On the Powerball Method

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Abstract

We propose a new method to accelerate the convergence of optimization algorithms. This method simply adds a power coefficient $\gamma \in [0, 1)$ to the gradient during optimization. We call this the Powerball method after the well-known Heavy-ball method by Polyak [1]. We analyze the convergence rate for the Powerball method for strongly convex functions and show that it has a faster convergence rate than gradient descent and Newton’s method in the initial iterations. We also demonstrate that the Powerball method provides a 10-fold speed up of the convergence of both gradient descent and L-BFGS on multiple real datasets as well as accelerates the computation for Pagerank vector.

1 Introduction

We consider minimizing a differentiable function $f(x): \mathbb{R}^n \rightarrow \mathbb{R}$ with iterative methods. Given a starting point $x(0) \in \mathbb{R}^n$, these methods compute

$$x(k+1) = x(k) - A_k^{-1} \nabla f(x(k)) \quad \text{for } k = 0, 1, \ldots.$$  (1)

Previous work has focused mainly on the choice of $A_k$. One choice is using a scalar step size $A_k = \alpha_k^{-1}$ with $\alpha_k > 0$, yielding the gradient descent method (due to Cauchy). Another widely adopted choice of $A_k$ is the Hessian matrix $\nabla^2 f(x(k))$, which is used by the notable Newton’s method.

In this paper, we propose the Powerball method, which applies a nonlinear element-wise transformation to the gradient by

$$x(k+1) = x(k) - A_k^{-1} \sigma_\gamma(\nabla f(x(k))), \quad \text{for } k = 0, 1, \ldots.$$  (2)

For any vector $x = (x_1, \ldots, x_n)^T$, the Powerball function $\sigma_\gamma$ is applied to all elements of $x$, that is $\sigma_\gamma(x) = (\sigma_\gamma(x_1), \ldots, \sigma_\gamma(x_n))^T$. For simplicity, we drop the subscript $\gamma$ and use $\sigma(x)$ to denote $\sigma_\gamma(x)$. The Powerball function $\sigma_\gamma(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$ has the form $\sigma(z) = \text{sign}(z)|z|^{\gamma}$ for $\gamma \in (0, 1)$, here sign$(z)$ returns the sign of $z$, or 0 if $z = 0$. We use a constant power coefficient $\gamma$ for all iterations.

We call the method with $A_k = \alpha_k$ in eq. (2) the gradient Powerball method and the method with $A_k = \nabla^2 f(x(k))$ the Newton Powerball method. We will also propose other Powerball variants of standard methods throughout the paper, for example, the stochastic gradient Powerball method, the L-BFGS Powerball method.

This paper is organized as follows. In Section 2, we shall provide intuition behind the Powerball method by viewing optimization algorithms as discretizations of ordinary differential equations (ODE). Furthermore, we analyze the convergence rate for the proposed Powerball method for strongly convex functions in Section 3. Moreover, we demonstrate the fast convergence of Powerball algorithms on real datasets and on Pagerank computation in Section 4 and Section 5. Finally, we conclude this paper with general discussion on applying insight in control and dynamical systems to optimization algorithms.

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2 Intuition from Ordinary Differential Equations (ODE)

Consider the algorithms presented in eq. (1) and eq. (2). If the index, or iteration number, of these algorithms is viewed as a discrete-time index, then these can be viewed as systems characterized by discrete-time dynamical systems. By taking this view, the convergence of an optimization method to a minimizer can be equivalently seen as the stability of a difference equation (i.e., the convergence of a dynamical system to an equilibrium) [15]. Particularly in [14], it has been observed that a natural formulation of the steepest descent algorithm for solving a least-squares matching problem leads to a differential equation. It has also been demonstrated in recent work by [3, 4, 5, 6] that insights from control theory and dynamical systems[12] can be used to analyze and design optimization algorithms.

The intuition of the gradient Powerball algorithm lies in the Euler discretization of the following ODE, for all $i = \{1, 2, \ldots, n\}$:

$$
\dot{x} = -\sigma(\nabla f(x)).
$$  

(3)

Remark 1. When $\gamma = 1$, eq. (3) becomes an ODE that corresponds to the gradient descent method.

Definition 1. A function $f$ is strongly convex with coefficient $m > 0$ if the function $g(x) \triangleq f(x) - \frac{m}{2} \|x\|^2$ is convex.

Proposition 1. For any strongly convex function $f$ with coefficient $m$ and $L$-Lipschitz gradient, the ordinary differential equation (eq. (3)) for $\gamma = 1$ converges to its equilibrium exponentially with rate $2m$.

Proof. We define a Lyapunov function for the system $\dot{x} = -\nabla f(x)$: $V(t) \triangleq \frac{1}{2} \sum_{i=1}^{n} \left| \frac{\partial f(x(t))}{x_i} \right|^2$, which satisfies $\frac{dV(t)}{dt} \leq -2mV(t)$ for some positive constant $K$. Then $V(t) \leq \exp(-2mt)V(0)$. \qed

Next, we will prove a finite-time stability result for the above ODE when $\gamma \in (0, 1)$ and compare it with the asymptotic stability [12] for the above ODE when $\gamma = 1$.

Proposition 2. For any strongly convex function $f$ with coefficient $m$ and $L$-Lipschitz gradient, the ordinary differential equation (eq. (3) for $\gamma \in (0, 1)$) converges to its equilibrium in finite time $T = \frac{((\gamma+1)V(0))^{1-\gamma}}{m(1-\gamma)}$, in which $V(t) \triangleq \frac{1}{\gamma+1} \sum_{i=1}^{n} \left| \frac{\partial f(x(t))}{x_i} \right|^\gamma$.

Proof. The proof relies on the following lemma:

Lemma 1. ([2], Theorem 1) Suppose that a function $V(t) : [0, \infty) \rightarrow [0, \infty)$ is differentiable (the derivative of $V(t)$ at 0 is defined as its Dini upper derivative), such that $\frac{dV(t)}{dt} + KV(t)$ is negative for all $t$, for some constant $K > 0$ and $0 < \gamma < 1$. Then $V(t)$ will reach zero at finite time $t^* \leq \frac{V^{1-\gamma}(0)}{K(1-\gamma)}$, and $V(t) = 0$ for all $t \geq t^*$.

Proof. Let $f(t)$ satisfy the following ODE $\frac{df(t)}{dt} = -Kf(t)$. Given any initial value $f(0) = V(0) > 0$, its unique solution is

$$
\begin{align*}
f(t) &= \begin{cases} 
-K(1-\gamma)t + V^{1-\gamma}(0) \frac{1}{1-\gamma} & t < \frac{V^{1-\gamma}(0)}{K(1-\gamma)} \\
0 & t \geq \frac{V^{1-\gamma}(0)}{K(1-\gamma)}
\end{cases}
\end{align*}
$$

Since $V(0) = f(0)$, by the Comparison Principle of differential equations in [2], we have $V(t) \leq f(t)$, $t \geq 0$. Hence, $V(t)$ will reach zero in time $\frac{V^{1-\gamma}(0)}{K(1-\gamma)}$. Since $V(t) \geq 0$ and $\frac{dV(t)}{dt} \leq 0$, $V(t)$ remains 0 once convergence. \qed
Next, we shall construct a Lyapunov function for eq. (3), which has a similar property in Lemma 1. Let \( y_i = \frac{\partial f(x)}{\partial x_i} \), and consider a nonnegative function \( V(t) = \frac{1}{T+1} \sum_{i=1}^{n} |y_i|^{\gamma+1} \). If we take the derivative of \( V(t) \) with respect to \( t \), then we have

\[
\frac{dV(t)}{dt} = \sum_{i=1}^{n} \frac{\partial V(t)}{\partial y_i} \frac{\partial y_i}{dt} = \sum_{i=1}^{n} \text{sign}(y_i) |y_i|^\gamma \left( \sum_{j=1}^{n} \frac{\partial y_j}{\partial x_j} \frac{\partial x_j}{dt} \right)
= -[\text{sign}(y_1)|y_1|^\gamma \ldots \text{sign}(y_n)|y_n|^\gamma] H(y_i) [\text{sign}(y_1)|y_1|^\gamma \ldots \text{sign}(y_n)|y_n|^\gamma]^T
\leq - m \sum_{i=1}^{n} |y_i|^{2\gamma} \leq -m(\gamma + 1) \frac{2^{\gamma}}{2^{\gamma}} V^{\frac{2\gamma}{\gamma+1}}(t).
\]

Equality (a) is due to the fact that \( \forall i, \frac{\partial y_i}{\partial y_i} = (\gamma + 1)\text{sign}(y_i)|y_i|^\gamma \). Inequality (b) is due to the Hessian \( H \triangleq \left[ \frac{\partial^2 f}{\partial x_i \partial x_j} \right] \geq mI \) for any strongly convex function \( f \). Inequality (c) holds using the fact that \( \sum_{i=1}^{n} |y_i|^{2\gamma} \geq (\sum_{i=1}^{n} |y_i|^{\gamma+1})^{\frac{2\gamma}{\gamma+1}}, \forall \gamma \in (0, 1) \).

Using Lemma 1, eq. (4) implies that there exists \( T = \frac{1-\gamma_0}{\gamma} \frac{1-\gamma}{1-\gamma_0}, \forall \gamma \in (0, 1) \) such that \( V(t) = 0 \) when \( t \geq T \). This implies that the system’s state is at its equilibrium.

Similarly, the intuition of the Newton Powerball method lies in the Euler discretization of the following ODE:

\[ x(t) = - (\nabla^2 f(x(k)))^{-1} \sigma(\nabla f(x(k))). \]

**Proposition 3.** For any twice differentiable function \( f \), the proposed continuous Newton Powerball method converges to an equilibrium point in finite time \( T = (\gamma + 1)\frac{1-\gamma}{1-\gamma_0}, \) in which \( V(t) \triangleq \frac{1}{T+1} \sum_{i=1}^{n} |\nabla f(x_i)|^{\gamma+1} \).

**Proof.** Consider a nonnegative function \( V(t) = \frac{1}{T+1} \sum_{i=1}^{n} |\nabla f_i(x)|^{\gamma+1} \), similar to the proof of Proposition 2, if we take the derivative of \( V(t) \) with respect to \( t \) and then we have \( \forall t \) and \( \gamma \in (0, 1) \)

\[
\frac{dV(t)}{dt} = -\|[\nabla f_1(x)]^T \ldots [\nabla f_n(x)]^T\|_2^2 \leq -((\gamma + 1) \frac{2^{\gamma}}{2^{\gamma}} V^{\frac{2\gamma}{\gamma+1}}(t)).
\]

Applying Lemma 1 leads to the result.

**Remark 2.** The Lipschitz constant \( L \) does not appear in the convergence rate in the above analysis which is different from the standard analysis for optimization algorithms [19]. However, L-Lipschitz gradient assumption is essential because it guarantees the existence and uniqueness of the solution of any considered ODE [18].

Through analyzing the continuous versions of optimization algorithms and viewing convergence of continuous optimization algorithms as stability of dynamical systems, we can apply Lyapunov theory from control theory to gain insight about the underlying optimization algorithms. What remains is to derive an analogous proof for discrete-time dynamical systems, or equivalently for optimization algorithms. As pointed out by Su, Boyd and Candes [4], the translation of ODE theory to optimization algorithms involves parameter tuning (for example, step-size) and tedious calculations. We shall derive, in the following section, the convergence rate for Powerball methods for strongly convex functions so that we can compare it with rates for standard methods.

3 Convergence Analysis

Given the intuition in the previous section, we shall derive the convergence rate for Powerball methods in eq. (2) for strongly convex functions. Gradient Powerball method will be focused in section. Definitions and derivations for the line search Powerball algorithm and Newton Powerball method can be found in the Appendix B. We shall demonstrate that the following gradient Powerball method converges faster in the initial iterations than gradient descent for strongly convex functions:

\[ x(k + 1) = x(k) - \alpha_k \sigma(\nabla f(x(k))), \quad \text{for } k = 0, 1, \ldots \]

Before presenting the main theorem, we would like to introduce a lemma which is essential for proving the theorem.
Lemma 2. Consider a sequence of nonnegative real numbers $v(k)$ such that
\[ v(k + 1) \leq v(k) - \alpha_k C v^\gamma(k), \quad \text{for } k = 0, 1, \ldots, \] (8)
where $C > 0$, $0 < \alpha_k < v^{1-\gamma}(k)/C$, for $k = 0, 1, \ldots,$ and $0 < \gamma < 1$. Then $v(k) \leq h(k)$ for $k = 0, 1, \ldots$, in which
\[ h(k) = -C(1 - \gamma) \left( \sum_{i=1}^{k} \alpha_i \right) + v^{1-\gamma}(0) \] (9)

Proof. We prove $v(k) \leq h(k)$ by induction. First of all, $v(0)$ satisfies the bound. Next, assume this bound is true for $v(k)$; we will prove it is true for $v(k+1)$:
\[ v^{1-\gamma}(k+1) \leq (v(k) - \alpha_{k+1} C v^\gamma(k))^{1-\gamma} = v^{1-\gamma}(k) \left( 1 - \alpha_{k+1} C v^{\gamma-1}(k) \right)^{1-\gamma} \]
\[ \leq v^{1-\gamma}(k) \left( 1 - \alpha_{k+1} C (1 - \gamma) v^{\gamma-1}(k) \right) \leq -C(1 - \gamma) \left( \sum_{i=1}^{k+1} \alpha_i \right) + v^{1-\gamma}(0) = h^{1-\gamma}(k+1). \]

Inequality (a) used the fact that $(1-x)^n \leq 1 - ax$ for any $0 \leq x, a \leq 1$. Inequality (b) used the induction assumption. □

Remark 3. One can view $v(k)$ in the lemma as a Lyapunov function of the discrete-time dynamical system of eq. (7). The proof technique for Theorem 1 relies on the construction of $v(k)$ and shows that the constructed $v(k)$ satisfies a similar inequality as eq. (8). Therefore, the dynamical system has a finite-time stability property.

Theorem 1. For any strongly convex function $f$ (with coefficient $m$) with $L$-Lipschitz gradient ($x \triangleq \frac{1}{m}$), the proposed gradient Powerball method reaches an $\varepsilon$ neighborhood of its global minimizer when the number of iterations $N$ satisfies
\[ N \geq \kappa C_f \left( \frac{V^{1-\gamma}(0)}{\varepsilon} \right)^{1-\gamma} \] (10)

Here $C_f \triangleq \frac{4\gamma}{1-\gamma}$ and a constructed Lyapunov function has the following form $V(k) \triangleq \sum_{i=1}^{N} \left| \frac{\partial f(x(k))}{\partial x} \right|^\gamma$.

Proof. See Appendix A. □

Fig. 1 computes the optimal $\gamma$ for the derived bound in eq. (10) when varying the approximated accuracy $\varepsilon$ for a fixed $V(0) = 1$. It shows that when $\varepsilon$ is not too small, a smaller $\gamma$ uses a smaller number of iterations to drive the norm of the gradient to the approximated accuracy.

Figure 1: The optimal $\gamma$ for different desired accuracy using a lower bound derived of $N$ in Theorem 1. When the desired approximated accuracy $\varepsilon$ is not too small, a smaller $\gamma$ has a faster convergence rate. In other words, in the initial iterations of the optimization algorithm, we can choose a small $\gamma$ in the gradient Powerball method to accelerate the gradient descent method.

Remark 4. Let $2\delta = 1 - \gamma$, when $\delta$ is close to 0, then the bound for $N$ becomes (by Taylor series) $N \geq \frac{2\kappa \left( \frac{V(0)}{\varepsilon} \right)^{\delta-1}}{\delta} \approx 2\log \left( \frac{V(0)}{\varepsilon} \right)$. It recovers the $O(\kappa \log(1/\varepsilon))$ convergence rate for gradient descent.
3.1 Variants of Powerball methods

In this subsection, we consider the following three variants of the proposed Powerball method. We shall only propose the variants and run a number of experiments in the later sections without deriving the corresponding convergence rates.

3.1.1 One-bit gradient descent method

From Fig. 1, it is natural to consider a special case when $\gamma = 0$. It has a low communication cost for optimizing strongly convex functions: it reduces the communication bandwidth requirement for the data exchanges [13] since only the sign for every element of the gradient computation is needed.

The one-bit gradient descent method has the following form (simply let $\gamma = 0$)
\[
x(k + 1) = x(k) - \alpha_k^{-1} \text{sign}(\nabla f(x(k))) \quad \text{for} \ k = 0, 1, \ldots.
\]
We observe a faster convergence rate on real datasets in Appendix C, and shall prove its convergence rate in future work.

3.1.2 Stochastic convergence rate on real datasets in Appendix C, and shall prove its convergence rate in future work.

4 Experiments

To evaluate the Powerball methods, we collected three datasets, which are listed in Table 1. RCV1 is a Reuters news classification dataset\(^1\). KDD10 is sampled from the KDD Cup 2010\(^2\), whose goal is to measure students’ performance. CTR is a sampled ad click-through rate dataset\(^3\).

We used the logistic regression with $\ell_2$-regularization as the objective function. Given a list of example pairs $\{(y_i, x_i)\}_{i=1}^{n}$, the goal is to solve the following minimization problem
\[
\min_w \sum_{i=1}^{n} \log(1 + \exp(-y_i(x, w))) + \lambda \|w\|_2^2.
\]

\(^1\)http://about.reuters.com/researchandstandards/corpus/
\(^2\)https://pslcdatashop.web.cmu.edu/KDDCup/
\(^3\)http://data.dmlc.ml

Algorithm 1 L-BFGS Powerball method

\begin{algorithm}
\begin{algorithmic}[1]
\STATE $g_k = \nabla f(x(k)), q = \sigma(g_k)$
\FOR {$i = k - 1, k - 2, \ldots, k - m$}
\STATE $\alpha_i = \rho \sigma_i \gamma_i, q = q - \alpha_i y_i, \quad H_k^0 = y_k^T s_{k-1}/y_{k-1}^T y_{k-1}, \quad z = H_k^0 q.$
\ENDFOR
\FOR {$i = k - m, k - m + 1, \ldots, k - 1$}
\STATE $\beta_i = \rho \sigma_i \gamma_i, \quad z = z + s_i (\alpha_i - \beta_i).$
\ENDFOR
\STATE Stop with $H_k g_k = z$
\end{algorithmic}
\end{algorithm}

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We used the logistic regression with $\ell_2$-regularization as the objective function. Given a list of example pairs $\{(y_i, x_i)\}_{i=1}^{n}$, the goal is to solve the following minimization problem
\[
\min_w \sum_{i=1}^{n} \log(1 + \exp(-y_i(x, w))) + \lambda \|w\|_2^2.
\]
We used $\lambda = 1$ for KDD10 and CTR while $\lambda = 0$ for RCV1.

Both gradient descent and L-BFGS [7] are compared with gradient Powerball method and L-BFGS Powerball method. The step size in both methods is chosen by standard backtracking line search. The weight $w$ is initialized according to a normal distribution $N(0, 0.01)$. We repeat each experiment 10 times and report the averaged results. The codes are available in the http://yy311.github.io.

4.1 Performance of Powerball methods for different $\gamma$ values

We first study the effect of varying $\gamma$. We choose four $\gamma$ values from a set $\{1, 0.7, 0.4, 0.1\}$, where for $\gamma = 1$ we obtain standard gradient descent.

The convergence of different optimization algorithms for each $\gamma$ are shown in Fig. 2. As can be seen, when a $\gamma < 1$ is applied to the gradient in every steps, it can significantly accelerate the convergence as compared with the standard gradient descent method. Especially, on both KDD10 and CTR datasets, less than 10 iterations with $\gamma = 0.1$ can result an objective value even less than the one for gradient descent method with 100 iterations.

The results for L-BFGS Powerball method comparison ($m = 5$) are shown in Fig. 3, which are similar to the observations for gradient Powerball method. We observe from Fig. 7 in the Appendix, as we expected, that the L-BFGS Powerball algorithm outperforms the L-BFGS algorithm in the initial iterations while L-BFGS algorithm outperforms the L-BFGS Powerball algorithm in the later iterations.

4.2 Adaptive $\gamma$ Powerball methods

Inspired by the theoretical results and Fig. 1, we propose a simple $\gamma$ scheduling method, named adaptive $\gamma$, which increases $\gamma$ during the optimization. More specifically, we specify both initial and final $\gamma$ values $\gamma_0, \gamma_1$ and the maximal number of iterations is $N$. At iteration $k$, we choose $\gamma$ by $\gamma = \gamma_0 + (\gamma_1 - \gamma_0) \frac{k}{K}$. By doing so, we can combine the property of faster convergence of the Powerball Method in the initial iterations and the faster convergence of the standard methods in the later iterations.

We fixed $\gamma_0 = 0.1$ and $\gamma_1 = 0.9$, and compared adaptive $\gamma$ with the fixed $\gamma$ approach. We use the relative objective $$\frac{f_{\text{fixed}} - f_{\text{adaptive}}}{f_{\text{adaptive}}} \times 100$$ as the metric, and show the results for L-BFGS Powerball method in Fig. 4. As can be seen, the adaptive $\gamma$ is comparable or even outperforms the best fixed $\gamma$ strategy at both the beginning and the end of the optimization.

5 Application to Pagerank computation

The Pagerank algorithm [17] formed the basis of the Google search engine; it ranks the webpages in world-wide web by its importance. Mathematically, we model the world-wide web as a directed unweighted graph denoted by $G = (V, E)$, where $V = \{v_1, \ldots, v_n\}$ is the set of $n$ nodes and $E \subseteq V \times V$ is the set of edges. $A_{n \times n}$ is the corresponding adjacency matrix for the webs, with the element on $i^{th}$ row and $j^{th}$ column $A[i, j] = 1$ when there is a link from $j$ to $i$, and $A[i, j] = 0$ when there is no edge from $j$ to $i$. This Pagerank vector $\rho$ is obtained as an eigenvector of the Google matrix $G$ built on the basis of the directed links between webpages:
$$G = \rho S + (1 - \rho)E / n .$$

Here $S$ is the matrix constructed from the adjacency matrix $A[i, j]$ of the directed links of the network of size $n$. Namely, $S[i, j] = A[i, j] / \sum_k A[k, j]$ if $\sum_k A[k, j] > 0$, and $S[i, j] = 1/n$ if all elements in the column $j$ of $A$ are zero. The last term

| name       | # examples  | # features | # nonzero entries |
|------------|-------------|------------|-------------------|
| RCV1       | $2.0 \times 10^4$ | $4.7 \times 10^4$ | $1.5 \times 10^6$ |
| KDD10      | $2.0 \times 10^5$ | $6.4 \times 10^5$ | $7.4 \times 10^6$ |
| CTR        | $2.2 \times 10^5$ | $6.2 \times 10^5$ | $1.3 \times 10^7$ |

Table 1: Standard benchmark datasets for classification.
Figure 2: We apply Gradient Powerball method ($\gamma < 1$) and gradient descent method ($\gamma = 1$) to minimize eq. (13) on three datasets. Left: RCV1, middle KDD10, right: CTR. We observe that Gradient Powerball method with $\gamma$ less than 1 can significantly accelerate the convergence. Especially, on both KDD10 and CTR datasets, the objective value of eq. (13) that Gradient Powerball method achieved using 10 iterations (with $\gamma = 0.1$) would require 100 iterations for the standard gradient descent method.

Figure 3: We apply L-BFGS Powerball method ($\gamma < 1$) and L-BFGS ($\gamma = 1$) to minimize eq. (13) on three datasets. We observe a similar result as the comparison of the gradient Powerball method with the gradient descent method. Left: RCV1, middle News20, right: CTR.

Figure 4: The relative objective function between the adaptive $\gamma$ and fixed $\gamma$ scheme discussed in Section 4.2. A positive relative objective function value means adaptive $\gamma$ has a faster convergence rate a. Left: RCV1, middle KDD10, right: CTR.
in eq. (14) with uniform matrix $E[i,j] = 1$ describes the probability $1 - \rho$ of a random surfer propagating along the network to jump randomly to any other node. In practice, Google uses the Power Iterative method (Algorithm 2) to compute the Pagerank vector and $\rho$ is chosen to be 0.85.

Algorithm 2 Power Iterative Method

Given $G$, generate a random vector $x_0$, specify $\epsilon > 0$

for $k = 1: \infty$
do

$x(k) = Gx(k - 1)$.

if $\|x(k) - x(k - 1)\| \leq \epsilon$ then

Stop and return $x(k)$.

end if

end for

Return: Pagerank vector $g = x(k)$

If we apply the Powerball method to the power iterative algorithm by changing the matrix multiplication $x(k) = Gx(k - 1)$ to the following step

$$x_i(k + 1) = x_i(k) + \alpha_k \text{sign} \left( \sum_j G_{ij}x_j(k) \right) \left\| \sum_j G_{ij}x_j(k) \right\|^\gamma; \text{ and } x(k + 1) = \frac{x(k + 1)}{\|x(k + 1)\|_1}.$$  \hspace{1cm} (15)

In which $\alpha_k$ is chosen in a way similar to backtracking line searching. We can similarly observe a faster convergence and here are some numerical simulations in Fig. 5.

6 Discussion

It is generally known that dynamical systems [12] can offer new insight to optimization methods [3, 4, 5] by viewing optimization algorithms as evolution of dynamical systems. Using intuition from finite-time stability of ordinary differential equations [2], we generalize the idea to the discrete schemes for optimization and demonstrate that the proposed methods can accelerate the process in the initial iterations. When it comes to large-scale optimization problems, initial iterations are crucial given computation constraints.

Future work lies in the extension of the Powerball idea to accelerated methods [8]. Motivated by [2], a continuous accelerated Powerball method has the following form

$$m\ddot{x} + \sigma \gamma \dot{x} + \sigma f(\nabla f(x)) = 0, \hspace{1cm} (16)$$

similar to momentum method, in here, $m$ can be viewed as mass of an object [19]. We will provide a discretization scheme and derive the convergence rate for strongly convex functions in a later note.

Matlab code is available from http://yy311.github.io.
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A Proof of Theorem 1

Consider the following nonnegative real numbers \( g \left( \frac{\partial f(x(k))}{\partial x} \right) \equiv \sum_{\gamma+1} 1 \cdot \frac{\partial f(x(k))}{\partial x} \). We derive the following expression using Taylor theorem

\[
g \circ \nabla f(x(k+1)) = g \circ \nabla f(x(k)) - \alpha_k (\sigma \circ \nabla f(z(k)))^T H(z(k)) \sigma \circ \nabla f(x(k))
\]

\[
\leq g \circ \nabla f(x(k)) - \alpha_k \left( 1 - L \alpha_k \gamma \max \left| \frac{\partial f(z(k))}{\partial x} \right|^{\gamma-1} \right) (\sigma \circ \nabla f(x(k)))^T H(z(k)) \sigma \circ \nabla f(x(k))
\]

\[
\leq g \circ \nabla f(x(k)) - \alpha_k \left( 1 - L \alpha_k \gamma \max \left| \frac{\partial f(z(k))}{\partial x} \right|^{\gamma-1} \right)
\]

\[
m(\gamma+1) \cdot 2^{\gamma} g \circ \nabla f(x(k))^{2^{\gamma}}.
\]

Here \( z(k) = tx(k) + (1-t)x(k) = x(k) - \alpha_k \sigma \circ \nabla f(x(k)) \) for some \( 0 < t < 1 \).

Equality (a) is due to \( \frac{\partial y}{\partial x} = (\gamma+1) \text{sign}(y)_y^\gamma \).

Inequality (b) is derived based on \( \sigma \circ \nabla f(z(k)) - \sigma \circ \nabla f(x(k)) = -\alpha_k \left( \int^1_0 J(z(k)) \sigma \circ \nabla f(x(k)) \right) \), in which \( J(z(k)) = \gamma \text{diag} \left( \sigma_{\gamma-1} \frac{\partial f(z(k))}{\partial x} \right) \) for \( z(k) = (1-t)z(k) + t \circ x(k) \) \((0 < t < 1) \). Since \( LI \geq H(z(k)) \geq ml \),

\[
\frac{\sigma \circ \nabla f(x(k))}{\sigma \circ \nabla f(x(k))^T H(z(k)) \sigma \circ \nabla f(x(k))} \leq \left\| H^{-1}(z(k)) \left( \int^1_0 J(z(k)) \sigma \circ \nabla f(x(k)) \right) \right\|_2 \]

\[
= \left\| \left( \int^1_0 J(z(k)) \sigma \circ \nabla f(x(k)) \right) \right\|_2 \leq \int^1_0 \left\| J(z(k)) \sigma \circ \nabla f(x(k)) \right\| dt
\]

\[
\leq L \left( \min_{\gamma} \min_{\gamma} \left| \frac{\partial f(z(k))}{\partial x} \right|^{\gamma-1} \right) \leq L \max_{\gamma} \gamma \left| \frac{\partial f(x(k))}{\partial x} \right|^{\gamma-1}.
\]

Inequality (c) holds using the following facts: a) the Hessian \( H(z(k)) \geq ml \) for a strongly convex function \( f \); b) for any \( y_i \in \mathbb{R} : \sum^n_{i=1} |y_i|^2 \geq (\sum^n_{i=1} |y_i|^{\gamma+1})^{\frac{2^{\gamma}}{4\gamma}} \), \( \forall \gamma \in (0, 1) \); and c) we can choose \( \alpha_k \) such that \( 1 - L \alpha_k \gamma \max_{\gamma} \left| \frac{\partial f(x(k))}{\partial x} \right|^{\gamma-1} > 0 \).

Indeed if we set \( \alpha_k = \frac{1}{2L \gamma \max_{\gamma} \left| \frac{\partial f(x(k))}{\partial x} \right|^{\gamma-1}} \), then the righthand side in eq. (17) satisfies

\[
\text{RHS} \leq g \circ \nabla f(x(k)) - \frac{(\gamma+1) \cdot 2^{\gamma} g \circ \nabla f(x(k))^{2^{\gamma}}}{4\gamma \max_{\gamma} \left| \frac{\partial f(x(k))}{\partial x} \right|^{\gamma-1}} \leq g \circ \nabla f(x(k)) - \frac{(\gamma+1) \cdot 2^{\gamma} g \circ \nabla f(x(k))^{2^{\gamma}}}{4\gamma \max_{\gamma} \left| \frac{\partial f(x(k))}{\partial x} \right|^{\gamma-1}}.
\]

Let \( V(k) \triangleq (\gamma+1)g \circ \nabla f(x(k)) \), then the above equation writes

\[
V(k+1) \leq V(k) - \frac{V^{2^{\gamma}}(k)}{4\gamma \max_{\gamma} \left| \frac{\partial f(x(k))}{\partial x} \right|^{\gamma-1}}.
\]

Let \( \lambda \triangleq \min_{\gamma} \min_{\gamma} \left| \frac{\partial f(x(k))}{\partial x} \right| \) and \( C_\gamma = \frac{4^2}{\gamma} \), we can then apply Lemma 2

\[
V(N) \leq \left( -\frac{N}{\lambda^2} \cdot \frac{1}{C_\gamma} + V^{1-\gamma}(0) \right)^{\frac{1}{1-\gamma}}.
\]
If we would like to control the error between the gradient at iteration $N$ close to 0, we can impose that as $\lambda = \epsilon$, then the number of steps $N$ should satisfy, in the worst case

$$N \geq \kappa \lambda^{-1} C_{\gamma} (V^{1+\gamma}(0) - V^{1+\gamma}(N)) \geq \kappa C_{\gamma} \left( \frac{V^{1+\gamma}(0)}{\epsilon} \right)^{1-\gamma} - 1 .$$

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B Newton Powerball method and line search Powerball method

B.1 Newton Powerball Method

For any twice differentiable function $f$, we have the following Newton Powerball method with $\beta_k$ step size

$$x(k + 1) = x(k) - \beta_k (\nabla^2 f(x(k)))^{-1} \sigma_{\gamma}(\nabla f(x(k))) \quad \text{for } k = 0, 1, \ldots$$

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We make a slightly stronger assumption than the standard assumption [9] for deriving the quadratic convergence rate of Newton’s method and will relax this in future work.

Assumption 1. Assume that $x^*$ is a point that $\nabla f(x^*) = 0$, suppose that there exists a neighborhood of $x^*$ such that $H(x) \approx H(y)$ for any $\|x - x^*\|_2 \leq \delta$ and $\|y - x^*\|_2 \leq \delta$.

Proposition 4. For any twice differentiable function $f$ (with L-Lipschitz gradient) under Assumption 4, if we initialize the method (19) using an $x(0)$ such that $\|x(0) - x^*\| \leq \delta$, then the Newton Powerball method converges an $\epsilon$-neighborhood of a minimizer when the number of iterations

$$N \geq LC_{\gamma} \left( \frac{V^{1+\gamma}(0)}{\epsilon} \right)^{1-\gamma} - 1 .$$

Proof. Similar to the previous proof, consider the following continuously increasing differentiable function

$$g \left( \frac{\partial f(x(k))}{\partial x} \right) = \sum g_i \left( \frac{\partial f(x(k))}{\partial x} \right) = \sum \frac{1}{\gamma + 1} \left| \frac{\partial f(x(k))}{\partial x} \right|^{\gamma - 1} .$$

For any $i$, (using Taylor theorem)

$$g_i \circ \nabla f(x(k + 1)) = g_i \circ \nabla f(x(k)) - \beta_k \sigma_{\gamma} \circ \nabla f(z(k)) H_i (z(k)) H^{-1} (x(k)) \sigma_{\gamma} \circ \nabla f(x(k)) ,$$

here $z(k) = tx(k) + (1 - t)x(k) = x(k) - \beta_k t H^{-1} (x(k)) \sigma_{\gamma} \circ \nabla f(x(k))$ for some $0 < t < 1$ and $H_i$ is the $i$th row of the Hessian. We can sum up over all $i$ and assume that $H(z(k)) H(x(k)) \approx I$ when $k$ is large enough

$$g \circ \nabla f(x(k + 1)) - g \circ \nabla f(x(k))$$

$$= -\beta_k \sigma_{\gamma} \circ \nabla f(z(k))^T \sigma_{\gamma} \circ \nabla f(x(k))$$

$$= -\beta_k \sigma_{\gamma} \circ \nabla f(x(k))^T \sigma_{\gamma} \circ \nabla f(x(k)) + \beta_k \sigma_{\gamma} \circ \nabla f(x(k))^T (\gamma + 1) H(z(k)) J(z(k)) \sigma_{\gamma} \circ \nabla f(x(k)) .$$

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$$\leq -\beta_k \left( 1 - \beta_k L_{\gamma} (\gamma + 1) \max \left| \frac{\partial f(x(k))}{\partial x_i} \right|^{\gamma - 1} \right) \sigma_{\gamma} \circ \nabla f(x(k))^T \sigma_{\gamma} \circ \nabla f(x(k)) .$$

Last inequality is due to

$$\left| \sigma_{\gamma} \circ \nabla f(x(k))^T H(z(k)) J(z(k)) \sigma_{\gamma} \circ \nabla f(x(k)) \right| \sigma_{\gamma} \circ \nabla f(x(k))^T \sigma_{\gamma} \circ \nabla f(x(k)) \right| \leq ||H(z(k))||_2 ||J(z(k))||_2 .$$

We can derive similar bounds for Newton Powerball method and obtain $N \geq LC_{\gamma} \left( \frac{V^{1+\gamma}(0)}{\epsilon} \right)^{1-\gamma} - 1 .$

Remark 5. The coefficient for strongly convex function $m$ (in Definition 1) is normally very small, i.e., $m < 1$, this proposition shows that Newton Powerball method has a faster convergence rate than Gradient Powerball method when $x(k)$ gets close to $x^*$ in the later iterations.
For any twice differentiable function $f$ with $L$-Lipschitz gradient, we now propose the line search Powerball method as follows:

1. pick a descent direction $-\text{sign}(\nabla f(x(k)))\nabla f(x(k))[T];$
2. pick a step size $\alpha_k = \arg\min_{\tau \geq 0} \{ f(x(k)) - \tau\text{sign}(\nabla f(x(k)))\nabla f(x(k))[T] \};$
3. update $x(k + 1) = x(k) - \alpha_k \sigma(\nabla f(x(k))).$

Next, we shall analyze its convergence rate. Applying Taylor theorem, for some $z(k)$

$$f(x(k) - \tau\nabla f(x(k)))\nabla f(x(k))[T] = f(x(k)) - \tau\nabla f(x(k))^T\text{sign}(\nabla f(x(k)))\nabla f(x(k))[T] + \frac{1}{2}\tau^2(\text{sign}(\nabla f(x(k)))\nabla f(x(k))[T]^T\nabla^2 f(z(k))\text{(sign}(\nabla f(x(k)))\nabla f(x(k))[T]).$$

Again, by defining $V(k) = \sum \left| \frac{\partial f(x(k))}{\partial x_i} \right|^{T+1}$ and

$$\alpha_k = \frac{(\gamma + 1)V(k)}{(\gamma + 1)|V(x(k))|} \text{sup}_{|y| \leq 1}(\text{sign}(\nabla f(x(k)))\nabla f(x(k))[T]^T\nabla^2 f(z(k))\text{(sign}(\nabla f(x(k)))\nabla f(x(k))[T].$$

Let $x(k + 1) = x(k) - \alpha_k \text{sign}(\nabla f(x(k)))\nabla f(x(k))[T]$, then

$$f(x(k + 1)) - f(x(k)) = -\tau \frac{V^2(k)}{2(|V f(x(k))[T]|)^T \nabla^2 f(z(k))\nabla f(x(k))[T] \leq \frac{V^2(k)}{2Ln^{1+\gamma}}.$$}

The above inequality holds using the fact that, for all $\gamma \in (0, 1)$, $\sup_{i,j=1,2,...,n} \frac{|x_i|^\gamma}{(\sum_{i=1}^n |x_i|)^{1+\gamma}} = n^{-\frac{\gamma}{1+\gamma}}.$

We can rearrange this and obtain

$$(f(x_0) - f(x_N))\sqrt{N}^{-\gamma} \geq \sum_{k=0}^{N-1} \frac{V^2(k)}{2Ln^{1+\gamma}} \geq \sum_{k=0}^{N-1} \frac{\min_{0 \leq k \leq N} ||V f(x(k))||_2^2}{N^{1+\gamma}}.$$}

Using a fact that

$$\frac{L}{2}||x_0 - x_0||^2 \geq \min_{0 \leq k \leq N} \frac{||V f(x(k))||_2^2}{N^{1+\gamma}}.$$

Since $\nabla f$ is Lipschitz continuous, we obtain the following convergence rate

$$\frac{L||x_0 - x_0||^\frac{1-\gamma}{\gamma}}{\sqrt{N}} \geq \sqrt{\frac{(f(x_0) - f(x_N))\sqrt{N}^{-\gamma}}{N}} \geq \frac{\min_{0 \leq k \leq N} ||V f(x(k))||_2^2}{N^{1+\gamma}}.$$}

### C Experiments

In this section, we shall run some further experiments using gradient descent and quasi-Newton methods for a larger number of iterations. The cost function is chosen as a standard logistic regression with $l_2$ regularization in eq. (15).

We observe in Fig. 6 that the 1-bit gradient descent (when $\gamma = 0$) has a faster convergence rate than the standard gradient method and gradient Powerball method (C) in solving the logistic regression on CTR.

We observe from Fig. 7, as we expected, that the L-BFGS Powerball algorithm outperforms the L-BFGS algorithm in the initial iterations while L-BFGS algorithm outperforms the L-BFGS Powerball algorithm in the later iterations.
Figure 6: We use gradient descent versus gradient Powerball method to solve the logistic regression with $l_2$ regularization on CTR. For strongly convex functions, we observe that the 1−bit gradient descent method has a faster convergence rate than the gradient Powerball method when choosing $1 > \gamma > 0$.

Figure 7: We use L-BFGS versus L-BFGS Powerball to solve the logistic regression with $l_2$ regularization on CTR. The L-BFGS Powerball algorithm outperforms the L-BFGS algorithm in the initial iterations ($< 150$ iterations) while L-BFGS algorithm outperforms the L-BFGS Powerball algorithm in the later iterations ($> 150$ iterations).