Hecke symmetries: an overview of Frobenius properties

Serge Skryabin

Introduction

Hecke symmetries are solutions to the braid version of the quantum Yang-Baxter equation satisfying the additional condition \((R - q \cdot \text{Id})(R + \text{Id}) = 0\). This notion was introduced by Gurevich [12] as a generalization of involutive symmetries considered earlier by Lyubashenko [14]. Hecke symmetries were used to construct nonstandard quantum groups.

With each Hecke symmetry \(R\) on a vector space \(V\) one associates two graded algebras with quadratic defining relations which are viewed as analogs of the symmetric and exterior algebras of \(V\). We will denote these algebras by \(S(V, R)\) and \(\Lambda(V, R)\), and their homogeneous components by \(S_i(V, R)\) and \(\Lambda_i(V, R)\). A Hecke symmetry \(R\) is said to be even if \(\Lambda(V, R)\) is finite dimensional and its highest degree nonzero homogeneous component is 1-dimensional. In this case Gurevich proved nondegeneracy of bilinear pairings between the components of such an algebra, which can be rephrased by saying that \(\Lambda(V, R)\) is a Frobenius algebra. This fact has important consequences.

However, most of previous work on Hecke symmetries assumed the technical condition that the parameter \(q\) of the Hecke relation is not a nontrivial root of unity. All arguments in [12] exploit projections onto certain subspaces of the tensor powers of the base space, while the definition of those projections involves division by \(q\)-integers. Because of that initial setup it has gone unnoticed that Gurevich’s nondegeneracy result is actually true without any restriction on \(q\), and the additional condition on Hecke symmetries used in [19] is also not needed. We will show a bit more:

Theorem 2.8. Suppose that \(\dim V > 1\) and \(\dim \Lambda_n(V, R) = 1\) for some \(n > 0\). Then \(\Lambda_i(V, R) = 0\) for all \(i > n\) and the algebra \(\Lambda(V, R)\) is Frobenius.

It will be convenient to deal first with an equivalent result for the quadratic dual \(S(V, R)^!\) of the algebra \(S(V, R)\). This will be done in Theorem 2.5. In the root of unity case it is not clear whether the algebras \(S(V, R)^!\) and \(\Lambda(V, R)\) always have the same Hilbert series. For this reason we distinguish between the algebra \(\Lambda(V, R)\) defined as a factor algebra of the tensor algebra \(\mathcal{T}(V)\) and another algebra \(\Upsilon(V, R)\) whose homogeneous components \(\Upsilon^{(k)}\) are subspaces of the tensor powers \(V \otimes^k\). The two algebras \(S(V, R)^!\) and \(\Upsilon(V, R)\) are Frobenius simultaneously.

Every Frobenius algebra admits a so-called Nakayama automorphism determined uniquely up to an inner automorphism. In the case of a connected graded Frobenius algebra we fix the choice of such an automorphism by requesting it to respect the grading. Let \(\psi\) be the 1-component of the Nakayama automorphism of \(\Upsilon(V, R)\), and let \(\psi\) be the linear operator on \(V\) dual to the 1-component of the Nakayama automorphism of \(S(V, R)^!\). In section 3 we describe interrelations between \(\varphi, \psi\) and
yet another operator $\theta \in GL(V)$ which encodes the braiding

$$V \otimes \Upsilon^{(n)} \to \Upsilon^{(n)} \otimes V$$

arising from $R$. This operator is implicitly present in Gurevich’s commutation formulas [12, Prop. 5.7]. Here we assume that $\Upsilon^{(n)}$ is the last nonzero homogeneous component of $\Upsilon(V, R)$. One of our main results is

**Theorem 3.8.** The operators $\varphi, \psi, \theta$ pairwise commute and $\psi = q^{-n-1} \varphi \theta^2$. Moreover, they lie in the subgroup of $GL(V)$ consisting of all invertible linear operators $\chi : V \to V$ such that $\chi \otimes \chi$ commutes with $R$.

It turns out that the formula $XQ = PY$ in a preprint of Ewen and Ogievetsky [9, eq. (35)] is the matrix form of the equality $\theta \varphi = \psi \theta$ satisfied with $\overline{\theta} = q^{n+1} \theta^{-1}$. The authors of [9] consider the 3-dimensional case and call $Q$ and $P$ the characteristic matrices of plane and coplane. They correspond to $\varphi$ and $\psi$. The matrices $X$ and $Y$ are responsible for the commutation law obeyed by the quantum determinant. They correspond, up to scalar multiples, to $\theta$ and $\overline{\theta}$. Interrelations between those 4 matrices formed a basis for the classification of quantum analogs of the group $GL(3)$.

Actually, Ewen and Ogievetsky postulated that a quantum $GL(3)$ should have a left and a right actions on two graded algebras with the Hilbert series equal to that of the polynomial algebra in 3 indeterminates, and the algebra generated by the matrix elements of each action should have the same Hilbert series as the polynomial algebra in 9 indeterminates. Existence of a Yang-Baxter operator satisfying a quadratic relation was deduced from the imposed conditions.

On the other hand, when $q$ is not a root of unity, it is now known that the graded algebras associated with Hecke symmetries have expected Hilbert series [13]. Moreover, the graded algebra $S(V, R)$ is Koszul [12], and therefore the Frobenius property of its dual $S(V, R)^!$ ensures that $S(V, R)$ is Gorenstein of finite global dimension (see [5, Prop. 4.1] and [20, Prop. 5.10]). This places $S(V, R)$ in the class of algebras introduced by Artin and Schelter [1] and suggests the following

**Definition.** A quantum $GL(3)$ is any Hopf algebra obtained as the Hopf envelope of the Faddeev-Reshetikhin-Takhtajan bialgebra associated with a Hecke symmetry $R$ on a 3-dimensional vector space $V$ such that the algebra $S(V, R)$ is Artin-Schelter regular of global dimension 3.

Two papers of Ohn [16] and [17] dealt with the classification of quantum analogs of $SL(3)$. The initial postulate there was that the representation theory for a quantum $SL(3)$ should be identical to that of the ordinary $SL(3)$, including complete reducibility among other things. Each quantum $SL(3)$ corresponds to a quantum $GL(3)$ with central quantum determinant. This happens precisely when $\theta$ is a scalar operator. All cases where $q$ is a nontrivial root of unity are excluded right away [16, Prop. 3.2]. So Ohn’s classification does not cover all quantum $GL(3)$ groups.

The preprint of Ewen and Ogievetsky provides a workable method for determining all quantum $GL(3)$’s and the associated Yang-Baxter operators. It requires heavy calculations needed to solve several sets of equations. However, details of those calculations are not presented in [9]. This makes it difficult to verify final conclusions.
It seems that the classification of quantum groups of $GL(3)$ type has not been analyzed in later publications.

We would like to know for each Artin-Schelter regular graded quadratic algebra $A$ generated by its degree 1 component $A_1 = V$ all Hecke symmetries $R$ such that $S(V, R) = A$. Twists used in [9] are not well suited to answer this question as they change the algebra. There are many algebras to consider, but this is not the purpose of the present paper. Theorem 5.1 discussed below serves as an illustration of general technique.

The Artin-Schelter regular quadratic algebras of global dimension 3 have been subdivided into several types. Those of elliptic type $A$ form a particularly interesting class consisting of the Sklyanin algebras $Skl_3(a, b, c)$ for a known set of parameters $a, b, c$. As to the precise definition of type $A$ we follow [2].

**Theorem 5.1.** Let $R$ be a Hecke symmetry on a 3-dimensional vector space $V$ over a field of characteristic $\neq 2, 3$. Then $S(V, R)$ cannot be any Artin-Schelter regular algebra of type $A$.

We will give a full proof of this theorem, allowing arbitrary $q$. In the proof we have to examine various possibilities for $\theta$. In section 4 we describe automorphisms of the algebras $Skl_3(a, b, c)$ under a suitable restriction on $a, b, c$ in terms of the Hessian group $G$ of order 216. The auxiliary results there permit us to consider only 4 types of linear operators $\theta$.

Theorem 5.1 shows that some Artin-Schelter regular quadratic algebras of global dimension 3 are not associated with any quantum $GL(3)$. One can expect abundant examples of such behaviour in higher dimensions. It would be very interesting to find a conceptual explanation as to why this happens. The proof of Theorem 5.1 is obtained by brute force.

In Manin’s approach to quantum groups [15] one considers the Hopf algebra universally coacting on a pair of graded algebras. In [7] the authors take two $N$-Koszul Artin-Schelter regular graded algebras $A, A'$ and assume a left coaction on $A$ and a right coaction on $A'$. Alternatively, one may use right coactions on $A'$ and on the dual of $A$. However, without further constraints on the two graded algebras the resulting Hopf algebra is likely to be too small to qualify for a true quantum $GL$. Hecke symmetries produce properly compatible pairs of algebras.

In the present paper we consistently use various relations satisfied in the Iwahori-Hecke algebras of type $A$. Not much of the theory of Hecke algebras is really needed. Several simple facts concerning antisymmetrizers are mentioned in section 1, and there we also introduce some notation. A thorough treatment of Hecke algebras can be found in [10]. Combinatorial properties of Coxeter groups are discussed in [4]. A standard reference on quadratic algebras is another book [18].

### 1. Antisymmetrizers in the Hecke algebras of type $A$

Let $k$ be the ground field. The Hecke algebra $H_n = H_n(q)$ of type $A_{n-1}$ with parameter $q \in k$ is presented by generators $T_1, \ldots, T_{n-1}$ and relations

\[
T_iT_jT_i = T_jT_iT_j \quad \text{when } |i - j| = 1, \quad T_iT_j = T_jT_i \quad \text{when } |i - j| > 1, \\
(T_i - q)(T_i + 1) = 0 \quad \text{for } i = 1, \ldots, n - 1.
\]
It has a standard basis \( \{ T_\sigma \mid \sigma \in \mathfrak{S}_n \} \) indexed by elements of the symmetric group \( \mathfrak{S}_n \). Recall that \( \mathfrak{S}_n \) is a Coxeter group with respect to its generating set

\[
\mathfrak{B}_n = \{ \tau_1, \ldots, \tau_{n-1} \}
\]

of basic transpositions \( \tau_i = (i, i+1) \). The length \( \ell(\sigma) \) of a permutation \( \sigma \in \mathfrak{S}_n \) is the smallest number of factors in the expressions of \( \sigma \) as product of basic transpositions. It can be computed as

\[
\ell(\sigma) = \# \{ (i, j) \mid 1 \leq i < j \leq n, \sigma(i) > \sigma(j) \}.
\]

The longest element of \( \mathfrak{S}_n \) has length

\[
\ell_n = n(n-1)/2.
\]

So \( 0 \leq \ell(\sigma) \leq \ell_n \) for all \( \sigma \in \mathfrak{S}_n \). Denote by \( e \) the identity permutation. We have \( T_e = 1 \), \( T_{\tau_i} = T_i \) and

\[
T_\pi T_\sigma = T_{\pi\sigma} \quad \text{for all } \pi, \sigma \in \mathfrak{S}_n \text{ such that } \ell(\pi\sigma) = \ell(\pi) + \ell(\sigma).
\]

Parabolic subgroups of \( \mathfrak{S}_n \) known as Young subgroups are generated by subsets of \( \mathfrak{B}_n \). If \( \mathfrak{S}' \) is any Young subgroup, then each left or right coset of \( \mathfrak{S}' \) in \( \mathfrak{S}_n \) contains a unique element of minimal length called the distinguished coset representative. Distinguished representatives of all respective cosets form the subsets

\[
D(\mathfrak{S}_n/\mathfrak{S}') = \{ \sigma \in \mathfrak{S}_n \mid \sigma(i) < \sigma(i+1) \text{ for all } i \text{ such that } \tau_i \in \mathfrak{S}' \},
\]

\[
D(\mathfrak{S}'\setminus\mathfrak{S}_n) = \{ \sigma \in \mathfrak{S}_n \mid \sigma^{-1} \in D(\mathfrak{S}_n/\mathfrak{S}') \}
\]

of the group \( \mathfrak{S}_n \). It is well known that

\[
\ell(d\sigma) = \ell(d) + \ell(\sigma) \quad \text{for all } \sigma \in \mathfrak{S}' \text{ and } d \in D(\mathfrak{S}_n/\mathfrak{S}'),
\]

\[
\ell(\sigma d) = \ell(d) + \ell(\sigma) \quad \text{for all } \sigma \in \mathfrak{S}' \text{ and } d \in D(\mathfrak{S}'\setminus\mathfrak{S}_n).
\]

For such \( \sigma \) and \( d \) we always have, respectively, \( T_{d\sigma} = T_d T_\sigma \) and \( T_{\sigma d} = T_\sigma T_d \).

The antisymmetrizer

\[
y_n = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\ell(\sigma)} q^{\ell(s) - \ell(\sigma)} T_\sigma \in \mathcal{H}_n \quad (\ell_n = n(n-1)/2)
\]

factorizes as product of partial antisymmetrizers in many ways. Since each element of \( \mathfrak{S}_n \) lies in only one left coset and in only one right coset of \( \mathfrak{S}' \), we get

\[
y_n = y(\mathfrak{S}_n/\mathfrak{S}') y(\mathfrak{S}') = y(\mathfrak{S}') y(\mathfrak{S}'\setminus\mathfrak{S}_n)
\]

where \( y(\mathfrak{S}') = \sum_{\sigma \in \mathfrak{S}'} (-1)^{\ell(\sigma)} q^{\ell(\mathfrak{S}') - \ell(\sigma)} T_\sigma \), \( \ell(\mathfrak{S}') = \max \{ \ell(\sigma) \mid \sigma \in \mathfrak{S}' \} \).
In particular, \( D \) elements of \( H \), note that \( T \) holds for each generator such that \( y \). Hence, taking \( S \) \( \geq k \), for some \( i \) and \( j \). In this way \( i \) \( i \) \( j \) \( \in \) \( S \) are defined for all pairs of integers \( 1 \leq i, j \leq n \). The corresponding elements of \( H_n \) are

\[
T_{i \curvearrowright j} = \begin{cases} 
T_i T_{i+1} \cdots T_{j-1} & \text{for } i < j, \\
1 & \text{for } i = j, \\
T_{i-1} \cdots T_{j+1} T_j & \text{for } i > j.
\end{cases}
\]

Note that \( \ell(i \curvearrowright j) = |j - i| \), and \((i \curvearrowright k)(k \curvearrowright j) = i \curvearrowright j \) whenever either \( i \leq k \leq j \) or \( i \geq k \geq j \). In this case \( T_{i \curvearrowright k} T_{k \curvearrowright j} = T_{i \curvearrowright j} \).

The group \( S_{n-1} \) is identified with the subgroup of those permutations \( \sigma \in S_n \) that leave \( n \) fixed. Then \( H_{n-1} \) is identified with the subalgebra of \( H_n \) spanned by all \( T_\sigma \) with \( \sigma \in S_{n-1} \). Note that

\[
D(S_n/S_{n-1}) = \{n \cap i \mid 1 \leq i \leq n\}, \quad D(S_n \setminus S_n) = \{n \cap i \mid 1 \leq i \leq n\}.
\]

Taking \( S' = S_{n-1} \) in the previous discussion, we get inductive formulas

\[
y_n = \sum_{i=1}^{n} (-1)^{n-i} q^{i-1} T_{i \cap n} y_{n-1} = y_{n-1} \cdot \sum_{i=1}^{n} (-1)^{n-i} q^{i-1} T_{n \cap i}.
\]

Taking \( S' = \{e, \tau_i\} \) for some \( i, 1 \leq i < n \), we deduce that there exist \( y', y'' \in H_n \) such that

\[
y_n = y' \cdot (T_i - q) = (T_i - q) \cdot y''.
\]

Hence \( y_n \) is annihilated by \( T_{i+1} \) on both sides, i.e., \( T_{i} y_n = y_n T_i = -y_n \). Since this holds for each generator \( T_i \) of \( H_n \), we see that \( y_n \) spans a 1-dimensional ideal of \( H_n \), and

\[
T_\sigma y_n = y_n T_\sigma = (-1)^{\ell(\sigma)} y_n \quad \text{for all } \sigma \in S_n.
\]

We will use standard notation for the \( q \)-integers and \( q \)-factorials:

\[
[n]_q = \sum_{i=1}^{n} q^{i-1}, \quad [n]!_q = \prod_{k=1}^{n} [k]_q.
\]

There is a combinatorial formula \( \sum_{\sigma \in S_n} q^{\ell(\sigma)} = [n]!_q \) proved easily by induction on \( n \). It follows that

\[
y_n^2 = \sum_{\sigma \in S_n} q^{\ell(\sigma)} y_n = \sum_{\sigma \in S_n} q^{\ell(\sigma)} y_n = [n]!_q y_n.
\]

If \( [n]_q \neq 0 \), then \( [n]!_q^{-1} y_n \) is an idempotent in \( H_n \).
Lemma 1.1. Given a left $\mathcal{H}_n$-module $M$, put
\[ \Sigma(M) = \sum_{i=1}^{n-1} \{ u \in M \mid T_i u = q u \} \quad \text{and} \quad \Upsilon(M) = \bigcap_{i=1}^{n-1} (T_i - q) M. \]

Then $y_n M \subset \Upsilon(M)$ and $y_n \Sigma(M) = 0$. So the action of $y_n$ on $M$ induces a linear map $M/\Sigma(M) \to \Upsilon(M)$.

If $[n]_q \neq 0$, then this map is bijective, and $M = \Upsilon(M) \oplus \Sigma(M)$.

Proof. The first part of the lemma is a consequence of the factorizations of $\nu_n$ with respect to the 2 element subgroups $\{ e, \tau_i \}$ (see above).

Next, the relation $(T_i - q)(T_i + 1) = 0$ implies that $(T_i + 1)u = 0$ for all $u \in \Upsilon(M)$ and $(T_i + 1)u \in \Sigma(M)$ for all $u \in M$. In other words, $\Upsilon(M)$ is a submodule and $M/\Sigma(M)$ a factor module of $M$ on which each $T_i \in \mathcal{H}_n$ acts as the minus identity operator. Hence $y_n u = [n]_q u$ for all $u \in \Upsilon(M)$ and $y_n u - [n]_q u \in \Sigma(M)$ for all $u \in M$. If $[n]_q \neq 0$, we get
\[ \Upsilon(M) = y_n M, \quad \Sigma(M) = \{ u \in M \mid y_n u = 0 \}, \]

and the rest is also clear. \qed

Lemma 1.2. Let $M$ be a left $\mathcal{H}_n$-module, let $\Upsilon(M)$ be as in Lemma 1.1, and let $\mathcal{S}'$ be a Young subgroup of $\mathcal{S}_n$. Put $T' = \{ j \mid \tau_j \in \mathcal{S}' \}$. Then
\[ y(\mathcal{S}_n/\mathcal{S}') u \in \Upsilon(M) \quad \text{for each} \quad u \in \bigcap_{j \in T'} (T_j - q) M. \]

Proof. Put $D = D(\mathcal{S}_n/\mathcal{S}')$. We have to show that the element
\[ y(\mathcal{S}_n/\mathcal{S}') u = \sum_{d \in D} (-1)^{\ell(d)} q^{\ell(\mathcal{S}')} - \ell(d) T_d u \]
lies in $(T_i - q) M$ for each $i = 1, \ldots, n - 1$.

Each permutation $d \in D$ has the property that $d(j) < d(j + 1)$ for all $j \in T'$. If $\tau_i d \notin D$, then $\tau_i d(j) > \tau_i d(j + 1)$ for some $j \in T'$, but this happens precisely when $i = d(j)$ and $i + 1 = d(j + 1)$. In this case $\tau_i d = d r_j$, and also $T_i T_d = T_r d T_i$ since $\ell(\tau_i d) = \ell(d) + 1$. Since $u = (T_j - q) u'$ for some $u' \in M$, we get
\[ T_d u = (T_i - q) T_d u' \in (T_i - q) M. \]

If $\tau_i d \in D$, then either $\ell(\tau_i d) = \ell(d) + 1$ or $\ell(d) = \ell(\tau_i d) + 1$. This yields
\[ y(\mathcal{S}_n/\mathcal{S}') u = \sum_{d \in D \setminus \{ \tau_i d \}} (-1)^{\ell(d)} q^{\ell(\mathcal{S}')} - \ell(d) T_d u + \sum_{\{ d \in D \mid \tau_i d \notin D, \ell(d) < \ell(\tau_i d) \}} (-1)^{\ell(d)} q^{\ell(\mathcal{S}')} - \ell(d - 1) (q - T_i) T_d u \in (T_i - q) M, \]
as required. \qed

For two nonnegative integers $k, n$ denote by $\mathcal{S}_{k,n}$ the Young subgroup of $\mathcal{S}_{k+n}$ consisting of all permutations that leave stable the subset $\{ 1, \ldots, k \}$ and the subset $\{ k + 1, \ldots, k + n \}$. We assume that $\mathcal{S}_0$ is a trivial group and $\mathcal{S}_{0,n} = \mathcal{S}_{n,0} = \mathcal{S}_n$. Clearly $\mathcal{S}_{k,n} \cong \mathcal{S}_k \times \mathcal{S}_n$, and $\mathcal{S}_{k,n}$ is generated by the set $\{ \tau_i \in \mathcal{S}_n \mid i \neq k \}$.

The set $D(\mathcal{S}_{k+n}/\mathcal{S}_{k,n})$ consists of all permutations in $\mathcal{S}_{k+n}$ which are increasing on both $\{ 1, \ldots, k \}$ and $\{ k + 1, \ldots, k + n \}$. 6
Lemma 1.3. Let $\rho$ be the longest permutation in the set $\mathcal{D}(\bar{\mathcal{S}}_{k+n}/\mathcal{S}_{k,n})$. Then
\[
\rho = ((1+n) \cap 1)((2+n) \cap 2) \cdots ((k+n) \cap k),
\]
\[
T_\rho = T_{(1+n)\cap 1}T_{(2+n)\cap 2} \cdots T_{(k+n)\cap k},
\]
\[
T_iT_\rho = T_{\tau_i\rho} = \begin{cases} T_iT_{k+i} & \text{when } 1 \leq i < n, \\ T_iT_{i-n} & \text{when } n < i < k+n. \end{cases}
\]

Proof. The permutation $\rho$ maps the set $\{1, \ldots, k\}$ onto $\{1+n, \ldots, k+n\}$ and the set $\{k+1, \ldots, k+n\}$ onto $\{1, \ldots, n\}$ preserving the natural order on these sets. The displayed expression for $\rho$ is easily checked straightforwardly. Also,
\[
\tau_i\rho = \begin{cases} \rho \tau_{k+i} & \text{when } 1 \leq i < n, \\ \rho \tau_{i-n} & \text{when } n < i < k+n. \end{cases}
\]
since $\rho$ maps $\{k+i, k+i+1\}$ (respectively, $\{i-n, i-n+1\}$) onto $\{i, i+1\}$.

The respective equalities in $\mathcal{H}_{k+n}$ are obtained by observing that $\ell(\rho) = kn$ and $\ell(\tau_i\rho) = \ell(\rho) + 1$ when $i \neq n$. \hfill \Box

The partial antisymmetrizers associated with the Young subgroups $\mathcal{S}_{k,l}$ will be used most frequently. We will use abbreviations
\[
y_{k,l} = y(\mathcal{S}_{k,l}), \quad y_{k+l/k,l} = y(\mathcal{S}_{k+l}/\mathcal{S}_{k,l}), \quad y_{k,l\backslash k+l} = y(\mathcal{S}_{k,l}\backslash \mathcal{S}_{k+l}).
\]
Thus for $n = k + l$ we have $y_n = y_{k,l} \cdot y_{k,l} = y_{k,l} \cdot y_{k,l\backslash n}$.

Recall that $\bar{\mathcal{S}}_{n-1}$ has been identified with the subgroup $\mathcal{S}_{n-1,1}$ of $\bar{\mathcal{S}}_n$. We will write $y_{n/n-1}$ for the element $y(\mathcal{S}_n/\mathcal{S}_{n-1}) \in \mathcal{H}_n$.

Lemma 1.4. Let $1 \leq k \leq n$ and $c = (n+1) \cap k \in \mathcal{S}_{n+1}$. Then
\[
y_{n+1/k,n+1-k} = q^k y_{n/k,n-k} + (-1)^{n+1-k} y_{n/k-1,n+1-k} T_c.
\]

Proof. Recall that
\[
y_{n+1/k,n+1-k} = \sum_{d \in \mathcal{D}(\mathcal{S}_{n+1}/\mathcal{S}_{k,n+1-k})} (-1)^{\ell(d)} q^{k(n+1-k)-\ell(d)} T_d.
\]
For each $d \in \mathcal{D}(\mathcal{S}_{n+1}/\mathcal{S}_{k,n+1-k})$ either $d(n+1) = n+1$ or $d(k) = n+1$. In the first of these two cases $d \in \mathcal{S}_n$, and in fact $d \in \mathcal{D}(\mathcal{S}_n/\mathcal{S}_{k,n})$. In the second case $d = d'c$ where $d' \in \mathcal{D}(\mathcal{S}_n/\mathcal{S}_{k-1,n+1-k})$, and we also have $T_d = T_{d'} T_c$ since $c \in \mathcal{D}(\mathcal{S}_n/\mathcal{S}_{n+1})$, while $d' \in \mathcal{S}_n$. Note that $\ell(d) = \ell(d') + n + 1 - k$. Splitting the above sum into two, we arrive at the desired conclusion. \hfill \Box

For each $h \in \mathcal{H}_n$ we denote by $h^{(k)} \in \mathcal{H}_{k+n}$ the image of $h$ under the algebra homomorphism $\mathcal{H}_n \to \mathcal{H}_{k+n}$ such that $T_i \mapsto T_{k+i}$ for $i = 1, \ldots, n-1$. We will call $h^{(k)}$ the $k$ step shift of $h$.

Let similarly $\sigma^{(k)} \in \mathcal{S}_{k+n}$ be the image of $\sigma \in \mathcal{S}_n$ under the group homomorphism $\mathcal{S}_n \to \mathcal{S}_{k+n}$ such that $\tau_i \mapsto \tau_{k+i}$ for $i = 1, \ldots, n-1$. Then $\sigma^{(k)}$ leaves $1, \ldots, k$ fixed and permutes $k+1, \ldots, k+n$ in exactly the same way as $\sigma$ permutes $1, \ldots, n$. Hence $\ell(\sigma^{(k)}) = \ell(\sigma)$ and $T_{\sigma^{(k)}} = T_{\sigma^{(k)}} \in \mathcal{H}_{k+n}.$

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Lemma 1.5. We have $y_{k,l} = y_k y_l^{(k)}$.

Proof. The equality follows from the explicit formula defining $y(\mathcal{G}_{k,l})$ since each permutation in $\mathcal{G}_{k,l}$ is written uniquely as the product $\pi \sigma^{(k)}$ where $\pi \in \mathcal{S}_k$ and $\sigma \in \mathcal{S}_l$. Clearly $\ell(\pi \sigma^{(k)}) = \ell(\pi) + \ell(\sigma)$ and $T_{\pi \sigma^{(k)}} = T_\pi T_\sigma$. □

Lemma 1.6. We have $y_{k+l+m/k+l,m} \cdot y_{k+l/m,k,l+m} = y_{k+l+m/k,l+m} \cdot y_{k+l/m,k,l+m}^{(k)}$.

Proof. Both products are equal to $y(\mathcal{G}_{k+l+m/k,l,m})$ where $\mathcal{G}_{k,l,m}$ is the Young subgroup of $\mathcal{S}_{k+l+m}$ consisting of all permutations that leave stable the 3 subsets

$I_1 = \{1, \ldots, k\}, \quad I_2 = \{k+1, \ldots, k+l\}, \quad I_3 = \{k+l+1, \ldots, k+l+m\}$.

This is explained as follows. The permutations in $D(\mathcal{G}_{k+l+m/k,l,m})$ are increasing on each of these subsets. Such a permutation is the product $\pi \sigma$ where $\pi$ is increasing on $I_1 \cup I_2$ and on $I_3$, while $\sigma$ fixes all elements of $I_3$ and is increasing on $I_1$ and on $I_2$. Then $\pi \in D(\mathcal{G}_{k+l+m/k,l,m})$ and $\sigma \in D(\mathcal{G}_{k+l/m,k,l,m})$ where $\mathcal{G}_{k,l}$ is identified with a subgroup of $\mathcal{S}_{k+l+m}$. We have $\ell(\pi \sigma) = \ell(\pi) + \ell(\sigma)$ and $T_{\pi \sigma} = T_\pi T_\sigma$.

On the other hand, each permutation in $D(\mathcal{G}_{k+l+m/k,l,m})$ can be written as $\pi \sigma$ where $\pi$ is increasing on $I_1$ and on $I_2 \cup I_3$, while $\sigma$ fixes all elements of $I_1$ and is increasing on $I_2$ and on $I_3$. Then $\pi \in D(\mathcal{G}_{k+l+m/k,l,m})$ and $\sigma = \sigma^{(k)}$ for some $\sigma' \in D(\mathcal{G}_{l+m/k,l,m})$. Again, $\ell(\pi \sigma) = \ell(\pi) + \ell(\sigma)$ and $T_{\pi \sigma} = T_\pi T_\sigma$. □

2. Nondegeneracy of multiplications

Let $V$ be a finite dimensional vector space over $k$. A Hecke symmetry on $V$ is a linear operator $R : V \otimes V \to V \otimes V$ satisfying the braid equation

$$(R \otimes \text{Id}_V)(\text{Id}_V \otimes R)(R \otimes \text{Id}_V) = (\text{Id}_V \otimes R)(R \otimes \text{Id}_V)(\text{Id}_V \otimes R)$$

and the quadratic Hecke relation

$$(R - q \cdot \text{Id}_{V \otimes V})(R + \text{Id}_{V \otimes V}) = 0 \quad \text{where} \quad 0 \neq q \in k.$$ 

In the case when $q \neq -1$ such an operator is diagonalizable with two eigenvalues, but we allow the value $q = -1$ as well.

The Hecke symmetry $R$ gives rise to a representation of $\mathcal{H}_n$ in the $n$th tensor power of $V$ such that $T_i$ acts on $V^{\otimes n}$ as the linear operator $R^{(n)}_i = \text{Id}_V^{\otimes(i-1)} \otimes R \otimes \text{Id}_V^{\otimes(n-i-1)}$.

In this way $V^{\otimes n}$ becomes a left $\mathcal{H}_n$-module. For each $m \geq n$ we identify $\mathcal{H}_n$ with the subalgebra of $\mathcal{H}_m$ generated by $\{T_i \mid i < n\}$, and $V^{\otimes m}$ will be always regarded as a left $\mathcal{H}_n$-module with respect to the action $h(a \otimes b) = ha \otimes b, \quad h \in \mathcal{H}_n, \quad a \in V^{\otimes n}, \quad b \in V^{\otimes(m-n)}$.

Note that the $k$ step shift $h^{(k)} \in \mathcal{H}_{k+n}$ of $h$ (see the definition preceding Lemma 1.5) acts as follows:

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\[ h^{(k)}(a \otimes b) = a \otimes hb, \quad h \in \mathcal{H}_n, \ a \in V^{\otimes k}, \ b \in V^{\otimes m}, m \geq n. \]

Denote by \( \mathbb{T}(V) \) the tensor algebra of \( V \). It is a graded algebra with homogeneous components \( T_n(V) = V^{\otimes n}, \ n \geq 0 \). The multiplication in \( \mathbb{T}(V) \) is defined by the rule \( ab = a \otimes b \) for homogeneous elements \( a, b \). This permits us to omit the sign \( \otimes \) when writing tensors.

We define the \( R \)-symmetric algebra \( S(V, R) \) and the \( R \)-skewsymmetric algebra \( \Lambda(V, R) \) as the factor algebras of \( \mathbb{T}(V) \) by the graded ideals generated, respectively, by the subspaces

\[
\text{Im} \left( R - q \cdot \text{Id} \right) \subset V^{\otimes 2} \quad \text{and} \quad \text{Ker} \left( R - q \cdot \text{Id} \right) \subset V^{\otimes 2}.
\]

Their homogeneous components will be denoted by \( S_n(V, R) \) and \( \Lambda_n(V, R) \).

Let \( I \) be the graded ideal of \( \mathbb{T}(V) \) such that \( \Lambda(V, R) = \mathbb{T}(V)/I \). Then \( I_0 = I_1 = 0 \), \( I_2 = \text{Ker} \left( R - q \cdot \text{Id} \right) \), and

\[
I_n = \sum_{i=1}^{n-1} V^{\otimes (i-1)} \otimes I_2 \otimes V^{\otimes (n-i-1)} = \sum_{i=1}^{n-1} \left\{ u \in V^{\otimes n} \mid T_i u = q u \right\}
\]

for \( n > 2 \). Put \( \mathcal{Y}^{(0)} = k, \ \mathcal{Y}^{(1)} = V, \ \mathcal{Y}^{(2)} = \text{Im} \left( R - q \cdot \text{Id} \right) \), and

\[
\mathcal{Y}^{(n)} = \bigcap_{i=1}^{n-1} (T_i - q)V^{\otimes n} = \bigcup_{i=1}^{n-1} V^{\otimes (i-1)} \otimes \mathcal{Y}^{(2)} \otimes V^{\otimes (n-i-1)}
\]

for \( n > 2 \). Thus \( I_n = \Sigma(V^{\otimes n}) \) and \( \mathcal{Y}^{(n)} = \mathcal{Y}(V^{\otimes n}) \) in the notation of Lemma 1.1. For each \( n \) we identify \( \mathbb{T}_n(V^*) \) with the dual of the vector space \( T_n(V) \) using the bilinear pairing

\[
\langle f_1 \otimes \ldots \otimes f_n, v_1 \otimes \ldots \otimes v_n \rangle = \prod f_i(v_i).
\]

Under this pairing

\[
(\mathcal{Y}^{(n)})^\perp = \sum_{i=1}^{n-1} (V^*)^{\otimes (i-1)} \otimes (\mathcal{Y}^{(2)})^\perp \otimes (V^*)^{\otimes (n-i-1)}.
\]

Therefore \( \sum_{n=0}^{\infty} (\mathcal{Y}^{(n)})^\perp \) is a graded ideal of the tensor algebra \( \mathbb{T}(V^*) \) generated by homogeneous elements of degree 2. The factor algebra by this ideal is nothing else but the quadratic dual \( \mathbb{S}(V, R)^\perp \) of \( \mathbb{S}(V, R) \). Thus the \( k \)th homogeneous component of \( \mathbb{S}(V, R)^\perp \) may be identified with the dual space \( (\mathcal{Y}^{(n)})^* \).

Let \( R^* : (V^*)^{\otimes 2} \to (V^*)^{\otimes 2} \) be the linear operator dual to \( R \). It is a Hecke symmetry on \( V^* \). Since \( (\mathcal{Y}^{(2)})^\perp = \text{Ker}(R^* - q \cdot \text{Id}) \), we have

\[
\mathbb{S}(V, R)^\perp = \Lambda(V^*, R^*),
\]

and so

\[
\Lambda_n(V^*, R^*) = \mathbb{T}_n(V^*)/(\mathcal{Y}^{(n)})^\perp \cong (\mathcal{Y}^{(n)})^*.
\]

We will also consider the subspaces

\[
\mathcal{Y}^{(k,n)} = \mathcal{Y}^{(k)} \otimes \mathcal{Y}^{(n)} = \bigcap_{\{i \mid 1 \leq i < k+n, \ i \neq k\}} (T_i - q)V^{\otimes (k+n)}.
\]

If \( u \in \mathcal{Y}^{(k,n)} \), then \( T_i u = -u \) whenever \( 1 \leq i < k + n \) and \( i \neq k \). Note also that \( \mathcal{Y}^{(k+n)} \subset \mathcal{Y}^{(k,n)} \).
Lemma 2.1. Assume that $k \geq 1$. Let $c = (k + n) \in S_{k+n}$, and let $\rho$ be the longest permutation in the set $D(S_{k+n}/S_{k,n})$. Then

$$T_\rho \Upsilon^{(k,n)} \subset \Upsilon^{(k-1,n)} \otimes V, \quad T_\rho \Upsilon^{(k,n)} \subset \Upsilon^{(n,k)}, \quad y_{k+n/k,n} \Upsilon^{(k,n)} \subset \Upsilon^{(k+n)}.$$ 

Proof. Let $u \in \Upsilon^{(k,n)}$. For $1 \leq i < k-1$ we have $\tau_i c = c \tau_i$ and $\ell(\tau_i c) = \ell(c) + 1$, whence $T_i T_\rho = T_{\tau_i} c = T_i T_i$. Writing $u = (T_i - q)w$ for some $w \in V^{\otimes (k+n)}$, we get

$$T_\rho u = T_\rho (T_i - q)w = (T_i - q)T_\rho w.$$ 

If $k \leq i < k + n - 1$, then $\tau_i c = c \tau_i + 1$ and $\ell(\tau_i c) = \ell(c) + 1$, which implies that $T_i T_\rho = T_{\tau_i} c = T_i T_i + 1$. Since $k < i + 1 < k + n$, we can write $u = (T_{i+1} - q)w$ for some $w \in V^{\otimes (k+n)}$, whence

$$T_\rho u = T_{i+1} T_\rho(T_i - q)w = (T_i - q)T_{\rho} w.$$ 

It follows that

$$T_\rho u \in \bigcap_{\{i \mid 1 \leq i < k+n-1, i \neq k-1\}} (T_i - q) V^{\otimes (k+n)} = \Upsilon^{(k-1,n)} \otimes V.$$ 

The inclusion $T_\rho u \in \Upsilon^{(n,k)}$ is proved in a similar way by making use of the formulas for $T_i T_\rho$ given in Lemma 1.3.

The inclusion $y_{k+n/k,n} \in \Upsilon^{(k+n)}$ is a special case of Lemma 1.2 applied to the $H_{k+n}$-module $M = \mathbb{T}_{k+n}(V)$ and the Young subgroup $S_{k,n}$ of $S_{k+n}$.

The Hecke symmetry $R$ gives rise to a braiding on a certain monoidal subcategory of the category of vector spaces. This was worked out by Lyubashenko [14] in the case of symmetries with parameter $q = 1$. A special role played by the element $T_\rho \in H_{k+n}$ stems from the fact that its action on $\mathbb{T}_{k+n}(V)$ gives the braiding (commutation in the language of [12]) between the spaces $V^{\otimes k}$ and $V^{\otimes n}$.

Lemma 2.2. The vector space $\Upsilon(V,R) = \bigoplus_{k=0}^{\infty} \Upsilon^{(k)}$ is a graded associative unital algebra with respect to the multiplication $\ast$ defined by the formula

$$a \ast b = y_{k+l/k,l}(ab), \quad a \in \Upsilon^{(k)}, \quad b \in \Upsilon^{(l)}.$$ 

The assignments $u \mapsto y_k u$ for $u \in V^{\otimes k}$, $k \geq 0$ define a homomorphism of algebras $\mathbb{T}(V) \mapsto \Upsilon(V,R)$ which factors through $\Lambda(V,R)$.

Proof. By Lemma 2.1 $a \ast b \in \Upsilon^{(k+l)}$. Thus the multiplication is well defined, and

$$(a \ast b) \ast c = y_{k+l+m/k+l,m} y_{k+l/k,l} y_{k+l+m/k+l,m}(abc),$$

$$a \ast (b \ast c) = y_{k+l+m/k,l+m} y_{k+l+m/k,l+m}(abc)$$

for $a \in \Upsilon^{(k)}$, $b \in \Upsilon^{(l)}$, $c \in \Upsilon^{(m)}$. The associativity law $(a \ast b) \ast c = a \ast (b \ast c)$ follows from Lemma 1.6. Also, $1 \ast b = b$ and $a \ast 1 = a$ since $y_{k+l/k,l} = 1$ whenever either $k = 0$ or $l = 0$.

By Lemma 1.1 the linear operator by which $y_k$ acts on $V^{\otimes k}$ has images in $\Upsilon^{(k)}$ and vanishes on the component $I_k$ of the ideal $I \subset \mathbb{T}(V)$ which defines $\Lambda(V,R)$. Hence the map $\mathbb{T}(V) \mapsto \Upsilon(V,R)$ is well defined and factors as required. If $u \in V^{\otimes k}$
and \( w \in V^\otimes l \), then

\[
(y_k u) \star (y_l w) = y_{k+l/k,l} y_k y_l (u w) = y_{k+l/k,l} y_{k,l} (u w) = y_{k+l} (u w)
\]

by Lemma 2.15.

**Lemma 2.3.** Suppose that \( \mathcal{Y}^{(n+1)} = 0 \) for some \( n > 0 \). Let \( 1 \leq k \leq n \), and let \( \rho \in \mathfrak{S}_{k+n} \) be the longest permutation in the set \( \mathcal{D}(\mathfrak{S}_{k+n}/\mathfrak{S}_{k,n}) \). Then

\[
y_{n/k,n-k} u = (-1)^{kn-k} q^{-k(k+1)/2} T_{\rho} u \quad \text{for all} \quad u \in \mathcal{Y}^{(k,n)}.
\]

In the case \( k = n \) this yields \( T_{\rho} u = q^{n(n+1)/2} u \) for \( u \in \mathcal{Y}^{(n,n)} \).

**Proof.** Let \( u \in \mathcal{Y}^{(k,n)} \). Note that \( \mathcal{Y}^{(k,n)} \subset \mathcal{Y}^{(k,n+1-k)} \otimes T_{k-1}(V) \). Therefore

\[
y_{n+1/k,n+1-k} u \in \mathcal{Y}^{(n+1)} \otimes T_{k-1}(V) = 0
\]

by Lemma 2.1. In view of Lemma 1.4 this can be rewritten as

\[
y_{n/k,n-k} u = (-1)^{n-k} q^{-k} y_{n/k-1,n+1-k} T_c u
\]

where \( c = (n+1) \setminus k \). Now we take \( \pi = (n+k) \setminus (n+1) \) and apply \( T_{\pi} \) to both sides of the last equality. As \( T_{\pi} \) lies in the subalgebra of \( \mathcal{H}_{n+k} \) generated by the set \( \{ T_i \mid n < i < n+k \} \), this element commutes with all elements of \( \mathcal{H}_n \). In particular, \( T_{\pi} \) commutes with \( y_{n/r,n-r} \) for \( r \leq n \). We get

\[
T_{\pi} y_{n/k,n-k} u = (-1)^{k-1} y_{n/k,n-k} u
\]

since \( T_{\pi} u = (-1)^{k-1} u \), and

\[
T_{\pi} y_{n/k-1,n+1-k} T_c u = y_{n/k-1,n+1-k} T_{\pi} T_c u = y_{n/k-1,n+1-k} T_{(n+k) \setminus k} u
\]

since \( T_{\pi} T_c = T_{\pi c} = T_{(n+k) \setminus k} \). Thus

\[
y_{n/k,n-k} u = (-1)^{n-1} q^{-k} y_{n/k-1,n+1-k} T_{(n+k) \setminus k} u.
\]

If \( k = 1 \), then \( \rho = (n+k) \setminus k \), and we get the desired formula. For \( k > 1 \) we use induction on \( k \). Note that \( T_{(n+k) \setminus k} u \in \mathcal{Y}^{(k-1,n)} \otimes V \) by Lemma 2.1. By the induction hypothesis we may assume that

\[
y_{n/k-1,n+1-k} T_{(n+k) \setminus k} u = (-1)^{(k-1)(n-1)} q^{-(k-1)k/2} T_{\rho'} T_{(n+k) \setminus k} u
\]

where \( \rho' \) is the longest permutation in the set \( \mathcal{D}(\mathfrak{S}_{k+n-1}/\mathfrak{S}_{k-1,n}) \). Lemma 1.3 shows that \( \rho = \rho' \cdot ((n+k) \setminus k) \) and \( T_{\rho} = T_{\rho'} T_{(n+k) \setminus k} \). This yields the conclusion.

If \( k = n \), then \( \mathfrak{S}_{k,n-k} = \mathfrak{S}_n \), and so \( y_{n/k,n-k} = 1 \). \qed

**Lemma 2.4.** Suppose that \( \dim \mathcal{Y}^{(n)} = 1 \) and \( \mathcal{Y}^{(n+1)} = 0 \) for some \( n > 0 \). Fix a nonzero element \( t \in \mathcal{Y}^{(n)} \). For each \( k = 0, \ldots, n \) the bilinear pairing

\[
\beta_k : \mathcal{Y}^{(k)} \times \mathcal{Y}^{(n-k)} \to k
\]
defined by the rule $\beta_k(u, w) t = u \ast w = y_{n/k,n-k}(uw)$ for $u \in \Upsilon^{(k)}$ and $w \in \Upsilon^{(n-k)}$ is nondegenerate.

There are linear bases $u_1, \ldots, u_d$ for $\Upsilon^{(k)}$ and $w_1, \ldots, w_d$ for $\Upsilon^{(n-k)}$ such that

$$t = w_1 u_1 + \cdots + w_d u_d.$$  

Proof. Since $\Upsilon^{(n)} \subset \Upsilon^{(n-k)}$, there is an expression $t = w_1 u_1 + \cdots + w_d u_d$ for some linearly independent elements $u_1, \ldots, u_d \in \Upsilon^{(k)}$ and $w_1, \ldots, w_d \in \Upsilon^{(n-k)}$. Suppose that $a \in \Upsilon^{(k)}$ is such that $\beta_k(a, w_i) = 0$ for all $i = 1, \ldots, d$. Then

$$y_{n/k,n-k}(at) = \sum y_{n/k,n-k}(aw_i) u_i = \sum \beta_k(a, w_i) tu_i = 0.$$  

On the other hand, $y_{n/k,n-k}(at) = (-1)^{kn-k} q^{-(k+1)/2} T_p(at)$ by Lemma 2.3 since $a \in \Upsilon^{(k)}$. Hence $T_p(at) = 0$, but this entails $a = 0$ since $T_p$ is an invertible element of $H_{k+n}$. It follows that $\beta_k$ has zero left radical, and also

$$\dim \Upsilon^{(k)} \leq d \leq \dim \Upsilon^{(n-k)}.$$  

Replacing $k$ with $n - k$, we deduce that $\dim \Upsilon^{(n-k)} \leq \dim \Upsilon^{(k)}$ as well. Hence we must have equalities in the displayed line above. This shows that $\beta_k$ is nondegenerate and the chosen linearly independent collections of elements form bases for $\Upsilon^{(k)}$ and $\Upsilon^{(n-k)}$.

**Theorem 2.5.** If $\dim \Upsilon^{(n)} = 1$ and $\Upsilon^{(n+1)} = 0$ then the algebras $\Upsilon(V, R)$ and $S(V, R)\mathbf{l}$ are Frobenius. Vanishing of $\Upsilon^{(n+1)}$ is actually a consequence of the condition $\dim \Upsilon^{(n)} = 1$ provided that $\dim V > 1$.

Proof. A connected graded algebra $A = k \oplus A_1 \oplus A_2 \cdots$ is Frobenius if and only if there exists $n \geq 0$ such that $\dim A_n = 1$, $A_{n+1} = 0$ and all multiplication maps

$$A_k \otimes A_{n-k} \to A_n$$

are nondegenerate in the sense that no nonzero element of $A_k$ annihilates $A_{n-k}$ on the left or on the right. For the algebra $\Upsilon(V, R)$ this nondegeneracy condition is equivalent to the first conclusion of Lemma 2.4. Similarly, the second conclusion of Lemma 2.4 implies that $S(V, R)\mathbf{l}$ is Frobenius. Indeed, the multiplication maps

$$(\Upsilon^{(l)})^* \otimes (\Upsilon^{(k)})^* \to (\Upsilon^{(l+k)})^*$$

in the algebra $S(V, R)\mathbf{l}$ are dual to the inclusion maps $\Upsilon^{(l+k)} \hookrightarrow \Upsilon^{(l)} \otimes \Upsilon^{(k)}$. Take $l = n - k$. If $\xi \in (\Upsilon^{(k)})^*$ is such that $(\Upsilon^{(n-k)})^* \xi = 0$, then

$$\sum \eta(w_i) \xi(u_i) = (\eta \otimes \xi)(t) = 0$$

for all $\eta \in (\Upsilon^{(n-k)})^*$. We must have $\xi(u_i) = 0$ for all $i$, whence $\xi = 0$.

Suppose now that $\dim \Upsilon^{(n)} = 1$, but $\Upsilon^{(n+1)} \neq 0$. We will show that this is possible only when $\dim V = 1$. Let $0 \neq t \in \Upsilon^{(n)}$. Since $\Upsilon^{(n+1)} = \Upsilon^{(n)} V \cap \Upsilon^{(n)}$, each element of $\Upsilon^{(n+1)}$ can be written as $ta$ and as $bt$ for some vectors $a, b \in V$. But for
\( a \neq 0 \) the equality \( ta = bt \) in the tensor algebra \( \mathbb{T}(V) \) can only be satisfied when \( t \) is a scalar multiple of \( a^n \). It follows that \( \mathcal{Y}^{(n)} \) and \( \mathcal{Y}^{(n+1)} \) are spanned, respectively, by the tensors \( a^n \) and \( a^{n+1} \) for some \( 0 \neq a \in V \).

Since \( a^n \in \mathcal{Y}^{(n)} \subset \mathcal{Y}^{(1,n-1)} \), we must have \( a^{n-1} \in \mathcal{Y}^{(n-1)} \). Now \( va^{n-1} \in \mathcal{Y}^{(1,n-1)} \) and \( va^n \in \mathcal{Y}^{(1,n)} \) for all \( v \in V \). By Lemma 2.1

\[
y_{n/1,n-1}(va^{n-1}) \in \mathcal{Y}^{(n)} = ka^n \quad \text{and} \quad y_{n+1/1,n}(va^n) \in \mathcal{Y}^{(n+1)} = ka^{n+1}.
\]

Recall that \( y_{n+1/1,n} = qy_{n/1,n-1} + (-1)^n T_c \) where \( c = (n+1) \sim 1 \) (see Lemma 1.4). Hence

\[
T_c(va^n) = (-1)^ny_{n+1/1,n}(va^n) - (-1)^nq y_{n/1,n-1}(va^{n-1})a \in \mathcal{Y}^{(n+1)}.
\]

Since \( \mathcal{Y}^{(n+1)} \) is an \( \mathcal{H}_{n+1} \)-submodule of \( T_{n+1}(V) \) and \( T_c \) is an invertible element of \( \mathcal{H}_{n+1} \), we deduce that

\[
va^n \in T_c^{-1}\mathcal{Y}^{(n+1)} = \mathcal{Y}^{(n+1)} = ka^{n+1}.
\]

Hence all vectors \( v \in V \) are scalar multiples of \( a \), i.e., \( V = \mathbb{k}a \). \( \square \)

**Proposition 2.6.** Suppose that \( \dim \mathcal{Y}^{(n)} = 1 \) and \( \mathcal{Y}^{(n+1)} = 0 \). Then the following conditions are equivalent:

(i) the action of \( y_n \) on \( V^\otimes n \) is nonzero,

(ii) the algebra \( \mathcal{Y}(V, R) \) is generated by its component \( \mathcal{Y}^{(1)} = V \),

(iii) the algebra \( \mathcal{Y}(V, R) \) is isomorphic to a factor algebra of \( \Lambda(V, R) \).

**Proof.** Clearly assertion (ii) is equivalent to surjectivity of the homomorphism \( \mathbb{T}(V) \to \mathcal{Y}(V, R) \) extending the identity map \( V \to V \). This homomorphism was described in Lemma 2.2. Its surjectivity means that \( y_k V^\otimes k = \mathcal{Y}^{(k)} \) for all \( k \). In particular, (ii) \( \Rightarrow \) (i).

Conversely, suppose that (i) holds. Then \( y_n V^\otimes n \) is a nonzero subspace of the one-dimensional space \( \mathcal{Y}^{(n)} \). Hence \( y_n V^\otimes n = \mathcal{Y}^{(n)} \). For \( k < n \) put \( W_k = y_k V^\otimes k \), which is a subspace of \( \mathcal{Y}^{(k)} \). Since \( y_n = y_{n-k,k} y_{n-k,k\setminus n} \), we get

\[
\mathcal{Y}^{(n)} = y_n V^\otimes n \subset y_{n-k,k} V^\otimes n = y_{n-k} V^\otimes (n-k) \otimes y_k V^\otimes k = W_{n-k} \otimes W_k,
\]

and so \( t = \sum w_i u_i \in W_{n-k} \otimes W_k \), in the notation of Lemma 2.4. It follows that \( u_i \in W_k \) for all \( i \). Hence \( W_k = \mathcal{Y}^{(k)} \) since \( \mathcal{Y}^{(k)} \) is spanned by \( \{u_i\} \). This shows that (i) \( \Rightarrow \) (ii).

Next, (ii) \( \Rightarrow \) (iii) since the homomorphism of algebras \( \mathbb{T}(V) \to \mathcal{Y}(V, R) \) factors through \( \Lambda(V, R) \). Finally, (iii) \( \Rightarrow \) (ii) since the algebra \( \Lambda(V, R) \) is generated by its component of degree 1. \( \square \)

It can be seen that conditions (i)–(iii) of Proposition 2.6 are satisfied at least when \( n \) is small or under some restrictions on \( q \). For a later reference we record here just one easy conclusion:

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Corollary 2.7. Suppose that \( \dim V > 1 \) and \( \dim \Upsilon^{(3)} = 1 \). Then the action of \( y_3 \) on \( V^\otimes 3 \) is nonzero.

Proof. We have seen in Theorem 2.5 that \( \Upsilon^{(4)} = 0 \). Note that \( y_{2/1,1} = y_2 = q - T_1 \) and \( (q - T_1) V^\otimes 2 = \text{Im}(q \cdot \text{Id} - R) = \Upsilon^{(2)} \). This means that \( V \star V = \Upsilon^{(2)} \). Also, \( V \star \Upsilon^{(2)} = \Upsilon^{(3)} \) by Lemma 2.4 applied with \( n = 3 \). Thus the algebra \( \Upsilon(V,R) \) is generated by \( V \).

Theorem 2.8. Suppose that \( \dim V > 1 \) and \( \dim \Lambda_n(V,R) = 1 \) for some \( n > 0 \). Then \( \Lambda_i(V,R) = 0 \) for all \( i > n \) and the algebra \( \Lambda(V,R) \) is Frobenius.

Proof. Since \( \Lambda(V,R) \cong S(V^*, R^*)^! \), the desired conclusion is just a reformulation of Theorem 2.5 with \( R \) changed to \( R^* \).

If \([n]_q \neq 0\), then for all \( k \leq n \) the map \( \Lambda_k(V,R) \to \Upsilon^{(k)} \) induced by the action of \( y_k \) on \( V^\otimes k \) is bijective. In this case \( \dim \Lambda_n(V,R) = \dim \Upsilon^{(n)} \), and if this dimension is equal to 1, then \( \Lambda(V,R) \) is a Frobenius algebra isomorphic to \( \Upsilon(V,R) \).

Question 2.9. Is it true without any restriction on \( q \) that \( \Lambda(V,R) \cong \Upsilon(V,R) \) under the assumptions of either Theorem 2.5 or Theorem 2.8?

3. Three fundamental operators

Let \( R \) be a Hecke symmetry with parameter \( q \) on a finite dimensional vector space \( V \) over a field \( k \). We will be concerned with certain linear operators \( \theta, \varphi, \psi \) defined in terms of \( R \) in the case when the two algebras \( \Upsilon(V,R) \) and \( S(V,R)^! \) are Frobenius. By Theorem 2.5 this happens precisely when

\[ \dim \Upsilon^{(n)} = 1 \quad \text{and} \quad \Upsilon^{(n+1)} = 0 \]

for some \( n > 0 \). We make this assumption throughout the whole section, and we fix a nonzero tensor \( t \in \Upsilon^{(n)} \subset V^\otimes n \).

In general, for a Frobenius connected graded algebra \( A = \bigoplus_{k \geq 0} A_k \) with the last nonzero component \( A_n \) there is a distinguished choice of Nakayama automorphism. This automorphism \( \nu \) preserves the grading and is characterized by the property

\[ ba = \nu(a)b \quad \text{whenever} \quad a \in A_k \text{ and } b \in A_{n-k} \text{ for some } k. \]

It commutes with every automorphism of \( A \) which preserves the grading of \( A \).

If \( A \) is generated by its degree 1 component \( A_1 = V \), then \( A \cong \Upsilon(V)/I \) where \( I \) is a graded ideal of \( \Upsilon(V) \). Since \( I_n \) has codimension 1 in \( \Upsilon_n(V) \), there is a linear function \( f \in \Upsilon_n(V)^* \) such that \( \text{Ker} f = I_n \). The characteristic property of \( \nu \) translates into the twisted cyclicity property for the function \( f \):

\[ f(wv) = f(\varphi(v)w), \quad v \in V, \ w \in \Upsilon_{n-1}(V), \]

where \( \varphi : V \to V \) is the restriction of \( \nu \) to \( A_1 \). Nondegeneracy of the multiplication map \( A_k \times A_{n-k} \to A_n \) implies that the \( k \)th homogeneous component \( I_k \) of the ideal \( I \) coincides with the left radical of the bilinear pairing \( \Upsilon_k(V) \times \Upsilon_{n-k}(V) \to k \) defined by the rule \((u,w) \mapsto f(uw)\). Thus such an algebra \( A \) is completely determined by the function \( f \).

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This description of graded Frobenius algebras generated in degree 1 by means of functions is well known (see, e.g., [5, Prop. 8.1]). In a more recent terminology one refers to $f$ as a preregular form or, regarding $f$ as an element of $\mathbb{T}_n(V^*)$, a twisted potential (see [8]).

Further on in this section we will denote by $\nu$ the Nakayama automorphism of the algebra $\mathcal{Y}(V, R)$ and by $\varphi$ its restriction to the degree 1 component $\mathcal{Y}^{(1)} = V$.

In terms of the pairings introduced in Lemma 2.4 we have

$$\beta_{n-k}(b, a) = \beta_k(\nu(a), b), \quad a \in \mathcal{Y}^{(k)}, \quad b \in \mathcal{Y}^{(n-k)}.$$ 

It will be seen in the proof of Theorem 3.8 that the action of $\nu$ on the $k$th homogeneous component $\mathcal{Y}^{(k)}$ is given by the linear operator $\varphi^{\otimes k} \in GL(V^{\otimes k})$, but this is not clear right now. In the case when $\mathcal{Y}(V, R)$ is generated by $V$, we can associate to the algebra $\mathcal{Y}(V, R)$ a linear function $f \in \mathbb{T}_n(V^*)$ with the twisted cyclicity property (see Proposition 3.12).

The graded Frobenius algebra $\mathbb{S}(V, R)^!$ is generated by its degree 1 component $V^*$. Therefore its Nakayama automorphism is determined by action on elements of degree 1. We denote by $\psi : V \to V$ the linear operator such that the dual operator $\psi^* : V^* \to V^*$ is the restriction of the Nakayama automorphism of $\mathbb{S}(V, R)^!$.

The linear function $\mathbb{T}_n(V^*) \to k$ associated with $\mathbb{S}(V, R)^!$ may be identified with the tensor $t$, and the twisted cyclicity property of $t$ means that whenever $t = \sum x_i t_i$ for some elements $x_i \in V$ and $t_i \in \mathbb{T}_{n-1}(V)$ we also have

$$t = \sum t_i \psi(x_i).$$

It has been seen in Lemma 2.4 that $t$ can be expressed as $\sum x_i t_i$ with $\{x_i\}$ being a linear basis for $V$ and $\{t_i\}$ a linear basis for $\mathbb{T}^{(n-1)}$.

**Proposition 3.1.** There are linear operators $\theta, \overline{\theta} \in GL(V)$ such that

$$T_{(n+1)\cap 1}(vt) = t \theta(v), \quad T_{1\cap (n+1)}(tv) = \overline{\theta}(v)t$$

for all $v \in V$. Moreover, $\overline{\theta} = q^{n+1} \theta^{-1}$.

**Proof.** By Lemma 2.1 $T_{(n+1)\cap 1}$ maps $\mathcal{Y}^{(1,n)} = V \otimes \mathcal{Y}^{(n)}$ into $\mathcal{Y}^{(n,1)} = \mathcal{Y}^{(n)} \otimes V$. Similarly, $T_{1\cap (n+1)}$ maps $\mathcal{Y}^{(n,1)}$ into $\mathcal{Y}^{(1,n)}$. Since the 1-dimensional space $\mathcal{Y}^{(n)}$ is spanned by $t$, we have $\mathcal{Y}^{(1,n)} = Vt$ and $\mathcal{Y}^{(n,1)} = tV$. Hence $\theta$ and $\overline{\theta}$ are well defined.

We will check that $\theta \circ \overline{\theta} = q^{n+1} \text{Id}$.

Lemma 2.1 shows also that $y_{n+1/n} \mathcal{Y}^{(n,1)} \subset \mathcal{Y}^{(n+1)} = 0$. Since

$$y_{n+1/n} = \sum_{i=1}^{n+1} (-1)^{n+1-i} q^{i-1} T_{i\cap (n+1)},$$

it follows that $T_{1\cap (n+1)} u = \sum_{i=2}^{n+1} (-1)^{i} q^{i-1} T_{i\cap (n+1)} u$, and therefore

$$T_{(n+1)\cap 1} T_{1\cap (n+1)} u = \sum_{i=2}^{n+1} (-1)^{i} q^{i-1} T_{(n+1)\cap 1} T_{i\cap (n+1)} u$$
for all \( u \in \mathcal{Y}^{(n,1)} \). For \( 2 \leq i \leq n + 1 \) easy computations in the group \( \mathfrak{S}_{n+1} \) yield
\[
((n+1) \curvearrowright 1) (i \curvearrowright (n+1)) = (i-1) \curvearrowright (n+1) (i \curvearrowright 1)
\]
\[
= ((i-1) \curvearrowright n)(n \curvearrowright (n+1)(n \curvearrowright 1)
\]
\[
= ((i-1) \curvearrowright (n+1))(n \curvearrowright 1).
\]

Note also that
\[
T_{(n+1) \curvearrowright 1} T_{i \curvearrowright (n+1)} = T_{((n+1) \curvearrowright 1)(i \curvearrowright (n+1))} = T_{(i-1) \curvearrowright (n+1)} T_{n \curvearrowright 1}
\]
and \( T_{n \curvearrowright 1} u = (-1)^{n-1} u \) for all \( u \in \mathcal{Y}^{(n,1)} \). Hence
\[
T_{(n+1) \curvearrowright 1} T_{1 \curvearrowright (n+1)} u = \sum_{i=1}^{n} (-1)^{n-i} q^i T_{i \curvearrowright (n+1)} u = q^{n+1} u - q^{y_{n+1}/n} u = q^{n+1} u.
\]

Taking \( u = tv \), we get \( \theta(\overline{\theta}(v)) = q^{n+1} v \), as claimed. □

**Lemma 3.2.** Let \( \rho \in \mathfrak{S}_{k+n} \) be the longest permutation in the set \( \mathcal{D}(\mathfrak{S}_{k+n}/\mathfrak{S}_{k,n}) \) where \( k \) is any positive integer. Then
\[
T_{\rho}(ut) = t \theta^k(u), \quad T_{\rho^{-1}}(tu) = \overline{\theta}^k(u) t \quad \text{for all } u \in V^\otimes k.
\]

**Proof.** If \( k = 1 \), then \( \rho = (n+1) \curvearrowright 1 \), and the formulas in the statement of Lemma 3.2 are those used in the definition of \( \theta \) and \( \overline{\theta} \). Let \( k > 1 \). It suffices to prove the formulas for \( u = av \) where \( a \in V^\otimes (k-1) \) and \( v \in V \). Let \( \rho' \) be the longest permutation in the set \( \mathcal{D}(\mathfrak{S}_{k+n-1}/\mathfrak{S}_{k-1,n}) \). Proceeding by induction on \( k \) we may assume that
\[
T_{\rho'}(at) = t \theta^\otimes (k-1)(a), \quad T_{\rho'^{-1}}(ta) = \overline{\theta}^\otimes (k-1)(a) t.
\]

By Lemma 1.3 \( \rho = \rho' c \) and \( T_{\rho} = T_{\rho'} T_c \) where \( c = (n+k) \curvearrowright k \). Then \( \rho^{-1} = c^{-1} \rho'^{-1} \) where \( c^{-1} = k \curvearrowright (n+k) \). Since \( \ell(\rho) = \ell(\rho') + \ell(c) \) and the lengths of permutations do not change after taking inverses, we also have \( T_{\rho^{-1}} = T_{c^{-1}} T_{\rho'^{-1}} \). Noting that \( T_c \) and \( T_{c^{-1}} \) act on \( V^\otimes (k+n) \), respectively, as
\[
\text{Id}_V^\otimes (k-1) \otimes T_{(n+1) \curvearrowright 1} \quad \text{and} \quad \text{Id}_V^\otimes (k-1) \otimes T_{1 \curvearrowright (n+1)},
\]
we get
\[
T_{\rho}(ut) = T_{\rho'}(at \theta(v)) = t \theta^\otimes (k-1)(a) \theta(v) = t \theta^k(u) \quad \text{and}
\]
\[
T_{\rho^{-1}}(tu) = T_{c^{-1}} T_{\rho'^{-1}}(tv) = T_{c^{-1}}(\overline{\theta}^\otimes (k-1)(a) tv) = \overline{\theta}^\otimes (k-1)(a) \overline{\theta}(v) t = \overline{\theta}^k(u) t,
\]
as claimed. □

**Corollary 3.3.** The linear operator \( \theta \otimes \theta \) on \( V^{\otimes 2} \) commutes with \( R \). In particular, \( \theta \) extends to automorphisms of \( \Lambda(V,R) \) and \( \mathfrak{S}(V,R) \).

**Proof.** Lemma 3.2 with \( k = 2 \) shows that \( T_{\rho}(ut) = t (\theta \otimes \theta)(u) \) for all \( u \in V^{\otimes 2} \). By Lemma 1.3 \( T_{\rho} T_1 = T_{n+1} T_{\rho} \). Hence
Corollary 3.4. We have $\theta^\otimes n(t) = q^{n(n+1)/2} t$.

Proof. Take $k = n$ and $u = t \in \Upsilon(n)$ in Lemma 3.1.2. Since $ut = t^2 \in \Upsilon(n,n)$, we know from Lemma 3.1 that $T^u_p(ut) = q^{n(n+1)/2} t^2$. This yields the formula for $\theta^\otimes n(t)$. ∎

We want to use both $\theta$ and $\bar{\theta}$. For this we will need also a right hand version of Lemma 2.3:

Lemma 3.5. Let $1 \leq k \leq n$, and let $\rho \in \mathfrak{S}_{k+n}$ be the longest permutation in the set $\mathcal{D}(\mathfrak{S}_{k+n}/\mathfrak{S}_{k,n})$. Then

$$y_{n/n-k,k}^{(k)} u = (-1)^{k-n-k} q^{-k(k+1)/2} T^{\rho-1} u \quad \text{for all } u \in \Upsilon(n,k)$$

where $y_{n/n-k,k}^{(k)} \in \mathcal{H}_{k+n}$ is the $k$ step shift of $y_{n/n-k,k}$.

Proof. We follow the proof of Lemma 2.3. Here we use the identity

$$y_{n+1/n+1-k,k} = q^k y_{n/n-k,k} + (-1)^{n+1-k} y_{n/n+1-k,k-1} T_{1 \curvearrowright (n+2-k)}$$

in $\mathcal{H}_{n+1}$ (cf. Lemma 1.4). After taking the $k-1$ step shifts of elements it gives

$$y_{n/n-k,k}^{(k)} u = (-1)^{n-k} q^{-k} y_{n/n+1-k,k-1} T_{k \curvearrowright (n+1)} u$$

since $y_{n+1/n+1-k,k}^{(k)} u = 0$. Applying $T_{1 \curvearrowright k}$ to both sides, we get

$$y_{n/n-k,k}^{(k)} u = (-1)^{n-1} q^{-k} y_{n/n+1-k,k-1} T_{1 \curvearrowright (n+1)} u.$$

For $k = 1$ this is the desired equality with $\rho^{-1} = 1 \curvearrowright (n+1)$. For $k > 1$ we use induction on $k$ noting that $T_{1 \curvearrowright (n+1)} u \in V \otimes \Upsilon(n,k-1)$ and

$$T^{\rho-1} = T_{k \curvearrowright (n+k)} \cdots T_{2 \curvearrowright (n+2)} T_{1 \curvearrowright (n+1)} = T_{(\rho')^{-1}}(1) T_{1 \curvearrowright (n+1)}$$

where $\rho'$ is the longest permutation in the set $\mathcal{D}(\mathfrak{S}_{k+n-1}/\mathfrak{S}_{k-1,n})$. ∎

Proposition 3.6. Given $1 \leq k \leq n$, let $u_i \in \Upsilon(k)$, $w_i \in \Upsilon(n-k)$ be elements such that

$$t = \sum u_i w_i = \sum w_i \psi^\otimes k(u_i).$$

Then

$$\theta^\otimes k(a) = (-1)^{kn-k} q^{k(k+1)/2} \sum \beta_k(a, w_i) \psi^\otimes k(u_i),$$

and

$$\bar{\theta}^\otimes k(a) = (-1)^{kn-k} q^{k(k+1)/2} \sum \beta_n(w_i, a) u_i$$

for all $a \in \Upsilon(k)$. In particular,
\[ \theta(v) = (-1)^{n-1}q \sum \beta_1(v, t_i) \psi(x_i), \quad \overline{\theta}(v) = (-1)^{n-1}q \sum \beta_{n-1}(t_i, v)x_i \]

for all \( v \in V \) where \( x_i \in V \) and \( t_i \in \mathcal{Y}^{(n-1)} \) are such that \( t = \sum x_i t_i = \sum t_i \psi(x_i) \).

**Proof.** The second equality in the requested expression for \( t \) follows from the twisted cyclicity of \( t \), and such an expression exists since \( \mathcal{Y}^{(n)} \subset \mathcal{Y}^{(k,n-k)} \). By Lemmas 3.2 and 2.3

\[ t \theta^\otimes_k(a) = T_{\rho}(at) = (-1)^{kn-k} q^{k(k+1)/2} y_{n/k,n-k}(at). \]

Substituting \( t = \sum w_i \psi^\otimes_k(u_i) \), we get

\[ y_{n/k,n-k}(at) = \sum y_{n/k,n-k}(aw_i) \psi^\otimes_k(u_i) = t \cdot \sum \beta_k(a, w_i) \psi^\otimes_k(u_i) \]

by the definition of \( \beta_k \) in Lemma 2.4. The formula for \( \theta^\otimes_k \) follows after cancelling the common factor \( t \). The formula for \( \overline{\theta}^\otimes_k \) is derived similarly, using Lemma 3.5 to expand \( T_{\rho^{-1}}(ta) \).

**Corollary 3.7.** We have \( \theta^\otimes_k(\nu(a)) = (\psi \overline{\theta})^\otimes_k(a) \) for all \( a \in \mathcal{Y}^{(k)} \).

**Proof.** Since \( \beta_k(\nu(a), w_i) = \beta_{n-k}(w_i, a) \), Proposition 3.6 yields

\[ \theta^\otimes_k(\nu(a)) = \sum (-1)^{kn-k} q^{k(k+1)/2} \beta_{n-k}(w_i, a) \psi^\otimes_k(u_i) = \psi^\otimes_k(\overline{\theta}^\otimes_k(a)), \]

as required.

**Theorem 3.8.** The operators \( \varphi, \psi, \theta \) pairwise commute and \( \psi = q^{-n-1} \varphi \theta^2 \). Moreover, they lie in the subgroup of \( GL(V) \) consisting of all invertible linear operators \( \chi : V \to V \) such that \( \chi \otimes \chi \) commutes with \( R \).

**Proof.** Recall that \( \varphi \) is the degree 1 component of \( \nu \). Therefore the case \( k = 1 \) of Corollary 3.7 gives \( \theta \varphi = \psi \overline{\theta} \). Since \( \overline{\theta} = q^{n+1} \theta^{-1} \), we get \( \psi = q^{-n-1} \theta \varphi \theta \).

Now \( \theta^\otimes_k \varphi^\otimes_k = (\theta \varphi)^\otimes_k = (\psi \overline{\theta})^\otimes_k \) for each \( k > 0 \). Comparing this with the identity of Corollary 3.7, we deduce that

\[ \nu(a) = \varphi^\otimes_k(a) \quad \text{for all} \quad a \in \mathcal{Y}^{(k)}. \]

Let \( u, v \in V \). Since \( \nu \) is an automorphism of the algebra \( \mathcal{Y}(V, R) \), we get

\[ \varphi^{\otimes 2}(u \ast v) = \varphi(u) \ast \varphi(v). \]

But \( u \ast v = y_2(uv) = (q - T_1)(uv) \). Similarly, \( \varphi(u) \ast \varphi(v) = (q - T_1)(\varphi(u) \varphi(v)) \).

Therefore the displayed identity above means precisely that \( \varphi^{\otimes 2} \) commutes with the operator \( q \cdot \text{Id} - R \). Then \( \varphi^{\otimes 2} \) commutes with \( R \). In Corollary 3.4 we have seen that \( \theta^{\otimes 2} \) commutes with \( R \). Then so too does \( \psi^{\otimes 2} \).

The fact that \( \theta^{\otimes 2} \) commutes with \( R \) implies that for each \( k > 0 \) the linear operator \( \theta^{\otimes k} \) is an \( H_k \)-module endomorphism of \( V^{\otimes k} \). As a consequence, \( \mathcal{Y}^{(k)} \) is stable under \( \theta^{\otimes k} \), and for \( a \in \mathcal{Y}^{(k)} \) and \( b \in \mathcal{Y}^{(l)} \) we have

\[ \theta^{\otimes (k+l)}(a \ast b) = \theta^{\otimes (k+l)}(y_{k+l,k,l}(ab)) = y_{k+l,k,l}(\theta^{\otimes k}(a) \theta^{\otimes l}(b)) = \theta^{\otimes k}(a) \ast \theta^{\otimes l}(b). \]
Thus the linear operators $\theta^{\otimes k}|_{\Upsilon^{(k)}}$, $k \geq 0$, define an automorphism of the algebra $\Upsilon(V, R)$ preserving the grading. Any such an automorphism commutes with the Nakayama automorphism $\nu$. In particular, $\theta \varphi = \varphi \theta$. The rest is clear. \hfill $\square$

**Proposition 3.9.** For $1 \leq k \leq n$ denote by $\xi_k$ and $\eta_k$, respectively, the restrictions of the linear operators $(\psi^{-1}\theta)^{\otimes k}$ and $(\psi^{-1}\phi)^{\otimes k}$ to the subspace $\Upsilon^{(k)} \subset V^{\otimes k}$. Then

$$\text{tr } \xi_k = \text{tr } \eta_k = (-1)^{kn-k} q^{k(k+1)/2} \left[\begin{array}{c} n \\ k \end{array}\right]_q$$

where $\left[\begin{array}{c} n \\ k \end{array}\right]_q$ is the $q$-binomial coefficient. In particular,

$$\text{tr } \psi^{-1}\theta = \text{tr } \psi\phi = (-1)^{n-1} q [n]_q.$$

**Proof.** Put $C = (-1)^{kn-k} q^{k(k+1)/2}$. In the notation of Proposition 3.6 we have

$$\begin{align*}
\xi_k(a) &= (\psi^{-1})^{\otimes k} \theta^{\otimes k}(a) = C \sum \beta_k(a, w_i) u_i, \\
\eta_k(a) &= \psi^{\otimes k} \phi^{\otimes k}(a) = C \sum \beta_{n-k}(w_i, a) \psi^{\otimes k}(u_i)
\end{align*}$$

for $a \in \Upsilon^{(k)}$. Note that for each $u \in \Upsilon^{(k)}$ and $g \in (\Upsilon^{(k)})^*$ the linear operator on $\Upsilon^{(k)}$ defined by the rule $a \mapsto g(a) u$, $a \in \Upsilon^{(k)}$, has trace $g(u)$. This yields

$$\begin{align*}
\text{tr } \xi_k &= C \sum \beta_k(u_i, w_i), \\
\text{tr } \eta_k &= C \sum \beta_{n-k}(w_i, \psi^{\otimes k}(u_i)),
\end{align*}$$

i.e., by the definition of bilinear pairings in Lemma 2.4,

$$\begin{align*}
(\text{tr } \xi_k) t &= C y_{n/k,n-k}(\sum u_i w_i) = C y_{n/k,n-k} t, \\
(\text{tr } \eta_k) t &= C y_{n/n-k,k}(\sum w_i \psi^{\otimes k}(u_i)) = C y_{n/n-k,k} t.
\end{align*}$$

Since $T_{\sigma} t = (-1)^{\ell(\sigma)}$ for each $\sigma \in S_n$, we have

$$y_{n/k,n-k} t = \sum_{\sigma \in D(S_n/\bar{S}_{k,n-k})} q^{k(n-k)-\ell(\sigma)} t = \sum_{\sigma \in D(S_n/\bar{S}_{k,n-k})} q^{\ell(\sigma)} t = \left[\begin{array}{c} n \\ k \end{array}\right]_q t.$$ 

Evaluation of $y_{n/n-k,k} t$ gives the same result. Hence $\text{tr } \xi_k = \text{tr } \eta_k = C \left[\begin{array}{c} n \\ n-k \end{array}\right]_q$. \hfill $\square$

**Corollary 3.10.** Let $\xi_k$ be defined as in Proposition 3.9. Then

$$\text{tr } \xi_{n-k} = q^{(n-2k)(n+1)/2} \text{tr } \xi_k.$$ 

**Proof.** This follows from the formulas for traces in Proposition 3.9 in view of the equality $\left[\begin{array}{c} n \\ n-k \end{array}\right]_q = \left[\begin{array}{c} n \\ k \end{array}\right]_q$. \hfill $\square$

**Corollary 3.11.** Suppose that $n$ is odd, say $n = 2m + 1$, and $\theta = \lambda \psi$ where $\lambda \in k$. If the characteristic of $k$ does not divide the dimension of $\Upsilon^{(m)}$, then $\lambda = q^{m+1}$. 

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Proof. Since $\psi^{-1}\theta = \lambda \cdot \text{Id}$, we have $\text{tr} \xi_k = \lambda^k \dim \Upsilon^{(k)}$ for each $k$. Hence for $k = m$ the equality of Corollary 3.10 is written as

$$\lambda^{m+1} \dim \Upsilon^{(m+1)} = q^{m+1} \lambda^m \dim \Upsilon^{(m)}.$$  

Since the bilinear pairing $\beta_m : \Upsilon^{(m)} \times \Upsilon^{(m+1)} \to \mathbb{k}$ is nondegenerate, the two spaces here have equal dimension, and the conclusion follows. □

Proposition 3.12. Suppose that $y_n V^\otimes n \neq 0$. Let $f \in T_n(V)^*$ be a linear function such that

$$\text{Ker } f = \{ u \in T_n(V) \mid y_n u = 0 \}. $$

Put $I' = \bigoplus I'_k$ where

$$I'_k = \{ u \in T_k(V) \mid f(uw) = 0 \text{ for all } w \in T_{n-k}(V) \}, \quad k = 0, \ldots, n.$$ 

Then $f$ satisfies the twisted cyclicity condition

$$f(uw) = f(\varphi(v)w), \quad v \in V, \ w \in T_{n-1}(V),$$

and $I'$ is a graded ideal of $T(V)$ such that $I'_0 = I'_1 = 0, I'_2$ coincides with the component $I_2$ of the ideal $I$ defining the algebra $\Lambda(V, R)$, and $T(V)/I'$ is a Frobenius algebra isomorphic to $\Upsilon(V, R)$. If $[n]!_q \neq 0$, then $I' = I$.

Proof. By Proposition 2.6 the algebra $\Upsilon(V, R)$ is generated by $\Upsilon^{(1)} = V$. Hence

$$\Upsilon(V, R) \cong T(V)/K$$

where $K$ is the kernel of the homomorphism $T(V) \to \Upsilon(V, R)$ described in Lemma 2.2. This homomorphism is defined by the actions of the antisymmetrizers $y_k$. In particular, $\text{Ker } f$ coincides with the $n$th homogeneous component of the ideal $K$. So it follows that $f$ is exactly the linear function, unique up to a scalar multiple, associated with the algebra $\Upsilon(V, R)$, as discussed at the beginning of this section. It satisfies the twisted cyclicity, as stated. Since $I'_k$ is the left radical of the pairing

$$T_k(V) \times T_{n-k}(V) \to \mathbb{k}, \quad (u, w) \mapsto f(uw),$$

we deduce that $I' = K$. Since $y_0 = y_1 = 1$, we get $I'_0 = I'_1 = 0$. By Lemma 2.2 $I \subset I'$. We have $I'_2 = \{ u \in V^\otimes 2 \mid y_2 u = 0 \} = I_2$ since $y_2$ acts on $V^\otimes 2$ by means of the linear operator $q \cdot \text{Id} - R$.

Suppose that $[n]!_q \neq 0$. Then $I'_k = I_k$ for all $k \leq n$ by Lemma 1.1. In particular, $\dim \Lambda_n(V, R) = 1$, and then $I_k = T_k(V)$ for all $k > n$, as is seen in Theorem 2.8 (in the trivial case $\dim V = 1$ the equality $\Upsilon^{(n+1)} = 0$ is possible only when $n = 1$ and $I_2 = I_2' = T_2(V)$). Hence $I'_k = I_k$ for all $k$. □

Under a restriction on $q$ the formulas of Proposition 3.6 can be rewritten in terms of the function $f$. We thus obtain a coordinate-free interpretation of Gurevich’s commutation formulas [12, Prop. 5.7]. In the next proposition we use a normalization of $f$ which permits us to consider the case $[n]_q = 0$. The coefficients in [12] correspond to a different normalization condition $f(t) = 1$ which requires $[n]!_q \neq 0$.  

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**Proposition 3.13.** Suppose that $[n-1]_q \neq 0$. Let $f \in \mathbb{T}_n(V)^*$ be the linear function defined by the rule

$$y_n u = [n-1]_q f(u)t \quad \text{for } u \in \mathbb{T}_n(V).$$

Then $f(t) = [n]_q$ and $f$ satisfies all conclusions of Proposition 3.12.

Writing $t = \sum x_i t_i$ for some elements $x_i \in V$ and $t_i \in \Upsilon(n^{-1})$, we have

$$\theta(v) = (-1)^{n-1}q \sum f(v t_i) \psi(x_i) \quad \text{and} \quad \overline{\theta}(v) = (-1)^{n-1}q \sum f(t_i v) x_i.$$

for all $v \in V$.

**Proof.** Let $v \in V$ and $w \in \Upsilon(n^{-1})$. Then $y_{n-1} w = [n-1]_q w$, and therefore

$$[n-1]_q f(vw)t = y_n(vw) = v \ast (y_{n-1} w) = [n-1]_q v \ast w = [n-1]_q \beta_1(v, w)t.$$

It follows that $f(vw) = \beta_1(v, w)$. Similarly, $f(uw) = \beta_{n-1}(w, v)$. Hence the formulas for $\theta$ and $\overline{\theta}$ repeat those of Proposition 3.6.

Since $\beta_1 \neq 0$, we see that $f \neq 0$. This means that $y_n V^\otimes n \neq 0$, and Proposition 3.12 does apply. That $f(t) = [n]_q$ is clear since $y_n t = [n]_q t$. □

**Remark.** Let $\tau : V^\otimes 2 \rightarrow V^\otimes 2$ be the usual flip of tensorands $v_1 \otimes v_2 \mapsto v_2 \otimes v_1$. Then $R^{op} = \tau R \tau$ is a Hecke symmetry with the same parameter $q$ such that the identity map on $V$ extends to antiisomorphisms of algebras

$$\Lambda(V, R) \rightarrow \Lambda(V, R^{op}), \quad \Upsilon(V, R) \rightarrow \Upsilon(V, R^{op}), \quad S(V, R) \rightarrow S(V, R^{op}).$$

When $R$ is replaced by $R^{op}$ the operators $\varphi, \psi, \theta$ change to $\varphi^{-1}, \psi^{-1}, \overline{\theta}$.

**4. Regular algebras of type $A$ and the Hessian group**

The Artin-Schelter regular graded algebras with 3 generators of degree 1 and 3 defining relations of degree 2 were classified by Artin, Tate and Van den Bergh [2] in terms of pairs $(E, \sigma)$ where $E$ is either the projective plane $\mathbb{P}^2$ or a cubic divisor in $\mathbb{P}^2$, and $\sigma$ is an automorphism of $E$. The point scheme $E$ parametrizes so-called point modules for the corresponding algebra, while $\sigma$ can be interpreted as transformation of point modules by shift of degrees. In this classification the algebras of type $A$ are distinguished by the properties that $E$ is a smooth cubic curve and $\sigma$ is a translation by a point of the abelian variety Pic$_0 E \cong E$ [2, 4.13].

Automorphisms of the graded algebra $A$ corresponding to $(E, \sigma)$ are extensions of the linear operators $\theta$ acting on the 3-dimensional component $V = A_1$ such that the induced transformation $\theta'$ of the projective plane $\mathbb{P}^2 = \mathbb{P}(V^*)$ leaves $E$ stable and the restriction of $\theta'$ to $E$ commutes with $\sigma$ [3, Prop. 8.8]. However, this description of automorphisms is not quite explicit, and moreover in the present paper we do not use the shift $\sigma$ and the whole geometric approach of [2].

For an algebra of type $A$ Proposition 4.1 describes automorphisms in terms of the Hessian group $G \subset PGL(V^*)$ of order 216. We refer to the book of Brieskorn and Knörrer [6] for details on this group. In the proof of Theorem 5.1 it will be
important that all operators $\theta$ whose images $\theta'$ lie in some conjugacy class of $G$ can be treated in the same way. This reduces the number of cases to consider.

In this section the base field $k$ will be assumed to be algebraically closed of characteristic $\neq 2, 3$. We will be concerned with a family of graded quadratic algebras introduced by Artin and Schelter [1]. An algebra $A$ in this family has 3 generators $x_1, x_2, x_3$ and 3 defining relations

$$ax_{i+1}x_{i-1} + bx_{i-1}x_{i+1} + cx_i^2 = 0, \quad i = 1, 2, 3,$$

where $(a, b, c) \in k^3 \setminus \{(0, 0, 0)\}$ is a triple of parameters and the indices 1, 2, 3 are viewed as elements of $\mathbb{Z}/3\mathbb{Z}$. Thus $A = T(V)/J$ where $V$ is the 3-dimensional vector space spanned by $x_1, x_2, x_3$ and $J$ is the ideal of $T(V)$ generated by $t_1, t_2, t_3$ where

$$t_i = ax_{i+1}x_{i-1} + bx_{i-1}x_{i+1} + cx_i^2 \in T_2(V).$$

This algebra is denoted $\text{SkK}_{3}(a, b, c)$ by association with Sklyanin’s elliptic algebras of global dimension 4.

The algebra $A$ is Artin-Schelter regular precisely when its quadratic dual $A^!$ is a Frobenius algebra. This happens when at least 2 of the parameters $a, b, c$ are nonzero and the two equalities $a^3 = b^3 = c^3$ do not hold simultaneously. We will always assume that these conditions are satisfied. In this case $J_2 V \cap VJ_2 \subset T_3(V)$ is the 1-dimensional space $(t)$ spanned by the tensor

$$t = \sum (ax_{i-1}x_1x_{i+1} + bx_{i-1}x_3x_{i+1} + cx_i^3) = \sum x_it_i = \sum t_ix_i.$$

The image of $t$ in the symmetric algebra $S(V) \cong k[x_1, x_2, x_3]$ is the polynomial

$$t^S = 3(a + b)x_1x_2x_3 + c(x_1^3 + x_2^3 + x_3^3).$$

The equation $t^S = 0$ defines a curve in the projective plane $\mathbb{P}(V^*)$ associated with the dual space $V^*$. This curve is smooth when $c \neq 0$ and $(a + b)^3 + c^3 \neq 0$. The point scheme of $A$ is the curve defined by a different equation

$$abc(x_1^3 + x_2^3 + x_3^3) = (a^3 + b^3 + c^3)x_1x_2x_3.$$

The 3-dimensional Artin-Schelter regular quadratic algebras of type $A$ are precisely the algebras $\text{SkK}_{3}(a, b, c)$ with a smooth point scheme. Here smoothness amounts to

$$abc \neq 0 \quad \text{and} \quad (a^3 + b^3 + c^3)^3 \neq 27a^3b^3c^3.$$

Note that

$$(a^3 + b^3 + c^3)^3 - 27a^3b^3c^3 = \prod_{i=1}^{3} \prod_{j=1}^{3} (\varepsilon^i a + \varepsilon^j b + c)$$

$$= (a + b)^3 + c^3)((a + \varepsilon b)^3 + c^3)((a + \varepsilon^2 b)^3 + c^3)$$

where $\varepsilon \in k$ is a primitive cube root of 1. Hence $(a + b)^3 + c^3 \neq 0$ for each algebra of type $A$, and the curve defined by the equation $t^S = 0$ is also smooth.
Consider the linear pencil $\mathcal{L}$ of cubic curves in $\mathbb{P}(V^*)$ defined by the equations
\[ \alpha x_1 x_2 x_3 + \beta (x_1^3 + x_2^3 + x_3^3) = 0, \quad \alpha, \beta \in k. \]

Singular curves in $\mathcal{L}$ are unions of 3 lines, while smooth ones are elliptic. The curves in $\mathcal{L}$ have 9 common points with homogeneous coordinates
\[
(0 : 1 : -1), \quad (0 : 1 : -\varepsilon), \quad (0 : 1 : -\varepsilon^2), \\
(-1 : 0 : 1), \quad (-\varepsilon : 0 : 1), \quad (-\varepsilon^2 : 0 : 1), \\
(1 : -1 : 0), \quad (1 : -\varepsilon : 0), \quad (1 : -\varepsilon^2 : 0)
\]
where $\varepsilon$ is a primitive cube root of 1, and these 9 points are precisely the inflection points of all nonsingular curves in $\mathcal{L}$.

The Hessian group $G$ consists of all projective linear transformations of $\mathbb{P}(V^*)$ permuting the 9 points listed above. If $g \in G$ and $C \in \mathcal{L}$, then $g(C)$ is a cubic passing through the same 9 points, and such a curve must be again a member of $\mathcal{L}$. Therefore $G$ can be described as the group of projective linear transformations permuting the curves in $\mathcal{L}$.

Suppose that $C \in \mathcal{L}$ is nonsingular. Then the Hessian group $G$ contains each transformation $g \in PGL(V^*)$ such that $g(C) = C$ since the set of inflection points of $C$ is invariant under automorphisms of the curve. In other words, the group
\[ G_C = \{ g \in PGL(V^*) \mid g(C) = C \} \]
is a subgroup of $G$. The index of $G_C$ in $G$ is equal to the cardinality of the orbit of $C$ with respect to the action of $G$ on $\mathcal{L}$. It is known that this number is 12 except when the $j$-invariant of $C$ is either 0 or 1728, in which cases there are, respectively, 4 and 6 curves in the orbit. Thus
\[ (G : G_C) = \begin{cases} 4 & \text{if } j(C) = 0, \\ 6 & \text{if } j(C) = 1728, \\ 12 & \text{otherwise.} \end{cases} \]

For convenience we make use of the canonical isomorphism $GL(V^*) \cong GL(V)$ which allows us to represent projective linear transformations of $\mathbb{P}(V^*)$ by linear operators acting on $V$.

Let $\mathbb{F}_3$ be the 3 element field. It is known that $G$ is isomorphic to the semidirect product of the additive group $\mathbb{F}_3^2$ by $SL_2(\mathbb{F}_3)$. Thus $G$ contains a normal subgroup $T$ of order 9. It is generated by 2 transformations represented, respectively, by the linear operators
\[ x_1 \mapsto x_2, \quad x_2 \mapsto x_3, \quad x_3 \mapsto x_1, \quad \text{and} \quad x_1 \mapsto \varepsilon x_1, \quad x_2 \mapsto \varepsilon^2 x_2, \quad x_3 \mapsto x_3. \]

Each curve in the pencil $\mathcal{L}$ is stable under the action of $T$, and one can see that smooth curves have no points fixed by a nontrivial element of $T$. This means that $T$ acts on each elliptic curve in $\mathcal{L}$ by translations. Accordingly, we call $T$ the subgroup of translations in $G$.

Now $G/T \cong SL_2(\mathbb{F}_3)$. The group $SL_2(\mathbb{F}_3)$ has a center of order 2. The nontrivial central element of $G/T$ has a representative in $GL(V)$ given by the assignments
\begin{align*}
x_1 & \mapsto x_2, \quad x_2 \mapsto x_1, \quad x_3 \mapsto x_3.
\end{align*}

Each curve in \(\mathcal{L}\) is stable under this transformation as well. Denote by \(Z\) the subgroup of \(G\) containing \(T\) such that \(Z/T\) is the center of \(G/T\). Then \(Z\) has order 18, and \(G/Z\) is a group of order 12 isomorphic to the alternating group on 4 letters. For each smooth curve \(C \in \mathcal{L}\) we get \(G_C = Z\) except for two cases when the stabilizer of \(C\) is larger:

\[
(G_C : Z) = \begin{cases} 
3 & \text{if } j(C) = 0, \\
2 & \text{if } j(C) = 1728.
\end{cases}
\]

Denote by \(\text{Aut } A\) the group of those automorphisms of \(A\) which preserve the grading of this algebra. Since \(A\) is generated by \(A_1 = V\), we may identify this group with a subgroup of \(GL(V)\).

**Proposition 4.1.** Suppose that the base field \(k\) is algebraically closed of characteristic \(\neq 2, 3\). Let \(a, b, c \in k\) be such that the algebra \(A = \text{Sk}_3(a, b, c)\) is Artin-Schelter regular and the corresponding cubic polynomial \(t^3 \in \mathcal{S}(V)\) is irreducible. So the curve \(C \subset \mathcal{P}(V^*)\) defined by the equation \(t^3 = 0\) is elliptic.

Then the image \(G_A\) of the group \(\text{Aut } A\) in \(\text{PGL}(V^*)\) is a subgroup of the Hessian group \(G\). If \(a \neq b\), then \(G_A = T\), the subgroup of translations in \(G\). If \(a = b\), then \(G_A = G_C\), the stabilizer of \(C\) in \(G\). In the latter case \(j(C) \neq 0\), and

\[
(G_A : T) = \begin{cases} 
4 & \text{if } j(C) = 1728, \\
2 & \text{otherwise}.
\end{cases}
\]

**Proof.** The tensor \(t\) is invariant under cyclic permutations of tensorands. It can be written as the sum \(t = t^+ + t^-\) of a symmetric and an alternating tensors. Explicitly,

\[
t^+ = \frac{a + b}{2} \sum_{\pi \in S_3} x_{\pi 1} x_{\pi 2} x_{\pi 3} + c \sum x_i^3, \quad t^- = \frac{a - b}{2} \sum_{\pi \in S_3} \text{sgn}(\pi)x_{\pi 1} x_{\pi 2} x_{\pi 3}.
\]

Since the space of defining relations \(J_2 = \langle t_1, t_2, t_3 \rangle\) is recovered from the tensor \(t\), a linear operator \(\theta \in GL(V)\) extends to an automorphism of \(A\) if and only if the 1-dimensional subspace \(\langle t \rangle \subset V^*\) spanned by \(t\) is stable under the operator \(\theta^\otimes 3\), if and only if \(t^+\) and \(t^-\) lie in one eigenspace of \(\theta^\otimes 3\). Since char \(k \neq 2, 3\), the space of symmetric tensors in \(V^*\) is isomorphic as a \(GL(V)\)-module to the homogeneous component \(S_3(V)\) of \(\mathcal{S}(V)\). Hence \(\theta^\otimes 3(t^+) \in \langle t^+ \rangle\) if and only if \(\theta t^3 \in \langle t^3 \rangle\). This condition on \(\theta\) means precisely that the curve \(C\) is invariant under the respective projective transformation \(\theta^\prime\). In other words, \(\theta^\prime \in G_C\). This shows that \(G_A \subset G_C\).

Note that \(t^- = 0\) if and only if \(a = b\). If \(a = b\), then \(t = t^+\). In this case \(t\) is stable under \(\theta^\otimes 3\) for any linear operator \(\theta \in GL(V)\) such that \(\theta^\prime \in G_C\). Therefore \(G_A = G_C\). In particular, \(T \subset G_A\). Since \(A\) is assumed to be Artin-Schelter regular, we have \(a \neq 0\) and \(a^3 \neq c^3\). Recall from \([2, (10.15)]\) that the cubic curve defined by the equation \(x_1^3 + x_2^3 + x_3^3 + 6\kappa x_1 x_2 x_3 = 0\) has \(j\)-invariant

\[
j = -2^{12} \cdot 3^3 (\kappa^3 - 1)^3 \kappa^3 / (8\kappa^3 + 1)^3.
\]

So \(j = 0\) if and only if either \(\kappa = 0\) or \(\kappa^3 = 1\). For the curve \(C\) here we have \(\kappa = a/c\), and it follows that \(j(C) \neq 0\). The index \((G_C : T)\) is twice the index \((G_C : Z)\) since \((Z : T) = 2\).
Suppose now that \( a \neq b \). In this case \( t^- \neq 0 \) and \( \theta t^- = (\det \theta)t^- \) for each linear operator \( \theta \in GL(V) \). In order that \( \langle t \rangle \) be stable under \( \theta^{S_3} \), it is necessary and sufficient that
\[
\theta t^S = (\det \theta)t^S.
\]
Thus \( G_A \) consists of all elements \( \theta' \in G \) whose representatives in \( GL(V) \) satisfy this condition. It does hold for the two linear operators representing generators of \( T \). Hence \( T \subset G_A \).

If \( G_A \neq T \), then we can find \( \theta' \in G_A \) with the property that the image of \( \theta' \) in \( G/T \cong SL_2(\mathbb{F}_3) \) has prime order, say \( p \). Since \( |G/T| = 24 \), there are two possibilities. If \( p = 2 \), then the coset \( \theta'T \) is the generator of the center \( Z/T \) of \( G/T \) (see the next lemma), and we must have \( Z \subset G_A \). However, the linear operator \( x_1 \mapsto x_2, \quad x_2 \mapsto x_1, \quad x_3 \mapsto x_3 \)
representing one of such elements has \( \det \theta = -1 \), while it fixes all polynomials in the 2-dimensional subspace \( L \subset S_3(V) \) spanned by \( x_1x_2x_3 \) and \( x_1^3 + x_2^3 + x_3^3 \). If \( p = 3 \), then \( \theta' \) and \( T \) generate a Sylow 3-subgroup of \( G \), say \( P \). We claim that in this case the eigenspace
\[
L_\theta = \{ f \in L \mid \theta f = (\det \theta)f \}
\]
is a 1-dimensional subspace of \( L \) spanned by a reducible polynomial, and therefore \( t^S \notin L_\theta \). Since Sylow 3-subgroups of \( G \) are conjugate to each other and since \( L \) is stable under all linear operators representing elements of \( G \), it suffices to check the claim assuming that \( P \) is one particular Sylow 3-subgroup. Moreover, since all elements of \( T \) are represented by linear operators acting on the whole \( L \) as the multiplications by their determinants, it suffices to consider any particular element \( \theta' \in P \) not lying in \( T \). So we take \( \theta' \) represented by the linear operator
\[
x_1 \mapsto \varepsilon x_1, \quad x_2 \mapsto x_2, \quad x_3 \mapsto x_3
\]
where \( \varepsilon \) is a primitive cube root of 1. Then \( \det \theta = \varepsilon \) and \( L_\theta = \langle x_1x_2x_3 \rangle \). Indeed, \( \theta \) has two eigenvalues on \( L \) since \( x_1^3 + x_2^3 + x_3^3 \) is fixed by \( \theta \). Thus we get a contradiction both for \( p = 2 \) and for \( p = 3 \). \( \square \)

**Lemma 4.2.** All elements of order 2 in \( G \) are conjugate to each other, and the same holds for the elements of order 4. The nonidentity elements of \( T \) are conjugate to each other in the group \( G \).

**Proof.** We view elements \( g \in SL_2(\mathbb{F}_3) \) as linear transformations of the vector space \( \mathbb{F}_3^2 \). If \( g \) has order 2, then \( g^2 = \text{Id} \). Such an operator is diagonalizable with eigenvalues equal to \(-1\) or \(1\). Since \( g \neq \text{Id} \) and \( \det g = 1 \), we must have \( g = -\text{Id} \). In other words, there is only one element of order 2 in \( SL_2(\mathbb{F}_3) \). Suppose that \( g \) has order 4. Then \( g^2 = -\text{Id} \). Taking any nonzero vector \( w_1 \in \mathbb{F}_3^2 \) and \( w_2 = gw_1 \), we get \( gw_2 = -w_1 \). This means that \( g \) has matrix
\[
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\]
in a suitable basis of \( \mathbb{F}_3^2 \). Any two such operators are conjugate in \( GL_2(\mathbb{F}_3) \), but in fact they are conjugate in \( SL_2(\mathbb{F}_3) \). This follows from the fact that the centralizer of \( g \) in \( GL_2(\mathbb{F}_3) \) contains a linear operator with determinant \(-1\), e.g., the operator with matrix
\[
\begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix}
\]
in the previously considered basis of \( \mathbb{F}_3^2 \). Thus the elements of order 4 form one conjugacy class in \( SL_2(\mathbb{F}_3) \).

Let \( \pi : G \to SL_2(\mathbb{F}_3) \) be the homomorphism with kernel \( T \). If \( g \in G \) has order 2 or 4, then \( \pi(g) \) has the same order. Since no nonzero vector in \( \mathbb{F}_3^2 \) is fixed by \( \pi(g) \), we have \( gtg^{-1} \neq t \), and so \( gt \neq tg \), for all \( t \in T \). But then the map \( t \mapsto t^{-1}gt \) gives a bijection of \( T \) onto \( gT \). Hence the conjugacy class of \( g \) contains the coset \( gT \). It follows that \( g \) is conjugate to some element \( h \in G \) if and only if \( \pi(g) \) is conjugate to \( \pi(h) \). Thus in \( G \) there is one conjugacy class containing all elements of order 2 and one conjugacy class containing all elements of order 4.

The claim concerning conjugacy of elements of \( T \) follows from the fact that \( SL_2(\mathbb{F}_3) \) acts transitively on nonzero vectors of \( \mathbb{F}_3^2 \).

**Corollary 4.3.** Each element \( \theta' \in G_A \) has order \( \leq 4 \), and if the order of \( \theta' \) is 3, then \( \theta' \in T \).

**Proof.** If \( \theta' \notin T \), then it follows from Proposition 4.1 that the coset \( \theta'T \in G/T \) has order 2 or 4. Since \( |T| = 9 \) is prime to 2, such a coset contains an element of order 2 or 4. But we have seen in the proof of Lemma 4.2 that all elements in such a coset belong to one conjugacy class, and so they have the same order. □

**Lemma 4.4.** The Hessian group \( G \) acts on the projective plane \( \mathbb{P}^2 \) in such a way that any linear operator \( \tau \in GL(V) \), representing an element \( \tau' \in G \) extends to an isomorphism \( \mathrm{Skl}_3(a, b, c) \to \mathrm{Skl}_3(\tilde{a}, \tilde{b}, \tilde{c}) \) whenever \( \tau'(a : b : c) = (\tilde{a} : \tilde{b} : \tilde{c}) \). If \( R \) is a Hecke symmetry on \( V \) such that \( S(V, R) = \mathrm{Skl}_3(a, b, c) \), then
\[
R_{\tau} = (\tau \otimes \tau) \circ R \circ (\tau^{-1} \otimes \tau^{-1})
\]
is a Hecke symmetry satisfying \( S(V, R_{\tau}) = \mathrm{Skl}_3(\tilde{a}, \tilde{b}, \tilde{c}) \), and if \( \theta \) and \( \theta' \) are the linear operators given by Proposition 3.1 for \( R \) and for \( R_{\tau} \), then \( \theta_{\tau} = \tau \theta \tau^{-1} \).

**Proof.** Let \( W \subset T_3(V) \) be the subspace with a basis formed by the 3 tensors
\[
w_1 = \sum x_ix_{i+1}x_{i+1}, \quad w_2 = \sum x_{i+1}x_ix_{i-1}, \quad w_3 = \sum x_i^3.
\]
We have \( W = W^+ \oplus W^- \) where \( W^+ \) is the intersection of \( W \) with the subspace of symmetric tensors, and \( W^- \) is the 1-dimensional subspace of alternating tensors in \( T_3(V) \). The subspace \( W^+ \) is mapped isomorphically onto the 2-dimensional subspace \( L \subset S_3(V) \) spanned by the polynomials \( x_1x_2x_3 \) and \( x_1^3 + x_2^3 + x_3^3 \). Since \( L \) is stable under the action of \( \tau \) on \( S_3(V) \), while \( W^- \) is a \( GL(V) \)-submodule of \( T_3(V) \), we get \( \tau^S(W) = W \). Hence there is a homomorphism \( G \to PGL(W) \) which defines an action of \( G \) on \( \mathbb{P}^2 = \mathbb{P}(W) \).

Now \( a, b, c \) are the coordinates of the tensor \( t = aw_1 + bw_2 + cw_3 \) associated with the algebra \( \mathrm{Skl}_3(a, b, c) \) in the chosen basis of \( W \). We have \( \tau'(a : b : c) = (\tilde{a} : \tilde{b} : \tilde{c}) \)

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where \( \hat{a}, \hat{b}, \hat{c} \) are the coordinates of the tensor \( \tau^{\otimes 3}(t) \in W \). Since the space \( J_2 \) of defining relations of \( \text{Skl}_3(a, b, c) \) is the smallest subspace of \( T_2(V) \) with the property that \( t \in J_2 V \), the operator \( \tau^{\otimes 2} \) maps it onto a similar subspace defined for \( \hat{a}, \hat{b}, \hat{c} \). This means that \( \tau \) extends to an isomorphism \( \text{Skl}_3(a, b, c) \to \text{Skl}_3(\hat{a}, \hat{b}, \hat{c}) \).

Clearly \( R_\tau \) satisfies the same Hecke relation and the braid relation satisfied by \( R \), and \( \tau \) extends to an isomorphism \( \mathbb{S}(V, R) \to \mathbb{S}(V, R_\tau) \). Hence the defining relations of the algebra \( \mathbb{S}(V, R_\tau) \) are those of \( \text{Skl}_3(\hat{a}, \hat{b}, \hat{c}) \). Finally, the equality \( \theta_\tau = \tau \theta \tau^{-1} \) is deduced from the fact that the operator \( \tau^{\otimes 4} \) intertwines the two representations of \( \mathcal{H}_4 \) in \( T_4(V) \) arising from \( R \) and from \( R_\tau \).

**Remark.** If \( \tau \in GL(V) \) is such that \( \tau \otimes \tau \) commutes with \( R \), then the \( \tau \)-twist of \( R \) is the Hecke symmetry \( (\tau \otimes \text{Id}_V) \circ R \circ (\tau^{-1} \otimes \text{Id}_V) \). In [9] Hecke symmetries were determined up to a twist. However, we will not use such twists.

### 5. Algebras of type A are not associated with Hecke symmetries

The final result announced in the introduction is probably true over an arbitrary base field. However, it will be proved under the assumption that \( \text{char } k \neq 2, 3 \). When \( \text{char } k = 2 \) or \( 3 \), there are differences in the description of automorphisms of the Artin-Schelter regular algebras of type A. Moreover, over a field of characteristic 3 the elliptic curve with the \( j \)-invariant 0 does not admit a Hessian normal form, and therefore Artin-Schelter regular algebras with such a point scheme are not realized in the family of algebras \( \text{Skl}_3(a, b, c) \).

**Theorem 5.1.** Let \( R \) be a Hecke symmetry on a 3-dimensional vector space \( V \) over a field \( k \) of characteristic \( \neq 2, 3 \). Then \( \mathbb{S}(V, R) \) cannot be any Artin-Schelter regular algebra of type A.

**Proof.** Suppose that \( \mathbb{S}(V, R) = \text{Skl}_3(a, b, c) \) in some fixed basis \( x_1, x_2, x_3 \) of \( V \), and assume that this algebra is Artin-Schelter regular of type A. Extending the base field \( k \) we may assume it to be algebraically closed.

Recall that the space of defining relations of the \( R \)-symmetric algebra \( \mathbb{S}(V, R) \) is \( \Upsilon^{(2)} = \text{Im}(R - q \cdot \text{Id}) \). Hence

\[
\Upsilon^{(2)} = \langle t_1, t_2, t_3 \rangle, \quad \Upsilon^{(3)} = (\Upsilon^{(2)} \otimes V) \cap (V \otimes \Upsilon^{(2)}) = \langle t \rangle
\]

where \( t_1, t_2, t_3 \) and \( t \) are as in section 4.

The operator \( \theta \) introduced in Proposition 3.1 extends to an automorphism of \( \text{Skl}_3(a, b, c) \), and by Proposition 4.1 its image \( \theta' \) in \( PGL(V^*) \) lies in the Hessian group \( G \). By Lemma 4.4 there exists then a similar Hecke symmetry with a changed triple of parameters \( a, b, c \) and with \( \theta' \) changed to any element in the same conjugacy class of \( G \). Therefore we may assume \( \theta' \) to be any representative of this conjugacy class. By Lemma 4.2 and Corollary 4.3 there are at most 4 conjugacy classes of \( G \) that can contain \( \theta' \). Accordingly, there are 4 cases to consider.

In each case we first use Proposition 3.9 to determine possible values of \( q \). It will be seen that \( q \neq -1 \), i.e., \( [2]_q \neq 0 \). Then we will use the relations between \( t, f, \varphi, \psi, \theta \) established in Theorem 3.8 and Proposition 3.13. Note that \( \psi = \text{Id} \), and therefore \( \varphi = q^4 \theta^{-2} \). The operator \( \theta \) is expressed as follows:

\[
\theta(x_j) = q \sum_i f(x_j t_i) x_i.
\]
Any Hecke symmetry with the prescribed algebra \( \mathcal{S}(V,R) \) is completely determined by the function \( f : V^\otimes 3 \rightarrow \mathbb{k} \) which is not yet known. However, for each possible \( \theta \) we find the values \( f(x_1t_i) \) from the previous expression. Combining them with the twisted cyclicity condition

\[
f(v_1v_2v_3) = f(\varphi(v_3)v_1v_2), \quad v_1, v_2, v_3 \in V,
\]

we get a set of linear equations for the values of \( f \) at the degree 3 monomials in \( x_1, x_2, x_3 \). The functions obtained by solving these equations may depend on several additional parameters. However, fulfillment of all linear equations arising from Proposition 3.13 does not guarantee the existence of a Hecke symmetry, and we have to look further.

By Proposition 3.12 the generating space \( I_2 \) of the ideal \( I \subset T(V) \) defining the algebra \( \Lambda(V,R) \) is determined by \( f \) as

\[
I_2 = \{ u \in T_2(V) \mid f(uV) = 0 \} = \{ u \in T_2(V) \mid f(Vu) = 0 \}.
\]

The projection \( P \) of \( V^\otimes 2 \) onto \( \Upsilon^{(2)} \) with respect to the direct sum decomposition \( V^\otimes 2 = I_2 \oplus \Upsilon^{(2)} \) is expressed as

\[
P(w) = \sum f(\bar{x}_i w)t_i, \quad w \in V^\otimes 2,
\]

where \( \bar{x}_1, \bar{x}_2, \bar{x}_3 \) is the basis of \( V \) dual to the basis \( t_1, t_2, t_3 \) of \( \Upsilon^{(2)} \) with respect to the pairing \( V \times \Upsilon^{(2)} \rightarrow \mathbb{k} \) induced by \( f \), i.e., \( f(\bar{x}_it_j) = 1 \) for \( i = j \) and 0 otherwise. Since \( I_2 \) and \( \Upsilon^{(2)} \) are the eigenspaces of \( R \) corresponding to the eigenvalues \( q \) and \(-1\), we get

\[
R = -P + q(\text{Id} - P) = q\text{Id} - (1 + q)P.
\]

The braid equation for \( R \) is equivalent to the following equation for \( P \):

\[
(P \otimes \text{Id}_V)(\text{Id}_V \otimes P)(P \otimes \text{Id}_V) - \frac{q}{(1 + q)^2}(P \otimes \text{Id}_V)
= (\text{Id}_V \otimes P)(P \otimes \text{Id}_V)(\text{Id}_V \otimes P) - \frac{q}{(1 + q)^2}(\text{Id}_V \otimes P).
\]

Note that \( P \otimes \text{Id} \) and \( \text{Id} \otimes P \) are projections (idempotent linear operators) mapping \( V^\otimes 3 \) onto \( \Upsilon^{(2)} \otimes V = \Upsilon^{(2,1)} \) and \( V \otimes \Upsilon^{(2)} = \Upsilon^{(1,2)} \), respectively. If the displayed equality is true, then the operators in the left and right hand sides of the formula must have images in \( \Upsilon^{(2,1)} \cap \Upsilon^{(1,2)} = \Upsilon^{(3)} \). Thus

\[
(\text{Id} \otimes P)(P \otimes \text{Id})u - q(1 + q)^{-2}u \in \Upsilon^{(3)} \quad \text{for all } u \in \Upsilon^{(1,2)}.
\]

This can be rephrased by saying that \( \text{Id} \otimes P \) and \( P \otimes \text{Id} \) induce linear maps between the vector spaces \( \Upsilon^{(2,1)}/\Upsilon^{(3)} \) and \( \Upsilon^{(1,2)}/\Upsilon^{(3)} \) which are inverse to each other up to a scalar multiple. This condition leads to a set of quadratic equations on the parameters involved in the construction of \( f \). It will be shown that in each case no solution is possible, and so the braid equation for \( R \) cannot be satisfied.
Case 1: $\theta'$ is the identity element. In this case $\theta = \lambda \text{Id}$ for some $\lambda \in \mathbb{k} \setminus \{0\}$. Then $\text{tr} \theta = 3\lambda$, and Proposition 3.9 yields $3\lambda = q(1 + q + q^2)$. By Corollary 3.11 $\lambda = q^2$. Hence $q(1 - 2q + q^2) = 0$, and we find $q = 1$. It follows that $\theta = \text{Id}$, and $\varphi = \text{Id}$ as well.

The expression of $\theta$ in terms of the function $f$ gives $f(x_j t_i) = 0$ for $i \neq j$, and $f(x_j t_i) = 1$ for all $i$. On the other hand, the cyclicity property $f(v_1 v_2 v_3) = f(v_3 v_1 v_2)$ allows us to write $$f(x_{i-1} t_i) = a f(x_{i-1} x_{i+1} x_{i-1}) + b f(x_{i-1} x_{i+1}) + c f(x_{i-1} x_i^2) = (a + b) f(x_{i+1} x_i^2) + c f(x_{i-1} x_i^2).$$

The equalities $f(x_{i-1} t_i) = 0$ give a system of linear equations for the values of $f$ at 3 monomials:

$$(a + b) f(x_2 x_3^2) + c f(x_3 x_1^2) = 0,$$

$$(a + b) f(x_3 x_1^2) + c f(x_1 x_2^2) = 0,$$

$$(a + b) f(x_1 x_2^2) + c f(x_2 x_3^2) = 0.$$ Its matrix of coefficients has determinant

$$\begin{vmatrix} a + b & c & 0 \\ 0 & a + b & c \\ c & 0 & a + b \end{vmatrix} = (a + b)^3 + c^3 \neq 0.$$ Hence the system admits only the zero solution, i.e., $f(x_{i-1} x_i^2) = 0$ for all $i$. Similarly, the equalities $f(x_{i+1} t_i) = 0$ force $f(x_{i+1} x_i^2) = 0$ for all $i$.

By cyclicity of $f$ we have $f(x_{i-1} x_{i+1} x_{i-1}) = a'$ and $f(x_{i+1} x_{i-1}) = b'$ for some $a', b' \in \mathbb{k}$ which do not depend on $i$. The equalities $f(x_j t_i) = 1$ are now written as

$$a a' + b b' + c f(x_i^3) = 0.$$ It follows that $f(x_i^3) = c'$ for all $i$ where $c' \in \mathbb{k}$ is such that $aa' + bb' + cc' = 1$.

The projection $P$ is given by the formula $P(w) = \sum f(x_i w) t_i$ for $w \in V^{\otimes 2}$, i.e.,

$$P(x_{i+1} x_{i-1}) = a' t_i, \quad P(x_{i-1} x_{i+1}) = b' t_i, \quad P(x_i^3) = c' t_i.$$ The projections $\text{Id} \otimes P$ and $P \otimes \text{Id}$ induce linear maps between 3 pairs of subspaces of $\mathbb{Y}^{(2,1)}$ and $\mathbb{Y}^{(1,2)}$. One pair is formed by the subspaces $\langle t_3 x_1, t_1 x_2, t_2 x_3 \rangle \subset \mathbb{Y}^{(2,1)}$ and $\langle x_1 t_3, x_2 t_1, x_3 t_2 \rangle \subset \mathbb{Y}^{(1,2)}$, and the matrices of linear maps in the respective bases are

$$\begin{pmatrix} ab' & ca' & bc' \\ bc' & ab' & ca' \\ ca' & bc' & ab' \end{pmatrix}, \quad \begin{pmatrix} ba' & cb' & ac' \\ ac' & ba' & cb' \\ cb' & ac' & ba' \end{pmatrix}.$$ Since $\langle x_1 t_3, x_2 t_1, x_3 t_2 \rangle \cap \mathbb{Y}^{(3)} = 0$, the composite map $(\text{Id} \otimes P)(P \otimes \text{Id})$, when restricted to $\langle x_1 t_3, x_2 t_1, x_3 t_2 \rangle$, must be a scalar multiplication. This leads to equations

$$bc b' c' + cac' a' + aba' b' = \kappa,$$

$$bca'^2 + cab'^2 + abc'^2 = 0,$$

$$a^2 b c' + b^2 c a' + c^2 a b' = 0.$$
where $\kappa = q(1 + q)^{-2} = 1/4$ (the precise value of $\kappa$ turns out to be irrelevant). The second pair of subspaces $\langle \tau_2 X_1, \tau_2 X_2, \tau_1 X_3 \rangle$ and $\langle \tau_1 X_2, \tau_3 X_3, \tau_3 X_1 \rangle$ produces the same result. Finally, the restrictions of $\text{Id} \otimes P$ and $P \otimes \text{Id}$, respectively, to the subspaces $\langle \tau_1 X_1, \tau_2 X_2, \tau_3 X_3 \rangle$ and $\langle \tau_1 X_1, \tau_2 X_2, \tau_3 X_3 \rangle$ give linear maps with the matrices

$$
\left( \begin{array}{ccc}
cc' & bb' & aa' \\
bb' & cc' & aa' \\
cc' & bb' & aa'
\end{array} \right), \quad \left( \begin{array}{ccc}aa' & bb' & cc' \\
bb' & cc' & aa' \\
cc' & bb' & cc'
\end{array} \right).
$$

Since $(\text{Id} \otimes P)(P \otimes \text{Id})(x_i t_i) - \kappa x_i t_i$ must be a scalar multiple of $t$, we get

$$
a^2 a'^2 + b^2 b'^2 + c^2 c'^2 - \kappa = b c b' c' + c a c' a' + a b a' b'.
$$

Combining the last equation with the previous three, we arrive at a system of three homogeneous quadratic equations

$$
\begin{align*}
bc a'^2 + ca b'^2 + ab c'^2 &= 0, \\
ab^2 b' c' + b^2 c' a' + c^2 a' b' &= 0, \\
2a^2 a'^2 + b^2 b'^2 + c^2 c'^2 &= 2bc b' c' + 2ca c' a' + 2ab a' b',
\end{align*}
$$

which we aim to solve in $a', b', c'$. Note that these are the same equations as in the two papers of Ohn [16, (10.5)] and [17, (5)]. A nonzero solution exists if and only if the resultant of the system vanishes. The resultant of the three polynomials

$$
F_1 = bc X^2 + ca Y^2 + ab Z^2, \\
F_2 = a^2 Y Z + b^2 Z X + c^2 X Y, \\
F_3 = a^2 X^2 + b^2 Y^2 + c^2 Z^2 - 2bc Y Z - 2ca Z X - 2ab X Y
$$

can be computed as a certain determinant of order 6 by the formula of Sylvester (see [11, Ch. 3, section 4D]). We need three other quadratic polynomials $D_1$, $D_2$, $D_3$ determined uniquely only modulo the linear span of $F_1$, $F_2$, $F_3$. One has

$$
D_1 = \det(l_{ij})_{1 \leq i, j \leq 3}
$$

where $l_{ij} \in \mathbb{k}$ and $l_{2j}$, $l_{3j}$ are linear forms such that $F_j = l_{1j} X^2 + l_{2j} Y + l_{3j} Z$. In the case considered here we may take

$$
D_1 = \begin{vmatrix}
bc & 0 & a^2 \\
ac & c^2 X + a^2 Z & b^2 Y - 2ab X - 2bc Z \\
ab Z & b^2 X & c^2 Z - 2ac X
\end{vmatrix} = 2abc(b^3 - c^3) X^2 + a^2 b(c^3 - a^3) Z^2 + b^2 c(a^3 - b^3) X Y + bc^2(2b^3 + c^3 - 3a^3) X Z.
$$

The polynomials $D_2$ and $D_3$ are defined similarly using expressions of $F_j$, respectively, as $l_{1j} X + l_{2j} Y^2 + l_{3j} Z$ and as $l_{1j} X + l_{2j} Y + l_{3j} Z^2$. For $D_2$, $D_3$ we take
By Sylvester’s formula the resultant \( \text{Res}(F_1, F_2, F_3) \) is equal to the determinant

\[
\begin{vmatrix}
bc & 0 & a^2 & 2abc(b^3 - c^3) & b^2c(a^3 - b^3) & 0 \\
ac & 0 & b^2 & 2abc(c^3 - a^3) & ac^2(b^3 - c^3) & 0 \\
ab & 0 & c^2 & a^2b(c^3 - a^3) & 0 & 2abc(a^3 - b^3) \\
0 & a^2 & -2bc & 0 & ac^2(b^3 - c^3) & ab^2(2a^3 + b^3 - 3c^3) \\
0 & b^2 & -2ac & bc^2(2b^3 + c^3 - 3a^3) & 0 & a^2b(c^3 - a^3) \\
0 & c^2 & -2ab & b^2c(a^3 - b^3) & a^2c(2c^3 + a^3 - 3b^3) & 0
\end{vmatrix}
\]

whose columns are composed of the coefficients of \( X^2, Y^2, Z^2, YZ, ZX, XY \) in \( F_1, F_2, F_3, D_1, D_2, D_3 \). A machine computation gives the following result:

\[
a^2b^2c^2(a^{18} + b^{18} + c^{18}) + 6a^2b^2c^2(a^{15}b^3 + a^{15}c^3 + a^3b^{15} + a^3c^{15} + b^{15}c^3 + b^3c^{15}) \\
+ 15a^2b^2c^2(a^{12}b^6 + a^{12}c^6 + a^6b^{12} + a^6c^{12} + b^{12}c^6 + b^6c^{12}) \\
+ 20a^2b^2c^2(a^9b^9 + a^9c^9 + b^9c^9) - 24a^5b^5c^5(a^9 + b^9 + c^9) \\
- 102a^5b^5c^5(a^6b^3 + a^6c^3 + a^3b^6 + a^3c^6 + b^6c^3 + b^3c^6) + 495a^8b^8c^8.
\]

Expressing this in contracted form, we conclude that

\[
\text{Res}(F_1, F_2, F_3) = a^2b^2c^2((a^3 + b^3 + c^3)^3 - 27a^3b^3c^3)^2 \neq 0.
\]

Thus we do not get any solution.

**Case 2: \( \theta' \) has order 3.** We may assume that

\[
\theta(x_i) = \lambda^i x_i, \quad i = 1, 2, 3,
\]

where \( \varepsilon \) is a primitive cube root of 1 and \( \lambda \in \mathbb{k} \setminus \{0\} \). Since \( \text{tr} \theta = 0 \), Proposition 3.9 shows that \( 1 + q + q^2 = 0 \), i.e., \( q \) is a primitive cube root of 1 too.

Since \( \theta^{\otimes 3}(t) = \lambda^3 t \), the identity \( \theta^{\otimes 3}(t) = \eta^9 \) yields \( \lambda^3 = \eta^6 \). Hence \( \lambda = \eta^2 \varepsilon \) where \( \omega \in \{1, \varepsilon, \varepsilon^2\} \). Renumbering the generators \( x_i \), we may assume that \( \omega = 1 \), and so \( \lambda = \eta^2 \). Then

\[
\varphi(x_i) = q^{-2} \eta^{-2}(x_i) = \varepsilon^{-2i} x_i = \varepsilon^i x_i.
\]

The cyclicity of \( f \) is expressed as \( f(wx_i) = \varepsilon^i f(x_i) \) for \( w \in T_2(V) \). In particular, \( f(x_3) = \varepsilon^i f(x_3) \) and \( f(x_i x_j x_k) = \varepsilon^{i+j+k} f(x_i x_j x_k) \). It follows that

\[
f(x_3) = f(x_2^3) = 0, \quad f(x_i x_j x_k) = 0 \quad \text{whenever } i + j + k \equiv 0 \pmod{3}.
\]

Let \( f(x_1 x_2 x_3) = f(x_3 x_1 x_2) = qa' \), \( f(x_2 x_1 x_3) = f(x_3 x_2 x_1) = qb' \), \( f(x_3) = qc' \). Then \( f(x_2 x_3 x_1) = \varepsilon qa' \) and \( f(x_1 x_3 x_2) = \varepsilon^2 qb' \).

The formula expressing \( \theta \) in terms of \( f \) shows that \( f(x_j t_i) = 0 \) whenever \( i \neq j \), and \( f(x_i t_i) = q \varepsilon^i \) for each \( i \). Therefore the projection \( P \) is written as
\[ P(w) = q^{-1} \sum \varepsilon^{-i} f(x_i w) t_i, \quad w \in T_2(V). \]

So
\[ P(x_1^2) = 0, \quad P(x_2 x_3) = \varepsilon^2 a' t_1, \quad P(x_3 x_2) = \varepsilon b' t_1, \]
\[ P(x_2^2) = 0, \quad P(x_3 x_1) = \varepsilon^2 a' t_2, \quad P(x_1 x_3) = \varepsilon b' t_2, \]
\[ P(x_3^2) = c' t_3, \quad P(x_1 x_2) = a' t_3, \quad P(x_2 x_1) = b' t_3. \]

The linear maps between \( \langle t_3 x_1, t_1 x_2, t_2 x_3 \rangle \) and \( \langle x_1 t_3, x_2 t_1, x_3 t_2 \rangle \) induced by the projections \( \text{Id} \otimes P \) and \( P \otimes \text{Id} \) have matrices
\[
\begin{pmatrix}
ab' & ca' & bc' \\
0 & \varepsilon cb' & \varepsilon^2 ca' \\
\varepsilon^2 ca' & 0 & \varepsilon ab'
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
ba' & cb' & ac' \\
0 & \varepsilon^2 ba' & \varepsilon cb' \\
\varepsilon cb' & 0 & \varepsilon^2 ba'
\end{pmatrix}.
\]

We compute
\[
(\text{Id} \otimes P)(P \otimes \text{Id})(x_1 t_3) = (ab' + \varepsilon bcb') x_1 t_3 + c^2 a'b' x_2 t_1 + \varepsilon^2 (bca'^2 + cab'^2) x_3 t_2.
\]

Since \( (\text{Id} \otimes P)(P \otimes \text{Id}) \) must be the multiplication by \( q(1 + q)^{-2} \), we deduce that \( bca'^2 + cab'^2 = c^2 a'b' = 0 \). Therefore \( a' = b' = 0 \) since \( abc \neq 0 \). But then
\[ (\text{Id} \otimes P)(P \otimes \text{Id})(x_1 t_3) = 0 \neq q(1 + q)^{-2} x_1 t_3, \]

a contradiction.

**Case 3:** \( a = b \) and \( \theta' \) has order 2. We may assume that
\[ \theta(x_1) = \lambda x_2, \quad \theta(x_2) = \lambda x_1, \quad \theta(x_3) = \lambda x_3 \]

for some \( \lambda \in k \setminus \{0\} \). Then \( t_1 \mapsto \lambda^2 t_2, t_2 \mapsto \lambda^2 t_1, t_3 \mapsto \lambda^2 t_3 \) under the action of \( \theta \circ 2 \). We see that \( \text{tr} \theta = \lambda \) and \( \text{tr} \theta \circ 2 |_{G(a)} = \lambda^2 \), whence
\[ \lambda = q(1 + q + q^2), \quad \lambda^2 = q^3(1 + q + q^2) \]

by Proposition 3.9. It follows that \( \lambda = q^2 = -1 \) and \( \varphi = q^4 \theta^{-2} = \text{Id} \).

The expression of \( \theta \) in terms of the function \( f \) yields
\[ f(x_1 t_2) = f(x_2 t_1) = f(x_3 t_3) = q^{-1} \lambda = q \]

and \( f(x_i t_i) = 0 \) whenever \( (i, j) \neq (1, 2), (2, 1), (3, 3) \). Using the cyclicity of \( f \) as in Case 1, we get
\[ 2af(x_3 x_1^2) + cf(x_1 x_2^2) = f(x_1 t_2) = q, \]
\[ 2af(x_1 x_2^2) + cf(x_2 x_3^2) = f(x_2 t_1) = 0, \]
\[ 2af(x_2 x_3^2) + cf(x_3 x_1^2) = f(x_3 t_1) = 0 \]

(note that \( 2a = a + b \)). This is a system of linear equations in 3 unknown values of \( f \). Its matrix of coefficients has determinant
\[ d = 8a^3 + c^3 = (a + b)^3 + c^3 \neq 0. \]
Hence the system admits a unique solution

\[ f(x_2 x_3^2) = -2q a c d^{-1}, \quad f(x_3 x_1^2) = 4q a^2 d^{-1}, \quad f(x_1 x_2^2) = q c^2 d^{-1}. \]

In a similar way the values \( f(x_2 t_1) = q, \ f(x_1 t_3) = f(x_3 t_2) = 0 \) lead to a system of equations yielding

\[ f(x_1 x_3^2) = -2q a c d^{-1}, \quad f(x_3 x_2^2) = 4q a^2 d^{-1}, \quad f(x_2 x_1^2) = q c^2 d^{-1}. \]

Next, setting \( f(x_1 x_2 x_3) = qa' \) and \( f(x_2 x_1 x_3) = qb' \), we get

\[
\begin{align*}
qa(a' + b') + cf(x_1^2) &= f(x_1 t_1) = 0, \\
qa(a' + b') + cf(x_2^2) &= f(x_2 t_2) = 0, \\
qa(a' + b') + cf(x_3^2) &= f(x_3 t_3) = q.
\end{align*}
\]

It follows that

\[ f(x_1^2) = f(x_2^2) = qc', \quad f(x_3^2) = qc'' \]

for \( c', c'' \in \mathbb{k} \) such that \( a(a' + b') + cc' = 0 \) and \( c'' = c' + c^{-1} \).

The projection \( P \) with \( \text{Im} \ P = \Upsilon^{(2)} \) and \( \text{Ker} \ P = I_2 \) is given by

\[ P(w) = q^{-1}(f(x_2 w)t_1 + f(x_1 w)t_2 + f(x_3 w)t_3). \]

So

\[
\begin{align*}
P(x_1^2) &= d^{-1}(c_1^2 t_1 + dc t_2 + 4a_1^2 t_3), \\
P(x_2^2) &= d^{-1}(dc' t_1 + c_2^2 t_2 + 4a_2^2 t_3), \\
P(x_3^2) &= d^{-1}(-2act_1 - 2act_2 + dc'' t_3), \\
P(x_2 x_3) &= d^{-1}(4a_1^2 t_1 + da' t_2 - 2act_3), \\
P(x_3 x_1) &= d^{-1}(da' t_1 + 4a_2^2 t_2 - 2act_3), \\
P(x_1 x_2) &= d^{-1}(c_1^2 t_1 + c_2^2 t_2 + da' t_3), \\
P(x_2 x_1) &= d^{-1}(c_1^2 t_1 + c_2^2 t_2 + db' t_3)
\end{align*}
\]

The projection \( \text{Id} \otimes P \) gives by restriction a linear map \( \Upsilon^{(2,1)} \to \Upsilon^{(1,2)} \). Its matrix with respect to the bases \( t_1 x_1, t_2 x_1, t_3 x_1, \ldots, t_3 x_3 \) and \( x_1 t_1, x_1 t_2, x_1 t_3, \ldots, x_3 t_3 \) of those two spaces is \( d^{-1} M \) where

\[
M = \begin{pmatrix}
  c^3 & daa' & a c^2 & c^3 & 4a^3 & dac' & dcb' & -2a^2 c & 4a^3 \\
dcc' & 4a^3 & a c^2 & c^3 & 4a^3 & dac' & a c^2 & -2a^2 c & 4a^3 \\
-2a^2 c & -2a^3 c & dcb' & daa' & -2a^2 c & 4a^3 & -2a^2 c & dac'' & -2a^2 c \\
adac' & c^3 & a c^2 & 4a^3 & dac' & a c^2 & -2a^2 c & 4a^3 & dac'' \\
-2a^2 c & daa' & 4a^3 & -2a^2 c & dac' & a c^2 & 4a^3 & dcb' & -2a^2 c \\
ac^2 & ac^2 & dac' & dac' & dac' & ac^2 & 4a^3 & dcb' & -2a^2 c \\
-2a^2 c & -2a^3 c & 4a^3 & dac' & dac' & dac' & ac^2 & 4a^3 & dcb' \\
dac' & 4a^3 & ac^2 & dac' & dac' & dac' & ac^2 & 4a^3 & dcb' \\
-2a^2 c & -2a^3 c & 4a^3 & dac' & dac' & dac' & ac^2 & 4a^3 & dac'
\end{pmatrix}
\]

The other projection \( P \otimes \text{Id} \) gives a map in the opposite direction with the matrix

\( d^{-1} N \) where \( N \) is obtained from \( M \) by interchanging \( a' \) and \( b' \).
This interrelation between $M$ and $N$ is explained by the fact that $d^{-1}N$ coincides with the matrix of the linear map $\Upsilon^{(2,1)} \to \Upsilon^{(1,2)}$ obtained by restriction of the linear operator $\text{Id}_V \otimes \tau P\tau$ where $\tau : V^\otimes 2 \to V^\otimes 2$ is the flip of tensorands. Note that all elements of $\Upsilon^{(2)}$ are fixed by $\tau$. The linear operator $\tau P\tau$ is the projection of $V^\otimes 2$ onto $\Upsilon^{(2)}$ corresponding to the Hecke symmetry $R^{op} = \tau R\tau$, and the latter is associated with the linear function $f^{op}$ on $V^\otimes 3$ whose values on the monomials in $x_1, x_2, x_3$ are given by the same formulas as the values of $f$, but with $a'$ and $b'$ interchanged. This observation explains also that whenever some relation between the parameters $a, c, a', b', c'$ must hold to satisfy the Hecke symmetry conditions, the companion relation with $a'$ and $b'$ interchanged must hold too.

Since $(\text{Id} \otimes P)(P \otimes \text{Id})u \equiv q(1 + q)^{-2}u \pmod{\Upsilon^{(3)}}$ for all $u \in \Upsilon^{(3)}$, the product $MN$ must have zero entries everywhere except the 1st, 5th, 9th rows and the principal diagonal. We will only need a few values in the 1st column of the matrix $MN$. Computing its entries in positions (2,1), (4,1), (7,1) and (8,1), we get

\[
\begin{align*}
\quad & d(cc' + ab')(c^3 + 4a^3) + 4a^3c^3 + (dab')^2 = 0, \\
\quad & d(aa' + cc')c^3 + 4a^3c + 4a^3(d(ab' + cc') + (dab')(dab')) = 0, \\
\quad & ac^3 + ac^2d(cc' + 2aa' + ab') + d^2a^2c'b = 0, \\
\quad & ac^5 + d^2acc^2 + 2a^4c^2 + ac^2(-dab' - daa') = 0.
\end{align*}
\]

Making substitutions $cc' + ab' = -aa'$ and $aa' + cc' = -ab'$ in the first two equalities, we rewrite them as

\[
\begin{align*}
(\text{d}aa' - c^3)(\text{d}aa' - 4a^3) & = 0, & (\text{d}aa' - c^3)(\text{d}ab' - 4a^3) & = 0.
\end{align*}
\]

As we have noted, the equalities remain true with $a'$ and $b'$ interchanged. Thus

\[
\begin{align*}
(\text{d}ab' - c^3)(\text{d}ab' - 4a^3) & = 0, & (\text{d}ab' - c^3)(\text{d}aa' - 4a^3) & = 0.
\end{align*}
\]

It follows that either $\text{d}aa' = d\text{ab}' = c^3$ or $\text{d}aa' = d\text{ab}' = 4a^3$. In any case $a' = b'$ since $d \neq 0$ and $a \neq 0$. Multiplying (3) by $a^{-1}c$ and substituting $b' = a'$, $cc' = -2aa'$, we arrive at

\[
(c^3 - d\text{aa}')c^3 + 2d\text{aa'}) = 0.
\]

If $\text{d}aa' \neq c^3$, then $\text{d}aa' = 4a^3$, but in this case $c_3 + 2d\text{aa'} = c^3 + 8a^3 = d \neq 0$, and we get a contradiction. Hence $\text{d}aa' = d\text{ab}' = c^3$ and $d\text{cc'} = -2c^3$. But then (4) reduces to $3ac^3 + 24a^4c^2 = 0$, i.e., $3ac^2d = 0$, again in contradiction with $ac \neq 0$ and $d \neq 0$.

**Case 4:** $a = b$ and $\theta'$ has order 4. We may assume that

\[
\theta(x_j) = \lambda \sum_{i=1}^{3} \varepsilon^{ij} x_i, \quad j = 1, 2, 3,
\]

where $\lambda \in \mathbb{R} \setminus \{0\}$ and $\varepsilon$ is a primitive cube root of 1. Since $a = b$, the tensor $t$ is symmetric, and its image in the symmetric algebra of $V$ is the polynomial

\[
t^S = c(x_1^3 + x_2^3 + x_3^3) + 6ax_1x_2x_3 \in \mathbb{S}(V).
\]

34
Put $\kappa = ac^{-1}$, so that $a = \kappa c$. The condition that $\theta$ is the degree 1 component of an automorphism of $A$ means precisely that $\theta$ sends $t^S$ to its scalar multiple. Since

$$\theta t^S = 3(c + 2a)\lambda^3(x_1^3 + x_2^3 + x_3^3) + 18(c - a)\lambda^3 x_1 x_2 x_3,$$

we deduce that $\kappa = (1 - \kappa)/(1 + 2\kappa)$, i.e., $2\kappa^2 + 2\kappa = 1$ (the curve defined by the equation $t^S = 0$ has then the $j$-invariant 1728). In this case $\theta t^S = (3 + 6\kappa)\lambda^3 t^S$, and therefore $\theta^{\otimes 3}(t) = (3 + 6\kappa)\lambda^3 t$. The identity $\theta^{\otimes 3}(t) = q^6 t$ entails

$$(3 + 6\kappa)\lambda^3 = q^6.$$

Making use of relations $\varepsilon^3 = 1$ and $1 + \varepsilon + \varepsilon^2 = 0$, we get

$$\theta^{\otimes 2}(t_j) = a \theta(x_{j+1}) \theta(x_{j-1}) + a \theta(x_{j-1}) \theta(x_{j+1}) + c \theta(x_j)^2$$

$$= \lambda^2 c \sum_{m=1}^3 \sum_{n=1}^3 (\kappa \varepsilon^{m(j+1)+n(j-1)} + \kappa \varepsilon^{m(j-1)+n(j+1)} + \varepsilon^{(m+n)j}) x_m x_n$$

$$= \lambda^2 c \sum_{i=1}^3 \varepsilon^{2ij} ((1 - \kappa)x_{i+1}x_{i-1} + (1 - \kappa)x_{i-1}x_{i+1} + (1 + 2\kappa)x_i^2)$$

$$= \lambda^2 (1 + 2\kappa) \sum_{i=1}^3 \varepsilon^{2ij} t_i.$$

Computation of traces gives $\text{tr} \theta = \lambda (1 + 2\varepsilon) = \lambda (\varepsilon - \varepsilon^2)$ and

$$\text{tr} \theta^{\otimes 2}|_{T(2)} = \lambda^2 (1 + 2\kappa)(1 + 2\varepsilon^2) = \lambda^2 (1 + 2\kappa)(\varepsilon^2 - \varepsilon).$$

By Proposition 3.9

$$\lambda(\varepsilon - \varepsilon^2) = q(1 + q + q^2), \quad \lambda^2 (1 + 2\kappa)(\varepsilon^2 - \varepsilon) = q^3 (1 + q + q^2).$$

Since $\lambda \neq 0$ and $\varepsilon \neq \varepsilon^2$, comparison of the last two equalities yields $(1 + 2\kappa)\lambda = -q^2$. Since $(1 + 2\kappa)^2 = 3$, it follows that

$$(3 + 6\kappa)\lambda^3 = ((1 + 2\kappa)\lambda)^3 = -q^6,$$

which contradicts a relation found earlier.

$\square$

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