INTERVAL MAPS QUASI-SYMMETRICALLY CONJUGATED TO
A PIECEWISE AFFINE MAP

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ABSTRACT. Consider a multimodal interval map \( f \) of \( C^3 \) with non-flat critical points. We establish several characterizations of the map \( f \) is quasi-symmetrically conjugated to a piecewise affine map in the case \( f \) is topologically exact and all of its periodic points are hyperbolic repelling. In particular, we give a negative answer to a question posted by Henk Bruin in [Bru96].

1. Introduction

Multimodal interval maps are often topologically conjugate to piecewise linear maps. So in a topological sense, such a multimodal map is the same as a piecewise linear map. In a metric sense, however, the difference is clear. Some metric similarities still occur, when the conjugacy satisfies certain constraints.

Let \( I \) be a compact interval of \( \mathbb{R} \). A homeomorphism \( h : I \to I \) on the interval is quasi-symmetric if there exists constant \( K \geq 1 \) such that for all \( x \in I \) and all \( \varepsilon > 0 \) with \( x \pm \varepsilon \) in \( I \) we have
\[
\frac{1}{K} \leq \frac{|h(x + \varepsilon) - h(x)|}{|h(x) - h(x + \varepsilon)|} \leq K.
\]

The notion of quasi-symmetry gained importance since Sullivan [Sul86] proved the following rigidity result: Let \( f_a(x) = 1 - ax^2 \), if \( f_a \) and \( f_b \) are quasi-symmetrically conjugate and do not have a periodic attractor, then \( a = b \).

In this paper, we give several characterizations of a multimodal interval map that is quasi-symmetrically conjugate to a piecewise affine map. To state our main results, let us be more precise. Let \( I \) be a compact interval of \( \mathbb{R} \). A non-injective continuous map \( f : I \to I \) is multimodal, if there is a finite partition of \( I \) into subintervals on each of which \( f \) is injective. A multimodal map \( f : I \to I \) is topologically exact, if for every open subset \( U \) of \( I \) there is an integer \( n \geq 1 \) such that \( f^n(U) = I \).

A turning point of a multimodal map \( f : I \to I \) is a point in \( I \) at which \( f \) is not locally injective. For a differentiable multimodal map \( f : I \to I \), a point of \( I \) is critical of \( f \) if the derivative of \( f \) vanishes at it. In what follows, denote by \( \text{Crit}(f) \) the set of critical points of \( f \), denote by \( \text{Crit}'(f) \) the turning points of \( f \), and put \( \text{CV}(f) := f(\text{Crit}(f)) \).

A \( C^1 \) multimodal map \( f : I \to I \) is of class \( C^3 \) with non-flat critical points, if

- The map \( f \) is of class \( C^3 \) outside \( \text{Crit}(f) \);

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For each critical point $c$ of $f$ there exists a number $\ell_c > 1$ and diffeomorphisms $\phi$ and $\psi$ of $\mathbb{R}$ of class $C^3$, such that $\phi(c) = \psi(f(c)) = 0$, and such that on a neighborhood of $c$ on $I$, we have $|\psi \circ f| = |\phi|^{\ell_c}$.

Recall that for an integer $n \geq 1$, a periodic point $p$ of $f$ of period $n$ is hyperbolic repelling if $|Df^n(p)| > 1$, and that a critical point $c \in \text{Crit}(f)$ is called recurrent if $c \in \omega(c)$, where $\omega(c)$ denote the $\omega$-limit set of $c$ that is the set of accumulation points of the forward orbit of $c$.

The topological entropy $h_{\text{top}}(f)$ of $f$ is equal to the supremum of the metric entropies $h_{\mu}(f)$ taken over all $f$-invariant Borel probability measures $\mu$, see for example [Wal82]. A $f$-invariant Borel probability measure $\mu$ such that $h_{\mu}(f) = h_{\text{top}}(f)$ is called a maximal entropy measure. It is well-known that a multimodal interval map $f : I \to I$ that is topologically exact has a unique maximal entropy measure $\mu_f$. Moreover, the topological support of $\mu_f$ is equal to $I$, and the Jacobian of $\mu_f$ is constant equal to $\exp(h_{\text{top}}(f))$.

For a point $x$ in $I$, $r > 0$, an integer $m \geq 1$, and each $j$ in $\{0, 1, \ldots, m-1\}$, let $W_j$ be the pull-back of $B(f^m(x), r) \cap I$ by $f^{m-j}$ containing $f^j(x)$. The criticality of $f^m$ at $x$ with respect to $r$ is defined as the following number

$$\# \{j \in \{0, 1, \ldots, m-1\} : W_j \cap \text{Crit}(f) \neq \emptyset\}.$$  

Moreover, the map $f$ is said to be semi-hyperbolic, if there exist constants $r > 0$ and $D \geq 1$ such that for every $x$ in $I$ and each integer $n \geq 1$ the criticality of $f^n$ at $x$ with respect to $r$ is at most $D$.

The main result of this paper is the following theorem.

**Theorem 1.** Let $f : I \to I$ be a multimodal map of class $C^3$ with non-flat critical points and with all periodic points hyperbolic repelling. If $f$ is topologically exact, then the following statements are equivalent.

1. $f$ is semi-hyperbolic;
2. $f$ has no recurrent critical points;
3. The maximal entropy measure of $f$ is doubling;
4. $f$ is quasi-symmetrically conjugate to a piecewise affine function with slope equal to $\pm \exp(h_{\text{top}}(f))$.

Recall that a Borel measure $\mu$ on a metric space $(X, \text{dist})$ is said to be doubling, if there are constants $C_* > 0$ and $r_* > 0$ such that for each $x$ in $X$ and $r$ in $(0, r_*)$ we have

$$\mu(B(x, 2r)) \leq C_* \mu(B(x, r)).$$

The concept of semi-hyperbolicity was introduced by Carleson, Jones and Yoccoz to characterize those complex polynomials whose basin of attraction of infinity is a John domain [CJY94]. In the context of complex polynomials, they proved the equivalence between (1) and (2) of Theorem 1. See also [Ym99] [Mh11] for an extension to complex rational maps. The proof in the case of interval maps is simpler (Lemma 2 in [2.1]), thanks to the backward Lyapunov stability of interval maps (Lemma 1 in [2.1]). The equivalence between conditions (3) and (4) of Theorem 1 is a simple consequence of the theory of Parry and Milnor and Thurston [Par66] [MT88], see Proposition 1 in [2.2]. The implication (2) $\Rightarrow$ (4) is new even for unimodal map. It gives an alternative proof of [Bru96, Theorem 1] when combined with [Sch11]. The implication (4) $\Rightarrow$ (2) is also new, when
restricted to unimodal maps it answers a question posted by Bruin [Bru96, Question], see also [BB04]. Are there non-Misiurewicz maps that are quasi-symmetrically conjugate to tent-maps? Recall that a unimodal interval map $f$ satisfies the Misiurewicz condition if the critical point of $f$ is not periodic, and has a forward orbit which stays away from itself. We notice that for unimodal maps the Misiurewicz condition coincides with semi-hyperbolicity, but these conditions are different for maps with several critical points.

The core of the proof of Theorem 1 is the proof of the implications (1) $\Rightarrow$ (3) (Proposition 2 in §3) and (3) $\Rightarrow$ (2) (Proposition 3 in §4). The proof of these implications follow in general lines used in the proof of the correspondent implications of [RL10, Theorem A], and of the existence of nice couples with arbitrarily large modulus for TCE maps, see [PRL07]. However, the proofs are different of various places, mainly due to the fact that interval maps are not open in general.

Proof of Theorem 1. The equivalence between (1) and (2) is shown in Lemma 2. That (1) implies (3) follows from Proposition 2. That (3) implies (2) is Proposition 3. The equivalence between (3) and (4) is shown in Proposition 1. □

1.1. Notations. Throughout the rest of this paper, let $I = [0,1]$ endowed with the distance induced by the absolute value $|\cdot|$ on $\mathbb{R}$, and denote by $\mathcal{A}$ the collection of interval maps of class $C^3$ with non-flat critical points and with all periodic points hyperbolic repelling. Put

$$\ell_{\max} := \max \{\ell_c : c \in \text{Crit}(f) \} \quad \text{and} \quad \ell_{\min} := \min \{\ell_c : c \in \text{Crit}(f) \}.$$ 

For an interval $J$ contained in $I$, we denote by $|J|$ its length and for $\eta > 0$ we denote by $\eta J$ the open interval of $\mathbb{R}$ of length $(1 + 2\eta)|J|$ that has the same middle point as $J$.

For each $x \in I$ and $r > 0$ we denote by $B_I(x,r)$ the ball of $I$ centered at $x$ and of radius $r$ defined as follows: $B_I(x,r) := \{y \in I : |x - y| < r \}$.

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2. Non-recurrence and quasi-symmetry

In this section, we prove the equivalences between (1) and (2), (3) and (4) in Theorem 1.

2.1. Semi-hyperbolicity and non-recurrent critical points. Given a multimodal map $f : I \to I$, an integer $n \geq 1$ and a subset $J$ of $I$, a connected component of $f^{-n}(J)$ will be called a pull-back of $J$ by $f^n$. The following general facts of multimodal maps is used for several time in what follows, see for example [RL12b] for a proof.

Lemma 1 (Lemma A.2, [RL12b]). Let $f : I \to I$ be an interval map in $\mathcal{A}$ that is topologically exact. Then for every $\kappa > 0$ there is $\delta > 0$ such that for every $x$ in $I$, every integer $n \geq 1$, and every pull-back $W$ of $B(x,\delta)$ by $f^n$, we have $|W| < \kappa$. 

In the complex setting, the following result was shown by Carleson, Jones and Yoccoz in [CJY94], see also [Yin99, Mih11]. Thanks to Lemma 1 the proof in the real case is much simpler than that in the complex setting.

**Lemma 2.** $f$ is semi-hyperbolic if and only if it has no recurrent critical points.

**Proof.** First we assume that $f$ has no recurrent critical points. Then there exists $\delta_1 > 0$ such that for each $c \in \text{Crit}(f)$ and each integer $n \geq 1$ we have $f^n(c) \notin B(c, \delta_1)$. Reducing $\delta_1$ if necessary we assume that

$$\inf\{|c_1 - c_2| : c_1, c_2 \in \text{Crit}(f), c_1 \neq c_2\} > \delta_1.$$ 

Let $\delta$ be the constant given by Lemma 1 with $\kappa = \delta_1$. It follows that for each $x$ in $I$, each integer $m \geq 1$ and each $c \in \text{Crit}(f)$, there is at most one $j$ in $\{0, \cdots, m-1\}$ such that the pull-back $B(f^m(x), \delta)$ by $f^m$ containing $f^j(x)$ contains $c$, since otherwise there is $k \geq 1$ such that $f^k(c) \in B(c, \delta_1)$. Hence, the criticality of $f^n$ at $x$ with respect to $\delta$ is at most $\#\text{Crit}(f)$. This implies $f$ is semi-hyperbolic.

Now let us assume $f$ is semi-hyperbolic. Then there exist constants $r > 0$ and $D \geq 1$ such that for every $x$ in $I$ and each integer $n \geq 1$ the criticality of $f^n$ at $x$ with respect to $r$ is at most $D$. Arguing by contradiction, assume there is a recurrent critical point $c$. Then we can define inductively a sequence $(n_i)_{i=1}^{\infty}$ of positive integers and a sequence $\{W_i\}_{i=0}^{\infty}$ of open intervals such that $W_0 = B(c, r/2)$, and for each $i \geq 1$ we have $c \in W_i$, $f^{n_i}(c) \in W_{i-1}$ and $W_i$ is the pull-back of $W_{i-1}$ by $f^{n_i}$ containing $c$. Put $S := n_1 + \cdots + n_D + 1$. For each $x \in W_{D+1}$, we have $B(c, r/2) \subset B(f^S(x), r)$. It follows that the criticality of $f^S$ at $x$ with respect to $r$ is at least $D + 1$. We obtain a contradiction, and so $f$ has no recurrent critical points.

2.2. Maximal entropy measure and the piecewise affine modal. The following lemma is a direct consequence of the compactness arguments, we omit the proof.

**Lemma 3.** Let $\mu$ be a non-atomic Borel measure whose support is equal to $I$. Then for each $\delta > 0$ we have $\inf_{x \in I} \mu(B_I(x, \delta)) > 0$.

**Lemma 4.** Let $\mu$ be a non-atomic Borel measure whose support is equal to $I$. Then $\mu$ is doubling if and only if there exists $C > 1$ such that for each $x$ in $I$ and each $\varepsilon > 0$ with $x \pm \varepsilon$ in $I$, we have

$$C^{-1} \mu((x, x + \varepsilon)) \leq \mu((x - \varepsilon, x)) \leq C \mu((x, x + \varepsilon)).$$

**Proof.** Assume that $\mu$ is doubling with constants $C_* > 1$ and $r_* > 0$. For each $\varepsilon > 0$ and each $x$ in $I$ with $x \pm \varepsilon$ in $I$, we have

$$(x, x + \varepsilon) \subset B(x - \frac{\varepsilon}{2}, 2\varepsilon) \text{ and } (x - \varepsilon, x) \subset B(x + \frac{\varepsilon}{2}, 2\varepsilon).$$

To prove the desired inequality, we consider the following two cases:

(i) If $\varepsilon$ is in $(0, r_*]$. Then by the doubling property of $\mu$ we obtain

$$\mu((x, x + \varepsilon)) \leq \mu(B_I(x - \frac{\varepsilon}{2}, 2\varepsilon)) \leq C_*^2 \mu(B_I(x - \frac{\varepsilon}{2}, 2\varepsilon)) = C_*^2 \mu((x - \varepsilon, x)),$$

$$\mu((x - \varepsilon, x)) \leq \mu(B_I(x + \frac{\varepsilon}{2}, 2\varepsilon)) \leq C_*^2 \mu(B_I(x + \frac{\varepsilon}{2}, 2\varepsilon)) = C_*^2 \mu((x, x + \varepsilon)).$$

This proves the desired inequality with $C_1 := C_*^2$. 


(ii) If $\varepsilon > r_*$, put
\[ C_+ := \inf\{\mu(J) : J \subset I \text{ is an interval with } |J| \geq r_*\}. \]
Then by Lemma 3, we have $C_+ > 0$. It follows that
\[ \mu((x, x + \varepsilon)) \leq 1 \leq C_+^{-1}\mu((x - r_*, x)) \leq C_+^{-1}\mu((x - \varepsilon, x)) \]
and
\[ \mu((x - \varepsilon, x)) \leq 1 \leq C_+^{-1}\mu((x + r_*, x)) \leq C_+^{-1}\mu((x, x + \varepsilon)). \]
This proves again the desired inequality with $C := C_+^{-1}$.

To prove the converse statement, fix $x$ in $I$ and $r > 0$. First we prove that
\[ (1) \quad \mu((x - 2r, x - r) \cap I) \leq C\mu((x - r, x) \cap I). \]
In fact, if $(x - 2r, x - r) \cap I = \emptyset$, then the inequality (1) is trivial. Otherwise, put $s := \max(|x - 2r, x - r| \cap I)$. Then $s \leq r$ and by the assumption with $x$ replaced by $x - r$ and $\varepsilon$ replaced by $s$, we have
\[ \mu((x - 2r, x - r) \cap I) = \mu((x - r - s, x - r)) \leq C\mu((x - r, x - r + s)) \leq C\mu((x - r, x) \cap I). \]
The same argument gives us
\[ \mu((x, x + r) \cap I) \geq C^{-1}\mu((x + r, x + 2r) \cap I). \]
Therefore, we have
\[
\mu(B_I(x, 2r)) = \mu(B(x, 2r) \cap I)
= \mu((x - 2r, x - r) \cap I) + \mu([x - r, x + r] \cap I) + \mu((x + r, x + 2r) \cap I)
\leq C\mu((x - r, x) \cap I) + \mu([x - r, x + r] \cap I) + C\mu((x, x + r) \cap I)
\leq (1 + C)\mu(B(x, r) \cap I) = (1 + C)\mu(B_I(x, r)),
\]
and proves $\mu$ is doubling with $C_* = 1 + C$.

As noted before, the following well-known lemma is used several times throughout the rest of this article, see for example [Hof79, Hof81, MT88].

**Lemma 5.** Let $f : I \to I$ be a multimodal interval map that is topologically exact. Then there is a unique maximal entropy measure $\mu_f$ of $f$. Moreover, the topological support of $\mu_f$ is equal to $I$, $\mu_f$ is non-atomic and the Jacobian of $\mu_f$ is constant equal to $\exp(h_{\text{top}}(f))$.

**Proposition 1.** Let $f : I \to I$ be a multimodal map in $\mathcal{A}$ that is topologically exact, and let $\mu_f$ be the maximal entropy measure given by Lemma 5. Then the following statements are equivalent.

1. $f$ is quasi-symmetrically conjugate to a piecewise affine map from the interval $I = [0, 1]$ to itself with slope equal to $\pm \exp(h_{\text{top}}(f))$ everywhere;
2. The measure $\mu_f$ is doubling.

Recall that $\mu_f$ is doubling if there is $C_* > 1$ and $r_* > 0$ such that for each $x$ in $I$ and each $r$ in $(0, r_*)$ we have
\[ \mu_f(B_I(x, 2r)) \leq C_*\mu_f(B_I(x, r)). \]
Proof of Proposition.\[^{2}\] Put $s := \exp(h_{\text{top}}(f))$ and for each $x$ in $I$, put $h(x) := \mu_f([0, x])$. Note that $\mu_f$ is non-atomic and the support of $\mu_f$ is equal to $I$. Then we have that $h : I \to I$ is continuous bijective function, so a homeomorphism. Defining $F := h \circ f \circ h^{-1}$, we have $h \circ f = F \circ h$, and $F$ maps $I$ to itself. Furthermore, a completely analogous arguments as that in the proof of [MT88, Theorem 7.4] shows that the map $F : I \to I$ is piecewise affine with slope equal to $\pm s$ everywhere. Conversely, suppose there is an increasing homeomorphism $\tilde{h} : I \to I$ and a piecewise linear function $\tilde{F} : I \to I$ with slope equal to $\pm s$ everywhere such that $\tilde{h} \circ f = \tilde{F} \circ h$. Denote by Leb the Lebegue measure on $I$ and put $\mu := (\tilde{h}^{-1})_*\text{Leb}$. Then we have $\mu(I) = 1$, and for every Borel set $A$ of $I$ on which $f$ is injective we have

$$\mu(f(A)) = (\tilde{h}^{-1})_*\text{Leb}(f(A)) = \text{Leb}(\tilde{h}(f(A))) = \text{Leb}(\tilde{F}(\tilde{h}(A))) = s\tilde{h}(A) = s\text{Leb}(\tilde{h}(A)) = s(\tilde{h}^{-1})_*\text{Leb}(A) = s\mu(A).$$

It follows that the Jacobian of $\mu$ satisfies $\text{Jac}(\mu) = s = \exp(h_{\text{top}}(f))$. By Rokhlin’s entropy inequality, see [PU10, §2.9] or [Par69, §10] for example, we have

$$h_{\mu}(f) \geq \int_I \log \text{Jac}(\mu) \, d\mu = \int_I h_{\text{top}}(f) \, d\mu = h_{\text{top}}(f).$$

This gives us $\mu$ is a maximal entropy measure of $f$. Hence, by Lemma 5 we have $\mu = \mu_f$. Furthermore, by the definition of $\mu$ we have

$$\mu([0, x]) = \text{Leb}(\tilde{h}([0, x])) = |\tilde{h}(0) - \tilde{h}(x)|.$$

It follows that

$$\tilde{h}(x) = \mu([0, x]) = \mu_f([0, x]) = h(x)$$

Therefore, to end the proof, it is enough to prove that $h$ is quasi-symmetric if and only if $\mu_f$ is doubling.

By the definition we know that $h$ is quasi-symmetric if and only if there is $M > 1$ such that for each $x \in I$ and $\varepsilon > 0$ with $x \pm \varepsilon$ in $I$ we have

$$M^{-1} \leq \frac{|h(x + \varepsilon) - h(x)|}{|h(x) - h(x - \varepsilon)|} \leq M.$$  

This, by the definition of $h$, is equivalent to the following

(2) \hspace{1cm} M^{-1} \leq \frac{|\mu_f([0, x + \varepsilon]) - \mu_f([0, x])|}{|\mu_f([0, x])| - \mu_f([0, x - \varepsilon])} \leq M.$$

Note that $\mu_f$ is non-atomic. Then by Lemma 3 we have that the equality (2) above holds if and only if $\mu_f$ is doubling, and complete the proof. \hfill $\square$

3. Semi-hyperbolicity and doubling

The main goal of this section is prove the following proposition which gives the implication (1) $\Rightarrow$ (3) of Theorem 1.

Proposition 2. Let $f : I \to I$ be a multimodal map in $\A$ that is topologically exact, and let $\mu_f$ be the maximal entropy measure given by Lemma 5. Assume that $f$ is semi-hyperbolic, then the measure $\mu_f$ is doubling.

The proof of this proposition, depending on several lemmas, is given at the end of this section.

Let $\tau > 0$ and let $J_1 \subset J_2$ be two intervals of $I$. We say that $J_1$ is $\tau$-well inside $J_2$, if both components of $J_2 \setminus J_1$ have length at least $\tau |J|$. 


Lemma 6 (Theorem A, [LS10]). Let \( f : I \to I \) be a multimodal map in \( \mathcal{A} \). Then for each \( \tau > 0 \) there exist \( C > 1 \) and \( \xi > 0 \) satisfying the following. Let \( T \subset J \) be an open interval, \( J \) a closed subinterval of \( T \) and an integer \( s \geq 1 \) such that the following hold:

1. \( f^s : T \to f^s(T) \) is a diffeomorphism;
2. \( |f^s(T)| \leq \xi \);
3. \( f^s(J) \) is \( \tau \)-well inside \( f^s(T) \).

Then for each pair \( x \) and \( y \) of points in \( J \) we have

\[
\frac{|Df^s(x)|}{|Df^s(y)|} \leq C.
\]

Furthermore, \( C \to 1 \) as \( \tau \to +\infty \).

A chain is a sequence of open intervals \( \{G_i\}_{i=0}^s \) such that for each \( 0 \leq i < s \), \( G_i \) is a component of \( f^{-1}(G_{i+1}) \). The criticality of the chain is the number of \( i \)'s such that \( G_i \) contains a critical point.

The following is a version of the Koebe principle for non-diffeomorphic pull-backs, see for example [LS08] for a proof.

Lemma 7 (Lemma 4.1, [LS08]). Let \( f : I \to I \) be a multimodal map in \( \mathcal{A} \), then there exists \( \eta > 0 \) such that for each \( \tau > 0 \) and \( N \geq 1 \) there are \( \tau' > 0 \) and \( C > 0 \) with the following property. Let \( \{H_j\}_{j=0}^s \) and \( \{H'_j\}_{j=0}^s \) be chains such that \( H_j \subset H'_j \) for all \( 0 \leq j \leq s \). Assume that \( |H_0| < \eta \), \( H'_s \) is \( \eta \)-well inside \( H_0 \) and that the criticality of the chain \( \{H_j\}_{j=0}^s \) is at most \( N \). Then the following holds:

1. \( H'_0 \) is \( \tau' \)-well inside \( H_0 \);
2. for each \( x \in H'_0 \),

\[
|Df^s(x)| \leq C \frac{H'_s}{H_0}.
\]

Moreover, for a fixed \( N \), \( \log \tau'/\log \tau \) tends to a positive constant as \( \tau \to +\infty \).

Lemma 8. Let \( f : I \to I \) be a multimodal map in \( \mathcal{A} \). Then for each \( N \geq 1 \) and \( \tau > 0 \) there exist \( \delta' > 0 \) and \( C_1 > 1 \) such that the following holds. Let \( J_0 \subset \tilde{J}_0 \) be two subintervals of \( I \), and let \( m \geq 1 \) be an integer. If \( (1 + 2\tau)|f^m(\tilde{J}_0)| < \delta', \tau f^m(\tilde{J}_0) \subset I \) and the number of \( i \)'s such that the pull-back of \( \tau f^m(\tilde{J}_0) \) by \( f^m \) containing \( f^i(\tilde{J}_0) \) intersects \( \text{Crit}(f) \) is at most \( N \), then we have

\[
C_1^{-1} \left( \frac{|f^m(\tilde{J}_0)|}{|f^m(J_0)|} \right)^{N-1} \leq \frac{|\tilde{J}_0|}{|J_0|} \leq C_1 \frac{|f^m(\tilde{J}_0)|}{|f^m(J_0)|}.
\]

Proof. By the non-flatness of critical points, there are \( \kappa > 0 \) and \( M > 0 \) such that for every \( c \in \text{Crit}(f) \) and every interval \( J \) contained in \( B(c, \kappa) \) we have

\[
M^{-1} |J|^{\varepsilon_c} \leq |f(J)| \leq M|J|^{\varepsilon_c}.
\]

Reducing \( \kappa \) if necessary, we can assume that for any two distinct critical \( c \) and \( c' \) we have \( |c - c'| > \kappa \). By Lemma 4 there is \( \delta > 0 \) such that for every interval \( J \) of \( I \) with \( |J| < \delta \), every integer \( n \geq 1 \), every pull-back \( W \) of \( J \) by \( f^n \) has length at most \( \kappa \). Let \( \eta > 0 \), \( \tau' > 0 \) and \( C > 0 \) be the constants given by Lemma 7 with \( N \) and \( \tau \), and let \( \delta' > 1 \) and \( \xi > 0 \) be the constants given by Lemma 6 for \( \tau' \). Put \( \delta' := \min\{\kappa, \delta, \xi, \eta\} \). To prove the lemma, let \( J_0 \subset \tilde{J}_0 \) be two subintervals of \( I \) and let \( m \geq 1 \) be an integer such that \( (1 + 2\tau)|f^m(\tilde{J}_0)| < \delta', \tau f^m(\tilde{J}_0) \subset I \).
and the number of $i$'s such that the pull-back of $\tau f^m(J_0)$ by $f^{m-i}$ containing $f^i(J_0)$ intersects $\text{Crit}(f)$ is at most $N$. Moreover, let $\{H_i\}^m_{i=0}$ be the chain such that $H_m = \tau f^m(J_0)$ and $H_0$ is the pull-back of $\tau f^m(J_2)$ by $f^m$ containing $J_2$. Let $0 \leq n_1 < n_2 < \cdots < n_s < n_{s+1} = m$ be all the integers $i$ with $H_i \cap \text{Crit}(f) \neq \emptyset$. Then $s \leq N$, and by Lemma 7 for every $i$ in $\{1, \cdots , s+1\}$ we have $f^{n_i}(J_0)$ is $\tau$-well inside in $H_{n_i}$. Therefore, by Lemma 6 for every $i$ in $\{1, \cdots , s\}$ we have

$$C' \frac{|f^{n_i+1}(J_0)|}{|f^{n_i+1}(J_0)|} \geq \frac{|f^{n_i+1}(J_0)|}{|f^{n_i+1}(J_0)|} \geq \frac{1}{C'} \frac{|f^{n_i+1}(J_0)|}{|f^{n_i+1}(J_0)|}$$

and

$$C' \frac{|f^{n_i}(J_0)|}{|f^{n_i}(J_0)|} \geq \frac{|f^{n_i}(J_0)|}{|f^{n_i}(J_0)|} \geq \frac{1}{C'} \frac{|f^{n_i}(J_0)|}{|f^{n_i}(J_0)|}.$$ 

On the other hand, by (3) for every $i$ in $\{1, \cdots , s\}$ we have

$$M^{2/\ell_{\min}} \frac{|f^{n_i+1}(J_0)|}{|f^{n_i+1}(J_0)|} \geq \frac{|f^{n_i+1}(J_0)|}{|f^{n_i+1}(J_0)|} \geq M^{-2/\ell_{\min}} \left( \frac{|f^{n_i+1}(J_0)|}{|f^{n_i+1}(J_0)|} \right)^{1/\ell_{\max}}.$$ 

It follows that

$$C'^{(1+N)} M^{-\frac{2N}{\ell_{\min}}} \left( \frac{|f^m(J_0)|}{|f^m(J_0)|} \right)^{\frac{\ell_{\max}}{\ell_{\min}}} \leq \frac{|J_0|}{|J_0|} \leq C'^{(1+N)} M^{\frac{2N}{\ell_{\min}}} \left( \frac{|f^m(J_0)|}{|f^m(J_0)|} \right).$$

This proves the lemma with $C_1 := C'^{(1+N)} M^{\frac{2N}{\ell_{\min}}}$. 

Proof of Proposition 8. Since $f$ is semi-hyperbolic, then exist constants $r' > 0$ and $D \geq 1$ such that for every $x$ in $I$ and each integer $n \geq 1$ the criticality of $f^n$ at $x$ with respect to $r'$ is at most $D$. Let $C_1 > 1$ and $r' > 0$ be the constants given by Lemma 8 with $\tau = 2$ and $N = D$. Put $M := \sup I |Df|$ and $r_0 := \min \{r', \tau'\}$. Fix $x \in I$ and a sufficiently small $r > 0$. Let $m$ be the minimal integer such that $|f^m(B_I(x, 2r))| \geq r_0/6M$. Since $f$ is topologically exact, such integer $m$ exists. Note that $|f^{m-1}(B_I(x, 2r))| < r_0/(6M)$, so $|f^m(B_I(x, 2r))| \leq r_0/6$. By our choice of $r_0$ we know that the number of $i$'s such that the pull-back of $2f^m(B_I(x, 2r))$ by $f^{m-i}$ containing $f^i(B_I(x, 2r))$ intersects $\text{Crit}(f)$ is at most $D$, and

$$|(1 + 4)|f^m(B_I(x, 2r))| < r'.$$

Therefore, by Lemma 8 we have

$$2 \geq \frac{|B_I(x, 2r)|}{|B_I(x, r)|} \geq C_1^{-1} \left( \frac{|f^m(B_I(x, 2r))|}{|f^m(B_I(x, r))|} \right)^{\frac{\ell_{\max}}{\ell_{\min}}}.$$ 

This implies $|f^m(B_I(x, r))| \geq (2C_1)^{\ell_{\max}(6M)^{-1}} r_0$. Moreover, by Lemma 8 we have

$$\delta' := \inf_{x \in I} \mu_f(B_I(x, (2C_1)^{\ell_{\max}(12M)^{-1}} r_0)) > 0.$$ 

On the other hand, by Lemma 11 we have

$$\mu_f(B_I(x, r)) \geq \exp(-mh_{top}(f)) \mu_f(f^m(B_I(x, r)))$$

and

$$\mu_f(B_I(x, 2r)) \leq 2^D \exp(-mh_{top}(f)) \mu_f(f^m(B_I(x, 2r))).$$ 

It follows that

$$\frac{\mu_f(B_I(x, 2r))}{\mu_f(B_I(x, r))} \leq 2^D \frac{\mu_f(f^m(B_I(x, 2r)))}{\mu_f(f^m(B_I(x, r)))}. $$
Hence,
\[ \mu_f(B_l(x, 2r)) \leq 2^D \mu_f(f^m(B_l(x, 2r))) \mu_f(B_l(x,r)) \leq 2^D \mu_f(B_l(x,r)) \]
This implies \( \mu_f \) is doubling.

4. Doubling implies non-recurrent critical points

In this section, our main goal is to prove the following proposition. The proof, which is given at the end of this section, depends on several lemmas.

Proposition 3. Let \( f : I \to I \) be a multimodal map in \( \mathcal{A} \) that is topologically exact, and let \( \mu_f \) be the maximal entropy measure given by Lemma 5. Assume the measure \( \mu_f \) of \( f \) is doubling, then \( f \) has no recurrent critical points.

We start with the following observation, see for example [RL10] for a proof.

Lemma 9 (Lemma 1, [RL10]). Let \((X, \text{dist})\) be a compact metric space and let \( \mu \) be a doubling measure on \( X \). Then there are constants \( C > 0 \) and \( \alpha > 0 \) such that for each sufficiently small \( r > 0 \) and each \( x \in X \) we have
\[ \mu(B(x,r)) \geq Cr^\alpha. \]

Throughout the rest of this section, fix a multimodal interval map \( f : I \to I \) in \( \mathcal{A} \) that is topologically exact, and let \( \mu_f \) be its maximal entropy measure given by Lemma 5. In particular, \( \mu_f \) is non-atomic and its support is equal to \( I \).

4.1. preliminaries.

Lemma 10. If \( \mu_f \) is doubling, then for each \( M' > 1 \) there is \( \varepsilon > 0 \) such that the following holds. For each pair of adjacent subintervals \( L \) and \( R \) of \( I \) with \( |L| \geq \varepsilon |R| \), we have \( \mu_f(L) \geq M' \mu_f(R) \).

Proof. Let \( C > 1 \) be the constants given by Lemma 4. For any \( M' > 1 \), let \( m \) be the minimal integer such that \( C^{-1}(1 + C^{-1})^m \geq M' \), and put \( \varepsilon := 2^m \). For each pair of adjacent subintervals \( L \) and \( R \) of \( I \) with \( |L| \geq \varepsilon |R| \), denote by \( s \) the common endpoint of \( L \) and \( R \), and for every \( i \in \{0, 1, \cdots, m\} \) put \( L_i := L \cap B(a, 2^i |R|) \). By Lemma 4 we know \( \mu_f(L_0) \geq C^{-1} \mu_f(R) \), and for each \( i \in \{0, 1, \cdots, m - 1\} \) we have
\[ \mu_f(L_{i+1}) \geq (1 + C^{-1}) \mu_f(L_i). \]
It follows inductively that
\[ \mu_f(L) \geq \mu_f(L_m) \geq (1 + C^{-1})^m \mu_f(L_0) \geq C^{-1}(1 + C^{-1})^m \mu_f(R) \geq M' \mu_f(R). \]
This completes the proof.

Lemma 11. There exist \( \tau > 0 \) such that the following holds. For each open subinterval \( V \) of \( I \) with \( |V| < \tau \), every integer \( m \geq 1 \) and each pull-back \( W \) of \( V \) by \( f^m \), if we denote by \( D \) the number of those \( j \in \{0, \cdots, m - 1\} \) such that the connected component of \( f^{-(m-j)}(V) \) containing \( f^j(W) \) intersects \( \text{Crit}(f) \), then
\[ \exp(-mh_{\text{top}}(f)) \mu_f(f^m(W)) \leq \mu_f(W) \leq 2^D \exp(-mh_{\text{top}}(f)) \mu_f(f^m(W)) \leq 2^D \exp(-mh_{\text{top}}(f)) \mu_f(V). \]
Proof. Let \( \kappa' > 0 \) be such that for each pair of critical points \( c_1 \neq c_2 \) in \( \text{Crit}(f) \) we have \( |c_1 - c_2| > \kappa' \), and \( \tau \) the constant given by Lemma 1 with \( \kappa = \kappa' \). We will prove the assertion holds for such \( \tau \). In fact, fix an open subinterval \( V \) of \( I \) with \( |V| < \tau \), an integer \( m \geq 1 \) and a pull-back \( W \) of \( V \) by \( f^m \). For each \( j \) in \( \{0, \ldots , m\} \), let \( \hat{W}_j \) be the connected component of \( f^{(m-j)}(V) \) containing \( f^j(V) \).

Let \( 0 \leq n_1 < n_2 < \cdots < n_D \leq m - 1 \) be the integers of those \( j \) in \( \{0, \ldots , m - 1\} \) such that \( \hat{W}_j \) intersects \( \text{Crit}(f) \), and put \( n_{D+1} = m \). By Lemma 5 for every \( i \) in \( \{0, 1, \cdots , D\} \) we have

\[
\mu_f(f^{n_{i+1}}(W)) = \exp((n_{i+1} - n_i - 1)\mu_{\top}f)\mu_f(f^{n_{i+1}}(W)).
\]

On the other hand, by our choice of \( \tau \) we have that each of \( W \) and \( f^{n_i}(W) \), \( i \in \{1, \cdots , D\} \), contains at most one critical point, so

\[
2^{-1} \exp(n_1\mu_{\top}f)\mu_f(W) \leq \mu_f(f^{n_1}(W)) \leq \exp(n_1\mu_{\top}f)\mu_f(W),
\]

and

\[
2^{-1} \exp(\mu_{\top}f)\mu_f(f^{n_i}(W)) \leq \mu_f(f^{n_1+1}(W)) \leq \exp(\mu_{\top}f)\mu_f(f^{n_1}(W)).
\]

Combining (4), (5) and (6), we obtain

\[
\exp(-m\mu_{\top}f)\mu_f(f^{m}(W)) \leq \mu_f(W) \leq 2^D \exp(-m\mu_{\top}f)\mu_f(f^{m}(W)),
\]

and complete the proof. \( \square \)

Lemma 12. There is \( \delta_* > 0, M > 1 \) and \( \tau_* > 0 \) such that for each \( \tau > \tau_* \) the following holds. Let \( T \) be a subinterval of \( I \) of the length at most \( \delta_* \), and let \( K \subset J \) be two subintervals of \( T \) such that both of connected components of \( J \setminus K \) have the length at least \( |K| \), and such that \( J \) is \( \tau_- \)-well inside in \( T \). For each \( n \geq 1 \) and every pull-back \( J_n \) of \( K \) by \( f^n \) containing a critical point \( c \), let \( J_n \) be the pull-back of \( J \) by \( f^n \) containing \( K_n \), and let \( J_{n-1} \) and \( K_{n-1} \) be the pull-back of \( J \) and \( K \) by \( f^{n-1} \) containing \( f(K_n) \), respectively. If \( f^{n-1} \) maps diffeomorphically a neighborhood of \( f(J_n) \) onto \( T \) and

\[
\frac{\max\{|f^n(c) - x| : x \in \partial K\}}{\min\{|f^n(c) - x| : x \in \partial K\}} \leq 2,
\]

then for each connected component \( W \) of \( J_n \setminus K_n \), we have \( f^n(W) \) is one of connected components of \( J \setminus K \), and

\[
\frac{|W|}{|K|} \leq M \left( \frac{|f^n(W)|}{|K|} \right)^{1/\ell_{\min}}.
\]

Proof. Fix a sufficiently large number \( \tau_* > 1 \), and let \( C \) and \( \delta_* \) be the constants given by Lemma 6 with \( \tau = \tau_* \). Enlarging \( \tau_* \) if necessary, we assume that \( C \leq 3/2 \).

Reducing \( \delta_* \) if necessary, we also assume that each pull-back of each interval of length at most \( \delta_* \) contains at most one critical point. By the hypothesis, each connected component of \( J_{n-1} \setminus K_{n-1} \) is mapped diffeomorphically onto one of connected component of \( J \setminus K \). Moreover, by our choice of \( \delta_* \) we know that each connected component \( W \) of \( J_n \setminus K_n \) is mapped diffeomorphically onto one of connected component of \( J_{n-1} \setminus K_{n-1} \). This proves that \( f^n(W) \) is one of connected component of \( J \setminus K \).
The desired inequality follows immediately from Koebe distortion and the non-flatness of critical points. In fact, by Lemma [6] we have
\[
\frac{|f(W)|}{|K_{n-1}|} \leq \frac{3}{2} \frac{|f^n(W)|}{|K|}.
\]
On the other hand, by the non-flatness of critical points, there is a constant $C$ depending only on $f$ such that
\[
\frac{|W|}{|K_n|} \leq C \left( \frac{|f(W)|}{|K_{n-1}|} \right)^{1/\ell_c}.
\]
This proves the desired inequality with $M = (3/2)^{1/\ell_{\min}} C$. \hfill \Box

4.2. Topologically Collect-Eckmann condition. Let $f : I \to I$ be a multimodal map and fix $r > 0$. Recall that given an integer $n \geq 1$, the criticality of $f^n$ at a point $x$ of $I$ with respect to $r$ is the number of those $j \in \{0, \ldots, n-1\}$ such that the connected component of $f^{(n-j)}(B(f^n(x), r))$ containing $f^j(x)$ contains a critical point of $f$ in $\text{Crit}^r(f)$. We say that $f$ satisfies the Topological Collet-Eckmann (TCE) condition, if for some choice of $r > 0$ there are constants $D \geq 1$ and $\theta$ in $(0, 1)$, such that the following property holds: For each point $x$ in $I$ the set $G_x$ of all those integers $m \geq 1$ for which the criticality of $f^m$ at $x$ is less than or equal to $D$, satisfies
\[
\liminf_{n \to +\infty} \frac{1}{n} \#(G_x \cap \{1, \ldots, n\}) \geq \theta.
\]
Clearly, every semi-hyperbolic interval map satisfies the TCE condition. The Topological Collet-Eckmann condition was first introduced in [NP98]. We will use the following fact that let $f : I \to I$ be a multimodal interval map in $\mathcal{A}$ that is topologically exact, then the TCE condition is characterized by each of the following conditions, see for example [RL12a, Corollary C] for a proof,

1. Exponential Shrinking of Components condition (ESC). There are constants $\delta > 0$ and $\lambda > 1$ such that for every interval $J$ contained in $I$ that satisfies $|J| \leq \delta$, the following holds: For every integer $n \geq 1$ and every connected component $W$ of $f^{-n}(J)$ we have $|W| \leq \lambda^{-n}$.

2. Uniform hyperbolicity on periodic orbits. There is $\lambda > 1$ such that for each integer $n \geq 1$ and each repelling periodic point $p$ of period $n$ we have $|Df^n(p)| \geq \lambda^n$.

The following proposition gives another characterization of the TCE condition, see for example [RL12a] for a proof; see also [RL10, Theorem B] for a similar result where $f$ is a rational map.

Lemma 13 (Remark 6.2, [RL12a]). Let $f : I \to I$ be a multimodal interval map in $\mathcal{A}$ that is topologically exact, and let $\mu_f$ be the maximal entropy measure of $f$. Then $f$ satisfies the TCE condition if and only if there are constants $r_0 > 0, \alpha > 0$ and $C > 0$ such that for all $x$ in $I$ and $r$ in $(0, r_0)$ we have
\[
\mu_f(B(x, r)) \geq Cr^\alpha,
\]

4.3. Nice sets and nice couples. Recall that an open subset $V$ of $I$ is a nice set for $f$, if for every integer $n \geq 0$ we have $f^n(\partial V) \cap V = \emptyset$, and each connected component of $V$ contains exactly one critical point of $f$. In this case, for each $c$ in $\text{Crit}(f)$ we denote by $V^c$ the connected component of $V$ containing $c$. A nice couple for $f$ is a pair of nice sets $(\hat{V}, V)$ such that $\hat{V} \subset V$, and such that for
every integer \( n \geq 1 \) the set \( f^n(\partial V) \) is disjoint from \( \hat{V} \). Moreover, for a nice nice couple \((\hat{V}, V)\), we define the *modulus of \((\hat{V}, V)\) as*

\[
\text{mod}(\hat{V}, V) := \min \{ \text{mod}(\hat{V}^c, V^c) : c \in \text{Crit}(f) \}
\]

where

\[
\text{mod}(\hat{V}^c, V^c) := \sup \{ \tau > 0 : V^c \text{ is } \tau\text{-well inside } \hat{V}^c \}.
\]

**Lemma 14.** Assume that \( f \) has arbitrarily small nice couples of arbitrarily large modulus. Then for each recurrent critical point \( c_0 \) in \( \text{Crit}(f) \), \( \kappa \in (0, 1) \), \( N \geq 2 \) and each \( r_* > 0 \) there is \( c \in \text{Crit}(f) \), \( r \in (0, r_*) \) and an integer \( m \geq 1 \), such that \( f^m(c_0) \in B_1(c, r) \) and such that the pull-back \( \hat{U} \) (resp. \( U \)) of \( B_1(c, r) \) (resp. \( B_1(c, kr) \)) containing \( c_0 \) satisfies the following properties.

1. \(|\hat{U}| < r_*
2. The criticality of \( f^m \) at \( x \) with respect to \( kr \) is equal to \( N \);
3. \( f^m \) maps diffeomorphically each connected component of \( \hat{U} \setminus U \) onto one of the connected component of \( B_1(c, r) \setminus B_1(c, kr) \).

**Proof.** See [RL10, Lemma 5] for a proof. There it is proved for rational maps, but the proof can be adapted to yield the lemma.

We also use the following lemma. In the case \( f \) is a complex rational map, it is [PRL07, Proposition 4.2]. The proof applies without changes to the case where \( f \) is an interval map.

**Lemma 15.** Assume that the map \( f \) satisfies the TCE condition. Then for every \( \delta > 0 \) and \( \tau > 0 \) there is a nice couple \((\hat{V}, V)\) of modulus at least \( \tau \) satisfying \( \hat{V} \subset B(\text{Crit}(f), \delta) \).

**Proof of Proposition 3** Arguing by contradiction, we assume that \( f \) has a recurrent critical point \( c_0 \). Let \( \delta_* > 0 \), \( M > 1 \) and \( \tau_* > 0 \) be the constants given by Lemma 4 and let \( \delta_0 \) be the constant given by Lemma 11 with \( \kappa = \delta_* \). Reducing \( \delta_0 \) if necessary we assume \( \delta_0 < \delta_* \). Moreover, in view of Lemma 9 and Lemma 13 we know that \( f \) satisfies the TCE condition. Therefore, by Lemma 15 we know that the map \( f \) satisfies the hypothesis of Lemma 14. Let \( C \) and \( r_0 \) be the constants given by Lemma 4. Put

\[
\ell_0 := \sum_{i=0}^{+\infty} \frac{1}{\ell^i_{\text{min}}}, \quad M_0 := M^{\ell_0}, \quad M_1 := \sum_{i=0}^{2M_0} C^i
\]

and let \( \varepsilon_1 > 0 \) be the constants given by Lemma 10 with \( M' = 4M_1 \), and let \( N \geq 2 \) be the integer such that \( \varepsilon_1^{1/\ell^N_{\text{min}}} \leq 2 \). Moreover, let \( \varepsilon_2 > 0 \) be the constants given by Lemma 10 with \( M' = 2^{N+1} \). Choose a sufficiently large \( \varepsilon_3 > 0 \) so that the constant \( \tau' \) given by Lemma 4 for \( \tau = \varepsilon_3 \) and \( N \) is at least \( \tau_* \).

Now let \( U, V, m, r, c \) be given by Lemma 14 for \( N \) and \( c_0 \) as above and with \( r_* = \min \{ r_0, \delta_0 \} \) and

\[
\kappa = \frac{1}{(1 + 2\varepsilon_3)(1 + 3\varepsilon_2 + 4\varepsilon_2\varepsilon_1)}.
\]

By the definition, we have that \( \hat{U} \) and \( U \) are the connected components of \( f^{-m}(B_1(c, r)) \) and \( f^{-m}(B_1(c, kr)) \) containing \( c_0 \), respectively. Put

\[
r_1 := (1 + 2\varepsilon_2)k\tau \text{ and } r_2 := (1 + 2\varepsilon_2 + 2\varepsilon_1\varepsilon_2)kr.
\]
Let $\hat{L}_1$ and $\hat{R}_1$ be the left-hand and right-hand connected components of $B_I(c, r_1) \setminus B_I(c, \kappa r)$, respectively. Let $\hat{L}_2$ and $\hat{R}_2$ be the left-hand and right-hand connected components of $B_I(c, r_2) \setminus B_I(c, r_1)$, respectively. The by the definitions of $r_1$ and $r_2$, and Lemma 10 we have

\begin{equation}
\frac{\mu_f(\hat{L}_2)}{\mu_f(\hat{L}_1)} \geq 4M_1 \quad \text{and} \quad \frac{\mu_f(\hat{R}_2)}{\mu_f(\hat{R}_1)} \geq 4M_1,
\end{equation}

and

\begin{equation}
\frac{\min\{\mu_f(\hat{L}_1), \mu_f(\hat{R}_1)\}}{\mu_f(B_I(c, \kappa r))} \geq 2^{N+1}.
\end{equation}

Let $B$ and $\hat{B}$ be the connected component of $B_I(c, r_1)$ and $B_I(c, r_2)$ by $f^m$ containing $c_0$, respectively. It follows that $f^m$ maps diffeomorphically each connected component of $\hat{B} \setminus U$ onto one of the connected components of $B_I(c, \kappa r)$. In particular, if letting $L_N$ be left-hand connected component of $\hat{B} \setminus B$, then by Lemma 5 we have $\mu_f(L_N) = \exp(-mh_{\text{top}}(f))\mu_f(f^m(L_N))$, and by Lemmas 5 and 11

\begin{equation}
\mu_f(B) = \mu_f(B \setminus U) + \mu_f(U) \\
\leq 2\exp(-mh_{\text{top}}(f))\mu_f(f^m(B \setminus U)) + 2^{N+1}\exp(-mh_{\text{top}}(f))\mu_f(B_I(c, \kappa r))
\end{equation}

This, together with (5) and (7), gives us

\begin{equation}
\frac{\mu_f(L_N)}{\mu_f(B)} \geq \frac{\exp(-mh_{\text{top}}(f))\mu_f(f^m(L_N))}{3\exp(-mh_{\text{top}}(f))\mu_f(f^m(B \setminus U))} = \frac{\mu_f(f^m(L_N))}{3\mu_f(f^m(B \setminus U))} \geq 4M_1/3.
\end{equation}

On the other hand, by Lemma 12 we know that

\begin{equation}
\frac{|L_N|}{|B|} \leq M^{1+\frac{1}{r_{\min}}+\cdots+\frac{1}{r_{\min}} \frac{f^m(L_N)}{|B_I(c, \kappa r)|}^{1/r_{\min}}} \leq M_{\varepsilon_1}^{1/\varepsilon_1} \leq 2M_0.
\end{equation}

Therefore, by Lemma 4 inductively we have

\begin{equation}
\mu_f(L_N) \leq (C + C^2 + \cdots + C^{2M_0})\mu_f(B) \leq M_1\mu_f(B).
\end{equation}

This contradicts the inequality (10), and completes the proof of the proposition. \Box

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