Complementarity, distillable secret key, and distillable entanglement

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We consider controllability of two conjugate observables $Z$ and $X$ by two parties with classical communication. The ability is specified by two alternative tasks, (i) agreement on $Z$ and (ii) preparation of an eigenstate of $X$ with use of an extra communication channel. We prove that their feasibility is equivalent to that of key distillation if the extra channel is quantum, and to that of entanglement distillation if it is classical. This clarifies the distinction between two entanglement measures, distillable key and distillable entanglement.

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When two remote parties Alice and Bob want to communicate a message over a public channel without disclosing it to a third party Eve, it is sufficient for them to have a resource called a (secret) key, which is a random number that is shared by Alice and Bob secretly from Eve. In quantum mechanics, a (log $d$)-bit key is described by a tripartite state

$$\tau_{ABE} \equiv d^{-1} \sum_{i=0}^{d-1} |ii\rangle\langle ii|_{AB} \otimes \rho_E$$

with arbitrary state $\rho_E$, where we assume Hilbert space $H_A \otimes H_B \otimes H_E$ of systems held by the three parties, with a standard basis $\{|i\rangle_A\}$ for $H_A$ and $\{|i\rangle_B\}$ for $H_B$. Quantum key distribution (QKD) protocols enable production of the key through communication over a quantum channel and an authenticated public channel. Strictly speaking, they do not provide the state $\tau_{ABE}$ but a state $\rho_{ABE}$ very close to $\tau_{ABE}$. The imperfection is often measured using the trace norm as

$$\delta_{\text{key}} \equiv \|\rho_{ABE} - \tau_{ABE}\|.$$  (2)

It is not an easy task to bound $\delta_{\text{key}}$ against Eve with unconditional power by considering all of her options in a QKD protocol. Hence we often invoke the fact that Alice and Bob could have done a different (virtual) protocol instead of the actual protocol, at least from Eve’s point of view.

One of successful approaches is to take an entanglement distillation protocol (EDP) as the virtual protocol, which tries to produce a (log $d$)-bit maximally entangled state (MES)

$$|\phi^{\text{mes}}\rangle_{AB} \equiv d^{-1/2} \sum_{i=0}^{d-1} |ii\rangle_{AB}. \quad (3)$$

Once its feasibility is proved, the security of the QKD protocol immediately follows since the task of entanglement distillation is stronger than that of key distillation. In fact, rather unexpectedly, it was shown that it is often strictly stronger, and distillable entanglement $E_D(\rho_{AB})$ is strictly smaller than distillable key $K_D(\rho_{AB})$. This implies that the security of a QKD protocol is not necessarily provable by a reduction to an EDP, and distillation of a wider class of states were proposed to restore the applicability.

On the other hand, the first proof of unconditional security by Mayers took a quite different approach. He considered a virtual protocol concerning an observable that is “conjugate” to the key. In contrast to the EDP approach, here neither the real protocol nor the virtual one alone can prove the security. Security follows from the fact that Alice and Bob can freely choose between the two protocols, which cannot be executed at the same time. This complementarity approach has been refined to achieve the simplicity comparable to the EDP approach. In addition, it has a unique practical advantage of low demand on the characterization of apparatuses. Recently, this has lead to the security proof of efficient QKDs using practical sources and detectors.

In this paper, we first show that this complementarity scenario is not merely a tool to prove the security, but captures exactly what the key distillation is, by proving that there exists a corresponding complementarity task whenever key can be distilled. Then we also show that a slightly different complementarity task, aimed at the same goal but with the available resource restricted, is equivalent to entanglement distillation. These results imply that the distillable key $K_D(\rho_{AB})$ and the distillable entanglement $E_D(\rho_{AB})$ have nice alternative definitions in the complementarity scenario, which clarifies the physical meaning of the difference between the two quantities.

We first formulate the complementarity scenario essentially used in the latest version of the security arguments, which here we call complementary control of a (log $d$)-bit observable. We consider a pair of protocols, the...
primary and the secondary, between which Alice and Bob can choose to execute. The two protocols are roughly described as follows. In the primary protocol, they communicate over a classical channel, and then Alice measures a $(\log d)$-bit local observable $Z$, while Bob tries to guess its outcome. In the secondary protocol, they perform the same classical communication, but after that Alice tries to prepare an eigenstate of an observable $X$, which is conjugate of $Z$. In doing so, we allow Bob to help Alice through an extra quantum (or classical) channel.

More precisely, we require that the choice between the alternative protocols can be postponed after the end of the classical communication. At this point, we assume that the standard basis $\{|i\rangle_A\}_{i=0,\ldots,d-1}$ of $\mathcal{H}_A$ corresponds to the observable $Z$. If they choose the primary protocol, Alice measures $\mathcal{H}_A$ (system $A$) on $\{|i\rangle_A\}_{i=0,\ldots,d-1}$ and Bob conducts a local operation on his entire systems, resulting in the state of systems $AB$ being $\sum_{ij} p_{ij} |ij\rangle \langle ij|_{AB}$. The error in this protocol is given by

$$\delta_Z \equiv 1 - \sum_i p_{ii}. \quad (4)$$

If they choose the secondary protocol, Alice and Bob cooperate over the extra channel in order to prepare system $A$ in state $|0_X\rangle_A \equiv d^{-1/2} \sum_i |i\rangle_A$. When they end up in state $\sigma_A$, we define its error by

$$\delta_X \equiv 1 - A(|0_X\rangle\langle 0_X|)_{A} = A\left(\sigma_A\right). \quad (5)$$

Of course, it would be meaningless if we allowed Alice to discard the contents of system $A$ and prepare $|0_X\rangle_A$ from scratch. In order to claim that they really have created an eigenstate of $X$, conjugate of $Z$, we require that their operation over the extra channel commute with the observable $Z$, namely, it preserves every eigenstate $|j\rangle_A$. We call it the nondisturbing condition.

We now show two theorems implying that this scenario is essentially equivalent to key distillation, as depicted in Fig. 1. In the proofs, we use the fidelity $F(\rho, \sigma) \equiv \| \sqrt{\rho} \sqrt{\sigma} \|^2$ as well as the trace distance. Both measures are monotone under quantum operations, and they are related by $2(1 - \sqrt{F}) \leq \| \rho - \sigma \| \leq 2\sqrt{1 - F}$ [12]. The fidelity is useful because of the existence of extensions $|\phi_+\rangle$ and $|\phi_+\rangle$ satisfying $|\langle \phi_+ | \phi_+ \rangle|^2 = F$, whereas the trace distance obeys the triangle inequality. Eq. (5) can be written as $F(\sigma_A, |0_X\rangle\langle 0_X|) = 1 - \delta_X$.

The security argument in [10] is essentially given by the following theorem.

**Theorem 1.** If complementary control of a $(\log d)$-bit observable with errors $\delta_Z$ and $\delta_X$ is possible with an extra quantum channel, then the primary protocol yields a $(\log d)$-bit key with imperfection $\delta_{\text{key}} \leq 2\delta_Z + 2\sqrt{\delta_X}$.

Proof. Suppose that Alice and Bob run the primary or the secondary protocol in the presence of Eve, leading to the final states $\rho_{ABE} = \sum_{ij} p_{ij} |ij\rangle \langle ij|_{AB} \otimes \rho_E^{(ij)}$ or $\sigma_{AE}$, respectively. Suppose that, after the secondary protocol, (a) we measure system $A$ on the basis $\{|i\rangle_A\}$ to obtain $\sigma'_{AE} = \sum_i q_i |i\rangle_A \otimes \rho_E^{(i)}$, and then (b) copy the outcome onto system $B$, resulting in $\rho_{ABE} = \sum_i q_i |\langle i|_{AB} \otimes \rho_E^{(i)}$.

We will show that $\rho_{ABE}$ and $\tau_{ABE}$ are both close to $\sigma'_{ABE}$.

Since $\sigma_A \equiv \text{Tr}_E (\sigma_{AE})$ satisfies Eq. (5), there exists a state $\tau_{AE} \equiv |0_X\rangle_0 \otimes \rho_E$ with $F(\sigma_{AE}, \tau_{AE}) = 1 - \delta_X$, and hence $\|\sigma_{AE} - \tau_{AE}\| \leq 2\sqrt{\delta_X}$. If we apply the steps (a) and (b) to state $\tau_{AE}$, the final state is an ideal key $\tau_{ABE} = d^{-1} \sum_i |ii\rangle_{AB} \otimes \rho_E$. Therefore, $\|\sigma'_{ABE} - \tau_{ABE}\| \leq 2\sqrt{\delta_X}$.

Thanks to the nondisturbing condition, $\sigma'_{AE} = \text{Tr}_B (\rho_{ABE})$, and hence $\sigma'_{ABE} = \sum_i p_{ij} |ii\rangle_{AB} \otimes \rho_E^{(ij)}$. Then, direct calculation leads to $\|\sigma'_{ABE} - \rho_{ABE}\| = 2\delta_Z$, proving Theorem 1.

Next, we show that the opposite direction is also true if there is no restriction to Eve’s power. Here, we assume the following for Eve with no restriction. Let us represent the entire data transmitted over the public communication by variable $\omega$. We assume that Alice, Bob, and Eve each has the record of $\omega$, and hence $\mathcal{H}_E$ is decomposed as $\mathcal{H}_E = \oplus_\omega \mathcal{H}_{E}^{(\omega)}$. In principle, by using large auxiliary systems $A'$ and $B'$, Alice and Bob can do the same key distillation coherently without discarding any subsystems. We assume that Eve can collect everything that is not possessed by Alice and Bob. This ensures that the final state for a particular value of $\omega$ is a pure state $|\Phi^{(\omega)}_{E'AB'}\rangle$, and the overall state is $\rho_{ABE' \omega} = \oplus_\omega \rho_{ABE'}^{(\omega)} |\Phi^{(\omega)}_{E'}\rangle$. Tracing out systems $A'B'$ gives state $\rho_{ABE} = \oplus_\omega \rho_{ABE'}^{(\omega)}$.

Now we can prove the following theorem.

**Theorem 2.** If a $(\log d)$-bit key with imperfection $\delta_{\text{key}}$ can be distilled against Eve with no restriction, then complementary control of a $(\log d)$-bit observable with an ex-
extra quantum channel is possible with errors $\delta_Z \leq \delta_{\text{key}}/2$ and $\delta_X \leq \delta_{\text{key}} - (\delta_{\text{key}}/2)^2$.

Proof. We regard the key distillation protocol as the primary protocol. Then $\delta_Z \leq \delta_{\text{key}}/2$ is trivial. Before stating the secondary protocol, we need the following observations. In the assumption $||\rho_{ABE} - \tau_{ABE}|| = \delta_{\text{key}}$, $\tau_{ABE}$ may not be a direct sum over $\omega$. But we can define such a state $\tau_{ABE} = \oplus_\omega \rho_{\omega}^E \rho_{\omega}^A \rho_{\omega}^B$ by applying decoherence to $\tau_{ABE}$. Since the same decoherence operation does not alter $\rho_{ABE}$, we have $||\rho_{ABE} - \tau_{ABE}|| \leq \delta_{\text{key}}$, or $F(\rho_{ABE}, \tau_{ABE}) \geq (1 - \delta_{\text{key}}/2)^2$.

Then, there exists an extension of $\tau_{ABE}$ taking the form of $\tau'_{ABE \rightarrow AB' \rightarrow E} = \oplus_\omega \rho_{\omega}^E \rho_{\omega}^A \rho_{\omega}^B$, satisfying $F(\rho_{ABE \rightarrow AB' \rightarrow E}, \tau'_{ABE \rightarrow AB' \rightarrow E}) \geq (1 - \delta_{\text{key}}/2)^2$.

Since $\tau_{\omega}' = d^{-1} \sum_i |i\rangle \langle i| \otimes \rho_{\omega}^E$, state $|\Phi_{\tau}^E\rangle$ must be written in the form of $d^{-1/2} \sum_i |i\rangle \langle i| \otimes \rho_{\omega}^E$, with $\text{Tr}_{AB'}(\rho_{\omega}^A \rho_{\omega}^B) \rho_{\omega}^E = \rho_{\omega}^E$, which is independent of $i$. This implies the existence of unitaries $U_{AB'}^{(\omega)}$ satisfying $U_{AB'}^{(\omega)} |\Phi_{\omega}^E\rangle = |\Phi_{\omega}^E\rangle$. If we define $U_{AB'}^{(\omega)} = \sum_i |i\rangle \langle i| A \otimes U_{AB'}^{(\omega)}$, we see

$$U_{AB'}^{(\omega)} |\Phi_{\omega}^E\rangle = d^{-1/2} \sum_i |i\rangle A |\Phi_{\omega}^E\rangle B \otimes E.$$  

where the state of system $A$ is $|0_x\rangle_A$.

Hence we can construct the secondary protocol as follows: After the coherent version of the key distillation protocol, using the record of $\omega$, Alice and Bob apply $U_{AA'BB'}^{(\omega)}$ using an extra quantum channel. The form of $U_{AB'}^{(\omega)}$ obviously satisfies the nondisturbing condition. If the state after the key distillation protocol was $\tau'_{\omega}$, the protocol would produce $|0_x\rangle_A$ exactly. Thus, for state $\rho_{ABE \rightarrow AB' \rightarrow E}$, the output $\sigma_A$ should satisfy $A(0_x \sigma_A |0_x\rangle_A) \geq (1 - \delta_{\text{key}}/2)^2$, namely, $\delta_X \leq \delta_{\text{key}} - (\delta_{\text{key}}/2)^2$.

The two theorems indicate that the complementarity scenario is a powerful tool for QKD, namely, there is no fundamental limitation in applying the scenario to prove the security of QKD protocols. They also show that distillable key $K_D(\rho_{AB})$ of a bipartite state $\rho_{AB}$ can be also defined in the complementarity scenario. Let us introduce the asymptotic yield of complementary control $Y_Q$, where the subscript signifies that the extra channel is quantum. We define $Y_Q(\rho_{AB})$ to be the supremum of real numbers $y$ with which the following statement holds true.

Starting with $\rho_{AB}$, complementarity control of a $(\log d_n)$-bit observable is possible with errors $(\delta_Z^{(n)}, \delta_X^{(n)})$, where $\delta_Z^{(n)} \rightarrow 0$, $\delta_X^{(n)} \rightarrow 0$, and $\log d_n / n \rightarrow y$ for $n \rightarrow \infty$. With this definition, Theorem 1 implies $K_D(\rho_{AB}) \geq Y_Q(\rho_{AB})$ while Theorem 2 implies $K_D(\rho_{AB}) \leq Y_Q(\rho_{AB})$, leading to

$$K_D(\rho_{AB}) = Y_Q(\rho_{AB}).$$  

Next, let us consider a slightly different scenario, in which Alice and Bob are allowed to communicate only classically in the secondary protocol. Then we can find a close connection to distillation of the $(\log d)$-bit maximally entangled state defined in Eq. (3), as depicted in Fig. 2. For output state $\rho_{AB}$, we measure the imperfection in the distillation task by

$$\delta_{\text{ent}} \equiv \|\rho_{AB} - \tau_{\text{ent}}^\text{AB}\|$$  

with $\tau_{\text{ent}}^\text{AB} \equiv |\phi^{\text{mes}}\rangle \langle \phi^{\text{mes}}|$. Then we can prove the following.

**Theorem 3.** If complementary control of a $(\log d)$-bit observable with errors $\delta_Z$ and $\delta_X$ is possible with an extra classical channel, then it is possible to distill a $(\log d)$-bit maximally entangled state with imperfection $\delta_{\text{ent}} \leq 4\sqrt{\delta_Z (1 - \delta_Z) + 2\delta_X}$.

Proof. In the primary protocol, Alice and Bob’s operation after the classical communication can be coherently done by enlarging the size of systems $A'B'$, namely, it can be done by a unitary operation $V = V_{AA'} \otimes V_{BB'}$. Let $\sigma_{AA'BB'}$ be the state after the application of $V$. The error $\delta_Z$ implies that

$$\text{Tr}[(Q_{AB}^m \sigma_{AA'BB'})] = 1 - \delta_Z$$  

with $Q_{AB}^m \equiv \sum_i |ii\rangle \langle ii|_{AB}$ is the projection onto the subspace with no errors.

From the state $\sigma_{AA'BB'}$, Alice and Bob can undo the unitary by applying $V^{-1}$, going back to the state just after the classical communication. From here, Alice and Bob can choose to conduct the secondary protocol using the extra classical channel to produce state $\rho_A = \Lambda(\sigma_{AA'BB'})$, where we write the whole quantum operation starting from $V^{-1}$ by a CPTP map $\Lambda$. The error $\delta_X$ implies that

$$\|\Lambda(\sigma_{AA'BB'}) - |0_x\rangle \langle 0_x|_A\| \leq 2\sqrt{\delta_X}. $$  

We construct a distillation protocol as follows. Alice and Bob conduct the coherent version of the primary protocol, resulting in $\sigma_{AA'BB'}$. Bob further uses an
auxiliary system $C$ with dimension $d$, prepared in state $\sigma_C \equiv |0\rangle\langle 0|_C$. He copies the contents of system $B$ onto system $C$ by unitary $U_B^{\mathrm{csv}}: |j0\rangle_{BC} \mapsto |jj\rangle_{BC}$, resulting in state

$$\sigma'_{AA'B'B'C} = U_B^{\mathrm{csv}}(\sigma_{AA'B'B'} \otimes \sigma_C)U_B^{\mathrm{csv}\dagger}. \quad (10)$$

Alice and Bob then apply $\Lambda$ on systems $AA'B'B'$ using the extra classical channel to achieve the final state $\rho'_{AC} = \Lambda(\sigma'_{AA'B'B'C})$.

We now prove that $\rho'_{AC}$ is close to $\tau^\text{ent}_{AC}$. Consider the state defined by

$$\sigma''_{AA'B'B'C} = U_B^{\mathrm{csv}}(\sigma_{AA'B'B'} \otimes \sigma_C)U_B^{\mathrm{csv}\dagger}. \quad (11)$$

with $U_B^{\mathrm{csv}}: |j0\rangle_{AC} \mapsto |jj\rangle_{AC}$. Using Eq. 8 and the obvious relation $U_B^{\mathrm{csv}}G_{AB} = U_B^{\mathrm{csv}\dagger}G_{AB}$, we can show [13] that $||\sigma'_{AA'B'B'C} - \sigma''_{AA'B'B'C}|| \leq 4\sqrt{d}(1 - \delta_Z)$ and hence

$$||\rho'_{AC} - \Lambda(\sigma''_{AA'B'B'C})|| \leq 4\sqrt{d}(1 - \delta_Z). \quad (12)$$

On the other hand, the non-disturbing condition implies that there is no difference whether we apply $U_B^{\mathrm{csv}}$ before or after the application of $\Lambda$. This leads to

$$\Lambda(\sigma''_{AA'B'B'C}) = U_B^{\mathrm{csv}}(\Lambda(\sigma_{AA'B'B'}) \otimes \sigma_C)U_B^{\mathrm{csv}\dagger}. \quad (13)$$

Then, using Eq. 2, we have

$$||\tau^\text{ent}_{AC} - \Lambda(\sigma''_{AA'B'B'C})|| \leq 2\sqrt{d}. \quad (14)$$

Combined with Eq. (12), it proves Theorem 3.

The opposite direction is trivial, and it is stated as follows (proof omitted).

**Theorem 4.** If a $(\log d)$-bit maximally entangled state with imperfection $\delta_{\text{ent}}$ can be distilled, then complementary control of a $(\log d)$-bit observable with an extra classical channel is possible with errors $\delta_Z \leq \delta_{\text{ent}}/2$ and $\delta_X \leq \delta_{\text{ent}} - (\delta_{\text{ent}}/2)^2$.

If we define the asymptotic yield $Y_C(\rho_{AB})$ with an extra classical channel as we defined $Y_Q$ before, Theorems 3 and 4 lead to

$$E_D(\rho_{AB}) = Y_C(\rho_{AB}), \quad (15)$$

which shows that the distillable entanglement $E_D$ also has an alternative definition in the complementarity scenario. Together with Eq. 9, now we see that distillable key and distillable entanglement can be regarded as achievable yields of the same task, carried out under different conditions. This gives a clear distinction between the two entanglement measures. Both are related to the potential to carry out two mutually exclusive tasks concerning a pair of conjugate observables $Z$ and $X$, using the same classical communication. One task is to share the value of $Z$, and the other one is to drive the state into an eigenstate of $X$. The latter task naturally requires additional communication, and this is where the difference between the two quantities shows up. If we insist that it also must be classical and hence both tasks are feasible with only classical communication, the achievable size of the observables tallies with the distillable entanglement. If we place no such requirement, then the achievable size matches the distillable key. This may be understandable because if the key is actually distilled, the task for $X$ is never carried out and hence there is no concern about what resources are required to carry it out.

We have seen that the complementarity scenario can explain two of the few operationally-defined entanglement measures, which shows its significance in understanding quantum entanglement. It is interesting to see whether we can also define yet another operationally-defined measure, entanglement cost [17] in a complementarity scenario. The task of the complimentary control defined here is merely one of many possible ways to quantify abilities related to the concept of complementarity, and it is worth seeking other tasks, for example, the ones retaining the symmetry between two conjugate observables.

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