The area operator and fixed area states in conformal field theories

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The fixed area states are constructed by gravitational path integrals in previous studies. In this paper we show the dual of the fixed area states in conformal field theories (CFTs). These CFT states are constructed by using spectrum decomposition of reduced density matrix \( \rho_A \) for a subsystem \( A \). For 2 dimensional CFTs we directly construct the bulk metric, which is consistent with the expected geometry of the fixed area states. For arbitrary pure geometric state \( |\psi\rangle \) in any dimension we also find the consistency by using the gravity dual of Rényi entropy. We also give the relation of parameters for the bulk and boundary state. The pure geometric state \( |\psi\rangle \) can be expanded as superposition of the fixed area states. Motivated by this, we propose an area operator \( \hat{A}^v \). The fixed area state is the eigenstate of \( \hat{A}^v \), the associated eigenvalue is related to Rényi entropy of subsystem \( A \) in this state. The Ryu-Takayanagi formula can be expressed as the expectation value \( \langle \psi|\hat{A}^v|\psi\rangle \) divided by \( 4G \), where \( G \) is the Newton constant. We also show the fluctuation of the area operator in the geometric state \( |\psi\rangle \) is suppressed in the semiclassical limit \( G \to 0 \).

INTRODUCTION

AdS/CFT provides us a way to understand the nature of the bulk spacetime by the CFT living on the boundary. One of interesting topics in AdS/CFT is the exact duality relation between quantum states in the Hilbert space of the boundary CFT and the ones in the bulk. For some quantum states in the CFT we expect they can be effectively described by the classical geometries in the limit \( G \to 0 \).

The geometry is associated with the entanglement entropy (EE) \( S(\rho_A) \) of a boundary subregion \( A \) by the well-known Ryu-Takayanagi formula for the bulk metric with time reflection symmetry,

\[
S(\rho_A) = \frac{\text{Area}(\gamma_A)}{4G},
\]

where \( \gamma_A \) is the minimal surface in the bulk that is homology to \( A \), \( \rho_A \) denotes the reduced density matrix of \( A \). For general bulk spacetime one should take \( \gamma_A \) to be Hubeny-Rangamani-Takayanagi (HRT) surface. The RT formula shows the secret relation between spacetime and intrinsic entanglement of underlying degree of freedoms of quantum gravity.

The area law like relation is generalized to the holographic Rényi entropy by Dong. The Rényi entropy, defined as \( S_n(\rho_A) := \frac{\log \text{tr} \rho_A^n}{n-1} \), is one parameter generalization of entanglement entropy. The gravity dual of Rényi entropy is given by

\[
n^2 \partial_n \left( \frac{n-1}{n} S_n(\rho_A) \right) = \frac{\text{Area}(B_n)}{4G},
\]

where \( B_n \) deotes the cosmic brane with the tension \( T_n = \frac{n-1}{4nG} \). The cosmic brane backreacts on the geometry by creating conical defect with opening angle \( \theta = \frac{\pi n}{n} \).

The Rényi entropies contain more information on the density matrix \( \rho_A \). Actually one could construct the spectrum decomposition once knowing the Rényi entropy for all the index \( n \). For the pure state \( |\psi\rangle \) we have Schmidt decomposition \( |\psi\rangle = \sum_i e^{-\frac{b^v}{2} t_i} |i\rangle |\bar{i}\rangle^v \), \( |i\rangle^v \) and \( |\bar{i}\rangle^v \) are Schmidt basis of \( A \) and its complementary part \( \bar{A} \). The reduced density matrix \( \rho^v_A \) has the spectrum decomposition \( \rho^v_A = \sum_i e^{-b^v t_i} |i\rangle \langle i| \) with \( t_i \in [0, +\infty) \), where \( e^{-b^v} \) is the maximal eigenvalue of \( \rho^v_A \). By definition we have \( b^v = \lim_{n \to \infty} S_n(\rho^v_A) \). The modular Hamiltonian \( H^v_A := -\log \rho^v_A \) satisfies \( H^v_A |i\rangle^v = (t_i + b)|i\rangle^v \). For quantum field theory the spectra of \( H^v_A \) should be continuous. By definition of Rényi entropy we have the relation \( \sum_i e^{-n(b^v + t_i)} = \int_0^\infty dt \mathcal{P}^v(t) e^{-n(b^v + t)} = e^{(1-n)S_n(\rho^v_A))} \), where \( \mathcal{P}^v(t) := \sum_k \delta(t_k - t) \) is the density of eigenstates. By an inverse Laplace transformation in the variable \( n \) we can obtain \( \mathcal{P}^v(t) \), see also \([2,10]\). With this we can construct the state,

\[
|\Phi_i^v\rangle := \frac{1}{\sqrt{\mathcal{P}^v(t)}} \sum_k |k\rangle^v |\bar{k}\rangle^v \delta(t_k - t). 
\]

The reduced density matrix of \( A \) is \( \rho^v_{t,A} = \frac{1}{\mathcal{P}^v(t)} \sum_k |k\rangle^v \langle k| \delta(t_k - t) \). One of the interesting result is the pure geometric state \( |\psi\rangle \) can be approximated by the state \( \rho^v_{t,A} \) with \( t = S(\rho^v_A) - S(\rho^v_A) \) for the holographic CFTs. Moreover, this state has flat spectra, thus the Rényi entropy is independent with \( n \). The so-called fixed area states constructed in \([11]\) also show the same property. Their approach is based on the gravitational path integral with inserting a cosmic brane fixed to be on the RT surface. Roughly, we can take the condition that Rényi entropies don’t depend on \( n \) as the definition of the fixed area state. They also has nice interpretation by the quantum error-correction code of AdS/CFT. More discussions on the fixed area states can be found in \([12,13]\).
One of the motivation of this paper is to show the state $|\Phi\rangle_t^\otimes n$ is exactly dual to the fixed area state for any $t \sim O(c)$. For the vacuum case in AdS$_3$ and $A$ being an interval, we show this conclusion by direct constructions of the bulk geometry and calculation of the Rényi entropy in the bulk. For more general cases, combination of the proposal of holographic Rényi entropy \cite{2}, we show the state $|\Phi\rangle_t^\otimes n$ corresponds to the geometry by inserting a cosmic brane $B_{n^*}$ with tension $\mu_{n^*} = \frac{2}{4\pi G}$, where $n^*$ can be solved by an equation involving $S_n(\rho_{\bar{A}})$ and $t$.

With these results we conclude any pure geometric state can be taken as superposition of fixed area states. The coefficients of the superposition are associated with the area of the cosmic brane $B_{n^*}$. Motivated by this, we introduce an area operator $A$, for which the fixed area states are its eigenstates. The expectation value of $A$ divided by $4G$ in the geometric state gives the EE. This can be seen as a quantum version of RT formula.

**VACUUM STATE: AN EXPLICIT EXAMPLE**

Consider a 2-dimensional CFT with central charge $c$ on complex plane with the coordinate $(w, \bar{w}) := (x + i\tau, x - i\tau)$. In this section we will remove the superscript “$\psi$” to indicate the quantities are defined for vacuum state. For an interval $A = [-R, R]$ in the vacuum state the Rényi entropy is universal for 2D CFTs \cite{19}, given by

$$S_n(\rho_A) = (1 + \frac{1}{n})b$$

with $b := \lim_{n \to \infty} S_n(\rho_A) = \frac{2}{3} \log \frac{2b}{c}$. We can obtain the density of eigenstates with respect to $t$ \cite{8}

$$\mathcal{P}(t) = \delta(t) + \frac{1}{t} I_1(2\sqrt{bt})H(t),$$

where $I_n(z)$ is the modified Bessel function of the first kind, $H(t)$ is the Heaviside step function.

In the CFT side the EE of the state $\rho_{t,A}$ is given by $\log \mathcal{P}(t)$. For the holographic CFT $b \sim O(c) \gg 1$, taking $t$ to be the order of $c$. The density of state $\mathcal{P}(t) \approx \frac{w^{2\sqrt{bt}}}{\sqrt{4\pi(2b)}}$. The EE is $\log \mathcal{P}(t) \approx 2\sqrt{bt} + O(\log c)$.

By construction the Rényi entropy of the state $\rho_{t,A}$ is same as the EE, which is an important feature for the so-called fixed area states \cite{11}. In the following we would like to show the state $|\Phi\rangle_t$ is a fixed area state by explicitly constructing the bulk geometry. Using a similar method as in \cite{20}, we can get the expectation value of stress energy tensor $T(w)$ in the state $|\Phi\rangle_t$ \cite{10}

$$\langle T(w) \rangle_t = \frac{cR^2}{6(R^2 - w^2)^2} \left(1 - \frac{t}{b}\right).$$

Similarly, one could get $\langle \bar{T}(\bar{w}) \rangle_t$ by replacing $w$ with $\bar{w}$ in the above expression. For pure state the Rényi entropy satisfies $S_n(\rho_A) = S_n(\rho_{\bar{A}})$, where $\rho_{\bar{A}}$ is the reduced density matrix of the complementary part of $A$. The singularity at the ending points of interval $A$ is associated with the conical defect as we will show soon. The bulk solution is fixed by the one-point function of $T(w)$, the bulk geometry dual to $|\Phi\rangle_t$ is

$$ds^2 = \frac{dy^2}{y^2} + \frac{L^4_4}{2} dw^2 + \frac{\bar{L}}{2} d\bar{w}^2 + \left(\frac{1}{y^2} + \frac{\bar{y}^2}{4L^4_4\bar{L}}\right) dw d\bar{w},$$

where $L_t := -\frac{d^2}{d\langle T(w) \rangle_t}$, $\bar{L}_t := -\frac{d^2}{d\langle \bar{T}(\bar{w}) \rangle_t}$. The above solution has singularity in the coordinate $(y, w, \bar{w})$. By a conformal transformation $\xi = \left(\frac{R^2 + w}{R^2 - w}\right)^\alpha$, $\xi = \left(\frac{R^2 + w}{R^2 - w}\right)^\beta$ with $\alpha := \sqrt{\frac{2}{b}}$, we have $\langle T(\xi) \rangle = \langle \bar{T}(\bar{\xi}) \rangle = 0$. At the points $\xi = 0$, $\beta$ has conical defect with opening angle $\theta = 2\alpha\pi$. The dual bulk solution is the Poincaré coordinate $ds^2 = \frac{dy^2 + dz d\bar{z}}{y^2}$ with a conical defect line $\gamma$.

With the geometry \cite{6} one could find the geodesic line $\gamma_A$ connecting the ending points of $A$ and evaluate the holographic EE by using the RT formula \cite{11}. The details of the calculations can be found in Appendix A. The result is

$$S_A(\rho_{t,A}) = \frac{L\gamma_A}{4G} = \frac{\alpha c}{3} \log \frac{2R}{c} = 2\sqrt{bt},$$

where we have used the Brown-Henneaux relation $c = \frac{3}{2G}$. The result is exactly consistent with the CFT result to the leading order in $1/G$.

Consider the $n$-replica state $\rho^n_{t,A}$, the one-point function $tr(\rho^n_{t,A} T(w))$ is given by the same formula as \cite{5}. But now $w$ is the coordinate on the $n$-sheet Riemann surface $\mathcal{R}_n$. Adopting polar coordinates near the ending points of $A$, we have $w - R \approx r e^{i\theta}$ with $\theta \sim \theta + 2n\pi$. Using the same conformal transformation $w \to \xi = \left(\frac{w + R}{w - R}\right)^\alpha$, $\mathcal{R}_n$ is mapped to the $\xi$-plane with the conical defect with opening angle $\theta_n = 2n\alpha\pi$. Therefore, the dual bulk geometry $\mathcal{M}_n$ for $\mathcal{R}_n$ is the Poincaré coordinate with a conical defect line $\gamma$. Moreover, $\mathcal{M}_n$ can be constructed by cyclically gluing $n$-copy geometry together along the defect line $\gamma$. In \cite{11} the fixed area states are constructed in the way as we have stated above. The conical defect line can be realized by inserting codimension-2 cosmic branes (lines in AdS$_3$). The tension of the cosmic brane $\mu_n$ is associated with the parameter $\alpha$ by the relation $\mu_n = \frac{1}{4\pi G} \frac{2}{n\alpha} \frac{2}{22}$.

We can show the defect line $\gamma = \gamma_A$ by using the requirement that $S_n(\rho_{t,A}) = S(\rho_{t,A})$. To evaluate the Rényi entropy $S_n(\rho_{t,A})$ we need to know the bulk action $I_{bulk}(n)$, which includes the on shell action $I_{g}(n)$ of the geometry $\mathcal{M}_n$ and the brane action $I_{b}(n)$. We show the details of the calculations in Appendix B. The result is

$$I_{bulk}(n) = I_{g}(n) + I_{b}(n),$$
with
\[ I_b(n) = \frac{(n\alpha - 1)\gamma}{4G}, \]
where \( L_\gamma \) is the length of the defect line \( \gamma \). The Rényi entropy is
\[ S_n(\rho_{t,A}) = \frac{I_{\text{bulk}}(n) - nI_{\text{bulk}}(1)}{n - 1} = \frac{L_\gamma}{4G}. \]

Comparing with the holographic EE result \( (7) \) we have \( L_\gamma = L_{\gamma_A} \). This means the defect line \( \gamma \) coincides with the geodesic line \( \gamma_A \).

We expect the states \( (3) \) are dual to the fixed area states only for \( t \sim O(c) \) in the holographic CFTs. For \( t \sim O(1) \) or \( t \ll c \sim b \), the one-point function of \( T \) is still given by \( (3) \). It seems we could construct the geometry for these states. But the density of state \( \mathcal{P}(t) \) no long scales as \( e^{\frac{c}{2b}} \), thus the EE log \( \mathcal{P}(t) \) in these states are not of \( O(c) \). We don’t expect they have well-defined bulk geometry.

**PURE GEOMETRIC STATES AS SUPERPOSITION OF FIXED AREA STATES**

For the holographic CFT with large central charge \( c \), our results in the above section show exactly that the state \( |\Phi \rangle_t \) can be explained as fixed area state if the \( t \) is of the order of \( c \). This gives us a new way to understand geometric states by decomposing them into fixed area states.

To be more precise we have
\[ |0\rangle = \sum_i e^{-\frac{b}{2}t+i} |i\rangle |\bar{i}\rangle = \int_0^\infty dt \sqrt{\mathcal{P}(t)} e^{-\frac{b}{2}t} |\Phi \rangle_t. \]

The reduced density matrix of \( A \) is
\[ \rho_A = \int_0^\infty dt e^{-b_\gamma t} \mathcal{P}(t) \rho_{t,A}. \]

Actually, \( (12) \) is just the spectrum decomposition of the operator \( \rho_A \). \( P_t := \mathcal{P}(t) \rho_{t,A} \) are projections into the Hilbert subspace with respect to the spectrum \( e^{-b-t} \). The states \( |\Phi \rangle_t \) are fixed area states if \( t \sim O(c) \). However, the contributions from \( t \ll c \) are usually exponentially suppressed in the large \( c \) limit. We can safely take the vacuum state of a holographic CFT as superposition of fixed area states by introducing a lower cut-off of the integral \( (11) \).

For arbitrary pure geometric state \( |\psi \rangle \), the reduced density matrix of subsystem \( A \) can be expressed as
\[ \rho_A^\psi = \int_0^\infty dt \mathcal{P}^\psi(t) e^{-b_\gamma t} \rho_{t,A}. \]

The density of eigenstates \( \mathcal{P}^\psi(t) \) is given by
\[ \mathcal{P}^\psi(t) = L^{-1} \left[ e^{nb_t+(1-n)S_n(\rho_A^\psi)} \right](t) \]
\[ = \frac{1}{2\pi i} \int_{\gamma_0-i\infty}^{\gamma_0+i\infty} dne^{sn}, \]

with
\[ s_n := (n\alpha + 1 - n)S_n(\rho_A^\psi), \]

where the \( \gamma_0 \) is chosen for the convergence of the integration. \( S_n(\rho_A^\psi) \) is the Rényi entropy of subsystem \( A \) in the state \( |\psi \rangle \). In general, it’s hard to evaluate the Rényi entropy for arbitrary states. For holographic theories, \( S_n(\rho_A^\psi) \) is expected to be of order \( O(G^{-1}) \) in the semi-classical limit \( G \to 0 \). For \( t \sim O(G^{-1}) \) we can evaluate the integral \( (13) \) by saddle point approximation. That is to solve the equation
\[ \partial_n s_n = (t + b) - \partial_n[(1 - n)S_n(\rho_A^\psi)] = 0. \]

In general, \( (16) \) is a complicated equation for \( n \). Assume the solutions exist. If we have more than one solution, we should take the one that minimizes \( s_n \). With the solution \( n^* = n^*(t) \) we have
\[ s_{n^*} = \left[ S_n(\rho_A^\psi) + (n - 1) \partial_n S_n(\rho_A^\psi) \right]_{n=n^*}. \]

Using Dong’s proposal of holographic Rényi entropy \( (2) \) we have
\[ s_{n^*} = \frac{\text{Area}(B_{n^*})}{4G}. \]

Therefore, the density of eigenstates is given by
\[ \mathcal{P}^\psi(t) \propto e^{-\frac{\text{Area}(B_{n^*})}{4G}}. \]

By definition the Rényi entropy of the state \( \rho_{t,A}^\psi \) is independent with \( n \), given by
\[ S_n(\rho_{t,A}^\psi) = \log \mathcal{P}^\psi(t) \simeq \frac{\text{Area}(B_{n^*})}{4G}. \]

Our results show the states \( |\Psi\rangle^\psi \) have same property as the fixed area state. \( (16) \) and \( (20) \) give the dual relation between the parameter \( t \) and the bulk fixed area, that is the area of the cosmic brane \( \text{Area}(B_n) \). Supposed the geometry dual to \( |\psi\rangle^\psi \) is \( M_\psi \). According to Dong’s proposal of Rényi entropy the tension of the codimension-2 cosmic brane \( B_n \) is \( \mu_n = \frac{c}{2b} \). To obtain the geometry dual to the fixed area state \( |\Phi\rangle^\psi \) one should insert a codimension-2 cosmic brane with tension \( \mu = \frac{c}{4G} \), where \( n^* \) is the solution of the equation \( (16) \). If the equation has more than one solution, we should take the one that minimizes the function \( s_n \). The cosmic brane backreacts on the geometry \( M_\psi \) and creates a conical defect with opening angle \( \theta := 2\pi\alpha_t = 2\pi - 8\pi G \mu_{n^*} \). The location of the cosmic brane coincides with the RT surface for subregion \( A \) in the backreacted geometry. The role of the cosmic
brane is like a sharp projection that maps the original geometry $\mathcal{M}_\psi$ to the fixed area geometry. The above results are consistent with the discussion in \cite{11} by using the gravitational path integral.

As a check of the above statement, let’s consider the vacuum state in AdS$_3$. Taking the Rényi entropy $S_n(\rho_A) = (1 + \frac{1}{n})b$ into the equation \cite{10}, we have the solution $n^* = \sqrt{b/t}$. The tension of the cosmic line is $\mu_t = \frac{1}{n^*} \sqrt{1 - \frac{1}{\sqrt{b/t}}}$ and the opening angle of the conical defect line is $\theta = 2\pi \sqrt{b/t}$. The results are exactly consistent with our direct calculations in last section.

By using the expression of $\mathcal{P}^\psi(t)$, arbitrary pure geometric state $|\psi\rangle$ can be seen as superposition of a series of the fixed area states,

$$|\psi\rangle = \int_0^\infty dt \sqrt{p_t^\psi} |\Phi_t^\psi\rangle,$$

where $p_t^\psi := e^{-\frac{\text{Area}B_n^{\psi*}}{4G} - b^\psi - t}$. Like the vacuum case we expect the contributions from small $t$ ($t \ll c$) of the above integration are negligible.

**PROBABILITY OF THE FIXED AREA STATES**

The quantum error correction code interpretation of AdS/CFT suggests the coefficients $p_t^\psi$ of \cite{21} can be associated with the on-shell action $I_t^\psi$ of the corresponding fixed area states $|\Phi_t^\psi\rangle$ \cite{10,11,17}. The expected relation is $p_t^\psi = e^{-I_t^\psi}$. Using the result \cite{21}, we have

$$I_t^\psi = b\psi + t - \frac{\text{Area}B_n^{\psi*}}{4G},$$

which depends on the parameter $t$. $p_t^\psi$ can be explained as the probability for the geometric state $|\psi\rangle$ to be the fixed area state $|\Phi_t^\psi\rangle$.

For the vacuum case $|0\rangle$, $b\psi = b$ and $\frac{\text{Area}B_n^{\psi*}}{4G} = 2\sqrt{bt}$, the action $I_t = b(1 - \sqrt{\frac{b}{t}})^2 = b(1 - \alpha)^2$, which is consistent with \cite{8} and the results in \cite{17}. The probability distribution $p_t := e^{-I_t}$ has maximal value at $t = b$. In the semiclassical limit $G \to 0$ or $c \to \infty$, the distribution will approach a delta function $\delta(t-b)$. Therefore, $\rho_A$ can be approximated by the fixed area state $\rho_{c=b}$. One could check the EE of $\rho_A$ is same as $\rho_{c=b}$ in the leading order of $c$. Taking $t = b$ into \cite{10} we get same geometry as the vacuum AdS$_3$. However, we could find other probes that could distinguish the two states, see more discussions in \cite{10}. This means the superposition among the fixed area states is important to understand the full properties of the geometry dual to $|\psi\rangle$.

We can also consider the unnormalized $n$-copy state

$$(\rho_A)^n = \int_0^\infty dt p_t^n (\rho_t)^n \simeq \int_0^\infty dt \sqrt{\frac{b}{t}} e^{-n(b+t)+2\sqrt{bt}} \rho_{t,A}.$$

It can be shown $(\rho_A)^n \simeq e^{-(n-\frac{4}{n})b_\rho t - \frac{1}{4G}}$. This means the geometry of $n$-copy state is approximated by the fixed area state with $t = \frac{b}{n}$, which is the spacetime inserting a cosmic brane with tension $\frac{1}{4G}$. It is a consistent check with Dong’s proposal of holographic Rényi entropy.

In general, $I_t^\psi$ is proportional to $1/G$. In the semiclassical limit $G \to 0$, we expect the probability $p_t^\psi$ has maximal value at $\tilde{t}$, which is fixed by the equation $\partial_t I_t^\psi|_{t=\tilde{t}} = 0$. It is not easy to find $\tilde{t}$ by solving \cite{18} and \cite{10}. Motivated by the vacuum case, we can fix $\tilde{t}$ by requiring the EE of $\rho_{t=\tilde{t}}^\psi$ is equal to the EE of $\rho_A^\psi$. This leads to the one-point functions of local operators $O$ in states $\rho_A^\psi$ are equal to the ones in $\rho_{t=\tilde{t}}^\psi$ in the semiclassical limit $c \to \infty$. This leads to the result

$$\int_0^\infty dt p_t^\psi \to \int_0^\infty dt \delta(t - \tilde{t})$$

in the semiclassical limit $c \to \infty$ or $G \to 0$.

**GEOMETRY AND AREA OPERATOR**

The fixed area states $|\Phi_t^\psi\rangle$ can be taken as the basis of a given pure geometric state $|\psi\rangle$. We may introduce an operator $A^\psi$, which is expected to satisfy the following conditions:

1. Positive semidefinite Hermitian and state-dependent operator \cite{23}.
2. Fixed area states are its eigenstates.
3. Located in subsystem $A$ or $\tilde{A}$
4. Its expectation value in geometric state $|\psi\rangle$ divided by $4G$ gives the RT formula \cite{24} and its fluctuation in $|\psi\rangle$ is supressed in the semiclassical limit $G \to 0$.

The area operator $A^\psi$ can be constructed by spectrum decomposition. The modular Hamiltonian $H_A^\psi$ has the spectrum decomposition as $H_A^\psi = \int_0^\infty dt (t+b^\psi)P_t^\psi$, where $P_t^\psi := \mathcal{P}^\psi(t)\rho_{t,A}^\psi$. According to the operator theory \cite{22}, we can define the new operators

$$F(H_A^\psi) := \int_0^\infty dt F(t+b^\psi)P_t^\psi,$$

where $F(x)$ is the functions of $x$ \cite{26}. The operators satisfy $F(H_A^\psi)|\Phi_t^\psi\rangle = F(t+b^\psi)|\Phi_t^\psi\rangle$. The area operator can be defined as

$$\hat{A}^\psi = s(H_A^\psi - b^\psi) = \int_0^\infty ds(t)P_t^\psi,$$

where $s(t) := \frac{s}{s_{n^*}}$, $s_{n^*}$ is given by \cite{17}. If we further use the holographic proposal of Rényi entropy, the area operator is

$$\hat{A}^\psi = \int_0^\infty dt \text{Area}(B_n^\psi)P_t^\psi,$$
where we used \( \langle \hat{A}^\psi \rangle_\psi = \frac{\langle \hat{A}^\psi \rangle_\psi}{\sqrt{2}} \). It is obvious that \( \hat{A}^\psi |\Phi_i^\psi \rangle = \text{Area}(B_{n^i}) |\Phi_i^\psi \rangle \), Area\((B_{n^i}) \) is the area of the bulk RT surface for the geometry dual to the fixed area state \( |\Phi_i^\psi \rangle \). The expectation value of \( \hat{A} \) in \( |\psi \rangle \) is

\[
\langle \hat{A}^\psi \rangle_\psi = \int_0^\infty dt \rho^\psi \text{Area}(B_{n^i}) = \int_0^\infty d\tau e^{-t^G} \text{Area}(B_{n^i}).
\]

According to (24), we have

\[
\langle \hat{A}^\psi \rangle_\psi \to \int_0^\infty dt \delta(t - \bar{t}) \text{Area}(B_{n^i}) = \text{Area}(B_1), \quad (28)
\]
in the semiclassical limit \( G \to 0 \). Area\((B_1) \) is just the area of the RT surface in the geometry dual to \( |\psi \rangle \). The RT formula can be expressed by area operator as

\[
S(\rho^\psi_A) = \frac{\langle \hat{A}^\psi \rangle_\psi}{4G}. \quad (29)
\]

By using the definition of the EE \( S(\rho^\psi_A) = -tr(\rho^\psi_A \log \rho^\psi_A) \), we have a nice result

\[
\langle \psi | (H_A^\psi - \frac{\hat{A}^\psi}{4G}) |\psi \rangle = \int_0^\infty d\tau e^{-t^G} (t + b^\psi - \frac{\text{Area}(B_{n^i})}{4G}),
\]

\[
\to (\bar{t} + b^\psi - \frac{\text{Area}(B_1)}{4G}) = 0, \quad (30)
\]
in the limit \( G \to 0 \). This can seen as the bulk dual of the modular Hamiltonian to the leading order in the \( 1/G \) expansion\([27]\).

To characterize the fluctuation of the area operator in the state \( |\psi \rangle \), we can define the uncertainty of the area operator \( \Delta \hat{A}^\psi \) as \( \sqrt{\langle (\hat{A}^\psi)^2 \rangle_\psi - \langle \hat{A}^\psi \rangle^2_\psi} \). By using (24), we can show \( \langle \Delta \hat{A}^\psi \rangle_\psi = 0 \) in the limit \( G \to 0 \).

We show the results for vacuum state in Appendix C.

**DISCUSSION**

In this paper we only focus on the pure geometric state. Some important modifications are necessary to generalize the results to the mixed states.

In the last section we only consider the leading order result in the expansion of gravitational coupling \( G \). The RT formula would receive correction at higher orders in \( G \)[23]. That would be interesting to consider the higher orders corrections, which is important to understand the quantum nature of spacetime.

Our constructed area operator is expressed as superposition of projectors in CFTs. It may be possible to find its bulk dual by reconstruction of the bulk operators in entanglement wedge\([13, 20]\).

In the last part of the paper, we show the uncertainty of the area operator is vanishing to the leading order in \( G \). This is the expected feature for geometric state, for which the quantum fluctuation should be suppressed.

This property is similar to the constraints of geometric states\([30]\), which are expressed as conditions for connected correlation functions of stress energy tensor. It would be interesting to find their possible connections.

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The bulk metric in the coordinate \((u, z, \bar{z})\) is
\[
ds^2 = \frac{du^2}{u^2} + \frac{L(z)}{2} dz^2 + \frac{L(z)}{2} d\bar{z}^2 + \left(1 + \frac{u^2}{4} L(z) \bar{L}(z)\right) dz d\bar{z},
\]
with
\[
L(z) = \left\{ f(z); \bar{z} \right\} := \frac{3 f''(z)^2 - 2 f'(z) f'''(z)}{2 f'(z)^2},
\]
\[
\bar{L}(z) = \{ \bar{f}(z); \bar{z} \} := \frac{3 \bar{f}''(\bar{z})^2 - 2 \bar{f}'(\bar{z}) \bar{f}'''(\bar{z})}{2 \bar{f}'(\bar{z})^2},
\]
where \(\{ f(z); \bar{z} \}\) is the Schwarzian derivative. Using the transformation law of the stress energy tensor of the boundary theory, we have
\[
L(z) = -\frac{2}{\pi} \text{tr}(\rho T(z)) \quad \text{and} \quad \bar{L}(z) = -\frac{2}{\pi} \text{tr}(\rho \bar{T}(\bar{z})),
\]
where \(\rho\) is the state of the boundary CFT.

We can use the above coordinate transformation associated the conformal mapping \(\xi \to w\) to evaluate the geodesic line homology to \(A\). By the conformal mapping \(\xi = g(w) := \left(\frac{R+w}{R-w}\right)^{\alpha}\), the ending point of \(A = w_2 = R\) is mapped to \(\xi = \infty\). To regularize the coordinate in the \(\xi\)-plane we choose the ending points of \(A\) as \(w_1 = -R+\epsilon\) and \(w_2 = R - \epsilon\), where \(\epsilon\) is the UV cut-off. The associated bulk coordinate transformation is given by \(w \to y, f(z) \to g(w)\) and \(\bar{g}(\bar{w}) \to \bar{g}(\bar{w})\).

By RT formula the EE of subsystem \(A\) is related to the geodesic line \(\gamma_A\) connecting \((y,w) = (\epsilon, w_1)\) and \((y, w) = (\epsilon, w_2)\). Their images in the Poincaré coordinate are \((\eta, \bar{\eta}) = (\eta_1, g(w_1))\) and \((\eta, \bar{\eta}) = (\eta_2, g(w_2))\), where \(\eta_1(\xi)\) is obtained by using the third equation of \(22\) and taking \(u \to \epsilon\), \(f(z) \to g(w)|_{w=w_1(\xi)}\). The length geodesic line \(L(\eta_1, g(w_1))\) and \((\eta_2, g(w_2))\) is
\[
L_{\gamma_A} = \log \frac{\xi_1 - \xi_2 (\xi_1 - \xi_2)}{\eta_1 \eta_2}.
\]

With some calculations we have
\[
S_A = \frac{L_{\gamma_A}}{4G} = \frac{\alpha}{2G} \log \frac{2R}{\epsilon} = \frac{\alpha e}{3} \log \frac{2R}{\epsilon},
\]
where we use the Brown-Henneaux relation \(c = \frac{3}{G}\).

In the main text we use the condition \(S_A(\rho_{t,A}) = S(\rho_{t,A})\) to get \(\gamma = \gamma_A\). We could obtain \(\gamma = \gamma_A\) more directly on the \(z\)-plane with the coordinate transformation \(z = \frac{w}{w_0}\). This is a global conformal transformation which maps the interval \(A\) to the half line \([0, +\infty)\). We can work out the corresponding bulk metric
\[
ds^2 = \frac{du^2}{u^2} + \frac{\alpha^2 - 1}{4\xi^2} dz^2 + \frac{\alpha^2 - 1}{4\bar{z}^2} d\bar{z}^2 + \left(1 + \frac{\alpha^2 - 1}{16 u^2}\right) dz d\bar{z},
\]
where $u$ is the holographic coordinate. With a further conformal map $\xi = z^{-\alpha}$ we arrive at the $\xi$-plane. Actually, the transformation $\xi = \left(\frac{R+u}{R-w}\right)^{\alpha}$ is given by the combined two maps $w \to z \to \xi$. For the conformal mapping $\xi = \left(\frac{R+u}{R-w}\right)^{\alpha}$, the bulk metric transforms to (10). For the conformal mapping $\xi = z^{-\alpha}$, $\xi = z^{-\alpha}$, we get the bulk metric (37). With a conformal mapping $z = \frac{R+u}{R-w}$ the bulk metric (10) transforms to (37).

By symmetry the conical defect line $\gamma$ should be the $u$-axis, which is same as the geodesic line $\gamma_A$.

**Appendix B: Action for $M_n$**

To evaluate Rényi entropy we need to calculate the on-shell action of $M_n$. We will focus on 3D. The action consists of three parts,

$$I_g = I_{EH} + I_{GH} + I_{ct},$$

with

$$I_{EH} = -\frac{1}{16\pi G} \int d^3x \sqrt{g} (R + 2),$$

$$I_{GH} = -\frac{1}{8\pi G} \int_{bdy} \sqrt{h} k,$$

$$I_{ct} = \frac{1}{16\pi G} \int_{bdy} \sqrt{h},$$

where “bdy” means the boundary of the bulk. The general metric of AdS$_3$ is given by (33). The boundary is taken to be the surface $u = \epsilon$. The term $I_{EH}$ involves the integration over the bulk, we should take an IR cutoff $u = u_{IR}$ to regularize it. We will follow the strategy of $\mathbb{R}$ to fix $u_{IR}$ by the condition $det(g) = 0$, which leads to the solution

$$u_{IR} = \frac{\sqrt{2}}{(L(z)\bar{L}(\bar{z}))^{1/4}}.$$  

With some calculations we have

$$I_{EH} = -\int d\bar{z} \bar{z} \left[ -\frac{1}{16\pi G} + \frac{\sqrt{L(z)\bar{L}(\bar{z})}}{16\pi G} \right],$$

$$I_{GH} = -\int d\bar{z} \bar{z} \left[ \frac{1}{8\pi G} \right], I_{ct} = \int d\bar{z} \bar{z} \frac{1}{16\pi G}$$

The total action is

$$I_g = \frac{-1}{16\pi G} \int d\bar{z} \bar{z} \sqrt{L(z)\bar{L}(\bar{z})}.$$  

To construct the fixed area state the cosmic brane is necessary. The cosmic brane backreacts the original geometry, thus the one-point function of stress energy tensor $T_{\mu\nu}$ should depends on the tension of cosmic brane. The action also contains the contributions from the cosmic brane, which is related to the opening angle of the conical defect and the area of the cosmic brane. For metric with conical defect the Ricci scalar contains a delta function. A new term associated with the length of the cosmic line is

$$I_{brane} = -\mu \int d\gamma,$$

where $\mu$ is the tension of the cosmic brane. The above results can be used to calculate the on-shell gravity action of the fixed area states. Consider the interval $A = [-R, R]$ on the $w$-plane. The geometry is given by the metric (36). The on-shell action of the geometry is given by

$$I_g(M_1) = -\frac{1}{16\pi G} \int dwd\bar{w} \sqrt{L_t L_1},$$

$$= -\frac{1}{16\pi G} \left( 1 - \frac{t}{b} \right) \int dwd\bar{w} \frac{2R^2}{|R-w|^2|R-\bar{w}|^2},$$

The above integration has singularity at the ending point of $A$. The integration of (44) is evaluated in the $w$-plane with cut-off $|w - R| \geq \epsilon$ and $|w + R| \geq \epsilon$. The same integration is done in section 3.2 in [9]. The integral can be reduced to two integrated terms around the cut-off circle around the ending point of $A$. The result is

$$I_g(M_1) = -\frac{1}{64\pi G} \left( 1 - \frac{t}{b} \right) \left[ \int_{w = -R} \int_{w = R} d\theta \epsilon (\chi \partial_\theta \chi) \right]$$

$$= \frac{1}{4G} \left( 1 - \frac{t}{b} \right) \log \frac{2R}{\epsilon} = (1 - \alpha^2)b,$$  

where $\chi := \frac{1}{2} \log \frac{w + R}{w - R}$, $\alpha = \sqrt{\frac{b}{2}}$ and $b = \frac{1}{4\pi} \log \frac{2R}{\epsilon}$. The geometry $M_1$ has conical defect at the line $\gamma$. The tension of the cosmic brane is $\mu = \frac{1}{4\gamma}$. The action from the cosmic line is

$$I_{brane} = \frac{1 - \alpha^2}{4\gamma} L_\gamma,$$  

where $L_\gamma$ is the length of the cosmic line. For the fixed area states the action of $M_n$ is generally expected to be

$$I(M_n) = nI_g(M_1) + \frac{n \Delta \theta_n - 2\pi}{8\pi G} A,$$  

where $I_g$ denotes the contributions from the geometry, $A$ is the area of the cosmic brane. The second term of the right hand side of (48) is the contribution from cosmic brane. The ansatz of the action of $M$ leads to the result that Rényi entropy in independent with $n$.

For the n-copy spacetime $M_n$, the action from the geometry is just $n$ times $I_g(M_1)$ since the range of $\theta$ in (46) is $2\pi$. The tension of the cosmic line is $\mu_n = \frac{1 - \frac{n}{4\gamma}}{4\gamma}$. The total action is

$$I_{tot}(M_n) = nI_g(M_1) + \frac{n\alpha - 1}{4\gamma} L_\gamma,$$  

which is consistent with the ansatz (48).
Appendix C: Area operator for vacuum state

We can study more details of the area operator for the vacuum case \( |\psi\rangle = |0\rangle \). The area operator is given by

\[
\hat{A} = 4G \int_0^\infty dt (2\sqrt{bt}) P_t,
\]

(50)

where we have used the fact \( \text{Area}(B) \approx 2\sqrt{bt} \). The expectation value of \( \hat{A} \) in vacuum is

\[
\langle \hat{A} \rangle = \int_0^\infty dt P(t)e^{-b-t(2\sqrt{bt})},
\]

\[
= \sqrt{\pi}b^{3/2}e^{-b/2} \left[ I_0\left(\frac{b}{2}\right) + I_1\left(\frac{b}{2}\right) \right] \simeq 8Gb.
\]

(51)

In the last step we use \( I_n(x) \simeq \frac{x^n}{\sqrt{2\pi x}} \) for large \( x \). The result is consistent with the classical RT formula \( S_A = \frac{\langle \hat{A} \rangle}{4G} = 2b \).

The expectation value of \( \hat{A}^2 \) is

\[
\langle \hat{A}^2 \rangle = (4G)^2 \int_0^\infty dt P(t)e^{-b-t(2\sqrt{bt})}^2 \simeq 64(Gb)^2
\]

(52)

Thus the uncertainty of the operator \( \hat{A} \) is vanishing to the leading order in \( G \),

\[
\langle \Delta \hat{A} \rangle = \sqrt{\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2} = 0.
\]

(53)

Motivated by the moments methods of a random variable, we can further check the third moment of the area operator defined as

\[
\langle \Delta \hat{A} \rangle_3 := \langle (\hat{A} - \langle \hat{A} \rangle)^3 \rangle
\]

\[
= \langle \hat{A}^3 \rangle - 2\langle \hat{A} \rangle \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^3.
\]

(54)

With some calculations we find

\[
\langle \Delta \hat{A} \rangle_3 = 512(Gb)^2G.
\]

(55)

The above result is of order \( O(G) \). In the semiclassical limit \( G \to 0 \), we have \( \langle \Delta \hat{A} \rangle_3 \to 0 \). One could show the \( n \)-th moments \( \langle \Delta \hat{A} \rangle_n := \langle (\hat{A} - \langle \hat{A} \rangle)^n \rangle \) are vanishing in the limit \( G \to 0 \).