes obvious that $\tilde{M}$ can be written as a product of two symmetries. To be more precise, if $\hat{\alpha}_1$ is the symmetry across the plane passing through $P_1, P_g$, and the center of the sphere (Fig. 2), then $M = \hat{\alpha}_1 \hat{\alpha}_2$, where $\hat{\alpha}_2$ is another symmetry.

Tedious but straightforward computations show that one of the liftings $\hat{\alpha}_1 \in M_g^\pm$ of $\hat{\alpha}_1$ has the following representation as an automorphism of $\pi_1(S_g)$:

$$\hat{\alpha}_1 : t_i \mapsto t_i^{-1} s_i^{t_1} \cdots s_i^{t_1} t_i$$

for $i = 1, \ldots, g - 2$,

$$t_{g-1} \mapsto t_{g-1}^{-1}, \quad s_{g-1} \mapsto s_{g-1} \cdots s_1 t_1, \quad t_g \mapsto t_g^{-1} t_g^{-1}, \quad s_g \mapsto s_g^{-1}.$$

To obtain the above representation we proceed as follows: take a generator $u$ of $\pi_1(S_g)$, find the image $\tilde{u}$ of $u$ under projection $S_g \to S_{g-1}$, find $\hat{\alpha}_1(\tilde{u})$, lift back $\hat{\alpha}_1(\tilde{u})$ to $S_g$ and finally express the obtained loop as a product of generators $t_1, s_1, \ldots, t_g, s_g$ of $\pi_1(S_g)$.

We would like to point out that although the above procedure is a bit subtle, it is quite simple to verify that the obtained formulas are correct. In fact, it is enough to check that $\hat{\alpha}_1^2 = 1$ and $\hat{\alpha}_1 R$ is conjugate to $R^{-1}$. Moreover, the representation of $\hat{\alpha}_2 = \hat{\alpha}_1 M$ is given by the following formulas:

$$\hat{\alpha}_2 : t_i \mapsto (t_g^{-1} s_g^{-1} \cdots s_{g-1}^{-1} t_{g-1}^{-1}) t_g^{-1} \cdots t_2^{-1}$$

for $i = 1, \ldots, g - 2$,

$$t_{g-1} \mapsto t_{g-1}^{-1} t_g^{-1} t_g^{-1} s_{g-1}^{-1}, \quad t_g \mapsto t_g^{-1} t_g^{-1}, \quad s_g \mapsto s_g^{-1}.$$

It is straightforward to verify that $\hat{\alpha}_2^2$ is an identity in $\text{Out}(\pi_1(S_g))$.

**Theorem 3.1.** For each $g \geq 1$, the extended mapping class group $M_g^\pm$ is generated by three symmetries.
Proof. As observed in the introduction, the result is well known for \( g = 1 \), but for the sake of completeness let us prove this in more geometric way. Since \( \mathcal{M}_1 = \langle U_1, C_1 \rangle \) (Fig. 1) and \( \tau U_1 \tau^{-1} = U_1^{-1}, \tau C_1 \tau^{-1} = C_1^{-1} \), the group \( \mathcal{M}_1^\pm \) is generated by the symmetries \( \tau, \tau U_1, \tau C_1 \).

Now suppose that \( g \geq 2 \). Let \( \varepsilon_1 \) and \( \varepsilon_2 = \varepsilon_1 M \) be the symmetries defined above. Since \( \varepsilon_1(t_g^{-1}) = t_g^{-1} \) we have \( \varepsilon_1 C_g^{-1} \varepsilon_1 = C_g^{-1} \), i.e., \( \varepsilon_3 = \varepsilon_1 C_g^{-1} \) is a symmetry. In particular \( \langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle \supset \langle \varepsilon_1 \varepsilon_2, \varepsilon_1 \varepsilon_3 \rangle = \langle M, C_g^{-1} \rangle \). But by [7] the latter group is equal to \( \mathcal{M}_g \). Since \( \langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle \) contains orientation reversing element, this proves that \( \langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle = \mathcal{M}_g^\pm \).

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References

[1] J. Birman, Automorphisms of the fundamental group of a closed, orientable 2-manifold, Proc. Amer. Math. Soc. 21 (1969) 351–354.
[2] J. Birman, H. Hilden, On mapping class groups of closed surfaces as covering spaces, in: Advances in the Theory of Riemann Surfaces, in: Ann. of Math. Stud., vol. 66, Princeton University Press, Princeton, NJ, 1971, pp. 81–115.
[3] T. Brendle, B. Farb, Every mapping class group is generated by 3 torsion elements and by 7 involutions, Preprint 2003.
[4] G. Gromadzki, M. Stukow, Involving symmetries of Riemann surfaces to a study of the mapping class group, Publ. Mat., in press.
[5] S. Humphries, Generators for the mapping class group, in: Topology of Low-Dimensional Manifolds, in: Lecture Notes in Math., vol. 722, Springer, 1979, pp. 44–47.
[6] S. Kerckhoff, The Nielsen realization problem, Ann. of Math. 117 (1983) 235–265.
[7] M. Korkmaz, Generating the surface mapping class group by two elements, Preprint, 2003.
[8] C. Maclachlan, Moduli space is simply-connected, Ann. of Math. 109 (1979) 85–86.
[9] W. Magnus, A. Karass, D. Solitar, Combinatorial Group Theory, Interscience, New York, 1966.
[10] J. McCarthy, A. Papadopoulos, Involutions in surface mapping class groups, Enseign. Math. 33 (1987) 275–290.
[11] B. Wajnryb, Mapping class group of a surface is generated by two elements, Topology 35 (1996) 377–383.
[12] A. Wiman, Über die hyperelliptischen Kurven und diejenigen vom Geschlecht \( p = 3 \), welche eindeutige Transformationen in sich zulassen, Bihang Till. Kongl. Svenska Vetenskaps-Akademien Handl. 21 (1895) 1–23.