SOME GLOBAL MINIMIZERS OF A SYMPLECTIC DIRICHLET ENERGY

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ABSTRACT. The variational problem for the functional $F = \frac{1}{2} \int_M \|\varphi^* \omega\|^2$ is considered, where $\varphi : (M, g) \to (N, \omega)$ maps a Riemannian manifold to a symplectic manifold. This functional arises in theoretical physics as the strong coupling limit of the Faddeev-Hopf energy, and may be regarded as a symplectic analogue of the Dirichlet energy familiar from harmonic map theory. The Hopf fibration $\pi : S^3 \to S^2$ is known to be a locally stable critical point of $F$. It is proved here that $\pi$ in fact minimizes $F$ in its homotopy class and this result is extended to the case where $S^3$ is given the metric of the Berger’s sphere. It is proved that if $\varphi^* \omega$ is coclosed then $\varphi$ is a critical point of $F$ and minimizes $F$ in its homotopy class. If $M$ is a compact Riemann surface, it is proved that every critical point of $F$ has $\varphi^* \omega$ coclosed. A family of holomorphic homogeneous projections into Hermitian symmetric spaces is constructed and it is proved that these too minimize $F$ in their homotopy class.

1. INTRODUCTION

Given a smooth map $\phi : M \to N$ between Riemannian manifolds, a natural notion of the energy of $\phi$ is the Dirichlet energy,

$$E = \frac{1}{2} \int_M \|d\phi\|^2.$$  

The variational problem for $E$, whose critical points are harmonic maps, has been heavily studied for many years. If we replace the metric on $N$ by a symplectic form $\omega$, a natural analogue of $E$ is

$$F = \frac{1}{2} \int_M \|\phi^* \omega\|^2,$$

which one can regard as a kind of symplectic Dirichlet energy. In this paper we study the variational problem for $F$, focussing particularly on the problem of obtaining global minimizers of $F$.

Our original motivation comes from theoretical physics. If $N$ is a Kähler manifold, both $E$ and $F$ are defined, and the variational problem for $E + \alpha F$, where $\alpha$ is a positive constant called the coupling constant, is known to physicists as the Faddeev-Hopf (or Faddeev-Skyrme) model. The case of most interest is $M = \mathbb{R}^3$, $N = S^2$. This variational problem, originally proposed by Faddeev in the 1970s, lay dormant for lack of computational power until the 1990s, but has been the subject of intense numerical study in recent years [3, 8, 12, 16]. Its critical points, which are interpreted as topological solitons, have been proposed as models of gluon flux tubes in hadrons (particles composed of quarks). There has been some analytic progress on this model too. Kapitanski and Vakulenko proved a rather curious topological energy bound for maps $\mathbb{R}^3 \to S^2$ [18], Kapitanski and Auckly proved weak existence (i.e. existence in some Sobolev space with low regularity) of global minimizers in every
homotopy class of maps $M \to S^2$, with $M$ a compact oriented 3-manifold \cite{1}, and Ward obtained some exact results for the case $S^3 \to S^2$ \cite{19}.

The variational problem for $F$, which we shall consider here, can be interpreted as the Faddeev-Hopf model in the strong coupling limit $\alpha \to \infty$. This has two key similarities with Yang-Mills theory: it is conformally invariant in dimension 4 and it possesses an infinite dimensional symmetry group (the symplectic diffeomorphisms of $N$). As in Yang-Mills, the most physically relevant choice of $M$ is $S^4$, regarded as the conformal compactification of Euclideanized spacetime $\mathbb{R}^4$. It is an interesting open question whether there is a critical point of $F$ in the generator of $\pi_4(S^2)$. Such a critical point would be interpreted as an instanton in the strongly coupled Faddeev-Hopf model on Minkowski space.

This variational problem seems to have received remarkably little attention. Ferreira and De Carli \cite{6} analyzed the case $M = S^3 \times \mathbb{R}$, with a Lorentzian metric, and $N = \mathbb{C}$, $S^2$ or the hyperbolic plane, working within a particular rotationally invariant ansatz. The first systematic development of the variational calculus was made in \cite{15}. We begin by briefly reviewing some results from that paper. Throughout, all maps are smooth, $(M, g)$ is a Riemannian manifold and $(N, \omega)$ is a symplectic manifold. In the case where $N$ is Kähler, $\omega$ is the Kähler form $\omega(\cdot, \cdot) = h(J\cdot, \cdot)$, $h$ is the metric and $J$ is the almost complex structure. The coderivative on differential forms will be denoted $\delta$.

**Theorem 1.1.** \cite{15} For any variation $\varphi_t$ of $\varphi : M \to N$ with variational vector field $X \in \Gamma(\varphi^{-1}TN)$ we have

$$\frac{d}{dt}F(\varphi_t)|_{t=0} = \int_M \omega(X, d\varphi(\# \delta \varphi^* \omega)) \ast 1.$$  

In particular, $\varphi$ is a critical point of $F$ if and only if

$$d\varphi(\# \delta \varphi^* \omega) = 0.$$  

**Theorem 1.2.** \cite{15} A Riemannian submersion $\varphi : M \to N$ from a Riemannian manifold to a Kähler manifold is a critical point of $F$ if and only if it has minimal fibres, i.e., if and only if it is a harmonic map.

A well known harmonic Riemannian submersion is the Hopf map

$$\pi : S^3 \to S^2,$$

where $S^3$ is the sphere in $\mathbb{R}^4$ of radius 1, and $S^2$ the sphere in $\mathbb{R}^3$ of radius $1/2$. As a harmonic map, $\pi$ is well known to be unstable, as are all harmonic maps from $S^3$ \cite{20}. We studied in \cite{15} the second variation of $F$ for maps into Kähler manifolds, and found the associated Jacobi operator.

**Theorem 1.3.** \cite{15} Let $N$ be Kähler and $\varphi : M \to N$ be a critical point of $F$. Then the Hessian of $\varphi$ is

$$H_{\varphi}(X, Y) = \int_M h(X, \mathcal{L}_\varphi Y) \ast 1,$$

where

$$\mathcal{L}_\varphi Y = -J \{ \nabla^\varphi_z Y + d\varphi(\# \delta \varphi^* (Y \lrcorner \omega)) \} \quad \text{and} \quad Z_\varphi = \# \delta \varphi^* \omega.$$
By a careful calculation of the spectrum of this operator for the Hopf map, we proved the following result, which was conjectured by Ward [19].

**Theorem 1.4.** [15] The Hopf map $\pi : S^3 \to S^2$ is stable for the full Faddeev-Hopf functional $E + \alpha F$ if and only if $\alpha \geq 1$.

In particular, the Hopf map is a stable critical point of $F$. In this paper, we strengthen this result to show that the Hopf map in fact minimizes $F$ in its homotopy class. The same is true for the Hopf map from the Berger’s spheres

$$\pi : (S^3, g_t) \to S^2,$$

for all $0 < t \leq 1$; see Example 2.2 for the definition of Berger’s spheres. It is interesting to note that a slightly stronger version of Theorem 1.4 (namely, that $\pi$ is a local minimizer of $E + \alpha F$ when $\alpha > 1$) was obtained independently by Isobe [11] using rather different methods. It remains an open question whether the Hopf map globally minimizes $E + \alpha F$ for some $\alpha$.

As proved in [15], we have

$$H_{\varphi}(X, X) = \int_M \omega(X, \nabla^\varphi Z_\varphi X) + \|d\varphi^*(X \omega)\|^2_{L^2}(X \in \Gamma(\varphi^{-1}TN)).$$

In particular, when $\varphi^*\omega$ is coclosed (so $Z_\varphi = 0$), $\varphi$ is a stable critical point of $F$. In this paper, we strengthen this by showing that, if $\varphi^*\omega$ is coclosed, then $\varphi$ actually minimizes $F$ in its homotopy class.

**2. Global minimizers**

Denote by $S^3$ the unit sphere in $\mathbb{R}^4$ of radius 1, and by $S^2$ the sphere in $\mathbb{R}^3$ of radius $1/2$.

**Theorem 2.1.** The Hopf map $\pi : S^3 \to S^2$ minimizes $F$ in its homotopy class.

The proof makes use of the Hopf invariant of a map $\varphi : S^3 \to S^2$. Recall that this is defined as the number

$$H(\varphi) = \int_{S^3} d\alpha \wedge \alpha,$$

where $d\alpha = \varphi^*\omega$, and $\omega$ is the volume form of $S^2$. It is well known that this is independent of the choice of $\alpha$, and depends only on the homotopy class of $\varphi$, see e.g. [5].

**Proof.** As above, assume that $\varphi : S^3 \to S^2$ is any map and write $\varphi^*\omega = d\alpha$. By the Hodge decomposition, we may assume that $\alpha$ is coexact. Then

$$F(\varphi) = \frac{1}{2}\langle d\alpha, d\alpha \rangle_{L^2} = \frac{1}{2}\langle \alpha, \Delta \alpha \rangle_{L^2} \geq \frac{\lambda_1}{2}\|\alpha\|^2_{L^2},$$

where $\lambda_1$ is the first eigenvalue of the Hodge-Laplace operator on coexact 1-forms on $S^3$. It is known that $\lambda_1 = 4$, see e.g., [9] page 270]. Hence

$$\|\alpha\|^2_{L^2} \leq \frac{1}{2}F(\varphi).$$
By Cauchy-Schwarz,
\[ H(\varphi) \leq \|\varphi^* \omega\|_{L^2} \|\alpha\|_{L^2} \leq \frac{1}{\sqrt{2}} \|\varphi^* \omega\|_{L^2} \sqrt{F(\varphi)} = F(\varphi). \]

For the Hopf map \( \pi : S^3 \to S^2 \), it is well known that (see e.g. [10, page 102])
\[ d \ast \pi^* \omega = 2\pi^* \omega, \]
so that
\[ H(\pi) = F(\pi). \]
Thus, if \( \varphi \) and \( \pi \) are homotopic, then
\[ F(\pi) = H(\pi) = H(\varphi) \leq F(\varphi). \]

Example 2.2. Consider again the Hopf map \( \pi : S^3 \to S^2 \). We may write the metric \( g \) on \( S^3 \) as
\[ g = g_V + g_H, \]
where \( V \) is the distribution tangent to the fibres of \( \pi \), and \( H \) its orthogonal complement. For \( 0 < t < 1 \), the 3-dimensional Berger’s sphere is the Riemannian manifold \((S^3, g_t)\), where
\[ g_t = t^2 g_V + g_H. \]
It is easy to see that
\[ d \ast \pi^* \omega = 2t \pi^* \omega. \]
For \( 0 < t < 1 \), a simple calculation shows that the sectional curvature of \( g_t \) is bounded below by \( t^2 \). Hence, the minimal eigenvalue for the Hodge-Laplace operator on coexact 1-forms with respect to \( g_t \) is bounded below by \( 4t^2 \), see [9, page 270].

Thus, for any map \( \varphi \) homotopic to \( \pi \), a calculation similar to that above gives
\[ F(\pi) = tH(\pi) = tH(\varphi) \leq F(\varphi). \]
So \( \pi \) still minimizes \( F \) in its homotopy class, as a map from the Berger’s sphere to \( S^2 \).

Next we consider another class of maps which minimize \( F \).

Theorem 2.3. Let \( M \) be compact and \( \varphi : M \to N \) have \( \varphi^* \omega \) coclosed. Then \( \varphi \) minimizes \( F \) in its homotopy class.

Proof. Let \( \varphi : M \to N \) have \( \varphi^* \omega \) coclosed, \( \psi : M \to N \) be homotopic to \( \varphi \) and \( \varphi_t \) be a smooth homotopy from \( \varphi \) to \( \psi \). By the homotopy lemma [7]
\[ \frac{d}{dt} \langle \varphi_t^* \omega, \varphi_t^* \omega \rangle_{L^2} = \langle d(\varphi_t^* X \lrcorner \omega), \varphi_t^* \omega \rangle_{L^2} = \langle \varphi_t^* X \lrcorner \omega, \delta \varphi_t^* \omega \rangle_{L^2} = 0. \]
Hence, by Cauchy-Schwartz,
\[ F(\varphi) = \frac{1}{2} \langle \psi^* \omega, \varphi^* \omega \rangle_{L^2} \leq \sqrt{F(\psi)F(\varphi)}, \]
so that
\[ F(\varphi) \leq F(\psi). \]
Remark 2.4. Note that if $\delta \varphi^* \omega = 0$ then $\varphi^* \omega$ is harmonic, and thus minimizes the $L^2$-norm in its cohomology class. If $\psi$ is homotopic to $\varphi$, then $\psi^* \omega$ is in this cohomology class, and this gives an alternative proof of the above result.

Corollary 2.5. Any critical immersion from a compact Riemannian manifold to a symplectic manifold minimizes $F$ in its homotopy class.

Clearly, any symplectomorphism on a compact symplectic Riemannian manifold is a minimizer in its homotopy class. In particular, we have

Corollary 2.6. The identity map on a compact symplectic Riemannian manifold minimizes $F$ in its homotopy class.

We next prove that if $M$ is a compact Riemann surface, all critical points of $F$ have $\varphi^* \omega$ coclosed.

Theorem 2.7. Let $M$ be a compact, oriented, 2-dimensional Riemannian manifold and $N$ be a symplectic manifold. Then every critical point $\varphi : M \to N$ of $F$ has $\varphi^* \omega$ coclosed (and hence minimizes $F$ in its homotopy class).

Proof. For any $\varphi : M \to N$,

$$\varphi^* \omega = f * 1,$$

where $* 1$ is the volume form on $M$ and $f$ is a real function on $M$. Then $\delta \varphi^* \omega = * df$, so that

$$\sharp \delta \varphi^* \omega = J \nabla f,$$

where $J$ is the Hermitian structure on $M$ associated with the orientation.

Assume now that $\varphi$ is critical. Then $d \varphi (J \nabla f) = 0$ so that

$$0 = (\varphi^* \omega)(J \nabla f, \nabla f) = -f |\nabla f|^2.$$

It follows that $\nabla f = 0$ everywhere, so $f$ is constant. (Assume, to the contrary, that $(\nabla f)(p) \neq 0$ for some $p \in M$. Then there is some neighbourhood of $p$ on which $\nabla f \neq 0$. But then $f = 0$ on this neighbourhood, so $(\nabla f)(p) = 0$, a contradiction.) Hence $\varphi^* \omega$ is coclosed. \qed

For our next result, recall that a submersion $\varphi : (M, g) \to (N, h, J)$ from a Riemannian manifold to an almost Hermitian manifold gives rise to an $f$-structure $f$ on $M$, such that $\ker f = \ker d \varphi$ and the restriction of $f$ to $(\ker d \varphi)^\perp$ corresponds to $J$ under the identification $(\ker d \varphi)^\perp \cong \varphi^* TN$ (see [14] for the definition of an $f$-structure). The map is said to be pseudo horizontally (weakly) conformal (PHWC) if $f$ is skew-symmetric with respect to the metric $g$ on $M$. In particular, if $M$ also carries an almost Hermitian structure with respect to which $\varphi$ is holomorphic, then $\varphi$ is PHWC. See e.g. [2, 13] for other characterizations of PHWC maps.

Proposition 2.8. Assume that $\varphi : (M, g) \to (N, h, J)$ is a PHWC submersion from a Riemannian manifold to an almost Hermitian manifold, with associated $f$-structure $f$. Then

$$\delta \varphi^* \omega = f \text{div} \varphi^* \omega - \sum_a h(\varphi_* f e_a, \nabla d \varphi(e_a, \cdot )),$$

where in the last term we sum over a local orthonormal frame for $TM$.\[5\]
Proof. For any vector field $Y$ on $M$ we have
\[
\delta \varphi^* \omega(Y) = - \sum_a (e_a \varphi^* \omega(e_a, Y) - \varphi^* \omega(\nabla_{e_a} e_a, Y) - \varphi^* \omega(e_a, \nabla_{e_a} Y))
\]
\[
= - \sum_a (h(\nabla_{e_a} \varphi_* f e_a - \varphi_* f \nabla_{e_a} e_a, \varphi_* Y) + h(\varphi_* f e_a, \nabla \delta \varphi(e_a, Y)))
\]
\[
= - \sum_a h(\nabla \delta \varphi(e_a, f e_a), \varphi_* Y) - h(\varphi_* \text{div} f, \varphi_* Y)
\]
\[
- \sum_a h(\varphi_* f e_a, \nabla \delta \varphi(e_a, Y))
\]

To see that the first term is zero, we can choose the local orthonormal frame such that $e_1, \ldots, e_n$ span $\ker d \varphi = \ker f$, while $e_{n+1}, \ldots, e_m$ span the orthogonal complement of $\ker d \varphi$. Evaluating the sum for the two local orthonormal frames $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_m\}$ and $\{e_1, \ldots, e_n, f e_{n+1}, \ldots, f e_m\}$, shows that this amounts to zero. Furthermore, the second term equals $f \text{div} f \varphi^* \omega(Y)$, and the proof is finished. \hfill\Box

Assume that $\varphi : (M, g) \to (N, h)$ is a submersion between two Riemannian manifolds, and denote by $\mathcal{V}$ the vertical distribution $\ker d \varphi$ and by $\mathcal{H}$ the horizontal distribution $\mathcal{V}^\perp$ on $M$. Recall that $\varphi$ is said to be horizontally conformal if there is a positive function $\lambda$ on $M$, the dilation of $\varphi$, such that
\[
h(\varphi_* X, \varphi_* Y) = \lambda^2 g(X, Y) \quad (X, Y \in \Gamma(\mathcal{H})).
\]
If $\lambda$ is constant, $\varphi$ is said to be horizontally homothetic. Clearly, any horizontally conformal submersion to an almost Hermitian manifold is PHWC.

**Corollary 2.9.** Assume that $\varphi : (M, g) \to (N, h, J)$ is a horizontally homothetic submersion from a Riemannian manifold to an almost Hermitian manifold. Then
\[
\delta \varphi^* \omega = - \lambda^2 \delta \text{div} f,
\]
where $\lambda$ is the dilation of $\varphi$ and $f$ the associated $f$-structure.

**Proof.** It is well known that $\nabla d \varphi(X, Y) = 0$ when $X$ and $Y$ are horizontal. Thus, if $Y$ is horizontal, we have
\[
\delta \varphi^* \omega(Y) = h(J \varphi_* f \text{div} f, \varphi_* Y) = - \lambda^2 g(\text{div} f, Y).
\]
On the other hand, if $Y$ is vertical, then
\[
\delta \varphi^* \omega(Y) = - \sum_a h(\varphi_* f e_a, \nabla \delta \varphi(e_a, Y)) = \sum_a h(\varphi_* f e_a, \varphi_* \nabla_{e_a} Y)
\]
\[
= - \lambda^2 \sum_a g(\nabla_{e_a} f e_a, Y) = - \lambda^2 g(\text{div} f, Y).
\]
\hfill\Box

We will use this result to construct a family of homogeneous projections into Hermitian symmetric spaces, all of which minimize $F$ in their homotopy class. Let $g^C$ be a semi-simple complex Lie algebra, $g$ be a compact real form of $g^C$ and $h \subset g$ be a Cartan subalgebra with complexification $h^C \subset g^C$. Denote by $\Delta = \Delta_+ \cup \Delta_-$ the set of roots and its decomposition into positive and negative roots, after a choice
of a positive Weyl chamber. Let \( \Pi \subset \Delta^+ \) be the set of simple roots. For any subset \( \Pi_0 \subset \Pi \), we can construct a parabolic subalgebra as

\[
p_0 = h^C + \sum_{\alpha \in [\Pi_0]} g_\alpha + \sum_{\alpha \in \Delta^+ \setminus [\Pi_0]} g_\alpha,
\]

where \([\Pi_0]\) is the set of roots in the span of \( \Pi_0 \).

Let \( G^C \) be a complex Lie group with Lie algebra \( g^C \), and \( G, P_0 \) and \( K_0 \) be Lie subgroups of \( G^C \) with Lie algebras \( g, p_0 \) and \( \mathfrak{f}_0 = g \cap p_0 \), respectively. Let \( m_0 \) be the Killing orthogonal complement to \( k_0 \), so that

\[
m_0 = \sum_{\alpha \in \Delta^+ \setminus [\Pi_0]} (g_\alpha + g_{-\alpha}) \cap g.
\]

We may identify \( m_0 \) with the tangent space of \( G/K_0 \) at the identity coset. Now \( G/K_0 \) has a \( G \)-invariant integrable complex structure, for which the \((1,0)\) and \((0,1)\) spaces at the identity coset are

\[
m_0^{1,0} = \sum_{\alpha \in \Delta^+ \setminus [\Pi_0]} g_\alpha, \quad m_0^{0,1} = \sum_{\alpha \in \Delta^+ \setminus [\Pi_0]} g_{-\alpha}.
\]

It is well known that minus the Killing form of \( g \) equips \( G/K_0 \) with a Riemannian metric for which this complex structure is Hermitian and cosymplectic (i.e. the Kähler form is coclosed), see e.g. [17].

Now, for any nested pair of subsets \( \Pi_0 \subset \Pi'_0 \subset \Pi \) of simple roots, we get, with obvious notation, a homogeneous fibration

\[
\varphi : G/K_0 \to G/K'_0,
\]

and this map is clearly a holomorphic Riemannian submersion with totally geodesic fibres, see e.g., [4, page 257]. Hence, in the case where \( G/K'_0 \) is Kähler, \( \varphi \) is a critical point of \( F \). From now on, we will assume that \( G/K'_0 \) is a Hermitian symmetric space, so that the complex structure is in fact Kähler.

**Proposition 2.10.** With the notation and conventions introduced above, the homogeneous fibration \( \varphi : G/K_0 \to G/K'_0 \) minimizes \( F \) in its homotopy class.

**Proof.** It is, by Theorem 2.3 Corollary 2.9 and \( G \)-invariance, enough to show that \( \text{div } f = 0 \) at the identity coset. It follows easily from the formula for the Levi-Civita connection in a reductive homogeneous space, see [4] page 183, that

\[
\langle \nabla_X Y, Z \rangle = 0,
\]

for all \( X, Y \in g_{\pm \alpha} \) and \( Z \in g_{\pm \beta} \), for any \( \alpha, \beta \in \Delta^+ \). Since \( f \) is \( G \)-invariant and preserves the subspace \((g_\alpha + g_{-\alpha}) \cap g\) of \( m_0 \), we have

\[
\langle (\nabla_X f)X, Z \rangle = \langle \nabla_X fX, Z \rangle - \langle f \nabla_X X, Z \rangle = 0,
\]

for all \( X \in g_{\pm \alpha} \) and \( Z \in g_{\pm \beta} \), for any \( \alpha, \beta \in \Delta^+ \). From the above orthogonal decomposition of \( m_0 \), we see that \( \text{div } f = 0 \). \( \square \)

**Example 2.11.** Let \( n_1, \ldots, n_k \) be positive integers, and \( n = n_1 + \cdots + n_k \). Define \( M \) as the space of decompositions

\[
\mathbb{C}^n = V_1 \oplus \cdots \oplus V_k,
\]
where \( \dim V_i = n_i \), and \( V_i \perp V_j \) whenever \( i \neq j \). If we denote by \( Gr_{n_1}(\mathbb{C}^n) \) the Grassmannian of \( n_1 \)-dimensional subspaces of \( \mathbb{C}^n \), we have an obvious map

\[
\varphi : M \to Gr_{n_1}(\mathbb{C}^n), \quad \varphi(V_1 \oplus \cdots \oplus V_k) = V_1.
\]

By considering both \( M \) and \( Gr_{n_1}(\mathbb{C}^n) \) as homogeneous \( SU(n) \)-spaces, it follows easily that \( \varphi \) is a homogenous projection of the type considered above, and hence a minimizer of \( F \) in its homotopy class.

**Remark 2.12.** Let \( \varphi : M \to N \) be a holomorphic map between almost Hermitian manifolds. It is well known that if \( M \) and \( N \) are almost Kähler then \( \varphi \) is harmonic and minimizes the Dirichlet energy \( E \) in its homotopy class, see e.g., [7] for a proof. In fact, this proof immediately generalizes to the case where \( M \) is only cosymplectic. Hence, if, in addition, \( \varphi^* \omega \) is coclosed, we see that \( \varphi \) minimizes the full Faddeev-Hopf energy \( E + \alpha F \) for all \( \alpha > 0 \). This applies to the homogeneous fibrations \( G/K_0 \to G/K_0' \) defined above.

**References**

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