LINEAR STABILITY ANALYSIS OF THE COUETTE FLOW FOR THE TWO DIMENSIONAL NON-ISENTROPIC COMPRESSIBLE EULER EQUATIONS

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Abstract. This note is devoted to the linear stability of the Couette flow for the non-isentropic compressible Euler equations in a domain $\mathbb{T} \times \mathbb{R}$. Exploiting the several conservation laws originated from the special structure of the linear system, we obtain a Lyapunov type instability for the density, the temperature, the compressible part of the velocity field, and also obtain an inviscid damping for the incompressible part of the velocity field.

1. Introduction and the main result

Inspired by [1] and [2], in the present paper, we are interested in the long-time asymptotic behaviour of the linearized two dimensional non-isentropic compressible Euler equations in a domain $\mathbb{T} \times \mathbb{R}$. The governing equations (in non-dimensional variables) are

\[
\frac{\partial \rho}{\partial t} + u \cdot \nabla \rho + \rho \text{div } u = 0, \tag{1.1}
\]

\[
\rho \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) = -\frac{\nabla P}{\gamma M^2}, \tag{1.2}
\]

\[
P = \rho \vartheta, \tag{1.3}
\]

\[
\rho \left( \frac{\partial \vartheta}{\partial t} + u \cdot \nabla \vartheta \right) = -(\gamma - 1) \text{div } u, \tag{1.4}
\]

where for $(x, y) \in \mathbb{T} \times \mathbb{R}$ and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, $u$ is the velocity vector, $\rho$ is the density, $P$ is the pressure, $\vartheta$ is the temperature, $\gamma$ is the ratio of specific heats, and $M$ is the Mach number of the reference state. The question of stability of Couette flows has a long history, see [1], [2], [6], [7], [8], [11], [12], [13], [15] for the compressible fluids and see [3], [4], [5], [9], [10], [14], [16] for the incompressible fluids.

The Couette flow,

\[
u_{sh} = \begin{pmatrix} y \\ 0 \end{pmatrix}, \quad \varrho_{sh} = 1, \quad \vartheta_{sh} = 1, \tag{1.5}
\]

is clearly a stationary solution of (1.1)–(1.4). Our goal is to understand the stability and long-time behavior of perturbations near the Couette flow.

Denote

\[
\rho = \varrho - \varrho_{sh}, \quad \vartheta = \vartheta - \vartheta_{sh}, \quad v = u - u_{sh}.
\]

The linearized system around the homogeneous Couette flow read as follows

\[
\begin{aligned}
\partial_t \rho + y \partial_x \rho + \text{div } v &= 0, \\
\partial_t v + y \partial_x v + \begin{pmatrix} vy \\ 0 \end{pmatrix} + \frac{1}{\gamma M^2} (\nabla \rho + \nabla \vartheta) &= 0, \\
\partial_t \vartheta + y \partial_x \vartheta + (\gamma - 1) \text{div } v &= 0.
\end{aligned} \tag{1.6}
\]
Define
\[ \alpha = \text{div } \mathbf{v}, \quad \omega = \nabla^\perp \cdot \mathbf{v}, \quad \text{with } \nabla^\perp = (-\partial_y, \partial_x)^T, \]
according to the Helmholtz projection operators, we have
\[ \mathbf{v} = (v^x, v^y)^T \stackrel{\text{def}}{=} \mathbb{P}[\mathbf{v}] + \mathbb{Q}[\mathbf{v}] \tag{1.7} \]
with
\[ \mathbb{P}[\mathbf{v}] \stackrel{\text{def}}{=} \nabla^\perp \Delta^{-1} \omega, \quad \mathbb{Q}[\mathbf{v}] \stackrel{\text{def}}{=} \nabla \Delta^{-1} \alpha. \tag{1.8} \]
From the above definition, one can infer that
\[ v^y = \partial_y (\Delta^{-1}) \alpha + \partial_x (\Delta^{-1}) \omega, \tag{1.9} \]
hence, we can rewrite (1.6) in terms of \((\rho, \alpha, \omega, \theta)\) that
\[
\begin{cases}
\partial_t \rho + y \partial_x \rho + \alpha = 0, \\
\partial_t \alpha + y \partial_x \alpha + 2 \partial_x (\partial_y (\Delta^{-1}) \alpha + \partial_x (\Delta^{-1}) \omega) + \frac{1}{\gamma M^2} (\Delta \rho + \Delta \theta) = 0, \\
\partial_t \omega + y \partial_x \omega - \alpha = 0, \\
\partial_t \theta + y \partial_x \theta + (\gamma - 1) \alpha = 0.
\end{cases} \tag{1.10}
\]
Obviously, the above system (1.10) is a closed system regarding of \((\rho, \alpha, \omega, \theta)\).

Now, we can state the main result of the present paper.

**Theorem 1.1.** Let \((\rho^{in}, \omega^{in}, \theta^{in}) \in H^1_x(\mathbb{T}) H^2_y(\mathbb{R})\) and \(\alpha^{in} \in H^{-\frac{1}{2}}_x(\mathbb{T}) L^2_y(\mathbb{R})\) be the initial data of (1.10) with
\[
\int_T \rho^{in} \, dx = \int_T \omega^{in} \, dx = \int_T \alpha^{in} \, dx = \int_T \theta^{in} \, dx = 0.
\]
Then
\[
\| \mathbb{P}[\mathbf{v}]^x(t) \|_{L^2} \leq C M \left( \frac{\| \rho^{in} + \theta^{in} \|_{H^1_x} + \| \alpha^{in} \|_{H^2_y} + \| \rho^{in} + \theta^{in} + \gamma \omega^{in} \|_{H^1_x H^2_y}}{\gamma} \right)
\]
\[
+ \frac{C}{(t)} \left( \frac{\| \rho^{in} + \theta^{in} + \gamma \omega^{in} \|_{H^1_x H^2_y}}{\gamma} \right), \tag{1.11}
\]
\[
\| \mathbb{P}[\mathbf{v}]^y(t) \|_{L^2} \leq C M \left( \frac{\| \rho^{in} + \theta^{in} \|_{H^1_x H^2_y} + \| \alpha^{in} \|_{H^2_y} + \| \rho^{in} + \theta^{in} + \gamma \omega^{in} \|_{H^1_x H^2_y}}{\gamma} \right)
\]
\[
+ \frac{C}{(t)^2} \left( \frac{\| \rho^{in} + \theta^{in} + \gamma \omega^{in} \|_{H^1_x H^2_y}}{\gamma} \right). \tag{1.12}
\]
Moreover,
\[
\| \mathbb{Q}[\mathbf{v}](t) \|_{L^2} + \frac{\gamma}{M} \| \rho(t) \|_{L^2} + \frac{\gamma}{M} \| \theta(t) \|_{L^2}
\]
\[
\leq C \left( \frac{\| \rho^{in} + \theta^{in} \|_{L^2} + \| \alpha^{in} \|_{H^{-1}} + \frac{\| \rho^{in} + \theta^{in} + \gamma \omega^{in} \|_{H^1_x}}{\gamma}}{M} \right) \tag{1.13}
\]

**Remark 1.2.** In a forthcoming paper, we consider the linear stability of the Couette flow for the non-isentropic compressible Euler equations in three dimensional.
Remark 1.3. Our stability analysis coincides with the isentropic compressible Euler equations discussed in [1] and [2] if we neglect the effect of the temperature. Moreover, following a similar argument as [1] and [2], we can also obtain the lower bound for the density, the temperature and the compressible part of the velocity field as

\[ \|Q[v]\|_{L^2} + \frac{1}{M} \|\rho + \theta\|_{L^2} \geq \langle t \rangle^{\frac{1}{2}} C_{in} \]

if the initial data up to a nowhere dense set.

2. The proof of the main theorem

First of all, we are concerned with the dynamics of the \( x \)-averages of the perturbations. In order to reveal the distinction between the zero mode case \( k = 0 \) and the nonzero modes \( k \neq 0 \). We define

\[ f_0(y) \overset{\text{def}}{=} \frac{1}{2\pi} \int_{\mathbb{T}} f(x, y) \, dx, \quad f_{\neq}(x, y) \overset{\text{def}}{=} f(x, y) - f_0(y), \quad \tag{2.1} \]

which represents the projection onto 0 frequency and the projection onto non-zero frequencies. Let

\[ \hat{f}(k, \eta) = \frac{1}{2\pi} \int_{\mathbb{T} \times \mathbb{R}} e^{-i(kx + \eta y)} f(x, y) \, dx \, dy, \]

then we define \( f \in H_{x}^{s_1} \bigl[H_{y}^{s_2} \bigr] \) if

\[ \|f\|^2_{H_{x}^{s_1} \bigl[H_{y}^{s_2} \bigr]} = \sum_{k} \langle k \rangle^{2s_1} \langle \eta \rangle^{2s_2} |\hat{f}(k, \eta)|^2 \, d\eta < +\infty. \]

Moreover, we also denote the usual \( H^s(\mathbb{T} \times \mathbb{R}) \) space as

\[ \|f\|^2_{H^s} = \sum_{k} \langle k \rangle^{2s} |\hat{f}(k, \eta)|^2 \, d\eta. \]

Integration in \( x \) equations in (1.10), one infer that

\[
\begin{aligned}
\partial_t \rho_0 &= -\alpha_0, \\
\partial_x \alpha_0 &= -\frac{1}{\gamma M^2} \partial_{yy} \rho_0 - \frac{1}{\gamma M^2} \partial_{yy} \theta_0, \\
\partial_t \omega_0 &= \alpha_0, \\
\partial_t \theta_0 &= - (\gamma - 1) \alpha_0.
\end{aligned} \quad \tag{2.2}
\]

From the above equation (2.2), we can further get \( \alpha_0, \rho_0 + \theta_0 \) satisfy the following wave equations:

\[
\begin{aligned}
\partial_{tt} \alpha_0 &= - \frac{1}{M^2} \partial_{yy} \alpha_0 = 0, \quad \text{in } \mathbb{R}, \\
\partial_{tt} (\rho_0 + \theta_0) &= - \frac{1}{M^2} \partial_{yy} (\rho_0 + \theta_0) = 0, \quad \text{in } \mathbb{R}.
\end{aligned} \quad \tag{2.3} \tag{2.4}
\]

Hence, given \( \rho_0^{in} = \alpha_0^{in} = \theta_0^{in} = \omega_0^{in} = 0 \), according to the explicit representation formula for (2.3), (2.4), we can get for all \( t \geq 0 \)

\[ \rho_0(t) = \alpha_0(t) = \theta_0(t) = \omega_0(t) = 0. \]

Consequently, in our analysis we can decouple the evolution of the \( k = 0 \) mode from the rest of the perturbation. Let us consider the following coordinate transform

\[
\begin{pmatrix}
x \\
y
\end{pmatrix} \mapsto \begin{pmatrix}
X \\
Y
\end{pmatrix} = \begin{pmatrix}
x - yt \\
y
\end{pmatrix}.
\]
Under the new coordinate transform, the differential operators change as follows
\[ \partial_x = \partial X, \quad \partial_y = \partial Y - t \partial X, \quad \Delta = \Delta_L = \partial_{XX} + (\partial_Y - t \partial_X)^2. \]

Define
\[ R(t, X, Y) = \rho(t, X + ty, Y), \quad A(t, X, Y) = \alpha(t, X + ty, Y), \]
\[ \Omega(t, X, Y) = \omega(t, X + ty, Y), \quad \Theta(t, X, Y) = \theta(t, X + ty, Y). \]

Then, the linear system (1.10) reduces to the following system in the new coordinates
\[
\begin{align*}
\partial_t R &= -A, \\
\partial_t A &= -2 \partial_X (\partial_Y - t \partial_X)(\Delta_L^{-1})A - 2 \partial_{XX}(\Delta_L^{-1})\Omega - \frac{1}{\gamma M^2} (\Delta_L R + \Delta_L \Theta), \\
\partial_t \Omega &= A, \\
\partial_t \Theta &= -(\gamma - 1)A.
\end{align*}
\]

(2.5)

From (1.10), we also know that
\[
\begin{align*}
(\partial_t + y \partial_x)(\rho + \omega) &= 0, \\
(\partial_t + y \partial_x)(\theta + (\gamma - 1)\omega) &= 0.
\end{align*}
\]

(2.6)

In particular, (2.6) implies that \(\rho + \omega\) and \(\theta + (\gamma - 1)\omega\) are transported by the Couette flow. Hence, if we further define
\[ \beta(t, X, Y) \overset{\text{def}}{=} R(t, X, Y) + \Omega(t, X, Y), \quad \Gamma(t, X, Y) \overset{\text{def}}{=} \Theta(t, X, Y) + (\gamma - 1)\Omega(t, X, Y), \]
then, we have
\[
\begin{align*}
\partial_t \beta &= 0, \\
\partial_t \Gamma &= 0,
\end{align*}
\]

which implies that
\[ \beta = \beta^{in} = \rho^{in} + \omega^{in}, \quad \Gamma = \Gamma^{in} = \theta^{in} + (\gamma - 1)\omega^{in}. \]

(2.8)

Moreover, one can infer from (2.7) and (2.8) that
\[
\begin{align*}
\Omega(t, X, Y) &= \beta^{in}(X, Y) - R(t, X, Y), \\
\Omega(t, X, Y) &= \frac{\Gamma^{in}(X, Y) - \Theta(t, X, Y)}{\gamma - 1},
\end{align*}
\]

which gives
\[ \Omega(t, X, Y) = \frac{\beta^{in}(X, Y) + \Gamma^{in}(X, Y)}{\gamma} - \frac{R(t, X, Y) + \Theta(t, X, Y)}{\gamma}. \]

(2.9)

To accelerate the proof, we continue to introduce the following notation
\[ \delta = \frac{R + \Theta}{\gamma}. \]

(2.10)

In view of (1.9), (2.9) and (2.10), we get
\[ V_y = (\partial_Y - t \partial_X)\Delta_L^{-1} A + \partial_X \Delta_L^{-1} \Omega = (\partial_Y - t \partial_X)\Delta_L^{-1} A + \frac{1}{\gamma} \partial_X \Delta_L^{-1} (\beta^{in} + \Gamma^{in}) - \frac{1}{\gamma} \partial_X \Delta_L^{-1} (R + \Theta) = (\partial_Y - t \partial_X)\Delta_L^{-1} A + \frac{1}{\gamma} \partial_X \Delta_L^{-1} (\beta^{in} + \Gamma^{in}) - \partial_X \Delta_L^{-1} \delta. \]
As a result, (2.5) reduces to the following system
\[
\begin{aligned}
\partial_t \delta &= - A, \\
\partial_t A &= - \frac{1}{M^2} \Delta_L + 2 \partial_{XX}(\Delta_L^{-1}) \delta - \frac{2}{\gamma} \partial_{XX}(\Delta_L^{-1})(\beta^{in} + \Gamma^{in}).
\end{aligned}
\] (2.11)

Compared to (2.5), the above system (2.11) is a closed $2 \times 2$ system only involving $\delta$ and $A$. Moreover, the above system (2.11) is almost the same as the isentropic compressible Euler system discussed in [1], [2]. Hence, most of the following argument can be founded in [1], [2], we present them here for reader’s convenience.

We define the symbol associated to $-\Delta_L$ as
\[ p(t, k, \eta) = k^2 + (\eta - kt)^2, \]
and denote the symbol associated to the operator $2 \partial_X(\partial_Y - t \partial_X)$ as
\[ (\partial_{tt} \mu)(t, k, \eta) = -2k(\eta - kt). \]

Taking the Fourier transform in (2.11), we have
\[
\begin{aligned}
\partial_t \hat{\delta} &= - \hat{A}, \\
\partial_t \hat{A} &= \frac{\partial_t p}{p} \hat{A} + \left( \frac{p}{M^2} + \frac{2k^2}{p^3} \right) \hat{\delta} - \frac{2k^2}{\gamma p} (\hat{\beta}^{in} + \hat{\Gamma}^{in}).
\end{aligned}
\] (2.12)

Motivated by [1], [2], we introduce the following metric
\[ Z(t) = \begin{pmatrix} Z_1(t) \\ Z_2(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{M p^\beta} \hat{\delta}(t) \\ \frac{1}{p^\beta} \hat{A}(t) \end{pmatrix}. \]

By a direct computation we find that $Z(t)$ satisfy
\[
\begin{aligned}
\frac{d}{dt} Z(t) &= L(t) Z(t) + F(t)(\hat{\beta}^{in} + \hat{\Gamma}^{in}), \\
Z(0) &= Z^{in}
\end{aligned}
\] (2.13)
where
\[ L(t) = \begin{bmatrix} -\frac{\partial_t p}{4p} & -\sqrt{p} M \\ 2 \sqrt{p} M + \frac{2Mk^2}{p^{3/2}} & \frac{\partial_t p}{4p} \end{bmatrix}, \quad F(t) = \begin{pmatrix} 0 \\ -\frac{2k^2}{\gamma p^{7/4}} \end{pmatrix}. \] (2.14)

and
\[ Z^{in} = \left( \frac{1}{M(k^2 + \eta^2)^{\beta}} \hat{\beta}^{in}, \frac{1}{(k^2 + \eta^2)^{\beta}} \hat{A}^{in} \right)^T. \] (2.15)

Observe that if we are able to get a uniform bound on $|Z|$, in view of the weight on $\delta$ and $A$, we can obtain the desired time decay. This point can be illustrated by the following lemma.

**Lemma 2.1.** Let $p = -\Delta_L = k^2 + (\eta - kt)^2$, then for any function $f \in H^{s+2\beta}(\mathbb{T} \times \mathbb{R})$, it holds that
\[
\| p^{-\beta} f \|_{H^s} \leq C \left( \frac{1}{t} \right)^{2\beta} \| f \|_{H^{s+2\beta}} \quad \| p^\beta f \|_{H^s} \leq C (t)^{2\beta} \| f \|_{H^{s+2\beta}},
\] (2.16)
for any $\beta > 0$. 

Proof. The bounds (2.16) follows just by Plancherel Theorem and the basic inequalities for japanese brackets $\langle k, \eta \rangle \leq C \langle \eta - \xi \rangle \langle k, \xi \rangle$. \hfill \Box

Next, we use the following key lemma to study the homogeneous problem associated to (2.13).

**Lemma 2.2.** Let $Z(t)$ be a solution to (2.13) with $\hat{\beta}^{in} + \hat{\Gamma}^{in} = 0$. Define

$$a(t) = \frac{1}{4} \frac{\partial_p}{p}, \quad b(t) = \sqrt{\frac{p}{M}}, \quad d(t) = \frac{2}{\gamma} \sqrt{\frac{p}{M}} + \frac{2Mk^2}{p^{3/2}}.$$ 

and

$$E(t) = \left( \sqrt{\frac{d}{b}} |Z_1|^2 \right)(t) + \left( \sqrt{\frac{b}{d}} |Z_2|^2 \right)(t) + 2 \left( \frac{a}{\sqrt{db}} \text{Re}(Z_1 \bar{Z}_2) \right)(t).$$

Then, there exists constants $c_1, C_1, c_2, C_2 > 0$ independent of $k, \eta$ such that

$$c_1 E(0) \leq E(t) \leq C_1 E(0),$$

and

$$c_2 |Z^{in}| \leq |Z(t)| \leq C_2 |Z^{in}|.$$ 

Proof. The proof of this lemma can be founded in [1], [2], here, we omit the details. \hfill \Box

To prove our theorem, we must use the integrated form of the solutions to (2.13). That is to say, we can write the solutions of (2.13) by

$$Z(t) = \Phi_L(t, 0) \left( Z^{in} + \int_0^t \Phi_L(0, s) F(s)(\hat{\beta}^{in} + \hat{\Gamma}^{in}) ds \right),$$

where $\Phi_L$ is the solution operator related to the equation

$$\frac{d}{dt} Z(t) = L(t) Z(t).$$

According to Lemma 2.2 there holds

$$\int_0^\infty |\Phi_L(t, s) F(s)| \, ds \leq C \int_0^\infty |F(s)| \, ds$$

$$\leq \frac{C}{|k|^2} \int_0^\infty \frac{ds}{(1 + (\eta/k - s)^2)^{\frac{3}{2}}}$$

$$\leq \frac{C}{\gamma |k|^2}.$$ 

(2.18)

Hence, for any $t \geq 0$, using Lemma 2.2 once again, we deduce from (2.17) and (2.18) that

$$|\hat{Z}(t, k, \eta)| \lesssim |Z^{in}(k, \eta)| + \frac{1}{\gamma} |\hat{\beta}^{in}(k, \eta) + \hat{\Gamma}^{in}(k, \eta)|.$$ 

(2.19)

Now, we begin to give the details of the estimates (1.13) – (1.12). We first deduce from (2.9), (2.10) that

$$\Omega(t, X, Y) = \frac{\beta^{in}(X, Y) + \Gamma^{in}(X, Y)}{\gamma} - \delta,$$

which implies

$$|\hat{\Omega}(t, k, \eta)| \leq |\hat{\delta}(t, k, \eta)| + \frac{1}{\gamma} |\hat{\beta}^{in} + \hat{\Gamma}^{in}|(k, \eta).$$

(2.20)
As a result, it follows from (2.13) that
\[
\|\mathbb{P}[v]^x(t)\|_{L^2}^2 = \|(\partial_y \Delta^{-1} \omega)(t)\|_{L^2}^2 = \sum_k \int \left( \frac{(\eta - kt)^2}{p^2} \right) d\eta
\]
\[
\leq C \sum_k \int \left( \frac{(\eta - kt)^2}{p^3/2} \left| \hat{\delta}(t) \right|^2 + \frac{(\eta - kt)^2}{p^2} \right) d\eta
\]
\[
\leq C \sum_k \int \left( \frac{(\eta - kt)^2}{p^3/2} \left| \vec{z} \right|^2 + \frac{(\eta - kt)^2}{p^2} \right) d\eta.
\]
In view of (2.19), Lemma 2.1 and the definition in (2.8) and (2.15), we further get
\[
\|\mathbb{P}[v]^x(t)\|_{L^2}^2 \leq C \sum_k \int \left( \frac{M^2}{\langle t \rangle} \left| \hat{\delta}(t) \right|^2 + \frac{1}{p^2} \right) d\eta
\]
\[
\leq C \frac{M^2}{\langle t \rangle} \left( \left| \frac{\rho^{in} + \theta^{in}}{\gamma M} \right|^2 + \left| \frac{\alpha^{in}}{\gamma} \right|^2 + \left| \frac{\rho^{in} + \theta^{in} + \gamma \omega^{in}}{\gamma} \right|^2 \right)
\]
\[
+ C \frac{M^2}{\langle t \rangle^3} \left| \frac{\rho^{in} + \theta^{in} + \gamma \omega^{in}}{\gamma} \right|^2 H^{-1} H^2.
\]
In a similar manner, we deal with \(\mathbb{P}[v]^y\) as follows
\[
\|\mathbb{P}[v]^y(t)\|_{L^2}^2 = \|\partial_y \Delta^{-1} \omega\|_{L^2}^2
\]
\[
\leq C \sum_k \int \left( \frac{M^2}{\langle t \rangle^3} \left| \hat{\delta}(t) \right|^2 + \frac{1}{p^2} \right) d\eta
\]
\[
\leq C \frac{M^2}{\langle t \rangle^3} \left( \left| \frac{\rho^{in} + \theta^{in}}{\gamma M} \right|^2 + \left| \frac{\alpha^{in}}{\gamma} \right|^2 + \left| \frac{\rho^{in} + \theta^{in} + \gamma \omega^{in}}{\gamma} \right|^2 \right)
\]
\[
+ C \frac{M^2}{\langle t \rangle^4} \left| \frac{\rho^{in} + \theta^{in} + \gamma \omega^{in}}{\gamma} \right|^2 H^{-1} H^2.
\]
Finally, we have to bound the compressible part of the velocity, the density and the temperature.

On the one hand, we infer from the Helmholtz decomposition that
\[
\|Q[v](t)\|_{L^2}^2 + \frac{1}{M^2} \|\rho(t) + \theta(t)\|_{L^2}^2
\]
\[
= \|\partial_t \Delta^{-1} \alpha(t)\|_{L^2}^2 + \|\partial_y \Delta^{-1} \alpha(t)\|_{L^2}^2 + \frac{1}{M^2} \|\rho(t) + \theta(t)\|_{L^2}^2
\]
\[
= \sum_k \int \left( \frac{|\hat{\alpha}(t)|^2(t, k, \eta)}{k^2 + \eta^2} + \frac{1}{M^2} \left| \hat{\rho}(t) + \hat{\theta}(t) \right|^2(t, k, \eta) \right) d\eta
\]
\[
= \sum_k \int \left( \frac{|\hat{\alpha}(t)|^2(t, k, \eta)}{p} + \frac{1}{M^2} \left| \hat{R}(t) + \hat{\Theta}(t) \right|^2(t, k, \eta) \right) d\eta
\]
\[
= \sum_k \int \left( \frac{|\hat{\alpha}(t)|^2(t, k, \eta)}{p} + \frac{\gamma^2}{M^2} \left| \hat{\delta}(t) \right|^2(t, k, \eta) \right) d\eta.
\]
As a result, we can further deduce from (2.19), Lemma 2.1 and the fact that \( p \leq \langle t \rangle^2 \langle k, \eta \rangle^2 \) that

\[
\| Q[v](t) \|_{L^2}^2 + \frac{1}{M^2} \| \rho(t) + \theta(t) \|_{L^2}^2 \\
= \sum_k \int \sqrt{p} \left( \frac{|\hat{A}(t)|^2}{|p^{3/4}|} + \frac{|\hat{\delta}(t)|^2}{|Mp^{1/4}|} \right) d\eta \\
= \sum_k \int \sqrt{p} |\hat{Z}(t)|^2 d\eta \\
\leq C(t) \left( \| Z^\text{in} \|_{H^{1/2}_x}^2 + \left\| \frac{\rho^\text{in} + \theta^\text{in} + \gamma \omega^\text{in}}{\gamma} \right\|_{H^{1/2}_x}^2 \right) \\
\leq C(t) \left( \| \frac{\rho^\text{in} + \theta^\text{in}}{\gamma M} \|_{L^2}^2 + \alpha^\text{in} \|_{H^{-1}_x}^2 + \left\| \frac{\rho^\text{in} + \theta^\text{in} + \gamma \omega^\text{in}}{\gamma} \right\|_{H^{1/2}_x}^2 \right).
\] (2.23)

On the other hand, by (1.10), there holds

\[
(\partial_t + y\partial_x)((\gamma - 1)\rho - \theta) = 0
\]

which implies that

\[
(\gamma - 1)\rho - \theta = (\gamma - 1)\rho^\text{in} - \theta^\text{in}.
\]

Moreover, we have

\[
\left\| \frac{(\gamma - 1)\rho(t) - \theta(t)}{M} \right\|_{L^2}^2 = \left\| \frac{(\gamma - 1)\rho^\text{in} - \theta^\text{in}}{M} \right\|_{L^2}^2.
\]

Let

\[
\begin{cases}
  x_1 = \frac{(\gamma - 1)\rho - \theta}{M}, \\
  y_1 = \frac{\rho + \theta}{M},
\end{cases}
\]

which gives

\[
\begin{cases}
  \frac{\gamma}{M} \rho = x_1 + y_1, \\
  \frac{\gamma}{M} \theta = (\gamma - 1)x_1 - y_1.
\end{cases}
\]

Hence

\[
\frac{\gamma^2}{M^2} \| \rho(t) \|_{L^2}^2 = \left( \left\| \frac{(\gamma - 1)\rho - \theta}{M} \right\|_{L^2}^2 + \left\| \frac{\rho + \theta}{M} \right\|_{L^2}^2 \right) \\
\leq C \left\| \frac{(\gamma - 1)\rho^\text{in} - \theta^\text{in}}{M} \right\|_{L^2}^2 \\
+ C(t) \left( \left\| \frac{\rho^\text{in} + \theta^\text{in}}{\gamma M} \right\|_{L^2}^2 + \alpha^\text{in} \|_{H^{-1}_x}^2 + \left\| \frac{\rho^\text{in} + \theta^\text{in} + \gamma \omega^\text{in}}{\gamma} \right\|_{H^{1/2}_x}^2 \right),
\]
and
\[
\frac{\gamma^2}{M^2} \| \theta(t) \|_{L^2}^2 = (\gamma - 1)^2 \left( \frac{(\gamma - 1) \rho - \theta}{M} \right)^2 + \left( \frac{\rho + \theta}{M} \right)^2 \\
\leq C(\gamma - 1)^2 \left( \frac{(\gamma - 1) \rho^{in} - \theta^{in}}{M} \right)^2 \\
+ C \langle t \rangle \left( \left( \frac{\rho^{in} + \theta^{in}}{\gamma M} \right)^2 + \| \alpha^{in} \|^2_{H^{-1}} + \left( \frac{\rho^{in} + \theta^{in} + \gamma \omega^{in}}{\gamma} \right)^2 \right).
\]

Consequently, we have completed the proof of the Theorem 1.1.

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