In the nonrelativistic case I found that whenever the relation \( mc^2/e^2 < X(\alpha, g_m) \) is satisfied, where \( \alpha \) is a flux, \( g_m \) is magnetic moment, and \( X(\alpha, g_m) \) is some function that is nonzero only for \( g_m > 2 \) (note that \( g_m = 2.00232 \) for the electron), then the matter is unstable against formation of the flux \( \alpha \). The result persists down to \( g_m = 2 \) provided the Aharonov-Bohm potential is supplemented with a short range attractive potential. I also show that whenever a bound state is present in the spectrum it is always accompanied by a resonance with the energy proportional to the absolute value of the binding energy. In the relativistic case one again finds the resonance when the bound state is present but the instability disappear as long as the minimal coupling is considered. For the Klein-Gordon equation with the Pauli coupling which exists in (2+1) dimensions without any reference to a spin the matter is again unstable for \( g_m > 2 \). The results are obtained by calculating the change of the density of states induced by the Aharonov-Bohm potential. The Krein-Friedel formula for this long-ranged potential is shown to be valid when supplemented with zeta function regularization.
In this letter a change of the density of states in the whole space is calculated for the Schrödinger, the Klein-Gordon, and the Dirac equations with the Aharonov-Bohm (AB) potential $A_r = 0, A_\phi = \Phi/2\pi r$. This enables to discuss the stability properties of matter against the spontaneous creation of a magnetic field. In the relativistic case and for the normal magnetic moment we reconfirm previous result, known as the diamagnetic inequality [1] that the matter is stable. The latter was proven under the assumption of minimal coupling which implicitly assumes the normal magnetic moment. However, in the nonrelativistic case a window exists for the magnetic moment $g_m > 2$ in which the inequality is violated leading to the instability of matter against a magnetic field formation. The reason is the formation of bound states which decouple from the Hilbert space by taking away negative energy.

Note that one has the unitary equivalence between a spin 1/2 charged particle in a 2D magnetic field and a spin 1/2 neutral particle with an anomalous magnetic moment in a 2D electric field [2]. One also has a formal similarity between the scattering of electron in the AB potential and in the spacetime of a gravitational vortex in 2 + 1 dimensions [3]. Moreover, one encounters the AB potential (of non-magnetic origin) in the cosmic-string scenarios [4] and our results apply to the this cases as well.

1. The Schrödinger equation.- By using the separation of variables, assuming $e = -|e|$, the total Hamiltonian is written as a direct sum of channel radial Hamiltonians $H_l$ [5, 6]

$$H_l = -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{\nu^2}{r^2},$$

where $\nu = |l + \alpha|$, $\alpha$ being the total flux $\Phi$ in the units of the flux quantum $\Phi_0 = \hbar c/|e|$, with the spectrum given by

$$\psi_l(r, \phi) = J_{|l+\alpha|}(kr)e^{i\phi},$$

with $k = \sqrt{2mE/\hbar^2}$. The eigenfunction expansion for the Green function in the polar coordinates $\mathbf{x} = (r_x, \phi_x)$ is

$$G(\mathbf{x}, \mathbf{y}, E) = \frac{m}{\pi \hbar^2} \int_0^\infty \frac{kdk}{q^2 - k^2} \sum_{i=-\infty}^{\infty} e^{i(\phi_x - \phi_y)} J_{|l+\alpha|}(kr_x) J_{|l+\alpha|}(kr_y).$$

The normalization is chosen such that $-(1/\pi)\text{Im} \text{Tr} G(\mathbf{x}, \mathbf{x} + i\epsilon) G(\mathbf{x}, \mathbf{x} - i\epsilon)$ gives the two-dimensional free density of states $\rho_o(E) = (mV/2\pi \hbar^2)dE$ for $\alpha \in \mathbb{Z}$. In this case the sum in
(3) can be taken exactly by means of Graf’s addition theorem ([4], relation 9.1.79). By taking the integral (assuming \( q^2 = q_x^2 + i \epsilon \), and \( r_x < r_y \)) one finds
\[
G(x, y, E) = \frac{m}{\pi \hbar^2} \int_0^\infty \frac{kdk}{q^2 - k^2} Jo(k|x - y|) = -i \frac{m}{2\hbar^2} H_1^1(q|x - y|). \tag{4}
\]
One can also take the integral
\[
\int_0^\infty \frac{kdk}{q^2 + i \epsilon - k^2} J_\nu(kr_x)J_\nu(kr_y) = -\frac{\pi i}{2} J_\nu(qr_x)H_\nu^1(qr_y) \tag{5}
\]
at first (which is valid for nonintegral \( \nu \) too) and then to take the remaining sum by Graf’s theorem with the same result (4). From the ‘formal scattering’ point of view taking the residuum at \( k = |q| + i \epsilon \) corresponds to choosing the outgoing boundary conditions. From the point of view of \( L^2(R^2) \) it corresponds to taking the boundary value of the resolvent operator on the upper side of the cut at \([0, \infty)\) in the complex energy plane [8]. The limiting value of the resolvent operator on the lower side of the cut is the complex conjugate of (4), the discontinuity across the cut \( G(x, x, E_+) - G(x, x, E_-) = -i(m/\hbar^2) \), and
\[
-\frac{1}{\pi} \text{Im} G(x, x, E_+) = m/2\pi \hbar^2 \tag{6}
\]
which confirms our normalization.

Whenever \( \alpha \notin \mathbb{Z} \) Graf’s theorem cannot be used. To proceed further with this case we use the fact that (4) has an analytic continuation on the imaginary axis in the complex momentum plane
\[
\int_0^\infty \frac{kdk}{q^2 + i \epsilon - k^2} J_\nu(kr_x)J_\nu(kr_y) = \frac{\pi i}{2} J_\nu(iqr_x)H_\nu^1(iqr_y) = I_\nu(qr_x)K_\nu(qr_y), \tag{7}
\]
where \( I_\nu \) and \( K_\nu \) are modified Bessel functions. To sum over \( l \) one uses the integral representation of these functions and following the steps of [9] one can separate the \( \alpha \)-dependent contribution
\[
G_\alpha(x, x, M) - G_o(x, x, M) = \frac{m}{\hbar^2} \frac{\sin(\eta \pi)}{(2\pi)^2} \int_{-\infty}^{\infty} d\theta \int_{-\infty}^{\infty} d\omega e^{-Mr_x(\cosh \theta + \cosh \omega)} \frac{e^{\eta(\theta - \omega)}}{1 + e^{\theta - \omega}}, \tag{8}
\]
where \( M = -i \sqrt{2mE/\hbar^2} \) and \( \eta \) is the nonintegral part of \( \alpha \), \( 0 \leq \eta < 1 \). After taking the trace over spatial coordinates, using formulæ 3.512.1 and 8.334.3 of [10], and returning back on the real momentum axis one finally finds
\[
\text{Tr} [G_\alpha(x, x, E) - G_o(x, x, E)] = -\frac{1}{2} \eta(1 - \eta) \frac{1}{E} \tag{9}
\]
which gives the change of the density of states in the whole space

$$\rho_\alpha(E) - \rho_\alpha(E) = -\frac{1}{2} \eta(1 - \eta) \delta(E)$$

(10)

that confirms [11], where it was obtained in the context of anyonic physics.

The channel Hamiltonians $H_l$ for which $|l + \alpha| < 1$ admit a one-parametric family of self-adjoint extensions [6, 8]. We shall consider the situation with bound states

$$\psi_l(r, \phi) = K_{|l + \alpha|}(\kappa l r) e^{i\phi}$$

(11)
of energy $E_l = -(\hbar^2/2m)\kappa_l^2$ at $l = -n$ and $l = -n - 1$ channels, with $n = [\alpha]$ denoting the nearest integer smaller than $\alpha$. In the presence of the bound states the scattering states (2) have to be modified

$$\psi_l(r, \phi) \to \psi_l(r, \phi) = (J_{|l + \alpha|}(kr) - A_l J_{-|l + \alpha|}(kr)) e^{i\phi}$$

(12)

This is because $H_l$ has to be necessary a symmetric operator what already determines $A_l$ to be $A_l = (k/\kappa_l)^{2|l|}$, i.e., energy dependent. To calculate the change of the integrated density of states in the whole space we make use of the Krein-Friedel formula [12]

$$N_\alpha(E) - N_\alpha(E) = -\frac{i}{2 \pi} \ln \det S,$$

(13)

S being the total on-shell S-matrix. Despite the AB potential is long-ranged we have found that the Krein-Friedel formula when combined with $\zeta$-function regularization can still be used. The radial part of the general solution (12) behaves as

$$R_l(r) \sim \text{const} \left( e^{-ikr} + \frac{1 - A_l e^{i\pi|l+\alpha|}}{1 - A_l e^{-i\pi|l+\alpha|}} e^{-i\pi(|l+\alpha|+1/2)} e^{-ikr} \right) \quad (r \to \infty),$$

(14)

which determines the $l$-th channel $S_l$-matrix to be $S_l = e^{2i\delta_l}$ with

$$\delta_l = \frac{1}{2} \pi(|l| - |l + \alpha|) + \arctan \left( \frac{\sin(|l + \alpha| \pi)}{\cos(|l + \alpha| \pi) - A_l^{-1}} \right).$$

(15)

Without the presence of bound states ($A_l = 0$)

$$\ln \det S = \sum_{l=-\infty}^{\infty} 2i\delta_l$$

$$= i\pi \sum_{l=-\infty}^{\infty} (|l| - |l + \alpha|) = i\pi \left[ 2 \sum_{l=1}^{\infty} l^{-s} - \sum_{l=0}^{\infty} (l + \eta)^{-s} - \sum_{l=1}^{\infty} (l - \eta)^{-s} \right] \bigg|_{s=-1}$$

$$= i\pi \left[ 2\zeta_R(s) - \zeta_H(s, \eta) - \zeta_H(s, 1 - \eta) \right] \bigg|_{s=-1} = -i\pi \eta(1 - \eta),$$

(16)
where \( \zeta_R \) and \( \zeta_H \) are the Riemann and the Hurwitz \( \zeta \)-function. The use of the Krein-Friedel formula (13) then independently confirms (10). Under the presence of bound states the contribution of scattering states to \( N_\alpha \) for \( E \geq 0 \) is

\[
N_\alpha(E) - N_\alpha(0) = -\frac{1}{2} \eta(1 - \eta) + \frac{1}{\pi} \arctan \left( \frac{\sin(\eta \pi)}{\cos(\eta \pi) - (|E_{-n}|/E)\eta} \right) - \frac{1}{\pi} \arctan \left( \frac{\sin(\eta \pi)}{\cos(\eta \pi) + (|E_{-n-1}|/E)^{(1-\eta)}} \right),
\]

(17)

where \( E_{-n} \) and \( E_{-n-1} \) are the binding energies in \( l = -n \) and \( l = -n - 1 \) channels. Note that for \( 0 < \eta < 1/2 \) we have a resonance at

\[
E = \frac{|E_{-n}|}{\cos(\eta \pi)^{1/\eta}} > 0.
\]

(18)

The phase shift \( \delta_{-n}(E) \) (13) changes by \( \pi \) in the direction of increasing energy and the integrated density of states (17) has a sharp increase by one. For \( 1/2 < \eta < 1 \) the resonance is shifted to the \( l = -n - 1 \) channel. \( \eta = 1/2 \) is a special point since resonances are in both channels at infinity. Since we always are in front of the resonances the contribution of \( \arctan \) terms in (17) does not vanish as \( E \to \infty \) but instead gives \(-1\).

Different self-adjoint extensions correspond to different physics inside the flux tube [13]. To identify the physics which underlines them we have considered the situation when the AB potential is regularized by a uniform magnetic \( B \) field within the radius \( R \). In order for a bound state to exist the matching equation for the exterior and interior solutions in the \( l \)-th channel,

\[
x \frac{K'_{|l+\alpha|}(x)}{K_{|l+\alpha|}(x)} = -\alpha + |l| + \alpha |l| + l + 1 + (x^2/2\alpha) \frac{1F_1 \left( \frac{|l|+l+3}{2} + (x^2/4\alpha), |l|+2, \alpha \right)}{1F_1 \left( \frac{|l|+l+1}{2} + (x^2/4\alpha), |l|+1, \alpha \right)},
\]

(19)

with \( 1F_1(a, b, c) \) the Kummer hypergeometric function [3], has to have a solution \( x_l = \kappa_l R \neq 0 \). However, since the l.h.s. decreases from \(-|l+\alpha| \) to \(-\infty \) as \( x \to \infty \) and the r.h.s. is always positive one finds that it is impossible unless it is an attractive potential \( V(r) \) inside the flux tube,

\[
V(r)|_{r \leq R} = -\frac{\hbar^2}{2m^2 R^2} \frac{\alpha}{c(R)},
\]

(20)

with \( c(R) = 2(1+\epsilon(R)) \), \( \epsilon(R) > 0 \), and \( \epsilon(R) \to 0 \) as \( R \to 0 \) [14]. This amounts to changing \( x^2/2\alpha \) by \( x^2/2\alpha - c/2 \) on the r.h.s. of (19). The attractive potential can be either put in
by hands or, when the Pauli equation is considered, as arising in the magnetic momentum coupling of electrons with spin opposite to the direction of magnetic field \( B \). The case of gyromagnetic ratio \( g_m = 2 \) \( (\epsilon(R) \equiv 0) \) is a critical point at which \((\ref{eqn:gyromagnetic})\) has a solution at \( x = 0 \) for \( l = -n \). It is known that there are \( |\alpha[| -1 \) zero modes in this case, \( |\alpha| \) the nearest integer larger than \( \alpha \) \([13]\). The result does only depend on the total flux \( \alpha \) and not on a particular distribution of a magnetic field \( B \).

Whenever \( g_m > 2 \) (or \( g_m = 2 \) with an attractive potential \( V(r) = -\epsilon/R^2, \ \epsilon > 0 \) arbitrary small) the bound states may occur in the spectrum. They correspond to solutions \( x_l > 0 \) of \((\ref{eqn:gyromagnetic})\). In contrast to the zero modes their number does depend on a particular distribution of the magnetic field \( B \) but it is less than or equals to \([16]\)

\[
#_b = 1 + n + 2[\alpha(g_m - 2)/4]
\]

with \([\cdot]\) as above. The bound is saturated when one uses the cylindrical shell regularization of the AB potential \([14, 17]\). Note that in this case the energy \( E_B \) of magnetic field is infinite for any \( R \neq 0 \) in contrast to homogeneous field regularization when

\[
E_B = \frac{1}{2} \pi B^2 R^2 = \frac{\Phi^2}{2\pi R^2}.
\]

\[2\]. Energy calculations.- Up to \( g_m = 2 \) no bound state is present in the spectrum and the change of the density of the scattering states is still given by \((\ref{eqn:scattering})\). Zero modes which may occur for \( g_m = 2 \) are regular at the origin and do not change phase shifts except for \( l = -n \) and \( l = -n - 1 \) channels where they cause phase shift flip \( (A_l^{-1} = 0 \) in \((\ref{eqn:gyromagnetic})\) \( (cf. [17]) \) but in sum they cancel. Therefore the energy of magnetic field \((\ref{eqn:energy})\) tends to infinity as \( R \to 0 \) (when the total flux \( \Phi \) is kept fixed) the matter is stable with regard to a spontaneous creation of the AB field.

When \( g_m > 2 \) then bound states may occur in the spectrum. Their energy is

\[
E_l = -\frac{\hbar^2}{2m} \frac{r^2}{R^2},
\]

and tends to \(-\infty \) as \( 1/R^2 \) when \( R \to 0 \), in the same way as the magnetic field energy \((\ref{eqn:energy})\) goes to \( \infty \). The bound states decouple in the \( R \to 0 \) limit from the Hilbert space \( L^2(R^2) \) \( (A_l \to 0 \) in \((\ref{eqn:scattering})\) in the limit) and take away the nonperiodicity of the spectrum with regard to \( \alpha \to \alpha \pm 1 \) which persists for any finite \( R \). What is left behind is nothing but
the conventional AB problem with the change of the density of states (10). We recall that to have a finite energy bound state in the spectrum it has to be the attractive potential (20) inside the flux tube.

Bound states solutions \( x_l \) for the homogeneous field regularization determine the function \( X(\alpha, g_m) \),

\[
X(\alpha, g_m) = \frac{1}{4\pi\alpha^2} \sum_{l} x_l^2(\alpha, g_m) \geq 0. \tag{24}
\]

By comparing the coefficients in front of \( 1/R^2 \) in (22) and (23) one finds that whenever

\[
mc^2/e^2 < X(\alpha, g_m) \tag{25}
\]

the total energy of field and matter altogether goes to \(-\infty\) as \( R \to 0 \).

3. The Dirac and the Klein-Gordon equations.- A connection with the Schrödinger equation is established by noticing that for the Dirac spinor to belong to the spectrum its up/down radial component has to be an eigenfunction with an eigenvalue \( k^2 \) of

\[
H_l = -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{\nu^2}{r^2} \pm g_m \frac{\alpha}{r} \delta(r), \tag{26}
\]

with \( g_m = 2 \) and \( (\nu = l+\alpha/\nu = l+1+\alpha) \) for the (up/down) component \[7\]. The dispersion relation for \( k \) is now changed to \( k = (1/\hbar c)\sqrt{E^2 - m^2c^4} \). The two-component solutions of the massive Dirac equation have only one degree of freedom which is reflected in the equality of up and down phase shifts \[9\]. For the conventional AB problem \( \delta^u_l = \delta^d_l = \pi\alpha/2 \) if \( l = -n-1 \), and \( \delta^u_l = \delta^d_l = \pi(|l|-|l+\alpha|)/2 \) otherwise. Using the results of \[13, 18\] one finds that for a fixed real \( k \neq 0 \) the spectrum is symmetric with respect to the origin, in accord with supersymmetry \[18\]. Thus the contribution of scattering states to the density of states is given by

\[
\rho_\alpha(E) - \rho_o(E) = -\frac{1}{2} \eta(1-\eta) \delta(E^2 - m^2c^4). \tag{27}
\]

The only asymmetry can occur for the threshold \[18\] and bound states. Depending on the orientation of \( B \) the threshold states occur either only at the upper or only at the lower threshold \[18\]. Their number equals to the number \( |\alpha|-1 \) of zero modes of the Schrödinger equation \[13, 18\].

By considering different self-adjoint extensions one finds that, in contrast to the Schrödinger equation, bound states can only occur in a single channel \( l = -n-1 \ (l = -n \).
for positive charge) \[13\]. In the presence of the bound state of energy \(E\) in the \(l = -(n+1)\)-th channel,

\[
\Psi_{E;-n-1}(t, r, \phi) = \frac{1}{N} \left( \sqrt{\frac{mc^2 + EK_{1-\eta}(kr)}{mc^2 - EK_\eta(kr)}} e^{i(n+1)\phi} e^{-i\varepsilon t/\hbar} \right)
\]

(28)

where \(\kappa = (1/\hbar c)\sqrt{m^2c^4 - E^2}\), the scattering state with energy \(E > mc^2\) is

\[
\Psi_{E;-n-1}(t, r, \phi) = \left( \begin{array}{c} \chi^1(r) \\ \chi^2(r)e^{i\phi} \end{array} \right) e^{-i(n+1)\phi} e^{-i\varepsilon t/\hbar},
\]

(29)

\[
\chi_{-n-1}(r) = \frac{1}{N} \left( \sqrt{\frac{E + mc^2}{i\sqrt{E - mc^2}}[\sin \mu J_{1-\eta}(kr) + (-1)^{n+1} \cos \mu J_{-(1-\eta)}(kr)]} \right)
\]

(30)

\[
N_{a}(E) - N_{o}(E) = -\frac{1}{2} \eta(1 - \eta) - \frac{1}{\pi} \arctan \frac{\sin(\eta\pi)}{\cos(\eta\pi) + (-1)^n \tan \mu}.
\]

(33)

By comparing (30) with (12) one finds a change of the conventional phase shifts

\[
\delta = -\arctan \frac{\sin(\eta\pi)}{\cos(\eta\pi) + (-1)^n \tan \mu}.
\]

(32)

When the threshold state \(E = mc^2\) is present \(\tan \mu = 0\) and one has a phase shift flip, in accord with \[17\]. By the Krein-Friedel formula the contribution of scattering states with \(E > mc^2\) to the integrated density of states is then

\[
N_{a}(E) - N_{o}(E) = -\frac{1}{2} \eta(1 - \eta) - \frac{1}{\pi} \arctan \frac{\sin(\eta\pi)}{\cos(\eta\pi) + (-1)^n \tan \mu}.
\]

Because of (31) there is again a typical resonance in the relativistic case for \(0 < \eta < 1/2\) which, however, for \(1/2 < \eta < 1\) disappear. When \(\eta = 1/2\) the resonance is shifted to infinity. For negative magnetic field the resonance appears for \(1/2 < \eta < 1\) and disappears for \(0 < \eta < 1/2\).

For \(-E < -mc^2\) the scattering states are given by \(\Psi_{E,l}(t, r, \phi) = \Psi_{E,l}^*(t, r, \phi)|_{m \rightarrow -m}\) supplemented with \(\tan \mu \rightarrow -\tan \mu\). The former relation is a consequence of \(h_m^*|_{m \rightarrow -m} = -h_m\) for the radial part \(h_m\) of the Dirac operator and can be verified also by direct calculations. The latter ensures the same boundary conditions at the origin as for the energy \(E\) scattering states, i.e., the both sets are in the same Hilbert space. Formally, the
bound states with the energy $\pm E$ and the scattering states with $\pm \mu$ are eigenstates of $h_m$, but more precise look shows that they belong to different self-adjoint extensions of $h_m$.

In contrast to the Schrödinger equation it is impossible to find a bound state with the homogeneous field regularization and our argument with the decoupling of bound states does not work. Whenever the magnetic moment coupling induces an attractive $V(r) = -g_m\alpha/R^2$ potential inside the flux tube for the down component then it induces a repulsive $-V$ potential for the up component. But it is known that for the bound state to exist it has to be at least the attractive potential (20). An arbitrary weak attractive potential cannot lead to bound states (cf. [17]). The same situation persists with the cylindrical shell regularization. Nevertheless, in the presence of the threshold state the total energy of the system with filled Dirac sea changes by infinite positive amount which reflects the stability of the system.

Note that (27) is nothing but radial part of the Klein-Gordon Hamiltonian with the Pauli coupling [19] and the relevant results for it are a simple consequence of the above calculations.

4. Discussion, three dimensional case.- We have derived the quantum-mechanical criterion of instability (25) which only involves fundamental parameters of matter and, in particular, shows an instability of massless charged particles (cf. [20] which claims they cannot exist in nature as they are completely locally screened in the process of formation). We did not find any instability of minimally-coupled relativistic matter without the Abelian Chern-Simons term (cf. [21] which claims that in its presence magnetic field is spontaneously generated).

In 3D the function $X(\alpha, g_m)$ is replaced by $X(\alpha, g_m)/L$, with $L$ the length of the flux string. Therefore the above instability may survive up to 3D provided the density of states for a long flux ring preserves essential features of the AB potential. However, in the case of the 3D Dirac equation any question about the instability due to the symmetry of its spectrum is pointless (cf. the suggestion of [22] for ‘flux spaghetti’ vacuum in the spirit of [23] as a mechanism for avoiding the divergence of perturbative QED). With the AB potential the pairs of components $(\chi^1, \chi^4)$, and $(\chi^3, \chi^2)$ of the four-spinor in the standard representation combine to the components of two bispinors which satisfy 2D Dirac equations with the opposite sign of mass. Because $h^*_m = -h_{-m}$ for the radial part
of the 2D Dirac Hamiltonians (cf. (3) of [13]), the symmetry of the spectrum (including
the threshold and bound states) of the 3D Dirac equation with respect to the origin is
restored, in accord with the general result [18]. A more general discussion of this and
related problems will be given elsewhere [24].

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