UNIFORM BOUNDS FOR SUMS OF KLOOSTERMAN SUMS OF HALF INTEGRAL WEIGHT

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Abstract. For $m, n > 0$ and $mn < 0$ we estimate the sums
$$\sum_{c \leq x} S(m, n, c, \chi) / c,$$
where the $S(m, n, c, \chi)$ are Kloosterman sums attached to a multiplier $\chi$ of weight $1/2$ on the full modular group. Our estimates are uniform in $m, n$ and $x$ in analogy with the bounds for the case $mn < 0$ due to Ahlgren–Andersen, and those of Sarnak–Tsimerman for the trivial multiplier when $m, n > 0$. In the case $mn < 0$, our estimates are stronger in the $mn$-aspect than those of Ahlgren–Andersen. We also obtain a refinement whose quality depends on the factorization of $24m - 23$ and $24n - 23$ as well as the best known exponent for the Ramanujan–Petersson conjecture.

1. Introduction and statement of results

The classical Kloosterman sum
$$S(m, n, c) := \sum_{d \, (\text{mod } c)} e\left( \frac{md + nd}{c} \right), \quad e(x) := \exp(2\pi i x)$$
plays a central part in analytic number theory. For applications, see [11, 25] for example.

In this paper we study generalised Kloosterman sums $S(m, n, c, \chi)$ attached to the Dedekind eta multiplier $\chi$ of weight $1/2$. These are given by
$$S(m, n, c, \chi) := \sum_{\substack{0 \leq a, d < c \\atop \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})}} \bar{\chi}\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) e\left( \frac{\bar{m}a + \bar{d}d}{c} \right), \quad \bar{m} := m - \frac{23}{24}. \quad (1.1)$$

Kloosterman sums with general multipliers have been studied by Bruggeman [4], Goldfeld–Sarnak [10] and Pribitkin [19], amongst many others.

For the ordinary Kloosterman sums $S(m, n, c)$, Linnik [16] and Selberg [27] conjectured that there should be considerable cancellation in the sums
$$\sum_{c \leq x} S(m, n, c) / c. \quad (1.2)$$

Sarnak and Tsimerman [24] proposed a modified version of Linnik’s and Selberg’s conjecture with an $\varepsilon$-“safety valve” in $m$ and $n$. In particular, the refined conjecture for (1.2) is
$$\sum_{c \leq x} S(m, n, c) / c \ll_{\varepsilon} (|mn| x)^{\varepsilon}.$$
One obtains the “trivial bound” for (1.2) by applying Weil’s bound \[30\]

\[|S(m, n, c)| \leq \tau(c)(m, n, c)^{\frac{1}{2}} \sqrt{c},\]

where \(\tau(c)\) is the number of divisors of \(c\). This yields

\[\sum_{c \leq x} \frac{S(m, n, c)}{c} \ll \tau((m, n)) x^{\frac{1}{4}} \log x.\]

Still the best known bound in the \(x\) aspect was obtained by Kuznetsov \[15\], who proved for \(m, n > 0\) that

\[\sum_{c \leq x} \frac{S(m, n, c)}{c} \ll_{m, n} x^{\frac{1}{6}} (\log x)^{\frac{1}{3}}.\]  

(1.3)

Sarnak and Tsimerman \[24\] refined Kuznetsov’s method and made the dependence on \(m\) and \(n\) explicit. They proved that for \(m, n > 0\) we have

\[\sum_{c \leq x} \frac{S(m, n, c)}{c} \ll (x^{\frac{1}{4}} + (mn)^{\frac{1}{3}} + (m + n)^{\frac{1}{3}}(mn)^{\frac{1}{6}})(xmn)^{\varepsilon} \]  

(1.4)

where \(\theta\) is any admissible exponent in the Ramanujan–Petersson conjecture for the coefficients of weight zero Maass cusp forms. By work of Kim and Sarnak \[12, Appendix 2\], the exponent \(\theta = 7/64\) is available. Ganguly and Sengupta \[9\] have generalised the results of Sarnak and Tsimerman to sums over \(c\) that are divisible by a fixed integer \(q\).

Kiral \[13\] obtained estimates in the case \(mn < 0\) using the opposite sign Kloosterman zeta function. He obtained the bound

\[\sum_{c \leq x} \frac{S(m, n, c)}{c} \ll x^{\frac{1}{6} + \varepsilon}((m, n)^{\varepsilon} + (mn)^{\theta}) + x^{\varepsilon}(mn)^{\frac{1}{6} + \varepsilon},\]

where \(\theta\) is as above.

For the Kloosterman sum \(S(m, n, c, \chi)\), Ahlgren and Andersen \[1, Theorem 1.3\] proved that for \(mn < 0\) we have

\[\sum_{c \leq x} \frac{S(m, n, c, \chi)}{c} \ll_{\varepsilon} (x^{\frac{1}{4}} + |mn|^{\frac{1}{6}})|mn|^\varepsilon \log x.\]  

(1.5)

They obtained a stronger result for sums of Kloosterman sums \(S(1, n, c, \chi)\) when \(n < 0\) \[1, Theorem 9.1\]. This leads to an improvement in error term \[1, Theorem 1.1\] when one truncates Rademacher’s formula \[21, 22\] for the partition function \(p(n)\).

Our first Theorem improves the \(mn\)-aspect in (1.5).

**Theorem 1.1.** Let \(m > 0\) and \(n < 0\) be integers. Then we have

\[\sum_{c \leq x} \frac{S(m, n, c, \chi)}{c} \ll_{\varepsilon} \left(x^{\frac{1}{4}} + \frac{|m|^{\frac{1}{4}}}{|mn|^{\frac{1}{10}}} + |mn|^{\frac{19}{4}}(mn)^{\varepsilon} \log^2 x.\]

We also consider the case when \(m, n > 0\). Let \(\mathcal{P} := \left\{\frac{k(3k \pm 1)}{2} : k \in \mathbb{Z}\right\}\), the set of generalized pentagonal numbers.

**Theorem 1.2.** Let \(m, n > 0\) be integers be such that \(m - 1 \notin \mathcal{P}\) or \(n - 1 \notin \mathcal{P}\). Then we have

\[\sum_{c \leq x} \frac{S(m, n, c, \chi)}{c} \ll_{\varepsilon} \left(x^{\frac{1}{4}} + (mn)^{\frac{1}{4}}\right)(mn)^{\varepsilon} \log^2 x.\]
Remark 1.1. When both \(m, n > 0\) are such that \(m - 1 \in \mathcal{P}\) and \(n - 1 \in \mathcal{P}\), we have the asymptotic formula

\[
\sum_{c \leq x} \frac{S(m, n, c, \chi)}{c} = C(m, n)x^{\frac{1}{2}} + O_{m,n}(x^{\frac{1}{6}}),
\]

for some constant \(C(m, n)\). See [2, Theorem 8].

We also obtain refined bounds which recognise the arithmetic of \(24m - 23\) and \(24n - 23\). In analogy with the result of Sarnak and Tsimerman, these depend on progress toward the Ramanujan–Petersson conjecture.

Theorem 1.3. Let \(m > 0, n < 0, m_0\) and \(n_0\) be integers such that \(24m - 23 = m_0^2s\) and \(24n - 23 = n_0^2t\) with \(s\) and \(t\) square-free integers. Then

\[
\sum_{c \leq x} \frac{S(m, n, c, \chi)}{c} \ll \varepsilon \left( x^{\frac{1}{6}} + |mn|^{\frac{1}{3}} + |mnst|^{\frac{1}{4} + \theta} \right) |mn|^{\varepsilon} \log^3 x,
\]

where \(\theta\) represents the best admissible exponent toward the Ramanujan–Petersson conjecture.

Theorem 1.4. Let \(m, n > 0\) be integers such that \(m - 1 \notin \mathcal{P}\) or \(n - 1 \notin \mathcal{P}\). Suppose \(m_0, n_0\) are integers such that \(24m - 23 = m_0^2s\) and \(24n - 23 = n_0^2t\) with \(s\) and \(t\) square-free integers. Then

\[
\sum_{c \leq x} \frac{S(m, n, c, \chi)}{c} \ll \varepsilon \left( x^{\frac{1}{6}} + (st)^{\frac{1}{4}} + (st)^{\frac{1}{4} + \varepsilon} \right) (mn)^{\varepsilon} \log^3 x,
\]

where \(\theta\) represents the best admissible exponent toward the Ramanujan–Petersson conjecture.

The proofs of Theorems 1.1–1.4 depend on generalisations of Kuznetsov’s trace formula due to Proskurin [20] and Ahlgren–Andersen [1]. These are given in Sections 5 and 6. This formula transfers the task at hand to that of establishing bounds for sums involving the coefficients of half integral weight holomorphic and Maass cusp forms.

To bound the contribution from holomorphic forms we appeal to Petersson’s formula. We also use the Shimura lift for half integer weight forms and Deligne’s bound to obtain bounds in terms of the factorisation of \(24m - 23\) and \(24n - 23\). Details can be found in Sections 3 and 7.

To bound the contribution from the Maass cusp forms we modify a dyadic argument in the spectral parameter that appears in [1, 24]. Our new treatment involves estimating an initial segment for the spectral parameter. This relies on an averaged bound of Duke [6] due to Ahlgren and Andersen [1]. We will also make use of a forthcoming mean value estimate for the coefficients of Maass cusp forms due to Andersen and Duke [3]. These bounds can be found in Section 8. We also appeal to a Shimura lift for half integral weight Maass cusp forms [1] and progress toward the Ramanujan–Petersson conjecture to obtain bounds in terms of the factorisation of \(24m - 23\) and \(24n - 23\). These tools appear in Section 4.

The proofs of Theorems 1.1–1.4 can be found in Sections 9–12 respectively.

2. Preliminaries

We give only a concise background related to the case we treat. More details can be found in [1] and [7] for example. Let \(\mathbb{H}\) denote the upper-half plane and \(\Gamma := \text{SL}_2(\mathbb{Z})\) denote the
full modular group. We have the usual action of \( SL_2(\mathbb{R}) \) on \( \mathbb{H} \) given by

\[
\gamma \tau = \frac{a \tau + b}{c \tau + d}, \quad \text{for} \quad \tau \in \mathbb{H} \quad \text{and} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}).
\]

For \( \gamma \in SL_2(\mathbb{R}) \) we define the weight \( k \) slash operator by

\[
f|_k \gamma := j(\gamma, z)^{-k} f(\gamma z), \quad j(\gamma, z) := \frac{cz + d}{|cz + d|} = e^{i \text{arg}(cz+d)},
\]

where the argument is always chosen in \((-\pi, \pi]\). The weight \( k \) Laplacian is defined by

\[
\Delta_k := y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - iky \frac{\partial}{\partial x}.
\]

A real analytic function \( f : \mathbb{H} \to \mathbb{C} \) is an eigenfunction of \( \Delta_k \) with eigenvalue \( \lambda \) if

\[
\Delta_k f + \lambda f = 0.
\]

We write

\[
\lambda = \frac{1}{4} + r^2,
\]

where \( r \) is the spectral parameter.

Let \( \eta \) denote Dedekind’s eta function, defined by

\[
\eta(\tau) := e\left( \frac{\tau}{24} \right) \prod_{n=1}^{\infty} \left( 1 - e(n\tau) \right) \quad \text{Im}(\tau) > 0.
\]

Furthermore, let \( \chi \) be the multiplier of weight 1/2 on \( SL_2(\mathbb{Z}) \), defined by

\[
\eta(\gamma \tau) = \chi(\gamma) \sqrt{c \tau + d} \eta(\tau), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).
\]

Rademacher [23] proved that

\[
\chi(\gamma) = \sqrt{-ie^{-\pi is(d,c)}} e\left( \frac{a + d}{24c} \right),
\]

where \( s(d, c) \) is the Dedekind sum

\[
s(d, c) := \sum_{r=1}^{c-1} \frac{r}{c} \left( \frac{dr}{c} - \left\lfloor \frac{dr}{c} \right\rfloor - \frac{1}{2} \right).
\]

In what follows let \( \nu \) be a general multiplier of weight \( k \). A function \( f : \mathbb{H} \to \mathbb{C} \) is automorphic of weight \( k \) and multiplier \( \nu \) for \( \Gamma_0(N) \) if

\[
f|_k \gamma = \nu(\gamma) f, \quad \text{for all} \quad \gamma \in \Gamma_0(N).
\]

Let \( \mathcal{A}_k(N, \nu) \) denote the space of such functions. If \( f \in \mathcal{A}_k(N, \nu) \) is a smooth eigenfunction of \( \Delta_k \) which satisfies the growth condition

\[
f(\tau) \ll y^{\sigma} + y^{1-\sigma},
\]

for some \( \sigma \) and all \( \tau \in \mathbb{H} \), then it is called a Maass form. Let \( \mathcal{A}_k(N, \nu, r) \) denote the vector space of Maass forms with spectral parameter \( r \). For \( f \in \mathcal{A}_k(N, \nu, r) \), let \( \alpha_{\nu} \in \mathbb{R} \) be such
that $f(\tau + 1) = e(-\alpha_\nu) f(\tau)$. We define $n_\nu := n - \alpha_\nu$ for all $n \in \mathbb{Z}$. Such an $f$ has Fourier expansion

$$f(\tau) = \sum_{n = -\infty}^{\infty} a(n) W_{\frac{2}{3} \text{sgn}(n), \nu}(4\pi|n_\nu|y),$$

where $W_{\kappa, \mu}$ denotes the $W$-Whittaker function.

Let $L_k(N, \nu)$ denote the $L_2$-space of automorphic functions with respect to the Petersson inner product

$$\langle f, g \rangle := \int_{\mathbb{H}/\Gamma_0(N)} f(\tau) \overline{g(\tau)} \, d\mu, \quad d\mu := \frac{dx dy}{y^2}.$$ 

The space $L_k(N, \nu)$ admits a complete spectral resolution with respect to $-\Delta_k$ [1, 7]. The spectrum of $L_k(N, \nu)$ on $\Delta_k$ consists of an absolutely continuous spectrum of multiplicity equal to the number of singular cusps, and a discrete spectrum of finite multiplicity. When $\langle k, \nu \rangle = (1/2, \chi)$ and we work on the full modular group $\Gamma$, there is no continuous spectrum because the only cusp is non-singular.

Let $S_k(N, \nu)$ denote the orthogonal complement in $L_k(N, \nu)$ to the space generated by the Eisenstein series. The spectrum of $-\Delta_k$ on $S_k(N, \nu)$ is countable and contained in $[\lambda_0(k), \infty)$, where $\lambda_0(k) \geq 0$ denotes the lowest eigenvalue. The spectrum has no limit points except for $\infty$. The eigenfunctions of $\Delta_k$ in $S_k(N, \nu)$ are called \textit{Maass cusp forms}. We will mostly be working with the space $S_{\frac{1}{2}}(1, \chi)$, so we fix an orthonormal basis $\{u_j\}$ with corresponding spectral parameters $\rho_j$ and with Fourier series given by

$$u_j(\tau) = \sum_{n \neq 0} \rho_j(n) W_{\frac{2}{3}\text{sgn}(n), \rho_j}(4\pi|n_\nu|y)e(n_\nu). \tag{2.1}$$

### 3. Hecke theory for holomorphic cusp forms

We briefly review Hecke theory for holomorphic cusp forms of half integral weight. For $N, k \in \mathbb{N}$, let $S_{\frac{1}{2}+k}(4N, \Psi)$ denote the space of holomorphic cusp forms of weight $1/2 + k$ on $\Gamma_0(4N)$ with Dirichlet character $\Psi$ (with respect to the $\Theta$–multiplier $\nu_\Theta$ [14, p. 148]). For all primes $p \nmid 4N$, the action of the Hecke operator $T_p^2$ on $f := \sum_{r = 1}^{\infty} a(r)e(r\tau) \in S_{\frac{1}{2}+k}(N, \Psi)$ is given by [17, Definition 3.1]

$$T_p^2(f)(z) := \sum_{n = 1}^{\infty} \left( a_f(p^n) + \frac{12}{p} \left( \frac{-1}{p} \right) p^{k-1} a_f(n) + p^{2k-1} a_f(n/p^2) \right) e(n\tau) \in S_{\frac{1}{2}}(4N, \Psi). \tag{3.1}$$

There are Hecke operators $T_n$ for all integers $n$ such that $(n, 4N) = 1$. For $v \in \mathbb{N}$, the operators $T_{p^v}$ are polynomials in the $T_p^2$. If $(nm, 4N) = 1$ and $(n, m) = 1$ then

$$T_n^2 T_m^2 = T_{nm^2}. \tag{3.2}$$

Let $\tilde{S}_{\frac{1}{2}+k}(1, \chi)$ denote the space of holomorphic cusp forms of weight $1/2 + k$ with respect to the multiplier $\chi$. Letting $\chi_{12} := \left( \frac{12}{\cdot} \right)$, we have the map

$$L : \tilde{S}_{\frac{1}{2}}(1, \chi) \to S_{\frac{1}{2}}(576, \chi_{12}), \quad L(f)(\tau) := f(24\tau).$$
For primes $p \nmid 6$, there are Hecke operators $\tilde{T}_p^2$ on $\tilde{S}_2(1, \chi)$ such that

$$L(\tilde{T}_p^2 g) = T_{p^2}(Lg).$$

(3.3)

All $\tilde{T}_n^2$ are determined using (3.3) and the above facts concerning $T_n^2$.

We recall the Shimura correspondence for half-integral weight holomorphic cusp forms.

**Lemma 3.1.** [17, Theorem 3.14] Let $N, k \in \mathbb{N}$ and $\Psi$ be a character modulo $4N$. Suppose that $g(\tau) := \sum_{n=1}^{\infty} a(n)e(n\tau) \in S_{k+\frac{1}{2}}(4N, \Psi)$. Let $t$ be a positive square-free integer, and define the Dirichlet character $\Psi_t$ by $\Psi_t(n) := \Psi(n)\left(\frac{-1}{n}\right)^k\left(\frac{t}{n}\right)$. Define $b_t(n) \in \mathbb{C}$ by

$$\sum_{n=1}^{\infty} \frac{b_t(n)}{n^s} := L(s - k + 1; \Psi_t) \sum_{n=1}^{\infty} a(tn^2) n^s.$$

Then

$$\text{Sh}_t(g) := \sum_{n=1}^{\infty} b_t(n)e(n\tau) \in M_{2k}(2N, \Psi^2).$$

Moreover, if $k \geq 2$, then $\text{Sh}_t(g)$ is a cusp form. We have

$$\text{Sh}_t(T_{p^2}g) = T_p(\text{Sh}_t(g)),$$

where $T_p$ denotes the usual Hecke operator on $M_{2k}(2N, \Psi^2)$.

### 4. Hecke theory for Maass cusp forms

We discuss Hecke theory for the spaces $S_0(N, 1)$ and $S_2(N, \chi)$. For each $n$ coprime to $N$, the Hecke operator $T_n$ acts on a Maass cusp form $f \in S_0(N, 1)$ [7, Section 6] by

$$(T_n f)(\tau) = \frac{1}{\sqrt{n}} \sum_{ad=n} \sum_{b \pmod{d}} f\left(\frac{a\tau + b}{d}\right) \in S_0(N, 1).$$

(4.1)

Each $T_n$ commutes with $\Delta_0$, so $T_n$ is an endomorphism of $S_0(N, 1)$.

These operators satisfy

$$T_m T_n = \sum_{d|m,n} T_{mnd^{-2}}.$$

Consider an orthonormal basis of Maass cusp forms $\{v_j\}$ for $S_0(N, 1)$ that are also Hecke eigenforms for all $T_n$ with $n$ coprime to $N$. Suppose that each $v_j$ has Fourier expansion [7, (6.14) and (6.15)]

$$v_j(\tau) = \sum_{n \neq 0} \tilde{\rho}_j(n) W_{0, it_j}(4\pi|n|y)e(nx),$$

where $t_j$ is the spectral parameter attached to $v_j$. The $H_{\theta}$-hypothesis asserts that

$$\lambda_j(n) \ll \varepsilon n^{\theta + \varepsilon},$$

where the $\lambda_j(n)$ are the Hecke–Maass eigenvalues defined by

$$T_n v_j = \lambda_j(n)v_j.$$

The Ramanujan–Petersson conjecture asserts that $H_0$ is true. The best known result is due to Kim and Sarnak [12, Appendix 2], who showed that the exponent $\theta = 7/64$ is available. Applying $T_n$ to the Fourier expansion of $v_j$ we see that [7, (6.14),(6.15)]

$$\tilde{\rho}_j(n) = \lambda_j(|n|)\tilde{\rho}_j(\text{sgn}(n)) |n|^{-1/2}.$$

(4.2)
The Hecke operators $T_{p^2}$ for $p \nmid 6$ are defined on $S_{1/2}(1, \chi)$ [1, Section 2.6] by

$$T_{p^2}f = \frac{1}{p} \left[ \sum_{b \mod p^2} e\left(\frac{-b}{24}\right)f_{1/2}\left(\frac{1}{p} \frac{b}{p} \frac{1}{p} \frac{1}{p}\right) + e\left(\frac{p-1}{8}\right) \sum_{h=1}^{p-1} e\left(\frac{-hp}{24}\right)f_{1/2}\left(\frac{1}{p} \frac{h}{p} \frac{1}{p} \frac{1}{p}\right)\right].$$

Each $T_{p^2}$ commutes with $\Delta_{1/2}$, so $T_{p^2}$ is an endomorphism of $S_{1/2}(1, \chi, r)$.

Ahlgren and Andersen [1] developed a Shimura type correspondence between Maass cusp forms of weight $1/2$ on $\Gamma_0(N)$ with the eta multiplier twisted by a Dirichlet character and Maass cusp forms of weight 0. Here we provide details only in the simplest case. In this setting it is most convenient to write the expansion of $f \in S_{1/2}(1, \chi, r)$ in the form

$$f(\tau) = \sum_{n \neq 0} a(n)W_{sgn(n)}(ir)\left(\frac{\pi|n|y}{12}\right)e\left(\frac{nx}{24}\right). \quad (4.3)$$

**Theorem 4.1.** [1, Theorem 5.1] Suppose that $G \in S_{1/2}(1, \chi, r)$ with $r \neq i/4$ and Fourier expansion given by (4.3). Let $t \equiv 1 \pmod{24}$ be a square-free positive integer and define $b_t(n) \in \mathbb{C}$ by the relation

$$\sum_{n=1}^{\infty} b_t(n) = L\left(s + 1, \left(\frac{t}{\bullet}\right)\right) \sum_{n=1}^{\infty} \left(\frac{12}{n}\right) a(tn^2) n^{s-1/2}. \quad (4.4)$$

Then the function $S_t$ defined by

$$(S_tG)(\tau) := \sum_{n=1}^{\infty} b_t(n)W_{0,2ir}(4\pi ny)\cos(2\pi nx)$$

is an even Maass cusp form in $S_0(6, 1, 2r)$. For any prime $p \geq 5$ we have

$$T_{p^2}S_t(G) = \left(\frac{12}{p}\right)S_t(T_{p^2}G).$$

**Remark 4.1.** Using Theorem 4.1, Ahlgren and Andersen rule out the existence of exceptional eigenvalues in $S_{1/2}(1, \chi)$. If $S_{1/2}(1, \chi, r) \neq \{0\}$, then either $r = i/4$ or $r > 1.9$. Note that $r_0 = i/4$ corresponds to the minimal eigenvalue $\lambda_0 = \frac{3}{16}$. This is achieved by the unique normalised cusp form

$$u_0(\tau) := \sqrt{\frac{3}{\pi}}(6y)^{1/2}\eta(\tau).$$

The Fourier coefficients $\rho_0(m)$ of $u_0$ are non-zero only when $m - 1 \in \mathcal{P}$. See [2, pg. 435].

### 5. Kuznetsov–Proskurin formula

Here we develop some tools for the case $m, n > 0$. Let $\phi \in C^4([0, \infty))$ be such that

$$\phi(0) = \phi'(0), \quad \phi(t) \ll \varepsilon t^{-1-\varepsilon} \quad \text{and} \quad \phi^{(j)}(t) \ll \varepsilon t^{-2-\varepsilon} \quad \text{for} \quad j = 1, 2, 3, 4, \quad (5.1)$$

as $t \to \infty$ for some fixed $\varepsilon > 0$. We define the auxiliary integrals

$$\dot{\phi}(r) := \int_0^\infty J_{r-1}(y)\phi(y)\frac{dy}{y}, \quad (5.2)$$

where $J_r(y)$ is the Bessel function of the first kind.
\[ \hat{\phi}(r) := \pi^2 e^{3\pi i/4} \int_0^\infty \frac{(\cos \pi (\frac{1}{4} + ir) J_{2ir}(y) - \cos \pi (\frac{1}{4} - ir) J_{-2ir}(y)) \phi(y) \frac{dy}{y}}{\text{sh}(\pi r) \text{ch}(2\pi r) \Gamma(\frac{1}{4} + ir) \Gamma(\frac{1}{4} - ir)}, \quad (5.3) \]

where \( J_\nu \) for \( \nu \in \mathbb{C} \) denotes the \( J \)-Bessel function.

Using a trigonometric identity, we write the integrand occurring in \( \hat{\phi}(r) \) in the more convenient form

\[ \frac{1}{\sqrt{2}} \phi(y) \left( \cos(\pi ir)(J_{2ir}(y) - J_{-2ir}(y)) - \sin(\pi ir)(J_{2ir}(y) + J_{-2ir}(y)) \right). \quad (5.4) \]

For each integer \( l \geq 1 \), let \( B_l \) denote an orthonormal basis for \( \tilde{S}_{l+2}(1, \chi) \) and \( \mathcal{S} := \bigcup_{l=1}^\infty B_l \).

Suppose each \( f \in \mathcal{S} \) has Fourier expansion given by

\[ f(\tau) := \sum_{n=1}^\infty a_f(n) e(\tilde{n} \tau), \]

and weight denoted by \( w(f) \). Let \( \{u_j\} \) be an orthonormal basis for \( \mathcal{S}_{12}(1, \chi) \) with Fourier expansion given by (2.1). For \( m, n > 0 \), Proskurin’s formula [20, p. 3888] asserts that

\[ \sum_{c \geq 1} S(m, n, c, \chi) \phi\left( \frac{4\pi \sqrt{m \tilde{n}}}{c} \right) = U + V, \quad (5.5) \]

where

\[ U := \sum_{f \in \mathcal{S}} \frac{4\Gamma(w(f)) e^{\pi iw(f)/2}}{(4\pi)^{w(f)} (m \tilde{n})^{w(f)-1}} a_f(m) a_f(n) \hat{\phi}(w(f)), \quad (5.6) \]

\[ V := 4\sqrt{m \tilde{n}} \sum_{j \geq 0} \frac{\rho_j(m) \rho_j(n)}{\text{ch}(\pi r_j)} \hat{\phi}(r_j). \quad (5.7) \]

Given \( a, x > 0 \), choose a parameter \( T > 0 \) such that \( T \leq x/3 \), \( T \asymp x^{1-\delta} \) with \( 0 < \delta < 1/2 \).

Now we choose a smooth \( \phi = \phi_{a,x,T} : [0, \infty) \to [0, 1] \) satisfying

- The conditions in (5.1)
- \( \phi(t) = 1 \) for \( \frac{a}{2x} \leq t \leq \frac{a}{x} \)
- \( \phi(t) = 0 \) for \( t \leq \frac{a}{2x+2T} \) and \( t \geq \frac{a}{x-T} \)
- \( \phi'(t) \ll \left( \frac{a}{x-T} - \frac{a}{x} \right)^{-1} \ll \frac{x^2}{T} \).
- \( \phi \) and \( \phi' \) are piecewise monotone on a fixed number of intervals.

Here we provide bounds for some useful expressions involving \( \phi \) and \( \hat{\phi} \).

**Lemma 5.1.** Let \( \phi = \phi_{a,x,T} \) be as above. For \( a := 4\pi \sqrt{m \tilde{n}} \) we have

\[ \sum_{l=1}^\infty \left| \left( \frac{1}{2} + 2l \right) \hat{\phi}\left( \frac{1}{2} + 2l \right) \right| \ll 1 + \frac{\sqrt{mn}}{x}. \quad (5.8) \]
Proof. For the reader’s convenience we sketch the argument that appears on [24, pp. 630–632]. When \( x \geq 4\pi\sqrt{mn} \), the support of \( \phi \) is contained in \([0, 3/2]\), and it is immediate from the decay of the Bessel function \([8, (10.14.4)]\) that

\[
\hat{\phi}\left(\frac{1}{2} + 2l\right) \ll \frac{1}{l(\frac{1}{2} + 2l)}.
\]

Thus the left hand side of (5.8) is bounded by \( O(1) \).

Now consider the case when \( x \leq 4\pi\sqrt{mn} \). In what follows we write \( k := 1/2 + 2l \) for convenience. We treat each integral (5.2) occurring in the summand (5.8) according to the transitional ranges of the \( J \)-Bessel function. In the range \( 0 \leq y \leq k - k^\frac{1}{3} \), \( J_k(y) \) is exponentially small and the contribution is \( O(1) \). In the range \( k - k^\frac{1}{3} \leq y \leq k + k^\frac{1}{3} \) one can appeal to the asymptotics for \( J_k(y) \) on [29, pp 249] or [24, (22) and (23)]. Using these asymptotics and the fact that the support of \( \phi \) is contained in the interval \([0, 6\pi\sqrt{mn}/x]\), we see that there are at most \( O(\sqrt{mn}/x) \) choices of \( k \) for which the transitional range is present on the left hand side of (5.8). Each \( k\hat{\phi}(k) \) is \( O(1) \) for all \( k \) in the transitional range using the asymptotics for \( J_{k\pm1/3} \) and \( K_{1/3} \) in \([8, (10.7.3)\) and \((10.30.2)\)] and hence the total contribution from all such \( k \) to (5.8) is \( O(\sqrt{mn}/x) \). We are now left to bound the contribution for the range \( y \geq k + k^\frac{1}{3} \). For this one can follow the argument in [24, pp. 631–632] starting with the asymptotic in [24, Eqn (52)]. The contribution in this last case is \( O(\sqrt{mn}/x) \).

Lemma 5.2. Suppose that \( a, x, T \) are as above and that \( \phi = \phi_{a,x,T} \). Then we have

\[
\hat{\phi}(r) \ll \begin{cases} 
\min \left( r^{-1}, r^{-2} \frac{x}{T} \right) & \text{if } r \geq \max \left( \frac{a}{x}, 1 \right) \\
r^{-1} & \text{if } r \geq 1.
\end{cases}
\]

Proof. Applying the triangle inequality to (5.4), it is sufficient to bound \( \hat{\phi}(2ir + 1) \). Sarnak and Tsimerman [24, pp 629–630] prove that

\[
\frac{\text{ch}(\pi r)}{\text{sh}(2\pi r)} |\hat{\phi}(2ir + 1)| \ll \begin{cases} 
r^{-\frac{1}{2}} & \text{for } |r| \geq 1 \\
\min \left( r^{-\frac{3}{2}}, r^{-\frac{5}{2}} \frac{x}{T} \right) & \text{for } |r| \geq \max \left( \frac{a}{x}, 1 \right).
\end{cases}
\]

The result follows by recalling the definition of \( \hat{\phi}(r) \) in (5.3) and observing that

\[
\frac{1}{|\Gamma(\frac{1}{4} + ir)|^2} \sim \frac{\sqrt{r}}{2\pi} e^{\pi r} \text{ as } r \to \infty,
\]

by Stirling’s formula.

6. Variant of Proskurin–Kuznetsov formula

We introduce the tools for the mixed sign case \( m > 0 \) and \( n < 0 \). Let \( \phi \) be as in Section 5. Define

\[
\tilde{\Phi}(r) := \cosh(\pi r) \int_0^\infty K_{2ir}(y) \phi(y) \frac{dy}{y},
\]

where \( K_{\nu} \) for \( \nu \in \mathbb{C} \) denotes the \( K \)-Bessel function. Let \( \{u_j\} \) be an orthonormal basis for \( S_{\frac{1}{2}}(1, \chi) \) with Fourier expansions given by (2.1). Then for \( m > 0 \) and \( n < 0 \), [1, Theorem 4.1] asserts that

\[
\mathcal{W} := \sum_{c \geq 1} S(m, n, c, \chi) \phi\left(4\pi\sqrt{mn}/c\right) = 8\sqrt{i}\sqrt{mn}|n| \sum_{j \geq 0} \frac{\rho_j(m)\rho_j(n)}{\text{ch}(\pi r_j)} \hat{\phi}(r_j). \tag{6.1}
\]
Lemma 6.1. [1, Theorem 6.1] Let \(a, x, T\) be as above and let \(\phi = \phi_{a, x, T}\). Then

\[
\Phi(r) \ll \begin{cases} 
    r^{-\frac{3}{2}}e^{-\frac{c}{r}} & \text{for } 1 \leq r \leq \frac{a}{8x} \\
    r^{-1} & \text{for } \max(1, \frac{a}{8x}) \leq r \leq \frac{a}{x} \\
    \min\left(r^{-\frac{3}{2}}, r^{-\frac{3}{2}}\right) & \text{for } r \geq \max\left(\frac{a}{x}, 1\right).
\end{cases}
\]

7. Bound for holomorphic forms

In this section we bound the \(U\) term in (5.5) uniformly in \(m, n\) and \(x\). To obtain bounds in terms of the square-free parts of \(24m - 23\) and \(24n - 23\), we exploit the Shimura correspondence and Deligne’s bound.

We first recall the half–integral weight Petersson formula [26, p. 89]. Let \(k, N \in \mathbb{N}\) and \(\{\psi_j := \sum_{r \geq 1} a_j(n)e(n\tau)\}_{j=1}^J\) be an orthonormal basis for \(S_{\frac{k}{2}+k}(4N, \Psi)\). Then

\[
\Gamma(k - \frac{1}{2}) (\frac{4\pi n}{c})^{k-1} \sum_{j=1}^J |a_j(n)|^2 = 1 + 2\pi i^{-\frac{1}{2} - k} \sum_{n \in \mathbb{N}} c^{-1}J_{k-\frac{1}{2}}(\frac{4\pi n}{c}) K_{\Psi}^{2k+1}(n, n, c),
\]

where

\[
K_{\Psi}^{2k+1}(m, n, c) := \sum_{d \mod c} \varepsilon_d^{-2k-1} \left(\frac{c}{d}\right) e\left(\frac{md + nd}{c}\right), \quad \varepsilon_d := \left(-\frac{1}{d}\right)^\frac{1}{2},
\]

is a twisted Kloosterman sum.

Lemma 7.1. Suppose \(m, n > 0\) are integers and \(\phi\) is as above with \(x \geq 1\) and \(a := 4\pi \sqrt{mn}\). Then for any \(\varepsilon > 0\) we have

\[
U \ll_\varepsilon (mn)^{\varepsilon} \left((mn)^{\frac{1}{2}} + \left(\frac{mn}{x}\right)^{\frac{3}{2}}\right).
\]

Furthermore, if \(24m - 23 = m_0^2 s\) and \(24n - 23 = n_0^2 t\) with \(s\) and \(t\) square-free, then

\[
U \ll_\varepsilon |m_0 n_0|^{\varepsilon} (st)^{\frac{1}{4} + \varepsilon} \left(1 + \left(\frac{mn}{x}\right)^{\frac{1}{2}}\right).
\]

Proof. Let \(\{f_{jl}\}_{1 \leq j \leq \dim S_{\frac{k}{2}+2}(1, \chi)}\) be an orthonormal Hecke eigenbasis with respect to \(T_p^2\) for all primes \(p \nmid 6\). Suppose that

\[
f_{jl}(\tau) = \sum_{r=1}^\infty a_{jl}(r)e\left((r - \frac{23}{24})\tau\right)
\]

and let

\[
g_{jl}(\tau) := f_{jl}(24\tau) \in S_{\frac{k}{2}+2}(576, \chi_{12}).\]

For fixed \(l\), the set \(\{f_{jl}\}\) injects into its image \(\{g_{jl}\} \subset S_{\frac{k}{2}+2}(576, \chi_{12})\) and \(\{g_{jl}\}\) is an orthogonal set consisting of Hecke eigenforms for all \(T_p\) with \(p \nmid 6\). Recall the Petersson inner product on \(S_{\frac{k}{2}+k}(N, \chi_{12})\),

\[
\langle f, g \rangle = \int_{\mathcal{F}_N} f(\tau) g(\bar{\tau}) y^{\frac{k}{2}+k} d\mu \quad \text{where} \quad \tau = x + iy, \quad d\mu = \frac{dx dy}{y^2},
\]

and \(\mathcal{F}_N\) is a fundamental domain for \(\mathbb{H}/\Gamma_0(N)\). For each \(l\), a computation shows that

\[
\{(24)^l \frac{1}{4} \{\Gamma : \Gamma_0(24, 24)]^{-\frac{1}{2}} g_{jl}\}\}
\]

(7.4)
is an orthonormal set of Hecke eigenforms. Note that
\[ \sum_j |a_{jt}(n)|^2 = \sum_j |c_{jt}(24n - 23)|^2. \] (7.5)

Applying the triangle and Cauchy–Schwarz inequalities to the right side of (5.6) and using (7.5) we obtain
\[ U \ll \sum_{l=1}^{\infty} \frac{\Gamma\left(\frac{1}{2} + 2l\right) \phi(l) \left(\frac{1}{2} + 2l\right)}{(4\pi)^{1+2l} \left(\frac{24n - 23}{y}\right)^{1+2l}} \left( \sum_j |c_{jt}(24m - 23)|^2 \right)^{1/2} \left( \sum_j |c_{jt}(24n - 23)|^2 \right)^{1/2}. \] (7.6)

The set in (7.4) can be extended to an orthonormal basis of \( S_{2+2l}(576, \chi_{12}) \). Applying (7.1) and the triangle inequality we obtain
\[ \sum_j |c_{jt}(24n - 23)|^2 \ll \frac{(24)^{-2l} \left(4\pi(24n - 23)\right)^{-1/2+2l}}{\Gamma\left(-\frac{1}{2} + 2l\right)} \times \left( 1 + 2\pi \sum_{c=0 \mod 576} \frac{|K_{\chi_{12}}^{d+1}(24n - 23, 24n - 23, c)|}{c} \left| J_{-\frac{1}{2}+2l} \left( \frac{4\pi(24n - 23)}{c} \right) \right| \right). \] (7.7)

Let \( \delta > 0 \) be fixed and small. To bound the right hand side of (7.7) we consider the cases \( c \leq n^{1+\delta} \) and \( c > n^{1+\delta} \). When \( c \leq n^{1+\delta} \), we use [31, Lemma 4]
\[ |K^{d+1}_{\chi_{12}}(24n - 23, 24n - 23, c)| \leq (24n - 23)^{\frac{1}{2}} \delta^{\frac{1}{2}} c^{\frac{1}{2}}, \] (7.8)

together with [8, (10.14.1)] to obtain
\[ \sum_{c=0 \mod 576} \frac{|K^{d+1}_{\chi_{12}}(24n - 23, 24n - 23, c)|}{c} \left| J_{-\frac{1}{2}+2l} \left( \frac{4\pi(24n - 23)}{c} \right) \right| \ll \epsilon n^{1+\frac{1}{2}+\epsilon}. \] (7.9)

In the case \( c \geq n^{1+\delta} \) we apply [8, (10.14.4)] and (7.8) in the following computation
\[ \sum_{c=0 \mod 576} \frac{|K^{d+1}_{\chi_{12}}(24n - 23, 24n - 23, c)|}{c} \left| J_{-\frac{1}{2}+2l} \left( \frac{4\pi(24n - 23)}{c} \right) \right| \ll \frac{1}{\Gamma\left(-\frac{1}{2} + 2l\right)} \sum_{c>n^{1+\delta}} \frac{|K^{d+1}_{\chi_{12}}(24n - 23, 24n - 23, c)|}{c} \left( \frac{4\pi(24n - 23)}{2c} \right)^{-\frac{1}{2}+2l} \ll \epsilon \left( \frac{48\pi)^2 n^{-1-\delta(\frac{1}{2}+2l)} \right) \left( \frac{4\pi(24n - 23)}{2c} \right)^{-\frac{1}{2}+2l} \ll \frac{1}{\Gamma\left(-\frac{1}{2} + 2l\right)} \sum_{c=0 \mod 576} \frac{1}{c^{1-\epsilon}} \sum_{c>n^{1+\delta}/d} \frac{1}{c'} \ll \epsilon n^{1+\frac{1}{2}+\epsilon}. \] (7.10)

Combining (7.6)–(7.10) we obtain
\[ U \ll \epsilon (mn)^{1+\epsilon} \sum_{l=1}^{\infty} \left( -\frac{1}{2} + 2l \right) \phi\left( \frac{1}{2} + 2l \right). \] (7.11)
Thus (7.2) follows from Lemma 5.1.

We now prove (7.3). Since $g_{jl} \in S_{ \frac{1}{2} + 2l}(576, \chi_{12})$ is an eigenform under the action of $T_n$ for all $n$ coprime to 6, we know that $\text{Sh}_t(g_{jl}) \in S_{4l}(288, 1)$ is an eigenform under the action of $T_n$ with the same eigenvalue. Denote this eigenvalue by $\lambda_{jl}(n)$. For each $l$ and $j$ define $b_l(n) \in \mathbb{C}$ by

$$\text{Sh}_t(f_{jl})(\tau) = \sum_{n=1}^{\infty} b_l(n) e(n \tau) \in S_{4l}(288, 1).$$

We also define the arithmetic functions

$$g(u) := c_{jl}(tu^2) \quad \text{and} \quad h(u) := u^{2l-1 - \frac{12l}{u}}.$$ (7.12)

The equality in Lemma 3.1 implies that $b_l = g \ast h$. Observe that $h$ is multiplicative and $h(1) = 1$, so $h$ has a multiplicative Dirichlet inverse. We have $h^{-1}(1) = 1$ and a computation for $p$ prime and $\alpha \in \mathbb{N}$ yields

$$h^{-1}(p^\alpha) = \begin{cases} -\left(\frac{12l}{p}\right) p^{2l-1} & \text{if } \alpha = 1 \\ 0 & \text{otherwise.} \end{cases}$$ (7.13)

Using the fact that the $\text{Sh}_t(g_{jl})$ are Hecke eigenforms of integral weight and that $b_l(1) = c_{jl}(t)$, we have

$$b_l(d) = \lambda_{jl}(d) b_l(1) = \lambda_{jl}(d) c_{jl}(t).$$

Thus

$$c_{jl}(tn_0^2) = \sum_{d|n_0} b_l(d) h^{-1}\left(\frac{n_0}{d}\right) = c_{jl}(t) \sum_{d|n_0} \lambda_{jl}(d) h^{-1}\left(\frac{n_0}{d}\right).$$ (7.14)

By Deligne’s bound [5] we have

$$|\lambda_{jl}(d)| \ll \varepsilon d^{2l-\frac{1}{2}+\varepsilon}.$$ (7.15)

Using (7.13) and (7.15), (7.14) becomes

$$|c_{jl}(24n - 23)| = |c_{jl}(tn_0^2)| \ll \varepsilon |c_{jl}(t)| n_0^{2l-\frac{1}{2}+\varepsilon}.$$

We may replace each summand on the right hand side of (7.5) with $|c_{jl}(t)| n_0^{2l-\frac{1}{2}+\varepsilon}$. Performing similar computations to those occurring in (7.6)–(7.10), we see that (7.11) becomes

$$U \ll \varepsilon (m_0 n_0)^{\frac{1}{2}+\varepsilon} \sum_{l=1}^{\infty} \left(\frac{1}{2} + 2l\right) \left| \tilde{\phi}\left(\frac{1}{2} + 2l\right) \right|.$$ (7.16)

Thus Lemma 5.1 implies (7.3).

8. Estimates for the coefficients of Maass cusp forms

We bound the quantities in (5.7) and (6.1) in the proofs of the main theorems by modifying the dyadic arguments of [24]. Here we collect the main inputs required for this argument. The first is a mean value estimate for the coefficients of weight $1/2$ Maass cusp forms due to Ahlgren–Andersen and also a forthcoming estimate of Andersen.
Proposition 8.1. [1, Theorem 1.5] Suppose \( \{u_j\} \) is an orthonormal basis for \( S_\frac{1}{2}(1, \chi) \) with spectral parameters \( r_j \) and Fourier expansion given by (2.1). If \( n < 0 \) then we have
\[
|n| \sum_{0 < r_j \leq x} \left| \frac{\rho_j(n)}{\text{ch}(\pi r_j)} \right|^2 = \frac{x^{\frac{3}{2}}}{5\pi^2} + O_x(x^{\frac{3}{2}} \log x + |n|^{\frac{1}{2} + \frac{\varepsilon}{2}}).
\]

Andersen and Duke improve the mean value estimate of [1, Theorem 1.5] when \( n > 0 \). This improvement will be important in the proofs of our main theorems. We state only a special case of their result.

Proposition 8.2. [3, Theorem 4.1] Suppose \( \{u_j\} \) is an orthonormal basis for \( S_\frac{1}{2}(1, \chi) \) with spectral parameters \( r_j \) and Fourier expansion given by (2.1). If \( n > 0 \) then we have
\[
\tilde{n} \sum_{x < r_j \leq 2x} \left| \frac{\rho_j(n)}{\text{ch}(\pi r_j)} \right|^2 \ll x^{\frac{1}{2}} + x^{-\frac{1}{2}}(\log x)^{\frac{1}{2} + \varepsilon} n^{\frac{1}{2} + \varepsilon}.
\]

The second main idea is the application of an averaged form of a pointwise bound due to Duke [6] for the Fourier coefficients of Maass cusp forms of half integral weight with multiplier \( (\frac{11}{2})_n \). This average bound was established by Ahlgren and Andersen [1, Theorem 8.1] using a modified version of Duke’s argument. We remove a hypothesis in their Theorem which restricts divisibility of \( n \) by arbitrarily high powers of 5 and 7. We note this allows the removal of hypothesis (1.6) in Theorems 1.1, 1.2 and 1.4 of [1].

Proposition 8.3. Let \( \{u_j\} \) be an orthonormal basis for \( S_\frac{1}{2}(1, \chi) \) with spectral parameters \( r_j \) and Fourier expansion given by (2.1). If all the \( u_j \) are Hecke eigenforms of \( T_{\nu^2} \), then for \( x > 0 \),
\[
|n| \sum_{0 < r_j \leq x} \left| \frac{\rho_j(n)}{\text{ch}(\pi r_j)} \right|^2 \ll x^{\frac{3}{2} + \varepsilon} n^{\frac{3}{2} + \varepsilon} x^{-\frac{7}{6} - \frac{\text{sgn}(n)}{2}}.
\]

Proof. Suppose \( 24n - 23 = n_0^2t \) with \( t \) square-free. Under the \( H_\theta \) hypothesis we can apply (11.4) below followed by [1, Theorem 8.1] to obtain
\[
|n| \sum_{0 < r_j \leq x} \left| \frac{\rho_j(n)}{\text{ch}(\pi r_j)} \right|^2 \ll x^{\frac{3}{2} + \varepsilon} n_0^{2\theta + \varepsilon} |t|^{\frac{3}{2} + \varepsilon} x^{-\frac{7}{6} - \frac{\text{sgn}(n)}{2}}.
\]

Since \( \theta = 7/64 \) is an acceptable exponent [12, Appendix 2] we see that
\[
|n_0|^{2\theta + \varepsilon} |t|^{\frac{3}{2} + \varepsilon} x^{-\frac{7}{6} - \frac{\text{sgn}(n)}{2}} \ll (n_0^{2\theta + \varepsilon} |t|^{\frac{3}{2} + \varepsilon} x^{-\frac{7}{6} - \frac{\text{sgn}(n)}{2}}) \ll x^{\frac{3}{2} + \varepsilon} x^{-\frac{7}{6} - \frac{\text{sgn}(n)}{2}}.
\]

This averaged bound allows us to optimise the dyadic argument in the spectral parameter, and is ultimately responsible for the improved \( mn \)-aspect occurring in Theorem 1.1.

9. Proof of Theorem 1.1

Proposition 9.1. Let \( m > 0 \) and \( n < 0 \) be integers. Then for \( x \geq |\tilde{m}n|^{\frac{77}{28}} \) we have
\[
\sum_{x \leq c \leq 2x} \frac{S(m, n, c, \chi)}{c} \ll x^{\frac{3}{2}} + \frac{|n|^{\frac{1}{2}}}{|mn|^{\frac{3}{28}}} + |mn|^{\frac{38}{28}} |mn|^{\varepsilon} \log^2 x.
\]
We show that Proposition 9.1 implies Theorem 1.1. Considering the sum
\[
\sum_{1 \leq c \leq X} \frac{S(m, n, c, \chi)}{c},
\]
the initial segment \(1 \leq c \leq \lfloor \tilde{m} \tilde{n}^{\frac{35}{2}} \rfloor\) contributes \(O_{\varepsilon}(\lfloor \tilde{m} \tilde{n}^{\frac{35}{2}} + \varepsilon \rfloor)\) by [1, (2.30)]. One then breaks the interval \(\lfloor \tilde{m} \tilde{n}^{\frac{35}{2}} \rfloor \leq c \leq X\) into \(O(\log X)\) dyadic intervals \(x \leq c \leq 2x\) with \(\lfloor \tilde{m} \tilde{n}^{\frac{35}{2}} \rfloor \leq x \leq X/2\), and then applies Proposition 9.1.

**Proof of Proposition 9.1.** Let \(\phi\) be a smooth test function with the properties listed in Section 5. Fix \(a := 4\pi\sqrt{\tilde{m}\tilde{n}}\) and let \(T > 0, 0 < \beta < 1/2\) both be chosen later. Suppose
\[
x \geq (\tilde{m}\tilde{n})^{\frac{1}{2} - \beta}.
\]
Using the Weil bound [1, Proposition 2.1] and the mean value bound for the divisor function we have
\[
\left| \sum_{c=1}^{\infty} \frac{S(m, n, \chi, c)}{c} \phi\left( \frac{a}{c} \right) - \sum_{x \leq c \leq 2x} \frac{S(m, n, \chi, c)}{c} \right| \leq \sum_{x-T \leq c \leq x} \frac{|S(m, n, \chi, c)|}{c} \ll_{\varepsilon} \frac{T \log x}{\sqrt{x}} |mn|^\varepsilon. \tag{9.2}
\]

Recall that we have
\[
\sum_{c=1}^{\infty} \frac{S(m, n, \chi, c)}{c} \phi\left( \frac{a}{c} \right) = \mathcal{W}. \tag{9.3}
\]

Let \(\{u_j\}\) be an orthonormal basis of Maass cusp forms for \(S_{\frac{1}{2}}(1, \chi)\) that are also a Hecke eigenbasis with respect to the \(T_p\) for all primes \(p \nmid 6\). Note that there is no contribution from \(r_0 = i/4\) since \(n\) is negative. For the initial segment \(0 < r_j \leq 4\pi(\tilde{m}\tilde{n})^{\beta}\), we apply Proposition 8.3 to obtain
\[
m|n| \sum_{0 < r_j \leq 4\pi(\tilde{m}\tilde{n})^{\beta}} \left| \frac{\rho_j(m)}{\chi(h r_j)} \right|^2 \cdot \sum_{0 < r_j \leq 4\pi(\tilde{m}\tilde{n})^{\beta}} \left| \frac{\rho_j(n)}{\chi(\pi r_j)} \right|^2 \ll_{\varepsilon} |mn|^{\frac{1}{2} + 10\beta + \varepsilon}. \tag{9.4}
\]

Then applying the Cauchy–Schwarz inequality, (9.4), Lemma 6.1 and the fact that \(r_j > 1.9\) [1, Corollary 5.3] yields the estimate
\[
\sqrt{\tilde{m}\tilde{n}} \sum_{0 < r_j \leq 4\pi(\tilde{m}\tilde{n})^{\beta}} \frac{\rho_j(m)\rho_j(n)}{\chi(h r_j)} \Phi(r_j) \ll_{\varepsilon} |mn|^{\frac{3}{2} + 5\beta + \varepsilon}. \tag{9.5}
\]

We now estimate the contribution to \(\mathcal{W}\) from the dyadic intervals \(A \leq r_j \leq 2A\) with \(A \geq 4\pi(\tilde{m}\tilde{n})^{\beta}\). Since we are assuming \(x \geq (\tilde{m}\tilde{n})^{\frac{1}{2} - \beta}\), we have \(A \geq \max (a/x, 1)\). Using
Lemma 6.1, Proposition 8.1 and Proposition 8.2 we have

$$\sqrt{\tilde{m}|\tilde{n}|} \sum_{A \leq r_j \leq 2A} \frac{\rho_j(m)\rho_j(n)}{\operatorname{ch}(\pi r_j)} \hat{\Phi}(r_j)$$

$$\ll \min \left( A^{-\frac{3}{2}}, A^{-\frac{3}{2}} \frac{x}{T} \right) \left( m \sum_{A \leq r_j \leq 2A} \frac{|\rho_j(m)|^2}{\operatorname{ch}(\pi r_j)} \right)^{\frac{1}{2}} \left( n \sum_{A \leq r_j \leq 2A} \frac{|\rho_j(n)|^2}{\operatorname{ch}(\pi r_j)} \right)^{\frac{1}{2}}$$

$$\ll_{\varepsilon} \min \left( A^{-\frac{3}{2}}, A^{-\frac{5}{2}} \frac{x}{T} \right) \left( A^2 + m^{\frac{1}{2}+\varepsilon} A^{-\frac{1}{2}} \right) \left( A^2 + n^{\frac{1}{2}+\varepsilon} A^{-\frac{1}{2}} \right)^{\frac{1}{2}}$$

$$\ll_{\varepsilon} \min \left( \sqrt{A}, \frac{x}{T\sqrt{A}} \right) \left( 1 + A^{-1} \log A(\frac{m^{\frac{1}{2}+\varepsilon} + |n|^{\frac{1}{4}+\varepsilon} + A^{-2}) \log A|mn|^{\frac{3}{4}+\varepsilon} \right)$$

$$\ll_{\varepsilon} \min \left( \sqrt{A}, \frac{x}{T\sqrt{A}} \right) \left( 1 + A^{-\frac{1}{2}} \log A \left( \frac{m^{\frac{1}{2}+\varepsilon} + |n|^{\frac{1}{4}+\varepsilon}}{|mn|^{\frac{7}{4}}} + A^{-\frac{1}{2}} \log A|mn|^{\frac{1}{4}-\frac{3}{2}+\varepsilon} \right) \right)$$

Summing over dyadic intervals \([2^j \cdot 4\pi(m|\tilde{n}|)^3, 2^{j+1} \cdot 4\pi(m|\tilde{n}|)^3]\) with \(j = 0, 1, 2, \ldots\) we see that when \(x \geq (m|\tilde{n}|)^{\frac{1}{2}-\beta}\), the following holds:

$$\sqrt{\tilde{m}|\tilde{n}|} \sum_{r_j \geq 4\pi(m|\tilde{n}|)^{\beta}} \frac{\rho_j(m)\rho_j(n)}{\operatorname{ch}(\pi r_j)} \hat{\Phi}(r_j) \ll_{\varepsilon} \left( \sqrt{\frac{x}{T}} + \frac{m^{\frac{1}{3}} + |n|^{\frac{1}{3}}}{|mn|^{\frac{3}{2}}} + |mn|^{\frac{1}{4}-\frac{3}{2}+\varepsilon} \right) |mn|^{\varepsilon} \log^2 x.$$

Combining (9.5) and (9.8) we obtain

$$\mathcal{W} \ll_{\varepsilon} \left( \sqrt{\frac{x}{T}} + |mn|^{\frac{3}{11} + 5\beta + \varepsilon} + \frac{m^{\frac{1}{3}} + |n|^{\frac{1}{3}}}{|mn|^{\frac{3}{2}}} + |mn|^{\frac{1}{4}-\frac{3}{2}+\varepsilon} \right) |mn|^{\varepsilon} \log^2 x.$$

To balance the \(x\)-aspect of (9.2) and (9.9) we choose \(T := x^{\frac{1}{4}}\). To balance the \(mn\)-aspect of (9.9) and the contribution from the initial segment for \(c\), we set \(\beta = \frac{1}{154}\). Combining (9.2) and (9.9) leads to the result.

10. Proof of Theorem 1.2

**Proposition 10.1.** Let \(m, n > 0\) be integers such that \(m - 1 \not\in \mathcal{P}\) or \(n - 1 \not\in \mathcal{P}\). Then for \(x \geq 4\pi \sqrt{mn} \) we have

$$\sum_{x \leq c \leq 2x} \frac{S(m, n, c, \chi)}{c} \ll_{\varepsilon} \left( x^{\frac{1}{5}} + (mn)^{\frac{1}{10}} \right) (mn)^\varepsilon \log x.$$

Proposition 10.1 implies Theorem 1.2. The initial segment \(1 \leq c \leq 4\pi \sqrt{\tilde{m}|\tilde{n}|}\) contributes \(O_{\varepsilon}(mn)^{\frac{3}{4}+\varepsilon}\) by [1, (2.30)] to (9.1). One then breaks the interval \(4\pi \sqrt{\tilde{m}|\tilde{n}|} \leq c \leq X\) into \(O(\log X)\) dyadic intervals \(x \leq c \leq 2x\) with \(4\pi \sqrt{\tilde{m}|\tilde{n}|} \leq x \leq X/2\), and applies Proposition 10.1.

**Proof of Proposition 10.1.** Let \(\phi\) be a smooth test function with the properties listed in Section 5. Fix \(a := 4\pi \sqrt{\tilde{m}|\tilde{n}|}\) and let \(T > 0\) be chosen later. Recall that (9.2) holds and that by (5.5) we have

$$\sum_{c=1}^{\infty} \frac{S(m, n, \chi, c)}{c} \phi \left( \frac{a}{c} \right) = \mathcal{U} + \mathcal{V}.$$

(10.1)
Note that there is no contribution from \( r_0 = i/4 \) by Remark 4.1 and the hypothesis on \( m \) and \( n \).

When \( x \geq 4\pi \sqrt{mn} \), Lemma 7.1 guarantees

\[
\mathcal{U} \ll_{\varepsilon} (mn)^{\frac{1}{4} + \varepsilon}. \tag{10.2}
\]

Let \( \{u_j\} \) be an orthonormal basis of Maass cusp forms for \( S_{\frac{1}{2}}(1, \chi) \). Applying Lemma 5.2 and Proposition 8.2 we see that for \( A \geq 1 \) (this is sufficient by Remark 4.1) we have

\[
\sqrt{mn} \left| \sum_{A \leq r_j \leq 2A} \frac{\rho_j(m)\rho_j(n)}{\text{ch}(\pi r_j)} \hat{\phi}(r_j) \right| \\
\ll \min \left( A^{-1}, A^{-2} \frac{x}{T} \right) \left( m \sum_{j \geq 1} \frac{|\rho_j(m)|^2}{\text{ch}(\pi r_j)} \right)^{\frac{1}{2}} \left( n \sum_{j \geq 1} \frac{|\rho_j(n)|^2}{\text{ch}(\pi r_j)} \right)^{\frac{1}{2}} \\
\ll_{\varepsilon} \min \left( A^{-1}, A^{-2} \frac{x}{T} \right) (A^{\frac{3}{2}} + m^{\frac{1}{2}+\varepsilon} A^{-\frac{1}{2}} |\log A|^{\frac{1}{2}+\varepsilon})^{\frac{1}{2}} (A^{\frac{3}{2}} + n^{\frac{1}{2}+\varepsilon} A^{-\frac{1}{2}} |\log A|^{\frac{1}{2}+\varepsilon})^{\frac{1}{2}} \\
\ll_{\varepsilon} \min \left( \sqrt{A}, \frac{x}{T \sqrt{A}} \right) (1 + A^{-1} |\log A| (m^{\frac{1}{4}+\varepsilon} + n^{\frac{1}{4}+\varepsilon}) + A^{-2} |\log A| (mn)^{\frac{1}{4}+\varepsilon}).
\]

Summing over dyadic intervals \([A, 2A]\) for \( A \geq \max \left( \frac{a}{x}, 1 \right) = 1 \) we obtain

\[
\mathcal{V} \ll_{\varepsilon} \left( \frac{\sqrt{x}}{T} + (mn)^{\frac{1}{4}+\varepsilon} \right) \log x. \tag{10.3}
\]

To balance the \( x \)-aspect of \((9.2)\) and \((10.3)\) we choose \( T := x^{\frac{2}{3}} \). Combining \((9.2)\), \((10.1)\), \((10.2)\) and \((10.3)\) leads to the result.

\[
11. \text{Proof of Theorem 1.3}
\]

The heart of the argument is to use Theorem 4.1 as a means to access the \( H_\theta \)-hypothesis.

**Proposition 11.1.** Let \( m > 0, n < 0 \) and \( m_0, n_0 \) be integers such that \( 24m - 23 = m_0 s \) and \( 24n - 23 = n_0 t \) with \( s \) and \( t \) square-free. If \( x \geq |\tilde{m}\tilde{n}|^{\frac{1}{3}} \), then under the \( H_\theta \)-hypothesis we have

\[
\sum_{x \leq c \leq 2x} S(m, n, c, \chi) \ll_{\varepsilon} \left( x^{\frac{1}{6}} + |mn|^{\frac{1}{12}} (\tilde{m}^{\frac{1}{6}} + |n|^{\frac{1}{6}}) + |mnst|^{\frac{1}{6} + \frac{\theta}{6} + \varepsilon} \right) |mn|^{\varepsilon} \log^2 x.
\]

As before, Proposition 11.1 implies Theorem 1.3 with the choice of initial segment \( 1 \leq c \leq |\tilde{m}\tilde{n}|^{\frac{1}{3}} \).

**Proof of Proposition 11.1.** Let \( \phi \) be a smooth test function with the properties listed in Section 5. Fix \( a := 4\pi \sqrt{|\tilde{m}| \tilde{n}} \) and let \( T > 0 \) be chosen later. Let \( \{u_j\} \) be an orthonormal basis for \( S_{\frac{1}{2}}(1, \chi) \) consisting of Hecke eigenforms of \( \mathcal{T}_p^2 \) for all primes \( p \nmid 6 \). We follow the proof of Proposition 9.1 to (9.3) and remind the reader that there is no contribution from \( r_0 = i/4 \) since \( n \) is negative.
To bound $W$, we will treat the spectral parameter separately on different ranges. Note that it is sufficient to consider $r_j \geq 1$ by Remark 4.1. The ranges are

$$1 \leq r_j \leq a/(8x),$$

$$\max (a/(8x), 1) \leq r_j \leq a/x,$$

$$r_j \geq \max (a/x, 1).$$

We first consider the case $r_j \geq \max (a/x, 1)$. For $A \geq \max (a/x, 1)$, we first prove

$$\sqrt{|m|/n} \sum_{A \leq r_j \leq 2A} \frac{\rho_j(m)\rho_j(n)}{\text{ch}(\pi r_j)} \Phi(r_j) \ll \varepsilon |m_0 n_0|^{\theta + \varepsilon} \min \{ \sqrt{A}, \frac{x}{T \sqrt{A}} \} \times (1 + A^{-1} \log A |s^{\frac{1}{3} + \varepsilon} + |t^{\frac{1}{3} + \varepsilon}| + A^{-2} \log A ||st||^{\frac{1}{3} + \varepsilon}). \quad (11.1)$$

To prove (11.1) we start with (9.6). Recalling the normalisations (2.1) and (4.3) we define

$$a_j(n) = \rho_j \left( \frac{n+23}{24} \right). \quad (11.2)$$

We write

$$S_t(u_j) := \sum_{n=1}^{\infty} b_t(n) W_{0,2r}(4\pi ny) \cos(2\pi nx) \in S_0(6,1,2r_j),$$

where $S_t$ denotes the lift for cusp forms in Theorem 4.1. The $S_t(u_j)$ are also eigenforms under the action of $T_n$. Let the eigenvalue be denoted by $\lambda_j(n)$. We define the arithmetic functions

$$g(u) := a_j(tu^2)u^{\frac{1}{3}} \left( \frac{12}{u} \right) \text{ and } h(u) := u^{-1} \left( \frac{t}{u} \right).$$

The equality in Theorem 4.1 implies that $b_t(\cdot, j) = g * h$. Observe that $h(1) = 1$, so arguing as in Section 7 gives

$$h^{-1}(p^\alpha) = \begin{cases} -\left( \frac{t}{u} \right) p^{-1} & \text{if } \alpha = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$|h^{-1}(u)| \leq u^{-1} \text{ for all } u \in \mathbb{N}. \quad (11.3)$$

Using (4.2) and the relation $b_t(1, j) = a_j(t)$, we have

$$b_t(d, j) = d^{-\frac{1}{2}} \lambda_j(d) b_t(1, j) = d^{-\frac{1}{2}} \lambda_j(d) a_j(t).$$

Thus

$$g_j(n_0^2) = \sum_{d|n_0} b_t(d, j) h^{-1} \left( \frac{n_0}{d} \right) = a_j(t) \sum_{d|n_0} d^{-\frac{1}{2}} \lambda_j(d) h^{-1} \left( \frac{n_0}{d} \right).$$

Applying the $H_\theta$–hypothesis and (11.3) we obtain

$$|g_j(n_0^2)| \ll \varepsilon |a_j(t)| n_0^{-\frac{1}{2} + \theta + \varepsilon}. \quad (11.4)$$

By the definition of $g$ and (11.2) we conclude that

$$|\rho_j(n)| \ll \varepsilon |\rho_j \left( \frac{t+23}{24} \right)| n_0^{-1+\theta+\varepsilon}. \quad (11.4)$$

Using (11.4) in (9.6), we see that (11.1) follows from Proposition 8.2.
Combining (9.7) and (11.1), for $A \geq \max \left( \frac{a}{x}, 1 \right)$ we have

$$\sqrt{m|\tilde{n}|} \sum_{A \leq r_j \leq 2A} \frac{\rho_j(m)\rho_j(n)}{\text{ch}(\pi r_j)} \Phi(r_j) \ll \epsilon \min \left( \sqrt{A}, \frac{x}{T\sqrt{A}} \right) \times$$

$$\min \left( 1 + A^{-1}|\log A|(m^{\frac{1}{4}+\epsilon} + |n|^{\frac{1}{4}+\epsilon}) + A^{-2}|\log A||mn|^{\frac{1}{4}}, \right.$$  

$$|m_0n_0|^\theta+\epsilon \left[ 1 + A^{-1}|\log A|(s^{\frac{1}{4}+\epsilon} + |t|^{\frac{1}{4}+\epsilon}) + A^{-2}|\log A||st|^{\frac{1}{4}+\epsilon} \right]. \quad (11.5)$$

Using the facts that for positive $B, C$ and $D$ we have

$$\min (B+C, D) \leq \min (B, D) + \min (C, D) \quad \text{and} \quad \min (B, C) \leq \sqrt{BC},$$

we simplify (11.5). This right side of (11.5) is

$$\ll \epsilon \min \left( \sqrt{A}, \frac{x}{T\sqrt{A}} \right) \left( 1 + |m_0n_0|^{\frac{\theta+\epsilon}{2}} |\log A| \left[ A^{-1}(m^{\frac{1}{4}+\epsilon} + |n|^{\frac{1}{4}+\epsilon})(s^{\frac{1}{4}+\epsilon} + |t|^{\frac{1}{4}+\epsilon}) + A^{-\frac{1}{2}}(m^{\frac{1}{4}+\epsilon} + |n|^{\frac{1}{4}+\epsilon})|st|^{\frac{1}{4}+\epsilon} + A^{-1}(mn)^{\frac{1}{4}+\epsilon} \right. \right.$$  

$$\left. + A^{-\frac{1}{2}}|mn|^{\frac{1}{4}+\epsilon}(s^{\frac{1}{4}+\epsilon} + |t|^{\frac{1}{4}+\epsilon}) + A^{-2}|mnt|^{\frac{1}{4}+\epsilon} \right). \quad (11.6)$$

Summing over all dyadic intervals $[A, 2A]$ with $A \geq \max \left( \frac{a}{x}, 1 \right)$ and ignoring the smallest terms we obtain

$$\sqrt{m|\tilde{n}|} \sum_{r_j \geq \max(\frac{a}{x}, 1)} \frac{\rho_j(m)\rho_j(n)}{\text{ch}(\pi r_j)} \Phi(r_j) \ll \epsilon \left( \sqrt{\frac{x}{T}} + |mnt|^{\frac{1}{4}+\epsilon}|m_0n_0|^{\frac{\theta+\epsilon}{2}} \right) \log^2 x$$

$$\ll \epsilon \left( \sqrt{\frac{x}{T}} + |mnt|^{\frac{1}{4}+\frac{\theta}{4}+\epsilon} \right) \log^2 x. \quad (11.7)$$

We now consider the range $\max \left( \frac{a}{8x}, 1 \right) \leq r_j \leq \frac{a}{x}$. Lemma 6.1 ensures that we can replace $\min \left( \sqrt{A}, \frac{x}{T\sqrt{A}} \right)$ in (11.5) and (11.6) with $A$. Recalling that $x \geq |\tilde{m}\tilde{n}|^{\frac{1}{4}}$, we have $a/x \leq 4\pi |\tilde{m}\tilde{n}|^{\frac{1}{8}}$. Summing over all dyadic intervals in this range and ignoring the smallest terms we obtain

$$\sqrt{m|\tilde{n}|} \sum_{max(\frac{a}{8x}, 1) < r_j \leq \frac{a}{x}} \frac{\rho_j(m)\rho_j(n)}{\text{ch}(\pi r_j)} \Phi(r_j) \ll \epsilon |mn|^{\frac{1}{4}+\epsilon} \left( m^{\frac{1}{8}} + |n|^{\frac{1}{8}} \right) + |mnt|^{\frac{1}{8}+\frac{\theta}{4}+\epsilon}. \quad (11.8)$$

By a similar argument, the same bound in (11.8) is also valid over the range $1 \leq r_j \leq a/8x$. Combining (11.7) and (11.8) we obtain

$$W \ll \epsilon \left( \sqrt{\frac{x}{T}} + |mn|^{\frac{1}{16}} \left( m^{\frac{1}{8}} + |n|^{\frac{1}{8}} \right) + |mnt|^{\frac{1}{8}+\frac{\theta}{4}} \right) |mn| \epsilon \log^2 x. \quad (11.9)$$

To balance the $x$-aspect of (9.2) and (11.9) we choose $T := x^{\frac{2}{3}}$, and the Proposition follows.
12. Proof of Theorem 1.4

Proposition 12.1. Let \( m, n > 0 \) be such that \( m - 1 \not\in \mathcal{P} \) or \( n - 1 \not\in \mathcal{P} \). Suppose \( m_0, n_0 \) be integers such that \( 24m - 23 = m_0^2s \) and \( 24n - 23 = n_0^2t \) with \( s \) and \( t \) square-free. If \( x \geq (st)^{1/3}(mn)^{1/3} \), then under the \( H_0 \)-hypothesis we have

\[
\sum_{x \leq c \leq 2x} \frac{S(m,n,c,\chi)}{c} \ll_{\varepsilon} x^{1/2} + (st)^{1/4} + (mn)^{1/6} + (mn)^{1/12} \log x.
\]

As before, Proposition 12.1 implies Theorem 1.4 with the choice of initial segment \( 1 \leq c \leq (st)^{1/3}(mn)^{1/3} \).

Proof of Proposition 12.1. First recall (5.5) and (9.2). Note that there is no contribution from \( r_0 = i/4 \) by Remark 4.1 and the hypothesis on \( m \) and \( n \). By Remark 4.1 it is sufficient to consider \( r_j \geq 1 \). In view of Lemma 7.1, we consider the ranges

\[
1 \leq r_j \leq a/x, \quad r_j \geq \max(a/x,1).
\]

In the latter range we establish (11.5) and (11.6) with \( \hat{\Phi} \) replaced with \( \hat{\phi} \). Thus

\[
\sqrt{mn} \left| \sum_{r_j \geq \max(a/x,1)} \frac{\rho_j(m)}{\rho_j(n)} \frac{\hat{\phi}(r_j)}{\text{ch}(\pi r_j)} \right| \ll_{\varepsilon} \left( \sqrt{\frac{x}{T}} + (mn)^{1/6} \log^2 x \right). \tag{12.1}
\]

When \( r_j \leq a/x \leq 4\pi(mn)^{1/2} \), Lemma 5.2 and the fact that \( r_j > 1.9 \) guarantees that we can replace \( \min(\sqrt{A},x/T\sqrt{A}) \) in (11.5) and (11.6) with \( \sqrt{A} \). Thus

\[
\sqrt{mn} \left| \sum_{0 < r_j \leq 4\pi(mn)^{1/2}} \frac{\rho_j(m)}{\rho_j(n)} \frac{\hat{\phi}(r_j)}{\text{ch}(\pi r_j)} \right| \ll_{\varepsilon} (mn)^{1/6} + \varepsilon. \tag{12.2}
\]

Combining (12.1) and (12.2) we have

\[
\mathcal{V} \ll_{\varepsilon} \left( \sqrt{\frac{x}{T}} + (mn)^{1/6} \right)(mn)^{\varepsilon} \log^2 x. \tag{12.3}
\]

We choose \( T := x^{2/3} \) to balance (9.2) and (12.3). When \( x \geq (st)^{1/3}(mn)^{1/3} \), Lemma 7.1 gives

\[
\mathcal{U} \ll_{\varepsilon} (st)^{1/3} + (st)^{1/2} + (mn)^{1/3} + \varepsilon. \tag{12.4}
\]

Combining (9.2), (12.3) and (12.4) finishes the proof.

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