A $q$-Analogue for Euler’s $\zeta(6) = \pi^6/945$

In honour of Prof. George Andrews on his 80th birthday

Ankush Goswami

Abstract. Recently, Sun [3] obtained a very nice $q$-analogue of Euler’s formula $\zeta(2) = \pi^2/6$. arXiv:1802.01473, 2018) obtained $q$-analogue of Euler’s formula for $\zeta(2)$ and $\zeta(4)$. Sun’s formulas were based on identities satisfied by triangular numbers and properties of Euler’s $q$-Gamma function. In this paper, we obtain a $q$-analogue of $\zeta(6) = \pi^6/945$. Our main results are stated in Theorems 2.1 and 2.2 below.

Mathematics Subject Classification. 11N25, 11N37, 11N60.

Keywords. $q$-Analogue, Triangular numbers.

1. Introduction

Recently, Sun [3] obtained a very nice $q$-analogue of Euler’s formula $\zeta(2) = \pi^2/6$.

**Theorem 1.1.** (Sun [3]) For a complex $q$ with $|q| < 1$, we have:

$$\sum_{k=0}^{\infty} \frac{q^k(1+q^{2k+1})}{(1-q^{2k+1})^2} = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^4}{(1-q^{2n-1})^4}. \quad (1.1)$$

Motivated by Theorem 1.1, the present author obtained the $q$-analogue of $\zeta(4) = \pi^4/90$ and noted that it was simultaneously and independently obtained by Sun in his subsequent revised paper.

**Theorem 1.2.** (Sun [3]) For a complex $q$ with $|q| < 1$, we have:

$$\sum_{k=0}^{\infty} \frac{q^{2k}(1+4q^{2k+1}+q^{4k+2})}{(1-q^{2k+1})^4} = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^8}{(1-q^{2n-1})^8}. \quad (1.2)$$

Furthermore, Sun commented that one does not know how to find $q$-analogue of Euler’s formula for $\zeta(6)$ and beyond, similar to Theorems 1.1 and 1.2. This further motivated the author to consider the problem, and indeed,
we obtained the $q$-analogue of $\zeta(6)$. As we shall see shortly, the $q$-analogue formulation of $\zeta(6)$ is more difficult as compared to $\zeta(2)$ and $\zeta(4)$ due to an extra term that shows up in the identity; however, in the limit as $q \uparrow 1$ (where $q \uparrow 1$ means $q$ is approaching 1 from inside the unit disk), this term vanishes. We also state the $q$-analogue of $\zeta(4) = \pi^4/90$, since we found it independently of Sun’s result; however, we skip the proof of this, since it essentially uses the same idea as Sun.

We emphasize here that the $q$-analogue of $\zeta(6) = \pi^6/945$ is the first non-trivial case where we notice the occurrence of an interesting extra term which essentially is the twelfth power of a well-known function of Euler (see Theorem 2.2). After obtaining this result, we obtained $q$-analogues of Euler’s general formula for $\zeta(2k), k = 4, 5, \ldots$ (see [1]). Each of these $q$-analogues has an extra term that arises from the general theory of modular forms all of which approach zero in the limit $q \uparrow 1$. The case $k = 3$ or the $q$-analogue of $\zeta(6)$ is special, since the extra term that we obtain in this case has a beautiful product representation, and has connections to well-known identities of Euler (see below).

2. Main Theorems

Theorem 2.1. For a complex $q$ with $|q| < 1$, we have:

$$\sum_{k=0}^{\infty} \frac{q^{2k} P_2(q^{2k+1})}{(1 - q^{2k+1})^4} = \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^8}{(1 - q^{2n-1})^8},$$

(2.1)

where $P_2(x) = x^2 + 4x + 1$. In other words, (2.1) gives a $q$-analogue of $\zeta(4) = \pi^4/90$.

Theorem 2.2. For a complex $q$ with $|q| < 1$, we have:

$$\sum_{k=0}^{\infty} \frac{q^k (1 + q^{2k+1}) P_4(q^{2k+1})}{(1 - q^{2k+1})^6} - \phi^{12}(q) = 256 q \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^{12}}{(1 - q^{2n-1})^{12}},$$

(2.2)

where $P_4(x) = x^4 + 236x^3 + 1446x^2 + 236x + 1$ and $\phi(q) = \prod_{n=1}^{\infty} (1 - q^n)$ is Euler’s function. In other words, (2.2) gives a $q$-analogue of $\zeta(6) = \pi^6/945$.

Remark 2.3. We note that $\phi^{12}(q)$ has a beautiful product representation and is uniquely determined by:

$$\phi^{12}(q) = \sum_{k=0}^{\infty} \frac{q^k (1 + q^{2k+1}) P_4(q^{2k+1})}{(1 - q^{2k+1})^6} - 256 q \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^{12}}{(1 - q^{2n-1})^{12}}.$$

(2.3)

In the general $q$-analogue formulation (see [1]), we do not have very elegant representations of these functions, although we obtain expressions for them similar to (2.3).
Remark 2.4. Since the coefficients in the $q$-series expansion of $\phi^{12}(q)$ are related to the pentagonal numbers by Euler’s pentagonal number theorem, and the coefficients of the product in the right-hand side of (2.2) are related to the triangular numbers, it will be worthwhile to understand the relationships of these coefficients via identity (2.2).

3. Some Useful Lemmas

Let $q = e^{2\pi i \tau}$, $\tau \in \mathcal{H}$ where $\mathcal{H} = \{ \tau \in \mathbb{C} : \text{Im}(\tau) > 0 \}$. Then, the Dedekind $\eta$-function defined by:

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad (3.1)$$

is a modular form of weight $1/2$. Also, let us denote by $\psi(q)$ the following sum:

$$\psi(q) = \sum_{n=0}^{\infty} q^{T_n}, \quad (3.2)$$

where $T_n = \frac{n(n+1)}{2}$ (for $n = 0, 1, 2, \ldots$) are triangular numbers. Then, we have the following well-known result due to Gauss:

Lemma 3.1.

$$\psi(q) = \prod_{n=1}^{\infty} \frac{1 - q^{2n}}{1 - q^{2n-1}}. \quad (3.3)$$

Thus, we have from Lemma 3.1 that:

$$\prod_{n=1}^{\infty} \frac{(1 - q^{2n})^{12}}{(1 - q^{2n-1})^{12}} = \psi^{12}(q) = \sum_{n=1}^{\infty} t_{12}(n) q^n, \quad (3.4)$$

where $t_{12}(n)$ is the number of ways of representing a positive integer $n$ as a sum of 12 triangular numbers. Next, we have the following well-known result of Ono, Robins and Wahl [2].

Theorem 3.2. Let $\eta^{12}(2\tau) = \sum_{k=0}^{\infty} a(2k+1) q^{2k+1}$. Then, for a positive integer $n$, we have:

$$t_{12}(n) = \frac{\sigma_5(2n+3) - a(2n+3)}{256}, \quad (3.5)$$

where

$$\sigma_5(n) = \sum_{d|n} d^5. \quad (3.6)$$
4. Proof of Theorem 2.2

Since $\zeta(6) = \frac{\pi^6}{945}$ has the following equivalent form:

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^6} = \frac{63}{64} \zeta(6) = \frac{\pi^6}{960},$$  
(4.1)

it will be sufficient to get the $q$-analogue of (4.1). Now, from $q$-analogue of Euler’s Gamma function, we know that:

$$\lim_{q \uparrow 1} (1 - q) \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^2}{(1 - q^{2n-1})^2} = \frac{\pi^2}{2},$$  
(4.2)

so that from (4.2), we have:

$$\lim_{q \uparrow 1} (1 - q)^6 \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^{12}}{(1 - q^{2n-1})^{12}} = \frac{\pi^6}{64}. $$  
(4.3)

Next, we consider the following infinite series

$$S_6(q) := \sum_{k=0}^{\infty} q^k \frac{(1 + q^{2k+1}) P_4(q^{2k+1})}{(1 - q^{2k+1})^6},$$  
(4.4)

where $P_4(x) = x^4 + 236x^3 + 1446x^2 + 236x + 1$.

By partial fractions, we have:

$$S_6(q) = \sum_{k=0}^{\infty} q^k \left\{ \frac{3840}{(1 - q^{2k+1})^6} - \frac{9600}{(1 - q^{2k+1})^5} + \frac{8160}{(1 - q^{2k+1})^4} - \frac{2640}{(1 - q^{2k+1})^3} + \frac{242}{(1 - q^{2k+1})^2} - \frac{1}{(1 - q^{2k+1})} \right\}.$$  
(4.5)

**Lemma 4.1.** With $S_6(q)$ represented by (4.5), we have:

$$S_6(q) = 256q \sum_{n=0}^{\infty} t_{12}(n) q^n + \phi^{12}(q). $$  
(4.6)

**Proof.** From (4.5), we have:

$$S_6(q) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} q^k \left\{ 3840 \binom{-6}{j} - 9600 \binom{-5}{j} + 8160 \binom{-4}{j} \right.$$  

$$- 2640 \binom{-3}{j} + 242 \binom{-2}{j} - \binom{-1}{j} \right\} (-q)^{j(2k+1)}$$  

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left\{ 32(j + 1)(j + 2)(j + 3)(j + 4)(j + 5) \right.$$  

$$- 400(j + 1)(j + 2)(j + 3)(j + 4) + 1360(j + 1)(j + 2)(j + 3)$$  

$$- 1320(j + 1)(j + 2) + 242(j + 1) - 1 \right\} q^{k+j}(2k+1)$$  




A $q$-Analogue for Euler’s $\zeta(6) = \frac{\pi^6}{945}$

\[
= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (2j + 1)^5 q^{\frac{(2j+1)(2k+1) - 1}{2}} \\
= \sum_{n=0}^{\infty} \sigma_5(2n + 1) q^n \\
= 1 + \sum_{n=1}^{\infty} \sigma_5(2n + 1) q^n \\
= 1 + q \sum_{n=0}^{\infty} \sigma_5(2n + 3) q^n.
\]

Also from (3.1), we have:

\[
\phi_{12}(q) = \eta_{12}(\tau) \\
= \sum_{n=0}^{\infty} a(2n + 1) q^n \\
= 1 + \sum_{n=1}^{\infty} a(2n + 1) q^n \\
= 1 + q \sum_{n=0}^{\infty} a(2n + 3) q^n.
\]

Thus, from above, we have:

\[
S_6(q) - \phi_{12}(q) = q \sum_{n=0}^{\infty} \{\sigma_5(2n + 3) - a(2n + 3)\} q^n \\
= 256 q \sum_{n=0}^{\infty} t_{12}(n) q^n,
\]

where the last step follows from Theorem 3.2. This completes the proof of Theorem 2.2. \qed

We also note that

\[
\lim_{q \uparrow 1} (1 - q)^6 (S_6(q) - \phi_{12}(q)) = \lim_{q \uparrow 1} (1 - q)^6 S_6(q) - \lim_{q \uparrow 1} (1 - q)^6 \phi_{12}(q) \\
= \sum_{k=0}^{\infty} \frac{3840}{(2k + 1)^6}, \tag{4.7}
\]

where $\lim_{q \uparrow 1} (1 - q)^6 \phi_{12}(q) = 0$ and $q \uparrow 1$ indicates $q \to 1$ from within the unit disk. Hence, combining Eqs. (4.1), (4.3), (4.7), and Lemma 4.1, Theorem 2.2 follows.

**Acknowledgements**

Open access funding provided by Johannes Kepler University Linz. I am grateful to Prof. Krishnaswami Alladi for carrying out discussions pertaining to the
function $\phi(q)$ and for his encouragement. I also thank Prof. George Andrews for going through my proof and providing me a few useful references.

**Open Access.** This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

**References**

[1] Goswami, A.: A $q$-analogue for Euler’s evaluations of the Riemann zeta function. Res. Number Theory 5(1), #Art. 3 (2019)

[2] Ono, K., Robins, S., Wahl, P.T.: On the representation of integers as sums of triangular numbers. Aequationes Math. 50(1-2), 73–94 (1995)

[3] Sun, Z.-W.: Two $q$-analogues of Euler’s formula $\zeta(2) = \pi^2/6$. arXiv:1802.01473 (2018)

Ankush Goswami
Department of Mathematics
University of Florida
Gainesville
Fl 32603
USA
e-mail: ankush04@ufl.edu

Present Address
Ankush Goswami
Research in Symbolic Computation (RISC)
Johannes Kepler University
Altenbergerstraße 69
4040 Linz
Austria
e-mail: agoswami@risc.jku.at

Received: 5 September 2018.
Accepted: 8 May 2019.