We discuss polynomials orthogonal with respect to a semi-classical generalized higher-order Freud weight

\[ \omega(x; t, \lambda) = |x|^{2\lambda+1} \exp(tx^2 - x^m), \quad x \in \mathbb{R}, \]

with parameters \( \lambda > -1, t \in \mathbb{R} \) and \( m = 2, 3, \ldots \). The sequence of generalized higher-order Freud weights for \( m = 2, 3, \ldots \), forms a hierarchy of weights, with associated hierarchies for the first moment and the recurrence coefficient. We prove that the first moment can be written as a finite partition sum of generalized hypergeometric \( 1_{\cdot}F_m \) functions and show that the recurrence coefficients satisfy difference equations which are members of the first discrete Painlevé hierarchy. We analyse the asymptotic behaviour of the recurrence coefficients and the limiting distribution of the zeros as \( n \to \infty \). We also investigate structure and other mixed recurrence relations satisfied by the polynomials and related properties.

1. Introduction

In this paper we consider polynomials orthogonal with respect to the generalised higher-order Freud weight

\[ \omega(x; t, \lambda) = |x|^{2\lambda+1} \exp(tx^2 - x^m), \quad x \in \mathbb{R}, \ m = 2, 3, \ldots, \]

with \( \lambda > -1 \) a parameter. The main goal of this article is to bring a comprehensive self-contained analysis of these polynomials when the parameter \( m \) takes integer values higher than 1 and for any values of \( \lambda > -1 \) and \( t \in \mathbb{R} \). The analysis for the particular cases of \( m = 2, 3 \) was considered in [1–6], with an emphasis on the study of the corresponding recurrence
coefficients. We significantly extend existing studies on Freud type weights whilst providing a coherent and consistent approach, using techniques which are also likely to be adopted in the study of other semi-classical type weights. Throughout, we link and explain the connections to the existing theory. After giving a short mathematical background in §2, in §3, we give a closed-form expression for the moments with respect to the weight (1.1), which corresponds to a finite partition sum of generalized hypergeometric $F_m$ functions. The corresponding recurrence coefficients in the three-term recurrence relation are investigated in §4. Therein, we prove a recursive method that gives nonlinear recurrence relations satisfied by these recurrence coefficients (proposition 4.4) and give them explicitly for the cases where $m = 4, 5$, whilst recovering the already known relations for $m = 2, 3$. We prove structure relations and mixed recurrence relations satisfied by generalized higher-order Freud polynomials in §5. The asymptotic behaviour of the recurrence coefficients proved in §4 determines the limiting distribution of the zeros, and this, as well as other properties of the zeros, is investigated in §6. We conclude with the quadratic decomposition of the generalized higher-order Freud weight in §7.

2. Mathematical background

Let $\nu$ be a positive measure on the real line for which the support is not finite and all the moments

$$
\mu_k = \int_{-\infty}^{\infty} x^k \, d\nu(x), \quad k = 0, 1, \ldots,
$$

exist. The corresponding monic orthogonal polynomial sequence $\{P_n(x)\}_{n \geq 0}$ is defined by

$$
\int_{-\infty}^{\infty} P_m(x) P_n(x) \, d\nu(x) = h_n \delta_{m,n}, \quad h_n > 0,
$$

where $\delta_{m,n}$ denotes the Kroneker delta. A fundamental property of orthogonal polynomials is that they satisfy a three-term recurrence relation of the form

$$
P_{n+1}(x) = (x - \alpha_n) P_n(x) - \beta_n P_{n-1}(x),
$$

with $\beta_n > 0$ and initial values $P_{-1}(x) = 0$ and $P_0(x) = 1$. The recurrence coefficients $\alpha_n$ and $\beta_n$ are given by the integrals

$$
\alpha_n = \frac{1}{h_n} \int_{-\infty}^{\infty} x P_n^2(x) \, d\nu(x) \quad \text{and} \quad \beta_n = \frac{1}{h_{n-1}} \int_{-\infty}^{\infty} x P_{n-1}(x) P_n(x) \, d\nu(x).
$$

Relevant for this article is the case of a measure that admits a representation via a positive weight function $\omega(x)$ on the real line as follows $d\nu(x) = \omega(x) \, dx$. Henceforth, we will only work with a weight function representation.

The coefficient $\beta_n$ in the recurrence relation (2.2) can also be expressed in terms of the Hankel determinant

$$
\Delta_n = \det \left[ \mu_{j+k} \right]_{j,k=0}^{n-1} =
\begin{vmatrix}
\mu_0 & \mu_1 & \ldots & \mu_{n-1} \\
\mu_1 & \mu_2 & \ldots & \mu_n \\
\vdots & \vdots & \ddots & \vdots \\
\mu_{n-1} & \mu_n & \ldots & \mu_{2n-2}
\end{vmatrix}, \quad n \geq 1,
$$

with $\Delta_0 = 1$ and $\Delta_{-1} = 0$, whose entries are given in terms of the moments (2.1) associated with the weight $\omega(x)$. Specifically

$$
\beta_n = \frac{\Delta_{n+1} \Delta_{n-1}}{\Delta_n^2}.
$$
The monic polynomial \( P_n(x) \) can be uniquely expressed as the determinant
\[
P_n(x) = \frac{1}{\Delta_n} \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}
\]
and the normalization constants as
\[
h_n = \frac{\Delta_{n+1}}{\Delta_n} \quad \text{and} \quad h_0 = \Delta_1 = \mu_0.
\] (2.5)

Also from (2.4) and (2.5), we see that the relationship between the recurrence coefficient \( \beta_n \) and the normalization constants \( h_n \) is given by
\[
h_n = \beta_n h_{n-1}.
\]

For symmetric weights, since \( \omega(x) = \omega(-x) \), it follows that \( \alpha_n = 0 \) in (2.2). Hence, for symmetric weights, the sequence of monic orthogonal polynomials \( \{P_n(x)\}_{n \geq 0} \), satisfy the three-term recurrence relation:
\[
P_{n+1}(x) = \alpha_n P_n(x) - \beta_n P_{n-1}(x).
\] (2.6)

The monic orthogonal polynomials \( P_n(x) \) associated with symmetric weights are also symmetric, i.e. \( P_n(-x) = (-1)^n P_n(x) \). This implies that each \( P_n \) contains only even or only odd powers of \( x \), and we can write
\[
P_{2n}(x) = x^{2n} + \sum_{k=1}^{n} c_{2n-2k} x^{2n-2k} = x^{2n} + c_{2n-2} x^{2n-2} + \cdots + c_{0}
\]
and
\[
P_{2n+1}(x) = x^{2n+1} + \sum_{k=1}^{n} (2n+1) c_{2n-2k+1} x^{2n-2k+1} = x^{2n+1} + c_{2n-1} x^{2n-1} + \cdots + c_{1} x.
\]

By substituting these expressions into the recurrence relation (2.6) and comparing the coefficients on each side, we obtain
\[
\beta_{2n} = c_{2n-2} - c_{2n-1} \quad \text{and} \quad \beta_{2n+1} = -\frac{c_{2n+1}}{c_{0}} = -\frac{P_{2n+2}(0)}{P_{2n}(0)}.
\] (2.7)

It follows from (2.1) that, for symmetric weights, \( \mu_{2k-1} = 0, k = 1, 2, \ldots \), and hence, it is possible to write the Hankel determinant \( \Delta_n \) given by (2.3) in terms of the product of two Hankel determinants obtained by matrix manipulation, interchanging columns and rows. The product decomposition, depending on \( n \) even or odd, is given by
\[
\Delta_{2n} = A_n B_n \quad \text{and} \quad \Delta_{2n+1} = A_{n+1} B_n,
\] (2.8)

where \( A_n \) and \( B_n \) are the Hankel determinants
\[
A_n = \begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2n-2} \\ \mu_2 & \mu_4 & \cdots & \mu_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{2n-2} & \mu_{2n} & \cdots & \mu_{4n-4} \end{vmatrix} \quad \text{and} \quad B_n = \begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2n} \\ \mu_4 & \mu_6 & \cdots & \mu_{2n+2} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{2n} & \mu_{2n+2} & \cdots & \mu_{4n-2} \end{vmatrix},
\] (2.9)

with \( A_0 = B_0 = 1 \). Consequently, for a symmetric weight, by substituting (2.8) into (2.4), the recurrence coefficient \( \beta_n \) is given by
\[
\beta_{2n} = \frac{A_{n+1} B_{n-1}}{A_n B_n} \quad \text{and} \quad \beta_{2n+1} = \frac{A_n B_{n+1}}{A_{n+1} B_n}.
\]
Semi-classical orthogonal polynomials are natural generalizations of classical orthogonal polynomials and were introduced by Shohat in [7]. Maroni provided a unified theory for semi-classical orthogonal polynomials (cf. [8,9]). The weights of classical orthogonal polynomials satisfy a first-order ordinary differential equation, the Pearson equation,

\[
\frac{d}{dx}(\sigma(x)\omega(x)) = \tau(x)\omega(x),
\]

(2.10)

where \(\sigma(x)\) is a monic polynomial of degree at most 2 and \(\tau(x)\) is a polynomial with degree 1. For semi-classical orthogonal polynomials, the weight function \(\omega(x)\) satisfies a Pearson equation (2.10) with either \(\deg(\sigma(x)) > 2\) or \(\deg(\tau(x)) \neq 1\) (cf. [8,10]). The generalized higher-order Freud weight given by (1.1) is a symmetric weight that satisfies the Pearson equation (2.10) with \(\sigma(x) = x\) and \(\tau(x) = 2(tx^2 - mx^{2m} + \lambda + 1)\) and therefore is a semi-classical weight.

3. Moments of the generalized higher-order Freud weights

The existence of the first moment \(\mu_0(t; \lambda, m)\) associated with the generalized higher-order Freud weight (1.1) follows from the fact that, at \(\infty\), the integrand behaves like \(\exp(-x^2)\) and, at \(x = 0\), the integrand behaves like \(x^\lambda\), which, for \(\lambda > -1\), is integrable.

**Theorem 3.1.** Let \(x \in \mathbb{R}, \lambda > -1, t \in \mathbb{R}\) and \(m = 2, 3, \ldots\) Then, for the generalized higher-order Freud weight (1.1), the first moment is given by

\[
\mu_0(t; \lambda, m) = \int_{-\infty}^{\infty} |x|^{2\lambda+1} \exp(tx^2 - x^{2m}) \, dx = \int_0^{\infty} s^{\lambda} \exp(ts - s^m) \, ds
\]

\[
= \frac{1}{m} \sum_{k=1}^{m} \frac{1}{(k-1)!} \frac{\lambda + k}{m} \times \frac{\Gamma\left(\frac{\lambda + k}{m}\right)}{\Gamma\left(\frac{\lambda}{m}\right)} \times 2 F_m\left(\frac{\lambda + k}{m}, 1; \frac{k + 1}{m}, \ldots, \frac{m + k - 1}{m}; \left(\frac{t}{m}\right)^m\right)
\]

\[
= \frac{1}{m} \left[ \frac{(\lambda + 1)}{m} \right] F_{m-1}\left(\frac{\lambda + 1}{m}, \frac{1}{m}, \ldots, \frac{m - 1}{m}; \left(\frac{t}{m}\right)^m\right)
\]

\[
+ \frac{1}{m} \sum_{k=2}^{m-1} \frac{1}{(k-1)!} \frac{\lambda + k}{m} \times \frac{\Gamma\left(\frac{\lambda + k}{m}\right)}{\Gamma\left(\frac{\lambda}{m}\right)} \times 1 F_{m-1}\left(\frac{\lambda + k}{m}, \frac{k + 1}{m}, \ldots, \frac{m - 1}{m}, \ldots, \frac{m + k - 1}{m}; \left(\frac{t}{m}\right)^m\right)
\]

\[
+ \frac{t^{m-1}}{m!} \left[ \frac{\lambda}{m} + 1 \right] F_m\left(\frac{\lambda}{m}, \frac{m + 1}{m}, \frac{m + 2}{m}, \ldots, \frac{2m - 1}{m}; \left(\frac{t}{m}\right)^m\right),
\]

where \(F_q(a; b; z)\) is the generalized hypergeometric function (cf. [11, eq. 16.2.1]).

**Proof.** By using the power series expansion of the exponential function, we obtain

\[
\mu_0(t; \lambda, m) = \int_{-\infty}^{\infty} |x|^{2\lambda+1} \exp(tx^2 - x^{2m}) \, dx = \int_0^{\infty} s^{\lambda} \exp(ts - s^m) \, ds
\]

\[
= \int_0^{\infty} s^{\lambda} \exp(-s^m) \sum_{n=0}^{\infty} \frac{(ts)^n}{n!} \, ds
\]

\[
= \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_0^{\infty} s^{\lambda + n} \exp(-s^m) \, ds
\]
\[
= \frac{1}{m} \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_{0}^{\infty} y^{(n+\lambda-m+1)/m} \exp(-y) \, dy
\]
\[= \frac{1}{m} \sum_{n=0}^{\infty} \frac{t^n}{n!} \Gamma \left( \frac{\lambda+n+1}{m} \right), \]
where \(\Gamma(x)\) denotes the Gamma function defined in [11, eq. 5.2.1], and the fourth equal sign follows from the Lebesgue’s dominated convergence theorem. By letting \(n = mk + j\) for \(j = 0, 1, \ldots, m - 1\), we can write
\[
\mu_0(t; \lambda, m) = \frac{1}{m} \sum_{k=0}^{m-1} \sum_{j=0}^{m-1} \Gamma \left( \frac{\lambda+j+1}{m} + k \right) \frac{t^{mk+j}}{(mk+j)!}.
\]
Using the Gauss multiplication formula [11, eq. 5.5.6] yields
\[
(mk+j)! = j! m^{mk} \prod_{\ell=1}^{m} \left( \frac{j+\ell}{m} \right)_k,
\]
where \((\cdot)_k\) denotes the Pochhammer symbol (cf. [11, §5.2(iii)]), while it follows from [11, eq. 5.5.1] that
\[
\Gamma \left( \frac{\lambda+j+1}{m} + k \right) = \left( \frac{\lambda+j+1}{m} \right)_k \Gamma \left( \frac{\lambda+j+1}{m} \right),
\]
and hence, we have
\[
\mu_0(t; \lambda, m) = \frac{1}{m} \sum_{k=0}^{m-1} \sum_{j=0}^{m-1} \frac{(\lambda+j+1/m)_k \Gamma((\lambda+j+1)/m)}{m^{mk}(j+1/m)_k(j+2/m)_k \cdots (j+m/m)_k} \frac{t^{mk+j}}{j!}
\]
\[= \frac{1}{m} \sum_{j=0}^{m-1} \frac{t^{j} \Gamma \left( \frac{\lambda+j+1}{m} \right)}{j!} \times 2F_m \left( \frac{\lambda+j+1}{m}, \frac{j+1}{m}, \frac{j+2}{m}, \ldots, \frac{m+j}{m}; \frac{t}{m} \right),
\]
as required. \(\square\)

Remark 3.2. In our earlier studies of semi-classical orthogonal polynomials, we proved special cases of theorems 3.1 and 3.3, namely, for \(m = 2\) in [4] and for \(m = 3, 4, 5\) in [3].

In the following theorem, we derive a differential equation satisfied by the first moment \(\mu_0(t; \lambda, m)\). It is often much easier to derive properties of a function from the differential equation it satisfies rather than from an integral representation or, as in this case, a sum of generalized hypergeometric functions.

**Theorem 3.3.** Let \(x \in \mathbb{R}, \lambda > -1, t \in \mathbb{R}\) and \(m = 2, 3, \ldots\). Then, for the generalized higher-order Freud weight (1.1), the first moment
\[
\mu_0(t; \lambda, m) = \int_{-\infty}^{\infty} |x|^{2\lambda+1} \exp(tx^2 - x^{2m}) \, dx = \int_{0}^{\infty} s^\lambda \exp(ts - s^m) \, ds,
\]
satisfies the ordinary differential equation:
\[
m \frac{d^n \varphi}{dt^n} - \lambda \frac{d \varphi}{dt} - \frac{\varphi}{t} = 0. \tag{3.1}
\]
Proof. Following [3,12], we look for a solution of (3.1) in the form
\[ \psi(t) = \int_0^\infty e^{st} v(s) \, ds. \] (3.2)

For (3.2) to satisfy (3.1), it is necessary that
\[ \frac{d^n \psi}{dt^n} - \frac{t}{m} \frac{d \psi}{dt} - \frac{\lambda + 1}{m} \psi = \int_0^\infty e^{st} \left( s^m - \frac{ts}{m} - \frac{\lambda + 1}{m} \right) v(s) \, ds = 0. \]

By using integration by parts, this is equivalent to
\[ \int_0^\infty e^{st} \left\{ s^m v(s) + \frac{1}{m} s \frac{d v}{d s} - \frac{\lambda + 1}{m} v(s) \right\} \, ds = 0, \]
under the assumption that \( \lim_{s \to \infty} s^m v(s) e^{st} = 0 \). Hence, for \( \psi(t) \) to be a solution of (3.1), we need to choose \( v(s) \) so that
\[ (ms^m - \lambda) v(s) + s \frac{d v}{d s} = 0. \]

One solution of this equation is \( v(s) = s^\lambda \exp(-s^m) \).

For the generalized higher-order Freud weight (1.1), the even moments can be written in terms of derivatives of the first moment, as follows:
\[ \mu_{2k}(t; \lambda, m) = \int_{-\infty}^{\infty} x^{2k} |x|^{2\lambda+1} \exp(tx^2 - x^{2m}) \, dx \]
\[ = \frac{d^k}{dt^k} \int_{-\infty}^{\infty} |x|^{2\lambda+1} \exp(tx^2 - x^{2m}) \, dx \]
\[ = \frac{d^k}{dt^k} \mu_0(t; \lambda, m), \quad k = 0, 1, 2, \ldots, \] (3.3)
where the interchange of integration and differentiation is justified by Lebesgue’s dominated convergence theorem. Furthermore, from the definition we have,
\[ \mu_{2k+2}(t; \lambda, m) = \mu_{2k}(t; \lambda + 1, m), \quad k = 0, 1, 2, \ldots, \] (3.4)
and therefore,
\[ \mu_{2k+2}(t; \lambda, m) = \mu_0(t; \lambda + k + 1, m), \quad k = 0, 1, 2, \ldots, \]
which illustrates the importance of the first moment.

4. Recurrence coefficients for generalized higher-order Freud weights

Theorem 4.1. For the generalized higher-order Freud weight (1.1), the recurrence coefficient \( \beta_n \) is given by
\[ \beta_{2n} = \frac{d}{dt} \ln \frac{B_n}{A_n} \quad \text{and} \quad \beta_{2n+1} = \frac{d}{dt} \ln \frac{A_{n+1}}{B_n}, \] (4.1)
with \( A_0 = B_0 = 1 \) and
\[ A_n = \text{Wr} \left( \frac{d \mu_0}{dt}, \ldots, \frac{d^{n-1} \mu_0}{dt^{n-1}} \right) \quad \text{and} \quad B_n = \text{Wr} \left( \frac{d \mu_0}{dt}, \frac{d^2 \mu_0}{dt^2}, \ldots, \frac{d^n \mu_0}{dt^n} \right), \] (4.2)
where
\[ \mu_0 = \mu_0(t; \lambda, m) = \int_0^\infty x^\lambda \exp(ts - s^m) \, dx, \]
and \( \text{Wr}(\varphi_1, \varphi_2, \ldots, \varphi_n) \) denotes the Wronskian given by

\[
\text{Wr}(\varphi_1, \varphi_2, \ldots, \varphi_n) = \begin{vmatrix}
\varphi_1 & \varphi_2 & \cdots & \varphi_n \\
\varphi_1^{(1)} & \varphi_2^{(1)} & \cdots & \varphi_n^{(1)} \\
\vdots & \vdots & \ddots & \vdots \\
\varphi_1^{(n-1)} & \varphi_2^{(n-1)} & \cdots & \varphi_n^{(n-1)}
\end{vmatrix},
\]

\[
\varphi_j^{(k)} = \frac{d^k \varphi_j}{dt^k}.
\]

**Proof.** It follows from (2.9) and (3.3) that \( A_n \) and \( B_n \) can be written in terms of the Wronskians given by (4.2). Furthermore,

\[
A_n \frac{dB_n}{dt} - B_n \frac{dA_n}{dt} = A_{n+1}B_{n-1} \quad \text{and} \quad B_n \frac{dA_{n+1}}{dt} - A_{n+1} \frac{dB_n}{dt} = A_{n+1}B_n
\]

(cf. [13, §6.5.1]) and (4.3), together with (2) yields (4.1).

**Theorem 4.2.** Let \( \omega_0(x) \) be a symmetric positive weight on the real line for which all the moments exist, and let \( \omega(x; t) = \exp(tx^2)\omega_0(x) \), with \( t \in \mathbb{R} \), is a weight such that all the moments of exist. Then the recurrence coefficient \( \beta_n(t) \) satisfies the Volterra, or the Langmuir lattice, equation:

\[
\frac{d\beta_n}{dt} = \beta_n(\beta_{n+1} - \beta_{n-1}).
\]

**Proof.** See, e.g. Van Assche [14, Theorem 2.4].

**Theorem 4.3.** For the generalized higher-order Freud weight (1.1), the associated monic polynomials \( P_n(x) \) satisfy the recurrence relation:

\[
P_{n+1}(x) = xP_n(x) - \beta_n(t; \lambda)P_{n-1}(x), \quad n = 0, 1, 2, \ldots,
\]

with \( P_{-1}(x) = 0 \) and \( P_0(x) = 1 \), where

\[
\beta_{2n}(t; \lambda) = \frac{A_{n+1}(t; \lambda)A_{n-1}(t; \lambda + 1)}{A_n(t; \lambda)A_n(t; \lambda + 1)} = \frac{d}{dt} \ln \frac{A_n(t; \lambda + 1)}{A_n(t; \lambda)}
\]

and

\[
\beta_{2n+1}(t; \lambda) = \frac{A_n(t; \lambda)A_{n+1}(t; \lambda + 1)}{A_{n+1}(t; \lambda)A_n(t; \lambda + 1)} = \frac{d}{dt} \ln \frac{A_{n+1}(t; \lambda)}{A_n(t; \lambda + 1)}.
\]

where \( A_n(t; \lambda) \) is the Wronskian given by (4.2) with

\[
\mu_0(t; \lambda, m) = \frac{1}{m} \sum_{k=1}^{m} \frac{t^{k-1}}{(k-1)!} \Gamma\left(\frac{\lambda + k}{m}\right)
\]

\[
\times \binom{\frac{\lambda + k}{m}, 1; \frac{k}{m}, \frac{k + 1}{m}, \ldots, \frac{m + k - 1}{m}; \left(\frac{t}{m}\right)^m}.\]

**Proof.** It follows from substituting (3.4) into the expression for \( B_n(t; \lambda) \) given in (4.2) that \( B_n = A_n(t; \lambda + 1) \) and then the result immediately follows from (2) and (4.1).

**Remark.** Nonlinear recursive relations

We follow the approach found in [9, §7] whose key results are summarized in [15, Proposition 3.1].
Note that for a given \( m \geq 1 \), we can write
\[
x^{2m} P_n(x) = \sum_{\ell=-m}^{m} C_{n, n+2\ell}^{(2m)} P_{n+2\ell},
\]  
(4.5)

where
\[
C_{n, n+2\ell}^{(2m)} = \frac{1}{h_{n+2\ell}} \int_{-\infty}^{\infty} x^{2m} P_{n+2\ell}(x) P_n(x) \omega(x) \, dx \quad \text{for} \quad \ell = -m, \ldots, m.
\]

Observe that \( C_{n, n+k}^{(2m)} = C_{n+k, n}^{(2m)} = 0 \) for \( |k| \geq 2m + 1 \) and
\[
C_{n, n+2\ell}^{(2m)} = \frac{h_n}{h_{n+2\ell}} C_{n+2\ell, n}^{(2m)} = \frac{1}{\beta_{n+1} \cdots \beta_{n+2\ell}} C_{n, n+2\ell}^{(2m)} \quad \text{for} \quad \ell = 1, \ldots, m.
\]

From the recurrence relation (2.6), it follows
\[
x^2 P_n(x) = P_{n+2} + (\beta_n + \beta_{n+1}) P_n + \beta_{n-1} \beta_n P_{n-2}, \quad n \geq 0.
\]  
(4.6)

In particular, one has \( C_{n, n}^{(2m)} = \beta_n + \beta_{n+1} \), \( C_{n, n-2}^{(2m)} = \beta_{n-1} \beta_n \) and \( C_{n, n+2}^{(2m)} = 1 \). The computation of the coefficients \( \{C_{n-2\ell, n}^{(2m)}\}_{\ell=0}^{m} \) can be derived from the coefficients \( \{C_{n, n}^{(2m)}\}_{\ell=0}^{m} \) as follows:
\[
C_{n-2\ell, n}^{(2m)} = \beta_{n+2} C_{n, n+2\ell}^{(2m)} + (\beta_n + \beta_{n+1}) C_{n, n-2\ell}^{(2m)} + C_{n, n-2\ell}^{(2m)} \quad \ell = 0, \ldots, m,
\]  
(4.7)

which is a direct consequence of (4.5) multiplied by \( x^2 \) and (4.6).

**Proposition 4.4.** The recurrence coefficient \( \beta_n \) for the generalized higher-order Freud weight (1.1) satisfies the discrete equation:
\[
2m V_n^{(2m)} - 2t \beta_n = n + \left( \lambda + \frac{1}{2} \right) [1 - (-1)^n],
\]  
(4.8)

where
\[
V_n^{(2m)} = C_{n, n}^{(2m-2)} + \beta_n C_{n, n}^{(2m-2)}.
\]  
(4.9)

Alternatively, (4.9) can be written as follows:
\[
V_n^{(2m)} = \frac{1}{h_{n-2}} \int_{-\infty}^{\infty} x^{2m-2} P_{n-2}(x) P_n(x) \omega(x) \, dx + \frac{\beta_n}{h_n} \int_{-\infty}^{\infty} x^{2m-2} P_n^2(x) \omega(x) \, dx.
\]

**Proof.** For any monic polynomial sequence \( \{P_n(x)\}_{n \geq 0} \), one can always write
\[
x \frac{dP_n}{dx}(x) = \sum_{j=0}^{n} \rho_{n,j} P_{n-j}(x), \quad \text{for} \quad n \geq 1,
\]
with \( \rho_{n,0} = n \). The assumption that \( \{P_n(x)\}_{n \geq 0} \) is orthogonal with respect to the semi-classical weight \( \omega(x) \) satisfying the differential equation (2.10) with \( \sigma(x) = x \) and \( \tau(x) = 2(tx^2 - mx^{2m} + \lambda + 1) \) gives, using integration by parts,
\[
\rho_{n,j} h_{n-j} = \int_{-\infty}^{\infty} x \frac{dP_n}{dx}(x) P_{n-j}(x) \omega(x) \, dx
\]
\[
= - \int_{-\infty}^{\infty} (\tau(x) P_{n-j}(x) + x \frac{dP_{n-j}}{dx}(x)) P_n(x) \omega(x) \, dx,
\]
where \( h_k = \int_{-\infty}^{\infty} P_k^2(x) \omega(x) \, dx > 0 \). Therefore, \( \rho_{n,j} = 0 \) for any \( j \geq 2m + 1 \) and the symmetry of the weight implies \( \rho_{n,j} = 0 \) for any \( j \) odd. Therefore, we have
\[
x \frac{dP_n}{dx}(x) = \sum_{\ell=0}^{m} \rho_{n,2\ell} P_{n-2\ell}(x), \quad \text{for} \quad n \geq 0.
\]  
(4.10)
Recall (4.6) to write
\[
\frac{1}{P_n} \int_{-\infty}^{\infty} x^2 P_n^2(x) \omega(x) \, dx = (\beta_n + \beta_{n+1})
\]
and hence,
\[
\frac{1}{P_{n-2}} \int_{-\infty}^{\infty} x^2 P_{n-2}(x) P_n(x) \omega(x) \, dx = \beta_n \beta_{n-1},
\]
and consequently, we have
\[
\rho_{n,2\ell} = \begin{cases} 
2m & \text{if } \ell = 0, \\
2m & \text{if } \ell = 1, \\
2m & \text{if } \ell = 2 \leq \ell \leq m - 1, \\
0, & \text{otherwise.}
\end{cases}
\]
Take \(\ell = 0\) in (4.7) and note that \(c_{n,n-2}^{(2m-2)} = c_{n-2,n}^{(2m-2)} \beta_n \beta_{n-1}\) to obtain
\[
c_{n,n}^{(2m)} = \beta_{n+1}(c_{n,n+2}^{(2m-2)} \beta_n + c_{n,n}^{(2m-2)}) + \beta_n(c_{n-2,n}^{(2m-2)} \beta_n + c_{n,n}^{(2m-2)}).
\]
The symmetric orthogonality recurrence relation (2.6) implies that
\[
P_{n+2}(x) P_n(x) = P_{n+1}^2(x) + \beta_n P_{n-1}(x) P_{n+1}(x) - \beta_{n+1} P_n^2(x),
\]
which gives the relation
\[
c_{n,n+2}^{(2m-2)} \beta_n + c_{n,n}^{(2m-2)} = c_{n-1,n+1}^{(2m-2)} \beta_n + c_{n+1,n+1}^{(2m-2)} (4.12)
\]
and consequently, we have
\[
c_{n,n}^{(2m)} = V_{n+1}^{(2m)} + V_{n}^{(2m)} \quad \text{where } V_{n}^{(2m)} = \beta_n(c_{n-2,n}^{(2m-2)} + c_{n,n}^{(2m-2)}).
\]

On the other hand, expressions for the coefficients \(\rho_{n,2}\) can be obtained through a purely algebraic way and therefore expressed recursively. For that, we differentiate with respect to \(x\) the recurrence relation (2.6) and use the structure relation (4.10) to obtain
\[
P_n(x) + \sum_{\ell=0}^{m} \rho_{n,2\ell} P_{n-2\ell}(x) = \frac{dP_{n+1}}{dx}(x) + \beta_n \frac{dP_{n-1}}{dx}(x).
\]
We multiply the latter by \(x\) and use again (4.10) and then (2.6) to obtain a linear combination of terms of \(\{P_n(x)\}_{n \geq 0}\), and this gives
\[
P_{n+1}(x) + \beta_n P_{n-1}(x) = \sum_{\ell=0}^{m} (\rho_{n+1,2\ell} - \rho_{n,2\ell} + \beta_n \rho_{n-1,2\ell-2} - \beta_{n-2\ell+2} \rho_{n,2\ell-2}) P_{n-2\ell+1}(x).
\]
Since the terms are linearly independent, we equate the coefficients of \(P_{n+1}, P_n, \ldots, P_{n-2m-1}\) to obtain
\[
\begin{align*}
\rho_{n,0} &= n, \\
\rho_{n+1,2} - \rho_{n,2} &= 2\beta_n, \\
\rho_{n+1,2\ell} - \rho_{n,2\ell} &= \beta_n \rho_{n-1,2\ell-2} - \beta_{n-2\ell+2} \rho_{n,2\ell-2}, & \text{for } \ell = 2, \ldots, m - 1, \\
\beta_{n-2m} \rho_{n,2m} &= \beta_n \rho_{n-1,2m}, & \text{for } j = m - 1.
\end{align*}
\]
We combine (4.11) with (4.14) to conclude that the first equation (when \(\ell = 0\)) gives
\[
m V_{n+1}^{(2m)} + m V_{n}^{(2m)} - l(\beta_n + \beta_{n+1}) = n + (\lambda + 1),
\]
which implies (4.8).

The expressions for \( V_n^{(2m)} \) can then be obtained recursively using (4.9), (4.7) and (4.13) to write

\[
V_n^{(2m)} = \beta_n \left( V_{n+1}^{(2m-2)} + V_n^{(2m-2)} \right) + (\beta_n + \beta_{n+1}) V_n^{(2m-2)} \\
- \beta_n (\beta_n + \beta_{n+1}) V_n^{(2m-4)} + V_{n+1}^{(2m-4)} \\
+ \beta_n \beta_{n-1} \left( V_{n-2}^{(2m-4)} + V_{n-1}^{(2m-4)} \right) \\
+ \beta_n \beta_{n-1} \beta_{n+1} \beta_n^{2} C_{n-2, n+2}^{(2m-4)}.
\]

Combining (4.12) with (4.13) gives

\[
V_n^{(2m)} - V_{n-1}^{(2m)} = \beta_n \left( V_{n+1}^{(2m-2)} + V_n^{(2m-2)} \right) - \beta_{n-1} \left( V_{n-1}^{(2m-2)} + V_{n-2}^{(2m-2)} \right).
\]

By using the latter relation, we replace the term \((\beta_n + \beta_{n+1}) V_n^{(2m-2)}\) in (4.15) to obtain

\[
V_n^{(2m)} = \beta_n \left( V_{n+1}^{(2m-2)} + V_n^{(2m-2)} + V_{n-1}^{(2m-2)} \right) + \beta_{n+1} V_n^{(2m-2)} \\
- \beta_{n+1} \beta_{n-1} \left( V_{n-2}^{(2m-4)} + V_{n-1}^{(2m-4)} \right) + \beta_{n+2} \beta_{n-1} \beta_n \beta_{n-1} C_{n-2, n+2}^{(2m-4)}.
\]

Consider \( n \to n - 1 \) and \( m \to m - 1 \) in the latter expression, and replace it in (4.16) and this yields

\[
V_n^{(2m)} = \beta_n \left( V_{n+1}^{(2m-2)} + V_n^{(2m-2)} + V_{n-1}^{(2m-2)} \right) + \beta_{n+1} \beta_n V_n^{(2m-4)} + \beta_n \beta_{n+1} V_n^{(2m-4)} \\
- \beta_{n+1} \beta_n \beta_{n-2} \left( V_{n-2}^{(2m-6)} + V_{n-3}^{(2m-6)} \right) + \beta_{n+1} \beta_n \beta_{n-1} \left( \beta_{n+1} \beta_n^2 C_{n-3, n+1}^{(2m-6)} + \beta_n^2 C_{n-2, n+2}^{(2m-6)} \right).
\]

If we replace the term \( V_{n-2}^{(2m-4)} \) by the corresponding expression given by the latter relation and successively continuing the process, then one can deduce the following expressions for \( V_n^{(2m)} \) as follows:

\[
V_n^{(2)} = \beta_n, \\
V_n^{(4)} = V_n^{(2)} \left( V_n^{(2)} + V_n^{(2)} + V_n^{(2)} \right), \\
V_n^{(6)} = V_n^{(2)} \left( V_n^{(4)} + V_n^{(4)} + V_n^{(4)} + V_n^{(2)} V_n^{(2)} \right).
\]

For higher orders, we compute the coefficients \( V_n^{(2m)} \) recursively as stated below. We opted for not giving the expressions in terms of \( \beta_n \) since those are rather long. For \( m = 4, 5 \), we have

\[
V_n^{(8)} = V_n^{(2)} \left( V_n^{(6)} + V_n^{(6)} + V_n^{(6)} \right) + V_n^{(4)} V_n^{(2)} \left( V_n^{(2)} + V_n^{(2)} \right) \\
+ V_n^{(2)} V_{n+1}^{(2)} V_{n-1}^{(2)} \left( V_n^{(2)} + V_n^{(2)} \right)
\]

and

\[
V_n^{(10)} = V_n^{(2)} \left( V_n^{(8)} + V_n^{(8)} + V_n^{(8)} \right) + V_n^{(6)} V_n^{(2)} V_n^{(2)} \left( V_n^{(2)} + V_n^{(2)} \right) \\
+ V_n^{(2)} V_{n+1}^{(2)} V_{n-1}^{(2)} \left( V_n^{(4)} + V_n^{(4)} \right) + V_n^{(2)} V_{n+1}^{(2)} V_{n-1}^{(2)} \\
\times \left\{ V_n^{(2)} + V_n^{(2)} V_n^{(2)} + \left( V_n^{(2)} + V_n^{(2)} \right) V_n^{(2)} + V_n^{(2)} V_n^{(2)} \right\}.
\]

Remark 4.5.

(i) For the case when \( \lambda = -\frac{1}{2} \), proposition 4.4 was proved by Benassi & Moro [16], using a result in [17]. Although it is straightforward to modify the proof presented therein for the
case when $\lambda \neq -\frac{1}{2}$, we hereby present an alternative approach purely depending on the structure relation of the semi-classical polynomials.

(ii) Equations such as (4.8) for recurrence relation coefficients are sometimes known as Laguerre-Freud equations [18,19]; see also [6,20–23].

(iii) When $m = 2$, the discrete equation is

$$4\beta_n(\beta_{n-1} + \beta_n + \beta_{n+1}) - 2t\beta_n = n + \left(\lambda + \frac{1}{2}\right)[1 - (-1)^n], \quad (4.17)$$

which is $dP_t$, and when $m = 3$, the discrete equation is

$$6\beta_n(\beta_{n-2} + \beta_{n-1} + \beta_{n+1}) - 2t\beta_n = n + (\lambda + \frac{1}{2})[1 - (-1)^n], \quad (4.18)$$

which is a special case of $dP^{(2)}_t$, the second member of the discrete Painlevé I hierarchy. For further information about the discrete Painlevé I hierarchy, see [24,25]. Equations (4.17) and (4.18) with $t = 0$ were derived by Freud [18]; see also [5,14]. Further, equations (4.17) and (4.18) with $\lambda = -\frac{1}{2}$ are also known as ‘string equations’ and arise in important physical applications such as two-dimensional quantum gravity, cf. [26–31].

(b) Asymptotics for the recurrence coefficients as $n \to \infty$

In 1976, Freud [18] conjectured that the asymptotic behaviour of recurrence coefficients $\beta_n$ in the recurrence relation (2.6) satisfied by monic polynomials $\{P_n(x)\}_{n \geq 0}$ orthogonal with respect to the weight

$$\omega(x) = |x|^\rho \exp(-|x|^m),$$

with $x \in \mathbb{R}$, $\rho > -1$, $m > 0$ could be described by

$$\lim_{n \to \infty} \frac{\beta_n}{n^{2/m}} = \left[\frac{\Gamma((1/2)m)\Gamma(1+(1/2)m)}{\Gamma(m+1)}\right]^{2/m}. \quad (4.19)$$

Freud stated the conjecture for orthonormal polynomials, proved it for $m = 2, 4, 6$ and also showed that (4.19) is valid whenever the limit on the left-hand side exists. Magnus [5] proved Freud’s conjecture for the case when $m$ is an even positive integer and also for weights

$$w(x) = \exp(-Q(x)),$$

where $Q(x)$ is an even degree polynomial with positive leading coefficient. We refer the readers to [32, §4.18] for a detailed history of solutions to Freud’s conjecture up to that point. The conjecture was settled by Lubinsky et al. in [33] as a special case of a more general result for recursion coefficients of exponential weights, see also [34]. In [35], Lubinsky & Saff introduced the class of very smooth Freud weights of order $\alpha$ with conditions on $Q$ that are satisfied when $Q$ is of the form $x^\alpha$, $\alpha > 0$. Associated with each weight in this class, one can define $a_n$ as the unique, positive root of the equation (cf. [33, p. 67] and the references therein)

$$n = \frac{2}{\pi} \int_0^1 a_n s Q'(a_n s) \sqrt{1-s^2} \, ds. \quad (4.20)$$

**Theorem 4.6.** Consider the generalized higher-order Freud weight (1.1). Then the recurrence coefficients $\beta_n$ associated with this weight satisfy

$$\lim_{n \to \infty} \frac{\beta_n(t; \lambda)}{n^{1/m}} = \frac{1}{4} \left(\frac{(m-1)!}{(1/2)_m}\right)^{1/m}.$$ 

**Proof.** Let $Q(x) = \frac{1}{2}x^{2m}$, then evaluating (4.20) yields $n = a_n^{2m}(1/2)_m/(m-1)!$ and the result is a straightforward consequence of the more general result in [33, Theorem 2.3] taking $W(x) = \exp(-Q(x))$, $w = |x|^{\lambda+1/2}$, $P(x) = \frac{1}{2}tx^2$ and $\Psi(x) = 1$. ■
Remark 4.7. Taking \( m = 2 \) in theorem 4.6, we recover [1, Corollary 4.2 (ii)] for the recurrence coefficients associated with the generalized quartic Freud weight \(|x|^{2\lambda+1} \exp(tx^2 - x^4)\), which satisfy
\[
\lim_{n \to \infty} \frac{\beta_n(t; \lambda)}{\sqrt{n}} = \frac{1}{\sqrt{12}},
\]
while, for \( m = 3 \), the recurrence coefficients associated with the generalized sextic Freud weight \(|x|^{2\lambda+1} \exp(tx^2 - x^6)\) satisfy
\[
\lim_{n \to \infty} \frac{\beta_n(t; \lambda)}{\sqrt{n}} = \frac{1}{\sqrt{60}},
\]
as shown in [3, Corollary 4.8].

5. Generalized higher-order Freud polynomials

(a) Differential equations

The second-order differential equations satisfied by generalized higher-order Freud polynomials can be obtained by using ladder operators as was done for the special cases \( m = 2 \) and \( m = 3 \) in [4, Theorem 6] and [2, Theorem 4.3], respectively. An alternative approach is given by Maroni in [8,9].

Proposition 5.1. The polynomial sequence \( \{P_n(x)\}_{n \geq 0} \) orthogonal with respect to the generalized higher-order Freud weight (1.1) is a solution to the differential equation:
\[
J(x; n) \frac{d^2 P_{n+1}}{dx^2} + K(x; n) \frac{d P_{n+1}}{dx} + L(x; n) P_{n+1}(x) = 0,
\]
where
\[
J(x; n) = xD_{n+1}(x), \\
K(x; n) = C_0(x)D_{n+1}(x) - x \frac{d D_{n+1}}{dx}(x) + D_{n+1}(x), \\
L(x; n) = W \left( \frac{1}{2}(C_{n+1}(x) - C_0(x)), D_{n+1}(x) \right) - D_{n+1}(x) \sum_{j=0}^{n} \frac{1}{\beta_j} D_j(x),
\]
with
\[
C_{n+1}(x) = -C_n(x) + \frac{2x}{\beta_n} D_n(x)
\]
and
\[
D_{n+1}(x) = -x + \frac{\beta_n}{\beta_{n-1}} D_{n-1}(x) + \frac{x^2}{\beta_n} D_n(x) - xC_n(x),
\]
subject to the initial conditions \( C_0(x) = -1 + 2(tx^2 - mx^{2m} + \lambda + 1) \), \( D_{-1}(x) = 0 \) and
\[
D_0(x) = 2x \left\{ m \sum_{j=1}^{m} \mu_{2j-2}(t, \lambda)x^{2m-2j} - t \mu_0(t, \lambda) \right\}.
\]

(b) Mixed recurrence relations

We first consider the connection formula between the corresponding sequences of generalized higher-order Freud orthogonal polynomials in the framework of Christoffel transformations when the measure is modified by multiplying with a polynomial. In our case, the measure is modified by a quadratic factor.
Theorem 5.2. Let \( \{p_n(x;\lambda)\}_{n \geq 0} \) be the sequence of monic generalized higher-order Freud polynomials orthogonal with respect to the weight (1.1), then, for \( m, n \) fixed,
\[
xP_{2n}(x;\lambda + 1) = P_{2n+1}(x;\lambda) \tag{5.1a}
\]
and
\[
x^2P_{2n-1}(x;\lambda + 1) = xP_{2n}(x;\lambda) - \left[ \beta_{2n}(\lambda) + \frac{P_{2n+1}(0;\lambda)}{P'_{2n-1}(0;\lambda)} \right] P_{2n-1}(x;\lambda). \tag{5.1b}
\]

Proof. Let \( p_n(x;\lambda + 1) \) be the polynomials associated with the even weight function:
\[
\omega(x;\lambda + 1) = |x|^{2\lambda+3} \exp(tx^2 - x^{2m}) = x^2\omega(x;\lambda), \quad m = 2, 3, \ldots
\]
The factor \( x^2 \) by which the weight \( \omega(x;\lambda) \) is modified has a double zero at the origin, and therefore, Christoffel’s formula (cf. [36, Theorem 2.5], [37, Theorem 2.7.1]), applied to the monic polynomials \( p_n(x;\lambda + 1) \), is
\[
x^2p_n(x;\lambda + 1) = \frac{1}{p'_n(0;\lambda)p_{n+1}(0;\lambda) - p'_n(0;\lambda)p_{n+1}(0;\lambda)} \begin{vmatrix}
p_n(x;\lambda) & p_{n+1}(x;\lambda) & p_{n+2}(x;\lambda) \\
p'_n(0;\lambda) & p'_{n+1}(0;\lambda) & p'_{n+2}(0;\lambda) \\
p'_n(0;\lambda) & 0 & 0
\end{vmatrix},
\]
Since the weight \( \omega(x;\lambda) \) is even, we have that \( p_{2n+1}(0;\lambda) = p'_{2n}(0;\lambda) \), while \( p_{2n}(0;\lambda) \neq 0 \) and \( p'_{2n+1}(0;\lambda) \neq 0 \), and hence,
\[
x^2p_n(x;\lambda + 1) = \frac{-1}{p'_n(0;\lambda)p_{n+1}(0;\lambda)} \begin{vmatrix}
p_n(x;\lambda) & p_{n+1}(x;\lambda) & p_{n+2}(x;\lambda) \\
0 & p'_{n+1}(0;\lambda) & p'_{n+2}(0;\lambda) \\
p'_n(0;\lambda) & 0 & 0
\end{vmatrix},
\]
for \( n \) odd, while, for \( n \) even,
\[
x^2p_n(x;\lambda + 1) = \frac{1}{p'_n(0;\lambda)p_{n+1}(0;\lambda)} \begin{vmatrix}
p_n(x;\lambda) & p_{n+1}(x;\lambda) & p_{n+2}(x;\lambda) \\
p'_n(0;\lambda) & 0 & 0 \\
p'_n(0;\lambda) & p'_{n+1}(0;\lambda) & p'_{n+2}(0;\lambda)
\end{vmatrix}.
\]
This yields
\[
x^2p_n(x;\lambda + 1) = p_{n+2}(x;\lambda) - a_n p_n(x;\lambda), \tag{5.2}
\]
where
\[
a_n = \begin{cases} 
\frac{p_{n+2}(0;\lambda)}{p'_n(0;\lambda)}, & \text{for } n \text{ even}, \\
\frac{p'_{n+2}(0;\lambda)}{p'_n(0;\lambda)}, & \text{for } n \text{ odd}.
\end{cases}
\]
By using the three-term recurrence relation (2.6) to eliminate \( p_{n+2}(x;\lambda) \) in (5.2), we obtain
\[
x^2p_n(x;\lambda + 1) = xp_{n+1}(x;\lambda) - (\beta_{n+1}(\lambda) + a_n)p_n(x;\lambda).
\]
It follows from (2.7) that, for \( n \) even, \( \beta_{n+1}(\lambda) + a_n = 0 \), and the result follows.

Theorem 5.3. For a fixed \( m = 2, 3, \ldots \), let \( \{p_n(x;\lambda)\}_{n \geq 0} \) be the sequence of monic generalized higher-order Freud polynomials orthogonal with respect to the weight (1.1). Then, for \( n \) fixed,
\[
p_{2n+1}(x;\lambda) = p_{2n+1}(x;\lambda + 1) + \beta_{2n}(\lambda + 1)p_{2n-1}(x;\lambda + 1) \tag{5.3a}
\]
and
\[
p_{2n}(x;\lambda) = p_{2n}(x;\lambda + 1) - \frac{\beta_{2n}(\lambda)\beta_{2n-1}(\lambda + 1)p'_{2n-1}(0;\lambda)}{p'_{2n+1}(0;\lambda)}p_{2n-2}(x;\lambda + 1). \tag{5.3b}
\]
Proof. Substitute (5.1a) into the three-term recurrence relation
\[ P_{2n+1}(x; \lambda) = xP_{2n}(x; \lambda) - \beta_{2n}(\lambda)P_{2n-1}(x; \lambda), \]  
(5.4)
to eliminate \( P_{2n+1}(x; \lambda) \) and obtain
\[ xP_{2n}(x; \lambda + 1) = xP_{2n}(x; \lambda) - \beta_{2n}(\lambda)P_{2n-1}(x; \lambda). \]
Let \( a_{2n} = P'_{2n+1}(0; \lambda)/P'_{2n-1}(0; \lambda) \). Substitute (5.1b) into (5.4) to eliminate \( P_{2n-1}(x; \lambda) \) and obtain
\[ xP_{2n}(x; \lambda + 1) = xP_{2n}(x; \lambda) - \frac{\beta_{2n}(\lambda)}{\beta_{2n}(\lambda) + a_{2n}} (xP_{2n}(x; \lambda) - x^2P_{2n-1}(x; \lambda + 1)). \]  
(5.5)

Simplification and rearrangement of terms in (5.5) yields
\[
\left( 1 - \frac{\beta_{2n}(\lambda)}{\beta_{2n}(\lambda) + a_{2n}} \right) P_{2n}(x; \lambda) = P_{2n}(x; \lambda + 1) - \frac{\beta_{2n}(\lambda)}{\beta_{2n}(\lambda) + a_{2n}} xP_{2n-1}(x; \lambda + 1),
\]
then, by using the three-term recurrence relation to eliminate \( xP_{2n}(x; \lambda + 1) \), we obtain
\[
\left( 1 - \frac{\beta_{2n}(\lambda)}{\beta_{2n}(\lambda) + a_{2n}} \right) P_{2n}(x; \lambda) = \left( 1 - \frac{\beta_{2n}(\lambda)}{\beta_{2n}(\lambda) + a_{2n}} \right) P_{2n}(x; \lambda + 1)
+ \frac{\beta_{2n}(\lambda)}{\beta_{2n}(\lambda) + a_{2n}} \beta_{2n-1}(\lambda + 1)P_{2n-2}(x; \lambda + 1),
\]
which simplifies to (5.3b). Substituting (5.1a) into the three-term recurrence relation
\[ P_{2n+1}(x; \lambda + 1) = xP_{2n}(x; \lambda + 1) - \beta_{2n}(\lambda + 1)P_{2n-1}(x; \lambda + 1), \]
yields (5.3a). \(\blacksquare\)

Theorem 5.3 gives the connection formula between the corresponding sequences of generalized higher-order Freud polynomials in the framework of Geronimus transformations, the inverse of a Christoffel transformation. For more on quadratic Geronimus transformations of a weight \( \omega(x) \), where \( (x^2 - c)v(x) = \omega(x) \), see [38]. The generalized Christoffel formula, where the weight is modified by a rational function, often referred to as an Uvarov transformation, can also be considered as the Darboux transformation of an integrable system (cf. [37,39]) and is considered in the framework of Gaussian quadrature rules in [40,41].

(c) Quasi-orthogonality for \( \lambda \in (-2, -1) \)

Theorem 5.3 yields the quasi-orthogonality of generalized higher-order Freud polynomials for \(-2 < \lambda < -1\).

Theorem 5.4. Suppose \(-2 < \lambda < -1\). For each fixed \( m = 2, 3, \ldots \), the generalized higher-order Freud polynomial \( P_n(x; \lambda) \) is quasi-orthogonal of order 2 on \( \mathbb{R} \) with respect to the weight
\[ |x|^{2\lambda+3} \exp(tx^2 - x^{2m}), \quad t \in \mathbb{R}. \]

Proof. Suppose \(-2 < \lambda < -1\), then \( \lambda + 1 > -1 \). When \( n \) is even, we have from (5.3b) that
\[ \int_{-\infty}^{\infty} x^k P_n(x; \lambda) |x|^{2\lambda+3} \exp(tx^2 - x^{2m}) \, dx \]
\[ = \int_{-\infty}^{\infty} x^k P_n(x; \lambda + 1) |x|^{2\lambda+3} \exp(tx^2 - x^{2m}) \, dx \]
\[ - \frac{\beta_n(\lambda)\beta_{n-1}(\lambda + 1)P'_{n-1}(0; \lambda)}{P'_{n+1}(0; \lambda)} \]
\[ \times \int_{-\infty}^{\infty} x^k P_{n-2}(x; \lambda + 1) |x|^{2\lambda+3} \exp(tx^2 - x^{2m}) \, dx, \]  
(5.6)
while, for \( n \) is odd, it follows from (5.3a) that
\[
\int_{-\infty}^{\infty} x^k P_n(x; \lambda)|x|^{2k+3} \exp \left( t x^2 - x^{2n} \right) \, dx = \int_{-\infty}^{\infty} x^k P_n(x; \lambda+1)|x|^{2k+3} \exp \left( t x^2 - x^{2n} \right) \, dx
\]
\[
+ \beta_n(\lambda+1) \int_{-\infty}^{\infty} x^k P_{n-2}(x; \lambda+1)|x|^{2k+3} \exp \left( t x^2 - x^{2n} \right) \, dx. \tag{5.7}
\]
Since \( \lambda + 1 > -1 \), it follows from the orthogonality of the generalized higher-order Freud polynomials that
\[
\int_{-\infty}^{\infty} x^k P_n(x; \lambda+1)|x|^{2k+3} \exp \left( t x^2 - x^{2n} \right) = 0 \quad \text{for } k = 0, \ldots, n-1,
\]
and we see that all the integrals on the right-hand side of (5.6) and (5.7) are equal to zero for \( k = 0, \ldots, n-3 \). \( \blacksquare \)

6. Zeros of generalized higher-order Freud polynomials

(a) Asymptotic zero distribution

The asymptotic behaviour of the recurrence coefficients of generalized higher-order Freud polynomials orthogonal with respect to (1.1), satisfying Freud’s conjecture, given by (4.19), is independent of the values of \( t \) and \( \lambda \). The asymptotic behaviour implies that the recurrence coefficients are regularly varying, irrespective of \( t \) and \( \lambda \). To consider the asymptotic distribution of the zeros of generalized higher-order Freud polynomials orthogonal with respect to the weight (1.1) as \( n \to \infty \), we use an appropriate scaling and apply the property of regular variation as detailed in [42].

**Theorem 6.1.** Let \( \phi(n) = n^{1/(2m)} \) and assume that \( n, N \) tend to infinity in such a way that the ratio \( n/N \to \ell \). Then, for the sequence of scaled monic polynomials \( P_{n,N}(x) = (\phi(n))^{-n} P_n(\phi(N)x) \) associated with the generalized higher-order Freud weight (1.1), the asymptotic zero distribution, as \( n \to \infty \), has density
\[
a_m(\ell) = \frac{2m}{c \pi (2m-1)} \left( 1 - \frac{x^2}{c^2} \right)^{1/2} {}_2F_1 \left( 1, 1 - m; \frac{3}{2} - m; \frac{x^2}{c^2} \right), \tag{6.1}
\]
where
\[
c = 2a \ell^{1/(2m)} \quad \text{with} \quad a = \frac{1}{2} \left( \frac{(m-1)!}{(1/2)_m} \right)^{1/(2m)},
\]
defined on the interval \((-2a \ell^{1/(2m)}, 2a \ell^{1/(2m)})\).

**Proof.** The scaled monic polynomials \( P_{n,N}(x) = (\phi(n))^{-n} P_n(\phi(N)x) \) associated with the generalized higher-order Freud weight (1.1) have recurrence coefficient \( \beta_{n,N}(t; \lambda) = \beta_n(t; \lambda)/(\phi(n))^2 \). Since \( \phi: \mathbb{R}^+ \to \mathbb{R}^+ \) and, for every \( \ell > 0 \), we have
\[
\lim_{x \to \infty} \frac{\phi(x\ell)}{\phi(x)} = \ell^{1/(2m)},
\]
where \( \phi \) is regularly varying at infinity with exponent of variation \( 1/2m \) (cf. [43]). Since it follows from (4.6) that
\[
\lim_{n/N \to \ell} \sqrt{n} \beta_{n,N}(t; \lambda) = \lim_{n/N \to \ell} \sqrt{n} \beta_n(t; \lambda) \phi(n)/(\phi(N))^2 = a \ell^{1/(2m)},
\]
the recurrence coefficients \( \beta_{n,N}(t; \lambda) \) are said to be regularly varying at infinity with index \( 1/2m \) (cf. [42, Section 4.5]). From the property of regular variation, using [42, Theorem 1.4], it follows
that the asymptotic zero distribution has density

\[
\frac{1}{\pi \ell} \int_0^\ell s^{-1/(2m)} \left( 2a - xs^{-1/(2m)} \right)^{-1/2} \left( 2a + xs^{-1/(2m)} \right)^{-1/2} ds
\]

\[
= \frac{m}{a \pi \ell} \int_0^{1/(2m)} y^{2m-2} \left( 1 - \left( \frac{x}{2ay} \right)^2 \right)^{-1/2} dy
\]

\[
= \frac{m}{a \pi \ell} \sum_{k=0}^\infty \frac{(1/2)_k}{k!} \left( \frac{x}{2a} \right)^{2k} \int_0^{1/(2m)} y^{2m-2k-2} dy
\]

\[
= \frac{m}{a \pi \ell^{1/(2m)}} \sum_{k=0}^\infty \frac{(1/2)_k}{k!} \frac{1}{2m-2k-1} \left( \frac{x}{2ae^{1/(2m)}} \right)^{2k}
\]

\[
= \frac{m}{a \pi (2m-1) \ell^{1/(2m)}} \sum_{k=0}^\infty \frac{(1/2)_k((1/2) - m)_k}{((3/2) - m)_k k!} \left( \frac{x}{2ae^{1/(2m)}} \right)^{2k}
\]

\[
= \frac{m}{a \pi (2m-1) \ell^{1/(2m)}} \frac{\Gamma(1/2, -m)}{\Gamma(3/2, -m)} \left( \frac{x}{2ae^{1/(2m)}} \right)^2.
\]

**Figure 1.** The zeros of $P_{nN}(x)$ (red) for $\lambda = 0.5$, $t = 1$, $m = 3$, $n = N = 10$ and $\ell = 1$ with the corresponding limiting distribution (6.1) (blue) and endpoints $(-2a, 0)$ and $(2a, 0)$ (green).

**Remark 6.2.** Note that the formula on [42, p. 189, line 22] should be $(1/t) \int_0^t (1/s^\lambda) \omega'_s [-2a, b+2a] (xs^{-\lambda}) ds$.

**(b) Bounds for the extreme zeros**

From the three-term recurrence relation (4.4), we obtain bounds for the extreme zeros of monic generalized higher-order Freud polynomials.
Figure 2. The limiting distribution of the zeros $a_{3}(\ell)$ for $\ell = 0.5$ (green), $\ell = 1$ (blue) and $\ell = 2$ (red).

**Theorem 6.3.** For each $n = 2, 3, \ldots$, the largest zero, $x_{1,n}$, of monic generalized higher-order Freud polynomials $P_{n}(x)$ orthogonal with respect to the weight (1.1), satisfies

$$0 < x_{1,n} < \max_{1 \leq k \leq n-1} \sqrt{c_{n}\beta_{k}(t;\lambda)},$$

where $c_{n} = 4 \cos^{2}\left(\frac{\pi}{n+1}\right) + \varepsilon$, $\varepsilon > 0$.

*Proof.* The upper bound for the largest zero $x_{1,n}$ follows by applying [44, Theorem 2 and 3], based on the Wall–Wetzel Theorem to the three-term recurrence relation (4.4). \[\square\]

(c) **Monotonicity of the zeros**

**Theorem 6.4.** Consider $0 < x_{[n/2],n} < \cdots < x_{2,n} < x_{1,n}$, the positive zeros of monic orthogonal polynomials $P_{n}(x)$ with respect to the generalized higher-order Freud weight (1.1), where $[k]$ denotes the largest integer less than or equal to $k$. Then, for $\lambda > -1$, $t \in \mathbb{R}$ and for a fixed value of $v$, $v \in \{1, 2, \ldots, \lfloor n/2 \rfloor\}$, the $v$-th zero $x_{n,v}$ increases when (i) $\lambda$ increases; and (ii) $t$ increases.

*Proof.* This follows from [2, Lemma 4.5], taking $C(x) = x$, $D(x) = x^2$, $\rho = 2\lambda + 1$ and $\omega_{0}(x) = \exp(-x^{2m})$. \[\square\]

(d) **Interlacing of the zeros**

Next, for fixed $\lambda > -1$, $t \in \mathbb{R}$ and $k \in (0, 1]$, we consider the relative positioning of the zeros of the monic generalized higher-order Freud polynomials $(P_{n}(x;\lambda))$ orthogonal with respect to the weight (1.1), and the zeros of $(P_{n}(x;\lambda + k))$, $k \in (0, 1]$, orthogonal with respect to the weight $\omega(x; t, \lambda) = \vert x \vert^{2k+2\lambda+1} \exp(tx^2 - x^{2m})$, $m = 2, 3, \ldots$.

The zeros of monic generalized higher-order Freud polynomials $(P_{n}(x;\lambda))$ orthogonal with respect to the symmetric weight (1.1) are symmetric around the origin. We denote the positive zeros of $P_{2n}(x;\lambda)$ by

$$0 < x_{n,2n}^{\lambda} < x_{n-1,2n}^{\lambda} < \cdots < x_{2,2n}^{\lambda} < x_{1,2n}^{\lambda},$$

and the positive zeros of $P_{2n+1}(x;\lambda)$ by

$$0 < x_{n,2n+1}^{\lambda} < x_{n-1,2n+1}^{\lambda} < \cdots < x_{2,2n+1}^{\lambda} < x_{1,2n+1}^{\lambda},$$

noting that $x_{n+1,2n+1} = 0$. 
The zeros of Equations (6.4), (6.7) and (6.9) yield (6.2). To prove (6.3), we note that by (5.1) and \( \ell \) order Freud polynomials monotonically increase as the parameter \( \lambda \) and, for \( m \) generalized higher-order Freud weight (1.1) for Freud orthogonal polynomials are interlacing, i.e.

\[
0 < x_{n,2n}^1 < x_{n,2n-1}^1 < x_{n,2n-2}^1 < \cdots < x_{1,2n-1}^1 < x_{1,2n}^1
\]

(6.4)

and

\[
0 < x_{n,2n}^0 < x_{n,2n+1}^0 < x_{n,2n-1}^0 < x_{n,2n-2}^0 < \cdots < x_{2,2n+1}^0 < x_{2,2n}^0 < x_{1,2n+1}^0
\]

(6.5)

On the other hand, we proved in theorem 6.4 that the positive zeros of generalized higher-order Freud polynomials monotonically increase as the parameter \( \lambda \) increases. This implies that, for each fixed \( \ell \in \{1, 2, \ldots, n\} \) and \( k \in (0, 1) \),

\[
x_{\ell,2n}^k < x_{\ell,2n}^{k+1} < x_{\ell,2n}^{k+2}
\]

(6.6)

and

\[
x_{\ell,2n-1}^k < x_{\ell,2n-1}^{k+1}
\]

(6.7)

Next, we prove that the zeros of \( P_{2n}(x; \lambda) \) interlace with those of \( P_{2n-1}(x; \lambda + 1) \). From (5.1b),

\[
P_{2n-1}(x; \lambda + 1) = \frac{x^2 P_{2n}(x; \lambda) - (\beta_{2n}(\lambda) + P'_{2n}(0; \lambda)/P'_{2n-1}(0; \lambda))P_{2n-1}(x; \lambda)}{x^2}.
\]

(6.8)

By evaluating (6.8) at consecutive zeros \( x_{\ell} = x_{\ell}^{(\ell)} \) and \( x_{\ell+1} = x_{\ell+1}^{(\ell)} \), \( \ell = 1, 2, \ldots, n - 1 \), of \( P_{2n}(x; \lambda) \), we obtain

\[
P_{2n-1}(x_{\ell}; \lambda + 1)P_{2n-1}(x_{\ell+1}; \lambda + 1)
\]

\[
= \frac{(\beta_{2n}(\lambda) + P'_{2n}(0; \lambda)/P'_{2n-1}(0; \lambda))^2 P_{2n-1}(x_{\ell}; \lambda)P_{2n-1}(x_{\ell+1}; \lambda)}{x^2 x_{\ell+1}^2} < 0,
\]

since the zeros of \( P_{2n}(x; \lambda) \) and \( P_{2n-1}(x; \lambda) \) separate each other. So there is at least one positive zero of \( P_{2n}(x; \lambda + 1) \) in the interval \( (x_{\ell}, x_{\ell+1}) \) for each \( \ell = 1, 2, \ldots, n - 1 \) since there are exactly \( n \) positive zeros, and this implies that

\[
0 < x_{n,2n}^1 < x_{n,2n-1}^1 < x_{n,2n-2}^1 < \cdots < x_{2,2n-1}^1 < x_{2,2n}^1 < x_{1,2n+1}^1 < x_{1,2n}^1.
\]

(6.9)

Equations (6.4), (6.7) and (6.9) yield (6.2). To prove (6.3), we note that by (5.1a), the \( n \) positive zeros of \( P_{2n}(x; \lambda + 1) \) and \( P_{2n+1}(x; \lambda) \) coincide, i.e. \( x_{\ell,2n}^{+1} = x_{\ell,2n+1}^{+1} \) for \( \ell \in \{1, 2, \ldots, n\} \), and the result follows using (6.5) and (6.6).

Figure 3 shows the interlacing of the zeros of polynomials orthogonal with respect to the generalized higher-order Freud weight (1.1) for \( m = 3 \) as described in (6.4) of theorem 6.5.
Figure 4. The zeros of $P_8(x; \lambda)$ (green), $P_8(x; \lambda + 0.5)$ (red), $xP_8(x; \lambda + 1)$ (blue) and $P_8(x; \lambda)$ (blue) for $\lambda = 1.5$ and $t = 2.3$.

Figure 4 illustrates the interlacing of the zeros of polynomials orthogonal with respect to the generalized higher-order Freud weight (1.1) for $m = 3$ as described in (6.5) of theorem 6.5.

7. Quadratic decomposition of the generalized higher-order Freud weight

We apply known results [45, Chapter 1, theorem 9.1] on the quadratic decomposition of any symmetric polynomials to this particular case of generalized higher-order Freud weights. Precisely, if

$$P_{2n}(x; t, \lambda) = B_n(x^2; t, \lambda) \quad \text{and} \quad P_{2n+1}(x; t, \lambda) = xR_n(x^2; t, \lambda),$$

for all $n \geq 0$, then from the recurrence relation (2.6), we have

$$B_{n+1}(x; t, \lambda) = R_{n+1}(x; t, \lambda) + \beta_{2n+2}R_n(x; t, \lambda)$$

and

$$xR_n(x; t, \lambda) = B_{n+1}(x; t, \lambda) + \beta_{2n+1}B_n(x; t, \lambda),$$

and this gives second-order recurrence relations for both $\{B_n\}_{n \geq 0}$ and $\{R_n\}_{n \geq 0}$ as follows:

$$B_{n+1}(x; t, \lambda) = (x - \beta_{2n} - \beta_{2n+1})B_n(x; t, \lambda) - \beta_{2n-1}\beta_{2n}B_{n-1}(x; t, \lambda), \quad n \geq 1,$$

$$B_1(x; t, \lambda) = x - \beta_1, \quad B_0(x; t, \lambda) = 1,$$

and

$$R_{n+1}(x; t, \lambda) = (x - \beta_{2n+2} - \beta_{2n+1})R_n(x; t, \lambda) - \beta_{2n+1}\beta_{2n}R_{n-1}(x; t, \lambda),$$

$$R_1(x; t, \lambda) = x - \beta_1 - \beta_2, \quad R_0(x; t, \lambda) = 1.$$

Furthermore, $\{B_n\}_{n \geq 0}$ and $\{R_n\}_{n \geq 0}$ satisfy the orthogonality relations:

$$\int_0^\infty B_k(x; t, \lambda)B_n(x; t, \lambda)x^{\lambda} \exp(tx - x^m) \, dx = h_{2n}(t, \lambda)\delta_{n,k},$$

and

$$\int_0^\infty R_k(x; t, \lambda)R_n(x; t, \lambda)x^{\lambda+1} \exp(tx - x^m) \, dx = h_{2n+1}(t, \lambda)\delta_{n,k}, \quad n, k \geq 0.$$
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