Ramanujan congruences for overpartitions with restricted odd differences

Michael Hanson and Jeremiah Smith

Abstract
We investigate Ramanujan congruences for the function $T(n)$, which counts the overpartitions of $n$ with restricted odd differences. In particular, we show that only one such congruence exists. Our method involves using the theory of modular forms to prove a more general theorem which bounds the number of primes possible for Ramanujan congruences in certain eta-quotients. This generalizes work done by Jonah Sinick. We also provide two congruences modulo 5 for $T(n)$.

1 Introduction
Perhaps one of the most famous results of Ramanujan was his collection of congruences for the partition function $p(n)$. For a positive integer $n$, $p(n)$ denotes the number of partitions of $n$, i.e. the number of ways to write $n$ as a non-increasing sum of positive integers. For example, there are five partitions of the integer 4, those being $4$, $3 + 1$, $2 + 2$, $2 + 1 + 1$, $1 + 1 + 1 + 1$, and so $p(4) = 5$. Here we define $p(0) := 1$ and $p(n) = 0$ when $n < 0$. It is common notation to write a partition $n = \lambda_1 \lambda_2 \ldots \lambda_r$ of $n$ in the concatenated form $\lambda_1 \lambda_2 \ldots \lambda_r$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$. Ramanujan’s famous result on $p(n)$ is the collection of congruences

$$\begin{align*}
\begin{cases}
p(5n + 4) & \equiv 0 \pmod{5}, \\
p(7n + 5) & \equiv 0 \pmod{7}, \\
p(11n + 6) & \equiv 0 \pmod{11}.
\end{cases}
\end{align*}$$

(1)

These are proven starting with the fact that the generating function for $p(n)$ takes the form $\sum_{n \geq 0} p(n) q^n = \prod_{n \geq 1} (1 - q^n)^{-1}$ with $q := e^{2\pi i z}$, which is essentially the inverted Dedekind’s eta-function $\eta(z) := q^{1/24} \prod_{n \geq 1} (1 - q^n)$. Ramanujan’s congruences for $p(n)$ all take the form $p(\ell n + a) \equiv 0 \pmod{\ell}$ for some prime $\ell$, hence all such congruences are called Ramanujan congruences. It is natural to ask whether there are other primes $\ell$ for which $p(n)$ has Ramanujan congruences. But as it turns out, work of Ahlgren and Boylan [1] show that the Ramanujan congruences (1) are the only ones for $p(n)$.

Ramanujan’s work has inspired many others to consider such congruences for other modified partition functions. In this work, we consider the modified partition function...
$t(n)$ that counts “overpartitions with restricted odd parts,” as originally described in [2]. An overpartition of $n$ is a partition of $n$ in which the final occurrence of a number may be overlined. We let $p(n)$ be the number of overpartitions of $n$. Thus, for example, $p(4) = 14$, with the 14 partitions given as

$$4, \overline{4}, 3 + 1, \overline{3} + 1, \overline{3} + \overline{1}, \overline{3} + \overline{1}, 2 + 2, 2 + \overline{2}, 2 + 1 + 1,$$

$$\overline{2} + 1 + 1, 2 + 1 + \overline{1}, \overline{2} + 1 + \overline{1}, 1 + 1 + 1 + 1, 1 + 1 + 1 + \overline{1}.$$

Our function of interest $\overline{t}(n)$ counts the number of overpartitions of $n$ with the following restrictions.

(i) The difference between two successive parts may be odd only if the larger part is overlined.

(ii) If the smallest part is odd, then it is overlined.

For example, using $n = 4$ again we have $\overline{t}(4) = 8$, with the 8 such partitions given as

$$4, \overline{4}, 3 + \overline{1}, \overline{3} + \overline{1}, 2 + 2, 2 + \overline{2}, \overline{2} + 1 + \overline{1}, \overline{2} + 1 + \overline{1}, 1 + 1 + 1 + \overline{1}.$$

The authors of [2] show that the generating function for $\overline{t}(n)$ is given by

$$\sum_{n \geq 0} \overline{t}(n)q^n = \frac{\eta(3z)}{\eta(2z)^2}.$$

This is a weakly holomorphic modular form of weight $-1/2$ for the congruence subgroup $\Gamma_1(144)$. Many congruences have been proven for $\overline{t}(n)$. In particular, congruences modulo 2, 3, and 5 have been found:

- **Theorem 1.1** of [8]: For $n \geq 0$ we have

  $$\overline{t}(n) \equiv \begin{cases} (-1)^{k+1} \pmod{3} & n = k^2 \text{ some } k, \\ 0 \pmod{3} & \text{else}. \end{cases}$$

- **Theorem 1.2** of [8]: For $n \geq 1$ we have

  $$\overline{t}(2n) \equiv \begin{cases} 1 \pmod{2} & n = (3k + 1)^2 \text{ some } k, \\ 0 \pmod{2} & \text{else}. \end{cases}$$

- **Theorem 1.1** of [10]: For all $\omega, n \geq 0$ we have

  $$\overline{t}(9^\omega(45n + 30)) \equiv 0 \pmod{5}.$$

More congruences can be found in [3,8,10,11]. We provide two more congruences modulo 5, proven in Sect. 7 using the theory of modular forms.

**Proposition 1** The following congruences hold for all $n \geq 0$:

$$\overline{t}(80n + 40) \equiv 0 \pmod{5}, \quad (2)$$

$$\overline{t}(80n + 60) \equiv 0 \pmod{5}. \quad (3)$$

**Remark 1** We note that although this result is proven using standard techniques from the theory of modular forms, these congruences do not appear in the current literature.
We bring this to the attention of combinatorialists who may like to prove this using only
manipulation of generating functions.

However, one might question if any Ramanujan congruences hold for \( t(n) \) (i.e. congru-
ences of the form \( t(\ell n + a) \equiv 0 \pmod{\ell} \) for a prime \( \ell \)). The results of [8] above show that
the only Ramanujan congruence mod 2 or 3 is \( t(3n + 2) \equiv 0 \pmod{3} \), and one can check
that there are no Ramanujan congruences mod 5. In fact, we have the following.

**Theorem 1** The only Ramanujan congruence for \( t(n) \) is

\[
t(3n + 2) \equiv 0 \pmod{3}.
\]

In order to prove Theorem 1, it suffices to show that there are no Ramanujan congru-
ences for primes \( \ell > 5 \). This is an immediate consequence of Theorem 3 below, which is
a generalization of Sinick’s Theorem 1.2 in [15]:

**Theorem 2** (Theorem 1.2 of [15]) Let \( S = (a_1, a_2, \ldots, a_j) \) be a sequence of positive integers
with \( j \) even and define \( c(n) \) by

\[
\prod_{n \geq 1} \prod_{i=1}^{j} \frac{1}{1 - q^{a_i n}} = \sum_{n \geq 0} c(n) q^n.
\]

Let \( N = \mathrm{lcm}(a_1, a_2, \ldots, a_j) \). If \( c(n) \) obeys a Ramanujan congruence modulo \( \ell \), then \( \ell \mid N \)
or \( \ell \leq \max(5, j + 4) \).

We extend this theorem to bound Ramanujan congruences for more general eta-
quotients in the following theorem.

**Theorem 3** Let \( \lambda = \lambda_1 \lambda_2 \ldots \lambda_r \) and \( \mu = \mu_1 \mu_2 \ldots \mu_s \) be partitions of \( u \in \mathbb{N} \) and \( v \in \mathbb{N} \),
respectively, where we assume without loss of generality that \( \lambda_i \neq \mu_j \) for all \( i, j \). Define

\[
\mathcal{h}(z) := \prod_{n \geq 1} \frac{(1 - q^{\lambda_1 n})(1 - q^{\lambda_2 n}) \cdots (1 - q^{\lambda_s n})}{(1 - q^{\mu_1 n})(1 - q^{\mu_2 n}) \cdots (1 - q^{\mu_s n})} := \sum_{n \geq 0} c(n) q^n.
\]

Let \( \gamma := \mathrm{lcm}(\lambda_1, \ldots, \lambda_r, \mu_1, \ldots, \mu_s) \), and let \( \gamma \) be the number of occurrences of the smallest
element of \( \{\lambda_1, \ldots, \lambda_r, \mu_1, \ldots, \mu_s\} \). Let \( \ell \) be prime such that \( \ell > \max(5, |r - s| + 4) \) and
\( \ell \nmid \gamma N \). Then the following statements hold.

(i) If \( r - s \) is even, then \( c(n) \) does not obey a Ramanujan congruence modulo \( \ell \).
(ii) If \( r - s \neq 1, 3 \) is odd and \( u \equiv v \pmod{\ell} \), then \( c(n) \) does not obey a Ramanujan
congruence modulo \( \ell \).

**Remark 2** The case \( r - s \neq 1, 3 \) is odd and \( u \equiv v \pmod{\ell} \) is not included in Theorem
3. We leave open the problem of whether one can establish an upper bound for primes \( \ell \)
that could admit a Ramanujan congruence in this case.

In Sect. 2, we provide a brief introduction to modular forms modulo \( \ell \), and we give a few
results within the theory that will be used in later sections. Section 3 provides motivation
and an outline for the proof of Theorem 3. Section 4 proves the most interesting and
essential result for the proof of Theorem 3. Section 5 disposes of some necessary calcu-
lations needed for Sect. 6 which completes the proofs of Theorems 3 and 1, respectively.
Section 7 proves Proposition 1. Section 8 provides a helpful example pertaining to the discussion following Proposition 3.

2 Modular forms modulo ℓ

In this section, we introduce the notion of “modular forms mod ℓ” as well as some preliminary results that will be used in later sections. We refer the reader to [13] for a more detailed account of this material.

We denote the C-vector space of weakly holomorphic modular forms of integer weight $k$ on the congruence subgroup $\Gamma_1(N)$ of $\text{SL}_2(\mathbb{Z})$ by $M^!_k(\Gamma_1(N))$. Let $M_k(\Gamma_1(N))$ denote the subspace of those forms which are holomorphic at the cusps of $\Gamma_1(N)$, and let $S_k(\Gamma_1(N))$ denote its corresponding subspace of cusp forms. Given $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$, define the usual weight-$k$ slash operator on holomorphic functions $f$ on the upper half plane $\mathbb{H}$ as

$$ f \mid_k M := (cz + d)^{-k} f \left( \frac{az + b}{cz + d} \right). $$

We will sometimes drop the subscript $k$ for notational convenience. Now define $\theta := \frac{1}{2\pi i} \frac{d}{dz} = q \frac{d}{dq}$, where $q := e^{2\pi iz}$, so that on Fourier series we have

$$ \theta \left( \sum_{n \geq 0} a(n)q^n \right) = \sum_{n \geq 0} na(n)q^n. $$

Also define the $m$-th $U$-operator on Fourier series as

$$ \sum_{n \geq 0} a(n)q^n \mid U_m := \sum_{n \geq 0} a(mn)q^n. $$

Eisenstein series are canonical examples of modular forms for $\text{SL}_2(\mathbb{Z})$, and they play an important role in Lemma 1 below. For even $k > 2$, the weight-$k$ Eisenstein series for $\text{SL}_2(\mathbb{Z})$ is

$$ E_k(z) := 1 - \frac{2k}{B_k} \sum_{n \geq 1} \sigma_{k-1}(n)q^n, $$

where $B_k$ is the $k$-th Bernoulli number, and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$. When $k = 2$, $E_2(z) := 1 - 24 \sum_{n \geq 1} \sigma_1(n)q^n$ is not a modular form, but rather a “quasi-modular form.” It has the transformation law

$$ E_2(z) \mid M = E_2(z) - \frac{6ic}{\pi(cz + d)}, $$

(4)

for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$. We also have the Delta function

$$ \Delta(z) = \eta(z)^{24} = q \prod_{n=1}^{\infty} (1 - q^n)^{24} \in M_{12}(\text{SL}_2(\mathbb{Z})), $$

which vanishes at the cusp $\infty$ of $\text{SL}_2(\mathbb{Z})$.

Given a modular form $f \in M_k(\Gamma_1(N)) \cap \mathbb{Z}[q]$, one can reduce the Fourier coefficients of $f$ modulo a prime $\ell$, giving an element $\tilde{f}$ of $\mathbb{F}_\ell[[q]]$. We call $\tilde{f}$ a modular form modulo $\ell$ for $\Gamma_1(N)$. The filtration of $f$ is defined as

$$ w_\ell(f) := \min\{k' : \tilde{f} \in \tilde{M}_{k'}(\Gamma_1(N))\}, $$
where
\[ \tilde{M}_k(\Gamma_1(N)) := \{ \tilde{g} : g \in M_k(\Gamma_1(N)) \}. \]

We will also refer to preimages of \( \tilde{f} \) under the reduction map as “modular forms modulo \( \ell \).”

By using (4), one can easily generalize Lemma 3 of [16] for \( N > 1 \) to conclude that if \( f \in M_k(\Gamma_1(N)) \) then \( 12\theta f - kE_{2f} \in M_{k+2}(\Gamma_1(N)) \). Theorem 2(i) from [16] implies that \( E_{\ell-1} \equiv 1 \pmod{\ell} \) and \( E_{\ell+1} \equiv 2 \pmod{\ell} \). These facts come together to prove Lemma 1.

**Lemma 1** (Lemma 2.1 of [15]) If \( f \in M_k(\Gamma_1(N)) \cap \mathbb{Z}[q] \), then defining \( R \) to be

\[ R := \left( \theta f - \frac{k}{12} E_{2f} \right) E_{\ell-1} + \frac{k}{12} E_{\ell+1} f, \tag{5} \]

\( R \) is a modular form of weight \( k + \ell + 1 \) such that \( R \equiv \theta f \pmod{\ell} \). In particular, \( \theta f \) is a modular form \( \pmod{\ell} \) for \( \Gamma_1(N) \). It follows that if \( \tilde{f} \not\equiv 0 \pmod{\ell} \), then \( w_{\ell}(\theta f) \leq w_{\ell}(f) + \ell + 1 \).

We will also need the following facts about filtrations.

**Lemma 2** Let \( N \geq 4 \), let \( f \in M_k(\Gamma_1(N)) \cap \mathbb{Z}[q] \) and \( g \in M_k(\Gamma_1(N)) \cap \mathbb{Z}[q] \), and let \( \ell \geq 5 \) be prime. Then:

(i) We have \( w_{\ell}(\theta f) = w_{\ell}(f) + \ell + 1 \) if and only if \( w_{\ell}(f) \not\equiv 0 \pmod{\ell} \).

(ii) If \( f \not\equiv 0 \pmod{\ell} \), then \( k_1 \equiv k_2 \pmod{\ell - 1} \).

(iii) If \( \ell \nmid N \) then for \( i \geq 0 \) we have \( w_{\ell}(f^i) = i \cdot w_{\ell}(f) \).

The proofs of (ii) and the reverse implication of (i) are given directly in Sect. 4 of [7]. The remaining facts are quick consequences of the results given in the same section.

The following elementary fact will be useful in Sect. 6: if \( f \in M_k(\Gamma_1(N)) \cap \mathbb{Z}[q] \) and \( \ell \) is a prime, then \( (f \mid U_{\ell})^\ell \equiv f - \theta^{\ell-1} f \pmod{\ell} \). It follows that

\[ f \mid U_{\ell} \equiv 0 \pmod{\ell} \iff \theta^{\ell-1} f \equiv f \pmod{\ell}. \]

### 3 Keys to the Proof of Theorem 3

The statement of Theorem 3 concerns the existence of Ramanujan congruences for a particular type of eta-quotient. We hope to apply the following proposition originally due to I. Kiming and J. Olsson [9] and then corrected by J. Sinick [15, Proposition 3.2].

**Proposition 2** Let \( \ell \geq 5 \) be prime and \( N \geq 4 \), \( \ell \nmid N \). Suppose that \( f(z) \in M_k(\Gamma_1(N)) \) has \( \ell \)-integral Fourier coefficients, \( w_{\ell}(f(z)) \not\equiv 0 \pmod{\ell} \), and \( \theta f(z) \not\equiv 0 \pmod{\ell} \). Suppose further that \( w_{\ell}(\theta^m f(z)) \geq w_{\ell}(f(z)) \) for any integer \( m \geq 0 \). Then if the Fourier coefficients \( d(n) \) of \( f(z) \) satisfy \( d((n + b) \equiv 0 \pmod{\ell} \), one of the following is true: \( b = 0 \), \( w_{\ell}(f(z)) \equiv (\ell + 1)/2 \pmod{\ell} \), or \( w_{\ell}(f(z)) \equiv (\ell + 3)/2 \pmod{\ell} \).

There is a glaring problem with naively applying this proposition to the eta-quotient in Theorem 3: the fact that the eta-quotient is not necessarily an integer weight holomorphic modular form. We will fix this by defining an integer weight modular form for each prime \( \ell \) for which a Ramanujan congruence exists \( \pmod{\ell} \) if and only if a Ramanujan congruence exists \( \pmod{\ell} \) for the given eta-quotient. This definition is given below.
Let $N, \lambda, \text{and} \mu$ be as in Theorem 3. Define

$$F_\ell(z) := \Delta(z)^{\ell t} \left( \frac{\Delta(\mu_1 z) \Delta(\mu_2 z) \cdots \Delta(\mu_\ell z)}{\Delta(\lambda_1 z) \Delta(\lambda_2 z) \cdots \Delta(\lambda_\ell z)} \right)^{\delta_\ell} = \sum_{n \geq 0} D(n)q^n,$$

where $\delta_\ell := \frac{\ell^2 - 1}{24}$, and where $t \geq 2$ is the smallest integer such that $F_\ell$ is holomorphic at the cusps of $\Gamma_1(N)$.

**Remark 3** Note that $F_\ell(z)$ is a modular form of weight $(\frac{(\ell^2 - 1)(\ell - 2)}{2} + 12\ell t)$ for $\Gamma_1(N)$. In what follows, we sometimes need $N \geq 4$. We can substitute $4N$ for $N$ without loss of generality when $N < 4$.

**Remark 4** We also note that $F_\ell(z)$ serves the same purpose as the modular form in equation (1.3) of [15]. Our form $F_\ell(z)$ differs from Sinick’s in that it requires an appropriate power of $\Delta(z)$ to account for any poles coming from the numerator of $h(z)$ in Theorem 3.

The fact that the Ramanujan congruences for $F_\ell(z)$ are in correspondence with those of $h(z) = \sum_{n \geq 0} c(n)q^n$ is the following lemma.

**Lemma 3** Let $u, v$ be as in Theorem 3. With notation as above, we have that $D(\ell n + b) \equiv 0 \pmod{\ell}$ if and only if $c(\ell n + a) \equiv 0 \pmod{\ell}$, where $b$ is defined by $24a \equiv 24b + (u - v) \pmod{\ell}$.

The proof of this will be given later in Sect. 5. Notice that if $u \equiv v \pmod{\ell}$, then $a \equiv b \pmod{\ell}$ (recall $\ell > 5$). Now that we have defined an appropriate modular form, we need to proceed by checking that $F_\ell$ satisfies the other assumptions of Proposition 2. The fact that $\theta F_\ell \equiv 0 \pmod{\ell}$ is a simple calculation (see Proposition 4) which will be done in Sect. 5.

We note that for the corresponding modular form in Sinick’s case, this is an immediate verification. On a technical note, the necessity of $\gamma$ in the statement of Theorem 3 comes from the proof of Proposition 4. The most difficult assumptions to verify in Proposition 2 are precisely those which deal with the filtrations. This will be accomplished in Sect. 4. By these calculations, we will see that the two congruences $w_\ell(F_\ell(z)) \equiv (\ell + 1)/2 \pmod{\ell}$ and $w_\ell(F_\ell(z)) \equiv (\ell + 3)/2 \pmod{\ell}$ are impossible. Lastly, in order to get the full strength of Theorem 3, we must dispose of the possibility that $b = 0$ in the statement of Proposition 2. This is a technical point which is resolved in Sect. 6.

**4 Calculating the filtrations**

In this section, we show that $w_\ell(\theta^mF_\ell) \geq w_\ell(F_\ell) = \frac{(\ell^2 - 1)(\ell - r)}{2} + 12\ell t$ by proving the more general Proposition 3.

**Proposition 3** Let $F \in M_k(\Gamma_1(N)) \cap \mathbb{Z}[q]^\ell$, $\ell \geq 5$ prime, $\ell \nmid N$, $\theta F \equiv 0 \pmod{\ell}$, and suppose that $F$ does not vanish on $\mathbb{H}$. Then $w_\ell(F) = k$ and $w_\ell(\theta^mF) \geq w_\ell(F)$ for any integer $m \geq 0$.

Following the discussion below Lemma 4.1 of [15], we enumerate the cosets of $\Gamma_1(N)$ in $\text{SL}_2(\mathbb{Z})$ by $[\ell]_{1 \leq \ell \leq 2d_N}$. Let $M_i$ be a representative of the $i$-th coset. Let $a_\ell$ be the cusp that $M_i$ sends to $\infty$. Denote the minimal period of $F | M_i$ by $t_i$. Then $F | M_i$ has a Fourier expansion in powers of $q_{t_i} := e^{2\pi i/\ell}t_i$, and the order of vanishing of $F$ at $a_\ell$ is the index of the first non-vanishing Fourier coefficient of $F$ in powers of $q_{t_i}$, denoted ord_{a_\ell}(F). Though
the coefficients of these \( q_i \)-Fourier expansions of \( F \) need not be integral, \([5, \S 12.3]\) tells us that they lie in \( \mathbb{Z}[\zeta_N] \) with \( \zeta_N \) a primitive \( N \)-th root of unity.

Instead of considering modular forms modulo \( \ell \), one may choose an algebraic number field \( L \) and look at forms \( g \in M_k(\Gamma_1(N)) \cap L[[q]] \). We can then reduce \( g \) modulo \( v \) for any prime \( v \in \mathcal{O}_L \) such that the \( v \)-adic valuation of \( g \) is 0. This allows us to define the notion of “modular form modulo \( v \),” and we can define the filtration \( w_v \) for nonvanishing forms (mod \( v \)) in the obvious way. Hence we can define the \( v \)-adic valuation of the corresponding power series to be the minimum of the \( v \)-adic valuations of the coefficients. Defining \( \overline{\text{ord}}_{\alpha_i}(f) \) to be the order of vanishing of \( f \) (mod \( v \)) at the cusp \( \alpha_i \) (this is well-defined; see for example Remark 2.4 of \([4]\)), we have Lemma 4.2 of \([15]\), stated below.

**Lemma 4** Let \( m \geq 1 \) be an integer and let \( v \) be a prime in \( \mathbb{Z}[\zeta_N] \) such that \( v \nmid 2, 3, N \). Let \( f(z) \) be a modular form for \( \Gamma_1(N) \) such that \( f(z) \mid M_i \) has coefficients in \( \mathbb{Q}(\zeta_N) \) and \( v \)-adic valuation 0. Let \( \alpha_i \) be a cusp of \( \Gamma_1(N) \). Then

\[
\overline{\text{ord}}_{\alpha_i}(\theta^m f) \geq \overline{\text{ord}}_{\alpha_i}(f).
\]

**Remark 5** Sinick’s proof states, “Since \( v \nmid N \) and \( f(z) \mid M_i \) has \( v \)-adic valuation 0, the Fourier expansion of \( \theta(f \mid M_i) \) has \( v \)-adic valuation 0.” However, this statement is false by taking a prime above 5 in \( \mathbb{Z}[\zeta_7] \) and considering \( E_4(z) \) as a modular form for \( \Gamma_1(7) \) for example. One needs the additional assumption that \( \theta f \not\equiv 0 \pmod{v} \).

**Proof of Proposition 3** We let \( M_i, \alpha_i, t_i, \) and \( v \in \mathbb{Z}[\zeta_N] \) be as above, \( v \) being a prime above \( \ell \). Since \( \theta F \not\equiv 0 \pmod{\ell} \) by assumption, \( F \not\equiv 0 \pmod{\ell} \). In particular, \( F \not\equiv 0 \pmod{v} \), and so Theorem 12.3.4 and Remark 12.3.5 of \([5]\) assert that \( F \mid M_i \not\equiv 0 \pmod{v} \). Define

\[
G(z) := \prod_{i=1}^{2d_N} (F \mid M_i),
\]

a modular form for \( \text{SL}_2(\mathbb{Z}) \) of weight \( 2d_N k \). As \( F \) is zero-free on \( \mathbb{H} \), so is \( G \), and so by the valence formula we have that \( G \) is a non-zero constant multiple of \( \Delta(z)^e \) with \( e := \frac{2dNk}{12} = \frac{dNk}{6} \). It follows that \( w_v(G) = 12e \) since if there existed a modular form of smaller weight which is congruent to \( G \), Sturm’s Theorem \([13, \text{Theorem 2.58}]\) would imply that it would have to vanish mod \( v \). Thus \( w_v(F) = k \), and since \( F \in \mathbb{Z}[[q]] \), we in fact have that \( w_v(F) = k \), as desired.

Now we show that \( w_v(\theta^m F) \geq w_v(F) \). Notice that \( \overline{\text{ord}}_{\infty}(G) = e \), whence \( \text{ord}_{\infty}(F \mid M_i) = \overline{\text{ord}}_{\alpha_i}(F) \). Thus

\[
\sum_{i=1}^{2d_N} \overline{\text{ord}}_{\alpha_i}(F) = \overline{\text{ord}}_{\alpha}(G) = e.
\]

Define

\[
H := \prod_{i=1}^{2d_N} (\theta^m F) \mid M_i.
\]

Notice that \( H \) is a modular form modulo \( v \) for \( \text{SL}_2(\mathbb{Z}) \) by the modified version of Lemma 1, where we replace \( \ell \) with \( v \). Then

\[
\overline{\text{ord}}_{\infty}(H) = \sum_{i=1}^{2d_N} \frac{\overline{\text{ord}}_{\alpha_i}(\theta^m F)}{t_i} \geq \sum_{i=1}^{2d_N} \frac{\overline{\text{ord}}_{\alpha_i}(F)}{t_i} = e,
\]
the inequality being a consequence of Lemma 4. Since \( \theta^m F \) is a modular form modulo \( v \) which does not vanish, Theorem 12.3.4 and Remark 12.3.5 of [5] assert that \( (\theta^m F) \mid M_l \neq 0 \) (mod \( v \)). This shows that \( H \neq 0 \) (mod \( v \)). Sturm’s Theorem [13, Theorem 2.58] now tells us that \( w_v(H) \geq 12e \), and so \( w_v(\theta^m F) = w_v(\theta^m F) \geq k = w_v(F) \), the first equality coming from the fact that \( F \) has integral Fourier coefficients.

\[ \square \]

5 Necessary calculations

In this section, we produce two calculations that are necessary for the application of Proposition 2.

**Proposition 4** We have that \( \theta F\ell \neq 0 \) (mod \( \ell \)).

**Proof** We note that by definition of \( \theta \) it is sufficient to compute the Fourier expansion modulo \( \ell \). We have

\[
F_{\ell}(z) = \Delta(z)^{\delta_{\ell}} \left( \frac{\Delta(1z)\Delta(\mu_2z)\cdots\Delta(\mu_{s}z)}{\Delta(\lambda_1z)\Delta(\lambda_2z)\cdots\Delta(\lambda_{r}z)} \right) = q^{\ell+u(1-u)\delta_{\ell}} \prod_{n=1}^{\infty} \left[ \left( 1 - q^{n}\right)^{\epsilon} \left( \prod_{i} \left( 1 - q^{\mu_{i} n} \right)^{\ell-1} \right) \left( \prod_{j} \left( \sum_{k=0}^{\infty} q^{\lambda_{j} k} \right)^{\ell-1} \right) \right].
\]

Using a geometric series expansion, we have

\[
F_{\ell}(z) = q^{\ell+u(1-u)\delta_{\ell}} \prod_{n=1}^{\infty} \left[ \left( 1 - q^{n}\right)^{\epsilon} \left( \prod_{i} \left( 1 - q^{\mu_{i} n} \right)^{\ell-1} \right) \left( \prod_{j} \left( 1 + q^{\lambda_{j} n} + \ldots \right) \left( 1 + (\ell^{2} - 1)q^{\lambda_{j} n} + \ldots \right) \right) \right] \equiv q^{\ell+u(1-u)\delta_{\ell}} \prod_{n=1}^{\infty} \left[ \left( 1 - q^{n}\right)^{\epsilon} \left( \prod_{i} \left( 1 + q^{\mu_{i} n} + \ldots \right) \left( 1 - q^{\lambda_{j} n} + \ldots \right) \right) \right] \pmod{\ell},
\]

where \( \alpha \) is the number of occurrences of \( \mu_{s} \) in \( \mu \) and \( \beta \) is the number of occurrences of \( \lambda_{r} \) in \( \lambda \). Hence,

\[
F_{\ell}(z) \equiv q^{\ell+u(1-u)\delta_{\ell}} (1 \pm \cdots \pm \gamma q^{m} \pm \cdots)\pmod{\ell},
\]

where \( m = \min(\mu_{s}, \lambda_{r}) \). So, by applying the theta operator,

\[
\theta F_{\ell}(z) \equiv (\ell^{t} + (v - u)\delta_{\ell}) q^{\ell^{t}+(v-u)\delta_{\ell}} \pm \cdots \pm \gamma (m + \ell^{t} + (v - u)\delta_{\ell}) q^{m+\ell^{t}+(v-u)\delta_{\ell}} \pm \cdots \pmod{\ell}.
\]

One of these coefficients does not vanish mod \( \ell \) because either \( (v - u)\delta_{\ell} \) is divisible by \( \ell \) or not, and from the fact that \( \ell \nmid vN, \) which implies \( \ell \nmid v m \), the result follows. \( \square \)
We also provide the proof of the correspondence of Ramanujan congruences of the eta-quotient to an integer weight modular form.

**Proof of Lemma 3** We have

\[
\sum_{n \geq 0} c(n)q^n = \prod_{n \geq 1} \frac{(1 - q^{\lambda_1n}) \cdots (1 - q^{\lambda_k n})}{(1 - q^{\mu_1n}) \cdots (1 - q^{\mu_l n})} = \prod_{n \geq 1} \left( \frac{(1 - q^{\mu_1n}) \cdots (1 - q^{\mu_l n})}{(1 - q^{\lambda_1n}) \cdots (1 - q^{\lambda_k n})} \right)^{-\ell^2} \frac{(1 - q^{\mu_1n}) \cdots (1 - q^{\mu_l n})}{(1 - q^{\lambda_1n}) \cdots (1 - q^{\lambda_k n})} \frac{\eta^{\delta_{\ell}}}{\eta^{\delta_{\ell}^2}} \frac{\Delta(\mu_1z) \cdots \Delta(\mu_l z)}{\Delta(\lambda_1z) \cdots \Delta(\lambda_k z)}.
\]

Hence,

\[
q^{\delta_{\ell}(u-v)} \left( \frac{\Delta(\mu_1z) \cdots \Delta(\mu_l z)}{\Delta(\lambda_1z) \cdots \Delta(\lambda_k z)} \right)^{\delta_{\ell}} = \left( \prod_{n \geq 1} \frac{(1 - q^{\mu_1n}) \cdots (1 - q^{\mu_l n})}{(1 - q^{\lambda_1n}) \cdots (1 - q^{\lambda_k n})} \right)^{\ell^2} \sum_{n \geq 0} c(n)q^n.
\]

Multiply both sides by \( \Delta(z)^{\ell^t} \) to get

\[
q^{\delta_{\ell}(u-v)} F_{\ell}(z) = \Delta(z)^{\ell^t} \left( \prod_{n \geq 1} \frac{(1 - q^{\mu_1n}) \cdots (1 - q^{\mu_l n})}{(1 - q^{\lambda_1n}) \cdots (1 - q^{\lambda_k n})} \right)^{\ell^2} \sum_{n \geq 0} c(n)q^n. \tag{7}
\]

Now we apply \( U_{\ell} \) to equation (7), reduce modulo \( \ell \), and multiply both sides by \( q^{-a} \) to get

\[
\sum_{n \geq 0} D(\ell n + \delta_{\ell}(u-v) + a)q^n = q^{-a} \Delta(z)^{\ell^t-1} \left( \prod_{n \geq 1} \frac{(1 - q^{\mu_1n}) \cdots (1 - q^{\mu_l n})}{(1 - q^{\lambda_1n}) \cdots (1 - q^{\lambda_k n})} \right)^{\ell^t} \sum_{n \geq 0} c(\ell n)q^n \quad (\text{mod } \ell).
\]

Applying Proposition (3) of [12] to the right-hand side, we get

\[
\sum_{n \geq 0} D(\ell n + \delta_{\ell}(u-v) + a)q^n \equiv 0 \quad (\text{mod } \ell) \iff \sum_{n \geq 0} c(\ell(n - \ell^t - 1) + a)q^n \equiv 0 \quad (\text{mod } \ell).
\]

Since \( c(n) = 0 \) for \( n < 0 \) this proves the lemma. \( \square \)
6 Proof of Theorem 3

By using the results in Sects. 4 and 5, we may apply Proposition 2 to $F_\ell$. Notice that the two congruences $w_\ell(F_\ell(z)) \equiv (\ell + 1)/2 \pmod{\ell}$ and $w_\ell(F_\ell(z)) \equiv (\ell + 3)/2 \pmod{\ell}$ are impossible given that $\ell > \max(5, |r - s| + 4)$ and also the fact that $r - s \neq 1, 3$. This shows that if $F_\ell$ has a Ramanujan congruence, then $b = 0$. Thus, checking that $h$ has no Ramanujan congruences modulo $\ell$ is equivalent to checking that $\sum_{n \geq 0} D(\ell n)q^n \equiv 0 \pmod{\ell}$. In the case that $u \equiv v \pmod{\ell}$, we have that $F_\ell = q^{\ell M(1 + \cdots)}$ for some $M$ and therefore $\sum_{n \geq 0} D(\ell n)q^n \equiv 0 \pmod{\ell}$.

The case $u \neq v$ and $s - r < 2N_0$ is more subtle, but luckily the proof of Theorem 1.2 in [15] works just as well in our situation. For the remainder of Sect. 6, we assume that $u \neq v \pmod{\ell}$ and $s - r < 2N_0$. We start by stating Proposition 5.1 of [15].

**Proposition 5** (Proposition 5.1 of [15]) Let $\ell \geq 5$ be prime and $N \geq 4, \ell \nmid N$. Suppose that $f(z) \in M_k(\Gamma_1(N))$ has $\ell$-integral Fourier coefficients, $w_\ell(f(z)) \neq 0 \pmod{\ell}$, and $\theta f \neq 0 \pmod{\ell}$. Suppose further that $w_\ell(\theta^{m}f(z)) \geq w_\ell(f(z))$. Let $i_1 < i_2 < \cdots < i_r$ be those $i \in \{0, 1, \ldots, \ell - 1\}$ for which $w_\ell(\theta^{i}f) \equiv 0 \pmod{\ell}$. Write $w_\ell(\theta^{i_1+1}f) = w_\ell(\theta^{i}f) + (\ell + 1) - s_j(\ell - 1)$. Write $k = w_\ell(f)$ and let $k_0 \in \{1, \ldots, \ell - 1\}$ be such that $k \equiv -k_0 \pmod{\ell}$. Then one of the four cases below holds:

1. $k \equiv 1 \pmod{\ell}$, $c = 1$, $i_1 = \ell - 1$, and $s_1 = \ell + 1$.
2. $k \equiv 2 \pmod{\ell}$, $c = 1$, $i_1 = \ell - 2$, and $s_1 = \ell + 1$.
3. $k \not\equiv 1 \pmod{\ell}$, $c = 1, i_1, i_2 = (k_0, \ell - 1)$, and $(s_1, s_2) = (k_0 + 1, \ell - k_0)$.
4. $k \not\equiv 1 \pmod{\ell}$, $c = 2$, $(i_1, i_2) = (k_0, \ell - 2)$, and $(s_1, s_2) = (k_0 + 2, \ell - k_0 - 1)$.

We have $w_\ell(f) = w_\ell(\theta^{i+1}f)$ if and only if case (II) or case (IV) holds.

We proceed by contradiction by assuming $D(\ell n) \equiv 0 \pmod{\ell}$ for all $n$ which implies that $\theta^{i-1}F_\ell \equiv F_\ell \pmod{\ell}$. So by the last statement in Proposition 5, we are in cases (II) or (IV). But in case (II) we have

$$w_\ell(F_\ell) = \frac{(s - r)(\ell^2 - 1)}{2} + 12\ell^t \equiv 2 \pmod{\ell} \iff r - s \equiv 4 \pmod{\ell},$$

which contradicts that $\ell > s - r + 4$. So we are in case (IV). Here, we have

$$k_0 \equiv -k \equiv \frac{s - r}{2} \pmod{\ell}.$$

Using Lemma 2, the identity $w_\ell(\theta^{i_1+1}f) = w_\ell(\theta^{i}f) + (\ell + 1) - s_j(\ell - 1)$ with $s_j = s_1 = k_0 + 2$ becomes

$$w_\ell(\theta^{k_0+1}F_\ell) = w_\ell(F_\ell) + (\ell + 1)(k_0 + 1) - (k_0 + 2)(\ell - 1)$$

$$= w_\ell(F_\ell) + 2k_0 + 3 - \ell. \quad (8)$$

Now, $2k_0 \equiv s - r \pmod{\ell}$. Since $s - r$ is even and $\ell > s - r \in \mathbb{N}_0$, we must have that

$$k_0 = \frac{s - r}{2}. $$

Thus, (8) becomes

$$w_\ell(F_\ell) + s - r + 3 - \ell.$$

But $\ell > s - r + 3$, so this implies that

$$w_\ell(\theta^{k_0+1}F_\ell) = w_\ell(F_\ell) + s - r + 3 - \ell < w_\ell(F_\ell),$$

contradicting Proposition 3. This concludes the proof of Theorem 3.
7 Proof of Proposition 1

In this section we use modular forms of half-integral weight. See [13] for background. We follow in the same theme as Ono [12]. We first define the eta-quotient

\[ f(z) := \frac{\eta(3z)}{\eta(2z)\eta(z)} \eta^{12}(80z) = \sum_{m \geq 0} b(m)q^m. \]

Using the appropriate theorems for eta-quotients (see for example [13, Theorem 1.64]), one can see that \( f \in M_{11/2}(\Gamma_1(1440)). \) To make this holomorphic at the cusps, we introduce the factor \( \eta(5z)/\eta(5z). \) We make the following definition:

\[ F(z) := f(z) \frac{\eta^5(z)}{\eta(5z)} = \sum_{m=0}^{\infty} c(m)q^m \in S_{15/2}(\Gamma_1(1440)). \]

Notice \( \eta^5(z)/\eta(5z) \equiv 1 \pmod{5}. \) So, \( F(z) \equiv f(z) \pmod{5}. \) Also, if \( a(n) \) for \( n \geq 0 \) are the Fourier coefficients of \( \eta^{12}(80z)/q^{40} \) then we have \( \eta^{12}(80z)/q^{40} = 1 + \sum_{m>0} a(80m)q^{80m}. \) Thus, if we can show that

\[ b(80n) \equiv 0 \pmod{5} \text{ for all } n \geq 1, \]  

then (2) holds. Since 80 \( \mid \) 1440, we have

\[ F(z) \mid U_{80} = \sum_{n=0}^{\infty} c(80n)q^n \in S_{15/2}(\Gamma_1(1440)). \]

Recall that \( b(80n) \equiv c(80n) \pmod{5}. \) By Sturm’s Criterion, we need to verify that \( c(80n) \equiv 0 \pmod{5} \) for \( 0 \leq n \leq \frac{15}{24}[\text{SL}_2(\mathbb{Z}) : \Gamma_0(1440)] + 1 = 2161. \) We do this by using SAGE and computing the series \( \sum_{m \geq 0} b(m)q^m \) up to \( 80 \cdot 2161 = 172,880 \) terms and then focusing on the coefficients whose index is divisible by 80. This proves the first congruence.

The second congruence is proved in a nearly identical fashion. First, we define the eta-quotient

\[ g(z) := \frac{\eta(3z)}{\eta(2z)\eta(z)} \eta^6(80z) = \sum_{m \geq 0} \beta(m)q^m \in M_{11/2}(\Gamma_1(2880)). \]

Define

\[ G(z) := g(z) \frac{\eta^5(z)}{\eta(5z)} \in S_{9/2}(\Gamma_1(2880)). \]

We notice that \( G(z) \equiv g(z) \pmod{5}. \) Also, letting \( a(n) \) for \( n \geq 0 \) be the Fourier coefficients of \( \eta^6(80z)/q^{20} \), we have \( \eta^6(80z)/q^{20} = q + \sum_{m>0} a(80m)q^{80m}. \) Thus, if we can show that

\[ \beta(80n) \equiv 0 \pmod{5} \text{ for all } n \geq 1, \]

then (3) holds. Since 80 \( \mid \) 2880, then we have

\[ G(z) \mid U_{80} = \sum_{n=0}^{\infty} \gamma(80n)q^n \in S_{9/2}(\Gamma_1(2880)). \]
Hence, \( \beta(80n) \equiv \gamma(80n) \pmod{5} \). By Sturm’s Criterion, we need to verify that \( \gamma(80n) \equiv 0 \pmod{5} \) for \( 0 \leq n \leq 9/24[SL_2(\mathbb{Z}) : \Gamma_0(2880)] + 1 = 2593 \). We do this by using SAGE and computing the series \( \sum_{m \geq 0} \beta(m)q^m \) up to \( 80 \cdot 2593 = 207440 \) terms and then focus on the coefficients whose index is divisible by 80. This completes the proof.

### 8 Example of Proposition 2

In this section, we give an explicit example of Proposition 3 by following the steps outlined in the proof. Let \( \ell = 7 \) and

\[
h(z) = \frac{\eta(3z)}{\eta(2z)\eta(z)}.
\]

The corresponding \( F_7 \) as defined in (6) is

\[
F_7(z) = \Delta^{49}(z) \left( \frac{\eta(z)\eta(2z)}{\eta(3z)} \right)^{48},
\]

which is a holomorphic modular form for \( \Gamma_1(6) \) of weight 612. Note that \( [\Gamma_1(6) : SL_2(\mathbb{Z})] = 24 \). We need to find a set of coset representatives for \( \Gamma_1(6) \) in \( SL_2(\mathbb{Z}) \). The following lemma will make this easier.

**Lemma 5** Let \( \gamma, \gamma' \in SL_2(\mathbb{Z}) \), and for \( (x, y) \in (\mathbb{Z}/N\mathbb{Z})^2 \) let \( |(x, y)| \) denote the order of \( (x, y) \) in \( (\mathbb{Z}/N\mathbb{Z})^2 \). We have

\[
\Gamma_1(N)\gamma = \Gamma_1(N)\gamma' \iff |(c, d)| = |(c', d')| = N \text{ and } (c, d) \neq (c', d') \text{ in } (\mathbb{Z}/N\mathbb{Z})^2,
\]

where \( \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) and \( \gamma' = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \).

**Remark 6** One needs to be slightly careful when applying the above lemma. For instance, when \( N = 6 \), \( |(5, 5)| = 6 \) but since \( c \) and \( d \) in this case are not relatively prime, then one cannot construct a matrix \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) \). However, \( (5, 5) \equiv (5, -1) \pmod{6} \) and this problem is resolved. The fact that this can always be resolved is Lemma 3.8.4 of [6].

The above lemma states that all we must find is the pairs \( (c, d) \) in \( (\mathbb{Z}/6\mathbb{Z})^2 \) with order 6. Furthermore, by consulting the formulas for computing \( q \)-expansions of eta-quotients in [14] and using the fact that we are raising the eta-quotient to the 48th power, we notice that the \( q \)-expansions are only dependent on these two matrix entries. There are 24 such elements of \( (\mathbb{Z}/6\mathbb{Z})^2 \); however, not all of these elements produce unique \( q \)-expansions. This is because \( (c, d) \) and \((-c, -d)\) give rise to the same expansion and are not equivalent because \( 3 \nmid \gcd(c, d) = 1 \). Thus, there are twelve expansions to compute. For example,

\[
F_7(z) \mid \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \Delta^{51}(z) \frac{1}{2\pi i} q^{\frac{1}{2}} \left( \prod_{n=1}^{\infty} (1 - (-q^{1/2})^n) \right)^{48} 3^{24} \zeta_3 q^{-2/3} \left( \prod_{n=1}^{\infty} (1 - (\zeta_3 q^{1/3})^n) \right)^{-48}.
\]

By multiplying all of these expansions together, we obtain a formula for \( G \) as in the proof of Proposition 3,

\[
G(z) = \frac{3^{432} \Delta^{1224}(z)\Delta^{16}(2z)}{2^{384} q^4} \prod_{n=1}^{\infty} \frac{(1 - q^{n/2})^{384}(1 - (-q^{1/2})^n)^{384}}{(1 - q^{n/3})^{288}(1 - (\zeta_3 q^{1/3})^n)^{288}(1 - (\zeta_5 q^{1/3})^n)^{288}}.
\]
which (as expected) is equal to $C \Delta^{1224}(z)$ where $C = \frac{3432}{2384}$. We will also show that $H$ (as in the proof of Proposition 3 with $m = 1$) does not vanish mod 7. To accomplish this, we use the fact $\theta(F \mid M) \equiv (\theta F) \mid M \pmod{7}$ which comes from the proof of Lemma 4.2 in [15]. Using SAGE, we obtain

$$H \equiv 2q^{1252} + 5q^{1253} + q^{1254} + 2q^{1255} + 2q^{1256} + \cdots \pmod{7}$$

which is clearly not equivalent to 0. We now see that $\tilde{\text{ord}}_{\infty}(H) = 1252 \geq 1224$.

**Data availability** Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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**References**

1. Ahlgren, S., Boylan, M.: Arithmetic properties of the partition function. Invent. Math. 153, 487–502 (2003)
2. Bringmann, K., Dousse, J., Lovejoy, J., Mahlburg, K.: Overpartitions with restricted odd differences. Electron. J. Comb. 22(3), 16 (2015)
3. Chem, S., Hao, L.: Congruences for two restricted overpartitions. Proc. Math. Sci. 129, 1–16 (2019)
4. Dewar, M.: Non-existence of Ramanujan congruences in modular forms of level four. Can. J. Math. 63(6), 1284–1306 (2011)
5. Diamond, F., Im, J.: Modular forms and modular curves. In: Seminar on Fermat’s Last Theorem (Toronto, ON, 1993–1994). CMS Conference Proceedings, vol. 17, pp. 39–133. American Mathematical Society, Providence, RI (1995)
6. Diamond, F., Shurman, J.: A First Course in Modular Forms, 1st edn. Springer, Berlin (2005)
7. Gross, B.: A tameness criterion for Galois representations associated to modular forms (mod p). Duke Math. J. 61, 445–517 (1990)
8. Hirschhorn, M.D., Sellers, J.A.: Congruences for overpartitions with restricted odd differences. Ramanujan J. (2019)
9. Kiming, I., Olsson, J.B.: Congruences like Ramanujan’s for powers of the partition function. Arch. Math. 59, 348–360 (1992)
10. Lin, B.L.S., Liu, J., Wang, A.Y.Z., Xiao, J.: Infinite families of congruences for overpartitions with restricted odd differences. Bull. Aust. Math. Soc. 102(1), 59–66 (2020)
11. Naika, M.S.M., Greesh, D.S.: Congruences for overpartitions with restricted odd differences. Afrika Matematika 30, 1–21 (2019)
12. Ono, K.: Congruences for Frobenius partitions. J. Number Theory 57, 170–180 (1996)
13. Ono, K.: The web of modularity: arithmetic of the coefficients of modular forms and $q$-series. In: CBMS Regional Conference Series in Mathematics, vol. 102. Published for the Conference Board of the Mathematical Sciences, Washington, DC. American Mathematical Society, Providence (2004)
14. Ryan, N.C., Scherr, Z., Sirelli, N., Treeneer, S.: Congruences Satisfied by Eta-quotients. Number Theory (2019)
15. Snicker, J.: Ramanujan congruences for a class of eta quotients. Int. J. Number Theory 6(4), 835–847 (2010)
16. Swinnerton-Dyer, H.P.F.: On $l$-adic representations and congruences for coefficients of modular forms. II. In: Modular Functions of One Variable, V (Proceedings of Second International Conference, University of Bonn, Bonn, 1976). Lecture Notes in Mathematics, vol. 601, pp. 63–90. Springer, Berlin (1977)

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