A STRONGER CONCEPT OF K-STABILITY

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Abstract. In this paper, by introducing a wider class of one-parameter group actions for test configurations, we have a stronger version of the definition of K-stability. This allows us to obtain some key step of [12] in proving that constant scalar curvature polarization implies K-stability for polarized algebraic manifolds.

1. Introduction

In this paper, we fix once for all a polarized algebraic manifold \((M, L)\) consisting of a connected projective algebraic manifold \(M\) of dimension \(n\), defined over \(\mathbb{C}\), and a very ample line bundle \(L\) over \(M\). Take a 1-dimensional algebraic torus \(T_0 := \mathbb{C}^*\) which acts on the affine space

\[ A^1 = \{ z \in \mathbb{C} \} \]

by multiplication of complex numbers with some positive weight \(\alpha\), so that the action is given by\(^1\)

\[ \mathbb{C}^* (= T_0) \times A^1 \to A^1, \quad (t, z) \to t^\alpha z. \]

Let \((\mathcal{M}, \mathcal{L})\) be a test configuration (cf. Donaldson [2]) for \((M, L)\), so that we have a \(T_0\)-equivariant projective morphism

\[ \pi : \mathcal{M} \to A^1 (= \{ z \in \mathbb{C} \}) \]

of an irreducible reduced algebraic variety \(\mathcal{M}\), defined over \(\mathbb{C}\), onto the affine space \(A^1\), where \(\mathcal{L}\) is a relatively very ample invertible sheaf on

\(^1\) In Donaldson’s definition of a test configuration, \((\mathcal{M}_z, \mathcal{L}_z) \cong (M, L^z), z \neq 0\), for some exponent \(\alpha\), where the weight is chosen to be 1. However for our definition, the weight is \(\alpha\) and the exponent is 1. When \(T_0\) is replaced by its unramified cover of degree \(\alpha\) in Donaldson’s definition, the pair of the weight and the exponent changes from \((1, \alpha)\) to \((\alpha, 1)\), yielding our definition equivalent to Donaldson’s.
the fiber space \( \mathcal{M} \) over \( \mathbb{A}^1 \). Then the restriction of the pair \( (\mathcal{M}, \mathcal{L}) \) to each fiber \( \mathcal{M}_z := \pi^{-1}(z) \) admits a holomorphic isomorphism

\[
(\mathcal{M}_z, \mathcal{L}_z) \cong (M, L), \quad z \neq 0,
\]

where the \( T_0 \)-action on \( \mathcal{M} \) lifts to a \( T_0 \)-linearization of \( \mathcal{L} \). We now consider the identity component

\[
\mathcal{P} := \text{Aut}^0(\mathcal{M})^{T_0}
\]

of the group of all holomorphic automorphisms of \( \mathcal{M} \) commuting with the \( T_0 \)-action on \( \mathcal{M} \). Note that every element of \( \mathcal{P} \) maps fibers of \( \pi \) to fibers of \( \pi \), inducing multiplication by a complex number on the base space \( \mathbb{A}^1 \). For the Lie algebra \( \mathfrak{p} \) of \( \mathcal{P} \), in view of the \( T \)-equivariance of \( \pi \), we can choose an element \( X_0 \neq 0 \) in \( \mathfrak{p} \) generating holomorphically the \( T_0 \)-action on \( \mathcal{M} \) such that \( \exp (2\pi \sqrt{-1} X_0/\alpha) = \text{id}_\mathcal{M} \) and that

\[
\pi_* X_0 = \alpha z \frac{\partial}{\partial z},
\]

where \( T_0 \) is often identified with \( \{ \exp(sX_0) ; s \in \mathbb{C} \} \). For the fiber space \( \mathcal{M} \) over \( \mathbb{A}^1 \), consider the subgroup \( \text{Aut}(\mathcal{M}, \mathbb{A}^1)^{T_0} \) in \( \mathcal{P} \) of all fiber-preserving elements in \( \mathcal{P} \). Let \( \mathcal{Q} = \mathcal{Q}(\mathcal{M}) \) be its identity component

\[
\mathcal{Q} := \text{Aut}^0(\mathcal{M}, \mathbb{A}^1)^{T_0}.
\]

Then by restricting each automorphism in \( \mathcal{Q} \) to the fiber \( \mathcal{M}_1 \) \( (= \pi^{-1}(1) = M) \), \( \mathcal{Q} \) is viewed as a closed algebraic subgroup

\[
\mathcal{Q} \hookrightarrow \text{Aut}^0(M) \quad (= \text{Aut}^0(\mathcal{M}_1)),
\]

where injectivity follows from the commutativity of the elements in \( \mathcal{Q} \) with the \( T_0 \)-action on \( \mathcal{M} \). Define \( H = H(\mathcal{M}) \) as the intersection

\[
H := \mathcal{Q} \cap G,
\]

where \( G \) is the maximal connected linear algebraic subgroup of \( \text{Aut}^0(M) \). Let \( \mathfrak{g} \) and \( \mathfrak{h} = \mathfrak{h}(M) \) be the Lie algebras of \( G \) and \( H = H(\mathcal{M}) \), respectively. By fixing a Kähler form \( \omega \) in the class \( c_1(L)_{\mathbb{R}} \), we say that an element \( Y \) of \( \mathfrak{g} \) is Hamiltonian if

\[
(1.1) \quad i(Y) \omega = \bar{\partial} f_{\omega,Y}
\]
for some real-valued smooth function \( f_{\omega,Y} \in C^\infty(M) \) with normalization condition \( \int_M f_{\omega,Y} \omega^n = 0 \). Then for every Hamiltonian element \( Y \) of \( \mathfrak{g} \), the closure in \( G \) of the holomorphic one-parameter group generated by \( Y \) is an algebraic torus. In view of the inclusion \( \mathfrak{h} \subset \mathfrak{g} \), we define a subset \( \mathfrak{S} = \mathfrak{S}(M) \) of \( \mathfrak{h} \) as

\[ \mathfrak{S} : \text{ the set of all Hamiltonian elements in } \mathfrak{h}. \]

Since the \( H \)-action on \( \mathcal{M} \) lifts to an \( H \)-linearization of some positive integral multiple of \( \mathcal{L} \), there exists a finite unramified cover \( \tilde{H} \) of \( H \) such that the \( \tilde{H} \)-action on \( \mathcal{M} \) induced by

\[ H \subset \mathcal{Q} = \text{Aut}^0(\mathcal{M}, \mathbb{A}^1)^{T_0} \]

lifts to an \( \tilde{H} \)-linearization of \( \mathcal{L} \). Given a element \( Y \) of \( \mathfrak{S} \), the closure \( \mathcal{S} := \overline{\tau_Y} \) in \( \tilde{H} \) of the holomorphic one-parameter group

\[ \tau_Y := \{ \exp(tY) \in \tilde{H} ; t \in \mathbb{C} \} \]

is an algebraic torus. Put \( \delta_Y := \text{dim}_\mathbb{C} \overline{\tau_Y} \). If \( \delta_Y \leq 1 \), then \( \overline{\tau_Y} \) and \( \tau_Y \) coincide, and in this case, we say that \( Y \) is quasi-regular. On the other hand, if \( \delta_Y > 1 \), then \( Y \) is said to be irregular. These concepts are analogous to the quasi-regularity or irregularity in Sasakian geometry.

Fix an element \( Y \) of \( \mathfrak{S} \). Then \( X := X_0 + Y \in \mathfrak{p} \) generates a holomorphic one-parameter group

\[ T := \{ \exp(tX) ; t \in \mathbb{C} \} \]

in \( \Sigma := \mathcal{S} \times T_0 \) by \( \exp(tX) = \exp(tY) \cdot \exp(tX_0) \). Here \( \exp(tY) \) is regarded as an element of \( \mathcal{S} \), while \( \exp(tX_0) \) sits in \( T_0 \). Since the \( T_0 \)-action on \( \mathcal{M} \) lifts to a \( T_0 \)-linearization of \( \mathcal{L} \), a similar lifting is true also for the \( T \)-action. Hence, whether \( Y \) is quasi-regular or irregular, \( \Sigma \) and
its subgroup $T$ act on the vector bundles\footnote{\text{Let $h_{m;1}$ be a Hermitian metric on the vector space $(E_m)_z$ at $z = 1$. By [4] and [17], we have a $\Sigma$-equivariant (and hence $T$-equivariant) trivialization for $E_m$, \begin{equation}
abla E_m \cong \mathbb{A}^1 \times (E_m)_0, \end{equation} taking $h_{m;1}$ to a Hermitian metric $h_{m;0}$ on $(E_m)_0$ which is preserved by the action of the compact torus $(T_0)_c \cong S^1$ in $T_0$. For the maximal compact subgroup $S_c$ of $S$, by taking an average, we may also assume that both $h_{m;0}$ and $h_{m;1}$ are $S_c$-invariant.}}

$$E_m := \pi_* \mathcal{L}^m, \quad m = 1, 2, \ldots,$$

over $\mathcal{M}$. Here vector bundles and invertible sheaves are used interchangeably throughout this paper. Put $N_m := \dim H^0(M, \mathcal{O}_M(L^m))$. Replacing $\mathcal{L}$ by its suitable positive integral multiple if necessary, we may assume that $\dim H^0(M_0, \mathcal{L}_0^m)$ is $N_m$ for all $m$ and that

$$\otimes^m H^0(\mathcal{M}_z, \mathcal{L}_z) \to H^0(\mathcal{M}_z, \mathcal{L}_z^m), \quad m = 1, 2, \ldots,$$

are surjective for all $z \in \mathbb{A}^1$ (cf. [9], Remark 4.6). In view of

$$\pi_* X = \alpha z \frac{\partial}{\partial z},$$

the projective morphism $\pi : \mathcal{M} \to \mathbb{A}^1$ is always $T$-equivariant, though $T$ is not isomorphic to $\mathbb{C}^*$ in irregular cases. Then a triple $(\mathcal{M}, \mathcal{L}, X)$ is called a \textit{generalized test configuration} of $(M, L)$, if $Y := X - X_0$ belongs to $\mathcal{S} = \mathcal{S}(\mathcal{M})$. Note that a test configuration (cf. [2]) of $(M, L)$ in an ordinary sense corresponds to the triple $(\mathcal{M}, \mathcal{L}, X_0)$, i.e., to the case $Y = 0$. Now for the triple $(\mathcal{M}, \mathcal{L}, X)$, we consider the weight

$$w_m = w_m(\mathcal{M}, \mathcal{L}, X)$$

of the $T$-action on $\det(E_m)_0$, i.e., the trace of the endomorphism on the vector space $(E_m)_0$ induced by $X$. Now for $m \gg 1$, we have the expansion

$$\frac{w_m}{m N_m} = F_0 + F_1 m^{-1} + F_2 m^{-2} + \ldots,$$

with coefficients $F_i = F_i(\mathcal{M}, \mathcal{L}, X) \in \mathbb{R}$. We now say that $(M, L)$ is $K$-\textit{semistable in a strong sense} if

$$F_1(\mathcal{M}, \mathcal{L}, X) \leq 0$$
for all generalized test configurations \((\mathcal{M}, \mathcal{L}, X)\) of \((M, L)\). Moreover, a K-semistable \((M, L)\) is said to be 
\textit{K-stable in a strong sense} if, for every generalized test configurations \((\mathcal{M}, \mathcal{L}, X)\) of \((M, L)\),
\[
\mathcal{M} = \mathbb{A}^1 \times M \text{ if and only if } F_1(\mathcal{M}, \mathcal{L}, X) \text{ vanishes},
\]
where \(T\) does not necessarily act on the second factor \(M\) of \(\mathbb{A}^1 \times M\) trivially. Note that K-stability in a strong sense implies K-stability in an ordinary sense, and vice versa (cf. [13]). In this paper, by effectively using the concept of generalized test configurations, we shall complete the proof of the following:

**Main Theorem.** A polarized algebraic manifold \((M, L)\) is K-stable if the class \(c_1(L)_{\mathbb{R}}\) admits a Kähler metric of constant scalar curvature.

In Case 1 of Section 5 of [12], some key step in the proof of Main Theorem above is not given. We here give the omitted proof by showing that, given a Kähler metric in \(c_1(L)_{\mathbb{R}}\) of constant scalar curvature, the vanishing of \(F_1(\mathcal{M}, \mathcal{L})\) yields a suitable generalized test configuration \((\mathcal{M}', \mathcal{L}', X')\), from which \(\mathcal{M} = \mathcal{M}' \cong \mathbb{A}^1 \times M\) easily follows.

2. **A generalized test configuration associated to \(W\)**

In this section, we fix a test configuration \((\mathcal{M}, \mathcal{L})\) of \((M, L)\). To each Hamiltonian element \(W\) in \(\mathfrak{g}\), we associate a generalized test configuration \((\mathcal{M}', \mathcal{L}', X')\) of \((M, L)\) as follows:

Since the \(T_0\)-action on \(E_m\) preserves the fiber \((E_m)_0\) of \(E_m\) over the origin, we write the associated representation on \((E_m)_0\) as
\[
\Psi_{m,x_0}(=\Psi_{m,x_0;\mathcal{M},\mathcal{L}}): T_0 \rightarrow \text{GL}((E_m)_0).
\]
For each \(\hat{t} \in T_0\), we set \(t := \hat{t}^{N_m} \in T_0\), and define an algebraic group homomorphism \(\Psi^\text{SL}_{m,x_0}(=\Psi^\text{SL}_{m,x_0;\mathcal{M},\mathcal{L}}): T_0 \rightarrow \text{SL}((E_m)_0)\) by
\[
\Psi^\text{SL}_{m,x_0}(\hat{t}) := \frac{\Psi_{m,x_0}(t)}{\det(\Psi_{m,x_0}(t))}, \quad \hat{t} \in T_0.
\]
Let \(\Pi_m: \text{GL}((E_m)_0) \rightarrow \text{PGL}((E_m)_0)\) be the natural projection induced by the contragradient representation. For each \(s \in \mathbb{C}\), put \(\hat{s} := s/N_m\)
and $z := e^s$. In view of the identification of $T_0$ with $\mathbb{C}^*$, by setting

$$\mu_{m,s} := (\Pi_m \circ \Psi_{m,X_0}^{SL})(z),$$

we define $M_s := \mu_{m,s}(M)$, $s \in \mathbb{C}$, where $M$ is regarded as a submanifold of $\mathbb{P}^*((E_m)_0)$ by the inclusion

$$(2.1) \quad M = \mathcal{M}_1 \subset \mathbb{P}^*((E_m)_1) \cong \mathbb{P}^*((E_m)_0).$$

Here the last isomorphism is induced by the $T_0$-equivariant trivialization $E_m \cong \mathbb{A}^1 \times (E_m)_0$ in (1.2). Hence identifying the projective bundle $\mathbb{P}^*(E_m)$ with the product $\mathbb{A}^1 \times \mathbb{P}^*((E_m)_0)$, we obtain

$$(2.2) \quad \mathcal{M}_z = \{z\} \times M_s, \quad s \in \mathbb{C},$$

where $\mathcal{M}$ is regarded as a subvariety of $\mathbb{P}^*(E_m)$. Since $W$ is a Hamiltonian element of $\mathfrak{g}$, we see that

$$Y' := -W$$

is also Hamiltonian. For the holomorphic one-parameter group $\tau_{Y'}$, generated by $Y'$ in $G$, the closure $\mathcal{S}' := \tau_{Y'}$, in $G$ (later on, $\mathcal{S}'$ will be replaced by its finite unramified cover) is an algebraic torus. Put

$$\left\{ \begin{array}{l}
\hat{\kappa}_z := \Psi_{1,X_0}(z), \\
\kappa_z := \mu_{1,s} = \Pi_1 \circ \Psi_{1,X_0}(z),
\end{array} \right.$$  

for all $z (= e^s) \in \mathbb{C}^*$, where $m = 1$. Now for $m = 1$ and $z = e^s$, we define a subset $U$ of $(\mathbb{A}^1 \setminus \{0\}) \times \mathbb{P}^*((E_m)_0)$ by

$$U := \bigcup_{0 \neq z \in \mathbb{A}^1} \{z\} \times M_s \left( = \bigcup_{s \in \mathbb{C}} \{z\} \times M_s \right) = \mathcal{M} \setminus \mathcal{M}_0.$$  

The algebraic group $\Sigma' := \mathcal{S}' \times \mathbb{C}^* (= \mathcal{S}' \times T_0)$ acts biregularly on $U$ by

$$(2.3) \quad \Sigma' \times U \to U, \quad ((\theta,t),(z,p)) \mapsto (t z, \kappa_{t z} \theta \kappa^{-1}_z p),$$

where $(\theta,t) \in \mathcal{S}' \times \mathbb{C}^*$ and $(z,p) \in U \subset (\mathbb{A}^1 \setminus \{0\}) \times \mathbb{P}^*((E_1)_0)$. For each $\theta \in \mathcal{S}'$, let $\hat{\theta}$ denote the element of $\text{GL}((E^*_1)_0)$ induced by $\theta$ via the identification

$$(E^*_1)_0 \cong (E^*_1)_1 \cong \frac{H^0(M, \mathcal{O}_M(L))^*}{6}.$$
obtained from (1.2) applied to \( m = 1 \). Then (2.3) is induced by the \( \Sigma' \)-action on \( (\mathbb{A}^1 \setminus \{0\}) \times (E_1^*)_0 \) defined by

\[ ((\theta, t), (z, q)) \mapsto (tz, \hat{k}_t \hat{\theta} \hat{k}_z^{-1} q), \]

where \( (\theta, t) \in S' \times \mathbb{C}^* (= \Sigma') \) and \( (z, q) \in (\mathbb{A}^1 \setminus \{0\}) \times (E_1^*)_0 \). Since the line bundle \( \mathcal{O}_{\mathbb{P}^*(E_1)}(-1) \) is viewed as the blowing-up of \( \mathbb{A}^1 \times (E_1^*)_0 \) along \( \mathbb{A}^1 \times \{0\} \), we see that the \( \Sigma' \)-action in (2.3) naturally lifts to a \( \Sigma' \)-linearization of \( L = \mathcal{O}_{\mathbb{P}^*(E_1)}(1)|_U \) over \( U = \mathcal{M} \setminus \mathcal{M}_0 \) induced by (2.4). Let \( \text{pr}_1 : (\mathbb{A}^1 \setminus \{0\}) \times \mathbb{P}^*((E_1)_0) \rightarrow \mathbb{A}^1 \setminus \{0\} \) and \( \text{pr}_2 : (\mathbb{A}^1 \setminus \{0\}) \times \mathbb{P}^*((E_1)_0) \rightarrow \mathbb{P}^*((E_1)_0) \) be the projections to the first factor and to the second factor, respectively. Take a \( \Sigma' \)-equivariant compactification \( \hat{U} \) (cf. [23]) of the algebraic variety \( U \) such that the restriction of \( \text{pr}_1 \) to \( U \) extends to a \( \Sigma' \)-equivariant proper morphism

\[ \hat{\text{pr}} : \hat{U} \rightarrow \mathbb{P}^1(\mathbb{C}) \]

onto \( \mathbb{P}^1(\mathbb{C}) := \mathbb{A}^1 \cup \{\infty\} \). Here the first factor \( S' = S' \times T_0 \) acts on \( \mathbb{P}^1(\mathbb{C}) \) trivially. By restricting \( \hat{\text{pr}} \) to \( \hat{U} := \hat{U} \setminus \hat{\text{pr}}^{-1}(\{\infty\}) \), we obtain a \( \Sigma' \)-equivariant proper morphism

\[ \hat{\text{pr}} : \hat{U} \rightarrow \mathbb{A}^1. \]

Replacing \( \hat{U} \) by its suitable \( \Sigma' \)-equivariant desingularization, we may assume without loss of generality that \( \hat{U} \) is smooth and sits over \( \mathcal{M} \) by a \( T_0 \)-equivariant modification

\[ \iota : \hat{U} \rightarrow \mathcal{M}. \]

Fix a general hyperplane \( \mathcal{H} \) on \( \mathbb{P}^*((E_1)_0) \). We also fix a divisor \( \hat{\mathcal{H}} \) on \( \hat{U} \) such that \( \hat{\mathcal{H}} - \mathcal{H}' \) is effective and that

\[ \text{Supp}(\hat{\mathcal{H}} - \mathcal{H}') \subset \hat{\text{pr}}^{-1}(0), \]

where \( \mathcal{H}' \) denotes the irreducible reduced divisor on \( \hat{U} \) obtained as the closure of \( \text{pr}_2^{-1}(\mathcal{H}) \cap U \) in \( \hat{U} \). Then the divisor \( \hat{\mathcal{H}} \) defines a line bundle \( \hat{\mathcal{L}} := \mathcal{O}_{\hat{U}}(\hat{\mathcal{H}}) \) over \( \hat{U} \) extending \( \mathcal{L}|_U \). In view of [16], p.35, the \( \Sigma' \)-action on \( \hat{U} \) lifts to a \( \Sigma' \)-linearization of some positive integral multiple \( \hat{\mathcal{L}}^\beta \) of \( \hat{\mathcal{L}} \). We then consider the unramified cover

\[ \Sigma' \rightarrow \Sigma', \quad \sigma \mapsto \sigma^\beta. \]
Given $\Sigma'$ on the right-hand side, replacing this by $\Sigma'$ on the left-hand side (and hence the weight $\alpha$ in the introduction is replaced by $\alpha \beta$, so that the exponent can be chosen to be 1), we may assume from the beginning that the $\Sigma'$-action on $\hat{U}$ lifts to a $\Sigma'$-linearization of $\hat{L}$. For each $z \in \mathbb{A}^1 \setminus \{0\} \subset \mathbb{P}^1(\mathbb{C})$, let $\hat{L}_z$ denote the restriction of $\hat{L}$ to the fiber $U_z := \hat{pr}^{-1}(z)$ over $z$, and we define $\mathcal{M}'_z \subset \mathbb{P}^*((\hat{pr}_*\hat{L})_z)$ by

$$\mathcal{M}'_z := \{z\} \times \Phi|_{\hat{L}_z}(U_z),$$

where $\Phi|_{\hat{L}_z}$ is the projective embedding of $U_z$ associated to the complete linear system $|\hat{L}_z|$ for the line bundle $\hat{L}_z$. Then $U$ is viewed as the subset

$$U' := \bigcup_{0 \neq z \in \mathbb{A}^1} \mathcal{M}'_z$$

of $\mathbb{P}^*((\hat{pr}_*\hat{L})$. Let $\mathcal{M}'$ be the $\Sigma'$-invariant subvariety $\bar{U}'$ of $\mathbb{P}^*((\hat{pr}_*\hat{L})$ obtained as the closure of $U'$ in $\mathbb{P}^*((\hat{pr}_*\hat{L})$, where $U'$ is $\Sigma'$-equivariantly identified with $U$ above. The projection

$$\pi' : \mathcal{M}' \to \mathbb{A}^1,$$

induced by the natural projection $\mathbb{P}^*((\hat{pr}_*\hat{L}) \to \mathbb{A}^1$ is a $\Sigma'$-equivariant projective morphism with a relatively very ample invertible sheaf $\mathcal{L}'$, where $\mathcal{L}'$ denotes the restriction of $\mathcal{O}_{\mathbb{P}^*((\hat{pr}_*\hat{L})}(1)$ to $\mathcal{M}'$. In view of the $\Sigma'$-action on $\hat{L}$, note here that $\Sigma'$ acts on $\mathcal{L}'$ covering the $\Sigma'$-action on $\mathcal{M}'$. By replacing $\hat{U}$ by its $\Sigma'$-equivariant modification if necessary, we may assume that a $\Sigma'$-equivariant birational surjective morphism

$$\iota' : \hat{U} \to \mathcal{M}'$$

exists. We now observe that $\mathcal{M}' \setminus \mathcal{M}'_0 = U' \cong U = \mathcal{M} \setminus \mathcal{M}_0$, where $\mathcal{M}_0$ denotes the scheme-theoretic fiber of $\pi'$ over the origin. Then

$$(\mathcal{M}'_z, \mathcal{L}_z) \cong (M, L), \quad z \in \mathbb{A}^1 \setminus \{0\}.$$

Hence $(\mathcal{M}', \mathcal{L}')$ with the $\mathbb{C}^*$-action is a test configuration for the polarized algebraic manifold $(M, L)$, where $\mathbb{C}^* (= T_0)$ acts on $\mathcal{M}'$ as the second factor of $\Sigma' = S' \times \mathbb{C}^*$. Then by the notation in the introduction,

$$S' \subset Q(\mathcal{M}') \cap G = H(\mathcal{M}').$$

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Since $Y'$ belongs to the Lie algebra of $\mathcal{S}'$, we see that $Y' \in \mathfrak{h}(\mathcal{M}')$. Moreover $Y'$ is Hamiltonian, and hence

$$Y' \in \mathfrak{S}(\mathcal{M}').$$

Even for $\mathcal{M}'$ in place of $\mathcal{M}$, we mean by $X_0$ the generator as in the introduction for the $T_0$-action. Then by setting $X' := X_0 + Y'$, we have a generalized test configuration $(\mathcal{M}', \mathcal{L}', X')$ of $(M, L)$.

For the generalized test configuration $(\mathcal{M}', \mathcal{L}', X')$ defined above, we set $T' := \{\exp(tX'); t \in \mathbb{C}\}$, and consider the vector bundle

$$E'_m := \pi'_*(\mathcal{L}')^m, \quad m = 1, 2, \ldots,$$

over $\mathbb{A}^1$. Note that $\Sigma'$ acts on $E'_m$ covering the natural $\Sigma'$-action on $\mathbb{A}^1$. Then as in (1.2), we have a $\Sigma'$-equivariant trivialization for $E'_m$,

$$(2.5) \quad E'_m \cong \mathbb{A}^1 \times (E'_m)_0.$$

Consider the line bundle $\mathcal{C} := (i')^*\mathcal{L} \otimes (i^*\mathcal{L})^{-1}$ over $\hat{U}$. Then $\mathcal{C}$ is expressible as $\mathcal{O}_{\hat{U}}(\mathcal{D})$ for a divisor $\mathcal{D}$ on $\hat{U}$ satisfying $\text{Supp}(\mathcal{D}) \subset \hat{\text{pr}}^{-1}(0)$ set-theoretically. Then the direct image sheaf $\Gamma := \hat{\text{pr}}_\ast\mathcal{C}$ over $\mathbb{A}^1$ is invertible. Note that the $T_0$-action on $\hat{U}$ naturally lifts to a $T_0$-linearization of $\mathcal{C}$. Take a $T_0$-equivariant trivialization

$$\Gamma \cong \mathbb{A}^1 \times \Gamma_0,$$

where $\Gamma_0$ is the fiber of $\Gamma$ over the origin. Fix an element $0 \neq \xi \in \Gamma_0$. Then we have a multiplicative algebraic character $\chi : T_0 \rightarrow \mathbb{C}^*$ such that $g \cdot \xi = \chi(g)\xi$ for all $g \in T_0$. Hence the nonvanishing section $\gamma_\xi$ of $\Gamma$ associated to $\mathbb{A}^1 \times \{\xi\}$ satisfies

$$(2.6) \quad g \cdot \gamma_\xi = \chi(g)\gamma_\xi, \quad g \in T_0.$$

Now by $H^0(\mathbb{A}^1, \Gamma) \cong H^0(\hat{U}, \mathcal{C})$, regard $\gamma_\xi$ as an element of $H^0(\hat{U}, \mathcal{C})$, and we denote by $(\gamma_\xi)$ the associated divisor on $\hat{U}$. Then $\mathcal{D}$ above can be chosen as $(\gamma_\xi)$. In view the isomorphisms

$$\begin{cases}
H^0(\mathbb{A}^1, E_m) \cong H^0(\mathcal{M}, \mathcal{L}^m) \cong H^0(\hat{U}, i^*\mathcal{L}^m), \\
H^0(\mathbb{A}^1, E'_m) \cong H^0(\mathcal{M}', \mathcal{L}'^m) \cong H^0(\hat{U}, (i')^*\mathcal{L}'^m),
\end{cases}$$

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the linear isomorphism $H^0(\hat{U}, v^*L^m) \cong H^0(\hat{U}, (t')^*L^m)$, $e \leftrightarrow \gamma^m \cdot e$, induces a vector bundle isomorphism

\[(2.7) \quad E_m \cong E'_m, \quad m = 1, 2, \ldots,\]

where in terms of this isomorphism, the $T_0$-actions on $E_m$ and $E'_m$ differ just by $\chi^m$, i.e., $\Psi_{m,x_0;M',\mathcal{L}'}(g) = \chi(g)^m \Psi_{m,x_0;M,\mathcal{L}}(g)$ for all $g \in T_0$. Hence we obtain

\[(2.8) \quad \Psi_{m,x_0;M,\mathcal{L}} = \Psi_{m,x_0;M',\mathcal{L}'};
\]

and we often write them as $\Psi_{m,x_0}$ for simplicity. Then by the definition of $\mu_{m,s}$ with $z = e^s$, it follows that

\[(\Pi_m \circ \Psi_{m,x_0;M,\mathcal{L}})(z) = \mu_{m,s} = \Pi_m \circ \Psi_{m,x_0;M',\mathcal{L}'}(z), \]

for all $s \in \mathbb{C}$. For the time being, put $m = 1$. Since $\mathcal{M}_1 = M = M'_1$ in $\mathbb{P}^*(E_1)_0$, we see from (2.2) that $\mathcal{M}_z = \{z\} \times M_s = M'_z$ for all $z \in \mathbb{C}^*$, where $M_s$ is the subvariety $\mu_{1,s}(M)$ in $\mathbb{P}^*((E_1)_0) (= \mathbb{P}^*((E'_1)_0))$. Thus

\[(2.9) \quad \mathcal{M} = \mathcal{M}'\]

as the closure of $\mathcal{M} \setminus \mathcal{M}_0 = \mathcal{M}' \setminus \mathcal{M}'_0$ in the complex variety $\mathbb{A}^1 \times \mathbb{P}^*((E_1)_0) = \mathbb{A}^1 \times \mathbb{P}^*((E'_1)_0)$.

Note that both the $T'$-action and the $S'$-action and on $E'_m$ preserves $(E''_m)_0$. We write the associated representation on $(E''_m)_0$ as

\[(2.10) \quad \left\{ \begin{array}{l}
\Psi_{m,x_0'}(= \Psi_{m,x_0';M',\mathcal{L}'}): T' \to \text{SL}((E''_m)_0) (= \text{SL}((E'_m)_0)), \\
\Psi_{m,s'}(= \Psi_{m,s';M',\mathcal{L}'}): S' \to \text{SL}((E''_m)_0) (= \text{SL}((E'_m)_0)).
\end{array} \right.\]

By $T_0 \cong \{1\} \times T_0$ and $S' \cong S' \times \{1\}$, we view $T_0$, $T'$, $S'$ as subgroups of $\Sigma' = S' \times T_0$. Hence $\Psi_{m,x_0'}$, $\Psi_{m,x_0'}$, $\Psi_{m,s'}$, $\Psi_{m,s'}$ are the restrictions of

$\Psi_{m,x_0'}(= \Psi_{m,x_0';M',\mathcal{L}'}): \Sigma' \to \text{SL}((E''_m)_0) (= \text{SL}((E'_m)_0)).$

to these subgroups. Since the $T_0$-action and the $S'$-action on $(E''_m)_0$ commute, and since every $\exp(sX')$, $s \in \mathbb{C}$, in $T'$ is expressible as $\exp(sX_0) \cdot \exp(sY')$, we obtain

\[(2.11) \quad \psi_s \cdot \varphi_s = \Psi_{m,x_0'}(\exp(sX')) = \varphi_s \cdot \psi_s,
\]

by setting $\varphi_s := \Psi_{m,x_0'}(\exp(sX_0))$ and $\psi_s := \Psi_{m,s'}(\exp(sY'))$. Note that $\mu_{m,s} = \Pi_m(\varphi_s)$. For every $Y \in \mathfrak{g}(\mathcal{M}') (= \mathfrak{g}(\mathcal{M}))$, by setting $X :=
First, we consider the case \( Y = Y' \). Then by \( X = X' \),

\[
\mathcal{V}^m(\mathcal{M}', \mathcal{L}', X') = (\Pi_m \circ \Psi_{m,X'})_*(X').
\]

Next let \( Y = 0 \), so that \( X = X_0 \). Then by (2.8),

\[
(2.12) \quad \mathcal{V}^m(\mathcal{M}', \mathcal{L}', X_0) = (\Pi_m \circ \Psi_{m,X_0})_*(X_0) = \mathcal{V}^m(\mathcal{M}, \mathcal{L}, X_0).
\]

3. **Proof of Main Theorem**

In this section, fixing a polarized algebraic manifold \((M, L)\) as in the introduction, we assume that \( c_1(L) \) admits a Kähler metric

\[
\omega_\infty = c_1(L, h_\infty)
\]

of constant scalar curvature. Here \( h_\infty \) is a suitably chosen Hermitian metric for \( L \). We here retain the notation in the preceding sections. Let \( \omega_{\text{FS}} \) denote the Fubini-Study metric for \( \mathbb{P}^*((E_m)_0) = \mathbb{P}^{|m|} (\mathbb{C}) \), where we identify \( \mathbb{P}^*((E_m)_0) \) with \( \mathbb{P}^{|m|} (\mathbb{C}) \) by an orthonormal basis for the Hermitian metric \( h_{m,0} \) (cf. (1.2)) on \((E_m)_0\). For each \( s \in \mathbb{C} \), we consider the orthogonal complement \( TM_s^\perp \) of \( TM_s \) in \( T\mathbb{P}^*((E_m)_0)|_{M_s} \) in terms of the Fubini-Study metric \( \omega_{\text{FS}} \) on \( \mathbb{P}^*((E_m)_0) \). By writing \( h_{m,0} \) simply as \( h_m \), and following [12], we may choose a sequence \( h_m, m = 1, 2, \ldots, \) converging in \( C^\infty \) to \( h_\infty \) on \( M \hookrightarrow \mathbb{P}^*((E_m)_0) \) (cf. (2.1)).

By fixing a Hamiltonian element \( W \) in \( \mathfrak{g} \), we consider the associated generalized test configuration \((\mathcal{M}', \mathcal{L}', X')\) as in the preceding section, where \( W \) will be specified later. Recall that \( \mathcal{M}'_s = \{z\} \times M_s \), where \( z = e^s \). Restricting \( \mathcal{V}^m(\mathcal{M}', \mathcal{L}', X') \) to \( M_s, s \in \mathbb{C} \), we can write

\[
(3.1) \quad \mathcal{V}^m(\mathcal{M}', \mathcal{L}', X')|_{M_s} = \mathcal{V}^m_{TM_s}(\mathcal{M}', \mathcal{L}', X') + \mathcal{V}^m_{T\mathcal{M}'_s}(\mathcal{M}', \mathcal{L}', X'),
\]

where \( \mathcal{V}^m_{TM_s}(\mathcal{M}', \mathcal{L}', X') \) and \( \mathcal{V}^m_{T\mathcal{M}'_s}(\mathcal{M}', \mathcal{L}', X') \) are smooth sections of \( TM_s \) and \( T\mathcal{M}'_s^\perp \), respectively. Put \( \omega_s := (\mu_{m,s}^* \omega_{\text{FS}}/m)|_M \). For the subspace \( \mathfrak{r} := H^0(M, \mathcal{O}_M(TM)) \) of \( \mathfrak{w} := H^0(M, C^\infty(TM)) \), let \( \mathfrak{r}_s^\perp \) denote its orthogonal complement in terms of the Hermitian pairing

\[
< W_1, W_2 >_s := \int_M (W_1, W_2)_{\omega_s} \omega^n_s, \quad W_1, W_2 \in \mathfrak{w},
\]
where \((W_1, W_2)\) is the pointwise Hermitian pairing of \(W_1\) and \(W_2\) by the Kähler metric \(\omega\) on \(M\). Now, as in [12], \(V_{TM_s}^m = V_{TM_s}^m(\mathcal{M}', \mathcal{L}', X')\) is written as a sum

\[
V_{TM_s}^m = V_{m,s}^0 + V_{m,s}^\bullet,
\]

where \(V_{m,s}^0 = V_{m,s}(\mathcal{M}', \mathcal{L}', X')\) and \(V_{m,s}^\bullet = V_{m,s}(\mathcal{M}', \mathcal{L}', X')\) belongs to \((\mu_{m,s})_* \mathfrak{r}\) and \((\mu_{m,s})_* \mathfrak{r}_s^\perp\), respectively. For \(X_0\), \(Y'\) and \(X'\) regarded as elements of the Lie algebra of \(\Sigma\), we consider the \(\Sigma\)-action on \((E_m')_0 (= (E_m)_0)\). Then by \(X_0^{(m)}\), \(Y^{(m)}\) and \(X^{(m)}\), we mean the holomorphic vector fields on the projective space \(\mathbb{P}^*((E_m')_0) (= \mathbb{P}^*((E_m)_0))\) induced by the infinitesimal actions of \(X_0\), \(Y'\) and \(X'\), respectively. We now observe that \(X^{(m)} = V^m(\mathcal{M}', \mathcal{L}', X')\) and that \(X_0^{(m)} = V^m(\mathcal{M}', \mathcal{L}', X_0)\). Hence by \(X^{(m)} = X_0^{(m)} + Y^{(m)}\) and (2.12), it follows that

\[
\left\{ \begin{array}{c}
V^m(\mathcal{M}', \mathcal{L}', X') - Y^{(m)} = X_0^{(m)} \\
= V^m(\mathcal{M}', \mathcal{L}', X_0) = V^m(\mathcal{M}, \mathcal{L}, X_0).
\end{array} \right.
\]

By (2.1), we identify \(M\) with \(\mathcal{M}_1' (= \mathcal{M}_1)\) sitting in \(\mathbb{P}^*((E_m')_0) (= \mathbb{P}^*((E_m)_0))\). Let us now put

\[
A_m := Y^{(m)}|_M \quad \text{and} \quad B_m := X_0^{(m)}|_M.
\]

Then \(A_m\) coincides with the holomorphic vector field \(Y' = -W\) on \(M\), and by (3.3), \(B_m\) coincides with \(V^m(\mathcal{M}, \mathcal{L}, X_0)|_M\). The element \(\varphi_s\) in \(\text{SL}((E_m)_0) (= \text{SL}((E_m')_0))\) (cf. (2.11)) induces a projective linear transformation on \(\mathbb{P}^*((E_m')_0) (= \mathbb{P}^*((E_m)_0))\), taking \(M\) to \(M_s\). This mapping of \(M\) onto \(M_s\) is written also as \(\varphi_s\) by abuse of terminology. Then for each \(s \in \mathbb{C}\), we can write

\[
Y^{(m)}|_{M_s} = (\varphi_s)_* A_m = -(\varphi_s)_* W,
\]

\[
X_0^{(m)}|_{M_s} = (\varphi_s)_* B_m = (\varphi_s)_* \{ V^m(\mathcal{M}, \mathcal{L}, X_0)|_M \},
\]

where (3.4) follows from the commutativity of the \(S'\)-action and \(T_0\)-action. Now by (3.4), \(Y^{(m)}|_{M_s}\) is a holomorphic vector field on \(M_s\), and hence by (3.3), we obtain

\[
\left\{ \begin{array}{c}
V_{TM_s}^m(\mathcal{M}', \mathcal{L}', X') = V_{TM_s}^m(\mathcal{M}, \mathcal{L}, X_0) \\
V_{m,s}(\mathcal{M}', \mathcal{L}', X') = V_{m,s}(\mathcal{M}, \mathcal{L}, X_0),
\end{array} \right.
\]
where we used the identification $\mathcal{M}' = \{z\} \times M_s = \mathcal{M}_z$ (cf. (2.9)) with $z = e^s$. Then by (3.3), (3.4) and (3.5), $\mathcal{V}^m(\mathcal{M}', \mathcal{L}', X')_{|M_s} (= X^{(m)}_{|M_s})$ is expressible as

$$(3.7) \quad \mathcal{V}^m(\mathcal{M}', \mathcal{L}', X')_{|M_s} = (\varphi_s)_*\{-W + \mathcal{V}^m(\mathcal{M}, \mathcal{L}, X_0)_{|M_s}\}.$$ 

Then for a subsequence $\{m(j)\,; j = 1, 2, \ldots\}$ of $\{m = 1, 2, \ldots\}$, and a sequence $\{s(j) \in \mathbb{R}; j = 1, 2, \ldots\}$ as in Section 5 in [12] such that

$$-C_0 (\log m(j)) q(j) \leq s(j) \leq 0 \quad \text{and} \quad q(j) = 1/m(j),$$

we obtain (cf. (5.19) and (5.20) of [12])

$$(3.8) \quad \int_M |\mathcal{V}_{TM^+}^{m(j)}(j)|^2_{\omega(j)} \omega(j)^n = O \left( \frac{q(j)}{\log m(j)} \right),$$

$$(3.9) \quad \int_M |\mathcal{V}^\bullet(j)|^2_{\omega(j)} \omega(j)^n = O \left( \frac{1}{\log m(j)} \right),$$

where $C_0$ is a sufficiently small positive real constant independent of $j$, and we define $\omega(j)$, $N(j)$, $\mathcal{V}_{TM^+}(j)$, $\mathcal{V}^\bullet(j)$, $\mathcal{V}^{\circ}(j)$ by

$$\begin{cases}
\omega(j) &:= \{q(j) \mu^*_{FS}\}_{|M}, \quad N(j) := N_{m(j)}, \\
\mathcal{V}_{TM^+}(j) &:= (\mu_j^{-1})_* \mathcal{V}^m_{TM^+}(\mathcal{M}_j, \mathcal{L}, X_0) = (\mu_j^{-1})_* \mathcal{V}^m_{TM^+}(\mathcal{M}', \mathcal{L}', X'), \\
\mathcal{V}^\bullet(j) &:= (\mu_j^{-1})_* \mathcal{V}^\bullet_{m(j), s(j)}(\mathcal{M}, \mathcal{L}, X_0) = (\mu_j^{-1})_* \mathcal{V}^\bullet_{m(j), s(j)}(\mathcal{M}', \mathcal{L}', X'), \\
\mathcal{V}^{\circ}(j) &:= (\mu_j^{-1})_* \mathcal{V}^{\circ}_{m(j), s(j)}(\mathcal{M}, \mathcal{L}, X_0),
\end{cases}$$

with $\mu_j := \mu_{m(j), s(j)}$. Put $\mathcal{V}^{\circ}(j)' := (\mu_j^{-1})_* \mathcal{V}^{\circ}_{m(j), s(j)}(\mathcal{M}', \mathcal{L}', X')$. In view of $\mu_{m,j} = \Pi_m(\varphi_s)$, we can write $(\varphi_s)_*$ in (3.7) as $(\mu_{m,j})_*$. Then by (3.7) applied to $s(j)$, it follows from (3.6) that

$$(3.10) \quad \mathcal{V}^{\circ}(j)' = \mathcal{V}^{\circ}(j) - W.$$

In (5.1) of [12], the basis $\{\tau_1, \tau_2, \ldots, \tau_{N(j)}\}$ for $(E_{m(j)})_0$ can be chosen in such a way that, in terms of this basis, the $\Sigma'$-action (as well as the $T_0$-action) on $(E_m)_0$ is also diagonalizable. Then $\eta(j) = \eta(j)(\mathcal{M}, \mathcal{L}, X_0) \in C^\infty(M_\mathbb{R})$ as in [12] satisfies

$$(3.11) \quad i_{\mathcal{V}(j)} \omega(j) = \partial \eta(j),$$

where $\mathcal{V}(j) := \{(\mu_j^{-1})_* \mathcal{V}(\mathcal{M}, \mathcal{L}, X_0)\}_{|M} (= \mathcal{V}^m(\mathcal{M}, \mathcal{L}, X_0)_{|M})$. By (3.7), we see that $\eta'(j) := \eta(j) - f_{\omega(j), W} \in C^\infty(M_\mathbb{R})$ (cf. (1.1)) satisfies

$$(3.11) \quad i_{\mathcal{V}(j)} \omega(j) = \partial \eta'(j).$$
where $\mathcal{V}(j) := \{(\mu^{-1}_j)^* \mathcal{V}(\mathcal{M}', \mathcal{L}', X')\}_{j=1}^\infty$. Recall that, as $j \to \infty$, the Kähler metric $\omega(j)$ converges in $C^\infty$ to the metric $\omega_\infty$ in $c_1(L)_R$ of constant scalar curvature. Moreover, for some $\eta_\infty \in C^\infty(M)_R$,

$$\eta(j) \to \eta_\infty \text{ in } L^2(M, \omega_\infty), \quad \text{as } j \to \infty,$$

where $\eta_\infty$ is a ‘Hamiltonian function’ for a holomorphic vector field on $M$ (cf. [12]). Then $W \in \mathfrak{g}$ is specified as follows. Following [12], by using the notation in (1.1), let $W$ be the unique element of $\mathfrak{g}$ such that

$$f_{\omega_\infty, W} = \eta_\infty.$$

Since $\omega(j)$ converges to $\omega_\infty$ in $C^\infty$, by (3.12) and (3.13), the definition of $\eta'(j)$ implies the convergence

$$\eta'(j) \to 0 \text{ in } L^2(M), \quad \text{as } j \to \infty.$$

By the identification of $(E_{m(j)})_1 = (E'_{m(j)})_1$ with $(E_{m(j)})_0 = (E'_{m(j)})_0$ (cf. (2.5)), we consider the basis $\{\tau'_1, \tau'_2, \ldots, \tau'_{N(j)}\}$ for

$$H^0(M, \mathcal{O}_M(L^{m(j)})) = (E_{m(j)})_1$$

corresponding to the basis $\{\tau_1, \tau_2, \ldots, \tau_{N(j)}\}$ for $(E_{m(j)})_0$. By the representation in the first line of (2.10), we can write

$$X' \cdot \tau_\alpha = e'_\alpha(j) \tau_\alpha, \quad \alpha = 1, 2, \ldots, N(j),$$

for some real numbers $e'_\alpha(j)$ with $\Sigma_{\alpha=1}^{N(j)} e'_\alpha(j) = 0$. By setting

$$\xi(j) := \frac{\Sigma_{\alpha=1}^{N(j)} e'_\alpha(j) |z_\alpha|^2}{m(j) \Sigma_{\alpha=1}^{N(j)} |z_\alpha|^2} \quad \text{and} \quad \bar{\xi}(j) := \frac{\int_{M(j)} \xi(j) \{q(j) \omega_{FS}\}^n}{\int_{M(j)} \{q(j) \omega_{FS}\}^n},$$

we consider $e_\alpha(j) := e'_\alpha(j) - m(j)\bar{\xi}(j)$, where $\mathbb{P}^*(E_{m(j)})_0$ is identified with the projective space $\mathbb{P}^{N(j)}(\mathbb{C}) = \{(z_1 : z_2 : \cdots : z_{N(j)})\}$ by the basis $\{\tau_1, \tau_2, \ldots, \tau_{N(j)}\}$. Hereafter, let $C_k$, $k=1,2,\ldots$, denote positive real constants independent of $j$ and $\alpha$. Since $|e'_\alpha(j)| \leq C_1 m(j)$ for all $\alpha$ and $j$, the inequality $|\xi(j)| \leq C_1$ holds for all $j$, so that

$$|\bar{\xi}(j)| \leq C_1 \quad \text{for all } j.$$
In particular \(|e_n(j)| \leq C_2 m(j)|\) for all \(j\). Put \(h(j) := h_{m(j)}\). In view of the definition of \(\eta(j)\) in [12], we can write \(\eta'(j)\) in (3.11) as
\[
\eta'(j) = \frac{\sum_{\alpha=1}^{N(j)} e_{\alpha}(j) |\tau_{\alpha}'|_{h(j)}^2 \exp\{2s(j)e_{\alpha}(j)\}}{m(j)\sum_{\alpha=1}^{N(j)} |\tau_{\alpha}'|_{h(j)}^2 \exp\{2s(j)e_{\alpha}(j)\}},
\]
since \(\int_M \eta'(j)\omega(j)^n = 0\). Note that the functions \(\eta'(j), j = 1, 2, \ldots,\) are uniformly bounded on \(M\). We further define uniformly bounded functions \(\zeta'(j), j = 1, 2, \ldots,\) on \(M\) by
\[
\zeta'(j) := \frac{\sum_{\alpha=1}^{N(j)} e_{\alpha}(j)^2 |\tau_{\alpha}'|_{h(j)}^2 \exp\{2s(j)e_{\alpha}(j)\}}{m(j)^2\sum_{\alpha=1}^{N(j)} |\tau_{\alpha}'|_{h(j)}^2 \exp\{2s(j)e_{\alpha}(j)\}}.
\]
Define bounded sequences \(\{\lambda_j\}, \{\kappa_j\}\) of real numbers by setting
\[
\lambda_j := \int_M \eta'(j)^2 \omega(j)^n \geq 0 \quad \text{and} \quad \kappa_j := \int_M \zeta'(j) \omega(j)^n \geq 0.
\]
If necessary, we replace \(\{\lambda_j\}, \{\kappa_j\}\) by respective subsequences with a common sequence of indices. Then we may assume that both \(\{\lambda_j\}\) and \(\{\kappa_j\}\) converge. Put
\[
\lambda_{\infty} := \lim_{j \to \infty} \lambda_j \quad \text{and} \quad \kappa_{\infty} := \lim_{j \to \infty} \kappa_j.
\]
By the Cauchy-Schwarz inequality, we have \(\eta'(j)^2 \leq \zeta'(j)\) and hence \(\kappa_j \geq \lambda_j\) for all \(j\). In particular \(\kappa_{\infty} \geq \lambda_{\infty}\). We now claim that there exists a positive constant \(C\) independent of \(j\) such that
\[
0 \leq \kappa_j - \lambda_j \leq C/m(j) \quad \text{and hence} \quad \kappa_{\infty} = \lambda_{\infty}.
\]
\(\text{Proof.}\) For each \(j = 1, 2, \ldots,\), we define \(I_j^s\) and \(I_j^{s'}\) by
\[
\left\{\begin{array}{l}
I_j^s := \int_M |\mathcal{V}^{\sigma}(j)|_{\omega(j)}^2 \omega(j)^n, \\
I_j^{s'} := \int_M |\mathcal{V}^{\sigma}(j)|_{\omega'(j)}^2 \omega'(j)^n.
\end{array}\right.
\]
Since it suffices to consider Case 1 of Section 5 in [12], we may assume that \(I_j^s, j=1,2,\ldots,\) are bounded. Then by (3.10), \(I_j^{s'}, j=1,2,\ldots,\) are also bounded. Since \(I_j' := \int_M |\mathcal{V}(j)|_{\omega(j)}^2 \omega(j)^n\) is written as
\[
I_j' = I_j^{s'} + \int_M |\mathcal{V}(j)|_{\omega(j)}^2 \omega(j)^n + \int_M |\mathcal{V}_{TM^L}(j)|_{\omega(j)}^2 \omega(j)^n,
\]
we see from (3.8) and (3.9) that \(I_j', j=1,2,\ldots,\) are also bounded, i.e., \(0 \leq I_j \leq C\) for some \(C\) as above. On the other hand, it is easily checked
that \( I'_j = m(j)(\kappa_j - \lambda_j) \). Hence \( 0 \leq \kappa_j - \lambda_j \leq C/m(j) \). By letting \( j \to \infty \), we obtain \( \kappa_\infty = \lambda_\infty \), as required. \( \square \)

Now by (3.14), \( \lambda_\infty = 0 \), and hence from (3.17), it follows that

(3.18) \( \kappa_\infty = 0 \).

If \( \Psi_{SL,m,X}' \) in (2.10) is nontrivial for \( m = 1 \), then from the next section, we see that \( \kappa_\infty > 0 \) in contradiction to (3.18). Hence, \( \Psi_{SL,1,X}' \) is trivial. Then the argument leading to (2.9) shows that \( \mathcal{M}' = \{ z \} \times \mathcal{M}_s \), where \( \mathcal{M}_s \) is the subvariety \( \mu_1.s(M) = (\varphi_s \psi_s)(M) \) in \( \mathbb{P}^s((E_1)_0) \). Thus by the triviality of \( \Psi_{SL,1,X}' \), we see from (2.11) that the generalized test configuration \( (\mathcal{M}', \mathcal{L}', X') \) is trivial, i.e., \( \mathcal{M}' = \mathbb{A}^1 \times \mathcal{M} \) with \( T' \) acting on the second factor trivially. Therefore by (2.8) and (2.11), we conclude that the original test configuration \( (\mathcal{M}, \mathcal{L}) \) is a product configuration, i.e., \( \mathcal{M} = \mathbb{A}^1 \times \mathcal{M} \), as required. \( \square \)

4. \( \kappa_\infty > 0 \) if \( \Psi_{SL,1,X}' \) is nontrivial

In this section, we consider the set \( \Delta_j^+ \) of all \( \alpha \in \{1, 2, \ldots, N(j)\} \) such that \( e_\alpha(j) \geq 0 \). Similarly, let \( \Delta_j^- \) be the set of all \( \alpha \in \{1, 2, \ldots, N(j)\} \) such that \( e_\alpha(j) < 0 \). Put

\[
\nu_\alpha(j) := \int_M \frac{|\tau'_\alpha|^2(h(j)) \exp\{2s(j)e_\alpha(j)\}}{\sum_{\alpha=1}^{N(j)} |\tau'_\alpha|^2(h(j)) \exp\{2s(j)e_\alpha(j)\}} \omega(j)^n, \\
\hat{e}_\alpha(j) := e_\alpha(j)/m(j),
\]

for \( \alpha = 1, 2, \ldots, N(j) \). Then \( \sum_{\alpha=1}^{N(j)} \nu_\alpha(j) = 1 \) and \( |\hat{e}_\alpha(j)| \leq C_2 \). Since \( \int_M \eta'(j) \omega(j)^n = 0 \) and \( \kappa_j = \sum_{\alpha=1}^{N(j)} \nu_\alpha(j) e_\alpha(j)^2/m(j)^2 \), we obtain

(4.1) \( \sum_{\alpha \in \Delta_j^+} \nu_\alpha(j) \hat{e}_\alpha(j) = -\sum_{\alpha \in \Delta_j^-} \nu_\alpha(j) \hat{e}_\alpha(j), \)

(4.2) \( \kappa_j = \sum_{\alpha=1}^{N(j)} \nu_\alpha(j) \hat{e}_\alpha(j)^2. \)

Let \( r_j \) be the left-hand side (= right-hand side) of (4.1). From now on until the end of this section, we assume that \( \Psi_{SL,1,X}' \) is nontrivial. Now we claim the following inequality:

(4.3) \( r_\infty := \lim_{j \to \infty} r_j > 0. \)
On the other hand, by $0 \leq \Sigma_{\alpha \in \Delta_j} \nu_\alpha(j) \leq 1$, the Cauchy-Schwarz inequality together with (4.2) implies that

$$r_j^2 = \{\Sigma_{\alpha \in \Delta_j^+} \nu_\alpha(j) \hat{e}_\alpha(j)\}^2 \leq \{\Sigma_{\alpha \in \Delta_j^+} \nu_\alpha(j)\}\{\Sigma_{\alpha \in \Delta_j^+} \nu_\alpha(j) \hat{e}_\alpha(j)^2\}$$

$$\leq \Sigma_{\alpha \in \Delta_j^+} \nu_\alpha(j) \hat{e}_\alpha(j)^2 \leq \kappa_j.$$

We here let $j \to \infty$. Hence, once (4.3) is proved, it follows that

$$0 < r_\infty^2 \leq \kappa_\infty,$$

as required. Thus the proof of $\kappa_\infty > 0$ is reduced to showing (4.3).

**Proof of (4.3).** For each $\theta \in \mathbb{R}$ with $0 \leq \theta \leq C_2$, we consider the set $\Delta_j^{\theta^+}$ of all $\alpha \in \{1, 2, \ldots, N(j)\}$ such that $\hat{e}_\alpha(j) \geq \theta$. Consider also the set $\Delta_j^{\theta^-}$ of all $\alpha \in \{1, 2, \ldots, N(j)\}$ such that $\hat{e}_\alpha(j) < \theta$. For each $\alpha \in \{1, 2, \ldots, N(j)\}$, we put $\hat{e}_\alpha(j) := \hat{e}_\alpha(j) - \theta$ and $\epsilon_\alpha(j) := m(j) \hat{e}_\alpha(j) = e_\alpha(j) - m(j) \theta$. We now define nonnegative real-valued functions $V^{\theta,+}(j), V^{\theta,-}(j), Y^{\theta,+}, Y^{\theta,-}, Z^{\theta,+}(j), Z^{\theta,-}(j)$ on $M$ by

$$V^{\theta,+}(j) := \Sigma_{\alpha \in \Delta_j^{\theta^+}} \hat{e}_\alpha(j)^2 |\tau_{\alpha h(j)}|^2 \exp\{2s(j) \epsilon_\alpha(j)\},$$

$$V^{\theta,-}(j) := \Sigma_{\alpha \in \Delta_j^{\theta^-}} \hat{e}_\alpha(j)^2 |\tau_{\alpha h(j)}|^2 \exp\{2s(j) \epsilon_\alpha(j)\},$$

$$Y^{\theta,+}(j) := \Sigma_{\alpha \in \Delta_j^{\theta^+}} \hat{e}_\alpha(j) |\tau_{\alpha h(j)}|^2 \exp\{2s(j) \epsilon_\alpha(j)\},$$

$$Y^{\theta,-}(j) := \Sigma_{\alpha \in \Delta_j^{\theta^-}} \{-\hat{e}_\alpha(j)\} |\tau_{\alpha h(j)}|^2 \exp\{2s(j) \epsilon_\alpha(j)\},$$

$$Z^{\theta,+}(j) := \Sigma_{\alpha \in \Delta_j^{\theta^+}} |\tau_{\alpha h(j)}|^2 \exp\{2s(j) \epsilon_\alpha(j)\},$$

$$Z^{\theta,-}(j) := \Sigma_{\alpha \in \Delta_j^{\theta^-}} |\tau_{\alpha h(j)}|^2 \exp\{2s(j) \epsilon_\alpha(j)\},$$

$$Z(j) := \Sigma_{\alpha = 1}^{N(j)} |\tau_{\alpha h(j)}|^2 \exp\{2s(j) \epsilon_\alpha(j)\} = Z^{\theta,+}(j) + Z^{\theta,-}(j).$$

Then by the Cauchy-Schwarz inequality, we have

$$\Theta_+ := \frac{V^{\theta,+}(j)Z^{\theta,+}(j) - Y^{\theta,+}(j)^2}{Z(j)^2} \geq 0,$$

$$\Theta_- := \frac{V^{\theta,-}(j)Z^{\theta,-}(j) - Y^{\theta,-}(j)^2}{Z(j)^2} \geq 0.$$
Hence by $\kappa_j - \lambda_j = \int_M \{\zeta'(j) - \eta'(j)^2\} \omega(j)^n \leq C/m(j)$ (cf. (3.17)), and also by

$$0 \leq \zeta'(j) - \eta'(j)^2 = \Theta_+ + \Theta_- + \frac{V^{\theta,+}(j)Z^{\theta,-}(j) + V^{\theta,-}(j)Z^{\theta,+}(j) + 2Y^{\theta,+}(j)Y^{\theta,-}(j)}{Z(j)^2},$$

we see the following inequality:

$$(4.4) \quad 0 \leq \int_M \frac{V^{\theta,+}(j)Z^{\theta,-}(j)}{Z(j)^2} \omega(j)^n \leq \frac{C}{m(j)},$$

$$(4.5) \quad 0 \leq \int_M \frac{V^{\theta,-}(j)Z^{\theta,+}(j)}{Z(j)^2} \omega(j)^n \leq \frac{C}{m(j)}.$$

We further define nonnegative real-valued functions $R^{\theta,+}(j), R^{\theta,-}(j), R(j), S^{\theta,+}(j), S^{\theta,-}(j), U^{\theta,+}(j), U^{\theta,-}(j)$ on $M$ by

$$R^{\theta,+}(j) := \sum_{\alpha \in \Delta_j^+} |\sigma'_\alpha h(j)|,$$

$$R^{\theta,-}(j) := \sum_{\alpha \in \Delta_j^-} |\sigma'_\alpha h(j)|,$$

$$R(j) := \sum_{\alpha = 1}^{\Delta(j)} |\sigma'_\alpha h(j)| = R^{\theta,+}(j) + R^{\theta,-}(j),$$

$$S^{\theta,+}(j) := \sum_{\alpha \in \Delta_j^+} \hat{\epsilon}_\alpha (j) |\sigma'_\alpha h(j)|,$$

$$S^{\theta,-}(j) := \sum_{\alpha \in \Delta_j^-} \{ -\hat{\epsilon}_\alpha (j) \} |\sigma'_\alpha h(j)|,$$

$$U^{\theta,+}(j) := \sum_{\alpha \in \Delta_j^+} \hat{\epsilon}_\alpha (j)^2 |\sigma'_\alpha h(j)|,$$

$$U^{\theta,-}(j) := \sum_{\alpha \in \Delta_j^-} \hat{\epsilon}_\alpha (j)^2 |\sigma'_\alpha h(j)|.$$

Now by $-C_0 (\log m(j)) q(j) \leq s(j) \leq 0$ and $|\hat{\epsilon}_\alpha (j)| \leq 2C_2$, we have

$$m(j)^{-4C_0 C_2} \leq \exp\{2s(j) \epsilon_\alpha (j)\} \leq m(j)^{4C_0 C_2}.$$
Since $C_0$ is sufficiently small, we may assume $C_0 C_2 \leq \varepsilon/16$ for a positive constant $\varepsilon \ll 1$ independent of $j$ and $\theta$. Hence

\[
\begin{align*}
0 \leq m(j)^{-\varepsilon/4} Z(j) & \leq R(j) \leq m(j)^{\varepsilon/4} Z(j), \\
0 \leq m(j)^{-\varepsilon/4} S^{\theta,+}(j) & \leq Y^{\theta,+}(j) \leq S^{\theta,+}(j), \\
0 \leq S^{\theta,-}(j) & \leq Y^{\theta,-}(j), \\
0 \leq m(j)^{-\varepsilon/4} U^{\theta,+}(j) & \leq V^{\theta,+}(j), \\
0 \leq U^{\theta,-}(j) & \leq V^{\theta,-}(j), \\
0 \leq Z^{\theta,+}(j) & \leq R^{\theta,+}(j) \leq m(j)^{\varepsilon/4} Z^{\theta,+}(j), \\
0 \leq m(j)^{-\varepsilon/4} Z^{\theta,-}(j) & \leq R^{\theta,-}(j) \leq Z^{\theta,-}(j).
\end{align*}
\]

Then by (4.4) and (4.5),

\[
0 \leq \int_M \frac{U^{\theta,+}(j) R^{\theta,-}(j)}{R(j)^2} \omega(j)^n \leq \frac{C}{m(j)^{1-\varepsilon}},
\]

(4.7)

\[
0 \leq \int_M \frac{U^{\theta,-}(j) R^{\theta,+}(j)}{R(j)^2} \omega(j)^n \leq \frac{C}{m(j)^{1-\varepsilon}},
\]

(4.8)

while by $S^{\theta,+}(j)^2 \leq U^{\theta,+}(j) R^{\theta,+}(j)$ and $S^{\theta,-}(j)^2 \leq U^{\theta,-}(j) R^{\theta,-}(j)$, we have the inequalities

\[
\begin{align*}
\left(\frac{S^{\theta,+}(j) R^{\theta,-}(j)}{R(j)^2}\right)^2 & \leq \frac{S^{\theta,+}(j)^2 R^{\theta,-}(j)}{R^{\theta,+}(j) R(j)^2} \leq \frac{U^{\theta,+}(j) R^{\theta,-}(j)}{R(j)^2}, \\
\left(\frac{S^{\theta,-}(j) R^{\theta,+}(j)}{R(j)^2}\right)^2 & \leq \frac{S^{\theta,-}(j)^2 R^{\theta,+}(j)}{R^{\theta,-}(j) R(j)^2} \leq \frac{U^{\theta,-}(j) R^{\theta,+}(j)}{R(j)^2}.
\end{align*}
\]

(4.9)

(4.10)

It now follows from (4.7) and (4.9) that

\[
\int_M \frac{S^{\theta,+}(j) R^{\theta,-}(j)}{R(j)^2} \omega(j)^n \leq \frac{C_3}{m(j)^{(1-\varepsilon)/2}},
\]

(4.11)

where $C_k$, $k = 3, 4, \ldots$, are positive constants independent of $j$, $\alpha$ and also $\theta$. Similarly by (4.8) and (4.10), we obtain

\[
\int_M \frac{S^{\theta,-}(j) R^{\theta,+}(j)}{R(j)^2} \omega(j)^n \leq \frac{C_4}{m(j)^{(1-\varepsilon)/2}}.
\]

(4.12)
Put $\delta := 1/\log m(j)$. Consider the set $\Delta^\theta_\delta$ of all $\alpha \in \{1, 2, \ldots, N(j)\}$ such that $\theta \leq \hat{e}_\alpha(j) < \theta + \delta$, i.e., $0 \leq \hat{e}_\alpha(j) \leq \delta$. By setting

$$
\left\{ \begin{array}{l}
Y^{\theta, \delta} := \sum_{\alpha \in \Delta^\theta_\delta} \hat{e}_\alpha(j)|\tau'_\alpha h(j)|^2 \exp\{2s(j)\epsilon_\alpha(j)\}, \\
\tilde{Y}^{\theta, \delta} := \sum_{\alpha \in \Delta^\theta_\delta} \hat{e}_\alpha(j)|\tau'_\alpha h(j)|^2 \exp\{2s(j)\epsilon_\alpha(j)\},
\end{array} \right.
$$

we consider the nonnegative real-valued functions $\rho^{\theta, \delta}(j), \tilde{\rho}^{\theta, \delta}(j), \rho^{\theta,-}(j), \tilde{\rho}^{\theta,-}(j), j = 1, 2, \ldots, \text{on} \ M$ defined by

$$
\rho^{\theta, \delta}(j) := \frac{Y^{\theta, \delta}(j)}{Z(j)}, \quad \tilde{\rho}^{\theta, \delta}(j) := \frac{\theta \delta^{-1}\tilde{Y}^{\theta, \delta}(j)}{R(j)},
$$

$$
\rho^{\theta,-}(j) := \frac{Y^{\theta,-}(j)}{Z(j)}, \quad \tilde{\rho}^{\theta,-}(j) := \frac{S^{\theta,-}(j)}{R(j)}.
$$

We first study the function $\rho^{\theta, \delta}(j)$ and integrate it over $M$. Note that $\theta \delta^{-1}\hat{e}_\alpha(j) \leq \theta \leq \hat{e}_\alpha(j)$ for all $\alpha \in \Delta^\theta_\delta$. Hence

$$
\rho^{\theta, \delta}(j) - \tilde{\rho}^{\theta, \delta}(j) \geq \theta \delta^{-1}
\left( \frac{\tilde{Y}^{\theta, \delta}(j)}{Z(j)} - \frac{\tilde{Y}^{\theta, \delta}(j)}{R(j)} \right)
\geq \theta \delta^{-1}\frac{\tilde{Y}^{\theta, \delta}(j)\{R^{\theta,+}(j) - Z^{\theta,+}(j)\} + \tilde{Y}^{\theta, \delta}(j)\{R^{\theta,-}(j) - Z^{\theta,-}(j)\}}{Z(j)R(j)}
\geq -\theta \delta^{-1}\frac{Y^{\theta,+}(j)Z^{\theta,+}(j)}{Z(j)R(j)}.
$$

Integrating this over $M$, by (4.6) and (4.11), we obtain

$$
\int_M \rho^{\theta, \delta}(j)\omega(j)^n \geq \int_M \tilde{\rho}^{\theta, \delta}(j)\omega(j)^n - \theta \delta^{-1}\frac{C_5}{m(j)^{(1/2) - \varepsilon}}.
$$

Since $\sum_{\alpha \in \Delta^\theta_\delta} \nu_\alpha(j) \hat{e}_\alpha(j) = \int_M \rho^{\theta, \delta}(j)\omega(j)^n$, it follows that

$$
\sum_{\alpha \in \Delta^\theta_\delta} \nu_\alpha(j) \hat{e}_\alpha(j) \geq \int_M \tilde{\rho}^{\theta, \delta}(j)\omega(j)^n - \frac{C_6}{m(j)^{(1/2) - (3/2)\varepsilon}}.
$$

We next consider the function $\rho^{\theta,-}(j)$ and integrate it over $M$. Then

$$
\rho^{\theta,-}(j) - \tilde{\rho}^{\theta,-}(j)
\geq \frac{-S^{\theta,-}(j)Z^{\theta,+}(j) + \{Y^{\theta,-}(j)R^{\theta,-} - S^{\theta,-}(j)Z^{\theta,-}(j)\}}{Z(j)R(j)}.
$$
Put \( f(t) := \log(\sum_{\alpha \in \Delta} |\tau_{\alpha}^j|^{2} \exp\{-t \epsilon_{\alpha}(j)\}) \), and its second derivative \( \ddot{f}(t) \) with respect to \( t \) is nonnegative, wherever defined, and hence \( \ddot{f}(-2s(j)) \geq \ddot{f}(0) \), i.e., \( Y^{\theta,-}(j)R^{\theta,-} - S^{\theta,-}(j)Z^{\theta,-}(j) \geq 0 \). Thus
\[
\rho^{\theta,-}(j) - \dot{\rho}^{\theta,-}(j) \geq -S^{\theta,-}(j)Z^{\theta,+}(j) / Z(j)R(j).
\]

Integrating this over \( M \), by (4.6) and (4.12), we obtain
\[
(4.14) \quad \int_{M} \rho^{\theta,-}(j)\omega(j)^{n} \geq \int_{M} \dot{\rho}^{\theta,-}(j)\omega(j)^{n} - \frac{C_{7}}{m(j)^{(1/2) - \varepsilon}}.
\]

We now put \( a_{j}(\theta) := \int_{M} \rho^{\theta,-}(j)\omega(j)^{n} \) and \( \tilde{a}_{j}(\theta) := \int_{M} \dot{\rho}^{\theta,-}(j)\omega(j)^{n} \). Then by setting
\[
a_{\infty}(\theta) := \lim_{j \to \infty} a_{j}(\theta) \quad \text{and} \quad \tilde{a}_{\infty}(\theta) := \lim_{j \to \infty} \tilde{a}_{j}(\theta),
\]
we obtain from (4.14) the inequalities
\[
(4.15) \quad a_{\infty}(\theta) \geq \tilde{a}_{\infty}(\theta) \geq 0
\]
for all \( \theta \). Since \( a_{j}(0) = -\sum_{\alpha \in \Delta_{j}} \nu_{\alpha} \hat{e}_{\alpha}(j) = r_{j} \), it now follows from (4.3) and (4.15) that
\[
r_{\infty} = a_{\infty}(0) \geq \tilde{a}_{\infty}(0).
\]

Hence if \( \tilde{a}_{\infty}(0) > 0 \), then \( r_{\infty} > 0 \). It now suffices to consider the remaining case \( \tilde{a}_{\infty}(0) = 0 \). Consider the positive constants \( C_{9}, C_{10} \) \( C_{11} \) in Lemma 5.1. In view of \( C_{9} < C_{10} \), we put \( C_{8} := C_{10} - C_{9} \) for simplicity. For every \( a \in \mathbb{R} \), by using the Gauss symbol, let \( [a] \) denote the largest integer which does not exceed \( a \). Put \( \gamma_{j} := [C_{8}/\delta] \). Then
\[
C_{8} \log m(j) - 1 < \gamma_{j} \leq C_{8} \log m(j).
\]

Now by setting \( \theta_{\ell} := C_{9} + (\ell - 1)\delta \), we obtain
\[
C_{9} \leq \theta_{\ell} < \theta_{\ell} + \delta \leq C_{10}, \quad \ell = 1, 2, \ldots, \gamma_{j}.
\]

Hence by applying Lemma 5.1 to \( \theta = \theta_{\ell} \), we see from (4.13) and (5.2) the following inequality:
\[
\sum_{\alpha \in \Delta_{\theta_{\ell}^{\theta}} \nu_{\alpha}(j) \hat{e}_{\alpha}(j) \geq C_{11} \theta_{\ell} \delta - \frac{C_{6}}{m(j)^{(1/2) - (3/2)\varepsilon}}, \quad \ell = 1, 2, \ldots, \gamma_{j}.
\]
By summing these up, we obtain
\[ r_j = \sum_{\alpha \in \Delta_j} \nu_{\alpha}(j) \hat{e}_{\alpha}(j) \geq \sum_{\alpha \in \Delta_j^{\gamma_j}} \sum_{\ell=1} C_6 \delta \gamma_j \ell \cdot \nu_{\alpha}(j) \hat{e}_{\alpha}(j) \]
\[ \geq C_9 \delta \gamma_j \sum_{\ell=1} \frac{C_6 \gamma_j}{m(j)^{(1/2)-(3/2)\varepsilon}} \geq C_9 \delta \gamma_j \sum_{\ell=1} \frac{C_6 \gamma_j}{m(j)^{(1/2)-(3/2)\varepsilon}} \]
\[ \geq C_9 C_{11} C_8 \log m(j) - \frac{1}{\log m(j)} - C_6 C_8 \log m(j) \frac{m(j)^{(1/2)-(3/2)\varepsilon}}{m(j)^{(1/2)-(3/2)\varepsilon}} \]

Then letting \( j \to \infty \), we now conclude that
\[ r_{\infty} = \lim_{j \to \infty} r_j \geq C_8 C_9 C_{11} > 0, \]
as required.

5. Appendix

In this appendix, under the same assumption as in the preceding section (hence \( \Psi_{1, X'} \) is nontrivial), we shall show the following:

**Lemma 5.1.** If \( \bar{a}_{\infty}(0) = 0 \), then there exist positive real constants \( C_9, C_{10}, C_{11} \) independent of \( j \) satisfying \( C_9 < C_{10} \) such that

\[ (5.2) \quad \int_{M} \tilde{\rho}^\theta \delta(j) \omega(j)^n \geq C_11 \theta \delta \]

for all \( \theta \in \mathbb{R} \) with \( C_9 \leq \theta < \theta + \delta \leq C_{10} \).

**Proof.** As in (3.15), we choose a basis \( \{ \tau_1, \tau_2, \ldots, \tau_{N_1} \} \) for \( (E_1)_0 = H^0(M_0, L_0) \) such that

\[ X' \cdot \tau_\alpha = e'_\alpha \tau_\alpha, \quad \alpha = 1, 2, \ldots, N_1, \]

where \( e'_\alpha \) are real numbers satisfying \( \sum_{\alpha=1}^{N_1} e'_\alpha = 0 \). Since \( \dim_{\mathbb{C}} M_0 = n \), the very ampleness of \( L_0 \) implies that \( N_1 \geq n+1 \). Then we may assume without loss of generality that \( e'_1 \leq e'_2 \leq \cdots \leq e'_{n+1} \) and that \( \tau_1, \tau_2, \ldots, \tau_{n+1} \) are transcendental over \( \mathbb{C} \) in the graded algebra

\[ \bigoplus_{m=1}^\infty H^0(M_0, L_0^m). \]

For later purposes, consider the first positive integer \( k \) satisfying the inequality \( e_k < e_{k+1} \). Note that \( k \leq n \). Since \( \Psi_{1, X'} \) is nontrivial, there is an irreducible component \( \mathcal{F} \) of \( M_0 \) with subscheme structure in \( M_0 \).
such that $T'$ acts on the scheme $\mathcal{F}$ nontrivially. Hence, in view of the fact that the generically finite rational map $\tau: \mathcal{F} \to \mathbb{P}^n(\mathbb{C})$ defined by

$$\tau(p) := (\tau_1(p) : \tau_2(p) : \cdots : \tau_{n+1}(p)), \quad p \in \mathcal{F},$$

is $T'$-invariant, the set $\{c_\alpha'; \alpha = 1, 2, \ldots, n+1\}$ contains at least two distinct real numbers. In particular,

$$e'_1 < e'_{n+1}.$$

Let $B(j)$ be the set of all $b = (b_1, b_2, \ldots, b_{n+1}) \in \mathbb{Z}^{n+1}$ such that

$$b_1 \geq 0, \ b_2 \geq 0, \ldots, \ b_{n+1} \geq 0, \ b_1 + b_2 + \cdots + b_{n+1} = m(j).$$

Then the weight $w(b)$ of the action of $X'$ on $\tau^b := \tau_1^{b_1} \tau_2^{b_2} \cdots \tau_{n+1}^{b_{n+1}}$ is

$$w(b) = k_j + \Sigma_{\alpha=1}^{n+1} b_\alpha e'_\alpha, \quad b \in B(j),$$

where $k_j$ is a real constant independent of the choice of $b$ such that the sequence $k_j/m(j)$, $j = 1, 2, \ldots$, is bounded. Here $k_j$ is chosen in such a way that the representation matrix of $X'$ on $(E_{m(j)})_0$ is traceless. In view of (3.16), replacing

$$c(j) := \frac{k_j}{m(j)} - \bar{\xi}(j), \quad j = 1, 2, \ldots,$$

by its subsequence if necessary, we may assume without loss of generality that $c(j)$ converges to some real number $c(\infty)$ as $j \to \infty$. For each $\theta \in \mathbb{R}$ with $0 \leq \theta \leq C_2$, we define $\epsilon_{b(j)_\theta}$ and $\hat{\epsilon}_{b(j)_\theta}$ by

$$\left\{ \begin{array}{l}
\epsilon_{b(j)_\theta} := w(b) - m(j)\bar{\xi}(j) - m(j)\theta \\
\hat{\epsilon}_{b(j)_\theta} := \epsilon_{b(j)_\theta}/m(j).
\end{array} \right.$$

Note that $\epsilon_{\text{max}}(j)_\theta := \max_{b \in B(j)} \epsilon_{b(j)_\theta}$ is exactly $m(j)\{c(j) + e'_{n+1} - \theta\}$. Put $\eta := c(\infty) + e'_{n+1}$. Then one of the following two cases occurs:

Case 1: $\eta \leq 0$, \quad Case 2: $\eta > 0$.

First, we consider Case 1, i.e., let $\eta \leq 0$. Let $\varepsilon_1 > 0$ be a sufficiently small real number independent of the choice of $j$. Then

$$\text{(5.3)} \quad \epsilon_{\text{max}}(j)_0 - \varepsilon_1 m(j) = \left\{ \eta - c(\infty) + c(j) - \varepsilon_1 \right\} m(j) \leq \left\{ -c(\infty) + c(j) - \varepsilon_1 \right\} m(j) \leq 0$$
for \( j \gg 1 \). Since \( \int_{M} |\tau'_{\alpha} h(j)| \omega(j)^n \) is 1 for all \( \alpha \) and \( j \), then from

\[
\tilde{a}_j(0) = \int_{M} \tilde{\rho}^0(j) \omega(j)^n = \int_{M} \frac{\sum_{\alpha \in \Delta_j} \{-e_\alpha(j)\}|\tau'_{\alpha} h(j)|}{\sum_{\alpha = 1}^{\Sigma^{N(j)}_n}|\tau'_{\alpha} h(j)|} \omega(j)^n,
\]

and also from

\[
(5.4) \quad \Sigma_{\alpha = 1}^{N(j)}|\tau'_{\alpha} h(j)| = C_{12} m(j)^n \{1 + O(1/m(j))\},
\]

it now follows that

\[
(5.5) \quad \tilde{a}_j(0) = \frac{C_{13} \{1 + O(1/m(j))\} \Sigma_{\alpha \in \Delta_j} \{-e_\alpha(j)\}}{m(j)^{n+1}}.
\]

Let \( \Sigma_- \{-e_b(j)_0\} \) denote the sum of all \(-e_b(j)_0\) with \( b \) running through the set of all \( b \) in \( B(j) \) such that \( e_b(j)_0 < 0 \). Then

\[
(5.6) \quad \Sigma_{\alpha \in \Delta_j} \{-e_\alpha(j)\} \geq \Sigma_- \{-e_b(j)_0\} \geq \Sigma_{b \in B(j)} \{-e_b(j)_0\}
\]

\[
\geq \Sigma_{b \in B(j)} \{e_{\text{max}}(j)_0 - e_b(j)_0\} - \Sigma_{b \in B(j)} e_{\text{max}}(j)_0
\]

\[
\geq \Sigma_{b \in B(j)} \{e_{\text{max}}(j)_0 - e_b(j)_0\} - \Sigma_{b \in B(j)} \varepsilon_1 m(j),
\]

where the last inequality follows from \((5.3)\). The first term in the last line is estimated as follows:

\[
(5.7) \quad \Sigma_{b \in B(j)} \{e_{\text{max}}(j)_0 - e_b(j)_0\}
\]

\[
\geq \sum_{b \in B(j)} \sum_{\alpha = 1}^{\Sigma^{n+1}_b} b_\alpha (e'_{n+1} - e'_\alpha) \geq (e'_{n+1} - e'_1) \sum_{b \in B(j)} b_1
\]

\[
= \frac{e'_{n+1} - e'_1}{(n-1)!} \sum_{b_1 = 0}^{m(j)} b_1 \Pi_{\ell = 1}^{n-1} \{m(j) - b_1 + \ell\}
\]

\[
= \frac{e'_{n+1} - e'_1}{(n-1)!} \sum_{b_1 = 0}^{m(j)} b_1 \{(m(j) - b_1)^{n-1} + O(m(j)^{n-2})\}
\]

\[
= \frac{e'_{n+1} - e'_1}{(n+1)!} m(j)^{n+1} + O(m(j)^n).
\]

Since \( \sum_{b \in B(j)} \varepsilon_1 m(j) = (\varepsilon_1/n!) \Pi_{\ell = 0}^{n} (m(j) + \ell) \), it now follows from \((5.5), (5.6)\) and \((5.7)\) that

\[
\tilde{a}_j(0) \geq C_{13} \left\{ \frac{e'_{n+1} - e'_1}{(n+1)!} - \frac{\varepsilon_1}{n!} \right\} + O(1/m(j)).
\]

Since \( \varepsilon_1 \) is sufficiently small, we obtain in this case

\[
\tilde{a}_\infty(0) = \lim_{j \to \infty} \tilde{a}_j(0) \geq C_{13} \left\{ \frac{e'_{n+1} - e'_1}{(n+1)!} - \frac{\varepsilon_1}{n!} \right\} > 0,
\]
in contradiction to the assumption $\tilde{a}_\infty(0) = 0$.

Hence, it suffices to consider Case 2. In this case by $\eta > 0$, using the Gauss symbol, we consider the nonnegative integer $p := \lfloor (e'_{n+1} - e'_1) / \eta \rfloor$. We then choose positive constants $C_9$ and $C_{10}$ by

$$C_9 := \eta - \frac{e'_{n+1} - e'_1}{p + 3} \quad \text{and} \quad C_{10} := \eta - \frac{e'_{n+1} - e'_1}{p + 4}.$$  

Take an arbitrary real number $\theta$ such that $C_9 \leq \theta < \theta + \delta \leq C_{10}$. Then for some $p_\theta \in \mathbb{R}$ satisfying $p + 3 \leq p_\theta < p + 4$, we can write $\theta$ as

$$\theta = \eta - \frac{e'_{n+1} - e'_1}{p_\theta}.$$  

Again by using the Gauss symbol, we put

$$v_{j,\theta} := \frac{c(j) - c(\infty)}{e'_{n+1} - e'_1} + \frac{1}{p_\theta},$$

$$m' := \lfloor m(j) v_{j,\theta} \rfloor.$$

Here $1/(p + 5) \leq v_{j,\theta} \leq 2/5$ for all $j \geq j_0$, where $j_0$ is a sufficiently large positive integer independent of the choice of $\theta$. Put $C_{14} := (e'_{n+1} - e'_1)/5$ for simplicity. For $j \gg 1$, let $B'(j)$ denote the subset of $B(j)$ consisting of all $b = (b_1, b_2, \ldots, b_{n+1}) \in B(j)$ such that $b_1, b_2, \ldots, b_{n+1}$ are expressible as

$$\begin{cases} 
  b_1 = m' + (1/5)m(j) - \Sigma_{\alpha=2}^{n+1} \beta_\alpha, \\
  b_\alpha = \beta_\alpha, \quad \alpha = 2, 3, \ldots, n, \\
  b_{n+1} = (4/5)m(j) - m' + \beta_{n+1},
\end{cases}$$

for some $\beta_2, \beta_3, \ldots, \beta_{n+1}$, where $\beta = (\beta_2, \beta_3, \ldots, \beta_{n+1})$ runs through the subset $B''(j)$ of $\mathbb{Z}_{\geq 0}^n$ consisting of all $\beta \in \mathbb{Z}^n$ such that

$$\begin{cases} 
  \beta_2 \geq 0, \beta_3 \geq 0, \ldots, \beta_{n+1} \geq 0, \Sigma_{\alpha=2}^{n+1} \beta_\alpha \leq (1/5)m(j), \\
  C_{14} m(j) \leq \Sigma_{\alpha=k+1}^{n+1} \beta_\alpha (e'_\alpha - e'_1) \leq (C_{14} + \delta) m(j) - (e'_{n+1} - e'_1),
\end{cases}$$

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where for $k = 1$, the condition $\Sigma_{\alpha=2}^k \beta_\alpha \leq (1/5)m(j)$ is assumed to be void. Then for each $b = (b_1, b_2, \ldots, b_{n+1}) \in B'(j)$,
\[
\epsilon_b(j)_\theta = \epsilon_{\text{max}}^\dagger(j)_\theta - \{\epsilon_{\text{max}}^\dagger(j)_\theta - \epsilon_b(j)_\theta\}
\]
\[
= m(j)\{\eta - c(\infty) + c(j) - \theta\} - \Sigma_{\alpha=1}^{n+1} b_\alpha (e''_{\alpha+1} - e'_\alpha)
\]
\[
= m(j) v_{j,\theta} (e''_{n+1} - e'_1) - \Sigma_{\alpha=1}^{n+1} b_\alpha (e''_{\alpha+1} - e'_\alpha)
\]
\[
= \{m(j) v_{j,\theta} - m' - (1/5)m(j)\} (e''_{n+1} - e'_1) + \Sigma_{\alpha=2}^{n+1} \beta_\alpha (e''_\alpha - e'_1)
\]
\[
\geq -C_{14} m(j) + \Sigma_{\alpha=k+1}^{n+1} \beta_\alpha (e''_\alpha - e'_1) \geq 0,
\]
where we used the inequality $m(j)v_{j,\theta} - m' \geq 0$ (cf. (5.9)). Again by (5.9), $m(j)v_{j,\theta} - m' < 1$ and hence, for each $b \in B'(j)$
\[
\epsilon_b(j)_\theta < \{1 - (1/5)m(j)\} (e''_{n+1} - e'_1) + \Sigma_{\alpha=2}^{n+1} \beta_\alpha (e''_\alpha - e'_1)
\]
\[
\leq e''_{n+1} - e'_1 - C_{14} m(j) + \Sigma_{\alpha=k+1}^{n+1} \beta_\alpha (e''_\alpha - e'_1) \leq m(j)\delta.
\]
Hence by (5.4) and $-C_0(\log m(j))q(j) \leq s(j) \leq 0$, it follows that
\[
\int_M \rho^{\theta,\delta}(j)\omega(j)^n = C_{15} \theta^{\delta^{-1}} \frac{1 + O(1/m(j))}{m(j)^n} \int_M \tilde{Y}^{\theta,\delta}\omega(j)^n
\]
\[
= C_{15} \theta^{\delta^{-1}} \frac{1 + O(1/m(j))}{m(j)^n} \Sigma_{\alpha \in \Delta_j^{\theta,\delta}} \hat{e}_\alpha(j) \exp\{2s(j)\epsilon_\alpha(j)\}
\]
\[
\geq C_{16} \theta^{\delta^{-1}} \frac{1 + O(1/m(j))}{m(j)^n} \Sigma_{b \in B'(j)} \hat{\epsilon}_b(j)_\theta \exp\{2s(j)\epsilon_b(j)_\theta\}
\]
\[
\geq C_{16} \theta^{\delta^{-1}} \frac{1 + O(1/m(j))}{m(j)^n} \Sigma_{b \in B'(j)} \hat{\epsilon}_b(j)_\theta,
\]
where the last inequality follows from $2s(j)\epsilon_b(j)_\theta \geq -2C_0$. Put
\[
\hat{x}(\beta) := \frac{\Sigma_{\alpha=k+1}^{n+1} \beta_\alpha (e''_\alpha - e'_1) - C_{14} m(j)}{m(j)}
\]
for simplicity. Now by $\epsilon_b(j)_\theta \geq \hat{x}(\beta)m(j)$, we see that
\[
(5.10) \quad \int_M \rho^{\theta,\delta}(j)\omega(j)^n \geq C_{16} \theta^{\delta^{-1}} \frac{1 + O(1/m(j))}{m(j)^n} \Sigma_{\beta \in B'(j)} \hat{x}(\beta).
\]
For $j \to \infty$, as far as the growth order of $\Sigma_{\beta \in B'(j)} \hat{x}(\beta)$ with respect to $m(j)$ is concerned, we may assume without loss of generality
\[
e''_{k+1} - e'_1 = e''_{k+2} - e'_1 = \cdots = e''_{n+1} - e'_1
\]
Then a computation similar to (5.7) allows us to write \( \sum_{\beta \in B''(j)} \hat{x}(\beta) \) as

\[
C_{17} m(j)^{k-1} \cdot \{\delta m(j)\}^2 m(j)^{n-k-1} + \text{lower order term in } m(j) \\
= C_{17} \, \delta^2 \, m(j)^{n} + \text{lower order term in } m(j).
\]

This together with (5.10) implies (5.2), as required. \( \square \)

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