CONVERGENCE AND STRONG SUMMABILITY OF THE TWO–DIMENSIONAL VILENKIN-FOURIER SERIES

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Abstract. In this paper we investigate convergence and strong summability of the two-
dimensional Vilenkin-Fourier series in the martingale Hardy spaces.

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1. Introduction

The definitions and notations used in this introduction can be found in our next Section.

It is known [16, p. 125] that the two-dimensional Vilenkin systems are not Schauder bases
in $L_1(G_m^2)$. Moreover, (see [1] and [25]) there exists a function $f \in H_p^\square(G_m^2)$,
such that the corresponding partial sums are not bounded in $L_p(G_m^2)$, for all $0 < p \leq 1$.

However, Weisz [47] proved that if $\alpha \geq 0$, $0 < p \leq 1$ and $f \in H_p(G_m^2)$, then there exists
an absolute constant $c_p$, depending only on $p$, such that

$$\sup_{n,m \geq 2} \left( \frac{1}{\log n \log m} \right)^{[p]} \sum_{2^{-\alpha} \leq k/l \leq 2^{\alpha}, (k,l) \leq (n,m)} \frac{\|S_{k,l}f\|_p^p}{(kl)^{2-p}} \leq c_p \|f\|_{H_p^\square(G_m^2)},$$

where $[p]$ denotes the integer part of $p$. Moreover, in [36] it was proved that the rate of
sequence $(kl)^{2-p}$ ($0 < p < 1$) in inequality (1) can not be improved, which gives sharpness
for $\alpha > 0$.

In the case when $\alpha = 0$ and $0 < p \leq 1$ it follows that if $f \in H_p(G_m^2)$, then there exists
an absolute constant $c$, such that

$$\sup_{n \geq 2} \frac{1}{\log n^{2|p|}} \sum_{k=0}^{n} \frac{\|S_{k,k}f\|_p^p}{k^{4-2p}} \leq c \|f\|_{H_p^\square(G_m^2)},$$

For the two-dimensional Walsh-Fourier series Goginava and Gogoladze [14] for $p = 1$ and
Tephnadze [32] for $0 < p < 1$ generalized inequality (2) and proved that there exists an
absolute constant $c_p$, depending only on $p$, such that

$$\sum_{n=1}^{\infty} \frac{\|S_{n,n}f\|_p}{n^{3-2p} \log^{2|p|} n} \leq c_p \|f\|_{H_p^\square(G_m^2)},$$

for all $f \in H_p^\square(G_m^2)$, where $[p]$ denotes integer part of $p$. Moreover, in [32] and [33] there
were proved that the sequence $\{1/k^{3-2p} \log^{2|p|} (k+1) : k \in \mathbb{N}\}$ in inequality (3) is sharp.

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Strong summability of the one and two-dimensional Vilenkin-Fourier (Walsh-Fourier) series can be found in Baramidze, Persson, Tephnadze and Wall [2], Blahota [4], [7], Belinskii [3], Gát [13], Goginava and Gogoladze [15], Smith [29], Simon [27, 28], Blahota and Tephnadze [5], [6], Tephnadze [35], Tutberidze [41].

Concerning approximation properties of Fourier series in the classical and real Hardy spaces only few results are known. We refer to the papers by Oswald [23], Kryakin and Trebels [17], Storoienko [30], [31]. For martingale Hardy spaces approximation properties of some general summability methods were investigated in Fridli, Manchanda and Siddiqi [12] (see also [10], [11]), Tephnadze [34], [37], [38], [39], Nagy [19], [20], [21], Weiss [43], [44]. In [24] it was proved that if $0 < p < 1$, $2^{-\alpha} \leq m/n \leq 2^\alpha$ and

$$
\omega_{H^p_{\bar{\mu}}(G^2_m)}(\frac{1}{2^k}, f) = o\left(\frac{1}{2k(2/p-2)}\right), \text{ as } k \to \infty,
$$

then

$$
\|S_{m,n} f - f\|_{H^p_{\bar{\mu}}(G^2_m)} \to 0, \text{ as } m, n \to \infty.
$$

Moreover, there exists a martingale $f \in H^p(G^2_m)$, such that

$$
\omega_{H^p_{\bar{\mu}}(G^2_m)}(\frac{1}{2^k}, f) = O\left(\frac{1}{2k(2/p-2)}\right), \text{ as } k \to \infty
$$

and

$$
\|S_{m,n} f - f\|_{w_{\text{weak}}-L^p(G^2_m)} \not\to 0 \text{ as } m, n \to \infty.
$$

$0 < p < 1$ and $2^{-\alpha} < m/n \leq 2^\alpha$.

The main aim of this paper is to generalize inequality (1) for bounded Vilenkin systems. We also prove that the sequence $(kl)^{2-p}$ in inequality (1) can not be improved. Moreover, we find necessary and sufficient conditions for modulus of continuity of the two-dimensional Vilenkin-Fourier series, which provide convergence of partial sums in $H^p_{\bar{\mu}}(G^2_m)$ norm.

This paper is organized as follows: in order not to disturb our discussions later on some definitions and notations and some important propositions are presented in Section 2. The main results with proofs can be found in Section 3. Moreover, in Section 4 we will state some interesting open problems and conjectures of this research area.

### 2. Preliminary

Denote by $\mathbb{N}_+$ the set of positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Let $m := (m_0, m_1, \ldots)$ be a sequence of positive integers not less than 2. Denote by $Z_{m_k} := \{0, 1, \ldots, m_k - 1\}$ the additive group of integers modulo $m_k$.

Define the group $G_m$ as the complete direct product of the groups $Z_{m_i}$ with the product of the discrete topologies of $Z_{m_i}$’s.

In this paper we discuss bounded Vilenkin groups, i.e. in the case when $\sup_{n \in \mathbb{N}} m_n < \infty$.

The direct product $\mu$ of the measures

$$
\mu_k(\{j\}) := 1/m_k, \ (j \in Z_{m_k})
$$

is the Haar measure on $G_m$ with $\mu(G_m) = 1$.

The elements of $G_m$ are represented by sequences

$$x := (x_0, x_1, \ldots, x_j, \ldots), \ (x_j \in Z_{m_j}).$$
It is easy to give a base of neighbourhoods of $G_m$:

$$I_0(x) := G_m, \quad I_n(x) := \{y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\} \quad (x \in G_m, \ n \in \mathbb{N}).$$

Denote $I_n := I_n(0)$, for $n \in \mathbb{N}_+$ and $e_n := (0, \ldots, 0, x_n = 1, 0, \ldots) \in G_m, \ (n \in \mathbb{N}).$

It is evident that

$$\bigcup_{s=0}^{N-1} I_s \setminus I_{s+1} = N \setminus \bigcup_{s=0}^{N-1} I_s \setminus I_{s+1}. \quad (1)$$

If we define the so-called generalized number system based on $m$ in the following way:

$$M_0 := 1, \quad M_{k+1} := m_k M_k \quad (k \in \mathbb{N}),$$

then every $n \in \mathbb{N}$ can be uniquely expressed as

$$n = \sum_{j=0}^{\infty} n_j M_j, \quad \text{where} \quad n_j \in Z_{m_j} \quad (j \in \mathbb{N})$$

and only a finite number of $n_j$’s differ from zero. Let $|n|$ denote the largest integer $j$ for which $n_j \neq 0$.

Denote by $\mathbb{N}_{n_0}$ the subset of positive integers $\mathbb{N}_+$, for which $n_0 = 1$. Then for every $n \in \mathbb{N}_{n_0}$, $M_k < n < M_{k+1}$ can be written as

$$n = 1 + \sum_{j=1}^{k} n_j M_j,$$

where $n_j \in \{0, \ldots, m_j - 1\}, \ (j = 1, \ldots, k - 1)$ and $n_k \in \{1, \ldots, m_k - 1\}$.

For any $\alpha > 0$ we get that

$$\sum_{\{n:M_k \leq n \leq 2^\alpha M_k, \ n \in \mathbb{N}_{n_0}\}} 1 = \frac{2^\alpha M_k - M_k}{m_0} \geq c M_k,$$

where $c$ is an absolute constant.

The norms (quasi-norm) of the spaces $L_p(G_m^2)$ and weak $L_p(G_m^2)$ are respectively defined by

$$\|f\|_p := \left(\int_{G_m^2} |f|^p \, d\mu \times d\mu\right)^{1/p}, \quad \|f\|_{\text{weak}-L_p} := \sup_{\lambda > 0} \lambda \mu(f > \lambda)^{1/p} \quad (0 < p < \infty).$$

Next, we introduce on $G_m^2$ an orthonormal system which is called the Vilenkin system. At first, we define the complex-valued function $r_k(x) : G_m \to \mathbb{C}$, the generalized Rademacher functions by

$$r_k(x) := \exp(2\pi i x_k/m_k), \quad (i^2 = -1, \ x \in G_m, \ k \in \mathbb{N}).$$

Now, define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on $G_m$ as:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_n^{n_k}(x), \quad (n \in \mathbb{N}).$$

We define the two-dimensional Vilenkin system as a kronecker product of two Vilenkin systems. The Vilenkin system is orthonormal and complete in $L_2(G_m^2)$ (for details see [1] and [12]). Specifically, we call this system the Walsh-Paley system, when $m \equiv 2$. 
The rectangular partial sum of the 2-dimensional Vilenkin-Fourier series of function \( f \in L_2(G_n^2) \) is defined as follows:

\[
S_{M,N} f (x, y) := \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \hat{f}(i, j) \psi_i(x) \psi_j(y),
\]

where the numbers

\[
\hat{f}(i, j) = \int_{G_n^2} f(x, y) \psi_i(x) \psi_j(y) \, d\mu(x) \, d\mu(y)
\]
is said to be the \((i, j)\)-th Vilenkin-Fourier coefficient of the function \( f \).

It is well-known that (for details see e.g. [25])

\[
S_{M,N} f (x, y) = \int_{G_n^2} f(x, y) D_M(x - t) D_N(y - s) \, d\mu(x) \, d\mu(y),
\]

where

\[
D_n(x) = \sum_{i=0}^{n-1} \psi_i(x)
\]
is called as \( n \)-th Dirichlet Kernel. Recall that

\[
D_M_n(x) = \begin{cases} 
M_n, & \text{if } x \in I_n, \\
0, & \text{if } x \notin I_n.
\end{cases}
\]

It is also known that (see [1])

\[
D_{s M_n} = D_M_n \sum_{k=0}^{s-1} \psi_{k M_n} = D_M_n \sum_{k=0}^{s-1} r_{M_n}^k \quad \text{and} \quad D_n = \psi_n \left( \sum_{j=0}^{\infty} D_{M_j} \sum_{u=m_j - n_j}^{m_j - 1} r_{M_n}^u \right).
\]

Moreover, the following estimation holds true:

\[
\int_{I_N} |D_n(x - t)| \, d\mu(t) \leq \frac{c M_s}{M_N}, \quad x \in I_s \setminus I_{s+1}, \quad s = 0, \ldots, N - 1.
\]

For our investigation we need the following estimates of the two-dimensional Dirichlet kernels of independent interest:

**Lemma 1.** Let \( m, n \in \mathbb{N} \). Then, for every \( 0 < \varepsilon \leq 1 \), there exists an absolute constant \( c \), such that

\[
\int_{I_N \times I_N} |D_m(x - t) D_n(y - s)| \, d\mu(t) \, d\mu(s) \leq \frac{c m^\varepsilon M_s}{M_{N+\varepsilon}}, \quad (x, y) \in I_N \times (I_s \setminus I_{s+1}), \quad s = 0, \ldots, N - 1
\]

and

\[
\int_{I_N \times I_N} |D_m(x - t) D_n(y - s)| \, d\mu(t) \, d\mu(s) \leq \frac{c m^\varepsilon M_s}{M_{N+\varepsilon}}, \quad (x, y) \in (I_s \setminus I_{s+1}) \times I_N, \quad s = 0, \ldots, N - 1.
\]

Moreover, let \((x, y) \in (I_{s_1} \setminus I_{s_1+1}) \times (I_{s_2} \setminus I_{s_2+1})\), \( s_1, s_2 = 0, \ldots, N - 1 \). Then there exists an absolute constant \( c \), such that

\[
\int_{I_N} |D_n(x - t) D_m(y - s)| \, d\mu(t) \, d\mu(s) \leq \frac{c M_{s_1} M_{s_2}}{M_N^2}.
\]
Proof. Since \(|D_m(x)| \leq m\) and \(|D_m(x)| \leq M_s\), for \(x \in I_s \setminus I_{s+1}\), by using (8) we obtain that
\[
\int_{I_N} |D_m(x-t)| \, d\mu(t) \leq m^\varepsilon \sum_{s=N}^\infty \int_{I_s \setminus I_{s+1}} |D_m(x-t)|^{1-\varepsilon} \, d\mu(t)
\]
\[
\leq cm^\varepsilon \sum_{s=N}^\infty M_s^{1-\varepsilon} d\mu(t) \leq cm^\varepsilon \sum_{s=N}^\infty M_s^{-\varepsilon} \leq \frac{cm^\varepsilon}{M_N^\varepsilon}.
\]
Therefore, by using inequality (8) we obtain that
\[
\int_{I_N \times I_N} |D_m(x-t) D_n(y-s)| \, d\mu(t) \, d\mu(s)
\]
\[
\leq \int_{I_N} |D_m(x-t)| \, d\mu(t) \int_{I_N} |D_n(y-s)| \, d\mu(s) \leq \frac{cm^\varepsilon M_2}{M_N^{1+\varepsilon}}.
\]
The proof of the second estimation is quite analogous to the proof of Lemma 1. So, we leave out the details.
To prove the third estimate we apply inequality (8) to obtain that
\[
\int_{I_N \times I_N} |D_m(x-t) D_n(y-s)| \, d\mu(t) \, d\mu(s)
\]
\[
\leq \int_{I_N} |D_m(x-t)| \, d\mu(t) \int_{I_N} |D_n(y-s)| \, d\mu(s) \leq \frac{cM_{s_1}M_{s_2}}{M_N^2}.
\]
The proof is complete. \(\square\)

We also consider the following maximal operators \(\tilde{S}^{*,p}\) and \(\widetilde{\Sigma}^*\) defined by
\[
(9) \quad \tilde{S}^{*,p} f = \sup_{m,n \geq 1} \frac{|S_{m,n} f|}{(m+n)^{2/p-2}}, \\
\quad \widetilde{\Sigma}^* := \sup_{n \in \mathbb{N}} |S_{M_n,M_n}|.
\]
The \(\sigma\)-algebra generated by the 2-dimensional \(I_n(x) \times I_n(y)\) square of measure \(M_n^{-1} \times M_n^{-1}\) will be denoted by \(F_{n,n} (n \in \mathbb{N})\). Denote by \(f = (f_{n,n} \, n \in \mathbb{N})\) one-parameter martingales with respect to \(F_{n,n} (n \in \mathbb{N})\) (for details see e.g. [22], [6], [45] and [46]).

The maximal function of a martingale \(f\) is defined by \(f^* = \sup_{n \in \mathbb{N}} |f_{n,n}|\).
Let \(f \in L_1 (G_m^2)\). Then the maximal function is given by
\[
f^*(x,y) = \sup_{n \in \mathbb{N}} \frac{1}{\mu(I_n(x) \times I_n(y))} \int_{I_n(x) \times I_n(y)} f(s,t) \, d\mu(s) \, d\mu(t), \quad \text{where} \quad (x,y) \in G_m^2.
\]
The two-dimensional Hardy space \(H_p^2(G_m^2)\) (\(0 < p < \infty\)) consists of all martingale for which
\[
\|f\|_{H_p^2} := \|f^*\|_p < \infty.
\]
If \(f \in L_1 (G_m^2)\), then it is easy to show that the sequence \((S_{M_n,M_n} f : n \in \mathbb{N})\) is a martingale. If \(f = (f_{n,n}, n \in \mathbb{N})\) is a martingale, then the Vilenkin-Fourier coefficients must be defined
in a slightly different manner:

\[ \hat{f}(i,j) := \lim_{k \to \infty} \int_{G^2_m} f_{k,k}(x,y) \overline{\psi_i}(x) \overline{\psi_j}(y) \, d\mu(x) \, d\mu(y). \]

It is known (for details see e.g. Weisz [45]) that the following holds true for the bounded two-dimensional Vilenkin-Fourier series:

**Proposition 1.** Let \( g \in L^1(G^2_m) \) and \( f := (E_{n}g : n \in \mathbb{N}) \) be regular martingale. Then the Vilenkin-Fourier coefficients of \( f \in L^1(G^2_m) \) are the same as those of the martingale \( (S_{M_n,M_n}f : n \in \mathbb{N}) \) obtained from \( f \).

Moreover, \( H^p_p(G^2_m) \) \((0 < p \leq 1)\) norm is calculated by

\[ \|f\|_{H^p_p(G^2_m)} = \left\| \sup_{n \in \mathbb{N}} |S_{M_n,M_n}g| \right\|_p. \]

A bounded measurable function \( a \) is a \( p \)-atom, if there exists a two-dimensional cube \( I^2 = I \times I \), such that

\[ \int_{I^2} a \, d\mu = 0, \quad \|a\|_{\infty} \leq \mu(I^2)^{-1/p}, \quad \text{supp} \,(a) \subseteq I^2. \]

In order to prove our main results we need the following lemma of Weisz (for details see e.g. Weisz [45])

**Proposition 2.** A martingale \( f \) is in \( H^p_p(G^2_m) \) \((0 < p \leq 1)\) if and only if there exist a sequence \( (a_k, k \in \mathbb{N}) \) of \( p \)-atoms and a sequence \( (\mu_k, k \in \mathbb{N}) \) of real numbers such that

\[ \sum_{k=0}^{\infty} \mu_k S_{M_n,M_n}a_k = f_{n,n}, \quad \text{a.e.}, \]

and

\[ \sum_{k=0}^{\infty} |\mu_k|^p < \infty, \]

Moreover,

\[ \|f\|_{H^p_p(G^2_m)} \sim \inf \left( \sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p}, \]

where the infimum is taken over all decompositions of \( f \) of the form (10).

Definition several variable Hardy spaces and real Hardy spaces and related theorems of atomic decompositions of these spaces can be found in Fefferman and Stein [9] (see also Later [18], Torchinsky [40], Wilson [48]).

3. **Main results**

Our first main result reads:
Theorem 1. a) Let $0 < p < 1$ and $f \in H^p_p(G^2_m)$. Then the maximal operator $\tilde{S}^*_p$ defined by (9) is bounded from the martingale Hardy space $H^p_p(G^2_m)$ to the space $L_p(G^2_m)$.

b) (Sharpness) Let $0 < p < 1$ and $\varphi : \mathbb{N} \to [1, \infty)$ be a non-decreasing function, satisfying the condition

$$\sup_{m,n \in \mathbb{N}} \frac{(m+n)^{2/p-2}}{\varphi(m,n)} = +\infty.$$  

Then

$$\sup_{m,n \in \mathbb{N}} \left\| \frac{S_{m,n}f}{\varphi(m,n)} \right\|_{\text{weak-}L_p(G^2_m)} = \infty.$$  

Proof. Since $\tilde{S}^*_p$ is bounded from $L_\infty$ to $L_\infty$ by using Proposition 2 we conclude that the proof of part a) will be complete, if we show that

$$\int_{I \times I} \left| \tilde{S}^*_p a(x,y) \right|^p d\mu(x) d\mu(y) \leq c < \infty, \quad \text{as} \quad 0 < p < 1,$$

for every $p$-atom $a$, where $I \times I$ denotes the support of the atom.

Let $a$ be an arbitrary $p$-atom with support $I \times I$ and $\mu(I \times I) = M_N^2$. We may assume that $I \times I = I_N \times I_N$, where $I_N := I_N(0)$. It is easy to see that $S_{m,n}a = 0$ when $m \leq M_N$ and $n \leq M_N$. Therefore we can suppose that either $m > M_N$ or $n > M_N$. Since $\|a\|_\infty \leq M_N^{2/p}$ we find that

$$|S_{m,n}a| \leq \int_{I_N \times I_N} |a(t_1,t_2)| |D_{m,n} (x + t_1, y + t_2)| d\mu(t_1) d\mu(t_2)$$

$$\leq \|a\|_\infty \int_{I_N \times I_N} |D_{m,n} (x + t_1, y + t_2)| d\mu(t_1) d\mu(t_2)$$

$$\leq M_N^{2/p} \int_{I_N \times I_N} |D_{m,n} (x + t_1, y + t_2)| d\mu(t_1) d\mu(t_2).$$

Let $0 < p < 1$ and $(x,y) \in I_N \times (I_{s_2} \setminus I_{s_2+1})$. We choose $\varepsilon$, so that $2/p - 2 - \varepsilon > 0$ and then from Lemma 1 and (12) it follows that

$$\frac{|S_{m,n}a(x,y)|}{(m+n+1)^{2/p-2}} \leq \frac{M_N^{2/p} M_N^2 n^\varepsilon}{(m+n+1)^{2/p-2} M_N^{\varepsilon+1}} \leq \frac{M_N^{2/p-1-\varepsilon} M_{s_2}}{(m+n+1)^{2/p-2-\varepsilon}} \leq M_{s_2} M_N.$$

According to (11) and (13) we have that

$$\int_{I_N \times I_N} \left| \tilde{S}^*_p a \right|^p d\mu \times d\mu = \sum_{s_2=0}^{N-1} \int_{I_N \times (I_{s_2} \setminus I_{s_2+1})} \left| \tilde{S}^*_p a \right|^p d\mu \times d\mu \leq \sum_{s_2=0}^{N-1} \frac{M_{s_2}^p}{M_{s_2}} < c_p < \infty.$$

If we apply (11), (12) and Lemma 1 analogously to (14) we obtain that

$$\int_{I_N \times I_N} \left| \tilde{S}^*_p a \right|^p d\mu \times d\mu = \sum_{s_1=0}^{N-1} \int_{(I_{s_1} \setminus I_{s_1+1}) \times I_N} \left| \tilde{S}^*_p a \right|^p d\mu \times d\mu \leq \sum_{s_1=0}^{N-1} \frac{M_{s_1}^p}{M_{s_1}} < c_p < \infty.$$

Let $0 < p < 1$ and $(x,y) \in (I_{s_1} \setminus I_{s_1+1}) \times (I_{s_2} \setminus I_{s_2+1})$. By using Lemma 1 we get that

$$\frac{|S_{m,n}a(x,y)|}{(m+n+1)^{2/p-2}} \leq \frac{M_N^{2(1/p-1)} M_{s_1} M_{s_2}}{(m+n+1)^{2/p-2}} \leq M_{s_1} M_{s_2}.$$
Hence,

\[
(17) \quad \left\lfloor \frac{1}{I_{N} \times I_{N}} \left| \tilde{S}_{p} \right|^{p} d\mu \times d\mu \right\rfloor = \sum_{s_{1}=0}^{N-1} \sum_{s_{2}=0}^{N-1} \int_{(I_{s_{1}} \setminus I_{s_{1}+1}) \times (I_{s_{2}} \setminus I_{s_{2}+1})} \left| \tilde{S}_{p} \right|^{p} d\mu \times d\mu
\]

\[
\leq \sum_{s_{1}=0}^{N-1} \frac{M_{s_{1}}^{p}}{M_{s_{2}}^{p}} \sum_{s_{2}=0}^{N-1} \frac{M_{s_{2}}^{p}}{M_{s_{2}}^{p}} < c_{p} < \infty.
\]

Since

\[
I_{N} \times I_{N} = (I_{N} \times I_{N}) \cup (I_{N} \times I_{N}) \cup (I_{N} \times I_{N}),
\]

by combining (14), (15) and (17) we get that (11) holds for every $p$-atom and the proof of part a) is complete.

Now, we prove the second part of the theorem. Let $\varphi : \mathbb{N}^{2} \to [1, \infty)$ be a non-decreasing function and $\{\alpha_{k} : k \in \mathbb{N}\}$ be a sequence of natural numbers satisfying the condition

\[
\lim_{k \to \infty} \frac{(M_{\alpha_{k}} + 1)^{2/p-2}}{\varphi (2^{\alpha_{k}} + 1, 1)} = +\infty.
\]

For $k \in \mathbb{N}_{+}$ set

\[
f_{k}(x, y) = \left( D_{M_{\alpha_{k}+1}}(x) - D_{M_{\alpha_{k}}}(x) \right) D_{M_{\alpha_{k}}}(y).
\]

It is evident that

\[
\tilde{f}_{k}(i, j) = \begin{cases} 1, & \text{if } (i, j) \in \bigcup_{l=0}^{k} \{M_{\alpha_{l}}, ..., M_{\alpha_{l+1}} - 1\} \times \{0, ..., M_{\alpha_{l+1}} - 1\}, \\ 0, & \text{otherwise.} \end{cases}
\]

Therefore,

\[
(18) \quad S_{l,j}(f_{k}; x, y) = \left\lfloor \begin{array}{l}
(D_{i}(x) - D_{M_{\alpha_{k}}}(x)) D_{j}(y), \\
\tilde{f}_{k}(x, y)
\end{array} \right\rfloor,
\]

\[
\left\lfloor \begin{array}{l}
\text{if } (i, j) \in \bigcup_{l=0}^{k} \{M_{\alpha_{l}}, ..., M_{\alpha_{l+1}} - 1\} \times \{1, ..., M_{\alpha_{l+1}} - 1\}, \\
\text{if } i \geq M_{\alpha_{k}+1} \text{ and } j \geq M_{\alpha_{k}+1}, \\
0, & \text{otherwise.} \\
\end{array} \right\rfloor
\]

From (18) it follows that

\[
(19) \|f_{k}\|_{H_{p}^{\varphi}} = \left\lfloor \sup_{a \in \mathbb{N}} S_{M_{a, M_{a}}}(f_{k}) \right\rfloor_{p} = \left\lfloor \left( D_{M_{\alpha_{k}+1}}(x) - D_{M_{\alpha_{k}}}(x) \right) D_{M_{\alpha_{k}}}(y) \right\rfloor_{p} \leq c_{p}M_{\alpha_{k}}^{2(1-1/p)}.
\]

Let $(x, y) \in G_{m}^{2}$. Moreover, (18) also implies that

\[
\frac{|S_{M_{\alpha_{k}+1},1}(f_{k}; x, y)|}{\varphi (M_{\alpha_{k}+1}, 1)} = \frac{|(D_{M_{\alpha_{k}+1}}(x) - D_{M_{\alpha_{k}}}(x)) D_{1}(y)|}{\varphi (M_{\alpha_{k}+1}, 1)} = \frac{|w_{M_{\alpha_{k}}}(x) w_{0}(y)|}{\varphi (M_{\alpha_{k}+1}, 1)} = \frac{1}{\varphi (M_{\alpha_{k}+1}, 1)}.
\]
Hence, by also using (19), we find that
\[
\frac{1}{\varphi(M_{\alpha_k} + 1,1)} \left( \mu \left\{ (x, y) \in C_m^2 : \frac{S_{M_{\alpha_k} + 1,1}(f_k x, y)}{\varphi(M_{\alpha_k} + 1,1)} \geq \frac{1}{\varphi(M_{\alpha_k} + 1,1)} \right\} \right)^{1/p} \|f_k\|_{H_p^r(G_m^2)}
\]
\[
\geq \frac{1}{\varphi(M_{\alpha_k} + 1,1) M_{\alpha_k}^{2(1 - 1/p)}} \geq \frac{(M_{\alpha_k} + 1)^{2/p - 2}}{\varphi(M_{\alpha_k} + 1,1)} \rightarrow \infty, \quad \text{as} \quad k \rightarrow \infty.
\]

The proof is complete. □

We also apply Theorem 1 to obtain that the following is true:

**Theorem 2.** Let \( 0 < p < 1, f \in H_p^r(G_m^2), 2^{-\alpha} < m/n \leq 2^\alpha \) and \( 2^k < m, n \leq 2^{k+1+|\alpha|} \). Then there exists an absolute constant \( c_p \), such that

\[
\|S_{m,n} f - f\|_{H_p^r(G_m^2)} \leq c_p M^{2/p - 2}_k \omega_{H_p^r(G_m^2)} \left( \frac{1}{M_k} \right). 
\]

**Proof.** Without lost a generality we may assume that \( n < m \). According to Theorem 1 we can conclude that

\[
\|S_{m,n} f\|_p \leq c_p^1 (m + n)^{2/p - 2} \|f\|_{H_p^r(G_m^2)} \leq c_p^2 M^{2/p - 2}_k \|f\|_{H_p^r(G_m^2)} \leq c_p^3 M^{2/p - 2}_k \|f\|_{H_p^r(G_m^2)}.
\]

Since \( 2^{-\alpha} \leq m/n \leq 2^\alpha \) we obtain that

\[
M_k < m, n, M_{[n]}, M_{[n]+1}, \ldots, M_{[m]} \leq M_{k+1+|\alpha|}, \quad (|m| - |n| + 1) \leq \alpha + 2
\]

and

\[
\|S_{M_i,n} f\|_p \leq c_p^3 M^{2/p - 2}_k \|f\|_{H_p^r(G_m^2)}, \quad \text{where} \quad |n| \leq i \leq |m|.
\]

Let us consider the following martingale \( f_\#: (S_{2^k,2^k} S_{m,n} f, k \in \mathbb{N}_+) \). By a simple calculation we get that

\[
f_\# = \left( S_{M_0,M_0} f, S_{M_{[n]},M_{[n]}, f}, S_{M_{[n]+1},n} f, \ldots, S_{M_{[m]},n} f, S_{m,n} f, \ldots, S_{m,n} f, \ldots \right).
\]

By using Proposition 1 we immediately get that

\[
\|S_{m,n} f\|_{H_p^r(G_m^2)} \leq \sup_{0 \leq t < |n|} \|S_{M_t,n} f\|_p + \sum_{i=|n|}^{|m|} \|S_{M_i,n} f\|_p + \|S_{m,n} f\|_p
\]
\[
\leq \|S_{\#} f\|_p + (|m| - |n| + 1) M^{2/p - 2}_k \|f\|_{H_p^r(G_m^2)}
\]
\[
\leq c_p^5 \|f\|_{H_p^r(G_m^2)} + c_p^6 M^{2/p - 2}_k \|f\|_{H_p^r(G_m^2)} \leq c_p^3 M^{2/p - 2}_k \|f\|_{H_p^r(G_m^2)}.
\]

Hence,

\[
\|S_{m,n} f - f\|_{H_p^r(G_m^2)} \leq \|S_{m,n} f - S_{M_k,M_k} f\|_{H_p^r(G_m^2)} + \|S_{M_k,M_k} f - f\|_{H_p^r(G_m^2)}
\]
\[
= \|S_{m,n} (S_{M_k,M_k} f - f)\|_{H_p^r(G_m^2)} + \|S_{M_k,M_k} f - f\|_{H_p^r(G_m^2)}
\]
\[
\leq c_p^2 (M_k^{-2p} + 1) \omega_{H_p^r(G_m^2)} \left( \frac{1}{M_k} \right).
\]
and
\[ \|S_{m,n}f - f\|_{H^p_p(G_n^m)}^p \leq c_p M_k^{2/p - 2} \omega_{H^p_p(G_n^m)} \left( \frac{1}{M_k} f \right). \]

The proof is complete. \qed

**Theorem 3.** a) Let \( 0 < p < 1, f \in H^p_p(G_n^m), 2^{-\alpha} \leq m/n \leq 2^\alpha \) and
\[ \omega_{H^p_p(G_n^m)} \left( \frac{1}{M_k} f \right) = o \left( \frac{1}{M_k^{2/p - 2}} \right), \text{ as } k \to \infty. \]
Then
\[ \|S_{m,n}f - f\|_{H^p_p(G_n^m)} \to 0, \text{ as } m, n \to \infty. \]

b) (Sharpness) Let \( 0 < p < 1 \) and \( 2^{-\alpha} < m/n \leq 2^\alpha \). Then there exists a martingale \( f \in H^p_p(G_n^m) \), such that
\[ \omega_{H^p_p(G_n^m)} \left( \frac{1}{M_k} f \right) = O \left( \frac{1}{M_k^{2/p - 2}} \right), \text{ as } k \to \infty \]
and
\[ \|S_{m,n}f - f\|_{L^p_p(G_n^m)} \not\to 0 \text{ as } m, n \to \infty. \]

**Proof.** Let \( 0 < p < 1, f \in H^p_p(G_n^m), 2^{-\alpha} \leq m/n \leq 2^\alpha \) and
\[ \omega_{H^p_p(G_n^m)} \left( \frac{1}{M_k} f \right) = o \left( \frac{1}{M_k^{2/p - 2}} \right), \text{ as } k \to \infty. \]
By using Theorem 1 we immediately get that
\[ \|S_{m,n}f - f\|_{H^p_p(G_n^m)} \to \infty, \text{ as } n \to \infty \]
and the proof of part a) is complete.

Let
\[ f_{n,n} = \sum_{\{k; \alpha_k + 1 < n\}} \lambda_k a_k, \]
where
\[ \lambda_k = M_k^{-\frac{2}{p-2}} \]
and
\[ a_k(x,y) = M_{\alpha_k}^{2/p-2} \left( D_{M_{\alpha_k+1}}(x) - D_{M_{\alpha_k}}(x) \right) \left( D_{M_{\alpha_k+1}}(y) - D_{M_{\alpha_k}}(y) \right). \]
Since
\[ S_{M_n:M_n} a_k = \begin{cases} a_k, & \alpha_k + 1 < n, \\ 0, & \alpha_k + 1 \geq n, \end{cases} \]
\[ \text{supp}(a_k) = I_{\alpha_k}^2, \quad \int_{I_{\alpha_k}^2} a_k d\mu = 0, \quad \|a_k\|_\infty \leq M_{\alpha_k}^{2/p} = (\text{supp } a_k)^{-1/p} \]
from Lemma 2 and the fact that \( \sum_{k=0}^\infty |\mu_k|^p < \infty \), we conclude that \( f \in H^p_p(G_n^m) \).
Moreover, for all $k \in \mathbb{N}_+$,

\begin{align}
(20) \quad f - S_{M_n,M_n} f &= \left( f^{(1)} - S_{M_n,M_n} f^{(1)}, \ldots, f^{(n)} - S_{M_n,M_n} f^{(n)}, \ldots, f^{(n+k)} - S_{M_n,M_n} f^{(n+k)} \right) \\
&= (0, \ldots, 0, f^{(n+1)} - f^{(n)}, \ldots, f^{(n+k)} - f^{(n)}, \ldots) = \left( 0, \ldots, 0, \sum_{i=n}^{n+k} a_i(x,y) \right)
\end{align}

is a martingale and (20) is its atomic decomposition. By using Lemma 2 we find that

\begin{align}
\omega_{H^p(G^2_{\infty})} \left( \frac{1}{M_n}, \hat{f} \right) := \| f - S_{M_n,M_n} f \|_{H^p(G^2_{\infty})} \leq \sum_{i=n}^{\infty} \frac{1}{M_i^{2/p-2}} \leq \frac{c}{M_n^{2/p-2}}.
\end{align}

It is easy to show that

\begin{align}
\hat{f}(i,j) = \begin{cases} 
1, & \text{if } (i,j) \in \{M_{\alpha_k}, \ldots, M_{\alpha_k+1} - 1 \}^2, \ k \in \mathbb{N}, \\
0, & \text{if } (i,j) \notin \bigcup_{k=1}^{\infty} \{M_{\alpha_k}, \ldots, M_{\alpha_k+1} - 1 \}^2.
\end{cases}
\end{align}

Hence,

\begin{align}
(21) \quad S_{M_{\alpha_k+1},M_{\alpha_k+1}} f (x,y) = S_{M_{\alpha_k},M_{\alpha_k}} f (x,y) + w_{M_{\alpha_k}} (x) w_{M_{\alpha_k}} (y) =: I + II.
\end{align}

It is obvious that $|II| = \left| w_{M_{2\alpha_k}} (x) w_{M_{\alpha_k}} (y) \right| = 1$ and

\begin{align}
(22) \quad \|II\|_{L^p(G^2_{\infty})} \geq \frac{1}{2^p} \mu \left\{ (x,y) \in G^2_m : |II| \geq \frac{1}{2} \right\} \geq \frac{1}{2^p} \mu \left( G^2_m \right) \geq \frac{1}{2^p}.
\end{align}

Since (for details see e.g. Weisz [45] and [46])

\begin{align}
\| f - S_{M_n,M_n} f \|_{L^p(G^2_{\infty})} \to 0, \quad \text{as } n \to \infty.
\end{align}

According to (21) and (22) we obtain that

\begin{align}
\limsup_{k \to \infty} \| f - S_{M_{\alpha_k+1},M_{\alpha_k+1}} f \|_{L^p(G^2_{\infty})} &\geq \limsup_{k \to \infty} \|II\|_{L^p(G^2_{\infty})} - \limsup_{k \to \infty} \| f - S_{M_{\alpha_k},M_{\alpha_k}} f \|_{L^p(G^2_{\infty})} \geq c > 0.
\end{align}

The proof is complete. \(\square\)

**Theorem 4.** a) Let $0 < p < 1$, $f \in H^p(G^2_{\infty})$. Then there exists an absolute constant $c_p$, depending only of $p$, such that

\begin{align}
\sum_{\{(k,l) : 2^{-\alpha} \leq k,l \leq 2^\alpha \}} \left\| S_{k,l} f \right\|_p^p \leq c_p \| f \|_{H^p(G^2_{\infty})}^p.
\end{align}

b) Let $0 < p < 1$ and $\Phi : \mathbb{N}^2 \to [1,\infty)$ be non-decreasing, nonnegative function, satisfying the condition

\begin{align}
(23) \quad \lim_{m,n \to \infty} \Phi(m,n) = +\infty.
\end{align}
Then there exists a martingale $f \in H^2_p(G_m^2)$ such that

$$\sum_{\{(k,l) : 2^{-\alpha} \leq k/l \leq 2^\alpha\}} \|S_{k,l}f\|_p^p \Phi(k,l) \frac{1}{(kl)^{2-p}} = \infty.$$ 

Proof. If we follow similar steps of the proof of Theorem 1 then we obtain that either $m > M_N$ or $n > M_N$, but in this case under condition $2^{-\alpha} \leq m/n \leq 2^\alpha$ we can conclude that $M_{N-1-[\alpha]} < m, n \leq M_{N+1-[\alpha]}$. If we apply Proposition 2 we only have to prove that

$$\sum_{2^{-\alpha} \leq k/l \leq 2^\alpha, \ k,l > M_{N-1-[\alpha]}} \|S_{k,l}a\|_p^p = \sum_{2^{-\alpha} \leq k/l \leq 2^\alpha, \ k,l > M_{N-1-[\alpha]}} \left| \frac{S_{k,l}a}{(kl)^{2/p-1}} \right|^p < c_p < \infty.$$ 

Let $0 < p < 1$ and $(x,y) \in I_N \times (I_{s_2} \backslash I_{s_2+1})$. We choose $\varepsilon$, so that $2/p - 1 - \varepsilon > 0$. Since

$$(24) \quad \frac{|S_{k,l}a(x,y)|}{(kl)^{2/p-1}} \leq \frac{M_2^{2/p} M_{s_2} k_p^e}{(kl)^{2/p-1} M_N^{p-1}} \leq \frac{M_2^{2/p-1}}{M_N^{p-1}} \frac{1}{(kl)^{2/p-1}}< c_p \frac{M_1^{1-p-\varepsilon}}{k_2^{2-p-\varepsilon}} \frac{1}{M_N^{1-p-\varepsilon}} \frac{1}{k_2^{2-p}} \sum_{s_2=0}^{N-1} M_{s_2}^p < c_p \frac{M_1^{1-p-\varepsilon}}{k_2^{2-p-\varepsilon}}.$$ 

According to (1) and (24) we get that

$$(25) \quad \int_{I_N \times I_N} \left| \frac{S_{k,l}a}{(kl)^{2/p-1}} \right|^p \ d\mu \times d\mu \leq \sum_{s_2=0}^{N-1} \int_{I_N \times (I_{s_2} \backslash I_{s_2+1})} \left| \frac{S_{k,l}a}{(kl)^{2/p-1}} \right|^p \ d\mu \times d\mu \leq \frac{M_1^{1-p-\varepsilon}}{k_2^{2-p-\varepsilon}} \frac{1}{M_N^{1-p-\varepsilon}} \frac{1}{k_2^{2-p}} \sum_{s_2=0}^{N-1} M_{s_2}^p < c_p \frac{M_1^{1-p-\varepsilon}}{k_2^{2-p-\varepsilon}}.$$ 

By using (25) we find that

$$\sum_{2^{-\alpha} \leq k/l \leq 2^\alpha, \ k,l \geq M_{N-1-[\alpha]}} \int_{I_N \times I_N} \left| \frac{S_{k,l}a}{(kl)^{2/p-1}} \right|^p \ d\mu \times d\mu \leq c_p \frac{M_1^{1-p-\varepsilon}}{M_N^{1-p-\varepsilon}} \frac{1}{k_2^{2-p-\varepsilon}} \sum_{s_2=0}^{N-1} M_{s_2}^p < c_p \frac{M_1^{1-p-\varepsilon}}{k_2^{2-p-\varepsilon}}.$$ 

Let $0 < p < 1$ and $(x,y) \in (I_{s_1} \backslash I_{s_1+1}) \times I_N$ and $\varepsilon$ be real number, so that $2/p - 1 - \varepsilon > 0$. Analogously, we can prove that

$$(26) \quad \int_{I_N \times I_N} \left| \frac{S_{k,l}a}{(kl)^{2/p-1}} \right|^p \ d\mu \times d\mu \leq \frac{M_1^{1-p-\varepsilon}}{k_2^{2-p} k_2^{2-p}} \sum_{s_1=0}^{N-1} M_{s_1}^p < c_p \frac{M_1^{1-p-\varepsilon}}{k_2^{2-p}}.$$ 

It follows that

$$\sum_{2^{-\alpha} \leq k/l \leq 2^\alpha, \ k,l \geq M_{N-1-[\alpha]}} \int_{I_N \times I_N} \left| \frac{S_{k,l}a}{(kl)^{2/p-1}} \right|^p \ d\mu \times d\mu \leq c_p \frac{M_1^{1-p-\varepsilon}}{M_N^{1-p-\varepsilon}} \frac{1}{k_2^{2-p}} \sum_{s_2=0}^{N-1} M_{s_2}^p < c_p \frac{M_1^{1-p-\varepsilon}}{k_2^{2-p}}.$$ 

...
Let $0 < p < 1$ and $(x, y) \in (I_{s_1} \setminus I_{s_1 + 1}) \times (I_{s_2} \setminus I_{s_2 + 1})$. Then by using Lemma 1 we get that
\[(27) \quad \left( \frac{|S_{m,n}a(x, y)|}{(mn)^{2/p-1}} \right)^p \leq \frac{M_N^{2-2p}M_{s_1}M_{s_2}^p}{(mn)^{2-p}}.\]

In view of (4) and (27) we can conclude that
\[(28) \quad \int_{I_N \times I_N} \left( \frac{|S_{m,n}a|}{(mn)^{2/p-1}} \right)^p d\mu \times d\mu = \frac{M_N^{2-2p}}{(mn)^{2-p}} \sum_{s_1=0}^{N-1} \sum_{s_2=0}^{N-1} \int_{(I_{s_1} \setminus I_{s_1 + 1}) \times (I_{s_2} \setminus I_{s_2 + 1})} \left| S_{x,y}^a \right|^p d\mu \times d\mu \leq \frac{M_N^{2-2p}}{(mn)^{2-p}}.\]

Hence,
\[
\sum_{2^{-\alpha} \leq k \leq 2^{\alpha}, \ k,l \geq M_{N-1-\alpha}} \frac{1}{k^{2-2p}} \sum_{a \in \{\lambda_k \in [\alpha+1, \alpha+2] \cap \mathbb{Z} \}} \frac{1}{l^{2-2p}} \leq c_p \frac{M_N^{2-2p}}{M_{N-1-\alpha}^{2-2p}} \leq c_p < \infty.
\]

The proof of part a) is complete.

Under the condition (23) there exists an increasing sequence of positive integers $\{\alpha_k: k \geq 0\}$ such that $\alpha_0 \geq 2$, $\alpha_k + [\alpha] + 1 < \alpha_{k+1}$ and

\[(29) \quad \sum_{k=0}^{\infty} \Phi^{-p/4} (M_{\alpha_k}, M_{\alpha_k}) < \infty.
\]

Let
\[f_{n,n} = \sum_{\{k: \alpha_k + [\alpha] + 1 < n\}} \lambda_k \alpha_k,
\]
where $\lambda_k := (m_{\alpha_k} \ldots m_{\alpha_k + [\alpha]})^{2/p-2} \Phi^{-1/4} (M_{\alpha_k}, M_{\alpha_k})$ and
\[a_k(x, y) := M_{\alpha_k + [\alpha] + 1}^{2/p-2} D_{M_{\alpha_k + [\alpha] + 1}} (x) - D_{M_{\alpha_k}} (x) \left( D_{M_{\alpha_k + [\alpha] + 1}} (y) - D_{M_{\alpha_k}} (y) \right).\]

Since
\[S_{2^n,2^n} \alpha_k = \left\{ \begin{array}{ll} a_k, & \alpha_k + [\alpha] + 1 < n, \\ 0, & \alpha_k + [\alpha] + 1 \geq n, \end{array} \right. \]
\[
\text{supp}(a_k) = I_{\alpha_k}^2, \quad \int_{I_{\alpha_k}^2} \alpha_k d\mu = 0, \quad \|a_k\|_{\infty} \leq M_{\alpha_k}^{2/p} = (\mu(\text{supp} a_k))^{-1/p}
\]
from Proposition 1 and (29) we obtain that $f \in H^{2/p}_{p}(G_m)$. It is obvious that
\[\hat{f}(i, j) = \left\{ \begin{array}{ll} \frac{M_{\alpha_k}^{2/p-2}}{\Phi^{1/4}(M_{\alpha_k}, M_{\alpha_k})}, & (i, j) \in \left\{ M_{\alpha_k}, \ldots, M_{\alpha_k + [\alpha] + 1} - 1 \right\}^2, \\ 0, & (i, j) \notin \bigcup_{k=1}^{\infty} \left\{ M_{\alpha_k}, \ldots, M_{\alpha_k + [\alpha] + 1} - 1 \right\}^2. \end{array} \right. \]

Let $M_{\alpha_k} < m, n < M_{\alpha_k + [\alpha] + 1}$. Then
According to (5), (23) and (31)-(33) we can conclude that

\[
\begin{align*}
&\geq M^{2/p-2} \frac{\mu((G \setminus I_1))}{\Phi^{1/4}(M_k, M_k)} \geq \frac{C_p M^{2/p-2}}{\Phi^{1/4}(M_k, M_k)}. \\
&\geq \frac{M^{2/p-2}}{2 \Phi^{1/4}(M_k, M_k)} \left(\mu\left((x, y) \in (G \setminus I_1) \times (G \setminus I_1) : |S_{m,n}f(x, y)| \geq \frac{M^{2/p-2}}{2 \Phi^{1/4}(M_k, M_k)}\right)\right)^{1/p}
\end{align*}
\]

Let \((x, y) \in (G \setminus I_1) \times (G \setminus I_1), m, n \in \mathbb{N}_{\alpha},\) such that \(M_k < m, n < 2^\alpha M_k.\) Since \(\alpha_k \geq 2\) \((k \in \mathbb{N}),\) if we combine (6)-(7) it follows that \(D_{M_k}(x) = D_{M_k}(y) = 0\) and

\[
|II| = \frac{M^{2/p-2}}{\Phi^{1/4}(M_k, M_k)} |w_m(x) w_1(x) D_1(x) w_n(y) D_1(y)| = \frac{M^{2/p-2}}{\Phi^{1/4}(M_k, M_k)}.
\]

By applying (6) and the condition \(\alpha_n \geq 2\) \((n \in \mathbb{N})\) for \(I\) we have that

\[
|I| = \sum_{\eta=0}^{k-1} \frac{M^{2/p-2}}{\Phi^{1/4}(M_{\alpha+1}, M_{\alpha+1})} \left(\frac{D_{M_{\alpha+1}}(x) - D_{M_{\alpha+1}}(x)}{D_{M_{\alpha+1}}(y) - D_{M_{\alpha+1}}(y)}\right) = 0.
\]

By combining (31) and (32) for \(M_k < m, n < 2^\alpha M_k\) we get that

\[
\|S_{m,n}f\|_{\text{weak-}L_p(G_m^2)} \\
\geq \frac{M^{2/p-2}}{2 \Phi^{1/4}(M_k, M_k)} \left(\mu\left((x, y) \in (G \setminus I_1) \times (G \setminus I_1) : |S_{m,n}f(x, y)| \geq \frac{M^{2/p-2}}{2 \Phi^{1/4}(M_k, M_k)}\right)\right)^{1/p}
\]

According to (5), (23) and (31)-(33) we can conclude that

\[
\sum_{\{k,l: 2^{-\alpha} \leq k, l \leq 2^\alpha\}} \frac{\|S_{m,n}f\|^p_{\text{weak-}L_p} \Phi(m, n)}{(mn)^{2-p}} \geq \sum_{M_k < m, n \leq 2^\alpha M_k} \frac{\|S_{m,n}f\|^p_{\text{weak-}L_p} \Phi(m, n)}{(mn)^{2-p}}
\]
\( (35) \geq \frac{c_p \Phi(M_{\alpha_k}, M_{\alpha_k})}{M_{\alpha_k}^{4-2p}} \sum_{M_{\alpha_k} < m, n \leq 2^\alpha M_{\alpha_k}} \|S_{m,n}f\|_{\text{weak}-L_p}^p \)

\[ \geq \frac{c_p \Phi(M_{\alpha_k}, M_{\alpha_k})}{M_{\alpha_k}^{4-2p}} \sum_{M_{\alpha_k-1} < m, n \leq 2^\alpha M_{\alpha_k-1}, n \in \mathbb{N}_0} \|S_{m,n}f\|_{\text{weak}-L_p}^p \]

\[ \geq \frac{c_p \Phi(M_{\alpha_k}, M_{\alpha_k})}{M_{\alpha_k}^{4-2p}} \frac{M_{\alpha_k}^{2-2p}}{\Phi^{1/4}(M_{\alpha_k}, M_{\alpha_k})} \sum_{M_{\alpha_k-1} < m, n \leq 2^\alpha M_{\alpha_k-1}, n \in \mathbb{N}_0} 1 \]

\[ \geq \frac{c_p \Phi^{3/4}(M_{\alpha_k}, M_{\alpha_k})}{M_{\alpha_k}^2} M_{\alpha_k-1}^{2-2p} \geq c_p \Phi^{3/4}(M_{\alpha_k}, M_{\alpha_k}) \to \infty, \text{ as } k \to \infty. \]

The proof is complete. \(\square\)

4. OPEN PROBLEMS

In this chapter we state some open problems, which can be interesting for the researchers, who work in this area. First one reads as:

Conjecture 1. Let \( f \in H_1^\square(G_2^2), \) \( 2^{-\alpha} < m/n \leq 2^\alpha. \) Then there exists an absolute constant \( c, \) such that

\[ \|S_{m,n}f - f\|_{H_1^\square(G_2^2)} \leq c k^2 \omega_{H_1^\square(G_2^2)} \left( \frac{1}{M_k}, f \right), \]

where \( S_{m,n}f \) denotes \((m, n)\)-th partial sum of the two-dimensional Vilenkin-Fourier series of \( f. \)

It is also interesting if we can prove that the following is true:

Conjecture 2. a) Let \( f \in H_p^\square(G_2^2), \) \( 2^{-\alpha} \leq m/n \leq 2^\alpha \) and

\[ \omega_{H_p^\square(G_2^2)} \left( \frac{1}{M_k}, f \right) = o \left( \frac{1}{k^2} \right), \text{ as } k \to \infty. \]

Then

\[ \|S_{m,n}f - f\|_{H_p^\square(G_2^2)} \to 0, \text{ as } m, n \to \infty. \]

b) (Sharpness) Let \( 2^{-\alpha} < m/n \leq 2^\alpha. \) Then there exists a martingale \( f \in H_1^\square(G_2^2), \) such that

\[ \omega_{H_p^\square(G_2^2)} \left( \frac{1}{M_k}, f \right) = O \left( \frac{1}{k^2} \right), \text{ as } k \to \infty \]

and

\[ \|S_{m,n}f - f\|_1 \not\to 0 \text{ as } m, n \to \infty, \]

where \( S_{m,n}f \) denotes \((m, n)\)-th partial sum of the two-dimensional Vilenkin-Fourier series of \( f. \)

Strong summability result for the two-dimensional Vilenkin-Fourier series in the case when \( p = 1 \) and \( 2^{-\alpha} \leq k/l \leq 2^\alpha \) is also open problem:
Conjecture 3. Let $f \in H_1(G^2_m)$. Then there exists an absolute constant $c$, such that

$$\sup_{n,m \geq 2} \frac{1}{\log n \log m} \sum_{2^{-\alpha} \leq k,l \leq 2^\alpha, (k,l) \leq (n,m)} \frac{\|S_{k,l}f\|_{H_1(G^2_m)}}{kl} \leq c \|f\|_{H_1(G^2_m)},$$

where $S_{k,l}f$ denotes $(k,l)$-th partial sum of the two-dimensional Vilenkin-Fourier series of $f$.

We also state some interesting open problems without any conjectures

Problem 1. For any $f \in H_p(G^2_m)$ ($0 < p \leq 1$), is it possible to find strong convergence theorems for partial sums $S_{m,n}$ with respect to the two-dimensional Vilenkin-Fourier series without any restriction on indexes $(m,n)$?

Problem 2. For any $f \in H_p(G^2_m)$ ($0 < p \leq 1$), is it possible to find necessary and sufficient conditions in terms of the two-dimensional modulus of continuity of martingale $f \in H_p(G^2_m)$ ($0 < p \leq 1$), for which

$$\|S_{k,j,l}f - f\|_{H_p(G^2_m)} \to 0, \text{ as } j \to \infty,$$

where $S_{k,j,l}f$ denotes $(k,j,l)$-th partial sum of the two-dimensional Vilenkin-Fourier series of $f$?

Problem 3. For any $f \in H_p(G^2_m)$ ($0 < p \leq 1$), is it possible to find necessary and sufficient conditions for the indexes $(k_j,l_j)$, for which

$$\|S_{k_j,l_j}f - f\|_{H_p(G^2_m)} \to 0, \text{ as } j \to \infty,$$

where $S_{k_j,l_j}f$ denotes $(k_j,l_j)$-th partial sum of the two-dimensional Vilenkin-Fourier (Walsh-Fourier) series of $f$?

Problem 4. For any $f \in H_p(G^2_m)$ ($0 < p \leq 1$), is it possible to find necessary and sufficient conditions for the indexes $(k_j,l_j)$, for which

$$\sup_{j \in \mathbb{N}} \left\| S_{k_j,l_j}f \right\|_{H_p(G^2_m)} \leq c_p \|f\|_{H_p(G^2_m)},$$

where $S_{k_j,l_j}f$ denotes $(k_j,l_j)$-th partial sum of the two-dimensional Vilenkin-Fourier (Walsh-Fourier) series of $f$?

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