Abstract

In this work, we investigate weighted $\alpha\beta$-Statistical approximation properties of $q$-Durrmeyer-Stancu operators. Also, give some corrections in limit of $q$-Durrmeyer-Stancu operators defined in [1] and discuss their convergence properties.

Keywords: Durrmeyer operators; Korovkin type theorems; Rate of the weighted $\alpha\beta$-statistical convergent

2000 Mathematics Subject Classification: primary 41A25, 41A30, 41A36.

1. Introduction

The concept of statistical convergence has been defined by Fast [2] and studied by many other authors. It is well known that every statistically convergent sequence is ordinary convergent, but the converse is not true, examples and some related work can be found in [3–8]. The idea $\alpha\beta$-statistical convergence was introduced by Aktünglu in [9] as follows:

Let $\alpha(n)$ and $\beta(n)$ be two sequences positive number which satisfy the following conditions

(i) $\alpha$ and $\beta$ are both non-decreasing,

(ii) $\beta(n) \geq \alpha(n),$

(iii) $\beta(n) - \alpha(n) \to \infty$ as $n \to \infty$

and let $\wedge$ denote the set of pairs $(\alpha, \beta)$ satisfying (i)-(iii). For each pair $(\alpha, \beta) \in \wedge$, $0 < \gamma \leq 1$ and $K \in \mathbb{N}$, we define $\delta^{\alpha,\beta}(K, \gamma)$ in the following way

\[
\delta^{\alpha,\beta}(K, \gamma) = \lim_{n \to \infty} \frac{|K \cap P_n^{\alpha,\beta}|}{(\beta(n) - \alpha(n) + 1)^\gamma},
\]

where $P_n^{\alpha,\beta}$ in the closed interval $[\alpha(n), \beta(n)]$. A sequence $x = (x_k)$ is said to be $\alpha\beta$-statistically convergent of order $\gamma$ to $\ell$ or $S^{\alpha,\beta}_\gamma$-convergent, if

\[
\delta^{\alpha,\beta}(\{k : |x_k - \ell| \leq \epsilon\}, \gamma) = \lim_{n \to \infty} \frac{|\{k \in P_n^{\alpha,\beta} : |x_k - \ell| \geq \epsilon\}|}{(\beta(n) - \alpha(n) + 1)^\gamma} = 0.
\]
The concept of weighted $\alpha \beta$-statistically convergent was developed by Karakaya and Karaisa [10]. Let $s = (s_k)$ be a sequence of non-negative real numbers such that $s_0 > 0$ and

$$S_n = \sum_{k \in \mathbb{P}_{n,\beta}} s_k \rightarrow \infty, \text{ as } n \rightarrow \infty \text{ and } z_n^\gamma(x) = \frac{1}{S_n} \sum_{k \in \mathbb{P}_{n,\beta}} s_k x_k.$$ 

A sequence $x = (x_k)$ is said to be weighted $\alpha \beta$-statistically convergent of order $\gamma$ to $\ell$ or $S^\gamma_{\alpha \beta}$-convergent, if for every $\epsilon > 0$

$$\delta^{\alpha, \beta}(\{k : s_k |x_k - \ell| \geq \epsilon\}, \gamma) = \lim_{n \rightarrow \infty} \frac{1}{S_n}|\{k : s_k |x_k - \ell| \geq \epsilon\}| = 0$$

and denote $S^\gamma_{\alpha \beta} - \lim x = \ell$ or $x_k \rightarrow \ell[S^\gamma_{\alpha \beta}]$, where $S^\gamma_{\alpha \beta}$ denotes the set of all weighted $\alpha \beta$-statistically convergent sequences of order $\gamma$.

The $q$-Bernstein operators were introduced by Phillips in [11] and they generalized the well-known Bernstein operators. A survey of the obtained results and references concerning $q$-Bernstein operators can be found in [12]. It is worth mentioning that the first generalization of the Bernstein operators based on $q$-integers was obtained by Lupşa [13]. The Durrmeyer type modification of $q$-Bernstein operators were established by Gupta [14], and it’s local approximation, global approximation and simultaneous approximation properties were discussed in [15], we refer some of the important papers in this direction as [16, 23]. Also, better approximation properties were established by Gupta and Sharma [24]. Stancu type generalization of the $q$-Durrmeyer operators were discussed by Mishra and Patel [1, 22], which define for $f \in C([0,1])$ as

$${D_n}^{\omega, \vartheta}_{n,q} = [n+1]_{q} \int_{0}^{1} f \left( \frac{[n]_{q} t + \vartheta}{[n]_{q} + \vartheta} \right) p_{nk}(q, qt) d_{q} t$$

$${D_n}^{\omega, \vartheta}_{n,q} = \sum_{k=0}^{\infty} A_{n,k}(f) p_{nk}(q, x); 0 \leq x \leq 1,$$

where $p_{nk}(q, x) = \binom{n}{k}_{q} x^{k}(1-x)^{n-k}$. We have used notations of $q$-calculus as given in [26]. Along the paper, $C([a,b])$ denote by set of continuous functions on interval $[a,b]$ and $\|h\|_{C([a,b])}$ represents the sup-norm of the function $h \in C([a,b])$.

In this work, we establish $\alpha \beta$-statistical convergence for operators [11]. Also, in section 3 we discuss convergence results of limit of $q$-Durrmeyer-Stancu operators [11].

**Lemma 1** ([11]). We have

$${D_n}^{\omega, \vartheta}_{n,q}(1; x) = 1, \quad {D_n}^{\omega, \vartheta}_{n,q}(t; x) = \frac{[n]_{q} + \vartheta [n+2]_{q} + qx[n]_{q}^{2}}{[n+2]_{q} ([n]_{q} + \vartheta)}$$

and

$${D_n}^{\omega, \vartheta}_{n,q}(t^2; x) = \frac{q^3 [n]_{q}^{3} ([n]_{q} - 1) x^2 + ((q(1 + q^2) + 2\vartheta q^4) [n]_{q}^{2} + 2\vartheta q[3]_{q}[n]_{q} x + (1 + q + 2\vartheta q^3) [n]_{q}^{2} + 2\vartheta [3]_{q} [n]_{q} x + (1 + q + 2\vartheta q^3) [n]_{q}^{2} + 2\vartheta [3]_{q} [n]_{q}) x}{([n]_{q} + \vartheta)^2 [n+2]_{q} [n+3]_{q}}$$

**Remark 1.** By simple computation, we can find the central moments.
Now, \( \delta_n(x) = D_{n,q}^{\alpha \beta}(t - x; x) = \left( \frac{q[n]^2}{n + 2}_q [n]_q + \vartheta \right) x + \frac{[n]_q + \varpi [n + 2]_q}{n + 2}_q [n]_q + \vartheta \).

\( \gamma_n(x) = D_{n,q}^{\alpha \beta}((t - x)^2; x) = \frac{q[n]^3}{n + 2}_q - 2q[n]^2_2 + 3q[n]^2_2 [n + 3]_q [n]_q + \vartheta + [n + 2]_q [n + 3]_q [n]_q + \vartheta \)

\( \frac{[n]_q + \varpi [n + 2]_q}{n + 2}_q [n]_q + \vartheta \) \]

\( + \frac{(1 + q) [n]_q + 2\varpi [n]_q [n + 3]_q - (2[n]_q + 2\varpi [n + 2]_q) [n + 3]_q [n]_q + \vartheta}{(n + 2)_q [n + 3]_q} \]

\( + \frac{(n + 2)_q [n + 3]_q}{(n + 2)_q [n + 3]_q} \vartheta \).  

2. \( \alpha \beta \)-Statistical Convergence

**Theorem 1** ([10]). Let \( (L_k) \) be a sequence of positive linear operator from \( C([a,b]) \) into \( C([a,b]) \). Then for all \( f \in C([a,b]) \)

\[ \tilde{S}^\gamma_{\alpha \beta} \lim_{k \to \infty} \| L_k(f, x) - f(x) \|_{C([a,b])} = 0 \]

if and only if

\[ \tilde{S}^\gamma_{\alpha \beta} \lim_{k \to \infty} \| L_k(x^i, x) - x^i \|_{C([a,b])} = 0, \quad i = 0, 1, 2. \]

Let \( \{q_n\} \) be a sequence in the interval \([0,1]\) satisfying

\[ \tilde{S}^\gamma_{\alpha \beta} \lim_{k \to \infty} q_n = 1, \quad \tilde{S}^\gamma_{\alpha \beta} \lim_{k \to \infty} (q_n)^n = a \in (0,1), \quad \tilde{S}^\gamma_{\alpha \beta} \lim_{k \to \infty} \frac{1}{n}_q = 1 \]  \hspace{1cm} (2.1)

**Theorem 2.** Let \( \{q_n\} \) be a sequence satisfying (2.1) and \( D_{n,q}^{\alpha \beta} \) as defined in (1.1). For any \( f \in C([0,1]) \), we have

\[ \tilde{S}^\gamma_{\alpha \beta} \lim_{k \to \infty} \| D_{n,q}^{\alpha \beta}(f, x) - f(x) \|_{C([0,1])} = 0. \]

**Proof:** By Theorem 1 it is enough to prove that

\[ \tilde{S}^\gamma_{\alpha \beta} \lim_{k \to \infty} \| D_{n,q}^{\alpha \beta}(x^j, x) - x^j \|_{C([0,1])} = 0, \quad j = 0, 1, 2 \]  \hspace{1cm} (2.2)

From the \( D_{n,q}^{\alpha \beta}(1, x) = 1 \), it is easy to obtain that

\[ \tilde{S}^\gamma_{\alpha \beta} \lim_{k \to \infty} \| D_{n,q}^{\alpha \beta}(1, x) - 1 \|_{C([0,1])} = 0. \]

Now,

\[ |D_{n,q}^{\alpha \beta}(t; x) - x| \leq \frac{|q[n]^2 - [n + 2]_q [n]_q + \vartheta|}{[n + 2]_q [n]_q + \vartheta} + \frac{|[n]_q + \varpi [n + 2]_q|}{[n + 2]_q [n]_q + \vartheta} \]

\[ = \frac{|n]_q (q[n]^3 - [n + 2]_q - \vartheta [n + 2]_q|}{[n + 2]_q [n]_q + \vartheta} + \frac{|[n]_q + \varpi [n + 2]_q|}{[n + 2]_q [n]_q + \vartheta} \]

\[ \leq \frac{|n]_q (1 + q^{n+1})}{[n + 2]_q [n]_q + \vartheta} + \frac{\vartheta}{[n]_q + \vartheta} + \frac{|[n]_q + \varpi [n + 2]_q|}{[n + 2]_q [n]_q + \vartheta} \]
Using equation (2.1), we get
\[ S^\gamma_{\alpha\beta} - \lim_{k \to \infty} \frac{[n]_q(1 + q^n + 1)}{[n + 2]_q([n]_q + \vartheta)} = 0; \quad S^\gamma_{\alpha\beta} - \lim_{k \to \infty} \frac{\vartheta}{[n]_q + \vartheta} = 0 \]
and
\[ S^\gamma_{\alpha\beta} - \lim_{k \to \infty} \frac{[n]_q + \varpi [n + 2]_q}{[n + 2]_q([n]_q + \vartheta)} = 0 \]
Define the following sets:
\[ A = \{ n \in \mathbb{N} : \| D_{n,q}^{\varpi,\vartheta} (t ; x) - x \|_{C([0,1])} \geq \epsilon \}; \quad A_1 = \{ n \in \mathbb{N} : \| \frac{[n]_q(1 + q^n + 1)}{[n + 2]_q([n]_q + \vartheta)} \geq \frac{\epsilon}{3} \}; \]
\[ A_2 = \{ n \in \mathbb{N} : \| \frac{\vartheta}{[n]_q + \vartheta} \geq \frac{\epsilon}{3} \}, \quad A_3 = \{ n \in \mathbb{N} : \| \frac{[n]_q + \varpi [n + 2]_q}{[n + 2]_q([n]_q + \vartheta)} \geq \frac{\epsilon}{3} \}, \]
Then, we obtain \( A \subset A_1 \cup A_2 \cup A_3 \), which implies that \( \delta^\alpha_{\gamma,\beta} (A) \leq \delta^\alpha_{\gamma,\beta} (A_1) + \delta^\alpha_{\gamma,\beta} (A_2) + \delta^\alpha_{\gamma,\beta} (A_3) \) and hence
\[ S^\gamma_{\alpha\beta} - \lim_{k \to \infty} \| D_{n,q}^{\varpi,\vartheta} (t, x) - x \|_{C([0,1])} = 0. \]
Similarly, we have
\[ |D_{n,q}^{\varpi,\vartheta} (t^2 ; x) - x^2| \leq \frac{q^3 [n]_q^3 ([n]_q - 1)}{([n]_q + \vartheta)^2 [n + 2]_q [n + 3]_q} - 1 \]
\[ + \frac{(q(1 + q)^2 + 2 \varpi q^4) [n]_q^3 + 2 \varpi q [n]_q [n]_q^2}{([n]_q + \vartheta)^2 [n + 2]_q [n + 3]_q} \]
\[ + \frac{(1 + q + 2 \varpi q^3) [n]_q^2 + 2 \varpi [n]_q [n]_q^2}{([n]_q + \vartheta)^2 [n + 2]_q [n + 3]_q} + \frac{\varpi^2}{([n]_q + \vartheta)^2} \]
\[ \leq \frac{q^3 [n]_q^4 (1 - q^n)}{([n]_q + \vartheta)^2 [n + 2]_q [n + 3]_q} + \frac{q(1 + q)^2 + 2 \varpi q^4) [n]_q^3}{([n]_q + \vartheta)^2 [n + 2]_q [n + 3]_q} \]
\[ + \frac{2 \varpi q [n]_q [n]_q^2}{([n]_q + \vartheta)^2 [n + 2]_q [n + 3]_q} + \frac{1 + q + 2 \varpi q^3 [n]_q^2}{([n]_q + \vartheta)^2 [n + 2]_q [n + 3]_q} \]
\[ + \frac{2 \varpi [n]_q [n]_q^2}{([n]_q + \vartheta)^2 [n + 2]_q [n + 3]_q} + \frac{\varpi^2}{([n]_q + \vartheta)^2}. \]
Again, using $\tilde{S}_{\alpha,\beta}^{\gamma} - \lim_{k \to \infty} q_{n} = 1$, $\tilde{S}_{\alpha,\beta}^{\gamma} - \lim_{k \to \infty} (q_{n})^{n} = a \in (0, 1)$, $\tilde{S}_{\alpha,\beta}^{\gamma} - \lim_{k \to \infty} \frac{1}{[n]_{q}} = 1$, we get

\[
\tilde{S}_{\alpha,\beta}^{\gamma} - \lim_{n \to \infty} \frac{q_{n}^{3}[n]_{q}^{4}(1 - q_{n}^{2})}{([n]_{q} + \vartheta)^{2}[n + 2][n + 3]_{q}} = 0,
\]
\[
\tilde{S}_{\alpha,\beta}^{\gamma} - \lim_{n \to \infty} \frac{(q + 1 + q)^{2} + 2q^{3}}{([n]_{q} + \vartheta)^{2}[n + 2][n + 3]_{q}} = 0,
\]
\[
\tilde{S}_{\alpha,\beta}^{\gamma} - \lim_{n \to \infty} \frac{2q^{3}3q[n]_{q}^{2}}{([n]_{q} + \vartheta)^{2}[n + 2][n + 3]_{q}} = 0,
\]
\[
\tilde{S}_{\alpha,\beta}^{\gamma} - \lim_{n \to \infty} \frac{(1 + q + 2q^{3})[n]_{q}^{2}}{([n]_{q} + \vartheta)^{2}[n + 2][n + 3]_{q}} = 0,
\]
\[
\tilde{S}_{\alpha,\beta}^{\gamma} - \lim_{n \to \infty} \frac{2q^{3}3[q]_{q}^{2}}{([n]_{q} + \vartheta)^{2}[n + 2][n + 3]_{q}} = 0.
\]

Now, consider the following sets:

\[
B_{1} := \left\{ n \in \mathbb{N} : \frac{q_{n}^{3}[n]_{q}^{4}(1 - q_{n}^{2})}{([n]_{q} + \vartheta)^{2}[n + 2][n + 3]_{q}} \geq \frac{\epsilon}{6} \right\},
\]
\[
B_{2} := \left\{ n \in \mathbb{N} : \frac{(q + 1 + q)^{2} + 2q^{3}}{([n]_{q} + \vartheta)^{2}[n + 2][n + 3]_{q}} \geq \frac{\epsilon}{6} \right\},
\]
\[
B_{3} := \left\{ n \in \mathbb{N} : \frac{2q^{3}3q[n]_{q}^{2}}{([n]_{q} + \vartheta)^{2}[n + 2][n + 3]_{q}} \geq \frac{\epsilon}{6} \right\},
\]
\[
B_{4} := \left\{ n \in \mathbb{N} : \frac{(1 + q + 2q^{3})[n]_{q}^{2}}{([n]_{q} + \vartheta)^{2}[n + 2][n + 3]_{q}} \geq \frac{\epsilon}{6} \right\},
\]
\[
B_{5} := \left\{ n \in \mathbb{N} : \frac{2q^{3}3[q]_{q}^{2}}{([n]_{q} + \vartheta)^{2}[n + 2][n + 3]_{q}} \geq \frac{\epsilon}{6} \right\},
\]
\[
B_{6} := \left\{ n \in \mathbb{N} : \frac{[n]_{q}}{([n]_{q} + \vartheta)^{2}} \geq \frac{\epsilon}{6} \right\}.
\]

Consequently, we obtain $B \subset B_{1} \cup B_{2} \cup B_{3} \cup B_{4} \cup B_{5} \cup B_{6}$, which implies that $\delta(B) \leq \sum_{i=1}^{6} \delta(B_{i})$. Hence, we get

\[
\tilde{S}_{\alpha,\beta}^{\gamma} - \lim_{k \to \infty} \| D_{n, q}^{\alpha, \beta}(t^{2}, x) - x^{2} \|_{C([0, 1])} = 0.
\]

This completes the proof of Theorem 2.

3. Limit $q$-Durrmeyer-Stancu operators

The authors found mistake in the proof part of [25, Theorem 2]. In [25, Sec. 4], authors defined the operators $D_{\infty, q}^{\alpha, \beta}$ [25, Eq. (4.2)], which depend on $[n]_{q}$ was mistaken. So, follow by [25, Theorem 2] the proof part have some errors. With this note we correctly define the operators and prove Theorem 2 of [25].

Here, we define the limit $q$-Durrmeyer-Stancu operators [1,11] as:
Let $q \in (0, 1)$ be fixed and $x \in [0, 1]$, the operators $D^q_{\infty, q}(f; x)$ is defined by

$$D^q_{\infty, q}(f; x) = \frac{1}{1 - q} \sum_{k=0}^{\infty} p_{\infty k}(q; x) q^{-k} \int_{0}^{1} f \left( \frac{t + (1 - q)\vartheta}{1 + (1 - q)\vartheta} \right) p_{\infty k}(q, qt) d_q t$$

(3.1)

$$= \sum_{k=0}^{\infty} A^q_{\infty k}(f) p_{\infty k}(q; x).$$

Using the fact that (see [27]), we have

$$\sum_{k=0}^{\infty} p_{\infty k}(q; x) = 1, \quad \sum_{k=0}^{\infty} (1 - q^k) p_{\infty k}(q; x) = x,$$

(3.2)

and

$$\sum_{k=0}^{\infty} (1 - q^k)^2 p_{\infty k}(q; x) = x^2 + (1 - q)x(1 - x).$$

(3.3)

Using (1.1) and (3.1), it is easy to prove that $D^q_{\infty, q}(1; x) = 1$, $D^q_{\infty, q}(t; x) = \frac{1 + q(x - 1) + \vartheta(1 - q)}{1 + \vartheta(1 - q)}$.

For $f \in C[0, 1], t > 0$, we define the modulus of continuity $\omega(f, t)$ as follows:

$$\omega(f, t) = \sup \{|f(x) - f(y)| : |x - y| \leq t, \quad x, y \in [0, 1]|.$$

**Theorem 3.** Let $0 < q < 1$ then for each $f \in C[0, 1]$ the sequence $\{D^q_{n, q}(f; x)\}$ converges to $D^q_{\infty, q}(f; x)$ uniformly on $[0, 1]$. Furthermore,

$$\|D^q_{n, q}(f) - D^q_{\infty, q}(f)\| \leq C^q_{n, q} \omega(f, q^n).$$

**Proof:** $D^q_{\infty, q}(f; x)$ and $D^q_{n, q}(f; x)$ reproduce constant function that is $D^q_{n, q}(1; x) = D^q_{\infty, q}(1; x) = 1$. Hence for all $x \in [0, 1]$, by definition of $D^q_{n, q}(f; x)$ and $D^q_{\infty, q}(f; x)$, we know that

$$|D^q_{n, q}(f; x) - D^q_{\infty, q}(f; x)| = \left| \sum_{k=0}^{n} A^q_{nk}(f) p_{nk}(q; x) - \sum_{k=0}^{\infty} A^q_{\infty k}(f) p_{\infty k}(q; x) \right|$$

$$= \left| \sum_{k=0}^{n} A^q_{nk}(f - f(1)) p_{nk}(q; x) - \sum_{k=0}^{\infty} A^q_{\infty k}(f - f(1)) p_{\infty k}(q; x) \right|$$

$$\leq \sum_{k=0}^{n} |A^q_{nk}(f - f(1)) - A^q_{\infty k}(f - f(1))| |p_{nk}(q; x)|$$

$$+ \sum_{k=n+1}^{\infty} |A^q_{\infty k}(f - f(1))| |p_{nk}(q; x) - p_{\infty k}(q; x)|$$

$$+ \sum_{k=n+1}^{\infty} |A^q_{\infty k}(f - f(1))| |p_{\infty k}(q; x)| = I_1 + I_2 + I_3.$$
By the well known property of modulus of continuity (see [28]), \(\omega(f, \lambda t) \leq (1 + \lambda)\omega(f, t), \lambda > 0\), we get

\[
|f(t) - f(1)| \leq \omega(f, 1 - t) \leq \omega(f, q^n) \left(1 + \frac{1 - t}{q^n}\right).
\]

Thus

\[
|A_{nk}(f - f(1))| = |[n + 1]_q \int_0^1 q^{-k} \left[f \left(\frac{[n]_q t + \vartheta}{[n]_q + \omega}\right) - f(1)\right] p_{nk}(qt) d_q t|
\]

\[
\leq [n + 1]_q \int_0^1 q^{-k} \left[f \left(\frac{[n]_q t + \vartheta}{[n]_q + \omega}\right) - f(1)\right] p_{nk}(qt) d_q t
\]

\[
\leq [n + 1]_q \int_0^1 q^{-k} [\omega(f, q^n) \left(1 + \frac{1}{q^n} \left(1 - \frac{[n]_q t + \vartheta}{[n]_q + \omega}\right)\right)] p_{nk}(qt) d_q t
\]

\[
\leq \omega(f, q^n) \left(1 + q^{-n} \left(1 - \frac{[n]_q [k + 1]_q - \vartheta[n + 2]_q}{[n + 2]_q ([n]_q + \omega)}\right)\right).
\]

Similarly,

\[
|A_{nk}(f - f(1))| = \left|\frac{q^{-k}}{1 - q} \int_0^1 \left(f \left(\frac{t + \vartheta(1 - q)}{1 + \omega(1 - q)}\right) - f(1)\right) p_{\infty k}(qt) d_q t\right|
\]

\[
\leq \frac{q^{-k}}{1 - q} \int_0^1 \omega(f, q^n) \left(1 + \frac{1}{q^n} \left(1 - \frac{t + \vartheta(1 - q)}{1 + \omega(1 - q)}\right)\right) p_{\infty k}(qt) d_q t
\]

\[
\leq \frac{q^{-k}}{1 - q} \int_0^1 \omega(f, q^n) \left(1 + \frac{1}{q^n} (1 - t) + \frac{1}{q^n} \frac{\omega - \vartheta}{1 + \omega(1 - q)}\right) p_{\infty k}(qt) d_q t
\]

\[
\leq \omega(f, q^n) \left(1 + q^{k+1-n} + \frac{q^{-n}(\omega - \vartheta)}{1 + \omega(1 - q)}\right).
\]

From [28, Eq.4.5], we have

\[
|p_{nk}(q; x) - p_{\infty k}(q; x)| \leq \frac{q^{n-k}}{1 - q} \left(p_{nk}(q; x) + p_{\infty k}(q; x)\right).
\]

(3.4)
Hence by using (3.4), we have

\[ |A_{nk}^{\varnothing,\varpi}(f - f(1)) - A_{\infty k}^{\varnothing,\varpi}(f - f(1))| \]
\[ \leq [n+1]_q \int_0^1 q^{-k} \left| f \left( \frac{[n]_q t + \varpi}{[n]_q + \varpi} \right) - f(1) \right| p_{nk}(q; t) dt + \frac{1}{1 - q} \int_0^1 q^{-k} \left| f \left( \frac{t + \varpi(1 - q)}{1 + \varpi(1 - q)} \right) - f(1) \right| p_{\infty k}(q; t) dt \]
\[ \leq [n+1]_q \int_0^1 q^{-k} \left| f \left( \frac{[n]_q t + \varpi}{[n]_q + \varpi} \right) - f(1) \right| p_{nk}(q; t) - p_{\infty k}(q; t) dt \]

+ \frac{1}{1 - q} \int_0^1 q^{-k} \left| f \left( \frac{t + \varpi(1 - q)}{1 + \varpi(1 - q)} \right) - f(1) \right| p_{\infty k}(q; t) dt + [n+1]_q \int_0^1 q^{-k} \left| f \left( \frac{[n]_q t + \varpi}{[n]_q + \varpi} \right) - f(1) \right| p_{\infty k}(q; t) dt

\[ \leq \omega(f, q^n) \left[ 2 q^{n-k} \left( 1 + q^{k+1-n} + \frac{q^{-n}(\varpi - \varpi)}{[n]_q + \varpi} \right) + \left( 1 + q^{k+1-n} + \frac{q^{-n}(\varpi - \varpi)}{1 + \varpi(1 - q)} \right) + \left( 1 + q^{k+1-n} + \frac{q^{-n}(\varpi - \varpi)}{[n]_q + \varpi} \right) \right]. \]

To estimate \( I_1, I_2 \) and \( I_3 \), we have

\[ I_1 \leq \frac{\omega(f, q^n)}{1 - q} \left( 8 + \frac{3(\varpi - \varpi)}{[n]_q + \varpi} + \frac{q^n(\varpi - \varpi)}{q^n(1 + \varpi(1 - q))} \right) \sum_{k=0}^n p_{nk}(q; x) \]

\[ = \frac{\omega(f, q^n)}{1 - q} \left( 8 + \frac{3(\varpi - \varpi)}{[n]_q + \varpi} + \frac{q^n(\varpi - \varpi)}{q^n(1 + \varpi(1 - q))} \right); \]

\[ I_3 = \sum_{k=n+1}^{\infty} |A_{\infty k}^{\varnothing,\varpi}(f - f(1))| p_{\infty k}(q; x) \]
\[ \leq \omega(f, q^n) \sum_{k=n+1}^{\infty} \left( 1 + q^{k+1-n} + \frac{q^{-n}(\varpi - \varpi)}{1 + \varpi(1 - q)} \right) p_{\infty k}(q; x) \]
\[ \leq \omega(f, q^n) \left( 2 + \frac{q^{-n}(\varpi - \varpi)}{1 + \varpi(1 - q)} \right); \]

\[ I_2 = \sum_{k=0}^{n} |A_{\infty k}^{\varnothing,\varpi}(f - f(1))| p_{nk}(q; x) - p_{\infty k}(q; x) | \]
\[ \leq \sum_{k=0}^{n} \left[ \omega(f, q^n) \left( 1 + q^{k+1-n} + \frac{q^{-n}(\varpi - \varpi)}{1 + \varpi(1 - q)} \right) \right] \left[ \frac{q^{n-k}}{1 - q} p_{nk}(q; x) + p_{\infty k}(q; x) \right] \]
\[ \leq \frac{2 \omega(f, q^n)}{1 - q} \left( 2 + \frac{q^{-n}(\varpi - \varpi)}{1 + \varpi(1 - q)} \right). \]

Combining the estimates \( I_1 - I_3 \), we conclude that \( \| D_{n,q}^{\varnothing,\varpi}(f) - D_{\infty,q}^{\varnothing,\varpi}(f) \| \leq C_{q}^{\varnothing,\varpi}(f, q^n) \).

This complete the proof of Theorem 3.

**Lemma 2.** (29). Let \( L \) be a positive linear operator on \( C([0,1]) \) which reproduces constant functions. If \( L(t, x) > x \forall x \in (0,1) \), then \( L(f) = f \) if and only if \( f \) is constant.

**Remark 2.** Since \( D_{\infty,q}^{\varnothing,\varpi}(t; x) = \frac{(1 + q(x - 1)) + \varpi(1 - q)}{1 + \varpi(1 - q)} > x \) for \( 0 < q < 1 \) consequence of Lemma 2, we have the
Theorem 4. Let $0 < q < 1$ be fixed and let $f \in C([0,1])$. Then $D_{\infty, q}^{\vartheta, \omega}(f; x) = f(x)$ for all $x \in [0,1]$ if and only if $f$ is constant.

Theorem 5. For any $f \in C([0,1])$, $\{D_{\infty, q}^{\vartheta, \omega}(f)\}$ converges to $f$ uniformly on $[0,1]$ as $q \to 1^-$. 

Proof: We know that the operators $D_{\infty, q}^{\vartheta, \omega}$ is positive linear operator on $C([0,1])$ and reproduce constant functions. Also, $D_{\infty, q}^{\vartheta, \omega}(t; x) \to x$ uniformly on $[0,1]$ as $q \to 1^-$ and $D_{\infty, q}^{\vartheta, \omega}(t^2; x) \to x^2$ uniformly on $[0,1]$ as $q \to 1^-$. Thus, Theorem 5 follows from Korovkin Theorem.

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