ONE-DIMENSIONAL STATISTICAL MECHANICS
for IDENTICAL PARTICLES:
the CALOGERO and ANYON CASES

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Abstract: The thermodynamics of particles with intermediate statistics interpolating between Bose and Fermi statistics is addressed in the simple case where there is one quantum number per particle. Such systems are essentially one-dimensional. As an illustration, one considers the anyon model restricted to the lowest Landau level of a strong magnetic field at low temperature, the generalization of this model to several particles species, and the one dimensional Calogero model. One reviews a unified algorithm to compute the statistical mechanics of these systems. It is pointed out that Haldane’s generalization of the Pauli principle can be deduced from the anyon model in a strong magnetic field at low temperature.

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1 Introduction

Identical particles with statistics continuously interpolating between Bose-Einstein and Fermi-Dirac statistics can exist in one and two dimensions [1]. In one dimension, these statistics are described by the Calogero model [2], in the sense that if the particles are not classically free, their asymptotic properties are however not affected by the interaction, up to a permutation. In two dimensions, they are described by the anyon model: in the singular gauge, free anyonic eigenstates pick up a phase $\pi\alpha$ when any pair of anyons are exchanged [3], where $\alpha$ is the statistical parameter. The spectrum of the Calogero model in a harmonic well is quite similar to those of the anyon model in a harmonic well projected on the lowest Landau level (LLL) of an external magnetic field [4].

It is now widely accepted that particles with intermediate (fractional) statistics should play a role in the fractional quantum Hall effect [5], a strong magnetic field, low temperature effect observed in certain bidimensional conductors. When an integer fraction $1/m$ of the LLL is filled by the electrons, the Hall conductivity exhibits a plateau at a value $1/m$ in unit of $\hbar c/e^2$, and the longitudinal resistivity vanishes. The trial eigenstates proposed by Laughlin for the groundstate and the excited states are sustained by numerical few-body computations [6]. On the Hall plateaux, the Coulomb repulsion lifts the Landau degeneracy and leads to a nondegenerate groundstate with a gap, which explains the absence of dissipation in the longitudinal transport. The excitations are described as quasiparticles or quasiholes localised on some defects of the sample. A Berry phase calculation shows that they obey intermediate statistics, i.e. they are anyons. More recently, these excitations have been shown to obey a generalisation of the Pauli principle first proposed by Haldane [7, 8]. The general relation between the anyon model and this generalised exclusion principle remains to be clarified. It has however been shown that it is obeyed by anyons in a strong magnetic field at low temperature [9, 10].

On the one hand, the statistical mechanics of an anyon gas in a strong magnetic field at low temperature has been analyzed [11] in a situation where the singular flux tubes carried by the anyons are anti parallel to the magnetic field. In this particular screening regime,
the Hilbert space is restricted to the Landau groundstate, with effectively one quantum number per particle, its orbital angular momentum. There exists a critical value of the filling factor, where the screening is complete, and for which the system is incompressible (nondegenerate with a gap), leaving a Bose condensate.

On the other hand, the statistical mechanics of the Calogero model has been recently analyzed by various ways [12], [13].

The present letter presents a thorough discussion of the anyon model in a strong magnetic field, and displays its intimate relation with the other concepts of intermediate statistics.

In section 2, the cluster coefficients for the Calogero and anyon models will be computed in a harmonic well, and will be shown to have the same form. Here, the harmonic well has to be understood as a long distance regulator, which, when it is properly taken to vanish, yields the correct thermodynamic limit. Indeed, in sections 3 and 4, the thermodynamic limit of the anyon and Calogero models will be considered. The general thermodynamical prescription, which is depending on the space dimension \( d \), is that, at order \( n \) in the power series expansion of the thermodynamical potential in the fugacity \( z \), one replace \( 1/(n\beta^2\omega^2)^{d/2} \to V/\lambda^d \) (\( V \) is the volume of the system and \( \lambda = \sqrt{2\pi\beta/m} \) is the thermal wavelength) [14]. An appendix follows where the thermodynamic limit of the multi- species statistical problem is considered.

## 2 Statistical mechanics of the anyon and Calogero models in a harmonic well

In the sequel, one will consider Bose-based statistics quantum mechanics : eigenstates will be symmetric and the statistical parameter \( \alpha \) by convention such that \( \alpha = 0 \) for Bose statistics, and \( \alpha = -1 \) for Fermi statistics. In the anyon model, \( \alpha \) measures the statistical flux carried by the anyons in unit of the flux quantum. The system is periodic in \( \alpha \) with period 2. In the Calogero model, \( \alpha \) is related to the coupling constant \( g \) of the
one dimensional Calogero interaction $g/x^2_{ij}$ by $g = \alpha(\alpha + 1)$.

In a harmonic well $\omega$, the spectrum of the anyon model in the LLL of an external magnetic field and the spectrum of the Calogero model are similar. As for ideal bosons, an $N$-body eigenstate happens to be entirely characterized by the number $n_\ell$ of 1-body eigenstates with a given orbital quantum number $\ell = 0, 1, \ldots \infty$ and energy $\epsilon_\ell = \epsilon_0 + \ell \varpi$, with the constraint $\sum_\ell n_\ell = N$. The $N$-body spectrum is nothing else but the sum of the 1-body spectrum $\sum_\ell n_\ell \epsilon_\ell$ shifted by the 2-body interaction energy $-\frac{1}{2}N(N-1)\varpi\alpha$. The anyon spectrum in the LLL corresponds to $\varpi = \omega_t - \omega_c$ ($\omega_t = \sqrt{\omega_c^2 + \omega^2}$, $\omega_c = |eB|/(2m)$) and $\epsilon_0 = \omega_t$, whereas the Calogero spectrum corresponds to $\varpi = \omega$ and $\epsilon_0 = \frac{1}{2}\omega$.

Thus, the net effect of $\alpha$ intermediate statistics is simply the shift of the $N$-body Bose spectrum by $-\frac{1}{2}N(N-1)\varpi\alpha$, which in turn affects the Boltzmann weight of the $N$-body bosonic partition function

$$Z_N = e^{\frac{1}{2} \beta N(N-1)\varpi\alpha} \prod_{n=1}^{N} \frac{e^{-\beta \epsilon_0}}{1 - e^{-\beta \varpi}}$$

(1)

The cluster expansion of the thermodynamical potential $\Omega \equiv -\ln \sum_N Z_N z^N = -\sum_n b_n z^n$ in power series of the fugacity $z = \exp \beta \mu$, where $\mu$ is the chemical potential, yields, in the limit where the harmonic attraction becomes small, i.e. $\varpi \to 0$, [11, 12]

$$b_n = \frac{1}{\beta \varpi} e^{-n\beta \epsilon_0} \frac{n-1}{n^2} \prod_{k=1}^{n-1} \frac{k + n\alpha}{k}$$

(2)

Clearly, the cases $\alpha = 0$ and $\alpha = -1$ coincide with the bosonic and fermionic cluster expansions. Note also that the polynomial expression (2) appears in some recent numerical estimation of the $N$-cycle brownian closed paths contribution to the $N$-anyon partition function [15].

Let us consider the probability $P(n_\ell)$ to have a given occupation number $n_\ell$. For bosons in a harmonic well, it is well known that $P_B(n_\ell) = (1 - ze^{-\beta \epsilon_\ell})(ze^{-\beta \epsilon_\ell})^{n_\ell}$. For particles with $\alpha$ statistics in a harmonic well, it should be defined as

$$P(n_{\ell\circ}) = e^{\Omega} \text{tr}_{n_{\ell\circ}} e^{-\beta(H_N -\mu N)}$$

(3)

where the trace is as usual understood as summation over the $n_\ell$’s but with the constraint that one of them, $n_{\ell\circ}$, is fixed.
The trace can be computed from its bosonic end value $e^{-\Omega_b}P_b(n_\ell)$ by the shift $z^N \rightarrow e^{1/2N(N-1)\omega_\alpha}z^N$ in the power series expansion in $z$. In terms of the intermediate statistics grand partition function $e^{-\Omega(z)}$, one deduces

$$P(n_\ell) = \{e^{-\Omega(ze^{n_\ell\beta\omega_\alpha})+\Omega(z)}e^{1/2n_\ell(n_\ell-1)\beta\omega_\alpha}e^{-n_\ell\beta\epsilon_{n_\ell}} - \{n_\ell \rightarrow n_\ell + 1\}$$

In the limit $\omega \rightarrow 0$, one approximates

$$-\lim_{\omega \rightarrow 0} \left( \Omega(ze^{n_\ell\beta\omega_\alpha}) - \Omega(z) \right) = \sum_{n=0}^{\infty} n_\ell \alpha \beta \omega_\alpha n_\ell z^n = n_\ell \alpha \ln y$$

where $y$ is solution of

$$y - z e^{-\beta_0} y^{1+\alpha} = 1$$

with $y \rightarrow 1$ when $z \rightarrow 0$. One directly finds

$$P(n_\ell) = \left( \frac{y}{y-1} e^{\beta(\epsilon_{n_\ell}-\epsilon_0)} - 1 \right) \left( \frac{y}{y-1} e^{\beta(\epsilon_{n_\ell}-\epsilon_0)} \right)^{-n_\ell-1}$$

$$\langle n_\ell \rangle = \sum_{n_\ell=0}^{\infty} n_\ell P(n_\ell) = \frac{1}{y - 1} e^{\beta(\epsilon_{n_\ell}-\epsilon_0)} - 1$$

The occupation numbers are essentially statistical independent in the approximation used in deriving (7). Indeed, the joint probability $P(n_\ell, n_m) \approx P(n_\ell)P(n_m)$ factorizes, which implies subleading correlations $\langle n_\ell n_m \rangle - \langle n_\ell \rangle \langle n_m \rangle$ compared with $\langle n_\ell \rangle \langle n_m \rangle$. Alternatively, correlations become important when a large number of occupation numbers is considered.

Note that one could as well have dealt with Fermi-based statistics quantum mechanics, meaning antisymmetric eigenstates and fermionic occupation numbers $n_\ell$ either 0 or 1. One then would have found $P(n_\ell) = (y-1)^{n_\ell} e^{-n_\ell\beta(\epsilon_{n_\ell}-\epsilon_0)}/(1 + (y-1)e^{-\beta(\epsilon_{n_\ell}-\epsilon_0)})$ and $\langle n_\ell \rangle = P(1)$.

One is now in position to consider the thermodynamic limit $\omega \rightarrow 0$ in (1,2). The prescription has already been given in the introduction. It is simply dictated by the natural requirement that when $\omega \rightarrow 0$, the thermodynamical potential of a system of $d$-dimensional harmonic oscillators coincide with those of a system of $d$-dimensional particles in a box of infinite volume. A qualitative justification is that in a neighbourhhood of any given point $\vec{r}$, one can always approximate $\sum_i 1/2m\omega^2 r_i^2$ by $\sum_i 1/2m\omega^2 r_i^2$. It follows that for
a system of particles confined in an infinitesimal volume \( d^d r \) around \( \vec{r} \), the local thermodynamic potential in the presence of a small harmonic attraction can be approximated by simply replacing \( z \rightarrow ze^{-\frac{1}{2} \beta m \omega^2 r^2} \) and \( V \rightarrow d^2 r \) in the infinite box thermodynamical potential. Since it is additive in the limit \( \omega \rightarrow 0 \), for the entire system one has

\[
- \int \frac{d^d r}{V} \sum_{n=1}^{\infty} b_n \left( ze^{-\frac{1}{2} \beta m \omega^2 r^2} \right)^n = - \sum_{n=1}^{\infty} \frac{\lambda^d}{n^{d/2}(\beta \omega)^{d/2}} b_n z^n
\]  

(9)

where \( d \) is the space dimension of the system. Thus the prescription \( 1/(n\beta^2 \omega^2)^{d/2} \rightarrow V/\lambda^d \) at each order \( z^n \) [14], that will be used in the following sections.

3 Statistical Mechanics of the Anyon gas in a strong \( B \)-field : the thermodynamic limit

The model : The \( N \)-anyon Hamiltonian (\( \hbar = 1 \) and \( c = 1 \)) in an external magnetic field \( B \) is

\[
H_N = \sum_{i=1}^{N} \frac{1}{2m} \left( \frac{1}{i} \frac{\partial}{\partial \vec{r}_i} - \alpha \sum_{j \neq i} \frac{\vec{k} \times \vec{r}_{ij}}{r_{ij}^2} - e \frac{B}{2} \vec{k} \times \vec{r}_i \right)^2
\]  

(10)

where \( \vec{k} \) is the unit vector perpendicular to the plane, and \( \vec{r}_{ij} = \vec{r}_i - \vec{r}_j \). The Hamiltonian being invariant under \((x_i, y_i, \alpha, \epsilon) \rightarrow (x_i, -y_i, -\alpha, -\epsilon)\), where \( \epsilon = eB/|eB| \), the spectrum and thus the partition function are invariant under \((\alpha, \epsilon) \rightarrow (-\alpha, -\epsilon)\). They depend only on \(|\alpha|, \epsilon \alpha \) and \( \omega_c \). By convention, and without loss of generality, one chooses \( \epsilon = +1 \) (in the opposite case one would simply change \( \alpha \rightarrow -\alpha \)). The shift \( \alpha \rightarrow \alpha + 2 \) in \( H_N \) amounts to the regular gauge transformation \( \psi \rightarrow \exp(-2i \sum_{i<j} \arg \vec{r}_{ij}) \psi \), which does not affect the symmetry of the eigenstates, implying that the spectrum is periodic in \( \alpha \) with period 2.

One finds an infinitely degenerate groundstate with energy \( N \omega_c \) and eigenstate basis (in the interval \( \alpha \in [-1, 0] \)) [17]

\[
\psi = \prod_{i<j} r_{ij}^{-\alpha} S \prod_i z_i^{\ell_i} \exp(-\frac{1}{2} m \omega_c \sum_i z_i \bar{z}_i), \quad \ell_i \geq 0
\]  

(11)

where \( S \) is a symmetrisation operator. If one leaves aside the anyonic prefactor \( \prod_{i<j} r_{ij}^{-\alpha} \), the \( N \)-anyon groundstate is a symmetrised product of 1-body Landau groundstates of
energy $\omega_c$ and orbital angular momentum $\ell_i$. One can argue that, starting from the Bose case $\alpha = 0$ modulo 2, the gap above the groundstate remains of order the cyclotron gap $2\omega_c$ when $\alpha \in [-1, 0]$ modulo 2, but is no more under control in the interval $\alpha \in [0, 1]$ modulo 2. In the limit $\alpha \to 0^+$ modulo 2, unknown non-linear states (only a small subset of excited states are known) have to merge in the groundstate basis to get a complete Landau basis. This result is sustained by various numerical and semi-classical considerations. It could even be true, as a numerical computation of the 4-anyon spectrum in a magnetic field seems to indicate, that the gap above the groundstate is exactly equal to the cyclotron gap for $\alpha \in [-1, 0]$ modulo 2. Thus when $\alpha \in [-1, 0]$ interpolates between Bose and Fermi statistics with singular flux tubes anti-parallel to the external $B$-field, the thermal probability $e^{-2\beta\omega_c}$ to have an excited state is negligible when the thermal energy $kT = 1/\beta$ is much smaller than the cyclotron gap. In what follows, one focuses on the thermodynamics of an anyon gaz precisely in these conditions where it is licit to consider anyons projected in the groundstate of the external magnetic field.

Statistical mechanics in the thermodynamic limit: In order to define a proper thermodynamic limit, i.e. to give a non ambiguous meaning to the infinite degeneracy of the spectrum and to the factorized volume in the thermodynamical potential, the system should be regularized at long distance. With a harmonic regulator, the eigenstates are still given by (11), but with $\omega_c \to \omega_t$. The harmonic confinement partially lifts the degeneracy, and, as already emphasized, the intermediate $\alpha$ statistics shifts the $N$-body Bose spectrum $\sum_i \epsilon_{\ell_i}$ by $-\frac{1}{2}N(N-1)(\omega_t - \omega_c)\alpha$, where the 1-body energy is $\epsilon_{\ell_i} = \omega_t + \ell_i(\omega_t - \omega_c)$. In 2 dimensions, the thermodynamic limit $\omega \to 0$ in the cluster expansion is $1/(n\beta^2\omega^2) \to V/\lambda^2$. One precisely finds a thermodynamical potential $\Omega = -\rho_L V \ln y$

$$\Omega = -\rho_L V \ln y$$

(12)

where $\rho_L = 2\beta\omega_c/\lambda^2$ is the Landau degeneracy per unit volume, and $y$ is the implicit solution of equation (3) with $\epsilon_0 = \omega_c$. One can check that the ideal bosonic and fermionic thermodynamical potentials are readily recovered when $\alpha = 0$ and $\alpha = -1$, since $y =$
\[ \frac{1}{1 - z e^{-\beta \omega_c}} \text{, and respectively } y = 1 + z e^{-\beta \omega_c}. \] The ratio of the mean anyon number to the Landau degeneracy is given by

\[ \nu \equiv \frac{\langle N \rangle}{N_L} = \frac{1}{y - 1 - \alpha} \quad \iff \quad y = 1 + \frac{\nu}{1 + \alpha \nu} \quad (13) \]

\( \nu \) is monotonically increasing with \( z \), as required. Its implicit definition directly follows

\[ z e^{-\beta \omega_c} = \frac{\nu}{(1 + \nu + \alpha \nu)^{1+\alpha}(1 + \alpha \nu)^{-\alpha}} \quad (14) \]

Standard thermodynamical functions as the pressure \( P = -\Omega/(V \beta) \), the magnetization \( M = -\partial \Omega/\partial B \), the internal energy \( U = \partial \Omega/\partial \beta + \mu \langle N \rangle \) and the entropy \( TS = U - \mu \langle N \rangle - \Omega/\beta \) follow from (12) and (13): the equation of state reads

\[ P \beta = \rho_L \ln \left( 1 + \frac{\nu}{1 + \alpha \nu} \right) \quad (15) \]

and the virial coefficients are

\[ a_n = \left( -\frac{1}{\rho_L} \right)^{n-1} \frac{1}{n} \{ (1 + \alpha)^n - \alpha^n \} \quad (16) \]

In the limit where the Boltzman weight \( \exp(-2 \beta \omega_c) \) can be neglected, both the exact second virial coefficient and the pressure at the first perturbative order in \( \alpha \) are correctly reproduced. We will comment later on the divergence of the pressure at the critical value \( \nu_{cr} = -1/\alpha \).

The magnetization per unit volume is

\[ \mathcal{M} = -\mu_B \rho + 2 \frac{\mu_B}{\lambda^2} \ln \left( 1 + \frac{\nu}{1 + \alpha \nu} \right) \quad (17) \]

where \( \mu_B \equiv |e|/2m \) is the Bohr magneton. Except near the singularity \( \nu = -1/\alpha \), the ratio of the logarithmic correction to the De Haas-Van Alphen magnetisation \( -\mu_B \rho \) is of order \( (\beta \omega_c)^{-1} \), and thus negligible.

The internal energy \( U = \langle N \rangle \omega_c \) and the entropy \( S \) reads

\[ TS = \frac{1}{\beta} \rho_L V \ln \left( \frac{(1 + \nu + \alpha \nu)^{1+\nu+\alpha \nu}}{\nu^\nu(1 + \alpha \nu)^{1+\alpha \nu}} \right) \quad (18) \]

\[ ^{3}\text{It can be verified a posteriori that the fluctuations } (\langle N^2 \rangle - \langle N \rangle^2)/\langle N \rangle^2 \text{ vanish in the thermodynamic limit.} \]
The expression (18) for the entropy has a natural interpretation. If one assumes that the small 1-body angular momenta $\ell_i$ are energetically preferred and that they are the only one to survive in the thermodynamic limit, one infers that the number $G$ of quantum states available per particle should be defined by $0 \leq \ell_i < G$. Since the bosonic $N$-anyon groundstate is the symmetrised product of 1-body eigenstates of momentum $\ell_i$, its degeneracy is $C_N^G G^{-1+G}$. Accordingly, since the entropy (18) is the logarithmic measure of the groundstate degeneracy, one has $S = k \ln C_{G-1+G}^{(N)}$, from which one deduces $(G) \simeq N_L + \alpha (N-1)$.

$G$ also appears in the mean occupation number $\langle n_\ell \rangle$, simply because it is independent on $\ell$ in the thermodynamic limit. One indeed verifies that $\langle N \rangle = \sum_\ell \langle n_\ell \rangle = \langle G \rangle \langle n_\ell \rangle$ implies $\langle G \rangle = N_L (1 + \alpha \nu)$.

Note also that if one assumes that the $N$-th anyon experiences an uniform effective magnetic field $B_{\text{eff}} = B + \alpha \phi_0 (N-1)/V$, i.e. the external magnetic field screened by the flux tubes carried by the other $N-1$ anyons \cite{20}, one simply recovers $N_{\text{eff}} = VB_{\text{eff}}/\phi_0 = N_L + \alpha (N-1)$. In this mean field point of view, the effective magnetic field $B_{\text{eff}}$ vanishes at the critical filling.

The mean field result given above can be strengthened by computing the degeneracy per particle directly in a box. To achieve this goal, one determines which eigenstates have their maxima contained inside a circular box of radius $R$. Let us consider the eigenstates (11) without bothering about symmetrisation to make the argument simpler. At the maximum of an eigenstate one has

$$
\frac{1}{2} \frac{\partial^2}{\partial \vec{r}_i^2} |\psi|^2 = \left( -\alpha \sum_{j \neq i} \frac{\vec{r}_{ij}}{r_{ij}^2} + \ell_i \frac{\vec{r}_i}{r_i^2} - m \omega_c \vec{r}_i \right) |\psi|^2 = 0 \quad (19)
$$

The density of particles is defined as $\rho(r) = \langle \psi | \sum_{j \neq i} \delta (\vec{r} - \vec{r}_j) |\psi \rangle / \langle \psi | \psi \rangle$, normalized to $\int d^2 r \rho(r) = N - 1$. It depends only on $r$ because $|\psi|^2$ is rotationally invariant. In a mean field approach, one replaces the summation in (19) over discrete indices $j$ by a continuous integral over the density

$$
- \alpha \int d^2 r' \rho(r') \left( \frac{\vec{r}_i - \vec{r}'}{(\vec{r}_i - \vec{r}')^2} + \ell_i \frac{\vec{r}_i}{r_i^2} - m \omega_c \vec{r}_i \right) = 0 \quad (20)
$$
Using the identity \( \vec{\partial}_i \int d^2r' \rho(r')(\vec{r}_i - \vec{r}')/(\vec{r}_i - \vec{r}')^2 = 2\pi \rho(r_i) \), one finds
\[
\ell_i = m_\omega r_i^2 + \alpha \int_0^{r_i} 2\pi r dr \rho(r) \tag{21}
\]

One deduces that \( \ell_i \) increases with \( r_i \), implying a maximum angular momentum corresponding to the size of the box \( R \). Thus, the maximum of the eigenstate will be inside the box if and only if
\[
0 \leq \ell_i < m_\omega R^2 + \alpha \int_0^{R} 2\pi r dr \rho(r) = N_L + \alpha(N - 1) \tag{22}
\]

In conclusion, one has
\[
G = N_L + \alpha(N - 1) \tag{23}
\]

which correctly reproduces the usual Landau degeneracy when \( \alpha = 0 \) (and also when \( N = 1 \)). The result (23) is convincing because it implies that the Bose-based \( N \)-anyon ground-state (11) is, in the singular gauge, the Landau groundstate of a fermionic system when \( \alpha = -1 \), as it should. Indeed, the basis (11) is then \( \{\prod_{i<j} z_{ij} S \prod_i z_{i}^{\ell_i} e^{-\frac{1}{2}m_\omega} \sum_i z_i \bar{z}_i, 0 \leq \ell_i < N_L - (N - 1)\} \), whereas the usual Fermi basis is \( \{A \prod_i z_{i}^{m_i} e^{-\frac{1}{2}m_\omega} \sum_i z_i \bar{z}_i, 0 \leq m_i < N_L\} \) (\( A \) the antisymmetrisation operator). Both basis are equivalent because any totally antisymmetrised polynomial in \( z_i \) of degree \( d_i \) can be written as the product of the Jastrow factor \( \prod_{i<j} z_{ij} \) and of a totally symmetrised polynomial of degree \( d_i - (N - 1) \), and reciprocally.

**Discussion :** (23) coincides with Haldane’s statistics definition \( \Delta G = \alpha \Delta N \) [7], which therefore appears as the analytical continuation in \( \alpha \) of the anyon statistics in a strong magnetic field. Actually, the entropy as well as other thermodynamical quantities can be recovered from (19) (see for example [10]).

When \( \alpha = 0 \), any value of \( \nu \) is allowed, due to Bose condensation. In the case of Fermi statistics \( \alpha = -1 \), Pauli exclusion implies that the LLL is completly filled when \( \nu = 1 \). At a particular negative \( \alpha \), anyons obey a generalised exclusion principle. Indeed, the groundstate exists as long as \( \langle G \rangle > 0 \), i.e. \( \nu \leq -1/\alpha \). The critical filling \( \nu_{cr} = -1/\alpha \) describes a non degenerate groundstate with all the \( \ell_i \)’s null, i.e. a minimum angular
momentum $L = -\frac{1}{2}(\langle N \rangle_{cr} - 1)\alpha$. In the singular jauge,

$$\psi' = \prod_{i<j} z_{ij}^{-\alpha} \exp(-\frac{m\omega_c}{2} \sum_{i} N_i \bar{z}_i)$$

when $\alpha = -1$, one recovers a Vandermonde determinant built from 1-body Landau eigenstates. Incidentally, in the Haldane statistics point of view, when $\alpha = -m$, the non degenerate groundstate coincides with the Laughlin eigenstate at the critical filling $\nu_{cr} = 1/m$.

Since transitions to excited levels are by construction forbidden, the pressure diverges and the entropy vanishes when the LLL is fully occupied (such that any additional particle is excluded). The system is in its non degenerate groundstate as one can also see by inspecting the large $z$ behavior of the thermodynamical potential $\Omega \to -e^{-\beta\langle N \rangle_{cr} \omega_c \langle N \rangle_{cr}}$. In this situation the gas is incompressible: the isothermal compressibility coefficient $\chi_T = -\frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_{T,B}$ vanishes at the critical filling.

On the contrary, when one analytically continues the thermodynamical quantities computed above through positive $\alpha$, any value of the filling factor is allowed since $G$ increases with the filling factor. The pressure becomes constant while the entropy diverges when $\nu \to \infty$.

4 Statistical Mechanics of the Calogero model in the thermodynamic limit

As stated in the introduction, the Calogero model can be viewed as an intermediate statistics model for 1-dimensional particles. Contrary to the anyon model, there is no periodicity in the statistical parameter $\alpha$. One also assumes $\alpha < 1/2$ in order to have square-integrable eigenstates. The Calogero Hamiltonian is

$$H = \frac{1}{2m} \sum_{i=1}^{N} \left( -\frac{\partial^2}{\partial x_i^2} + m^2 \omega^2 x_i^2 \right) + \frac{1}{m} \sum_{i<j} \frac{\alpha(1+\alpha)}{(x_i - x_j)^2}$$

A long distance harmonic regularor is again introduced to discretize the spectrum. One redefines $\tilde{\psi} = \prod_{i<j}(x_i - x_j)^{-\alpha} \tilde{\psi}$, where the Hilbert space of eigenstates $\tilde{\psi}$ can always be
chosen to be completely symmetric. In this way, a new Hamiltonian is obtained \cite{21}

\[ \tilde{H} = \sum_i (\tilde{a}_i^+ \tilde{a}_i + \frac{1}{2})\omega - \frac{1}{2}N(N-1)\alpha \omega \]  

(26)

where the creation and annihilation operators \( \tilde{a}_i^+ = \frac{1}{\sqrt{2}}(\sqrt{m\omega}x_i - D_i/\sqrt{m\omega}) \) and \( \tilde{a}_i = \frac{1}{\sqrt{2}}(\sqrt{m\omega}x_i + D_i/\sqrt{m\omega}) \), with \( D_i = \partial_i - \alpha \sum_{j\neq i}(1 - P_{ij})/(x_i - x_j) \) and \( P_{ij} \) the exchange operator, satisfy the commutation relations \([\tilde{a}_i, \tilde{a}_j^+] = \delta_{ij}(1 - \alpha \sum_{k=1}^N P_{ik}) + \alpha P_{ij} \). The groundstate \( \tilde{\psi}_o \) is solution of \( a_i \tilde{\psi}_o = 0 \) for all \( i \), and the excited states are obtained as \( \tilde{\psi}_{\{\ell_i\}} = S \prod_i a_i^{\ell_i} \tilde{\psi}_o \) with energy \( \sum_i (\ell_i + \frac{1}{2})\omega - \frac{1}{2}N(N-1)\alpha \omega \). Thus, as expected, \( \alpha \) statistics shifts the Bose spectrum by a constant.

One would now like to compute the thermodynamical potential in the thermodynamic limit, following a procedure analogous to the one used in section 3 for the anyon model. However, in one dimension, the thermodynamic limit \( \omega \to 0 \) has to be understood for a given coefficient cluster coefficient \( b_n \) as \( 1/(\sqrt{n}\beta\omega) \to V/\lambda \). Using the identity \( \frac{1}{\sqrt{n}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp \, e^{-n^{\beta\omega}/2m} \), one can easily resum the cluster expansion in the thermodynamic limit\(^4\) to get

\[ \Omega = -PV\beta = -\frac{V}{2\pi} \int dp \ln y \]  

(27)

Here, \( y \) is the solution of

\[ y - ze^{-\beta p^2/2m}y^{1+\alpha} = 1 \]  

(28)

where \( p^2/2m \) appears quite naturally as a continuous one dimensional energy spectrum in the \( \omega \to 0 \) limit. The thermodynamical potential for bosons (fermions) is correctly reproduced when \( \alpha = 0 \) since \( y = 1/(1 - ze^{-\beta^2/2m}) \) (respectively \( \alpha = -1 \) since \( y = 1 + ze^{-\beta^2/2m} \)). One deduces the density

\[ \rho = \frac{1}{2\pi} \int dp \, \frac{1}{y - 1 - \alpha} \]  

(29)

and the internal energy

\[ U = \frac{V}{2\pi} \int dp \, \frac{p^2}{2m} \frac{1}{y - 1 - \alpha} \]  

(30)

\(^4\)\( \Omega \) has also been derived via the Bethe ansatz directly in the thermodynamic limit \cite{2, 3}.
One can also compute the number $g(p)dp$ of quantum states at momentum $p$ available per particle and per unit volume. Since the local thermodynamical potential at momentum $p$ in (27) coincides with the anyonic thermodynamical potential (12) where $\omega_c$ is replaced by $p^2/2m$, this suggests that the occupation number $n_p$ can be deduced from the anyonic one, given in (8) with $\omega = 0$. One infers

$$n_p = y - 1$$

(31)

where $y$ is solution of (28). However, $\frac{V}{2\pi} \int dp \ n_p$ does not define the total number of particles because of possible corrections due to particle correlations. In fact, the information about these corrections is precisely described by $g(p)$. If one reminds that the local density of particle at momentum $p$ is nothing but $n_p \ g(p)dp$, one gets from (29)

$$g(p) = \frac{V}{2\pi} \frac{1}{1 - \alpha(y - 1)}$$

(32)

which coincides with the “hole density” of the Bethe ansatz [13]. As expected, the entropy can be rewritten as

$$S = k \frac{1}{2\pi} \int dp \ \ln C_{g(p)}^{n_p g(p)}$$

(33)

Note that the particular thermodynamical limit algorithm in one dimension has played a crucial role for the derivation of a thermodynamical potential, which coincides with the one obtained directly in the continuum via the Bethe ansatz. Any other prescription to define the thermodynamical limit would lead to different results [22].

5 Conclusion and Perspectives

In conclusion, the Calogero model and the anyon model in the LLL of an external magnetic field have been shown to be intimately connected, not only at the level of their spectrum, but also of their thermodynamical properties. By anyon model, one means here singular flux tubes anti-parallel to the $B$-field, i.e. $\alpha \in [-1, 0]$. Haldane’s statistics can be understood in this context, simply as the analytical continuation in $\alpha$ of the statistical mechanics of the anyon model in the strong $B$-field.
It would be certainly interesting to have more precise information on the thermodynamics of the anyon model when $\alpha \in [0, 1]$. To do so, one has necessarily to take into account, in one way or another, the excited states which join the groundstate when $\alpha = 0$. More interesting and richer physics, than the strictly 1-dimensional one, is to be expected in this regime.

**Appendix : Many anyons species in the thermodynamic limit**

One can consider different species of identical particles to define a generalized anyon model \[ H = \sum_{i=1}^{N} \frac{1}{2m_i} \left( \vec{p}_i - \sum_{j \neq i} \alpha_{ij} \frac{\vec{k} \times \vec{r}_{ij}}{r_{ij}^2} - \frac{1}{2} e_i B \vec{k} \times \vec{r}_i \right)^2 \] (34)

When all the $\alpha_{ij} = \alpha$, one recovers the anyon model. One again assumes $e_i B > 0$. The shift $\alpha_{ij} \rightarrow \alpha_{ij} + 1$ amounts to the regular gauge transformation $\psi \rightarrow \exp(-i \sum_{i<j} \arg \vec{r}_{ij}) \psi$ which does not affect the symmetry of the eigenstates. It follows that the spectrum is now periodic for each $\alpha_{ij}$ with period 1. Using the same lines of reasoning as above, one finds the groundstate for $\alpha_{ij} \in [-1, 0]$ ($\omega_{ci} = e_i B_i / 2m_i$)

\[ \psi = \prod_{i<j} r_{ij}^{-\alpha_{ij}} \prod_i \left( z_i^\ell \exp\left(-\frac{1}{2} m \omega_{ci} z_i \bar{z}_i \right) \right) \quad \ell_i \geq 0 \] (35)

Contrary to anyons, eigenstates are not totally symmetric anymore, implying that some excited eigenstates have to merge in the groundstate when $\alpha_{ij} \rightarrow -1$. A clear information on the gap above the groundstate is missing. However, one can still project the system on the LLL Hilbert space. Let us denote the number of particle in the specy $a$ by $N_a$. If one generalizes the mean field computation of the degeneracy per particle given above, one recovers the mutual statistics definition of Haldane

\[ G_a = N_{La} + \sum_b \alpha_{ab} \left( N_b - \delta_{ab} \right) \] (36)

with $N_{La} = e_a B_a V / 2\pi$, where $G_a$ satisfies $\Delta G_a = \sum_b \alpha_{ab} \Delta N_b$. The total degeneracy is thus $\prod_a C_{G_a - 1 + N_a}^{N_a}$ and the thermodynamical potential reads \[ \Omega = -PV\beta = - \sum_a N_{La} \ln y_a \] (37)
where $y_a$ is solution of $y_a - z e^{-\beta \omega_a} \prod_b y_b^{N_{La} \alpha_{ab} / N_{La}} = 1$ with $y_a \to 1$ when $z \to 0$. The filling factor $\nu_a = \langle N_a \rangle / N_{La}$ of the specy $a$ is determined by

$$y_a = 1 + \frac{\nu_a}{1 + \sum_b \alpha_{ab} \nu_b} \quad (38)$$

One finds for the magnetization and the entropy

$$M = \sum_a \left[ -\mu_{Ba} \rho_a + 2\mu_{Ba}^2 \ln \left( 1 + \frac{\nu_a}{1 + \sum_b \alpha_{ab} \nu_b} \right) \right] \quad (39)$$

$$S = \sum_a kN_{La} \ln \left( \frac{(1 + \nu_a + \sum_b \alpha_{ab} \nu_b)^{1+\nu_a+\sum_b \alpha_{ab} \nu_b}}{\nu_a^\nu_a (1 + \sum_b \alpha_{ab} \nu_b)^{1+\sum_b \alpha_{ab} \nu_b}} \right) \quad (40)$$

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