Analysis of a time-stepping discontinuous Galerkin method for modified anomalous subdiffusion problems

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Abstract

This paper analyzes a time-stepping discontinuous Galerkin method for modified anomalous subdiffusion problems with two time fractional derivatives of orders $\alpha$ and $\beta$ ($0 < \alpha < \beta < 1$). The stability of this method is established, the temporal accuracy of $O(\tau^{m+1-\beta/2})$ is derived, where $m$ denotes the degree of polynomials for the temporal discretization. It is shown that, even the solution has singularity near $t = 0^+$, this temporal accuracy can still be achieved by using the graded temporal grids. Numerical experiments are performed to verify the theoretical results.

Keywords: modified anomalous subdiffusion, discontinuous Galerkin method, stability, convergence.

1 Introduction

This paper considers the following modified anomalous fractional subdiffusion problem:

$$
\begin{cases}
\partial_t u - \left( \kappa_1 D_{0+}^{\alpha} + \kappa_2 D_{0+}^{\beta} \right) \partial_x^2 u = f & \text{in } \Omega \times (0, T), \\
u = 0 & \text{on } \partial\Omega, \\
u(\cdot, 0) = u_0 & \text{in } \Omega,
\end{cases}
$$

where $\Omega \subset \mathbb{R}$ is an open interval, $\kappa_1$ and $\kappa_2$ are two positive constants, $0 < T < \infty$, $0 < \alpha < \beta < 1$, $u_0 \in H_0^1(\Omega)$, and $f \in L^1(0, T; H^{-1}(\Omega))$.

A considerable amount of research has been devoted to the numerical treatment of time fractional diffusion problems, especially in the past decade. So far, most of the existing algorithms are classified as fractional difference methods, since they employ the $L^1$ formula, Grünwald-Letnikov discretization or fractional linear multi-step method to discretize the fractional derivatives. Despite their ease of implementation, the fractional difference methods are generally of...
temporal accuracy orders not greater than two; see [30, 13, 29, 3, 19, 34, 6, 32, 2, 5, 16, 20, 9, 15, 31, 12, 28, 10, 17] and the references therein. We also note that Gao et. al [10] designed a formula to approximate the Caputo fractional derivative of order \(\alpha\) \((0 < \alpha < 1)\) and applied this formula to numerically solve time fractional diffusion problems; however, the theory of stability and convergence was not established there. To improve the temporal accuracy, fractional spectral methods, namely those algorithms using spectral methods to discretize the fractional derivatives, were proposed; see [18, 33, 14]. Recently, Mustapha and Mclean ([22, 24, 23]) used the discontinuous Galerkin method to approximate the time fractional derivatives, and they proposed a class of methods that possess high-order temporal accuracy. Moreover, as the fractional difference methods, the numerical solutions of their methods are computed in a step by step fashion.

Due to the nonlocal property of the fractional derivatives, the computation and storage cost of an accurate numerical solution to a time fractional diffusion problem significantly exceeds that to a corresponding normal diffusion problem. Naturally, developing high-order accuracy algorithms, especially those with high-order temporal accuracy, is an efficient way to reduce the cost. However, as aforementioned, generally the best temporal accuracy order of the fractional difference methods is merely two. This motivates us to develop algorithms that possess high-order accuracy in both space and time while retaining the advantage of the fractional finite difference methods.

Following the work of [23] for fractional diffusion equations, we analyze a time-stepping discontinuous Galerkin method for problem (1.1). Firstly, we establish a new stability estimate. Secondly, we prove that the temporal accuracy is \(O(\tau^{m+1-\beta/2})\), and that if \(u\) has singularity near \(t = 0^{+}\), then this temporal accuracy can still be achieved by using graded temporal grids. We note that on appropriate graded temporal grids, [23] obtained the temporal accuracy \(O(\tau^{2-\beta/2})\) in the case of \(m = 1\) and the temporal accuracy \(O(\tau^{m+(1-\beta)/2})\) in the case of \(m \geq 2\).

The rest of this paper is organized as follows. Section 2 introduces some notations. Sections 3 and 4 establish the stability and convergence of the time-stepping discontinuous Galerkin method. Section 5 performs several numerical experiments to verify the theoretical results. Finally, Section 6 gives concluding remarks.

### 2 Notation

Let us first introduce the Riemann-Liouville fractional calculus operators.

**Definition 2.1.** For \(0 < \gamma < \infty\) and any \(v \in L^1(0, T; X)\), define

\[
(I_{0+}^{\gamma} v)(t) := \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} v(s) \, ds, \quad 0 < t < T,
\]

\[
(I_{T-}^{\gamma} v)(t) := \frac{1}{\Gamma(\gamma)} \int_t^T (s-t)^{\gamma-1} v(s) \, ds, \quad 0 < t < T,
\]

where \(\Gamma(\cdot)\) is the gamma function.
Definition 2.2. For $0 < \gamma < 1$, define
\[
D_{T^+}^{\gamma,X} := D_{I_{T^+}}^{1 - \gamma,X}, \\
D_{T^-}^{\gamma,X} := -D_{I_{T^-}}^{1 - \gamma,X},
\]
where $D$ is the first-order differential operator in the distribution sense.

Above $X$ is a Banach space and $L^1(0,T; X)$ is a standard $X$-valued Bochner $L^1$ space. For convenience, we shall simply use $I_{T^+}^{\gamma}$, $I_{T^-}^{\gamma}$, $D_{T^+}^{\gamma}$, and $D_{T^-}^{\gamma}$, without indicating the underlying Banach space $X$.

Next we introduce some vector valued spaces. Let $X$ be a separable Hilbert space with an inner product $\langle \cdot, \cdot \rangle_X$ and an orthonormal basis $\{e_j : j \in \mathbb{N}\}$, and let $O$ be an interval. For $0 < \gamma < \infty$, define
\[
H^{\gamma}(O; X) := \left\{ v \in L^2(O; X) : \sum_{j=0}^{\infty} \| (v, e_j)_X \|^2_{H^{\gamma}(O)} < \infty \right\}
\]
and equip this space with the norm
\[
\| \cdot \|_{H^{\gamma}(O; X)} := \left( \sum_{j=0}^{\infty} \| (v, e_j)_X \|^2_{H^{\gamma}(O)} \right)^{\frac{1}{2}},
\]
where $L^2(O; X)$ is an $X$-valued Bochner $L^2$ space. If $0 < \gamma < 1/2$, we also introduce the seminorm
\[
| \cdot |_{H^{\gamma}(O; X)} := \left( \sum_{j=0}^{\infty} | (v, e_j)_X |^2_{H^{\gamma}(O)} \right)^{\frac{1}{2}}.
\]

Here, $H^{\gamma}(O)$ is a standard Sobolev space (see [26]), and
\[
|v|_{H^{\gamma}(O)} := \left( \int_{\mathbb{R}} |\xi|^{2\gamma} |\mathcal{F}(v \chi_O)(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}
\]
for each $v \in H^{\gamma}(O)$ with $0 < \gamma < 1/2$, where $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the Fourier transform operator and $\chi_O$ is the indicator function of the interval $O$. For $v \in H^{\gamma}(O; X)$ with $i \in \mathbb{N}_{>0}$, define its $i$th weak derivative $v^{(i)}$ by
\[
v^{(i)} := \sum_{j=0}^{\infty} c_j^{(i)}(t)e_j, \quad t \in O,
\]
where $c_j := (v, e_j)_X$ and $c_j^{(i)}$ is its $i$th weak derivative. In particular, $v^{(1)}$ is abbreviated to $v'$.

Additionally, for $0 \leq \delta < 1$, define
\[
L^2_{\delta}(O; X) := \left\{ v \in L^1(O; X) : \|v\|_{L^2_{\delta}(O; X)} < \infty \right\},
\]
where
\[
\|v\|_{L^2_{\delta}(O; X)} := \left( \int_{O} |t|^\delta \|v(t)\|^2_X \, dt \right)^{\frac{1}{2}}.
\]
We note that in the context, standard Sobolev spaces (see \[O\]) and \(P_j(\Omega; X)\) on \(a\) partition of \(\Omega\) consisting of open intervals, and let \(h\) denote the maximum length of the elements in \(K_h\). We introduce a graded mesh subdivision of the temporal interval \((0, T)\). For \(j \in \mathbb{N}_{>0}\) and \(\sigma \geq 1\), we set
\[
\begin{aligned}
t_j &:= (j/J)^\sigma T \\
I_j &:= (t_{j-1}, t_j), \quad \tau_j := t_j - t_{j-1}
\end{aligned}
\] (2.1)
and use \(\tau\) to abbreviate \(\tau_j\). Define
\[
S_h := \{ v_h \in H_0^1(\Omega) : v_h|_K \in P_n(K), \forall K \in K_h \},
\]
\[
W_{h, \tau} := \{ V \in L^2(0, T; S_h) : V|_{I_j} \in P_n(I_j; S_h), \forall 1 \leq j \leq J \},
\]
where \(m \in \mathbb{N}\) and \(n \in \mathbb{N}_{>0}\). Moreover, for \(V \in W_{h, \tau}\) we introduce the following notation:
\[
\begin{aligned}
V_j^+ &:= \lim_{t \to t_j^+} V(t) \quad \text{for } 0 \leq j < J; \\
V_j^- &:= \lim_{t \to t_j^-} V(t) =: V(t_j) \quad \text{for } 0 < j \leq J, \text{ and } V_0 := 0;
\end{aligned}
\]
\[
\|V_j\| := V_j^+ - V_j^- \quad \text{for } 0 \leq j < J.
\]
We note that in the context, \(H^s(\Omega) (s \in \mathbb{Z})\) and \(H_0^s(\Omega) (s \in \mathbb{N}_{>0})\) denote two standard Sobolev spaces (see [26]).

Throughout this paper, we make the following conventions: each \(v \in L^1(\Omega \times (0, T))\) is regarded as an element of \(L^1(0, T; L^1(\Omega))\), also denoted by \(v\); the notation \(a \lesssim b\) means that there exists a positive constant \(C\) depending only on \(\alpha, \beta, m\) or \(n\) such that \(a \leq Cb\), and \(a \sim b\) means \(a \lesssim b \lesssim a\); by \(C_x\) we denote a positive constant that only depends on \(x\) and its value may differ at each of its occurrences; if \(\mathcal{O}\) is a Lebesgue measurable set of \(\mathbb{R}\) or \(\mathbb{R}^2\), then \(\langle \cdot, \cdot \rangle_{\mathcal{O}}\) means \(\int_{\mathcal{O}} \langle \cdot, \cdot \rangle\); if \(X\) is a Banach space, then \(\langle \cdot, \cdot \rangle_X\) denotes the duality pairing between \(X^*\) and \(X\).

3 Main Results

Let us first describe the time-stepping discontinuous Galerkin method to be analyzed as follows: seek \(U \in W_{h, \tau}\) such that
\[
\begin{aligned}
\langle U', V \rangle_{\Omega_T} + \sum_{j=0}^{J-1} \langle [U_j], V_j^+ \rangle_{\Omega_T} + \left( \left( \kappa_1 D_{0+}^\alpha + \kappa_2 D_{0+}^\beta \right) \partial_x U, \partial_x V \right)_{\Omega_T} \\
= \langle R_h u_0, V_0^+ \rangle_{\Omega_T} + \langle f, V \rangle_{L^2(0, T; H_0^1(\Omega))}
\end{aligned}
\] (3.1)
for all \(V \in W_{h, \tau}\), where \(\Omega_T := \Omega \times (0, T)\) and the projection operator \(R_h\) is defined by
\[
\langle \partial_x (v - R_h v), \partial_x v_h \rangle_{\Omega} = 0, \quad \forall v \in H_0^1(\Omega), \forall v_h \in V_h.
\]
Above \( U' \) is understood by
\[
U'|_{I_j} := (U|_{I_j})', \quad 1 \leq j \leq J.
\]

Then, assuming \( X \) to be a Banach space, we define an interpolation operator \( Q^X_v \) as follows [27, Chapter 12]: given \( v \in C((0,T]; X) \cap L^1(0,T; X) \), the interpolant \( Q^X_v \) fulfills, for each \( 1 \leq j \leq J \),
\[
\begin{cases}
(Q^X_v)|_{I_j} \in P_m(I_j; X), & \lim_{t \to t_{j-1}^-} (Q^X_v)(t) = v(t_j), \\
\int_{t_{j-1}}^{t_j} (v - Q^X_v) q \, dt = 0 & \text{for all } q \in P_{m-1}(I_j).
\end{cases}
\]

Below we will use \( Q \) instead of \( Q^X_v \) when no confusion will arise.

Now we are ready to state the main results of this paper, and, for convenience, we assume that \( u \) is the solution to problem \((1.1)\).

**Theorem 3.1.** The scheme \((3.1)\) admits a unique solution \( U \). In addition, if \( f \in L^2(0,T; H^{-1}(\Omega)) \), then
\[
\|U(t_j)\|_{L^2(\Omega)} + \sqrt{\kappa_1} |U|_{H^{n/2}(0,t_j; H^0_0(\Omega))} + \sqrt{\kappa_2} |U|_{H^{n/2}(0,t_j; H^0_0(\Omega))} 
\leq \|u_0\|_{H^0_0(\Omega)} + 1/\sqrt{\kappa_2} \|f\|_{L^2(0,t_j; H^{-1}(\Omega))} \tag{3.2}
\]
for each \( 1 \leq j \leq J \).

**Theorem 3.2.** If \( u' \in L^2(0,T; H^0_0(\Omega)) \), then
\[
\|\theta(t_j)\|_{L^2(\Omega)} + \sqrt{\kappa_1} |\theta|_{H^{n/2}(0,t_j; H^0_0(\Omega))} + \sqrt{\kappa_2} |\theta|_{H^{n/2}(0,t_j; H^0_0(\Omega))} 
\leq \eta_{j,1} + \eta_{j,2} + \eta_{j,3} \tag{3.3}
\]
for each \( 1 \leq j \leq J \), where \( \theta := U - Q_T R_h u \) and
\[
\begin{align*}
\eta_{j,1} &:= k_{min(2,n)} / \sqrt{\kappa_2} \|(I - R_h) u'\|_{L^2(0,t_j; H^0_0(\Omega))}, \\
\eta_{j,2} &:= \sqrt{\kappa_1} \left( \sum_{i=1}^{J} \tau_i^{2-\alpha} \inf_{0 \leq \delta < 1} \frac{t_i^{-\delta}}{1 - \delta} \|(u - Q_T u)'\|_{L^2(I_i; H^0_0(\Omega))}^2 \right) \frac{1}{2}, \\
\eta_{j,3} &:= \sqrt{\kappa_2} \left( \sum_{i=1}^{J} \tau_i^{2-\beta} \inf_{0 \leq \delta < 1} \frac{t_i^{-\delta}}{1 - \delta} \|(u - Q_T u)'\|_{L^2(I_i; H^0_0(\Omega))}^2 \right) \frac{1}{2}.
\end{align*}
\]

**Corollary 3.1.** If \( u \in H^{m+1}(0,T; H^1(\Omega)) \) and \( u' \in L^2(0,T; H^0_0(\Omega) \cap H^{n+1}(\Omega)) \),
then
\[
\|(u - U)(t_j)\|_{L^2(\Omega)} \leq \nu_{j,1} + \nu_{j,2}
\]
for each \( 1 \leq j \leq J \), where
\[
\begin{align*}
\nu_{j,1} &:= k_{min(2,n)+n} / \sqrt{\kappa_2} \|u'\|_{L^2(0,t_j; H^{n+1}(\Omega))} + H^{n+1} \|u(t_j)\|_{H^{n+1}(\Omega)}, \\
\nu_{j,2} &:= \left( \sqrt{\kappa_1} \tau_j^{1+\alpha/2} + \sqrt{\kappa_2} \tau_j^{1+m-\beta/2} \right) \|u\|_{H^{n+1}(0,t_j; H^1(\Omega))}.
\end{align*}
\]
Corollary 3.2. If \( u(x,t) = t^r \phi(x) \) for \( (x,t) \in \Omega_T \), with \( (\beta-1)/2 < r \leq m+1/2 \) and \( \phi \in H^3_\delta(\Omega) \cap H^{m+1}(\Omega) \), then
\[
\|(u-U)(t_j)\|_{L^2(\Omega)} \lesssim C_{\sigma,r} (\sqrt{h_1} + \sqrt{h_2}) \varepsilon_j \|\phi\|_{H^3_\delta(\Omega)} + h^{n+1} t_j^r \|\phi\|_{H^{m+1}(\Omega)}
\]
for all \( 1 \leq j \leq J \), where
\[
\varepsilon_j := \begin{cases} 
T^{(1-\sigma)(r+(1-\beta)/2)}_r \sigma(r+(1-\beta)/2) & \text{if } \sigma < \sigma^*, \\
(1 + \ln (j)) T^{r-m-1/2}_{m+1-\beta/2} & \text{if } \sigma = \sigma^*, \\
T^{r-m-1/2}_{m+1-\beta/2} & \text{if } \sigma > \sigma^*,
\end{cases}
\]
\( \sigma \) is the graded parameter in (2.1), and
\[
\sigma^* := \frac{2m + 2 - \beta}{2r + 1 - \beta} \geq 1. \quad (3.4)
\]

Remark 3.1. Due to the fact that
\[
\|U\|_{H^{\beta/2}(0,T;H^3_\delta(\Omega))} \lesssim C_T \|U\|_{H^{\beta/2}(0,T;H^3_\delta(\Omega))},
\]
by Theorems 3.1 and 3.2 we can also derive the stability and error estimates of \( U \) with respect to the norm on \( H^{\beta/2}(0,T;H^3_\delta(\Omega)) \).

Remark 3.2. If \( n \geq 2 \) and the condition of Corollary 3.1 is satisfied, then Theorem 3.2 implies
\[
\|(U - Q_h R_h u)(t_j)\|_{L^2(\Omega)} = O(h^{n+2} + r^{m+1-\beta/2}).
\]
Assume that \( K_h \) is quasi-uniform and \( \{x_i: 1 \leq i \leq N\} \) is the set of all nodes of \( K_h \). Using the standard result
\[
R_h u(x_i, t_j) = u(x_i, t_j), \quad 1 \leq i \leq N,
\]
we obtain
\[
\max_{1 \leq i \leq N} |U(x_i, t_j) - u(x_i, t_j)| = O(h^{n+1} + h^{-1} r^{m+1-\beta/2}).
\]
Therefore, if \( r \) is sufficiently small, then
\[
\max_{1 \leq i \leq N} |U(x_i, t_j) - u(x_i, t_j)| = O(h^{n+1}).
\]

Remark 3.3. Though the graded grids are assumed in (2.1), from the proofs in Section 4 it is easy to see that Theorem 3.1, Theorem 3.2, and Corollary 3.1 still hold for more general temporal grids, with \( \tau_j \) in Corollary 3.1 replaced by \( \max_{1 \leq i \leq j} \tau_i \).

Remark 3.4. First, Corollary 3.2 shows that if \( u \) has singularity near \( t = 0^+ \), then the graded grids in the time direction can improve the temporal accuracy to \( O(r^{m+1-\beta/2}) \) up to an factor \( \ln (j) \) provided that \( \sigma = \sigma^* \). Numerical results show that our estimates are sharp for \( \sigma \leq \sigma^* \). Second, theoretically we can not expect the optimal accuracy \( O(r^{m+1}) \) as \( \sigma \) increases. However, numerical tests indicate that the optimal convergence rate can also be obtained if
\[
\sigma = \sigma^{**} := \frac{2m + 2}{2r + 1 - \beta} > \sigma^*.
\]
Remark 3.5. We note that for the time stepping discontinuous Galerkin discretization of fractional diffusion problems, [23] obtained the temporal accuracy $O(\tau^{2-\beta/2})$ for $m = 1$ and $O(\tau^{m+(1-\beta)/2})$ for $m \geq 2$ on appropriate graded temporal grids.

The rest of this section will briefly discuss the singularity of the solution to problem (1.1) near $t = 0^+$. Let $\{\phi_j\}_{j=0}^\infty$ be an orthonormal basis of $L^2(\Omega)$ such that $\phi_j \in H^1_0(\Omega)$ and

$$-\partial_x^2 \phi_j = \lambda_j \phi_j \quad x \in \Omega,$$

where $\{\lambda_j\}_{j=0}^\infty \subset \mathbb{R}_{>0}$ is a non-decreasing sequence. For each $j \in \mathbb{N}$, define

$$u_j(t) := \langle u(x,t), \phi_j \rangle_\Omega, \quad 0 < t < T,$$

$$f_j(t) := \langle f(x,t), \phi_j \rangle_\Omega, \quad 0 < t < T.$$

Evidently, $u_j$ satisfies the fractional ordinary equation

$$u'_j + \lambda_j (\kappa_1 D^\alpha_{0+} + \kappa_2 D^\beta_{0+}) u_j = f_j \quad t \in (0,T),$$

subject to the initial value condition $u_j(0) = \langle u_0, \phi_j \rangle_\Omega$. An elementary computation yields

$$u_j + \lambda_j (\kappa_1 I^1_{0+} - \alpha + \kappa_2 I^1_{0+} - \beta) u_j = I_{0+} f_j + u_j(0) \quad \text{in } (0,T).$$

Suppose that $w_j$ satisfies

$$w'_j + \lambda_j (\kappa_1 I^1_{0+} - \alpha + \kappa_2 I^1_{0+} - \beta) w_j = 1 \quad \text{in } (0,T).$$

Using the famous Picard iterative process gives

$$w_j(t) = \sum_{p+q=r}^{\infty} \sum_{r=0} \binom{r}{p} \frac{(-\lambda_j \kappa_1)^p (-\lambda_j \kappa_2)^q p^{p(1-\alpha)+q(1-\beta)}}{\Gamma(1+p(1-\alpha)+q(1-\beta))}.$$ 

It is easy to verify that

$$u_j(t) = u_j(0)w_j(t) + \int_0^t w_j(t-s)f_j(s) \, ds, \quad 0 < t < T,$$

which indicates that the singularity part of $u_j$ belongs to

$$S := \text{span} \left\{ p^{p(1-\alpha)+q(1-\beta)} : p + q > 0, \ p, q \in \mathbb{N} \right\},$$

provided $f_j$ is sufficiently regular. Therefore, since

$$w' \in L^2_0(0,T) \quad \text{for all } w \in S,$$

the assumption $u' \in L^2_0(0,T;H^1_0(\Omega))$ is reasonable. We note that the relation (3.7) has been applied to ordinary differential equations with multi-term fractional derivatives ([11, 21]).
4  Proofs

4.1  Auxiliary Results

Let us first summarize some standard results.

**Lemma 4.1** ([4, 1]). If \( v \in H^1_0(\Omega) \cap H^{n+1}(\Omega) \), then
\[
\|(I - R_h)v\|_\Omega + h \|(I - R_h)v\|_{H^1_0(\Omega)} \lesssim h^{n+1} \|v\|_{H^{n+1}(\Omega)}.
\]

If \( v \in H^1_0(\Omega) \) and \( w \in H^1(\Omega) \), then
\[
\langle (I - R_h)v, w \rangle_\Omega \lesssim h_{\min\{2, n\}} \|(I - R_h)v\|_{H^1_0(\Omega)} \|w\|_{H^1(\Omega)}.
\]

If \( v \in H^{m+1}(I_j) \) with \( 1 \leq j \leq J \), then
\[
\|(I - Q_{\tau})v\|_{L^2(I_j)} + \tau_j \|(I - Q_{\tau})v\|_{H^1(I_j)} \lesssim \tau_j^{m+1} \|v\|_{H^{m+1}(I_j)}.
\]

**Lemma 4.2** ([26]). If \( v \in H^\gamma(\mathbb{R}) \) with \( 0 < \gamma < 1 \), then
\[
C_\gamma |v|_{H^\gamma(\mathbb{R})} \lesssim \left( \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|v(s) - v(t)|^2}{|s - t|^{1 + 2\gamma}} \, ds \, dt \right)^{\frac{1}{2}} \lesssim C_\gamma |v|_{H^\gamma(\mathbb{R})}.
\]

**Lemma 4.3** ([25, 7]). The following properties hold:

- If \( 0 < \gamma, \delta < \infty \), then
  \[
  I_{\gamma+} I_{\delta+} = I_{\gamma+\delta+}, \quad I_{\gamma-} I_{\delta-} = I_{\gamma+\delta}.
  \]

- If \( 0 < \gamma < \infty \) and \( u, v \in L^2(0, T) \), then
  \[
  \langle I_{\gamma+} u, v \rangle_{(0, T)} = \langle u, I_{\gamma+} v \rangle_{(0, T)}.
  \]

**Lemma 4.4.** If \( v, w \in H^{\gamma/2}(0, T) \) and \( D_{\gamma+}^0 v \in L^1(0, T) \) with \( 0 < \gamma < 1 \), then
\[
\begin{align*}
\langle D_{\gamma+}^0 v, v \rangle_{(0, t)} & = \cos(\gamma \pi / 2) |v|_{\gamma/2}^2(0, t), \quad (4.1) \\
\langle D_{\gamma+}^0 v, w \rangle_{(0, t)} & \leq \cos(\gamma \pi / 2) |v|_{\gamma/2}^2(0, t) |w|_{\gamma/2}^2(0, t), \quad (4.2)
\end{align*}
\]
for all \( 0 < t \leq T \).

The proof of the above lemma is contained in [8, Lemmas 2.2, 2.4 and 2.9].

The purpose of the rest of this subsection is to prove the following three lemmas.

**Lemma 4.5.** For \( 0 < t \leq T \), it holds that
\[
\int_0^t |(vw)(s)| \, ds \lesssim \|v\|_{L^2_\beta(0, t)} \|w|_{H^{\beta/2}(0, t)} \quad (4.3)
\]
for all \( v \in L^2_\beta(0, T) \) and \( w \in H^{\beta/2}(0, T) \).
Lemma 4.6. Let $0 < a < b < \infty$ and $0 < \gamma < 1$. If $v' \in L^2_b(a, b)$ with $0 \leq \delta < 1$ and $v(b) = 0$, then
\[
\int_a^b v^2(t)(t - a)^{-\gamma} \, dt \leq \frac{b^{-\delta}}{(1 - \delta)(1 - \gamma)}(b - a)^{2-\gamma} \|v'\|_{L^2_b(a, b)}^2, \quad (4.4)
\]
\[
\int_a^b v^2(t)(b - t)^{-\gamma} \, dt \leq \frac{b^{-\delta}}{(1 - \delta)(1 - \gamma)}(b - a)^{2-\gamma} \|v'\|_{L^2_b(a, b)}^2, \quad (4.5)
\]
\[
\int_a^b \int_a^b |v(s) - v(t)|^2 |s - t|^{-1-\gamma} \, ds \, dt \leq \frac{8b^{-\delta}}{1 - \delta}(b - a)^{2-\gamma} \|v'\|_{L^2_b(a, b)}^2, \quad (4.6)
\]
where $B(\cdot, \cdot)$ is the Beta function.

Lemma 4.7. For $0 < \gamma < 1$, if $v \in H^{\gamma/2}(0, T)$ and $v' \in L^1(0, T)$, then
\[
|I - Q_T|^2 v^2 H^{\gamma/2}(0, t_j) \lesssim C_\gamma \sum_{i=1}^{j} \tau_i^{2-\gamma} \inf_{0 < \delta < 1} \frac{t_i^{-\gamma}}{1 - \delta} \|(v - Q_T v)'\|_{L^2_{\tau_i}}^2, \quad (4.7)
\]
for each $1 \leq j \leq J$.

Lemma 4.8. If $v(t) := t^r, \quad 0 \leq t \leq T,$
with $0 < r \leq m + 1/2$, then
\[
\sum_{i=1}^{j} \tau_i^{2-\gamma} \inf_{0 < \delta < 1} \frac{t_i^{-\gamma}}{1 - \delta} \|(v - Q_T v)'\|_{L^2_{\tau_i}}^2 \lesssim C_{\gamma, \sigma, \tau}
\begin{cases}
T^{(1-\sigma)(2r+1-\gamma)} & \text{if } \sigma < \frac{2m+2-\gamma}{2r+1-\gamma}
(1 + \ln(j))T^{2r-1-2m+2\gamma} & \text{if } \sigma = \frac{2m+2-\gamma}{2r+1-\gamma}
T^{2r-1-2m+2\gamma} & \text{if } \sigma > \frac{2m+2-\gamma}{2r+1-\gamma},
\end{cases}
\quad (4.8)
\]
for each $1 \leq j \leq J$.

Proof of Lemma 4.5. By Lemma 4.2, extending $w$ to $\mathbb{R} \setminus (0, t)$ by zero gives
\[
\int_0^t \int_{-\infty}^0 w^2(s)(s - \tau)^{-1-\beta} \, d\tau \lesssim |w|_{H^{\beta/2}(\mathbb{R})}^2,
\]
which indicates
\[
\int_0^t s^{\beta}w^2(s) \, ds \lesssim |w|_{H^{\beta/2}(\mathbb{R})}^2.
\]
Therefore, the Cauchy-Schwarz inequality yields
\[
\int_0^t |(v w)(s)| \, ds \lesssim \left( \int_0^t s^\beta v^2(s) \, ds \right)^{1/2} \left( \int_0^t s^{\beta}w^2(s) \, ds \right)^{1/2}
\lesssim \|v\|_{L^2_b(0, t)} \|w\|_{H^{\beta/2}(\mathbb{R})},
\]
and hence the fact $|w|_{H^{\beta/2}(\mathbb{R})} = |w|_{H^{\beta/2}(0, t)}$ proves the lemma. \qed
Proof of Lemma 4.6. The proof below shall be brief, since the techniques used are standard (see Minkowski’s integral inequality and Hardy’s inequality). For \( a < t < b \), a simple computing gives

\[
|v(t)| \leq \int_t^b |v'(s)| \, ds \leq \left( \int_t^b s^{-\delta} \, ds \right) \frac{1}{\delta} \left( \int_t^b s^\delta |v'(s)|^2 \, ds \right)^{\frac{1}{2}} \leq \sqrt{\frac{b^{1-\delta} - t^{1-\delta}}{1 - \delta}} \|v'||_{L^2(a,b)} \leq \sqrt{\frac{b^{-\delta} (b - a)}{1 - \delta}} \|v'||_{L^2(a,b)}
\]

so that we obtain

\[
\int_a^b v^2(t)(t - a)^{-\gamma} \, dt \leq \frac{c(b - a)}{1 - \delta} \int_a^b (t - a)^{-\gamma} \, dt \|v'||_{L^2(a,b)}^2 = \frac{b^{-\delta} (b - a)^{2-\gamma}}{(1 - \delta)(1 - \gamma)} \|v'||_{L^2(a,b)}^2
\]

namely the estimate (4.4). Similarly, we have

\[
\int_a^b v^2(t)(b - t)^{-\gamma} \, dt \leq \frac{b^{-\delta} (b - a)}{1 - \delta} \int_a^b (b - t)^{-\gamma} \, dt \|v'||_{L^2(a,b)}^2 = \frac{b^{-\delta} (b - a)^{2-\gamma}}{(1 - \delta)(1 - \gamma)} \|v'||_{L^2(a,b)}^2
\]

namely the estimate (4.5). Finally, let us prove (4.6). Since

\[
\int_a^b \int_a^b \left| v(s) - v(t) \right|^2 |s - t|^{-1-\gamma} \, ds \, dt \\
= 2 \int_a^b \int_a^b \int_t^s \left| v'(\tau) \right|^2 (s - t)^{-1-\gamma} \, d\tau \, ds \, dt \\\n= 2 \int_a^b \int_a^b \int_0^1 \left| v'(t + \theta(s - t)) \right|^2 (s - t)^{1-\gamma} \, d\theta \, ds \, dt \\\n\leq 2(b - a)^{1-\gamma} \int_a^b \left( \int_0^1 \sqrt{\int_t^b \left| v'(t + \theta(s - t)) \right|^2 \, ds} \, d\theta \right)^2 \, dt \\\n= 2(b - a)^{1-\gamma} \int_a^b \left( \int_0^1 \sqrt{\int_t^{t+\theta(b-t)} \left| v'(\eta) \right|^2 \, d\eta} \theta^{-1} \, d\theta \right)^2 \, dt,
\]
the inequality (4.6) is a direct consequence of
\[
\int_a^b \left( \int_0^1 \sqrt{\int_t^{t+\theta(b-t)} |v'(\eta)|^2 \, d\eta} \, d\theta \right)^2 \, dt
\leq \int_a^b \left( \int_0^1 \sqrt{\int_t^{t+\theta(b-t)} \eta \, v'(\eta)^2 \, d\eta} \, d\theta \right)^2 \, dt
\]
\[
= \int_a^b \left( \int_0^1 \theta^{1/2} \sqrt{\int_t^{t+\theta(b-t)} \eta \, v'(\eta)^2 \, d\eta} \, d\theta \right)^2 \, dt
\]
\[
\leq \int_a^b \left( \int_0^1 \theta^{-1/2} \, d\theta \right)^2 \, dt \|v'\|^2_{L^2(a,b)}
\leq \frac{4b^{-\delta}(b-a)}{1-\delta} \|v'\|^2_{L^2(a,b)}.
\]
This lemma is thus proved. ■

Proof of Lemma 4.7. By Lemma 4.2 we only need to prove
\[
I_1 + I_2 + I_3 \leq C \gamma \sum_{j=1}^{j} \tau_i^{2-\gamma} \inf_{0 \leq \delta < 1} \frac{t_i^{-\delta}}{1-\delta} \|v - Q \tau_i v\|^2_{L^2(t_i)}.
\]
where
\[
I_1 = \sum_{i=1}^{j} \int_{t_{i-1}}^{t_i} dt \int_{t_{i-1}}^{t_i} |g(t) - g(s)|^2 |t - s|^{-1-\gamma} \, ds,
\]
\[
I_2 = \sum_{i=1}^{j} \sum_{l=i+1}^{j} \int_{t_{i-1}}^{t_i} dt \int_{t_{i-1}}^{t_l} |g(t) - g(s)|^2 |t - s|^{-1-\gamma} \, ds,
\]
\[
I_3 = \int_0^{t_j} |g(t)|^2 \left( \int_{t_j}^{t_j} (s-t)^{-1-\gamma} \, ds + \int_{t_j}^0 (t-s)^{-1-\gamma} \, ds \right) \, dt.
\]
A straightforward calculation gives
\[
\sum_{i=1}^{j} \sum_{l=i+1}^{j} \int_{t_{i-1}}^{t_i} dt \int_{t_{i-1}}^{t_l} g^2(t) |t - s|^{-1-\gamma} \, ds
\]
\[
= \frac{1}{2} \sum_{i=1}^{j} \sum_{l=i+1}^{j} \int_{t_{i-1}}^{t_i} g^2(t) \left( (t_{i-1} - t)^{-\gamma} - (t_i - t)^{-\gamma} \right) \, dt
\]
\[
\leq \frac{1}{2} \sum_{i=1}^{j} \int_{t_{i-1}}^{t_i} g^2(t) (t_i - t)^{-\gamma} \, dt.
\]
and
\[
\sum_{i=1}^{j} \sum_{l=i+1}^{\gamma} \int_{t_{l-1}}^{t_{l}} dt \int_{t_{l-1}}^{t_{l}} g^{2}(s) |t-s|^{1-\gamma} ds
= \frac{1}{\gamma} \sum_{i=1}^{j} \sum_{l=i+1}^{\gamma} \int_{t_{l-1}}^{t_{l}} g^{2}(s) ((s-t_{i})^{-\gamma} - (s-t_{l-1})^{-\gamma}) ds
\leq \frac{1}{\gamma} \sum_{l=2}^{j} \int_{t_{l-1}}^{t_{l}} g^{2}(s)(s-t_{l-1})^{-\gamma} ds.
\]

It follows
\[
\| I_2 \| \leq \frac{2}{\gamma} \sum_{i=1}^{j} \int_{t_{l-1}}^{t_{l}} g^{2}(t)((t_{l-1} - t)^{-\gamma} + (t - t_{l-1})^{-\gamma}) dt.
\]

Therefore, since it is evident that
\[
\| I_3 \| \leq \frac{1}{\gamma} \sum_{i=1}^{j} \int_{t_{l-1}}^{t_{l}} g^{2}(t)((t_{l-1} - t)^{-\gamma} + (t - t_{l-1})^{-\gamma}) dt,
\]

using Lemma 4.6 yields
\[
\| I_2 + I_3 \| \leq \frac{3}{\gamma} \sum_{i=1}^{j} \int_{t_{l-1}}^{t_{l}} g^{2}(t)((t_{l-1} - t)^{-\gamma} + (t - t_{l-1})^{-\gamma}) dt
\leq C \gamma \sum_{i=1}^{j} \tau_{i}^{2-\gamma} \inf_{0 \leq \delta < 1} \frac{t_{i}^{-\delta}}{1-\delta} \| (v - Q_{\tau}v)' \|_{L_{x}^{2}(I_{i})}^{2}.
\]

As using Lemma 4.6 also yields
\[
\| I_1 \| \leq C \gamma \sum_{i=1}^{j} \tau_{i}^{2-\gamma} \inf_{0 \leq \delta < 1} \frac{t_{i}^{-\delta}}{1-\delta} \| (v - Q_{\tau}v)' \|_{L_{x}^{2}(I_{i})}^{2},
\]

we readily obtain (4.9) and thus complete the proof of Lemma 4.7.

**Proof of Lemma 4.8.** Setting
\[
\delta_{0} := \begin{cases} 1 - r & \text{if } 0 < r < 1/2, \\ 1/2 & \text{if } r \geq 1/2, \end{cases}
\]

by a standard scaling argument we obtain
\[
\| (Q_{\tau}v)' \|_{L_{x}^{2}(I_{1})} \leq C_{r} \| v' \|_{L_{x}^{2}(I_{1})} \leq C_{r} \tau_{1}^{(2r+\delta_{0} - 1)/2}
\]

and hence
\[
\inf_{0 \leq \delta < 1} \frac{t_{1}^{-\delta}}{1-\delta} \| (v - Q_{\tau}v)' \|_{L_{x}^{2}(I_{1})}^{2} \leq \tau_{1}^{-\delta_{0}} \inf_{0 \leq \delta < 1} \frac{t_{1}^{-\delta}}{1-\delta} \| (v - Q_{\tau}v)' \|_{L_{x}^{2}(I_{1})}^{2} \leq C_{r} \tau_{1}^{2r-1}.
\]
Therefore, Lemma 4.1 implies
\[
\sum_{i=1}^{\tau_i} t_{i}^{2-\gamma} \inf_{0 \leq \delta < 1} \frac{t_{i}^{\delta}}{1 - \delta} \|(v - Q)v\|_{L^2(I_i)}^2 \\
\lesssim C_r t_i^{2r+1-\gamma} + \sum_{i=2}^{\tau_i} t_{i}^{2m+2-\gamma} \|u^{(m+1)}\|_{L^2(I_i)}^2.
\]

(4.10)

Since a simple computing yields
\[
\tau_i < \frac{2^{\sigma-1}}{\sigma} J^{-1/\sigma} t_{i-1}^{-1/\sigma}, \quad 2 \leq i \leq J,
\]
it follows
\[
\sum_{i=2}^{\tau_i} t_{i}^{2m+2-\gamma} \|u^{(m+1)}\|_{L^2(I_i)}^2 \\
\leq C_{\sigma,r} \left( J^{-1/\sigma} \right)^{2m+2-\gamma} \int_{t_1}^j t_{i}^{2r-\gamma-(2m+2-\gamma)/\sigma} dt \\
\leq C_{\gamma,\sigma,r} \begin{cases} 
T^{2r-1-\gamma} J^{-\sigma(2r+1-\gamma)} & \text{if } \sigma < \frac{2m+2-\gamma}{2r+1-\gamma}, \\
\ln \left( \frac{t_j}{t_1} \right) T^{2r+1-\gamma} J^{-2m-2+\gamma} & \text{if } \sigma = \frac{2m+2-\gamma}{2r+1-\gamma}, \\
t_j^{2r+1-\gamma-(2m+2-\gamma)/\sigma} T^{(2m+2-\gamma)/\sigma} J^{-2m-2+\gamma} & \text{if } \sigma > \frac{2-2m-\gamma}{1+2r-\gamma}.
\end{cases}
\]

(4.11)

Therefore, by (4.10) and (4.11) and the fact $T/J < \tau$, a direct computation yields (4.8) and thus concludes the proof of Lemma 4.8.

\[\square\]

4.2 Proofs of Theorems 3.1 and 3.2 and Corollaries 3.1 and 3.2

Since the proofs of Corollaries 3.1 and 3.2 are straightforward by Theorem 3.2 and Lemmas 4.1 and 4.8, this subsection only proves Theorems 3.1 and 3.2. For $1 \leq j \leq J$, set $\Omega_{t_j} := \Omega \times (0, t_j)$ and define
\[
s_j(V, W) := \sum_{i=0}^{j-1} \langle [V_j^\pm], W_j^\pm \rangle_\Omega, \quad \forall V, W \in W_{h,\tau}.
\]

Lemma 4.9. If $V \in W_{h,\tau}$ and $v \in L^2(\Omega)$, then
\[
\frac{1}{2} \left( \|V_j\|^2_{\Omega} + \|V_j^+\|^2_{\Omega} \right) \leq \langle V', V \rangle_{\Omega_{t_j}} + s_j(V, V),
\]
\[
\frac{1}{2} \left( \|V_j\|^2_{\Omega} - \|v\|^2_{\Omega} \right) \leq \langle V', V \rangle_{\Omega_{t_j}} + s_j(V, V) - \langle v, V_j^+ \rangle_{\Omega},
\]
for all $1 \leq j \leq J$.

The above lemma is contained in the proof of [27, Theorem 12.1].
Proof of Theorem 3.1. Since the stability result (3.2) indicates the unique existence of $U$, it suffices to prove the former. By Lemmas 4.4 and 4.9, inserting $V = U\chi_{(0,t_{j})}$ into (3.1) yields
\[
\frac{1}{2}\|U_{j}\|_{L^{2}(\Omega)}^{2} + \kappa_{1}\|U\|_{H^{3/2}(0,t_{j};H_{0}^{1}(\Omega))}^{2} + \kappa_{2}\|U\|_{H^{3/2}(0,t_{j};H_{0}^{1}(\Omega))}^{2} \\
\leq \frac{1}{2}\|R_{h}u_{0}\|_{L^{2}(\Omega)}^{2} + \langle f, U \rangle_{L^{\infty}(0,t_{j};H_{0}^{1}(\Omega))},
\]
so that Lemma 4.5 implies
\[
\|U_{j}\|_{L^{2}(\Omega)}^{2} + \kappa_{1}\|U\|_{H^{3/2}(0,t_{j};H_{0}^{1}(\Omega))}^{2} + \kappa_{2}\|U\|_{H^{3/2}(0,t_{j};H_{0}^{1}(\Omega))}^{2} \\
\lesssim \|R_{h}u_{0}\|_{L^{2}(\Omega)}^{2} + \|f\|_{L_{x}^{2}(0,t_{j};H^{-1}(\Omega))}^{2} \|U\|_{H^{3/2}(0,t_{j};H_{0}^{1}(\Omega))},
\]
which, together with the estimate
\[
\|R_{h}u_{0}\|_{L^{2}(\Omega)}^{2} \lesssim \|u_{0}\|_{H_{0}^{1}(\Omega)},
\]
proves (3.2) and thus concludes the proof of Theorem 3.1.

Proof of Theorem 3.2. By integration by parts, using (1.1) yields
\[
\langle f, \theta \rangle_{L^{\infty}(0,t_{j};H_{0}^{1}(\Omega))} = \langle u', \theta \rangle_{\Omega_{j}} + \left\langle (\kappa_{1}D_{0+}^{\alpha} + \kappa_{2}D_{0+}^{\beta}) \partial_{x}u, \partial_{x}\theta \right\rangle_{\Omega_{j}}.
\]
Moreover, substituting $V = \theta\chi_{(0,t_{j})}$ into (3.1) yields
\[
\langle f, \theta \rangle_{L^{\infty}(0,t_{j};H_{0}^{1}(\Omega))} = \langle U', \theta \rangle_{\Omega_{j}} + \left\langle (\kappa_{1}D_{0+}^{\alpha} + \kappa_{2}D_{0+}^{\beta}) \partial_{x}U, \partial_{x}\theta \right\rangle_{\Omega_{j}} + s_{j}(U, \theta) - \langle R_{h}u_{0}, \theta_{0}^{+} \rangle_{\Omega}.
\]
Consequently, it follows
\[
0 = \langle (u - U)', \theta \rangle_{\Omega_{j}} + \left\langle (\kappa_{1}D_{0+}^{\alpha} + \kappa_{2}D_{0+}^{\beta}) \partial_{x}(u - U), \partial_{x}\theta \right\rangle_{\Omega_{j}},
\]
s_{j}(u - U, \theta) - \langle (I - R_{h})u_{0}, \theta_{0}^{+} \rangle_{\Omega},
\]
and then a simple calculation gives
\[
\left\langle \theta', \theta \right\rangle_{\Omega_{j}} + \left\langle (\kappa_{1}D_{0+}^{\alpha} + \kappa_{2}D_{0+}^{\beta}) \partial_{x}\theta, \partial_{x}\theta \right\rangle_{\Omega_{j}} + s_{j}(\theta, \theta) = \mathbb{I}_{1} + \mathbb{I}_{2} + \mathbb{I}_{3},
\]
where $\rho := (I - Q_{+}R_{h})u$ and
\[
\mathbb{I}_{1} = \langle \rho', \theta \rangle_{\Omega_{j}} + s_{j}(\rho, \theta) - \langle (I - R_{h})u_{0}, \theta_{0}^{+} \rangle_{\Omega},
\]
\[
\mathbb{I}_{2} = \kappa_{1} \left\langle D_{0+}^{\alpha} \partial_{x}\rho, \partial_{x}\theta \right\rangle_{\Omega_{j}},
\]
\[
\mathbb{I}_{3} = \kappa_{2} \left\langle D_{0+}^{\beta} \partial_{x}\rho, \partial_{x}\theta \right\rangle_{\Omega_{j}}.
\]
Therefore, Lemma 4.9 implies
\[ \|\theta_j\|^2_{L^2(\Omega)} + \kappa_1 |\theta|^2_{H^{\alpha/2}(0,t_j;H^\alpha(\Omega))} + \kappa_2 |\theta|^2_{H^{\beta/2}(0,t_j;H^\beta(\Omega))} \lesssim I_1 + I_2 + I_3. \]

Let us first estimate $I_1$. By the definition of $Q_r$, using integration by parts gives
\[
\langle ((I - Q_r)R_h u')', \theta \rangle_{\Omega_{t_j}} + s_j ((I - Q_r)R_h u, \theta) = \sum_{i=1}^{j} \langle ((I - Q_r)R_h u)_{i-1}^+, \theta_{i-1}^+ \rangle_{\Omega} + \sum_{i=1}^{j-1} \langle (I - Q_r)R_h u)_{i-1}^+, \theta_{i}^+ \rangle_{\Omega}
\]
which implies
\[ I_1 = \langle u - R_h u', \theta \rangle_{\Omega_{t_j}} + s_j (u - R_h u, \theta) - \langle (I - R_h)u_{0}, \theta_0^+ \rangle_{\Omega} = \langle (I - R_h)u', \theta \rangle_{\Omega_{t_j}}. \]

Therefore, using Lemmas 4.1 and 4.5 yields
\[ I_1 \lesssim \eta_{j,1} \sqrt{2} |\theta|_{H^{\alpha/2}(0,t_j;H^\alpha(\Omega))}. \]

Then let us estimate $I_2$ and $I_3$. A straightforward calculation gives
\[
I_2 = \kappa_1 \langle D_{0}^\alpha u, \partial \tau (u - Q_r R_h u), \partial_x \theta \rangle_{\Omega_{t_j}} = \kappa_1 \langle D_{0}^\alpha u, \partial \tau (u - Q_r u), \partial_x \theta \rangle_{\Omega_{t_j}} = \kappa_1 \langle D_{0}^\alpha (I - Q_r) \partial \tau u, \partial_x \theta \rangle_{\Omega_{t_j}}, \]
so that Lemmas 4.4 and 4.7 imply
\[ I_2 \lesssim \eta_{j,2} \sqrt{2} \kappa_1 |\theta|_{H^{\alpha/2}(0,t_j;H^\alpha(\Omega))}. \]

Analogously, we obtain
\[ I_3 \lesssim \eta_{j,3} \sqrt{2} \kappa_1 |\theta|_{H^{\alpha/2}(0,t_j;H^\alpha(\Omega))}. \]

Finally, by the Young’s inequality with $\epsilon$, combining the above estimates for $I_1$, $I_2$ and $I_3$ yields (3.3). This concludes the proof of Theorem 3.2.

5 Numerical Experiments

This section investigates numerically the temporal accuracy of $U$. We set $\alpha = 0.2$, $\beta = 0.8$, $\kappa_1 = \kappa_2 = 1$, $\Omega = (0,1)$ and $T = 1$, and let
\[ u(x,t) := t^r \sin(\pi x), \quad (x,t) \in \Omega_T \]
be the solution to problem (1.1), where $r > 0$ is a constant. To ensure that the spatial discretization error is negligible compared with the temporal discretization error, we set $n = 3$ and $h = 1/32$. Additionally, define
\[
E_1(U) := \max_{1 \leq j \leq J} \left\| (u - U)(t_j) \right\|_{L^2(\Omega)} , \\
E_2(U) := \left\| (u - U)(T) \right\|_{L^2(\Omega)}. 
\]
Experiment 1. This experiment investigates the temporal accuracy of $U$ under the condition that $u$ is sufficiently regular and the temporal grid is equidistant ($\sigma = 1$). We set $r = 4$ and present the corresponding numerical results in Table 1. These numerical results show that $E_1(U) = O(\tau^{m+1})$, which exceeds the theoretical temporal accuracy $O(\tau^{m+0.6})$ indicated by Corollary 3.1.

| $J$  | $m = 0$ | $m = 1$ |
|------|---------|---------|
|      | $E_1(U)$ | Order | $E_1(U)$ | Order |
| 64   | 1.43e-2 | –      | 3.02e-5 | –      |
| 128  | 7.56e-3 | 0.92   | 7.20e-6 | 2.07   |
| 256  | 3.95e-3 | 0.93   | 1.71e-6 | 2.07   |
| 512  | 2.05e-3 | 0.95   | 4.08e-7 | 2.07   |
| 1024 | 1.06e-3 | 0.96   | 9.69e-8 | 2.07   |

Table 1: $r = 4, \sigma = 1$.

Experiment 2. This experiment investigates the temporal accuracy of $U$ under the condition that $u$ has singularity near $t = 0^+$ and the temporal grid is also equidistant. The corresponding numerical results are displayed in Tables 2 and 3, and they illustrate that $E_1(U) = O(\tau^{r+0.1})$ which agrees with Corollary 3.2. The numerical results also show that the theoretical accuracy $E_1(U) = O(\tau^{m+1-\beta/2})$ indicated by Corollary 3.1 is optimal with respect to the regularity of $u$. Furthermore, Tables 2 and 3 illustrate the following interesting result:

$E_2(U) = O(\tau^{m+1})$.

Therefore, if only $u(T)$ is concerned, then equidistant temporal grids are sufficient.
Table 2: \( m = 0, \sigma = 1 \).

| \( r \) | \( J \) | \( E_1(U) \) | Order | \( E_2(U) \) | Order |
|---------|--------|-------------|-------|-------------|-------|
| 0.2     | 16     | 2.32e-2     | –     | 5.29e-3     | –     |
|         | 32     | 1.83e-2     | 0.34  | 2.64e-3     | 1.00  |
| 0.5     | 64     | 1.46e-2     | 0.32  | 1.32e-3     | 1.00  |
|         | 128    | 1.19e-2     | 0.30  | 6.55e-4     | 1.01  |
|         | 256    | 9.73e-3     | 0.29  | 3.26e-4     | 1.01  |
| 0.8     | 16     | 2.09e-2     | –     | 1.09e-2     | –     |
|         | 32     | 1.34e-2     | 0.64  | 5.51e-3     | 0.99  |
|         | 64     | 8.75e-3     | 0.62  | 2.76e-3     | 0.99  |
|         | 128    | 5.78e-3     | 0.60  | 1.38e-3     | 1.00  |
|         | 256    | 3.85e-3     | 0.59  | 6.93e-4     | 1.00  |
| 1.5     | 16     | 1.59e-2     | –     | 1.55e-2     | –     |
|         | 32     | 8.41e-3     | 0.92  | 7.89e-3     | 0.97  |
|         | 64     | 4.45e-3     | 0.92  | 4.01e-3     | 0.98  |
|         | 128    | 2.37e-3     | 0.91  | 2.03e-3     | 0.98  |
|         | 256    | 1.27e-3     | 0.90  | 1.02e-3     | 0.99  |

Table 3: \( m = 1, \sigma = 1 \).

| \( r \) | \( J \) | \( E_1(U) \) | Order | \( E_2(U) \) | Order |
|---------|--------|-------------|-------|-------------|-------|
| 0.2     | 16     | 1.89e-3     | –     | 1.42e-5     | –     |
|         | 32     | 1.23e-3     | 0.62  | 3.10e-6     | 2.20  |
| 0.5     | 64     | 8.10e-4     | 0.60  | 6.97e-7     | 2.15  |
|         | 128    | 5.38e-4     | 0.59  | 1.60e-7     | 2.12  |
|         | 256    | 3.59e-4     | 0.58  | 3.70e-8     | 2.11  |
| 0.8     | 16     | 4.08e-4     | –     | 8.36e-6     | –     |
|         | 32     | 2.16e-4     | 0.92  | 1.86e-6     | 2.17  |
|         | 64     | 1.15e-4     | 0.90  | 4.26e-7     | 2.13  |
|         | 128    | 6.23e-5     | 0.89  | 9.90e-8     | 2.11  |
|         | 256    | 3.38e-5     | 0.88  | 2.30e-8     | 2.10  |
| 1.5     | 16     | 1.71e-4     | –     | 3.65e-5     | –     |
|         | 32     | 5.57e-5     | 1.62  | 8.36e-6     | 2.12  |
|         | 64     | 1.84e-5     | 1.60  | 1.95e-6     | 2.10  |
|         | 128    | 6.13e-6     | 1.59  | 4.57e-7     | 2.09  |
|         | 256    | 2.05e-6     | 1.58  | 1.08e-7     | 2.08  |

**Experiment 3.** This experiment investigates the temporal accuracy of \( U \) under the condition that \( u \) has singularity near \( t = 0^+ \) and the temporal grid is graded with different parameter \( \sigma > 1 \). We consider \( r = 0.2 \) and \( r = 0.4 \), and list the corresponding numerical results in Tables 4, 5, 6 and 7. For \( 1 < \sigma \leq \sigma^* \), the numerical results show that \( E_1(U) = O(\tau^{(r^0(r+0.1))}) \), which agrees with Corollary 3.2. Moreover, in the case of \( \sigma = \sigma^{**} \), the temporal accuracy \( E_1(U) = O(\tau^{m+1}) \) is observed. Here, we recall that \( \sigma^* \) and \( \sigma^{**} \) are defined by (3.4) and (3.5), respectively.
### Table 4: \( r = 0.2, \ m = 0 \).

| \( \sigma \backslash J \) | 16     | 32     | 64     | 128    | 256    |
|---------------------------|--------|--------|--------|--------|--------|
| 1.5                       | \( E_1(U) \) | 1.46e-02 | 1.07e-02 | 8.02e-03 | 6.04e-03 | 4.55e-03 |
| Order                     | –      | 0.45   | 0.42   | 0.41   | 0.41   |
| 2(\( \sigma^* \)) \n| \( E_1(U) \) | 1.05e-02 | 7.04e-03 | 4.82e-03 | 3.31e-03 | 2.26e-03 |
| Order                     | –      | 0.58   | 0.55   | 0.54   | 0.55   |
| 10(\( \sigma^{**} \)) \n| \( E_1(U) \) | 1.11e-02 | 5.87e-03 | 3.05e-03 | 1.57e-03 | 8.05e-04 |
| Order                     | –      | 0.92   | 0.95   | 0.96   | 0.96   |

### Table 5: \( r = 0.2, \ m = 1 \).

| \( \sigma \backslash J \) | 4      | 8      | 16     | 32     | 64     |
|---------------------------|--------|--------|--------|--------|--------|
| 16                        | \( E_1(U) \) | 2.44e-03 | 1.35e-03 | 7.40e-04 | 3.94e-04 | 2.01e-04 |
| Order                     | –      | 0.86   | 0.86   | 0.91   | 0.97   |
| 3(\( \sigma^* \)) \n| \( E_1(U) \) | 3.73e-03 | 1.01e-03 | 3.35e-04 | 1.10e-04 | 3.30e-05 |
| Order                     | –      | 1.89   | 1.59   | 1.61   | 1.73   |
| 20(\( \sigma^{**} \)) \n| \( E_1(U) \) | 5.13e-03 | 1.51e-03 | 3.81e-04 | 9.11e-05 | 2.33e-05 |
| Order                     | –      | 1.76   | 1.99   | 2.06   | 1.97   |

### Table 6: \( r = 0.4, \ m = 0 \).

| \( \sigma \backslash J \) | 16     | 32     | 64     | 128    | 256    |
|---------------------------|--------|--------|--------|--------|--------|
| 1.1                       | \( E_1(U) \) | 2.01e-02 | 1.34e-02 | 9.11e-03 | 6.27e-03 | 4.35e-03 |
| Order                     | –      | 0.58   | 0.56   | 0.54   | 0.53   |
| 1.2(\( \sigma^* \)) \n| \( E_1(U) \) | 1.73e-02 | 1.12e-02 | 7.42e-03 | 4.97e-03 | 3.34e-03 |
| Order                     | –      | 0.62   | 0.60   | 0.58   | 0.57   |
| 2(\( \sigma^{**} \)) \n| \( E_1(U) \) | 1.41e-02 | 7.31e-03 | 3.76e-03 | 1.92e-03 | 9.85e-04 |
| Order                     | –      | 0.94   | 0.96   | 0.97   | 0.97   |

### Table 7: \( r = 0.4, \ m = 1 \).

| \( \sigma \backslash J \) | 4      | 8      | 16     | 32     | 64     |
|---------------------------|--------|--------|--------|--------|--------|
| 2                         | \( E_1(U) \) | 2.64e-03 | 1.29e-03 | 6.59e-04 | 3.37e-04 | 1.71e-04 |
| Order                     | –      | 1.03   | 0.98   | 0.97   | 0.98   |
| 3(\( \sigma^* \)) \n| \( E_1(U) \) | 2.27e-03 | 6.69e-04 | 2.29e-04 | 7.73e-05 | 2.53e-05 |
| Order                     | –      | 1.76   | 1.55   | 1.57   | 1.61   |
| 4(\( \sigma^{**} \)) \n| \( E_1(U) \) | 3.28e-03 | 8.43e-04 | 2.02e-04 | 4.81e-05 | 1.26e-05 |
| Order                     | –      | 1.96   | 2.06   | 2.07   | 1.93   |
6 Conclusions

This paper analyzes a time-stepping discontinuous Galerkin method for the modified anomalous subdiffusion problem. We establish the stability of this method and prove that the temporal accuracy is $O(\tau^{m+1-\beta/2})$, and the numerical results confirm that this accuracy is optimal with respect to the regularity of $u$. Furthermore, if $u$ has singularity near $t = 0^+$, we prove that employing graded grids in the temporal discretization can improve the temporal accuracy to $O(\tau^{m+1-\beta/2})$, which is also verified by the numerical results.

However, further investigations are still needed.

- The numerical results illustrate that if $u$ is sufficiently regular, then
  \[
  \max_{1 \leq j \leq J} \|(u - U)(t_j)\|_{L^2(\Omega)} = O(\tau^{m+1}).
  \]

- Although $u$ has singularity near $t = 0^+$, the numerical results show that
  \[
  \|(u - U)(T)\|_{L^2(\Omega)} = O(\tau^{m+1}).
  \]

- The numerical results also illustrate that if $u$ has singularity near $t = 0^+$, then adopting graded grids in the temporal discretization can improve the temporal accuracy to $O(\tau^{m+1})$.

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