Frequency-limited Pseudo-optimal Model Order Reduction for Bilinear Systems

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Abstract—In this short note, we propose a simple model reduction approach, which satisfies a subset of the first-order optimality conditions for $H_2$-model reduction problems of the bilinear system. We give the detailed proofs of our results.

Index Terms—Pseudo-optimal.

I. INTRODUCTION

Consider a bilinear control system $\Sigma$ with the following state and output equations

$$
\Sigma : \begin{cases}
\dot{x}(t) = Ax(t) + \sum_{k=1}^{m} N_k x(t) u_k(t) + Bu(t) \\
y(t) = Cx(t)
\end{cases}
$$

(1)

where $A \in \mathbb{R}^{n \times n}$, $N_k \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{p \times n}$. $x(t)$, $u(t)$, $y(t)$ are states, control inputs, and outputs, respectively. The MOR problem under consideration is to obtain a ROM $\Sigma_r$ of $\Sigma$ such that $\Sigma_r$ accurately mimics $\Sigma$ when used as a surrogate. Let $\Sigma_r$ be represented by the following state and output equations

$$
\Sigma_r : \begin{cases}
\dot{x}_r(t) = A_r x_r(t) + \sum_{k=1}^{m} N_{rk} x_r(t) u_k(t) + B_r u(t) \\
y_r(t) = C_r x_r(t)
\end{cases}
$$

(2)

where $A_r \in \mathbb{R}^{r \times r}$, $N_{rk} \in \mathbb{R}^{r \times r}$, $B_r \in \mathbb{R}^{r \times m}$, and $C_r \in \mathbb{R}^{p \times r}$ such that $r << n$.

The controllability gramian $P_r$ and the observability gramian $Q_r$ of (2) are the solutions of the following generalized Lyapunov equations

$$
AP + PA^T + \sum_{k=1}^{m} N_{rk} P_{rk} N_{rk}^T + BB^T = 0
$$

(3)

$$
A^T Q_r + Q_r A_r + \sum_{k=1}^{m} N_{rk}^T Q_{rk} N_{rk} + C^T C_r = 0.
$$

(4)

Similarly, the controllability gramian $P_r$ and the observability gramian $Q_r$ of (2) are the solutions of the following generalized Lyapunov equations

$$
A_r P_r + P_r A_r^T + \sum_{k=1}^{m} N_{rk} P_{rk} N_{rk}^T + B_r B_r^T = 0
$$

(5)

$$
A_r^T Q_r + Q_r A_r + \sum_{k=1}^{m} N_{rk}^T Q_{rk} N_{rk} + C_r^T C_r = 0.
$$

(6)

The $H_2$-norm of $\Sigma - \Sigma_r$ is defined as

$$
||\Sigma - \Sigma_r||_{H_2}^2 = \text{tr}(CPCT) + \text{tr}(C_r P_r C_r^T) - 2\text{tr}(C \hat{P} C^T) + \text{tr}(B_r Q_r B_r) + 2\text{tr}(B_r^T \hat{Q} B)
$$

where $\hat{P}$ and $\hat{Q}$ solve the following generalized Sylvester equations (and $\text{tr}(\cdot)$ represents the trace)

$$
A \hat{P} + \hat{P} A + \sum_{k=1}^{m} N_k \hat{P} N_{rk}^T + BB^T = 0
$$

(7)

$$
A^T \hat{Q} + \hat{Q} A_r + \sum_{k=1}^{m} N_{rk}^T \hat{Q} N_{rk} + C^T C_r = 0.
$$

(8)

In some applications, it is desired to ensure that $\Sigma_r$ exhibits a superior accuracy within the desired frequency interval. Such a MOR problem is called the frequency-limited MOR problem. The MOR techniques for the frequency-limited MOR often use frequency-limited gramians. The frequency-limited gramians of (1) within the desired frequency interval $[0, \omega]$ are the solutions of the following generalized Lyapunov equations

$$
A P_\omega + P_\omega A^T + F_\omega (A) B B^T + F_\omega (A) \left( \sum_{k=1}^{m} N_k P_\omega N_k^T \right) + B B^T F_\omega (A)^T
$$

$$
+ \left( \sum_{k=1}^{m} N_k P_\omega N_k^T \right) F_\omega (A) = 0
$$

(9)

$$
A^T Q_\omega + Q_\omega A + F_\omega (A)^T C^T C + C^T F_\omega (A) + F_\omega (A)^T \left( \sum_{k=1}^{m} N_k^T Q_\omega N_k \right)
$$

$$
+ \left( \sum_{k=1}^{m} N_k^T Q_\omega N_k \right) F_\omega (A) = 0
$$

(10)

where

$$
F_\omega (A) = \text{Re} \left( \frac{i}{\pi} \ln(-j\omega I - A) \right).
$$

(11)
Similarly, the frequency-limited gramians of \( \Sigma \) within the desired frequency interval \([0, \omega]\) are the solutions of the following generalized Lyapunov equations

\[
A_r \hat{P}_r + \hat{P}_r A_r^T + F_\omega(A_r)B_r B_r^T + \sum_{k=1}^{m} N_{rk} \hat{P}_r N_{rk}^T + B_r B_r^T P_\omega(A_r)^T \\
+ \left( \sum_{k=1}^{m} N_{rk} \hat{P}_r N_{rk}^T \right) P_\omega(A_r)^T = 0
\]  

(12)

\[
A_r^T \hat{Q}_r + \hat{Q}_r A_r - F_\omega(A_r)^T C_r^T C_r \\
+ C_r^T C_r F_\omega(A_r) + F_\omega(A_r)^T \left( \sum_{k=1}^{m} N_{rk}^T \hat{Q}_r N_{rk} \right) \\
+ \left( \sum_{k=1}^{m} N_{rk}^T \hat{Q}_r N_{rk} \right) F_\omega(A_r) = 0.
\]  

(13)

The frequency-limited \( H_2 \)-norm of \( \Sigma - \Sigma_r \) is defined as

\[
\| \Sigma - \Sigma_r \|_{H_2, \omega}^2 = tr(CP_\omega C_r^T) + tr(Cr \hat{P}_r C_r^T) - 2tr(C \hat{P}_r C_r^T) = tr(B^T Q_r B) + tr(B_r^T \hat{Q}_r B_r) + 2tr(B_r^T \hat{Q}_r B)
\]

where \( \hat{P}_r \) and \( \hat{Q}_r \) solve the following generalized Sylvester equations

\[
A \hat{P}_r + \hat{P}_r A^T + F_\omega(A)B B^T + \sum_{k=1}^{m} N_{rk} \hat{P}_r N_{rk}^T + B B^T P_\omega(A)^T \\
+ \left( \sum_{k=1}^{m} N_{rk} \hat{P}_r N_{rk}^T \right) P_\omega(A)^T = 0
\]  

(14)

\[
A^T \hat{Q}_r + \hat{Q}_r A - F_\omega(A)^T C^T C_r \\
- C^T C_r F_\omega(A) + F_\omega(A)^T \left( \sum_{k=1}^{m} N_{rk}^T \hat{Q}_r N_{rk} \right) \\
+ \left( \sum_{k=1}^{m} N_{rk}^T \hat{Q}_r N_{rk} \right) F_\omega(A) = 0.
\]  

(15)

A. \( H_{2, \omega} \)-optimal MOR \([1]\)

In \([1]\), the first-order optimality conditions for the \( H_{2, \omega} \)-MOR problem are derived. Let \( \hat{R} \) and \( \hat{Q} \) solve the following generalized Sylvester equations

\[
A^T \hat{R} + \hat{R} A_r + \sum_{k=1}^{m} N_{rk}^T F_\omega(A) \hat{R} N_{rk} \\
+ \sum_{k=1}^{m} N_{rk}^T R F_\omega(A_r) N_{rk} - C^T C_r = 0
\]  

(16)

\[
A_r^T \hat{Q} + \hat{Q} A_r + \sum_{k=1}^{m} N_{rk}^T \hat{Q} F_\omega(A_r) N_{rk} \\
+ \sum_{k=1}^{m} N_{rk}^T F_\omega(A_r)^T \hat{Q} N_{rk} + C_r^T C_r = 0.
\]  

(17)

Let us define \( S_1, S_2, \) and \( \hat{W}_i \) as

\[
S_1 = \sum_{k=1}^{m} N_{rk} \hat{P}_r N_{rk}^T \hat{Q} + B_r B_r^T \hat{Q}
\]  

(18)

\[
S_2 = \sum_{k=1}^{m} N_{rk} \hat{P}_r N_{rk}^T R + B_r B_r^T R
\]  

(19)

\[
\hat{W}_i = Re \left[ \frac{1}{\pi} L(-A_r - j\omega I, S_i) \right]
\]  

(20)

where \( L(\cdot, \cdot) \) represents the Frechét derivative of the matrix logarithm. The ROM is a local optimum for the problem \( \| \Sigma - \Sigma_r \|_{H_{2, \omega}} \) if it satisfies the following conditions

\[
C_r \hat{P}_r = C \hat{P}_r,
\]  

(21)

\[
\hat{Q}_r B_r = -\hat{Q}_r B,
\]  

(22)

\[
R^T \hat{P}_r + \hat{P}_r^T \hat{W}_i = W_1^T + W_2^T,
\]  

(23)

\[
\sum_{k=1}^{m} \left( R^T F_\omega(A) N_{rk} \hat{P}_r \\
+ F_\omega(A_r)^T R N_{rk} \hat{Q}_r + F_\omega(A_r)^T \hat{Q} N_{rk} \hat{P}_r \right) = 0.
\]  

(24)

In \([1]\), an iterative algorithm is proposed, which tends to (approximately) achieve the optimality conditions \([21]\) - \([24]\). Let \( V_r \) and \( W_r \) are obtained by solving the following generalized Sylvester equations

\[
A V_r + V_r A_r^T + F_\omega(A) \left( \sum_{k=1}^{m} N_{rk} V_r N_{rk}^T \right) \\
+ \left( \sum_{k=1}^{m} N_{rk} V_r N_{rk}^T \right) F_\omega(A_r)^T = 0
\]  

(25)

\[
A^T W_r + W_r A_r + F_\omega(A)^T \left( \sum_{k=1}^{m} N_{rk}^T W_r N_{rk} \right) \\
+ \left( \sum_{k=1}^{m} N_{rk}^T W_r N_{rk} \right) F_\omega(A_r) = 0.
\]  

(26)

Then, the interim ROM is updated as

\[
\hat{A}_r = (W_r^T V_r)^{-1} W_r^T A V_r, \quad \hat{B}_r = (W_r^T V_r)^{-1} W_r^T B,
\]  

(27)

\[
\hat{N}_{rk} = (W_r^T V_r)^{-1} W_r^T N V_r, \quad \hat{C}_r = C V_r.
\]  

(28)

\( V_r \) and \( W_r \) are recomputed and the process continues until the relative change in the eigenvalues of \( \hat{A}_r \) stagnates.

II. MAIN WORK

In this section, we propose iteration-free algorithms, which can achieve a subset of the first-order optimality conditions for the problem \( \| \Sigma - \Sigma_r \|_{H_{2, \omega}} \) in a single run. The proposed algorithms only require an initial guess of the ROM for their execution. We basically generalize the ideas of the following references \([2], [3], [4]\), which consider the linear systems, for frequency-limited case of bilinear systems.
A. \(\mathcal{H}_{2,\omega}\)-pseudo-optimal MOR

Let \(V_r\) is computed by solving equation (25), and \(\tilde{P}_\omega\) solves the following generalized Lyapunov equation

\[
\tilde{A}_r \tilde{P}_\omega + \tilde{P}_\omega \tilde{A}_r^T + F_\omega(\tilde{A}_r) \tilde{B}_r \tilde{B}_r^T \\
+ F_\omega(\tilde{A}_r) \left( \sum_{k=1}^{m} \tilde{N}_{rk} \tilde{P}_\omega \tilde{N}_{rk}^T \right) + \tilde{B}_r \tilde{B}_r^T F_\omega(\tilde{A}_r)^T \\
+ \left( \sum_{k=1}^{m} \tilde{N}_{rk} \tilde{P}_\omega \tilde{N}_{rk}^T \right) F_\omega(\tilde{A}_r)^T = 0.
\]

(29)

Let the ROM is obtained as the following

\[
A_r = \tilde{P}_\omega^{-1} \tilde{A}_r \tilde{P}_\omega, \quad B_r = \tilde{P}_\omega^{-1} \tilde{B}_r, \\
N_{rk} = \tilde{P}_\omega^{-1} \tilde{N}_{rk} \tilde{P}_\omega, \quad C_r = CV_r.
\]

(30)

(31)

**Theorem 1:** If \(\Sigma_r\) is computed as in equations (30)-(31), the following statements hold

(i) \(\tilde{P}_\omega^{-1}\) is the frequency-limited controllability gramian \(\tilde{P}_\omega\) of \((A_r, N_{rk}, B_r)\), i.e. \(\tilde{P}_\omega = \tilde{P}_\omega^{-1}\).

(ii) \(C_r \tilde{P}_\omega = C \tilde{P}_\omega\).

(iii) \(||\Sigma - \Sigma_r||^2_{H_{2,\omega}} = tr(CP \omega C^T) - tr(C_r \tilde{P}_\omega C_r^T)\).

**Proof:**

(i) By pre- and post-multiplying equation (31) with \(\tilde{P}_\omega\), we get

\[
\tilde{P}_\omega A_r \tilde{P}_\omega + \tilde{P}_\omega \tilde{P}_\omega A_r^T \tilde{P}_\omega + \tilde{P}_\omega F_\omega(\tilde{A}_r) \tilde{B}_r \tilde{B}_r^T \tilde{P}_\omega \\
+ \tilde{P}_\omega F_\omega(\tilde{A}_r) \left( \sum_{k=1}^{m} N_{rk} \tilde{P}_\omega \tilde{N}_{rk}^T \tilde{P}_\omega \right) + \tilde{P}_\omega \tilde{B}_r \tilde{B}_r^T F_\omega(\tilde{A}_r)^T \tilde{P}_\omega \\
+ \left( \sum_{k=1}^{m} \tilde{P}_\omega N_{rk} \tilde{P}_\omega \tilde{N}_{rk}^T \tilde{P}_\omega \right) F_\omega(\tilde{A}_r)^T \tilde{P}_\omega = 0.
\]

By putting the value of \(A_r, N_{rk}, \) and \(B_r\) according to equations (30)-(31), and also by noting \(\tilde{P}_\omega F_\omega(\tilde{A}_r) \tilde{P}_\omega^{-1} = F_\omega(\tilde{A}_r)\) yield the following

\[
\tilde{A}_r \tilde{P}_\omega \tilde{P}_\omega \tilde{A}_r + \tilde{P}_\omega \tilde{P}_\omega \tilde{A}_r^T \tilde{P}_\omega + \tilde{P}_\omega F_\omega(\tilde{A}_r) \tilde{B}_r \tilde{B}_r^T \tilde{P}_\omega \\
+ \tilde{P}_\omega F_\omega(\tilde{A}_r) \left( \sum_{k=1}^{m} N_{rk} \tilde{P}_\omega \tilde{N}_{rk}^T \tilde{P}_\omega \right) + \tilde{P}_\omega \tilde{B}_r \tilde{B}_r^T F_\omega(\tilde{A}_r)^T \tilde{P}_\omega \\
+ \left( \sum_{k=1}^{m} \tilde{P}_\omega N_{rk} \tilde{P}_\omega \tilde{N}_{rk}^T \tilde{P}_\omega \right) F_\omega(\tilde{A}_r)^T \tilde{P}_\omega = 0.
\]

(ii) Consider the following

\[
AV_r \tilde{P}_\omega + V_r \tilde{P}_\omega A_r^T + F_\omega(\tilde{A}_r) \tilde{B}_r \tilde{B}_r^T \tilde{P}_\omega \\
+ F_\omega(\tilde{A}_r) \left( \sum_{k=1}^{m} N_{rk} \tilde{P}_\omega \tilde{N}_{rk}^T \tilde{P}_\omega \right) + \tilde{B}_r \tilde{B}_r^T F_\omega(\tilde{A}_r)^T \tilde{P}_\omega \\
= \left[ -V_r \tilde{A}_r^T - F_\omega(\tilde{A}_r) \left( \sum_{k=1}^{m} N_{rk} \tilde{P}_\omega \tilde{N}_{rk}^T \tilde{P}_\omega \right) \\
- \left( \sum_{k=1}^{m} N_{rk} \tilde{P}_\omega \tilde{N}_{rk}^T \tilde{P}_\omega \right) F_\omega(\tilde{A}_r)^T \tilde{P}_\omega \right. \\
+ F_\omega(\tilde{A}_r) \tilde{B}_r \tilde{B}_r^T \tilde{P}_\omega + \tilde{B}_r \tilde{B}_r^T F_\omega(\tilde{A}_r)^T \tilde{P}_\omega = 0.
\]

Due to uniqueness, \(V_r \tilde{P}_\omega = \tilde{P}_\omega, \quad CV_r \tilde{P}_\omega = C \tilde{P}_\omega, \) and \(C_r \tilde{P}_\omega = C \tilde{P}_\omega\).

(iii) It directly follows from (ii).

A ROM \(\Sigma_r\) which ensures (23) can also be obtained using \(W_r\). Let \(W_r\) is computed by solving equation (26), and let \(Q_\omega\) solves the following generalized Lyapunov equation

\[
\tilde{A}_r^T Q_\omega + Q_\omega \tilde{A}_r + F_\omega(\tilde{A}_r)^T \tilde{C}_r^T \tilde{C}_r \\
+ \tilde{C}_r^T \tilde{C}_r F_\omega(\tilde{A}_r) + F_\omega(\tilde{A}_r)^T \left( \sum_{k=1}^{m} \tilde{N}_{rk}^T Q_\omega \tilde{N}_{rk} \right) + \left( \sum_{k=1}^{m} \tilde{N}_{rk}^T Q_\omega \tilde{N}_{rk} \right) F_\omega(\tilde{A}_r)^T = 0.
\]

(32)

Let the ROM is obtained as the following

\[
A_r = \tilde{Q}_\omega \tilde{A}_r \tilde{Q}_\omega^{-1}, \quad B_r = W_r^T B, \\
N_{rk} = \tilde{Q}_\omega \tilde{N}_{rk} \tilde{Q}_\omega^{-1}, \quad C_r = -\tilde{C}_r \tilde{Q}_\omega^{-1}.
\]

(33)

(34)

**Theorem 2:** If \(\Sigma_r\) is computed as in equations (33)-(34), the following statements hold

(i) \(\tilde{Q}_\omega^{-1}\) is the frequency-limited observability gramian \(\tilde{Q}_\omega\) of \((A_r, N_{rk}, C_r)\), i.e. \(\tilde{Q}_\omega = \tilde{Q}_\omega^{-1}\).

(ii) \(\tilde{Q}_\omega B_r = -\tilde{Q}_\omega B_r\).

(iii) \(||\Sigma - \Sigma_r||^2_{H_{2,\omega}} = tr(B \tilde{Q}_\omega B) - tr(B_r \tilde{Q}_\omega B_r)\).

**Proof:**

(i) By pre- and post-multiplying equation (13)
with $\tilde{Q}_\omega$, we get
\[
\begin{align*}
\tilde{Q}_\omega A_r^T \tilde{Q}_\omega \tilde{Q}_\omega + \tilde{Q}_\omega \tilde{Q}_\omega A_r \tilde{Q}_\omega + \tilde{Q}_\omega F_\omega (A_r)^T C_r^T C_r \tilde{Q}_\omega \\
+ Q_\omega C_r^T C_r F_\omega (A_r) \tilde{Q}_\omega + Q_\omega F_\omega (A_r)^T \left( \sum_{k=1}^m N_{rk}^T \tilde{Q}_\omega \tilde{N}_{rk} \right) \tilde{Q}_\omega \\
+ \tilde{Q}_\omega \left( \sum_{k=1}^m N_{rk}^T \tilde{Q}_\omega \tilde{N}_{rk} \right) F_\omega (A_r) \tilde{Q}_\omega = 0
\end{align*}
\]
By putting the value of $A_r$, $N_{rk}$, $C_r$, and also by noting that $\tilde{Q}_\omega F_\omega (A_r)^T \tilde{Q}_\omega^{-1} = F_\omega (\tilde{A}_r)^T$, we get
\[
\begin{align*}
\bar{A}_r^T \tilde{Q}_\omega \tilde{Q}_\omega \bar{Q}_\omega + \tilde{Q}_\omega \tilde{Q}_\omega \bar{A}_r + F_\omega (\tilde{A}_r)^T C_r^T C_r \\
+ C_r^T C_r F_\omega (\tilde{A}_r) + F_\omega (\tilde{A}_r)^T \left( \sum_{k=1}^m \tilde{N}_{rk} \tilde{Q}_\omega \tilde{Q}_\omega \tilde{N}_{rk} \right) \\
+ \left( \sum_{k=1}^m \tilde{N}_{rk} \tilde{Q}_\omega \tilde{Q}_\omega \tilde{N}_{rk} \right) F_\omega (\tilde{A}_r) = 0. \quad (35)
\end{align*}
\]
Due to uniqueness, $\tilde{Q}_\omega \tilde{Q}_\omega \tilde{Q}_\omega = \tilde{Q}_\omega$, $\tilde{Q}_\omega \tilde{Q}_\omega = I$, and $\tilde{Q}_\omega = \tilde{Q}_\omega^{-1}$.

(ii) Consider the following

\[
\begin{align*}
-A_r^T \tilde{Q}_\omega W_r^T - \tilde{Q}_\omega W_r^T A - F_\omega (A_r)^T \left( \sum_{k=1}^m N_{rk}^T \tilde{Q}_\omega W_r^T N_k \right) \\
- F_\omega (A_r)^T C_r^T C - \left( \sum_{k=1}^m N_{rk}^T \tilde{Q}_\omega W_r^T N_k \right) F_\omega (A) - C_r^T C F_\omega (A) \\
= - \tilde{Q}_\omega^{-1} A_r^T \tilde{Q}_\omega W_r^T - \tilde{Q}_\omega \left[ - A_r^T W_r^T + C_r^T C F_\omega (A) \\
- \left( \sum_{k=1}^m N_{rk} W_r^T N_k \right) F_\omega (A) + F_\omega (\tilde{A}_r)^T C_r^T C \right] \\
- F_\omega (A_r)^T \left( \sum_{k=1}^m N_{rk} W_r^T N_k \right) \\
- \tilde{Q}_\omega^{-1} F_\omega (A_r)^T \left( \sum_{k=1}^m N_{rk} W_r^T N_k \right) + Q_\omega F_\omega (\tilde{A}_r)^T C_r^T C \\
- \tilde{Q}_\omega^{-1} \left( \sum_{k=1}^m N_{rk} W_r^T N_k \right) F_\omega (A) + \tilde{Q}_\omega^{-1} C_r^T C F_\omega (A) \\
= 0.
\end{align*}
\]
Due to uniqueness, $- \tilde{Q}_\omega W_r^T = \hat{Q}_\omega^T$, $\tilde{Q}_\omega W_r^T B = - \hat{Q}_\omega^T B$, and $\tilde{Q}_\omega B_r = - \hat{Q}_\omega B$.

(iii) It directly follows from (ii).

III. Conclusion

In this short note, we propose iteration-free algorithms which generate ROMs that satisfy a subset of the first-order optimality conditions for the frequency-limited MOR problem. We have proved that the ROM models are the pseudo-optimal ROMs of the original system.

REFERENCES

[1] K.-L. Xu and Y.-L. Jiang, “An approach to $h_\infty$ model reduction on finite interval for bilinear systems,” Journal of the Franklin Institute, vol. 354, no. 16, pp. 7429–7443, 2017.

[2] T. Wolf, “H 2 pseudo-optimal model order reduction,” Ph.D. dissertation, Technische Universität München, 2014.

[3] C. Varona et al., “On the $h_2$-pseudo-optimal bilinear model reduction,” in Applied Numerical Analysis Seminar, Virginia Tech, Blacksburg (VA), USA, 2017.

[4] U. Zaliznyak, V. Sreeram, and X. Du, “Frequency-limited pseudo-optimal rational krylov algorithm for power system reduction,” International Journal of Electrical Power & Energy Systems, vol. 118, p. 105798, 2020.