The Pion-Nucleon Coupling Constant
in QCD Sum Rules

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Abstract

The pion-nucleon coupling constant $g_{\pi N}$ is studied on the basis of the QCD sum rules. Both the Borel sum rules and the finite energy sum rules for $g_{\pi N}$ are used to examine the effects of higher dimensional operators (up to dim. 7) and $\alpha_s$ corrections in the operator product expansion. Agreement with the experimental number is reached only when $S_\pi/S_N$ is greater than one, where $S_\pi$ ($S_N$) is the continuum threshold for the $g_{\pi N}$ (nucleon) sum rule.
I. INTRODUCTION

The pion-nucleon coupling constant $g_{\pi N}$ is one of the most fundamental quantities in hadron physics. $g_{\pi N} = 13.4 \pm 0.1$ is obtained empirically using the $N - N$ scattering data, $\pi - N$ scattering data and the deuteron properties \[1\]. From the theoretical point of view, it is a great challenge to reproduce this value from the first principle namely the quantum chromodynamics (QCD). So far, there have been two attempts; one is based on the lattice QCD simulations (see ref. \[2\]) and the other is based on the QCD sum rules (QSR) \[3,4\]. The latter approach for $g_{\pi N}$, however, has not been explored in great detail beyond the leading order of the operator product expansion (OPE) \[3,4\] within the authors’ knowledge.\[1\] The purpose of this paper is to reexamine the problem using the currently available information on the higher dimensional operators in OPE and the $\alpha_s$ corrections to the Wilson coefficients.

To study $g_{\pi N}$ in QSR, two methods have been proposed so far: (i) method based on the two point function $\langle 0 | T\eta(x)\bar{q}(0) | \pi \rangle$, and (ii) method based on the three point function $\langle 0 | T\eta(x)\bar{q}\gamma_5q(y)\bar{\eta}(0) | 0 \rangle$, where $\eta(x)$ is the nucleon interpolating field. We will take the first approach (i) throughout this paper, since OPE in (ii) is problematic when the four momentum of the pion becomes soft. The lowest non-trivial order of OPE in (i) is known to give $g_{\pi N}^{\text{lowest}} = M_N/f_\pi$ when the continuum threshold is neglected \[4,5\]. The origin of the 25% disagreement of $g_{\pi N}^{\text{lowest}}$ from the Goldberger-Treiman (GT) relation $g_{\pi N} = g_A M_N/f_\pi$ may originate either from the higher dimensional operators, $\alpha_s$ corrections and the continuum threshold. We will examine $g_{\pi N}$ with all these ingredients.

In section II, we will examine QSR for the nucleon mass by adopting OPE up to dimension 7 operators with $O(\alpha_s)$ corrections, which is essential for the discussions in later sections. In section III, QSR for $g_{\pi N}$ is studied in close analogy with that for the nucleon mass. In section IV, Borel analyses for $g_{\pi N}$ are made, and the effect of the $\alpha_s$ corrections, higher dimensional operators and the continuum thresholds are studied. Section V is devoted to summary and concluding remarks.

II. SUM RULES FOR THE NUCLEON

Let’s consider the two point function,
\[ \Pi_{\alpha\beta}(q) = i \int d^4x e^{iq\cdot x} \langle 0 | T \eta_\alpha(x) \bar{\eta}_\beta(0) | 0 \rangle = \Pi_1(q) \hat{q}_{\alpha\beta} + \Pi_2(q) 1_{\alpha\beta}, \] (1)

with \( \hat{q} \equiv q \cdot \gamma \). For interpolating nucleon current, we use the Ioffe current [7]

\[ \eta(x) = \epsilon_{abc}(u^a(x)C\gamma_\mu u^b(x))\gamma_5\gamma^\mu d^c(x), \] (2)

where \( a, b \) and \( c \) are color indices and \( C \) is the charge conjugation operator.

The operator product expansion (OPE) at \( q^2 \to -\infty \) has a general form

\[ \int d^4x e^{iq\cdot x} T \eta(x) \bar{\eta}(0) = \sum_n C_n(q, \mu, \alpha_s(\mu^2)) O_n(\mu), \] (3)

where \( C_n \) denote the Wilson coefficients and \( O_n \) are the local gauge invariant operators. They depend on the renormalization scale \( \mu \) which separates the short distance dynamics in \( C_n \) and the long distance dynamics in \( O_n \). If one takes the vacuum expectation value of (3), it can be used for the sum rules of the nucleon, while if one takes the vacuum to pion matrix element, it can be used for the sum rules of \( g_{\pi N} \).

The Lorentz structure of \( O_n \) depends on the states one chooses to sandwich (3). For the nucleon sum rules, only the scalar operators contribute. OPE up to dimension 7 with \( \alpha_s \) corrections in the chiral limit can be extracted from refs. [8] and [9];

\[ \text{Re} \Pi(Q^2) = \left( \Pi_1^a + \Pi_1^b + \Pi_1^c \right) \hat{q} + \left( \Pi_2^d + \Pi_2^e + \Pi_2^f \right), \] (4)

\[ \Pi_1^a(Q^2) = -\frac{1}{64\pi^4} Q^4 \ln \frac{Q^2}{\mu^2} \left( 1 + \frac{71}{12} \frac{\alpha_s}{\pi} - \frac{1}{2} \frac{\alpha_s}{\pi} \ln \frac{Q^2}{\mu^2} \right), \] (5)

\[ \Pi_1^b(Q^2) = -\frac{1}{32\pi^2} \left( \frac{\alpha_s}{\pi} \langle G^2 \rangle \right) \ln \frac{Q^2}{\mu^2}, \] (6)

\[ \Pi_1^c(Q^2) = \frac{2}{3} \langle \bar{u}u \rangle^2 \left( 1 - \frac{\alpha_s}{\pi} \left( \frac{1}{3} \ln \frac{Q^2}{\mu^2} + \frac{5}{6} \right) \right), \] (7)

\[ \Pi_2^d(Q^2) = -\frac{\langle \bar{d}d \rangle}{4\pi^2} Q^2 \ln \frac{Q^2}{\mu^2} \left( 1 + \frac{3}{2} \frac{\alpha_s}{\pi} \right), \] (8)

\[ \Pi_2^e(Q^2) = 0, \] (9)

\[ \Pi_2^f(Q^2) = \frac{1}{18Q^2} \langle \frac{\alpha_s}{\pi} \langle G^2 \rangle \rangle \langle \bar{d}d \rangle, \] (10)

where \( Q^2 \equiv -q^2 \to \infty \) and \( \langle \cdot \rangle \) denotes the vacuum expectation value. The argument of \( \alpha_s \) is \( \mu^2 \) which is not written explicitly. The diagrammatic illustration of \( \Pi_1^a \sim \Pi_2^f \) is shown in Fig. 1.

Several remarks are in order here.

(a) We take the chiral limit \( (m_q = 0) \) throughout this paper. Small \( u, d \) quark mass does not change the essential conclusion of this paper.
(b) The fact that the Wilson coefficient of the dimension 5 operator $g\bar{q}\sigma \cdot Gq$ vanishes in $\Pi_2$ is a unique property of the Ioffe current $[7]$. Since this operator is already $O(g)$, we do not consider the $O(\alpha_s)$ correction to it.

(c) A small discrepancy between the formula in $[9]$ and that in $[10]$ for $\Pi^d_2$ has been recently resolved (see $[11]$) and the final result boils down to the form given in the above.

(d) We always assume the vacuum saturation when evaluating matrix elements of higher dimensional operators.

The correlation function $[4]$ satisfies the standard dispersion relation

$$\text{Re}\Pi_{1,2}(Q^2) = \frac{1}{\pi} \int \frac{\text{Im}\Pi_{1,2}(s)}{s+Q^2} ds + \text{subtraction.} \quad (11)$$

In QSR, $\text{Re}\Pi(Q^2)$ in the left hand side of (11) is calculated by OPE at large $Q^2$ as given in (4), while $\text{Im}\Pi(s)$ in the right hand side is parametrized by the nucleon pole and the phenomenological continuum. The pole part reads

$$\hat{q}\text{Im}\Pi^\text{pole}_1(s) + \text{Im}\Pi^\text{pole}_2(s) = \pi \lambda_N^2 (\hat{q} + M_N) \delta(s - M_N^2), \quad (12)$$

where $\lambda_N$ is defined as $\langle 0 | \bar{\eta} | N \rangle = \lambda_N u(p) \rangle$ with $u(p)$ being the nucleon Dirac spinor. We assume that, when $s > S_N$, the hadronic continuum reduces to the same form with that obtained by an analytic continuation of OPE;

$$\text{Im}\Pi^\text{cont}_1(s) = \pi \theta(s - S_N) \left[ \frac{s^2}{64\pi^4} \left\{ 1 + \frac{\alpha_s}{\pi} \left( \frac{71}{12} - \log\left( \frac{s}{\mu^2} \right) \right) \right\} + \frac{1}{32\pi^2} \left( \frac{\alpha_s}{\pi} G^2 \right) \right], \quad (13)$$

$$\text{Im}\Pi^\text{cont}_2(s) = -\pi \theta(s - S_N) \frac{\langle \bar{d}d \rangle}{4\pi^2} s \left( 1 + \frac{3}{2} \frac{\alpha_s}{\pi} \right). \quad (14)$$

Substituting (4) and (12,13,14) into the dispersion relation (11), one can generate sum rules for the resonance parameters. We will write here the Borel transformed version of the sum rules in which the higher dimensional operators and the effect of the continuum can be relatively suppressed. (See Appendix A for useful formula.)

$$4\pi^4 \lambda_N^2 e^{-M_N^2/M^2} = \frac{M^6}{8} E_2(x) \left( 1 + \frac{53}{12} + \gamma_E \frac{\alpha_s(M^2)}{\pi} \right)$$

$$+ b M^2 \frac{32}{3} E_0(x) + \frac{a_u^2}{6} \left( 1 - \frac{5}{6} - 3 \frac{1}{\gamma_E} \frac{\alpha_s(M^2)}{\pi} \right), \quad (15)$$

$$4\pi^4 \lambda_N^2 M_N e^{-M_N^2/M^2} = \frac{a_d}{4} M^4 E_1(x) \left( 1 + \frac{3}{2} \frac{\alpha_s(M^2)}{\pi} \right) - \frac{a_d b}{72}. \quad (16)$$

Here $M^2$ is the Borel mass, $E_n = 1 - (1 + x + \cdots + \frac{x^n}{n!}) e^{-x}$ with $x = S_N/M^2$, $\gamma_E$ is the Euler constant (0.5772· · ·) and
\[ a_q \equiv -4\pi^2 \langle \bar{q}q \rangle, \quad b \equiv 4\pi^2 \langle \frac{\alpha}{\pi} G^2 \rangle. \] (17)

Note that we chose \( \mu^2 \) to be \( M^2 \) which is a typical scale of the system after the Borel transform. We call eq. (15) (eq. (16)) as “even” (“odd”) sum rule since it contains only even (odd) dimensional operators. As for the running coupling constant, we use a simplest one-loop form,

\[ \alpha_s(M^2) = \frac{4\pi}{9\log(M^2/\Lambda^2)}, \] (18)

with \( \Lambda^2 = (0.174\text{GeV})^2 \) which is obtained to reproduce \( \alpha_s(1) \simeq 0.4 \) [12].

Eq.(14) and Eq. (15) will be used when we analyse the sum rules for \( g_{\pi N} \). One can derive formula for \( M_N \) as a function of \( M \) in three ways; (i) the ratio of (15) and (16), (ii) the ratio of (15) and its logarithmic derivative with respect to \( M^2 \) and (iii) the ratio of (16) and its logarithmic derivative with respect to \( M^2 \). We have made an extensive Borel stability analyses for these three cases. We found that the higher dimensional operator and the \( \alpha_s \) corrections improve the Borel stability as well as the prediction for \( M_N \). However, there is a wide range of \( S_N \) which can reproduce the experimental nucleon mass in the Borel analyses. In later sections, we will use the finite energy sum rules (FESR) to fix \( S_N \).

In Fig.1, shown are the Borel curves for three cases (i)-(iii) with \( S_N = 1.601\text{GeV}^2 \) obtained by the FESR with \( \langle \bar{q}q \rangle = -(0.2402\text{GeV})^3 \) and \( \langle \frac{\alpha}{\pi} G^2 \rangle = 0.012\text{GeV}^4 \). The solid, dashed and dash-dotted curves correspond to cases (i), (ii) and (iii), respectively. The curve in case (iii) is the most stable of three cases and reproduces the experimental value. The others do not show good stability. Also, three curves are rather sensitive to the change of \( S_N \).

III. SUM RULES FOR THE \( \pi - N \) COUPLING CONSTANT

In this section, we look at the two-point correlation function,

\[ \Pi_{\alpha\beta}^\pi(q) = i \int d^4xe^{iq\cdot x}\langle 0|T\eta_\alpha(x)\bar{\eta}_\beta(0)|\pi^0(p = 0)\rangle, \] (19)

where \( |\pi^0(p = 0)\rangle \) is a neutral pion state with vanishing four momentum (the soft pion). Since we are working in the chiral limit, this soft pion is simultaneously on shell.

Since we are taking the matrix element between the vacuum and the soft pion in (19), only the pseudo-scalar operator \( \bar{q}i\gamma_5q \) survives in OPE for \( T\eta(x)\bar{\eta}(0) \). The pseudo-vector operator \( \bar{q}\gamma_\mu\gamma_5q \) vanishes since \( \langle 0|\bar{q}\gamma_\mu\gamma_5q|\pi^0(p)\rangle \propto p_\mu \to 0 \). Thus the relevant terms in OPE read as follows:
\[ \Pi^\pi(Q^2) = \Pi_{\text{dim}3}(Q^2) + \Pi_{\text{dim}5}(Q^2) + \Pi_{\text{dim}7}(Q^2), \quad (20) \]
\[ \Pi_{\text{dim}3}(Q^2) = -i\gamma_5 \frac{\left< 0 | \bar{d} i \gamma_5 d | \pi \right>}{4\pi^2} \left( 1 + \frac{3}{2} \frac{\alpha_s}{\pi} \right) Q^2 \ln \frac{Q^2}{\mu^2}, \quad (21) \]
\[ \Pi_{\text{dim}5}(Q^2) = 0, \quad (22) \]
\[ \Pi_{\text{dim}7}(Q^2) = i\gamma_5 \frac{1}{18Q^2} \left( \frac{\alpha_s}{\pi} G^2 \right) \left< 0 | \bar{d} i \gamma_5 d | \pi \right> \cdot \cdot \cdot \quad (23) \]

\( \mu^2 \) dependence of \( \alpha_s \) is again implicit in the above equations.

The diagrams corresponding to (21), (22) and (23) are Fig.1(d), 1(e) and 1(f) respectively. The Wilson coefficients with \( O(\alpha_s) \) corrections in (21), (22) and (23) turn out to be identical to (8), (9) and (10) respectively. This can be explicitly checked by carrying out OPE with the background field method in the fixed point gauge. An alternative way to see this is the plain wave method. As an illustration, let’s consider dimension 3 operator (Fig.1(d)) and expand eq.(1) in terms of \( \bar{d} d \) and \( \bar{d} i \gamma_5 d \).

\[ i \int d^4 x e^{iq \cdot x T(\eta_\alpha(x) \bar{\eta}_\beta(0)) = C_s(1)_{\alpha\beta} \bar{d} d + C_{ps}(i\gamma_5)_{\alpha\beta} \bar{d} i \gamma_5 d + \cdots \quad (24) \]

where \( C_{s,ps} \) is the Wilson coefficient with \( s(ps) \) denoting scalar (pseudoscalar). Note that \( \cdots \) are vector, pseudovector and tensor operators. Note also that \( \bar{u} u \) and \( \bar{u} i \gamma_5 u \) do not arise for the Ioffe current. We sandwich eq.(24) by free quark states to extract the Wilson coefficients. Applying the projections \( 1_{\alpha\beta} \) and \( (i\gamma_5)_{\alpha\beta} \) to eq.(24), one gets

\[ C_s = \frac{L_1(q^2 \rightarrow -\infty)}{\left< p | \bar{d} d | p \right>}, \quad L_1 = \frac{i}{4} \int d^4 x e^{iq \cdot x} \left< p | T(\eta(x) \bar{\eta}(0)) | p \right> \quad (25) \]
\[ C_{ps} = \frac{L_{i\gamma_5}(q^2 \rightarrow -\infty)}{\left< p | \bar{d} i \gamma_5 d | p \right>}, \quad L_{i\gamma_5} = \frac{-i}{4} \int d^4 x e^{iq \cdot x} \left< p | T(\eta(x)i\gamma_5 \bar{\eta}(0)) | p \right> \quad (26) \]

with \( | p \rangle \) being a quark state with momentum \( p \). Since massless QCD does not flip chirality in perturbation theory, \( \left< p | \bar{d} d | p \right> \sim \left< p | \bar{d} i \gamma_5 d | p \right> \) and \( L_1 \sim L_{i\gamma_5} \) except for trivial kinematical factors. Thus \( C_s = C_{ps} \) is obtained even when \( \alpha_s \) corrections are included.

\( \Pi^\pi(Q^2) \) satisfies the dispersion relation

\[ \text{Re} \Pi^\pi(Q^2) = \frac{1}{\pi} \int \frac{\text{Im} \Pi^\pi(s)}{s + Q^2} ds + \text{subtraction}. \quad (27) \]

The hadronic imaginary part \( \text{Im} \Pi^\pi(s) \) has a nucleon pole and the continuum. The pole part is parametrized by assuming the \( \pi - N - N \) vertex \( \mathcal{L}_{\text{int}} = g_{\pi NN} \bar{N} i \gamma_5 \vec{\tau} \cdot \vec{\pi} N \) as was done in [4,5]. The continuum part is extracted from the analytic continuation of OPE. In total,
\[ \text{Im} \Pi^\pi(s) = i \gamma_5 \pi^N g_{\pi N} \delta(s - M_N^2) \]
\[ -i \gamma_5 \pi s \left( 1 + \frac{3 \alpha_s}{2 \pi} \right) \theta(s - S_\pi) \langle 0 | \bar{d} i \gamma_5 d | \pi^0 \rangle, \tag{28} \]

where \( S_\pi \) is the continuum threshold. Note that \( S_\pi \) does not have to be the same with \( S_N \).

Putting Eq.(28) and Eq.(20) into (27) and making the Borel improvement, one obtains the following sum rule:

\[ g_{\pi N} = \frac{e^{M_N^2/M^2}}{\lambda_N^2} \left\{ \frac{M^4}{4 \pi^2} E_1 \left( \frac{S_\pi}{M^2} \right) \left( 1 + \frac{3 \alpha_s}{2 \pi} \right) \right\} \left( -\frac{1}{f_\pi} \langle \bar{d} d \rangle \right), \tag{29} \]

where \( f_\pi \) is the pion decay constant (93 MeV), and \( \langle 0 | \bar{d} i \gamma_5 d | \pi^0 \rangle \) has been rewritten by the soft pion theorem

\[ \langle 0 | \bar{d} i \gamma_5 d | \pi^0(p = 0) \rangle = \frac{1}{f_\pi} \langle \bar{d} d \rangle. \tag{30} \]

One can get rid of the coefficient \( e^{M_N^2/M^2}/\lambda_N^2 \) in (29) by using the nucleon sum rules (13) or (16). Thus one arrives at two different sum rules for \( g_{\pi N} \):

\[ g_{\pi N}^{\text{even}} = \frac{\pi^2}{M^6 8 E_2 \left( \frac{S_N}{M^2} \right)} \left\{ \frac{M^4}{4 \pi^2} E_1 \left( \frac{S_\pi}{M^2} \right) \left( 1 + \frac{3 \alpha_s(M^2)}{2 \pi} \right) \right\} \left( -\frac{1}{f_\pi} \langle \bar{d} d \rangle \right), \tag{31} \]

which is obtained from the “even” sum rule (13) for the nucleon, and

\[ g_{\pi N}^{\text{odd}} = \frac{M_N \left\{ E_1 \left( \frac{S_\pi}{M^2} \right) \left( 1 + \frac{3 \alpha_s(M^2)}{2 \pi} \right) \right\} \left( -\frac{1}{f_\pi} \langle \bar{d} d \rangle \right)}{f_\pi \left\{ E_1 \left( \frac{S_N}{M^2} \right) \left( 1 + \frac{3 \alpha_s(M^2)}{2 \pi} \right) \right\}}, \tag{32} \]

which is obtained from the “odd” sum rule (13) for the nucleon. For \( M_N \) in eq.(32), we just take the experimental number instead of reexpressing \( M_N \) by the Borel mass \( M \) through the nucleon sum rule. Even if one uses \( M_N(M) \) in eq.(32), the Borel curve for \( g_{\pi N}^{\text{odd}} \) is not affected so much as far as one adopts the odd nucleon sum rule (case (iii) in section II). Note here that, if one assumes \( S_\pi = S_N \), Eq.(32) gives \( g_{\pi N}^{\text{odd}} = M_N/f_\pi \), namely the GT relation with \( g_A = 1 \).

It is in order here to remark some difference of the present work from that of ref. [45]. First of all, we have carried out OPE up to dimension 7 both for \( g_{\pi N} \) and the nucleon mass, while only the lowest dimension operator, namely \( \bar{q} \gamma_5 q \), is taken into account for the \( g_{\pi N} \) sum rules in ref. [45]. Secondly, \( \alpha_s \) corrections to the Wilson coefficients are not taken into account in [45]. Thirdly, the continuum thresholds \( S_\pi \) and \( S_N \) are completely neglected in [45] for the \( g_{\pi N} \) sum rules.
IV. BOREL ANALYSIS FOR THE $\pi - N$ COUPLING CONSTANT

A. The determination of the thresholds $S_\pi$ and $S_N$

As we have mentioned in section II, it is difficult to determine $S_N$ from the Borel sum rules of the nucleon. In this section, we will utilize the finite energy sum rules (FESR) to give a constraint on $S_N$ as well as $S_\pi$.

The FESR has a general form [13],

$$\int_0^{S_0} ds\ s^n\text{Im}\Pi_{OPE}(s) = \int_0^{S_0} ds\ s^n\text{Im}\Pi_{phen.}(s),$$

where $\text{Im}\Pi_{OPE}(s)$ is an imaginary part obtained by the analytic continuation of $\Pi$ in OPE, $\text{Im}\Pi_{phen.}(s)$ is the phenomenological imaginary part and $S_0$ is the continuum threshold (either $S_N$ or $S_\pi$). For the nucleon, by using $\Pi_1$ with $n = 0, 1$ and $\Pi_2$ with $n = 0$, one gets [9]

$$64\pi^4\lambda_N^2 = \left(1 + \frac{25}{4}\frac{\alpha_s}{\pi}\right)\frac{S_N^3}{3} + 2\pi^2\left(\frac{\alpha_s}{\pi}\right)^2 + \frac{128}{3}\pi^4\langle\bar{q}q\rangle^2\left(1 - \frac{5}{6}\frac{\alpha_s}{\pi}\right),$$

$$64\pi^4\lambda_N^2 M_N = -8\pi^2\langle\bar{q}q\rangle\left(1 + \frac{3}{2}\frac{\alpha_s}{\pi}\right) S_\pi^2 + \frac{32}{9}\pi^4\langle\bar{q}q\rangle\left(\frac{\alpha_s}{\pi}G^2\right),$$

$$64\pi^4\lambda_N^2 M_N^2 = \left(1 + \frac{37}{6}\frac{\alpha_s}{\pi}\right)\frac{S_N^4}{4} + \pi^2\left(\frac{\alpha_s}{\pi}G^2\right) S_N^2 - \frac{128}{9}\pi^4\langle\bar{q}q\rangle^2\left(1 - \frac{5}{6}\frac{\alpha_s}{\pi}\right),$$

The renormalization point here is chosen as $\mu^2 = S_N$.

By using the standard values of the condensates $\langle\bar{q}q\rangle(1\text{GeV}^2) = -(225 \pm 25\text{MeV})^3$ and $\langle\frac{2}{3}G^2\rangle = 0.012\text{GeV}^4$, we solved Eq.(34) $\sim$ Eq.(36) numerically. Table 1 shows the results for four different values of $\langle\bar{q}q\rangle$.

| $\langle\bar{q}q\rangle$ | $(-0.250\text{GeV})^3$ | $(-0.240\text{GeV})^3$ | $(-0.225\text{GeV})^3$ | $(-0.200\text{GeV})^3$ |
|-------------------------|---------------------|---------------------|---------------------|---------------------|
| $S_N(\text{GeV}^2)$     | 1.77                | 1.60                | 1.34                | 0.887               |
| $\lambda_N(\text{GeV}^3)$ | 0.0267             | 0.0235             | 0.0187             | 0.0118             |
| $M_N(\text{GeV})$      | 0.997               | 0.940               | 0.845               | 0.645               |

Table 1: $S_N, \lambda_N, M_N$ obtained from Eq.(34) $\sim$ (36) with four different values of $\langle\bar{q}q\rangle$. $\langle\bar{q}q\rangle = -(0.2402\text{GeV})^3$ reproduces the nucleon mass.

From this table, we choose $S_N = 1.34 - 1.77$ as physical range where the nucleon mass is reasonable reproduced within $\pm 10\%$ errors. Our $S_N$ is smaller than that usually used in the literatures [6–8,13]. However, the $\alpha_s$ corrections are not
taken into account in these references. The effect of the $\alpha_s$ correction to the spectral parameters can be explicitly seen by expanding the solutions of Eq. (34) up to linear in $\alpha_s$;

$$\lambda_N^2 = \lambda_0^2 \left( 1 - 26.2(-16\pi^2\langle \bar{q}q \rangle)^{-\frac{1}{2}} \left(\frac{\alpha_s}{\pi} G^2\right) \right) (1 - 3.08\frac{\alpha_s}{\pi}),$$

(37)

$$S_N = S^0_N \left( 1 - 21.2(-16\pi^2\langle \bar{q}q \rangle)^{-\frac{1}{2}} \left(\frac{\alpha_s}{\pi} G^2\right) \right) (1 - 3.26\frac{\alpha_s}{\pi}),$$

(38)

$$M_N = M^0_N \left( 1 - 18.6(-16\pi^2\langle \bar{q}q \rangle)^{-\frac{1}{2}} \left(\frac{\alpha_s}{\pi} G^2\right) \right) (1 - 1.94\frac{\alpha_s}{\pi}),$$

(39)

where $\lambda_0^2 = 4\langle \bar{q}q \rangle^2$, $S^0_N = (640\pi^4\langle \bar{q}q \rangle^2)^{\frac{1}{2}}$ and $M^0_N = (-\frac{25}{2}\pi^2\langle \bar{q}q \rangle)^{\frac{1}{2}}$, which are the solutions when the $\alpha_s$ corrections and the gluon condensate are neglected. (37)-(39) show that the $\alpha_s$ corrections tend to reduce the observables by considerable amount particularly in $S_N$.

Next, we estimate $S_\pi$ by taking the $n = 0$ FESR of $\Pi^\pi$;

$$\lambda^2_N g_{\pi N} = \left\{ \frac{1}{8\pi^2} S^2_\pi \left( 1 + \frac{3}{2} \frac{\alpha_s(S_\pi)}{\pi} \right) - \frac{1}{18} \left(\frac{\alpha_s}{\pi} G^2\right) \right\} \left( -\frac{1}{f_\pi} \langle \bar{dd} \rangle \right).$$

(40)

Since the FESR is rather sensitive to the structure of the continuum compared to the Borel sum rule and we do not know much about the detailed structure of the continuum for (19), we just limit ourselves to the $n = 0$ sum rule (local duality relation) for safety. To roughly evaluate the range of $S_\pi$, we simply put $g_{\pi N} = 13.4$ in (40) with $\lambda_N$ being determined in the nucleon FESR. The result is given in Table 2 for three different values of the condensate:

| $\langle \bar{q}q \rangle (GeV^3)$ | $S_N (GeV^2)$ | $S_\pi (GeV^2)$ |
|-----------------------------------|---------------|----------------|
| set 1 -0.250$^3$                  | 1.77          | 1.98           |
| set 2 -0.240$^3$                  | 1.60          | 1.85           |
| set 3 -0.225$^3$                  | 1.34          | 1.62           |

Table 2: $S_N, S_\pi$ with three different values of $\langle \bar{q}q \rangle$. $S_\pi$ is obtained by substituting $g_{\pi N} = 13.4$ into (40).

From Table 2, one finds that $S_\pi$ is always greater than $S_N$. This is consistent with the Borel sum rule $g_{\pi N}^{odd}$ in (32) which tells us that $g_{\pi N} > M_N/f_\pi$ only when $S_\pi > S_N$. In subsections below, we will examine the Borel stability of $g_{\pi N}$ with the parameter sets obtained in Table 2.
B. Borel analysis for $g_{\pi N}^{even}$

In Fig.3, $g_{\pi N}^{even}$ is shown as a function of $M^2$. The solid, dashed and dash-dotted curves correspond to set 1, set 2 and set 3 in Table 2, respectively. $g_{\pi N}^{even}$ in Fig.3(a) includes the $\alpha_s$ corrections to the Wilson coefficients, while they are neglected in Fig.3(b) except for the gluon condensate.

$g_{\pi N}^{even}$ has a sizable $M^2$ variation and good Borel stability is not seen in Fig.3. By comparing Fig.3(a) with Fig.3(b), one finds that the $\alpha_s$ corrections improve the Borel stability only slightly. The effect of the higher dimensional operator to the Borel curve is also small. In fact, $\frac{2\pi^2}{9M^2}(\frac{\alpha_s}{\pi}G^2)/\{E_1(\frac{S_{\pi}}{M^2})(1 + \frac{3}{2}\frac{\alpha_s}{\pi})\}$, which is a ratio of the dimension 3 term and the dimension 7 term in eq.(29), is about 4% at $M^2 \sim 1$GeV$^2$. (Note that dimension 5 terms do not arise for the Ioffe current.)

Although $g_{\pi N}^{even}$ in eq.(31) is proportional to $\langle \bar{q}q \rangle$, three curves in Fig.3, which correspond to different values of $\langle \bar{q}q \rangle$, almost overlap with each other. This is because the change of $\langle \bar{q}q \rangle$ is compensated by the changes of $S_{\pi,N}$.

In Fig.4, $g_{\pi N}^{even}$ for set 2 is shown with $S_{\pi}$ being changed by $\pm 10\%$. The solid, dashed and dash-dotted curves correspond to $S_{\pi} = 1.85 \times 1.1, 1.85$ and $1.85 \times 0.9$, respectively. $g_{\pi N}^{even}$ increases as $S_{\pi}$ increases, which is consistent with the prediction of FESR in eq.(40).

C. Borel analysis for $g_{\pi N}^{odd}$

Fig.5(a),(b) show $g_{\pi N}^{odd}$ as a function of $M^2$. The solid, dashed, dash-dotted curves correspond to set 1, set 2, set 3, respectively. $g_{\pi N}^{odd}$ in Fig.5(a) includes $\alpha_s$ corrections to the Wilson coefficients, while they are neglected in Fig.5(b) except for the gluon condensate.

$g_{\pi N}^{odd}$ has apparently better Borel stability than $g_{\pi N}^{even}$, but still sizable $M^2$ variation is seen. The $\alpha_s$ corrections do not affect the Borel stability much, since the same $\alpha_s$ correction appears both in numerator and denominator in eq.(32).

$g_{\pi N}^{odd}$ is rather sensitive to the change of the parameter sets, in particular the $S_{\pi}/S_N$. To see the effect of $S_{\pi,N}$ on $g_{\pi N}^{odd}$ in more detail, we expand eq.(32) up to $O(\frac{1}{M^2})$:

$$g_{\pi N}^{odd} \simeq \frac{M_N}{f_\pi} \left( \frac{S_{\pi}^2 - \frac{b_9}{9}}{S_{\pi}^2} \right) \left\{ 1 - \frac{2}{3M^2} \left( \frac{S_{\pi}^3}{S_{\pi}^2 - \frac{b_9}{9}} - \frac{S_N^3}{S_N^2 - \frac{b_9}{9}} \right) \right\}$$

$$\simeq \frac{M_N}{f_\pi} \left( \frac{S_{\pi}}{S_N} \right)^2 \left\{ 1 - \frac{2}{3M^2}(S_{\pi} - S_N) \right\} , \quad (41)$$
where we have neglected small $\alpha_s$ corrections and used the fact $S_{\pi N}^2 \gg b/9$. The approximate formula eq.(41) is in good agreement with the exact one eq.(32) in 10% for $M^2 > 1.0\text{GeV}^2$.

When $S_{\pi} > S_N$, $M^2$ independent term in eq.(41) gives $g_{\pi N}^{\text{odd}} = (S_{\pi}/S_N)^2(M_N/f_{\pi})$ which is larger than $M_N/f_{\pi}$. The leading $1/M^2$ correction reduces $g_{\pi N}^{\text{odd}}$ slightly. The experimental value for $g_{\pi N} = 13.4$ is obtained when $M^2 = 1.6\text{GeV}^2$ for the parameter set 3.

In Fig.6, $g_{\pi N}^{\text{odd}}$ for set 2 is shown with $S_{\pi}$ being changed by $\pm10\%$. The solid, dashed and dash-dotted curves correspond to $S_{\pi} = 1.85 \times 1.1, 1.85$ and $1.85 \times 0.9$, respectively. $g_{\pi N}^{\text{odd}}$ increases as $S_{\pi}$ increases, which is consistent with the prediction of FESR in eq.(40) and also with the approximate formula (41).

D. Comparison of $g_{\pi N}^{\text{even}}$ and $g_{\pi N}^{\text{odd}}$

As we have already mentioned, the Borel stability for $g_{\pi N}^{\text{odd}}$ is better than $g_{\pi N}^{\text{even}}$. This is consistent with the fact that the Borel curve for $M_N$ is most stable in “odd” sum rule (case (iii) in Fig.2). Also the absolute value of $g_{\pi N}^{\text{odd}}$ is larger than $g_{\pi N}^{\text{even}}$ for appropriate range of $M^2$: e.g. $g_{\pi N}^{\text{odd}} = 11.3 - 12.8$ versus $g_{\pi N}^{\text{even}} = 9.02 - 9.10$ at $M^2 = 1\text{GeV}^2$.

In previous subsections, $S_{\pi} = 1.85\text{GeV}^2$ has been used as a standard value to study the Borel stability. Alternative way to make the Borel analysis is to eliminate $S_{\pi}$ from eq.(31) and eq.(32) using eq.(40), and then to solve $g_{\pi N}$ self-consistently for each $M^2$. By this procedure, we found that there exists no solution satisfying eq.(31) and eq.(40) simultaneously, while there exists solutions of eq.(32) and eq.(40) which are given in Fig.7. This result again confirms that $g_{\pi N}^{\text{odd}}$ is better starting point to study the $\pi - N$ coupling constant than $g_{\pi N}^{\text{even}}$.

V. CONCLUSION

In this article we have made extensive Borel and FESR analyses of $g_{\pi N}$ by taking into account higher dimensional operators, $\alpha_s$ corrections and the continuum threshold. None of them has been considered in the previous analyses which led to $g_{\pi N} = M_N/f_{\pi}$ [4,5].

What we have found are summarized as follows:
(a) The higher dimensional operators up to dim. 7 and the $\alpha_s$ corrections play no crucial role for the Borel stability of $g_{\pi N}$.
(b) $g_{\pi N}^{\text{odd}}$ is more appropriate for examining $g_{\pi N}$ than $g_{\pi N}^{\text{even}}$, since the former has better
Borel stability. This fact is also consistent with the fact that “odd” sum rule for $M_N$ has a best stability.

(c) $g_{\pi N}$ is most sensitive to the ratio $S_\pi/S_N$, and both the FESR and Borel sum rules tell us that $S_\pi/S_N > 1$ is a crucial ingredient to reproduce the experimental $g_{\pi N}$.\[1\]

We also found that the Borel stability of $g_{\pi N}^{\text{odd}}$, even if dim. 7 operator is taken into account, is not satisfactory enough to determine the pion-nucleon coupling constant precisely. For the nucleon sum rule, there have been some attempts to improve the Borel stability such as the modification of the Ioffe current \[14\] and the inclusion of the instantons \[15\]. In particular, instantons improve the nucleon Borel sum rules considerably at low $M^2$ region, so it will be an interesting problem to study $g_{\pi N}$ with instanton contribution in the future.

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\(^2\)This point may have some relation to the Adler-Weisberger sum rule \[16\] which tells us that $g_A > 1$ (or equivalently $g_{\pi N} > M_N/f_\pi$) is obtained only when the continuum contribution in the $\pi - N$ channel is taken into account.
APPENDIX A:

The Borel transform is defined as,

\[ \hat{B} = \frac{(-1)^n(Q^2)^n}{(n-1)!} \left( \frac{d}{dQ^2} \right)^n, \quad Q^2 = -q^2, \quad (A.1) \]

with \( Q^2 \to \infty, n \to \infty \), and \( \frac{Q^2}{n} = M^2 \) being fixed.

When \( \hat{B} \) is applied to the correlation function \( \Pi(Q^2) = \frac{1}{\pi} \int \frac{Im\Pi(s)}{s+Q^2}, \) it leads to

\[ \hat{B}\Pi(Q^2) = \frac{1}{\pi M^2} \int ds \text{ Im}\Pi(s)e^{-sM^2} \quad (A.2) \]

This shows that the Borel transform tends to suppress the high energy contribution.

Some useful formula are

\[ \hat{B} \left( \frac{1}{Q^2} \right)^k = \frac{1}{(k-1)!} \left( \frac{1}{M^2} \right)^k, \quad (A.3) \]
\[ \hat{B}(Q^2)^k \log Q^2 = (-1)^{k+1} \Gamma(k+1)(M^2)^k, \quad (A.4) \]
\[ \hat{B} \frac{1}{s+Q^2} = \frac{1}{M^2} e^{-sM^2}, \quad (A.5) \]
\[ \hat{B} \frac{\log Q^2}{Q^2} = \frac{1}{M^2} (\log M^2 - \gamma_E), \quad (A.6) \]
\[ \hat{B}(\log Q^2)^2 = -2 \log M^2 + 2\gamma_E, \quad (A.7) \]
\[ \hat{B}Q^2(\log Q^2)^2 = 2M^2(\log M^2 - \gamma_E + 1), \quad (A.8) \]
\[ \hat{B}(Q^2)^2(\log Q^2)^2 = (M^2)^2(-4 \log M^2 + 4\gamma_E - 6). \quad (A.9) \]
Figure Captions

Fig. 1
OPE up to dimension 7 operators for the correlation of Ioffe current. Wavy lines denote gluon lines, broken lines denote the quark/ gluon condensate.

Fig. 2
\( M_N \) (nucleon mass) as a function of the Borel mass squared \( M^2 \). The solid, dashed, dash-dotted lines correspond to the cases (i), (ii) and (iii), respectively. 
\[ \langle \bar{q}q \rangle = -(0.240 GeV)^3, \langle (\alpha_s) G^2 \rangle = 0.012 GeV^4 \] and \( S_N = 1.60 GeV^2 \) are used.

Fig. 3(a),(b)
\( g_{\pi N}^{even} \) as a function of \( M^2 \). The solid, dashed and dash-dotted lines correspond to set 1, set 2 and set 3, respectively. \( \alpha_s \) corrections are taken into account in Fig.3(a), while they are neglected in Fig.3(b) except for gluon condensate.

Fig. 4
\( g_{\pi N}^{even} \) with \( \langle \bar{q}q \rangle = -(0.240 GeV)^3 \) as a function of \( M^2 \). The solid, dashed and dash-dotted lines correspond to \( S_\pi = 1.85 \times 1.1, 1.85 \times 0.9 \), respectively.

Fig. 5(a),(b)
\( g_{\pi N}^{odd} \) as a function of \( M^2 \). The solid, dashed and dash-dotted lines correspond to set 1, set 2 and set 3, respectively. \( \alpha_s \) corrections are taken into account in Fig.5(a), while they are neglected in Fig.5(b) except for gluon condensate.

Fig. 6
\( g_{\pi N}^{odd} \) with \( \langle \bar{q}q \rangle = -(0.240 GeV)^3 \) as a function of \( M^2 \). The solid, dashed and dash-dotted lines correspond to \( S_\pi = 1.85 \times 1.1, 1.85 \times 0.9 \), respectively.

Fig. 7
\( g_{\pi N} \) as a function of \( M^2 \). \( S_\pi \) in eq.(40) with \( \lambda_N \) being determined in the nucleon FESR is used for \( S_\pi \) in eq.(32). The solid, dashed and dash-dotted lines correspond to \( g_{\pi N}^{odd} \) with \( \langle \bar{q}q \rangle = -(0.25 GeV)^3, -(0.240 GeV)^3 \) and \( -(0.225 GeV)^3 \), respectively.
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Fig. 1
Fig. 2

$M_N$

$M^2 (\text{GeV}^2)$
Fig. 3(a)
Fig. 3(b)
Fig. 5(a)

$g_{\pi NN}$

$M^2$(GeV$^2$)
Fig. 5(b)

\[ g_{\pi NN}(M^2(\text{GeV}^2)) \]
$g_{\pi NN}$ vs $M^2 (\text{GeV}^2)$

Fig. 7