Error estimates for splitting methods based on AMF-Runge-Kutta formulas for the time integration of advection diffusion reaction PDEs.

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Abstract

The convergence of a family of AMF-Runge-Kutta methods (in short AMF-RK) for the time integration of evolutionary Partial Differential Equations (PDEs) of Advection Diffusion Reaction type semi-discretized in space is considered. The methods are based on very few inexact Newton Iterations of Approximate Matrix Factorization splitting-type (AMF) applied to the Implicit Runge-Kutta formulas, which allows very cheap and inexact implementations of the underlying Runge-Kutta formula. Particular AMF-RK methods based on Radau IIA formulas are considered. These methods have given very competitive results when compared with important formulas in the literature for multidimensional systems of non-linear parabolic PDE problems. Uniform bounds for the global time-space errors on semi-linear PDEs when simultaneously the time step-size and the spatial grid resolution tend to zero are derived. Numerical illustrations supporting the theory are presented.

Key words: Evolutionary Advection-Diffusion-Reaction Partial Differential equations, Approximate Matrix Factorization, Runge-Kutta Radau IIA methods, Finite Differences, Stability and Convergence.
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1 Introduction

We consider numerical methods for the time integration of a family of Initial Value Problems in ODEs

\[ y'_h(t) = f_h(t, y_h(t)), \quad y_h(0) = u_{0,h}, \quad 0 \leq t \leq t^*, \quad y_h, f_h \in \mathbb{R}^{m(h)}, \quad h \to 0^+, \]

(1.1)

coming from the spatial semi-discretization of an \( l \)-dimensional Advection Diffusion Reaction problem in time dependent Partial Differential Equations (PDEs), with prescribed Boundary Conditions and an Initial Condition. Here \( h \) denotes a small positive parameter associated with the spatial resolution and usually \( l = 2, 3, \ldots \).

The typical PDE problem with Dirichlet boundary conditions is given by (\( \Omega \) is a bounded open connected region in \( \mathbb{R}^l \), \( \partial \Omega \) its boundary and \( \nabla \) is the gradient operator)

\[
\begin{align*}
  u_t(x, t) &= -\nabla \cdot (a(x, t) u(x, t)) + \nabla \cdot (\bar{d}(x, t) \cdot \nabla u(x, t)) + r(u, x, t), \\
  u(x, t) &= g_1(x, t), \quad (x, t) \in \partial \Omega \times [0, t^*]; \\
  u(x, 0) &= g_2(x), \quad x \in \Omega,
\end{align*}
\]

(1.2)

which is assumed to have some diffusion (\( \bar{d}_j(x, t) \geq d_0 > 0 \), \( j = 1, \ldots, l \)), namely that it is not of pure hyperbolic type, and it is also assumed that some adequate spatial discretization based on Finite Differences or Finite Volume is applied to obtain the system \( [ ] \). Some stiffness in the reaction part \( r(u, x, t) \) is also allowed. The treatment of Systems of PDEs do not involve more difficulty for our analysis but for simplicity of presentation we prefer to confine ourselves to the case of one PDE.

We denote by \( u_h(t) \) the solution of the PDE problem confined to the spatial grid (or well to the \( h \)-space related). It will be tacitly assumed that the PDE problem admits a smooth solution \( u(x, t) \) in the sense that continuous partial derivatives in all variables up to some order \( p \) exist and are continuous and uniform bounded on \( \Omega \times [0, t^*] \) and that \( u(x, t) \) is continuous on \( \bar{\Omega} \times [0, t^*] \) \( (\bar{\Omega} = \Omega \cup \partial \Omega) \). It is also assumed that the spatial discretization errors

\[
\sigma_h(t) := u'_h(t) - f_h(t, u_h(t))
\]

(1.3)

satisfy in the norm considered,

\[
\|\sigma_h(t)\| \leq C h^r, \quad (C \geq 0, \ r > 0), \quad 0 \leq t \leq t^*, \quad h \to 0.
\]

(1.4)

In general \( C, C' \) or \( C^* \) will refer to some constants that maybe different at each occurrence but that all of them are independent of \( h \to 0 \) and from the
time-stepsize $\tau \to 0$. The vector norm used is arbitrary as long as it is defined for vectors of any dimension. For square matrices the norm used is the induced operator norm, $\|A\| = \sup_{v \neq 0} \|Av\|/\|v\|.$

In spite of most of our results apply in general, we will provide specific results for weighted Euclidean norms of type

$$\|(v_j)_{j=1}^N\| = N^{-1/2}\|(v_j)_{j=1}^N\|_2.$$ 

It should be noted that in this case we have for any square matrix $A$ that,

$$\|A\| = \|A\|_2, \quad \forall A \in \mathbb{R}^{N,N}, \ N = 1, 2, 3, \ldots.$$ 

We assume some natural splitting for $f_h$ (directional or other),

$$f_h(t, y) = \sum_{j=1}^df_{j,h}(t, y), \quad \text{(1.5)}$$

which provides some natural splitting for the Jacobian matrix at the current point $(t_n, y_n)$,

$$J_h = \sum_{j=1}^d J_{j,h}, \quad J_h := \frac{\partial f_h(t_n, y_n)}{\partial y}, \quad J_{j,h} := \frac{\partial f_{j,h}(t_n, y_n)}{\partial y}. \quad \text{(1.6)}$$

This goal of the paper is to analyze the convergence order of the Method of Lines (MoL) approach for time-dependent PDEs of Advection Reaction Diffusion PDEs, with the main focus on the time integration of the large ODE systems resulting of the spatial PDE-semidiscretization, where some stiffness is assumed (parabolic dominant problems with stiff reaction terms) and the time integrators are based on very few iterations of splitting type (Approximate Matrix Factorization and Newton-type schemes) applied to highly stable Implicit Runge-Kutta methods. It should be remarked that the underlying Implicit Runge-Kutta method is never solved up to convergence, hence the convergence study does not follows from the results collected in classical references about finite difference methods such as $[14,10,17,13,9]$. The kind of approach to be considered here has interest since it is easily applicable to general systems of PDEs as we will see later on and it is reasonably cheap for non-linear problems in general (although we give convergence results for semilinear problems only) when some splitting of the function $f_h$ and its Jacobian is available and the split terms can be handled efficiently. In particular a method based on three AMF-iterations of the two-stage Radau IIA method $[1]$ has shown to be competitive $[7]$ when compared with some standard PDE-solvers such as VODPK $[2,3]$ in some interesting non-linear diffusion reaction problems widely considered in the literature. We also present two new methods based on the 2-stage Radau IIA, by performing just one or two iterations.
of splitting type, respectively. The method based on two iterations is one of the very few one-step methods of splitting type we have seen in the literature that has order three in PDE-sense for the time integration.

The rest of the paper is organized as follows. In section 2 we introduce the AMF\(_q\)-RK methods, and special attention is paid to some methods based on Radau IIA formulas. In section 3, the convergence for semilinear PDEs is studied in detail. The local and global errors are studied for the AMF\(_q\)-RK splitting methods based on some general Runge-Kutta methods. Section 4 is devoted to some applications of the convergence results to 2D and 3D-parabolic PDEs.

Henceforth, for simplicity in the notations, we omit in many cases the \(h\)-dependence of some vectors such as \(f_h, f_{j,h}\) and of some matrices such as \(J_h\) and \(J_{j,h}\) \((j = 1, \ldots, d)\). It should be clear from the context which ones are \(h\)-dependent. Besides, we will refer to the identity matrix as \(I\) when its dimension is clear from the context.

### 2 AMF-IRK methods

For the integration of the ODEs (1.1), we consider as a first step an implicit \(s\)-stage Runge-Kutta method with a nonsingular coefficient matrix \(A = (a_{ij})_{i,j=1}^s\) and a weight vector \(b = (b_j)_{j=1}^s\). The method is given by the compact formulation (below \(\otimes\) denotes the Kronecker product of matrices \(A \otimes B = (a_{ij}B)\), \(A = (a_{ij}), B = (b_{ij})\))

\[
Y_n = e \otimes y_n + \tau (A \otimes I_m) F(Y_n), \\
y_{n+1} = \varpi y_n + (b^T \otimes I_m) Y_n, \\
c \equiv (c_j)_{j=1}^s := Ae, \quad e = (1, \ldots, 1)^T \in \mathbb{R}^s, \quad b^T := b^T A^{-1}, \quad \varpi = 1 - b^T e, \\
Y_n = (Y_{n,j})_{j=1}^s \in \mathbb{R}^{ms}, \quad F(Y_n) = (f(t_n + \tau c_j, Y_{n,j}))_{j=1}^s \in \mathbb{R}^{ms}.
\]

(2.1)

It should be noted that we have replaced the usual formulation at the stepping point \(y_{n+1} = y_n + \tau (b^T \otimes I_m) F(Y_n)\) by the equivalent in (2.1), which has some computational advantages for stiff problems when the algebraic system for the stages is not exactly solved.

A typical Quasi-Newton iteration to solve the stage equations above is given by (below, \(J = \partial f/\partial y(t_n, y_n)\) is the exact Jacobian at the step-point \((t_n, y_n)\)),

\[
[I_{ms} - A \otimes \tau J] \Delta^\nu = D_n^{-\nu}, \quad Y_n^\nu = Y_n^{\nu-1} + \Delta^\nu, \quad \nu = 1, 2, \ldots,
\]

(2.2)
where

\[ D_n^{\nu-1} \equiv D(t_n, \tau, y_n, Y_n^{\nu-1}) := e \otimes y_n - Y_n^{\nu-1} + \tau(A \otimes I_m)F(Y_n^{\nu-1}). \]  

(2.3)

A cheaper iteration of Newton-type when the matrix \( A \) has a multipoint spectrum has been considered in [6,12] (denoted as Single-Newton iteration)

\[ [I_{ms} - T_\nu \otimes \tau J] \Delta^{\nu} = D_n^{\nu-1}, \quad Y_n^{\nu} = Y_n^{\nu-1} + \Delta^{\nu}, \quad \nu = 1, 2, \ldots, q \]  

(2.4)

where

\[ T_\nu = \gamma S_\nu (I - L_\nu)^{-1} S_\nu^{-1}, \quad \gamma > 0, \]

\[ S_\nu \in \mathbb{R}^{s,s} \text{ are regular matrices and} \]

\[ L_\nu \in \mathbb{R}^{s,s} \text{ are strictly lower triangular matrices}. \]

After some simple manipulations, by using standard properties of the Kronecker product, this iteration can be rewritten in the equivalent form,

\[ [I_s \otimes (I_m - \gamma \tau J)] E^{\nu} = ((I_s - L_\nu)S_\nu^{-1} \otimes I_m) D_n^{\nu-1} + (L_\nu \otimes I_m) E^{\nu}, \]

\[ Y_n^{\nu} = Y_n^{\nu-1} + (S_\nu \otimes I_m) E^{\nu}, \quad \nu = 1, 2, \ldots, q. \]  

(2.6)

To reduce the algebra cost, we use the Approximate Matrix Factorization \[8\] in short AMF, with \( J \equiv J_h \) and \( J_j \equiv J_{j,h} \) given in (1.6),

\[ \Pi_d := \prod_{j=1}^{d} (I_m - \gamma \tau J_j) = (I_m - \gamma \tau J) + \mathcal{O}(\tau^2), \]  

(2.7)

and replace in (2.6) \( (I_m - \gamma \tau J) \) by \( \Pi_d \), which yields the AMF\(_q\)-RK method based on the underlying Runge-Kutta method

\[ [I_s \otimes \Pi_d] E^{\nu} = ((I_s - L_\nu)S_\nu^{-1} \otimes I_m) D_n^{\nu-1} + (L_\nu \otimes I_m) E^{\nu}, \]

\[ Y_n^{\nu} = Y_n^{\nu-1} + (S_\nu \otimes I_m) E^{\nu}, \quad \nu = 1, 2, \ldots, q \]  

(2.8)

Our starting point for the convergence analysis in the next section takes into account that the AMF\(_q\)-RK method can be rewritten in the equivalent form \[9\]

\[ [I \otimes I - T_\nu \otimes \tau P] (Y_n^{\nu} - Y_n^{\nu-1}) = D(t_n, \tau, y_n, Y_n^{\nu-1}), \quad 1 \leq \nu \leq q \]

\[ Y_n^{0} = e \otimes y_n, \quad y_{n+1} = \varpi y_n + (\beta^T \otimes I_m) Y_n^{q}. \]  

(2.9)
where the matrix $P$ plays a primary role

$$P := (\gamma\tau)^{-1}(I - \Pi_d)$$
$$= J + (-\gamma\tau)\sum_{j<k} J_j J_k + (-\gamma\tau)^2 \sum_{j<k<l} J_j J_k J_l + \ldots + (-\gamma\tau)^{d-1} J_1 J_2 \cdots J_d. \quad (2.10)$$

### 2.1 AMF$_q$-RK methods based on the 2 stage Radau IIA formula

We are going to deserve special attention to AMF$_q$-RK methods based on the 2 stage Radau IIA formula \[\text{[1]}\]. This formula has coefficient Butcher tableau given by

$$\begin{array}{c|ccc}
\begin{array}{c}
\alpha
\end{array} & \begin{array}{c}
A
\end{array} & \begin{array}{c}
b^T
\end{array} \\
\hline
1/3 & 5/12 & -1/12 \\
3/4 & 1/4 & 1/4
\end{array}$$

This is a collocation method (stage order is two) possessing good stability properties, such as $L$-stability (i.e. $A$-stability plus $R(\infty) = 0$, with $R(z)$ being the linear stability function of the method), and has order of convergence three (in ODE sense), not only on non-stiff problems but also in many kinds of stiff problems \[\text{[4]}\]. These properties for the underlying Runge-Kutta method are convenient, since the family of ODEs \[\text{[1]}\] involves stiffness in most of cases, due to the diffusion terms and possibly to the reaction part, and it is expected that the methods to be built on inherit part of the good properties of the original Runge-Kutta method.

The next three AMF$_q$-Rad methods have coefficient matrices ($L_\nu$, $S_\nu$ and $T_\nu$) and eigenvalue $\gamma$ of the form

$$T_\nu = \gamma S_\nu (I_2 - L_\nu)^{-1} S_\nu^{-1}, \quad S_\nu = \begin{pmatrix} 1 & s_\nu \\ 0 & 1 \end{pmatrix}, \quad L_\nu = \begin{pmatrix} 0 & 0 \\ l_\nu & 0 \end{pmatrix}, \quad \gamma = \sqrt{\det(A)} = 1/\sqrt{6}. \quad (2.11)$$

AMF$1$-Rad was derived in \[\text{[3]}\] by looking for good stability properties and order two (ODE sense). In particular the method is $A(\pi/2)$-stable for a 2-splitting (see in Definition \[\text{[1]}\] below, the concept of stability for a $d$-splitting), $A(0)$-stable for any $d$-splitting and has stability wedges close to $\theta_d = \pi/(2(d - 1))$ for $d = 3, 4$. The method is based on one iteration ($q = 1$) and was required to fulfil $(A - T_1)c = 0$ and it has coefficients given by

$$s_1 = -\frac{3 + 2\sqrt{6}}{9}, \quad l_1 = \frac{3}{4}(-12 + 5\sqrt{6}). \quad (2.12)$$
AMF$^2$-Rad was derived in [3] by looking for good stability properties and order three (ODE sense). The method is A($\pi/2$)-stable for a 2-splitting, A(0)-stable for any $d$-splitting and A($\pi/6$)-stable for $d = 3, 4$. The method is based on two iterations ($q = 2$) and their matrices $T_1$ and $T_2$ were required to satisfy $(A-T_1)c = 0$ and $e_2^T T_2^{-1} (A-T_2) = 0^T$, $e_2^T = (0, 1)$, respectively. Its coefficients are uniquely given by

$$s_1 = \frac{3 + 2\sqrt{6}}{9}, \quad l_1 = \frac{3}{4}(-12 + 5\sqrt{6})$$

$$s_2 = \frac{5 - 2\sqrt{6}}{9}, \quad l_2 = \frac{3\sqrt{6}}{4}. \quad (2.13)$$

AMF$^3$-Rad was derived in [27] by looking for good stability properties and order three (ODE sense). The method is A($\pi/2$)-stable for a 2-splitting, A(0)-stable for any $d$-splitting and close to A($\theta_d$)-stable for $d = 3, 4$ with $\theta_d = \pi/(2(d - 1))$. The method is based on three iterations ($q = 3$) and their matrices $T = T_1 = T_2 = T_3$ were required to satisfy $e_2^T T^{-1} (A-T) = 0^T$. Its coefficients are uniquely given by

$$s_1 = s_2 = s_3 = \frac{5 - 2\sqrt{6}}{9}, \quad l_1 = l_2 = l_3 = \frac{3\sqrt{6}}{4}. \quad (2.14)$$

In [7], a variable-stepsize integrator based on the AMF$^3$-Rad method was successfully tested on several interesting 2D and 3D advection diffusion reaction PDEs by exhibiting good performances in comparison with state-of-the-art codes like VODPK [2, 3] and RKC [16, 19] and its implicit-explicit counterpart, IRKC [15, 18]. The other two methods, AMF$q$-Rad ($q = 1, 2$), were introduced later [5] after carefully analyzing the PDE errors on semilinear problems and with the purpose of reducing the number of iterations w.r.t. AMF$^3$-Rad.

### 3 Convergence for semilinear problems

For our convergence analysis we consider AMF$q$-RK methods applied to the ODE problems coming from the spatial discretizations of semilinear PDE problems of type (1.2) where the advection and diffusion vectors $a(x, t)$ and $\vec{d}(x, t)$ are both constant and the reaction part has the form

$$r(u, x, t) = \kappa u + g(x, t), \quad \kappa \text{ being a constant, } x \in \Omega \subseteq \mathbb{R}^l. \quad (3.1)$$

In this way, the ODE systems have the form

$$y_h'(t) = f_h(t, y_h) := J_h y_h(t) + g_h(t), \quad y_h(0) = u_{0, h}^\ast, \quad h \to 0^+, \quad J_h = \sum_{j=1}^d J_{j, h}, \quad t \in [0, t^\ast]. \quad (3.2)$$
Here, the exact solution of the PDE confined to the spatial grid \( u_h(t) = u(x,t) \), is assumed to satisfy (1.3) and (1.4). Thus, we focus on the global errors of the MoL approach, where the spatial discretization is carried out first by using finite differences (or finite volumes) and then the time discretization is performed by using AMF\(_q\)-RK methods. It is important to remark that we will not pursue the details of the spatial semidiscretizations but rather it is assumed that the spatial semidiscretizations are stable and provides spatial discretization errors satisfying (1.4). We shall provide uniform bounds for the global errors of the MoL approach (\( y_h(t) \) henceforth denotes the numerical solution of the MoL approach) in the sense

\[
\epsilon_{n,h} := u_h(t_n) - y_h(t_n) = \mathcal{O}(\tau^{p_1}) + \mathcal{O}(h^\alpha \tau^{p_2}), \quad h \to 0^+, \tau \to 0^+, \tag{3.3}
\]

which is meant that there exist constants \( C_1, C_2, p_1, p_2, \alpha \) (all of them independent on \( h \) and \( \tau \)) so that in the norm considered,

\[
\|\epsilon_{n,h}\| \leq C_1 \tau^{p_1} + C_2 h^\alpha \tau^{p_2}, \quad h \to 0^+, \tau \to 0^+ \quad \text{holds.}
\]

In our convergence analysis we need that all the matrices \( J_{j,h} \) pairwise commute and that they can be brought to the following decomposition (it has some resemblance with the Jordan’s decomposition, but it is a little more general)

\[
J_{j,h} = \Theta_h \Lambda_{j,h} \Theta_h^{-1}, \quad \text{Cond}(\Theta_h) := \|\Theta_h\| \cdot \|\Theta_h^{-1}\| \leq C, \quad h \to 0^+, \quad 1 \leq j \leq d,
\]

\[
\Lambda_{j,h} = \text{BlockDiag}(\Lambda_{j,h}^{(1)}, \Lambda_{j,h}^{(2)}, \ldots, \Lambda_{j,h}^{(\vartheta_h)}), \quad \Lambda_{j,h}^{(l)} = \lambda_{j,h}^{(l)} I + E_h^{(l)}, \quad \text{Re} \lambda_{j,h}^{(l)} \leq 0,
\]

\[
\dim(E_h^{(l)}) \leq N, \quad \|E_h^{(l)}\|_\infty \leq C', \quad l = 1, 2, \ldots, \vartheta_h \quad (h \to 0^+).
\]

\( E_h^{(l)} \) are all of them strictly lower triangular matrices.

(3.4)

Another important approach for the convergence analysis of the MoL method (mainly concerned with the time integration) is based on the pseudo-spectra analysis of the matrix \( J_h \) [13] and the related matrices \( J_{j,h} \). That analysis is of more general scope but it is much more difficult to make and as we will see below, our analysis is enough for some interesting kind of semilinear problems and it is expected that the results extend to most of the non-linear problems of parabolic dominant type.

Next, we consider a standard 3D semilinear-PDEs problem where the assumptions in (3.4) are fulfilled.

### 3.1 An example

Consider the semilinear PDE-problem (1.2) with \( x \in \Omega = (0,1)^3 \), with constant vectors, \( a(x,t) = (a_j)^3_{j=1}, \quad d(x,t) = (d_j)^3_{j=1}, \quad d_j > 0 \ (j = 1, 2, 3) \) and
Consider the spatial semidiscretization by using second order central differences and spatial resolution \( h = 1/(N + 1) \). This yields a semilinear ODE systems of dimension \( m = N^3 \) of the form (3.2) for \( d = 3 \). The matrices \( J_{j,h} \) are given by

\[
J_{1,h} = I_N \otimes I_N \otimes T_1, \quad J_{2,h} = I_N \otimes T_2 \otimes I_N, \quad J_{3,h} = T_3 \otimes I_N \otimes I_N
\]

where \( T_l = \text{Tridiag}(\alpha_l, \delta_l, \beta_l) \in \mathbb{R}^{N,N} \), \( l = 1, 2, 3 \),

\[
\alpha_l = h^{-2}(\bar{d}_l - 2^{-1}h a_l), \quad \beta_l = h^{-2}(\bar{d}_l + 2^{-1}h a_l), \quad \delta_l = h^{-2}(-2\bar{d}_l + h^2 \kappa),
\]

and the vector \( g_h(t) \) includes the reaction part \( g(x,t) \) plus the boundary conditions. It is straightforward to see that the \( J_{l,h} \) pairwise commute. Moreover, by assuming Cell-Pécelt numbers \([9, \text{p. 67, formula (3.42)} \] and the vector \( g_h(t) \) includes the reaction part \( g(x,t) \) plus the boundary conditions. It is straightforward to see that the \( J_{l,h} \) pairwise commute. Moreover, by assuming Cell-Pécelt numbers \([9, \text{p. 67, formula (3.42)} \] and the vector \( g_h(t) \) includes the reaction part \( g(x,t) \) plus the boundary conditions. It is straightforward to see that the \( J_{l,h} \) pairwise commute. Moreover, by assuming Cell-Pécelt numbers \([9, \text{p. 67, formula (3.42)} \]

\[
\frac{h|a_l|}{\bar{d}_l} < 2, \quad l = 1, 2, 3,
\]

from \([11, \text{section 2}] \) it follows that their spectral decomposition has the form

\[
\mathcal{T}_l = \text{Tridiag}(\alpha_l, \delta_l, \beta_l) = V_l \Lambda_l V_l^{-1}, \quad V_l = D_l U, \quad l = 1, 2, 3,
\]

\[
\Lambda_l = \text{Diag}(\lambda_{l,k})_{k=1}^{N}, \quad \lambda_{l,k} = \delta_l + 2\sqrt{\alpha_l \beta_l} \cos \frac{k\pi}{N+1},
\]

\[
U = \left( \frac{2}{N+1} \right)^{1/2} \left( \sin \frac{k\pi}{N+1} \right)_{k=1}^{N} \text{ is an orthogonal matrix and }
\]

\[
D_l = \left( \frac{N+1}{2} \right)^{1/2} \text{Diag}\left( (\alpha_l/\beta_l)^{k/2} \right)_{k=1}^{N}.
\]

From here we conclude that all the matrices can be brought to the spectral decomposition in \([3,4] \) having negative eigenvalues and with matrix \( \Theta_h = V_3 \otimes V_2 \otimes V_1 \). Observe that

\[
\|\Theta_h\|_2 \|\Theta_h^{-1}\|_2 = \prod_{l=1}^{3} \|V_l\|_2 \|V_l^{-1}\|_2 = \prod_{l=1}^{3} \|D_l\|_2 \|D_l^{-1}\|_2
\]

\[
= \prod_{l=1}^{3} \left( \frac{2\bar{d}_l + h|a_l|}{2\bar{d}_l - h|a_l|} \right)^{N/2} \leq \prod_{l=1}^{3} \left( \frac{2\bar{d}_l + h|a_l|}{2\bar{d}_l - h|a_l|} \right)^{1/(2h)}
\]

\[
\approx \exp \left( \sum_{l=1}^{3} \frac{|a_l|}{2\bar{d}_l} \right) \text{ as } h \to 0.
\]

### 3.2 Analysis of the Truncation Errors

The \( \text{AMF}_p - \text{RK} \) method applied on problem (1.1) can be expressed in the simple one-step format \( y_{n+1} = \phi_f(t_n, y_n, \tau) \), \( n \geq 0 \). Thus, the time-space global errors

\[
\|\phi_f(t_n, y_n, \tau)\|_2 \leq \sum_{l=1}^{3} \frac{|a_l|}{2\bar{d}_l} \text{ as } h \to 0.
\]
\[ \epsilon_n = u_h(t_n) - y_n \] satisfy
\[ \epsilon_{n+1} := u_h(t_{n+1}) - \phi_f(t_n, y_n, \tau) \]
\[ = (u_h(t_{n+1}) - \phi_f(t_n, u_h(t_n), \tau)) + (\phi_f(t_n, u_h(t_n), \tau) - \phi_f(t_n, y_n, \tau)) \]
\[ = l(t_n, \tau, h) + [\partial \phi_f / \partial y]_n (u_h(t_n) - y_n), \]
where
\[ [\partial \phi_f / \partial y]_n = \int_0^1 \frac{\partial \phi_f}{\partial y} (t_n, u_h(t_n) + (\theta - 1)\epsilon_n, \tau) d\theta, \]
and the time-space local errors are defined by
\[ l_n \equiv l(t_n, \tau, h) := u_h(t_{n+1}) - \phi_f(t_n, u_h(t_n), \tau). \]
(3.7)

Then, we have for the time-space global errors \( \epsilon_n \) the recurrence
\[ \epsilon_{n+1} = [\partial \phi_f / \partial y]_n \cdot \epsilon_n + l_n, \quad n = 0, 1, 2, \ldots, t^*/\tau - 1. \]
(3.8)

In order to get a better understanding of the latter recurrence, we next introduce the following matrix operators (\( P \) is defined in (2.10))
\[ Q_\nu = (I \otimes I - T_\nu \otimes \tau P)^{-1}, \quad M_\nu = Q_\nu (A \otimes \tau J - T_\nu \otimes \tau P), \quad \nu \geq 1, \quad Q_0 = I. \]
(3.9)

**Lemma 1** The time-space global errors provided by the AMF\(_q\)-RK method when applied to the problem (3.2) satisfy the recurrence
\[ \epsilon_{n+1} = R_q(\tau J, \tau P) \cdot \epsilon_n + l_n, \quad n = 0, 1, 2, \ldots, t^*/\tau - 1, \]
(3.10)
where \( l_n \) stands for the time-space local error defined in (3.7) and
\[ R_q(\tau J, \tau P) = \infty I + (B^T \otimes I) \left( Q_q + \sum_{j=q}^1 \prod_{i=q}^j M_i Q_{j-1} \right) (e \otimes I), \]
(3.11)
with \( Q_\nu, M_\nu \) (\( \nu \geq 1 \)) given by (3.2). Moreover, the function \( R_q(\tau J, \tau P) \) fulfills
\[ R_q(\tau J, \tau P) - I = (B^T \otimes I) \left( Q_q + \sum_{j=q}^1 \prod_{i=q}^j M_i Q_{j-1} \right) - \prod_{i=q}^1 M_i) (c \otimes \tau J). \]
(3.12)

**Remark 1** It must be observed that commutativity does not hold in general, thus \( \prod_{j=q}^1 M_j \equiv M_q M_{q-1} \cdots M_1 \). On the other hand, \( R_q(\cdot) \) can be seen as the linear stability function of the method. The identity (3.12) for the function \( R_q(\cdot) - I \) will play a major role in a favorable propagation of the local errors in a similar way as indicated in Lemma 2.3 in [9, p.162].

**Proof of Lemma 1** Our first step is to analyze the operator \([\partial \phi_f / \partial y]_n\) for the semilinear problem (3.2). Taking into account that the method is defined
by (2.9), then we are led to compute \( \frac{\partial y_{n+1}}{\partial y_n} \) with \( y_{n+1} = \varpi y_n + (b^T \otimes)Y^q_n \). At this end, by taking derivatives with regard to \( y_n \) in the iteration (2.9), it holds that
\[
(I \otimes I - T_{\nu} \otimes \tau P) \left( \frac{\partial Y_{\nu}^n}{\partial y_n} - \frac{\partial Y_{\nu}^{n-1}}{\partial y_n} \right) = \frac{\partial D(t_n, \tau, y_n, Y_{\nu}^{n-1})}{\partial y_n}
= e \otimes I + (-I \otimes I + A \otimes \tau J) \frac{\partial Y_{\nu}^{n-1}}{\partial y_n}.
\]

From here, after some simple manipulations it follows that,
\[
\frac{\partial Y_{\nu}^n}{\partial y_n} = Q_{\nu}(e \otimes I) + M_{\nu} \frac{\partial Y_{\nu}^{n-1}}{\partial y_n}, \quad (\nu = 1, 2, \ldots, q),
\]
and we deduce both (3.11) and (3.10) from (3.8) and (3.14).

In order to prove (3.12), we first take into account that \( R_{\nu}(\cdot) = \varpi I + (b^T \otimes) \frac{\partial Y_{\nu}^n}{\partial y_n} \), where \( Z_{\nu} = \frac{\partial Y_{\nu}^n}{\partial y_n} - e \otimes I \). Then, from the recurrence (3.13), it follows after some simple calculations that \( Z_{\nu} = M_{\nu} Z_{\nu-1} + Q_{\nu}(c \otimes \tau J), \quad (\nu = 1, 2, \ldots, q), \)
with \( Z_0 = 0 \). From here, we deduce \( Z_{\nu} = \left( Q_{\nu} + \sum_{j=q}^{1} \left( \prod_{i=q}^{j} M_i \right) Q_{j-1} \right) (c \otimes \tau J), \)
and this directly gives (3.12).  

Remark 2 For a given rational function of two complex variables
\[
\zeta(z, w) = \sum_{i,j=0}^{m_1} \alpha_{ij} z^i w^j \sum_{i,j=0}^{m_2} \beta_{ij} z^i w^j 
= \left( \sum_{i,j=0}^{m_1} \alpha_{ij} z^i w^j \right) \left( \sum_{i,j=0}^{m_2} \beta_{ij} z^i w^j \right)^{-1},
\]
we define the associated mapping \( \zeta(Z, W) \) for two arbitrary commuting matrices \( Z \) and \( W \) just by replacing \( z \) by \( Z \) and \( w \) by \( W \) whenever the denominator yields a regular matrix. Sometimes we are given the rational mapping \( \zeta(Z, W) \) first and then we define the rational complex function just by replacing the matrices \( Z \) and \( W \) by the complex variables \( z \) and \( w \), respectively. The above definitions are straightforward extended to functions and mappings of more than two complex variables.
We will be mainly concerned with the case in which \( z = \tau J \) and \( w = \tau P \), where \( J \) and \( P \) are defined in (3.2) and (2.10), respectively. It should be noticed that for instance the \((i, j)\)-element of the matrix \( M_\nu \), see (3.9), would be given by
\[
M_{ij}(\tau J, \tau P) = (e_i^T \otimes I_s) (I_s \otimes I_m - T_\nu \otimes \tau P)^{-1} (A \otimes \tau J - T_\nu \otimes \tau P) (e_j \otimes I)
\]
where \( e_j \) denotes the \( j \)-vector of the canonical basis in \( \mathbb{R}^s \) and the corresponding complex function is
\[
M_{ij}(z, w) = e_i^T (I_s - w T_\nu)^{-1} (z A - w T_\nu) e_j.
\]

Another important point is that despite of we are considering cases with a \( d \)-splitting for \( J \) as indicated in (3.2), the replacement of every \( \tau J_j \) by the complex variable \( z_j \) and the definition of
\[
z := \sum_{k=1}^{d} z_k,
\]
\[
w := \gamma^{-1} \left( 1 - \prod_{k=1}^{d} (1 - \gamma z_k) \right),
\]
simplifies the study to the case of two complex variables \( z \) and \( w \) or well to the case of mappings acting on the two matrices \( \tau J \) and \( \tau P \).

It is worth to mention that our rational mappings and related complex functions are all well defined whenever \( \Re z_k \leq 0 \) for \( k = 1, 2, \ldots, d \) and \( d \) arbitrary, because the existence of the matrix inverse \((I - w T_\nu)^{-1}\) is guaranteed if and only if \((1 - \gamma w)^{-1} = \prod_{k=1}^{d} (1 - \gamma z_k)^{-1}\) exists. It is easily seen the existence of the late expression by virtue of \( \gamma > 0 \) and that all the eigenvalues of the matrices \( J_j, \ (j = 1, \ldots, d) \) have a non-positive real part. Moreover, for any \( \nu = 1, \ldots, q \) and any \( d \geq 1 \), we next prove that
\[
\sup_{\Re z_k \leq 0, \ k=1,\ldots,d} |Q_\nu(z, w)| < +\infty,
\]
\[
\sup_{\Re z_k \leq 0, \ k=1,\ldots,d} |M_\nu(z, w)| < +\infty,
\]
simplified in (3.16).

This see this, observe that \((T_\nu - \gamma I)^s = 0\) and that
\[
Q_\nu(z, w) = (I - w T_\nu)^{-1} = ((1 - w \gamma) I - w (T_\nu - \gamma I))^{-1}
\]
\[
= (1 - w \gamma)^{-1} \left( I - \frac{w}{1- w \gamma} (T_\nu - \gamma I) \right)^{-1} = (1 - w \gamma)^{-1} \sum_{j=0}^{s-1} \left( \frac{w}{1- w \gamma} \right)^j (T_\nu - \gamma I)^j
\]
and
\[
M_\nu(z, w) = Q_\nu(z, w) (z A - w T_\nu) = \frac{z}{1- w \gamma} \left( \sum_{j=0}^{s-1} \left( \frac{w}{1- w \gamma} \right)^j (T_\nu - \gamma I)^j \right) A
\]
\[
- \frac{w}{1- w \gamma} \left( \sum_{j=0}^{s-1} \left( \frac{w}{1- w \gamma} \right)^j (T_\nu - \gamma I)^j \right) T_\nu.
\]
Hence the boundedness of $Q_\nu(z, w)$ and $M_\nu(z, w)$ follows from the boundedness of

$$\left| \frac{1}{1-w'^\gamma} \right| = \left| \prod_{k=1}^{d} (1 - \gamma z'_k)^{-1} \right| \leq 1,$$

$$\left| \frac{w}{1-w'^\gamma} \right| = \gamma^{-1} \left| 1 - \frac{1}{1-w'^\gamma} \right| \leq \gamma^{-1} (1+1) = 2 \gamma^{-1},$$

and from the next lemma.

\begin{lemma}
For any $d = 2, 3, \ldots$, and $z$ and $w$ defined in (3.16), we have that

$$\sup_{\text{Re} z_k \leq 0} \left| \frac{z}{1-\gamma w} \right| = \gamma^{-1} \left| 1 - \frac{1}{1-\gamma w} \right| \leq \gamma^{-1} \left( \frac{(d-1)^{d-1}}{d^{d-2}} \right)^{1/2}.$$

\end{lemma}

\textbf{Proof.} The third equality below follows from the Maximum Modulus principle, which says that the Maximum Modulus is reached at the boundary of the open region for complex analytical functions,

$$\sup_{\text{Re} z_k \leq 0} \left| \frac{z}{1-\gamma w} \right| = \gamma^{-1} \sup_{\text{Re} z_k \leq 0} \left| \frac{\gamma z}{1-\gamma w} \right| = \gamma^{-1} \sup_{\text{Re} z_k \leq 0} \left| \frac{u_1 + u_2 + \ldots + u_d}{\prod_{k=1}^{d} (1-u_k)} \right|$$

$$= \gamma^{-1} \left( \prod_{k=1}^{d} \sqrt{1+(y'_k)^2} \right) \leq \gamma^{-1} \max_{x_k \geq 0} \left( \frac{(x_1 + x_2 + \ldots + x_d)^2}{\prod_{k=1}^{d} (1+(x_k)^2)} \right)^{1/2}.$$

The computation of the extrema by making zero the gradient of the real function of several variables $(x_1, \ldots, x_d)$ gives the maximum for $x_1 = x_2 = \ldots = x_d = (d-1)^{-1/2}$. The proof follows after substituting above this value.

\begin{definition}
A method of the form (2.9) is said to be $A(\theta)$-stable for a $d$-splitting, if and only if

$$|R_q(z, w)| \leq 1, \quad \forall z, w \text{ given by (3.16)} \text{ whenever } z_k \in \mathcal{W}(\theta), \ k = 1, 2, \ldots, d,$$

where (we consider that the argument of a non-null complex number ranges in $[-\pi, \pi]$)

$$\mathcal{W}(\theta) := \{ u \in \mathbb{C} : u = 0 \text{ or } |\text{arg}(-u)| \leq \theta \}.$$  \hspace{1cm} (3.18)

\end{definition}

3.3 Analysis of the Local Errors

Next, we study the time-space local errors $l_n$ given by (3.7). We will see that the time-space local error $l_n$ is composed of two terms, $l_n^{[2]}$ related to the predictor used in the AMF$^q$-RK method and $l_n^{[1]}$ related to the quadrature associated with the underlying Runge-Kutta method.
Lemma 3. If the linear system has continuous derivatives $u_h^{(k)}(t)$ up to order $p + 1$ in $[0,t^*]$ and the underlying RK method has stage order $\ell \geq 1$ ($\ell \leq p$), i.e.
\[ A c^{j-1} = j^{-1} c^j, \quad B^T c^{j-1} = j^{-1}, \quad j = 1, 2, \ldots, \ell. \]
Then, the local error $l_n$ in (3.7) of the AMF$_q$-RK method is given by
\[
l_n = l_n^{[1]} + l_n^{[2]},
\]
\[
l_n^{[1]} := (B^T \otimes I) \left( Q_q + \sum_{j=q}^{1} (\Pi_{r=1}^{j} M_i) Q_{j-1} - \Pi_{r=1}^{1} M_i \right) \hat{D}_n + \delta_n, \tag{3.19}
\]
\[
l_n^{[2]} := (B^T \otimes I)(\Pi_{r=1}^{1} M_i) \triangle u_h(t_n),
\]
where
\[
\triangle u_h(t_n) := (u_h(t_n + c_i \tau) - u_h(t_n))_{i=1}^{s} = \sum_{j=1}^{p} \frac{\tau^j}{j!} (c^j \otimes I) u_h^{(j)}(t_n) + \frac{\tau^{p+1}}{p!} \left( \int_{0}^{1} (1 - \theta)^p \phi(\theta) u_h^{(p+1)}(t_n + \theta \tau) d\theta \right)_{i=1}^{s}. \tag{3.20}
\]
and (we use, $(x)_+ := x$ if $x \geq 0$ and $(x)_+ := 0$ otherwise)
\[
\hat{D}_n = \sum_{j=\ell+1}^{p} \frac{\tau^j}{j!} \left( (c^j - jA c^{j-1}) \otimes u_h^{(j)}(t_n) \right) + \tau^{p+1} \int_{0}^{1} \left( \varphi(\theta) \otimes u_h^{(p+1)}(t_n + \theta \tau) \right) d\theta +
\]
\[
\tau (A \otimes I) \left( \sigma_h(t_n + c_i \tau) \right)_{i=1}^{s}, \quad \varphi(\theta) = \frac{1}{p!} \left( (c_i - \theta)_+^p - p \sum_{j=1}^{s} a_{ij}(c_j - \theta)_+^{p-1} \right)_{i=1}^{s},
\]
\[
\delta_n = \sum_{j=\ell+1}^{p} \frac{\tau^j}{j!} (1 - B^T c^j) u_h^{(j)}(t_n) + \tau^{p+1} \int_{0}^{1} \phi(\theta) u_h^{(p+1)}(t_n + \theta \tau) d\theta,
\]
\[
\phi(\theta) = \frac{1}{p!} \left( (1 - \theta)^p - \sum_{j=1}^{s} B_j (c_j - \theta)_+^p \right).
\tag{3.21}
\]

Proof. Let us define
\[
\hat{D}_n := (u_h(t_n + c_i \tau))_{i=1}^{s} - e \otimes u_h(t_n) - \tau (A \otimes I)(f_h(t_n + c_i \tau, u_h(t_n + c_i \tau))_{i=1}^{s}. \tag{3.22}
\]
From (1.3), it follows that
\[
\hat{D}_n = (u_h(t_n + c_i \tau))_{i=1}^{s} - (u_h(t_n))_{i=1}^{s} - \tau (A \otimes I)(u_h'(t_n + c_i \tau) - \sigma_h(t_n + c_i \tau))_{i=1}^{s}. \tag{3.23}
\]
Now, by using the Taylor expansion with integral remainder (below $\zeta(x)$ denotes a generic function having $r + 1$-continuous derivatives in an adequate interval)
\[
\zeta(t_n + x) = \sum_{l=0}^{r} \frac{x^l}{l!} \zeta^{(l)}(t_n) + \frac{x^{r+1}}{r!} \int_{0}^{1} (1 - \theta)^r \zeta^{(r+1)}(t_n + \theta x) d\theta, \tag{3.24}
\]

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and applying it conveniently to \( u_h(t_n + c_i\tau) \) and \( u_h(t_n + c_i\tau) \) in (3.23) with \( r = p \) and \( r = p - 1 \) respectively, we deduce after some computations, the expression for \( \hat{D}_n \) in (3.21). Observe that order stage \( \ell \) for the Runge-Kutta method implies that \( \beta^j - j Ac^j = 0, \beta^j T c^j - 1 = 0, j = 1, \ldots, \ell \). The expression for \( \delta_n \) is obtained in a similar way, but taking into account that this time we define,

\[
\delta_n := u_h(t_n + \tau) - \varpi u_h(t_n) - \sum_{j=1}^s \beta_j u_h(t_n + c_j\tau). \tag{3.25}
\]

Let us now take \( \hat{U}_n := (u_h(t_n + c_i\tau))_{i=1}^s \) and \( \Delta^\nu_n := \hat{U}_n - U^\nu_n \), where \( U^\nu_n \) are the iterates obtained by the scheme (2.9) when the predictor \( U^0_n = e \otimes u_h(t_n) \) is taken on the exact solution of the PDE at \( t_n \), i.e. \( y_n = u_h(t_n) \). This gives as solution, see (2.9)

\[
y_{n+1} = \varpi u_h(t_n) + (\beta^T \otimes I) U^0_n. \tag{3.26}
\]

From (3.25) and (3.26) it follows

\[
l_n = u_h(t_{n+1}) - y_{n+1} = (\beta^T \otimes I) \Delta^q_n + \delta_n. \tag{3.27}
\]

In order to compute \( \Delta^q_n \) we insert the expression for \( U^\nu_n \) in (2.9). It follows for the semi-linear problem (3.2) that

\[
(I \otimes I - T \otimes \tau P)(\Delta^\nu_n - \Delta^\nu_{n-1}) = -D(t_n, \tau, u_h(t_n), U^\nu_{n-1})
\]

\[
= -(I \otimes I - A \otimes \tau J) \Delta^\nu_{n-1} + \hat{D}_n, \quad (\nu = 1, 2, \ldots, q).
\]

This implies that \( \Delta^\nu_n = M^\nu \Delta^\nu_{n-1} + Q^\nu \hat{D}_n, 1 \leq \nu \leq q \), and from this recurrence

\[
\Delta^q_n = \left( Q_q + \sum_{j=q+1}^1 \prod_{i=q}^j M_i \right) \hat{D}_n + \prod_{i=q}^1 M_i \Delta^0_n,
\]

with \( \Delta^0_n = \Delta u_h(t_n) \) in (3.20). Now, from this expression and from (3.27) the formula (3.19) follows.

\[\square\]

**Theorem 1** Consider a family of matrices \( \{J_{k,h}\}_{k=1}^d \) and \( P_h, h \to 0^+ \), as given in (3.2) and (2.10), respectively. Assume that (3.4) holds and that

\[
\bigcup_{k=1}^d \operatorname{Spect}(J_{k,h}) \subseteq W(\theta), \quad (h \to 0^+) \tag{3.28}
\]

is fulfilled for some \( \theta \in [0, \pi/2] \). Let \( L(z, w) \) be a complex rational function satisfying

\[
\sup_{z_k \in W(\theta), k=1,2,\ldots,d} |L(z, w)| \leq 1, \quad z \text{ and } w \text{ given by (3.16)}.
\]

Then, we have that

\[
\|L(\tau J, \tau P)^n\| \leq C^*, \quad 0 \leq n\tau \leq t^*, \quad (\tau, h \to 0^+).
\]
\textbf{Proof.} For simplicity of notations, we omit the sub-index \( h \) in the matrices. By virtue of (3.2), (3.4) and (3.15) it follows that
\[
\| (L(\tau J, \tau P))^n \| = \| \Theta \cdot (L(\tau \Lambda, \tau \Upsilon))^n \cdot \Theta^{-1} \| \leq C \| (L(\tau \Lambda, \tau \Upsilon))^n \|, \quad n \geq 1,
\]
where
\[
\tau \Lambda := \sum_{k=1}^{d} \tau \Lambda_k, \quad \Lambda_k = \text{Block-Diag}(\Lambda_k^{(1)}, \Lambda_k^{(2)}, \ldots, \Lambda_k^{(q)}),
\]
\[
\tau \Upsilon := \gamma^{-1} \left( I - \prod_{k=1}^{d} (I - \gamma \tau \Lambda_k) \right).
\]
By defining \( \tau \Lambda^{(l)} := \sum_{k=1}^{d} \tau \Lambda_k^{(l)} \) and \( \tau \Upsilon^{(l)} := \gamma^{-1} \left( I - \prod_{k=1}^{d} (I - \gamma \tau \Lambda_k^{(l)}) \right) \), for the norm considered it follows that
\[
\| (L(\tau \Lambda, \tau \Upsilon))^n \| = \max_{l=1, \ldots, d} \| (L(\tau \Lambda^{(l)}, \tau \Upsilon^{(l)}))^n \|, \quad n \geq 1.
\]
Consider any diagonal block \( \Lambda_k^{(l)} = \lambda_k^{(l)} I + E \) \( (E \equiv E^{(l)} \) for simplicity of notation. Observe that all the matrices \( E \) are strictly lower triangular and they have uniform bounded entries and uniform bounded dimensions, hence all of them are nilpotent with nilpotency index \( \leq N \) and define
\[
z_k = \tau \lambda_k^{(l)}, \quad 1 \leq k \leq d, \quad z = \sum_{k=1}^{d} z_k, \quad w = \gamma^{-1} \left( 1 - \prod_{k=1}^{d} (1 - \gamma z_k) \right),
\]
and it follows that,
\[
L(\tau \Lambda^{(l)}, \tau \Upsilon^{(l)}) = L \left( \sum_{k=1}^{d} (z_k I + \tau E), \gamma^{-1} (I - \prod_{k=1}^{d} (I - \gamma (z_k I + \tau E)) \right).
\]
By defining the function of \( d \) complex variables,
\[
\psi(w_1, \ldots, w_d) := L \left( \sum_{k=1}^{d} w_k, \gamma^{-1} (1 - \prod_{k=1}^{d} (1 - \gamma w_k)) \right),
\]
we get that \( L(\tau \Lambda^{(l)}, \tau \Upsilon^{(l)}) = \psi(z_1 I + \tau E, \ldots, z_d I + \tau E) \). Then, by using the Taylor expansion for \( \psi \) around \( \tau = 0 \) and taking into the nilpotency of the matrix \( E \), we deduce that,
\[
\psi(z_1 I + \tau E, \ldots, z_d I + \tau E) = \psi(z_1, \ldots, z_d) I + \sum_{l=1}^{N-1} \frac{\tau^l}{l!} E^l \sum_{i_1+i_2+\ldots+i_d=l} \left( \frac{\partial \psi}{\partial^{i_1} z_1 \cdots \partial^{i_d} z_d} \right) (z_1, z_2, \ldots, z_d).
\]
Now, since \( L(z, w) \equiv L(z_1, \ldots, z_d) \) and all its partial derivatives up to order \( N \) are uniformly bounded on the wedge \( \mathcal{W}(\theta) \), we can write that
\[
\psi(z_1 I + \tau E, \ldots, z_d I + \tau E) = \psi(z_1, \ldots, z_d) I + \tau L_{\tau,h}^*, \quad \| L_{\tau,h}^* \| \leq C^*, \quad (\tau, h \to 0^+).
\]
From here we get for $0 \leq \tau n \leq t^*$ that
\[ \| (\psi(z_1 I + \tau E, \ldots, z_d I + \tau E))^n \| = \| (L(z, w) I + \tau L_{r,h})^n \| \leq (1 + \tau C^*)^n \leq \exp(t^* C^*). \]
\[ \Box \]

### 3.4 Some mappings and definitions

For a given mapping $\zeta(X, Y) \in \mathbb{C}^{m,m}$ where $X$ and $Y$ are two arbitrary square complex matrices of order $m$ we define some associated mappings in the following way,

\[ \zeta^{[1]}(X, Y) := (\zeta(X, Y) - \zeta(X, X)) (Y - X)^{-1}, \quad \text{whenever } \det(Y - X) \neq 0, \]
\[ \zeta^{[1]}(X, X) := \lim_{\epsilon \to 0} \zeta^{[1]}(X, X + \epsilon I), \quad \text{whenever the limit exists.} \tag{3.29} \]

In a recursive form, when $\det(Y - X) \neq 0$ and $\zeta^{[l]}(X, X)$ exists, we continue by defining

\[ \zeta^{[l+1]}(X, Y) := \left( \zeta^{[l]}(X, Y) - \zeta^{[l]}(X, X) \right) (Y - X)^{-1}, \]
\[ \zeta^{[l+1]}(X, X) := \lim_{\epsilon \to 0} \zeta^{[l+1]}(X, X + \epsilon I), \quad l = 1, 2, \ldots, l^*. \tag{3.30} \]

By assuming $\det(Y - X) \neq 0$ and the existence of $\zeta^{[l]}(X, X)$, $l = 1, 2, \ldots, l^*$, it is straightforward to show by induction that

\[ \zeta(X, Y) = \sum_{l=0}^{l^*} \zeta^{[l]}(X, X) (Y - X)^l + \zeta^{[l^*+1]}(X, Y) (Y - X)^{l^*+1}. \tag{3.31} \]

We have considered for convenience that $\zeta^{[0]}(X, Y) := \zeta(X, Y)$. It should be noted that the commutativity of the matrices $X$ and $Y$ is neither necessary in the definitions above nor in the formula (3.31).

To have a practical meaning of the mapping $\zeta^{[l]}(X, Y)$ we show next that assuming $\zeta(x, y)$ has $l^*$ continuous partial derivatives regarding the second variable, then it holds that

\[ \zeta^{[l]}(X, X) = \frac{1}{l!} \frac{\partial^l \zeta(x, y)}{\partial y^l}(X, X), \quad l = 1, 2, \ldots, l^*. \tag{3.32} \]

To see (3.32), we use the induction. For $l = 0$ it is true for convenience. For $l = 1$ it is true since

\[ \zeta^{[l]}(X, X) = \lim_{\epsilon \to 0} \zeta^{[l]}(X, X + \epsilon I) = \lim_{\epsilon \to 0} \epsilon^{-1} (\zeta(X, X + \epsilon I) - \zeta(X, X)) = \frac{\partial \zeta}{\partial y}(X, X). \]
Assume it is true up to \( l \), we show it for \( l + 1 \) by using (3.31) in the second equality and the induction in the third equality below. The L’Hospital formula for limits (for the indetermination 0/0) is used \( l + 1 \) times in the fourth equality,

\[
\zeta^{[l+1]}(X, X) = \lim_{\epsilon \to 0} \zeta^{[l+1]}(X, X + \epsilon I) = \lim_{\epsilon \to 0} \frac{\zeta(X, X + \epsilon I) - \sum_{j=0}^{l} \zeta^{[j]}(X, X)(\epsilon I)^j}{(\epsilon I)^{l+1}} = \lim_{\epsilon \to 0} \frac{\zeta(X, X + \epsilon I)}{(l + 1)!} \frac{\partial^{l+1} \zeta(x, y)}{\partial y^{l+1}}(X, X + \epsilon I) = \frac{1}{(l + 1)!} \frac{\partial^{l+1} \zeta(x, y)}{\partial y^{l+1}}(X, X)
\]

These results can be trivially extended to vectors (and matrices), namely \((\zeta_{ij}(X, Y)) \in \mathbb{C}^{q_1 m, q_2 m}\), by applying them to each component \( \zeta_{ij}(X, Y) \in \mathbb{C}^{m, m} \). Sometimes we will make use of this kind of vectors as we will see in the next section.

### 3.5 Bounds for the local errors

The forthcoming convergence results for AMF\(_q\)-RK methods are based in the Lemma II.2.3 \([9, \text{p. 162}]\), which can be stated as follows

**Lemma 4** Assume that the global errors \( \epsilon_n \equiv \epsilon_n(\tau; h) \), of a one-step method satisfy the recursion (3.10), where the local errors \( l_n \) can be split (uniformly on \( h \) and \( \tau \)) as

\[
l_n = (R_q(\tau J, \tau P) - I) \phi(t_n) \tau^n h^\alpha + \tau O(\tau^n h^\beta), \quad n = 0, 1, \ldots, t^*/\tau - 1, \ (3.33)
\]

where the function \( \phi(t) \) and its first derivative regarding \( t \) are uniformly bounded, then the stability condition

\[
\sup_{1 \leq n \leq t^*/\tau} \frac{\|R_q(\tau J, \tau P)^n\|}{\tau \to 0^+, h \to 0^+} \leq C,
\]  

implies that the global errors uniformly fulfil

\[
\epsilon_n = O(\tau^n h^\alpha) + O(\tau^n h^\beta), \quad n = 1, \ldots, t^*/\tau, \quad \tau \to 0^+, h \to 0^+. \quad (3.35)
\]

**General Assumptions on the semilinear problem.**

To bound the local errors and consequently the global errors we henceforth assume that the exact PDE solution \( u_h(t) \) confined to the spatial grid and the semilinear problem (3.23) fulfil (1.3)-(1.4), (3.4) and (3.28) for some \( \theta \in [0, \pi/2] \), and that the following hypotheses (related the matrices \( J \) and \( P \)) hold
for some constants (not necessarily positive) $\alpha_l$, $\beta_l$ and $\eta$ and some nonnegative integer $l^*$, whenever $h \to 0^+$ and $\tau \to 0^+$.

\begin{align}
\tag{P1}
\begin{cases}
(P - J)^l u_h^{(k)}(t) = \tau^l h^{\alpha_l} O(1), \\
(P - J)^{l+1} u_h^{(k)}(t) = \tau^{l+1} h^{\beta_l+1} J O(1)
\end{cases}
\end{align}

\begin{align}
\tag{P2}
J^0 u_h^{(k)}(t) = O(1), \quad k = 1, 2, \ldots, p + 1, \text{ for some } \eta.
\end{align}

It should be noticed that always $\alpha_0 = 0$, because the derivatives (up to some order) of the exact solution are uniformly bounded, i.e. $u_h^{(k)}(t) = O(1)$, $t \in [0, t^*]$, $k = 0, 1, \ldots, p + 1$.

**Theorem 2** Assume that the Runge-Kutta method has stage order $\ell$ and that

\begin{align}
\sup_{z_k \in W(\theta), k = 1, 2, \ldots, d} |z/(R_q(z, w) - 1)| \leq C, \quad z \text{ and } w \text{ given by (3.16).} \quad (3.37)
\end{align}

Then for the AMF$_q$-RK method we have that,

\begin{align}
\lim_{\tau \to 0} l_n^{[1]} = O(\tau^{r}) + O(\tau^{\ell+1}), \quad (\tau \to 0, h \to 0),
\end{align}

and

\begin{align}
l_n^{[1]} = \tau^{r} O(1) + \tau^{\ell+1} R_q(\tau J, \tau P) - I \left( O(1) + \tau h^{\beta_1} O(1) \right), \quad \tau \to 0, h \to 0.
\end{align}

**Proof.** According to Lemma 3 the term $l_n^{[1]}$ of the local error is given by,

\begin{align}
l_n^{[1]} = \xi(\tau J, \tau P) \hat{D} n + \delta_n, \quad (3.38)
\end{align}

where

\begin{align}
\xi(\tau J, \tau P) := (B^T \otimes I) \left[ Q_q(\tau J, \tau P) + \frac{1}{j-q} \left( \prod_{i=q}^{j} M_i(\tau J, \tau P) \right) Q_{j-1}(\tau J, \tau P) - \prod_{i=q}^{j-1} M_i(\tau J, \tau P) \right]
\end{align}

From Remark 2 we have that ($e_j$ denotes the $j$-vector of the canonical basis)

\begin{align}
\sup_{Re z_k \leq 0, k = 1, \ldots, d} |\xi(z, w)e_j| \leq C, \quad (j = 1, \ldots, s), \quad z, w \text{ given by (3.16).}
\end{align}

From Theorem 1 this implies that

\begin{align}
\max_{j=1, \ldots, s} \|\xi(\tau J, \tau P)(e_j \otimes I)\| \leq C', \quad \tau \to 0^+, \quad h \to 0^+.
\end{align}

Then, from (3.21) in Lemma 3 the first bound for $l_n^{[1]}$ follows.
For the second bound, we separate in (3.38) the $\tau^{\ell+1}$-term from the others, take into account (3.39) and Lemma 3, we get

$$l_n^{[1]} = \frac{\tau^{\ell+1}}{(\ell+1)!} \left( \xi(\tau J, \tau P) \left( (\ell+1) A c^{\ell} - (\ell+1) I \right) + (1 - B^T c^{\ell+1}) I \right) u_h^{(\ell+1)}(t_n) + \mathcal{R},$$

where \( \mathcal{R} = \mathcal{O}(\tau^{\ell+2}) + \mathcal{O}(\tau^r). \)

Next, we define the mapping (assume that \( J \) is regular only to simplify the proof)

$$v(\tau J, \tau P) := (R_q(\tau J, \tau P) - I)^{-1} \left( \xi(\tau J, \tau P) \left( (\ell+1) A c^{\ell} - (\ell+1) I \right) + (1 - B^T c^{\ell+1}) I \right).$$

By using the assumption (3.37), the bounds in Remark 2 and Lemma 2, it is not very difficult to see that

$$\sup_{z \in W(\theta)} |v(z, z)| < +\infty, \quad \sup_{z \in W(\theta)} |zv^{[1]}(z, w)| < +\infty, \quad z \text{ and } w \text{ given by (3.16).}$$

Then, from (3.40) it follows that,

$$l_n^{[1]} = \frac{\tau^{\ell+1}}{(\ell+1)!} (R(\tau J, \tau P) - I) v(\tau J, \tau P) u_h^{(\ell+1)}(t_n) + \mathcal{R}$$

$$= \frac{\tau^{\ell+1}}{(\ell+1)!} (R(\tau J, \tau P) - I) \left( v(\tau J, \tau J) + v^{[1]}(\tau J, \tau P) (\tau P - \tau J) \right) u_h^{(\ell+1)}(t_n) + \mathcal{R}$$

$$= \mathcal{R} + \frac{\tau^{\ell+1}}{(\ell+1)!} (R(\tau J, \tau P) - I) v(\tau J, \tau J) u_h^{(\ell+1)}(t_n)$$

$$+ \frac{\tau^{\ell+1}}{(\ell+1)!} (R(\tau J, \tau P) - I) v^{[1]}(\tau J, \tau P) (\tau J) (J^{-1}(P - J)) u_h^{(\ell+1)}(t_n)$$

$$= \mathcal{R} + \frac{\tau^{\ell+1}}{(\ell+1)!} (R(\tau J, \tau P) - I) \mathcal{O}(1) + \frac{\tau^{\ell+1}}{(\ell+1)!} (R(\tau J, \tau P) - I) \mathcal{O}(\tau h^{\beta_1}).$$

(3.43)

For the analysis of the local error term $l_n^{[2]}$ in (3.19), we define the mappings

$$\psi_q(\tau J, \tau X) := (\beta^T \otimes I) \prod_{j=q}^1 M_j(\tau J, \tau X) \in \mathbb{C}^{m,sm},$$

$$\zeta_q(\tau J, \tau X) := (R_q(\tau J, \tau X) - I)^{-1} \psi_q(\tau J, \tau X) \in \mathbb{C}^{m,sm},$$

and their associated vector complex functions

$$\psi_q(z, w) := \beta^T \prod_{j=q}^1 (I - wT_j)^{-1}(zA - wT_j) \in \mathbb{C}^{1,s},$$

$$\zeta_q(z, w) := (R_q(z, w) - 1)^{-1} \psi_q(z, w) \in \mathbb{C}^{1,s}.$$

(3.45)

These mappings will play a major role in the proof of the convergence results. It must be remarked that whereas $\|\psi_q(z, w)\|_2$ is uniformly bounded when $z$ and $w$ are given by (3.16), the vector $\zeta_q(z, w) = \mathcal{O}(z^{-1})$ as $z \to 0$ due to the...
fact that (see (3.12))

\[ R_q(z, w) - 1 = B^T \left( Q_q(z, w) + \sum_{j=q}^{n} (\prod_{i=q}^{j} M_i(z, w)) Q_{j-1}(z, w) - \prod_{i=q}^{n} M_i(z, w) \right) cz. \]  

(3.46)

Hence \( \zeta_q(z, w) \) is not bounded in general for \( z \) and \( w \) given by (3.16). However, \( \zeta_q(z, z) \) is uniformly bounded as long as \( R_q(z, z) - 1 \neq 0 \) for \( z \in W(\theta) \setminus \{0\} \).

From (3.19), by using (3.31), we deduce that,

\[
\begin{align*}
  l_n^{[2]} &= (R_q(\tau J, \tau P) - I) \sum_{j=0}^{l_n} \zeta_q^{[j]}(\tau J, \tau J)(I_s \otimes (\tau (P - J))^j) \Delta_n(t_n) \\
  &\quad + (R_q(\tau J, \tau P) - I) \zeta_q^{[l_n+1]}(\tau J, \tau P)(I_s \otimes (\tau (P - J))^{l_n+1}) \Delta_n(t_n).
\end{align*}
\]

(3.47)

Next, we provide some convergence results for different kind of AMFq-RK methods, which depends on the Runge-Kutta method on which the AMFq-RK is based on. We start with Theorem 3 that meets applications for DIRK methods (Diagonally Implicit Runge-Kutta) and SIRK methods (Single Implicit Runge-Kutta) and then with Theorems 4, 5 and 6 which meet applications in the AMFq-Rad methods presented in section two. Of course, the assumptions (P1)-(P2) will be always assumed for some integers \( l^* \geq 0, \ell \geq 1, p \geq 1 \).

**Theorem 3** If \( T_\nu = A, \nu = 1, \ldots, q \), with the Runge-Kutta coefficient matrix \( A \) having unique eigenvalue \( \gamma > 0 \) (with multiplicity \( s \)), then the local errors \( (l_n = l_n^{[1]} + l_n^{[2]} ) \) fulfill

\[
\begin{align*}
  l_n^{[1]} &= O(\tau h^\ell) + \tau^{\ell+1}(R(\tau J, \tau P) - I)(O(1) + O(\tau h^{\beta_1})), \\
  l_n^{[2]} &= \tau^{2l+2} h^{\beta_1} (R(\tau J, \tau P) - I) O(1), \quad l = 0, 1, \ldots, \bar{l}, \\
  \bar{l} &= \max\{0, \min\{q - 2, l^*\}\},
\end{align*}
\]

\( (\tau \to 0^+, h \to 0^+) \).

If the method is \( A(\theta) \)-stable for a d-splitting and (3.28) holds, then for any \( l = 0, 1, \ldots, \bar{l}, \) the global errors fulfill (whenever \( \tau \to 0^+ \) and \( h \to 0^+ \) ) that,

\[
\epsilon_{n,h} = O(h^r) + \tau^\ell \min\{1, \max\{\tau, \tau^{2} h^{\beta_1}\}\} O(1) + O(\tau^{2l+2} h^{\beta_1+1}); \quad n = 1, 2, \ldots, t^*/\tau.
\]

**Proof.** The expression of \( l_n^{[1]} \) was seen in Theorem 2. In order to show the expression for \( l_n^{[2]} \), we start by deducing from (3.44) and (3.12) that

\[
\begin{align*}
  \zeta_q(z, w) &= (R_q(z, w) - 1) - 1 B^T ((I - wA)^{-1} A)^q (z - w)^q, \\
  R_q(z, w) - 1 &= B^T ((I - wA)^{-1} A)^q (z - w)^q - I (zA - I) cz.
\end{align*}
\]

(3.48)
From (3.32) we have that 
\[ \zeta_{q}^{[l]}(z, z) = \frac{1}{l!} \frac{\partial^{l} \zeta_{q}}{\partial u^{l}}(z, z). \]
From here and from (3.48) it follows that
\[ \zeta_{q}^{[l]}(z, z) = 0, \quad l = 0, 1, \ldots, \tilde{l}. \]
From (3.47) by taking \( \tilde{l} \) as upper index, for any \( l = 0, 1, \ldots, \tilde{l} \), we have that
\[ l_{n}^{[2]} = (R_{q}(\tau J, \tau P) - I) \zeta_{q}^{[l_{n}^{[2]}]}(\tau J, \tau P)(I_{s} \otimes (\tau(P - J)))l^{[2]} \Delta_{h}(t_{n}) \]
\[ = \tau^{l}(R_{q}(\tau J, \tau P) - I) \left( \zeta_{q}^{[l]}(\tau J, \tau P)(I_{s} \otimes \tau J) \right) (I_{s} \otimes J^{-1}(P - J))l^{[2]} \Delta_{h}(t_{n}) \]
\[ = \tau^{l}(R(\tau, \tau J, \tau P) - I)O(1)(I_{s} \otimes J^{-1}(P - J))l^{[2]} + \tau^{2}O(1) \]
\[ = \tau^{2l + 2}h^{\beta_{l} + 1}(R(\tau J, \tau P) - I)O(1). \]

To see the bound for the global errors we apply Lemma 4. The bounds for the local errors \( l_{n}^{[2]} \) have been obtained above (see also Theorem 2 for \( l_{n}^{[3]} \)). The boundedness of the powers of \( R_{q}(\tau J, \tau P) \) as indicated in (3.34) follows from Theorem 1 by taking into account the \( A(\theta) \)-stability of the method for the \( \nu \)-splitting and that (3.28) holds. Now from Lemma 4 the proof is accomplished.

**Theorem 4** For AMFq-RK methods with \( \gamma > 0 \) and satisfying \( (A - T_{1})c = 0 \), we have that
\[ l_{n}^{[2]} = \tau^{2}(R(\tau J, \tau P) - I)(O(1) + h^{\beta_{l}}O(1)), \quad (\tau \to 0^{+}, \ h \to 0^{+}). \]

Additionally if the method is \( A(\theta) \)-stable for a \( d \)-splitting and (3.28) holds, then for \( \tau \to 0^{+} \) and \( h \to 0^{+} \), the global errors fulfill
\[ \epsilon_{n,h} = O(h^\gamma + \tau^\delta \min\{1, \max\{\tau, \tau^2 h^{\beta_{l}}\}\}O(1) + \tau^2(O(1) + h^{\beta_{l}}O(1)) ; \quad n = 1, 2, \ldots, t^{*}/\tau. \]

**Proof.** The expression of \( l_{n}^{[3]} \) was seen in Theorem 2. In order to show the expression for \( l_{n}^{[2]} \), from (3.47) by setting \( l^{*} = 0 \) we get that (observe that \( \zeta_{q}(z, z)c = 0 \) because \( (A - T_{1})c = 0 \). This expression is used in the third equality below)
\[ l_{n}^{[2]} = (R_{q}(\tau J, \tau P) - I)\zeta_{q}(\tau J, \tau J)\Delta_{h}(t_{n}) \]
\[ + (R_{q}(\tau J, \tau P) - I)\zeta_{q}^{[1]}(\tau J, \tau P)(I_{s} \otimes (\tau(P - J)))\Delta_{h}(t_{n}) \]
\[ = (R_{q}(\tau J, \tau P) - I)\zeta_{q}(\tau J, \tau J)(\tau c \otimes I + \tau^{2}O(1)) \]
\[ + (R_{q}(\tau J, \tau P) - I)\left( \zeta_{q}^{[1]}(\tau J, \tau P)(I \otimes \tau J) \right) (I_{s} \otimes (J^{-1}(P - J)))\tau O(1) \]
\[ = (R(\tau J, \tau P) - I)(\tau^{2}O(1)) + (R(\tau J, \tau P) - I)O(1) \left( \tau^{2}h^{\beta_{l}}O(1) \right). \]
This provides the bound for the local errors $l_n^{[2]}$. The boundedness of the powers of $R_q(\tau J, \tau P)$ as indicated in (3.34) follows from Theorem 1 by taking into account the $A(\theta)$-stability of the method for the $d$-splitting and that (3.28) holds. Now, from the bounds for the local error and from Lemma 4 the proof follows.

\textbf{Theorem 5} For AMF$_q$-RK methods with $\gamma > 0$ and satisfying

$$\sup_{\text{Re } z \leq 0, z \neq 0} \| z^{-\eta} \zeta_q(z, z) \|_2 < +\infty,$$

with $\eta$ given in (P2) we have that

$$l_n^{[2]} = (R_q(\tau J, \tau P) - I) \left( O(\tau^{1+\eta}) + O(\tau^2 h^{\beta_1}) \right), \quad (\tau \to 0^+, \ h \to 0^+).$$

Additionally if the method is $A(\theta)$-stable for a $d$-splitting and (3.28) holds, then for $\tau \to 0^+$ and $h \to 0^+$, the global errors fulfil

$$\epsilon_{n,h} = O(h^r) + \min\{1, \max\{\tau, \tau^2 h^{\beta_1}\}\} O(\tau^{\ell}) + O(\tau^{1+\eta}) + O(\tau^2 h^{\beta_1}); \ n = 1, 2, \ldots, t^*/\tau.$$ 

\textbf{Proof.} The expression of $l_n^{[1]}$ was seen in Theorem 2. In order to show the expression for $l_n^{[2]}$, from (3.47) by setting $l^* = 0$ we get that

$$l_n^{[2]} = (R_q(\tau J, \tau P) - I) \zeta_q(\tau J, \tau J) \Delta_h(t_n)$$

$$+ (R_q(\tau J, \tau P) - I) \zeta_q^{[1]}(\tau J, \tau P) (I_s \otimes (\tau(P - J))) \Delta_h(t_n)$$

$$= (R_q(\tau J, \tau P) - I) \zeta_q(\tau J, \tau J) \left( \tau O(1) \right)$$

$$+ (R_q(\tau J, \tau P) - I) \left( \zeta_q^{[1]}(\tau J, \tau P) (I_s \otimes (\tau(P - J))) \right) \left( I_s \otimes (J^{-1}(P - J)) \right) \left( \tau O(1) \right)$$

$$= (R_q(\tau J, \tau P) - I) \left( \zeta_q(\tau J, \tau J) (I \otimes (\tau(J)^{-\eta})) (I \otimes (\tau J)^{\eta}) \right) \left( \tau O(1) \right)$$

$$+ (R(\tau J, \tau P) - I) \left( \tau^2 h^{\beta_1} O(1) \right)$$

$$= R_q(\tau J, \tau P) - I) \left( O(1) \right) \left( \tau^{n+1} I \otimes J^n O(1) \right)$$

$$+ (R(\tau J, \tau P) - I) \left( \tau^2 h^{\beta_1} O(1) \right)$$

$$= (R(\tau J, \tau P) - I) \left( O(\tau^{1+\eta}) + O(\tau^2 h^{\beta_1}) \right).$$

This provides the bound for the local errors $l_n^{[2]}$. The rest of the proof follows as in the previous theorems. 

\textbf{Theorem 6} For AMF$_q$-RK methods with $\gamma > 0$ and

$$(A - T_1)c = 0, \ \sup_{\text{Re } z \leq 0, z \neq 0} \| z^{-\eta} \zeta_q(z, z) \|_2 < +\infty,$$
with \( \eta \) given in (P2) and assuming (P1) for \( l^* = 1 \), we have that

\[
\| l^2_n \| = (R(\tau, J, \tau P) - I) \left( \mathcal{O}(\tau^{2+\eta}) + \mathcal{O}(\tau^3 h^{\alpha_1}) + \mathcal{O}(\tau^4 h^{\beta_2}) \right), \ (\tau \to 0^+, \ h \to 0^+).
\]

Additionally if the method is \( A(\theta) \)-stable for a \( d \)-splitting and (3.28) holds, then the global errors fulfil

\[
\epsilon_{n,h} = \mathcal{O}(h^n) + \min\{1, \max\{\tau, \tau^2 h^{\beta_1}\}\} \mathcal{O}(\tau^\ell) + \mathcal{O}(\tau^{2+\eta}) + \mathcal{O}(\tau^3 h^{\alpha_1}) + \mathcal{O}(\tau^4 h^{\beta_2}),
\]

\( n = 1, 2, \ldots, t^*/\tau, \quad (\tau \to 0^+, \ h \to 0^+). \)

**Proof.** In order to show the expression for \( l^2_n \), from (3.47) by setting \( l^* = 1 \) we get that

\[
l^2_n = (R_q(\tau, J, \tau P) - I) \left( \zeta_q(\tau, J, \tau J) + \zeta_q^{[1]}(\tau J, \tau J)(I \otimes \tau(P - J)) \right) \Delta_h(t_n)
\]

\[
+ (R_q(\tau, J, \tau P) - I)\zeta_q^2(\tau, J, \tau P)(I_s \otimes \tau^2(P - J)^2)\Delta_h(t_n)
\]

\[
= (R_q(\tau, J, \tau P) - I) \left( \zeta_q(\tau, J, \tau J) + \zeta_q^{[1]}(\tau J, \tau J)(I \otimes \tau(P - J)) \right) ((\tau_c \otimes \tau)u_h(t_n) + \tau^2 \mathcal{O}(1))
\]

\[
+ (R_q(\tau, J, \tau P) - I)\zeta_q^2(\tau, J, \tau P)(I_s \otimes \tau^2(P - J)^2)(\tau \mathcal{O}(1))
\]

\[
= (R_q(\tau, J, \tau P) - I) \left( \tau^2 \zeta_q(\tau, J, \tau J)\mathcal{O}(1) + \tau\zeta_q^{[1]}(\tau J, \tau J)(I \otimes \tau(P - J)\mathcal{O}(1)) \right)
\]

\[
+ (R_q(\tau, J, \tau P) - I) \left( \zeta_q^{[2]}(\tau J, \tau P)(I_s \otimes \tau J^{-1}(P - J)^2) \mathcal{O}(1) \right)
\]

\[
= (R_q(\tau, J, \tau P) - I) \left( \tau^2 (\zeta_q(\tau, J, \tau J) - \eta)(\tau^\eta J^\eta\mathcal{O}(1) + \mathcal{O}(\tau^3 h^{\alpha_1})) \right)
\]

\[
+ (R_q(\tau, J, \tau P) - I) \left( \mathcal{O}(1) \tau^2 \tau J^{-1}(P - J)^2\mathcal{O}(1) \right)
\]

\[
= (R(\tau, J, \tau P) - I) \left( \mathcal{O}(\tau^{2+\eta}) + \mathcal{O}(\tau^3 h^{\alpha_1}) + \mathcal{O}(\tau^4 h^{\beta_2}) \right).
\]

This provides the bound for the local errors \( l^2_n \). The rest of the proof follows as in the previous theorems. \( \square \)

4 Application of the convergence results for Dirichlet Boundary Conditions in parabolic problems

Let us next consider the 2D semi-linear diffusion-reaction model (\( \varepsilon \) is a positive constant)

\[
u_t = \varepsilon(u_{xx} + u_{yy}) + g(x, y, t), \ (x, y) \in (0, 1)^2, \ t \in [0, 1], \ \varepsilon > 0, \quad (4.1)
\]

with prescribed Dirichlet boundary conditions and an initial condition. The PDE is discretized on uniform spatial meshes \((x_i, y_j) = (ih, jh), \ h = N^{-1}, 1 \leq n \leq N, 1 \leq m \leq M\).
on the grid a semi-discrete system (1.2) with dimension using second-order central differences, we obtain for the exact solution of (4.1) enough when \((x, y, t)\) variable. We shall assume that the exact solution of the PDE (4.1) is regular a row-wise ordering, where \(u_{i,j}(t) := u(x_i, y_j, t)\) for \(0 \leq i, j \leq N\). Then, by using second-order central differences, we obtain for the exact solution of (4.1) on the grid a semi-discrete system (1.2) with dimension \(m = (N - 1)^2\)

\[
u'_h(t) = \varepsilon J u_h(t) + g_h(t) + \sigma_h(t) + \varepsilon h^{-2} u_{\Gamma_h}(t),\]

where

\[
J := J_1 + J_2, \quad J_1 = I_{N-1} \otimes B_{N-1}, \quad J_2 = B_{N-1} \otimes I_{N-1},
\]

\[
B_{N-1} = h^{-2} \text{TriDiag}(1, -2, 1) \in \mathbb{R}^{(N-1) \times (N-1)}, \quad h = 1/N.
\]

Moreover, \(g_h(t) = (g(x_i, y_j, t))_{i,j=1}^{N-1}, \|\sigma_h(t)\|_{2,h} = \mathcal{O}(h^2)\) \((0 \leq t \leq t^*)\), whereas \(u_{\Gamma_h}(t)\) contains the values of the exact solution on the boundary, i.e.,

\[
u_{\Gamma_h}(t) = u_h^{(0,y)}(t) \otimes e_1 + u_h^{(1,y)}(t) \otimes e_{N-1} + e_1 \otimes u_h^{(x,0)}(t) + e_{N-1} \otimes u_h^{(x,1)}(t),
\]

with \(u_h^{(0,y)}(t) = (u_i(t))_{j=1}^{N-1}, \ u_h^{(1,y)}(t) = (u_{i,j}(t))_{j=1}^{N-1}, \ u_h^{(x,0)}(t) = (u_{i,0}(t))_{i=1}^{N-1}\) and \(u_h^{(x,1)}(t) = (u_{i,N}(t))_{i=1}^{N-1}\). Above, \(e_1, \ldots, e_{N-1}\) denotes the canonical basis in \(\mathbb{R}^{N-1}\).

For the proof of the convergence results we need the lemma 5 and the lemma 6 given below. These lemmas can be derived from the material in [9, pp. 96-300] (see from Lemma 6.1 to Lemma 6.5). Lemmas 5 and 6 supply sharp values for the constants \(\alpha_l, \beta_l\) and \(\eta\) appearing in the \(P\)-assumptions of section 3. These constants together with the convergence theorems provide specific orders of convergence of the MoL approach for several AMF_RK methods, in particular for the AMF_q-Rad methods presented in section 2.

The norm considered here for vectors, is the weighed Euclidean norm

\[
\|(v_{ij})_{i,j=1}^{N-1}\|_{2,h} := \sqrt{\frac{1}{N^2} \sum_{i,j=1}^{N-1} |v_{ij}|^2} = h\|(v_{ij})_{i,j=1}^{N-1}\|_2,
\]

and for matrices the corresponding operator norm.

**Lemma 5** Assume that exact solution \(u(x, y, t)\) of the 2D-PDE problem (4.1) has as many continuous partial derivatives as needed in the analysis in \((x, y, t) \in [0, 1]^2 \times [0, t^*]\). Then for \(k = 1, 2 \ldots\) and \(\omega < \frac{1}{4}\) we have that,

\[
\|J^\omega u_h^{(k)}(t)\|_{2,h} = \mathcal{O}(1), \quad \text{and moreover}
\]

\[
\|J^{1+\omega} u_h^{(k)}(t)\|_{2,h} = \mathcal{O}(1), \quad \text{whenever} \ u_{\Gamma_h}^{(1)}(t) \equiv 0.
\]
Lemma 6 Assume that exact solution \( u(x, y, t) \) of the 2D-PDE problem (4.1) has as many continuous partial derivatives as needed in the analysis in \((x, y, t) \in [0, 1]^2 \times [0, t^*]\). Then, for \( l = 0, 1, \ldots \) we have that,

\[
\left\| (P - J)^l u_h^{(k)}(t) \right\|_{2,h} = O(\tau^l h^{\alpha_l}), \quad \left\| J^{-1}(P - J)^l u_h^{(k)}(t) \right\|_{2,h} = O(\tau^l h^{\beta_l}),
\]

where

\[
\alpha_l = \begin{cases} 
- \max\{0, 3 + 4(l - 2)\}, & \text{if } u^{(1)}_{f_h}(t) \equiv 0, \\
- \max\{0, 3 + 4(l - 1)\}, & \text{otherwise},
\end{cases}
\]

and

\[
\beta_l = \begin{cases} 
- \max\{0, 1 + 4(l - 2)\}, & \text{if } u^{(1)}_{f_h}(t) \equiv 0, \\
- \max\{0, 1 + 4(l - 1)\}, & \text{otherwise}.
\end{cases}
\]

We next give a convergence theorem for 2D-parabolic PDEs when the MoL approach with AMF\(_q\)-Rad methods in section 2 are applied to the time discretization. The results still hold for 3D-parabolic problems (even for \( d \geq 3 \) and Time-Independent Dirichlet boundary conditions, but the proof requires some extra length to be included here.

Theorem 7 The global errors (GE) in the weighted Euclidean norm of the MoL approach for the 2D-PDE (4.1) when the spatial semi-discretization is carried out with second order central differences and the time integration is performed with AMF\(_q\)-RK methods, are given in Table 1. There, \( \bar{o} = \min\{1, \tau^2 h^{-1}\} \) and \( O(\tau^{2.25}) \) is meant for \( O(\tau^\mu) \) where \( \mu < 2.25 \) is any constant.

| \( (\tau \to 0^+, h \to 0^+) \) | GE (Time-Indep.) | GE (Time-Dep.) |
|---------------------------------|----------------|----------------|
| AMF\(_1\)-Rad | \( O(h^2) + O(\tau^2) \) | \( O(h^2) + O(\bar{o}) \) |
| AMF\(_2\)-Rad | \( O(h^2) + O(\tau^3) + \tau^2 O(\bar{o}) \) | \( O(h^2) + O(\bar{o}) \) |
| AMF\(_3\)-Rad | \( O(h^2) + O(\tau^{2.25}) \) | \( O(h^2) + O(\bar{o}) \) |

Table 1
Global error estimates in the weighted Euclidean norm for Time-Dependent Dirichlet boundary conditions (in short Time-Dep.) and Time-Independent Dirichlet boundary conditions (in short Time-Indep.).

Proof. In all cases we have that the stage order of the underlying Runge-Kutta Radau IIA method is \( \ell = 2 \) and the order of the spatial semi-discretization is \( r = 2 \). Moreover, all the three methods AMF\(_q\)-Rad \((q = 1, 2, 3)\) are A(\( \pi/2 \))-stable for a 2-splitting as it is shown in [3] for the cases \( q = 1 \) and \( q = 2 \) and in [7] for the case \( q = 3 \). Also, it should be noticed that (3.28) holds.

We start with the AMF\(_1\)-Rad method. We have for the case of Time-Independent Dirichlet Boundary conditions that the derivative regarding \( t \) vanishes on
boundary points \((x, y) \in \Gamma_h\), i.e. \(u^{(1)}_{\Gamma_h}(t) \equiv 0\). From Lemma 6 we get that \(\alpha_1 = 0\) and \(\beta_1 = 0\). Then the bound for the global errors follows from Theorem 4. For the case of Time-Dependent Dirichlet Boundary conditions, from Lemma 6, we have that \(\alpha_1 = -3\) and \(\beta_1 = -1\). Then, the bound for the global errors follows from Theorem 4. The bound also applies to the AMF_2-Rad method for Time-Dependent Dirichlet BCs, because this method fulfills the assumptions in Theorem 4.

For the case of the AMF_2-Rad method and Time-Independent Dirichlet BCs we apply Theorem 4 for the case \(\varrho = 1\) and Theorem 6 with \(l^* = 1\) for the case \(\varrho = \tau^2 h^{-1}\). Observe that from Lemma 6 we have that \(\alpha_1 = 0\) and \(\beta_1 = 0\) and \(\beta_2 = -1\). Moreover the AMF_2-Rad method fulfills all the assumptions in Theorem 4 by taking \(\eta = 1\), see also Lemma 5.

For the case of the AMF_3-Rad method and Time-Independent Dirichlet BCs we apply Theorem 5 with any \(\eta < 1.25\), see Lemma 5. Observe that in this case \(\alpha_1 = 0,\ \beta_1 = 0\). Then from Theorem 5 the global errors are of size \(O(h^2) + O(\tau^2)\). The proof that the order can be increased up to \(O(h^2) + O(\tau^{2.25})\) requires some extra technical details that we have omitted for simplicity. The case of Time-Dependent Dirichlet BCs follows from Theorem 5 too, but in this case \(\beta_1 = -1\).

\[4.1\] Numerical Experiments

We have performed some numerical experiments on two 2D-PDE and 3D-PDE problems of parabolic type in order to illustrate the convergence results presented in former sections for the AMF_q-Rad methods.

1. **Problem 1** is the 2D-PDE problem (4.11) with diffusion parameter \(\varepsilon = 0.1\) and Dirichlet Boundary Conditions and an Initial Condition so that

\[
u(x, y, t) = 10x(1-x)y(1-y)e^t + \beta e^{2x-y-t}, \quad (4.8)\]

is the exact solution. The case \(\beta = 0\) provides Time-Independent Boundary conditions and no spatial error \((\sigma_h(t) \equiv 0,\) due to the polynomial nature of the exact solution). The case \(\beta = 1\) provides Time-Dependent boundary conditions and spatial discretizations errors of order two.

2. **Problem 2** is the 3D-PDE problem (4.9) with diffusion parameter \(\varepsilon = 0.1\)

\[
u_t(x, y, z, t) = \varepsilon \Delta u(x, y, z, t) + g(x, y, z, t), \quad t \in [0, 1], \quad x = (x, y, z) \in (0, 1)^3 \subset \mathbb{R}^3, \quad (4.9)\]
and Dirichlet Boundary Conditions and an Initial Condition so that
\[
u(x, y, t) = 64x(1 - x)y(1 - y)z(1 - z)e^t + \beta e^{2x-y-z-t}, \quad (4.10)\]
is the exact solution. Again, the case \(\beta = 0\) provides Time-Independent
Boundary conditions and no spatial error and the case \(\beta \neq 0\) provides
Time-Dependent boundary conditions and spatial discretizations errors
of order two.

On the end-point of the time interval \(t^* = 1\), in the weighted Euclidean norm
we have computed as specified in (4.11), the global errors \(\epsilon_2(h, \tau)\) (\(y_{\text{met}}(t^*)\)
denotes the numerical solution at \(t^*\) by the method considered), the number
of significant figures of the global errors \(\delta_2(h, \tau)\) and the estimated order of
the global errors \(p(h, \tau)\) as powers of \(h\) when \(r = \tau/h\) is kept constant and
both \(\tau\) and \(h\) tend to zero.

\[
\begin{align*}
\epsilon_2(h, \tau) &:= \|u_h(t^*) - y_{\text{met}}(t^*)\|_{2,h}, \quad \delta_2(h, \tau) = -\log_{10} \epsilon_2(h, \tau) \\
p(h, \tau) &= (\delta_2(h/2, \tau/2) - \delta_2(h, \tau))/\log_{10} 2. \quad (4.11)
\end{align*}
\]

In the Tables 2, 3 and 4 we have considered for each \(h\) the time-steps size \(\tau = qh\)
for the corresponding \(\text{AMF}_q\)-Rad method (\(q = 1, 2, 3\)), so that all the methods
make use of the same number of \(f\)-evaluations and similar CPU times in
the computations. In those tables we have displayed the number of significant
figures in the global errors \(\delta_2(h, \tau)\) and in brackets the estimated orders \(p(h, \tau)\)
of each method.

From Theorem 7, the global errors are expected to be of size \(h^\mu\) (observe that
\(\tau/h\) is kept constant) where:

1. for the \(\text{AMF}_1\)-Rad method, \(\mu = 2\) if Time-Independent BCs are consid-
ered and \(\mu = 1\) if Time-Dependent BCs are imposed. This nicely fits with
the results displayed in Table 2 (Time-Independent BCs) and in Table 3
(Time-Dependent BCs) for the 2D-PDE problem. Moreover, the conver-
gence order is still \(\mu = 2\) in the 3D-PDE problem for Time-Independent
BCs as it can be seen in Table 4.

2. For the \(\text{AMF}_2\)-Rad method, \(\mu = 3\) if Time-Independent BCs are consid-
ered and \(\mu = 1\) if Time-Dependent BCs are imposed. This fits well with
the results displayed in Table 2 (Time-Independent BCs) and in Table 3
(Time-Dependent BCs) for the 2D-PDE problem. Moreover, the conver-
gence order is also \(\mu = 3\) in the 3D-PDE problem for Time-Independent
BCs as it can be observed in Table 4.

3. For the \(\text{AMF}_3\)-Rad method, \(\mu = 2.25\) if Time-Independent BCs are con-
sidered and \(\mu = 1\) if Time-Dependent BCs are imposed. This can be
observed in Table 2 (Time-Independent BCs) and in Table 3 (Time-
Dependent BCs) for the 2D-PDE problem. Moreover, the convergence
order also approaches to \( \mu = 2.3 \) in the 3D-PDE problem for Time-Independent BCs as shown in Table 4.

| \( h \) | \( \text{AMF}_1\)-Rad (p) \( \tau/h = 1 \) | \( \text{AMF}_2\)-Rad (p) \( \tau/h = 2 \) | \( \text{AMF}_3\)-Rad (p) \( \tau/h = 3 \) |
|------|----------------|----------------|----------------|
| \( \frac{1}{24} \) | \( \delta_2 = 3.74 \) (2.03) | \( \delta_2 = 4.94 \) (2.82) | \( \delta_2 = 4.90 \) (3.56) |
| \( \frac{1}{48} \) | \( \delta_2 = 4.35 \) (2.03) | \( \delta_2 = 5.79 \) (2.89) | \( \delta_2 = 5.67 \) (2.42) |
| \( \frac{1}{96} \) | \( \delta_2 = 4.96 \) (1.99) | \( \delta_2 = 6.66 \) (2.92) | \( \delta_2 = 6.40 \) (2.36) |
| \( \frac{1}{192} \) | \( \delta_2 = 5.56 \) (1.99) | \( \delta_2 = 7.54 \) (2.93) | \( \delta_2 = 7.11 \) (2.29) |
| \( \frac{1}{384} \) | \( \delta_2 = 6.16 \) (2.03) | \( \delta_2 = 8.42 \) (2.96) | \( \delta_2 = 7.80 \) (2.29) |
| \( \frac{1}{768} \) | \( \delta_2 = 6.77 \) (--) | \( \delta_2 = 9.31 \) (--) | \( \delta_2 = 8.49 \) (--) |

Table 2
Significant correct digits \( (l_{2,h},\text{-norm}) \) for the 2D-PDE problem with Time-Independent Dirichlet BCs \( (\beta = 0) \).

In brackets the estimated orders of convergence (by halving both the spatial resolution \( h \) and the time-stepizes \( \tau \) and taking ratio \( r = \tau/h \)).

As a conclusion we can say that the convergence results presented in Theorem 7 seem to be sharp for 2D-parabolic problems and that they still hold for \( d \)-D-parabolic problems \( (d > 2) \) when Time-Independent boundary conditions
are considered. The proof of this fact requires some additional work and is not presented here. On the other hand, the convergence results are very poor when Time-Dependent Boundary conditions are considered. However, in such a situation we have developed a very simple technique (Boundary Correction Technique) to recover the convergence order as if Time-Independent Boundary conditions were considered. The explanation of the Boundary Correction Technique and the proof of the convergence orders requires some extra length and will be the objective of another paper.

It is also important to remark that although we have considered in Theorem 7 second-order central differences for the spatial discretization, the convergence results also hold for most of the usual spatial discretizations as long as they are stable and consistent with order \( r \geq 1 \). Numerical experiments carried by the authors seem to indicate that the convergence results also hold for many classes of non-linear problems.

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