RATIONAL EQUIVALENCE OF CUSPS

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Abstract. We prove that two cusps of the same dimension in the Baily-Borel compactification generate the same line in the rational Chow group for the following three types of modular varieties: orthogonal modular varieties, Siegel modular varieties and modular varieties of unitary type. This gives a higher dimensional analogue of the Manin-Drinfeld theorem.

1. Introduction

The classical theorem of Manin and Drinfeld ([12], [5], [6]) asserts that the difference of two cusps is torsion in the Picard group of the modular curve for a congruence subgroup of $\text{SL}_2(\mathbb{Z})$. In this paper we prove an analogue of this result for cusps in the Baily-Borel compactification of some higher dimensional classical modular varieties. In higher dimension cusps are algebraic cycles of various codimension. We wish to clarify their contribution to the Chow group of the Baily-Borel compactification.

The modular varieties of our object are of the following three types:

(1) modular varieties of orthogonal type attached to rational quadratic forms of signature $(2, n)$, which have only 0-dimensional and 1-dimensional cusps;

(2) Siegel modular varieties attached to rational symplectic forms; and

(3) modular varieties of unitary type, including the Picard modular varieties, attached to Hermitian forms over imaginary quadratic fields.

In Cartan’s classification of irreducible Hermitian symmetric domains, these correspond to type IV, III, I respectively. Below, by a cusp we mean the closure, in the Baily-Borel compactification of the modular variety, of the image of a rational boundary component of the Hermitian symmetric domain.

Our main results are the following.

Theorem 1.1 (orthogonal case). Let $\Lambda$ be an integral quadratic lattice of signature $(2, n)$, $\Gamma$ a congruence subgroup of the orthogonal group $O^+(\Lambda)$, and $X_\Gamma$ the Baily-Borel compactification of the modular variety defined by

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Let $Z_1, Z_2$ be two cusps of $X_\Gamma$ of the same dimension, say $k \in \{0, 1\}$. Assume that $n \geq 3$ if $k = 1$. Then we have $\mathbb{Q}[Z_1] = \mathbb{Q}[Z_2]$ in the rational Chow group $CH_k(X_\Gamma)_\mathbb{Q} = CH_k(X_\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}$ of $X_\Gamma$.

**Theorem 1.2** (symplectic case). Let $\Lambda$ be an integral symplectic lattice, $\Gamma$ a congruence subgroup of the symplectic group $\text{Sp}(\Lambda)$, and $X_\Gamma$ the Satake-Baily-Borel compactification of the Siegel modular variety defined by $\Gamma$. If $Z_1, Z_2$ are two cusps of $X_\Gamma$ of the same dimension, say $k$, then $\mathbb{Q}[Z_1] = \mathbb{Q}[Z_2]$ in $CH_k(X_\Gamma)_\mathbb{Q}$.

**Theorem 1.3** (unitary case). Let $K$ be an imaginary quadratic field, $\Lambda$ a Hermitian lattice over $O_K$, $\Gamma$ a congruence subgroup of the unitary group $U(\Lambda)$, and $X_\Gamma$ the Baily-Borel compactification of the modular variety defined by $\Gamma$. If $Z_1, Z_2$ are two cusps of $X_\Gamma$ of the same dimension, say $k$, then $\mathbb{Q}[Z_1] = \mathbb{Q}[Z_2]$ in $CH_k(X_\Gamma)_\mathbb{Q}$.

Note that for 0-dimensional cusps, the equality $\mathbb{Q}[Z_1] = \mathbb{Q}[Z_2]$ is equivalent to $[Z_1] = [Z_2]$ in $CH_0(X_\Gamma)_\mathbb{Q}$, or in other words, $[Z_1] - [Z_2]$ is torsion in the integral Chow group $CH_0(X_\Gamma)$. We also notice that when $\Lambda$ has rank $\geq 4$ in the symplectic case, every finite-index subgroup of $\text{Sp}(\Lambda)$ is a congruence subgroup ($13$, $2$).

The case $(n, k) = (2, 1)$ in Theorem [1.1] is indeed an exception. We have self products of modular curves as typical examples of $X_\Gamma$ in $n = 2$, for which two transversal boundary curves are not homologically equivalent. On the other hand, we should note that some consideration in the case $n = 2$ is necessary for our proof for the case $n \geq 3$.

The proof of Theorems [1.1]–[1.3] is based on the same simple idea. We connect $Z_1$ and $Z_2$ by a chain of sub modular varieties or their products, through the interior or the boundary, and use induction on the dimension of modular varieties. This eventually reduces the problem to the Manin-Drinfeld theorem for modular curves. The actual argument requires case-by-case construction depending on the combinatorics of rational boundary components. We need to argue the three cases separately, though the symplectic and the unitary cases are similar. Theorem [1.1] is proved in §2, Theorem [1.2] in §3, and Theorem [1.3] in §4.

As a consequence of these results, we obtain higher dimensional analogues of modular units ([9]) as higher Chow cycles on the modular varieties $Y_\Gamma = \Gamma \backslash \mathcal{D}$ before compactification. Indeed, the localization exact sequence of higher Chow groups ([3])

$$
\cdots \to CH_k(Y_\Gamma, 1)_\mathbb{Q} \to CH_k(X_\Gamma - Y_\Gamma)_\mathbb{Q} \to CH_k(X_\Gamma)_\mathbb{Q} \to CH_k(Y_\Gamma)_\mathbb{Q} \to 0
$$

tells us that for each pair $\{Z_1, Z_2\}$ of cusps of dimension $k$, there exists an element of $CH_k(Y_\Gamma, 1)_\mathbb{Q}$ which is mapped to $Z_1 - \alpha Z_2$ in $CH_k(X_\Gamma - Y_\Gamma)_\mathbb{Q}$ for a suitable $\alpha \in \mathbb{Q}$. By our proof, when $Z_1, Z_2$ are not top dimensional, $Z_1 - \alpha Z_2$
already vanishes in $\text{CH}_i(X_{\Gamma} - Y_{\Gamma})_\mathbb{Q}$ in many cases. When nonzero (e.g., for top dimensional cusps), the higher Chow cycle is essentially made out of modular units on modular curves.

Throughout this paper $\Gamma(N)$ stands for the principal congruence subgroup of $\text{SL}_2(\mathbb{Z})$ of level $N$, and $X(N) = \Gamma(N)\backslash \mathbb{H}$ the (compactified) modular curve for $\Gamma(N)$. In §2 and §3 for a free $\mathbb{Z}$-module $\Lambda$ of finite rank, we denote by $\Lambda^\vee = \text{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ its dual $\mathbb{Z}$-module and denote $\Lambda_F = \Lambda \otimes_{\mathbb{Z}} F$ for $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$. For a $\mathbb{Q}$-vector space $V$ we also write $V^\vee = \text{Hom}_{\mathbb{Q}}(V, \mathbb{Q})$ and $V_F = V \otimes_{\mathbb{Q}} F$ when no confusion is likely to occur.

2. The orthogonal case

In this section we prove Theorem 1.1. We first recall orthogonal modular varieties (cf. [14], [10]). Let $\Lambda$ be a free $\mathbb{Z}$-module of rank $2 + n$ equipped with a nondegenerate symmetric bilinear form $(\ , ) : \Lambda \times \Lambda \to \mathbb{Z}$ of signature $(2, n)$. Let

$$Q_\Lambda = \{[C\omega] \in \mathbb{P}\Lambda_{\mathbb{C}} \mid (\omega, \omega) = 0\}$$

be the isotropic quadric in $\mathbb{P}\Lambda_{\mathbb{C}}$. The open set of $Q_\Lambda$ defined by $(\omega, \bar{\omega}) > 0$ consists of two connected components, and the Hermitian symmetric domain $D_\Lambda$ attached to $\Lambda$ is defined as one of them. This choice is equivalent to the choice of an orientation of a positive definite plane in $\Lambda_{\mathbb{R}}$.

Let $O(\Lambda)$ be the orthogonal group of $\Lambda$, namely the group of isomorphisms $\Lambda \to \Lambda$ preserving the quadratic form. We write $O^+(\Lambda)$ for the subgroup of $O(\Lambda)$ preserving the component $D_\Lambda$. For a natural number $N$ let $O^+(\Lambda, N) < O^+(\Lambda)$ be the kernel of the reduction map $O^+(\Lambda) \to \text{GL}(\Lambda/N\Lambda)$. A subgroup $\Gamma$ of $O^+(\Lambda)$ is called a congruence subgroup if it contains $O^+(\Lambda, N)$ for some level $N$.

There are two types of rational boundary components of $D_\Lambda$: 0-dimensional components and 1-dimensional components. 0-dimensional components correspond to isotropic $\mathbb{Q}$-lines $I$ in $\Lambda_{\mathbb{Q}}$: we take the point $p_I = [I_{\mathbb{C}}] \in Q_\Lambda$ in the closure of $D_\Lambda$ for each such $I$. 1-dimensional components correspond to isotropic $\mathbb{Q}$-planes $J$ in $\Lambda_{\mathbb{Q}}$: we take the connected component of $\mathbb{P}J_{\mathbb{C}} - \mathbb{P}J_{\mathbb{R}} = \mathbb{H} \sqcup \mathbb{H}$, say $\mathbb{H}_J$, that is in the closure of $D_\Lambda$. The union

$$D_\Lambda^* = D_\Lambda \sqcup \bigsqcup_{\dim J = 2} \mathbb{H}_J \sqcup \bigsqcup_{\dim I = 1} p_I$$

is equipped with the Satake topology ([1], [4]). By Baily-Borel ([1], [4]), the quotient space $X_{\Gamma} = \Gamma \backslash D_\Lambda^*$ has the structure of a normal projective variety and contains $\Gamma \backslash D_\Lambda$ as a Zariski open set.
In §2.1 we prove Theorem 1.1 for 0-dimensional cusps, and in §2.2 for 1-dimensional cusps. Throughout this section \( U \) stands for the rank 2 unimodular hyperbolic lattice with Gram matrix \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). The symbol \( \Lambda_1 \perp \Lambda_2 \) stands for the orthogonal direct sum of quadratic lattices (or spaces) \( \Lambda_1, \Lambda_2 \), while \( \Lambda_1 \oplus \Lambda_2 \) just stands for the direct sum of \( \Lambda_1, \Lambda_2 \) as \( \mathbb{Z} \)-module (or linear space) and does not mean that \( (\Lambda_1, \Lambda_2) \equiv 0 \).

### 2.1. 0-dimensional cusps

In this subsection we prove Theorem 1.1 for 0-dimensional cusps. Let \( I_1 \neq I_2 \) be two isotropic lines in \( \Lambda_Q \) and \( p_1, p_2 \in X_{\Gamma} \) the corresponding 0-dimensional cusps. We consider separately the cases where \( (I_1, I_2) \equiv 0 \) or \( (I_1, I_2) \not\equiv 0 \). In the former case \( p_1 \) and \( p_2 \) are joined by a boundary curve, while in the latter case they are joined by a modular curve through the interior of \( X_{\Gamma} \).

#### 2.1.1. The case \( (I_1, I_2) \equiv 0 \)

We first assume that \( (I_1, I_2) \equiv 0 \). The direct sum \( J = I_1 \oplus I_2 \) is an isotropic plane in \( \Lambda_Q \). Let \( \mathbb{H}_J' = \mathbb{H}_J \cup \bigsqcup_{\gamma \in \Gamma} p_\gamma \) and \( \Gamma_J \subset \text{SL}(J) \) be the image of the stabilizer of \( J \) in \( \Gamma \). We have a generically injective morphism \( f : X_J \to X_{\Gamma} \) from the modular curve \( X_J = \Gamma_J \backslash \mathbb{H}_J' \) onto the 1-dimensional cusp associated to \( J \).

**Claim 2.1.** \( \Gamma_J \) is a congruence subgroup of \( \text{SL}(J_Z) \) where \( J_Z = J \cap \Lambda \).

**Proof.** There exists a rank 2 isotropic sublattice \( J_Z' \) in \( \Lambda_Q \) such that \( J_Z' \cong (J_Z)^\vee \) by the pairing. The lattice \( \Lambda_1 = J_Z \oplus J_Z' \) is isometric to \( U \perp U \). We set \( \Lambda_2 = (\Lambda_1)^\perp \cap \Lambda \) and \( \Lambda' = \Lambda_1 \perp \Lambda_2 \). Recall that \( \Gamma \) contains \( \text{O}^+(\Lambda, N) \) for some level \( N \). Since both \( \Lambda \) and \( \Lambda' \) are full lattices in \( \Lambda_Q \), we can find natural numbers \( N_1, N_2 \) such that

\[
N_1 \Lambda' \subset N \Lambda \subset \Lambda \subset N_2^{-1} \Lambda'.
\]

If we set \( N' = N_1 N_2 \), this tells us that

\[
(2.1) \quad \text{O}^+(\Lambda', N') \subset \text{O}^+(\Lambda, N) \subset \Gamma
\]

inside \( \text{O}(\Lambda_Q) = \text{O}(\Lambda'_Q) \). Now we have the embedding

\[
\text{SL}_2(\mathbb{Z}) \simeq \text{SL}(J_Z) \hookrightarrow \text{O}^+(\Lambda'), \quad \gamma \mapsto (\gamma|_{J_Z}) \oplus (\gamma^\vee|_{J_Z}) \oplus \text{id}_{\Lambda_2}.
\]

Since this maps \( \Gamma(N') \) into \( \text{O}^+(\Lambda', N') \), we see that \( \Gamma_J \) contains \( \Gamma(N') \). \( \square \)

Let \( q_1, q_2 \) be the cusps of \( X_J \) corresponding to \( I_1, I_2 \) respectively. By this claim we can apply the Manin-Drinfeld theorem to \( X_J \). Therefore \( [q_1] = [q_2] \) in \( CH_0(X_J)_{\mathbb{Q}} \). Since \( f(q_1) = p_1 \) and \( f(q_2) = p_2 \), we obtain

\[
[p_1] = f_*[q_1] = f_*[q_2] = [p_2]
\]

in \( CH_0(X_{\Gamma})_{\mathbb{Q}} \).
2.1.2. The case \((I_1, I_2) \neq 0\). Next we assume that \((I_1, I_2) \neq 0\). In this case 
\(I_1 \oplus I_2\) is isometric to \(U_\mathbb{Q}\). Its orthogonal complement has signature 
\((1, n-1)\). If this contains an isotropic line, say \(I_3\), we could apply the result of §2.1.1 
to \(I_1 \oplus I_3\) and to \(I_3 \oplus I_2\), thus obtaining \([p_1] = [p_2]\) via \(I_3\). This assumption 
is always satisfied when \(n \geq 5\) ([15]), but not always when \(n \leq 4\). Below 
we use another construction.

We choose a vector \(v\) of positive norm from \((I_1 \oplus I_2)^\perp\) and put \(\Lambda'_\mathbb{Q} = I_1 \oplus 
I_2 \oplus \mathbb{Q}v\). Then \(\Lambda'_\mathbb{Q}\) has signature \((2, 1)\). Let \(\mathcal{D}_{\Lambda'}\) be the Hermitian symmetric 
domain attached to \(\Lambda'_\mathbb{Q}\). We have the natural inclusion \(\mathcal{D}'_{\Lambda'} \subset \mathcal{D}^*_{\Lambda'}\) which is 
compatible with the embedding of orthogonal groups

\[
\iota : \mathcal{O}^+(\Lambda'_\mathbb{Q}) \hookrightarrow \mathcal{O}^+(\Lambda_\mathbb{Q}), \quad \gamma \mapsto \gamma \oplus \text{id}_{\Lambda'_\mathbb{Q}}.
\]

Claim 2.2. There is a subgroup \(\Gamma' \subset \mathcal{O}^+(\Lambda'_\mathbb{Q})\) such that 
\(\iota(\Gamma') \subset \Gamma\) and that 
\(X' = \Gamma'\backslash \mathcal{D}^*_{\Lambda'}\) is naturally isomorphic to \(X(N)\) for some level \(N\).

Proof. Let \(\Lambda_1 = U \perp \langle 2 \rangle\). Then \(\Lambda'_\mathbb{Q}\) is isometric to the scaling of \((\Lambda_1)_\mathbb{Q}\) by 
some positive rational number. This gives canonical isomorphisms \(\mathcal{D}^*_{\Lambda'} \simeq 
\mathcal{D}^*_{\Lambda_1}\) and \(\mathcal{O}^+(\Lambda'_\mathbb{Q}) \simeq \mathcal{O}^+(\Lambda_1)_\mathbb{Q}\). The group \(\mathcal{O}^+(\Lambda_1)_\mathbb{Q}\) is related to \(\text{SL}_2(\mathbb{Q})\) 
by the following well-known construction (cf. [11] §2.4). Let \(V \subset M_2(\mathbb{Q})\) be the space of \(2 \times 2\) matrices with trace 0, equipped with the symmetric 
form \((A, B) = \text{tr}(AB)\). Then \(V \cap M_2(\mathbb{Z})\) is isometric to \(\Lambda_1\). By conjugation 
\(\text{SL}_2(\mathbb{Q})\) acts on \(V\). This defines a homomorphism

\[
\varphi : \text{SL}_2(\mathbb{Q}) \rightarrow \mathcal{O}^+(V) = \mathcal{O}^+(\Lambda_1)_\mathbb{Q}
\]

with \(\text{Ker}(\varphi) = \{\pm I\}\). (We have \(\text{Im}(\varphi) = \text{SO}^+(V)\), but we do not need this 
fact.) It is readily checked that \(\varphi(\Gamma(N)) \subset \mathcal{O}^+(\Lambda_1, N)\) for every level \(N\). Furthermore, \(\varphi\) is compatible with the Veronese isomorphism

\[
\mathbb{H}^* \rightarrow \mathcal{D}^*_{\Lambda_1}, \quad \tau \mapsto e + \tau v_0 - \tau^2 f,
\]

where \(e, f\) are the standard basis of \(U\) and \(v_0\) is a generator of \(\langle 2 \rangle\). Now by 
the same argument as (2.1), there exists a level \(N\) such that the embedding \(\iota\) 
maps \(\mathcal{O}^+(\Lambda_1, N)\) into \(\Gamma\). This proves our claim. \(\square\)

Let \(q_1, q_2\) be the cusps of \(X'\) corresponding to the isotropic lines \(I_1, I_2\) of 
\(\Lambda'_\mathbb{Q}\). By this claim we have a finite morphism \(f : X' \rightarrow X_\Gamma\) which sends 
\(q_1\) to \(p_1\) and \(q_2\) to \(p_2\). By the Manin-Drinfeld theorem for \(X'\) we have 
\([q_1] = [q_2]\) in \(CH_0(X')_\mathbb{Q}\). Sending this equality by \(f_\ast\), we obtain \([p_1] = [p_2]\) 
in \(CH_0(X_\Gamma)_\mathbb{Q}\). This finishes the proof of Theorem [11] for 0-dimensional cusps.

2.2. 1-dimensional cusps. In this subsection we prove Theorem [11] for 
1-dimensional cusps.
2.2.1. Preliminaries in $n = 2$. Although the case $n = 2$ is not included in Theorem [1.1] for 1-dimensional cusps, we need to study a special example in $n = 2$ as preliminaries for the proof for the case $n \geq 3$. We consider the lattice $2U = U \perp U$. Let $e_1, f_1$ be the standard basis of the first copy of $U$, and $e_2, f_2$ be that of the second $U$. Let $J'_1 = \mathbb{Q}e_2 \oplus \mathbb{Q}e_1$ and $J'_2 = \mathbb{Q}f_2 \oplus \mathbb{Q}f_1$, which are isotropic planes in $2U$. We take an arbitrary natural number $N$ and consider the modular surface $S(N) = O^*(2U, N) \backslash \mathcal{D}_{2U}$. Let $C_1, C_2$ be the boundary curves of $S(N)$ associated to $J'_1, J'_2$ respectively.

**Lemma 2.3.** We have $\mathbb{Q}[C_1] = \mathbb{Q}[C_2]$ in $CH_1(S(N))_\mathbb{Q}$.

**Proof.** Recall that we have the Segre isomorphism

$$\mathbb{H} \times \mathbb{H} \to \mathcal{D}_{2U}, \quad (\tau_1, \tau_2) \mapsto e_1 - \tau_1 \tau_2 f_1 + \tau_1 e_2 + \tau_2 f_2.$$  

This extends to $\mathbb{H}^* \times \mathbb{H}^* \to \mathcal{D}_{2U}^*$, and maps the boundary components $\mathbb{H} \times (\tau_2 = 0), \mathbb{H} \times (\tau_2 = i\infty)$ of $\mathbb{H}^* \times \mathbb{H}^*$ to the boundary components $\mathbb{H}^1, \mathbb{H}^2$ of $\mathcal{D}_{2U}^*$ respectively.

Let $J'_3 = \mathbb{Q}f_2 \oplus \mathbb{Q}e_1$ and $J'_4 = \mathbb{Q}e_2 \oplus \mathbb{Q}f_1$. By the pairing we identify $J'_2 \cong (J'_1)^\vee$ and $J'_4 \cong (J'_3)^\vee$. Then we define an embedding

$$\text{SL}_2(\mathbb{Q}) \times \text{SL}_2(\mathbb{Q}) = \text{SL}(J'_1) \times \text{SL}(J'_3) \hookrightarrow O^*(2U)$$

by sending $\gamma_1 \in \text{SL}(J'_1)$ to $(\gamma_1|_{J'_1}) \oplus (\gamma_1^\vee|_{J'_1})$ and $\gamma_3 \in \text{SL}(J'_3)$ to $(\gamma_3|_{J'_3}) \oplus (\gamma_3^\vee|_{J'_3}).$ This embedding of groups is compatible with the isomorphism (2.2) of domains, and it maps $\Gamma(N) \times \Gamma(N)$ into $O^*(2U, N).$ We thus obtain a finite morphism $f : X(N) \times X(N) \to S(N)$ which maps the boundary curves

$$C'_1 = X(N) \times (\tau_2 = 0), \quad C'_2 = X(N) \times (\tau_2 = i\infty)$$

of $X(N) \times X(N)$ onto $C_1, C_2$ respectively. By the Manin-Drinfeld theorem for the second copy of $X(N)$, we have $[C'_1] = [C'_2]$ in $CH_1(X(N) \times X(N))_\mathbb{Q}$. Sending this equality by $f$, we obtain the assertion. \(\square\)

2.2.2. The case $J_1 \cap J_2 = \{0\}$. We go back to the proof of Theorem [1.1]. Let $\Lambda$ have signature $(2, n)$ with $n \geq 3$. Let $J_1 \neq J_2$ be two isotropic planes in $\Lambda_{\mathbb{Q}}$ and $Z_1, Z_2 \subset X_\Gamma$ the corresponding 1-dimensional cusps. We first consider the case where $J_1 \cap J_2 = \{0\}$. In this case the pairing between $J_1$ and $J_2$ is perfect because $J_1^\perp / J_1$ is negative definite. The direct sum $\Lambda'_\mathbb{Q} = J_1 \oplus J_2$ is isometric to $2U$. We can take an isometry $2U \mathbb{Q} \to \Lambda'_\mathbb{Q}$ which maps $J'_1, J'_2$ to $J_1, J_2$ respectively. This gives an embedding of orthogonal groups

$$O^+(2U) \cong O^+(\Lambda'_\mathbb{Q}) \hookrightarrow O^+(\Lambda_{\mathbb{Q}}), \quad \gamma \mapsto \gamma \circ \text{id}_{\Lambda'_\mathbb{Q}},$$

which is compatible with the embedding $\mathcal{D}_{2U} \cong \mathcal{D}_{\Lambda'} \subset \mathcal{D}_{\Lambda}$ of domains. By the same argument as (2.1), we can find a level $N$ such that the embedding (2.3) maps $O^+(2U, N)$ into $\Gamma$. We thus obtain a finite morphism $f : S(N) \to \cdots$
X_\Gamma with \( f(C_1) = Z_1 \) and \( f(C_2) = Z_2 \). Sending the equality \( \mathbb{Q}[C_1] = \mathbb{Q}[C_2] \) of Lemma 2.3 by \( f_\ast \), we obtain \( \mathbb{Q}[Z_1] = \mathbb{Q}[Z_2] \) in \( CH_1(X_\Gamma)_\mathbb{Q} \).

2.2.3. The case \( J_1 \cap J_2 \neq \{0\} \). We next consider the case where \( J_1 \cap J_2 \neq \{0\} \). Let \( I = J_1 \cap J_2 \) and choose splittings \( J_1 = I \oplus I_1 \) and \( J_2 = I \oplus I_2 \). Since \( (I_1, I_2) \neq 0 \), we have \( I_1 \oplus I_2 \simeq U_\mathbb{Q} \). Let \( \Lambda'_Q = I_1 \oplus I_2 \) and \( \Lambda''_Q = (\Lambda'_Q)^\perp \). Then \( \Lambda''_Q \) has signature \((1, n-1)\). Since \( n-1 \geq 2 \) and \( \Lambda''_Q \) contains at least one isotropic line \( I \), then \( \Lambda''_Q \) contains infinitely many isotropic lines. We can choose isotropic lines \( I_3, I_4 \) in \( \Lambda''_Q \) such that \( I, I_3, I_4 \) are linearly independent. Put \( J_3 = I_4 \oplus I_2 \) and \( J_4 = I_3 \oplus I_1 \). Then \( J_3, J_4 \) are isotropic of dimension 2 and we have

\[
J_1 \cap J_3 = \{0\}, \quad J_3 \cap J_4 = \{0\}, \quad J_4 \cap J_2 = \{0\}.
\]

If \( Z_i \subset X_\Gamma \) is the 1-dimensional cusp corresponding to \( J_i \), we can apply the result of §2.2.2 successively and obtain

\[
\mathbb{Q}[Z_1] = \mathbb{Q}[Z_3] = \mathbb{Q}[Z_4] = \mathbb{Q}[Z_2]
\]

in \( CH_1(X_\Gamma)_{\mathbb{Q}} \). This finishes the proof of Theorem 1.1 for 1-dimensional cusps.

3. The symplectic case

In this section we prove Theorem 1.2. We first recall Siegel modular varieties (cf. \[8\], \[10\]). Let \( \Lambda \) be a free \( \mathbb{Z} \)-module of rank \( 2g \) equipped with a nondegenerate symplectic form \((\cdot, \cdot) : \Lambda \times \Lambda \to \mathbb{Z} \). Let \( Sp(\Lambda) \) be the symplectic group of \( \Lambda \), namely the group of isomorphisms \( \Lambda \to \Lambda \) preserving the symplectic form. For a natural number \( N \) we write \( Sp(\Lambda, N) \) for the kernel of the reduction map \( Sp(\Lambda) \to GL(\Lambda/\Lambda N) \). A subgroup \( \Gamma \) of \( Sp(\Lambda) \) is called a congruence subgroup if it contains \( Sp(\Lambda, N) \) for some level \( N \). When \( g \geq 2 \), every finite-index subgroup of \( Sp(\Lambda) \) is a congruence subgroup (\[13\], \[2\]).

Let

\[
LG_\Lambda = \{[V] \in G(\mathcal{L}, \Lambda_\mathbb{C}) | (\cdot, \cdot)|_V \equiv 0 \}
\]

be the Lagrangian Grassmannian parametrizing \( g \)-dimensional (= maximal) isotropic \( \mathbb{C} \)-subspaces of \( \Lambda_\mathbb{C} \). The Hermitian symmetric domain attached to \( \Lambda \) is defined as the open locus \( \mathcal{D}_\Lambda \subset LG_\Lambda \) of those \([V]\) such that the Hermitian form \( i(\cdot, \cdot)|_V \) on \( V \) is positive definite.

Rational boundary components of \( \mathcal{D}_\Lambda \) correspond to isotropic \( \mathbb{Q} \)-subspaces \( I \) of \( \Lambda_\mathbb{Q} \). To each such \( I \) we associate the locus \( \mathcal{D}_I \subset LG_\Lambda \) of those \([V]\) which contains \( I \) and for which \( i(\cdot, \cdot)|_V \) is positive semidefinite with kernel \( I_\mathbb{C} \). If we consider the rational symplectic space \( \Lambda'_Q = I^\perp / I \),
then $\mathcal{D}_I$ is canonically isomorphic to the Hermitian symmetric domain $\mathcal{D}_{\Lambda'}$ attached to $\Lambda'_Q$ by mapping $[V] \in \mathcal{D}_I$ to $[V/I_c] \in \mathcal{D}_{\Lambda'}$. The union

$$\mathcal{D}'_{\Lambda} = \mathcal{D}_{\Lambda} \sqcup \bigcup_{I \in \Lambda'_Q} \mathcal{D}_I$$

is equipped with the Satake topology ([1], [4], [8]). By Baily-Borel [1], the quotient space $X_\Gamma = \Gamma \backslash \mathcal{D}'_{\Lambda}$ has the structure of a normal projective variety and contains $\Gamma \backslash \mathcal{D}_{\Lambda}$ as a Zariski open set.

Theorem [1, 2] is proved by induction on $g$. The case $g = 1$ follows from the Manin-Drinfeld theorem. Let $g \geq 2$. Assume that the theorem is proved for every congruence subgroup of $\operatorname{Sp}(\Lambda')$ for every symplectic lattice $\Lambda'$ of rank $< 2g$. We then prove the theorem for $\Gamma < \operatorname{Sp}(\Lambda)$ with $\Lambda$ rank $2g$.

Let $I_1 \neq I_2$ be two isotropic $\mathbb{Q}$-subspaces of $\Lambda'_Q$ of the same dimension, say $g'$, and $Z_1, Z_2 \subset X_\Gamma$ the corresponding cusps. If we write $g'' = g - g'$, then $Z_i$ has dimension $k = g''(g'' + 1)/2$. We consider the following three cases separately:

1. $I_1 \cap I_2 \neq \{0\}$; or
2. the pairing between $I_1$ and $I_2$ is perfect; or
3. $I_1 \cap I_2 = \{0\}$ but the pairing between $I_1$ and $I_2$ is not perfect.

The case (1) is studied in §3.1 where $Z_1$ and $Z_2$ are joined by a modular variety in the boundary. The case (2) is studied in §3.2 where $Z_1$ and $Z_2$ are joined by a product of two modular varieties (when $k > 0$) or by a chain of boundary modular varieties (when $k = 0$). The remaining case (3) is considered in §3.3 where we combine the results of (1) and (2).

3.1. The case $I_1 \cap I_2 \neq \{0\}$. Assume that $I_1 \cap I_2 \neq \{0\}$. We write $I = I_1 \cap I_2$. In this case $\mathcal{D}_{I_1}, \mathcal{D}_{I_2}$ are in the boundary of $\mathcal{D}_I$. We set $\Lambda''_Q = I^\perp/I$, $I'_1 = I_1/I$ and $I''_2 = I_2/I$. Then $I'_1, I''_2$ are isotropic subspaces of $\Lambda''_Q$. The isomorphism $\mathcal{D}_I \rightarrow \mathcal{D}_{\Lambda''}$ extends to $\mathcal{D}'_I \rightarrow \mathcal{D}'_{\Lambda''}$ and maps $\mathcal{D}_I$ to $\mathcal{D}'_I$. The stabilizer of $I$ in $\Gamma$ acts on $\Lambda''_Q$ naturally. Let $\Gamma_I < \operatorname{Sp}(\Lambda''_Q)$ be its image in $\operatorname{Sp}(\Lambda''_Q)$. By a similar argument as Claim 2.1, $\Gamma_I$ is a congruence subgroup of $\operatorname{Sp}(\Lambda')$ for some lattice $\Lambda' \subset \Lambda''_Q$. If we put $X_I = \Gamma_I \backslash \mathcal{D}'_{\Lambda''}$, we have a generically injective morphism $f : X_I \rightarrow X_\Gamma$ onto the $I$-cusp.

Let $Z'_1, Z'_2 \subset X_I$ be the cusps of $X_I$ corresponding to $I'_1, I''_2 \subset \Lambda''_Q$ respectively. By our hypothesis of induction, we have $\mathbb{Q}[Z'_1] = \mathbb{Q}[Z'_2]$ in $CH_k(X_I)_\mathbb{Q}$. Since $f(Z'_1) = Z_i$, sending this equality by $f$, gives $\mathbb{Q}[Z'_1] = \mathbb{Q}[Z_2]$ in $CH_k(X_\Gamma)_\mathbb{Q}$. This proves Theorem [1, 2] in the case $I_1 \cap I_2 \neq \{0\}$.

3.2. The case $(I_1, I_2)$ perfect. Next we consider the case where the pairing between $I_1$ and $I_2$ is perfect. We shall distinguish the case $k = 0$ (i.e., $g' = g$) and the case $k > 0$ (i.e., $g' < g$).
3.2.1. The case $k = 0$. First let $g' = g$. In this case we have $\Lambda_{\mathbb{Q}} = I_1 \oplus I_2$. We choose a proper subspace $J_1 \neq \{0\}$ of $I_1$. We put $J_2 = J_1^\perp \cap I_2$ and $I_3 = J_1 \oplus J_2$. Then $I_3$ is isotropic of dimension $g$. By construction we have $I_1 \cap I_3 \neq \{0\}$ and $I_1 \cap I_2 \neq \{0\}$. Therefore we can apply the result of §3.1 to $I_1$ vs $I_3$ and to $I_3$ vs $I_2$. If $Z_3 \in X_\Gamma$ is the 0-dimensional cusp associated to $I_3$, this gives $[Z_1] = [Z_3] = [Z_2]$ in $CH_0(X_\Gamma)_{\mathbb{Q}}$.

3.2.2. The case $k > 0$. Next let $g' < g$. We set $\Lambda'_{\mathbb{Q}} = I_1 \oplus I_2$, which by our assumption is a nondegenerate rational symplectic space of dimension $2g'$. Then $\Lambda_{\mathbb{Q}}'' = (\Lambda'_{\mathbb{Q}})^+$ is also nondegenerate of dimension $2g''$ and we have $\Lambda_{\mathbb{Q}} = \Lambda_{\mathbb{Q}}' \perp \Lambda_{\mathbb{Q}}''$. Let $\mathcal{D}_{\Lambda'}$, $\mathcal{D}_{\Lambda''}$ be the Hermitian symmetric domains attached to $\Lambda_{\mathbb{Q}}'$, $\Lambda_{\mathbb{Q}}''$ respectively. We have the embedding of domains

\[(3.1) \quad \mathcal{D}_{\Lambda'} \times \mathcal{D}_{\Lambda''} \hookrightarrow \mathcal{D}_{\Lambda}, \quad (V', V'') \mapsto V' \oplus V''.\]

This is compatible with the embedding of groups

\[(3.2) \quad \text{Sp}(\Lambda'_{\mathbb{Q}}) \times \text{Sp}(\Lambda''_{\mathbb{Q}}) \hookrightarrow \text{Sp}(\Lambda_{\mathbb{Q}}), \quad (\gamma', \gamma'') \mapsto \gamma' \oplus \gamma''.\]

Since $I_1, I_2$ are maximal isotropic subspaces of $\Lambda'_{\mathbb{Q}}$, they correspond to the 0-dimensional rational boundary components $[(I_1)_c], [(I_2)_c]$ of $\mathcal{D}_{\Lambda'}$. Then (3.1) extends to $\mathcal{D}_{\Lambda'} \times \mathcal{D}_{\Lambda''} \hookrightarrow \mathcal{D}_{\Lambda}$ and maps $[(I_1)_c] \times \mathcal{D}_{\Lambda''}$ to $D_{\Lambda}$.

We take some full lattices $\Lambda' \subset \Lambda'_{\mathbb{Q}}$ and $\Lambda'' \subset \Lambda''_{\mathbb{Q}}$. By the same argument as (2.1), we can find a level $N$ such that (3.2) maps $\text{Sp}(\Lambda', N) \times \text{Sp}(\Lambda'', N)$ into $\Gamma$. If we put $X' = \text{Sp}(\Lambda', N) \backslash \mathcal{D}_{\Lambda'}$ and $X'' = \text{Sp}(\Lambda'', N) \backslash \mathcal{D}_{\Lambda''}$, we thus obtain a finite morphism $f : X' \times X'' \to X_\Gamma$. Let $p_1, p_2 \in X'$ be the 0-dimensional cusps corresponding to $I_1, I_2 \subset \Lambda'_{\mathbb{Q}}$ respectively. If we set

$$Z'_i = p_i \times X'' \subset X' \times X'',$$

the above consideration shows that $f(Z'_i) = Z_i$.

Since $g' < g$, we have $[p_1] = [p_2]$ in $CH_0(X')_{\mathbb{Q}}$ by our assumption of induction. Taking pullback by $X' \times X'' \to X'$, we obtain $[Z'_1] = [Z'_2]$ in $CH_k(X' \times X'')_{\mathbb{Q}}$. Finally, taking pushforward by $f$, we obtain $\mathbb{Q}[Z_1] = \mathbb{Q}[Z_2]$ in $CH_k(X_\Gamma)_{\mathbb{Q}}$. Thus Theorem 1.2 is proved in the case $(I_1, I_2)$ is perfect.

3.3. The remaining case. Finally we consider the remaining case, namely $I_1 \cap I_2 = \{0\}$ but the pairing between $I_1$ and $I_2$ is not perfect. Let $J_1 \subset I_1$, $J_2 \subset I_2$ be the kernels of the pairing between $I_1$ and $I_2$. We choose splittings $I_1 = J_1 \oplus K_1$ and $I_2 = J_2 \oplus K_2$. Then $\dim J_1 = \dim J_2$ and the pairing between $K_1$ and $K_2$ is perfect. We set $\Lambda_{\mathbb{Q}}' = K_1 \oplus K_2$ and $\Lambda_{\mathbb{Q}}'' = (\Lambda_{\mathbb{Q}}')^\perp$, which are nondegenerate subspaces of $\Lambda_{\mathbb{Q}}$ with $\Lambda_{\mathbb{Q}} = \Lambda_{\mathbb{Q}}' \perp \Lambda_{\mathbb{Q}}''$. By definition $J_1$ and $J_2$ are isotropic subspaces of $\Lambda_{\mathbb{Q}}''$ with $J_1 \cap J_2 = \{0\}$ and $(J_1, J_2) \equiv 0$. We can take another isotropic subspace $J_0$ of $\Lambda_{\mathbb{Q}}''$ of the same dimension as $J_1, J_2$ such that the pairings $(J_0, J_1)$ and $(J_0, J_2)$ are perfect. We set $I_3 = J_0 \oplus K_2$ and $I_4 = J_0 \oplus K_1$. Then $I_3, I_4$ are isotropic subspaces of $\Lambda_{\mathbb{Q}}$ of the same dimension
as $I_1, I_2$. By construction the pairings $(I_1, I_3)$ and $(I_2, I_4)$ are perfect, and we have $I_3 \cap I_4 \neq \{0\}$. Then we can apply the result of §3.2 to $I_1$ vs $I_2$ and to $I_3$ vs $I_4$, and when $\dim K_i > 0$ the result of §3.1 to $I_3 \cap I_4$. (When $\dim K_i = 0$, so that $I_3 = I_4$, the latter process is skipped.) If $Z_3, Z_4$ are the cusps of $X_\Gamma$ associated to $I_3, I_4$ respectively, this shows that

$$\mathbb{Q}[Z_1] = \mathbb{Q}[Z_3] = \mathbb{Q}[Z_4] = \mathbb{Q}[Z_2]$$

in $CH_k(X_\Gamma)_\mathbb{Q}$. This completes the proof of Theorem 1.2.

4. The unitary case

In this section we prove Theorem 1.3. We first recall modular varieties of unitary type (cf. [7], [10]). Let $K = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field with $R = O_K$ its ring of integers (or more generally an order in $K$). By a Hermitian lattice over $R$ we mean a finitely generated torsion-free $R$-module $\Lambda$ equipped with a nondegenerate Hermitian form $(, ) : \Lambda \times \Lambda \to R$. We denote $\Lambda_K = \Lambda \otimes_R K$ and $\Lambda_\mathbb{C} = \Lambda \otimes_R \mathbb{C}$, which are Hermitian spaces over $K, \mathbb{C}$ respectively and in which $\Lambda$ is naturally embedded. We may assume without loss of generality that the signature $(p, q)$ of $\Lambda$ satisfies $p \leq q$.

Let $U(\Lambda)$ be the unitary group of $\Lambda$, namely the group of $R$-linear isomorphisms $\Lambda \to \Lambda$ preserving the Hermitian form. This is the same as $K$-linear isomorphisms $\Lambda_K \to \Lambda_K$ preserving the lattice $\Lambda$ and the Hermitian form. We write $SU(\Lambda)$ for the subgroup of $U(\Lambda)$ of determinant 1. For a natural number $N$ we write $U(\Lambda, N)$ for the kernel of the reduction map $U(\Lambda) \to GL(\Lambda/N\Lambda)$. A subgroup $\Gamma$ of $U(\Lambda)$ is called a congruence subgroup if it contains $U(\Lambda, N)$ for some level $N$.

Let $Gr_\Lambda = G(p, \Lambda_\mathbb{C})$ be the Grassmannian parametrizing $p$-dimensional $\mathbb{C}$-linear subspaces of $\Lambda_\mathbb{C}$. The Hermitian symmetric domain $\mathcal{D}_\Lambda$ attached to $\Lambda$ is defined as the open locus

$$\mathcal{D}_\Lambda = \{ [V] \in Gr_\Lambda \mid (, )_V > 0 \}$$

where restriction of the Hermitian form to $V$ is positive definite. When $p = 0$, this is one point; when $p = 1$, this is a ball in $\mathbb{P}\Lambda_\mathbb{C} = \mathbb{P}^q$.

Rational boundary components of $\mathcal{D}_\Lambda$ correspond to isotropic $K$-subspaces $I$ of $\Lambda_K$. For each such $I$ we associate the locus $\mathcal{D}_I \subset Gr_\Lambda$ of those $V$ which contains $I$ and for which $(, )_V$ is positive semidefinite with kernel $I_\mathbb{C}$. If we consider $\Lambda'_K = I^\perp / I$, this is a nondegenerate $K$-Hermitian space of signature $(p - r, q - r)$ where $r = \dim_K I_i$, and $\mathcal{D}_I$ is naturally isomorphic to the Hermitian symmetric domain attached to $\Lambda'_K$ by sending $[V] \in \mathcal{D}_I$ to $[V/I]$.

The union

$$\mathcal{D}'_\Lambda = \mathcal{D}_\Lambda \cup \bigsqcup_{i \in \Lambda_K} \mathcal{D}_I$$
is equipped with the Satake topology ([1], [4]). By Baily-Borel [1], the quotient space \( X = \Gamma \backslash \mathcal{D}_\Lambda \) has the structure of a normal projective variety and contains \( \Gamma \backslash \mathcal{D}_\Lambda \) as a Zariski open set.

The proof of Theorem 1.3 proceeds by induction on \( q \). The case \( q = 1 \) is the Manin-Drinfeld theorem: we explain this in §4.1. The inductive argument is done in §4.2. Since this is similar to the symplectic case, we will be brief in §4.2.

4.1. On the case \( q = 1 \). Let \( q = 1 \). Then \( r = p = q = 1 \), so \( \Lambda_K \) is the (unique) \( K \)-Hermitian space of signature \((1, 1)\) containing an isotropic vector, and \( \mathcal{D}_\Lambda \) is the unit disc in \( \mathbb{P} \Lambda_C = \mathbb{P}^1 \). The group \( \text{SU}(\Lambda_K) \) is naturally isomorphic to \( \text{SL}_2(\mathbb{Q}) \), and \( \Gamma \cap \text{SU}(\Lambda) \) is mapped to a conjugate of a congruence subgroup of \( \text{SL}_2(\mathbb{Z}) \) under this isomorphism. This is a very classical fact, but since we could not find a suitable reference for the second assertion, below we supplement a self-contained account for the convenience of the reader. Theorem 1.3 in the case \( q = 1 \) then follows from the Manin-Drinfeld theorem, because we have a natural finite morphism from \( X_{\Gamma \cap \text{SU}(\Lambda)} \) to \( X_\Gamma \).

We embed \( K = \mathbb{Q}(\sqrt{-D}) \) into the matrix algebra \( M_2(\mathbb{Q}) \) by sending \( \sqrt{-D} \) to \( J_D = \begin{pmatrix} 0 & -D \\ 1 & 0 \end{pmatrix} \). Left multiplication by \( J_D \) makes \( M_2(\mathbb{Q}) \) a \( 2 \)-dimensional \( K \)-linear space. We have a \( K \)-Hermitian form on \( M_2(\mathbb{Q}) \) defined by

\[
(A, B) = \text{tr}(AB^*) + \sqrt{-D}^{-1}\text{tr}(J_DAB^*), \quad A, B \in M_2(\mathbb{Q}),
\]

where for \( B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) we write \( B^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \). We denote \( \Lambda_K = M_2(\mathbb{Q}) \) when we want to stress this \( K \)-Hermitian structure. Then \( \Lambda_K \) has signature \((1, 1)\) and contains an isotropic vector, e.g., \( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \). Right multiplication by \( \text{SL}_2(\mathbb{Q}) \) on \( M_2(\mathbb{Q}) \) is \( K \)-linear and preserves this Hermitian form. This defines a homomorphism

\[
(4.1) \quad \text{SL}_2(\mathbb{Q}) \to \text{SU}(\Lambda_K)
\]

which in fact is an isomorphism (see e.g., [16] §2).

Let \( \Lambda \subset \Lambda_K \) be a full \( R \)-lattice. We shall show that for every level \( N \) the image of \( \text{SU}(\Lambda, N) = \text{SU}(\Lambda_K) \cap U(\Lambda, N) \) by (4.1) is conjugate to a congruence subgroup of \( \text{SL}_2(\mathbb{Z}) \). Let

\[
O = \{ X \in M_2(\mathbb{Q}) \mid \Lambda X \subset \Lambda \}.
\]

This is an order in \( M_2(\mathbb{Q}) \) (see [11] §2.2). Then \( \text{SU}(\Lambda) = O^1 \), where for any subset \( S \) of \( M_2(\mathbb{Q}) \) we write \( S^1 = S \cap \text{SL}_2(\mathbb{Q}) \). Take a maximal order
$O_{\text{max}}$ of $M_2(\mathbb{Q})$ containing $O$. Since $O$ is of finite index in $O_{\text{max}}$, there exists a natural number $N_0$ such that $N_0O_{\text{max}} \subset O$. Therefore

$$I + NN_0O_{\text{max}} \subset I + NO \subset O \subset O_{\text{max}}.$$  

Since $(I + NO)^I \subset \text{SU}(\Lambda, N)$, this implies that

$$(I + NN_0O_{\text{max}})^I \subset \text{SU}(\Lambda, N) \subset \text{SU}(\Lambda) \subset O_{\text{max}}^I.$$  

Since every maximal order of $M_2(\mathbb{Q})$ is conjugate to $M_2(\mathbb{Z})$, there exists $g \in \text{GL}_2(\mathbb{Q})$ such that

$$\Gamma(NN_0) \subset \text{Ad}_g(\text{SU}(\Lambda, N)) \subset \text{Ad}_g(\text{SU}(\Lambda)) \subset \text{SL}_2(\mathbb{Z}).$$  

This proves our claim.

4.2. Inductive step. Let $q \geq 2$. Suppose that Theorem 1.3 is proved for all Hermitian lattices of signature $(p', q')$ with $p' \leq q' < q$. We then prove the theorem for Hermitian lattices of signature $(p, q)$ with $p \leq q$. Since the argument is similar to the symplectic case, we will just indicate the outline. Let $I_1 \neq I_2$ be two isotropic $K$-subspaces of $\Lambda_K$ of the same dimension, say $r$, and $Z_1, Z_2 \subset X_K$ the associated cusps. We make the following classification:

1. $I_1 \cap I_2 \neq \{0\}$; or
2. the pairing between $I_1$ and $I_2$ is perfect; or
3. $I_1 \cap I_2 = \{0\}$ but the pairing between $I_1$ and $I_2$ is not perfect.

1. This is similar to §3.1. In this case $Z_1$ and $Z_2$ are joined by the cusp associated to $I_1 \cap I_2$, to which we can apply the hypothesis of induction.

2. The case $r < q$ is similar to §3.2.2. If we set $\Lambda_K' = I_1 \oplus I_2$ and $\Lambda_K'' = (\Lambda_K')^\perp$, these are nondegenerate of signature $(r, r)$ and $(p - r, q - r)$ respectively. Then $Z_1$ and $Z_2$ are joined by the embedding $\mathcal{D}_{\Lambda_K'} \times \mathcal{D}_{\Lambda_K''} \hookrightarrow \mathcal{D}_\Lambda$. We can apply the hypothesis of induction to $\mathcal{D}_{\Lambda_K'}$.

The case $r = q$ is similar to §3.2.1. We have $r = p = q$ and $\Lambda_K = I_1 \oplus I_2$. We can interpolate $Z_1$ and $Z_2$ by a third 0-dimensional cusp by taking a proper subspace $J_1 \neq \{0\}$ of $I_1$ and setting $I_3 = J_1 \oplus (J_1^\perp \cap I_2)$. Then we can apply the result of the case (1) to $I_1 \text{ vs } I_3$ and to $I_3 \text{ vs } I_2$.

3. This is similar to §3.3. We take splittings $I_1 = J_1 \oplus K_1$ and $I_2 = J_2 \oplus K_2$ such that $(J_1, I_2) \equiv 0$, $(J_2, I_1) \equiv 0$ and $(K_1, K_2)$ perfect. We choose an isotropic subspace $J_0$ from $(K_1 \oplus K_2)^\perp$ with $(J_1, J_0)$ and $(J_2, J_0)$ perfect, and put $I_3 = J_0 \oplus K_2$ and $I_4 = J_0 \oplus K_1$. Then we apply the case (2) to $I_1 \text{ vs } I_3$ and to $I_4 \text{ vs } I_2$, and the case (1) to $I_3 \text{ vs } I_4$ when $K_i \neq \{0\}$. This proves Theorem 1.3.
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