Exact renormalization group equation for the Lifshitz critical point

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Abstract

An exact renormalization equation (ERGE) accounting for an anisotropic scaling is derived. The critical and tricritical Lifshitz points are then studied at leading order of the derivative expansion which is shown to involve two differential equations. The resulting estimates of the Lifshitz critical exponents compare well with the $O(\epsilon^2)$ calculations. In the case of the Lifshitz tricritical point, it is shown that a marginally relevant coupling defies the perturbative approach since it actually makes the fixed point referred to in the previous perturbative calculations $O(\epsilon)$ finally unstable.

Key words: Exact renormalization group, Derivative expansion, Lifshitz Critical point

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An exact renormalization group (RG) equation (ERGE) is a continuous version of the (originally discrete) Wilson RG transformation of an action $S[\phi]$ [1]. It results in an integro-differential equation for the action on which one may apply nonperturbative approximations or truncations such as an expansion in powers of the derivative of the field $\phi$ (for an introductory review see [2]). In a preceding paper [3] I have argued that the study of the first terms of the derivative expansion of an ERGE may favorably compete with the customary perturbative approach provided that one does not expect an accuracy larger than that obtained from the $\epsilon$-expansion developed up to, say, $O(\epsilon^2)$ and even, sometimes, one could show that the perturbative approach has failed. The Lifshitz critical point [4] (for reviews see [5,6,7]) is an excellent example to test these claims. The previous calculations relative to this system have been done up to $O(\epsilon^2)$ only. They have shown the difficulty of estimating some Feynman graph integrals, various results have coexisted for a long time and a controversy still persists (see the recent review in [7]). Moreover,
the tricritical side involves several operators among which one is marginal at the physical dimension and has not been (cannot be) included in the previous perturbative studies [8,9].

The object of the present study is threefold. In a first time I write down the ERGE adapted to the study of the Lifshitz point. The main difficulty in that case is the existence of two distinct correlation lengths which induce an anisotropic scaling. In a second time I derive the leading order of the derivative expansion applied to the ERGE. In a third time I numerically study the differential equations that follow using the shooting method as in [3].

1 Derivation of the ERGE

A Lifshitz critical point is a multicritical point at which a disordered, a homogeneous ordered and a (spatially) modulated ordered phases meet [4]. The simplest way of illustrating the theoretical conditions of realization of a Lifshitz point is to look at the Landau form of the Euclidean action (Hamiltonian) relevant to the problem:

\[ S[\phi]_{\text{Landau}} = \int d^d x \left[ (\partial_\perp \phi)^2 + \rho (\partial_\parallel \phi)^2 + \tau \phi^2 + u \phi^4 + \sigma (\partial_\parallel^2 \phi)^2 \right] \quad (1) \]

in which \( \phi(x) \) is a scalar field (the generalization to a \( n \)-vector field is straightforward), \( \partial_\perp \) and \( \partial_\parallel \) stand for the derivatives along two directions \( (x_\parallel \text{ and } x_\perp) \) in the \( x \)-space of dimension \( d \). The subspace \( x_\parallel \) involves \( m \) components while the remaining \( x_\perp \) has \( (d-m) \) components. The Lifshitz critical point corresponds to both \( \tau = 0 \) and \( \rho = 0 \) (in the Landau approximation, in general \( \tau = \tau_0 \) and \( \rho = \rho_0 \)). Usually one would have named this point a tricritical point however the qualifier tricritical associated to Lifshitz has been reserved to the case where \( u \) vanishes also with \( \tau \) and \( \rho \) —in that case other terms must be added [9,10] to (1), such as \( c_1 \int d^d x \phi^6 \) and \( c_2 \int d^d x (\partial_\parallel^2 \phi)^2 \phi^2 \).

Lifshitz criticality is characterized by an anisotropic scaling. The expected scaling property of the two-point vertex function (in the momentum space) is as follows [7] \( (s \to 0) \):

\[
\begin{align*}
\Gamma^{(2)}(sq_\perp, 0) & \sim s^{2-\eta_{L2}} \\
\Gamma^{(2)}(0, sq_\parallel) & \sim s^{4-\eta_{L4}} \\
\Gamma^{(2)}(sq_\perp, s^\theta q_\parallel) & \sim s^{2-\eta_{L2}}
\end{align*}
\]

in which \( \eta_{L2} \) and \( \eta_{L4} \) vanish in the Landau approximation and \( \theta = \frac{2-\eta_{L2}}{4-\eta_{L4}} \). The
The expected scaling property (at the fixed point) is [11,5]:

$$\Gamma^{(2)} \left( s^{\lambda_x \rho}; s^{\lambda_y \mu}; s q \perp, s_{q \parallel} \right) = s^{2-n_{L^2}} \Gamma^{(2)} \left( \mu_{t}, \mu_{\rho}; q \perp, q \parallel \right)$$  \hspace{1cm} (5)

in which $\mu_{t}$ and $\mu_{\rho}$ are the two relevant parameters (scaling fields) associated to $\tau$ and $\rho$ respectively, $\lambda_{\tau} = 1/\nu_{L^2}$ and $\lambda_{\rho} = \varphi/\nu_{L^2}$ where $\nu_{L^2}$ is the critical exponent of the “perpendicular” correlation length ($\xi \perp$) and $\varphi$ the crossover exponent controlling the crossing between the Lifshitz and ordinary critical points.

The adaptation of the Wilson-Polchinski ERGE to the problem is in fact an easy task: it is a straightforward application of the rules stated in [3] to the case involving two sets of scales ($\parallel$ and $\perp$).

Let me consider the following general action with a double cutoff function $P$:

$$S[\phi] = \frac{1}{2} \int q \phi q^{-1} \frac{q_{\perp}^2}{\Lambda_{\perp}^2} \frac{q_{\parallel}^2}{\Lambda_{\parallel}^2} \ell_{\perp,\parallel}$$

with $q_{\perp} = q_1, \ldots, q_m, q_{\perp} = q_{m+1}, \ldots, q_d, m \in [0,d]$. The two running cutoffs $\Lambda_{\perp}$ and $\Lambda_{\parallel}$ are related to two independent momentum scales of reference $\Lambda_{0,\perp}$ and $\Lambda_{0,\parallel}$ via: $\Lambda_{\perp} = \ell_{\perp} \Lambda_{0,\perp}$ and $\Lambda_{\parallel} = \ell_{\parallel} \Lambda_{0,\parallel}$ with the anisotropy relation:

$$\ell_{\parallel} = \ell_{\perp}^{\theta}$$  \hspace{1cm} (7)

I underline the fact explained in [3] that, contrary to the current use, the cutoff function must not depend only on the ratio $q^2/\Lambda^2$ but also explicitly on $\ell$ (here on $\ell_{\parallel}$ and $\ell_{\perp}$) in order to correctly account for the “history” of the many scale effect which little by little starting from some large momentum scale $\Lambda_0$ (here $\Lambda_{0,\parallel}$ and $\Lambda_{0,\perp}$) finally induces the anomalous scaling observed at small momenta [here given by eq. (4)].

In the following, for the sake of simplifying the notations, I shall use $\ell_{\perp} = \ell = e^{-t}$.

Due to the anisotropy relation (7) the explicit double dependence of $P$ on $\ell_{\perp}$ and $\ell_{\parallel}$ reduces to a dependence on the unique renormalization parameter $\ell$, so that in the following I shall write $P(\frac{q_{\perp}^2}{\Lambda_{\perp}^2}, \frac{q_{\parallel}^2}{\Lambda_{\parallel}^2}; \ell)$.

As explained in [3] and repeated above, the explicit $\ell$-dependence in $P$ is essential. Technically speaking, it is required to get a fixed point relevant to the physics under study while the resulting ERGE does not depend explicitly on $\ell$. Specifically, the “physics” which one is presently interested in is merely the expected scaling behavior (4) and thus, following the same considerations.
as in [3] one easily shows that the \( \ell \)-dependence of \( P \) must factorize:

\[
P(\frac{q_\perp^2}{\Lambda^2}, \frac{q_\parallel^2}{\Lambda^2}; \ell) = \ell^\varpi \tilde{P}(\frac{q_\perp^2}{\Lambda^2}, \frac{q_\parallel^2}{\Lambda^2})
\]  

(8)

with

\[
\varpi = 1 - \eta_{L2}/2
\]  

(9)

Then the ERGE adapted to the anisotropic scaling of interest follows (\( \dot{S} = dS/dt = -\ell dS/d\ell = -\theta \ell || dS/d\ell || \)):

\[
\dot{S} = G_{dl} S - \int q \left( \varpi \tilde{P} - q_\perp^2 \tilde{P}_\perp - \theta q_\parallel^2 \tilde{P}_\parallel \right) \left[ \frac{\delta S}{\delta \phi_q \delta \phi_{-q}} - \frac{\delta^2 S}{\delta \phi_q \delta \phi_{-q}} - 2 \tilde{P}^{-1} \phi_q \frac{\delta S}{\delta \phi_q} \right]
\]  

(10)

in which \( q, q_\perp, q_\parallel \) are dimensionless and \( \tilde{P}'_{\perp,\parallel} = d\tilde{P}(q_\perp, q_\parallel) / dq_{\perp,\parallel} \) and:

\[
G_{dl} S = - \int q \phi_q \left( \frac{d_m}{2} + \varpi + q_\perp \cdot \partial_{q_\perp} + \theta q_\parallel \cdot \partial_{q_\parallel} \right) \frac{\delta S}{\delta \phi_q}
\]  

(11)

\[
d_m = d + m (\theta - 1)
\]  

(12)

Notice that in the case of the ordinary critical point (\( m = 0 \)) the above equations give back the ordinary Wilson-Polchinski ERGE of [3] while in the isotropic Lifshitz case (\( m = d \)) the renormalization parameter to be considered is \( \ell_\parallel \) instead of presently \( \ell = \ell_\perp \) and one must account for (7) to get the ERGE adapted to the isotropic Lifshitz point from the eqs (10-12) [practical rules: set formally \( \theta = 1 \) and change \( \varpi \rightarrow \frac{\varpi}{\theta} \)].

2 Leading order of the derivative expansion

The object of this section is to obtain the leading order of the derivative expansion of eqs (10-12). To this end it is worth reminding the reader with the usual case of the ordinary critical point (\( m = 0 \)). In that case, the leading order of the derivative expansion is named the local potential approximation and is associated with the critical exponent \( \eta = 0 \). One knows that it consists of a truncation of the ERGE to actions of the type:

\[
S_{LPA} = \int d^4 x \left[ Z (\partial \phi)^2 + V (\phi) \right]
\]  

(13)
with $Z$ a constant independent of $\phi$ and of $t (= - \ln \ell)$. Eq (10) then reduces to a single equation for the unique function $V$ [12]:

$$
\dot{V} = dV - \left( \frac{d}{2} - \varpi \right) \phi V' - \varpi \left( V'^2 - V'' \right)
$$

(14)

and the fact that $Z$ is constant (i.e. $\dot{Z} = 0$) implies the condition $\varpi = 1$ (i.e. $\eta = 0$). The next order in the derivative expansion is then obtained by allowing $Z$ to depend on $\phi$ and $\eta$ no longer vanishes.

By analogy, the lowest approximation in the case of the isotropic Lifshitz point ($m = d$), leads to assign the role played by $Z$ to the coefficient $A$ of a term $(\partial^2 \phi)^2$ in the action $S$ while $Z$ is no longer constrained:

$$
S_{\text{Liso}} = \int d^d x \left[ Z(\phi) (\partial \phi)^2 + V(\phi) + A (\partial^2 \phi)^2 \right]
$$

(15)

There are two equations ($\dot{Z}$ and $\dot{V}$) and the condition $A = \text{const}$ implies that $\eta_L = 0$.

In the case of an $m$-axial Lifshitz point, the leading order of the derivative expansion is a mixing of the two previous cases such that the approximation corresponds to both $\eta_L = 0$ and $\eta_L = 0$. Hence, the action is limited to the following form:

$$
S_{\text{Lm}} = \int d^d x \left[ Z_\perp (\partial_\perp \phi)^2 + Z_\parallel (\partial_\parallel \phi)^2 + V(\phi) + A (\partial_\parallel^2 \phi)^2 \right]
$$

(16)

yielding two equations for $Z_\parallel (\phi)$ and $V(\phi)$ while $Z_\perp$ and $A$ are constants independent of $\phi$ and $t$ what implies $\varpi = 1$ and $\theta = 1/2$ (i.e., $\eta_L = \eta_{L4} = 0$).

Following [3], before writing down these equations let me introduce some convenient modifications. First, in the complete equation, I consider a redundant transformation of the field $\tilde{\phi}_q = \psi(q^2) \dot{\phi}_q$ in which $\psi(q^2)$ is arbitrary except the normalization $\psi(0) = 1$, then I subtract the high temperature fixed point $\frac{1}{2} \int_{p} \left( \tilde{P} \psi^2 \right)^{-1} \phi_0 \phi_{-p}$ from the action. Two other useful modifications are introduced in the derivative expansion. First I rescale the field $\phi = I_0^{1/2} \tilde{\phi}$ and the potential $V = I_0 \dot{V}$ where $I_0 = \int_{q} \left( \varpi \tilde{P} - q_\perp^2 \tilde{P}_\perp - q_\parallel^2 \tilde{P}_\parallel \right) \psi^2$ and instead of $V$, I consider the equation for $v_1 = d\dot{V}/d\tilde{\phi}$. Finally, for the sake of unified notations I rename $\dot{Z}_\parallel$ as $v_2$. With these new definitions, the RG equations in the leading order of the derivative expansion read:

$$
\dot{v}_1 = v_1'' + d_\varphi v_1 - \left( \tilde{d}_\varphi \tilde{\phi} + 2\varpi v_1 \right) v_1' + P_1 v_2'
$$

(17)

$$
\dot{v}_2 = v_2'' - 2(\varpi + \theta) v_2 - \left( \tilde{d}_\varphi \tilde{\phi} + 2\varpi v_1 \right) v_2' + v_1' \left( P_2 v_1' - 2\theta \psi_0' - 4\varpi v_2 \right)
$$

(18)
in which the prime means the derivative with respect to $\tilde{\phi}$, $d_{\phi} = \frac{d\bar{m}}{d\bar{w}} = \frac{d\bar{m}}{2} + \bar{w}$, $\psi_{||0}' = d\psi/dq_{||}|_{q=0}$, $P_1 = 2\frac{d\bar{m}}{d\bar{w}}$ with $I_1 = \int_q q^2 \left(\bar{w}\bar{P} - \bar{q}^2 \tilde{P}' - q^2 \tilde{P}'\right) \psi^2$

and $P_2 = -\left[\tilde{P}'(\bar{w} - 1) + 2\bar{w}\psi_{||0}'\right]$ with $\tilde{P}'|_{q=0} = d\tilde{P}/dq_{||}|_{q=0}$.

In the following section, in the same manner as in [3], I look for the fixed point solutions of eqs (17,18) ($\dot{v}_1 = \dot{v}_2 = 0$) which satisfy the following regular behavior at large $\bar{w}$:

$$v_{1\text{asy}} = G_1 \tilde{\phi}^{\theta_1} + \theta_1 G_2 \tilde{\phi}^{\theta_2 - 1} + \cdots \quad (19)$$

$$v_{2\text{asy}} = -\psi_{||0}' \theta_1 G_1 \tilde{\phi}^{\theta_1 - 1} + \cdots + G_2 \tilde{\phi}^{\theta_2} + \cdots \quad (20)$$

in which $G_1$ and $G_2$ are arbitrary constants, $\theta_1 = \frac{d\bar{m} - 2\bar{w}}{d\bar{m} + 2\bar{w}}$ and $\theta_2 = -4\frac{\bar{w} - \bar{w}}{d\bar{m} + 2\bar{w}}$.

I am interested also in the study of the equations linearized in the vicinity of a fixed point $v_i^\ast$ (eigenvalue equations). By setting $v_i = v_i^\ast + \varepsilon e^{\lambda t} g_i$ (with $t = -\ln \ell$) and retaining the linear term in $\varepsilon$, the eigenvalue equations read:

$$g_1'' = (\lambda - d_{\phi} + 2\bar{w}v_1') g_1 + \left(\tilde{d}_{\phi} \bar{\phi} + 2\bar{w}v_1' \right) g_1' - P_1 g_2' \quad (21)$$

$$g_2'' = \left[\lambda + 2 \left(\bar{w} + \theta + 2\bar{w}v_1''\right)\right] g_2 + \left(\tilde{d}_{\phi} \bar{\phi} + 2\bar{w}v_1' \right) g_2'$$

$$+ 2\bar{w}v_2'' g_1 + 2 \left(2\bar{w}v_2' - P_2 v_1'' + \theta \psi_{||0}' \right) g_1' \quad (22)$$

the interesting solutions of which must satisfy the following large $\bar{w}$ behavior (with $\kappa_1 = \frac{d\bar{m} - 2\bar{w} - 2\lambda}{d\bar{m} + 2\bar{w}}$, $\kappa_2 = -2\frac{2\bar{w} + 2\bar{w} + \lambda}{d\bar{m} + 2\bar{w}}$):

$$g_{1\text{asy}} = S_1 \tilde{\phi}^{\kappa_1} + \cdots \quad (23)$$

$$g_{2\text{asy}} = -\psi_{||0}' S_1 \tilde{\phi}^{\kappa_1 - 1} + S_2 \tilde{\phi}^{\kappa_2} + \cdots \quad (24)$$

Notice that, since $\ell$ is chosen to be the scaling parameter in the $\perp$ direction ($\ell = \ell_{\perp}$) then the eigenvalues $\lambda$ stand for those associated with the $\perp$ direction. Specifically, the highest positive eigenvalue will be associated to $1/\nu_{L2}$.

3 Numerical study

3.1 Preliminary

Before looking specifically at the numerical study of (17,18) for $m = 1$ (uni-axial Lifshitz systems), it is worthwhile to make some general remarks and
(1) Eqs (17,18) differ from those of the ordinary critical point up to $O(\partial^2)$ (see [3]) essentially for $\theta \neq 1$ and $d$ replaced by $d_m$ [eq (12)]. Since $\theta = 1/2$ in the order presently considered they are essentially the same if one considers $d_m$ as an effective value of $d$. This remark is important since one must find a fixed point with one supplementary direction of instability compared to the ordinary critical point [the Lifshitz point is a multicritical —tricritical indeed— point, see eq (5)] starting with practically the same equations! I explain below how this is possible.

(2) The classical dimensional analysis of the Lifshitz system is identical to that of the ordinary system provided that one changes $d$ into $d_m$ with $\theta = 1/2$. So, for the Lifshitz criticality, the ordinary upper critical dimension $d^{(u)} = 4$ becomes $d^{(u)}_m = 4$ that is to say $d^{(u)}_L = 4 + \frac{m}{2}$ (hence the $\epsilon_L$-expansion with $\epsilon_L = 4 - d + \frac{m}{2}$); and, for the Lifshitz tricriticality, the ordinary upper critical dimension $d^{(u)}_{tri} = 3$ becomes $d^{(u)}_m = 3 + \frac{m}{2}$ (hence the $\epsilon_{Lt}$-expansion with $\epsilon_{Lt} = 3 - d + \frac{m}{2}$).

(3) Below I study the uniaxial Lifshitz system ($m = 1$). Since at the leading order of the derivative expansion one has $\theta = 1/2$, then the effective value of $d$ (in the sense of remark 1) is $d_m = 2.5$. One knows that in the ordinary case for $d < 3$ several fixed points can co-exist. One thus expects the same in the present study allowing for the presence of several fixed points controlling different critical points.

(4) The Lifshitz critical and tricritical points may be studied from equations (17,18), only the fixed point considered is different in the two cases (it is a question of number of unstable directions which differentiates the two). Then one understands well why Nicoll et al [8] had not the right number of couplings in their $\epsilon$-expansion: they had considered only one equation [similar to eq (17) with $v_2$ lacking]. In fact as stressed in [9] they had forgotten the coupling associated with the operator $(\partial_{||}\phi)^2\phi^2$. In the present study such a term is included within the function $v_2$ (i. e. $Z_{||}(\phi)$) and eq (18). Notice that $Z_{||}(\phi)$ induces supplementary terms in the action and especially $(\partial_{||}\phi)^2\phi^4$ which has not been considered even in [9] while it is associated with a marginal coupling at the physical dimension $d = 3$ ($d_m = 2.5$).

In order to numerically study eqs (17,18) I must choose the cutoff function $\tilde{P}$ and the redundant function $\psi$. But, at this low order of approximation, the precise forms of these functions have no great signification. In fact I have explicitly considered (similarly to [3]):
\begin{align*}
\tilde{P}(q_{\parallel}^2, q_{\perp}^2) &= e^{-a(q_{\parallel}^4 + q_{\perp}^2)} \\
\psi(q^2) &= \frac{1}{1 + bq^2}
\end{align*} 

(25) 

(26)

with \(a\) and \(b\) two parameters. The fourth power of \(q_{\parallel}\) in the cutoff function has been imposed after having observed that no fixed point exists if \(\tilde{P}_{\parallel}' \sim q_{\parallel}^2\) at small \(q_{\parallel}\). With this cutoff function, it appears that the results are insensitive to the value of \(a\). As for \(b\), its actual significance is to play with the breakdown of the reparametrization invariance introduced by the approximation, since at this low order the invariance cannot be studied (because both \(Z_\perp\) and \(A\) are fixed), I set \(b = 0\) in the following. The remaining of the numerical study follows the same lines as in [3]: I consider the shooting method starting from a value \(\tilde{\phi}_0\) at which point the large \(\tilde{\phi}\) behavior of the regular solutions of eqs (17,18) is imposed then I try, with the Newton-Raphson algorithm, to find solutions that reach the origin \(\tilde{\phi} = 0\) with some conditions satisfied. For the fixed point, the conditions are conditions of \(Z_2\)-symmetry:

\begin{align*}
\psi_1(0) &= 0 \\
\psi_2'(0) &= 0
\end{align*} 

(27) 

(28)

I find at least two fixed points of interest (other fixed points coexist but they are less stable): one corresponds to \(G_1 = -2.385\) and the second one to \(G_1 = -5.031\). For all the fixed points, \(G_2 = 0\) and \(v_2\) vanishes exactly.

The first fixed point is like the customary fixed point that one encounters in studying the ordinary critical point and which is called the Wilson-Fisher fixed point [13]. However, presently it controls the Lifshitz critical point and possesses one direction of instability more that the Wilson-Fisher fixed point (see below). This is possible since eqs (17,18) are like the \(O(\partial^2)\)-equations of the ordinary case in which \(\eta\) would be arbitrarily fixed to zero. In that case the reparametrization invariance is strongly violated and the zero-eigenvalue usually associated with the (satisfied) invariance becomes a positive eigenvalue.

The second fixed point is candidate for representing the Lifshitz tricritical fixed point, the study of the eigenvalue equations will give the answer.

3.2 Study of the eigenvalue equations

3.2.1 First fixed point and the Lifshitz critical point

I look for the eigenvalue equations (21,22) for the first fixed point and with the \(Z_2\)-symmetry imposed (i.e. \(g_1(0) = 0\) and \(g'_2(0) = 0\)) and I find two positive
eigenvalues:

\[ \lambda_1 = 1.5785, \quad \lambda_2 = 1.1557 \]  

yielding:

\[ \nu_{L2} = \frac{1}{\lambda_1} = 0.6335, \quad \varphi = \frac{\lambda_2}{\lambda_1} = 0.7322 \]  

The first negative eigenvalue is found to be:

\[ \lambda_3 = -0.8916 \]  

giving the correction exponent \( \omega_{L2} \):

\[ \omega_{L2} = -\lambda_3 = 0.8916 \]

Let me remind you the best results at order \( O(\epsilon^2) \) [14] once \( \epsilon = 3/2 \) is set in the expansions: \( \nu_{L2} = 0.625, \varphi = 0.625, \omega_{L2} = 1.5 \) at \( O(\epsilon) \); \( \nu_{L2} = 0.709, \varphi = 0.677, \omega_{L2} = 0.414 \) at \( O(\epsilon^2) \) (see the other kinds of estimates in [14]). One sees that, regarding the very low order of the approximation, the present results are very good. But one advantage of the ERGE is that one obtains easily the eigenvalues in the \( Z_2 \)-asymmetric case, it suffices to impose the conditions \( g'_1(0) = 0 \) and \( g_2(0) = 0 \) and one finds:

\[
\begin{align*}
\tilde{\lambda}_1 &= 2.25 = \frac{d_m + 2 - \eta_{L2}}{2} \\
\tilde{\lambda}_2 &= 0.25 = \frac{d_m - 2 + \eta_{L2}}{2} \\
\tilde{\lambda}_3 &= -0.5821 = -\tilde{\omega}_{L2}
\end{align*}
\]

One observes that, since \( \tilde{\omega}_{L2} < \omega_{L2} \) the corrections to scaling are, in fact, dominated by the asymmetry. This is a new result.

### 3.2.2 Second fixed point and the supposed Lifshitz tricritical point

In the same way as above, in the symmetric case I find the following eigenvalues:

\[
\begin{align*}
\lambda_1 &= 1.984398, \quad \lambda_2 = 1.050316, \quad \lambda_3 = 0.885100, \\
\lambda_4 &= 0.068527, \quad \lambda_5 = -1.352355
\end{align*}
\]

with four positive answers instead of only three if the fixed point had to control a Lifshitz tricritical point as expected (the other fixed points have even more
positive eigenvalues). The $O (\epsilon)$ estimates gave for the three positive eigenvalues [9]:

\begin{align*}
\lambda_1 &= 1.86 - 1.87, \\
\lambda_2 &= 0.857 - 0.875, \\
\lambda_3 &= 0.643 - 0.737
\end{align*}

according to whether one resums the series or its inverse. The highest negative eigenvalue was estimated to be $\lambda_{y_1} = -1$ [15]. Hence the intruder eigenvalue in (36) is $\lambda_4$ the closest to zero. A simple dimensional analysis allows to understand how it can appear and why it has not been accounted for in the $\epsilon$-expansion framework.

The two terms $c_1 \int d^d x \phi^6$ and $c_2 \int d^d x \left( \partial_\parallel \phi \right)^2 \phi^2$, mentioned at the beginning of section 1 as contributing to the action relevant to the study of the Lifshitz tricritical point in the framework of the $\epsilon$-expansion [9,10], have couplings of respective classical dimension $[c_1] = 2 (3 - d_m)$ and $[c_2] = 3 - d_m$ and thus can (must) be considered together has small quantities with $\epsilon_{Lt} = 3 - d_m$. They indeed destabilize the gaussian fixed point and they are at the origin of a new nontrivial fixed point of order $\epsilon_{Lt}$ in the same way as the $\phi^4$ coupling $u$ destabilizes the gaussian fixed point when $d = 4 - \epsilon$. A difference is that it is a tricritical point with the effect of $u$ inhibited. In the ordinary case, only the $\phi^6$ coupling has to be considered, moreover, since it has the dimension $3 - d$, it is marginal at the physical dimension (the coupling parameter is small) and the study may be done perturbatively: the gaussian fixed point is actually not destabilized and the effective exponents are classical up to logarithms. In the present case of the Lifshitz tricritical point, the physical dimension is not $d_m = 3$ but instead $d_m = 2.5$ and the fixed point of interest is no longer the Gaussian fixed point but a nontrivial fixed point (the second fixed point of the present study). If only the two couplings $c_1$ and $c_2$ had to be considered —as effectively supposed in [9]— then one could still use the perturbative approach as in the well known case of the Wilson-Fisher fixed point [13] relevant to the study of the ordinary critical point. Unfortunately, at the physical dimension $d_m = 2.5$ a third term appears to have a marginal coupling, it is $c_3 \int d^d x \left( \partial_\parallel \phi \right)^2 \phi^4$ with $[c_3] = 5 - 2d_m$. One does not know whether it is marginally relevant or marginally irrelevant but one thing is certain: the answer cannot be done perturbatively because at the dimension where its effect is small ($d_m = 2.5$), the fixed point of interest (created by the effects of $c_1$ and $c_2$) is no longer close to the Gaussian fixed point since this would imply $d_m \approx 3$. A nonperturbative study is obliged in that case and the present one indicates that $c_3$ is marginally relevant at the fixed point of interest which is thus unstable. Consequently, if this result is maintained at higher orders of the derivative expansion, the so-called Lifshitz tricritical point is a point of higher criticality. However, since $\lambda_4$ is small, the observation of almost universal Lifshitz tricritical behavior is possible.

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References

[1] K. G. Wilson and J. Kogut, Phys. Rep. 12C, 77 (1974).
[2] C. Bagnuls and C. Bervillier, Phys. Rep. 348, 91 (2001).
[3] C. Bervillier, hep-th/040525.
[4] R. M. Hornreich, M. Luban and S. Shtrikman, Phys. Rev. Lett. 35, 1678 (1975).
[5] R. M. Hornreich, J. Magn. Magn. Mat. 15-18, 387 (1980).
[6] W. Selke, in Phase Transitions and Critical Phenomena Vol. 15, p. 1, Ed. by C. Domb and J. Lebowitz (Acad. Press, London, 1992).
[7] H. W. Diehl, Acta Phys. Slov. 52, 271 (2002).
[8] J. F. Nicoll, G. F. Tuthill, T. S. Chang and H. E. Stanley, Phys. Lett. 58A, 1 (1976).
[9] A. Aharony, E. Domany, R. M. Hornreich, T. Schneider and M. Zannetti, Phys. Rev. B32, 3358 (1985). R. Dengler, Phys. Lett. 108A, 269 (1985).
[10] N. F. Tonchev and D. I. Uzunov, Physica A134, 265 (1985).
[11] J. Sak and G. S. Grest, Phys. Rev. B17, 3602 (1978).
[12] I would like to draw reader’s attention on the fact that if one performs the changes $V \rightarrow -H$, $\dot{V} \rightarrow -\partial H/\partial l$, and $\theta = 1/\sigma$ for $m = d$ in eq (14) then one should obtain (for example) eq (1.2) of [16] which was intended to generalize the local potential approximation of the Wilson equation to critical systems with “propagator exponent” $\sigma$. One may easily check that the two equations differ indeed by the factor $\varpi$ appearing in front of the “coarse-graining” term $(V' - V'')$ in (14). This is a direct consequence of the omission of re-normalization factor $\zeta$ of the field as stressed in [3].
[13] K. G. Wilson and M. E. Fisher, Phys. Rev. Lett. 28, 240 (1972).
[14] H. W. Diehl and M. Shpot, Phys. Rev. B62, 12338 (2000). M. Shpot and H. W. Diehl, Nucl. Phys. B612, 340 (2001).
[15] A. Aharony, E. Domany and R. M. Hornreich, Phys. Rev. B36, 2006 (1987).
[16] J. F. Nicoll, T. S. Chang and H. E. Stanley, Phys. Rev. A13, 1251 (1976).