Upper critical dimension in the scaling theory of localization

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Abstract

It is argued that the Thouless number \( g(L) \) is not the only parameter relevant in scale transformations, and that the second parameter connected with off-diagonal disorder should be introduced. A two-parameter scaling theory is suggested that explains a phenomenon of the upper critical dimension from the viewpoint of scaling ideas.

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The one-parameter scaling hypothesis [1] provides the basis for the contemporary theory of localization. Its justification is still actual [2] and may require more accurate definitions of the scaling variables as well as lead to a restriction of the range of applicability. Here we discuss modifications of the scaling hypothesis which are inevitable in high dimensions.

The scaling consideration is usually applied only for space dimensions \( d \) being in the interval between \( d_{c1} \) and \( d_{c2} \), the lower and the upper critical dimensions [3]. There are no doubts that \( d_{c1} = 2 \) [4], whereas the value of \( d_{c2} \) was disputable for many years [4, 5, 6]. Recently it was established in the author’s series of papers [7] that \( d_{c2} = 4 \) for the problem of the density of states (defined by the average Green function \( \langle G \rangle \)). The value \( d_{c2} = 4 \) is distinguished from the viewpoint of renormalizability: the theory is nonrenormalizable for dimensions \( d > 4 \) and the cutoff is necessary on the atomic scale for high momenta. Since the atomic scale cannot be left out of consideration, no scale-invariance is possible. The same argument can also be applied to the problem of conductivity defined by the correlator \( \langle G^R G^A \rangle \). This is confirmed by author’s ”symmetry theory” [8], that reproduces on a rigorous level the results of Vollhardt and Wölfle [9] and gives the values of
critical exponents (claimed to be exact in Ref. 5), which are consistent with
the one-parameter scaling theory only for \( d < 4 \).

On the other hand, the recent assertions that \( d_c^2 = \infty \) have some
grounds: \textit{there is no place for the upper critical dimension in the one-parameter
scaling theory}. Overall, there are certain drawbacks in the existing physical
picture of localization. Another question to be cleared out is related to
the mechanism responsible for violation of the Wegner relation \( s = (d - 2)\nu \)
for \( d > 4 \) \([3, 4, 5]\). The present paper is aimed to fill these gaps in our
knowledge.

The scaling theory \([1]\) is based on Thouless’s scaling considerations \([10]\).
A disordered system, that is described by the Anderson model with the over-
lap integrals \( J \) between the nearest neighbours and with the spread \( W \) of cite
levels, is divided into blocks of size \( L \). In the absence of interaction between
the blocks, each of them has a random system of energy levels with a char-
acteristic spacing \( \Delta(L) \sim J(a_0/L)^d \), where \( a_0 \) is the lattice constant. When
the interaction between the blocks is "switched on", the states of neighbour-
ning blocks become coupled and the corresponding matrix elements become
nonzero. This hybridization is the most essential for the states nearest in
energy, and should be taken into account first of all. If the level nearest to
the considered energy \( E \) is selected in each block, we obtain an effective An-
derson model with the spread of levels \( W(L) \sim \Delta(L) \) and overlap integrals
\( J(L) \) determined by corresponding matrix elements. This model describes
the system on the scale larger than \( L \), and is characterized by the Thouless
parameter

\[
g(L) = \frac{J(L)}{W(L)} ,
\]

which is also equal to the dimensionless conductance of the block \([1]\). Repe-
tition of the same consideration for the effective Anderson model constitutes
the principal algorithm for evaluation of \( g(bL) \) with integer \( b \) when value
\( g(L) \) is known: \( g(bL) = F(b, g(L)) \). Taking the limit \( b \to 1 \) in this relation,
results in the Gell-Mann and Low equation \([1]\)

\[
\frac{d \ln g}{d \ln L} = \beta(g) .
\]

For \( d > 2 \), there exists a phase transition point \( g_c \), defined by the condi-
tion \( \beta(g_c) = 0 \), and, in a vicinity of the transition, the conductivity \( \sigma \) and
localization length $\xi$ have the following behavior

$$\sigma \sim (g_0 - g_c)^s, \quad \xi \sim (g_c - g_0)^{-\nu}.$$  \hspace{1cm} (3)

Here $g_0$ is the value of $g(L)$ on the scale $L \sim a_0$, and the critical exponents are given by $1/\nu = g_c\beta'(g_c)$ and $s = (d-2)\nu$. \textsuperscript{[1]}

The above consideration relies heavily on the assumption that $g(L)$ is the only relevant parameter in the scale transformations. We shall show that, in general, it is not the case. In order to see it, let us assume that the typical wave function of localized states has the form

$$|\Psi(r)| \sim \begin{cases} r^{-\zeta}, & r \ll \xi \\ \exp(-r/\xi), & r \gg \xi \end{cases},$$  \hspace{1cm} (4)

where the exponent $\zeta$ goes to infinity for large $d$. There are some reasons for such an assumption: (i) the optimal fluctuation method \textsuperscript{[11]} results in Eq. (4) with $\zeta = d-2$ in the range of deep localization, and (ii) an analogous behavior for the critical region can be guessed from the $d$-dependence of the exponent $\eta$ of density correlator \textsuperscript{[12]}. A large value of $\zeta$ means that the eigenfunctions of separate blocks in Thouless’s construction are well localized on the scale $L \ll \xi$ (Fig. 1). Consequently, strong off-diagonal disorder appears: f.e. the overlap integral between the states 1 and 2 is much smaller than one between states 3 and 4. With the increase of $\zeta$ we approach the well-known situation of topological disorder in the system of impurities with exponential overlap (Ref. 13). So a catastrophe, viz. localization due to the pure off-diagonal disorder, becomes possible. It can even occur for $W(L) = 0$, when the Thouless parameter is infinite and cannot play any role. Therefore, it is reasonable to suggest that hybridization of the block states is determined by some other parameter connected with the off-diagonal disorder.

Let us suppose that a disordered system is characterized by two parameters

$$g(L) = \frac{J(L)}{W(L)}, \quad \varphi(L) = \frac{\delta J(L)}{J(L)},$$ \hspace{1cm} (5)

where $\delta J(L)$ is the fluctuation of overlap integrals. The boundary $AB$ (Fig. 2) between localized and extended states should then be situated at $g \sim 1$ for $\varphi = 0$ and go to infinity at some critical point $\varphi_c$, in accordance with the possibility of localization due to the pure off-diagonal disorder. In the course of scale transformations, one point of the $(g, \varphi)$-plane turns into another
point of this plane, and one point of line $AB$ turns into another point of this line.

To return to the one-parameter scaling, it is sufficient to assume the existence of a fixed point $F$, that is stable on the critical surface $(AB)$ and unstable beyond it (Ref. 3, Ch. 6). The point $F$ is of saddle-type and characterized by two asymptotes, $AB$ and $CD$ (Fig. 2,a). The movement in the $(g, \varphi)$-plane can be roughly divided into two stages: relaxation to line $CD$ on some scale $L_0$ and evolution along $CD$ on the scale $\xi$, which is arbitrarily large near the phase transition. For $L \gg L_0$, the whole $(g, \varphi)$-plane reduces to line $CD$, and the position on the latter is uniquely determined by the Thouless parameter $g(L)$. Thereby we return to the usual picture of localization, and we assume it to be valid for low dimensions.

Let us now suppose that for high dimensions there is no fixed point on the critical line $AB$ (Fig. 2,b). If a system is in a critical point, then it moves upwards along this line (the movement downwards means that the off-diagonal disorder dissappears asymptotically, and contradicts to the previous arguments). Consequently, this implies that in the critical point parameter $g(L)$ increases (in contrast to $g(L) = \text{const}$ in the previous scenario): it does not mean that degree of gybridization grows but indicates that the diagonal disorder transforms to off-diagonal one. In the metallic phase, $g(L)$ is represented by a more rapid dependence $\sim \sigma L^{d-2}$ (Ref. 1), and, in the localized phase, it exhibits non-monotonic behavior, i.e. increasing for $L \lesssim \xi$ and decreasing for $L \gtrsim \xi$.

The first scenario is changed by the second one at some critical value of $d$, which we identify with $d_{c2}$. To obtain the phenomenological description of such a bifurcation we introduce a new variable $h = F(g, \varphi)$, so that in the $(g, h)$-plane the line $AB$ has behavior $g \sim h$ for large $g, h$ (the critical line will then have regular projections on both axes), and the other asymptote $CD$ becomes vertical (this would simplify equations). In the case of the two relevant parameters, $g(L)$ and $h(L)$, the following relations can be written down by following the usual line of reasoning (cf. Eq. (2)):

$$
\frac{d \ln g}{d \ln L} = \beta(g, h) \quad , \quad \frac{d \ln h}{d \ln L} = \gamma(g, h) \quad .
$$

In the region of large $g$ and $h$, where the fixed point $F$ is situated for $d$ close
to $d_{c2}$, Eqs. (6) take the form

$$
\frac{d \ln g}{d \ln L} = (d - 2) + \frac{Ah}{g} + \frac{Bh^2}{g^2} + \frac{Ch^3}{g^3} + \ldots \equiv (d - 2) + \tilde{\beta} \left( \frac{g}{h} \right) , \quad (7)
$$

$$
\frac{d \ln h}{d \ln L} = \mu + \frac{b}{h} , \quad (7b)
$$

where $\mu$ changes the sign at the point $d = d_{c2}$,

$$
\mu = \alpha (d - d_{c2}) , \quad d \to d_{c2} \quad (8)
$$

and the following inequalities are satisfied: $\alpha > 0, b > 0$, and $A < 0$. Indeed, at constant $h$ the function $\beta(g, h)$ should have all the properties discussed in Ref. 1, and it should then be expanded accordingly: $\beta(g, h) = (d - 2) + A_1(h)/g + A_2(h)/g^2 + \ldots$. In addition, the coefficients $A_n(h)$ should have expansion in $1/h$ beginning with $h^n$ in order to yield a root $g_c \sim h$.

Keeping the leading terms with respect to $h$ results in Eq. (7a). For $d > d_{c2}$, function $\gamma(g, h)$ should provide the indefinite growth of $h$, which, however, should not be faster than that of $g$. This gives inequality $0 < \gamma(g, h) < d - 2$, suggesting that the expansion of $\gamma(g, h)$ in $1/g$, $1/h$ begins with zero-order: $\gamma(g, h) = \mu + a/g + b/h + \ldots$. In the case of the vertical asymptote $CD$, the fixed point $h_c$ is independent of $g$, and thereby $a = 0$. The fixed point should be stable for $d < d_{c2}$ and absent for $d > d_{c2}$. This requires that $b > 0$ and $\mu$ to change the sign in the point $d_{c2}$.

The system of Eqs. (7) can be easily investigated. For $d < d_{c2}$, Eq. 7b has a fixed point $h_c = b/|\mu|$ and replacement $g \to gh_c$ in Eq. (7) results in the one-parameter scaling description with critical exponents given by

$$
1/\nu = g_c \tilde{\beta}'(g_c) , \quad s = \nu(d - 2) , \quad (d - 2) + \tilde{\beta}(g_c) = 0 . \quad (9)
$$

For $d > d_{c2}$, we have $h(L) \sim L^\mu$ at large $h$, and the replacement $g \to gL^\mu$ in Eq. 7b gives

$$
1/\nu = g_c \tilde{\beta}'(g_c) , \quad (10a)
$$

$$
s = \nu(d - 2 - \mu) , \quad (10b)
$$

$$
(d - 2 - \mu) + \tilde{\beta}(g_c) = 0 . \quad (10c)
$$

For $L \lesssim \xi$, the Thouless parameter can be written down as follows:

$$
g(L) = g_c + (g_0 - g_c) (L/a_0)^{1/\nu} , \quad d < d_{c2} \quad (11a)
$$
\[ g(L) = g_c \left( \frac{L}{a_0} \right)^\mu + (g_0 - g_c) \left( \frac{L}{a_0} \right)^{\mu + 1/\nu} , \quad d > d_{c2} . \quad (11b) \]

In the critical point \( g(L) \) grows as \( L^\mu \) for \( d > d_{c2} \), thereby leading to the violation of the Wegner relation (see Eq. (10b)). In general, critical exponents as functions of \( d \) have cusps at \( d = d_{c2} \). Usually the critical exponents are independent of \( d \) for \( d > d_{c2} \). According to Eq. (10b) this would become possible for \( \mu = d + \text{const} \), which results together with Eq. (8) in

\[ \mu = d - d_{c2} . \quad (12) \]

The results obtained can be compared with the symmetry theory \cite{8} that gives the same critical exponents as in Ref. 9:

\[
\nu = \frac{1}{(d - 2)} , \quad s = 1 \quad \text{for} \quad 2 < d < 4 \quad , \]

\[
\nu = 1/2 \quad , \quad s = 1 \quad \text{for} \quad d > 4 . \quad (13) \]

The Wegner relation \( s = \nu(d - 2) \) is valid only for \( d < 4 \) implying that \( d_{c2} = 4 \). To obtain the result analogous to Eqs. (11), we find from Ref. 8 the diffusion constant \( D_L \) of a finite block of size \( L \). It is determined by the diffusion coefficient \( D(\omega, q) \) of the infinite system: \( D_L \sim D(\frac{i D_L}{L^2}, \frac{1}{L}) \).

Using Eqs. (112) and (116a) of Ref. 8 and \( g(L) \propto D_L L^{d-2} \) one can finally obtain the following relation:

\[ g(L) = g_c \left( \frac{L}{a_0} \right)^{d-2-1/\nu} + (g_0 - g_c) \left( \frac{L}{a_0} \right)^{d-2} . \quad (15) \]

This result coincides with Eq. (11) if Eqs. (12) and (13) are taken into account. Such an agreement is nontrivial because the symmetry theory \cite{8} is based on completely different principles without any reference to the scaling ideas.

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\footnote{This relation is not valid in the localized phase for \( L > \xi \) due to the nonlocal response}
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Figure 1:

Figure 2: Flow diagram in the \((g, \varphi)\) plane: (a) in the case of existence of the fixed point \(F\) on the critical surface \(AB\), (b) in the case of its absence.