AN AERATED TRIANGULAR ARRAY OF INTEGERS

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Abstract
Congruences modulo prime powers involving generalized Harmonic numbers are known [12], [5]. While looking for similar congruences, we have encountered simple, but not so well-known identities for the Stirling cycle numbers and a curious triangular array of numbers indexed with positive integers \( n, k \), involving the Bernoulli and Stirling cycle numbers. It is shown that these numbers are all integers and that they vanish when \( n - k \) is odd. These integers do not seem to have been previously investigated.

1. Introduction
Let \( n \) and \( k \) be non-negative integers and let the generalized Harmonic numbers \( H_n^{(k)} \) and \( G_n^{(k)} \) be defined as

\[
H_n^{(k)} := \sum_{j=1}^{n} \frac{1}{j^k} \quad \text{and} \quad G_n^{(k)} := \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \frac{1}{i_1 i_2 \cdots i_k},
\]

with \( H_n^{(1)} = G_n^{(1)} = \sum_{j=1}^{n} \frac{1}{j} = H_n = G_n \); \( H_n^{(0)} = n \) and \( G_n^{(0)} = 1 \). It is known [8] that \( \left\lceil \frac{n+1}{k+1} \right\rceil = n! G_n^{(k)} \left\lfloor \frac{n}{k} \right\rfloor \) being the Stirling cycle number (unsigned Stirling number of first kind), so that the Harmonic and Stirling cycle numbers are inter-related by the convolution

\[
k \left\lfloor \frac{n+1}{k+1} \right\rfloor = -\sum_{j=0}^{k-1} (-1)^{k-j} H_n^{(k-j)} \left\lfloor \frac{n+1}{j+1} \right\rfloor,
\]

which is obtained as a direct application of the well-known [7] relation between elementary symmetric polynomials and power sums.

Extended congruences for the Harmonic numbers \( H_n^{(k)} \), modulo any power of a prime \( p \) are known [12], [5]. Our initial motivation for the work reported in the present paper is to look for similar congruences modulo prime powers, involving
\( G_{p-1}^{(k)} \), or the Stirling cycle numbers \([p_{k+1}]\), instead of \( H_{p-1}^{(k)} \). We will show that such similar congruences for \( G_{p-1}^{(k)} \) do exist, but that they are just the particular prime instances of not very well-known but elementary identities for the Stirling cycle numbers. We will eventually introduce a new triangular array of integers, involving the Bernoulli and Stirling cycle numbers.

2. Notation and preliminaries

In addition to what was exposed in the previous introduction, further notation that we use throughout this paper is presented in this section, along with classical results which we will need. Most of these results can be found in textbooks like [9] and they are given hereafter without proof. In the following, \( g, h, i, j, k, \ell, m, n \) denote integers, \( p \) a prime number, and \( x \) or \( t \) denote the argument in a generating function. Let \( f(x) \) be a formal series in powers of \( x \), we denote \([x^n]f(x)\) the coefficient of \( x^n \) in \( f(x) \) and \( D_m f(x) \) is the \( m \)-order derivative of \( f(x) \) with respect to \( x \). If \( x \) is a real number, we denote \( \lfloor x \rfloor \) the largest integer smaller or equal to \( x \), \( \lceil x \rceil \) the smallest integer larger or equal to \( x \). We will use the Iverson bracket notation: \([\mathcal{P}] = 1 \) when proposition \( \mathcal{P} \) is true, and \([\mathcal{P}] = 0 \) otherwise.

The binomial coefficients \( \binom{n}{k} \), are defined by \( \sum_k \binom{n}{k} x^k = (1 + x)^n \), whatever the sign of integer \( n \). They obviously vanish when \( k < 0 \). When \( n > 0 \), we have \( \binom{-n}{k} = (-1)^k \binom{n+k-1}{n-1} \). They are easily obtained by the basic recurrence relation \( \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \) and they obey the “hockey stick” identity

\[
\binom{n}{k+1} = \sum_{j \geq 1} \binom{n-j}{k}, \tag{2.1}
\]

The Stirling cycle numbers \( [n]_k \), \( n \geq 0 \) may be defined by the horizontal generating function

\[
\sum_{k} [n]_k x^k = \prod_{j=0}^{n-1} (x + j), \tag{2.2}
\]

where an empty product is meant to be 1. They obviously vanish when \( k < 0 \) and \( k > n \). They are easily obtained by the basic recurrence \( [n]_k = (n-1)[n-1]_k + [n-1]_{k-1} \), valid for \( n \geq 1 \), with \( [0]_k = [k = 0] \). They also obey the generalized recurrence relation

\[
\frac{n+1}{m+1} = \sum_{h \geq 0} \binom{h+m}{h} \binom{n}{h+m}. \tag{2.3}
\]
Let \( \{ \binom{n}{k} \} \), \( n \geq 0 \), be the partition (or second kind) Stirling number. They also vanish when \( k < 0 \) and \( k > n \). Their basic recurrence is \( \{ \binom{n}{k} \} = k \{ \binom{n-1}{k} \} + \{ \binom{n-1}{k-1} \} \) for \( n \geq 1 \), with \( \{ \binom{0}{k} \} = [k=0] \). They also obey a generalized recurrence relation

\[
\{ \binom{n+1}{m+1} \} = \sum_{j=0}^{\infty} \binom{n}{j} \binom{j}{m}
\]

and they have an explicit expression

\[
\{ \binom{n}{k} \} = \frac{(-1)^{n-k}}{k!} \sum_{j=0}^{\infty} (-1)^{j} \binom{k}{j} j^n.
\]

Let \( B_h \) be the Bernoulli number \( (B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, \ldots \) and \( B_{2h+1} = 0 \) for \( h > 0 \)). We also introduce the Bernoulli polynomials \( B_n[x] \). We will use the following classical properties of the Bernoulli numbers and polynomials. The Bernoulli numbers have the exponential generating function

\[
\frac{t}{e^{t} - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.
\]

By definition, the Bernoulli polynomials are

\[
B_n[x] := \sum_{k=0}^{\infty} \binom{n}{k} B_k x^{n-k},
\]

and their exponential generating function is

\[
\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n[x] \frac{t^n}{n!},
\]

so that we have \( B_n[0] = B_n \) and the translation formula:

\[
B_n[x + y] = \sum_{k=0}^{\infty} y^{n-k} \binom{n}{k} B_k [x]
\]

and

\[
B_n[1] = (-1)^{n} B_n.
\]

and

\[
[n = 1] = \sum_{k=0}^{n-1} \binom{n}{k} B_k
\]
Moreover, the Bernoulli numbers are related to the Stirling partition numbers \([9]\) by
\[
B_k = \sum_{m \geq 0} (-1)^m \frac{m!}{m+1} \binom{k}{m}
\]  (2.9)

We will also make use of the Von Staudt-Clausen theorem which states that the denominator of \(B_k\) in reduced form, is the product of all primes \(p\) such that \(p - 1\) divides \(k\). In particular, any prime may divide the denominator of a Bernoulli number once at most.

We shall need the Legendre formula which states that the highest power of a prime \(p\) which divides \(j!\) is \(\frac{j - s_p(j)}{p-1}\), where \(s_p(j)\) be the sum of the standard base-\(p\) digits of \(j\) and the Kummer theorem which says that the highest power of \(p\) which divides \(\binom{n}{k}\) is the number of carries when doing the addition \(k + (n - k)\) in base \(p\). In particular, we will make use of two consequences of Kummer theorem: (i) we have \(\binom{2n}{n} \equiv \binom{n}{k} \mod 2\) and (ii) \(\binom{n}{k}\) is even when \(n\) is even and \(k\) is odd. We recall the Wilson theorem which states that \((p-1)! \equiv -1 \mod p\) for any prime \(p\), and some other well-known congruences, valid for any prime \(p\):
\[
\binom{p}{j} \equiv [j = 0 \text{ or } j = p] \pmod{p},
\]  (2.10)
\[
\binom{p}{j} \equiv [j = 1 \text{ or } j = p] \pmod{p}.
\]  (2.11)

Moreover,
\[
\binom{p - 1}{j} \equiv (-1)^j [0 \leq j \leq p - 1] \pmod{p},
\]
\[
\sum_{p - 1 \geq j \geq 1} j^k \equiv (-1)[p - 1 \text{ divides } k] \pmod{p},
\]
so that, from (2.5), when \(n > 0\), we have
\[
\binom{n}{p - 1} \equiv [p - 1 \text{ divides } n] \pmod{p}.
\]  (2.12)

### 3. Lemmas

Some lemmas are also going to be used. Since they may not be as well-known as the classic results of the previous section, they are given hereafter with proofs, for the sake of self-containment.
Lemma 3.1. For the parity of the Stirling numbers of the first kind, we have
\[
\binom{n}{k} \equiv \left( \left\lfloor \frac{n}{2} \right\rfloor \right) \left( k - \left\lceil \frac{n}{2} \right\rceil \right) \pmod{2}.
\] (3.1)

Proof. We reproduce the proof from [13]. Starting from (2.2), we have
\[
\sum_{k=0}^{n} \binom{n}{k} x^k = \prod_{k=0}^{n-1} (x + k)
\equiv x(x + 1)x(x + 1) \cdots \pmod{2}
\equiv x\left\lfloor \frac{x}{2} \right\rfloor (x + 1)\left\lceil \frac{x}{2} \right\rceil \pmod{2},
\]
hence
\[
\binom{n}{k} \equiv \left[ \left\lfloor \frac{x}{2} \right\rfloor \left( x + 1 \right)\left\lceil \frac{x}{2} \right\rceil \right] \pmod{2}
\equiv \binom{k - \left\lfloor \frac{n}{2} \right\rfloor}{\left\lceil \frac{n}{2} \right\rceil} \pmod{2}.
\]

Lemma 3.2. Let \( n, m, k \geq 0 \) be integers, the following identity holds
\[
\left\{ \begin{array}{c} n \\ m \end{array} \right\} = \sum_{j=0}^{k} \sum_{i=0}^{k} \binom{k}{j} \binom{n-k}{m-i} (-1)^{i+j} m^{k-j}.
\] (3.2)

Proof. This is Proposition 2.2 in [10]. Equation (3.2) can be rewritten as
\[
\left\{ \begin{array}{c} n+k \\ m \end{array} \right\} = \sum_{i=0}^{k} (-1)^{i} P_{k,i}(m) \left\{ \begin{array}{c} n \\ m-i \end{array} \right\},
\] (3.3)
where \((P_{k,i}; k, i \geq 0)\), is a family of polynomials of degree \(\max(0, k-i)\), defined as
\[
P_{k,i}(m) := \sum_{j=0}^{k} (-1)^{j} \binom{k}{j} \binom{j}{i} m^{k-j}.
\] (3.4)

We give a proof of Equation (3.3) by induction on \( k \). It is clear that \( P_{k,i} = 0 \), when \( i > k \). Equation (3.3) is trivially true for \( k = 0 \), for all \( n, m \geq 0 \), since \( P_{0,0} = 1 \).
Our induction hypothesis is that Equation (3.3) holds for \( k \) and for all \( m, n \). Then
\[
\binom{n+k+1}{m} = m \binom{n+k}{m} + \binom{n+k}{m-1}
\]
\[
= \sum_{i=0}^{k} (-1)^i mP_{k,i}(m) \binom{n}{m-i} + \sum_{i=0}^{k} (-1)^i P_{k,i}(m-1) \binom{n}{m-(i+1)}
\]
\[
= \sum_{i=0}^{k+1} (-1)^i (mP_{k,i}(m) - P_{k,i-1}(m-1)) \binom{n}{m-i}.
\]
Then, there remains to show that \( mP_{k,i}(m) - P_{k,i-1}(m-1) = P_{k+1,i}(m) \). We have
\[
mP_{k,i}(m) - P_{k,i-1}(m-1)
\]
\[
= \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{j}{i} m^{k-j+1} - \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{j}{i-1} (m-1)^{k-j}
\]
\[
= \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{j}{i} m^{k-j+1} - \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{j}{i-1} \sum_{h=0}^{h} \binom{k-j}{h} m^h (-1)^{k-j-h}
\]
\[
= \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{j}{i} m^{k-j+1} - \sum_{h=0}^{h} m^h (-1)^h \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{j}{i-1} \binom{k-j}{h} (-1)^{k-j}
\]
\[
= \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{j}{i} m^{k-j+1} - \sum_{h=0}^{h} m^h (-1)^h \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{j}{i-1} \binom{k-j}{h} \frac{1}{h!(k-j-h)!} (-1)^{k-j}
\]
\[
= \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{j}{i} m^{k-j+1} - \sum_{h=0}^{h} m^h (-1)^h \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{j}{i-1} \binom{k-j}{h} \frac{1}{h!(k-j-h)!} (-1)^{k-j}
\]
\[
= \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{j}{i} m^{k-j+1} - \sum_{h=0}^{h} m^h (-1)^h \sum_{j=0}^{k} \binom{k}{h} m^{k-h} (-1)^h \binom{h+1}{i} \text{ by 2.21}
\]
\[
= \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{j}{i} m^{k-j+1} + \sum_{h=0}^{h} \binom{k}{h} m^{k-h+1} (-1)^h \binom{h}{i}.
\]
That is
\[ mP_{k,i}(m) - P_{k,i-1}(m-1) = \binom{0}{i} m^{k+1} + \sum_{j \geq 1} (-1)^j \binom{k+1}{j} \binom{i}{j} m^{k-j+1} \]
\[ = \sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j} \binom{i}{j} m^{k-j} = P_{k+1,i}(m). \]

\[ \square \]

**Lemma 3.3.** Let \( p \) be prime and \( n \geq 0, q \geq 1 \), we have
\[ \binom{n}{pq} \equiv [p-1 \text{ divides } n-q] \binom{n-q}{p-1} \binom{q-1}{pq} \pmod{p}. \]  
(3.5)

**Proof.** This is also quite well-known. See for instance [3], [6] and [4] for the more general case where the modulus is a prime power. The proof generally involves the generating function of the Stirling partition numbers. Here, we give a different proof. We shall first recall and prove another known [10] congruence:
\[ \binom{n}{m} \equiv \binom{n-p}{m-p} + \binom{n-p+1}{m} \pmod{p}. \]  
(3.6)

Indeed, when letting \( k = p \) into Equation (3.2) and accounting for (2.10) and (2.11), we have
\[ \binom{n}{m} = \sum_{j=0}^{p} \sum_{i=0}^{p} (-1)^{i+j} m^{p-j} \binom{p}{j} \binom{i}{m-i} \binom{n-p}{m} \pmod{p} \]
\[ \equiv m^p \binom{n-p}{m} + \sum_{i \geq 0} (-1)^{p+i} \binom{i}{m} \binom{n-p}{i} \pmod{p} \]
\[ \equiv m^p \binom{n-p}{m} + (-1)^{p+1} \binom{n-p}{m-1} + \binom{n-p}{m-p} \pmod{p} \]
\[ \equiv m^p \binom{n-p}{m} + \binom{n-p}{m-1} + \binom{n-p}{m-p} \pmod{p} \]
\[ \equiv m \binom{n-p+1}{m} + \binom{n-p}{m} \pmod{p}. \]

Now, for the proof of Lemma 3.3 we write (3.6) with \( n-jp \) for \( n \) and \( (q-j)p \) for \( m \). That is
\[ \binom{n-jp}{(q-j)p} \equiv \binom{n-(j+1)p}{(q-(j+1))p} + \binom{n-(j+1)p+1}{(q-j)p} \pmod{p}, \]
then, summing from \( j = 0 \) to \( j = q - 1 \) and telescoping, we obtain
\[
\left\{ \frac{n}{pq} \right\} \equiv [n = qp] + \sum_{j=1}^{q} \left\{ \frac{n - qp - (p - 1) + jp}{jp} \right\} \pmod{p}. \tag{3.7}
\]

We fix \( p \) and we are going to prove Lemma 3.3 by induction on \( n \). It is easy to see directly that it is true for \( n = 0, n = 1 \) or \( n = 2 \). We suppose (induction hypothesis) that for any \( r \geq 1 \) and all \( N < n \), \( \{N\} \equiv [p - 1 \text{ divides } N - r] \left(\frac{N}{p-1} - 1\right) \pmod{p} \). Actually, we don’t need the induction hypothesis to see that (3.5) is true if \( n > pq \). When \( n < pq \), \( \left\{ \frac{n}{pq} \right\} = 1 \) and \( [p - 1 \text{ divides } n - q] = 0 \) since \( n - q < q(p - 1) \leq p - 1 \).

When \( n = pq \), \( \left\{ \frac{n}{pq} \right\} = 1 \) and \( [p - 1 \text{ divides } n - q] \left(\frac{pq}{pq} - 1\right) = (\frac{q-1}{q-1}) = 1 \). So we only need to consider the case \( n > pq \). Since \( j \leq q \), we have \( n - (q - j + 1)p + 1 \leq n - p + 1 < n \) and then, by the induction hypothesis, we have
\[
\left\{ \frac{n - qp - (p - 1) + jp}{jp} \right\} \equiv [p - 1 \text{ divides } n - q] \left(\frac{n - qp - (p - 1) + jp - j}{p - 1} - 1\right) \pmod{p}
\]
\[
\equiv [p - 1 \text{ divides } n - q] \left(\frac{n - q}{p - 1} + j - 2 - q\right) \pmod{p}
\]
\[
\equiv [p - 1 \text{ divides } n - q] \left(\frac{n - q}{p - 1} + j - 2 - q\right) \pmod{p}.
\]

Then, accounting for \( n > pq \), Equation (3.7) becomes
\[
\left\{ \frac{n}{pq} \right\} \equiv [p - 1 \text{ divides } n - q] \sum_{j=1}^{q} \left(\frac{n - q}{p - 1} + j - 2 - q\right) \pmod{p}.
\]

If \( pq < n < pq + p - 1 \), \( \left(\frac{pq + j - 2 - q}{pq - 1 - q}\right) = 0 \) and \( p - 1 \) does not divides \( n - q \), then
\[
\left\{ \frac{n}{pq} \right\} \equiv 0 \equiv [p - 1 \text{ divides } n - q] \left(\frac{n - q}{p - 1} - 1\right) \pmod{p}.
\]

If \( n \geq pq + p - 1 \), by \((2.1)\) we have \( \sum_{j=1}^{q} \left(\frac{pq + j - 2 - q}{pq - 1 - q}\right) = \left(\frac{pq - 1}{pq - 1}\right) = \left(\frac{n - q}{n - q}\right) \), then
\[
\left\{ \frac{n}{pq} \right\} \equiv [p - 1 \text{ divides } n - q] \left(\frac{n - q}{n - q} - 1\right) \pmod{p}.
\]

\qed
Lemma 3.4. Let $p$ be prime, for any $n \geq 0$ and $g, k \geq 0$, we have

$$\sum_{i \geq 1} \binom{i - 1}{k} \left[ i(p - 1) + k + 1 - g \right]^n \equiv \left\{ \begin{array}{ll} g & \text{if } g \equiv 0 \pmod{(k+1)p} \\ (k+1)p - n & \text{if } g \equiv (k+1)p \pmod{p} \end{array} \right. \quad (3.8)$$

Proof. To the author’s knowledge, this congruence involving both kinds of Stirling numbers does not seem to have been published already. Let

$$a_n(g, k) := \sum_{i \geq 1} \binom{i - 1}{k} \left[ i(p - 1) + k + 1 - g \right]^n$$

$$= \sum_{i=k+1}^{n+g-k-1} \binom{i - 1}{k} \left[ i(p - 1) + k + 1 - g \right]^n.$$

Lemma 3.4 is true when $n = 0$, since

$$a_0(g, k) = \sum_{i \geq 1} \binom{i - 1}{k} \left[ i(p - 1) + k + 1 - g \right]^0$$

$$= \left[ p - 1 \text{ divides } g - k - 1 \right] \left( \frac{g-k-1}{p-k-1} - 1 \right)$$

$$= \left\{ \begin{array}{ll} g & \text{if } g \equiv 0 \pmod{(k+1)p} \\ (k+1)p & \text{if } g \equiv (k+1)p \pmod{p} \end{array} \right. \quad \text{from Lemma 3.3}$$

Now, we suppose (induction hypothesis) that for a given $n$, for all $g, k \geq 0$, we have

$$a_n(g, k) \equiv \left\{ \begin{array}{ll} g & \text{if } g \equiv 0 \pmod{(k+1)p} \\ (k+1)p - n & \text{if } g \equiv (k+1)p \pmod{p} \end{array} \right. \quad (mod \ p).$$

Then

$$a_{n+1}(g, k) = \sum_{i=k+1}^{n+g-k} \binom{i - 1}{k} \left[ i(p - 1) + k + 1 - g \right]^{n+1}$$

$$= n \sum_{i=k+1}^{n+g-k} \binom{i - 1}{k} \left[ i(p - 1) + k + 1 - g \right]^n$$

$$+ \sum_{i=k+1}^{n+g-k} \binom{i - 1}{k} \left[ i(p - 1) + g - k \right]^n$$

$$= n \sum_{i=k+1}^{n+g-k} \binom{i - 1}{k} \left[ i(p - 1) + k + 1 - g \right]^n$$

$$+ \sum_{i=k+1}^{n+g-k} \binom{i - 1}{k} \left[ i(p - 1) + g - k \right]^n.$$
Indeed, the first sum on the right-hand side may be limited to \( \frac{n + g - k - 1}{p - 1} \) because even though \( i = \frac{n + g - k}{p - 1} \) may be integer, the corresponding summand is zero because \( \left[ \frac{n}{n+1} \right] = 0 \). Then

\[
a_{n+1}(g, k) = n \cdot a_n(g, k) + a_n(g + 1, k)
\]

\[
\equiv n \left\{ \frac{g}{(k+1)p-n} \right\} + \left\{ \frac{g+1}{(k+1)p-n} \right\} \pmod{p}
\]

\[
\equiv -(k+1)p-n \left\{ \frac{g}{(k+1)p-n} \right\} + \left\{ \frac{g+1}{(k+1)p-n} \right\} \pmod{p}
\]

\[
\equiv -(k+1)p-n \left\{ \frac{g}{(k+1)p-n} \right\} + \left\{ \frac{g}{(k+1)p-n} \right\} \pmod{p}
\]

\[
+ (k+1)p-n \left\{ \frac{g}{(k+1)p-n} \right\} + \left\{ \frac{g}{(k+1)p-n} \right\} \pmod{p}
\]

\[
\equiv \left\{ \frac{g}{(k+1)p-\left(n+1\right)} \right\} \pmod{p}.
\]

\[
\square
\]

**Lemma 3.5.** Let \( n, k \) be non-negative integers such that \( k \leq n \) and \( Q_k \) a function such that \( Q_k(n) = \left[ \frac{n}{n-k} \right] \). Then \( Q_k \) is a polynomial function of degree \( 2k \). Moreover, \( Q_k(-1) = 1 \) and, if \( k > 0 \) then \( 0, 1, \ldots, k \) are roots of the polynomial \( Q_k \).

**Proof.** This can be found in [9]. Our proof is by induction on \( k \). Suppose that \( Q_{k-1} \) is a polynomial function of degree \( 2k-2 \), then by the fundamental recurrence relation for Stirling cycle numbers, we have:

\[
Q_k(n+1) - Q_k(n) = nQ_{k-1}(n)
\]

Then

\[
Q_k(n+1) - Q_k(1) = \sum_{j=1}^{n} jQ_{k-1}(j)
\]

The right-hand side in the above is a polynomial function of \( n \) of degree \( 1+2k-2+1 \) by a classical result about the sum of powers, and then \( Q_k(n) \) is a polynomial function of \( n \) of degree \( 2k \). And since \( Q_0(n) = 1 \) is a polynomial function of degree
0, this completes the proof that \( Q_k(n) \) is indeed a polynomial function of degree 2k. It is clear that \( Q_k(k) = [k] = 0 \) for \( k > 0 \). Since \( Q_k(n + 1) - Q_k(n) = nQ_{k-1}(n) \), letting \( n = k - 1 \), we get \( Q_k(k - 1) = 0 \), and then by iteration of the same process with \( n = k - 2, k - 3, \ldots \) till \( n = 0 \), we see that 0, 1, ..., \( k \) are roots of \( Q_k \). Now, since we have \( Q_k(0) = Q_k(-1) = -1 \cdot Q_{k-1}(-1) \), we see that \( Q_k(-1) = Q_{k-1}(-1) \) for any \( k > 0 \). Then \( Q_k(-1) = 1 \) for any \( k \geq 0 \) since \( Q_0(-1) = 1 \).

4. Two identities for the Stirling cycle numbers

In this section, we will demonstrate two identities for the Stirling cycle numbers.

**Theorem 4.1.** Let \( m, n \) be non-negative integers. We have

\[
\binom{n + 1}{m + 1} = (-1)^{n-m} \sum_{h=0}^{\infty} \binom{h + m}{m} \binom{n}{h + m} (-n)^h. \tag{4.1}
\]

Moreover, if \( n > 0 \),

\[
\binom{n}{m} = (-1)^{n-m} \sum_{h=0}^{\infty} \binom{h + m - 1}{m - 1} \binom{n}{h + m} (-n)^h. \tag{4.2}
\]

In more symmetric formulations, these two identities also read

\[
(-1)^{n-m} \binom{n + 1}{m + 1} (-n)^m = \sum_h \binom{h}{m} \binom{n}{h} (-n)^h \tag{4.3}
\]

and, if \( n > 0 \),

\[
(-1)^{n-m} \binom{n}{m} (-n)^m = \sum_h \binom{h - 1}{m - 1} \binom{n}{h} (-n)^h. \tag{4.4}
\]

**Remark.** In spite of their similarity to (2.3), these identities do not seem to be very well-known. They are not in [9] where quite many finite sums, recurrences and convolutions involving Stirling numbers are reported. Our equation (4.1) may be obtained as a particular case of Theorem 3 in [2]. An identity equivalent to our equation (4.2) is obtained incidentally in [1], where it is not even labelled. Another identity, equivalent to our equation (4.2) is the equation (18) in [11], where it is said to be new.

**Proof of Theorem 4.1.** Like in [1], our proof will highlight that (4.1) and (4.2) are actually closely related to the convolution identity (1.1) between the Harmonic and Stirling cycle numbers. Let \( f_n(x) := \prod_{h=0}^{n-1} (x - h) \). We are going to show that,
for \( m \geq 1, \)

\[
\frac{D^m f_n(x)}{m!} = - \sum_{h=0}^{m-1} (-1)^{m-h} \frac{D^h f_n(x)}{h!} \sum_{j=0}^{n-1} \frac{1}{(x-j)^{m-h}}. \tag{4.5}
\]

It is true for \( m = 1, \) since \( D f_n(x) = \sum_{j=0}^{n-1} \prod_{k\neq j} (x - h) = f_n(x) \sum_{j=0}^{n-1} \frac{1}{x-j}. \) We suppose (induction hypothesis) that is true for some \( m, \) then

\[
(m + 1) \frac{D^{m+1} f_n(x)}{(m+1)!} = D \frac{D^m f_n(x)}{m!} = \frac{1}{m} D \left( m \frac{D^m f_n(x)}{m!} \right)
= \frac{1}{m} \left( \sum_{h=0}^{m-1} (-1)^{m-h} \frac{D^h f_n(x)}{h!} \sum_{j=0}^{n-1} \frac{1}{(x-j)^{m-h}} \right)
+ \frac{1}{m} \left( \sum_{h=0}^{m-1} (-1)^{m-h} \frac{D^h f_n(x)}{h!} (m-h) \sum_{j=0}^{n-1} \frac{1}{(x-j)^{m+1-h}} \right)
= \frac{1}{m} \sum_{h=1}^{m} (-1)^{m-h} \frac{D^h f_n(x)}{(h-1)!} \sum_{j=0}^{n-1} \frac{1}{(x-j)^{m+1-h}}
+ \frac{1}{m} \left( m \sum_{h=0}^{m-1} (-1)^{m-h} \frac{D^h f_n(x)}{h!} \sum_{j=0}^{n-1} \frac{1}{(x-j)^{m+1-h}} \right)
- \frac{1}{m} \sum_{h=1}^{m} (-1)^{m-h} \frac{D^h f_n(x)}{(h-1)!} \sum_{j=0}^{n-1} \frac{1}{(x-j)^{m+1-h}}
= \frac{1}{m} \left( \sum_{h=0}^{m} \frac{D^m f_n(x)}{(m-1)!} \sum_{j=0}^{n-1} \frac{1}{x-j} \right)
+ \frac{1}{m} \left( m \sum_{h=0}^{m-1} (-1)^{m-h} \frac{D^h f_n(x)}{h!} \sum_{j=0}^{n-1} \frac{1}{(x-j)^{m+1-h}} \right)
= \sum_{h=0}^{m} (-1)^{m-h} \frac{D^h f_n(x)}{h!} \sum_{j=0}^{n-1} \frac{1}{(x-j)^{m+1-h}}.
\]

This establishes the validity of (4.5). Now, when \( x = n, \) (4.5) reads

\[
m! \frac{D^m f_n(n)}{m!} = - \sum_{h=0}^{m-1} (-1)^{m-h} \frac{D^h f_n(n)}{h!} H_{n-h}^{(m-h)}. \tag{4.6}
\]

This is the same recurrence as in (1.1), with the same initial value, since by definition \( f_n(n) = n! = [n+1]. \) Then

\[
\frac{D^n f_n(n)}{m!} = \left[ \frac{n+1}{m+1} \right].
\]
On the other hand, from (2.2), we have
\[ f_n(x) = \sum_{h=0}^{n} \left[ \binom{n}{h} \right] (-1)^{n-h} x^h, \]
then
\[ D^m f_n(x) = m! \sum_{h=0}^{n} \binom{n}{h} \left[ \binom{h}{m} \right] (-1)^{n-h} x^{h-m}, \]
and
\[ D^m f_n(n) = \sum_{h=0}^{n} \binom{n}{h} (-1)^{n-h} n^{h-m}. \]

Hence
\[ \binom{n+1}{m+1} = \sum_{h=0}^{n} \frac{n}{h} \left( \binom{h}{m} \right) (-1)^{n-h} n^{h-m} \]
\[ = \sum_{h=0}^{n} (-1)^{n-m} \left[ \frac{n}{h+m} \right] \left( \binom{h}{m} \right) (-n)^h. \]

This completes the proof of (4.1). Now, for the proof of (4.2), we also use an induction argument, but on \( m \) and backward. Our induction hypothesis is
\[ \binom{n+1}{m+1} = (-1)^{n-(m+1)} \sum_{h=0}^{n-(m+1)} \left( \binom{h+m}{m} \right) \left[ \frac{n}{h+m+1} \right] (-n)^h. \]

Then
\[ \binom{n}{m+1} = (-1)^{n-m} \sum_{h=1}^{n-m} \left( \binom{h+m-1}{m} \right) \left[ \frac{n}{h+m} \right] (-n)^{h-1}. \]

Hence
\[ n \binom{n}{m+1} = (-1)^{n-m} \sum_{h=1}^{n-m} \left( \binom{h+m-1}{m} \right) \left[ \frac{n}{h+m} \right] (-n)^{h-1}. \]

We subtract the latter equation from (4.1), and we obtain
\[ \binom{n+1}{m+1} - n \binom{n}{m+1} = (-1)^{n-m} \sum_{h=1}^{n-m} \left( \binom{h+m}{m} - \binom{h+m-1}{m} \right) \left[ \frac{n}{h+m} \right] (-n)^h. \]

That is
\[ \binom{n}{m} = (-1)^{n-m} \sum_{h=1}^{n-m} \left( \binom{h+m-1}{m-1} \right) \left[ \frac{n}{h+m} \right] (-n)^h. \]

To finish the proof, we just need that (4.2) be true for \( m = n \), which is obvious.

5. Extended congruences for the Harmonic numbers \( G_{p-1}^{(j+1)} \)

**Theorem 5.1.** Let \( k \geq 0 \) an integer and \( p \) a prime number, we have
\[ G_{p-1}^{(k)} = (-1)^k \sum_{j=0}^{k} (-1)^j \binom{j+k}{j} G_{p-1}^{(k+j)} p^j. \]  
\[ (5.1) \]
In particular, when \( k = 0 \), we have
\[
\sum_{j \geq 0} (-1)^j G_{p-1}^{(j+1)} p^j = 0. \tag{5.2}
\]

Proof. Letting \( n = p \) a prime number, and \( m = k + 1 \) in (4.2), and dividing throughout by \((p - 1)!\) provides the desired result. \( \Box \)

Recall [5] that when \( k \geq 1 \), the generalized Harmonic numbers \( H_{p-1}^{(k)} \) admit the following \( p \)-adically converging expansion:
\[
H_{p-1}^{(k)} = (-1)^k \sum_{j \geq 0} \binom{j + k - 1}{j} H_{p-1}^{(k+j)} p^j. \tag{5.3}
\]

It is interesting to point out the similarity of (5.3) and (5.1), but also some differences. Contrary to (5.3), the sum on the right-hand side of (5.1) is finite. It is actually limited to \( j = p - 1 - k \); we also notice that the sign alternates in (5.1) and that there is a slight difference in the binomial coefficient.

It is also known [12] that, for odd prime \( p \),
\[
\sum_{j \geq 0} \binom{j + 2k}{2k} B_j H_{p-1}^{(2k+j+1)} (-p)^j = 0, \tag{5.4}
\]
the convergence of the series being understood \( p \)-adically. More precisely [5] when \( p \geq 5 \), the following congruence was shown:
\[
\sum_{j=0}^{2n+1} \binom{j + 2k}{2k} B_j H_{p-1}^{(2k+j+1)} (-p)^j \equiv 0 \pmod{p^{2n+3}}. \tag{5.5}
\]

Now, we look for an equation similar to (5.4), but for the Stirling cycle numbers. In the case where \( k = 0 \), we have, for \( p \geq 5 \)
\[
\sum_{j=0}^{2n+1} B_j H_{p-1}^{(j+1)} (-p)^j \equiv 0 \pmod{p^{2n+3}}. \tag{5.6}
\]

For the lowest values of \( n, n = 0, 1, 2..., \) these congruences read
\[
H_{p-1} + \frac{p}{2} H_{p-1}^{(2)} \equiv 0 \pmod{p^3},
\]
\[
H_{p-1} + \frac{p}{2} H_{p-1}^{(2)} + \frac{p^2}{6} H_{p-1}^{(3)} \equiv 0 \pmod{p^5},
\]
\[
H_{p-1} + \frac{p}{2} H_{p-1}^{(2)} + \frac{p^2}{6} H_{p-1}^{(3)} - \frac{p^4}{30} H_{p-1}^{(5)} \equiv 0 \pmod{p^7}, \ldots, \text{respectively.}
\]
A clue for our search of the analogous to (5.4) is obtained by making use of (1.1) in order to recursively compute \( H_{p-1}^{(i+1)} \) as function of the \( G_{p-1}^{(i+1)} \), with \( i \leq j \), then substituting \( H_{p-1}^{(j+1)} \) in the above congruences and finally reducing modulo \( p^{2n+3} \) as much as possible, by accounting for any previous congruence involving the \( G_{p-1}^{(i+1)} \). In doing so, it is found that, for \( p \geq 5 \)

\[
G_{p-1} - pG_{p-1}^{(2)} \equiv 0 \pmod{p^3},
\]

\[
G_{p-1} - pG_{p-1}^{(2)} + \frac{p^2}{2}G_{p-1}^{(3)} \equiv 0 \pmod{p^5},
\]

\[
G_{p-1} - pG_{p-1}^{(2)} + \frac{p^2}{2}G_{p-1}^{(3)} - \frac{p^4}{6}G_{p-1}^{(5)} \equiv 0 \pmod{p^7},
\]

\[
G_{p-1} - pG_{p-1}^{(2)} + \frac{p^2}{2}G_{p-1}^{(3)} - \frac{p^4}{6}G_{p-1}^{(5)} + \frac{p^6}{6}G_{p-1}^{(7)} \equiv 0 \pmod{p^9} \ldots \text{etc.}
\]

The calculations become increasingly laborious as \( n \) increases, but we are able to guess that

\[
\sum_{j=0}^{2n+1} (j + 1)B_{j}G_{p-1}^{(j+1)}p^j \equiv 0 \pmod{p^{2n+3}}. \quad (5.7)
\]

In an even broader generalization for the Stirling cycle numbers, we anticipate that, for odd prime \( p \) and \( i > 0 \),

\[
\sum_{j \geq 0} B_{j} \left( \begin{array}{c} j + 2i \vspace{1ex} \\ j \end{array} \right) \left[ \begin{array}{c} p \\ j + 2i \end{array} \right] p^j = 0. \quad (5.8)
\]

Note that the sum in (5.8) is actually finite.

For performing numerical verifications of (5.8), we now introduce the number \( A_{n,k} \).

**Definition.** Let \( n, k \) be positive integers, we define the number \( A_{n,k} \) by

\[
A_{n,k} := \sum_{h \geq 0} B_{h} \left( \begin{array}{c} k + h - 1 \\ h \end{array} \right) \left[ \begin{array}{c} n \\ h + k \end{array} \right] n^h. \quad (5.9)
\]

It is clear from this definition that \( A_{n,k} \) is zero when \( k > n \) and that \( A_{n,n} = 1 \). The first terms of the sequence \( (A_{n,k}) \) are computed numerically and displayed in the following table:
Table 1: The triangular array $A_{n,k}$ for $n, k$ in the range 1 to 10.

It is striking that these numbers seem to be zero when $n - k$ is odd, which, if true, would imply the validity of (5.5) and (5.7). This will be demonstrated in the next section. It is also striking that they seem to be all integers. This will be shown in Section 7.

6. The value of $A_{n,k}$ when $n - k$ is odd

Theorem 6.1. Let $n, k$ be positive integers and

$$A_{n,k} := \sum_{h \geq 0} B_h \binom{k+h-1}{h} \left[ \begin{array}{c} n \\ h+k \end{array} \right] n^h,$$

then $A_{n,k} = (-1)^{n-k} A_{n,k}$. Equivalently, $A_{n,k} = 0$ when $n - k$ is odd.

Proof. We will make use of (2.3) in the definition of $A_{n,k}$, so that

$$A_{n,k} = \sum_{h \geq 0} B_h \binom{k+h-1}{h} \left[ \begin{array}{c} n \\ h+k \end{array} \right] n^h$$

$$= \sum_{h \geq 0} B_h \binom{k+h-1}{h} \sum_{g \geq 0} \binom{g}{h+k-1} \left[ \begin{array}{c} n-1 \\ g \end{array} \right] n^h$$

$$= \sum_{g \geq 0} \binom{n-1}{g} \sum_{h \geq 0} B_h \binom{k+h-1}{h} \binom{g}{h+k-1} n^h.$$
But, it is easy to see that \((k+\frac{h-1}{h})(\frac{g}{h+k-1}) = (\frac{g-1}{h})\), so that

\[
A_{n,k} = \sum_{g \geq 0} \binom{g}{k-1} \left[ \sum_{h \geq 0} B_h \left( \frac{g - k + 1}{h} \right) h \right] \sum_{h \geq 0} B_h \left( \frac{g - k + 1}{h} \right) \left( \frac{1}{n} \right)^{g-k+1-h}
\]

\[
= \sum_{g \geq 0} \binom{g}{k-1} \left[ \sum_{h \geq 0} B_h \left( \frac{g - k + 1}{h} \right) h \right] \left( \frac{1}{n} \right)^{g-k+1-h}
\]

\[
= \sum_{g \geq 0} \binom{g}{k-1} \left[ \sum_{h \geq 0} B_h \left( \frac{g - k + 1}{h} \right) h \right] \left( \frac{1}{n} \right)^{g-k+1-h}
\]

\[
= \sum_{g \geq 0} \binom{g}{k-1} \left[ n^{g-k+1} \sum_{h \geq 0} B_h \left( \frac{g - k + 1}{h} \right) \left( \frac{1}{n} \right)^{g-k+1-h} \right]
\]

Now, using the translation formula (2.6) and (2.7), we have

\[
B_{g-k+1} \left[ 1 - \frac{n-1}{n} \right] = \sum_{h \geq 0} \binom{g - k + 1}{h} B_h[1] \left( \frac{n-1}{n} \right)^{g-k+1-h}
\]

\[
= - \sum_{h \geq 0} B_h \left( \frac{g - k + 1}{h} \right) (-1)^{g+k} \left( \frac{n-1}{n} \right)^{g-k+1-h}
\]

so that

\[
A_{n,k} = - \sum_{g \geq 0} \binom{g}{k-1} \left[ n^{g-k+1} \sum_{h \geq 0} B_h \left( \frac{g - k + 1}{h} \right) \left( \frac{1}{n} \right)^{g-k+1-h} \right]
\]

\[
= - \sum_{h \geq 0} B_h n^h \sum_{g \geq 0} \binom{g}{h} \binom{g - k + 1}{h} (-1)^{g+k} \left[ \frac{n-1}{g} \right] (n-1)^{g-k+1-h}
\]

\[
= - \sum_{h \geq 0} B_h n^h \sum_{g \geq 0} \binom{h + k - 1}{h} \binom{g}{h + k - 1} (-1)^{g+k} \left[ \frac{n-1}{g} \right] (n-1)^{g-k+1-h}
\]

\[
= - \sum_{h \geq 0} B_h \left( \frac{h + k - 1}{h} \right) n^h \sum_{g \geq 0} \binom{g}{h + k - 1} (-1)^{g+k} \left[ \frac{n-1}{g} \right] (n-1)^{g-k+1-h}
\]

\[
= - \sum_{h \geq 0} B_h \left( \frac{h + k - 1}{h} \right) n^h \sum_{g \geq h+k-1} \binom{g}{h + k - 1} (-1)^{g+k} \left[ \frac{n-1}{g} \right] (n-1)^{g-k+1-h}
\]

\[
= \sum_{h \geq 0} B_h \left( \frac{h + k - 1}{h} \right) n^h \sum_{g \geq h+k-1} \binom{h + k - 1 + g}{h + k - 1} (-1)^{h+g} \left[ \frac{n-1}{g + h + k - 1} \right] (n-1)^{h+g}
\]

\[
= (-1)^{n-k} \sum_{h \geq 0} B_h \left( \frac{h + k - 1}{h} \right) n^h \left[ \frac{n}{h + k} \right] = (-1)^{n-k} A_{n,k}.
\]

In the above derivation, after inversion of the summations, we have again make use of the identity \(\binom{g}{k} \binom{h+k-1}{h} = \binom{h+k-1}{h+k-1} \binom{h}{h+k-1}\), then we have reindexed the inner sum, then we have pulled out the factor \((-1)^{n-k}\) so as to be able to make use of the identity [4.1] from Lemma 4.1. □
7. Integrality of $A_{n,k}$

**Theorem 7.1.** Let $n, k$ be positive integers, $B_h$ a Bernoulli number, $[n]_k$ an unsigned Stirling number of first kind, and

$$A_{n,k} := \sum_{h \geq 0} B_h \binom{k + h - 1}{h} \left[ \frac{n}{h + k} \right] n^h,$$

then $A_{n,k}$ is a triangular array of integers.

**Proof.** In the definition of $A_{n,k}$, we replace the Bernoulli numbers by the expression (2.9) in terms of Stirling numbers of second kind, so that

$$A_{n,k} = \sum_{h \geq 0} \sum_{m \geq 0} (-1)^m \frac{m!}{m+1} \left\{ \frac{h}{m} \right\} \binom{k + h - 1}{k - 1} \left[ \frac{n}{h + k} \right] n^h.$$  \hspace{1cm} (7.1)

We split the sum in four parts: $m + 1 = 1$, $m + 1 = 4$, $m + 1 > 4$ composite, and $m + 1$ prime, so that

$$A_{n,k} = \sum_{h \geq 0} B_h \binom{k + h - 1}{h} \left[ \frac{n}{h + k} \right] n^h - \sum_{m \geq 0} (-1)^m \frac{m!}{m+1} \sum_{h \geq 0} \left\{ \frac{h}{m} \right\} \binom{k + h - 1}{k - 1} \left[ \frac{n}{h + k} \right] n^h$$

$$+ \sum_{m+1>4 \text{ composite}} (-1)^m \frac{m!}{m+1} \sum_{h \geq 0} \left\{ \frac{h}{m} \right\} \binom{k + h - 1}{k - 1} \left[ \frac{n}{h + k} \right] n^h$$

$$+ \sum_{p \text{ prime}} (-1)^{p-1} \frac{(p-1)!}{p} \sum_{h \geq p-1} \left\{ \frac{h}{p-1} \right\} \binom{k + h - 1}{k - 1} \left[ \frac{n}{h + k} \right] n^h.$$  \hspace{1cm} (7.1)

The first term on the right-hand side of (7.1) is clearly an integer. We now show that the second term on the right-hand side of (7.1) is also integer. It is obvious that $\left\{ \frac{h}{3} \right\} \binom{k + h - 1}{h + k} n^h$ is even when $n$ is even, since $h \geq 3$. It is also even when $h$ is even ($h = 2g$, $g \geq 1$). Indeed, by the explicit expression (2.5), we have $\left\{ \frac{h}{3} \right\} = 3^{h-1} + 1 - 2^h$ which is even, since $g \geq 1$. Now we suppose that $n$ is odd and $h$ is odd ($n = 2m + 1$ and $h = 2g + 1$). By Kummer theorem $\binom{k + h - 1}{k - 1}$ is even when $k > 0$ is even and $h$ is odd, since there is at least one carry when doing the addition $h + (k - 1)$ in base 2. So it suffices to consider the case where $k$ is odd ($k = 2q + 1$) and we now need to prove that $Q_{m,g}$ is even, with

$$Q_{m,g} := \sum_{g \geq 1} \left\{ \frac{2g + 1}{3} \right\} \binom{2q + 2g + 1}{2q} \left[ \frac{2m + 1}{2q + 2} \right] (2m + 1)^{2g+1}.$$
Since \((2m + 1)^{2q + 1} \equiv 1 \bmod 2\) and \(\left\{2^{q+1}\right\} = \frac{2^{q+1}}{2} - 2^{2q} \equiv 1 \bmod 2\), we have

\[
Q_{m,q} \equiv \sum_{g \geq 1} \left(\frac{2q + 2g + 1}{2q}\right) \left[\frac{2m + 1}{2g + 2q + 2}\right] \pmod{2}
\]

\[
\equiv \sum_{g \geq 1} \left(\frac{2q + 2g + 1}{2q}\right) \left[\frac{2m}{2g + 2q + 1}\right] \pmod{2}
\]

\[
\equiv \sum_{g \geq 1} \left(\frac{2q + 2g}{2q}\right) \left[\frac{2m}{2g + 2q + 1}\right] \pmod{2}
\]

\[
\equiv \sum_{g \geq 1} \left(\frac{2q + 2g}{2q}\right) \left[\frac{2m}{2g + 2q + 1}\right] \pmod{2}
\]

\[
\equiv \sum_{g \geq 1} \left(\frac{2q + 2g}{2q}\right) \left(\frac{m}{2m - 2g - 2q - 1}\right) \pmod{2}
\]

\[
\equiv \sum_{g=1}^{m-1} \left(\frac{2g}{2q}\right) \left(\frac{m}{2m - 2g - 1}\right) \pmod{2}
\]

by Kummer theorem.

If \(m\) is even, \(\left(\frac{m}{2m - 2g - 1}\right) \equiv 0 \bmod 2\) by Kummer theorem, then \(Q_{m,q} \equiv 0 \bmod 2\). So it suffices to prove that \(Q_{2\ell+1,q}\) is even \((\ell \geq 0)\). We have

\[
Q_{2\ell+1,q} \equiv \sum_{g=q+1}^{2\ell} \left(\frac{2g}{2q}\right) \left(\frac{2\ell + 1}{4\ell - 2g + 1}\right) \pmod{2}
\]

\[
\equiv \sum_{g=q+1}^{2\ell} \left(\frac{2g}{2q}\right) \left(\left(\frac{2\ell}{4\ell - 2g + 1}\right) + \left(\frac{2\ell}{4\ell - 2g}\right)\right) \pmod{2}
\]

\[
\equiv \sum_{g=q+1}^{2\ell} \left(\frac{2g}{2q}\right) \left(\frac{2\ell}{4\ell - 2g}\right) \pmod{2}
\]

by Kummer theorem.

\[
\equiv \sum_{g=q+1}^{2\ell} \left(\frac{g}{q}\right) \left(\frac{\ell}{2\ell - g}\right) \pmod{2}
\]

again by Kummer theorem.

\[
\equiv \sum_{g=q}^{2\ell} \left(\frac{g}{q}\right) \left(\frac{\ell}{2\ell - g}\right) - \left(\frac{\ell}{2\ell - q}\right) \pmod{2}
\]

\[
\equiv \sum_{g=0}^{2\ell-q} \left(\frac{g + q}{q}\right) \left(\frac{\ell}{2\ell - g - q}\right) - \left(\frac{\ell}{2\ell - q}\right) \pmod{2}
\]

Now we have \(\sum_{g \geq 0} \left(\frac{g+q}{q}\right) x^g = (1 - x)^{-q-1}\) and \(\sum_{g \geq 0} \left(\frac{\ell}{g}\right) x^g = (1 + x)^\ell\), so that

\[
\frac{(1 + x)^\ell}{(1 - x)^{q+1}} - (1 + x)^\ell = \sum_{h \geq 0} \sum_{g=0}^{h} \left(\frac{g + q}{q}\right) \left(\frac{\ell}{h - g}\right) x^h - \sum_{h \geq 0} \left(\frac{\ell}{h}\right) x^h,
\]
hence
\[ Q_{2\ell+1,q} \equiv [x^{2\ell-q}] \left( \frac{1+x}{1-x} \right)^\ell - (1+x)^\ell \quad (\text{mod } 2) \]
\[ \equiv [x^{2\ell-q}] \left( (1-x)^{\ell-q-1} - (1+x)^\ell \right) \quad (\text{mod } 2). \]

If \( q + 1 \leq \ell \) then \( 2\ell - q \geq \ell + 1 \), hence \( 2\ell - q > \ell \) and \( 2\ell - q > \ell - q - 1 \geq 0 \) hence
\[ Q_{2\ell+1,q} \equiv (-1)^q \left( \frac{\ell-q-1}{2\ell-q} \right) - \left( \frac{\ell}{2\ell-q} \right) \quad (\text{mod } 2) \]
\[ \equiv 0 - 0 = 0 \quad (\text{mod } 2). \]

And if \( q + 1 > \ell \),
\[ Q_{2\ell+1,q} \equiv \left( \frac{q - \ell + 2\ell - q}{2\ell - q} - \left( \frac{\ell}{2\ell - q} \right) \right) \quad (\text{mod } 2) \]
\[ \equiv \left( \frac{\ell}{2\ell - q} \right) - \left( \frac{\ell}{2\ell - q} \right) = 0 \quad (\text{mod } 2). \]

The third term on the right-hand side of (7.1) is also integer. Indeed, if \( m + 1 \) is composite and \( m + 1 > 4 \) then \( m + 1 \) divides \( m! \). For suppose \( m + 1 > 4 \) is composite and not the square of a prime \( p \), then there exists integers \( m_1, m_2 \) such that \( 2 \leq m_1 < m_2 < m \), and \( m + 1 = m_1m_2 \). Then \( m + 1 \) obviously divides \( m! \). And if \( m + 1 > 4 \) is a squared prime, \( m + 1 = p^2 \), with \( p \geq 3 \), then \( m = (p-1)p + (p-1) \) and the sum of the base-\( p \) digits of \( m \) is \( 2p - 2 \) so that by Legendre formula \( p^{\frac{p^2-1}{p-1}} = p^{p-1} \) divides \( m! \), then \( p^2 \) divides \( m! \), since \( p \geq 3 \).

Finally, in order to demonstrate that \( \mathcal{A}_{n,k} \) is integer, it suffices to show that for any integers \( n, k \geq 0 \) and any prime \( p \)
\[ \sum_{h=p-1}^{n-k} \left\{ \binom{k+h-1}{k-1} \binom{n}{h+k} \right\} n^h \equiv 0 \quad (\text{mod } p). \]

This is obvious when \( p \) divides \( n \), and it suffices to show that this is true when \( p \) and \( n \) are coprime. By Fermat little theorem, and by (2.12), we then see that we just need to show that for any \( k \geq 1 \),
\[ (p, n) \text{ coprime} \implies \sum_{p-1 \text{ divides } h} \binom{k+h-1}{k-1} \binom{n}{h+k} \equiv 0 \quad (\text{mod } p). \quad (7.2) \]

In Equation (3.8) from Lemma 3.4, we let \( g = 0 \) and \( i(p-1) = h \) and we introduce \( k - 1 \) instead of \( k \). Then, we have
\[ \sum_{h=p-1}^{p-1 \geq 1} \binom{k+h-1}{k-1} \binom{n}{h+k} \equiv \left\{ \begin{array}{ll} 0 & \text{if } kp - n \in \{1, 2, \ldots, p-1\} \\ kp - n & \text{otherwise} \end{array} \right\} \quad (\text{mod } p). \]
Now
\[
\left(\frac{h}{p-1} - 1\right) \left(\frac{h}{p-1} - 2\right) \cdots \left(\frac{h}{p-1} - k + 1\right) = \left(\frac{h}{h+1} - (-h-2) \cdots (-h-(k-1))\right) \left(\frac{h}{h+1} - (-h-1)\right) \equiv (-1)^{k-1} \binom{k+h-1}{k-1} \pmod{p}.
\]

Then
\[
(-1)^{k-1} \sum_{p-1 \text{ divides } h} \binom{k+h-1}{k-1} \left[\begin{array}{c}
 n \\
 h+k
\end{array}\right] \equiv [n = kp] \pmod{p},
\]
which imply the validity of (7.2).

\[\square\]

8. Discussion and Questions

Apart from their appearance in the above investigation of congruences modulo prime powers for the Stirling cycle numbers, we don’t know the mathematical interest of the integers $A_{n,k}$. It is quite a pity that our demonstration of Theorem 7.1 has to be that long and technical, because technicalities may easily conceal much of mathematical signification. Any recurrence that would allow to compute the entry in the above triangular array from entries from previous lines would be more insightful, and would probably lead to more direct proofs for Theorems 6.1 and 7.1. Also, finding for $A_{n,k}$ generating functions of any sort (or at least functional equations for such functions) would certainly help. The ultimate problem lies in giving a combinatorial signification to these numbers without their signs, should any exist.

We can still point out some similarities with the Stirling numbers. We have
\[
A_{n,n-k} = \sum_{h=0}^{k} B_h \binom{n - 1 - (k - h)}{h} \left[\begin{array}{c}
 n \\
 n - (k - h)
\end{array}\right] n^h,
\]
where the binomial coefficient is a polynomial in $n$ of degree $h$, and $\left[\begin{array}{c}
 n - (k - h) \\
 n - (k - h)
\end{array}\right]$ is known [3] or Lemma 3.5 to be a polynomial in $n$ of degree $2(k-h)$. Therefore $A_{n,n-k}$ is also a polynomial in $n$ of degree $2k$. For instance, we have $A_{n,n-2} = \frac{-n-1}{4} \binom{n}{3} = (-1)^{n+1}$, whereas $\left[\begin{array}{c}
 n - 2 \\
 n - 2
\end{array}\right] = \frac{3n-1}{4} \binom{n}{3}$. We then see that $A_{n,n-2} - \left[\begin{array}{c}
 n \\
 n-2
\end{array}\right] = -n(n)$. More generally, we will have the following theorem.

**Theorem 8.1.** Let $n, k$ be positive integers, $B_h$ a Bernoulli number and $\left[\begin{array}{c}
 n \\
 k
\end{array}\right]$ an unsigned Stirling number of first kind, then
\[
A_{n,k} \equiv \frac{1 + (-1)^{n-k} \left[\begin{array}{c}
 n \\
 k
\end{array}\right]}{2} \pmod{n}.
\]
Proof. From (4.2), we have if \( n > 0, \)
\[
\frac{n}{k} - (-1)^{n-k} \frac{n}{k} \frac{n}{k+1} + \frac{(-1)^{n-k} kn}{2} \frac{n}{k+1} = \frac{(-1)^{n-k}}{2} \sum_{h \geq 2} \left( h + k - 1 \right) \frac{n}{h+k} (-n)^h.
\]
Hence
\[
-(n-k) \frac{n}{k} + \frac{n}{k} \equiv \frac{kn}{2} \frac{n}{k+1} \pmod{n}.
\]
On the other hand, from the definition of \( A_{n,k}, \) we have
\[
A_{n,k} - \frac{n}{k} + \frac{kn}{2} \frac{n}{k+1} = \sum_{h \geq 2} \frac{B_h}{h} \left( k+h-1 \right) \frac{n}{h+k} n^h.
\]
Then
\[
A_{n,k} - \frac{1 + (-1)^{n-k}}{2} \frac{n}{k} \equiv n \sum_{h \geq 2} nB_h \left( k+h-1 \right) \frac{n}{h+k} n^{h-2} \pmod{n}.
\]
But each summand in the sum on the right-hand side is \( p \)-integral for all prime \( p \) that divides \( n. \) To see this, we make use of the Von Staudt-Clausen theorem whereby \( p \) may divide the denominator of \( B_h \) once at most. Then \( nB_h \) is \( p \)-integral and then the right-hand side is 0 modulo \( n. \)

As a corollary to Theorem 8.1, we have a Wilson-like theorem for \( A_{n,1}, \) illustrating the similarity between the first column in the above Table and the factorial \((n-1)!.\)

Theorem 8.2. Let \( n \) be a positive integer, we have
\[
A_{n,1} + [n \text{ is an odd prime}] \equiv 0 \pmod{n}.
\]

Proof. From Theorem 8.1, we have \( A_{n,1} \equiv \frac{1 + (-1)^{n-1}}{2} \frac{n}{1} \pmod{n}. \) But \( \frac{n}{1} = (n-1)! \) then \( A_{n,1} \equiv \frac{1 + (-1)^{n-1}}{2} (n-1)! \pmod{n}. \) If \( n \) is an odd prime, by the Wilson theorem we have \( A_{n,1} \equiv -1 \pmod{n}; \) otherwise, if \( n \) is even, clearly \( A_{n,1} \equiv 0 \pmod{n}, \) and if \( n \) is an odd composite, we have already seen that \( n \) divides \((n-1)!, \) so that we also have \( A_{n,1} \equiv 0 \pmod{n}. \)

Theorem 8.3. Let \( k > 0 \) and \( P_k \) be the polynomial such that \( P_k(n) = A_{n,n-k}, \) then \(-1, 0, \ldots, k \) are \( k+2 \) roots of \( P_k(x).\)

Proof. It is known from Lemma 3.5 that \( 0, 1, \ldots, j \) are roots of the polynomial
function $Q_j(x)$ such that $Q_j(n) = \left[ \begin{array}{c} n \\ n-j \end{array} \right]$, and $Q_j(-1) = 1$. We have

$$P_k(x) = \sum_{h=0}^{k} B_h \binom{x-1-(k-h)}{h} Q_{k-h}(x)x^h$$

$$P_k(u) = \sum_{h=0}^{k} B_h \binom{u+h-1-k}{h} Q_{k-h}(u)u^h$$

$$= \sum_{h=0}^{k} B_h \frac{(u+h-1-k)\cdots(u-k)}{h!} Q_{k-h}(u)u^h$$

Let $0 \leq u \leq k$, if $0 \leq u \leq k - h$, then $Q_{k-h}(u) = 0$, then

$$P_k(u) = \sum_{h=k-u+1}^{k} B_h \frac{(u-k)\cdots(u-k+h-1)}{h!} Q_{k-h}(u)u^h$$

$$= \sum_{h=k-u+1}^{k} (-1)^h B_h \frac{(k-u)\cdots(k-u+h+1)}{h!} Q_{k-h}(u)u^h$$

If $u = 0$, clearly $P_k(u) = 0$, because of the factor $u^h$ and since $h \geq 1$. Moreover, if $0 < u \leq k$, for any $h$ in the set \{ $k-u+1, \cdots, k$ \} the product $(k-u)\cdots(k-u+h+1)$ must vanish because we see that it has one factor which is zero, and then we also have $P_k(u) = 0$. Finally, if $u = -1$,

$$P_k(-1) = \sum_{h=0}^{k} B_h \frac{(h-2-k)\cdots(-1-k)}{h!}(-1)^h Q_{k-h}(-1)$$

$$= \sum_{h=0}^{k} B_h \frac{(k+2-h)\cdots(k+1)}{h!}$$

$$= \sum_{h=0}^{k} \binom{k+1}{h} B_h = 0 \quad \text{(by 2.8)}.$$ 

Finally, by classical formulas for inverse matrix computation, since $(A_n,k)$ is a lower triangular matrix of integers with the main diagonal full of 1s (lower unitriangular matrix), it is inversible, and its inverse $(A_{n,k}')$ is also a lower unitriangular matrix of integers. And also by these rules, since $A_n,k = 0$ when $n - k$ is odd, it could be shown that the integers $A_{n,k}'$ also vanish when $n - k$ is odd. The triangular array for the $A_{n,k}'s$ is computed numerically by matrix inversion, and displayed hereafter:
We see that $A_{n,n-2}' = -A_{n,n-2}$, which is also quite easily obtained by the rules of matrix inversion for such an aerated lower unitriangular matrix. This can be compared to the well-known $\binom{n}{n-1} = \binom{n}{n-1}$ for Stirling numbers. An explicit expression similar to (5.9), but for $A_{n,k}'$, is needed and the same questions as for $A_{n,k}$, are raised for $A_{n,k}'$: give recurrence relations, generating functions and combinatorial interpretations.

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