Generalized Frobenius Manifolds with Non-flat Unity and Integrable Hierarchies

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Abstract

For any generalized Frobenius manifold with non-flat unity, we construct a bi-hamiltonian integrable hierarchy of hydrodynamic type which is an analogue of the Principal Hierarchy of a Frobenius manifold. We show that such an integrable hierarchy, which we also call the Principal Hierarchy, possesses Virasoro symmetries and a tau structure, and the Virasoro symmetries can be lifted to symmetries of the tau-cover of the integrable hierarchy. We derive the loop equation from the condition of linearization of actions of the Virasoro symmetries on the tau function, and construct the topological deformation of the Principal Hierarchy of a generalized semisimple Frobenius manifold with non-flat unity. We also give two examples of generalized Frobenius manifolds with non-flat unity which are shown to be closely related to the well-known integrable hierarchies: the Volterra hierarchy, the q-deformed KdV hierarchy and the Ablowitz-Ladik hierarchy.

Keywords. Frobenius manifold; Principal Hierarchy; Tau structure; Virasoro symmetry

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The notion of Frobenius manifold was introduced by Dubrovin in [13] as a coordinate free form of the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) associativity equations satisfied by the primary free energies of 2D topological field theories (TFTs) [10, 48, 49]. It has played a central role in the study of Gromov-Witten theory, singularity theory, mirror symmetry, integrable systems and some other research subjects of mathematical physics in the past thirty years. The main motivation for the study of Frobenius manifolds in [13] is to reconstruct a complete 2D TFT starting from the corresponding Frobenius manifold, i.e., starting from a solution of the WDVV associativity equations. The reconstruction of the genus zero part of a 2D TFT was given in [12]. In this reconstruction, a bihamiltonian integrable hierarchy of hydrodynamic type, known as the Principal Hierarchy of the corresponding Frobenius manifold, plays a crucial role. This integrable hierarchy possesses a tau structure which enables one to assign a tau function to any of its solution, and the logarithm of the tau function of a particular solution selected by the string equation yields the genus zero free energy of the 2D TFT. The reconstruction of a complete 2D TFT was given in [21, 23] under the assumption that the corresponding Frobenius manifold is semisimple. It is achieved by performing a certain quasi-Miura transformation to the Principal Hierarchy, which linearizes the action of the Virasoro symmetries of the Principal Hierarchy on the tau function of the resulting deformed integrable hierarchy. Such a deformed integrable hierarchy is called the topological deformation of the Principal Hierarchy, and its tau function which is selected by the string equation yields the partition function of the 2D TFT when the Virasoro conjecture for this 2D TFT [25] holds true.

The topological deformation of the Principal Hierarchy of a semisimple Frobenius manifold is a bihamiltonian integrable hierarchy [23, 35]. For the one dimensional case, the associated integrable hierarchy is the KdV hierarchy which controls the 2D topological gravity as we learn from Witten’s conjecture and its proof by Kontsevich [33, 49]. Other well-known bihamiltonian integrable hierarchies that are topological deformations of the Principal Hierarchies of certain semisimple Frobenius manifolds include the extended Toda hierarchy [8] and the Drinfeld-Sokolov hierarchy associated with the untwisted affine Lie algebras of ADE type [11], they correspond to the Frobenius manifolds that are associated respectively with the Gromov-Witten invariants of the projective line [24, 28, 43, 50], and the FJRW invariants of simple singularities of ADE type [26, 30].

In this paper, we study the relationship of a class of generalized Frobenius manifold with bihamiltonian integrable hierarchies. By Dubrovin’s definition [14], on a Frobenius manifold $M$ there is defined a flat metric, and on each of its tangent space there is a commutative associative algebra structure with unity, and moreover the unit vector field is required to be flat w.r.t. the flat metric. In applications of the notion of Frobenius mani-
ifold to the study of singularity theory, Gromov-Witten theory, quantum K-theory and the theory of integrable systems, certain weaker Frobenius manifold structures also arise and play important roles. One class of weak Frobenius manifolds, called F-manifolds, was introduced and studied by Manin and Hertling in [31]. On such a weak Frobenius manifold the existence of a flat metric is replaced by a certain weaker condition imposed on the commutative associative algebra structure of the tangent spaces. By imposing a certain compatible flat connection and a flat unity on an F-manifold, one obtains the notion of flat F-manifold [29, 41], whose relationship with integrable hierarchies of hydrodynamic type and their deformations are studied in [37, 38, 4]. The weak Frobenius manifolds that we study in this paper are defined by Dubrovin’s definition of Frobenius manifolds but without the flatness condition of the unit vector fields, and they will be called generalized Frobenius manifolds with non-flat unity.

Our study of generalized Frobenius manifolds with non-flat unity is mainly motivated by the relationship of the Ablowitz-Ladik hierarchy, a well-known bihamiltonian integrable hierarchy, with the equivariant Gromov-Witten invariants of the resolved conifold, as it was revealed by Brini in [5]. Brini conjectured in [5] that a particular tau function of the Ablowitz-Ladik hierarchy yields the generating function of the equivariant Gromov-Witten invariants of the resolved conifold with anti-diagonal action, and verified his conjecture at the genus one approximation. In [6, 7] Brini and his collaborators showed that the dispersionless limit of the Ablowitz-Ladik hierarchy belongs to the Principal Hierarchy of a generalized Frobenius manifold with non-flat unity, and that the almost dual of this generalized Frobenius manifold coincides with the non-conformal Frobenius manifold associated with the equivariant Gromov-Witten invariants of the resolved conifold with anti-diagonal action. It is natural to ask whether there is an analogue of the construction of the topological deformation of the Principal Hierarchy of a usual semisimple Frobenius manifold for the one of a generalized semisimple Frobenius manifold with non-flat unity? Such a construction could lead to a quasi-Miura transformation which relates the dispersionless Ablowitz-Ladik hierarchy with the dispersionfull one, and provides support of Brini’s conjecture of the relation of this integrable hierarchy with the equivariant Gromov-Witten invariants of the resolved conifold.

Other motivation of our study comes from the relation between certain special cubic Hodge integrals and the Volterra hierarchy (also called the discrete KdV hierarchy) that is established in [17, 18]. This relation is given by the identification of the generating function of these Hodge integrals with a certain tau function of the Volterra hierarchy. The Volterra hierarchy has a bihamiltonian structure, from the hydrodynamic limit of which we can obtain a 1-dimensional generalized Frobenius manifold with non-flat unity. The re-construction of the Volterra hierarchy from this generalized Frobenius manifold would help us to have a deeper understanding of the above-mentioned relation between Hodge integrals and integrable hierarchies.

For a generalized Frobenius manifold with non-flat unity, we can construct its Principal Hierarchy as it is done for a usual Frobenius manifold, and we can prove that the Principal Hierarchy possesses Virasoro symmetries and a tau structure. To construct the topological deformation of the Principal Hierarchy by using the condition of linearization of the actions of the Virasoro symmetries on the tau function, we need to lift the Virasoro symmetries of the Principal Hierarchy to its tau-cover and to represent
the actions of the Virasoro symmetries on the tau function in terms of certain linear differential operators. However, due to the non-flatness of the unit vector field we can not achieve this in a direct way. One important point in our solution of this problem is the introduction of an additional infinite number of commuting flows to the Principal Hierarchy. These additional flows include the one given by the translation along the spatial variable \( x \), and they commute with the original flows of the Principal Hierarchy. Then we show that the the Virasoro symmetries can be lifted to the tau-cover of the extended integrable hierarchy which we still call the Principal Hierarchy of the generalized Frobenius manifold, and their actions on the tau function can be represented in terms of certain linear differential operators which satisfy the Virasoro commutation relations. Notably, in our study of properties of the Virasoro symmetries of the Principal Hierarchy of an \( n \)-dimensional generalized Frobenius manifold \( M \) with non-flat unity, we used the structure of a special \((n + 2)\)-dimensional Frobenius manifold which has a flat unity and has \( M \) as its Frobenius submanifold.

We derive the loop equation of a generalized Frobenius manifold with non-flat unity by using the condition of linearization of the actions of the Virasoro symmetries on the tau function. A solution of the loop equation of a generalized semisimple Frobenius manifolds with non-flat unity yields a quasi-Miura transformation which gives a certain deformation of the Principal Hierarchy. For the two examples of generalized Frobenius manifolds that we present in this paper, such deformations of the Principal Hierarchies are closely related to Hodge integrals and the equivariant Gromov-Witten invariants of the resolved conifold with anti-diagonal action, so we call them the topological deformations of the Principal Hierarchies.

The paper is organized as follows. In Section 2 we present some properties of an \( n \)-dimensional generalized Frobenius manifold \( M \) with non-flat unity. In Section 3, we give the definition of the Principal Hierarchy of \( M \). In Section 4, we introduce an \((n + 2)\)-dimensional Frobenius manifold associated with \( M \). In Section 5, we construct the tau-cover of the Principal Hierarchy. In Sections 6-8, we introduce the notion of periods of \( M \) and in terms of which we present the Virasoro symmetries of the tau-cover of \( M \). In Sections 9 and 10, we derive the loop equation of \( M \). In Section 11, we present two examples of generalized Frobenius manifolds with non-flat unity, and relate the topological deformations of their Principal Hierarchies to the well-known integrable hierarchies: the Volterra hierarchy, the q-deformed KdV hierarchy and the Ablowitz-Ladik hierarchy. In Section 12 we give some concluding remarks. In the Appendix we give the proof of the Theorem 7.7.

2 Generalized Frobenius manifolds

Let us begin with the description of the class of generalized Frobenius manifolds that we are to study following Dubrovin’s definition of Frobenius manifolds [14].

Definition 2.1. A smooth or analytic manifold \( M^n \) is called a generalized Frobenius manifold with non-flat unity if it is endowed on each of its tangent spaces a Frobenius algebra structure \( A = (\cdot, \langle , \rangle, e) \), i.e., a commutative associative algebra (over \( \mathbb{R} \) or \( \mathbb{C} \)) with multiplication \( \cdot \) and unity \( e \), and a nondegenerate bilinear form \( \langle , \rangle \), which is
invariant w.r.t. the multiplication; this Frobenius algebra structure depends smoothly or analytically on the point of $M$, and is required to satisfy the following conditions.

**FM1.** The bilinear form $\langle \cdot, \cdot \rangle$ yields a flat metric $\eta$ on $M$, and $\nabla e \neq 0$.

**FM2.** Define a 3-tensor $c$ on $M$ by $c(u, v, w) := \langle u \cdot v, w \rangle$, then the 4-tensor

$$\nabla c : (z, u, v, w) \mapsto \nabla_z c(u, v, w), \quad \forall z, u, v, w \in \text{Vect}(M)$$

is symmetric, here $\nabla$ is the Levi-Civita connection of $\eta$.

**FM3.** There exists an Euler vector field $E$ on $M$ such that $\nabla \nabla E = 0$, $\nabla E$ is diagonalizable, and

$$\mathcal{L}_E(a \cdot b) = \mathcal{L}_E a \cdot b + a \cdot \mathcal{L}_E b + a \cdot b \quad \text{(2.1)}$$

Here the constant $d$ is called the charge of $M$.

Notably, in the above definition we only replace the flatness condition $\nabla e = 0$ of the unit vector field from Dubrovin’s original definition of a Frobenius manifold by the condition $\nabla e = 0$. In what follows, we will called a generalized Frobenius manifold with non-flat unity a generalized Frobenius manifold.

Let us first recall some basic properties of a Frobenius manifold that are still possessed by a generalized Frobenius manifold [14]. We fix a system of flat coordinates $v^1, \ldots, v^n$ of the flat metric in a neighborhood $U$ of a certain point $v_0 \in M$, and denote

$$\eta_{\alpha\beta} = \langle \partial_\alpha, \partial_\beta \rangle, \quad (\eta^{\alpha\beta}) = (\eta_{\alpha\beta})^{-1},$$

where $\partial_\alpha = \frac{\partial}{\partial v^\alpha}$. The structure constants of the Frobenius algebras $c_{\gamma}^{\alpha\beta} = c_{\gamma}^{\alpha\beta}(v)$ are defined by

$$\partial_\alpha \cdot \partial_\beta = c_{\gamma}^{\alpha\beta} \partial_\gamma, \quad \alpha, \beta = 1, \ldots, n,$$

here and in what follows summation over repeated upper and lower Greek indices range from 1 to $n$ is assumed.

From the definition of a generalized Frobenius manifold it follows the existence of a function $F = F(v)$, called the potential of the Frobenius manifold, such that

$$\frac{\partial^3 F}{\partial v^\alpha \partial v^\beta \partial v^\gamma} \eta_{\gamma \xi} \partial_\alpha \partial_\beta = \eta_{\xi \gamma} \partial_\alpha \partial_\beta,$$

$$\partial E F = (3 - d) F + \frac{1}{2} A_{\alpha\beta} v^\alpha v^\beta + B_\alpha v^\alpha + C,$$

where $A_{\alpha\beta}, B_\alpha, C$ are certain constants, and the functions $c_{\gamma}^{\alpha\beta}$ satisfy the associativity equation

$$c_{\gamma}^{\xi} c_{\xi\beta}^{\delta} = c_{\gamma}^{\xi} c_{\xi\alpha}^{\delta}, \quad \forall \alpha, \beta, \gamma, \delta = 1, \ldots, n. \quad \text{(2.3)}$$

Denote

$$\mu = \frac{2 - d}{2} - \nabla E, \quad \text{(2.4)}$$

then from [22] we have

$$\mu \eta + \eta \mu = 0. \quad \text{(2.5)}$$
Since $\nabla E$ is diagonalizable, we can choose the flat coordinates $v^1, \ldots, v^n$ such that
\[
\mu := \text{diag}(\mu_1, \ldots, \mu_n),
\]
and the Euler vector field $E$ can be represented as
\[
E = E^\alpha \partial_\alpha = \sum_{\alpha=1}^{n} \left( 1 - \frac{d}{2} - \mu_\alpha \right) v^\alpha + r^\alpha \partial_\alpha.
\]
Moreover, the constants $r^\alpha \neq 0$ only if $\mu_\alpha = 1 - \frac{d}{2}$.

As for a usual Frobenius manifold, we define the intersection form $g$ of $M$ by
\[
(\omega_1, \omega_2) = i_E (\omega_1 \cdot \omega_2), \quad \forall \omega_1, \omega_2 \in T^*_v M,
\]
where the multiplication on $T^*_v M$ is defined via the isomorphism between $T_v M$ and $T^*_v M$ induced by the bilinear form $(\cdot, \cdot)$. In the flat coordinates we have
\[
g^{\alpha\beta}(v) = (dv^\alpha, dv^\beta)_v = E^\gamma c^{\alpha\beta}_\gamma,
\]
and the contravariant coefficients of the Levi-Civita connection of $g$ can be represented in the form
\[
\Gamma^{\alpha\beta}_\gamma = \left( \frac{1}{2} - \mu_\beta \right) c^{\alpha\beta}_\gamma,
\]
where $c^{\alpha\beta}_\gamma = \eta^{\alpha\lambda} c^{\beta}_\lambda$. As it is shown in [14], the intersection form $g$ and the flat metric $\eta$ yield a flat pencil of metrics $g^{\alpha\beta} - \lambda \eta^{\alpha\beta}$ on $M$.

**Proposition 2.1.** On a generalized Frobenius manifold $M$ there exists a function $\varphi(v)$ which satisfies the relations
\[
e^\alpha = \eta^{\alpha\beta} \frac{\partial \varphi}{\partial v^\beta}, \quad E^\alpha = g^{\alpha\beta} \frac{\partial \varphi}{\partial v^\beta}.
\]
Here $e^\alpha, E^\alpha$ are the coefficients of the unit vector field $e = e^\alpha \partial_\alpha$ and the Euler vector field $E = E^\alpha \partial_\alpha$.

**Proof.** To prove the existence of a function $\varphi$ satisfying the first relation of (2.11), it suffices to show that
\[
\partial^\alpha e^\beta = \partial^\beta e^\alpha,
\]
where $\partial^\alpha = \eta^{\alpha\beta} \partial_\beta$. In fact, from $e \cdot \partial_\alpha = \partial_\alpha$ it follows that $e^\gamma c^{\beta}_\gamma = \delta^\beta_\alpha$, so we have
\[
0 = e^\gamma \partial^\alpha \left( e^\xi c^{\beta}_\gamma \right) = e^\gamma \left( \left( \partial^\alpha e^\xi \right) c^{\beta}_\gamma + e^\xi e^{\alpha\beta} \right) = \partial^\alpha e^\beta + e^\gamma e^\xi c^{\alpha\beta}_\gamma,
\]
which yields the required relation
\[
\partial^\alpha e^\beta = -e^\gamma e^\xi c^{\alpha\beta}_\gamma = \partial^\beta e^\alpha.
\]
Here we use the notation $c^{\alpha\beta}_\gamma = \partial_\xi c^{\alpha\beta}_{\gamma\xi}$. The validity of the second relation of (2.11) follows from the fact that
\[
g^{\alpha\beta} \frac{\partial \varphi}{\partial v^\beta} = E^\xi c^{\alpha\beta}_{\gamma\xi} \eta^{\gamma\beta} \frac{\partial \varphi}{\partial v^\beta} = E^\xi c^{\alpha\beta}_{\gamma\xi} e^\gamma = E^\xi c^{\alpha\beta}_{\gamma\xi} e^\gamma = E^\xi \delta^\alpha_\xi = E^\alpha.
\]
The proposition is proved. \(\square\)
Proposition 2.2. For a generalized Frobenius manifold the following identities hold true:

\[ [E, e] = -e, \quad \nabla(E, e) = (1 - d)e. \]  

(2.14)

Proof. The first identity of (2.14) follows from (2.1). By using this identity and (2.12) we obtain

\[ \eta_{\alpha\beta} E^\alpha \partial^\gamma e^\beta = \eta_{\alpha\beta} E^\alpha \partial^\beta e^\gamma = -e^\gamma + e^\alpha \partial^\alpha E^\gamma \]

\[ = -e^\gamma + e^\gamma \left( \frac{2 - d}{2} - \mu_\gamma \right) = -\left( \frac{d}{2} + \mu_\gamma \right) e^\gamma. \]  

(2.15)

Thus from the relation (2.5) it follows that

\[ \partial^\gamma (E, e) = (\partial^\gamma E^\alpha) e^\beta \eta_{\alpha\beta} + E^\alpha (\partial^\gamma e^\beta) \eta_{\alpha\beta} = \eta_{\gamma\xi} (\partial^\xi E^\alpha) e^\beta \eta_{\alpha\beta} - \left( \frac{d}{2} + \mu_\gamma \right) e^\gamma, \]

hence the second identity of (2.14) also holds true. The Proposition is proved. \[ \square \]

From the second identity of (2.14) we see that the function \( \varphi \) that is described in Proposition 2.1 can be taken as

\[ \varphi = \frac{(E, e)}{1 - d} \]

when \( d \neq 1 \).

As for a usual Frobenius manifold, on a generalized Frobenius manifold \( M \) we also have the deformed flat connection \( \tilde{\nabla} \) defined by

\[ \tilde{\nabla} u v = \nabla u v + z u \cdot v, \quad \forall u, v \in \text{Vect}(M). \]

It can be extended to a flat affine connection on \( M \times \mathbb{C}^* \) by defining

\[ \tilde{\nabla} u \left( \frac{d}{dz} \right) = 0, \quad \tilde{\nabla} \frac{d}{dz} = 0, \quad \tilde{\nabla} \frac{d}{\partial z} v = \partial_z v + E \cdot v - \frac{1}{z} \mu v, \]

here \( u, v \) are vector fields on \( M \times \mathbb{C}^* \) with zero components along \( \mathbb{C}^* \). We can choose a system of deformed flat coordinates \( \tilde{v}^1(v, z), \ldots, \tilde{v}^n(v, z) \) of \( M \) which have the form

\[ (\tilde{v}_1(v, z), \ldots, \tilde{v}_n(v, z)) = (\theta_1(v, z), \ldots, \theta_n(v, z)) z^u z^R, \]

(2.16)

and such that

\[ \zeta_\alpha = \frac{\partial \tilde{v}_\alpha}{\partial v^\gamma} dv^\gamma + 0 dz, \quad \alpha = 1, \ldots, n \]

yield a basis of solutions of the system \( \tilde{\nabla} \zeta = 0 \). Here the constant matrices \( \mu, R \) are the monodromy data of \( M \) at \( z = 0 \), with \( \mu \) given by (2.4) and \( R = R_1 + \cdots + R_m \) (for a certain \( m \in \mathbb{Z}_+ \)) satisfies the relations

\[ (R_s)_{\beta}^\alpha = 0 \quad \text{if} \quad \mu_\alpha - \mu_\beta \neq s, \]

(2.17)

\[ R_s^T \eta = (-1)^{s+1} \eta R_s, \quad \forall s \geq 1. \]

(2.18)
The functions $\theta_\alpha(v; z)$ are analytic near $z = 0$, and can be represented in the form

$$\theta_\alpha(v; z) = \sum_{p \geq 0} \theta_{\alpha,p}(v) z^p, \quad \alpha = 1, \ldots, n.$$ 

Denote

$$\xi_{\alpha,p} = \xi_{\alpha,p}^\gamma \partial_\gamma = \nabla \theta_{\alpha,p}, \quad \alpha = 1, \ldots, n, \ p \geq 0,$$ 

then these vector fields satisfy the recursion relations

$$\partial_\alpha \xi_{\beta,p+1} = c_{\alpha \beta}^\gamma \xi_{\gamma,p},$$ 

and the quasi-homogeneity condition

$$\partial_E \xi_{\alpha,p} = (p + \mu_\alpha - \mu_\gamma) \xi_{\alpha,p} + \sum_{s=1}^{p} (R_s)^\gamma_\alpha \xi_{\gamma,p-s}.$$ 

We also impose the following normalization conditions on the deformed flat coordinates:

$$\xi_{\alpha}(v; 0) = \xi_{\alpha,0} = \partial_{\alpha},$$

$$\langle \xi_{\alpha}(v; -z), \xi_{\beta}(v; z) \rangle = \eta_{\alpha \beta},$$

where

$$\xi_{\alpha}(v; z) = \sum_{p \geq 0} \xi_{\alpha,p} z^p, \quad \alpha = 1, \ldots, n.$$ 

In general, the equations (2.20)–(2.23) do not uniquely determine the vector fields $\xi_{\alpha,p}$, since we have the following ambiguity in determining $\theta_{\alpha,p+1}$ from $\theta_{\alpha,p}$ for $\alpha = 1, \ldots, n$ by using the recursion relations (2.20):

$$\xi_{\alpha,p+1} \mapsto \xi_{\alpha,p+1} + a_{\alpha,p+1},$$

where the constant vectors $a_{\alpha,p+1} = a_{\alpha,p+1}^\gamma \partial_\gamma$ satisfy the conditions

$$(p + 1 + \mu_\alpha - \mu_\gamma)a_{\alpha,p+1}^\gamma = 0, \quad a_{\alpha,p+1}^\gamma \eta_{\gamma \beta} = (-1)^p a_{\beta,p+1}^\gamma \eta_{\gamma \alpha}.$$ 

In what follows, we will fix a system of solutions $\xi_{\alpha,p}$ of the equations (2.20)–(2.23), from which we will also fix a choice of the functions $\theta_{\alpha,p}$ satisfying (2.19) in the next section.

### 3 The Principal Hierarchy

For a usual Frobenius manifold $M^n$ with flat unity, the functions $\theta_{\alpha,p}$ yield a hierarchy of integrable Hamiltonian systems

$$\frac{\partial v'^\alpha}{\partial p^\beta,q} = \eta^\alpha_\gamma \frac{\partial}{\partial x} \left( \frac{\partial \theta_{\beta,q+1}}{\partial v'^\gamma} \right) = \eta^\alpha_\gamma \frac{\partial^2 \theta_{\beta,q+1}}{\partial v'^\gamma \partial v'^\xi} v'^x, \quad 1 \leq \alpha, \beta \leq n, \ q \geq 0$$

defined on the jet space $J^\infty(M)$, and it is called the Principal Hierarchy of the Frobenius manifold. For a generalized Frobenius manifold we also have such an integrable Hierarchy.
which possesses a bihamiltonian structure, a tau structure and an infinite number of Virasoro symmetries, just as for a Frobenius manifold with flat unity. However, due to the non-flatness of the unit vector field, the flow given by the translation along the spatial variable $x$ does not belong to this hierarchy, and we are no longer able to lift the actions of the Virasoro symmetries to the tau function of this integrable hierarchy. In order to solve this problem, we propose to introduce an additional set of commuting flows to this integrable hierarchy, and we call the integrable hierarchy (3.1) together with these additional flows the Principal Hierarchy of the generalized Frobenius manifold. We construct these additional flows in this section.

In this and the subsequent sections, we will denote by $M$ an $n$-dimensional generalized Frobenius manifold.

**Lemma 3.1.** There exist $\xi_{0,p} \in \text{Vect}(M)$ for $p \geq 0$, and constants $r^\alpha_p \in \mathbb{C}$ for $p \geq 1$, $\alpha = 1, 2, \ldots, n$ such that

\[
\xi_{0,0} = e, \quad \xi_{0,1} = v^\alpha \partial_\alpha,
\]

\[
\partial_\alpha \xi_{0,p} = c^\gamma_{\alpha\beta} \xi_{0,p-1}, \quad \alpha, \gamma = 1, \ldots, n, \quad p \geq 1.
\]

\[
\partial_E \xi_{0,p} = \left( p - \frac{d}{2} - \mu \right) \xi_{0,p} + \sum_{s=1}^{p} r^\lambda_s \xi_{\lambda,p-s}, \quad p \geq 0,
\]

and the constants $r^\alpha_p$ satisfy the conditions

\[
r^\alpha_1 = r^\alpha,
\]

\[
r^\alpha_p \neq 0 \quad \text{only if} \quad \mu_\alpha + \frac{d}{2} = p.
\]

Here $r^\alpha$ are defined by the expression (2.7) of the Euler vector field $E$.

**Proof.** Let us prove the existence of the vector fields $\xi_{0,p}$ and the constants $r^\alpha_p$ by induction on $p$. From (2.14) and the expression (2.7) of the Euler vector field it follows that (3.2)–(3.6) hold true for $p = 0, 1$. Assume that we have already found the vector fields $\xi_{0,p}$ and the constants $r^\alpha_p$ for $1 \leq p \leq k$ and $1 \leq \alpha \leq n$ satisfying the relations (3.2)–(3.6), then by using the associativity equations (2.3) we know the existence of a vector field $\xi_{0,k+1} = \xi^\gamma_{0,k+1} \partial_\gamma$ satisfying the recursion relation (3.3) for $p = k + 1$. From the relation

\[
\partial_E \left( \partial_\alpha \cdot \xi_{0,k} \right)^\lambda
\]

\[
= \left( \partial_E \xi_{0,k}^\lambda \right) + c^\lambda_{\alpha\beta} \partial_E \xi_{0,k}^\beta
\]

\[
= \left( \mu_\alpha + \mu_\beta - \mu_\lambda + \frac{d}{2} \right) c^\lambda_{\alpha\beta} \xi_{0,k}^\beta + c^\lambda_{\alpha\beta} \left( k - \frac{d}{2} - \mu_\beta \right) \xi_{0,k}^\beta + \sum_{s=1}^{k} r^\gamma_s \xi_{\gamma,k-s}^\beta
\]

\[
= (k + \mu_\alpha - \mu_\lambda) c^\lambda_{\alpha\beta} \xi_{0,k}^\beta + \sum_{s=1}^{k} r^\gamma_s \xi_{\gamma,k-s}^\beta
\]

\[
= (k + \mu_\alpha - \mu_\lambda) \partial_\alpha \xi_{0,k+1}^\lambda + \sum_{s=1}^{k} r^\gamma_s \partial_\gamma \xi_{\gamma,k+1-s}^\lambda,
\]
it follows that
\[ \partial_\alpha (\partial_E \xi_{0,k+1}) = \left( 1 - \frac{d}{2} - \mu_\alpha \right) \partial_\alpha \xi_{0,k+1} + \partial_E (\partial_\alpha \cdot \xi_{0,p}) \]
\[ = \left( k + 1 - \frac{d}{2} - \mu \right) \partial_\alpha \xi_{0,k+1} + \sum_{s=1}^{k} r_s^\xi \partial_\alpha \xi_{\varepsilon,k+1-s}. \]

So there exist constants \( r_{k+1}^\alpha \) such that
\[ \partial_E \xi_{0,p+1} = \left( k + 1 - \frac{d}{2} - \mu \right) \partial_\alpha \xi_{0,k+1} + \sum_{s=1}^{k+1} r_s^\xi \partial_\alpha \xi_{\varepsilon,k+1-s} + r_{k+1}^\alpha \partial_e. \]

The last identity can be written in the form
\[ \partial_E \xi_{0,k+1}^\alpha = \left( k + 1 - \frac{d}{2} - \mu \right) \xi_{0,k+1}^\alpha + \sum_{s=1}^{k} r_s^\xi \xi_{\varepsilon,k+1-s} + r_{k+1}^\alpha \]
so we can adjust \( \xi_{0,k+1}^\alpha \), if needed, by adding a constant term so that \( r_{k+1}^\alpha \) satisfies the condition (3.6) with \( p = k + 1 \). The lemma is proved.

When we use the recursion relation (3.3) to find \( \xi_{0,p}^\alpha \) from \( \xi_{0,p-1}^\alpha \), there is the ambiguity \( \xi_{0,p} \mapsto \xi_{0,p} + a_{0,p} \), here the constant vector fields \( a_{0,p} \) satisfy the condition
\[ (p - \frac{d}{2} - \mu) a_{0,p} = 0. \]

In what follows we will fix a choice of the vector fields \( \xi_{0,p} \) for \( p \geq 0 \) that satisfy the requirement of the above lemma.

**Remark 3.2.** For a Frobenius manifold with flat unity, one can choose the flat coordinates such that \( e = \frac{\partial}{\partial v} \), and \( r^\alpha = (R_1)^\alpha_1. \) In such a case, one can choose
\[ \xi_{0,p}(v) := \xi_{1,p}(v), \quad p \geq 0, \]
and the constants \( r_k^\xi := (R_k)_1^1. \)

**Lemma 3.3.** Let \( \xi_{0,-p} = \xi_{0,-p}^\gamma \partial_\gamma \) for \( \forall p \geq 1 \) be vector fields on \( M \) defined recursively by
\[ \xi_{0,0} = e, \quad \xi_{0,-p} = \partial_e \xi_{0,-p+1}, \quad p \geq 1. \]

Then the following relations hold true:
\[ \partial_\alpha \xi_{0,-p}^\gamma = c_{\alpha \beta}^\gamma \xi_{0,-p-1}^\beta, \quad \alpha, \gamma = 1, \ldots, n, \quad p \geq 0, \quad (3.8) \]
\[ \partial_E \xi_{0,-p} = \left( -p - \frac{d}{2} - \mu \right) \xi_{0,-p}, \quad p \geq 0. \quad (3.9) \]
Proof. Let us prove the validity of (3.8) and (3.9) by induction on \( p \geq 0 \). By using (2.12)–(2.14) we have

\[
\partial_c \xi_{0,-k}^\gamma = \partial_c \partial_e \xi_{0,-k+1}^\gamma = \partial_c \left( e^\lambda \partial_\lambda \xi_{0,-k+1}^\gamma \right)
\]

so (3.8) and (3.9) hold true when \( p = 0 \). Now we assume that they also hold true for \( 0 \leq p \leq k - 1 \), then we have

\[
\partial_c \xi_{0,-k}^\gamma = \partial_c \partial_e \xi_{0,-k+1}^\gamma = \partial_c \left( e^\lambda \partial_\lambda \xi_{0,-k+1}^\gamma \right)
\]

and

\[
\partial_E \xi_{0,0} = \partial_E e^\gamma \partial_\gamma = -e + e^\gamma \partial_\gamma E = -e + e^\gamma \left( 1 - \frac{d}{2} - \mu \right) \partial_\gamma
\]

so (3.8) and (3.9) hold true when \( p = 0 \). Now we assume that they also hold true for \( 0 \leq p \leq k - 1 \), then we have

\[
\partial_c \xi_{0,-k}^\gamma = \partial_c \partial_e \xi_{0,-k+1}^\gamma = \partial_c \left( e^\lambda \partial_\lambda \xi_{0,-k+1}^\gamma \right)
\]

and

\[
\partial_E \xi_{0,-k} = \partial_E \partial_c \xi_{0,-k+1} = \partial_E \partial_c \left( e^\gamma \partial_\gamma \xi_{0,-k+1} \right)
\]

hence (3.8) and (3.9) also hold true when \( p = k \). The lemma is proved.

Remark 3.4. For a Frobenius manifold with flat unity, the vector fields \( \xi_{0,-p} \) are trivial. Now let us fix a choice of the functions \( \theta_{\alpha,p} \) for \( \alpha = 1, \ldots, n, p \geq 0 \) that satisfy (2.19) by the following proposition.

Proposition 3.5. Let \( \xi_{\alpha,p} \in \text{Vect}(M), \alpha = 1, \ldots, n, p \geq 0 \) be a fixed solution of the equations (2.20)–(2.23), and \( \xi_{0,p} \in \text{Vect}(M), p \geq 0 \) satisfy the conditions of Lemma 3.1 then the functions \( \theta_{\alpha,p} \) defined by

\[
\theta_{\alpha,p} = \sum_{k=0}^{p} (-1)^k \langle \xi_{0,k+1}, \xi_{\alpha,p-k} \rangle, \quad 1 \leq \alpha \leq n, \ p \geq 0
\]
satisfy the relation (2.19), and they also satisfy the quasi-homogeneity condition

$$\partial_E \theta_{\alpha, p} = \left( p + 1 - \frac{d}{2} + \mu_\alpha \right) \theta_{\alpha, p} + \sum_{s=1}^{p} \theta_{\varepsilon, p-s} (R_s)_{\alpha}^{\varepsilon} + (-1)^{p} r_p^{\varepsilon}_{p+1} \eta_{\alpha},$$  \hspace{1cm} (3.11)$$

where the constants $r_p^{\varepsilon}$ are given by Lemma 3.1.

**Proof.** By using the recursion relations (2.20) and (3.3) we obtain

$$\partial_\beta \theta_{\alpha, p} = \sum_{k=0}^{p} (-1)^k \left( \langle \partial_\beta \cdot \xi_{0,k}, \xi_{\alpha,p-k} \rangle + \langle \xi_{0,k+1}, \theta_{\alpha,p-k} \rangle \right)$$

$$= \langle \partial_\beta \cdot \xi_{0,0}, \xi_{\alpha,p} \rangle = \langle \partial_\beta \cdot \xi_{\alpha,p} \rangle = \eta_{\beta \gamma} \xi_{\alpha,p},$$

so (2.19) holds true. To check the quasi-homogeneity condition, we use (2.5), (2.21), (2.23), and (3.4) to arrive at

$$\partial_E \theta_{\alpha, p} = \sum_{k=0}^{p} (-1)^k \left( \langle \partial_E \xi_{0,k+1}, \xi_{\alpha,p-k} \rangle + \langle \xi_{0,k+1}, \partial_E \xi_{\alpha,p-k} \rangle \right)$$

$$= \sum_{k=0}^{p} (-1)^k \left( (k + 1 - \frac{d}{2} - \mu) \xi_{0,k+1}, \xi_{\alpha,p-k} \right) + \sum_{s=1}^{k+1} r_s^{\varepsilon} \langle \xi_{\varepsilon,k+1-s}, \xi_{\alpha,p-k} \rangle$$

$$+ \sum_{k=0}^{p} (-1)^k \left( \langle \xi_{0,k+1}, (p-k+\mu_\alpha - \mu) \xi_{\alpha,p-k} \rangle + \sum_{s=1}^{p-k} (R_s)_{\alpha}^{\varepsilon} \langle \xi_{0,k+1}, \xi_{\varepsilon,p-k-s} \rangle \right)$$

$$= \left( p + 1 - \frac{d}{2} + \mu_\alpha \right) \sum_{k=0}^{p} (-1)^k \langle \xi_{0,k+1}, \xi_{\alpha,p-k} \rangle$$

$$+ \sum_{s=1}^{p+1} r_s^{\varepsilon} \sum_{k=s-1}^{p} (-1)^k \langle \xi_{\varepsilon,k+1-s}, \xi_{\alpha,p-k} \rangle + \sum_{s=1}^{p} (R_s)_{\alpha}^{\varepsilon} \sum_{k=0}^{p-s} \langle \xi_{0,k+1}, \xi_{\varepsilon,p-k-s} \rangle$$

$$= \left( p + 1 - \frac{d}{2} + \mu_\alpha \right) \theta_{\alpha, p} + \sum_{s=1}^{p} (R_s)_{\alpha}^{\varepsilon} \theta_{\varepsilon,p-s} + (-1)^{p} r_p^{\varepsilon}_{p+1} \eta_{\alpha}.$$

The proposition is proved.

**Proposition 3.6.** There exist functions $\{\theta_{0,p}\}_{p \in \mathbb{Z}}$ on $M$ which satisfy the relation

$$\nabla \theta_{0,p} = \xi_{0,p}, \hspace{1cm} p \in \mathbb{Z}$$  \hspace{1cm} (3.12)$$

and the quasi-homogeneity condition

$$\partial_E \theta_{0,p} = (p - d + 1) \theta_{0,p} + \sum_{s=1}^{p} r_s^{\varepsilon} \theta_{\varepsilon,p-s} + c_p, \hspace{1cm} p \in \mathbb{Z},$$  \hspace{1cm} (3.13)$$

where we assume that $\theta_{\varepsilon,p} := 0$ if $p < 0$ and $\varepsilon \neq 0$, and $c_p$ are certain constants which are possibly non-zero only if $p = d - 1 \in \mathbb{Z}$ and $p$ is even.
Proof. The existence of functions $\theta_{0,p}$ for $p \in \mathbb{Z}$ satisfying (3.12) and (3.13) follows from the recursion relations (3.3), (3.8) and the associativity equations (2.3). In particular, when $p = 2q + 1$ we can take

$$\theta_{0,p} = \begin{cases} \frac{1}{2} \sum_{k=0}^{p-1} (-1)^k \langle \xi_0, p-k, \xi_{0,k+1} \rangle, & p > 0, \\ \frac{1}{2} \sum_{k=0}^{-p-1} (-1)^k \langle \xi_0, p+1+k, \xi_0, -k \rangle, & p < 0, \end{cases} \quad (3.14)$$

then it follows from (3.4) and (3.9) that $c_p = 0$. When $p$ is even and $p \neq d - 1$, we can adjust $\theta_{0,p}$ by adding a constant so that $c_p$ vanishes in the relation (3.13). The proposition is proved.

From (3.10), (3.14) and (2.22), (3.2), (3.7) we know that $\theta_{0,0} = \varphi$, $\theta_{0,1} = \frac{1}{2} v^\gamma v_\gamma$, $\theta_{0,-1} = \frac{1}{2} e^\gamma e_\gamma$, $\theta_{\alpha,0} = v_\alpha$, $\alpha = 1, \ldots, n$. (3.15)

Here $\varphi$ is defined in Proposition 2.1. When $d \neq 1$, we can adjust the choice of $\varphi$ by adding to it a constant so that $\theta_{0,0} = \varphi$ satisfies the quasi-homogeneity condition (3.13) with $c_0 = 0$.

Now we are ready to introduce the Principal Hierarchy of the generalized Frobenius manifold $M$.

**Definition 3.1.** The Principal Hierarchy of $M$ is given by the following family of evolutionary PDEs:

$$\begin{cases} \frac{\partial v_\alpha}{\partial t_{\beta,q}} = \eta^\gamma \frac{\partial}{\partial x} \left( \frac{\partial \theta_{\beta,q+1}}{\partial v_\gamma} \right), & 1 \leq \beta \leq n, q \geq 0, \\ \frac{\partial v_0}{\partial t_{0,q}} = \eta^\gamma \frac{\partial}{\partial x} \left( \frac{\partial \theta_{0,q+1}}{\partial v_0} \right), & q \in \mathbb{Z}, \end{cases} \quad (3.16)$$

where $\theta_{\alpha,p} \geq 0$, $\theta_{0,q} \in \mathbb{Z}$ are defined in Propositions 3.3, 3.6.

From the associativity equations (2.3) it follows that the flows of the Principal Hierarchy pairwise commute, and from (3.15) we know that the Principal Hierarchy contains the flow given by the translation along the spatial variable $x$, i.e.,

$$\frac{\partial v_\alpha}{\partial t^{0,0}} = v_\alpha^x,$$

so in what follows we will identify the time variable $t^{0,0}$ with the spatial variable $x$, i.e.,

$t^{0,0} = x$.

We note that the Principal Hierarchy can also be written in the form

$$\begin{cases} \frac{\partial v}{\partial t^{\alpha,p}} = \partial_x \nabla \theta_{\alpha,p+1} = \nabla \theta_{\alpha,p} \cdot v_x, & 1 \leq \alpha \leq n, p \geq 0, \\ \frac{\partial v}{\partial t^{0,q}} = \partial_x \nabla \theta_{0,p+1} = \nabla \theta_{0,q} \cdot v_x, & q \in \mathbb{Z}, \end{cases} \quad (3.17)$$

where $v := (v^1, \ldots, v^n)^T$, and $v_x := (v_x^1, \ldots, v_x^n)^T$. 

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On the jet space $J^\infty(M)$ we also have a bihamiltonian structure given by the compatible Hamiltonian operators $P_a = (P_a^{\alpha\beta})$, $a = 1, 2$ with

$$P_1^{\alpha\beta} = \eta^{\alpha\beta} \partial_x, \quad P_2^{\alpha\beta} = g^{\alpha\beta} \partial_x + \Gamma^{\alpha\beta}_\gamma v^\gamma_x,$$

(3.18)

here the intersection form $\eta^{\alpha\beta}$ and the contravariant coefficients of the Levi-Civita connection of $g$ are given in (2.9) and (2.10). The Principal Hierarchy (3.16) is a bihamiltonian hierarchy with respect to this bihamiltonian structure, i.e., its flows are Hamiltonian systems of the form

$$\frac{\partial v^\alpha}{\partial t_j,q} = P_1^{\alpha\beta} \frac{\delta H}{\delta v^\beta}, \quad \alpha = 1, \ldots, n, \quad (j,q) \in \mathcal{I},$$

and they satisfy the bihamiltonian recursion relations

$$P_2^{\alpha\gamma} \frac{\delta H_{\beta,q-1}}{\delta v^\gamma} = (q + \frac{1}{2} + \mu_\beta) P_1^{\alpha\gamma} \frac{\delta H_{\beta,q}}{\delta v^\gamma} + \sum_{s=1}^{q} (R_s)_\lambda^{\alpha\gamma} P_1^{\lambda\gamma} \frac{\delta H_{\lambda,q-s}}{\delta v^\gamma},$$

$$P_2^{\alpha\gamma} \frac{\delta H_{0,p-1}}{\delta v^\gamma} = (p + \frac{1}{2} - \frac{d}{2}) P_1^{\alpha\gamma} \frac{\delta H_{0,p}}{\delta v^\gamma} + \sum_{s=1}^{p} r^\lambda P_1^{\alpha\gamma} \frac{\delta H_{\lambda,p-s}}{\delta v^\gamma},$$

for $\alpha, \beta = 1, \ldots, n$, $q \geq 0$, $p \in \mathbb{Z}$. Here the index set $\mathcal{I}$ is defined by

$$\mathcal{I} = \left(\{1, \ldots, n\} \times \mathbb{Z}_{\geq 0}\right) \cup \left(\{0\} \times \mathbb{Z}\right),$$

(3.19)

and the Hamiltonians are given by

$$H_{j,q} = \int \theta_{j,q+1}(v(x)) \, dx.$$

4 The associated $(n+2)$-dimensional Frobenius manifold

In this section, we are to associate with $M$ an $(n+2)$-dimensional Frobenius manifold $\tilde{M}$ with flat unity following the construction of Mironov and Taimanov that was given in [42]. This Frobenius manifold structure will be used in our consideration of the Virasoro symmetries of the Principal Hierarchy of $M$ and their linearization.

**Lemma 4.1** (see Lemma 3 of [42]). Let $F = F(v^1, \ldots, v^n)$, $E = E^\alpha \partial_\alpha$ be the potential and the Euler vector field of $M$, and $d$ be the charge of $M$, then the function

$$\tilde{F}(v^0, v^1, \ldots, v^n, v^{n+1}) := \frac{1}{2}(v^0)^2 v^{n+1} + \frac{1}{2} v^0 v^a v_a + F(v^1, \ldots, v^n)$$

yields a Frobenius manifold structure of charge $d$ on $\tilde{M} = \mathbb{C} \times M \times \mathbb{C}$ with flat coordinates $v^0, v^1, \ldots, v^n, v^{n+1}$ and flat unity $\tilde{\eta} = \frac{\partial}{\partial v^0}$. The flat metric and the Euler vector field are given by

$$\tilde{\eta} := \begin{pmatrix} \eta \\ 1 \end{pmatrix}$$

(4.1)
and
\[ \tilde{E} = v^0 \frac{\partial}{\partial v^0} + E + ((1 - d)v^{n+1} + c_0) \frac{\partial}{\partial v^{n+1}}, \]  
(4.2)

where the constant \( c_0 \) is defined in Lemma 3.3, and \( v^0, v^{n+1} \) are the coordinates of the first and the last components \( C \) of \( \tilde{M} \).

**Remark 4.2.** The Frobenius manifold \( \tilde{M} \) is not semisimple, since the vector field \( \frac{\partial}{\partial v^{n+1}} \) is nilpotent.

**Remark 4.3.** The map
\[ \nu : M \rightarrow \tilde{M}, \quad (v^1, \ldots, v^n) \mapsto (0, v^1, \ldots, v^n, \theta_{0,0}(v^1, \ldots, v^n)) \]
preserves the multiplication and the flat metrics of the (generalized) Frobenius manifolds, so we can view \( M \) as a Frobenius submanifold of \( \tilde{M} \).

The monodromy data \( \tilde{\mu}, \tilde{R} = \tilde{R}_1 + \tilde{R}_2 + \ldots \) of the Frobenius manifold \( \tilde{M} \) at \( z = 0 \) can be represented in terms of that of \( M \) as follows:

\[ \tilde{\mu} = \begin{pmatrix} \mu_0 & \mu \\ \mu_{n+1} & \mu \end{pmatrix}, \quad \tilde{R}_s = \begin{pmatrix} 0 & 0 & 0 \\ r_s & R_s & 0 \\ r_{s-1}^\dagger & r_s^\dagger & 0 \end{pmatrix}, \quad s \geq 1, \]
(4.3)

where \( \mu \) is defined in (2.4),
\[ \mu_0 = -\frac{d}{2}, \quad \mu_{n+1} = \frac{d}{2}, \]
and \( r_s, r_s^\dagger \) are given by the column and row vectors
\[ r_s = (r_s^1, r_s^2, \ldots, r_s^n)^T, \quad r_s^\dagger = (r_{s,1}, \ldots, r_{s,n}) \]
respectively with
\[ r_{s,\alpha} := (-1)^{s+1} r_{s}^{\beta} \eta_{\alpha\beta}. \]

It is easy to check that the matrices \( \tilde{\eta}, \tilde{\mu} \) and \( \tilde{R}_s \) satisfy the relations (2.5), (2.17) and (2.18). The following proposition justifies our assertion that \( \tilde{\mu}, \tilde{R} \) are indeed the monodromy data of the Frobenius manifold \( \tilde{M} \) at \( z = 0 \).

**Proposition 4.4** (47). Let \( \tilde{M} \) be the \((n+2)\)-dimensional Frobenius manifold associated with \( M \), and \( \theta_{0,p}, \theta_{0,0} \) be the functions on \( M \) defined in the previous sections. Introduce the following function on \( \tilde{M} \):

\[ \tilde{\theta}_{0,p} = \sum_{k=0}^{p-1} \frac{(v^0)^k}{k!} \theta_{0,p-k} + v^{n+1} \frac{(v^0)^p}{p!}, \quad \tilde{\theta}_{n+1,p} = \frac{(v^0)^{p+1}}{(p+1)!}, \quad p \geq 0, \]
(4.5)

\[ \tilde{\theta}_{\alpha,p} = \sum_{k=0}^{p} \frac{(v^0)^k}{k!} \theta_{\alpha,p-k}, \quad 1 \leq \alpha \leq n, \quad p \geq 0. \]
(4.6)

Then
\[ (\tilde{\theta}_0(z), \ldots, \tilde{\theta}_{n+1}(z)) \tilde{\mu} \tilde{R} \]
(4.7)
yields a system of deformed flat coordinates of $\tilde{M}$, where

$$\tilde{\theta}_i(z) := \sum_{p \geq 0} \tilde{\theta}_{i,p} z^p, \quad 0 \leq i \leq n + 1. \quad (4.8)$$

Moreover, the functions $\tilde{\theta}_{i,p}$ satisfy the relations

$$\tilde{\theta}_{i,0} = \sum_{j=0}^{n+1} \tilde{\eta}_{ij} v^j, \quad \tilde{\theta}_{i,p} = \partial^2 \tilde{\theta}_{i,p+1} / \partial v^0, \quad 0 \leq i \leq n + 1, \quad p \geq 0. \quad (4.9)$$

Proof. To prove that (4.7) gives a system of deformed flat coordinates of $\tilde{M}$, we need to show the validity of the following relations:

$$\frac{\partial^2 \tilde{\theta}_{i,p+1}}{\partial v^i \partial v^j} = \sum_{k=0}^{n+1} \tilde{c}_{ij} \frac{\partial \tilde{\theta}_{i,p}}{\partial v^k}, \quad 0 \leq i, j, \ell \leq n + 1, \quad p \geq 0,$$

$$\partial_E \tilde{\nabla} \tilde{\theta}_{i,p} = (p + \tilde{\mu}_i - \tilde{\mu}) \tilde{\nabla} \tilde{\theta}_{i,p} + \sum_{s=1}^{p} \tilde{\nabla} \tilde{\theta}_{j,p-s}(\tilde{R}_s)^j_i, \quad 0 \leq i \leq n + 1, \quad p \geq 0,$$

where $E$ is given by (4.2), $\tilde{\nabla} \tilde{\theta}_{i,p}$ are the gradients of $\tilde{\theta}_{i,p}$ with respect to the metric $\tilde{\eta}$, and

$$\tilde{c}_{ij} = \sum_{k=0}^{n+1} \tilde{\eta}_{ik} \frac{\partial \tilde{\nabla} \tilde{F}}{\partial v^l \partial v^j \partial v^k}, \quad 0 \leq i, j, k \leq n + 1.$$

These relations can be proved by a straightforward calculation, so we omit it here. The proposition is proved. \(\square\)

Remark 4.5. Let $\tilde{f}$ be a function on $\tilde{M}$. Define the restriction of $\tilde{f}$ to $M$ by

$$\tilde{f}_{| M} := v^* (\tilde{f}), \quad (4.10)$$

where the map $v: M \hookrightarrow \tilde{M}$ is introduced in Remark 4.3. Then it is clear that

$$v^0_{| M} = 0, \quad v^a_{| M} = v^a, \quad v^{n+1}_{| M} = \theta_{0,0}, \quad (4.11)$$

and

$$\tilde{\theta}_{0,p}_{| M} = \theta_{0,p}, \quad \tilde{\theta}_{a,p}_{| M} = \theta_{a,p}, \quad \tilde{\theta}_{n+1,p}_{| M} = 0, \quad p \geq 0. \quad (4.12)$$

We note that the construction of $\tilde{\theta}_{i,p}$ and $\tilde{R}$ given above does not involve the functions $\theta_{0,-p}$ and the constants $c_{-p}$ for $p > 0$, and we also have the following identities which can be easily proved:

$$\frac{\partial}{\partial v^0} \left( \tilde{\theta}_{0}(z), \tilde{\theta}_{\gamma}(z), \tilde{\theta}_{n+1}(z) \right)_{| M} = (z\theta_{0}(z), z\theta_{\gamma}(z), 1), \quad (4.13)$$

$$\frac{\partial}{\partial v^a} \left( \tilde{\theta}_{0}(z), \tilde{\theta}_{\gamma}(z), \tilde{\theta}_{n+1}(z) \right)_{| M} = \left( \frac{\partial \theta_{0}(z)}{\partial v^a} - e_{\alpha}, \frac{\partial \theta_{\gamma}(z)}{\partial v^a}, 0 \right), \quad (4.14)$$

where $\theta_{0}(z) = \sum_{p=0}^{\infty} \theta_{0,p} z^p$. 

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5 Tau-cover of the Principal Hierarchy

In this section, we introduce the tau structure and tau-cover of the Principal Hierarchy of the generalized Frobenius manifold \( M \) following [14, 23].

Let \( \{ \theta_{i,p} \mid (i,p) \in I \} \) be the set of functions on \( M \) given by Propositions 3.5, 3.6, where the index set \( I \) is defined by (3.19). We introduce a set of functions \( \{ \Omega_{i,p,j,q} \mid (i,p),(j,q) \in I \} \) on \( M \) as follows:

\[
\begin{align*}
\Omega_{\alpha,p;\beta,q} &= \sum_{k=0}^{q} (-1)^k \langle \nabla \theta_{\alpha,p+k+1}, \nabla \theta_{\beta,q-k} \rangle, \quad (5.1) \\
\Omega_{0,p';\beta,q} &= \sum_{k=0}^{q} (-1)^k \langle \nabla \theta_{0,p'+k+1}, \nabla \theta_{\beta,q-k} \rangle, \quad (5.2) \\
\Omega_{\alpha,p;0,q'} &= \sum_{k=0}^{p} (-1)^k \langle \nabla \theta_{\alpha,p-k}, \nabla \theta_{0,q'+k+1} \rangle, \quad (5.3) \\
\Omega_{0,p';0,q'} &= \left\{ \begin{array}{ll}
\sum_{k=0}^{p'} (-1)^k \langle \nabla \theta_{0,p'-k}, \nabla \theta_{0,q'+k+1} \rangle + (-1)^{p'} \theta_{0,p'+q'} & , \quad p' \geq 0, \\
\sum_{k=0}^{p'-1} (-1)^k \langle \nabla \theta_{0,p'+k+1}, \nabla \theta_{0,q'-k} \rangle + (-1)^{p'} \theta_{0,p'+q'} & , \quad p' < 0.
\end{array} \right. \quad (5.4)
\end{align*}
\]

Here \( \alpha, \beta = 1, \ldots, n \), \( p,q \in \mathbb{Z}_{\geq 0} \) and \( p', q' \in \mathbb{Z} \). We also define \( \Omega_{i,p,j,q} := 0 \) if \( i \neq 0 \), \( p < 0 \) or \( j \neq 0 \), \( q < 0 \).

Proposition 5.1. The functions \( \Omega_{i,p,j,q} \) defined above satisfy the following identities:

\[
\begin{align*}
\Omega_{0,0;i,p} &= \theta_{i,p}, \quad \Omega_{i,p;j,q} = \Omega_{j,q;i,p}, \quad \nabla \Omega_{i,p;j,q} = \nabla \theta_{i,p} \cdot \nabla \theta_{j,q}, \quad (5.5)
\end{align*}
\]

and for any solution \( v(t) = (v^1(t), \ldots, v^n(t)) \) of the Principal Hierarchy (3.16), we have

\[
\frac{\partial \theta_{i,p}}{\partial t^q} = \partial_{q} \Omega_{i,p;j,q}, \quad (5.6)
\]

here the independent variables \( v^\alpha \) of the functions \( \theta_{i,p} \) and \( \Omega_{i,p;j,q} \) are understood to be substituted by \( v^\alpha(t) \), and we denote \( t = (t^p)_{(i,p) \in I} \).

Proof. The identities in (5.5) follow from (2.23), (3.14) and the recursion relations (2.20), (3.3) and (3.8). The equations in (5.6) also follow from these recursion relations and the definition of the Principal Hierarchy. We omit the details of the verification here.

We note that the functions \( \Omega_{\alpha,p;\beta,q} \) coincide with the ones defined by the following formula of Dubrovin [14]:

\[
\Omega_{\alpha,\beta}(z,w) = \frac{\langle \nabla \theta_{\alpha}(z), \nabla \theta_{\beta}(w) \rangle - \eta_{\alpha\beta}}{z+w}, \quad (5.7)
\]

where the generating functions \( \Omega_{\alpha,\beta}(z,w) \) are defined by

\[
\Omega_{\alpha,\beta}(z,w) := \sum_{p,q \geq 0} \Omega_{\alpha,p;\beta,q} z^p w^q, \quad 1 \leq \alpha, \beta \leq n. \quad (5.8)
\]

Now let us introduce the tau-cover of the Principal Hierarchy of \( M \) following [19].
Definition 5.1. The system of PDEs
\[
\frac{\partial f}{\partial t^{i,p}} = f_{i,p}, \quad \frac{\partial f_{i,p}}{\partial t^{j,q}} = \Omega_{i,p,j,q}, \quad \frac{\partial v^{\alpha}}{\partial t^{j,q}} = \eta^{\alpha \gamma} \partial_x \Omega_{\gamma,0,j,q}
\] (5.9)
is called the tau-cover of the Principal Hierarchy (3.16) of \( M \), where \((i, p), (j, q) \in I\) and \(\alpha = 1, \ldots, n\).

For a given solution
\[
\{ f(t), f_{i,p}(t), v^{\alpha}(t) \mid (i, p) \in I, \alpha = 1, \ldots, n \}
\]
of (5.9), we call the function \( \tau(t) = e^f \) the tau function of the solution \( v(t) = (v^1(t), \ldots, v^n(t)) \) of the Principal Hierarchy (3.16). We have the following relation of the tau function with the associated solution of the Principal Hierarchy:
\[
v^{\alpha} = \eta^{\alpha \gamma} \frac{\partial^2 \log \tau}{\partial t^0 \partial t^\gamma, 0} = \eta^{\alpha \gamma} \frac{\partial^2 \log \tau}{\partial x \partial t^\gamma, 0}.
\] (5.10)

We can specify a particular class of solutions of the tau-cover (5.9) of the Principal Hierarchy following the approach of [14, 23]. To this end, we fix a point \( v_0 \in M \) and a set of constants \( c^{i,p}, (i, p) \in I \) with only finitely many of them being nonzero. These constants are also required to satisfy the conditions that
\[
c^{0,0} c^{\alpha,0}(v_0) + c^{\alpha,0} = - \sum_{(i, p) \in I, p \neq 0} c^{i,p} \nabla^{\alpha} \theta_{i,p}(v_0)
\]
and the matrix
\[
\left( \sum_{(i, p) \in I} c^{i,p} \frac{\partial^2 \theta_{i,p}}{\partial v^{\alpha} \partial v^{\beta}}(v_0) \right)
\]
is invertible. Then from [14, 23] we know that the Euler-Lagrange equation
\[
\sum_{(i, p) \in I} \tilde{t}^{i,p} \nabla \theta_{i,p}(v(t)) = 0
\] (5.11)
yields a solution \( v(t) \) of the Principal Hierarchy (3.16) with \( v(t)|_{t=0} = v_0 \).

Here we denote
\[
\tilde{t}^{i,p} = t^{i,p} - c^{i,p}, \quad (i, p) \in I,
\]
and we identify \( t^{0,0} \) with the spatial variable \( x \). For such a solution \( v(t) \) of the Principal Hierarchy, let us introduce, following [14], the genus zero free energy
\[
\mathcal{F}_0(t) := \frac{1}{2} \sum_{(i, p), (j, q) \in I} \tilde{t}^{i,p} \tilde{t}^{j,q} \Omega_{i,p,j,q}(v(t)),
\] (5.12)
then we have a solution of the tau-cover (5.9) of the Principal Hierarchy given by \( v(t) \) and
\[
f(t) = \mathcal{F}_0(t), \quad f_{i,p}(t) = \sum_{(j, q) \in I} \tilde{t}^{j,q} \Omega_{i,p,j,q}(v(t)).
\]
6 Periods of Generalized Frobenius manifolds and Laplace-type integrals

In this section we introduce the notion of periods of the generalized Frobenius manifold $M$, which is a slight modification of the one for a usual Frobenius manifold [14, 15, 23]. We will use a certain basis of periods of $M$ to represent the Virasoro symmetries of the tau-cover of the associated Principal Hierarchy in the next section.

For any complex parameter $\lambda \in \mathbb{C}$ we denote

$$
\Sigma_\lambda := \left\{ v \in M \mid \det (g^{\alpha \beta} - \lambda \eta^{\alpha \beta}) |_v = 0 \right\},
$$

where $g = (g^{\alpha \beta})$ is the intersection form of $M$ defined in (2.8), (2.9), then the inverse matrix $(g^{\alpha \beta}) := (g^{\alpha \beta} - \lambda \eta^{\alpha \beta})^{-1}$ defines a flat metric $(\cdot , \lambda)$ on $M \setminus \Sigma_\lambda$. In the flat coordinates $v^1, \ldots, v^n$ of $M$, the contravariant components of the Levi-Civita connection $\nabla(\lambda)$ of this metric are given by

$$
\nabla(\lambda)^\alpha d v^\beta = (g^{\alpha \gamma} - \lambda \eta^{\alpha \gamma}) \nabla(\lambda)_\gamma d v^\beta = \Gamma^{\alpha \beta \gamma} d v^\gamma,
$$

where $\Gamma^{\alpha \beta \gamma}$ are the contravariant coefficients of the Levi-Civita connection of $g$ given in (2.10). We can find a system of flat local coordinates $p_1(v; \lambda), \ldots, p_n(v; \lambda)$ of $(\cdot , \lambda)$ on an open subset of $M \setminus \Sigma_\lambda$ which satisfy the equations

$$
\partial_\gamma \nabla p_\alpha = 0, \quad \alpha = 1, \ldots, n.
$$

These equations can be represented in the form

$$
\partial_\gamma \nabla p_\alpha = -(U - \lambda I)^{-1} C_\gamma \left( \mu + \frac{1}{2} \right) \nabla p_\alpha,
$$

where $\nabla p_\alpha$ are the gradients of $p_\alpha$ with respect to the metric $\eta$, and $U, C_\gamma$ are the operators of multiplication by the vector fields $E$ and $\partial_\gamma$ respectively, i.e.,

$$
U_\beta^\alpha = E^{\gamma \epsilon} c^\alpha_{\gamma \beta}, \quad (C_\gamma)_\beta^\alpha = c^\alpha_{\gamma \beta}.
$$

It can be verified directly that the functions $p_\alpha$ can be chosen in such a way that the following equations

$$
\frac{\partial}{\partial \lambda} \nabla p_\alpha = (U - \lambda I)^{-1} \left( \mu + \frac{1}{2} \right) \nabla p_\alpha
$$

also hold true. The equations (6.3), (6.5) are called the Gauss-Manin equations of the generalized Frobenius manifold $M$. Solutions of the Gauss-Manin equations are called periods of $M$.

Let $p_1(v; \lambda), \ldots, p_n(v, \lambda)$ be a basis of periods of $M$, then the entries

$$
G^{\alpha \beta} = \left( \frac{\partial}{\partial p_\alpha}, \frac{\partial}{\partial p_\beta} \right)_\lambda
$$
of the Gram matrix \( (G^{\alpha \beta}) \) are constants, and that of the inverse \( (G_{\alpha \beta}) = (G^{\alpha \beta})^{-1} \) of the Gram matrix are given by

\[
G_{\alpha \beta} = \nabla p^T_\alpha \eta (U - \lambda I) \nabla p_\beta.
\]

We note that the Gauss-Manin equations also imply that the periods satisfy the relations

\[
\frac{\partial}{\partial \lambda} \nabla p_\alpha = -\partial_e \nabla p_\alpha,
\]

where \( e \) is the unit vector field of \( M \). For a usual Frobenius manifold with flat unity, the periods \( p_\alpha \) can be chosen in a way such that \( \frac{\partial}{\partial \lambda} p_\alpha = -\partial_e p_\alpha \); however, this may not hold true for a generalized Frobenius manifold because of the non-flatness of the unity.

Let us introduce, following [16, 23], the regularized periods

\[
\left( p_1^{(\nu)}(v; \lambda), \ldots, p_n^{(\nu)}(v; \lambda) \right) = \lim_{\nu \to 0} \left( p_1^{(\nu)}(v), \ldots, p_n^{(\nu)}(v; \lambda) \right),
\]

here the Laplace integral in the right hand side of (6.7) is understood as an analytic continuation via Gamma functions by the following formula:

\[
\int_0^\infty e^{-\lambda z} z^{p+\mu+\nu+\frac{1}{2}} R^R d z = \sum_{s \geq 0} \left[ e^{R \partial_\lambda} \right]_s \Gamma \left( p + \mu + \nu + s + \frac{1}{2} \right) \lambda^{-(p+\mu+\nu+s+\frac{1}{2})} \lambda^{-R}
\]

for \( p \in Z_{\geq 0}, \nu \in \mathbb{C} \). The components \( [P(\mu, R)]_s \) for any polynomial \( P(\mu, R) \) of \( \mu, R \) are given by

\[
[P(\mu, R)]_s := \sum_{\lambda \in \text{Spec} \mu} \pi_{\lambda+s} P(\mu, R) \pi_{\lambda},
\]

where \( \pi_{\lambda}: V \to V_{\lambda} \) is the projection of \( V = \mathbb{C}^n \) to the eigenspace \( V_{\lambda} \) of the operator \( \mu \).

In the non-resonant case where the spectrum of \( \mu \) of \( M \) contains no half integers, a bases of periods of the generalized Frobenius manifold \( M \) can be chosen by using Laplace-type integrals as follows [16, 23]:

\[
(p_1(v; \lambda), \ldots, p_n(v; \lambda)) = \lim_{\nu \to 0} \left( p_1^{(\nu)}(v), \ldots, p_n^{(\nu)}(v; \lambda) \right),
\]

and the associated Gram matrix is given by

\[
(G^{\alpha \beta}) := -\frac{1}{2\pi} \left( e^{\pi i R e^{\pi i \mu}} + e^{-\pi i R e^{-\pi i \mu}} \right) \eta^{-1}.
\]

In the next section we also need to use the periods of the \((n+2)\)-dimensional Frobenius manifold \( \tilde{M} \) associated to \( M \). Recall that the monodromy data \( \tilde{\mu} \) and \( \tilde{R} \) of \( \tilde{M} \) are defined in (4.3), and the functions \( \tilde{\theta}_{0,p}, \ldots, \tilde{\theta}_{n+1,p} \) for \( \tilde{M} \) are defined in (4.8). If the spectrum of \( \tilde{\mu} \) contains no half integers, a basis of periods of \( \tilde{M} \) can be chosen as follows:

\[
(\tilde{p}_0, \tilde{p}_1, \ldots, \tilde{p}_n, \tilde{p}_{n+1}) = \lim_{\nu \to 0} \left( \tilde{p}_0^{(\nu)}, \tilde{p}_1^{(\nu)}, \ldots, \tilde{p}_n^{(\nu)}, \tilde{p}_{n+1}^{(\nu)} \right),
\]
where the regularized periods are defined as in (6.7) by
\[
(\tilde{p}_0^{(\nu)}, \tilde{p}_1^{(\nu)}, \ldots, \tilde{p}_n^{(\nu)}, \tilde{p}_{n+1}^{(\nu)}) = \int_{0}^{+\infty} \frac{dz}{\sqrt{z}} e^{-\lambda z} \left( \tilde{\theta}_0(z), \tilde{\theta}_1(z), \ldots, \tilde{\theta}_n(z), \tilde{\theta}_{n+1}(z) \right) \eta z^\mu + \nu z^R, \tag{6.13}
\]
and the associated Gram matrix is given by
\[
(G^{ij}) = \frac{1}{2\pi} \left( e^{\pi i R} + e^{-\pi i R} \right) \eta^{-1}, \tag{6.14}
\]
here \(\eta\) is defined in (4.1).

The following formula on Laplace-type integrals is useful.

**Lemma 6.1** \([23, 36]\). Let \(\{a_p^{(\alpha)} \mid p \in \mathbb{Z}, \alpha = 1, 2, \ldots, n\}\) be a family of formal variables in a certain \(\mathbb{C}\)-algebra, and \(a_p := (a_p^{(1)}, \ldots, a_p^{(n)})^T\). Define the row vector-valued functions
\[
\phi^{(\nu)}(a; \lambda) := \int_{0}^{\infty} \frac{dz}{\sqrt{z}} e^{-\lambda z} \sum_{p \in \mathbb{Z}} a_p^T \eta z^p z^R, \tag{6.15}
\]
then the following identity holds true:
\[
-\frac{1}{2} \frac{\partial \phi^{(\nu)}(a; \lambda)}{\partial \lambda} G^{\alpha\beta}(\nu) \frac{\partial \phi^{(-\nu)}(a; \lambda)}{\partial \lambda} = \frac{1}{2} \sum_{p,q \in \mathbb{Z}, s \geq 0} \frac{1}{\lambda^{p+q+s+3}} a_p^T \eta N_{p,q}(s; \nu) a_q, \tag{6.16}
\]
where
\[
(G^{\alpha\beta}(\nu)) = \frac{1}{2\pi} \left( e^{\pi i R} e^{\pi i \nu} + e^{-\pi i R} e^{-\pi i \nu} \right) \eta^{-1}, \tag{6.17}
\]
\[
N_{p,q}(s; \nu) = \frac{1}{\pi} \left[ e^{R \eta} \right]_s \left( \Gamma(p + \mu + \nu + s + \frac{3}{2}) \cos \left( \pi (\mu + \nu) \right) \Gamma(q - \mu - \nu + \frac{3}{2}) \right). \tag{6.18}
\]

We note that \(N_{p,q}(s; \nu)\) are polynomials in \(\nu\), so the left hand side of (6.16) is well-defined for any generalized Frobenius manifold, even though \(\phi^{(\nu)}\) may be singular at \(\nu = 0\) in the resonant case when the spectrum of \(\mu\) contains a half integer.

We also introduce the following \((n + 2)\)-dimensional row vector-valued function for the \((n + 2)\)-dimensional Frobenius manifold \(\tilde{M}\) associated to \(M\) as follows:
\[
\tilde{\phi}^{(\nu)}(a; \lambda) := \int_{0}^{\infty} \frac{dz}{\sqrt{z}} e^{-\lambda z} \sum_{p \in \mathbb{Z}} a_p^T \tilde{\eta} z^p \tilde{\eta} z^R, \tag{6.19}
\]
where \(a_p = (a_p^0, \ldots, a_p^{n+1})^T\), and \(\{a_p^i \mid p \in \mathbb{Z}, 0 \leq i \leq n + 1\}\) is a family of formal variables in a certain \(\mathbb{C}\)-algebra. The corresponding \((n + 2)\)-dimensional version of (6.16) also holds true.
7 Virasoro symmetries for the Principal Hierarchy

Now let us study Virasoro symmetries of the Principal Hierarchy (3.16). Recall that a
symmetry of the hierarchy (3.16) is an evolutionary system
\[
\frac{\partial v}{\partial s} = S(v, v_x, \ldots; t)
\]
that commutes with all the flows \( \frac{\partial}{\partial t_{\alpha,p}} (1 \leq \alpha \leq n, p \geq 0) \) and \( \frac{\partial}{\partial t_{0,p}} (p \in \mathbb{Z}) \), i.e.,
\[
\left[ \frac{\partial}{\partial s}, \frac{\partial}{\partial t_{\alpha,p}} \right] = 0, \quad 1 \leq \alpha \leq n, p \geq 0, \quad q \in \mathbb{Z}.
\]
All symmetries of the hierarchy form a Lie algebra with respect to the commutator.

**Theorem 7.1.** The Principal Hierarchy (3.16) admits the Galilean symmetry
\[
\frac{\partial v}{\partial s_{-1}} = e + \sum_{(i,p) \in \mathcal{I}} t^{i,p+1} \frac{\partial v}{\partial t_{i,p}}, \quad (7.1)
\]
where \( e = e^\alpha \partial_\alpha \) is the unit vector field of \( M \).

**Proof.** Let us first show that
\[
\left[ \frac{\partial}{\partial s_{-1}}, \frac{\partial}{\partial t_{\beta,q}} \right] v^\lambda = 0, \quad 1 \leq \beta \leq n, \quad q \geq 0, \quad 1 \leq \lambda \leq n.
\]
From the relations
\[
\left( \frac{\partial}{\partial s_{-1}} \circ \frac{\partial}{\partial t^{\beta,q}} \right) v^\lambda = \sum_{s \geq 0} \left( \frac{\partial}{\partial v^{\beta,s}} \frac{\partial v^\lambda}{\partial t^{\beta,q}} \right) \frac{\partial s}{\partial v^\lambda} \left( e^\gamma + \sum_{\alpha,p} t^{\alpha,p+1} \frac{\partial v^\gamma}{\partial t_{\alpha,p}} + \sum_{p \in \mathbb{Z}} t^{0,p} \frac{\partial v^\gamma}{\partial t_{0,p-1}} \right)
\]
\[
= e^\gamma \frac{\partial v^\lambda}{\partial t^{\beta,q}} + e^\gamma \frac{\partial^2 v^\lambda}{\partial t^{\beta,q} \partial t^{\beta,q}} + \sum_{\alpha,p} t^{\alpha,p+1} \frac{\partial^2 v^\lambda}{\partial t_{\alpha,p} \partial t^{\beta,q}} + \sum_{p \in \mathbb{Z}} t^{0,p} \frac{\partial^2 v^\lambda}{\partial t_{0,p-1} \partial t^{\beta,q}}
\]
\[
= \sum_{\alpha,p} t^{\alpha,p+1} \frac{\partial^2 v^\lambda}{\partial t^{\beta,q} \partial t^{\beta,q}} + \sum_{p \in \mathbb{Z}} t^{0,p} \frac{\partial^2 v^\lambda}{\partial t_{0,p-1} \partial t^{\beta,q}}
\]
\[
+ e^\gamma \frac{\partial v^\lambda}{\partial t^{\beta,q}} + 2e^\gamma \partial_\gamma \partial^\lambda \theta_{\beta,q+1},
\]
and
\[
\left( \frac{\partial}{\partial t^{\beta,q}} \circ \frac{\partial}{\partial s_{-1}} \right) v^\lambda = \frac{\partial v^\lambda}{\partial t^{\beta,q}} + \frac{\partial^2 v^\lambda}{\partial t^{\beta,q} \partial t^{\beta,q}} + \sum_{\alpha,p} t^{\alpha,p+1} \frac{\partial^2 v^\lambda}{\partial t_{\alpha,p} \partial t^{\beta,q}} + \sum_{p \in \mathbb{Z}} t^{0,p} \frac{\partial^2 v^\lambda}{\partial t_{0,p-1} \partial t^{\beta,q}}
\]
it follows that we only need to show the validity of the identity
\[
e^\gamma \frac{\partial v^\lambda}{\partial t^{\beta,q}} + 2e^\gamma \partial_\gamma \partial^\lambda \theta_{\beta,q+1} = \frac{\partial v^\lambda}{\partial t^{\beta,q}} + \frac{\partial v^\lambda}{\partial t^{\beta,q-1}}, \quad (7.2)
\]
which holds true since
\[ e^\gamma \frac{\partial}{\partial v^\gamma} \frac{\partial v^\lambda}{\partial \beta_{q+1}} + 2e^\gamma \partial_x \theta_{\beta_{q+1}} = e^\gamma (\partial_x \theta_{\beta_{q+1}}) + 2e^\gamma \partial_x \theta_{\beta_{q+1}} \]
\[ = e^\gamma \left( \partial_x c^s \theta_{\beta_q} + (\partial_s e^\gamma) c^s \theta_{\beta_q} \right) = \partial_x (e^\gamma \theta_{\beta_q}) - e^\gamma c^s \theta_{\beta_q} \]
\[ = \frac{\partial v^\lambda}{\partial \beta_{q-1}} + (\partial^s e^\gamma) c^s \theta_{\beta_q} \]
\[ = \frac{\partial v^\lambda}{\partial \beta_{q-1}} + e^\gamma \lambda \frac{\partial v^\lambda}{\partial \beta_{q-1}}. \]

By performing a similar calculation, we can check that
\[ \left[ \frac{\partial}{\partial s_{-1}}, \frac{\partial}{\partial t_{0,q}} \right] v^\lambda = 0, \quad q \in \mathbb{Z}. \]

The theorem is proved. □

The following theorem shows that the Galilean symmetry of the Principal Hierarchy can be lifted to its tau-cover \((5.9)\).

**Theorem 7.2.** The tau-cover \((5.9)\) admits the following Galilean symmetry:
\[
\frac{\partial f}{\partial s_{-1}} = \sum_{(i,p)\in I} t^{i,p+1} f_{i,p} + \frac{1}{2} \eta_{\alpha \beta} t^{\alpha,0} t^{\beta,0}, \quad (7.3)
\]
\[
\frac{\partial f_{i,p}}{\partial s_{-1}} = f_{i,p-1} + \sum_{(j,q)\in \mathbb{I}} t^{j,q+1} \Omega_{i,p;j,q} + \eta_{\alpha \beta} t^{\alpha,0} s_p^{\beta,0}, \quad (i,p)\in I, \quad (7.4)
\]
\[
\frac{\partial v^\gamma}{\partial s_{-1}} = e^\gamma + \sum_{(i,p)\in I} t^{i,p+1} \frac{\partial v^\gamma}{\partial t^{i,p}}, \quad \gamma = 1, \ldots, n. \quad (7.5)
\]

Here \(f_{i,p} := 0\) if \(i \neq 0\) and \(p < 0\).

**Proof.** We need to check that
\[
\left[ \frac{\partial}{\partial t^{j,q}}, \frac{\partial}{\partial s_{-1}} \right] f_{i,p} = 0, \quad \left[ \frac{\partial}{\partial t^{\alpha}}, \frac{\partial}{\partial s_{-1}} \right] f_{i,p} = 0, \quad (i,p), (j,q) \in I. \]

The first set of relations obviously hold true, so we only need to check the validity of the second set of relations. In fact, we have
\[
\left( \frac{\partial}{\partial t^{j,q}} \circ \frac{\partial}{\partial s_{-1}} \right) f_{i,p} = \Omega_{i,p-1;j,q} + \Omega_{i,p;j,q-1} + \sum_{(k,r)\in I} t^{j,q+1} \partial \Omega_{i,p;k+r} + \eta_{\alpha \beta} t^{\alpha,0} s_p^{\beta,0} \]
\[
\left( \frac{\partial}{\partial s_{-1}} \circ \frac{\partial}{\partial t^{j,q}} \right) f_{i,p} = \frac{\partial}{\partial s_{-1}} \Omega_{i,p;j,q} = \frac{\partial v^\alpha}{\partial s_{-1}} \partial \alpha \Omega_{i,p;j,q}
\]
\[
= (e^\alpha + \sum_{(k,r)\in I} t^{k,r+1} \frac{\partial v^\alpha}{\partial t^{k,r}}) \partial \alpha \Omega_{i,p;j,q}
\]

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\[
\frac{\partial}{\partial t} \Omega_{i,p,j,q} + \sum_{(k,r) \in I} t^{k,r+1} \frac{\partial \Omega_{i,p,j,q}}{\partial t^{k,r}},
\]
here we assume that
\[
\Omega_{i,p,j,q} := 0, \quad i \neq 0, \quad p < 0.
\]
From the relations
\[
\frac{\partial \Omega_{i,p,j,q}}{\partial t^{k,r}} = \frac{\partial^3 f}{\partial t^{i,p} \partial t^{j,q} \partial t^{k,r}} = \frac{\partial \Omega_{i,p,j,q}}{\partial t^{k,r}}
\]
it follows that we only need to check the validity of the identities
\[
\partial_t \Omega_{i,p,j,q} = \Omega_{i,p-1,j,q} + \Omega_{i,p,j,q-1} + \eta \delta_i^a \delta_j^b \delta_0^c \delta_0^d, \quad (i,p) \in I, \quad j,q) \in I. \tag{7.6}
\]
By using (2.20), (3.3) and (3.7) we know that
\[
\frac{\partial}{\partial s} \theta_{i,p} = \nabla \theta_{i,p},
\]
which, together with (5.1)-(5.4), lead to the identities (7.6). The theorem is proved.

**Remark 7.3.** If the flows \(\frac{\partial}{\partial t^p}\), \(p \in \mathbb{Z}\) are not included in the Principal Hierarchy, then we still have the symmetry
\[
\frac{\partial v}{\partial s} = e^\gamma + x \frac{\partial e^\gamma}{\partial v} \frac{\partial}{\partial x} + \sum_{p \geq 0} t^{p+1} \frac{\partial v}{\partial t^p}.
\]
However, in this case we are not able to lift this symmetry to the tau-cover of the Principal Hierarchy.

By applying the bihamiltonian recursion operator
\[
\mathcal{R} := \mathcal{P}_2 \mathcal{P}^{-1} = \mathcal{U} + \mathcal{C}(v_x) \left( \frac{1}{2} + \mu \right) \frac{\partial}{\partial x} \tag{7.7}
\]
to the symmetry \(\frac{\partial v}{\partial s-1}\), we obtain the following symmetry of the Principal Hierarchy:
\[
\frac{\partial v}{\partial s_0} := \mathcal{R} \frac{\partial v}{\partial s} = E + \sum_{p \geq 0} \left( p + \mu \alpha + \frac{1}{2} \right) t^{\alpha,p} \frac{\partial v}{\partial t^{\alpha,p}} + \sum_{p \in \mathbb{Z}} \left( p - \frac{d}{2} + \frac{1}{2} \right) \frac{\partial v}{\partial t^p} + \sum_{p \geq 0} \sum_{s \geq 1} (R_s)_{\epsilon}^\alpha t^{\alpha,p} \frac{\partial v}{\partial t^{\epsilon,p-s}} + \sum_{p \in \mathbb{Z}} \sum_{s \geq 1} p^\epsilon t^{\epsilon,p-s} \frac{\partial v}{\partial t^{\epsilon,p-s}}.
\]
Iterating this procedure we obtain a series of symmetries
\[
\frac{\partial v}{\partial s_{m+1}} := \mathcal{R} \frac{\partial v}{\partial s_m}, \quad m \geq -1
\]
of the Principal Hierarchy. However, such symmetries involve non-local terms in general. To eliminate such non-locality, we are to lift these symmetries to the tau-cover \((5.9)\) of the Principal Hierarchy. To this end, we first introduce some notations.
Let us denote
\[
\tilde{S}^{(\nu)}_i = \tilde{S}^{(\nu)}(t, \frac{\partial f}{\partial t}; \lambda)
\]
\[
= \left( \int_0^\infty \frac{dz}{\sqrt{z}} e^{-\lambda z} \left[ \sum_{p \geq 0} \left( f_{0,p}, f_{\bullet,p}, (-1)^{p+1} t_{n+1}^{-p-1} \right) z^p \right] \right)_i
\]
\[
+ \left( \int_0^\infty \frac{dz}{\sqrt{z}} e^{-\lambda z} \left[ \sum_{p \geq 0} (-1)^p \left( -1 \right)^{p+1} f_{0,-p-1}, t_{\bullet}, t_{n+1}^p \right) z^{-p-1} \right] \right)_i
\]
(7.8)
for 0 \leq i \leq n + 1. Here \( \lambda \) is a formal parameter, and the row vectors \( f_{\bullet,p} \) and \( t_{\bullet}^p \) are defined by
\[
f_{\bullet,p} = (f_{1,p}, \ldots, f_{n,p}), \quad t_{\bullet}^p = (t_{1}^p, t_{2}^p, \ldots, t_{n}^p)
\]
with \( t_{n+1}^p := \eta_{\alpha\beta} t_{\beta,p} \) and \( t_{n+1}^p := t_{0,p} \). Similar notations of vectors will be used in the rest part of the paper. We also introduce the \((n + 2)\)-dimensional row vector
\[
\tilde{S}^{(\nu)} = (\tilde{S}^{(\nu)}_0, \tilde{S}^{(\nu)}_1, \tilde{S}^{(\nu)}_{n+1}).
\]
Notably, the notation \( \tilde{S}^{(\nu)}(t, \frac{\partial f}{\partial t}; \lambda) \) is just obtained from \( \tilde{g}^{(\nu)}(a; \lambda) \) defined in (6.19) by the following substitution:
\[
a_{\alpha}^p \mapsto \begin{cases} 
(-1)^{p+1} t_{0,-p-1}, & i = 0; \\
\eta^{\alpha\beta} f_{\beta,p}, & 1 \leq i \leq n, p \geq 0; \\
(-1)^{p+1} t_{i,-p-1}, & 1 \leq i \leq n, p < 0; \\
f_{0,p}, & i = n + 1.
\end{cases}
\]
Recall that the number of non-zero elements of the series \{\(c_p\)\}_{p \in \mathbb{Z}}\ defined in Proposition 3.6 is at most one, and \( c_p \neq 0 \) for some \( p < 0 \) only if the charge \( d \) of the generalized Frobenius manifold \( M \) is a negative odd integer and in this case \( p = d - 1 \). Now let us introduce the notation \( C_{p,q}(\lambda) \) for \( p, q \in \mathbb{Z} \) as follows: \( C_{p,q}(\lambda) \neq 0 \) only if the charge \( d \) of \( M \) is a negative odd integer and \( p + q \leq d - 1 \). In this case we set
\[
C_{p,q}(\lambda) = \sum_{m \geq -1} \frac{C_{m;p,q}}{\lambda^{m+2}}
\]
\[
= \frac{(-1)^{p+1} t_{0,-p-1}}{2^{d+1-p-q}} \sum_{k=0}^{d-1-p-q} (-1)^k \left( p - \frac{d}{2} + \frac{1}{2} \right)^k \left( q - \frac{d}{2} + \frac{1}{2} \right)^{[d-1-p-q-k]}; \quad (7.9)
\]
otherwise we set \( C_{p,q}(\lambda) = 0 \). Here \( x^{[0]} = 1 \), and
\[
x^{[k]} := x(x + 1) \cdots (x + k - 1), \quad x \in \mathbb{C}, \ k \geq 1. \quad (7.10)
\]
It is straightforward to see that the constant \( C_{m;p,q} \neq 0 \) only if \( p + q + m = d - 1, m \geq 0 \) and \( d \) is a negative odd integer.

For any formal power series \( A(\lambda) := \sum_{m \in \mathbb{Z}} A_m \lambda^m \in R[\lambda, \frac{1}{\lambda}] \) in \( \lambda \) over a certain ring \( R \), we denote
\[
[A(\lambda)]_- := \sum_{m < 0} A_m \lambda^m.
\]
Theorem 7.4. The flows \( \frac{\partial}{\partial s} m, m \geq -1 \) defined by the following formulae are symmetries of the tau-cover (5.9) of the Principal Hierarchy of \( M \):

\[
\frac{\partial}{\partial s} = \sum_{m \geq -1} \frac{1}{\lambda^{m+2}} \frac{\partial}{\partial s_{m}},
\]

(7.11)

\[
\frac{\partial f}{\partial s} = -\frac{1}{2} \lim_{\nu \to 0} \left[ \frac{\partial \tilde{S}^{(\nu)}_i}{\partial \lambda} \tilde{G}^{ij}(\nu) \frac{\partial \tilde{S}^{(-\nu)}_j}{\partial \lambda} \right] - \sum_{p,q \in \mathbb{Z}} C_{p,q}(\lambda) t^{0,p} t^{0,q},
\]

(7.12)

\[
\frac{\partial f_{i,p}}{\partial s} = \frac{\partial}{\partial t^{i,p}} \frac{\partial f}{\partial s},
\]

(7.13)

\[
\frac{\partial \nu^\alpha}{\partial s} = \lim_{\nu \to 0} \left[ \frac{\partial \tilde{S}^{(\nu)}_i}{\partial \lambda} \tilde{G}^{ij}(\nu) \left( \frac{\partial}{\partial \nu} \tilde{p}^{(\nu)}_j \right) \right]_{M} - \left( \frac{1}{E - \lambda e} \right)^\alpha.
\]

(7.14)

Here the restriction of functions on \( \widetilde{M} \) to \( M \) is defined in (4.11); \( \tilde{p}^{(\nu)}_i \) are the regularized periods of \( \widetilde{M} \) defined by (6.13), and

\[
\dot{\xi}^{(\nu)}(\lambda) = \sum_{p \geq 0} \int_{0}^{\infty} e^{-\lambda z} \theta_{0,-p} z^{-d/2 - \nu - 1/2} \, dz
\]

(7.15)

\[
= \sum_{p \geq 0} \theta_{0,-p} \lambda^{p+d/2 - \nu} \Gamma \left( -p - d/2 + \nu + 1/2 \right);
\]

(7.16)

the vector field

\[
\frac{1}{E - \lambda e} := -\sum_{m \geq -1} \frac{1}{\lambda^{m+2}} E^{m+1}
\]

is the inverse of \( E - \lambda e \) with respect to the multiplication of \( M \).

We present the proof of the above theorem in the non-resonant case. In this case the spectrum of \( \tilde{\mu} \) does not contain half integers, and the limits

\[
\tilde{S}_i = \lim_{\nu \to 0} \tilde{S}^{(\nu)}_i, \quad \tilde{G}^{ij} = \lim_{\nu \to 0} \tilde{G}^{ij}(\nu), \quad \tilde{p}_i = \lim_{\nu \to 0} \tilde{p}^{(\nu)}_i, \quad \xi(\lambda) = \lim_{\nu \to 0} \xi^{(\nu)}(\lambda)
\]

(7.17)

exist. The proof for the general case is similar, so we omit the details here. In order to prove the theorem, we need some lemmas.

Lemma 7.5. If the spectrum of \( \tilde{\mu} \) contains no half integers, then

\[
\frac{\partial}{\partial \nu} \frac{\partial \tilde{S}}{\partial \lambda} = -\tilde{\partial}_{\lambda}(\tilde{p}_0, \tilde{p}_0, \tilde{p}_{n+1}) \bigg|_{M} - \partial_{\lambda}(\xi(\lambda), 0, 0),
\]

(7.18)

\[
\frac{\partial}{\partial \nu} \frac{\partial \tilde{S}}{\partial \lambda} = -\tilde{\partial}_{\lambda}(\tilde{p}_0, \tilde{p}_0, \tilde{p}_{n+1}) \bigg|_{M} - (\psi(\lambda), 0, 0),
\]

(7.19)

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where \( \tilde{p}_i \) are periods of \( \tilde{M} \) defined in (6.12), which are functions on \( \tilde{M} \), we regard them as functions on \( M \) by the restriction defined in (4.11); and

\[
\xi(\lambda) = \sum_{p \geq 0} \int_0^\infty e^{-\lambda z} \theta_{0,-p} z^{-p-d/2 - 1/2} \, dz = \sum_{p \geq 0} \theta_{0,-p} \lambda^{p+1} \frac{d}{2} \frac{1}{2} \Gamma \left( -p + \frac{1}{2} \right),
\]

\[
\psi(\lambda) = \sum_{p \geq 0} \int_0^\infty e^{-\lambda z} \theta_{0,-p-1} z^{-p-d/2 - 1/2} \, dz = \sum_{p \geq 0} \theta_{0,-p-1} \lambda^{p+\frac{d}{2} - 1} \frac{1}{2} \Gamma \left( -p + \frac{1}{2} \right).
\]

Proof. From the relation between \( \tilde{\theta}_i(z) \) and \( \theta_\alpha(z) \) that is given in (4.5), (4.6), (4.13), (4.14), and the relation \( \Omega_{\alpha,0; i, p} = \partial_\alpha \theta_{i+p+1} \) for \( (i, p) \in \mathcal{I} \), it follows that

\[
\partial_{\alpha, 0} \frac{\partial \tilde{S}}{\partial \lambda} = - \partial_{\alpha, 0} \int_0^\infty \frac{dz}{\sqrt{z}} e^{-\lambda z} \sum_{p \geq 0} (f_{0,p}, f_{\bullet, p}, (-1)^{p+1} t_{0,-p-1}) z^{p+1} \tilde{p}_z \tilde{R}
\]

\[
- \partial_{\alpha, 0} \int_0^\infty \frac{dz}{\sqrt{z}} e^{-\lambda z} \sum_{p \geq 0} (-1)^p \left((-1)^p f_{0,-p-1}, t_{\bullet, p}, t_{\bullet, n+1, p} \right) z^{-p} \tilde{p}_z \tilde{R}
\]

\[
= - \int_0^\infty \frac{dz}{\sqrt{z}} e^{-\lambda z} \sum_{p \geq 0} (\Omega_{\alpha, 0; 0, p}, \Omega_{\alpha, 0; p, 0}) z^{p+1} \tilde{p}_z \tilde{R}
\]

\[
- \int_0^\infty \frac{dz}{\sqrt{z}} e^{-\lambda z} \sum_{p \geq 0} (\Omega_{\alpha, 0, -p-1}, \eta_{\alpha, \bullet, p, 0}) z^{-p} \tilde{p}_z \tilde{R}
\]

\[
= - \int_0^\infty \frac{dz}{\sqrt{z}} e^{-\lambda z} \sum_{p \geq 0} (\partial_\alpha \theta_0(z) - e_\alpha, \partial_\alpha \theta_{\bullet}(z), 0) z^{p+1} \tilde{R}
\]

\[
- \int_0^\infty \frac{dz}{\sqrt{z}} e^{-\lambda z} \sum_{p \geq 0} (\partial_\alpha \theta_0(z), 0, 0) \left( \left( z^{p+\frac{d}{2} + \frac{1}{2}} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right) \right) dz
\]

\[
= - \int_0^\infty \frac{dz}{\sqrt{z}} e^{-\lambda z} \left( \tilde{\partial}_\alpha \tilde{\theta}_0(z), \tilde{\partial}_\alpha \tilde{\theta}_{\bullet}(z), \tilde{\partial}_\alpha \tilde{\theta}_{n+1}(z) \right) \bigg|_{M^n} \tilde{p}_z \tilde{R} - \partial_\alpha (\xi(\lambda), 0, 0)
\]

so the identity (7.18) holds true. We can prove the identity (7.19) in a similar way. The lemma is proved.

The following lemma shows that the flows given in (7.12)-(7.13) are compatible with the relation \( \nu^\alpha = \eta^{\alpha\beta} \partial_\beta \partial_\delta \partial_\sigma f \).

Lemma 7.6. If the spectrum of \( \tilde{\mu} \) contains no half integers, then the following relations hold true:

\[
- \frac{1}{2} \eta^{\alpha\beta} \partial_\beta \partial_\alpha \left( \frac{\partial \tilde{S}_1}{\partial \lambda} G^{ij} \partial_\lambda \tilde{S}_j \right) = \left( \frac{\partial \tilde{S}_1}{\partial \lambda} G^{ij} \partial_\lambda \tilde{p}_j \right) + \left( \partial_\alpha \xi(\lambda) \tilde{G}^{n+1,0} \frac{\partial \tilde{S}_2}{\partial \lambda} \right) - \left( \frac{1}{E - \lambda} \right)^\alpha. \tag{7.20}
\]
Proof. From the definitions of \( \tilde{\mu} \) and \( \tilde{R} \) given in (13) it follows that
\[
e^{\pi i \tilde{\mu}} = \begin{pmatrix} e^{-\frac{\pi d}{2}} & e^{\pi i \mu} \\ e^{\frac{\pi d}{2}} & * \end{pmatrix}, \quad e^{\pi i \tilde{R}} = \begin{pmatrix} 1 & 0 & 0 \\ * & e^{\pi i R} & 0 \\ * & * & 1 \end{pmatrix},
\]
so the Gram matrix \( \tilde{G}^{ij} \) defined in (5.14) has the form
\[
\tilde{G} = \begin{pmatrix} 0 & 0 & -\frac{1}{\pi} \cos \frac{\pi d}{2} \\ 0 & G & * \\ -\frac{1}{\pi} \cos \frac{\pi d}{2} & * & * \end{pmatrix}.
\]
In particular, \( \tilde{G}^{0,0} = \tilde{G}^{\alpha,\alpha} = \tilde{G}^{0,0} = 0 \), and \( \tilde{G}^{n+1,0} = \tilde{G}^{0,n+1} = -\frac{1}{\pi} \cos \frac{\pi d}{2} \). Thus we have
\[
-\frac{1}{2} \partial_x \partial_{n,0} \left[ \frac{\partial \tilde{S}_1}{\partial \lambda} \tilde{G}^{ij} \frac{\partial \tilde{S}_1}{\partial \lambda} \right] - \partial_x \left( \tilde{\partial}_0 \tilde{\partial}_i \right)_{M,0} \tilde{G}^{ij} \frac{\partial \tilde{S}_1}{\partial \lambda} = 0,
\]
where
\[
\tilde{\partial}_0 \tilde{\partial}_i = \left. \tilde{\partial}_0 \tilde{\partial}_i \right|_M \tilde{G}^{ij} \left. \frac{\partial \tilde{S}_1}{\partial \lambda} \right|_M = 0.
\]
By using the relations
\[
\left( \tilde{\partial}_0 \tilde{\partial}_n \right)_{M,0} = 0, \quad \left( \tilde{\partial}_0 \tilde{\partial}_n \right)_{M} = \lambda^{-\frac{d}{2} - \frac{1}{2}} \Gamma \left( \frac{d}{2} + \frac{1}{2} \right)
\]
we obtain
\[
- \left[ \left( \tilde{\partial}_0 \tilde{\partial}_n \right)_{M} \tilde{G}^{n+1,0} \psi(\lambda) \right] - \left[ \partial_0 \xi(\lambda) \tilde{G}^{0,n+1} \left( \tilde{\partial}_0 \tilde{\partial}_n \right)_{M} \right] =
\]
\[
= \frac{1}{\pi} \cos \frac{\pi d}{2} \sum_{p \geq 0} \partial_0 \theta_{0,-p} \lambda^{p+\frac{d}{2} - \frac{1}{2}} \Gamma \left( -p - \frac{d}{2} + \frac{1}{2} \right) \cdot \lambda^{-\frac{d}{2} - \frac{1}{2}} \Gamma \left( \frac{d}{2} + \frac{1}{2} \right) = e_\alpha \lambda.
\]
On the other hand, from the definition of periods \( \tilde{\rho}_i \) of the \( (n+2) \)-dimensional Frobenius manifold \( \tilde{M} \) it follows that
\[
\left( \tilde{\partial}_0 \tilde{\rho}_i \right) \tilde{G}^{ij} \left( \tilde{\partial}_0 \tilde{\rho}_j \right) = \left( (\tilde{g}^{ij} - \lambda \tilde{\eta}^{ij})^{-1} \right)_{\alpha,0},
\]
with \( \tilde{\eta}^{ij} = \tilde{b}^{ij} - \lambda \tilde{a}^{ij} \).

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where
\[
(\tilde{g}^{ij}) = \begin{pmatrix} 0 & 0 & v^0 \\ 0 & g^{\alpha\beta} + v^0 \eta^{\alpha\beta} & E^* \eta^{-1} \\ v^0 & E^* \eta^{-1} & (1-d)^{n+1} + c_0 \end{pmatrix}
\]
is the intersection form of \( \tilde{M} \). We observe that the operator \( \tilde{U} = (\sum_{k=0}^{n+1} \tilde{g}^{ik} \tilde{\eta}_{kj}) \) of multiplication by the Euler vector field \( \tilde{E} \) of \( \tilde{M} \) satisfies the relation
\[
\left( \tilde{U} - \lambda \tilde{I} \right)^{-1} |_M = \begin{pmatrix} -\frac{1}{\lambda} & 0 & 0 \\ \frac{1}{\lambda} (U - \lambda I)^{-1} E^* & (U - \lambda I)^{-1} & 0 \\ \frac{1}{\lambda^2} X_{n+1,0} & \frac{1}{\lambda} E^* (U - \lambda I)^{-1} & -\frac{1}{\lambda} \end{pmatrix},
\]
where \( \tilde{I} \) is the \((n+2) \times (n+2)\) identity matrix, and
\[
X_{n+1,0} := E^* (U - \lambda I)^{-1} E^* - (1-d) \theta_{0,0} - c_0.
\]
Thus we arrive at
\[
\left[ \partial_\alpha \tilde{p}_i \mid M \tilde{G}^{ij} \left( \partial_\alpha \tilde{p}_j \right) \right]_{M} = \left( \tilde{g}^{ij} - \lambda \tilde{\eta}^{ij} \right)^{-1} |_{M} = \left( \tilde{\eta} (\tilde{U} - \lambda \tilde{I})^{-1} \right) |_{M} \alpha,0
\]
\[
= \frac{1}{\lambda} \left[ \eta (U - \lambda I)^{-1} E^* \right] |_{\alpha} = \frac{1}{\lambda} \left[ E^* (U - \lambda I)^{-1} \right] \alpha = \left( \frac{e}{\lambda} + \frac{1}{E - \lambda e} \right) \alpha.
\]
Here we use the identities
\[
(U - \lambda I)^{-1} = \eta ((U - \lambda I)^{-1})^T \eta^{-1},
\]
\[
\frac{1}{\lambda} E^* (U - \lambda I)^{-1} = \frac{1}{\lambda} \left( \frac{E}{E - \lambda e} \right)_* = \left( \frac{e}{\lambda} + \frac{1}{E - \lambda e} \right)_*.
\]
the second one of which follows from \((U - \lambda I) \frac{E}{E - \lambda e} = E\). Hence we have
\[
-\frac{1}{2} \partial_\alpha \partial_\eta^{a,0} \left[ \frac{\partial \tilde{S}_i}{\partial \lambda} \tilde{G}^{ij} \frac{\partial \tilde{S}_j}{\partial \lambda} \right]_{M} = \left[ \partial_\alpha \left( \tilde{p}_i \right) \mid M^n \tilde{G}^{ij} \frac{\partial \tilde{S}_j}{\partial \lambda} \right]_{M} - \left( \frac{e}{\lambda} + \frac{1}{E - \lambda e} \right) \alpha
\]
\[
+ \left[ \partial_\alpha \xi (\lambda) \tilde{G}^{0,n+1} \frac{\partial \tilde{S}_{n+1}}{\partial \lambda} \right]_{M} + \frac{e_\alpha}{\lambda}
\]
\[
= \left[ \partial_\alpha \left( \tilde{p}_i \right) \mid M^n \tilde{G}^{ij} \frac{\partial \tilde{S}_j}{\partial \lambda} \right]_{M^n} + \left[ \partial_\alpha \xi (\lambda) \tilde{G}^{0,n+1} \frac{\partial \tilde{S}_{n+1}}{\partial \lambda} \right]_{M^n} - \left( \frac{1}{E - \lambda e} \right) \alpha.
\]
The lemma is proved. \( \Box \)
Theorem 7.7. For $m \geq -1$ and $(i, p), (j, q) \in \mathcal{I}$, the following identities hold true:

\[
\frac{\partial^{-1}}{\partial \varepsilon^{\infty}} \Omega_{i, p; j, q} := - \sum_{m \geq -1} \frac{1}{\lambda^{m+2}} \partial_{E^{m+1}} \Omega_{i, p; j, q} = \lim_{\nu \to 0} \left[ \left( \frac{\partial}{\partial t^{v_{i, p}}} \frac{\partial}{\partial t^{k, \ell}} \right) \tilde{S}_{j, q}^{(\nu)} \left( \frac{\partial}{\partial t^{v_{j, q}}} \frac{\partial}{\partial t^{k, \ell}} \right) \right] - 2 \delta_{i, 0} \delta_{j, 0} C_{p, q}(\lambda), \tag{7.22}
\]

where $C_{p, q}(\lambda)$ are defined by (7.20).

The proof of this theorem is given in the Appendix.

Now we start to prove Theorem 7.4 in the non-resonant case.

Proof of theorem 7.4. To show that (7.12)–(7.14) are symmetries of the tau-cover (5.9) of the Principle Hierarchy, we need to verify the following commutation relations:

\[
\left[ \frac{\partial}{\partial s}, \frac{\partial}{\partial t^{k, \ell}} \right] f = 0, \quad \left[ \frac{\partial}{\partial s}, \frac{\partial}{\partial t^{k, \ell}} \right] f_{i, p} = 0, \quad \left[ \frac{\partial}{\partial s}, \frac{\partial}{\partial t^{k, \ell}} \right] v^\alpha = 0 \tag{7.23}
\]

for all $(i, p), (k, \ell) \in \mathcal{I}$ and $\alpha = 1, 2, \ldots, n$. From (7.13) we see that the first set of commutation relations $\left[ \frac{\partial}{\partial s}, \frac{\partial}{\partial t^{k, \ell}} \right] f = 0$ hold true trivially, and from (7.14) we have

\[
\frac{\partial}{\partial s} \frac{\partial f_{i, p}}{\partial t^{k, \ell}} = \frac{\partial}{\partial s} \Omega_{i, p; k, \ell} = \gamma \Omega_{i, p; k, \ell} \frac{\partial}{\partial s} = \Omega_{i, p; k, \ell} \frac{\partial}{\partial s} \gamma \Omega_{i, p; k, \ell} \frac{\partial}{\partial s} \gamma
\]

\[
=: \Omega_{i, p; k, \ell} \left( \frac{\partial}{\partial s} \gamma G^{v_{i, p}} \left( \frac{\partial}{\partial s} \gamma G^{v_{j, q}} \right) \right) \partial_{s, q} \Omega_{i, p; k, \ell} \frac{\partial}{\partial s} \gamma \Omega_{i, p; k, \ell} \frac{\partial}{\partial s} \gamma
\]

\[
\quad - \Omega_{i, p; k, \ell} \frac{\partial}{\partial s} \gamma \Omega_{i, p; k, \ell} \frac{\partial}{\partial s} \gamma
\]

On the other hand, by using Theorem 7.4 we obtain

\[
\frac{\partial}{\partial t^{k, \ell}} \frac{\partial f_{i, p}}{\partial s} = \frac{\partial}{\partial t^{k, \ell}} \frac{\partial}{\partial t^{k, \ell}} \left( \frac{1}{2} \left[ \frac{\partial}{\partial s} \gamma G^{v_{i, p}} \left( \frac{\partial}{\partial s} \gamma G^{v_{j, q}} \right) \right] + \sum_{s, \mu \in \mathbb{Z}} C_{s, \mu}(\lambda) t^{i, p} t^{j, q} \right)
\]

\[
\quad - \frac{\partial}{\partial t^{k, \ell}} \left[ \left( \frac{\partial}{\partial t^{k, \ell}} \frac{\partial}{\partial t^{k, \ell}} \right) \tilde{S}_{j, q} \left( \frac{\partial}{\partial t^{k, \ell}} \frac{\partial}{\partial t^{k, \ell}} \right) \right] - 2 \delta_{i, 0} \delta_{j, 0} C_{p, q}(\lambda)
\]

\[
\quad = \left( \frac{\partial}{\partial t^{k, \ell}} \frac{\partial}{\partial t^{k, \ell}} \right) \tilde{S}_{j, q} \left( \frac{\partial}{\partial t^{k, \ell}} \frac{\partial}{\partial t^{k, \ell}} \right) + 2 \delta_{i, 0} \delta_{j, 0} C_{p, q}(\lambda)
\]

Thus by using the relation

\[
\frac{\partial}{\partial t^{k, \ell}} \frac{\partial}{\partial t^{k, \ell}} \frac{\partial}{\partial s} \tilde{S}_{j, q} = \int_0^\infty dz e^{-s \lambda} \sum_{q \geq 0} \left( \frac{\partial^2 f_{0, q}}{\partial t^{v_{i, p}} \partial t^{k, \ell}} - \frac{\partial^2 f_{0, q}}{\partial t^{v_{j, q}} \partial t^{k, \ell}} \right) z^{q+1} \tilde{\mu}_{z} \tilde{R}
\]

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we arrive at
\[ - \int_0^\infty \frac{dz}{\sqrt{z}} e^{-\lambda z} \sum_{q \geq 0} \left( \frac{\partial^2 f_{0,q-1}}{\partial t^2 \partial t^{k \ell}}(0,0) \right) z^{-q} z^R \]
\[ = - \int_0^\infty \frac{dz}{\sqrt{z}} e^{-\lambda z} \sum_{q \geq 0} \left( \frac{\partial \Omega_{i,p,k \ell}}{\partial t^0} \frac{\partial \Omega_{i,p,k \ell}}{\partial t^0} ; 0,0 \right) z^{q+1} z^R \]
\[ - \int_0^\infty \frac{dz}{\sqrt{z}} e^{-\lambda z} \sum_{q \geq 0} \left( \frac{\partial \Omega_{i,p,k \ell}}{\partial t^0} ; 0,0 \right) z^{-q} z^R \]
\[ = - \partial_{\gamma \Omega_{i,p,k \ell}} (\epsilon \gamma \tilde{p}_0, \epsilon \gamma \tilde{p}_\bullet, \epsilon \gamma \tilde{p}_{n+1}) + (\partial_q \xi(\lambda), 0, 0) \right) \]}

we arrive at \[ \frac{\partial}{\partial \lambda} \frac{\partial}{\partial t^{i \ell}} f_{i, p} = 0 \], so the second set of relations of (7.23) is proved. In particular, we have
\[ \left[ \frac{\partial}{\partial s}, \frac{\partial}{\partial t^{i \ell}} \right] f_{\beta, 0} = 0, \quad \forall (i, p) \in I, \beta = 1, 2, \ldots, n. \] 

(7.24)

Finally, from (7.20) and (7.24) it follows that
\[ \frac{\partial}{\partial t^{i \ell}} \frac{\partial}{\partial s} = \eta^{\alpha \beta} \frac{\partial}{\partial t^{i \ell}} \frac{\partial}{\partial s} f_{\beta, 0} = \eta^{\alpha \beta} \frac{\partial}{\partial s} \left( \frac{\partial}{\partial t^{i \ell}} f_{\beta, 0} \right) \]
\[ = \eta^{\alpha \beta} \frac{\partial}{\partial s} \left( \frac{\partial}{\partial t^{i \ell}} f_{\beta, 0} \right) = \frac{\partial}{\partial s} \left( \eta^{\alpha \beta} \frac{\partial}{\partial s} \Omega_{i,p,k \ell} \right) = \frac{\partial}{\partial s} \frac{\partial^2}{\partial t^{i \ell}} f_{i, p}, \]

which leads to \[ \left[ \frac{\partial}{\partial \lambda}, \frac{\partial}{\partial t^{i \ell}} \right] v^\alpha = 0 \]. The theorem is proved. □

Remark 7.8. It is easy to see from Lemma 6.1 that the symmetries \[ \frac{\partial f}{\partial s_m}, m \geq -1 \] given in Theorem 7.2 can be represented in the form
\[ \frac{\partial f}{\partial s_m} = \sum_{(i,p),(j,q) \in I} a_{i,p,j,q}^{m} \frac{\partial f}{\partial t^{i \ell}} \frac{\partial f}{\partial t^{j \ell}} + \sum_{(i,p),(j,q) \in I} b_{i,p,j,q}^{m} \frac{\partial f}{\partial t^{i \ell}} + \sum_{(i,p),(j,q) \in I} c_{m,i,p,j,q} \frac{\partial f}{\partial t^{i \ell}} + \sum_{p,q \in \mathbb{Z}} C_{m,p,q} t^{p \ell} p^q, \]

(7.25)

for some constants \[ a_{i,p,j,q}^{m}, b_{i,p,j,q}^{m}, c_{m,i,p,j,q} \in \mathbb{C} \]. We will also use the following generating series:
\[ a_{i,p,j,q}^{m}(\lambda) := \sum_{m \geq -1} a_{m}^{i,p,j,q} \lambda^{m+2}, \quad b_{i,p,j,q}^{m}(\lambda) := \sum_{m \geq -1} b_{m}^{i,p,j,q} \lambda^{m+2}, \quad c_{m,i,p,j,q}(\lambda) := \sum_{m \geq -1} c_{m,i,p,j,q} \lambda^{m+2}. \]

(7.26)

In general, let \{ \Phi^{i,p}, \Psi_{j,q} \mid (i, p), (j, q) \in I \} be a family of formal variables in some \[ \mathbb{C} \]-algebra, and let us denote
\[ \widehat{S}(\nu)(\Phi, \Psi; \lambda) := \widehat{S}(\nu)(t, \frac{\partial f}{\partial t}; \lambda) \]

\[ \left. \left|_{t^{i \ell} \rightarrow \Phi_{i,p}, \tilde{t}^{j \ell} \rightarrow \tilde{\Psi}_{j,q}} \right| \]

then the following identity holds true:
\[ - \frac{1}{2} \lim_{\nu \rightarrow 0} \left[ \frac{\partial \widehat{S}(\nu)(\Phi, \Psi; \lambda)}{\partial \nu} \widehat{S}(\nu)(\Phi, \Psi; \lambda) \right] \]

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\[
\sum_{(i,p),(j,q)\in I} a_{i,p}^{i,j,q} \Psi_{i,p} \Psi_{j,q} + \sum_{(i,p),(j,q)\in I} b_{j,q}^{i,j,q} (\lambda) \Phi_{i,p}^q \Psi_{i,p} + \sum_{(i,p),(j,q)\in I} c_{i,p}^{j,q} (\lambda) \Phi_{i,p} \Phi_{j,q}.
\]

(7.27)

**Definition 7.1.** We call the constants \(a_{i,p}^{i,j,q}\), \(b_{j,q}^{i,j,q}\), \(c_{i,p}^{j,q}\) and \(C_{m,p,q}\) that appear in (7.25) and (7.9) the extended Virasoro coefficients of \(M\).

The extended Virasoro coefficients can be calculated explicitly by using the formulae given in (6.16)–(6.18). The following properties of these coefficients can be verified by straightforward calculation:

\[
a_{i,p}^{i,j,q} = 0, \quad \text{if } i = 0 \text{ or } j = 0,
\]

(7.28)

\[
b_{m,i,p}^{0,q} \neq 0, \quad \text{only if } i = 0 \text{ and } q = p + m,
\]

(7.29)

\[
b_{m,i,p}^{i,q} = \delta_{p,m}^q \left( p + \mu_i + \frac{1}{2} \right),
\]

(7.30)

here \((i,p),(j,q)\in I\) and \(m \geq -1\), and the symbol \(x^{[m]}\) is defined in (7.10).

### 8 The Virasoro operators

We are to construct in this section a collection of Virasoro operators in terms of the monodromy data \(\eta, \mu, R\) of \(M\) at \(z = 0\), the constant vectors \(r_s = (r_1^s, \ldots, r_n^s)^T\), \(s \geq 1\) and the constants \(c_p, p \in \mathbb{Z}\) that are specified in Lemma 3.1 and Proposition 3.6.

Our construction of the Virasoro operators for \(M\) is based on the one given in [22, 23] for the \((n+2)\)-dimensional Frobenius manifold \(\tilde{M}\) that is introduced in Lemma 4.1. Let \(a_p^i, 0 \leq i \leq n + 1, p \in \mathbb{Z}\) and \(1\) be the generators of a Heisenberg algebra \(\mathcal{H}\), which satisfy the commutation relations

\[
[1, a_p^i] = 0, \quad [a_p^i, a_q^j] = (-1)^p \eta^{ij} \delta_{p+q+1,0} \cdot 1.
\]

(8.1)

Introduce the normal ordering as follows:

\[
: a_p^i a_q^j : = \begin{cases} a_p^i a_q^j & \text{if } p \geq 0, \quad q < 0, \\ a_p^i a_q^j & \text{other cases}, \end{cases}
\]

then the Virasoro operators of \(\tilde{M}\) are defined by

\[
\tilde{L}(\lambda) = \sum_{m \geq -1} \tilde{L}_m \chi^{m+2} = -\frac{1}{2} \lim_{\nu \to 0} : \frac{\partial \tilde{\phi}(\nu)}{\partial \lambda} \tilde{G}^{ij}(\nu) \frac{\partial \tilde{\phi}(-\nu)}{\partial \lambda} : + \frac{1}{4 \lambda^2} \text{tr} \left( \frac{1}{4} - \mu^2 \right),
\]

(8.2)

where \(\tilde{\phi}(\nu) = \tilde{\phi}(\nu)(a; \lambda)\) is given in (6.19).
These operators satisfy the commutation relations

\[ [\tilde{L}_m, \tilde{L}_n] = (m - n)\tilde{L}_{m+n}, \quad m, n \geq -1. \]

By putting

\[ a^i_p := \begin{cases} 
\eta^{ij} \frac{\partial}{\partial \xi^j} , & p \geq 0, \\
(-1)^{p+1} t^{-p-1} , & p < 0,
\end{cases} \quad (8.3) \]

for \(0 \leq i, j \leq n+1, p \in \mathbb{Z}\), we obtain a realization of the Heisenberg algebra in the space of functions of the variables \(t^i_p, 0 \leq i \leq n+1, p \geq 0\), and a collection of linear differential operators \(\tilde{L}_m\) acting on this space.

Let us proceed to define the Virasoro operators for the generalized Frobenius manifold \(M\) by modifying the expressions (8.2) of \(\tilde{L}\) and by using a different realization of the Heisenberg algebra \(H\). This realization of the Heisenberg algebra is given by

\[ a^i_p := \begin{cases} 
(-1)^{p+1} t^0_{-p-1} , & i = 0, p \in \mathbb{Z}, \\
\eta^{i\beta} \frac{\partial}{\partial \xi^\beta} , & i = 1, \ldots, n, p \geq 0, \\
(-1)^{p+1} t^i_{-p-1} , & i = 1, \ldots, n, p < 0, \\
\frac{\partial}{\partial \xi^i} , & i = n+1, p \in \mathbb{Z}.
\end{cases} \quad (8.4) \]

**Definition 8.1.** The Virasoro operators \(L_m, m \geq -1\) of the generalized Frobenius manifold \(M\) are defined by the following formula:

\[
L(t, \frac{\partial}{\partial t}; \lambda) = \sum_{m \geq -1} \frac{L_m(t, \frac{\partial}{\partial t})}{\lambda^{m+2}} \\
= -\frac{1}{2} \lim_{\nu \to 0} : \frac{\partial}{\partial \lambda} \tilde{G}^{ij}(\nu) \frac{\partial}{\partial \lambda} \tilde{G}^{ij}(-\nu) : + \sum_{p,q \in \mathbb{Z}} C_{p,q}(\lambda) t^{0, p} t^{0, q} \quad (8.5)
\]

where \(\tilde{G}^{(\nu)} = \tilde{G}^{(\nu)}(a; \lambda)\) is defined in \((6.19)\), with \(a_p = (a^0_p, \ldots, a^{n+1}_p)^T\) having the realization \((8.4)\), series \(C_{p,q}(\lambda)\) are defined in \((7.9)\), and the normal ordering \(:\cdot:\) means to put the differential operator terms on the right.

It is important to note that the \(n \times n\) matrix \(\mu\) that appears in \((8.5)\) is not the \((n + 2) \times (n + 2)\) matrix \(\tilde{\mu}\). From the above definition it follows that the Virasoro operators \(L_m, m \geq -1\) have the form

\[
L_m(t, \frac{\partial}{\partial t}) = \sum_{(i,p),(j,q) \in \mathcal{I}} a^{i,p,j,q}_m \frac{\partial^2}{\partial t^i \partial t^j \partial \xi^p \partial \xi^q} + \sum_{(i,p),(j,q) \in \mathcal{I}} b^{j,q}_{m,i,p} t^{i,p} \frac{\partial}{\partial t} + \sum_{(i,p),(j,q) \in \mathcal{I}} c_{m,j,p,q} t^{i,j,q} + \sum_{p,q \in \mathbb{Z}} C_{m,p,q} t^{0,p} t^{0,q} \\
+ \frac{1}{4} \delta_{m,0} \text{tr} \left( \frac{1}{4} - \mu^2 \right) \cdot 1, \quad (8.6)
\]
where the extended Virasoro coefficients $a_{m,i,p}^{i,p;j,q}$, $b_{m,i,p}^{j,q}$, $c_m;i,p;j,q$ and $C_m;p,q$ are specified in Definition 7.1. By using (6.16) and (6.18), we can also obtain the following formulae for $L_m(t,\frac{\partial}{\partial t})$ (see [22]):

$$L_m(t,\frac{\partial}{\partial t}) = \frac{1}{2} \sum_{p,q\in\mathbb{Z}} (-1)^{p+1} :a_T^{p,\eta} \left[ P_m(\tilde{\mu} - p, \tilde{R})\right]_{m-1-q-p} :a_p : + \sum_{p,q\in\mathbb{Z}} C_{m,p,q} t^{0,p} t^{0,q} + \frac{1}{4} \delta_{m,0} \text{tr} \left( \frac{1}{4} - \mu^2 \right) 1, \quad m \geq -1.$$

Here $a_p = (a_p^0,\ldots,a_p^{n+1})^T$ have the realization (8.4), and $\tilde{\eta},\tilde{\mu},\tilde{R}$ are the monodromy data of $\tilde{M}^{n+2}$ at $z = 0$ given by (4.1) and (4.3). The $(n+2) \times (n+2)$ matrices $P_m$ have the expressions

$$P_m(\tilde{\mu},\tilde{R}) := \begin{cases} e^{\tilde{R}_s} \prod_{j=0}^{m} \left( x + \tilde{\mu} + j - \frac{1}{2} \right) & m \geq 0, \\
1 & m = -1, \end{cases}$$

and their components $[P_m]_k$ are defined as in (6.9).

**Proposition 8.1.** The Virasoro operators $L_m$, $m \geq -1$ of the generalized Frobenius manifold $M$ satisfy the following commutation relations:

$$[L_m, L_k] = (m - k)L_{m+k}, \quad m, k \geq -1. \quad (8.8)$$

**Proof.** We can prove the commutation relations of the Virasoro operators in the same way as it is done in [22, 23] for the Virasoro operators of a usual Frobenius manifold. As for the additional term $\sum_{p,q\in\mathbb{Z}} C_{m,p,q} t^{0,p} t^{0,q}$ in $L_m$, we note that the only term in $L_k$ which has non-trivial commutator with $\sum_{p,q\in\mathbb{Z}} C_{m,p,q} t^{0,p} t^{0,q}$ is

$$\sum_{p\in\mathbb{Z}} b_{k,0,p}^{0,p+k,0,p} \frac{\partial}{\partial t^{0,p+k}}$$

because of (7.28) and (7.30). The commutator

$$\left[ \sum_{p,q\in\mathbb{Z}} C_{m,p,q} t^{0,p} t^{0,q}, \sum_{p\in\mathbb{Z}} b_{k,0,p}^{0,p+k,0,p} \frac{\partial}{\partial t^{0,p+k}} \right]$$

can be calculated directly by using the explicit formulae (7.9) and (7.30). We omit the details here. The proposition is proved. \(\square\)

At the end of this section, we provide the explicit formulae of $L_{-1}$, $L_0$, $L_1$ and $L_2$.

Recall $\tilde{R}_{s;1} := \tilde{R}_s = \begin{pmatrix} 0 & R_s \\
r_s & 0 \\
c_{s-1} & r_s^T & 0 \end{pmatrix}$. It is easy to check that for $k \geq 2$,

$$\tilde{R}_{s;k} := [\tilde{R}_s]^k = \begin{pmatrix} 0 & \langle R_s \rangle_{s;k} & R_{s;k} \\
\langle r_s R \rangle_{s;k} & \langle r_s R \rangle_{s;k} & \langle r_s R \rangle_{s;k} \end{pmatrix}.$$  \(8.9\)
where $R_{s,k} := [R^k]_s$, and

\[
\langle Rr \rangle_{s,k} = \sum_{\ell \geq 1} R_{s,k-1} r_{s-\ell},
\]

\[
\langle r^l R \rangle_{s,k} = \sum_{\ell \geq 1} r_{s-\ell} R_{s,k-1},
\]

\[
\langle r^l Rr \rangle_{s,k} = \sum_{p+q+s} r^l_p R_{q,k-2} r_{s}.
\]

By using the above notations and our assumption that $\frac{\partial}{\partial R^\alpha} = 0$ if $\alpha \neq 0$ and $p < 0$, we can write down the explicit expressions of the operators $L_m$ for $m = -1, 0, 1, 2$ as follows:

\[
L_{-1} = \sum_{p \geq 0} t^{\alpha,p+1} \frac{\partial}{\partial t^{\alpha,p}} + \sum_{p \in \mathbb{Z}} t^{0,p+1} \frac{\partial}{\partial t^{0,p}} + \frac{1}{2} \eta_{\alpha \beta} t^{\alpha,0} t^{\beta,0}.
\]

\[
L_0 = \sum_{p \geq 0} \left( p + \mu_\alpha + \frac{1}{2} \right) t^{\alpha,p} \frac{\partial}{\partial t^{\alpha,p}} + \sum_{p \in \mathbb{Z}} \left( p - \frac{d}{2} + \frac{1}{2} \right) t^{0,p} \frac{\partial}{\partial t^{0,p}}
\]

\[
+ \sum_{p \geq 0} \sum_{s \geq 1} (R_s)_{\alpha} t^{\alpha,p} \frac{\partial}{\partial t^\varepsilon, p-s} + \sum_{p \in \mathbb{Z}} \sum_{s \geq 1} t^{\varepsilon, p} \frac{\partial}{\partial t^\varepsilon, p-s}
\]

\[
+ \frac{1}{2} \sum_{p, q \geq 0} (-1)^p \eta_{\alpha \varepsilon} (R_{p+q+1})_{\beta} t^{\alpha,p} t^{\beta,q}
\]

\[
+ \sum_{p \geq 0} \sum_{s \geq 1} (-1)^p (r_s)_{\alpha} t^{0,p,s-1} t^{\alpha,p} + \frac{1}{2} \sum_{p, q \in \mathbb{Z}} (-1)^p c_{p+q} t^{0,p} t^{0,q}
\]

\[
+ \frac{1}{4} t t \left( \frac{1}{4} - \mu^2 \right) \cdot 1. \tag{8.10}
\]

\[
L_1 = \frac{1}{2} \left( \frac{1}{4} - \mu^2 \right)^\alpha \eta_{\alpha \beta} \frac{\partial^2}{\partial t^\varepsilon, \partial t^\varepsilon, 0}
\]

\[
+ \sum_{p \geq 0} \left( p + \mu_\alpha + \frac{1}{2} \right) \left( p + \mu_\alpha + \frac{3}{2} \right) t^{\alpha,p} \frac{\partial}{\partial t^{\alpha,p+1}}
\]

\[
+ \sum_{p \in \mathbb{Z}} \left( p - \frac{d}{2} + \frac{1}{2} \right) \left( p - \frac{d}{2} + \frac{3}{2} \right) t^{0,p} \frac{\partial}{\partial t^{0,p+1}}
\]

\[
+ 2 \sum_{p \geq 0, s \geq 1} (R_s)_{\alpha} (p + \mu_\alpha + 1) t^{\alpha,p} \frac{\partial}{\partial t^\varepsilon, p+1-s}
\]

\[
+ 2 \sum_{p \in \mathbb{Z}, s \geq 1} r^\varepsilon_s \left( p - \frac{d}{2} + 1 \right) t^{\alpha,p} \frac{\partial}{\partial t^\varepsilon, p+1-s}
\]

\[
+ \sum_{p \geq 0, s \geq 2} (R_s)_{\alpha} t^{\alpha,p} \frac{\partial}{\partial t^\varepsilon, p+1-s} + \sum_{p \in \mathbb{Z}, s \geq 2} \langle Rr \rangle_{s,2} t^{0,p} \frac{\partial}{\partial t^\varepsilon, p+1-s}
\]

\[
+ \sum_{p, q \geq 0} (-1)^q (p + \mu_\alpha + 1) (R_{p+q+2})_{\alpha} \eta_{\varepsilon \beta} t^{\alpha,p} t^{\beta,q}
\]

\[
\tag{8.11}
\]
\[ L_2 = \frac{1}{2} \left( -3 \mu_\alpha^2 + 3 \mu_\alpha + \frac{1}{4} \right) (R_1)^{\alpha \beta} \partial^2_\alpha \partial^2_\beta \eta^\alpha \eta^\beta \]
\[ + \left( \frac{1}{2} - \mu_\alpha \right) \left( \frac{3}{2} - \mu_\alpha \right) \eta^\alpha \eta^\beta \partial^2_\alpha \partial^2_\beta + \sum_{p \geq 0} \left( p + \mu_\alpha + \frac{1}{2} \right) \left( p + \mu_\alpha + \frac{5}{2} \right) t^{\alpha \beta} \partial^2_\alpha \partial^2_\beta + \sum_{p \in \mathbb{Z}, s \geq 2} \left( p - \frac{d}{2} + \frac{1}{2} \right) \left( p - \frac{d}{2} + \frac{5}{2} \right) t^{\alpha \beta} \partial^2_\alpha \partial^2_\beta + \sum_{p \geq 0, s \geq 2} \left( p + \mu_\alpha + \frac{3}{2} \right) \left( R_{s; 2} \right)^{s} \partial^2_\alpha \partial^2_\beta + \sum_{p \in \mathbb{Z}, s \geq 2} \left( p - \frac{d}{2} + \frac{3}{2} \right) \left( R_{s; 2} \right)^{s} \partial^2_\alpha \partial^2_\beta + \sum_{p \geq 0, s \geq 3} \left( R_{s; 3} \right)^{s} \partial^2_\alpha \partial^2_\beta + \sum_{p \in \mathbb{Z}, s \geq 3} \left( R_{s; 3} \right)^{s} \partial^2_\alpha \partial^2_\beta + \sum_{p \geq 0, s \geq 0} \left( -1 \right)^{s+p+1} \left[ \left( R_{p+q+3; 3} \right)^{s} + 3 \left( p + \mu_\alpha + \frac{3}{2} \right) \left( R_{p+q+3; 2} \right)^{s} \right] \left( r_s \right)_4 \partial^0, s-3-p, p, q + \sum_{p \geq 0, s \geq 0} \left( -1 \right)^{s+p+1} \left( r^4 R \right)_{p+q+3; 2} \partial^0, s-3-p, p, q + \sum_{p \geq 0, s \geq 0} \left( -1 \right)^{s+p+1} \left( r^4 R \right)_{s; 3} \partial^0, s-3-p, p, q + \sum_{p \in \mathbb{Z}, s \geq 3} \left( r^4 R \right)_{p+q+3; 3} \partial^0, s-3-p, p, q + \frac{1}{4} \sum_{p, q \in \mathbb{Z}} \left( -1 \right)^{p} \left( r^4 R \right)_{p+q+3; 2} \partial^0, p, q + \frac{1}{2} \sum_{p, q \in \mathbb{Z}} \left( -1 \right)^{p} \left( r^4 R \right)_{p+q+3; 3} \partial^0, p, q. \ (8.12) \]
\[
+ \frac{1}{2} \sum_{p,q \in \mathbb{Z}} (-1)^p e_{p+q+2} \left[ \left( p + \frac{1}{2} - \frac{d}{2} \right) \left( p + \frac{3}{2} - \frac{d}{2} \right) + \left( q + \frac{1}{2} - \frac{d}{2} \right) \left( q + \frac{3}{2} - \frac{d}{2} \right) \right] e^{p,q}.
\]

\section{Linearization of the Virasoro symmetries and the loop equation}

Let us consider the integrable hierarchy

\[
\frac{\partial w_\alpha}{\partial t_i,p} = K_{i,p}^\alpha(w; w_x, w_{xx}, \ldots), \quad (i,p) \in \mathcal{I}, \quad \alpha = 1, 2, \ldots, n \tag{9.1}
\]

that is obtained from the Principal Hierarchy (3.16) by using a certain quasi-Miura transformation of the form

\[
v_\alpha \mapsto w_\alpha = v_\alpha + \eta_\alpha \gamma \partial_x \partial_{\gamma,j,q},
\]

where \( F^{[k]} = F^{[k]}(v; v_x, \ldots, v^{(m_k)}) \) are functions on the jet space \( J^\infty(M^n) \).

Denote \( F = \varepsilon^{-2} f + \Delta F, \quad \Delta F = \sum_{k \geq 1} \varepsilon^{k-2} F^{[k]}, \) \( \tag{9.3} \)

then the quasi-Miura transformation \( \ref{9.2} \) yields a deformation of the tau-cover \( \ref{5.9} \) of the Principal Hierarchy of \( M \) which has the form

\[
\mathcal{E} \frac{\partial F}{\partial t_i,p} = F_{j,q}, \quad \mathcal{E} \frac{\partial F_{i,p}}{\partial t_j,q} = \hat{\Omega}_{i,p;j,q}, \quad \frac{\partial w_\alpha}{\partial t_i,p} = \eta_\alpha \gamma \partial_x \hat{\Omega}_{\gamma,j,q},
\]

here the functions \( \hat{\Omega}_{i,p;j,q} = \hat{\Omega}_{i,p;j,q}(w; w_x, \ldots) \) are given by

\[
\hat{\Omega}_{i,p;j,q} = \left( \Omega_{i,p;j,q}(v) + \varepsilon^2 \frac{\partial^2 \Delta F}{\partial t^{i,p} \partial t^{j,q}} \right) \bigg|_{v \rightarrow v(w,w_x,\ldots)}.
\]

This deformation of the tau-cover of the Principal Hierarchy also possesses the following Virasoro symmetries:

\[
\frac{\partial F}{\partial s} = \varepsilon^{-2} \frac{\partial f}{\partial s} + \sum_{k \geq 1} \varepsilon^{k-2} \sum_{r \geq 1} \frac{\partial F^{[k])}}{\partial \varepsilon^{r,s}} \partial_{x}^{r} \partial_{s}^{s},
\]

\[
\frac{\partial F_{i,p}}{\partial s} = \varepsilon \frac{\partial}{\partial t^{i,p}} \frac{\partial F}{\partial s},
\]

\[
\frac{\partial w_\alpha}{\partial s} = \eta_\alpha \gamma \varepsilon^2 \partial_x \partial_{\gamma,j,q} \frac{\partial F}{\partial s},
\]

where \( \bar{\partial} = \sum_{m \geq -1} \frac{1}{\lambda^{m+2}} \frac{\partial}{\partial \lambda_m}. \)

We say that the quasi-Miura transformation \( \ref{9.2} \) linearizes the Virasoro symmetries of the Principal Hierarchy of \( M \) if the actions of these symmetries on the tau function

\[
\tau = \exp F \tag{9.4}
\]
of the deformed Principal Hierarchy (9.1) are given by

\[ \frac{\partial \tau}{\partial s} = L_m(\varepsilon^{-1} t, \varepsilon \frac{\partial}{\partial t}; \lambda), \quad m \geq -1, \quad (9.5) \]

where the Virasoro operators \( L_m(\varepsilon^{-1} t, \varepsilon \frac{\partial}{\partial t}; \lambda) \) are given by (8.6). We note that the linearization condition (9.5) can be rewritten as

\[
\frac{\partial F}{\partial s} = \varepsilon^2 \sum_{(i,p),(j,q) \in \mathcal{I}} a^{i,p,j,q}(\lambda) \left( \frac{\partial^2 F}{\partial t^{i,p} \partial t^{j,q}} + \frac{\partial F}{\partial t^{i,p}} \frac{\partial F}{\partial t^{j,q}} \right) \\
+ \varepsilon^{-2} \sum_{(i,p),(j,q) \in \mathcal{I}} b^{i,p,j,q}(\lambda) \frac{\partial F}{\partial t^{i,p} \partial v^{j,q}} + \varepsilon^2 \sum_{(i,p),(j,q) \in \mathcal{I}} c_{i,p,j,q}(\lambda) t^{i,p} v^{j,q} \\
+ \varepsilon^{-2} \sum_{p,q \in \mathbb{Z}} C_{p,q}(\lambda) t^{p} v^{q} + \frac{1}{4\lambda^2} \text{tr} \left( \frac{1}{4} - \mu^2 \right), \quad (9.6)
\]

which lead to the following lemma.

**Lemma 9.1.** The quasi-Miura transformation (9.2) linearizes the Virasoro symmetries of the Principal Hierarchy of \( M \) if and only if the function \( \Delta F = \sum_{k \geq 1} \varepsilon^{k-2} F^{[k]} \) defined on the jet space of \( M \) satisfies the equation

\[
\frac{\partial \Delta F}{\partial s} = \mathcal{D}(\Delta F) + \sum_{(i,p),(j,q) \in \mathcal{I}} a^{i,p,j,q}(\lambda) \frac{\partial^2 f}{\partial t^{i,p} \partial t^{j,q}} \\
+ \varepsilon^2 \sum_{(i,p),(j,q) \in \mathcal{I}} a^{i,p,j,q}(\lambda) \left( \frac{\partial^2 \Delta F}{\partial t^{i,p} \partial t^{j,q}} + \frac{\partial \Delta F}{\partial t^{i,p}} \frac{\partial \Delta F}{\partial t^{j,q}} \right) + \frac{1}{4\lambda^2} \text{tr} \left( \frac{1}{4} - \mu^2 \right), \quad (9.7)
\]

where the linear operator \( \mathcal{D} \) is defined by

\[
\mathcal{D} = \sum_{(i,p),(j,q) \in \mathcal{I}} 2a^{i,p,j,q}(\lambda) \frac{\partial f}{\partial t^{i,p} \partial t^{j,q}} + \sum_{(i,p),(j,q) \in \mathcal{I}} b^{i,p,j,q}(\lambda) t^{i,p} \frac{\partial}{\partial t^{j,q}}. \quad (9.8)
\]

and

\[
\frac{\partial \Delta F}{\partial s} = \sum_{r \geq 0} \frac{\partial \Delta F}{\partial v^{\gamma}} \frac{\partial v^{\gamma}}{\partial s}. \]

The following lemma gives a more explicit formula for the linear operator \( \mathcal{D} \).

**Lemma 9.2.** Let \( \tilde{M} \) be the \((n+2)\)-dimensional Frobenius manifold associated with \( M \), and let \( \{\tilde{p}_0, \tilde{p}_1, \ldots, \tilde{p}_{n+1}\} \) be any basis of periods of \( \tilde{M} \) with Gram matrix \( \tilde{G} = (\tilde{G}^{ij}) \). Then, for any function \( \Delta F \) defined on the jet space of \( M \) we have

\[
\mathcal{D}(\Delta F) = \frac{\partial \Delta F}{\partial s} + \sum_{s \geq 0} \frac{\partial \Delta F}{\partial v^{r,s}} \frac{1}{E - \lambda e} \gamma - \frac{1}{\lambda} \sum_{s \geq 1} \frac{\partial \Delta F}{\partial v^{r,s}} s \partial_x^s e^\gamma \\
+ \sum_{s \geq 1} \frac{\partial \Delta F}{\partial v^{r,s}} \sum_{k=1}^s \left( \begin{array}{c} s \\ k \end{array} \right) \left( (\partial_x^{k-1} \partial_y \tilde{p}_k) \tilde{G}^{ij} (\partial_x^{k+1} \partial_y \tilde{p}_j) \right) \bigg|_M. \quad (9.9)
\]
We note that the formula (9.9) is independent of the choice of periods \{\tilde{p}_i\} of \tilde{M}, in particular, we can choose periods \tilde{p}_i as in (6.12) if the spectrum of \tilde{\mu} contains no half integers.

**Proof.** We give the proof of the lemma for the case when the spectrum of \tilde{\mu} contains no half integers, for the general case the proof is similar. Let \S_i = \S_i(t, \frac{\partial F}{\partial t}; \lambda) be defined in (7.8) and (7.17). From the relation (7.27) between Laplace-type integrals and the extended Virasoro coefficients it follows that

\[
D(\Delta F) = \left[ \frac{\partial \S_i}{\partial \lambda} G^{ij} \left( \int_0^\infty \frac{dz}{\sqrt{z}} e^{-\lambda z} \sum_{p \geq 0} \left( \frac{\partial \Delta F}{\partial t^0, p}, \frac{\partial \Delta F}{\partial t^* p}, 0 \right) z^{p+1} \tilde{\mu} \tilde{R} \right) \right]_j
\]

By using (7.21)–(7.24) and (7.19), we can represent the first term of the right hand side of (9.10) as follows:

\[
\left[ \frac{\partial \S_i}{\partial \lambda} G^{ij} \left( \int_0^\infty \frac{dz}{\sqrt{z}} e^{-\lambda z} \sum_{p \geq 0} \left( \frac{\partial \Delta F}{\partial t^0, p}, \frac{\partial \Delta F}{\partial t^* p}, 0 \right) z^{p+1} \tilde{\mu} \tilde{R} \right) \right]_j
= \sum_{s \geq 0} \frac{\partial \Delta F}{\partial v^s} \left[ \frac{\partial \S_i}{\partial \lambda} G^{ij} \left( \int_0^\infty \frac{dz}{\sqrt{z}} e^{-\lambda z} \sum_{p \geq 0} \partial_z^s \left( \frac{\partial \Delta F}{\partial t^0, p}, \frac{\partial \Delta F}{\partial t^* p}, 0 \right) z^{p+1} \tilde{\mu} \tilde{R} \right) \right]_j
\]

Similarly, by using (7.21) and (7.16) we can represent the second term of the right hand
The lemma is proved.

Hence by using (7.14) we arrive at

$$\frac{\partial S_i}{\partial \lambda} \left( \int_0^\infty \frac{e^{-\lambda z}}{\sqrt{\pi}} \sum_{p \geq 0} \left( \frac{\partial \Delta F}{\partial \mu_{0,-p-1}}(0) \right) z^{-p} \tilde{\rho}_z \right)_{j,i} = \left[ \frac{\partial S_i}{\partial \lambda} \tilde{G}^{n+1} \right]_{j,i}$$

$$= \sum_{s \geq 0} \frac{\partial \Delta F}{\partial \mu_{s\gamma,s}} \left[ \frac{\partial \tilde{S}_{n+1}}{\partial \lambda} \tilde{G}^{n+1} \int_0^\infty e^{-\lambda z} \sum_{p \geq 0} \frac{\partial \Delta F}{\partial \mu_{0,-p-1}} z^{-p} \tilde{\rho}_z \, dz \right]$$

$$= \sum_{s \geq 0} \frac{\partial \Delta F}{\partial \mu_{s\gamma,s}} \left[ \frac{\partial \tilde{S}_{n+1}}{\partial \lambda} \tilde{G}^{n+1} \int_0^\infty e^{-\lambda z} \sum_{p \geq 0} \frac{\partial \Delta F}{\partial \mu_{0,-p-1}} z^{-p} \tilde{\rho}_z \, dz \right]$$

$$= \sum_{s \geq 0} \frac{\partial \Delta F}{\partial \mu_{s\gamma,s}} \left[ \frac{\partial \tilde{S}_{n+1}}{\partial \lambda} \tilde{G}^{n+1} \int_0^\infty e^{-\lambda z} \sum_{p \geq 0} \frac{\partial \Delta F}{\partial \mu_{0,-p-1}} z^{-p} \tilde{\rho}_z \, dz \right]$$

Hence by using (7.14) we arrive at

$$\mathcal{D}(\Delta F) = \sum_{s \geq 0} \frac{\partial \Delta F}{\partial \mu_{s\gamma,s}} \left[ \frac{\partial \tilde{S}_{n+1}}{\partial \lambda} \tilde{G}^{n+1} \left( \frac{\partial \gamma}{\partial \lambda} \right) \right]_{M} - \frac{1}{\lambda} \sum_{s \geq 1} \frac{\partial \Delta F}{\partial \mu_{s\gamma,s}} \tilde{\rho}_s \tilde{\rho}_e$$

$$= \sum_{s \geq 0} \frac{\partial \Delta F}{\partial \mu_{s\gamma,s}} \left[ \frac{\partial \tilde{S}_{n+1}}{\partial \lambda} \tilde{G}^{n+1} \left( \frac{\partial \gamma}{\partial \lambda} \right) \right]_{M} - \frac{1}{\lambda} \sum_{s \geq 1} \frac{\partial \Delta F}{\partial \mu_{s\gamma,s}} \tilde{\rho}_s \tilde{\rho}_e$$

The lemma is proved.
We introduce, following [23], the star product map \(*\) as follows:

\[
(\sum_{k \geq 0} \theta_{i_k, p_k} \lambda^{-k+s}) \ast (\sum_{\ell \geq 0} \theta_{j_\ell, q_\ell} \lambda^{-\ell+t}) = \sum_{m \geq 0} \left( \sum_{k=0}^{m} \Omega_{i_k, p_k, j_{m-k}, q_{m-k}} \right) \lambda^{-m+s+t} \tag{9.11}
\]

for all \((i_k, p_k), (j_\ell, q_\ell) \in \mathcal{I}\), and \(s, t \in \mathbb{C}\). In particular, \(\theta_{i, p} \ast \theta_{j, q} = \Omega_{i, j, p, q}\). Notice that the regularized periods \(p^{(s)}_a(\lambda)\) are linear combinations of terms of the form \(\sum_{k \geq 0} \theta_{i, p} \lambda^{-k+s}\)
for some \(s \in \mathbb{C}\), so \(\frac{\partial p^{(s)}_a}{\partial \lambda} \ast \frac{\partial p^{(-s)}_a}{\partial \lambda}\) are well-defined.

**Theorem 9.3.** The linearization condition (9.5) holds true if and only if \(\Delta F\) satisfies the following differential equations:

\[
\sum_{s \geq 0} \frac{\partial \Delta F}{\partial v_{i,p}^s} \frac{\partial}{\partial e^\gamma} \left( \frac{1}{E - \lambda e} \right)^\gamma - \frac{1}{\chi} \sum_{s \geq 1} \frac{\partial \Delta F}{\partial v_{i,p}^s} s \frac{\partial e^\gamma}{\partial v_{i,p}^s} \\
+ \sum_{s \geq 1} \frac{\partial \Delta F}{\partial v_{i,p}^s} \sum_{k=1}^{s} \left( \left( (\partial^{k-1}_x \partial \alpha_i, \tilde{G}^{ij} (\partial^{k+1}_x \partial \gamma p_j)) \right) \right) \bigg|_M \\
= \varepsilon^2 \sum_{k \geq 0} \frac{\partial \Delta F}{\partial v_{i,p}^{k+1}} \frac{\partial}{\partial p_{p_\alpha}} \left( (\partial^{k+1}_x \partial^\alpha p_\alpha) G^{p_{p_\alpha}} (\partial^\alpha p_\alpha) \right) \\
+ \frac{\partial \Delta F}{\partial v_{i,p}^{k+1}} \left( \nabla \frac{\partial p_{p_\alpha}}{\partial \lambda} \cdot \nabla \frac{\partial p_{p_\beta}}{\partial \lambda} \cdot v_x \right)^\alpha G^{p_{p_\alpha}} \\
+ \frac{1}{2} G^{\alpha, \beta} \frac{\partial p_{p_\alpha}}{\partial \lambda} \frac{\partial p_{p_\beta}}{\partial \lambda} - \frac{1}{4 \mu^2} \text{tr} \left( \frac{1}{4 \mu^2} \right), \tag{9.12}
\]

where the star product map \(*\) is defined as in (9.11), \(\{\tilde{p}_i\}\) is any basis of periods of \(\tilde{M}\) with Gram matrix \(\tilde{G} = (\tilde{G}^{ij})\), and \(\{p_\alpha\}\) is any basis of periods of \(M\) with Gram matrix \(G = (G^{\alpha, \beta})\).

Let us note that the equation (9.12) is independent of the choice of the periods \(\{p_\alpha\}\) and \(\{\tilde{p}_i\}\), so in the non-resonant case we can choose \(p_\alpha\) and \(\tilde{p}_i\) as (6.10), (6.12) respectively.

**Proof.** We give the proof of the theorem under the assumption that the spectrum of \(\tilde{\mu}\) contains no half integers, the proof for the general case is similar. By using Lemma 9.1, 9.2 we know that the linearization condition (9.5) is equivalent to the equation

\[
0 = \sum_{s \geq 0} \frac{\partial \Delta F}{\partial v_{i,p}^s} \frac{\partial}{\partial e^\gamma} \left( \frac{1}{E - \lambda e} \right)^\gamma - \frac{1}{\chi} \sum_{s \geq 1} \frac{\partial \Delta F}{\partial v_{i,p}^s} s \frac{\partial e^\gamma}{\partial v_{i,p}^s} \\
+ \sum_{s \geq 1} \frac{\partial \Delta F}{\partial v_{i,p}^s} \sum_{k=1}^{s} \left( \left( (\partial^{k-1}_x \partial \alpha_i, \tilde{G}^{ij} (\partial^{k+1}_x \partial \gamma p_j)) \right) \right) \bigg|_M \\
+ \varepsilon^2 \sum_{(i,p), (j,q) \in \mathcal{I}} a_{i,p, j,q}^{(\alpha, \beta)} (\lambda) \left( \frac{\partial \Delta F}{\partial v_{i,p}^{k+1}} \frac{\partial}{\partial v_{j,q}^\alpha} + \frac{\partial^2 \Delta F}{\partial v_{i,p}^{k+1} \partial v_{j,q}^\alpha} \right)
\]

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\[ + \sum_{(i,p),(j,q) \in I} a_{i,p;j,q}(\lambda) \frac{\partial^2 F_0}{\partial t^{i,p} \partial t^{j,q}} + \frac{1}{4\lambda^2} \text{tr} \left( \frac{1}{4} - \mu^2 \right). \]

From (7.27) it follows that
\[
\varepsilon^2 \sum_{(i,p),(j,q) \in I} a_{i,p;j,q}(\lambda) \frac{\partial^2 F}{\partial t^{i,p} \partial t^{j,q}} \partial_\lambda \partial_\lambda \partial F_{ij} = -\varepsilon^2 \sum_{k,\ell \geq 0} \frac{\partial \Delta F}{\partial t^\alpha k} \frac{\partial \Delta F}{\partial t^\beta \ell} \left( \left. \left( \partial_{x^k} \tilde{G}^{ij} \left( \partial_{x^\ell} \tilde{p}_j \right) \right) \right|_M \right) \]
\[
= -\varepsilon^2 \sum_{k,\ell \geq 0} \frac{\partial \Delta F}{\partial t^\alpha k} \frac{\partial \Delta F}{\partial t^\beta \ell} \left( \partial_{x^k} \partial_{x^\ell} \tilde{p}_o \right) \left( \partial_{x^k} \partial_{x^\ell} \partial_{v_\omega} \right) \left( \partial_{x^k} \partial_{x^\ell} \partial_{v_\rho} \right) \]

In a similar way, we can verify that
\[
\varepsilon^2 \sum_{(i,p),(j,q) \in I} a_{i,p;j,q}(\lambda) \frac{\partial^2 F}{\partial t^{i,p} \partial t^{j,q}} \partial_\lambda \partial_\lambda \partial F_{ij} = -\varepsilon^2 \sum_{k,\ell \geq 0} \frac{\partial \Delta F}{\partial t^\alpha k} \frac{\partial \Delta F}{\partial t^\beta \ell} \left( \partial_{x^k} \partial_{x^\ell} \tilde{p}_o \right) \left( \partial_{x^k} \partial_{x^\ell} \partial_{v_\omega} \right) \left( \partial_{x^k} \partial_{x^\ell} \partial_{v_\rho} \right) \]

and
\[
\sum_{(i,p),(j,q) \in I} a_{i,p;j,q}(\lambda) \frac{\partial^2 f}{\partial t^{i,p} \partial t^{j,q}} = \sum_{(i,p),(j,q) \in I} a_{i,p;j,q} \theta_{i,p} \theta_{j,q} = -\frac{1}{2} \left( \partial_{\lambda^\alpha} \partial_{\lambda^\beta} \right) \left( \partial_{\lambda^\alpha} \partial_{\lambda^\beta} \right). \]

The theorem is proved. \(\square\)

### 10 Quasi-periods and the simplified loop equation

The goal of this section is to simplify the term
\[
\left( \left. \left( \partial_{x^k} \partial_{x^\ell} \tilde{p}_o \right) \tilde{G}^{ij} \left( \partial_{x^k} \partial_{x^\ell} \tilde{p}_j \right) \right) \right|_M
\]
in the left hand side of the loop equation (9.12). To this end, we are to choose the periods of \(\tilde{M}^{n+2}\) and of \(M^n\) in a certain special way, since the loop equation is independent of the choice of these periods.
Proposition 10.1. There exists a basis $p_1(v;\lambda), \ldots, p_n(v;\lambda)$ of periods of $M$ which satisfy the following quasi-homogeneous conditions:

$$
\begin{align*}
\left(\partial_E + \lambda \frac{\partial}{\partial \lambda}\right) \nabla p_{\alpha} + \left(\mu + \frac{1}{2}\right) \nabla p_{\alpha} &= 0, \\
\left(\partial_E + \lambda \frac{\partial}{\partial \lambda}\right) \frac{\partial p_{\alpha}}{\partial \lambda} + \frac{1 + d}{2} \frac{\partial p_{\alpha}}{\partial \lambda} &= 0.
\end{align*}
$$

(10.1), (10.2)

Such periods of $M$ are said to be quasi-homogeneous.

Proof. Suppose $p(v;\lambda)$ is a solution of the Gauss-Manin equations (6.3) and (6.5), then

$$
\begin{align*}
\partial_E \nabla p &= E^\alpha \partial_\alpha \nabla p = -U(U - \lambda I)^{-1} \left(\mu + \frac{1}{2}\right) \nabla p \\
&= -\left(\mu + \frac{1}{2}\right) \nabla p - \lambda(U - \lambda I)^{-1} \left(\mu + \frac{1}{2}\right) \nabla p \\
&= -\lambda \frac{\partial}{\partial \lambda} \nabla p - \left(\mu + \frac{1}{2}\right) \nabla p,
\end{align*}
$$

thus the equation (10.1) holds true automatically for $p(v;\lambda)$. From this equation it follows, for $\forall \alpha = 1, \ldots, n$, that

$$
\begin{align*}
\partial^\alpha \left[\left(\partial_E + \lambda \frac{\partial}{\partial \lambda}\right) \nabla p\right] &= \left(\frac{2}{2} - \mu + \mu_{\alpha}\right) \frac{\partial}{\partial \lambda} \partial^\alpha p + \left(\partial_E + \lambda \frac{\partial}{\partial \lambda}\right) \frac{\partial}{\partial \lambda} \partial^\alpha p \\
&= \left(-\frac{d}{2} + \mu_{\alpha}\right) \frac{\partial}{\partial \lambda} \partial^\alpha p + \frac{\partial}{\partial \lambda} \left[\left(\partial_E + \lambda \frac{\partial}{\partial \lambda}\right) \partial^\alpha p\right] = \partial^\alpha \left(-\frac{d}{2} + 1 \partial p\right),
\end{align*}
$$

therefore, there exists a function $c = c(\lambda)$ such that

$$
\begin{align*}
\left(\partial_E + \lambda \frac{\partial}{\partial \lambda}\right) \frac{\partial p}{\partial \lambda} + \frac{1 + d}{2} \frac{\partial p}{\partial \lambda} &= c(\lambda).
\end{align*}
$$

Let $y = y(\lambda)$ be a solution of the following ODE of Euler-type:

$$
\lambda y'(\lambda) + \frac{1 + d}{2} y(\lambda) = c(\lambda).
$$

Consider the substitution $p \mapsto \hat{p} := p - \int y(\lambda) d\lambda$, then (10.2) holds true for $\hat{p}$ which still satisfies the Gauss-Manin equations (6.3) and (6.5). The proposition is proved.

We note that solutions of the Gauss-Manin equations (6.3), (6.5) and the equation (10.2) have the following ambiguity

$$
p(v;\lambda) \mapsto p(v;\lambda) + a \lambda \frac{1 + d}{2} + b,
$$

(10.3)

where $a, b$ are any constants. We also note that the periods given by the Laplace-type integrals (6.10) are not necessarily quasi-homogeneous.
Remark 10.2. If $M$ has flat unity $e = \frac{\partial}{\partial e}$, then $\mu_1 = -\frac{d}{2}$. It is easy to show that any period $p(v; \lambda)$ of $M$ which satisfies the condition $\frac{\partial}{\partial e} p = -\partial_e p$ is quasi-homogeneous. Thus the quasi-homogeneous conditions (10.1) and (10.2) generalize the condition $\frac{\partial}{\partial e} p = -\partial_e p$ for periods of a usual Frobenius manifold.

Remark 10.3. If a period $p(v; \lambda)$ of $M$ is quasi-homogeneous, then for $\forall \alpha = 1, 2, \ldots, n$ we have

$$ \partial^\alpha \left[ \left( \partial_E + \lambda \frac{\partial}{\partial \lambda} + \frac{d-1}{2} \right) p \right] = \frac{\partial}{\partial \lambda} \left[ \left( \partial_E + \lambda \frac{\partial}{\partial \lambda} + \frac{d-1}{2} \right) p \right] = 0, \quad (10.4)$$

which yields

$$ \left( \partial_E + \lambda \frac{\partial}{\partial \lambda} + \frac{d-1}{2} \right) p = \text{const.} \quad (10.5)$$

If the charge $d \neq 1$, then we can adjust the parameter $b$ in (10.3) such that the right hand side of (10.5) equals zero.

Now we are to construct a basis $\tilde{p}_0, \ldots, \tilde{p}_{n+1}$ of periods of $\tilde{M}$ from a basis of quasi-homogeneous periods of $M$ such that $\frac{\partial}{\partial \lambda} \tilde{p}_i = -\partial_e \tilde{p}_i$ for $i = 0, 1, \ldots, n + 1$.

Lemma 10.4. Let $p_1(v; \lambda), \ldots, p_n(v; \lambda)$ be a basis of periods of $M$, and $Q = Q(v; \lambda)$ be a function on $M \times \mathbb{C}$. Then the functions $\tilde{p}_0, \ldots, \tilde{p}_{n+1}$ defined by

$$ \tilde{p}_0(v^0, v, v^{n+1}; \lambda) = Q(v; \lambda - v^0) + v^{n+1}(\lambda - v^0)^{d-1}, \quad (10.6) $$

$$ \tilde{p}_\alpha(v^0, v, v^{n+1}; \lambda) = p_\alpha(v; \lambda - v^0), \quad (10.7) $$

$$ \tilde{p}_{n+1}(v^0, v, v^{n+1}; \lambda) = \begin{cases} (\lambda - v^0) & \text{if } d \neq 1 \\ \log(\lambda - v^0) & \text{if } d = 1 \end{cases} \quad (10.8) $$

form a basis of periods of $\tilde{M}$ if and only if the periods $p_1, \ldots, p_n$ of $M$ are quasi-homogeneous, and the function $Q(v; \lambda)$ satisfies the following system of equations:

$$ (\mathcal{U} - \lambda \mathcal{I}) \partial_\gamma \nabla Q = -\mathcal{C}_\gamma \left( \mu + \frac{1}{2} \right) \nabla Q + \frac{d-1}{2} \lambda^{\frac{d-3}{2}} \partial_\gamma, \quad (10.9) $$

$$ (\mathcal{U} - \lambda \mathcal{I}) \frac{\partial}{\partial \lambda} \nabla Q = \left( \mu + \frac{1}{2} \right) \nabla Q - \frac{d-1}{2} \lambda^{\frac{d-3}{2}} E, \quad (10.10) $$

$$ \left( \partial_E + \lambda \frac{\partial}{\partial \lambda} \right) \nabla Q + \left( \mu + \frac{1}{2} \right) \nabla Q = 0, \quad (10.11) $$

$$ \left( \partial_E + \lambda \frac{\partial}{\partial \lambda} \right) \frac{\partial Q}{\partial \lambda} + \frac{d+1}{2} \frac{\partial Q}{\partial \lambda} = 0. \quad (10.12) $$

Proof. It is straightforward to check that $\tilde{p}_0, \ldots, \tilde{p}_{n+1}$ satisfy the Gauss-Manin equations of $\tilde{M}$, we omit the details here.

Since the homogeneous part of the system of equations (10.9) and (10.10) coincides with the system of Gauss-Manin equations (6.3) and (6.5), if $Q(v; \lambda)$ is a solution of the equations (10.9) and (10.10) then $Q + p$ is also a solution of these equations for any
period $p$ of $M$. Notably, the system of equations (10.11) and (10.12) coincides with that of the equations (10.1) and (10.2). Therefore, for the particular case when $d = 1$, the function $Q(v, \lambda)$ can be chosen as any quasi-homogeneous period of $M$.

We proceed to prove the existence of a function $Q(v, \lambda)$ satisfying the equations (10.9)–(10.12).

**Lemma 10.5.** There exists a vector field $Y(v; \lambda) = Y^\alpha(v; \lambda) \partial_\alpha$ on $M$, such that

\[
(U - \lambda I) \partial_\gamma Y = -C_\gamma \left( \mu + \frac{1}{2} \right) Y + \frac{d - 1}{2} \lambda^{\frac{d-1}{2}} \partial_\gamma,
\]

\[
(U - \lambda I) \frac{\partial}{\partial \lambda} Y = \left( \mu + \frac{1}{2} \right) Y - \frac{d - 1}{2} \lambda^{\frac{d-1}{2}} E
\]

hold true for $\forall \gamma = 1, 2, \ldots, n$.

**Proof.** Let $p_1(v; \lambda), \ldots, p_n(v; \lambda)$ be a basis of periods of $M$ with Gram matrix $G = (G^{\alpha\beta})$, and

\[
Y = -\lambda^{\frac{d-1}{2}} P G \left( \partial_E + \lambda \frac{\partial}{\partial \lambda} \right) p_*^T,
\]

where

\[
P := (\nabla p_1, \ldots, \nabla p_n), \quad p_* = (p_1, p_2, \ldots, p_n).
\]

Then by using (10.4) and properties of $P, G$ it is straightforward to check that the vector field $Y$ satisfies the equations (10.13) and (10.14). The lemma is proved.

**Theorem 10.6.** Let $p_1(v; \lambda), \ldots, p_n(v; \lambda)$ be a basis of quasi-homogeneous periods of $M$, and the vector field $Y = Y(v; \lambda)$ be given by (10.15). Then there exists a function $Q = Q(v; \lambda)$ on $M \times \mathbb{C}$ such that $\nabla Q = Y$, and (10.11), (10.12) hold true. This function together with (10.6)–(10.8) yields a basis of periods of $\tilde{M}$ which satisfy the condition $\frac{\partial}{\partial \lambda} \tilde{p}_i = -\partial_\lambda \tilde{p}_i$ and moreover, the associated Gram matrix $\tilde{G} = (\tilde{G}^{ij})$ has the form

\[
\tilde{G} = \begin{pmatrix}
0 & 0 & \frac{2}{1-d} \\
0 & G & 0 \\
\frac{2}{1-d} & 0 & * \\
\end{pmatrix}, \quad \text{if} \quad d \neq 1,
\]

\[
\tilde{G} = \begin{pmatrix}
0 & 0 & 1 \\
0 & G & 0 \\
1 & 0 & * \\
\end{pmatrix}, \quad \text{if} \quad d = 1.
\]

**Proof.** By using (10.5) it is easy to see that the vector field $Y$ is a gradient field for a certain function $Q_0(v; \lambda)$, which satisfies the equations (10.9) and (10.10). Then we can adjust $Q_0$ by adding a certain function $c = c(\lambda)$ so that $Q = Q_0 + c(\lambda)$ satisfies the equations (10.9)–(10.12). It is easy to check that the associated Gram matrix $\tilde{G}$ has the form (10.16) or (10.17). The theorem is proved.

**Definition 10.1.** Let $p_1(v; \lambda), \ldots, p_n(v; \lambda)$ be a basis of quasi-homogeneous periods of $M$, then we call the function $Q(v; \lambda)$ determined by Theorem 10.6 the quasi-period of $M$ associated with $p_1, \ldots, p_n$. 

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Remark 10.7. If $M$ has flat unity $e = \frac{\partial}{\partial \omega}$, then for any basis of periods $p_1, \ldots, p_n$ of $M$ satisfying $\partial_e p_\alpha = -\frac{\partial p_\alpha}{\partial \lambda}$ we have

$$Y = -\lambda^{\frac{d-1}{2}} PG \left( \partial_E + \lambda \frac{\partial}{\partial \lambda} \right) p_\alpha^T = -\lambda^{\frac{d-1}{2}} PG \partial_{E-\lambda e} p_\alpha^T$$

$$= -\lambda^{\frac{d-1}{2}} PGP^T (E - \lambda e) = -\lambda^{\frac{d-1}{2}} (U - \lambda)^{-1} (E - \lambda e)$$

therefore the associated quasi-period $Q(v; \lambda)$ can be chosen as

$$Q = -\lambda^{\frac{d-1}{2}} v_1 + c(\lambda)$$

for a certain function $c(\lambda)$.

Now we are ready to simplify the loop equation (9.12).

Theorem 10.8. The loop equation (9.12) of $M$ can be represented in the form

$$\sum_{s \geq 0} \frac{\partial \Delta F_{\gamma \sigma}}{\partial v_{\gamma \sigma}} \left( \frac{1}{E - \lambda e} \right)^\gamma + \sum_{s \geq 1} \frac{\partial \Delta F_{\alpha \beta}}{\partial v_{\alpha \beta}} s \delta^s_\alpha \left( (\partial^\gamma p_\alpha) G^{\alpha \beta} \left( \partial_e + \frac{\partial}{\partial \lambda} \right) p_\beta \right)$$

$$- \sum_{s \geq 1} \frac{\partial \Delta F_{\alpha \beta}}{\partial v_{\alpha \beta}} \sum_{k=1}^s \left( \frac{s}{k} \right) \left( \partial^k \partial^\gamma p_\alpha \right) G^{\alpha \beta} \left( \partial^{k+1-\gamma} p_\beta \right)$$

$$= \frac{\varepsilon^2}{2} \sum_{k, \ell \geq 0} \left( \frac{\partial \Delta F_{\alpha \beta}}{\partial v_{\alpha \beta}} \right) (\partial^k p_\omega) G^{\alpha \beta} \left( \partial^{k+1-\gamma} p_\beta \right)$$

$$+ \frac{\varepsilon^2}{2} \sum_{k \geq 0} \frac{\partial \Delta F_{\alpha \beta}}{\partial v_{\alpha \beta}} \partial_{\gamma x} \left( \nabla \frac{\partial p_\omega}{\partial \lambda} \cdot \nabla \frac{\partial p_\rho}{\partial \lambda} \cdot v_x \right) G^{\alpha \beta}$$

$$+ \frac{1}{2} G^{\alpha \beta} \frac{\partial p_\alpha}{\partial \lambda} + \frac{\partial p_\beta}{\partial \lambda} - \frac{1}{4 \lambda^2} \text{tr} \left( \frac{1}{4} - \mu^2 \right), \quad (10.18)$$

where $p_1, \ldots, p_n$ is any basis of periods of $M$ and $(G^{\alpha \beta})$ is the associated Gram matrix.

Proof. Since the loop equation (9.12) is independent of the choice of periods $\{p_\alpha\}$ and $\{\tilde{p}_i\}$, we first fix a basis of quasi-homogeneous periods $\{p_\alpha\}$ of $M$, and then choose the basis of periods $\{\tilde{p}_i\}$ of $M$ by using the associated quasi-period $Q(v; \lambda)$ and the formulae (10.6)–(10.8). From the form (10.16) or (10.17) of the Gram matrix $(\tilde{G}^{ij})$ associated with $\{\tilde{p}_i\}$, it follows that the left hand side of the loop equation (9.12) can be represented as

$$\sum_{s \geq 0} \frac{\partial \Delta F_{\gamma \sigma}}{\partial v_{\gamma \sigma}} \left( \frac{1}{E - \lambda e} \right)^\gamma - \frac{1}{\lambda} \sum_{s \geq 1} \frac{\partial \Delta F_{\alpha \beta}}{\partial v_{\alpha \beta}} s \delta^s_\alpha e^\gamma - \frac{1}{\lambda^{\frac{d+1}{2}}} \sum_{s \geq 1} \frac{\partial \Delta F_{\alpha \beta}}{\partial v_{\alpha \beta}} s \delta^s_\alpha e^\gamma$$

$$- \sum_{s \geq 1} \frac{\partial \Delta F_{\alpha \beta}}{\partial v_{\alpha \beta}} \sum_{k=1}^s \left( \frac{s}{k} \right) \left( \partial^k \partial^\gamma p_\alpha \right) G^{\alpha \beta} \left( \partial^{k+1-\gamma} p_\beta \right).$$
Since $Y = \nabla Q$ is given by (10.15), and $P\partial_{E-\lambda e} p_{\gamma,\alpha}^T = e$, we have

$$-\frac{1}{\lambda} \sum_{s \geq 1} \frac{\partial \Delta F}{\partial v_{\gamma,s}} s^s \partial^s_x e^\gamma - \frac{1}{\lambda^{x+1}} \sum_{s \geq 1} \frac{\partial \Delta F}{\partial v_{\gamma,s}} s^s \partial^s_x \partial^\gamma Q$$

$$= -\frac{1}{\lambda} \sum_{s \geq 1} \frac{\partial \Delta F}{\partial v_{\gamma,s}} s^s \partial^s_x e^\gamma + \frac{1}{\lambda} \sum_{s \geq 1} \frac{\partial \Delta F}{\partial v_{\gamma,s}} s^s \partial^s_x \left[ PG \left( \partial_E + \frac{\partial}{\partial \lambda} \right) p_{\gamma,\alpha}^T \right]_{\gamma}$$

$$= \frac{1}{\lambda} \sum_{s \geq 1} \frac{\partial \Delta F}{\partial v_{\gamma,s}} s^s \partial^s_x \left[ P \partial_{E-\lambda e} p_{\gamma,\alpha}^T - e + \lambda PG \left( \partial_e + \frac{\partial}{\partial \lambda} \right) p_{\gamma,\alpha}^T \right]$$

$$= \sum_{s \geq 1} \frac{\partial \Delta F}{\partial v_{\gamma,s}} s^s \partial^s_x \left[ PG \left( \partial_e + \frac{\partial}{\partial \lambda} \right) p_{\gamma,\alpha}^T \right]$$

$$= \sum_{s \geq 1} \frac{\partial \Delta F}{\partial v_{\gamma,s}} s^s \partial^s_x \left[ \left( \partial^\gamma p_{\alpha} \right) G_{\alpha\beta} \left( \partial_e + \frac{\partial}{\partial \lambda} \right) p_{\beta} \right],$$

hence the equation (9.12) is equivalent to (10.18) if \{p_{\alpha}\} are quasi-homogeneous. On the other hand, from the expression of the equation (10.18) we see that this equation is independent of the choice of the basis of periods of $M$. The theorem is proved.\[\square\]

**Remark 10.9.** If $M$ has flat unity $e = \partial_e$, then the loop equation (10.18) coincides with the one given in [23], because in this case we can choose a basis of periods $\{p_{\alpha}\}$ of $M$ such that $\partial_e p_{\alpha} = -\frac{\partial}{\partial \lambda} p_{\alpha}$.

We will give proof of the existence and uniqueness (up to the addition of constants) of solutions of the loop equation for a generalized semisimple Frobenius manifold with non-flat unity in subsequent publications. Such a solution of the loop equation yields a quasi-Miura transformation (9.2) which gives a deformation (9.1) of the Principal Hierarchy of the generalized semisimple Frobenius manifold. In the next section, we will present two examples of generalized semisimple Frobenius manifolds, and show that the associated deformed Principal Hierarchies are closely related to the Hodge integrals and the equivariant Gromov-Witten invariants of the resolved conifold with anti-diagonal action, so we call such deformations of the Principal Hierarchy the topological deformations.

### 11 Two examples

In this section we present two examples of generalized Frobenius manifolds of dimension $n = 1$ and $2$ respectively, and establish their relations with the well-known integrable hierarchies: the Volterra hierarchy, the q-deformed KdV hierarchy and the Ablowitz-Ladik hierarchy.

**Example 11.1.** Let us consider the 1-dimensional generalized Frobenius manifold $M$ with potential

$$F = \frac{1}{12} v^4,$$ (11.1)
here we denote the flat coordinate $v^1$ of $M$ by $v$. The flat metric, the unity and the Euler vector field are given respectively by

$$\eta = 1, \quad e = \frac{1}{2v} \partial_v, \quad E = \frac{1}{2} v \partial_v.$$  

(11.2)

This generalized Frobenius manifold has charge $d = 1$, and $\mu = R = 0$.

The Hamiltonian densities $\theta_{\alpha,p}$ for the Principal Hierarchy are given by

$$\theta_{0,0} = \frac{1}{2} \log v,$$

$$\theta_{0,p} = \frac{2^{p-1}(2p-2)!!}{(2p)!} v^{2p}, \quad \theta_{0,-p} = (-1)^{p+1} \frac{(2p-1)!}{2^{p+1}(2p+1)!!} v^{-2p}, \quad p \geq 1,$$

$$\theta_{1,p} = \frac{2^p (2p-1)!!}{(2p+1)!} v^{2p+1}, \quad p \geq 0.$$

They satisfy the quasi-homogeneity conditions (3.11) and (3.13) with

$$r^c_p = 0, \quad c_p = \frac{1}{4} \delta_{p,0}.$$

The first few flows of the Principal Hierarchy (3.16) have the expressions

$$\frac{\partial v}{\partial t^{0,-3}} = -\frac{15}{8} v^5 x, \quad \frac{\partial v}{\partial t^{0,-2}} = \frac{3}{4} v^4 x, \quad \frac{\partial v}{\partial t^{0,-1}} = -\frac{1}{2} v^2 x,$$

$$\frac{\partial v}{\partial t^{0,0}} = v_x, \quad \frac{\partial v}{\partial t^{0,1}} = 2v^2 x, \quad \frac{\partial v}{\partial t^{0,2}} = \frac{4}{3} v^4 x,$$

$$\frac{\partial v}{\partial t^{1,0}} = 2v v_x, \quad \frac{\partial v}{\partial t^{1,1}} = 2v^3 x, \quad \frac{\partial v}{\partial t^{1,2}} = v^5 x.$$

The tau function $\tau^{[0]} = e^f$ of the Principal Hierarchy is defined by

$$\frac{\partial^2 \log \tau^{[0]}}{\partial t^{\alpha,p} \partial t^{\beta,q}} = \Omega_{\alpha,p;\beta,q} = \partial_x^{-1} \frac{\partial \theta_{\alpha,p}}{\partial t^{\beta,q}}.$$

In particular, we have

$$\log v = 2 \frac{\partial^2 \log \tau^{[0]}}{\partial x^2}, \quad v = \frac{\partial^2 \log \tau^{[0]}}{\partial x \partial t^{1,0}}.$$

We have the Virasoro operators

$$L_{-1} = \sum_{p \in \mathbb{Z}} t^{0,p+1} \frac{\partial}{\partial t^{0,p}} + \sum_{p \geq 0} t^{1,p+1} \frac{\partial}{\partial t^{1,p}} + \frac{1}{2} t^{1,0} t^{1,0},$$

$$L_0 = \sum_{p \in \mathbb{Z}} p t^{0,p} \frac{\partial}{\partial t^{0,p}} + \sum_{p \geq 0} (p + \frac{1}{2}) t^{1,p} \frac{\partial}{\partial t^{1,p}} + \frac{1}{8} \sum_{p \in \mathbb{Z}} (-1)^p p t^{0,p} t^{0,-p} + \frac{1}{16}$$

$$L_m = \frac{1}{2} \sum_{p=0}^{m-1} \frac{(2p+1)!!(2m-2p-1)!!}{2^{m+1}} \frac{\partial^2}{\partial t^{1,p} \partial t^{1,m-1-p}}.$$
\[ + \sum_{p \geq 1} \frac{(p + m)!}{(p - 1)!} \left( t_0^p \frac{\partial}{\partial t_0^{p+m}} + (-1)^{m+1} t_0^{p-m} \frac{\partial}{\partial t_0^{-p}} \right) \]

\[ + \sum_{p \geq 0} \frac{(2p + 2m + 1)!!}{2^{m+1}(2p - 1)!!} t^p \frac{\partial}{\partial t_0^{p+m}} \]

\[ + \frac{1}{4} \sum_{p \geq 1} (-1)^{p+m} \frac{(p + m)!}{(p - 1)!} \left( \sum_{k=p}^{p+m} \frac{1}{k} \right) t_0^p t_0^{p-m-p} \]

\[ + \frac{1}{8} \sum_{p=0}^{m} (-1)^m p! (m-p)! t_0^{p-m-p}, \quad \forall m \geq 1. \]

The generalized Frobenius manifold \( M \) has a quasi-homogeneous period

\[ p(v, \lambda) = \log(v + \sqrt{v^2 - \lambda}) \]

with the associated Gram matrix \((G^{\alpha\beta}) = 1\) and the star product

\[ \frac{\partial p}{\partial \lambda} * \frac{\partial p}{\partial \lambda} = \frac{1}{8\lambda^2} - \frac{1}{8(v^2 - \lambda)^2}. \]

Thus the loop equation \([10.18]\) of \( M \) reads

\[ \sum_{s \geq 0} \frac{\partial \Delta F}{\partial v(s) \partial x^s} \frac{1}{2v(v^2 - \lambda)} \sum_{s \geq 1} \frac{\partial \Delta F}{\partial v(s) \partial x^s} \left[ \frac{1}{2\lambda} \left( \frac{1}{\sqrt{v^2 - \lambda}} - \frac{1}{v} \right) \right] \]

\[ + \sum_{s \geq 1} \frac{\partial \Delta F}{\partial v(s)} \sum_{k=1}^{s} \left( \frac{k^{s-k-1}}{2\lambda} \left( \frac{v}{\sqrt{v^2 - \lambda}} - 1 \right) \left( \frac{k^{s+1-k}}{\sqrt{v^2 - \lambda}} \right) \right) \]

\[ = \frac{1}{2} \varepsilon^2 \sum_{k,\ell \geq 0} \left( \frac{\partial \Delta F}{\partial v(k) \partial v(\ell)} + \frac{\partial^2 \Delta F}{\partial v(k) \partial v(\ell)} \right) \left( \frac{v^2 v_x}{(v^2 - \lambda)^3} - \frac{1}{16(v^2 - \lambda)^2} \right) \]

\[ + \frac{1}{2} \varepsilon^2 \sum_{k \geq 0} \frac{\partial \Delta F}{\partial v(k)} \frac{v^2 v_x}{(v^2 - \lambda)^3} - \frac{1}{16(v^2 - \lambda)^2}. \]

This equation has a unique solution, up to the addition of constant series \(a_1 \varepsilon^{-1} + a_2 + a_3 \varepsilon + \ldots, \)

\[ \Delta F = \sum_{k \geq 1} \varepsilon^{k-2} F[k] = \sum_{g \geq 1} \varepsilon^{2g-2} F_g(v, v_x, \ldots, v^{(3g-2)}). \]

The coefficient of \( \varepsilon^0 \) of the loop equation yields the equation

\[ \frac{1}{(v^2 - \lambda)^2} \left( \frac{1}{16} - \frac{3}{2} v_x \frac{\partial F_1}{\partial v_x} \right) + \frac{1}{2v^2(v^2 - \lambda)} \left( v \frac{\partial F_1}{\partial v} - 2v_x \frac{\partial F_1}{\partial v_x} \right) = 0, \]

which leads to

\[ F_1 = \frac{1}{24} \log v_x + \frac{1}{12} \log v. \]

One can solve the loop equation recursively to obtain \( F_g, g \geq 2. \) For example, we have

\[ F_2 = \frac{v_{xx}}{120v} - \frac{v_x^2}{120v^2} + \frac{v_x^4}{576v^2} + \frac{37v(3)}{2880v_x} + \frac{v_x^3}{180v^2} - \frac{11v_x^2}{960v^2} = 7v_x v_{xx} v(v^3). \]
Let us note that after the replacement
\[ v \rightarrow e^{-\frac{1}{2}\varepsilon}, \quad \varepsilon \rightarrow -\frac{i}{\sqrt{2}}\varepsilon, \]
the function \( F_1(v, v_x) + \varepsilon^2 F_2(v, v_x, \ldots, v^{(4)}) \) coincides, up to the addition of a constant, with the function \( H_1(v, v_x) + \varepsilon^2 H_2(v_x, v_{xx}, v^{(3)}, v^{(4)}) \) given in (26), (27) of [20] as the generating function for the genus one and two special cubic Hodge integrals, see also [17] [18].

Now let us now consider the topological deformation (9.1) of the Principal Hierarchy [3] [16] obtained via the quasi-Miura transformation (9.2), i.e.,
\[ w = v + \varepsilon^2 \frac{\partial^2 F_1}{\partial x \partial t^{1.0}} + \varepsilon^4 \frac{\partial^2 F_2}{\partial x \partial t^{1.0}} + \ldots. \]

In order to represent the deformed integrable hierarchy in a simple form, we use the unknown function
\[
U = \frac{1}{2} \Lambda - \Lambda^{-1} \sqrt{2\varepsilon} \frac{\partial}{\partial x} \left( v + \varepsilon^2 \frac{\partial^2 F_1}{\partial x \partial t^{1.0}} + \varepsilon^4 \frac{\partial^2 F_2}{\partial x \partial t^{1.0}} + \ldots \right) = w - \frac{\varepsilon^2}{3} w_{xx} + \frac{\varepsilon^4}{30} w^{(4)} - \frac{\varepsilon^6}{630} w^{(6)} + \ldots, \tag{11.3}
\]
where
\[ \Lambda = e^{\sqrt{2\varepsilon} \partial_x} = e^{\varepsilon \partial_x}, \quad \varepsilon = \sqrt{2\xi}. \]

Then, at the approximation up to \( \varepsilon^6 \), we can verify that \( U \) satisfies the following equation
\[
\frac{\partial U}{\partial t^{1.0}} = \frac{4U}{\sqrt{2\xi} \Lambda + 1} U = \frac{4U}{\varepsilon \Lambda + 1} U. \tag{11.4}
\]

This equation is just the \( q \)-deformed KdV equation introduced in [27], and it can be represented as the following Lax equation:
here the Lax operator is given by

\[ L = \Lambda^2 + U\Lambda + 1 . \]  

(11.5)

We can verify that the \( t^{1,1} \)-flow and the \( t^{1,2} \)-flow of the deformed Principal Hierarchy can be represented, also at the approximation up to \( \epsilon^6 \), by the Lax equations

\[
\begin{align*}
\frac{\partial L}{\partial t^{1,1}} &= \frac{32}{3\epsilon} \left( \left( L^2 \right)_+ - \frac{3}{2} \left( L^2 \right)_+ , L \right), \\
\frac{\partial L}{\partial t^{1,2}} &= -\frac{256}{15\epsilon} \left( \left( L^2 \right)_+ - \frac{5}{2} \left( L^2 \right)_+ + \frac{15}{8} \left( L^2 \right)_+ , L \right).
\end{align*}
\]

Let us proceed to consider the flows \( \frac{\partial}{\partial t^{0,-1}} \), \( p \in \mathbb{Z} \) of the deformed Principal Hierarchy. Introduce the unknown function

\[
W = -\left( \Lambda^\frac{1}{2} + \Lambda^{-\frac{1}{2}} \right) \log U = -2 \log v + \epsilon^2 w_1(v, v_x, \ldots) + \ldots
\]

\[ = w + \epsilon^2 w_1(v, v_x, \ldots) + \ldots, \]  

(11.6)

then at the approximation up to \( \epsilon^6 \) we know that \( W \) satisfies the Volterra equation (also called the discrete KdV equation)

\[
\frac{\partial W}{\partial t^{0,-1}} = -\frac{1}{4\epsilon} \left( \Lambda - \Lambda^{-1} \right) e^W,
\]

which is a well-known discrete integrable system [32, 40] originally introduced for the description of the population dynamics [38, 39, 45]. The relation of this equation to the special cubic Hodge integrals is given in [17, 18], which is in accordance with the aforementioned relation between the function \( F_1 + \epsilon^2 F_2 \) and the generating function of the special cubic Hodge integrals. In terms of the Lax operator

\[ L = \Lambda + e^W \Lambda^{-1}, \]  

(11.7)

the Volterra equation can be represented by the Lax equation

\[
\frac{\partial L}{\partial t^{0,-1}} = -\frac{1}{4\epsilon} \left( \mathcal{L}^2 \right)^+ , L \right).
\]

The \( t^{0,-2} \)-flow and the \( t^{0,-3} \)-flow can also be represented, at the approximation up to \( \epsilon^6 \), by the Lax equations

\[
\frac{\partial L}{\partial t^{0,-2}} = \frac{1}{16\epsilon} \left( \left( \mathcal{L}^4 \right)^+ , L \right), \quad \frac{\partial L}{\partial t^{0,-3}} = -\frac{1}{32\epsilon} \left( \left( \mathcal{L}^6 \right)^+ , L \right).
\]

We have the following conjecture which relates the topological deformation of the Principal Hierarchy with the q-deformed KdV hierarchy and the Volterra hierarchy.

**Conjecture 11.1.** The \( \frac{\partial}{\partial t^{1,k}} \) -flows and the \( \frac{\partial}{\partial t^{0,-m}} \) -flows of the topological deformation of the Principal Hierarchy of the generalized Frobenius manifold defined by (11.1), (11.2), for \( k \geq 0 \) and \( m \geq 1 \), can be represented by the following Lax equations:

\[
\frac{\partial L}{\partial t^{1,k}} = \sum_{\ell=0}^{k} (-1)^{k+\ell+1} \frac{2^{3k-\ell+2}}{\ell!(2k-2\ell+1)!\epsilon} \left[ \left( \mathcal{L}^{k-\ell+\frac{1}{2}} \right)^+ + L \right], \quad k \geq 0
\]
\[
\frac{\partial \mathcal{L}}{\partial t^0_{m}} = (-1)^m \frac{(m-1)!}{2^{2m} \epsilon} \left[(\mathcal{L}^{2m})_+, \mathcal{L}\right], \quad m \geq 1.
\]

Here the Lax operators \( \mathcal{L}, \mathcal{L} \) are defined by (11.3), (11.5) and (11.6), (11.7).

For the flows \( \frac{\partial}{\partial v^p} \), \( p \geq 0 \) we have not found their relation with known integrable hierarchies yet.

**Example 11.2.** We consider the 2-dimensional generalized Frobenius manifold with potential

\[
F = \frac{1}{2} (v^1)^2 v^2 + v^1 e^{v^2} + \frac{1}{2} (v^1)^2 \log v^1.
\]  

(11.8)

The flat metric, the unity and the Euler vector field are given respectively by

\[
\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e = \frac{v^1 \partial_{v^1} - \partial_{v^2}}{v^1 - e^{v^2}}, \quad E = v^1 \partial_{v^1} + \partial_{v^2}.
\]  

(11.9)

It has change \( d = 1 \) and monodromy data

\[
\mu = \begin{pmatrix} -\frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}, \quad R = R_1 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}.
\]

This generalized Frobenius manifold and its relation to the dispersionless Ablowitz-Ladik hierarchy are presented in [6], see also [34].

The Hamiltonian densities \( \theta_{2,p}, p \geq 0 \) and \( \theta_{0,-p}, p \geq 1 \) for the Principal Hierarchy are given by

\[
\theta_{2,p} = \frac{1}{(p+1)!} \sum_{k=0}^{p+1} \binom{p+1}{k} \binom{p+k}{k} e^{k v^2} (v^1 - e^{v^2})^{p+1-k},
\]

\[
\theta_{0,0} = v^2 - \log(v^1 - e^{v^2})
\]

\[
\theta_{0,-p} = \frac{(-1)^p (p-1)!}{(v^1 - e^{v^2})^{2p}} \theta_{2,p-1}, \quad p \geq 1,
\]

and the first few \( \theta_{0,p} \) and \( \theta_{1,p} \) for \( p \geq 0 \) have the expressions

\[
\theta_{0,1} = v^1 v^2,
\]

\[
\theta_{0,2} = \frac{1}{2} (v^2 + 1)(v^1)^2 + (v^2 - 1)e^{v^2} v^1,
\]

\[
\theta_{0,3} = \frac{1}{12} v^1 \left[ (2v^2 + 3)(v^1)^2 + 6e^{v^2}(2v^2 - 1)v^1 + e^{2v^2}(6v^2 - 9) \right],
\]

\[
\theta_{1,0} = v^2,
\]

\[
\theta_{1,1} = (v^2 + \log v^1) v^1 + (e^{v^2} - v^1),
\]

\[
\theta_{1,2} = (v^2 + \log v^1) \theta_{2,1} + \frac{1}{4} \left( e^{2v^2} - 4e^{v^2} v^1 - (v^1)^2 \right),
\]

\[
\theta_{1,3} = (v^2 + \log v^1) \theta_{2,2} + \frac{1}{18} \left( e^{3v^2} - 9e^{2v^2} v^1 - 27e^{v^2}(v^1)^2 - (v^1)^3 \right).
\]
We have the Virasoro operators
\[
L_{-1} = \sum_{p \geq 0} \left( t^{1,p+1} \frac{\partial}{\partial t^{1,p}} + t^{2,p+1} \frac{\partial}{\partial t^{2,p}} \right) + \sum_{p \in \mathbb{Z}} t^{0,p+1} \frac{\partial}{\partial t^{0,p}} + t^{1,0} t^{2,0},
\]
\[
L_0 = \sum_{p \in \mathbb{Z}} pt^{0,p} \frac{\partial}{\partial t^{0,p}} + \sum_{p \geq 1} p \left( t^{1,p} \frac{\partial}{\partial t^{1,p}} + t^{2,p-1} \frac{\partial}{\partial t^{2,p-1}} \right)
+ \sum_{p \geq 1} \left( t^{0,p} \frac{\partial}{\partial t^{2,p-1}} + 2t^{1,p} \frac{\partial}{\partial t^{2,p-1}} \right) + \sum_{p \geq 0} (-1)^p t^{0,-p} t^{1,p} + t^{1,0} t^{1,0},
\]
\[
L_{m \geq 1} = \sum_{p=1}^{m-1} p!(m-p)! \frac{\partial^2}{\partial t^{1,p+1} \partial t^{2,m-p-1}}
+ \sum_{p \geq 1} \frac{(p+m)!}{(p-1)!} \left( t^{0,p} \frac{\partial}{\partial t^{0,p+m}} + (-1)^{m+1} t^{0,-p-m} \frac{\partial}{\partial t^{0,-p}} \right)
+ t^{1,p} \frac{\partial}{\partial t^{1,p+m}} + t^{2,p-1} \frac{\partial}{\partial t^{2,p+m-1}}
+ \sum_{p=0}^{m-1} (-1)^p p! (m-p)! t^{0,-p} \frac{\partial}{\partial t^{2,-p+m-1}}
+ \sum_{p \geq 1} \frac{(p+m)!}{(p-1)!} \left( \sum_{k=p}^{p+m} \frac{1}{k} \right) t^{0,p} \frac{\partial}{\partial t^{2,p+m-1}}
+ 2m! t^{1,0} \frac{\partial}{\partial t^{2,m-1}} + 2 \sum_{p \geq 1} \frac{(p+m)!}{(p-1)!} \left( \sum_{k=p}^{p+m} \frac{1}{k} \right) t^{1,p} \frac{\partial}{\partial t^{2,p+m-1}}
+ (-1)^m m! t^{0,-m} t^{1,0} + \sum_{p \geq 1} (-1)^{p+m} \frac{p+m}{(p-1)!} \left( \sum_{k=p}^{p+m} \frac{1}{k} \right) t^{0,-p-m} t^{1,p}.
\]

The intersection form of the generalized Frobenius manifold has the expression
\[
(g^{\alpha \beta}) = \begin{pmatrix}
2v^1 e^{v^2} & v^1 + e^{v^2} \\
v^1 + e^{v^2} & 2
\end{pmatrix}.
\]

We have the following basis of quasi-homogeneous periods
\[
p_1 = \log \left( v^1 - \lambda - e^{v^2} + \sqrt{D} \right) - \frac{v^2}{2}, \quad p_2 = v^2
\]
with the associated Gram matrix
\[
(G^{\alpha \beta}) = \begin{pmatrix}
-2 & 0 \\0 & \frac{1}{2}
\end{pmatrix}.
\]

It can be verified that
\[
\frac{1}{E - \lambda e} = \frac{v^1 (3e^{v^2} + v^1 - \lambda) \partial_{v^1} - (e^{v^2} + 3v^1 - \lambda) \partial_{v^2}}{(v^1 - e^{v^2}) D},
\]

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where \( D = (\lambda - v^1 - e^{v^2})^2 - 4v^1e^{v^2} \). Hence the loop equation (10.18) reads

\[
\sum_{s \geq 0} \left( \frac{\partial \Delta F}{\partial v^{1,s}} \frac{3e^{v^2} + v^1 - \lambda}{(v^1 - e^{v^2})D} \right) - \frac{\partial \Delta F}{\partial v^{2,s}} \frac{e^{v^2} + 3v^1 - \lambda}{(v^1 - e^{v^2})D} \\
- \frac{1}{\lambda} \sum_{s \geq 1} \sum_{k \geq 1} \frac{(s)}{k} \left( \frac{\partial^k \Delta F}{\partial v^1, \partial v^{2,s}} \right) \frac{1}{\sqrt{D}} \left( \frac{\partial \Delta F}{\partial v^{1,s}} \frac{\partial x^{k+1}}{\partial v^1 \frac{\lambda - v^1 - e^{v^2}}{\sqrt{D}} + \frac{\partial \Delta F}{\partial v^{2,s}} \frac{\partial x^{k+1}}{\partial v^1 \frac{2}{\sqrt{D}}} \right) \\
\right)
\]

\[
= - \frac{1}{4} \sum_{k,\ell \geq 0} \left( \frac{\partial \Delta F}{\partial v^{1,k}} \frac{\partial \Delta F}{\partial v^{1,\ell}} \right) \frac{\partial^k \Delta F}{\partial v^1, \partial v^{1,\ell}} + \frac{\partial \Delta F}{\partial v^{2,k}} \frac{\partial \Delta F}{\partial v^{2,\ell}} \frac{\partial^k \Delta F}{\partial v^{2,k} \partial v^{2,\ell}} \frac{1}{\sqrt{D}} \cdot \frac{\partial x^{k+1}}{\partial v^1 \frac{2}{\sqrt{D}}}
\]

\[
- \varepsilon^2 \sum_{k \geq 0} \left( \frac{\partial \Delta F}{\partial v^{2,k}} \frac{\partial x^{k+1}}{\partial v^1} \frac{e^{v^2} \left( K v^1_x - 2 \lambda v^1 + \left( (e^{v^2} - v^1)^2 - \lambda^2 \right) v^2_x \right)}{D^3} \right)
\]

\[
- \frac{\partial \Delta F}{\partial v^{2,k}} \frac{\partial x^{k+1}}{\partial v^1} \frac{\left( 2 \left( (e^{v^2} - v^1)^2 - \lambda^2 \right) v^1_x - K v^2_x \right)}{D^3} \frac{1}{D^2},
\]

where

\[
K = (v^1 + e^{v^2})D + 8\lambda v^1 e^{v^2}.
\]

The loop equation has a unique solution

\[
\Delta F = \sum_{k \geq 1} \varepsilon^{k-2} F[k] = \sum_{g \geq 1} \varepsilon^{2g-2} F_g(v^1, v^2, v^1_x, v^2_x, \ldots, v^{1,3g-2}, v^{2,3g-2})
\]

up to the addition of constant series \( b_1 \varepsilon^{-1} + b_2 + b_3 \varepsilon + \ldots \). The coefficient of \( \varepsilon^0 \) of the loop equation yields the following equation for \( F_1 \):

\[
\frac{\partial F_1}{\partial v^1} \frac{v^1(3e^{v^2} + v^1 - \lambda)}{(v^1 - e^{v^2})D} - \frac{\partial F_1}{\partial v^2} \frac{e^{v^2} + 3v^1 - \lambda}{(v^1 - e^{v^2})D} + \frac{\partial F_1}{\partial v^1} \frac{v^1(3e^{v^2} + v^1 - \lambda)}{(v^1 - e^{v^2})D} \\
- \frac{\partial F_1}{\partial v^2} \frac{e^{v^2} + 3v^1 - \lambda}{(v^1 - e^{v^2})D} - \frac{1}{\lambda} \left( \frac{\partial F_1}{\partial v^1} \frac{v^1}{v^1 - e^{v^2}} - \frac{\partial F_1}{\partial v^2} \frac{1}{v^1 - e^{v^2}} \right) \\
- \frac{\lambda + v^1 - e^{v^2}}{2\lambda \sqrt{D}} \left( \frac{\partial F_1}{\partial v^1} \frac{\lambda - v^1 - e^{v^2}}{\sqrt{D}} + \frac{\partial F_1}{\partial v^2} \frac{2}{\sqrt{D}} \right) + v^1e^{v^2} = 0,
\]

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from which it follows that

\[ \mathcal{F}_1 = \frac{1}{24} \log \left( (v_1^1)^2 - v^1 e^{v^2} (v_2^2)^2 \right) + \frac{1}{12} \log \left( v^1 - e^{v^2} \right) - \frac{1}{8} \log v^1 - \frac{1}{24} v^2. \]

This function coincides, up to the addition of a constant, with the following genus one free energy for the equivariant Gromov-Witten invariants of the resolved conifold with anti-diagonal action [5]:

\[ \tilde{\mathcal{F}}_1 = \frac{1}{24} \log \left( v'(x)^2 + \frac{\lambda^2 e^{w(x)}}{1 - e^{w(x)}} w'(x)^2 \right) + \frac{1}{12} \text{Li}_1 \left( e^{w(x)} \right) - \frac{w(x)}{24} \]

after putting \( \lambda = 1 \) and taking the change of variables

\[ v^1 = e^v (e^w - 1), \quad v^2 = v + w, \]

which relates the flat coordinates of the Frobenius manifold associated with the Ablowitz-Ladik hierarchy and its almost dual [6].

Now let us consider the relation of the deformed Principal Hierarchy with the positive flows of the Ablowitz-Ladik hierarchy defined by \[1, 2, 6, 34, 44\]

\[ \frac{\partial L}{\partial t_k} = \frac{1}{(k + 1)!} [(L^{k+1})_+, L], \quad k \geq 0, \tag{11.10} \]

where the Lax operator \( L \) has the expression

\[ L = (1 - QA^{-1})^{-1}(\Lambda - P) = \Lambda + Q - P + Q(\Lambda - Q)^{-1}(Q - P) \]

with \( \Lambda = e^{\epsilon \partial_x} \). The first and the second positive flows of the Ablowitz-Ladik hierarchy have the expressions \[34\]

\begin{align*}
\varepsilon P_{t_0} &= P(Q^+ - Q), \quad \varepsilon Q_{t_0} = Q(Q^+ - Q^- - P + P^-), \\
\varepsilon P_{t_1} &= \frac{1}{2} P(PQ - PQ^+ + P^+ - P^+Q + Q^+Q^+ + Q^+Q^- - QQ^- - Q^2), \\
\varepsilon Q_{t_1} &= \frac{1}{2} Q(P^2 - P^2 - P^2Q + 2PQ^+ - PQ + P^2Q - 2PQ^+ - Q^+Q^- - Q^+Q^- - Q^+Q^- - Q^2). \\
\end{align*}

Here we use the notion \( P^\pm = \Lambda^\pm P, Q^\pm = \Lambda^\pm Q \). Introduce the new unknown functions

\[ U = Q - P, \quad W = \log Q, \]

and let

\begin{align*}
w^1 &= U - \frac{\varepsilon}{2} U_x + \frac{\varepsilon^2}{12} U_{xx}, \\
w^2 &= W - \frac{\varepsilon}{2} \partial_x \log U + \frac{\varepsilon^2}{12} \frac{\partial}{\partial x} \left( \frac{2U_x - (2e^W + U)W_x}{U} \right) \tag{11.11}
\end{align*}
\[ + \varepsilon^3 \frac{\partial}{12 \partial x} \left( \frac{(UU_{xx} - U^2_x)\xi^W}{U^3} \right), \quad \text{(11.12)} \]

then the first positive flow of the Ablowitz-Ladik hierarchy can be represented in the form

\[
\frac{\partial w^1}{\partial t_0} = e^{w^2} w^1_x + e^{w^2} w^1 w_x^2 + \varepsilon^{\frac{2}{3}} w^2 \left( 5(w_0^1)^3 + 4e^{w^2}(w_0^1)^2 w_x^2 - 5w^1(1)^2 w_x^2 - 8e^{w^2} w^1_1(1)^2 w_x^2 \right) + 8e^{w^2}(1)(w_x^2)^3 - 10w^1_1 w_x^1 w_x^1 - 4e^{w^2} w_x^2 w_x^2 w_x^1 + 6(w_0^1)^2 w_x^1 w_x^1 \\
- 4e^{w^2} w^1 w_x^1 w_x^2 + 2(w_0^1)^2 w_x^2 w_x^2 + 16e^{w^2}(1)(1)^2 w_x^2 w_x^1 + 2(w_0^1)^3 w_x^2 w_x^2 \\
+ 6(w_0^1)^2 w_x^1 w_x^2 + 4e^{w^2}(1)(1)^2 w_x^2 w_x^2 + 2(w_0^1)^3 w_x^2 w_x^2 \right) + O(\varepsilon^4),
\]

\[
\frac{\partial w^2}{\partial t_0} = w^1_x + e^{w^2} w_x^2 + \varepsilon^{\frac{2}{3}} w^2 \left( 6(w_0^1)^3 - 3w^1_1(1)^2 w_x^2 - 8w^1_1 w_x^1 w_x^1 + 2(w_0^1)^2 w_x^1 w_x^1 \\
+ 2(w_0^1)^3 w_x^2 w_x^2 + 2(w_0^1)^2 w_x^1 w_x^2 + 2(w_0^1)^1 w_x^2 w_x^2 \right) + O(\varepsilon^4).
\]

This flow coincides, at the approximation up to \( \varepsilon^4 \), with the \( \frac{\partial}{\partial x^3} \)-flow of the deformed Principal Hierarchy \( (9.1) \), where the unknown functions \( w^1, w^2 \) are related to the unknown functions \( v^1, v^2 \) of the Principal Hierarchy by the quasi-Miura transformation \( (9.2) \), i.e.,

\[
w^1 = \varepsilon^2 \frac{\partial^2 \log \tau}{\partial x \partial t_0^2} = v^1 \quad \text{and} \quad \frac{\partial \Delta F}{\partial x \partial t_0^2}, \quad w^2 = \varepsilon^2 \frac{\partial^2 \log \tau}{\partial x \partial t_1^1} = v^2 \quad \text{and} \quad \frac{\partial \Delta F}{\partial x \partial t_1^1}, \quad \text{(11.13)}
\]

where the \( \tau \) function is defined by

\[
\log \tau = \varepsilon^{-2} f + \sum_{g \geq 1} \varepsilon^{2g-2} \mathcal{F}_g = \varepsilon^{-2} f + \Delta \mathcal{F}.
\]

We can also verify such a relation of the second positive flow \( \frac{\partial}{\partial x^1} \) of the Ablowitz-Ladik hierarchy with the \( \frac{\partial}{\partial x^3} \)-flow of the deformed Principal Hierarchy.

We also obtain the explicit expression of the function \( \mathcal{F}_2 \) which is quite long, so we do not present here. By using the expression of \( \mathcal{F}_2 \) and by adding certain \( \varepsilon^4 \)-terms at the right hand sides of \( (11.11), \) \( (11.12) \), we also check the validity of the above-mentioned relation of the first two positive flows of the Ablowitz-Ladik hierarchy and the \( \frac{\partial}{\partial x^p} \)-flows \( (p = 0, 1) \) of the topological deformation of the Principal Hierarchy at the approximation up to \( \varepsilon^4 \). Actually, we conjecture that under the Miura-type transformation given by the relation

\[
U = \sum_{k=0}^{\infty} \frac{\varepsilon^k}{(k + 1)!} \partial_x^k w^1, \quad \text{(11.14)}
\]
\[ W = \left( \sum_{k=0}^{\infty} \frac{(-1)^k \varepsilon^k}{(k+1)!} \left( \frac{\partial}{\partial x} \right)^k \right) \left( \sum_{\ell=0}^{\infty} \frac{\varepsilon^{\ell}}{\ell (\ell + 1)!} \left( \frac{\partial}{\partial t^{1,0}} \right)^{\ell} \right) w^2, \quad (11.15) \]

the positive flows \( \frac{\partial}{\partial t_k} \) of the Ablowitz-Ladik hierarchy (11.10) coincide with the flows \( \frac{\partial}{\partial t^{2,k}} \) of the topological deformation of the Principal Hierarchy. In other words, we have the following conjecture.

**Conjecture 11.2.** Let \( \tau \) be a tau function of the topological deformation of the Principal Hierarchy of the generalized Frobenius manifold defined by (11.8), (11.9), then the following special two-point functions

\[ U = \varepsilon (\Lambda - 1) \frac{\partial \log \tau}{\partial t^{2,0}}, \quad W = (1 - \Lambda^{-1}) \left( e^{\varepsilon \frac{\partial}{\partial x}} - 1 \right) \log \tau \quad (11.16) \]

satisfy the positive flows of the Ablowitz-Ladik hierarchy (11.10) if we identify \( t^{2,k} \) with \( t_k \) for \( k \geq 0 \), here \( \Lambda = e^{\varepsilon \frac{\partial}{\partial x}} \).

From (11.13) we see that the relations (11.14), (11.15) follow from (11.16). This representation of the unknown functions \( U, W \) in terms the tau function was obtained by D. Yang and C. Zhou in their study of the relation between the equivariant Gromov-Witten invariants of the resolved conifold with anti-diagonal action and the Ablowitz-Ladik hierarchy [46].

### 12 Conclusion

In this paper we study the relationship between a class of generalized Frobenius manifolds with non-flat unit vector fields and integrable hierarchies. For any such generalized Frobenius manifold, we construct an analogue of the Principal Hierarchy of a usual Frobenius manifold which contains infinitely many additional flows. We show that this Principal Hierarchy has a tau-cover, and possesses Virasoro symmetries which can be lifted to the tau-cover of the integrable hierarchy. The condition of linearization of the actions of the Virasoro symmetries on the tau function enables us to derive the loop equation for the generalized Frobenius manifold, and the unique solution of the loop equation (in the semisimple case) yields a quasi-Miura transformation which gives the topological deformation of the Principal Hierarchy.

In subsequent publications, we will prove the existence and uniqueness of solutions of the loop equation of a generalized semisimple Frobenius manifold with non-flat unity, and study the bihamiltonian structure of the topological deformation of the Principal Hierarchy. We will also study more examples of such generalized Frobenius manifolds and their relations to integrable systems. Notably, even in the 1-dimensional case we have other examples apart from the one that is given in the present paper, which may have close relations to some important integrable hierarchies. For example, we have the 1-dimensional generalized Frobenius manifold with

\[ F = e^v, \quad E = \partial_v, \quad e = e^{-v} \partial_v, \]
its charge and flat metric are given by $d = 2$ and $\eta = 1$, and its monodromy data at $z = 0$ are given by $\mu = R = 0$. The Hamiltonian densities $\theta_{i,p}$ of the Principal Hierarchy have the expressions

$$
\theta_{0,p} = \begin{cases}
\frac{1}{2} v^2 (v - H_{p-1}) e^{(p-1)v} & p = 1 \\
\frac{1}{(p-1)!} (v - H_{p-1}) e^{(p-1)v} & p \geq 2
\end{cases}
$$

$$
\theta_{0,-p} = \frac{(-1)^{p+1} p!}{p+1} e^{-(p+1)v}, \quad p \geq 0
$$

$$
\theta_{1,p} = \begin{cases}
v & p = 0 \\
\frac{1}{p!} v^p & p \geq 1
\end{cases}
$$

where $H_p = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{p}$. The loop equation has a solution $\Delta F = F_1 + \varepsilon^2 F_2 + \cdots$

with

$$
F_1 = \frac{1}{24} \log v + \frac{1}{12} v,
$$

$$
F_2 = \frac{49e^v v_x^2}{5760} + \frac{e^v v_{xx}}{64} - \frac{11e^{2v} v_{xxx}}{1920v_x^2} + \frac{e^v v_x^{3}}{360v_x^4} + \frac{37e^{3v} v_{xxxx}}{5760v_x^5} - \frac{7e^v v_{xxxx} v_{xxx}}{1920v_x^{3}} + \frac{e^v v_{xxxxx}}{1152v_x^{4}}.
$$

We will study the topological deformation of the corresponding Principal Hierarchy in subsequent works.

Before we list other examples of 1-dimensional generalized Frobenius manifolds, let us first introduce the following notion.

**Definition 12.1.** Two generalized Frobenius manifolds with non-flat unity $M$ and $M'$ are called equivalent, if there exists a smooth or analytic homeomorphism $h: M \rightarrow M'$, such that

$$
dh(E) = E', \quad h^*(\eta') = \eta, \quad h^*(c') = kc
$$

for some nonzero constant $k$, where $E, \eta$ and $E', \eta'$ are the Euler vector fields, the flat metrics of $M$ and $M'$ respectively, and the 3-tensors $c, c'$ of $M$ and $M'$ are defined as in Definition 2.1.

From the above definition, it follows that if $M, M'$ are equivalent, then their charges are equal, i.e., $d = d'$, and $dh(e) = ke'$. We can classify the 1-dimensional generalized Frobenius manifolds with non-flat unity and get the following result.

**Proposition 12.1.** There are four equivalence classes of 1-dimensional generalized Frobenius manifolds with non-flat unity, and their representatives are given in the following table:

| charge | Euler vector field | potential | unit vector field |
|--------|-------------------|-----------|------------------|
| $d = 2$ | $E = v \partial_v$ | $F = e^v$ | $e = r^{d-2} e^{-\frac{d}{2}} \partial_v$ |
| $d = 3$ | $E = -\frac{1}{2} v \partial_v$ | $F = \log v$ | $e = \frac{1}{2} v^3 \partial_v$ |
| $d = 4$ | $E = -v \partial_v$ | $F = v \log v$ | $e = -v^2 \partial_v$ |
| $d \neq 2, 3, 4$ | $E = \frac{2-d}{2} v \partial_v$ | $F = v^{\frac{d-2}{d-4}}$ | $e = -\frac{1}{4} \frac{(d-2)^3}{(d-3)(d-4)} v^{\frac{d}{d-2}} \partial_v$ |

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Here \( r \in \mathbb{C} \setminus \{0\} \) is a parameter.

We remark that the 1-dimensional generalized Frobenius manifold with \( d = 3 \) and \( F = \log v \) arises in the study of properties of a negative spin version of Witten’s r-spin class, and its relation to integrable hierarchies [9].

Finally, we remark that the genus zero free energy \( F_0 \) defined by (5.12) satisfies the genus zero Virasoro constraints

\[
\sum_{(i,p),(j,q) \in \mathcal{I}} a_{i,p;j,q}^{i,p} \frac{\partial F_0}{\partial t^i} \frac{\partial F_0}{\partial t^j} + \sum_{(i,p),(j,q) \in \mathcal{I}} b_{m;i,p}^{i,p} \frac{\partial F_0}{\partial t^i} \frac{\partial F_0}{\partial t^j} + \sum_{(i,p),(j,q) \in \mathcal{I}} c_{m;i,p;j,q}^{i,p} \frac{\partial F_0}{\partial t^i} \frac{\partial F_0}{\partial t^j} + \sum_{(i,p),(j,q) \in \mathcal{I}} \tilde{\delta}_{i,0} \delta_{j,0} \bar{C}_{m;0} \frac{\partial F_0}{\partial t^i} \frac{\partial F_0}{\partial t^j} = 0
\]

for \( m \geq -1 \), this fact can be easily proved by using the formulae (A.1) and the Euler-Lagrange equation (5.11). For a generalized semisimple Frobenius manifold \( M \), we have a unique solution

\[
\Delta F = \sum_{k \geq 1} \varepsilon^{k-2} F^k = \sum_{g \geq 1} \varepsilon^{2g-2} F_g(v,v_x,...,v^{(3g-2)}),
\]

up to the addition of a constant series \( \sum_{g \geq 1} a_g \varepsilon^{2g-2} \), of the loop equation. Then from the condition of linearization of the Virasoro symmetries of the Principal Hierarchy of \( M \) it follows that the tau function

\[
\tau = e^{\varepsilon^{-2} F_0 + \Delta F}
\]

satisfies the Virasoro constraints

\[
L_m (\varepsilon^{-1} \partial, \varepsilon \frac{\partial}{\partial t}) \tau = 0, \quad m \geq -1.
\]

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**A Proof of the Theorem 7.7**

In this appendix, we prove the Theorem 7.7. From the definition (7.8) of \( \tilde{S}^{(0)}_i \) and the relation (7.27) between the Laplace-type integrals and the extended Virasoro coefficients, it follows that

\[
\lim_{\nu \to 0} \left[ \left( \frac{\partial}{\partial t^i} \frac{\partial \tilde{S}^{(0)}_i}{\partial \lambda} \right) \tilde{G}_{i'j'} (\nu) \left( \frac{\partial}{\partial t^{j'}} \frac{\partial \tilde{S}^{(-0)}_{j'}}{\partial \lambda} \right) \right] - 2 \delta_{i,0} \delta_{j,0} \bar{C}_{p,q} (\lambda)
\]
\[-2 \sum_{(i',p',j',q') \in \mathcal{I}} a^{i',j',q'}(\lambda)\Omega_{i',p',j',q'} \Omega_{i',p',j',q} \]
\[- \sum_{(i',p',j',q') \in \mathcal{I}} \left( b^{i',j',q'}(\lambda)\Omega_{i',p',j',q'} \delta_{j, q}^{j'} + b^{i',j',q'}(\lambda)\Omega_{j',q',j, q} \delta_{i, p}^{j'} \right) \]
\[-2 \sum_{(i',p',j',q') \in \mathcal{I}} c^{i',j',q'}(\lambda) \delta_{i, p}^{i'} \delta_{j, q}^{j'} - 2\delta_{i, 0} \delta_{j, 0} C_{p,q}(\lambda), \]
\[-2 \sum_{(i',p') \in \mathcal{I}} a^{k,r,s}(\lambda)\Omega_{i',p',k,r} \Omega_{i',p',s,j,q} \]
\[- \sum_{(k,r) \in \mathcal{I}} \left( b^{k,r}(\lambda)\Omega_{k,r,j, q} + b^{k,r}(\lambda)\Omega_{i,p;k,r} \right) - 2c_{i,p,c,j,q}(\lambda) - 2\delta_{i,0} \delta_{j,0} C_{m,p,q}(\lambda), \]

so the result of the Theorem \ref{thm:main} is equivalent to the validity of the following identities:

\[ \partial_{E^{m+1}} \Omega_{i,p,c,j,q} = 2 \sum_{(k,r),(l,s) \in \mathcal{I}} a^{k,r,s}(\lambda)\Omega_{i,p,k,r} \Omega_{l,s,j,q} \]
\[ + \sum_{(k,r) \in \mathcal{I}} \left( b^{k,r}(\lambda)\Omega_{k,r,j, q} + b^{k,r}(\lambda)\Omega_{i,p;k,r} \right) \]
\[ + 2c_{m,i,p,j,q} + 2\delta_{i,0} \delta_{j,0} C_{m,p,q}(\lambda) \quad (A.1) \]

for all \((i, p), (j, q) \in \mathcal{I} \) and \(m \geq -1\), where \(a^{i,p,j,q}(\lambda), b^{k,r}(\lambda), c_{m,i,p,j,q} \) and \(C_{m,p,q}(\lambda) \) are the extended Virasoro coefficients given by Definition \ref{def:extendedVirasoro} and they are explicitly given for \(m = -1, 0, 1, 2\) by the coefficients of \(L_m\) that are presented in \ref{eq:virasoro}. From the commutation relations

\[ [E^{m+1}, E^{k+1}] = (k - m)E^{m+k+1}, \quad \forall m, k \geq -1 \]

that is proved in \cite{31} and \cite{83}, it follows that we only need to show the validity of the identities \ref{eq:virasoro} for \(m = -1, 0, 1, 2\). We can prove this fact in a similar way as it is done in \cite{22} for a usual Frobenius manifold. The main subtlety that we need to deal with for a generalized Frobenius manifold is in the computation of the derivatives of the functions \(\Omega_{0,p,0,q}\) and \(\Omega_{0,0,0,q}\) along powers of the Euler vector field.

To simplify the presentation of the proof of \ref{eq:virasoro}, we first introduce some notations. Define the \((n+1) \times (n+1)\) matrices

\[ \hat{\mu} = \begin{pmatrix} \mu_0 \\ \mu \end{pmatrix}, \quad \hat{R}_s = \begin{pmatrix} 0 \\ r_s \\ R_s \end{pmatrix}, \quad s \geq 1, \]

where \(\mu_0 = -\frac{q}{h}\), and denote \(\hat{R} = \sum_{s \geq 1} \hat{R}_s\). We also introduce the \(n \times (n+1)\) matrix-valued functions

\[ \Theta_p(z) := \sum_{k \geq 0} \left( \nabla \theta_{0,p+k}, \nabla \theta_{1,p+k}, \ldots, \nabla \theta_{n,p+k} \right) z^k, \quad p \in \mathbb{Z}, \]

here \(\theta_{i,p} = 0\) if \(i \neq 0\) and \(p < 0\), and the \(n \times n\) matrix-valued function

\[ \Theta(w) = \sum_{k \geq 0} \left( \nabla \theta_{1,k}, \ldots, \nabla \theta_{n,k} \right) w^k. \]
By using these notations, we can represent the quasi-homogeneous conditions (2.21), (3.4) and (3.9) as follows:

\[ \partial E \Theta(w) = \Theta(w)D_w - \mu \Theta(w), \quad (A.2) \]
\[ \partial E \hat{\Theta}_p(w) = \left( p - \mu + w \frac{d}{dw} \right) \hat{\Theta}_p(w) + \sum_{l \geq 0} \hat{\Theta}_{p-l}(w) \hat{B}_l, \quad (A.3) \]

where

\[ \hat{B}_0 := \hat{\mu}, \quad \hat{B}_l := \hat{R}_l \quad \text{for} \ l \geq 1, \]

and the action of the right operator \( D_w \) on an \( n \times n \) matrix function \( X(w) = \sum_{k \geq 0} X_k w^k \) is defined by

\[ X(w)D_w = w \frac{dX(w)}{dw} + X(w)B(w), \quad B(w) = \mu + \sum_{s \geq 1} R_s w^s. \]

On the other hand, from the relations (2.20), (3.3) and (3.8) it follows that

\[ \partial Y \Theta(w) = w C(Y) \Theta(w), \quad (A.4) \]
\[ \partial Y \hat{\Theta}_p(w) = C(Y) \hat{\Theta}_p - \hat{\mu} \quad (A.5) \]

for any vector field \( Y = Y^\alpha \partial_\alpha \), where \( (C(Y))^{\alpha}_{\beta} = Y^\varepsilon c^\alpha_{\varepsilon \beta} \). In particular, let \( Y \) be the Euler vector field \( E \), then we have

\[ \partial E \Theta(w) = w \mathcal{U} \Theta(w), \quad (A.6) \]
\[ \partial E \hat{\Theta}_p(w) = \mathcal{U} \hat{\Theta}_{p-1}(w), \quad (A.7) \]

where \( \mathcal{U} = (\mathcal{U}_\beta^\alpha) = (E^\gamma c^\alpha_{\beta \gamma}) \) is the operator of multiplication by the Euler vector field \( E \).

Now we introduce the \( (n + 1) \times n \) matrix-valued functions \( \hat{\Omega}_p(w), p \in \mathbb{Z}, w \in \mathbb{C} \) as follows:

\[ \left( \hat{\Omega}_p(w) \right)_{i,\beta} = \sum_{q \geq 0} \Omega_{i,p;\beta,q} w^q, \quad 0 \leq i \leq n, \ 1 \leq \beta \leq n, \]

here \( \Omega_{i,p;\beta,q} = 0 \) if \( i \neq 0 \) and \( p < 0 \). By using the relation (2.23) we know that the functions \( \Omega_{i,p;j,q} \) defined in (5.1)–(5.2) can be represented in the form

\[ \hat{\Omega}_p(w) = \hat{\Theta}_{p+1}(w) \eta \Theta(w) + (-1)^p \chi_{p < 0} \left( \begin{array}{c} 0 \\ \eta \end{array} \right) w^{-p-1}, \quad p \in \mathbb{Z}, \quad (A.8) \]

where we use the function

\[ \chi_P := \begin{cases} 1, & \text{if } P \text{ is true}, \\ 0, & \text{if } P \text{ is false}, \end{cases} \]

for any proposition \( P \). We also denote

\[ \tilde{\hat{\Omega}}_p(w) := \hat{\Theta}_{p+1}(w) \eta \Theta(w), \quad p \in \mathbb{Z}, \]

then we have

\[ \hat{\Omega}_p(w) = \tilde{\hat{\Omega}}_p(w) + (-1)^p \chi_{p < 0} \left( \begin{array}{c} 0 \\ \eta \end{array} \right) w^{-p-1}. \]
Here we emphasize that the wide tilde used in the above introduced notation does not mean that it is one for the \( (n + 2) \)-dimensional Frobenius manifold.

In terms of the notations that we just introduced, we need to compute the following derivatives:
\[
\partial_E \hat{\Omega}_p(w), \quad \partial_E \Omega_{0,p,0,q}, \quad p, q \in \mathbb{Z}, \ s = 0, 1, 2, 3.
\]

Recall that the derivatives \( \partial_e \Omega_{i,p,j,q} \) are already computed in (7.6). From the relations (5.5) it follows that
\[
\partial_e \Omega_{i,p,j,q} = \langle e, \nabla \Omega_{i,p,j,q} \rangle = \langle e, \nabla \theta_{i,p} \cdot \nabla \theta_{j,q} \rangle = \langle \nabla \theta_{i,p}, \nabla \theta_{j,q} \rangle,
\]
hence we have the identities
\[
\langle \nabla \theta_{i,p}, \nabla \theta_{j,q} \rangle = \Omega_{i,p-1;j,q} + \Omega_{i,p;j,q-1} + \eta_{\alpha \beta} \delta^\alpha_i \delta^\beta_j \delta_{p,0} \delta_{q,0}, \quad (i, p), (j, q) \in \mathcal{I}. \tag{A.9}
\]
In fact, the above identities holds true for \( (i, p), (j, q) \in \{0, 1, \ldots, n\} \times \mathbb{Z} \), since in our conventions we have \( \Omega_{\alpha,p,j,q} = 0 \) if \( \alpha \neq 0 \) and \( p < 0 \).

Now let us start to compute \( \partial_E \hat{\Omega}_p(w) \).

**Proposition A.1.** The following identities hold true:
\[
\partial_E \hat{\Omega}_p(w) = \left( p + \hat{\mu} + \frac{1}{2} \right) \hat{\Omega}_p(w) + \sum_{s \geq 1} \hat{R}_s T \hat{\Omega}_{p-s}(w)
+ \hat{\Omega}_p(w) \left( D_w + \frac{1}{2} \right), \quad p \in \mathbb{Z}. \tag{A.10}
\]

**Proof.** By using the identities (A.2), (A.3), (A.8) and the relation (2.5), we have
\[
\partial_E \hat{\Omega}_p(w) = \left( \partial_E \hat{\Theta}_{p+1}^T(-w) \right) \eta \Theta(w) + \hat{\Theta}_{p+1}^T(-w) \eta \partial_E \Theta(w)
= \left[ \left( p + 1 - w \frac{d}{dw} \right) \hat{\Theta}_{p+1}^T(-w) \right] \eta \Theta(w) - \hat{\Theta}_{p+1}^T(-w) \mu \eta \Theta(w)
+ \sum_{l \geq 0} \hat{R}_l T \hat{\Theta}_{p+1-l}^T(-w) \eta \Theta(w) + \hat{\Theta}_{p+1}^T(-w) \eta \left( \Theta(w) D_w - \mu \Theta(w) \right)
= \left( p + \hat{\mu} + 1 \right) \Theta_{p+1}^T(-w) \eta \Theta(w) + \sum_{s \geq 1} \hat{R}_s T \Theta_{p+1-s}^T(-w) \eta \Theta(w)
+ \left[ \Theta_{p+1}^T(-w) \eta \Theta(w) \right] D_w
= \left( p + \hat{\mu} + \frac{1}{2} \right) \hat{\Omega}_p(w) + \sum_{s \geq 1} \hat{R}_s T \hat{\Omega}_{p-s}(w) + \hat{\Omega}_p(w) \left( D_w + \frac{1}{2} \right).
\]
The proposition is proved. \( \square \)

We proceed to compute \( \partial_E \hat{\Omega}_p(w) \). Denote
\[
\hat{\Theta}_p = \hat{\Theta}_p(w) \big|_{w=0} = \left( \nabla \theta_{0,p}, \nabla \theta_{1,p}, \ldots, \nabla \theta_{n,p} \right), \quad p \in \mathbb{Z}.
\]

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Then it is clear that
\[
\frac{1}{w} \tilde{\Theta}^T_{p-1}(-w) + \tilde{\Theta}^T_{p}(-w) = \frac{1}{w} \tilde{\Theta}^T_{p-1},
\]
\[
\partial_E \tilde{\Theta}^T_{p+1} = (p + 1) \tilde{\Theta}^T_{p+1} - \tilde{\Theta}^T_{p+1}\mu + \sum_{l \geq 0} \tilde{B}^T_l \tilde{\Theta}^T_{p+1-l},
\]
where \( \tilde{B}_0 = \hat{\mu}, \tilde{B}_s = \hat{R}_s, s \geq 1 \). We first prove a useful lemma.

**Lemma A.2.** For \( \forall p \in \mathbb{Z} \), we have the following identity:
\[
\frac{1}{w} \tilde{\Theta}^T_p \eta \mu \Theta(w) = -\left( p + \hat{\mu} \right) \tilde{\Omega}_p(w) - \sum_{s \geq 1} \tilde{R}^s \tilde{\Omega}_{p-s}(w) + \left( \tilde{\Omega}_{p-1}(w) \frac{1}{w} \right) (D_w + 1).
\] (A.13)

In particular, for \( p, q \in \mathbb{Z} \), \( \alpha, \beta \in \{1, \ldots, n\} \), we have
\[
\nabla \theta^T_{0,p+1} \eta \mu \nabla \theta_{\beta,q} = \left( q + \mu_\beta \right) \Omega_{0,p;\beta,q} - \left( p + 1 - \frac{d}{2} \right) \Omega_{0,p+1;\beta,q-1}
\]
\[
+ \sum_{s \geq 1} \Omega_{0,p;\beta,q-s}(R_s)_\beta - \sum_{s \geq 1} r_s^q \Omega_{p+1-s;\beta,q-1}
\]
\[
+ (-1)^p \chi_{q \geq 1} \chi_{p+q+1} \eta \beta,
\]
(A.14)
\[
\nabla \theta^T_{\alpha,p+1} \eta \mu \nabla \theta_{\beta,q} = \left( q + \mu_\beta \right) \Omega_{\alpha,p;\beta,q} - \left( p + 1 + \mu_\alpha \right) \Omega_{\alpha,p+1;\beta,q-1}
\]
\[
+ \sum_{s \geq 1} (R_s)_\beta \Omega_{\alpha,p;\beta,q-s} - \sum_{s \geq 1} (R_s)_\alpha \Omega_{p+1-s;\beta,q-1}
\]
\[
+ (-1)^p \eta \alpha (R_{p+q+1})_\beta \chi_{p} \chi_{q} \geq 1.
\] (A.15)

**Proof.** From the identities [A.6], [A.7], [A.11] and the relation \( U^T \eta = \eta U \) it follows that
\[
\partial_E \tilde{\Omega}_p(w) = \tilde{\Theta}^T_p(-w) U^T \eta \Theta(w) + \tilde{\Theta}^T_{p+1}(-w) \eta \partial_E \Theta(w)
\]
\[
= \left[ \frac{1}{w} \tilde{\Theta}^T_p(-w) + \tilde{\Theta}^T_{p+1}(-w) \right] \eta \partial_E \Theta(w)
\]
\[
= \frac{1}{w} \tilde{\Theta}^T_p \eta \left( \Theta(w) D_w - \mu \Theta(w) \right)
\]
\[
= - \frac{1}{w} \Theta^T_p \eta \mu \Theta(w) + \left( \frac{1}{w} \Theta^T_p \eta \Theta(w) \right) (D_w + 1),
\]
which yields
\[
\frac{1}{w} \tilde{\Theta}^T_p \eta \mu \Theta(w) = \left( \frac{1}{w} \Theta^T_p \eta \Theta(w) \right) (D_w + 1) - \partial_E \tilde{\Omega}_p(w)
\]
\[
= \left( \frac{1}{w} \tilde{\Omega}_{p-1}(w) + \tilde{\Omega}_p(w) \right) (D_w + 1) - \partial_E \tilde{\Omega}_p(w).
\]
Then by comparing the above identity with [A.10] we arrive at [A.13]. The lemma is proved. \( \square \)
Proposition A.3. We have the following identities:

\[
\partial_{E^2} \tilde{\Omega}_p(w) = \tilde{\Omega}_p(0) \left( \frac{1}{4} - \mu^2 \right) \eta^{-1} \Omega(0, w) \\
+ \left( p + \tilde{\mu} + \frac{1}{2} \right) \left( p + \tilde{\mu} + \frac{3}{2} \right) \tilde{\Omega}_{p+1}(w) \\
+ \sum_{l \geq 1} 2 \left( p + \tilde{\mu} + 1 \right) \tilde{R}^T_l \tilde{\Omega}_{p+1-l}(w) + \sum_{l \geq 2} \tilde{R}^T_{l-2} \tilde{\Omega}_{p+1-l}(w) \\
+ \left[ \tilde{\Omega}_p(w) \frac{1}{w} \left( D_w + \frac{1}{2} \right) \left( D_w + \frac{3}{2} \right) \right]_+, \quad \mu \in \mathbb{Z}.
\]

(A.16)

Here \( \tilde{R}_{l,2} = |\tilde{R}^T_l| \), and the definition of component \([-l]_l\) is obtained from (0.9) by replacing the \( n \times n \) matrices \( \mu, R \) with the \( (n+1) \times (n+1) \) matrices \( \tilde{\mu}, \tilde{R} \). The \( n \times n \) matrix-valued formal power series \( \Omega(z, w) \) in \( z, w \) is given by

\[
\left( \Omega(z, w) \right)_{\alpha \beta} = \sum_{p,q \geq 0} \Omega_{\alpha,p;\beta,q} z^p w^q,
\]

(A.17)

and for any Laurent series \( A = \sum_{k \geq s} A_k w^k \) we denote \( [A]_+ = \sum_{k \geq 0} A_k w^k \).

Proof. By using the identities (A.4) - (A.7), (A.11), (A.12) and the relation \( U^T \eta = \eta U \) we have

\[
\partial_{E^2} \tilde{\Omega}_p(w) = \partial_{E^2} \left( \tilde{T}_{p+1}^T(\cdot) \eta \Theta(w) \right) \\
= \tilde{T}_{p+1}^T(\cdot) (U^T)^2 \eta \Theta(w) + w \tilde{T}_{p+1}^T(\cdot) \eta U \Theta(w) \\
= \left( \frac{1}{w} \partial_{E^2} \tilde{T}_{p+1}(-\cdot) + \partial_{E^2} \tilde{T}_{p+1}^T(\cdot) \right) \eta \left( \partial_{E} \Theta(w) \right) \\
= \left( \frac{1}{w} \partial_{E^2} \tilde{T}_{p+1} \right) \eta \left( \partial_{E} \Theta(w) \right) \\
= \frac{1}{w} \left( (p+1) \tilde{T}_{p+1}^T - \tilde{T}_{p+1}^T \mu + \sum_{l \geq 0} \tilde{B}^T_l \tilde{T}_{p+1-l}^T \right) \eta \left( \Theta(w) D_w - \mu \Theta(w) \right).
\]

Then from (A.13) it follows that

\[
\partial_{E^2} \tilde{\Omega}_p(w) = \frac{1}{w} \tilde{T}_{p+1}^T \mu \eta \theta(w) + (p+1)^2 \tilde{\Omega}_{p+1}(w) \\
+ \sum_{l \geq 0} \tilde{B}^T_l (2p + 2 - l) \tilde{\Omega}_{p+1-l}(w) + \sum_{l \geq 0} \tilde{B}^T_{l-2} \tilde{\Omega}_{p+1-l}(w) + \tilde{\Omega}_p(w) \frac{1}{w} (D_w + 1)^2 \\
= \frac{1}{w} \tilde{T}_{p+1}^T \mu \eta \theta(w) + \left( p + \tilde{\mu} + 1 \right)^2 \tilde{\Omega}_{p+1}(w) \\
+ 2 \sum_{s \geq 1} \left( p + \tilde{\mu} + 1 \right) \tilde{R}^T_s \tilde{\Omega}_{p+1-s}(w) + \sum_{s \geq 2} \tilde{R}^T_{s-2} \tilde{\Omega}_{p+1-s}(w) + \tilde{\Omega}_p(w) \frac{1}{w} (D_w + 1)^2
\]

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for all \( p \) and \( \hat{\omega} \), the proposition is proved.

By using these relations we get

\[
\partial w \Omega_p(w) = \frac{1}{w} \Theta_p^T(u \eta \mu \Theta(w) + (p + \hat{\mu} + \frac{1}{2}) \left( p + \hat{\mu} + \frac{3}{2} \right) \bar{\hat{\omega}}_{p+1}(w)
\]

\[
+ 2 \sum_{s \geq 1} \left( p + \hat{\mu} + 1 \right) \hat{R}_s^T \bar{\hat{\omega}}_{p+1-s}(w) + \sum_{s \geq 2} \hat{R}_{s,2}^T \bar{\hat{\omega}}_{p+1-s}(w)
\]

\[
+ \bar{\hat{\omega}}_p(w) \frac{1}{w} \left( D_w + \frac{1}{2} \right) \left( D_w + \frac{3}{2} \right) + \frac{1}{4} \left( \frac{1}{w} \Omega_p(w) + \bar{\hat{\omega}}_p(w) \right),
\]

where \( \hat{B}_{i,j} := [\hat{B}^2]_{i,j} = [(\hat{\mu} + \hat{R})^2]_{i,j} \). On the other hand, from the relations

\[
\Theta_p = \Omega_{p-1}(0) \eta^{-1},
\]

\[
\Theta(w) = I + w \eta^{-1} \Omega(0, w)
\]

for all \( p \in \mathbb{Z} \) and \( (A.9) \), it follows that

\[
\frac{1}{w} \left( \bar{\hat{\omega}}_p(w) - \bar{\hat{\omega}}_p(0) \right) + \bar{\hat{\omega}}_{p+1}(w) = \frac{1}{w} \Theta_p^T(u \eta \mu \Theta(w) - I) = \bar{\hat{\omega}}_p(0) \eta^{-1} \Omega(0, w),
\]

and

\[
\frac{1}{w} \Theta_p^T(u \eta \mu \Theta(w) = \frac{1}{w} \tilde{\omega}_p(0) \eta^{-1} \mu \eta \tilde{\omega}_p(0) \eta^{-1} \Omega(0, w) + I
\]

\[
= -\bar{\hat{\omega}}_p(0) \mu \eta^{-1} \Omega(0, w) - \frac{1}{w} \bar{\hat{\omega}}_p(0) \mu^2.
\]

By using these relations we get

\[
\partial_{E^3} \hat{\Omega}_p(w)
\]

\[
= \bar{\hat{\omega}}_p(0) \left( \frac{1}{4} - \mu^2 \right) \eta^{-1} \Omega(0, w) + \left( p + \hat{\mu} + \frac{1}{2} \right) \left( p + \hat{\mu} + \frac{3}{2} \right) \bar{\hat{\omega}}_{p+1}(w)
\]

\[
+ \sum_{l \geq 1} 2 (p + \hat{\mu} + 1) \hat{R}_l^T \bar{\hat{\omega}}_{p+1-l}(w) + \sum_{l \geq 2} \hat{R}_{l,2}^T \bar{\hat{\omega}}_{p+1-l}(w)
\]

\[
+ \left[ \frac{1}{w} \bar{\hat{\omega}}_p(0) \left( \frac{1}{4} - \mu^2 \right) \right] + \bar{\hat{\omega}}_p(w) \frac{1}{w} \left( D_w + \frac{1}{2} \right) \left( D_w + \frac{3}{2} \right)
\]

\[
= \bar{\hat{\omega}}_p(0) \left( \frac{1}{4} - \mu^2 \right) \eta^{-1} \Omega(0, w) + \left( p + \hat{\mu} + \frac{1}{2} \right) \left( p + \hat{\mu} + \frac{3}{2} \right) \bar{\hat{\omega}}_{p+1}(w)
\]

\[
+ \sum_{l \geq 1} 2 (p + \hat{\mu} + 1) \hat{R}_l^T \bar{\hat{\omega}}_{p+1-l}(w) + \sum_{l \geq 2} \hat{R}_{l,2}^T \bar{\hat{\omega}}_{p+1-l}(w)
\]

\[
+ \left[ \bar{\hat{\omega}}_p(w) \frac{1}{w} \left( D_w + \frac{1}{2} \right) \left( D_w + \frac{3}{2} \right) \right].
\]

The proposition is proved. \( \square \)

Now let us start to prepare the computation of \( \partial_{E^3} \hat{\Omega}_p(w) \). Introduce the \( n \times n \) matrix

\[
C := \frac{d\Theta(w)}{dw} \bigg|_{w=0} = \eta^{-1} \Omega(0, 0) = (\nabla \theta_{1,1}, \ldots, \nabla \theta_{n,1}),
\]

(A.18)
then it is clear that \((\partial_a C)_\gamma^\beta = c^\beta_{\alpha\gamma}\), so by using (2.21) we arrive at
\[
U = (1 - \mu)C + C\mu + R_1.
\] (A.19)

We also have, for any vector field \(Y = Y^\alpha \partial_\alpha\), the relation
\[
\partial_Y C = C(Y),
\] (A.20)

where \(C(Y)\) is defined as in (A.4). In particular, we have
\[
\partial_E C = U, \quad \partial_{E^2} C = U^2.
\]

**Lemma A.4.** The following relations hold true for \(i = 0, 1, \ldots, n\) and \(p \in \mathbb{Z}^n\):
\[
(C \nabla \theta_{i,p})^\gamma = C^\gamma_{\epsilon \delta} \partial_{\epsilon \delta} \theta_{i,p} = \eta^{-1} \Omega_{\epsilon_{p+1}} \delta_{p+1,0} \delta_{\beta} \partial_{\beta} \eta_{-1},
\] (A.21)

where the \(n \times 1\) matrix \(\Omega_{\epsilon_{p+1}}\) is given by \((\Omega_{\epsilon_{p+1}}, \ldots, \Omega_n)\) and vector field \(\partial_{\beta}\) is regarded as column vector \((0, \ldots, 1, \ldots, 0)^T\), where the number 1 is the \(\beta\)th component.

**Proof.** From the identity (2.23) it follows that
\[
\partial_{\alpha} \theta\beta,1 = \partial_{\beta} \theta\alpha,1, \quad 1 \leq \alpha, \beta \leq n.
\]

Then, for any fixed \(1 \leq \gamma \leq n\), by using the relation (A.9) and the definition of \(\Omega_{i,p}\) we obtain
\[
(C \nabla \theta_{i,p})^\gamma = C^\gamma_{\epsilon \delta} \partial_{\epsilon \delta} \theta_{i,p} = \eta^{-1} \Omega_{\epsilon_{p+1}} \delta_{p+1,0} \delta_{\beta} \partial_{\beta} \eta_{-1},
\]

hence the relation (A.21) holds true. The lemma is proved. \(\square\)

We can rewrite the relation (A.21) in matrix form
\[
\hat{\Theta}_{p}^{T}C = \hat{\Theta}_{p+1}^{T} + \partial_u \tilde{\Omega}_{-1}(0) \eta_{-1},
\] (A.22)
\[
\hat{\Theta}_{p}^{T}(w)C = \hat{\Theta}_{p+1}^{T}(w) + \sum_{k \geq 0} (\partial_w \tilde{\Omega}_{p+k-1}(0)) \eta_{-1} w^k,
\] (A.23)
\[
C\Theta(w) = \frac{1}{w}(\Theta(w) - I) + w \eta^{-1} \partial_z \Omega(0, w)
\] (A.24)

here the differentials \(\partial_w, \partial_z\) are defined by
\[
\partial_u \tilde{\Omega}_{-1}(0) = \frac{d}{dw} \tilde{\Omega}_{-1}(w) \bigg|_{w=0}, \quad \partial_z \Omega(0, w) = \frac{d}{dz} \Omega(z, w) \bigg|_{z=0},
\]

and \(\Omega(z, w)\) is defined in (A.17).
Lemma A.5. For ∀ \( p \in \mathbb{Z} \) the following relations hold true:

\[
\hat{\Theta}_p^T \eta \mu C = -(p + \hat{\mu}) \hat{\Omega}_p(0) - \sum_{s \geq 1} \hat{R}_s^T \hat{\Omega}_{p-s}(0)
\]

\[
+ \partial_w \hat{\Omega}_{p-1}(0)(\mu + 1) + \hat{\Omega}_{p-1}(0) R_1,
\]

\[
C^T \mu \eta \Theta(w) = -\eta \Theta(w) (D_w - 1) \frac{1}{w} + w(1 + \mu) \partial_z \Omega(0, w)
\]

\[
+ R_1^T \eta \Theta(w) - (1 + \mu) \eta \cdot \frac{1}{w}.
\]

Proof. By using the identities (A.13), we have

\[
\hat{\Theta}_p^T \eta \mu \Theta(w) = -(p + \hat{\mu}) \hat{\Omega}_p(w) w - \sum_{s \geq 1} \hat{R}_s^T \hat{\Omega}_{p-s}(w) w + \hat{\Omega}_{p-1}(w) D_w.
\]

which, together with the definition of \( C \) given in (A.18), yield the result of the lemma.

Lemma A.6. We have the following relations for ∀ \( p \in \mathbb{Z} \):

\[
\partial_{E^2} \hat{\Theta}_p^T(w) = \left( p + \hat{\mu} - \frac{1}{2} + w \frac{d}{d w} \right) \left( p + \hat{\mu} + \frac{1}{2} + w \frac{d}{d w} \right) \hat{\Theta}_{p+1}(w)
\]

\[
+ 2 \sum_{s \geq 1} \left( p + \hat{\mu} + \frac{d}{d w} \right) \hat{R}_s^T \hat{\Theta}_{p+1-s}(w) + \sum_{s \geq 2} \hat{R}_s^T \hat{\Theta}_{p+1-s}(w)
\]

\[
+ \sum_{k \geq 0} \partial_w \hat{\Omega}_{p-k+1}(0) \eta^{-1} \left( \mu - \frac{1}{2} \right) \left( \mu - \frac{3}{2} \right) w^k
\]

\[
- 2 \hat{\Theta}_p^T(w) R_1^T (\mu - 1) + \hat{\Theta}_p^T(w) \left( \frac{1}{4} - \mu^2 \right) C^T,
\]

\[
\partial_{E^2} \Theta(w) = \Theta(w) \left( D_w - \frac{1}{2} \right) \left( D_w - \frac{3}{2} \right) \frac{1}{w} - \frac{1}{w} \left( \frac{1}{2} - \mu \right) \left( \frac{3}{2} - \mu \right)
\]

\[
+ w \left( \frac{1}{2} - \mu \right) \left( \frac{3}{2} - \mu \right) \eta^{-1} \partial_z \Omega(0, w)
\]

\[
- 2 (\mu - 1) R_1 \Theta(w) + C \left( \frac{1}{4} - \mu^2 \right) \Theta(w).
\]

Proof. Let us prove this lemma by computing \( \partial_{E^2} \partial_{E} \hat{\Theta}_p(w) \) in two different ways. On the one hand, by using (A.19) and (A.20) we obtain

\[
\partial_{E^2} \partial_{E} \hat{\Theta}_p(w)
\]

\[
= \partial_{E} (U \hat{\Theta}_{p-1}(w)) = [(1 - \mu) U + U \mu] \hat{\Theta}_{p-1}(w) + U^2 \hat{\Theta}_{p-2}(w)
\]

\[
= \partial_{E^2} \hat{\Theta}_{p-1}(w) + 2 (1 - \mu) \partial_{E} \hat{\Theta}_{p}(w) - (1 - \mu) U \hat{\Theta}_{p-1}(w) + U \mu \hat{\Theta}_{p-1}(w)
\]

\[
= \partial_{E^2} \hat{\Theta}_{p-1}(w) + 2 (1 - \mu) \partial_{E} \hat{\Theta}_{p}(w)
\]

\[
- (1 - \mu) [(1 - \mu) C + C \mu + R_1] \hat{\Theta}_{p-1}(w)
\]

\[
+ [(1 - \mu) C + C \mu + R_1] \mu \hat{\Theta}_{p-1}(w)
\]
= \partial_E^2 \hat{\Theta}_{p-1}(w) + 2(1 - \mu)\partial_E \hat{\Theta}_p(w) - (1 - \mu)^2C \hat{\Theta}_{p-1}(w) \\
+ 2(\mu - 1)R_1 \hat{\Theta}_{p-1}(w) + C\mu^2 \hat{\Theta}_{p-1}(w),

from which it follows that

\partial_E^2 \hat{\Theta}_{p-1}(w) = \partial_E \partial_E \hat{\Theta}_p(w) - 2(1 - \mu)\partial_E \hat{\Theta}_p(w) + (1 - \mu)^2C \hat{\Theta}_{p-1}(w) \\
- 2(\mu - 1)R_1 \hat{\Theta}_{p-1}(w) - C\mu^2 \hat{\Theta}_{p-1}(w).

(A.29)

On the other hand, by using the relation (A.3) we have

\partial_E \partial_E \hat{\Theta}_p^T(w)
\begin{align*}
&= \partial_E \left[ \left( p + \hat{\mu} + w \frac{d}{dw} \right) \hat{\Theta}_p^T(w) + \sum_{s \geq 1} \hat{R}_s \hat{\Theta}_{p-s}^T(w) - \hat{\Theta}_p^T(w)\mu \right] \\
&= \left( p + \hat{\mu} + w \frac{d}{dw} \right) \left[ \left( p + \hat{\mu} + w \frac{d}{dw} \right) \hat{\Theta}_p^T(w) + \sum_{s \geq 1} \hat{R}_s \hat{\Theta}_{p-s}^T(w) - \hat{\Theta}_p^T(w)\mu \right] \\
&\quad + \sum_{s \geq 1} \hat{R}_s^T \left[ \left( p - s + \hat{\mu} + w \frac{d}{dw} \right) \hat{\Theta}_{p-s}^T(w) + \sum_{l \geq 1} \hat{R}_l \hat{\Theta}_{p-s-l}^T(w) - \hat{\Theta}_{p-s}^T(w)\mu \right] \\
&\quad - \left[ \left( p + \hat{\mu} + w \frac{d}{dw} \right) \hat{\Theta}_p^T(w) + \sum_{s \geq 1} \hat{R}_s \hat{\Theta}_{p-s}^T(w) - \hat{\Theta}_p^T(w)\mu \right] \mu \\
&= \left( p + \hat{\mu} + w \frac{d}{dw} \right)^2 \hat{\Theta}_p^T(w) + 2 \sum_{s \geq 1} \left( p + \hat{\mu} + w \frac{d}{dw} \right) \hat{R}_s \hat{\Theta}_{p-s}^T(w) \\
&\quad - 2 \left( p + \hat{\mu} + w \frac{d}{dw} \right) \hat{\Theta}_p^T(w)\mu - 2 \sum_{s \geq 1} \hat{R}_s \hat{\Theta}_{p-s}^T(w)\mu \\
&\quad + \sum_{s \geq 2} \hat{R}_{s-2} \hat{\Theta}_{p-s}^T(w) + \hat{\Theta}_p^T(w)\mu^2.
\end{align*}

Combining this relation with (A.29) and (A.23), we arrive at the relation (A.27). The relation (A.28) can be proved in a similar way, and we omit the details here. The lemma is proved.

Proposition A.7. For \( \forall p \in \mathbb{Z} \), we have

\begin{align*}
\partial_E \tilde{\Omega}_p(w) &= \tilde{\Omega}_p(0) \left( \frac{1}{2} - \mu \right) \left( \frac{3}{2} - \mu \right) \eta^{-1} \left( \frac{1}{2} - \mu \right) \partial_2 \Omega(0, w) \\
&\quad + \partial_\omega \tilde{\Omega}_p(0) \left( \frac{1}{2} - \mu \right) \eta^{-1} \left( \frac{1}{2} - \mu \right) \left( \frac{3}{2} - \mu \right) \Omega(0, w) \\
&\quad + \tilde{\Omega}_p(0) \left( -3\mu^2 + 3\mu + \frac{1}{4} \right) R_1 \eta^{-1} \Omega(0, w) \\
&\quad + \left( p + \hat{\mu} + \frac{1}{2} \right) \left( p + \hat{\mu} + \frac{3}{2} \right) \left( p + \hat{\mu} + \frac{5}{2} \right) \tilde{\Omega}_{p+2}(w)
\end{align*}
Thus we have

\[ + \sum_{s \geq 1} \left[ 3 \left( p + \hat{\mu} + \frac{1}{2} \right)^2 + 6 \left( p + \hat{\mu} + \frac{1}{2} \right) + 2 \right] \hat{R}_s^T \hat{\Omega}_{p+2-s}(w) \]

\[ + \sum_{s \geq 2} 3 \left( p + \hat{\mu} + \frac{3}{2} \right) \hat{R}_s^T \hat{\Omega}_{p+2-s}(w) + \sum_{s \geq 3} \hat{R}_s^T \hat{\Omega}_{p+2-s}(w) \]

\[ + \left[ \hat{\Omega}_p(w) \frac{1}{w^2} \left( D_w + \frac{1}{2} \right) \left( D_w + \frac{3}{2} \right) \right]_+ . \quad (A.30) \]

**Proof.** From the proof of the Propositions A.1, A.3 and the relations (A.4)–(A.5), it follows that

\[ \partial_E \hat{\Omega}_p(w) = \hat{\Theta}_p^T(w) \eta \mathcal{U} \Theta(w) + w \hat{\Theta}_{p+1}^T(-w) \eta \mathcal{U} \Theta(w) \]

\[ = \hat{\Theta}_p^T(\eta \mathcal{U} \Theta(w)), \quad (A.31) \]

\[ \partial_E^2 \hat{\Omega}_p(w) = \frac{1}{w} \left( \partial_E \hat{\Theta}_{p+1}^T(w) \right) \eta \left( \partial_E \Theta(w) \right) \]

\[ = \hat{\Theta}_p^T \eta \mathcal{U} \Theta(w) = \hat{\Theta}_p^T \eta \mathcal{U}^2 \Theta(w). \quad (A.32) \]

On the other hand, by using (A.19)–(A.20) we obtain

\[ \partial_E \mathcal{U} = (1 - \mu) \mathcal{U}^2 + \mathcal{U}^2 \mu. \]

Thus we have

\[ \partial_E \hat{\Omega}_p(w) \]

\[ = \hat{\Theta}_p^T(-w)(\mathcal{U}^T)^3 \eta \Theta(w) + w \hat{\Theta}_{p+1}^T(-w) \eta \mathcal{U} \Theta(w) \]

\[ = \hat{\Theta}_p^T(-w)(\mathcal{U}^T)^2 \eta \Theta(w) + w \hat{\Theta}_{p+1}^T(-w) \eta \mathcal{U} \cdot \mathcal{U} \Theta(w) \]

\[ = \partial_E^2 \left( \hat{\Theta}_{p+1}^T(-w) \eta \mathcal{U} \Theta(w) \right) - \hat{\Theta}_{p+1}^T(-w) \eta \left( \partial_E \mathcal{U} \right) \Theta(w) \]

\[ = \sum_{k \geq 0} (-1)^k \partial_E^2 \hat{\Theta}_{p+1+k}(w) \cdot w^k - \hat{\Theta}_{p+1}^T(-w) \eta \left[ (1 - \mu) \mathcal{U}^2 + \mathcal{U}^2 \mu \right] \Theta(w) \]

\[ = \sum_{k \geq 0} (-1)^k \partial_E^2 \hat{\Theta}_{p+1+k}(w) \cdot w^k - 2 \hat{\Theta}_{p+1}^T(-w) \eta \mathcal{U}^2 \Theta(w) \]

\[ + \hat{\Theta}_{p+1}^T(-w) \eta \left( \mu + \frac{1}{2} \right) \mathcal{U}^2 \Theta(w) + \hat{\Theta}_{p+1}^T(-w)(\mathcal{U}^T)^2 \eta \left( \frac{1}{2} - \mu \right) \Theta(w) \]

\[ = \sum_{k \geq 0} \partial_E^2 \left( \partial_E - 2 \hat{\Theta}_{p+1+k}(w) \cdot (-w)^k + \frac{1}{w} \hat{\Theta}_{p+1}^T(-w) \eta \left( \mu + \frac{1}{2} \right) \partial_E \Theta(w) \right. \]

\[ + \left. \partial_E \hat{\Theta}_{p+2}^T(-w) \left( \mu + \frac{1}{2} \right) \eta \Theta(w) \right). \]

Then the relation (A.30) follows from (A.10), (A.13), (A.16), (A.22)–(A.28) and a straightforward calculation. The theorem is proved.

Finally, let us compute \( \partial_E \Omega_{0,p+q} \) for \( s = 0, 1, 2, 3 \) and \( p, q \in \mathbb{Z} \).

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Proposition A.8. The following relations hold true for $p, q \in \mathbb{Z}$:

\[
\partial_{\Omega} \Omega_{0,p;0,q} = \Omega_{0,p-1,0,q} + \Omega_{0,p;0,q-1}.
\]  
\[\text{(A.33)}\]

\[
\partial_{E} \Omega_{0,p;0,q} = (p + q + 1 - d) \Omega_{0,p;0,q} + \sum_{s \geq 1} r_s^\epsilon \Omega_{\epsilon,p-s,0,q} + \sum_{s \geq 1} r_s^\epsilon \Omega_{\epsilon,0,p-c,q-s} + (-1)^p c_{p+q}.
\]  
\[\text{(A.34)}\]

\[
\partial_{E^2} \Omega_{0,p;0,q} = \Omega_{0,p;0,0} \left( \frac{1}{4} - \mu \right) \eta^{-1} \Omega_{\epsilon,0,0,q} + \left( p - \frac{d}{2} + \frac{1}{2} \right) \left( -d + \frac{3}{2} \right) \Omega_{0,p+1;0,q} + \left( q - \frac{d}{2} + \frac{1}{2} \right) \left( q - \frac{d}{2} + \frac{3}{2} \right) \Omega_{0,p;0,q+1} + 2 \left( p - \frac{d}{2} + 1 \right) \sum_{s \geq 1} r_s^\epsilon \Omega_{\epsilon,p+1-s,0,q} + 2 \left( q - \frac{d}{2} + 1 \right) \sum_{s \geq 1} r_s^\epsilon \Omega_{\epsilon,0,p-c,q+1-s} + \sum_{s \geq 2} (R_r)^\epsilon r_{s,2} \Omega_{\epsilon,p+1-s,0,q} + \sum_{s \geq 2} (R_r)^\epsilon r_{s,2} \Omega_{\epsilon,0,p-c,q+1-s} + (-1)^p c_{p+q+1,0}. 
\]  
\[\text{(A.35)}\]
+ (-1)^p c_{p+q+2} \left[ \left( p - \frac{d}{2} + \frac{1}{2} \right) \left( q - \frac{d}{2} + \frac{3}{2} \right) \right] \left( r_{\alpha}^x \sigma_{\alpha}^{0} \right) \right] = \left( p - \frac{d}{2} + \frac{1}{2} \right) \left( q - \frac{d}{2} + \frac{3}{2} \right),}

(A.36)

where we denote the $n \times 1$ matrix $(\Omega_{t,\cdot:p,1,q}, \ldots, \Omega_{t,\cdot:p,n,q})$ by $\Omega_{t,\cdot:p,\cdot,q}$, and denote the $1 \times n$ matrix $(\Omega_{t,\cdot:j,q}, \ldots, \Omega_{t,\cdot:p,j,q})^T$ by $\Omega_{t,\cdot:p,j,q}$.

Proof. The relation (A.33) is given by (7.3), and that of (A.34) can be easily verified by using (5.4), (3.13), (3.4) and (A.9), so we are left to compute the derivatives $\partial_E \Omega_{t,\cdot:0,0,q}$ and $\partial_E \Omega_{t,\cdot:p,0,q}$.

From the relations given in (5.5), it follows that

\[
\partial_E \Omega_{t,\cdot:0,0,q} = \langle E, \nabla \Omega_{t,\cdot:0,0,q} \rangle = \langle E, \nabla \theta_{0,p} \cdot \nabla \theta_{0,q} \rangle = \langle E \cdot \nabla \theta_{0,p}, \nabla \theta_{0,q} \rangle
\]

\[
= \nabla \theta_{0,p}^T \eta \nabla \theta_{0,q} = (\partial_E \nabla \theta_{0,p+1}) \eta \nabla \theta_{0,q},
\]

(A.37)

then by using the relation (4.31) we get another way to compute $\partial_E \Omega_{t,\cdot:0,0,q}$. Comparing the resulting formula with (A.33), we obtain

\[
\nabla \theta_{0,p+1}^T \eta \nabla \theta_{0,q} = \left( q - \frac{d}{2} \right) \Omega_{t,\cdot:0,0,q} - \left( p + 1 - \frac{d}{2} \right) \Omega_{t,\cdot:p,0,q-1} + \sum_{s \geq 1} r_{\alpha}^x \Omega_{t,\cdot:0,p+1-s,0,q-1} - \sum_{s \geq 1} r_{\alpha}^x \Omega_{t,\cdot:p+1-s,0,0,q} + (-1)^p c_{p+q+2}.
\]

(A.38)

Then, by using (5.5), (3.4) and (3.9) we have

\[
\partial_E \Omega_{t,\cdot:p,0,q}
\]

\[
= \langle E^2, \nabla \theta_{0,p} \cdot \nabla \theta_{0,q} \rangle = \langle E \cdot \nabla \theta_{0,p}, E \cdot \nabla \theta_{0,q} \rangle = \langle \partial_E \nabla \theta_{0,p+1}, \partial_E \nabla \theta_{0,q+1} \rangle
\]

\[
= \left[ \nabla \theta_{0,p+1}^T \left( p + 1 - \frac{d}{2} - \mu \right) + \sum_{s \geq 1} r_{\alpha}^x \nabla \theta_{0,p+1-s}^T \right]
\]

\[
x \eta \left[ \left( q + 1 - \frac{d}{2} - \mu \right) \nabla \theta_{0,q+1} + \sum_{s \geq 1} r_{\alpha}^x \nabla \theta_{\lambda,q+1-s} \right]
\]

\[
= \left( p + 1 - \frac{d}{2} \right) \left( q + 1 - \frac{d}{2} \right) \langle \nabla \theta_{0,p+1}, \nabla \theta_{0,q+1} \rangle
\]

\[
+ \left[ - \left( p + 1 - \frac{d}{2} \right) + \left( q + 1 - \frac{d}{2} \right) \right] \nabla \theta_{0,p+1}^T \eta \nabla \theta_{0,q+1} + \nabla \theta_{0,p+1}^T \eta \nabla \theta_{0,q+1}
\]

\[
+ \left( q + 1 - \frac{d}{2} \right) \sum_{s \geq 1} r_{\alpha}^x \langle \nabla \theta_{\alpha,p+1-s}, \nabla \theta_{0,q+1} \rangle + \left( p + 1 - \frac{d}{2} \right) \sum_{s \geq 1} r_{\alpha}^x \langle \nabla \theta_{0,p+1}, \nabla \theta_{\alpha,q+1-s} \rangle
\]

\[
- \sum_{s \geq 1} r_{\alpha}^x \nabla \theta_{0,p+1-s}^T \eta \nabla \theta_{\lambda,q+1-s} + \sum_{s \geq 1} r_{\alpha}^x \nabla \theta_{0,p+1-s}^T \eta \nabla \theta_{\lambda,q+1-s}
\]

\[
+ \sum_{k,l \geq 1} r_{kl}^x \langle \nabla \theta_{\alpha,p+1-k-l}, \nabla \theta_{\lambda,q+1-l} \rangle,
\]

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from this and the relations given in (A.9), (A.14), (A.15) and (A.38) we arrive at the formula (A.35).

Finally, let us compute \( \partial E \Omega_{0,p+0,q} \). By using Lemma (A.4) and the way that is used to prove Lemma (A.3), it is easy to show that

\[
C \nabla \theta_{0,p} = \nabla \theta_{0,p+1} + \eta^{-1} \Omega_{1,0,p-1} \tag{A.39}
\]

\[
\nabla \theta_{0,p}^T \eta C = \nabla \theta_{0,p+1}^T \eta + \Omega_{0,p-1,1,1}, \tag{A.40}
\]

\[
\nabla \theta_{0,p}^T \eta C = \Omega_{0,p-1,1,1} (1 + \mu) - \left( p - \frac{d}{2} \right) \nabla \theta_{0,p+1}^T \eta
\]

\[
+ \nabla \theta_{0,p}^T \eta R - \sum_{s \geq 1} r^s \nabla \theta_{s,p+1-s} - (1)^p \chi_{s \geq 0, (r_{p+1}, \eta)} \tag{A.41}
\]

Then by using the formulae given in (3.4), (3.9), (5.5) and the relations (A.19), (A.31), (A.37), we get

\[
\partial E \Omega_{0,p+0,q} = \left( E^3, \nabla \theta_{0,p} \cdot \nabla \theta_{0,q} \right) = \nabla \theta_{0,p}^T \eta C \nabla \theta_{0,q} = (\partial E \nabla \theta_{0,p+1}) \eta \mathcal{U} (\partial E \nabla \theta_{0,q+1})
\]

\[
= \left[ \nabla \theta_{0,p+1}^T \left( p + 1 - \frac{d}{2} - \mu \right) + \sum_{s \geq 1} r^s \nabla \theta_{s,p+1-s} \right] \eta \mathcal{U}
\]

\[
= \left[ q + 1 - \frac{d}{2} - \mu \right] \nabla \theta_{0,q+1} + \sum_{s \geq 1} r^s \nabla \theta_{s,q+1-s}
\]

\[
= \left( p + 1 - \frac{d}{2} \right) \left( q + 1 - \frac{d}{2} \right) \nabla \theta_{0,p+1}^T \eta \mathcal{U} \nabla \theta_{0,q+1}
\]

\[
- \left( q + 1 - \frac{d}{2} \right) \nabla \theta_{0,p+1}^T \eta \mathcal{U} \nabla \theta_{0,q+1} - \left( p + 1 - \frac{d}{2} \right) \nabla \theta_{0,p+1}^T \eta \mathcal{U} \nabla \theta_{0,q+1}
\]

\[
+ \sum_{s \geq 1} r^s \nabla \theta_{s,p+1-s} \eta \mathcal{U} \nabla \theta_{s,q+1-s} + \left( p + 1 - \frac{d}{2} \right) \sum_{s \geq 1} r^s \nabla \theta_{s,p+1-s} \eta \mathcal{U} \nabla \theta_{s,q+1-s}
\]

\[
+ \sum_{k,l \geq 1} r^p \lambda_{k,l} \nabla \theta_{k,p+1-k} \eta \mathcal{U} \nabla \theta_{l,q+1-l}
\]

\[
= \left( p + 1 - \frac{d}{2} \right) \left( q + 1 - \frac{d}{2} \right) \partial E \Omega_{0,p+1,0,q+1}
\]

\[
+ \sum_{s \geq 1} r^s \nabla \theta_{s,p+1} \eta \mathcal{U} \nabla \theta_{s,q+1-s} - \sum_{s \geq 1} r^s (\partial E \nabla \theta_{s,p+2-s}) \eta \mu \nabla \theta_{s,q+1}
\]

\[
+ \nabla \theta_{0,p+1}^T \eta \left[ (1 - \mu) C + C \mu + R \right] \nabla \theta_{0,q+1}
\]

\[
+ \left( q + 1 - \frac{d}{2} \right) \sum_{s \geq 1} r^s \nabla \theta_{s,p+1-s} \eta \mathcal{U} \nabla \theta_{s,q+1-s}
\]

\[
+ \left( p + 1 - \frac{d}{2} \right) \sum_{s \geq 1} r^s \partial E \Omega_{0,p+1,s,q+1-s}
\]

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\[ + \sum_{k,l \geq 1} r_k^* T^l \partial E \Omega_{\varepsilon, p+1-k; \lambda, q+1-l}. \tag{A.42} \]

On the other hand, for any parameter \( x, y \in \mathbb{C} \) we have

\[
\left( q + 1 - \frac{d}{2} \right) \nabla \theta^T_{0, p+1} \eta \mu \nabla \theta_{0, q+1} \\
= \left( q - \frac{d}{2} + x \right) \nabla \theta^T_{0, p+1} \eta \mu (\partial E \nabla \theta_{0, q+2}) \\
+ (1 - x) \nabla \theta^T_{0, p+1} \eta \mu [(1 - \mu) C + \eta \mu + R_1] \nabla \theta_{0, q+1} \\
= \left( q - \frac{d}{2} + x \right) \nabla \theta^T_{0, p+1} \eta \mu (\partial E \nabla \theta_{0, q+2}) + (1 - x) \nabla \theta^T_{0, p+1} \eta \mu C \nabla \theta_{0, q+1} \\
- (1 - x) \nabla \theta^T_{0, p+1} \eta \mu^2 (C \nabla \theta_{0, q+1}) + (1 - x) (\nabla \theta^T_{0, p+1} \eta \mu C) \mu \nabla \theta_{0, q+1} \\
+ (1 - x) \nabla \theta^T_{0, p+1} \eta \mu R_1 \nabla \theta_{0, q+1}.
\]

We take \( x = \frac{5}{4} \), \( y = \frac{3}{5} \), and replace the second term after the last equality of (A.42) by the expression that appears after the last equality of the above formula. Then by using the relations given in (2.21), (A.10), (A.14), (A.15), (A.34), (A.38), and (A.39)–(A.41) we can verify the validity of the formula (A.36). The proposition is proved.

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