ON DEGENERATE HAMBURGER MOMENT PROBLEM AND EXTENSIONS OF POSITIVE SEMIDEFINITE HANKEL BLOCK MATRICES

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Abstract. In this paper we consider two related objects: singular positive semidefinite Hankel block–matrices and associated degenerate truncated matrix Hamburger moment problems. The description of all solutions of a degenerate matrix Hamburger moment problem is given in terms of a linear fractional transformation. The case of interest is the Hamburger moment problem whose Hankel block–matrix admits a positive semidefinite Hankel extension.

This is the corrected version of the original paper [2]. The work was inspired by V. Dubovoj’s paper [4] containing the first systematic study of degenerate matricial interpolation problems. Another source of inspiration must have been the paper by R. Curto and L. Fialkow [3] but I was not aware of it then. The original paper contained several errata and the author is very grateful to A. Ben-Artzi and H. Woerdeman for indicating them. A short proof in Section 5 fixes these incorrectnesses. The remaining four sections are mostly the same as in [2].

1. Introduction

The objective of this article is to describe the solutions of a degenerate truncated matrix Hamburger moment problem HMP. We start with a set of Hermitian matrices $s_0, \ldots, s_{2n} \in \mathbb{C}^{m \times m}$ and let $K_n$ denote the Hankel block matrix

$$K_n = (s_{i+j})_{i,j=0}^n. \quad (1.1)$$

Let $Z(K_n)$ denote the set of all solutions of the associated truncated Hamburger moment problem, i.e., the set of nondecreasing right continuous $m \times m$ matrix-valued functions $\sigma(\lambda)$ such that

$$\int_{-\infty}^{\infty} \lambda^k d\sigma(\lambda) = s_k \quad (k = 0, \ldots, 2n - 1) \quad (1.2)$$

and

$$\int_{-\infty}^{\infty} \lambda^{2n} d\sigma(\lambda) \leq s_{2n}. \quad (1.3)$$

As in the scalar case (see [1: §2.1]) $Z(K_n)$ is nonempty if and only if $K_n$ is positive semidefinite and, moreover, by a theorem of H. Hamburger and R. Nevanlinna [1: §3.1], the formula

$$w(z) = \int_{-\infty}^{\infty} \frac{d\sigma(\lambda)}{\lambda - z} \quad (1.4)$$

establishes a one-to-one correspondence between $Z(K_n)$ and the class $R(K_n)$ of $\mathbb{C}^{m \times m}$–valued functions $w(z)$ analytic and with positive semidefinite imaginary part in the upper half plane $\mathbb{C}_+$ such that uniformly in the angle $\{ z = re^{i\theta} : \varepsilon \leq \theta \leq \pi - \varepsilon, \varepsilon > 0 \},$

$$\lim_{z \to \infty} \left\{ z^{2n+1} w(z) + \sum_{k=0}^{2n} s_k z^{2n-k} \right\} \geq 0. \quad (1.5)$$

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This correspondence reduces the HMP problem to a boundary interpolation problem of finding all $C^{m \times m}$–valued Pick functions $w$ (which by definition are analytic and with positive semidefinite imaginary part in $\mathbb{C}_+$) with prescribed asymptotic behaviour (1.5) at infinity.

In this paper we follow the Potapov’s method of the fundamental matrix inequality [9]. The starting point is the following theorem which describes the set $R(K_n)$ in terms of a matrix inequality (see [9, §1] for the proof).

**Theorem 1.1.** Let $w$ be a $C^{m \times m}$–valued function analytic in $\mathbb{C}_+$. Then $w$ belongs to $R(K_n)$ if and only if it satisfies the inequality

$$
\begin{pmatrix}
K_n & (I - zF_{m,n})^{-1}(Uw(z) + M) \\
(w(z)^*U^* + M^*)(I - \bar{z}F_{m,n}^*)^{-1} & \frac{w(z) - w(z)^*}{z - \bar{z}}
\end{pmatrix} \geq 0 \tag{1.6}
$$

for every $z \in \mathbb{C}_+$, where

$$
F_{m,n} = \begin{pmatrix}
0_m & \cdots & 0 \\
I_m & \ddots & 0 \\
\vdots & \ddots & \ddots \\
0 & \cdots & 0 \ I_m & 0_m
\end{pmatrix} \in \mathbb{C}^{m(n+1) \times m(n+1)} \tag{1.7}
$$

is the matrix of the $m$-dimensional shift in $\mathbb{C}_+^{m(n+1)}$ and where $U, M \in \mathbb{C}^{m(n+1) \times m}$ are given by

$$
U = \begin{pmatrix}
I_m \\
0 \\
\vdots \\
0
\end{pmatrix}, \quad M = F_{m,n}K_nU = \begin{pmatrix}
0 \\
0 \\
\vdots \\
0
\end{pmatrix} \begin{pmatrix}
s_0 \\
0 \\
\vdots \\
s_{n-1}
\end{pmatrix}. \tag{1.8}
$$

The matrix $K_n$ (the so–called *Pick matrix* of the HMP) satisfies the following Lyapunov identity

$$
F_{m,n}K_n - K_nF_{m,n}^* = MU^* - UM^* \tag{1.9}
$$

which can be easily verified with help of (1.1), (1.7) and (1.8).

The HMP is called *nondegenerate* if its Pick matrix $K_n$ is strictly positive and it is termed *degenerate* if $K_n$ is singular and positive semidefinite. The parametrization of all solutions to the inequality (1.6) for the case $K_n > 0$ was obtained in [9] and will be recalled in Theorem 1.3 below. To formulate this theorem we first introduce some needed definitions and notations. We will denote by $W$ the class of $\mathbb{C}^{2m \times 2m}$–valued meromorphic functions $\Theta$ which are $J$–unitary on $\mathbb{R}$ and $J$–expansive in $\mathbb{C}_+$:

$$
\Theta(z)J\Theta(z)^* = J \quad (z \in \mathbb{R}), \quad \Theta(z)J\Theta(z)^* \geq J \quad (z \in \mathbb{C}_+) \tag{1.10}
$$

where

$$
J = \begin{pmatrix}
0 & iI_m \\
-iI_m & 0
\end{pmatrix}. \tag{1.11}
$$
Definition 1.2. A pair \( \{p, q\} \) of \( \mathbb{C}^{m \times m} \)-valued functions meromorphic in \( \mathbb{C} \backslash \mathbb{R} \) is called a Nevanlinna pair if

\[
\begin{align*}
(i) & \quad \det (p(z)^* p(z) + q(z)^* q(z)) \neq 0 \quad \text{(the nondegeneracy of the pair)} \\
(ii) & \quad \frac{q(z)^* p(z) - p(z)^* q(z)}{z - \bar{z}} = (p(z)^*, q(z)^*) \frac{J}{i(z - \bar{z})} \left( \begin{array}{c} p(z) \\ q(z) \end{array} \right) \geq 0 \quad (\exists z \neq 0).
\end{align*}
\] (1.12)

A pair \( \{p, q\} \) is said to be equivalent to the pair \( \{p_1, q_1\} \) if there exists a \( \mathbb{C}^{m \times m} \)-valued function \( \Omega \) (\( \det \Omega(z) \neq 0 \)) meromorphic in \( \mathbb{C} \backslash \mathbb{R} \) such that \( p_1 = p \Omega \) and \( q_1 = q\Omega \). The set of all \( m \times m \) matrix valued Nevanlinna pairs will be denoted by \( \mathbf{N}_m \).

Theorem 1.3. Let \( K_n \) be a strictly positive matrix given by (1.7) and let \( F_{m,n}, U \) and \( M \) be defined by (1.7), (1.8). Then

1. The function
   \[
   \Theta(z) = \left( \begin{array}{cc} \theta_{11}(z) & \theta_{12}(z) \\ \theta_{21}(z) & \theta_{22}(z) \end{array} \right) = I_{2m} + z \left( \begin{array}{cc} M^* \\ -U^* \end{array} \right)(I - zF_{m,n}^*)^{-1}K_n^{-1}(U, M) \quad (1.13)
   \]
   belongs to the class \( \mathbf{W} \).

2. The formula
   \[
   w(z) = (\theta_{11}(z)p(z) + \theta_{12}(z)q(z))(\theta_{21}(z)p(z) + \theta_{22}(z)q(z))^{-1} \quad (1.14)
   \]
gives all the solutions \( w(z) \) to the inequality (1.6) when \( \{p, q\} \) varies in \( \mathbf{N}_m \).

3. Two pairs \( \{p(z), q(z)\} \) and \( \{p_1(z), q_1(z)\} \) lead by (1.14) to the same function \( w(z) \) if and only if these pairs are equivalent.

The degenerate scalar HMP is simple: \( \mathcal{R}(K_n) \) consists of the unique rational function \( w(z) \) (this follows immediately from (1.6)). In the degenerate matrix case, the description of \( \mathcal{R}(K_n) \) depends on the degeneracy of \( K_n \), but we still have a parametrization of all the solutions as a linear fractional transformation \( (1.13) \) with the coefficient matrix \( \Theta \) from the class \( \mathbf{W} \) and for a suitable choice of parameters \( \{p, q\} \) (see Theorem 1.6 below). To construct the coefficient matrix of the degenerate HMP, we follow the method of V. Dubovoj which was applied in [4] to the degenerate Schur problem. Note that if \( \det \theta_{22} \neq 0 \), the transformation \( (1.14) \) can be written as

\[
   w(z) = \psi_{11}(z) + \psi_{12}(z)p(z)(\psi_{22}(z)p(z) + q(z))^{-1}\psi_{21} \quad (1.15)
\]
where

\[
   \psi_{11} = \theta_{11}\theta_{22}^{-1}, \quad \psi_{12} = \theta_{11} - \theta_{12}\theta_{22}^{-1}\theta_{22}, \quad \psi_{21} = \theta_{22}^{-1}, \quad \psi_{22} = \theta_{22}^{-1}\theta_{21} \quad (1.16)
\]
and it turns out that the function \( \Psi(z) = \left( \begin{array}{cc} \psi_{11}(z) & \psi_{12}(z) \\ \psi_{21}(z) & \psi_{22}(z) \end{array} \right) \) is a Pick function (i.e. analytic and with positive semidefinite imaginary part in \( \mathbb{C}_+ \)). If \( \det \theta_{22} \equiv 0 \), formulas \( (1.16) \) make no sense, but nevertheless the set \( \mathcal{R}(K_n) \) can be parametrized by the transformation \( (1.15) \) with a coefficient matrix \( \Psi \) from the Pick class. This \( \Psi \) can be constructed as a characteristic function of certain unitary colligation associated with the initial data \( \{s_j\} \) of the problem. This approach (see [8]) is much more stable with respect to a possible degeneracy of the Pick matrix \( K_n \). The degenerate HMP will be discussed in some more detail in Section 2.
2. Positive semidefinite Hankel extensions of Hankel block matrices

Let $\mathcal{H}_{m,n}$ be the set of all positive semidefinite Hankel block matrices of the form (1.1). We say that a matrix $K_n \in \mathcal{H}_{m,n}$ admits a positive semidefinite Hankel extension if there exist Hermitian matrices $s_2^{n+1}, s_2^{n+2} \in \mathbb{C}^{m \times m}$ such that the block matrix $K_{n+1} = (s_{i+j})_{i,j=0}^{n+1}$ is still positive semidefinite. The class of such matrices will be denoted by $\mathcal{H}_{m,n}^+$:

$$\mathcal{H}_{m,n}^+ = \{ K_n \in \mathcal{H}_{m,n} : (s_{i+j})_{i,j=0}^{n+1} \geq 0 \text{ for some } s_1 = s_1^* \text{ and } s_2 = s_2^* \}.$$ (2.1)

In the scalar case ($m = 1$) every positive semidefinite Hankel matrix admits a positive semidefinite Hankel extension and therefore, $\mathcal{H}_{1,n}^+ = \mathcal{H}_{1,n}$. For $n \geq 2$, $\mathcal{H}_{m,n}^+$ is a proper subset of $\mathcal{H}_{m,n}$ as can be seen from the example

$$K_2 = \begin{pmatrix} s_0 & s_1 \\ s_1 & s_2 \end{pmatrix}, \quad s_0 = s_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad s_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

We introduce two more subsets of $\mathcal{H}_{m,n}$:

$$\tilde{\mathcal{H}}_{m,n} := \{ K_n \in \mathcal{H}_{m,n} : \text{P Ker} K_n^{-1} = 0 \}$$ (2.2)

and

$$\hat{\mathcal{H}}_{m,n} := \{ K_n \in \mathcal{H}_{m,n} : s_{2n} = \int_{-\infty}^{\infty} \lambda^{2n} d\sigma(\lambda) \text{ for some } \sigma \in Z(K_n) \}.$$ (2.3)

Thus, $\hat{\mathcal{H}}_{m,n}$ consists of all matrices $K_n \in \mathcal{H}_{m,n}$, the associated truncated Hamburger moment problem admits an “exact” solution $\sigma$ such that

$$\int_{-\infty}^{\infty} \lambda^k d\sigma(\lambda) = s_k \quad (k = 0, \ldots, 2n),$$ (2.4)

that is, with equality for the last assigned moment $s_{2n}$ rather than inequality (1.3). In (2.2) and in what follows, $\text{P Ker} K$ denotes the orthogonal projection onto the kernel of $K$. We will show below that

$$\mathcal{H}_{m,n}^+ = \tilde{\mathcal{H}}_{m,n} = \hat{\mathcal{H}}_{m,n}$$ (2.5)

which will provide therefore, several equivalent characterizations of Hankel block matrices admitting positive semidefinite Hankel extensions. The following two propositions can be easily verified.

Lemma 2.1. The block matrix $T = (t_{ij})_{i,j=0}^{n} \quad (t_{ij} \in \mathbb{C}^{r \times l})$ is Hankel if and only if

$$F_{l,n}^*(F_{l,n} T - TF_{r,n}^*)F_{r,n} = 0$$ (2.6)

where $F$ is a shift matrix defined via (1.7).

Lemma 2.2. Let $K, V \in \mathbb{C}^{N \times N}$ and $A \in \mathbb{C}^{N \times r}$ be matrices such that $K = K^*$ and det $V \neq 0$. Then, $\text{P Ker} K A = \text{P Ker} V K V^* V A$.

Given a $K \geq 0$, let $Q$ be a matrix such that

$$QKQ^* > 0 \quad \text{and} \quad \text{rank} QKQ^* = \text{rank} K.$$ (2.7)

We define the pseudoinverse matrix $K^{-1}$ by

$$K^{-1} = K^* (QKQ^*)^{-1} Q.$$ (2.8)

Since the pseudoinverse matrix depends on the choice of $Q$, it is not uniquely defined.
Lemma 2.3. For every choice of $K^{[-1]}$,\[ I - KK^{[-1]} = \left( I - KK^{[-1]} \right) P_{KerK}. \] (2.9)

**Proof:** By (2.7), every vector $f$ can be decomposed as $f = g + hQ$ for some $g \in Ker\ K$ and $h \in \mathbb{C}^{1 \times \text{rank}K}$. Therefore,\[ f \left( I - KK^{[-1]} \right) = \left( g + hQ \right) \left( I - KQ(QK)^{-1}Q \right) = g \]
which implies (2.9). \[ \square \]

Lemma 2.4. The block matrix $\left( \begin{array}{cc} K & B \\ B^* & C \end{array} \right)$ is positive semidefinite if and only if $K \geq 0$, $P_{KerKB} = 0$ and $R = C - B^*K^{[-1]}B \geq 0$.
Moreover, if $\left( \begin{array}{cc} K & B \\ B^* & C \end{array} \right) \geq 0$, then the matrix $R$ does not depend on the choice of $K^{[-1]}$.

**Proof:** The first assertion of lemma follows from the factorization\[ \left( \begin{array}{cc} K & B \\ B^* & C \end{array} \right) = \left( \begin{array}{cc} I & 0 \\ B^* & K^{[-1]} \end{array} \right) \left( \begin{array}{cc} K & I \end{array} \right) B \]
which in view of (2.9), is valid if and only if $P_{KerKB} = 0$.
Furthermore, let $C$ admit two different representations $C = R_i + B^*K^{[-1]}_iB$ ($i = 1, 2$). Then\[ R_1 - R_2 = B^* \left( K^{[-1]}_2 - K^{[-1]}_1 \right) B. \] (2.10)

In view of (2.9),\[ K \left( K^{[-1]}_2 - K^{[-1]}_1 \right) B = \left\{ \left( I - KK^{[-1]}_1 \right) - \left( I - KK^{[-1]}_2 \right) \right\} P_{KerKB} = 0. \]
Since $\left( \begin{array}{cc} K & B \\ B^* & C \end{array} \right) \geq 0$, then also $B^* \left( K^{[-1]}_2 - K^{[-1]}_1 \right) B = 0$ which both with (2.10) implies $R_1 = R_2$. \[ \square \]

Lemma 2.5. Let $K_n \in \mathcal{H}_{m,n}$ and let $\mathcal{L}$ be the subspace of $\mathbb{C}^{1 \times m}$ given by\[ \mathcal{L} = \{ f \in \mathbb{C}^{1 \times m} : \left(f_0, \ldots, f_{n-2}, f\right) \in KerK_{n-1} \ for \ some \ f_0, \ldots, f_{n-2} \in \mathbb{C}^{1 \times m} \}. \] (2.11)
Then $K_n$ belongs to $\bar{\mathcal{H}}_{m,n}$, that is $P_{KerK_{n-1}} \left( \begin{array}{c} s_{n+1} \\ \vdots \\ s_{2n} \end{array} \right) = 0,$ (2.12)
if and only if the block $s_{2n}$ is of the form\[ s_{2n} = (s_n, \ldots, s_{2n-1})K_n^{[-1]}(s_n, \ldots, s_{2n-1})^* + R \] (2.13)
for some positive semidefinite matrix $R \in \mathbb{C}^{m \times m}$ which vanishes on the subspace $\mathcal{L}$ and does not depend on the choice of $K^{[-1]}_{n-1}$.

**Proof:** Since $K_n \geq 0$, then by Lemma 2.4,\[ s_{2n} - (s_n, \ldots, s_{2n-1})K_n^{[-1]}(s_n, \ldots, s_{2n-1})^* \geq 0 \]
and therefore, \( s_{2n} \) admits a representation \((2.13)\) for some \( R \geq 0 \). Moreover, since \( K_n \geq 0 \), then for every vector \((f_0, \ldots, f_{n-1})\) from \( \text{Ker} K_{n-1} \)

\[
(f_0, \ldots, f_{n-1}) \begin{pmatrix} s_1 & \cdots & s_n \\ \vdots & \ddots & \vdots \\ s_{n-1} & \cdots & s_{2n-1} \end{pmatrix} = 0
\]

and therefore,

\[
f_{n-1}(s_n, \ldots, s_{2n-1}) = -(f_0, \ldots, f_{n-2}) \begin{pmatrix} s_1 & \cdots & s_n \\ \vdots & \ddots & \vdots \\ s_{n-1} & \cdots & s_{2n-2} \end{pmatrix} = -(0, f_0, \ldots, f_{n-2}) K_{n-1}. \tag{2.14}
\]

Thus,

\[
f_0 s_{n+1} + \ldots + f_{n-2} s_{2n-1} + f_{n-1}(s_n, \ldots, s_{2n-1}) K_{n-1}^{-1}(s_n, \ldots, s_{2n-1})^* = (0, f_0, \ldots, f_{n-2}) \left( I - K_{n-1} K_{n-1}^{-1} \right) (s_n, \ldots, s_{2n-1})^* = (0, f_0, \ldots, f_{n-2}) \left( I - K_{n-1} K_{n-1}^{-1} \right) P_{\text{Ker} K_{n-1}}(s_n, \ldots, s_{2n-1})^* = 0 \tag{2.15}
\]

where the first equality holds due to \((2.11)\), the second follows by \((2.9)\) and the last one holds since \( K_n \geq 0 \) and therefore, \( P_{\text{Ker} K_{n-1}}(s_n, \ldots, s_{2n-1})^* = 0 \). Comparing \((2.15)\) with \((2.13)\) gives

\[
f_0 s_{n+1} + \ldots + f_{n-1} s_{2n} = f_{n-1} R. \tag{2.16}
\]

It remains to show that \( R \) vanishes on the subspace \( \mathcal{L} \) if and only if \((2.12)\) holds. To this end, let us observe that condition \((2.12)\) means that \( f_0 s_{n+1} + \ldots + f_{n-1} s_{2n} = 0 \) for every vector \((f_0, \ldots, f_{n-1}) \in \text{Ker} K_{n-1} \). The latter is equivalent, in view of \((2.10)\) and \((2.11)\), to \( f_{n-1} R = 0 \) for all \( f_{n-1} \in \mathcal{L} \). By Lemma 2.4, the matrix \( R = s_{2n} - (s_n, \ldots, s_{2n-1}) K_{n-1}^{-1}(s_n, \ldots, s_{2n-1})^* \) does not depend on the choice of \( K_{n-1}^{-1} \).

\[ \square \]

**Lemma 2.6.** Let \( \mathcal{H}^+_{m,n}, \widetilde{\mathcal{H}}_{m,n} \) and \( \check{\mathcal{H}}_{m,n} \) be the classes defined in \((2.1)-(2.3)\). Then

\[
\mathcal{H}^+_{m,n} \subseteq \widetilde{\mathcal{H}}_{m,n} \subseteq \check{\mathcal{H}}_{m,n}. \tag{2.17}
\]

**Proof:** Let \( K_{n+1} \) be a positive semidefinite Hankel extension of \( K_n \). Since \( K_{n+1} \geq 0 \), by the solvability criterion for the associated Hamburger moment problem, the set \( \mathcal{Z}(K_{n+1}) \) is nonempty. Furthermore, for every \( \sigma \in \mathcal{Z}(K_{n+1}) \)

\[
\int_{-\infty}^{\infty} \lambda^k d\sigma(\lambda) = s_k \quad (k = 0, \ldots, 2n + 1) \text{ and } \int_{-\infty}^{\infty} \lambda^{2n+2} d\sigma(\lambda) \leq s_{2n+2}
\]

and therefore, \( K_n \in \check{\mathcal{H}}_{m,n} \) which proves the first containment in \((2.17)\).

Now let us assume that \( K_n \) belongs to \( \check{\mathcal{H}}_{m,n} \) and let \( d\sigma \) be the measure satisfying conditions \((2.4)\). Then

\[
K_n = \int_{-\infty}^{\infty} (I_m, \ldots, \lambda^n I_m)^* d\sigma(\lambda) (I_m, \ldots, \lambda^n I_m). \tag{2.18}
\]

Let \( f = (f_0, \ldots, f_{n-1}) \in \mathbb{C}^{1 \times mn} \) be a vector from \( \text{Ker} K_{n-1} \). Then \( \int_{-\infty}^{\infty} f(\lambda) d\sigma(\lambda) f(\lambda)^* = 0 \), where

\[
f(\lambda) = f_0 + \lambda f_1 + \ldots + \lambda^{n-1} f_{n-1} = f(I_m, \ldots, \lambda^{n-1} I_m)^*. \tag{2.19}
\]
In particular, for every choice of $-\infty < a < b < +\infty$,
\[ \int_a^b f(\lambda) d\sigma(\lambda) f(\lambda)^* = 0. \quad (2.20) \]

Let $g \in \mathbb{C}^{1 \times m}$ be an arbitrary nonzero vector. By the Cauchy inequality,
\[ \int_a^b f(\lambda) d\sigma(\lambda) \lambda^{n+1} g^* \leq \left( \int_a^b f(\lambda) d\sigma(\lambda) f(\lambda)^* \int_a^b \lambda^{2n+2} g d\sigma(\lambda) g^* \right)^{\frac{1}{2}} \]
which in view of (2.21) implies $\int_a^b f(\lambda) d\sigma(\lambda) \lambda^{n+1} g^* = 0$. Since $a, b \in \mathbb{R}$ and $g \in \mathbb{C}^{1 \times m}$ are arbitrary, then
\[ \int_{-\infty}^{\infty} f(\lambda) d\sigma(\lambda) \lambda^{n+1} I_m = 0 \]
which on account of (2.21)–(2.19) can be rewritten as
\[ f(s_{n+1}, \ldots, s_{2n})^* = 0. \quad (2.21) \]

Thus, every vector $f \in \text{Ker} K_{n-1}$ satisfies (2.22) or in other words, $P_{\text{Ker} K_{n-1}} (s_{n+1}, \ldots, s_{2n})^* = 0$ and therefore, $K_n \in \hat{H}_{m,n}$, which completes the proof of the second inclusion in (2.17). □

In connection with the last lemma we consider the following question: to describe all matrices $s \in \mathbb{C}^{m \times m}$ such that $s = \int_{-\infty}^{\infty} \lambda^{2n} d\sigma(\lambda)$ for some $\sigma \in \mathcal{Z}(K_n)$.

**Lemma 2.7.** Let $K_n \geq 0$ be a block matrix of the form (1.1) with the block $s_{2n}$ of the form
\[ s_{2n} = (s_n, \ldots, s_{n-1}) K_{n-1}^{-1} (s_n, \ldots, s_{n-1})^* + R \quad (2.22) \]
for some matrix $R \geq 0$ (which does not depend on the choice of $K_{n-1}^{-1}$) and let $s \in \mathbb{C}^{m \times m}$ be defined by
\[ s = \int_{-\infty}^{\infty} \lambda^{2n} d\sigma(\lambda) \quad (2.23) \]
for some $\sigma \in \mathcal{Z}(K_n)$. Then there exists a positive semidefinite matrix $R_0 \leq R$ which vanishes on the subspace $\mathcal{L}$ defined by (2.11) and such that
\[ s = (s_n, \ldots, s_{2n-1}) K_{n-1}^{-1} (s_n, \ldots, s_{2n-1})^* + R_0 \quad (0 \leq R_0 \leq R \quad \text{and} \quad R_0|_{\mathcal{L}} = 0). \quad (2.24) \]

**Proof:** Let $s$ be of the form (2.23) for some $\sigma \in \mathcal{Z}(K_n)$. We introduce the Hankel block matrix
\[ \tilde{K}_n = \begin{pmatrix} s_0 & \ldots & s_{n-1} & s_n \\ \vdots & & \vdots \\ s_{n-1} & \ldots & s_{2n-1} & s \end{pmatrix} \quad (2.25) \]
which differs from $\bar{K}_n$ only by the block $\tilde{s}_{2n} = s$. Thus, $\tilde{K}_n \in \hat{H}_{m,n}$. Therefore, $\tilde{K}_n \in \hat{H}_{m,n}$, by Lemma 2.6. By Lemma 2.5, the block $\tilde{s}_{2n} = s$ admits representation (2.24) for some $R_0 \geq 0$ vanishing on $\mathcal{L}$. The inequality $R_0 \leq R$ follows from (1.3) and (2.22)–(2.24). □

**Lemma 2.8.** Let $K_n \in \hat{H}_{m,n}$ be of the form (1.1), let $\mathcal{L}$ be the subspace given by (2.11), let $s_{2n}, s$ and $\tilde{K}_n$ be matrices defined by (2.22), (2.24) and (2.25) respectively, and let the positive semidefinite $R_0 : \mathbb{C}^m \rightarrow \mathbb{C}^m$ be defined by
\[ R_0 h = \begin{cases} 0 & \text{for } h \in \mathcal{L}, \\ R h & \text{for } h \in \mathcal{L}^\perp. \end{cases} \quad (2.26) \]
Then the Hamburger moment problems associated with the sets of matrices \( \{s_0, \ldots, s_{2n-1}, s_{2n}\} \) and \( \{s_0, \ldots, s_{2n-1}, s\} \) have the same solutions: \( \mathcal{Z}(K_n) = \mathcal{Z}(\hat{K}_n) \).

**Proof:** Let \( \sigma \) belong to \( \mathcal{Z}(K_n) \). By Lemma 2.7 the matrix \( \hat{s} = \int_{-\infty}^{\infty} \lambda^{2n} d\sigma(\lambda) \) admits a representation \( (2.22) \) with a positive semidefinite matrix \( \hat{R}_0 \leq R \) vanishing on \( L \). In view of \( (2.26) \), \( \hat{R}_0 \leq R_0 \). Therefore, \( \hat{s} \leq s \) and \( \sigma \in \mathcal{Z}(\hat{K}_n) \). So, \( \mathcal{Z}(K_n) \subseteq \mathcal{Z}(\hat{K}_n) \). The converse inclusion follows from the inequality \( s \leq s_{2n} \).

**Remark 2.9.** By Lemmas 2.4 and 2.8 we can assume without loss of generality that the Pick matrix of the HMP belongs to \( \mathcal{H}_{m,n} \).

Otherwise we replace the block \( s_{2n} \) (which is necessarily of the form \( (2.22) \)) by the block \( \hat{s}_{2n} = s \) defined by \( (2.24), (2.26) \). By Lemma 2.8, \( \hat{K}_n \in \mathcal{H}_{m,n} \) and we describe the set \( \mathcal{Z}(\hat{K}_n) \) of solutions of this new moment problem, which coincides, by Lemma 2.8, with \( \mathcal{Z}(K_n) \).

3. The Coefficient Matrix of the Problem

The coefficient matrix \( \Theta \) of the nondegenerate HMP given by the formula \( (1.13) \) is the matrix polynomial of \( \text{deg } \Theta = n+1 \) and \( (1.13) \) is a realization of \( \Theta \) with state space equal \( \mathbb{C}^{m(n+1)} \). In this section we obtain some special decomposition (see formula \( (5.13) \)) of the state space which will allow us to construct the analogue of \( (1.13) \) for \( K_n \) not strictly positive (formula \( (5.23) \)). The idea is simple: to replace in \( (1.13) \) the inverse of the matrix \( K_n \) (which does not exist for the degenerate case) by its pseudoinverse. However after this replacement the function \( \Theta \) may lose its \( J \)-properties \( (1.10) \) which are essential for the description \( (1.14) \) to be in force. This suggests the following question: is there exist a pseudoinverse matrix \( K_n^{-1} \) of the form \( (2.8) \) such that the function

\[
\Theta(z) = I_{2m} + z \begin{pmatrix} M^* & 0 \\ -U^* & (I - zF^*_{m,n})^{-1}K_n^{-1} \end{pmatrix} (U, M)
\]

still belongs to the class \( W \)? We show in Lemmas 3.2 and 3.3 below that such a pseudoinverse exists if (and in fact, only if) the Pick matrix \( K_n \) belongs to the class \( \mathcal{H}_{m,n} \). Recall that for the degenerate matricial Schur problem such a pseudoinverse always exists (see \( [4] \)).

**Lemma 3.1.** Let \( T_n = (t_{i+j})_{i,j=0}^{n} \in \mathcal{H}_{l,n} \) \( (t_i \in \mathbb{C}^{l \times l}) \), let \( t_0 > 0 \) and let \( \hat{T}_{n-1} \) be the block matrix defined as

\[
\hat{T}_{n-1} = D_n^{-1} \{ S - T_n t_0^{-1} T_n^* \} D_n^{-*} \tag{3.1}
\]

where

\[
D_n = \begin{pmatrix}
0 & \ldots & 0 \\
t_0 & \ddots & \vdots \\
0 & \ddots & \ddots \\
t_1 & \ldots & t_n
\end{pmatrix}, \quad S = (t_{i+j})_{i,j=1}^{n}, \quad T_n = \begin{pmatrix}
t_1 \\
\vdots \\
t_n
\end{pmatrix} \tag{3.2}
\]

Then \( \hat{T}_{n-1} \) is a Hankel block matrix:

\[
\hat{T}_{n-1} = (\hat{t}_{i+j})_{i,j=0}^{n-1}
\]

and moreover, \( \hat{T}_{n-1} \in \widetilde{\mathcal{H}}_{l,n-1} \).

**Proof:** Let \( F_{l,n-1} \) be the matrix defined via formula \( (1.7) \) and let

\[
\tilde{U} := (I_l, 0, \ldots, 0)^* \in \mathbb{C}^{ln \times l}. \tag{3.4}
\]

We begin with the identities

\[
D_n F_{l,n-1} = F_{l,n-1} D_n, \quad \tilde{U} F_{l,n-1} = 0, \quad D_n \tilde{U} - F_{l,n-1} T_n = \tilde{U} t_0 \tag{3.5}
\]
and
\[ F_t t_{n-1} (S - T_n t_0^{-1} T_n^*) - (S - T_n t_0^{-1} T_n^*) F_{t_{n-1}} = T_n t_0^{-1} \tilde{U}^* D_n^* - D_n \tilde{U} t_0^{-1} T_n^* \]
which follow immediately from (3.7), (3.2) and (3.4). Using these identities we get
\[ F_{t_{n-1}} (F_t t_{n-1} \tilde{T}_n - \tilde{T}_n F_{t_{n-1}}) F_{t_{n-1}} \]
\[ = F_{t_{n-1}} D_n^{-1} \left\{ F_{t_{n-1}} (S - T_n t_0^{-1} T_n^*) - (S - T_n t_0^{-1} T_n^*) F_{t_{n-1}} \right\} D_n^{-*} F_{t_{n-1}} \]
\[ = F_{t_{n-1}} D_n^{-1} \left\{ T_n t_0^{-1} \tilde{U}^* D_n^* - D_n \tilde{U} t_0^{-1} T_n^* \right\} D_n^{-*} F_{t_{n-1}} \]
\[ = F_{t_{n-1}} D_n^{-1} T_n t_0^{-1} \tilde{U}^* F_{t_{n-1}} - F_{t_{n-1}} \tilde{U} t_0^{-1} T_n^* F_{t_{n-1}} = 0 \]
and (3.3) follows by Lemma 2.1. Since \( D_n \) is invertible, the factorization formula
\[ T_n = \begin{pmatrix} I & 0 \\ T_n t_0^{-1} & D_n \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \tilde{T}_{n-1} \end{pmatrix} \begin{pmatrix} I & t_0^{-1} T_n^* \\ 0 & D_n \end{pmatrix} \]
(3.6)
implies that \( \tilde{T}_{n-1} \geq 0 \) and thus, \( \tilde{T}_{n-1} \in \mathcal{H}_{t_{n-1}} \). It remains to verify that
\[ P_{Ker \tilde{T}_{n-2}} \begin{pmatrix} \tilde{t}_n \\ \vdots \\ \tilde{t}_{n-2} \end{pmatrix} = 0. \]
(3.7)
To this end, we first observe that
\[ P_{Ker \tilde{T}_{n-1}} (T_n, S) = 0 \]
(3.8)
since \( T_n \geq 0 \). Using the factorization of \( T_{n-1} \) similar to (3.7) we obtain
\[ \begin{pmatrix} s_0 & 0 \\ 0 & \tilde{T}_{n-2} \end{pmatrix} = \begin{pmatrix} I & 0 \\ -D_n^{-1} T_{n-1} t_0^{-1} & D_n^{-1} \end{pmatrix} T_{n-1} \begin{pmatrix} I & -t_0^{-1} T_n^* D_n^{-*} \\ 0 & D_n^{-*} \end{pmatrix} \]
(3.9)
where \( D_{n-1} \) and \( T_{n-1} \) are defined via (3.2). Upon applying Lemma 2.2 to the matrices
\[ K = T_{n-1}, \quad V = \begin{pmatrix} I & 0 \\ -D_n^{-1} T_{n-1} t_0^{-1} & D_n^{-1} \end{pmatrix} \quad \text{and} \quad A = (T_n, S), \]
and making use of (3.8), (3.9) we obtain
\[ P_{Ker \tilde{T}_{n-2}} D_n^{-1} (-T_{n-1} t_0^{-1}, I_{mn}) (T_n, S) = 0. \]
(3.10)
From the block decomposition \( D_n = \begin{pmatrix} t_0 & 0 \\ T_{n-1} & D_n^{-1} \end{pmatrix} \) we have
\[ D_n^{-1} = \begin{pmatrix} t_0^{-1} & 0 \\ -D_{n-1} t_0^{-1} & D_n^{-1} \end{pmatrix}. \]
(3.11)
Substituting (3.11) into (3.1) we obtain
\[ \begin{pmatrix} \tilde{t}_1 & \cdots & \tilde{t}_n \\ \vdots \\ \tilde{t}_{n-1} & \cdots & \tilde{t}_{2n-2} \end{pmatrix} = (0, I_{n(n-1)}) \tilde{T}_{n-1} = D_{n-1}^{-1} (-T_{n-1} t_0^{-1}, I_{mn}) (S - T_n t_0^{-1} T_n^*) D_n^{-*}. \]
The last equality both with (3.10) implies
\[ P_{Ker \tilde{T}_{n-2}} \begin{pmatrix} \tilde{t}_1 & \cdots & \tilde{t}_n \\ \vdots \\ \tilde{t}_{n-1} & \cdots & \tilde{t}_{2n-2} \end{pmatrix} = 0. \]
and, in particular, (3.7), which completes the proof of lemma.

\[ \square \]

**Lemma 3.2.** Let \( K_n \in \hat{\mathcal{H}}_{m,n} \) and let rank \( K_n = r \). Then there exists \( Q \in \mathbb{C}^{r \times (n+1)m} \) such that

\[ QK_nQ^* > 0, \quad \text{rank } QK_nQ^* = \text{rank } K_n, \quad QF_{m,n} = NQ \]  

(3.12)

for the shift \( F_{m,n} \) defined by (1.7) and some matrix \( N \in \mathbb{C}^{r \times r} \). In other words, there exists a subspace \( Q = \text{Ran } Q = \{ y \in \mathbb{C}^{m(n+1)} : y = fQ \text{ for some } f \in \mathbb{C}^r \} \) coinvariant with respect to \( F_{m,n} \) and such that

\[ \mathbb{C}^{m(n+1)} = Ker K + Q. \]  

(3.13)

**Proof:** We prove this lemma by induction. Let \( n = 0 \) and let rank \( s_0 = l \leq m \). Then there exists a unitary matrix \( v \in \mathbb{C}^{l \times m} \) such that

\[ vs_0v^* = \begin{pmatrix} t_0 & 0 \\ 0 & 0 \end{pmatrix} \quad (t_0 > 0), \]  

(3.14)

and the matrix

\[ g = (I_l, 0)v \in \mathbb{C}^{l \times m} \]  

(3.15)

(considered as \( Q \)) clearly satisfies (3.12).

Let us suppose that the statement of the lemma holds for all integers up to \( n - 1 \). Let as above, rank \( s_0 = l \) and let \( v \) and \( g \) be matrices defined by (3.14), (3.15). Since \( K_n \in \hat{\mathcal{H}}_{m,n} \), we have \( Ker s_0 \subseteq Ker s_i \) for \( i = 1, \ldots, 2n \), and then we have from (3.14),

\[ vs_i vr = \begin{pmatrix} t_i & 0 \\ 0 & 0 \end{pmatrix} \quad (t_i \in \mathbb{C}^{l \times l}; \ i = 1, \ldots, 2n). \]  

(3.16)

In more detail, representations (3.16) for \( i = 1, \ldots, 2n - 1 \) follow from positivity of \( K_n \) along with its Hankel structure. Since \( K_n \) belongs to \( \hat{\mathcal{H}}_{m,n} \), equality (2.12) holds. Upon substituting decompositions (3.14) and (3.16) (for \( i = 1, \ldots, 2n - 1 \)) into (2.12), one can easily see that \( s_{2n} \) is necessarily of the form \( vs_{2n} v^* = \begin{pmatrix} t_{2n} & \gamma \\ 0 & 0 \end{pmatrix} \) for some \( \gamma \in \mathbb{C}^{l \times (m-l)} \). Since \( s_{2n} \) is Hermitian, \( \gamma = 0 \) and representation (3.16) for \( s_{2n} \) follows.

From (3.14)–(3.16) we obtain that \( g_{2n} = t_i \) (i.e., \( 0, \ldots, 2n \)) and

\[ T_n = (t_{i+1})^{2n}_{i=0} = G_nK_nG_n^*, \quad \text{rank } T_n = \text{rank } K_n, \]  

(3.17)

where \( G_n \) is the \((n+1)l \times (n+1)m \) matrix defined by

\[ G_n = \begin{pmatrix} g & 0 \\ 0 & \ddots \\ 0 & g \end{pmatrix}. \]  

(3.18)

Since \( K_n \in \hat{\mathcal{H}}_{m,n} \), then it is readily seen that \( T_n \in \hat{\mathcal{H}}_{l,n} \). Let \( \hat{T}_{n-1}, D_n \) and \( T_n \) be matrices defined by (3.1), (5.2). Multiplying \( K_n \) on the left by the matrix

\[ \Phi = \begin{pmatrix} I_l \\ -D_n^{-1}T_n^{-1}D_n^{-1} \end{pmatrix}. \]  

(3.19)

and by \( \Phi^* \) on the right we obtain, on account of of (3.18) and (5.0),

\[ \Phi K_n \Phi^* = \begin{pmatrix} t_{0} & 0 \\ 0 & \hat{T}_{n-1} \end{pmatrix}. \]  

(3.20)
By Lemma 3.1, \( \hat{T}_{n-1} \in \tilde{H}_{l,n-1} \), and it follows from (3.17), (3.19) and (3.20) that \( \text{rank} \: \hat{T}_{n-1} = \text{rank} \: K_n - \text{rank} \: t_0 = r - l \). Therefore, by the induction hypothesis, there exist matrices \( \tilde{Q} \in \mathbb{C}^{(r-l) \times l} \) and \( \tilde{N} \in \mathbb{C}^{(r-l) \times (r-l)} \) such that
\[
\tilde{Q} \hat{T}_{n-1} \tilde{Q}^* > 0 \quad \text{and} \quad \tilde{Q}F_{l,n} = \tilde{N} \tilde{Q}.
\] (3.21)
We show that the matrices
\[
Q = \begin{pmatrix} I_l & 0 \\ 0 & \tilde{Q} \end{pmatrix} \Phi \in \mathbb{C}^{r \times (n+1)}, \quad N = \begin{pmatrix} 0, \tilde{Q}t_0^{-1} & 0 \\ 0 & \tilde{N} \end{pmatrix} \in \mathbb{C}^{r \times r} \tag{3.22}
\]
(where \( \tilde{U} \) is the matrix given by (3.2)) satisfy (3.12). Indeed, by (3.20)–(3.22)
\[
\text{rank} \: QK_nQ^* = \text{rank} \: t_0 + \text{rank} \: \tilde{Q} \hat{T}_{n-1} \tilde{Q}^* = l + (r - l) = \text{rank} \: K_n.
\] We next make use of (3.19)–(3.21) and of the block decompositions
\[
G_n = \begin{pmatrix} g & 0 \\ 0 & G_{n-1} \end{pmatrix} \quad \text{and} \quad F_{m,n} = \begin{pmatrix} 0 & 0 \\ 0 & F_{m,n-1} \end{pmatrix}
\]
to compute
\[
QF_{m,n} = \begin{pmatrix} \tilde{Q}D_{n-1}G_{n-1} \tilde{U} & \tilde{Q}D_{n-1}G_{n-1}F_{m,n-1} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \tilde{Q}D_{n-1} \tilde{U}g & \tilde{Q}D_{n-1}F_{l,n-1}G_{n-1} \\ 0 & 0 \end{pmatrix}
\]
and
\[
NQ = \begin{pmatrix} 0 & 0 \\ (\tilde{Q} \tilde{U} - \tilde{N} \tilde{Q}D_{n-1} \tilde{T}_n) \tilde{t}_0^{-1} g & \tilde{N} \tilde{Q}D_{n-1}G_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ (\tilde{Q}D_{n-1}(D_n \tilde{U} - F_{l,n-1} \tilde{T}_n) \tilde{t}_0^{-1} g & \tilde{Q}F_{l,n-1}D_n^{-1}G_{n-1} \end{pmatrix}.
\]
We now invoke equalities (3.5) to verify that the right hand side matrices in the two last formulas coincide. Thus, \( QF_{m,n} = NQ = 0 \), and the matrices \( Q \) and \( N \) defined by (3.22) satisfy (3.12). This completes the proof.

In what follows, the indeces will be omitted and by \( K \) and \( F \) we mean matrices \( K_n \) and \( F_{m,n} \) given by (1.1) and (1.7) respectively.

**Lemma 3.3.** Let \( K \in \tilde{H}_{m,n} \), let \( Q \) be any matrix satisfying (3.12) and let \( F, U, M, J \) and \( K^{[-1]} \) be matrices given by (1.7), (1.8), (1.11) and (2.8). Then the \( \mathbb{C}^{2m \times 2m} \)–valued function
\[
\Theta(z) = I_{2m} + z \begin{pmatrix} M^* \\ -U^* \end{pmatrix} (I - zF^*)^{-1} K^{[-1]}(U, M)
\] (3.23)
is of the class \( W \) and moreover,
\[
\Theta(z)^*J\Theta(z) - J = i(\bar{z} - z) \begin{pmatrix} U^* \\ M^* \end{pmatrix} K^{[-1]}(I - \bar{z}F)^{-1} K(I - zF^*)^{-1} K^{[-1]}(U, M), \tag{3.24}
\]
\[
J - \Theta(z)^*J\Theta^{-1}(z) = i(\bar{z} - z) \begin{pmatrix} U^* \\ M^* \end{pmatrix} (I - \bar{z}F^*)^{-1} K^{[-1]}(I - zF)^{-1} (U, M).
\] (3.25)
Observe that the two first relations in (3.12) enable us to construct the pseudoinverse matrix $K[-1]$ according to (2.8) and the third equality guarantees (3.24) and (3.25) to be in force.

**Proof:** Using (3.23), (1.11) and (1.9) we have

$$\Theta(z)^*J\Theta(z) - J = i \begin{pmatrix} U^* & M^* \end{pmatrix} L(z)(U, M)$$

where

$$L(z) = |z|^2 K[-1](I - \bar{z}F)^{-1} \{MU^* - UM^*\} (I - zF^*)^{-1} K[-1]$$

$$+ \bar{z}K[-1](I - \bar{z}F)^{-1} - z(I - zF^*)^{-1} K[-1]$$

$$= (\bar{z} - z)K[-1](I - \bar{z}F)^{-1} K(I - zF^*)^{-1} K[-1]$$

$$+ \bar{z}K[-1](I - \bar{z}F)^{-1}(I - KK[-1]) - z(I - K[-1]K)(I - zF^*)^{-1} K[-1].$$

It follows from (3.12) that $QF^j = N^jQ$ $(j = 0, 1, \ldots)$ which both with (2.8) implies

$$K[-1]F^j \left(I - KK[-1]\right) = Q^*(QKQ^*)^{-1}N^jQ \left(I - QK(QKQ^*)^{-1}Q\right) = 0$$

for $j = 0, 1, \ldots$ Since $(I - zF^*)^{-1} = \sum_{j=0}^{\infty} z^j F^j$, then also

$$K[-1](I - zF)^{-1} \left(I - KK[-1]\right) \quad (z \in \mathbb{C}).$$

Substituting (3.24) into (3.24) and (3.27) into (3.24), we obtain (3.24). Similarly,

$$\Theta(z)J\Theta(z)^* - J = i \begin{pmatrix} M^* & -U^* \end{pmatrix} (I - zF^*)^{-1} \tilde{L}(z)(I - \bar{z}F)^{-1}(M, -U)$$

where

$$\tilde{L}(z) = \bar{z}(I - zF^*)K[-1] - zK[-1](I - \bar{z}F) - |z|^2 K[-1] \{MU^* - UM^*\} K[-1]$$

$$= (\bar{z} - z)K[-1] + |z|^2 K[-1] F(I - KK[-1]) - |z|^2 (I - K[-1]K) F^* K[-1].$$

Using (3.28) for $j = 1$ we obtain from (3.31) that $\tilde{L}(z) = (\bar{z} - z)K[-1]$ and by (3.30),

$$\Theta(z)J\Theta(z)^* - J = i(\bar{z} - z) \begin{pmatrix} M^* & -U^* \end{pmatrix} (I - zF^*)^{-1} K[-1](I - \bar{z}F)^{-1}(M, -U).$$

Relations (1.10) follow from (3.32) and thus, $\Theta \in W$. Since it $\Theta$ is $J$-unitary on $\mathbb{R}$, then by the symmetry principle, $\Theta^{-1} = J\Theta(z)^*J$ which both with (3.32) leads to

$$J - \Theta(z)^*J\Theta^{-1}(z) = J(J - \Theta(z)J\Theta(z)^*)J$$

$$= i(z - \bar{z}) \begin{pmatrix} M^* & -U^* \end{pmatrix} (I - zF^*)^{-1} K[-1](I - \bar{z}F)^{-1}(M, -U)J$$

and implies (3.25).
4. Parametrization of all solutions

In this section we parametrize the set \( \mathcal{R}(K_n) \) of all solutions of the degenerate HMP in terms of a linear fractional transformation. The following theorem can be found in \([7, 9]\).

**Theorem 4.1.** Let \( \Theta = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \) be the block decomposition of a \( \mathbb{C}^{2m \times 2m} \)-valued function \( \Theta \in W \) into four \( \mathbb{C}^{m \times m} \)-valued blocks. Then all \( \mathbb{C}^{m \times m} \)-valued analytic in \( \mathbb{C} \setminus \mathbb{R} \) solutions \( w \) to the inequality

\[
(w(z)^*, I_m) \frac{\Theta(z)^{-1}(z)}{i(z - \bar{z})} \begin{pmatrix} w(z) \\ I_m \end{pmatrix} \geq 0
\]  

are parametrized by the formula

\[
w(z) = (\theta_{11}(z)p(z) + \theta_{12}(z)q(z))(\theta_{21}(z)p(z) + \theta_{22}(z)q(z))^{-1}
\]

when the parameter \( \{p, q\} \) varies in the set \( N_m \) of all Nevanlinna pairs and satisfies

\[
\det (\theta_{21}(z)p(z) + \theta_{22}(z)q(z)) \neq 0;
\]

Moreover, two Nevanlinna pairs lead via (4.3) to the same function \( w \) if and only if these pairs are equivalent.

**Lemma 4.2.** Let \( \{p, q\} \in N_m \) be a Nevanlinna pair. Then

\[
\det (p(z) + iq(z)) \neq 0,
\]

the function

\[
S(z) = (p(z) - iq(z))(p(z) + iq(z))^{-1}
\]

is a \( \mathbb{C}^{m \times m} \)-valued contraction in \( \mathbb{C}_+ \) and moreover, two different pairs lead by (4.2) to the same function \( w \) if and only if these pairs are equivalent.

The proof is given in \([7]\). Observe that by (4.1), every Nevanlinna pair \( \{p, q\} \) satisfies the dual nondegeneracy property (compare with Definition 4.2)

\[
\det (p(z)p(z)^* + q(z)q(z)^*) \neq 0.
\]

**Lemma 4.3.** Let \( \{p, q\} \in N_m \) be a Nevanlinna pair such that \((I_\nu, 0) p(z) \equiv 0 \ (\nu \leq m)\). Then \( \{p, q\} \) is equivalent to a pair

\[
\left\{ \begin{pmatrix} 0 & 0 \\ 0 & \bar{p}(z) \end{pmatrix}, \begin{pmatrix} I_\nu & 0 \\ 0 & \bar{q}(z) \end{pmatrix} \right\}
\]

for some \( \{\bar{p}, \bar{q}\} \in N_{m-\nu} \).

**Proof:** By the assumption assumption, \( p \) and \( q \) are of the form

\[
p(z) = \begin{pmatrix} 0 & 0 \\ p_{21}(z) & p_{22}(z) \end{pmatrix}, \quad q(z) = \begin{pmatrix} q_{11}(z) & q_{12}(z) \\ q_{21}(z) & q_{22}(z) \end{pmatrix}
\]

and in view of (4.6), \( \text{rank} (q_{11}(z), q_{12}(z)) = m \) at almost all \( z \in \mathbb{C}_+ \). Multiplying \((q_{11}(z), q_{12}(z))\) by an appropriate unitary matrix \( U \) on the right we obtain

\[
(q_{11}(z), q_{12}(z)) U = (\bar{q}_{11}(z), \bar{q}_{12}(z)), \quad \det \bar{q}_{11}(z) \neq 0.
\]

The pair \( \{p, q\} \) is equivalent to the pair \( \{p_1, q_1\} \) defined as

\[
\begin{pmatrix} p_1(z) \\ q_1(z) \end{pmatrix} = \begin{pmatrix} p(z) \\ q(z) \end{pmatrix} U \Phi(z) \quad \text{where} \quad \Phi(z) = \begin{pmatrix} \bar{q}_{11}(z) & 0 \\ 0 & -\bar{q}_{11}(z) \bar{q}_{12}(z) \end{pmatrix} \begin{pmatrix} I_{m-\nu} & 0 \\ 0 & \bar{p}(z) \end{pmatrix}
\]

It follows from (4.8) that the functions \( p_1 \) and \( q_1 \) are of the form

\[
p_1(z) = \begin{pmatrix} 0 & 0 \\ \bar{p}_1(z) & \bar{p}(z) \end{pmatrix}, \quad q_1(z) = \begin{pmatrix} I_\nu & 0 \\ \bar{q}_1(z) & \bar{q}(z) \end{pmatrix}
\]
and it remains to show that \{p_1, q_1\} is equivalent to the pair defined in (1.7). Indeed, (1.9)
implies that \{\tilde{p}, \tilde{q}\} ∈ \mathbb{N}_{m−ν}\} and therefore, \det (\tilde{p}(z) + i\tilde{q}(z)) \neq \mathbb{N}_{m−ν}\}.
Substituting the pair (1.9) into (4.5) gives
\[
S(z) = \frac{(p_1(z) - iq_1(z))(p_1(z) + iq_1(z))^{-1}}{-I}
\]
\[
= \left( \begin{array}{cc} -iI & 0 \\ \tilde{p}_1 - i\tilde{q}_1 & \tilde{p} - i\tilde{q} \end{array} \right) \left( \begin{array}{cc} iI & 0 \\ \tilde{p}_1 + i\tilde{q}_1 & \tilde{p} + i\tilde{q} \end{array} \right)^{-1}
\]
\[
= \left( \begin{array}{cc} i(\tilde{p} - i\tilde{q})(\tilde{p} + i\tilde{q})^{-1}(\tilde{p}_1 + i\tilde{q}_1) - i(\tilde{p}_1 - i\tilde{q}_1) & 0 \\ -I & 0 \end{array} \right)
\]
\[
= \left( \begin{array}{cc} -I & 0 \\ (\tilde{p} - i\tilde{q})(\tilde{p} + i\tilde{q})^{-1} \end{array} \right)
\]
(to obtain the last equality we used the following: if the function \( S = \left( \begin{array}{cc} s_1 & 0 \\ s_2 & -I \end{array} \right) \) is contractive valued, then \( s_2 \equiv 0 \). It is easily seen that the pair (1.7) being substituted into (4.5), leads to
the same function \( S \). By Lemma 4.2, the pairs (4.5) and (4.9) are equivalent.

**Lemma 4.4.** Let \( R ∈ \mathbb{C}^{l×2m} \) be a \( J \)-neutral matrix (i.e. \( RJR = 0 \)) and let \( \text{rank} \ R = ν ≤ \text{min}(m, l) \). Then there exists a \( J \)-unitary matrix \( Ψ \) and an invertible \( T \) such that
\[
TRΨ = \left( \begin{array}{cc} I_ν & 0 \\ 0 & 0 \end{array} \right).
\] (4.10)

**Proof:** Since \( \text{rank} \ R = ν \), there exists an invertible matrix \( T \) such that
\[
TR = \left( \begin{array}{cc} \tilde{R} \\ 0_{(m−ν)×2m} \end{array} \right)
\] (4.11)
where \( \tilde{R} \) is a full rank \( J \)-neutral matrix. Let us endow the space \( \mathbb{C}^{l×2m} \) with the indefinite inner product \([x, y] = yJx^*\). By (1.11), the subspace
\[
\mathcal{G} = \{g ∈ \mathbb{C}^{l×2m} : g = (\tilde{g}, 0) \text{ for some } \tilde{g} ∈ \mathbb{C}^{l×ν}\}
\]
is \( J \)-neutral. The subspace \( \mathcal{F} = \{f ∈ \mathbb{C}^{l×2m} : f = \tilde{g}\tilde{R}, \tilde{g} ∈ \mathbb{C}^{l×ν}\} \) \( J \)-neutral as well. Let us introduce the operator \( \tilde{Ψ} : \mathcal{F} → \mathcal{G} \) by \( \tilde{g}\tilde{R}\tilde{Ψ} = (\tilde{g}, 0) \). Since \( \mathcal{F} \) and \( \mathcal{G} \) are \( J \)-neutral and \( \dim \mathcal{F} = \dim \mathcal{G} \), the operator \( \tilde{Ψ} \) is \( J \)-isometric and has equal defect numbers. Furthermore, \( \tilde{Ψ} \) is invertible and therefore, it admits a \( J \)-unitary extension \( Ψ \) to all of \( \mathbb{C}^{l×2m} \). The matrix \( Ψ \) of this extended operator in the standard basis is \( J \)-unitary and satisfies \( \tilde{R}\tilde{Ψ} = (I_ν, 0) \) which both with (1.11) implies (4.10).

**Remark 4.5.** Let \( R = (R_1, R_2) ∈ \mathbb{C}^{l×2m} \) be a \( J \)-neutral matrix: \( R_1R_2^* - R_2R_1^* = 0 \). Then \( \text{rank} \ R = \text{rank} \ (R_1 + iR_2) \). Indeed,
\[
\text{rank} \ (R_1 + iR_2) = \text{rank} \ (R_1 + iR_2)(R_1 + iR_2)^* = \text{rank} \ (R_1R_2^* + R_2R_1^*) = \text{rank} \ RR^* = \text{rank} \ R.
\]

The following theorem is the degenerate analogue of Theorem 1.3.

**Theorem 4.6.** Let the Pick matrix \( K_n \) of the HMP be in the class \( \mathcal{H}_{m,n} \) and let \( Θ \) be the \( \mathbb{C}^{2m×2m} \)–valued function defined by (3.23). Then, there exists a \( J \)-unitary matrix \( Ψ ∈ \mathbb{C}^{2m×2m} \) such that...
(1) All the functions \( w \in \mathcal{R}(K_n) \) are obtained by the formula
\[
  w(z) = (a_{11}(z)p(z) + a_{12}(z)q(z))(a_{21}(z)p(z) + a_{22}(z)q(z))^{-1}
\]
with the coefficient matrix \( A(z) = (a_{ij}(z)) = \Theta(z) \Psi \in \mathcal{W} \) when the parameter \( \{p, q\} \)
varies in the set of all Nevanlinna pairs of the form
\[
  \{p(z), q(z)\} = \left\{ \begin{pmatrix} 0 & 0 \\ \tilde{p}(z) & \tilde{q}(z) \end{pmatrix}, \quad \begin{pmatrix} I & 0 \\ 0 & \tilde{q}(z) \end{pmatrix} \right\}
\]
where \( \tilde{p}, \tilde{q} \in \mathbb{N}_{m-\mu} \) and \( \nu \) is the integer given by
\[
  \nu = \text{rank} \{ (I_m, is_0, \ldots, is_{n-1})P_{\ker K_n} \}.
\]

(2) Two pairs lead to the same function \( w \) if and only if they are equivalent.

**Proof:** According to Theorem 1.1 the set \( \mathcal{R}(K_n) \) coincides with the set of all solutions to the inequality (1.10) which is equivalent, by Lemma 2.4, to the following system
\[
  \frac{w(z) - w(z)^*}{z - \bar{z}} - (Uw(z) + M)^*(I - zF)^{-*}K^{-1}(I - zF)^{-1}(Uw(z) + M) \geq 0,
\]
and is equivalent, in view of (3.25) with the function \( \Theta \) defined by (4.12) which is of the class \( \mathcal{W} \) by Lemma 3.3. According to Theorem 4.1, all solutions \( w \) to the inequality (4.14) are parametrized by the linear fractional transformation (4.12) when the parameter \( \{p, q\} \) varies in the set \( \mathbb{N}_m \) of all Nevanlinna pairs and satisfies (4.3). It remains to choose among these solutions all functions \( w \) which satisfy also identity (4.15). The rest of the proof is broken into four steps which we now specify.

**Step 1:** The function \( w(z) \) of the form (4.2) satisfies the identity (4.15) if and only if the corresponding parameter \( \{p, q\} \) satisfies
\[
  P_{\ker K}(I - zF)^{-1}Uw(z) + M) \equiv 0.
\]

**Step 2:** If a pair \( \{p, q\} \in \mathbb{N}_m \) satisfies (4.16) then it also satisfies (4.13).

**Step 3:** If a pair \( \{p, q\} \in \mathbb{N}_m \) satisfies (4.16) then it is equivalent to some pair \( \{p_1, q_1\} \) of the form
\[
  \begin{pmatrix} p_1(z) \\ q_1(z) \end{pmatrix} = \Psi \begin{pmatrix} 0 & 0 \\ \tilde{p}(z) & \tilde{q}(z) \end{pmatrix} \sim \begin{pmatrix} p(z) \\ q(z) \end{pmatrix}
\]
for some \( J \)-unitary matrix \( \Psi \in \mathbb{C}^{2m \times 2m} \) which depends only on \( K_n \) and a pair \( \{\tilde{p}, \tilde{q}\} \in \mathbb{N}_{m-\nu} \) where \( \nu = \text{rank} P_{\ker K}(U, M) = \text{rank} P_{\ker K}(U + iM) \).

**Proof of Step 1:** Let \( \Theta = (\theta_{ij}) \) be the function defined by (3.23) and let \( w \) be a function of the form (4.2) for some pair \( \{p, q\} \in \mathbb{N}_m \) which satisfies (4.3). Then
\[
  \begin{pmatrix} w(z) \\ I \end{pmatrix} = \Theta(z) \begin{pmatrix} p(z) \\ q(z) \end{pmatrix} (\theta_{21}(z)p(z) + \theta_{22}(z)q(z))^{-1}
\]
and therefore, identity (4.15) is equivalent to
\[
  P_{\ker K}(I - zF)^{-1}(U, M)\Theta(z) \begin{pmatrix} p(z) \\ q(z) \end{pmatrix} \equiv 0.
\]
we get

\[ K(I - zF^*)^{-1} - (I - zF)^{-1}K = z(I - zF)^{-1}(KF^* - FK)(I - zF^*)^{-1} \]

we get

\[ (I - zF)^{-1}(U, M)\Theta(z) = K(I - zF^*)^{-1}K[I^{-1}](U, M) + (I - zF)^{-1}\left\{I - 1K[I^{-1}]\right\}(U, M).\]

Substituting the latter equality into (4.18) gives

\[ P_{KerK}(I - zF)^{-1}\{I - 1K[I^{-1}]\}(Up(z) + Mq(z)) \equiv 0 \]

which on account of (2.9), can be written as

\[ \{I + zP_{KerK}F(I - zF)^{-1}(I - 1K[I^{-1}])\}P_{KerK}(Up(z) + Mq(z)) \equiv 0. \] (4.19)

Since the matrix \( \{I + zP_{KerK}F(I - zF)^{-1}(I - 1K[I^{-1}])\} \) is nondegenerate, (4.19) implies (4.16).

**Proof of Step 2:** Let a pair \( \{p, q\} \in \mathbb{N}_m \) satisfy the condition (4.16). We introduce the pair

\[ \left( \begin{array}{c} p_0(z) \\ q_0(z) \end{array} \right) = \Theta(z) \left( \begin{array}{c} p(z) \\ q(z) \end{array} \right) \] (4.20)

and show that \( \det q_0(z) \neq 0 \). Indeed, suppose that the point \( \lambda \in \mathbb{C}_+ \) and the nonzero vector \( h \in \mathbb{C}^m \) are such that \( \det \Theta(\lambda) \neq 0 \) and

\[ q_0(\lambda)h = 0. \] (4.21)

Since \( h^*(p(\lambda)^*, q(\lambda)^*) \Theta(\lambda)^*J\Theta(\lambda) \left( \begin{array}{c} p(\lambda) \\ q(\lambda) \end{array} \right) h = h^*(p(\lambda)^*, q(\lambda)^*) \right\{J - \Theta(\lambda)^*J\Theta(\lambda)\} \left( \begin{array}{c} p(\lambda) \\ q(\lambda) \end{array} \right) h, \)

due to (1.12). Substituting (3.24) into this last inequality leads us to

\[ K(I - \lambda F^*)^{-1}K[I^{-1}](Up(\lambda) + Mq(\lambda))h = 0. \] (4.22)

It follows from (3.23) and (4.20) that

\[ p_0(\lambda) = p(\lambda) + \lambda M^*(I - \lambda F^*)^{-1}K[I^{-1}](Up(\lambda) + Mq(\lambda)). \] (4.23)

Since \( M = F K U \) (see (1.18)), then \( \lambda M^*(I - \lambda F^* )^{-1} = U^*K(I - \lambda F^*)^{-1} - U^*K \). Substituting this last equality into (4.23) and taking into account (2.9), (4.10), (4.22) and the evident equalities \( U^*U = I_m \) and \( U^*M = 0 \) we receive

\[ p_0(\lambda)h = p(\lambda)h - U^*K[I^{-1}](Up(\lambda) + Mq(\lambda))h + U^*K(I - zF^*)^{-1}K[I^{-1}](Up(\lambda) + Mq(\lambda))h \]

\[ = U^*(I - K[I^{-1}])(Up(\lambda) + Mq(\lambda))h + (I - UU^*)p(\lambda) - U^*Mq(\lambda) \]

\[ = U^*(I - K[I^{-1}])P_{KerK}(Up(\lambda) + Mq(\lambda))h = 0. \]

Since \( \det \Theta(\lambda) \neq 0 \), the equality \( p_0(\lambda)h = 0 \) both with (4.20) and (4.21) implies

\[ \left( \begin{array}{c} p(\lambda) \\ q(\lambda) \end{array} \right)h = \Theta(\lambda)^{-1} \left( \begin{array}{c} p_0(\lambda) \\ q_0(\lambda) \end{array} \right)h = 0 \]

and since \( \lambda \) is an arbitrary point, the latter equality contradicts to the nondegeneracy of the pair \( \{p, q\} \).

**Proof of Step 3:** Using (1.9) we obtain that the matrix \( P_{KerK}(U, M) \) is \( J \)-neutral:

\[ P_{KerK}(U, M)J \left( \begin{array}{c} U^* \\ M^* \end{array} \right) P_{KerK} = iP_{KerK}(KF^* - FK)P_{KerK} = 0. \]
Thus, by Remark 4.5,
\[ \mu = \text{rank} \left( P_{\text{Ker} K}(U, M) \right) = \text{rank} \left( P_{\text{Ker} K}(U + iM) \right) = \text{rank} \left\{ (I_m, is_0, \ldots, is_{n-1})P_{\text{Ker} K_n} \right\}. \]
According to Lemma 4.4, there exist a \( J \)-unitary matrix \( \Psi \) and an invertible \( T \) such that
\[ TP_{\text{Ker} K}(U, M) \Psi = \begin{pmatrix} I_\nu & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.24} \]
Let \( \{p_2, q_2\} \) be the pair defined by
\[ \left( \begin{array}{c} p(z) \\ q(z) \end{array} \right) = \Psi \left( \begin{array}{c} p_2(z) \\ q_2(z) \end{array} \right). \tag{4.25} \]
On account of (4.24) and (4.25), condition (4.3) can be rewritten as \( (I_\nu, 0) p_2(z) \equiv 0 \) and by Lemma 4.3, the pair \( \{p_2, q_2\} \) is equivalent to some pair of the form (4.7), i.e.,
\[ \left( \begin{array}{c} p(z) \\ q(z) \end{array} \right) = \Psi \left( \begin{array}{c} p_2(z) \\ q_2(z) \end{array} \right) \sim \Psi \begin{pmatrix} 0 & 0 \\ I_\nu & 0 \end{pmatrix} \begin{pmatrix} 0 & \bar{p}(z) \\ \bar{q}(z) \end{pmatrix} = \begin{pmatrix} p_1(z) \\ q_1(z) \end{pmatrix} \]
which completes the proof of Step 3.

Substituting (4.17) into (4.2) and taking into account that the equivalent pairs lead under the linear fractional transformation to the same function \( w(z) \), we finish the proof of theorem. \( \square \)

By Remark 2.9, the condition \( K_n \in \tilde{H}_{m,n} \) is not restrictive and hence, the received in Theorem 3.3 description is applicable to the general situation \( K_n \in H_{m,n} \).

5. Correction of erratae in [2]

The following result was formulated in [2] (see Lemmas 2.5, 2.10 and 2.11 there).

**Lemma 5.1.** Let \( K_n = (s_{i+j})_{i,j=0}^n \in H_{m,n} \) and let \( \mathcal{L} \) be the subspace of \( \mathbb{C}^{1 \times m} \) given in (2.11). The following are equivalent:

1. \( K_n \) admits a positive semidefinite Hankel extension.
2. \( P_{\text{Ker} K_{n-1}} \begin{pmatrix} s_{n+1} \\ \vdots \\ s_{2n} \end{pmatrix} = 0. \)
3. The block \( s_{2n} \) is of the form
   \[ s_{2n} = (s_n, \ldots, s_{2n-1})K_{n-1}^{-1}(s_n, \ldots, s_{2n-1})^* + R \]
   for some positive semidefinite matrix \( R \in \mathbb{C}^{m \times m} \) which vanishes on the subspace \( \mathcal{L} \) and does not depend on the choice of \( K_{n-1}^{-1} \).
4. The associated truncated Hamburger moment problem admits an “exact” solution \( \sigma \) such that
   \[ \int_{-\infty}^{\infty} \lambda^k d\sigma(\lambda) = s_k \quad (k = 0, \ldots, 2n). \]

The proofs of implications (1) \( \Rightarrow \) (4) \( \Rightarrow \) (2) \( \Leftrightarrow \) (3) presented in [2] are correct; they are reproduced in Lemmas 2.10 and 2.11 above. To complete the proof, it suffices to justify (2) \( \Rightarrow \) (1), that is, in our current terminology, to show that
\[ \tilde{H}_{m,n} \subseteq H_{m,n}. \tag{5.2} \]
This inclusion together with (2.17) implies that all three classes introduced in Section 2 coincide.

**Proof of (5.2):** Let \( K_n \in \tilde{H}_{m,n} \). Plug in the Nevanlinna pair \( \{p, q\} = \{0, I_m\} \) (which is certainly of the form (4.13)) into formula (4.12) to get a solution \( w(z) = a_{12}(z)a_{22}(z)^{-1} \) from
This Pick function $w$ is rational (since $A$ is) and takes Hermitian values at every real point at which it is analytic (since $A$ is $J$-unitary on $\mathbb{R}$). Then the measure $\sigma$ from the Herglotz representation (1.4) of $w$ is rational (since $A$ is) and takes Hermitian values at every real point at which it is analytic (since $A$ is $J$-unitary on $\mathbb{R}$). Then the measure $\sigma$ from the Herglotz representation (1.4) of $w$ is finitely atomic and therefore, the integrals
\[ \int_{-\infty}^{\infty} \lambda^N d\sigma(\lambda) \]
exists for every $N \geq 0$. Since this measure solves the associated HMP, it satisfies (1.2) and (1.3). By virtue of (2.18), the Hankel block matrix
\[ \tilde{K}_n = \begin{pmatrix} K_{n-1} & s_n & s_{n+1} \\ \vdots & \vdots & \vdots \\ s_{2n-1} & s_{2n} \\ s_n & \cdots & s_{2n-1} \\ s_{n+1} & \cdots & s_{2n} \end{pmatrix} \]
is positive semidefinite, where we have set
\[ s = \int_{-\infty}^{\infty} \lambda^2 d\sigma(\lambda), \quad s_{2n+1} = \int_{-\infty}^{\infty} \lambda^{2n+1} d\sigma(\lambda), \quad s_{2n+2} = \int_{-\infty}^{\infty} \lambda^{2n+2} d\sigma(\lambda). \]
The Hankel block matrix $K_{n+1} := (s_{i+j})_{i,j=0}^{n+1}$ extends $K_n$ and is positive semidefinite. Indeed, by (1.3), we have $K_{n+1} \geq \tilde{K}_n \geq 0$. Thus $K_n \in \mathcal{H}_{m,n}^+$ which completes the proof. □

**Remark 5.2.** The proof of implication $(2) \Rightarrow (1)$ presented in [2] does not rely on interpolation Theorem 4.6. The extending matrices $s_{2n+1}$ and $s_{2n+2}$ were constructed directly in terms of the given $s_0, \ldots, s_{2n}$. Unfortunately, the construction turned out to be wrong. The author was very glad to learn that correct explicit proofs of the above implication have been recently obtained [3, 10].

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