Lie models of simplicial sets and representability of the Quillen functor

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Abstract
Extending the model of the interval, we explicitly define for each $n \geq 0$ a free complete differential graded Lie algebra $\mathfrak{L}_n$ generated by the simplices of $\Delta^n$, with desuspended degrees, in which the vertices are Maurer-Cartan elements and the differential extends the simplicial chain complex of the standard n-simplex. The family $\{\mathfrak{L}_n\}_{n \geq 0}$ is endowed with a cosimplicial differential graded Lie algebra structure which we use to construct two adjoint functors

$$\text{SimpSet} \xrightarrow{\langle \cdot \rangle} \text{DGL}$$

given by $\langle L \rangle = \text{DGL}(\mathfrak{L}_n, L)$ and $\mathfrak{L}(K) = \lim_{\to} \mathfrak{L}_n$. This new tools let us extend Quillen rational homotopy theory approach to any simplicial set $K$ whose path components are non necessarily simply connected.

We prove that $\mathfrak{L}(K)$ contains a model of each component of $K$. When $K$ is a 1-connected finite simplicial complex, the Quillen model of $K$ can be extracted from $\mathfrak{L}(K)$. When $K$ is connected then, for a perturbed differential $\partial_a$, $H_0(\mathfrak{L}(K), \partial_a)$ is the Malcev Lie completion of $\pi_1(K)$. Analogous results are obtained for the realization $\langle L \rangle$ of any complete DGL.

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Introduction

In [18], R. Lawrence and D. Sullivan raise the following observation and subsequent general questions: the rational singular chains on a cellular complex are naturally endowed with a structure of cocommutative, coassociative infinity coalgebra and hence, taking the commutators of a “generalized bar construction” it should give rise to a complete differential graded Lie algebra (DGL henceforth). What is the topological and geometrical meaning of this DGL? Are there closed formulae for its differential? Allowing 1-cells, what is the relation of this DGL with the fundamental group of the given complex? In the same reference they carefully construct such a DGL for the interval.

Here we attack these problems for any finite simplical complex $K$. In fact, we construct in a functorial way a complete differential graded Lie algebra $\mathcal{L}(K) = (\hat{\mathcal{L}}(V_K), \partial)$ which is the only (up to isomorphism) free complete differential graded Lie algebra for which $V_K$ together with the linear part $\partial_1$ of $\partial$ is the desuspension of the simplicial chain complex of $K$, and any generator of degree $-1$, (i.e., any 0-simplex) is a Maurer-Cartan element. The departure point is the construction of a family $\mathcal{L}_n := \mathcal{L}(\Delta^n)$ of such DGL’s for the standard simplices, together with coface and codegeneracy operators, for which we prove:

**Theorem A.** The family $\mathcal{L}_\bullet := \{\mathcal{L}(\Delta^n), \partial\}_{n \geq 0}$ form a cosimplicial differential graded Lie algebra.

We also prove uniqueness of such a family as Theorem 2.8 and give as examples, explicit models $\mathcal{L}_2$ and $\mathcal{L}_3$ for the triangle and the tetrahedron in Proposition 2.7 and Example 2.9 respectively.

On the one hand, this cosimplicial structure let us finally define,

$$\mathcal{L}(K) = \lim_{\longrightarrow} \mathcal{L}_\bullet,$$

for any simplicial set $K$. Whenever $K$ is in particular a finite simplicial complex, $\mathcal{L}(K)$ is trivially isomorphic to a sub DGL of a certain $\mathcal{L}_n$ as $K \subset \Delta^n$ for some $n$.

On the other hand, the cosimplicial DGL $\mathcal{L}_\bullet$ defines a functor $\langle \cdot \rangle$ from the category $\text{DGL}$ of complete differential graded Lie algebras to the category $\text{SimpSet}$ of simplicial sets via

$$\langle L \rangle = \text{DGL}(\mathcal{L}_\bullet, L).$$

Then, we prove:
**Theorem B.** The following are adjoint functors

\[ \text{SimpSet} \xrightarrow{\iota} \text{DGL}. \]

These functors not only shed light on our original questions but also let us develop an Eckmann-Hilton dual of the Sullivan geometrical approach to rational homotopy theory, even for non simply connected complexes. Here, \( \mathfrak{L}_\bullet \) plays the dual role of the simplicial differential graded algebra \( \mathcal{A}_\bullet = A_{PL}(\Delta^\bullet) \) of PL-forms on the standard simplices \([10, 27, 29]\). Recall that the Quillen approach \([25]\) is based on the construction of several equivalences between homotopy categories joining the homotopy category of rational 2-reduced simplicial sets and that of 1-reduced differential graded Lie algebras.

Our work, although using a completely different method, it extends Quillen theory in several ways.

First of all, if \( X \) is a simply connected simplicial set, we denote by \( Y \) the associated 2-reduced simplicial set. In the text, we abuse notation by denoting \( \lambda(X) \) the image of \( Y \) by the Quillen construction. We prove:

**Theorem C.** Let \( K \) be a 1-connected finite simplicial complex. Then, for any vertex \( a \in K \), there is a quasi-isomorphism of differential graded Lie algebras,

\[ (\mathfrak{L}(K), \partial_a) \xrightarrow{\simeq} \lambda(K). \]

In particular, \( H_n(\mathfrak{L}(K), \partial_a) \cong \pi_{n+1}(K) \otimes \mathbb{Q}, \ n \geq 1. \)

Here \( \partial_a = \partial + \text{ad}_a \) is the twisting of the original differential \( \partial \) of \( \mathfrak{L}(K) \) via the Maurer-Cartan element \( a \).

On the other hand, concerning the realization functor, we show that, for a non negatively graded DGL \( L = L_{\geq 0} \), there are explicit isomorphisms,

\[ \pi_n(L) \cong H_{n-1}(L), \ n \geq 1, \]

in which the group structure on \( H_0(L) \) is given by the Baker-Campbell-Hausdorff product (see Proposition \([15]\))

Moreover, recall the dual Sullivan geometric realization \([10, 27]\) of a commutative differential graded algebra \( A \) given by

\[ \langle A \rangle_S = \text{CDGA}(A, \mathcal{A}_\bullet). \]
Then:

**Theorem D.** If $L$ is a non negatively graded finite type DGL, then there is a homotopy equivalence $\langle L \rangle \simeq \langle \mathcal{C}^*(L) \rangle_S$.

Here, $\mathcal{C}^*$ denotes the cochain functor. In particular our realization also equals Quillen realization for finite type 1-reduced DGL’s in view of the main result in [19]. Moreover, in [13] Getzler has constructed another functor $\langle \cdot \rangle_G$ from DGL to SimpSet,

$$\langle L \rangle_G = \text{MC}(L \otimes \mathcal{A}_\bullet),$$

in which MC denotes the set of Maurer-Cartan elements. By [13 Proposition 1.1], $\langle L \rangle_G = \langle \mathcal{C}^*(L) \rangle_S$ when $L$ is a non negatively graded finite type DGL, and thus, for these DGL’s, all realizations coincide (see also [2 Cor. 1.3]).

Finally, we answer the last question which we begin with and explicitly describe the relation between the fundamental group of a complex and its associated DGL:

**Theorem E.** If $K$ is a connected finite simplicial complex, then $H_0(\mathfrak{L}(K), \partial_a)$ is the Malcev Lie completion of the fundamental group of $K$.

This result implies that, unlike the classical Quillen approach and their known interesting extensions [5, 17], our model functor reflects geometrical properties of non-nilpotent spaces.

We also point out that our constructions apply to non necessarily connected complexes. Indeed, for any finite complex $K$, the connected components of $K_+$, the disjoint union of $K$ and a point, correspond to the classes of Maurer-Cartan elements of $\mathfrak{L}(K)$ modulo the gauge action [4]. Moreover, if $a$ is a non-zero Maurer-Cartan element, then $(\mathfrak{L}(K), \partial_a)$, is a “model” of the corresponding path component of $K$.

On the other hand, in Theorem 4.6 we prove that, for a general DGL $L$, the components of the realization $\langle L \rangle$ correspond also to the classes of Maurer-Cartan elements of $L$ modulo the gauge action. Moreover, if $a$ is a Maurer-Cartan element, the corresponding component of $\langle L \rangle$ is the realization $\langle L(a) \rangle$ of the “localization of $L$ at $a$”. In other words, $\langle L \rangle \simeq \bigcup_{a \in \text{MC}(L)} \langle L(a) \rangle$. This extends [6 Theorem 5.5] to any (complete) DGL.

We remark that the definition and main properties of the model and realization functor purposely stay in the Lie side and do not rely upon any connection to commutative algebras via cochain functors. The main reason is the following: in forthcoming work [3], we endow the category of complete DGL’s with a model structure which makes the model and realization functors a Quillen pair. This structure geometrically shapes complete Lie algebras as it is the result of “transfering” the
classical model structure on simplicial sets. With this, weak equivalences and fibrations properly contain the weak equivalences and fibrations of the model structure defined in [17] via the cochain functor.

This paper is organized as follows:

Section 1 is devoted to recall the Baker-Campbell-Hausdorff product and its connection with the basic properties of Lawrence-Sullivan model $\mathcal{L}_1$ for the interval [18]. In Section 2 we extend this construction and build the sequence $\{\mathcal{L}_n\}_{n \geq 0}$ of compatible models for $\Delta^n$. We establish some technical properties and also prove uniqueness.

In Section 3 we prove the existence of compatible symmetric models, i.e., models $(\mathcal{L}_n, \partial)$ that are $\Sigma_{n+1}$-DGL with an action that permutes the inclusions of the different copies of $\mathcal{L}_{n-1}$. This leads to the existence of the cosimplicial DGL structure on $\mathcal{L}_*$. All of this proves Theorem A.

In Sections 4 and 5 we define the model $\mathcal{L}$ and realization $\langle \cdot \rangle$ functors, we prove Theorem B, and give some properties and examples.

Section 6 is devoted to introduce the material on differential graded Lie coalgebras and the transfer theorem that we use in Section 7 to build a sequence of compatible models from a transfer process applied to a classical diagram of the form

$$\phi \bigcirc A_{PL}(K) \xrightarrow{p} C^*(K).$$

Using the uniqueness Theorem 2.8 we establish relations between the classical rational models of Sullivan and Quillen and the models constructed in the previous sections. In particular we prove Theorem C as Theorem 7.4.

Section 8 contains the correspondence between our realization and Sullivan’s realization. Theorem D corresponds to Theorem 8.1 Finally, Theorem E is proved in Section 9.

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The **BCH product and the Lawrence-Sullivan model for the interval**

Throughout this paper we assume that \( \mathbb{Q} \) is the base field. A graded Lie algebra consists of a \( \mathbb{Z} \)-graded vector space \( L = \bigoplus_{p \in \mathbb{Z}} L_p \) together with a bilinear product called the Lie bracket denoted by \([ - , - ]\) such that \([ x , y ] = - ( -1)^{|x||y|} [y, x] \) and

\[
( -1 )^{|x||z|} [x, [y, z]] + ( -1 )^{|y||x|} [y, [z, x]] + ( -1 )^{|z||y|} [z, [x, y]] = 0.
\]

Here \(|x|\) denotes the degree of \( x \).

A differential graded Lie algebra is a graded Lie algebra \( L \) endowed with a linear derivation \( \partial \) of degree \(-1\) such that \( \partial^2 = 0 \). It is called free if \( L \) is free as a Lie algebra, \( L = \mathbb{L}(V) \) for some graded vector space \( V \).

The completion \( \hat{L} \) of a graded Lie algebra \( L \) is

\[
\hat{L} = \lim_{\leftarrow n} L / L_n
\]

where \( L^1 = L \), \( L_n = [L, L^{n-1}] \) for \( n \geq 2 \), and the limit is taken on the topology arising from this filtration. A Lie algebra \( L \) is called complete if \( L \) is isomorphic to its completion. From now on, and unless explicitly stated otherwise, by a DGL we mean a complete differential graded Lie algebra.

A Maurer-Cartan element is an element \( a \in L_{-1} \) such that \( \partial a + \frac{1}{2} [a, a] = 0 \). Denote by \( \text{MC}(L) \) the set of Maurer-Cartan elements. These are preserved by DGL morphisms. In particular, if \( L = (\hat{L}(V), \partial) \) is a complete free DGL, and \( \theta \) is a derivation satisfying \( \theta(V) \subset \hat{L}_{\geq 2}(V) \) and \( [\theta, \partial] = 0 \), then \( e^\theta = \sum_{n \geq 0} \frac{\theta^n}{n!} \) is an automorphism of \( L \) and so, if \( a \in \text{MC}(L) \), then \( e^\theta(a) \) is also a Maurer-Cartan element.

Given \( (\hat{L}(V), \partial) \) a complete free DGL and \( v \in V \), we will often write \( \partial v = \sum_{n \geq 1} \partial_n v \) where \( \partial_n v \in \mathbb{L}^n(V) \).

\[ 5 \text{ The } \mathcal{L} \text{-model of } K \]

\[ 6 \text{ Differential graded Lie coalgebras and the Transfer Theorem} \]

\[ 7 \text{ The cosimplicial structure via a transfer} \]

\[ 8 \text{ Representability of the Quillen realization functor} \]

\[ 9 \text{ The Malcev completion of the fundamental group} \]
Let \((L, \partial)\) be a DGL and \(a \in \text{MC}(L)\). Then, the derivation, \(\partial_a = \partial + \text{ad}_a\) is again a differential on \(L\).

Given \(L\) a complete DGL, the \textit{gauge action} \(\mathcal{G}\) of \(L_0\) on \(\text{MC}(L)\) determines an equivalence relation among Maurer-Cartan elements defined as follows (see for instance [21, §4]): given \(x \in L_0\) and \(a \in \text{MC}(L)\),

\[
x \mathcal{G} a = e^{\text{ad}_x}(a) - \frac{e^{\text{ad}_x} - 1}{\text{ad}_x}(\partial x).
\]

Here and from now on, 1 inside an operator will denote the identity. Explicitly,

\[
x \mathcal{G} a = \sum_{i \geq 0} \frac{\text{ad}_x^i(a)}{i!} - \sum_{i \geq 0} \frac{\text{ad}_x^i(\partial x)}{(i + 1)!}.
\]

We denote the quotient set by \(\widehat{\text{MC}}(L) = \text{MC}(L)/\mathcal{G}\). Geometrically [13, 18], interpreting Maurer-Cartan elements as points in a space, one thinks of \(x\) as a flow taking \(x \mathcal{G} a\) to \(a\) in unit time. In topological terms [6], the points \(a\) and \(x \mathcal{G} a\) are in the same path component.

Let \(L\) be a complete Lie algebra concentrated in degree 0. We denote by \(UL\) its enveloping algebra, by \(I_L\) its augmentation ideal and by \(\widehat{UL}\) and \(\widehat{I_L}\) the completions of \(UL\) and \(I_L\) with respect to the powers of \(I_L\),

\[
\widehat{UL} = \varprojlim_n UL/I_L^n, \quad \widehat{I_L} = \varprojlim_n I_L/I_L^n.
\]

Denote finally by \(G_L = \{ x \in \widehat{UL} \mid \Delta(x) = x \otimes x \}\) the group of grouplike elements in \(\widehat{UL}\). Moreover, the injection of \(L\) in the set of primitive elements in \(\widehat{UL}\) is an isomorphism and the functions \(\exp\) and \(\log\) give inverse bijections between \(L\) and \(G_L\). This induces a product on \(L\), called the \textit{Baker-Campbell-Hausdorff product}, \(\text{BCH product}\) henceforth, defined by

\[
a * b = \log(\exp(a) \cdot \exp(b)).
\]

Note that \(a * (-a) = 0\). Therefore, \(-a\) is the inverse of \(a\) for the \(\text{BCH product}\) and we also use the notation \(-a = a^{-1}\).

As the law in \(G_L\) is associative, the \(\text{BCH product}\) is also associative. An explicit form of the product is given by the Baker-Campbell-Hausdorff formula

\[
a * b = a + b + \frac{1}{2}[a, b] + \frac{1}{12}[a, [a, b]] - \frac{1}{12}[b, [a, b]] + \cdots
\]
It follows from the Jacobi identity that in the Lie algebra of derivations of $L$ we have $\text{ad}_{a*b} = \text{ad}_a \circ \text{ad}_b$. Hence $e^{\text{ad}_{a*b}} = e^{\text{ad}_a} \circ e^{\text{ad}_b}$.

Note that the BCH product is compatible with the gauge action on $\text{MC}(L)$; i.e., if $y \in L_0$ and $a \in \text{MC}(L)$, we have

$$(x * y)Ga = xG(yG a).$$

We also need the following property.

**Proposition 1.1.** Let $L$ be a complete DGL and let $x, y \in L_0$. Then,

$$x * y * (-x) = e^{\text{ad}_x}(y).$$

With the previous convention, the formula also reads

$$x * y * x^{-1} = e^{\text{ad}_x}(y).$$

**Proof.** First, we note that, in $\hat{UL}$,

$$e^{\text{ad}_x}(y) = e^x ye^{-x}.$$  

Indeed,

$$e^{\text{ad}_x}(y) = \sum_{n=0}^{\infty} \frac{\text{ad}_x^n(y)}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i=0}^{n} (-1)^i \binom{n}{i} x^{n-i} y x^i
= \sum_{i=0}^{\infty} \sum_{n=i}^{\infty} \frac{x^{n-i}}{(n-i)!} \frac{(-x)^i}{i!} = e^x ye^{-x}.$$  

Replacing $y$ by $e^y$, we deduce

$$e^{\text{ad}_x}(e^y) = e^x e^y e^{-x}. \quad (1)$$

In a second step, we prove the equality

$$\left(e^{\text{ad}_x}(y^n) = \left(e^{\text{ad}_x}(y)\right)^n. \quad (2)$$

On the left hand side, the term of length $k$, $k \geq 0$, in $x$ equals

$$\frac{\text{ad}_x^k(y^n)}{k!}. \quad (3)$$
In the right hand side, this term is
\[ \sum_{k_1 + \cdots + k_n = k} \frac{\text{ad}_x^{k_1}(y)}{k_1!} \cdots \frac{\text{ad}_x^{k_n}(y)}{k_n!}. \]

As \( \text{ad}_x \) is a derivation, both terms coincide and the equality (2) is proved. Therefore,
\[ e^{\text{ad}_x(y)} = \sum_{n \geq 0} \frac{(e^{\text{ad}_x(y)})^n}{n!} = \sum_{n \geq 0} \frac{\text{ad}_x^n(y)^n}{n!} = e^{\text{ad}_x(y)}. \] (3)

Finally, the proposition follows from
\[ x \ast y \ast (-x) = \log(e^x e^y e^{-x}) = \text{[1]} \log(e^{\text{ad}_x(y)}) = \text{[3]} \log(e^{\text{ad}_x(y)}) = e^{\text{ad}_x(y)}. \]

\[ \square \]

The construction of a model for the interval was first introduced in [18].

**Definition 1.2.** The Lawrence-Sullivan model for the interval, LS-interval henceforth, is the complete DGL
\[ (\mathcal{L}, \partial) = (\hat{\mathcal{L}}(a, b, x), \partial), \]
in which \( a \) and \( b \) are Maurer-Cartan elements, \( x \) is of degree 0 and
\[ \partial x = \text{ad}_x b + \sum_{n=0}^{\infty} \frac{B_n}{n!} \text{ad}_x^n(b - a) = \text{ad}_x b + \frac{\text{ad}_x}{e^{\text{ad}_x} - 1}(b - a), \] (4)
where the \( B_n \)'s are the Bernoulli numbers.

Using the identity
\[ \left( \frac{-x}{e^{-x} - 1} \right) = x + \left( \frac{x}{e^x - 1} \right), \]
we may also write
\[ \partial x = \text{ad}_x a + \frac{\text{ad}_{-x}}{e^{-\text{ad}_x} - 1}(b - a). \] (5)

Let \( (\hat{\mathcal{L}}(a_0, a_1, a_2, x_1, x_2), \partial) \) be two glued LS-models of the interval. That is, \( a_0, a_1 \) and \( a_2 \) are Maurer-Cartan elements, \( \partial x_1 = \text{ad}_{x_1}(a_1) + \frac{\text{ad}_{x_1}}{e^{\text{ad}_{x_1} - 1}}(a_1 - a_0) \) and \( \partial x_2 = \text{ad}_{x_2}(a_2) + \frac{\text{ad}_{x_2}}{e^{\text{ad}_{x_2} - 1}}(a_2 - a_1) \). These two models give a model for the subdivision of an interval, as follows.
Theorem 1.3. [IS Theorem 2] The map

\[ γ : (\hat{L}(a, b, x), \partial) \rightarrow (\hat{L}(a_0, a_1, a_2, x_1, x_2), \partial), \]

defined by \( γ(a) = a_0, \ γ(b) = a_2 \) and \( γ(x) = x_1 * x_2 \), is a DGL morphism.

Finally, we will make use of the following results (for the first cf. also [24]).

Theorem 1.4. Let \((\hat{L}(a, b, x), \partial')\) be a complete Lie algebra in which \(a, b\) are Maurer-Cartan elements and the linear part of the differential satisfies \(\partial'_1 x = b - a\). Then \((\hat{L}(a, b, x), \partial') = (\mathfrak{L}_1, \partial)\).

Proof. Let \(\partial = \sum_{n \geq 1} \partial_n\), with \(\mathrm{Im} \partial_n \subset \mathbb{L}^n(a, b, x)\), be the differential in \(\mathfrak{L}_1\). We show that \(\partial_n x = \partial'_n x\) for any \(n \geq 1\). For \(n = 1\) this is trivially true. For \(n = 2\), by using \(\partial_1 \partial'_2 + \partial'_2 \partial_1 = 0\), a short computation shows that

\[ \partial_2 x = \partial'_2 x = \frac{1}{2}[x, a + b]. \]

Assume \(\partial_m x = \partial'_m x\), for \(1 \leq m < n\). From \(\partial^2 = \partial'^2 = 0\) and the induction hypothesis we deduce that \(\partial_n x - \partial'_n x\) is a decomposable \(\partial_1\)-cycle of degree \(-1\). Since \(H_*(\hat{L}(a, b, x), \partial_1) = \mathbb{L}(a)\) and \(\hat{L}(a, b, x)_0 = \mathbb{Q}x\), we conclude that \(\partial_n x - \partial'_n x = 0\). \(\square\)

Proposition 1.5. Let \(L\) be a complete DGL which contains an LS-interval \((\hat{L}(a, b, x), \partial)\). Then, for any \(v \in L\),

\[ \partial_a e^{ad_x}(v) = e^{ad_x}(\partial_b v). \]

In other words, the map

\[ e^{ad_x} : (L, \partial_b) \rightarrow (L, \partial_a) \]

is an isomorphism of DGL’s.

Proof. In fact,

\[ \partial_b e^{-ad_x}(v) = e^{-ad_x}(\partial_b v) + (-1)^{|v|} e^{-ad_x} \frac{e^{ad_x} - 1}{ad_x}(\partial_b x) \]

\[ = e^{-ad_x}(\partial_b v + (-1)^{|v|} \mathrm{ad}_v (b - a)) \]

\[ = e^{-ad_x}(\partial_a v), \]

where the first equality comes from [IS Lemma 1] and the second is a direct computation using (4). \(\square\)
2 Sequences of compatible models of $\Delta$

For each $n \geq 0$, we consider the standard $n$-simplex $\Delta^n$,

$$\Delta^n_p = \{ (i_0, \ldots, i_p) \mid 0 \leq i_0 < \cdots < i_p \leq n \}, \quad \text{if } p \leq n,$$

and $\Delta^n_p = \emptyset$ if $p > n$. Let $(\hat{\mathbb{L}}(s^{-1}\Delta^n), d)$ be the complete free DGL on the desuspended rational simplicial chain complex on $\Delta^n$,

$$da_{i_0 \cdots i_p} = \sum_{j=0}^{p} (-1)^j a_{i_0 \cdots \hat{i}_j \cdots i_p}. \quad (6)$$

Here, $a_{i_0 \cdots i_p}$ denotes the generator of degree $p - 1$ represented by the $p$-simplex $(i_0, \ldots, i_p) \in \Delta^n_p$. Henceforth, unless explicitly needed, we drop the desuspension sign to avoid unnecessary notation and write simply $(\hat{\mathbb{L}}(\Delta^n), d)$.

For each $0 \leq i \leq n$ consider the $i$-th coface affine map $\delta_i : \Delta^{n-1} \to \Delta^n$, defined on the vertices by,

$$\delta_i(j) = \begin{cases} j, & \text{if } j < i, \\ j+1, & \text{if } j \geq i. \end{cases}$$

We use the same notation for the induced DGL morphism,

$$\delta_i : (\hat{\mathbb{L}}((\Delta^{n-1}), d) \longrightarrow (\hat{\mathbb{L}}(\Delta^n), d), \quad (7)$$

defined by

$$\delta_i(a_{j_0 \cdots j_p}) = a_{\ell_0 \cdots \ell_p} \quad \text{with} \quad \ell_k = \begin{cases} j_k, & \text{if } j_k < i, \\ j_k + 1, & \text{if } j_k \geq i. \end{cases}$$

Finally, we denote by $\check{\Delta}^n$ and $\Lambda^n_i$ the boundary of $\Delta^n$ and the $i$-horn obtaining by removing the $i$-th coface from $\Delta^n$.

**Definition 2.1.** A sequence of compatible models of $\Delta$ is a family $\{(\mathcal{L}_n, \partial) = (\hat{\mathbb{L}}(\Delta^n), \partial)\}_{n \geq 0}$ of DGL’s satisfying the following properties:

1. For each $i = 0, \ldots, n$, the generator $a_i \in \Delta^n_0$ is a Maurer-Cartan element, $\partial a_i = -\frac{1}{2}[a_i, a_i]$.

2. The linear part $\partial_1$ of $\partial$ is precisely $d$ as in (6).

3. For each $i = 0, \ldots, n$, the coface maps, $\delta_i : (\hat{\mathbb{L}}((\Delta^{n-1}), \partial) \longrightarrow (\hat{\mathbb{L}}(\Delta^n), \partial)$, are DGL morphisms.
Each element \((\hat{L}(\Delta^n), \partial)\) of this sequence is called a model of \(\Delta^n\), which is thus implicitly endowed with models of \(\Delta^q, q < n\), satisfying the compatibility condition (3) above.

**Definition 2.2.** A sequence of models \(\{(\hat{L}(\Delta^n), \partial)\}_{n \geq 0}\) is called inductive if, for \(n \geq 2\), we have

\[
\partial a_0 a_{0...n} \in \hat{L}(\Delta^n).
\] (8)

**Theorem 2.3.** There exists sequences of compatible inductive models of \(\Delta\).

Its proof uses the following key proposition: Let \((\hat{L}(V \oplus W), \partial)\) be a complete free DGL in which \(V\) and \(W\) are of finite dimension. We denote by \(I\) the ideal generated by \(W\), and by \(\partial\) the linear part of the differential.

**Proposition 2.4.** With the above notations, if \(\partial(I) \subset I\) and \(H(W, \partial) = 0\), then the projection \((\hat{L}(V \oplus W), \partial) \rightarrow (\hat{L}(V), \partial)\) is a quasi-isomorphism.

Moreover, every cycle in \(I \cap \hat{L}^{\geq n}(V \oplus W)\) is a boundary \(\alpha = \partial \beta\) with \(\beta \in I \cap \hat{L}^{\geq n}(V \oplus W)\).

**Proof.** Let \(K\) be the kernel of the projection \((\hat{L}(V \oplus W), \partial) \rightarrow (\hat{L}(V), \partial)\). Since \(H(W, \partial) = 0\), \(K\) is acyclic. Now let \(\alpha \in I\) be a \(\partial\)-cycle. There exists \(n\) such that \(\alpha = \alpha_n + \beta_{n+1}\), with \(\alpha_n \in I \cap \hat{L}^n(W \oplus V)\) and \(\beta_{n+1} \in I \cap \hat{L}^{>n}(W \oplus V)\). From \(\partial \alpha = 0\), we deduce \(\partial \alpha_n = 0\) and since \(\alpha_n \in K \cap \hat{L}^n(W \oplus V)\) and \(H(K, \partial) = 0\), there is an element \(\gamma_n \in K \cap \hat{L}^n(W \oplus V) = I \cap \hat{L}^n(W \oplus V)\) such that \(\alpha_n = \partial \gamma_n\). Thus, we have \(\alpha - \partial \gamma_n \in I \cap \hat{L}^{>n}(W \oplus V)\). By iterating this process, we obtain a sequence \((\gamma_j)_{j \geq n}\), such that \(\gamma_j \in I \cap \hat{L}^j(W \oplus V)\) and \(\alpha = \partial \left( \sum_{j \geq n} \gamma_j \right)\).

**Corollary 2.5.** If \((\hat{L}(\Delta^n), \partial)\) is an inductive model, then:

(i) \(H(\hat{L}(\Delta^n), \partial) = H(\hat{L}(\Delta^i), \partial) = 0\) for any \(i = 0, \ldots, n\).

(ii) \(H(\hat{L}(\Delta^n), \partial) = \hat{L}(\Omega), \; \text{with} \; \Omega = \partial a_{0...n}\).

**Proof.** (i) This follows from Proposition 2.4. Let \(W\) be the vector space generated by the elements \(a_i - a_0\) and the elements of higher degree. We verify directly that \(H(W, \partial) = 0\) and that the ideal generated by \(W\) is a differential ideal.

(ii) Write \((\hat{L}(\Lambda^n), \partial) = (\hat{L}(Qa_0 \oplus W), \partial)\), with \(W\) as above.

Let \((L, \partial) = (\hat{L}(Qa_0 \oplus Qu \oplus W), \partial)\) be the complete DGL, with \(|u| = n-2\), \(\partial u = 0\) and \(\partial = \partial a_0\) on \(Qa_0 \oplus W\). This DGL satisfies the hypotheses of Proposition 2.4 and by Lemma 2.6

\[
H(L, \partial) = H(\hat{L}(a_0, u), \partial) = \hat{L}(u).
\] (9)
We extend the canonical inclusion \((\hat{L}(\Lambda^n_\Lambda), \partial_{a_0}) \to (\hat{L}(\hat{n}), \partial_{a_0})\) to a morphism,
\[
f : (L, \partial) \to (\hat{L}(\hat{n}), \partial_{a_0}),
\]
given by \(f(u) = \partial_{a_0} a_{0...n}\), which is well defined thanks to the inductive hypothesis. Observing its behavior on the generators we deduce that \(f\) is a DGL isomorphism. Using (9), we get
\[
\hat{L}(u) \cong \hat{L}(\Omega) = H(\hat{L}(\hat{n}), \partial_{a_0}).
\]

**Lemma 2.6.** Let \((\hat{L}(\Lambda a \oplus V), \partial)\) be a DGL in which \(a\) is a Maurer-Cartan element, \(\dim V < \infty\) and \(\partial(V) \subset \hat{L}(V)\). Then, the injection \((\hat{L}(V), \partial) \hookrightarrow (\hat{L}(\Lambda a \oplus V), \partial)\) is a quasi-isomorphism.

**Proof.** Denote by \(J\) the ideal generated by \(V\). Since \(V\) is finite dimensional,
\[
J = \hat{L}(\text{ad}_a^q v, v \text{ basis of } V \text{ and } q \geq 0).
\]

We then denote by \(d_1\) the linear part of the differential \(d\) in \(J\).

We claim that for \(v \in V\), and \(q\) odd,
\[
d_1 \text{ad}_a^q(v) = -\text{ad}_a^{q+1}(v) + (-1)^q \text{ad}_a^q(\partial_1 v).
\]
This holds for \(q = 1\) because \(d_1[a, v] = -\frac{1}{2}[[a, a], v] - [a, \partial_1 v] = -\text{ad}_a^2(v) - [a, \partial_1 v].\)

Suppose this is true for \(q - 2\). Then, since \(\text{ad}_a^q(v) = \frac{1}{q}[[a, a], \text{ad}_a^{q-2}(v)]\), we have
\[
d_1 \text{ad}_a^q(v) \overset{?}{=} \frac{1}{2} d_1[[a, a], \text{ad}_a^{q-2}(v)] = -\frac{1}{2} [[a, a], \text{ad}_a^{q-1}(v)] + (-1)^q \text{ad}_a^q(\partial_1 v)
\]
\[
= -\text{ad}_a^{q+1}(v) + (-1)^q \text{ad}_a^q(\partial_1 v).
\]
The result is then a consequence of Proposition 2.4. □

**Proposition 2.7.** The model of the triangle is given by
\[
\mathcal{L}_2 = (\hat{L}(\Delta^2), \partial) = (\hat{L}(a_0, a_1, a_2, a_{01}, a_{02}, a_{012}), \partial),
\]
in which \((\hat{L}(a_0, a_1, a_{01}), \partial), (\hat{L}(a_1, a_2, a_{12}), \partial)\) and \((\hat{L}(a_0, a_2, a_{02}), \partial)\) are LS-intervals and where their inclusions in \(\mathcal{L}_2\) give the coface maps. Moreover, the differential of \(a_{012}\) is defined by
\[
\partial_{a_0} a_{012} = a_{01} * a_{12} * a_{02}^{-1}.
\]
Proof. The composition of morphisms as in Theorem 1.3 defines the DGL morphism \( \psi : (\mathbb{L}(a, b, x), d) \to (\hat{\mathbb{L}}(a_0, a_1, a_2, a_{01}, a_{02}, a_{021}), \partial) \) in which \( \psi(a) = \psi(b) = a_0 \) and \( \psi(x) = a_{01} \ast a_{12} \ast a_{021}^{-1} \). Hence,

\[
\partial(a_{01} \ast a_{12} \ast a_{021}^{-1}) = \partial \psi(x) = \psi(\partial x) = \psi \left( \text{ad}_x(b) + \sum_{k \geq 0} \frac{B_k}{k!} \text{ad}_x^k(b - a) \right) = \text{ad}_{a_{01} \ast a_{12} \ast a_{021}^{-1}}(a_0).
\]

Finally observe that the linear part of \( \partial(a_{012}) \) is \( a_{12} - a_{02} + a_{01} \). In geometrical terms, this expression draws the border of \( \Delta^2 \) starting from the base point \( a_0 \).

Proof of Theorem 2.3. We proceed by induction on \( n \). Let \( (\mathfrak{L}_0, \partial) = (\hat{\mathbb{L}}(a_0), \partial) \) in which \( a_0 \) is a Maurer-Cartan element. On the other hand, let \( \mathfrak{L}_1 = (\hat{\mathbb{L}}(a_0, a_1), \partial) \) be the LS-interval and let \( \mathfrak{L}_2 \), the first one satisfying the inductive statement, be the model of the triangle as in Proposition 2.7.

Assume that \( \mathfrak{L}_q = (\hat{\mathbb{L}}(\Delta^q), \partial) \) is defined for \( q < n \), as in the statement, with \( n \geq 3 \). The condition (3) of Definition 2.1 defines the differential \( \partial \) on each face of \( \Delta^n \). In particular, by induction, \( \partial_a a_{0...n-1} \in \hat{\mathbb{L}}(\Lambda_n^a) \). As \( H(\hat{\mathbb{L}}(\Lambda_n^a), \partial_a) = 0 \) by Corollary 2.5(i), there exists \( \Gamma \in \hat{\mathbb{L}}(\Lambda_n^a) \) such that \( |\Gamma| = n - 2 \) and \( \partial_a a_{0...n-1} = \partial_a \Gamma \).

We set,

\[
\partial_a a_{0...n} = (-1)^n (a_{0...n-1} - \Gamma).
\]

By construction, this model satisfies the conditions (1) and (3) of Definition 2.1. Now denote by \( \partial_1 a_{0...n} \) the linear part of \( \partial_a a_{0...n} \) and by \( \Gamma_1 \) the linear part of \( \Gamma \). Since \( \partial_1^2 = 0 \), \( \partial_1 \Gamma_1 = \partial_1 (a_{0...n-1}) \). Let \( \omega \) be the difference \( \omega = (-1)^{n-1} \Gamma_1 - \sum_{i=0}^{n-1} (-1)^i a_{0...\hat{i}...n} \). Since \( \partial_1 (\sum_{i=0}^{n} (-1)^i a_{0...\hat{i}...n}) = 0 \),

\[
\partial_1 \omega = (-1)^{n-1} \partial_1 (a_{0...n-1}) + (-1)^n \partial_1 (a_{0...n-1}) = 0.
\]

Thus, \( \omega \) is a \( \partial_1 \)-cycle of degree \( n - 2 \) in \( \hat{\mathbb{L}}(\Lambda_n^a) \). Hence, there is a linear element, \( \gamma \), of degree \( n - 1 \) in \( \hat{\mathbb{L}}(\Lambda_n^a) \) such that \( \partial_1 \gamma = \omega \). As \( \hat{\mathbb{L}}(\Lambda_n^a) \) is generated by elements of degree \( \leq n - 2 \), we have \( \gamma = 0 \) and \( \omega = 0 \). Therefore, we get the expected linear differential,

\[
\partial_1 a_{0...n} = (-1)^n a_{0...n-1} + \sum_{i=0}^{n-1} (-1)^i a_{0...\hat{i}...n-1}.
\]

We now prove the uniqueness up to isomorphism of the sequences of models.
Theorem 2.8. Two sequences \( \{(\tilde{L}(\Delta^n), \partial)\}_{n \geq 0} \) and \( \{(\tilde{L}(\Delta^n), \partial')\}_{n \geq 0} \) of compatible models of \( \Delta \) are isomorphic: for \( n \geq 0 \), there are DGL isomorphisms,

\[
\varphi_n : (\tilde{L}(\Delta^n), \partial) \xrightarrow{\sim} (\tilde{L}(\Delta^n), \partial'),
\]

which commute with the coface maps \( \delta_i \), for \( i = 0, \ldots, n \),

\[
(\tilde{L}(\Delta^n), \partial) \xrightarrow{\varphi_n} (\tilde{L}(\Delta^n), \partial') \xrightarrow{\delta_i} (\tilde{L}(\Delta^i), \partial)
\]

and such that \( \text{Im} (\varphi_n - \text{id}) \subset \tilde{L}^{\geq 2}(\Delta^n) \).

Proof. Obviously there is only one choice for \( (\tilde{L}(\Delta^0), \partial) \) while \( (\tilde{L}(\Delta^1), \partial) \) is also uniquely determined by Theorem 2.4. Suppose \( n \geq 2 \) and \( \varphi_m \) defined for all \( m < n \). Then, the condition (3) of Definition 2.1 determines \( \varphi_n \) on every generator of \( \tilde{L}(\Delta^n) \) except on \( x = a_{0 \ldots n} \).

We may assume without losing generality that the sequence \( \{(\tilde{L}(\Delta^n), \partial)\}_{n \geq 0} \) is inductive. Hence, as \( \partial_{a_0} x \in \tilde{L}(\Delta^n) \), the hypothesis \( \text{Im} (\varphi_i - \text{id}) \subset \tilde{L}^{\geq 2}(\Delta^i) \) for \( i < n \), implies that \( \varphi_n \partial_{a_0} x - \partial'_{a_0} x \) is a decomposable element. Since this is a \( \partial'_{a_0} \)-cycle, there exists \( \omega \in \tilde{L}(\Delta^n) \), \(|\omega| = n - 1\), such that \( \partial'_{a_0} \omega = \varphi_n \partial_{a_0} x - \partial'_{a_0} x \). The linear part, \( \omega_1 \), of \( \omega \) is a \( \partial_1 \)-cycle and there exists a linear element \( \gamma \in \tilde{L}(\Delta^n) \), \(|\gamma| = n \), such that \( \partial_1' \gamma = \omega_1 \). By degree reasons, we have \( \gamma = 0 \) and \( \omega_1 = 0 \). Thus, we set

\[
\varphi_n(x) = x + \omega.
\]

As \( \omega \in \tilde{L}^{\geq 2}(\Delta^n) \), the condition \( \text{Im} (\varphi_n - \text{id}) \subset \tilde{L}^{\geq 2}(\Delta^n) \) is fulfilled. Moreover, the coface maps being linear, the square \( (\Box) \) is commutative.

Example 2.9. The model of the tetrahedron

Let \( (L, \partial) \) be a complete DGL, and \( e_1, \ldots, e_n \in L_1 \). We form the DGL \( (L', \partial') = (\tilde{L}(e_i, u_i), \partial') \) where \( \partial'(u_i) = 0 \) and \( \partial'(e_i) = u_i \) and the DGL morphism \( \gamma : (L', \partial') \to (L, \partial) \) defined by \( \gamma(e_i) = e_i \) and \( \gamma(u_i) = \partial e_i \). Now the product \( u_1 \ast \cdots \ast u_n \) is a linear combination of Lie brackets and in each of them we replace one and only one \( u_i \) by the corresponding \( e_i \). This defines an element \( A_{e_1 \ldots e_n} \in L' \), and we define \( B_{e_1 \ldots e_n} = \gamma(A_{e_1 \ldots e_n}) \). By construction we have

\[
\partial B_{e_1 \ldots e_n} = \partial e_1 \ast \cdots \ast \partial e_n.
\]
and its linear part is precisely $\sum_{i=1}^{n} e_i$.

For the model of the tetrahedron, we observe that via Proposition 2.7,

$$\partial_{a_0} B_{a_{012}, a_{023}, -a_{013}} = \partial_{a_0} a_{012} \ast \partial_{a_0} a_{023} \ast (-\partial_{a_0} a_{013}) = a_{01} \ast a_{12} \ast a_{23} \ast a_{13}^{-1} \ast a_{01}^{-1}.$$  

On the other hand, a direct computation, using successively Proposition 1.5, the model of the triangle described in Proposition 2.7 and Proposition 1.1, gives

$$\partial_{a_0} (e^{\text{ad}_{a_01}} a_{123}) = e^{\text{ad}_{a_01}} \partial_{a_1} a_{123} = e^{\text{ad}_{a_01}} (a_{12} \ast a_{23} \ast a_{13}^{-1}) = a_{01} \ast a_{12} \ast a_{23} \ast a_{13}^{-1} \ast a_{01}^{-1}.$$  

Finally, define

$$\partial_{a_0} a_{0123} = e^{\text{ad}_{a_01}} a_{123} - B_{a_{012}, a_{023}, -a_{013}}.$$  

We finish by finding base points in the barycenter of the simplex.

**Proposition 2.10.** Given $(\hat{L}(\Delta^n), \partial)$ a symmetric model of $\Delta^n$, there exists a Maurer-Cartan element $a$ whose linear part is the barycentre of $\Delta^n$ and such that, for $n \geq 2$, $\text{Im} \partial_a \subset \hat{L}(\Delta^n)$.

**Proof.** Consider the sequence of Maurer-Cartan elements $x_0, \ldots, x_n$ defined by

$$x_0 = a_n, \quad x_r = \frac{a_{rn}}{n+1} \hat{g} x_{r-1}.$$  

Denote by $u_1$ the linear part of an element $u$. Since for $y \in L_0$ and $u \in \text{MC}(L)$, $(y \hat{g} u)_1 = u_1 - (dy)_1$, we have $(x_n)_1 = \sum_{i=0}^{n} \frac{a_i}{n+1}$.

The last part of the statement is a consequence of the behavior of the differential on inductive models.  

3 Symmetric models of $\Delta$ and the cosimplicial structure

Let $a_{i_0 \ldots i_p}$ be a generator of $\hat{L}(\Delta^n)$ and $\sigma \in \Sigma_{p+1}$ of signature $\varepsilon_{\sigma}$. We set

$$a_{i_{\sigma(0)} \ldots i_{\sigma(p)}} = \varepsilon_{\sigma} a_{i_0 \ldots i_p}.$$  

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With this notation, we define an action of the symmetric group $\Sigma_{n+1}$ on the generators of $\hat{\mathbb{L}}(\Delta^n)$ by the rule,

$$\sigma \cdot a_{i_0...i_p} = a_{\sigma(i_0)\ldots\sigma(i_p)}.$$ 

We extend it to brackets by $\sigma \cdot [a, b] = [\sigma \cdot a, \sigma \cdot b]$ and get an action on $\hat{\mathbb{L}}(\Delta^n)$.

**Definition 3.1.** A sequence $\{(\hat{\mathbb{L}}(\Delta^n), \partial)\}_{n \geq 0}$ of compatible models of $\Delta$ is called symmetric if for the above action each $(\hat{\mathbb{L}}(\Delta^n), \partial)$ is a $\Sigma_{n+1}$-DGL, that is, a DGL whose bracket and differential are compatible with the $\Sigma_{n+1}$ action.

Since $(\mathbb{L}_n, \partial_1)$ is $\Sigma_{n+1}$-DGL, we have

**Lemma 3.2.**

$$H_q(\hat{\mathbb{L}}(\Delta^n)^{\Sigma_{n+1}}, \partial_1) = \begin{cases} \mathbb{Q}[\sum_{n+1}a_i] & \text{if } q = -1, \\ 0 & \text{if } q \geq 0. \end{cases}$$

**Proof.** This is clear for $n = 0$. In general $H(\hat{\mathbb{L}}(\Delta^n)^{\Sigma_{n+1}}, \partial_1)$ injects into $H(\hat{\mathbb{L}}(\Delta^n), \partial_1) = \mathbb{L}(a_0)$. Suppose $\alpha$ is a symmetric cycle whose image is in the same homology class than $a_0$, then $\alpha$ is homologous to the symmetrization $\overline{a_0} = \frac{1}{(n+1)!} \sum_{\sigma \in \Sigma_{n+1}} \sigma \cdot a_0$.

**Theorem 3.3.** There exists a sequence of symmetric models of $\Delta$.

**Proof.** The model of $\Delta^0$ is trivally symmetric. Let $\sigma$ be the generator of $\Sigma_2$ and observe directly from (4) and (5) that in the LS-interval, $\Sigma_1 = (\hat{\mathbb{L}}(a_0, a_1, a_{01}), \partial)$, the morphism defined by $\sigma(a_0) = a_1$, $\sigma(a_1) = a_0$ and $\sigma(a_{01}) = -a_{01}$ commutes with $\partial$. We deduce then directly that $\Sigma_1$ is symmetric.

Assume $(\hat{\mathbb{L}}(\Delta^{n-1}), \partial)$ to be symmetric with $n \geq 2$ and observe that, by construction $(\mathbb{L}(\Delta^n), \partial)$ is a $\Sigma_{n+1}$-DGL. It remains only to define $d(a_{0...n})$ in order that $\sigma d(a_{0...n}) = d\sigma(a_{0...n})$ for all permutation $\sigma$.

Write $x = a_{0...n}$ and $\partial = \partial_1 + \partial_2 + \ldots$ where $\partial_q$ increases the length of the brackets by $q - 1$. By definition $\partial_1 x = \sum_{i=0}^n (-1)^i a_{0...i}...a_{i...n}$ is symmetric, and we suppose by induction that, for all $r < q$, the elements $\partial_r x$ have been defined, are symmetric and satisfy

$$\sum_{i+j=r+1} \partial_i \partial_j x = 0.$$
Then $\sum_{i=2}^{q} \partial_{i} \partial_{q+1-i}x$ is a decomposable symmetric $\partial_{1}$-cycle. By Lemma 3.2 there exists a symmetric element $\omega_{q}$ such that

$$
\sum_{i=2}^{q} \partial_{i} \partial_{q+1-i}x = \partial_{1}\omega_{q}.
$$

We set $\partial_{q}x = \omega_{q}$ and check easily that the induction step is attained. \qed

**Theorem 3.4.** Any sequence $\{\mathfrak{L}_{n}\}_{n \geq 0}$ of models of $\Delta$ admits a cosimplicial DGL structure for which the cofaces are the usual ones.

**Proof.** Assume first that the models $\mathfrak{L}_{n}$ are symmetric. For $0 \leq i \leq n$, denote as usual by $\sigma_{i} : \{0, \ldots, n+1\} \to \{0, \ldots, n\}$ the morphism defined by

$$
\sigma_{i}(j) = \begin{cases} 
  j & \text{if } j \leq i, \\
  j - 1 & \text{if } j > i.
\end{cases}
$$

Then, we define a DGL morphism, denoted in the same way,

$$
\sigma_{i}(a_{\ell_{0} \ldots \ell_{q}}) = \begin{cases} 
  a_{\sigma_{i}(\ell_{0}) \ldots \sigma_{i}(\ell_{q})} & \text{if } \sigma_{i}(\ell_{0}) < \cdots < \sigma_{i}(\ell_{q}), \\
  0 & \text{otherwise}.
\end{cases}
$$

When the elements $\sigma_{i}(\ell_{j})$ are different, then $\sigma_{i}$ extends to an element of $\Sigma_{n+2}$ and $\partial\sigma_{i}(a_{\ell_{0} \ldots \ell_{q}}) = \sigma_{i}\partial(a_{\ell_{0} \ldots \ell_{q}})$.

In the case the sequence $\ell_{0}, \ldots, \ell_{q}$ contains the elements $i$ and $i+1$, then denote by $\tau$ the permutation $\tau = (i, i+1)$. We have $\sigma_{i} \circ \tau = \sigma_{i}$. Thus,

$$
\sigma_{i}\partial(a_{\ell_{0} \ldots \ell_{q}}) = \sigma_{i}\tau\partial(a_{\ell_{0} \ldots \ell_{q}}) = -\sigma_{i}\partial(a_{\ell_{0} \ldots \ell_{q}}).
$$

Therefore $\sigma_{i}\partial(a_{\ell_{0} \ldots \ell_{q}}) = 0 = \partial\sigma_{i}(a_{\ell_{0} \ldots \ell_{q}})$, and thus the $\sigma_{i}$’s are DGL morphisms. The cosimplicial identities are trivially satisfied.

Now let $\{\mathfrak{L}'_{n}\}_{n \geq 0}$ be another sequence of compatible models. By Theorem 2.8 we have compatible isomorphisms

$$
\varphi_{n} : \mathfrak{L}'_{n} \xrightarrow{\cong} \mathfrak{L}_{n}.
$$

We define then the codegeneracies as $\varphi^{-1}_{n} \sigma_{i} \varphi_{n}$. Since the $\varphi_{n}$’s commute with the cofaces, the cosimplicial identities are also satisfied in this case. \qed
4 The realization functor and its adjoint

Based on a sequence $\mathfrak{L}_\bullet$ of compatible models of $\Delta$ with the cosimplicial structure given by Theorem 3.4, we define a pair of adjoint functors,

$$\xymatrix{ \text{SimpSet} & \ar[r]^-{\cdot} & \text{DGL} \ar[l]_-{\mathfrak{L}} }$$

between the categories of simplicial sets and complete DGL’s.

**Definition 4.1.** Let $L \in \text{DGL}$. The *realization* of $L$ is the simplicial set,

$$\langle L \rangle_\bullet = \text{DGL}(\mathfrak{L}_\bullet, L).$$

On the other hand, let $\Delta$ be the category whose objects are the sets $[n] = \{0, \ldots, n\}$, $n \geq 0$, and whose morphisms are monotone maps. Now, let $I: \Delta \to \text{SimpSet}$ the functor that associates to $[n]$ the simplicial set $\Delta^n$ whose $p$-simplices are the sequences $0 \leq i_0 \leq \cdots \leq i_p \leq n$. Observe that, by construction, $\mathfrak{L}_\bullet$ is a functor from $\Delta$ to DGL.

**Definition 4.2.** The functor *model* $\mathfrak{L}: \text{SimpSet} \to \text{DGL}$ is defined as the left Kan extension of $\mathfrak{L}_\bullet$ along $I$,

$$\xymatrix{ \Delta & \text{SimpSet} \ar[l]^-{I} \\ \text{DGL} \ar[ur]_{\mathfrak{L}_\bullet} }$$

The DGL $\mathfrak{L}(K)$ is thus the colimit of $\mathfrak{L}_\bullet$ over the comma category $I \downarrow K$,

$$\mathfrak{L}(K) = \text{Lan}_I \mathfrak{L}_\bullet(K) = \lim_{f: \Delta^n \to K} \mathfrak{L}_n.$$

For simplicity, we write

$$\mathfrak{L}(K) = \lim_K \mathfrak{L}_\bullet,$$

and refer to it as the $\mathfrak{L}$-*model* of the simplicial set $K$.

In the case $K$ is a finite simplicial complex, then $K \subset \Delta^n$ for some $n$, and $\mathfrak{L}(K)$ is trivially isomorphic to the complete sub DGL $(\hat{\mathfrak{L}}(V), \partial) \subset \mathfrak{L}_n$ where $(V, \partial_1)$ is the desuspension of the chain complex of $K$. 

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Theorem 4.3. The functors \( \mathcal{L} \) and \( \langle \cdot \rangle \) are adjoint. More precisely, for any simplicial set \( K \) and any complete differential graded Lie algebra \( L \), there is a bijection,

\[
\text{SimpSet}(K, \langle L \rangle) \cong \text{DGL}(\mathcal{L}(K), L).
\]

Proof. The result follows from classical properties of commutation of limits with hom functors, i.e.,

\[
\text{DGL}(\mathcal{L}(K), L) = \lim_{\rightarrow k} \text{DGL}(\mathcal{L}_n, L) = \lim_{\rightarrow k} \text{SimpSet}(\Delta^n, \langle L \rangle)
\]

\[
= \text{SimpSet}(\lim_{\rightarrow k} \Delta^n, \langle L \rangle) = \text{SimpSet}(K, \langle L \rangle).
\]

We now interpret the homotopy groups of the realization of a DGL and its path component.

Proposition 4.4. For any DGL, \( (L, \partial) \), there is a natural bijection \( \pi_0 \langle L \rangle \cong \tilde{\text{MC}}(L) \).

Proof. By [5, Proposition 3.1], two Maurer-Cartan elements \( z_0, z_1 \in \text{MC}(L) \) are gauge equivalent if there is a map \( \varphi : \mathcal{L}_1 = (\bar{\mathcal{L}}(a, b, x), \partial) \to L \) with \( \varphi(a) = z_0 \) and \( \varphi(b) = z_1 \). By Definition 4.1, \( \langle L \rangle_0 \) is the set of Maurer-Cartan elements of \( L \), and \( \langle L \rangle_1 \) is the set of DGL morphisms from the LS-interval \( \mathcal{L}_1 \) to \( L \). This implies the result.

Proposition 4.5. Let \( (L, \partial) \) be a non negatively graded DGL. Then, \( \langle L \rangle \) is a connected simplicial set and there are natural group isomorphisms

\[
\pi_n \langle L \rangle \cong H_{n-1}(L, d), \quad n \geq 1,
\]

in which \( H_0(L, d) \) is considered with the group structure given by the Baker-Campbell-Hausdorff product.

Proof. By Proposition 4.4, \( \langle L \rangle \) is connected. The coface maps, \( \delta_j : \mathcal{L}_{n-1} \to \mathcal{L}_n \), induce the face maps

\[
d_i = \text{DGL}(\delta_j, L) : \langle L \rangle_n \to \langle L \rangle_{n-1}.
\]

We denote \( \ker d_j = \{ f : (\mathcal{L}_n, \partial) \to (L, \partial) \mid d_j f = 0 \} \). Recall that

\[
\pi_n \langle L \rangle = \cap_{i=0}^n \ker d_i / \sim
\]

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where $f \sim g$ if there is $h \in \langle L \rangle_{n+1}$ such that $d_n h = f$, $d_{n+1} h = g$ and $d_i h = 0$ for $i < n$. We denote by $\overline{f}$ the element of $\pi_n(L)$ represented by $f$. Define,

$$\varphi: \pi_n(L) \xrightarrow{\cong} H_{n-1}(L), \quad \varphi(\overline{f}) = [f(a_0\ldots n)],$$

and observe that, for any $\overline{f} \in \pi_n(L)$, the morphism $f$ vanishes in any $p$-simplex of $\Delta^n$, with $0 \leq p < n$. Hence, it is uniquely determined by $f(a_0\ldots n)$. Straightforward computations show that $\varphi$ is a well defined isomorphism for $n \geq 2$ and a bijection for $n = 1$.

To check that for $n = 1$ this bijection is in fact an isomorphism of groups choose $\alpha, \beta \in \pi_1(L)$, and consider $h \in \langle L \rangle_2 = \text{DGL}(\mathfrak{L}_2, L)$ given by

$$h(a_{01}) = g(a_{01}), \quad h(a_{12}) = f(a_{01}), \quad h(a_{02}) = f(a_{01}) \ast g(a_{01}), \quad h(a_{012}) = 0,$$

being $f, g \in \langle L \rangle_1 = \text{DGL}(\mathfrak{L}_1, L)$ representing $\alpha$ and $\beta$ respectively. Note that, since $L$ is non negatively graded the image of any 0-simplex vanishes for every morphism in $\langle L \rangle$.

Now, in view of the model of $\Delta^2$ given in Proposition 2.7, $h$ is a well defined morphism for which $d_0 h = f$ and $d_2 h = g$. Hence, by definition of the product in $\pi_1(L)$, $\alpha \cdot \beta$ is represented by $d_1 h$. Finally,

$$\varphi(\alpha \cdot \beta) = d_1 h(a_{01}) = h \delta_1(a_{02}) = h(a_{02}) = f(a_{01}) \ast g(a_{01}) = \varphi(\alpha) \ast \varphi(\beta).$$

$\square$

For any differential graded Lie algebra $L$ and any Maurer-Cartan element $z \in \text{MC}(L)$ consider the localization of $L$ at $z$ which is the DGL,

$$L(z) = (L, \partial_z)/(L_{<0} \oplus M) \cong L_{\geq 0} \oplus (L \cap \ker \partial_z)$$

where $M$ is a complement of $\ker \partial_z$ in $L_0$. Observe that the injection of $(L(z), d_z) \hookrightarrow (L, d_z)$ induces an isomorphism in homology in degrees $\geq 0$.

**Theorem 4.6.** $\langle L \rangle \cong \bigcup_{z \in \text{MC}(L)} \langle L(z) \rangle$.

**Proof.** As we know, the components of $\langle L \rangle$ are identified with $\widehat{\text{MC}}(L)$. Via this identification, the component of a given $z \in \widehat{\text{MC}}(L)$ is of the same homotopy type as the reduced simplicial set which we denote by $\langle L \rangle_z$ whose $n$-simplices are the DGL morphisms $f: \mathfrak{L}_n \to L$ such that $f(a_i) = z$ for any 0-simplex $a_i$, $i = 0, \ldots, n$.

The simplicial set $\langle L \rangle_z$ has only one 0-simplex $\overline{z}: \mathfrak{L}_0 \to L$ and its degeneracies are the maps $\overline{z}: \mathfrak{L}_n \to L$ such that $\overline{z}(a_i) = z$ for all $i$ and which vanish on all
generators of non-negative degrees. Observe that, for any \( n \geq 1 \), \( \pi_n(\langle L \rangle_z, \overline{x}) \) is the quotient space \( E_n/\sim \), where \( E_n \) denotes the set of DGL morphisms \( f : \mathcal{L}_n \to L \) such that \( d_i f = \overline{x} \) for all \( i \). When \( f \in E_n \), \( f(a_0...n) \) is a \( \partial_z \)-cycle which defines an isomorphism \( \pi_n(\langle L \rangle_z, \overline{x}) \cong H_{n-1}(L, \partial_z) \) that is in turn induced by the simplicial set weak equivalence

\[
\psi : \langle L \rangle_z \xrightarrow{\sim} \langle L(\hat{z}) \rangle, \quad \psi(f)(a_i) = 0, \quad \psi(f)(ai_0...i_q) = f(ai_0...i_q) \text{ for } q > 0.
\]

\[
\square
\]

5 The \( \mathcal{L} \)-model of \( K \)

We first prove that the homology of \( \mathcal{L}(K) \) for the differential \( \partial_a \) depends only on the component of \( a \).

**Proposition 5.1.** Let \( K \) be a finite simplicial complex and let \( a \) be a 0-simplex. Denote by \( K_a \) the component of \( a \) in \( K \) and denote also by \( a \) the corresponding generator of degree \(-1\) in \( \mathcal{L}(K) \). Then, the injection \( (\mathcal{L}(K_a), \partial_a) \xrightarrow{\sim} (\mathcal{L}(K), \partial_a) \) is a quasi-isomorphism.

**Proof.** Write \( K = K_a \amalg K' \), \( \mathcal{L}(K_a) = \widehat{\mathbb{L}}(V \oplus \mathbb{Q}a) \) and \( \mathcal{L}(K') = \widehat{\mathbb{L}}(W) \). Recall (see for instance [29] Proposition 6.2.(7)) that the ideal \( I \) in \( \mathbb{L}(V \oplus W \oplus \mathbb{Q}a) \) generated by \( V \oplus W \) is the free DGL \( J = \mathbb{L}(V \oplus W) \), where

\[
\overline{V} = \{ \text{ad}^n_a (v_i), n \geq 0 \}, \quad \text{with } (v_i) \text{ a basis of } V; \quad \overline{W} = \{ \text{ad}^n_a (w_j), n \geq 0 \}, \quad \text{with } (w_j) \text{ a basis of } W.
\]

Since \( V \oplus W \) is finite dimensional and \( a \) has degree \(-1\), the kernel \( J \) of the projection \( (\widehat{\mathbb{L}}(V \oplus W \oplus \mathbb{Q}a), \partial_a) \to (\mathbb{L}(a), \partial_a) \) is the DGL \( (\widehat{\mathbb{L}}(V \oplus W), \partial) \). Denote by \( d_1 \) the linear part of the differential \( \partial \) in \( J \). We claim that for \( w \in W \) and \( q \) even, \( \text{ad}_a^q(w) = \text{ad}_a^{q+1}(w) + (-1)^q \text{ad}_a^q(d_1 w) \). Suppose this is true for \( q - 2 \). Then, since \( \text{ad}_a^q(v) = \frac{1}{q}[[a, a], \text{ad}_a^{q-2}(v)] \) we have

\[
\begin{align*}
d_1 \text{ad}_a^q(v) &= \frac{1}{2} d_1 [[a, a], \text{ad}_a^{q-2}(v)] = \frac{1}{2} [[a, a], \text{ad}_a^{q-1}(v) + (-1)^q \text{ad}_a^{q-2}(d_1 v)] \\
&= \text{ad}_a^{q+1}(v) + (-1)^q \text{ad}_a^q(d_1 v).
\end{align*}
\]

Therefore the homology \( H(\overline{W}, \partial_1) = 0 \) and the injection \( (\widehat{\mathbb{L}}(V), \partial_a) \xrightarrow{\sim} (\mathcal{L}(K), \partial_a) \) is a quasi-isomorphism. This implies the result. \[
\square
\]

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Definition 5.2. Let $K$ be a simplicial set. A model of $K$ is a free complete differential Lie algebra $(\hat{L}(Z),\partial)$ connected to $L(K)$ by a sequence of quasi-iso- morphisms. Such model is called minimal if the linear part of the differential is null, i.e., $\partial_1 = 0$.

We precise the behaviour of some models in the connected case.

Let $K$ be a finite connected simplicial complex. Then $(L(K),\partial) = (\hat{L}(V),\partial)$, where $(V,\partial_1)$ is the desuspension of the chain complex of $K$. Let $a_0,\ldots,a_n$ be the 0-simplices of $K$ viewed as generators of $L^{-1}$. We write $W = \bigoplus_{q \geq -1} W_q$ with $W_q = V_q$ for $q \geq 0$ and $W_{-1}$ the vector space generated by the elements $a_i - a_0$.

Proposition 5.3. Let $K$ be a finite connected simplicial complex. With the previous notation, the differential $\partial$ in $L(K)$ can be chosen so that $\partial a_0(W) \subset \hat{L}(W)$ and there is a quasi-isomorphism, 

$$(L(K),\partial_{a_0}) \xrightarrow{\cong} (\hat{L}(W),\partial_{a_0}).$$

Moreover, if we denote by $(L(K)/(a_0),\overline{\partial})$ the quotient of $(L(K),\partial)$ by the ideal generated by $a_0$, then there is an isomorphism, 

$$(L(K)/(a_0),\overline{\partial}) \xrightarrow{\cong} (\hat{L}(W),\partial_{a_0}).$$

Proof. We prove by induction on $n$ that $\partial_{a_0}(W_n) \subset \hat{L}(W)$. By the naturality of the differential in $L(K)$ we have only to prove it when $K = \Delta^n$. It is obvious for $n = -1$ and directly deduced for $n = 0$ from the definition of the differential of the LS-interval. Suppose that $\partial_{a_0}(W_q) \subset \hat{L}(W)$ for $q < n - 1$ and $n \geq 2$. From the induction hypothesis, we get 

$$(\hat{L}(\hat{\Delta}^n),\partial_{a_0}) = (L(a_0),\partial_{a_0}) \hat{\Pi} (\hat{L}(Z),\partial_{a_0}),$$

where $Z$ is the graded vector space generated by the elements $\{a_i - a_0 | i > 0\}$, together with the generators of degree $\geq 0$. Here, $\hat{\Pi}$ denotes the coproduct in the category of DGL’s which is obtained by completing the non-complete coproduct. Since $H(L(a_0),\partial_{a_0}) = 0$, the inclusion of the second factor is a quasi-isomorphism, 

$$(\hat{L}(Z),\partial_{a_0}) \xrightarrow{\cong} (L(a_0),\partial_{a_0}) \hat{\Pi} (\hat{L}(Z),\partial_{a_0}). \quad (12)$$

On the other hand, in view of Corollary 2.5, 

$$H(\hat{L}(\hat{\Delta}^n),\partial_{a_0}) = L(u),$$

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where \(|u| = n - 2\). With (12), we can choose \(u \in \hat{\mathcal{L}}(Z)\), and set \(\partial_{a_0}(a_{0...n}) = u\). Since \(W = Z \oplus \mathbb{Q}a_{0...n}\), the inclusion \(\partial_{a_0}(W) \subset \hat{\mathcal{L}}(W)\) is established. Then, it follows that

\[
\mathcal{L}(K), \partial_{a_0} \xrightarrow{\sim} (\mathbb{L}(a_0), \partial_{a_0}) \hat{\Pi} (\hat{\mathcal{L}}(W), \partial_{a_0}).
\]

By Lemma 2.6 this implies that \((\mathcal{L}(K), \partial_{a_0})\) is quasi-isomorphic to \((\hat{\mathcal{L}}(W), \partial_{a_0})\).

Moreover on the quotient \(\mathcal{L}(K)/\mathcal{I}\) the induced differentials \(\partial_{a_0}\) coincide. This implies the isomorphism of the statement.

**Corollary 5.4.** Let \(K\) be a connected finite simplicial complex. Then, \((\mathcal{L}(K), \partial_{a_0})\) has a minimal model of the form \((\hat{\mathcal{L}}(R), \partial)\), with \(R = R_{\geq 0} \simeq s^{-1}\tilde{H}(K; \mathbb{Q})\).

**Proof.** Denote by \(T\) a maximal tree contained in the graph formed by the 0-simplices and the 1-simplices of \(K\). Since \(K\) is connected, this tree contains all the vertices. Moreover, in \(\mathcal{L}(K)\), the ideal \(\mathcal{I}\), generated by the \(\{a_i - a_0 \mid i > 0\}\) together with the generators of dimension 0 corresponding to the edges of \(T\), is acyclic. Therefore, the quotient \(\mathcal{L}(K)/\mathcal{I}\) is quasi-isomorphic to a DGL of the form

\[
(\mathbb{L}(a_0), \partial_{a_0}) \hat{\Pi} (\hat{\mathbb{L}}(R), \partial),
\]

with \(R = R_{\geq 0}\).

Now, suppose that for some \(x \in R\) we have \(\partial_1 x = y \neq 0\). Then, replace \(y\) by \(\partial x\), and by Proposition 2.4, taking the quotient by the ideal generated by \(x\) and \(y\) gives a quasi-isomorphism \((\hat{\mathbb{L}}(R), \partial) \xrightarrow{\sim} (\hat{\mathbb{L}}(R)/(x,y), \partial)\). We continue in the same way and finally obtain a minimal complete free DGL quasi-isomorphic to \((\mathcal{L}(K), \partial_{a_0})\).

**Example 5.5.** If \(K\) is a connected finite simplicial complex of dimension 1, then \((\mathcal{L}(K), \partial_a) \simeq (\hat{\mathbb{L}}(V), 0)\) with \(V = s^{-1}H_1(K)\).

**Example 5.6.** The minimal model of a finite simplicial set \(K\) of the homotopy type of \(S^1 \lor S^2\) is of the form \((\hat{\mathbb{L}}(a,b), 0)\), with \(|a| = 0\) and \(|b| = 1\). In particular, \(\tilde{H}(\mathcal{L}_K)\) is the completion of \(\mathbb{L}(a,b)\) for the adjoint action of \(a\) on \(b\).

### 6 Differential graded Lie coalgebras and the Transfer Theorem

In this, and in the next section, we show how the homotopy transfer theorem and the structure of free Lie coalgebras let us also associate a DGL to a finite simplicial
complex $K$. The uniqueness theorem (Theorem 2.8) will then identify this DGL with our construction $\mathcal{L}(K)$. Following this approach we may understand the relation between the combinatorial properties of $\mathcal{L}(K)$ with the algebra of PL-forms on $K$.

Recall that a graded Lie coalgebra is a graded vector space, $V$, with a co-multiplication $\Delta: V \rightarrow V \otimes V$ that satisfies two properties,

$$(1 + \tau) \circ \Delta = 0 \quad \text{and} \quad (1 + \sigma + \sigma^2) \circ (1 \otimes \Delta) \circ \Delta = 0.$$ 

Here, $\tau: V \otimes V \rightarrow V \otimes V$ is the graded permutation and $\sigma: V \otimes V \otimes V \rightarrow V \otimes V \otimes V$ is the graded cyclic permutation [12, 22].

Any graded coalgebra $(C, \Delta)$ admits a Lie coalgebra structure defined by $\Delta_L = \Delta - \tau \circ \Delta$. If $T^c(V)$ denotes the tensor coalgebra on a graded vector space $V$, then $\Delta_L$ induces a Lie coalgebra structure on the indecomposables for the shuffle product, i.e., on the quotient of $T^c(V)$ by the shuffle products. This quotient is called the free Lie coalgebra on $V$ and is denoted by $\mathbb{L}^c(V)$. We denote by $\text{DGLC}$ the category of differential graded Lie coalgebras and by $\text{CDGC}$ the category of differential connected graded cocommutative coalgebras. If $\text{Vect}$ denotes the category of graded vector spaces, there is a natural isomorphism

$$\text{Vect}(\Gamma, V) = \text{DGLC}((\Gamma, \mathbb{L}^c(V)))$$

when $V$ is graded vector space and $\Gamma$ a graded Lie coalgebra [12, §4.2.1]. The graded universal coenveloping coalgebra of a graded Lie coalgebra $L$ is a coalgebra, denoted $U^cL$ with the property that for any graded coalgebra $C$, we have

$$\text{DGLC}(C, L) \cong \text{CDGC}(C, U^cL).$$

In particular, there is a coalgebra isomorphism $U^cL(V^\sharp) \cong T^c(V)$. If $(T^c(V), d)$ is a differential graded coalgebra such that, for every $n$, the part $d_n: \otimes^n V \rightarrow V$ of the differential vanishes on the $(p, q)$-shuffles with $p + q = n$, then $d$ makes of $(\mathbb{L}^c(V), d)$ a differential graded Lie coalgebra.

Clearly, the dual of a differential Lie coalgebra is a differential Lie algebra. Moreover, the following holds.

**Lemma 6.1.** For any finite type vector space, $V$, we have $(\mathbb{L}^c(V))^\sharp = \mathbb{L}(V^\sharp)$.

**Proof.** From $\mathbb{L}^c(V) = \lim_{\rightarrow n} (\mathbb{L}^c)^{\leq n}(V)$, we deduce,

$$\text{Hom}(\lim_{\rightarrow n} (\mathbb{L}^c)^{\leq n}(V), \mathbb{Q}) = \lim_{\leftarrow n} (\text{Hom}(\mathbb{L}^c)^{\leq n}(V), \mathbb{Q})$$

$$= \lim_{\leftarrow n} \mathbb{L}(V^\sharp)/\mathbb{L}^{> n}(V^\sharp) = \mathbb{L}(V^\sharp).$$

$\square$
Let \( f: (T^c(V), d) \to (T^c(W), d) \) be a morphism of differential graded coalgebras. We denote by \( d_1 \) the linear part of the differentials, \( d_1(V) \subset V, d_1(W) \subset W \), and we write \( f = f_0 + f_1 + \cdots \) with \( f_i(T^c)^r(V) \subset (T^c)^{r-i}(W) \).

**Proposition 6.2.** With the above notations, suppose \( f_0: (V, d_1) \cong (W, d_1) \) is a quasi-isomorphism. Then,

(i) \( f \) is a quasi-isomorphism.

(ii) Moreover, if \( f \) and the differentials \( d \) vanish on the decomposables for the shuffle product, then the induced map \( f: (\mathbb{L}^c(V), d) \cong (\mathbb{L}^c(W), d) \) is also a quasi-isomorphism.

**Proof.** (i) First of all, by the Künneth Theorem,

\[
f_0^{\otimes n}: (T^n(V), d_1) \to (T^n(W), d_1)
\]

is a quasi-isomorphism for each \( n \). Therefore, by the five lemma,

\[
f_{\leq q} = f_0 + \cdots + f_q: (T^{\leq q}(V), d) \to (T^{\leq q}(W), d)
\]

is a quasi-isomorphism for all \( q \geq 1 \). This implies the surjectivity of \( H(f) \). To prove the injectivity suppose \( H(f)(\alpha) = 0 \). Write \( \alpha = [a], a \in T^{\leq q}(V) \) and \( f(a) = db, b \in T^{\leq r}(W) \), for some \( r \geq q \). Then since \( f_{\leq r} \) is a quasi-isomorphism, \( \alpha = 0 \).

(ii) Recall that \( \mathbb{L}^c(V) \) is a retract of \( T^c(V) \), obtained by identifying \( \mathbb{L}^c(sV) \) with the image of the first Eulerian idempotent \( e^{(1)} \), i.e., see [12, Theorem 4.2.3],

\[
\mathbb{L}^c(sV) = e^{(1)}T^c(sV).
\]

Since this retraction is natural with respect to linear maps, it commutes with the differentials \( d_1 \) and with \( f_0 \). Therefore,

\[
\mathbb{L}^c(f_0): (\mathbb{L}^c(V), d_1) \to (\mathbb{L}^c(W), d_1)
\]

is a retract of \( T^c(f_0) \) and is thus a quasi-isomorphism. We deduce in the same way that \( f: (\mathbb{L}^c(V), d) \cong (\mathbb{L}^c(W), d) \) is a quasi-isomorphism. \( \square \)

The Quillen correspondence \( \mathcal{L} \) and \( \mathcal{C} \) (generalized by Neisendorfer in [20]) between \( \text{DGL} \) and \( \text{CDGC} \) dualizes in a pair of adjoint functors between \( \text{DGLC} \) and the
category CDGA of augmented commutative differential graded algebras, that we can summarize in the following commutative diagram \[26, \text{Theorem 4.17} \],

\[
\begin{array}{c}
\text{CDGC} \\ \uparrow (-)^t
\end{array}
\begin{array}{c}
\text{DGL} \\ \downarrow (-)^t
\end{array}
\begin{array}{c}
\text{CDGA} \\ \uparrow (-)^t
\end{array}
\begin{array}{c}
\text{DGLC}
\end{array}
\]

Moreover, the adjunction maps, \( \mathcal{L} \mathcal{C}(L) \cong L \) and \( \mathcal{C} \mathcal{L}(C) \cong \mathcal{C} \mathcal{L}(C) \), are quasi-isomorphisms, see [23, Proposition 4.1]. In the same way, we have also quasi-isomorphisms, \( \mathcal{A} \mathcal{E}(A) \cong A \) and \( E \cong \mathcal{E} \mathcal{A}(E) \).

The functor \( \mathcal{A} \) is defined as follows. Let \((E,d)\) be a differential graded Lie coalgebra. Then, \( \mathcal{A}(E) = (\wedge sE,D) \) with

\[
D(sx) = \frac{1}{2} \sum_i (-1)^{|x_i|} sx_i \wedge sx_i' - sdx,
\]

with \( \Delta x = \sum_i x_i \otimes x'_i \). In particular, if \((\wedge Z,d)\) is the minimal model of \( \mathcal{A}(E) \), then \( Z \cong sH(E,d) \).

Recall now that an \( A_\infty \)-algebra structure on a graded vector space \( A \) consists in a sequence of linear maps \( m_n : A^\otimes n \to A \), \( n \geq 1 \), of degree \( 2 - n \) such that for all \( r,s,t \geq 1 \)

\[
\sum_{r+s+t=n} (-1)^{r+s+t} m_{r+s+t+1}(id^\otimes r \otimes m_s \otimes id^\otimes t) = 0.
\]

Now to each \( m_n \) is associated a canonical map of degree 1, \( d_n : (sA)^\otimes n \to sA \). The set of \( d_n \) induces a coderivation \( d \) on the coalgebra \( T^c(sA) \). It is well known that \( A \) is an \( A_\infty \)-algebra if and only if \((T^c(sA),d)\) is a differential graded coalgebra. Moreover, in the case \( A \) is a differential graded algebra, we recover the bar construction on \( A \), \((T^c(sA),d)\).

A morphism of \( A_\infty \)-algebras \( f : A \to B \) consists of a sequence of morphisms \( f_n : A^\otimes n \to B \) such that the induced morphism \( F : (T^c(sA),d) \to (T^c(sB),d) \) is a morphism of differential graded coalgebras.

An \( A_\infty \)-algebra structure on \( A \) is commutative if the differential \( d \) vanishes on the shuffle products in \( T^c(sA) \). In this case, \( d \) induces a differential on the quotient \( \mathbb{L}^c_A \), and we have a differential graded Lie coalgebra, denoted \((\mathbb{L}^c(sA),d)\). When \( f \) is a morphism between commutative \( A_\infty \)-algebras, that preserves the shuffle products, then \( f \) restricts to a morphism of Lie coalgebras, \((\mathbb{L}^c(sA),d) \to (\mathbb{L}^c(sA'),d')\).
A unital $A_\infty$-algebra is an $A_\infty$-algebra endowed with an element $1_A$ of degree 0 such that $m_1(1_A) = 0$, $m_2(1_A, a) = a = m_2(a, 1_A)$ for all $a \in A$ and such that for all $i > 2$ and all $a_1, \ldots, a_i \in A$, the product $m_i(a_1, \ldots, a_i)$ vanishes if one of the $a_i$ equals $1_A$. A morphism of $A_\infty$-algebras, $f: A \to B$ is unital if $f(1_A) = 1_B$ and for $n > 1$ $f_n(a_1, \ldots, a_n) = 0$ if one of the $a_i$ equals $1_A$. A unital $A_\infty$-algebra is augmented if it is endowed with a unital morphism $\varepsilon: A \to \mathbb{Q}$ such that $\varepsilon(1_A) = 1$. The augmentation ideal $\mathfrak{A} = \ker \varepsilon$ inherits then an $A_\infty$-algebra structure. A morphism of augmented $A_\infty$-algebras is a unital morphism $f: A \to B$ such that $\varepsilon_B \circ f = \varepsilon_A$. It induces a morphism between the augmentation ideals.

In particular, the functor $E$ is extended to any augmented $A_\infty$-algebra $A$ as the graded Lie coalgebra $E(A) = (Lc(sA), D)$.

In our setting, a transfer CDGA diagram is a diagram of the form

$$\mathfrak{D}: \quad \phi \bigcirc (A, d) \xrightarrow{p} (V, d)$$

where $(A, d)$ is an augmented CDGA, $(V, d)$ is an augmented (cochain) differential graded vector space, $p$ and $i$ are quasi-isomorphisms preserving the augmentations, $pi = id_V$, and $\phi$ is a chain homotopy between $id_A$ and $ip$, i.e., $\phi d + d\phi = id_A - ip$, which satisfies $\phi i = p\phi = \phi^2 = 0$.

**Theorem 6.3.** The diagram $\mathfrak{D}$ induces quasi-isomorphisms of differential graded Lie coalgebras,

$$I: (Lc(sV), D) \xrightarrow{\simeq} (Lc(sA), D),$$

where $I_1 = si$ and $D_1 = sd$ and

$$I: (Lc(sV), D) \xrightarrow{\simeq} (Lc(sA), D).$$

Moreover, since $(A, d)$ is commutative, the differential graded Lie coalgebra $(Lc(sA), D)$ is quasi-isomorphic to $E(A, d)$.

**Proof.** In [16], Kontsevich and Soibelman proves that there is an augmented $A_\infty$-structure on $V$, $(T^c(sV), D)$, and a morphism of $A_\infty$-algebras, $I: (T^c(sV), D) \to (T^c(sA), D)$, such that the linear part of $I$ and the linear part of the differentials $D$ are the suspensions of the original ones. Moreover, by construction, $I(sV) \subset T^c(sA)$. The theorem follows then directly from Proposition 6.2. \qed
The cosimplicial structure via a transfer

Let $K$ be a finite simplicial complex. We denote by $C^*(K)$ the dual of the chain complex of $K$, with rational coefficients, and by $A_{PL}(K)$ the algebra of rational PL-forms on $K$, [10, 27].

**Theorem 7.1.** [8, 9, 13] There is a CDGA transfer diagram of the form,

$$D: \phi \bigcirc A_{PL}(K) \xrightarrow{p_\bullet} C^*(K).$$

In particular there is a simplicial CDGA transfer diagram,

$$D_\bullet: \phi_\bullet \bigcirc A_{PL}(\Delta^\bullet) \xrightarrow{p_\bullet} C^*(\Delta^\bullet),$$

where $p_\bullet$, $i_\bullet$ and $\phi_\bullet$ preserve the simplicial structure.

The transfer diagram $D_\bullet$ is explicitly described for instance in [13, §3]. As usual, $A_{PL}(\Delta^n) = \Lambda(t_0, \ldots, t_n, dt_0, \ldots, dt_n)/\left( \sum t_i - 1, \sum dt_i \right)$ and the maps $p_\bullet$ and $i_\bullet$ are defined as follows:

Let $\alpha_{i_0\ldots i_k}$ be the basis for $C^*(\Delta^n)$ defined by

$$\langle \alpha_{i_0\ldots i_k}, a_{j_0\ldots j_k} \rangle = \begin{cases} (-1)^{k(k-1)/2} & \text{if } (j_0, \ldots, j_k) = (i_0, \ldots, i_k), \\ 0 & \text{otherwise}. \end{cases}$$

Then, $i_n(\alpha_{i_0\ldots i_k})$ is the Whitney elementary form $\omega_{i_0\ldots i_k}$ defined by

$$\omega_{i_0\ldots i_k} = k! \sum_{j=0}^k (-1)^j t_{i_j} dt_{i_0} \ldots \widehat{dt_{i_j}} \ldots dt_{i_k}.$$

Since $d(\omega_{i_0\ldots i_k}) = \sum q \omega_{q_{i_0\ldots i_k}}$, the map $i_n: C^*(\Delta^n) \to A_{PL}(\Delta^n)$ is a morphism of cochain complexes. The map $p_n: A_{PL}(\Delta^n) \to C^*(\Delta^n)$ is defined by

$$p_n(\omega) = \sum_{k=0}^n \sum_{i_0 \cdots < i_k} \alpha_{i_0\ldots i_k} I_{i_0\ldots i_k}(\omega),$$

with

$$I_{i_0\ldots i_k}(t_{i_1}^{a_1} \ldots t_{i_k}^{a_k} dt_{i_1} \ldots dt_{i_k}) = \frac{a_1! \cdots a_k!}{(a_1 + \cdots + a_k + k)!}.$$
and 0 otherwise. In particular, $I_{i_0...i_k}(\omega_{i_0...i_k}) = 1$.

Since the morphisms $p_\bullet$ and $i_\bullet$ preserve the simplicial structures they preserve sub simplicial complexes. So, if $K$ is a finite simplicial complex then $K \subset \Delta^n$ and the diagram $\mathfrak{D}$ is obtained by restriction.

**Corollary 7.2.** For a finite simplicial complex $K$, we have a quasi-isomorphism of differential graded Lie coalgebras,

$$I: (L^c(sC^*(K)), d) \xrightarrow{\sim} (L^c(sA_{PL}(K)), d).$$

Let $a_0$ be a base point. The associated augmentation of $A_{PL}(K)$ is defined by

$$\varepsilon(t_0) = 1, \quad \varepsilon(t_i) = 0 \quad \text{for } i \neq 0.$$

An augmentation in the cochain complex $C^*(K)$ is also defined by $\varepsilon(a_0) = 1$ and $\varepsilon(a_i) = 0$ for $i > 0$. In the transfer diagram the morphisms $p_n$ and $i_n$ preserve the augmentations because $p_n(t_0) = a_0$ and $i_n(a_0) = t_0$. Therefore, the morphism $I$ induces a quasi-isomorphism

$$(L^c(sC^*(K)), d) \xrightarrow{\sim} E(A_{PL}(K)).$$

**Theorem 7.3.** With the above notations,

(i) The dual of the simplicial differential graded Lie coalgebra $(L^c(sC^*(\Delta^\bullet)), D)^{\#}$ is a sequence of compatible models of $\Delta$.

(ii) The differential graded Lie algebras $(\hat{\mathfrak{L}}^c(sC^*(K)), d)^{\#}$ and $\mathfrak{L}(K)$ are isomorphic.

**Proof.** (i) Note first that, by construction, the linear part of the differential is the good one. Thus by Theorem 2.8 we need to check both that, for all $n \geq 0$, each vertex in $(L^c(sC^*(\Delta^\bullet)), D)^{\#}$ is a Maurer-Cartan element, and that the cofaces are as in (7).

On the one hand, By naturality it is enough to check that the only vertex $a_0$ in $(L^c(sC^*(\Delta^0)), D)^{\#}$ is a Maurer-Cartan element. Note that $A_{PL}(\Delta^0) = \mathbb{Q} \cdot e = C^*(\Delta^0)$, with $e^2 = e$ the unit of the algebra. Now in the bar construction [10, Page 269], $d([se|se]) = -[se]$. The anti-symmetrisation of the diagonal gives $\Delta_L([se|se]) = 2[se, se]$. In the dual Lie algebra this gives $d(a_0) = -\frac{1}{2} [a_0, a_0]$.

On the other hand, a simple inspection shows that the faces on $C^*(\Delta^\bullet)$ become the usual cofaces when taking the dual of the free lie coalgebra.

(ii) This is a direct consequence of the fact that $K \subset \Delta^n$ for some $n$ and $\mathfrak{L}(K)$ is isomorphic to a sub DGL $(\hat{\mathfrak{L}}(V), \partial) \subset \mathfrak{L}_n$ where $(V, \partial_1)$ is the suspension of the chain complex of $K$. \qed
Next, we connect the functor $\mathcal{L}$ with the classical Quillen functor $\lambda$: $\text{SimpSet}_1 \to \text{DGL}$ from the category of 2-reduced simplicial sets [25] and with Sullivan minimal models [27].

**Theorem 7.4.** Let $K$ be a connected finite simplicial complex with Sullivan minimal model $(\wedge V, d)$. Then:

(i) For any vertex $a$, $(\mathcal{L}(K), \partial_a)$ is quasi-isomorphic to $(\mathcal{E}(\wedge V, d))^{\sharp}$.

(ii) In particular, if $K$ is 1-connected, then $(\mathcal{L}(K), \partial_a)$ is quasi-isomorphic to $\lambda(K)$.

**Proof.** (i) We have a sequence of quasi-isomorphisms

$$\mathcal{E}(\wedge V, d) \xrightarrow{\simeq} \mathcal{E}(A_{PL}(K)) \xrightarrow{\simeq} (\mathcal{L}(sC^*(K)), d).$$

By taking duals we get a series of DGL quasi-isomorphisms

$$\mathcal{E}(\wedge V, d)^{\sharp} \xrightarrow{\simeq} \mathcal{E}(A_{PL}(K))^{\sharp} \xrightarrow{\simeq} (\mathcal{L}(sC^*(K)), d)^{\sharp}.$$

Now remark that $(\mathcal{L}(sC^*(K)), d)^{\sharp} = (\mathcal{E}(K)/(a_0, \overline{\partial}))$ and recall from Proposition 5.3 that $(\mathcal{L}(K)/(a_0, \overline{\partial})) \cong (\mathcal{L}(K), \partial_{a_0})$.

(ii) When $K$ is 1-connected, the minimal model $(\wedge V, d)$ of $K$ is of finite type. Thus, it is the dual of a differential graded coalgebra $(\wedge V, d)^{\sharp}$, and

$$\mathcal{L}((\wedge V, d)^{\sharp}) = (\mathcal{E}(\wedge V, d))^{\sharp}.$$

The result is then a direct consequence of a theorem of Majewski [19] who connects $\lambda(K)$ with $\mathcal{L}((\wedge V, d)^{\sharp})$ by a sequence of quasi-isomorphisms. 

**8 Representability of the Quillen realization functor**

We have proved that $\mathcal{L}(K)$ is homotopy equivalent to the Quillen construction $\lambda(K)$, when $K$ is 2-reduced of finite type. We study now the connection between the realization functor $\langle - \rangle$ introduced in Section 4 with the composite of the functors, $\mathcal{C}^* = (-)^{\sharp} \circ \mathcal{C}: \text{DGL}_f \to \text{CDGA}$ and $\langle - \rangle_S: \text{CDGA} \to \text{SimpSet}$, where $\text{DGL}_f$ is the full subcategory of $\text{DGL}$ of finite type DGL. The first functor is the classical cochain functor defined on the category of Lie algebras of finite type and the second one is the Sullivan’s realization functor defined by $\langle A \rangle_S = \text{CDGA}(A, A_{PL}(\Delta^*))$. 

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Theorem 8.1. Let \((L, d)\) be a finite type DGL with \(H_q(L, d) = 0\) for \(q < 0\). Then, there is a homotopy equivalence of simplicial sets,

\[
\langle L \rangle \simeq \langle C^*(L) \rangle_S.
\]

Proof. Given the simplicial differential graded Lie coalgebra \(L_c^\bullet = (\mathbb{L}^c(sC^*(\Delta^\bullet)), D)\) of Proposition 7.3 and \(L \in \text{DGL}_f\), we define the simplicial set \(\langle L \rangle' := \text{DGLC}(L^\bullet, L_c^\bullet)\).

A coalgebra morphism \(L^\bullet \to L_n^c\) is completely determined by its projection on the indecomposables \(L^\bullet \to sC^*(\Delta^n)\) and an algebra morphism \(L_n \to L_c^n\) by its restriction \(s^{-1}C_n(\Delta^n) \to L\). Since \(L\) is finite type, \(\langle L \rangle'_n = \langle L \rangle_n\). Also, as taking dual is compatible with the simplicial structure, the dual process induces a bijection of simplicial sets \((-)^\sharp: \langle L \rangle'_\bullet \to \langle L \rangle^\bullet\).

The quasi-isomorphism \(I: L_c^\bullet \to E(\text{A}_{PL}(\Delta^\bullet))\) of Proposition 7.3 induces a morphism of simplicial sets \(\Psi: \text{DGLC}(L^\bullet, L_c^\bullet) \to \text{DGLC}(L^\bullet, E\text{A}_{PL}(\Delta^\bullet))\).

On the other hand the functor \(\mathcal{A}\) is equipped with a natural evaluation morphism,

\[
\epsilon_{\mathcal{A}}: \mathcal{A}\mathcal{E}(A, d) = (\wedge s\mathbb{L}^c(sA), \partial) \to (A, d),
\]

which vanishes on \((\mathbb{L}^c)^{\ge 2}(\overline{A})\). We then obtain a sequence of simplicial maps,

\[
\begin{array}{ccc}
\text{DGLC}(L^\bullet, L_c^\bullet) & \xrightarrow{\Psi} & \text{DGLC}(L^\bullet, E\text{A}_{PL}(\Delta^\bullet)) \\
& \downarrow \mathcal{A} & \downarrow \epsilon_{\mathcal{A}} \\
\text{CDGA}(\mathcal{A}(L^\bullet), E\text{A}_{PL}(\Delta^\bullet)) & \xrightarrow{\epsilon_{\mathcal{A}}} & \text{CDGA}(E^*(L), A_{PL}(\Delta^\bullet)) = \langle E^*(L) \rangle_S.
\end{array}
\]

Recall now that in the transfer, \(i_n(\alpha_{0, \ldots, n}) = \omega_{0, \ldots, n} = n! dt_1 \cdots dt_n\). Therefore, the linear part of \(L_c^n \to E\text{A}_{PL}(\Delta^n)\) is \(i_n\). As \(\pi_n(E^*(L))\) can be identified with the linear maps \(\text{Hom}(H_{n-1}(L), \mathbb{Q}) \to \mathbb{Q} \omega_{0, \ldots, n}\) (see [11 §1.7]), this composition induces an isomorphism on the homotopy groups.

Recall now that a group \(G\) is \(\mathbb{Q}\)-complete if each \(G^n/\mathbb{Q}G^{n+1}\) is a \(\mathbb{Q}\)-vector space and \(G = \lim_{\leftarrow n} G/\mathbb{Q}G^n\). Here \(G = G^1 \supset G^2 \supset \cdots\) is the lower central series of \(G\). That
is, $G^2$ is the subgroup generated by the commutators $[a, b]$ and $G^n$ is the subgroup generated by the iterated commutators $[g_1, [g_2, \ldots [g_{n-1}, g_n], \ldots]$. A nilpotent space $X$ is called a rational nilpotent space if its fundamental group is $\mathbb{Q}$-complete, if each $\pi_n(X)$ is a finite dimensional $\mathbb{Q}$-vector space for $n \geq 2$, and if $\pi_1(X)$ acts nilpotently on $\pi_n(X)$ for $n \geq 2$.

**Proposition 8.2.** Let $K$ be a finite connected simplicial complex. Then, the simplicial set $\langle \mathit{L}(K), \partial_n \rangle$ is the projective limit of a tower of rational nilpotent spaces.

**Proof.** Write $(\mathit{L}(K), \partial_n) = (\overset{\longrightarrow}{\bigoplus} V^\ast, d)$. Then

$$(\mathit{L}(K), \partial_n) = \lim_{\longrightarrow} (\overset{\longrightarrow}{\bigoplus} V^\ast / \mathbb{L}^n, \partial_n).$$

Since the realization functor is a right adjoint, it commutes with limits and

$$\langle \mathit{L}(K), \partial_n \rangle = \lim_{\longrightarrow} \langle (\overset{\longrightarrow}{\bigoplus} V^\ast / \mathbb{L}^n, \partial_n) \rangle.$$

Now, since $(\overset{\longrightarrow}{\bigoplus} V^\ast / \mathbb{L}^n, \partial)$ is a finite dimensional nilpotent Lie algebra, its geometric realization is a rational nilpotent space in view, for instance, of Theorem 8.1.

9 The Malcev completion of the fundamental group

This section is entirely devoted to the proof of the following result.

**Theorem 9.1.** Let $K$ be a connected, finite simplicial complex. Then, $H_0(\mathit{L}(K), \partial_n)$ is the Malcev Lie completion of the fundamental group $\pi_1(K)$.

**Proof.** Let $(\land V, d)$ be the minimal Sullivan model of $K$. Since $K$ is connected, $V = V^{>0}$. The graded vector space $L_k(\land V, d) = (V^{k+1})^\ast$ equipped with the quadratic part of the differential $d$ inherits the structure of a graded Lie algebra [11, §2], whose component $L_0(\land V, d)$ is the Lie algebra associated to the Malcev completion of $\pi_1(K)$ [11, Theorem 7.5].

Suppose first that $V$ is a finite type vector space. In that case $(\land V, d)$ is the dual of a commutative differential graded coalgebra $(C, d)$. Thus, by [23], $H_0(\mathcal{L}(C, d)) \cong L_0(\land V, d)$. On the other hand, the natural map $\mathcal{L}(C, d) \to (\land (\land V, d))^\ast$ is the inclusion

$$\varphi: (\mathit{L}(s^{-1}C), d) \hookrightarrow (\overset{\longrightarrow}{\bigoplus} (s^{-1}C), d).$$

Finally, since $(s^{-1}V)^\ast$ injects in the homology of the two DGL’s, and is in an isomorphism in both cases, the injection $\varphi$ is a quasi-isomorphism.
In the general case, \((\wedge V, d)\) is the increasing union of finite type CDGA’s of the form \((\wedge V(n), d)\), i.e.,
\[
(\wedge V, d) = \lim_n (\wedge V(n), d),
\]
where
\[
L_0(\wedge V(n), d) = L_0(\wedge V, d)/L^n_0(\wedge V, d),
\]
\[
L^2_0(\wedge V, d) = [L_0(\wedge V, d), L_0(\wedge V, d)],
\]
\[
L^n_0(\wedge V, d) = [L_0(\wedge V, d), L^{n-1}_0(\wedge V, d)], \quad n > 2.
\]
Moreover,
\[
L_0(\wedge V, d) = \lim_n L_0(\wedge V(n), d)/L^n_0(\wedge V, d).
\]

We write
\[
(\hat{L}(W(n)), \partial) = E(\wedge V(n), d)^2.
\]
It follows that \(E(\wedge V, d) = \lim_n E(\wedge V(n), d)\) and
\[
(\mathcal{L}_K, \partial) = \lim_n (E(\wedge V(n), d))^2 = \lim_n (\mathcal{L}(W(n)), \partial).
\]

By construction \(W(n) = W(n)_{\geq 0}\) is a finite type graded vector space. We write \(Y_n = \hat{L}(W(n))_0, X_n = \hat{L}(W(n))_1\) and \(X^n_n = L^*(W(n))_1\). We denote by \(\rho_n : X_n \to X_{n-1}\) and \(\sigma_n : Y_n \to Y_{n-1}\) the surjective morphisms induced by the inclusions \(\wedge V(n-1) \hookrightarrow \wedge V(n)\). For \(r < q\) we also denote by \(\rho_{qr} : X_q \to X_r\) the composition \(\rho_{r+1} \circ \cdots \circ \rho_q\).

We consider then the short exact sequences of towers of vector spaces

\[
\begin{array}{ccc}
0 & \to \partial(X_n) & \to \partial(X_{n-1}) & \to \cdots \\
\vdots & \downarrow & \downarrow & \downarrow \\
\cdots & \to Y_n & \to Y_{n-1} & \to \cdots \\
\vdots & \downarrow & \downarrow & \downarrow \\
\cdots & \to H_0(\hat{L}(W(n)), \partial) & \to H_0(\hat{L}(W(n-1)), \partial) & \to \cdots \\
0 & \to \lim_n \partial(X_n) & \to \lim_n Y_n & \to \lim_n H_0(\hat{L}(W(n)), \partial) & \to \lim_n \partial(X_n).
\end{array}
\]

This induces an exact sequence
\[
0 \to \lim_n \partial(X_n) \to \lim_n Y_n \to \lim_n H_0(\hat{L}(W(n)), \partial) \to \lim_n \partial(X_n).
\]
Since every $\rho_n : X_n \to X_{n-1}$ is surjective, the induced map $\partial(X_n) \to \partial(X_{n-1})$ is also surjective and so $\lim \partial(X_n) = 0$.

Now, by the next Lemma, $\lim \partial(X_n) = \partial(\lim X_n)$. Therefore,

$$H_0(\mathcal{L}(K), \partial_a) \cong \lim_{n} H_0(\widehat{\mathcal{L}}(W(n)), \partial) = \lim_{n} L_0(\bigwedge V, d)/L_0^n(\bigwedge V, d)$$

is the Malcev Lie completion of $\pi_1(K)$. □

**Lemma 9.2.** With the notation in the previous proof, let $a = \lim(a_n) \in Y$. If for each $n$, $a_n = \partial c_n$ for some $c_n$, then there is an element $b = \lim(b_n) \in X$ such that $\partial b_n = a_n$ for all $n$.

**Proof.** The element $a_1$ can be written as $a_1 = \sum_{q=p}^{\infty} a_q^p$ with $a_q^p \in \mathbb{L}^q(W(1))_0$ and $a_q^p \neq 0$. For fixed $n$ we denote by $f(n)$ the maximum integer such that there is an element $e \in \widehat{\mathbb{L}}_{\geq 1}^n(W(n))_1$ with $\partial e = a_n$. Since $d \circ \rho_{n1}(e) = a_1$, we have $f(n) \leq p$. As we can replace $e_1, \ldots, e_{n-1}$ by the images of $e_n$, the function $f(n)$ is decreasing and we denote by $r$ its limit.

We construct a sequence $\beta^r = (\beta^r_p)$ with the following properties:

(i) $\beta^r_p \in X^r$ and $\rho_p(\beta^r_p) = \beta^r_{p-1}$.

(ii) There exist elements $e_p \in X_p$ with $\partial e_p = a_p$ and $e_p - \beta^r_p \in X^r_p$.

Suppose we have constructed $\beta^r_1, \ldots, \beta^r_{n-1}$ with the two preceding properties for $p < n$ and

(iii) for all $q$ there exists an element $e_q \in X^q_{n-1}$ with $\partial e_q = a_q$ and $\rho_{q(n-1)}(e_q) - \beta^r_{n-1} \in X^r_{n-1}$.

Then, we denote by $q^r_{n-1} : X_{n-1} \to X^r_{n-1}$ the projection on the component of length degree $r$ and we write

$$W_q = q^r_{n} \rho_{qn} \left( [\partial^{-1}(a_q) \cap X^r_{n-1}] \cap (q^r_{n-1} \circ \rho_{q(n-1)})^{-1}(\beta^r_{n-1}) \right).$$

The $W_q$ form a decreasing sequence of affine subspaces of $X^r_n$. Since $X^r_n$ is finite dimensional and the $W_q$ are non empty, their intersection is not empty and we choose an element $\beta^r_n$ in this intersection. This ends the construction of $\beta^r = (\beta^r_n)$.

Now consider $\alpha' = \alpha - \partial \beta^r$. We proceed in the same way for $\alpha'$ and note that by construction there is an element $e'_n = e_n - \beta^r_n \in \mathbb{L}^r_{n+1}$ with $\partial e'_n = a'_n$. This gives an element $\beta^{r+1}$, and by induction elements $\beta^q$ for $q \geq r$. Now $\sum_{q} \beta^q = (\sum_{q} \beta^q_n)$ is a well defined element in $\lim_{n} \widehat{\mathbb{L}}(W(n))$ and by construction $\partial(\sum_{q} \beta^q) = a$. □
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