Research Article

Existence and Uniqueness for a System of Caputo-Hadamard Fractional Differential Equations with Multipoint Boundary Conditions

S. Nageswara Rao, Ahmed Hussein Msral, Manoj Singh, and Abdullah Ali H. Ahmadini

Department of Mathematics, Faculty of Science, Jazan University, Jazan, Saudi Arabia

Correspondence should be addressed to S. Nageswara Rao; snrao@jazanu.edu.sa

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In this paper, we study existence and uniqueness of solutions for a system of Caputo-Hadamard fractional differential equations supplemented with multi-point boundary conditions. Our results are based on some classical fixed point theorems such as Banach contraction mapping principle, Leray-Schauder fixed point theorems. At last, we have presented two examples for the illustration of main results.

1. Introduction

In recent years, fractional differential equations (FDE) gain enormous attention among scientists due to the applications which were not possible with ordinary or partial differential equations of integer order. FDEs becomes a very successful tool in modeling anomalous diffusion and fractal-like nature. Agrawal discusses diffusion and heat equations of fractional order in [1–3]. Agrawal et al., Baleanu, and others investigated the boundary value problems for fractional differential equations [4]. Fractional dynamic models, fractional control systems, fractional population dynamics models, and fractional fluid dynamics all involve at least one ordinary or partial fractional derivative.

Fractional differential equations have several kinds of fractional derivatives, such as Riemann-Liouville fractional derivative, Caputo fractional derivative, and Grunwald-Letnikov fractional derivative. Another kind of fractional derivative is Hadamard type which was introduced in 1892 [5]. This derivative differs from various derivatives in the sense that the kernel of the integral in the definition of Hadamard derivative contains logarithmic function of arbitrary exponent. A detailed description of Hadamard fractional derivative and integral can be found in [6]. The readers who are interested in the subject of fractional calculus is referred to the books by Kilbas et al. [7], Podlubny [8], Miller and Ross [9], Samko et al. [10], Diethelm [11], and Zhou [12] and the references therein.

Coupled systems of fractional differential equations play a key role in developing differential models such as the synchronization of chaotic systems [13–15], anomalous diffusion [16, 17], disease models [18, 19], ecological models [20], Lorenz system [21], and nonlocal thermoelectricity systems [22, 23]. For recent theoretical results on the topic, we refer the reader to a series of papers [24–37] and the references cited therein. Ahmad and Ntouyas [32, 33] discussed some fractional integral boundary value problems involving Hadamard fractional differential equations/systems and obtained the existence and uniqueness of solutions by applying the Banach fixed point theorem and Leray–Schauder alternative, respectively.

In [35], the authors investigated the existence and uniqueness of solutions for the coupled system of nonlinear fractional differential equations with three-point boundary conditions.

where $1 < \alpha, \beta < 2, p, q, \gamma > 0, 0 < \eta < 1, \alpha - q \geq 1, \beta - p \geq 1, \gamma n^{p-1} < 1, y n^{q-1} < 1$ and $D^\alpha, D^\beta$ are the standard Riemann-Liouville fractional derivatives and $f, g : [0, 1] \times R \times R \rightarrow R$ are given continuous functions.

Recently, Alsulami et al. [36] established the existence and uniqueness results for a nonlinear coupled system of Caputo type fractional differential equations supplemented with nonseparated coupled boundary conditions.

$$
\begin{align*}
\mathcal{D}^\alpha u(t) &= f(t, u(t), v(t)), t \in [a, b], \\
\mathcal{D}^\beta v(t) &= g(t, u(t), v(t)), t \in [a, b],
\end{align*}
$$

with multipoint boundary conditions

$$
\begin{align*}
u(a) &= \lambda_1 v(b), \lambda_2 \mathcal{D}^\alpha u(b) = \mu_2 \sum_{k=0}^n \mathcal{D}^\delta u(\eta_k), \\
\nu(a) &= \mu_1 u(b), \lambda_3 \mathcal{D}^\beta v(b) = \mu_3 \sum_{k=0}^n \mathcal{D}^\delta v(\xi_k),
\end{align*}
$$

where $\alpha, \beta \in (1, 2), \gamma_i, \delta_i, \theta_i \in (0, 1), i = 1, 2, \eta_i, \xi_i, \xi_i \in R, f, g : [a, b] \times R \times R \rightarrow R$ are appropriately chosen functions.

The paper is organized as follows. In Sect. 2, we present some preliminary concepts of fractional calculus. Sect. 3 contains main results concerning the existence and uniqueness of solutions for the given problem (3), (4). The Leray-Schauder alternative theorem is applied to prove existence, while the uniqueness result was obtained via the Banach contraction mapping principle. Finally, we also discuss some examples for illustration of the existence-uniqueness results.
Lemma 6 [38]. Let \( q \geq 0 \) and \( n = [q] + 1 \). If \( x(t) \in AC^n_{\alpha}[a, b] \), then the Caputo-Hadamard fractional differential equation \( ^cD^\alpha_x x(t) = 0 \) has a solution:

\[
x(t) = \sum_{k=0}^{n-1} c_k \left( \log \frac{t}{\alpha} \right)^k,
\]

(11)

and the following formula holds:

\[
^cD^\alpha_x x(t) = x(t) + \sum_{k=0}^{n-1} c_k \left( \log \frac{t}{\alpha} \right)^k,
\]

(12)

where \( c_k \in \mathbb{R}, k = 1, 2, \ldots, n - 1 \).

Now, we present an auxiliary lemma for boundary value problem of linear fractional differential equation with Caputo-Hadamard derivative.

Lemma 7. Let \( \Delta = (\lambda_1, \lambda_2, \gamma, \beta, \alpha, \delta, \Gamma) (2 - \gamma_1, 2 - \gamma_2) (2 - \beta) \), \( \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6 \). Let \( x, y \in AC^n_\alpha[a, b] \). Then, the solution of the linear Caputo-Hadamard fractional differential system

\[
\begin{cases}
^cD^\alpha_x u(t) = x(t), t \in [a, b], I < \alpha \leq 2, \\
^cD^\beta_x v(t) = y(t), t \in [a, b], I < \beta \leq 2, \\
u(a) = \lambda_1 v(b), \lambda_2^c D^\alpha_x u(b) = \mu_2 \sum_{\eta=1}^{N} D^\beta_x v(\eta), \\
v(a) = \mu_1 u(b), \lambda_3^c D^\alpha_x v(b) = \mu_3 \sum_{\xi=1}^{M} D^\beta_x v(\xi),
\end{cases}
\]

(13)
is equivalent to the system of integral equations

\[
u(t) = \frac{\mu_1}{\Delta} \left[ \sum_{j=1}^{N} \mu_2 D^\beta_x v(\eta_j) \right] \frac{\lambda_1}{\Gamma(2 - \gamma_1)} + \frac{\mu_2}{\Gamma(2 - \beta)} \left[ \sum_{j=1}^{N} \lambda_2 D^\alpha_x u(\eta_j) \right] \frac{\Gamma(2 - \gamma_1)}{\Gamma(2 - \beta)} \]

\[
+ \frac{\lambda_1}{\Gamma(2 - \gamma_1)} \left[ \mu_2 D^\beta_x v(\eta_1) \right] \frac{\lambda_2}{\Gamma(2 - \gamma_1)} + \frac{\mu_2}{\Gamma(2 - \beta)} \left[ \sum_{j=1}^{N} \lambda_2 D^\alpha_x u(\eta_j) \right] \frac{\Gamma(2 - \gamma_1)}{\Gamma(2 - \beta)} \]

\[
- \frac{\lambda_2}{\Gamma(2 - \gamma_1)} \left[ \mu_2 D^\beta_x v(\eta_1) \right] \frac{\lambda_1}{\Gamma(2 - \gamma_1)} + \frac{\mu_2}{\Gamma(2 - \beta)} \left[ \sum_{j=1}^{N} \lambda_2 D^\alpha_x u(\eta_j) \right] \frac{\Gamma(2 - \gamma_1)}{\Gamma(2 - \beta)} \]

\[
+ \frac{\lambda_1}{\Gamma(2 - \gamma_1)} \left[ \mu_2 D^\beta_x v(\eta_1) \right] \frac{\lambda_2}{\Gamma(2 - \gamma_1)} + \frac{\mu_2}{\Gamma(2 - \beta)} \left[ \sum_{j=1}^{N} \lambda_2 D^\alpha_x u(\eta_j) \right] \frac{\Gamma(2 - \gamma_1)}{\Gamma(2 - \beta)} \]

\[
+ \int_{a}^{t} \frac{\lambda_1}{\Gamma(2 - \gamma_1)} \left[ \mu_2 D^\beta_x v(\eta_1) \right] \frac{\Gamma(2 - \gamma_1)}{\Gamma(2 - \beta)} x(s) \frac{ds}{\gamma_1}
\]

(14)

where

\[
B_1 = \int_{a}^{b} \frac{\left( \ln(\text{bs}) \right)^{\alpha-1}}{\Gamma(\alpha)} x(s) \frac{ds}{s}
\]

\[
B_2 = \sum_{i=0}^{N} \int_{a}^{b} \frac{\left( \ln(\text{bs}) \right)^{\beta-1}}{\Gamma(\beta)} y(s) \frac{ds}{s}
\]

\[
B_3 = \sum_{i=0}^{N} \int_{a}^{b} \frac{\left( \ln(\text{bs}) \right)^{\gamma-1}}{\Gamma(\gamma)} x(s) \frac{ds}{s}
\]

(16)

Proof. We apply Lemma 6 that the general solution of the Caputo-Hadamard fractional differential equation in (13) can be written as:

\[
u(t) = c_0 + c_1 \left( \ln \frac{t}{\alpha} \right) + \int_{a}^{t} \frac{\left( \ln \left( \text{ts} \right) \right)^{\alpha-1}}{\Gamma(\alpha)} x(s) \frac{ds}{s}
\]

(17)

\[
v(t) = d_0 + d_1 \left( \ln \frac{t}{\alpha} \right) + \int_{a}^{t} \frac{\left( \ln \left( \text{ts} \right) \right)^{\beta-1}}{\Gamma(\beta)} y(s) \frac{ds}{s}
\]

(18)

where \( c_i, d_i, i = 0, 1, \) are arbitrary real constants. From (17) and (18) we have

\[
^cD^\alpha_x u(t) = \left( \ln \frac{t}{\alpha} \right) \left( \ln \frac{t}{\alpha} \right) + \int_{a}^{t} \frac{\left( \ln \left( \text{ts} \right) \right)^{\alpha-1}}{\Gamma(\alpha)} x(s) \frac{ds}{s}
\]

(19)

\[
^cD^\beta_x v(t) = \left( \ln \frac{t}{\alpha} \right) \left( \ln \frac{t}{\alpha} \right) + \int_{a}^{t} \frac{\left( \ln \left( \text{ts} \right) \right)^{\beta-1}}{\Gamma(\beta)} y(s) \frac{ds}{s}
\]

(20)

\[
^cD^\alpha_x v(t) = \left( \ln \frac{t}{\alpha} \right) \left( \ln \frac{t}{\alpha} \right) + \int_{a}^{t} \frac{\left( \ln \left( \text{ts} \right) \right)^{\beta-1}}{\Gamma(\beta)} y(s) \frac{ds}{s}
\]

(21)
\[ d_0 = \frac{1}{1 - \mu_1 \lambda_1} \left[ \frac{\lambda_1 \mu_2 \mu_3}{\Delta T(2 - \delta_1)} \sum_{i=1}^{N} \left( \ln \frac{b_i}{a} \right) \right] + \mu_2 \mu_3 \sum_{i=1}^{M} \left( \ln \frac{\xi_i}{a} \right) \]

and

\[ d_i = \frac{1}{1 - \mu_1 \lambda_1} \left[ \frac{\lambda_1 \mu_2 \mu_3}{\Delta T(2 - \delta_1)} \sum_{i=1}^{N} \left( \ln \frac{b_i}{a} \right) \right] + \mu_2 \mu_3 \sum_{i=1}^{M} \left( \ln \frac{\xi_i}{a} \right) \]

Inserting the values of \( c_i, d_i, i = 0, 1 \) in (17) and (18), which leads to the solution system (14), (15). The converse follows by direct computation. The proof is completed.

### 3. Existence and Uniqueness Results

This section is concerned with the main results of the paper. First of all, we fix our terminology. Let \( \mathbb{E} = C([a, b], R), a > 0 \) be the Banach space of all continuous functions from \([a, b]\) to \(R\). Space \( X = \{ u(t): u(t) \in C^1([a, b], R) \} \) endowed with the norm \( \|u\| = \sup \{ |u(t)|, t \in [a, b] \} \) is a Banach space. In addition, let \( Y = \{ v(t): v(t) \in C([a, b], R) \} \) with the norm \( \|v\| = \sup \{ |v(t)|, t \in [a, b] \} \). It is obvious that product space \( (X \times Y, \| (u, v) \|) \) is a Banach space with the norm \( \| (u, v) \| = \|u\| + \|v\| \).

In view of Lemma 7, we introduce an operator \( \mathcal{F} : X \times Y \to X \times Y \) as follows:

\[ \mathcal{F}(u, v)(t) = (\mathcal{F}_1(u, v)(t), \mathcal{F}_2(u, v)(t)), \]

where

\[ \mathcal{F}_1(u, v)(t) = \frac{b_v}{a} \left[ \frac{\mu_1 \mu_2 \mu_3}{\Delta T^2(2 - \delta_1)} \sum_{i=1}^{N} \left( \ln \frac{b_i}{a} \right) \right] + \mu_2 \mu_3 \sum_{i=1}^{M} \left( \ln \frac{\xi_i}{a} \right) \]

and

\[ \mathcal{F}_2(u, v)(t) = \frac{b_v}{a} \left[ \frac{\mu_1 \mu_2 \mu_3}{\Delta T^2(2 - \delta_1)} \sum_{i=1}^{N} \left( \ln \frac{b_i}{a} \right) \right] + \mu_2 \mu_3 \sum_{i=1}^{M} \left( \ln \frac{\xi_i}{a} \right) \]
and

\[ F_{j}(x, u(t)) = \frac{\mu_{j} \lambda_{j} \ln(b/a)^{\gamma_{j}}}{(2 - \delta_{i}) \Gamma(1 - \mu_{j} \lambda_{j})} + \frac{\mu_{j} \lambda_{j} \ln(b/a)^{\gamma_{j}}}{(2 - \delta_{i}) \Gamma(1 - \mu_{j} \lambda_{j})} \]

Here,

\[ B_{1j} = \int_{a}^{b} \frac{\ln(b/a)^{\gamma_{j}}}{T(\alpha)} f(s, u(s), v(s)) \frac{ds}{s}, \quad A_{1j} \]

\[ B_{2j} = \int_{a}^{b} \frac{\ln(b/a)^{\gamma_{j}}}{T(\beta)} g(s, u(s), v(s)) \frac{ds}{s}, \quad A_{2j} \]

\[ B_{3j} = \int_{a}^{b} \frac{\ln(b/a)^{\gamma_{j}}}{T(\alpha - \delta_{j})} f(s, u(s), v(s)) \frac{ds}{s}, \quad A_{3j} \]

For computational convenience, we set

\[ K_{1} = \left| \frac{\mu_{j} \lambda_{j}}{\alpha} \right| \left[ \frac{\mu_{j} \lambda_{j} (\ln(b/a)^{\gamma_{j}})}{(2 - \delta_{i}) \Gamma(1 - \mu_{j} \lambda_{j})} \right] \]

\[ K_{2} = \left| \frac{\mu_{j} \lambda_{j} (\ln(b/a)^{\gamma_{j}})}{\alpha} \right| \left[ \frac{\mu_{j} \lambda_{j} (\ln(b/a)^{\gamma_{j}})}{(2 - \delta_{i}) \Gamma(1 - \mu_{j} \lambda_{j})} \right] \]

\[ K_{3} = \left| \frac{\mu_{j} \lambda_{j} (\ln(b/a)^{\gamma_{j}})}{\alpha} \right| \left[ \frac{\mu_{j} \lambda_{j} (\ln(b/a)^{\gamma_{j}})}{(2 - \delta_{i}) \Gamma(1 - \mu_{j} \lambda_{j})} \right] \]

\[ K_{4} = \left| \frac{\mu_{j} \lambda_{j} (\ln(b/a)^{\gamma_{j}})}{\alpha} \right| \left[ \frac{\mu_{j} \lambda_{j} (\ln(b/a)^{\gamma_{j}})}{(2 - \delta_{i}) \Gamma(1 - \mu_{j} \lambda_{j})} \right] \]

Now, we are in a position to present our main results. The methods used to prove the existence and uniqueness solutions of boundary value problem (3), (4) via Banach's contraction principle.

**Theorem 8.** Suppose that \( f, g : [a, b] \times R \times R \rightarrow R \) are continuous functions. In addition, we assume that:

(H1) there exist constants \( m_{1}, n_{i}, i = 1, 2 \), such that for all \( t \in [a, b] \) and \( u_{i}, v_{i} \in R, i = 1, 2 \), we have

\[ |f(t, u_{1}, v_{1}) - f(t, u_{2}, v_{2})| \leq m_{1}|u_{1} - u_{2}| + m_{2}|v_{1} - v_{2}|, \]

\[ |g(t, u_{1}, v_{1}) - g(t, u_{2}, v_{2})| \leq n_{1}|u_{1} - u_{2}| + n_{2}|v_{1} - v_{2}|. \]

(34)
Then, the system (3), (4) has a unique solution on \([a, b]\), if

\[
(K_1 + K_3)(m_1 + m_2) + (K_2 + K_4)(n_1 + n_2) < 1. \tag{35}
\]

Proof. Define sup \(f(t, 0, 0) = \sigma_1 < \infty\) and sup \(g(t, 0, 0) = \sigma_2 < \infty\) and \(r > 0\) such that

\[
r > \frac{(K_1 + K_3) \sigma_1 + (K_2 + K_4) \sigma_2}{1 - ((K_1 + K_3)(m_1 + m_2) + (K_2 + K_4)(n_1 + n_2))}. \tag{36}
\]

Now, we show that \(\mathcal{B} B \subset B_s\), where \(B_s = \{(u, v) \in X \times Y : \|\mathcal{M}_a(u, v)\| \leq r\} \).

By assumption (H1), for \((u, v) \in B_s, t \in [a, b]\), we have that

\[
|f(t, u(t), v(t))| \leq |f(t, u(t), v(t)) - f(t, 0, 0)| + |f(t, 0, 0)| \\
\leq m_1 |u(t)| + m_2 |v(t)| + \sigma_1 \leq m_1 \|u\| + m_2 \|v\| + \sigma_1,
\]

which leads to

\[
|\mathcal{M}_a(u, v)(t)| \leq \left[ m_1 \|u\| |\mathcal{M}_a(u, v)(t)| \right] \\
+ \left[ m_2 \|v\| |\mathcal{M}_a(u, v)(t)| \right] \\
+ \left[ m_1 |u(t)| + m_2 |v(t)| + \sigma_1 \right].
\]

Hence,

\[
\|\mathcal{M}_a(u, v)(t)\| \leq \left[ m_1 \|u\| |\mathcal{M}_a(u, v)(t)| \right] \\
+ \left[ m_2 \|v\| |\mathcal{M}_a(u, v)(t)| \right] \\
+ \left[ m_1 |u(t)| + m_2 |v(t)| + \sigma_1 \right].
\]

In the same way, we can obtain that

\[
\|\mathcal{M}_a(u, v)(t)\| \leq \frac{m_1 |u(t)| + m_2 |v(t)| + \sigma_1}{1 - (m_1 + m_2) \|u\| + (m_2 + m_1) \|v\| + \sigma_1},
\]

which implies \(\mathcal{B} B \subset B_s\). Next, we show that operator \(\mathcal{M}_a\) is contraction mapping. For any \((u_1, v_1), (u_2, v_2) \in X \times Y\) and for any \(t \in [a, b]\), we obtain

\[
\|\mathcal{M}_a(u_1, v_1)(t) - \mathcal{M}_a(u_2, v_2)(t)\| \leq \frac{m_1 \|u_1 - u_2\| + m_2 \|v_1 - v_2\|}{1 - (m_1 + m_2) \|u\| + (m_2 + m_1) \|v\| + \sigma_1}.
\]

(37)
Therefore, we get the following inequality
\[
\|\mathcal{T}_1(u_1, v_1)(t) - \mathcal{T}_1(u_2, v_2)(t)\| \leq (K_1(m_1 + n_2)) + K_2(n_1 + n_2)(\|u_1 - u_2\| + \|v_1 - v_2\|).
\]

(43)

Similarly,
\[
\|\mathcal{T}_2(u_1, v_1)(t) - \mathcal{T}_2(u_2, v_2)(t)\| \leq (K_3(m_1 + m_2)) + K_4(n_1 + n_2)(\|u_1 - u_2\| + \|v_1 - v_2\|).
\]

(44)

From inequalities (43) and (44), it yields
\[
\|\mathcal{T}(u_1, v_1)(t) - \mathcal{T}(u_2, v_2)\| \leq ((K_1 + K_3)(m_1 + m_2)) + (K_2 + K_4)(n_1 + n_2)(\|u_1 - u_2\| + \|v_1 - v_2\|).
\]

(45)

Since \((K_1 + K_3)(m_1 + m_2) + (K_2 + K_4)(n_1 + n_2) < 1\), therefore, \(\mathcal{T}\) is a contraction operator. So, by applying Banach’s fixed point theorem, the operator \(\mathcal{T}\) has a unique fixed point in \(\mathcal{B}_r\). Hence, there exists a unique solution of problem (3), (4) on \([a, b]\).

Now, we prove our second existence result via the Leray-Schauder alternative.

**Lemma 9** (Leray-Schauder alternative [39]). Let \(F : E \rightarrow E\) be a completely continuous operator (i.e., a map restricted to any bounded set in \(E\) is compact). Let
\[
\epsilon(F) = \{x \in E : x = \lambda F(x) \text{ for some } 0 < \lambda < 1\}.
\]

(46)

Then, either the set \(\epsilon(F)\) is unbounded or \(F\) has at least one fixed point.

**Theorem 10.** Assume that:

(H2) \(f, g : [a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}\) are continuous functions and there exist real constants \(a_i, b_i \geq 0\) (\(i = 0, 1, 2\)) and \(a_0 > 0, b_0 > 0\) such that \(\forall x_i \in \mathbb{R} (i = 1, 2)\) we have
\[
\begin{align*}
|f(t, x_1, x_2)| &\leq a_0 + a_1|x_1| + a_2|x_2|, \\
|g(t, x_1, x_2)| &\leq b_0 + b_1|x_1| + b_2|x_2|.
\end{align*}
\]

(47)

If \((K_j + K_3)a_j + (K_j + K_4)b_j < 1 \text{ and } (K_j + K_3)a_j + (K_2 + K_4)b_j < 1\) then system (3), (4) has at least one solution on \([a, b]\).

**Proof.** By the continuity of functions \(f, g\) on \([a, b] \times \mathbb{R} \times \mathbb{R}\), the operator \(\mathcal{T}\) is continuous. Now, we show that the operator \(\mathcal{T} : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{X} \times \mathbb{Y}\) is completely continuous. Let \(\Omega \subset \mathbb{X} \times \mathbb{Y}\) be bounded. Then, there exist two positive constants, \(M_1\) and \(M_2\), such that
\[
|f(t, u(t), v(t))| \leq M_1, |g(t, u(t), v(t))| \leq M_2 \forall (u, v) \in \Omega.
\]

(48)

Then, for any \((u, v) \in \Omega\), we have
\[
\|\mathcal{T}_1(u, v)(t)\| \leq \left[ M_1 \left( \int_{\Omega} |f(t, u(t), v(t))| \right) \right] \left( \sum_{i=1}^{N} (\ln(|\eta_i| \alpha))^{2-\delta_i} \right) + \left[ M_2 \left( \int_{\Omega} |g(t, u(t), v(t))| \right) \right] \left( \sum_{i=1}^{N} (\ln(|\eta_i| \alpha))^{1-\delta_i} \right) + \left[ \int_{\Omega} |(\ln(|\eta_i| \alpha))^{2-\delta_i} \right] \left( \sum_{i=1}^{N} (\ln(|\eta_i| \alpha))^{1-\delta_i} \right)
\]

(49)

which yields,
\[
\|\mathcal{T}_1(u, v)\| \leq K_1M_1 + K_2M_2.
\]

(50)

In the same way, we can obtain that \(\|\mathcal{T}_2(u, v)\| \leq K_1M_1 + K_2M_2\). Hence, from the above inequalities, we get that the operator \(\mathcal{T}\) is uniformly bounded, since \(\|\mathcal{T}_1(u, v)\| \leq (K_1 + K_3)M_1 + (K_2 + K_4)M_2\).

Next, we show that \(\mathcal{T}\) is equicontinuous. For any \((u, v) \in \Omega\) and \(t_1, t_2 \in [a, b]\) with \(t_1 < t_2\). Then, we have
\[
\|\mathcal{T}_1(u, v)(t_1) - \mathcal{T}_1(u, v)(t_2)\| \leq M_1|t_2| \left( \int_{\Omega} |f(t, u(t), v(t))| \right) \left( \sum_{i=1}^{N} (\ln(|\eta_i| \alpha))^{2-\delta_i} \right) + \left( \sum_{i=1}^{N} (\ln(|\eta_i| \alpha))^{1-\delta_i} \right) \left( \int_{\Omega} |g(t, u(t), v(t))| \right) + \left( \sum_{i=1}^{N} (\ln(|\eta_i| \alpha))^{2-\delta_i} \right) \left( \sum_{i=1}^{N} (\ln(|\eta_i| \alpha))^{1-\delta_i} \right)
\]

(51)
Then, we have

\[
\|u(t)\| \leq K_1(a_0 + a_1\|u + a_2\|\|v\|) + K_2(b_0 + b_1\|u + b_2\|\|v\|)
\]

\[
= K_1a_0 + K_2b_0 + (K_1a_1 + K_2b_1)\|u\| + (K_1a_2 + K_2b_2)\|v\|
\]

\[
\|v(t)\| \leq K_3(a_0 + a_1\|u\| + a_2\|v\|) + K_4(b_0 + b_1\|u\| + b_2\|v\|)
\]

\[
= K_3a_0 + K_4b_0 + (K_3a_1 + K_4b_1)\|u\| + (K_3a_2 + K_4b_2)\|v\|
\]

which implies that

\[
\|u\| + \|v\| \leq (K_1 + K_3)a_0 + (K_2 + K_4)b_0
\]

\[
+ \{(K_1 + K_3)a_1 + (K_2 + K_4)b_1\|u\|
\]

\[
+ \{(K_1 + K_3)a_2 + (K_2 + K_4)b_2\|v\|
\].

Consequently,

\[
\|\langle u, v \rangle\| \leq \frac{(K_1 + K_3)a_0 + (K_2 + K_4)b_0}{K_0},
\]

where

\[
K_0 = \min \{ 1 - [(K_1 + K_3)a_1 + (K_2 + K_4)b_1], 1
\]

\[
- [(K_1 + K_3)a_2 + (K_2 + K_4)b_2] \}
\]

which proves that \( \epsilon \) is bounded. Therefore, by applying Lemma 9, the operator \( \mathcal{T} \) has at least one fixed point in \( \Omega \). Therefore, we deduce that the boundary value problem (3), (4) has at least one solution on \([a, b]\).

4. Some Examples

In this section, we give an example to illustrate our main results.

Example 11. Consider the following system of Caputo-Hadamard boundary value problem:

\[
\mathcal{D}_t^{\alpha/2} u(t) = f(t, u(t), v(t)), t \in [1, e],
\]

\[
\mathcal{D}_t^{\alpha/2} v(t) = g(t, u(t), v(t)), t \in [1, e],
\]

\[
 u(1) = v(e), 1/2\mathcal{D}_t^{1/2} u(1) = 1/3\mathcal{D}_t^{1/3} v(3/2) + 1/3\mathcal{D}_t^{1/3} v(4/3),
\]

\[
 v(1) = 2u(e), 1/4\mathcal{D}_t^{1/4} v(e) = 1/5\mathcal{D}_t^{1/5} u(5/3) + 1/5\mathcal{D}_t^{1/5} u(5/4).
\]

Here, \( a = \beta = 3/2, \ a = 1, \ b = e, \ \gamma_1 = 1/2, \ \gamma_2 = 1/4, \ \delta_1 = 1/3, \delta_2 = 1/5, \ N = M = 2, \eta_1 = 3/2, \eta_2 = 4/3, \xi_1 = 5/3, \xi_2 = 5/4, \lambda_1 = 1, \lambda_2 = 1/2, \lambda_3 = 1/4, \mu_1 = 2, \mu_2 = 1/3, \mu_3 = 1/5 \). By simple calculation, we found that \( \Delta = 0.078172, K_1 = 10.36402, K_2 = 8.38734, K_3 = 11.58173, K_4 = 11.18721 \).

(i) Let two nonlinear functions \( f, g : [1, e] \times R \times R \rightarrow R \) be given by

\[
u(t) = \lambda\mathcal{T}_1(u, v)(t), v(t) = \lambda\mathcal{T}_2(u, v)(t).
\]

(54)
\[ f(t, x, y) = \frac{1}{15\sqrt{24 + t^2}} \frac{|x|}{1 + |x(t)|} + \sin y(t) \frac{t^3}{64 + t^2} + \frac{1}{2}, \quad (60) \]

\[ g(t, x, y) = \frac{\sin (|x|)}{124 + t^3} + \frac{\tan^{-1}(y)}{120t^3 + 2} + \frac{2}{3}. \quad (61) \]

Note that
\[
|f(t, x_1, x_2) - f(t, y_1, y_2)| \leq \frac{1}{75} |x_1 - x_2| + \frac{1}{63} |y_1 - y_2|,
\]
\[
|g(t, x_1, x_2) - g(t, y_1, y_2)| \leq \frac{1}{125} |x_1 - x_2| + \frac{1}{122} |y_1 - y_2|,
\]

we obtain \((K_1 + K_2)(1/75 + 1/65) + (K_3 + K_4)(1/125 + 1/122) = 0.9552435311 < 1\). Thus, all the conditions of Theorem 8 are satisfied. Problem (59) with (60) and (61) has a unique solution on \([1, e]\).

(ii) Let two nonlinear functions \(f, g : [1, e] \times R \times R \to R\) be given by
\[
f(t, x, y) = \frac{e^{-2t}}{4} + \frac{x^2 \cos^2 t}{39(1 + |x|)} + \frac{|y|^4 \sin^2 t}{45(1 + y^3)}, \quad (63)\]
\[
g(t, x, y) = \frac{2}{t^2 + 2} + \frac{\sin x}{12(t + 4)} + \frac{\tan^{-1} y}{14(3 + t^3)}. \quad (64)\]

Note that
\[
|f(t, x, y)| \leq \frac{1}{4} + \frac{1}{39} |x| + \frac{1}{45} |y|, \quad (65)\]
\[
|g(t, x, y)| \leq \frac{2}{3} + \frac{1}{66} |x| + \frac{1}{56} |y|.
\]

We get \(a_1 = 1/39, a_2 = 1/45, b_1 = 1/60, b_2 = 1/56\). By simple calculation, we have \((K_1 + K_2)a_1 + (K_3 + K_4)b_1 = 0.8960930769 < 1\) and \((K_1 + K_2)a_2 + (K_3 + K_4)b_2 = 0.8434206667 < 1\). By Theorem 10, the coupled boundary value problem (59) with (63) and (64) has at least one positive solution on \([1, e]\).

5. Conclusions

In this paper, we studied existence and uniqueness of solutions for the system of Caputo-Hadamard fractional differential equations with multipoint boundary conditions. The existence theory of solutions of a Caputo-Hadamard system using a variety of fixed point theorems. The Leray-Schauder alternative was applied to prove existence, while the uniqueness result was obtained via the Banach contradiction mapping principle. Finally, we have given two examples to demonstrate our result.

Data Availability

No data were used to support this study.

Conflicts of Interest

There is no competing interest among the authors regarding the publication of the article.

Authors’ Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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