The Casimir effect for a massless scalar field in the $n$ dimensional Einstein universe with Dirichlet boundary conditions on a sphere

Patrick Moylan

Institute of Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, 72 Tsarigradsko Chaussee, 1784 Sofia, Bulgaria
E-mail: pjm11@psu.edu

Abstract. We use the representation theory of $SO(2,n)$ to determine the renormalized vacuum energy for a massless scalar field in the $n$-dimensional Einstein universe subject to Dirichlet boundary conditions on a sphere of maximum radius. The problem is an exactly solvable one. This is in remarkable contrast to the analogous problem in flat $n$ dimensional Minkowski space where, except for the lowest dimensional case ($n = 2$), there is no known exactly solvable method of solution for any radius of the spherical boundary. For $n = 4$ our results agree with those of Bayen and Özcın, Class. Quant. Grav., 10 (1993) L115-L121. We use our results to obtain some qualitative information about the Casimir effect for spherical boundaries of smaller radii, and we comment on how one may apply these results to obtain information about the corresponding problem in Minkowski space.

1. The conformal group in $n$ dimensions and preliminaries

Consider the quadratic form $Q(x)$ defined on $\mathbb{R}^{n+2}$ by $Q(x) = x_{-1}^2 + x_{0}^2 - x_{1}^2 - x_{2}^2 - \ldots - x_{n}^2$ where $x = (x_{-1}, x_{0}, x_{1}, \ldots, x_{n}) \in \mathbb{R}^{n+2}$. Thus $\mathbb{R}^{n+2}$ equipped with the metric defined by $Q(x)$ is $n + 2$ dimensional Minkowski space. We denote $n$ dimensional Minkowski space by $M_0$.

Let $G = SO(2,n)$ denote the connected component of the group of linear transformations of $\mathbb{R}^{n+2}$ preserving the symmetric bilinear form which is associated to $Q(x)$ by polarization. We shall call $G$ the $n$-dimensional conformal group, and we denote the universal cover of $G$ by $G^\sim = SO(2,n)^\sim$. Let $\mathcal{G}$ be the Lie algebra of $G$. $\mathcal{G}$ is identified with the set of all matrices $(a_{ij})$ ($-1 \leq i, j \leq n$) such that $a_{ii} = 0$ ($0 \leq i \leq n$), $a_{ij} = -a_{ji}$ ($1 \leq i \leq j \leq n$), $a_{ij} = a_{j,i}$ ($1 \leq j \leq 4$), $a_{-1,j} = a_{j,-1}$ ($1 \leq j \leq 4$) and $a_{-1,0} = -a_{0,-1}$. We define subalgebras $\mathcal{K}, \mathcal{H}, A, \mathcal{N}_+$ and $\mathcal{N}_-$ as follows. Let $E_{ij}$ be the matrix such that the $(i,j)$ component is equal to 1 and the other components are all equal to 0. Let $L_{ij} = E_{ij} - E_{ji}$ ($1 \leq i \leq j \leq n$), $L_{0i} = E_{i,0} + E_{0,i}$ ($1 \leq i \leq n$), $L_{-1,i} = E_{i,-1} + E_{-1,i}$ ($1 \leq i \leq n$) and $L_{-10} = E_{-1,0} - E_{0,-1}$. Let $\mathcal{K}$ be the subalgebra spanned by: $L_{ij}$ ($1 \leq i, j \leq n$) and $L_{-10}; \mathcal{H}$ be the subalgebra spanned

1 Dedicated to Professor Arno R. Böhm on the occasion of his 70th birthday. (Paper presented at the 5th International Symposium on Quantum Theory and Symmetries, Valladolid, Spain, July 22-28, 2007)
2 Permanent address: Physics Dept., The Pennsylvania State University, Abington College, Abington, PA 19001 USA
by: \( L_{ij} \) (\( 1 \leq i, j \leq n - 1 \)), \( L_{0i} \) (\( 1 \leq i \leq n - 1 \)) and \( L_{-1,0} \). \( A \) be the subalgebra spanned by \( L_{-1,n} \) and \( \mathcal{N}_{+} (\mathcal{N}_{-}) \) be the subalgebra spanned by \( P_{i} = L_{n,i} + L_{-1,i} \) (\( 0 \leq i \leq n - 1 \)) (\( \tilde{P}_{i} = L_{n,i} - L_{-1,i} \) (\( 0 \leq i \leq n - 1 \))). Denote the analytic subgroups of \( G \) corresponding to \( K, \mathcal{H}, \mathcal{A}, \mathcal{N}_{+} \) and \( \mathcal{N}_{-} \) by \( K, \mathcal{H}, \mathcal{A}, \mathcal{N}_{+} \) and \( \mathcal{N}_{-} \), respectively. The group elements corresponding to the subalgebras \( \mathcal{N}_{+} \) and \( \mathcal{N}_{-} \) spanned by the \( P_{i} \) and \( \tilde{P}_{i} \) are in \((n + 2) \times (n + 2)\) matrix form:

\[
\exp \{ x^{i}(L_{n,i} \pm L_{-1,i}) \} = \begin{bmatrix} 1 - \frac{1}{2}q(x) & x & \pm \frac{1}{2}q(x) \\ -x & I_{n} & \pm x \\ \mp \frac{1}{2}q(x) & \pm x & 1 + \frac{1}{2}q(x) \end{bmatrix},
\]

where \( x = (x_{0}, x_{1}, \ldots, x_{n}) \), \( x^{1} = (x_{0}, -x_{1}, \ldots, -x_{n}) \), \( q(x) = x_{0}^{2} - x_{1}^{2} - \ldots - x_{n-1}^{2} \) and \( I_{n} \) is the \( n \times n \) identity matrix. We note that \( H = SO(2, n - 1) \), and we have an Iwasawa like decomposition of \( G \) i.e. the map \( H' \times \mathcal{A} \times \mathcal{N}_{+} \rightarrow G \) is an injective diffeomorphism onto an open, dense subset of \( G \), where \( H' \) is \( SO(2, n - 1) \) [1].

Consider the \( n + 1 \) dimensional isotropic cone in \( \mathbb{R}^{n+2} \) defined by

\[
C = \{ x \in \mathbb{R}^{n+2}|Q(x) = 0 \}.
\]

Let \( \mathbb{R}^{n+2} \) and \( C^{*} \) be the sets of nonzero elements in \( \mathbb{R}^{n+2} \) and \( C \), respectively. Let \( P \subset G \) be the stabilizer subgroup of \( e = e_{-1} + e_{0} \) where \( e_{i} = (0, 0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^{n+2} \) (i.e. \( e_{i} \) is the vector with 1 in the \( i \)th slot and zeros elsewhere). \( P = SO(1, n - 1) \times_{s} \mathcal{N}_{+} \), where \( x_{s} \) denotes semi-direct product, i.e. \( P \) is the \( n \) dimensional Poincaré group [2]. The orbit of \( e \) under \( G \) is \( C^{*} \) [2]. Hence \( C^{*} \cong G/P \). We let \( S^{n-1} \) denote the \((n - 1)\)-sphere:

\[
S^{n-1} = \{ u = (u_{1}, u_{2}, \ldots, u_{n}) \in \mathbb{R}^{n}|u_{1}^{2} + \ldots u_{n}^{2} = 1 \}
\]

We define the upper and lower hemispheres of \( S^{n-1} \) as:

\[
S_{\pm}^{n-1} = \{ s \in S^{n-1} | u_{n} > 0 \}.
\]

and the \( \Sigma^{n-1} \) is:

\[
\Sigma^{n-1} = \partial S_{+}^{n-1} = \partial S_{-}^{n-1}
\]

Spherical coordinates on \( S^{n-1} \) are as follows:

\[
u = (\sin(\rho) \omega, \cos(\rho)) \in S^{n-1} \text{ with } \omega \in S^{n-2}.
\]

Let \( \text{Proj}(\mathbb{R}^{n+2}) \) be the real projective variety of all one dimensional subspaces in \( \mathbb{R}^{n+2} \). We have the map

\[
X^{*} \rightarrow \text{Proj}(\mathbb{R}^{n+2}) \quad x \rightarrow \vec{x}.
\]

Let \( M \subset \text{Proj}(\mathbb{R}^{n+2}) \) be the image of \( C^{*} \) under (1.7). The above defined action of \( SO(2, n)^{\sim} \) on \( \mathbb{R}^{n+2} \) induces an action of \( SO(2, n)^{\sim} \) on \( \text{Proj}(\mathbb{R}^{n+2}) \). Since \( C \) is stable under the action of \( SO(2, n)^{\sim} \), \( M \) is stable under the action of \( SO(2, n)^{\sim} \) on \( \text{Proj}(\mathbb{R}^{n+2}) \). \( SO(2, n) \) and therefore \( SO(2, n)^{\sim} \) are transitive on \( C^{*} \), hence \( SO(2, n) \) and \( SO(2, n)^{\sim} \) are transitive on \( M \). \( M \) is naturally diffeomorphic to \((S^{1} \times S^{n-1})/\mathbb{Z}_{2} \) where the \( \mathbb{Z}_{2} \) action is the product of antipodal maps on \( S^{1} \) and \( S^{n-1} \). Denote \( S^{1} \times S^{n-1} \) by \( M \). \( K = SO(2) \times \mathbb{R}(n) \) acts transitively on \( M \), and \( M \) is the homogeneous space \( M \cong K/K_{0} \) where \( K_{0} = SO(1) \times SO(n - 1) \). The universal cover \( \tilde{M} \) of \( M \) is the \((n \text{ dimensional}) \) Einstein universe. (See [4] for the definition in four dimensions.) Since \( M \) is the conformal compactification of \( M_{0} \) [2], the Einstein universe is the universal cover of the conformal compactification of Minkowski space.
2. Line bundles and invariant differential operators

The line bundle $L^1(M)$ over $M$ associated with the character $\lambda \to |\lambda|^{-s}$ of $\mathcal{B}^s$ is the bundle whose fibre over $\pi$ is the set of all pairs $(x, |\lambda|^s) \in C \times \mathcal{A}$, $(s \in \mathcal{A})$. Denote by $\Gamma^s(M)$ the space of smooth sections of $L^s(M)$. There is a unique isomorphism between $\Gamma^s(M)$ and the space of smooth functions $f : C^* \to \mathcal{A}$ which satisfy the homogeneity condition $f(\lambda x) = |\lambda|^{-s} f(x)$. $\Gamma^s(M)$ is an $SO_0(2, n)^*$ module with respect to the representation $\pi_s$ defined by $(\pi_s(\gamma)) f(x) = f(g^{-1} x)$, $g \in SO_0(2, n)$, $x \in C^*$, and $g^{-1} x$ denotes the action of $g^{-1}$ on $x \in C^*$. We denote the associated representation of the Lie algebra $so(2, n)$ by $d\pi_s$.

Let $D \subset \mathbb{R}^{n+2}$ be open and be stable under multiplication by $\mathbb{R}$ and $C \subset D$. Define

$$\Gamma^s(D) = \{ f \in C^\infty(D) \mid f(\lambda x) = |\lambda|^{-s} f(x) ; \ x \in D^* , \ \lambda \in \mathbb{R} \} . \quad (2.1)$$

Let $x_1$ be the natural linear coordinates on $\mathbb{R}^{n+2}$ introduced above. Important operators on $\Gamma^s(D)$ are:

$$S = \Sigma_{i=1}^n x_i \frac{\partial}{\partial x_i} , \ Q = \Sigma_{i=1}^n \epsilon_i x_i^2 I , \ \Delta = \Sigma_{i=1}^n \epsilon_i \frac{\partial^2}{\partial x_i^2} . \quad (2.2)$$

$\epsilon_i$ is +1 for $i = -1$ or 0 and −1 for $i = 1, \ldots, n$. $(I = \text{Identity operator on } \Gamma^s(D).)$ We have:

$$[\Delta, Q] = 4\{S + \frac{n+2}{2} I\} , \ [S, \Delta] = -2\Delta , \ [S, Q] = 2Q , \quad (2.3)$$

and we also have Euler's theorem on homogeneous functions:

$$S \phi = -s \phi \ \text{for} \ \phi \in \Gamma^s(D) . \quad (2.4)$$

Now, we explicitly let $D = \{(x, y) \mid x = (x_1, x_0) \in \mathbb{R}^2, \ y = (y_1 \ldots y_n) \in \mathbb{R}^n, (x, y) = (0, 0) \text{ or } x \neq 0, y \neq 0 \}$, and we have the following results [3]:

**Lemmas:** (i) $\Delta : \Gamma^s(D) \to \Gamma^{s+2}(D)$.

(ii) Let $s = -2 + \frac{n+2}{2}$, then, if $\phi \in \Gamma^s(D)$ vanishes on $C^*$, so does $\Delta \phi$.

**Corollary:** Let $\phi \in \Gamma^s(M)$ with $s = -2 + \frac{n+2}{2}$ and let $\phi_e$ be an extension of $\phi$ to $\Gamma^s(D)$, then $\Delta \phi_e$ is independent of the extension.

Since $\Delta \phi_e$ is independent of the extension, we obtain:

**Proposition:** For $s = -2 + \frac{n+2}{2}$ we have that $\Delta$ induces a map $\Delta_M : \Gamma^s(M) \to \Gamma^{s+2}(M)$ which intertwines for the $G$ actions on $\Gamma^s(M)$ and $\Gamma^{s+2}(M)$.

To define $\Delta_M$ we choose $\phi \in \Gamma^s(M)$ and extend to $\Gamma^s(D)$, we then apply $\Delta$ and restrict $\Delta \phi_e$ to $C^*$. It is clear from the above that only for

$$s = -2 + \frac{n+2}{2}$$

we obtain a well-defined operator $\Delta_M$ on $\Gamma^s(M)$ from this procedure, and we shall henceforth only consider such values of $s$.

3. The harmonic component of $\Gamma^s(M)$

Let $C^k_M(M)$ be the space of all $K$ finite elements of $C^\infty(M)$ for which $\phi(w) = (-1)^s \phi(-w)$ $(w \in M)$, and define extensions to $D$ by $(\phi_e \otimes \psi_e)(rv, \tilde{ru}) = \phi_e(rv) \psi_e(\tilde{ru})$ $(v \in S^1, u \in S^{n-1})$ and $r, \tilde{r} \in \mathbb{R}^+$ with $\phi_e(rv) = r^{-\frac{s}{2}} \phi(v)$, and $\psi_e(\tilde{ru}) = \tilde{r}^{-\frac{s}{2}} \psi(u)$ where $\phi \otimes \psi \in C^k_M(M)$. Now $(r, v) \in \mathbb{R}^+ \times S^1 = \mathbb{R}^2 \setminus \{0\}$ and $(\tilde{r}, u) \in \mathbb{R}^+ \times S^{n-1} = \mathbb{R}^n \setminus \{0\}$ and, in terms of $(r, v, \tilde{r}, u)$, $\Delta$ may be expressed as:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_S1 - \left( \frac{\partial^2}{\partial \tilde{r}^2} + \frac{n-1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} + \frac{1}{\tilde{r}^2} \Delta_{S^{n-1}} \right) . \quad (3.1)$$
where $\Delta_{S^1}$ and $\Delta_{S^{n-1}}$ are the Laplacians on $S^1$ and $S^{n-1}$, respectively. Thus, using $s = -2 + \frac{n+2}{2}$ we have

$$
(\Delta \phi_c(vu)\psi_c(\overline{ru}))|_{C^*} = \frac{1}{r^2} \left\{ \left( \frac{n-2}{2} \right)^2 + \Delta_{S^1} - \Delta_{S^{n-1}} \right\} \phi(vu)\psi(ru) . \tag{3.2}
$$

Now let $H \subset \Gamma^s(M)$ be the kernel of $\Delta_M$ and let $H'$ be the cokernel, then we have an exact sequence

$$
0 \rightarrow H \rightarrow \Gamma^s(M) \rightarrow \Gamma^{s+2}(M) \rightarrow H' \rightarrow 0
$$

and the subspace $H_K$ of elements of $H$ in $C^*_K(M)$ is an $so(2,n)$ module, with the representation of $so(2,n)$ on $H_K$ being $d\pi_s$.

4. Quantization of a real massless scalar field on $\tilde{M}$

We have shown above that there is a representation of $SO_0(2,n)$ on $C^\infty(\tilde{M})$ and Ker $\Delta_M$ (the harmonic component of $\Gamma^v(M)$) defines a subrepresentation. This harmonic component splits into two irreducible components, and Kobayashi and Orsted [5] show that, for $n \geq 3$, they are spaces of positive and negative energy massless, spin zero fields on $\tilde{M}$ (the minimal representations) [3]), and, indeed for $n = 4$, they are described in the paper by Barut and Böhm [6] as representations for massless, spin zero particles. They are just the spin zero, singleton representations of $SO(2,n)$ considered by Angelopoulos and Laoues [7], and, for $n = 3$ they reduce to the positive and negative energy Rac representations of $SO_0(2,3)$.

Let $\Delta_j$ denote the covariant derivative determined by the semi-Riemannian metric $g$ on $\tilde{M}$, then the energy operator for a massless, scalar field on $\tilde{M}$ is: [8]

$$
H = \int_{S^{n-1}} \frac{1}{2} \left\{ \sum_{j=1}^n (\Delta_j \phi)^* (\Delta_j \phi) + \left( \frac{n-2}{2R} \right)^2 |\phi|^2 \right\} du . \tag{4.1}
$$

where $du = R^{n-1} \sin^{n-2}(\rho) d\rho \wedge dw$ is the volume form on $S^{n-1}$, where $d\omega$ denotes the volume form on $S^{n-2}$. ($R$ is the radius of the Einstein universe, which, in the above, we had set equal to one.)

Recall the space of spherical harmonics of degree $\sigma$ on $S^{n-1}$ is defined to be

$$
\mathcal{H}_\sigma = \left\{ f \in C^\infty(S^{n-1}) | \Delta_{S^{n-1}} f = -\sigma(\sigma + n - 2) f \right\} . \tag{4.2}
$$

The spherical harmonics are

$$
Y_{\sigma \ell \{m\}}(v) = N(k, \ell, \{m\}) \sin^{\ell} \rho \ C_{\sigma-\ell}^{\ell+\frac{n-2}{2}}(\cos \rho) \ Y^{(m)}{\ell \{m\}}(\theta_1, \theta_2, \ldots, \theta_{n-2}) \tag{4.3}
$$

where $\theta_1, \theta_2, \ldots, \theta_{n-2}, \rho$ are spherical coordinates of the point $v \in S^{n-1}(\omega = (\theta_1, \theta_2, \ldots, \theta_{n-2}) \in S^{n-2})$, $\{m\}$ is used for the other labels and $N(k, \ell, \{m\})$ is the normalization factor for the spherical harmonics. For $\phi \in$ Ker$\Delta_M$ we have from (3.2) that

$$
\Delta_M \phi = \left\{ \frac{\partial^2}{\partial \tau^2} - \Delta_{S^{n-1}} + \left( \frac{n-2}{2} \right)^2 \right\} \phi = 0 . \tag{4.4}
$$

For $\phi$ a K-finite function of the form $\phi_{\sigma \ell \{m\}}(u,v) = e^{ivr}Y_{\sigma \ell \{m\}}(v)$ which are in Ker$\Delta_M$ we obtain

$$
\nu^2 - \sigma(\sigma + n - 2) - \left( \frac{n-2}{2} \right)^2 = 0 . \tag{4.5}
$$
The standard quantization of a real, massless scalar field on \( \tilde{M} \) is carried out in a similar way to the quantization of a real, massless scalar field on \( M_0 \) [8]. Let \( c \) be the speed of light and \( \hbar \) Planck’s constant, then, for a point \( x \in \tilde{M} \) with coordinates \( (t = \frac{R}{c} \tau, \omega, \rho) \), the quantum field \( \phi(x) \) is given by [8]:

\[
\phi(t, \omega, \rho) = \sum_{\sigma, \ell, \{m\}} \left( \frac{\hbar}{E_\sigma} \right)^{1/2} \left\{ a_{\sigma \ell \{m\}} e^{-i\nu_\sigma t} Y_{\sigma \ell \{m\}}(\rho, \omega) + a^\dagger_{\sigma \ell \{m\}} e^{i\nu_\sigma t} Y_{\sigma \ell \{m\}}^*(\rho, \omega) \right\}, \tag{4.6}
\]

where \( \nu_\sigma \) is specified by eqn. (4.5), and the \( a_{\sigma \ell \{m\}} \) and \( a^\dagger_{\sigma \ell \{m\}} \) satisfy the following relations on the Fock space:

\[
[a_{\sigma \ell \{m\}}, a^\dagger_{\sigma' \ell' \{m'\}}] = \delta_{\sigma,\sigma'} \delta_{\ell,\ell'} \delta_{\{m\}, \{m'\}} \tag{4.7a}
\]

and

\[
[a_{\sigma \ell \{m\}}, a_{\sigma' \ell' \{m'\}}] = [a^\dagger_{\sigma \ell \{m\}}, a^\dagger_{\sigma' \ell' \{m'\}}] = 0 . \tag{4.7b}
\]

By a calculation, which uses eqn. (4.4), we may rewrite the Hamiltonian as

\[
H = \int_{S^{n-1}} \frac{1}{2} \left\{ (\partial_t \phi)^* (\partial_t \phi) - \phi^* \partial^2 \phi \right\} du . \tag{4.8}
\]

5. The Casimir energy of a massless scalar field on \( \tilde{M} \) which vanishes on \( \Sigma^{n-1} \)

From eqns. (1.4), (1.5) and (1.6) we obtain: \( S^{n-1}_\rho = \left\{ v \in S^{n-1} \mid \rho > \frac{\pi}{2} \right\} \), \( S^{n-1}_\rho = \left\{ v \in S^{n-1} \mid \rho < \frac{\pi}{2} \right\} \), and \( \Sigma^{n-1} = \left\{ v \in S^{n-1} \mid \rho = \frac{\pi}{2} \right\} \). We want to determine the Casimir energy of a quantized massless, scalar field on \( \tilde{M} \) subject to vanishing of the field on the equator \( \Sigma^{n-1} \).

In order to do this we need the following result [9]:

**Proposition:** For \( \nu > 0 \)

\[
C_k^{(\nu)} \left( \cos \left( \frac{\pi}{2} \right) \right) = 0 \iff k \text{ is odd} . \tag{5.1}
\]

Using eqn. (4.6) and this result in eqn. (4.8), we are able to compute the vacuum expectation value of the energy operator (eqn. (4.8)), to obtain the following for the total zero point energy of the problem:

\[
E = \frac{\hbar c}{2R} \sum_{\sigma = 0}^{\infty} \sqrt{\sigma(\sigma + n - 2) + \left( \frac{n-2}{2} \right)^2}, \tag{5.2}
\]

where the sum over \( \ell \) goes from zero to \( \sigma \), and the ranges of summations for the \( \{m\} \)'s follow from the theory of spherical harmonics of degree \( \ell \) on \( S^{n-2} \).

In order to perform the (finite) sums over \( \ell \) and the \( \{m\} \)'s we need to recall a little about the representation theory of \( SO(n-1) \) on \( S^{n-2} \) and its relationship to spherical harmonics in \( n-1 \) dimensions. Let \( \tau \) be the left regular representation of \( SO(n-1) \) on \( C^\infty(S^{n-2}) \cong C^\infty(SO(n-1)/SO(n-2)) \). Denote by \( \Lambda \) the set of all integers for \( n = 3 \), and the set of all nonnegative integers for \( n > 3 \). The zonal spherical function \( \omega_{\ell} \) with height \( \ell \) (\( \ell \in \Lambda \)) is given by \( C_{\ell n^2}^{\infty} (\theta_{n-2}) \) for \( n > 3 \) and \( (x_1 + ix_2)^\ell \) for \( n = 3 \) [10]. For any \( \ell \in \Lambda \) we denote by \( \mathcal{H}_\ell \) the subspace of \( C^\infty(SO(n-1)/SO(n-2)) \), which is spanned by the elements \( \tau(k)\omega_{\ell} \)
(k ∈ K). Then, as is well-known, τ decomposes into the direct sum of inequivalent, irreducible representations of SO(n − 1) i.e. $C^\infty(S^{n-2}) \cong \bigoplus_{\ell} \mathcal{H}_\ell$, and the spaces $\mathcal{H}_\ell$ are identified as spaces of harmonic, homogeneous polynomials on $R^{n-1}$ of degree $\ell$ i.e the spaces of spherical harmonics of degree $\ell$. Since an arbitrary homogeneous polynomial of degree $\ell$ in $n − 1$ variables depends on
\[
\binom{\ell + n - 2}{\ell}
\]
arbitrary constants and the condition that the polynomial be harmonic gives
\[
\binom{\ell + n - 4}{\ell - 2}
\]
restrictions, the number of linearly independent spherical harmonics $Y_\ell^{(m)}(\theta_1, \theta_2, \ldots, \theta_{n-2})$ of degree $\ell$ on $S^{n-2}$ (i.e. the dimension of $\mathcal{H}_\ell$) is:
\[
N(\ell, n) = \binom{\ell + n - 2}{\ell} - \binom{\ell + n - 4}{\ell - 2} = \frac{\Gamma(\ell + n - 3)(2\ell + n - 3)}{\Gamma(n - 2)\Gamma(\ell + 1)} \quad (5.3)
\]
for $n > 3$ and one for $n = 3$. (It follows from the representation theory of SO(2) that for $n = 3$ the space $\mathcal{H}_\ell$ is one dimensional.) Substitution of eqn. (5.3) into (5.2) gives (for $n > 3$):
\[
E = \frac{\hbar c}{2R} \sum_{\sigma = 0}^{\infty} \sqrt{\sigma(\sigma + n - 2) + \left(\frac{n - 2}{2}\right)^2} \sum_{\ell = 0}^{\sigma} \frac{\Gamma(\ell + n - 3)(2\ell + n - 3)}{\Gamma(n - 2)\Gamma(\ell + 1)} \quad (5.4)
\]
We may perform the finite sums over $\ell$ for $\ell$ even ($\sigma$ odd) and $\ell$ odd ($\sigma$ even), respectively, and we obtain
\[
\frac{\Gamma(\sigma - 2 + n)}{\Gamma(\sigma)\Gamma(n - 1)}
\]
in both cases. Thus, after a little algebra, eqn. (5.4) becomes
\[
E = \frac{\hbar c}{4R} \sum_{\sigma = 0}^{\infty} (n - 2 + 2\sigma) \frac{\Gamma(\sigma - 2 + n)}{\Gamma(\sigma)\Gamma(n - 1)} \quad (5.5)
\]
Due to the difference in multiplicities of $\mu$’s for $n = 3$ and $n > 3$ the much simpler $n = 3$ case must be treated, with minor alterations, separately. This is an easy exercise which we leave to the reader. Our final result for the Casimir energy in this case ($n = 3$) is recorded below in the Table. In the Table we have also recorded the well known case of one spatial dimension for completeness [11].

We must regularize the above divergent series (eqn. (5.5)). For this purpose we introduce the exponential cutoff $e^{-\frac{\pi}{2}n(2\sigma + n - 2)}$ in eqn. (5.5) to obtain:
\[
E^{\text{reg}}(\alpha, n) = \frac{\hbar c}{4R} \sum_{\sigma = 0}^{\infty} (n - 2 + 2\sigma) \frac{\Gamma(\sigma - 2 + n)}{\Gamma(\sigma)\Gamma(n - 1)} e^{-\frac{\pi}{2}n(2\sigma + n - 2)} = \frac{1}{4R} n e^{-\frac{\pi}{2n}} \left( 1 - e^{-\frac{\pi}{n}} \right)^{n+1} + 2 \left( \frac{n - 1}{n} \right) (1 - e^{-\frac{\pi}{n}})^n e^{-\frac{\pi}{n}} \quad (5.6)
\]
By using the Euler-Maclaurin sum formula we may extract out of this formula the finite part of $E^{\text{reg}}(\alpha, n)$ which does not depend upon $\alpha$. This is the Casimir energy. We have tabulated our results for various $n$ in Table 1.
Table 1. Casimir energy of a massless scalar field in the \( n \) dimensional Einstein universe with Dirichlet boundary conditions on \( \Sigma^{n-1} \)

| spatial dimension ( \( n - 1 \) ) | Casimir Energy / \( \left( \frac{\hbar c}{4R} \right) \) |
|---------------------------------|---------------------------------|
| 1                              | \(-\frac{1}{72}\)               |
| 2                              | \(-\frac{1}{98}\)               |
| 3                              | \(+\frac{1}{380}\)              |
| 4                              | \(+\frac{17}{7680}\)            |
| 5                              | \(-\frac{31}{120960}\)         |
| 6                              | \(-\frac{367}{774144}\)        |
| 7                              | \(+\frac{289}{725760}\)        |
| 8                              | \(+\frac{27859}{265420800}\)   |
| 9                              | \(-\frac{317}{45619200}\)      |
| 10                             | \(-\frac{1295803}{54499737600}\) |

We also may extract the Casimir energy out of (5.5) by using the zeta regularization technique [12]. For example, if \( n = 4 \), we obtain from eqn. (5.5)

\[
E = \frac{\hbar c}{4R} \left( \sum_{\sigma=0}^{\infty} (\sigma + 1)^2 \sigma \right).
\]  

(5.7)

We rewrite this as

\[
E = \frac{\hbar c}{4R} \left( \sum_{\sigma=0}^{\infty} (\sigma + 1)^3 - \sum_{\sigma=0}^{\infty} (\sigma + 1)^2 \right).
\]  

(5.8)

Now we use

\[
\zeta(-3) = \sum_{\sigma=0}^{\infty} (\sigma + 1)^3 = \frac{1}{120} \quad (5.9a)
\]

and

\[
\zeta(-2) = \sum_{\sigma=0}^{\infty} (\sigma + 1)^2 = 0 \quad (5.9b)
\]

in eqn. (5.8) to obtain

\[
E = +\frac{1}{480} \frac{\hbar c}{R} \quad (5.10)
\]

which is the result obtained in ref. [13]
6. Conclusions

Our results may be compared with the corresponding results for the Casimir energy of a massless scalar field in an n dimensional Minkowski space with Dirichlet boundary conditions on a sphere $S^{n-2}$ of radius $R$ in $\mathbb{R}^{n-1}$ ($n-1$ spatial dimensions) (c.f. ref. [14]). Notice that we obtain the same type of behavior for alteration in signs in steps of two, and in odd spatial dimensions our results agree qualitatively with the corresponding results for the corresponding problem in Minkowski space (Casimir energy of a massless scalar field in n dimensional Minkowski space with Dirichlet boundary conditions on a sphere). However, in contrast to our results for the Einstein universe, calculations for the Casimir energy in Minkowski spaces of odd dimensions (even spatial dimensions) give infinite answers [14], [15]! This is a clear manifestation of the greater regularity properties obtained by working with quantum field theory in the Einstein universe [16].

For the physically important case of three spatial dimensions we obtain from the Table that the Casimir energy is $+\frac{\hbar c}{350 R}$. Notice the all important sign: the energy is positive, as it is for the corresponding problem in Minkowski space. We can explain why both results (our result and the corresponding result for the Casimir energy of a scalar field with Dirichlet boundary conditions on a sphere in Minkowski space) are positive as follows. First, we argue that the Casimir energy with Dirichlet boundary conditions on spheres of slightly smaller radii in the Einstein universe must also be positive and greater than $+\frac{\hbar c}{350 R}$. This is because the equator $\Sigma^3$ is the sphere of largest radius in the Einstein universe, and, therefore, the Casimir energy must be a minimum for $\Sigma^3$. If the energy were not a minimum, then, on the one hand, there would be an outward force on $\Sigma^3$ centered at the "north pole" i.e. the point $\hat{u} = (0, 0, 0, R) \in S^3$. On the other hand, by invariance of conformal inversion, we can equally view the problem from the diametrically opposite point $\hat{u}' = (0, 0, 0, -R) \in S^3$, and from this point of view the Casimir force will be an inward force towards the point $\hat{u}$. The only way out of this paradox is to insist that the Casimir energy be a minimum for $\Sigma^3$. (Arguments of this nature, which exploit the symmetry of the Einstein universe under conformal inversion, have been used before; in particular, Landau and Lifschitz ([17]) use this symmetry together with Gauss' law to show that the total charge of the Einstein universe must be zero.) Hence, by the physical requirement that the observable energy must be a continuous function, we conclude that the Casimir energy must be positive for a sphere of cosmic size in the four dimensional Einstein universe and smaller in radius than $\Sigma^3$. (We have also obtained the same conclusion in a much more rigorous way, namely by applying second order perturbation theory to the image of the Einstein energy under scaling [18].) If we relate the Einstein universe and the Minkowski space via a Lorentzian version of stereographic projection (the flat or "Minkowski space" parallelization in the terminology of ref. [19], we can conclude from this result that also for a sphere of cosmic size in Minkowski space-time the Casimir energy must be positive. Then, using a scaling argument [18], we conclude that the Casimir energy of a massless scalar field in four dimensional Minkowski space with Dirichlet boundary conditions on a sphere of radius $R$ is positive.

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