Liouville Type Theorems for the Planar Stationary MHD Equations with Growth at Infinity

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Abstract. For the two dimensional steady MHD equations, we prove that Liouville type theorems hold if the velocity is growing fast at infinity. The main obstacle comes from the nonlinear terms, since the vorticity system of the MHD equations has no maximum principle unlike the Navier–Stokes equations. As a corollary, we obtain that all solutions of the 2D Navier–Stokes equations satisfying $\nabla u \in L^p(\mathbb{R}^2)$ with $1 < p < \infty$ are constants, which is sharp since there exist some non-trivial linear solutions like the Couette flow in the sense of $\nabla u \in L^\infty(\mathbb{R}^2)$.

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1. Introduction

Consider the incompressible MHD equations on the whole space $\mathbb{R}^2$ as follows:

$$\begin{cases}
-\Delta u + u \cdot \nabla u + \nabla \pi = b \cdot \nabla b, \\
-\Delta b + u \cdot \nabla b = b \cdot \nabla u, \\
d\text{div } u = 0, \text{div } b = 0,
\end{cases}$$

(1)

and the Dirichlet energy integral is defined as the following:

$$D(u, b) = \int_{\mathbb{R}^2} |\nabla u|^2 + |\nabla b|^2 \, dx.$$  

(2)

When $b = 0$ in (1), it is the 2D Navier–Stokes equations. Let us recall some known results of Navier–Stokes equations on this issue. For example, Gilbarg–Weinberger proved the above Liouville type theorem by assuming the Dirichlet energy integral of (2) with $b = 0$ is finite in [11], where they made use of the fact that the vorticity function satisfies a nice elliptic equation to which a maximum principle applies. The assumption on the boundedness of the Dirichlet energy integral can be relaxed to $\nabla u \in L^p(\mathbb{R}^2)$ with some $p \in (\frac{6}{5}, 3]$ by Bildhauer–Fuchs–Zhang [1]. If $u$ is bounded, a Liouville theorem being more in the spirit of the classical one for entire analytic functions was obtained by Koch–Nadirashvili–Seregin–Sverak [13] as a byproduct of their work on the non-stationary case. The above results also can be generalized to the shear thickening flows, for example seeing [4–6,12,19,20]. In addition, the equation of (1) was also considered in an exterior domain like the existence and asymptotic behavior of solutions, such as referring to [3,9,10,14–17].

Furthermore, along with [13], Fuchs–Zhong in [7] showed the velocity field $u$ satisfying the stationary Navier–Stokes equations on the entire plane must be constant under the growth condition $\limsup |x|^{-\alpha} |u(x)| < \infty$ as $|x| \to \infty$ for some $\alpha \in [0, 1/7)$. Later, Bildhauer–Fuchs–Zhang [1] proved the component can be improved to $\alpha < \frac{1}{3}$.
However, for the two dimensional stationary MHD equations, the similar Liouville type theorems seem to be more difficult, since the maximum principle for the vorticity system is not available to the best of my knowledge. In [18], the author and Y. Wang obtained some Liouville type theorems by assuming $D(u, b) < \infty$ in (2) or $u \in L^\infty$, where the smallness conditions of the magnetic field are added. Here we go on this topic in this direction. Since all the exact solutions of (1) with $b = 0$ we know are polynomials, it seems that the smooth solutions below linear growth are trivial. A natural question is as follows:

**What happens if the velocity is growing at infinity?**

Note that the vorticity equations are as follows. Let $w = \partial_2 u_1 - \partial_1 u_2$ and $h = \partial_2 b_1 - \partial_1 b_2$, then
\[
\begin{aligned}
-\Delta w + u \cdot \nabla w &= b \cdot \nabla h, \\
-\Delta h + u \cdot \nabla h &= b \cdot \nabla w + H
\end{aligned}
\]  
(3)

where
\[
H = 2\partial_2 b_2(\partial_2 u_1 + \partial_1 u_2) + 2\partial_1 u_1(\partial_2 b_1 + \partial_1 b_2).
\]

The main difficulty comes from the terms $b \cdot \nabla w$, $H$ etc., which are not vanishing by the usual energy method. This is why we have to assume the smallness of some norm of $b$ holds. Moreover, if the velocity is largely growing as in [1], i.e. there exist two constants $\alpha > 0$ and $c_0 > 0$ such that
\[
|u(x)| \leq c_0 (1 + |x|)^\alpha, \quad \forall \, x \in \mathbb{R}^2,
\]  
(4)

then this case is more complicated. In fact, using the same arguments as in [1], the term
\[
C(q)L^{2\alpha}\int_{\mathbb{R}^2} h^2 |b|^2 w^{2q-4}(\eta^2)dx
\]
seems to be not controlled (see (8) in the second subsection). To overcome it, we introduce the decay condition of $b$:
\[
|b(x)| \leq c_0 (1 + |x|)^{\beta}, \quad \forall \, x \in \mathbb{R}^2,
\]  
(5)

where $\beta < 0$. Another new observation is to consider the local energy estimate in an annular domain, which is similar as the idea of Littlewood-Paley decomposition.

Next we state our first result:

**Theorem 1.1.** Let $(u, b, \pi)$ be a smooth solution of the 2D MHD equations (1) defined over the entire plane satisfying the growth estimates (4) with $\alpha < 1/3$ and (5) with $\beta < -\alpha$. Then, there exists $\varepsilon_0 = \varepsilon_0(\alpha, \beta, c_0) > 0$ such that, if
\[
\|b\|_{L^1(\mathbb{R}^2)} + \|\nabla |b|^{1/2}\|_{L^1(\mathbb{R}^2)} \leq \varepsilon_0,
\]
then $u$ and $\pi$ are constants and $b \equiv 0$.

**Remark 1.** The above result generalized the Liouville type theorem in [1,7,13] to the MHD case. It is worth mentioning that we don’t know whether this condition of $\alpha < 1/3$ is sharp. As far as the author knows, even for the Navier–Stokes equation, it is the best result at present.

It follows from the above theorem that

**Theorem 1.2.** Let $(u, b, \pi)$ be a smooth solution of the 2D MHD equations (1) defined over the entire plane satisfying the growth estimates $\nabla u \in L^{q_0}(\mathbb{R}^2)$ for $1 < q_0 < \infty$, and $\nabla b \in L^\infty(\mathbb{R}^2)$. Then, there exists $\varepsilon = \varepsilon(q_0, \|\nabla u\|_{L^{q_0}}, \|\nabla b\|_{L^\infty}) > 0$ such that, if
\[
\|b\|_{L^1(\mathbb{R}^2)} + \|\nabla |b|^{1/3}\|_{L^1(\mathbb{R}^2)} \leq \varepsilon,
\]
then $u$ and $\pi$ are constants and $b \equiv 0$.

When $b$ vanishes, the 2D Navier–Stokes equations follows from (1).

**Corollary 1.3.** Let $(u, \pi)$ be a smooth solution of the 2D NS equations defined over the entire plane satisfying the growth estimates $\nabla u \in L^q(\mathbb{R}^2)$ for some $1 < q < \infty$. Then $u$ and $\pi$ are constants.
Remark 2. The above result generalized the Liouville type theorem by Gilbarg–Weinberger in [11] for \( q = 2 \). Moreover, this is the best estimate in a sense, since there are counter-examples, whose gradient \( \nabla u \) belongs to \( L^{\infty}(\mathbb{R}^2) \) (for example, the Couette flow \((x_2,0)\)).

We need the following lemma in the proof.

**Lemma 1.4** (Theorem II.9.1 [8]). Let \( \Omega \subset \mathbb{R}^2 \) be an exterior domain and let
\[
\nabla f \in L^p(\Omega),
\]
for some \( 2 < p < \infty \). Then
\[
\lim_{|x| \to \infty} \frac{|f(x)|}{|x|^{\frac{p-2}{p}}} = 0,
\]
uniformly.

Throughout this article, \( C(\alpha_1, \cdots, \alpha_n) \) denotes a constant depending on \( \alpha_1, \cdots, \alpha_n \), which may be different from line to line.

2. Proof of Theorem 1.1

In this section, we are aimed to prove Theorem 1.1 by following the same route in [1]. Different from the arguments in [1], we consider the local energy estimates in an annular domain and obtain the \( L^q \) estimates of the vorticity.

First, we prove the following proposition.

**Proposition 2.1.** Let \((u, b, \pi)\) be a smooth solution of the 2D MHD equations (1) defined over the entire plane satisfying the growth estimates (4) with \( 0 < \alpha < \frac{1}{3} \) and (5) with \( \beta < -\alpha \). Then
\[
\|\nabla u\|_{L^q(\mathbb{R}^2)} + \|\nabla b\|_{L^q(\mathbb{R}^2)} \leq C(\alpha, \beta, q, c_0) < \infty
\]
holds for any \( q > q_0 \), where
\[
q_0 = \max \left\{ \frac{2}{1 - 3\alpha}, \frac{-1}{\alpha + \beta}, -\frac{1}{2\beta} \right\}.
\]

**Proof of Proposition 2.1.** Let \( \eta(x) \in C^\infty_c(2R) \) be a cut-off function on an annular domain with \( 0 \leq \eta \leq 1 \) satisfying
\[
\eta(x) = \begin{cases} 
1, & x \in B_R \setminus B_{R/2}, \\
0, & x \in B_{2R} \cup B_{R/4}.
\end{cases}
\]

Write \( w^{2q} = (w^2)^q \). Then for \( q \geq 2, \ell \geq q \), we have
\[
\int_{\mathbb{R}^2} w^{2q} \eta^{2\ell} dx = \int_{\mathbb{R}^2} (\partial_2 u_1 - \partial_1 u_2) w^{2q-2} w \eta^{2\ell} dx
\]
\[
= \int_{\mathbb{R}^2} (u_2, -u_1) \cdot \nabla [w^{2q-2} w \eta^{2\ell}] dx
\]
\[
\leq (2q - 1) \int_{\mathbb{R}^2} \|u\| \|\nabla w\| w^{2q-2} \eta^{2\ell} dx + 2\ell \int_{\mathbb{R}^2} \|u\| \|\nabla \eta\| |w|^{2q-1} \eta^{2\ell-1} dx
\]
\[
\leq \frac{1}{2} \int_{\mathbb{R}^2} w^{2q} \eta^{2\ell} dx + C(q) \int_{\mathbb{R}^2} \|u\|^2 |\nabla w|^2 w^{2q-4} \eta^{2\ell} dx + 2\ell \int_{\mathbb{R}^2} \|u\| \|\nabla \eta\| |w|^{2q-1} \eta^{2\ell-1} dx
\]
Similarly, we have

\[
\int_{\mathbb{R}^2} h^2 \eta^2 \bar{f} \, dx \leq C(q) \int_{\mathbb{R}^2} |b|^2 |\nabla h|^2 h^{2q-4} \eta^2 \bar{f} \, dx + 4\ell \int_{\mathbb{R}^2} |b| |\nabla \eta| |h|^{2q-1} \eta^2 \bar{f} \, dx
\]

Due to the growth estimates (4) and (5), we have

\[
\int_{\mathbb{R}^2} w^2 \eta^2 \bar{f} + h^2 \eta^2 \bar{f} \, dx
\]

\[
\leq C(q) R^{2\alpha} \int_{\mathbb{R}^2} |\nabla w|^2 w^{2q-4} \eta^2 \bar{f} \, dx + C(q) R^{2\beta} \int_{\mathbb{R}^2} |\nabla h|^2 h^{2q-4} \eta^2 \bar{f} \, dx
\]

\[
+ C(\ell) R^{\alpha-1} \int_{\mathbb{R}^2} |w|^{2q-4} \eta^2 \bar{f} \, dx + C(\ell) R^{\beta-1} \int_{\mathbb{R}^2} |h|^{2q-4} \eta^2 \bar{f} \, dx
\]

(7)

On the other hand, multiply \( \eta^2 w h^{q-4} \) and \( \eta^2 h^{q-4} \) on both sides of (3), and we have

\[
I \doteq (2q-3) \int_{\mathbb{R}^2} |\nabla w|^2 w^{2q-4} \eta^2 \bar{f} \, dx
\]

\[
= \frac{1}{2q-2} \int_{\mathbb{R}^2} w^{2q-2} \triangle (\eta^2) \, dx + \frac{1}{2q-2} \int_{\mathbb{R}^2} w^{2q-2} u \cdot \nabla (\eta^2) \, dx
\]

\[
+ \int_{\mathbb{R}^2} b \cdot \nabla h |w|^{2q-4} \eta^2 \bar{f} \, dx
\]

\[
\leq \frac{1}{2q-2} \int_{\mathbb{R}^2} w^{2q-2} \triangle (\eta^2) \, dx + \frac{1}{2q-2} \int_{\mathbb{R}^2} w^{2q-2} u \cdot \nabla (\eta^2) \, dx
\]

\[
+ \frac{1}{2} I + C(q) \int_{\mathbb{R}^2} h^2 |\eta^2 w| \, dx - \int_{\mathbb{R}^2} w^{2q-4} w b \cdot \nabla (\eta^2) \, dx
\]

(8)

and similarly

\[
II \doteq (2q-3) \int_{\mathbb{R}^2} |\nabla h|^2 h^{2q-4} \eta^2 \bar{f} \, dx
\]

\[
\leq \frac{1}{2q-2} \int_{\mathbb{R}^2} h^{2q-2} \triangle (\eta^2) \, dx + \frac{1}{2q-2} \int_{\mathbb{R}^2} h^{2q-2} u \cdot \nabla (\eta^2) \, dx
\]

\[
- \int_{\mathbb{R}^2} w |h|^{2q-4} h b \cdot \nabla (\eta^2) \, dx + \frac{1}{2} II + C(q) \int_{\mathbb{R}^2} w^2 |\eta^2 w| \, dx
\]

\[
+ C \int_{\mathbb{R}^2} |\nabla u| |\nabla b| |h|^{2q-3} (\eta^2) \, dx
\]

(9)

Then it follows from (7), (8) and (9) that

\[
\int_{\mathbb{R}^2} w^{2q} \eta^2 \bar{f} + h^2 \eta^2 \bar{f} \, dx
\]

\[
\leq C(q, \ell) R^{\alpha-1} \left( R^{\alpha-1} \int_{\mathbb{R}^2} w^{2q-2} (\eta^2 f^2) \, dx + R^{2\alpha} \int_{\mathbb{R}^2} w^{2q-2} (\eta^2 f^2) \, dx + \int_{\mathbb{R}^2} |w|^{2q-1} (\eta^2 f^2) \, dx \right)
\]

\[
+ C(q, \ell) \left( R^{2\alpha+2\beta} \int_{\mathbb{R}^2} w^{2q-4} (\eta^2 f^2) \, dx + R^{2\alpha-1+\beta} \int_{\mathbb{R}^2} |w|^{2q-3} |h| (\eta^2 f^2) \, dx \right)
\]

\[
+ C(q, \ell) R^{\beta-1} \left( R^{\beta-1} \int_{\mathbb{R}^2} h^{2q-2} (\eta^2 f^2) \, dx + R^{\alpha+\beta} \int_{\mathbb{R}^2} h^{2q-2} (\eta^2 f^2) \, dx + \int_{\mathbb{R}^2} |h|^{2q-1} (\eta^2 f^2) \, dx \right)
\]

\[
+ C(q) \left( R^{4\beta} \int_{\mathbb{R}^2} w^2 h^{2q-4} (\eta^2 f^2) \, dx + R^{-1+3\beta} \int_{\mathbb{R}^2} |w|^{2q-3} (\eta^2 f^2) \, dx \right)
\]

\[
+ C(q) R^{2\beta} \int_{\mathbb{R}^2} |\nabla u| |\nabla b| |h|^{2q-3} (\eta^2 f^2) \, dx = I_1 + \cdots + I_5
\]

(10)
**Estimate of $I_5$.** For a smooth vector-valued function $F \in C^2_0(\Omega)$, by applying the decomposition inequality of $L^p$ norm [2] for $q > 1$ we have

$$\|
abla F\|_{L^q(\Omega)} \leq C(n, q) \left(\|\text{div } F\|_{L^q(\Omega)} + \|
abla \times F\|_{L^q(\Omega)}\right).$$

(11)

Hence, by choosing $F = u_\ell \eta_\ell^5$ or $b_\ell \eta_\ell^5$ we get

$$(\int_{\mathbb{R}^2} |\nabla u|^{2q} \eta^{2\ell} dx)^{\frac{1}{2q}} \leq C(q, \ell) \left(\int_{\mathbb{R}^2} |u|^{2q} \eta^{2\ell} \nabla \eta|^{2q} dx\right)^{\frac{1}{2q}} + C(q, \ell) \left(\int_{\mathbb{R}^2} |w|^{2q} \eta^{2\ell} dx\right)^{\frac{1}{2q}}$$

$$\leq C(q, \ell) R^{-1+\alpha+\frac{1}{q}} + C(q, \ell) \left(\int_{\mathbb{R}^2} |w|^{2q} \eta^{2\ell} dx\right)^{\frac{1}{2q}}$$

and

$$\int_{\mathbb{R}^2} |\nabla u| |\nabla b||h^{2q-3}(\eta^{2\ell})dx$$

$$\leq C(q, \ell) R^{\frac{1}{2q}} \left(R^{\alpha+\frac{1}{q}-1} + \left(\int_{\mathbb{R}^2} |w|^{2q} \eta^{2\ell} dx\right)^{\frac{1}{2q}}\right) \left(R^{3+\frac{1}{q}-1} + \left(\int_{\mathbb{R}^2} |h|^{2q} \eta^{2\ell} dx\right)^{\frac{1}{2q}}\right)$$

Then for the term $I_5$, Young inequality implies that

$$I_5 \leq C(q) R^{2\beta} \int_{\mathbb{R}^2} |\nabla u| |\nabla b||h^{2q-3}(\eta^{2\ell})dx$$

$$\leq C(\delta, q, \ell) R^{2\beta} (-2+\alpha+3\beta+\frac{1}{q}) + C(\delta, q, \ell) R^{3(1+\alpha+2\beta+\frac{1}{q})} + \delta \left(\int_{\mathbb{R}^2} |w|^{2q} \eta^{2\ell} dx\right)$$

$$+ C(\delta, q, \ell) R^{2+4\beta q} + \delta \left(\int_{\mathbb{R}^2} |h|^{2q} \eta^{2\ell} dx\right)$$

where $\delta > 0$, to be decided.

**Estimate of $I_1$.** Noting $\ell \geq q$, by Young inequality we have

$$I_1 = C(\ell, q) R^{2\alpha-2} \int_{\mathbb{R}^2} w^{2q-2} (\eta^{2\ell-2}) dx + C(\ell, q) R^{3\alpha-1} \int_{\mathbb{R}^2} w^{2q-2} (\eta^{2\ell-1}) dx$$

$$+ C(\ell, q) R^{\alpha-1} \int_{\mathbb{R}^2} w^{2q-1} \eta^{2\ell-1} dx = I_{11} + \cdots + I_{13},$$

where

$$I_{11} \leq \delta \int_{\mathbb{R}^2} w^{2q} \eta^{2(2\ell-2) - \frac{1}{q-1}} dx + C(\delta, \ell, q) R^{2+q(2\alpha-2)},$$

$$I_{12} \leq \delta \int_{\mathbb{R}^2} w^{2q} \eta^{2(2\ell-1) - \frac{1}{q-1}} dx + C(\delta, \ell, q) R^{2+q(3\alpha-1)},$$

and

$$I_{13} \leq \delta \int_{\mathbb{R}^2} w^{2q} \eta^{2(\ell-1) - \frac{1}{q-1}} dx + C(\delta, \ell, q) R^{2+q(2\alpha-1)}.$$

**Estimate of $I_2$.** Similar to the calculation of the term $I_1$, we get

$$I_2 \leq \delta \int_{\mathbb{R}^2} (w^{2q} + h^{2q}) \eta^{2\ell} dx + C(\delta, \ell, q) R^{2+q(2\alpha+2\beta)}$$

$$+ \delta \int_{\mathbb{R}^2} (w^{2q} + h^{2q}) \eta^{2(\ell-1) - \frac{1}{q-1}} dx + C(\delta, \ell, q) R^{2+q(2\alpha-1+\beta)}.$$
Estimate of $I_3$. By Hölder and Young inequalities we have
\[
I_3 \leq \delta \int_{\mathbb{R}^2} h^{2q} \eta^{(2\ell-2)} \frac{2q}{\pi^{\ell\tau}} dx + C(\delta, \ell, q) R^{2+q(2\beta-2)} + \delta \int_{\mathbb{R}^2} h^{2q} \eta^{(2\ell-1)} \frac{2q}{\pi^{\ell\tau}} dx + C(\delta, \ell, q) R^{2+q(\alpha-1+2\beta)} + \delta \int_{\mathbb{R}^2} h^{2q} \eta^{(2\ell-1)} \frac{2q}{\pi^{\ell\tau}} dx + C(\delta, \ell, q) R^{2+2q(\beta-1)}
\]

Estimate of $I_4$. Similar to the calculation of the term $I_3$, we get
\[
I_4 \leq \delta \int_{\mathbb{R}^2} (w^{2q} + h^{2q}) \eta^{(2\ell)} dx + C(\delta, \ell, q) R^{2+q(4\beta)} + \delta \int_{\mathbb{R}^2} (w^{2q} + h^{2q}) \eta^{(2\ell-1)} \frac{2q}{\pi^{\ell\tau}} dx + C(\delta, \ell, q) R^{2+q(-1+3\beta)}
\]

Note that $\eta^{(2\ell-1)} \frac{2q}{\pi^{\ell\tau}} \leq \eta^{2\ell}$ and $\eta^{(2\ell-1)} \frac{2q}{\pi^{\ell\tau}} \leq \eta^{2\ell}$ for $\ell = q \geq q_0$. Hence, firstly taking $\ell = q$ and $\delta < \frac{1}{32}$; secondly, for fixed $\alpha < \frac{1}{3}$ with $\beta < -\alpha$, we take the minimum $q_0$ satisfying the following conditions
\[
2 + q(2\alpha - 2) \leq 0, \quad 2 + q(3\alpha - 1) \leq 0, \quad 2 + 2q(\alpha - 1) \leq 0,
\]
and
\[
2 + 4\beta q \leq 0, \quad 2 + q(2\alpha + 2\beta) \leq 0.
\]

Obviously, $q_0$ is as in (6). And for any $q > q_0$, we write
\[
\gamma_0 = \max\{2 + q(3\alpha - 1), 2 + 4\beta q, 2 + q(2\alpha + 2\beta)\} < 0.
\]

Then we get
\[
\int_{\mathbb{R}^2} w^{2q} \eta^{2\ell} + h^{2q} \eta^{2\ell} dx \leq C(\ell, q) \left[ R^{2+q(2\alpha-2)} + R^{2+q(3\alpha-1)} + R^{2+2q(\alpha-1)} \right] + C(\ell, q) R^{2+4\beta q} + C(\ell, q) R^{2+q(2\alpha+2\beta)}
\]

Choose $R = 2^{k+1}$ with $k \in \mathbb{N}$ such that
\[
\int_{2^k \leq |x| \leq 2^{k+1}} w^{2q} + h^{2q} dx \leq C(\alpha, \beta, q) 2^{k\gamma_0}
\]

Consequently, we get
\[
\int_{\mathbb{R}^2 \setminus B_1} w^{2q} + h^{2q} dx \leq C(\alpha, \beta, q, c_0) < \infty, \quad (12)
\]

for any $q > q_0$.

Arguments for the estimate in $B_1$. Firstly,
\[
\int_{\mathbb{R}^2} w^{2q} + h^{2q} dx < \infty, \quad q > q_0,
\]
due to the regularity of the solutions. Secondly, by (11) we have
\[
\int_{B_R} |\nabla u|^{2q} + |\nabla b|^{2q} dx \leq C(q) \int_{\mathbb{R}^2} w^{2q} + h^{2q} dx + C(q) R^{-2q} \int_{B_{2R}} (|u| + |b|)^{2q} dx,
\]
and thus
\[
\int_{\mathbb{R}^2} |\nabla u|^{2q} + |\nabla b|^{2q} dx \leq C(q) \int_{\mathbb{R}^2} w^{2q} + h^{2q} dx < \infty, \quad q > q_0, \quad (13)
\]
where we used the growth estimates (4) and (5). Finally, for the non-negative cut-off function \( \eta_1 \), i.e.

\[
\eta_1(x) = \begin{cases} 
1, & x \in B_1, \\
0, & x \in B_2^c,
\end{cases}
\]
satisfying \(|\nabla \eta_1| + |\nabla^2 \eta_1| \leq C\), one can also obtain the similar estimate to (10) with \( R = 1 \) and arrive at

\[
\int_{\mathbb{R}^2} w^{2q}\eta_1^{2q} + h^{2q}\eta_1^{2q} \, dx \leq C(q) \int_{\mathbb{R}^2} (|\nabla u| + |\nabla b|)^{2q - 2}\eta_1^{2q - 2} + (|\nabla u|^2 + |\nabla b|)^{2q - 1}\eta_1^{2q - 1} \, dx,
\]

which can be controlled by

\[
C(q)\int_{\mathbb{R}^2} (|\nabla (u\eta_1)| + |\nabla (b\eta_1)|)^{2q - 2} + (|\nabla (u\eta_1)| + |\nabla (b\eta_1)|)^{2q - 1} \, dx + C(q, c_0),
\]

where we used \( q = \ell \),

\[
|\nabla u\eta_1| = (|\nabla (u)\eta_1| = |\nabla (u\eta_1)| - \nabla \eta_1 \otimes u) \leq |\nabla (u\eta_1)| + |\nabla \eta_1| |u|,
\]

and \(|u| + |b| \leq C(c_0)\) in \( B_2 \). Using (11) and Hölder inequality, we get

\[
\int_{\mathbb{R}^2} w^{2q}\eta_1^{2q} + h^{2q}\eta_1^{2q} \, dx \leq \frac{1}{2} \int_{\mathbb{R}^2} w^{2q}\eta_1^{2q} + h^{2q}\eta_1^{2q} \, dx + C(q, c_0),
\]

which and (12) imply that

\[
\int_{\mathbb{R}^2} w^{2q} + h^{2q} \, dx \leq C(\alpha, \beta, q, c_0) < \infty,
\]

for any \( q > q_0 \). And the required inequality follows by using (11), (4) and (5) again.

Thus the proof of Proposition 2.1 is complete. \( \square \)

**Lemma 2.2.** Let \((u, b, \pi)\) be a smooth solution of the 2D MHD equations (1) defined over the entire plane satisfying the growth estimates (4) with \( 0 < \alpha < \frac{1}{3} \). Moreover, we assume that \( b \) satisfies (5) with \( \beta < -\alpha \). Then

\[
\|\nabla(|w|^{q-1})\|_{L^2(\mathbb{R}^2)} + \|\nabla(|h|^{q-1})\|_{L^2(\mathbb{R}^2)} \leq C(\alpha, \beta, q, c_0) < \infty,
\]

where

\[
q > q_0 + 1 = \max \left\{ \frac{2}{1 - 3\alpha}, -\frac{1}{\alpha + \beta}, -\frac{1}{2\beta} \right\} + 1.
\]

**Proof of Lemma 2.2.** On the other hand, let \( \phi(x) \in C_c^\infty(B_R) \) and \( 0 \leq \phi \leq 1 \) satisfying

\[
\phi(x) = \begin{cases} 
1, & x \in B_R, \\
0, & x \in B_2^c_R
\end{cases}
\]

Using similar estimates as in (8) and (9), multiply \( \phi^{2q}w^{2q-4}w \) and \( \phi^{2q}h^{2q-4}h \) on both sides of (3) with \( q > 2 \), and we have

\[
I' = (2q - 4) \int_{\mathbb{R}^2} |\nabla w|^2 w^{2q-4} \phi^{2q}w \, dx
\]

\[
\leq \frac{1}{2q - 2} \int_{\mathbb{R}^2} w^{2q-2} \Delta (\phi^{2q}) \, dx + \frac{1}{2q - 2} \int_{\mathbb{R}^2} w^{2q-2} u \cdot \nabla (\phi^{2q}) \, dx
\]

\[
+ C(q) \int_{\mathbb{R}^2} h^2 |b|^2 w^{2q-4}(\phi^{2q}) \, dx - \int_{\mathbb{R}^2} w^{2q-4} \phi h \cdot \nabla (\phi^{2q}) \, dx
\]

\[
= I'_1 + \cdots + I'_4,
\]

(14)
and
\[ II' = (2q - 4) \int_{\mathbb{R}^2} |\nabla h|^2 h^{2q-4} \phi^{2q} \, dx \]
\[ \leq \frac{1}{2q - 2} \int_{\mathbb{R}^2} h^{2q-2} \triangle (\phi^{2q}) \, dx + \frac{1}{2q - 2} \int_{\mathbb{R}^2} h^{2q-2} u \cdot \nabla (\phi^{2q}) \, dx \]
\[ \quad - \int_{\mathbb{R}^2} w h^{2q-4} h b \cdot \nabla (\phi^{2q}) \, dx + C(q) \int_{\mathbb{R}^2} w^2 |b|^2 h^{2q-4} (\phi^{2q}) \, dx \]
\[ + C \int_{\mathbb{R}^2} |\nabla u| |\nabla b| h^{2q-3} (\phi^{2q}) \, dx \]
\[ = II'_1 + \cdots + II'_5 \]  

(15)

Since
\[ ||\nabla u||_{2\tilde{q}} + ||\nabla b||_{2\tilde{q}} < \infty, \]

for any \( \tilde{q} > q_0 \) by Proposition 2.1, we have
\[ ||\nabla (|w|^{q-1})||_{L^2(B_R)}^2 + ||\nabla (|h|^{q-1})||_{L^2(B_R)}^2 \]
\[ \leq C \int_{B_{2R}} |w|^{2q-2} + |h|^{2q-2} + |\nabla u| |\nabla b| |h|^{2q-3} \, dx < \infty, \]

for any \( q > q_0 + 1 \). Then the proof is complete.

**Proof of Theorem 1.1.** For \( q > q_0 + 1 \), we still consider the inequalities (14) and (15). Now we estimate the term \( I'_3 \):
\[ I'_3 \leq C(q) \int_{\mathbb{R}^2} |b|^2 w^{2q-2} (\phi^{2q}) \, dx + \int_{\mathbb{R}^2} |b|^2 h^{2q-2} (\phi^{2q}) \, dx = I'_{31} + I'_{32}. \]

In details,
\[ I'_{31} = \int_{\mathbb{R}^2} |b|^2 w^{2q-2} (\phi^{2q}) \, dx \]
\[ \leq \left( \int_{\mathbb{R}^2} |b|^{2p} \, dx \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^2} |\tilde{w}|^{2p} \, dx \right)^{\frac{1}{p}} \]
\[ \leq C(q) ||b||_{L^p(\mathbb{R}^2)} ||w||_{\infty} ||\tilde{w}||_{\frac{2q}{q-1}} \||\nabla \tilde{w}||_{2-\theta}^2 \]

where \( \theta := \frac{2q}{p(q-1)} \), \( \tilde{w} = |w|^{q-1} \) and we used Hölder inequality, Lemma 2.2, and Gagliardo-Nirenberg inequality( for example, see Lemma II.3.3 in [8]). So when \( p = 4q \),
\[ \theta = \frac{1}{2(q-1)} \]

Taking \( p_1 = 8q - 2 \), it follows from (13), which is also valid for \( b \), that
\[ ||b||_{\infty} \leq ||b||_{L^1(\mathbb{R}^2)} \||\nabla b||_{L^{2p_1}(\mathbb{R}^2)} \leq C(q) ||b||_{L^1(\mathbb{R}^2)} \||h||_{L^{p_1}(\mathbb{R}^2)} \]

Write \( \tilde{h} = |h|^{q-1} \). By Gagliardo–Nirenberg inequality we have
\[ I'_{31} \leq C(q) ||b||_{L^p(\mathbb{R}^2)} ||b||_{L^{2p_1}(\mathbb{R}^2)} \||h||_{L^{p_1}(\mathbb{R}^2)} \||\nabla h||_{\frac{2q}{q-1}} \||\nabla \tilde{w}||_{2-\theta}^2 \]
\[ \leq C(q) ||b||_{L^p(\mathbb{R}^2)} ||b||_{L^{2p_1}(\mathbb{R}^2)} \||h||_{L^{p_1}(\mathbb{R}^2)} \||\nabla \tilde{h}||_{\frac{2q-1}{q-2}} \||\nabla \tilde{w}||_{2-\theta}^2 \]
\[ \leq C(q) ||b||_{L^p(\mathbb{R}^2)} ||b||_{L^{2p_1}(\mathbb{R}^2)} \||\nabla \tilde{h}||_{\frac{2q-1}{q-2}} \||\nabla \tilde{w}||_{2-\theta}^2 , \]
since

\[
\left( \frac{p_1 - 2q}{p_1(q - 1)} \right) \left( \frac{2p_1}{3p_1 - 2} \right) = \theta.
\]  

(16)

Due to (5), we have

\[
\|b\|_{L^{p}(\mathbb{R}^2)} \leq C(\beta, q)\|b\|_{L^{1}(\mathbb{R}^2)}^{\frac{1}{p}}
\]

Hence, using Proposition 2.1, there exists a positive number \(\varepsilon_2 = \varepsilon_2(\alpha, \beta, q, c_0)\) such that if \(\|b\|_{L^{1}(\mathbb{R}^2)} \leq \varepsilon_2\), then

\[
I_{31}' \leq C(q, \alpha, \beta, c_0)\|b\|_{L^{1}(\mathbb{R}^2)}^{\frac{1}{p}} \frac{1}{p} \|\nabla \tilde{h}\|^2 \|\nabla \tilde{w}\|^2
\leq \frac{1}{16} \|\nabla \tilde{h}\|^2 + \|\nabla \tilde{w}\|^2
\]

The terms \(I_{32}'\) and \(I_{4}'\) are similar, hence we have

\[
I_{3} + I_{4}' = \int_{\mathbb{R}^2} w^2 |b|^2 h^2 q^{-2} (\phi^{2q}) dx \leq \frac{1}{8} \|\nabla \tilde{h}\|^2 + \|\nabla \tilde{w}\|^2
\]

Next we estimate the term \(I_{5}'\). Using (11) to estimate the \(L^p\) norm of \(\nabla b\) with that of \(h\) and Gagliardo-Nirenberg inequality again, we have

\[
I_{5}' = \int_{\mathbb{R}^2} \|\nabla u\|_{L^p(\mathbb{R}^2)} |\nabla b| h^{2q - 3} (\phi^{2q}) dx
\leq \|\nabla u\|_{2p(q - 1)} \|\nabla b\|_{2p(q - 1)(2q - 2)} \|h^{2q - 2}\|_{2p(q - 1)(2q - 2)} \|\nabla u\|_{L^{p}(\mathbb{R}^2)}
\leq C(q) \|h\|_{L^{p}(\mathbb{R}^2)} \left( \|\nabla \tilde{u}\|_{L^{p}(\mathbb{R}^2)} \|\nabla \tilde{w}\|_{L^{p}(\mathbb{R}^2)} \right)
\]

where \(p = 4q\) and

\[
\theta = \frac{2q}{p(q - 1)} = \frac{1}{2(q - 1)}
\]

Taking \(p_1 = 8q - 2\) and \(\gamma = \frac{p_1 - 2}{2q(p_1 - 2)}\) we have

\[
C(q) \|h\|_{L^{p}(\mathbb{R}^2)} \leq C(q) \left( \int_{\mathbb{R}^2} |h|^{p - 2q}\|\nabla\|_{L^{p}(\mathbb{R}^2)} \right)^{\frac{p_1 - p'(1 - \gamma)}{p_1}} \|\nabla h\|_{L^{p_1}(\mathbb{R}^2)}^{\frac{2p_1}{2q}} \|\tilde{h}\|_{L^{p_1}(\mathbb{R}^2)}^{\frac{2q - 1}{2q}} \|\nabla \tilde{h}\|_{L^{p_1}(\mathbb{R}^2)}^{\frac{2p_1}{2q}}
\]

where \(\tilde{h} = |h|^{q - 1}\), and

\[
\|h\|_{L^{p_1}(\mathbb{R}^2)}^{\frac{p - 2q}{2q}} = \|h\|_{L^{p_1}(\mathbb{R}^2)}^{\frac{p - 2q}{2q}}
\]

Since

\[
\frac{p'}{p - 1} = \frac{p}{p - 1} \cdot \frac{p_1 - 2}{p_1 - 2} = \frac{4q(8q - 4)}{12q^2 - 8q + 1}
\]
thence by (16) we have

\[
I_5' \leq C(q) \left( \| h \|_{L^1(\mathbb{R}^2)} \right)^{\frac{1}{2}} \left( \| h \|_{L^2(\mathbb{R}^2)} \right)^{\frac{1}{2}} \| h \|_{L^{\frac{2q}{q-2}}(\mathbb{R}^2)} \left( \| \nabla h \|_{L^{\frac{2q}{q-2}}(\mathbb{R}^2)} \right)^{\frac{1}{2}}
\]

where we used Hölder inequality, due to

\[
\frac{1}{3} < \frac{4q^2 - 2q}{12q^2 - 8q + 1} < 2q
\]

for \( q > 2 \). Hence there exists a positive number \( \| | h |^{\frac{1}{2}} \|_{L^1(\mathbb{R}^2)} \leq \varepsilon_3(\alpha, \beta, q, c_0) \) such that

\[
I_5' \leq \frac{1}{16} \left( \| \nabla h \|_{L^2}^2 + \| \nabla \tilde{u} \|_{L^2}^2 \right)
\]

Recalling the inequalities (14) and (15), using the growth (4), (5) and the above estimates, by Proposition 2.1 and Lemma 2.2 we get

\[
\int_{\mathbb{R}^2} |\nabla w|^2 w^{2q - 4} \phi^{2q} dx + \int_{\mathbb{R}^2} |\nabla h|^2 h^{2q - 4} \phi^{2q} dx
\]

\[
\leq C(\alpha, \beta, q, c_0) \left( R^{-2} + R^{\alpha - 1} + R^{\alpha - 1} \right)
\]

and \( R \to \infty \) implies that

\[
\nabla(|w|^{q - 1}) \equiv 0, \quad \nabla(|h|^{q - 1}) \equiv 0,
\]

which yields that

\[
w \equiv C, \quad h \equiv C,
\]

and it follows from Proposition 2.1 that \( C \equiv 0 \) and \( u, b \) are constants.

The proof of Theorem 1.1 is complete by taking \( \varepsilon_0 = \min \{ \varepsilon_2, \varepsilon_3 \} \).

\[\square\]

3. Proof of Theorem 1.2

Proposition 3.1. Let \((u, b, \pi)\) be a smooth solution of the 2D MHD equations (1) defined over the entire plane satisfying the growth estimates \( \nabla u \in L^{\infty} (\mathbb{R}^2) \) for \( 2 < q_0 < \infty \), and \( \nabla b \in L^{\infty} (\mathbb{R}^2) \). Then there exists \( \varepsilon_1 > 0 \) such that, if

\[
\| b \|_{L^1(\mathbb{R}^2)} + \| h \|_{L^1(\mathbb{R}^2)}^{\frac{1}{3}} \leq \varepsilon_1(\varepsilon_0, \| \nabla u \|_{L^{\infty}}, \| \nabla b \|_{L^{\infty}}),
\]

then

\[
\nabla u \in L^p(\mathbb{R}^2), \nabla b \in L^p(\mathbb{R}^2),
\]

for any \( p \geq q_0 \).

Proof of Proposition 3.1. By Lemma 1.4, there exists \( R > 0 \) such that

\[
|u(x)| \leq (1 + |x|)^{1 - \frac{\alpha}{2} \varepsilon_0}, \quad |x| > R, \quad (17)
\]

since \( \nabla u \in L^{q_0}(\mathbb{R}^2) \), and we also have \( b(x) \in L^p(\mathbb{R}^2) \) for any \( 1 \leq p \leq \infty \) by Gagliardo-Nirenberg inequality satisfying

\[
\| b \|_{L^p(\mathbb{R}^2)} \leq C(p, \| \nabla b \|_{L^{\infty}}) \quad (18)
\]

Moreover, by (11) we have

\[
\int_{B_R} |\nabla b|^p dx \leq C(q) \int_{\mathbb{R}^2} h^p dx + C(q) R^{-p} \int_{B_{2R}} |b|^p dx, \quad p > 1
\]
by letting \( R \to \infty \), it follows that
\[
\int_{\mathbb{R}^2} |\nabla b|^p \, dx \leq C(q) \int_{\mathbb{R}^2} h^p \, dx \leq C(p, \|\nabla b\|_{\infty}), \quad \forall \, p > 1,
\]
where we have used (18) and \( |h|^{1/3} \in L^1 \cap L^{\infty} \).

Recalling the inequalities (14) and (15) with \( q - 1 = \frac{q_0}{2} \), we have
\[
\|\nabla(|w|^{q-1})\|^2_{L^2(B_R)} + \|\nabla(|h|^{q-1})\|^2_{L^2(B_R)}
\leq C(q_0, \|\nabla b\|_{\infty}) \int_{B_{2R} \setminus B_R} \left( R^{-\frac{2}{q_0}} + |b| R^{-1} \right) (|w|^{q_0} + |h|^{q_0}) \, dx
\]
\[+ C(q_0, \|\nabla b\|_{\infty}) \int_{B_{2R}} |b|^2 (|w|^{q_0} + |h|^{q_0}) + |\nabla u||\nabla b|^{q_0} \, dx < \infty,
\]
where we used (17) for \( R \) large enough. Thus
\[
\nabla \left( |w|^{\frac{q_0}{2}} \right), \nabla \left( |h|^{\frac{q_0}{2}} \right) \in L^2(\mathbb{R}^2),
\]
since \( b \in L^{\infty} \) in (18), \( \nabla b \in L^{\infty} \), \( \nabla u \in L^{q_0} \) form known conditions and \( \nabla b \in L^{q_0} \) in (19). Recall that \( \nabla b \in L^p(\mathbb{R}^2) \), for any \( p > q_0 \) due to (19), then in order to prove Proposition 3.1, it suffices to obtain the following estimate:
\[
\nabla u \in L^p(\mathbb{R}^2),
\]
for any \( p > q_0 \). Firstly, with the help of (21) and \( \nabla u \in L^{q_0} \) we get \( w \in L^p(\mathbb{R}^2) \) for any \( p > q_0 \) due to Gagliardo–Nirenberg inequality. Secondly, using (11) again we have
\[
\int_{B_R} |\nabla u|^p \, dx \leq C(q) \int_{\mathbb{R}^2} u^p \, dx + C(q) R^{-p} \int_{B_{2R}} |u|^p \, dx,
\]
where the last term can be controlled due to (17)
\[
R^{-p} \int_{B_{2R}} |u|^p \, dx \leq C(n) R^{-p+2+p(1-\frac{2}{p})} \leq C(n) R^{2+p(-\frac{2}{p})} \to 0,
\]
as \( R \to \infty \) if \( p > q_0 \). Finally, (22) holds. The proof is complete. \( \square \)

Proof of Theorem 1.2. Case I: \( q_0 > 2 \). One can make the same arguments with the two terms of \( I_{31}^j \) and \( II_5^j \) as in the proof of Theorem 1.1 by noting that (20) and (17).

Case II: \( q_0 = 2 \). At this time, \( \nabla u \in L^2 \) and \( \nabla b \in L^2 \) due to (19). We refer to Theorem 1.1 of [18], which is an immediate corollary.

Case III: \( 1 < q_0 < 2 \). At this time, \( u \in L^{\frac{2q_0}{2-q_0}}(\mathbb{R}^2) \) by Sobolev inequality and \( b \in L^p \) for any \( p > 1 \) due to (18). Thus, we refer to Theorem 1.2 of [18], which is also an immediate corollary.

The proof is complete. \( \square \)

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Declarations

Conflict of interest The author states that there is no conflict of interest.

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