Abstract

Most work in mechanism design assumes that buyers are risk neutral, while some considers risk aversion arising due to a non-linear utility for money. Yet behavioral studies have established that real agents exhibit various risk attitudes, which, beyond violating risk-neutrality, cannot be captured by any expected utility model. We initiate the study of revenue-optimal mechanism design under behavioral models beyond expected utility theory. We adopt a model from prospect theory, which arose to explain these behavioral results. In our model, an event which occurs with probability \( x \leq 1 \) is worth strictly less than \( x \) times the value of the event when it occurs with certainty.

We specifically examine three questions. First, how does risk aversion affect the use of randomization within mechanism design? Second, in dynamic settings where the buyer’s lack of information about future values allows the seller to extract more revenue, how does risk aversion affect the ability of the seller to exploit the uncertainty of future events? Finally, is it possible to obtain approximation guarantees for revenue that are robust to the specific risk model?

We present three main results. First, we characterize optimal mechanisms in the single-shot setting as menus of binary lotteries. Second, we show that under a reasonable bounded-risk-aversion assumption, posted pricing obtains a constant risk-robust approximation to the optimal revenue. Third, in contrast to this positive result, we show that in dynamic settings it is not possible to achieve any constant factor approximation to revenue in a risk-robust manner.
1 Introduction

A large fraction of the algorithmic mechanism design literature is based on the assumption that agents have quasilinear utilities and are expected utility maximizers. This model is mathematically simple and allows for a number of beautiful characterizations of optimal and near-optimal mechanisms. However, it often does not hold up in practice. Behavioral experiments show that people often display a risk-averse or a risk-seeking attitude. For example, most people prefer getting $50 with certainty as opposed to $100 with probability 1/2. This sort of behavior can be explained within von Neumann-Morgenstern expected utility theory (EUT) [von Neumann and Morgenstern, 1944] via a non-linear utility for money. For instance, the above behavior arises if the agent’s utility for $100 is less than twice the utility for $50. In particular, in this model risk-averse behavior is equivalent to the utility being a concave function of value, and risk-seeking behavior corresponds to utility being a convex function of value.

Expected utility theory has successfully explained certain kinds of risk attitudes. Yet, experiments have uncovered behavior which it fails to capture.\(^1\) Prospect theory (PT) [Kahneman and Tversky, 1979, Tversky and Kahneman, 1992] explains these behaviors by hypothesizing that risk attitudes arise not only from a non-linear utility for money but also from agents’ perception of uncertain outcomes. For example, a buyer who is averse to uncertain outcomes may value obtaining $100 with probability 1/2 at less than half his value for $100. Thus, aversion to uncertainty provides an alternate explanation for the above example, with or without a concave utility for money.

In general, prospect theory incorporates both a non-linear utility for value and a non-linear weighting of probability. The buyer’s net utility from a random value is then the weighted expectation of the non-linear utility function applied to the value. We aim to isolate the effect of this non-linear weighting of probability on mechanism design, and we therefore consider a simpler special case in which utility is quasilinear but the buyer maximizes weighted expected utility.

In our model, if the agent gets a utility \(u\) with probability \(x\), his net utility from this uncertain outcome is given by \(u \times y(x)\). The weighting function \(y: [0, 1] \rightarrow [0, 1]\), with \(y(0) = 0\) and \(y(1) = 1\). It encodes how strongly the agent dislikes uncertainty. If \(y\) is convex, the agent displays risk-averse behavior—the more uncertain an event is, the more the agent discounts his value. On the other hand, concavity captures risk-seeking behavior. We focus on risk aversion and assume throughout the paper that the weighting function is convex.

More generally, we define the notion of weighted expectation, a.k.a. risk-averse expectation, of a random variable taking on many different values. Our definition has a clean mathematical formulation in terms of the random variable’s distribution. For example, just like the expectation of a non-negative random variable with c.d.f. \(F\) can be written as an integral over \(1 - F(x)\), the weighted expectation is an integral over \(y(1 - F(x))\). Our definition has a number of interesting

\(^1\)Consider, for example, the Allais paradox [Allais, 1953]. Subjects are asked to choose between two options: option A has a reward of $1M with certainty; option B rewards $1M with probability 89%, increases the reward to $5M with probability 10%, but rewards nothing with the remaining 1% probability. Most people prefer the first option of getting a high reward with certainty as opposed to getting nothing with a small probability even if the latter is counterbalanced by a much larger reward with some probability. Subjects are then asked to choose between two other options that are modifications of the first two: option C has a reward of $1M with 11% probability and nothing otherwise; option D rewards $5M with probability 10%, or nothing with the remaining 90% probability. Each of C and D are obtained from options A and B by reducing the probability of receiving the $1M reward by a fixed amount and replacing it with no reward. The paradox is that in the second case, most people choose option D over option C. No assignment of utilities to the two amounts can explain this “switch” in preferences across the two experiments. This paradox can be explained through prospect theory, including the special case that we study.
and appealing properties. For example, for a constant $c$ and random variable $X$, the weighted expectation of $c + X$ is just $c$ plus the weighted expectation of $X$. In other words, events that happen with certainty don’t affect the agent’s evaluation of uncertain events. On the other hand, if $X_1$ and $X_2$ are independent Bernoulli random variables, then the weighted expectation of $X_1 + X_2$ can be much larger (but no smaller) than the sum of the two weighted expectations—the sum is more concentrated than the random variables individually, and so its utility is discounted to a lesser extent.

The main goal of this paper is to understand how risk attitudes arising from aversion to uncertainty affect revenue-optimal mechanism design. We consider two aspects of this. First, randomization is an important tool in the design of optimal mechanisms. How does aversion to uncertainty affect the extent to which randomization helps? Second, in many mechanism design settings, the seller can exploit the buyer’s lack of knowledge about certain parameters (e.g., other agents’ values or the buyer’s own future value) to extract more revenue. Does uncertainty aversion help or hinder such revenue extraction?

**Single-shot mechanism design**

We begin our investigation with the simplest mechanism design setting, namely a monopolist selling a single item to a single buyer. When the buyer is risk-neutral, it is well known that the revenue-optimal mechanism is a deterministic mechanism, namely a posted price. Relative to a risk-neutral buyer, an uncertainty-averse buyer reacts to a deterministic posted price the same way, but undervalues mechanisms that employ randomness. Therefore one might expect that the optimal single-item mechanism for an uncertainty-averse buyer continues to be a posted price. In fact, the mechanism can exploit the buyer’s aversion to uncertainty to extract more revenue for certain outcomes. As the risk aversion grows, the seller can extract nearly all of the buyer’s value.

What do optimal mechanisms look like? We think of mechanisms as menus where each option corresponds to a (correlated) random allocation and random payment. We show that revenue-optimal menus are composed of binary lotteries, where each lottery sells the item at a certain price with some probability. In particular, it doesn’t help the mechanism to offer menu options with multiple random outcomes. Given this format, we can express the payment as a function of the allocation rule in the form of a payment identity. Unfortunately, this payment identity is not linear in the allocation rule, and so does not permit a closed form solution for the optimal mechanism.

The theory of optimal mechanism design is often criticized for being too detail-oriented, and consequently impractical. Designing optimal mechanisms in the settings we consider is even less practical and realistic in this regard—the seller not only needs to know the buyer’s value distribution, but also his weighting function exactly. Indeed it is difficult to imagine that a buyer can describe his own weighting function in any manner other than reacting to options presented to him. We therefore consider the problem of designing mechanisms that achieve revenue guarantees robust to the precise risk model.

Formally, given a family of weighting functions, we ask for a single mechanism that for every weighting function in the family is approximately optimal with respect to the revenue-optimal mechanism specific to that weighting function. While it appears challenging to obtain a constant-
factor risk-robust approximation\footnote{In Section 3.3 we give an $O(\log \log H)$ risk-robust approximation under certain assumptions, where $H$ is the ratio of the maximum to the minimum value in the buyer’s value distribution.} for arbitrary families of weighting functions, we obtain a simple approximation under a boundedness condition on the buyer’s risk profile. Specifically, we show that as long as there is some probability $x = 1 - \Theta(1)$ at which the buyer’s weight is $y(x) = \Theta(1)$, then Myerson’s optimal posted pricing mechanism achieves an $O(1)$ approximation to the optimal revenue. In other words, the only “bad” case for revenue maximization, where the optimal revenue is far greater than that achievable from a risk-neutral buyer, is when the buyer values any event with probability bounded away from 1 at a weight arbitrarily close to 0.

Dynamic mechanism design

Consider the following two-stage mechanism design problem. The seller has one item to sell in each stage. During the first stage, the buyer knows his value for the first item, but not for the second item. At this time, the seller may charge the buyer a higher or lower price for the first item in exchange for a better or worse second-stage mechanism respectively. In the second stage, the buyer’s value for the second item is realized. The seller follows through with his commitment and sells the second item according to the mechanism promised in the first stage. Several recent works have studied dynamic mechanism design settings with risk-neutral buyers.

The setting we consider is a special case of that introduced by Papadimitriou et al. [2016] and subsequently studied by Ashlagi et al. [2016] and Mirrokni et al. [2016]. Ashlagi et al. show that in the optimal mechanism the seller generally charges higher prices in the first stage in exchange for a higher expected utility promised to the buyer in the second stage. In fact in some cases it becomes possible to extract the smaller of the buyer’s expected values for the two items\footnote{To be precise, if the buyer’s values are denoted $v_1$ and $v_2$ respectively, these mechanisms can extract $E[\min(v_1, E_y[v_2])]$ plus the single-shot revenues in the two stages.}. Ashlagi et al.’s mechanisms have an unusual format, however. The mechanisms offer a menu to the buyer in which the buyer’s net utility from every menu option is zero (even accounting for future gains). Since the buyer is now indifferent between all of the menu options, he by default picks the most expensive one he can afford. Observe that the revenue guarantee of this mechanism is quite fragile with respect to the buyer’s risk attitude. Since different menu options have different amounts of uncertainty involved, if the buyer’s risk attitude changes, the options are no longer all equivalent and the mechanism loses a lot of revenue. Is a risk-robust approximation achievable in the dynamic setting?

We first show that the kinds of mechanisms and revenue guarantees that Ashlagi et al. obtain in the risk-neutral setting continue to hold in risk-averse settings, when the seller knows the buyer’s weighting function. We focus on the simple class of posted pricing mechanisms where each menu option offers the buyer a fixed price in each stage, but the second-stage price is a decreasing function of the first-stage price. Beyond the single-shot revenues in both stages, posted price mechanisms can obtain an additional revenue of $O(E[\min(v_1, E_y[v_2])])$, where $E_y[v_2]$ is the risk-averse or weighted expectation of $v_2$, but not much more.

We then explore whether it is possible to extract a constant fraction of $E[\min(v_1, E_y[v_2])]$ via posted price mechanisms in a risk-robust manner. We construct a family of value distributions and a family of weighting functions such that the buyer’s risk-averse expectation of his second-stage value exhibits a large range under different weighting functions, although all of the weighting functions satisfy the boundedness condition discussed previously. We then show that for any constant $\alpha > 1$, $E[\min(v_1, E_y[v_2])]$
there exists a value distribution in the family such that no posted pricing mechanism can obtain an α risk-robust approximation to revenue with respect to all of the weighting functions we consider.

The high level take-away from these results is that in settings where the seller seeks to exploit the buyer’s lack of information and the inherent uncertainty in future outcomes for gains in revenue, the seller needs to have precise information about the buyer’s attitude towards risk, without which such revenue extraction is not possible.

A summary of our results

Our main results can be summarized as follows.

• Optimal mechanisms in the single-shot setting are menus of binary lotteries. See Theorem 1.

• In the single-shot setting, optimal mechanisms can obtain much more revenue than Myerson’s mechanism; however, under a natural condition bounding the extent of risk-aversion, Myerson’s mechanism extracts a constant fraction of the optimal revenue. See Theorems 2 and 4.

• For the dynamic two-stage setting, we present an upper bound as well as a 2-approximation to the revenue achievable using posted-price mechanisms. See Theorems 5 and 6.

• We exhibit an example that shows that it is impossible to obtain any constant factor risk-robust approximation to revenue in the two-stage setting, even if the buyer’s risk aversion is bounded. See Section 4.2.

Related work

Although empirically successful, prospect theory as defined by Kahneman and Tversky [1979] suffers from a number of weaknesses, rectified in a series of works subsumed by the cumulative prospect theory\textsuperscript{5} of Tversky and Kahneman [1992]. Our model is readily seen to be a special case of Kothiyal et al. [2011]’s extension of cumulative prospect theory to continuous values. See [Machina, 1987] for a survey of non-EUT theories.

Despite the success of prospect theory, expected utility theory remains the standard in mechanism design, where a large body of work studies revenue-optimal mechanism design. For example, Hu et al. [2010] compare different auction formats. More relevant, Matthews [1983] and Maskin and Riley [1984] characterize the optimal mechanism under certain expected-utility models; our Theorems 1 and 2 (characterization of the optimal mechanism and full welfare extraction under extreme risk aversion, respectively) have close analogues in their work. Unsurprisingly, these characterizations are complex and work in limited settings.

Non-EUT models have thus far attracted less attention in the mechanism design community. Fiat and Papadimitriou [2010] study existence and computation of equilibria in a variety of models. Easley and Ghosh [2015] explore the implications of a realistic, prospect-theoretic behavioral model in contract design. To our knowledge, we are the first to consider revenue maximization within any non-EUT model.

Much recent work in algorithmic mechanism design has explored revenue guarantees that are robust to finer details of the model. To our knowledge, however, only three works consider robustness to the buyer’s risk attitude. Dughmi and Peres [2012] show that truthful-in-expectation

\textsuperscript{5}In modern usage, “prospect theory” is understood to mean this improved theory and its extensions.
mechanisms can be implemented almost as-is in a manner robust to risk attitudes. Fu et al. [2013] and Chawla et al. [2016] provide risk-robust revenue guarantees in (different) stylized settings; Fu et al.’s mechanism is additionally independent of the buyers’ value distributions. Their techniques are unrelated to ours.

2 A model for risk aversion

The premise of prospect theory is that the agent fundamentally misvalues uncertain events. While prospect theory allows for both risk-averse and risk-seeking attitudes, we consider only risk-averse buyers. Thus, if the agent gains value \( v \) with probability \( x \), his risk-averse utility from this uncertain event is \( y(x)v \) where \( y(x) \leq x \). The function \( y \) is called an uncertainty weighting function. Prospect theory requires that the weighting function satisfy the following properties: (1) \( y : [0, 1] \to [0, 1] \), and (2) \( y(0) = 0 \) and \( y(1) = 1 \). Because we are interested in risk-averse behavior, we additionally assume (3) \( y \) is weakly increasing and convex.

Given such a weighting function \( y \), we next describe how to compute the risk-averse expectation, \( \mathbb{E}_y \), of a random variable \( V \) that denotes the (uncertain/random) value that an agent gains. Our definition is designed to satisfy the following additivity axiom:

For any constant \( c \), \( \mathbb{E}_y[c + V] = c + \mathbb{E}_y[V] \).

The additivity axiom implies, in particular, that the utility of an agent from participating in a mechanism does not depend on the wealth of the agent, but rather only depends on how much the agent gains or loses in the mechanism. Accordingly, we express the risk-averse expectation in the form of increments from a base value. Suppose, for example, that an agent gains a value of $1 with probability 1/3 and $2 with probability 1/3. Then we observe that the agent gets an increment of $1 over his base value with probability 1/3 + 1/3 and a further increment of $(2-1) with probability 1/3. So, the risk-averse expectation of his value, a.k.a. his risk-averse utility, is \( 1 \times y(2/3) + 1 \times y(1/3) \). Formally, the risk-averse expectation of a non-negative random variable is defined as follows.

**Definition 1.** Let \( Z \) be a random variable supported over \([0, \infty)\) with c.d.f. \( F \). Then the risk-averse expectation of the random variable with respect to weighting function \( y \) is

\[
\mathbb{E}_y[Z] = \int_0^\infty y(1 - F(z))dz.
\]

The weighting function for losses

When the random variable \( Z \) takes on negative values, we need to take extra care in defining its risk-averse expectation. Once again, following the additivity axiom stated above, for \( L = \inf\{z : F(z) > 0\} \), we define

\[
\mathbb{E}_y[Z] = L + \int_L^\infty y(1 - F(z))dz.
\]

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6Prospect theory also allows nonlinear utility for value, and is thus a strict generalization of expected utility theory; we focus on inherent aversion to uncertainty and therefore assume a linear utility function.
It is convenient to express the contributions of the negative values that $Z$ takes to the risk-averse expectation in the form of decrements from the base value of 0. Accordingly, we obtain the following\footnote{It is well known that the quantile of a draw from a distribution is itself distributed uniformly between 0 and 1. We note that our model can be interpreted as sampling the distribution by drawing a quantile according to a non-uniform distribution supported on $[0, 1]$, although we do not make use of this observation.} equivalent definition:

**Definition 2.** Let $Z$ be a random variable supported over $(-\infty, \infty)$ with c.d.f. $F$. Then the risk-averse expectation of the random variable with respect to weighting function $y$ is

$$E_y[Z] = -\int_{-\infty}^{0} (1 - y(1 - F(z)))dz + \int_{0}^{\infty} y(1 - F(z))dz.$$ 

See Figure 1.

![Figure 1: The application of a weighting function $y$ to a c.d.f. $F(z)$. Left: The expected value of a random variable $Z$ drawn from $F$ is equal to the difference between the shaded areas: area above the curve on the positive axis adds to the expectation, and area below the curve on the negative axis subtracts. Right: After applying the weighting function $y$ as in Definition 2, the positive area decreases, whereas the negative area increases. The original c.d.f. is shown as a dotted line for comparison.](image)

Some examples

**Example 1.** If $y(x) = x$ for all $x \in [0, 1]$, then the risk-averse expectation is exactly the expectation of the distribution. I.e., the agent is risk-neutral.

**Example 2.** If the distribution of $Z$ is a point mass at $z$, its risk-averse expectation is exactly $z$.

**Example 3.** If the distribution of $Z$ is symmetric around 0, that is, for all $z > 0$, $F(-z) = 1 - F(z)$, then $E[Z] = 0$. The risk-averse expectation, on the other hand, is non-positive: \( \int_{0}^{\infty} (y(1 - F(z))) + y(1 - F(-z)) - 1)dz = \int_{0}^{\infty}(y(1 - F(z)) + y(F(z)) - 1)dz \leq \int_{0}^{\infty}(1 - F(z) + F(z) - 1)dz = 0. \)

Quantifying risk aversion

The buyer’s risk attitude in our model is described by a function rather than by one (or a few) parameters. While this leads to a rich set of behaviors, it makes it challenging to understand, for example, whether one weighting function is more risk-averse than another. We argue that a natural
measure for the extent of risk aversion is the gap between the function and the $x$ and $y$ axes. In other words, if the function “hugs” the axes $x = 0$ and $y = 1$, then the buyer is highly risk-averse, whereas if the function is bounded away from the axes, then the buyer is less risk-averse. See Figure 2.

![Figure 2: A $\beta$-bounded weighting function: the area of the shaded rectangle is $\beta$, which is the maximal area of any rectangle contained under the curve. This area gives a measure of the aversion to uncertainty: a smaller $\beta$ corresponds to stronger aversion to risk.](image)

Figure 2: A $\beta$-bounded weighting function: the area of the shaded rectangle is $\beta$, which is the maximal area of any rectangle contained under the curve. This area gives a measure of the aversion to uncertainty: a smaller $\beta$ corresponds to stronger aversion to risk.

**Definition 3.** A weighting function is $\beta$-bounded if there exists an $x \in (0, 1)$ such that $y(x)(1-x) \geq \beta$, in other words, we can fit a rectangle with area $\beta$ under the $y$ curve.

We will see in Section 3.3 that $\beta$-boundedness affects the revenue approximation achievable for the given weighting function.

### 3 Single-shot revenue maximization

We begin by considering the “single-shot” setting in which the seller wants to sell a single item to the buyer. The buyer’s value $v$ for the item is drawn from known distribution $F$. When the buyer is risk-neutral, it is well known that the optimal mechanism is deterministic, and in particular is a posted-price mechanism. Observe that a risk-averse buyer obtains the same utility from a posted-price as a risk-neutral one, so over the class of posted-price mechanisms, the optimal one remains the same regardless of the buyer’s risk attitude. We will refer to the optimal posted price as the Myerson price, and to the corresponding mechanism as Myerson’s mechanism. We denote the revenue obtained by Myerson’s mechanism as

$$\text{MYE}(F) = \max_p \{p(1 - F(p))\}.$$ 

Given that a risk-averse buyer derives less utility from a randomized outcome than a risk-neutral buyer does, one might conclude that the optimal mechanism for a risk-averse buyer continues to be deterministic. However, it is interesting to note that, as our next example shows, this is not the case. The seller exploits the buyer’s risk aversion to price higher allocations superlinearly.
Example 4. Suppose \( v \sim U[0,1] \) and \( y(x) = x^2 \). The optimal deterministic mechanism offers a price of \( 1/2 \) and earns revenue of \( 1/4 \) in expectation.

Suppose we offer an additional option: the binary lottery \((1/2, 3/8)\) allocates the item to the buyer with probability \( 1/2 \) and if the item is allocated, charges the price \( 3/8 \).

The buyer chooses the deterministic option if \( (v - 1/2) \geq (v - 3/8)(1/2)^2 \); that is, if \( v \geq 13/24 \).

Otherwise, he chooses the second option as long as \( v \geq 3/8 \). The expected revenue is therefore

\[
\frac{1}{2} \left( 1 - \frac{13}{24} \right) \left( \frac{1}{2} \right) + \frac{3}{8} \left( \frac{13}{24} - \frac{3}{8} \right) \left( \frac{1}{2} \right) = \frac{25}{96} > \frac{1}{4}.
\]

3.1 Incentive-Compatible Mechanisms

A single-shot mechanism can be described by the allocation it makes and the prices it charges to the buyer as a function of the buyer’s value. Let \( X(v) \) and \( P(v) \) denote these functions, with \( X(v) \in \{0,1\} \) and \( P(v) \in \mathbb{R}_{\geq 0} \). Observe that \( X(v) \) and \( P(v) \) are random variables and may be correlated. Then, the buyer’s risk-averse utility from the mechanism’s outcome is given by \( \mathbf{E}_y[vX(v) - P(v)] \).

We say that a mechanism with allocation and pricing functions \((X, P)\) is incentive-compatible (IC) for a buyer with weighting function \( y \) if for all possible values \( v, v' \) of the buyer, it holds that

\[
\mathbf{E}_y[vX(v) - P(v)] \geq \mathbf{E}_y[vX(v') - P(v')] .
\]

It is without loss of generality to express an incentive-compatible mechanism in the form of a menu, \( \mathcal{M} \), with each menu option, a.k.a. lottery, corresponding to a particular random (allocation, payment) pair, \((X, P)\). Then, the allocation and payment of a buyer with value \( v \) and weighting function \( y \) is given by the utility-maximizing menu option:

\[
(X_y(v), P_y(v)) = \arg\max_{(X, P) \in \mathcal{M}} \mathbf{E}_y[vX(v) - P(v)].
\]

The revenue of the mechanism is correspondingly given by

\[
\text{Rev}_y, F(\mathcal{M}) = \mathbf{E}_{v \sim F}[\mathbf{E}_y[P_y(v)]]
\]

where \( F \) denotes the distribution from which \( v \) is drawn.

Let \( \text{OPT}(y, F) \) denote the optimal revenue achievable by an incentive-compatible mechanism from selling an item to a buyer with weighting function \( y \) and value drawn from \( F \):

\[
\text{OPT}(y, F) = \max_{\mathcal{M}} \text{Rev}_y, F(\mathcal{M}).
\]

We will drop the argument/subscript \( y \) from the above definitions when it is clear from the context.

3.1.1 Buyer’s utility function and binary lotteries

Lotteries or menu options in a mechanism can have many different outcomes—they can charge a random price when the item is not allocated, and a different random price when the item is allocated. When a buyer purchases such a lottery, his risk-averse utility as a function of his value depends on which of the outcomes of the lottery bring him negative utility and which ones bring positive utility.

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\(^8\)We assume that any ties are broken in favor of menu options with a higher expected price.
Example 5. Consider, for example, a lottery with three outcomes, and a buyer with weighting function \( y(x) = x^2 \). With probability 1/2, \( X = P = 0 \); With probability 1/4, \( X = 1 \) and \( P = 1 \); with probability 1/4, \( X = 1 \) and \( P = 2 \). The risk-averse utility of this lottery is plotted on the left in Figure 3. A buyer with value \( v \in [0,1) \) gets a negative utility from both of the second and third options. His risk-averse utility is given by

\[
-2 + (v) y(1) + (1) y(3/4) + (1 - v) y(1/2) = \frac{3}{4} v - \frac{19}{16}.
\]

A buyer with value \( v \in [1,2) \) gets a negative utility from the third option alone. His risk-averse utility is given by

\[
-2 + (v) y(1) + (2 - v) y(3/4) + (v - 1) y(1/4) = \frac{1}{2} v - \frac{15}{16}.
\]

Finally, a buyer with value \( v \geq 2 \) gets positive utility from all of the options. His risk-averse utility is given by

\[
(v - 2) y(1/2) + (1) y(1/4) = \frac{1}{4} v - \frac{7}{16}.
\]

In general, the utility \( u_y(v, (X, P)) \) that a buyer derives from a lottery \((X, P)\) is a concave function. See Appendix A for a proof.

Lemma 1. For any profile \( y \) and lottery \((X, P)\), \( u_y(v, (X, P)) \) is a concave function of \( v \). The slope of this function lies between \( 1 - y(1 - x) \) and \( y(x) \), where \( x = \Pr[X = 1] \).

A particularly simple form of a lottery is one which has a binary outcome: with some probability the buyer is allocated the item and charged a deterministic price; with the remaining probability, both the allocation and price are 0. We denote such a binary lottery by the pair \((x, p)\) where \( x \in [0,1] \) is the probability of allocation and \( p \) is the price charged upon allocation. The buyer’s utility function for a binary lottery has a convenient form—it is linear for \( v \geq p \). In particular, when \( v > p \), \( u_y(v, (x, p)) = y(x)(v - p) \). See Figure 3.
3.2 Optimal Risk-Averse Mechanisms

When the buyer is risk-neutral, the utility that the buyer receives as a function of his value in any IC mechanism is a concave function. This property allows for convenient analysis of optimal mechanisms. For a risk-averse buyer, this is no longer necessarily true. For example, if the mechanism offers a single lottery with more than two outcomes, as in Example 5, the buyer’s utility function is concave. In general, the buyer’s utility function is the supremum over concave functions.

We now study properties of revenue optimal mechanisms for a single buyer. We will show that revenue optimal mechanisms can be described by a menu composed of binary lotteries, and always induce a convex utility curve. We also observe that payment of such a mechanism is explicitly determined by the allocation, and that optimal revenue weakly increases with risk aversion.

**Theorem 1.** For any revenue-optimal IC mechanism \((X, P)\) in the single-shot setting, the buyer’s utility function \(u_y(v, (X_v, P_v))\) is convex and nondecreasing. Furthermore, there exists an optimal ex-post IR mechanism that can be described as a menu of binary lotteries.

The proof of this theorem is deferred to Appendix A. At a high-level, our proof proceeds as follows. We start with an arbitrary IC mechanism, and consider the buyer’s utility function induced by this mechanism. We then take the lower convex envelope of this function. We show that every point on this curve can be supported by the utility curve of a binary lottery which has an expected payment as least that of the menu option it replaces in the original mechanism. At points where the lower convex envelope is strictly below the original utility curve, the new mechanism obtains strictly more revenue. See Figure 4 in the appendix.

**Payment Identity.** Consider a mechanism that offers the buyer a menu of binary lotteries. The utility of a menu option \((x, p)\) for a buyer with value \(v\) is \(y(x)(v - p)\). This form permits a standard payment identity analysis, and gives the following identity for any IC mechanism:

\[
y(x(v))p(v) = y(x(v))v - \int_0^v y(x(z)) dz.
\]

Unfortunately, unlike the risk-neutral setting, this payment identity does not lead to an expression for the mechanism’s revenue that is linear in the allocation function, and so does not allow a Myerson-type theorem characterizing optimal mechanisms.

3.2.1 Optimal revenue approaches social welfare

As we observed earlier, the optimal revenue in general exceeds Myerson’s revenue when the buyer is risk-averse. But to what extent? We now show that as long as the buyer is sufficiently risk-averse, the seller can extract nearly the entire expected value of the buyer as revenue, regardless of the buyer’s value distribution. This stands in contrast to the risk-neutral setting where for some distributions (e.g. the equal revenue distribution), the revenue-welfare gap can be unbounded. More generally, Lemma 2 in the following subsection shows that the revenue of every IC mechanism composed of binary lotteries increases weakly with increased risk aversion.

**Theorem 2.** For every \(\varepsilon > 0\) and \(H > 1\), if the buyer’s weighting function satisfies \(y(1 - \varepsilon) \leq 2^{-H/\varepsilon}\), there exists a mechanism that for any value distribution \(F\) supported over \([1, H]\) obtains revenue at least \(1 - O(\varepsilon)\) times the buyer’s expected value \(E_{v \sim F}[v]\).
follows by recalling that $y$.

Then Rev suggests that this is challenging: for some
value distributions supported over $[1, H]$, as we
vary the buyer’s risk profile, the optimal revenue can vary by a factor as large as $\Theta(\log H)$.

Formally, we say that a mechanism defined by menu $M$ achieves an $\alpha$ risk-robust approximation to revenue for value distribution $F$ over class $Y$ of risk profiles, if for all $y \in Y$,

$$\text{Rev}_{y,F}(M) \geq \alpha \text{OPT}(y,F).$$

Achieving a risk-robust approximation to revenue requires understanding how the revenue of a mechanism changes as the buyer’s risk profile changes. Ideally, since the optimal revenue tends to increase as the buyer gets more and more risk averse, we would require that the revenue of our robust mechanism also increases in tandem. We show that this is indeed true for mechanisms composed of binary lotteries under a certain assumption about how risk aversion increases.

### 3.3.1 An $O(\log \log H)$ risk-robust approximation to revenue.

Consider a family of weighting functions $Y$. We say that $Y$ is non-crossing if for all pairs of functions $y_1$ and $y_2$ in $Y$, for all $x_1, x_2 \in [0, 1]$, $y_1(x_1) \geq y_2(x_1)$ implies $y_1(x_2) \geq y_2(x_2)$. In other words, one function always lies above the other – they never cross. We express this relationship between the functions as $y_1 \succeq y_2$. We say that the family $Y$ is monotone if for all pairs of functions $y_1 \succeq y_2$ in $Y$, $y_2(x)/y_1(x)$ is monotone non-decreasing in $x$. In other words, $y_2$ is relatively more risk-averse at small probabilities than at large probabilities.

**Lemma 2.** Let $Y$ be a monotone non-crossing family of weighting functions, and let $y_1 \succeq y_2$ be any two weighting functions in $Y$. Then, for any IC mechanism $M$ composed of binary lotteries, we have $\text{Rev}_{y_2,F}(M) \geq \text{Rev}_{y_1,F}(M)$. 

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This lemma follows by observing that a buyer with risk profile \( y_2 \) selects a menu option in \( \mathcal{M} \) that is no cheaper than the menu option a buyer with the same value but risk profile \( y_1 \) selects. A formal proof is given in Appendix A. We also show in the appendix that the monotonicity property of the family \( \mathcal{Y} \) is necessary to obtain this revenue monotonicity.

An implication of this lemma is that if the optimal revenues under weighting functions \( y_1 \) and \( y_2 \) are close, then a mechanism that is approximately optimal for \( y_1 \) continues to be approximately optimal for \( y_2 \). This observation allows us to develop a mechanism that obtains a risk-robust \( O(\log \log H) \) approximation to revenue over monotone non-crossing families of weighting functions. Observe that this approximation factor is exponentially smaller than the range of optimal revenues for the different risk profiles.

Our mechanism chooses \( \log \log H \) representative weighting functions from the given family, and then picks a random one of the optimal menus from each family. We note that this is only an existential and not a computationally efficient result. See Appendix A for a proof.

**Theorem 3.** Let \( \mathcal{Y} \) be a monotone non-crossing family of weighting functions and let \( F \) be a value distribution supported on \([0, H]\). Then there exists a mechanism \( \mathcal{M} \) that for any weighting function in \( \mathcal{Y} \) achieves an \( O(\log \log H) \) approximation to revenue. Formally, for all \( y \in \mathcal{Y} \),

\[
\text{Rev}_{y,F}(\mathcal{M}) \geq \Omega\left(\frac{1}{\log \log H}\right) \text{OPT}(y,F).
\]

### 3.3.2 Risk-robust approximation via Myerson’s mechanism

Theorem 3 is unsatisfying for two reasons. One, finding the mechanism that achieves the revenue guarantee in the theorem appears challenging. Second, the theorem works only for certain families of weighting functions, and not for arbitrary ones.

We now consider risk-robust approximation from a different viewpoint. We observe that obtaining the high revenue guaranteed by Theorem 2 requires the buyer to heavily discount any probabilities that are bounded away from 1. Such extreme risk aversion is unrealistic. We therefore focus on weighting functions that map some probabilities bounded away from 1 (i.e. \( x = 1 - \Theta(1) \)) to weights bounded away from 0 (i.e. \( y(x) = \Theta(1) \)). In other words, the weighting function is \( \beta \)-bounded for some \( \beta = \Theta(1) \) (Definition 3). We show that for such weighting functions, Myerson’s mechanism already achieves an approximation to the optimal revenue. Of course, Myerson’s mechanism is defined independently of the buyer’s risk profile. This, therefore, implies a risk-robust approximation.

**Theorem 4.** For any \( \beta \)-bounded convex weighting function \( y \), and any value distribution \( F \), we have

\[
\text{Mye}(F) \geq \beta \text{OPT}(y,F).
\]

To understand the intuition behind this theorem, consider an \( \beta \)-bounded convex weighting function \( y \), and let \( \mathcal{M} = (x(v), p(v)) \) denote any mechanism composed of binary lotteries. We now make two observations. First, this mechanism cannot extract too much revenue from low values. Intuitively, this is because if most of its revenue came from buyers who purchase the low-probability allocations, it could extract much more revenue from these buyers by selling to them with higher probability. Second, at the high types, the allocation probability \( x(t) \) is large, consequently the buyer faces less uncertainty, and so the mechanism behaves nearly like the optimal risk-neutral mechanism. We now formalize this intuition.
Lemma 3. Let $\mathcal{M} = (x(v), p(v))$ denote an IC mechanism composed of binary lotteries for value distribution $F$ and weighting function $y$. Then, for any type $t$,

$$\int_0^t f(v)x(v)p(v)dv \leq x(t)\text{OPT}(y, F).$$

Proof. We examine an alternate mechanism that increases the probability of allocation to types below $t$ by a factor of $\frac{1}{x(t)}$. Consider the alternate mechanism $\hat{\mathcal{M}}$ with allocation rule $\hat{x}(b) = \frac{x(b)}{x(t)}$ if $b < t$, and 1 otherwise, and payment rule $\hat{p}(b) = p(b)$. Note that this mechanism is not truthful; in particular, an agent with value $v$ may choose a menu option $(\hat{x}(b), \hat{p}(b))$ for $b \neq v$ under this mechanism. We will show, however, that an agent will never deviate to a bid below his true value, and will continue to pay a price at least as high as in $\mathcal{M}$.

For convenience, we abuse notation and write $y(x(v))$ as $y(v)$ and $y(\hat{x}(v))$ as $\hat{y}(v)$. Because $\mathcal{M}$ is incentive-compatible, for any $v$ and $w < v$, we have that $u(v, v) = (v - p(v))y(v) \geq (v - p(v))y(w) = u(v, w)$. Since $y$ is positive, increasing, and convex as a function of $x$, $\hat{y}(v) = y(x(v)/x(t))$ is nondecreasing in $v$ for $v \leq t$. Therefore

$$\frac{\hat{y}(w)}{\hat{y}(v)}(v - p(w)) \leq \frac{y(w)}{y(v)}(v - p(v)) \leq (v - p(v)),$$

so $\hat{u}(v, w) = \hat{y}(w)(v - p(w)) \leq \hat{y}(v)(v - p(v)) = \hat{u}(v, v)$, and the buyer will not underreport. Let $b(v)$ be the optimal bid for buyer $v$ in mechanism $\mathcal{M}$; then

$$\text{Rev}(\hat{\mathcal{M}}) \geq \int_0^t f(v)x(\hat{x}(b(v)))p(b(v))dv \geq \frac{1}{x(t)} \int_0^t f(v)x(v)p(v)dv,$$

and the lemma follows. \hfill \Box

Lemma 4. Let $\mathcal{M} = (x(v), p(v))$ denote an IC mechanism composed of binary lotteries for value distribution $F$ and weighting function $y$. Then, for any type $t$,

$$\int_1^\infty f(v)x(v)p(v)dv \leq \frac{1}{y(t)}\text{MYE}(F).$$

Proof. Let $\hat{p}(v) = p(v)y(x(v))$. Then the payment identity in Equation (1) implies that the mechanism $(y(x(v)), \hat{p}(v))$ is IC for a risk-neutral buyer. Therefore, noting that $x(v) \leq 1$ and $y(v) \geq y(t)$ for all $v \geq t$, we have

$$\int_t^\infty f(v)x(v)p(v)dv = \int_t^\infty f(v)\frac{x(v)}{y(v)}\hat{p}(v)dv \leq \frac{1}{y(t)} \int_t^\infty f(v)\hat{p}(v)dv \leq \frac{1}{y(t)} \int_0^\infty f(v)\hat{p}(v)dv \leq \frac{1}{y(t)}\text{MYE}(F).$$

We can now combine the two lemmas into a proof for Theorem 4.

**Proof of Theorem 4.** Let $\mathcal{M} = (x(v), p(v))$ be an optimal mechanism for value distribution $F$ and weighting function $y$. Let $t$ be any type at which $(1 - x(t))y(x(t)) \geq \beta$. By the definition of $\beta$-boundedness, such a type exists.
By Lemma 3, we have
\[ \text{OPT} \leq \left( \frac{1}{1 - x(t)} \right) \int_t^\infty f(v) x(v) p(v) dv, \]
which, using Lemma 4, gives
\[ \leq \frac{1}{(1 - x(t))y(t)} \text{MYE}(F) \leq \frac{1}{\beta} \text{MYE}(F). \]

4 Two-stage revenue maximization

We now turn to a setting where the seller has two items to sell to the buyer in succession. We will denote the buyer’s values for the two items by \( v_1 \) and \( v_2 \). The values are drawn from independent distributions with c.d.f.s \( F_1 \) and \( F_2 \), and the buyer’s value for receiving both the items is \( v_1 + v_2 \). The mechanism proceeds in two stages. In the first stage, the buyer reveals \( v_1 \) and the seller sells item 1 to the buyer at a certain price. At this time, neither the buyer nor the seller knows the buyer’s value for the second item. The buyer learns \( v_2 \) at the start of stage two. The seller then uses a second mechanism, that may depend on the buyer’s report of \( v_1 \), to sell the second item to the buyer.

**Properties of two-stage mechanisms.** A two-stage mechanism is *incentive-compatible* if for any report \( v_1 \) during the first stage, the second-stage mechanism is incentive-compatible with respect to \( v_2 \), and the combination of the first- and second-stage mechanisms is incentive-compatible with respect to \( v_1 \). We further impose the constraint of *ex-post IR* which states that the price charged to the buyer in either stage cannot exceed the ex-post value obtained by the buyer in that stage.

Recall that \( \text{OPT}(y, F_1) \) and \( \text{OPT}(y, F_2) \) denote the optimal revenue that the seller can obtain by selling items 1 and 2 respectively via independent mechanisms (such as posting a fixed price in each stage). We will denote by \( \text{OPT}(y, F) \) the optimal revenue achievable by an incentive-compatible ex-post IR mechanism for the two stages combined, where \( F = F_1 \times F_2 \) denotes the joint distribution over \( (v_1, v_2) \).

Of course, \( \text{OPT}(y, F) \geq \text{OPT}(y, F_1) + \text{OPT}(y, F_2) \), but in fact, the former can be much larger than the latter sum. Observe that the second-stage mechanism can depend on the buyer’s first-stage report. This gives the seller some flexibility in extracting more revenue in the first stage. Ashlagi et al. [2016] show, in particular, that for risk-neutral buyers the seller can charge a premium on the first stage in exchange for more utility in the second stage, which in some settings allows the seller to extract almost the entire second-stage social welfare as revenue. We will show that a similar result is achievable under our model of risk aversion.

A two-stage mechanism can be described without loss of generality as a menu of options with each option being a three-tuple \( (X, P, M) \), where \( X \) is an indicator variable representing the allocation of item 1 to the agent, \( P \) is the price to be paid in stage one, and \( M \) is an incentive-compatible mechanism for the second stage. Let \( U(v_2, M) \) be a random variable denoting the ex-post utility that the agent obtains from mechanism \( M \) in stage two. Then, the buyer’s risk-averse utility from the menu option \( (X, P, M) \) in stage one is given by
\[ E_y[v_1X - P + U(v_2, M)] \]
where the weighted expectation is taken over the randomness in the mechanisms as well as the randomness in the agent’s stage two value. In stage one, the agent chooses the menu option that maximizes his risk-averse utility.

Observe that once again $X, P,$ and $M$ can be arbitrarily correlated. In particular, the contribution of the second-stage mechanism $M$ to the buyer’s risk-averse utility in stage one can depend not only on the chosen menu option and his expected $v_2$, but also on his actual value $v_1$. This makes it challenging to reason about the choices of the agent and account for the contribution of the agent’s second-stage utility to the first-stage revenue, as we see in the following example.

**Example 6.** Consider a menu option that with probability $x$ allocates the first item to the bidder and charges him $p$, and then always gives the second item away for free. Then with probability $1 - F_2(v_2)$, the buyer gets utility of at least $v_2$. Suppose $v_2 \sim U\{1, 2\}$. Let $y(x) = x^2$ for all $x$. We will compute the utility of the buyer from this menu option at different first-stage values. Observe that although the menu option, in particular the second-stage mechanism, stays the same, the contribution of the first- and second-stage mechanisms to the buyer’s utility vary as $v_1$ varies.

Case 1: $v - p \geq 2$. Getting the first item is worth more than any second-stage utility alone. Then his utility from the mechanism is:

$$y(1)1 + y((1 - x)/2)(2 - 1) + y(x)(v - p - 2) + y(x/2)(2)$$

$$= x^2(v - p) + 1 + \frac{(1 - x)^2}{4} - \frac{3x^2}{2}.$$ 

Case 2: $v - p \in (1, 2)$. Getting the second item when his value is high is worth more than just getting the first item. His utility from this mechanism is:

$$y(1)1 + y((1 - x)/2)(v - p - 1) + y(1/2)(2 - (v - p)) + y(x/2)(v - p)$$

$$= \frac{x(x - 1)}{2}(v - p) + \frac{3}{2} - \frac{(1 - x)^2}{4}.$$ 

In the case where $p = 1, x = 1/2, y = x^2,$ and $v$ is 4 in the first case and 2.5 in the second, we get that the first case has utility $12/16 + 11/16 = 23/16$ and the second case has utility $-3/16 + 23/16 = 5/4$.

We focus on a simple and practical class of mechanisms, namely posted-price mechanisms, that in addition to $\text{OPT}(y, F_1) + \text{OPT}(y, F_2)$ can in some cases obtain an additional $E_y[v_2]$ in revenue, matching results known for the risk-neutral setting.

### 4.1 Posted-price mechanisms and their revenue properties

A two-stage posted-price mechanism is specified by a menu, where each menu option is a pair of prices $(p_1, p_2)$. If the buyer selects this menu option, he is offered item 1 at a price of $p_1$ and promised item 2 at a price of $p_2$. Observe that the buyer makes this choice knowing $v_1$ but not knowing $v_2$. The buyer would potentially be willing to pay a higher price for item 1 if in return he is promised a lower price for item 2. Accordingly, the undominated menu options\(^9\) correspond to higher first-stage prices being coupled with lower second-stage prices and vice versa.

\(^9\)A menu option is dominated by another if the buyer prefers the latter to the former regardless of his value for the first item.
In the remainder of this section, we use the notation $\mathcal{PP}_\ell$ to represent a two-stage posted-price mechanism that offers menu options $(p, \ell(p))$ for every price $p$ in some range, where $\ell(\cdot)$ is a non-increasing function mapping the first-stage price to the corresponding second-stage price.\footnote{Observe that this notation captures menus with a finite number of options. In particular, if the function $\ell(p)$ is constant over a range of prices $p$, then all options other than the smallest price in that range are dominated, and effectively not present in the menu.}

**The buyer’s optimization problem.** Fix a posted-price mechanism $\mathcal{PP}_\ell$, and consider a buyer with uncertainty weighting function $y$ and first-stage value $v_1$. Observe that if the buyer purchases the menu option $(p, \ell(p))$, he gets utility of $v_1 - p$ with certainty, and expects to obtain some (random) utility from the second-stage posted price of $\ell(p)$. The risk-averse expectation of the buyer’s second-stage utility from posted price $p_2$ can be written as

$$U(p_2) := E_y[\max(0, v_2 - p_2)] = \int_0^\infty y(1 - F_2(z + p_2))dz = \int_{p_2}^\infty y(1 - F_2(z))dz.$$  

Accordingly, the buyer’s risk-averse utility in the first stage from purchasing option $(p, \ell(p))$ is

$$v_1 - p + U(\ell(p)).$$

The menu option $(p, \ell(p))$ gives the buyer the same utility as offering an “effective price” of $p - U(\ell(p))$ in a single-shot mechanism. Accordingly, the buyer chooses the menu option corresponding to the minimal $p - U(\ell(p))$ over all prices that he can afford, that is, with $p \leq v_1$. Then without loss of generality, the mechanism contains menu options with effective prices that are non-increasing in the first-stage price, as otherwise they would be dominated. We assume without loss of generality that the buyer breaks ties across menu options with equal effective prices in favor of the largest price.

**Example 7.** Consider a buyer with $v_1, v_2 \sim U[0, 1]$ and uncertainty weighting function $y(x) = x^2$. Consider the mechanism that offers menu options $(0, 1), (\frac{1}{6}, \frac{1}{2})$, and $(\frac{1}{3}, 0)$. Note that $U(\ell(p)) = \int_0^1 (1 - v_2)^2dv_2$. Then $U(1) = 0$, $U(0) = \frac{1}{3}$, and $U(\frac{1}{2}) \approx 0.04$. This gives

- Buyer’s utility from option $(0, 1) = v - 0 + U(1) = v$
- Buyer’s utility from option $\left(\frac{1}{6}, \frac{1}{2}\right) = v - \frac{1}{6} + U\left(\frac{1}{2}\right) \approx v - 0.13$
- Buyer’s utility from option $\left(\frac{1}{3}, 0\right) = v - \frac{1}{3} + U(0) = v$

Then a buyer with $v \in \left[0, \frac{1}{3}\right]$ will purchase the option $(0, 1)$; a buyer with $v \in \left[\frac{1}{3}, 1\right]$ will purchase the option $\left(\frac{1}{6}, 1\right)$; and no buyer will purchase the option $\left(\frac{1}{3}, 0\right)$, as it is dominated by the other options with cheaper effective prices of $0 < 0.13$.

**The seller’s revenue.** We now present an upper bound on the revenue achievable via two-stage posted-price mechanisms.

**Theorem 5.** The revenue of any two-stage posted-price mechanism for a buyer with value distribution $F_1 \times F_2$ and uncertainty weighting function $y$ is bounded by

$$\text{MYE}(F_1) + \text{MYE}(F_2) + E_{v_1 \sim F_1}[\min(v_1, E_y[v_2])].$$
Proof. We will account for the seller’s revenue in the two stages separately. Observe first that regardless of the buyer’s first-stage value, the revenue obtained by the seller in the second stage is no more than $\text{MYE}(F_2)$.

Now let’s consider the seller’s first-stage revenue. Let $p_{\text{min}}$ denote the smallest price offered in stage one. Because the “effective price” is non-increasing as a function of the first-stage price, we have $p - U(\ell(p)) \leq p_{\text{min}} - U(\ell(p_{\text{min}}))$, hence

$$p \leq p_{\text{min}} - U(\ell(p_{\text{min}})) + U(\ell(p)) \leq p_{\text{min}} + U(0),$$

where $U(0) = E_y[v_2]$ is the risk-averse expectation of the buyer’s second-stage value.

On the other hand, the buyer never pays more than $v_1$ in the first stage. Therefore, the seller’s first-stage revenue, when the buyer’s first-stage value is $v_1 \geq p_{\text{min}}$, is bounded by $\min(v_1, p_{\text{min}} + U(0))$. We can now bound the seller’s first-stage revenue by

$$E_{v_1 \sim F_1}[\min(v_1, p_{\text{min}} + U(0))] \leq p_{\text{min}}(1 - F_1(p_{\text{min}})) + E_{v_1 \sim F_1}[\min(v_1, U(0))] \leq \text{MYE}(F_1) + E_{v_1 \sim F_1}[\min(v_1, U(0))].$$

We will now show that there exists a simple posted-pricing mechanism that achieves a 2-approximation to the upper bound in Theorem 5.

Theorem 6. For the two-stage setting described above, there exists a posted-price mechanism $PP_\ell$ that obtains revenue at least

$$\frac{1}{2} \left( \text{MYE}(F_1) + \text{MYE}(F_2) + E_{v_1 \sim F_1}[\min(v_1, E_y[v_2])]. \right)$$

Proof. Charging the optimal single-shot posted-price in each stage already obtains revenue $\text{MYE}(F_1) + \text{MYE}(F_2)$. We will now describe a posted-price mechanism $PP_\ell$ that obtains revenue at least $E_{v_1 \sim F_1}[\min(v_1, U(0))]$. The intuition that a mechanism can achieve this is as follows: if every menu option charges a price $p$ but guarantees utility equal to $p$ back in the next stage, then the buyer will be willing to pay any price subject to ex-post IR. The better of these two mechanisms achieves the bound stated in the lemma.

The mechanism $PP_\ell$ offers menu options $(p, \ell(p))$ with $\ell(p) = U^{-1}(p)$ for all $p \in [0, U(0)]$. Observe that since $U$ is continuous and ranges from $U(\infty) = 0$ to $U(0)$, for every $p$ in the range $[0, U(0)]$, a second-stage price $\ell(p) = U^{-1}(p)$ exists, and therefore the mechanism is properly defined.

Furthermore, for every menu option, $(p, \ell(p))$, we have $p - U(\ell(p)) = p - p = 0$. So all menu options bring the same effective utility to the buyer on the first stage, and by default the buyer purchases the most expensive one that he can afford. Consequently, the seller’s first-stage revenue in given by $\min(v_1, U(0))$, and the theorem follows.

4.2 Risk-robust approximation

We now turn to risk-robust approximation in the two-stage setting. Observe that the results of Section 3 already imply that we can obtain a risk-robust approximation to the single-shot revenue achievable in each stage independently, when the buyer’s weighting function is bounded (Definition 3). Can we obtain a risk-robust approximation to the last term in the bound given by Lemma 5, namely, $E[\min(v_1, E_y[v_2])]$?
In this section we argue that this last term cannot be extracted via a posted-pricing mechanism in a risk-robust manner even if all of the possible weighting functions for the buyer are bounded.\footnote{Observe, of course, that if the risk averse expectation of the buyer’s second-stage value does not differ much across the different weighting functions, then we can use ideas from Section 3 to extract the optimal revenue in a risk-robust manner.} This fact leads to Theorem 7.

**Theorem 7.** No posted-price mechanism can obtain a constant-factor risk-robust approximation to revenue in the two-stage dynamic setting. This continues to hold even if all of the relevant weighting functions are $\Theta(1)$-bounded.

At a high-level, the idea behind our construction is as follows. We choose the family of weighting functions and the second-stage value distribution in such a way that although all of the weighting functions satisfy the boundedness property, they cover a large range of weighted expectations for the second-stage value, placing different constraints on the first-stage menu. Intuitively, in order to extract enough revenue, the seller must offer a menu with many different prices, indeed a continuum of first-stage prices. Then, to incentivize the buyer to pay as high a price as he can afford in the first stage, the seller must provide a discount over the second stage’s price. The extent of discount provided depends on the most risk-averse profile for which the effective menu contains the corresponding option. As the buyer goes from being very risk-averse to almost risk-neutral, the seller needs to offer a bigger and bigger discount for higher and higher first-stage prices, and eventually runs out of discounts to offer.

We now make this formal. Consider a two-stage mechanism design setting where the buyer’s value for the first stage is distributed according to the unbounded equal revenue distribution, that is, $F_1(v_1) = 1 - 1/v_1$ for $v_1 \geq 1$, and his second-stage value is distributed according to the equal revenue distribution bounded at $e^n$, that is, $F_2(v_2) = 1 - 1/v_2$ for $v_2 \in [1, e^n]$. We will consider a family $\mathcal{Y}$ of weighting functions parameterized by $\epsilon \in [0, 1]$ as follows.

$$
y_{\epsilon}(x) = \begin{cases} x^2 & \text{for } x \in [0, \epsilon] \\
(1 + \epsilon)x - \epsilon & \text{for } x \in [\epsilon, 1]
\end{cases}
$$

In words, $y_{\epsilon}(x)$ is equal to $x^2$ up to $\epsilon$, and then rises linearly to $y_{\epsilon}(1) = 1$. Observe that each function in this family is convex and at least $1/8$-bounded.

Suppose that $M = \{(p, \ell(p))\}_{p \in P}$ for some $P \subset \mathbb{R}_{\geq 0}$ is a menu that achieves a risk-robust $c$-approximation, $c > 1$, with respect to the family $\mathcal{Y}$. Let $P_{\epsilon} \subseteq P$ index the “effective menu” when the buyer’s weighting function is $y_{\epsilon}$, i.e., the set of first-stage prices corresponding to menu options that the buyer actually purchases at some value. Any option indexed by $p \in P \setminus P_{\epsilon}$ is dominated.

Recall that $U_{\epsilon}(\ell(p))$ is the risk-averse utility that a buyer with weighting function $y_{\epsilon}$ obtains from the second-stage mechanism when he chooses first-stage price $p$:

$$
U_{\epsilon}(\ell(p)) = \begin{cases} \int_{\ell(p)}^{e^n} y_{\epsilon}(1/v_2)dv_2 & \text{for } \ell(p) \geq 1 \\
1 - \ell(p) + \int_{1}^{e^n} y_{\epsilon}(1/v_2)dv_2 & \text{otherwise.}
\end{cases}
$$

We make the following observations about effective menus. See Appendix A.3 for proofs.

**Lemma 5.** For the setting described above, if $\{(p, \ell(p))\}_{p \in P}$ gives a risk-robust $c$-approximation, the following properties hold without loss of generality:
1. For all \( p \in P \), \( \ell(p) \geq 1 \).

2. For any \( \varepsilon \), \( P_\varepsilon \supseteq [1, p_\varepsilon] \) where \( p_\varepsilon \) is defined such that

\[
1 + \mathbb{E}_{v \sim F_1} [\min(v, p_\varepsilon)] = \frac{1}{c} \mathbb{E}_{v \sim F_1} [\min(v, U_\varepsilon(0))].
\]  

(2)

3. For every \( \varepsilon \), \( p_\varepsilon = \alpha_c U_\varepsilon(0)^{1/c} \) for some constant \( \alpha_c > 0 \) depending only on \( c \).

4. For every \( \varepsilon \), the left derivative of \( U_\varepsilon(\ell(p)) \) with respect to \( p \) at \( p = p_\varepsilon \) must be \( \geq 1 \).

Informally, the first property holds because second-stage prices below 1 lose as much second-stage revenue as they gain on the first stage; the second property is a consequence of minimizing the required second-stage discounts; the third follows by solving Equation (2); the fourth follows from the assumption that prices in \( P_\varepsilon \) are undominated.

We can now derive a differential equation for the function \( \ell(p) \). First,

\[
\frac{dU_\varepsilon(\ell(p))}{dp} = \begin{cases} 
-y_\varepsilon(1/\ell(p))\ell'(p) & \text{when } \ell(p) \geq 1 \\
-\ell'(p) & \text{otherwise.}
\end{cases}
\]

Then, Lemma 5 (4) implies

\[
-\ell'(p) \geq \begin{cases} 
\frac{1}{y_\varepsilon(1/\ell(p))} & \text{when } \ell(p) \geq 1 \\
1 & \text{otherwise},
\end{cases}
\]  

(3)

where, for any price \( p \), \( \varepsilon_p \) is the value of \( \varepsilon \) for which \( p = p_\varepsilon \) as given by Equation (2). We have the following two boundary conditions:

\[
\ell(1) \leq e^n \quad \text{and} \quad \ell(p) > 1 \quad \text{for } p < p_0.
\]  

(4)

(5)

The first enforces that the second-stage price at \( p = 1 \) be no more than the maximum value that \( v_2 \) takes. The second follows from Lemma 5 (1).

We use \( \bar{\ell}(p) \) to denote the solution to Equations (3) and (4), each with inequality replaced by equality. So \( \bar{\ell}(p) \) gives an upper bound on \( \ell(p) \). We will show that, for large enough \( n \), a solution to (3) which satisfies (4) cannot also satisfy (5).

The following claim will be useful; a proof appears in Appendix A.3.

**Lemma 6.** For all \( \varepsilon \), \( U_\varepsilon(0) \geq \min\{\ln 1/\varepsilon, n\} \).

We are now ready to prove Theorem 7.

**Proof of Theorem 7.** By Lemma 5 (3), \( p_\varepsilon = \alpha_c U_\varepsilon(0)^{1/c} \). Fix \( \varepsilon^* = e^{-(2/\alpha_c)^c} \), and let \( N = (2/\alpha_c)^c \). Then, by Lemma 6, for all \( n > N \),

\[
p_{\varepsilon^*} \geq \alpha_c (\ln 1/\varepsilon^*)^{1/c} = 2.
\]

Note that \( \varepsilon^* \leq \varepsilon_p \leq 1 \) for all \( p \in [1, 2] \).
We now show that \( \tilde{\ell}(2) \) is at most \( 1/\varepsilon^\ast \) for all \( n > N \). Suppose not. Then, for all \( p \in [1,2] \), 
\( \tilde{\ell}(p) \geq \tilde{\ell}(2) \) and \( \varepsilon_p \geq \varepsilon^\ast \) implies that \( \hat{\ell}(p) > 1/\varepsilon_p \), or \( 1/\hat{\ell}(p) < \varepsilon_p \). So by the definition of \( y_\varepsilon \) we have 
\( y_\varepsilon(1/\hat{\ell}(p)) = \hat{\ell}(p)^{-2} \). Thus the differential equation (3) simplifies to 
\( -\hat{\ell}'(p) = \hat{\ell}(p)^2 \) for \( p \in [1,2] \).
The solution, incorporating the boundary condition \( \hat{\ell}(1) = e^n \), is 
\[
\hat{\ell}(p) = \frac{1}{e^{-n} + p - 1},
\]
and so \( \hat{\ell}(2) = 1/(e^{-n} + 1) < 1 \), a contradiction.

But if \( \tilde{\ell}(2) \) is bounded above by \( 1/\varepsilon^\ast \), we can argue that \( \ell(p) \leq \tilde{\ell}(p) = 1 \) at some \( p < p_0 \), contradicting the boundary condition (5). Specifically, Equation (3) gives \( \ell'(p) \leq -1 \) for all \( \ell(p) \geq 1 \), 
so \( \ell(1/\varepsilon^\ast + 1) \leq \tilde{\ell}(2) - (1/\varepsilon^\ast - 1) \leq 1 \). On the other hand, by Lemma 5 (3), 
\( p_0 \propto n^{1/c} \), so for large enough \( n \) we have \( p_0 > 1/\varepsilon^\ast + 1 \).

\[\square\]

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A Deferred proofs

A.1 Utility curves and a characterization of optimal single-shot mechanisms

The proofs in this section lead up to the conclusion of Theorem 1: both the optimal mechanism and the risk-averse utility that the buyer gets from it have a simple structure.

**Lemma 7.** For any profile \( y \) and lottery \((X, P)\), \( u_y(v, (X, P)) \) is a concave function of \( v \). The slope of this function lies between \( 1 - y(1 - x) \) and \( y(x) \), where \( x = \Pr[X = 1] \).

**Proof.** We assume that \((X, P)\) satisfies ex-post IR, meaning \( \Pr[P = 0 | X = 0] = 1 \). Let \( F_P(p) = \Pr[P \leq p | X = 1] \). We further assume that \( F_P \) is differentiable, with p.m.f. \( f_P \). We make these assumptions for simplicity; both can be relaxed. By definition,

\[
u_y(v, (X, P)) = -\int_0^\infty (1 - y(\Pr[vX - P \geq -z]))dz + \int_0^\infty y(\Pr[vX - P \geq z])dz.
\]

For \( z \geq 0 \),

\[
\Pr[vX - P \geq -z] = 1 - \Pr[X = 1] \Pr[P \geq v + z | X = 1] = 1 - x(1 - F_P(v + z)),
\]

and

\[
\Pr[vX - P \geq z] = 1 - \Pr[X = 1] \Pr[P \geq v - z | X = 1] - \Pr[X = 0]\]

\[
= 1 - x(1 - F_P(v - z)) - (1 - x)
\]

\[
= xF_P(v - z).
\]

Therefore,

\[
u_y'(v) = \int_0^\infty y'(1 - x(1 - F_P(v + z)))xf_P(v + z)dz + \int_0^\infty y'(xF_P(v - z))xf_P(v - z)dz
\]

\[
= y(1 - x + xF_P(v + z))|_{z=0}^{\infty} - y(xF_P(v - z))|_{z=0}^{\infty}
\]

\[
= 1 - y(1 - x + xF_P(v)) + y(xF_P(v)).
\]

Note that when \( F_P(v) = 1 \) (i.e., when \( v \) is greater than the maximum price charged), \( u_y'(v) = y(x) \). Similarly, when \( F_P(v) = 0 \), \( u_y'(v) = 1 - y(1 - x) \). In either case, \( u_y'(v) \geq 0 \).

Finally, to show concavity, we show \( u_y''(v) \leq 0 \):

\[
u_y''(v) = x f_P(v) \left[ y'(x f_P(v)) - y'(1 - x + x f_P(v)) \right].
\]

The first term, \( x f_P(v) \), is always nonnegative. Note that \( 1 - x + x f_P(v) \geq x f_P(v) \). Since \( y \) is convex and increasing, \( y'(1 - x + x f_P(v)) \geq y'(x f_P(v)) \), and so \( u_y''(v) \leq 0 \).

**Lemma 8.** Fix any (allocation, payment) pair \((X, P)\). Let \((x, p)\) be the lottery that sells with probability \( x = \Pr[X = 1] \) and charges \( p = E[P] / x \). Then, assuming payments are non-negative, \( u_y(v, (x, p)) \geq u_y(v, (X, P)) \) for all \( v \).
Proof. First observe that $E[Xv - P] = x(v - p)$, but we can also write it as

$$E[Xv - P] = \int_0^\infty \Pr[Z > z] \, dz - \int_{-\infty}^0 (1 - \Pr[Z > z]) \, dz$$

where $Z = Xv - P$ and it may range from $-\infty$ to $\infty$. Then

$$u_y(v, (x, p)) = y(x)(v - p) = \frac{y(x)}{x} x(v - p) = \frac{y(x)}{x} E[Xv - P].$$

Recall that $y(x)/x$ is non-decreasing in $x$ by convexity of $y$. Assuming that all payments are non-negative, then if $Z > 0$, it must be that $X = 1$, hence $\Pr[Z > 0] \leq x$. Then for the positive values that $Xv - P$ takes on, we have

$$\int_0^\infty \frac{y(x)}{x} \Pr[Z > z] \, dz \geq \int_0^\infty \frac{y(\Pr[Z > z])}{\Pr[Z > z]} \Pr[Z > z] \, dz = \int_0^\infty y(\Pr[Z > z]) \, dz.$$

For the negative values that $Xv - P$ takes on, by monotonicity of $y(x)/x$ in $x$ and because the utility is negative, we have

$$-\frac{y(x)}{x} \int_{-\infty}^0 (1 - \Pr[Z > z]) \, dz \geq -\int_{-\infty}^0 \frac{y(\Pr[Z > z])}{\Pr[Z > z]} (1 - \Pr[Z > z]) \, dz$$

$$\geq -\int_{-\infty}^0 (1 - y(\Pr[Z > z])) \, dz$$

where the the second inequality follows from $-\frac{y(\Pr[Z > z])}{\Pr[Z > z]} \geq -1$. Hence, all together,

$$u_y(v, (x, p)) = \frac{y(x)}{x} E[Xv - P] \geq \int_0^\infty y(\Pr[Z > z]) \, dz - \int_{-\infty}^0 (1 - y(\Pr[Z > z])) \, dz = u_y(v, (X, P)).$$

\[\square\]

Lemma 9. Fix $y$, and let $(X, P)$ be any (allocation, payment) pair. For any lottery $(x, p)$ such that there exists $v$ with $0 \leq u_y(v, (x, p)) \leq u_y(v, (X, P))$ and $x \geq \Pr[X = 1]$, the expected revenue of $(x, p)$ is at least as large as the expected revenue of $(X, P)$. If $u_y(v, (x, p)) < u_y(v, (X, P))$, then the revenue is strictly larger.

Proof. Let $x' = \Pr[X = 1]$ and $p' = E[P]/x'$. Note that the expected revenue from $(x', p')$ is exactly $E[P]$. By Lemma 8, $u_y(v, (X, P)) \leq u_y(v, (x', p'))$. Since $0 \leq u_y(v, (x, p)) \leq u_y(v, (x', p'))$, we have

$$y(x')(v - p') = u_y(v, (x', p')) \geq u_y(v, (x, p)) = y(x)(v - p).$$

Since $x \geq x'$ by assumption, $y(x) \geq y(x')$. Therefore $p \geq p'$, and so $xp \geq x'p' = E[P]$. Note that if $u_y(v, (X, P)) > u_y(v, (x, p))$, then $xp > E[P]$. \[\square\]
For any IC/IR mechanism \((X, P)\) defined on the interval \([a, b]\), let \(u(v)\) be the lower convex envelope of \(u_y(v, (X_v, P_v))\). That is, \(u\) is the maximal convex function upper bounded by the utility curve.

**Definition 4.** For any convex function \(f : I \to \mathbb{R}\), the subdifferential of \(f\) at \(x \in I\) is
\[
\partial f(x) = \{ m : f(x') - f(x) \ge m(x' - x) \ \forall x' \in I \}.
\]
Likewise, for any concave function \(g : I \to \mathbb{R}\), the superdifferential of \(g\) at \(x \in I\) is
\[
\partial g(x) = \{ m : g(x') - g(x) \le m(x' - x) \ \forall x' \in I \}.
\]

Let \(\partial^* f(x) = \max\{ m \in \partial f(x) \}\) be the maximal slope of a line tangent to \(f\) at \(x\). Similarly, define \(\partial^* g(x) = \min\{ m \in \partial g(x) \}\) to be the minimal slope of a line tangent to \(g\) at \(x\).

**Lemma 10.** \(\partial u(v) \subseteq [0, 1]\) for all \(v \in [a, b]\).

**Proof.** First, \(u(v)\) is an nondecreasing function of \(v\) because, by Lemma 1, \(u_y(v, (X, P))\) is nondecreasing for all \((X, P)\) in \(\mathcal{M}\). So \(\partial u(v) \subseteq [0, \infty)\) for all \(v\).

Let \(v^*\) be any value in \([a, b]\). Since \(u(v)\) is the lower convex envelope of \(u_y(v, (X_v, P_v))\), there exists \(v_0 \le v^*\) such that for all \(v' > v^*\),
\[
\partial^* u(v^*) \le \frac{u_y(v', (X_v', P_v')) - u_y(v_0, (X_{v_0}, P_{v_0}))}{v' - v_0} \le \frac{u_y(v', (X_v', P_v')) - u_y(v_0, (X_{v_0}, P_{v_0}))}{v' - v_0} \le \frac{v' - v_0}{v' - v_0} = 1
\]
The second inequality follows by the definition of \((X_{v_0}, P_{v_0})\), and the third follows from Lemma 1 together with the fact that \(y(\Pr[P_v = 1]) \le 1\). \(\square\)

**Theorem 1.** For any revenue optimal IC mechanism \((X, P)\) in the single-shot setting, the buyer's utility function \(u_y(v, (X_v, P_v))\) is convex and nondecreasing. Furthermore, there exists an optimal ex-post IR mechanism that can be described as a menu of binary lotteries.

**Proof.** First, we show that we can find a menu of lotteries \((x(v), p(v))\) which obtain a utility curve equal to \(u\). Fix \(v_0 \in [a, b]\). Let \(m_0 = \partial^* u(v_0)\). Note that if \(m_0 = 0\), then \(\mathbb{E}[X_{v_0}] = 0\) by Lemma 1, and so \(\mathbb{E}[P_{v_0}] = 0\) by IR. So assume that \(m_0 > 0\). Let \(p = v_0 - u(v_0)/m_0\) and let \(x = y(m_0)\). Since \(y(a) \le a\) and \(u(v)\) is convex and nondecreasing, \(y(v_0) \le m_0 v_0\), so \(p \ge 0\). By Lemma 10, \(m_0 \in [0, 1]\), so \(x\) is a well-defined probability. So \((x, p)\) is a feasible lottery with utility curve tangent to \(u\) at \(v_0\): \(u_y(v_0, (x, p)) = (v_0 - p)y(x) = u(v_0)\) and \(u'_y(v_0, (x, p)) = m_0\).

It remains to show that \(xp \ge \mathbb{E}[P_{v_0}]\). For ease of notation, let \(x_0 = \Pr[X_{v_0} = 1]\). We will show that the conditions of Lemma 9 are satisfied: namely, that \(x \ge x_0\) and there exists \(v\) such that \(0 \le u_y(v, (x, p)) \le u_y(v, (X_{v_0}, P_{v_0}))\). The latter condition is satisfied at \(v_0\) by construction, and the second inequality is strict if \(u(v_0) < u_y(v, (X_{v_0}, P_{v_0}))\). \(\square\)
To show \( x \geq x_0 \), it suffices to show \( m_0 \geq y(x_0) \). Suppose \( u(v_0) = u_y(v_0, (X_{v_0}, P_{v_0})) \). It follows that

\[
m_0 = \partial^* u(v_0) \\
\geq \partial^* u_y(v_0, (X_{v_0}, P_{v_0})) \\
\geq y(x_0).
\]

The first inequality holds because \( u(v') \geq u_y(v', (X_{v_0}, P_{v_0})) \) for all \( v' \geq v_0 \). The second follows by Lemma 1.

Otherwise, if \( u(v_0) < u_y(v_0, (X_{v_0}, P_{v_0})) \), there exists a point\(^{12} \) \( v' > v_0 \) such that \( u(v') = u_y(v', (X_{v_0}, P_{v_0})) \) and \( u(v') = u(v_0) + m_0(v' - v_0) \). Thus,

\[
u_y(v', (X_{v_0}, P_{v_0})) = u(v_0) + m_0(v' - v_0) \\
\leq u_y(v_0, (X_{v_0}, P_{v_0})) + m_0(v' - v_0).
\]

Rearranging and appealing to Lemma 1, we have

\[
m_0 \geq \frac{u_y(v', (X_{v_0}, P_{v_0})) - u_y(v_0, (X_{v_0}, P_{v_0}))}{v' - v_0} \\
\geq y(x_0)
\]

\[\square\]

### A.2 Single-shot risk robust revenue

**Lemma 11.** Let \( \mathcal{Y} \) be a monotone non-crossing family of weighting functions, and let \( y_1 \geq y_2 \) be any two weighting functions in \( \mathcal{Y} \). Then, for any IC mechanism \( \mathcal{M} \) composed of binary lotteries, we have \( \text{Rev}_{y_2,F}(\mathcal{M}) \geq \text{Rev}_{y_1,F}(\mathcal{M}) \).

**Proof.** Fix a value \( v \). We abuse notation and write \( y_1(v) \) to mean \( y_1(x(v)) \), and similarly with profile \( y_2 \) and bid \( b \). By incentive-compatibility of \( \mathcal{M} \), \( y_1(v)(v - p(v)) \geq y_1(b)(v - p(b)) \) for all \( b \). By

\[\text{Possibly } v' = b. \text{ Note that } y(b) = u_y(b, (X_b, P_b)).\]
assumption and monotonicity of \( x(v) \) in \( v \), for any \( b < v \), \( y_2(v)/y_1(v) \geq y_2(b)/y_1(b) \). Multiplying these inequalities gives

\[
\frac{y_2(v)}{y_1(v)} y_1(v)(v-p(v)) \geq \frac{y_2(b)}{y_1(b)} y_1(b)(v-p(b)),
\]

or equivalently,

\[
y_2(v)(v-p(v)) \geq y_2(b)(v-p(b)).
\]

Therefore, a buyer \( v \) will not underreport his value, and so the revenue of \( M \) can only increase under profile \( y_2 \).

The following example shows that the condition of monotonicity is necessary for Lemma 2.

**Example 8.** Suppose \( F \) is a point mass at \( v = 1 \), and \( M \) is a menu consisting of two lotteries: \( x_1 = 1 - 2\epsilon/3 \), \( p_1 = 3/4 \); and \( x_2 = 1/2 \), \( p_2 = \epsilon \). Let \( y_1(x) = \max(3x/2 - 1/2, x^2) \) and \( y_2(x) = x^2 \). Observe that \( y_1 \) and \( y_2 \) are non-crossing but not monotone.

Under \( y_1 \), the buyer chooses the first lottery:

\[
y_1(x_1)(1-p_1) = \frac{3}{2}(1-2\epsilon/3)\frac{1}{4} = (1/4)(1-\epsilon) = y_1(x_2)(1-p_2).
\]

The revenue under \( y_1 \) is therefore \( \approx 3/4 \).

However, under \( y_2 \), the buyer chooses the second lottery:

\[
y_2(x_1)(1-p_1) = (1-3\epsilon/2)^2\frac{1}{4} < \frac{1}{4}(1-\epsilon) = y_2(x_2)(1-p_2).
\]

The revenue under \( y_2 \) is therefore \( \epsilon/2 \).

**Theorem 3.** Let \( \mathcal{Y} \) be a monotone non-crossing family of weighting functions and let \( F \) be a value distribution supported on \([0, H]\). Then there exists a mechanism \( M \) that for any weighting function in \( \mathcal{Y} \) achieves an \( O(\log \log H) \) approximation to revenue. Formally, for all \( y \in \mathcal{Y} \),

\[
\text{Rev}_{y,F}(M) \geq \Omega \left( \frac{1}{\log \log H} \right) \text{OPT}(y, F).
\]

**Proof.** Since \( \mathcal{Y} \) is a non-crossing family of functions, the relation \( \geq \) defines a total ordering over the functions. We say that \( y_1 \) is larger than \( y_2 \) if \( y_1 \geq y_2 \).

Let \( n \) be a constant to be determined later. For \( i \in \{0, \cdots, n\} \), let \( k_i = E[v]/(\log H)^{i/n} \). Observe that \( k_0 \) is the buyer’s expected value and \( k_n \) is a lower bound on the revenue of Myerson’s mechanism. Therefore, for all \( y \in \mathcal{Y} \), we have \( k_n \leq \text{OPT}(y, F) \leq k_0 \).

Let \( \mathcal{Y}_i = \{ y \in \mathcal{Y} : k_i \leq \text{OPT}(y, F) < k_{i-1} \} \), and let \( y_i \) be the largest (i.e. least risk-averse) weighting function in \( \mathcal{Y}_i \). Define \( M_i \) to be the revenue-optimal mechanism for \( y_i \), that is, \( \text{Rev}_{y_i,F}(M_i) = \text{OPT}(y_i, F) \).
We now claim that a mechanism that randomly chooses one of the mechanisms \( M_i \) to offer to the agent achieves the desired risk robust approximation.

Consider some \( y \in \mathcal{Y} \) and suppose that this weighting function belongs to the set \( \mathcal{Y}_i \). With probability \( 1/n \), we choose to run the mechanism \( M_i \). Now we observe:

\[
\text{OPT}(y, F) \leq (\log H)^{1/n} \text{OPT}(y_i, F) = (\log H)^{1/n} \text{REV}_{y_i, F}(\mathcal{M}_i) \leq (\log H)^{1/n} \text{REV}_{y, F}(\mathcal{M}_i) \quad \text{(by the definition of } \mathcal{M}_i \text{)}
\]

Therefore, we get an approximation factor of \( n(\log H)^{1/n} \), which is minimized at \( n = \log \log H \). \( \square \)

A.3 Lower bound for risk-robust approximation

We will now prove Lemma 5.

**Lemma 12.** For the setting described in Section 4.2, if \( \{(p, \ell(p))\}_{p \in P} \) gives a risk-robust \( c \)-approximation, the following properties hold without loss of generality:

1. For all \( p \in P \), \( \ell(p) \geq 1 \).
2. For any \( \varepsilon \), \( P_\varepsilon \supseteq [1, p_\varepsilon] \) where \( p_\varepsilon \) is defined such that

\[
1 + E_{v \sim F_1}[\min(v, p_\varepsilon)] = \frac{1}{c} E_{v \sim F_1}[\min(v, U_\varepsilon(0))].
\]

3. For every \( \varepsilon \), \( p_\varepsilon = \alpha_c U_\varepsilon(0)^{1/c} \) for some constant \( \alpha_c > 0 \) depending only on \( c \).
4. For every \( \varepsilon \), the left derivative of \( U_\varepsilon(\ell(p)) \) with respect to \( p \) at \( p = p_\varepsilon \) must be \( \geq 1 \).

**Proof.** We prove the statements in sequence:

1. Suppose there is a menu option \( (p, \ell(p)) \) for \( p \in P \) with \( \ell(p) < 1 \). Consider replacing this menu option with the option \( (p + 1 - \ell(p), 1) \). Observe that the buyer’s risk-averse utility under the two options is identical—relative to the original option, the buyer loses an additive amount of \( 1 - \ell(p) \) in his first-stage utility but gains the same additive amount of \( 1 - \ell(p) \) in his second-stage utility in the new menu option. On the other hand, the seller’s revenue under the two options is also identical—the seller’s first-stage revenue is higher by an additive \( 1 - \ell(p) \) amount under the new option, but his second-stage revenue is lower by the same additive \( 1 - \ell(p) \) amount. Therefore, without loss of generality, we may replace \( (p, \ell(p)) \) with the new option \( (p + 1 - \ell(p), 1) \) without affecting the buyer’s utility or the seller’s revenue.

2. We show below that the effective menu \( M_\varepsilon \) must be of the form \([1, p]\). Then, the revenue from such an effective menu is \( 1 + E_{v \sim F_1}[\min(v, p)] \) because the buyer purchases the option with price \( \min(v, p) \) in the first stage, and the mechanism gets a fixed revenue of 1 in the second stage. In order to obtain a \( c \)-approximation, this quantity must be at least \( \frac{1}{c} E_{v \sim F_1}[\min(v, U_\varepsilon(0))] \), implying, by the definition of \( p_\varepsilon \) that \( p \geq p_\varepsilon \).
3. This follows by recalling that for any $p$ in $\mathcal{M}_\varepsilon$, and any $p' < p < 1/\varepsilon$, it must be the case that the effective price for $p'$ is at least as large:

$$p' - U_\varepsilon(\ell(p')) \geq p - U_\varepsilon(\ell(p)).$$

However, as $U_\varepsilon(\ell(p)) = \int_{\ell(p)}^{1/\varepsilon} \frac{1 - F_2(v_2) - \varepsilon}{1 - \varepsilon} dv_2$, then

$$U_\varepsilon(\ell(p)) - U_\varepsilon(\ell(p')) = \int_{\ell(p)}^{\ell(p')} \frac{1 - F_2(v_2) - \varepsilon}{1 - \varepsilon} dv_2 \geq \int_{\ell(p)}^{\ell(p')} \frac{1 - F_2(v_2) - \varepsilon'}{1 - \varepsilon'} dv_2 = U_\varepsilon'(\ell(p)) - U_\varepsilon'(\ell(p')) \geq p - p'.$$

Hence the effective price of $p - U_\varepsilon(\ell(p))$ is preferable to any smaller first-stage price under $\varepsilon$; a buyer with value $p$ would prefer this menu option to all others.

4. For any $p < p_\varepsilon$, because $p_\varepsilon$ is in the effective menu, then for all $p < p_\varepsilon$,

$$p_\varepsilon - U_\varepsilon(\ell(p_\varepsilon)) \leq p - U_\varepsilon(\ell(p))$$

hence,

$$\lim_{p \to p_\varepsilon} \frac{U_\varepsilon(\ell(p_\varepsilon)) - U_\varepsilon(\ell(p))}{p_\varepsilon - p} \geq 1.$$
Proof of Lemma 6. For any $\varepsilon \geq e^{-n}$,

$$U_\varepsilon(0) = 1 + \int_1^{e^n} y_\varepsilon(1/v) dv$$

$$= 1 + \int_1^{1/\varepsilon} \left[ \frac{1}{v}(1 + \varepsilon) - \varepsilon \right] dv + \int_{1/\varepsilon}^{e^n} \frac{1}{v^3} dv$$

$$= 1 + (1 + \varepsilon) \ln \frac{1}{\varepsilon} - \varepsilon \left( \frac{1}{\varepsilon} - 1 \right) + \varepsilon - e^{-n}$$

$$= 2\varepsilon - e^{-n} + (1 + \varepsilon) \ln \frac{1}{\varepsilon}.$$ 

Since $\varepsilon \geq e^{-n}$, this is at least $\ln \frac{1}{\varepsilon}$. If $\varepsilon < e^{-n}$, a similar argument shows $U_\varepsilon(0) \geq n$. \hfill \Box