RELATIVITY WITHOUT RELATIVITY

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We give a derivation of general relativity (GR) and the gauge principle that is novel in presupposing neither spacetime nor the relativity principle. We consider a class of actions defined on superspace (the space of Riemannian 3-geometries on a given bare manifold). It has two key properties. The first is symmetry under 3-diffeomorphisms. This is the only postulated symmetry, and it leads to a constraint linear in the canonical momenta. The second property is that the Lagrangian is constructed from a ‘local’ square root of an expression quadratic in the velocities. The square root is ‘local’ because it is taken before integration over 3-space. It gives rise to quadratic constraints that do not correspond to any symmetry and are not, in general, propagated by the Euler–Lagrange equations. Therefore these actions are internally inconsistent. However, one action of this form is well behaved: the Baierlein–Sharp–Wheeler (BSW [1]) reparametrisation-invariant action for GR. From this viewpoint, spacetime symmetry is emergent. It appears as a ‘hidden’ symmetry in the (underdetermined) solutions of the Euler-Lagrange equations, without being manifestly coded into the action itself. In addition, propagation of the linear diffeomorphism constraint together with the quadratic square-root constraint acts as a striking selection mechanism beyond pure gravity. If a scalar field is included in the configuration space, it must have the same characteristic speed as gravity. Thus Einstein causality emerges. Finally, self-consistency requires that any 3-vector field must satisfy Einstein causality, the equivalence principle and, in addition, the Gauss constraint. Therefore we recover the standard (massless) Maxwell equations.

I. INTRODUCTION

Traditionally the 3+1 dynamical formulation of GR, with its associated Hamiltonian and Lagrangian structures, is obtained by projection from spacetime. Geometrodynamics has a four-dimensional symmetry from the start. In contrast, the classic study of Hojman, Kuchař, and Teitelboim [2] starts with a three-dimensional fully constrained Hamiltonian and requires its constraint algebra to close by reproducing the standard Dirac algebra for GR. In this way, they recover the ADM Hamiltonian. However, by insisting on the specific structure ‘constants’ of the Dirac algebra, they still presuppose spacetime. In fact, this is unnecessary. We shall show that mere closure of the algebra is enough to obtain GR.

We shall begin by explaining how the Baierlein–Sharp–Wheeler action, which is at the centre of our investigation, arose from the ADM formalism [3]. In the ADM approach, one starts from spacetime (a pseudo-Riemannian 4-manifold). One observes that the ‘surfaces’ of constant label time are spacelike (Riemannian) 3-geometries. Geometrodynamics is the evolution of these 3-geometries. Choosing 3-coordinates, one gets a smooth curve of 3-metrics, \( g_{ij}(x^i, t) \), \( i, j = 1, 2, 3 \) with \( x^i \) the coordinates. The \( g_{ij} \)’s depend on the slice (labelled by \( t \)), on the coordinates and on the point on the slice. The coordinates can be transformed because only the 3-geometry matters, not the 3-metric. Moreover, the transformation can be changed freely from slice to slice as well as from point to point.

The key concepts in the ADM formalism are the 3-metrics \( g_{ij} \), the lapse \( N \) and the shift \( N^i \). The lapse measures the rate of change of proper time w.r.t. the label time, while the shift determines how the coordinates are laid down on the successive 3-geometries. Prior to the transition to the Hamiltonian, the standard Hilbert–Einstein action for matter-free GR is rewritten, after divergence terms have been omitted, in the 3+1 form

\[
I = \int \sqrt{g} N \left[ R + K^{ij} K_{ij} - \text{tr} K^2 \right] d^3 x. \tag{1.1}
\]

Here \( R \) is the three-dimensional scalar curvature, and \( K_{ij} = -(1/2N)(\partial g_{ij}/\partial t - N_{ij} - N_{ji}) \) is the extrinsic curvature. From here the transition made by BSW [1] is trivial. They first replaced \( K_{ij} \) in the action by \( k_{ij} = \partial g_{ij}/\partial t - N_{ij} - N_{ji} \), the unnormalised normal derivative, to give

\[

1
\]
\[
I = \int \sqrt{g} \left[ NR + \frac{1}{4N} (k^{ij} k_{ij} - \text{tr} k^2) \right] \, d^4x. \tag{1.2}
\]

They varied this action with respect to the lapse and found an algebraic expression for it,

\[
N = \sqrt{\frac{k^{ij} k_{ij} - \text{tr} k^2}{4R}}. \tag{1.3}
\]

This, of course, is clearly consistent with the ADM Hamiltonian constraint. In turn, this expression for \( N \) is substituted back into Eq.(1.1) to obtain the BSW Lagrangian

\[
I = \int \sqrt{g} \sqrt{R} \sqrt{k^{ij} k_{ij} - \text{tr} k^2}. \tag{1.4}
\]

There is a large class of four-dimensionally generally covariant theories of matter fields coupled to gravity in which the Lagrangian is a sum of potential-type terms (like \( R \)) + quadratic products of the ‘proper’ time derivatives, all multiplied by \( N \). In all of them, one can repeat the process above, solve for the lapse algebraically and pass trivially to the BSW form.

The BSW action has attracted relatively little attention. It was initially proposed, especially by Wheeler, as the starting point of a method of solving the initial-value problem of GR. The original BSW paper formuluates the ‘thin-sandwich problem’, which will be mentioned later. However, in this paper, we propose to take (1.4) as paradigmatic for the dynamics of the universe. We shall recall some basic concepts and identify the two key properties of BSW-type actions. Then, in view of its relative unfamiliarity, we shall give a fairly extended outline of our approach, which we call the 3-space approach.

Wheeler \cite{wheeler} called the set of Riemannian 3-geometries on a given topology superspace, which is the ADM configuration space and the space we shall use here. However, we actually believe that conformal superspace, which is obtained by quotienting superspace by three-dimensional conformal transformations, may well be the true configuration space for gravity \cite{conformal}, but in this paper we work in superspace. The fundamental object we consider is a ‘curve in superspace’, a sequence of 3-geometries, labelled by some parameter \( \lambda \). We use \( \lambda \) rather than \( t \) because in our approach time emerges from geodesic-type curves on configuration spaces. This timeless approach is motivated below and in \cite{conformal}.

For well-known technical reasons, we cannot work directly with superspace; to integrate and perform other key mathematical operations, we must introduce coordinates. Therefore the actual objects we consider are curves of 3-metrics \( g_{ij}(x^i, \lambda) \), for which we define an action \( A \). It must not depend on the choice of coordinates, i.e., \( A \) must be truly a function on curves in superspace rather than just on curves in Riem, the space of 3-metrics. This leads us to the first key property of the BSW-type actions.

We know that \( A \) will depend on \( g_{ij} \), its spatial derivatives, and also on its ‘velocity’ \( \partial g_{ij}/\partial \lambda \). Moreover, \( A \) must be coordinate invariant. In the case of \( \lambda \)-independent transformations, all we need ensure is that the integrand of \( A \) is a 3-scalar, i.e., that we use covariant, rather than ordinary, derivatives and that the indices upstairs and downstairs match. However, we must also consider \( \lambda \)-dependent transformations, i.e., ones that differ from slice to slice. They have the form \( x^i \to x^i(x^j, \lambda) \) with inverse \( x^j(x^i, \lambda) \). We now find

\[
\frac{\partial g_{ij}}{\partial \lambda} = \frac{\partial x^a}{\partial x^i} \frac{\partial x^b}{\partial x^j} \left( \frac{\partial g_{ab}}{\partial \lambda} - \xi_{a;b} - \xi_{b;a} \right)
\]

where \( \xi^a = -\partial x^a/\partial \lambda \), so the \( \lambda \) derivative does not transform as a 3-tensor. This is a general problem that arises whenever the ‘points’ of a configuration space have an internal (gauge) symmetry. Indeed, it will be argued in \cite{conformal} that this is the defining characteristic of any gauge theory. In the present case, the ‘gauge’ corresponds to coordinate transformations on space. We overcome this problem by having the metric velocity always appear as part of the combination \( k_{ab} = \partial g_{ab}/\partial \lambda - \xi_{a;b} - \xi_{b;a} \), where we regard \( \xi^i \) as an arbitrary 3-vector field and minimize the action with respect to it as well as with respect to the metric. This process of adjusting for the gauge freedom is universal and is appropriately called best matching. It was introduced to implement the conviction of Leibniz, Mach and many others that motion is relative \cite{relativity}. We will write \( \xi_{a;b} + \xi_{b;a} = (K \xi)_{ab} \), because this expression is the Killing form of the vector \( \xi^a \). It is also the Lie derivative of \( g_{ij} \) along \( \xi^i \). A further point is that we are constructing geodesic-type actions, which requires \( A \) to be reparametrisation invariant (see below). This will be guaranteed because \( \xi^i \) is actually a velocity, so that the combination \( k_{ij} = \partial g_{ab}/\partial \lambda - \xi_{a;b} - \xi_{b;a} \) is linear in \( 1/\partial \lambda \).

Initially, we consider actions which depend only on the metric and its velocity (and the vector \( \xi \)). We will later extend the configuration space to include a scalar function \( \phi(x^i, \lambda) \) and a 3-vector field \( A^i(x^i, \lambda) \). The velocities of these fields will also include best-matching corrections. Since they are based on the Lie derivative, all such correction
terms have a universal nature. However, their universal common origin manifests itself in remarkably different ways because each different bosonic field has its own different Lie derivative. Eqns. (1.1)–(1.4) together with the expression for the extrinsic curvature show how best matching is implemented in GR through the Lie derivative. We shall show how this fact coupled with the second key property of BSW-type actions leads to the novel derivation of GR.

We now turn to this second key property of BSW actions, the square roots, which make them similar to the action of Jacobi’s principle of classical mechanics. In fact, as Lanczos notes in his beautiful book on the variational principles of mechanics [3], Jacobi-type square-root actions were effectively used by Euler and Lagrange in their application of Maupertuis’s principle. However, it was Jacobi who clarified the mathematics and cleanly separated two different principles. The first is Hamilton’s principle based on the familiar $T - V$ Lagrangian. It gives dynamical curves in configuration space together with the speed at which they are traversed with respect to an independent external time variable. The second is the principle now known as Jacobi’s principle, which merely gives the dynamical orbit in configuration space independently of the speed at which it is traversed. It is, in fact, a geodesic principle based on the square root of a form quadratic in the particle displacements. In the Kepler problem, it yields the elliptical planetary orbits. We shall be interested in their generalization to general dynamical systems.

It is important to distinguish between dynamical orbits determined by a Jacobi-type principle and group orbits, generated in the configuration space by the action of a gauge group. For us, the most important example of a group orbit arises from coordinate transformations on our 3-spaces. These correspond to 3-diffeomorphisms. All 3-metrics in $\text{Riem}$ that can be carried into each other by 3-diffeomorphisms lie on one orbit of the three-dimensional diffeomorphism group. Since the notion of group orbits is well known, it is enough to mention it.

However, it is necessary to say why Jacobi-type actions are needed in the attempt to treat the universe as an isolated dynamical system, as we do here. Newton took time as an independent existent. But it is read from clocks isolated dynamical system that can be carried into each other by 3-diffeomorphisms lie on one orbit of the three-dimensional diffeomorphism group. All 3-metrics in $\text{Riem}$ that can be carried into each other by 3-diffeomorphisms lie on one orbit of the three-dimensional diffeomorphism group. Since the notion of group orbits is well known, it is enough to mention it.

In Sec. 2, we present Jacobi’s principle for point particles. The Jacobi action is the product of the square root of a potential term multiplied by the square root of a kinetic term. We shall discuss two different Jacobi-type actions. One is the standard, ‘good’, one; the other is a nonstandard, ‘bad’, one. In the first, the quadratic kinetic terms of the particles are summed, and then we take the square root of the sum. In the second, we take the root before we sum. We have found, to our surprise, that the ‘bad’ choice, which is realized in the BSW action (1.4), is a good idea. It leads to a massively over-constrained action but, almost by magic, selects GR, in the BSW form, as the only nontrivial case that works. We should like to mention here that, so far as we know, the first person to realise that idea. It leads to a vastly over-constrained action but, almost by magic, selects GR, in the BSW form, as the only nontrivial case that works. We should like to mention here that, so far as we know, the first person to realise that idea. It leads to a vastly over-constrained action but, almost by magic, selects GR, in the BSW form, as the only nontrivial case that works. We should like to mention here that, so far as we know, the first person to realise that idea. It leads to a vastly over-constrained action but, almost by magic, selects GR, in the BSW form, as the only nontrivial case that works. We should like to mention here that, so far as we know, the first person to realise that idea. It leads to a vastly over-constrained action but, almost by magic, selects GR, in the BSW form, as the only nontrivial case that works. We should like to mention here that, so far as we know, the first person to realise that idea. It leads to a vastly over-constrained action but, almost by magic, selects GR, in the BSW form, as the only nontrivial case that works. We should like to mention here that, so far as we know, the first person to realise that idea. It leads to a vastly over-constrained action but, almost by magic, selects GR, in the BSW form, as the only nontrivial case that works. We should like to mention here that, so far as we know, the first person to realise that idea.

Thus, we take a reparametrisation-invariant and coordinate-invariant Jacobi action of the form

$$A = \int d\lambda \int \sqrt{g} \sqrt{P} \sqrt{T} d^3x.$$  \hspace{1cm} (1.5)

Formally, this is defined on the Cartesian product of $\text{Riem}$ (the space of three-metrics) and the space of the fields $\xi$ considered earlier. However, the intention is to turn it into an action defined on superspace by varying w.r.t. $\xi$. The parameter $\lambda$ labels the ‘points’ on trial curves in $\text{Riem}$, $g$ is the determinant of $g_{ij}$, the ‘potential’ $P$ is a 3-scalar constructed from $g_{ij}$ and its spatial derivatives, and $T$ is a kinetic term.

We assume that $T$ is quadratic in the ‘corrected metric velocities’ $k_{ij}$. Then either the ‘good’ or the ‘bad’ square root makes the Lagrangian linear in $1/d\lambda$ and reparametrisation invariant because of the $d\lambda$ outside. Dirac [3] noted that any action that is homogeneous of degree one in the velocities will have canonical momenta that are homogeneous of degree zero. They must then necessarily satisfy a constraint that is primary, i.e., follows without any variation already from the mere form of the momenta. Now Noether’s theorem (Part 2) tells us that reparametrisation invariance, which for general actions of the form (1.3) just allows a uniform scaling of the velocities over the entire manifold, will yield one constraint. In contrast, the ‘bad’ actions of the form (1.3) generate an independent constraint at each space point. This mismatch between the constraints and the symmetry will be crucial.

We now seek stationary points of the action $A$ (1.5) as a functional of $g_{ij}$ and the gauge field $\xi^i$. We first compute $p^i$, the momentum conjugate to $g_{ij}$. This is the variation of $A$ w.r.t. $\partial g_{ij}/d\lambda$ which, in turn, is identical to the
variation w.r.t. $k_{ij}$. The variation of $A$ w.r.t. $\xi$ is the variation w.r.t. $k_{ij}$ (which is just $p^{ij}$), times the variation of $k_{ij}$ w.r.t. $\xi$, which is just the gradient of the variation of $\xi$. We integrate once and get the standard ADM diffeomorphism constraint, $p^{ij} = 0$. This result has nothing to do with the square root or any other structure of the action except the use of ‘best matching’ to implement diffeomorphism invariance. As we add extra fields, we will get extra velocities, but each will have the appropriate Lie-derivative correction with respect to the same vector $\xi$. The striking universality of the results obtained later arise because all fields in nature are subject to the same diffeomorphisms generated by the one single field $\xi$.

Variation of the terms that involve the extra non-metric fields will generate source terms in the diffeomorphism constraint, but the universality of the Lie-derivative prescription ensures that their form will be independent of the Lagrangian’s detailed structure. As a result, the diffeomorphism constraint seems to be kinematic in origin, reflecting merely the 3-tensorial nature of the considered fields. This is wrong. The diffeomorphism constraint reflects one of the two key properties of the dynamics, namely, that its purpose is to make the action depend solely on the group orbits that constitute the points of superspace.

Because the coordinates are more or less freely specifiable at each point of the 3-manifold, the resulting symmetry (an arbitrary 3-vector at each point) matches the constraint (the vanishing of a 3-tensor divergence at each point). Therefore, we have no reason to expect that the evolution will fail to propagate this constraint.

In the terminology of Bergmann adopted by Dirac [10], the diffeomorphism constraint is a secondary constraint because it arises from variation (the associated primary constraint is the identical vanishing of the momentum conjugate to $\xi$ because $A$ does not depend on $\partial \xi / \partial \lambda$). However, as noted above, there is already a nontrivial primary constraint in the system. It follows from the mere definition of $p^{ij}$ that a quadratic expression of the same basic form as the Hamiltonian constraint of GR vanishes identically at each point of the 3-manifold. However, if we put the square root needed to construct a geodesic-type action outside the integral sign, we get an integral rather than a point constraint. Then the single resulting constraint per manifold matches the one degree of freedom per manifold due to the reparametrisation invariance. There is no longer an independent constraint at each point of the manifold. This is why we say the ‘bad’ action (1.5) is over-constrained, and why it is no surprise that, in general, it does not generate sensible dynamics. We call the Hamiltonian-type constraint the square-root constraint because it arises from the placing of the square root.

Our initial motivation for studying the actions (1.5) was because the BSW [1] action for GR has this form. In Secs. IV and V we give a fairly complete account of the BSW action, especially the propagation of its constraints. We then make the Legendre transformation from velocities to momenta and rediscover the ADM Hamiltonian with its well-known complete freedom to ‘push forward the evolution of spacetime’ by different amounts across the instantaneous spacelike 3-manifold. This remarkable property of GR, which goes by various names including foliation invariance and ‘many-fingered time’, is thus the ‘hidden symmetry’ of the BSW action. It shows that an over-constrained system can nevertheless be viable.

Section VI contains our first new result. We consider a general action of the form (1.5) on superspace. Specifically, we assume: 1) $T$ is the most general quadratic kinetic form allowed by a natural locality requirement; and 2) the potential term $P$ is an arbitrary scalar function of $g_{ij}$ and its spatial derivatives. We check the propagation of the constraints by the equations of motion and find that in general the square-root constraint is not propagated. We need a ‘hidden symmetry’ in the Euler–Lagrange equations to match the extra constraints on the initial data. From this over-constrained perspective, it is remarkable that nevertheless there exists a handful of actions (3) that do propagate the constraints. In them, $T$ must have the standard DeWitt form (a result already obtained some years ago by Giulini [4] and $P = DR + \Lambda$, where $D$ and $\Lambda$ are constants and $R$ is the scalar 3-curvature. By an overall scaling, we can set $D = -1, 0, +1$. The cases $D = +1$ and $D = -1$ give Lorentzian and Euclidean GR, respectively, both with $\Lambda$ as a possible cosmological constant. The $D = 0$ case is called strong coupling or strong gravity and is the limit of standard GR for an infinite gravitational constant [1]. One can further argue that the $D = 1$ case is the only one for which the equations of motion can be written in hyperbolic form and thus is the only viable candidate for a dynamical system. All other actions force the 3-metric to be flat and its velocity to vanish. Unlike the square-root constraint, the diffeomorphism constraint is always propagated, as expected.

In Sec. VII, we see if the kinetic and potential terms for a scalar field $\phi$ can be added to BSW gravity in the Lorentzian case, which has an emergent ‘light cone’. Best matching uniquely determines the kinetic term, but there is wide freedom in the potential, for which we make the ansatz

$$R \rightarrow R - \frac{C}{4} g^{ab} \partial_a \phi \partial_b \phi + \sum_n A_n \phi^n. \tag{1.6}$$

This is the standard (minimal-coupling) potential energy (with an arbitrary constant $C$) and a polynomial with constants $A_n$. The polynomial, which could in fact be replaced by an arbitrary function of $\phi$ (but not its derivatives), gives no difficulty, but we must have $C = 1$ for the constraint algebra to close. Therefore, the canonical speed of $\phi$
must match with the speed of the metric disturbances: the scalar field must respect the gravitational light cone. This result surprised us even more than the recovery of GR from pure 3-space dynamics, especially since the manner of its derivation showed clearly that we had found a possible universal method of generating the SR light cone. (Until we extend our approach to include fermions, we cannot claim that 3-diffeomorphism invariance coupled with the local square-root is the origin of the universal light cone.)

The result we have just described shows that minimal coupling (and with it Einsteinian free fall) arises naturally in the BSW approach. In Sec. VII, we also discuss a general range of ‘dilaton’ theories and show that, in many cases, they can be written in the local square root form. Brans–Dicke theory is a special case of such theories. We express its ‘vacuum’ form as a BSW-type action. We further show that we can self-consistently couple in a massive scalar field in the Brans–Dicke frame. Therefore, it seems that Brans–Dicke theory is consistent with our approach.

Life becomes even more interesting when we try to couple in a 3-vector field, which we denote by \( A_a \) (Sec. VIII). The kinetic term is again uniquely fixed. We consider a fairly general potential energy expression for \( A_a \):

\[
U_A = C_1 A_{a;b} A^{a;b} + C_2 A_{a;b} A^{b;a} + C_3 A_{a;a} A^{a;b} + \sum_k B_k (A^a A_a)^k,
\]

where \( C_i \) and \( B_k \) are constants. We include, thus, all possible terms quadratic in the first derivative plus a general undifferentiated polynomial term. We find that, at this level, the polynomial terms do not hinder propagation of the square-root constraint but that we must have \( C_1 = -1/4, C_2 = +1/4, C_3 = 0 \). Therefore, the potential-energy term becomes \(-\langle \text{curl} A \rangle^2 / 4 \). This means that here too we recover the common light cone for the vector field from our putative universal method.

But now there is an even more striking result. As we shall see, the structure of the 3-vector field is significantly more complicated than in the scalar case. It generates not only the above three conditions on the three constants \( C_i \), but also an extra secondary constraint requiring the divergence of the vector momentum to vanish, essentially the Gauss constraint. This too must propagate, and we can easily show it does so only when all the polynomial terms vanish. This means we have derived standard Maxwell electrodynamics. We find this result truly striking. It suggests that gravity, the light cone, and massless electrodynamics all arise from the local square root and the action of the three-dimensional Lie derivative.

In a companion paper [12] we show that the 3-metric field, the vector field and scalar fields can interact among themselves only in the form of complex scalar fields with U(1) gauge coupling to the vector field and both scalar fields minimally coupled to gravity. Thus we obtain classical gauge theory. A paper just completed by Edward Anderson and one of us [13] shows that in the BSW framework 3-vector fields can interact among themselves only as Yang–Mills fields minimally coupled to gravity.

The 3-space approach uses nothing but manifold geometry, the mathematics that arose when Euclid’s fifth axiom was seen to be independent and could be omitted. It uses topology and bosonic fields: scalar, vector, and 3-metric. As geometrical objects, these are on a par. But their individual properties lead to a hierarchy. The 3-metric is first among equals. Through its covariant derivative and determinant \( g \), it creates tensor calculus and the densities that permit integration and variation. It creates the highroad of dynamics [4]. Moreover, the Lie derivatives impose a chain that arises from the intricate core of the metric Killing form: gravity, light cone, gauge theory. The effectiveness of geometry in establishing this structure and logic in dynamics is impressive. Galileo said: “He who attempts natural philosophy without geometry is lost.” He meant 3-geometry. Was spacetime an accident?

## II. SQUARE ROOT ACTIONS

We first formulate Newtonian particle mechanics without time, using Jacobi’s principle. It describes the orbits of conservative dynamical systems in configuration space without reference to motion. Jacobi’s principle is very well known to N-body specialists, who geometrize motion in configuration space by exploiting the Riemannian metric defined there by the kinetic energy of conservative mechanical systems [8]. However, it is virtually unknown among relativists, who use spacetime to geometrize motion. We have here a ‘cultural divide’ that we hope the present paper, which suggests that the configuration space is more fundamental than spacetime, will help to overcome.

For \( N \) particles, labelled by \( (m) = 1, \ldots, N \), of masses \( m_{(m)} \) moving in a potential \( V(x^1, x^2, \ldots, x^N) \), the Jacobi action is [8]

\[
I_G = 2 \int \sqrt{-\nabla V} \sqrt{T} d\lambda,
\]
where the constant $E$ is the total energy, $V$ is the potential energy, $\lambda$ labels the points on trial curves and

$$T = \sum_{(m)=1}^N \frac{m_{(m)} \, dx_{(m)i}^i \, dx_{(m)i}}{2 \, d\lambda \, d\lambda}$$

is the kinetic energy but with $\lambda$ in place of Newtonian time. The subscript G on $I_G$ distinguishes the ‘good’ action from the ‘bad’ one introduced later. The action (2.1) is timeless since the label $\lambda$ could be omitted and the mere displacements $dx_{(m)i}$ employed, as is reflected in the invariance of $I_G$ under the reparametrization

$$\lambda \to f(\lambda). \quad (2.2)$$

Jacobi’s principle describes all Newtonian motions of one $E$ as geodesics on configuration space. Its square roots are characteristic, indeed essential, and are central in the 3-space approach. They fix the structure of the action’s canonical momenta, which are

$$p_{(m)i} = \frac{\delta L}{\delta \left( \frac{dx_{(m)i}}{d\lambda} \right)} = m_{(m)} \sqrt{E - V} \frac{dx_{(m)i}}{d\lambda} \quad (2.3)$$

Because $T$ (quadratic in the velocities) occurs under the square root in the denominator while the velocity occurs linearly in the numerator, the canonical momenta are homogeneous of degree zero. They resemble direction cosines, which, if squared and added, give 1. We have

$$\frac{p_{(m)i} \, p_{(m)i}}{2m_{(m)}} = \frac{E - V \, m_{(m)} \, dx_{(m)i}^i \, dx_{(m)i}}{2 \, d\lambda \, d\lambda}.$$  

If we sum the kinetic energies, they give $T$, cancelling the numerator’s $T$. This gives the reparametrisation, or square-root, identity

$$\sum_{(m)=1}^N \frac{p_{(m)i} \, p_{(m)i}}{2m_{(m)}} = \frac{E - V}{T} \times T = E - V. \quad (2.4)$$

Equation (2.4) seems to express energy conservation, but it is actually an identity true on all curves in the configuration space and not merely on-shell. In the Hamiltonian formalism, (2.4) becomes a quadratic constraint.

The Euler–Lagrange equations are

$$\frac{dp_{(m)i}}{d\lambda} = \frac{\delta L}{\delta x_{(m)i}} = -\sqrt{\frac{T}{E - V}} \, d_{(m)i} V, \quad (2.5)$$

where $\lambda$ is still arbitrary. The solutions of Eq. (2.5) are, as they must be, parametrised curves in configuration space. Within generalized Hamiltonian dynamics, there is no guarantee that the Euler–Lagrange equations used to propagate the canonical momenta will do so in such a way that the identity (2.4) will be maintained. This is why (2.4), and any other identity like it, is to be regarded as a constraint whose propagation must always be checked. In the present case, the Euler–Lagrange equations do conserve the constraint:

$$\sum_{(m)=1}^N \frac{p_{(m)i} \, dp_{(m)i}}{m_{(m)} \, d\lambda} = -\sum_{(m)=1}^N \frac{p_{(m)i} \, \sqrt{\frac{T}{E - V}} \, d_{(m)i} V}{m_{(m)}} = -\sum_{(m)=1}^N \sqrt{\frac{E - V \, dx_{(m)i}^i \, dx_{(m)i}}{T \, \sqrt{E - V}}} \, d_{(m)i} V = -\frac{dV}{d\lambda}. \quad (2.6)$$

But this is not the end since $\lambda$ is still free. If we choose it such that

$$\frac{T}{E - V} = 1 \Rightarrow T = E - V \quad (2.7)$$

then Eqs. (2.3) and (2.8) become

$$p_{(m)i} = m_{(m)} \frac{dx_{(m)i}^i}{d\lambda}, \quad \frac{dp_{(m)i}}{d\lambda} = -d_{(m)i} V. \quad (2.8)$$
and we recover Newton’s second law with respect to this special \( \lambda \), which has the same properties as Newton’s absolute time. However, Eq. (2.7), which is usually taken to express energy conservation, becomes, in the absence of an external time, the definition of ‘Newtonian’ time. Indeed, this emergent time is the astronomers’ operational ephemeris time \( \tilde{\mathbf{E}} \).

We now consider the ‘bad’ (suffix B) Jacobi action with square root and summation swapped:

\[
I_B = 2 \int \sqrt{E-V} \sum_{(m)=1}^N \sqrt{T_{(m)} d\lambda},
\]

where \( E, V \) and \( \lambda \) have their previous meanings, and

\[
T_{(m)} = \frac{m_{(m)}}{2} \frac{dx_{(m)i}^l}{d\lambda} \frac{dx_{(m)i}}{d\lambda}
\]

is the \( m^{th} \) particle’s kinetic energy.

We will generalise this slightly by bringing the ‘\( E-V \)’ inside the summation sign. Then each particle can have its own energy and potential, and we have

\[
I_B' = 2 \int \sum_{(m)=1}^N \sqrt{E_{(m)} - V_{(m)}} \sqrt{T_{(m)} d\lambda},
\]

In Jacobi’s action, the ‘distance’ in configuration space is an \( N \)-dimensional Pythagorean sum, i.e., the square root of a sum of squares (with mass-weighted ‘legs’). In the new action, we add the \( ds \)’s along a line. In the ‘good’ \( ds \), the change due to a change in one leg depends also on what all the other particles are doing. This linkage is not present in the ‘bad’ \( ds \).

Now to the point. Instead of just one in the ‘good’ case, there are now \( N \) independent identities, one for each particle. They arise just due to the placing of the square root. For this one core mathematical relationship, we shall use various names: reparametrization or square-root identity; Hamiltonian, quadratic or square-root constraint. The new identities are

\[
\frac{p_{(m)i}^l}{2m_{(m)}} = \frac{E_{(m)} - V_{(m)}}{T_{(m)}} \times T_{(m)} = E_{(m)} - V_{(m)}.
\]

The Euler–Lagrange equations

\[
\frac{dp_{(m)i}^l}{d\lambda} = \frac{\delta \mathcal{L}}{\delta x_{(m)i}^l} = - \sum_{(m)=1}^N \frac{\sqrt{T_{(m)} d_{(m)}^l V_{(m)}}}{\sqrt{E_{(m)} - V_{(m)}}}
\]

do not now in general preserve the constraints (2.12). There are extra secondary constraints, one for each particle:

\[
\frac{d}{d\lambda} \left[ \frac{p_{(m)i}^l p_{(m)i}^l}{2m_{(m)}} - E_{(m)} + V_{(m)} \right] = p_{(m)i} \frac{dp_{(m)i}^l}{d\lambda} + \frac{dp_{(m)i}^l}{d\lambda} + \frac{dV_{(m)}}{d\lambda} = \frac{p_{(m)i}}{m_{(m)}} \sum_{(n)=1}^N \frac{\sqrt{T_{(n)} d_{(n)}^l V_{(n)}}}{\sqrt{E_{(n)} - V_{(n)}}} + \sum_{(n)=1}^N \frac{d_{(n)}^l V_{(m)}}{m_{(n)}} \sqrt{E_{(m)} - V_{(m)}}.
\]

Here, we inverted the expression (2.11) for the momenta to find \( dx_{(m)i}/d\lambda \) and substituted to obtain the last line. This does not vanish identically and thus is a new secondary constraint. We cannot have such proliferation, so are there any cases for which the expressions (2.14) vanish?

The easiest way to understand their implications is to consider special initial data. Suppose particle (1) is moving and all the others are instantaneously at rest. Take Eq. (2.14) for \( (m) = (2) \). If it vanishes
This requires \( d^i_{(1)} V^{(2)} = 0 \). Taking different \((m)\) and \((n)\), we find \( d^i_{(m)} V^{(n)} = 0 \forall n \neq m\), i.e., \( V^{(n)} \) can only depend on \( x^i_{(n)} \) and not on the other particles’ coordinates. Then from Eq.(2.14) we find that all the secondary constraints are identically satisfied, with no restriction on \( V^{(n)} \). The particles can each move in their own external potential but cannot interact among themselves.

Physically, the system starts off with one global reparametrisation invariance. However, each particle wants to be independently reparametrisable. To see this, we seek an equation to propagate the kinetic energy \( T^{(m)} \) of any one particle. None exists. The direction of each particle is fixed, but its speed is arbitrary. To see this another way, note [Eq.(2.13)] that the momenta are homogeneous of degree zero in the velocities. They depend only on the direction of motion of each particle and not at all on its speed. The Euler–Lagrange equations (2.13) govern the momenta, which are directions, and thus say nothing about speeds. This is why the system makes sense only when the particles do not interact. We may also note that the local square-root action is the sum of \( N \) independent single-particle Jacobi actions. The particles cannot interact; but if they do not interact each can be independently reparametrised.

The ‘bad’ Hamiltonian is illuminating. It is just the sum of the constraints, each with a Lagrange multiplier:

\[
H_B = \sum_{(m)=1}^N N^{(m)} \left( \frac{p^{(m)i}_i}{2m^{(m)}} - E^{(m)} + V^{(m)} \right).
\]

Variation w.r.t. the multipliers \( N^{(m)} \) gives the constraints. Hamilton’s equations are

\[
\frac{dp^{(m)i}_i}{dt} = -N^{(m)} a^{(m)i}_i V^{(m)}, \quad \frac{dx^{(m)i}_i}{dt} = N^{(m)} \left( \frac{p^{(m)i}_i}{2m^{(m)}} \right).
\]

These are exactly the equations of motion [2.11] and [2.13] from \( I_B \) with the identification \( N^{(m)} = \sqrt{T^{(m)}/E^{(m)} - V^{(m)}} \). Once again the kinetic energy is seen to be undetermined.

The ‘bad’ action [2.10] evidently has interesting and important properties: The system is over-constrained and the constraints do not, in general, propagate; the ‘defect’ acts as a filter, picking out the ‘simplest’ action; the special action has a ‘hidden’ symmetry (not explicitly exhibited in the action); the ‘bad’ Hamiltonian is just a sum of the constraints. Alternatively, one could start with a Hamiltonian such as Eq.(2.16) which is a sum of constraints, each of which is quadratic in the (undifferentiated) momenta. Now all one has to do is repeat the BSW reduction procedure. First one constructs a Lagrangian by means of a Legendre transformation. This will contain the Lagrange multipliers. These are eliminated and one ends up back with Eq.(2.10). All these properties carry over to the extension of the ‘bad’ Jacobi action [2.10] to a field theory, in particular to gravity.

### III. REPARAMETRISATION-ININVARIANT ACTIONS AND GRAVITY

As we already said at the start, 40 years ago Dirac [9] and Arnowitt, Deser and Misner (ADM) [3] showed that GR could be treated as a dynamical system on the configuration space of Riemannian 3-geometries with fixed topology (called superspace by Wheeler [4]). We work in this paper with compact manifolds without boundary; the extension to more general topologies (e.g., asymptotically flat) is straightforward.

We now begin the programme outlined in the introduction. We shall consider a class of Jacobi-type actions on superspace. We will show that the requirement that the equations of motion be strongly local (in a sense to be defined) and self consistent drastically restricts the actions. In fact, standard GR with locally Lorentz-covariant matter fields that interact through the gauge principle emerges naturally. We begin with matter-free superspace and obtain pure geometrodynamics.

For definiteness, we take the compact manifold to be \( S^3 \). We will actually work in \( Riem \), the space of suitably smooth Riemannian metrics \( g_{ij}(x), x \in S^3 \). The equivalence classes \( \{g_{ij}\} \) of the metrics \( g_{ij} \) under suitably smooth 3-diffeomorphisms on \( Riem \) are the points in superspace. Consider a smooth family of smooth diffeomorphisms \( x^i(x', \theta) \), where the label \( \theta \) distinguishes the members of the family and the diffeomorphism goes to the identity map as \( \theta \to 0 \). The general infinitesimal 3-diffeomorphism of \( g_{ij}(x) \) is generated by the Killing form by

\[
\frac{\partial g_{ij}}{\partial \theta} \bigg|_{\theta = 0} = \mathcal{L}_\xi g_{ij} = (K \xi)_{ij} = \nabla_i \xi_j + \nabla_j \xi_i, \quad (3.1)
\]
where \( \xi^i(x) = \partial x'^i/\partial \theta \) evaluated at \( \theta = 0 \), and \( \mathcal{L} \) indicates the Lie derivative. The Killing form, with its multi-index covariant derivatives \( \nabla_j \), is the most intricate Lie derivative we shall encounter; propagation of its intricacy creates gravity and imposes deep universal structure on matter coupled to gravity. The Killing form drives it all through the square-root filter.

We seek a measure on superspace like the ‘bad’ Jacobi action \( (2.10) \). Consider two metrics on, say, \( S^3 \), which we call \( g^{(1)} \) and \( g^{(2)} \). We take coordinates with the same range, so that we can identify points on the two manifolds with the same coordinate values. We can compute, at each point, the difference between the metrics as \( dg_{ab} = g^{(2)}_{ab} - g^{(1)}_{ab} \), where we take \( dg \) to be small but finite. We want a difference linearly proportional to \( dg \), something like \( \sqrt{(dg)^2} \).

There are two natural squares of a symmetric two-index tensor. One is the sum of the squares of the individual elements, while the other is the square of the trace. We include both with an as yet undetermined coefficient, and, maintaining maximum generality with strong locality, we consider the object

\[
\tau = G^{abcd}[dg_{ab}][dg_{cd}],
\]

\[
G^{abcd} = g^{ac}g^{bd} - Ag^{ab}g^{cd}
\]

where \( A \) is the undetermined constant coefficient. Clearly, \( \sqrt{\tau} \) is a natural measure of the difference of \( g^{(1)} \) and \( g^{(2)} \) at one coordinate point. To find the total difference between \( g^{(1)} \) and \( g^{(2)} \), we need to integrate over \( S^3 \). To obtain a more general measure, we can modulate \( \sqrt{\tau} \) by some spatial function of the metric, which we call \( \sqrt{P} \), to give

\[
\Delta(g^{(1)}, g^{(2)}) = \int \sqrt{g} \sqrt{P} \sqrt{\tau} d^3x.
\]

Let us assume that all the unlabelled \( g \)'s are \( g^{(1)} \)'s and integrate over the first manifold. The second manifold is an equally good base that gives an answer whose slight difference is immaterial in the limit. It is clear that \( (3.4) \) is the natural generalization to a field theory of the ‘bad’ Jacobi action \( (2.10) \). We have merely replaced summation over particles by integration over a field.

This is a distance on \( Riem \). To get a distance on superspace, we make different coordinate transformations on each of the two manifolds, getting \( g^{(1)} \)'s and \( g^{(2)} \)'s. We recalculate \( \Delta(g^{(1)}, g^{(2)}) \) and, in the key step, find the minimum of \( \Delta \) over all possible transformation pairs. The minimum, provided it exists, will be a function on superspace. In fact, we only need to make transformations on the second manifold, because identical transformations on the two manifolds leaves \( \Delta \) unchanged.

To transform this finite-distance function \( \Delta \) into an infinitesimal, consider a curve of metrics with label \( \lambda \) that interpolates \( g^{(1)} \) and \( g^{(2)} \). Evaluate \( \partial \Delta/\partial \lambda \) at any \( \lambda \) and get

\[
\frac{\partial \Delta}{\partial \lambda} = \int \sqrt{g} \sqrt{P} \sqrt{T} d^3x,
\]

with

\[
T = G^{abcd} \left[ \frac{\partial g_{ab}}{\partial \lambda} \right] \left[ \frac{\partial g_{cd}}{\partial \lambda} \right],
\]

(3.5)

with \( G^{abcd} \) given by \( (3.3) \).

To take into account the \( \lambda \)-dependent coordinate transformations, we add a Killing form to \( \partial g_{ab}/\partial \lambda, \partial g_{ab}/\partial \lambda \rightarrow \partial g_{ab}/\partial \lambda - (K\xi)_{ab} \), where \( \xi \) is \( a \ priori \) an arbitrary 3-vector field. The ‘distance’ along the curve between \( g^{(1)} \) and \( g^{(2)} \) is now

\[
\Delta(g^{(1)}, g^{(2)}) = \text{extremum with respect to } \xi \text{ of } \int d\lambda \int d^3x \sqrt{g} \sqrt{P} \sqrt{T}.
\]

The extremalization in \( (3.6) \) w.r.t. \( \xi \), which we call best matching \( [], \) makes \( \Delta \) into a measure on superspace and not just on \( Riem \). We use the word measure, as opposed to metric, because we are actually about to create a degenerate (albeit highly interesting) structure. The idea, motivated in \( \{ 6 \} \) by an idea that goes back to Poincaré, would be to have a true metric on superspace with geodesics as dynamical curves. However, because we choose the ‘bad’ action, we will find that although a best-matching action is in principle defined on all curves in superspace there are many curves joining two given points in superspace that have the same action. This degeneracy would not have occurred had we opted for the ‘good’ action. The consequence of our ‘bad’ choice coupled with the square-root filter
is to acquire an extra symmetry, which, unlike the 3-diffeomorphism symmetry on the configuration space, acts on the phase space.

It is worth relating this to the number of physical degrees of freedom of the gravitational field, which is universally agreed to be two if one rules out tensor–scalar gravity of Brans–Dicke type. Now, since the $3 \times 3$ symmetric tensor $g_{ij}$ has six independent components of which three correspond to coordinate freedom, a true geodesic theory on superspace must have three physical degrees of freedom per space point. Our ‘bad’ choice introduces an extra scalar symmetry and takes us down to two physical degrees of freedom. Our readers might begin to suspect that we are merely taking them on a roundabout route to standard GR.

To counter this worry, we first note that even if it is indirect the way is new. We get to the goal with much reduced kinematics – there is no time and no presupposed Minkowskian spacetime structure in the small. More significantly, we get new insights into the origin of the universal light cone and gauge theory. We should also like to draw attention to our earlier paper [5], in which we apply best matching and a local square root on conformal superspace (CS). Since the 3-vector symmetry of the 3-diffeomorphisms is augmented on CS by the scalar symmetry of 3-conformal transformations, the six degrees of freedom on $Riem$ are directly reduced to $6 - 3 - 1 = 2$ on CS. Remarkably, the use of the local square root in this case still acts as a filter of theories but does not reduce the physical degrees of freedom any further. There is no extra phase-space symmetry associated with the theory on CS. Since the coupling to matter on CS has still to be worked out in detail, we shall not explore this route further in the present paper. But it does hint at interesting possibilities (see also [7]).

Returning to the task in hand, the variational principle on superspace, the final step is to seek the curve between $g^{(1)}$ and $g^{(2)}$ that minimizes the best-matching ‘distance’ between the two geometries. This is achieved by varying w.r.t. the metric $g_{ij}$.

Note that $T$ as written is the most general ultralocal (i.e., with the supermetric $G^{abcd}$ dependent only on $g_{ij}$ and not on its spatial derivatives) quadratic form in $dg_{ij}$ up to an overall constant (which cannot affect the resulting geodesic curves). As we noted, the first term is the sum of squares and the second the square of the trace. The relative contributions of these two scalar terms has yet to be determined, hence the coefficient $A$. For $A = +1$, $G^{abcd}$ is the DeWitt supermetric.

The above analysis shows that $\xi$ in the action is the rate of change of a three-dimensional coordinate transformation, i.e., it is a ‘velocity’. Moreover, the action does not depend on the associated ‘position’ (three-dimensional coordinate transformation). Thus, it appears that $\xi$ should be treated as a ‘cyclic’ or ‘ignorable’ coordinate $\xi$, rather than as a Lagrange multiplier (a ‘position’ without a ‘velocity’). In fact, it will be shown in [8] that $\xi$, like all such variables used to implement best matching, is neither a multiplier nor a cyclic coordinate but a suî generis variable for which the variation at the end points is not fixed. The same analysis will show that in most cases, including the one considered here, the treatment as a multiplier is valid, though there are cases in which it is not. In this paper, we shall therefore regard $\xi$ as a multiplier, which matches the standard treatment in gauge theory.

We conclude this section by underlining the presence of the local square root in (3.6), i.e., the square roots are taken before the integration over space. This is the analogue of the ‘bad’ Jacobi action. The analogue of the ‘good’ action with ‘global’ square roots is

$$ I = \sqrt{\int d^3x \sqrt{g}P} \sqrt{\int d^3x \sqrt{g}T}. \quad (3.7) $$

Such an action yields geometrodynamics with three freedoms per point. Moreover, the global square roots do not ‘filter’. The one basic form (3.7) allows many instantiations. In contrast, mere consistency applied to candidates for $P$ and $T$ in (3.6) and its obvious generalization to scalar and 3-vector fields on $Riem$ leads directly to matter-free gravity (Sec. V), to matter fields minimally coupled to gravity and obeying its light cone (Secs. VI and VII) and to the gauge principle and massless electrodynamics (Sec. VII). These are all latent in (3.6). We must now make them appear.

IV. THE BSW ACTION AND ITS HAMILTONIAN FORM

The Baierlein–Sharp–Wheeler (BSW) action [1] has the form

$$ A_{BSW} = \int d\lambda \int \sqrt{g} \sqrt{R} \sqrt{T} d^3x, \quad (4.1) $$

where the ‘kinetic energy’ $T$ is
\[
T = (g^{ac}g^{bd} - g^{ab}g^{cd}) \left[ \frac{\partial g_{ab}}{\partial \lambda} - (K \xi)_{ab} \right] \left[ \frac{\partial g_{cd}}{\partial \lambda} - (K \xi)_{cd} \right],
\]

the monotonic \( \lambda \) labels the 3-metrics \( g_{ij}(x, \lambda) \) on a curve in \( \text{Riem} \), \( R \) is the scalar curvature of \( g_{ij} \), and the other symbols have been explained.

Although it was not found in this way, the BSW action can clearly be derived from the ‘distance’ \( A_{BSW} \) with \( A = +1 \) and \( R \) as the ‘potential’ \( P \). We first show that \( A_{BSW} \) leads to the Dirac–ADM Hamiltonian. The Lagrangian density is \( \mathcal{L} = \sqrt{g(R + T)} \), and the canonical momenta conjugate to \( g_{ij} \) are

\[
p^{ij} = \frac{\delta \mathcal{L}}{\delta \left( \frac{\partial g_{ij}}{\partial \lambda} \right)} = \sqrt{\frac{gR}{T}} (g^{ic}g^{jd} - g^{ij}g^{cd}) \left[ \frac{\partial g_{cd}}{\partial \lambda} - (K \xi)_{cd} \right].
\]

We immediately observe their homogeneity of degree zero in the ‘corrected velocities’ \( \partial g_{ab}/\partial \lambda - (K \xi)_{ab} \), which occur linearly in the numerator and quadratically under the square root in the denominator \( T \). They define a [‘local’] direction in superspace, as opposed to the direction and speed of ordinary momenta. The BSW action is therefore timeless, determining only paths without speed in superspace. Just as squared direction cosines sum to 1, the BSW constraint;

\[
\text{variation w.r.t. } \xi^i \text{ as a multiplier gives}
\]

\[
2 \left( \sqrt{\frac{gR}{T}} (g^{ic}g^{jd} - g^{ij}g^{cd}) \left[ \frac{\partial g_{cd}}{\partial \lambda} - (K \xi)_{cd} \right] \right)_{ij} = 0.
\]

This, on using Eq.\((4.3)\), gives us the secondary constraint

\[
\sqrt{g} H^i = p^{ij}_{,i} = 0,
\]

which consists of three conditions per space point. We also have the as yet undetermined vector \( \xi \), with three components per point. We wish to interpret \((4.3)\) as an equation for \( \xi \). The solution of this equation is the well-known thin-sandwich problem \([12]\). Suppose we can find \( \xi \) as a function of \( g_{ij} \) and \( \partial g_{ij}/\partial \lambda \). We can then substitute this back into \((4.1)\) to get a measure on superspace that is determined solely by the 3-metric. The auxiliary multiplier \( \xi \) will have been eliminated, and we should have a well-posed initial-value problem for \( g_{ij} \) and its \( \lambda \) derivative.

This is what the logic of the 3-space approach suggests. In fact, the thin-sandwich problem is burdened with difficulties \([13]\), and one should see our Lagrangian as heuristic rather than practical. Fortunately, it leads unambiguously to a Hamiltonian form – the Dirac–ADM fully constrained Hamiltonian – that is much more tractable and in which, as York showed \([16]\), one can genuinely obtain a well-posed initial-value problem. Our motto is therefore: “Conceptualize in the configuration space, calculate in the [constrained] phase space.”

The entire evolution dynamics is in the Euler–Lagrange equations for \( g_{ij} \):

\[
\frac{\partial p^{ij}}{\partial \lambda} = \frac{\delta \mathcal{L}}{\delta g_{ij}} = -\sqrt{\frac{gT}{4R}} (R^{ij} - g^{ij}R) - \sqrt{\frac{T}{gR}} \left( p^{im}p_{mj} - \frac{1}{2} p^{ij} \right) + \sqrt{\frac{gT}{4R}} \left( \frac{\partial p^{ij}}{\partial \lambda} - g^{ij} \nabla^2 \sqrt{\frac{gT}{4R}} \right) + \mathcal{L}_\xi p^{ij},
\]

where \( \mathcal{L}_\xi \) stands for the Lie derivative along \( \xi^i \).

The square-root identity forces the standard Hamiltonian to vanish identically, as it does for all Lagrangians homogeneous of degree one in the velocities. Using Dirac’s generalized Hamiltonian dynamics \([9]\), we consider

\[
\mathcal{H} = \int \sqrt{g}(NH + N_i H^i) d^3x
\]

where \( N \) and \( N_i \) are position-dependent multipliers, and \( H \) and \( H^i \) are the constraints \((4.4)\) and \((4.6)\), respectively. Variation w.r.t. \( N \) and \( N_i \) imposes \( H = 0 \) and \( H^i = 0 \).

The expression \((4.8)\) is exactly the Dirac–ADM Hamiltonian; \( H \) is the Hamiltonian constraint; \( H^i \) is the momentum constraint; \( N \) is the lapse; \( N^i \) is the shift. The standard equations of motion that it yields are
\[ \frac{\partial g_{ab}}{\partial \lambda} = 2 \frac{N}{\sqrt{g}} \left( p_{ab} - \frac{1}{2} g_{ab} p \right) + N_{;a;b} + N_{b;a} \]  
(4.9)\\
\[ \frac{\partial p^{ij}}{\partial \lambda} = -\sqrt{g} N \left( R^{ij} - \frac{1}{2} g^{ij} R \right) + \frac{N g^{ij}}{2 \sqrt{g}} \left( p^{ab} p_{ab} - \frac{1}{2} p^2 \right) - \frac{2N}{\sqrt{g}} \left( p^{im} p_{m}^{\lambda} - \frac{1}{2} p^{ij} \right) + \frac{\sqrt{g}}{2} \left( N^{;ij} - g^{ij} \nabla^2 N \right) + \mathcal{L}_{N;} p^{ij}. \]  
(4.10)

Inverting (4.3) to obtain \( \partial g_{ij}/\partial \lambda \) in terms of the momenta,
\[ \frac{\partial g_{ij}}{\partial \lambda} = \sqrt{\frac{T}{gR}} \left( p_{ij} - \frac{1}{2} g_{ij} p \right) + \xi_{ij} + \xi_{ij;}, \]  
(4.11)
and comparing (L7) with (4.11) and (4.11) with (4.9), we get the identifications
\[ N = \sqrt{\frac{T}{4R}}, \quad N^i = \xi^i. \]  
(4.12)

In fact, even after these identifications, the dynamical equations (4.7) and (4.10) differ by a multiple of the Hamiltonian constraint. This has no conceptual significance because the constraints vanish, but such differences can be important in numerical work. Moreover, if in the ADM Hamiltonian one chooses \( N/\sqrt{g} \) instead of \( N \) as the independent variable, then, as emphasized by York [17], the extra term vanishes, and one gets complete agreement. This shows that the BSW and ADM solution curves are closely related.

However, they are not quite identical. The ADM lapse and shift are entirely free functions, while the BSW logic calls for the solution of (4.5) for \( \xi \).

V. PROPAGATION OF THE BSW CONSTRAINTS

For consistency, the evolution equations of constrained theories must propagate the constraints \( \lambda \): initially zero, they must remain so. Thus, we must have \( \partial H/\partial \lambda = 0 \) and \( \partial H^i/\partial \lambda = 0 \) by virtue of (4.7) and the definition (L3). Since constraint propagation is crucial throughout this paper, we shall exhibit it in some detail for BSW. We begin with the momentum constraint \( H^i = 0 \), differentiating it w.r.t. \( \lambda \) and using the BSW (not ADM) evolution equations (L7)–(L11) to replace the \( \lambda \) derivatives of \( g_{ij} \) and \( p^{ij} \). To simplify (and make the ADM connection), we write \( N \) in place of \( \sqrt{T/4R} \). After cancellations and rearrangements, we are left with
\[ \frac{\partial}{\partial \lambda} \left( p^{ij}_{;j} \right) = \frac{4}{\sqrt{g}} \left[ \sqrt{g} R - \frac{1}{\sqrt{g}} \left( p^{ab} p_{ab} - \frac{1}{2} p^2 \right) \right]^{ij} + N^{;ij} \sqrt{g} \left[ R - g^{-1} \left( p^{ab} p_{ab} - \frac{1}{2} p^2 \right) \right]^{ij} - \frac{2N}{\sqrt{g}} \left( p^{im} - \frac{1}{2} g^{im} p \right) p_{m;b}^{\lambda} + \mathcal{L}_\xi \left( p^{ij}_{;j} \right). \]  
(5.1)

Thus, \( \partial/\partial \lambda (p^{ij}_{;j}) \) vanishes weakly \( \mathfrak{B} \): if the constraint holds initially, it will propagate. For the Hamiltonian constraint
\[ \frac{\partial}{\partial \lambda} \left[ gR - \left( p^{ab} p_{ab} - \frac{1}{2} p^2 \right) \right] = 4N^a_{;a} p^{ab} + 2N p^{ab}_{;ab} + \mathcal{L}_\xi \left[ gR - \left( p^{ab} p_{ab} - \frac{1}{2} p^2 \right) \right], \]  
(5.2)
so this constraint too propagates. (In actual fact, when expressed in Lagrangian, as opposed to Hamiltonian, form, the expressions within the square parentheses in (5.2) vanish identically. This vanishing is precisely the content of the square-root identities. However, both here and in the equations below, we prefer to show these terms explicitly, if for no other reason than that they must be present in the Hamiltonian formulation, in which, as we already noted, the Lagrangian identity becomes a true Hamiltonian constraint.)

In the light of the definition (L3), the evolution equations (L7) appear to give six equations for the six \( \partial^2 g_{ij}/\partial \lambda^2 \). The fact that (L3) propagate the four constraints tells us that only two of (L3) are true evolution equations. Therefore, the system is highly under-determined. In particular, we expect that \( \partial (p^{ij}_{;j})/\partial \lambda = 0 \) should be used to solve for \( \partial \xi/\partial \lambda \). Since the constraint is preserved, we end up with no restriction on \( \partial \xi/\partial \lambda \). A similar situation is true for \( \partial N/\partial \lambda \).

The upshot is that \( N \) and \( \xi \) are fixed on the initial slice by the initial data, but their evolution is free. Hence the freedom in the ADM lapse and shift is effectively shared by the BSW action, and both generate Einsteinian gravity.
In the local-square-root particle model, discussed in Section II, in the special case where the constraints are propagated the action reduces to a sum of single particle Jacobi actions. Therefore the action is locally reparametrisation invariant even though it is expressed in terms of a global parameter. This is not true in the BSW action. The spatial derivatives in the scalar curvature are evaluated at a fixed ‘time’. If we make a ‘local’ parameter change we will change the surfaces of constant time and thus change the derivatives in a complicated fashion. Therefore the BSW action remains only globally reparametrisation invariant.

VI. UNIQUENESS OF BSW

We have seen that constraint propagation is important. Many have sought conditions under which GR can be derived. Two main strategies have been followed. The older classical arguments, reviewed by Hojman, Kuchar, and Teitelboim (HKT) [2], relied on four-dimensional general covariance coupled with simplicity restrictions in a Lagrangian framework. These essentially select the Hilbert action uniquely (up to an arbitrary cosmological constant). More recently, Teitelboim [3] started from a Hamiltonian viewpoint and deduced matter-free GR by postulating: 1) that the Hamiltonian should have the local form (4.8); 2) that $H$ and $H^i$ should depend only on the 3-metric $g_{ij}$ and its conjugate momentum $p^{ij}$; and 3) that the resulting dynamics should satisfy an embeddability criterion proposed by Wheeler: “If one did not know the Einstein–Hamilton–Jacobi equation, how might one hope to derive it straight off from plausible first principles, without ever going through the formulations of the Einstein field equations themselves? The central starting point in the proposed derivation would necessarily seem to be ‘embeddability’ [in a four-dimensional pseudo-Riemannian spacetime].”

As Teitelboim noted in his PhD thesis [19], this is an extremely restrictive condition. Developing an approach of Dirac [3], he showed that embeddability imposes a strict requirement on the Poisson-bracket relations between $H$ and $H^i$. They must satisfy the so-called Dirac algebra. In [2], HKT then sought theories in which the manner in which the constraints close ensures embeddability and showed (again with certain simplicity requirements) that GR is the unique theory that does so.

As our first new result, we show that embeddability is a much stronger condition than one needs. The constraint algebra need not close in a specific way. It is merely necessary that it close. As we shall see, this opens up an entirely new derivation of relativity – both the special and the general theory – in which no a priori assumption of geometrodynamical evolution of spacelike hypersurfaces in a four-dimensional pseudo-Riemannian spacetime is made.

We can derive relativity without relativity merely by postulating an action based on a metric ‘distance’ of the form (3.6) and requiring that its constraints propagate.

Best matching, which ensures 3-diffeomorphism invariance, automatically leads to a momentum constraint of the form (4.6). The local square root leads to a local square-root identity like (4.4), which becomes a quadratic Hamiltonian constraint like (4.5). We are led naturally to a local Hamiltonian of the form (4.8). Both constraints strongly restrict the possible Lagrangians (or Hamiltonians) through the condition of constraint propagation.

We make no attempt at an exhaustive analysis and employ a pedestrian technique. We suspect our various individual results could be obtained more elegantly in a unified manner but think it premature to seek it at this stage, since there are several extensions of the method, which we shall mention at the end of the paper, that should first be explored. In the meanwhile, our individual results show the potential of the 3-space approach.

We start with the simplest modification of the BSW Lagrangian: changing the coefficient $A$ in the supermetric from the DeWitt value $A = 1$. The inverse to the supermetric is $g_{ae}g_{bf} - \frac{A}{3A-1}g_{ab}g_{ef}$ because

$$\left[g_{ae}g_{bf} - \frac{A}{3A-1}g_{ab}g_{ef}\right]\left[g^{ac}g^{bd} - Ag^{ab}g^{cd}\right] = \delta^e_c\delta^d_f. \quad (6.1)$$

We define

$$B = \frac{2A}{3A-1} \quad (6.2)$$

because when $A = 1$ we also have $B = 1$.

Hence we start with a modified BSW action

$$A'_{BSW} = \int d\lambda \int \sqrt{g}\sqrt{R\sqrt{T}}d^3x \quad (6.3)$$

with
\[
T = (g^{ac}g^{bd} - Ag^{ab}g^{cd}) \left( \frac{\partial g_{ab}}{\partial \lambda} - (K\xi)_{ab} \right) \left( \frac{\partial g_{cd}}{\partial \lambda} - (K\xi)_{cd} \right).
\] (6.4)

The canonical momenta conjugate to \( g_{ij} \) are
\[
p^{ij} = \frac{\delta L}{\delta \left( \frac{\partial g_{ij}}{\partial \lambda} \right)} = \sqrt{\frac{gR}{T}} (g^{jc}g^{id} - Ag^{ij}g^{cd}) \left[ \frac{\partial g_{cd}}{\partial \lambda} - (K\xi)_{cd} \right].
\] (6.5)

This can be inverted to give
\[
\frac{\partial g_{ij}}{\partial \lambda} = \sqrt{\frac{T}{gR}} (p_{ij} - \frac{B}{2} g_{ij} p) + \xi_{i;j} + \xi_{j;i}.
\] (6.6)

The square-root identity becomes
\[
gH = -p^{ij}p_{ij} + \frac{B}{4} p^2 + gR = 0,
\] (6.7)

while the linear constraint that arises from varying w.r.t. \( \xi \) is unchanged:
\[
\sqrt{g} H^i = p^{ij;i} = 0.
\] (6.8)

The evolution is again in the Euler–Lagrange equations for \( g_{ij} \):
\[
\frac{\partial p^{ij}}{\partial \lambda} = \frac{\delta L}{\delta \left( \frac{\partial g_{ij}}{\partial \lambda} \right)} = -\sqrt{\frac{gT}{4R}} (R_{ij} - g_{ij} R) - \sqrt{\frac{T}{gR}} (p^{im}p_{m} - \frac{B}{2} pp^{ij}) + \left( \sqrt{\frac{gT}{4R}} - g^{ij} \nabla^2 \sqrt{\frac{gT}{4R}} + L_{\xi} p^{ij} \right).
\] (6.9)

We use equations (6.6) and (6.9) to evolve the constraints, obtaining
\[
\frac{\partial}{\partial \lambda} \left( p^{ij;i} \right) = \frac{1}{2} N \left[ \sqrt{gR} - \frac{1}{\sqrt{g}} \left( p^{ab}p_{ab} - \frac{B}{2} p^2 \right) \right] + N^{ij} \sqrt{g} \left[ R - g^{-1} \left( p^{ab}p_{ab} - \frac{B}{2} p^2 \right) \right] - 2N \left( p^{im} - \frac{B}{2} g^{jm} p \right) p_{m;i} + L_{N;i} \left( p^{ij} ; i \right).
\] (6.10)

The \( \lambda \) derivative of the momentum constraint, being proportional to itself, vanishes weakly, and so the constraint propagates. However, for the Hamiltonian constraint
\[
\frac{\partial}{\partial \lambda} \left[ \sqrt{gR} - \frac{1}{\sqrt{g}} \left( p^{ab}p_{ab} - \frac{B}{2} p^2 \right) \right] = 4N^{ac}p^{b}_{ab} + 2Np^{ab}_{;ab} + \left( \frac{3B-2}{2} \right) Np \left[ \sqrt{gR} - \frac{1}{\sqrt{g}} \left( p^{ab}p_{ab} - \frac{B}{2} p^2 \right) \right] + (2B - 2) N\nabla^2 p + (4B - 4) N^{ij}p_{;i}
\]
\[+ L_{N;i} \left( \sqrt{gR} - \frac{1}{\sqrt{g}} \left( p^{ab}p_{ab} - \frac{B}{2} p^2 \right) \right).
\] (6.11)

The right hand side of (6.11) does not vanish weakly. It is clear that the trace \( p \) must satisfy the secondary constraint
\[
p = \text{constant}.
\] (6.12)

This is the well-known constant-mean-curvature (CMC) gauge condition (6.6), and it severely restricts the initial data. It also forms a second-class constraint (6.9) with the Hamiltonian, with which it does not commute. When we evolve this constraint, we get the standard CMC slicing condition:
\[
\nabla^2 N - RN = C,
\] (6.13)

where \( C \) is a spatial constant, essentially half \( \partial p/\partial \lambda \). This is yet another restriction on the initial data, and these results show that we cannot connect arbitrary 3-geometries, i.e., points on superspace, by curves that extremalize the action based on the measure (6.6) unless \( A = 1 \).

Thus, a consistent BSW-type action with \( A \neq 1 \) does not exist on superspace. We note, however, that the secondary constraints just obtained, particularly the special case \( p = 0 \) of (6.12), arise naturally if gravity is treated on conformal superspace (CS) (6.6, 6.7). We have already mentioned our belief that gravity will only be properly understood when
The Euler–Lagrange equations are

and we get the standard momentum constraint

also mention that our $A \neq 1$ result was already obtained by Giulini some years ago $[10]$. Characteristically, both ‘escape routes’ are indicated by our method. We have seen that the sole apparent freedom (the value of $A$) in the kinetic term $T$ is illusory, and that best matching and consistency fix it uniquely. We now apply the same technique to the ‘potential’ $P$. One modification works. We first show that $P = \Lambda + DR$ gives a consistent theory with $\Lambda$ as the cosmological constant and $D$ is another constant.

Keeping the uniquely determined DeWitt $T$, we now modify the BSW action to

$$A'_{\text{BSW}} = \int d\lambda \int \sqrt{g} \sqrt{DR + \Lambda} \sqrt{T} d^3x.$$  (6.14)

The momenta conjugate to $g_{ij}$ are

$$p^i_j = \frac{\delta L}{\delta (\partial g_{ij})/\partial \lambda)} = \frac{g(DR + \Lambda)}{T} (g^{ij} g^{jd} - g^{jd} g^{d}i) \left[ \frac{\partial g_{cd}}{\partial \lambda} - (K)_{cd} \right].$$  (6.15)

This can be inverted to give

$$\frac{\partial g_{ij}}{\partial \lambda} = \sqrt{\frac{g(DR + \Lambda)}{T}} \left( p_{ij} - \frac{1}{2} g_{ij} p \right) + \xi_{ij} + \xi_{ji}.$$  (6.16)

The square-root identity becomes

$$gH = -p^i_j p_{ij} + \frac{1}{2} p^2 + g(DR + \Lambda) = 0,$$  (6.17)

and we get the standard momentum constraint

$$\sqrt{g} H^2 = p^i_{ji} = 0.$$  (6.18)

The Euler–Lagrange equations are

$$\frac{\partial p^i_j}{\partial \lambda} = \frac{\delta L}{\delta (\partial g_{ij}) / \partial \lambda)} = - \sqrt{\frac{g^2}{4(DR + \Lambda)}} \left( DR_{ij} - g_{ij} DR - g_{ij} \Lambda \right) - \sqrt{\frac{T}{g(DR + \Lambda)}} \left( p^m_i p^j_m - \frac{1}{2} p^j p^i \right) + \left( D \sqrt{\frac{g^2}{4(DR + \Lambda)}} - g_{ij} D \nabla^2 \sqrt{\frac{g^2}{4(DR + \Lambda)}} \right) + L_N \xi_{ij}.$$  (6.19)

We use (6.16) and (6.19) to evolve the constraints, noting that now $N = \sqrt{T/4(DR + \Lambda)}$. We get

$$\frac{\partial}{\partial \lambda} \left( p^i_{ji} \right) = \frac{1}{2} N \left[ \sqrt{g}(DR + \Lambda) - \frac{1}{\sqrt{g}} \left( p^{ab} p_{ab} - \frac{1}{2} p^2 \right) \right]^i + N^i \sqrt{g} \left[ 4(DR + \Lambda) - g^{-1} \left( p^{ab} p_{ab} - \frac{1}{2} p^2 \right) \right]$$

$$- \frac{2}{\sqrt{g}} \left( p^m_i - \frac{1}{2} g^{im} p \right) p^b_{mb} + L_{N} \left( \xi_{ij} \right).$$  (6.20)

Thus, as expected, the momentum constraint propagates. For the Hamiltonian constraint we have

$$\frac{\partial}{\partial \lambda} \left[ \sqrt{g}(DR + \Lambda) - \frac{1}{\sqrt{g}} \left( p^{ab} p_{ab} - \frac{1}{2} p^2 \right) \right] =$$

$$4 D N^{a} p^{b}_{ab} + 2 D N p^{ab}_{ab} + \frac{1}{2} N p \left[ \sqrt{g}(DR + \Lambda) - \frac{1}{\sqrt{g}} \left( p^{ab} p_{ab} - \frac{1}{2} p^2 \right) \right]$$

$$+ L_{N} \left[ \sqrt{g}(DR + \Lambda) - \frac{1}{\sqrt{g}} \left( p^{ab} p_{ab} - \frac{1}{2} p^2 \right) \right].$$  (6.21)

The right-hand side of (6.21) also vanishes weakly.

As we mentioned in the introduction, we can by multiplying by an overall scaling factor set $D = +1, -1, \text{or } 0$. The first case is standard general relativity with a cosmological constant, the second case is euclidean gravity, again with a cosmological constant, and the third is strong gravity.
We now try to modify \( P \) more radically. Keeping to our pedestrian approach, we consider two special but illuminating cases. The first is \( P = R^\alpha \) with \( \alpha \) a constant. Then, with the DeWitt \( T \), we have

\[
A_{\text{BSW}} = \int d\lambda \int \sqrt{g} \sqrt{R^\alpha} \sqrt{T} d^3 x.
\]

(6.22)

The dynamical equations are

\[
p^{ij} = \frac{\delta \mathcal{L}}{\delta \left( \frac{\partial g_{ij}}{\partial \lambda} \right)} = \sqrt{\frac{g R^\alpha}{T}} (g^{ic} g^{jd} - g^{ij} g^{cd}) \left[ \frac{\partial g_{cd}}{\partial \lambda} - (K \xi)_{cd} \right].
\]

(6.23)

The momenta conjugate to \( g_{ij} \) are

\[
p^{ij} = \frac{\partial g_{ij}}{\partial \lambda} = \sqrt{\frac{T}{g R^\alpha}} \left( p_{ij} - \frac{1}{2} g_{ij} p \right) + \xi_{ij} + \xi_{;ij},
\]

so we define \( N = \sqrt{\frac{T}{4R^\alpha}} \). The square-root identity becomes

\[
gH = -p^{ij} p_{ij} + \frac{1}{2} p^2 + g R^\alpha = 0,
\]

(6.25)

and we get the standard momentum constraint

\[
\sqrt{g} H^i = p^{ij} ; i = 0.
\]

(6.26)

The dynamical equations are

\[
\frac{\partial p^{ij}}{\partial \lambda} = \frac{\delta \mathcal{L}}{\delta g_{ij}} = \sqrt{g} N \alpha^{-1} R^{\alpha-1} (\alpha R^{ij} - g^{ij} R) - 2 N (p^{im} p_{m}^{j} - \frac{1}{2} p^{ij} p)
\]

\[
+ \alpha \sqrt{g} (\left[ N R^{\alpha-1} \right]^{ij} - g^{ij} \nabla^2 N R^{\alpha-1}) + L_{\xi} p^{ij}.
\]

(6.27)

We use (6.24) and (6.27) to evolve the constraints. Once again, the Hamiltonian constraint does not propagate:

\[
\frac{\partial}{\partial \lambda} \left[ \sqrt{g} R^\alpha - \frac{1}{\sqrt{g}} \left( p^{ab} p_{ab} - \frac{1}{2} p^2 \right) \right] = 4 \alpha R^{\alpha-1} N \alpha^{-1} p_{ab}^{ab} + 2 \alpha R^{\alpha-1} N p^{ab}_{ab}
\]

\[
+ \frac{1}{2} N p \left[ \sqrt{g} R^\alpha - \frac{1}{\sqrt{g}} (p^{ab} p_{ab} - \frac{1}{2} p^2) \right]
\]

\[
- 4 \alpha N \alpha^{-1} R^{\alpha-1} p_{ab} - 2 \alpha N (R^{\alpha-1})^{;ab} p_{ab}
\]

\[
+ L_{\xi} \left[ \sqrt{g} R^\alpha - \frac{1}{\sqrt{g}} (p^{ab} p_{ab} - \frac{1}{2} p^2) \right].
\]

(6.28)

This does not vanish weakly for any \( \alpha \) except \( \alpha = 1 \). We get the extra constraint \( R = \text{constant} \). Conserving this gives yet another, unpleasant, equation. It is difficult to conceive any solution of this system except static flat space.

Another choice we tested was

\[
P = C_1 R^2 + C_2 R^{ab} R_{ab} + C_3 \nabla^2 R,
\]

(6.29)

where \( C_1, C_2, C_3 \) are arbitrary constants. If \( g_{ij} \) is taken to have dimensions \((\text{length})^2\), then \( R \) will be \((\text{length})^{-2}\). No scalar with dimensions \((\text{length})^{-3}\) can be constructed from \( g_{ij} \). The only geometric scalars that have dimension \((\text{length})^{-4}\) are the three in expression (6.23). The other two obvious candidates, the square of the Riemann tensor and \( R^{ij;i} \), need not appear. The three-dimensional Riemann tensor can be written as a sum of the Ricci tensor and \( R \), and the divergence of the Ricci tensor can be eliminated using the Bianchi identity.

We repeat the calculation, evolving the square-root constraint. This leads to an explosion of unpleasant non-cancelling terms that arise from the extra terms in \( P \).

One soon sees that the same problems will arise for all possible extra terms. We conclude that BSW is the unique consistent matter-free theory on superspace based on a ‘distance’ of the form (1.5). We believe that this is a new result. In many respects, our calculations repeat those of HKT [4]. The novelty is our weaker assumption. The HKT assumptions are: 1) there is a local Hamiltonian constraint, quadratic in the momenta; 2) there is a local momentum constraint; 3) the Poisson bracket of these constraints reflects embeddability. Our local square root is equivalent to 1); best matching is equivalent to 2); constraint propagation on its own replaces 3). There is no need to presuppose spacetime. Already latent in (1.5), it is laid bare by consistency.
VII. SCALAR FIELD INTERACTING WITH GRAVITY

There exist matter-free solutions of Einstein’s equations on $S^3$. Thus, there is an emergent light-cone structure in the 3-space approach. Besides the pure-gravity light cone and 4-covariance (which we have recovered), the most basic relativistic facts are the universality of free fall and the universal light cone (all matter fields respecting the gravity cone). If (3.6) is the basis of relativity, both of these further features should be implied by it.

Let us see how a real scalar field $\phi$ can be introduced. First, best matching essentially fixes the form in which $\phi$ enters the kinetic term $T$. For $\phi$ is ‘painted’ onto the 3-geometries described by the 3-metrics $g_{ij}$, so that the correction to its ‘naive’ velocity $\partial \phi / \partial \lambda$, like the correction $K\xi$ to the metric velocity $\partial g_{ij} / \partial \lambda$ induced by (3.1), is predetermined. It is the scalar product of $\xi$ with the spatial gradient of $\phi$: the matter is ‘dragged along’ with the geometry by the diffeomorphisms. Technically, the correction term $\phi; i \xi^i$ is just the Lie derivative of $\phi$ along $\xi$ just as $K\xi$ is the Lie derivative of $g_{ij}$. The modified $T$ is

$$T = (g^{ac} g^{bd} - g^{ab} g^{cd}) \left[ \frac{\partial g_{ab}}{\partial \lambda} - (K\xi)_{ab} \right] \left[ \frac{\partial g_{cd}}{\partial \lambda} - (K\xi)_{cd} \right] + \left[ \frac{\partial \phi}{\partial \lambda} - \phi; i \xi^i \right]^2. \quad (7.1)$$

As here, the coefficient of the scalar kinetic term can always be set to 1 by absorbing a constant into $\phi$. The obvious modifications to the potential $P$ are

$$R \rightarrow R - \frac{C}{4} g^{ab} \phi; a \phi; b + \sum_n A_n \phi^n. \quad (7.2)$$

The first addition is the standard scalar-field term that gives rise to wave propagation. It has the same dimensions, (length)$^{-2}$, as $R$. If the constant $C \neq +1$, then $\phi$ will not have the same light cone as gravity and local Lorentz invariance will be violated. The second addition is a general polynomial non-derivative self-interaction term for $\phi$. For $n = 2$ and $A_2 = m^2/4$, we get the standard mass term for $\phi$. We need not demand that $n$ be an integer. We have dropped $\Lambda$, but it can easily be restored. We include neither the higher-order metric terms excluded in Sec. VI nor higher-order metric–scalar interactions. We expect that these too can be eliminated.

The metric momenta are

$$p^{ij} = \frac{\delta L}{\delta (\frac{\partial g_{ij}}{\partial \lambda})} = \sqrt{g} \frac{g^{ic} g^{jd} - g^{ij} g^{cd}}{T_g + T_\phi} \left[ \frac{\partial g_{cd}}{\partial \lambda} - (K\xi)_{cd} \right], \quad (7.3)$$

where $U_\phi$ is the $\phi$ potential term. Inversion of (7.3) gives

$$\frac{\partial g_{ij}}{\partial \lambda} = 2N \sqrt{g} \left( p_{ij} - \frac{1}{2} g_{ij} p \right) + \xi_{ij} + \xi_{i; j}. \quad (7.4)$$

where we define $2N = \sqrt{T_g + T_\phi / R + U_\phi}$. The momentum conjugate to $\phi$ is

$$\pi = \frac{\delta L}{\delta (\frac{\partial \phi}{\partial \lambda})} = \sqrt{g} \frac{g (R + U_\phi)}{T_g + T_\phi} \left[ \frac{\partial \phi}{\partial \lambda} - \phi; i \xi^i \right]. \quad (7.5)$$

Its inversion gives

$$\frac{\partial \phi}{\partial \lambda} = \frac{2N \pi}{\sqrt{g}} + \phi; i \xi^i. \quad (7.6)$$

The square-root identity becomes

$$gH = -p^{ij} p_{ij} + \frac{1}{2} b^2 - \pi^2 + g (R + U_\phi) = 0, \quad (7.7)$$

and variation w.r.t. $\xi$ gives the momentum constraint

$$\sqrt{5} H^j = p^{ij} ; i - \frac{1}{2} \pi^{ij} = 0. \quad (7.8)$$
The Euler–Lagrange equations for \( g_{ij} \) and \( \phi \) are

\[
\frac{\partial p_{ij}}{\partial \lambda} = \frac{\delta L}{\delta g_{ij}} = -\sqrt{g} N \left( R^{ij} - g^{ij} R \right) - 2\sqrt{g} \left( p^{im} p_{mj} - \frac{1}{2} p^{ij} \right) + \sqrt{g} \left( N g^{ij} - g^{ij} \nabla^2 N \right) + \sqrt{\frac{2}{N}} \nabla^i \phi \nabla^j \phi - \frac{\sqrt{2}}{N} g^{ab} \phi_a \phi_b g^{ij} + \sqrt{g} N \sum_n A_n \phi^n g^{ij} + \mathcal{L}_\xi p^{ij} \tag{7.9}
\]

\[
\frac{\partial \pi}{\partial \lambda} = \frac{\delta L}{\delta \phi} = \sqrt{\frac{2}{N}} (N \nabla^i \phi) ; i + \sqrt{g} N \sum_n n A_n \phi^{n-1} + \mathcal{L}_\xi \pi. \tag{7.10}
\]

We can now combine (7.6) and (7.10) to give

\[
\frac{\partial}{\partial \lambda} \left( \sqrt{\frac{g}{N}} \left[ \frac{\partial \phi}{\partial \lambda} - \phi, \pi^i \right] \right) = -\sqrt{2} C (N \nabla^i \phi) ; i - 2 \sqrt{g} N \sum_n n A_n \phi^{n-1} - 2 \mathcal{L}_\xi \pi = 0 \tag{7.11}
\]

If we set \( \sqrt{g} = N = 1, n = 2, A_2 = m^2/4, \xi = 0 \), this reduces to \( \partial^2 \phi / \partial^2 \lambda - C \nabla^2 \phi - m^2 \phi = 0 \), i.e., the wave equation for a scalar field with mass \( m \) and canonical speed \( \sqrt{C} \).

The new Hamiltonian and momentum constraints (7.7) and (7.8) are the 00 and 0i Einstein field equations. Note that \( \pi \phi^{ij} \) from (7.8) completes the square of \( \pi^2 + 1/4 (\nabla \phi)^2 \) from (7.5). The 1/2 in (7.8) arises because the Hamiltonian constraint has 16 \( \pi \phi \) terms, while the momentum constraint has 8 \( \pi J \) terms. These are obtained because of our best-matching ‘dragging’ of \( \phi \), which leads to minimal coupling and the equivalence principle. Note that the divergence \( p^{ij} \) of the gravitational momenta no longer vanishes but equals the \( \phi \) ‘current’ \( 1/2 \phi \pi^{ij} \) (the \( \phi \) momentum density), while the Hamiltonian constraint picks up (twice) the energy density.

We now come to the next constraint-propagation result. The momentum constraint propagates, as always, and the polynomial self-interaction terms (including the mass term) in the quadratic constraint cause no difficulty; the extra terms in its \( \lambda \) derivative cancel. But the coefficient \( C \) produces non-vanishing terms. We have

\[
\frac{\partial}{\partial \lambda} \left( \sqrt{\frac{g}{N}} \left[ \frac{\partial \phi}{\partial \lambda} - \phi, \pi^i \right] \right) = -\sqrt{2} C (N \nabla^i \phi) ; i - 2 \sqrt{g} N \sum_n n A_n \phi^{n-1} + \mathcal{L}_\xi \pi. \tag{7.12}
\]

Although most terms in (7.12) vanish weakly, the last three, all proportional to \( 1 - C \), do not. They generate a secondary constraint. This forms a second-class constraint set with the Hamiltonian and so generates yet another constraint. In fact, the Hamiltonian constraint will not propagate unless \( C = 1 \). But this means that \( \phi \) is forced to respect the metric light cone.

Moreover, the mechanism that enforces this will generate the universal metric–matter light cone – for bosons at least, we have not yet considered fermions. The mechanism has several hinges, but the linkage is unbreakable. The key is the presence of the scalar curvature \( R \) in the square-root constraint. The \( \lambda \) derivative of this constraint therefore contains \( \partial R / \partial \lambda \). Now purely by kinematics

\[
\frac{\partial R}{\partial \lambda} = \left[ \frac{\partial g_{ij}}{\partial \lambda} \right]^{ij} - \nabla^2 \left[ g^{ij} \frac{\partial g_{ij}}{\partial \lambda} \right] - R^{ij} \frac{\partial g_{ij}}{\partial \lambda} = \left[ \frac{2N}{\sqrt{g}} \pi^{ij} \right]_{;ij} + \ldots, \tag{7.13}
\]

where we use Eq.(7.4) to replace the time derivative of \( g \) with the momentum. The final expression in Eq.(7.13) will contain terms with \( \pi^{ij} \). We now subtract \( \pi \phi^{ij} \) to generate the momentum constraint and obtain the first two terms on the right hand side of Eq.(7.12). This is the first hinge. The extra \( \pi \phi^{ij} \) terms now appear as the \( C \)-independent parts of the last three terms in Eq.(7.13). The \( \pi \phi^{ij} \) terms with the factor \( C \) arise from two different sources. The \( (\nabla \phi)^2 \) term in the Hamiltonian constraint has an explicit \( C \) in front of it. The time derivative of this term, combined with Eq.(7.4) gives the term in the middle of the three. The other two terms arise from the time derivative of the \( \pi^2 \) in the square root constraint. To evaluate this we use the Euler-Lagrange equation for \( (\phi, \pi) \), Eq.(7.10); the first term in this has a \( C \) in it. This gives the first and third terms. This is the second hinge.

Therefore, the constraint-propagation condition contains identical terms with and without the coefficient. This is the final hinge in the light-cone generating mechanism, for it makes it inevitable that constraint propagation will
enforce $C = 1$. The universality of the mechanism is ensured by the universal nature of the momentum constraint: any field of whatever tensor rank whose velocity appears in the Lagrangian with best-matched correction will be represented as a source term in the momentum constraint in a form solely determined by its Lie derivative. We shall see in the next section that this mechanism has even more implications.

For the scalar field, we have thus derived the correct light-cone behaviour of Lorentz-invariant field theory and can see that this will hold uniformly. We believe that this is a new result. We can also show that a derivative coupling term in (7.2) cannot be included consistently in the potential of our Lagrangian.

It turns out that the fact that the ‘mass’ term in the scalar field can be chosen quite arbitrarily, as we have shown above, is part of a wider freedom. It turns out that a large class of what might be called ‘dilatonic’ theories can be written selfconsistently in BSW form. Let us consider a theory which has three constituents. One is the gravitational field, represented by a three-metric, $g_{ij}$; one is a massless scalar field, $\phi$ (the dilaton); and the third is a ‘massive’ scalar field, $\chi$. The key point is that we assume that the ‘mass’ of $\chi$ is some arbitrary function of $\phi$, $m^2 = f(\phi)$. We further assume that both scalar fields are minimally coupled to the metric and that the kinetic energy takes the simplest possible form.

This means that we assume the potential term is

$$P = R - \frac{1}{4} g^{ab} \phi, a \phi, b - \frac{1}{4} (g^{ab} \phi, a \chi, b - f(\phi) \chi^2),$$

and the kinetic term is

$$T = (g^{ac} g^{bd} - g^{ab} g^{cd}) \left[ \frac{\partial g_{ab}}{\partial \lambda} - (K \xi)_{ab} \right] \left[ \frac{\partial g_{cd}}{\partial \lambda} - (K \xi)_{cd} \right] + \left[ \frac{\partial \phi}{\partial \lambda} - \phi, a \xi^a \right]^2 + \left[ \frac{\partial \chi}{\partial \lambda} - \chi, i \xi^i \right]^2.$$  

(7.15)

The square-root identity becomes

$$gH = -p^{ij} p_{ij} + \frac{1}{2} p^2 - \pi^2 - \chi^2 + g(R + U_\phi + U_\chi) = 0,$$

(7.16)

and variation w.r.t. $\xi$ gives the momentum constraint

$$\sqrt{g} H^j = p^{ij}; - \frac{1}{2} \chi^2 \xi^i - \frac{1}{2} \pi^2 \xi^j = 0.$$  

(7.17)

We need to consider the propagation of the square-root constraint. We know that most of the terms will take care of themselves and we really need only focus on the derivative of $f(\phi)$. One place is in the Euler–Lagrange equation for $\phi$

$$\frac{\partial \pi_{\phi}}{\partial \lambda} = \frac{\delta L}{\delta \phi} = \frac{\sqrt{g}}{2} (N \nabla^i \phi)_j + \frac{\sqrt{g} N}{4} f'(\phi) \chi^2 + L_\xi \pi_{\phi}.$$  

(7.18)

This will feed into

$$- \frac{\partial \pi_{\phi}^2}{\partial \lambda} = - \frac{\sqrt{g} N \pi_{\phi}}{2} f'(\phi) \chi^2 + \ldots.$$  

(7.19)

The other place is in the time derivative of $U_\chi$ where we get

$$\frac{\partial f(\phi) g \chi^2}{\partial \lambda} = \frac{g f'(\phi) \chi^2}{4} \frac{\partial \phi}{\partial \lambda} = \frac{g f'(\phi) \chi^2}{4} \frac{2 N \pi_{\phi}}{\sqrt{g}} = \frac{\sqrt{g} N \pi_{\phi}}{2} f'(\phi) \chi^2,$$

(7.20)

where we used Eq. (7.4) to replace the time derivative of $\phi$ with its momentum. The two terms obviously cancel. It is clear that we can replace the mass term in $\chi$ with a polynomial and have each coefficient be a different function of $\phi$.

Possibly the best known dilaton theory is Brans–Dicke theory [34]. It is interesting to see how this can be written in BSW form. Let us start with the simplest possible local square root form with just gravity and a minimally coupled massless scalar field. This means that we choose the potential and kinetic terms as in Eqs. (7.14) and (7.13) with $\chi \equiv 0$. We now perform a ‘point’ transformation and change variables to $(\gamma, \Phi)$ via:

$$g_{ab} = \Phi \gamma_{ab}, \quad \phi = -\sqrt{4 \omega + 6 \log \Phi}.$$  

(7.21)

The action changes to
\[
\sqrt{g}\sqrt{P} = \sqrt{\gamma} \sqrt{\Phi^2 R - \omega \gamma^{ij} \partial_i \Phi \partial_j \Phi - 2\Phi \nabla^2 \Phi} \times \sqrt{(\gamma^{ac}g^{bd} - \gamma^{ab}g^{cd}) \left[ \frac{\partial \gamma^{ab}}{\partial \xi^a} - (K\xi)_{ab} \right] \left[ \frac{\partial \gamma^{cd}}{\partial \xi^c} - (K\xi)_{cd} \right] - \frac{1}{\Phi} \gamma^{cd} \left[ \frac{\partial \gamma^{ab}}{\partial \xi^a} - (K\xi)_{ab} \right] \left[ \frac{\partial \Phi}{\partial \xi^d} - \Phi \partial_d \xi^d \right] + \frac{\partial \Phi}{\partial \xi^d} - \Phi \partial_d \xi^d \right) - \frac{1}{\Phi} \gamma^{cd} \left[ \frac{\partial \gamma^{ab}}{\partial \xi^a} - (K\xi)_{ab} \right] \left[ \frac{\partial \Phi}{\partial \xi^d} - \Phi \partial_d \xi^d \right] + \frac{\partial \Phi}{\partial \xi^d} - \Phi \partial_d \xi^d \right)^2, \tag{7.22}
\]

where \( \xi^i = \xi^i \) and \( \bar{\xi}^i = \Phi^{-1} \xi^i \). We vary this w.r.t. \( \partial \gamma^{cd}/\partial \lambda \) and \( \partial \Phi/\partial \lambda \) to get the momenta conjugate to \( \gamma^{cd} \) and \( \Phi \). These are

\[
p_{\gamma}^{cd} = \sqrt{\gamma} \frac{P}{\gamma} \left\{ (\gamma^{ac}g^{bd} - \gamma^{ab}g^{cd}) \left[ \frac{\partial \gamma^{ab}}{\partial \lambda} - (K\xi)_{ab} \right] - \frac{2}{\Phi} \gamma^{cd} \left[ \frac{\partial \Phi}{\partial \lambda} - \Phi \partial_d \xi^d \right] \right\} \tag{7.23}
\]

and

\[
\pi_\Phi = \sqrt{\gamma} \frac{P}{\gamma} \left\{ 4\omega \left[ \frac{\partial \Phi}{\partial \lambda} - \Phi \partial_d \xi^d \right] - 2 \gamma^{cd} \left[ \frac{\partial \gamma^{ab}}{\partial \lambda} - (K\xi)_{ab} \right] \right\}. \tag{7.24}
\]

Then the square-root constraint simply becomes the standard 3+1 Brans–Dicke Hamiltonian constraint and similarly for the momentum constraint. The square-root identity becomes

\[
\gamma H = -p_{\gamma}^{ij} p_{ij} + \frac{1}{2} p^2 - \frac{1}{4(4\omega + 6)} (p - \Phi \pi_\Phi)^2 + \Phi^2 R - \omega \gamma^{ij} \partial_i \Phi \partial_j \Phi - 2\Phi \nabla^2 \Phi = 0, \tag{7.25}
\]

and variation w.r.t. \( \xi \) gives the momentum constraint

\[
\sqrt{\gamma} H^3 = p_{\gamma;i} - \frac{1}{2} \pi_\Phi \Phi ; i = 0. \tag{7.26}
\]

This constraint algebra will close because it is just a transformed version of a minimally coupled scalar field.

There is thus a one-to-one correspondence between solutions of the Einstein equations with a minimally coupled massless scalar field and the solutions of the ‘vacuum’ Brans–Dicke equations. The true difference between the two theories is whether the Einstein metric \( g \) or the Brans–Dicke metric \( \gamma \) determines the geodesics. One way to test this is to add a second minimally-coupled scalar field.

Therefore, consider the Brans–Dicke action \( \gamma H \) and minimally couple a second, constant mass, scalar field \( \chi \) to it. To do this, we add \( \Delta P = -\Phi^2 \frac{1}{2} (\gamma^{ij} \chi_i \chi_j - m^2 \chi^2) \) to the potential term and add \( \Delta T = \partial \gamma^{ij}/\partial \lambda - \chi_i \xi^i \right)^2 \) to the kinetic term. Rather than computing in the Brans–Dicke frame it is easier to see what is happening in the Einstein frame. Let us now undo the transformation \( \gamma H \). The Brans–Dicke field translates into Einstein gravity with a minimally-coupled scalar field. The kinetic term is of the standard form,

\[
T = (g^{ac}g^{bd} - g^{ab}g^{cd}) \left[ \frac{\partial g_{ab}}{\partial \lambda} - (K\xi)_{ab} \right] \left[ \frac{\partial g_{cd}}{\partial \lambda} - (K\xi)_{cd} \right] + \left[ \frac{\partial \phi}{\partial \lambda} - \phi \partial_d \xi^d \right]^2 + \left[ \frac{\partial \chi}{\partial \lambda} - \chi \partial_d \xi^d \right]^2, \tag{7.27}
\]

the additional potential term becomes

\[
P = R - \frac{1}{4} g^{ab} \phi_{,a} \phi_{,b} - \frac{1}{4} \left( g^{ab} \chi_{,a} \chi_{,b} - e^{2\phi \gamma^{ab} \xi^2} \right). \tag{7.28}
\]

This is a special case of the dilaton theories we discussed earlier. Therefore the constraint algebra will close both in the Einstein frame and in the Brans–Dicke frame. Therefore Brans–Dicke fits very naturally into the BSW framework.

The conclusion of this section must be that, when we consider scalar fields coupled to gravity and insist on a local square root action, we recover causality. The characteristic speeds of the scalar fields must coincide with that of gravity. Other than that, there is a large freedom in the detailed form of the scalar fields and in the way they interact. In the next section we deal with vector fields. There we discover that the detailed form of the vector field is rigidly prescribed. The only theory that fits our framework is massless electrodynamics, we recover the Maxwell equations unambiguously.

**VIII. THREE-VECTORS FIELD INTERACTING WITH GRAVITY**

Since the kinetic and potential terms of different fields are simply added separately to the potential and \( T \) and do not mix unless an interaction between them is introduced explicitly, we can treat different fields (scalar, 3-vector,
and vary it to get

\[ \partial \mathcal{A}_a / \partial \lambda - \mathcal{L}_{\xi \mathcal{A}} = \partial \mathcal{A}_a / \partial \lambda - g^{bc} \mathcal{A}_{a;b} \xi_c - g^{bc} \xi_{b;a} \mathcal{A}_c. \]  

(8.1)

Therefore we add a term

\[ T_A = g^{ad} \left( \partial \mathcal{A}_a / \partial \lambda - g^{bc} \mathcal{A}_{a;b} \xi_c - g^{bc} \xi_{b;a} \mathcal{A}_c \right) \left( \partial \mathcal{A}_d / \partial \lambda - g^{bc} \mathcal{A}_{d;b} \xi_c - g^{bc} \xi_{b;d} \mathcal{A}_c \right) \] 

(8.2)

to the metric kinetic energy.

The additions \( U_A \) to the potential are equally obvious:

\[ U_A = C_1 A_{a;b} A^{a;b} + C_2 A_{a;b} A^{b;a} + C_3 A_{a;b} A^{c;d} + \sum_k B_k (A^a A_a)^k \]

\[ = (C_1 g^{ab} g^{cd} + C_2 g^{ad} g^{bc} + C_3 g^{ac} g^{bd}) A_{a;c} A_{b;d} + \sum_k B_k (g^{ab} A_a A_b)^k. \]

(8.3)

Hence we begin with the modified BSW action (which we call \( A_A \))

\[ A_A = \int d\lambda \int \sqrt{g} R + U_A \sqrt{g}^2 + \mathcal{T}^3 \] 

and vary it to get

\[ p^{ij} = \frac{\delta \mathcal{L}}{\delta \left( \frac{\partial \mathcal{A}_i / \partial \lambda}{\sqrt{g}} \right)} = \sqrt{g} (g^{R + U_A} \right) ((g^{ic} g^{id} - g^{ij} g^{cd}) \left[ \partial g_{cd} / \partial \lambda - (K \xi)_{cd} \right). \]

(8.5)

This can be inverted to give

\[ \frac{\partial g_{ij}}{\partial \lambda} = 2N \sqrt{g} \left( p_{ij} - \frac{1}{2} g_{ij} p \right) + \xi_{ij} + \xi_{ji}. \]

(8.6)

where we define \( 2N = \sqrt{g} / \sqrt{T_A + R + U_A} \). The momentum conjugate to \( A_a \) is

\[ \pi^a = \frac{\delta \mathcal{L}}{\delta \left( \frac{\partial A_a / \partial \lambda}{\sqrt{g}} \right)} = \sqrt{g} (g^{R + U_A} \right) g^{ab} \left[ \partial A_b / \partial \lambda - \mathcal{L}_{\xi A_b} \right]. \]

(8.7)

This also can be inverted to give

\[ \frac{\partial A_b}{\partial \lambda} = 2N \pi_b / \sqrt{g} + \mathcal{L}_{\xi A_b}. \]

(8.8)

The square-root identity becomes

\[ g H = -p^{ij} p_{ij} + \frac{1}{2} p^2 - \pi^a \pi_a + g(R + U_A) = 0, \]

(8.9)

and variation w.r.t. \( \xi \) gives the momentum constraint

\[ \sqrt{g} H^j = p^{ij} ;_{i} - \frac{1}{2} \left( \pi^c A_{c}^{ij} - \pi^b A_{b}^{ij} \right) = p^{ij} ;_i - \frac{1}{2} \left( \pi^c \left[ A_{c}^{ij} - A^{ij}_{c} \right] - \pi^b A^b \right) = 0. \]

(8.10)

The Euler–Lagrange evolution equations for \( g_{ij} \) and \( A_a \) are

\[ \frac{\partial p^{ij}}{\partial \lambda} = \frac{\delta \mathcal{L}}{\delta g_{ij}} = -\sqrt{g} (R^{ij} - g^{ij} R) - \frac{2N}{\sqrt{g}} (p^{im} p_{m} - \frac{1}{2} p p^{ij}) + \sqrt{g} \left[ \partial A_{a} / \partial \lambda - \mathcal{L}_{\xi A_b} \right]. \]

(8.11)

\[ \frac{\partial \pi^i}{\partial \lambda} = \frac{\delta \mathcal{L}}{\delta A_i} = -\frac{\sqrt{g} N}{\sqrt{g}} \left[ C_1 A^{i[a} A^{j]a} + A^{a[i} A^{j]}_{b} \right] + C_2 A^{i[a} A^{j]}_{b} A^{k} + C_3 A^{a[i} A^{b]}_{j} + \sum_k B_k (A^a A_a)^k + \mathcal{L}_{\xi} p^{ij}. \]

(8.12)
We now check for propagation of the constraints (8.3) and (8.10) under the evolution by (8.6), (8.8), (8.11) and (8.12). As expected, the momentum constraint propagates. In the Hamiltonian constraint, the simple self-interaction terms (with coefficients $B_k$) with no derivatives of either $A_a$ or $g_{ij}$ give no problem at this stage. However, $\partial H/\partial \lambda$ contains terms that are not proportional to the constraints. The full expression is

$$\frac{\partial}{\partial \lambda} \left[ \sqrt{g} (R + U_A) - \frac{1}{\sqrt{g}} \left( p^{ab} p_{ab} - \frac{1}{2} p^2 + \pi^a \pi_a \right) \right] =$$

$$2 N^a \left( 2 p^b_{;ab} - \pi^c \left[ A_{c;a} - A_{a;c} \right] + \pi^b_{;c} A^c \right) + N \left( 2 p^{ab}_{;a} - \pi^c \left[ A_{c;b} - A_{b;c} \right] + \pi^a_{;b} A^b \right)_{;b} +$$

$$\frac{1}{2} N p \left[ \sqrt{g} (R + U_A) - \frac{1}{\sqrt{g}} \left( p^{ab} p_{ab} - \frac{1}{2} p^2 + \pi^a \pi_a \right) \right] + L N \left[ \sqrt{g} (R + U_A) - \frac{1}{\sqrt{g}} \left( p^{ab} p_{ab} - \frac{1}{2} p^2 + \pi^a \pi_a \right) \right] +$$

$$\frac{4 C_i + 1}{N} \left( N^2 \pi^a A_{a;b} \right)_{;b} + \frac{4 C_i - 1}{N} \left( N^2 \pi^a A_{b;a} \right)_{;b} - \frac{1}{N} \left( N^2 \pi^a A_{a;b} \right)_{;b} + \frac{4 C_i}{N} \left( N^2 \pi^a A_{b:a} \right)_{;a}$$

$$- 2 \frac{C_i + C_2}{N} \left( 2 N^2 A_{(b;cd)} A^f (p_{f}^b - \frac{1}{2} p_{d}^b) \right)_{;d} - \left( N^2 A_{(b;cd)} A^f (p_{f}^b - \frac{1}{2} p_{d}^b) \right)_{;f}$$

$$- 2 \frac{C_i}{N} \left( 2 N^2 (p_{cd;} A)_{g} A^f (p_{g}^b - \frac{1}{2} p_{d}^b) \right)_{;d} - \left( N^2 (p_{cd;g} A)_{f} A^f (p_{f}^b - \frac{1}{2} p_{d}^b) \right)_{;f}. \quad (8.13)$$

The last six terms do not vanish weakly. They have to vanish, otherwise we get a family of secondary constraints which eliminates all the degrees of freedom. We can (and must!) eliminate most of them by choosing $C_1 = -C_2 = -1/4, C_3 = 0$ (which means that $U_A = - (\text{curl} A)^2/4$). This still leaves us with $\pi^b_{;b} = 0$ as a new, secondary, constraint which we cannot eliminate. This is none other than the Gauss constraint.

Propagation of this new constraint requires

$$\frac{\partial}{\partial \lambda} \pi^b_{;b} = +2 \sqrt{g} \left[ N \sum_{k} k B_k (A^A A_a)^{k-1} A^A \right]_{;i} + L \xi \pi_i \pi_{;i}. \quad (8.14)$$

The only way to ensure propagation is to set $B_k = 0 \forall k$. Therefore, the previously unproblematic undifferentiated potential terms are incompatible with the extra constraint. In our scheme, a scalar field can have mass (or power-law nonlinear coupling, say $\phi^4$) but a 3-vector field cannot. Only the Maxwell 3-vector field minimally coupled to gravity and locked to its light cone survives.

Since only curl $A$ appears in the potential, addition of a time-independent scalar to $A_a, A_a \rightarrow A_a + \partial_a \Lambda$, does not change the action. This is the analogue of a time-independent 3-coordinate transformation. If, more appropriately, we take a time-dependent $\Lambda$, we need to add the extra term $-\partial_{\lambda} \Phi$ to (8.1). The role of $\Phi$ is to absorb the $\partial A / \partial \lambda$ term that arises from the $\partial A / \partial \lambda$ term. The $\xi$ terms play just the same role in compensating time-dependent coordinate transformations.

It is now easy to build the divergence constraint $\pi^b_{;b} = 0$ into the action. We simply regard $\Phi$ as an independent variable and vary w.r.t. it, just as with $\xi$. Thus we regard $A_a$ as a gauge field, so that $\pi^b_{;b} = 0$ becomes the Gauss constraint.

The full correction to (8.1) is now

$$\frac{\partial A_a}{\partial \lambda} - \frac{\partial A_a}{\partial \lambda} - \Phi_{;a} - L \xi A_a = \frac{\partial A_a}{\partial \lambda} - g^{bc} A_{a;b} \xi_c - g^{bc} \xi_{b;a} A_c - \Phi_{;a}, \quad (8.15)$$

and we minimize w.r.t. $\Phi$ in exactly the same way as w.r.t. $\xi$. This ‘gauge best matching’ yields the Gauss constraint. Thus, the only way we have found to propagate the constraints of a metric–vector BSW-type action is to take

$$U_A = \frac{1}{4} (A^a A_{b;a} - A^{a;b} A_{b;a}) = -\frac{1}{4} \left( \text{curl} A \right)^2$$

$$T = T_0 + T_A = (g^{ac} g^{bd} - g^{ab} g^{cd}) \left[ \frac{\partial g_{ab}}{\partial \lambda} - (K \xi)_{ab} \right] \left[ \frac{\partial g_{cd}}{\partial \lambda} - (K \xi)_{cd} \right]$$

$$+ g^{ad} \left[ \frac{\partial A_d}{\partial \lambda} - g^{bc} A_{d;b} \xi_c - g^{bc} \xi_{b;d} A_c - \Phi_{;d} \right] \left[ \frac{\partial A_d}{\partial \lambda} - g^{bc} A_{d;b} \xi_c - g^{bc} \xi_{b;d} A_c - \Phi_{;d} \right], \quad (8.16)$$

which is simply Einstein–Maxwell. There is, however, a technical point to note here. The Lie-corrected velocity that appears in the square parentheses in the final line of (8.16) does not appear in the form generally encountered in the literature. However, we can rewrite

$$\frac{\partial A_a}{\partial \lambda} - g^{bc} A_{a;b} \xi_c - g^{bc} \xi_{b;a} A_c - \Phi_{;a} = \frac{\partial A_a}{\partial \lambda} - g^{bc} A_{a;b} \xi_c + g^{bc} A_{b;a} \xi_c - g^{bc} A_{b;a} \xi_c - \Phi_{;a}$$

$$= \frac{\partial A_a}{\partial \lambda} - g^{bc} A_{a;b} \xi_c - [\xi C_{a}]_{;a} - \Phi_{;a} = \frac{\partial A_a}{\partial \lambda} - g^{bc} A_{a;b} \xi_c - \Psi_{;a}. \quad (8.17)$$
Here, we have replaced the gauge variable $\Phi$ by a new scalar variable $\Psi = \Phi + \xi^c A_c$. This means that in Eq. (8.17) we now have a term linear in the undifferentiated shift which is just the familiar minimally-coupled Maxwell field tensor. The residual scalar part that results from this manipulation gets absorbed in the original gauge variable $\Phi$, which is the reason why we denote this new composite scalar by $\Psi$. Two points are worth mentioning here: 1) The Lie derivative of a 3-vector field becomes ambiguous if the field is gauged, as is reflected in the heterogeneous gauge–diffeomorphism nature of $\Psi$; 2) we are obtaining highly sophisticated theories – Einsteinian gravity and Maxwell minimally coupled to gravity – essentially uniquely out of the 3-space approach but always only 'by the skin of our teeth'.

It is illuminating to consider why a massive vector field, which is a perfectly good generally covariant theory, is not allowed in the 3-space approach. As Giulini pointed out to us [21], the ADM decomposition for the massive vector field is quite different from the massless case. While the momentum can be written as

$$\pi_a = \sqrt{g} \left[ \frac{\partial A_a}{\partial \xi} - F_{ab} N^b - A_{0,a} \right]$$

which looks identical to the last term in Eq. (8.17), there is a major difference in that, while $\Psi$ is a true gauge degree of freedom and can be chosen arbitrarily, $A_0$ in the massive vector field case is fixed. There exists a primary constraint that the momentum conjugate to $A_0$, $\pi^0$, vanishes. The conservation of this constraint leads to

$$\pi^i_0 + \frac{\sqrt{g}}{N} m^2 [A_0 - N^a A_a] = 0.$$  \hspace{1cm} (8.19)

This is quite different from the massless case, for which, of course, $m = 0$, so that the second term is zero and we get a constraint that is homogeneous (and linear) in the canonical momenta and contains neither $A_0$ nor the shift. It is this restriction to the canonical momenta that limits the physical degrees of freedom and indicates that the action is defined on curves in the space that is the product of superspace and the transverse degrees of freedom of the vector field. This matches our ideal of a geodesic-type theory. In the massive case, the restriction is lifted since the part homogeneous in the canonical momenta is now equal to a term containing what were previously purely gauge variables. In whatever way one attempts to interpret this extra term, it is clear that the massive vector field does not belong to the class of theories in which best matching with a local square root is applied to \textit{bona fide} three-dimensional geometrical objects.

Thus, we find that a non-gauge vector field cannot be coupled in any simple manner to the BSW action. For the reasons to be explained in the conclusions, we think it would be premature to attempt a rigorous no-go theorem, but we feel the provisional result is already remarkable and even hints at a partial unification of gravity and electromagnetism. The fact is that in pure geometrodynamics the BSW action is essentially unique, and we have found only one vector field that couples to it: Maxwell. Since our approach exploits 3-geometry to the maximal extent possible but nothing else, we can say that Maxwellian theory is uniquely inherent in Riemannian 3-geometries. We have already noted that, within our scheme, the attempt to couple a single 3-vector field to scalar fields leads to the standard U(1) gauge coupling [12], and the attempt to let a collection of 3-vector fields interact among themselves leads to Yang-Mills gauge theory [13].

Our result also gives further insight into the origin of Lorentz invariance. We begin with the field $A_0$, which is as unashamedly three dimensional as the 3-metric. How does the full panoply of the 4-potential $A_\nu, \nu = 1, 2, 3, 4$, and the electromagnetic field tensor $F_{\mu\nu}$ arise? The answer is that the extra ('time') elements arise from the combined effect of the Lie-derivative best-matching correction to the 'bare' 3-velocity $\partial A_\nu / \partial \lambda$ and from having to propagate the new square-root constraint. Best matching and the local square root do it all. It is particularly striking that the universal light cone and gauge theory arise from one and the same Lie-derivative correction mechanism. For the scalar field, the terms generated by the Lie correction in the momentum constraint fix one constant in the constraint-propagation condition and lock it to the gravity light cone. For the 3-vector field, they fix three coefficients, with the same effect, and impose the Gauss constraint. So, in a way, light cone and gauge are the same thing. Both derive from best-matched gravity.

Some results of this section are only partly new, since Teitelboim [18] showed that his postulates (see the end of Sec. VI) enforce gauge coupling. As in the case of the HKT result [2], we obtain his result with a significantly weaker assumption, and we also obtain the light cone and reveal its intimate connection to gauge theory.

**IX. CONCLUDING REMARKS**

Hitherto it has always seemed that four-dimensional general spacetime covariance is the very essence of GR. Many physicists have expressed strong reservations about the 3+1 Hamiltonian formalism. It is held to be against the spirit
of general covariance and incapable of encompassing the wide range of topologies allowed by GR. The restriction to globally hyperbolic spacetimes – a necessary condition for the Hamiltonian treatment – is often severely criticized.

We believe that the present work, if it can be successfully extended to the fermionic sector and withstands critical peer review, puts these issues in a different light. The fact is that, in a choice between two different theoretical schemes, there must always be a preference (in the absence of compelling experimental evidence) for the one that is more restrictive and, hence, makes stronger predictions. There are two respects in which the 3-space approach gives tighter predictions than the Einstein–Minkowski approach: it rules out many fields (the massive vector field for example) and, within the possibilities that remain, rules out many solutions, e.g., solutions with closed timelike curves, which are not globally hyperbolic. These are potential benefits.

Here we should also mention the possibility of advancing to genuinely new physics. Our present paper does not actually predict any new theory. It merely slims down a class of theories long known to exist (all generally covariant theories) and, within the restricted class, rules out exotic scenarios (time travel for example). However, given that the treatment of fermions in curved space is so delicate, we do not rule out the possibility that the extension to the fermionic sector (if it succeeds) will bring further insights. There is also the possibility [5], already mentioned in the paper, of extension of the idea of best matching to conformal superspace (CS). As yet, we have performed this extension only for the matter-free case, but the results so far obtained are promising. They suggest the existence of a theory virtually identical to GR except for the elimination of scale as a dynamical variable.

If a full theory can be developed on CS, the implications are far reaching. In the truly scale-invariant theory that will result, the Hubble red shift cannot be explained by the ‘stretching of space’, as it is at present. For the theory on CS is designed precisely to eliminate such ‘stretching’ as a degree of freedom. According to the present standard model, the universe is both expanding and simultaneously changing its ‘shape’, i.e., becoming more ‘clumpy’. The expansion is used to explain the Hubble red shift. In a scale-invariant theory, only change of shape is physically meaningful, and it must explain the Hubble law though a gravitational red shift induced by ‘clumping’ [6], [22]. Thus, the 3-space approach does have the potential to lead to very different cosmological predictions.

Finally, it is worth mentioning that the local square root creates a theory that seems to be maximally sensitive to all properties of Riemannian 3-geometries, as can be seen by comparing the global form (3.7) with the local form (3.6). The product of two global integrals in (3.7) cannot be as sensitive as the local form. Indeed, it seems to us that the twin principles of best matching and the local square root may implement the Cartesian ideal of explaining all dynamics by geometry.

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