Little Groups and Statistics of Branes

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Abstract

The little groups (i.e. the subgroups of Lorentz group, leaving invariant given configurations of tensorial charges) of unitary irreps of superstring/M-theory superalgebras are considered. It is noted, that in the case of \((n - 1)/n\) (maximal supersymmetric) BPS configuration in any dimensions the non-zero supercharge is neutral w.r.t. the algebra of little group, which means that all members of supermultiplet are in the same representation of that algebra and hence of (generalized with tensorial charges) Poincare algebra. This situation is similar to two-dimensional case and shows that usual spin-statistics connection statement is insufficient in the presence of branes, because different little groups can appear. We discuss the rules for definition of statistics for representations of generalized Poincare, and note that a geometric quantization method seems to be most relevant for that purpose.

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1 Introduction

Many features of modern superstring theories can be deduced directly from considerations of their supersymmetry algebras \([1]\). Most general among them is the M-theory \(N = 1, d = 11\) superalgebra

\[
\{Q, \bar{Q}\} = \gamma^\mu P_\mu + \gamma^{\mu\nu} Z_{\mu\nu} + \gamma^{\mu
u\lambda\rho\sigma} Z_{\mu\nu\lambda\rho\sigma},
\]

(1)

\(\mu, \nu, ... = 0, 1, 2, ..10\).

(plus relations, including Lorentz generators) where \(Q\) is a Majorana spinor. We shall consider also the simpler analog in \(d = 4\), the \(N = 1\) supersymmetry algebra

\[
\{Q, \bar{Q}\} = \gamma^\mu P_\mu + \frac{1}{2} \gamma^{\mu\nu} Z_{\mu\nu},
\]

(2)

\(\mu, \nu, ... = 0, 1, 2, 3\).

with Majorana \(Q\).

Let’s construct one of the simplest (“particle”) representation of (2), i.e. the representation for which the vector \(P_\mu\) is non-zero, and all tensorial charges are zero. At first steps of construction of (unitary) irreps for such an algebras one have to fix the values of all Casimirs, constructed from \(P_\mu, Z_{\mu\nu}, ...\), and take the particular point on that orbit. Let’s take the point \(P_\mu = (m, 0, 0, ...), Z = 0\). The stabilizer (i.e. little group) of this point on the orbit is \(SO(3)\). The algebra (2) becomes, in a two-component notations (of e.g. \([2]\)):

\[
\begin{align*}
\{Q_A, \bar{Q}_{\bar{B}}\} &= m \delta_{AB} \\
\{Q_A, Q_B\} &= 0 \\
\{\bar{Q}_A, \bar{Q}_{\bar{B}}\} &= 0
\end{align*}
\]

(3)

where \(\bar{Q}_A\) is a Hermitian conjugate to \(Q_A\), so one of them can be considered as creation, and second one as annihilation operators. The representation of an algebra (3) can be constructed by taking a “vacuum” \(|s\rangle\) in any unitary representation of \(SO(3)\) with spin \(s\), which is annihilated by operators \(Q\), then applying the creation operators \(\bar{Q}\) as many times as possible, and finally inducing the representation to the whole super-Poincare group. So the whole supermultiplet before induction will be a collection of few irreducible
representations of SO(3), transforming one into another under an action of $Q, \bar{Q}$. More exactly, $Q, \bar{Q}$ will transform states with integer spins into those with half-integer and vice-versa, because $Q, \bar{Q}$ itself have spin one-half. This is in agreement with spin-statistics connection at $d = 4$, because $Q, \bar{Q}$ are fermionic and flip the statistics.

Algebra (3) is an algebra of 2 pairs of fermionic creation-annihilation operators, so the number of states in the supermultiplet described is maximal: $2^2$, composed of 2 fermions plus 2 bosons.

The number of states becomes less in the so called shortened (BPS) supermultiplets, playing an important role in these theories. Their existence is based on a specific features of (1), (2) and similar superalgebras. For our model example (2) such a multiplet appears e.g. for $P_\mu = (1, 0, 0, 1), Z_{\mu\nu} = 0$, which corresponds to massless particle. Then (2) becomes

$$\begin{align*}
\{Q_1, \bar{Q}_1\} &= 2 \\
\{Q_2, \bar{Q}_2\} &= 0 \\
\{Q_A, Q_B\} &= 0 \\
\{\bar{Q}_A, \bar{Q}_B\} &= 0
\end{align*}$$

(4)

The little group for this massless particle case is a semidirect product of $SO(2)$ on a group of two-dimensional translation, i.e. it is a two-dimensional Euclidean Poincare. Supercharge $Q_1$ has spin (helicity) $1/2$ w.r.t. the $SO(2)$ (which acts as multiplication by a phase factor on $Q_1$), so it is transforming half-integer helicity states into integer ones and vise-versa. Due to relations (4) the half of components of operator $\bar{Q}$ are represented by zero, so there is only one creation operator, the number of states is $1 + 1$, half of them fermionic, with half-integer helicity and another half bosonic, with integer helicity.

Another BPS multiplet appears for BPS membrane, i.e. the charges configuration $P_\mu = (m, 0, 0, 0), Z_{12} = -Z_{21} = m$, other components of $Z_{\mu\nu}$ zero. Situation is similar to that of massless particle. The little group evidently is an $SO(2)$ group of rotations around $z$ axis. Non-zero $Q$ have a spin (helicity) $\pm 1/2$ w.r.t. this little group, so states connected by $Q$ have helicities, differing by $1/2$. These facts can be deduced from the main relation (3), which in this case acquire a form (in two-component notations, only non-zero relations are written):
The little group of rotation around $z$ axis, with generator $\sigma^{12}$ (in notations of [2]), acts on a $Q$ spinors as multiplication on a phase factor for $Q_1$ and opposite phase factor multiplication for $Q_2$. Let’s introduce a combinations

$$a = Q_2 + i\bar{Q}_1$$
$$\bar{a} = Q_2 - iQ_1$$
$$b = Q_2 - i\bar{Q}_1$$
$$\bar{b} = Q_2 + iQ_1$$

It is easy to check using (5) that operators $b, \bar{b}$ anticommutate with all $Q$ operators, and hence will be represented by zero matrixes. Operators $a, \bar{a}$ are usual fermionic creation-annihilation operators. Dimensionality of representation is 1+1. According to abovementioned properties of $Q$ under little group rotations, we see that operators $a, \bar{a}$ have a helicity $\pm 1/2$.

Taking different ”vacuums” $|s>$ with spin $s$ we shall obtain an irreps of susy algebra (6), which can be called ”BPS membranes with spins”. (One may ask for a Lagrangians for that branes. Recall that in similar situation for particles with spins one has to introduce an internal degrees of freedom, e.g. internal sphere $S^2$. One can expect something similar here, although we are not aware of any such considerations. Particularly, we don’t know classical solutions for supergravities, which can be identified with ”branes with spin”.)

The problem can be formulated at this stage. The usual spin-statistics theorem is derived for usual $d = 4$ Poincare algebra, and claims that states with integer spins have a Bose statistics, and states with half-integer spins - Fermi one. In higher dimensions spin (or helicity)is substituted by some representation of little groups, which are of $SO$ type (more exactly, $SO$ groups are their compact part), and spin-statistics statement is that spin-tensor representations are fermionic and purely tensorial representations - of bosonic type. This can be deduced, particularly, by dimensional reduction, assuming that statistics is not being changed by that process. This is enough to
describe spin-statistics connection for usual Poincare (super)algebras. The problem is that in the presence of brane charges the little groups, and representations of $Q$ with respect to these little groups can be different. In that case one have to generalize the statement of spin-statistics connection. So, the problem is: for all brane superalgebras (1), (2), etc., find out, for all (physical) orbits, the corresponding little groups and representations of $Q$ with respect to this little groups. Then one has to assign Fermi or Bose or both statistics to each of these representations. Actually situation certainly will be different in comparison with standard at $d=4$, in that it is possible, that the same representation of Poincare algebra can have both types of statistics, as we shall see below. This resembles the $d=2$ situation, when there is no spin, and the same Poincare algebra’s representations can have both statistics. The assignment of Fermi-Bose statistics has to be in agreement with the fermionic nature of $Q$, i.e. the fact that it is changing statistics. It is not clear now what kinds of little groups are possible in different dimensions.

2 \ N=1, \ d=4, \ 3/4 \ BPS \ Supermultiplet

Now we shall present a promised example at $d=4$, in which usual spin-statistics connection is not applicable. That is the BPS representation, maintaining maximal $(n-1)/n$ supersymmetry, first described at $d=4$ in [3], and called preons in [4]. Let’s take $P_\mu$ and $Z_{\mu\nu}$ such that $(\Gamma^i P_i + \Gamma^{ij} Z_{ij}) \alpha \beta = \lambda_\alpha \lambda_\beta$. Then (2) becomes

\[
\{Q_\alpha, Q_\beta\} = \lambda_\alpha \lambda_\beta \\
\alpha, \beta, ... = 1, 2, 3, 4.
\]

This configuration satisfies positivity restriction, i.e. the eigenvalues of r.h.s. matrix are non-negative. For this orbit the algebra of little group is $T^2$, i.e. two-dimensional Abelian subalgebra of Lorentz algebra. We are interested what are the representations of non-zero $Q$-s w.r.t this algebra. Assuming that only first component of $\lambda_\alpha$ is non-zero we can construct the minimal representation of (2), by representing all component of $Q$ except first one by zero, and first one by two by two matrix. Then it is evident that Lorentz generators which leave $\lambda_\alpha$ invariant, will leave invariant $Q_\alpha$, also.
Explicitly, in two-component notations, these statements look as follows. Anticommutators are
\[
\begin{align*}
\{Q_A, \bar{Q}_B\} &= \lambda_A \lambda_{\bar{B}} \\
\{Q_A, Q_B\} &= \lambda_A \lambda_B \\
\{\bar{Q}_A, \bar{Q}_B\} &= \bar{\lambda}_A \bar{\lambda}_{\bar{B}}
\end{align*}
\] (8)

Take for definiteness \(\lambda_A = (1, 0)\), then little algebra has two non-zero elements. Let \(\omega_{\mu\nu} M_{\mu\nu}\) be the general element of Lorentz algebra, where \(M_{\mu\nu}\) are generators. Then one element of little group has non-zero parameters \(\omega_{01} = \omega_{13} = -\omega_{10} = -\omega_{31}\) only, another one \(\omega_{02} = \omega_{23} = -\omega_{20} = -\omega_{312}\) only. These two elements are commuting with each other. So, little algebra is a two-dimensional Abelian algebra, which acts trivially on \(\lambda_A = (1, 0)\).

Algebra (8) becomes (non-zero anticommutators only):
\[
\begin{align*}
\{Q_1, \bar{Q}_1\} &= 1 \\
\{Q_1, Q_1\} &= 1 \\
\{\bar{Q}_1, \bar{Q}_1\} &= 1
\end{align*}
\] (9)

Introducing combinations \(b = (Q_1 + \bar{Q}_1)/\sqrt{2}\) and \(b_1 = (Q_1 - \bar{Q}_1)/\sqrt{2}\), we see that \(b_1\) commute with all \(Q\)-s and should be represented by zero matrix. Hermitian operator \(b\) satisfies a relation
\[
\{b, b\} = 2
\] (10)

This algebra can be represented in two-dimensional space by taking \(b\) as two by two antidiagonal matrix with non-zero elements equal to 1. The dimensionality of representation is \(2 = 1 + 1\). The difference with above representations for e.g. massless particle, is in that in this case we can choose representation real, with real dimensionality \(1 + 1\). This is possible because operator \(b\) is Hermitian and doesn’t change phase under little group transformation, as is the case for operators \(a, \bar{a}\) for massless particle. So, as expected, in this case with 3 supersymmetries surviving [3], the supermultiplet becomes even shorter than for massless particle or membrane.

Thus, for the orbit considered, the non-zero components of supercharge are neutral w.r.t. the corresponding little algebras. In other words, for any representation \(|s\rangle\) the \(b|s\rangle\) is the same representation of the little algebra. So, \(b\) is not changing any ”spin”, but is changing the statistics, due to its fermionic nature. Evidently, this situation is similar to two dimensional one for usual Poincare algebra: both fermions and bosons are realizing the same representation.
3 Conclusion: Higher Dimensions, Geometric Quantization

These considerations evidently are applicable for all dimensions, provided r.h.s. of anticommutator of supercharges is of rank one, i.e. equal to $\lambda_\alpha \lambda_\beta$, which means that the maximum number of supersymmetries is maintained. In all that cases non-zero supercharges are neutral w.r.t. the algebra of little group, and hence members of these supermultiplets are in the same representation of generalized Poincare algebra. For other configurations of branes situation can be more complicated, as discussed above. Particularly, the list of possible little groups has to be derived. The rules for defining the statistics of different states (unitary irreps of these little groups) can be the following. First, one can try to define them directly. The most relevant seems to be the derivation [3] of spin-statistic relation in the framework of geometric quantization method, which doesn’t require features, absent in our case, such as brane’s quantum field theory. Instead, a consideration of Hamiltonian actions [4] of bosonic subalgebra of super Poincare groups (1), (2), etc, is required. Second, one can use reasonable assumption that statistics is not changing under dimensional reduction. Of course, for the cases with usual little groups, i.e. those for non-zero $P_\mu$ only, we can use usual spin-statistics relation. Combination of these approaches should permit one to find all possible spin-statistics relations for unitary irreps of generalized Poincare algebras.

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