Warped Convolutions: A Novel Tool in the Construction of Quantum Field Theories*

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Abstract

Recently, Grosse and Lechner introduced a novel deformation procedure for non–interacting quantum field theories, giving rise to interesting examples of wedge–localized quantum fields with a non–trivial scattering matrix. In the present article we outline an extension of this procedure to the general framework of quantum field theory by introducing the concept of warped convolutions: given a theory, this construction provides wedge–localized operators which commute at spacelike distances, transform covariantly under the underlying representation of the Poincaré group and admit a scattering theory. The corresponding scattering matrix is nontrivial but breaks the Lorentz symmetry, in spite of the covariance and wedge–locality properties of the deformed operators.

1 Introduction

Recent advances in algebraic quantum field theory have led to purely algebraic constructions of quantum field models on Minkowski space, both classical and noncommutative [2–5,8,11–15], many of which cannot be constructed by the standard methods of constructive quantum field theory. Some of these models are local and free, some are local and have nontrivial S-matrices, and yet others manifest only certain remnants of locality, though these remnants suffice to enable the computation of nontrivial two–particle S-matrix elements.

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In a recent paper [8], Grosse and Lechner have presented an infinite family of quantum fields which, taken as a whole, are wedge–local and Poincaré covariant and which have nontrivial scattering. They produce this family by deforming the free quantum field in a certain manner, motivated by the desire to understand the field as being defined on noncommutative Minkowski space. However, as they point out, one can forget the original motivation and view the resulting deformed fields as being defined on classical Minkowski space. It is, however, essential to the arguments of [8] that the free field is deformed.

In this paper we present a generalization of their deformation which can be applied to any Minkowski space quantum field theory in any number of dimensions. This deformation results in a one parameter family of distinct field algebras which are wedge–local and covariant under the representation of the Poincaré group associated with the initial, undeformed theory. It turns out that also the $S$–matrix changes under this deformation, and the deformed $S$–matrix breaks the Lorentz symmetry, in spite of the Lorentz covariance of the deformed theory. When taking the free quantum field as the initial model, our deformation coincides with that of Grosse and Lechner.

The deformation in question involves an apparently novel operator–valued integral, whose mathematical definition requires some care. Apart from the operators which are to be integrated, it involves a unitary representation of the additive group $\mathbb{R}^d$, $d \geq 2$, satisfying certain properties which arise naturally when considering relativistic quantum field theories on two (or higher) spacetime dimensional Minkowski space. We outline in Sec. 2 the intriguing properties of this integral; proofs will be given elsewhere. In Sec. 3 we apply these results to quantum field theories to obtain the results mentioned above. Finally, in Sec. 4 we indicate some paths of further investigation suggested by these results.

2 Warped convolutions

In order to draw attention to what may be regarded as the mathematical core of the deformation studied in this paper, we consider a quite general setting which covers both the case of Wightman Quantum Field Theory considered in [8] and the case of Algebraic Quantum Field Theory [9].

We shall assume the existence of a strongly continuous unitary representation $U$ of the additive group $\mathbb{R}^d$, $d \geq 2$, on some separable Hilbert space $\mathcal{H}$. The joint spectrum of the generators $P$ of $U$ is denoted by $\text{sp} U$ and will be further specified in the following section. Let $\mathcal{D}$ be the dense subspace of vectors in $\mathcal{H}$ which transform smoothly under the action of $U$, cf. [6]. We consider the set $\mathfrak{F}$ of all operators $F$ which have $\mathcal{D}$ in their domain of definition and are smooth under the adjoint action $\alpha_x(F) = U(x)FU(x)^{-1}$ of $U$ in the following sense: for each $F \in \mathfrak{F}$ there is some $n \in \mathbb{N}$ such that the operator valued function $x \mapsto (1 + |P|^2)^{-n} \alpha_x(F)(1 + |P|^2)^{-n}$ is arbitrarily often differentiable in norm, where $|P|^2$ denotes the sum of the squares of the generators of $U$. It is easily seen that $\mathfrak{F}$ is a unital $*$–algebra.
Within this framework one can establish a deformation procedure for the elements of $\mathfrak{F}$. The basic ingredient in this construction is the spectral resolution $E$ of the unitary group $U$,

$$U(x) = e^{ipx} = \int e^{ipx} \, dE(p), \quad x \in \mathbb{R}^d,$$

where the inner product on $\mathbb{R}^d$ is arbitrary here and will be fixed later. Given any skew-symmetric $d \times d$-matrix $Q$, i.e. $q Q p = -p Q q$ for $p, q \in \mathbb{R}^d$, one can give meaning to the operator valued integrals for any $F \in \mathfrak{F}$

$$Q F = \int \alpha_{Qp}(F) \, dE(p), \quad F_Q = \int dE(p) \alpha_{Qp}(F). \quad (2.1)$$

These left and right integrals are defined on the domain $\mathcal{D}$ in the sense of distributions. Moreover, the resulting operators are smooth with regard to the adjoint action of $U$ in the sense explained above; hence $Q F, F_Q \in \mathfrak{F}$. We omit the proof and only note that the above integrals may be viewed as warped (by the matrix $Q$) convolutions of $F$ with the spectral measure $dE$.

The above integrals have a number of remarkable properties, which are crucial for their application to quantum field theory. We begin by noting the at first sight surprising fact that the left and right integrals coincide.

**Lemma 2.1** Let $F \in \mathfrak{F}$. Then $QF = F_Q$.

The proof of this lemma requires the proper treatment of expressions such as $dE(p)F dE(q)$ (which are not product measures) as well as the discussion of subtle domain problems. We therefore forego here a rigorous argument. Yet, in order to display the significance of the skew symmetry of the matrix $Q$ for the result, we indicate the various steps in the proof. Making use several times of the relation $dE(p)f(P) = dE(p)f(p) = f(P)dE(p)$, which holds for any continuous function $f$, we get the following chain of equalities, which are justified below.

$$F_Q = \int dE(p) \alpha_{Qp}(F)$$

$$= \int dE(p) U(Qp)F U(Qp)^{-1} dE(q)$$

$$= \int \int dE(p) U(Qp)F U(Qp)^{-1} dE(q)$$

$$= \int \int dE(p) F e^{ipQq} dE(q)$$

$$= \int \int dE(p) e^{ipQq} F dE(q)$$

$$= \int \int dE(p) U(Qq)F U(Qq)^{-1} dE(q)$$

$$= \int \alpha_{Qq}(F) dE(q) = q F.$$
In the second equality we made use of $\int dE(q) = 1$, in the third one we relied on the fact that the preceding expression can be rewritten as a double integral, and in the fourth one we used the skew symmetry of $Q$, implying $dE(p) e^{-iPQp} = dE(p)$ and $e^{-iPQp} dE(q) = e^{-iqQp} dE(q) = e^{iPQq} dE(q)$. The fifth equality then follows, since $e^{iPQq}$ is a c-number, and the sixth one is a consequence of $dE(p) e^{iPQq} = dE(p) e^{iPQq}$ and $dE(q) = e^{-iPQq} dE(q)$. In the last step we made use once again of $\int dE(p) = 1$.

It can be inferred from the defining relations (2.1) that $(QF) \supset F^* Q$. Thus, as an immediate consequence of the preceding lemma, one finds that the operation of taking adjoints commutes with the warped convolution in the following sense.

**Lemma 2.2** Let $F \in \mathcal{F}$. Then $F^* Q \supset F^* Q$.

It is also noteworthy that $(F_{Q_1})_{Q_2} = F_{Q_1 + Q_2}$, for any $F \in \mathcal{F}$ and skew symmetric matrices $Q_1, Q_2$. In the next lemma we exhibit commutation properties of certain specific elements of $\mathcal{F}$, which are preserved by the deformation procedure. The shape of the spectrum $\text{sp} U$ of the unitary group $U$, which coincides with the support of the spectral measure $dE$, enters in the formulation of this result.

**Lemma 2.3** Let $F, G \in \mathcal{F}$ be such that

$$\alpha_{Q_p}(F) \alpha_{-Q_q}(G) = \alpha_{-Q_q}(G) \alpha_{Q_p}(F)$$

for all $p, q \in \text{sp} U$. Then,

$$F_Q G_{-Q} = G_{-Q} F_Q.$$

Again, the rigorous proof of this result is plagued by technicalities and will not be given here. But the following formal steps, which are explained below, display the basic facts underlying the argument.

$$F_Q G_{-Q} = \int dE(p) \alpha_{Q_p}(F) \int dE(q) \alpha_{-Q_q}(G)$$

$$= \int dE(p) \alpha_{Q_p}(F) \int \alpha_{-Q_q}(G) dE(q)$$

$$= \int \int dE(p) \alpha_{Q_p}(F) \alpha_{-Q_q}(G) dE(q)$$

$$= \int \int dE(p) \alpha_{-Q_q}(G) \alpha_{Q_p}(F) dE(q)$$

$$= \int \int dE(p) U(-Qq) GU(-Qq)^{-1} U(Qp) FU(Qp)^{-1} dE(q)$$

$$= \int \int dE(p) e^{-iPQq} GU(Qq + Qp) F e^{-iqQp} dE(q)$$

$$= \int \int dE(p) U(-Qp) GU(Qp + Qq) F U(-Qq) dE(q)$$

$$= \int \int dE(p) \alpha_{-Q_p}(G) \alpha_{Q_q}(F) dE(q)$$

$$= \int dE(p) \alpha_{-Q_p}(G) \int dE(q) \alpha_{Q_q}(F) = G_{-Q} F_Q.$$
In the second equality use was made of Lemma 2.1, the third equality relies on the fact that the preceding product of operators can be presented as a double integral, and in the fourth equality the commutation properties of the operators $F, G$ were exploited. The adjoint action of $U$ is written out explicitly in the fifth equality, and in the sixth equality the group law for $U$ as well as the relations $dE(p) e^{-iPq} = dE(p) e^{-iQq}$ and $e^{-iPq} dE(q) = e^{-iQp} dE(q)$ were used. The step to the seventh equality is accomplished by noting that the phase factors in the preceding expression cancel in view of the skew symmetry of $Q$, which also implies $dE(p) = dE(p) e^{-iPq}$, $dE(q) = e^{-iPq} dE(q)$. In the eighth equality the various unitaries are recombined into the form of adjoint actions, and in the subsequent equality the double integral is reexpressed as a product of simple integrals; Lemma 2.1 is then used once again.

We conclude this discussion of the warped convolution with a remark on its covariance properties. Let $\mathcal{L}$ be a matrix group acting isometrically (with regard to the chosen inner product) on $\mathbb{R}^d$ and let $\mathcal{P} = \mathcal{L} \ltimes \mathbb{R}^d$ be the semidirect product of the two groups. We assume that the unitary representation $U$ of $\mathbb{R}^d$ can be extended to a representation of $\mathcal{P}$, denoted by the same symbol. Denoting the elements of $\mathcal{P}$ by $\lambda = (\Lambda, x)$, one then has $U(\lambda) U(y) = U(\Lambda y) U(\lambda)$ and consequently $U(\lambda) dE(p) = dE(\Lambda p) U(\lambda)$. It follows from standard arguments that $\mathfrak{F}$ is stable under the action of $\mathcal{P}$ given by $\alpha_\lambda (F) = U(\lambda) F U(\lambda)^{-1}$. Moreover,

$$
U(\lambda) \left( \int \alpha_{Qp}(F) dE(p) \right) U(\lambda)^{-1} = \int \alpha_{\Lambda Qp}(U(\lambda) F U(\lambda)^{-1}) dE(\Lambda p) \\
= \int \alpha_{\Lambda Q A^{-1} p}(U(\lambda) F U(\lambda)^{-1}) dE(p).
$$

Note that the matrix $\Lambda Q A^{-1}$ is again skew symmetric with regard to the chosen inner product. We state the above result in the form of a lemma for later reference.

**Lemma 2.4** Let $F \in \mathfrak{F}$, let $Q$ be any skew symmetric matrix and let $\lambda = (\Lambda, x)$ be any element of $\mathcal{P}$. Then

$$
\alpha_\lambda (F_Q) = \left( \alpha_\lambda (F) \right)_{\Lambda QA^{-1}}.
$$

With these results we have laid the foundation for the application of warped convolutions to quantum field theory.

## 3 Deformations of quantum field theories

We turn now to the discussion of local quantum field theories in Minkowski space and their deformations. Identifying $d$–dimensional Minkowski space with the manifold $\mathbb{R}^d$, the Lorentz inner product is given in proper coordinates by $xy = x_0 y_0 - \sum_{i=1}^{d-1} x_i y_i$. Any given quantum field theory on $\mathbb{R}^d$ may then be described as follows: there is a continuous unitary representation $U$ of the Poincaré
group $\mathcal{P} = \mathcal{L} \ltimes \mathbb{R}^d$ on a separable Hilbert space $\mathcal{H}$, where $\mathcal{L}$ is the identity component of the group of Lorentz transformations and $\mathbb{R}^d$ the group of spacetime translations. The joint spectrum of the generators $P$ of the abelian subgroup $U \upharpoonright \mathbb{R}^d$ is contained in the closed forward lightcone $V_+ = \{ p \in \mathbb{R}^d : p_0 \geq |p| \}$ and there is a, up to a phase unique, unit vector $\Omega \in \mathcal{H}$, representing the vacuum, which is invariant under the action of $U$.

We assume that the underlying local field operators and observables generate a unital $^*$–algebra $\mathfrak{A} \subset \mathfrak{F}$, where $\mathfrak{F}$ is the algebra of smooth operators with respect to the translations $U \upharpoonright \mathbb{R}^d$ introduced in the preceding section. In the Wightman setting of quantum field theory this assumption obtains if the underlying fields satisfy polynomial energy bounds [6]. In the framework of local quantum physics, where one deals with von Neumann algebras of bounded operators, one has to proceed to weakly dense subalgebras of elements smooth with respect to the action of the translation subgroup. So in both settings this assumption does not impose any significant restriction of generality and covers all models of interest.

The detailed structure of the theory is of no relevance here. What matters, however, is the assumption that one can identify all fields and observables which are localized in certain specific wedge–shaped regions, called wedges, for short. We fix a standard wedge (see Figure 1)

$$\mathcal{W}_0 \doteq \{ x \in \mathbb{R}^d : x_1 \geq |x_0| \}$$

and note that in $d > 2$ dimensions all other wedges $\mathcal{W}$ can be obtained from $\mathcal{W}_0$ by suitable Poincaré transformations, $\mathcal{W} = \lambda \mathcal{W}_0$, $\lambda \in \mathcal{P}$. In $d = 2$ dimensions this statement only holds true if one also includes the spacetime reflections in $\mathcal{P}$.

![Figure 1: A wedge $\mathcal{W}$, its causal complement $\mathcal{W}'$ and their common edge](image)

Denoting by $\mathcal{W} = \{ \mathcal{W} \subset \mathbb{R}^d \}$ the set of all wedges in $\mathbb{R}^d$, we consider for any given $\mathcal{W} \in \mathcal{W}$ the $^*$–algebra $\mathfrak{A}(\mathcal{W}) \subset \mathfrak{A}$ generated by all fields and observables localized in $\mathcal{W}$. We call the algebras $\mathfrak{A}(\mathcal{W})$ wedge–algebras. It is apparent from
the definition that $\mathfrak{A}(W_1) \subset \mathfrak{A}(W_2)$ whenever $W_1 \subset W_2$, i.e. isotony holds. The covariance, locality and Reeh–Schlieder property of the underlying theory can then be expressed in terms of the wedge algebras as follows:

(a) Covariance: $\alpha_\lambda(\mathfrak{A}(W)) = U(\lambda)\mathfrak{A}(W)U(\lambda)^{-1} = \mathfrak{A}(\lambda W)$ for all $W \in \mathcal{W}$ and $\lambda \in \mathcal{P}$.

(b) Locality: $\mathfrak{A}(W') \subset \mathfrak{A}(W')'$, $W \in \mathcal{W}$, where $W'$ denotes the closure of the causal complement of $W$ and $\mathfrak{A}(W)'$ the relative commutant of $\mathfrak{A}(W)$ in $\mathcal{F}$.

(c) Reeh–Schlieder property: $\Omega$ is cyclic for any $\mathfrak{A}(W)$, $W \in \mathcal{W}$.

We mention as an aside that these assumptions also cover quantum field theories on non–commutative Minkowski space (Moyal space), as considered for example in [8]. These spaces are described by non–commuting coordinates $X_\mu$, $X_\nu$ satisfying the commutation relations

$$[X_\mu, X_\nu] = i \theta_{\mu\nu}1,$$

where $\theta_{\mu\nu} = -\theta_{\nu\mu}$ are real constants. If the dimension of the spacetime satisfies $d > 2$, there exist lightlike coordinates $X_\pm$ with $[X_+, X_-] = 0$ which can thus be simultaneously diagonalized. Hence fields and observables on such spaces can be localized in wedges $W$, yet they are dislocalized along the directions of the edges of these wedges. The wedge algebras are in general sufficient to reconstruct the algebras corresponding to arbitrary causally closed regions $R$. These are given by

$$\mathfrak{A}(R) = \bigcap_{W \supset R} \mathfrak{A}(W)$$

and inherit from the wedge algebras both locality and covariance properties. Yet in theories on non–commutative Minkowski space, where fields and observables cannot be localized in bounded regions, the corresponding algebras are trivial and consequently do not manifest the Reeh–Schlieder property.

Given a theory as described above, we can now apply the deformation procedure established in the preceding section. To this end, we fix the standard wedge $W_0$ and pick a corresponding $d \times d$–matrix $Q_\kappa$, which with respect to the chosen proper coordinates has the form

$$Q_\kappa = \begin{pmatrix} 0 & \kappa & 0 & \cdots & 0 \\ \kappa & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

for some fixed $\kappa > 0$. Note that this matrix is skew symmetric with respect to the Lorentz inner product. The following basic facts pointed out in [8] are crucial for the subsequent construction.
(i) Let \( \lambda = (\Lambda, x) \in \mathcal{P} \) be such that \( \lambda \mathcal{W}_0 \subset \mathcal{W}_0 \). Then \( \Lambda Q_\kappa \Lambda^{-1} = Q_\kappa \).

(ii) Let \( \lambda' = (\Lambda', x') \in \mathcal{P} \) be such that \( \lambda' \mathcal{W}_0 \subset \mathcal{W}_0' \). Then \( \Lambda' Q_\kappa \Lambda'^{-1} = -Q_\kappa \).

(iii) \( Q_\kappa V_+ = \mathcal{W}_0 \).

It is an immediate consequence of (i) that for any two Poincaré transformations \( \lambda_i = (\Lambda_i, x_i), \ i = 1, 2 \), such that \( \lambda_1 \mathcal{W}_0 = \lambda_2 \mathcal{W}_0 \), one has \( \Lambda_1 Q_\kappa \Lambda_1^{-1} = \Lambda_2 Q_\kappa \Lambda_2^{-1} \). Indeed, \( \Lambda_2^{-1} \lambda_1 = (\Lambda_2^{-1} \Lambda_1, \Lambda_2^{-1} (x_1 - x_2)) \) maps \( \mathcal{W}_0 \) onto itself, hence \( \Lambda_2^{-1} \Lambda_1 Q_\kappa \Lambda_1^{-1} \Lambda_2 = Q_\kappa \).

After these preparations we can now proceed from the given family of wedge algebras to a new “deformed” family with the help of the warped convolutions introduced in the preceding section. For \( \mathcal{W} \in \mathcal{W} \) the corresponding deformed algebras \( \mathfrak{A}_\kappa(\mathcal{W}) \) are defined as follows.

**Definition 3.1** Let \( \mathcal{W} \in \mathcal{W} \) and let \( \lambda = (\Lambda, x) \in \mathcal{P} \) be such that \( \mathcal{W} = \lambda \mathcal{W}_0 \). The associated algebra \( \mathfrak{A}_\kappa(\mathcal{W}) \) is the polynomial algebra generated by all warped operators \( A_{\Lambda Q_\kappa \Lambda^{-1}} \) with \( A \in \mathfrak{A}(\mathcal{W}) \).

Note that according to the preceding remarks this definition is consistent, since it does not depend on the particular choice of the Poincaré transformation \( \lambda \) mapping \( \mathcal{W}_0 \) onto \( \mathcal{W} \). Moreover, by Lemma 2.2, each \( \mathfrak{A}_\kappa(\mathcal{W}) \) is a *–algebra. We will show that the algebras \( \mathfrak{A}_\kappa(\mathcal{W}) \) have all desired properties of wedge algebras in a quantum field theory.

The isotony of the algebras \( \mathfrak{A}_\kappa(\mathcal{W}) \) is a consequence of the fact that if \( \mathcal{W}_1 \subset \mathcal{W}_2 \), these wedges can be mapped onto each other by a pure translation. Hence there are Poincaré transformations \( \lambda_i = (\Lambda, x_i), \ i = 1, 2 \), with the same \( \Lambda \) mapping \( \mathcal{W}_0 \) onto \( \mathcal{W}_1 \) and \( \mathcal{W}_2 \), respectively. As \( \mathfrak{A}_\kappa(\mathcal{W}_1) \), \( \mathfrak{A}_\kappa(\mathcal{W}_2) \) are generated by the operators \( A_{\Lambda Q_\kappa \Lambda^{-1}} \) with \( A \in \mathfrak{A}(\mathcal{W}_1) \) and \( A \in \mathfrak{A}(\mathcal{W}_2) \), respectively, the isotony of the original wedge algebras implies \( \mathfrak{A}_\kappa(\mathcal{W}_1) \subset \mathfrak{A}_\kappa(\mathcal{W}_2) \) whenever \( \mathcal{W}_1 \subset \mathcal{W}_2 \).

For the proof of covariance we make use of Lemma 2.3. Let \( \mathcal{W} = \lambda \mathcal{W}_0 \mathcal{W}_0 \) with \( \lambda_\mathcal{W} = (\Lambda_\mathcal{W}, x_\mathcal{W}) \) and let \( \lambda = (\Lambda, x) \). As the original theory is covariant, one has \( \alpha_{\lambda_\mathcal{W}}(\mathfrak{A}(\mathcal{W}_0)) = \mathfrak{A}(\mathcal{W}) \) and consequently the algebra \( \mathfrak{A}_\kappa(\mathcal{W}) \) is generated by the operators \( (\alpha_{\lambda_\mathcal{W}}(A))_{\Lambda \mathcal{W} Q_\kappa \Lambda^{-1}} \), \( A \in \mathfrak{A}(\mathcal{W}_0) \). Now by Lemma 2.4

\[
U(\lambda) \left( \alpha_{\lambda_\mathcal{W}}(A) \right)_{\Lambda \mathcal{W} Q_\kappa \Lambda^{-1}} U(\lambda)^{-1} = \left( \alpha_{\lambda_\mathcal{W}}(A) \right)_{\Lambda_\mathcal{W} Q_\kappa \Lambda^{-1} \Lambda^{-1}},
\]

and the operators on the right hand side of this equality are, for \( A \in \mathfrak{A}(\mathcal{W}_0) \), the generators of the algebra \( \mathfrak{A}_\kappa(\lambda \mathcal{W}) \). Thus \( \alpha_\lambda(\mathfrak{A}_\kappa(\mathcal{W})) \subset \mathfrak{A}_\kappa(\lambda \mathcal{W}) \). Replacing in this inclusion \( \lambda \) by \( \lambda^{-1} \) and \( \mathcal{W} \) by \( \lambda \mathcal{W} \) and making use of the fact that \( \alpha_{\lambda^{-1}} = \alpha_{\lambda^{-1}} \), one obtains \( \mathfrak{A}_\kappa(\lambda \mathcal{W}) \subset \alpha_{\lambda}(\mathfrak{A}_\kappa(\mathcal{W})) \). Hence \( \alpha_{\lambda}(\mathfrak{A}_\kappa(\mathcal{W})) = \mathfrak{A}_\kappa(\lambda \mathcal{W}) \), i.e. the deformed algebras satisfy the condition of covariance as well.
Turning to the proof of locality, we first restrict attention to the wedge $\mathcal{W}_0$. According to fact (iii) mentioned above, one has $Q_\kappa V_+ = \mathcal{W}_0$; hence $\mathcal{W}_0 + Q_\kappa p \subseteq \mathcal{W}_0$ and consequently $\mathcal{W}_0' \subseteq (\mathcal{W}_0 + Q_\kappa p)'$ for $p \in V_+$. Since $V_+$ is a cone, this implies $(\mathcal{W}_0' - Q_\kappa q) \subseteq (\mathcal{W}_0 + Q_\kappa p)'$ for all $p, q \in V_+$. It then follows from the covariance and locality properties of the original algebras that for any pair of operators $A \in \mathfrak{A}(\mathcal{W}_0)$ and $B \in \mathfrak{A}(\mathcal{W}_0')$ one has (denoting the pure translations $(1, x) \in \mathcal{P}$ by $x$)

$$[\alpha_{Q_\kappa p}(A), \alpha_{-Q_\kappa q}(B)] = 0, \quad p, q \in V_+.$$ 

According to Lemma 2.3, this implies $[A_{Q_\kappa}, B_{-Q_\kappa}] = 0$. Now if $\lambda = (\Lambda, x)$ is any Poincaré transformation such that $\lambda \mathcal{W}_0 = \mathcal{W}_0'$, it follows from fact (ii) mentioned above that $\Lambda Q_\kappa \Lambda^{-1} = -Q_\kappa$. Hence the operators $B_{-Q_\kappa}, B \in \mathfrak{A}(\mathcal{W}_0')$, generate the algebra $\mathfrak{A}_\kappa(\mathcal{W}_0')$, and similarly the operators $A_{Q_\kappa}, A \in \mathfrak{A}(\mathcal{W}_0)$, generate the algebra $\mathfrak{A}_\kappa(\mathcal{W}_0)$. So we obtain the inclusion $\mathfrak{A}_\kappa(\mathcal{W}_0') \subseteq \mathfrak{A}_\kappa(\mathcal{W}_0)'$. By the Poincaré covariance of the deformed algebras, established in the preceding step, it is then clear that $\mathfrak{A}_\kappa(\mathcal{W}') \subseteq \mathfrak{A}_\kappa(\mathcal{W})'$ for all $\mathcal{W} \in \mathfrak{W}$.

It remains to establish the Reeh–Schlieder property of the deformed algebras. According to Lemma 2.1, one has $A_\varphi = \varphi A$ for any skew symmetric matrix $Q$. Hence $A_\varphi \Omega = \varphi A_\Omega = \int \alpha_{Q_\varphi}(A)dE(p)\Omega = A_\Omega$, since $\Omega$ is invariant under space-time translations. Thus $\mathfrak{A}_\varphi(\mathcal{W})\Omega \supset \mathfrak{A}(\mathcal{W})\Omega$ for any $\mathcal{W} \in \mathfrak{W}$, so the Reeh–Schlieder property of the deformed wedge algebras is inherited from the original algebras. We summarize these findings in a theorem.

**Theorem 3.2** Let $\mathfrak{A}(\mathcal{W}) \subset \mathfrak{F}, \mathcal{W} \in \mathfrak{W}$, be a family of wedge algebras having the Reeh–Schlieder property and satisfying the conditions of isotony, covariance, and locality. Then the family of deformed algebras $\mathfrak{A}_\kappa(\mathcal{W}) \subset \mathfrak{F}, \mathcal{W} \in \mathfrak{W}$, introduced in Definition 3.1 also has these properties.

This theorem establishes that the deformation procedure outlined above can be applied to any quantum field theory. If one starts with the wedge algebras in a free field theory, one arrives at the deformed theories considered in [8], as can be seen by explicit computations. But one may equally well take as a starting point any rigorously constructed model, such as the self–interacting $\mathcal{P}(\varphi^4)$–theories in $d = 2$ dimensions or the $\varphi^4$–theory in $d = 3$ dimensions [7]. In all of these cases, the warped convolution produces a true deformation of the underlying theory, in the sense that the scattering matrix changes.

To exhibit this fact, let us assume that the underlying theory describes a single scalar massive particle. Then the spectrum of $U \upharpoonright \mathbb{R}^d$ has the form

$$\text{sp} U \upharpoonright \mathbb{R}^d = \{0\} \cup \{p : p_0 = \sqrt{p^2 + m^2}\} \cup \{p : p_0 \geq \sqrt{p^2 + M^2}\},$$

with $M > m > 0$. In the present general setting of wedge–local operators one can then define two–particle scattering states as in Haag–Ruelle–Hepp scattering theory [1]. To see this, we fix the standard wedge $\mathcal{W}_0$ and pick operators $A \in \mathfrak{A}(\mathcal{W}_0)$ and $A' \in \mathfrak{A}(\mathcal{W}_0')$ which interpolate between the vacuum vector $\Omega$ and
single particle states of mass $m$. We then proceed to the deformed operators $A_{Q_\kappa} \in \mathfrak{A}_\kappa(\mathcal{W}_0), A'_{-Q_\kappa} \in \mathfrak{A}_\kappa(\mathcal{W}_0')$ and note that these operators have the same interpolation properties as the original ones, recalling that $A_{Q_\kappa} \Omega = A\Omega, A'_{-Q_\kappa} \Omega = A'\Omega$.

Next, we pick test functions $f, f' \in \mathcal{S}(\mathbb{R}^d)$ whose Fourier transforms $\tilde{f}, \tilde{f}'$ have compact supports in small neighborhoods of points on the isolated mass shell in $\text{sp} U \uparrow \mathbb{R}^d$ which do not intersect with the rest of the spectrum. With the help of these functions and the above operators we define

$$A_{Q_\kappa}(f_t) = \int dx f_t(x) a_x(A_{Q_\kappa}),$$

where the functions $f_t \in \mathcal{S}(\mathbb{R}^d), t \in \mathbb{R}$, are given by

$$x \mapsto f_t(x) = (2\pi)^{-d/2} \int dp \tilde{f}(p) e^{i(p_0 - \omega p) t} e^{-ipx} \tag{3.2}$$

with $\omega_p = (p^2 + m^2)^{1/2}$. Similarly, one defines the operators $A'_{-Q_\kappa}(f'_t)$. Bearing in mind the support properties of $\tilde{f}, \tilde{f}'$ and the preceding remark about the action of the deformed operators on the vacuum vector, it follows that $A_{Q_\kappa}(f_t) \Omega = A(f_0) \Omega$ and $A'_{-Q_\kappa}(f'_t) \Omega = A'(f_0') \Omega$ are single particle states which do not depend on $t$.

The operators $A_{Q_\kappa}(f_t), A'_{-Q_\kappa}(f'_t)$ can be used to construct incoming, respectively outgoing, two–particle scattering states. Yet in the present case of wedge–localized operators this construction requires a proper adjustment of the support properties of the Fourier transforms of $f, f'$. Introducing the notation $\Gamma(g) = \{(1, p/\omega_p) : p \in \text{supp} \tilde{g}\}$ for the velocity support of a test function $g$ and writing $g_1 \succ g_2$ whenever the set $\Gamma(g_1) - \Gamma(g_2)$ is contained in the interior of the wedge $\mathcal{W}_0$, one relies on the following facts. According to a result of Hepp [10], the essential supports of the functions $f_t, f'_t$ are, for asymptotic $t$, contained in $t \Gamma(f), t \Gamma(f')$, respectively. Moreover, the regions $\mathcal{W}_0 + t \Gamma(f)$ and $\mathcal{W}_0' + t \Gamma(f')$ are spacelike separated for $t < 0$ ($t > 0$) if $f' \succ f$ $(f \succ f')$, respectively. Because of the covariance and locality properties of the deformed wedge–algebras, one can then establish by standard arguments [1] the existence of the strong limits

$$\lim_{t \to -\infty} A_{Q_\kappa}(f_t) A'_{-Q_\kappa}(f'_t) \Omega = |A(f) \Omega \otimes_{\kappa} A'(f') \Omega|^{\text{in}} \quad \text{for } f' \succ f$$

$$\lim_{t \to \infty} A_{Q_\kappa}(f_t) A'_{-Q_\kappa}(f'_t) \Omega = |A(f) \Omega \otimes_{\kappa} A'(f') \Omega|^{\text{out}} \quad \text{for } f \succ f'.$$

The limit vectors have all properties of a symmetric tensor product of the single particle states $A(f) \Omega, A'(f') \Omega$. In particular, they do not depend on the specific choice of operators $A, A'$ and test functions $f, f'$ within the above limitations. Because of the Reeh–Schlieder property of the wedge algebras, it is also clear that these vectors form a basis in the respective asymptotic two–particle spaces.
In order to exhibit the dependence of the tensor products on the deformation parameter $\kappa$, we note that for $f' > f$

$$|A(f)\Omega \otimes_\kappa A'(f')\Omega|^{\text{in}} = \lim_{t \to -\infty} A_{Q_\kappa}(f_t)A'_{Q_\kappa}(f'_t)\Omega$$

$$= \lim_{t \to -\infty} \int dE(p) \alpha_{Q_\kappa p}(A(f_t))A'(f'_t)\Omega$$

$$= \int dE(p) |U(Q_\kappa p)A(f)\Omega \otimes A'(f')\Omega|^{\text{in}},$$

where the third equality follows from the fact that the limit can be pulled under the integral and the symbol $\otimes$ denotes the tensor product in the original theory. Similarly, one obtains for $f > f'$

$$|A(f)\Omega \otimes_\kappa A'(f')\Omega|^{\text{out}} = \int dE(p) |U(Q_\kappa p)A(f)\Omega \otimes A'(f')\Omega|^{\text{out}}.$$

These relations between the scattering states in the original and in the deformed theory become more transparent if one proceeds to improper single particle states of sharp momentum $p = (\sqrt{p^2 + m^2}, p)$, $q = (\sqrt{q^2 + m^2}, q)$. There one has

$$|p \otimes_\kappa q|^{\text{in}} = e^{i|pQ_\kappa q|} |p \otimes q|^\text{in}$$

$$|p \otimes_\kappa q|^{\text{out}} = e^{-i|pQ_\kappa q|} |p \otimes q|^\text{out}.$$

The scattering states in the deformed theory depend on the matrix $Q_\kappa$ through the choice of the wedge $\mathcal{W}_0$ and thus break the Lorentz symmetry in $d > 2$ dimensions. This can be understood if one interprets the wedge–local operators as members of a theory on non–commutative Minkowski space, where the Lorentz symmetry is broken [8].

The kernels of the elastic scattering matrices in the deformed and undeformed theory are related by

$$^{\text{out}} \langle p \otimes_\kappa q|p' \otimes_\kappa q'|^{\text{in}} = e^{i|pQ_\kappa q|+|p'Q_\kappa q'|}^{\text{out}} \langle p \otimes q|p' \otimes q'|^{\text{in}}.$$

Thus they differ from each other, showing that the deformed and undeformed theories are not isomorphic. Yet since the difference is only a phase factor, the collision cross sections do not change under these deformations. Hence the effects of the deformation, such as the asymptotic breakdown of Lorentz invariance, could only be seen in certain specific arrangements such as time delay experiments.

4 Concluding remarks

In the present article we have presented a generalization of the deformation procedure of free quantum field theories, established by Grosse and Lechner [8], to the general setting of relativistic quantum field theory. Even though the new theories which emerge in this way may not be of direct physical relevance, the results are
of methodical interest. For they reveal yet again the significance of the wedge algebra in the algebraic approach to the construction of models.

From the algebraic point of view the problem of constructing a quantum field theory presents itself as follows. Given the stable particle content in the situation to be described, one first constructs a corresponding Fock space and representation $U$ of the Poincaré group $\mathcal{P}$. A theory with this particle content is then obtained by fixing a wedge $\mathcal{W}_0$, say, and exhibiting a $\mathfrak{g}$-algebra $\mathfrak{G} \subset \mathfrak{F}$ which can be interpreted as the algebra generated by fields and observables localized in $\mathcal{W}_0$. It thus has to satisfy the conditions

(a) $\alpha_\lambda(\mathfrak{G}) \subset \mathfrak{G}$ whenever $\lambda\mathcal{W}_0 \subset \mathcal{W}_0$ for $\lambda \in \mathcal{P}$.

(b) $\alpha_{\lambda'}(\mathfrak{G}) \subset \mathfrak{G}'$ whenever $\lambda'\mathcal{W}_0 \subset \mathcal{W}_0'$ for $\lambda' \in \mathcal{P}$.

Any algebra $\mathfrak{G}$ satisfying these conditions is the germ of a quantum field theory in the following sense: setting $\mathfrak{A}(\mathcal{W}) = \alpha_\lambda(\mathfrak{G})$, where $\lambda \in \mathcal{P}$ is such that $\mathcal{W} = \lambda\mathcal{W}_0$ for given $\mathcal{W} \in \mathfrak{W}$, it is an immediate consequence of the assumed properties of $\mathfrak{G}$ that the definition of the wedge algebras $\mathfrak{A}(\mathcal{W})$ is consistent and satisfies the conditions of isotony, covariance and locality. As explained above, the algebras corresponding to arbitrary causally closed regions can then consistently be defined by taking intersections of wedge algebras. Conversely, any asymptotically complete quantum field theory with the given particle content fixes an algebra $\mathfrak{G}$ with the above properties. Thus any quantum field theory can in principle be presented in this way. However, at present a dynamical principle by which the algebras $\mathfrak{G}$ can be selected is missing.

Nevertheless, this algebraic approach has already proven to be useful in the construction of interesting examples of quantum field theories. For instance, the existence of an infinity of models in $d = 2$ spacetime dimensions with non-trivial scattering matrix was established in this setting in [11–13], thereby solving a long-standing problem in the so-called form factor program of quantum field theory, cf. [15] and references quoted there. Wedge algebras associated with a nonlocal field in $d \geq 2$ spacetime dimensions were used in [5] to construct local observables manifesting non-trivial scattering. Wedge algebras were also used in [2] for the construction of quantum field theories describing massless particles with infinite spin, cf. also [14] for a construction of operators in these theories with somewhat better localization properties.

The idea of deforming given wedge algebras in order to arrive at new theories is a quite recent development in the algebraic approach and sheds new light on the constructive problems in quantum field theory. One may expect that the particular deformation procedure considered here is only an example of a richer family of similar constructions. Moreover, these methods can also be transferred to quantum field theories on curved spacetimes with a sufficiently big isometry group.

It is an intriguing question in this context to find manageable criteria which allow one to decide whether the intersections of wedge algebras are non-trivial. In
such a criterion based on the modular structure was put forward. Unfortunately, it is only meaningful in $d = 2$ spacetime dimensions. In the examples of deformed theories in $d > 2$ spacetime dimensions discussed here, it can be shown that the algebras corresponding to bounded spacetime regions are trivial. But, viewing the deformed theory as living on non–commutative Minkowski space [8], one may expect that the algebras corresponding to the intersection of two opposite wedges are non–trivial. It would be of conceptual interest to establish this conjecture.

References

[1] H.-J. Borchers, D. Buchholz and B. Schroer, Polarization–free generators and the S-matrix, Commun. Math. Phys., 219, 125–140 (2001).

[2] R. Brunetti, D. Guido and R. Longo, Modular localization and Wigner particles, Rev. Math. Phys., 14, 759–785 (2002).

[3] D. Buchholz and G. Lechner, Modular nuclearity and localization, Ann. Henri Poincaré, 5, 1065–1080 (2004).

[4] D. Buchholz and S.J. Summers, Stable quantum systems in Anti-de Sitter space: Causality, independence and spectral properties, J. Math. Phys., 45, 4810–4831 (2004).

[5] D. Buchholz and S.J. Summers, String– and brane–localized causal fields in a strongly nonlocal model, J. Phys. A, 40, 2147–2163 (2007).

[6] K. Fredenhagen and J. Hertel, Local algebras of observables and pointlike localized fields, Commun. Math. Phys., 80, 555–561 (1981).

[7] J. Glimm and A. Jaffe, Quantum Physics. A Functional Integral Point of View, New York: Springer, 1987.

[8] H. Grosse and G. Lechner, Wedge–local quantum fields and noncommutative Minkowski space, JHEP, 0711, 012 (2007).

[9] R. Haag, Local Quantum Physics, Berlin: Springer-Verlag, 1992.

[10] K. Hepp: On the connection between Wightman and LSZ quantum field theory, pp. 135–246 in: Brandeis University Summer Institute in Theoretical Physics 1965, “Axiomatic Field Theory”, (M. Chretien and S. Deser eds.), Gordon and Breach 1966.

[11] G. Lechner, Polarization-free quantum fields and interaction, Lett. Math. Phys., 64, 137–154 (2003).

[12] G. Lechner, On the existence of local observables in theories with a factorizing S-matrix, J. Phys. A, 38, 3045–3056 (2005).

[13] G. Lechner, Construction of quantum field theories with factorizing S-matrices, Commun. Math. Phys., 277, 821–860 (2008).

[14] J. Mund, B. Schroer and J. Yngvason, String–localized quantum fields and modular localization, Commun. Math. Phys., 268, 621–672 (2006).

[15] B. Schroer, Modular localization and the bootstrap–formfactor program, Nucl. Phys. B, 499, 547–568 (1997).