NON-COMMUTATIVE HOM-POISSON ALGEBRAS

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ABSTRACT. A Hom-type generalization of non-commutative Poisson algebras, called noncommutative Hom-Poisson algebras, are studied. They are closed under twisting by suitable selfmaps. Hom-Poisson algebras, in which the Hom-associative product is commutative, are closed under tensor products. Through (de)polarization Hom-Poisson algebras are equivalent to admissible Hom-Poisson algebras, each of which has only one binary operation. Multiplicative admissible Hom-Poisson algebras are Hom-power associative.

1. Introduction

A Poisson algebra \((A,\{,\},\mu)\) consists of a commutative associative algebra \((A,\mu)\) together with a Lie algebra structure \(\{,\}\), satisfying the Leibniz identity:
\[
\{\mu(x,y),z\} = \mu(\{x,z\},y) + \mu(x,\{y,z\}).
\]
Poisson algebras are used in many fields in mathematics and physics. In mathematics, Poisson algebras play a fundamental role in Poisson geometry \([18]\), quantum groups \([4,5]\), and deformation of commutative associative algebras \([8]\). In physics, Poisson algebras are a major part of deformation quantization \([11]\), Hamiltonian mechanics \([3]\), and topological field theories \([17]\). Poisson-like structures are also used in the study of vertex operator algebras \([6]\).

One way to generalize Poisson algebras is to omit the commutativity requirement. Such a structure is called a non-commutative Poisson algebra. When the associative product happens to be commutative, one has a Poisson algebra. Some classification results of finite dimensional non-commutative Poisson algebras can be found in \([12]\). One can also think of a non-commutative Poisson algebra as a special case of a Leibniz pair \([3]\).

The purpose of this paper is to study a twisted generalization of non-commutative Poisson algebras, called non-commutative Hom-Poisson algebras. In a non-commutative Hom-Poisson algebra \(A\), there is a linear self-map \(\alpha\) (the twisting map) and two binary operations \(\{,\}\) (the Hom-Lie bracket) and \(\mu\) (the Hom-associative product). The associativity, the Jacobi identity, and the Leibniz identity in a non-commutative Poisson algebra are replaced by their Hom-type (i.e., \(\alpha\)-twisted) analogues in a non-commutative Hom-Poisson algebra. In particular, \((A,\mu,\alpha)\) is a Hom-associative algebra \([3]\), and \((A,\{,\},\alpha)\) is a Hom-Lie algebra \([10]\). If the twisting map is the identity map, then a non-commutative Hom-Poisson algebra reduces to a non-commutative Poisson algebra.

Most of our results are about Hom-Poisson algebras, which are non-commutative Hom-Poisson algebras in which the Hom-associative products are commutative. Hom-Poisson algebras were defined in \([15]\) by Makhlouf and Silvestrov. It is shown in \([15]\) that Hom-Poisson algebras play the...
same role in the deformation of commutative Hom-associative algebras as Poisson algebras do in the deformation of commutative associative algebras. Other Hom-type algebras are studied in [13, 14] and [19] - [26] and the references therein.

The rest of this paper is organized as follows. In section 2 non-commutative Hom-Poisson algebras are defined. It is shown that, just like Poisson algebras, Hom-Poisson algebras are closed under tensor products in a non-trivial way (Theorem 2.9). It should be noted that Hom-Lie algebras are not closed under tensor products in any non-trivial way. In section 3 it is shown that non-commutative Hom-Poisson algebras are closed under suitable twistings by weak morphisms (Theorem 3.2). This is a unique feature for Hom-type algebras, as non-commutative Poisson algebras are not closed under such twistings. Using a special case of this result, several classes of (non-commutative) Hom-Poisson algebras are constructed.

In section 4 it is shown that Hom-Poisson algebras are equivalent to admissible Hom-Poisson algebras, each of which has one twisting map and only one binary operation. The correspondence between Hom-Poisson algebras and admissible Hom-Poisson algebras is the Hom-version of the correspondence between Poisson algebras and admissible Poisson algebras [9, 16]. In section 5 it is shown that multiplicative admissible Hom-Poisson algebras are Hom-power associative.

2. Basic properties of non-commutative Hom-Poisson algebras

In this section, we introduce (non-commutative) Hom-Poisson algebras and study tensor products of Hom-Poisson algebras.

2.1. Notations. We work over a fixed field \( k \) of characteristic 0. If \( V \) is a \( k \)-module and \( \mu: V^\otimes 2 \to V \) is a bilinear map, then \( \mu^{\text{op}}: V^\otimes 2 \to V \) denotes the opposite map, i.e., \( \mu^{\text{op}} = \mu \tau \), where \( \tau: V^\otimes 2 \to V^\otimes 2 \) interchanges the two variables. For a linear self-map \( \alpha: V \to V \), denote by \( \alpha^n \) the \( n \)-fold composition of \( n \) copies of \( \alpha \), with \( \alpha^0 \equiv \text{Id} \).

Let us begin with the basic definitions regarding Hom-algebras.

**Definition 2.2.**  
(1) By a **Hom-module** we mean a pair \((A, \alpha)\) in which \( A \) is a \( k \)-module and \( \alpha: A \to A \) is a linear map, called the twisting map.

(2) By a **Hom-algebra** we mean a triple \((A, \mu, \alpha)\) in which \((A, \alpha)\) is a Hom-module and \( \mu: A^\otimes 2 \to A \) is a bilinear map, called the multiplication. A Hom-algebra \((A, \mu, \alpha)\) and the corresponding Hom-module \((A, \alpha)\) are often abbreviated to \( A \).

(3) A Hom-algebra \((A, \mu, \alpha)\) is said to be **multiplicative** if \( \alpha \mu = \mu \alpha^{\otimes 2} \). It is called **commutative** if \( \mu = \mu^{\text{op}} \).

Unless otherwise specified, an algebra \((A, \mu)\) with \( \mu: A^\otimes 2 \to A \) is also regarded as a Hom-algebra \((A, \mu, \text{Id})\) with identity twisting map. Given a Hom-algebra \((A, \mu, \alpha)\), we often use the abbreviation
\[
\mu(x, y) = xy
\]
for \( x, y \in A \).

Let us now recall the Hom-type generalizations of associative and Lie algebras from [13].

**Definition 2.3.** Let \((A, \mu, \alpha)\) be a Hom-algebra.
(1) The **Hom-associator** $a_{A}: A^{\otimes 3} \to A$ is defined as
\[
a_{A} = \mu(\mu \otimes \alpha - \alpha \otimes \mu).
\]
(2.3.1)

(2) The Hom-algebra $A$ is called a **Hom-associative algebra** if it satisfies the **Hom-associative identity**
\[
a_{A} = 0.
\]
(2.3.2)

(3) The **Hom-Jacobian** $J_{A}: A^{\otimes 3} \to A$ is defined as
\[
J_{A} = \mu(\mu \otimes \alpha)(\text{Id} + \sigma + \sigma^{2}),
\]
(2.3.3)

where $\sigma(x \otimes y \otimes z) = z \otimes x \otimes y$. The sum $(\text{Id} + \sigma + \sigma^{2})$ is called a **cyclic sum**.

(4) The Hom-algebra $A$ is called a **Hom-Lie algebra** if it satisfies the **Hom-Jacobi identity**
\[
J_{A} = 0.
\]
(2.3.4)

When the twisting map $\alpha$ is the identity map, the above definitions reduce to the usual definitions of the associator, an associative algebra, the Jacobian, and a Lie algebra. Examples of Hom-associative and Hom-Lie algebras can be found in [13, 19, 20]. In terms of elements $x, y, z \in A$, the Hom-associator and the Hom-Jacobian are
\[
\begin{align*}
a_{A}(x, y, z) &= (xy)\alpha(z) - \alpha(x)(yz), \\
J_{A}(x, y, z) &= (xy)\alpha(z) + (zx)\alpha(y) + (yz)\alpha(x).
\end{align*}
\]

Let us recall the definition of a non-commutative Poisson algebra [12].

**Definition 2.4.** A **non-commutative Poisson algebra** $(A, \{,\}, \mu)$ consists of

(1) a Lie algebra $(A, \{,\})$ and
(2) an associative algebra $(A, \mu)$

such that the Leibniz identity
\[
\{x, yz\} = \{x, y\}z + y\{x, z\}
\]
is satisfied for all $x, y, z \in A$. A **Poisson algebra** is a non-commutative Poisson algebra $(A, \{,\}, \mu)$ in which $\mu$ is commutative. A **morphism** of non-commutative Poisson algebras is a linear map that is a morphism of the underlying Lie algebras and associative algebras.

In a non-commutative Poisson algebra $(A, \{,\}, \mu)$, the Lie bracket $\{,\}$ is called the Poisson bracket, and $\mu$ is called the associative product. The Leibniz identity says that $\{, -\}$ is a derivation with respect to the associative product. It can be rewritten in element-free form as
\[
\{,\}(\text{Id} \otimes \mu) = \mu(\{,\} \otimes \text{Id} + (\text{Id} \otimes \{,\}))(1\ 2),
\]
where $(1\ 2)(x \otimes y \otimes z) = y \otimes x \otimes z$.

Hom-Poisson algebras are first introduced in [14] by Makhlouf and Silvestrov. We now define the Hom-type generalization of a non-commutative Poisson algebra.

**Definition 2.5.** A **non-commutative Hom-Poisson algebra** $(A, \{,\}, \mu, \alpha)$ consists of

(1) a Hom-Lie algebra $(A, \{,\}, \alpha)$ and
(2) a Hom-associative algebra $(A, \mu, \alpha)$
such that the **Hom-Leibniz identity**

\[
\{ \{ \alpha \otimes \mu \} \} = \mu (\{ \{ \} \} \otimes (\alpha \otimes \{ \}))(1 \ 2)
\] (2.5.1)

is satisfied. A **Hom-Poisson algebra** is a non-commutative Hom-Poisson algebra \((A, \{ \}, \mu, \alpha)\) in which \(\mu\) is commutative \([14]\).

In a non-commutative Hom-Poisson algebra \((A, \{ \}, \mu, \alpha)\), the operations \(\{ \} \) and \(\mu\) are called the **Hom-Poisson bracket** and the **Hom-associative product**, respectively. In terms of elements \(x, y, z \in A\), the Hom-Leibniz identity says

\[
\{ \alpha(x), yz \} = \{ x, y \} \alpha(z) + \alpha(y) \{ x, z \},
\] (2.5.2)

where as usual \(\mu(x, y)\) is abbreviated to \(xy\). By the anti-symmetry of the Hom-Poisson bracket \(\{ \} \), the Hom-Leibniz identity is equivalent to

\[
\{ xy, \alpha(z) \} = \{ x, z \} \alpha(y) + \alpha(x) \{ y, z \}.
\]

A (non-commutative) Poisson algebra is exactly a multiplicative (non-commutative) Hom-Poisson algebra with identity twisting map.

Let us now provide some basic properties of non-commutative Hom-Poisson algebras. Every associative algebra \((A, \mu)\) has a non-commutative Poisson algebra structure in which the Poisson bracket is the commutator bracket. The following result is the Hom-type analogue of this observation.

**Proposition 2.6.** Let \((A, \mu, \alpha)\) be a Hom-associative algebra. Then

\[ A^- = (A, [,] = \mu - \mu^{op}, \mu, \alpha) \]

is a non-commutative Hom-Poisson algebra.

**Proof.** It is proved in \([13]\) (Proposition 1.6) that \((A, [,], \alpha)\) is a Hom-Lie algebra. Indeed, the commutator bracket \([,] \) is obviously anti-symmetric. One can write out all twelve terms (in terms of \(\mu\)) in the Hom-Jacobian of \(A^-\) and observe that their sum is zero. To check the Hom-Leibniz identity \((2.5.2)\) for \(A^-\), we compute as follows:

\[
[x, y] \alpha(z) + \alpha(y)[x, z] - \{ \alpha(x), yz \} = (xy)\alpha(z) - (yx)\alpha(z) + \alpha(y)(xz) - \alpha(y)(zx) - \alpha(x)(yz) + (yz)\alpha(x)
\]

\[
= a_{sA}(x, y, z) - a_{sA}(y, x, z) + a_{sA}(y, z, x).
\]

Since \(a_{sA} = 0\), we conclude that \(A^-\) satisfies the Hom-Leibniz identity. \(\square\)

Next we study tensor products of Hom-Poisson algebras. For motivation, recall that Lie algebras are not closed under tensor products. The same is true for Hom-Lie algebras. On the other hand, Poisson algebras are closed under tensor products. In the rest of this section, we show that Hom-Poisson algebras are also closed under tensor products.

We need the following preliminary result.

**Lemma 2.7.** Let \((A, \mu, \alpha)\) be a commutative Hom-associative algebra. Then the expression \((xy)\alpha(z)\) is invariant under permutations of \(x, y, z \in A\). In other words, we have

\[ \mu(\mu \otimes \alpha) = \mu(\mu \otimes \alpha) \pi \]

for all the permutations \(\pi\) on three letters.
Proof. Pick \(x, y, z \in A\). Then
\[
(xy)\alpha(z) = (yx)\alpha(z) = \alpha(y)(xz) = (xz)\alpha(y).
\]
So the expression \((xy)\alpha(z)\) is invariant under the transpositions \((1 \ 2)\) and \((2 \ 3)\). The proof is complete because these two transpositions generate the symmetric group on three letters. \(\square\)

**Definition 2.8.** Let \((A, \{\}, \mu, \alpha)\) be a quadruple in which \((A, \alpha)\) is a Hom-module and \(\{\}, \mu: A^{\otimes 2} \to A\) are bilinear operations. Define its **Hom-associator** as \(as_A\) and **Hom-Jacobian** \(J_A\) using \(\mu\) and \(\{\}, \) respectively, i.e.,
\[
\begin{align*}
as_A &= \mu(\mu \otimes \alpha - \alpha \otimes \mu), \\
J_A &= \{\} \{\} \otimes (Id + \sigma + \sigma^2),
\end{align*}
\]
where \(\sigma(x \otimes y \otimes z) = z \otimes x \otimes y\).

Now we are ready to prove that Hom-Poisson algebras are closed under tensor products.

**Theorem 2.9.** Let \((A_i, \{\}, \mu_i, \alpha_i)\) be Hom-Poisson algebras for \(i = 1, 2\), and let \(A = A_1 \otimes A_2\). Define the operations \(\alpha: A \to A\) and \(\mu, \{\} : A^{\otimes 2} \to A\) by:
\[
\begin{align*}
\alpha &= \alpha_1 \otimes \alpha_2, \\
\mu(x_1 \otimes x_2, y_1 \otimes y_2) &= \mu_1(x_1, y_1) \otimes \mu_2(x_2, y_2), \\
\{x_1 \otimes x_2, y_1 \otimes y_2\} &= \{x_1, y_1\}_1 \otimes \mu_2(x_2, y_2) + \mu_1(x_1, y_1) \otimes \{x_2, y_2\}_2
\end{align*}
\]
for \(x_i, y_i \in A_i\). Then \((A, \{\}, \mu, \alpha)\) is a Hom-Poisson algebra.

**Proof.** That \((A, \mu, \alpha)\) is a commutative Hom-associative algebra follows from the commutativity and Hom-associativity of both \(\mu_i\). Also, the commutativity of the \(\mu_i\) and the anti-symmetry of the \(\{\}\), imply the anti-symmetry of \(\{\}\). It remains to prove the Hom-Jacobi identity and the Hom-Leibniz identity in \(A\).

To simplify the typograph, we abbreviate \(\mu_1, \mu_2,\) and \(\mu\) using juxtaposition and drop the subscripts in \(\{\}, i\) and \(\alpha_i\). Pick \(x = x_1 \otimes x_2, y = y_1 \otimes y_2,\) and \(z = z_1 \otimes z_2 \in A\). Then
\[
\begin{align*}
\{\{x, y\}, \alpha(z)\} &= \{\{x_1, y_1\} \otimes x_2y_2, \alpha(z_1) \otimes \alpha(z_2)\} + \{x_1y_1 \otimes \{x_2, y_2\}, \alpha(z_1) \otimes \alpha(z_2)\} \\
&= \{\{x_1, y_1\} \otimes (x_2y_2), \alpha(z_1) \otimes \alpha(z_2)\} + \{x_1y_1 \otimes \{x_2, y_2\}, \alpha(z_1) \otimes \alpha(z_2)\} \\
&+ \{x_1y_1, \alpha(z_1)\} \otimes \{x_2, y_2\} \alpha(z_2) + \{x_1y_1 \alpha(z_1) \otimes \{x_2, y_2\}, \alpha(z_2)\}
\end{align*}
\]
The cyclic sum over \(x, y, z\) of the term \(a\) is 0 by Lemma 2.7 and the Hom-Jacobi identity in \(A_1\). Likewise, the cyclic sum over \(x, y, z\) of the term \(d\) is 0 by Lemma 2.7 and the Hom-Jacobi identity in \(A_2\). It follows that the Hom-Jacobi \(J_A(x, y, z)\) consists of only the cyclic sum over \(x, y, z\) of the terms \(b\) and \(c\). Using the Hom-Leibniz identity in \(A_1\), the terms \(b\) and \(c\) become
\[
\begin{align*}
b &= \{x_1, y_1\} \alpha(z_1) \otimes \{x_2y_2, \alpha(z_2)\} \\
&= \{x_1, y_1\} \alpha(z_1) \otimes \alpha(x_2) \{y_2, z_2\} + \{x_1, y_1\} \alpha(z_1) \otimes \{x_2, z_2\} \alpha(y_2), \\
c &= \{x_1y_1, \alpha(z_1)\} \otimes \{x_2, y_2\} \alpha(z_2) \\
&= \alpha(x_1) \{y_1, z_1\} \otimes \{x_2, y_2\} \alpha(z_2) + \{x_1, z_1\} \alpha(y_1) \otimes \{x_2, y_2\} \alpha(z_2).
\end{align*}
\]
Therefore, the Hom-Jacobian of $A$ applied to $x \otimes y \otimes z$ is:

$$J_A(x, y, z) = \{x_1, y_1\} \alpha(z_1) \otimes \alpha(x_2) \{y_2, z_2\} + \{x_1, y_1\} \alpha(x_2) \{y_2, z_2\} \alpha(z_1)$$

Using the anti-symmetry of $\{,\}$ and the commutativity of $\mu$, observe that the following sums are 0: $b_1 + c_6$, $b_2 + c_3$, $b_3 + c_2$, $b_4 + c_5$, $b_5 + c_4$, and $b_6 + c_1$. This shows that $(A, \{,\}, \alpha)$ satisfies the Hom-Jacobi identity $J_A = 0$.

Finally, we check the Hom-Leibniz identity in $A$. Using the Hom-associativity and the Hom-Leibniz identity in the $A_i$ and Lemma 2.7, we compute as follows:

$$\{xy, \alpha(z)\} = \{x_1y_1 \otimes x_2y_2, \alpha(z_1) \otimes \alpha(z_2)\}$$

$$= \{x_1y_1, \alpha(z_1)\} \otimes (x_2y_2) \alpha(z_2) + (x_1y_1) \alpha(z_1) \otimes \{x_2y_2, \alpha(z_2)\}$$

$$= \alpha(x_1) \{y_1, z_1\} \otimes \alpha(x_2) \{y_2, z_2\} + \{x_1, z_1\} \alpha(y_1) \otimes \{x_2, z_2\} \alpha(y_2)$$

$$= \alpha(x_1) \{y_1, z_1\} \otimes \alpha(x_2) \{y_2, z_2\} + \{x_1, z_1\} \alpha(y_1) \otimes \{x_2, z_2\} \alpha(y_2)$$

$$= \alpha(x) \{y, z\} + \{x, z\} \alpha(y).$$

This shows that $A$ satisfies the Hom-Leibniz identity. \hfill \Box

Setting $\alpha_i = Id_{A_i}$ in Theorem 2.9, we recover the following well-known result about Poisson algebras.

**Corollary 2.10.** Let $(A_i, \{,\}, \mu_i)$ be Poisson algebras for $i = 1, 2$, and let $A = A_1 \otimes A_2$. Define the operations $\mu, \{,\}: A \otimes 2 \to A$ by:

$$\mu(x_1 \otimes x_2, y_1 \otimes y_2) = \mu_1(x_1, y_1) \otimes \mu_2(x_2, y_2),$$

$$\{x_1 \otimes x_2, y_1 \otimes y_2\} = \{x_1, y_1\}_1 \otimes \mu_2(x_2, y_2) + \mu_1(x_1, y_1) \otimes \{x_2, y_2\}_2$$

for $x_i, y_i \in A_i$. Then $(A, \{,\}, \mu)$ is a Poisson algebra.

3. **Twistings of non-commutative Hom-Poisson algebras**

In this section, we first observe that non-commutative Hom-Poisson algebras are closed under twisting by suitable self-maps. A special case of this observation is that non-commutative Poisson algebras give rise to multiplicative non-commutative Hom-Poisson algebras via twisting by self-morphisms (Corollary 3.4). To study whether these twistings of non-commutative Poisson algebras
Actually give rise to non-commutative Poisson algebras, we introduce the concept of \textit{rigidity} in Definition 3.5. Twistings and rigidity of some classes of Poisson algebras are then studied.

\textbf{Definition 3.1.} Let $(A, \{,\}, \mu, \alpha)$ be a quadruple in which $(A, \alpha)$ is a Hom-module and \{,\}, $\mu: A \otimes^2 \to A$ are bilinear operations.

1. $A$ is \textbf{multiplicative} if
   \[ \alpha\{,\} = \{,\} \alpha^\otimes^2 \quad \text{and} \quad \alpha \mu = \mu \alpha \otimes^2. \]
2. Let $(B, \{,\}_B, \mu_B, \alpha_B)$ be another such quadruple. A \textbf{weak morphism} $f: A \to B$ is a linear map such that
   \[ f\{,\} = \{,\}_B f^\otimes^2 \quad \text{and} \quad f \mu = \mu_B f \otimes^2. \]

A \textbf{morphism} $f: A \to B$ is a weak morphism such that $f \alpha = \alpha_B f$.

Note that a quadruple $(A, \{,\}, \mu, \alpha)$ is multiplicative if and only if the twisting map $\alpha: A \to A$ is a morphism.

The following result says that (non-commutative) Hom-Poisson algebras are closed under twisting by self-weak morphisms.

\textbf{Theorem 3.2.} Let $(A, \{,\}, \mu, \alpha)$ be a (non-commutative) Hom-Poisson algebra and $\beta: A \to A$ be a weak morphism. Then

\[ A_\beta = (A, \{,\}_\beta = \beta\{,\}, \mu_\beta = \beta \mu, \beta \alpha) \]

is also a (non-commutative) Hom-Poisson algebra. Moreover, if $A$ is multiplicative and $\beta$ is a morphism, then $A_\beta$ is a multiplicative (non-commutative) Hom-Poisson algebra.

\textbf{Proof.} If $\mu$ is commutative, then clearly so is $\mu_\beta$. The rest of the proof applies whether $\mu$ is commutative or not.

Since $\beta$ is compatible with $\mu$ and \{,\}, the Hom-associators and the Hom-Jacobians of $A$ and $A_\beta$ are related as:

\[ \beta^2 a_A = (\beta \mu)(\beta \mu \otimes \beta \alpha - \beta \alpha \otimes \beta \mu) = a_{A_\beta} \]

and

\[ \beta^2 J_A = (\beta\{,\})(\beta\{,\} \otimes \beta \alpha)(\text{Id} + \sigma + \sigma^2) = J_{A_\beta}. \]

This implies that $(A, \mu_\beta, \beta \alpha)$ is a Hom-associative algebra and that $(A, \{,\}_\beta, \beta \alpha)$ is a Hom-Lie algebra. Likewise, applying $\beta^2$ to the Hom-Leibniz identity (2.5.1) in $A$, we obtain

\[ (\beta\{,\})(\beta \alpha \otimes \beta \mu) = (\beta \mu)(\beta\{,\} \otimes \beta \alpha + (\beta \alpha \otimes \beta\{,\})(1 2)), \]

which is the Hom-Leibniz identity in $A_\beta$.

For the multiplicativity assertion, suppose $A$ is multiplicative and $\beta$ is a morphism. Let $\nu$ denote either $\mu$ or \{,\}. Then

\[ (\beta \alpha)\nu_\beta = (\beta \alpha)(\beta \nu) = \beta \nu \alpha \otimes^2 \beta \otimes^2 = \nu_\beta(\alpha \beta) \otimes^2 = \nu_\beta(\beta \alpha) \otimes^2. \]
This shows that $A_{\beta}$ is multiplicative. □

Two special cases of Theorem 3.2 follow. The following result says that each multiplicative (non-commutative) Hom-Poisson algebra gives rise to a derived sequence of multiplicative (non-commutative) Hom-Poisson algebras.

**Corollary 3.3.** Let $(A,\{\cdot,\},\mu,\alpha)$ be a multiplicative (non-commutative) Hom-Poisson algebra. Then

$$A^n = \left( A, \{\cdot,\}^{(n)} = \alpha^n\{\cdot,\}, \mu^{(n)} = \alpha^n\mu, \alpha^{n+1} \right)$$

is a multiplicative (non-commutative) Hom-Poisson algebra for each integer $n \geq 0$.

**Proof.** The multiplicativity of $A$ implies that $\alpha^n : A \to A$ is a morphism. By Theorem 3.2 $A_{\alpha^n} = A^n$ is a multiplicative (non-commutative) Hom-Poisson algebra. □

The following observation is the $\alpha = \text{Id}$ special case of Theorem 3.2. It gives a twisting construction of multiplicative (non-commutative) Hom-Poisson algebras from (non-commutative) Poisson algebras. A result of this form was first given by the author in [20] for $G$-Hom-associative algebras. This twisting construction highlights the fact that the usual category of (non-commutative) Poisson algebras is *not* closed under twisting by self-morphisms, which, in view of Theorem 3.2, is in strong contrast with the category of (non-commutative) Hom-Poisson algebras. This is a major conceptual difference between (non-commutative) Hom-Poisson algebras and (non-commutative) Poisson algebras.

**Corollary 3.4.** Let $(A,\{\cdot,\},\mu)$ be a (non-commutative) Poisson algebra and $\beta : A \to A$ be a morphism. Then

$$A_{\beta} = \left( A, \{\cdot,\}_{\beta} = \beta\{\cdot,\}, \mu_{\beta} = \beta\mu, \beta \right)$$

is a multiplicative (non-commutative) Hom-Poisson algebra.

In the setting of Corollary 3.4, two natural questions arise: Is the triple

$$A'_{\beta} = \left( A, \{\cdot,\}_{\beta} = \beta\{\cdot,\}, \mu_{\beta} = \beta\mu \right) \quad (3.4.1)$$

a (non-commutative) Poisson algebra? If so, is it isomorphic to $A$ itself? To study these questions, we introduce the following concepts.

**Definition 3.5.** Let $(A,\{\cdot,\},\mu)$ be a non-commutative Poisson algebra.

1. Given a morphism $\beta : A \to A$, the triple $A'_{\beta}$ in (3.4.1) is called the $\beta$-twisting of $A$. A *twisting of $A$* is a $\beta$-twisting of $A$ for some morphism $\beta : A \to A$.
2. The $\beta$-twisting $A'_{\beta}$ of $A$ is called *trivial* if

$$\{\cdot,\}_{\beta} = 0 = \mu_{\beta}.$$ 

$A'_{\beta}$ is called *non-trivial* if either $\{\cdot,\}_{\beta} \neq 0$ or $\mu_{\beta} \neq 0$.
3. $A$ is called *rigid* if every twisting of $A$ is either trivial or isomorphic to $A$.

The reader is cautioned not to confuse the above notion of rigidity with Gerstenhaber’s [8].

Every non-commutative Poisson algebra $A$ has at least one trivial twisting $A'_{\beta}$, which is obtained by taking $\beta = 0$. As we will see below, it is possible for a $\beta$-twisting to be trivial even if $\beta$ is not the zero map. On the other hand, if a non-commutative Poisson algebra $A$ has either a non-zero
Lie bracket or a non-zero associative product, then the $Id_A$-twisting is non-trivial. As we will see below, it is possible for a $\beta$-twisting to be isomorphic to $A$ even if $\beta$ is not the identity map.

The following result gives some basic criteria for non-rigidity.

**Proposition 3.6.** Let $(A, \{\cdot,\cdot\}, \mu)$ be a non-commutative Poisson algebra. Suppose there exists a morphism $\beta: A \to A$ such that either:

1. $\mu \beta = \beta \mu$ is not associative or
2. $\{\cdot,\cdot\}_\beta = \beta \{\cdot,\cdot\}$ does not satisfy the Jacobi identity.

Then $A$ is not rigid.

**Proof.** The $\beta$-twisting $A'_\beta$ is non-trivial, since otherwise $\mu \beta$ would be associative and $\{\cdot,\cdot\}_\beta$ would satisfy the Jacobi identity. For the same reason, the $\beta$-twisting $A'_\beta$ cannot be isomorphic to $A$. □

Using Proposition 3.6 we now consider some classes of non-commutative Poisson algebras and study their (non-)rigidity.

**Example 3.7 (Free associative algebras are not rigid).** Let $A = (k(S), \mu)$ be the free unital associative algebra on a non-empty set $S$. Equip $A$ with the non-commutative Poisson algebra structure in which the Poisson bracket is the commutator bracket (Proposition 2.6). We claim that $A$ is not rigid. By Proposition 3.6 it suffices to show that $\mu \alpha = \alpha \mu$ is not associative for some morphism $\alpha$ on $A$. Pick an element $X \in S$, and let $\alpha: A \to A$ be the morphism determined for $Y \in S$ by

$$
\alpha(Y) = \begin{cases} 
1 + X & \text{if } Y = X, \\
Y & \text{if } Y \neq X.
\end{cases}
$$

For $n \geq 1$, we have

$$
\alpha^n(X) = n + X.
$$

The associator of $\mu_\alpha$ applied to $(X, X, \alpha(X))$ is:

$$
\mu_\alpha(\mu_\alpha(X, X), \alpha(X)) - \mu_\alpha(X, \mu_\alpha(X, \alpha(X)))
= (\alpha^2(X))^3 - \alpha(X)\alpha^2(X)\alpha^3(X)
= (2 + X)^3 - (1 + X)(2 + X)(3 + X)
\neq 0.
$$

So $\mu_\alpha$ is not associative, and $A$ is not rigid. □

**Example 3.8 (Matrix algebras are not rigid).** Let $A = (M_n(k), \mu)$ be the associative algebra of $n \times n$ matrices over $k$ for some $n \geq 2$. Equip $A$ with the non-commutative Poisson algebra structure in which the Poisson bracket is the commutator bracket (Proposition 2.6). We claim that $A$ is not rigid. By Proposition 3.6 it suffices to show that $\mu_\alpha = \alpha \mu$ is not associative for some morphism $\alpha$ on $A$.

To construct such a morphism, consider the diagonal matrix

$$
D = \begin{pmatrix}
1/2 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}.
$$
The inverse of $D$ is the same as $D$ except that its $(1,1)$-entry is 2. There is a morphism $\alpha: A \to A$ given by

$$
\alpha((a_{ij})) = D(a_{ij})D^{-1} = 
\begin{pmatrix}
  a_{11} & a_{12}/2 & \cdots & a_{1n}/2 \\
  2a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  2a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}.
$$

From here on, a matrix $(a_{ij}) \in A$ with $a_{ij} = 0$ whenever $i \geq 3$ or $j \geq 3$ is abbreviated to its upper left $2 \times 2$ submatrix

$$
\begin{pmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{pmatrix}.
$$

Consider such a matrix $X = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in A$.

Then a quick computation shows that

$$
\mu_\alpha(\mu_\alpha(X, X), \alpha(X)) = \alpha^2(X^3) = \begin{pmatrix} 3 & 1/2 \\ 8 & 1 \end{pmatrix},
$$

whereas

$$
\mu_\alpha(X, \mu_\alpha(X, \alpha(X))) = \alpha(X\alpha(X)\alpha^2(X)) = \begin{pmatrix} 5 & 3/8 \\ 6 & 1/4 \end{pmatrix}.
$$

So $\mu_\alpha$ is not associative, and $A$ is not rigid.

\begin{example}[Hom-Poisson algebras from linear Poisson structures]
Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, and let $(S(\mathfrak{g}), \mu)$ be its symmetric algebra. If \{e_i\}_{i=1}^n is a basis of $\mathfrak{g}$, then $S(\mathfrak{g})$ is the polynomial algebra $k[e_1, \ldots, e_n]$. Suppose the structure constants for $\mathfrak{g}$ are given by

$$
[e_i, e_j] = \sum_{k=1}^n c_{ij}^k e_k.
$$

Then the symmetric algebra $S(\mathfrak{g})$ becomes a Poisson algebra with the Poisson bracket

$$
\{F, G\} = \frac{1}{2} \sum_{i,j,k=1}^n c_{ij}^k e_k \left( \frac{\partial F}{\partial e_i} \frac{\partial G}{\partial e_j} - \frac{\partial F}{\partial e_j} \frac{\partial G}{\partial e_i} \right) (3.9.1)
$$

for $F, G \in S(\mathfrak{g})$. This Poisson algebra structure on $S(\mathfrak{g})$ is called the \textbf{linear Poisson structure}.

Note that $[e_i, e_j] = [e_i, e_j]$.

Let $\alpha: \mathfrak{g} \to \mathfrak{g}$ be a Lie algebra morphism. It extends to an associative algebra morphism $\alpha: S(\mathfrak{g}) \to S(\mathfrak{g})$ determined by

$$
\alpha(F(e_1, \ldots, e_n)) = F(\alpha(e_1), \ldots, \alpha(e_n)).
$$

Moreover, we claim that $\alpha$ respects the Poisson bracket (3.9.1), i.e., $\alpha$ is a morphism of Poisson algebras on $S(\mathfrak{g})$. Indeed, suppose that

$$
\alpha\{F, G\} = \{\alpha(F), \alpha(G)\} \quad \text{and} \quad \alpha\{H, G\} = \{\alpha(H), \alpha(G)\}
$$

for some $F, G, H \in S(\mathfrak{g})$. Then the Leibniz identity

$$
\{FH, G\} = \{F, G\}H + F\{H, G\}
$$

holds.

\end{example}
which is not 0 in the symmetric algebra $S$ and the linear Poisson structure on $g$ algebra morphism on morphism of Poisson algebras. By Corollary 3.4 there is a multiplicative Hom-Poisson algebra $(\lambda g, \mu g)$ is not rigid if $\mu$ extended map, which is a Poisson algebra morphism. By Proposition 3.6, the Poisson algebra in the sense of Definition 3.5.

To show that $(\lambda g, \mu g)$ is not associative. We have

$$\lambda (\alpha \mu g, \alpha g) = \lambda (\alpha \mu, \alpha g)$$

Therefore, given a Lie algebra morphism $\alpha: g \to g$, the extended map $\alpha: S(g) \to S(g)$ is a morphism of Poisson algebras. By Corollary 3.4 there is a multiplicative Hom-Poisson algebra

$$S(g) = (S(g), \{\}, \alpha = \alpha \{\}, \mu = \alpha \mu, \alpha)$$

which reduces to the linear Poisson structure $S(g)$ if $\alpha = \text{Id}$. □

**Example 3.10** ($S(\mathfrak{s}l_2)$ is not rigid). In this example, we show that the symmetric algebra $(S(\mathfrak{s}l_2), \mu)$ on the Lie algebra $\mathfrak{s}l_2$, equipped with the linear Poisson structure $(3.9.1)$, is not rigid in the sense of Definition 3.5.

The Lie algebra $\mathfrak{s}l_2$ has a basis $\{e, f, h\}$, with respect to which the Lie bracket is given by

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$  

To show that $S(\mathfrak{s}l_2) = k[e, f, h]$ is not rigid, consider the Lie algebra morphism $\alpha: \mathfrak{s}l_2 \to \mathfrak{s}l_2$ given by

$$\alpha(e) = \lambda e, \quad \alpha(f) = \lambda^{-1} f, \quad \alpha(h) = h,$$

where $\lambda \in k$ is a fixed scalar with $\lambda \neq 0, 1$. As in Example 3.3, denote by $\alpha: S(\mathfrak{s}l_2) \to S(\mathfrak{s}l_2)$ the extended map, which is a Poisson algebra morphism. By Proposition 3.6, the Poisson algebra $S(\mathfrak{s}l_2)$ is not rigid if $\mu = \alpha \mu$ is not associative. We have

$$\mu (\mu (e, h), h) - \mu (e, \mu (h, h)) = \alpha^2 (e) \alpha^2 (h) \alpha (h) - \alpha (e) \alpha^2 (h) \alpha (h) = (\lambda^2 - \lambda)eh^2,$$

which is not 0 in the symmetric algebra $S(\mathfrak{s}l_2)$ because $\lambda \neq 0, 1$. Therefore, $\mu$ is not associative, and the linear Poisson structure on $S(\mathfrak{s}l_2)$ is not rigid. □
Example 3.11 (Hom-Poisson algebras from Poisson manifolds). In this example, we discuss how multiplicative Hom-Poisson algebras arise from Poisson manifolds, which include symplectic manifolds and Poisson-Lie groups. The reader is referred to, e.g., [4, 18] for discussion of Poisson manifolds. The ground field in this example is the field of real numbers.

Let $M$ be a Poisson manifold. This means that $M$ is a smooth manifold and that the commutative associative algebra $(\mathcal{C}^\infty(M), \mu)$ (under point-wise multiplication) of smooth $\mathbb{R}$-valued functions on $M$ is equipped with a Poisson algebra structure

$$\{,\} : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M).$$

Let $N$ be another Poisson manifold. A Poisson map $\varphi : M \to N$ between two Poisson manifolds is a smooth map such that

$$\{f, g\}_\varphi = \{f \varphi, g \varphi\}$$

for all $f, g \in \mathcal{C}^\infty(N)$. Given such a Poisson map, the induced map $\varphi^* : \mathcal{C}^\infty(N) \to \mathcal{C}^\infty(M), \varphi^*(f) = f\varphi$ is a morphism of Poisson algebras.

Let $\varphi : M \to M$ be a Poisson map. By Corollary 3.4 the morphism $\varphi^* : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ yields a multiplicative Hom-Poisson algebra

$$\mathcal{C}^\infty(M)_{\varphi^*} = (\mathcal{C}^\infty(M), \{,\}_{\varphi^*} = \varphi^*\{,\}, \mu_{\varphi^*} = \varphi^*\mu, \varphi^*).$$

If $\varphi = \text{Id}_M$, then $\mathcal{C}^\infty(M)_{\text{Id}^*}$ is the original Poisson algebra $\mathcal{C}^\infty(M)$.

We now give sufficient conditions that guarantee that the Poisson algebra $\mathcal{C}^\infty(M)$ of smooth functions on $M$ is not rigid in the sense of Definition 3.3.

**Corollary 3.12.** Let $M$ be a Poisson manifold. Suppose there exist a Poisson map $\varphi : M \to M$, $x \in M$, and $f \in \mathcal{C}^\infty(M)$ such that the matrix

$$\Phi = \begin{pmatrix} f(\varphi^2(x)) & f(\varphi(x)) \\ f(\varphi^3(x)) & f(\varphi^2(x)) \end{pmatrix}$$

has non-zero trace and non-zero determinant. Then the Poisson algebra $\mathcal{C}^\infty(M)$ is not rigid.

**Proof.** Using Proposition 3.6 and the notations in Example 3.11, it suffices to show that $\mu_{\varphi^*}$ is not associative. That the matrix $\Phi$ has non-vanishing trace and determinant means

$$f(\varphi^2(x)) \neq 0 \quad \text{and} \quad (f(\varphi^2(x)))^2 \neq f(\varphi(x))f(\varphi^3(x)). \quad (3.12.1)$$

Using the conditions in (3.12.1), we have:

$$\mu_{\varphi^*}(\mu_{\varphi^*}(f, f), f \varphi)(x) = (f(\varphi^2(x)))^3$$

$$\neq f(\varphi(x))f(\varphi^2(x))f(\varphi^3(x))$$

$$= \mu_{\varphi^*}(f, \mu_{\varphi^*}(f, f \varphi))(x).$$

This shows that $\mu_{\varphi^*}$ is not associative, so $\mathcal{C}^\infty(M)$ is not rigid.

The following example illustrates how the non-rigidity criterion in Corollary 3.12 can be applied.
Example 3.13: \((C^\infty(\mathbb{R}^{2n}) \text{ is not rigid})\) The Euclidean space \(\mathbb{R}^{2n}\) is a Poisson manifold with Poisson structure

\[
\{f, g\} = \sum_{i,j=1}^{n} \left( \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_{i+n}} - \frac{\partial f}{\partial x_{i+n}} \frac{\partial g}{\partial x_i} \right)
\]

for \(f, g \in C^\infty(\mathbb{R}^{2n})\). Using Corollary 3.12 we show that the Poisson algebra \(C^\infty(\mathbb{R}^{2n})\) is not rigid in the sense of Definition 3.5.

Consider the map \(\varphi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}\) defined as

\[
\varphi(a_1, \ldots, a_{2n}) = (a_1 + c_1, \ldots, a_{2n} + c_{2n}),
\]

where the \(c_j \in \mathbb{R}\) are fixed scalars, not all of which are zero, say, \(c_i \neq 0\). The map \(\varphi\) is a Poisson map by Chain Rule. Let \(f \in C^\infty(\mathbb{R}^{2n})\) be the function

\[
f(a_1, \ldots, a_{2n}) = a_i,
\]

and let 0 be the origin in \(\mathbb{R}^{2n}\). For each \(k \geq 0\), we have

\[
\varphi^k(0) = (kc_1, \ldots, kc_{2n}).
\]

So

\[
f(\varphi^2(0)) = 2c_i \neq 0
\]

and

\[
f(\varphi^2(0))^2 - f(\varphi(0))f(\varphi^3(0)) = (2c_i)^2 - c_i(3c_i)
\]

\[
= c_i^2 \neq 0.
\]

Therefore, by Corollary 3.12 the Poisson algebra \(C^\infty(\mathbb{R}^{2n})\) is not rigid. \(\square\)

Example 3.14 (Every Poisson structure on the Heisenberg algebra is rigid). Over the ground field of complex numbers, the Poisson algebra structures on the Heisenberg algebra are classified by Goze and Remm in [9] (section 2). In this example, we show that these Poisson algebras are all rigid in the sense of Definition 3.5.

Let us first recall the Goze-Remm classification of Poisson algebra structures on the Heisenberg algebra [9]. Let \(\mathcal{H}\) be the complex Heisenberg algebra, which is the three-dimensional complex Lie algebra with a basis \(\{X, Y, Z\}\) such that

\[
[X, Y] = Z, \quad [X, Z] = 0 = [Y, Z].
\]

The isomorphism classes of Poisson algebra structures on \(\mathcal{H}\) are divided into two families. First there is a one-parameter family of Poisson algebra structures on \(\mathcal{H}\),

\[
\mathcal{P}_{3,1}(\zeta) = \{XY = \zeta Z\},
\]

where \(\zeta\) is any complex number. The notation above means that in the Poisson algebra \(\mathcal{P}_{3,1}(\zeta)\), the commutative associative product satisfies

\[
XY = \zeta Z = YX,
\]

and the unspecified binary products of basis elements are all zero. The only other isomorphism class of Poisson algebra structure on \(\mathcal{H}\) is the Poisson algebra

\[
\mathcal{P}_{3,2} = \{X^2 = Z\}.
\]

We aim to show that the Poisson algebras \(\mathcal{P}_{3,1}(\zeta)\) and \(\mathcal{P}_{3,2}\) are all rigid in the sense of Definition 3.5.
Consider the Lie algebra morphisms on the Heisenberg algebra. It follows from the Heisenberg algebra relations (3.14.1) that a linear map \( \alpha : H \to H \) is a Lie algebra morphism if and only if its matrix with respect to the basis \( \{X, Y, Z\} \) is

\[
\alpha = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & b \end{pmatrix}
\]

with \( b = a_{11}a_{22} - a_{21}a_{12} \), \( 3.14.3 \)

where the \( a_{ij} \) are arbitrary complex numbers. If \( \alpha \) is a morphism on either \( P_{3,1}(\zeta) \) or \( P_{3,2} \), then we obtain further relations among the \( a_{ij} \).

(i) We first show that \( P_{3,1}(\zeta) \) is rigid. If \( \alpha : P_{3,1}(\zeta) \to P_{3,1}(\zeta) \) is a Poisson algebra morphism, then applying \( \alpha \) to the relations (3.14.2) yields

\[
\zeta b = \zeta (a_{11}a_{22} + a_{21}a_{12}),
\]

so

\[
\zeta a_{21}a_{12} = 0.
\]

(3.14.4)

Likewise, applying \( \alpha \) to the relations

\[
X^2 = 0 = Y^2
\]

in \( P_{3,1}(\zeta) \) yields

\[
\zeta a_{11}a_{21} = 0 = \zeta a_{12}a_{22}.
\]

(3.14.5)

Conversely, if a Lie algebra morphism \( \alpha : H \to H \) satisfies (3.14.4) and (3.14.5), then it is a Poisson algebra morphism on \( P_{3,1}(\zeta) \). We consider the two cases: \( \zeta = 0 \) and \( \zeta \neq 0 \).

- If \( \zeta = 0 \), then (3.14.4) and (3.14.5) do not impose further relations on \( \alpha \) in (3.14.3). Since \( P_{3,1}(0) \) has a trivial associative product \( \mu \), every \( \alpha \)-twisting of \( P_{3,1}(0) \) also has a trivial Hom-associative product \( \mu_\alpha \). With \( \alpha \) as in (3.14.3), the induced Hom-Lie bracket as in Corollary 3.4 is determined by

\[
[X, Y]_\alpha = bZ.
\]

If \( b = 0 \), then \( [\ , \]_\alpha = 0 \) and the \( \alpha \)-twisting

\[
P_{3,1}(0)'_\alpha = (P_{3,1}(0), [\ , \]_\alpha, \mu_\alpha)
\]

of \( P_{3,1}(0) \) is trivial. If \( b \neq 0 \), then the map

\[
P_{3,1}(0) \to P_{3,1}(0)'_\alpha \quad \text{with} \quad \begin{cases} 
X \mapsto X, \\
Y \mapsto Y, \\
Z \mapsto bZ
\end{cases}
\]

is an isomorphism of Poisson algebras. In other words, the Poisson algebra \( P_{3,1}(0) \) is rigid.

- If \( \zeta \neq 0 \), then the relations (3.14.4) and (3.14.5) become

\[
a_{21}a_{12} = a_{11}a_{21} = a_{12}a_{22} = 0.
\]
This means that $\alpha$ takes one of the following three forms:

$$\alpha_1 = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{11}a_{22} \end{pmatrix},$$

$$\alpha_2 = \begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & 0 & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix} \quad \text{with} \quad a_{12} \neq 0,$$

$$\alpha_3 = \begin{pmatrix} 0 & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix} \quad \text{with} \quad a_{21} \neq 0.$$

For $\alpha = \alpha_2$ or $\alpha_3$, we have $\alpha(Z) = 0$, so $[,]_\alpha = 0 = \mu_\alpha$. This implies that the $\alpha_2$-twisting and the $\alpha_3$-twisting of $P_{3,1}(\zeta)$ (with $\zeta \neq 0$) are both trivial.

For $\alpha = \alpha_1$, if $a_{11} = 0$ or $a_{22} = 0$, then $\alpha(Z) = 0$. This implies that $[,]_\alpha = 0 = \mu_\alpha$, so the $\alpha_1$-twisting of $P_{3,1}(\zeta)$ (with $\zeta \neq 0$) is trivial when $a_{11}a_{22} = 0$. On the other hand, suppose $a_{11}a_{22} \neq 0$. Since

$$\mu_\alpha(X,Y) = \zeta a_{11}a_{22}Z = \mu_\alpha(Y,X),$$

$$[X,Y]_\alpha = a_{11}a_{22}Z,$$

the map

$$P_{3,1}(\zeta) \rightarrow P_{3,1}(\zeta)'_\alpha \quad \text{with} \quad \begin{cases} X \mapsto X \\ Y \mapsto Y \\ Z \mapsto a_{11}a_{22}Z \end{cases}$$

is an isomorphism of Poisson algebras. We have shown that when $\zeta \neq 0$, every twisting of $P_{3,1}(\zeta)$ is either trivial or isomorphic to $P_{3,1}(\zeta)$ itself. Therefore, $P_{3,1}(\zeta)$ is rigid.

(ii) Next we show that $P_{3,2}$ is rigid. Suppose $\alpha : P_{3,2} \rightarrow P_{3,2}$ is a morphism of Poisson algebras. Applying $\alpha$ to the relations

$$Y^2 = 0 \quad \text{and} \quad X^2 = Z,$$

we obtain

$$a_{32} = 0 \quad \text{and} \quad a_{11}a_{22} = a_{11}^2. \quad (3.14.6)$$

Conversely, if a Lie algebra morphism $\alpha : H \rightarrow H$ satisfies $(3.14.6)$, then it is a Poisson algebra morphism on $P_{3,2}$. Therefore, the Poisson algebra morphisms $\alpha$ on $P_{3,2}$ are

$$\alpha_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix}$$

and

$$\alpha_5 = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{11} & 0 \\ a_{31} & a_{32} & a_{11}^2 \end{pmatrix} \quad \text{with} \quad a_{11} \neq 0.$$

Since $\alpha_4(Z) = 0$, we have $[,]_{\alpha_4} = 0 = \mu_{\alpha_4}$, so the $\alpha_4$-twisting of $P_{3,2}$ is trivial.

For $\alpha = \alpha_5$, we have $\alpha(Z) = a_{11}^2 Z$ and

$$\mu_\alpha(X,X) = a_{11}^2 Z = [X,Y]_\alpha.$$
So the map
\[ \mathcal{P}_{3,2} \to (\mathcal{P}_{3,2})'_{\alpha} \]
with
\[
\begin{align*}
X & \mapsto X, \\
Y & \mapsto Y, \\
Z & \mapsto a_{11}^2 Z
\end{align*}
\]
is an isomorphism of Poisson algebras. We have shown that every twisting of \( \mathcal{P}_{3,2} \) is either trivial or isomorphic to \( \mathcal{P}_{3,2} \) itself. Therefore, \( \mathcal{P}_{3,2} \) is rigid. □

4. Admissible Hom-Poisson algebras

A Poisson algebra has two binary operations, the Lie bracket and the commutative associative product. It is shown in [8, 14] that Poisson algebras can be described using only one binary operation via the polarization-depolarization process. The purpose of this section is to extend this alternative description of Poisson algebras to Hom-Poisson algebras. In other words, we will show that a Hom-Poisson algebra can be described using only the twisting map and one binary operation.

We first define the Hom-algebras that correspond to Hom-Poisson algebras.

**Definition 4.1.** Let \((A, \mu, \alpha)\) be a Hom-algebra. Then \(A\) is called an **admissible Hom-Poisson algebra** if it satisfies
\[
as_A(x, y, z) = \frac{1}{3} \left\{ (xz)\alpha(y) - (zx)\alpha(y) + (yz)\alpha(x) - (yx)\alpha(z) \right\}
\]
for all \(x, y, z \in A\), where \(as_A\) is the Hom-associator \((2.3.1)\) of \(A\).

As usual in [1.1.1] the product \(\mu\) is denoted by juxtapositions of elements in \(A\). An admissible Hom-Poisson algebra with \(\alpha = Id\) is exactly an **admissible Poisson algebra** as defined in [9].

To compare Hom-Poisson algebras and admissible Hom-Poisson algebras, we need the following function, which generalizes a similar function in [16].

**Definition 4.2.** Let \((A, \mu, \alpha)\) be a Hom-algebra. Define the quadruple
\[
P(A) = \left( A, \{ \cdot \} = \frac{1}{2}(\mu - \mu^{op}), \bullet = \frac{1}{2}(\mu + \mu^{op}), \alpha \right),
\]
called the **polarization** of \(A\). We call \(P\) the **polarization function**.

The following result says that admissible Hom-Poisson algebras, and only these Hom-algebras, give rise to Hom-Poisson algebras via polarization. It is the Hom-version of [16] (Example 2).

**Theorem 4.3.** Let \((A, \mu, \alpha)\) be a Hom-algebra. Then the polarization \(P(A)\) is a Hom-Poisson algebra if and only if \(A\) is an admissible Hom-Poisson algebra.

The proof will be given below. Assuming Theorem 4.3 for the moment, first we observe that the polarization function is actually a bijection from admissible Hom-Poisson algebras to Hom-Poisson algebras. To prove this statement, we introduce the following function.

**Definition 4.4.** Let \((A, \{ \cdot \}, \bullet, \alpha)\) be a quadruple in which \((A, \alpha)\) is a Hom-module and \(\{ \cdot \}, \bullet : A \otimes A \to A\) are binary operations. Define the Hom-algebra
\[
P^-(A) = (A, \mu = \{ \cdot \} + \bullet, \alpha),
\]
called the **depolarization** of \(A\). We call \(P^-\) the **depolarization function**.
The following result says that there is a bijective correspondence between admissible Hom-Poisson algebras and Hom-Poisson algebras via polarization and depolarization. It is the Hom-version of [9] (Proposition 3).

**Corollary 4.5.** The polarization and the depolarization functions

\[ P: \{ \text{admissible Hom-Poisson algebras} \} \leftrightarrow \{ \text{Hom-Poisson algebras} \}; P^- \]

are the inverses of each other.

**Proof.** If \((A, \mu, \alpha)\) is an admissible Hom-Poisson algebra, then \(P(A)\) is a Hom-Poisson algebra by Theorem 4.3. We have \(P^-(P(A)) = A\) because

\[ \mu = \frac{1}{2}(\mu - \mu^{op}) + \frac{1}{2}(\mu + \mu^{op}). \]

Conversely, suppose \((A, \{,\}, \bullet, \alpha)\) is a Hom-Poisson algebra. To see that \(P^-(A)\) is an admissible Hom-Poisson algebra, note that the anti-symmetry of \(\{,\}\) and the commutativity of \(\bullet\) imply that

\[ \{,\} = \frac{1}{2}((\{,\} + \bullet) - (\{,\} + \bullet)^{op}), \]

\[ \bullet = \frac{1}{2}((\{,\} + \bullet) + (\{,\} + \bullet)^{op}). \]

So the Hom-algebra \(P^-(A)\) has the property that \(P(P^-(A)) = A\), which is a Hom-Poisson algebra. It follows from Theorem 4.3 that \(P^-(A)\) is an admissible Hom-Poisson algebra. Since \(P^-P\) and \(PP^-\) are both identity functions, \(P\) and \(P^-\) are the inverses of each other. \(\square\)

It should be noted that both the polarization and the depolarization functions preserve multiplicativity. So the polarization of a multiplicative admissible Hom-Poisson algebra is a multiplicative Hom-Poisson algebra. Conversely, the depolarization of a multiplicative Hom-Poisson algebra is a multiplicative admissible Hom-Poisson algebra.

We now prove Theorem 4.3 with a series of Lemmas. To prove the “if” part of Theorem 4.3, we need some preliminary results. The following observation says that admissible Hom-Poisson algebras are Hom-flexible. It is the Hom-version of [9] (Proposition 4). The notion of Hom-flexibility is a Hom-type generalization of the usual definition of flexibility and was first introduced in [13].

**Lemma 4.6.** Every admissible Hom-Poisson algebra \((A, \mu, \alpha)\) is Hom-flexible, i.e.,

\[ a_{\text{as}}(x, y, z) + a_{\text{as}}(z, y, x) = 0 \] \hspace{1cm} (4.6.1)

for all \(x, y, z \in A\).

**Proof.** The required identity (4.6.1) follows immediately from the defining identity (4.1.1), in which the right-hand side is anti-symmetric in \(x\) and \(z\). \(\square\)

Next we observe that in an admissible Hom-Poisson algebra the cyclic sum of the Hom-associator is trivial.

**Lemma 4.7.** Let \((A, \mu, \alpha)\) be an admissible Hom-Poisson algebra. Then

\[ s_{\text{as}}(x, y, z) \overset{\text{def}}{=} a_{\text{as}}(x, y, z) + a_{\text{as}}(z, y, x) + a_{\text{as}}(y, z, x) = 0 \] \hspace{1cm} (4.7.1)

for all \(x, y, z \in A\).
Proof. Using the defining identity (4.1.1), we have:
\[
\begin{align*}
\text{as}_A(x, y, z) &= \frac{1}{3} ((yz)\alpha(x) + (xz)\alpha(y) - (yx)\alpha(z) - (zx)\alpha(y)) \\
&= -\frac{1}{3} ((zy)\alpha(x) - (yz)\alpha(x) + (xy)\alpha(z) - (xz)\alpha(y)) \\
&\quad + \frac{1}{3} ((xy)\alpha(z) - (yx)\alpha(z) + (zy)\alpha(x) - (zx)\alpha(y)) \\
&= -\text{as}_A(z, x, y) + \text{as}_A(x, z, y) \\
&= -\text{as}_A(z, x, y) - \text{as}_A(y, z, x).
\end{align*}
\]

The last equality above follows from Hom-flexibility (Lemma 4.6). Therefore, we conclude that 
\(S_A = 0\). □

Next we show that the polarization of an admissible Hom-Poisson algebra is commutative Hom-associative.

**Lemma 4.8.** Let \((A, \mu, \alpha)\) be an admissible Hom-Poisson algebra. Then
\[
\left( A, \bullet = \frac{1}{2} (\mu + \mu^{op}), \alpha \right)
\]
is a commutative Hom-associative algebra.

Proof. It is obvious that \(\bullet = (\mu + \mu^{op})/2\) is commutative. To show that the Hom-asscociator
\[
\text{as}_{P(A)} = \bullet \otimes \alpha - \alpha \otimes \bullet
\]
is trivial, pick \(x, y, z \in A\). We write \(\mu\) using juxtaposition of elements in \(A\). Expanding \(\text{as}_{P(A)}\) in terms of \(\mu\), we have:
\[
\begin{align*}
4\text{as}_{P(A)}(x, y, z) &= (xy)\alpha(z) + (yx)\alpha(z) + \alpha(z)(xy) + \alpha(z)(yx) \\
&\quad - \alpha(x)(yz) - \alpha(x)(zy) - (yz)\alpha(x) - (zy)\alpha(x) \\
&= \text{as}_A(x, y, z) - \text{as}_A(z, y, x) + (yx)\alpha(z) - (yz)\alpha(x) \\
&\quad - \text{as}_A(z, x, y) + (zx)\alpha(y) + \text{as}_A(x, z, y) - (zx)\alpha(y)
\end{align*}
\]
(4.8.1)

Using (4.1.1) and Hom-flexibility (Lemma 4.6), we can combine six of the eight terms above as follows:
\[
\begin{align*}
\text{as}_A(x, y, z) - \text{as}_A(z, y, x) + (yx)\alpha(z) - (yz)\alpha(x) + (zx)\alpha(y) - (xz)\alpha(y) \\
&= \text{as}_A(x, y, z) + \text{as}_A(x, y, z) - 3\text{as}_A(x, y, z) \\
&= -\text{as}_A(x, y, z)
\end{align*}
\]
(4.8.2)

By Lemmas 4.6 and 4.7, the other two terms in the last line in (4.8.1) become:
\[
\begin{align*}
-as_A(z, x, y) + \text{as}_A(x, z, y) &= -\text{as}_A(z, x, y) - \text{as}_A(y, z, x) \\
&= \text{as}_A(x, y, z).
\end{align*}
\]
(4.8.3)

Using (4.8.2) and (4.8.3) in (4.8.1), we conclude that \(\text{as}_{P(A)} = 0\). □

Now we observe that the polarization of an admissible Hom-Poisson algebra is a Hom-Lie algebra.
Lemma 4.9. Let \((A, \mu, \alpha)\) be a Hom-algebra. Then
\[
4J_{P(A)}(x, y, z) = as_A(x, y, z) + as_A(z, x, y) + as_A(y, z, x) - as_A(y, x, z) - as_A(z, y, x) - as_A(x, z, y)
\]
for all \(x, y, z \in A\), where \(J_{P(A)}\) is the Hom-Jacobian \((2.3.3)\) of the polarization \(P(A)\) \((4.2.1)\). Moreover, if \(A\) is an admissible Hom-Poisson algebra, then
\[
\left( A, \{, \} = \frac{1}{2}(\mu - \mu^{op}), \alpha \right)
\]
is a Hom-Lie algebra.

**Proof.** Since \(\{, \} = (\mu - \mu^{op})/2\), one can expand
\[
4J_{P(A)} = 4\{, \}(\{, \} \otimes \alpha)(Id + \sigma + \sigma^2)
\]
in terms of \(\mu\). The resulting twelve terms are then written in terms of the Hom-associator \(as_A\). The result is the right-hand side of \((4.9.1)\).

For the second assertion, first note that \(\{, \} \) is clearly anti-symmetric. Next observe that the identity \((4.9.1)\) can be rewritten as
\[
4J_{P(A)}(x, y, z) = S_A(x, y, z) - S_A(y, x, z),
\]
where \(S_A\) is the cyclic sum of the Hom-associator defined in \((4.7.1)\). Since \(S_A = 0\) in an admissible Hom-Poisson algebra (Lemma 4.7), we conclude that \(J_{P(A)} = 0\). In other words, \((A, \{, \}, \alpha)\) satisfies the Hom-Jacobi identity. \(\Box\)

The following result says that the polarization of an admissible Hom-Poisson algebra satisfies the Hom-Leibniz identity \((2.5.2)\).

Lemma 4.10. Let \((A, \mu, \alpha)\) be a Hom-algebra. Then the polarization \(P(A)\) satisfies
\[
4\left( \{\alpha(x), y \cdot z\} - \{x, y\} \cdot \alpha(z) - \alpha(y) \cdot \{x, z\} \right) = as_A(x, y, z) + as_A(z, x, y) + as_A(y, z, x) - as_A(y, x, z) - as_A(z, y, x) - as_A(x, z, y)
\]
for all \(x, y, z \in A\). Moreover, if \(A\) is an admissible Hom-Poisson algebra, then the polarization \(P(A)\) satisfies the Hom-Leibniz identity.

**Proof.** Since \(\{, \} = (\mu - \mu^{op})/2\) and \(\cdot = (\mu + \mu^{op})/2\), the left-hand side of \((4.10.1)\) can be expanded in terms of \(\mu\) into twelve terms. The result can be written in terms of the Hom-associator \(as_A\), which turns out to be the right-hand side of \((1.10.1)\).

For the second assertion, suppose that \(A\) is an admissible Hom-Poisson algebra. Then Hom-flexibility (Lemma 4.6) implies that the right-hand side of \((4.10.1)\) is 0. We conclude that
\[
\{\alpha(x), y \cdot z\} = \{x, y\} \cdot \alpha(z) + \alpha(y) \cdot \{x, z\},
\]
which is the Hom-Leibniz identity in the polarization \(P(A)\). \(\Box\)

Next we show that only admissible Hom-Poisson algebras can give rise to Hom-Poisson algebras via polarization.

Lemma 4.11. Let \((A, \mu, \alpha)\) be a Hom-algebra such that the polarization \(P(A)\) is a Hom-Poisson algebra. Then \(A\) is an admissible Hom-Poisson algebra.
Moreover, the Hom-Leibniz identity in \( P \) and (4.11.1) imply that
\[
as_A(x, y, z) = as_A(y, x, z) - as_A(y, z, x) + as_A(z, y, x) + as_A(x, z, y).
\] (4.11.1)

Moreover, the Hom-Leibniz identity in \( P(A) \) and (4.10.1) imply that
\[
as_A(x, y, z) = as_A(y, x, z) - as_A(y, z, x) + as_A(z, y, x) - as_A(x, z, y).
\] (4.11.2)

Adding (4.11.1) and (4.11.2) and dividing the result by 2, we obtain
\[
as_A(x, y, z) = as_A(y, x, z) - as_A(y, z, x),
\] (4.11.3)

which we will use in a moment.

On the other hand, since \( \mu = \{, \} + \bullet \), we can expand the Hom-associator \( as_A \) in terms of \( \{, \} \) and \( \bullet \) as follows:
\[
as_A(x, y, z) = (xy)\alpha(z) - \alpha(x)(yz)
\]
\[
= \{\{x, y\}, \alpha(z)\} + \{x \bullet y, \alpha(z)\} + \{x, y\} \bullet \alpha(z) + (x \bullet y) \bullet \alpha(z)
\]
\[
- \{\alpha(x), \{y, z\}\} - \{\alpha(x), y \bullet z\} - \alpha(x) \bullet \{y, z\} - \alpha(x) \bullet (y \bullet z)
\] (4.11.4)

Since the polarization \( P(A) \) is assumed to be a Hom-Poisson algebra, we have:
\[
0 = as_{P(A)}(x, y, z) = (x \bullet y) \bullet \alpha(z) - \alpha(x) \bullet (y \bullet z),
\]
\[
0 = \{x, y\} \bullet \alpha(y) - \alpha(y) \bullet \{x, z\}
\]
\[
= \{x \bullet y, \alpha(z)\} - \alpha(x) \bullet \{y, z\} - \{\alpha(x), y \bullet z\} + \{x, y\} \bullet \alpha(z),
\] (4.11.5)

\[
\{\{x, z\}, \alpha(y)\} = \{\{x, y\}, \alpha(z)\} - \{\alpha(x), \{y, z\}\}.
\]

Using the identities (4.11.1) in (4.11.4), we obtain:
\[
4as_A(x, y, z) = 4\{\{x, z\}, \alpha(y)\}
\]
\[
= (xz)\alpha(y) - (zx)\alpha(y) - \alpha(y)(xz) + \alpha(y)(zx)
\]
\[
= (xz)\alpha(y) - (zx)\alpha(y) + as_A(y, x, z) - (yx)\alpha(z) - as_A(y, z, x) + (yz)\alpha(x)
\]
\[
= (xz)\alpha(y) - (zx)\alpha(y) + (yz)\alpha(x) - (yx)\alpha(z) + as_A(x, y, z),
\]

where the last equality follows from (4.11.3). Finally, subtracting \( as_A(x, y, z) \) in the above calculation and dividing the result by 3, we obtain the desired identity (4.11.1). \( \square \)

**Proof of Theorem 4.3.** If \( A \) is an admissible Hom-Poisson algebra, then Lemmas 4.8, 4.9, and 4.10 imply that the polarization \( P(A) \) is a Hom-Poisson algebra. The converse is Lemma 4.11. \( \square \)

5. **Hom-power associativity**

The purpose of this section is to show that multiplicative admissible Hom-Poisson algebras are Hom-power associative. Power associativity of admissible Poisson algebras is proved in [1] (Proposition 6).

Let us first recall the definition of a Hom-power associative algebra from [26].

**Definition 5.1.** Let \( (A, \mu, \alpha) \) be a Hom-algebra, \( x \in A \), and \( n \) be a positive integer.
(1) The $n$th Hom-power $x^n \in A$ is defined inductively as
\[ x^1 = x, \quad x^n = x^{n-1} \alpha^{n-2}(x) \] (5.1.1) for $n \geq 2$.

(2) For positive integers $i$ and $j$, define
\[ x^{i,j} = \alpha^{j-1}(x)^i \alpha^{i-1}(x^j). \] (5.1.2)

(3) $A$ is called $n$th Hom-power associative if
\[ x^n = x^{n-i,i} \] (5.1.3)
for all $x \in A$ and $i \in \{1, \ldots, n-1\}$.

(4) $A$ is called Hom-power associative if $A$ is $n$th Hom-power associative for all $n \geq 2$.

By definition
\[ x^2 = xx, \quad x^n = x^{n-1,1} \] for all $n \geq 2$.

If the twisting map $\alpha$ is the identity map, then
\[ x^n = x^{n-1}x, \quad x^{i,j} = x^ix^j, \] and $n$th Hom-power associativity reduces to
\[ x^n = x^{n-i}x^i \] (5.1.4)
for all $x \in A$ and $i \in \{1, \ldots, n-1\}$. Therefore, Hom-powers and (nth) Hom-power associativity become Albert’s right powers and (nth) power associativity \[1, 2\] if $\alpha = Id$. Examples of Hom-power associative algebras include multiplicative right Hom-alternative algebras and non-commutative Hom-Jordan algebras. Other results for Hom-power associative algebras can be found in \[26\].

By definition, power associativity involves infinitely many defining identities, namely, (5.1.4) for all $n$. A well-known result of Albert \[1\] says that an algebra $(A, \mu)$ is power associative if and only if it is third and fourth power associative, i.e., the condition (5.1.4) holds for $n = 3$ and 4. Moreover, for (5.1.4) to hold for $n = 3$ and 4, it is necessary and sufficient that
\[ (xx)x = x(xx) \quad \text{and} \quad ((xx)x)x = (xx)(xx) \]
for all $x \in A$. The Hom-versions of these statements are also true. More precisely, the author proved in \[26\] that a multiplicative Hom-algebra $(A, \mu, \alpha)$ is Hom-power associative if and only if it is third and fourth Hom-power associative, which in turn is equivalent to
\[ x^2 \alpha(x) = \alpha(x)x^2 \quad \text{and} \quad x^4 = \alpha(x^2)\alpha(x^2) \] (5.1.5)
for all $x \in A$.

The following result is the Hom-version of \[1\] (Proposition 6).

**Theorem 5.2.** Every multiplicative admissible Hom-Poisson algebra is Hom-power associative.

**Proof.** As discussed above, by a result in \[26\] it suffices to prove the two equalities in (5.1.3). Hom-flexibility (Lemma \[4,6\]) implies that
\[ 0 = a_{sA}(x, x, x) = x^2 \alpha(x) - \alpha(x)x^2, \]
which proves the first identity in (5.1.5). To prove the other equality in (5.1.7), note that Hom-flexibility implies that:

\[ 0 = as_A(\alpha(x), x^2, \alpha(x)) = (\alpha(x)x^2)\alpha^2(x) - \alpha^2(x)(x^2\alpha(x)). \]

Together with the first identity in (5.1.5), we have:

\[ x^4 \overset{\text{def}}{=} (x^2\alpha(x))\alpha^2(x) = (\alpha(x)x^2)\alpha^2(x) \]

\[ = \alpha^2(x)(x^2\alpha(x)) = \alpha^2(x)(\alpha(x)x^2). \]

Using multiplicativity and (5.2.1), the defining identity (4.1.1) applied to \((\alpha(x), \alpha(x), x^2)\) says that:

\[ 0 = 3as_A(\alpha(x), \alpha(x), x^2) - (\alpha(x)x^2)\alpha^2(x) + (x^2\alpha(x))\alpha^2(x) \]

\[ = 3\{ (\alpha(x)\alpha(x))x^2 - \alpha^2(x)(\alpha(x)x^2) \} - x^4 + (\alpha(x)\alpha(x))(\alpha(x))x^2 \]

\[ = 4(\alpha(x^2)\alpha(x^2) - 4x^4). \]

We have proved the second identity in (5.1.5). □

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