Logarithmic spreading of out-of-time-ordered correlators without many-body localization

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Out-of-time-ordered correlators (OTOCs) describe information scrambling under unitary time evolution, and provide a useful probe of the emergence of quantum chaos. Here we calculate OTOCs for a model of disorder-free localization whose exact solubility allows us to study long-time behaviour in large systems. Remarkably, we observe logarithmic spreading of correlations, qualitatively different to both thermalizing and Anderson localized systems. Rather, such behaviour is normally taken as a signature of many-body localization, so that our findings for an essentially non-interacting model are surprising. We provide an explanation for this unusual behaviour, and suggest a novel Loschmidt echo protocol as a probe of correlation spreading.

Out-of-time-ordered correlators have proven to be useful in quantifying the spreading of correlations and entanglement in many-body quantum dynamics. While originally introduced in the context of quasiclassical approaches to quantum systems [1], recently they have received renewed interest due to their connections with the emergence of quantum chaotic behaviour [2–4]. This has generated a flurry of activity on OTOCs in the studies of entanglement and information scrambling in integrable [5–7], thermalizing [4] and many-body-localized (MBL) systems [8–10], where OTOCs have been calculated for a number of important models including the transverse field Ising model [5], Luttinger-liquids [6], and random unitary circuits [11–14]. Beyond these examples, the calculation of OTOCs is a difficult task. Nonetheless, some generic features of OTOCs are emerging.

We consider OTOCs \( \langle \langle \hat{A}(t), \hat{B} \rangle \rangle = \langle \langle [\hat{A}(t), \hat{B}] \rangle \rangle / 2 \) where \( \hat{A}, \hat{B} \) are two local operators in their Heisenberg representation, and in the following we assume that expectation values are taken with respect to pure quantum states. For local operators, the infinite temperature OTOC is bounded, having a light-cone causality structure and exponentially suppressed correlations outside the light-cone [15]. However, in many paradigmatic models the behaviour differs. For example, in generic thermalizing systems and for random unitary circuits the OTOC saturates to a non-zero value within the light-cone [4], whereas for particular spin components for the transverse-field Ising model the OTOC decays back to zero [5].

Localized systems provide a distinct setting where correlations do not spread linearly. For example, in non-interacting disordered systems correlations (including OTOCs) don’t spread beyond the localization length [8, 9], while in many-body localized systems they extend beyond the localization length, albeit logarithmically slowly [9, 16, 17].

Here we show that the situation in disordered systems is even more complex: in particular, many-body localization is not a necessary prerequisite for logarithmic OTOC spreading. We demonstrate this in a disorder-free localization model that we have introduced recently [18–21], which consists of spinless fermions coupled via a minimal coupling to dynamical Ising spins. Here, the OTOCs are accessible to large scale numerics and unambiguously demonstrate previously unobserved spreading of correlations beyond the localization length in an essentially non-interacting localized system. While some of the OTOCs for the fermions corresponds to those discussed previously in the context of Anderson localization [8, 9], the spin degrees of freedom yield a richer set of correlators, which in fact correspond to novel fermion correlators.

The main results of this paper are threefold: (i). OTOCs in our model can be expressed in terms of a double Loschmidt echo, which can be reduced to disorder-averaged correlators in a model of Anderson localization via a non-linear mapping to free-fermions. (ii). at short times we find power-law growth of the OTOCs, similar to the behaviour found for integrable models, and in contrast to the exponential growth found in semiclassical and large-N models [22–24]. (iii). most remarkably, we observe logarithmic spreading of OTOCs at long times – typically associated with many-body localization. To our knowledge this is the first example of such spreading in a model that has a free-fermions representation. This not only adds a facet to our knowledge of OTOC behavioural types, but also imposes further constraints on their systematic classification.

Setup of the problem. First, we review our model of disorder-free localization [18–20], which is a model of spinless fermions \( \hat{f}_j \) living on lattice sites (here we consider a 1D lattice with open boundary conditions) which are minimally coupled to spin-1/2s, \( \hat{\sigma}_j \), defined on the links between neighbouring sites \( j \) and \( j+1 \). The model is described by the Hamiltonian

\[
\hat{H} = - \sum_{j=1}^{N-1} J_j \hat{\sigma}_j^z (\hat{f}_j^\dagger \hat{f}_{j+1} + \text{H.c.}) - \sum_{j=2}^{N-1} h_j \hat{\sigma}_{j-1}^+ \hat{\sigma}_j^-, \tag{1}
\]

where \( N \) is the number of lattice sites, \( J_j \) and \( h_j \) are the fermion tunnelling amplitude and spin-couplings correspondingly, which we assume to be position independent, \( J_j = J \) and \( h_j = h \).


In the following we study OTOCs for the spins,
\[C_{\alpha\beta}^{\alpha\beta}(t) = \frac{1}{2} \langle \Psi | [\hat{\sigma}_{j}^x(t), \hat{\sigma}_{j}^\alpha(t)] | \Psi \rangle = 1 - \text{Re}[F_{\alpha\beta}^{\alpha\beta}(t)], \tag{2}\]
where \(\alpha, \beta \in \{x, z\}\) and \(|\Psi\rangle\) is an initial state of the spins and \(\hat{\sigma}_{j}^\alpha\) are Pauli operators for the fermions described by the Hamiltonian in Eq. (2) corresponds to a global quantum quench \([25, 26]\). Note that both the Hamiltonian and the initial state are translationally invariant.

**Double Loschmidt echo.** It is instructive to rewrite the correlator \(F_{\alpha\beta}^{\alpha\beta}(t)\) in terms of an average which is similar to a Loschmidt echo. Using standard commutation relations for the spin-operators we have \(\hat{\sigma}_{j}^\alpha e^{i \hbar \hat{H}_t} = e^{i \hbar \hat{H}_t} \hat{\sigma}_{j}^\alpha\), where \(\hat{H}_t\) is the Hamiltonian with redefined couplings \(h_j, h_{j+1} \rightarrow -h_j, -h_{j+1}\) for \(\alpha = z\), and for \(\alpha = x\) we have \(J_j \rightarrow -J_j\). Hence, we can write the correlator in terms of a double Loschmidt echo
\[F_{\alpha\beta}^{\alpha\beta}(t) = \langle \Psi | e^{-i \hbar \hat{H}_t} e^{i \hbar \hat{H}_t} e^{i \hbar \hat{F}_t} e^{-i \hbar \hat{F}_t} | \Psi \rangle. \tag{4}\]
This correlator can be interpreted in a generalized Keldysh formalism with a contour that folds forward and backward in time twice \([27, 28]\).

**Free-fermion mapping.** The calculations of the OTOCs are made possible by an exact mapping to free-fermions \([20]\) resulting from the identification of \(N - 2\) conserved charges \(q_j = \hat{\sigma}_{j-1}^z \hat{\sigma}_{j}^x (1 - \hat{\sigma}_j^x)\), having eigenvalues \(\pm 1\), which commute amongst themselves and with the Hamiltonian of Eq. (1). The mapping proceeds by a duality transformation for the spins \([29]\) defining new spin degrees of freedom \(\hat{\tau}\) on lattice sites via
\[\hat{\tau}_{j}^x = \hat{\sigma}_{j-1}^z \hat{\sigma}_{j}^x, \quad \hat{\tau}_{j}^z = \hat{\sigma}_{j-1}^z \hat{\sigma}_{j}^x \hat{\sigma}_{j+1}^z = \hat{\sigma}_{j}^z. \tag{5}\]
After introducing new fermions, \(\hat{c}_{j} = \hat{\tau}_{j}^x \hat{f}_{j}\), the Hamiltonian acting in each of the charge sectors, labelled by \([q_j] = \pm 1\), can be written as
\[\hat{H}(q) = -\sum_{j} J_{j} (\hat{c}_{j}^\dagger \hat{c}_{j+1} + \text{h.c.}) + 2h_{j} q_{j} (\hat{c}_{j}^\dagger \hat{c}_{j} - 1/2). \tag{6}\]
For a given charge configuration this Hamiltonian corresponds to a tight-binding model with a binary potential.

In terms of conserved charges, our initial state assumes the form \([18, 20]\)
\[|\Psi\rangle = \frac{1}{\sqrt{2^{N-2}}} \sum_{[q_j] = \pm 1} |q_0 q_3 \cdots q_{N-1}\rangle \otimes |\psi\rangle, \tag{7}\]
with \(|\psi\rangle\) the same Slater determinant for \(\hat{c}\) fermions as for \(\hat{f}\) fermions. Equation (4) in terms of free-fermions reads
\[F_{\alpha\beta}^{\alpha\beta}(t) = \frac{1}{2^{N-2}} \sum_{[q_j] = \pm 1} \langle \psi | e^{i \hbar \hat{F}_t} e^{-i \hbar \hat{F}_t} |\psi\rangle, \tag{8}\]
where the sum is over all \(2^{N-2}\) charge configurations. This is our first key result, that the spin OTOCs reduce to disorder-averaged double Loschmidt echo for free-fermions. Since the Hamiltonian of Eq. (6) is bilinear in fermion operators, the expectation values in every charge sector can be efficiently computed using determinants, as explained in the Appendix of Ref. \([20]\). We note that the OTOCs for the fermion density operators, \(\rho_j = \hat{c}_{j}^\dagger \hat{c}_{j}\), correspond exactly to disorder-averaged density correlators for the free-fermions, see e.g. \([8, 9]\).

Below we present the results of numerical evaluation of the spin OTOCs. All results shown are for 1D lattices with \(N = 60\) sites and open boundary conditions. Calculations are performed using Eq. (8) where instead of summing over all charge sectors, we use 4,000 randomly chosen configurations. We fix \(j\) to be the central bond so that the correlators become functions of the distance \(r\) from this bond.

**Short-Time Behaviour.** In all cases that we studied (quenches for different components of spin and values of \(h\)) we observe power-law growth of the OTOC, as shown, e.g., in Fig. 1, consistent with the discussion in Refs. \([5, 6]\). This power-law behaviour can be extracted from the Baker-Campbell-Hausdorff expansion for the time evolution of the operators,
\[\hat{\sigma}_{j}^\alpha(t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} [\hat{H}, \hat{\sigma}_{j}^\alpha]_n. \tag{9}\]
At leading order in this expansion the OTOC behaves as \(r^{2n}/(nt)^2\), where \(n\) is given by the smallest value for which \([\hat{H}, \hat{\sigma}_{j}^\alpha]_n, \hat{\sigma}_{j}^\beta\] does not vanish. Since \(\hat{H}\) is a local Hamiltonian, the operator \([\hat{H}, \hat{\sigma}_{j}^\alpha]_n\) has finite support proportional to \(n\), and the lowest-order contribution to the OTOCs arises when \(n\) is of the order of the distance between spins \(r\). This analysis agrees with the observed short-time behaviour. In particular, we find that \(C_{\alpha\beta}^{\alpha\beta}(t)\) has the asymptotic form \(\sim (Jt)^{2r+2}\), see Fig. 1.

The authors of Ref. \([5]\) suggested that similar arguments
Figure 2. Spreading of the OTOC $C_{zz}^{(r)}(t)$ after a quantum quench. (a) results for $h = 0.2J$ on a logarithmic scale. (inset) Short-time behaviour on a linear scale. Dashed line indicates the linear light-cone with Lieb-Robinson velocity $v_{LR} = 2J$. (b) Results for $h = 0.8J$. (inset) Long time behaviour of correlators on a log-scale showing exponential tails.

hold for any OTOC of local operators whose time evolution is generated by a local Hamiltonian. However, this is in contrast to the exponential growth observed in models with semiclassical limits [22–24], and is our second key result.

Spreading of correlations. In Figs. 2–4 we present the correlation spreading in the four distinct spin OTOCs. First, let us discuss the behaviour of $C_{zz}^{(r)}(t)$. For small values of Ising coupling $h = 0.2J$, shown in Fig. 2(a), we find a linear light-cone at short times (see inset), which agrees with the Lieb-Robinson bound with velocity $v = 2J$. At longer times the system shows a stationary distribution of OTOCs with an oscillatory structure. The latter comes as a result of our choice of fermion initial state ("Friedel oscillations" at the Fermi-momentum of a half-filled band). For the larger value of $h = 0.8J$, the initial linear spreading halts quickly, and we find only short-range correlations at long times, see Fig. 2(b). The inset shows a spatial distribution of the correlations at long times, which decays exponentially with separation.

This exponential decay of spatial correlations can be understood from Eq. (8). For each charge configuration this corresponds to a series of local bond quenches between the forward and backward evolution, i.e., the Hamiltonians in this expression differ only in the sign of the hopping coefficient $J_j$ on particular bonds. However, as we show in the SM, these changes in sign can be gauged away, or in other words the spectrum of the free-fermion Hamiltonian (6) is independent of the signs of hoppings, and the only way that the correlator can decay is via particle transport between the sites $j$ and $l$. However, this transport is exponentially suppressed in the separation $r$ for typical disordered charge configurations due to the Anderson localization of the fermions.

Next we discuss the logarithmic spreading of the $C_{zz}^{(r)}(t)$ correlator, which is one of the main results of our paper. The logarithmic behaviour is evident from the linear form of the contours on a semi-log plot, shown in Fig. 3(a). Note that for the value $h = 0.8J$ shown in this figure, the single-particle localization length for the fermions is $\lambda \approx 2.15$, much smaller than the system size and the scale of correlation spreading.

This logarithmic spreading is a result of the local potential quenches appearing in this correlator (see Eq. (8)), which cannot be gauged away – unlike the bond quenches considered above. The quenches change the spectra of the Hamiltonians
in the forward and backward time evolution so that its time-dependent part can decay due to destructive interference of terms that differ in energy. In the SM we use the Lehmann representation to show that this quantity is the sum of terms that decay when $\Delta E t \sim 1$, where $\Delta E = E_\lambda - E_\lambda + E_\lambda - E_\lambda$, with $|\lambda\rangle$ a many-body eigenstate, $E_\lambda$ the corresponding energy, and $E_\lambda,j, E_\lambda,j$ are the perturbed energies due to changes in the potential. When $\Delta E \propto e^{-r/\xi}$ decays exponentially with the separation due to localization of the single-particle eigenstates, we obtain the logarithmic spreading of correlations, as seen in Fig. 3(a).

At long times the $C^{\sigma_z}(t)$ OTOC has power-law behaviour, as can be seen in Fig. 3(b), which shows the time-dependent piece $F^\sigma_z(t)$. The exponent appears to be approximately independent of the separation as we find by scaling the time by $Jt \rightarrow \alpha'Jt$ such that the curves coincide with each other. The value $\alpha = 0.5066$ and the exponent $\beta = -1.3$ of the power-law are found empirically. The authors of Refs. [30, 31] also found power-law decay of the Loschmidt echo for localized systems. Similar power-law decay was also observed in the context of OTOCs in a many-body localized system [32], the transverse-field Ising model [5], as well as for the XY spin-chain and symmetric Kitaev chain in Ref. [7].

Finally, we consider two inequivalent OTOCs involving different spin components $\hat{\sigma}^x$ and $\hat{\sigma}^z$, namely $C^{\sigma_z^x}(t)$ and $C^{\sigma_z^z}(t)$, see Fig. 4. These correlators show qualitatively similar behaviour. For short separations we find nearly time independent contours signifying localization behaviour. However, for larger separations we observe additional spreading of correlations. While the analytic arguments presented above do not apply to this case of mixed-component correlators, the logarithmic spreading appears to be a more general feature.

Discussion. We have studied four distinct OTOCs for the spins in a model of disorder-free localization. These present a remarkably rich phenomenology beyond that which has been observed for Anderson-localized systems. We show that OTOCs in our model can be mapped onto a disorder-averaged double Loschmidt echo. Perhaps unsurprisingly, we do not find exponential growth of the OTOC that has been attributed to chaotic behaviour. Instead we find short-time power-law growth consistent with that found for other integrable models [5–7]. While our model does not contain the ingredients of many-body localization, we find correlation spreading which is logarithmic, and in some cases the model shows a complete lack of correlation spreading.

We suggest that the logarithmic spreading arises as a result of the double Loschmidt echo form of the spin correlators, which are unlike the usual correlators appearing in fermion models with quenched disorder. When the local perturbations in the double Loschmidt echo correlator change the energy spectrum we get logarithmic spreading. We note that a similar slow spreading of free-fermion OTOCs has been observed in Ref. [7] due to a different mechanism, namely a non-local form of the operators in the computational basis. In cases where the effect of changes of bond signs can be gauged away, we find that the correlations are exponentially localized, as is also observed for standard fermion correlators in quenched disorder models of Anderson localization, see e.g. Ref. [9]. Although these quantities arise as natural spin OTOCs in our model, we propose that the resulting double Loschmidt echo form of the correlators may be useful in the studies of correlation spreading more generally.

Our model provides an ideal setting for further studies of OTOC phenomenologies because of a free-fermion mapping which allows one to access large system sizes and gain analytical insights, and which can also be used for thermal or infinite-temperature OTOCs. We leave a full investigation of the initial state and temperature dependence of the OTOCs to future work. It is worth stressing that the gauge-invariance of the spectrum under bond quenches is not a special feature of our model, but applies more generally to, e.g., a $\mathbb{Z}_2$ representation of the Hubbard model [20]. Unfortunately, in these cases one is facing severe computational limitations on system sizes and time scales. Remarkably, there are also prospects of simulating OTOCs in experiments [33, 34]. These precisely controlled systems may provide access to strongly-correlated physics beyond numerical capabilities.

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**Supplemental Material**

In this supplemental material we provide explicit details of the arguments used in the main text to understand the observed behaviour for the $C^{t,t}_r(t)$ and $C^{t,t}_{\bar{r}}(t)$ OTOCs. Please see the appendices of Refs. [18, 20] for details of the determinant method used for our numerical simulations.

**Exponential localization of $C^{t,t}_r(t)$**

The time-dependent part of the $C^{t,t}_r(t)$ OTOC is

$$F^{x x}_{\bar{r}}(t) = \frac{1}{2^{N-2}} \sum_{|q_j| = \pm 1} \langle \phi | e^{i H^q_j(t) \hat{t}^q_j} \hat{R}_e e^{-i H^q_j(t) \hat{t}^q_j} e^{i H^q_j(t) \hat{t}^q_j} | \psi \rangle,$$  \hspace{1cm}(10)

where for $H^q_j(t)$ we have $J_j \rightarrow -J_j$, for $\hat{R}_{\bar{r}}^q(t)$ we have $J_j, J_l \rightarrow -J_j, -J_l$, and for $\hat{R}_e^q(t)$ we have $J_l \rightarrow -J_l$, all relative to $\hat{H}(q)$. These four Hamiltonians differ locally by the sign of the relevant tunnelling parameter, i.e., local bond quenches. We proceed by noting that the transformation $J_j \rightarrow -J_j$ is equivalent to $\hat{c}_i \rightarrow -\hat{c}_i$, for $i \leq j$. This transformation can be implemented by the unitary string operator $\hat{R}_j = \prod_{l<i}(-1)^{\hat{n}_l}$ or equivalently by $\hat{R}_{\bar{r}} = \prod_{l>j}(-1)^{\hat{n}_l}$. For instance, $e^{i H^q_j(t)} = \hat{R}_e e^{i H^q_j(t)} \hat{R}_{\bar{r}}$. Importantly, this means that the spectrum is unchanged by the local bond quenches.

Let us, without loss of generality, consider $l > j$, then using the string operators we can rewrite the time-dependent piece of the OTOC (4) as

$$F^{x x}_{\bar{r}}(t) = \frac{1}{2^{N-2}} \sum_{|q_j| = \pm 1} \langle \phi | e^{i H^q_j(t) \hat{t}^q_j} \hat{R}_e e^{-i H^q_j(t) \hat{t}^q_j} e^{i H^q_j(t) \hat{t}^q_j} \hat{R}_e | \psi \rangle,$$  \hspace{1cm}(11)

where we have used the facts that $[\hat{R}_e, \hat{R}_{\bar{r}}] = 0$ and $(\hat{R}_{\bar{r}})^2 = 1$. The further means that the OTOC can be written as

$$C^{x x}_{\bar{r}}(t) = \frac{1}{2^{N-1}} \sum_{|q_j| = \pm 1} \langle \phi | [\hat{R}_e(t), \hat{R}_{\bar{r}}]^2 | \psi \rangle,$$  \hspace{1cm}(12)

where $\hat{R}_e(t) = e^{i H^q_j(t) \hat{t}^q_j} e^{-i H^q_j(t) \hat{t}^q_j}$, which implicitly depend on the charge configuration. Note the extra prefactor of $1/2$ coming from the definition of the OTOC in Eq. (2). This OTOC is therefore reduced to a disorder-averaged OTOC of parity operators for the free fermion Hamiltonian (6). We can understand the observed behaviour by considering the operator commutator for random configurations of the potential, i.e., for the individual terms of the sum in Eq. (12).
where $\Delta$ gives it in terms of the many-body eigenstates and energies. This Eq. (4). We can use the Lehmann decomposition to express for a typical charge configuration, which are averaged over in Ref. [30] for the long-time behaviour of the standard Fig. 3(a) using perturbative arguments similar to those long-time behaviour is observed in Fig. 2(b), where we find exponential tails for the spatial distribution of correlations.

Logarithmic spreading of $C_{zz}(t)$. Whereas $\xi_{xx}$ corresponds to a disorder-averaged double Loschmidt echo procedure with local bond quenches, the quenches in the $C_{zz}(t)$ OTOC are of the local potential. Importantly, this type of quench does not preserve the spectrum. The time-dependent part of the $C_{zz}(t)$ OTOC is

$$F_{zz}(t) = \frac{1}{2N^2} \sum_{|q|=\pm 1} \langle \psi | e^{i\hat{H}(q)t} e^{-i\hat{H}(q)\mu}\psi e^{i\hat{H}(q)\mu} e^{-i\hat{H}(q)\mu}|\psi\rangle,$$

for a typical charge configuration, which are averaged over in Eq. (4). We can use the Lehmann decomposition to express it in terms of the many-body eigenstates and energies. This gives

$$L_{zz}(t) = \sum_{\lambda, \mu, \nu, \rho} \langle \lambda | \langle \mu | \nu \rangle \langle \nu | \lambda \rangle \xi_{\mu,\nu,\lambda,\rho} | \lambda \rangle | \mu \rangle \xi_{\mu,\nu,\lambda,\rho} | \nu \rangle \psi e^{i\Delta E t}.$$

where $\Delta E = E_\lambda - E_\mu + E_\nu - E_\rho$. In this expansion, the states $|\lambda\rangle, |\mu\rangle, |\nu\rangle, |\lambda\rangle$ are many-body eigenstates of the Hamiltonians $\hat{H}(q), \hat{H}_j(q), \hat{H}_{\mu}(q), \hat{H}_{\nu}(q)$, respectively, and $E_\lambda, E_\mu, E_\nu, E_\rho$ are the corresponding energy eigenvalues. We then proceed by first making the approximation that the wavefunctions are only locally perturbed and, in particular, we assume that $\langle \mu | \nu \rangle \approx \delta_{\mu,\nu}$, and similarly for all the eigenstate overlaps appearing in Eq. (15), and we neglect the modification of the eigenvectors due to the perturbation. This is justified by the fact that the single-particle eigenstates are localized and the local potential perturbation only locally changes the eigenstates, and in particular the exponential profile is preserved.

With this approximation taken into account, the expression (15) reduces to

$$L_{zz}(t) \approx \sum_{\lambda} \langle \psi | \lambda \rangle \langle \lambda | \psi \rangle e^{i\Delta E t},$$

where $\Delta E(\lambda) = E_\lambda - E_{\mu_0} + E_{\nu_0} - E_{\lambda_0}$. The deviation of $L_{zz}(t)$ from 1 is thus determined by $\Delta E(\lambda)$, wherein the energies $E_{\lambda_0}, E_{\mu_0}, E_{\nu_0}$ are the perturbed energies corresponding to the same eigenstate $|\lambda\rangle$. We estimate this energy difference using a second-order perturbation expansion.

For convenience, let us state the result of perturbation theory for the eigenvalues up to second order (see, e.g., Ref. [35]). Consider a Hamiltonian $\hat{H}$ and a perturbation $eV$. In our case, $V$ takes the form $V_j = \pm \hat{n}_j \pm \hat{n}_{j+1}$, where $\hat{V} = \pm \hat{n}_j \pm \hat{n}_{j+1}$, with $\epsilon = 2h$. The energies of the perturbed Hamiltonian $\hat{H} + \hat{V}$ are then given by

$$E_{\lambda} = E_{\lambda_0} + e\langle \lambda | \hat{V} | \lambda \rangle + e^2 \sum_{\mu \neq \lambda} \frac{|\langle \lambda | \mu \rangle|^2}{E_{\lambda_0} - E_{\mu_0}} + O(e^3),$$

where $|\lambda\rangle, |\mu\rangle$ are unperturbed eigenstates, and $E_{\lambda_0}, E_{\mu_0}$ are the unperturbed energy eigenvalues. The first-order corrections to the energies cancel in the energy difference $\Delta E(\lambda)$ and the leading-order correction from second-order is

$$\Delta E(\lambda) \approx \pm 8\hbar^2 \text{Re} \sum_{\mu \neq \lambda} \frac{|\langle \lambda | \mu \rangle|^2}{E_{\lambda_0} - E_{\mu_0}} \frac{1}{E_{\lambda_0} - E_{\mu_0}},$$

where the signs are determined by the particular charge configuration. Since for a typical disorder configuration the system is Anderson localized, the energy eigenvalues are uncorrelated and the difference $E_{\lambda_0} - E_{\mu_0}$ is a random function of the states $|\lambda\rangle, |\mu\rangle$. The operator

$$\sum_{\mu \neq \lambda} \frac{|\mu\rangle \langle \mu |}{E_{\lambda_0} - E_{\mu_0}},$$

is therefore a random diagonal operator in the basis of eigenstates. Furthermore, for a given localized eigenstate $|\lambda\rangle$, the density correlations in Eq. (18) are exponentially decaying with the separation $r$. The energy difference to second-order will therefore take the functional form

$$\Delta E(\lambda) \sim \pm 8\hbar^2 C(|E_{\mu_0}|, |\lambda\rangle) e^{-r/\xi_\lambda},$$

where $C(|E_{\mu_0}|, |\lambda\rangle)$ is a random function of the energies of the many-body states of $\hat{H}(q)$ with dimensions of inverse energy, $r$ is the separation between the bonds $j$ and $l$, and the length scale $\xi$ is a function of the single-particle localization lengths. Plugging this back into Eq. (16) we find that the correlator deviates from 1 when $\hbar^2 C e^{-r/\xi} \sim 1$. Rearranging this expression we find the relationship

$$r \sim \ln \left( \frac{\hbar^2 C}{\xi} \right).$$
which defines the light-cone for the OTOC. Averaging over $|\lambda \rangle$ and the charge configurations $\{q_j\}$ results in a logarithmic light-cone, as observed in Fig. 3(a).

While these arguments are perturbative, we also see this logarithmic light-cone beyond the perturbative regime. We argue that they can be extended due to the decreasing localization length with increasing $\hbar$. This means that the orthogonality assumptions, e.g., $\langle \lambda \mu \rangle \approx \delta_{\lambda \mu}$, remain valid. The energy corrections to all orders are also correlators of density operators and random diagonal operators and therefore decay exponentially with the separation $r$. 