Toponogov Comparison Theorem for Open Triangles*†

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in memory of the late professor Detlef Gromoll

Abstract

The aim of our article is to generalize the Toponogov comparison theorem to a complete Riemannian manifold with smooth convex boundary. A geodesic triangle will be replaced by an open (geodesic) triangle standing on the boundary of the manifold, and a model surface will be replaced by the universal covering surface of a cylinder of revolution with totally geodesic boundary. As an application of our comparison theorem, we will prove splitting theorems of two types.

1 Introduction

Cohn-Vossen is one of pioneers in global differential geometry. More than seventy years ago, he investigated the relationship between the total curvature and the Riemannian structure of complete open surfaces. He has given big influence to many geometers who research in global differential geometry, although he studied only 2-dimensional manifolds in [CV1] and [CV2]. For example, Cohn-Vossen proved the following theorem known as a splitting theorem:

Theorem 1.1 ([CV2 Satz 5]) If a complete Riemannian 2-manifold has non-negative Gaussian curvature and admits a straight line, then its universal covering space is isometric to Euclidean plane.

Toponogov ([T2]) generalized this splitting theorem for any dimensional complete Riemannian manifolds with non-negative sectional curvature by making use of the Toponogov comparison theorem ([T1]). It is well known that the Toponogov comparison theorem has produced many great classical results, e.g., the maximal diameter theorem by Toponogov ([T1]), the structure theorem with positive sectional curvature by Gromoll and Meyer ([GM]), and the soul theorem with non-negative sectional curvature by Cheeger and Gromoll ([CG]). Besides the Toponogov comparison theorem, some techniques originating from Euclidean geometry also play a key role in the references above. The techniques

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such as drawing a circle or a geodesic polygon, and joining two points by a minimal geodesic segment are very powerful in the comparison geometry. Cohn-Vossen first introduced such techniques into global differential geometry (see [CV1] and [CV2]). The Toponogov comparison theorem enables us to make use of such a technique in the comparison geometry.

In 2003, Itokawa, Machigashira, and Shiohama generalized the Toponogov comparison theorem by means of the radial sectional curvature. Their result contains the original Toponogov comparison theorem as a corollary (see [IMS, Theorem 1.3]). The model surface in the original Toponogov comparison theorem is a complete 2-dimensional manifold of constant Gaussian curvature, but in [IMS], the model surface is replaced by a von Mangoldt surface of revolution. Here, a von Mangoldt surface of revolution is, by definition, a complete surface of revolution homeomorphic to Euclidean plane whose radial curvature function is non-increasing on $[0, \infty)$. Very familiar surfaces such as paraboloids or 2-sheeted hyperboloids are typical examples of a von Mangoldt surface of revolution. Hence, it is natural to employ a von Mangoldt surface of revolution as a model surface. The reason why a von Mangoldt surface of revolution is used as a model surface lies in the following property of the surface:

**Theorem 1.2 (Tn, Main Theorem)** The cut locus of a point on a von Mangoldt surface of revolution is empty or a subray of the meridian opposite to the point.

It would be impossible to prove [IMS, Theorem 1.3] for general surfaces of revolution, because the cut locus of the surface appears as an obstruction, when we draw a geodesic triangle in the model surface. For example, the proof of [KT2, Lemma 4.10] suggests such an obstruction. In [KT2], the present authors very recently generalized [IMS, Theorem 1.3] for a surface of revolution admitting a sector which has no pair of cut points.

Our purpose in this article is to establish the Toponogov comparison theorem for Riemannian manifolds with convex boundary from the radial curvature geometry’s standpoint.

Now we will introduce the radial curvature geometry for manifolds with boundary. We first introduce our model, which will be later employed as a reference surface of comparison theorems in complete Riemannian manifolds with boundary. Let

\[ \tilde{\mathcal{M}} := (\mathbb{R}, d\tilde{x}^2) \times_m (\mathbb{R}, d\tilde{y}^2) \]

be a warped product of two 1-dimensional Euclidean lines $(\mathbb{R}, d\tilde{x}^2)$ and $(\mathbb{R}, d\tilde{y}^2)$, where the warping function $m : \mathbb{R} \to (0, \infty)$ is a positive smooth function satisfying $m(0) = 1$ and $m'(0) = 0$. Then we call

\[ \tilde{X} := \{ \tilde{p} \in \tilde{\mathcal{M}} \mid \tilde{x}(\tilde{p}) \ge 0 \} \]

a model surface. Since $m'(0) = 0$, the boundary

\[ \partial \tilde{X} := \{ \tilde{p} \in \tilde{X} \mid \tilde{x}(\tilde{p}) = 0 \} \]
of $\tilde{X}$ is totally geodesic. The metric $\tilde{g}$ of $\tilde{X}$ is expressed as

$$\tilde{g} = d\tilde{x}^2 + m(\tilde{x})^2 d\tilde{y}^2$$

(1.1)
on $[0, \infty) \times \mathbb{R}$. The function $G \circ \tilde{\mu} : [0, \infty) \to \mathbb{R}$ is called the radial curvature function of $\tilde{X}$, where we denote by $G$ the Gaussian curvature of $\tilde{X}$, and by $\tilde{\mu}$ any ray emanating perpendicularly from $\partial \tilde{X}$ (Notice that such a $\tilde{\mu}$ will be called a $\partial \tilde{X}$-ray). Remark that $m : [0, \infty) \to \mathbb{R}$ satisfies the differential equation

$$m''(t) + G(\tilde{\mu}(t))m(t) = 0$$

with initial conditions $m(0) = 1$ and $m'(0) = 0$. We define a sector

$$\tilde{X}(\theta) := \tilde{y}^{-1}((0, \theta))$$

in $\tilde{X}$ for each constant number $\theta > 0$. Since a map $(\tilde{\mu}, \tilde{\nu}) \to (\tilde{\mu}, \tilde{\nu} + c), c \in \mathbb{R}$, over $\tilde{X}$ is an isometry, $\tilde{X}(\theta)$ is isometric to $\tilde{y}^{-1}(c, c + \theta)$ for all $c \in \mathbb{R}$. Note that the $n$-dimensional model surfaces are defined similarly, and, as seen in [MS, Theorem 1.1], we may completely classify them by taking half spaces of spaces in $[\text{MS}, 1.1]$. Hereafter, let $(X, \partial X)$ denote a complete Riemannian $n$-dimensional manifold $X$ with smooth boundary $\partial X$. We say that $\partial X$ is convex, if all eigenvalues of the shape operator $A_\xi$ of $\partial X$ are non-negative in the inward vector $\xi$ normal to $\partial X$. Notice that our sign of $A_\xi$ differs from [S]. That is, for each $p \in \partial X$ and $v \in T_p \partial X$,

$$A_\xi(v) = - (\nabla_v N)$$

holds. Here, we denote by $N$ a local extension of $\xi$, and by $\nabla$ the Riemannian connection on $X$.

For a positive constant $\ell$, a unit speed geodesic segment $\mu : [0, \ell] \to X$ emanating from $\partial X$ is called a $\partial X$-segment, if

$$d(\partial X, \mu(t)) = t$$

on $[0, \ell]$. If $\mu : [0, \ell] \to X$ is a $\partial X$-segment for all $\ell > 0$, we call $\mu$ a $\partial X$-ray. Here, we denote by $d(\partial X, \cdot)$ the distance function to $\partial X$ induced from the Riemannian structure of $X$. Notice that a $\partial X$-segment is orthogonal to $\partial X$ by the first variation formula, and so a $\partial X$-ray is too.

For any fixed two points $p, q \in X \setminus \partial X$, an open triangle

$$(\partial X, p, q) = (\gamma, \mu_1, \mu_2)$$

in $X$ is defined by two $\partial X$-segments $\mu_i : [0, \ell_i] \to X, i = 1, 2$, a minimal geodesic segment $\gamma : [0, d(p, q)] \to X$, and $\partial X$ such that

$$\mu_1(\ell_1) = \gamma(0) = p, \quad \mu_2(\ell_2) = \gamma(d(p, q)) = q.$$

In this article, whenever an open triangle $(\partial X, p, q) = (\gamma, \mu_1, \mu_2)$ in $X$ is given,
always means that the minimal geodesic segment \( \gamma \) is the opposite side to \( \partial X \) emanating from \( p \) to \( q \), and that the \( \partial X \)-segments \( \mu_1, \mu_2 \) are sides emanating from \( \partial X \) to \( p, q \), respectively.

\((X, \partial X)\) is said to have the radial curvature (with respect to \( \partial X \)) bounded from below by that of \((\tilde{X}, \partial \tilde{X})\) if, for every \( \partial X \)-segment \( \mu : [0, \ell] \to X \), the sectional curvature \( K_X \) of \( X \) satisfies

\[
K_X(\sigma_t) \geq G(\tilde{\mu}(t))
\]

for all \( t \in [0, \ell) \) and all 2-dimensional linear spaces \( \sigma_t \) spanned by \( \mu'(t) \) and a tangent vector to \( X \) at \( \mu(t) \). For example, if the Riemannian metric of \( \tilde{X} \) is \( d\tilde{x}^2 + d\tilde{y}^2 \), or \( d\tilde{x}^2 + \cosh^2(\tilde{x}) d\tilde{y}^2 \), then \( G(\tilde{\mu}(t)) = 0 \), or \( G(\tilde{\mu}(t)) = -1 \), respectively. Furthermore, the radial curvature may change signs wildly (e.g., [KT1, Example 1.2], [KT3]).

Our main theorem is now stated as follows:

**Toponogov’s Comparison Theorem for Open Triangles.**

Let \((X, \partial X)\) be a complete connected Riemannian \( n \)-dimensional manifold \( X \) with smooth convex boundary \( \partial X \) whose radial curvature is bounded from below by that of a model surface \((\bar{X}, \partial \bar{X})\) with its metric \((\tilde{L}, \tilde{T})\). Assume that \( \bar{X} \) admits a sector \( \bar{X}(\theta_0) \) which has no pair of cut points. Then, for every open triangle \((\partial X, p, q) = (\gamma, \mu_1, \mu_2)\) in \( X \) with

\[
d(\mu_1(0), \mu_2(0)) < \theta_0,
\]

there exists an open triangle \((\partial \bar{X}, \hat{p}, \hat{q}) = (\hat{\gamma}, \hat{\mu}_1, \hat{\mu}_2)\) in \( \bar{X}(\theta_0) \) such that

\[
d(\partial \bar{X}, \hat{p}) = d(\partial X, p), \quad d(\hat{p}, \hat{q}) = d(p, q), \quad d(\partial \bar{X}, \hat{q}) = d(\partial X, q)
\]

and that

\[
\angle p \geq \angle \hat{\gamma}, \quad \angle q \geq \angle \hat{\gamma}, \quad d(\mu_1(0), \mu_2(0)) \geq d(\hat{\mu}_1(0), \hat{\mu}_2(0)).
\]

Furthermore, if

\[
d(\mu_1(0), \mu_2(0)) = d(\hat{\mu}_1(0), \hat{\mu}_2(0))
\]

holds, then

\[
\angle p = \angle \hat{\gamma}, \quad \angle q = \angle \hat{\gamma}
\]

hold. Here \( \angle p \) denotes the angle between two vectors \( \gamma'(0) \) and \(-\mu'(d(\partial X, p))\) in \( T_pX \).

Notice that we do not assume that \( \partial X \) is connected in our main theorem. Moreover, remark that the opposite side \( \gamma \) of \((\partial X, p, q) = (\gamma, \mu_1, \mu_2)\) does not meet \( \partial X \) (Lemma 0.1 in Section 6). A related result for our main theorem is [MS, Theorem 3.4] of Mashiko and Shiohama. In [MS], they treat a pair \((M, N)\) of a complete connected Riemannian manifold \( M \) and a compact connected totally geodesic hypersurface \( N \) of \( M \) such that the radial curvature with respect to \( N \) is bounded from below by that of the model \(((a, b) \times_m N, N)\), where \((a, b)\) denotes an interval, in their sense. Thus, our Toponogov comparison theorem for open triangles is applicable to the pair \((M, N)\), because the radial curvature with respect to \( N \) is bounded from below by that of our model \(((0, \infty), d\tilde{x}^2) \times_m (\mathbb{R}, d\tilde{y}^2)\).
There are many examples of model surfaces satisfying the assumption on our main theorem. For example, it is clear that a model surface $(\tilde{X}, \partial \tilde{X})$ with its metric $d\tilde{x}^2 + d\tilde{y}^2$, or $d\tilde{x}^2 + \cosh^2(\tilde{x}) d\tilde{y}^2$ has no pair of cut points in a sector $\tilde{X}(\theta)$ for each constant $\theta > 0$, respectively. Moreover, we have another example of model surfaces which have no pair of cut points in a sector:

**Example 1.3** Let $\tilde{M} := (\mathbb{R}, dt^2) \times_m (S^1, d\theta^2)$ be a warped product of a 1-dimensional Euclidean line $(\mathbb{R}, dt^2)$ and a unit circle $(S^1, d\theta^2)$ satisfying the next three conditions:

(C–1) The warping function $m : \mathbb{R} \rightarrow (0, \infty)$ is a smooth even function satisfying $m(0) = 1$ and $m'(0) = 0$.

(C–2) The radial curvature function $G(\tilde{\mu}(t)) = -m''(t)/m(t)$ is non-increasing on $[0, \infty)$.

(C–3) $m'(t) \neq 0$ on $\mathbb{R} \setminus \{0\}$.

Tamura ([Tm]) proved that the cut locus of a point $\tilde{p} \in \tilde{M}$ with $\theta(\tilde{p}) = 0$ is the union of the meridian $\theta = \pi$ opposite to $\theta = 0$ and a subarc of the parallel $t = -t(\tilde{p})$. Now, we introduce the Riemannian universal covering surface $\hat{M} := (\mathbb{R} \times_m \mathbb{R}, d\tilde{x}^2 + m(\tilde{x})^2 d\tilde{y}^2)$ of $(\tilde{M}, dt^2 + m(t)^2 d\theta^2)$. It follows from Tamura’s theorem above that the half space $\tilde{X} := (0, \infty) \times_m \mathbb{R}, d\tilde{x}^2 + m(\tilde{x})^2 d\tilde{y}^2$ of $\hat{M}$ has no pair of cut points in a sector $\tilde{X}(\theta)$ for each constant $\theta > 0$. For example, a model surface with its metric $d\tilde{x}^2 + (e^{-\tilde{x}^2})^2 d\tilde{y}^2$ is one of such models.

As an application of the Toponogov comparison theorem for open triangles, we will present splitting theorems of two types in Sections [10] and [11], respectively. The following theorem is one of them:

**Theorem 1.4** (Corollary [10.6] and Proposition [10.7] in Section [10])

Let $(X, \partial X)$ be a complete non-compact connected Riemannian $n$-dimensional manifold $X$ with smooth convex boundary $\partial X$ whose radial curvature is bounded from below by that of a model surface $(\tilde{X}, \partial \tilde{X})$ with its metric (1.1). Assume that $X$ admits at least one $\partial X$-ray.

(ST–1) If $(\tilde{X}, \partial \tilde{X})$ satisfies

$$\int_0^\infty \frac{1}{m(t)^2} dt = \infty,$$

then $X$ is isometric to $(0, \infty) \times_m \partial X$. In particular, $\partial X$ is the soul of $X$, and the number of connected components of $\partial X$ is one.

(ST–2) If $(\tilde{X}, \partial \tilde{X})$ satisfies

$$\liminf_{t \rightarrow \infty} m(t) = 0,$$

then $X$ is diffeomorphic to $(0, \infty) \times \partial X$. In particular, the number of connected components of $\partial X$ is one.

5
The Toponogov comparison theorem for open triangles in a weak form (Proposition 9.2) will be applied in the proof of Theorem 1.4 (see Section 10). The assumption on the existence of a ∂X-ray is very natural, because we may find at least one ∂X-ray if ∂X is compact. If the model X is Euclidean (i.e., m ≡ 1), then the (ST–1) holds. Hence, Theorem 1.4 extends one of Burago and Zalgaller’ splitting theorems to a wider class of metrics than those described in [BZ, Theorem 5.2.1], i.e., we mean that they assumed that sectional curvature is non-negative everywhere.

In Section 11 we will prove another splitting theorem (Theorem 11.6) for a complete connected Riemannian manifold with disconnected smooth compact convex boundary whose radial curvature is bounded from below by 0.

The body of this article is divided into twelve sections, inclusive of this introduction, as follows:

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In the following sections, all geodesics will be normalized, unless otherwise stated.

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2 The Sketch from Section 3 to Section 6

Here, we sketch in the organization from Sections 3 to 6, because we need many lemmas for proving our main theorem.

Throughout this section, let \((X, \partial X)\) be a complete connected Riemannian \(n\)-manifold \(X\) with smooth convex boundary \(\partial X\) whose radial curvature is bounded from below by that of a model surface \((\tilde{X}, \partial \tilde{X})\).

Our main purpose between Sections 3 to 5 is to prove the following lemma, which is one of fundamental lemmas to establish the Toponogov comparison theorem for open triangles (Theorem 8.4):

**Lemma on Thin Open Triangles.** For every thin open triangle \((\partial X, p, q)\) in \(X\), there exists an open triangle \((\partial \tilde{X}, \tilde{p}, \tilde{q})\) in \(\tilde{X}\) such that

\[
d(\partial \tilde{X}, \tilde{p}) = d(\partial X, p), \quad d(\tilde{p}, \tilde{q}) = d(p, q), \quad d(\partial \tilde{X}, \tilde{q}) = d(\partial X, q)
\]

and that

\[
\angle p \geq \angle \tilde{p}, \quad \angle q \geq \angle \tilde{q}.
\]

Thin open triangles are defined as follows:

**Definition 2.1 (Thin Open Triangle)** An open triangle \((\partial X, p, q) = (\gamma, \mu_1, \mu_2)\) in \((X, \partial X)\) is called a thin open triangle, if

(TOT–1) the opposite side \(\gamma\) of \((\partial X, p, q)\) to \(\partial X\) is contained in a normal convex neighborhood in \(X \setminus \partial X\), and

(TOT–2) \(L(\gamma) < \text{inj}(\tilde{q}_s)\) for all \(s \in [0, d(p, q)]\),

where \(L(\gamma)\) denotes the length of \(\gamma\), and \(\tilde{q}_s\) denotes a point in \(\tilde{X}\) with

\[
d(\partial \tilde{X}, \tilde{q}_s) = d(\partial X, \gamma(s))
\]

for each \(s \in [0, d(p, q)]\).

Here, the injectivity radius \(\text{inj}(\tilde{p})\) of a point \(\tilde{p} \in \tilde{X}\) is the supremum of \(r > 0\) such that, for any point \(\tilde{q} \in \tilde{X}\) with \(d(\tilde{p}, \tilde{q}) < r\), there exists a unique minimal geodesic segment joining \(\tilde{p}\) to \(\tilde{q}\). Remark that, for each point \(\tilde{p} \in \tilde{X} \setminus \partial \tilde{X}\),

\[
\text{inj}(\tilde{p}) > d(\partial \tilde{X}, \tilde{p})
\]

holds, if \(\tilde{p}\) is sufficiently close to \(\partial \tilde{X}\).

Hence, Sections 3 and 4 are set up to prove Lemma on thin open triangles (Lemma 5.8): In Section 3, we investigate the relationship between minimal geodesic segments in
3 The Focal Cut Locus of \( \partial X \)

Our purpose of this section is to investigate the relationship (Lemma 3.5) between minimal geodesic segments in a complete connected Riemannian manifold with smooth boundary and the focal cut locus of the boundary. In particular, it will be clarified, by using Lemma 3.5 in Section 5 that the cut locus of the manifold is not an obstruction at all when we draw a corresponding open triangle in a model surface for each open triangle in the manifold.

Throughout this section, let \((X, \partial X)\) denote a complete connected Riemannian \(n\)-manifold \(X\) with smooth boundary \(\partial X\).

First, we will recall the definitions of \(\partial X\)-Jacobi fields, focal loci of \(\partial X\), and cut loci of \(\partial X\), which are used throughout this article.

**Definition 3.1 (\(\partial X\)-Jacobi Field)** Let \(\mu : [0, \infty) \rightarrow X\) be a unit speed geodesic emanating perpendicularly from \(\partial X\). A Jacobi field \(J_{\partial X}\) along \(\mu\) is called a \(\partial X\)-Jacobi field, if \(J_{\partial X}(0) \in T_{\mu(0)}\partial X\), \(J_{\partial X}'(0) + A_{\mu'(0)}(J_{\partial X}(0)) \in (T_{\mu(0)}\partial X)\perp\).

Here \(J'\) denotes the covariant derivative of \(J\) along \(\mu\), and \(A_{\mu'(0)}\) denotes the shape operator of \(\partial X\).

**Definition 3.2 (Focal Locus of \(\partial X\))** A point \(\mu(t_0), t_0 \neq 0\), is called a focal point of \(\partial X\) along a unit speed geodesic \(\mu : [0, \infty) \rightarrow X\) emanating perpendicularly from \(\partial X\), if there exists a non-zero \(\partial X\)-Jacobi field \(J_{\partial X}\) along \(\mu\) such that \(J_{\partial X}(t_0) = 0\). The focal locus \(\text{Foc}(\partial X)\) of \(\partial X\) is the union of the focal points of \(\partial X\) along all of the unit speed geodesics emanating perpendicularly from \(\partial X\).

**Definition 3.3 (Cut Locus of \(\partial X\))** Let \(\mu : [0, \ell_0] \rightarrow X\) be a \(\partial X\)-segment. The end point \(\mu(\ell_0)\) of \(\mu([0, \ell_0])\) is called a cut point of \(\partial X\) along \(\mu\), if any extended geodesic \(\bar{\mu} : [0, \ell_1] \rightarrow X\) of \(\mu\), \(\ell_1 > \ell_0\), is not a \(\partial X\)-segment anymore. The cut locus \(\text{Cut}(\partial X)\) of \(\partial X\) is the union of the cut points of \(\partial X\) along all of the \(\partial X\)-segments.
Set
\[ \text{FC}(\partial X) := \text{Foc}(\partial X) \cap \text{Cut}(\partial X). \]
We then call \( \text{FC}(\partial X) \) the focal cut locus of \( \partial X \).

From the similar argument in [IT1], we have

**Lemma 3.4** (see [IT1, Lemma 2]) The Hausdorff dimension of \( \text{FC}(\partial X) \) is at most \( n-2 \). In particular,
\[ \mathcal{H}_{n-1}(\text{FC}(\partial X)) = 0. \]
Here \( \mathcal{H}_{n-1} \) denotes the \((n-1)\)-dimensional Hausdorff measure.

An open neighborhood \( U(q) \) of \( q \in X \) is called a normal convex neighborhood of \( q \), if, for any points \( q_1, q_2 \in U(q) \), there exists a unique minimal geodesic segment \( \sigma \) joining \( q_1 \) to \( q_2 \) such that the segment \( \sigma \) is contained in \( U(q) \). Then, it follows from Lemma 3.4 that

**Lemma 3.5** Assume that \( p \not\in \text{Foc}(\partial X), \ q \not\in \text{Cut}(p), \text{ and } \gamma([0,d(p,q)]) \cap \partial X = \emptyset \), where \( \gamma \) denotes the minimal geodesic segment joining \( p \) to \( q \). Then, for each \( v \in S_{q}^{n-1} := \{ v \in T_{q}X \ | \ ||v|| = 1 \} \), there exists a sequence
\[ \{ \gamma_i : [0,\ell_i] \rightarrow X \}_{i \in \mathbb{N}} \]
of minimal geodesic segments \( \gamma_i \) emanating from \( p = \gamma_i(0) \) convergent to \( \gamma \) such that
\[ \gamma_i([0,\ell_i]) \cap \text{FC}(\partial X) = \emptyset \]
and
\[ \lim_{i \to \infty} \frac{1}{\| \exp_{q}^{-1}(\gamma_i(\ell_i)) \|} \exp_{q}^{-1}(\gamma_i(\ell_i)) = v. \]
Here \( \exp_{q}^{-1} \) denotes the local inverse of the \( \exp_{q} \) on a normal convex neighborhood \( U(q) \) of \( q \) disjoint from \( \partial X \).

**Proof.** Let \( \{ q_j \}_{j \in \mathbb{N}} \) denote a sequence of points \( q_j \in U(q) \) convergent to \( q \) such that
\[ q_j \not\in \text{Cut}(p), \quad \alpha_j([0,d(p,q_j)]) \cap \partial X = \emptyset, \]
and
\[ \lim_{j \to \infty} \frac{1}{\| \exp_{q}^{-1}(q_j) \|} \exp_{q}^{-1}(q_j) = v. \]
Here \( \alpha_j : [0,d(p,q_j)] \rightarrow X \) denotes the minimal geodesic segment emanating from \( p = \alpha_j(0) \) to \( q_j \). We will prove that, for each \( q_j \), there exists a sequence
\[ \{ \gamma_i^{(j)} : [0,\ell_i^{(j)}] \rightarrow X \}_{i \in \mathbb{N}} \]
of minimal geodesic segments $\gamma_i^{(j)}$ emanating from $p = \gamma_i^{(j)}(0)$ convergent to $\alpha_j$ such that

$$\gamma_i^{(j)}([0, \ell_i^{(j)}]) \cap \text{FC}(\partial X) = \emptyset \quad (3.1)$$

It is sufficient to prove the existence of the sequence $\gamma_i^{(j)}$ for each $j \in \mathbb{N}$, because it is easy to prove the existence of the sequence $\{\gamma_i : [0, \ell_i] \rightarrow X\}_{i \in \mathbb{N}}$ in our lemma by taking a subsequence of $\{\gamma_i^{(j)} : [0, \ell_i^{(j)}] \rightarrow X\}_{i, j \in \mathbb{N}}$.

Choose any $q_j$ and fix it. Since $p$ is not a focal point of $\partial X$, there exists a normal convex neighborhood $B_{2\varepsilon}(p)$ of $p$ with radius $2\varepsilon$ such that

$$B_{2\varepsilon}(p) \cap \text{Foc}(\partial X) = \emptyset. \quad (3.2)$$

Since $q_j$ is not a cut point of $p$, there exist two numbers $\ell_j > d(p, q_j)$, $\theta_j > 0$, and a neighborhood $U_j$ around $q_j$ such that $U_j$ is diffeomorphic to $V_{\alpha_j^{(j)}(0)}(\theta_j) \times (\varepsilon, \ell_j)$. Here we set

$$V_{\alpha_j^{(j)}(0)}(\theta_j) := \{w_j \in T_p X \mid \|w_j\| = 1, \angle(w_j, \alpha_j^{(j)}(0)) < \theta_j\}.$$ 

Here, the diffeomorphism $\Phi_j$ from $V_{\alpha_j^{(j)}(0)}(\theta_j) \times (\varepsilon, \ell_j)$ onto $U_j$ is given by

$$\Phi_j(w_j, s) := \exp_p(s w_j).$$

Since $\Phi_j^{-1}$ is Lipschitz, the map $\Pi_j := \mathcal{P}_j \circ \Phi_j^{-1} : U_j \rightarrow V_{\alpha_j^{(j)}(0)}(\theta_j)$ is also Lipschitz, where $\mathcal{P}_j : V_{\alpha_j^{(j)}(0)}(\theta_j) \times (\varepsilon, \ell_j) \rightarrow V_{\alpha_j^{(j)}(0)}(\theta_j)$ denotes the projection to the first factor. Therefore, it follows from Lemma 3.4 that

$$\mathcal{H}_{n-1}(\Pi_j(U_j \cap \text{FC}(\partial X))) = 0.$$ 

This implies that there exists a sequence $\{w_i^{(j)}\}_{i \in \mathbb{N}}$ of elements $w_i^{(j)} \in V_{\alpha_j^{(j)}(0)}(\theta_j)$ convergent to $\alpha_j^{(j)}(0)$ such that

$$w_i^{(j)} \not\in \Pi_j(U_j \cap \text{FC}(\partial X)) \quad (3.3)$$

for each $i \in \mathbb{N}$. Let $\{\ell_i^{(j)}\}_{i \in \mathbb{N}}$ be a sequence of numbers $\ell_i^{(j)} \in (0, \ell_j)$ convergent to $d(p, q_j)$. By setting

$$\gamma_i^{(j)}(s) := \exp_p(s w_i^{(j)}), \quad s \in [0, \ell_i^{(j)}],$$

for each $i \in \mathbb{N}$, it follows from (3.2) and (3.3) that we get a sequence of minimal geodesic segments $\gamma_i^{(j)}$ emanating from $p = \gamma_i^{(j)}(0)$ convergent to $\alpha_j$ satisfying (3.1). \hfill \Box

## 4 Length of $\partial X$-segments in Variations

Our purpose of this section is to prove a comparison theorem (Lemma 4.5) of the Rauch type on length of $\partial X$-segments in variations of a $\partial X$-segment, by using the second variation formula and the Warner comparison theorem. As a result, readers might be surprised by, and would realize, as Gromoll once suggested, that we may still understand a global matter on a Riemannian manifold by the second variation, because Lemma on thin open triangles (Lemma 3.8), proved by Lemmas 3.5 and 4.5, plays an important role in the proof of the Toponogov comparison theorem for open triangles (see Section 3).
Throughout this section, let \((X, \partial X)\) denote a complete connected Riemannian \(n\)-manifold \(X\) with smooth \textbf{convex} boundary \(\partial X\) whose radial curvature is bounded from below by the radial curvature function \(G\) of a model surface \((\tilde{X}, \partial \tilde{X})\) with its metric \((1,1)\).

Take any point \(r \in X \setminus (\partial X \cup \text{Foc}(\partial X))\), and fix it. Then there exists a positive number \(\varepsilon_0 := \varepsilon_0(r)\) such that

\[
B_{2\varepsilon_0}(r) \cap (\text{Foc}(\partial X) \cup \partial X) = \emptyset, \tag{4.1}
\]

where \(B_{2\varepsilon_0}(r)\) denotes the normal convex neighborhood of \(r\) with radius \(2\varepsilon_0\). Take any point \(p \in B_{\varepsilon_0}(r)\), and fix it. Let \(\mu : [0, \ell] \to X\) denote a \(\partial X\)-segment to \(p = \mu(\ell)\). By \((4.1)\), we may find a number \(\varepsilon_1 \in (0, \varepsilon_0]\) independent of the choice of \(p\) and an open neighborhood \(U\) around \(\ell \mu'(0)\) such that

\[
\exp^\perp : U \to B_{\varepsilon_1}(p)
\]

is a diffeomorphism. Here \(\exp^\perp\) denotes the normal exponential map on the normal bundle of \(\partial X\). Let \(\xi : \mathbb{R} \to S_{p}^{n-1}\) be a unit speed geodesic on \(S_{p}^{n-1}\) emanating from \(\mu'(\ell) = \xi(0)\), where \(S_{p}^{n-1} := \{v \in T_pX \mid \|v\| = 1\}\). Notice that

\[
\angle(\mu'(\ell), \xi(\theta)) = |\theta|
\]

for all \(\theta \in [-\pi, \pi]\). From now on, we assume that the curve \(\xi\) and its parameter value \(\theta \in [-\pi, \pi]\) are also fixed. Then, we get a minimal geodesic segment \(c\) emanating from \(p = c(0)\) defined by

\[
c(s) := \exp_p(s \xi(\theta))
\]

for all \(s \in (-\varepsilon_1, \varepsilon_1)\). Thus, we get a geodesic variation \(\varphi : [0, \ell] \times (-\varepsilon_1, \varepsilon_1) \to X\) of \(\mu\) defined by

\[
\varphi(t, s) := \exp^\perp \left( \frac{t}{\ell} v(s) \right),
\]

where we set \(v(s) := (\exp^\perp |_{\mu})^{-1}(c(s))\). For each \(s \in (-\varepsilon_1, \varepsilon_1)\), \(c(s)\) is joined by a geodesic segment \(\varphi_s(\cdot, s) := \varphi(\cdot, s)\) emanating perpendicularly from \(\partial X\). By setting

\[
J_{\partial X}(t) := \frac{\partial \varphi}{\partial s}(t, 0),
\]

we get a \(\partial X\)-Jacobi field \(J_{\partial X}\) along \(\mu\). It is clear that

\[
J_{\partial X}(\ell) = c'(0). \tag{4.2}
\]

Then, we first get

\[\textbf{Lemma 4.1} \text{ For each } t \in [0, \ell], \text{ a vertical component } Y_{\partial X}(t) \text{ of } J_{\partial X}(t) \text{ with respect to } \mu'(t) \text{ is given by}
\]

\[
Y_{\partial X}(t) := J_{\partial X}(t) - \frac{\cos \theta}{\ell} t \mu'(t).
\]

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Proof. Since $J_{\partial X}$ is a Jacobi field along $\mu$, there exist constant numbers $a$ and $b$ satisfying

$$\langle J_{\partial X}(t), \mu'(t) \rangle = at + b$$

for all $t \in [0, \ell]$. Since $J_{\partial X}(0)$ is orthogonal to $\mu'(0)$, we see $b = 0$. Furthermore, by (4.2), we see

$$a = \cos \frac{\theta}{\ell}.$$ 

Thus, we get

$$\langle J_{\partial X}(t), \mu'(t) \rangle = \frac{\cos \theta}{\ell} t$$

for all $t \in [0, \ell]$. Hence, the Jacobi field $Y_{\partial X}$ along $\mu$ defined by

$$Y_{\partial X}(t) := J_{\partial X}(t) - \frac{\cos \theta}{\ell} t \mu'(t)$$

is orthogonal to $\mu'(t)$ on $[0, \ell]$.

In this article, we denote by

$$\mathcal{I}^f_{\partial X}(V, W) := I_\ell(V, W) - \langle A_{\mu}(V(0)), W(0) \rangle$$

the index form with respect to $\mu|[0, \ell]$ for piecewise $C^\infty$ vector fields $V, W$ along $\mu|[0, \ell]$, where we set

$$I_\ell(V, W) := \int_0^\ell \left\{ \langle V', W' \rangle - \langle R(\mu', V) \mu', W \rangle \right\} dt,$$

which is a symmetric bilinear form. The following lemma is clear from the first and second variation formulas and Lemma 4.1.

Lemma 4.2

$$L'(0) = \cos \theta$$

and

$$L''(0) = \mathcal{I}^f_{\partial X}(Y_{\partial X}, Y_{\partial X})$$

hold. Here $L(s)$ denotes the length of the geodesic segment $\varphi_s(\cdot)$ emanating perpendicularly from $\partial X$.

Now, choose any sufficiently small number $\lambda > 0$ and fix it. Let $(\tilde{X}_\lambda, \partial \tilde{X}_\lambda)$ denote a model surface with its metric

$$\tilde{g}_\lambda = d\tilde{x}^2 + m_\lambda(\tilde{x})^2 d\tilde{y}^2$$

on $[0, \infty) \times \mathbb{R}$. Here the positive smooth function $m_\lambda$ satisfies the differential equation

$$m_\lambda'' + (G - \lambda)m_\lambda = 0, \quad m_\lambda(0) = 1, \quad m_\lambda'(0) = 0,$$

where $G$ denotes the radial curvature function of $(\tilde{X}, \partial \tilde{X})$. Thus, the radial curvature of $(X, \partial X)$ is greater than $G_\lambda := G - \lambda$. Take any point $\tilde{p}$ in $\tilde{X}_\lambda \setminus \partial \tilde{X}_\lambda$ satisfying

$$d(\partial \tilde{X}_\lambda, \tilde{p}) = d(\partial X, p) = d(\partial X, \mu(\ell)) = \ell.$$
Throughout this section, we fix the $\tilde{\mu}$.

Let $\tilde{\mu}_\lambda: [-\ell, \ell] \to \tilde{X}_\lambda$ denote a $\partial\tilde{X}_\lambda$-segment to $\tilde{\mu}$, and let $\tilde{E}_\lambda$ denote a unit parallel vector field along $\tilde{\mu}_\lambda$ orthogonal to $\tilde{\mu}_\lambda$. Then, we define a $\partial\tilde{X}_\lambda$-Jacobi field $\tilde{Z}_\lambda$ along $\tilde{\mu}_\lambda$

$$\tilde{Z}_\lambda(t) := \frac{1}{m_\lambda(\ell)} m_\lambda(t) \tilde{E}_\lambda(t).$$

Furthermore, by the same definition above, we also denote by $I_\ell(\cdot, \cdot)$ the symmetric bilinear form for piecewise $C^\infty$ vector fields along $\tilde{\mu}_\lambda|_{[0,\ell]}$. Then, we have

**Lemma 4.3**

$$I_\ell(\tilde{Z}_\lambda, \tilde{Z}_\lambda) \geq I_{\partial X}^\ell(Z_{\partial X}, Z_{\partial X}) + \frac{\lambda}{m_\lambda(\ell)^2} \int_0^\ell m_\lambda(t)^2 dt$$

holds for all $\partial X$-Jacobi field $Z_{\partial X}$ along $\mu$ orthogonal to $\mu$ with $\|Z_{\partial X}(\ell)\| = 1$.

**Proof.** From the argument in the proof of the Warner comparison theorem [W], we may prove this lemma. For completeness, we will give a proof here: Let $E$ be a unit parallel vector field along $\mu$ orthogonal to $\mu$ such that

$$E(\ell) = Z_{\partial X}(\ell),$$

where $Z_{\partial X}$ denotes a $\partial X$-Jacobi field along $\mu$ orthogonal to $\mu$. Set

$$W(t) := \frac{1}{m_\lambda(\ell)} m_\lambda(t) E(t).$$

Since $K_X(\sigma_t) \geq G(\tilde{\mu}(t)) > G_\lambda(\tilde{\mu}_\lambda(t)) = G(\tilde{\mu}(t)) - \lambda$, we have

$$I_\ell(\tilde{Z}_\lambda, \tilde{Z}_\lambda) = \int_0^\ell \left\{ \left\langle \tilde{Z}_\lambda', \tilde{Z}_\lambda' \right\rangle - G_\lambda(\tilde{\mu}_\lambda(t)) \| \tilde{Z}_\lambda \|^2 \right\} dt$$

$$= \int_0^\ell \left\{ \left\langle W', W' \right\rangle - (G(\tilde{\mu}(t)) - \lambda) \| W \|^2 \right\} dt$$

$$\geq \int_0^\ell \left\{ \left\langle W', W' \right\rangle - K_X(\sigma_t) \| W \|^2 \right\} dt + \lambda \int_0^\ell \| W \|^2 dt$$

$$= I_\ell(W, W) + \frac{\lambda}{m_\lambda(\ell)^2} \int_0^\ell m_\lambda(t)^2 dt. \quad (4.3)$$

Since $Z_{\partial X}$ is the $\partial X$-Jacobi field with $Z_{\partial X}(\ell) = E(\ell) = W(\ell)$, it follows from [S, Lemma 2.10 in Chapter III] that

$$I_\ell(W, W) - \left\langle A_{\mu'(0)}(W(0)), W(0) \right\rangle = \mathcal{I}_{\partial X}^\ell(W, W) \geq \mathcal{I}_{\partial X}^\ell(Z_{\partial X}, Z_{\partial X}). \quad (4.4)$$

Since $\left\langle A_{\mu'(0)}(W(0)), W(0) \right\rangle \geq 0$, we get, by (4.3) and (4.4),

$$I_\ell(\tilde{Z}_\lambda, \tilde{Z}_\lambda) \geq \mathcal{I}_{\partial X}^\ell(Z_{\partial X}, Z_{\partial X}) + \left\langle A_{\mu'(0)}(W(0)), W(0) \right\rangle + \frac{\lambda}{m_\lambda(\ell)^2} \int_0^\ell m_\lambda(t)^2 dt$$

$$\geq \mathcal{I}_{\partial X}^\ell(Z_{\partial X}, Z_{\partial X}) + \frac{\lambda}{m_\lambda(\ell)^2} \int_0^\ell m_\lambda(t)^2 dt.$$

$\Box$
Let \( \tilde{c}_\lambda : (-\varepsilon_1, \varepsilon_1) \to \tilde{X}_\lambda \) denote the minimal geodesic segment emanating from \( \tilde{p} = \tilde{c}_\lambda(0) \) corresponding to the minimal geodesic segment \( c(s) = \exp_p(s \xi(\theta)), s \in (-\varepsilon_1, \varepsilon_1) \) in \( B_{\varepsilon_1}(p) \subset X \). Without loss of generality, we may assume that \( B_{\varepsilon_1}(<\tilde{p}) \cap \partial \tilde{X} = \emptyset \). We consider a geodesic variation \( \tilde{\varphi}^{(\lambda)} : [0, \ell] \times (-\varepsilon_1, \varepsilon_1) \to \tilde{X}_\lambda \) of \( \tilde{\mu}_\lambda \) defined by

\[
\tilde{\varphi}^{(\lambda)}(t, s) := \exp^\perp (\frac{t}{\ell} \tilde{v}_\lambda(s)),
\]

where we set \( \tilde{v}_\lambda(s) := (\exp^\perp)^{-1}(\tilde{c}_\lambda(s)) \). By setting \( \tilde{J}_\lambda(t) := \partial \tilde{\varphi}^{(\lambda)}(t, 0) \), we get a \( \partial \tilde{X}_\lambda \)-Jacobi field \( \tilde{J}_\lambda \) along \( \tilde{\mu}_\lambda \). As well as above, \( \tilde{J}_\lambda(\ell) = \tilde{c}_\lambda'(0) \) holds, and the Jacobi field \( \tilde{Y}_\lambda \) along along \( \tilde{\mu}_\lambda \) defined by

\[
\tilde{Y}_\lambda(t) := \tilde{J}_\lambda(t) - \frac{\cos \theta}{\ell} t \tilde{\mu}_\lambda'(t)
\]
is orthogonal to \( \tilde{\mu}_\lambda'(t) \) on \([0, \ell]\).

**Lemma 4.4** There exists a number \( \lambda_0 := \lambda_0(\ell_0, \varepsilon_0) > 0 \) depending on \( \ell_0 \) and \( \varepsilon_0 \) such that, for any \( \lambda \in (0, \lambda_0) \), any unit speed geodesic \( \xi \) on \( S_p^{n-1} \) emanating from \( \mu'(\ell) \), and any \( \theta \in (0, \pi) \),

\[
I_\ell(\tilde{Y}_\lambda, \tilde{Y}_\lambda) - \mathcal{I}_{\partial X}(Y_{\partial X}, Y_{\partial X}) \geq \lambda C_1 \sin^2 \theta
\]
holds. Here \( C_1 \) is a constant number given by

\[
C_1 := \frac{1}{2m(\ell_0)^2} \int_0^{\ell_0} m(t)^2 \, dt,
\]
where \( \ell_0 := d(\partial X, r) \).

**Proof.** Since

\[
\tilde{Y}_\lambda(\ell) = \tilde{c}_\lambda'(0) - \cos \theta \tilde{\mu}_\lambda'(\ell) = \pm \sin \theta \cdot \tilde{E}_\lambda(\ell) = \pm \sin \theta \cdot \tilde{Z}_\lambda(\ell),
\]
we see

\[
\tilde{Y}_\lambda(t) = \pm \sin \theta \cdot \tilde{Z}_\lambda(t)
\]
on \([0, \ell]\). Hence, we have

\[
I_\ell(\tilde{Y}_\lambda, \tilde{Y}_\lambda) = \sin^2 \theta \cdot I_\ell(\tilde{Z}_\lambda, \tilde{Z}_\lambda). \tag{4.5}
\]

Similarly, we have, by Lemma [4.1],

\[
Y_{\partial X}(t) = \sin \theta \cdot Z_{\partial X}(t)
\]
for some \( \partial X \)-Jacobi field \( Z_{\partial X} \) along \( \mu \) orthogonal to \( \mu \) with \( \|Z_{\partial X}(\ell)\| = 1 \). Hence, we have

\[
\mathcal{I}_{\partial X}(Y_{\partial X}, Y_{\partial X}) = \sin^2 \theta \cdot \mathcal{I}_{\partial X}(Z_{\partial X}, Z_{\partial X}). \tag{4.6}
\]
By combining (4.5) and (4.6), we get, by Lemma 4.3,

\[ I_\ell(\tilde{Y}_\lambda, \tilde{Y}_\lambda) - I_\ell(\tilde{Z}_\lambda, \tilde{Z}_\lambda) = \sin^2 \theta \left\{ I_\ell(\tilde{Z}_\lambda, \tilde{Z}_\lambda) - I_\ell(\tilde{Y}_\lambda, \tilde{Y}_\lambda) \right\} \]

\[ \geq \frac{\lambda \sin^2 \theta}{m_\lambda(\ell)^2} \int_0^\ell m_\lambda(t)^2 \, dt . \] (4.7)

On the other hand, since \( \lim_{\lambda \downarrow 0} m_\lambda(t) = m(t) \) and \( |\ell - \ell_0| < \varepsilon_0 \), we may find a number \( \lambda_0 > 0 \) such that

\[ \frac{1}{m_\lambda(\ell)^2} \int_0^\ell m_\lambda(t)^2 \, dt > \frac{1}{2m(\ell_0)^2} \int_0^{\ell_0} m(t)^2 \, dt \] (4.8)

for all \( \lambda \in (0, \lambda_0) \). From (4.7) and (4.8), we have proved this lemma. \( \Box \)

**Lemma 4.5 (Key Lemma)** For each \( \lambda \in (0, \lambda_0) \), there exists a number \( \delta_1 := \delta_1(\lambda) \in (0, \varepsilon_0) \) such that, for any \( p \in B_{\varepsilon_0}(r) \), any unit speed geodesic \( \xi \) on \( \mathbb{S}^{n-1}_p \) emanating from \( \mu'(\ell) \), any \( \theta \in [0, \pi] \), and any \( \lambda \in (0, \lambda_0) \),

\[ L(s) \leq \tilde{L}_\lambda(s) \]

holds for all \( s \in [0, \delta_1] \), and equality occurs if and only if \( s = 0 \), \( \theta = 0 \), or \( \theta = \pi \). Here \( \tilde{L}_\lambda(s) \) denotes the length of the geodesic segment \( \tilde{\varphi}_s(\cdot) = \tilde{\varphi}(\cdot, s) \) emanating perpendicularly from \( \partial \tilde{X}_\lambda \) to \( \tilde{c}_\lambda(s) \).

**Proof.** Although the angle \( \theta \) has been fixed in the arguments above of this section, we consider here that \( \theta \) is a variable. Hence, we denote \( L(s) \) by \( L(s, \theta) \), which is a smooth function of two variables \( s \) and \( \theta \) and depends smoothly on \( p, \xi(0) \), and \( \xi'(0) \). Furthermore, we define the reminder term \( R(s, \theta) \) of the Taylor expansion of \( L(s, \theta) \) about \( s = 0 \) by

\[ R(s, \theta) := L(s, \theta) - \left\{ L(0, \theta) + L'(0, \theta) s + \frac{1}{2!} L''(0, \theta) s^2 \right\} \] (4.9)

where we set

\[ L'(0, \theta) := \frac{\partial L}{\partial s}(0, \theta) \quad \text{and} \quad L''(0, \theta) := \frac{\partial^2 L}{\partial s^2}(0, \theta) . \]

From (4.9), Lemma 4.2, and the equation (4.6) in the proof of Lemma 4.4, we have

\[ L(s, \theta) = \ell + s \cos \theta + \frac{s^2}{2} I_\ell(\tilde{Y}_\lambda, \tilde{Y}_\lambda) + R(s, \theta) \] (4.10)

\[ = \ell + s \cos \theta + \frac{s^2 \sin^2 \theta}{2} I_\ell(\tilde{Z}_\lambda, \tilde{Z}_\lambda) + R(s, \theta) . \] (4.11)

It is clear that

\[ R(0, \theta) = \frac{\partial R}{\partial s}(0, \theta) = \frac{\partial^2 R}{\partial s^2}(0, \theta) = 0 \] .

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Hence, there exists a smooth function $R_1(s, \theta)$ depending smoothly on $p, \xi(0),$ and $\xi'(0)$ such that
\[ R(s, \theta) = R_1(s, \theta)s^3. \] (4.12)

Since $B_{2\varepsilon_0}(r) \cap \text{Foc}(\partial X) = \emptyset$, the geodesic $\varphi_s(\cdot)$ is locally minimal for each $s \in (-\varepsilon_1, \varepsilon_1)$. Hence, we may assume that the triangle inequalities
\[ L(s, \theta) \leq \ell + s = L(s, 0) \] (4.13)
and
\[ L(s, \pi - \theta) \geq \ell - s = L(s, \pi) \] (4.14)
hold for all sufficiently small $|\theta|$ and all $s \in [0, \varepsilon_1)$. The equations (4.13) and (4.14) mean that, for each $s \in (0, \varepsilon_1)$, the function $L(s, \cdot)$ attains a local maximum (respectively, minimum) at $\theta = 0$ (respectively, $\theta = \pi$). Hence, by (4.11) and (4.12),
\[ \frac{\partial R_1}{\partial \theta}(s, 0) = \frac{\partial R_1}{\partial \theta}(s, \pi) = 0 \] (4.15)
for each $s \in [0, \varepsilon_1)$. Since $R_1(s, 0) = R_1(s, \pi) = 0$ holds on $[0, \varepsilon_1)$, we see, by (4.15), that there exists a smooth function $R_2(s, \theta)$ such that
\[ R_1(s, \theta) = R_2(s, \theta)\theta^2(\pi - \theta)^2. \] (4.16)

By (4.12) and (4.16), we have
\[ R(s, \theta) = R_2(s, \theta)\theta^2(\pi - \theta)^2s^3. \] (4.17)
for all $\theta \in [0, \pi]$ and all $s \in [0, \varepsilon_1)$. On the other hand, since $R_2$ depends continuously on $p, \xi(0),$ and $\xi'(0)$, there exists a constant $C_2 > 0$ such that
\[ |R_2(s, \theta)| \leq C_2 \] (4.18)
holds for all $p \in B_{\varepsilon_0}(r)$, all $\xi$ on $S_p^{n-1}$, all $\theta \in [0, \pi]$, and all $s \in [0, \varepsilon_1/2]$. Thus, by (4.17) and (4.18), we obtain
\[ |R(s, \theta)| \leq C_2 \theta^2(\pi - \theta)^2s^3 \] (4.19)
for all $p \in B_{\varepsilon_0}(r)$, all $\xi$ on $S_p^{n-1}$, all $\theta \in [0, \pi]$ and all $s \in [0, \varepsilon_1/2]$. Combining (4.10) and (4.19), we get
\[ L(s, \theta) \leq \ell + s \cos \theta + \frac{s^2}{2}I_\lambda(Y_{\lambda}, Y_{\lambda}) + C_2 \theta^2(\pi - \theta)^2s^3. \] (4.20)

By applying the same argument above for $\tilde{L}(s) = \tilde{L}(s, \theta)$, there exists a constant $C_3 > 0$ such that
\[ \tilde{L}(s, \theta) \geq \ell + s \cos \theta + \frac{s^2}{2}I_\lambda(\tilde{Y}_{\lambda}, \tilde{Y}_{\lambda}) - C_3 \theta^2(\pi - \theta)^2s^3 \] (4.21)
holds for all $\theta \in [0, \pi]$ and all $s \in [0, \varepsilon_1/2]$. From Lemma 4.4, (4.20), and (4.21), it follows that

$$
\bar{L}_\lambda(s, \theta) - L(s, \theta) \geq \frac{s^2}{2} \left\{ I_\ell(\tilde{Y}_\lambda, \tilde{Y}_\lambda) - I_\ell^\theta(Y_{\partial X}, Y_{\partial X}) \right\} - (C_3 + C_2)\theta^2(\pi - \theta)^2s^3
$$

$$
\geq \frac{\lambda C_1 \sin^2 \theta}{2} s^2 - (C_3 + C_2)\theta^2(\pi - \theta)^2s^3
$$

$$
\geq \frac{\lambda C_1 \sin^2 \theta}{2} s^2 - 2C_4 \theta^2(\pi - \theta)^2s^3 \quad (4.22)
$$

holds for all $\theta \in [0, \pi]$ and all $s \in [0, \varepsilon_1/2]$. Here we set $C_4 := \max\{C_2, C_3\}$. Since

$$
\frac{x}{\sin x} < \frac{\pi}{2}
$$

for all $x \in (0, \pi/2)$,

$$
\frac{\theta}{\sin \theta} \cdot (\pi - \theta) < \frac{\pi}{2} \cdot (\pi - \theta) < \frac{\pi^2}{2} \quad (4.23)
$$

holds on $(0, \pi/2)$, and

$$
\frac{\pi - \theta}{\sin \theta} \cdot \theta = \frac{\pi - \theta}{\sin(\pi - \theta)} \cdot \theta < \frac{\pi}{2} \cdot \theta < \frac{\pi^2}{2} \quad (4.24)
$$

also holds on $(\pi/2, \pi)$. Hence, by (4.23) and (4.24), we see

$$
\frac{\theta(\pi - \theta)}{\sin \theta} < \frac{\pi^2}{2} \quad (4.25)
$$

on $(0, \pi)$. If we define

$$
\delta_1 := \min \left\{ \frac{\varepsilon_1}{2}, \frac{\lambda C_1}{\pi C_4} \right\} \left( \leq \frac{\varepsilon_1}{2} < \varepsilon_0 \right).
$$

then, by (4.25),

$$
\frac{\lambda C_1 \sin^2 \theta}{2} s^2 - 2C_4 \theta^2(\pi - \theta)^2s^3 = \left( \frac{s \cdot \sin \theta}{2} \right)^2 \left[ \lambda C_1 - 4C_4 \left\{ \frac{\theta(\pi - \theta)}{\sin \theta} \right\} \right]^2 s
$$

$$
> \left( \frac{s \cdot \sin \theta}{2} \right)^2 \left( \lambda C_1 - \pi^4 C_4 s \right)
$$

$$
\geq 0 \quad (4.26)
$$

holds for all $s \in [0, \delta_1]$ and all $\theta \in (0, \pi)$. Therefore, by (4.22) and (4.26), the proof is complete. \(\square\)

5 Thin Open Triangles

Throughout this section, let $(\tilde{X}, \partial \tilde{X})$ denote a model surface with its metric (1.1).
Lemma 5.1 Let $\tilde{\mu} : [0, \ell] \to \tilde{X}$ be a $\partial \tilde{X}$-segment. Then, for each $0 < s < \min\{\text{inj}(\tilde{\mu}(\ell)), \ell\}$, the function $d(\partial \tilde{X}, \exp_{\tilde{\mu}(s)}(s \tilde{\xi}(\theta)))$ is strictly increasing on $[0, \pi]$. Here $\tilde{\xi} : \mathbb{R} \to S^1_{\tilde{\mu}(\ell)} := \{ \tilde{v} \in T_{\tilde{\mu}(\ell)}\tilde{X} \mid \|\tilde{v}\| = 1 \}$ denotes a unit speed geodesic segment on $S^1_{\tilde{\mu}(\ell)}$ emanating from $-\tilde{\mu}'(\ell) = \tilde{\xi}(0)$.

Proof. This lemma is clear from the first variation formula. 

The next lemma is a direct consequence of the Clairaut relation ([SST, Theorem 7.1.2]) and the first variational formula:

Lemma 5.2 For each constant $c \geq 0$, and each point $\tilde{p} \in \tilde{X}$, $d(\tilde{p}, \tilde{t}_c(s))$ is strictly increasing on $[\tilde{y}(\tilde{p}), \infty)$. Here $\tilde{t}_c(s) := (c, s) \in \tilde{X}$ denote the arc of $\tilde{x} = c$.

By Lemma 5.2, we have

Lemma 5.3 Let $(\partial \tilde{X}, \tilde{p}_1, \tilde{q}_1) = (\tilde{\gamma}_1, \tilde{\mu}_1^{(1)}, \tilde{\mu}_1^{(2)}), (\partial \tilde{X}, \tilde{p}_2, \tilde{q}_2) = (\tilde{\gamma}_2, \tilde{\mu}_2^{(1)}, \tilde{\mu}_2^{(2)})$ be open triangles in $\tilde{X}$ such that

\[ d(\partial \tilde{X}, \tilde{q}_1) = d(\partial \tilde{X}, \tilde{p}_2), \quad (5.1) \]

and that

\[ \angle \tilde{q}_1 + \angle \tilde{p}_2 \leq \pi. \quad (5.2) \]

If

\[ d(\tilde{p}_1, \tilde{q}_1) + d(\tilde{p}_2, \tilde{q}_2) < \text{inj}(\tilde{p}_1), \]

then there exists an open triangle $(\partial \tilde{X}, \tilde{p}, \tilde{q})$ such that

\[ d(\partial \tilde{X}, \tilde{p}) = d(\partial \tilde{X}, \tilde{p}_1), \quad d(\tilde{p}, \tilde{q}) = d(\tilde{p}_1, \tilde{q}_1) + d(\tilde{p}_2, \tilde{q}_2), \quad d(\partial \tilde{X}, \tilde{q}) = d(\partial \tilde{X}, \tilde{q}_2), \quad (5.3) \]

and that

\[ \angle \tilde{p}_1 \geq \angle \tilde{p}. \quad (5.4) \]

Proof. By (5.1), we may assume that $(\partial \tilde{X}, \tilde{p}_2, \tilde{q}_2)$ is adjacent to $(\partial \tilde{X}, \tilde{p}_1, \tilde{q}_1)$ as a common side $\tilde{\mu}_2^{(1)} = \tilde{\mu}_1^{(2)}$, and that $\tilde{y}(\tilde{p}_1) < \tilde{y}(\tilde{q}_1) = \tilde{y}(\tilde{p}_2) < \tilde{y}(\tilde{q}_2)$. Choose any number

\[ a \in (d(\tilde{p}_1, \tilde{q}_1) + d(\tilde{p}_2, \tilde{q}_2), \text{inj}(\tilde{p}_1)), \]

and fix it. We will introduce geodesic polar coordinates $(r, \theta)$ around $\tilde{p}_1$ on $B_a(\tilde{p}_1)$ such that $\theta = 0$ on $\tilde{\mu}_1^{(1)} \cap B_a(\tilde{p}_1)$, and that $0 < \theta(\tilde{q}_2) \leq \theta(\tilde{q}_1) \leq \pi$. Notice that

\[ \tilde{q}_2 \in B_a(\tilde{p}_1). \]

In fact, from the triangle inequality,

\[ d(\tilde{p}_1, \tilde{q}_2) \leq d(\tilde{p}_1, \tilde{q}_1) + d(\tilde{q}_1, \tilde{q}_2) = d(\tilde{p}_1, \tilde{q}_1) + d(\tilde{p}_2, \tilde{q}_2) < a. \]
Thus, by (5.6) and (5.8), we have
\[ \angle \tilde{q}_1 + \angle \tilde{p}_2 < \pi. \] (5.5)
Hence,
\[ \tilde{q}_2 \in \mathcal{A}(\tilde{p}_1), \]
where \( \mathcal{A}(\tilde{p}_1) \) is a domain defined by
\[ \mathcal{A}(\tilde{p}_1) := B_a(\tilde{p}_1) \cap \theta^{-1}(0, \theta(\tilde{q}_1)). \]
Let \( \tilde{\tau} : [\tilde{y}(\tilde{q}_2), \infty) \rightarrow \tilde{X} \) be an arc of \( \tilde{x} = \tilde{x}(\tilde{q}_2) \) emanating from \( \tilde{q}_2 = \tilde{\tau}(\tilde{y}(\tilde{q}_2)) \in \mathcal{A}(\tilde{p}_1) \) given by
\[ \tilde{\tau}(s) := (\tilde{x}(\tilde{q}_2), s). \]
By (5.5), we get
\[ d(\tilde{p}_1, \tilde{\tau}(\tilde{y}(\tilde{q}_2))) < d(\tilde{p}_1, \tilde{q}_1) + d(\tilde{q}_1, \tilde{q}_2) = d(\tilde{p}_1, \tilde{q}_1) + d(\tilde{p}_2, \tilde{q}_2) < a. \]
Since \( \lim_{s \to \infty} d(\tilde{p}_1, \tilde{\tau}(s)) = \infty \), it follows from the intermediate value theorem that there exists a number \( s_0 \in (\tilde{y}(\tilde{q}_2), \infty) \) satisfying
\[ d(\tilde{p}_1, \tilde{\tau}(s_0)) = a, \]
and furthermore that there exists a number \( s_1 \in (\tilde{y}(\tilde{q}_2), s_0) \) satisfying
\[ d(\tilde{p}_1, \tilde{\tau}(s_1)) = d(\tilde{p}_1, \tilde{q}_1) + d(\tilde{q}_1, \tilde{q}_2) = d(\tilde{p}_1, \tilde{q}_1) + d(\tilde{p}_2, \tilde{q}_2). \] (5.6)
We will prove that the subarc \( \tilde{\tau}|_{[\tilde{y}(\tilde{q}_2), s_1]} \) is contained in \( \mathcal{A}(\tilde{p}_1) \). Suppose that there exists a number \( s_2 \in (\tilde{y}(\tilde{q}_2), s_1] \) such that
\[ \tilde{\tau}(s_2) \not\in \mathcal{A}(\tilde{p}_1). \]
Since the subarc \( \tilde{\tau}|_{[\tilde{y}(\tilde{q}_2), s_2]} \) lies in \( B_a(\tilde{p}_1) \), there exists \( s_3 \in (\tilde{y}(\tilde{q}_2), s_2] \) such that
\[ \theta(\tilde{\tau}(s_3)) = \theta(\tilde{q}_1). \] (5.7)
Since \( \tilde{y}(\tilde{q}_2) < s_3 \), we have, by Lemma 5.2
\[ d(\tilde{q}_1, \tilde{q}_2) < d(\tilde{q}_1, \tilde{\tau}(s_3)). \] (5.8)
By (5.7), we see that the geodesic extension \( \tilde{\sigma} : [0, d(\tilde{p}_1, \tilde{\tau}(s_3))] \rightarrow \tilde{X} \) of \( \tilde{\tau}_1 \) meets \( \tilde{\tau} \) at \( \tilde{\tau}(s_3) = \tilde{\sigma}(d(\tilde{p}_1, \tilde{\tau}(s_3))) \). Notice that the geodesic segment \( \tilde{\sigma} \) is minimal, since
\[ \tilde{\tau}(s_3) \in B_a(\tilde{p}_1) \subset B_{\min(\tilde{p}_1)}(\tilde{p}_1). \]
Thus, by (5.6) and (5.8), we have
\[ d(\tilde{p}_1, \tilde{\tau}(s_3)) = d(\tilde{p}_1, \tilde{q}_1) + d(\tilde{q}_1, \tilde{\tau}(s_3)) > d(\tilde{p}_1, \tilde{q}_1) + d(\tilde{q}_1, \tilde{q}_2) = d(\tilde{p}_1, \tilde{\tau}(s_1)). \] (5.9)
On the other hand, since \( s_3 < s_1 \), it follows from Lemma 5.2 that
\[
d(\tilde{p}_1, \tilde{\tau}(s_3)) < d(\tilde{p}_1, \tilde{\tau}(s_1)).
\] (5.10)

The equation (5.10) contradicts the equation (5.9). Therefore, we have proved that the subarc \( \tilde{\tau}|_{\tilde{y}(\tilde{q}_2), s_1} \) is contained in \( A(\tilde{p}_1) \).

Since the minimal geodesic segment \( \tilde{\gamma} : [0, d(\tilde{p}_1, \tilde{\tau}(s_1))] \rightarrow \tilde{X} \) joining \( \tilde{p}_1 \) to \( \tilde{\tau}(s_1) \) lies in the closure of \( A(\tilde{p}_1) \),
\[
\angle \tilde{p}_1 \geq \angle (\tilde{\gamma}'(0), -\frac{d\tilde{\mu}_1^{(1)}}{dt}(d(\partial \tilde{X}, \tilde{p}_1)))
\]
holds. Hence, it is clear that the open triangle \( (\partial \tilde{X}, \tilde{p}, \tilde{q}) := (\partial \tilde{X}, \tilde{p}_1, \tilde{\tau}(s_1)) \) satisfies (5.3) and (5.4) in our lemma.

Hereafter, let \( (X, \partial X) \) be a complete connected Riemannian \( n \)-manifold \( X \) with smooth convex boundary \( \partial X \) whose radial curvature is bounded from below by that of \( (\tilde{X}, \partial \tilde{X}) \).

Let \( \lambda_0 \) denote the positive number guaranteed in Lemma 4.4. Choose any number \( \lambda \in (0, \lambda_0) \) and fix it. In the following, for the \( \lambda \), we also denote by \( (\tilde{X}_\lambda, \partial \tilde{X}_\lambda) \) a model surface with its metric \( \tilde{g}_\lambda = d\tilde{x}^2 + m_\lambda(\tilde{x})^2 d\tilde{y}^2 \) on \([0, \infty) \times \mathbb{R}\). Here the positive smooth function \( m_\lambda \) satisfies the differential equation
\[
m_\lambda'' + (G - \lambda)m_\lambda = 0
\]
with initial conditions \( m_\lambda(0) = 1 \) and \( m_\lambda'(0) = 0 \), where \( G \) denotes the radial curvature function of \( (\tilde{X}, \partial \tilde{X}) \). Then, the next lemma is clear from Lemmas 4.5 and 5.1:

**Lemma 5.4** Let \( p \) be a point in \( X \setminus (\partial X \cup \text{Foc}(\partial X)) \), and \( \delta_1(p) \) the number \( \delta_1 \) guaranteed in Lemma 4.3 to the point \( r := p \). Then, for any \( q \in X \) with
\[
d(p, q) < \delta_1(p),
\]
there exists an open triangle \( (\partial \tilde{X}_\lambda, \tilde{p}, \tilde{q}) \) in \( \tilde{X}_\lambda \) corresponding to the triangle \( (\partial X, p, q) \) in \( X \) such that
\[
d(\partial \tilde{X}_\lambda, \tilde{p}) = d(\partial X, p), \quad d(\tilde{p}, q) = d(p, q), \quad d(\partial \tilde{X}_\lambda, \tilde{q}) = d(\partial X, q)
\] (5.11)
and that
\[
\angle p \geq \angle \tilde{p}, \quad \angle q \geq \angle \tilde{q}.
\] (5.12)

By Lemmas 5.3 and 5.4, we have
Lemma 5.5 For every thin open triangle \((\partial X, p, q) = (\gamma, \mu_1, \mu_2)\) in \(X\) with
\[
\gamma \cap \text{Foc}(\partial X) = \emptyset,
\]
there exists an open triangle \((\partial \tilde{X}_\lambda, \tilde{p}, \tilde{q})\) in \(\tilde{X}_\lambda\) such that
\[
d(\partial \tilde{X}_\lambda, \tilde{p}) = d(\partial X, p), \quad d(\tilde{p}, \tilde{q}) = d(p, q), \quad d(\partial \tilde{X}_\lambda, \tilde{q}) = d(\partial X, q)
\]
and that
\[
\angle p \geq \angle \tilde{p}, \quad \angle q \geq \angle \tilde{q}.
\]

Proof. It is sufficient to prove that
\[
\max S = d(p, q),
\]
where \(S\) denotes the set of all \(s \in [0, d(p, q)]\) such that there exists an open triangle \((\partial \tilde{X}_\lambda, \tilde{p}, \tilde{r}(s))\) \(\subset \tilde{X}_\lambda\) corresponding to the triangle \((\partial X, p, \gamma(s))\) \(\subset X\) satisfying (5.13) and (5.14) for \(q = \gamma(s)\). From Lemma 5.4 it is clear that \(S\) is non-empty. Supposing that
\[
s_0 := \max S < d(p, q),
\]
we will get a contradiction. Since \(s_0 \in S\), there exists an open triangle \((\partial \tilde{X}_\lambda, \tilde{p}_1, \tilde{q}_1)\) \(\subset \tilde{X}_\lambda\) corresponding to \((\partial X, p, \gamma(s_0))\) \(\subset X\) such that (5.13) and (5.14) hold for \(q = \gamma(s_0)\), \(\tilde{p} = \tilde{p}_1\), and \(\tilde{q} = \tilde{q}_1\). In particular,
\[
\angle p \geq \angle \tilde{p}_1, \quad \angle (\partial X, \gamma(s_0), p) \geq \angle \tilde{q}_1,
\]
where \(\angle (\partial X, \gamma(s_0), p)\) denotes the angle between two sides joining \(\gamma(s_0)\) to \(\partial X\) and \(p\) forming the triangle \((\partial X, p, \gamma(s_0))\). Let \(\delta(\gamma(s_0))\) denote the number \(\delta_1\) guaranteed in Lemma 4.5 to the point \(r := \gamma(s_0)\). Choose a sufficiently small number
\[
0 < \varepsilon_1 < \min \{\delta(\gamma(s_0)), d(p, q) - s_0\},
\]
and fix it. By Lemma 5.4 we have an open triangle \((\partial \tilde{X}_\lambda, \tilde{p}_2, \tilde{q}_2)\) \(\subset \tilde{X}_\lambda\) corresponding to \((\partial X, \gamma(s_0), \gamma(s_0 + \varepsilon_1))\) \(\subset X\) such that (5.13) and (5.14) hold for \(p = \gamma(s_0), q = \gamma(s_0 + \varepsilon_1), \tilde{p} = \tilde{p}_2,\) and \(\tilde{q} = \tilde{q}_2\). In particular,
\[
\angle (\partial X, \gamma(s_0), \gamma(s_0 + \varepsilon_1)) \geq \angle \tilde{p}_2, \quad \angle \gamma(s_0 + \varepsilon_1) \geq \angle \tilde{q}_2.
\]
Since
\[
\angle (\partial X, \gamma(s_0), p) + \angle (\partial X, \gamma(s_0), \gamma(s_0 + \varepsilon_1)) = \pi,
\]
we get, by (5.16) and (5.17),
\[
\angle \tilde{q}_1 + \angle \tilde{p}_2 \leq \pi.
\]
Since \((\partial X, p, q)\) is a thin open triangle,
\[
\min\{\text{inj}(\tilde{p}_1), \text{inj}(\tilde{q}_2)\} \geq L(\gamma) \geq d(\tilde{p}_1, \tilde{q}_1) + d(\tilde{p}_2, \tilde{q}_2)
\]

holds. Thus, if we apply Lemma 5.3 twice for the pair \((\partial \tilde{X}_\lambda, \tilde{p}_1, \tilde{q}_1)\) and \((\partial \tilde{X}_\lambda, \tilde{p}_2, \tilde{q}_2)\), we get two open triangles \((\partial \tilde{X}_\lambda, \tilde{p}, \tilde{\gamma}(s_0 + \varepsilon_1))\) and \((\partial \tilde{X}_\lambda, \tilde{p}, \tilde{\gamma}(s_0 + \varepsilon_1))\) in \(\tilde{X}_\lambda\) such that
\[
\angle \tilde{p}_1 \geq \angle \tilde{p}, \quad \angle \tilde{q}_2 \geq \angle \tilde{\gamma}(s_0 + \varepsilon_1).
\] (5.18)

Since both triangles \((\partial \tilde{X}_\lambda, \tilde{p}, \tilde{\gamma}(s_0 + \varepsilon_1))\) and \((\partial \tilde{X}_\lambda, \tilde{p}, \tilde{\gamma}(s_0 + \varepsilon_1))\) are isometric, we obtain
\[
\angle \tilde{p} = \angle \tilde{\gamma}(s_0 + \varepsilon_1).
\] (5.19)

Hence, by \((5.1.6), (5.1.7), (5.1.8),\) and \((5.1.9),\) both open triangles \((\partial X, p, \gamma(s_0 + \varepsilon_1))\) and \((\partial \tilde{X}_\lambda, \tilde{p}, \tilde{\gamma}(s_0 + \varepsilon_1))\) satisfy \((5.1.3)\) for \(q = \gamma(s_0 + \varepsilon_1)\) and \(\tilde{q} = \tilde{\gamma}(s_0 + \varepsilon_1)\) and
\[
\angle p \geq \angle \tilde{p}, \quad \angle (s_0 + \varepsilon_1) \geq \angle \tilde{\gamma}(s_0 + \varepsilon_1).
\]
This implies that
\[s_0 + \varepsilon_1 \in S.\]
This contradicts the fact that \(s_0\) is the maximum of \(S\). Hence \((5.1.5)\) holds.

\[\square\]

**Lemma 5.6** For every thin open triangle \((\partial X, p, q) = (\gamma, \mu_1, \mu_2)\) in \(X\) with \(p \notin \text{Foc}(\partial X)\),
\[
d(\partial \tilde{X}_\lambda, \tilde{p}) = d(\partial X, p), \quad d(\tilde{p}, \tilde{q}) = d(p, q), \quad d(\tilde{X}_\lambda, \tilde{q}) = d(\partial X, q)
\] (5.21)
and that
\[
\angle p \geq \angle \tilde{p}, \quad \angle q \geq \angle \tilde{q}.
\] (5.22)

**Proof.** Let \((\partial X, p, q) = (\gamma, \mu_1, \mu_2)\) be a thin open triangle in \(X\) satisfying \((5.20)\), and we fix it. Since \(p\) is not a focal point of \(\partial X\), and \(q\) is not a cut point of \(p\), it follows from Lemma 5.5 that there exists a sequence \(\{\gamma_i : [0, \ell_i] \rightarrow X\}_{i \in \mathbb{N}}\) of minimal geodesic segments \(\gamma_i\) emanating from \(p = \gamma_i(0)\) convergent to the opposite side \(\gamma\) of \((\partial X, p, q)\) to \(\partial X\) such that
\[
\gamma_i([0, \ell_i]) \cap \text{FC}(\partial X) = \emptyset,
\]
and that
\[
\lim_{i \to \infty} \frac{1}{\| \exp_{\gamma}^{-1}(\gamma_i(\ell_i)) \|} \exp_{\gamma}^{-1}(\gamma_i(\ell_i)) = -\gamma'\ell
\]
where \(\ell := d(p, q)\). Then, we may find a sufficiently large \(i_0 \in \mathbb{N}\) such that \((\partial X, p, \gamma_i(\ell_i)) = (\gamma_i, \mu_1, \eta_i)\) is a thin open triangle in \(X\) for each \(i \geq i_0\). Here each \(\eta_i\) is a \(\partial X\)-segment to \(\gamma_i(\ell_i)\). Choose any \(i \geq i_0\) and fix it. By Lemma 5.3, there exists an open triangle \((\partial \tilde{X}_\lambda, \tilde{p}, \tilde{\gamma}_i(\ell_i)) = (\tilde{\gamma}_i, \tilde{\mu}_1, \tilde{\eta}_i) \subset \tilde{X}_\lambda\) corresponding to \((\partial X, p, \gamma_i(\ell_i))\) such that \((5.1.4)\) hold for \(q = \gamma_i(\ell_i)\), and
\[
\angle (-\mu'_i(d(\partial X, p)), \gamma'_i(0)) \geq \angle (-\tilde{\mu}'_i(d(\partial X, p)), \tilde{\gamma}'(0)),
\] (5.23)
\[
\angle (\eta'_i(d(\partial X, \gamma_i(\ell_i))), \gamma'_i(\ell_i)) \geq \angle (\tilde{\eta}'_i(d(\partial X, \gamma_i(\ell_i))), \tilde{\gamma}'(\ell_i)).
\] (5.24)
Since \( \lim_{i \to \infty} \gamma_i'(0) = \gamma'(0) \),
\[
\angle p = \lim_{i \to \infty} \angle(-\mu'_1(d(\partial X, p)), \gamma_i'(0)). \tag{5.25}
\]
On the other hand,
\[
\angle q \geq \lim \sup_{i \to \infty} \angle(\eta_i'(d(\partial X, \gamma_i(\ell_i))), \gamma_i'(\ell_i)) \tag{5.26}
\] holds by [IT3 Lemma 2.1]. Then, from (5.23), (5.24), (5.25), (5.26), it follows that
\[
\angle p \geq \lim_{i \to \infty} \angle(-\mu_1'(d(\partial X, p)), \tilde{\gamma}_i'(0)),
\]
and that
\[
\angle q \geq \lim_{i \to \infty} \angle(\eta_i'(d(\partial X, \gamma_i(\ell_i))), \tilde{\gamma}_i'(\ell_i)).
\]
By taking the limit of the sequence \((\partial \tilde{X}_\lambda, \tilde{p}, \tilde{\gamma}_i(\ell_i)) = (\tilde{\gamma}_i, \tilde{\mu}_1, \tilde{\mu}_2) \subset \tilde{X}_\lambda\) corresponding to \((\partial X, p, q) \subset X\) such that (5.21) and (5.22) hold.

**Lemma 5.7** For every thin open triangle \((\partial X, p, q) = (\gamma, \mu_1, \mu_2)\) in \(X\), there exists an open triangle \((\partial \tilde{X}_\lambda, \tilde{p}, \tilde{q})\) in \(\tilde{X}_\lambda\) such that
\[
d(\partial \tilde{X}_\lambda, \tilde{p}) = d(\partial X, p), \quad d(\tilde{p}, \tilde{q}) = d(p, q), \quad d(\partial \tilde{X}_\lambda, \tilde{q}) = d(\partial X, q) \tag{5.27}
\]
and that
\[
\angle p \geq \angle \tilde{p}, \quad \angle q \geq \angle \tilde{q}. \tag{5.28}
\]

**Proof.** Take any sufficiently small \(\varepsilon > 0\) such that
\[
q \notin \text{Cut}(p_\varepsilon),
\]
where we set \(p_\varepsilon := \mu_1(d(\partial X, p) - \varepsilon)\). Let \(\mu_\varepsilon\) denote the restriction of \(\mu_1\), i.e.,
\[
\mu_\varepsilon(t) := \mu_1(t)
\]
on \([0, d(\partial X, p) - \varepsilon]\). Without loss of generality, we may assume that the open triangle \((\partial X, p_\varepsilon, q) = (\gamma_\varepsilon, \mu_\varepsilon, \mu_2)\) is thin. Here \(\gamma_\varepsilon : [0, \ell_{\varepsilon}] \to X\) denotes the minimal geodesic segment emanating from \(p_\varepsilon = \gamma_\varepsilon(0)\) to \(q = \gamma_\varepsilon(\ell_{\varepsilon})\). Since \(p_\varepsilon \notin \text{Foc}(\partial X)\), it follows from Lemma 5.6 that there exists an open triangle \((\partial \tilde{X}_\lambda, \tilde{p}_\varepsilon, \tilde{q}) = (\tilde{\gamma}_\varepsilon, \tilde{\mu}_\varepsilon, \tilde{\mu}_2) \subset \tilde{X}_\lambda\) corresponding to \((\partial X, p_\varepsilon, q)\) such that (5.21) holds for \(p = p_\varepsilon\), and that
\[
\angle(-\mu_1'(d(\partial X, p_\varepsilon)), \gamma'_\varepsilon(0)) \geq \angle(-\mu'_2(d(\partial X, p_\varepsilon)), \gamma'_\varepsilon(\ell_{\varepsilon})),
\]
\[
\angle(\mu_1'(d(\partial X, q)), \gamma'_\varepsilon(0)) \geq \angle(\mu_2'(d(\partial X, q)), \gamma'_\varepsilon(\ell_{\varepsilon})).
\]
Since \(\lim_{\varepsilon \to 0} \gamma_\varepsilon = \gamma\), we have
\[
\angle p = \lim_{\varepsilon \to 0} \angle(-\mu'_1(d(\partial X, p_\varepsilon)), \gamma'_\varepsilon(0))
\]
and
\[
\angle q = \lim_{\varepsilon \to 0} \angle(\mu'_2(d(\partial X, q)), \gamma'_\varepsilon(\ell_{\varepsilon})).
\]
If \(\varepsilon\) goes to 0, we therefore get an open triangle \((\partial \tilde{X}_\lambda, \tilde{p}, \tilde{q}) = (\tilde{\gamma}, \tilde{\mu}_1, \tilde{\mu}_2) \subset \tilde{X}_\lambda\) corresponding to \((\partial X, p, q) \subset X\) such that (5.27) and (5.28) hold. \(\square\)
By taking the limit of $\lambda$, it follows from Lemma 5.7 that we have the lemma on thin open triangles:

**Lemma 5.8 (Lemma on Thin Open Triangles)** For every thin open triangle $(\partial X, p, q)$ in $X$, there exists an open triangle $(\partial \tilde{X}, \tilde{p}, \tilde{q})$ in $\tilde{X}$ such that

$$d(\partial \tilde{X}, \tilde{p}) = d(\partial X, p), \quad d(\tilde{p}, \tilde{q}) = d(p, q), \quad d(\partial \tilde{X}, \tilde{q}) = d(\partial X, q)$$  \hspace{1cm} (5.29)

and that

$$\angle p \geq \angle \tilde{p}, \quad \angle q \geq \angle \tilde{q}. \hspace{1cm} (5.30)$$

**Remark 5.9** From Section 4 and this section, it has been clarified that we can prove Lemma on Thin Open Triangles by the second variation formula and the Warner comparison theorem. But, we cannot do by the first variation formula and the Berger comparison theorem.

### 6 The Opposite Side to $\partial X$ of an Open Triangle

In Definition 2.11, the opposite side to $\partial X$ of a thin open triangle is defined not to meet the boundary. In this section, we will show that the opposite side to $\partial X$ of any open triangle on any complete connected Riemannian manifold $X$ with smooth boundary $\partial X$ does not meet $\partial X$, if $\partial X$ is convex.

**Lemma 6.1** Let $(X, \partial X)$ be a complete connected Riemannian $n$-dimensional manifold $X$ with smooth boundary $\partial X$ whose radial curvature is bounded from below by that of a model surface $(\tilde{X}, \partial \tilde{X})$. If $\partial X$ is convex, then, for any open triangle $(\partial X, p, q) = (\gamma, \mu_1, \mu_2)$ in $X$, $\gamma$ does not meet $\partial X$.

**Proof.** Suppose that $\gamma$ intersects $\partial X$ at $\gamma(s_0)$ for some $s_0 \in (0, d(p, q))$. Without loss of generality, we may assume that

$$\gamma((0, s_0)) \cap \partial X = \emptyset.$$ 

Since $\gamma$ intersects $\partial X$ at $\gamma(s_0)$, $\gamma$ is tangent to $\partial X$ at $\gamma(s_0)$.

It is well-known that each point of $\tilde{X}$ admits a normal convex neighborhood. Hence, there exists a constant $C_0 > 0$ such that

$$\text{inj}(\tilde{q}_s) > C_0$$

for all $s \in [0, s_0]$, where $\tilde{q}_s$ denotes a point in $\tilde{X}$ satisfying

$$d(\partial \tilde{X}, \tilde{q}_s) = d(\partial X, \gamma(s)).$$

By this property, we may choose a number $s_1 \in (0, s_0)$ in such a way that

$$L(\gamma|_{[s_1, s_0]}) = s_0 - s_1 < \text{inj}(\tilde{q}_s)$$
for all \( s \in [s_1, s_0] \). Therefore, for each \( s_2 \in [s_1, s_0] \), any open triangle \((\partial X, \gamma(s_2), \gamma(s_3))\) in \( X \) is thin, if \( s_3 \in [s_1, s_0] \setminus \{s_2\} \) is sufficiently close to \( s_2 \).

Let \( S \) denote the set of all \( s \in (s_1, s_0) \) such that there exists an open triangle \((\partial \tilde{X}, \tilde{\gamma}(s_1), \tilde{\gamma}(s)) \subset \tilde{X} \) corresponding to the triangle \((\partial X, \gamma(s_1), \gamma(s)) \subset X \) satisfying
\[
d(\partial \tilde{X}, \tilde{\gamma}(s_1)) = d(\partial X, \gamma(s_1)), \ d(\tilde{\gamma}(s_1), \tilde{\gamma}(s)) = s - s_1, \ d(\partial \tilde{X}, \tilde{\gamma}(s)) = d(\partial X, \gamma(s)),
\]
and that
\[
\angle \gamma(s_1) \geq \angle \tilde{\gamma}(s_1), \quad \angle \gamma(s) \geq \angle \tilde{\gamma}(s).
\]
By Lemma 5.8 the set \( S \) is non-empty. By the similar argument in the proof of Lemma 5.5 we see that
\[
\sup S = s_0.
\]
Hence, there exists a decreasing sequence \( \{\varepsilon_i\}_{i \in \mathbb{N}} \) convergent to 0 such that
\[
s_0 - \varepsilon_i \in S
\]
for all \( i \in \mathbb{N} \). For each \( i \in \mathbb{N} \), there exists an open triangle \((\partial \tilde{X}, \tilde{\gamma}(s_1), \tilde{\gamma}(s_0 - \varepsilon_i)) \subset \tilde{X} \) corresponding to the triangle \((\partial X, \gamma(s_1), \gamma(s_0 - \varepsilon_i)) \subset X \) such that (6.1) and (6.2) hold for \( \gamma(s) = \gamma(s_0 - \varepsilon_i) \). Since \( \gamma \) is tangent to \( \partial X \) at \( \gamma(s_0) \),
\[
\lim_{s \to s_0} \angle \gamma(s) = \frac{\pi}{2}.
\]
Thus, by (6.2) for \( \gamma(s) = \gamma(s_0 - \varepsilon_i) \), we get
\[
\limsup_{i \to \infty} \angle \tilde{\gamma}(s_0 - \varepsilon_i) \leq \frac{\pi}{2}.
\]
If
\[
\liminf_{i \to \infty} \angle \tilde{\gamma}(s_0 - \varepsilon_i) < \frac{\pi}{2}
\]
holds, the opposite side to \( \partial \tilde{X} \) of \((\partial \tilde{X}, \tilde{\gamma}(s_1), \tilde{\gamma}(s_0 - \varepsilon_i)) \) meets \( \partial \tilde{X} \) for some \( s_2 \in (s_1, s_0) \) sufficiently close to \( s_0 \), which contradicts the fact that \( \partial \tilde{X} \) is totally geodesic. Hence, \[
\lim_{i \to \infty} \angle \tilde{\gamma}(s_0 - \varepsilon_i) = \frac{\pi}{2}
\]
holds. Thus, the opposite side to \( \partial \tilde{X} \) of the limit open triangle \((\partial \tilde{X}, \tilde{\gamma}(s_1), \tilde{\gamma}(s_0 - \varepsilon_i)) \) as \( i \to \infty \) is tangent to \( \partial \tilde{X} \). This is also a contradiction, since \( \partial \tilde{X} \) is totally geodesic. Therefore, \( \gamma \) does not meet \( \partial X \).

By the same argument in the proof of [KT2], Lemma 5.1, we have that

**Lemma 6.2** For any complete connected Riemannian manifold \( X \) with smooth boundary \( \partial X \), there exists a locally Lipschitz function \( G(t) \) on \([0, \infty)\) such that the radial curvature of \( X \) is bounded from below by that of the model surface with radial curvature function \( G(t) \).

It is clear from Lemmas 6.1 and 6.2 that

**Proposition 6.3** Let \( X \) be a complete connected Riemannian manifold \( X \) with smooth boundary \( \partial X \). If \( \partial X \) is convex, then the opposite side to \( \partial X \) of any open triangle on \( X \) does not meet \( \partial X \).
7 Alexandrov’s Convexity

Our purpose of this section is to establish the Alexandrov convexity (Lemma 7.3). In the proof of Lemma 7.3, we may understand that it is a very important property that the opposite side of an open triangle to the boundary in a model surface is unique (i.e., we can not prove the equation (7.17) in the proof of Lemma 7.3 without this property).

In order to prove Lemma 7.3, we have to treat a non-differentiable Lipschitz function. It follows from Dini’s theorem ([D]) that, for any Lipschitz function \( f \) on \([a,b]\), \( f \) is differentiable almost everywhere, and

\[
\int_a^b f'(t)dt = f(b) - f(a)
\]

holds. Note that the Cantor-Lebesgue function \( g \) on \([0,1]\) is differentiable almost everywhere, but

\[
0 = \int_0^1 g'(t)dt < g(1) - g(0) = 1
\]

Cohn-Vossen applied in [CV1] and [CV2] these properties above to global differential geometry.

Throughout this section, let \((\tilde{X}, \partial\tilde{X})\) denote a model surface with its metric \((1.1)\).

It follows from Lemma 5.2 that

\[
\lim_{s \to s_0} \frac{d(\tilde{\mu}(a), \tilde{\tau}_c(s)) - d(\tilde{\mu}(a), \tilde{\tau}_c(s_0))}{s - s_0} \geq 0
\]

(7.1)

for each \( s_0 > \hat{y}(\tilde{\mu}(0)) \). The following two lemmas are useful for proving that the function \( D \) defined in Lemma 7.3 is locally Lipschitz. In the first lemma, we will prove that the left-hand term of the equation \((7.1)\) is strictly positive:

\textbf{Lemma 7.1} For each \( \partial\tilde{X} \)-ray \( \tilde{\mu} : [0,\infty) \to \tilde{X} \) and each number \( a_0, c_0 > 0, s_0 > \hat{y}(\tilde{\mu}(0)) \), there exist numbers \( \varepsilon_1 > 0 \) and \( \delta > 0 \) such that

\[
|d(\tilde{\mu}(a), \tilde{\tau}_c(s)) - d(\tilde{\mu}(a), \tilde{\tau}_c(s_0))| \geq |s - s_0| \cdot m(c) \cdot \sin \varepsilon_1
\]

(7.2)

holds for all \( a \in (a_0 - \delta, a_0 + \delta), c \in (c_0 - \delta, c_0 + \delta), \) and \( s \in (s_0 - \delta, s_0 + \delta) \).

\textbf{Proof.} We choose a positive number \( \delta \) less than

\[
\min\{a_0, c_0, s_0 - \hat{y}(\tilde{\mu}(0))\}
\]

and fix it. Let \( a, c, s \) be any numbers in \( (a_0 - \delta, a_0 + \delta), (c_0 - \delta, c_0 + \delta), \) and \( (s_0 - \delta, s_0 + \delta) \), respectively. Since no minimal geodesic segment joining \( \tilde{\mu}(a) \) to \( \tilde{\tau}_c(s) \) is perpendicular to \( \tilde{\tau}_c \), there exists a positive number \( \varepsilon_1 \in (0, \pi/2) \) such that

\[
\Phi(\hat{\gamma}, s) := \angle(\tilde{\tau}'_c(s), \tilde{\tau}'(d(\tilde{\mu}(a), \tilde{\tau}_c(s)))) \leq \frac{\pi}{2} - \varepsilon_1
\]

(7.3)
holds for all \( a \in (a_0 - \delta, a_0 + \delta) \) \( c \in (c_0 - \delta, c_0 + \delta) \), \( s \in (s_0 - \delta, s_0 + \delta) \), and minimal geodesic segments \( \tilde{\gamma} \) joining \( \tilde{\mu}(a) \) to \( \tilde{\tau}_c(s) \). By \([133] \) Lemma 2.1 and \([73] \) that, 
\[
\liminf_{s \downarrow s_0} \frac{d(\tilde{\mu}(a), \tilde{\tau}_c(s)) - d(\tilde{\mu}(a), \tilde{\tau}_c(s_0))}{d(\tilde{\tau}_c(s), \tilde{\tau}_c(s_0))} \geq \sin \varepsilon_1.
\]
This equation implies \([72] \) (See the proof of \([KT2, Lemma 4.2] \) about the detail of this proof).

Notice that, for given a triple \( (a, b, c) \) of positive numbers \( a, b, c \), there exists an open triangle \( (\partial \tilde{X}, \tilde{p}, \tilde{q}) = (\tilde{\gamma}, \tilde{\mu}_1, \tilde{\mu}_2) \) in \( \tilde{X} \) with \( \angle \tilde{p}, \angle \tilde{q} \in (0, \pi) \) satisfying \( a = d(\partial \tilde{X}, \tilde{p}), b = d(\tilde{p}, \tilde{q}), \) and \( c = d(\partial \tilde{X}, \tilde{q}) \) if and only if 
\[(a, b, c) \in T := \{(a, b, c) \in \mathbb{R}^3 \mid a, c > 0, |a - c| < b\}.
\]
By Lemma \(5.2 \) the existence of such a triangle \( (\partial \tilde{X}, \tilde{p}, \tilde{q}) = (\tilde{\gamma}, \tilde{\mu}_1, \tilde{\mu}_2) \) is unique up to an isometry except for the opposite side \( \tilde{\gamma} \) to \( \partial \tilde{X} \). Hence, 
\[
\Theta(a, b, c) := |\tilde{g}(\tilde{\mu}_1(0)) - \tilde{g}(\tilde{\mu}_2(0))|
\]
is a well-defined function on the set \( T \).

**Lemma 7.2** The function \( \Theta(a, b, c) \) is locally Lipschitz.

**Proof.** Choose any point \( (a_0, b_0, c_0) \in T \), and fix it. Let \( \tilde{\mu}_1 : [0, \infty) \rightarrow \tilde{X} \) be the \( \partial \tilde{X} \)-ray with \( \tilde{g}(\tilde{\mu}_1(0)) = 0 \). Moreover, we choose the \( \partial \tilde{X} \)-ray \( \tilde{\mu}_2 : [0, \infty) \rightarrow \tilde{X} \) in such a way that 
\[
d(\tilde{\mu}_1(a_0), \tilde{\mu}_2(c_0)) = b_0
\]
and \( \tilde{g}(\tilde{\mu}_2(0)) > 0 \). By setting \( \tilde{p}_0 := \tilde{\mu}_1(a_0) \) and \( \tilde{q}_0 := \tilde{\mu}_2(c_0) \), we hence get an open triangle \( (\partial \tilde{X}, \tilde{p}_0, \tilde{q}_0) \subset \tilde{X} \) with \( \angle \tilde{p}_0, \angle \tilde{q}_0 \in (0, \pi) \) satisfying 
\[
a_0 = d(\partial \tilde{X}, \tilde{p}_0), \quad b_0 = d(\tilde{p}_0, \tilde{q}_0), \quad c_0 = d(\partial \tilde{X}, \tilde{q}_0).
\]
First we will prove that 
\[
|\Theta(a_0 + \Delta a, b_0, c_0) - \Theta(a_0, b_0, c_0)| \leq \frac{1}{m(c_0) \sin \varepsilon_1} |\Delta a|
\]
for all \( \Delta a \in \mathbb{R} \) with \( |\Delta a| < \delta \). Here the numbers \( \varepsilon_1 \) and \( \delta \) are the constants guaranteed to \( a_0, c_0 \), and \( s_0 := \tilde{g}(\tilde{\mu}_2(0)) > 0 \) in Lemma \(7.1 \). Let \( \tilde{\tau}_{c_0} : \mathbb{R} \rightarrow \tilde{X} \) be the arc \( \tilde{x} = c_0 \). Then, we may find a point \( \tilde{q}_{\Delta a} \) on \( \tilde{\tau}_{c_0} \) satisfying 
\[
b_0 = d(\tilde{p}_{\Delta a}, \tilde{q}_{\Delta a}),
\]
where \( \tilde{p}_{\Delta a} := \tilde{\mu}_1(a_0 + \Delta a) \). Thus, we also get an open triangle \( (\partial \tilde{X}, \tilde{p}_{\Delta a}, \tilde{q}_{\Delta a}) \subset \tilde{X} \) with \( \angle \tilde{p}_{\Delta a}, \angle \tilde{q}_{\Delta a} \in (0, \pi) \) satisfying 
\[
a_0 + \Delta a = d(\partial \tilde{X}, \tilde{p}_{\Delta a}), \quad b_0 = d(\tilde{p}_{\Delta a}, \tilde{q}_{\Delta a}), \quad c_0 = d(\partial \tilde{X}, \tilde{q}_{\Delta a}).
\]

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Let \( \tilde{\mu}_{\Delta a} \) denote the side of \((\partial \tilde{X}, \tilde{\nu}_{\Delta a}, \tilde{q}_{\Delta a})\) joining \( \partial \tilde{X} \) to \( \tilde{q}_{\Delta a} \). By definition,

\[
\Theta(a_0 + \Delta a, b_0, c_0) = \tilde{y}(\tilde{\mu}_{\Delta a}(0)),
\]

and

\[
\Theta(a_0, b_0, c_0) = \tilde{y}(\tilde{\mu}_2(0)) = s_0.
\]

Here we may assume that \( \tilde{y}(\tilde{\mu}_{\Delta a}(0)) > 0 \). It is clear from (7.5) and (7.6) that

\[
|\Delta a \Theta| = |\tilde{y}(\tilde{\mu}_{\Delta a}(0)) - s_0|,
\]

where we set

\[
\Delta a \Theta := \Theta(a_0 + \Delta a, b_0, c_0) - \Theta(a_0, b_0, c_0).
\]

Thus, by (7.7), the length \( |\Delta s| \) of the subarc of \( \tilde{r}_{\Delta a} \) with end points \( \tilde{q}_{\Delta a} \) and \( \tilde{q}_0 \) is equal to

\[
|\Delta s| = m(c_0) \cdot |\tilde{y}(\tilde{\mu}_{\Delta a}(0)) - s_0| = m(c_0) \cdot |\Delta a \Theta|.
\]

It follows from Lemma 7.1 and (7.8) that

\[
|d(\tilde{p}_0, \tilde{q}_0) - d(\tilde{p}_0, \tilde{q}_{\Delta a})| \geq |s_0 - \tilde{y}(\tilde{\mu}_{\Delta a}(0))| \cdot m(c_0) \cdot \sin \varepsilon_1 = |\Delta s| \sin \varepsilon_1.
\]

Since

\[
b_0 = d(\tilde{p}_0, \tilde{q}_0) = d(\tilde{p}_{\Delta a}, \tilde{q}_{\Delta a}),
\]

we get, by (7.9),

\[
|d(\tilde{p}_{\Delta a}, \tilde{q}_{\Delta a}) - d(\tilde{p}_0, \tilde{q}_{\Delta a})| \geq |\Delta s| \sin \varepsilon_1.
\]

By the triangle inequality,

\[
|\Delta a| = d(\tilde{p}_0, \tilde{q}_{\Delta a}) \geq |d(\tilde{p}_{\Delta a}, \tilde{q}_{\Delta a}) - d(\tilde{p}_0, \tilde{q}_{\Delta a})|.
\]

By combining the equations (7.8), (7.10), and (7.11), we obtain (7.4). Since \( \Theta(a, b, c) = \Theta(c, b, a) \) for all \((a, b, c) \in T \), it is clear that

\[
|\Theta(a_0, b_0, c_0 + \Delta c) - \Theta(a_0, b_0, c_0)| \leq \frac{1}{m(c_0) \sin \varepsilon_1} |\Delta c|
\]

for all \( \Delta c \in \mathbb{R} \) with \( |\Delta c| < \delta \). We omit the proof of the following equation, since the proof is similar to that of (7.4):

\[
|\Theta(a_0, b_0 + \Delta b, c_0) - \Theta(a_0, b_0, c_0)| \leq \frac{1}{m(c_0) \sin \varepsilon_1} |\Delta b|
\]

for all \( \Delta b \in \mathbb{R} \) with \( |\Delta b| < \delta \). Therefore, the function \( \Theta(a, b, c) \) is locally Lipschitz at \((a_0, b_0, c_0) \in T \) by (7.4), (7.12), and (7.13). \(\square\)

**Lemma 7.3 (Alexandrov’s Convexity for Open Triangles)**

Let \((X, \partial X)\) be a complete connected Riemannian \(n\)-dimensional manifold \(X\) with smooth convex boundary \(\partial X\) whose radial curvature is bounded from below by that of \((\tilde{X}, \partial \tilde{X})\), and let \((\partial X, p, q) = (\gamma, \mu_1, \mu_2)\) be a non-degenerate open triangle in \(X\), i.e.,

\[\angle p, \angle q \in (0, \pi).\]
Assume that, for each open triangle 
\[(\partial X, \mu_1(at), \mu_2(ct)) = (\gamma_t, \mu_1|_{[0,at]}, \mu_2|_{[0,ct]}), \quad t \in (0,1],\]
where \(a = d(\partial X, p)\) and \(c = d(\partial X, q)\), there exists a unique open triangle
\[(\partial \tilde{X}, \tilde{\mu}_1^{(t)}(at), \tilde{\mu}_2^{(t)}(ct)) = (\tilde{\gamma}_t, \tilde{\mu}_1|_{[0,at]}, \tilde{\mu}_2|_{[0,ct]}),\]
up to an isometry in \(\tilde{X}\) such that
\[d(\partial \tilde{X}, \tilde{\mu}_1^{(t)}(at)) = d(\partial X, \mu_1(at)), \quad d(\partial \tilde{X}, \tilde{\mu}_2^{(t)}(ct)) = d(\partial X, \mu_2(ct)),\] (7.14)
\[d(\tilde{\mu}_1^{(t)}(at), \tilde{\mu}_2^{(t)}(ct)) = d(\mu_1(at), \mu_2(ct)),\] (7.15)
and that
\[\angle \mu_1(at) \geq \angle \tilde{\mu}_1^{(t)}(at), \quad \angle \mu_2(ct) \geq \angle \tilde{\mu}_2^{(t)}(ct).\] (7.16)

Then, the function
\[D(t) := |\tilde{y}(\tilde{\mu}_1^{(t)}(0)) - \tilde{y}(\tilde{\mu}_2^{(t)}(0))|\]
is locally Lipschitz on \((0,1)\), and non-increasing on \([0,1]\).

Proof. We will state the outline of the proof, since the proof is very similar to [KT2, Lemma 4.4]. If we define a Lipschitz function \(\varphi\) on \([0,1]\) by
\[\varphi(t) := d(\mu_1(at), \mu_2(ct)),\]
then, the function \(D(t)\) is equal to \(\Theta(at, \varphi(t), ct)\). Hence \(D(t)\) is locally Lipschitz on \((0,1]\) by Lemma 7.2. From Dini’s theorem [D] (cf. [H, Section 2.3], [WZ, Theorem 7.29]), the function \(D(t)\) is differentiable for almost all \(t \in (0,1)\). Let \(t_0 \in (0,1)\) be any number where \(D(t)\) is differentiable. Then, by the assumption, we have an open triangle
\[(\partial \tilde{X}, \tilde{\mu}_1^{(t_0)}(at_0), \tilde{\mu}_2^{(t_0)}(ct_0)) = (\tilde{\gamma}_{t_0}, \tilde{\mu}_1|_{[0,at_0]}, \tilde{\mu}_2|_{[0,ct_0]}),\]
corresponding to the triangle
\[(\partial X, \mu_1(at_0), \mu_2(ct_0)) = (\gamma_{t_0}, \mu_1|_{[0,at_0]}, \mu_2|_{[0,ct_0]}),\]
such that \((7.14), (7.15), (7.16)\) hold. Since \((\partial X, \mu_1(at_0), \mu_2(ct_0))\) is non-degenerate, we may assume, without loss of generality, that
\[0 = \tilde{y}′(\tilde{\mu}_1^{(t_0)}(at_0)) < \tilde{y}′(\tilde{\mu}_2^{(t_0)}(ct_0)).\]
Let \(\tilde{\mu}, \tilde{\eta} : [0, \infty) \rightarrow \tilde{X}\) be \(\partial \tilde{X}\)-rays passing through \(\tilde{\mu}_1^{(t_0)}(at_0) = \tilde{\mu}(at_0), \tilde{\mu}_2^{(t_0)}(ct_0) = \tilde{\eta}(ct_0)\), respectively. We define a function
\[\tilde{\psi}(t) := d(\tilde{\mu}(at), \tilde{\eta}(ct)).\]
Since \(\tilde{\gamma}_{t_0}\) is unique, we may prove that the function \(\tilde{\psi}(t)\) is differentiable at \(t = t_0\), and that
\[\tilde{\psi}'(t_0) = \cos(\angle \tilde{\mu}_1^{(t_0)}(at_0)) + \cos(\angle \tilde{\mu}_2^{(t_0)}(ct_0)).\] (7.17)
Indeed, let $\tilde{z}_0$ and $\tilde{z}_i$ denote the midpoint of $\tilde{\gamma}_{t_0}$ and $\tilde{\mu}(at)\tilde{\eta}(ct)$, respectively. Here, $\tilde{\mu}(at)\tilde{\eta}(ct)$ denotes a minimal geodesic segment joining $\tilde{\mu}(at)$ to $\tilde{\eta}(ct)$. Since there exists a unique minimal geodesic segment joining $\tilde{\mu}_1^{(t_0)}(at_0) = \tilde{\mu}(at_0)$ to $\tilde{\mu}_2^{(t_0)}(ct_0) = \tilde{\eta}(ct_0)$,

$$\lim_{t\to t_0} \tilde{z}_t = \tilde{z}_0$$

(7.18)

holds. By the triangle inequality, we have

$$\tilde{\psi}(t) - \tilde{\psi}(t_0) \leq d(\tilde{\mu}(at), \tilde{z}_0) + d(\tilde{\eta}(ct), \tilde{z}_0) - d(\tilde{\mu}(at_0), \tilde{z}_0) - d(\tilde{\eta}(ct_0), \tilde{z}_0)$$

and

$$\tilde{\psi}(t) - \tilde{\psi}(t_0) \geq d(\tilde{\mu}(at), \tilde{z}_t) + d(\tilde{\eta}(ct), \tilde{z}_t) - d(\tilde{\mu}(at_0), \tilde{z}_t) - d(\tilde{\eta}(ct_0), \tilde{z}_t)$$

Hence,

$$\limsup_{t\downarrow t_0} \frac{\tilde{\psi}(t) - \tilde{\psi}(t_0)}{t - t_0} \leq \limsup_{t\downarrow t_0} \frac{d(\tilde{\mu}(at), \tilde{z}_0) - d(\tilde{\mu}(at_0), \tilde{z}_0)}{t - t_0} + \limsup_{t\downarrow t_0} \frac{d(\tilde{\eta}(ct), \tilde{z}_0) - d(\tilde{\eta}(ct_0), \tilde{z}_0)}{t - t_0}$$

(7.19)

and

$$\liminf_{t\downarrow t_0} \frac{\tilde{\psi}(t) - \tilde{\psi}(t_0)}{t - t_0} \geq \liminf_{t\downarrow t_0} \frac{d(\tilde{\mu}(at), \tilde{z}_t) - d(\tilde{\mu}(at_0), \tilde{z}_t)}{t - t_0} + \liminf_{t\downarrow t_0} \frac{d(\tilde{\eta}(ct), \tilde{z}_t) - d(\tilde{\eta}(ct_0), \tilde{z}_t)}{t - t_0}$$

(7.20)

hold. From the first variation formula, we have

$$\limsup_{t\downarrow t_0} \frac{d(\tilde{\mu}(at), \tilde{z}_0) - d(\tilde{\mu}(at_0), \tilde{z}_0)}{t - t_0} = \cos(\angle \tilde{\mu}_1^{(t_0)}(at_0))$$

(7.21)

and

$$\limsup_{t\downarrow t_0} \frac{d(\tilde{\eta}(ct), \tilde{z}_0) - d(\tilde{\eta}(ct_0), \tilde{z}_0)}{t - t_0} = \cos(\angle \tilde{\mu}_2^{(t_0)}(ct_0))$$

(7.22)

By imitating the proof of [IT3 Lemma 2.1], we obtain

$$\liminf_{t\downarrow t_0} \frac{d(\tilde{\mu}(at), \tilde{z}_t) - d(\tilde{\mu}(at_0), \tilde{z}_t)}{t - t_0} = \cos(\angle \tilde{\mu}_1^{(t_0)}(at_0))$$

(7.23)

and

$$\liminf_{t\downarrow t_0} \frac{d(\tilde{\eta}(ct), \tilde{z}_t) - d(\tilde{\eta}(ct_0), \tilde{z}_t)}{t - t_0} = \cos(\angle \tilde{\mu}_2^{(t_0)}(ct_0))$$

(7.24)

In the above equations, notice (7.18). Combining (7.19), (7.20), (7.21), (7.22), (7.23), and (7.24), we have

$$\lim_{t\downarrow t_0} \frac{\tilde{\psi}(t) - \tilde{\psi}(t_0)}{t - t_0} = \cos(\angle \tilde{\mu}_1^{(t_0)}(at_0)) + \cos(\angle \tilde{\mu}_2^{(t_0)}(ct_0))$$

(7.25)
By the same way, we also see
\[ \lim_{t \to t_0} \frac{\tilde{\psi}(t) - \tilde{\psi}(t_0)}{t - t_0} = \cos(\angle \tilde{\mu}_1(t_0)(at_0)) + \cos(\angle \tilde{\mu}_2(t_0)(ct_0)). \] 
(7.26)

From (7.25) and (7.26), we hence get (7.17). As well as above, since \( \phi \) is differentiable at \( t = t_0 \), we also get
\[ \phi'(t_0) = \cos(\angle \mu_1(at_0)) + \cos(\angle \mu_2(ct_0)). \] 
(7.27)

By (7.16), (7.17), and (7.27), we get
\[ \phi'(t_0) \leq \tilde{\psi}'(t_0). \]

Hence, we conclude that \( D'(t_0) \leq 0 \) (see the proof of [KT2, Lemma 4.4]). Thus,
\[ D'(t) \leq 0 \]
for almost all \( t \in (0, 1) \). This implies that \( D(t) \) is non-increasing, since \( D(t) \) is locally Lipschitz. \( \square \)

**Remark 7.4** As pointed out in [KT2, Remark 4.5], it is a very important property that \( D(t) \) is locally Lipschitz. Without this property, we can not conclude that \( D(t) \) is non-increasing.

## 8 Toponogov’s Comparison Theorem

Our purpose of this section is to prove our main theorem, i.e., the Toponogov comparison theorem for open triangles (Theorem 8.4), by using new techniques established in [KT2, Section 4] and Lemmas 5.8, 5.2, and 7.3.

Throughout this section, let \((\tilde{X}, \partial \tilde{X})\) denote a model surface with its metric (1.1).

**Lemma 8.1** Let \((\partial \tilde{X}, \tilde{p}_1, \tilde{q}_1) = (\tilde{\gamma}_1, \tilde{\mu}_1^{(1)}, \tilde{\mu}_2^{(1)})\), \((\partial \tilde{X}, \tilde{p}_2, \tilde{q}_2) = (\tilde{\gamma}_2, \tilde{\mu}_1^{(2)}, \tilde{\mu}_2^{(2)})\) be open triangles in \( \tilde{X} \) such that
\[ d(\partial \tilde{X}, \tilde{q}_1) = d(\partial \tilde{X}, \tilde{p}_2), \]  
(8.1)
and that
\[ \angle \tilde{q}_1 + \angle \tilde{p}_2 \leq \pi. \]  
(8.2)
If there exists an open triangle \((\partial \tilde{X}, \tilde{p}, \tilde{q}) = (\tilde{\gamma}, \tilde{\mu}_1, \tilde{\mu}_2)\) in a sector \( \tilde{X}(\theta_0) \), which has no pair of cut points, satisfying
\[ d(\partial \tilde{X}, \tilde{p}) = d(\partial \tilde{X}, \tilde{p}_1), \quad d(\tilde{p}, \tilde{q}) = d(\tilde{p}_1, \tilde{q}_1) + d(\tilde{p}_2, \tilde{q}_2), \quad d(\partial \tilde{X}, \tilde{q}) = d(\partial \tilde{X}, \tilde{q}_2), \]  
(8.3)
then
\[ \angle \tilde{p}_1 \geq \angle \tilde{p}, \quad \angle \tilde{q}_2 \geq \angle \tilde{q}. \]  
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Proof. By (8.1), we may assume that \((\partial \tilde{X}, \tilde{p}_2, \tilde{q}_2)\) is adjacent to \((\partial \tilde{X}, \tilde{p}_1, \tilde{q}_1)\) so as to have a common side \(\tilde{\mu}_2^{(1)} = \tilde{\mu}_1^{(2)}\), i.e., \(\tilde{q}_1 = \tilde{p}_2\). We may also assume that
\[
0 = \tilde{y}(\tilde{p}_1) < \tilde{y}(\tilde{q}_1) = \tilde{y}(\tilde{p}_2) < \tilde{y}(\tilde{q}_2).
\]
Furthermore, we may assume that
\[
\tilde{p} = \tilde{p}_1 \quad \text{and} \quad \tilde{y}(\tilde{q}) > 0.
\]
Remark that \(\tilde{\mu}_1 = \tilde{\mu}_1^{(1)}\). If \(\angle \tilde{q}_1 + \angle \tilde{p}_2 = \pi\) holds, then there is nothing to prove. Thus, by (8.2), we may assume that
\[
\angle \tilde{q}_1 + \angle \tilde{p}_2 < \pi.
\]
Hence, from the triangle inequality and (8.3), we see
\[
d(\tilde{p}_1, \tilde{q}_2) < d(\tilde{p}, \tilde{q}) = d(\tilde{p}_1, \tilde{q}). \tag{8.4}
\]
Since \(\tilde{q} \in \tilde{X}(\theta_0)\), it follows from Lemma 5.2 and (8.4) that
\[
\tilde{y}(\tilde{q}_2) < \tilde{y}(\tilde{q}) < \theta_0.
\]
Thus, \(\tilde{\gamma}_2\) lies in \(\tilde{X}(\theta_0)\). Since \(\tilde{X}(\theta_0)\) has no pair of cut points, the geodesic extension \(\tilde{\sigma}\) of \(\tilde{\gamma}_1\) does not intersect the side \(\tilde{\gamma}_2\) except for \(\tilde{q}_1\). We will prove that \(\tilde{\sigma}\) does not intersect \(\tilde{\tau}(\tilde{y}(\tilde{q}_2), \tilde{y}(\tilde{q}))\), where \(\tilde{\tau}\) denotes \(\tilde{\tau}(t) := (\tilde{x}(\tilde{q}_2), t) \in \tilde{X}\). Suppose that \(\tilde{\sigma}\) intersect \(\tilde{\tau}(\tilde{y}(\tilde{q}_2), \tilde{y}(\tilde{q}))\) at a point \(\tilde{\sigma}(s_0)\). From Lemma 5.2, we have
\[
d(\tilde{q}_1, \tilde{q}_2) < d(\tilde{q}_1, \tilde{\sigma}(s_0)). \tag{8.5}
\]
Notice that \(\tilde{\sigma}(s_0) \neq \tilde{q}_2\), since \(\tilde{\gamma}_2\) does not meet \(\tilde{\sigma}\) except for \(\tilde{q}_1\). Thus, by (8.3) and (8.5),
\[
d(\tilde{p}_1, \tilde{q}) < d(\tilde{p}_1, \tilde{q}_1) + d(\tilde{q}_1, \tilde{\sigma}(s_0)) = d(\tilde{p}_1, \tilde{\sigma}(s_0)).
\]
Hence, by applying Lemma 5.2 again, we get
\[
\tilde{y}(\tilde{q}) < \tilde{y}(\tilde{\sigma}(s_0)).
\]
This is impossible, since
\[
\tilde{\sigma}(s_0) \in \tilde{\tau}(\tilde{y}(\tilde{q}_2), \tilde{y}(\tilde{q})).
\]
Therefore, we have proved that \(\tilde{\sigma}\) does not intersect \(\tilde{\tau}(\tilde{y}(\tilde{q}_2), \tilde{y}(\tilde{q}))\).

If the extension \(\tilde{\sigma}\) intersects \(\partial \tilde{X}\) at a point \(\tilde{\sigma}(s_1)\) in \(\tilde{X}(\theta_0)\), then we denote by \(\tilde{A}(\theta_0)\) the domain bounded by \(\tilde{\mu}_1\) and \(\tilde{\sigma}([0, s_1])\). If \(\tilde{\sigma}\) does not intersect \(\partial \tilde{X}\) in \(\tilde{X}(\theta_0)\), then \(\tilde{\sigma}\) intersects the \(\partial \tilde{X}\)-ray \(\tilde{y} = \theta_0\) at a point \(\tilde{\sigma}(s_2)\). In this case, \(\tilde{A}(\theta_0)\) denotes the domain bounded by \(\tilde{\mu}_1\), \(\tilde{\sigma}([0, s_2])\), and the \(\partial \tilde{X}\)-segment to \(\tilde{\sigma}(s_2)\). By the argument above, the point \(\tilde{q}\) lies in the domain \(\tilde{A}(\theta_0)\). Hence, the opposite side \(\tilde{\gamma}\) of \((\partial \tilde{X}, \tilde{p}, \tilde{q})\) to \(\partial \tilde{X}\) must lie in the closure of \(\tilde{A}(\theta_0)\), since \(\tilde{X}(\theta_0)\) has no pair of cut points. In particular, it is now clear that
\[
\angle \tilde{p}_1 \geq \angle \tilde{p}.
\]
By repeating the same argument above for the pair of open triangles \((\partial \tilde{X}, \tilde{q}_2, \tilde{p}_2)\) and \((\partial \tilde{X}, \tilde{q}_1, \tilde{p}_1)\), we also get \(\angle \tilde{q}_2 \geq \angle \tilde{q}\). \qed
Lemma 8.2 If an open triangle \((\partial X, p, q) = (\gamma, \mu_1, \mu_2)\) in \(X\) admits an open triangle \((\partial \tilde{X}, \tilde{p}, \tilde{q}) = (\tilde{\gamma}, \tilde{\mu}_1, \tilde{\mu}_2)\) in a sector \(\tilde{X}(\theta_0)\) satisfying
\[
d(\partial \tilde{X}, \tilde{p}) = d(\partial X, p), \quad d(\tilde{p}, \tilde{q}) = d(p, q), \quad d(\partial \tilde{X}, \tilde{q}) = d(\partial X, q), \tag{8.6}
\]
then, for any \(s \in (0, d(p, q))\), there exists an open triangle \((\partial \tilde{X}, \tilde{p}, \tilde{\gamma}(s))\) in \(\tilde{X}(\theta_0)\) satisfying (8.6) for \(q = \sigma(s)\) and \(\tilde{q} = \tilde{\sigma}(s)\).

Proof. It is clear from Lemmas 5.8 and 5.2. See also the proof of [KT2, Lemma 4.9]. \(\square\)

Proposition 8.3 Let \((\partial X, p, q) = (\gamma, \mu_1, \mu_2)\) be an open triangle in \(X\). Then, there exists an open triangle \((\partial \tilde{X}, \tilde{p}, \tilde{q}) = (\tilde{\gamma}, \tilde{\mu}_1, \tilde{\mu}_2)\) in \(\tilde{X}\) satisfying
\[
d(\partial \tilde{X}, \tilde{p}) = d(\partial X, p), \quad d(\tilde{p}, \tilde{q}) = d(p, q), \quad d(\partial \tilde{X}, \tilde{q}) = d(\partial X, q). \tag{8.7}
\]
Furthermore, if the \((\partial \tilde{X}, \tilde{p}, \tilde{q})\) lies in a sector \(\tilde{X}(\theta_0)\), which has no pair of cut points, then
\[
\angle p \geq \angle \tilde{p}, \quad \angle q \geq \angle \tilde{q}. \tag{8.8}
\]

Proof. Since \(\mu_1\) (respectively \(\mu_2\)) is the \(\partial X\)-segment to \(p\) (respectively to \(q\)), we obtain
\[
c \leq a + b, \quad a \leq b + c.
\]
Here we set \(a := d(\partial X, p), b := d(p, q), \) and \(c := d(\partial X, q)\). Hence, we have
\[
|a - c| \leq b.
\]
Choose any point \(\tilde{p} \in \tilde{X}\) satisfying \(d(\partial \tilde{X}, \tilde{p}) = a\), and fix the point. Since the function
\[
d(\tilde{p}, \tilde{\tau}_c(s)) = |a - c|
\]
at \(s = \tilde{y}(\tilde{p})\) and
\[
\lim_{s \to \infty} d(\tilde{p}, \tilde{\tau}_c(s)) = \infty,
\]
we may find a number \(s_0 \geq \tilde{y}(\tilde{p})\) such that
\[
d(\tilde{p}, \tilde{\tau}_c(s_0)) = b.
\]
Here \(\tilde{\tau}_c\) denotes the arc \(\tilde{x} = c\), i.e., \(\tilde{\tau}_c(s) = (c, s) \in \tilde{X}\). Putting \(\tilde{q} := \tilde{\tau}_c(s_0)\), we therefore find a triangle \((\partial \tilde{X}, \tilde{p}, \tilde{q})\) satisfying (8.7).

Hereafter, we assume that the \((\partial \tilde{X}, \tilde{p}, \tilde{q})\) lies in the sector \(\tilde{X}(\theta_0)\). Let \(S\) be the set of all \(s \in (0, d(p, q))\) such that there exists an open triangle \((\partial \tilde{X}, \tilde{p}, \tilde{\gamma}(s)) \subset \tilde{X}(\theta_0)\) corresponding to the triangle \((\partial X, p, \gamma(s)) \subset X\) satisfying (8.7) and (8.8) for \(q = \gamma(s)\) and \(\tilde{q} = \tilde{\gamma}(s)\). Since \((\partial X, p, \gamma(\varepsilon)) \subset X\) is a thin open triangle in \(X\) for any sufficiently small \(\varepsilon > 0\), it
follows from Lemma 5.8 that $S$ is non-empty. Since there is nothing to prove in the case where $\sup S = d(p, q)$, we then suppose that 
\[ s_1 := \sup S < d(p, q). \]

Since $s_1 \in S$, there exists an open triangle $(\partial X, \tilde{p}_1, \tilde{q}_1) \subset \tilde{X}(\theta_0)$ corresponding to the triangle $(\partial X, p, \gamma(s_1)) \subset X$ such that (8.7) and (8.8) hold for $q = \gamma(s_1), \tilde{p} = \tilde{p}_1$, and $\tilde{q} = \tilde{q}_1$. Choose any $\varepsilon_1 \in (0, d(p, q) - s_1)$ in such a way that the open triangle $(\partial X, \gamma(s_1), \gamma(s_1 + \varepsilon_1)) \subset X$ is thin. From Lemma 5.8 there exists an open triangle $(\partial X, \tilde{p}_2, \tilde{q}_2) \subset X$ corresponding to the $(\partial X, \gamma(s_1), (s_1 + \varepsilon_1)) \subset X$ such that (8.7) and (8.8) hold for $p = \gamma(s_1), q = \gamma(s_1 + \varepsilon_1), \tilde{p} = \tilde{p}_2$, and $\tilde{q} = \tilde{q}_2$. It is clear that the pair of open triangles $(\partial X, \tilde{p}_1, \tilde{q}_1)$ and $(\partial X, \tilde{p}_2, \tilde{q}_2)$ satisfy (8.11) and (8.12) in Lemma 8.1. For this pair, it is clear from Lemma 8.2 that there exists an open triangle $(\partial X, \tilde{p}, \tilde{q}) \subset \tilde{X}(\theta_0)$ such that (8.3) holds for $\tilde{p} = \tilde{p}$ and $\tilde{q} = \tilde{q}$. This implies that $s_1 + \varepsilon_1 \in S$. This therefore contradicts the fact that $s_1 = \sup S$. □

**Theorem 8.4 (Toponogov’s Comparison Theorem for Open Triangles)**

Let $(X, \partial X)$ be a complete connected Riemannian $n$-dimensional manifold $X$ with smooth convex boundary $\partial X$ whose radial curvature is bounded from below by that of a model surface $(\tilde{X}, \partial \tilde{X})$ with its metric (1.7). Assume that $\tilde{X}$ admits a sector $\tilde{X}(\theta_0)$ which has no pair of cut points. Then, for every open triangle $(\partial X, p, q) = (\gamma, \mu_1, \mu_2)$ in $X$ with 
\[ d(\mu_1(0), \mu_2(0)) < \theta_0, \]

there exists an open triangle $(\partial \tilde{X}, \tilde{p}, \tilde{q}) = (\tilde{\gamma}, \tilde{\mu}_1, \tilde{\mu}_2)$ in $\tilde{X}(\theta_0)$ such that 
\[ d(\partial \tilde{X}, \tilde{p}) = d(\partial X, p), \quad d(\tilde{p}, \tilde{q}) = d(p, q), \quad d(\partial \tilde{X}, \tilde{q}) = d(\partial X, q) \quad (8.9) \]

and that 
\[ \angle p \geq \angle \tilde{p}, \quad \angle q \geq \angle \tilde{q}, \quad d(\mu_1(0), \mu_2(0)) \geq d(\tilde{\mu}_1(0), \tilde{\mu}_2(0)). \quad (8.10) \]

Furthermore, if 
\[ d(\mu_1(0), \mu_2(0)) = d(\tilde{\mu}_1(0), \tilde{\mu}_2(0)) \]

holds, then 
\[ \angle p = \angle \tilde{p}, \quad \angle q = \angle \tilde{q} \]

hold.

**Proof.** Since the claim of our theorem is trivial for degenerate open triangles, we assume that the open triangle $(\partial X, p, q)$ is not degenerate. Here, we make use of the same notations used in Lemma 7.3 and its proof.

Applying the triangle inequality to the open triangle $(\partial X, \mu_1(at), \mu_2(ct)) \subset X$, we see 
\[ \varphi(t) - (a + c)t \leq d(\mu_1(0), \mu_2(0)) \leq \varphi(t) + (a + c)t \quad (8.11) \]

for all $t \in (0, 1]$, where $a := d(\partial X, p), c := d(\partial X, q)$, and $\varphi(t) := d(\mu_1(at), \mu_2(ct))$. By the first assertion of Proposition 8.3 for each $t \in (0, 1]$, we may find an open triangle 
\[ (\partial \tilde{X}, \tilde{\mu}_1^{(t)}(at), \tilde{\mu}_2^{(t)}(ct)) = (\tilde{\gamma}_1^{(t)}, \tilde{\mu}_1^{(t)}||_{0, at}, \tilde{\mu}_2^{(t)}||_{0, ct}) \subset \tilde{X} \]
which has the same side lengths as the $(\partial X, \mu_1(at), \mu_2(ct))$. Thus, as well as (8.11), we see
\[ \varphi(t) - (a+c)t \leq d(\tilde{\mu}_1^{(t)}(0), \tilde{\mu}_2^{(t)}(0)) \leq \varphi(t) + (a+c)t \] (8.12)
for all $t \in (0,1]$. From (8.11) and (8.12), we obtain
\[ d(\mu_1(0), \mu_2(0)) - 2(a+c)t \leq d(\tilde{\mu}_1^{(t)}(0), \tilde{\mu}_2^{(t)}(0)) \leq d(\mu_1(0), \mu_2(0)) + 2(a+c)t \] (8.13)
for all $t \in (0,1]$. Since $d(\mu_1(0), \mu_2(0)) < \theta_0$, it follows from (8.13) that there exists a number $\varepsilon_1 > 0$ such that
\[ d(\tilde{\mu}_1^{(t)}(0), \tilde{\mu}_2^{(t)}(0)) < \theta_0 \] (8.14)
holds on $(0, \varepsilon_1)$. Hence,
\[ (\partial \tilde{X}, \tilde{\mu}_1^{(t)}(at), \tilde{\mu}_2^{(t)}(ct)) \subset \tilde{X}(\theta_0) \] (8.15)
for each $t \in (0, \varepsilon_1)$. By the second assertion of Proposition 8.3, we get
\[ \angle \mu_1(at) \geq \angle \tilde{\mu}_1^{(t)}(at), \quad \angle \mu_2(ct) \geq \angle \tilde{\mu}_2^{(t)}(ct) \] (8.16)
for each $t \in (0, \varepsilon_1)$. Since $\tilde{X}(\theta_0)$ has no pair of cut points, it follows from (8.15) that the opposite side $\tilde{\gamma}$ of $(\partial \tilde{X}, \tilde{\mu}_1^{(t)}(at), \tilde{\mu}_2^{(t)}(ct))$ to $\partial \tilde{X}$ is unique for all $t \in (0, \varepsilon_1)$. From Lemma 7.3 and (8.14), it follows that the function $D(t) = d(\tilde{\mu}_1^{(t)}(0), \tilde{\mu}_2^{(t)}(0))$ is non-increasing on $(0, \varepsilon_1)$ and $D(t) < \theta_0$ holds on $(0, \varepsilon_1)$. Thus, we finally see that $D(t)$ is non-increasing on $(0,1]$, $D(t) < \theta_0$ holds on $(0,1]$, and (8.16) holds on $(0,1]$. In particular, setting
\[ (\partial \tilde{X}, \tilde{p}, \tilde{q}) = (\tilde{\gamma}, \tilde{\mu}_1, \tilde{\mu}_2) := (\partial \tilde{X}, \tilde{\mu}_1^{(1)}(a), \tilde{\mu}_2^{(1)}(c)) \subset \tilde{X}(\theta_0), \]
we get
\[ \angle p \geq \angle \tilde{p}, \quad \angle q \geq \angle \tilde{q}. \] (8.17)
Moreover, by (8.13),
\[ D(t) = d(\tilde{\mu}_1^{(t)}(0), \tilde{\mu}_2^{(t)}(0)) \leq d(\mu_1(0), \mu_2(0)) + 2(a+c)t \] (8.18)
holds on $(0,1]$. Since $D(t)$ is non-increasing on $(0,1]$, we have, by (8.13),
\[ D(1) = d(\tilde{\mu}_1(0), \tilde{\mu}_2(0)) \leq d(\mu_1(0), \mu_2(0)) + 2(a+c)t \]
on $(0,1]$. Hence we get
\[ d(\mu_1(0), \mu_2(0)) \geq d(\tilde{\mu}_1(0), \tilde{\mu}_2(0)). \] (8.19)
By (8.17) and (8.19), the open triangle $(\partial \tilde{X}, \tilde{p}, \tilde{q})$ is therefore an open triangle satisfying conditions (8.9) and (8.10).

Assume that
\[ d(\mu_1(0), \mu_2(0)) = d(\tilde{\mu}_1(0), \tilde{\mu}_2(0)) = D(1) \]
holds. By (8.18),
\[ D(t) \leq 2(a+c)t + D(1) \]
holds on $(0,1]$. Thus, we get
\[ \lim_{t \to 0} D(t) \leq D(1). \]
Hence, $D(t)$ must be constant on $(0,1]$, since $D(t)$ is non-increasing on $(0,1]$. From the proof of Lemma 7.3, it follows that $\angle \mu_1(at) = \angle \tilde{\mu}_1^{(t)}(at)$ and $\angle \mu_2(ct) = \angle \tilde{\mu}_2^{(t)}(ct)$ hold on $(0,1]$. In particular, we obtain $\angle p = \angle \tilde{p}$ and $\angle q = \angle \tilde{q}$. □
9 Generalized Open Triangles

In the following, we will prove the Toponogov comparison theorem for open triangles in a weak form (Proposition 9.2), where we do not demand any assumption on a sector.

We first define generalized open triangles in a model surface:

**Definition 9.1 (Generalized Open Triangles)** A generalized open triangle

\[(\partial \tilde{X}, \hat{p}, \hat{q}) = (\hat{\gamma}, \hat{\mu}_1, \hat{\mu}_2)\]

in a model surface \((\tilde{X}, \partial \tilde{X})\) is defined by two \(\partial \tilde{X}\)-segments \(\hat{\mu}_i : [0, \ell_i] \to \tilde{X}, i = 1, 2\), and a geodesic segment \(\hat{\gamma}\) emanating from \(\hat{p}\) to \(\hat{q}\) such that

\[
\hat{\mu}_1(\ell_1) = \hat{\gamma}(0) = \hat{p}, \quad \hat{\mu}_2(\ell_2) = \hat{\gamma}(d(\hat{p}, \hat{q})) = \hat{q},
\]

and that \(\hat{\gamma}\) is a shortest arc joining \(\hat{p}\) to \(\hat{q}\) in the compact domain bounded by \(\hat{\mu}_1, \hat{\mu}_2\), and \(\hat{\gamma}\).

Using Lemma 5.8, we may prove the Toponogov comparison theorem for open triangles in a weak form:

**Proposition 9.2** Let \((X, \partial X)\) be a complete connected Riemannian \(n\)-dimensional manifold \(X\) with smooth convex boundary \(\partial X\) whose radial curvature is bounded from below by that of a model surface \((\tilde{X}, \partial \tilde{X})\). Then, for every open triangle \((\partial X, p, q) = (\gamma, \mu_1, \mu_2)\) in \(X\), there exists a generalized open triangle \((\partial \tilde{X}, \hat{p}, \hat{q}) = (\hat{\gamma}, \hat{\mu}_1, \hat{\mu}_2)\) in \(\tilde{X}\) such that

\[
d(\partial \tilde{X}, \hat{p}) = d(\partial X, p), \quad d(\partial \tilde{X}, \hat{q}) = d(\partial X, q),
\]

and

\[
d(\partial X, q) - d(\partial X, p) \leq d(\hat{p}, \hat{q}) \leq L(\hat{\gamma}) \leq d(p, q),
\]

and that

\[
\angle p \geq \angle \hat{p}, \quad \angle q \geq \angle \hat{q}.
\]

Here \(L(\hat{\gamma})\) denotes the length of \(\hat{\gamma}\).

**Proof.** Let \(s_0 := 0 < s_1 < \cdots < s_{k-1} < s_k := d(p, q)\) be a subdivision of \([0, d(p, q)]\) such that, for each \(i \in \{1, \ldots, k\}\), the open triangle \((\partial X, \gamma(s_{i-1}), \gamma(s_i))\) is thin. It follows from Lemma 5.8 that, for each triangle \((\partial X, \gamma(s_{i-1}), \gamma(s_i))\), there exists an open triangle \(\tilde{\Delta}_i := (\partial X, \tilde{\gamma}(s_{i-1}), \tilde{\gamma}(s_i))\) in \(\tilde{X}\) such that

\[
d(\partial \tilde{X}, \tilde{\gamma}(s_{i-1})) = d(\partial X, \gamma(s_{i-1})),
\]

\[
d(\tilde{\gamma}(s_{i-1}), \tilde{\gamma}(s_i)) = d(\gamma(s_{i-1}), \gamma(s_i)),
\]

\[
d(\partial \tilde{X}, \tilde{\gamma}(s_i)) = d(\partial X, \gamma(s_i)),
\]

and...
and that
\[ \angle(\partial X, \gamma(s_{i-1}), \gamma(s_i)) \geq \angle(\partial \tilde{X}, \tilde{\gamma}(s_{i-1}), \tilde{\gamma}(s_i)), \quad (9.7) \]
\[ \angle(\partial X, \gamma(s_i), \gamma(s_{i-1})) \geq \angle(\partial \tilde{X}, \tilde{\gamma}(s_i), \tilde{\gamma}(s_{i-1})). \quad (9.8) \]

Here \( \angle(\partial X, \gamma(s_{i-1}), \gamma(s_i)) \) denotes the angle between two sides joining \( \gamma(s_{i-1}) \) to \( \partial X \) and \( \gamma(s_i) \) forming the triangle \( (\partial X, \gamma(s_{i-1}), \gamma(s_i)) \). Under this situation, draw \( \tilde{\Delta}_1 = (\partial \tilde{X}, \tilde{p}, \tilde{\gamma}(s_i)) \) in \( \tilde{X} \) satisfying \( (9.4), (9.5), (9.6), (9.7), (9.8) \) for \( i = 1 \). Inductively, we draw an open triangle \( \tilde{\Delta}_{i+1} = (\partial \tilde{X}, \tilde{\gamma}(s_i), \tilde{\gamma}(s_{i+1})) \) in \( \tilde{X} \), which is adjacent to \( \tilde{\Delta}_i \) so as to have the \( \partial \tilde{X} \)-segment to \( \tilde{\gamma}(s_i) \) as a common side. Since
\[ \angle(\partial X, \gamma(s_i), \gamma(s_{i+1})) + \angle(\partial X, \gamma(s_i), \gamma(s_{i+1})) = \pi, \]
for each \( i = 1, 2, \ldots, k - 1 \), we get, by \( (9.7) \) and \( (9.8) \),
\[ \angle(\partial \tilde{X}, \tilde{\gamma}(s_i), \tilde{\gamma}(s_{i-1})) + \angle(\partial \tilde{X}, \tilde{\gamma}(s_i), \tilde{\gamma}(s_{i+1})) \leq \pi \quad (9.9) \]

and
\[ \angle p \geq \angle(\partial \tilde{X}, \tilde{\gamma}(s_0), \tilde{\gamma}(s_1)), \quad \angle q \geq \angle(\partial \tilde{X}, \tilde{\gamma}(s_k), \tilde{\gamma}(s_{k-1})). \quad (9.10) \]

Then, we get a domain \( D \) bounded by two \( \partial \tilde{X} \)-segments \( \tilde{\mu}_0, \tilde{\mu}_k \) to \( \tilde{\gamma}(s_0), \tilde{\gamma}(s_k) \), respectively, and \( \tilde{\eta} \), where \( \tilde{\eta} \) denotes the broken geodesic consisting of the opposite sides of \( \tilde{\Delta}_i \) \( (i = 1, 2, \ldots, k) \) to \( \partial \tilde{X} \). Since the domain \( D \) is locally convex by \( (9.9) \), there exists a minimal geodesic segment \( \tilde{\gamma} \) in the closure of \( D \) joining \( \tilde{\gamma}(s_0) \) to \( \tilde{\gamma}(s_k) \). From \( (9.10) \), it is clear that the generalized open triangle \( (\tilde{\gamma}, \tilde{\mu}_0, \tilde{\mu}_k) \) has the required properties in our proposition. \( \square \)

10 Application, I

In this section, we will prove Theorem 1.4 (Corollary 10.6 and Proposition 10.7) as an application of Proposition 9.2.

From the similar argument in the proof of [ST] Lemma 3.1, one may prove

**Lemma 10.1** Let
\[ f''(t) + K(t)f(t) = 0, \quad f(0) = 1, \quad t \in [0, \infty), \]
\[ m''(t) + G(t)m(t) = 0, \quad m(0) = 1, \quad m'(0) = 0, \quad t \in [0, \infty), \]
be two ordinary differential equations with \( K(t) \geq G(t) \) on \( [0, \infty) \).

(L-1) If \( f > 0 \) on \( (0, \infty) \), \( f'(0) = 0 \), and
\[ \int_0^\infty \frac{1}{m(t)^2} dt = \infty, \]
then \( K(t) = G(t) \) on \( [0, \infty) \).
If \( m > 0 \) on \((0, \infty)\), \( f'(0) < 0 \), and
\[
\int_0^\infty \frac{1}{m(t)^2} dt = \infty,
\]
then there exists \( t_0 \in (0, \infty) \) such that \( f > 0 \) on \([0, t_0)\) and \( f(t_0) = 0 \).

Hereafter, let \((X, \partial X)\) be a complete non-compact connected Riemannian \(n\)-manifold \(X\) with smooth convex boundary \(\partial X\) whose radial curvature is bounded from below by that of a model surface \((\tilde{X}, \partial \tilde{X})\) with its metric \([1.1]\). Moreover,

we assume that \(X\) admits at least one \(\partial X\)-ray.

By Lemma \([10.1]\) we have

**Lemma 10.2** Let \(\mu : [0, \infty) \rightarrow X\) be a \(\partial X\)-ray. If \((\tilde{X}, \partial \tilde{X})\) satisfies
\[
\int_0^\infty \frac{1}{m(t)^2} dt = \infty,
\]
then, \(\mu(0)\) is the geodesic point in \(\partial X\), i.e., the second fundamental form vanishes at the point.

**Proof.** Let \(E\) be a unit parallel vector field along \(\mu\) such that
\[
A_{\mu'}(E(0)) = \lambda E(0), \tag{10.1}
\]
\[
E(t) \perp \mu'(t). \tag{10.2}
\]

Here \(\lambda\) denotes an eigenvalue of the shape operator \(A_{\mu'}(0)\) of \(\partial X\). Since \(\partial X\) is convex,
\[
\lambda \geq 0.
\]

Consider a smooth vector field \(Y(t) := f(t)E(t)\) along \(\mu\) satisfying
\[
f''(t) + K_X(\mu'(t), E(t))f(t) = 0,
\]
with initial conditions
\[
f(0) = 1, \quad f'(0) = -\lambda. \tag{10.3}
\]

Here \(K_X(\mu'(t), E(t))\) denotes the sectional curvature with respect to the 2-dimensional linear space spanned by \(\mu'(t)\) and \(E(t)\) at \(\mu(t)\). Notice that \(Y\) satisfies
\[
Y(0) \in T_{\mu(0)}\partial X, \quad Y'(0) + A_{\mu'(0)}(Y(0)) = 0 \in (T_{\mu(0)}\partial X)^\perp.
\]

by \([10.1], [10.2],\) and \([10.3]\). Suppose that
\[
\lambda > 0.
\]

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Since \( f'(0) < 0 \) and
\[
\int_0^\infty \frac{1}{m(t)^2} dt = \infty.
\]
it follows from (L–2) in Lemma 10.1 that there exists \( t_0 \in (0, \infty) \) such that \( f > 0 \) on \([0, t_0)\) and
\[
f(t_0) = 0, \tag{10.4}
\]
i.e.,
\[
Y(t) \neq 0, \quad t \in [0, t_0)
\]
and \( Y(t_0) = 0 \). Since
\[
\langle R(\mu'(t), Y(t))\mu'(t), Y(t) \rangle = f(t)^2 \langle R(\mu'(t), E(t))\mu'(t), E(t) \rangle = -f''(t)f(t),
\]
we have, by (10.3) and (10.4),
\[
I_{t_0}(Y, Y) = \int_0^{t_0} \frac{d}{dt}(f'f) dt = f(t_0)f'(t_0) - f(0)f'(0) = \lambda \tag{10.6}
\]
Thus, by (10.4), (10.3), and (10.6),
\[
\mathcal{T}_{\partial X}^0(Y, Y) = I_{t_0}(Y, Y) - \langle A_{\mu(0)}(Y(0)), Y(0) \rangle = \lambda - \lambda = 0. \tag{10.7}
\]
On the other hand, since \( \partial X \) has no focal point along \( \mu \), for any non-zero vector field \( Z \) along \( \mu \) satisfying \( Z(0) \in T_{\mu(0)}\partial X \) and \( Z(t_0) = 0 \),
\[
\mathcal{T}_{\partial X}^0(Z, Z) > 0 \tag{10.8}
\]
holds (cf. Lemma 2.9 in [S, Chapter III]). Thus, by (10.7) and (10.8), \( Y \equiv 0 \) on \([0, t_0)\]. This is a contradiction to (10.5). Therefore, \( \lambda = 0 \), i.e., \( \mu(0) \) is the geodesic point in \( \partial X \). \( \square \)

Here we want to go over some fundamental tools on \((\tilde{X}, \partial \tilde{X})\) : A unit speed geodesic \( \tilde{\gamma} : [0, a) \rightarrow \tilde{X} \) \((0 < a \leq \infty)\) is expressed by \( \tilde{\gamma}(s) = (\tilde{x}(\tilde{\gamma}(s)), \tilde{y}(\tilde{\gamma}(s))) =: (\tilde{x}(s), \tilde{y}(s)) \). Then, there exists a non-negative constant \( \nu \) depending only on \( \tilde{\gamma} \) such that
\[
\nu = m(\tilde{x}(s))^2|\tilde{y}'(s)| = m(\tilde{x}(s)) \sin \angle(\tilde{\gamma}'(s), (\partial/\partial \tilde{x})\tilde{\gamma}(s)). \tag{10.9}
\]
This (10.9) is a famous formula – the Clairaut relation. The constant \( \nu \) is called the Clairaut constant of \( \tilde{\gamma} \). Remark that, by (10.9), \( \nu > 0 \) if and only if \( \tilde{\gamma} \) is not a \( \partial \tilde{X} \)-ray, or its subarc. Since \( \tilde{\gamma} \) is unit speed, we have, by (10.9),
\[
\tilde{x}'(s) = \pm \sqrt{\frac{m(\tilde{x}(s))^2 - \nu^2}{m(\tilde{x}(s))}}. \tag{10.10}
\]
By (10.10), we see that \( \tilde{x}'(s) = 0 \) if and only if \( m(\tilde{x}(s)) = \nu \). Moreover, by (10.10), we have that, for a unit speed geodesic \( \tilde{\gamma}(s) = (\tilde{x}(s), \tilde{y}(s)), s_1 \leq s \leq s_2 \), with the Clairaut constant \( \nu \),
\[
s_2 - s_1 = \phi(\tilde{x}'(s)) \int_{\tilde{x}(s_1)}^{\tilde{x}(s_2)} \frac{m(t)}{\sqrt{m(t)^2 - \nu^2}} dt. \tag{10.11}
\]
if \( \tilde{x}'(s) \neq 0 \) on \([s_1, s_2] \). Here, \( \phi(\tilde{x}'(s)) \) denotes the sign of \( \tilde{x}'(s) \). Furthermore, we have a lemma with respect to the length \( L(\tilde{\gamma}) \) of \( \tilde{\gamma} \):
Lemma 10.3  Let \( \tilde{\gamma} : [0,s_0] \to \overline{X} \setminus \partial \overline{X} \) denote a unit speed geodesic segment with Clairaut constant \( \nu \). Then, \( L(\tilde{\gamma}) \) is not less than
\[
t_2 - t_1 + \frac{\nu^2}{2} \int_{t_1}^{t_2} \frac{1}{m(t) \sqrt{m(t)^2 - \nu^2}} \, dt,
\]
where we set \( t_1 := \tilde{x}(0) \) and \( t_2 := \tilde{x}(s_0) \).

**Proof.** We may assume that \( t_2 > t_1 \), otherwise (10.12) is non-positive. Let \([s_1,s_2]\) be a sub-interval of \([0,s_0]\) such that
\[
\tilde{x}'(s) \neq 0
\]
on \((s_1,s_2)\). By (10.11),
\[
L(\tilde{\gamma}|_{[s_1,s_2]}) = s_2 - s_1 = \left| \int_{\tilde{x}(s_1)}^{\tilde{x}(s_2)} \frac{m(t)}{\sqrt{m(t)^2 - \nu^2}} \, dt \right|.
\]
Since \( \tilde{x}'(s) \neq 0 \) for all \( s \in (s_1,s_2) \) with \( \tilde{x}(s) \in [t_1,t_2] \), we may choose the numbers \( s_1 \) and \( s_2 \) in such a way that
\[
\tilde{x}(s_1) = t_1, \quad \tilde{x}(s_2) = t_2
\]
or
\[
\tilde{x}(s_1) = t_2, \quad \tilde{x}(s_2) = t_1.
\]
Thus, we see that
\[
L(\tilde{\gamma}) \geq \int_{t_1}^{t_2} \frac{m(t)}{\sqrt{m(t)^2 - \nu^2}} \, dt.
\]
Since
\[
\frac{m(t)}{\sqrt{m(t)^2 - \nu^2}} \geq 1 + \frac{\nu^2}{2m(t) \sqrt{m(t)^2 - \nu^2}},
\]
we have, by (10.13),
\[
L(\tilde{\gamma}) \geq t_2 - t_1 + \frac{\nu^2}{2} \int_{t_1}^{t_2} \frac{1}{m(t) \sqrt{m(t)^2 - \nu^2}} \, dt.
\]
\( \square \)

**Lemma 10.4** For any \( q \in \text{Cut}(\partial X) \cap (X \setminus \partial X) \) and any \( \varepsilon > 0 \), there exists a point in \( \text{Cut}(\partial X) \cap B_{\varepsilon}(q) \) which admits at least two \( \partial X \)-segments.

The above lemma is well-known in the case of the cut locus of a point (see [B]), which can be proved similarly. However we will give a proof of the lemma totally different from it (see Section 12).

**Proposition 10.5** Let \( \mu_0 : [0, \infty) \to X \) be a \( \partial X \)-ray guaranteed by the assumption above. If \((\overline{X}, \partial \overline{X})\) satisfies
\[
\int_{0}^{\infty} \frac{1}{m(t)^2} \, dt = \infty,
\]
\( (10.14) \)
or
\[
\liminf_{t \to \infty} m(t) = 0, \quad (10.15)
\]
than, any point of \( X \) lies in a unique \( \partial X \)-ray. In particular, \( \partial X \) is totally geodesic in the case where \((10.14)\) is satisfied.

**Proof.** Choose any point \( q \in X \setminus \partial X \) not lying on \( \mu_0 \). Let \( \mu_1 : [0, d(\partial X, q)] \to X \) denote a \( \partial X \)-segment with \( \mu_1(d(\partial X, q)) = q \). For each \( t > 0 \), let \( \gamma : [0, d(q, \mu_0(t))] \to X \) denote a minimal geodesic segment emanating from \( q \) to \( \mu_0(t) \). From Proposition \( 9.2 \) and the triangle inequality, it follows that there exists a generalized open triangle \( (\partial \tilde{X}, \tilde{\mu}_0(t), \tilde{q}) = (\tilde{\gamma}_t, \tilde{\mu}_0(t), \tilde{\mu}_1) \) in \( \tilde{X} \) corresponding to the triangle \( (\partial X, \mu_0(t), q) = (\gamma_t, \mu_0(0), \mu_1) \) in \( X \) such that
\[
d(\partial \tilde{X}, \tilde{\mu}_0(t)) = t, \quad d(\partial \tilde{X}, \tilde{q}) = d(\partial X, q), \quad (10.16)
\]
and
\[
L(\tilde{\gamma}_t) \leq d(\mu_0(t), q) \leq t + d(q, \mu_0(0)) \quad (10.17)
\]
and that
\[
\angle(\partial X, q, \mu_0(t)) \geq \angle(\partial \tilde{X}, \tilde{q}, \tilde{\mu}_0(t)). \quad (10.18)
\]
Here \( \angle(\partial X, q, \mu_0(t)) \) denotes the angle between two sides \( \mu_1 \) and \( \gamma_t \) joining \( q \) to \( \partial X \) and \( \mu_0(t) \) forming the triangle \( (\partial X, \mu_0(t), q) \). From Lemma \( 10.3 \), \( 10.16 \), and \( 10.17 \), we get
\[
t + d(q, \mu_0(0)) \geq L(\tilde{\gamma}_t)
\]
\[
\geq t - d(\partial X, q) + \frac{\nu_t^2}{2} \int_{d(\partial X, q)}^{t} \frac{1}{m(t)} \sqrt{m(t)^2 - \nu_t^2} \, dt \quad (10.19)
\]
where \( \nu_t \) denotes the Clairaut constant of \( \tilde{\gamma}_t \). By \( 10.19 \),
\[
d(\partial X, q) + d(q, \mu_0(0)) \geq \frac{\nu_t^2}{2} \int_{d(\partial X, q)}^{t} \frac{1}{m(t)} \, dt \quad (10.20)
\]
First, assume that \((\tilde{X}, \partial \tilde{X})\) satisfies \((10.14)\). Then, it is clear from \( 10.20 \) that
\[
\lim_{t \to \infty} \nu_t = 0.
\]
Hence, by \( 10.9 \), we have
\[
\lim_{t \to \infty} \angle(\partial \tilde{X}, \tilde{q}, \tilde{\mu}_0(t)) = \pi. \quad (10.21)
\]
By \( 10.18 \) and \( 10.21 \), \( \gamma_\infty := \lim_{t \to \infty} \gamma_t \) is a ray emanating from \( q \) such that
\[
\angle(\gamma_\infty(0), -\mu_1'(d(\partial X, q))) = \pi.
\]
This implies that \( q \) lies on a unique \( \partial X \)-segment. Therefore, by Lemma \( 10.4 \), \( q \) lies on a \( \partial X \)-ray. Now, it is clear from Lemma \( 10.2 \) that \( \partial X \) is totally geodesic.

Second, assume that \((\tilde{X}, \partial \tilde{X})\) satisfies \((10.15)\). Then, there exists a divergent sequence \( \{t_i\}_{i \in \mathbb{N}} \) such that
\[
\lim_{t_i \to \infty} m(t_i) = 0. \quad (10.22)
\]
From (10.9), we see
\[ \nu_i \leq m(t_i), \quad (10.23) \]
where \( \nu_i \) denotes the Clairaut constant of \( \gamma_{t_i} \). Hence, by (10.22) and (10.23),
\[ \lim_{t \to \infty} \nu_t = 0 \]
holds. Now, it is clear that there exist a limit geodesic \( \gamma_\infty \) of \( \{ \gamma_{t_i} \} \) such that \( \gamma_\infty \) is a ray emanating from \( q \) and satisfies \( \angle (\gamma_\prime_\infty(0), -\mu_1'(d(\partial X, q))) = \pi \). Therefore, by Lemma 10.4, \( q \) lies on a \( \partial X \)-ray.

By Proposition 10.5, there does not exist a cut point of \( \partial X \). Therefore, it is clear that

**Corollary 10.6** If \((\tilde{X}, \partial \tilde{X})\) satisfies (10.14), or (10.15), then \( X \) is diffeomorphic to \([0, \infty) \times \partial X\).

Furthermore, we may reach stronger conclusion than Corollary 10.6:

**Proposition 10.7** If \((\tilde{X}, \partial \tilde{X})\) satisfies
\[ \int_0^\infty \frac{1}{m(t)^2} dt = \infty, \]
then, for every \( \partial X \)-ray \( \mu : [0, \infty) \to X \), the radial curvature \( K_X \) satisfies
\[ K_X(\sigma_t) = G(\tilde{\mu}(t)) \quad (10.24) \]
for all \( t \in [0, \infty) \) and all 2-dimensional linear space \( \sigma_t \) spanned by \( \mu'(t) \) and a tangent vector to \( X \) at \( \mu(t) \). In particular, \( X \) is isometric to the warped product manifold \([0, \infty) \times_m \partial X\) of \([0, \infty) \) and \( (\partial X, g_{\partial X}) \) with the warping function \( m \). Here \( g_{\partial X} \) denotes the induced Riemannian metric from \( X \).

**Proof.** Take any point \( p \in \partial X \), and fix it. By Proposition 10.5, we may take a \( \partial X \)-ray \( \mu : [0, \infty) \to X \) emanating from \( p = \mu(0) \). Suppose that
\[ K_X(\sigma_{t_0}) > G(\tilde{\mu}(t_0)) \quad (10.25) \]
for some linear plane \( \sigma_{t_0} \) spanned by \( \mu'(t_0) \) and a unit tangent vector \( v_0 \) orthogonal to \( \mu'(t_0) \). If we denote by \( E(t) \) the parallel vector field along \( \mu \) satisfying \( E(t_0) = v_0 \), then \( E(t) \) is unit and orthogonal to \( \mu'(t_0) \) for each \( t \). We define a non-zero vector field \( Y(t) \) along \( \mu \) by \( Y(t) := f(t) E(t) \), where \( f \) is the solution of the following differential equation
\[ f''(t) + K_X(\mu'(t), E(t)) f(t) = 0 \quad (10.26) \]
with initial condition \( f(0) = 1 \) and \( f'(0) = 0 \). Here \( K_X(\mu'(t), E(t)) \) denotes the sectional curvature of the plane spanned by \( \mu'(t) \) and \( E(t) \). It follows from (10.25) and (L–1) in Lemma 10.1 that there exists \( t_1 > 0 \) such that \( f(t_1) = 0 \). From (10.26), we get
\[ I_{t_1}(Y, Y) = \int_0^{t_1} \frac{d}{dt} (ff')dt = 0 \quad (10.27) \]
Since \( \partial X \) is totally geodesic by Proposition 10.5, \( A_{\mu'(0)}(E(0)) = 0 \). Thus, by (10.27), \( T_{\partial X}^1(Y,Y) = 0 \) holds. On the other hand, \( T_{\partial X}^1(Y,Y) > 0 \) holds, since there is no focal point of \( \partial X \) along \( \mu \). This is a contradiction. Therefore, we get the first assertion (10.24).

Now it is clear that the map \( \varphi : [0, \infty) \times_m \partial X \longrightarrow X \) defined by

\[
\varphi(t, q) := \exp^+(tv_q)
\]
gives an isometry from \([0, \infty) \times_m \partial X \) onto \( X \). Here \( v_q \) denotes the inward pointing unit normal vector to \( \partial X \) at \( q \in \partial X \). \( \square \)

11 Application, II

In this section, we will prove another splitting theorem (Theorem 11.6) as an application of Theorem 8.4.

Throughout this section, let \((X, \partial X)\) be a complete connected Riemannian \( n \)-manifold \( X \) with disconnected smooth compact convex boundary \( \partial X \) whose radial curvature is bounded from below by 0. Under the hypothesis, we may assume

\[
\partial X = \bigcup_{i=1}^{k} \partial X_i, \quad k \geq 2.
\]

Here each \( \partial X_i \) denotes a connected component of \( \partial X \) and is compact. Set

\[
\ell := \min\{d(\partial X_i, \partial X_j) \mid 1 \leq i, j \leq k, i \neq j\}.
\]

Then let \( \partial X_1, \partial X_2 \) denote the connected components of \( \partial X \) satisfying

\[
d(\partial X_1, \partial X_2) = \ell.
\]

The proof of the next lemma is standard:

**Lemma 11.1** Let \( \mu \) denote a minimal geodesic segment in \( X \) emanating from \( \partial X_1 \) to \( \partial X_2 \). Then, there does not exist any other \( \partial X \)-segment to \( \mu(\ell/2) \) than \( \mu|_{[0, \ell/2]} \) and \( \mu|_{[\ell/2, \ell]} \). Furthermore, each midpoint \( \mu(\ell/2) \) is not a focal point of \( \partial X \) along \( \mu \).

Hereafter, the half plane

\[
\mathbb{R}^2_+ := \{ \tilde{p} \in \mathbb{R}^2 \mid \tilde{x}(\tilde{p}) \geq 0 \}
\]

with Euclidean metric \( d\tilde{x}^2 + d\tilde{y}^2 \) will be used as the model surface for \((X, \partial X)\).

**Lemma 11.2** Any point in \( X \) lies on a minimal geodesic segment emanating from \( \partial X_1 \) to \( \partial X_2 \) of length \( \ell \). In particular, \( \partial X \) consists of \( \partial X_1 \) and \( \partial X_2 \).
Proof. Since $X$ is connected, it is sufficient to prove that the subset $\mathcal{O}$ of $X$ is open and closed, where $\mathcal{O}$ denotes the set of all points $r \in X$ which lies on a minimal geodesic segment emanating from $\partial X_1$ to $\partial X_2$ of length $\ell$. Since it is trivial that $\mathcal{O}$ is closed, we will prove that $\mathcal{O}$ is open.

Choose any point 

$$r \in \mathcal{O},$$

and fix it. Thus, $r$ lies on a minimal geodesic segment $\mu_1 : [0, \ell] \rightarrow X$ emanating from $\partial X_1$ to $\partial X_2$. Set 

$$p := \mu_1(\ell/2).$$

Let $S$ be the equidistant set from $\partial X_1$ and $\partial X_2$, i.e.,

$$S := \{q \in X \mid d(\partial X_1, q) = d(\partial X_2, q)\}. \tag{11.1}$$

It follows from Lemma 11.1 that 

$$S \cap B_{\varepsilon_1}(p) \subset \text{Cut}(\partial X),$$

if $\varepsilon_1 > 0$ is chosen sufficiently small. Choose any point 

$$q \in S \cap B_{\varepsilon_1}(p) \setminus \{p\},$$

and also fix it. Let $\eta_i$, $i = 1, 2$, denote a $\partial X$-segment to $q$ such that $\eta_1(0) \in \partial X_1$ and $\eta_2(0) \in \partial X_2$, respectively. Moreover, let $\gamma : [0, d(p, q)] \rightarrow X$ denote a minimal geodesic segment emanating from $p$ to $q$. Since

$$\angle(\gamma'(0), -\mu_1'(\ell/2)) + \angle(\gamma'(0), \mu_1'(\ell/2)) = \pi,$$

we may assume, without loss of generality, that

$$\angle(\gamma'(0), -\mu_1'(\ell/2)) \leq \frac{\pi}{2}. \tag{11.2}$$

It follows from Theorem 8.4 that there exists an open triangle $(\partial \mathbb{R}^2_+, \tilde{\gamma}, \tilde{\eta}_1)$ in $\mathbb{R}^2_+$ corresponding to the triangle $(\partial X_1, p, q) = (\gamma, \mu_1|_{[0, \ell/2]}, \eta_1)$ such that

$$d(\partial \mathbb{R}^2_+, \tilde{\gamma}) = \ell/2, \quad d(\tilde{\gamma}, \tilde{\eta}_1) = d(p, q), \quad d(\partial \mathbb{R}^2_+, \tilde{\eta}_1) = d(\partial X_1, q), \tag{11.3}$$

and

$$\angle(\gamma'(0), -\mu_1'(\ell/2)) = \angle p \geq \angle \tilde{\gamma}, \quad \angle q \geq \angle \tilde{\eta}_1. \tag{11.4}$$

By (11.2) and $\angle p \geq \angle \tilde{p}$ of (11.4), we have

$$\angle \tilde{\gamma} \leq \frac{\pi}{2}. \tag{11.5}$$

Since our model is $\mathbb{R}^2_+$, it follows from the two equations $d(\partial \mathbb{R}^2_+, \tilde{\gamma}) = \ell/2$, $d(\partial \mathbb{R}^2_+, \tilde{\eta}_1) = d(\partial X_1, q)$ of (11.3), and (11.5) that

$$d(\partial X_1, q) = d(\partial \mathbb{R}^2_+, \tilde{\eta}_1) \leq \frac{\ell}{2}. \tag{11.6}$$
On the other hand, the broken geodesic segment defined by combining $\eta_1$ and $\eta_2$ is a curve joining $\partial X_1$ to $\partial X_2$. This implies that length of the broken geodesic segment is not less than that of $\mu_1$. Thus,

$$2L(\eta_1) = L(\eta_1) + L(\eta_2) \geq \ell,$$  \hspace{1em} (11.7)

where $L(\cdot)$ denotes the length of a curve. Since $L(\eta_1) = d(\partial X_1, q)$, we have, by (11.7), that

$$d(\partial X_1, q) \geq \frac{\ell}{2}. \hspace{1em} (11.8)$$

By (11.6) and (11.8),

$$d(\partial X_1, q) = d(\partial X_2, q) = \frac{\ell}{2}.$$  

Therefore, we have proved that any point $q \in S \cap B_{\varepsilon_1}(p)$ is the midpoint of a minimal geodesic segment emanating from $\partial X_1$ to $\partial X_2$ of length $\ell$. Furthermore, by Lemma 11.1 each point of $S \cap B_{\varepsilon_1}(p)$ is not a focal point of $\partial X$. It is therefore clear that any point sufficiently close to the point $r \in \mathcal{O}$ is a point of $\mathcal{O}$, i.e, $\mathcal{O}$ is open. \hfill \Box

**Remark 11.3** From Lemmas 11.1 and 11.2 it is clear that

$$\text{Cut}(\partial X) = \{ p \in X \mid d(\partial X, p) = \ell/2 \} = S \hspace{1em} (11.9)$$

and that

$$d(\partial X, p) \leq \frac{\ell}{2} \hspace{1em} (11.10)$$

for all $p \in X$. Here $S$ is the equidistant set defined by (11.1). Thus, from the proof of Lemma 11.2 we see that

$$\angle p = \angle q = \frac{\pi}{2}$$

holds for all $p, q \in \text{Cut}(\partial X)$.

**Lemma 11.4** \textit{Cut}(\partial X) \textit{is totally geodesic}.

**Proof.** Let $p, q$ be any mutually distinct points of \text{Cut}(\partial X), and fix them. Moreover, let $\gamma : [0, d(p, q)] \rightarrow X$ denote a minimal geodesic segment emanating from $p$ and $q$. If we prove that

$$\gamma(t) \in \text{Cut}(\partial X)$$

for all $t \in [0, d(p, q)]$, then our proof is complete. Suppose that

$$\gamma(t_0) \notin \text{Cut}(\partial X) \hspace{1em} (11.11)$$

for some $t_0 \in (0, d(p, q))$. By (11.9), we have that

$$d(\partial X, \gamma(t_0)) \neq \frac{\ell}{2}, \hspace{1em} (11.12)$$

and that

$$d(\partial X, p) = d(\partial X, q) = \frac{\ell}{2}. \hspace{1em} (11.13)$$
The equations (11.10) and (11.12) imply that
\[ d(\partial X, \gamma(t_0)) < \frac{\ell}{2}. \tag{11.14} \]

Without loss of generality, we may assume that
\[ d(\partial X, \gamma(t_0)) = \min\{d(\partial X, \gamma(t)) \mid 0 \leq t \leq d(p, q)\}. \tag{11.15} \]

By Remark 11.3, (11.11), and (11.15), we obtain the open triangle \((\partial X, p, \gamma(t_0))\) satisfying
\[ \angle p = \frac{\pi}{2}, \quad \angle \gamma(t_0) = \frac{\pi}{2}. \tag{11.16} \]

From Theorem 8.4, (11.13), (11.14), and (11.16), we thus get an open triangle \((\partial \mathbb{R}_+^2, \tilde{p}, \tilde{\gamma}(t_0))\) in \(\mathbb{R}_+^2\) corresponding to the triangle \((\partial X, p, \gamma(t_0))\) such that
\[ d(\partial \mathbb{R}_+^2, \tilde{p}) = \frac{\ell}{2}, \quad d(\partial \mathbb{R}_+^2, \tilde{\gamma}(t_0)) < \frac{\ell}{2}, \]
and that
\[ \angle \tilde{p} \leq \frac{\pi}{2}, \quad \angle \tilde{\gamma}(t_0) \leq \frac{\pi}{2}. \]

This is a contradiction, since our model is \(\mathbb{R}^2_+\). Therefore, \(\gamma(t) \in \text{Cut}(\partial X)\) holds for all \(t \in [0, d(p, q)]\). \(\square\)

**Lemma 11.5** For each \(t \in (0, \ell/2)\), the level set
\[ H_i(t) := \{ p \in X \mid d(\partial X_i, p) = t \}, \quad i = 1, 2, \]

is totally geodesic, and \(H_1(t)\) is totally geodesic for all \(t \in (0, \ell)\).

**Proof.** Take any \(t \in (0, \ell/2)\), and fix it. Let \(p, q\) be any mutually distinct points in \(H_1(t)\), and also fix them. Let \(\mu_1, \mu_2 : [0, \ell] \rightarrow X\) denote minimal geodesic segment emanating from \(\partial X_1\) to \(\partial X_2\) and passing through \(\mu_1(t) = p, \mu_2(t) = q\), respectively. Thus, we have an open triangle \((\partial X_1, p, q) = (\gamma_t, \mu_1|_{[0, t]}, \mu_2|_{[0, t]}),\) where \(\gamma_t : [0, d(p, q)] \rightarrow X\) denotes a minimal geodesic segment emanating from \(p\) to \(q\). If we prove
\[ \angle p = \angle q = \frac{\pi}{2}, \tag{11.17} \]
then we see, by similar argument in the proof of Lemma 11.4, that \(H_1(t)\) is totally geodesic. Thus, we will prove (11.17) in the following.

By Theorem 8.4, there exists an open triangle \((\partial \mathbb{R}_+^2, \tilde{p}, \tilde{q}) = (\tilde{\gamma}_t, \tilde{\mu}_1|_{[0, t]}, \tilde{\mu}_2|_{[0, t]}\) in \(\mathbb{R}_+^2\) corresponding to the triangle \((\partial X_1, p, q)\) such that
\[ d(\partial \mathbb{R}_+^2, \tilde{p}) = d(\partial \mathbb{R}_+^2, \tilde{q}) = t, \quad d(\tilde{p}, \tilde{q}) = d(p, q) \tag{11.18} \]
and that
\[ \angle p \geq \angle \tilde{p}, \quad \angle q \geq \angle \tilde{q}. \tag{11.19} \]
Since our model is $\mathbb{R}^2_+$, the equation $d(\partial \mathbb{R}^2_+, \hat{p}) = d(\partial \mathbb{R}^2_+, \hat{q})$ of (11.18) implies that
\[ \angle \hat{p} = \angle \hat{q} = \frac{\pi}{2}. \] (11.20)
Thus, by (11.19) and (11.20), we have
\[ \angle p \geq \frac{\pi}{2}, \quad \angle q \geq \frac{\pi}{2}. \] (11.21)
On the other hand, by Lemma 11.4, $\text{Cut}(\partial X)$ is totally geodesic, i.e., all eigenvalues of the shape operator of $\text{Cut}(\partial X)$ are 0 in the vector normal to $\text{Cut}(\partial X)$. Since the radial vector of any $\text{Cut}(\partial X)$-segment is parallel to that of a $\partial X$-segment, $\text{Cut}(\partial X)$ has also non-negative radial curvature. Therefore, we can apply Theorem 8.4 to the open triangle $(\text{Cut}(\partial X), p, q) = (\gamma_t, \mu_1|_{\ell/2}, \mu_2|_{\ell/2})$.

Thus, by Theorem 8.4, there exists an open triangle $(\partial \mathbb{R}^2_+, \hat{p}, \hat{q}) = (\hat{\gamma}_t, \hat{\mu}_1|_{\ell/2}, \hat{\mu}_2|_{\ell/2})$ in $\mathbb{R}^2_+$ corresponding to the triangle $(\text{Cut}(\partial X), p, q)$ such that
\[ d(\partial \mathbb{R}^2_+, \hat{p}) = d(\partial \mathbb{R}^2_+, \hat{q}) = \ell/2 - t, \quad d(\hat{p}, \hat{q}) = d(p, q) \] (11.22)
and that
\[ \pi - \angle p \geq \angle \hat{p}, \quad \pi - \angle q \geq \angle \hat{q}. \] (11.23)
As well as above, the equations (11.22) and (11.23) imply
\[ \pi - \angle p \geq \frac{\pi}{2}, \quad \pi - \angle q \geq \frac{\pi}{2}, \] since our model is $\mathbb{R}^2_+$. Thus, we have
\[ \angle p \leq \frac{\pi}{2}, \quad \angle q \leq \frac{\pi}{2}. \] (11.24)
By (11.21) and (11.24), we therefore get (11.17). By the same argument above, one may prove that $H_2(t)$ is also totally geodesic for all $t \in (0, \ell/2)$. Since $H_1(t) = H_2(\ell - t)$, $H_1(t)$ is totally geodesic for all $t \in (0, \ell)$.

**Theorem 11.6** Let $(X, \partial X)$ be a complete connected Riemannian $n$-dimensional manifold $X$ with disconnected smooth compact convex boundary $\partial X$ whose radial curvature is bounded from below by 0. Then, $X$ is isometric to $[0, \ell] \times \partial X_1$ with Euclidean product metric of $[0, \ell]$ and $\partial X_1$, where $\partial X_1$ denotes a connected component of $\partial X$. In particular, $\partial X_1$ is the soul of $X$. □
Proof. Let \( \Phi : [0, \ell] \times \partial X_1 \rightarrow X \) denote the map defined by
\[
\Phi(t, p) := \exp_{\perp}(tv_p),
\]
where \( v_p \) denotes the inward pointing unit normal vector to \( \partial X_1 \) at \( p \in \partial X_1 \). We will prove that the \( \Phi \) is an isometry. From Lemma 11.2, it is clear that \( \Phi \) is a diffeomorphism.

Let \( \mu_1 : [0, \ell] \rightarrow X \) denote any minimal geodesic segment emanating from \( \partial X_1 \) to \( \partial X_2 \), and fix it. Choose a minimal geodesic segment \( \mu_2 : [0, \ell] \rightarrow X \) emanating from \( \partial X_1 \) to \( \partial X_2 \) sufficiently close \( \mu_1 \), so that, for each \( t \in (0, \ell) \), \( \mu_1(t) \) is joined with \( \mu_2(t) \) by a unique minimal geodesic segment \( \gamma_t \). Since each level hypersurface \( H_1(t) \) is totally geodesic by Lemma 11.5, \( \gamma_t \) meets \( \mu_1 \) and \( \mu_2 \) perpendicularly at \( \mu_1(t) \) and \( \mu_2(t) \), respectively. Therefore, by the first variation formula,
\[
\frac{d}{dt}d(\mu_1(t), \mu_2(t)) = 0,
\]
holds for all \( t \in (0, \ell) \). Thus, \( d(\mu_1(t), \mu_2(t)) = d(\mu_1(0), \mu_2(0)) \) holds for all \( t \in [0, \ell] \). This implies that
\[
\left\| d\Phi_{(t, p)} \left( \frac{\partial}{\partial x_i} \right) \right\| = \left\| d\Phi_{(0, p)} \left( \frac{\partial}{\partial x_i} \right) \right\| \quad (11.25)
\]
for all \( t \in [0, \ell] \). Here \( (x_1, x_2, \ldots, x_{n-1}) \) denotes a system of local coordinates around \( p : = \mu_1(0) \) with respect to \( \partial X_1 \). Since
\[
d\Phi_{(0, p)} \left( \frac{\partial}{\partial x_i} \right) = \left( \frac{\partial}{\partial x_i} \right)_{(0, p)},
\]
we get, by (11.25),
\[
\left\| d\Phi_{(t, p)} \left( \frac{\partial}{\partial x_i} \right) \right\| = \left\| \left( \frac{\partial}{\partial x_i} \right)_{(0, p)} \right\| = \left\| \left( \frac{\partial}{\partial x_i} \right)_{p} \right\|. \quad (11.26)
\]
It is clear that
\[
d\Phi_{(t, p)} \left( \frac{\partial}{\partial x_i} \right) \perp d\Phi_{(t, p)} \left( \frac{\partial}{\partial x_0} \right), \quad i = 1, 2, \ldots, n - 1, \quad (11.27)
\]
and
\[
\left\| d\Phi_{(t, p)} \left( \frac{\partial}{\partial x_0} \right) \right\| = 1 \quad (11.28)
\]
for all \( t \in [0, \ell] \). Here \( x_0 \) denotes the standard local coordinate system for \([0, \ell] \). By (11.26), (11.27), (11.28), \( \Phi \) is an isometry. \( \square \)

Remark 11.7 Notice that non-negative radial curvature does not always mean non-negative sectional curvature (cf. [KT2, Example 5.6]). Although Theorem 11.6 extends one of Burago and Zalgaller’ splitting theorems to a wider class of metrics than those described in [BZ, Theorem 5.2.1], Ichida [I] and Kasue [K] obtain the same conclusion of Theorem 11.6 under weaker assumptions, i.e., the mean curvature (with respect to the inner normal direction) of boundary are non-negative, and that Ricci curvature is non-negative everywhere.
12 Appendix

In this section, we will give the proof of Lemma 10.4, where we do not demand any curvature assumption on a manifold.

**Lemma 12.1** Let $(X, \partial X)$ be a complete connected Riemannian $n$-manifold $X$ with smooth boundary $\partial X$. For any $q \in \text{Cut}(\partial X) \cap (X \setminus \partial X)$ and any $\varepsilon > 0$, there exists a point in $\text{Cut}(\partial X) \cap B_{\varepsilon}(q)$ which admits at least two $\partial X$-segments.

**Proof.** Suppose that the cut point $q$ admits a unique $\partial X$-segment $\mu_q$ to $q$. Then, $q$ is the first focal point of $\partial X$ along $\mu_q$. For each $p \in \partial X$, we denote by $v_p$ the inward pointing unit normal vector to $\partial X$ at $p \in \partial X$. And let $U$ be a sufficiently small open neighborhood around $d(\partial X, q)\mu_q'(0)$ in the normal bundle $\mathcal{N}_{\partial X}$ of $\partial X$, so that there exists a number $\lambda(v_p) \in (0, \infty)$ such that $\exp_{\perp}^\perp(\lambda(v_p)v_p)$ is the first focal point of $\partial X$ for each $\lambda(v_p)v_p \in U$. Set

$$ k := \liminf_{v_p \to \mu_q'(0)} \nu(v_p), $$

where $\nu(v_p) := \dim \ker(d \exp_{\perp}^\perp)_{\lambda(v_p)v_p}$. Since $U$ is sufficiently small, we may assume that $\nu(v_p) \geq k$ on $U_k := \{w/\|w\| \in U \}$. It is clear that, for each integer $m \geq 0$, the set

$$ \{v_p \in U_k | \text{rank}(d \exp_{\perp}^\perp)_{\lambda(v_p)v_p} \geq m\} $$

is open in $U_k$. Hence, by [IT2] Lemma 1, $\lambda$ is smooth on the open set

$$ \{v_p \in U_k | \nu(v_p) \leq k\} = \{v_p \in U_k | \nu(v_p) = k\} \subset U_k. $$

Since

$$ (d \exp_{\perp}^\perp)_{\lambda(v_p)v_p} : T_{\lambda(v_p)v_p} \mathcal{N}_{\partial X} \to T_{\exp_{\perp}^\perp(\lambda(v_p)v_p)}X $$

is a linear map depending smoothly on $v_p \in U_k$, there exists a non-zero vector field $W$ on $U_k$ such that

$$ W_{v_p} \in \ker(d \exp_{\perp}^\perp)_{\lambda(v_p)v_p} $$

on $U_k$. Here, we assume that

$$ \ker(d \exp_{\perp}^\perp)_{\lambda(v_p)v_p} \subset T_{v_p}U_k $$

by the natural identification.

Assume that that there exists a sequence $\{\mu_i : [0, \ell_i] \to X\}$ of $\partial X$-segments convergent to $\mu_q$ such that $\mu_i(\ell_i) \in \text{Cut}(\partial X)$ and $\mu_i(\ell_i) \notin \text{Foc}(\partial X)$ along $\mu_i$. Then it is clear that each $\mu_i(\ell_i)$ admits at least two $\partial X$-segments. Hence, we have proved our lemma in this case.
Assume that
\[ \exp^\perp(\lambda(v_p)v_p) \in \text{Cut}(\partial X) \]
for all \( v_p \in \mathcal{U}_\lambda \). Let \( \sigma(s), \ s \in (-\delta, \delta), \) be the local integral curve of \( W \) on \( \mathcal{U}_\lambda \) with \( \mu_q'(0) = \sigma(0) \). Hence,
\[ (d\exp^\perp)_{\lambda(\sigma(s))}\sigma(s)(\sigma'(s)) = 0 \]
on \( (-\delta, \delta) \). From [IT1, Lemma 1], it follows from that
\[ \exp^\perp(\lambda(\sigma(s))\sigma(s)) = \exp^\perp(\lambda(\sigma(0))\sigma(0)) = q. \]
Hence \( q \) is a point in \( \text{Cut}(\partial X) \) admitting at least two \( \partial X \)-segments. \( \square \)

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