A CONVERGENCE ANALYSIS OF THE PERTURBED COMPOSITIONAL GRADIENT FLOW: AVERAGING PRINCIPLE AND NORMAL DEVIATIONS

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Abstract. We consider in this work a system of two stochastic differential equations named the perturbed compositional gradient flow. By introducing a separation of fast and slow scales of the two equations, we show that the limit of the slow motion is given by an averaged ordinary differential equation. We then demonstrate that the deviation of the slow motion from the averaged equation, after proper rescaling, converges to a stochastic process with Gaussian inputs. This indicates that the slow motion can be approximated in the weak sense by a standard perturbed gradient flow or the continuous-time stochastic gradient descent algorithm that solves the optimization problem for a composition of two functions. As an application, the perturbed compositional gradient flow corresponds to the diffusion limit of the Stochastic Composite Gradient Descent (SCGD) algorithm for minimizing a composition of two expected-value functions in the optimization literatures. For the strongly convex case, such an analysis implies that the SCGD algorithm has the same convergence time asymptotic as the classical stochastic gradient descent algorithm. Thus it validates, at the level of continuous approximation, the effectiveness of using the SCGD algorithm in the strongly convex case.

1. Introduction. In this work we target at analyzing a system of two stochastic differential equations called the perturbed compositional gradient flow, which takes the form

\[
\begin{aligned}
&dy(t) = -\varepsilon y(t)dt + \varepsilon E_{w(t)}g_w(x(t))dt + \varepsilon \Sigma_1(x(t))dW^1_t, \quad y(0) = y_0, \\
&dx(t) = -\eta \nabla_{w(t)}g_w(x(t))\nabla_{f_v}(y(t))dt + \eta \Sigma_2(x(t), y(t))dW^2_t, \quad x(0) = x_0.
\end{aligned}
\]

(1)

Here \((w, v)\) follows a certain distribution on an index set \(D\); \(f_v : \mathbb{R}^m \to \mathbb{R}\) and \(g_w : \mathbb{R}^n \to \mathbb{R}^m\) are assumed to be in \(C^{(4)}\); the vector \(\nabla_{f_v}(y)\) is the gradient column \(m\)-vector of \(f_v\) evaluated at \(y\) and the matrix \(\nabla g_w(x)\) is the \(n \times m\) matrix formed

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by the gradient column \( n \)-vector of each of the \( m \) components of \( g_w \) evaluated at \( x; \varepsilon > 0 \) and \( \eta > 0 \) are two small parameters. We assume that the functions \( f \) and \( g \) are supported on some compact subsets of \( \mathbb{R}^m \) and \( \mathbb{R}^n \), respectively.

The two Brownian motions \( W^1_t \) and \( W^2_t \) are independent standard Brownian motions moving in the spaces \( \mathbb{R}^m \) and \( \mathbb{R}^n \), respectively. Here the diffusion matrix \( \Sigma_1(x) \) satisfies

\[
\Sigma_1(x)\Sigma_1^T(x) = \mathbb{E} \left[ (g_w(x) - \mathbb{E}g_w(x)) (g_w(x) - \mathbb{E}g_w(x))^T \right],
\]

and the diffusion matrix \( \Sigma_2(x, y) \) satisfies

\[
\Sigma_2(x, y)\Sigma_2^T(x, y) = \mathbb{E} \left[ (\nabla g_w(x) \nabla f_v(y) - \mathbb{E}\nabla g_w(x) \nabla f_v(y)) \cdot (\nabla g_w(x) \nabla f_v(y) - \mathbb{E}\nabla g_w(x) \nabla f_v(y))^T \right],
\]

and both matrices are assumed to be non-degenerate for any choice of \( x \in \mathbb{R}^n \) and \( y \in \mathbb{R}^n \).

1.1. Coupled fast–slow dynamics and averaging principle. It turns out that, by an appropriate choice of the step size parameters, the perturbed compositional gradient flow exhibits a fast–slow dynamics. To see this, we perform a change into

\[
t \rightarrow \frac{t}{\eta}
\]

and we know that the vectors \( X^{\varepsilon, \eta}(t), Y^{\varepsilon, \eta}(t) = (x(t/\eta), y(t/\eta)) \), that

\[
\begin{align*}
dY^{\varepsilon, \eta}(t) &= -\frac{\varepsilon}{\eta} Y^{\varepsilon, \eta}(t) dt + \frac{\varepsilon}{\eta} \mathbb{E} g_w(X^{\varepsilon, \eta}(t)) dt + \frac{\varepsilon}{\sqrt{\eta}} \Sigma_1(X^{\varepsilon, \eta}(t)) dW^1_t, \\
Y^{\varepsilon, \eta}(0) &= y_0, \\
dX^{\varepsilon, \eta}(t) &= -\mathbb{E} \nabla g_w(X^{\varepsilon, \eta}(t)) \nabla f_v(Y^{\varepsilon, \eta}(t)) dt + \sqrt{\eta} \Sigma_2(X^{\varepsilon, \eta}(t), Y^{\varepsilon, \eta}(t)) dW^2_t, \\
X^{\varepsilon, \eta}(0) &= x_0.
\end{align*}
\]

We will set the vectors

\[
B_1(X) = \mathbb{E} g_w(X) \in \mathbb{R}^m
\]

and

\[
B_2(X, Y) = -\mathbb{E} \nabla g_w(X) \nabla f_v(Y) \in \mathbb{R}^n,
\]

and the matrices

\[
A_1(X) = \Sigma_1(X)\Sigma_1^T(X) \in \mathbb{R}^{m} \otimes \mathbb{R}^{m},
\]

and

\[
A_2(X, Y) = \Sigma_2(X, Y)\Sigma_2^T(X, Y) \in \mathbb{R}^{n} \otimes \mathbb{R}^{n}.
\]

From our assumption on \( f \) and \( g \) we know that the vectors \( B_1(X) \), \( B_2(X, Y) \) and the matrices \( A_1(X) \) and \( A_2(X, Y) \) contain bounded coefficients together with their first derivatives, so that these quantities are also uniformly Lipschitz continuous with respect to their arguments.

One can write system (2) as

\[
\begin{align*}
dY^{\varepsilon, \eta}(t) &= -\frac{\varepsilon}{\eta} Y^{\varepsilon, \eta}(t) dt + \frac{\varepsilon}{\eta} B_1(X^{\varepsilon, \eta}(t)) dt + \frac{\varepsilon}{\sqrt{\eta}} \Sigma_1(X^{\varepsilon, \eta}(t)) dW^1_t, \quad Y^{\varepsilon, \eta}(0) = y_0, \\
dX^{\varepsilon, \eta}(t) &= B_2(X^{\varepsilon, \eta}(t), Y^{\varepsilon, \eta}(t)) dt + \sqrt{\eta} \Sigma_2(X^{\varepsilon, \eta}(t), Y^{\varepsilon, \eta}(t)) dW^2_t, \quad X^{\varepsilon, \eta}(0) = x_0.
\end{align*}
\]

\(^1\) A further discussion of this assumption is provided in Remark 3 of Section 5.
Fix $\varepsilon > 0$ and let $\eta \to 0$. Thus $\frac{\varepsilon}{\eta} \to \infty$ as $\eta \to 0$. Then system (7) is a standard fast–slow system of stochastic differential equations. In fact, the $Y$–component of the system (2) can be written as

$$dY^{\varepsilon, \eta}(t) = -\frac{\varepsilon}{\eta} Y^{\varepsilon, \eta}(t) dt + \frac{\varepsilon}{\eta} E_{g_w}(X^{\varepsilon, \eta}(t)) dt + \sqrt{\varepsilon} \left( \frac{\varepsilon}{\eta} \right)^{1/2} \Sigma_1(X^{\varepsilon, \eta}(t)) dW_t^1,$$

$$Y^{\varepsilon, \eta}(0) = y_0,$$

so that as $\frac{\varepsilon}{\eta} \to \infty$, the $Y$ motion is running at a fast speed the following Ornstein–Uhlenbeck process (OU process for short, see [21, Exercise 5.5])

$$dY^{\varepsilon, \eta}(t) = -\eta Y^{\varepsilon, \eta}(t) dt + E_{g_w}(X) dt + \sqrt{\varepsilon} \Sigma_1(X) dW^2_t, \quad Y^{\varepsilon, \eta}(0) = y_0.$$  

(8)

The invariant measure $\mu^{X^{\varepsilon, \eta}}(dY)$ of the (multidimensional) OU process $Y^{\varepsilon, \eta}(t)$ is a Gaussian measure with mean $E_{g_w}(X)$ and covariance matrix $\frac{\varepsilon}{2} \Sigma_1(X) \Sigma_1^T(X)$:

$$\mu^{X^{\varepsilon, \eta}}(dY) \sim \mathcal{N} \left( E_{g_w}(X), \frac{\varepsilon}{2} \Sigma_1(X) \Sigma_1^T(X) \right).$$  

(9)

Let us introduce the operator

$$q(X, Y) = q(X, Y) = \int_{\mathbb{R}^m} q(X, Y) \mu^{X^{\varepsilon, \eta}}(dY),$$  

(10)

where $q(X, Y)$ can be scalar, vector or matrix–valued functions with arguments $X$ and $Y$.

As the fast motion $Y^{\varepsilon, \eta}(t)$ process is running at a high speed, the process $X^{\varepsilon, \eta}(t)$ in (7) plays the role of the slow motion. That is to say, $X^{\varepsilon, \eta}(t)$ changes very little, and thus could be viewed as frozen, during a small time interval in which $Y^{\varepsilon, \eta}(t)$ is running very fast. Roughly speaking, in the dynamics of $X^{\varepsilon, \eta}(t)$, the fast component $Y^{\varepsilon, \eta}(t)$ can be replaced by the invariant measure of $\mu^{X^{\varepsilon, \eta}(t), \varepsilon}$ with frozen $X^{\varepsilon, \eta}(t)$. This heuristic supports the following asymptotic picture: as $\varepsilon > 0$ fixed and $\eta \to 0$, thus $\frac{\varepsilon}{\eta} \to \infty$, one can approximate the slow process $X^{\varepsilon, \eta}(t)$ in (7) by an averaged process $X^\varepsilon(t)$ satisfying

$$dX^\varepsilon(t) = B_2(X^\varepsilon(t), Y)(X^\varepsilon(t)) dt, \quad X^\varepsilon(0) = x_0.$$  

(11)

The approximation of $X^{\varepsilon, \eta}(t)$ by $X^\varepsilon(t)$ is the content of the classical averaging principle and was discussed in many literatures (see e.g. [16], [15], [10], Chapter 7). In this paper we will show that (see Proposition 1), as $\varepsilon > 0$ is fixed and set $\eta \to 0$, for $0 \leq t \leq T$ we have

$$\sup_{0 \leq t \leq T} E|X^{\varepsilon, \eta}(t) - X^\varepsilon(t)|^2_{\mathbb{R}^n} \to 0.$$  

(12)

This justifies the approximation of the averaged motion $X^\varepsilon(t)$ to the slow process $X^{\varepsilon, \eta}(t)$.

It turns out, as we will prove quantitatively in Lemma A.1 below, that when $\varepsilon \to 0$,

$$q(X, Y) - q(X, E_{g_w}(X)) \approx O(\sqrt{\varepsilon}).$$  

(13)

Therefore as $\varepsilon \to 0$, by (4) we see that

$$B_2(X, Y)(X) \approx -E \nabla g_w(X) \nabla f_v(E_{g_w}(X)) + O(\sqrt{\varepsilon}).$$
Thus as \( \varepsilon \to 0 \), the process \( X^\varepsilon(t) \) approximates another process \( \bar{X}(t) \) that solves an ordinary differential equation:

\[
d\bar{X}(t) = -\mathbf{E}\nabla g_w(\bar{X}(t)) \nabla f_v(\mathbf{E} g_w(\bar{X}(t))) dt , \quad \bar{X}(0) = x_0 ,
\]

with an error of \( O(\sqrt{\varepsilon}) \). In fact, equation (14) can be viewed as a gradient flow, which is the perturbed gradient flow with no stochastic noise terms [13]. Such averaging principle can hence explain why we call (1) the perturbed compositional gradient flow.

1.2. A sharper rate via normal deviation. One major drawback of the classical averaging principle is that, the approximation \( X^{\varepsilon,\eta}(t) \to X^\varepsilon(t) \) as \( \eta \to 0 \) in (12) can only identify the deterministic drift, and thus the small diffusion part in the equation for \( X^{\varepsilon,\eta}(t) \) vanishes as \( \eta \to 0 \). To overcome this difficulty, let us consider the deviation \( X^{\varepsilon,\eta}(t) - X^\varepsilon(t) \) and we rescale it by a factor of \( \sqrt{\eta} \). Thus we consider the process

\[
Z^{\varepsilon,\eta}(t) = \frac{X^{\varepsilon,\eta}(t) - X^\varepsilon(t)}{\sqrt{\eta}} .
\]

We will show that (see Proposition 2), as \( \eta \to 0 \), the process \( Z^{\varepsilon,\eta}(t) \) converges weakly to random process \( Z^\varepsilon t \). The process \( Z^\varepsilon t \) has its deterministic drift part and is driven by two mean 0 Gaussian processes carrying explicitly calculated covariance structures. This implies that, roughly speaking, from (15) we can expand

\[
X^{\varepsilon,\eta}(t) \overset{D}{\approx} X^\varepsilon(t) + \sqrt{\eta} Z^\varepsilon t ,
\]

as \( \eta \to 0 \). Here \( \overset{D}{\approx} \) means approximate equality of probability distributions. In fact, such approximate expansions have been introduced in the classical program under the context of stochastic climate models (see [12], [1, equation (4.8)]), and in physics this is also known as the Van Kampen’s approximation (see [26]).

Therefore by (13), (14) and (16) we know that the slow motion \( X^{\varepsilon,\eta}(t) \) in (7) (or (2)) has an expansion around the GD algorithm in (14):

\[
X^{\varepsilon,\eta}(t) \overset{D}{\approx} \bar{X}(t) + O(\sqrt{\varepsilon}) + \sqrt{\eta} Z^\varepsilon t .
\]

Let us introduce the process \( X^{\varepsilon,\eta}(t) \) as the following fast time-scale version of the perturbed gradient flow [13], [14]:

\[
dX^{\varepsilon,\eta}(t) = -\mathbf{E}\nabla g_w(X^{\varepsilon,\eta}(t)) \nabla f_v(\mathbf{E} g_w(X^{\varepsilon,\eta}(t))) dt + \sqrt{\eta} dZ^\varepsilon t , \quad X^{\varepsilon,\eta}(0) = x_0 .
\]

From (14) and (18), we know that

\[
\bar{X}(t) + \sqrt{\eta} Z^\varepsilon t \overset{D}{\approx} X^{\varepsilon,\eta}(t) + O(\sqrt{\varepsilon}) + O(\sqrt{\eta}) .
\]

So that by (17) we further have

\[
X^{\varepsilon,\eta}(t) \overset{D}{\approx} X^\varepsilon(t) + O(\sqrt{\varepsilon}) + O(\sqrt{\eta}) .
\]

From the perspective of mathematical techniques, there are two classical approaches to averaging principle and normal deviations. One is the classical Khasminskii’s averaging method [16]. This method chooses an intermediate time scale \( \Delta \to 0 \) such that \( \Delta \to \infty \). This intermediate time scale enables the analysis of averaging procedure by using a fast motion with frozen slow component. To

\[\text{Remark 1 of Section 5.}\]
demonstrate its effectiveness, in this work we exploit this method to do our averaging analysis. Another less intuitive method is the corrector method, which relies on the solution of an auxiliary Poisson equation. Upon obtaining appropriate a–priori estimates for this Poisson equation, one can reduce the averaging principle or normal deviations to the analysis of an Itô’s formula. Since we are working in the case when fast motion \( Y^\varepsilon,\eta(t) \) is an OU process, when applying the corrector method, we are mostly close to the set–up of [23] (see also [22, 24, 7]). Our analysis of the normal deviations will be following the corrector method and based on a–priori bounds provided in [23].

1.3. Connection with stochastic compositional gradient descent algorithm. In the field of statistical optimization, the stochastic composition optimization problem of the following form has been of tremendous interests in both theory and application:

\[
\min_x (E f_v \circ E g_w)(x). \tag{20}
\]

Here \( x \in \mathbb{R}^q, f \circ g \equiv f(g(x)) \) denotes the composite function, and \((v, w)\) denotes a pair of random variables. [29] has shown that the optimization problem (20) includes many important applications in statistical learning and finance, such as reinforcement learning, statistical estimation, dynamic programming and portfolio management.

Let us consider the following version of Stochastic Composite Gradient Descent (SCGD) algorithm in [29, Algorithm 1] whose iteration takes the form

\[
\begin{align*}
    y_{k+1} &= (1 - \varepsilon)y_k + \varepsilon g_{w_k}(x_k), \\
    x_{k+1} &= x_k - \eta \nabla g_{w_k}(x_k) \nabla f_{v_k}(y_{k+1}), & y_0 \in \mathbb{R}^m, \\
    x_0 \in \mathbb{R}^n. & \tag{21}
\end{align*}
\]

Here \((w_k, v_k)\) is taken as i.i.d. random vectors following some distribution \( \mathcal{D} \) over the parameter space; \(^3 f_{v_k} : \mathbb{R}^m \to \mathbb{R} \) and \( g_{w_k} : \mathbb{R}^n \to \mathbb{R}^m \) are functions indexed by the aforementioned random vectors; the vector \( \nabla f_{v_k}(y_{k+1}) \) is the gradient column \( m \)-vector of \( f_{v_k} \) evaluated at \( y_{k+1} \) and the matrix \( \nabla g_{w_k}(x_k) \) is the \( n \times m \) matrix formed by the gradient column \( n \)-vector of each of the \( m \) components of \( g_{w_k} \) evaluated at \( x_k \). The SCGD algorithm (21) is a provably effective method that solves (20); see early optimization literatures on the convergence and rates of convergence analysis in [8, 29]. However, the convergence rate of SCGD algorithm and its variations is not known to be comparable to its SGD counterpart [29, 30]. To drill further into this algorithm we consider the coupled diffusion process (1) which is a continuum version, as both \( \varepsilon, \eta \to 0 \) and \( \varepsilon/\eta \to \infty \), of the SCGD algorithm (21). We copy in below the perturbed compositional gradient flow (1) for convenience:

\[
\begin{align*}
    dy(t) &= -\varepsilon y(t)dt + \varepsilon E g_{w_u}(x(t))dt + \varepsilon \Sigma_1(x(t))dW^1_t, & y(0) = y_0, \\
    dx(t) &= -\eta E \nabla g_{w_u}(x(t))\nabla f_{v_u}(y(t))dt + \eta \Sigma_2(x(t), y(t))dW^2_t, & x(0) = x_0. & \tag{22}
\end{align*}
\]

Here \((w, v)\) is taken to be distributed as \( \mathcal{D} \), and \( f_{v} : \mathbb{R}^m \to \mathbb{R} \) and \( g_{w} : \mathbb{R}^n \to \mathbb{R}^m \) are assumed to be in \( C^4 \). Without loss of generality, when considering an optimization problem (20), we can assume that the functions \( f \) and \( g \) are supported on some compact subsets of \( \mathbb{R}^m \) and \( \mathbb{R}^n \), respectively\(^4\). Also for convenience, let us further assume that the \( w \) and \( v \) in the \((w, v)\)-pair drawn from \( \mathcal{D} \) are independent. We

\(^3\)Often in optimization for finite samples, the parameter space is chosen as some finite, discrete index set \( \{1, \ldots, N_1\} \times \{1, \ldots, N_2\} \), and \( \mathcal{D} \) is the uniform distribution over such index set. We extend this setting to any distribution over general parameter space.

\(^4\)See Remark 3 of Section 5.
do not believe this assumption is necessary, see discussions in [29, 30]; however it
does simplify our analysis since \((W_1^1, W^2_1)\) in the perturbed compositional gradient
flow \((22)\) can be chosen as an independent pair of Brownian motions which in turn
simplifies the proof.

Recall that \((X^{\varepsilon, \eta}(t), Y^{\varepsilon, \eta}(t)) = (x(t/\eta), y(t/\eta)).\) In the case where the objective
function \((E f_{\varepsilon} \circ E_{g_{\eta}})(x)\) is strongly convex, \(X^{\varepsilon, \eta}(t)\) in \((18)\) enters a basin containing
the minimizer of \((20)\) in finite time \(T > 0\), so that \((19)\) implies \(X^{\varepsilon, \eta}(t)\) in \((2)\) enters
a basin containing the minimizer of \((20)\) also in finite time \(T > 0\). Such heuristic
analysis validates, in the sense of convergence, the effectiveness of using the per-
turbed compositional gradient flow to solve \((20)\) in the strongly convex case. Such
argument can be generalized to the convex case and omitted due to the limitation
of space.

It is worth pointing out that in an early probability literature [23], the authors
have briefly mentioned in its introductory part the potential application of averag-
ing principle to the analysis of stochastic approximation algorithms. In contrast, in
the classical literature on stochastic approximation algorithms (see [4], [5], [19]),
the techniques of normal deviations have been addressed under the context of weak con-
vergence to diffusion processes in the discrete setting. For example, [4, Chap. 4, Part
II] analyzed the asymptotic behavior of a board class of single-equation adaptive
algorithms including SGD. Moreover, [19, Chap. 8] discussed the idea of multiple
timescale analysis for stochastic approximation algorithms; see also [5, Chap. 6]
for a connection to averaging principle for constant stepsiz algorithms. However,
these mathematical theories focus on the long-time asymptotic analysis instead of
convergence rates, which is vital in many recent applications. The current work
serves as an attempt on convergence rates using one algorithmic example (SCGD)
and can be viewed as a further contribution along this line of research thread.

**Organization.** The paper is organized as follows. In Section 2 we will show the
averaging principle that justifies the convergence of \(X^{\varepsilon, \eta}(t)\) to \(X^*(t)\) as \(\eta \to 0.\) In
Section 3 we will consider the rescaled deviation \(Z^{\varepsilon, \eta}_t = (X^{\varepsilon, \eta}(t) - X^*(t))/\sqrt{\eta}\)
and we show that as \(\eta \to 0\) it converges weakly to the process \(Z^*_t.\) This justifies \((16).\nIn Section 4 we show the approximation \((19)\) and we justify the effectiveness of
using SCGD in the strongly convex case. In Section 5 we discuss further problems,
remarks and generalizations.

**Notational Conventions.** For an \(n\)-vector \(v = (v_1, \ldots, v_n)\) we define the norm

\[ |v|_{\mathbb{R}^n} = (v_1^2 + \ldots + v_n^2)^{1/2}. \]

We also denote \([v]_k = v_k\) for \(k = 1, 2, \ldots, n.\) For any \(n \times n\) matrix \(\sigma \in \mathbb{R}^n \otimes \mathbb{R}^n,\) let
us define the norm

\[ \|\sigma\|_{\mathbb{R}^n \otimes \mathbb{R}^n} = \left( \sum_{i,j=1}^n \sigma_{ij}^2 \right)^{1/2}. \]

If \(q\) is a vector or a matrix, then \(|q|_{\text{norm}}\) denotes either \(|q|_{\mathbb{R}^n}\) when \(q\) is an \(n\)-vector,
or \(\|q\|_{\mathbb{R}^n \otimes \mathbb{R}^n}\) if \(q\) is an \(n \times n\) matrix. The standard inner product in \(\mathbb{R}^n\) is denoted
as \((\cdot, \cdot)_{\mathbb{R}^n}.\)

The spaces \(C^{(i)}(D), i = 0, 1, \ldots\) (and \(C(D) = C^{(0)}(D)\)) are the spaces of \(i\)-times
continuously differentiable functions on a domain \(D (D\) can be the whole space). For
a function \(f \in C^{(i)}(D)\) we define \(\|f\|_i\) to be the \(C^{(i)}(D)\) norm of \(f\) on \(D.\) In case we
need to highlight the target space, we also use \(C^{(i)}(D, M)\) that refers to functions
in the space $C^{(i)}(D)$ that are mapped into $M$. If $f \in \text{Lip}(D)$ is Lipschitz continuous on $D$, then $[f]_{\text{Lip}}$ is the Lipschitz seminorm $[f]_{\text{Lip}} = \sup_{x,y \in D} \frac{|f(x) - f(y)|_{\text{norm}}}{|x - y|_{\text{norm}}}$. In the case of vector or matrix valued functions, the Lipschitz norm is then defined to be the largest Lipschitz norm for its corresponding component functions.

Throughout the paper, capital $X(t), Y(t), X(t)$, etc., are quantities for the time rescaled process (2), and small $x(t), y(t), \tilde{x}(t)$, etc., are quantities for the original process (1). The constant $C$ denotes a positive constant that varies from line to line. Sometimes, to emphasize the dependence of this constant on other parameters, $C = C(\bullet)$ may also be used. For notational convenience, we use simultaneously, e.g., $X(t)$ or $X_t$ to denote a stochastic process.

2. The convergence of $X^{\varepsilon,\eta}(t)$ to $X^{\varepsilon}(t)$: Averaging principle. In this section we are going to show the convergence of $X^{\varepsilon,\eta}(t)$ to $X^{\varepsilon}(t)$ as $\eta \to 0$ by arguing as in the classical averaging principle (see [15, 16]).

Our first Lemma is about $L^2$–boundedness of the system $(X^{\varepsilon,\eta}_t, Y^{\varepsilon,\eta}_t)$ in (7).

Lemma 2.1. For any $T > 0$ and $0 < \eta < 1$ there exist some constant $C = C(T, \varepsilon) > 0$ such that

$$\sup_{0 \leq t \leq T} E|X^{\varepsilon,\eta}_t|^2_{\mathbb{R}^n} \leq C(1 + |x_0|^2_{\mathbb{R}^n}),$$

and

$$\sup_{0 \leq t \leq T} E|Y^{\varepsilon,\eta}_t|^2_{\mathbb{R}^n} \leq C(1 + |y_0|^2_{\mathbb{R}^n}).$$

Proof. This Lemma can be derived in the same way as in [6, Lemma 4.2]. In fact, we can write the equation (7) for $X^{\varepsilon,\eta}_t$ in an integral form as

$$X^{\varepsilon,\eta}_t = x_0 + \int_0^t B_2(X^{\varepsilon,\eta}_s, Y^{\varepsilon,\eta}_s)ds + \sqrt{\eta} \int_0^t \Sigma_2(X^{\varepsilon,\eta}_s, Y^{\varepsilon,\eta}_s)dW^2_s.$$

Therefore

$$E|X^{\varepsilon,\eta}_t|^2_{\mathbb{R}^n} \leq C \left(|x_0|^2_{\mathbb{R}^n} + E \left| \int_0^t B_2(X^{\varepsilon,\eta}_s, Y^{\varepsilon,\eta}_s)ds \right|^2_{\mathbb{R}^n} + \eta E \left| \int_0^t \Sigma_2(X^{\varepsilon,\eta}_s, Y^{\varepsilon,\eta}_s)dW^2_s \right|^2_{\mathbb{R}^n} \right).$$

For a matrix valued random function $\sigma(t) = \sigma(\omega, t)$ adapted to the filtration of $W_t$ we have (see [17, (3.12) and (3.13)])

$$E \left| \int_0^t \sigma(t)dW_t \right|^2_{\mathbb{R}^n} = \int_0^t E\|\sigma(t)\|^2_{\mathbb{R}^n \otimes \mathbb{R}^n} dt.$$

Therefore we obtain (23).

We can write the solution $Y^{\varepsilon,\eta}_t$ in (7) in mild form as

$$Y^{\varepsilon,\eta}_t = e^{-\tilde{\eta}t}y_0 + \frac{\varepsilon}{\eta} \int_0^t e^{-\tilde{\eta}(t-s)} B_1(X^{\varepsilon,\eta}_s)ds + \frac{\varepsilon}{\sqrt{\eta}} \int_0^t e^{-\tilde{\eta}(t-s)} \Sigma_1(X^{\varepsilon,\eta}_s)dW^1_s.$$

Set $\Gamma(t) = \frac{\varepsilon}{\sqrt{\eta}} \int_0^t e^{-\tilde{\eta}(t-s)} \Sigma_1(X^{\varepsilon,\eta}_s)dW^1_s$ and $\Lambda(t) = Y^{\varepsilon,\eta}_t - \Gamma(t)$. Then we have

$$d\Lambda(t) = -\frac{\varepsilon}{\eta} \Lambda(t) + B_1(X^{\varepsilon,\eta}_t) dt, \quad \Lambda(0) = y_0.$$
Proof. Let us first recall the auxiliary process in which the constant $C$ where the constant $C > 0$ so that

$$\int_0^t e^{\frac{\mu^2}{2} s} \Lambda_1(X^{\varepsilon, \eta}_s) ds \leq \frac{\varepsilon}{2\eta} |\Lambda(t)|^2_{\mathbb{R}^m} + C\varepsilon \frac{\eta}{\eta}.$$ 

Therefore by Gronwall inequality we know that for $0 \leq t \leq T$ we have

$$|\Lambda(t)|^2_{\mathbb{R}^m} \leq C e^{-\frac{\varepsilon}{2\eta} t} |y_0|_{\mathbb{R}^m}^2 + 2CT \leq C(1 + |y_0|_{\mathbb{R}^m}^2).$$

It remains to estimate $E|\Gamma(t)|^2_{\mathbb{R}^m}$. Again, by (25) we have

$$E |\Gamma(t)|^2_{\mathbb{R}^m} = \frac{\varepsilon^2}{\eta} e^{-\frac{\mu^2}{4} t} \int_0^t e^{\frac{\mu^2}{2} s} \|\Omega_1(X^{\varepsilon, \eta}_s, \eta)\|_{\mathbb{R}^n}^2 ds \leq \frac{C\varepsilon}{2}.$$ 

Thus we obtain

$$E|Y^{\varepsilon, \eta}_t|^2_{\mathbb{R}^m} \leq C(E |\Lambda(t)|^2_{\mathbb{R}^m} + E |\Gamma(t)|^2_{\mathbb{R}^m}) \leq C(1 + |y_0|_{\mathbb{R}^m}^2),$$

which is (24).

The next Lemma summarizes basic facts about the process $\eta^{X, \varepsilon}$ defined in (8).

**Lemma 2.2.** Let the process $\eta^{X, \varepsilon}(t)$ defined in (8) start from $\eta^{X, \varepsilon}(0) = y \in \mathbb{R}^m$. Then for any function $\varphi : \mathbb{R}^m \rightarrow \mathbb{R}$, for some $\delta > 0$ we have

$$\left|E \varphi(\eta^{X, \varepsilon}(t)) - \int_{\mathbb{R}^m} \varphi(Y) \mu^{X, \varepsilon}(dY)\right| \leq C e^{-\delta t} (1 + |y|_{\mathbb{R}^m}) |\varphi|_{Lip},$$

where the constant $C > 0$ may depend on $\varepsilon$, but is independent of $X$.

Moreover, for some constant $C > 0$ we have

$$E \left[ \frac{1}{T} \int_t^{t+T} B_2(X, \eta^{X, \varepsilon}(s)) ds - \int_{\mathbb{R}^m} B_2(X, Y) \mu^{X, \varepsilon}(dY) \right]_{\mathbb{R}^m}^2 \leq \frac{C}{T} \left[ \sqrt{\varepsilon} \mu \right]_{Lip} (1 + |y|_{\mathbb{R}^m}) + |B_2(X, 0)|_{\mathbb{R}^m}^2.$$ 

Finally

$$E|\eta^{X, \varepsilon}(t)|^2_{\mathbb{R}^m} \leq C(1 + e^{-2t} |y|_{\mathbb{R}^m}^2),$$

and

$$\int_{\mathbb{R}^m} |y|_{\mathbb{R}^m}^2 \mu^{X, \varepsilon}(dY) \leq C < \infty,$$

in which the constant $C$ may depend on $\varepsilon$ but is independent of $X$.

**Proof.** Let us first recall the auxiliary process $\eta^{X, \varepsilon}(t)$ in (8). From (8) we have

$$d(\eta^{X, \varepsilon}(t) - E g_w(X)) = -(\eta^{X, \varepsilon}(t) - E g_w(X)) dt + \sqrt{\varepsilon} \Sigma_1(X) dW^1_t,$$

so that

$$\eta^{X, \varepsilon}(t) - E g_w(X) = (\eta^{X, \varepsilon}(0) - E g_w(X)) e^{-t} + \sqrt{\varepsilon} \Sigma_1(X) \int_0^t e^{-(t-s)} dW^1_s.$$ 

Let $Z(t) = \int_0^t e^{-(t-s)} dW^1_s$ be the OU process satisfying the stochastic differential equation

$$dZ(t) = -Z(t) dt + dW^1_t,$$

$Z(0) = 0 \in \mathbb{R}^m$.

Thus we have the explicit representation

$$\eta^{X, \varepsilon}(t) = E g_w(X) + (\eta^{X, \varepsilon}(0) - E g_w(X)) e^{-t} + \sqrt{\varepsilon} \Sigma_1(X) Z(t).$$
Let $\mu(dY) \sim \mathcal{N} \left(0, \frac{1}{2}I_m\right)$ be the invariant measure of $Z(t)$, where $I_m$ is the identity matrix in $\mathbb{R}^m$. Then we have the exponential mixing estimate, that for $\delta > 0$ we have

$$\left| \mathbb{E} \varphi(Z(t)) - \int_{\mathbb{R}^m} \varphi(Y) \mu(dY) \right| \leq C e^{-\delta t} |\varphi|_{\text{Lip}(\mathbb{R}^m)}. \tag{32}$$

This, together with (31), as well as the boundedness of $\mathbb{E} g_w(X)$ in terms of $X$, imply (27).

From (32), by the same argument as in [6, Lemma 2.3], we obtain

$$\mathbb{E} \left| \frac{1}{T} \int_t^{t+T} B_2(X, Z(s))ds - \int_{\mathbb{R}^m} B_2(X, Y)\mu(dY) \right|^2_{\mathbb{R}^m} \leq \frac{C}{T} (|B_2|_{\text{Lip}} + |B_2(X, Y)|_{\mathbb{R}^m})^2. \tag{33}$$

From here, by making use of the representation (31), we obtain (28). Moreover, by (31) we infer that

$$\mathbb{E}| Y^x,\eta(t) |^2_{\mathbb{R}^m} \leq C(1 + e^{-2t} + e^{-2|\eta||X,\epsilon(0)|^2_{\mathbb{R}^m}}) + C e \mathbb{E}|Z(t)|^2_{\mathbb{R}^m} \leq C(1 + e^{-2t}) + \frac{C}{2}(1 - e^{-2t})$$

which is (29).

Finally, (30) is a result of (9) and the fact that $\mathbb{E} g_w(X)$ and $\Sigma_1(X)$ are uniformly bounded in $X$. \hfill \square

Now we will derive the averaging principle following the classical method in [16]. Let $T > 0$. Let us consider a partition of the time interval $[0, T]$ into intervals of the same length $\Delta > 0$. Let us introduce the auxiliary processes $\hat{Y}^x,\eta, \hat{X}^x,\eta$ by means of the relations

$$\hat{Y}^x,\eta_t = Y^x,\eta_t - \frac{\epsilon}{\eta} \int_{k\Delta}^{t} \hat{Y}^x,\eta_s ds + \frac{\epsilon}{\eta} \int_{k\Delta}^{t} B_1(X^x,\eta_s) ds + \frac{\epsilon}{\sqrt{\eta}} \int_{k\Delta}^{t} \Sigma_1(X^x,\eta_s) dW^1_s, \quad t \in [k\Delta, (k+1)\Delta], \tag{34}$$

$$\hat{X}^x,\eta_t = x_0 + \int_0^t B_2(X^x,\eta_s) ds + \sqrt{\eta} \int_0^t \Sigma_2(X^x,\eta_s) dW^2_s. \tag{35}$$

**Lemma 2.3.** The interval length $\Delta = \Delta(\eta)$ can be chosen such that $\eta^{-1}\Delta(\eta) \to \infty$, $\Delta(\eta) \to 0$ as $\eta \to 0$ and for any small $0 < \kappa < 1$ we have

$$\mathbb{E}| Y^x,\eta_t - \hat{Y}^x,\eta_t |^2_{\mathbb{R}^m} \leq C e^{2\epsilon \eta^{1-\kappa}} \to 0 \tag{36}$$

uniformly in $x_0 \in \mathbb{R}^n$, $y_0 \in \mathbb{R}^m$ and $t \in [0, T]$. 
Proof. In fact we can write, for \( t \in [k\Delta, (k+1)\Delta] \), that
\[
\mathbb{E}|Y_{t \epsilon}^\eta - \hat{Y}_{t \epsilon}^\eta|^2_{\mathbb{R}^m} = E \left[ \frac{\epsilon}{\eta} \int_{k\Delta}^t (Y_s^\epsilon - \hat{Y}_s^\epsilon) ds + \frac{\epsilon}{\eta} \int_{k\Delta}^t [B_1(X_s^{\epsilon, \eta}) - B_1(X_{k\Delta}^{\epsilon, \eta})] ds \
+ \frac{\epsilon}{\sqrt{\eta}} \int_{k\Delta}^t [\Sigma_1(X_s^{\epsilon, \eta}) - \Sigma_1(X_{k\Delta}^{\epsilon, \eta})] dW_s^1 \right]^2_{\mathbb{R}^m}.
\]
\[
= C\epsilon^2 \Delta \int_{k\Delta}^t \mathbb{E}|Y_s^\epsilon - \hat{Y}_s^\epsilon|^2_{\mathbb{R}^m} ds + C\epsilon^2 \Delta \int_{k\Delta}^t \mathbb{E}|X_s^\epsilon - X_{k\Delta}^\epsilon|^2_{\mathbb{R}^m} ds \
+ C\epsilon^2 \Delta \int_{k\Delta}^t \mathbb{E}|X_s^\epsilon - X_{k\Delta}^\epsilon|^2_{\mathbb{R}^m} ds.
\]
It follows from the boundedness of the coefficients of the stochastic equation (7) for \( X^{\epsilon, \eta} \) that we have
\[
\mathbb{E}|X_s^\epsilon - X_{k\Delta}^\epsilon|^2_{\mathbb{R}^m} \leq C|s - k\Delta|^2 \leq C\Delta^2,
\]
for \( 0 < \Delta < 1 \) and \( s \in [k\Delta, (k+1)\Delta] \). Therefore we have
\[
\mathbb{E}|Y_{t \epsilon}^\eta - \hat{Y}_{t \epsilon}^\eta|^2_{\mathbb{R}^m} \leq C\epsilon^2 \left( \frac{\Delta}{\eta^2} + \frac{1}{\eta} \right) \Delta^2 + C\epsilon^2 \Delta \int_{k\Delta}^t \mathbb{E}|Y_s^\epsilon - \hat{Y}_s^\epsilon|^2_{\mathbb{R}^m} ds.
\]
By Gronwall’s inequality this implies that we have, for each \( k \) and every \( t \in [k\Delta, (k+1)\Delta] \), that
\[
\mathbb{E}|Y_{t \epsilon}^\eta - \hat{Y}_{t \epsilon}^\eta|^2_{\mathbb{R}^m} \leq C\epsilon^2 \left( \frac{\Delta}{\eta^2} + \frac{1}{\eta} \right) \Delta^2 \exp \left( C\epsilon^2 \Delta^2 \right) .
\]
We then pick
\[
\Delta = \Delta(\eta) = \eta \sqrt{\ln(\eta^{-1})}
\]
and we conclude (36) by making use of the asymptotic that for any small \( a > 0 \) fixed, we have
\[
\frac{\ln \eta^{-1}}{\eta^a} \to 0, \quad \frac{\sqrt{\ln(\eta^{-1})}}{\ln(\eta^{-1})} \to 0 \quad \text{as} \quad \eta \to 0.
\]
In fact, we have
\[
C\epsilon^2 \left( \frac{\Delta}{\eta^2} + \frac{1}{\eta} \right) \Delta^2 \exp \left( C\epsilon^2 \Delta^2 \right) \leq C\epsilon^2 (\eta^{-\frac{3}{4}} + \eta^{-\frac{5}{4}}) \eta \cdot \eta^{-\frac{1}{4}} \exp(C\epsilon^2 \ln(\eta^{-a})) \
\leq C\epsilon^2 (\eta^{-\frac{3}{4}} + \eta^{-\frac{5}{4}}) \eta \cdot \eta^{-\frac{1}{4}} \eta^{-a} C\epsilon^2 ,
\]
and thus we can pick \( \kappa = \left( \frac{3}{4} + C\epsilon^2 \right) a > 0 \) to be any small positive number. \( \square \)

Lemma 2.4. For any small \( 0 < \kappa < 1 \) we have
\[
\sup_{0 \leq t \leq T} \mathbb{E}|X_t^{\epsilon, \eta} - \hat{X}_t^{\epsilon, \eta}|^2_{\mathbb{R}^n} \leq C(T^2 + T\epsilon^2 + 1)\eta^{1-\kappa} \to 0.
\]

Proof. By (7) we know that we can write the process \( X_t^{\epsilon, \eta} \) as an integral equation
\[
X_t^{\epsilon, \eta} = x_0 + \int_0^t B_2(X_s^{\epsilon, \eta}, Y_s^{\epsilon, \eta}) ds + \sqrt{\eta} \int_0^t [\Sigma_2(X_s^{\epsilon, \eta}, Y_s^{\epsilon, \eta})] dW_s^2 .
\]
Comparing (35) and (40) we see that
\[
\mathbb{E}|X_t^{\epsilon, \eta} - \hat{X}_t^{\epsilon, \eta}|^2_{\mathbb{R}^n} \leq C E \left[ \int_0^t [B_2(X_s^{\epsilon, \eta}, Y_s^{\epsilon, \eta}) - B_2(X_{s/\Delta}^{\epsilon, \eta}, \hat{Y}_s^{\epsilon, \eta})] ds \right]^2_{\mathbb{R}^n} \
+ \eta C E \left[ \int_0^t [\Sigma_2(X_s^{\epsilon, \eta}, Y_s^{\epsilon, \eta}) - \Sigma_2(X_{s/\Delta}^{\epsilon, \eta}, \hat{Y}_s^{\epsilon, \eta})] dW_s^2 \right]^2_{\mathbb{R}^n}.
\]
and so that, by (36), the Cauchy–Schwarz inequality and Lipschitz continuity of $B_2(X, Y)$ with respect to $X, Y$ we know that

$$\begin{align*}
\sup_{0 \leq t \leq T} E|X_t^{e, \eta} - \hat{X}_t^{e, \eta}|^2_{\mathbb{R}^n} & \leq CT \sum_{k=0}^{(k+1)\Delta} \left( E|X_s^{e, \eta} - X_{k\Delta}^{e, \eta}|^2 + E|Y_s^{e, \eta} - \hat{Y}_s^{e, \eta}|^2 \right) ds + C\eta + C\Delta \\
& \leq CT^2 \Delta^2 + CT \int_0^T E|Y_s^{e, \eta} - \hat{Y}_s^{e, \eta}|^2 ds + C\eta + C\Delta \\
& \leq CT^2 \Delta^2 + CT^2 \epsilon^2 \eta^{1-\kappa} + C\eta + C\Delta.
\end{align*}$$

From the asymptotic that for any small $a > 0$ fixed, we have $\frac{\ln \eta^{-1}}{\eta^{-a}} \to 0$, we have

$$\Delta \leq \eta^{1-\frac{\eta}{2}}.$$ 

This implies that

$$CT^2 \Delta^2 + CT^2 \epsilon^2 \eta^{1-\kappa} + C\eta + C\Delta \leq CT^2 \epsilon^2 \eta^{-\frac{\eta}{2}} + CT^2 \epsilon^2 \eta^{1-\kappa} + C\eta + C\eta^{1-\frac{\eta}{2}}.$$

Thus for possibly another small $\kappa > 0$

$$\sup_{0 \leq t \leq T} E|X_t^{e, \eta} - \hat{X}_t^{e, \eta}|^2_{\mathbb{R}^n} \leq C(T^2 + T^2 \epsilon^2 + 1)\eta^{1-\kappa} \to 0$$

as $\epsilon \to 0$. \hfill \Box

**Proposition 1.** For any $T > 0$ and $\epsilon > 0, \eta > 0$ small enough, for $0 \leq t \leq T$ and any small $0 < \kappa < 1$ we have

$$E|X_t^{e, \eta} - X_t^\ast|^2_{\mathbb{R}^n} \leq \frac{C}{\epsilon^{\frac{1}{2}} \sqrt{\ln(\eta^{-1})}},$$

for some constant $C = C(T) > 0$.

**Proof.** By the defining equation (34) of the process $\hat{Y}_t^{e, \eta}$ we know that we have $\hat{Y}_t^{e, \eta} = \eta X_t^{e, \eta}$ with $\eta_0 \hat{X}_t^{e, \eta} = Y_t^{e, \eta}$. This, together with Lemma 2.2 estimate (28) imply that

$$\begin{align*}
\frac{1}{\Delta^2} \left| \int_{k\Delta}^{(k+1)\Delta} \left[ B_2(X_s^{e, \eta}, \hat{Y}_s^{e, \eta}) - B_2(X_s^{e, \eta}, Y_s^{e, \eta}) \right] ds \right|^2_{\mathbb{R}^n} & \leq \frac{C}{\eta} \left( 1 + \epsilon + \epsilon E|X_{k\Delta}^{e, \eta}|^2_{\mathbb{R}^n} + \epsilon E|Y_{k\Delta}^{e, \eta}|^2_{\mathbb{R}^n} \right). 
\end{align*}$$

By making use of (36), (37), (39), (42), (82) we have

$$\begin{align*}
E|\hat{X}_t^{e, \eta} - X_t^\ast|^2_{\mathbb{R}^n} & = E \left| \int_0^t \left[ B_2(X_s^{e, \eta}, \hat{Y}_s^{e, \eta}) - B_2(X_s^{e, \eta}, Y_s^{e, \eta}) \right] ds + \sqrt{\eta} \int_0^t \Sigma_2(\hat{X}_s^{e, \eta}, \hat{Y}_s^{e, \eta}) dW_s^2 \right|^2_{\mathbb{R}^n} \\
& \leq CE \left| \sum_{k=0}^{[t/\Delta]} \int_{k\Delta}^{(k+1)\Delta} \left[ B_2(X_s^{e, \eta}, \hat{Y}_s^{e, \eta}) - B_2(X_s^{e, \eta}, Y_s^{e, \eta}) \right] ds \right|^2_{\mathbb{R}^n} \\
& \quad + CE \left| \int_0^t \left[ B_2(X_s^{e, \eta}, Y_s^{e, \eta}) - B_2(X_s^{e, \eta}, Y_s^{e, \eta}) \right] ds \right|^2_{\mathbb{R}^n} \\
& \quad + CE \left| \int_0^t \left[ B_2(X_s^{e, \eta}, Y_s^{e, \eta}) - B_2(X_s^{e, \eta}, \hat{X}_s^{e, \eta}) \right] ds \right|^2_{\mathbb{R}^n}.
\end{align*}$$
Corrector method. Similar techniques can be found in [18]. To apply this method, a-priori estimates of an auxiliary Poisson equation is needed, and there are various previous works dedicated to obtaining these estimates. In the paper [11], the authors considered the case when diffusion matrix is a scalar multiple of the identity matrix, and the Poisson equation there corresponds to hypo-elliptic diffusions. Our analysis relies more on estimates obtained in [23] (also see [22], [24]).

By using the corrector method in the next section, it is possible to remove \( \kappa \) in the estimate (41), and obtain a little bit better upper bound \( C \left( \frac{\eta^2}{\varepsilon^2} + \eta \right) \).

However, the Khasminskii’s method we use here is more intuitive. See Lemma 3.5 for a precise statement and proof. Also see Remark 2 in Section 5 for further discussion.

Remark 1. By using the corrector method in the next section, it is possible to remove \( \kappa \) in the estimate (41), and obtain a little bit better upper bound \( C \left( \frac{\eta^2}{\varepsilon^2} + \eta \right) \).

However, the Khasminskii’s method we use here is more intuitive. See Lemma 3.5 for a precise statement and proof. Also see Remark 2 in Section 5 for further discussion.

3. Normal deviations. In this section we consider normal deviations of the process \( X^{\varepsilon, \eta}(t) \) from the averaged motion \( X^\varepsilon(t) \). The method we use here is the corrector method. Similar techniques can be found in [18]. To apply this method, a-priori estimates of an auxiliary Poisson equation is needed, and there are various previous works dedicated to obtaining these estimates. In the paper [11], the authors considered the case when diffusion matrix is a scalar multiple of the identity matrix, and the Poisson equation there corresponds to hypo-elliptic diffusions. Our analysis relies more on estimates obtained in [23] (also see [22], [24]).
As in (15), we define
\[ Z^{\epsilon,\eta}(t) = \frac{X^{\epsilon,\eta}(t) - X^{\epsilon}(t)}{\sqrt{\eta}}. \]

By (7) and (11) we see that the process \( Z^{\epsilon,\eta}(t) \) satisfies the integral equation
\[ Z^{\epsilon,\eta}(t) = \frac{1}{\sqrt{\eta}} \int_0^t \left[ B_2(X^{\epsilon,\eta}_s, Y^{\epsilon,\eta}_s) - B_2(X^{\epsilon}_s, Y^{\epsilon}_s) \right] (X^{\epsilon}_s) ds + \int_0^t \Sigma_2(X^{\epsilon,\eta}_s, Y^{\epsilon,\eta}_s) dW_s^2. \]

Set
\[ U^{\epsilon,\eta}(t) = \frac{1}{\sqrt{\eta}} \int_0^t \left[ B_2(X^{\epsilon,\eta}_s, Y^{\epsilon,\eta}_s) - B_2(X^{\epsilon}_s, Y^{\epsilon}_s) \right] (X^{\epsilon}_s) ds \]
and
\[ V^{\epsilon,\eta}(t) = \int_0^t \Sigma_2(X^{\epsilon,\eta}_s, Y^{\epsilon,\eta}_s) dW_s^2. \]

Let us introduce the infinitesimal generator of the OU process \( \eta^{X,\epsilon}(t) \) in (8) as the operator
\[ \mathcal{L}^{X,\epsilon} f(Y) = \frac{\epsilon}{2} \nabla_Y \cdot (A_1(X) \nabla_Y f(Y)) + (B_1(X) - Y) \cdot \nabla_Y f(Y), \]
and consider the auxiliary Poisson equations
\[ \mathcal{L}^{X,\epsilon} u_k(X, Y) = [B_2]_k(X, Y) - \left[ B_2(X, Y') \right]_{k}(X) \]
for \( k = 1, 2, \ldots, n. \)

Since the variables \((X, Y) \in \mathbb{R}^n \times \mathbb{R}^m\), we are going to single out a unique solution \( u_k(X, Y) \) to (47) by putting a restriction that for each \( k = 1, 2, \ldots, n \)
\[ \int_{\mathbb{R}^n} u_k(X, Y) \mu^{X,\epsilon}(dY) = 0. \]

We have the following a–priori bounds for the solution \( u_k(X, Y) \) based on [23].

**Lemma 3.1.** The solution \( u_k(X, Y) \in C^{(2)}(\mathbb{R}^n \times \mathbb{R}^m) \). Moreover, there exist some integer \( p > 0 \) and constant \( C_k > 0 \) such that for each \( k = 1, 2, \ldots, n \) we have
\[ |u_k(X, Y)| + |\nabla_X u_k(X, Y)|_{\mathbb{R}^n} + |\nabla^2_X u_k(X, Y)|_{\mathbb{R}^n \times \mathbb{R}^m} + |\nabla_Y u(X, Y)|_{\mathbb{R}^m} \leq C_k (1 + |Y|_{\mathbb{R}^m}^p). \]

**Proof.** For simplicity of notations we will suppress the index \( k \) throughout the proof. Set \( f(X, Y) = - \left( [B_2]_k(X, Y) - \left[ B_2(X, Y') \right]_{k}(X) \right) \). The Poisson equation (47) can be written as
\[ \mathcal{L}^{X,\epsilon} u(X, Y) = -f(X, Y). \]

An explicit representation of the solution \( u(X, Y) \) can be formulated as
\[ u(X, Y) = \int_0^\infty dt \int_{\mathbb{R}^m} dY' f(X, Y') p_t(Y, Y'; X). \]
Here \( p_t(Y, Y'; X) \) is the (parabolic) fundamental solution (transition probability density) corresponding to the operator \( \mathcal{L}^{X,\epsilon} \), i.e.,
\[ \frac{\partial}{\partial t} p_t(Y, Y'; X) = \mathcal{L}^{X,\epsilon} p_t(Y, Y'; X), \quad p_0(Y, Y'; X) = \delta(Y' - Y). \]
The fact that $u_k(X, Y) \in C^{(2)}(\mathbb{R}^n \times \mathbb{R}^m)$ is a consequence of Theorem 3 in [23]. Let us define

$$p_t(Y, f; X) = \int_{\mathbb{R}^m} f(X, Y') p_t(Y, Y'; X) dY'$$

$$= \int_{\mathbb{R}^m} f(X, Y') [p_t(Y, Y'; X) - p_\infty(Y'; X)] dY' .$$

(51)

Here $p_\infty(Y; X)$ is the density function for the invariant measure $\mu^{X, \varepsilon}(dY)$ in (9). Notice that the way we define the averaging operator in (10) guarantees that

$$\int_{\mathbb{R}^m} f(X, Y') p_\infty(Y'; X) dY' = 0 ,$$

(52)

which thus leads to the validity of the second equality in (51).

From (50), and combining (51), we can write

$$u(X, Y) = \int_0^1 dt p_t(Y, f; X) + \int_1^\infty dt p_t(Y, f; X) .$$

(53)

Let us define $p^{(j)}_t(Y, Y'; X) = \nabla_X^j p_t(Y, Y'; X)$ to be the $j$-derivative of $p_t(Y, Y'; X)$ with respect to $X$, and it is a tensor with $j$ indices. In a similar fashion, from (51), we define $p^{(j)}_t(Y, f; X) = \nabla_X^j \left( \int_{\mathbb{R}^m} f(X, Y') p_t(Y, Y'; X) dY' \right)$. Thus $p^{(1)}_t(Y, f; X)$ is an $n$-dimensional vector, and $p^{(2)}_t(Y, f; X)$ is an $n \times n$ matrix. Thus we have

$$\nabla_X u(X, Y) = \int_0^1 dt p^{(1)}_t(Y, f; X) + \int_1^\infty dt p^{(1)}_t(Y, f; X) ,$$

(54)

$$\nabla_X^2 u(X, Y) = \int_0^1 dt p^{(2)}_t(Y, f; X) + \int_1^\infty dt p^{(2)}_t(Y, f; X) ,$$

(55)

$$\nabla_Y u(X, Y) = \int_0^1 dt \nabla_Y p_t(Y, f; X) + \int_1^\infty dt \nabla_Y p_t(Y, f; X) .$$

(56)

By the estimates (14) and (15) from Theorem 2 in [23], taking into account the centering condition (52), we see that for any $k > 0$ there exist $C, p > 0$ such that for all $t \geq 1$ we have

$$|p^{(1)}_t(Y, f; X)|_{\mathbb{R}^n} \leq C \frac{1 + |Y|_{\mathbb{R}^m}^p}{(1 + t)^k} ;$$

(57)

$$|p^{(2)}_t(Y, f; X)|_{\mathbb{R}^n \times \mathbb{R}^n} \leq C \frac{1 + |Y|_{\mathbb{R}^m}^p}{(1 + t)^k} ;$$

(58)

and

$$|\nabla_Y p_t(Y, f; X)|_{\mathbb{R}^m} \leq C \frac{1 + |Y|_{\mathbb{R}^m}^p}{(1 + t)^k} .$$

(59)

Now we consider standard estimates, including derivatives with respect to $X$, for the integral $\int_t^1 p_s(Y, f; X) ds$ which solves a Cauchy problem for a parabolic equation in the region $[0, 1] \times \mathbb{R}^m$ (with an initial value at $t = 1$). In fact, let

$$v(t, Y; X) := \int_t^1 p_s(Y, f; X) ds = \int_t^1 ds \int_{\mathbb{R}^m} f(X, Y') p_s(Y, Y'; X) dY' ,$$

(51)
then by using Duhamel’s principle we see that \( \tilde{v}(t, Y; X) = \tilde{v}(1 - t, Y; X) \), where \( \tilde{v}(s, Y; X) \) is the solution of the Cauchy problem

\[
\frac{\partial \tilde{v}}{\partial s} - \mathcal{L}^{X} \tilde{v} = f(X, Y) , \quad \tilde{v}(0, Y; X) = 0 .
\]

We can apply standard parabolic estimates to the solution \( \tilde{v} \) so that by taking into account the compactly supportedness of \( f(X, Y) \) in terms of \( X \) and \( Y \), we get

\[
|\nabla_{X} \tilde{v}(1, Y; X)|_{\mathbb{R}^{n}} + |\nabla_{Y}^{2} \tilde{v}(1, Y; X)|_{\mathbb{R}^{n} \otimes \mathbb{R}^{n}} + |\nabla_{Y} \tilde{v}(1, Y; X)|_{\mathbb{R}^{m}} \leq C < \infty .
\]

(60)

By applying (57), (58), (59) to the second integrals in (54), (55), (56), and (60) to the first integrals in (54), (55), (56), we conclude (49).

\[\Box\]

The next Lemma is a reproduction of Proposition 4.2 estimate (4.5) in [6].

**Lemma 3.2.** For any \( p \geq 1 \) and any \( T > 0 \), there exist constant \( C = C(p, T) > 0 \), such that for all \( x_{0} \in \mathbb{R}^{n}, y_{0} \in \mathbb{R}^{m} \) we have

\[
\int_{0}^{T} E|X_{t}^{\varepsilon, \eta}|_{\mathbb{R}^{n}}^{p} dt \leq C(1 + |y_{0}|_{\mathbb{R}^{m}}^{p}) .
\]

(61)

**Proof.** First of all, we can write the equation (7) for \( X_{t}^{\varepsilon, \eta} \) in an integral form as

\[
X_{t}^{\varepsilon, \eta} = x_{0} + \int_{0}^{t} B_{2}(X_{s}^{\varepsilon, \eta}, Y_{s}^{\varepsilon, \eta})ds + \sqrt{\eta} \int_{0}^{t} \Sigma_{2}(X_{s}^{\varepsilon, \eta}, Y_{s}^{\varepsilon, \eta})dW_{s}^{2}.
\]

Therefore we know that

\[
E|X_{t}^{\varepsilon, \eta}|_{\mathbb{R}^{n}}^{p} \leq C(p, t) \left( |x_{0}|_{\mathbb{R}^{n}}^{p} + E \left| \int_{0}^{t} B_{2}(X_{s}^{\varepsilon, \eta}, Y_{s}^{\varepsilon, \eta})ds \right|_{\mathbb{R}^{n}}^{p} + \eta^{p/2} E \left| \int_{0}^{t} \Sigma_{2}(X_{s}^{\varepsilon, \eta}, Y_{s}^{\varepsilon, \eta})dW_{s}^{2} \right|_{\mathbb{R}^{n}}^{p} \right) .
\]

By making use of the identity (25), as well as the Burkholder–Davis–Gundy inequality (see [25, Corollary IV.4.2]), we have

\[
E \left| \int_{0}^{t} \Sigma_{2}(X_{s}^{\varepsilon, \eta}, Y_{s}^{\varepsilon, \eta})dW_{s}^{2} \right|_{\mathbb{R}^{n}}^{p} \leq C(p, t) \left( \int_{0}^{t} E\left| \Sigma_{2}(X_{s}^{\varepsilon, \eta}, Y_{s}^{\varepsilon, \eta}) \right|_{\mathbb{R}^{n} \otimes \mathbb{R}^{n}}^{p} ds \right)^{p/2}.
\]

(62)

From the mild form (26) of the process \( Y_{t}^{\varepsilon, \eta} \) we have

\[
Y_{t}^{\varepsilon, \eta} = e^{-\frac{\varepsilon}{\eta} t} y_{0} + \frac{\varepsilon}{\eta} \int_{0}^{t} e^{-\frac{\varepsilon}{\eta}(t-s)} B_{1}(X_{s}^{\varepsilon, \eta})ds + \frac{\varepsilon}{\sqrt{\eta}} \int_{0}^{t} e^{-\frac{\varepsilon}{\eta}(t-s)} \Sigma_{1}(X_{s}^{\varepsilon, \eta})dW_{s}^{1}.
\]

Set \( \Gamma(t) = \frac{\varepsilon}{\sqrt{\eta}} \int_{0}^{t} e^{-\frac{\varepsilon}{\eta}(t-s)} \Sigma_{1}(X_{s}^{\varepsilon, \eta})dW_{s}^{1} \) and \( \Lambda(t) = Y_{t}^{\varepsilon, \eta} - \Gamma(t) \). Then we have

\[
d\Lambda(t) = -\frac{\varepsilon}{\eta} \Lambda(t) + B_{1}(X_{t}^{\varepsilon, \eta})dt , \quad \Lambda(0) = y_{0} ,
\]
which gives
\[
\frac{1}{p} \frac{d}{dt} |\Lambda(t)|_{R^n}^p = \left( \Lambda(t), -\frac{\epsilon}{\eta} |\Lambda(t) + B_1(X_{t}^{\epsilon, \eta})| \right)_{R^n}^p |\Lambda(t)|_{R^n}^{p-2} \\
\leq -\frac{\epsilon}{\eta} |\Lambda(t)|_{R^n}^p - \frac{\epsilon}{\eta} |\Lambda(t)|_{R^n}^{p-2} \Lambda(t), B_1(X_{t}^{\epsilon, \eta})|_{R^n} \\
\leq -\frac{\epsilon}{\eta} |\Lambda(t)|_{R^n}^p + \frac{\eta}{\eta} \left( |\Lambda(t)|_{R^n}^p + \frac{1}{p} |B_1(X_{t}^{\epsilon, \eta})|_{R^n}^p \right) \\
= -\frac{\epsilon}{\eta} |\Lambda(t)|_{R^n}^p + \frac{C_{p, \epsilon}}{\eta}.
\]

Therefore by Gronwall inequality we know that for \(0 \leq t \leq T\) we have
\[
|\Lambda(t)|_{R^n}^p \leq C e^{-\frac{\epsilon}{\eta} t} |y_0|_{R^n}^p + 2CT \leq C(1 + |y_0|_{R^n}^p).
\]

It remains to estimate \(E|\Gamma(t)|_{R^n}^p\). Again, by the identity (25) as well as the Burkholder–Davis–Gundy inequality, we have
\[
E|\Gamma(t)|_{R^n}^p = \frac{e^p}{\eta} \epsilon^{-2} \left( \int_0^t e^{\frac{2\epsilon}{\eta} s} E\|\Sigma_1(X_{s}^{\epsilon, \eta})\|_{2, R^n}^2 ds \right)^{p/2} \leq C \epsilon^{p/2}.
\]

Thus we obtain
\[
E|Y_{t}^{\epsilon, \eta}|_{R^n}^p \leq C(E|\Lambda(t)|_{R^n}^p + E|\Gamma(t)|_{R^n}^p) \leq C(1 + |y_0|_{R^n}^p),
\]
which leads to (61) by integrating on \([0, T]\).

The next Lemma is about how averaging principle is used to evaluate integrals.

**Lemma 3.3.** Let \(K(X, Y)\) be a Lipschitz continuous function in \(X, Y\) such that \(|K(X_1, Y_1) - K(X_2, Y_2)| \leq C(|X_1 - X_2|_{R^n} + |Y_1 - Y_2|_{R^n})\) and \(|K(X, Y)| \leq C(1 + |Y|_{R^n}^p)\) for some \(C, p > 0\). For each \(\epsilon > 0\) fixed, as \(\eta \to 0\) we have
\[
E \left| \int_0^t K(X_{s}^{\epsilon, \eta}, Y_{s}^{\epsilon, \eta}) ds - \int_0^t ds \int_{R^n} K(X_{s}^{\epsilon, \eta}, Y) \mu_{\epsilon, X_{s}^{\epsilon, \eta}} (dY) \right| \to 0 .
\]

**Proof.** By the estimates (36) and (41) and making use of the fact that \(K(X, Y)\) is Lipschitz continuous in \(X\) and \(Y\), we have
\[
E \left| \int_0^t K(X_{s}^{\epsilon, \eta}, Y_{s}^{\epsilon, \eta}) ds - \int_0^t K(X_{s}^{\epsilon, \eta}, \hat{Y}_{s}^{\epsilon, \eta}) ds \right| \\
\leq C \epsilon \left( \int_0^t \left| X_{s}^{\epsilon, \eta} - X_{s}^{\epsilon, \eta} \right|_{R^n} + \left| Y_{s}^{\epsilon, \eta} - \hat{Y}_{s}^{\epsilon, \eta} \right|_{R^n} \right) ds \\
\leq C \int_0^t \left( E|X_{s}^{\epsilon, \eta} - X_{s}^{\epsilon, \eta}|_{R^n}^2 \right)^{1/2} + \left( E|Y_{s}^{\epsilon, \eta} - \hat{Y}_{s}^{\epsilon, \eta}|_{R^n}^2 \right)^{1/2} \right] ds ,
\]
so that
\[
E \left| \int_0^t K(X_{s}^{\epsilon, \eta}, Y_{s}^{\epsilon, \eta}) ds - \int_0^t K(X_{s}^{\epsilon, \eta}, \hat{Y}_{s}^{\epsilon, \eta}) ds \right| \to 0
\]
as \(\eta \to 0\). Here the process \(\hat{Y}_{s}^{\epsilon, \eta}\) is defined as in (34). Reasoning as in (42), we have, that for each \(k = 0, 1, \ldots\) and each interval \([k \Delta, (k + 1) \Delta], \Delta > 0\),
\[
E \left| \int_{k \Delta}^{(k+1) \Delta} K(X_{s}^{\epsilon, \eta}, \hat{Y}_{s}^{\epsilon, \eta}) ds - \int_{k \Delta}^{(k+1) \Delta} ds \int_{R^n} K(X_{s}^{\epsilon, \eta}, Y) \mu_{\epsilon, X_{s}^{\epsilon, \eta} \Delta} (dY) \right|^2 \\
\leq C \Delta \eta \left( \frac{1}{\epsilon} + 1 + |x_0|_{R^n}^2 + |y_0|_{R^n}^2 \right).
\]
so that if we divide the interval \([0, t]\) into intervals of size \(\Delta = \eta \sqrt{\ln(\eta^{-1})}\), with the same argument as in Proposition 1 we derive

\[
\mathbf{E} \left| \int_0^t K(X_s^\varepsilon, \hat{Y}_s^\varepsilon, \eta) ds - \int_0^t ds \int_{\mathbb{R}^m} K(X_s^\varepsilon, Y) \mu^{\varepsilon, X_s^\varepsilon, \eta} (dY) \right| \to 0
\]

as \(\eta \to 0\). Reasoning as in Lemma A.2, and making use of the fact that \(|K(X, Y)| \leq C(1 + |Y|^{p \mathbb{R}_m})\) for some \(C, p > 0\), we have

\[
\mathbf{E} \left| \int_{k\Delta}^{(k+1)\Delta} ds \int_{\mathbb{R}^m} K(X_{s^k}^\varepsilon, Y) \mu^{\varepsilon, X_{s^k}^\varepsilon, \eta} (dY) - \int_{k\Delta}^{(k+1)\Delta} ds \int_{\mathbb{R}^m} K(X_s^\varepsilon, Y) \mu^{\varepsilon, X_s^\varepsilon} (dY) \right|
\]

\[
\leq C \Delta \max_{k\Delta \leq s \leq (k+1)\Delta} \mathbf{E} |X_{X_s^\varepsilon}^\varepsilon - X_{s}^\varepsilon|^p_{\mathbb{R}^n}
\]

\[
\leq C \Delta \max_{k\Delta \leq s \leq (k+1)\Delta} \left( \mathbf{E} |X_{X_s^\varepsilon}^\varepsilon - X_{s}^\varepsilon|^2_{\mathbb{R}^n} + \mathbf{E} |X_{s}^\varepsilon - X_{s}^\varepsilon|^2_{\mathbb{R}^n} \right)^{1/2}.
\]

From here, using (37) and (41), and summing over all intervals of the form \([k\Delta, (k+1)\Delta]\) for \(k = 0, 1, \ldots, N - 1\) we know that

\[
\mathbf{E} \left| \int_0^t ds \int_{\mathbb{R}^m} K(X_s^\varepsilon, Y) \mu^{\varepsilon, X_s^\varepsilon, \eta} (dY) - \int_0^t ds \int_{\mathbb{R}^m} K(X_s^\varepsilon, Y) \mu^{\varepsilon, X_s^\varepsilon} (dY) \right| \to 0
\]

as \(\eta \to 0\). Finally (65), (66) and (67) conclude (64).

Set

\[
U_1^{\varepsilon, \eta}(t) = \frac{1}{\sqrt{\eta}} \int_0^t \left| B_2(X_s^\varepsilon, \eta, Y_s^\varepsilon) - B_2(X_s^\varepsilon, \eta, Y_s^\varepsilon) \right|^2 (X_s^\varepsilon, \eta) ds,
\]

\[
U_2^{\varepsilon, \eta}(t) = \frac{1}{\sqrt{\eta}} \int_0^t \left| B_2(X_s^\varepsilon, \eta, Y_s^\varepsilon) - B_2(X_s^\varepsilon, \eta, Y_s^\varepsilon) \right|^2 (X_s^\varepsilon, \eta) ds.
\]

The next Lemma characterizes the weak convergence of \(U_1^{\varepsilon, \eta}(t)\) as \(\eta \to 0\) to a mean zero Gaussian process.

**Lemma 3.4.** For each \(\varepsilon > 0\), as \(\eta \to 0\), for \(0 \leq t \leq T\), the family of processes \(U_1^{\varepsilon, \eta}(t)\) converges weakly to a Gaussian process \(N_1^t(t)\) with mean 0 and covariance matrix \(\mathbf{A}^\varepsilon(t) = (a_{i,j}^\varepsilon(t))_{1 \leq i, j \leq n}\), so that

\[
a_{i,j}^\varepsilon(t) = \int_0^t \nabla^T u_i(X_s^\varepsilon, Y) (X_s^\varepsilon) \Sigma_1(X_s^\varepsilon) \Sigma_1^T(X_s^\varepsilon) \nabla^T u_j(X_s^\varepsilon, Y) (X_s^\varepsilon) ds.
\]

**Proof.** Let \(u_k(X, Y)\) be the solution to (47), \(k = 1, 2, \ldots, n\). Let us then apply Itô’s formula to \(u_k(X^\varepsilon, Y^\varepsilon, \eta)\), and we get

\[
u_k(X^\varepsilon, Y^\varepsilon, \eta) - u_k(X^\varepsilon, Y^\varepsilon, \eta) = \int_0^t \nabla X u_k(X^\varepsilon, Y^\varepsilon, \eta) \cdot B_1(X^\varepsilon, Y^\varepsilon, \eta) ds
\]
Therefore $U_1^{\varepsilon, \eta} = \int_0^t \left( \frac{\eta}{\varepsilon} \left( [B_2]_k(X_s^{\varepsilon, \eta}, Y_s^{\varepsilon, \eta}) - [B_2](X_0^{\varepsilon, \eta}, Y_0^{\varepsilon, \eta}) \right) + \frac{\eta}{2} \int_0^t \nabla X u_k(X_s^{\varepsilon, \eta}, Y_s^{\varepsilon, \eta}) \cdot \nabla X u_k(X_s^{\varepsilon, \eta}, Y_s^{\varepsilon, \eta}) ds \right) dW_s^1.$

From the above identity and the Poisson equation (47) we obtain that

\[
\int_0^t \left( \int_0^t \left( \frac{\eta}{\varepsilon} \left( [B_2]_k(X_s^{\varepsilon, \eta}, Y_s^{\varepsilon, \eta}) - [B_2](X_0^{\varepsilon, \eta}, Y_0^{\varepsilon, \eta}) \right) + \frac{\eta}{2} \int_0^t \nabla X u_k(X_s^{\varepsilon, \eta}, Y_s^{\varepsilon, \eta}) \cdot \nabla X u_k(X_s^{\varepsilon, \eta}, Y_s^{\varepsilon, \eta}) ds \right) dW_s^1 \right) ds
\]

\[
= \int_0^t \left( \int_0^t \left( \int_0^t \left( \frac{\eta}{\varepsilon} \left( [B_2]_k(X_s^{\varepsilon, \eta}, Y_s^{\varepsilon, \eta}) - [B_2](X_0^{\varepsilon, \eta}, Y_0^{\varepsilon, \eta}) \right) + \frac{\eta}{2} \int_0^t \nabla X u_k(X_s^{\varepsilon, \eta}, Y_s^{\varepsilon, \eta}) \cdot \nabla X u_k(X_s^{\varepsilon, \eta}, Y_s^{\varepsilon, \eta}) ds \right) dW_s^1 \right) ds
\]

\[
\cdot \Sigma_1(X_s^{\varepsilon, \eta}) dW_s^1.
\]

Therefore

\[
[U_1^{\varepsilon, \eta}]_{k}(t) = \int_0^t \left( \frac{\eta}{\varepsilon} \left( [B_2]_k(X_s^{\varepsilon, \eta}, Y_s^{\varepsilon, \eta}) - [B_2](X_0^{\varepsilon, \eta}, Y_0^{\varepsilon, \eta}) \right) + \frac{\eta}{2} \int_0^t \nabla X u_k(X_s^{\varepsilon, \eta}, Y_s^{\varepsilon, \eta}) \cdot \nabla X u_k(X_s^{\varepsilon, \eta}, Y_s^{\varepsilon, \eta}) ds \right) dW_s^1.
\]

Here

\[
(I) = \int_0^t \left( \frac{\eta}{\varepsilon} \left( [B_2]_k(X_s^{\varepsilon, \eta}, Y_s^{\varepsilon, \eta}) - [B_2](X_0^{\varepsilon, \eta}, Y_0^{\varepsilon, \eta}) \right) + \frac{\eta}{2} \int_0^t \nabla X u_k(X_s^{\varepsilon, \eta}, Y_s^{\varepsilon, \eta}) \cdot \nabla X u_k(X_s^{\varepsilon, \eta}, Y_s^{\varepsilon, \eta}) ds \right) dW_s^1.
\]

\[
(II) = -\frac{\eta}{2} \cdot \frac{\eta}{\varepsilon} \int_0^t \nabla X u_k(X_s^{\varepsilon, \eta}, Y_s^{\varepsilon, \eta}) \cdot \nabla X u_k(X_s^{\varepsilon, \eta}, Y_s^{\varepsilon, \eta}) ds ,
\]

\[
(III) = \int_0^t \nabla Y u_k(X_s^{\varepsilon, \eta}, Y_s^{\varepsilon, \eta}) \cdot \Sigma_1(X_s^{\varepsilon, \eta}) dW_s^1.
\]
We conclude from Lemma 3.1 that
\[ E(|I|^2) \leq \frac{C \eta}{\varepsilon^2} \left( 1 + |x_0|^2 + |y_0|^2 + \left( E \int_0^t |Y_s^T|^p_{L^p} \, ds \right)^2 \right). \]

Moreover, by combining Lemma 3.1 as well as (25) we also see that
\[ E(|II|^2) \leq \frac{C \eta^2}{\varepsilon^2} \left( 1 + \left( E \int_0^t |Y_s^T|^p_{L^p} \, ds \right)^2 \right). \]

Making use of Lemma 3.2 the estimate (61), we know that
\[ E(|I|^2 + |II|^2) \rightarrow 0 \]
as \( \eta \rightarrow 0 \).

Now we look at (III). In fact, the term
\[ (III) = M_{k,n}^k(t) := \int_0^t \nabla_Y u_k(X_s^\varepsilon, Y_s^\varepsilon) \cdot \Sigma_1(X_s^\varepsilon, \eta) \, dW_s^1 \]
is a martingale with mean 0 and quadratic variation
\[ \langle M_{k,n}^k, M_{k,n}^k \rangle_t = \alpha_{k,n}^k(t) \]

\[ = \int_0^t \nabla_Y^T u_k(X_s^\varepsilon, Y_s^\varepsilon) \Sigma_1(X_s^\varepsilon, \eta) \Sigma_1^T(X_s^\varepsilon, \eta) \nabla_Y u_k(X_s^\varepsilon, Y_s^\varepsilon) \, ds. \]

Set
\[ \alpha_{k,n}^k(t) = \int_0^t \nabla_Y^T u_k(X_s^\varepsilon, Y_s^\varepsilon) \Sigma_1(X_s^\varepsilon) \Sigma_1^T(X_s^\varepsilon) \nabla_Y u_k(X_s^\varepsilon, Y_s^\varepsilon) \, ds. \]

Making use of Lemma 3.3, we know that as \( \eta \rightarrow 0 \), for any \( R > 0 \) we have the convergence
\[ m^\varepsilon, \eta(R) := E \left( \int_0^t \nabla_Y^T u_k(X_s^\varepsilon, Y_s^\varepsilon) \Sigma_1(X_s^\varepsilon, \eta) \Sigma_1^T(X_s^\varepsilon, \eta) \nabla_Y u_k(X_s^\varepsilon, Y_s^\varepsilon) 1_{|Y_s^\varepsilon|_{L^p} \leq R} \, ds \right) \]

\[ - E \left( \int_0^t \nabla_Y^T u_k(X_s^\varepsilon, Y_s^\varepsilon) \Sigma_1(X_s^\varepsilon) \Sigma_1^T(X_s^\varepsilon) \nabla_Y u_k(X_s^\varepsilon, Y_s^\varepsilon) 1_{|Y_s^\varepsilon|_{L^p} \leq R} \, ds \right) \]

\[ \rightarrow 0 \quad (73) \]
as \( \eta \rightarrow 0 \).

Therefore by using (49) for \( \nabla_Y u \) we have the estimate
\[ E|\alpha_{k,n}^k(t) - \alpha_{k,n}^k(t)| \leq m^\varepsilon, \eta(R) \]

\[ + E \left( \int_0^t \nabla_Y^T u_k(X_s^\varepsilon, Y_s^\varepsilon) \Sigma_1(X_s^\varepsilon, \eta) \Sigma_1^T(X_s^\varepsilon, \eta) \nabla_Y u_k(X_s^\varepsilon, Y_s^\varepsilon) 1_{|Y_s^\varepsilon|_{L^p} > R} \, ds \right) \]

\[ - E \left( \int_0^t \nabla_Y^T u_k(X_s^\varepsilon, Y_s^\varepsilon) \Sigma_1(X_s^\varepsilon) \Sigma_1^T(X_s^\varepsilon) \nabla_Y u_k(X_s^\varepsilon, Y_s^\varepsilon) 1_{|Y_s^\varepsilon|_{L^p} > R} \, ds \right) \]

\[ \leq m^\varepsilon, \eta(R) + C \int_0^t E \left( 1 + |Y_s^\varepsilon|_{L^p}^p \right) 1_{|Y_s^\varepsilon|_{L^p} > R} \, ds \]

\[ + E \left( \int_0^t \nabla_Y^T u_k(X_s^\varepsilon, Y_s^\varepsilon) \Sigma_1(X_s^\varepsilon) \Sigma_1^T(X_s^\varepsilon) \nabla_Y u_k(X_s^\varepsilon, Y_s^\varepsilon) 1_{|Y_s^\varepsilon|_{L^p} > R} \, ds \right) \]

\[ \leq m^\varepsilon, \eta(R) + \rho(R). \]
By Cauchy–Schwarz inequality we can estimate \( \left( E(1 + |Y_s|^p) \right) \leq E(1 + |Y_s|^p) \mathbf{P}(1_{|Y_s| > R}) \to 0 \) as \( R \to \infty \). Thus we see that \( X_s \) converges weakly in \( C([0,T];\mathbb{R}^n) \) to a Gaussian process \( \{ N_t \}_{t \in [0,T]} \) with mean 0 and variance \( \sigma^2 \).

In a same fashion, we can define for any \( i,j = 1,2,...,n \), that

\[
\alpha^e_{i,j}(t) = \int_0^t \nabla_i u_k(X_{s}, Y_{s}) (X_{s}) \Sigma_1 (X_{s}) \Sigma_1^T (X_{s}) \nabla_j Y_s (X_{s}) (X_{s}) ds
\]

Let \( \mathcal{A}(t) = (\alpha^e_{i,j}(t))_{1 \leq i,j \leq n} \). With the same reasoning as above, we can show that \( U^e_{1,2}(t) \) converges weakly to \( N^e_1(t) \) with covariance matrix \( \mathcal{A}(t) \).

In regards to the remark made after Proposition 1, we can improve the estimate (41) in the following lemma.

**Lemma 3.5.** For any \( T > 0 \) and \( \varepsilon > 0, \eta > 0 \) small enough, for \( 0 \leq t \leq T \) and any small \( 0 < \kappa < 1 \) we have

\[
E|X_t^\varepsilon - X_t^\eta|^2_{\mathbb{R}^n} \leq C \left( \frac{\eta^2}{\varepsilon^2} + \eta \right),
\]

for some constant \( C = C(T) > 0 \).

**Proof.** We can write the equation (7) for \( X_t^\varepsilon, \eta \) in integral form as

\[
X_t^\varepsilon, \eta = x_0 + \int_0^t B_2(X_s^\varepsilon, Y_s^\eta) ds + \sqrt{\eta} \int_0^t \Sigma_2(X_s^\varepsilon, Y_s^\eta) dW_s^2.
\]

By (11), we also have

\[
X_t^\varepsilon = x_0 + \int_0^t B_2(X_s^\varepsilon, Y_s^\varepsilon) ds.
\]

Therefore we can write

\[
X_t^\varepsilon, \eta - X_t^\varepsilon
\]

\[
= \int_0^t [B_2(X_s^\varepsilon, Y_s^\eta) - B_2(X_s^\varepsilon, Y_s^\varepsilon)] ds
\]

\[
+ \sqrt{\eta} \int_0^t \Sigma_2(X_s^\varepsilon, Y_s^\varepsilon) dW_s^2.
\]

We can estimate, by (71) and same methods in the proof of Lemma 3.4, that for \( 0 \leq t \leq T \) and some constant \( C = C(T) \) we have

\[
E|I|^2_{\mathbb{R}^n} \leq C \left( \frac{\eta^2}{\varepsilon^2} + \eta \right).
\]

By Lemma A.2 we know that

\[
E|II|^2_{\mathbb{R}^n} \leq C \int_0^t E|X_s^\varepsilon - X_s^\eta|^2_{\mathbb{R}^n} ds.
\]

It is straightforward to have

\[
E|III|^2_{\mathbb{R}^n} \leq C \eta.
\]
Combining the above three estimates and make use of Gronwall’s inequality, we arrive at (74).

**Proposition 2.** As \( \eta \to 0 \) the process \( Z^\epsilon_{\eta}(t) \) converges weakly on the interval \([0, T]\) and in the space \( C([0, T]; \mathbb{R}^n) \) to the process \( Z^\epsilon(t) \) defined by the following equation

\[
Z^\epsilon_i = \int_0^t M(X^\epsilon_s)Z^\epsilon_s ds + N^\epsilon_i(t) + N^\epsilon(t),
\]

(75)

where \( N^\epsilon_i(t) \) and \( N^\epsilon(t) \) are two Gaussian processes with means 0 and explicitly calculated covariances, and \( M(X) \) is an \( n \times n \) matrix function.

**Proof.** Apparently, from (43), (44), (45), (68) and (69) we know that we have the decomposition

\[
Z^\epsilon_{\eta}(t) = U^\epsilon_{\eta}(t) + U^\epsilon_{\eta}(t) + V^\epsilon_{\eta}(t).
\]

By using Lemma 3.3, we know that the process \( V^\epsilon_{\eta}(t) \) converges weakly as \( \eta \to 0 \) to a Gaussian process \( N^\epsilon_2(t) \) with covariance matrix

\[
A^\epsilon(t) = \left( \int_0^t \Sigma(X^\epsilon_s, Y) \Sigma(X^\epsilon_s, Y)^T ds \right)_{1 \leq i, j \leq n}.
\]

Lemma 3.4 provides the weak convergence of \( U^\epsilon_{\eta}(t) \) to \( N^\epsilon_1(t) \) with covariance matrix \( A^\epsilon(t) \) as in (70).

Finally, the weak convergence of \( U^\epsilon_{\eta}(t) \) to \( \int_0^t M(X^\epsilon_s)Z^\epsilon_s ds \) can be obtained by doing a standard Taylor expansion argument as in the proof of Theorem 3.1 of [15]. In fact, set \( M(X) = \nabla_X [B_2(X, Y) \cdot (X)] \). From (69) we have

\[
U^\epsilon_{\eta}(t) - \int_0^t M(X^\epsilon_s)Z^\epsilon_{\eta} ds = \frac{1}{\sqrt{\eta}} \int_0^t \left[ B_2(X^\epsilon_s, Y) (X^\epsilon_s - \sqrt{\eta} M(X^\epsilon_s) Z^\epsilon_{\eta}) \right] ds
\]

\[
= \frac{1}{\sqrt{\eta}} \int_0^t \left[ B_2(X^\epsilon_s + \sqrt{\eta} Z^\epsilon_{\eta}, Y) (X^\epsilon_s + \sqrt{\eta} Z^\epsilon_{\eta}) - B_2(X^\epsilon_s, Y) (X^\epsilon_s) - \sqrt{\eta} M(X^\epsilon_s) Z^\epsilon_{\eta} \right] ds.
\]

Therefore by boundedness of second derivatives of of \( B_2(X, Y) \) with respect to \( X \) and reasoning as in Lemma A.2, we have

\[
\mathbb{E} \left| U^\epsilon_{\eta}(t) - \int_0^t M(X^\epsilon_s)Z^\epsilon_{\eta} ds \right|_{\mathbb{R}^n} \leq C \sqrt{\eta} \int_0^t \mathbb{E} |Z^\epsilon_s|_{\mathbb{R}^n}^2 ds.
\]

(76)

By making use of Lemma 3.5, as well as Lemma A.2, we have

\[
\mathbb{E} \left| U^\epsilon_{\eta}(t) \right|_{\mathbb{R}^n}^2 = \frac{1}{\eta} \mathbb{E} \left| \int_0^t \left[ B_2(X^\epsilon_s, Y) (X^\epsilon_s - \sqrt{\eta} M(X^\epsilon_s) Z^\epsilon_{\eta}) \right] ds \right|_{\mathbb{R}^n}^2
\]

\[
\leq \frac{C}{\eta} \int_0^t \mathbb{E} |X^\epsilon_s - X^\epsilon_{\eta}|_{\mathbb{R}^n}^2 ds
\]

\[
\leq \frac{C}{\eta} \int_0^t \mathbb{E} |X^\epsilon_s - X^\epsilon_{\eta}|_{\mathbb{R}^n}^2 ds.
\]

(76)
and

\[
E[U_2^{\varepsilon,\eta}(t + h) - U_2^{\varepsilon,\eta}(t)]^2 = \frac{1}{\eta} E \int_t^{t+h} \left| B_2(X_s^{\varepsilon,\eta}, Y) (X_s^{\varepsilon,\eta}) - B_2(X_s^{\varepsilon,\eta}, Y) (X_s^{\varepsilon,\eta}) \right|^2 ds \\
\leq \frac{C h}{\eta} \int_t^{t+h} E|X_{s}^{\varepsilon,\eta} - X_{s}^{\varepsilon,\eta}|^2 ds \\
\leq \frac{C h^2}{\eta},
\]

which then imply \( E[Z_t^{\varepsilon,\eta}]^2 \leq C < \infty \), as well as the tightness of the family \( Z_t^{\varepsilon,\eta} \) in \( C([0, T]; \mathbb{R}^n) \). From the weak convergence of \( U_1^{\varepsilon,\eta}(t) \) and \( V^{\varepsilon,\eta}(t) \) to Gaussian processes, together with (76), we conclude this Proposition.

4. Error estimate of SCGD from averaged SGD: Justification of the approximation.

We briefly mention the justification of using SGD (18) to approximate \( X^{\varepsilon,\eta}(t) \) in (7). From Lemma A.1 we know that as \( \varepsilon \to 0 \), by (4),

\[
B_2(X, Y) (X) \approx -\nabla g_w(X) \nabla f_v(E g_w(X)) + O(\sqrt{\varepsilon}).
\]

Thus as \( \varepsilon \to 0 \), the process \( X^{\varepsilon}(t) \) approximates another process \( \tilde{X}(t) \) that solves an ordinary differential equation (14):

\[
d\tilde{X}(t) = -E \nabla g_w(\tilde{X}(t)) \nabla f_v(E g_w(\tilde{X}(t))) dt, \quad \tilde{X}(0) = x_0,
\]

with an error of \( O(\sqrt{\varepsilon}) \). In fact, equation (14) can be viewed as a continuous version of the Gradient Descent (GD) algorithm, which directly solves (20).

Furthermore, from Proposition 2 we know that, as \( \eta \to 0 \), the process \( Z^{\varepsilon,\eta}(t) \) converges weakly to random process \( Z_t^{\varepsilon} \). The process \( Z_t^{\varepsilon} \) has its deterministic drift part and is driven by two mean 0 Gaussian processes carrying explicitly calculated covariance structures. This implies that, roughly speaking, from (15) we have an expansion of the type (16):

\[
X^{\varepsilon,\eta}(t) \overset{D}{\approx} X^{\varepsilon}(t) + \sqrt{\eta} Z_t^{\varepsilon},
\]

as \( \eta \to 0 \). Here \( \overset{D}{\approx} \) means approximate equality of probability distributions.

Therefore by (13), (14) and (16) we know that the slow motion \( X^{\varepsilon,\eta}(t) \) in (7) (or (2)) has an expansion around the GD algorithm in (14):

\[
X^{\varepsilon,\eta}(t) \overset{D}{\approx} \tilde{X}(t) + O(\sqrt{\varepsilon}) + \sqrt{\eta} Z_t^{\varepsilon}.
\]

Let us introduce the process \( X^{\varepsilon,\eta}(t) \) as the following continuous version of the Stochastic Gradient Descent (SGD) algorithm, as in (18):

\[
dX^{\varepsilon,\eta}(t) = -E \nabla g_w(X^{\varepsilon,\eta}(t)) \nabla f_v(E g_w(X^{\varepsilon,\eta}(t))) dt + \sqrt{\eta} dZ_t^{\varepsilon}, \quad X^{\varepsilon,\eta}(0) = x_0.
\]

From (14) and (18) and using Gronwall’s inequality, we know that

\[
\tilde{X}(t) - X^{\varepsilon,\eta}(t) \approx O(\sqrt{\eta}).
\]

So that by (17) we further have

\[
X^{\varepsilon,\eta}(t) \overset{D}{\approx} X^{\varepsilon,\eta}(t) + O(\sqrt{\varepsilon}) + O(\sqrt{\eta}).
\]

The above is a justification of the approximation of SGD to the process \( X_1^{\varepsilon,\eta} \) in (7). In the strongly convex case, \( X_1^{\varepsilon,\eta}(t) \) in (18) enters a small neighborhood containing the minimizer of (20) in finite time \( T > 0 \), so that (19) implies \( X^{\varepsilon,\eta}(t) \) in (2) enters a basin containing the minimizer of (20) also in finite time \( T > 0 \).
This validates the effectiveness of using the SCGD algorithm in the strongly convex case.

5. Remarks and generalizations. (a) For general fast–slow systems of stochastic differential equations, strong approximation theorems are available (see [2], [3]). Let us introduce a diffusion approximation $X^{ε,η}(t)$ of $X^{ε}(t)$ in (11) by the stochastic differential equation:

$$dX^{ε,η}(t) = B_2 (X^{ε,η}(t), Y) (X^{ε,η}(t)) dt + \sqrt{η} \Sigma(X^{ε,η}(t)) dW_t^2, \quad X^{ε,η}(0) = x_0.$$  \hfill (77)

Here $\Sigma(X) \in \mathbb{R}^n \otimes \mathbb{R}^n$ is some appropriately chosen non–degenerate noise matrix. The method of Bakhtin–Kifer (see [3]) provides a more refined diffusion approximation analysis than the classical averaging principle. Roughly speaking, we have for $0 \leq t \leq T$,

$$E |X^{ε,η}(t) - X^{ε,η}(t)|^2_{\mathbb{R}^n} \leq C η^{1+δ},$$  \hfill (78)

for some $C = C(T) > 0$ and small $δ > 0$.

Since we have Lemma A.1, we shall also consider the diffusion limit $X^0(t)$ under the following SGD algorithm:

$$dX^0(t) = -\nabla g_w(X^0(t)) \nabla f_w(X^0(t)) dt + \sqrt{η} \Sigma(X^0(t)) dW_t^2, \quad X^0(0) = x_0,$$$$

where $\Sigma(X) \in \mathbb{R}^n \otimes \mathbb{R}^n$ is some appropriately chosen non–degenerate noise matrix.

The above diffusion limit of SGD algorithm aims at directly solving the optimization problem (20). The convergence time analysis in terms of $η$ follows standard results in SGD convergence analysis. By using standard technique in the theory of stochastic differential equations, we have, roughly speaking, for $0 \leq t \leq T$,

$$E |X^{ε,η}(t) - X^0(t)|^2_{\mathbb{R}^n} \leq C(ε + η),$$  \hfill (80)

for some $C = C(T) > 0$ and small $δ > 0$.

Combining (78) and (80), we obtain an error bound, that for $0 \leq t \leq T$,

$$E |X^{ε,η}(t) - X^0(t)|^2_{\mathbb{R}^n} \leq C(ε + η + η^{1+δ}),$$  \hfill (81)

for some $C = C(T) > 0$ and small $δ > 0$.

However, the method provided by Bakhtin–Kifer can only cover the case when fast motion is moving on a compact space. Yet in our case the fast motion $Y^{ε,η}(t)$ is an OU process for frozen $X$. Thus estimates (78) and henceforth (81) are only conjectures and have to be addressed in a future work.

(b) By using the corrector method as we did in Section 3, it is possible to show that the order of approximation in Proposition 1 can be improved to be $C \left( \frac{η^2}{ε^2} + η \right)$, as in Lemma 3.5. However, our normal deviation analysis indicates that the order of approximation of $X^{ε}(t)$ to $X^{ε,η}(t)$ in mean square sense has to be of order $O(η)$ (see (16)). This is because the latter approximation is only in the weak sense, and the former approximation is in the strong ($L^2$) sense. Thus the weak approximation can achieve better convergence rates. On the other hand, by making use of Dambis–Dubins–Schwarz theorem (see [25, Theorem 1.6]), as well as the Hölder continuity of the Brownian motion path, it is also possible to show strong approximations in the normal deviation analysis (see [11]).

(c) We have assumed that the functions $f_w : \mathbb{R}^m \rightarrow \mathbb{R}$ and $g_w : \mathbb{R}^n \rightarrow \mathbb{R}^m$ are supported on some compact subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. This leads to the fact that the drift vector fields and diffusion matrix fields $B_1(X), B_2(X, Y), A_1(X), A_2(X, Y)$ in (3), (4), (5), (6) contain bounded coefficients together with their first
derivatives. Such an assumption is essential for our arguments in deriving the
averaging principle and normal deviation results. In practical situations, as we are
dealing with the optimization problem (20), we are only interested in the dynamics
of the corresponding algorithm trajectories that approach the minimizer. Thus
we can take a large ball in the Euclidean space containing this minimizer, and
we eliminate the trajectories outside this ball. By making use of large d eviation
estimates, this leads to the fact that all error bounds or approximation results in our
work could only be understood to be valid with high probability (with probability
close to 1). The slogan of deriving approximation or convergence results with high
probability is consistent with standard results in the statistical machine learning
literature (see [27]).

Appendix A. Two technical lemmas. The following lemma characterizes quan-
titatively the convergence \( q(X, Y) \rightarrow q(X, E_{g_w}(X)) \) as \( \varepsilon \rightarrow 0 \). Recall that the ob-
ject \( q \) in (10) may be a scalar, a vector, or a matrix. Let, in general, the compon-
ents of \( q \) be \( q_{ij} \). Let

\[
\sup_{X,Y} \left( \sum_{k=1}^{n} \sum_{i,j} \frac{\partial q_{ij}}{\partial X_k}(X,Y), \sum_{l=1}^{m} \sum_{i,j} \frac{\partial q_{ij}}{\partial Y_l}(X,Y) \right) \leq M.
\]

Lemma A.1. We have

\[
|q(X,Y) - q(X, E_{g_w}(X))|_{\text{norm}} \leq C \sqrt{\varepsilon},
\]

where the constant \( C > 0 \) depends on \( M \), and the norm \( |q|_{\text{norm}} \) is a vector (matrix)

norm if \( q \) is a vector (matrix), respectively.

Proof. The Gaussian measure \( \mu^{X,\varepsilon}(dY) \) in (9) has a density function (see [20, The-
orem 1.2.9])

\[
\mu^{X,\varepsilon}(dY) = \frac{\exp \left( -\frac{1}{2} (Y - E_{g_w}(X))^T \left( \frac{\varepsilon}{2} \Sigma_1(X) \Sigma_1(X)^T \right)^{-1} (Y - E_{g_w}(X)) \right)}{(2\pi)^{m/2} \left( \frac{\varepsilon}{2} \right)^{m/2} \left( \det(\Sigma_1(X) \Sigma_1(X)^T) \right)^{1/2}} dY.
\]

Let the density function

\[
\rho(Z) = \frac{\exp \left( -\frac{1}{2} Z^T (\Sigma_1(X) \Sigma_1(X)^T)^{-1} Z \right)}{(2\pi)^{m/2} \left( \det(\Sigma_1(X) \Sigma_1(X)^T) \right)^{1/2}},
\]

so that

\[
\int_{\mathbb{R}^m} \rho(Z) dZ = 1.
\]

Then we have

\[
\int_{\mathbb{R}^m} q(X,Y) \mu^{X,\varepsilon}(dY) = \int_{\mathbb{R}^m} \frac{\exp \left( -\frac{1}{2} (Y - E_{g_w}(X))^T \left( \frac{\varepsilon}{2} \Sigma_1(X) \Sigma_1(X)^T \right)^{-1} (Y - E_{g_w}(X)) \right) q(X,Y)}{(2\pi)^{m/2} \left( \frac{\varepsilon}{2} \right)^{m/2} \left( \det(\Sigma_1(X) \Sigma_1(X)^T) \right)^{1/2}} dY
\]

\[
= \int_{\mathbb{R}^m} \rho(Z) q(X, E_{g_w}(X) + \sqrt{\frac{\varepsilon}{2}} Z) dZ.
\]
From here we have
\[
\int_{\mathbb{R}^m} q(X, Y) \mu^{X,\varepsilon}(dY) - q(X, E_{g_w}(X)) = \int_{\mathbb{R}^m} \rho(Z) \left[ q \left( X, E_{g_w}(X) + \sqrt{\frac{\varepsilon}{2}} Z \right) - q(X, E_{g_w}(X)) \right] dZ ,
\]
so that
\[
\left| \int_{\mathbb{R}^m} q(X, Y) \mu^{X,\varepsilon}(dY) - q(X, E_{g_w}(X)) \right|_{\text{norm}} \leq \sqrt{\frac{\varepsilon}{2}} M \int_{\mathbb{R}^m} |Z| \rho(Z) dZ \leq C\sqrt{\varepsilon} .
\]

The following Lemma is about regularity properties with respect to $X$ of the $q(X, Y)$ operator.

**Lemma A.2.** For any $X_1, X_2 \in \mathbb{R}^n$ and some $C > 0$ we have
\[
\| q(X_1, Y) - q(X_2, Y) \|_{\text{norm}} \leq C|X_1 - X_2|_{\mathbb{R}^n} ,
\]
where the constant $C > 0$ depends on $M$, and the norm $|q|_{\text{norm}}$ is a vector (matrix) norm if $q$ is a vector (matrix), respectively.

**Proof.** The Gaussian measure $\mu^{X,\varepsilon}(dY)$ in (9) has a density function (see [20, Theorem 1.2.9])
\[
\mu^{X,\varepsilon}(dY) = \frac{\exp \left( \frac{1}{2} (Y - E_{g_w}(X))^T \left( \frac{\varepsilon}{2} \Sigma_1(X) \Sigma_1(X)^T \right)^{-1} (Y - E_{g_w}(X)) \right)}{(2\pi)^{\frac{m}{2}}} \left( \frac{\varepsilon}{2} \right)^{m/2} |\det(\Sigma_1(X) \Sigma_1(X)^T)|^{1/2} dY .
\]

Let the density function for the standard normal distribution $\mathcal{N}(0, I_m)$ be
\[
\mu(dN) = \frac{1}{(2\pi)^{\frac{m}{2}}} \exp \left( \frac{-1}{2} N^T N \right) dN ,
\]
so that
\[
\int_{\mathbb{R}^m} \mu(dN) = 1 .
\]

Let the random variable $N$ follow the standard normal distribution $\mathcal{N}(0, I_m)$, so that $N$ has the distribution $\mu(dN)$ on $\mathbb{R}^m$. If $Y$ is a random variable that follows the distribution $\mu^{X,\varepsilon}(dY)$, then $Y = \sigma(X)N + E_{g_w}(X)$, where $\sigma(X)$ is a non-singular $m \times m$ matrix such that $\sigma(X)\sigma^T(X) = (\Sigma_1(X)\Sigma_1(X)^T)^{-1}$ (see [20, Theorem 1.2.9]). By [9, §3.2, Theorem 2.1], the matrix $\sigma(X)$ can be chosen to be symmetric and Lipschitz continuous. Since $\Sigma_1(X)\Sigma_1(X)^T$ is compactly supported, the matrix $\sigma(X)$ can also be chosen to be compactly supported.

Then we have
\[
q(X_1, Y) - q(X_2, Y) = \int_{\mathbb{R}^m} q(X_1, Y) \mu^{X_1,\varepsilon}(dY) - \int_{\mathbb{R}^m} q(X_2, Y) \mu^{X_2,\varepsilon}(dY)
\]
\[
= \int_{\mathbb{R}^m} q(X_1, \sigma(X_1)N + E_{g_w}(X_1))\mu(dN) - \int_{\mathbb{R}^m} q(X_2, \sigma(X_2)N + E_{g_w}(X_2))\mu(dN) .
\]
From here we have
\[
|q(X_1, Y) - q(X_2, Y)|_{\text{norm}} \leq C |X_1 - X_2|_{\mathbb{R}^n}^\ast \int_{\mathbb{R}^m} |N|_{R^m} \mu(dN) \leq C |X_1 - X_2|_{\mathbb{R}^n}.
\]

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