Analysis on controlling complex networks based on dominating sets

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Abstract. Recent works have emphasized the importance of controlling complex networks and have provided novel analytical approaches based on the maximum matching among others. The analysis leads to the conclusion that the driver nodes tend to avoid the high-degree nodes. Therefore, it implies that apparently inhomogeneous networks are more difficult to control. Here we present an approach to control complex networks from the perspective of the minimum dominating set (MDS). We apply the definition of MDS to complex networks with scale-free structure \( P(k) \propto k^{-\gamma} \) and assume that a node controls itself and each of its adjacent nodes through its links. Our theoretical calculations and simulations demonstrate that the more heterogeneous a network's degree distribution is, the smaller the fraction of individuals, devices or molecules required to control the entire system is. Here, we have extended our recent analytical derivations and we now provide a more accurate analysis for the case \( \gamma < 2 \). Moreover, we have also performed additional computer simulations to study in more detail the dependence of the MDS with the network size and the average degree below the phase transition at \( \gamma = 2 \).

1. Introduction
The emergence of large complex networks structures in physical, biological and technological systems poses the question as to whether we can effectively control and direct the system’s entire dynamics at will. Recent works have investigated the controllability in complex networks [1, 2]. In particular, Liu et al., has used a maximum matching algorithm to investigate network control. As a result, homogeneous (i.e., random networks) can be controlled easily because the number of nodes that need to be controlled externally is small. However, there are several ways to define controllability in complex networks. Nepusz and Vicsek [3, 4] have analyzed this problem from the angle of edge dynamics. In parallel, we have recently introduced the minimum dominating set (MDS) approach to control complex networks, which conceptually has similarities with the controlling link dynamics [3]. First, our theoretical findings suggest that scale-free networks with small scaling exponent values (\( \gamma < 2 \)), where high-degree nodes are present, require relatively few nodes to be controlled [5]. Conversely, networks with large degree exponent values (\( \gamma > 2 \)) or faster exponential decay, where hubs are weakly connected or almost absent, require more nodes to be fully regulated. In this work, we extend our previous analytical derivations and provide a more accurate analysis for the case \( \gamma < 2 \). Moreover, we also perform

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additional computer simulations to study in more detail the dependence of the MDS with the network size and the average degree below $\gamma = 2$.

2. Computation of the MDS
A set $S \subseteq V$ of nodes in a graph $G = (V, E)$ is a dominating set if every node $v \in V$ is either an element of $S$ or is adjacent to an element of $S$ [6]. Since the domination problem is a classical NP-complete decision problem in computational complexity theory [6], it is not plausible that there exists a polynomial time algorithm that finds a smallest dominating set for arbitrary graphs. Here, we reduce this problem to the computation of a binary integer programming problem. We then minimize a linear function $f(x)$ subject to linear constraints [5]. Each node $i$ is assigned to a binary integer variable $x_i$ that takes values 1 or 0. A node $i$ that belongs to the dominating set will take the value $x_i = 1$.

3. Theoretical analysis
3.1. Mathematical analysis of MDS size in scale-free networks
We here briefly mention the main results of our previous analysis and present our latest findings for the case $\gamma < 2$.

Let $G(V, E)$ be an undirected random graph with node set $V$ and edge set $E$ such that the degree distribution follows a power-law $k^{-\gamma}$. We let $n = |V|$. Our results show the following:

- if $\gamma > 2$, the size of the minimum dominating set is $\Theta(n)$,
- if $\gamma < 2$, the expected size of the minimum dominating set is $o(n)$.

This result means that a phase transition occurs at $\gamma = 2$. Here we remind that $f(n)$ is $\Theta(g(n))$ if it is both $O(g(n))$ and $\Omega(g(n))$, that is, $f(n)$ is $\Theta(g(n))$ if $k_1 \cdot g(n) \leq f(n) \leq k_2 \cdot g(n)$ holds for sufficiently large $n$ where $k_1$ and $k_2$ are some positive constants. Moreover, $f(n)$ is $o(g(n))$ if for every $\epsilon > 0$, $|f(n)| \leq |g(n)| \cdot \epsilon$ holds for sufficiently large $n$. Hereafter, we focus on the case $\gamma < 2$, and provide a more accurate analysis.

3.2. On Upper Bound of the Size of the Minimum Dominating Set
Here we give a more accurate estimate of an upper bound of the size of MDS in a scale-free network. We first assume that the degree distribution follows $\alpha k^{-\gamma}$ with cut off at $k = n$, where $1 < \gamma < 2$. From $\alpha n \int_1^n k^{-\gamma} dk = n$, we have $\alpha \approx \gamma - 1$.

We select the nodes with degree from $n^\beta$ to $n$ as a core part of a dominating set (this core part is denoted by $DS$), where $\beta < 1$ is a positive constant depending on $\gamma$. Then, the number of nodes $N_{DS}$ in $DS$ is given by

$$N_{DS} = \alpha n \int_{n^\beta}^{n} k^{-\gamma} dk = O(n^{1-\beta(\gamma-1)})$$

On the other hand, the total number of edges $E_G$ is given by

$$E_G = \frac{\alpha n}{2} \int_{1}^{n} k \cdot k^{-\gamma} dk = \frac{\gamma - 1}{2(2-\gamma)} \cdot n \cdot (n^{2-\gamma} - 1).$$

Let $E_{DS}$ be the number of edges covered by $DS$. A lower bound is estimated by

$$E_{DS} \geq \frac{\alpha n}{2} \int_{n^\beta}^{n} k \cdot k^{-\gamma} dk = \frac{\gamma - 1}{2(2-\gamma)} \cdot n \cdot (n^{2-\gamma} - n^{\beta(2-\gamma)})$$
Therefore, the probability that an arbitrary edge is not covered by $DS$ is

$$\frac{E_G - E_{DS}}{E_{DS}} \leq \frac{n^{\beta(2-\gamma)} - 1}{n^{2-\gamma} - n^{\beta(2-\gamma)}} \approx \frac{n^{\beta(2-\gamma)}}{n^{2-\gamma}} = n^{(\beta-1)(2-\gamma)}.$$

Let $G \ominus DS$ denote the set of nodes in $G$ that are not dominated by $DS$. Since a node is dominated by $DS$ if at least one edge connecting to the node is covered by $DS$, the expected number of nodes (denoted by $N_{G \ominus DS}$) of $G \ominus DS$ is bounded as

$$N_{G \ominus DS} \leq O(n \cdot n^{(\beta-1)(2-\gamma)}) = O(n^{1+(\beta-1)(2-\gamma)}).$$

Here, we note that $DS \cup (G \ominus DS)$ becomes a dominating set. Therefore, the size of a minimum dominating set is bounded by

$$|DS \cup (G \ominus DS)| \leq O(n^{1-\beta(\gamma-1)} + n^{1+(\beta-1)(2-\gamma)}).$$

In order to find $\beta$ minimizing this order (recall that we can choose $\beta$ depending on $\gamma$), we let

$$1 - \beta(\gamma - 1) = 1 + (\beta - 1)(2 - \gamma),$$

which results in $\beta = 2 - \gamma$. Therefore, an upper bound of the size of the dominating set is estimated as

$$O(n^{1+(2-\gamma)(1-\gamma)}).$$

For example, if $\gamma = 1.5$, this number is $O(n^{0.75})$, which means that the size of MDS is $o(n)$. It is interesting to note that this value takes the minimum when $\gamma = 1.5$.

### 4. Computational experiments on MDS size in scale-free networks

To analyze in detail the dependence of the scaling exponent and mean degree on MDS size, we generated a variety of scale-free networks by using an algorithm that generates random simple connected graphs with prescribed degree sequence [7]. Here, we extend our previous results by investigating the behaviour of the MDS size on the network size and the average degree below $\gamma = 2$. Our previous results indicate that the MDS size increases as $\gamma$ increases. Moreover, the MDS generally decreases when the average degree increases for values of $\gamma > 2$. Conversely, when $\gamma < 2$, the MDS slightly increases or becomes constant as the average degree increases. In particular, it seems to take a minimum value at $\gamma = 1.5$. Here, we study simultaneously the effect of average degree and network size on the MDS size and how it changes for values around $\gamma = 1.5$ and below $\gamma = 2$. Figure 1 shows the dependence of the MDS size on the systems size. The results show that, for $\gamma < 1.5$, denser networks are more difficult to control. This tendency is also observed when the system size grows. However, the networks show a different behaviour for $\gamma > 1.5$. In this case, our results show that the MDS size tends to decreases when average degree increases (data not shown due to limited space).

### 5. Conclusion

In sharp contrast to a recent work on controllability that uses a maximum matching [1], our analysis demonstrates that the MDS tends to target highly connected nodes, whereas the previous study suggested that driver nodes tend to avoid high-degree nodes. Our approach, however, can be connected to Liu et al., by assuming that every edge in a network is bi-directional.
and every node in MDS can control all of its outgoing links separately. Then, the network is structurally controllable by selecting the nodes in MDS as the driver nodes [5]. Though Müller and Schuppert [2] recently suggested that iPS cells can be controlled by a few driver nodes, they did not show general results or consider structural properties of networks. Moreover, our approach has some conceptual similarities with controlling edge dynamics [3], although we are tackling the problem from MDS view points rather than using switchboard-dynamics and its mapping to the line graph [8].

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