The Average Singular Value of a Complex Random Matrix Decreases with Dimension

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Abstract. We prove that the average singular value $\alpha(d)$ of a complex valued $d \times d$ matrix $\frac{1}{\sqrt{d}} X$ with random i.i.d., $N(0,1)$ entries, decreases monotonically with $d$ to the limit given by the Marchenko-Pastur distribution. The result follows from a simple recurrence relation for $\alpha(d)$ which may have independent interest. The monotonicity of $\alpha(d)$ has been recently conjectured by Bandeira, Kennedy and Singer in their study of the Little Grothendieck problem over the unitary group $U_d$ [3], a combinatorial optimization problem. The proof of the recurrence relation involves a connection with the theory of Turán determinants. We also discuss some applications to the problem that originally motivated the conjecture.

1. Introduction

Let $X$ be a $d \times d$ complex random matrix with random i.i.d. entries $N(0,1)$. Let

$$\sigma_k \left( \frac{1}{\sqrt{d}} X \right)$$

denote the $k$th singular value of $\frac{1}{\sqrt{d}} X$. Then the average singular value of $\frac{1}{\sqrt{d}} X$ is denoted by $\alpha_C(d)$ and given by

$$\alpha_C(d) = \mathbb{E} \left[ \frac{1}{d} \sum_{k=1}^{d} \sigma_k \left( \frac{1}{\sqrt{d}} X \right) \right].$$

The following is our main result and answers in the positive the complex case of Conjecture 8 in [3].

Theorem 1. Let $\alpha_C(d)$ be the average singular value of a complex valued $d \times d$ matrix $\frac{1}{\sqrt{d}} X$, with random i.i.d. $N(0,1)$ entries. Then, for all $d > 1$,

$$\alpha_C(d + 1) \leq \alpha_C(d).$$

As in [3], one can explicitly compute

$$\alpha_C(1) = \sqrt{\frac{\pi}{4}}$$

and use the Marchenko-Pastur distribution [5] with density

$$mp(x) = \frac{1}{2\pi x} \sqrt{x(4-x)} \mathbf{1}_{[0,4]},$$

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which gives the limit distribution of the eigenvalues of a Wishart matrix, in order to obtain
\[
\lim_{d \to \infty} \alpha_C (d) = \int_0^4 \sqrt{x} \frac{1}{2\pi x} \sqrt{x(4-x)} = \frac{8}{3\pi},
\]
leading to the following optimal global estimates for the singular values. The lower bound provides the optimal estimate for the complex case in [3 Theorem 7].

**Corollary 1.** Let \( \alpha_C (d) \) be the average singular value of a complex valued \( d \times d \) matrix \( \frac{1}{\sqrt{d}} X \), with random i.i.d., \( N(0, 1) \) entries. Then, for all \( d \geq 1 \),
\[
\frac{8}{3\pi} < \alpha_C (d) \leq \sqrt{\frac{\pi}{4}}.
\]

The singular values of \( \frac{1}{\sqrt{d}} X \) are the square-roots of the eigenvalues of the Wishart matrix \( \frac{1}{d} W = \frac{1}{d} XX^T \),
\[
\sigma_k \left( \frac{1}{\sqrt{d}} X \right) = \sqrt{\lambda_k \left( \frac{1}{d} W \right)}
\]
which are known to be distributed according to the Laguerre ensemble of order 0. More precisely, the eigenvalue distribution of such eigenvalues in the bulk of the spectrum is given by the first-order marginal [3]:
\[
P(x) = \frac{1}{d} e^{-x} \sum_{n=0}^{d-1} [L_n(x)]^2
\]
where \( L_n(x) = L_n^{(0)}(x) \) is the Laguerre polynomial, defined, for general parameter \( \alpha \), as
\[
L_n^\alpha(x) = \sum_{k=0}^{n} (-1)^k \binom{n + \alpha}{n - k} \frac{x^k}{k!}
\]
By combining the above definitions, one can write \( \alpha_C (d) \) explicitly in terms of Laguerre polynomials:
\[
\alpha_C (d) = \frac{1}{d} \sum_{n=0}^{d-1} \frac{1}{d^{2}} \int_0^\infty x^{\frac{3}{2}} e^{-x} [L_n(x)]^2 \, dx
\]
The core of the proof is the following Proposition, which reduces the proof of (1.1) to the estimation of two simple integrals involving Laguerre polynomials. First define the constants
\[
\delta_d = (d + 1)^{-\frac{1}{2}} - d^{-\frac{1}{2}} < 0,
\]
\[
\tilde{\delta}_d = (d + 1)^{-\frac{1}{2}} - d^{-\frac{1}{2}} < 0,
\]
and observe that
\[
\delta_d - \tilde{\delta}_d \leq 0.
\]
The key ingredient of the proof is the following recurrence relation for the average singular value as a function of dimension, which has independent interest, since it offers a computational method for evaluating \( \alpha_C (d) \):
Proposition 1. Let $\alpha_C(d)$ be the average singular value of a complex valued $d \times d$ matrix $\frac{1}{\sqrt{d}} X$, with random i.i.d., $N(0,1)$ entries. Then, for all $d \geq 1$,

$$
\alpha_C(d + 1) = \alpha_C(d) + \int_0^\infty x^{\frac{1}{2}} e^{-x} \left[ \delta_d(L_d(x))^2 - \tilde{\delta}_d L_d^{(1)}(x)L_{d-1}^{(1)}(x) \right] dx.
$$

The main insight leading to the proof of Proposition 1 is a connection to the theory of Turán determinants of orthogonal polynomials. The Turán determinant of a family of sequences $\{P_n\}$, of is an expression of the form:

$$
\begin{vmatrix}
P_n & P_{n-1} \\
P_{n+1} & P_n
\end{vmatrix} = P_n^2 - P_{n+1}P_{n-1}.
$$

Turán determinants of orthogonal polynomials were first studied by Turán [8]. In [7] Szegö found a manifold of connections with other mathematical areas. Our proof of Proposition 1 starts with an idea of Szász [6] to construct a recurrence relation for such determinants. We will see that the eigenvalues intensity (1.2) can be related to a Turán determinant in terms of the Laguerre polynomials $L_d^{(1)}(x)$.

2. Proof of Theorem 1

We rewrite (1.3) as

$$
\alpha_C(d) = \frac{1}{d^{\frac{3}{2}}} \int_0^\infty x^{\frac{1}{2}} e^{-x} \sum_{n=0}^{d-1} [L_n(x)]^2 dx
$$

and define

$$
\Delta_d(x) = \frac{1}{d^{\frac{3}{2}}} \sum_{n=0}^{d-1} [L_n(x)]^2.
$$

so that

$$
\alpha_C(d) = \int_0^\infty x^{\frac{1}{2}} e^{-x} \Delta_d(x) dx.
$$

In view of Proposition 1, (1.1) is equivalent to

$$
(2.1) \quad \delta_d \int_0^\infty x^{\frac{1}{2}} e^{-x} [L_d(x)]^2 dx + \tilde{\delta}_d \int_0^\infty x^{\frac{3}{2}} e^{-x} L_d^{(1)}(x)L_{d-1}^{(1)}(x) dx \leq 0.
$$

To complete the proof, we show that (2.1) holds, resorting to the following Lemma, whose proof is given in the last section.

Lemma 1.

$$
\int_0^\infty x^{3/2} e^{-x} L_d^{(1)}(x)L_{d-1}^{(1)}(x) dx \leq \frac{3d}{4\pi(d - 3/2)(d - 1/2)^{3/2}}.
$$
We already know from the proof of Lemma 19 in [3] that, for $d \geq 2$,
\[
\int_0^\infty x^\frac{d}{2} e^{-x} [L_d(x)]^2 \, dx \geq (d + 1)^\frac{1}{2}.
\]
Combining this estimate with Proposition 1 and Lemma 1, yields:
\[
\alpha_C (d + 1) - \alpha_C (d) = \delta_d \int_0^\infty x^\frac{d}{2} e^{-x} [L_d(x)]^2 \, dx + \tilde{\delta}_d \int_0^\infty x^\frac{d}{2} e^{-x} L_d^{(1)}(x)L_{d-1}^{(1)}(x) \, dx
\]
\[
\leq \delta_d (d + 1)^\frac{1}{2} - \tilde{\delta}_d \frac{3d}{4 \pi (d - 3/2)(d - 1/2)^{3/2}}
\]
\[
\leq (\delta_d - \tilde{\delta}_d)(d + 1)^\frac{1}{2}
\]
\[
\leq 0.
\]
Using $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(5/2) = \frac{3}{4}\sqrt{\pi}$.

3. Proof of Proposition 1

The key observation is to recognize $d^{\frac{1}{2}} \Delta_d(x)$ as a Turán determinant for $L_d^{(1)}(x)$. We start by using the Christoffel-Darboux formula, valid for any sequence of orthogonal polynomials [2, Theorem 5.2.4]:
\[
d^{\frac{1}{2}} \Delta_d(x) = \frac{1}{d} \sum_{n=0}^{d-1} L_n(x)L_n(y) = \frac{L_{d-1}(x)L_d(y) - L_d(x)L_{d-1}(y)}{x - y}.
\]
Setting $x = y$ in (3.1) we obtain
\[
d^{\frac{1}{2}} \Delta_d(x) = \frac{1}{d} \sum_{n=0}^{d-1} [L_n(x)]^2 = L_{d-1}(x)\frac{d}{dx}L_d(x) - L_d(x)\frac{d}{dx}L_{d-1}(x).
\]
Using the formulas [2] (6.2.16), (6.2.18]
\[
\left\{ \begin{array}{l}
\frac{d}{dx}L_d(x) = -L_d^{(1)}(x) \\
L_d(x) = L_d^{(1)}(x) - L_{d-1}^{(1)}(x)
\end{array} \right. \)
then $d^{\frac{1}{2}} \Delta_d(x)$ becomes a Turán determinant for $L_d^{(1)}(x)$, the Laguerre polynomials of parameter $\alpha = 1$:
\[
d^{\frac{1}{2}} \Delta_d(x) = \left[ L_{d-1}^{(1)}(x) \right]^2 - L_{d-2}^{(1)}(x)L_d^{(1)}(x).
\]
Then we use the following observation, after Otto Szász [6]: one can use the recurrence formula [2, (6.2.5)]
\[
L_{d+1}^{(1)}(x) = \left( 2 - \frac{x}{d+1} \right) L_d^{(1)}(x) - L_{d-1}^{(1)}(x),
\]
twice, one to eliminate $L_{d+1}^{(1)}$ from $\Delta_{d+1}(x)$ and another to eliminate $L_{d-2}^{(1)}$ from $\Delta_d(x)$. We first remove $L_{d+1}^{(1)}$ from $\Delta_{d+1}(x)$, leading to

\begin{equation}
(d + 1)^2 \Delta_{d+1}(x) = \left[ L_d^{(1)}(x) \right]^2 + \left[ L_{d-1}^{(1)}(x) \right]^2 - \left( 2 - \frac{x}{d+1} \right) \left( 2 - \frac{x}{d} \right) L_d^{(1)}(x)L_{d-1}^{(1)}(x);
\end{equation}

then, we remove $L_{d-2}^{(1)}$ from $\Delta_d(x)$, after replacing $d \to d - 1$ in (3.4) and using the resulting recurrence formula. This leads to

\begin{equation}
d^2 \Delta_d(x) = \left[ L_d^{(1)}(x) \right]^2 + \left[ L_{d-1}^{(1)}(x) \right]^2 - \left( 2 - \frac{x}{d} \right) L_d^{(1)}(x)L_{d-1}^{(1)}(x).
\end{equation}

Subtracting (3.6) from (3.5),

\begin{equation}
\Delta_{d+1}(x) - \Delta_d(x) = \left( (d + 1)^{-\frac{1}{2}} - d^{-\frac{1}{2}} \right) \left( \left[ L_d^{(1)}(x) \right]^2 + \left[ L_{d-1}^{(1)}(x) \right]^2 \right) - \left[ (d + 1)^{-\frac{1}{2}} \right] \left( 2 - \frac{x}{d+1} \right) - d^{-\frac{1}{2}} \left( 2 - \frac{x}{d} \right)
\end{equation}

Now recognizing the factor $\left( L_d^{(1)}(x) - L_{d-1}^{(1)}(x) \right)^2$, (3.7)-(3.8) can be rewritten as

\begin{equation}
\Delta_{d+1}(x) - \Delta_d(x) = \delta_d \left( \left[ L_d^{(1)}(x) \right]^2 + \left[ L_{d-1}^{(1)}(x) \right]^2 \right) - 2\delta_dL_d^{(1)}(x)L_{d-1}^{(1)}(x)
\end{equation}

\begin{equation}
- \left( d^{-\frac{3}{2}} - (d+1)^{-\frac{3}{2}} \right) xL_d^{(1)}(x)L_{d-1}^{(1)}(x)
\end{equation}

\begin{equation}
= \delta_d \left( L_d^{(1)}(x) - L_{d-1}^{(1)}(x) \right)^2 + \tilde{\delta}_d xL_d^{(1)}(x)L_{d-1}^{(1)}(x),
\end{equation}

taking into account the definitions of the constants $\delta_d, \tilde{\delta}_d$. Then, using (3.2) again, we obtain the desired simple recurrence relation for $\Delta_d(x)$:

\begin{equation}
\Delta_{d+1}(x) - \Delta_d(x) = \delta_d \left[ L_d(x) \right]^2 + \tilde{\delta}_d xL_d^{(1)}(x)L_{d-1}^{(1)}(x).
\end{equation}

Finally we use

\begin{equation}
\alpha_C(d + 1) - \alpha_C(d) = \int_0^\infty x^\frac{1}{2} e^{-x} \left[ \Delta_{d+1}(x) - \Delta_d(x) \right] dx.
\end{equation}

to see that (3.9) implies the equivalence between (1.1) and (2.1).

4. Application in combinatorial optimization

Theorem 1 has an immediate application to the little Grothendieck problem over the unitary group $U_d$ ($U \in U_d$ if and only if $UU^H = U^H U = I_{d \times d}$), the problem considered in [3] that originally motivated the conjecture: given $C \in \mathbb{C}^{dn \times dn}$ a complex valued semidefinite matrix, find

\[
\max_{U_1, \ldots, U_n \in U_d} \sum_{i=1}^n \sum_{j=1}^n \text{tr} \left( C_{ij} U_i U_j^H \right).
\]
In [3], an algorithm called Orthogonal-Cut for solving the above problem is presented, together with the following bounds.

**Theorem 2.** [3] Let $C \succeq 0$ and complex. If $W_1, ..., W_n \in U_d$ are the random output of the Unitary version of the Orthogonal-cut algorithm, then

$$
E \left[ \sum_{i=1}^{n} \sum_{j=1}^{n} \text{tr} \left( C_{ij} W_i W_j^H \right) \right] \geq \alpha_C (d)^2 \max_{U_1, ..., U_n \in U_d} \sum_{i=1}^{n} \sum_{j=1}^{n} \text{tr} \left( C_{ij}^H U_i U_j^H \right).
$$

**Remark 1.** Combining this with Corollary 1 leads to the $d$-independent inequality

$$
E \left[ \sum_{i=1}^{n} \sum_{j=1}^{n} \text{tr} \left( C_{ij} W_i W_j^H \right) \right] \geq \left( \frac{8}{3\pi} \right)^2 \max_{U_1, ..., U_n \in U_d} \sum_{i=1}^{n} \sum_{j=1}^{n} \text{tr} \left( C_{ij}^H U_i U_j^H \right).
$$

**Remark 2.** Theorem 1 shows, as suggested in [3], that the approximation ratio in the algorithm gets worse as the dimension increases.

5. **Proof of Lemma 1**

We start with the formula [9, formula (16), p. 330], valid for $p, \alpha, \beta > -1$:

$$
\int_{0}^{\infty} x^p e^{-x} L_m^{(\alpha)}(x) L_n^{(\beta)}(x) dx = \frac{\Gamma(p+1)}{\min\{m,n\}} \sum_{i=0}^{\min\{m,n\}} (-1)^{m+n} \binom{p-\alpha}{m-i} \binom{p-\beta}{n-i} \binom{p+i}{i}.
$$

Formula (5.1) can be written in terms of a hypergeometric function $\mathbf{3F2}$ as follows

$$
\int_{0}^{\infty} x^p e^{-x} L_m^{(\alpha)}(x) L_n^{(\beta)}(x) dx = \frac{\Gamma(p+1)}{\min\{m,n\}} \sum_{n=0}^{d-1} \binom{(a_1)_n (a_2)_n ... (a_p)_n}{(n+1)! (b_1)_n (b_2)_n ... (b_q)_n} \frac{z^n}{n!}.
$$

We find it more convenient to use the hypergeometric notation (see [2] for general definitions and properties of hypergeometric functions):

$$
_\mathbf{pFq} \left( \frac{a_1, a_2, ..., a_p}{b_1, ..., b_q} ; z \right) = \sum_{n=0}^{d-1} \frac{(a_1)_n (a_2)_n ... (a_p)_n}{(n+1)! (b_1)_n (b_2)_n ... (b_q)_n} \frac{z^n}{n!}.
$$

and setting $\alpha = \beta = 1, p = 3/2, n = d$ and $m = d - 1$ gives, using

$$
\binom{a}{b} = \frac{\Gamma(a+1)}{\Gamma(b+1) \Gamma(a-b+1)},
$$

$$
\int_{0}^{\infty} x^p e^{-x} L_m^{(\alpha)}(x) L_n^{(\beta)}(x) dx = \frac{\Gamma(p+1)}{\min\{m,n\}} \sum_{n=0}^{d-1} \binom{(a_1)_n (a_2)_n ... (a_p)_n}{(n+1)! (b_1)_n (b_2)_n ... (b_q)_n} \frac{z^n}{n!}.
$$
\[
\int_0^\infty x^{3/2}e^{-x}L_d^{(1)}(x)L_{d-1}^{(1)}(x)dx = \left( \frac{d}{d-1} \right) \left( \frac{d-3/2}{d} \right) \Gamma \left( \frac{5}{2} \right) \ _3F_2 \left( \frac{-d-1,5/2,3/2}{2,3/2-d} ; 1 \right) = \frac{\Gamma \left( \frac{5}{2} \right) \Gamma \left( \frac{d-1}{2} \right)}{\Gamma(d)} \sum_{n=0}^{d-1} \frac{(-d-1)_n(5/2)_n(3/2)_n}{(n+1)!(3/2-d)n!}
\]

Now, recall the formulas [1, page 255]:

\[
(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}
\]

\[
\frac{1}{\sqrt{n}} \leq \frac{\Gamma(n)}{\Gamma(n+1/2)} \leq \frac{1}{\sqrt{n-1/2}}
\]

\[
\sqrt{n} \leq \frac{\Gamma(n+1)}{\Gamma(n+1/2)} \leq \sqrt{n+1/2}.
\]

From the second one we have

\[
(5.2) \quad \frac{(5/2)_n}{(n+1)!} = \frac{\Gamma(n+5/2)}{\Gamma(5/2)\Gamma(n+2)} \geq \frac{\sqrt{n+3/2}}{\Gamma(5/2)}
\]

and

\[
(5.3) \quad \frac{(5/2)_n}{(n+1)!} \leq \frac{\sqrt{n+2}}{\Gamma(5/2)}
\]

We need an upper bound for the following series

\[
\int_0^\infty x^{3/2}e^{-x}L_d^{(1)}(x)L_{d-1}^{(1)}(x)dx = \frac{\Gamma(5/2)\Gamma(d-1/2)}{\Gamma(1/2)\Gamma(d)} \sum_{n=0}^{d-1} \frac{(-d-1)_n(5/2)_n(3/2)_n}{(n+1)!(3/2-d)_nn!}
\]

Now, we can bound the negative terms (the last two) of the series using (5.2) and the positive ones using (5.3):

\[
= \sum_{n=0}^{d-3} \frac{(-d-1)_n(5/2)_n(3/2)_n}{(n+1)!(3/2-d)_nn!} + \sum_{n=d-2}^{d-1} \frac{(-d-1)_n(5/2)_n(3/2)_n}{(n+1)!(3/2-d)_nn!}
\]

\[
\leq \frac{\sqrt{d-1}}{\Gamma(5/2)} \sum_{n=0}^{d-3} \frac{(-d-1)_n(3/2)_n}{(3/2-d)_nn!} + \frac{\sqrt{d-1/2}}{\Gamma(5/2)} \sum_{n=d-2}^{d-1} \frac{(-d-1)_n(3/2)_n}{(3/2-d)_nn!}
\]

\[
\leq \frac{\sqrt{d-1/2}}{\Gamma(5/2)} \sum_{n=0}^{d-3} \frac{(-d-1)_n(3/2)_n}{(3/2-d)_nn!} + \frac{\sqrt{d-1/2}}{\Gamma(5/2)} \sum_{n=d-2}^{d-1} \frac{(-d-1)_n(3/2)_n}{(3/2-d)_nn!}
\]

\[
= \frac{\sqrt{d-1/2}}{\Gamma(5/2)} \ _2F_1 \left( -d-1,3/2,3/2-d; 1 \right).
\]
Now we sum the $2F_1$ using Gauss formula:

$$2F_1\left(\begin{array}{c} a, b \\ c \end{array}; 1 \right) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}.$$ 

and obtain the estimate

$$\int_0^\infty x^{3/2}e^{-x}L_d^{(1)}(x)L_{d-1}^{(1)}(x)dx \leq \frac{\sqrt{d - 1/2}}{(d - 1/2)\Gamma(5/2)} \frac{\Gamma(3/2 - d)\Gamma(1)}{\Gamma(5/2)\Gamma(-d)}.$$ 

Finally,

$$\frac{\Gamma(d - 1/2)}{\Gamma(d)} = \frac{d}{d - 1/2} \frac{\Gamma(d + 2)}{\Gamma(d + 1)} \leq \frac{d}{(d - 1/2)\sqrt{d}} = \frac{\sqrt{d}}{d - 1/2}.$$ 

and

$$\frac{\Gamma(3/2 - d)}{\Gamma(-d)} = \frac{d}{(d - 1/2)(d - 3/2)} \frac{\Gamma(d + 2)}{\Gamma(d + 1)} \leq \frac{d}{(d - 1/2)(d - 3/2)\sqrt{d}} \leq \frac{\sqrt{d}}{(d - 1/2)(d - 3/2)}.$$ 

Combining this with (5.4) gives

$$\int_0^\infty x^{3/2}e^{-x}L_d^{(1)}(x)L_{d-1}^{(1)}(x)dx = \frac{\Gamma(5/2).\Gamma(d - 1/2)}{\Gamma(5/2)\Gamma(1/2)} \frac{\Gamma(3/2 - d)}{\Gamma(5/2)\Gamma(-d)} \leq \frac{1}{\sqrt{d - 1/2}} \frac{\sqrt{d}}{(d - 1/2)(d - 3/2)\Gamma(1/2)} \Gamma(-d) \frac{\sqrt{d}}{\Gamma(5/2)\Gamma(d)} \leq \frac{1}{\sqrt{d - 1/2}} \frac{\sqrt{d}}{(d - 1/2)(d - 3/2)\Gamma(1/2)} \frac{\sqrt{d}}{(d - 1/2)(d - 3/2)\Gamma(1/2)} \frac{\sqrt{d}}{\Gamma(5/2)\Gamma(d)} \leq \frac{d}{4\pi(d - 3/2)(d - 1/2)^{3/2}}.$$ 

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REFERENCES

[1] M. Abramowitz, I. A. Stegun, *Handbook of Mathematical functions with Formulas, Graphs and Mathematical Tables*. Dover, New York, 1964.

[2] R. Askey, G. Andrews, R. Roy, *Special Functions*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1999, XVI+664 pp.

[3] A. S. Bandeira, C. Kennedy, A. Singer, *Approximating the Little Grothendieck Problem over the Orthogonal and Unitary Groups*, Math. Programming Series A, 2016, online published.

[4] O. Leveque, *Random matrices and communication systems: Wishart random matrices: marginal eigenvalue distribution* (online available).

[5] V. A. Marchenko, L. A. Pastur, *Distribution of eigenvalues for some sets of random matrices*, Mat. Sb. (N.S.), 72, 507-536.

[6] O. Szász, *Identities and inequalities concerning orthogonal polynomials and Bessel functions*, J. Analyse Math. (1951) 116-134.

[7] G. Szego, *On an inequality of P. Turan concerning Legendre Polynomials*, Bull. Amer. Math. Soc. 54 (1948), 401-405.

[8] P. Turán, *On the zeros of the polynomials of Legendre*, Casopis pro Pestorani Matematik y a Fysiky 75 (1950), 113 122.

[9] Z. X. Wang, D. R. Guo, *Special Functions*, World Scientific, Singapore (1989).

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