Abstract.
Using exact expressions for the Ising form factors, we give a new very simple proof that the spin-spin and disorder-disorder correlation functions are governed by the Painlevé III non linear differential equation. We also show that the generating function of the correlation functions of the descendents of the spin and disorder operators is a $N$-soliton, $N \to \infty$, $\tau$-function of the sinh-Gordon hierarchy. We discuss a relation of our approach to isomonodromy deformation problems, as well as further possible generalizations.
1 Introduction.

In the scaling limit, the Ising model is described by a free Majorana fermion $\psi(x)$ with mass $m \equiv (T - T_C)$. However, the order and disorder operators $\sigma(x)$ and $\mu(x)$ are non local in terms of these free fermions. This shows up in the very striking result by T.T. Wu, B.M. McCoy, C.A. Tracy and E. Barouch \cite{1} for the correlation functions of $\sigma(x)$ and $\mu(x)$. Let $r$ be the radial distance, then:

\[
\begin{align*}
&\left( <\mu(r)\mu(0)> \right) = \left( \frac{\cosh \frac{1}{2} \chi(s)}{\sinh \frac{1}{2} \chi(s)} \right) \exp \left( -\frac{1}{4} \int_s^\infty du \ u \left[ \left( \frac{d\chi}{du} \right)^2 - \sinh^2 \chi \right] \right) \quad (1)
\end{align*}
\]

where $s$ is the scaling variable $s = \frac{mr}{2}$. $\chi$ is a solution of the radial sinh-Gordon equation:

\[
\frac{d^2 \chi}{ds^2} + \frac{1}{s} \frac{d\chi}{ds} = 2 \sinh(2\chi)
\]

Setting $\eta = e^{-\chi}$, this is equivalent to the Painlevé III equation:

\[
\frac{\eta''}{\eta} = \left( \frac{\eta'}{\eta} \right)^2 - \frac{1}{s} \left( \frac{\eta'}{\eta} \right) + \eta^2 - \frac{1}{\eta^2}
\]

We present a new (as far as we know) very simple proof of it, based on the analysis of form factors.

More generally, to compute exact correlation functions in 2D integrable quantum field theories remains a challenge despite recent progresses \cite{2}. These progresses are based on a fine analysis of the algebraic Bethe ansatz. On the other side, it has recently been argued that the two-dimensional integrable QFT can be solved by only using their infinite quantum group symmetries \cite{3, 4}. The simplest examples of such symmetries are provided by the Yangian symmetries of massive current algebras or by the quantum $sl(2)$ loop symmetry of the sine-Gordon model. These symmetries act more simply on the asymptotic states and on the momentum variables. Therefore, they lead more directly to algebraic methods for computing field form factors. The next question which naturally arises in this approach consists in knowing whether it is possible to reconstruct and to characterize, e.g. by differential equations, the correlation functions from the knowledge of the form factors in the spirit of the bootstrap program \cite{5}. In the following, we show by examining the simplest possible model, namely the Ising model, that it is not hopeless to expect a positive answer to this question.
2 Ising Form Factors.

As is well known [6], the Ising model is a $\mathbb{Z}_2$-invariant lattice model with nearest neighbour interactions between $\mathbb{Z}_2$ spins. It undergoes a second order phase transition at some critical temperature $T_c$. In the scaling limit near its critical point it is equivalent to a free Majorana fermion with mass $m \sim (T - T_c)$. It possesses an infinite number of local integrals of motion in involution which have odd spins. The asymptotic states are the free fermions and therefore the scattering is trivial:

$$S = -1$$

Among the fields of primary interest are the spin field $\sigma$, its “dual” field called the disorder field $\mu$, and the fermion $\psi$. We refer to [7] for their lattice definition.

Form factors are matrix elements of field operators. They satisfy algebraic relations, usually called form factor axioms [8, 9], which depend on the locality of the fields and on the sectors to which the fields belong. Due to its $\mathbb{Z}_2$ symmetry, the sectors of the Ising model are labelled by couples of indices $[a; b]$ with $a, b = 0, 1$. In each sector $[a; b]$, the first index $a$ refers to a representation of the group $\mathbb{Z}_2$ whereas the second index $b$ refers to an element in $\mathbb{Z}_2$. The fields $\Phi_{[a;b]}(x,t)$ in sectors $[a;b]$ satisfy the following equal-time braiding relations:

$$\Phi_{[a_1;b_1]}(x_1; t) \Phi_{[a_2;b_2]}(x_2; t) = (-1)^{a_2b_1} \Phi_{[a_2;b_2]}(x_2; t) \Phi_{[a_1;b_1]}(x_1; t) \quad ; \quad x_1 > x_2$$

The identity operator is in the sector $[0;0]$, the spin field $\sigma$ in $[1;0]$, the disorder operator $\mu$ in $[0;1]$, and the fermion $\psi$, which is defined by the

\[ More generally, in a statistical model invariant under a finite group $G$, one naturally defines spin fields $\sigma^\rho$, which take values in some representation $\rho$ of $G$, disorder fields $\mu_g$, where the label $g$ are elements of $G$, and parafermions $\psi^\rho_g$, which are defined by the operator product expansion $\psi^\rho_g \sim \mu_g \sigma^\rho$. An element $h \in G$ acts on the parafermions by $\psi^\rho_g \to \rho(h)\psi^\rho_g$. The parafermions $\psi^\rho_g$ span the sectors $[\rho; g]$ which form representations of the quantum double $\mathcal{D}(G)$ of functions on $G$. The braiding relations of the parafermions are given by the universal $R$-matrix of $\mathcal{D}(G)$:

$$ (\psi^{\rho_1}_{g_1}(x_1) \otimes 1)(1 \otimes \psi^{\rho_2}_{g_2}(x_2)) = (1 \otimes \rho_2(g_1)\psi^{\rho_2}_{g_1g_2^{-1}}(x_2))(\psi^{\rho_1}_{g_1}(x_1) \otimes 1) \quad ; \quad x_1 > x_2 $$

The Ising model is a particular case of that.
operator product expansion $\Psi \sim \mu \sigma$, is in the sector $[1; 1]$. In particular, the asymptotic particles, which are the fermions, are in the sector $[1; 1]$.

In a more algebraic way, the couples $[a; b]$ label the irreducible representations of the quantum double $D(Z_2)$ of functions on the group $Z_2$. In $D(Z_2)$, the universal $R$-matrix acts on two representations $[a_1; b_1]$ and $[a_2; b_2]$ as $R = (-1)^{a_2 b_1}$. The braiding matrices are therefore given by the universal $R$-matrix. Moreover, for the representation $[1; 1]$ corresponding to the fermions we find $R = -1$. We therefore have the relation (which is a complicated way to write a simple result):

$$S = R_{universal}$$

As expected from the general arguments \[4, 10\].

Let us denote by $|\theta_1, \cdots, \theta_n\rangle$ the $n$-particle states with energies $E_i = m \cosh \theta_i$ and momenta $P_i = m \sinh \theta_i$. The form factors for a field $\Phi_{[a; b]}$ in a sector $[a; b]$ are defined by :

$$F_{[a; b]}^{(n)}(\theta_1, \cdots, \theta_n) \equiv \langle 0|\Phi_{[a; b]}(0)|\theta_1, \cdots, \theta_n\rangle$$

(3)

By crossing symmetry, all the matrix elements of $\Phi_{[a; b]}$ are determined by $F_{[a; b]}^{(n)}$ :

$$\langle \theta_1, \cdots, \theta_p|\Phi_{[a; b]}(0)|\theta_{p+1}, \cdots, \theta_n\rangle = F_{[a; b]}^{(n)}(\theta_1 - i\pi, \cdots, \theta_p - i\pi, \theta_{p+1}, \cdots, \theta_n)$$

By $Z_2$-symmetry, the form factors $F_{[a; b]}^{(n)}$ are non-vanishing only for $a \equiv n \mod 2$.

The form factors axioms are usually written for local fields in the identity sector \[8, 9\]. Their generalizations for non-local fields in the other sectors is easy to find. In ref \[4, 10\], the generalized axioms were derived using quantum group symmetries. In the case of the Ising model, since all the sector are described by the $Z_2 \times Z_2$ indices $[a; b]$ and since the $S$-matrix is $S = -1$, they reduce to :

$$F_{[a; b]}^{(n)}(\theta_2, \theta_1, \theta_3, \cdots, \theta_n) = -F_{[a; b]}^{(n)}(\theta_1, \theta_2, \theta_3, \cdots, \theta_n)$$

$$F_{[a; b]}^{(n)}(\theta_1 - i2\pi, \theta_2, \cdots, \theta_n) = (-1)^{(1+a)(1+b)} F_{[a; b]}^{(n)}(\theta_1, \theta_2, \cdots, \theta_n)$$

$$\text{Res}_{\theta_1 = \theta_2 - i\pi} F_{[a; b]}^{(n)}(\theta_1, \theta_2, \theta_3, \cdots, \theta_n) = \left( (-1)^{a+b} - 1 \right) F_{[a; b]}^{(n-2)}(\theta_3, \cdots, \theta_n)$$

4
Here, we have used that $a \equiv n \ [\text{mod} \ 2]$. Because there is no bound state, the
form factors have no other pole in the physical strip, $0 \leq \Im \theta \leq \pi$.

In the sectors $[a; b]$ with $a + b \equiv 0 \ [\text{mod} \ 2]$, the form factors have no pole
at all in the physical strip. They correspond to those of fields in the sectors
of the identity and of the fermion. In the same way as in $[\text{II}],$ the set of
solutions to the equations (5) with $a + b \equiv 0 \ [\text{mod} \ 2]$ are found to be in
correspondence with the fields in the Neveu-Schwarz sector of the conformal
Ising theory.

The sectors with $a + b \equiv 1 \ [\text{mod} \ 2]$ are those of the spin and disorder fields.
There, the form factors have a simple pole in the physical strip. This pole
 corresponds to the particle-antiparticle scattering. The minimal solution to
eqs. (4) which possesses no zeroes in the physical strip is given by $[3, 12, 11]$ :

$$F_{\min}^{(n)}(\theta_1, \cdots, \theta_n) = \prod_{i<j} \tanh \left( \frac{\theta_i - \theta_j}{2} \right). \quad (5)$$

These minimal form factors with $n$ odd are identified as those of the spin
field $\sigma$, and with $n$ even as those of the disorder field $\mu$. Eqs. (3) determine
the minimal solution up to a multiplicative constant; we choose this constant
to be one by a normalization convention; in particular $\langle \mu \rangle = 1.$

As proved in $[3]$, all the solutions $F_{[a; b]}^{(n)}$ to eqs. (4) can be written as product
of the minimal solution times an auxiliary function, denoted $P^{(n)}(\theta_1, \cdots, \theta_n),$
which has no pole in the physical strip :

$$F_{[a; b]}^{(n)}(\theta_1, \cdots, \theta_n) = P^{(n)}(\theta_1, \cdots, \theta_n) F_{\min}^{(n)}(\theta_1, \cdots, \theta_n)$$

Moreover, from eqs. (3), it follows that the functions $P^{(n)}$ satisfy :

$$P^{(n)}(\theta_2, \theta_1, \theta_3, \cdots, \theta_n) = P^{(n)}(\theta_1, \theta_2, \theta_3, \cdots, \theta_n)$$

$$P^{(n)}(\theta_1 - i2\pi, \theta_2, \cdots, \theta_n) = P^{(n)}(\theta_1, \theta_2, \cdots, \theta_n)$$

$$P^{(n)}(\theta_2 + i\pi, \theta_2, \theta_3, \cdots, \theta_n) = P^{(n-2)}(\theta_3, \cdots, \theta_n) \quad (6)$$

In other words, $P^{(n)}$ are symmetric and periodic functions of $\theta_1, \cdots, \theta_n$ and
subject to the last constraints in eq. (3). It is easy to check that an infinite
set of solutions to eqs. (3) is given by :

$$P_{\{s_1, \cdots, s_M\}}^{(n)}(\theta_j)_{j=1, \cdots, n} = Q_{s_1}^{(n)}(\theta_j) \cdots Q_{s_M}^{(n)}(\theta_j)$$
where \( s_1, \cdots, s_M \), which label the solutions, are positive or negative odd integers. The functions \( Q_s \) are:

\[
Q_s^{(n)}(\theta_1, \cdots, \theta_n) = -\frac{m}{2} \sum_{j=1}^{n} e^{s\theta_j}
\]

As shown in [11], these solutions form a complete set of independent solutions to eqs. (6), and are in one-to-one correspondence with the fields in the Ramond sector of the Ising conformal theory. As a consequence, the form factors,

\[
F_{[1,0],\{s_1,\cdots, s_M\}}^{(2n+1)}(\theta_j)_{j=1,\cdots,2n+1} = P_{\{s_1,\cdots, s_M\}}^{(2n+1)}(\theta_j) F_{min}^{(2n+1)}(\theta_j)
\]

\[
F_{[0,1],\{t_1,\cdots, t_Q\}}^{(2n)}(\theta_j)_{j=1,\cdots,2n} = P_{\{t_1,\cdots, t_Q\}}^{(2n)}(\theta_j) F_{min}^{(2n)}(\theta_j)
\]

(7)

are identified with the form factors of the descendents \( \sigma_{s_1,\cdots, s_M} \) and \( \mu_{t_1,\cdots, t_Q} \) of the spin field \( \sigma \) and disorder field \( \mu \).

## 3 Ising correlation functions.

We will only consider the two-point correlation functions. If \( F_1^{(n)}(\theta_1, \cdots, \theta_n) \) and \( F_2^{(n)}(\theta_1, \cdots, \theta_n) \) are the form factors of two scalar fields \( \Phi_1(x,t) \) and \( \Phi_2(x,t) \), the Euclidean correlation function is then:

\[
\langle \Phi_1(x,t) \Phi_2(0,0) \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{+\infty} \prod_{j=1}^{n} \left( \frac{d\theta_j}{2\pi} e^{-mr\cosh\theta_j} \right) F_1^{(n)}(\theta_1, \cdots, \theta_n) F_2^{(n)}(\theta_1, \cdots, \theta_n)
\]

(8)

where \( r \) is the radial distance, \( r^2 = x^2 + t^2 \).

Let us first consider the spin-spin and disorder-disorder correlation functions. Define:

\[
C_\pm(r) = \langle \mu(r)\mu(0) \rangle \pm \langle \sigma(r)\sigma(0) \rangle
\]

From the formula for the form factors, we have:

\[
C_\pm(r) = \sum_{n=0}^{\infty} \frac{(\pm)^n}{n!} \int_{-\infty}^{+\infty} \prod_{j=1}^{n} \left( \frac{d\theta_j}{2\pi} e^{-mr\cosh\theta_j} \right) \prod_{i<j} \tanh^2 \left( \frac{\theta_i - \theta_j}{2} \right)
\]

(9)

We will study the sum of this series in the following section.
Consider now the correlation function of the descendents of the spin and disorder fields. Set:

\[ C_{\pm}^{\{s_p\};\{t_q\}}(x, t) = \langle \mu_{\{s_p\}}(x, t) \mu_{\{t_q\}}(0) \rangle \pm \langle \sigma_{\{s_p\}}(x, t) \sigma_{\{t_q\}}(0) \rangle \]

Since the operators \( \sigma_{\{s_p\}} \) and \( \mu_{\{t_q\}} \) have non-trivial Lorentz spins, the correlation functions do not only depend on the radial distance but on \( x \) and \( t \). We introduce the Euclidean coordinates \( z_{\pm} = x \pm it \). From the expression of the form factors, we have:

\[ C_{\pm}^{\{s_p\};\{t_q\}}(x, t) = \sum_{n=0}^{\infty} \frac{(\pm)^n}{n!} \int_{-\infty}^{+\infty} \prod_{j=1}^{n} \left( \frac{d\theta_j}{2\pi} e^{-\frac{m}{2}(z_s e^{\theta_j} + z_{-}\ e^{-\theta_j})} \right) \prod_{p,q} Q_{sp}(\theta_j) Q_{tq}(\theta_j) \prod_{i<j} \tanh^2 \left( \frac{\theta_i - \theta_j}{2} \right) \]

Here \( \{s_p\} \) refers to multi-indices \( \{s_{p1}, \cdots, s_{pM}\} \) and similarly \( \{t_q\} = \{t_{q1}, \cdots, t_{qK}\} \).

Let us introduce a generating function \( C_{\pm}^{\infty}(z_s) \) for these correlation functions. It depends on an infinite number of variables \( z_s \) with \( s = \pm 1, \pm 3, \cdots \); we denote \( z_{\pm1} \) by \( z_{\pm} \). It is specified by:

\[ C_{\pm}^{\{s_p\};\{t_q\}}(x, t) = \prod_{p,q} \frac{\partial}{\partial z_{sp}} \frac{\partial}{\partial z_{tq}} C_{\pm}^{\infty}(z_s) \big|_{z_s = 0; \ |s|\geq 3} \]

From the exact expressions (10) of the form factors of the descendents, we deduce:

\[ C_{\pm}^{\infty}(z_s) = \sum_{n=0}^{\infty} \frac{(\pm)^n}{n!} \int_{-\infty}^{+\infty} \prod_{j=1}^{n} \left( \frac{d\theta_j}{2\pi} X(\theta_j|z_s) \right) \prod_{i<j} \tanh^2 \left( \frac{\theta_i - \theta_j}{2} \right) \]

where the “potential” \( X(\theta|z_s) \) are given by:

\[ X(\theta|z_s) = \exp \left( -\frac{m}{2} \sum_{s=\pm 1, \pm 3, \cdots} z_s e^{\theta} \right) \]

In the next section, we show that the generating function \( C_{\pm}^{\infty}(z_s) \) can also be expressed in terms of a solution of the sinh-Gordon equation. More precisely we will show that it is a \( \tau \)-function for the affine sinh-Gordon hierarchy.

### 4 Proof of the differential equations.

The differential equation for the Ising correlation functions were originally proved by looking at the scaling limit of the lattice Ising model [1]. See also
Another approach was developed by the Kyoto school which is based on the study of isomonodromy deformation problems [14]. We present another proof based on form factors. Our proof relies on comparing the generating functions $C^\infty_\pm(z_s)$ with a $N \to \infty$ limit of the $N$-soliton $\tau$-functions of the affine sinh-Gordon model. Therefore, we need to do a small detour into the affine sinh-Gordon model. See ref. [15] for a recent study of this model. It is a Toda model over the affine Lie algebra $\hat{sl}(2)$. It involves two fields, which we denote by $\phi$ and $\xi$, whose equations of motion are:

$$
\partial_\nu \partial_\nu \phi = 8M^2 \sinh(2\phi)
$$
$$
\partial_\nu \partial_\nu \xi = 8M^2 \cosh(2\phi)
$$

Here $M$ is the classical sinh-Gordon mass. The vacuum solution to these equations is $\phi_{\text{vac}} = 0$ and $\xi_{\text{vac}} = 2M^2 r^2$ with $r$ the radial distance. Substracting the vacuum solution, we have:

$$
\partial_\nu \partial_\nu \phi = 8M^2 \sinh(2\phi)
$$
$$
\partial_\nu \partial_\nu \xi = 8M^2 \left[ \cosh(2\phi) - 1 \right]
$$

(11)

where $\xi = \xi - \xi_{\text{vac}}$. Once the field $\phi$ is known, $\xi$ is computed from it, if the boundary conditions have been specified.

The affine sinh-Gordon model possesses two $\tau$-functions $\tau_{\pm}$ which are defined by:

$$
\tau_+ \tau_- = \exp \left( -\xi \right)
$$
$$
\frac{\tau_+}{\tau_-} = \exp (-\phi)
$$

(12)

The $\tau$-functions $\tau_{\pm}^{(N)}$ for the $N$-soliton solutions are given by (See [16] and references therein):

$$
\tau_{\pm}^{(N)} = 1 + \sum_{p=1}^{N} (\pm)^p \sum_{k_1<k_2<\ldots<k_p} X_{k_1} \cdots X_{k_p} \prod_{k_i<k_j} \left( \frac{\mu_{k_i} - \mu_{k_j}}{\mu_{k_i} + \mu_{k_j}} \right)^2
$$

(13)

with

$$
X_i = a_i \exp \left( 2M \sum_s z_s \mu_i^s \right)
$$
Here $a_i$ and $\mu_i$ are the parameters of the N-soliton solutions. The variables $z_s$ correspond to all the commuting flows of the affine sinh-Gordon hierarchy. The $\tau$-functions $\tau_{\pm}^{(N)}$ are solutions of the quadratic differential equations:

$$\tau_{\pm} \partial_{z_+} \partial_{z_-} \tau_{\pm} - \partial_{z_+} \tau_{\pm} \partial_{z_-} \tau_{\pm} = -M^2 \tau_{\mp}^2 \quad (14)$$

These equations, which are the bilinear Hirota form of the affine sinh-Gordon model, are equivalent to the eqs. (11) for the fields $\phi$ and $\xi$. The claim is that $C_{\pm}^\infty(z_s)$ is a $N \to \infty$ limit of the functions $\tau_{\pm}^{(N)}$. The idea is to approximate the integrals appearing in the definition of $C_{\pm}^\infty$ by Riemann sums. Therefore, let us set,

$$M = \frac{m}{4}; \quad \mu_i = -\exp(\theta_i); \quad a_i = \frac{\Lambda}{2\pi N}$$

with $\Lambda$ an arbitrary constant. Taking the limit $N \to \infty$, $\Lambda$ fixed, gives:

$$\tau_{\pm}^{(N)} \to \sum_{n=0}^{\infty} \frac{(\pm)^n}{n!} \int_{-\Lambda}^{\Lambda} \prod_{j=1}^{n} \left( \frac{d\theta_j}{2\pi} X(\theta_j|z_s) \right) \prod_{i<j} \tanh^2 \left( \frac{\theta_i - \theta_j}{2} \right)$$

Finally, taking the limit $\Lambda \to \infty$ we exactly get the formula (14). Since the $\tau$-functions are solutions of the Hirota equations for any $N$, $a_i$ and $\mu_i$, it follows that $C_{\pm}^\infty$ is also a solution. Setting:

$$C_{\pm}^\infty(z_s) = \exp \frac{1}{2}(-\xi \mp \phi)$$

then $\phi$ and $\xi$ satisfy eqs. (14). Integrating these equations and taking into account the fact that $\phi$ and $\xi$ only depend on the radial distance proves the result of McCoy et al with the identification $\chi = -\phi$. The scaling variable $s$ is defined as $s = \frac{mr}{2}$. Notice that, as it becomes to be familiar in quantum integrable models [3], the quantum correlation functions are the $\tau$-functions for a hierarchy of classical integrable non-linear differential equations.

5 Miscellaneous remarks.

- Relation with isomonodromy deformation problems.
In [4], the Ising correlation functions were computed by using the Dirac equation for the fermions in an isomonodromy deformation problem. The fact that isomonodromy deformations are relevant to the computation of correlation functions can be understood from simple physical arguments. Indeed, monodromies of correlation functions reflect, and are actually equivalent to, the equal-time braiding relations among the fields. But the braiding relations are scale invariant and therefore renormalization group invariant. In other words, the renormalization group transformations are isomonodromy deformations.

Where do appear the isomonodromy deformations in our approach? We started from the algebraic equations defining the form factors, we solved them and then computed the correlations using the exact expression of the form factors. The point, making the relation with the isomonodromy deformations apparent, is that the form factor axioms depend both on the matrices encoding the braiding relations and on the $S$-matrix which also codes the braiding relations since it is given by the universal $R$-matrix. Moreover, the mass scale does not show up explicitly in the form factor axioms. Therefore, computing the form factors is just an algebraic way to solve an iso-braiding deformation problem.

• Generalizations.

Let us describe a way to generalize our approach to other situations. The proof that the correlation functions satisfy the affine sinh-Gordon equations relied on their identification with soliton $\tau$-functions of this hierarchy. This identification is simply the remark that the $\tau$-functions and the form factors are built from the same kernel $K(\theta_i, \theta_j) = \tanh(\theta_i - \theta_j/2)$. In the context of $\tau$-functions, this kernel is the expectation value of vertex operators from which the hierarchy can be reconstructed.

This suggests the following procedure. First, assuming that the form factors have been determined (e.g. using algebraic methods inherited from quantum groups), find vertex operator for which they are the expectation values,

$$F^{(n)}(\theta_1, \cdots, \theta_n)^2 = \langle V(\theta_1) \cdots V(\theta_n) \rangle$$

Here the expectation values are in an auxiliary Hilbert space on which the vertex operators are acting. Then, reconstruct the hierarchy from the knowledge of the vertex operators (e.g. using Casimirs in the vertex operator
algebra). The correlation functions will then be soliton-type $\tau$-functions of the corresponding hierarchy. The simplest models in this approach are those with diagonal $S$-matrices as we will describe elsewhere.

6 Appendix: Fredholm determinant.

Here, we show that the correlation functions can be written as Fredholm determinants. Indeed, using the identity,

$$
\prod_{i<j} \left( \frac{\mu_i - \mu_j}{\mu_i + \mu_j} \right)^2 = \det \left( \frac{2\sqrt{\mu_i \mu_j}}{\mu_i + \mu_j} \right)_{i,j=1,\ldots,n}
$$

the $\tau$-functions $C^\infty_\pm (z)$ becomes:

$$
C^\infty_\pm (z) = \sum_{n=0}^\infty \frac{\left( \pm \right)^n}{n!} \int_{-\infty}^{+\infty} \prod_{i=1}^{n} \left( \frac{d\theta_i}{2\pi} X(\theta_i|z) \right) \det \left( \frac{1}{\cosh(\frac{\theta_i - \theta_j}{2})} \right)_{i,j=1,\ldots,n}
$$

By definition, this is a Fredholm determinant,

$$
C^\infty_\pm (z) = \text{Det} \left( 1 \pm K \right)
$$

for an integral operator with kernel,

$$
K(\theta, \theta') = \sqrt{\frac{X(\theta|z)}{2\pi}} \frac{1}{\cosh\left(\frac{\theta - \theta'}{2}\right)} \sqrt{\frac{X(\theta'|z)}{2\pi}}
$$

It is interesting to note that all the dependence in the parameters $z$, as well as the mass scale, is concentrated in the potential $X(\theta|z)$; i.e. the heart of the kernel, $\cosh^{-1}\left(\frac{(\theta - \theta')}{2}\right)$, is independent of all these parameters. The property, that all the coupling constants and all the renormalization group dependence manifest themselves only in the potential part of the kernel, appears frequently when dealing with Fredholm determinant in physical context, see e.g. [4].

Finally, notice also that since the square of the Ising model is equivalent to the sine-Gordon model at the free field point, this implies that the correlation functions of the sine-Gordon theory at that particular point can be written as the square of Fredholm determinants.
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