Diagonal distance of quantum codes and hardness of the minimum distance problem

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Abstract

The diagonal distance or graph distance is an important parameter of a quantum error-correcting code that characterizes whether the code is degenerate or not. Degeneracy is a property unique to quantum codes, which allows quantum codes, unlike their classical counterparts, to correct more errors than they can uniquely identify. In the CWS framework introduced by Cross, Smith, Smolin and Zeng (2009), a quantum code is constructed using a classical code and a graph. It is known that the diagonal distance of such a code is upper bounded by $\delta + 1$, where $\delta$ is the minimum degree of the associated graph. In this paper, we give sufficient conditions on a graph such that a CWS code constructed from it has diagonal distance at least $\delta$, and in fact most of the graphs in our sufficient class achieve the upper bound of $\delta + 1$.

Using this result, first we give necessary conditions for a CWS code to be degenerate. Secondly, we prove hardness results for the problem of finding the distance of a CWS code. It is known that the distance of a CWS code is upper bounded by the distance of the associated classical code. Using the sufficient class of graphs from our first result, we construct a CWS code from a given classical code, with the distance of the CWS code being equal to the distance of the classical code. This allows us to translate well-known hardness results for computing the minimum distance in classical codes to quantum codes. Specifically, we show that exactly computing the distance of a CWS code is NP-complete, and multiplicatively or additively approximating it is NP-hard under polynomial-time randomized reductions. Our reduction from the classical problems to the quantum problems results in a non-degenerate quantum code, hence our result implies that the quantum problems remain NP-hard even with the promise that the code is non-degenerate. Moreover, using a mapping from stabilizer codes to CSS codes due to Bravyi, Terhal and Leemhuis (2010), we are able to show that the hardness results hold for CSS codes as well.

1 Introduction

In 1995, Shor [Sho95] showed that similar to classical computation, quantum computation can also be supported by an error correcting code. Soon after that, Calderbank and Shor [CS96] and Steane [Ste96] came up with a standard procedure to combine two classical error correcting codes to obtain a quantum error correcting code. In his seminal work, [Got97] Gottesman formalized the stabilizer set up, giving a group theoretic framework to the study of quantum error

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correction. Although quite a few non-stabilizer codes were introduced later [NVY18, RHSS97], many of the good and popular error correcting codes, such as toric codes [Kit03], surface codes that are useful for large computation [FMMC12] and recent quantum LDPC codes [HHO21], still come from the class of stabilizer codes. 

As far back as the works of Calderbank, Shor and Steane, one of the most important directions for constructing new quantum codes has been fruitful use of classical error correction. One common to obtain quantum codes is by using two classical codes such that one is contained in the dual of the other. Codeword stabilized (CWS) codes [CSSZ09] present another way to use classical codes. Codeword stabilized codes are quantum codes made out of two classical objects: a graph, and a code. The class of CWS codes contains all the stabilizer codes and also encompasses some non-stabilizer (or non-additive) quantum codes.

**Diagonal distance.** One of the main qualitative properties in which quantum codes differ from classical codes is their degeneracy. It allows a quantum code to correct more errors than it can uniquely identify. The Knill-Laflamme theorem [KL97] that characterizes the distance of a quantum code intrinsically also demonstrates this degenerate nature of errors in the quantum setting. According to the Knill-Laflamme condition for error correction, a quantum code with basis elements \{\ket{c_1}, \ket{c_2}, \ldots, \ket{c_k}\}, can uniquely identify all the errors \(E\) where \(\langle c_i | E | c_j \rangle = 0\) for \(i \neq j\). That is, a correctable error \(E\) should not have any off-diagonal entries in the basis of the codewords. However, \(E\) can afford to have non-zero diagonal entries \(\langle c_i | E | c_i \rangle\) as long as it is constant over the diagonal (for a fixed \(E\)). This property has no analogue in classical codes, as an error acting on a codeword will always take it to a different string (which may or may not be a codeword), which has no overlap with the original codeword. The minimum size of a Pauli error that has a non-zero diagonal entry \(\langle c_i | E | c_i \rangle\) is called the diagonal distance or graph distance\(^1\) of the code, and it characterizes whether the code is degenerate or non-degenerate. The minimum distance or simply distance of the code is the minimum size of a Pauli error which the code cannot detect. All quantum codes with distance less than diagonal distance are non-degenerate, whereas all codes with distance more than that are degenerate.

The distance of a CWS code is upper bounded by the distance of the classical code. Moreover, when the classical code uses all its components (i.e., the code is not trivially 0 in any of its components), a CWS code with a graph that has minimum degree \(\delta\), has both its distance and diagonal distance upper bounded by \(\delta + 1\). Degeneracy plays an important role in many problems related to quantum error correction, and hence the diagonal distance in an important parameter in these problems. For example, [LYGG08] gives a method to construct non-degenerate CWS codes by search. Specifically, their method fixes a graph and searches for good quantum codes with this fixed graph; to get a code with a large distance this way, a graph with a large diagonal distance should be used. Degeneracy also affects the decoding of quantum error correcting codes. In [FEMCGF21], it was shown that decoding strategies that work for sparse classical face additional challenges in the quantum domain because of degeneracy. [KL21] gave a belief propagation algorithm, which is a widely used method for decoding classical codes, that works for quantum codes by exploiting degeneracy appropriately.

**Hardness of problems related to error correction.** The two most important computational problems related to error-correcting codes are: the problem of decoding a codeword upon which an error has acted, and the problem of finding the minimum distance of a code. The decoding problem can be stated in many forms, but in syndrome decoding for linear codes, we

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\(^1\)In CWS codes, the diagonal distance is determined entirely by the graph associated with the code — hence the nomenclature. For this reason, we shall refer to the quantity as the diagonal distance of the code or the graph interchangeably.
are given an error syndrome. In the classical case, if the parity check matrix of the code is $H$, the syndrome of an error $e$ on codeword $x$ is $H(x + e) = He$. In maximum likelihood decoding, we are given an error syndrome $s$ and we want to find the most likely error that lead to it; if we assume there is an error in each component independently, this is the vector $u$ that satisfies $Hu = s$ and has minimum Hamming weight. The minimum distance problem of a code looks similar to the maximum likelihood decoding problem: here we are given the parity check matrix of the code and want to find the non-zero vector $u$ of minimum weight such that $Hu$ is the zero vector. Note that because of the non-zero vector requirement in the minimum distance problem, it is not simply a special case of the maximum likelihood decoding problem.

Both the maximum likelihood decoding problem and the minimum distance problem for classical codes (in their decision versions) are known to be NP-complete [BMvT78, Var97]. The first problem being NP-complete obviously closes the avenue for a generic decoding algorithm that works for all linear codes and errors of any weight. Of course this problem is trying to decode all linear codes, under all possible errors: good decoding algorithms exist for specific classes of codes, and when the error is promised to have weight at most $d/2$, where $d$ is the distance of the code. But actually finding the distance of a generic code in order to verify that the promise on the error is satisfied, is also hard. The minimum distance problem being NP-hard also rules out an obvious way of finding codes with good distance: if there was a polynomial time algorithm for the problem, one could generate parity check matrices at random and check if the associated code has good enough distance or not. Such an algorithm might still have been possible if one could efficiently compute the minimum distance of a code approximately rather than exactly. But in fact, (multiplicative and additive) approximate versions of the minimum distance problem have also been shown to be NP-hard under polynomial-time randomized reductions [DMS99].

Syndrome decoding for quantum stabilizer codes can be defined analogous to the classical case. In this case however, rather than finding the most likely error, finding the most likely error coset is desired [PC08]. In [HLG11], it was shown that using Hamming weight as the distance metric, decoding of stabilizer codes is are NP-hard, regardless of whether most likely error or most likely error coset is considered. In [Fuj12], it was shown that using symplectic weight as the distance metric, the decision version of the maximum likelihood decoding problem is NP-complete. [KL20] showed that this result holds even when one considers a smaller class of stabilizer codes with low full-rank check matrices, and when one restricts the error model to the depolarizing channel. [IP15] considered the quantum decoding problem taking degeneracy into account: they showed that the problem of degenerate quantum maximum likelihood decoding, which tries to find the most likely equivalence class of errors, all of which lead to the same syndrome, is in fact #P-complete.

As far as we are aware, the hardness of the quantum minimum distance problem has however not been studied. Note that showing that a minimum distance problem is hard can be quite different from showing a decoding problem is hard: in the classical case, there was a gap of nearly two decades between showing that the maximum likelihood decoding problem is hard and the minimum distance problem is hard. The results showing the NP-hardness of quantum decoding problems have all reduced the classical maximum likelihood decoding problem to the quantum problem. For example, in the formulation of [Fuj12], the decision version of the maximum likelihood decoding problem for stabilizer codes is stated as follows: given two $m \times n$ binary matrices $A$ and $A'$ such that $AA^T + A'A^T$ is the zero matrix, a syndrome $y$ of length $m$, do there exist vectors $x$ and $x'$ of length $n$ such that the Hamming weight of $x \lor x'$ is at most $w$, and $Ax + A'x' = y$? The decision version of the classical problem (is there a vector $u$ of weight at most $w$ such that $Hu = s$?) reduces to this easily by setting $A = H, y = s$, and $A'$ being the zero matrix. The minimum distance problem for stabilizer codes can similarly be stated as follows: given matrices $A$ and $A'$ such that $AA^T + A'A^T$ is the zero matrix, are
there vectors $x, x'$ of which at least one is non-zero, such that the Hamming weight of $x \lor x'$ is at most $w$, and $Ax + A'x'$ is the zero vector? However, it is not difficult to see that the classical minimum distance problem cannot be easily reduced to this problem; in particular, setting $A'$ to be the zero matrix does not help.

1.1 Our results

Our first result gives a sufficient condition for the diagonal distance of quantum CWS codes (or its associated graph) to be close to its maximum value of $\delta + 1$.

**Theorem 1.** Let $D_{\text{dist}}$ denote the diagonal distance of a graph or its associated CWS code. If $G$ is a graph with minimum degree $\delta$ and no 4-cycles, then $D_{\text{dist}}(G)$ is either $\delta$ or $\delta + 1$.

In fact we are able to further classify when the diagonal distance of such a graph is $\delta$ and when it is $\delta + 1$: for graphs without 4-cycles, only ones that have a particular very symmetric subgraph (that contains 3-cycles) have diagonal distance $\delta$, and the rest have diagonal distance $\delta + 1$. The description of this subgraph is rather technical, so we only provide a statement of this characterization in Lemma 2 later in the paper. However, recalling that distance of a CWS code is also upper bounded by $\delta + 1$ when the classical code uses all its components, and using Lemma 2, we are able to give the following necessary conditions for a CWS code to be degenerate.

**Theorem 2.** If a CWS code is degenerate, then at least one of the following is true:

(i) The graph $G$ associated with the CWS code has a cycle of length 3 or 4;

(ii) The classical code associated with the CWS code does not use its components corresponding to the vertices of $G$ which are of minimum degree.

By the Even Circuit Theorem [BS74], a graph with no 4-cycles can have at most $O(n^{3/2})$ edges, which means the minimum degree of such a graph is $O(\sqrt{n})$. In fact efficient constructions of such graphs are also known [PES66]. So the above two theorems let us construct non-degenerate codes with distance up to $O(\sqrt{n})$. Although one hopes to get linear distance codes there have been previous interesting codes of square-root distance (up to logarithmic factors) such as the toric code and LDPC codes [Kit03, EKZ20].

We then apply our first theorem to study the hardness of the minimum distance problem for CWS codes. The intuition here is as follows: the distance of a CWS code is upper bounded by the minimum of $\delta + 1$, and the distance of the associated classical code. If the diagonal distance achieves its upper bound of $\delta + 1$, and this is higher than the distance of the classical code, then the distance of the quantum code is in fact equal to the distance of the classical code. Thus we can reduce the classical minimum distance problem to the quantum minimum distance problem. In fact approximating the minimum distance of a classical code also reduces to approximating the minimum distance of a quantum code. The quantum codes we get in the reduction in either case are non-degenerate. So we have the following theorem, informally stated.

**Theorem 3.** The minimum distance problem for CWS codes is NP-complete. Multiplicatively or additively approximating the distance of a CWS code is also NP-hard under polynomial-time randomized reductions. The problems remain NP-hard even with the promise that the CWS code is non-degenerate.

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2In our reduction, we actually make use of a classical code that does not use most of its components; in such cases, the upper bound on the distance of the quantum code is in terms of the minimum degree of a vertex of the graph corresponding to a used component of the code, instead of the overall minimum degree. However, since the graph we use is close to regular, the upper bound in terms of minimum degree still holds.
The randomized reductions in this theorem are of the same type as those considered in the classical result [DMS99]; we expand on this more in Section 2. Also note that the additive factors in the additive approximation are different in the classical and quantum case: the additive approximation problem considered by [DMS99] is to distinguish between the cases that a classical code has distance at most \( t \) or at least \( t + \tau \cdot n \), for \( \tau > 0 \). Whereas we consider the problem of distinguishing between the CWS code having distance at most \( t \) or at least \( t + \tau \cdot \sqrt{n} \). This is because the quantum problem we reduce the classical problem to will have distance \( O(\sqrt{n}) \), due to the graph having to be free of 4-cycles.

Hardness of problems related to quantum error correction have so far been studied in the stabilizer formalism. A stabilizer code is described by the generators of its corresponding stabilizer group. However, as mentioned before, when the classical code of a CWS code is linear, stabilizer codes and CWS codes are equivalent. The problems that we show hardness results for are all stated in the CWS framework. However, the CWS description of a code can be efficiently converted to its stabilizer description, as we explain in more detail in Section 2. Therefore, our hardness results translate to the stabilizer framework as well.

Moreover, [BTL10] give an efficient way to map an \((n, k)\) stabilizer code with distance \( d \) to an \((4n, 2k)\) CSS code with distance \( d \). CSS (Calderbank-Shor-Steane) codes are a special class of stabilizer codes where the stabilizer group is the product of two subgroups \( S_X \) and \( S_Z \), each containing Pauli operators only of the \( X \)-type or \( Z \)-type respectively. Their advantage is that, like CWS codes, they can be constructed using classical codes: the elements of \( S_X \) and \( S_Z \) can be described as
\[
S_X = \{X(c) = X^{c_1} \otimes \ldots \otimes X^{c_n} : c \in C_1 \subseteq \{0, 1\}^n\}
\]
\[
S_Z = \{Z(c) = Z^{c_1} \otimes \ldots \otimes Z^{c_n} : c \in C_2 \subseteq \{0, 1\}^n\},
\]
where \( C_1 \) and \( C_2 \) are classical codes that are mutually orthogonal. The CSS code can thus be fully described by two mutually orthogonal classical codes.

Fact 1 ([BTL10], Corollary 1). There is a mapping that takes an \((n, k)\) stabilizer code with distance \( d \) and produces an \((4n, 2k)\) CSS code with distance \( 2d \), in time \( \text{poly}(n) \).

This mapping goes via an intermediate object called a Majorana fermion code, which we shall not describe in this paper. But the mapping allows us to reduce the (exact or approximate) minimum distance problem in stabilizer codes to the minimum distance problem in CSS codes. Hence our hardness result for stabilizer codes has the following corollary.

Corollary 4. The minimum distance problem for CSS codes is \( \text{NP-hard} \). Multiplicatively or additively approximating the distance of a CSS code is also \( \text{NP-hard} \) under polynomial-time randomized reductions.

Just like in the classical case, the results for the approximate minimum distance problems imply that there is no \( \text{RP} \) (class of problems having polynomial-time randomized algorithms with one-sided error) algorithm for approximating the minimum distance of a stabilizer or CSS code, and that there is no hope of finding a stabilizer code with good distance by randomly picking generators of the stabilizer group, or a CSS code by randomly picking rows of two parity-check matrices that are orthogonal.

Organization of the paper

In Section 2 we describe classical and quantum codes, in the CWS and stabilizer framework, in more detail, as well as state the exact and approximate versions of the minimum distance problem. In Section 3, we state Theorems 1 and 2 more formally and prove them. In Section 4 we state Theorem 3 more formally and prove it.
2 Preliminaries

2.1 Classical codes

A linear error-correcting code $C$ is a $k$-dimensional linear subspace of an $n$-dimensional space, for $k < n$. Elements of the code $C$ are called codewords. In this paper we shall only consider vector spaces over $\mathbb{F}_2$, which is the alphabet of the code.

The minimum distance, or simply distance, of a code is the minimum Hamming distance between two distinct codewords $u$ and $v$, i.e., the number of components in the vector $u \oplus v$ that have non-zero entries. For a linear code $C$, the minimum distance is the same as the minimum Hamming weight of a non-zero codeword (note that the zero vector is always a codeword), and it will be denoted by $\text{dist}(C)$. A $k$-dimensional linear code with distance $d$, sitting inside an $n$-dimensional subspace is denoted as $(n, k, d)$ code (when we do not explicitly want to refer to distance, we shall call it an $(n, k)$ code). Note that an error is a vector $e$ which takes a codeword $x$ to $x \oplus e$. The code does not detect the error $e$ if $x \oplus e$ is also a codeword; since the code is linear, this means that $e$ itself is a (non-zero) codeword. Therefore, $\text{dist}(C)$ is in fact the minimum Hamming weight of an error that is not detected by the code. A code may not actually be able to correct all the errors that it detects, but it can correct errors of weight up to $\lfloor (\text{dist}(C) - 1)/2 \rfloor$.

Since a code $C$ is a $k$-dimensional subspace of an $n$-dimensional space, it can be thought of as the kernel of a matrix $H \in \mathbb{F}_2^{(n-k) \times n}$, which is called its parity check matrix. The code is completely specified by this matrix; we shall use $C(H)$ to denote the code specified by parity check matrix $H$. The distance of a code is then the minimum Hamming weight of a non-zero vector $u$ such that $Hu = 0^{n-k}$. We shall formally define the minimum distance problem for a classical code in these terms.

2.2 Quantum codes

A quantum error-correcting code $Q$ is a $2^k$-dimensional subspace of $(\mathbb{C}^2)^\otimes n$, for $k < n$. Errors in this case are unitary operators acting on $(\mathbb{C}^2)^\otimes n$. The Pauli operators, given by

$$\{X(a)Z(b) = X^{a_1}Z^{b_1} \otimes \ldots \otimes X^{a_n}Z^{b_n} : a, b \in \{0, 1\}^n\},$$

where $X$ and $Z$ are the single qubit Pauli $X$ and $Z$ matrices, form a basis for matrices acting on $(\mathbb{C}^2)^\otimes n$. In particular, any error can be expanded in the Pauli basis. The support of a Pauli operator is the set of components $i$ such that $a_i \lor b_i = 1$. For a general error, its support is the union of the supports of the Pauli operators in its expansion. The size or weight of an error is then defined as the cardinality of its support.

Analogous to classical codes, distance of a quantum code is defined as the minimum $d$ such that the code can detect errors of size up to $d-1$, and correct errors of size up to $\lfloor (d - 1)/2 \rfloor$. The Knill-Laflamme condition of error correction [KL97] states that an error $E$ can be detected by a quantum code $Q$ spanned by a basis $\{|c_i\rangle\}_{i=1}^k$ iff for all $i, j$

$$\langle c_i | E | c_j \rangle = \delta_{ij} f(E)$$

where $f$ is a function that depends only on the error $E$ (and not $|c_i\rangle$, $|c_j\rangle$). Therefore, the distance $\text{qdist}(Q)$ of a code $Q$ is the minimum size of an error $E$ that violates (1).\footnote{Note that there may still be some errors of size greater than $\text{qdist}(Q) - 1$ that are detectable by the code, just not all such errors.}
For $i \neq j$, equation (1) ensures that an error acting on a codeword does not cause it to have overlap with another codeword, so that they cannot be perfectly distinguished — this condition is analogous to what is required for classical codes. The condition for $i = j$, requiring that all diagonal elements of an error with respect to the codewords should be the same in order for the code to be detectable, is unique to quantum codes. The value $f(E)$ of the diagonal elements determines an important property of quantum codes known as degeneracy.

- If $f(E) = 0$ for all errors $E$ of size up to $q\text{dist}(Q) - 1$, then the code is called non-degenerate.
- If there exists some error $E \neq 1$ of size up to $q\text{dist}(Q) - 1$ for which $f(E) \neq 0$, then the code is called degenerate.

In the rest of the paper, we shall only deal with CWS quantum codes. In these codes, the only allowed values of $f(E)$ are 0 and $\pm 1$, and $f(E) \neq 0$ means that the code is immune to that particular error.

### 2.2.1 CWS codes

CWS codes are a general framework of describing quantum codes that includes stabilizer codes [CSSZ09]. An $(n, k)$ CWS code is described by a graph $G$ with $n$ vertices and an $(n, k)$ classical error correcting code $C$. In general the classical code part of a CWS code does not have to be a linear code. But in this work, we shall only concern ourselves with the case when $C$ is linear. CWS codes with $C$ being linear are exactly equivalent to stabilizer codes [CSSZ09].

The idea behind CWS codes is that any Pauli error $X(a)Z(b)$ acting on a basis codeword of a CWS code is equivalent up to sign to an error of the form $Z(b')$ acting on the same codeword. These $Z$ errors can then be corrected by the classical error-correcting code part of the CWS code.\(^4\)

More specifically, let $\{u^i\}_{i=1}^n$ be the columns of the adjacency matrix $A_G$ of the graph $G$ associated with a CWS code. For a basis codeword $|c\rangle$ of a CWS code and $E = X(a)Z(b)$, $E|c\rangle = \pm Z(Cl_G(E))|c\rangle$ (the sign here depends on the error, but crucially, not on the codeword), where

$$Cl_G(E) = b \oplus \left( \bigoplus_{i=1}^n a_i u^i \right).$$

(2)

Intuitively speaking, this means that an $X$ error on the $i$-th qubit in $E$ propagates down the edges of the $i$-th vertex of the graph $G$, and acts as a $Z$ error on all the neighbours of the $i$-th vertex. Note that $Cl_G(E)$ can be the zero vector, even if $E$ is a nontrivial Pauli error.

We shall use the following result about CWS codes proved in [CSSZ09].

**Fact 2** (Theorem 3 in [CSSZ09]). A CWS code $(C, G)$, where $C$ is a classical code with codewords $\{c_j\}$ and $G$ is a graph whose adjacency matrix has columns $\{u^i\}_i$, detects a Pauli error $E$ if and only if the following conditions are met:

(i) If $Cl_G(E) \neq 0^n$, then it must be detectable by $C$ (i.e., it must not be a non-zero codeword of $C$)

(ii) If $Cl_G(E) = 0^n$, then $E$ must satisfy $Z(c_j)E = EZ(c_j) \forall j$.

\(^4\)The ‘obvious’ quantum encoding of a classical code as quantum states would be as computational basis elements. Then classical errors are actually of the form $X(a')$. However, if the encoding is done in the Hadamard basis instead, then the classical code can deal with errors of the form $Z(b')$. 


Obviously if the code detects Pauli errors from a set $E$, then it also detects errors which are linear combinations of Pauli errors from $E$. The distance of the code is then the smallest size of Pauli error $E \neq 1$ for which the conditions in Fact 2 are not satisfied. We shall use this characterization of distance in most of our analysis in this paper.

For CWS codes, another important parameter is the diagonal distance or graph distance $D \text{dist}(Q)$, which is defined to be the minimum size of a Pauli error $E \neq 1$ such that $C_{\delta}(E) = 0^n$. Unlike $q \text{dist}(Q)$, the diagonal distance of $Q$ is a property determined entirely by its associated graph $G$. Hence sometimes we shall talk about diagonal distance of a graph (without there necessarily being an associated code); with some abuse of notation, we shall also use $D \text{dist}(G)$ to refer to this.

The diagonal distance determines whether a CWS code is degenerate or not: if $D \text{dist}(Q) < q \text{dist}(Q)$, then the code is degenerate; otherwise it is non-degenerate. This characterization can be seen to be equivalent to our earlier characterization of degeneracy as follows: for a CWS code, the only possible values of the function $f(E)$ in the Knill-Laflamme condition are $\pm 1$, as stated earlier. Hence a code being degenerate means that there is a nontrivial error $E$ of size less than $q \text{dist}(Q)$ that takes any codeword $|c\rangle$ to $\pm |c\rangle$. From the definition of $C_{\delta}(E)$, $C_{\delta}(E)$ being the zero vector for a nontrivial $E$ of size less than $q \text{dist}(Q)$ also means that $E$ takes any codeword $|c\rangle$ to $\pm |c\rangle$. From both perspectives, a nondegenerate code is ‘immune’ to these errors.

The following upper bound on diagonal distance and distance of CWS codes holds for all graphs [KDP11]; we provide a proof for completeness.

**Fact 3.** Suppose $G$ is a graph with $\mindeg(G) = \delta$, and $C$ is a classical code that uses all its components, i.e., $\forall i \in [n], \exists c_i \in C \text{ s.t. } c_i \neq 0$. Then for the CWS code $Q = (C, G)$, $q \text{dist}(Q), D \text{dist}(Q) \leq \delta + 1$.

**Proof.** For an $i$ for which there exists $c \in C \text{ s.t. } c_i \neq 0$, we shall show that there exists an error $E$ of size $\deg(v_i) + 1$ ($v_i$ being the $i$-th vertex in $G$) such that $C_{\delta}(E) = 0^n$, and $Z(c)E \neq EZ(c)$. By Fact 2, this proves the upper bound on both $q \text{dist}(Q)$ and $D \text{dist}(Q)$.

Let $E = X(e^i)Z(u^i)$, where $u^i$ is the $i$-th row in $A_G$, corresponding to the neighbourhood of $v_i$; the size of $E$ is $\deg(v_i) + 1$, since $u^i$ has 1s in $\deg(v_i)$ locations, excluding the $i$-th location where $e^i$ has a 1. Clearly $C_{\delta}(E) = u^i \oplus u^i = 0^n$. Moreover, since $c$ has a 1 in the $i$-th location, $Z(c)$ anti-commutes with $X(e^i)$. Therefore we have, $Z(c)E = Z(c)X(e^i)Z(u^i) = -X(e^i)Z(c)Z(u^i) = -X(e^i)Z(u^i)Z(c) = -EZ(c)$. This completes the proof.

Using condition (i) of Fact 2, the above fact then has the following corollary.

**Fact 4.** Suppose $G$ is a graph with $\mindeg(G) = \delta$ and $C$ is a classical code that uses all its components. Then for $Q = (C, G)$, $q \text{dist}(Q) \leq \min\{\delta + 1, \text{dist}(C)\}$.

### 2.2.2 Stabilizer formalism

Stabilizer codes are the most commonly used framework for quantum codes. A stabilizer code is a linear subspace of $(\mathbb{C}^2)^\otimes n$ that is the simultaneous $+1$ eigenspace of all elements of a stabilizer group. A stabilizer group is an abelian subgroup of the Pauli group that does not contain $-1$. It can be shown that such a subgroup must always have a simultaneous $+1$ eigensate.

A stabilizer code is completely specified by a minimal set of generating elements of its corresponding stabilizer group. If the size of a minimal generating set for the stabilizer group is $n - k$, then the corresponding stabilizer code is $2^k$-dimensional. Elements of the Pauli group
are of the form $X(a)Z(b)$, along with an overall phase of $\pm 1$ or $\pm i$. Since a stabilizer group cannot have $-1$, none of its generators can have $\pm i$ as its phase, since such a generator would square to $-1$. So the only possibilities for the phases of the generators are $\pm 1$. As it turns out, changing the sign of one or more of the generators changes the stabilizer code to another code that is isomorphic to it, and so we do not need to keep track of the signs of these generators either. We shall represent each generator only by the string $a|b$. An $(n,k)$ stabilizer code is specified by a matrix $S \in \mathbb{F}_n^{(n-k) \times 2^n}$, where each row of the matrix is interpreted as a string of the form $a|b$, and corresponds to a generator. Two rows $a_1|b_1$ and $a_2|b_2$ of this matrix are required to satisfy $a_2b_1 - a_1b_2 = 0$, since the stabilizer group has to be abelian.

A less common representation of a stabilizer code is with a matrix $P \in \mathbb{F}_4^{(n-k) \times n}$. The elements of $\mathbb{F}_4$ can be represented as $0, 1, \omega$ and $\bar{\omega}$, which satisfy the relations

$$\omega^2 = \bar{\omega}, \quad \omega^3 = 1, \quad 1 + \omega = \bar{\omega}.$$ 

Here also each row $r$ corresponds to a single generator. If the generator is $X(a)Z(b)$, then the convention is

$$r_i = \begin{cases} 
0 & \text{if } a_i = b_i = 0 \\
1 & \text{if } a_i = 0, b_i = 1 \\
\omega & \text{if } a_i = 1, b_i = 0 \\
\bar{\omega} & \text{if } a_i = b_i = 1. 
\end{cases}$$

As we have mentioned before, a CWS code with a linear classical code is a stabilizer code. It can be shown [KDP11] that the $\mathbb{F}_4$ stabilizer representation of a CWS code $Q = (C, G)$, where $C$ has parity check matrix $H$ and $G$ has adjacency graph $A_G$, is given by

$$P = H(\omega I + A_G).$$  \hfill (3) 

The matrix obtained from (3) will have $1 + \omega$ as some of its entries, which can be replaced by $\bar{\omega}$ due to the relationship between the elements. From here converting to the $\mathbb{F}_2$ representation is also trivial.

2.3 The minimum distance problem

**Problem 1** (Classical minimum distance problem, MINDIST).

**Instance:** A matrix $H \in \mathbb{F}_n^{(n-k) \times n}$ and an integer $t > 0$

**Question:** Is there a non-zero vector $u \in \mathbb{F}_n^2$ such that Hamming weight $wt_H(u) \leq t$ and $Hu = 0^{n-k}$?

It is clear that Problem 1 is in NP; it was also shown to be NP-complete by Vardy.

**Fact 5** ([Var97]). **MINDIST is NP-hard.**

In some applications, one may not need to exactly compute the distance of a code, but an approximation of it may be enough. [DMS99] studied two kinds of approximate versions of MINDIST, multiplicative and additive. In the multiplicative version, we wish to approximate the distance of a code up to a constant factor $\gamma$; in the additive version, we wish to approximate it by an additive factor of $\tau \cdot n$. The decision versions of these approximation problems are stated as the following promise problems.\footnote{In [DMS99], the problems are stated in terms of the generator matrix of the code, instead of the parity check matrix. The generator matrix is a matrix whose row space is equal to the code subspace, and given the generator matrix one can efficiently compute the parity check matrix and vice versa. Hence the two representations are equivalent for our purposes.}
Problem 2 (Multiplicative approximate minimum distance problem, GAPDISTγ).

**INSTANCE:** A matrix $H \in \mathbb{F}^{(n-k)\times n}$, an integer $t > 0$ and an approximation factor $\gamma \geq 1$

**QUESTION:** Is $\text{dist}(C(H)) \leq t$ or $\gamma \cdot t$?

Problem 3 (Additive approximate minimum distance problem, GAPADDDISTγ).

**INSTANCE:** A matrix $H \in \mathbb{F}^{(n-k)\times n}$, an integer $t > 0$, and an approximation factor $\tau > 0$

**QUESTION:** Is $\text{dist}(C(H)) \leq t$ or $t + \tau \cdot n$?

It was shown in [DMS99] that GAPDIST and GAPADDDIST are NP-hard under polynomial-time reverse unfaithful randomized (RUR) reductions. These are probabilistic reductions where no-instances always map to no-instances, and yes-instances map to yes-instances with high probability. In particular, given a parameter $s$, the [DMS99] reduction maps a yes-instance correctly with probability $1 - 2^{-s}$ in time $\text{poly}(s)$.

Fact 6 ([DMS99], Theorems 22 and 32). For every $\gamma \geq 1$, GAPDISTγ is NP-hard under polynomial time RUR reductions with soundness error exponentially small in a security parameter. Moreover, there exists a $\tau > 0$ such that GAPADDDISTγ is NP-hard under polynomial time RUR reductions with the same soundness error.

Note that the above results imply that there is no RP algorithm GAPDISTγ or GAPADDDISTτ, with any $\gamma$ and the $\tau$ given by the above fact, unless NP = RP. In fact the result for GAPDIST can be amplified to get that for every $\epsilon > 0$, there is no RQP algorithm for GAPDISTγ with $\gamma = 2^{\log^{(1-\epsilon)} n}$, unless NP is contained in RQP (class of problems having randomized quasipolynomial time algorithms with one-sided error).

We define the quantum versions of the above classical problems analogously. This will be done in the framework of CWS codes described in 2.2. We note that any graph $G$ with $n$ vertices and $(n,k)$ classical code describes a valid CWS code. Therefore, any parity-check matrix $H \in \mathbb{F}_2^{(n-k)\times n}$ and a graph $G$ with $n$ vertices (along with parameter $t$) forms a valid instance of the exact minimum distance problem of a CWS code. The two approximate versions are of course promise problems just like in the classical case.

Problem 4 (Quantum minimum distance problem, QMINDIST).

**INSTANCE:** A matrix $H \in \mathbb{F}_2^{(n-k)\times n}$, a graph $G$ with $n$ vertices, and an integer $t > 0$

**QUESTION:** For the CWS code $Q = (C(H), G)$, is $\text{dist}(Q) \leq t$, i.e., is there a Pauli error $E \neq 1$ of size $t$ that does not satisfy the conditions of Fact 2?

Problem 5 (Multiplicative approximate quantum minimum distance problem, GAPQDISTγ).

**INSTANCE:** A matrix $H \in \mathbb{F}_2^{(n-k)\times n}$, a graph $G$ with $n$ vertices, an integer $t > 0$, and an approximation factor $\gamma \geq 1$

**QUESTION:** For the CWS code $Q = (C(H), G)$, is $\text{dist}(Q) \leq t$ or $\gamma \cdot t$?

Problem 6 (Additive approximate quantum minimum distance problem, GAPADDDISTτ).

**INSTANCE:** A matrix $H \in \mathbb{F}_2^{(n-k)\times n}$, a graph $G$ with $n$ vertices, an integer $t > 0$, and an approximation factor $\tau > 0$

**QUESTION:** For the CWS code $Q = (C(H), G)$, is $\text{dist}(Q) \leq t$ or $t + \tau \cdot \sqrt{n}$?

Like MINDIST, it is easy to see that QMINDIST is in NP. The two approximation problems, being promise problems, are not. The additive approximation problem GAPADDDISTτ is
defined with the additive factor being $\tau \cdot \sqrt{n}$ instead of $\tau \cdot n$ like in the classical case because that is what naturally arises in our reduction between the classical and quantum minimum distance problems. We can also define versions of these problems where the CWS code is promised to be non-degenerate, though we shall not explicitly name them.

Since the stabilizer formalism is more commonly used for quantum codes, one may wish to state the exact and approximate quantum minimum distance problems in this formalism. But as we saw in 2.2.2, given a CWS code with a parity check matrix $H$ and an adjacency matrix $A_G$, we can efficiently obtain either the $\mathbb{F}_2$ or $\mathbb{F}_4$ representation of the corresponding stabilizer code. Therefore, hardness of the minimum distance problems with inputs in the CWS representation implies hardness of the minimum distance problems with inputs in the stabilizer representation.

3 Diagonal distance and degeneracy

3.1 Lower bound on diagonal distance for 4-cycle free graphs

In this section we prove Theorem 1, formally stated below.

**Theorem 1.** If $G$ is a graph with $\text{mindeg}(G) = \delta \geq 2$ and no 4-cycles, then $D_{\text{dist}}(G) \geq \delta$.

Before proving this theorem, we shall introduce some terms. Note that for a graph $G$ with columns $\{u_i\}$,

$$b \oplus \left( \bigoplus_{i=1}^{n} a_i u^i \right) = \left( \bigoplus_{j=1}^{n} b_j e^j \right) \oplus \left( \bigoplus_{i=1}^{n} a_i u^i \right)$$

where $e^j$ denotes the vector with 1 only in the $j$-th location. The above expression is the sum of a subset of columns of the matrix $(\mathbb{I}|A_G)$. If $G$ has minimum degree $\delta \geq 2$, then each column of $(\mathbb{I}|A_G)$ has Hamming weight either 1 or at least $\delta$ (with there existing a column which has weight exactly $\delta$). We call a such a set of vectors a set with a $\delta$ degree gap. If $\text{Cl}_G(E) = 0^n$ for some $E$, then that means a subset of columns of $(\mathbb{I}|A_G)$ of size some $d$ (which corresponds to the size of $E$ being between $d$ and $d/2$) sums to the zero vector. This subset is obviously one with a degree gap $\delta$.

It is not difficult to see that a graph $G$ has no 4-cycles if and only if each pair of vertices has at most one common neighbour. For a vector $v$, let $\text{supp}(v)$ denote the set of components where $v$ has a non-zero entry. The condition that each pair of vertices has at most one neighbour can be stated in terms of the columns of $A_G$ as

$$|\text{supp}(u^i) \cap \text{supp}(u^j)| \leq 1 \quad \text{for } j \neq i.$$  

Equation (4) is also true for the columns of $(\mathbb{I}|A_G)$, since all the columns of $\mathbb{I}$ have support size 1 anyway. We call a set of vectors where all pairs satisfy (4) an at-most-one-matching (ATOM) set.

A subset of columns of of $(\mathbb{I}|A_G)$ that sums to zero is an ATOM set with degree gap $\delta$. We prove the following lemma for such a set of vectors, which then proves Theorem 1.

**Lemma 1.** Let $S$ be a non-empty set of binary vectors which is ATOM, has degree graph $\delta$ and its elements sum to zero. Let $S_1$ denote the set of vectors of weight 1 in $S$, and let $S_\delta = S \setminus S_1$ be the set of vectors which have weight at least $\delta$. Then we must have, $\max\{|S_1|, |S_\delta|\} \geq \delta$.
Proof of Theorem 1. Let \( S \) satisfying the conditions of Lemma 1 correspond to an error \( E = X(a)Z(b) \). \( S_1 \) containing a vector \( e^l \) means that \( b_i = 1 \); similarly, each vector in \( S_\delta \) corresponds to \( a_i = 1 \) for a unique location \( j \). If \( Cl_G(E) = 0^n \), then by Lemma 1, there are at least \( \delta \) locations where \( a_i = 1 \), or \( \delta \) locations where \( b_i = 1 \). Since the size of \( E \) is the number of locations where \( a_i \lor b_i = 1 \), this means that the size of \( E \) is at least \( \delta \). Thus the minimum size of \( E \) for which \( Cl_G(E) = 0^n \), i.e., \( Dist(G) \), is at least \( \delta \).

Proof of Lemma 1. The proof will be done by case analysis. We deal with some easy cases first.

Claim 1. For any \( S \) satisfying the conditions of Lemma 1, must have \(|S| \geq \delta + 1 \). \( S \) of size \( \delta + 1 \) is of one of the following forms:

1. \( S_\delta = \{v\}, S_1 = \cup_{i \in \text{supp}(v)} \{e^l\} \)
2. \( S_\delta = \{v^1, \ldots, v^{\delta+1}\} \) for vectors \( v^i \) of weight exactly \( \delta \), \( S_1 = \emptyset \).

Proof. All the weight 1 vectors in \( S_1 \) have 1s in different locations, and these 1s cannot cancel between themselves in order for the sum of vectors in \( S \) to be zero. Therefore, we must have \(|S_\delta| \geq 1 \). If \( S_\delta \) has exactly one vector, this vector has 1s in at least \( \delta \) locations, all of which have to be cancelled by a unique vector in \( S_1 \). This gives \( S_1 = \cup_{i \in \text{supp}(v)} \{e^l\} \), i.e., \(|S_1| \geq \delta \). This means \(|S| \geq \delta + 1 \). \(|S| = \delta + 1 \) is achieved when the vector in \( S_\delta \) has weight exactly \( \delta \).

For the second part, we shall prove that if \(|S| = \delta + 1 \) and \(|S_\delta| \geq 2 \), then \( S_1 = \emptyset \). When \(|S_\delta| \geq 2 \), consider two vectors \( v^1 \) and \( v^2 \) in \( S_\delta \). The vector \( v^1 \oplus v^2 \) has 1s in at least \( 2\delta - 2 \) locations, since each of \( v^1 \) and \( v^2 \) has 1s in at least \( \delta \) locations, of which at most one location is common. For \( S \) to sum to zero, all these 1s need to be cancelled by other vectors in \( S \), with each other vector being able to cancel at most two locations (one due to \( v^1 \) and one due to \( v^2 \)). If each vector cancels exactly 2 locations, then an additional \( \delta - 1 \) vectors in \( S \) can take the sum of \( S \) to zero. However, if \( S \) has a vector of weight 1, the weight 1 vector is able to cancel a 1 in only one location, and hence the sum cannot be zero. Therefore, in this case \( S \) cannot have any weight 1 vectors, i.e., \( S_1 = \emptyset \). Moreover, if any vector in \( S_\delta \) has weight more than \( \delta \), then the 1s cannot be cancelled by the other vectors in \( S_\delta \). Therefore, all the vectors in \( S_\delta \) must have weight exactly \( \delta \).

Claim 2. If \( S \) satisfies the conditions of Lemma 1 and \(|S| \geq 2\delta \), then \( \max\{|S_1|, |S_\delta|\} \geq \delta \).

Proof. If \( S \) has size \( \geq 2\delta + 1 \), then obviously at least one of \( S_1 \) and \( S_\delta \) has size \( \delta + 1 \). When \( S \) has size \( 2\delta \), again either one of them has size at least \( \delta + 1 \), or both \( S_1 \) and \( S_\delta \) have size exactly \( \delta \). In either case Lemma 1 is true.

Claim 3. If \( S \) satisfies the conditions of Lemma 1 and contains a vector \( v \) of weight more than 1 such that \( \text{supp}(v) \cap \text{supp}(e^l) = \emptyset \) for all vectors \( e^l \in S_1 \), then \(|S_\delta| \geq \delta + 1 \).

Proof. The vector \( v \) has 1 in at least \( \delta \) locations, and no weight 1 vector has a 1 in any of these locations. For the vectors in \( S \) to sum to zero, the 1s in \( v \) have to be cancelled by 1s from other vectors. Since \( S \) is ATOM, any vector can cancel only a single 1 in \( v \). Since all the vectors cancelling the 1s in \( v \) come from \( S_\delta \), this means that \(|S_\delta| \geq \delta + 1 \).

We are now left with the task of proving Lemma 1 for sets \( S \) that do not fall under any of Claims 1, 2, 3. This means that we need to consider \( \delta + 2 \leq |S| \leq 2\delta - 1 \). In the proof of Claim 1 we already saw that \(|S_1| \geq \delta \) if \(|S_\delta| = 1 \) (if the single vector in \( S_\delta \) has weight exactly \( \delta \), this corresponds to \( S \) being of size exactly \( \delta + 1 \); if the weight of this vector is higher, then it corresponds to \( S \) being of size \( \delta + 2 \) or more). Therefore, we only need to consider \(|S_\delta| \geq 2 \).
For $|S| \geq \delta + 2$, if $|S_1| \leq 1$, then we have $|S_\delta| \geq \delta + 1$, and therefore, we are already done. Therefore, we shall also consider $|S_1| \geq 2$. We shall also assume that $\delta \geq 3$, since if $\delta = 2$ and $|S| \geq \delta + 2$, we are already in the $|S| \geq 2\delta$ case of Claim 2, and therefore done. Finally, we can assume that the condition in Claim 3 is not true.

**Claim 4.** If $S$ is a set satisfying the conditions of Lemma 1, and additionally, the following are true:

1. $\delta \geq 3$, and $\delta + 2 \leq |S| \leq 2\delta - 1$;
2. $|S_1|, |S_\delta| \geq 2$;
3. For each vector $v^i \in S_\delta$, there is a vector $e_j \in S_1$ such that $|\text{supp}(v_j) \cap \text{supp}(e_j)| = 1$.

Then $\max\{|S_1|, |S_\delta|\} \geq \delta + 1$.

**Proof.** The proof will be by induction on the difference between the degree gap of the set $S$ and its size. The base case is $|S| = \delta + 2$, where the difference is 2. The proof for the base case and the induction step are quite similar, so we shall describe them together. For the induction step where the difference is $i + 1$, the induction hypothesis is that $\max\{|S'_1|, |S'_\delta|\} \geq \delta' + 1$ for all sets $S'$ of size up to $\delta' + i$ and degree gap $\delta'$, satisfying the conditions of the lemma (the hypothesis is obviously not true for $i = 1$, since in that case $\max\{|S'_1|, |S'_\delta|\}$ may be $\delta'$ or $\delta' + 1$).

Suppose every vector in $S_\delta$ intersects (i.e., has a 1 in a common location with) the same vector in $S_1$. Since $S_1$ contains at least two vectors, this means there must be a vector in $S_1$ that does not intersect with any vector in $S_\delta$. But a 1 in a vector in $S_1$ cannot be cancelled by a 1 in another vector in $S_1$, therefore this cannot happen. Therefore, there must exist distinct vectors $v^i, v^j \in S_\delta$ and $e^i, e^j \in S_1$ such that $v^i$ intersects with $e^i$ and $v^j$ with $e^j$. We can also assume that $v^i$ is the highest weight vector in $S$. We shall divide the proof into two cases: when $v^i$ has weight $\delta$ (this means that all vectors in $S_\delta$ have weight $\delta$), and when it has weight more than $\delta$.

**Case (i):** $\text{wt}_{\text{ATOM}}(v^i) = \delta$. Consider the vectors $v^i \oplus e^i$ and $v^j \oplus e^j$. They have weight $\delta - 1 \geq 2$, so they cannot be in the set $S$. We define the set $S' = \{v^i \oplus e^i, v^j \oplus e^j\} \cup S \setminus \{v^i, v^j, e^i, e^j\}$. $S'$ is ATOM, has degree gap $\delta' = \delta - 1$, and sums to zero if and only if $S$ sums to zero. The size of $S'$ is 2 less than that of $S$, and therefore, the difference between its size and degree gap is 1 less than that of $S$.

If $|S| = \delta + 2$, then $|S'| = \delta' + 1$, and by Claim 1 it either has $|S'_\delta| = 1$ and $|S'_1| = \delta'$, or $|S'_\delta| = \delta' + 1$ and $|S'_1| = 0$. But we in fact know the second case cannot happen as it would mean all the vectors in $S'_\delta$ have weight $\delta'$, whereas we know two vectors in $S'_\delta$ have weight $\delta'$, and the rest have weight $\delta' + 1$. Therefore we have $|S'_1| = \delta'$, which means that $|S_1| = \delta' + 2 = \delta + 1$.

If $|S| = \delta + i + 1$, then $|S'| = \delta' + i$. Then by the induction hypothesis, $\max\{|S'_1|, |S'_\delta|\} \geq \delta' + 1$. Now $|S_1| = |S'_1| + 2$ and $|S_\delta| = |S'_\delta|$, which means that if $|S_1| \geq |S_\delta| - 2$, then $|S_\delta| \geq \delta' + 3 = \delta + 2$. Otherwise, if $|S_\delta| = 2 > |S_1|$, then $|S_\delta| \geq \delta$. But we shall show that the $|S_\delta| = \delta$ case cannot happen. If $|S_\delta|$ is of size $\delta$, then every vector in $S_\delta$ must have at least one 1 in some location that cannot be cancelled by 1s from other vectors in $S_\delta$. Moreover, this location must be unique for each vector in $S_\delta$, because otherwise two vectors from $S_\delta$ would intersect in more than one location. Therefore, there must be a unique vector in $S_1$ that cancels this 1 for each vector in $S_\delta$. This gives $|S_1| \geq \delta$, which means $|S| \geq 2\delta$. This contradicts the assumption that $|S| \leq 2\delta - 1$, and therefore we must have $|S_\delta| \geq \delta + 1$ in this case.

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Note that we do not assume in the statement of Lemma 1 that $S$ is an ATOM set with degree gap $\delta$ of minimum size that sums to zero. Therefore, the lower bound of $\delta + 2$ here does not contradict Fact 3. Rather it shows that sets which satisfy this case are never the sets of minimum size.
Case (ii): $w_{14}(v^i) > \delta$. Consider the vector $v^i \oplus e^i$. It has weight $\geq \delta$ and intersects with $v^i$ in at least $\delta \geq 3$ many places. Therefore it cannot be in $S$: otherwise $S$ would not be ATOM. Consider $S' = \{v^i \oplus e^i\} \cup S \setminus \{v^i, e^i\}$. $S'$ is ATOM, has degree gap $\delta$, and sums to zero if and only if $S$ sums to zero. The size of $S'$ is 1 less than that of $S$, and therefore, the difference between its size and degree gap is 1 less than that of $S$.

If $|S| = \delta + 2$, then $|S'| = \delta + 1$, and it satisfies either $|S'| \geq \delta$, or $|S'| \geq \delta + 1$. In the first case we have $|S_1| = |S'_1| + 1 \geq \delta + 1$, and in the second case we have $|S_\delta| = |S'_\delta| \geq \delta + 1$.

If $|S| = \delta + i + 1$, then $|S| = \delta + i$. By the induction hypothesis, $\max\{|S'_1|, |S'_\delta|\} \geq \delta + 1$. This gives either $|S_1| \geq \delta + 2$, or $|S_\delta| \geq \delta + 1$.

Claims 1 to 4 cover all cases of Lemma 1. This completes the proof.

3.2 Necessary conditions for degeneracy

Theorem 1 and Fact 3 leave us with only two possible values of $D_{\text{dist}}(G)$ for a graph without 4-cycles: $\delta$ and $\delta + 1$. The following lemma further sharpens both these results to exactly classify when $D_{\text{dist}}(G)$ is $\delta$ and when it is $\delta + 1$.

Lemma 2. Let $G = ([n], E)$ be a graph with $\min\deg(G) = \delta$ and no 4-cycles. Then $D_{\text{dist}}(G) = \delta$ if and only if $\delta$ is even, and there exists $V \subseteq [n]$ such that $|V| = \delta$, satisfying

(i) All the vertices in $V$ are of degree $\delta$.

(ii) For every distinct pair of vertices $u, v$ in $V$, there exists a path of distance 2 between them in $G$. Moreover, all these paths are edge-disjoint.

(iii) For each $v \in V$, the set $V$ contains exactly one neighbor of $v$, which is not a neighbour of any other vector in $V$.

In all other cases, $D_{\text{dist}}(G) = \delta + 1$.

Proof. In the proof of Lemma 1, $\max\{|S_1|, |S_\delta|\}$ is $\delta$ in exactly two cases: if $S_1 = \{v\}$ and $S_1 = \cup_{j \in \supp(v)} \{e^j\}$ for a vector $v$ of weight $\delta$, or if $|S_1| = \delta$ and $|S_\delta| = \delta$. The first case corresponds to an error of the form $X(e^j)Z(v)$, if $v$ is the vector corresponding to the $i$-th vertex of $G$. Since $v$ does not have support on $i$, this leads to the size of $E$ being $\delta + 1$.

The second case is then the only remaining case that can give rise to $D_{\text{dist}}(G) = \delta$. As we argued in case (ii) in the proof of Claim 4, when $|S_\delta| = \delta$, each vector in $S_\delta$ must intersect with at least one unique vertex in $S_1$ (i.e., no other vector from $S_\delta$ intersects with the same vector in $S_1$). If $|S_1| = \delta$ that means that each vector in $S_\delta$ intersects with exactly one vector in $S_1$, which happens when every vector in $S_\delta$ has weight exactly $\delta$, and intersects with every other vector in $S_\delta$ (in one location).

The vectors in $S_\delta$ are the rows corresponding to some vertices $V \subseteq [n]$ of $G$. It is clear that all vertices in $V$ have degree $\delta$, which shows condition (i). Let $N(V)$ denote the neighbours of $V$, and $N'(V)$ denote the neighbours of $V$ that are cancelled by vectors in $S_1$ (so that $|N'(V)| = \delta$). Since any two of vectors in $S_\delta$ intersect at exactly one location (in $N(V)$), this means that every pair of vertices in $V$ has a path of length 2 between them. If two of these paths share an edge, which is incident on $i \in N(V)$, then that means three vectors from $S_\delta$ intersect at the same location $i$. But every vector from $S_\delta$ must intersect with every other vector in a unique location, otherwise all the 1s from that vector cannot be cancelled. Therefore, the paths cannot share an edge. This shows condition (ii).
An error corresponding to $S$ will be of the form $X(e^V)Z(e^{N'(V)})$, where $e^V$ is the vector which has 1s in the locations corresponding to $V$, and $e^{N'(V)}$ is defined similarly. If $N'(V) \neq V$, then that would make the size of $E$ more than $\delta$, since there is at least one location where it has a $Z$ where it does not have an $X$. Therefore we must have $N'(V) = V$, i.e., every vertex in $V$ has a neighbour in $V$, which is not a neighbour of any other vertices in $V$. In order for this to happen, every vector in $V$ can only have one neighbour in $V$. This shows condition (iii). Of course if $j$ is a neighbour of $i$, then $i$ is also a neighbour of $j$, i.e., the vertices in $V$ occur in pairs, and $\delta$ must be even. 

Using Lemma 2, we can now show necessary conditions for a CWS code to be degenerate in Theorem 2.

**Theorem 2.** If a CWS code $Q = (C, G)$, with the graph $G$ having minimum degree $\delta$, is degenerate, then at least one of the following is true:

(i) $G$ contains a cycle of length 4, or $\delta$ is even and $G$ contains $\delta/2$ cycles of length 3;

(ii) For all vertices $i \in V(G)$ which are of minimum degree, $c_i = 0$ for all codewords $c \in C$.

**Proof.** If $Q = (C, G)$ is degenerate, $C$ uses all bits in positions corresponding to the minimum degree vertices in $G$, and $G$ does not contain a cycle of length 4, then we shall show that $\delta$ is even and $G$ contains $\delta/2$ cycles of length 3. This suffices to prove the theorem.

From Fact 3 and Lemma 2, we know that in this case, $q\text{dist}(Q) \leq \delta + 1$. Also $D\text{dist}(Q) = \delta + 1$ unless $\delta$ is even and $G$ contains a subset $V$ of vertices satisfying the conditions in Lemma 2. In $V$, each vertex contains an edge to another vertex, and every pair of vertices has a path of length 2 (involving a vertex outside $V$) between them. Thus, each pair of vertices in $V$ which have an edge between them is part of a triangle in $G$, and there are $\delta/2$ such pairs. 

### 4 NP-hardness of quantum minimum distance

In this section, we prove Theorem 3, formally stated below.

**Theorem 3.** $\text{QMINDIST}$ is NP-complete. For every $\gamma \geq 1$, $\text{GAPQDIST}_\gamma$ is NP-hard under polynomial time RUR reductions with soundness error exponentially small in a security parameter. There exists $\tau > 0$ such that $\text{GAPADDQDIST}_\tau$ is NP-hard under polynomial time RUR reductions with the same soundness error. Versions of $\text{QMINDIST}$, $\text{GAPQDIST}_\gamma$ and $\text{GAPADDQDIST}_\tau$ where the CWS code is promised to be non-degenerate are also NP-hard.

Our proof strategy will be as follows: we shall reduce the minimum distance problem for classical codes to the minimum distance problem in CWS codes. First, we shall go from the given $(n, k)$ classical code $C$ to an $(m, k)$ code $C'$ such that $m = O(n^2)$ and $\text{dist}(C') = \text{dist}(C)$. This is just to ensure that $\text{dist}(C') = O(m^{1/2})$. Moreover, we shall also pick $m$ such that $m = p^2 + p + 1$ for some prime $p$. Such an $m$ can be efficiently found due to the following lemma.

**Lemma 3.** For $n > 7$, there exists a number of the form $p^2 + p + 1$ for some prime between $p$, in the interval $[n, 7n]$. Furthermore, given $n$, there exists an algorithm that finds this number in time $\text{poly}(n)$.

**Proof.** Let $q$ be the largest prime such that $q^2 + q + 1 < n$; such a prime always exists for $n \geq 7$. Let $p$ be the next prime after $q$. By Bertrand’s postulate, $p$ must lie in the interval $[q + 1, 2(q + 1)]$. Now consider the number $p^2 + p + 1$. It must satisfy $p^2 + p + 1 \leq 4(q +$
1)^2 + 2(q + 1) + 1 < 7(q^2 + q + 1) \leq 7n. Moreover, by definition \( p^2 + p + 1 \geq n \). Therefore, \( p^2 + p + 1 \) is a number of the required form in the internal \([n, 7n]\).

In order to find \( p \), the algorithm first finds the largest prime \( q \) such that \( q^2 + q + 1 < n \). This can be done by checking all the numbers up to \( \sqrt{n} \), which takes \( \text{poly}(n) \) time. Then the algorithm finds the next prime \( p \) and outputs it. This can be done by checking all the numbers up to \( 2(q + 1) \), which can also be done in \( \text{poly}(n) \) time.

Once we have \( C' \), we shall use it to construct a CWS code \( Q = (C', G) \). The graph \( G \) we use in \( Q \) will have no 4-cycles, so that we can apply Lemma 1. But we shall also need the graph to have high enough degree for each vertex. We shall use the following lemma to construct such a graph. The construction is the so-called orthogonal polarity graph, defined by Erdős, Rényi and Sós [PES66].

**Lemma 4.** For \( m = p^2 + p + 1 \) for some prime \( p \), there is an algorithm that, given \( m \), constructs a graph \( G \) with \( m \) vertices satisfying the following properties:

(i) Each vertex of \( G \) has degree \( p \) or \( p + 1 \);

(ii) \( G \) is free of 4-cycles.

Furthermore, the running time of the algorithm is \( \text{poly}(m) \).

**Proof.** We use the construction due to Erdős, Rényi and Sós [PES66]. Consider the set of elements \((x, y, z)\) in \( \mathbb{F}_p^3 \) where all three entries are not zero. We define an equivalence relation on these elements as follows: \((x, y, z)\) and \((\lambda x, \lambda y, \lambda z)\) for all \( \lambda \neq 0 \) are in the same equivalence class. The number of such equivalence classes is \( p^2 + p + 1 \), since we can represent each equivalence class by elements of the form \((1, y, z)\), \((0, 1, z')\) or \((0, 0, 1)\) for arbitrary \( y, z, z' \in \mathbb{F}_p \). The number of elements of the first type is \( p^2 \), the number of elements of the second type is \( p \), and there is only one element of the third type. The vertices of the graph \( G \) will correspond to these \( p^2 + p + 1 \) equivalence classes.

Two vertices \((x, y, z)\) and \((x', y', z')\) are connected by an edge in \( G \) if and only if they satisfy the condition

\[
xx' + yy' + zz' = 0. \tag{5}
\]

(Notice that if the pair of representative points of two equivalence classes satisfy this condition, then so does any other pair of points from the two classes.) Some vertices will satisfy \( x^2 + y^2 + z^2 = 0 \); we do not add self-loops for these vertices. We claim that the degree of vertices which do not satisfy \( x^2 + y^2 + z^2 = 0 \) is \( p + 1 \), which would make the degree of the rest of the vertices \( p \). To see this, we calculate the number of possible \((x', y', z')\) in \( \mathbb{F}_p^3 \) \(\setminus \{0^3\}\) which satisfy (5) for a fixed \((x, y, z)\). Without loss of generality, assume \( x \neq 0 \); the argument is similar for the case when \( x = 0 \) but \( y \neq 0 \) or \( z \neq 0 \). All points of the form \((-x^{-1} \cdot (yy' + zz'), y', z')\) where either \( y' \neq 0 \) or \( z' \neq 0 \) are solutions to (5). This gives \( p^2 - 1 \) possible solutions in \( \mathbb{F}_p^3 \) \(\setminus \{0^3\}\). However, since \((\lambda x', \lambda y', \lambda z')\) for \( \lambda \neq 0 \) satisfies (5) whenever \((x', y', z')\) does, we have overcounted by a factor of \( p - 1 \). Therefore, the number of equivalence classes of solutions (including possibly the equivalence class of \((x, y, z)\) itself) is \( p + 1 \), which makes the degree of vertices not satisfying \( x^2 + y^2 + z^2 = 0 \) \( p + 1 \).

Finally, to prove that \( G \) has no 4-cycles, we use the fact that a necessary and sufficient condition for a graph to be 4-cycle free is for no two vertices to have more than one common neighbour. For a vertex \((u, v, w)\) to be a common neighbour of two different vertices \((x, y, z)\)
and \((x', y', z')\), it must satisfy
\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
\begin{pmatrix}
x' \\
y' \\
z'
\end{pmatrix}
\begin{pmatrix}
u \\
v' \\
w
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0\end{pmatrix}.
\tag{6}
\]

Note that the matrix on the right-hand side of (6) is rank 2, since \((x, y, z)\) and \((x', y', z')\) belong to different equivalence classes. Therefore, it has a null space of dimension 1, and up to equivalence class, (6) can only have a single solution. Therefore, two vertices have exactly one common neighbour.

We have shown that the graph \(G\) satisfies the required properties, and now we only have to show that the algorithm constructing it runs in time \(\text{poly}(m)\). This is easy to see: given \(m = p^2 + p + 1\), the algorithm lists the representative points of all the equivalence classes, and for each pair of points, checks whether (5) is satisfied or not, and adds an edge accordingly. This can be done in \(\binom{m}{2}\) time.

For \(Q\) constructed using Lemmas 3 and 4, we shall show that \(\text{qdist}(Q) = \text{dist}(C') = \text{dist}(C)\), and hence if we can solve the minimum distance problem for \(Q\), we can also solve the minimum distance problem for \(C\), either exactly or approximately.

**Proof of Theorem 3.** Given an instance of \textsc{Mindist} or \textsc{Gapdist} or \textsc{Gapaddist}, consisting of \(H \in \mathbb{F}_2^{(n-k) \times n}\) (with \(n > 7\)), first by applying Lemma 3 we can find \(m \in [25n^2, 175n^2]\) which is of the form \(p^2 + p + 1\). Consider the parity check matrix \(H' \in \mathbb{F}_2^{(m-k) \times m}\) whose first \(n-k\) rows are the same as the rows of \(H\) in the first \(n\) columns, and have 0s in the rest of the; the other rows of \(H'\) are all 0s. Each codeword in \(C(H')\) is of length \(m\), and is a codeword of \(C(H)\) appended with \(m-n\) 0s. It is clear that \(\text{dist}(C(H)) = \text{dist}(C(H')) \leq \frac{m^{1/2}}{5}\). Moreover, all codewords of \(C(H')\) have Hamming weight at most \(\frac{m^{1/2}}{5}\).

Next, we construct a graph \(G\) with \(m\) vertices using Lemma 4. Consider the CWS code \(Q = (C(H'), G)\). We shall first prove that \(\text{qdist}(Q) \leq \text{dist}(C(H))\). Let \(c\) be a non-zero codeword of \(C(H')\) with \(\text{wt}_H(c) = \text{dist}(C(H))\). Now consider the error \(E = Z(c)\), which has weight \(\text{dist}(C(H))\). Clearly \(\text{Cl}_G(E) = c\), which \(C(H')\) cannot detect.

To prove \(\text{qdist}(Q) \geq \text{dist}(C(H))\), we need to show that all Pauli errors of size up to \(\text{dist}(C(H))-1\) are detected by \(Q\), i.e., they satisfy the conditions of Fact 2. Note that it is sufficient to show that for such an \(E\), \(\text{Cl}_G(E)\) is not a codeword of \(C(H')\). Firstly, for such an error, \(\text{Cl}_G(E)\) cannot be the zero codeword. This is because by Theorem 1 we have that the minimum size of \(E\), for which \(\text{Cl}_G(E) = 0^m\), is \(p \geq \frac{m^{1/2}}{\sqrt{2}}\), whereas \(\text{dist}(C(H)) \leq \frac{m^{1/2}}{5}\).

To show that \(\text{Cl}_G(E)\) is also not a non-zero codeword, we shall divide the errors into two cases: first when \(E\) is of the form \(Z(a)\), and second when it is not. For an error of size \(\text{dist}(C(H))-1\) of the form \(Z(a)\), \(\text{Cl}_G(E) = a\), whose weight is too low to be a codeword of \(C(H')\) by definition. In the second case, \(E\) has \(X\) in some set of components \(S\), such that \(|S| \leq \text{dist}(C(H))-1\). Suppose \(S\) contains a component \(i\). Then \(\text{Cl}_G(E)\) is of the form
\[
u^i \oplus a \oplus \bigoplus_{j \in S, j \neq i} u^j
\]
for some \(a\) such that \(\text{wt}_H(a) \leq \text{dist}(C(H)) - 1\). The \(u^j\)'s are columns of \(G\), and so they have 1s in \(\geq \frac{m^{1/2}}{\sqrt{2}}\) components each. Moreover, we have for \(j \neq i\),
\[
|\text{supp}(u^i) \cap \text{supp}(u^j)| \leq 1,
\]
which means that

$$\sum_{j \in S, j \neq i} |\text{supp}(u^i) \cap \text{supp}(u^j)| \leq \text{dist}(C(H)) - 1.$$  (7)

$\text{Cl}_G(E)$ may have 0 at a component at which $u^i$ has a 1 if $a$ has a 1 in that component, or one of the other $u^j$-s has a 1 in that component. Hence by (7), we have that

$$\text{wt}_H(\text{Cl}_G(E)) \geq \text{wt}_H(u^i) - \text{wt}_H(a) - \text{dist}(C(H)) + 1 \geq \frac{m^{1/2}}{\sqrt{2}} - 2(\text{dist}(C(H)) - 1) \geq \frac{4m^{1/2}}{15}.$$

Therefore, the weight of $\text{Cl}_G(E)$ is too high to be a codeword of $C(H')$.

Thus we have shown that $\text{dist}(C(H)) = \text{qdist}(Q)$. Given an instance $(H, t)$ of MIN DIST, our polynomial time construction gives us an instance of QMIN DIST with $H', G$ as described, and the same parameter $t$. If $(H, t)$ is an instance of GAP DIST, $(Q = (H', G), t)$ satisfies the promise $\text{qdist}(Q) \leq t$ or $> \gamma \cdot t$, and is therefore an instance of GAP QDIST. If $(H, t)$ is an instance of GAP ADD DIST, then $(Q = (H', G), t)$ satisfies the promise $\text{qdist}(Q) \leq t$ or $> t + \tau \cdot \sqrt{m}$. By Facts 5 and 6, this completes the proof of the first part. The hardness of the problems when restricted to non-degenerate codes follows because the quantum code $Q$ we have constructed satisfies $\text{qdist}(Q) \leq \text{Ddist}(Q)$, and hence is non-degenerate.

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