The sums of Rogers, Schur and Ramanujan and the Bose-Fermi correspondence in 1 + 1-dimensional quantum field theory

Rinat Kedem, Barry M. McCoy, and Ezer Melzer

Institute for Theoretical Physics
State University of New York
Stony Brook, NY 11794-3840

Abstract

We discuss the relation of the two types of sums in the Rogers-Schur-Ramanujan identities with the Bose-Fermi correspondence of massless quantum field theory in 1 + 1 dimensions. One type, which generalizes to sums which appear in the Weyl-Kac character formula for representations of affine Lie algebras and in expressions for their branching functions, is related to bosonic descriptions of the spectrum of the field theory (associated with the Feigin-Fuchs construction in conformal field theory). Fermionic descriptions of the same spectrum are obtained via generalizations of the other type of sums. We here summarize recent results for such fermionic sum representations of characters and branching functions.

1 rinat or mccoy or melzer @max.physics.sunysb.edu
1. Introduction

The most important and difficult course taught in any university is philosophy. It is of overriding importance because everyone has a deeply rooted philosophy that rules their actions with an iron hand. It is the most difficult of all courses to learn because no two people are in complete agreement as to their philosophic principles. The inevitable response to a question in philosophy is that it is either trivially obvious or absolutely absurd. Unhappily, there is no agreement about what is obvious and what is absurd. The consequence is that almost nobody studies philosophy.

The second most difficult subject is physics. The difficulty is that the study of physics requires students to hold two competing philosophies in their minds at the same time and to form a synthesis. The two competing philosophies are empiricism on the one hand, as embodied in experiment and measurement, and rationalism or abstraction on the other hand, as embodied in mathematics and computation. Physics is neither the one nor the other but the Hegelian synthesis of both. More students take elementary courses in physics than study Aristotle, Aquinas, Kant and Hegel, but most do poorly and get bad grades.

In the past (say) 30 years great progress has been made in theoretical (or mathematical) physics. Yet because physics is a synthesis, the true understanding of the accomplishment is best made not by presenting one set of developments but rather by describing two parallel sets of developments, one loosely called mathematical and the other loosely called physical. The mathematical side of the developments we concentrate upon embodies Rogers-Ramanujan identities, modular forms, infinite-dimensional algebras (such as affine Kac-Moody and the Virasoro algebras) and their representation theory. The physical side involves statistical mechanics, quantum spin chains, quantum field theories (both conformal and massive), bosons and fermions.

Our ultimate goal here is to present the status of some of our recent results on fermionic sum representations for conformal field theory characters. This is done in section 4. However, we also wish to elucidate the position our results have within the larger tapestry of work of the last century in physics and mathematics. To that end we will present in sections 2 and 3, respectively, the elements of the mathematical and the physical sides out of which the synthesis is born. The elements have the names given in the previous paragraph. There is no accepted name for the synthesis that puts both the competing elements on an equal footing. We trust that this lack of a name will not be an impediment to the philosophic reader.
2. The mathematical ingredients

In 1894 Rogers [1] proved the following set of identities

\[ S_0 = \sum_{n=0}^{\infty} q^{n^2} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-1})(1-q^{5n-4})} = \frac{1}{(q)_{\infty}} \sum_{n=-\infty}^{\infty} (q^{n(10n+1)} - q^{(5n+2)(2n+1)}) \]

\[ S_1 = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-2})(1-q^{5n-3})} = \frac{1}{(q)_{\infty}} \sum_{n=-\infty}^{\infty} (q^{n(10n+3)} - q^{(5n+1)(2n+1)}) \]

where

\[ (q)_0 = 1 , \quad (q)_n = \prod_{k=1}^{n} (1-q^k) \quad \text{for} \quad n = 1, 2, 3, \ldots . \]

A second proof by Rogers [2] and two independent proofs by Schur [3] were given in 1917. Hardy tells us [4][5] that the equality of the left-hand sums with the products was independently conjectured by Ramanujan in 1913, with a proof due to him published in [4], and these equalities have subsequently come to be known as the Rogers-Ramanujan identities. There seems to be no commonly accepted term which refers to all three expressions in these identities of Rogers, Schur, and Ramanujan on the same footing.

The products and the right-hand sums in (2.1)-(2.2) may be directly expressed in terms of theta functions [1][4] and consequently it is readily seen that if one sets

\[ c_0(q) = q^{-1/60} S_0(q) , \quad c_1(q) = q^{11/60} S_1(q) \]

and defines \( \tau \) by \( q = e^{2\pi i \tau} \), the following linear transformation law is obtained:

\[ \begin{pmatrix} c_0(-1/\tau) \\ c_1(-1/\tau) \end{pmatrix} = \frac{2}{\sqrt{5}} \begin{pmatrix} \sin \frac{2\pi}{5} & \sin \frac{3\pi}{5} \\ \sin \frac{\pi}{5} & -\sin \frac{2\pi}{5} \end{pmatrix} \begin{pmatrix} c_0(\tau) \\ c_1(\tau) \end{pmatrix} . \]

This enables one to show that \( c_0(\tau) \) and \( c_1(\tau) \) form a two-dimensional representation of the modular group. This group has two generators

\[ T : \ \tau \rightarrow \tau + 1 , \quad S : \ \tau \rightarrow -1/\tau , \]

which satisfy the relations

\[ S^2 = (ST)^3 = 1 . \]

The second mathematical ingredient we need is the infinite-dimensional generalization of Lie algebras introduced by Kac [6] and Moody [7] in 1967. Our purpose here is not to
review this theory, which is presented in detail in [8], but merely to recall a few definitions.

In particular, given a simple Lie algebra $G$ of rank $r$ and dimension $d$ with structure constants $f^{abc}$, the untwisted affine Lie algebra $G^{(1)}$ (defined in terms of a generalized Cartan matrix [6][7]) is realized by the commutation relations

$$[J^a_m, J^b_n] = \sum_{c=1}^d i f^{abc} J^c_{m+n} + km\delta^{ab}\delta_{m,-n} \quad (m, n \in \mathbb{Z}, \ a, b = 1, \ldots, d),$$

(2.8)

where we use the basis and normalization conventions of [9]. Here $k$ is a central element, i.e. it commutes with every element of the algebra, and takes on a constant value in any given irreducible representation of $G^{(1)}$. The value of $2k/\psi^2$, where $\psi$ is the highest root of $G$ (which will be normalized to $\psi^2 = 2$ below), is then called the level of the representation and is a positive integer in the representations considered here.

It is worth noticing that the synthesis of the physical and the mathematical is already inherent in (2.8). This realization of the affine Kac-Moody algebras, which was derived from the definitions of [6][7] in the late 1970s, had been found earlier by Schwinger [10] in his analysis of relativistic invariance of gauge theories in 3+1 dimensions. Consequently the central element in (2.8), which plays a crucial role in physics applications as well as in representation theory, is at times [9] referred to as a Schwinger term.

Shortly after the construction of Kac-Moody algebras the Virasoro algebra was introduced in 1970 [11]. This algebra is defined by the commutation relations

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12} m(m^2 - 1)\delta_{m+n,0} \quad (m, n \in \mathbb{Z}),$$

(2.9)

where the normalization of the $L_n$ is chosen such that

$$[L_{\pm 1}, L_0] = \pm L_{\pm 1} , \quad [L_1, L_{-1}] = 2L_0 ,$$

(2.10)

and $c$ is a central element whose constant value in an irreducible representation is called the central charge.

Of great importance are the Virasoro characters

$$\chi_l(q) = q^{-c/24} \text{ Tr } q^{L_0} ,$$

(2.11)

where the trace is over an irreducible highest-weight representation $\mathcal{V}_l(c, \Delta_l)$ of the Virasoro algebra, and the factor $q^{-c/24}$ is inserted to guarantee linear behavior under the modular transformations (2.6). Such a representation $\mathcal{V}_l$, and hence its character, is characterized
by the central charge \( c \) and the highest weight \( \Delta_l \) which is the \( L_0 \)-eigenvalue of the highest-weight vector of \( \mathcal{V}_l \).

Of interest to us here are several cases of these characters, as well as characters of representations of various algebras which contain the Virasoro algebra as a subalgebra (such as superconformal algebras, \( \mathcal{W} \)-algebras, parafermionic algebras, and the already mentioned affine Lie algebras). The first case concerns the irreducible representations of the Virasoro algebra at central charge

\[
c = 1 - \frac{6(p - p')^2}{pp'}
\]

(2.12)

(where \( p \) and \( p' \) are coprime positive integers) and highest weights

\[
\Delta^{(p,p')}_{r,s} = \frac{(rp' - sp)^2 - (p - p')^2}{4pp'} \quad (r = 1, \ldots, p - 1; \quad s = 1, \ldots, p' - 1).
\]

(2.13)

Based on the work of Feigin and Fuchs [12], Rocha-Caridi [13] obtained the following expressions for the corresponding characters:

\[
\hat{\chi}^{(p,p')}_{r,s} = \frac{q^{c/24-\Delta^{(p,p')}_{r,s}}}{(q)^\infty} \chi^{(p,p')}_{r,s} = \sum_{k=-\infty}^{+\infty} (q^{k(rp' - sp)} - q^{(kp+s)(kp+r)}).
\]

(2.14)

As required from \( \Delta^{(p,p')}_{r,s} = \Delta^{(p,p')}_{p-r,p'-s} \), these characters have the symmetry

\[
\chi^{(p,p')}_{r,s} = \chi^{(p,p')}_{p-r,p'-s}.
\]

(2.15)

We note in particular that if \( (p, p') = (2, 5) \), then the two independent sums on the right-hand side of (2.14), namely with \( (r, s) \) set to \((1, 2)\) and \((1, 1)\), are identical with the two sums on the right-hand side of the Rogers-Schur-Ramanujan identities (2.1) and (2.2), respectively.

The second case of interest here is that of the affine Lie algebras \( G^{(1)} \). Now \( L_0 \), entering the definition of the characters (2.11) of level \( k \) representations of \( G^{(1)} \), is quadratic in the generators \( J_a^a \) of \( G^{(1)} \). In fact, all the Virasoro generators \( L_n \) can be obtained from the \( J_a^a \) via [9]-[18] the construction used by Sugawara [14] in the analysis of Schwinger terms [10] in non-abelian gauge theory:

\[
L_n = \frac{1}{2(k + g)} \sum_{a=1}^{r} \sum_{m=-\infty}^{+\infty} :J^a_{m+n}J^a_{-m}:,
\]

(2.16)
where \( g \) is the dual Coxeter number \( 4 \) of \( G \), and the normal ordered product \( :J^a_m J^b_n: \)
equals \( J^a_m J^b_n \) if \( m \leq n \) and \( J^b_n J^a_m \) otherwise. The corresponding Virasoro central charge is

\[
c = \frac{k \dim(G)}{k + g}.
\]

(2.17)

For a general algebra \( G^{(1)} \) and arbitrary integer level \( k \), the characters (2.11), where now the trace is taken over irreducible highest-weight representations of \( (G^{(1)})_k \), are given by

The Weyl-Kac formula [8][15]. In the case of \( G = A_1 \equiv su(2) \) and \( k = 1 \) there are two characters, which can be written in the particularly simple form (cf. [19])

\[
q^{1/24-\ell^2/4} \chi_l = \frac{1}{(q)^\infty} \sum_{n=-\infty}^{\infty} q^{n(n+\ell)} \quad (l = 0, 1).
\]

(2.18)

The most general case we need are the characters of the algebras which arise in coset constructions, introduced by Goddard, Kent and Olive [18]. The characters are then branching functions of affine Lie algebras [20][21]. A wide class of cosets is given by \( (G^{(1)}_k \times G^{(1)}_l)/(G^{(1)}_{k+l}) \); for \( l = 1 \) the corresponding branching functions are characters of the \( \mathcal{W}G \)-algebra [22] which reduces to the Virasoro algebra when \( G = A_1 \). As a particular example, the branching functions for \( (A^{(1)}_{N-1})_1 \times (A^{(1)}_{N-1})_1 \) (which is equivalent by level-rank duality [23] to the coset \( (G^{(1)}_N \times U(1))/(G^{(1)}_{N-1})_2 \) are [20][24]

\[
q^{c/24-h^l_m} b^l_m = \frac{1}{(q)^\infty} \left[ \left( \sum_{s \geq 0} \sum_{n \geq 0} - \sum_{s < 0} \sum_{n < 0} \right) (-1)^s q^{(s+1)/(1+s) + (l+1)n + (l+m)s + (N+2)(n+s)n} \right.

\[
+ \left( \sum_{s > 0} \sum_{n \geq 0} - \sum_{s \leq 0} \sum_{n < 0} \right) (-1)^s q^{(s+1)/(1+s) + (l+1)n + (l-m)s + (N+2)(n+s)n} \right],
\]

where

\[
c = \frac{2(N-1)}{N+2}, \quad h^l_m = \frac{l(l+2)}{4(N+2)} - \frac{m^2}{4N}.
\]

(2.19)

(2.20)

Here \( l = 0, 1, \ldots, N - 1 \), \( l - m \) is even, and the formulas are valid for \( |m| \leq l \) while for \( |m| > l \) one uses the symmetries

\[
b^l_m = b^l_{-m} = b^l_{m+2N} = b^l_{N-m}.
\]

(2.21)

We note that the right-hand sides of (2.14), (2.18), and (2.19) share the feature with the sums on the right of (2.1)-(2.2) that the denominator is a power of \( (q)^\infty \) and the
numerator is a power series in $q$ (with integer coefficients). Divided by the explicit power of $q$ on the left-hand sides, they also share the property with (2.4) that they can be seen to form representations of the modular group (2.6). These features hold for the general case of the Weyl-Kac character formula and the branching functions as obtained from it.

The above discussion shows that the right-hand side of the Rogers-Schur-Ramanujan identities (2.1)-(2.2) has a vast generalization in terms of characters of representations of infinite-dimensional algebras. It is thus natural to ask whether the remaining parts of these identities can also be generalized, thus yielding different expressions for such characters.

The first step in this direction was taken by Lepowsky and Milne [23] in 1978 when they showed that for $A_1^{(1)}$ and $A_2^{(2)}$ the Weyl-Kac formula, when suitably specialized, admits a product form. In 1981 Lepowsky and Wilson [26] found a way to obtain the sums on the left-hand side of (2.1)-(2.2) using a construction which they called $Z$-algebras, and thus provided a Lie-algebraic proof of the Rogers-Ramanujan identities.

A major generalization of these results is due to Lepowsky and Primc [27]. Extending the work of [26][28] on $Z$-algebras (for a recent review see [29]), they found in 1985 that the branching functions (2.19) can be written as

$$q^{c/24} \sum_{N-1 \equiv Q (\text{mod } N), \mathbf{m} = (m_1, \ldots, m_{N-1}) \text{ subject to restrictions}} \frac{q^{mC_{N-1}^{-1} \mathbf{m}' - A_l \cdot \mathbf{m}}}{(q)_{m_1} \cdots (q)_{m_{N-1}}},$$

where $Q$ is an integer (mod $N$), $\mathbf{m} = (m_1, \ldots, m_{N-1})$ is subject to the restriction

$$\sum_{a=1}^{N-1} a m_a \equiv Q \mod N,$$

$C_{N-1}$ is the Cartan matrix of the Lie algebra $A_{N-1}$ in the basis where

$$m^{C_{N-1}^{-1} \mathbf{m}' = \frac{1}{N} \left( \sum_{a=1}^{N-1} a(N-a)m_a^2 + 2 \sum_{1 \leq a < b \leq N-1} a(N-b)m_am_b \right)},$$

and $A_0=0$ while for $l = 1, \ldots, N-1$

$$A_l \cdot \mathbf{m} = -(mC_{N-1}^{-1})_l = -\left( \frac{N-l}{N} \sum_{a=1}^{l} a m_a + \frac{l}{N} \sum_{a=l+1}^{N-1} (N-a)m_a \right).$$

This representation is of the form of a $q$-series which generalizes the left-hand sums of (2.1) and (2.2) to multiple sums such as appear in the Gordon-Andrews identities [30][31].
Moreover, the sums in the Gordon-Andrews identities themselves have also been found \cite{32} to be the Virasoro characters (2.14) with $(p, p') = (2, 2n + 3)$. (We note that the analysis in \cite{32} of the corresponding representations of the Virasoro algebra leads directly to the product rather than the sum sides of the Gordon-Andrews identities.) In particular, \( \hat{\chi}_{1, n+1}^{(2, 2n+3)} \) is given by the right-hand side of (2.22) with no restrictions on the sum, \( A_{l} = 0 \) and \( C_{N-1}^{-1} \) replaced by \( (C_{n}')^{-1} \), where \( C_{n}' \) differs from \( C_{n} \) only in one entry which is \( (C_{n}')_{nn} = 1 \). All the other characters are obtained \cite{32} by adding suitable linear terms to the quadratic form in (2.22), leading to the full set of sums appearing in the Gordon-Andrews identities \cite{30} \cite{31}. When \( n=1 \) one has \( (p, p')=(2, 5) \) and as noted above the Virasoro characters reduce to the original sums on the left-hand side of the Rogers-Ramanujan identities (2.1)-(2.2).

However, until quite recently these were the only results known. The major purpose of this note is to summarize the recent progress in finding generalizations of the left-hand side of (2.1)-(2.2) for the Virasoro characters (2.14), the Weyl-Kac characters, and characters of the coset models discussed above. (Product formulas for characters have been recently discussed in \cite{33} \cite{34}.)

3. The physical ingredients

There are at least three physical starting points which will be used to form the synthesis with the mathematics of the previous section: two-dimensional classical statistical mechanics, one-dimensional quantum spin chains, and conformal field theory. The latter two lead to the concept of boson and fermion and to a relation between them that exists in 1+1 dimensions.

Consider first an \( M \)-body quantum spin chain with periodic boundary conditions. In the study of the spectra of \( M \)-body hamiltonians with local interactions and translational invariance, the eigenstates which lie a finite energy above the ground state energy as \( M \to \infty \) have the quasi-particle form for the energy

\[
E_{i} - E_{GS} = \sum_{\alpha} \sum_{j=1, \text{rules}}^{m_{\alpha}} e_{\alpha}(P_{\alpha}^{j})
\]  

(3.1)

and for the momentum

\[
P_{i} \equiv \sum_{\alpha} \sum_{j=1, \text{rules}}^{m_{\alpha}} P_{\alpha}^{j} \pmod{2\pi},
\]  

(3.2)
where $e_\alpha(P_j^\alpha)$ is called the single-particle excitation energy of type $\alpha$, and $m_\alpha$ are the numbers of such excitations in the eigenstate under consideration, which is labeled by $i$. The sum over $P_j^\alpha$ is subject to certain rules. If one of these rules is
\[ P_j^\alpha \neq P_k^\alpha \quad \text{for } j \neq k \text{ and all } \alpha, \tag{3.3} \]
the spectrum is called fermionic. If there is no such exclusion rule and an arbitrary number of coinciding $P_j^\alpha$ is allowed, then the spectrum is called bosonic.

The calculation of single-particle energies is extensively considered in condensed matter physics. When considered on a lattice they are often periodic functions defined in an appropriate Brillouin zone. By definition $e_\alpha(P)$ can never be negative. If all the $e_\alpha(P)$ are positive the system is said to have a mass gap. If some $e_\alpha(P)$ vanishes at some momentum (say 0) the system is said to be massless, and for $P \sim 0$ a typical behavior is
\[ e_\alpha(P) = v_\alpha |P| \tag{3.4} \]
where $v_\alpha$ is variously referred to as the fermi velocity, spin-wave velocity, speed of sound or speed of light.

In the statistical mechanics of many-body systems the most fundamental quantity is the partition function which is defined as
\[ Z = \text{Tr} \ e^{-H/k_BT}, \tag{3.5} \]
where $H$ is the hamiltonian, the trace is over all states of the system, $k_B$ is Boltzmann’s constant and $T$ is the temperature. More explicitly this may be written as
\[ Z = e^{-E_{GS}/k_BT} \sum_i e^{-(E_i-E_{GS})/k_BT} \tag{3.6} \]
where the sum is over all the eigenvalues $E_i$ of $H$ (with their multiplicities) and we have explicitly factored out the contribution of the ground state energy $E_{GS}$.

For a macroscopic system we are usually more interested in the free energy per site $f$ and the specific heat $C$, in the thermodynamic limit which is defined as
\[ \text{fixed } T > 0 \quad \text{and } \quad M \to \infty. \tag{3.7} \]
The free energy and the specific heat are then
\[ f = -k_BT \lim_{M \to \infty} \frac{1}{M} \ln Z, \quad C = -T \frac{\partial^2 f}{\partial T^2}. \tag{3.8} \]
Starting with the work of Bethe \[36\] in 1931 and Onsager \[37\] in 1944 it has been discovered that there is a very large number of one-dimensional quantum spin chains (and two-dimensional classical statistical mechanical systems) whose energy spectrum and partition function may be exactly studied by what are essentially algebraic means, starting with the existence of a family of commuting transfer matrices \[38\]

\[
[T(u), T(u')] = 0 .
\]  

(3.9)

This commutation relation generalizes the concept of integrability of classical mechanics. The search for solutions to this equation leads to the famous star-triangle \[37\][39] or Yang-Baxter equation \[38\][40] and is beyond the scope of this note. We remark, however, that these systems can be massive as well as massless, and have profound connections to the theory of non-linear differential equations \[41\].

The next physical ingredient we need is the approach to the study of conformal field theories introduced in 1984 by Belavin, Polyakov, and Zamolodchikov \[42\]. The original presentation is directly relevant for two-dimensional statistical mechanics. However, for our present purpose it is more convenient to formulate the theory in terms of one-dimensional quantum spin chains. In this formulation, conformal field theory deals with massless systems whose excitations are characterized by (2.25) where the \(v_\alpha\) are the same for all \(\alpha\), i.e. \(v_\alpha = v\). However, instead of the thermodynamic limit (3.7) we study the limit

\[
M \to \infty, \quad T \to 0 \quad \text{with} \quad MT \quad \text{fixed},
\]  

(3.10)

which we will refer to as the conformal limit. Defining the scaled partition function

\[
\hat{Z} \equiv \lim_{e_0 \to k_B T} e^{M e_0/k_B T} Z
\]  

(3.11)

in the conformal limit, where \(e_0 \equiv \lim_{M \to \infty} \frac{1}{M} E_{GS}\), \(\hat{Z}\) becomes a function of the dimensionless variable

\[
q = e^{-\frac{2\pi v}{M k_B T}}.
\]  

(3.12)

It is here that the first synthesis with mathematics takes place because it is found \[43\] that the partition function \(\hat{Z}\) (3.11) is expressed in terms of characters of representations of the Virasoro algebra (or possibly some extension of it) as

\[
\hat{Z} = \sum_{k,l} N_{kl} \chi_k(q) \chi_l(\bar{q}),
\]  

(3.13)
where the $N_{kl}$ are non-negative integers. In the two-dimensional statistical system $\bar{q}$ is the complex conjugate of $q$. In the quantum spin chain context $q = \bar{q}$ is of course real, given by (3.12), and the factorization corresponds to a decomposition into contributions coming from the right-moving and left-moving excitations separately. This factorization is sometimes called chiral decomposition, and the algebras of which $\chi_k$ are characters are referred to as chiral algebras. In the interpretation as a two-dimensional statistical system the modular transformation $S$ (2.6) corresponds to interchange of the vertical and horizontal axes. This interchange should leave the partition function invariant (when the boundary conditions in both directions are the same), and this invariance follows from the modular transformation properties of the $\chi_k$ if the $N_{kl}$ are suitably chosen [43] [44].

If there is only one length scale in the problem, the low-temperature specific heat computed from (3.8) should agree with the specific heat computed from $\hat{Z}$ of (3.13) in the limit $q \to 1^-$. Generically, if we set $q = e^{2\pi i \tau}$ and $\tilde{q} = e^{-2\pi i / \tau}$, we have

$$\chi_k \sim A_k \tilde{q}^{\bar{c}/24} = A_k e^{-\left(2\pi\right)^2 \bar{c} / \ln q} \quad \text{as} \quad q \to 1^-,$$

(3.14)

where $\bar{c}$ is independent of $k$ and the $A_k$ are positive constants independent of $q$. Using (3.14) in (3.13) one concludes that the low-temperature behavior of the specific heat is

$$C \sim \frac{\pi k_B \bar{c}}{3\nu} T.$$

(3.15)

The quantity $\bar{c}$ is known as the effective central charge, and since the $\chi_k$ form a representation of the modular group (2.7) (with $S$, in particular, transforming $q$ to $\tilde{q}$) we find that

$$\bar{c} = c - 24 \min_k \Delta_k.$$

(3.16)

The final piece of physical information we need is the concept of Bose-Fermi correspondence in 1+1 dimensions. In three space dimensions the concepts of bosons and fermions, whether defined through their spectra as we did above or in terms of commutation and anti-commutations relations (which are equivalent definitions due to the spin-statistics connection), are quite distinct. However, in 1+1 dimensions they are related. The earliest recognition of such a phenomenon was in the 1929 paper of Jordan and Wigner [45] and the transformation they found plays a key role in the 1949 solution of the Ising model by Kaufman [46]. In quantum field theory the most familiar example of the phenomenon is the relation between the massive Thirring model and the sine-Gordon model [47] [48] [49].
Mandelstam’s operator \[ \text{[48]} \] can be thought of as implementing in the continuum the Jordan-Wigner transformation that relates \[ \text{[50]} \] the spin chains underlying the two field theories. This Bose-Fermi correspondence has been extensively studied in the more general context of current algebras (see \[ \text{[52]} \] for a recent review), and in the massless case contact is made with affine Lie algebras \[ \text{[53]} \].

Our goal here is to indicate that this Bose-Fermi correspondence is of universal occurrence and that all conformal field theory characters have two types of sum representations, generalizing the right-hand (bosonic) sums of the Rogers-Schur-Ramanujan identities \( \text{[2.1]} \)-\( \text{[2.2]} \) and their left-hand (fermionic) sums. The remainder of this note will concern our recent discoveries of the fermionic counterparts for large classes of models for which only the bosonic forms have been known previously.

### 4. Fermionic sums for conformal field theory characters

The presentation of the previous two sections strongly suggests that for solvable one-dimensional quantum spin chains derived from two-dimensional statistical mechanical models characterized by commuting transfer matrices \( \text{[3.9]} \), it should be possible to derive the characters of the related chiral algebra by directly computing the energy levels and the partition function \( \hat{Z} \) \( \text{[3.11]} \), and then putting \( \hat{Z} \) in the form \( \text{[3.13]} \). Recently a great deal of progress has been made in this program for the critical 3-state Potts model \[ \text{[55]} \]-\[ \text{[56]} \]. A prominent feature of these methods, which utilize parametrization of the energy levels in terms of solutions to a set of Bethe equations, is that they always lead to spectra with the fermi exclusion rule \( \text{[3.3]} \) and never to bosonic spectra. Consequently, one obtains fermionic sum representations for the characters of the chiral algebra of the conformal field theory which describes the continuum limit of the spin chain.

The results of the 3-state Potts model computations strongly suggest further generalizations which were presented in \[ \text{[57]} \]-\[ \text{[59]} \]. The most general of these results is that all characters can be written in the form

\[
\hat{\chi} = \sum_i e^{-\hat{E}_i/k_B T},
\]

where

\[
\hat{E}_i = \sum_{\alpha=1}^n \sum_{j=1}^{m_\alpha} v P_{ij}^\alpha,
\]
$n$ is the number of types of quasi-particles, and the $m_\alpha$ specify the number of quasi-particles of type \(\alpha\) and will in general be subject to certain restrictions (such as being even or odd). In addition

\[
P^\alpha_j \in \left\{ P^\alpha_{\text{min}}(m), P^\alpha_{\text{min}}(m) + \frac{2\pi}{M}, P^\alpha_{\text{min}}(m) + \frac{4\pi}{M}, \ldots, P^\alpha_{\text{max}}(m) \right\}, \quad (4.3)
\]

with the further requirement that the fermi exclusion rule (3.3) holds, namely

\[
P^\alpha_j \neq P^\alpha_k \quad \text{for} \quad j \neq k \quad \text{and all} \quad \alpha. \quad (4.4)
\]

The $P^\alpha_{\text{min}}(m)$ and $P^\alpha_{\text{max}}(m)$ depend linearly on $m = (m_1, m_2, \ldots, m_n)$, with $P^\alpha_{\text{max}}(m)$ possibly infinite.

To make the sum (4.1) more transparent, define $Q_m(N; N')$ to be the number of additive partitions of $N \geq 0$ into $m$ distinct non-negative integers each less than or equal to $N'$ (and $Q_m(N)$ to be the number of partitions of $N$ into $m$ distinct non-negative integers), and recall the identity

\[
\sum_{N=0}^{\infty} Q_m(N; N') q^N = q^{m(m-1)/2} \left[ \frac{N' + 1}{m} \right]_q, \quad (4.5)
\]

where the $q$-binomial is defined (for integers $m, n$) by

\[
\left[ \begin{array}{c} n \\ m \end{array} \right]_q = \begin{cases} \frac{(q)_n}{(q)_m(q)_{n-m}} & \text{if} \quad 0 \leq m \leq n \\ 0 & \text{otherwise}. \end{cases} \quad (4.6)
\]

Taking $N' = \infty$ in (4.5) the corresponding expression for $Q_m(N)$ is obtained, namely

\[
\sum_{N=0}^{\infty} Q_m(N) q^N = q^{m(m-1)/2} \left( \frac{1}{(q)_m} \right). \quad (4.7)
\]

Thus if $P^\alpha_{\text{min}}$ and $P^\alpha_{\text{max}}$ are parametrized in terms of the (symmetric) $n \times n$ matrix $B$ and the $n$-component vectors $A$ and $u$ as

\[
P^\alpha_{\text{min}}(m) = \frac{2\pi}{M} \left[ \frac{1}{2} (m(B - 1))_\alpha - A_\alpha + 1 \right] \quad (4.8)
\]

and

\[
P^\alpha_{\text{max}}(m) = -P^\alpha_{\text{min}}(m) + \frac{2\pi}{M} \left( \frac{u}{2} - A \right)_{\alpha}, \quad (4.9)
\]
(and we note that if some \( u_{\alpha} = \infty \) the corresponding \( P_{\text{max}}^{\alpha} = \infty \) we find from (4.1)-(4.3), using (3.12), that

\[
\hat{\chi} = S_B \left[ \frac{Q}{A} \right] (u|q) \equiv \sum_{m \in \mathbb{Z}^n \text{ restrictions } Q} q^{\frac{1}{2} m B m' - \frac{1}{2} A \cdot m} \prod_{\alpha=1}^{n} \left[ (m(1 - B) + \frac{u}{\alpha})_{\alpha} \right] q . \tag{4.10}
\]

In the special case where all \( u_{\alpha} = \infty \) (and hence (4.7) is used exclusively in place of (4.5)) we find that (4.10) reduces to

\[
S_B \left[ \frac{Q}{A} \right] (q) \equiv \sum_{m_1, \ldots, m_n = 0 \text{ restrictions } Q} q^{\frac{1}{2} m B m' - \frac{1}{2} A \cdot m} \frac{(q)_{m_1} \cdots (q)_{m_n}}{(q)_m} . \tag{4.11}
\]

As the simplest example consider the case of \( n=1 \), \( P_{\min} = \pi/M \) or 0, and \( P_{\max} = \infty \), which describes what is called a free chiral fermion (with anti-periodic or periodic boundary conditions, respectively). From (4.8) this corresponds to \( B=1 \) and \( A=0 \) or 1 (with no restrictions \( Q \)), and the corresponding characters (4.11) are

\[
\hat{\chi}_{F}^{A} = \sum_{m=0}^{\infty} \frac{q^{m^2/2}}{(q)_m} , \quad \hat{\chi}_{F}^{B} = \sum_{m=0}^{\infty} \frac{q^{m(m-1)/2}}{(q)_m} . \tag{4.12}
\]

These free chiral fermion characters are to be contrasted with the character of a single chiral boson, computed from (4.1)-(4.3) with \( n=1 \), \( P_{\min} = 2\pi/M \) and \( P_{\max} = \infty \) but without any exclusion rule on the \( P_j \), leading to

\[
\hat{\chi}^{B} = \sum_{N=0}^{\infty} P(N) q^N = \prod_{n=1}^{\infty} \frac{1}{1 - q^n} = \frac{1}{(q)_{\infty}} , \tag{4.13}
\]

where \( P(N) \) is the number of additive partitions of \( N \) into an arbitrary number of (not necessarily distinct) positive integers.

Upon comparison of (4.13) and (4.12) with (2.1)-(2.2) we see that it is natural to refer to the left-hand sums in (2.1)-(2.2) as fermionic and the right-hand sides as bosonic. To complete the generalization of (2.1)-(2.2) for the case of a single free chiral fermion, to exhibit the Bose-Fermi correspondence, and to show the relation with the conformal field theory of the Ising model which is the Virasoro minimal model \( \mathcal{M}(3,4) \), we note from (83)-(86) of [61] and from [13] [33] that

\[
\sum_{m=0}^{\infty} \frac{q^{m^2/2}}{(q)_m} = \prod_{n>0} \frac{1}{1 - q^n} = q^{1/48} \chi_{1,1}^{(3,4)} \tag{4.14}
\]
\[
\sum_{m=1}^{\infty} \frac{q^{m^2/2}}{(q)_m} = q^{1/2} \prod_{n=1,4,6,7,9,10,12,15 \mod 16} \frac{1}{1-q^n} = q^{1/48} \chi^{(3,4)}_{1,3}
\]

(4.15)

\[
\sum_{m=0}^{\infty} \frac{q^{m(m-1)/2}}{(q)_m} = \sum_{m=1,3,5,7,9,11,13,15 \mod 16} \prod_{n=1}^{\infty} \frac{1}{1-q^{2n-1}} = q^{-1/24} \chi^{(3,4)}_{1,2}
\]

(4.16)

and thus

\[
\tilde{\chi}_A^F = \sum_{m=0}^{\infty} \frac{q^{m^2/2}}{(q)_m} = \prod_{n=1}^{\infty} (1+q^{n-1/2}) = q^{1/48} (\chi^{(3,4)}_{1,1} + \chi^{(3,4)}_{1,3})
\]

(4.17)

\[
\tilde{\chi}_P^F = \sum_{m=0}^{\infty} \frac{q^{m(m-1)/2}}{(q)_m} = 2 \prod_{n=1}^{\infty} (1+q^n) = 2q^{-1/24}\chi^{(3,4)}_{1,2}
\]

where bosonic sum representations for the Virasoro characters on the right-hand sides are given in (2.14).

We begin our presentation of fermionic sum representations for characters with an example where the Bose-Fermi correspondence (at the level of character formulas) is particularly easy to prove: the \(SU(2)\) Wess-Zumino-Witten model [54] [62]. The symmetry algebra of this conformal field theory is shown in [54] to be the affine \(su(2)\) Kac-Moody algebra denoted by \((A_1^{(1)})_k\) where \(k = 1, 2, \ldots\) is the level. At level one this theory was argued [63] to describe the conformal limit of the system originally studied by Bethe [36], the spin \(\frac{1}{2}\) Heisenberg anti-ferromagnetic chain

\[
H_{XXX} = \sum_{j=1}^{M} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \sigma_j^z \sigma_{j+1}^z)
\]

(4.18)

(where \(\sigma^i\) are the Pauli spin matrices, \(M\) is even, and periodic boundary conditions \(\sigma_{M+1}^i = \sigma_1^i\) are imposed).

One form of the two characters of the \(SU(2)\) level 1 theory was given in (2.18). Comparing this form of the character with the character of the free chiral boson (4.13) we see that (2.18) is interpreted in terms of a free chiral boson with an internal quantum number \(Q\) (called charge) that adds an extra term \(\frac{2\pi vQ^2}{M}\) to the total energy (4.2). The character \(q^{1/24}\chi_l\) with \(l=0\) (1) is obtained by summing over all charge sectors with \(Q\) an integer (half-odd-integer). We call this the bosonic form of the characters. Product formulas for the characters are readily obtained due to the fact that (2.18) are two Jacobi theta functions (divided by \((q)_{\infty}\)), namely

\[
q^{1/24}\chi_l(q) = (1+l) q^{l/2} \prod_{n=1}^{\infty} (1+q^n)(1+q^{2n+l-1})^2 \quad (l = 0, 1).
\]

(4.19)
However, it was shown by Faddeev and Takhtajan \[64\] that the spectrum of the spin chain (4.18) can be constructed from two fermionic excitations (forming an SU(2) doublet), and thus a representation of the character in the form of (4.11) with \(n=2\) should be possible. Indeed, we find that the two characters (2.18) have the representation

\[
q^{1/24} \chi_l(q) = \sum_{m_1, m_2 = 0}^{\infty} q^{(m_1 + m_2)^2/2} (q)_{m_1} (q)_{m_2}.
\]

(4.20)

When compared with (4.11) this gives \(B = \frac{1}{2} \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right) \) and \(A = 0\), and thus from (4.8) we see that the minimum momenta are

\[
P_{\text{min}}^{1}(m) = \pi M (1 - m_1 - m_2), \quad P_{\text{min}}^{2}(m) = \pi M (1 - m_2 - m_1).
\]

(4.21)

To prove the equality of (2.18) and (4.20) we first recall a relation due to Cauchy (eq. (2.2.8) of \[65\]), called the \(q\)-analogue of Kummer’s theorem:

\[
\sum_{n=0}^{\infty} \frac{q^{n^2 - n} z^n}{(q)_n} \prod_{j=1}^{\infty} (1 - z q^{-1})^{-1} = \prod_{m=0}^{\infty} (1 - z q^m)^{-1}.
\]

(4.22)

setting \(z = q^{k+1}\) and dividing by \((q)_k\) we obtain

\[
\sum_{n=0}^{\infty} \frac{q^{n^2 + nk}}{(q)_n (q)_{n+k}} = \frac{1}{(q)_{\infty}} \quad (k = 0, 1, 2, \ldots).
\]

(4.23)

We then write (4.20) as

\[
q^{1/24} \chi_l(q) = \delta_{l,0} \sum_{m=0}^{\infty} \frac{q^{m^2}}{(q)_m^2} + 2 \sum_{0 \leq m_1 < m_2, m_1 - m_2 \equiv l (\mod 2)} q^{(m_1 + m_2)^2/2} (q)_{m_1} (q)_{m_2},
\]

(4.24)

and set \(m_2 = m_1 + 2n - l\) to obtain

\[
q^{1/24} \chi_l(q) = \delta_{l,0} \sum_{m=0}^{\infty} \frac{q^{m^2}}{(q)_m^2} + 2 \sum_{n=1}^{\infty} \sum_{m_1 = 0}^{\infty} q^{m_1^2 + m_1 (2n - l) + 1/2 (2n - l)^2} (q)_{m_1} (q)_{m_1 + 2n - l}.
\]

(4.25)

Then, using (4.23) to reduce the sums over \(m\) and \(m_1\) we obtain

\[
q^{1/24} \chi_l(q) = \frac{1}{(q)_{\infty}} (\delta_{l,0} + 2 \sum_{n=1}^{\infty} q^{(n-\frac{1}{2})^2}) = \frac{1}{(q)_{\infty}} \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2}.
\]

(4.26)
as desired.

In fact, the equality of (2.18) and (4.20) follows from a more general identity. For $p$ and $p'$ relatively prime positive integers, $p \geq p'$, $Q = 0, 1, \ldots, p - 1$ and $Q' \in \mathbb{Z}_{2p'}$, define

$$G^{(p,p')}_{Q,Q'}(z,q) = \sum_{m_1,m_2=0}^{\infty} \sum_{m_1-m_2 = Q'(\mod 2p')} z^{p(m_1-m_2)+Q} q^{pp'(p(m_1-m_2)+Q)^2+m_1m_2} (q)_{m_1} (q)_{m_2} .$$

Let us also define

$$f_{a,b}(z,q) = \frac{1}{(q)_\infty} \sum_{j=-\infty}^{\infty} z^{j+\frac{b}{p}} q^{a(j+\frac{b}{p})^2} ,$$

which satisfy the periodicity properties

$$f_{a,b}(z,q) = f_{a,b+2a}(z,q) = f_{a,-b}(z,q) .$$

Then exactly the same proof as given above shows that

$$G^{(p,p')}_{Q,Q'}(z,q) = f_{pp',pQ'+Q}(z,q) .$$

The equality of (2.18) and (4.20) is just the case $p = p' = z = 1$ of (4.30).

Now recall that $q^{-1/24}f_{pp',pQ'+Q}(1,q)$ form the complete set of characters of the gaussian $c=1$ model with compactification radius $r = \sqrt{\frac{p}{2p'}}$ (in the conventions of \cite{67}). This model is the conformal field theory of the XXZ spin chain with the hamiltonian \cite{68}

$$H_{XXZ} = \sum_{j=1}^{M} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \cos \mu \ \sigma_j^z \sigma_{j+1}^z)$$

with

$$r = \frac{1}{\sqrt{2(1-\frac{2}{\pi})}} \quad (0 \leq \mu < \pi).$$

The fermi single-particle energies of this model are \cite{69} (for $0 \leq \mu \leq \frac{\pi}{2}$)

$$e(P) = \frac{2\pi \sin \mu}{\mu} \sin P \quad (0 \leq P \leq \pi).$$
and so the speed of sound is \( v = \frac{2\pi \sin \mu}{\mu} \). The conformal field theory prediction for the scaled partition function (3.11) of (4.31) (with \( M \) even, \( \sigma^i_{M+1} = \sigma^i_1 \), and \( r = \sqrt{\frac{24}{4p}} \)) is

\[
\hat{Z}_{XXZ} = \frac{(qq)^{-1/24}}{(q)_\infty(q)_\infty} \sum_{m,n=-\infty}^{\infty} q^{\frac{1}{2}(\frac{nr}{24}+nr)^2} q^{\frac{1}{2}(\frac{nr}{24}-nr)^2}
\]

\[
= (qq)^{-1/24} \sum_{p=0}^{p-1} \sum_{Q' = 0}^{2p-1} \hat{f}_{pp',pQ'+p'}Q(1,q) \hat{f}_{pp',pQ'-p'}Q(1,q).
\]

It is suggestive to interpret (4.30) as expressing the Bose-Fermi correspondence of the gaussian and Thirring models, which are the massless limits of the sine-Gordon and massive Thirring field theories [47][49]. To support this interpretation of the two types of fermionic excitations in (4.27) (carrying opposite charge, which is “measured” by the “fugacity” variable \( z \)) as the fermion and anti-fermion of the Thirring model, consider the characters at the Thirring decoupling point \( r=1 \), i.e. \((p,p')=(2,1)\), which corresponds to the XX point \( \mu = \pi/2 \) of (4.31). At this point we can rewrite

\[
G_Q^{(2,1)}(1,q) = q^2/8 \sum_{m_1=0}^{\infty} \frac{q^{m_1(m_1+Q)/2}}{(q)_{m_1}} \sum_{m_2=0}^{\infty} \frac{q^{m_2(m_2-Q)/2}}{(q)_{m_2}},
\]

and using the easily obtained identity

\[
\sum_{m=0}^{\infty} \frac{q^{m(m+1)/2}}{(q)_m} = \frac{1}{2} \sum_{m=0}^{\infty} \frac{q^{m(m-1)/2}}{(q)_m},
\]

we see that the four characters \( q^{-1/24}G_Q^{(2,1)}(1,q) \ (Q,Q' = 0,1) \) are simple quadratic combinations of the Ising characters (1.14)-(1.16), namely

\[
q^{-1/24}G_{0,0}^{(2,1)}(1,q) = (\chi_{1,1}^{(3,4)})^2 + (\chi_{1,3}^{(3,4)})^2, \quad q^{-1/24}G_{0,1}^{(2,1)}(1,q) = 2\chi_{1,1}^{(3,4)}\chi_{1,3}^{(3,4)}
\]

\[
q^{-1/24}G_{1,0}^{(2,1)}(1,q) = q^{-1/24}G_{1,1}^{(2,1)}(1,q) = (\chi_{1,2}^{(3,4)})^2.
\]

For points other than \((p,p')=(2,1)\) the two fermionic quasi-particles in (4.27) do not decouple, which is our interpretation of the fact that when bringing \( G_Q^{(p,p')}Q(1,q) \) to the form (1.11) the matrix \( B \) is not diagonal. Explicitly,

\[
B = \left( \begin{array}{cc} \frac{p}{2p'} & 1 - \frac{p}{2p'} \\ 1 - \frac{p}{2p'} & \frac{p}{2p'} \end{array} \right).
\]
We note also that the appropriate linear shift in (4.11) is $A = (-Q/p', Q/p')$, and hence the momentum restrictions (4.8) in this case are

$$P_{\text{min}}^1(m) = \frac{\pi}{M} \left[ (m_2 - m_1)(1 - \frac{p}{2p'}) + \frac{Q}{p'} + 1 \right]$$

$$P_{\text{min}}^2(m) = -\frac{\pi}{M} \left[ (m_2 - m_1)(1 - \frac{p}{2p'}) + \frac{Q}{p'} - 1 \right].$$

We now are finally in a position to summarize the status of fermionic representations of conformal field theory characters. All the needed notation has been introduced above and we may now proceed in a summary fashion. The results have been originally presented in [55]-[59] [70]-[71]. In particular we follow the presentation of [59]. As specified in the original papers some of the involved $q$-series identities are proven and others are conjectures verified to high orders in $q$.

Characters of these coset conformal field theories are of the form (4.11) with $n = r$ and $B = 2C^{-1}_{G_r}$, namely twice the inverse Cartan matrix of $G_r$. Denoting $S_{B}^{Q}(q)$ of (4.11) by $S_{G_r}^{Q}(q)$ in this subsection, the results in the various cases are as follows:

$G_r = A_n$: This is the original case of Lepowsky and Primc [27]: the sums (4.11) with $B = 2C^{-1}_{A_n-1}$ are (2.22)-(2.23), which provide fermionic sum representations for all the characters of the corresponding $Z_{n+1}$-parafermionic conformal field theory [72]. We merely note here that the linear shift term $A_l \cdot m$ of (2.25) can be obtained from $A_l \cdot m = 0$ by replacing $m_l$ by $m_l + \frac{1}{2}$ in the quadratic form in (4.11).

$G_r = D_n (n \geq 3)$: The corresponding conformal field theories are the points $r = \sqrt{n}$ on the $c=1$ gaussian line. Hence, as discussed earlier in this section, the characters are $q^{-1/24} f_{n,j}(1,q)$ of (4.28), with $j = 0, 1, \ldots, n$, for which a fermionic sum representation in terms of two quasi-particles was given in (4.27). Now (4.11) with $B = 2C^{-1}_{D_n}$, in the basis

$$mC^{-1}_{D_n}m^t = \sum_{\alpha=1}^{n-2} \alpha m_{\alpha}^2 + \frac{n}{4} (m_{n-1}^2 + m_n^2) + 2 \sum_{1 \leq \alpha < \beta \leq n-2} m_{\alpha} m_{\beta}
+ \sum_{\alpha=1}^{n-2} \alpha m_{\alpha} (m_{n-1} + m_n) + \frac{n-2}{2} m_{n-1} m_n,$$

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$G_r = D_n (n \geq 3)$: The corresponding conformal field theories are the points $r = \sqrt{n}$ on the $c=1$ gaussian line. Hence, as discussed earlier in this section, the characters are $q^{-1/24} f_{n,j}(1,q)$ of (4.28), with $j = 0, 1, \ldots, n$, for which a fermionic sum representation in terms of two quasi-particles was given in (4.27). Now (4.11) with $B = 2C^{-1}_{D_n}$, in the basis

$$mC^{-1}_{D_n}m^t = \sum_{\alpha=1}^{n-2} \alpha m_{\alpha}^2 + \frac{n}{4} (m_{n-1}^2 + m_n^2) + 2 \sum_{1 \leq \alpha < \beta \leq n-2} m_{\alpha} m_{\beta}
+ \sum_{\alpha=1}^{n-2} \alpha m_{\alpha} (m_{n-1} + m_n) + \frac{n-2}{2} m_{n-1} m_n,$$
provides a representation for the same characters in terms of \( n \) quasi-particles (the degenerate \( n=2 \) case coinciding with (4.27) with \((p,p')=(2,1)\), namely (4.35)). In particular,

\[
S_{D_n}^Q(p,q) = f_{n,n}(1,q) \tag{4.27}
\]

with \( Q = 0,1 \) indicating restriction of the summation in (4.11) to \( m_{n-1} + m_n \equiv Q \) (mod 2).

Note that due to the coincidence \( D_3 = A_3 \) the expressions (2.22) and (4.41) are related when \( n = 3 \) by (cf. [21][55])

\[
S_0^0 D_3 = S_0^0 A_3 + S_2^0 A_3 \quad \text{and} \quad S_1^0 D_3 = 2 S_1^0 A_3.
\]

\( G_r = E_6 \): The conformal field theory is the Virasoro minimal model [12] \( \mathcal{M}(6,7) \) of central charge \( c = \frac{6}{7} \) with the D-series [44] partition function. With a suitable labeling of roots we have

\[
C_{E_6}^{-1} = \begin{pmatrix}
  4/3 & 2/3 & 1 & 4/3 & 5/3 & 2 \\
  2/3 & 4/3 & 1 & 5/3 & 4/3 & 2 \\
  1 & 1 & 2 & 2 & 3 & 2 \\
  4/3 & 5/3 & 2 & 10/3 & 8/3 & 4 \\
  5/3 & 4/3 & 2 & 8/3 & 10/3 & 4 \\
  2 & 2 & 3 & 4 & 4 & 6
\end{pmatrix}
\]

and we find (cf. (2.14))

\[
S_0^0 E_6(q) = q^{c/24} \left[ \chi_{1,1}^{(6,7)}(q) + \chi_{5,1}^{(6,7)}(q) \right], \quad S_1^1 E_6(q) = q^{c/24} \chi_{3,1}^{(6,7)}(q),
\]

with the restrictions \( m_1 - m_2 + m_4 - m_5 \equiv Q \) (mod 3).

\( G_r = E_7 \): The conformal field theory is \( \mathcal{M}(4,5) \) of central charge \( c = \frac{7}{10} \). Now

\[
C_{E_7}^{-1} = \begin{pmatrix}
  3/2 & 1 & 3/2 & 2 & 2 & 5/2 & 3 \\
  1 & 2 & 2 & 3 & 3 & 4 & 3 \\
  3/2 & 2 & 7/2 & 3 & 4 & 9/2 & 6 \\
  2 & 2 & 3 & 4 & 4 & 5 & 6 \\
  2 & 3 & 4 & 4 & 6 & 6 & 8 \\
  5/2 & 3 & 9/2 & 5 & 6 & 15/2 & 9 \\
  3 & 4 & 6 & 6 & 8 & 9 & 12
\end{pmatrix}
\]

and we find

\[
S_0^0 E_7(q) = q^{c/24} \chi_{1,1}^{(4,5)}(q), \quad S_1^1 E_7(q) = q^{c/24} \chi_{3,1}^{(4,5)}(q),
\]

when the restrictions are \( m_1 + m_3 + m_6 \equiv Q \) (mod 2).
**Gr = E₈:** The coset in this case is equivalent to the Ising conformal field theory $\mathcal{M}(3, 4)$ of central charge $c = \frac{1}{2}$. Here

$$C_{E_8}^{-1} = \begin{pmatrix}
2 & 2 & 3 & 3 & 4 & 4 & 5 & 6 \\
2 & 4 & 4 & 5 & 6 & 7 & 8 & 10 \\
3 & 4 & 6 & 6 & 8 & 8 & 10 & 12 \\
3 & 5 & 6 & 8 & 9 & 10 & 12 & 15 \\
4 & 6 & 8 & 9 & 12 & 12 & 15 & 18 \\
4 & 7 & 8 & 10 & 12 & 14 & 16 & 20 \\
5 & 8 & 10 & 12 & 15 & 16 & 20 & 24 \\
6 & 10 & 12 & 15 & 18 & 20 & 24 & 30
\end{pmatrix}$$

and, without any restrictions in the sum (4.11),

$$S_{E_8}(q) = q^{c/24} \chi^{(3,4)}_{1,1}(q).$$

We further note that if $m_1$ in the quadratic form in (4.11) is replaced by $m_1 - \frac{1}{2}$, then one obtains (up to a power of $q$) $\widehat{\chi}^{(3,4)}_{1,1} + \widehat{\chi}^{(3,4)}_{1,2}$, and similarly replacing $m_2$ by $m_2 - \frac{1}{2}$ the combination $\widehat{\chi}^{(3,3)}_{1,1} + \widehat{\chi}^{(3,4)}_{1,2} + \widehat{\chi}^{(3,4)}_{1,3}$ is obtained.

This case has been considered in [70] and [71] where the identity characters in the corresponding generalized parafermion conformal field theory [73] are given by (4.11) (with suitable restrictions on the summation variables) by taking $B = C_{G_r} \otimes C_{A_n}^{-1}$, which is explicitly written in a double index notation as

$$B_{ab}^{\alpha\beta} = (C_{G_r})_{\alpha\beta}(C_{A_n}^{-1})_{ab} \quad \alpha, \beta = 1, \ldots, r, \quad a, b = 1, \ldots, n.$$  

When $G_r = A_1$, this reduces to the result (2.22) of [27].

**Unitary minimal models $\mathcal{M}(p, p+1)$:** For this and subsequent cases we must use the more general form of eq. (4.10). For $\mathcal{M}(p, p+1)$

$$B = \frac{1}{2} C_{A_{p-2}}, \quad u_1 = \infty,$$

and the $Q$-restriction is taken to be $m_a \equiv Q_a$ (mod 2). Defining

$$Q_{r,s} = (s - 1) \rho + (e_{r-1} + e_{r-3} + \ldots) + (e_{p+1-s} + e_{p+3-s} + \ldots)$$

we have

$$Q_{r,s} = (s - 1) \rho + e_{r-1} + e_{r-3} + \ldots + e_{p+1-s} + e_{p+3-s} + \ldots$$
where \( \rho = \sum_{a=1}^{p-2} e_a \) and \( (e_a)_b = \delta_{ab} \) for \( a = 1, \ldots, p - 2 \) and 0 otherwise, the conjecture for the Virasoro characters, whose bosonic sum representations are given in (2.14), is

\[
\hat{\chi}_{r,s}^{(p,p+1)}(q) = q^{-\frac{1}{2}(s-r)(s-r-1)} S_B \left[ \frac{Q_{r,s}}{e_{p-s}} \right] (e_r + e_{p-s}|q).
\]

Due to (2.21) another representation must also exist, namely

\[
\hat{\chi}_{r,s}^{(p,p+1)}(q) = q^{-\frac{1}{2}(s-r)(s-r-1)} S_B \left[ \frac{Q_{p-r,p+1-s}}{e_{s-1}} \right] (e_{p-r} + e_{s-1}|q).
\]

In this case \( B = C_{G_r}^{-1} \otimes C_{A_{k+l-1}} \), and the infinite entries of the vector \( u \) are \( u_\alpha^0 \) for all \( \alpha = 1, \ldots, r \), in the double index notation used in subsect. 4.2.

As an example with both \( k \) and \( l \) greater than 1, consider the case \( G=A_1 \) with \( l=2 \), the resulting series of theories labeled by \( k \) being the unitary \( N=1 \) superconformal series whose characters are given in a bosonic form in [18]. We find that the character corresponding to the identity superfield in these models is obtained by summing over \( m_1 \in \mathbb{Z}, m_a \in 2\mathbb{Z} \) for \( a = 2, \ldots, k+1 \).

Another example is the coset \( \frac{(E_8^{(1)})_k \times (E_8^{(1)})_{k+l}}{(G_1^{(1)})_{k+l}} \) of central charge \( c = \frac{21}{22} \), which is identified as the minimal model \( \mathcal{M}(11,12) \) (with the partition function of the \( E_6 \)-type). The corresponding sum (4.10), with \( A=0, u_1^\infty = 0 \) and \( u_a^0 = 0 \) for \( a = 2, \ldots, p-1 \), and the \( m_a \) are summed over all even non-negative integers.

The character \( \hat{\chi}_{(p-1)/2,(p+1)/2}^{(p,p+2)} \) (see (2.14)) corresponding to the lowest conformal dimension \( \Delta_{(p-1)/2,(p+1)/2}^{(p,p+2)} = -\frac{3}{4p(p+2)} \) in this model is given by (4.10) with \( B = \frac{1}{2} C_{(p-1)/2} \) (where \( C_n \) is defined at the end of sect. 2), \( A=0, u_1=\infty \) and \( u_a=0 \) for \( a = 2, \ldots, \frac{p-1}{2} \), and the \( m_a \) are summed over all even non-negative integers.
Minimal models $\mathcal{M}(p, kp + 1)$.

For $k=1$ these models are the ones considered in sect. 4.3, while for $p=2$ they were discussed in sect. 2. Here we consider the general case. The character $\hat{\chi}_{1,k}^{(p, kp+1)}$ corresponding to the lowest conformal dimension in the model is obtained from (4.10) with $B = (k + p - 3) \times (k + p - 3)$ matrix whose nonzero elements are given by $B_{ab} = 2(C'_{k-1})_{ab}$ and $B_{ka} = B_{ak} = a$ for $a, b = 1, 2, \ldots, k - 1$, and $B_{ab} = \frac{1}{2}[(C_{A_{p-2}})_{ab} + (k - 1)\delta_{ak}\delta_{bk}]$ for $a, b = k, k + 1, \ldots, k + p - 3$. Summation is restricted to even non-negative integers for $m_{k-1}, m_{k+p-3}$, the other $m_1, \ldots, m_{k-1}$ running over all non-negative integers, and $u_a = \infty$ for $a = 1, \ldots, k$ and 0 otherwise.

The case $p=3$ is special in that the fermionic sums are of the form (4.11) for any $k$. A slight modification of the matrix $B$ appropriate for $\mathcal{M}(3, 3k + 1)$, namely just setting $B_{kk} = \frac{k}{2}$ while leaving all other elements unchanged, gives the character $\hat{\chi}_{1,k}^{(3, 3k+2)}$ of $\mathcal{M}(3, 3k + 2)$.

Unitary $N=2$ superconformal series.

Expressions in a bosonic form for the characters of these models, of central charge $c = \frac{3k}{k+2}$, where $k$ is a positive integer, can be found in [73]. The identity character, given by $\chi_0^{(0)}(q) + \chi_0^{(2)}(q)$ in the notation of [73], can be obtained from (4.10) if one takes $B = \frac{1}{2}C_{Dk+2}$, $u_k = \infty$ (in the basis used in (4.40)) and all other $u_a$ set to zero, and $m_{k+1}, m_{k+2}$ running over all non-negative integers while all other $m_a$ summed only over the even non-negative integers.

$\mathbb{Z}_N$ parafermions.

The characters of these models are the branching functions $b^l_m$ given by (2.19), or by the fermionic representation (2.22). In sect. 4.3 we found another fermionic representation for the case $N=3$ which coincides with the minimal model $\mathcal{M}(5, 6)$ with the $D$-series partition function [13][14]. (The $b^l_m$ in this case are linear combinations of the $\chi_{r,s}^{(5,6)}$ of (2.14).) This latter form can be generalized to arbitrary $N$. For instance, $b^0_0$ is obtained from (4.11) by setting $B = \frac{1}{2}C_{D_N}$, $u_N = \infty$ (in the basis used in (4.40)) and all other $u_a$ set to zero, and $m_{N-1}, m_N$ running over all non-negative integers such that $m_{N-1} + m_N$ is even, while all other $m_a$ are restricted to be even.
We have now completed presenting the known results and conjectures for the fermionic sum representations of conformal field theory characters. From these results a number of questions and speculations arise.

First of all, it is clear that there are many cases where as yet we do not have any conjectures for the fermionic sums. The most obvious is the general Virasoro minimal model $\mathcal{M}(p,p')$. Furthermore, for many of the cases of section 4 not all the characters as yet have conjectured forms. Not to mention the fact that proofs of the various conjectures remain to be given.

It would be most useful, however, to turn the program around and to find the fermionic sum forms directly. For example, it would be highly desirable to determine mathematically which matrices $B$ in (4.10) lead to sums which form a representation of the modular group (2.6). It should be possible to answer this without reference to either the bosonic sum or the product representation. A related question is concerned with the analysis of the leading $q \to 1$ behavior (3.14) of the characters, which can be obtained \cite{57} \cite{58} \cite{70} \cite{71} \cite{74} \cite{75} from their fermionic sum representations. This analysis gives $\tilde{c}$ of (3.14) as a sum of the Rogers dilogarithm function \cite{76} evaluated at points determined by $B$, and the dilogarithm sum rules necessary to reproduce (3.16) are related \cite{75} to deep questions in different areas of mathematics.

We also want to call the readers attention to the fact repeatedly seen above that there may be several “different” fermionic representations for the same character. Such a statement is vague because the concept of “different” still remains to be defined. Nevertheless, as a suggestive specific example we consider the Ising model characters (4.14)-(4.16). These three characters were seen in sect. 4.1 to have a representation in terms of one quasi-particle if the algebra $A_1$ is used, and a representation in terms of eight quasi-particles if $E_8$ is used. The representation in terms of one quasi-particle is essentially the conformal limit of Kaufman’s representation \cite{16} of the general Ising model in zero magnetic field in terms of a single free fermion. Similarly, the representation in terms of eight fermionic quasi-particles is related to Zamolodchikov’s \cite{77} treatment of the Ising model at $T = T_c$ in a non-zero magnetic field. It is thus most suggestive to think that the various “different” quasi-particle representations have more than just a mathematical significance and give insight into the various integrable massive extensions of conformal field theories. One further piece of insight is that the single fermionic quasi-particle of the zero-field Ising model has a
direct interpretation in the nonlinear differential equations that determine the correlation functions [1], giving rise to a form-factor expansion of the correlation functions. This work has been recently extended in the context of $N=2$ supersymmetric theories [78], and it would seem that further extensions are possible.

We further mention the remarkable fact that the character formulas (2.14) and (2.19) occur not only in the study of the spectra of massless systems but also arise in the computation of the order parameters in off-critical RSOS models by means of corner transfer matrix techniques [79]-[81].

However, some of these remarks can be considered as technical in nature and speak to the mathematical side of the synthesis, but not to the synthesis itself. Since our ultimate focus is on the synthesis and not on computation, we wish to conclude with a few remarks of a more general nature.

A physicist, as opposed to a mathematician, has an almost inborn instinct to interpret results in some “physical” terms. Inevitably this process of interpretation involves the setting up of some categories (in the non-mathematical sense), and even as early as Aristotle it was realized that the names given to these categories are not mere labels but carry a great deal of philosophic content. This applies also to any attempt to “interpret” the physical “meaning” of the quasi-particle momentum exclusion rules and the fermionic sum representations presented in this paper. For example, Haldane [82] has attempted to interpret momentum exclusion rules similar in spirit to (4.21) in terms of spinons [83], “nonorthogonality of localized particle states”, and topological excitations. These words are not particularly precise and by their introduction focus attention on a particular aspect of the problem. But their introduction is necessitated by the indisputable fact that the excitations which obey exclusion rules discussed above cannot be described in the language of conventional second quantization.

In the physical context of the fractional quantum Hall effect Haldane credits Laughlin [84] with the realization that second quantization fails to appropriately describe the observed phenomena. However, need to invent new concepts is much more widespread than this particular area of condensed matter physics. In particle physics it has been recognized for at least 30 years that the conventional second-quantized quantum field theory which describes point-like particles has severe shortcomings. For example, the discovery of confinement in Quantum Chromo-Dynamics demonstrates the need to go beyond this concept. The original motivation for the construction of string theory was to understand the
strong interactions in what later became known as the confining phase of QCD (see [83] for a recent discussion). This need to go beyond second-quantized point field theory has been extensively investigated not only at the level of hadrons, but at the more fundamental level of unifying string interactions with quantized gravity. It is a most remarkable coincidence that the mathematics considered in this paper also occurs in these studies of string theory. Such a coincidence cannot be accidental and the fact that mathematicians, high-energy physicists, condensed matter physicists, and physicists studying statistical mechanics are all contemplating the same abstract object is a truly remarkable demonstration that the whole is much more than the sum of its parts. The synthesis will be achieved when language can be developed that incorporates all aspects of the phenomena at the same time.

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