Independent Approximates
enable closed-form parameter estimation
of heavy-tailed distributions

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Abstract – A new statistical method, Independent Approximates (IAs), is defined and proven to enable closed-form estimation of the parameters of heavy-tailed distributions. Given independent identically-distributed samples from a one-dimensional distribution, IAs are formed by partitioning samples into pairs, triplets, or nth-order groupings and retaining the median of those groupings that are approximately equal. The pdf of the IAs is proven to be the normalized nth-power of the original density. From this property, heavy-tailed distributions are proven to have well-defined means for their IA pairs, finite second moments for their IA triplets and a finite, well-defined (n-1)th-moment for the nth-grouping. Estimation of the location, scale, and shape (inverse of degree of freedom) of the generalized Pareto and Student’s t distributions are possible via a system of three equations. Performance analysis of the IA estimation methodology is conducted for the Student’s t distribution using between 1000 to 100,000 samples. Closed-form estimates of the location and scale are determined from the mean of the IA pairs and the variance of the IA triplets, respectively. For the Student’s t distribution, the geometric mean of the original samples provides a third equation to determine the shape, though its nonlinear solution requires an iterative solver. With 10,000 samples the relative bias of the parameter estimates is less than 0.01 and the relative precision is less than ±0.1. The theoretical precision is finite for a limited range of the shape but can be extended by using higher-order groupings for a given moment.

Keywords: complex adaptive systems, heavy-tailed distributions, statistical estimation

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Statements and Declarations

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Availability of code, data and material:
https://github.com/Photrek/Independent-Approximates

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1. Introduction

Statistical estimation of the parameters of a heavy-tailed distribution is an infamously difficult problem. As the rate of tail decay decreases, i.e. becomes heavier, the probability of outlier samples, which overwhelm statistical analysis, increases. As a result, the statistical moments either diverge to infinity or become undefined with the higher-order moments becoming unstable first. So for instance, the Student’s t-distribution, whose asymptotic probability density function (pdf) is \( f \sim |x|^{\nu-1} \) where \( \nu \) is the degree of freedom, has an undefined mean when \( \nu \leq 1 \) (the Cauchy distribution is \( \nu = 1 \)), a variance which diverges when \( \nu \leq 2 \), and a skewness (third moment) which is undefined for \( \nu \leq 3 \). Each successive moment requires higher degrees of freedom. Fig. 1 illustrates the contrast in the dispersion of samples for a standard Normal and a standard Cauchy random variable.

Nevertheless, the study of complex adaptive systems has shown that the nonlinear dynamics of both natural and man-made systems make heavy-tailed distributions ubiquitous and thus essential for accurate modeling of important signals and systems (Clauset et al. 2009). A few examples include the distribution of ecological systems (generalized Pareto) (Katz et al. 2005), the foraging patterns on birds (Levy) (Viswanathan et al. 1996), log-return of markets (Student’s t) (Pisarenko and Sornette 2006; Vilela et al. 2019), and the distribution of words in a language (Zipf’s Law) (Piantadosi 2014). Comprehensive surveys of statistical analysis of heavy-tailed distributions include Resnick (Resnick 2007), Kotz and Nadarajah (Kotz and Nadarajah 2004) and of modeling for complex systems include Aschwanden (Aschwanden 2011), Tsallis (Tsallis 2009), and Cirillo (Cirillo 2012). Despite the extensive investigation of heavy-tailed distributions and the importance of modeling the stochastic properties of nonlinear systems, estimation using closed-form solutions as opposed to iterative optimization has been limited.

This paper introduces a new statistical method that enables a closed-form approach to the estimation of heavy-tailed distributions. Given independent identically-distributed samples from a one-dimensional distributions, the Independent Approximates (IAs) are formed by selecting pairs, triplets, or higher-order groups which are approximately equal. Within each approximately equal grouping the median is retained as a subsample. The process of selecting

![Fig. 1 Comparison of the dispersion of samples between a) the Gaussian distribution which has a finite variance with no fluctuation and b) the Cauchy distribution which has infinite variance and is defined by a scale with fluctuations. For the Gaussian, samples are exceedingly rare beyond 5 times the standard deviation, while for the Cauchy, samples are still dense at 10 times the scale.](image-url)
IA pairs and triplets is illustrated in Fig. 2 for a Cauchy distribution. The groupings form a n-dimensional distribution. Evident from the figure is the fact the subsamples along the diagonal have a significantly faster rate of decay in the tails then original samples along each axes. In Section 3.2 a proof is provided that marginal probability density function (pdf) along the diagonal is the normalized $n^{th}$-power of the original pdf $f(x)$.

**Definition 1:** The $n^{th}$-power-density is

$$f_x^{(n)}(x) = f_x^n(x) = \frac{\int_{x \in X} f_x^n(x) \, dx}{\int_{x \in X} f_x(x) \, dx} \quad (1.1)$$

where $f$ is the pdf of the random variable $X$ and $n$ is the number of equal and independent samples.

The paper will show that pairs of IAs insures that the first moment of heavy-tailed distributions is defined, triplets insure that the second moment is defined, and that higher-order IAs ensure the definition of the moment one less than the partition size of the IAs. And importantly, there is a functional mapping between the moments of the IAs and the parameters of the original distribution. Limitations in the estimators are examined both theoretically and empirically. Three limitations are examined 1) while the moments are defined, the domain of finite variance of the estimators is limited, 2) a tolerance for accepting approximates must balance filtering unequal samples with choosing an adequate subsample set for statistical analysis, and 3) the need to iteratively perturb the grouping of independent samples to ensure an adequate subsample set constrains the speed of the algorithm.
The approach is grounded in the properties of the generalized Pareto distribution and the Student’s $t$-distribution (Von Mises 1981; Karian and Tanis 1999), but is comprehensive enough to support definitions and estimation for other important heavy-tailed distributions such as the alpha-stable distributions, the log-normal, and extreme value distributions. The focus will be on estimating the location, scale, and shape of these heavy-tailed distributions though the method can also be applied to estimating the skew and other properties.

The importance of raising heavy-tailed pdfs to a power has been carefully studied in the context of the escort density $\frac{f^n(x)}{\int f^n(x)}$ of nonextensive statistical mechanics (Tsallis 2009) and information geometry (Amari et al. 2012). While $q$-statistics emphasizes using the real number values of $q$ to define generalized moments for matching $q$-distributions, this paper will show that the integer values of $q$ are sufficient to enable the definition of statistical properties and leads to a closed-form estimator which does not require a priori knowledge of or iterative search for the rate of decay of the tail. Furthermore, since $q$ is a composite of three properties (Nelson et al. 2017; Nelson et al. 2019) (the power and dimension of the random variable, and the tail-shape of the distribution) attempts to model complex systems with $q$-statistics obscure the central role of the shape of heavy-tailed distributions. The importance of the shape parameter for the modeling of complex systems has been furthered by derivation of its connection to the amplitude of multiplicative noise and fluctuations of scale. For this reason, the parameter can also be referred to as the nonlinear statistical coupling (Nelson et al. 2017).

In Section 2 established methods for estimating heavy-tailed distributions are reviewed. In Section 3 a formal definition of Independent Approximates is established along with their relationship to the powers of probability density functions. In Section 4 theoretical derivations of the power distributions for the generalized Pareto and Student’s $t$ are completed. Section 5 documents the computational estimation of the Student’s $t$-distribution using the IA algorithm. Section 6 gives a concluding discussion regarding the significance of the methodology and future directions for the research.

2. Review of heavy-tailed estimation method

2.1. Families of heavy-tailed distributions

The Pareto distributions provide a good foundation for specifying heavy-tailed distributions but do not fully encompass the variety of distributions with slower than exponential tail decay. Two other important classes are the Student’s $t$, which generalizes the Gaussian distribution, and the Levy stable distributions, which is defined from the Fourier transform of stretched exponential functions. The scale-free or purely power-law distributions can be understood to be the region of heavy-tail distribution in which the variable $x$ is much greater than the scale of the distribution or equivalently the scale of the distribution approaches zero.

This paper will focus on the estimation of the location, scale, and shape of a heavy-tailed distribution. Distributions that include a skew parameter will be deferred for future research. So, for the generalized Pareto, the focus will be the Type II distributions, whose survival function (sf) is
where $F$ is the cumulative distribution function (CDF), $\mu \in (-\infty, \infty)$ is the location parameter, $\sigma > 0$ is the scale parameter, and $0 \leq \kappa \leq \infty$ is the tail-index or shape. A compact-support domain $-1 \leq \kappa < 0$ with faster than exponential decay can also be defined but is not the focus of this paper. The exponential case is $\kappa = 0$. For the heavy-tailed domain, the generalized Pareto pdf is

$$f(x) = \frac{1}{\sigma} \left( 1 + \frac{x - \mu}{\sigma} \right)^{-\frac{1}{\kappa}}, \kappa > 0.$$  \hspace{1cm} (2.2)

In the limit where either $x >> \sigma$ or $\sigma \to 0$ the Type II converges to the purely power-law Type I Pareto

$$1 - F(x) = \left( \frac{x}{\mu} \right)^{-\frac{2}{\kappa}}, x > \mu \hspace{1cm} (2.3)$$

where $\mu$ rather than the traditional $\sigma$ is used to specify the Type I Pareto since the scale has effectively gone to zero, i.e. the distribution is scale-free, and the $\mu$ specifies the location. For the remainder of the paper, the generalized (Type II) Pareto distribution refers to the Pareto distribution.

Likewise, with the Student’s $t$ an adjustment is made in the traditional specification of the distribution. In earlier research, the author showed that the shape parameter of heavy-tailed distributions is equal to a source of nonlinearity (Nelson and Umarov 2010; Nelson et al. 2017), e.g. the magnitude of fluctuations in superstatistics and the multiplicative noise in certain stochastic processes (Sornette 1998). For this reason, the shape is also referred to as the nonlinear statistical coupling or simply the coupling; and a family of distributions referred to as the Coupled Exponential Family (Nelson 2015) has been defined. In this paper, the distributions (Pareto and Student’s $t$) will be referred to by their traditional names, but the insights regarding the importance of the shape parameter will be emphasized. Thus, the Student’s $t$ will be defined using the shape parameter which is the reciprocal of the degree of freedom ($\nu = 1/\kappa$). The sf is

$$1 - F(x) = -\frac{1}{2} \frac{x \sqrt{\kappa}}{\sigma B \left( \frac{1}{2}, \frac{1}{2\kappa} \right)} F_1 \left( \frac{1}{2}, \frac{1}{2\kappa}; \frac{3}{2}; -\kappa \left( \frac{x - \mu}{\sigma} \right)^2 \right), \sigma \geq 0, \kappa > 0, \hspace{1cm} (2.4)$$

where $F_1$ is the hypergeometric function and the pdf is

$$f(x; \mu, \sigma, \kappa) = \frac{\sqrt{\kappa}}{\sigma B \left( \frac{1}{2\kappa}, \frac{1}{2} \right)} \left( 1 + \kappa \left( \frac{x - \mu}{\sigma} \right)^2 \right)^{-\frac{1}{2} \frac{1}{\kappa} - 1}, \sigma \geq 0, \kappa > 0. \hspace{1cm} (2.5)$$
Not all heavy-tailed distributions have an analytical expression for the pdf. An important example is the Lévy-Stable distribution which is instead defined by its characteristic function, a generalized Gaussian
\[
\phi_x(t) = \exp \left( it\mu - |\sigma t|^\alpha \right), \quad 0 < \alpha \leq 2
\]  
(2.6)

where \( \mu \) is the location, \( \sigma \) is the scale, \( \alpha \) is the stability parameter that affects the shape, and the skew parameter has been set to zero. The Fourier transform of (2.6) defines the pdf
\[
f_x(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(t) e^{-itx} dt.
\]  
(2.7)

The asymptotic tail of the Levy-stable distribution is \( |x|^{-(\alpha+1)} \), thus the shape is \( \kappa = \frac{1}{\alpha} \) and has a domain of \( \kappa \geq \frac{1}{\alpha} \), i.e. the domain of infinite variance. While it is anticipated that IAs could be used for estimation of the Lévy-Stable distributions, the analysis requires consideration of power-convolutions and is deferred for future research.

2.2. Estimation of heavy-tailed distributions

Much of the scientific literature related to complex systems and their underlying heavy-tailed distributions has focused on non-exponential tail decay while treating the main body of the distribution as a separate function. As such, many of the existing estimation methods focus exclusively on estimates of the power-law or Pareto type I tail. Fedotenkov (Fedotenkov 2018) provides an extensive review of estimation methods for Pareto or power-law tail decay. Many of the estimators reviewed by Fedotenkov are refinements of the maximum likelihood Hill estimator (Hill 1975) which compares the logarithm of the highest order statistics \( X_{(n)} \)...

\[
\hat{\kappa}_n^H(k) = \frac{1}{k} \sum_{i=0}^{k-1} \log \left( \frac{X(n-i)}{X(n-k)} \right).
\]  
(2.8)

A key component of the Hill and related estimators is the need to choose \( k \) such that it is a) sufficiently greater than the location plus the scale \( \mu + \sigma \) so that remaining order statistics are approximately power-law, i.e. linear on a log-log scale, and b) sufficiently smaller than \( n \) to provide an adequate number of samples. The Hill estimate is based on measuring the average distance on the log scale between the \( n-i \) order statistic and the \( n-k \) order statistic provides an estimate of the tail-index. Advanced variations (Stoev et al. 2011) expand upon this relationship to improve the regression analysis of the log-log relationship of the tail decay. A recent example that iteratively optimizes both the cut-off and the estimate is reported by Hanel, et al. (Hanel et al. 2017).

The nonextensive statistical mechanics community has focused on a family of distributions \( (q\text{-statistics}) \) that include the generalized Pareto and Student’s \( t \) distributions. Through a generalization of the logarithm and exponential functions that accounts for the non-exponential tail-decay estimator, methods have been developed that maximize the generalized log-likelihood (Shalizi 2007; Ferrari and Yang 2010; Nielsen 2013; Qin and Priebe 2013; Gayen and Kumar 2018). The generalized likelihood method weights the gradient of the log-likelihood by the power of the pdf (Ferrari and Yang 2010)
\[
\sum_{i=1}^{N} f_x^{-\phi} \nabla \log f_x(x, \theta) = 0,
\]  
(2.9)
where θ is the location, scale, and possibly other non-shape parameters of the distribution. The shape parameter κ is related to the q-statistics parameter by the expression \( q = 1 + \frac{\alpha \kappa}{1 + \kappa} \), where \( \alpha = 1 \) for the Pareto distributions and \( \alpha = 2 \) for the Student’s t-distribution. The generalized log-likelihood methods depend on a separate estimate of the tail such as the Hill-like estimators.

In prior research, the author with Umarov and Kon (Nelson et al. 2019) showed that given the location of a Student’s t-distribution a functional relationship exists between the scale and shape parameters using the log-average or geometric mean of the samples

\[
\hat{\sigma} = 2\sqrt{\kappa} \exp\left(\frac{1}{\kappa} H^{-1}\left[\frac{1}{\kappa} \right] \right) \prod_{i=1}^{N} \left| X_i - \mu \right|^{1/N},
\]

where \( H_n \) is the \( n^{th} \) harmonic number. Given the new estimators for the location and scale defined in the next section, this geometric mean estimator will be used to estimate the shape of the Student’s t. In the context of IAs, this estimator can be regarded as the 0th moment of the power distribution (i.e. no subsampling is necessary) and will be used to estimate the shape given estimates of the location and scale.

3. Independent Approximates

3.1. Significance of Power-Distributions

Heavy-tailed distributions have a limited number of moments, and potentially no moments, which are both defined and finite. The moments \( \mu_m \) are defined by the derivative at \( t=0 \) of the characteristic function \( \phi \) and equivalently the moment-generating function \( M \)

\[
\mu_m = E[X^m] = M_X^{(m)}(0) = i^m \phi_X^{(m)}(0) = i^m \frac{d^m E[e^{itX}]}{dt^m} \bigg|_{t=0}
\]

(3.1)

Given a heavy-tailed distribution with an asymptotic power between \( 1 < 1 + \nu < \infty \) or equivalently using the shape parameter \( 1 + \frac{1}{\kappa} < \infty \), raising the distribution to a power greater than 1 increases the rate of decay of the distribution thereby increasing the number of moments that are defined and finite. So, for instance, the standard Cauchy density \( \pi^{-1} (1 + x^2)^{-1} \) has a characteristic function \( e^{\frac{1}{|\kappa|}} \) whose derivatives at \( t=0 \) are discontinuous; thus, the moments are not defined. Raising the Cauchy density to the \( n^{th} \) power \( \pi^{-n} (1 + x^2)^{-n} \) results in a Student’s t density upon normalization. The shape of this power-density (1.1) is \( \kappa = \frac{1}{n-1} \) which is less than 1 for \( n > 2 \), and would then have a defined first moment, and is less than 0.5 for \( n > 3 \), and would then have a finite second moment.

The even moments of the Student’s t distribution expressed in terms of the shape \( \kappa \) are equal to \( E[X^m] = \left( \frac{1}{\kappa} \right)^m \prod_{i=1}^{m/2} \frac{2i-1}{1-2i\kappa}, \) \( m < \frac{1}{\kappa} \). With a shape of \( \kappa = \frac{1}{n-1} \) for the \( n^{th} \) power-density of the Cauchy, the moments less than \( n-1 \) now exist. From the perspective of the characteristic function this is because each convolution smooths the function at \( t=0 \) making successive derivatives continuous and thus defined.
3.2. Selecting Independent Approximate Subsamples

By itself raising a density to a power does not lead to a closed-form estimator of the distribution because the empirical distribution would first have to be estimated prior to applying the power. This circularity requires iterative methods for estimation. Nevertheless, a closed-form estimation can be achieved by noticing that given a joint distribution with $n$ independent dimensions, the marginal distribution along the diagonal with equal values has the required $n^\text{th}$-power-density.

**Lemma 1:** Given an $n$-dimensional random variable $X$ with identical and independent marginal densities equal to $f_{X_i}(x_i)$, the marginal density along the diagonal of $n$ equal values, has a density equal to $f^{(n)}(x) = \frac{\prod_{i=1}^{n} f_{X_i}(x)}{\int_{x \in X} f^{n}(x)}$.

Proof: The marginal distribution along the equally-valued diagonal is a normalization of the joint distribution with all $x_i = x$

$$f_X(x)_{x_i=x} = \prod_{i=1}^{n} f_{X_i}(x_i)_{x_i=x} = f^{n}_X(x). \quad (3.2)$$

Renormalizing the result to form the marginal density completes the proof. □

**Definition 2** (Independent-Equals) Random samples drawn from the marginal distribution of an $n$-dimensional random variable $X$ with independent dimensions are defined to be Independent-Equals and are symbolically represented as $X^{(n)} \sim f^{(n)}(x)$.

**Definition 3** (Independent-Approximates) Random samples drawn from a neighborhood of tolerance $\varepsilon$ of the independent-equals marginal distribution are defined as Independent-Approximates (IAs). The IAs samples are also represented as $X^{(n)} \sim f^{(n)}(x)$ with the distinction between approximate and equal being described in the context of either theoretical computations assuming equality or experimental computations implementing approximate methods. The neighborhood for selection of IAs is defined as $|x_i^{(n)} - x_j^{(n)}| \leq \varepsilon$ where $x_i$ are samples from the $n$ independent dimensions or equivalently partitions with $n$ independent samples.

Thus, estimators for the $n^\text{th}$-power of a density can be formed by partitioning $N$ independent identically distributed samples into subsets of length $n$ and then selecting those subsets which are approximately equal. The median of those subsets forms a set of Independent Approximates (IA) whose size will be labeled $N^{(n)}$. Furthermore, for even moments the required equality is for the absolute value of $x$, expanding the diagonals which can be utilized for the estimator.
Corollary 1: Given \( n \) independent identically distributed random variables with density \( f \), the marginal distribution of equal absolute value has a density equal to 

\[
\frac{f^{(n)}(|x|)}{\int_{-\infty}^{\infty} f^{(n)}(|x|) \, dx}. 
\]

The proof has the same structure of Theorem 1 with \( x \) replaced by \( |x| \).

Fig. 2 illustrates the selection of pairs and triplets of Independent Approximates (IA) for a standard Cauchy random variable. A tolerance of \( \pm 0.1 \) has a reasonable balance between filtering outliers and selecting approximates to the diagonal. As will be described in detail in the next section the pairs are selected along the diagonal of equal value and are used to measure the first moment. The triplets are selected along all the diagonals since their use for estimating the second moment is insensitive to the sign of numbers. The selected set of IAs becomes one sample, the mean of the pairs and the median of the triplets. In the next section, power moments are derived for the generalized Pareto and Student’s \( t \) distributions.

4. Power-density moments

The power-density moments or power-moments provide a mapping between estimates that can be measured given the reduction in the shape of the distribution and the original distribution.

Definition 4: (Power-moment) The \( m^{th} \) moment of the \( n^{th} \) power of density \( f_x(x) \) is defined as

\[
\mu_m^{(n)} = \frac{\int_{x \in X} x^m f_x^n(x) \, dx}{\int_{x \in X} f_x^n(x) \, dx}. \tag{4.1}
\]

4.1. The generalized Pareto distribution

Lemma 2: Given a one-sided (two-sided) generalized Pareto distribution with density

\[
f_x(x) = \begin{cases} 
\frac{1}{\sigma} \left(1 + \kappa \frac{x - \mu}{\sigma}\right)^{\frac{1}{\kappa+1}} ; & x \geq \mu \\
\frac{1}{\sigma} \left(1 + \kappa \left|\frac{x - \mu}{\sigma}\right|\right)^{\frac{1}{\kappa+1}} ; & -\infty \leq x \leq \infty 
\end{cases} \tag{4.2}
\]

The \( n^{th} \) moment exists and is finite for all shapes \( (\kappa \geq 0) \) if the distribution is raised to the \( (n + 1) \)-power and renormalized. The functional relationship between moments for the \( n^{th} \) power of the generalized Pareto distribution and the location, scale, and shape parameters are specified in Table 1 (Table 2).
Table 1: The $n^{th}$ moments for the (n+1)-power-density of the one-sided generalized Pareto distribution.

| Moment, Centered (Scaled) | One-sided Pareto Type II | Centered (Scaled) |
|--------------------------|-------------------------|-------------------|
| $\mu_1^{[2]}$, $x-\mu$  | $\mu + \frac{\sigma}{2}$ | $\frac{\sigma}{2}$ |
| $\mu_2^{[3]}$, $x-\mu$  | $\mu^2 + \frac{2\mu\sigma}{3+\kappa} + \frac{2\sigma^2}{3+\kappa}$ | $\frac{2\sigma^2}{3(3+\kappa)}$ |
| $\mu_3^{[4]}$, $x-\mu$  | $\mu^3 + \frac{3\mu^2\sigma}{4+2\kappa} + \frac{12\mu\sigma^2 + 3\sigma^3}{2(4+\kappa)(4+2\kappa)}$ | $\frac{3\sigma^3}{2(4+\kappa)(4+2\kappa)}$ |
| $\mu_4^{[5]}$, $x-\mu$  | $\mu^4 + \frac{4\mu^3\sigma}{5+3\kappa} + \frac{12\mu^2\sigma^2}{(5+2\kappa)(5+3\kappa)} + \frac{120\mu\sigma^3 + 24\sigma^4}{5(5+\kappa)(5+2\kappa)(5+3\kappa)}$ | $\frac{24\sigma^4}{(5+\kappa)(5+2\kappa)(5+3\kappa)}$ |
| $\mu_n^{[n+1]}$        | $\sum_{i=0}^{n} \frac{n!\mu^{n-i}\sigma^i}{(n-i)! \left(\frac{1+n\left(n-i\right)\kappa}{\kappa}\right)! \left(\frac{1+n\left(n-i\right)\kappa}{\kappa}\right)! \sigma^n}$ | $\kappa^{-n} \frac{n!\left(1+n\kappa\right)!}{\left(1+n\kappa+n\kappa-n\kappa\right)!} \sigma^n$ |

Table 2: The $n^{th}$ moments for the (n+1)-power-density of the two-sided generalized Pareto distribution.

| Moment, Centered (Scaled) | Two-sided Pareto Type II | Centered (Scaled) |
|--------------------------|------------------------|-------------------|
| $\mu_1^{[2]}$, $x-\mu$  | $\mu$                 | 0                 |
| $\mu_2^{[3]}$, $x-\mu$  | $\mu^2 + \frac{4\sigma^2}{3+\kappa}$ | $\frac{4\sigma^2}{3(3+\kappa)}$ |
| $\mu_3^{[4]}$, $x-\mu$  | $\mu^3 + \frac{12\mu\sigma^2 + 3\sigma^3}{4+2\kappa}$ | 0 |
| $\mu_4^{[5]}$, $x-\mu$  | $\mu^4 + \frac{24\mu^3\sigma^2}{(5+2\kappa)(5+3\kappa)} + \frac{48\sigma^4}{5(5+\kappa)(5+2\kappa)(5+3\kappa)}$ | $\frac{48\sigma^4}{(5+\kappa)(5+2\kappa)(5+3\kappa)}$ |
| $\mu_n^{[n+1]}$        | $\sum_{i=0}^{n} \left(1+(-1)^i\right) \frac{n!\mu^{n-i}\sigma^i}{(n-i)! \left(\frac{1+n\left(n-i\right)\kappa}{\kappa}\right)! \left(\frac{1+n\left(n-i\right)\kappa}{\kappa}\right)! \sigma^n}$ | $\left(1+(-1)^n\right) \frac{\kappa^{-n}n!\left(1+n\kappa\right)!}{\left(1+n\kappa+n\kappa-n\kappa\right)!} \sigma^n$ |
Proof: The normalized \((n+1)\)-power-density of a single-sided generalized Pareto density (Type II) is also a Pareto density with a modified scale and shape

\[
\mathcal{f}_X^{(n+1)}(x) = \frac{1}{\sigma^{n+1}(1+\kappa \frac{x-\mu}{\sigma})^{n+1}}.
\]

Equating the new exponent with the structure of the generalized Pareto distribution determines the modified shape

\[
\left(\frac{1}{\kappa'}+1\right) = \left(\frac{1}{\kappa}+1\right)(n+1)
\]

\[
\kappa' = \frac{\kappa}{1+n+n\kappa}.
\]

Equating the factors multiplying the variable determines the modified scale

\[
\frac{\kappa'}{\sigma'} = \frac{\kappa}{\sigma},
\]

\[
\sigma' = \frac{\sigma}{1+n+n\kappa}.
\]

The \(n^{th}\) moment of the Pareto distribution is restricted to the domain \(\kappa < \frac{1}{n}\), thus for the \(n+1\) power-distribution is the region of finite existence for the moments is

\[
\kappa' = \frac{\kappa}{1+n+n\kappa} < \frac{1}{n}
\]

\(0 < \frac{1}{n} < +1\).

Thus, the moment for the \(n+1\) power-density of the Pareto distribution exists and is finite for all values of the shape parameter. Solutions for the moment integrals

\[
\mu_n^{(n+1)} = \frac{\int_\mu^\infty x^n \left(1+\frac{\kappa}{\sigma} \frac{x-\mu}{\sigma}\right)^n \frac{1}{\sigma}\left(1+\frac{\kappa}{\kappa'} \frac{x-\mu}{\sigma}\right)^n dx}{\int_\mu^\infty \left(1+\frac{\kappa}{\sigma} \frac{x-\mu}{\sigma}\right) \frac{1}{\sigma}\left(1+\frac{\kappa}{\kappa'} \frac{x-\mu}{\sigma}\right)^n dx}
\]

are specified in Table 1. The \(n+1\) power-distribution for the two-sided generalized Pareto has the same modified shape and thus the \(n^{th}\) also exists for all shape values. The solutions for the two-sided Pareto are specified in Table 2. □
Corollary 2.1 Given the location of a one-sided generalized Pareto distribution, the scale is determined independently of the shape by the first-moment with the pdf raised to the second-power and normalized.

\[
\sigma = 2\mu_1^{(2)} = 2E^{(2)}[X - \mu] = 2 \frac{\int_0^\infty x \left(1 + \kappa \frac{x - \mu}{\sigma}\right)^{-\frac{1}{\kappa}+1} \, dx}{\int_0^\infty \left(1 + \kappa \frac{x - \mu}{\sigma}\right)^{-\frac{1}{\kappa}+1} \, dx}
\]  
(4.8)

Corollary 2.2 Given the location and scale of a one-sided generalized Pareto distribution, the shape is determined by second-moment with pdf raised to the third-power and normalized.

\[
\kappa = 4 \frac{\mu_2^{(3)}}{3E^{(3)}[(X - \mu)^2]} - 3 = 4 \frac{\int_0^\infty \left(1 + \kappa \frac{x - \mu}{\sigma}\right)^{-\frac{1}{\kappa}+1} \, dx}{3 \int_0^\infty x^2 \left(1 + \kappa \frac{x - \mu}{\sigma}\right)^{-\frac{1}{\kappa}+1} \, dx} - 3
\]  
(4.9)

Corollary 2.3 The location of a two-sided generalized Pareto distribution is determined by the first moment with the pdf raised to the second-power and normalized.

\[
\mu = \mu_1^{(2)} = E^{(2)}[X] = \frac{\int_{-\infty}^\infty x \left(1 + \kappa \left|\frac{x - \mu}{\sigma}\right|\right)^{-\frac{1}{\kappa}+1} \, dx}{\int_{-\infty}^\infty \left(1 + \kappa \left|\frac{x - \mu}{\sigma}\right|\right)^{-\frac{1}{\kappa}+1} \, dx}
\]  
(4.10)

Corollary 2.4 Given the location of a two-sided generalized Pareto distribution the scale is determined independently of the shape by applying Corollary 1.1 to the \(x > \mu\) side of the distribution (4.8). Likewise, the shape is determined by applying Corollary 1.2 to the \(x > \mu\) side (4.9).

4.2. The Student’s t distribution

Lemma 3: Given a non-centered and scaled Student’s t distribution (2.5) the generalized nth moment exists for all shapes \(\kappa \geq 0\) for the \((n+1)\)-power-distribution. The generalized moments for the Student’s t distribution are specified in Table 3.

Proof: The normalized \((n+1)\)-power-density of a Student’s t-density is also a Student’s t with a modified scale and shape, where the shape is the inverse of the degree of freedom. The power-density, neglecting the original normalization which cancels out, is

\[
f_x^{(n+1)}(x) = \frac{x^{\frac{1}{2}\kappa+1}(1 + \kappa \frac{x - \mu}{\sigma})^{-\frac{1}{\kappa}(\frac{1}{2}\kappa+1)\left|\frac{n+1}{\kappa}\right|}}{\int_{-\infty}^\infty \left(1 + \kappa \frac{x - \mu}{\sigma}\right)^{-\frac{1}{\kappa}(\frac{1}{2}\kappa+1)\left|\frac{n+1}{\kappa}\right|} \, dx}
\]  
(4.11)
Table 3: The $n^{th}$ moments for the (n+1)-power-density the Student’s $t$-distribution.

| Moment, Centered (Scaled) | Student’s t, n+1 |
|---------------------------|------------------|
|                           | Non-centered     | Centered (Scaled) |
| $\mu_1^{[2]}$, $\mu$     | $\mu$           | -                |
| $\mu_2^{(3)}$, $x-\mu$   | $\mu^2 + \frac{\sigma^2}{3}$ | $\frac{\sigma^2}{3}$ |
| $\mu_3^{(4)}$, $x-\mu$   | $\mu^3 + \frac{3\mu\sigma^2}{4+\kappa}$ | 0 |
| $\mu_4^{(5)}$, $\frac{x-\mu}{\sigma}$ | $\mu^4 + \frac{6\mu^2\sigma^2}{5+2\kappa} + \frac{3\sigma^4}{5(5+2\kappa)}$ | $\frac{3\sigma^4}{25+10\kappa}$ |
| $\mu_n^{(n+1)}$, $\frac{x-\mu}{\sigma}$ | Simplification Not Available | $\left(1+(-1)^n\right)\left(\frac{\sigma}{\sqrt{\kappa}}\right)^n \frac{n! \cdot n^{n-2} \kappa^{n-2}}{2 \cdot 2\pi \cdot \kappa^{n+1}(n-2)!}$ |

The modified shape parameter is the same as the Pareto power-distribution since the $\frac{1}{2}$ factor cancels out

$$\left(\frac{1}{\kappa'}+1\right) = \left(\frac{1}{\kappa}+1\right)(n+1)$$

(4.12)

$$\kappa' = \frac{\kappa}{1+n+n\kappa}.$$  

The modified scale is determined by the terms multiplying the variable

$$\frac{\kappa'}{\sigma^2} = \frac{\kappa}{\sigma^2}$$

(4.13)

$$\sigma' = \frac{\sigma}{\sqrt{1+n+n\kappa}}.$$  

Like the generalized Pareto distribution, existent, finite $n^{th}$ moments of the Student’s $t$-distribution are restricted to the domain $\kappa < \frac{1}{n}$, thus from Lemma 2 the $n+1$ power-density of the Student’s $t$ has a finite $n^{th}$ finite moment for all shapes. Solutions for the moment integrals

$$\mu_n^{(n+1)} = \int_{-\infty}^{\infty} x^n \left(1+\kappa\left(\frac{x-\mu}{\sigma}\right)^2\right)^{-\frac{1}{2}\left(\frac{1}{\kappa}+1\right)} dx$$

(4.14)

are specified in Table 3. □
General $m^{th}$-moment and $n$-power-density $\mu_m(n)$ serve a variety of purposes. In the next section, the variance of estimators will be shown to depend on twice the moment of the estimator $\mu_{2m}$. Here it is noted that further increments of the power-density, such as $\mu_n^{(n+2)}$ shown in Table 4, provide additional relationships between the power-moments and distribution parameters. The general power-moments for the Student’s $t$-distribution was computed using Mathematica

$$
\mu_m(n) = \int_{-\infty}^{\infty} x^m f^n(x) dx / \int_{-\infty}^{\infty} f^n(x) dx
$$

$$
= \left(1 + (-1)^m \right) \kappa^{m/2} \sigma^m \left(\frac{m-1}{2}\right) \left(\frac{n+(n-m-3)\kappa}{2\kappa}\right),
$$

for $\kappa < \frac{n}{1+m-n}$. (4.15)

From this general solution, the infinite domain when $n = m+1$ is evident.

| Table 4: The $n^{th}$ moments for the (n+2)-power-density of the Student’s $t$-distribution. |
|---|---|---|
| Moment, Centered (Scaled) | Student’s t, $n+2$ |
| $\mu_1^{(3)}$ | $\mu$ | - |
| $\mu_2^{(4)}$, $x-\mu$ | $\mu^2 + \frac{\sigma^2}{4+\kappa}$ | $\frac{\sigma^2}{4+\kappa}$ |
| $\mu_3^{(5)}$, $x-\mu$ | $\mu^3 + \frac{3\mu\sigma^2}{5+2\kappa}$ | 0 |
| $\mu_4^{(6)}$, $\frac{x-\mu}{\sigma}$ | $\mu^4 + \frac{2\mu^2\sigma^2}{2+\kappa} + \frac{\sigma^4}{(2+\kappa)(6+\kappa)}$ | $\frac{\sigma^4}{(6+\kappa)(2+\kappa)}$ |
| $\mu_n^{(n+2)}$, $\frac{x-\mu}{\sigma}$ | Simplification | Not Available |

5. Performance of the IA algorithm for estimation of the Student’s $t$-distribution

The Student’s $t$-distribution is used to demonstrate and evaluate the performance of the Independent Approximates algorithm. Because the subsampling decimates the samples by a factor of about a thousand, a baseline of 10,000 samples with 10 iterations of partitioning and selecting approximates is used. The generalized Box-Müller method (Nelson and Thistleton 2006 May 23) is used to generate random samples. So that the bias, precision, and accuracy of the estimators can be measured each estimate is computed twenty times. First, a more detailed description of the algorithm is provided.
5.1. Description of the IA Algorithm

Mathematica software implementing the IA algorithm is available at the Photrek Github repository (Nelson 2020a) and calls functions from the Nonlinear-Statistical-Coupling repository (Nelson 2020b). The program flow of the algorithm to compute the parameters of a heavy-tailed distribution consists of:

1) center the samples using the median and scale the samples using the 75% quantile minus the median. This ensures the tolerance selection will be consistent for a variety of distributions.
2) permute the samples and select IAs several times (e.g. 10) by
   a. If estimating location
      i. partition samples into pairs; (optional: with an overlapping offset fewer IA pairs will be missed but pairs sharing a sample will be correlated)
      ii. select pairs which are equal within a tolerance (e.g. 0.1), Fig. 2a
   b. If estimating scale
      i. partition samples into triplets; (optional: with an overlapping offset fewer IA triplets will be missed but triplets sharing a sample will be correlated)
      ii. since the 2\textsuperscript{nd} moment is insensitive to the sign of samples, approximates can be selected along three diagonals, however, each diagonal must be searched separately since taking the absolute value first creates a bias of values of not near the origin, Fig. 2b
   c. If estimating a higher moment, partition into groups of \( n \) and select either along the equal diagonal for an odd moment or along all the diagonals if an even moment
3) use the IA subsamples to compute the required moment of the power-distribution
4) compute the desired parameter of the original distribution. For the generalized Pareto or Student’s \( t \), the relationships are defined in Table 1 thru Table 4.

The selection of IAs is sensitive to the tolerance criteria for choosing partitions with approximately equal samples. Based on experiments with 0.01, 0.1, and 1 as tolerance parameters, a tolerance of 0.1 provides a good balance between selecting a sufficient number of samples while ensuring that the desired filtering for a distribution approximating \( f^n \) is achieved. Additional research is planned to quantify this trade-off. The 10 permutations to select IAs takes approximately 0.2 seconds for the pairs, and approximately 2 seconds for the triplets running in Mathematica on a MacBook Pro with a 2.3 GHz Intel Core i5 and 16 GB of memory. Using the Mathematica CloudEvaluation, which computes the computation of a 64-bit Linux x86 server, takes approximately 0.35 seconds for both selections. Given the equal speeds, the latency of the cloud is presumably a significant factor.

There are several options for estimating the shape of the Student’s \( t \) given estimates of the location and scale. For this study, the geometric mean estimator (2.10) defined by Nelson et. al (Nelson et al. 2019) will be used. Although this is not strictly a closed-form estimate, the root finder algorithms required are well established. Both use of the 4-power-density to measure \( \mu_2^{(4)} \) and the Hill-type estimators are alternatives that could provide a closed-form estimator.
Comparative performance of the speed and accuracy of these and other shape estimators will be explored in future research.

5.2. Expected Bias and Precision

The accuracy (root mean square error) of an estimate is a function of the bias (expected error of the estimate) and the precision (standard deviation of the estimates). While the \((n+1)\)-power-density is sufficient to define the \(n\)th moment and thus assures the bias will be finite, this does not assure that the precision is finite. Since the variance of the \(n\)th moment estimator is proportional to the \(2n\)th moment a \((2n+1)\)-power-density is required to assure the variance of the estimator is finite. Nevertheless, this initial investigation of the IA estimator for the \(n\)th moment utilizes the \((n+1)\)-power-density and is thus restricted in the domain of finite variance of the estimate. For the proofs, the subsamples will be assumed to be independent-equals rather than approximates. First, the bias of the location and scale has the following properties.

**Lemma 4:** a) Given \(N^{(2)}\) independent-equals samples \(X^{(2)}\) drawn from a 2-power Student’s \(t\)-distribution \(f^{(2)}(x)\) (i.e. from equal pairs of \(X \sim f(x)\)) the first moment, which estimates the location, is unbiased. b) Given \(N^{(3)}\) samples from a centered 3-power Student’s \(t\)-density \(f^{(3)}(x)\), the estimate of the scale \(\hat{\sigma} = 3(\hat{\mu}_{z}^{(3)} - \hat{\mu})\) has a bias of \(-\frac{1}{N^{(2)}}\).

Proof: a) Given Lemma 3’s proof that the first moment of the 2-power Student’s \(t\)-density exists and is equal to the location, the estimate of the location is unbiased since

\[
E\left[\frac{1}{N^{(2)}} \sum_{i=1}^{N^{(2)}} X^{(2)}_{i} - \mu\right] = \frac{1}{N^{(2)}} \sum_{i=1}^{N^{(2)}} E\left[X^{(2)}_{i}\right] - \mu = 0. \tag{5.1}
\]

b) Given the unbiased estimate \(\hat{\mu}\), the bias of the estimate \(\hat{\sigma}\) is given by

\[
E\left[\frac{3}{N^{(3)}} \sum_{i=1}^{N^{(3)}} \left(X^{(3)}_{i} - \mu\right)^{2}\right] - \sigma^{2} = \frac{3}{N^{(3)}} \sum_{i=1}^{N^{(3)}} E\left[\left(X^{(3)}_{i} - \hat{\mu}\right)^{2}\right] - \sigma^{2} = \frac{3}{N^{(3)}} \sum_{i=1}^{N^{(3)}} \left(E\left[\left(X^{(3)}_{i} - \mu\right)^{2}\right] - E\left[\left(\hat{\mu} - \mu\right)^{2}\right]\right) - \sigma^{2} \tag{5.2}
\]

\[
= \frac{3}{N^{(3)}} \sum_{i=1}^{N^{(3)}} \left(\frac{\sigma^{2}}{3} - \frac{\sigma^{2}}{3N^{(2)}}\right) - \sigma^{2} = -\frac{1}{N^{(2)}}\sigma.
\]

Note that the bias of the scale estimate is inversely proportional to the number of samples used to estimate the location which is \(N^{(2)}\). □

The variance properties of the estimator depend on moments twice that of the estimators \(\mu_{z}^{(n+1)}\). For the domain of (4.15), the variance of the estimator is finite for a restricted range of shape values. The variance of the location estimator depends on \(\{m=2, n=2\}\) which has a finite domain of \(\kappa < 2\). The variance of the scale estimator depends on \(\{m=4, n=3\}\) which has a finite
domain of $\kappa < \frac{3}{2}$. The following lemma proves these domains and the expected value of the estimate variances.

**Lemma 5:** a) Given $N^{(2)}$ independent-equal samples $X^{(2)}$ drawn from a 2-power Student’s $t$-density $f^{(2)}(x)$ the first moment, which estimates the location, has a precision of

$$\frac{\sigma}{\sqrt{(2-\kappa)N^{(2)}}}$$

if $\kappa < 2$. b) Given $N$ samples from a centered 3-power Student’s $t$-density $f^{(3)}(x)$, the estimate of the scale $\hat{\sigma} = 3(\hat{\mu}_2^{(3)} - \hat{\mu})$ has a precision of

$$\frac{3\sigma^2}{\sqrt{N^{(3)}}} \left(\frac{1}{3-2\kappa} + \frac{1}{(2-\kappa)N^{(2)}}\right)^{1/2}$$

if $\kappa < \frac{3}{2}$.

**Proof:** a) The variance of the location estimate is given by

$$\text{Var}[\hat{\mu}] = \text{Var}\left[\frac{1}{N^{(2)} N^{(2)}} \sum_{i=1}^{N^{(2)}} X^{(2)}_i\right]$$

$$= \frac{1}{(N^{(2)})^2} \left(\sum_{i=1}^{N^{(2)}} \text{Var}\left[X^{(2)}_i\right]\right)$$

(5.3)

The variance of the 2-power samples from a Student’s $t$-density is given by $\{m=2, n=2\}$ in (4.15)

$$\int_{-\infty}^{\infty} x^2 f^{(2)}(x)dx / \int_{-\infty}^{\infty} f^{(2)}(x)dx = \frac{2\kappa^{-1}\sigma^2 \left(\frac{1}{2}\right) \left(\frac{2-3\kappa}{2\kappa}\right)}{2\sqrt{\pi} \left(\frac{2-\kappa}{2\kappa}\right)!}$$

if $\kappa < 2$

$$= \frac{\sigma^2}{2-\kappa}$$

if $\kappa < 2$.

(5.4)

Thus, taking the square root the precision is $\frac{\sigma}{\sqrt{(2-\kappa)N^{(2)}}}$ if $\kappa < 2$. □
b) The precision of the scale estimate is given by

\[
\text{Var}[\hat{\sigma}] = \text{Var}\left[\frac{3}{N^{(3)}} \sum_{i=1}^{N^{(3)}} (X_i^{(3)} - \hat{\mu})^2\right] \equiv \frac{9}{(N^{(3)})^2} \sum_{i=1}^{N^{(3)}} \left(\text{Var}\left[X_i^{(3)}\right] + \text{Var}[\hat{\mu}]\right),
\]

(5.5)
since the cross-term \(\text{Var}[2X_i^{(3)}\hat{\mu}]\) is zero given that the \(X_i^{(3)}\) and \(X_i^{(2)}\) samples are independent.

The variance of the square of the samples from a 3-power Student’s \(t\)-density is given by

\[
\int_{-\infty}^{\infty} x^4 f^3(x) \, dx / \int_{-\infty}^{\infty} f^3(x) \, dx = \frac{2\kappa^{-2} \sigma^4 \left(\frac{3}{2}\right) \left(\frac{3 - 4\kappa}{2\kappa}\right)!}{2\sqrt{\pi} \left(\frac{3}{2\kappa}\right)!} \quad \text{if } \kappa < \frac{3}{2}
\]

\[
= \frac{\sigma^4}{3 - 2\kappa}.
\]

(5.6)
The variance of the square of the location estimate is the square of the variance of the location estimate \(\frac{\sigma^4}{(2\kappa) N^{(2)}}\) if \(\kappa < 2\). Thus, the precision of the scale estimate is

\[
\text{Prec}[\hat{\sigma}] = \sqrt{\text{Var}[\hat{\sigma}]} = \frac{3}{N^{(3)}} \left[N^{(3)} \frac{\sigma^4}{3 - 2\kappa} + N^{(3)} \frac{\sigma^4}{(2\kappa) N^{(2)}}\right]^{1/2} \equiv \frac{3\sigma^2}{\sqrt{N^{(3)}}} \left[\frac{1}{3 - 2\kappa} + \frac{1}{(2\kappa) N^{(2)}}\right]^{1/2} \quad \text{if } \kappa < \frac{3}{2}.
\]

(5.7)

5.3. Performance of the IA Algorithm

The parameters of the Student’s \(t\)-distribution are estimated using the IA algorithm by selecting approximate pairs for the location, approximate absolute value triplets for the scale, and geometric mean of all the samples to estimate the shape, which is the inverse of the degree of freedom. Figures 3-7 document the bias and precision of the IA estimation as the shape is increased from 0.25 to 0.4. This spans the domains of finite variance (0.25), the boundary of infinite variance (0.5), the Cauchy distribution (1) which is the focal point between exponential (0) and a delta function (\(\infty\)), a stressing case (2), and approaching the limits of generating and estimating reliable distributions (4). Each figure includes a) the mean and three times the
standard deviation for 20 measurements each of 25 different combinations of the location and scale, b) the mean and standard deviation for the shape and scale estimates as a function of the scale, and c) a 3D image of the bias and precision of all the measurements. In the 3D image, the location and scale measurements are scaled by $\sigma$. For the overall precision, the computation of the standard deviation over all the measurements is multiplied by $\sqrt{25}$ since it is the precision of each estimate rather than the group of estimates which is of interest. Table 5 summarizes the bias and precision performance.

**Table 5:** The measured bias and precision for the estimates of the location, scale, and shape of the Student’s $t$-distributions ranging from a shape of 0.25 to 4.00 with an initial sample size of 10,000. Up until a shape of 4, the bias for each estimate is less than 0.01 and the precision ranges from ±0.02 to ±0.11. The shape of 4 begins to show weaknesses but the precision is still within ±0.33.

| Generated Shape | Location | Scale | Shape |
|-----------------|----------|-------|-------|
| 0.25            | 0.000 ± 0.017 | 0.00 ± 0.04 | 0.00 ± 0.07 |
| 0.50            | 0.000 ± 0.020 | 0.00 ± 0.05 | 0.01 ± 0.07 |
| 1.00            | 0.001 ± 0.024 | 0.00 ± 0.06 | 0.00 ± 0.09 |
| 2.00            | 0.00 ± 0.04   | 0.00 ± 0.10 | 0.00 ± 0.11 |
| 4.00            | 0.00 ± 0.33   | 0.05 ± 0.18 | -0.06 ± 0.15 |

The effect of initial sample sizes is shown in Fig. 8 and summarized in Table 6. The IA algorithm is still effective with an initial sample size of 1,000 though the bias and precision are reduced relative to the 10,000 samples size. An initial sample of 100 will have cases where the subsample selection is too small for statistical estimation; however, adjustments in the algorithm may enable course estimates given a small sample size. Increasing the sample size to 100,000 shows an order of magnitude improvement (x10) in the precision over the 10,000 sample size performance when the shape is 0.25 and 1.00. For a shape of 4.00 the improvement is more modest (x2).

**Table 6:** The measured bias and precision for the estimates of the location, scale, and shape as a function of sample size and shape for the Student’s $t$-distribution. For the shapes 0.25 and 1.00, the difference in precision between 1,000 and 100,000 samples is approximately a factor of 10. For shape 4.00 is only improved by a factor of 2.

| Generated Shape | Sample Size | Location | Scale | Shape |
|-----------------|-------------|----------|-------|-------|
| 0.25            | 1,000       | 0.00 ± 0.05 | -0.03 ± 0.15 | 0.04 ± 0.19 |
| 0.25            | 100,000     | 0.000 ± 0.006 | 0.000 ± 0.012 | 0.000 ± 0.021 |
| 1.00            | 1,000       | -0.02 ± 0.07 | 0.00 ± 0.17 | 0.02 ± 0.22 |
| 1.00            | 100,000     | 0.000 ± 0.008 | 0.002 ± 0.021 | -0.002 ± 0.029 |
| 4.00            | 1,000       | 0.00 ± 0.29 | 0.02 ± 0.22 | -0.01 ± 0.21 |
| 4.00            | 100,000     | 0.00 ± 0.14 | 0.05 ± 0.11 | -0.04 ± 0.09 |

**Table 7** shows the number of pairs and triplets selected as a function of the initial sample size and the shape of the distribution. Each estimate is performed by the selection of pairs to estimate the location and triplets to estimate the scale. To ensure an adequate number of samples, a type of bootstrapping is utilized in which the original sample is permuted 10 times. The pairs and triplets are partitioned without an offset, which does create some correlation in the selection...
due to the overlap of groups; however, the alternative is to complete more permutations which is the slowest part of the algorithm. From the mean pairs and triplets selected the theoretical bias and precision are determined using Lemma 4 and Lemma 5, respectively. For the shape of 0.25 and 1.00, the theoretical and experimental bias and precision are similar. For a shape of 4, the precision is not finite, which is not fully reflected in the experimental measurements but could be shown by further increases in the sample size. A follow-up study is planned to examine the optimization of the speed and accuracy of the algorithm, which will examine more closely the effect of the tolerance, partition offset, and permutations.

**Table 7**: Relationship between original sample size, selection of pairs and triplets, and the bias and precision for the Student’s $t$ estimates. The mean and standard deviation of the pairs and triplets selected are computed over the 20 estimates and 25 different location and scale values. Each selection includes 10 permutations of the initial samples. The bias and precision of the location and scale are determined from Lemma 4 and Lemma 5, respectively using the mean pair and triplet values.

| Generated Shape | Sample Size | Pairs Selected | Triplets Selected | Theoretical Bias ± Precision |
|----------------|-------------|----------------|-------------------|------------------------------|
|                |             |                |                   | Location                     | Scale           |
| 0.25           | 1,000       | 9 ± 1          | 7 ± 1             | 0.00 ± 0.25                 | -0.1 ± 0.7      |
|                | 10,000      | 90 ± 3         | 71 ± 4            | 0.00 ± 0.08                 | -0.01 ± 0.23   |
|                | 100,000     | 893 ± 8        | 709 ± 13          | 0.000 ± 0.025               | 0.00 ± 0.07    |
| 1.00           | 1,000       | 8 ± 1          | 7 ± 2             | 0.00 ± 0.35                 | -0.1 ± 1.2     |
|                | 10,000      | 79 ± 2         | 68 ± 4            | 0.00 ± 0.11                 | 0.0 ± 0.4      |
|                | 100,000     | 794 ± 7        | 681 ± 14          | 0.000 ± 0.035               | 0.00 ± 0.12    |
| 4.00           | 1,000       | 11 ± 2         | 24 ± 8            | 0 ± $\infty$               | 0 ± $\infty$   |
|                | 10,000      | 118 ± 7        | 261 ± 38          | 0 ± $\infty$               | 0 ± $\infty$   |
|                | 100,000     | 1170 ± 26      | 2605 ± 128        | 0 ± $\infty$               | 0 ± $\infty$   |
Fig. 3. Student’s $t$ (Shape = 0.25) Estimation Performance Estimation performance for the location, scale, and shape (0.25) Student’s $t$ distribution with 10,000 initial samples. a) Location (mean) versus the scale for 25 different estimates each performed 20 times. Three times the standard deviation are shown as error bars. b) Scale versus the shape. The error bars are the standard deviation. b) Scale versus the shape. The error bars are the standard deviation of 20 estimates. c) Bias (estimate minus truth) and precision (standard deviation) for the three parameter estimations. The colors indicate the scale as in a and b. The location and scale error is divided by the scale, while the coupling error is not scaled. The average performance over all the results is in black and also as a text inset (location, scale, shape).
Fig. 4. **Student’s t (Shape = 0.5) Estimation Performance** Estimation performance for the location, scale, and shape (0.5) Student’s t distribution with 10,000 initial samples. a) Location (mean) versus the scale for 25 different estimates each performed 20 times. Three times the standard deviation are shown as error bars. b) Scale versus the shape. The error bars are the standard deviation. c) Bias (estimate minus truth) and precision (standard deviation) for the three parameter estimations. The colors indicate the scale as in a and b. The location and scale error is divided by the scale, while the coupling error is not scaled. The average performance over all the results is in black and also as a text inset (location, scale, shape).
Fig. 5. Student’s t (Shape = 1.0) Estimation Performance. Estimation performance for the location, scale, and shape (1.0) Student’s t distribution with 10,000 initial samples. a) Location (mean) versus the scale for 25 different estimates each performed 20 times. Three times the standard deviation are shown as error bars. b) Scale versus the shape. The error bars are the standard deviation. c) Bias (estimate minus truth) and precision (standard deviation) for the three parameter estimations. The colors indicate the scale as in a and b. The location and scale error is divided by the scale, while the coupling error is not scaled. The average performance over all the results is in black and also as a text inset (location, scale, shape).
Fig. 6. Student’s t (Shape = 2.0) Estimation Performance  Estimation performance for the location, scale, and shape (2.0) Student’s t distribution with 10,000 initial samples. a) Location (mean) versus the scale for 25 different estimates each performed 20 times. Three times the standard deviation are shown as error bars. b) Scale versus the shape. The error bars are the standard deviation. c) Bias (estimate minus truth) and precision (standard deviation) for the three parameter estimations. The colors indicate the scale as in a and b. The location and scale error is divided by the scale, while the coupling error is not scaled. The average performance over all the results is in black and also as a text inset (location, scale, shape).
Fig. 7. Student’s t (Shape = 4) Estimation Performance

Estimation performance for the location, scale, and shape (4.0) Student’s t distribution with 10,000 initial samples. a) Location (mean) versus the scale for 25 different estimates each performed 20 times. Three times the standard deviation are shown as error bars. b) Scale versus the shape. The error bars are the standard deviation. c) Bias and precision which together quantify the accuracy of the estimator. The colors indicate scale. The location and scale error is divided by the scale, while the coupling error is not scaled. The average performance over all the results is in black and is inset as (location, scale, shape).
Fig. 8. Student’s t estimation performance for 1000 and 100,000 samples. The bias and precision of mean, scale, and shape for a sample size of 1,000 (left) and 100,000 (right). The rows from top to bottom have shape values of 0.25, 1.0, and 4.0. The 100-fold increase in the sample size improves the precision of the estimate by approximately a factor of 10 for shape 0.25 and 1.0 and a factor 2 for the shape of 4.0.
6. Conclusion and Future Research

A closed-formed method for estimating heavy-tailed distributions has been discovered that relies on a filtering method that selects Independent Approximates. The IAs are selected by partitioning iid samples into partitions of size $n$ and selecting the median of only the groups approximately equal within a tolerance. The IAs are approximately distributed as the normalized power-density
\[ f_X^n(x) \left/ \int_{x \in X} f_X^n(x) \, dx \right. \]  
Because the power-density preserves a precise relationship to properties of $f$ while increasing the rate of decay of the tail, estimates of the parameters of $f$ are enabled. For integer values then, the power-density referred to as an escort probability in $q$-statistics is the density of a subsample of independent-approximates. Further research is suggested to specify the subsampling with a fractional power-density. The fractional power-density is conjectured to be either a non-equal diagonal or a fractal representation of partitions.

To the author’s knowledge, having completed a thorough literature search, this is the first report of such a filtering method to estimate heavy-tailed distributions. A proof applicable to distributions with an analytical density is completed for the Type II Pareto and Student’s $t$. Numerical experiments for the Student’s $t$ distribution demonstrate that estimates of the location, scale, and shape with relative biases below 0.01 and precision below $\pm 0.1$ are achieved with an initial sample size of $10,000$. Estimates with 1000 samples are also achieved and indicate that estimation with smaller sample sizes is possible but sensitive to parameters of the algorithm.

Many questions remain. The slowest part of the algorithm is the selection of IAs that requires permuting, partitioning, and subselecting samples; however, the partitions can be formed and tested for approximates independently lending itself to parallel computation. The use of quartets of IAs is expected to enable estimation of the heavy-tailed distributions with skew such as the Pareto Type IV. Proofs regarding the performance of the algorithm as a function of sample size, approximation tolerance, and the number of permutations used in the selection process would be an important contribution to the research. A trade-off exists between the use of larger partitions to insure the variance of the estimation is finite versus the subsample size available with larger partitions.

A variety of applications are anticipated as industrial processes still rely heavily on estimates of the mean and standard deviation even when the underlying distributions are non-Gaussian. For instance, in the financial sector, events such as the 2008 financial crisis are understood to have been exacerbated by methods such as the Black-Sholes options pricing (Goldstein and Taleb 2007; Black and Scholes) that relied on assumptions about independence and Gaussian processes. The Volatility Index (VIX) underlying pricing of many financial derivatives is closely related to the log-return standard deviation despite long-standing evidence that the variations in log-returns are heavy-tailed and thus not characterized by the standard deviation (Kapadia and Du 2011; Park et al. 2017; Zubillaga et al. 2019 May 10).

In turn, scientific investigations of complex systems have relied on a variety of iterative methods that are cumbersome to describe and implement. In medicine, vital signs such as heartbeats are known to follow fractal rhythms (Goldberger et al. 2002; Selvaraj et al. 2011); fast, accurate measures of the distributions of vital signs could improve diagnostics. In addition to estimation, signal processing applications particularly in digital media and aerospace instrumentation may be able to use the IA method to filter outliers while preserving core signal information.
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