Constructing uniform spaces

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Abstract. We exhibit geometric conditions that ensure a metric space is uniform.

Sisätieyhtenäisten avaruksien rakentaminen

Tiivistelmä. Esitämme joukon geometrisia ehtoja, jotka takaavat metrisen avaruuden sisätieyhtenäisyyden.

1. Introduction

Throughout this section $X$ is a rectifiably connected non-complete locally complete metric space. These are the minimal requirements for $X$ to support a quasihyperbolic distance $k = k_X$, and we dub $X$ a quasihyperbolic metric space; see Section 2 for precise definitions.

Roughly speaking, such an $X$ is a uniform metric space provided each pair of points can be joined by a path that moves away from the boundary of $X$ and whose length is comparable to the distance between the points. See §2.B for a precise definition.

The class of Euclidean uniform domains was introduced by Martio and Sarvas in [MS79] and has proven to be invaluable in geometric function theory, potential theory, geometric group theory, and especially for the “analysis in metric spaces” program; e.g., see [Geh87, Väi88, Jon81, Aik04, Aik06, BS07, CT95, CGN00, Gre01, BHK01, HSX08]. A finitely connected proper subdomain of the plane is uniform if and only if each boundary component is either a point or a quasicircle, but in general there are no such simple geometric criterion for uniformity.

Given their fundamental importance, it seems worthwhile to investigate two questions.

− When can we “poke holes” in a uniform space and still have a uniform space?

+ When can we “fill in” some boundary points of a uniform space and keep uniformity?

As a warm up, we have the following result, similar to [Her11, Prop. 2.3]. While surely not surprising to those well versed in uniform space theory, our discussion employs some tools that may not be well known. Again, see §2.B for definitions.

**Theorem A.** Let $X$ be a quasihyperbolic metric space. Let $o \in X$ be a fixed point and put $X_* := X \setminus \{o\}$. If $X_*$ is $C_*$-uniform, then for any $C > C_*$

\[(1.1) \quad X \text{ is } C\text{-uniform and } C_1\text{-annular quasiconvex at } o \]

where $C_1 = 2(C_* + 1)$. Conversely, if (1.1) holds, then $X_*$ is $C_*$-uniform with $C_* = C_*(C, C_1)$.
An open subspace $\Omega$ of $X$ is uniformly collared provided $X$ is uniform and there are disjointed open sets $U_i$ such that

$$B := X \setminus \Omega = \bigcup_i B_i \quad \text{where} \quad B_i := B \cap U_i$$

and such that each collar $\Omega_i := \Omega \cap U_i$ is a uniform space. This terminology was introduced in the Euclidean setting by Astala and Heinonen in [AH88]; see also [HK91]. We say that $\Omega$ is uniformly collared with fat collars provided it is uniformly collared and there is a positive constant $\Phi$ such that

$$\frac{1}{\Phi} \text{diam}(B_i) \leq \text{dist}(B_i, \partial X) \leq \Phi \text{dist}(B_i, \text{bd} U_i).$$

**Theorem B.** Let $\Omega$ be an open subspace of a $C_0$-uniform space $X$. Suppose $\Omega$ is uniformly collared with $C_1$-uniform $\Phi$-fat collars. Then $\Omega$ is $C$-uniform with $C = C(C_0, C_1, \Phi)$.

As an application of the above, we establish the following.

**Theorem C.** Let $X$ be a quasihyperbolic metric space. Let $\Omega := X \setminus A$ where $A \subset X$. Assume there is a constant $\kappa \in (0, 1)$ such that

$$\forall \ a \neq b \in A, \ k(a, b) \geq \kappa.$$  

If $\Omega$ is $C$-uniform, then for any $C_0 > C$

$$X \text{ is } C_0\text{-uniform and } C_1\text{-annular quasiconvex at each } a \in A$$

with $C_1 = 2(C + 1)$. Conversely, if both (1.4) and

$$\forall \ a \in A, \ B\left(a; \frac{\kappa}{4C_0} \text{dist}(a, \partial X)\right) \text{ is } C_2\text{-uniform}$$

hold, then $\Omega$ is $C$-uniform with $C = C(C_0, C_1, C_2, \kappa)$.

A special case of Theorem C, with $\kappa = 1/2$ and $A \subset X$ a countable subset of a Banach space uniform domain $X$, was proved in [HVW17]. Theorems B and C were established in the Euclidean setting in [Her89]; see also [Her87].

We can replace (1.3) by the (a priori stronger, but here equivalent) condition that for some positive constant $v$,

$$\forall \ a \neq b \in A, \ |a - b| \geq v \text{dist}(a, \partial X).$$

The condition (1.5) can be relaxed, e.g., to asking that there exist an $\varepsilon \in (0, \kappa/4C_0]$ such that for each $a \in A$ there is a $C_2$-uniform ball $B(a; r_a)$ with $r_a/\text{dist}(a, \partial X) \in [\varepsilon, \kappa/4C_0]$.

Our results inspire some natural questions.

(A) When is a metric space annular quasiconvex?

(B) What properties of a metric space ensure that its balls are uniform spaces?

(C) Which properties of Euclidean uniformly collared spaces (e.g., see [HK91]) have metric space analogs?

Theorems A, B, C are established in §§3.A, 3.B, 3.C respectively.

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1There should be a single uniformity constant for the collars.

2Banach spaces are annular quasiconvex at each point and balls are 2-uniform.
2. Metric space definitions

Our notation is relatively standard. We write $C = C(D, \ldots)$ to indicate a constant $C$ that depends only on the data $D, \ldots$. For real numbers $r$ and $s$, 

$$r \wedge s := \min\{r, s\} \quad \text{and} \quad r \vee s := \max\{r, s\}.$$ 

2.A. Metric space notation and terminology. Throughout this section $X$ is an arbitrary metric space with distance denoted $|x - y|$; this is not meant to imply that $X$ possesses any sort of linear or group structure. In this setting, all topological notions refer to the metric topology; here $\text{cl}(A), \text{bd}(A), \text{int}(A)$ are the topological closure, boundary, interior (respectively) of $A \subset X$.

The open ball, sphere, closed ball of radius $r$ centered at the point $a \in X$ are

$$B(a; r) := \{ x : |x - a| < r \}, \quad S(a; r) := \{ x : |x - a| = r \},$$

$$B[a; r] := B(a; r) \cup S(a; r).$$

The closed annular ring centered at $a$ with inner radius $r$ and outer radius $s$ is

$$A[a; r, s] := B[a; s] \setminus B(a; r) = \{ x : r \leq |x - a| \leq s \}.$$ 

Recall that every metric space can be isometrically embedded into a complete metric space. We let $\bar{X}$ denote the metric completion of the metric space $X$; thus $\bar{X}$ is the closure of the image of $X$ under such an isometric embedding. We call $\partial X := \bar{X} \setminus X$ the metric boundary of $X$. When $X$ is non-complete, $\delta(x) = \delta_X(x) := \text{dist}(x, \partial X)$ is the distance from a point $x \in X$ to the boundary $\partial X$ of $X$. Note that $\partial X$ is closed in $\bar{X}$ if and only if $\delta(x) > 0$ for all $x \in X$; e.g., this holds when $X$ is locally compact.

When $A \subset X$, there is a natural embedding $\bar{A} \hookrightarrow \bar{X}$ and $\text{bd}(A) \subset \partial A$. Here if $A \subset X$ is open and $X$ complete, then $\partial A = \text{bd}(A)$, but in general $\bar{A} = \bar{\text{cl}}(A)$ and $\partial A = \bar{\text{bd}}(A) \setminus A$ where $\bar{\text{cl}}$ and $\bar{\text{bd}}$ denote topological closure and boundary in $\bar{X}$.

A metric space $X$ is locally complete provided each point has an open neighborhood which is complete. When $X$ is non-complete, this is equivalent to requiring that $\delta(x) > 0$ for all $x \in X$, or, $\partial X$ is closed in $\bar{X}$, or, $X$ is open in $\bar{X}$.

2.A.1. Paths, arcs, & length. A path in $X$ is a continuous map $R \ni I \xrightarrow{\gamma} X$ where $I = I \subset R$ is an interval (called the parameter interval for $\gamma$) that may be closed or open or neither and finite or infinite. The trajectory of such a path $\gamma$ is $|\gamma| := \gamma(I)$ which we call a curve and often denote by just $\gamma$. When $I$ is closed and $I \neq \emptyset$, $\partial \gamma := \gamma(\partial I)$ denotes the set of endpoints of $\gamma$ and consists of one or two points depending on whether or not $I$ is compact. For example, if $I = [u, v] \subset R$, then $\partial \gamma = \{\gamma(u), \gamma(v)\}$. When $\partial \gamma = \{a, b\}$, we write $\gamma: a \leadsto b$ (in $X$) to indicate that $\gamma$ is a path (in $X$) with initial point $a$ and terminal point $b$; this notation is meant to imply an orientation—$a$ precedes $b$ on $\gamma$.

We call $\gamma$ a compact path if its parameter interval $I$ is compact. An arc $\alpha$ is an injective compact path. Given points $a, b \in |\alpha|$, there are unique $u, v \in I$ with $\alpha(u) = a$, $\alpha(v) = b$ and we write $\alpha[a, b] := |\alpha|_{uv}$. We also use this notation for a general path $\gamma$, but here $\gamma[a, b]$ denotes the unique injective subpath of $\gamma$ that joins $a, b$ obtained by using the last time $\gamma$ is at $a$ up to the first time $\gamma$ is at $b$.

When $\alpha: a \leadsto b$ and $\beta: b \leadsto c$ are paths that join $a$ to $b$ and $b$ to $c$ respectively, $\alpha \ast \beta$ denotes the concatenation of $\alpha$ and $\beta$; so $\alpha \ast \beta: a \leadsto c$. Of course, $|\alpha \ast \beta| = |\alpha| \cup |\beta|$. Also, the reverse of $\gamma$ is the path $\tilde{\gamma}$ defined by $\tilde{\gamma}(t) := \gamma(1 - t)$ (when $I_\gamma = [0, 1]$) and going from $\gamma(1)$ to $\gamma(0)$. Of course, $|\tilde{\gamma}| = |\gamma|$. 

Every compact path contains an arc with the same endpoints; see [Väi94].

The length of a compact path \([0, 1] \to X\) is defined in the usual way by

\[
\ell(\gamma) := \sup \left\{ \sum_{i=1}^{n} |\gamma(t_{i}) - \gamma(t_{i-1})| \mid 0 = t_{0} < t_{1} < \cdots < t_{n} = 1 \right\},
\]

\(\gamma\) is rectifiable when \(\ell(\gamma) < \infty\), and \(X\) is rectifiably connected provided each pair of points in \(X\) can be joined by a rectifiable path. Every rectifiable path can be parametrized with respect to its arclength [Väi71, p.5]. When \(\gamma\) is a rectifiable path, we tacitly assume its parameter interval is \(I_{\gamma} = [0, \ell(\gamma)]\) unless specifically stated otherwise.

Every rectifiably connected metric space \(X\) admits a natural intrinsic distance, its so-called (inner) length distance given by

\[
l(a, b) := \inf \{ \ell(\gamma) \mid \gamma: a \sim b \text{ a rectifiable path in } X \}.
\]

A metric space \((X, |\cdot|)\) is a length space provided for all points \(a, b \in X\), \(|a - b| = l(a, b)\); it is also common to call such a \(|\cdot|\) an intrinsic distance function. Notice that an \(l\)-geodesic \([x, y]\) is a shortest curve joining \(x\) and \(y\).

There are two useful properties of length spaces that we use repeatedly. First, for any open set \(U\) in a length space \(X\), we always have \(\text{dist}(x, \partial U) = \text{dist}(x, X \setminus U)\) for all points \(x \in U\). Second, \(X\) is also a length space. In fact, for all \(x \in X, \xi \in \partial X, \varepsilon > 0\) there is a path \(\gamma: x \leadsto \xi\) in \(X \cup \{\xi\}\) with \(\ell(\gamma) < |x - \xi| + \varepsilon\).

2.A.2. Quasihyperbolic distance. Recall that \(X\) is a quasihyperbolic metric space if it is non-complete, locally complete, and rectifiably connected. In such a space, \(\delta(x) = \delta_{X}(x) := \text{dist}(x, \partial X) > 0\) for all \(x \in X\), so \(\delta^{-1} ds\) is a conformal metric that we call the quasihyperbolic metric on \(X\). The length distance induced by the quasihyperbolic metric \(\delta^{-1} ds\) is dubbed the quasihyperbolic distance \(k = k_{X}\) in \(X\). In a locally compact quasihyperbolic space, this is a geodesic distance: there are always \(k\)-geodesics joining any two points in \(X\).

The following basic estimates for quasihyperbolic distance were first established for Euclidean domains by Gehring and Palka [GP76, 2.1]. For all \(a, b \in X\) and any rectifiable path \(\gamma: a \sim b\) in \(X\)

\[
\begin{align*}
(2.1a) \quad k(a, b) & \geq \log \left( 1 + \frac{l(a, b)}{\delta(a) \wedge \delta(b)} \right) \\
& \geq \log \left( 1 + \frac{|a - b|}{\delta(a) \wedge \delta(b)} \right) \\
& \geq \left| \log \frac{\delta(a)}{\delta(b)} \right|
\end{align*}
\]

which is a special case of the more general inequality

\[
(2.1b) \quad \ell_{k}(\gamma) \geq \log \left( 1 + \frac{\ell(\gamma)}{\text{dist}(|\gamma|, \partial X)} \right).
\]

2.B. Quasiconvex, annular quasiconvex, and uniform spaces. A rectifiable path \(\gamma: a \sim b\) is \(C\)-quasiconvex, \(C \geq 1\), if its length is at most \(C\) times the distance between its endpoints; i.e., if \(\gamma\) satisfies

\[
\ell(\gamma) \leq C |a - b|.
\]

A metric space is \(C\)-quasiconvex if each pair of points can be joined by a \(C\)-quasiconvex path. A \(1\)-quasiconvex metric space is a geodesic space, and a space is a length space if and only if it is \(C\)-quasiconvex for all \(C > 1\). By cutting out loops, we can always replace a \(C\)-quasiconvex path with a \(C\)-quasiconvex arc having the same endpoints; see [Väi94].
The inequalities in (2.1a) yield the following ‘local’ estimates for quasihyperbolic distances.

2.2. Fact. Let $X$ be a $C$-quasiconvex quasihyperbolic metric space. Then for all $x, a \in X$,
\[
k(x, a) \leq 1 \text{ or } \frac{|x - a|}{\delta(a)} \leq \frac{1}{2C} \implies \frac{1}{2} \frac{|x - a|}{\delta(a)} \leq k(x, a) \leq 2C \frac{|x - a|}{\delta(a)}.\]

2.B.1. Annular quasiconvexity. A metric space $X$ is $C$-annular quasiconvex at $p \in X$ provided it is connected and for all $r > 0$, points in $A[p; r, 2r]$ can be joined by $C$-quasiconvex paths lying in $A[p; r/C, 2Cr]$. We call $X$ $C$-annular quasiconvex if it is $C$-annular quasiconvex at each point. Examples of quasiconvex and annular quasiconvex metric spaces include Banach spaces and upper regular Loewner spaces; the latter includes Carnot groups and certain Riemannian manifolds with non-negative Ricci curvature; see [HK98, 3.13,3.18, Section 6]. Korte [Kor07] proved that doubling metric measure spaces that support a $(1, p)$-Poincaré inequality with sufficiently small $p$ are annular quasiconvex.

To the best of our knowledge, the notion of annular quasiconvexity was introduced in [Kor07] and [BHX08]; it was an essential ingredient in [HSX08]. A similar concept was employed in [Mac10].

2.B.2. Uniformity. Roughly speaking, a metric space is uniform when points in it can be joined by paths that are not “too long” and “move away” from the region’s boundary. More precisely, a quasihyperbolic metric space $X$ is $C$-uniform (for some constant $C \geq 1$) provided each pair of points can be joined by a $C$-uniform arc. Here a rectifiable $\gamma: a \rightarrow b$ is a $C$-uniform arc if and only if it is both a $C$-quasiconvex arc and a double $C$-cone arc; this latter condition means that
\[
\forall x \in |\gamma|, \quad \ell(\gamma[x, a]) \wedge \ell(\gamma[x, b]) \leq C \delta(x).
\]

Double cone arcs are often called cigar arcs. In [Väi88] Väisälä provides a description of various possible double cone conditions (which he calls length cigars, diameter cigars, distance cigars, and Möbius cigars). The work [Mar80] of Martio should also be mentioned.

To simplify an argument, we prevail upon the following characterization for uniform spaces established in [Her11, Prop. C].

2.4. Fact. A quasihyperbolic metric space is uniform if and only if it is plump and proximate points can be joined by uniform arcs. More precisely, if $X$ is $C$-plump and $3C$-proximate points can be joined by $B$-uniform arcs, then $X$ is $18B^2C$-uniform; conversely, if $X$ is $C$-uniform, then it is $4C$-plump.

Two points $x, y$ are $C$-proximate, for some constant $C > 0$, if $|x - y| \leq C[\delta(x) \wedge \delta(y)]$. If this holds, then also $(C + 1)^{-1} \leq \delta(x)/\delta(y) \leq C + 1$. A non-complete locally complete metric space $U$ is $C$-plump, $C \geq 1$, provided for each $x \in U$ and all $r \in (0, \text{diam } U)$
\[
(2.5) \quad \exists z \in B[x; r] \text{ with } \text{dist}(z, \partial U) \geq r/C.
\]

This terminology was introduced by Väisälä in [Väi88] and perhaps is understood best when $U$ is an open subspace of a length space $X$, for then (2.5) asserts that $\text{dist}(z, X \setminus U) \geq r/C$, so the ball $B(z; r/C)$, in $X$, is contained in $U$. 
3. Proofs

Here we establish Theorems A, B, C as stated in the Introduction. In each of these, $X$ is a given quasihyperbolic metric space.

3.A. Proof of Theorem A. Recall that $o \in X$ and $X_*:=X \setminus \{o\}$. We let $\delta_*$ denote distance to $\partial X_*$, so $\delta_*(x) := |x| \wedge \delta(x)$ where $|x| := |x-o|$.

To utilize Fact 2.4, we first verify the following.

3.1. Lemma. Suppose $X$ is $C$-uniform. Then $X_*:=X \setminus \{o\}$ is 12C-plump.

Proof. Let $a \in X_*$ and $r \in (0, \text{diam } X_*)$. We seek a point $z \in B[a; r]$ with $\delta_*(z) \geq r/12C$.

Pick $b \in X_*$ with $|a-b| \geq \frac{1}{2}\text{diam}(X_*)$. Let $\gamma: a \curvearrowright b$ be a $C$-uniform arc in $X$. Let $z_0$ be the arclength midpoint of $\gamma$. Then

$$\delta(z_0) \geq \frac{\ell(\gamma)}{2C} \geq \frac{|a-b|}{2C} \geq \frac{r}{4C}.$$  

Assume $|z_0-a| \leq r/2$. If $|z_0| \geq r/12C$, then $z_0$ is the sought after point. Suppose $|z_0| < r/12C$. By examining paths from $z_0$ to $\partial X$, we obtain a point $z_1 \in S(z_0; r/6C)$. It follows that $\delta_*(z_1) \geq r/12C$, and that

$$|z_1-a| \leq |z_1-z_0| + |z_0-a| \leq \frac{r}{6C} + \frac{r}{2} \leq r,$$

so $z_1$ is the sought after point.

Assume $|z_0-a| > r/2$. Pick $z_2 \in \gamma[z_0, a] \cap S(a; r/2)$. Then $\delta(z_2) \geq r/2C$. Thus if $|z_2| \geq r/6C$, then $z_2$ is the sought after point. Suppose $|z_2| < r/6C$. By examining paths from $z_2$ to $\partial X$, we obtain a point $z_3 \in S(z_2; r/3C)$. It follows that $\delta_*(z_3) \geq r/6C$, and that

$$|z_3-a| \leq |z_3-z_2| + |z_2-a| = \frac{r}{3C} + \frac{r}{2} \leq r,$$

so $z_3$ is the sought after point. \hfill \Box

Now we establish Theorem A. When $X_*$ is $C_*$-uniform, it is not hard to check that $X$ is $C$-uniform for any $C > C_*$ (this also follows from Theorem C) and the proof of (c) $\implies$ (a) in [Her11, Prop. 2.3] shows that $X$ is $2(C_*+1)$-annular quasiconvex at $o$.

For the converse, assume $X$ is $C$-uniform and $C_1$-annular quasiconvex at $o$. By Lemma 3.1 we know that $X_*$ is 12C-plump, so it suffices to show that there is a constant $B$ such that 36C-proximate points in $X_*$ can be joined by $B$-uniform arcs; then Fact 2.4 asserts that $X_*$ is $C_*$-uniform with $C_* = 216B^2C$.

Let $a, b \in X_*$ be 36C-proximate; so, $|a-b| \leq 36C(\delta_*(a) \wedge \delta_*(b))$. By relabeling, if necessary, we may assume $|a| \leq |b|$. Now $|a-b| \leq 36C|a|$. Let $\gamma_o: a \curvearrowright b$ be a $C$-uniform arc in $X$. Put $R:=|a|/10CC_1$. Then $\ell(\gamma_o) \leq C|a-b| \leq 36C^2|a|$, so

$$R \geq \left(180C^3C_1\right)^{-1}\frac{\ell(\gamma_o)}{2}.$$  

As $|b| \geq |a|$, $\{a, b\} \cap B(o; |a|) = \emptyset$. Suppose $\gamma_o \cap B(o; R) = \emptyset$. Then for each $x \in \gamma_o$, $|x| \geq R$ and we readily deduce that $\gamma_o$ is a $180C^3C_1$-uniform arc in $X_*$. Assume $\gamma_o \cap B(o; R) \neq \emptyset$. Let $a_o, b_o$ be the first, last points (respectively) of $\gamma_o$ in $S(o; R)$. Put $\gamma := \alpha \star \sigma \star \beta$ where $\alpha := \gamma_o[a, a_o]$, $\beta := \gamma_o[b_o, b]$ and where $\sigma: a_o \curvearrowright b_o$ is a $C_1$-quasiconvex arc in $A[o; R/2C_1, C_1R]$.

Note that as $\alpha, \beta$ both join the spheres $S(o; R), S(o; |a|)$, they each have length at least $|a| - R = (10CC_1 - 1)R \geq 9CC_1R$. It follows that $\delta(a_o) \vee \delta(b_o) \geq 9C_1R$. 

As in the proof of (c) $\implies$ (a) of [Her11, Prop. 2.3], we can show that $\gamma_o$ is $C_1$-annular quasiconvex at $o$. 

This completes the proof of (c) $\implies$ (a). \hfill \Box

Theorem A follows from (c) $\implies$ (a) and Fact 2.4.
Evidently, $\ell(\gamma) \leq C_1 \ell(\gamma_o) \leq CC_1|a - b|$, so to verify that $\gamma$ is a uniform arc it remains to corroborate the double cone arc condition. To begin, let $x \in \sigma$. Then

$$|x| \geq \frac{R}{2C_1} \geq (360C^3C_1^2)^{-1} \frac{\ell(\gamma_o)}{2} \geq (360C^3C_1^3)^{-1} \frac{\ell(\gamma)}{2}.$$  

Also,

$$9C_1 R \leq \delta(a_o) \leq \delta(o) + |a_o| = \delta(o) + R$$

so

$$\delta(x) \geq \delta(o) - C_1 R \geq (30C^3C_1)^{-1} \frac{\ell(\gamma)}{2}$$

and we see that the double cone condition holds for points in $\sigma$ with constant $360C^3C_1^3$.

It remains to examine points $x \in \alpha \cup \beta$. Evidently, $|x| \geq R \geq (180C^3C_1^2)^{-1} \ell(\gamma)/2$. Let $z_o$ be the arclength midpoint of $\gamma_o$; so, $z_o$ lies in $\alpha$ or $\beta$ or $\gamma_o \setminus (\alpha \cup \beta)$. If $z_o$ lies in $\gamma_o \setminus (\alpha \cup \beta)$, then $\alpha, \beta$ are both “short subarcs of $\gamma_o$” and we readily see that

$$x \in \alpha \implies \ell(\gamma[x, a]) = \ell(\gamma_o[x, a]) \leq C\delta(x)$$

and

$$x \in \beta \implies \ell(\gamma[x, b]) = \ell(\gamma_o[x, b]) \leq C\delta(x)$$

and thus the double cone condition holds.

Suppose $z_o \in \beta$. Here $\alpha$ is again a “short subarc” and we precede exactly as above for points $x \in \alpha$. Assume $x \in \beta$. When $x \in \beta[z_o, b] = \gamma_o[z_o, b]$ we can–again–precede as above. So, assume $x \in \beta[b_o, z_o]$. Here

$$\delta(x) \geq \frac{\ell(\gamma_o[x, a])}{C} \geq \frac{\ell(\alpha)}{C} \geq 9C_1 R \geq (20C^3C_1)^{-1} \frac{\ell(\gamma)}{2}.$$  

Thus the double cone condition holds for points in $\alpha \cup \beta$ with constant $180C^3C_1^2$. It now follows that $\gamma$ is a $B$-uniform arc with $B := 360C^3C_1^3$. \hfill $\Box$

**3.B. Proof of Theorem B.** Here we assume $\Omega$ is a uniformly collared subspace of a $C_o$-uniform space $X$ with $C_1$-uniform $\Phi$-fat $^3$ collars as described in the Introduction. Also, since the associated hypotheses and conclusions are all bi-Lipschitz invariant, we may and do assume that $X$ is a length space.

To join points, we start with a uniform arc in $X$. If this arc gets near some $B_i$, we replace (by “cutting and pasting”) an appropriate subarc with a uniform arc in $\Omega_i$.

Before immersing ourselves in the details, we discuss some basic immediate properties. First, since the open sets $U_i$ are disjointed, each $B_i$ is closed and, e.g., $\text{bd}(B) = \bigcup_i \text{bd}(B_i)$. Also, while $\partial B \subset \partial X$ may be empty or non-empty, for each $i$,

$$d_i := \text{dist}(B_i, \partial X) > 0.$$  

It’s not difficult to check that $\Omega_i := \Omega \cap U_i = U_i \setminus B_i$, and that $\text{bd}(\Omega_i) = \text{bd}(B_i) \cup \text{bd}(U_i)$ and this is a disjoint union.

Next, as $\Omega$ is open in $X$ and $X$ is open in $\bar{X}$, we (eventually) see that

$$\partial \Omega = \text{bd}(B) \cup \partial X$$  

(and this is a disjoint union)

$^3$We assume $\Phi \geq 2$.  

from which we deduce that for each \( x \in \Omega \),
\[
\text{dist}(x, \partial \Omega) = \text{dist}(x, B) \land \text{dist}(x, \partial X) = \inf_i \text{dist}(x, B_i) \land \text{dist}(x, \partial X).
\]

It also follows that for each \( x \in \Omega \):

\begin{align*}
(3.2a) & \quad \text{If } \exists i \text{ with } \text{dist}(x, B_i) \leq d_i/2, \text{ then } \text{dist}(x, \partial \Omega) = \text{dist}(x, B). \\
(3.2b) & \quad \text{If } \exists i \text{ with } \text{dist}(x, B_i) \leq d_i/2\Phi, \text{ then } \text{dist}(x, \partial \Omega) = \text{dist}(x, B_i). \\
(3.2c) & \quad \text{If } x \in \Omega_i, \text{ then } \text{dist}(x, \partial \Omega) \geq \text{dist}(x, \partial \Omega_i).
\end{align*}

Now define
\[
A_i := \left\{ x \in X \setminus B_i \mid \text{dist}(x, B_i) < \frac{d_i}{10\Phi} \right\} \quad \text{and} \quad A := \bigcup_i A_i.
\]

From (3.2b) we see that
\[
x \in \text{cl}(A_i) \implies \text{dist}(x, \partial \Omega) = \text{dist}(x, B_i)
\]
and similarly
\[
\text{dist}(x, \partial \Omega) = \text{dist}(x, \partial X) \implies x \not\in \text{cl}(A).
\]

Let \( a, b \in \Omega \) and let \( \gamma_o : a \curvearrowright b \) be a \( C_0 \)-uniform arc in \( X \). Suppose \( \gamma_o \cap A = \emptyset \). Let \( x \in \gamma_o \). Then for all \( i \),
\[
\text{dist}(x, \partial X) \leq \text{dist}(x, B_i) + \text{diam}(B_i) + \text{dist}(B_i, \partial X) \leq \text{dist}(x, B_i) + (\Phi + 1)d_i \leq (10\Phi(\Phi + 1) + 1) \text{dist}(x, B_i) \leq 20\Phi^2 \text{dist}(x, B_i),
\]
so \( \text{dist}(x, \partial \Omega) \geq (20\Phi^2)^{-1} \text{dist}(x, \partial X) \) and we deduce that \( \gamma_o \) is a \( 20C_0\Phi^2 \)-uniform arc in \( \Omega \).

Suppose \( \gamma_o \cap A \neq \emptyset \) and, for a moment, assume \( \{a, b\} \cap A = \emptyset \). Let \( J \) denote the set of all indeces \( i \) with \( \gamma_o \cap A_i \neq \emptyset \). For each \( j \in J \): let \( a_j, b_j \) be the first, last points of \( \gamma_o \) in \( \text{bd}(A_j) \); let \( \sigma_j : a_j \curvearrowright b_j \) be a \( C_1 \)-uniform arc in \( \Omega_j \); and, replace each subarc \( \gamma_o[a_j, b_j] \) with the corresponding \( \sigma_j \).

If \( a \in A \), say \( a \in A_{i_0} \): let \( a_{i_0} \) be the last point of \( \gamma_o \) in \( \text{bd}(A_{i_0}) \); let \( \alpha : a \curvearrowright a_{i_0} \) be a \( C_1 \)-uniform arc in \( \Omega_{i_0} \); and, replace the subarc \( \gamma_o[a, a_{i_0}] \) with \( \alpha \). Similarly, if \( b \in A_{j_0} \), we get a \( C_1 \)-uniform \( \beta : b_{j_0} \curvearrowright b \) in \( \Omega_{j_0} \) that replaces \( \gamma_o[b_{j_0}, b] \), where \( b_{j_0} \) is the first point of \( \gamma_o \) in \( \text{bd}(A_{j_0}) \).

We now have an arc \( \gamma : a \curvearrowright b \) in \( \Omega \) that has been obtained by replacing certain subarcs of \( \gamma_o \) with appropriate subarcs \( \sigma_j \) or \( \alpha \) or \( \beta \). As each of these new subarcs is \( C_1 \)-quasiconvex, we see that
\[
\ell(\gamma) \leq C_1 \ell(\gamma_o) \leq C_0C_1|a - b|,
\]
so \( \gamma \) is a \( C_0C_1 \)-quasiconvex arc. It remains to verify the double cone condition.

Let \( x \in \gamma \). Suppose \( x \notin \alpha \cup \beta \cup \bigcup_j \sigma_j \). As above, where \( \gamma_o \cap A = \emptyset \), we again see that \( \text{dist}(x, \partial \Omega) \geq \text{dist}(x, \partial X)/20\Phi^2 \) and the double cone condition holds with \( C = 20C_0C_1\Phi^2 \).

Suppose \( x \in \alpha \). If \( \ell(\gamma[x,a]) = \ell(\alpha[x,a]) \leq \ell(\alpha[x,a_o]) \), then \( \text{dist}(x, \partial \Omega) \geq \text{dist}(x, \partial \Omega_{i_0}) \geq C^{-1}_1 \ell(\gamma[x,a]) \) and the double cone condition holds with constant \( C = C_1 \).
Assume \( \ell(\alpha[x, a]) > \ell(\alpha[x, a_o]) \). Now
\[
\ell(\gamma[x, a]) = \ell(\alpha[x, a]) \leq \ell(\alpha) \leq C_1|a - a_o|
\]
\[
\leq C_1(\text{dist}(a, B_{i_o}) + \text{diam}(B_{i_o}) + \text{dist}(a_o, B_{i_o}))
\]
\[
\leq C_1(\Phi + 1)d_{i_o} \leq 2C_1\Phi d_{i_o}.
\]
If \( \ell(\alpha[x, a_o]) \geq d_{i_o}/20\Phi \), then
\[
\text{dist}(x, \partial\Omega) \geq \text{dist}(x, \partial\Omega_{i_o}) \geq C_1^{-1}\ell(\alpha[x, a_o]) \geq \frac{d_{i_o}}{20C_1\Phi} \geq \frac{\ell(\gamma[x, a])}{40C_1^2\Phi^2}.
\]
On the other hand, if \( \ell(\alpha[x, a_o]) < d_{i_o}/20\Phi \), then
\[
\frac{d_{i_o}}{10\Phi} = \text{dist}(a_o, B_{i_o}) = \text{dist}(a_o, \partial\Omega) \leq |x - a_o| + \text{dist}(x, \partial\Omega)
\]
\[
\leq \ell(\alpha[x, a_o]) + \text{dist}(x, \partial\Omega)
\]
and so now
\[
\text{dist}(x, \partial\Omega) \geq \frac{d_{i_o}}{20\Phi} \geq \frac{\ell(\gamma[x, a])}{40C_1^2\Phi^2}.
\]
Thus in all cases, when \( x \in \alpha \), the double cone condition holds with constant \( C = 40C_1^2\Phi^2 \).

A similar argument applies if \( x \in \beta \).

Finally, suppose \( x \in \sigma_j \) for some \( j \in J \). We demonstrate that
\[
\ell(\gamma[x, a]) \cap \ell(\gamma[x, b]) \leq 3C_0C_1\Phi d_j \quad \text{and} \quad d_j \leq 20C_1\Phi \text{ dist}(x, \partial\Omega)
\]
which gives the double cone condition with constant \( C = 60C_0C_1^2\Phi^2 \).

First, if
\[
\ell(\sigma_j[x, a_j]) \cap \ell(\sigma_j[x, b_j]) \geq \frac{d_j}{20\Phi},
\]
then \( \text{dist}(x, \partial\Omega) \geq \text{dist}(x, \partial\Omega_j) \geq d_j/20C_1\Phi \). Suppose
\[
\ell(\sigma_j[x, z]) < \frac{d_j}{20\Phi} \quad \text{for some} \quad z \in \{ a_j, b_j \}.
\]
Then
\[
\frac{d_j}{10\Phi} = \text{dist}(z, B_j) = \text{dist}(z, \partial\Omega) \leq |x - z| + \text{dist}(x, \partial\Omega) \leq \ell(\sigma_j[x, z]) + \text{dist}(x, \partial\Omega)
\]
whence again \( \text{dist}(x, \partial\Omega) \geq d_j/20C_1\Phi \).

Next, \( \ell(\gamma[x, a]) \leq \ell(\sigma_j) + \ell(\gamma[a_j, a]) \) and \( \ell(\gamma[x, b]) \leq \ell(\sigma_j) + \ell(\gamma[b_j, b]) \), so
\[
\ell(\gamma[x, a]) \cap \ell(\gamma[x, b]) \leq \ell(\sigma_j) + \ell(\gamma[a_j, a]) \cap \ell(\gamma[b_j, b])
\]
\[
\leq \ell(\sigma_j) + C_1(\ell(\gamma[a_j, a]) \cap \ell(\gamma[b_j, b])).
\]
Now
\[
\ell(\sigma_j) \leq C_1|a_j - b_j| \leq C_1(\text{dist}(a_j, B_j) + \text{diam}(B_j) + \text{dist}(b_j, B_j)) \leq C_1\left(\Phi + \frac{1}{5\Phi}\right)d_j.
\]
Also, for \( z \in \{ a_j, b_j \} \),
\[
\text{dist}(z, \partial\Omega) \leq \text{dist}(z, B_j) + \text{diam}(B_j) + \text{dist}(B_j, \partial\Omega) \leq \left(\Phi + 1 + \frac{1}{10\Phi}\right)d_j.
\]
If \( \ell(\gamma_\alpha[a_j, a]) \leq \ell(\gamma_\alpha[b_j, b]) \), then
\[
\ell(\gamma_\alpha[a_j, a]) = \ell(\gamma_\alpha[a_j, a]) \land \ell(\gamma_\alpha[a_j, b]) \leq C_0 \text{dist}(a_j, \partial X)
\]
and similarly if \( \ell(\gamma_\alpha[b_j, b]) \leq \ell(\gamma_\alpha[a_j, a]) \), then
\[
\ell(\gamma_\alpha[b_j, b]) = \ell(\gamma_\alpha[b_j, a]) \land \ell(\gamma_\alpha[b_j, b]) \leq C_0 \text{dist}(b_j, \partial X).
\]
Therefore
\[
\ell(\gamma_\alpha[a_j, a]) \land \ell(\gamma_\alpha[b_j, b]) \leq C_0 \left( \Phi + 1 + \frac{1}{10 \Phi} \right) d_j.
\]

It now follows that
\[
\ell([x, a]) \land \ell([x, b]) \leq C_1 \left( \Phi + \frac{1}{5 \Phi} \right) d_j + C_0 C_1 \left( \Phi + 1 + \frac{1}{10 \Phi} \right) d_j \leq 3C_0 C_1 \Phi d_j.
\]

Examining all our various constants we see that \( \gamma : a \sim b \) is a \( C_0 C_1 \)-quasiconvex double \( 60C_0 C_1^2 \Phi^2 \)-cone arc in \( \Omega \).

\[\square\]

3.C. Proof of Theorem C. Here \( \Omega = X \setminus A \) and (1.3) holds for some constant \( \kappa \in (0, 1) \).

First, suppose \( \Omega \) is \( C \)-uniform. The proof of \( (c) \implies (a) \) in [Her11, Prop. 2.3] shows that \( X \) is \( 2(C + 1) \)-annular quasiconvex at all points \( a \in A \). Let \( \varepsilon \in (0, 1) \). We verify that \( X \) is \( (C + \varepsilon) \)-uniform.

Since \( A \subset \partial \Omega \), \( X = \Omega \cup A \) is \( C' \)-quasiconvex for any \( C' > C \) which permits use of Fact 2.2.

Let \( a, b \in X \). Since we can join points in \( \Omega \) with \( C \)-uniform arcs, we may assume \( a, b \in A \); the case where one point lies in \( A \) and one in \( \Omega \) is similar and easier. We select \( a_0, b_0 \in \Omega \) sufficiently near \( a, b \), pick quasiconvex arcs \( \alpha : a \sim a_0, \beta : b_0 \sim b \) and a uniform arc \( \gamma_\alpha : a_0 \sim b_o \) in \( \Omega \), and then check that \( \gamma := \alpha \star \gamma_\alpha \star \beta \) is \( (C + \varepsilon) \)-uniform.

For each \( a \in A \), set \( \rho_a := \nu \delta(a) \) where \( \nu := \kappa/10C \). Employing (1.3) in conjunction with Fact 2.2 we see that the balls \( B(a; \rho_a) \), with \( a \in A \), are disjointed.

Fix points \( a_0 \in B(a; \varepsilon \rho_a/10C^2), b_0 \in B(b; \varepsilon \rho_b/10C^2) \) and let \( \alpha : a \sim a_0, \beta : b_0 \sim b \) be \( (C + \varepsilon) \)-quasiconvex arcs. Note that
\[
\ell(\alpha) \leq (C + \varepsilon)|a - a_0| \leq C + \varepsilon \frac{10C^2 \varepsilon \rho_a}{10C^2 \rho_a} \leq \frac{\varepsilon}{5C} \rho_a
\]
and similarly \( \ell(\beta) \leq \varepsilon \rho_b/5C \).

Let \( \gamma_\alpha : a_0 \sim b_0 \) be a \( C \)-uniform arc in \( \Omega \). Put \( \gamma := \alpha \star \gamma_\alpha \star \beta \); then \( \gamma : a \sim b \) in \( \Omega \cup \{a, b\} \). Since \( |a - b| \geq \rho_a + \rho_b \) and \( |a_0 - b_0| \leq |a - a_0| + |a - b| + |b - b_0| \), we see that
\[
\ell(\gamma) \leq (C + \varepsilon)(|a - a_0| + |b - b_0|) + C|a_0 - b_0|
\]
\[
\leq (2C + \varepsilon)(|a - a_0| + |b - b_0|) + C|a - b|
\]
\[
\leq \varepsilon \frac{2C + \varepsilon}{10C^2} (\rho_a + \rho_b) + C|a - b|
\]
\[
\leq (C + \varepsilon)|a - b|
\]
and so \( \gamma \) is \( (C + \varepsilon) \)-quasiconvex.

To establish the double cone condition, let \( x \in \gamma \), and let \( z_\delta \) be the arclength midpoint of \( \gamma_\alpha \). If \( x \in \alpha \), it is not difficult to check that \( \delta(x) \geq \ell(\alpha) \geq \ell([x, a]) \);
similarly for $x \in \beta$. Assume $x \in \gamma_o$, say $x \in \gamma_o[a_o, z_o]$. Here
\[
\delta(x) \geq \text{dist}(x, \partial \Omega) \geq C^{-1}\ell(\gamma_o[x, a_o]).
\]
If $\ell(\gamma_o[x, a_o]) \geq \rho_a/5$, then $\ell(\alpha) \leq (C/\varepsilon)\ell(\gamma_o[x, a_o])$, so
\[
\ell(\gamma[x, a]) = \ell(\gamma_o[x, a_o]) + \ell(\alpha) \leq \left(1 + \frac{C}{\varepsilon}\right)\ell(\gamma_o[x, a_o]) \leq (C + \varepsilon)\delta(x).
\]
Finally, if $\ell(\gamma_o[x, a_o]) \leq \rho_a/5$, then
\[
|x - a| \leq \frac{\rho_a}{5} + \frac{\varepsilon\rho_a}{10C^2} \leq \frac{2C^2 + \varepsilon}{10C^2} \nu\delta(a) \leq \frac{3\nu}{10}\delta(a)
\]
so
\[
\delta(x) \geq \left(1 - \frac{3\nu}{10}\right)\delta(a)
\]
whence
\[
\ell(\gamma[x, a]) \leq \frac{\rho_a}{5} + \frac{\varepsilon\rho_a}{5C} = \frac{C + \varepsilon}{5C} \nu\delta(a) \leq \frac{C + \varepsilon}{C} \frac{2\nu}{10 - 3\nu} \leq (1 + \varepsilon)\delta(x).
\]
For the converse, suppose $X$ is $C_0$-uniform and $C_1$-annular quasiconvex at each point of $A$. We show that $\Omega := X \setminus A$ is uniformly collared with fat collars and appeal to Theorem B.

For each $a \in A$, set $\rho_a := \nu_0\delta(a)$ where $\nu_0 := \kappa/4C_0$. Employing Fact 2.2 and (1.3) we see that the balls $B(a; \rho_a)$, with $a \in A$, are disjointed. By assumption, each $B(a; \rho_a)$ is $C_0$-uniform and $X$ is $C_1$-annularly quasiconvex at $a$; by the proof of Theorem A, $B_{v_0}(a; \rho_a) := B(a; \rho_a) \setminus \{a\}$ is $C_\ast$-uniform. Since $A = X \setminus \Omega = \bigcup_{a \in A}\{a\}$ with $\{a\} = \{a\} \cap B(a; \rho_a)$, we see that $\Omega$ is uniformly collared. Evidently, the collars $B_{\ast}(a; \rho_a)$ are $\Phi$-fat with $\Phi := 1/\nu_0 = 4C_0/\kappa$.

\[\square\]

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