A variational characterization of the optimal exit rate for controlled diffusions

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Abstract. The main result in this paper is a variational formula for the exit rate from a bounded domain for a diffusion process in terms of the stationary law of the diffusion constrained to remain in this domain forever. Related results on the geometric ergodicity of the controlled $Q$-process are also presented.

1. Introduction

A variational formulation for the principal eigenvalue of certain elliptic operators associated with diffusion processes was given in the celebrated article of Donsker and Varadhan [17]. A discrete counterpart for Markov chains appears in [16]. Subsequently, the present authors and their collaborators extended it to controlled Markov chains [1] and diffusions [3, 4, 9]; see also [5] for an overview. These articles also point out that these results are abstract counterparts of the Collatz–Wielandt formula for the Perron–Frobenius eigenvalue of irreducible non-negative matrices [14, 28].

The present article makes a connection between the study of principal eigenvalues and a different strand of research. This concerns quasi-stationarity which aims to study the ‘near-stationary’ behavior of a Markov process when it spends a long time in a sub-domain of its state space before exiting from it [15, 26]. An important notion in this context is the law of the process conditioned on never exiting a given domain. This is the so called $Q$-process, which in case of a diffusion, amounts to adding a drift that explodes at the boundary of the domain so as to confine the process to the domain in a precise manner. The $Q$-process has been studied for various classes of Markov processes, a landmark paper for the special case of diffusions being [27]. Some important recent contributions to the general case are [12, 13, 25]. As a part of our development, we re-prove some of their results for diffusions using stochastic calculus based arguments. Our main result is a new variational formula for the optimal eigenvalue of the aforementioned controlled eigenvalue problem in terms of the associated $Q$-process.

The principal eigenvalue is also the exit rate from the domain, that is, the asymptotic exponential decay rate of the tail of the distribution the first exit time $\tau$ from the domain, given by $\lim_{t \to \infty} \frac{1}{t} \log P(\tau > t)$. Exit time statistics and related themes have been of independent interest, particularly in the small noise limit, and have been extensively studied, e.g., in [10, 18–22]. Our focus, however, is different from that of these works.

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The article is organized as follows. The next subsection introduces the key notation. Section 2 introduces the controlled eigenvalue problem associated with a diffusion in a bounded domain. Section 3 introduces the $Q$-process whose law agrees with the law of the original diffusion conditioned on never exiting the prescribed domain. It also develops some key properties of this process such as ergodicity and the associated invariant probability measure. Section 4 states and proves our main result, a new representation theorem for the optimal eigenvalue in terms of the $Q$-process.

1.1. Notation. We denote by $\tau(A)$ the first exit time of the process $\{X_t\}_{t \geq 0}$ from the set $A \subset \mathbb{R}^d$, defined by

$$\tau(A) := \inf \{ t > 0 : X_t \notin A \} .$$

The complement, closure, and boundary of a set $A \subset \mathbb{R}^d$ are denoted by $A^c$, $\bar{A}$ and $\partial A$, respectively, and $1_A$ denotes its indicator function. The inner product of two vectors $x$ and $y$ in $\mathbb{R}^d$ is denoted as $\langle x, y \rangle$, $| \cdot |$ denotes the Euclidean norm, $x^T$ stands for the transpose of $x$, and $\operatorname{Tr} S$ denotes the trace of a square matrix $S$.

The term domain in $\mathbb{R}^d$ refers to a nonempty, connected open subset of the Euclidean space $\mathbb{R}^d$. For a domain $D \subset \mathbb{R}^d$, the space $C^k(D)$ ($C^k_b(D)$), $k \geq 0$, refers to the class of all real-valued functions on $D$ whose partial derivatives up to order $k$ exist and are continuous (and bounded), $C^k_c(D)$ denotes its subset consisting of functions that have compact support, and $C^k_0(D)$ the closure of $C^k_c(D)$. Also $C^k(D)$ denotes the subspace of $C^k(D)$ consisting of those functions that are positive on $D$. The space $L^p(D)$, $p \in [1, \infty)$, stands for the Banach space of (equivalence classes of) measurable functions $f$ satisfying $\int_D |f(x)|^p \, dx < \infty$, and $L^\infty(D)$ is the Banach space of functions that are essentially bounded in $D$. The standard Sobolev space of functions on $D$ whose generalized derivatives up to order $k$ are in $L^p(D)$, equipped with its natural norm, is denoted by $W^{k,p}(D)$, $k \geq 0$, $p \geq 1$. In general, if $\mathcal{X}$ is a space of real-valued functions on $D$, the space $\mathcal{X}_{\text{loc}}$ consists of all functions $f$ such that $f \varphi \in \mathcal{X}$ for every $\varphi \in C_c(\mathcal{X})$. This defines $W^{k,p}_{\text{loc}}(D)$.

We say that a continuous function $f : D \to \mathbb{R}$ is inf-compact if the sublevel set $\{ x \in D : f(x) \leq C \}$ is compact (or empty) for any $C \in \mathbb{R}$.

2. The controlled exit rate problem

We begin by recalling the controlled eigenvalue problem from [11]. We consider a controlled diffusion given by the $d$-dimensional Itô stochastic differential equation (SDE)

$$X_t = X_0 + \int_0^t m(X_s, U_s) \, ds + \int_0^t \sigma(X_s) \, dW_s , \quad t \geq 0 ,$$

where:

1. $m : \mathbb{R}^d \times \mathbb{U} \mapsto \mathbb{R}^d$ for a prescribed compact metric ‘control’ space $\mathbb{U}$, is continuous and locally Lipschitz in its first argument uniformly with respect to the second, and satisfies, for some constant $C$,

$$\langle m(x, u), x \rangle \leq C |x|^2 \quad \forall x, u \in \mathbb{R}^d \times \mathbb{U} ;$$

2. $\sigma : \mathbb{R}^d \mapsto \mathbb{R}^d$ is Lipschitz and satisfies:

$$|\sigma^T(x)y|^2 \geq c_0 |y|^2 \quad \forall x, y \in \mathbb{R}^d ,$$

for some $c_0 > 0$;

3. $X_0$ is prescribed in law with bounded moments;

4. $\{W_t\}_{t \geq 0}$ is a standard Brownian motion in $\mathbb{R}^d$ independent of $X_0$;
(5) \( \{U_t\}_{t \geq 0} \) is a \( U \)-valued process with measurable paths, satisfying the ‘non-anticipativity condition’: for all \( t > s \geq 0 \), \( W_t - W_s \) is independent of \( \mathcal{F}_s \), which is defined as the right-continuous completion of \( \sigma(X_r, U_r, r \leq s) \). We call such \( \{U_t\}_{t \geq 0} \) admissible, and let \( \mathcal{U} \) denote that class of these controls.

We shall use the relaxed control formulation, that is, \( U = \mathcal{P}(\mathcal{U}_0) \) where \( \mathcal{U}_0 \) is compact metric and \( \mathcal{P}(\mathcal{U}) \) is the compact Polish space of probability measures on \( \mathcal{U} \) with the Prokhorov topology, and furthermore, \( m \) is of the form

\[
m(x,u) = \int_{\mathcal{U}_0} m_0(x,y)u(dy)
\]

for some \( m_0 : \mathbb{R}^d \times \mathcal{U}_0 \to \mathbb{R}^d \) which is continuous and Lipschitz in its first argument uniformly with respect to the second. We also define the special control class of stationary controls wherein \( U_t = v(X_t) \) for some measurable \( v : \mathbb{R}^d \to \mathcal{U} \), identified with the map \( v \) by standard abuse of terminology and termed stationary control policy. We let \( \mathcal{U}_{\text{sm}} \) denote the class of these policies. Under any \( v \in \mathcal{U}_{\text{sm}} \), the SDE in (2.1) has a unique strong solution, and this is a strong Markov process [23]. We let \( \mathbb{P}_x^v \) and \( \mathbb{E}_x^v \) denote the probability measure and the expectation operator, respectively, on the canonical space of the process \( \{X_t\}_{t \geq 0} \) controlled by \( v \in \mathcal{U}_{\text{sm}} \) and with initial condition \( X_0 = x \). We extend this definition to the class of admissible controls, and use \( \mathbb{P}_x^U \) and \( \mathbb{E}_x^U \) for \( U \in \mathcal{U}_{\text{sm}} \). For \( v \in \mathcal{U}_{\text{sm}} \), we also use the simplified notation

\[
m_v(x) := m(x,v(x)).
\]

We introduce the following notation for the controlled extended generator of \( \{X_t\}_{t \geq 0} \):

\[
\mathcal{L} f(x,u) := \frac{1}{2} \text{Tr} \left( a(x) \nabla^2 f(x) \right) + \langle m(x,u), \nabla f(x) \rangle \quad \text{for } f \in C^2(\mathbb{R}^d),
\]

with

\[
a(x) := \sigma(x)\sigma^T(x), \quad x \in \mathbb{R}^d.
\]

Under a stationary policy \( v \) as above, we denote \( \mathcal{L} f(x,v(x)) \) as \( \mathcal{L}_v f(x) \).

Let \( D \) be a bounded domain with \( C^{2,1} \) boundary, which is kept fixed throughout the paper, and \( \tau \equiv \tau(D) \), the first exit time from \( D \) (see (1.1)). In this work, we consider the problem of minimizing the rate of exit from \( D \) over all admissible controls, that is

\[
\beta^*(x) := \inf_{U \in \mathcal{U}_{\text{sm}}} \left( -\limsup_{T \to \infty} \frac{1}{T} \log \mathbb{P}_x^U(\tau > T) \right). \tag{2.2}
\]

It turns out that \( \beta^* \) is independent of \( x \). Note that

\[
\mathbb{P}_x^U(\tau > T) = \mathbb{P}_x^U \left( X_t \in D, \ t \in [0,T] \right).
\]

Consider the process under a given control \( v \in \mathcal{U}_{\text{sm}} \). Then it is well known [11] that the rate of exit

\[
\beta_v := -\limsup_{T \to \infty} \frac{1}{T} \log \mathbb{P}_x^v(\tau > T) \tag{2.3}
\]

is equal to the principal eigenvalue of the operator \( \mathcal{L}_v \) on \( D \), which is defined as

\[
\lambda_v := \inf \{ \lambda \in \mathbb{R} : \exists \phi \in W^{2,d}_{\text{loc}}(D), \phi > 0 \text{ in } D, \mathcal{L}_v \phi + \lambda \phi \leq 0 \text{ in } D \}. \tag{2.4}
\]

It is also well known that \( \lambda_v \) is a simple eigenvalue, and its eigenvector, or eigenfunction, is the unique solution \( \Psi_v \) in \( W^{2,p}_{\text{loc}}(D) \cap C(D) \), for any \( p \geq d \), to the Dirichlet problem

\[
\mathcal{L}_v \Psi_v(x) + \lambda_v \Psi_v(x) = 0 \quad \text{a.e. } x \in D, \quad \Psi_v > 0 \quad \text{in } D, \quad \Psi_v = 0 \quad \text{on } \partial D. \tag{2.5}
\]
Uniqueness of the eigenfunction is, of course, up to scalar multiplication with a positive constant, and keeping that in mind, we use the term unique without any reference to a specific normalization. For such Dirichlet eigenvalue problems we refer to \((\lambda_v, \Psi_v)\) as an eigenpair.

Going back to the optimal rate \(\beta^*\) in (2.2), we define the semilinear operator \(\mathcal{G}^*\) by

\[
\mathcal{G}^* f(x) := \sup_{u \in \mathcal{U}} \mathcal{L} f(x,u),
\]

and denote its principal eigenvalue in \(D\) as \(\lambda^*\), which is defined exactly as in (2.4) by replacing \(\mathcal{L}_v\) with \(\mathcal{G}^*\). Then, as shown in [11], \(\beta^* = \lambda^* = \inf_{v \in \mathcal{U}_{sm}} \lambda_v\), and again, \(\lambda^*\) is a simple eigenvalue, and its eigenfunction \(\Psi^* \in C^2(D)\) is the unique solution to the Dirichlet problem

\[
\max_{u \in \mathcal{U}} \mathcal{L} \Psi^*(x,u) + \lambda^* \Psi^*(x) = 0 \text{ a.e. in } D, \\
\Psi^* > 0 \text{ in } D, \quad \Psi^* = 0 \text{ on } \partial D.
\]

We note here that the Lemma and Theorem in [11] actually state that \(\lambda^* = \inf_{v \in \mathcal{U}_{sm}} \lambda_v\), rather than \(\lambda^* = \beta^*\) which was claimed above. However, a close inspection shows that using the same argument in the first part of their proof we obtain the upper bound \(\lambda^* \leq \inf_{U \in \mathcal{U}} \beta_U\), with \(\beta_U\) defined as in (2.3) by replacing \(v\) with \(U \in \mathcal{U}\). So indeed, \(\lambda^* = \beta^*\) by the results in [11]. Let \(\mathcal{U}_{sm}^*\) denote the set of a.e. selectors from the minimizer in (2.7). Then, as shown in the Theorem in [11], \(v \in \mathcal{U}_{sm}^*\) if and only if \(\beta_v = \beta^*\).

3. The \(Q\)-process

Let \(v\) be a generic element of \(\mathcal{U}_{sm}\). Define \(\psi_v := \log \Psi_v\), with \(\Psi_v\) the eigenfunction in (2.5). The \(Q\)-process \(\{\tilde{X}_t\}_{t \geq 0}\) associated with the process \(\{X_t\}_{t \geq 0}\) in (2.1) on the domain \(D\) is given by the Itô stochastic differential equation

\[
d\tilde{X}_t = \tilde{m}_v(\tilde{X}_t) dt + \sigma(\tilde{X}_t) d\tilde{W}_t,
\]

with

\[
\tilde{m}_v(x) := m_v(x) + a(x) \nabla \psi_v(x).
\]

Note that its extended generator \(\tilde{\mathcal{L}}_v\) satisfies

\[
\tilde{\mathcal{L}}_v f := \frac{1}{2} \text{Tr}(a \nabla^2 f) + \langle \tilde{m}_v, \nabla f \rangle = \mathcal{L}_v f + \langle a \nabla \psi_v, \nabla f \rangle.
\]

By (2.5) and (2.7), we also have the identities

\[
\tilde{\mathcal{L}}_v \psi_v(x) - \frac{1}{2} \left| \sigma^T(x) \nabla \psi_v(x) \right|^2 = -\lambda_v, \quad x \in D,
\]

and

\[
\max_{u \in \mathcal{U}} \left( \mathcal{L} \Psi^*(x,u) + \frac{1}{2} \left| \sigma^T(x) \nabla \Psi^*(x) \right|^2 \right) = -\lambda^*, \quad x \in D,
\]

with \(\psi^* := \log \Psi^*\).

Remark 3.1. Since we often refer to the results in [2-4, 6, 7], we want to caution the reader that in all these papers, the principal eigenvalues are defined with the opposite sign. In particular, the converse of the monotonicity properties asserted in the above papers should be applied here, and convexity should be replaced by concavity.

Remark 3.2. When applying Itô’s formula in equations such as (3.3), in order to localize the martingales, we use the stopping time \(\tau\), which is defined as the first exit time from the set

\[
D_\epsilon := \{x \in D : \text{dist}(x, \partial D) > \epsilon\},
\]

where ‘dist’ denotes the Euclidean distance.
3.1. Ergodicity of the $Q$-process. It is well known that the $Q$-process is confined in $D$ and is geometrically ergodic \cite{12, 27}. This is in fact true for a more general class of models. In the theorem which follows we give an independent proof of this fact for diffusions using PDE theory, and stochastic calculus. In the process, we present some important techniques concerning the eigenvalue problem. An important inequality which we use, and which can be easily verified, is the following:

$$\text{if } \mathcal{L}_v \Phi + F \Phi \leq 0 \text{ in } D, \text{ then } \tilde{\mathcal{L}}_v \left( \frac{\Phi}{\psi} \right) \leq (\lambda_v - F) \frac{\Phi}{\psi} \text{ in } D.$$  \hspace{1cm} (3.5)

We also use the notation $A \in D$ to indicate that $\bar{A} \subset D$.

**Theorem 3.1.** For any $v \in \mathcal{U}_{\text{sm}}$, the $Q$-process in (3.1) is confined in $D$ and is geometrically ergodic. In particular, there exists $\mathcal{V}_v \in W^{2,p}_{\text{loc}}(D)$, $p > d$, a compact set $K \subset D$, and positive constants $C_v$ and $\rho_v$, such that

$$\tilde{\mathcal{L}}_v \mathcal{V}_v (x) \leq C_v \mathbb{1}_K (x) - \rho_v \mathcal{V}_v (x) \quad \forall x \in D,$$  \hspace{1cm} (3.6)

and $\mathcal{V}_v \psi_v$ is bounded from below away from 0 in $D$. As a result, $e^{-\psi_v}$ is integrable under the invariant probability measure $\bar{\mu}_v$ of the $Q$-process. Moreover,

$$\lambda_v = \frac{1}{2} \int_D \left| \sigma^T (x) \nabla \psi_v (x) \right|^2 \bar{\mu}_v (dx).$$  \hspace{1cm} (3.7)

**Proof.** In the interest of economy of notation, we drop the explicit dependence on $v$ from $\lambda_v$, $\mathcal{L}_v$, and $\Psi_v$. Let $B \in D$ be a nonempty open set. Then the principal eigenvalue of the operator $\mathcal{L} - \mathbb{1}_B$ on $D$, denoted as $\lambda_D (\mathcal{L} - \mathbb{1}_B)$, satisfies $\lambda_D (\mathcal{L} - \mathbb{1}_B) > \lambda$ \cite[Lemma 2.1 (b)]{7}. Hence, by the monotonicity and continuity of the principal eigenvalue with respect to the domain \cite[Lemma 2.2 (a)]{6} (see also the more general result in \cite[Corollary 2.1]{2}), there exists a bounded $C^{2,1}$ domain $D' \supseteq D$, such that

$$\lambda_D (\mathcal{L} - \mathbb{1}_B) > \lambda_D' (\mathcal{L} - \mathbb{1}_B) > \lambda.$$  

Note that Remark 3.1 applies to these assertions. Let $\Phi$ denote the principal eigenfunction of $\mathcal{L} - \mathbb{1}_B$ on $D'$. Then we have

$$\mathcal{L} \Phi + (\lambda_D' (\mathcal{L} - \mathbb{1}_B) - \mathbb{1}_B) \Phi = 0 \text{ in } D'.$$

Therefore, by (3.5), we obtain

$$\tilde{\mathcal{L}} \left( \frac{\Phi}{\Psi} \right) \leq \left( \mathbb{1}_B + \lambda - \lambda_D' (\mathcal{L} - \mathbb{1}_B) \right) \frac{\Phi}{\Psi}. $$  \hspace{1cm} (3.8)

Since $\lambda_D (\mathcal{L} - \mathbb{1}_B) > \lambda$, and $\inf_D \Phi > 0$ by the continuity and positivity of $\Phi$ on $D'$, the Foster–Lyapunov inequality in (3.6) is implied by (3.8) with $V := \frac{\Phi}{\Psi}$.

It is fairly standard to show that (3.6) implies that the $Q$-process is confined to $D$. Indeed, using the Itô formula, and with $\tau_\epsilon$ and $D_\epsilon$ as in Remark 3.2, we obtain

$$\tilde{\mathbb{E}}_x^\epsilon \left[ \mathcal{V}_v (\tilde{X}_{t \wedge \tau_\epsilon}) \right] \leq \mathcal{V}_v (x) + C_v t,$$

which implies by the Markov inequality that

$$\tilde{\mathbb{P}}_x (\tau_\epsilon < t) \leq \left( \inf_{\partial D_\epsilon} \mathcal{V}_v \right)^{-1} \left( \mathcal{V}_v (x) + C_v t \right).$$

Thus, since $\mathcal{V}_v$ is inf-compact, we have $\tilde{\mathbb{P}}_x (\tau_\epsilon < t) \to 0$ as $\epsilon \searrow 0$ for all $x \in D$ and $t > 0$.

Since $\mathcal{V} \geq c e^{-\psi}$ for some positive constant $c$, and $\mathcal{V}$ is inf-compact, it follows that $\frac{\psi (x)}{\mathcal{V} (x)}$ tends to 0 as $\text{dist} (x, \partial D) \to 0$. By \cite[Lemma 3.7.2]{8}, then we have

$$\tilde{\mathbb{E}}_x [\psi (X_{t \wedge \tau_\epsilon})] \xrightarrow{\epsilon \searrow 0} \tilde{\mathbb{E}}_x [\psi (X_t)], \quad \text{and} \quad \lim_{t \to \infty} \frac{1}{t} \tilde{\mathbb{E}}_x [\psi (X_t)] = 0$$  \hspace{1cm} (3.9)
for all $x \in D$. Thus applying Itô’s formula to $\psi$, using (3.3), Remark 3.2 and (3.9), we obtain (3.7).

The method of proof of Theorem 3.1 leads to another interesting result. Let $G_*$ denote the minimal operator (compare with (2.6))

$$G_* f(x) := \inf_{u \in U} Lf(x, u),$$

and denote its principal eigenpair in $D$ as $(\lambda_*, \psi_*)$. It is clear that $\lambda_* \geq \lambda^*$, and that for a generic drift $m$ we should have strict inequality. Let $\psi_* = \log \Psi_*$, which satisfies

$$\max_{u \in U} L\psi_*(x, u) + \frac{1}{2} \sigma^T(x) \nabla \psi_*(x)^2 = -\lambda^*, \quad x \in D. $$

Consider the controlled diffusion

$$dY_t = (m(Y_t, U_t) + a(Y_t) \nabla \psi_*(Y_t)) dt + \sigma(Y_t) dW_t. \quad (3.11)$$

Denote its extended controlled generator as $L^{\psi_*}$, that is,

$$L^{\psi_*} f(x, u) := \frac{1}{2} \text{Tr} \left( a(x) \nabla^2 f(x) \right) + \langle m(x, u) + a(x) \nabla \psi_*(x), \nabla f(x) \rangle \quad \text{for } f \in C^2(D).$$

As done earlier, under $v \in U_{sm}$, we denote $L^{\psi_*} f(x, v(x))$ as $L^{\psi_*} f(x)$.

We have the following result, which in a way extends Theorem 3.1.

**Theorem 3.2.** The process $\{Y_t\}_{t \geq 0}$ in (3.11) is confined in $D$, and is geometrically ergodic uniformly over $v \in U_{sm}$. In particular, there exists $\hat{V} \in C^2(D)$, a compact set $K \subset D$, and positive constants $C$ and $\rho$, such that

$$L^{\psi_*} \hat{V}(x, u) \leq C\mathbf{1}_K(x) - \rho \hat{V}(x) \quad \forall (x, u) \in D \times U, \quad (3.12)$$

and $\hat{V} \Psi_*$ is bounded from below away from 0 in $D$. As a result, $e^{-\psi_*}$ is integrable under the invariant probability measure $\mu_v$ of $\{Y_t\}_{t \geq 0}$ for any $v \in U_{sm}$. Moreover,

$$\lambda_* \geq \frac{1}{2} \int_D |\sigma^T(x) \nabla \psi_*(x)|^2 \tilde{\mu}_v(dx) \quad \forall v \in U_{sm}. \quad (3.13)$$

**Proof.** Let $D_\epsilon$ be as in Remark 3.2. It is clear that $\lambda_D(G^* - 2\lambda_* \mathbf{1}_{D_\epsilon})$ is nondecreasing in $\epsilon$, and converges to $\lambda^* + 2\lambda_* \epsilon$ as $\epsilon \searrow 0$. Therefore, we can select some $\epsilon > 0$ and a compactly supported function nonnegative function $h$ which is a smooth approximation of $\mathbf{1}_{D_\epsilon}$, such that $\lambda_D(G^* - 2\lambda_* h) > \lambda^* + \lambda_*$. This means that we can select a bounded $C^{2,1}$ domain $D' \ni D$ as in the proof of Theorem 3.1, such that $\lambda_D(G^* - 2\lambda_* h) > \lambda^* + \lambda_*$. Let $\Phi \in C^2(D')$ denote the associated principal eigenfunction in $D'$. Thus we have

$$G^* \Phi + (\lambda^* + \lambda_* - 2\lambda_* h) \Phi \leq 0 \quad \text{in } D',$$

which of course implies that

$$L \Phi(x, u) + (\lambda^* + \lambda_* - 2\lambda_* h(x)) \Phi(x) \leq 0 \quad \forall (x, u) \in D \times U. \quad (3.14)$$

On the other hand, using the eigenfunction $\Psi_*$ of the operator $G_*$ in (3.10), we obtain

$$L \Psi_*(x, u) + \lambda_* \Psi_*(x) \geq 0 \quad \forall (x, u) \in D \times U. \quad (3.15)$$

Combining (3.14) and (3.15), and using (3.5), we get

$$L^{\psi_*} \left( \frac{\Phi}{\Psi_*} \right)(x, u) \leq (2\lambda_* h(x) - \lambda^*) \frac{\Phi}{\Psi_*}(x).$$

This establishes (3.12), from which the integrability of $e^{-\psi_*}$ and the property that the process is confined in $D$ follow.
In analogy to (3.3) we have
\[ \mathcal{L}^v \psi_v(x) - \frac{1}{2} |\sigma^T(x) \nabla \psi_v(x)|^2 \geq -\lambda_v, \quad x \in D, \]
from which (3.13) follows. This completes the proof.

\[ \square \]

3.2. The quasi-stationary distribution. An application of the Itô formula to the first equation in (2.5) shows that
\[ e^{-\lambda_v t} e^{\psi_v(x)} = \mathbb{E}^v_x \left[ e^{\psi_v(X_t)} 1 \{ t < \tau \} \right], \quad x \in D. \]
Let
\[ M_t := \exp \left( \int_0^t (\sigma^T(X_s) \nabla \psi_v(X_s), dW_s) ds - \frac{1}{2} \int_0^t |\sigma^T(X_s) \nabla \psi_v(X_s)|^2 ds \right). \]
Let \( \mathbb{P}^v_x \) and \( \mathbb{E}^v_x \) denote the probability measure and the expectation operator, respectively, on the canonical space of the process \( \{ \tilde{X}_t \}_{t \geq 0} \) in (3.1) controlled by \( v \in \mathcal{U}_{sm} \), and with initial condition \( \tilde{X}_0 = x \). Then, for any bounded function \( g \) which is compactly supported in \( D \), we get
\[ \mathbb{E}^v_x \left[ g(X_t) 1 \{ t < \tau \} \right] = \mathbb{E}^v_x \left[ g(X_{t \wedge \tau}) 1 \{ t < \tau \} \right] \]
\[ = \mathbb{E}^v_x \left[ e^{-\lambda_v (t \wedge \tau)} g(X_{t \wedge \tau}) \exp \left( -\psi_v(X_{t \wedge \tau}) + \psi_v(x) \right) M_{t \wedge \tau} 1 \{ t < \tau \} \right] \]
\[ = \mathbb{E}^v_x \left[ e^{-\lambda_v (t \wedge \tau)} g(\tilde{X}_{t \wedge \tau}) \exp \left( -\psi_v(\tilde{X}_{t \wedge \tau}) + \psi_v(x) \right) 1 \{ t < \tau \} \right] \]
\[ = e^{-\lambda_v t} e^{\psi_v(x)} \mathbb{E}^v_x \left[ g(\tilde{X}_t) \exp \left( -\psi_v(\tilde{X}_t) \right) \right], \]
where in the third equality we use Girsanov’s theorem, and we drop the term \( 1 \{ t < \tau \} \) in the fourth equality since the process \( \{ \tilde{X}_t \}_{t \geq 0} \) is confined in \( D \). Then by monotone convergence we can extend (3.16) to all nonnegative bounded functions \( g \). Since \( \Psi_v \) vanishes on \( \partial D \), we also obtain from (3.16) using monotone convergence that
\[ \mathbb{E}^v_x \left[ h(X_t) \exp \left( \psi_v(X_t) \right) 1 \{ t < \tau \} \right] = e^{-\lambda_v t} e^{\psi_v(x)} \mathbb{E}^v_x \left[ h(\tilde{X}_t) \right], \]
for all bounded functions \( h \).

By (3.16), we obtain
\[ \mathbb{E}^v_x \left[ g(X_t) \mid t < \tau \right] = \frac{\mathbb{E}^v_x \left[ g(X_t) 1 \{ t < \tau \} \right]}{\mathbb{E}^v_x \left[ 1 \{ t < \tau \} \right]} = \frac{\mathbb{E}^v_x \left[ g(\tilde{X}_t) \exp \left( -\psi_v(\tilde{X}_t) \right) \right]}{\mathbb{E}^v_x \left[ \exp \left( -\psi_v(\tilde{X}_t) \right) \right]}. \]

Since the \( Q \)-process is ergodic, and \( e^{-\psi_v} \) is integrable under its invariant distribution \( \bar{\mu}_v \) by Theorem 3.1, then taking limits in (3.17) and using [24, Theorem 4.12], we obtain
\[ \lim_{t \to \infty} \mathbb{E}^v_x \left[ g(X_t) \mid t < \tau \right] = \frac{\int_D g(y) e^{-\psi_v(y)} \bar{\mu}_v(dy)}{\int_D e^{-\psi_v(y)} \bar{\mu}_v(dy)}. \]

We have the following theorem.

**Theorem 3.3.** It holds that
\[ \alpha_v(A) := \lim_{t \to \infty} \mathbb{P}^v_x(X_t \in A \mid t < \tau) = \frac{\int_D 1_A(y) e^{-\psi_v(y)} \bar{\mu}_v(dy)}{\int_D e^{-\psi_v(y)} \bar{\mu}_v(dy)}, \quad A \in \mathfrak{B}(D), \]
where \( \mathfrak{B}(D) \) denotes the class of Borel sets in \( D \). In addition,
\[ \bar{\mu}_v(A) = \frac{\int_D 1_A(y) e^{\psi_v(y)} \alpha_v(dy)}{\int_D e^{\psi_v(y)} \alpha_v(dy)}, \]
(3.20)
and

\[ \lim_{t \to \infty} e^{\lambda t} \mathbb{P}^\nu_x(t < \tau) = e^{\psi(x)} \int_D e^{-\psi(y)} \tilde{\mu}_\nu(dy), \quad (3.21) \]

Proof. Equation (3.19) is a direct consequence of (3.18) with \( g(\cdot) \equiv 1 \), and (3.20) follows by inverting (3.19). Lastly, (3.21) follows by taking limits in (3.16) with \( g(\cdot) \equiv 1 \). \( \square \)

The probability measure \( \alpha_v \) in Theorem 3.3 is known in the literature as a quasi-stationary distribution [12, 13, 26]. Equation (3.20) should be compared with [12, Theorem 1.3], and (3.21) with [12, Proposition 1.2].

The rate of convergence in (3.19) has been studied extensively in the literature (see the results and discussion in [12]). It follows from [12, 25] that for each \( v \in \Upsilon_{sm} \), there exist positive constants \( \kappa_v \) and \( \gamma_v \) such that

\[ \| \mathbb{P}^\nu_x(X_t \in \cdot | t < \tau) - \alpha_v(\cdot) \|_{TV} \leq \kappa_v e^{-\gamma_v t} \quad \forall (t, x) \in \mathbb{R}_+ \times D, \]

with \( \| \cdot \|_{TV} \) denoting the total variation norm.

4. Variational formulation

In this section we first derive a variational formula for the controlled eigenvalue that can be viewed as an abstract Collatz–Wielandt formula as in [1, 3, 4, 9] and then map it to an expression in terms of the \( Q \)-process in view of the foregoing developments.

For \( v \in \Upsilon_{sm} \), let

\[ A_{v,w} f(x) := L_v f(x) + \langle a(x)w, \nabla f(x) \rangle, \quad \text{for } f \in C^2(D) \cap C(D^c), \]

\[ \mathcal{H}_v := \left\{ \nu \in \mathcal{P}(D \times \mathbb{R}^d) : \int_{D \times \mathbb{R}^d} A_{v,w} f(x) \nu(dx, dw) = 0 \quad \forall f \in C^2(D) \cap C(D^c) \right\}, \]

where \( \mathcal{P}(D \times \mathbb{R}^d) \) denotes the set of probability measures on the Borel \( \sigma \)-algebra of \( D \times \mathbb{R}^d \). We refer to \( \mathcal{H}_v \) as the set of infinitesimal ergodic occupation measures of the operator \( A_{v,w} \). A measure \( \nu \in \mathcal{H}_v \) can be disintegrated into \( \mu(dx)\eta(dw \mid x) \), with \( \eta \) corresponding to a (randomized) stationary Markov control. We wish to emphasize that the diffusion with generator \( A_{v,w} \) under such a control \( \eta \) is not, in general, confined in \( D \). Therefore, the elements of \( \mathcal{H}_v \) are not necessarily ergodic occupation measures of a controlled diffusion process. Nevertheless, the measure \( \tilde{\mu}_\nu(dx)\delta_{\psi_v}(dw) \) lies in \( \mathcal{H}_v \) and is indeed an ergodic occupation measure of the controlled diffusion with generator \( A_{v,w} \).

We also define

\[ \mathcal{P}_{*,v} := \left\{ \nu \in \mathcal{P}(D \times \mathbb{R}^d) : \int_{D \times \mathbb{R}^d} \frac{|\sigma^T(x) \nabla \psi_v|^2}{1 + |\psi_v|} \nu(dx, dw) < \infty \right\}, \]

\[ \mathcal{P}_{\kappa,v} := \left\{ \nu \in \mathcal{P}(D \times \mathbb{R}^d) : \int_{D \times \mathbb{R}^d} |\sigma^T(x)w|^2 \nu(dx, dw) < \infty \right\}. \]

In the proof of the results which follow, we make use of a family of cutoff functions defined as follows.

**Definition 4.1.** For \( t > 0 \), we let \( \chi_t \) be a convex \( C^2(\mathbb{R}) \) function such that \( \chi_t(s) = s \) for \( s \geq -t \), and \( \chi_t(s) \) = constant for \( s \leq -te^2 \). Then \( \chi_t' \) and \( \chi_t'' \) are nonnegative. In addition, we select \( \chi_t \) so that

\[ \chi_t''(s) \leq -\frac{1}{s} \quad \text{for } s \in [-te^2, -t] \quad \text{and } t \geq 0. \]

This is always possible.
Lemma 4.1. For any \( v \in \mathcal{U}_{sm} \) we have
\[
\lambda_v = \inf_{\nu \in \mathcal{H}_v \cap \mathcal{P}_{s,v}} \frac{1}{2} \int_{D \times \mathbb{R}^d} |\sigma^T(x)w|^2 \nu(dx, dw).
\]

Proof. We let
\[
L(x, w) := \frac{1}{2} |\sigma^T(x)w|^2.
\]

Using the definition in (3.2), we have the identity
\[
\mathcal{A}_{v,w} \chi_t \psi_v(x) = L_v \chi_t \psi_v(x) + \frac{1}{2} \sigma^T(x) \nabla \psi_v(x)^2 + L(x, w) - \frac{1}{2} \sigma^T(x) (w - \nabla \psi_v(x))^2
\]
\[
= L(x, w) - \lambda_v - \frac{1}{2} \sigma^T(x) (w - \nabla \psi_v(x))^2,
\]
where the second equality follows from (3.3). Since
\[
\mathcal{A}_{v,w} \chi_t \psi_v(x) = \chi_t' \psi_v(x) \mathcal{A}_{v,w} \psi_v + \frac{1}{2} \chi_t'' \psi_v(x) |\sigma^T \nabla \psi_v|^2
\]
we obtain from (4.2) that
\[
\mathcal{A}_{v,w} \chi_t \psi_v(x) - \chi_t'' \psi_v(x) L(x, \nabla \psi_v(x)) = \chi_t'(\psi_v(x)) \left( L(x, w) - \frac{1}{2} \sigma^T(x)(w - \nabla \psi_v(x))^2 - \lambda_v \right).
\]

Let \( \nu \in \mathcal{H}_v \cap \mathcal{P}_{s,v} \), and without loss of generality assume \( \nu \in \mathcal{P}_{s,v} \). Then \( \int \mathcal{A}_{v,w} \chi_t \psi_v(x) \, d\nu = 0 \) by the definition of \( \mathcal{H}_v \), and
\[
\int_{D \times \mathbb{R}^d} \chi_t'(\psi_v(x)) L(x, \nabla \psi_v(x)) \nu(dx, dw) \xrightarrow{t \to \infty} 0
\]
by the definitions of \( \chi_t \) and \( \mathcal{P}_{s,v} \). Thus, integrating (4.3) with respect to \( \nu \) and letting \( t \to \infty \), using also monotone convergence, we obtain
\[
\lambda_v + \frac{1}{2} \int_{D \times \mathbb{R}^d} \left| \sigma^T(x) (w - \nabla \psi_v(x)) \right|^2 \nu(dx, dw) = \int_{D \times \mathbb{R}^d} L(x, w) \nu(dx, dw).
\]

As mentioned earlier, \( \tilde{\mu}_v(dx) \delta_{\psi_v}(dw) \in \mathcal{H}_v \), and also lies in \( \mathcal{P}_{s,v} \) by (3.7). Therefore, the result follows by (3.7) and (4.4). \( \square \)

We continue with a variational formula for \( \lambda^* \). Let
\[
\mathcal{A}f(x, u, w) := Lf(x, u) + \langle a(x)w, \nabla f(x) \rangle, \quad (x, u, w) \in D \times U \times \mathbb{R}^d.
\]

As in (4.2), we have
\[
\mathcal{A}f(x, u, w) = Lf(x, u) + \frac{1}{2} |\sigma^T(x) \nabla f(x)|^2 + L(x, w) - \frac{1}{2} \sigma^T(x) (w - \nabla f(x))^2
\]
for all \( f \in C^2(D) \), and thus we obtain
\[
\mathcal{A}^*f(x, u, w) - L(x, w) = L^*(x, u) + \frac{1}{2} |\sigma^T(x) \nabla^* f(x)|^2 - \frac{1}{2} \sigma^T(x) (w - \nabla^* f(x))^2
\]
\[
\leq -\lambda^* - \frac{1}{2} \sigma^T(x) (w - \nabla^* f(x))^2 \quad \forall (x, u, w) \in D \times U \times \mathbb{R}^d.
\]
Recall from subsection 1.1 that $C^2_+(D)$ denotes the set of functions in $C^2(D)$ which are positive on $D$. The starting point of the analysis is [3, Theorems 2.1 and 2.5] which assert that

$$-\lambda^* = \inf_{h \in C^2_+(D) \cap C(\overline{D})} \sup_{\mu \in \mathcal{P}(D)} \int_D \frac{G^*h(x)}{h(x)} \mu(dx)$$

$$= \sup_{\mu \in \mathcal{P}(D)} \inf_{h \in C^2_+(D) \cap C(\overline{D})} \int_D \frac{G^*h(x, u)}{h(x)} \mu(dx).$$

with $G^*$ as in (2.6). We claim that we may replace $C^2_+(D) \cap C(\overline{D})$ with $C^2_+(\overline{D})$ in the first equality of (4.6). To prove the claim first note that (4.6) implies that

$$-\lambda^* \leq \inf_{h \in C^2_+(\overline{D})} \sup_{\mu \in \mathcal{P}(D)} \int_D \frac{G^*h(x)}{h(x)} \mu(dx).$$

Now let $\epsilon > 0$ be arbitrary and $D' \supseteq D$ be a bounded $C^{2,1}$ domain such that $\lambda_D'(G^*) \geq \lambda^* - \epsilon$. Let $\Phi^*$ denote the principal eigenfunction of $G^*$ on $D'$. Then $\Phi^* \in C^2_+(\overline{D})$ and $-\lambda^* + \epsilon \geq \frac{G^*\Phi^*}{\Phi^*}$ on $\overline{D}'$, which implies equality in (4.7). Note that the infimum in (4.7) is not attained in $C^2_+(\overline{D})$, but working in this space allows us to obtain a representation formula which does not rely on $\mathcal{P}_{*,n}$ in (4.1) as is the case in Lemma 4.1.

We define

$$F(f, \pi) := \int_{D \times U \times \mathbb{R}^d} (Af(x, u, w) - L(x, w)) \pi(dx, du, dw)$$

for $f \in C^2(\overline{D})$ and $\pi \in \mathcal{P}(D \times U \times \mathbb{R}^d)$. Consider $h \in C^2_+(\overline{D})$, and let $f = \log h$. Then $f \in C^2(\overline{D})$, and using (4.5), a simple calculation shows that

$$\frac{G^*h(x)}{h(x)} = \sup_{(u, w) \in U \times \mathbb{R}^d} [Af(x, u, w) - L(x, w)].$$

Combining this with (4.7), for which we have already shown that equality holds, we obtain

$$-\lambda^* = \inf_{f \in C^2(\overline{D})} \sup_{\pi \in \mathcal{P}(D \times U \times \mathbb{R}^d)} F(f, \pi).$$

Define

$$\mathcal{M}_A := \left\{ \pi \in \mathcal{P}(D \times U \times \mathbb{R}^d) : \int_{D \times U \times \mathbb{R}^d} Af \, d\pi = 0 \quad \forall f \in C^2(\overline{D}) \right\}.$$ 

We have the following result.

**Lemma 4.2.** It holds that

$$-\lambda^* = \sup_{\pi \in \mathcal{P}(D \times U \times \mathbb{R}^d)} \inf_{f \in C^2(\overline{D})} F(f, \pi) = -\inf_{\pi \in \mathcal{M}_A} \int_{D \times U \times \mathbb{R}^d} L(x, w) \pi(dx, du, dw).$$

**Proof.** Let

$$\rho := \sup_{\pi \in \mathcal{P}(D \times U \times \mathbb{R}^d)} \inf_{f \in C^2(\overline{D})} F(f, \pi).$$

It follows by (4.8) and (4.10) that $\rho \leq -\lambda^*$. It is also clear that if $\pi \notin \mathcal{M}_A$ then $\inf_{f \in C^2(\overline{D})} F(f, \pi) = -\infty$, so we assume that $\pi \in \mathcal{M}_A$. Since the second and third terms in (4.9) are equal when $\pi \in \mathcal{M}_A$, this also shows that

$$\inf_{\pi \in \mathcal{M}_A} \int_{D \times U \times \mathbb{R}^d} L(x, w) \pi(dx, du, dw) \geq \lambda^*.$$
It remains to show equality in (4.11). Recall that $\Omega^*_\text{sm}$ denotes the set of selectors from the minimizer in (2.7). For $v^* \in \Omega^*_\text{sm}$, the $Q$-process is ergodic by Theorem 3.1, and thus its invariant measure $\tilde{\mu}_{v^*}$ satisfies
\[
\int_D \tilde{\mathcal{L}}_{v^*} f(x) \, \tilde{\mu}_{v^*}(dx) = 0 \quad \forall f \in C^2(\overline{D}).
\]
It is also clear that the measure $\pi^*$ defined by
\[
\pi^*(dx, du, dw) := \tilde{\mu}_{v^*}(dx) \delta_{v^*}(x)(du) \delta_{\psi^*}(x)(dw)
\]
satisfies
\[
\int_{D \times U \times \mathbb{R}^d} Af(x, u, w) \pi^*(dx, du, dw) = \int_D \tilde{\mathcal{L}}_{v^*} f(x) \, \tilde{\mu}_{v^*}(dx) = 0 \quad \forall f \in C^2(\overline{D}),
\]
which implies that $\pi^* \in \mathcal{M}_A$. On the other hand
\[
\lambda^* = \int_D L(x, \nabla \psi^*(x)) \, \tilde{\mu}_{v^*}(dx) = \int_{D \times U \times \mathbb{R}^d} L(x, u) \pi^*(dx, du, dw)
\]
by (3.7), which shows that the infimum $\inf_{\pi \in \mathcal{M}_A} \int L \, d\pi$ is attained at $\pi^*$. This completes the proof. \hfill \Box

We gather the results in (4.8) and Lemma 4.2 in the following theorem, which also characterizes the measures $\pi \in \mathcal{M}_A$ which attain the infimum in (4.9).

**Theorem 4.1.** We have
\[
-\lambda^* = \inf_{f \in C^2_{\text{loc}}(\overline{D})} \sup_{\pi \in \mathcal{P}(D \times U \times \mathbb{R}^d)} F(f, \pi) = \sup_{\pi \in \mathcal{P}(D \times U \times \mathbb{R}^d)} \inf_{f \in C^2_{\text{loc}}(\overline{D})} F(f, \pi)
\]
and
\[
\lambda^* = \min_{\pi \in \mathcal{M}_A} \frac{1}{2} \int_{D \times U \times \mathbb{R}^d} \left| \sigma^T(x)w \right|^2 \pi(dx, du, dw).
\]
In addition, any $\pi \in \mathcal{M}_A$ which attains the minimum in (4.13) has the form in (4.12) for some $v^* \in \Omega^*_\text{sm}$.

**Proof.** We need to prove the assertion in the second part of the theorem. Let $D_n \supseteq D$, $n \in \mathbb{N}$, be a decreasing sequence of bounded $C^{2,1}$ domains such that $\cap_{n \in \mathbb{N}} D_n = \overline{D}$. We denote the principal eigenpair of $G^*$ on $D_n$ by $(\lambda^*_n, \Phi_n) \in \mathbb{R} \times C^2(\overline{D}_n)$, and set $\lambda_n = \log \Phi_n$. Suppose $\hat{\pi} \in \mathcal{M}_A$ attains the minimum in (4.13). We disintegrate it as
\[
\hat{\pi}(dx, du, dw) = \hat{\mu}(dx) \hat{v}(du \mid x) \, \hat{\nu}(dw \mid u, x).
\]
Using (4.5), with $f = \phi_n \in C^2_{\text{loc}}(\overline{D})$, and integrating with respect to $\hat{\pi}$ we obtain
\[
-\lambda^* = \int_{D \times U \times \mathbb{R}^d} L(x, w) \hat{\pi}(dx, du, dw)
\]
\[
= \int_D \left( \mathcal{L}_{\hat{\nu}} \phi_n(x) + \frac{1}{2} |\sigma^T(x) \nabla \phi_n(x)|^2 \right) \hat{\mu}(dx)
\]
\[
- \int_{D \times U \times \mathbb{R}^d} \frac{1}{2} |\sigma^T(x)(w - \nabla \phi_n(x))|^2 \hat{\nu}(dw \mid u, x) \hat{\eta}(dx, du),
\]
with $\hat{\eta}(dx, du) = \hat{\mu}(dx) \hat{v}(du \mid x)$. Now
\[
\mathcal{L}_{\hat{\nu}} \phi_n(x) + \frac{1}{2} |\sigma^T(x) \nabla \phi_n(x)|^2 \leq -\lambda^*_n,
\]
and we know that $\lambda_n^* \nearrow \lambda^*$ as $n \to \infty$. By (4.14), we must have
\[
\limsup_{n \to \infty} \int_D \left( \mathcal{L}_n \phi_n(x) + \frac{1}{2} |\sigma(x)\nabla \phi_n(x)|^2 \right) \mu(dx) \geq -\lambda^*.
\] (4.16)
Since $\phi_n$ converges to $\psi^*$ in $C^{1,\alpha}(K)$ for any compact $K \subset D$ and any $\alpha \in (0, 1)$, and in view of (4.15), we can apply Fatou’s lemma to (4.16) to obtain
\[
\limsup_{n \to \infty} \int_D \left( \mathcal{L}_n \psi^*(x) + \frac{1}{2} |\sigma(x)\nabla \psi^*(x)|^2 \right) \mu(dx) = -\lambda^*.
\]
This shows that $\hat{v}(du \mid x) = v^*(x)$ a.e. on the support of $\hat{\mu}$ for some $v^* \in \mathcal{U}_{sm}$. Similarly, from the last term in (4.14), we obtain
\[
\int_{D \times \mathcal{U} \times \mathbb{R}^d} \frac{1}{2} |\sigma(x)(w - \nabla \psi^*(x))|^2 \tilde{w}(dw \mid u, x) \hat{\eta}(dx, du) = 0,
\]
which shows that $\tilde{w}(dw \mid u, x) = \delta_{\nabla \psi^*(x)}(dw)$ a.e. on the support of $\hat{\eta}(dx, du) = \mu(dx) \delta_{v^*(x)}(du)$.
Let
\[
\tilde{\pi}(dx, du, dw) = \mu(dx) \delta_{v^*(x)}(du) \delta_{\nabla \psi^*(x)}(dw).
\]
Since $\tilde{\pi}$ agrees with $\tilde{\pi}$ on the support of $\hat{\mu}$, we must have $\int_{D \times \mathcal{U} \times \mathbb{R}^d} A f \, d\tilde{\pi} = 0$ for all $f \in C^2(\mathcal{D})$, which implies that
\[
\int_{D \times \mathcal{U} \times \mathbb{R}^d} \tilde{\mathcal{L}}_{v^*} f(x) \mu(dx) = 0 \quad \forall f \in C^2(\mathcal{D}).
\]
The uniqueness of the invariant probability measure of the $Q$-process then implies that $\hat{\mu} = \hat{\mu}_{v^*}$, which combined with the argument in the preceding paragraph shows that $\hat{v} = v^*(x)$ and $\tilde{w} = \delta_{\nabla \psi^*(x)}(dw)$ a.e. $x \in D$. This completes the proof. \qed

One may view (4.13) as the abstract linear programming formulation of the ergodic control problem for the controlled diffusion $\{Z_t\}_{t \geq 0}$ in $D$ whose (controlled) extended generator is given by $A$ indexed by the control variables $u \in \mathcal{U}$ and $w \in \mathbb{R}^d$, with the objective of minimizing the ergodic cost
\[
\limsup_{T \nearrow \infty} \frac{1}{T} \mathbb{E} \left[ \frac{1}{2} \int_0^T |\sigma(Z_t)W_t|^2 \right.
\]
over all non-anticipative control processes $(U_t, W_t)_{t \geq 0}$ such that $\mathbb{P}(\tau_\epsilon < t) \to 0$ as $\epsilon \searrow 0$ for any $t > 0$, with $\tau_\epsilon$ as in Remark 3.1. The corresponding Hamilton–Jacobi–Bellman equation is
\[
\min_{(u, w) \in \mathcal{U} \times \mathbb{R}^d} \left( \mathcal{A} \Phi(x, u, w) + \frac{1}{2} |\sigma(x)w|^2 \right) - \beta = 0,
\]
where $\Phi \in C^2(D) \cap \{ f : \lim_{x \to \partial D} f(x) = -\infty \}$. Performing the minimization over $w$, we obtain
\[
\min_{u \in \mathcal{U}} \left( \mathcal{L} \Phi(x, u) - \frac{1}{2} |\sigma(x)\nabla \Phi(x)|^2 \right) - \beta = 0.
\]
Comparing with (3.4), we have $\Phi = -\psi^*$ and $\beta = \lambda^*$. Here we use the well-posedness of (3.4) which follows from the fact that the invertible smooth transformation $\Psi^* = e^{\psi^*}$ converts (3.4) into the well posed Dirichlet problem (2.7) with a unique solution in $C^2(D) \cap \overline{C(D)}$. Recall that $\mathcal{U}_{sm}$ denotes the set of a.e. selectors from the minimizer in (2.7), which is precisely the set of optimal stationary Markov controls, that is $\lambda = \lambda_v = \beta_v$ and $\Psi_v = \Psi^*$ for all $v \in \mathcal{U}_{sm}$. Therefore, under any optimal choice $v^* \in \mathcal{U}_{sm}$, the optimal choice of $\{W_t\}_{t \geq 0}$ is precisely $W_t = \nabla \psi^*(Z_t)$, $t \geq 0$. That is, the optimal process is the corresponding $Q$-process $\{\tilde{Z}_t\}_{t \geq 0}$ controlled by $v^*$. This equivalence leads to the following representation for the optimal eigenvalue (i.e., optimal exit rate) in terms of the $Q$-process.
Theorem 4.2. For any $v^* \in \mathcal{U}_{\text{sm}}^*$, the optimal exit rate $\lambda^*$ is given by

$$
\lambda^* = \frac{1}{2} \int_D |\sigma^T(x) \nabla \psi^*(x)|^2 \tilde{\mu}_{v^*}(dx)
$$

$$
= \inf_{v \in \mathcal{U}_{\text{sm}}} \frac{1}{2} \int_D |\sigma^T(x) \nabla \psi_v(x)|^2 \tilde{\mu}_{v}(dx)
$$

$$
= \frac{\int_D |\sigma^T(x) \nabla \psi^*_v(x)|^2}{\Psi_v^*(x) \alpha_{v^*}(dx)}
$$

$$
= \frac{\int_D |\sigma^T(x) \nabla \psi_v(x)|^2}{2 \int_D \Psi_v(x) \alpha_v(dx)}.
$$

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