Article

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Multiplicities result for non-homogeneous fractional Schrodinger–Kirchhoff-type equations in \( \mathbb{R}^n \)

DOI: 10.1515/anona-2015-0096
Received July 21, 2015; revised October 28, 2015; accepted December 16, 2015

Abstract: In this paper we consider the existence of multiple solutions for the non-homogeneous fractional \( p \)-Laplacian equations of Schrödinger–Kirchhoff type

\[
M \left( \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(z)|^p}{|x - z|^{n+ps}} \, dz \, dx \right) (-\Delta)_p^s u + V(x)|u|^{p-2}u = f(x, u) + g(x)
\]

in \( \mathbb{R}^n \), where \((-\Delta)_p^s \) is the fractional \( p \)-Laplacian operator with \( 0 < s < 1 < p < \infty \), \( ps < n \), \( f: \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R} \) is a continuous function, \( V: \mathbb{R}^n \rightarrow \mathbb{R}^+ \) is a potential function and \( g: \mathbb{R}^n \rightarrow \mathbb{R} \) is a perturbation term. Assuming that the potential \( V \) is bounded from bellow, that \( f(x, t) \) satisfies the Ambrosetti–Rabinowitz condition and some other reasonable hypotheses, and that \( g(x) \) is sufficiently small in \( L^p(\mathbb{R}^n) \), we obtain some new criterion to guarantee that the equation above has at least two non-trivial solutions.

Keywords: Fractional \( p \)-Laplacian, Schrödinger–Kirchhoff equation, Mountain Pass Theorem

MSC 2010: 35J35, 35J60

1 Introduction

The aim of this article is to study a Schrödinger–Kirchhoff-type equation with fractional \( p \)-Laplacian in \( \mathbb{R}^n \),

\[
M \left( \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(z)|^p}{|x - z|^{n+ps}} \, dz \, dx \right) (-\Delta)_p^s u + V(x)|u|^{p-2}u = f(x, u) + g(x),
\]

where \( 0 < s < 1 < p < \infty \), \( ps < n \), and the operator \((-\Delta)_p^s \) is the fractional \( p \)-Laplacian which may be defined along a function \( \varphi \in C_0^\infty(\mathbb{R}^n) \) as

\[
(-\Delta)_p^s u(x) = \lim_{\epsilon \rightarrow 0^+} \iint_{\mathbb{R}^n \setminus B(x, \epsilon)} \frac{|\varphi(x) - \varphi(z)|^{p-2}(\varphi(x) - \varphi(z))}{|x - z|^{n+ps}} \, dz,
\]

where \( x \in \mathbb{R}^n \), and \( B(x, \epsilon) = \{ y \in \mathbb{R}^n : |x - y| < \epsilon \} \). We invite the reader to check \([16, 19–21, 25, 35] \) and the references therein for further details on the fractional \( p \)-Laplace operator. The function \( g = g(x) \) can be viewed as a perturbation term.

When \( p = 2 \) and \( M = 1 \), equation (1.1) becomes the fractional Laplacian equation in \( \mathbb{R}^n \),

\[
(-\Delta)_p^s u + V(x)u = f(x, u),
\]

which has been by many researchers, see, for instance, \([2, 6–8, 11, 34] \) and the references therein for some results.

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In recent years, great attention has been paid on the study of problems involving the non-local fractional Laplacian or more general integro-differential operators. This type of operators arises in a quite natural way in many different applications, such as continuum mechanics, phase transition phenomena, population dynamics, minimal surfaces and game theory, as they are the typical outcome of stochastical stabilization of Lévy processes, see [3–5, 23] and the references therein. The literature on fractional operators and their applications is very extensive, see, for example, [1, 2, 6–8, 11–15, 24, 27–40]. For a short introduction to the fractional Laplacian and the fractional Sobolev spaces, the reader is referred to [10]. Furthermore, research has been done in recent years for the regional fractional Laplacian, where the scope of the operator is restricted to a variable region near each point. We mention the work by Guan [17] and Guan and Ma [18] where they study these operators, their relation with stochastic processes and they develop an integration by parts formula. We also mention the work by Ishii and Nakamura [21], where they studied the Dirichlet problem for regional fractional Laplacian modeled on the $p$-Laplacian and the recent work by Felmer and Torres [12, 13], where they considered the existence, symmetry properties and concentration phenomena of solutions of the non-linear Schrödinger equation with non-local regional diffusion. These regional operators present various interesting characteristics that make them very attractive from the point of view of mathematical theory of non-local operators.

On the other hand, Fiscella and Valdinoci in [15] first proposed a stationary Kirchhoff variational equation which models the non-local aspect of the tension arising from non-local measurements of the fractional length of the string. Indeed, problem (1.1) is a fractional version of a model, the so-called Kirchhoff equation, introduced by Kirchhoff in [22]. More precisely, Kirchhoff established a model given by the problem

$$
\frac{\partial^2 u}{\partial t^2} - \left( \frac{p_0}{h} + \frac{E}{2L} \int_0^L |\frac{\partial u}{\partial x}|^2 \, dx \right) \frac{\partial^2 u}{\partial x^2} = 0,
$$

(1.2)

where $\rho$ is the mass density, $p_0$ is the initial tension, $h$ is the area of the cross section, $E$ is the Young modulus of the material and $L$ is the length of the string, which extends the classical D’Alembert wave equation by considering the effects of the changes in the length of the strings during the vibrations. Note that non-local boundary problems like (1.2) can be used to model several physical and biological systems where $u$ describes a process, which depend on the average of itself, such as the population density [9]. A parabolic version of problem (1.2) can be used to describe the growth and movement of a particular species. The movement, modeled by the integral term, is assumed to be dependent on the energy of the entire system with $u$ being its population density. Alternatively, the movement of a particular species may be subject to the total population density within the domain (for instance, the spreading of bacteria) which gives rise to equations of the type

$$
u_t - \psi \left( \int_{\Omega} \frac{u}{|x|} \, dx \right) \Delta u = h(x, u).$$

It is worth pointing out that problem (1.2) received much attention only after Lions [26] proposed an abstract framework to it. For some motivation in the physical background for the fractional Kirchhoff model, we refer to [15, Appendix A]. Also, see for example [29–31, 37] for non-degenerate Kirchhoff-type problems, and [1, 36], for degenerate Kirchhoff-type problems in this direction.

Very recently in [32], Pucci, Xiang and Zhang considered problem (1.1) under the following assumptions:

For the Kirchhoff function $M$, suppose that:

$(M_1)$ $M \in C(\mathbb{R}_+^\ast)$ satisfies $\inf_{t \in \mathbb{R}_+^\ast} M(t) \geq a > 0$, where $a > 0$ is a constant.

$(M_2)$ There exists $\theta \in [1, \frac{n}{n-sp})$ such that

$$\theta \mathcal{M}(t) = \theta \int_0^t M(s) \, ds \geq tM(t) \quad \text{for all } t \in \mathbb{R}_+^\ast.$$

For the potential $V$, suppose that:

$(V_1)$ $V \in C(\mathbb{R}^n, \mathbb{R})$ and $\inf_{x \in \mathbb{R}^n} V(x) \geq V_0 > 0$, where $V_0 > 0$ is a constant.

$(V_2)$ There exists $h > 0$ such that

$$\lim_{|y| \to \infty} \text{meas}(x \in B_h(y) : V(x) \leq c) = 0 \quad \text{for any } c > 0.$$
For the nonlinearity $f$, suppose that:

- $(f_1)$ $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and there exist $q$, with $\theta p < q < p^*_s$, and $a_1 > 0$ such that $|f(x, t)| \leq a_1(1 + |t|^{q-1})$ for a.e. $x \in \mathbb{R}^n$ and all $t \in \mathbb{R}$.

- $(f_2)$ There exists $\mu > \theta p$ such that

$$\mu F(x, t) \leq tf(x, t) \quad \text{for all } x \in \mathbb{R}^n \text{ and all } t \in \mathbb{R}.$$ 

- $(f_3)$ $f(x, t) = o(|t|^{p-1})$ as $t \to 0$, uniformly for $x \in \mathbb{R}^n$.

- $(f_4)$ $\inf_{t \in \mathbb{R}} F(x, t) > 0$.

Under the assumptions $(M_1)$–$(M_2)$, $(V_1)$–$(V_2)$, $(f_1)$–$(f_4)$, the authors first establish a Batsch–Wang-type compact embedding theorem for the fractional Sobolev spaces. Multiplicity results are then obtained by using the Ekeland’s variational principle and the Mountain Pass Theorem. Also we mention the work by Xiang, Zhang and Ferrara [38], where they considered the non-homogeneous fractional $p$-Kirchhoff equations with concave-convex nonlinearities and have obtained the existence of two non-trivial entire solutions by applying the mountain pass theorem and Ekeland’s variational principle.

Inspired by these previous works, in this paper we consider problem (1.1) under some weaker assumptions. More precisely, we assume that the Kirchhoff function $M$ satisfies $(M_1)$–$(M_2)$. Furthermore, regarding the potential $V$ we only assume $(V_1)$ and for the functions $f$ we suppose that:

- $(H_1)$ $f \in C(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$ and there exists $p^*_s > \mu > \theta p$ such that

$$0 < \mu F(x, \zeta) \leq \zeta f(x, \zeta), \quad \text{for all } x \in \mathbb{R}^n \text{ and all } \zeta \in \mathbb{R} \setminus \{0\},$$

where $F(x, \zeta) = \int_{0}^{\zeta} f(x, s) \, ds$.

- $(H_2)$ There exists some positive continuous function $a : \mathbb{R}^n \to \mathbb{R}$ with

$$\lim_{|x| \to +\infty} a(x) = 0 \quad (1.3)$$

such that

$$|f(x, \zeta)| \leq a(x)|\zeta|^{p-1} \quad \text{for all } (x, \zeta) \in \mathbb{R}^n \times \mathbb{R}.$$ 

Before stating our main results, we introduce some useful notations. First of all, define the Gagliardo seminorm by

$$[u]_{s,p}^p = \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{nsp}} \, dx \, dy \right)^{1/p},$$

where $u : \mathbb{R}^n \to \mathbb{R}$ is a measurable function. Now, let the fractional Sobolev space be denoted as

$$W^{s,p}(\mathbb{R}^n) = \{ u \in L^p(\mathbb{R}^n) : u \text{ is measurable and } [u]_{s,p}^p < \infty \}$$

and assume that it is endowed with the norm

$$\|u\|_{s,p} = ([u]_{s,p}^p + \|u\|_p^p)^{1/p},$$

where the fractional critical exponent is defined by

$$p^*_s = \begin{cases} \frac{np}{n-sp} & \text{if } sp < n, \\ \infty & \text{if } sp \geq n. \end{cases}$$

Moreover, we consider the fractional Sobolev space with potential

$$X^s := \left\{ u \in W^{s,p}(\mathbb{R}^n) : \int_{\mathbb{R}^n} V(x)|u|^p \, dx < \infty \right\}$$

endowed with the norm

$$\|u\|_{X^s} = ([u]_{s,p}^p + \|V(x)^{1/p}u\|_p^p)^{1/p}.$$
We say that \( u \in X^s \) is a weak solution of problem (1.1) if
\[
M(\|u\|_{s,p}^p) \left\{ \int \frac{|u(x) - u(z)|^{p-2}(u(x) - u(z))(\varphi(x) - \varphi(z))}{|x - z|^{n+ps}} \, dz \, dx + \int \frac{V(x)|u(x)|^{p-2}u(x)\varphi(x)}{|x|^{n+ps}} \, dx \right\} = \int f(x,u)\varphi(x) \, dx + \int g(x)\varphi(x) \, dx
\]
for any \( \varphi \in X^s \).

From here on we set \( p' = p/(p - 1) \), the Hölder conjugate of \( p \). Our main result in this paper is the following theorem.

**Theorem 1.1.** Let \((M_1)-(M_2), (V_1), (H_1)-(H_2)\) hold and suppose that \( g \in L^{p'}(\mathbb{R}^n) \) and \( g \neq 0 \). Then there exists a constant \( \delta_0 > 0 \) such that problem (1.1) has at least two non-trivial solutions in \( X^s \), provided that \( \|g\|_{p'} \leq \delta_0 \).

**Remark 1.1.** We note that condition \((V_2)\) is used to establish some compact embedding theorems to guarantee that the (PS) condition holds, which is the essential step to obtain the existence of weak solutions of (1.1) via the Mountain Pass Theorem. In the present paper, we assume that \( V(x) \) is bounded from below but we could not obtain some compact embedding theorem. Therefore, one difficulty is to adapt some new technique to overcome this difficulty and test that the (PS) condition is verified, see Lemmas 3.1 and 3.2 below.

The remaining part of this paper is organized as follows. Some preliminary results are presented in Section 2. In Section 3, we are devoted to accomplishing the proof of Theorem 1.1.

## 2 Preliminaries

To study the fractional problem (1.1), the so-called fractional Sobolev spaces \( W^{s,p}(\mathbb{R}^n) \) with \( 0 < s < 1 \) are expended. If \( 1 < p < \infty \), as usual, the norm is defined through
\[
\|u\|_{s,p}^p = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy + \int_{\mathbb{R}^n} |u(x)|^p \, dx.
\]

We recall the Sobolev embedding theorem.

**Theorem 2.1** ([10]). Let \( s \in (0, 1) \) and \( p \in [1, \infty) \) be such that \( sp < n \). Then there exists a positive constant \( C = C(n, p, s) \) such that
\[
\|u\|_{L^{p^*_s}} \leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy,
\]
where \( p^*_s = \frac{np}{n-sp} \) is the so-called “fractional critical exponent”. Consequently, the space \( W^{s,p}(\mathbb{R}^n) \) is continuously embedded in \( L^{q}(\mathbb{R}^n) \) for any \( q \in [p, p^*_s] \). Moreover, the embedding \( W^{s,p}(\mathbb{R}^n) \hookrightarrow L^{q}_{loc}(\mathbb{R}^n) \) is compact for \( q \in [p, p^*_s] \).

Consider now the space \( X^s \) defined by
\[
X^s := \left\{ u \in W^{s,p}(\mathbb{R}^n) : V(x)|u|^p \, dx < \infty \right\},
\]
endowed with the norm
\[
\|u\|_{X^s} = \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) + u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy + \int_{\mathbb{R}^n} V(x)|u(x)|^p \, dx \right)^{1/p}.
\]

**Remark 2.1.** From \((V_1)\), Theorem 2.1 and the Hölder inequality, we have \( X^s \hookrightarrow L^{q}(\mathbb{R}^n) \) for \( p \leq q \leq p^*_s \) and \( X^s \hookrightarrow L^{q}_{loc}(\mathbb{R}^n) \) compactly for \( q \in [p, p^*_s] \). Moreover, there exists \( \mathcal{E}_q \) such that
\[
\|u\|_q \leq \mathcal{E}_q \|u\|_{X^s}. \tag{2.1}
\]
Now we introduce some more notations and necessary definitions. Let $\mathcal{B}$ be a real Banach space. If $I$ is a continuously Fréchet-differentiable functional defined on $\mathcal{B}$, we write $I \in C^1(\mathcal{B}, \mathbb{R})$. Recall that $I \in C^1(\mathcal{B}, \mathbb{R})$ is said to satisfy the (PS) condition if every sequence $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{B}$ for which $\{I(u_n)\}_{n \in \mathbb{N}}$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence in $\mathcal{B}$.

Moreover, let $B_r$ be the open ball in $\mathcal{B}$ with radius $r$ and centered at 0, and let $\partial B_r$ denote its boundary. Under the conditions of Theorem 1.1, we obtain the existence of the first weak solution of (1.1) using the following well-known Mountain Pass Theorem, see [33].

**Lemma 2.1.** [33, Theorem 2.2] Let $\mathcal{B}$ be a real Banach space and let $I \in C^1(\mathcal{B}, \mathbb{R})$ satisfy the (PS) condition. Suppose that $I(0) = 0$ and

(A1) there are constants $\rho, \eta > 0$ such that $I|_{\partial B_r} \geq \eta$,

(A2) there is an $e \in \mathcal{B} \setminus B_\rho$ such that $I(e) \leq 0$.

Then $I$ possesses a critical value $c \geq \eta$. Moreover, $c$ can be characterized as

$$c = \inf_{s \in [0,1]} \max_{g \in \Gamma} I(g(s)),$$

where

$$\Gamma = \{ g \in C([0,1], \mathcal{B}) : g(0) = 0, g(1) = e \}.$$

As far as the second solution is concerned, we obtain it using the minimizing method, which is contained in a small ball centered at 0.

### 3 Proof of Theorem 1.1

The aim of this section is to establish the proof of Theorem 1.1. For this purpose, we are going to establish the corresponding variational framework to obtain solutions of (1.1). To this end, define the functional $I : X^s \to \mathbb{R}$ by

$$I(u) = J(u) - H(u),$$

where

$$J(u) = \frac{1}{p} \left( M(\|u\|_{L^p}) + \|V(x)^{1/p}u\|_{L^p}^p \right),$$

$$H(u) = \int_{\mathbb{R}^n} F(x, u) + g(x)u(x) \, dx.$$ 

If $(M_1)$ and $(V_1)$ hold, then $J : X^s \to \mathbb{R}$ is of class $C^1(X^s)$ and

$$\langle J'(u), v \rangle = M(\|u\|_{L^p}) \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(z)|^{p-2}(u(x) - u(z))(v(x) - v(z))}{|x - z|^{n+ps}} \, dz \, dx$$

$$+ \int_{\mathbb{R}^n} V(x)|u(x)|^{p-2}u(x)v(x) \, dx$$

for all $u, v \in X^s$. Moreover, $J$ is weakly lower semi-continuous in $X^s$, see [32].

Now we prove our key lemma.

**Lemma 3.1.** Under the conditions of Theorem 1.1, $\Phi^\dagger$ is compact, i.e., $\Phi^\dagger(u_k) \to \Phi^\dagger(u)$ if $u_k \to u$ in $X^s$, where $\Phi : X^s \to \mathbb{R}$ is defined by

$$\Phi(u) = \int_{\mathbb{R}^n} F(x, u(x)) \, dx.$$ 

**Proof.** From $(H_2)$ it is obvious that

$$f(x, u) = o(|u|^{p-1}) \quad \text{as } |u| \to 0, \quad \text{a.e. } x \in \mathbb{R}^n$$

(3.3)

and

$$f(x, u) = o(|u|^{p^*_s-1}) \quad \text{as } |u| \to \infty, \quad \text{a.e. } x \in \mathbb{R}^n.$$ 

(3.4)
Therefore, by (3.3) and (3.4), for any $\epsilon > 0$, there exists $C_\epsilon$ such that
\[
|f(x, u)| \leq \epsilon |u|^{p-1} + C_\epsilon |u|^{p^*-1}
\quad \text{for all } u \in \mathbb{R} \text{ and a.e. } x \in \mathbb{R}^n.
\] (3.5)

Now assume that $u_n \to u$ in $X$. Then, there is some constant $K > 0$ such that
\[
\frac{1}{\varrho q} \|u_k\|_q \leq \|u_k\|_{X'} \leq K \quad \text{and} \quad \frac{1}{\varrho q} \|u\|_q \leq \|u\|_{X'} \leq K
\] (3.6)

for $k \in \mathbb{N}$. Furthermore, by Remark 2.1, given $R > 0$ up to a subsequence, we may assume that:

(i) $u_k \to u$ in $L^q(B(0, R))$ for $q \in [p, p^*_\epsilon)$ as $k \to \infty$,
(ii) $u_k(x) \to u(x)$ a.e. in $\mathbb{R}^n$ as $k \to \infty$,
(iii) there is $\psi \in L^q(B(0, R))$ such that $|u_k| \leq \psi(x)$ a.e in $\mathbb{R}^n$ and for all $k \in \mathbb{N}$.

Thus, by the previous inequality
\[
|f(x, u_k)| \leq \|a\|_\infty \psi^{p-1} \in L^q(B(0, R))
\] and
\[
\lim_{k \to \infty} |f(x, u_k(x)) - f(x, u(x))| = 0 \quad \text{a.e. in } B(0, R).
\]

Hence, by the Lebesgue Dominated Convergence Theorem we have
\[
\lim_{k \to \infty} \int_{B(0, R)} |f(x, u_k) - f(x, u)| \, dx = 0,
\] (3.7)

where $\mu' + \mu = \mu^\prime$.

On the other hand, from $(H_2)$, for any $\epsilon > 0$ there exists $R > 0$ such that
\[
|f(x, u)| \leq \epsilon |u|^{p-1} \quad \text{and} \quad |f(x, u_k)| \leq \epsilon |u_k|^{p^*-1}
\quad \text{for all } |x| > R.
\] (3.8)

Consequently, in view of (2.1), (3.6)–(3.8), for $k$ large enough, we have
\[
\|\Phi'(u_k) - \Phi'(u), v\| \leq \epsilon \int_{\mathbb{R}^n} |f(x, u_k) - f(x, u)| \, dx
\]
\[
\leq \epsilon \int_{B(0, R)} |f(x, u_k) - f(x, u)| \, dx + \epsilon \int_{B(0, R)^c} |f(x, u_k)| \, dx + \epsilon \int_{B(0, R)^c} |f(x, u)| \, dx
\]
\[
\leq \epsilon \|v\|_{L^p} + \epsilon \int_{B(0, R)} \frac{1}{\mu} |u_k|^{p^*-1} |v| \, dx + \epsilon \int_{B(0, R)^c} |u|^{p^*-1} |v| \, dx
\]
\[
\leq \epsilon \mathcal{E}_\mu \|v\|_{X'} + \epsilon \int_{B(0, R)} \left( \frac{\mu - 1}{\mu} |u_k|^p + \frac{1}{\mu} |v|^p r \right) \, dx + \epsilon \int_{B(0, R)^c} \left( \frac{\mu - 1}{\mu} |u|^p + \frac{1}{\mu} |v|^p r \right) \, dx
\]
\[
\leq \epsilon \mathcal{E}_\mu \|v\|_{X'} + \epsilon \frac{\mu - 1}{\mu} \int_{B(0, R)^c} (|u_k|^p + |u|^p) \, dx + \epsilon \frac{2}{\mu} \int_{B(0, R)^c} |v|^p \, dx.
\] (3.9)

Here, we apply the Young inequality
\[
ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad a, b > 0, \quad p, q > 1 \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} = 1.
\]

Consequently, we obtain that
\[
\|\Phi'(u_k) - \Phi'(u)\| = \sup_{|v|_{X'} = 1} \left| \int_{\mathbb{R}^n} (f(x, u_k) - f(x, u), v) \, dx \right|
\]
\[
\leq \epsilon \mathcal{E}_\mu + \frac{2\epsilon \mathcal{E}_\mu}{\mu} [(\mu - 1)K^\mu + 1],
\]

which yields that $\Phi'(u_k) \to \Phi'(u)$ as $u_k \to u$, so $\Phi'$ is compact.

\[\square\]
Remark 3.1. Under the conditions of Lemma 3.1, by the Hölder inequality we get
\[ \lim_{k \to \infty} \int_{\mathbb{R}^n} g(x)u_k(x) \, dx = \int_{\mathbb{R}^n} g(x)u(x) \, dx. \]
Therefore, \( H \) is weakly continuous in \( X^s \).

To prove Theorem 1.1 we first consider some lemmas.

Lemma 3.2. Under the conditions of Theorem 1.1, the functional \( I \) satisfies the (PS) condition.

Proof. Although the proof of this lemma is just the repetition of the process of [32, Lemma 6], we give the details for convenience of the reader.

Assume that \( \{u_k\}_{k \in \mathbb{N}} \subset X^s \) is a sequence such that \( |I(u_k)| \) is bounded and \( I'(u_k) \to 0 \) as \( k \to \infty \). Then there exists a constant \( C > 0 \) such that
\[ |I(u_k)| \leq C \quad \text{and} \quad \|I'(u_k)\|_{(X^s)'} \leq C \quad (3.10) \]
for every \( k \in \mathbb{N} \), where \( (X^s)^* \) is the dual space of \( X^s \).

Firstly, we show that \( \{u_k\}_{k \in \mathbb{N}} \) is bounded. In fact, in view of (M1)–(M2), (H1), (3.10) and using also that \( \mu > \theta p \geq p > 1 \), we obtain
\[ C + \frac{C}{\mu} \|u_k\|_{X^s} \geq I(u_k) - \frac{1}{\mu} I'(u_k) u_k \]
\[ = \frac{1}{p} M([u_k]_{s,p}^p) - \frac{1}{\mu} M([u_k]_{s,p}^p) [u_k]_{s,p} + \left( \frac{1}{p} - \frac{1}{\mu} \right) \|u_k\|_{X^s}^p \]
\[ \geq - \int_{\mathbb{R}^n} \left[ F(x, u_k(x)) - \frac{1}{p} f(x, u_k(x)) u_k(x) \right] \, dx - \frac{\mu - 1}{\mu} \int_{\mathbb{R}^n} g(x)u_k \, dx \]
\[ \geq \left( \frac{1}{p} \right) M([u_k]_{s,p}^p) [u_k]_{s,p} + \left( \frac{1}{p} - \frac{1}{\mu} \right) \|u_k\|_{X^s}^p \]
\[ \geq \frac{1}{\mu} \|u_k\|_{X^s}^p - \frac{\mu - 1}{\mu} \|\mathcal{S}_p\|_{L^p} \|u_k\|_{X^s} \]
where \( \mathcal{S}_p = \mathcal{S}_p \|\cdot\|_{L^p} \). Hence, \( \{u_k\}_{k \in \mathbb{N}} \) is bounded in \( X^s \). Then, the sequence \( \{u_k\}_{k \in \mathbb{N}} \) has a subsequence, again denoted by \( \{u_k\}_{k \in \mathbb{N}} \), and there exists \( u \in X^s \) such that
\[ u_k \rightharpoonup u \quad \text{weakly in} \quad X^s \]
which implies that
\[ \langle I'(u_k) - I'(u), u_k - u \rangle \to 0 \quad \text{as} \quad k \to \infty. \quad (3.11) \]

Furthermore, according to Lemma 3.1, we have
\[ H'(u_k) \rightharpoonup H'(u) \quad \text{as} \quad k \to \infty. \quad (3.12) \]

Let \( \varphi \in X^s \) be fixed and denote by \( B_\varphi \) the linear functional on \( X^s \) defined by
\[ B_\varphi(v) = \int_{\mathbb{R}^n} \frac{\varphi(x) - \varphi(z)}{|x-z|^{n+ps}} \frac{f(x,v(z))}{v(z)} \, dz \, dx \]
for all \( v \in X^s \). By the Hölder inequality, \( B_\varphi \) is continuous and hence
\[ \lim_{k \to \infty} (M([u_k]_{s,p}^p) - M([u]_{s,p}^p)) B_u(u_k - u) = 0, \quad (3.13) \]
since \( \{M([u_k]_{s,p}^p) - M([u]_{s,p}^p)\}_{k} \) is bounded in \( \mathbb{R} \).

Therefore, for \( k \) large enough we get
\[ o(1) = \langle I'(u_k) - I'(u), u_k - u \rangle \]
\[ = \int_{\mathbb{R}^n} \frac{\varphi(x) - \varphi(z)}{|x-z|^{n+ps}} \frac{f(x,v(z))}{v(z)} \, dz \, dx \]
\[ + \int_{\mathbb{R}^n} V(x)(|u_k|^{p-2} u_k - |u|^{p-2} u)(u_k - u) \, dx - \int_{\mathbb{R}^n} (f(x, u_k) - f(x, u))(u_k - u) \, dx + o(1), \]
that is
$$\lim_{k \to \infty} \left( M([u_k]_{L^p}) [B u_k(u_k - u) - B u(u_k - u)] + \int_{\mathbb{R}^n} V(x)|u_k|^{p-2} u_k - |u|^{p-2} u| (u_k - u) \, dx \right) = 0. $$

Note that
$$M([u_k]_{L^p}) [B u_k(u_k - u) - B u(u_k - u)] \geq 0 \quad \text{and} \quad V(x)|u_k|^{p-2} u_k - |u|^{p-2} u| (u_k - u) \geq 0$$
for all $k \in \mathbb{N}$ by convexity. Considering also (M_1) and (V_1), we have in particular
$$\lim_{k \to \infty} \left[ B u_k(u_k - u) - B u(u_k - u) \right]_x \geq 0 \quad \text{and} \quad \lim_{k \to \infty} V(x)(|u_k|^{p-2} u_k - |u|^{p-2} u)(u_k - u) \geq 0. \tag{3.14}$$

Now, by the well-known Simon inequalities [32], for $p \geq 2$, (M_1) and (3.14) as $k \to \infty$, one has
$$[u_k - u]_{L^p}^p \leq c_p [B u_k(u_k - u) - B u(u_k - u)] = o(1).$$
Similarly, by (V_1) and (3.14) as $k \to \infty$, we get
$$\|V(x)^{1/p}(u_k - u)\|_{L^p}^p \leq o(1).$$
In conclusion, $\|u_k - u\|_{X^\alpha} \to 0$ as $k \to \infty$, as required.

In the case $1 < p < 2$, since $u_k \to u$ weakly in $X^\alpha$ there exists $K > 0$ such that $\|u_k\|_{L^p} \leq K$ for all $k \in \mathbb{N}$. Now by the Simon inequality, the H"older inequality and (3.14) as $k \to \infty$, we have
$$[u_k - u]_{L^p}^p \leq C[B u_k(u_k - u) - B u(u_k - u)]^{p/2} = o(1), \tag{3.15}$$
where $C = 2C_p K^{(2-p)/2}$. Furthermore, by the Hölder inequality and (3.14) as $k \to \infty$
$$\|V(x)^{1/p}(u_k - u)\|_{L^p}^p \leq L \left( \int_{\mathbb{R}^n} V(x)(|u_k|^{p-2} u_k - |u|^{p-2} u)(u_k - u) \, dx \right)^{p/2}, \tag{3.16}$$
where $L = 2C_p K^{(2-p)/2}$. Hence, $\|u_k - u\|_{X^\alpha} \to 0$ as $k \to \infty$ also in the second case. Therefore, $I$ satisfies the (PS) condition, as stated.

**Lemma 3.3.** Under the conditions of Theorem 1.1, there exist $\rho > 0$ and $\alpha > 0$ such that
$$\inf_{\|u\|_{X^\alpha} = \rho} I(u) > \alpha.$$

**Proof.** By (3.5), for any $\epsilon > 0$, there exists $C_\epsilon$ such that
$$|F(x, u)| \leq \epsilon |u|^p + C_\epsilon |u|^{p_\alpha} \quad \text{for all } (x, u) \in \mathbb{R}^n \times \mathbb{R}.$$ 
Thus, using the Hölder inequality, we have
$$I(u) = \frac{1}{p} \left( M([u]_{L^p})^p + \|V(x)^{1/p} u\|_{L^p}^p \right) - \int_{\mathbb{R}^n} F(x, u(x)) \, dx - \int_{\mathbb{R}^n} g(x) u \, dx \geq \min_{\|u\|_{X^\alpha} = \rho} \frac{1}{p} \|u\|_{X^\alpha}^p - C \|u\|_{L^p}^p - \|g\|_{L^p}^p \|u\|_{L^p} \geq \min_{\|u\|_{X^\alpha} = \rho} \frac{1}{p} \|u\|_{X^\alpha}^p - C \|u\|_{L^p}^p = \|u\|_{X^\alpha} \left( \left( \frac{\min_{\|u\|_{X^\alpha} = \rho} \|u\|_{X^\alpha}^p - C \|u\|_{L^p}^p}{\min_{\|u\|_{X^\alpha} = \rho} \|u\|_{X^\alpha}^p - 2C \|u\|_{L^p}^p} \right) \right).$$
Taking $\epsilon = \min_{\|u\|_{X^\alpha} = \rho} \|u\|_{X^\alpha}^p$ and setting
$$\eta(t) = \min_{\|u\|_{X^\alpha} = \rho} \|u\|_{X^\alpha}^p - C \|u\|_{L^p}^p = \|u\|_{X^\alpha} \left( \left( \frac{\min_{\|u\|_{X^\alpha} = \rho} \|u\|_{X^\alpha}^p - C \|u\|_{L^p}^p}{\min_{\|u\|_{X^\alpha} = \rho} \|u\|_{X^\alpha}^p - 2C \|u\|_{L^p}^p} \right) \right).$$
we see that there exists $\rho > 0$ such that $\max_{t \in \mathbb{R}^+} \eta(t) = \eta(\rho) > 0$, since $p_\alpha^* > p > 1$ by (H_1). Taking $\delta_0 := \frac{\eta(\rho)}{2\epsilon^p}$, we obtain that
$$I(u) \geq \alpha = \frac{\rho \eta(\rho)}{2} > 0$$
for all $u \in X^\alpha$ with $\|u\|_{X^\alpha} = \rho$ and for all $g \in L^{p^*}(\mathbb{R}^n)$ with $\|g\|_{L^{p^*}} \leq \delta_0$. \qed
Lemma 3.4. Under the conditions of Theorem 1.1, let $\rho > 0$ be defined as in Lemma 3.3. Then, there exists an $e \in X^s$ with $\|e\|_{X^s} > \rho$ such that $I(e) < 0$.

Proof. First, we note that by $(H_1)$,
\[
\lim_{|u| \to \infty} \frac{F(x, u)}{|u|^{\beta_p}} = \infty.
\]

Fix $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\|\varphi\|_{X^s} = 1$. Then, for any $\epsilon > 0$ we have
\[
\lim_{s \to \infty} \int_{\text{supp}(\varphi)} \frac{F(x, s\varphi)}{s^{\beta_p}} \, dx \geq \frac{1}{\epsilon} \int_{\text{supp}(\varphi)} |\varphi(x)|^{\beta_p} \, dx.
\]

Since $\epsilon$ is arbitrary, by (3.17) we obtain
\[
\lim_{s \to \infty} \int_{\text{supp}(\varphi)} \frac{F(x, s\varphi)}{s^{\beta_p}} \, dx = \infty.
\]

Furthermore, by assumption $(M_2)$, we also get that
\[
\mathcal{M}(\xi) \leq \mathcal{M}(1)\xi^\theta \quad \text{for all} \quad \xi \geq 1.
\]

Consequently, by (3.18) and (3.19), we have as $s \to \infty$,
\[
\frac{I(s\varphi)}{s^{\beta_p}} = \frac{1}{ps^{\beta_p}} \left(\mathcal{M}([s\varphi]_{X^s}^p) + s^p \|V(x)^{1/p} \varphi\|_{L^p}^p\right) - \int_{\mathbb{R}^n} \frac{F(x, s\varphi)}{s^{\beta_p}} \, dx - \frac{1}{s^{\beta_p-1}} \int_{\mathbb{R}^n} g\varphi \, dx
\]
\[
\leq \frac{1}{ps^{\beta_p}} \left(\mathcal{M}(1)s^{\beta_p}[\varphi]_{X^s}^p + s^p \|V(x)^{1/p} \varphi\|_{L^p}^p\right) - \int_{\text{supp}(\varphi)} \frac{F(x, s\varphi)}{s^{\beta_p}} \, dx - \frac{1}{s^{\beta_p-1}} \int_{\mathbb{R}^n} g\varphi \, dx
\]
\[
\leq \max\{1, \mathcal{M}(1)\} \frac{1}{p} - \int_{\text{supp}(\varphi)} \frac{F(x, s\varphi)}{s^{\beta_p}} \, dx - \frac{1}{s^{\beta_p-1}} \int_{\mathbb{R}^n} g\varphi \, dx \to -\infty
\]

Hence, if $s_0$ is big enough and $e = s_0\varphi$, one gets $I(e) < 0$. 

Now we are in a position to give the proof of Theorem 1.1.

Proof of Theorem 1.1. Since $g \in L^{p'}(\mathbb{R}^n)$ and $g \neq 0$, we can choose a function $\chi \in C_0^\infty(\mathbb{R}^n) \subset X^s$ such that
\[
\int_{\mathbb{R}^n} g(x)\chi(x) \, dx > 0.
\]

Then, by $(H_1)$ we note that $F(x, t) \geq 0$ a.e. and
\[
I(\sigma \chi) \leq \frac{1}{p} \mathcal{M}([\sigma \chi]_{X^s}^p) + \frac{\sigma^p}{p} \int_{\mathbb{R}^n} V(x)|\chi|^p \, dx - \sigma \int_{\mathbb{R}^n} g(x)\chi(x) \, dx
\]
\[
\leq \frac{1}{p} \left(\sup_{\xi \in [0, \rho]} \mathcal{M}(\xi)\sigma^p[\chi]_{X^s}^p + \frac{\sigma^p}{p} \int_{\mathbb{R}^n} V(x)|\chi|^p \, dx - \sigma \int_{\mathbb{R}^n} g(x)\chi(x) \, dx < 0,
\]

for $\sigma > 0$ small enough, where $\rho$ is given in Lemma 3.3. Thus, we get
\[
c_1 = \inf\{I(u) : u \in \overline{B}_\rho\} < 0,
\]

where $B_\rho = \{u \in X^s : \|u\|_{X^s} < \rho\}$. By the Ekeland variational principle and Lemma 3.3, there exists a sequence \{uk\}_k \subset B_\rho such that
\[
c_1 \leq I(u_k) \leq c_1 + \frac{1}{k} \quad \text{and} \quad I(v) \geq I(u_k) - \frac{1}{k}\|v - u_k\|_{X^s}
\]

for all $v \in \overline{B}_\rho$. Then, a standard procedure gives that $\{uk\}_k$ is a bounded (PS) sequence of $I$. Therefore, Lemmas 3.2 and 3.3 imply that there exists a function $u_1 \in B_\rho$ such that
\[
I'(u_1) = 0 \quad \text{and} \quad I(u_1) = c_1 < 0.
\]
On the other hand, by Lemmas 3.3 and 3.4 and the Mountain Pass Theorem, there exists a sequence \( \{u_k\}_k \in X^e \) such that, as \( k \to \infty \),
\[
I(u_k) \to c_2 > 0 \quad \text{and} \quad I'(u_k) \to 0.
\]
In view of the proof of Lemma 3.2, we know that there exists a critical point \( u_2 \in X^e \) of \( I \). Moreover,
\[
I(u_2) = c_2 > 0 = K(0).
\]
Thus, \( u_2 \neq 0 \) and \( u_2 \neq u_1 \).

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