1. GENERALISED DIFFUSION AND WAVE EQUATIONS: RECENT ADVANCES

T. Sandev\textsuperscript{1,2}, R. Metzler\textsuperscript{3} and A. Chechkin\textsuperscript{3,4}

\textsuperscript{1}Research Center for Computer Science and Information Technologies, Macedonian Academy of Sciences and Arts, Bul. Krste Misirkov 2, 1000 Skopje, Macedonia
\textsuperscript{2}Institute of Physics, Faculty of Natural Sciences and Mathematics, Ss Cyril and Methodius University, Arhmedova 3, 1000 Skopje, Macedonia
\textsuperscript{3}Institute of Physics \& Astronomy, University of Potsdam, D-14776 Potsdam-Golm, Germany
\textsuperscript{4}Akhiezer Institute for Theoretical Physics, Kharkov 61108, Ukraine
e-mail: achechkin@kipt.kharkov.ua

We present a short overview of the recent results in the theory of diffusion and wave equations with generalised derivative operators. We give generic examples of such generalised diffusion and wave equations, which include time-fractional, distributed order, and tempered time-fractional diffusion and wave equations. Such equations exhibit multi-scaling time behaviour, which makes them suitable for the description of different diffusive regimes and characteristic crossover dynamics in complex systems.

KEY WORDS: Memory kernel, mean squared displacement, subordination
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1. INTRODUCTION

Diffusion equations with fractional time and space derivatives instead of the integer ones are widely used to describe anomalous diffusion processes where the mean squared displacement (MSD) scales as a power of time,

$$\langle x^2(t) \rangle \simeq t^\alpha. \quad (1.1)$$

Depending on the values of the anomalous diffusion exponent $\alpha$ one distinguishes the cases of subdiffusion for $0 < \alpha < 1$, normal Brownian diffusion for $\alpha = 1$, superdiffusion for $1 < \alpha < 2$, ballistic motion for $\alpha = 2$, and superballistic motion for $\alpha > 2$. Well-known examples of anomalous transport include subdiffusion in artificially crowded systems and protein-crowded lipid bilayer membranes [10, 11, 33], subdiffusive charge carrier motion in semiconductors [29], subdiffusive motion of submicron probes in living biological cells [8], superdiffusive tracer motion in chaotic laminar flows [32], diffusion in porous inhomogeneous media [35], and random search processes [34], to name but a few.

Modern microscopic techniques such as fluorescence correlation spectroscopy or advanced single particle tracking methods have led to the discovery of a multitude of anomalous diffusion processes in living biological cells and complex fluids, see e.g. the reviews [11, 9, 16, 18, 19, 28] and references therein. With the growing number of anomalous diffusion phenomena it became clear that a wide range of complex systems do not show a unique, mono-scaling behaviour, Eq. (1.1), but instead demonstrate transitions between different diffusion regimes in the course of time. Such observations put forward the idea that in order to capture the multi-scaling dynamics one may generalise the fractional differential operator in the fractional diffusion equation by more universal operators with
specific memory kernels. Here we analyse in detail the different versions of such generalised operators and the specific dynamical crossovers they effect. Special case of a power-law kernel recovers fractional derivative and respectively, the mono-scaling diffusion regime.

2. GENERALISED DIFFUSION EQUATIONS

2.1 Natural and modified forms
We consider generalised diffusion equations in the so-called natural and modified form as generalizations of the time fractional diffusion equations in the Caputo or Riemann-Liouville sense.

The generalised diffusion equation in the natural form is given by [22]

\[ \int_0^t \gamma(t-t') \frac{\partial W(x,t')}{\partial t'} dt' = \frac{\partial^2 W(x,t)}{\partial x^2}, \]  

(1.2)

where the memory kernel \( \gamma(t) \) stands on the left hand side of the equation. We consider zero boundary conditions at infinity, \( W(\pm\infty,t) = 0, \frac{\partial}{\partial x} W(\pm\infty,t) = 0 \), and initial conditions of the form

\[ W(x,t=0) = \delta(x). \]  

(1.3)

In turn, the modified form of the equation is given by [25, 26]

\[ \frac{\partial W(x,t)}{\partial t} = \frac{\partial}{\partial t} \int_0^t \eta(t-t') \frac{\partial^2 W(x,t')}{\partial x^2} dt', \]  

(1.4)

with the memory kernel \( \eta(t) \) on the right hand side of the equation. As it was shown in [25], these two equations are simply connected through the memory kernels in the form \( \hat{\gamma}(s) \rightarrow 1/[s\hat{\eta}(s)] \), where \( \hat{\gamma}(s) = \int_0^\infty e^{-st}\gamma(t) dt = \mathcal{L}[\gamma(t)] \) and \( \hat{\eta}(s) = \mathcal{L}[\eta(t)] \) are...
the Laplace transforms of the memory kernels $\gamma(t)$ and $\eta(t)$, respectively. These equations have been obtained from continuous
time random walk (CTRW) theory for finite variance of jump
lengths and generalised waiting time probability density functions
(PDFs) of the forms $\hat{\psi}(s) = \frac{1}{1+s\hat{\gamma}(s)}$ and $\hat{\psi}(s) = \frac{1}{1+[\hat{\eta}(s)]^{-1}}$, respectively.

In order to have well established stochastic processes encoded in
both equations, we need to prove that their solutions are normalised
and non-negative. The non-negativity of the solutions can be shown
by using the subordination approach \cite{4,14,15}. We will elaborate
on this approach for Eq. (1.2), this can be done for Eq. (1.4) in
the same way. By Fourier ($\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)e^{ikx} dx$) and Laplace
transformations of Eq. (1.2) one finds
\begin{equation}
\tilde{W}(k, s) = \hat{\gamma}(s) \int_{0}^{\infty} e^{-u(s\hat{\gamma}(s)+k^2)} du = \int_{0}^{\infty} e^{-uk^2} \hat{G}(u, s) du. \tag{1.5}
\end{equation}
Here the function $G$ is defined by
\begin{equation}
\hat{G}(u, s) = \hat{\gamma}(s)e^{-us\hat{\gamma}(s)} = -\frac{\partial}{\partial u} \frac{1}{s}e^{-us\hat{\gamma}(s)}. \tag{1.6}
\end{equation}
Therefore, the PDF $W(x, t)$ is given by \cite{14,15}
\begin{equation}
W(x, t) = \int_{0}^{\infty} \frac{e^{-\frac{x^2}{4u}}}{\sqrt{4\pi u}} G(u, t) du, \tag{1.7}
\end{equation}
which means that the function $G(u, t)$ is the PDF providing the
subordination transformation from time scale $t$ (physical time) to
time scale $u$ (operational time). The function $G(u, t)$ is normalized
with respect to $u$ for any $t$, i.e.,
\begin{equation}
\int_{0}^{\infty} G(u, t) du = \mathcal{L}^{-1}_s \left[ \int_{0}^{\infty} \hat{\gamma}(s)e^{-us\hat{\gamma}(s)} ds \right] = \mathcal{L}^{-1}_s \left[ \frac{1}{s} \right] = 1. \tag{1.8}
\end{equation}
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In order to prove the positivity of $W(x,t)$ according to the Bernstein theorem it is sufficient to show that the function $\hat{G}(u,s)$ is completely monotone on the positive real axis $s$ [30]. For that we only need to show that (i) the function $\hat{\gamma}(s)$ is completely monotone, and (ii) the function $s\hat{\gamma}(s)$ is a Bernstein function. If (ii) holds, then the function $e^{-s\hat{\gamma}(s)}$ is completely monotone as a composition of completely monotone and a Bernstein function. Moreover, $G(u,s)$ is completely monotone, as a product of two completely monotone functions, $e^{-s\hat{\gamma}(s)}$ and $\hat{\gamma}(s)$. Alternatively, one can check that $s\hat{\gamma}(s)$ is a complete Bernstein function, which is an important subclass of the Bernstein functions [30]. This condition is enough to ensure the complete monotonicity of $\hat{G}(u,s)$ due to the property of the complete Bernstein function: if $f(s)$ is a complete Bernstein function, then $f(s)/s$ is completely monotone [30]. The proof of the non-negativity of the solutions to the generalised diffusion equations with different memory kernels can be found in [25, 26] along with the list of properties of completely monotone, Bernstein, and complete Bernstein functions.

By analogy, the solution of Eq. (1.4) is non-negative if (i) the function $1/[s\hat{\eta}(s)]$ is completely monotone, and (ii) the function $1/\hat{\eta}(s)$ is a Bernstein function. Alternatively, one can prove the non-negativity of the solution if $1/\hat{\eta}(s)$ is a complete Bernstein function.

By solving both equations (1.2) and (1.4), one can find the corresponding MSDs, as follows [22, 25, 26]

$$\langle x^2(t) \rangle = 2\mathcal{L}^{-1} \left[ \frac{s^{-2}}{\hat{\gamma}(s)} \right], \quad (1.9)$$

$$\langle x^2(t) \rangle = 2\mathcal{L}^{-1} \left[ s^{-1}\hat{\eta}(s) \right], \quad (1.10)$$

from which we can analyze the diffusive behaviours for a given form of the memory kernel.
2.2 Particular examples

2.2.1. Standard diffusion equation  The case with $\gamma(t) = \delta(t)$, which means $\hat{\eta}(s) = 1/[s\hat{\gamma}(s)] = 1/s$, i.e., $\eta(t) = 1$, leads us both to the standard diffusion equation

$$\frac{\partial W(x,t)}{\partial t} = \frac{\partial^2 W(x,t)}{\partial x^2}, \quad (1.11)$$

for Brownian diffusion with linear time dependence of the MSD, $\langle x^2(t) \rangle = 2L^{-1}[s^{-2}] = 2t$. The waiting time PDF in the corresponding CTRW scheme is exponential, $\psi(t) = L^{-1}\left[\frac{1}{1+s}\right] = e^{-t}$, which in the long time limit ($s \to 0$) can be used as $\hat{\psi}(s) \simeq 1 - s$.

2.2.2. Mono-fractional diffusion equation  Another well known example is the case of a power-law memory kernel $\gamma(t) = t^{-\alpha}/\Gamma(1-\alpha)$, $0 < \alpha < 1$, from where it follows that $\hat{\eta}(s) = s^{-\alpha}$, i.e., $\eta(t) = t^{\alpha-1}/\Gamma(\alpha)$. These memory kernels yield two equivalent formulations of the time fractional diffusion equation, namely,

$$cD_{0+}^{\alpha}W(x,t) = \frac{\partial^2 W(x,t)}{\partial x^2}, \quad (1.12)$$

and

$$\frac{\partial W(x,t)}{\partial t} = RL\!D_{0+}^{\alpha-1}\frac{\partial^2 W(x,t)}{\partial x^2}, \quad (1.13)$$

where

$$cD_{t}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} \frac{f^{(n)}(t')}{(t-t')^{\alpha+1-n}} \, dt', \quad n-1 < \alpha < n,$$

$$RL\!D_{t}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{0}^{t} \frac{f(t')}{(t-t')^{\alpha+1-n}} \, dt', \quad n-1 < \alpha < n,$$

$$-6-$$
are the Caputo and Riemann-Lioville fractional derivatives, respectively \([13]\). Both derivatives for \(\alpha = n\) become ordinary derivatives, \(f^{(n)}(t)\). The corresponding MSD shows subdiffusive behaviour \(\langle x^2(t) \rangle = 2 \mathcal{L}^{-1} \left[ s^{-\alpha-1} \right] = 2 \frac{t^\alpha}{\Gamma(\alpha+1)}\), with the Mittag-Leffler waiting time PDF, \(\psi(t) = \mathcal{L}^{-1} \left[ \frac{1}{\Gamma(s)} \right] = t^{\alpha-1} E_{\alpha,\alpha}(t^\alpha)\). Here \(E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{\Gamma(\alpha n + \beta)}\) is the two parameter Mittag-Leffler function \([13]\), which has the following asymptotic \(E_{\alpha,\beta}(z) \simeq -\sum_{n=1}^{\infty} \frac{(-z)^n}{\Gamma(\beta - \alpha n)}\) for \(z \gg 1\). Here we note that in the long time limit the waiting time PDF is of power-law form, \(\psi(t) \simeq t^{-1-\alpha}\) [17].

2.2.3. Bi-fractional diffusion equation

Now we introduce a memory kernel with two power-law functions, \(\gamma(t) = B_1 t^{-\alpha_1}/\Gamma(1 - \alpha_1) + B_2 t^{-\alpha_2}/\Gamma(1 - \alpha_2), 0 < \alpha_1 < \alpha_2 < 1, B_1 + B_2 = 1\), which gives rise to the bi-fractional diffusion equation in natural form [3],

\[
B_1 C D_t^{\alpha_1} W(x, t) + B_2 C D_t^{\alpha_2} W(x, t) = \frac{\partial^2 W(x, t)}{\partial x^2}. \tag{1.16}
\]

Using the relation with the memory kernel \(\eta(t), \hat{\eta}(s) = 1/[s \hat{\gamma}(s)] = [B_1 s^{\alpha_1} + B_2 s^{\alpha_2}]^{-1}\), we find that \(\eta(t) = \frac{B_1 t^{\alpha_2-1}}{B_2} E_{\alpha_2-\alpha_1,\alpha_2} \left( -B_1 B_2 t^{\alpha_2-\alpha_1} \right)\), i.e., the equivalent representation to Eq. (1.16) in the modified form is given by

\[
\frac{\partial W(x, t)}{\partial t} = \frac{1}{B_2} \frac{\partial}{\partial t} \int_0^t (t - t')^{\alpha_2-1} \\
\times E_{\alpha_2-\alpha_1,\alpha_2} \left( -\frac{B_1}{B_2} [t - t']^{\alpha_2-\alpha_1} \right) \frac{\partial^2 W(x, t')}{\partial x^2} \, dt'. \tag{1.17}
\]
The bi-fractional diffusion equation in the natural form is a model to describe decelerating subdiffusion, since the MSD is given by \[3, 23\]

\[
\langle x^2(t) \rangle = \frac{2t^{\alpha_2}}{B_2} E_{\alpha_2-\alpha_1,\alpha_2+1} \left( -\frac{B_1}{B_2} t^{\alpha_2-\alpha_1} \right) \approx \begin{cases} 
2B_2 \frac{t^{\alpha_2}}{\Gamma(1+\alpha_2)}, & t \ll 1, \\
2B_1 \frac{t^{\alpha_1}}{\Gamma(1+\alpha_1)}, & t \gg 1.
\end{cases}
\]

(1.18)

The corresponding waiting time PDF is given by \[23\]

\[
\psi(t) = \mathcal{L}^{-1} \left[ \frac{1}{1 + B_1 s^{\alpha_1} + B_2 s^{\alpha_2}} \right] = \frac{t^{\alpha_2-1}}{B_2} \sum_{n=0}^{\infty} \frac{(-1)^n B_n}{B_2^n} t^{\alpha_2 n} E_{\alpha_2-\alpha_1,\alpha_2+n+1} \left( -\frac{B_1}{B_2} t^{\alpha_2-\alpha_1} \right).
\]

(1.19)

Here \(E_{\delta,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_n z^n}{\Gamma(\alpha n + \beta)} \frac{1}{n!}\) is the three parameter Mittag-Leffler function \[21\], where \((\delta)_n = \Gamma(\delta+n)/\Gamma(\delta)\) is the Pochhammer symbol. Its asymptotic expansions are given by \(E_{\delta,\alpha,\beta}(-t^\alpha) \approx \frac{1}{\Gamma(\beta)} - \frac{\Gamma(\beta) t^\alpha}{\Gamma(\alpha+\beta)} \exp \left( -\frac{\Gamma(\beta) t^\alpha}{\Gamma(\alpha+\beta)} \right)\) for \(t \ll 1\), and \(E_{\delta,\alpha,\beta}(-t^\alpha) = \frac{t^{-\alpha \delta}}{\Gamma(\delta)} \sum_{n=0}^{\infty} \frac{\Gamma(\delta+n)}{\Gamma(\alpha \delta + \alpha(n))} \frac{(-t^\alpha)^n}{n!}\) for \(0 < \alpha < 2\) and \(t \gg 1\) \[6, 23\].

In accordance to the previous case, we may introduce the bi-fractional diffusion equation in the modified form, where the memory kernel is given by \(\eta(t) = B_1 \frac{t^{\alpha_1-1}}{\Gamma(\alpha_1)} + B_2 \frac{t^{\alpha_2-1}}{\Gamma(\alpha_2)}\) with \(0 < \alpha_1 < \alpha_2 < 1\), \(B_1 + B_2 = 1\), i.e., \[5, 31\]

\[
\frac{\partial W(x,t)}{\partial t} = B_1 RL D_t^{1-\alpha_1} \frac{\partial^2 W(x,t)}{\partial x^2} + B_2 RL D_t^{1-\alpha_2} \frac{\partial^2 W(x,t)}{\partial x^2}.
\]

(1.20)

Since \(\hat{\gamma}(s) = 1/[s \hat{n}(s)] = 1/[s(B_1 s^{-\alpha_1} + B_2 s^{-\alpha_2})]\), we have \(\gamma(t) = t^{1-\alpha_1} E_{\alpha_2-\alpha_1,1-\alpha_1} \left( -\frac{B_2}{B_1} t^{\alpha_2-\alpha_1} \right)\), and the equivalent representation
of Eq. (1.20) in the natural form is given by a Mittag-Leffler memory kernel,
\[
\frac{1}{B_1} \int_0^t (t - t')^{-\alpha_1} E_{\alpha_2 - \alpha_1, 1 - \alpha_1} \left(-\frac{B_2}{B_1} [t - t']^{\alpha_2 - \alpha_1}\right) \frac{\partial W(x, t')}{\partial t'} \, dt' = \frac{\partial^2 W(x, t)}{\partial x^2}.
\] (1.21)

The bi-fractional diffusion equation in the modified form is a useful model for the description of accelerating diffusion since the MSD is given by [5, 23]
\[
\langle x^2(t) \rangle = 2B_1 t^{\alpha_1} E_{\alpha_2 - \alpha_1, \alpha_1 + 1}^{-1} \left(-\frac{B_2}{B_1} t^{\alpha_2 - \alpha_1}\right)
\approx \begin{cases} 
2B_1 \frac{t^{\alpha_1}}{\Gamma(1 + \alpha_1)}, & t \ll 1, \\
2B_2 \frac{t^{\alpha_2}}{\Gamma(1 + \alpha_2)}, & t \gg 1.
\end{cases}
\] (1.22)

2.2.4. Tempered time-fractional diffusion equation At the end of this section we show two other models that describe transitions from one to another diffusive behaviour. This can be achieved if one introduces an exponential cut-off of the power-law memory kernel of the form \( \gamma(t) = e^{-bt} t^{-\alpha} / \Gamma(1 - \alpha) \), \( 0 < \alpha < 1 \), where \( b > 0 \) is the truncation parameter. Therefore, we obtain the tempered fractional diffusion equation in the natural form,
\[
\frac{1}{\Gamma(1 - \alpha)} \int_0^t e^{-b(t - t')} (t - t')^{-\alpha} \frac{\partial W(x, t')}{\partial t'} \, dt' = \frac{\partial^2 W(x, t)}{\partial x^2}.
\] (1.23)

The corresponding equation in the modified form reads
\[
\frac{\partial W(x, t)}{\partial t} = \frac{\partial}{\partial t} \int_0^t (t - t')^{\alpha - 1} E_{1, \alpha}^{-(1 - \alpha)} (-b[t - t']) \frac{\partial^2 W(x, t')}{\partial x^2} \, dt',
\] (1.24)
since $\eta(t) = \mathcal{L}^{-1} \left[ \frac{\gamma^{-1}}{(s+b)^{\alpha-1}} \right] = t^{\alpha-1} E_{1,\alpha}^{\alpha-1}(-bt)$. Equation (1.24) is actually the diffusion equation
\[
\frac{\partial W(x,t)}{\partial t} = RL D_{1,-b,0+}^{1-\alpha,1-\alpha} \frac{\partial^2 W(x,t)}{\partial x^2}.
\] (1.25)
with the Prabhakar derivative ($0 < \mu < 1$, $\rho > 0$) [7]
\[
RL D_{\rho,\omega,0+}^{\gamma,\mu} f(x) = \frac{d}{dt} \int_0^t (t-t')^{-\mu} E_{\rho,1-\mu}^{-\gamma} (\omega [t-t']^\rho) f(t') \, dt'.
\] (1.26)

The MSD shows a crossover from subdiffusion to normal diffusion,
\[
\langle x^2(t) \rangle = 2t^{\alpha} E_{1,1+\alpha}^{\alpha-1} (-bt) \simeq \begin{cases} 
\frac{1}{\Gamma(1+\alpha)} t^\alpha, & t \ll 1, \\
2b^{1-\alpha} t, & t \gg 1. 
\end{cases}
\] (1.27)

Consider now the tempered fractional diffusion equation in the modified form with $\eta(t) = e^{-bt} t^{\alpha-1} / \Gamma(\alpha)$ ($0 < \alpha < 1$, $b > 0$), i.e.,
\[
\frac{\partial W(x,t)}{\partial t} = \frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^t e^{-b(t-t')} (t-t')^{\alpha-1} \frac{\partial^2 W(x,t')}{\partial x^2} \, dt'.
\] (1.28)
From $\hat{\eta}(s) = (s+b)^{-\alpha}$ it follows that $\hat{\gamma}(s) = 1/\{s(s+b)^{-\alpha}\}$, and $\gamma(t) = t^{-\alpha} E_{1,1-\alpha}^{-\alpha} (-bt)$, i.e., the corresponding equation in the natural form equivalent to (1.28) is given by
\[
\int_0^t (t-t')^{-\alpha} E_{1,1-\alpha}^{-\alpha} (-b[t-t']) \frac{\partial W(x,t')}{\partial t'} \, dt' = \frac{\partial^2 W(x,t)}{\partial x^2},
\] (1.29)
which can be presented with the regularized Prabhakar derivative [7]
\[
CD_{\rho,\omega,0+}^{\gamma,\mu} f(x) = \int_0^t (t-t')^{-\mu} E_{\rho,1-\mu}^{-\gamma} (\omega [t-t']^\rho) \frac{df(t')}{dt'} \, dt',
\] (1.30)

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as

\[ cD^{\alpha,\alpha}_{1,-b,0+}W(x,t) = \frac{\partial^2 W(x,t)}{\partial x^2}. \]  \hspace{1cm} (1.31)

The corresponding MSD shows a crossover from subdiffusion to a plateau value,

\[ \langle x^2(t) \rangle = 2t^\alpha E_{1,\alpha+1}^{\alpha+1}(-bt) \simeq \begin{cases} 2t^\alpha \Gamma(1+\alpha), & t \ll 1, \\ 2b^{-\alpha}, & t \gg 1. \end{cases} \]  \hspace{1cm} (1.32)

Here we note that different models based on the tempered versions of the generalised Langevin equation and fractional Brownian motion have been introduced recently, which also give similar crossovers from subdiffusion to normal diffusion [20]. Moreover, general diffusion equations on two dimensional structures have been analyzed and different diffusive regimes obtained [24].

3. GENERALISED DIFFUSION-WAVE EQUATION

3.1 Natural and modified forms

In analogy to the generalised diffusion equations in natural and modified forms, we now consider the generalised diffusion-wave equation

\[ \int_0^t \zeta(t-t') \frac{\partial^2 W(x,t)}{\partial t'^2} dt' = \frac{\partial^2 W(x,t)}{\partial x^2}, \]  \hspace{1cm} (1.33)

in the natural form with non-negative memory kernel \( \zeta(t) \), and similarly

\[ \frac{\partial^2 W(x,t)}{\partial t^2} = \frac{\partial^2}{\partial t^2} \int_0^t \xi(t-t') \frac{\partial^2 W(x,t')}{\partial x^2} dt'. \]  \hspace{1cm} (1.34)
in the modified form with non-negative memory kernel $\xi(t)$. In what follows we consider the natural form, only \[27\]. The boundary conditions at infinity are $W(\pm\infty, t) = 0$, $\frac{\partial}{\partial x}W(\pm\infty, t) = 0$, and the initial conditions are of the form

$$W(x, t = 0) = \delta(x), \quad \frac{\partial}{\partial t}W(x, t = 0) = 0.$$  
(1.35)

We here refer to \[27\] for discussion on the choice of the initial conditions.

Making the Fourier-Laplace transform of Eq. (1.33), and then inverse Fourier transform, we find

$$\hat{W}(x, s) = \frac{1}{2}\sqrt{\hat{\zeta}(s)} \exp \left(-s\sqrt{\hat{\zeta}(s)}|x|\right).$$  
(1.36)

From here one easily concludes that the PDF is normalized to 1, i.e., $\int_{-\infty}^{\infty} W(x, t) \, dx = 1$, since $\int_{-\infty}^{\infty} \hat{W}(x, s) \, dx = 1/s$. The non-negativity of the solution can be shown by applying the Bernstein theorem, i.e., by showing that the solution in the Laplace space is a completely monotone function \[30\]. To this end, solution (1.36) can be considered as a product of two functions, $\frac{1}{2}\sqrt{\hat{\zeta}(s)}$ and $\exp \left(-s\sqrt{\hat{\zeta}(s)}|x|\right)$, and it is sufficient to prove that both functions $\sqrt{\hat{\zeta}(s)}$ and $\exp \left(-s\sqrt{\hat{\zeta}(s)}|x|\right)$ are completely monotone.

Therefore, it is sufficient to show that $\sqrt{\hat{\zeta}(s)}$ is completely monotone, and $s\sqrt{\hat{\zeta}(s)}$ is a Bernstein function. The non-negativity of the solution can also be shown by proving that the function $\sqrt{\hat{\zeta}(s)}$ is a Stieltjes function, which is again completely monotone, or that $s\sqrt{\hat{\zeta}(s)}$ is a complete Bernstein function. The proof of
the non-negativity of the solutions of the generalised diffusion-wave equations with different memory kernels can be found in [27] along with the list of properties of completely monotone, Stieltjes, Bernstein, and complete Bernstein functions.

By solving Eq. (1.33) we find the MSD,

\[
\langle x^2(t) \rangle = \left\{-\frac{\partial^2}{\partial k^2} \mathcal{L}^{-1} \left[ \tilde{W}(k, s) \right](k, t) \right\}_{k=0} = 2 \mathcal{L}^{-1} \left[ \frac{1}{s^3 \zeta(s)} \right](t),
\]

(1.37)

from where we analyze the diffusive regimes depending on the memory kernel \( \zeta(t) \).

3.2 Particular cases

3.2.1. Standard wave equation The simplest case of Eq. (1.33) is the one with Dirac delta memory kernel \( \zeta(t) = \delta(t) \), which yields the classical wave equation

\[
\frac{\partial^2 W(x, t)}{\partial t^2} = \frac{\partial^2 W(x, t)}{\partial x^2}.
\]

(1.38)

The MSD (1.37) then becomes

\[
\langle x^2(t) \rangle = t^2,
\]

(1.39)

which reports ballistic motion.

3.2.2. Mono-fractional diffusion-wave equation The case with the power-law memory kernel \( \zeta(t) = \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha} \), \( 0 < \alpha < 2 \), yields

\[
CD_t^\alpha W(x, t) = \frac{\partial^2 W(x, t)}{\partial x^2},
\]

(1.40)
for $1 < \alpha < 2$, whereas for the case with $0 < \alpha < 1$ we get

$$
\frac{1}{\Gamma(2 - \alpha)} \int_0^t(t - t')^{1-\alpha} \frac{\partial^2 W(x, t')}{\partial t'^2} \, dt' = \frac{\partial^2 W(x, t)}{\partial x^2}.
$$

(1.41)

The MSD for the mono-fractional diffusion-wave equation reads

$$
\langle x^2(t) \rangle = 2 \frac{t^\alpha}{\Gamma(1 + \alpha)}.
$$

(1.42)

Since $0 < \alpha < 2$, the generalised diffusion-wave equation with power-law memory kernel describes both superdiffusive and subdiffusive processes. The case $\alpha = 1$ reduces to the classical diffusion equation for Brownian motion, i.e., $\langle x^2(t) \rangle = 2t$, whereas the case with $\alpha = 2$ yields ballistic diffusion, $\langle x^2(t) \rangle = t^2$.

3.2.3. Bi-fractional diffusion-wave equation

The next case we consider is the bi-fractional diffusion-wave equation with the memory kernel of the form $\eta(t) = \frac{\Gamma(-\alpha_1)}{\Gamma(2 - \alpha_1)} + \frac{\Gamma(-\alpha_2)}{\Gamma(2 - \alpha_2)}$, $B_1 + B_2 = 1$. The case with $1 < \alpha_1 < \alpha_2 < 2$ yields

$$
B_1 \, \mathcal{C}_D^{\alpha_1} W(x, t) + B_2 \, \mathcal{C}_D^{\alpha_2} W(x, t) = \frac{\partial^2 W(x, t)}{\partial x^2},
$$

(1.43)

where $\mathcal{C}_D^{\alpha_j}$ is the Caputo fractional derivative of the order $1 < \alpha_j < 2$ ($n = 2$), whereas the case $0 < \alpha_1 < \alpha_2 < 1$ yields equation

$$
\frac{B_1}{\Gamma(2 - \alpha_1)} \int_0^t(t - t')^{1-\alpha_1} \frac{\partial^2 W(x, t')}{\partial t'^2} \, dt' + \frac{B_2}{\Gamma(2 - \alpha_2)} \int_0^t(t - t')^{1-\alpha_2} \frac{\partial^2 W(x, t')}{\partial t'^2} \, dt' = \frac{\partial^2 W(x, t)}{\partial x^2}.
$$

(1.44)
The corresponding MSD then becomes
\[
\langle x^2(t) \rangle = \frac{2}{B_2} t^{\alpha_2} E_{\alpha_2-\alpha_1, \alpha_2+1} \left( -\frac{B_1}{B_2} t^{\alpha_1} \right),
\]
which means *decelerating superdiffusion* for \( 1 < \alpha_1 < \alpha_2 < 2 \), including crossover from superdiffusion to normal diffusion in the case \( 1 = \alpha_1 < \alpha_2 < 2 \), and *decelerating subdiffusion* for \( 0 < \alpha_1 < \alpha_2 < 1 \), including crossover from normal diffusion to subdiffusion for the case \( 0 < \alpha_1 < \alpha_2 = 1 \). Decelerating superdiffusion has indeed been observed, for example, in Hamiltonian systems with long-range interactions \[12\], and different biological systems \[2\].

### 3.2.4. Tempered time-fractional wave equation

Furthermore, we consider a truncated power-law memory kernel of the form \( \zeta(t) = e^{-bt t^{1-\alpha}/\Gamma(2-\alpha)} \), where \( b > 0 \), and \( 1 \leq \alpha < 2 \), corresponding to the following tempered fractional wave equation:
\[
\frac{1}{\Gamma(2-\alpha)} \int_0^t e^{-b(t-t')(t-t')^{1-\alpha}} \frac{\partial^2 W(x,t)}{\partial t'^2} \, dt' = \frac{\partial^2 W(x,t)}{\partial x^2}. \tag{1.46}
\]

For the MSD we get
\[
\langle x^2(t) \rangle = 2 RL I^3_t \left( e^{-bt} \frac{t^{-3+\alpha}}{\Gamma(-2+\mu)} \right) \simeq \begin{cases} 2 \frac{t^\alpha}{b^{2-\alpha}t^2}, & t \ll 1, \\ \frac{2}{B_t} \frac{t^\alpha}{\Gamma(1+\alpha)}, & t \gg 1, \end{cases} \tag{1.47}
\]
where
\[
RL I^\alpha_t f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-t')^{\alpha-1} f(t') \, dt', \quad \alpha > 0, \tag{1.48}
\]
is the Riemann-Liouville integral [13]. Thus, there is a crossover from superdiffusion to ballistic motion in the case with $1 < \alpha < 2$, and from normal diffusion to ballistic motion in the case with $\alpha = 1$. For the case of the diffusion-wave equation with Prabhakar derivative we address the reader to [27].

4. SUMMARY

We consider different stochastic processes governed by the generalised diffusion and diffusion-wave equations which contain the well known time fractional diffusion and wave equations as particular cases. Such processes demonstrate a rich multi-scaling behaviour which manifests itself in specific crossovers between different diffusion regimes in the course of time. We thus obtain a flexible tool which can be applied for the description of diverse diffusion phenomena in complex systems demonstrating crossover behaviours.

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