GAPS IN PROBABILITIES OF SATISFYING SOME COMMUTATOR-LIKE IDENTITIES

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ABSTRACT. We show that there is a positive constant $\delta < 1$ such that the probability of satisfying either the 2-Engel identity $[X_1, X_2, X_2] = 1$ or the metabelian identity $[[X_1, X_2], [X_3, X_4]] = 1$ in a finite group is either 1 or at most $\delta$.

1. INTRODUCTION

It is an old, elegant, well-known, and at the same time somewhat surprising result that the probability that two randomly chosen elements commute in a nonabelian finite group can not be arbitrarily close to 1. To be more precise (see [3]), commuting probability of no finite group can belong to the interval $(\frac{2}{3}, 1)$, and so there is a gap in the possible probability values. Following on this, many other, deeper results on the structure of the set of all possible values of the probability of satisfying the commutator identity have since emerged (see [2] and the references therein).

Recently, more general word maps on finite groups have been explored from a standpoint of a similar probabilistic flavour. Here, a word map on a group $G$ is a map $w: G^d \to G$ induced by substitution from a word $w \in F_d$ belonging to a free group of rank $d$. For a fixed element $g \in G$ of a finite group $G$, set

$$\mathbb{P}_{w=g}(G) = \frac{|w^{-1}(g)|}{|G|^d}$$
to be the probability that \( w(g_1, g_2, \ldots, g_d) = g \) in \( G \), where \( g_1, g_2, \ldots, g_d \) are chosen independently according to the uniform probability distribution on \( G \). Following recent breakthroughs on the values of these probabilities for finite simple groups (see [8] and [1]), applications have been developed also for infinite groups (see [5] for an approach via the Hausdorff dimension for residually finite groups), indicating how these probabilities of finite quotients of a given infinite group are tightly related to its algebraic structure (see [11], as well as [6] for a more geometric approach). This has been done mostly for the simple and longer commutator words, the results ultimately resting on the aforementioned probability gap and its stronger variants (see [7]).

The purpose of this paper is to investigate some words that are natural generalizations of the simple commutator. Our main result shows the existence of gaps in probabilities of satisfying these words.

**Main Theorem.** Let \( w \) be either the \( 2 \)-Engel or the metabelian word. There exists a constant \( \delta < 1 \) such that whenever \( w \) is not an identity in a finite group \( G \), we have \( \mathbb{P}_{w=1}(G) \leq \delta \).

The strategy of the proof is quite general and might be applied to some other words. It goes as follows. Suppose \( G \) is a finite group in which \( w \) is not an identity. Consider a chief factor \( G_1/G_2 = T^k \) of \( G \), where \( T \) is a simple group and \( k \geq 1 \).

If \( T \) can be chosen to be nonabelian, then \( T \) does not satisfy the word \( w \). For the purposes of our claim, we can replace \( G \) by its quotient \( G/C_G(G_1/G_2) \) and therefore assume that \( T^k \leq G \leq \text{Aut}(T^k) \) (see Section 2). Word probabilities in such groups have been studied extensively. As long as \( T \) is a large enough simple group, our claim follows from the works of Larsen and Shalev (see [5]). For small groups \( T \), it is necessary to bound their multiplicities \( k \) in \( G \). This is tightly related to the concept of varied coset identities of \( T \) (see [8]). By inspecting the required cases following Bors (see [1]), we are able to achieve the goal of bounding these multiplicities for both the \( 2 \)-Engel word and the metabelian word (see Subsection 2.2).

**Theorem.** The \( 2 \)-Engel and the metabelian word are multiplicity bounding.

The proof relies of inspecting fixed points of outer automorphisms of finite simple groups for the \( 2 \)-Engel word, whereas a more direct algebraic manipulation works for the metabelian word. Along the way, we also investigate coset probabilities of finite groups with respect to nonsolvable normal subgroups (see Subsection 2.10). We show that in infinite groups that possess
infinitely many nonabelian upper composition factors, the infimum of probabilities of satisfying the 2-Engel or metabelian word in its finite quotients is 0.

On the other hand, if all the chief factors of $G$ are abelian, then $G$ is solvable. In this case, the verbal subgroup $w(G)$ can be assumed to be the unique minimal normal subgroup of $G$, and so it is a vector space over a finite field. We proceed by analysing the linear representation of $G$ on this space. As long as this representation is nontrivial, we are able to provide a general procedure on how to establish a word probability gap (see Subsection 3.1). This is then executed for the 2-Engel word, where the only problematic elements are those acting quadratically on $w(G)$, and for the metabelian word, where the situation is simpler due to invariance of variables. In the case when the verbal subgroup is 1-dimensional, it is not possible to obtain any information from the representation alone. Here, we instead consider the restriction of the word map on the coordinate axes (see Subsection 4.1). An argument involving the analysis of whether or not such an induced map is trivial works for both the 2-Engel word and the metabelian one. Joint with the above Theorem, we conclude the validity of the Main Theorem.

The explicit and sharp value of $\delta$ in the Main Theorem could, in principle, be determined by examining the proof. For this, one would need to compute the probabilities $\mathbb{P}_{w=1}(G)$ for all finite groups $G$ with $T^k \leq G \leq \text{Aut}(T^k)$, where $T$ is a nonabelian finite simple group and both $|T|$ and $k$ are bounded in terms of $w$. The difficulty lies in finding good bounds. On the other hand, when restricting only to solvable groups, it follows from our proofs that, for the 2-Engel word, one can take $\delta = \frac{3}{4}$, equality being attained with the dihedral group $D_{16}$, and for the metabelian word, one has $\delta \leq \frac{29}{32}$, but this bound might not be sharp.

A word on the notation. The generators of the free group $F_d$ will be denoted by $X_1, \ldots, X_d$. The multiplicity of $X_i$ in $w \in F_d$ will be denoted by $\mu_w(X_i)$. The length of $w$ will be denoted by $\ell(w)$.

2. Nonsolvable groups

In this section, we deal with bounding the probability $\mathbb{P}_{w=1}(G)$ for finite nonsolvable groups $G$. We will repeatedly use the following reduction lemma.

**Lemma.** Let $w$ be a nontrivial word. Let $G$ be a finite group and $N$ its normal subgroup. Then $\mathbb{P}_{w=g}(G) \leq \mathbb{P}_{w=gN}(G/N)$ for every $g \in G$. 
We can therefore replace $G$ by its quotient $G/C(G)$ and whence reduce our claim to bounding the probability $P_{w=1}(G)$ in the case when $T^k \leq G \leq \text{Aut}(T^k)$ for a nonabelian finite simple group $T$.

2.1. Large simple groups.

**Lemma 2.1.1** ([5], Theorem 1.8). Let $G$ be a finite group such that $T^k \leq G \leq \text{Aut}(T^k)$ for some $k \geq 1$ and a finite nonabelian simple group $T$. Suppose $w$ is a nontrivial word. Then there exist constants $C = C(w)$, $\epsilon = \epsilon(w) > 0$ depending only on $w$ such that, if $|T| \geq C$ we have $P_{w=g}(G) \leq |T^k|^{-\epsilon}$.

As long as $|T| > C$, we therefore have $P_{w=1}(G) < C^{-\epsilon}$.

2.2. Multiplicity bounding words. A reduced word $w$ is called multiplicity bounding (see [1]) if, whenever $G$ is a finite group such that $P_{w=g}(G) > \rho$ for some $g \in G$, the multiplicity of a nonabelian simple group $S$ as a composition factor of $G$ can be bounded above by a function of only $\rho$ and $S$.

Whenever our word $w$ is multiplicity bounding, we can solve our problem for groups $G$ with $|T| \leq C$. Namely, for each of these nonabelian groups $T$, we either have that $P_{w=1}(G) \leq \frac{1}{2}$, in which case we are done, or we can assume that $P_{w=1}(G) > 1/2$. In the latter case, the multiplicity $k$ of $T$ is bounded above by a constant depending only on $w$ and $T$. This means that $|G| \leq |\text{Aut}(T^k)|$ is bounded above by a constant, and we therefore have an upper bound for $P_{w=1}(G)$ as well.

We have thus proved the following.

**Proposition 2.2.1.** Let $w \in F_d$ be a multiplicity bounding word. Then there exists a constant $\delta = \delta(w) < 1$ such that every nonsolvable finite group $G$ satisfies $P_{w=1}(G) \leq \delta$.

It therefore remains to deal with proving that the words we are interested in are indeed multiplicity bounding. In fact, we will prove that these words satisfy a stronger property.

2.3. Coset word maps and variations. Let $S$ be a nonabelian finite simple group. The word $w \in F_d$ defines a word map $S^d \to S$ by evaluation $(s_1, \ldots, s_d) \mapsto w(s_1, \ldots, s_d)$. Consider $S \leq \text{Aut}(S)$ and let $g_1, \ldots, g_d \in \text{Aut}(S)$. Then there is a corresponding coset word map $S^d \to S$ defined by $(s_1, \ldots, s_d) \mapsto w(s_1g_1, \ldots, s_dg_d)$.

We will also require the notion of a variation of $w$. This is a word $\tilde{w}$ obtained from $w$ by adding, for each $1 \leq i \leq d$, to each occurrence of
$X_i^\pm 1$ in $w$ a second index from the range $\{1, 2, \ldots, \mu_w(X_i)\}$. To each such variation $\tilde{w}$, we can associate a varied coset word map of $w$, which is just a coset word map of the variation $\tilde{w}$.

2.4. **Very strongly multiplicity bounding words.** A word $w$ is called very strongly multiplicity bounding (VSMB) if for all nonabelian finite simple groups $S$, none of the varied coset word maps of $w$ on $S$ is constant. Such words are multiplicity bounding (see [1, Proposition 2.9]).

**Example 2.4.1** ([1], Corollary 3.4). Long commutator words $\gamma_n(X_1, \ldots, X_n) = [X_1, \ldots, X_n]$ are all VSMB.

The following criterion for being VSMB will be of use.

**Lemma 2.4.2** ([1], Proposition 3.1, Proposition 6.1). The following words are VSMB.

(i) Words in which some variable occurs with multiplicity 1.

(ii) Words in which some variable occurs with multiplicity 2, provided that either $w = w_1X_dw_2X_dw_3$ or $w = w_1X_{\overline{d}}w_2X_{\overline{d}}w_3$, where $w_1, w_2, w_3$ are reduced and $\overline{w}_2$ is VSMB.

(iii) Words of length at most 8 excluding the power word $X_1^8$.

In order to verify that a given word is VSMB, it suffices to inspect only a limited set of simple groups. The following criterion will suffice here.

**Lemma 2.4.3** ([1], Proposition 4.9 (8)). Let $w \in F_d$ be a reduced word. Set $m = \max_i \mu_w(X_i)$. Then $w$ is VSMB as long as the word map of $w$ on $\text{PSL}_2(2)$ and $\text{Sz}(2)$ is not constant and none of the varied coset words maps on $\text{PSL}_2(p^n)$ for a prime $2 < p \leq m$ and $n$ a power of 2 is constant.

2.5. **Automorphisms of the relevant simple groups.** We will be inspecting coset word maps on the simple groups from Lemma 2.4.3. For this, we will need to understand cosets of inner automorphisms of these groups.

The automorphisms of $\text{PSL}_2(2^n)$ and $\text{Sz}(2^n)$ consist of inner automorphisms and field automorphisms. The field automorphisms are generated by the Frobenius automorphism $\sigma$ that extends the field automorphism $\mathbb{F}_{2^n} \to \mathbb{F}_{2^n}, x \mapsto x^2$. The order of $\sigma$ in $\text{Aut}(S)$ is equal to $n$, which is assumed to be a prime.

As for the groups $\text{PSL}_2(p^n)$ with $p$ odd, there is an additional outer automorphism $D \in \text{Aut}(S)$ induced by conjugation with the diagonal matrix

$$
\begin{pmatrix}
\omega & 0 \\
0 & 1
\end{pmatrix}.
$$
where \( \omega \) is a generator of \( \mathbb{F}_{2^n}^\times \). This automorphism fixes all the diagonal matrices. Moreover, it satisfies the relation \([D, \sigma] = D^{p-1}\). Therefore every element of the group \( \langle D, \sigma \rangle \) can be written uniquely as \( \sigma^i D^j \) with \( 0 \leq i < n \) and \( 0 \leq j < p^n - 1 \). The square \( D^2 \) is an inner automorphism, given as conjugation with the matrix
\[
\begin{pmatrix}
\omega^2 & 0 \\
0 & 1
\end{pmatrix} = \begin{pmatrix}
\omega & 0 \\
0 & \omega^{-1}
\end{pmatrix} \in \text{PGL}_2(p^n).
\]

We will require the following property of these outer automorphisms.

**Lemma 2.5.1.** Let \( 1 \neq \alpha \in \langle D, \sigma \rangle \leq \text{Aut}(\text{PSL}_2(p^n)) \) with \( p > 2 \) and \( n \) a power of 2. Then
\[
|\text{Fix}(\alpha)| \leq \frac{p^n}{2} (p^n - 1).
\]

**Proof.** Suppose \( \alpha = \sigma^i D^j \). We work in the cover \( \text{SL}_2(p^n) \). Fixed points of \( \alpha \) correspond to solutions of the system
\[
\begin{pmatrix}
a^p^i & b^p^i \\
c^p^i & d^p^i
\end{pmatrix} = \pm \begin{pmatrix}
a & b \omega^j \\
c \omega^{-i} & d
\end{pmatrix}
\]
for \( a, b, c, d \in \mathbb{F}_{p^n} \).

If \( i = 0 \), then a fixed point can only be a diagonal matrix, and the number of these in \( \text{PSL}_2(p^n) \) is at most \( (p^n - 1)/2 \). This also covers the case when \( n = 1 \).

Now assume that \( i > 0 \). Note that since \( n \) is a power of 2, \(-1\) is a square in \( \mathbb{F}_{p^n} \). The diagonal fixed points of \( \alpha \), i.e., the cases when \( b = c = 0 \), correspond to fixed points of \( \sigma^i \). On the other hand, as long as the matrix is not diagonal, i.e., \( bc \neq 0 \), a solution is possible if and only if \( j \) is divisible by \( p^i - 1 \), whence by \( p - 1 \). Write \( j = (p - 1)k \) for some \( k \), so that \( \alpha = \sigma^i D^{(p-1)k} = D^k \sigma^i D^{-k} \). In this situation, an element \( x \in \text{PSL}_2(p^n) \) is fixed under \( \alpha \) if and only if \( x D^k \) is fixed under \( \sigma^i \). All in all, we therefore have that
\[
|\text{Fix}(\alpha)| = |\text{Fix}(\sigma^i)| = |\text{Fix}(\sigma^{n/2})| = |\text{PSL}_2(p^{n/2})|. \]

**Lemma 2.5.2.** Let \( \alpha \in \text{Aut}(S) \). Set
\[
ad_{\alpha} : S \to S, \quad a \mapsto [a, \alpha].
\]
Then
\[
|\text{im} \ ad_{\alpha}| = \frac{|S|}{|\text{Fix}(\alpha)|}.
\]

**Proof.** For elements \( a, b \in S \), we have \( \text{ad}_{\alpha}(a) = \text{ad}_{\alpha}(b) \) if and only if \( ab^{-1} = (ab^{-1})^\alpha \), which is the same as saying that \( a \) and \( b \) belong to the same coset of \( \text{Fix}(\alpha) \). The claim follows immediately.
2.6. 2-Engel word: Variations. Let \( w = [X_1, X_2, X_2] \) be the 2-Engel word. In expanded form, this is

\[
w = X_2^{-1}X_1^{-1}X_2X_1X_2^{-1}X_1^{-1}X_2^{-1}X_1X_2X_2.
\]

If a group satisfies the word \( w \), it must be nilpotent. Whence the word map of \( w \) on \( \text{PSL}_2(2) \) ∼ \( S_3 \) and \( \text{Sz}(2) \) ∼ \( C_5 \rtimes C_4 \) is not constant. It follows from Lemma 2.4.3 that in order to verify that \( w \) is VSMB, we only need to consider the varied coset word maps of \( w \) on \( \text{PSL}_2(p^n) \) with \( p > 2 \) and \( n \) a power of 2.

It follows from Lemma 2.4.2 that every variation of \( w \) in which appearances of \( X_1^{\pm 1} \) are replaced by using more than one second index are VSMB. Therefore we only need to vary occurrences of the variable \( X_2 \). A general variation of \( w \) can therefore be assumed to be of the form

\[
\tilde{w} = Y_1^{-1}X^{-1}Y_2XY_3^{-1}X^{-1}Y_4^{-1}XY_5Y_6
\]

with some of the \( Y_i \) being potentially equal. Using Lemma 2.4.2, we can further reduce the words that need to be checked. As long as precisely two \( Y_i \)'s are equal, the variation itself is VSMB. Whence we can assume that either we are dealing with the original word or with a variations in which each \( Y_i \) is equal to a variable \( Z_1 \) or \( Z_2 \).

For any such word \( \tilde{w} \), let \( S = \text{PSL}_2(p^n) \) with \( p > 2 \) and \( n \) a power of 2; we only need to inspect these simple groups by Lemma 2.4.3. Consider the coset word map induced by elements \( x, y_1, \ldots, y_6 \) in \( \text{Aut}(S) \). Here we can assume that \( x, y_i \) belong to the subgroup of \( \text{Aut}(S) \) generated by the field and diagonal automorphisms. We want to show that this word map is not constant. To this end, assume the contrary. For any \( a, b_1, \ldots, b_6 \in S \), we therefore have

(2.6.1) \( \tilde{w}(ax, b_1y_1, \ldots, b_6y_6) = \tilde{w}(x, y_1, \ldots, y_k) \).

2.7. 2-Engel word: Inspecting the original. Let us first deal with the original 2-Engel word. Insert \( b_i = 1 \) into (2.6.1) and collect the left hand side to get that

(2.7.1) \( a^{-xy}a^{y^{-1}xy}a^{-xy^{-1}y^{-1}xy}a^{xy^{-1}y^{-1}xy} \equiv 1 \)

for all \( a \in S \). Cancelling \( xy \) and collecting, we obtain

\[
[a, y^{-1}][a, y]^{xy^{-1}y^{-1}} \equiv 1,
\]

and by additionally cancelling \( y^{-1} \), it follows that

\[
[a, y]^{-1}[a, y]^{xy^{-1}} \equiv 1.
\]
This can be rewritten as

\[(2.7.2) \quad [a, y, xyx^{-1}] \equiv 1,\]

which is the same as saying that

\[\text{im ad}_y \subseteq \text{Fix}(yx^{-1}).\]

Comparing the sizes, it now follows from Lemma 2.5.2 and Lemma 2.5.1 that, as long as \(y\) is nontrivial,

\[
\frac{p^n (p^{2n} - 1)}{2} = |\text{PSL}_2(p^n)| \leq |\text{Fix}(y)| \cdot |\text{Fix}(xyx^{-1})| \leq \frac{p^n (p^n - 1)^2}{4}
\]

This is impossible. Whence \(y = 1\) and in this case it is clear that (2.6.1) can not hold.

2.8. **2-Engel word: Inspecting the variations.** In each of the proper variations \(\tilde{w}\), we have a variable \(X\) and two other variables \(Z_1, Z_2\), each one with multiplicity 3. Consider (2.6.1) with the variables \(ax, by, cz\). Insert \(a = c = 1\). After collecting, we obtain

\[(2.8.1) \quad b^{\pm \alpha}b^{\pm \beta}b^{\pm \gamma} \equiv 1\]

for some fixed \(\alpha, \beta, \gamma \in \langle D, \sigma \rangle \leq \text{Aut}(S)\) depending on \(x, y, z\). Insert the diagonal matrix \(D^2 \in \text{PSL}_2(p^n)\) into the last equality. The automorphisms \(\alpha, \beta, \gamma\) act on it as powers of the Frobenius automorphism \(\sigma\). We obtain

\[(D^2)^{\pm p^i \pm p^j \pm p^k} \equiv 1\]

for some fixed \(0 \leq i, j, k < n\). Note, however, that \(D^2\) is of order \((p^n - 1)/2\). This number is even as long as \(n\) is a proper power of 2. On the other hand, the sum \(\pm p^i \pm p^j \pm p^k\) is always odd. Thus we are forced into the conclusion \(n = 1\). Therefore it suffices to verify that \(\tilde{w}\) can not satisfy an identity of the form (2.6.1) in the group \(\text{PSL}_2(3) \cong \text{Alt}(4)\). In this group, insert the element \((1\ 2)(3\ 4)\) into (2.8.1). This element is fixed by \(D\), which is represented as conjugation by \((1\ 2)\). The automorphisms \(\alpha, \beta, \gamma\) act as powers of \(D\), and whence we obtain

\[((1\ 2)(3\ 4))^{\pm 1 \pm 1} = 1\]

a contradiction which completes our analysis of the variations.

2.9. **Metabelian word.** In this section, we deal with the metabelian word \([[[X_1, X_2], [X_3, X_4]]]\). In expanded form, this is

\[X_2^{-1}X_1^{-1}X_2X_1X_4^{-1}X_3^{-1}X_4X_3X_1^{-1}X_2^{-1}X_1X_2X_3^{-1}X_4^{-1}X_3X_4.\]

By Lemma 2.4.2, any proper variation of the metabelian word is VSMB. Therefore it suffices to consider only the original word.
Let $S$ be a nonabelian finite simple group. Consider the coset word map induced by elements $x, y, z, t \in \text{Aut}(S)$. Assume that this map is constant on $S$. For any $a, b, c, d \in S$, we therefore have

$$[[ax, by], [cz, dt]] = [[x, y], [z, t]] = \text{const}. $$

Expand the first commutator to get

$$[[a, by]^x, [cz, dt]] = \text{const}. $$

and once again

$$[[a, by]^x, [cz, dt]][x, by][x, by] = \text{const}. $$

The second factor is constant, since it equal to the original word map with $a = 1$. We conclude that

$$[[a, by]^x, [cz, dt]][x, by][x, by] = \text{const}. $$

Inserting $a = 1$, we see that the value of the last word must in fact be equal to 1. Whence

$$([a, by]^x, [cz, dt])[x, by][x, by] = 1. $$

Expand the first commutator, now in the second variable, to obtain

$$[[a, y]^x[a, b]y^x, [cz, dt]] = 1, $$

and once again

$$[[a, y]^x[a, b]y^x, [cz, dt]] = 1. $$

We see that the first factor is trivial by inserting $b = 1$ into (2.9.1). It follows that

$$[a, b]y^x, [cz, dt] = 1. $$

Since $S$ is a perfect group, we now conclude that the element $[cz, dt]$ must fix the whole of $S$, and so $[z, t] = 1$. Whence $[z, t] = 1$, forcing $[c, t] = 1$ and similarly $[z, t] = 1$. This gives $z = t = 1$. A symmetric argument shows that $x = y = 1$. But now $S$ should satisfy the metabelian identity, a contradiction.

2.10. **Coset probabilities.** For a group $G$, elements $g_1, g_2, \ldots, g_d$ and a normal subgroup $N$ of $G$, denote

$$\mathbb{P}_{w=1}^{(g_1, g_2, \ldots, g_d)}(N) = \frac{|\{(n_1, n_2, \ldots, n_d) \in N^d | w(n_1 g_1, n_2 g_2, \ldots, n_d g_d) = 1\}|}{|N|^d}. $$

This is the coset probability of $N$ in $G$ of satisfying the word $w$. Taking $N = G$, we recover the ordinary probability of satisfying $w$ in $G$. As long as $N$ is not solvable, it is possible to universally bound this coset probability.
Proposition 2.10.1. Let $w$ be a VSMB word. There exists a constant $\mu < 1$ such that whenever $G$ is a finite group and $N$ its nonsolvable normal subgroup, we have, for all $g_1, \ldots, g_d \in G$,

$$\mathbb{P}_{w=1}^{(g_1,g_2,\ldots,g_d)}(N) \leq \mu \cdot 1_{w(g_1,g_2,\ldots,g_d)\in N}.$$ 

Proof. As long as $\mathbb{P}_{w=1}^{(g_1,g_2,\ldots,g_d)}(N) > 0$, we must have $w(g_1,g_2,\ldots,g_d) \in N$. This is the reason why we include the indicator function in the statement.

Let $T^k$ be a composition factor of $N$ with $T$ a nonabelian simple group. After replacing $G$ by a suitable quotient, we can assume that $C_G(T^k) = 1$, and so $T^k \leq G \leq \text{Aut}(T^k)$.

There exists $C > 0$ depending only on $w$ such that whenever $|T| \geq C$, we have (see [5, Theorem 4.5] together with [1, Lemma 2.7])

$$\mathbb{P}_{w=1}^{(g_1,g_2,\ldots,g_d)}(N) \leq \frac{1}{2}.$$ 

Therefore it suffices to consider only finitely many options for the simple group $T$.

As $w$ is assumed to be a VSMB word, no varied coset word map on $T$ is constant (see [1, Definition 2.8]). There now exists an $0 < \epsilon < 1$ depending only on $w$ and $T$ such that (see [1, Lemma 2.12])

$$|\{(n_1,n_2,\ldots,n_d) \in N^d \mid w(n_1g_1,n_2g_2,\ldots,n dg_d) = 1\}| \leq \epsilon^{[k/\ell(w)^2]|T|^{kd}}.$$ 

Therefore there exists an $\bar{C} > 0$ such that whenever $k \geq \bar{C}$, we have

$$\mathbb{P}_{w=1}^{(g_1,g_2,\ldots,g_d)}(N) \leq \frac{1}{2}.$$ 

Therefore it suffices to consider only finitely many options for the multiplicity $k$ of $T$.

Now, as $G \leq \text{Aut}(T^k)$, there are only finitely many options left for the group $G$. None of these groups satisfy a coset identity since $w$ is VSMB, and so each value $\mathbb{P}_{w=1}^{(g_1,g_2,\ldots,g_d)}(N)$ is smaller than 1. Thus we can take $\delta$ to the maximum of all these values and $\frac{1}{2}$.

A consequence of the existence of this bound is the following bound for the probability of satisfying $w$ when extending groups.

Corollary 2.10.2. Let $w$ be a VSMB word. There exists a constant $\mu < 1$ such that whenever $G$ is a finite group and $N$ its nonsolvable normal subgroup, we have

$$\mathbb{P}_{w=1}(G) \leq \mu \cdot \mathbb{P}_{w=1}(G/N).$$
Proof. Let $\mathcal{R}$ be a set of coset representatives of $N$ in $G$. We have
\[
P_{w=1}(G) = \frac{1}{|G|^d} \sum_{(r_1, r_2, \ldots, r_d) \in \mathcal{R}^d} |N|^d \cdot P_{w=1}^{(r_1, r_2, \ldots, r_d)}(N).
\]
Bounding the latter probability using Proposition 2.10.1, we obtain
\[
P_{w=1}(G) \leq \frac{1}{|G/N|^d} \sum_{(r_1, r_2, \ldots, r_d) \in \mathcal{R}^d} \mu \cdot \mathbb{1}_{w(r_1, r_2, \ldots, r_d) \in N} = \mu \cdot P_{w=1}(G/N),
\]
as claimed. □

Corollary 2.10.3. Let $w$ be a VSMB word. Let $G$ be a group with a chain $N_1 > N_2 > \ldots$ of normal subgroups of finite index in $G$ such that the consecutive factors $N_i/N_{i+1}$ are not solvable. Then
\[
\lim_{i \to \infty} P_{w=1}(G/N_i) = 0.
\]
Proof. Immediate by Corollary 2.10.2. □

3. Solvable groups with higher dimensional verbal subgroup

In this section, we deal with bounding the probability $P_{w=1}(G)$ for finite solvable groups $G$. As explained above, this is reduced to bounding $P_{w=1}(G)$ in the case when the verbal subgroup of $w$ in $G$, denoted throughout by $V$, is a minimal normal subgroup that is a vector space over a finite field $\mathbb{F}_p$, say of dimension $n$. We will exploit this action, so we assume throughout this section that $n > 1$.

3.1. General principle for bounding the probability. Let $\mathcal{R}$ be a set of coset representatives for $V$ in $G$. The probability of satisfying $w$ in $G$ can be expressed as
\[
P_{w=1}(G) = \frac{1}{|G|^d} \sum_{a_i \in V, r_i \in \mathcal{R}} \mathbb{1}_{w(a_1 r_1, \ldots, a_d r_d) = 1}.
\]
Each summand can be expanded as
\[
w(a_1 r_1, \ldots, a_d r_d) = \prod_{i=1}^d a_i^{w_i(r_1, \ldots, r_d)} \cdot w(r_1, \ldots, r_d)
\]
for some endomorphisms $w_i(r_1, \ldots, r_d) \in \text{End}(V)$. Set
\[
\text{BAD} = \{(r_1, \ldots, r_d) \in \mathcal{R} \mid \forall i. w_i(r_1, \ldots, r_d) \in C_G(V)\}.
\]
This set consists of those tuples of elements of $\mathcal{R}$ for which a summand above is independent of the values $a_i \in V$. Thus, these tuples are providing
a coset identity. Correspondingly, set \( \text{GOOD} = \mathcal{R}^d - \text{BAD} \). By first summing over the bad representatives, we have

\[
\frac{1}{|G|^d} \sum_{(r_1, \ldots, r_d) \in \text{BAD}} |V|^d \cdot \mathbb{1}_{w(r_1, \ldots, r_d) = 1} \leq \frac{|\text{BAD}|}{|\mathcal{R}|^d}.
\]

On the other hand, for a good tuple of representatives, at least one exponential endomorphism, say \( w_j(r_1, \ldots, r_d) \), acts nontrivially on \( V \). Its centralizer in \( V \) is therefore of codimension at most 1. In this case, we have

\[
|\{a_j \mid w(a_1 r_1, \ldots, a_d r_d) = 1\}| \leq |C_V(w_j(r_1, \ldots, r_d))| \leq \frac{|V|}{p},
\]

and it follows from this that by summing over the good representatives, we have

\[
\frac{1}{|G|^d} \sum_{(r_1, \ldots, r_d) \in \text{GOOD}} \sum_{a \in V} \mathbb{1}_{w(a_1 r_1, \ldots, a_d r_d) = 1} \leq \frac{1}{|G|^d} \sum_{(r_1, \ldots, r_d) \in \text{GOOD}} |V|^{d-1} \frac{|V|}{p} = \frac{|\text{GOOD}|}{p|\mathcal{R}|^d}.
\]

We can collect the two upper bounds to finally obtain

\[
\mathbb{P}_{w=1}(G) \leq \frac{|\text{BAD}|}{|\mathcal{R}|^d} + \frac{|\text{GOOD}|}{p|\mathcal{R}|^d}.
\]

Taking \( |\text{BAD}| + |\text{GOOD}| = |\mathcal{R}|^d \) into account, we can take the latter one step further and write

\[
\mathbb{P}_{w=1}(G) \leq \frac{1}{p} + \left( 1 - \frac{1}{p} \right) \frac{|\text{BAD}|}{|\mathcal{R}|^d} \leq \frac{1}{2} \left( 1 + \frac{|\text{BAD}|}{|\mathcal{R}|^d} \right).
\]

We will use this general principle for bounding the word probability. In order for it to give us a gap on word probability, we will need to show that for a given word \( w \), there is a gap on the relative size of the set \( \text{BAD} \) inside \( \mathcal{R}^d \).

### 3.2. 2-Engel word: Inspecting badness

In this section, we focus on the case of the 2-Engel word \([ax, by, by]\) with \( a, b \in V \) and \( x, y \in \mathcal{R} \). In order to obtain the equations for defining the \( \text{BAD} \) representatives, collect the word value in terms of \( a \) and \( b \). This can be done separately. We first get, as in (2.7.1) and simplified to (2.7.2),

\[
[a, y, xyx^{-1}] \equiv 1,
\]

so that the operator \((1 - y)(1 - xyx^{-1})\) acts trivially on \( V \). On the other hand, the condition that \([x, by, by]\) be constant can be translated by expanding commutators into

\[
[x, y, b]_y [x, b]_y^y, y] \equiv 1.
\]

Collecting the exponents at \( b \), we see that the operator \((1 - [x, y]) - (1 - x)(1 - y)\) must also annihilate everything on \( V \). Now, since \([x, y]\) commutes
with $y$ in its action on $V$, it follows that $(1 - x)(1 - y)$ also commutes with $y$, and whence $x(1 - y)$ also commutes with $y$. Thus we obtain

$$0 \equiv (1 - y)(1 - xy^{-1}) = (1 - y)x(1 - y)x^{-1} = (1 - y)^2.$$ 

The latter means that $y$ acts quadratically on $V$, i.e., for all $a \in V$ we have $[a, y, y] = 1$. Therefore

$$[a, y^p] = [a, y]^p = 1,$$

and so $y^p \in C_G(V)$. We have thus derived the inclusion

$$\text{BAD} \subseteq \mathcal{R} \times \{ y \in \mathcal{R} \mid y^p \in C_G(V) \}.$$ 

3.3. 2-Engel word: Bounding badness in a nontrivial action. Suppose that the action of $G$ on $V$ is nontrivial, that is $C_G(V) \neq G$. We can view $V$ as a modular irreducible representation of $G/C_G(V)$. Since $G/C_G(V)$ satisfies the 2-Engel word, it is nilpotent. Now, $G/C_G(V)$ cannot be a $p$-group, since the only irreducible representation of a $p$-group in characteristic $p$ is the trivial one. Whence the Sylow $p$-subgroup $P/C_G(V) \leq G/C_G(V)$ is proper. We can identify $\mathcal{R}$ with cosets of $V$ in $G$, and in this sense

$$\text{BAD} \subseteq \mathcal{R} \times (P/V).$$

This gives the desired bound for the relative size of bad representatives,

$$\frac{\lvert \text{BAD} \rvert}{\lvert \mathcal{R} \rvert^2} \leq \frac{|P|}{|V||\mathcal{R}|} = \frac{1}{|G : P|} \leq \frac{1}{2}.$$ 

3.4. Metabelian word: Inspecting badness. In this section, we focus on the case of the metabelian word $[[ax, by], [cz, dt]]$ with $a, b, c, d \in V$ and $x, y, z, t \in \mathcal{R}$. Collecting each word value separately, we obtain

$$(1 - x)y(1 - [z, t]) \equiv (1 - y)x(1 - [z, t]) \equiv 0$$

and similarly for the symmetric situation. Set $A = 1 - [z, t]$ and $B = 1 - [x, y]$. Thus $xyA = yA$ and $yxA = xA$. Note that $yx - xy = yxB$. Now, since $[x, y]$ and $[z, t]$ induce commuting operators on $V$, it follows that

$$(x - y)A = (yx - xy)A = yxBA = yxAB = xAB.$$ 

This gives

$$yA = xA - xAB = xA[x, y] = x[x, y]A = y^{-1}xyA = y^{-1}yA = A.$$ 

(Similarly we can derive other equalities.) This means that we have, for all $a \in V$,

$$[[a, y], [z, t]] = 1.$$ 

Thus we have the inclusion

$$\text{BAD} \subseteq \mathcal{R} \times \{(y, z, t) \in \mathcal{R}^3 \mid [V, y, [z, t]] = 1\}. $$
3.5. Metabelian word: Bounding badness in higher dimensions.

Suppose that the action of $G$ on $V$ is nontrivial, that is $C_V(G) \neq G$. We can identify $\mathcal{R}$ with cosets of $V$ in $G$. Let

$$Y_{[z,t]} = \{ y \in G/V \mid [V, y, [z, t]] = 1 \}.$$  

Note that for $y_1, y_2 \in Y_{[z,t]}$, we have

$$[a, y_1 y_2, [z, t]] = [a, y_2, [z, t]]^{[a, y_1]} \cdot [a, y_1, [z, t]]^{[a, y_1, y_2]} \cdot [[a, y_1], y_2, [z, t]] = 1,$$

and so $Y_{[z,t]}$ is a subgroup of $G$. As long as $[z, t]$ is not trivial in $G/C_V(V)$, this is a proper subgroup, since $[V, G] = V$. Set

$$\text{UGLY} = \{(z, t) \in (G/V)^2 \mid [z, t] \in C_G(V)\}.$$

Thus we can express

$$\text{BAD} = (\mathcal{R}^2 \times \text{UGLY}) \cup \left( \mathcal{R} \times \bigcup_{(z,t) \in \mathcal{R}^2 - \text{UGLY}} Y_{[z,t]} \right)$$

and compute, taking into account that $Y_{[z,t]}$ is of index at least 2 in $G/V$,

$$|\text{BAD}| \leq |\mathcal{R}|^2 |\text{UGLY}| + |\mathcal{R}| (|\mathcal{R}|^2 - |\text{UGLY}|) \frac{|G/V|}{2}.$$

Thus we obtain a bound for the badness ratio,

$$\frac{|\text{BAD}|}{|\mathcal{R}|^4} \leq \frac{1}{2} + \frac{1}{2} \frac{|\text{UGLY}|}{|\mathcal{R}|^2}.$$

As long as $G/C_G(V)$ is not abelian, we obtain the last bound

$$\frac{|\text{UGLY}|}{|\mathcal{R}|^2} = \frac{|\{(z, t) \in (G/C_G(V))^2 \mid [z, t] = 1\}|}{|G/C_G(V)|^2} \leq \frac{5}{8}.$$

Thus we are left with inspecting the case when $G/C_G(V)$ is abelian. But this means that its irreducible module $V$ is 1-dimensional, a case that will be dealt with in the following section.

4. Solvable groups with 1-dimensional verbal subgroup

In this case, every tuple in $\mathcal{R}^4$ is bad, and so our general procedure for bounding the word probability in terms of counting bad tuples does not work. We will therefore use the following principle.
4.1. **General principle for bounding the probability.** The probability of satisfying $w$ in $G$ can be expressed as

$$\mathbb{P}_{w=1}(G) = \frac{1}{|G|^d} \sum_{g_2, \ldots, g_d \in G} |\{g_1 \in G \mid w(g_1, \ldots, g_d) = 1\}|.$$

Let $\text{BAD} \subseteq G^{d-1}$ be a certain subset of tuples, and set correspondingly $\text{GOOD} = G^{d-1} - \text{BAD}$. Denote

$$C_w(g_2, \ldots, g_d) = \{g_1 \in G \mid w(g_1, \ldots, g_d) = 1\}.$$

**Assumption:** There exist absolute constants $0 < \delta_{\text{GOOD}}, \delta_{\text{BAD}} < 1$, depending only on $w$ and not on $G$, such that:

(i) $\forall (g_2, \ldots, g_d) \in \text{GOOD}, |C_w(g_2, \ldots, g_d)| \leq \delta_{\text{GOOD}} \cdot |G|$

(ii) $|\text{BAD}| \leq \delta_{\text{BAD}} \cdot |G|^{d-1}$

Under the above assumption, we can bound the word probability as follows. First of all, we let the sum expressing the word probability go over the good and the bad tuples separately,

$$\mathbb{P}_{w=1}(G) \leq \frac{1}{|G|^d} \sum_{g_2, \ldots, g_d \in \text{GOOD}} |C_w(g_2, \ldots, g_d)| + \frac{1}{|G|^d} \sum_{g_2, \ldots, g_d \in \text{BAD}} |G|$$

$$\leq \frac{1}{|G|^d} |\text{GOOD}| \delta_{\text{GOOD}} |G| + \frac{1}{|G|^d} |\text{BAD}| |G|.$$

Taking $|\text{BAD}| + |\text{GOOD}| = |G|^{d-1}$ into account, we can therefore write

$$\mathbb{P}_{w=1}(G) \leq \delta_{\text{GOOD}} + (1 - \delta_{\text{GOOD}}) \frac{|\text{BAD}|}{|G|^{d-1}},$$

we can bound the relative badness by assumption, and hence

$$\mathbb{P}_{w=1}(G) \leq \delta_{\text{GOOD}} + (1 - \delta_{\text{GOOD}}) \delta_{\text{BAD}},$$

which gives an absolute upper bound on the word probability in $G$.

4.2. **Long commutator.** Consider the long commutator word,

$$\gamma_d(X_1, \ldots, X_d) = [X_1, \gamma_{d-1}(X_2, \ldots, X_d)] = [X_1, [X_2, \ldots, X_d]].$$

We know that in a nonabelian group, the fiber over 1 of $\gamma_2(X_1, X_2)$ is of relative size at most $\frac{5}{8}$. Let us show by induction that a bound exists for all the long commutator words, giving a sample application of the general principle for bounding the probability. Let $G$ be a group that does not satisfy the word $\gamma_d$. Set

$$\text{BAD} = \{(g_2, \ldots, g_d) \in G^d \mid C_w(g_2, \ldots, g_d) = G\}$$

$$= \{(g_2, \ldots, g_d) \in G^d \mid \gamma_{d-1}(g_2, \ldots, g_d) \in Z(G)\}.$$
The size of the latter set is
\[ |\text{BAD}| = \left| \{(g_2, \ldots, g_d) \in (G/Z(G))^d \mid \gamma_{d-1}(g_2, \ldots, g_d) = 1\} \right| \cdot |Z(G)|. \]
The group $G/Z(G)$ does not satisfy the word $\gamma_{d-1}$. Therefore we can argue by induction that there is a constant $\delta_{BAD,d-1}$ with
\[ \left| \{(g_2, \ldots, g_d) \in (G/Z(G))^{d-1} \mid \gamma_{d-1}(g_2, \ldots, g_d) = 1\} \right| \leq \delta_{d-1}|G/Z(G)|^{d-1}. \]
Thus
\[ |\text{BAD}| \leq \delta_{d-1}|G|^{d-1}, \]
so we can take $\delta_{BAD,d} = \delta_{d-1}$. On the other hand, for a tuple $(g_2, \ldots, g_d) \notin \text{BAD}$, we have
\[ C_w(g_2, \ldots, g_d) = C_G(\gamma_{d-1}(g_2, \ldots, g_d)), \]
which is a proper subgroup of $G$, and whence
\[ |C_w(g_2, \ldots, g_d)| \leq \frac{1}{2}|G|. \]
Therefore we can take $\delta_{GOOD} = \frac{1}{2}$. This gives a bound for the probability,
\[ P_{\gamma_d=1}(G) = \delta_d \leq \frac{1}{2} + \frac{1}{2}\delta_{d-1}. \]
Since $\delta_2 = \frac{5}{8}$, we can derive inductively that we can take $\delta_d = 1 - \frac{3}{2^{d+1}}$.
In this case, the obtained bound is sharp, as can be seen for example by looking at dihedral groups.

4.3. 2-Engel word: Bounding two types of badness. We can assume that $V = \mathbb{F}_p$, and that $V$ is central in $G$. The case when $G$ acts nontrivially on $V$ has been dealt with in Section 3.3. Since $G/V$ is assumed to be 2-Engel, it is nilpotent, and so $G$ must also be nilpotent. As $V$ is the smallest normal subgroup of $G$, this implies that $G$ must in fact be a $p$-group. Moreover, $G/V$ is of nilpotency class at most 3, so $G$ is of nilpotency class at most 4.
We will use this fact freely in what follows.

Let $y \in G$ and consider the map
\[ \phi_y : G \to V, \quad a \mapsto [a, y, y]. \]
This may not be a homomorphism, but it does satisfy the following expansion law which will be of use:
\[ \phi_y(ab) = \phi_y(a) \cdot \phi_y(b) \cdot [a, y, b, y]. \]
We will be interested in two possible situations, depending on whether or not $\phi_y$ is a homomorphism.
4.3.1. The nice situation: \( \forall y \in G. \ [G, y, G, y] = 1 \). In this case, \( \phi_y \) is a homomorphism for all choices of \( y \in G \). Note that this means that \( \phi_y \) factors through the Frattini quotient \( G/\Phi(G) \), and so we can think of \( \phi_y \) as a linear functional over \( \mathbb{F}_p \), mapping into \( V = \langle z \rangle \). Let \( \{g_1, g_2, \ldots, g_d\} \) be a minimal generating set of \( G \). Every element \( a \in G \) has a unique expansion

\[
a \equiv \prod_{i=1}^{d} g_i^{\beta_i} \pmod{\Phi(G)}
\]

with \( \beta_i \in \mathbb{F}_p \).

Claim. If \( p = 3 \), then

\( \forall y \in G. \ [G, y, G, y] = 1 \iff \forall y \in G. \ [G', y, y] = 1 \),

and these two equivalent conditions imply that \( [G, G, G] \leq V \).

Proof. We know that \( [G, G, G, G]^3 = 1 \), and that \( [G, G, G]^3 \leq V \). So we have (see [4, Lemma 2.2(v)])

\[
[a, y, b, y] = [a, b, y, y]^{-1}.
\]

for all \( a, b \in G \). This proves the equivalence in the claim. As for the second part of the claim, suppose that \( \gamma_4(G) \neq 1 \). Thus \( \gamma_4(G) = V \). For any \( c \in G \), we have

\[
1 \equiv [c, ab, ab] \equiv [c, a, b][c, b, a] \pmod{\gamma_4(G)},
\]

and using the Jacobi identity modulo \( \gamma_4(G) \), we obtain

\[
[c, a, b] \equiv [a, b, c] \pmod{\gamma_4(G)}.
\]

Thus we have cyclic invariance of commutators of length 3. Now we have, by [9, Vol. 2, p. 43], that for any \( d \in G \),

\[
[[[a, b], c, d]] = [[[a, b], d, c]]^{-1} \text{ and } [[a, b], [c, d]] = [[a, b], c, d]^2.
\]

Using all the above, we can execute the computation

\[
[a, b, c, d]^2 = [a, b, [c, d]]
\]

\[
= [[a, b], [c, d]]^{-1}
\]

\[
= [c, d, a, b]^{-2}
\]

\[
= [c, d, a, b]
\]

\[
= [a, c, d, b]
\]

\[
= [a, c, b, d]^{-1}
\]

\[
= [c, a, b, d]
\]

\[
= [a, b, c, d],
\]

with \( \beta_i \in \mathbb{F}_p \).
giving \([a, b, c, d] = 1\). Whence indeed \(\gamma_4(G) = 1\). As \(\gamma_3(G)\) is central in \(G\), it must be cyclic, since \(V\) is the smallest normal subgroup of \(G\). But, since \(\gamma_3(G)\) is of exponent 3, it follows that \(\gamma_3(G) = V\), and the proof is complete. \(\square\)

We can now express the map \(\phi_y\) in terms of the generating set of \(G\). Note that \(\phi_y\) does not depend on the specific coset representative of \(y\) modulo \(\Phi(G)\). Set \([g_i, g_j, g_k] = z^{\gamma_{ijk}}\) with \(\gamma_{ijk} \in \mathbb{F}_p\). Then we have

\[
\phi_y(g_i) = z^{\sum_{j,k} \gamma_{ijk} \beta_j \beta_k}.
\]

Set

\[
\text{BAD} = \{ y \in G \mid \phi_y \equiv 1 \}.
\]

Therefore \(y \in \text{BAD}\) if and only if each of the \(d\) quadratic forms \(\sum_{j,k} \gamma_{ijk} \beta_j \beta_k\) vanishes. As \(G\) is assumed not to be 2-Engel, at least one of these forms is not identically equal to 0. This form can be diagonalized (see [10]) to a form

\[
\sum_{l=1}^{d'-1} \beta_l^2 + a_{d'} \beta_{d'}^2
\]

for some \(1 \leq d' \leq d\) and \(a_{d'} \in \mathbb{F}_p\). The number of zeros of this form can be bounded from above as follows. For each of \(\beta_2, \ldots, \beta_d\) there are at most \(p\) choices and, after fixing these, there are at most two possibilities for \(\beta_1\) if \(p \neq 2\), and at most one choice for \(\beta_1\) if \(p = 2\). This implies that, if \(p \neq 2\),

\[
|\text{BAD}| \leq 2p^{d-1}|\Phi(G)| = \frac{2|G|}{p} \leq \frac{2|G|}{3},
\]

and similarly, if \(p = 2\), then \(|\text{BAD}| \leq |G|/2\).

4.3.2. The other situation: \(\exists y \in G. [G, y, G, y] \neq 1\). Note that this case is only possible when \(p = 3\) (see [9, Theorem 7.15]). It follows from the claim the previous section that the restriction of the map \(\phi_y\) to \(G'\) is not always trivial. Set

\[
\text{BAD}' = \{ y \in G \mid \phi_y(G') \equiv 1 \}.
\]

The same argument as above with \(G\) replaced by \(G'\) gives the bound

\[
|\text{BAD}'| \leq \frac{2|G|}{3}.
\]

4.4. 2-Engel word: Bounding good fibers. In order to provide a bound for the word probability, we now need to ensure that as long as \(y\) does not belong to \(\text{BAD}\) (or \(\text{BAD}\)), we can bound the number of solutions of the equation

\[
[a, y, y] = 1
\]
for \( x \in G \). This is equivalent to saying that we want to provide a relative upper bound for the fiber \( \phi_y^{-1}(1) \). Note that the map \( \phi_y \) is not trivial in this situation. We will need to analyse two cases.

4.4.1. The nice situation: \( \forall y \in G. \ [G, y, G, y] = 1 \). In this case, let \( y \) be an element of \( G \) that is not \( \text{BAD} \). Thus \( \phi_y \) is a nontrivial homomorphism, and so it is surjective. Whence we can bound the fiber,

\[
|\phi_y^{-1}(1)| = |\ker \phi_y| = \frac{|G|}{p}.
\]

4.4.2. The other situation: \( \exists y \in G. \ [G, y, G, y] \neq 1 \). In this case, we will bound the relative size of \( \phi_y^{-1}(1) \) for \( y \) that is not \( \text{BAD}' \). If \( \phi_y^{-1}(1) \) generates a proper subgroup of \( G \), we can use the trivial bound of \( \frac{1}{2} \) for the relative size of the fiber in \( G \). Thus we can assume from now on that \( \langle \phi_y^{-1}(G) \rangle = G \).

Claim. \( \phi_y^{-1}(1) \cdot G' = G \).

Proof. Let \( a, b \in \phi_y^{-1}(1) \). We claim that there is an element \( g \in G' \) such that \( abg \in \phi_y^{-1}(1) \). Expanding based on (4.3.1), we obtain

\[
\phi_y(a \cdot b \cdot g) = \phi_y(a) \cdot \phi_y(b) \cdot \phi_y(g) \cdot [a, y, b, y].
\]

Since \( \phi_y \) is not trivial on \( G' \) in this situation, it is surjective, and so indeed we can find a \( g \in G' \) with \( \phi_y(g) = [a, y, b, y]^{-1} \). The claim now follows since the fiber of \( \phi_y \) over 1 generates \( G \).

Let \( R \) be a set of coset representatives of \( G' \) in \( G \). By the claim above, we can pick these so that \( R \subseteq \phi_y^{-1}(1) \). Now we show that the map \( \phi_y \) does not depend on these representatives.

Claim. Let \( a \in G' \) and \( r \in R \). Then

\[
\phi_y(ra) = \phi_y(a).
\]

Proof. Expanding based on (4.3.1), we obtain

\[
\phi_y(r \cdot a) = \phi_y(r) \cdot \phi_y(a) \cdot [r, y, a, y].
\]

As \( a \in G' \), the last commutator is trivial. Our claim follows from the fact that \( R \subseteq \phi_y^{-1}(1) \).

It follows from the claim that

\[
|\phi_y^{-1}(1)| = |R| \cdot |\phi_y^{-1}(1)|.
\]
The restriction \( \phi_y |_{G'} \) is, however, a nontrivial homomorphism. Therefore its fiber over 1 equals its kernel, which is of index 3 in \( G' \). In this situation, we therefore have

\[
|\phi_y^{-1}(1)| = \frac{|G|}{|G'|} \cdot \frac{|G'|}{3} = \frac{|G|}{3},
\]
just as in the previous situation.

4.5. **Metabelian word: Bounding two types of badness.** We are in the situation when \( G' \leq C_G(V) \). This means that \( V = G'' \) commutes with \( G' \), and so \( G' \) is nilpotent of class at most 2. Since \( V \) is the smallest normal subgroup of \( G \), this implies that \( G' \) must be a \( p \)-group.

Let \( y, z, t \in G \) and consider the map

\[
\phi_{y,z,t} : G \to V, \quad a \mapsto [[a, y], [z, t]].
\]
This may not be a homomorphism, but it does satisfy the following expansion law which will be of use:

\[
\phi_{y,z,t}(ab) = \phi_{y,z,t}(a) \cdot \phi_{y,z,t}(b) \cdot [[a, y, b], [z, t]].
\]

Set

\[
\text{BAD} = \{(y, z, t) \in G \mid \phi_{y,z,t} \equiv 1\}.
\]
With a fixed pair \((z, t)\), set

\[
S_{z,t} = \{y \in G \mid \phi_{y,z,t} \equiv 1\}.
\]
Note that for any \( a \in G \), we have

\[
\phi_{y_1y_2,z,t}(a) = [[a, y_1y_2], [z, t]] = \phi_{y_2,z,t}(a) \cdot \phi_{y_1,z,t}(a) \cdot \phi_{y_2,z,t}([a, y_1]),
\]
so that \( S_{z,t} \) is a subgroup of \( G \). Thus we either have that \( S_{z,t} = G \) or it is a proper subgroup of \( G \). The first case occurs if and only if \([z, t] \in C_G(G')\).

Set

\[
\text{UGLY} = \{(z, t) \in G^2 \mid [z, t] \in C_G(G')\}.
\]
Thus we can express

\[
\text{BAD} = (G \times \text{UGLY}) \cup \{(y, z, t) \in G^3 \mid (z, t) \notin \text{UGLY}, \ y \in S_{z,t}\},
\]
and so we have

\[
|\text{BAD}| = |G||\text{UGLY}| + \sum_{(z,t) \notin \text{UGLY}} |S_{z,t}|.
\]
When \((z, t)\) not an ugly pair, \( S_{z,t} \) is of index at least 2 in \( G \). Thus we can bound

\[
|\text{BAD}| \leq |G||\text{UGLY}| + (|G|^2 - |\text{UGLY}|)|G| = \frac{1}{2}|G|^3 + \frac{1}{2}|G||\text{UGLY}|.
\]
Note that as $G$ is assumed not to be metabelian, $G/C_G(G')$ is not abelian. This means that we can bound the relative number of ugly pairs, finally giving

$$|\text{BAD}| \leq \left( \frac{1}{2} + \frac{1}{2} \cdot \frac{5}{8} \right) |G|^3 = \frac{13}{16} |G|^3.$$  

We will also need to deal with the particular situation when $[G', G, G']$ is a nontrivial subgroup of $G$. This means that $G'$ is not contained in $C_G([G', G])$, and so $G/C_G([G', G])$ is not abelian. In this case, we will need to resort to the sets

$$\text{BAD}' = \{ (y, z, t) \in G^3 | \phi_{y, z, t}(G') = 1 \}$$

and

$$\text{UGLY}' = \{ (z, t) \in G^3 | [z, t] \in C_G([G', G]) \}.$$  

The same argument as above gives the same bound

$$|\text{BAD}'| \leq \frac{13}{16} |G|^3.$$  

4.6. **Metabelian word: Bounding good fibers.** In order to provide a bound for the word probability, we now need to ensure that as long as a tuple $(y, z, t)$ does not belong to BAD (or BAD'), we can bound the number of solutions of the equation

$$[[a, y], [z, t]] = 1$$

for $a \in G$. This is equivalent to saying that we want to provide a relative upper bound for the fiber $\phi_{y, z, t}^{-1}(1)$. Note that the map $\phi_{y, z, t}$ is not trivial in this situation. We will need to analyse two cases.

4.6.1. **The nice situation:** $[G', G, G'] = 1$. In this case, we take a tuple that is not BAD. Thus $\phi_{y, z, t}$ is a nontrivial homomorphism, and so it must be surjective. Whence we can bound the fiber,

$$|\phi_{y, z, t}^{-1}(1)| = |\ker \phi_{y, z, t}| = \frac{|G|}{p}.$$  

4.6.2. **The other situation:** $[G', G, G'] \neq 1$. In this situation, we need to bound the relative fiber size $\phi_{y, z, t}^{-1}(1)$ for a tuple $(y, z, t) \notin \text{BAD}'$. If $\phi_{y, z, t}^{-1}(1)$ generates a proper subgroup of $G$, we have the trivial bound of $\frac{1}{2}$ for the relative size of the fiber in $G$. Thus we can assume from now on that $\langle \phi_{y, z, t}^{-1}(1) \rangle = G$.

**Claim.** $\phi_{y, z, t}^{-1}(1) \cdot G' = G$. 

Proof. Let \( a, b \in \phi^{-1}_{y,z,t}(1) \). We claim that there is an element \( g \in G' \) such that \( abg \in \phi^{-1}_{y,z,t}(1) \). Expanding based on (4.5.1), we obtain
\[
\phi_{y,z,t}(a \cdot b \cdot g) = \phi_{y,z,t}(a) \cdot \phi_{y,z,t}(b) \cdot \phi_{y,z,t}(g) \cdot [[a, y, b], [z, t]].
\]
Since \( \phi_{y,z,t} \) is not trivial on \( G' \), it is surjective, and so indeed we can find a \( g \in G' \) with \( \phi_{y,z,t}(g) = [[a, y, b], [z, t]]^{-1} \). The claim now follows since the fiber of \( \phi_{y,z,t} \) over 1 generates \( G \). \( \square \)

Let \( R \) be a set of coset representatives of \( G' \) in \( G \). By the claim above, we can pick these so that \( R \subseteq \phi^{-1}_{y,z,t}(1) \). Now we show that the map \( \phi_{y,z,t} \) does not depend on these representatives.

Claim. Let \( a \in G' \) and \( r \in R \). Then
\[
\phi_{y,z,t}(ra) = \phi_{y,z,t}(a).
\]

Proof. Expanding based on (4.5.1), we obtain
\[
\phi_{y,z,t}(r \cdot a) = \phi_{y,z,t}(r) \cdot \phi_{y,z,t}(a) \cdot [[r, y, a], [z, t]].
\]
As \( a \in G' \), the last commutator is trivial. Our claim follows from the fact that \( R \subseteq \phi^{-1}_{y,z,t}(1) \). \( \square \)

It follows from the claim that
\[
|\phi^{-1}_{y,z,t}(1)| = |R| \cdot |\phi_{y,z,t}^{-1}(1)|.
\]
The restriction \( \phi_{y,z,t}|_{G'} \) is, however, a nontrivial homomorphism. Therefore its fiber over 1 equals its kernel, which is of index \( p \) in \( G' \). In this situation, we therefore have
\[
|\phi^{-1}_{y,z,t}(1)| = \frac{|G|}{|G'|} \cdot \frac{|G'|}{p} = \frac{|G|}{p},
\]
just as in the previous situation.

Thus we have shown that we can bound the relative size of \(|BAD|\) (or \(|BAD'|\), depending on whether or not \([G', G, G']\) is trivial) from above by \( \frac{1}{2} \).
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