Theory in functional principal components of discretely observed data

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December 27, 2022

Abstract

Convergence of eigenfunctions with diverging index is essential in nearly all methods based on functional principal components analysis. The main goal of this work is to establish the unified theory for such eigencomponents in different types of convergence based on discretely observed functional data. We obtain the moment bounds for eigenfunctions and eigenvalues for a wide range of the sampling rate and show that under some mild assumptions, the $L^2$ bound of eigenfunctions estimator with diverging indices is optimal in the minimax sense as if the curves are fully observed. This is the first attempt at obtaining an optimal rate for eigenfunctions with diverging index for discretely observed functional data. We propose a double truncation technique in handling the uniform convergence of function data and establish the uniform convergence of covariance function as well as the eigenfunctions for all sampling scheme under mild assumptions. The technique route proposed in this work provides a new tool in handling the perturbation series with discretely observed functional data and can be applied in most problems based on functional principal components analysis and models involving inverse issue.

Keywords: Kernel smoothing; perturbation series; phase transition; optimal convergence.
1 Introduction

With the rapid evolution of modern data collection technologies, functional data emerge ubiquitously and have been extensively developed over the past decades. In general, functional data are typically regarded as stochastic processes with certain smoothness conditions or realizations of Hilbert space valued random elements. These perspectives convey two essential natures of functional data, smoothness and infinite dimensionality, which distinguish functional data from high-dimensional and Euclidean data. For a comprehensive treatment on functional data, we recommend monographs by Ramsay and Silverman (2006), Ferraty and Vieu (2006), Horváth and Kokoszka (2012), Hsing and Eubank (2015) and Kokoszka and Reimherr (2017).

Although functional data provide information over a continuum, which is often time, or spatial location, real data are mostly collected in a discrete form with measurement errors. For instance, we usually use $n$ to denote the sample size and $N_i$ the number of observations for the $i$th subject. Thanks to the smoothness nature of functional data, having the number of observations per subject large is a blessing rather than a curse in contrast to the high-dimensional data (Hall et al., 2006; Zhang and Wang, 2016). There is an extensive literature on the nonparametric methods in accessing the smoothness nature of functional data, including kernel method (Yao et al., 2005a; Hall et al., 2006; Zhang and Wang, 2016) and various kinds of spline methods (Rice and Wu, 2001; Yao and Lee, 2006; Paul and Peng, 2009; Cai and Yuan, 2011).

Given a smoothing method, there are two typical strategies. When the observed points per subject are relatively dense, pre-smoothing each curve before further analysis is suggested by Ramsay and Silverman (2006) and Zhang and Chen (2007). Otherwise, pooling observations together from all subjects is more recommended when the sampling scheme is rather sparse (Yao et al., 2005a). Whether using the individual or pooling information affects convergence rates and phase transitions in estimating population quantities such as mean and covariance functions. When $N_i \geq O(n^{5/4})$ and the tuning parameter is optimally chosen per subject, the reconstructed curves by pre-smoothing are $\sqrt{n}$-consistency and thus, the estimated mean and covariance functions based on the pre-smoothed curves have the optimal parametric rate. In contrast, by borrowing information from all subjects, the pooling method only requires $N_i \geq O(n^{1/4})$ for mean and covariance estimation reaching optimal (Cai and Yuan, 2010, 2011; Zhang and Wang, 2016), which provides theoretical insight for the supremacy of pooling strategy over pre-smoothing.

However, estimating mean and covariance functions only reflects the smoothness nature of functional data but leaves the infinite dimensionality out of consideration. Due to the decaying eigenvalues, covariance operators for functional random objects are non-invertible and consequently, regularization is needed in models involving inverse issue with functional covariates, such as functional linear model (Yao et al., 2005b; Hall and Horowitz, 2007; Yuan and Cai, 2010), functional generalized linear model (Müller and Stadtmüller, 2005; Dou et al., 2012) and functional Cox model (Qu et al., 2016). Truncation on the leading functional principal components is a well developed approach to do regularization (Hall and Horowitz, 2007; Dou et al., 2012), and in order to suppress the model bias, the
number of principal components used in estimating the regression function should grow with sample size. Therefore, convergence rate of the estimated eigenfunctions with diverging indices is a fundamental issue in functional data analysis. It is not only interested in its own theoretical merits but also involved in most models with inverse issue and methodology based on principal component analysis.

For fully observed functional data, the seminal work Hall and Horowitz (2007) obtained the convergence rate $j^2/n$ for the $j$th eigenfunction, which has been proofed optimal in the minimax sense by Wahl (2020). Subsequently, this result becomes the keystone in establishing the optimal convergence in functional linear model (Hall and Horowitz, 2007) and functional generalized linear model (Dou et al., 2012). For discretely observed functional data, stochastic bounds for a fixed number of eigenfunctions are obtained by different methods. Specifically, applying the local linear smoother, Hall et al. (2006) showed that the $L^2$ rate of a fixed eigenfunction for finite $N_i$ is $O_p(n^{-4/5})$. For another line of research, under the reproducing kernel Hilbert space framework, Cai and Yuan (2010) claimed that eigenfunctions with fixed indices admit the same convergence rate as the covariance function, which is $O_p((n/\log n)^{-4/5})$. It is important to point out that even though both of their results are one-dimensional non-parametric rates (at most differ by a factor of $(\log n)^{4/5}$), the methodologies and techniques they used are completely disparate and a detailed discussion can be found in Section 2. Moreover, Paul and Peng (2009) proposed a reduced rank model and studied its asymptotic properties under a particular setting. To our best knowledge, when the data are discretely observed with noise contaminations, there is no progress in obtaining the convergence rate of eigenfunctions with diverging indices.

An intrinsic difference between estimating the diverging and fixed number of eigenfunctions lies in the infinite-dimensional nature of functional data. The decaying eigenvalues make it challenging in analyzing the eigenfunctions with diverging indices, which yields all the existing techniques failed. Nonetheless, the perturbation results used in Hall and Horowitz (2007) and Dou et al. (2012) are facilitated by the cross-sectional sample covariance, which reduces each term in the perturbation series to the principal components scores. When the trajectories are observed at discrete time points, this virtue no longer exists and a summability issue occurs due to the estimation bias, see Section 2 for further elaboration. Combination of smoothness and infinite dimensionality reveals the elevated difficulties in estimating a diverging number of eigenfunctions, which remains an open problem in functional data analysis.

In view of the aforementioned significance and difficulties, we present in this paper a unified theory in estimating a diverging number of eigenfunctions from discretely observed functional data. The main contributions of this paper are at least threefold. First, we establish the $L^2$ bound of the eigenfunctions and asymptotic normality of the eigenvalues with diverging indices. These rates could address all scenarios of sampling rates, and we proof that when $N_i$ reaches the magnitude $n^{1/4+\delta}$, these rates become optimal as if the curves are fully observed, where $\delta$ is determined by some smoothing parameters of functional data. Second, we propose a new technique in analyzing the uniform convergence for functional data, which overcomes the theoretical barriers in the existing literature. Under our new theoretical framework, uniform convergence rates of covariance and eigenfunctions are established under...
some mild assumptions. Besides, we study the uniform convergence of eigenfunctions with diverging indices, which was less known even for the fully observed case. Third, we bring up a new technique route in handling the perturbation series with discrete observed functional data, which can be applied in most of functional principal component analysis based problems and models involving inverse issue. Moreover, the results in this paper have a wide range of applications to improve the existing results of models involving inverse issue with discretely observed functional data. In summary, this work is not merely a theoretical progress, it builds the bridge between “ideal” and “reality” in models involving inverse issue, which has its own merits deserving further investigation.

The reminder of the paper is organized as follows. In Section 2, we shall give a synopsis of eigenfunction estimation problem in functional data. We present the $L^2$ convergence of eigenfunctions in Section 3, and discuss the uniform convergence of functional data in Section 4. Asymptotic normality of eigenvalues can be founded in Section 5. The proofs of our main results can be founded in Section 6 and 7. The proofs of ancillary results are collected together in the Appendix.

In what follows, we use $A_n = O_p(B_n)$ stands for $\mathbb{P}(A_n \leq MB_n) \geq 1 - \epsilon$ whereas $A_n = o_p(B_n)$ stands for $\mathbb{P}(A_n \leq \epsilon B_n) \to 0$ as $n \to \infty$ for each $\epsilon > 0$ and a positive constant $M$. A non-random sequence $a_n$ is said to be $O(1)$ if it is bounded and for each non-random sequence $b_n$, $b_n = o(a_n)$ stands for $b_n/a_n = O(1)$ and $b_n = o(a_n)$ stands for $b_n/a_n$ converges to zero. Throughout this paper, we use $\text{Const.}$ stands for a positive constant which may vary from place to place. The relation $a \lesssim b$ is used to indicate that $a \leq \text{Const.}b$ and the relation $\gtrsim$ can be defined analogously. We write $a \asymp b$ if $a \lesssim b$ and $b \lesssim a$. For $a \in \mathbb{R}$, we use $[a]$ to denote the largest integer smaller or equal to $a$. For a function $p(s) \in L^2[0, 1]$, where $L^2[0, 1]$ denotes the space of square-integrable functions on $[0, 1]$, we use $\|p\|^2$ to denote $\int_{[0, 1]}|p(s)|^2ds$ and $\|p\|_{\infty}$ for $\sup_{s\in[0,1]}|p(s)|$. For a function $A(s, t) \in L^2[0, 1]^2$, define $\|A\|_{HS}^2 = \int_{[0, 1]^2}A(s, t)^2d\sigma d\tau$ and $\|A\|_{(j)}^2 = \int_{[0, 1]}\{\int_{[0, 1]}A(s, t)\phi_j(s)ds\}^2dt$, where $\{\phi_j\}_{j=1}^\infty$ are the eigenfunctions of interest. We write $\int pq$ and $\int Apq$ for $\int p(u)q(u)du$ and $\iint A(u, v)p(u)q(v)dudv$ occasionally for brevity.

## 2 Eigenfunctions estimation for discretely observed data

Denote $X(t)$ a square integrable stochastic process on $[0, 1]$ and $X_i(t)$ independent and identically distributed copies of $X(t)$. The mean and covariance functions of $X(t)$ are $\mu(t) = \mathbb{E}\{X(t)\}$ and $C(s, t) = \mathbb{E}\{(X(s) - \mu(s))(X(t) - \mu(t))\}$, respectively. By the Mercer’s Theorem (Indritz, 1963, Chapter 4), $C(s, t)$ admits the spectral decomposition

$$C(s, t) = \sum_{k=1}^{\infty} \lambda_k \phi_k(s)\phi_k(t),$$

where $\lambda_1 > \lambda_2 > \ldots > 0$ are eigenvalues and $\{\phi_j\}_{j=1}^\infty$ are the corresponding eigenfunctions, which form a complete orthonormal system on $L^2[0, 1]$. For each $i$, the process $X_i$ admits the so-called Karhunen-Loève expansion

$$X_i(t) = \mu(t) + \sum_{j=1}^{\infty} \xi_{ik} \phi_k(t),$$

4
where $\xi_{ik} = \int_0^1 \{X_i(t) - \mu(t)\} \phi_k(t)dt$ are uncorrelated zero mean random variables with variance $\lambda_k$. In this paper, we focus on the eigenfunction estimation thus we assume $\mu(t) = 0$ without loss of generality.

However, having each $X_i(t)$ for all $t \in [0, 1]$ is only achievable in the theoretical analysis, and in practice, measurements are taken at $N_i$ discrete time points with noise contaminations. The actual observations for each $X_i$ are

$$\{(T_{ij}, X_{ij})|X_{ij} = X_i(T_{ij}) + \varepsilon_{ij}, j = 1, \ldots, N_i\},$$

where $\varepsilon_{ij}$’s are random copies of $\varepsilon$ with $\mathbb{E}(\varepsilon) = 0$ and $\text{Var}(\varepsilon) = \sigma_x^2$. We further assume that $T_{ij}$ are random copies of $T$, which follows the uniform distribution on $[0, 1]$, $X$, $T$ and $\varepsilon$ are mutually independent, and $N_i = N$ without loss of generality.

Kernel method has been widely developed as a smoothing approach in functional data analysis due to its attractive theoretical features (Yao et al., 2005a; Hall et al., 2006; Li and Hsing, 2010; Zhang and Wang, 2016). The main motivation of this paper is to establish a unified theory in estimating a diverging number of eigenfunctions for discretely observed functional data and thus, we adopt the local constant smoother to avoid distractions from complicated calculations. Denote $\delta_{ij} = X_{ij}X_{il} = \{X_i(T_{ij}) + \varepsilon_{ij}\} \{X_i(T_{il}) + \varepsilon_{il}\}$ and the covariance estimator is constructed through

$$\hat{C}(s, t) = \frac{1}{n} \sum_{i=1}^n \frac{1}{N(N-1)} \frac{1}{h^2} \sum_{1 \leq j \leq l \leq N} K\left(\frac{T_{ij} - s}{h}\right) K\left(\frac{T_{il} - t}{h}\right) \delta_{ijt},$$

where $K$ is a symmetric Lipschitz continuous density kernel on $[-1, 1]$. The function $\hat{C}(s, t)$ admits an empirical version of decomposition (1),

$$\hat{C}(s, t) = \sum_{k=1}^{\infty} \hat{\lambda}_k \hat{\phi}_k(s) \hat{\phi}_k(t),$$

where $\hat{\lambda}_k$ and $\hat{\phi}_k$ are estimators for $\lambda_k$ and $\phi_k$. We further assume $(\hat{\phi}_k, \phi_k) \geq 0$.

Before moving to our theoretical results, we shall give a comprehensive synopsis for the eigenfunction estimation problem in functional data analysis. We start with the following resolvent series (6) and illustrate how it is applied in statistical analysis,

$$\mathbb{E}(\|\hat{\phi}_j - \phi_j\|^2) \asymp \sum_{k \neq j} \mathbb{E}\{(\hat{C} - C)\phi_j, \phi_k\}^2 (\lambda_k - \lambda_j)^2.$$

Such kind of expansions can be found in Bosq (2000), Dou et al. (2012) and Li and Hsing (2010), see chapter 5 in Hsing and Eubank (2015) for details. Denote $\eta_j := \min_{k \neq j} |\lambda_k - \lambda_j|$ the eigengap of $\lambda_j$, a rough bound of $\mathbb{E}(\|\hat{\phi}_j - \phi_j\|^2)$ can be derived by (6) and the Bessel’s inequality directly

$$\mathbb{E}(\|\hat{\phi}_j - \phi_j\|^2) \leq \eta_j^{-2} \mathbb{E}(\|\hat{C} - C\|^2_{HS}),$$

which is exactly the bound in Bhatia et al. (1983). However, this bound is clearly suboptimal in twofold. First, $\eta_j$ is away from 0 when $j$ is regarded as a fixed number, and (7) indicates
that eigenfunctions admit the same convergence rates as the covariance function. This is counter-intuitive since \( \phi_j(s) = \lambda_j^{-1} \int C(s,t) \phi_j(t) dt \) and the integration usually brings extra smoothness (Cai and Hall, 2006). Thus, eigenfunctions should have a faster rate rather than the two-dimensional kernel smoothing rate of \( \| \hat{C} - C \|_{\text{HS}} \). Second, when \( j \) is diverging as the sample size tends to infinity, there is \( \eta_j \to 0 \) and \( \eta_j^{-2} \) appearing in (7) could not be ignored. Assume \( \lambda_j \propto j^{-\alpha} \), the rate of \( \mathbb{E}(\| \hat{\phi}_j - \phi_j \|^2) \) obtained by (7) is slower than \( j^{2\alpha+2}/n \), which is suboptimal by Wahl (2020). Therefore, in order to obtain a sharp rate of \( \mathbb{E}(\| \hat{\phi}_j - \phi_j \|^2) \), we should adopt to the original perturbation series (6) rather than its approximation (7).

When each trajectory \( X_i(t) \) is observed for all \( t \in [0,1] \), which refers to the fully observed case, the cross-sectional sample covariance \( \hat{C}(s,t) = n^{-1} \sum_{i=1}^n X_i(s) X_i(t) \) is a canonical estimator of \( C(s,t) \). Then the numerators in each term of equation (6) can be reduced to the principal components scores under some mild assumptions, e.g. \( \mathbb{E}\{(\hat{C} - C)\phi_j, \phi_k)^2} \leq n^{-1}\lambda_j \lambda_k \) (Hall and Horowitz, 2007; Dou et al., 2012). Then \( \mathbb{E}(\| \hat{\phi}_j - \phi_j \|^2) \) is bounded by \( (\lambda_j/n) \sum_{k \neq j} \lambda_k / (\lambda_k - \lambda_j)^2 \). Assuming the polynomial decay of eigenvalues, the aforementioned summation is dominated by \( \lambda_j/n \sum_{|j|\geq k \geq 2} \lambda_k / (\lambda_k - \lambda_j)^2 \), which is \( O(j^2/n) \) and optimal in the minimax sense (Wahl, 2020), see Lemma 7 in Dou et al. (2012) for detailed elaboration.

However, we emphasize that when tackling with discretely observed functional data, all the existing literature use a bound analogous to (7), which excludes the diverging indices as a consequence. Specifically, the result in Cai and Yuan (2010) is a direct application of bound (7) and their one-dimensional rate is inherited from the covariance estimator, which is assumed in a tensor product space that is much smaller than the \( L^2[0,1]^2 \). In contrast, the one-dimensional rates obtained by Hall et al. (2006) and Li and Hsing (2010) utilize a detailed calculation on the approximation of perturbation series (6). However, all these results are based on the assumption that \( \eta_j \) is bounded away from zero, which indicates that \( j \) must be a fixed constant. When the data are discretely observed, how to utilize perturbation series (6) effectively is the key in obtaining a sharp bound of eigenfunctions with diverging indices.

The main challenges come from quantifying the summation (6) without the fully observed sample covariance. For pre-smoothing method, the reconstructed \( \hat{X}_i \) achieves a \( \sqrt{n} \) convergence in the \( L^2 \) sense when each \( N_i \) reaches the magnitude \( n^{5/4} \) and thus, the estimated covariance function \( \hat{C}(s,t) = n^{-1} \sum_{i=1}^n \hat{X}_i(s) \hat{X}_i(t) \) has \( \| \hat{C} - C \|_{\text{HS}} = O_p(n^{-1/2}) \). However, this does not guarantee the optimal convergence of a diverging number of eigenfunctions by the existing techniques. The numerators in each term of (6) are no longer the principal component scores and such a complicated form makes it infeasible to quantify this infinite summation when \( |\lambda_k - \lambda_j| \to 0 \). Besides, for pooling method, it is also highly nontrivial to sum up all \( \mathbb{E}\{(\hat{C} - C)\phi_j, \phi_k)^2} \) with respect to \( j, k \), see comments below Theorem 1 for further illustration.

3 \( L^2 \) convergence

Based on the aforementioned issues, we propose a new technique to handle the perturbation series (6) for the discretely observed functional data. We shall make the following regularity
assumptions.

A.1 $X(t)$ has finite fourth moment $\int \mathbb{E}(X^4(t))dt < \infty$; $\mathbb{E}(\xi_j^4) \leq \lambda_j^2$ for all $j$ and $\sigma_X^2 < \infty$.

A.2 The eigenvalues $\lambda_j$ are decreasing with $\text{Const.} j^{-a} \geq \lambda_j \geq \lambda_{j+1} + (a/\text{Const.}) j^{-a-1}$ for $a > 1$ and each $j \geq 1$.

A.3 For each $j \in \mathbb{N}^+$, the eigenfunctions $\phi_j$ satisfies $\sup_{t \in [0,1]} |\phi_j(t)| = O(1)$ and

$$\sup_{t \in [0,1]} |\phi_j^{(k)}(t)| \leq j^{c/2} \sup_{t \in [0,1]} |\phi_j^{(k-1)}(t)|, \quad \text{for } k = 1, 2,$$

where $c$ is a positive constant and assume $\phi_j(0) = \phi_j(1), \phi_j^{(1)}(0) = \phi_j^{(1)}(1)$ without loss of generality.

Assumptions (A.1) and (A.2) are commonly adopted in the functional data literature (Yao et al., 2005a; Hall and Horowitz, 2007; Cai and Yuan, 2010; Dou et al., 2012). The number of eigenfunctions that can be well estimated for exponential decaying eigenvalues is at the order of $\log n$ and thus, polynomial decaying eigenvalues are of major interest. To propose a good estimation for a specific eigenfunction, it is necessary to take its smoothness into consideration. In general, the frequency of $\phi_j$ is higher for larger $j$, which requires a smaller bandwidth to capture its local variation. Assumption (A.3) characterizes the frequency increment of a specific eigenfunction via the smoothness of its derivatives, and for some common used bases, Fourier, Legendre and wavelet basis for instance, $c = 2$. In Hall et al. (2006), the authors assume that $\max_{1 \leq j \leq r} \max_{s=0,1,2} \sup_{t \in [0,1]} |\phi^{(s)}_j(t)| \leq \text{Const.}$, which is achievable only for fixed $r$ and assumption (A.3) can be regarded as its generalized version. The boundary assumption on $\phi_j$ and its first order derivative eliminates the edge effect caused by the local constant smoother for convenience, and can be relaxed with more technicality. The following theorem establishes the bound of $\mathbb{E}((\hat{C} - C)\phi_j, \phi_k)^2$, which is fundamental in most problems with functional covariates that involving inverse issues, such as eigenfunction estimation with diverging index, estimation and prediction in functional linear models.

**Theorem 1.** Under assumptions (A.1) to (A.3), and $h^4 j^{2a+2c} = O(1)$, denote $\Delta = \hat{C} - C$. For all $1 \leq k \leq 2j$

$$\mathbb{E}((\Delta \phi_j, \phi_k)^2) \leq \frac{1}{n} \left( j^{-a} k^{-a} + \frac{j^{-a} + k^{-a}}{N} + \frac{1}{N^2} \right) + h^4 k^{2c-2a}$$

and

$$\sum_{k=j+1}^{\infty} \mathbb{E}((\Delta \phi_j, \phi_k)^2) \leq \frac{1}{n} \left( j^{1-2a} \frac{1}{N} + \frac{h^{-1} j^{-a} + j^{1-a}}{h N^2} + \frac{1}{h N^2} \right) + h^4 j^{1+2c-2a}.$$

When $\hat{C}$ is obtained from a kernel type smoother similar to (4), the convergence rate of $||\Delta||^2_{\text{HS}}$ should be a two-dimensional rate with variance $n^{-1}\{1 + (Nh)^{-2}\}$ (Zhang and Wang, 2016). The first assertion of Theorem 1 reveals that the convergence rate of $\mathbb{E}((\Delta \phi_j, \phi_k)^2)$
is a degenerated kernel smoothing rate and the variance terms \((Nh)^{-1}, (Nh)^{-2}\) vanish after twice integrations. Similarly, one can show that \(\mathbb{E}(\|\Delta\|^2(j))\) admits a one-dimensional rate with variance \(n^{-1}(1 + (Nh)^{-1})\). However, by Bessel’s equality, there is \(\sum_{n=1}^{\infty} \mathbb{E}((\Delta \phi_j, \phi_k)^2) = \mathbb{E}(\|\Delta\|^2(j))\) and \(\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \mathbb{E}((\Delta \phi_j, \phi_k)^2) = \mathbb{E}(\|\Delta\|^2_{HS})\), which indicates that one cannot sum up all \(\mathbb{E}((\Delta \phi_j, \phi_k)^2)\) with respect to all \(j, k\) directly due to the estimation bias. In the fully observed case, the summation \(\sum_{1\leq j} \lambda_k \lambda_j/(\lambda_k - \lambda_j)^2\) is dominated by \(\sum_{|j|\leq 2j} \lambda_k \lambda_j/(\lambda_k - \lambda_j)^2\) (Hall and Horowitz, 2007; Dou et al., 2012). This inspires us that the convergence rate caused by the inverse issue can be captured by the summation on the set \(\{k \leq 2j\}\), and the tail sum on \(\{k > 2j\}\) can be treated as a unity. The second assertion in Theorem 1 reveals that \(\sum_{k=j+1}^{\infty} \mathbb{E}((\Delta \phi_j, \phi_k)^2)\) admits a one-dimensional kernel smoothing rate. The terms appearing in the variance term with \(j^{-a}\) are facilitated by the additional integration and the term involves \(j^c\) is due to the increasing local variation for larger \(j\).

The following theorem is one of the main results of this work. It characterizes the \(L^2\) convergence of the estimated eigenfunctions with diverging indices. Given a integer \(m\), we use \(\Omega_m(n, N, h)\) to denote the set of all possible realizations for which \(\|\Delta\|_{HS} < \eta_m/2\) with sample size \(n\), sampling rate \(N\) and bandwidth \(h\).

**Theorem 2.** Under assumptions (A.1) to (A.3), for \(m \in \mathbb{N}_+\) satisfies

\[
\textbf{M.1} \quad \frac{m^{2a+2}}{n} \to 0, \quad \frac{m^{2a+2}}{nNh^2} \to 0, \quad h^4m^{2a+2} \to 0 \quad \text{and} \quad h^4m^{2a+2c} = O(1).
\]

Then \(\mathbb{P}\{\Omega_m(n, N, h)\} \to 1\) and on \(\Omega_m(n, N, h)\)

\[
\mathbb{E}(\|\hat{\phi}_j - \phi_j\|^2) \leq \frac{j^2}{n} \left\{ 1 + \left(\frac{j^a}{N}\right)^2 \right\} + \frac{j^a}{nNh} \left( 1 + \frac{j^a}{N} \right) + h^4j^{2c+2}, \quad j=1, \ldots, m. \tag{8}
\]

The integer \(m\) in Theorem 2 can be regarded as the largest number of eigenfunctions that can be well estimated based on the observed data and tuning parameter. Note that under Assumption (M.1), \(m\) could diverge to infinity and the upper bound of \(m\) is jointly determined by the observed data \((n, N)\), smoothing strategy \((h)\) and the decaying eigengap \((\eta_j\) or \(a\)). Specifically, the growing sample size and sampling rate make the covariance function to be well estimated and thus the eigenfunctions. The frequency of \(\phi_j\) is higher for larger \(j\), which requires smaller \(h\) to capture its local variations. For larger \(a\), the eigengap \(\eta_j\) shrinks to 0 rapidly, which makes it more difficult to distinguish the adjacent eigenvalues. The \(m\) in Assumptions (M.1) could cover the most cases in the subsequent analyses, such as functional linear regression problems.

Theorem 2 is a good illustration for both infinite dimensionality and smoothness nature of functional data. To understand this result, note that \(j^2/n\) is the optimal rate in the fully observed case while remaining terms appearing in the right-hand of (8) can be viewed as the contamination caused by the discrete observations and measurement errors. Specifically, the term involving \(h^4\) represents the smoothing bias, and \((Nh)^{-1}\) is a typical one-dimensional kernel smoothing variance. The terms containing \(N^{-1}\) are owing to the discrete approximation and those involving \(j\) with its positive powers arise from the decaying eigengaps for increasing number of eigen components. Rather than stochastic bounds (in the form of \(O_p\))
obtained by Hall et al. (2006), Paul and Peng (2009) and Cai and Yuan (2010), equation (8) establishes a moment bound and is more feasible for subsequent analyses involving inverse issue, where \( \sum_{j=1}^{m} \| \hat{\phi}_j - \phi_j \|^2 \) is commonly appeared with \( m \to \infty \).

Theorem 2 is a unified result for eigenfunction estimation and has no restriction on the sampling rate \( N \). As the phase transitions of mean and covariance functions studied in Cai and Yuan (2011) and Zhang and Wang (2016), we hope to derive a systematic partition for eigenfunctions estimators with diverging indices. The following corollary discusses the convergence rate of \( \mathbb{E}(\| \hat{\phi}_j - \phi_j \|^2) \) under different sampling scheme after selecting the optimal bandwidth \( h \).

**Corollary 1.** Under assumptions (A.1) to (A.3), given \( m \in \mathbb{N}_+ \) satisfies (M.1), For each \( j \leq m \) and let \( h_{opt}(j) = (nN)^{-1/5} j^{(a-2c-2)/5} (1 + j^a/N)^{1/5} \),

(a) If \( N \geq \text{Const.} j^a \),
\[
\mathbb{E}(\| \hat{\phi}_j - \phi_j \|^2) \leq \frac{j^2}{n} + \frac{j^{(4a+2c+2)/5}}{(nN)^{4/5}}.
\]

In addition, if \( N \geq n^{1/4} j^{a+c/2+2} \), \( \mathbb{E}(\| \hat{\phi}_j - \phi_j \|^2) \leq j^2/n \).

(b) If \( N = o(j^a) \),
\[
\mathbb{E}(\| \hat{\phi}_j - \phi_j \|^2) \leq \frac{j^{2a+2}}{nN^2} + \frac{j^{(8a+2c+2)/5}}{(nN^2)^{4/5}}.
\]

To appreciate this result, in case of the common used bases where \( c = 2 \), the convergence rate for the \( j \)th eigenfunction reaches optimal as if the curves are fully observed when \( N > \max\{j^a, n^{1/4} j^{a-1}\} \). When \( j \) is fixed, the phase transition occurs at \( n^{1/4} \), which is same as the mean and covariance function and consistent with the results in Hall et al. (2006) and Cai and Yuan (2010). For \( n \) subjects, the largest index of eigenfunction that can be well estimated is smaller than \( m_{max} := n^{1/(2a+2)} \), which is a direct consequence from Assumption (M.1). It can be check that \( m_{max}^a = n^{1/4} m_{max}^{a-1} \) and in this case, the phase transition occurs at \( n^{1/4+(a-1)/(2a+2)} \), which can be interpreted from two aspects. On one hand, more observations per subject are needed in order to obtain the optimal convergence for eigenfunctions with diverging indices compared to the mean and covariance estimation, which reveals the elevated difficulties caused by the infinite dimensionality. On the other hand, \( n^{1/4+(a-1)/(2a+2)} \) is slightly larger than \( 1/4 \), which provides the merits of the pooling method as well as our theoretical analysis.

When \( j \) is fixed and \( N \) is finite, the convergence rate derived by Corollary 1 becomes \( (nh)^{-1} + h^4 \), which is a typical one-dimensional kernel smoothing rate and reaches optimal at \( h \approx n^{-1/5} \). This result is consistent with those in Hall et al. (2006) and optimal in the minimax sense. When \( j \) is not fixed but diverging to infinity, the knowing lower bound \( j^2/n \) for fully observed data is derived by applying the van Tree’s inequality on the special orthogonal group (Wahl, 2020). For discretely observed functional data, there are no lower bounds for the eigenfunctions with diverging indices and it is highly non-trivial to extend the results in Wahl (2020) to the discrete case. Lower bound of eigenfunctions with diverging
indices for discretely observed data is beyond the scope of this paper and remains an open problem deserving further investigation.

It is worth noting that if we focus on the convergence rate of a specific eigenfunction, the optimal bandwidth in estimating the $j$th eigenfunction is varying with the index $j$. When applying Theorem 2 in subsequent analyses such as function linear regression models, the optimal bandwidth is chosen by $h_{opt}(m_{\text{max}})$, where $m_{\text{max}}$ is the largest index of eigenfunctions used in constructing the regression estimator. When $a \leq 2c + 2$, which is the common case of most settings, the optimal bandwidth becomes smaller for growing $m_{\text{max}}$ to capture the local variations. When $a > 2c + 2$, that is, the eigenvalues are decaying rapidly with respect to the frequency increment, the optimal bandwidth is larger for growing $m_{\text{max}}$ due to the smoothness of the underlying trajectories.

By similar arguments, the following proportion states the convergence rate of $|\hat{\lambda}_j - \lambda_j|^2$ and a detailed discussion for the asymptotic normality of eigenvalues can be found in Section 5.

**Proposition 3.** Under assumptions (A.1) to (A.3), for $m \in \mathbb{N}_+$ satisfies

$$M.2 \quad \sqrt{n}(m^a + N)[m^2n^{-1}(1 + (j^a/N)^2) + m^a(nNh)^{-1}(1 + m^a/N) + h^4m^{2c+2}] = o(1) \quad \text{and} \quad h(m^{2c} + m^a) = o(1).$$

Then $\mathbb{P}\{\Omega_m(n, N, h)\} \to 1$ and on $\Omega_m(n, N, h)$, $\mathbb{E}(\hat{\lambda}_j/\lambda_j - 1)^2 \leq n^{-1}(1 + j^{2a}/N^2) + h^4j^{2c}$.

### 4 Uniform convergence

In classical nonparametric regression with independent observations, there are a number of results for uniform convergence of kernel type estimators (Bickel and Rosenblatt, 1973; Hardle et al., 1988; Härdle, 1989). For functional data with in-curve dependence, Yao et al. (2005a) obtained the uniform convergency for nonparametric functions based on local linear smoother, which plays a central role in obtaining the theoretical results of the functional principal component analysis. More recently, Li and Hsing (2010) studied the strong uniform convergence for functional data and showed that these rates depend on both sample size and the number of observations per subject. Based on Li and Hsing (2010), Zhang and Wang (2016) studied the uniform convergence rates of mean and covariance function under different weighting and sampling schemes. Despite extensive studies on this topic, there are still some unsolved issues of uniform convergence in functional data analysis.

To address the in-curve dependence, a common used technique is using Bernstein inequality to obtain a uniform bound over a finite grid of the domain, which grows increasingly dense with $n \to \infty$, and then showing these two bounds are asymptotic equivalent (Li and Hsing, 2010; Zhang and Wang, 2016). Due to the un-compactness of functional data, truncation on the observed $X_{ij} \leq A_n$ is needed in order to apply the Bernstein inequality, where $A_n \to \infty$ as $n \to \infty$. The truncation number $A_n$ is selected to balance the two terms appearing in the denominator of Bernstein inequality. However, there is a trade-off of $A_n$ and additional moment assumptions on $X$ and $\varepsilon$ are needed to guarantee the difference between the truncated and original estimator is negligible. For the covariance estimation, assuming the sampling
rate satisfies $N \asymp n^{7/4}$ for $\tau \in [0, 1)$, the state-of-the-art results need to assume that the $\beta$th moments of $X(t)$ and $\varepsilon$ are finite, where $\beta = 6/(1-\tau)$ (Li and Hsing, 2010; Zhang and Wang, 2016). When the sampling scheme is relatively dense and $N$ is close to $n^{1/4}$, $\beta$ diverges to infinity. Moreover, when $N \asymp n^{1/4}$, which refers to the dense and ultra dense cases, the bias between the truncated and original estimator is always the dominating term and the optimal convergence rate cannot be reached by the existing techniques.

Before moving to the uniform convergence of eigenfunctions, we first resolve the issues raised in the previous paragraph. To address the limitations in the existing literature and get a unified rates for all possible sampling scenarios, we propose a double truncation technique to handle uniform convergence rate of the variance term. We shall make the following assumptions. Assumption (U.1) is the additional moments assumption needed for the uniform convergence (Li and Hsing, 2010; Zhang and Wang, 2016). Under Assumption (U.2), the bias term $\|\hat{E}C - C\|_{\text{HS}}^2$ is simplified to $h^4$, which makes the results more concise.

**THEOREM 4.** Under the assumptions (A.1) to (A.3) and (U.1),

$$
\mathbb{E} \left\{ \sup_{s,t \in [0,1]} |\hat{C}(s,t) - \mathbb{E}\hat{C}(s,t)| \right\} \lesssim \sqrt{\frac{\ln n}{n} \left( 1 + \frac{1}{Nh} \right) + \left| \frac{\ln n}{N} \right|^{\frac{1}{\alpha}} \left( 1 + \frac{\ln n}{Nh} \right)^{2-\frac{2}{\alpha}} h^{-\frac{2}{\alpha}}}. \tag{11}
$$

**COROLLARY 2.** Under the assumptions of Theorem 4 and (U.2). If $\alpha > 3$,

(1). When $N/(n/\log n)^{1/4} \to 0$ and $h \asymp (nN^2/\log n)^{-1/6}$,

$$
\mathbb{E} \sup_{s,t \in [0,1]} |\hat{C}(s,t) - C(s,t)| \lesssim \left( \frac{\log n}{nN^2} \right)^{\frac{1}{3}}.
$$

(2). When $N/(n/\log n)^{1/4} \to \text{Const.}$ and $h \asymp (n/\log n)^{1/4}$

$$
\mathbb{E} \sup_{s,t \in [0,1]} |\hat{C}(s,t) - C(s,t)| \lesssim \sqrt{\frac{\log n}{n}}.
$$

(3). When $N/(n/\log n)^{1/4} \to \infty$, $h = o(n/\log n)^{1/4}$ and $Nh \to \infty$,

$$
\mathbb{E} \sup_{s,t \in [0,1]} |\hat{C}(s,t) - C(s,t)| \lesssim \sqrt{\frac{\log n}{n}}.
$$
Theorem 4 establishes the uniform convergence rate of the variance term of $\|\tilde{C} - C\|_{HS}$ for all sampling rate $N$. It is interesting to compare our results with those obtained by Li and Hsing (2010) and Zhang and Wang (2016). The last term in the right hand side of (11) is the truncation bias, which is smaller than those in Li and Hsing (2010) and Zhang and Wang (2016) due to the double truncation technique. As a consequence, Corollary 2 shows the truncation bias is dominated by the main term if $\alpha > 3$, which is much milder compared with the moment assumptions in Li and Hsing (2010) and Zhang and Wang (2016). Under the assumption $\alpha > 3$, Corollary 2 establishes the uniform convergence rate for both sparse and dense functional data. When $1 < \alpha \leq 3$, which refers to the case that the sixth moment of $X(t)$ or $\varepsilon$ is infinite, the additional term caused by truncation is the dominating term. In summary, by proposing the double truncation technique, we fix the issues mentioned above in the original proofs of Li and Hsing (2010) and Zhang and Wang (2016), obtain the uniform convergence rate for covariance function in all scenarios of the sampling rate and show the optimal rates for the dense functional data can be derived as a special case.

The following theorem gives the uniform convergence for estimated eigenfunctions.

**Theorem 5.** Under the assumptions (A.1) to (A.3), and (U.1). Given a $j_{\max} \in \mathbb{N}$ satisfies $h_{j_{\max}} \leq 1$, $h_{j_{\max}}^{a} \leq 1$ and $\mathbb{P}(\Omega_u) \to 1$ with $\Omega_u := \{\|\Delta\|_{HS} \leq \eta_j^{2max}/2, \|\Delta\|_{\infty} \leq \eta_j^{2max} \leq 1, \|\Delta\|_{HS} \leq 1\}$. On $\Omega_u$ and for all $j \leq j_{\max}$

$$
\mathbb{E}(\|\hat{\phi}_j - \phi_j\|_{\infty}) \lesssim \frac{j}{\sqrt{n}} (\sqrt{\ln n} + \ln j) \left\{ 1 + \frac{j^{a}}{N} + \sqrt{\frac{j^{a-1}}{Nh}} \left( 1 + \sqrt{\frac{j^{a}}{N}} \right) \right\}
+ j^{a} \frac{\ln n}{n} \frac{1}{h^{1/2}} \left( \frac{\ln n}{Nh} \right)^{1/2} h^{-\frac{a}{\alpha}} + h^{2} j^{c+1} \log j.
$$

There contributions of Theorem 5 are at least twofold. First, by the assumptions and definition of $\Omega_u$, the maximum number of eigenfunctions that can be well estimated depends on $n$, $N$ and $h$. When $n \to \infty$, $h \to 0$, $j_{\max}$ can diverge to infinity, which is the main contribution of Theorem 5 compared with the existing results in Li and Hsing (2010). This is the first result of uniform convergence for eigenfunctions with diverging indices and provides a useful tool in the theoretical analysis of models involving inverse problems. Second, by the double truncation technique, the truncation bias is smaller than those in Li and Hsing (2010) and Zhang and Wang (2016), which allows our result valid for all sampling rate. For fixed $j_{\max}$, Corollary 3 discusses the uniform convergence rate under different ranges of $N$. When $\alpha > 5/2$, the last term in equation (12), which is the truncation bias, is dominated by the first two terms for all scenarios of $N$. If $N/(n/\log n)^{1/4} \to 0$, $\|\hat{\phi}_j - \phi_j\|_{\infty}$ admits a typical one dimensional kernel smoothing rate that only differs to $\log n$, which is consistent with the results in Li and Hsing (2010). The superiority of the double truncation technique is reflected in comparing our moments assumption with those in Li and Hsing (2010), which only works for sparse data where $N = o(n^{1/4})$.

**Corollary 3.** Under the assumptions of Theorem 5 and (U.2). If $j$ is fixed and $\alpha > 5/2$,
(1). When \( N/(n/\log n)^{1/4} \to 0 \) and \( h \asymp (nN/\log n)^{-1/5} \),
\[
\mathbb{E} \sup_{t \in [0,1]} |\hat{\phi}_j(t) - \phi_j(t)| \lesssim \left( \frac{\log n}{nN} \right)^{\frac{2}{5}}.
\]

(2). When \( N/(n/\log n)^{1/4} \to \text{Const.} \) and \( h \asymp (n/\log n)^{1/4} \)
\[
\mathbb{E} \sup_{t \in [0,1]} |\hat{\phi}_j(t) - \phi_j(t)| \lesssim \sqrt{\frac{\log n}{n}}.
\]

(3). When \( N/(n/\log n)^{1/4} \to \infty \), \( h = o(n/\log n)^{1/4} \) and \( Nh \to \infty \),
\[
\mathbb{E} \sup_{t \in [0,1]} |\hat{\phi}_j(t) - \phi_j(t)| \lesssim \sqrt{\frac{\log n}{n}}.
\]

The following corollary gives the optimal uniform convergence for eigenfunction with diverging indices under some mild assumptions. Different to the \( \mathcal{L}^2 \) convergence, the maximum number of eigenfunction that can be well estimated in \( \| \cdot \|_\infty \) norm is smaller and determined by the moment assumption \( \alpha \). When \( \alpha > 5/2 \), \( j_{\max} \) is diverging as the sample size \( n \to \infty \). To the best knowledge of authors, this is the first result of eigenfunctions with diverging indices in uniform convergence.

**Corollary 4.** Under the assumptions of Theorem 5 and (U.2). Given \( \alpha > 5/2 \), denote \( j_{\max} = \min\{n^{1/2\alpha}, n^{\alpha/2\alpha-1}\} \). If \( N \geq j_{\max}^{\alpha}, h^2 j_{\max}^{-\alpha} \leq n^{-1} \) and \( Nh \geq j_{\max}^{\alpha-1} \),
\[
\mathbb{E} \sup_{t \in [0,1]} |\hat{\phi}_j(t) - \phi_j(t)| \lesssim \frac{j}{\sqrt{n}} (\sqrt{\log n} + \ln j), \text{ for all } j \leq j_{\max}.
\]

## 5 Asymptotic normality of eigenvalues

Structure of the covariance matrix has been extensively studied in multivariate analysis and high dimensional statistic. The distribution of eigenvalues plays an important role in statistical learning and is of significant interest in the high dimensional setting. The development of random matrix theory provides a systematic tool in deriving the distribution of eigenvalues of squared matrix (Anderson et al., 2010; Pastur and Shcherbina, 2011), and has been successfully applied in many statistical problems, such as signal detection (Nadler et al., 2011; Onatski, 2009; Bianchi et al., 2011); spiked covariance models (Johnstone, 2001; Paul, 2007; El Karoui, 2007; Ding and Yang, 2021; Bao et al., 2022) and hypothesis testing (Bai et al., 2009; Chen and Qin, 2010; Zheng, 2012). For a comprehensive treatment on random matrix in statistics, we recommend the monograph of Bai and Silverstein (2010) and the review paper by Paul and Aue (2014).

Despite the success of random matrix in high dimensional statistics, it is not straightforward to apply it to functional data analysis. If the observations are taken place at the
same \{T_j\}_{j=1}^N \text{ for all } i, \text{ which refers to the common fixed design, one can obtain an estimator for } \Sigma_T = \text{Cov}(\tilde{X}_i, \tilde{X}_i) + \sigma_X^2 I_N, \text{ where } \tilde{X}_i = (X_i(T_1), \ldots, X_i(T_N))^\top. \text{ Note that } \tilde{X}_i \text{ can be regarded as random vector, but the adjacent elements in } \tilde{X}_i \text{ are highly correlated as } N \text{ increasing due to the smoothness nature of functional data. This violates the independence assumption required in the most settings of random matrix. Although it is not appropriate to apply random matrix in functional data directly, similar techniques such as perturbation and resolvent that are widely used in random matrix inspire us to adopt Theorem 1 to obtain the asymptotic normality of eigenvalues of functional data.}

In the context of functional data, variables of interest become the eigenvalues of the covariance operator. By Weyl’s inequality, one can obtain a rough bound \(|\hat{\lambda}_k - \lambda_k| \leq \|\Delta\|_{\text{HS}},\) which is suboptimal in two aspects. First, \(|\hat{\lambda}_k - \lambda_k| \text{ should admit a degenerated kernel smoothing rate with variance } n^{-1} \text{ after twice integration, while } \|\Delta\|_{\text{HS}} \text{ has a two dimensional kernel smoothing rate with variance } (nN^2h^2)^{-1}. \text{ Second, due to the infinite dimensionality of functional data, the eigenvalues } \{\lambda_k\} \text{ converge to zero. Therefore, a general bound for all eigenvalues provides limited information for those with larger indices. For fully observed functional data, Hall and Hosseini-Nasab (2006) and Hall and Hosseini-Nasab (2009) provided expansions of } \hat{\lambda}_k - \lambda_k \text{ in increasing powers of } n^{-1/2}, \text{ Wahl (2019) and Jirak and Wahl (2023) obtained the relative bounds and asymptotic normality for eigenvalues with diverging indices. For discretely observed functional data, Hall et al. (2006) derived the asymptotic normality for a fixed number of eigenvalues. However, the study for eigenvalues with diverging number is much less.}

In view of the aforementioned issues, we applied our obtained perturbation results to derive the asymptotic normality of eigenvalues with diverging indices, which has a wide application in inference problems of functional data. Before presenting our results, we shall make the following assumption.

\textbf{N.1} \quad \mathbb{E}(\|X\|^6) < \infty \text{ and } \mathbb{E}(\epsilon^6) < \infty.

\textbf{N.2} \quad \text{For any sequence } j_1, \ldots, j_4, \mathbb{E}(\xi_{j_1}\xi_{j_2}\xi_{j_3}\xi_{j_4}) = 0 \text{ unless each index } j_k \text{ is repeated.}

Assumption (N.1) gives the additional moment condition needed in deriving the asymptotic normality (Zhang and Wang, 2016). Assumption (N.2) is standard in functional principal components analysis (Cai and Hall, 2006; Hall and Hosseini-Nasab, 2009), which simplifies the moment calculation.

\textbf{Theorem 6.} \quad \text{Under assumptions (A.1) to (A.3), (N.1) and (N.2), for } m \in \mathbb{N}_+, \text{ satisfies (M.2), } \mathbb{P}\{\Omega_m(n, N, h)\} \to 1 \text{ and on for all } j \leq m, 

\begin{align*}
\Sigma_n^{-\frac{1}{2}} \left( \frac{\hat{\lambda}_j - \lambda_j}{\lambda_j} - 2\lambda_j\sigma_X^2 h^2 \int_h^{1-h} \phi_j^{(2)}(u)\phi_j(u)du + o(j^{-a}h^2) \right) \xrightarrow{d} \mathcal{N}(0, 1),
\end{align*}

where

\begin{align*}
\Sigma_n &= \frac{1}{n} \left[ \frac{(N-2)(N-3)}{N(N-1)} \frac{\mathbb{E}(\xi_{j}^4)}{\lambda_j^2} + \frac{4(N-2)}{N(N-1)} \frac{\mathbb{E}(\xi_{j}^2(\|X\phi_j\|^2 + \sigma_X^2))}{\lambda_j^2} \\
&\quad + \frac{2}{N(N-1)} \frac{\mathbb{E}((\|X\phi_j\|^2 + \sigma_X^2)^2)}{\lambda_j^2} - 1 \right].
\end{align*}
Note that $\lambda_j \to 0$ as $j \to \infty$, we need to do regularization such that the eigenvalues serve on a common scale of variabilities. After twice integration, $(\hat{\lambda}_j - \lambda_j)/\lambda_j$ admits a degenerated kernel smoothing rate and for fixed $j$, $(\hat{\lambda}_j - \lambda_j)/\lambda_j$ is $\sqrt{n}$ consistency for small enough $h$. For slowly diverging $j$, three types of asymptotic normalities emerge from Theorem 6 depending on the order of $N$.

**COROLLARY 5.** Under the assumption of Theorem 6,

(a) If $N\lambda_j \to \infty$, $\sqrt{n}\lambda_j h^2 \int_h^{1-h} \phi_j^{(2)}(u)\phi_j(u)du \to 0$,

\[ \sqrt{n} \left( \frac{\hat{\lambda}_j - \lambda_j}{\lambda_j} \right) \xrightarrow{d} N \left( 0, \frac{\mathbb{E}(\xi_j^4) - \lambda_j^2}{\lambda_j^2} \right). \]

(b) If $N\lambda_j \to C_1$,

\[ \sqrt{n} \left( \frac{\hat{\lambda}_j - \lambda_j}{\lambda_j} - 2\lambda_j \sigma_K^2 h^2 \int_h^{1-h} \phi_j^{(2)}(u)\phi_j(u)du \right) \xrightarrow{d} N \left( 0, \frac{\mathbb{E}(\xi_j^4) - \lambda_j^2}{\lambda_j^2} + \frac{4\mathbb{E}(X\phi_j)^2 + 2\mathbb{E}(\|X\phi_j\|^2 + \sigma_X^2)}{C_1\lambda_j} + \frac{2\mathbb{E}(\|X\phi_j\|^2 + \sigma_X^2)}{C_1^2} \right). \]

(c) If $N\lambda_j \to 0$,

\[ \sqrt{n}N\lambda_j \left( \frac{\hat{\lambda}_j - \lambda_j}{\lambda_j} - 2\lambda_j \sigma_K^2 h^2 \int_h^{1-h} \phi_j^{(2)}(u)\phi_j(u)du \right) \xrightarrow{d} N \left( 0, \frac{2N}{N-1} \mathbb{E}(\|X\phi_j\|^2 + \sigma_X^2) \right). \]

6 Proofs of theorems in Section 3

In this section, we will give the proofs of theorems in Section 3. In order to avoid the summability issues mentioned in Section 2, we propose a new approach to treat the dominating terms in the summation.

To begin with, denote $T_h f(x) = h^{-1} \int K((x-y)/h)f(y)dy$ and it is clear that $T_h$ is a self-adjoint and bounded operator. We first give the bound for $\|T_h \phi_k - \phi_k\|^2$,

\[
\begin{align*}
|T_h \phi_k(v) - \phi_k(v)| & = \left| \frac{1}{h} \int K \left( \frac{v-t}{h} \right) \phi_k(t) dt - \phi_k(v) \right| \\
& = \left| \int_{-1}^{1} K(u) \left\{ \phi_k(v) - hu\phi_k^{(1)}(v) + \frac{h^2u^2}{2} \phi_k^{(2)}(v^*) \right\} du - \phi_k(v) \right| \\
& \lesssim h^2 |\phi_k^{(2)}(t)|_{\infty} \lesssim h^2 k^c,
\end{align*}
\]

where $v^* \in [v-hu, v]$ and the last inequality comes form assumption (A.3). Then there is

\[ \|T_h \phi_k - \phi_k\|^2 \lesssim h^4 k^{2c} \text{ for all } k. \]  

(13)
6.1 Proof of Theorem 1

PROOF. Recall $\delta_{ijl} = X_{ij}X_{il} = (X_i(T_{ij}) + \varepsilon_{ij})(X_i(T_{il}) + \varepsilon_{il})$ and note that

$$
(\hat{C}\phi_j, \phi_k) = \frac{1}{n} \sum_{i = 1}^{n} \frac{1}{N(N-1)} \sum_{l_1,l_2} \delta_{i,l_1}T_h\phi_j(T_{il_1})T_h\phi_k(T_{il_2}).
$$

Denote $A_i(\phi_j, \phi_k) = \sum_{l_1,l_2} \delta_{i,l_1}T_h\phi_j(T_{il_1})T_h\phi_k(T_{il_2})$, then

$$
\text{Var}((\Delta\phi_j, \phi_k)) = \text{Var}((\hat{C}\phi_j, \phi_k)) \leq \frac{1}{n N^2(N-1)^2} \mathbb{E}A_i^2(\phi_j, \phi_k).
$$

The second moment of each $A_i$ can be decomposed as

$$
\mathbb{E}A_i^2(\phi_j, \phi_k) = 4! \binom{N}{4} A_{i1}(\phi_j, \phi_k) + 3! \binom{N}{3} A_{i2}(\phi_j, \phi_k) + 2! \binom{N}{2} A_{i3}(\phi_j, \phi_k)
$$

with

$$
A_{i1}(\phi_j, \phi_k) = \mathbb{E}\{(X_i, T_h\phi_j)^2(X_i, T_h\phi_k)^2\},
$$
$$
A_{i2}(\phi_j, \phi_k) = 2\mathbb{E}\{(X_i, T_h\phi_j)(X_i, T_h\phi_k)(X_i, T_h\phi_j, X_i, T_h\phi_k)\}
\quad + \mathbb{E}\{\{(X_i, T_h\phi_j)^2(\|X_iT_h\phi_j\|^2 + \sigma^2_X T_h\phi_k^2)\}
\quad + \mathbb{E}\{\{(X_i, T_h\phi_k)^2(\|X_iT_h\phi_j\|^2 + \sigma^2_X T_h\phi_k^2)\}
\quad := A_{i21}(\phi_j, \phi_k) + A_{i22}(\phi_j, \phi_k) + A_{i23}(\phi_j, \phi_k)
$$
$$
A_{i3}(\phi_j, \phi_k) = \mathbb{E}\{\|X_iT_h\phi_j\|^2 + \sigma^2_X T_h\phi_k^2\}(\|X_iT_h\phi_k\|^2 + \sigma^2_X T_h\phi_k^2)
\quad + \mathbb{E}\{(X_iT_h\phi_j, X_iT_h\phi_k) + \sigma^2_X (T_h\phi_j, T_h\phi_k)\}
\quad := A_{i31}(\phi_j, \phi_k) + A_{i32}(\phi_j, \phi_k).
$$

By AM-GM inequality,

$$
A_{i21}(\phi_j, \phi_k) \leq 2\mathbb{E}\{(X, T_h\phi_j)(X, T_h\phi_k)\|X_iT_h\phi_j\|X_iT_h\phi_k\}
\quad \leq \mathbb{E}\{(X, T_h\phi_j)^2\|X_iT_h\phi_k\|^2 + (X, T_h\phi_k)^2\|X_iT_h\phi_j\|^2\}
\quad \leq A_{i22}(\phi_j, \phi_k) + A_{i23}(\phi_j, \phi_k).
$$

Similarly $A_{i32}(\phi_j, \phi_k) \leq A_{i31}(\phi_j, \phi_k)$ and thus $A_{i2}(\phi_j, \phi_k) \leq 2\{A_{i22}(\phi_j, \phi_k) + A_{i23}(\phi_j, \phi_k)\}$, $A_{i3}(\phi_j, \phi_k) \leq 2A_{i31}(\phi_j, \phi_k)$. In summary,

$$
\mathbb{E}(\Delta\phi_j, \phi_k)^2 \leq (\mathbb{E}(\Delta\phi_j, \phi_k))^2
\quad + \frac{1}{n} \left\{ A_{i1}(\phi_j, \phi_k) + \frac{A_{i22}(\phi_j, \phi_k) + A_{i23}(\phi_j, \phi_k)}{N} + \frac{A_{i31}(\phi_j, \phi_k)}{N^2} \right\}.
$$

(14)

The following lemmas give the bounds for $\|\mathbb{E}\Delta\|^2_{HS}$ and the fourth moment of $(X, T_h\phi_k)$, which are useful in qualifying the bias and variance terms in equation (14) and their proofs can be found in the Appendix.
**Lemma 7.** Under assumptions (A.1) to (A.3),
\[
\|E\Delta\|_{HS}^2 \leq \text{Const.} \left\{ \begin{array}{ll}
h^4, & 2a - 2c > 1, \\
h^4 \ln h, & 2a - 2c = 1, \\
h^2(2a-1)/c, & 2a - 2c < 1. \end{array} \right.
\]

**Lemma 8.** Under assumptions (A.1) to (A.3) and \(h^4 j^{2c+2a} = O(1)\), there are \(E(\|X\|^4) < \infty\), \(E((X, T_h \phi_k)^4) \leq k^{-2a}\) for \(1 \leq k \leq 2j\) and
\[
E \left| \sum_{k>j} |(X, T_h \phi_k)|^2 \right| \leq \text{Const.} j^{-2a}.
\]

**Step 1: bound the bias term.**

In the following, we first give the bounds for the bias term \((E(\Delta \phi_j, \phi_k))^2\) for each \(1 \leq k \leq 2j\) and the summation \(\sum_{k=j+1}^\infty (E(\Delta \phi_j, \phi_k))^2\). Note that
\[
E(\Delta \phi_j, \phi_k) = (CT_h \phi_j, T_h \phi_k) - (C \phi_j, \phi_k) \\
= (C(T_h \phi_j - \phi_j, T_h \phi_k - \phi_k) + (C(T_h \phi_j - \phi_j), \phi_k) + (C \phi_j, T_h \phi_k - \phi_k).
\]

For the first part in equation (15),
\[
|(C(T_h \phi_j - \phi_j), T_h \phi_k - \phi_k)| \leq \lambda_1 \|T_h \phi_j - \phi_j\| \|T_h \phi_k - \phi_k\| \leq \text{Const.} h^2 j^c h^2 k^c.
\]

As \((T_h \phi_j, \phi_k) = (T_h \phi_k, \phi_j)\), there is
\[
|(T_h \phi_j - \phi_j, \phi_k)| = |(\phi_j, T_h \phi_k - \phi_k)| \leq \|\phi_j\| \|T_h \phi_k - \phi_k\| \leq \text{Const.} h^2 k^c.
\]

Thus the last two terms in equation (15) are bounded by
\[
|(C \phi_j, T_h \phi_k - \phi_k)| \leq \lambda_j \|\phi_j, T_h \phi_k - \phi_k\| \leq \text{Const.} j^{-a} h^2 k^c,
\]
\[
|(C(T_h \phi_j - \phi_j), \phi_k)| \leq \lambda_k \|T_h \phi_j - \phi_j, \phi_k\| \leq \text{Const.} k^{-a} h^2 k^c.
\]

Then we conclude that if \(1 \leq k \leq 2j\) and under the assumption \(h^4 j^{2c+2a} = O(1)\),
\[
|E(\Delta \phi_j, \phi_k)| \leq C(h^2 j^c h^2 k^c + j^{-a} h^2 k^c + k^{-a} h^2 k^c) \leq \text{Const.} h^2 k^{c-a}.
\]

Furthermore, by Lemma 7, \(h^4 j^{2c+2a} = O(1)\) and the fact \(\min(4, 2(2a - 1)/c) > 4(2a - 1)/(2a + 2c)\), we deduce that \(\|T_h C - C\|_{HS}^2 \leq \text{Const.} j^{1-2a}\). Then the summation of the first term in equation (15) over \(k > j\) can be bounded by
\[
\sum_{k>j} (C(T_h \phi_j - \phi_j), T_h \phi_k - \phi_k)^2 \leq \|T_h C - C\|_{HS}^2 \|T_h \phi_j - \phi_j\|^2 \leq \text{Const.} j^{1-2a} h^4 j^{2c}.
\]
Furthermore, note that

\[
(C(T_h \phi_j - \phi_j), \phi_k) + (C \phi_j, T_h \phi_k - \phi_k) = \lambda_k(T_h \phi_j - \phi_j, \phi_k) + \lambda_j(\phi_j, T_h \phi_k - \phi_k)
\]

\[
= (\lambda_k + \lambda_j)(T_h \phi_j - \phi_j, \phi_k).
\]

There is

\[
\sum_{k>j}|(C(T_h \phi_j - \phi_j), \phi_k) + (C \phi_j, T_h \phi_k - \phi_k)|^2 = \sum_{k>j}(\lambda_k + \lambda_j)^2(T_h \phi_j - \phi_j, \phi_k)^2
\]

\[
\leq (2\lambda_j)^2 \sum_{k>j}(T_h \phi_j - \phi_j, \phi_k)^2 \leq (2\lambda_j)^2\|T_h \phi_j - \phi_j\|^2 \leq \text{Const.} j^{-2a} h^4 j^{2c}.
\]

Then under assumptions (A.1) to (A.3), and \( h^4 j^{2c+2a} = O(1) \), there is

\[
\sum_{k=j+1}^{\infty} \left( \mathbb{E}(\Delta \phi_j, \phi_k) \right)^2 \leq \text{Const.} h^4 j^{2c+1-2a}. \quad (17)
\]

**Step 2: bound the variance term.**

Next, we shift to variance terms \( A_{i1}, A_{i22}, A_{i23}, A_{i31} \) for each \( 1 \leq k \leq 2j \) and theirs corresponding summations on \( k \geq j \). We start with the bounds for \( A_{i1} \). By Lemma 8 and Cauchy-Schwarz inequality

\[
A_{i1}(\phi_j, \phi_k) = \mathbb{E}(\langle X, T_h \phi_j \rangle^2 \langle X, T_h \phi_k \rangle^2)
\]

\[
\leq (\mathbb{E}(X, T_h \phi_j)^{4} \mathbb{E}(X, T_h \phi_k)^{4})^{1/2} \leq \text{Const.} j^{-a} k^{-a} \quad \text{and}
\]

\[
\sum_{k>j} A_{i1}(\phi_j, \phi_k) = \mathbb{E}\left( \langle X, T_h \phi_j \rangle^2 \sum_{k>j} \langle X, T_h \phi_k \rangle^2 \right)
\]

\[
\leq \left( \mathbb{E}(X, T_h \phi_j)^{4} \mathbb{E}\left| \sum_{k>j} \langle X, T_h \phi_k \rangle^2 \right|^2 \right)^{1/2} \leq \text{Const.} j^{1-2a}. \quad (18)
\]

For \( A_{i22} \). By Lemma 8, (A.3) and Cauchy-Schwarz inequality,

\[
A_{i22}(\phi_j, \phi_k) = \mathbb{E}\left\{ \langle X, T_h \phi_j \rangle^2 \left( \|XT \phi_k\|^2 + \sigma_X^2 \|T \phi_k\|^2 \right) \right\}
\]

\[
\leq \|T \phi_k\|^2 \mathbb{E}\left\{ \langle X, T_h \phi_j \rangle^2 \left( \|X\|^2 + \sigma_X^2 \right) \right\}
\]

\[
\leq \|\phi_k\|^2 \mathbb{E}\left\{ \mathbb{E}(X, T_h \phi_j)^{4} \mathbb{E}(\|X\|^2 + \sigma_X^2) \right\}^{1/2}
\]

\[
\leq \text{Const.} \left( \mathbb{E}(X, T_h \phi_j)^{4} \right)^{1/2} \leq \text{Const.} j^{-a}. \quad (19)
\]

For the summation \( \sum_{k>j} A_{i22}(\phi_j, \phi_k) \), note that

\[
\sum_{k=1}^{\infty} |T_h \phi_k(x)|^2 = \frac{1}{h^2} \int \left| K\left( \frac{x - y}{h} \right) \right|^2 dy \leq \text{Const.} h^{-1} \quad \text{for all} \ x \in [0, 1].
\]
Thus,
\[
\sum_{k=j}^{\infty} A_{i22}(\phi_j, \phi_k) = \mathbb{E}\left\{ (X, T_h \phi_j)^2 \sum_{k=j}^{\infty} (\|XT_h \phi_k\|^2 + \sigma_X^2 \|T_h \phi_k\|^2) \right\}
\]
\[
\leq \sum_{k=1}^{\infty} \|T_h \phi_k(x)\|^2 \mathbb{E}\left\{ (X, T_h \phi_j)^2 (\|X\|^2 + \sigma_X^2) \right\}
\]
\[
\leq \text{Const.} h^{-1} (\mathbb{E}(X, T_h \phi_j)^4)^{1/2} \leq \text{Const.} h^{-1-j^a}. \tag{20}
\]

For \(A_{i23}\), by symmetry and Lemma 8, there is
\[
A_{i23}(\phi_j, \phi_k) = A_{i22}(\phi_k, \phi_j) \leq \|T_h \phi_j\|_\infty^2 \mathbb{E}\left\{ (X, T_h \phi_k)^2 (\|X\|^2 + \sigma_X^2) \right\}
\]
\[
\leq \text{Const.} (\mathbb{E}(X, T_h \phi_k)^4)^{1/2} \leq \text{Const.} k^{-a}, \quad \forall 1 \leq k \leq 2j,
\tag{21}
\]
and
\[
\sum_{k>j}^{\infty} A_{i23}(\phi_j, \phi_k) \leq \|T_h \phi_j\|_\infty^2 \mathbb{E}\left\{ \sum_{k>j}^{\infty} (X, T_h \phi_k)^2 (\|X\|^2 + \sigma_X^2) \right\}
\]
\[
\leq \text{Const.} \left( \mathbb{E}\left( \sum_{k>j}^{\infty} |(X, T_h \phi_k)|^2 \right)^2 \mathbb{E}(\|X\|^2 + \sigma_X^2)^2 \right)^{1/4} \leq \text{Const.} j^{1-a}. \tag{22}
\]

For the last term \(A_{i31}\),
\[
A_{i31}(\phi_j, \phi_k) = \mathbb{E}\left\{ (\|XT_h \phi_j\|^2 + \sigma_X^2 \|T_h \phi_j\|^2)(\|XT_h \phi_k\|^2 + \sigma_X^2 \|T_h \phi_k\|^2) \right\}
\]
\[
\leq \text{Const.} \mathbb{E}(\|X\|^2 + \sigma_X^2)^2 \leq \text{Const.},
\]
and
\[
\sum_{k=j}^{\infty} A_{i31}(\phi_j, \phi_k)
\]
\[
\leq \mathbb{E}\left\{ (\|XT_h \phi_j\|^2 + \sigma_X^2 \|T_h \phi_j\|^2) \sum_{k=1}^{\infty} (\|XT_h \phi_k\|^2 + \sigma_X^2 \|T_h \phi_k\|^2) \right\}
\]
\[
\leq \mathbb{E}\left\{ (\|XT_h \phi_j\|^2 + \sigma_X^2 \|T_h \phi_j\|^2) \int \{X^2(u) + \sigma_X^2\} \sum_{k=1}^{\infty} |T_h \phi_k(u)|^2 du \right\}
\]
\[
\leq \mathbb{E}\left\{ (\|XT_h \phi_j\|^2 + \sigma_X^2 \|T_h \phi_j\|^2)(\|X\|^2 + \sigma_X^2) \frac{\|K\|^2}{h} \right\}
\]
\[
\leq \text{Const.} h^{-1} \mathbb{E}(\|X\|^2 + \sigma_X^2)^2 \leq \text{Const.} h^{-1}. \tag{23}
\]

Combine equation (18) to (23), under assumptions (A.1) to (A.3) and \(h^4 j^{2c+2a} = O(1)\), there are
\[
A_{i1}(\phi_j, \phi_k) \leq \text{Const.} j^{-a} k^{-a}, \quad A_{i22}(\phi_j, \phi_k) \leq \text{Const.} j^{-a},
\]
\[
A_{i23}(\phi_j, \phi_k) \leq \text{Const.} k^{-a}, \quad A_{i31}(\phi_j, \phi_k) \leq \text{Const.} \text{ for } 1 \leq k \leq 2j,
\]
\[
\sum_{k=j+1}^{\infty} A_{i1}(\phi_j, \phi_k) \leq \text{Const.} j^{1-2a}, \quad \sum_{k=j+1}^{\infty} A_{i22}(\phi_j, \phi_k) \leq \text{Const.} j^{-a} h^{-1}, \tag{24}
\]
\[
\sum_{k=j+1}^{\infty} A_{i23}(\phi_j, \phi_k) \leq \text{Const.} j^{1-a}, \quad \sum_{k=j+1}^{\infty} A_{i31}(\phi_j, \phi_k) \leq \text{Const.} h^{-1}.
\]

The proof of Theorem 1 is complete by combing equations (17), (24) and (14).
6.2 Proof of Theorem 2

Proof. By the proof of Theorem 5.1.8 in Hsing and Eubank (2015), for \( j \in \Omega(n, N, h) \), we have the following expansion,

\[
\hat{\phi}_j - \phi_j = \sum_{k \in \mathcal{J}} \int \frac{(\hat{C} - C) \phi_j \phi_k}{(\lambda_j - \lambda_k)} \phi_k + \sum_{k \in \mathcal{J}} \int \frac{(\hat{C} - C)(\hat{\phi}_j - \phi_j) \phi_k}{(\lambda_j - \lambda_k)} \phi_k + \sum_{k \in \mathcal{J}} \int \frac{(\hat{C} - C) \phi_j \phi_k}{(\lambda_j - \lambda_k)} \phi_k
\]

\[
\hat{\phi}_j - \phi_j \leq \sum_{k \in \mathcal{J}} \int \frac{(\hat{C} - C)(\hat{\phi}_j - \phi_j) \phi_k}{(\lambda_j - \lambda_k)} \phi_k + \sum_{k \in \mathcal{J}} \int \frac{(\hat{C} - C) \phi_j \phi_k}{(\lambda_j - \lambda_k)} \phi_k,
\]

such kind of expansion can also be found in Hall and Hosseini-Nasab (2006) and Li and Hsing (2010). We first show that \( \mathbb{E}\|\hat{\phi}_j - \phi_j\|^2 \) is dominated by the \( L^2 \) norm of the first term in the right hand side of equation (25). By Bessel’s inequality, we see that

\[
\mathbb{E}\left\| \sum_{k \in \mathcal{J}} \int \frac{(\hat{C} - C)(\hat{\phi}_j - \phi_j) \phi_k}{(\lambda_j - \lambda_k)} \phi_k \right\|^2 \leq \mathbb{E}\|\hat{C} - C\|_{HS}^2 \|\hat{\phi}_j - \phi_j\|^2 \leq \frac{1}{16} \mathbb{E}\|\hat{\phi}_j - \phi_j\|^2,
\]

where the last equality comes from the fact \( \eta_j^{-1} \|\hat{C} - C\| < 1/2 \) on \( \Omega_n(n, N, h) \). Similarly,

\[
\mathbb{E}\sum_{k \in \mathcal{J}} \sum_{s=1}^{\infty} \frac{(\lambda_j - \lambda_k)^s}{(\lambda_j - \lambda_k)^{s+1}} \left\{ \int (\hat{C} - C) \phi_j \phi_k \right\}^2 \leq \frac{8}{9} \mathbb{E}\left[ \frac{\|\hat{C} - C\|_{HS}^2}{\eta_j^2} \sum_{k \in \mathcal{J}} \left\{ \int (\hat{C} - C) \phi_j \phi_k \right\}^2 + \frac{\|\hat{C} - C\|_{HS}^4}{\eta_j^4} \left\|\hat{\phi}_j - \phi_j\right\|^2 \right]
\]

\[
\leq \frac{2}{9} \mathbb{E}\sum_{k \in \mathcal{J}} \left\{ \int (\hat{C} - C) \phi_j \phi_k \right\}^2 + \frac{1}{18} \mathbb{E}\|\hat{\phi}_j - \phi_j\|^2.
\]

Combining (25) to (27) and the fact \( \|\int (\hat{\phi}_j - \phi_j) \phi_j\| = 1/2 \|\hat{\phi}_j - \phi_j\|^2 \), \( \mathbb{E}\|\hat{\phi}_j - \phi_j\|^2 \) is dominated by the first term in the right hand side of equation (25). The proof is complete by

\[
\mathbb{E}\|\hat{\phi}_j - \phi_j\|^2 \leq \sum_{k \in \mathcal{J}} \mathbb{E}\left[ \frac{\Delta \phi_j \phi_k}{(\lambda_j - \lambda_k)^2} \right] + \sum_{k \in \mathcal{J}} \mathbb{E}\left[ \frac{\Delta \phi_j \phi_k}{(\lambda_j - \lambda_k)^2} \right] + \sum_{k \in \mathcal{J}} \mathbb{E}\left[ \frac{\Delta \phi_j \phi_k}{(\lambda_j - \lambda_k)^2} \right]
\]

\[
\leq \text{Const.} \left\{ h^4 j^{2c+2} + \frac{j^2}{n} \left\{ 1 + j^a \right\} + \left( \frac{h^4 j^{2c+2}}{N} \right) \right\}
\]

\[
\leq \text{Const.} \left\{ h^4 j^{2c+1} + \frac{j^a}{Nh} + \frac{j^a}{N^2 h} \right\}
\]

\[
\leq \text{Const.} \left\{ \frac{j^2}{n} \left\{ 1 + \left( \frac{j^a}{N} \right)^2 \right\} + \frac{j^a}{Nh} \left( 1 + \frac{j}{N} \right) + h^4 j^{2c+2} \right\},
\]

where the first inequality comes from Theorem 1 and Lemma 7 in Dou et al. (2012).
7 Proofs of Theorem 5

To obtain the uniform convergence rate for functional data, a common used technique is to obtain a uniform bound over a finite grid of the domain, which grows increasingly dense with $n \to \infty$ (Li and Hsing, 2010; Zhang and Wang, 2016). We generalize this methodology to the eigenfunction case and let $\chi_1(\rho) = \{n^p 1_{(1/n^p, 1/(n^p+1)]} | j \in \mathbb{Z} \cap [0, n^p) \}$, $(\rho \in \mathbb{Z}_+)$. Then

$$\sup_{s \in [0, 1]} |\hat{\phi}_j(s) - \phi_j(s)| \leq \sup_{g \in \chi_1(\rho)} |(\hat{\phi}_j - \phi_j, g)| + D_1 + D_2$$

where

$$D_1 = \sup_{s_1, s_2 \in [0, 1]} |\hat{\phi}_j(s_1) - \hat{\phi}_j(s_2)|, \quad D_2 = \sup_{s_1, s_2 \in [0, 1]} |\phi_j(s_1) - \phi_j(s_2)|.$$

As $\lambda_j \hat{\phi}_j(s) = \hat{C}\hat{\phi}_j(s) = \int \hat{C}(s, t)\hat{\phi}(t)dt$, and by the definition of $\hat{C}$ and Lipschitz continuous of the kernel function $K$, one has $|\hat{C}(s_1, t) - \hat{C}(s_2, t)| \leq \text{const.} Z_1 |s_1 - s_2|h^3$, where

$$Z_1 := \frac{1}{n^N(N-1)} \sum_{i=1}^n \sum_{1 \leq t_1 \neq t_2 \leq N} |\delta_{it_1t_2}| \text{ and } \mathbb{E}Z_1 = \mathbb{E}|\delta_{it_1t_2}| \leq \text{const.}.$$

Then

$$|\hat{\phi}_j(s_1) - \hat{\phi}_j(s_2)| \leq \lambda_j^{-1} \int |\hat{C}(s_1, t) - \hat{C}(s_2, t)||\hat{\phi}_j(t)|dt$$

$$\leq \text{const.} \lambda_j^{-1} Z_1 \frac{|s_1 - s_2|}{h^3} \int |\hat{\phi}_j(t)|dt \leq \text{const.} \lambda_j^{-1} Z_1 |s_1 - s_2|h^3$$

$$\leq \text{const.} j^a Z_1 |s_1 - s_2|h^3,$$

where the last inequality holds on the high probability set $\Omega_u$. Under Assumption (A.3), we also have $|\hat{\phi}_j(s_1) - \hat{\phi}_j(s_2)| \leq \text{const.} j^{c/2}|s_1 - s_2|$. Then we have $D_2 \leq \text{const.} j^{c/2} n^{-p}$, $D_1 \leq \text{const.} j^a Z_1 n^{-p} h^{-3}$, $\mathbb{E}D_1 \leq \text{const.} j^a n^{-p} h^{-3}$ and

$$\mathbb{E}|D_1 + D_2| \leq \text{const.} (j^a h^{-3} + j^{c/2}) n^{-p}.$$

By using the perturbation technique again, the following Lemma decompose $\hat{\phi}_j - \phi_j$ into two parts, where $\phi_{j,0}$ can be bounded by Theorem 1 and $\phi_{j,1}$ is the dominating term that requires subsequent analyses.

**Lemma 9.** Under assumptions (A.1) to (A.3), (U.1), and $h^4 j^{2a+2c} = O(1)$, we have $\hat{\phi}_j - \phi_j = \phi_{j,0} + \lambda_j^{-1} \phi_{j,1}$ with

$$\|\phi_{j,0}\|_\infty = O_p \left( \frac{j^{\log j}}{\sqrt{n}} + \frac{1}{\sqrt{nN}} \left( j^{\frac{a+1}{2}} \log j + h^{-\frac{a}{2}} j^{\frac{a+1}{2}} \right) + \frac{j^{a+1}}{\sqrt{nhN}} + \frac{1}{\sqrt{nN}} j^{a+1} \log j + h^2 j^{c+1} \log j \right)$$

$$+ \frac{j^{a+1}}{\sqrt{nhN}} + \frac{1}{\sqrt{nN}} j^{a+1} \log j + h^2 j^{c+1} \log j$$

and $\phi_{j,1} = \sum_{k=2j}^{\infty} \langle \Delta \phi_j, \phi_k \rangle \phi_k$. 

21
For fixed $g \in \chi_1(\rho)$ we have $\|g\|_1 = 1, (\hat{\phi}_j - \phi_j, g) = (\hat{\phi}_j, g) \leq \|\hat{\phi}_j\|_{\infty} + \hat{\lambda}_j^{-1}|(\phi_j, g)|$ by Lemma 9. For all $g \in \mathcal{L}^2$ denote $P_{\geq m}g := \sum_{k=m}^{\infty} (g, \hat{\phi}_k), P_{\leq m}g := \sum_{k=1}^{m-1} (g, \hat{\phi}_k)$, and $g = P_{\leq m}g + P_{\geq m}g$. Thus,

$$\langle \phi_j, g \rangle = \sum_{k=2j}^{\infty} \langle \Delta \phi_j, \phi_k \rangle \langle \phi_k, g \rangle = (\Delta \phi_j, P_{\geq 2j}g) = (\hat{C} \phi_j, P_{\geq 2j}g).$$

The first term in equation (29) becomes

$$\sup_{g \in \chi_1(\rho)} |\langle \phi_j, g \rangle| \leq \|\phi_j\|_{\infty} + \hat{\lambda}_j^{-1} \sup_{g \in \chi_1(\rho)} \|\hat{C} \phi_j, P_{\geq 2j}g\|$$

and

$$\sup_{g \in \chi_1(\rho)} |(\hat{C} \phi_j, P_{\geq 2j}g)| \leq \sup_{g \in \chi_1(\rho)} |(\hat{C} - \mathbb{E} \hat{C}) \phi_j, P_{\geq 2j}g\| + \sup_{g \in \chi_1(\rho)} |(\mathbb{E} \hat{C} \phi_j, P_{\geq 2j}g)|.$$  

For the bias term in equation (32), we have the following lemma.

**Lemma 10.** Under assumptions (A.1) to (A.3) and (U.1), for all $g \in \chi_1(\rho)$,

(i) $\|P_{\leq m}g\|^2 \leq m$, $\|P_{\leq m}g\|_{\infty} \leq m$ and $\|P_{\geq m}g\|_1 \leq m^{1/2}$.

(ii) $\sup_{g \in \chi_1(\rho)} |(\mathbb{E} \hat{C} \phi_j, P_{\geq 2j}g)| \leq \text{Const.} h^2 j^{c_{n+1}}.$  

### 7.1 Bound the variance term in equation (32)

In this subsection, we shall use the double truncation technique to bound the variance term in equation (32). To be generic, for all $f, g \in \mathcal{L}^2$, recall that

$$\langle \hat{C} f, g \rangle = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{N(N-1)} \sum_{i_1 \neq i_2} \delta_{i_1 i_2} T_h f(T_{i_1}) T_h g(T_{i_2}).$$

Let $A_i(f, g) = \sum_{i_1 \neq i_2} \delta_{i_1 i_2} T_h f(T_{i_1}) T_h g(T_{i_2})$ then

$$\langle \hat{C} f, g \rangle = \frac{1}{n} \sum_{i=1}^{n} \frac{A_i(f, g)}{N(N-1)}, \quad \mathbb{E} \langle \hat{C} f, g \rangle = \frac{\mathbb{E} A_i(f, g)}{N(N-1)}$$

By the un-compactness of functional data, $\delta_{i_1 i_2}$ is unbounded random variable and we first truncate on $\delta_{i_1 i_2}$. Denote

$$\bar{C}(s, t) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{N(N-1)} \frac{1}{h^2} \sum_{1 \leq i_1 \neq i_2 \leq N} K\left(\frac{T_{i_1} - s}{h}\right) K\left(\frac{T_{i_2} - t}{h}\right) \delta_{i_1 i_2} 1_{[|\delta_{i_1 i_2}| \leq A_n]}$$

where $A_n$ is a positive constant we will define later and then,

$$\sup_{g \in \chi_1(\rho)} |(\hat{C} - \mathbb{E} \hat{C}) \phi_j, P_{\geq 2j}g\| \leq \sup_{g \in \chi_1(\rho)} |(\hat{C} - \mathbb{E} \hat{C}) \phi_j, P_{\geq 2j}g\| + E_1 + \mathbb{E} E_1.$$
with
\[ E_1 = \sup_{g \in \chi_1(\rho)} \frac{1}{N} \sum_{i=1}^{n} \frac{1}{N(N-1)} \sum_{1 \leq i_1 < i_2 \leq N} |T_h \phi_j(T_{i_1} T_{i_2})| \delta_{i_1,i_2}^2 \mathbf{1}_{\{(|\phi_j| - A_n) > A_n \}}. \]

For the first term in equation (34), let \( \hat{A}_i(f, g) = \sum_{i_1 < i_2} \delta_{i_1,i_2}^2 \mathbf{1}_{\{(|\phi_j| - A_n) > A_n \}} T_h f(T_{i_1}) T_h g(T_{i_2}) \) and we obtain a trivial bound for \( A_i(f, g) \), which is the same as those in Li et al. (2010) and Zhang and Wang (2016).

\[ |\hat{A}_i(f, g)| \leq A_n N^2 J_i[T_h f] J_i[T_h g] \text{ with } J_i[\varphi] := \frac{1}{N} \sum_{i=1}^{N} |\varphi(T_{i})|. \] (35)

However, the bound (35) is not applicable for the dense and ultra dense case. To address this, we propose a double truncation technique, that is, truncation on \( \tilde{A}_i(f, g) \) with \( A_n N^2 \|f\|_\infty M \), where \( M \) is a positive number defined later. We start with the probability bound \( \mathbb{P}(J_i[\varphi] > M) \) for all \( \varphi \), note that \( \mathbb{E}|\varphi(T_{i})| = \|\varphi\|_1 \), \( |\varphi(T_{i})| \leq \|\varphi\|_\infty \) and \( \mathbb{E}|\varphi(T_{i})|^2 = \|\varphi\|^2 \). For fixed \( M \geq 4 \|\varphi\|_1 \) and \( M' = M - 4 \|\varphi\|_1 \), by Bernstein inequality,

\[ \mathbb{P}(J_i[\varphi] > M) = \mathbb{P}\left( \sum_{1 \leq i \leq N} (|\varphi(T_{i})| - \|\varphi\|_1) > M'N \right) \leq \exp\left( -\frac{(M'N)^2}{2} \|\varphi\|_1^2 + \|\varphi\|_\infty M'N/3 \right) \leq \exp\left( -\frac{(M'N)^2}{2 \|\varphi\|_\infty} \right). \] (36)

For \( M \geq 4 \|T_h g\|_1 \), let \( \hat{A}_i^M(f, g) = \hat{A}_i(f, g) \mathbf{1}_{\{\hat{A}_i(f, g) > A_n N^2 \|f\|_\infty M \}} \). As \( \hat{J}_i[T_h f] \leq \|T_h f\|_\infty \leq \|f\|_\infty \), we have \( |\hat{A}_i(f, g)| \leq A_n N^2 J_i[T_h f] J_i[T_h g] \leq A_n N^2 \|f\|_\infty J_i[T_h g] \), thus

\[ \mathbb{P}(|\hat{A}_i(f, g)| > A_n N^2 \|f\|_\infty M) \leq \mathbb{P}(J_i[T_h g] > M) \leq \exp\left( -\frac{MN}{2 \|T_h g\|_\infty} \right). \]

Let

\[ C^M_\ast(f, g) = \frac{1}{n} \sum_{i=1}^{n} \hat{A}_i^M(f, g) \text{ and } C^{\ast M}_\ast(f, g) = \frac{1}{n} \sum_{i=1}^{n} \hat{A}_i(f, g) \mathbf{1}_{\{\hat{A}_i(f, g) > A_n N^2 \|f\|_\infty M \}}, \]

we have \( \langle \hat{C} f, g \rangle = C^M_\ast(f, g) + C^{\ast M}_\ast(f, g) \) and

\[ \langle (\hat{C} - \mathbb{E}\hat{C}) f, g \rangle \leq |C^M_\ast(f, g) - \mathbb{E}[C^M_\ast(f, g)]| + C^{\ast M}_\ast(f, g) + \mathbb{E}[C^{\ast M}_\ast(f, g)]. \]

Denote

\[ M_0 := \sup_{g \in \chi_1(\rho)} \|T_h P_{\leq 2j} g\|_\infty, \quad M_1 := \sup_{g \in \chi_1(\rho)} \|T_h P_{\leq 2j} g\|_1, \quad M_2 := \sup_{g \in \chi_1(\rho)} \mathbb{E}|\hat{A}_i(\phi_j, P_{\leq 2j} g)|^2. \]
Then for $M \geq 4M_1$, $M_1 := \sup_{g \in \chi_1(\rho)} \|T_h P_{\geq 2j} g\|_1,$

$$
\sup_{g \in \chi_1(\rho)} |((\tilde{C} - \mathbb{E}\tilde{C})\phi_j, P_{\geq 2j} g)| 
\leq \sup_{g \in \chi_1(\rho)} |C_\star^M(\phi_j, P_{\geq 2j} g) - \mathbb{E}[C_\star^M(\phi_j, P_{\geq 2j} g)]| + F_1 + \mathbb{E}F_1
$$

with

$$
F_1 = \sup_{g \in \chi_1(\rho)} C_\star^{2M}(\phi_j, P_{\geq 2j} g).
$$

By Bernstein inequality we have (for $t > 0$)

$$
\mathbb{P}\left( |C_\star^M(f, g) - \mathbb{E}[C_\star^M(f, g)]| > t \right)
= \mathbb{P}\left( \left| \sum_{i=1}^n (\tilde{A}_i^M(f, g) - \mathbb{E}[\tilde{A}_i^M(f, g)]) \right| > nN(N-1)t \right)
\leq 2 \exp\left( -\frac{n^2N^2(N-1)^2t^2/2}{\sum_{i=1}^n \mathbb{E}(A_i^M(f, g) - \mathbb{E}[A_i^M(f, g)])^2 + 2A_nN^2\|f\|_\infty M\|g\|_\infty M(N-1)t/3} \right)
\leq 2 \exp\left( -\frac{n\mathbb{E}[A_i^M(f, g)]^2 + 2A_nN^2\|f\|_\infty M\|g\|_\infty M(N-1)t/3}{n^2N^2(N-1)^2t^2/2} \right).
$$

We obtain the following explication bound,

$$
\mathbb{E}\left[ \sup_{g \in \chi_1(\rho)} |C_\star^M(\phi_j, P_{\geq 2j} g) - \mathbb{E}[C_\star^M(\phi_j, P_{\geq 2j} g)]| \right]
\leq \int_0^\infty \min \left\{ 1, 2n^\rho \exp\left( -\frac{n^2N^2(N-1)^2t^2/2}{nM_2 + 2A_nN^2\|\phi_j\|_\infty M\|g\|_\infty M(N-1)t/3} \right) \right\} dt
\leq \text{Const.} \left[ \sqrt{\frac{\rho n M_2}{n^4}} + \frac{M}{n\rho\ln n} \right].
$$

The following lemma bound the truncated terms $E_1$ and $F_1$ in (34) and (37).

**LEMMA 11.** Under assumptions (A.1) to (A.3) and (U.1)

$$
\mathbb{E}|E_1| \leq \frac{1}{hA_n^{\alpha-1}} \quad \text{and} \quad \mathbb{E}|F_1| \leq \frac{n^\rho M_2^{1/2}}{N(N-1)} \exp\left( -\frac{MN}{4M_0} \right).
$$

Combine equation (34), (37) and Lemma 11, we have

$$
\sup_{g \in \chi_1(\rho)} |((\tilde{C} - \mathbb{E}\tilde{C})\phi_j, P_{\geq 2j} g)|
\leq \text{Const.} \left[ \sqrt{\frac{\rho n M_2}{n^4}} + \frac{M}{n\rho\ln n} \right] + \text{Const.} \frac{2n^\rho M_2^{1/2}}{hA_n^{\alpha-1} + \frac{N(N-1)}{4M_0}} \exp\left( -\frac{MN}{4M_0} \right).$$
7.2 Bound for $M_2$ and choice of $M$ and $A_n$

Recall that $M_2 := \sup_{g \in X(\rho)} \mathbb{E} \tilde{A}_i(\phi_j, P_{2j}g)^2$, with $\tilde{A}_i(f, g) = \sum_{l_1 \neq l_2} \delta_{l_1l_2} 1_{\{\delta_{l_1l_2} > A_n\}} T_h f(T_{l_1}) T_h g(T_{l_2})$, which implies that $M_2$ is depend on $A_n$. Consider the un-truncated version of $M_2$, $M^n_2 := \sup_{g \in X(\rho)} \mathbb{E} A_i(\phi_j, P_{2j}g)^2$, where $A_i(f, g) = \sum_{l_1 \neq l_2} \delta_{l_1l_2} T_h f(T_{l_1}) T_h g(T_{l_2})$. We first study the difference between $M_2$ and $M^n_2$. Start with the difference between $A_i(f, g)$ and $\tilde{A}_i(f, g)$ for all $f, g \in \mathcal{L}^2$,

$$
|A_i(f, g) - \tilde{A}_i(f, g)|^2 = \left| \sum_{l_1 \neq l_2} \delta_{l_1l_2} \chi_{\{\delta_{l_1l_2} > A_n\}} T_h f(T_{l_1}) T_h g(T_{l_2}) \right|^2 
$$

$$
\leq \sum_{l_1 \neq l_2} \delta_{l_1l_2} \chi_{\{\delta_{l_1l_2} > A_n\}} \left| T_h f(T_{l_1}) T_h g(T_{l_2}) \right|^2 
$$

$$
\leq A_n^{-\alpha} Z_{2,i} N^4 J_i \left[ |T_h f|^2 \right] J_i \left[ |T_h g|^2 \right] \leq A_n^{-\alpha} Z_{2,i} N^4 \| f \|_{\infty}^2 \| T_h g \|_{\infty} J_i \left[ |T_h g| \right],
$$

where

$$
Z_{2,i} := \frac{1}{N(N-1)} \sum_{1 \leq l_1 < l_2 \leq N} |\delta_{l_1l_2}|^\alpha \text{ with } \mathbb{E} Z_{2,i} = \mathbb{E} |\delta_{l_1l_2}|^\alpha \leq \text{Const..}
$$

On the set $\{ J_i[T_h g] \leq M \}$, we have

$$
\mathbb{E} \left[ |A_i(f, g) - \tilde{A}_i(f, g)|^2 1_{\{ J_i[T_h g] \leq M \}} \right] \leq \mathbb{E} \left[ A_n^{-\alpha} Z_{2,i} N^4 \| f \|_{\infty}^2 \| T_h g \|_{\infty} M \right] 
$$

$$
\leq \text{Const.} A_n^{-\alpha} N^4 \| f \|_{\infty}^2 \| T_h g \|_{\infty} M. \tag{41}
$$

On the set $\{ J_i[T_h g] > M \}$, under the condition $M \geq 4 \| T_h g \|_1$,

$$
\mathbb{E} \left[ |A_i(f, g) - \tilde{A}_i(f, g)|^2 1_{\{ J_i[T_h g] > M \}} \right] 
$$

$$
\leq \mathbb{E} \left[ |A_i(f, g) - \tilde{A}_i(f, g)|^\alpha \right]^{\frac{2}{\alpha}} \left[ \mathbb{P} \left( J_i[T_h g] > M \right) \right]^{1 - \frac{2}{\alpha}} 
$$

$$
\leq \| f \|_{\infty}^\alpha \| T_h g \|_{\infty}^\alpha N^{2\alpha} \mathbb{E} Z_{2,i} \cdot \exp \left( -\frac{(1 - 2/\alpha) MN}{2 \| T_h g \|_{\infty}} \right) \tag{42}
$$

$$
\leq \text{Const.} \| f \|_{\infty}^2 \| T_h g \|_{\infty}^2 N^4 \exp \left( -\frac{(\alpha - 2) MN}{2 \alpha \| T_h g \|_{\infty}} \right),
$$

where the second last inequality is from equation (36) and the last inequality is by $|A_i(f, g) - \tilde{A}_i(f, g)|^\alpha \leq \| f \|_{\infty} \| T_h g \|_{\infty} N^{2\alpha} Z_{2,i}$. Combining equation (41), (42) and take $f = \phi_j, g = P_{2j}g$,

$$
\mathbb{E} A_i(\phi_j, P_{2j}g) - \tilde{A}_i(\phi_j, P_{2j}g)^2 
$$

$$
\leq \mathbb{E} \left[ |A_i(\phi_j, P_{2j}g) - \tilde{A}_i(\phi_j, P_{2j}g)|^2 1_{\{ J_i[T_h g] \leq M \}} \right] 
$$

$$
+ \mathbb{E} \left[ |A_i(\phi_j, P_{2j}g) - \tilde{A}_i(\phi_j, P_{2j}g)|^2 1_{\{ J_i[T_h g] > M \}} \right] 
$$

$$
\leq \text{Const.} A_n^{-\alpha} N^4 \| \phi_j \|_{\infty}^2 \| T_h P_{2j}g \|_{\infty} M 
$$

$$
+ \text{Const.} \| \phi_j \|_{\infty}^2 \| T_h P_{2j}g \|_{\infty}^2 N^4 \exp \left( -\frac{(\alpha - 2) MN}{2 \alpha \| T_h P_{2j}g \|_{\infty}} \right) \leq \text{Const.} A_n^{-\alpha} N^4 h^{-1} M + \text{Const.} h^{-2} N^4 \exp \left( -\frac{(\alpha - 2) MN}{2 \alpha M_0} \right). \tag{43}
$$
Thus,
\[
M_2^{1/2} - (M_2^0)^{1/2} \leq \sup_{g \in \chi_1(\rho)} \left[ \mathbb{E} |A_i(\phi_j, P_{2jg}) - \tilde{A}_i(\phi_j, P_{2jg})|^2 \right]^{1/2}
\]
\[
\leq \text{Const.} A_n^{1-\alpha/2} N^2 |M|h^{1/2} + \text{Const.} h^{-1} N^2 \exp \left( -\frac{(\alpha - 2) MN}{4\alpha M_0} \right).
\]

(44)

Combine equation (32), (33), (39) and (44), we have
\[
\mathbb{E} \left[ \sup_{g \in \chi_1(\rho)} |\langle \tilde{C}\phi_j, P_{2jg} \rangle| \right] \leq \text{Const.} \left[ \sqrt{\rho \ln n \frac{M_0^2}{n^4}} + A_n \frac{\rho \ln n + h^2 j^{c+1-a}}{M n} \right] + \text{Const.} \frac{\rho \ln n}{h A_n^{-1}} + \text{Const.} \frac{\rho \ln n}{h A_n^{-1}}.
\]

(45)

Take \( M = 4M_1 + 4 \frac{M_0}{\alpha-2} \rho \ln n + \frac{\rho M_0}{\alpha-2} \rho \ln n \), \( A_n = n^{1/\alpha} |M| h \rho \ln n |1/\alpha|, (h \geq n^{-\alpha}) \) then,
\[
\mathbb{E} \left[ \sup_{g \in \chi_1(\rho)} |\langle \tilde{C}\phi_j, P_{2jg} \rangle| \right] \leq \sqrt{\frac{\rho \ln n}{n}} \sqrt{\frac{M_0^2}{N^2}} + |j^{-a}| + \frac{\rho \ln n}{n} \left| j^{1/2} \right| + \frac{\ln n}{Nh} \left| j^{1/2} \right| \left( h^{-\alpha} \right)
\]

(45)

Here we use \( M_1 \leq \text{Const.} j^{1/2} \) and \( M_0 \leq \text{Const.} h^{-1} \), which follow from \( \|T_h P_{2jg}\|_1 \leq \|P_{2jg}\|_1 \leq \text{Const.} j^{1/2} \), \( \|T_h P_{2jg}\|_\infty \leq \text{Const.} h^{-1} \) for all \( g \in \chi_1(\rho) \). The following lemma gives the bound of \( M_0 \).

**Lemma 12.** Under assumptions (A.1) to (A.3), and (U.1)

\[
M_2^0 \leq N^4 j^{2-2a} + N^3 (j^{-a} h^{-1} + j^{2-a}) + N^2 h^{-1}.
\]

(46)

Combine equation (29), (30), (31), (45) and (46), on the high probability set \( \Omega_u \),
\[
\mathbb{E}(\|\hat{\phi}_j - \phi_j\|_\infty) \leq \frac{j}{\sqrt{n}} \left( \sqrt{\ln n} + \ln j \right) \left\{ 1 + \frac{j^a}{N} + \frac{j^{a-1}}{Nh} \left( 1 + \frac{j^a}{N} \right) \right\}
\]
\[
+ j^a \left| \frac{\ln n}{n} \right| \left( \frac{j}{\sqrt{n}} \right) \left| j^{1/2} \right| + \frac{\ln n}{Nh} \left| j^{1/2} \right| \left| h^{-\alpha} \right| + j^{c+1} \log j
\]
\[
+ (j^a h^{-3} + j^{cl/2}) n^{-\rho}, \quad (\rho > 0, c > 0).
\]

and the proof is complete by choosing large enough \( \rho \).
8 Conclusion

In this paper, we focus on the convergence rate of eigenfunctions with diverging indices for discretely observed functional data. We propose new techniques to handle the perturbation series and establish sharp bounds for eigencomponents in different types of convergence. We systematically extend the partition “dense” and “sparse” defined for mean and covariance functions to functional principal components analysis. The results in this paper provide useful tools in handling functional principal component analysis based problems and models involving inverse issue. An immediate application is functional linear model and one could apply our results directly in the plug-in method (Hall and Horowitz, 2007) to establish the convergence rate and phase transition. For complex regression models like functional generalized linear model and functional cox model, the approaches in this paper serve as the fundamental keystone for further explorations.

Another contribution of this paper is the double truncation technique in handling the uniform convergence. Due to the un-compactness of functional data, truncation is always needed to use the Bernstein inequality in getting the uniform convergence. The existing results of uniform convergence for covariance estimation require a strong moment condition on $X(t)$ and only work for the sparse function data where $N = o_p(n^{1/4})$. We propose new theoretical route and establish an improved bound for the truncated bias, which guarantees the uniform convergence of covariance and eigenfunctions for all sampling schemes under mild conditions. These asymptotic properties play a direct role in all kinds of statistical inference involving functional data (Yao et al., 2005a; Li and Hsing, 2010).

Because of the extra smoothness by integration, the eigenfunctions admit a one dimensional kernel smoothing rate. Inspired by this, a potential estimator of the covariance function could be

$$\tilde{C}(s, t) := \sum_{j=1}^{J_n} \hat{\lambda}_j \hat{\phi}_j(s) \hat{\phi}_j(t).$$

As mentioned in Li and Hsing (2010), it seems to be possible to choose $J_n \to \infty$ so that $\tilde{C}(s, t)$ has a faster rate. However, the truncated bias is dominating term in the most cases and $J_n$ should be large to balance the trade-off between bias and variance. Thus, the overall rate of $\||\tilde{C} - C||_{HS}$ is still slower than $\||\hat{C} - C||_{HS}$ even though the eigenpairs are well estimated. This also happens in the functional reduce rank model (Paul and Peng, 2009), where strong assumptions are needed to handle the truncation bias. The impacts of discrete observations and measurements error for reduce rank model in the general case is unknown but the results in this paper might be useful for further explorations.

Appendix
S.1 Proofs of the ancillary results

S.1.1 Proof of Theorem 4

**Proof of Theorem 4.** Let \( \chi(\rho) = \{ n^{-\rho}(i, j) : i, j \in \mathbb{Z} \cap (0, n^\rho) \} \) be an equally sized mesh on \([0, 1]^2\) with grid size \( n^{-\rho} \) by \( n^{-\rho} \). Then,

\[
\sup_{s, t \in [0, 1]} |\dot{C}(s, t) - \mathbb{E}\dot{C}(s, t)| \leq \sup_{(s, t) \in \chi(\rho)} |\dot{C}(s, t) - \mathbb{E}\dot{C}(s, t)| + D_1 + D_2 \tag{47}
\]

where

\[
D_1 = \sup_{(s_1, t_1), (s_2, t_2) \in [0, 1]^2 : |s_1 - s_2|, |t_1 - t_2| \leq n^{-\rho}} |\dot{C}(s_1, t_1) - \dot{C}(s_2, t_2)|,
\]

\[
D_2 = \sup_{(s_1, t_1), (s_2, t_2) \in [0, 1]^2 : |s_1 - s_2|, |t_1 - t_2| \leq n^{-\rho}} |\mathbb{E}\dot{C}(s_1, t_1) - \mathbb{E}\dot{C}(s_2, t_2)|.
\]

Recall \( \delta_{ijl} = X_{ij}X_{il} = \{ X_i(T_{ij}) + \varepsilon_{ij} \} \{ X_i(T_{il}) + \varepsilon_{il} \} \) and for all \( s_1, s_2, t_1, t_2 \in [0, 1] \)

\[
|\dot{C}(s_1, t_1) - \dot{C}(s_2, t_2)|
\leq \frac{1}{n} \frac{1}{N(N-1)} \frac{1}{h^2} \sum_{i=1}^{n} \sum_{1 \leq i_1 \neq i_2 \leq N} |\delta_{i_1 i_2}| \sup \left| K \left( \frac{T_{i_1} - s_1}{h} \right) - K^* \left( \frac{T_{i_1} - s_2}{h} \right) \right| K \left( \frac{T_{i_2} - t_1}{h} \right)
\]

\[+ \frac{1}{n} \frac{1}{N(N-1)} \frac{1}{h^2} \sum_{i=1}^{n} \sum_{1 \leq i_1 \neq i_2 \leq N} |\delta_{i_1 i_2}| \sup \left| K \left( \frac{T_{i_1} - t_1}{h} \right) - K^* \left( \frac{T_{i_1} - t_2}{h} \right) \right| K \left( \frac{T_{i_2} - s_2}{h} \right)
\]

\[\leq \frac{1}{n} \frac{1}{N(N-1)} \frac{1}{h^2} \sum_{i=1}^{n} \sum_{1 \leq i_1 \neq i_2 \leq N} \frac{|\delta_{i_1 i_2}|}{h} \left( |s_1 - s_2| + |t_1 - t_2| \right) = Z_1 \frac{|s_1 - s_2| + |t_1 - t_2|}{h^3},
\]

where

\[
Z_1 := \frac{1}{n} \frac{1}{N(N-1)} \sum_{i=1}^{n} \sum_{1 \leq i_1 \neq i_2 \leq N} |\delta_{i_1 i_2}| \text{ and } \mathbb{E}Z_1 = \mathbb{E}|\delta_{i_1 i_2}| \leq \text{Const.}.
\]

We start with the first term in the right hand side of equation (47), in order to use the Bernstein inequality, we need to do truncation on \( \dot{C}(s, t) \). Denote

\[
\tilde{C}(s, t) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{N(N-1)} \frac{1}{h^2} \sum_{1 \leq i_1 \neq i_2 \leq N} K \left( \frac{T_{i_1} - s}{h} \right) K \left( \frac{T_{i_2} - t}{h} \right) \delta_{i_1 i_2} 1_{(|\delta_{i_1 i_2}| \leq A_n)}
\]

where \( A_n \) is a positive constant we will define later. Then

\[
\sup_{(s, t) \in \chi(\rho)} |\dot{C}(s, t) - \mathbb{E}\dot{C}(s, t)| \leq \sup_{(s, t) \in \chi(\rho)} |\tilde{C}(s, t) - \mathbb{E}\tilde{C}(s, t)| + E_1 + E_2 \tag{48}
\]

with

\[
E_1 = \sup_{(s, t) \in \chi(\rho)} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{N(N-1)} \frac{1}{h^2} \sum_{1 \leq i_1 \neq i_2 \leq N} K \left( \frac{T_{i_1} - s}{h} \right) K \left( \frac{T_{i_2} - t}{h} \right) |\delta_{i_1 i_2}| 1_{(|\delta_{i_1 i_2}| > A_n)},
\]
\[ E_2 = \sup_{(s,t) \in \chi(\rho)} \mathbb{E} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{N(N-1)} h^2 \sum_{1 \leq l_1 \leq l_2 \leq N} K \left( \frac{T_{il_1} - s}{h} \right) K \left( \frac{T_{il_2} - t}{h} \right) |\delta_{il_1} \delta_{il_2}| 1_{|\delta_{il_1} \delta_{il_2}| > A_n}. \]

To bound \( E_1 \) and \( E_2 \), denote

\[ Z_2 := \frac{1}{n} \sum_{i=1}^{n} Z_{2,i}, \text{ with } Z_{2,i} := \frac{1}{N(N-1)} h^2 \sum_{1 \leq l_1 \neq l_2 \leq N} |\delta_{il_1} \delta_{il_2}|^{\alpha}, \]

and \( \mathbb{E}Z_2 = \mathbb{E}Z_{2,i} = \mathbb{E}|\delta_{il_1} \delta_{il_2}|^{\alpha} \leq \text{Const. by Assumption (U.1)}. \) Then

\[ \mathbb{E}|E_1 + E_2| \leq 2\mathbb{E}|E_1| \]
\[ \leq 2\text{Const.} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{N(N-1)} h^2 \sum_{1 \leq l_1 \neq l_2 \leq N} |\delta_{il_1} \delta_{il_2}|^{\alpha} 1_{|\delta_{il_1} \delta_{il_2}| > A_n} \]
\[ \leq \text{Const.} \frac{Z_2}{h^2 A_n^{\alpha-1}} \leq A_n^{-\alpha} h^{-2}. \]

For each \((s,t) \in \chi(\rho)\), denote

\[ L_i(s,t) = \frac{1}{n} \frac{1}{N(N-1)} h^2 \sum_{1 \leq l_1 \leq l_2 \leq N} K \left( \frac{T_{il_1} - s}{h} \right) K \left( \frac{T_{il_2} - t}{h} \right) \delta_{il_1} \delta_{il_2} 1_{|\delta_{il_1} \delta_{il_2}| \leq A_n}. \]

Then, \( \tilde{C}(s,t) = \sum_{i=1}^{n} L_i(s,t), \mathbb{E}|L_i(s,t)|^2 \leq n^{-2} \{ 1 + (Nh)^{-2} \}. \) A trivial bound for \( L_i(s,t) \) can be derived by

\[ |L_i(s,t)| \leq A_n \frac{1}{n} \frac{1}{N(N-1)} h^2 \sum_{1 \leq l_1 \leq l_2 \leq N} K \left( \frac{T_{il_1} - s}{h} \right) K \left( \frac{T_{il_2} - t}{h} \right) \]
\[ \leq A_n \frac{1}{n} \frac{1}{N(N-1)} h^2 \left| \sum_{1 \leq l \leq N} K \left( \frac{T_{il} - s}{h} \right) \right| \left| \sum_{1 \leq l \leq N} K \left( \frac{T_{il} - t}{h} \right) \right| \]
\[ = A_n \frac{1}{n} \frac{N}{N-1} J_i(s) J_i(t). \]

Note that this bound is not applicable for the dense and ultra dense case and we need to do an additional truncation. Start with the probability bound of \( J_i(s) > M \), note that \( \mathbb{E}h^{-1}K((T_{il} - s)/h) = 1, 0 \leq h^{-1}K((T_{il} - s)/h) \leq 2/h, \) for fixed \( M \geq 5, M' = M - 1, \) by Bernstein equality,

\[ \mathbb{P} \left( J_i(s) > M \right) = \mathbb{P} \left( \frac{1}{N} \sum_{1 \leq l \leq N} Y_l > M \right) = \mathbb{P} \left( \sum_{1 \leq l \leq N} (Y_l - 1) > M'N \right) \]
\[ \leq \exp \left( -\frac{(M'N)^2/2}{N} \mathbb{E}Y_l - 1^2 + \frac{2}{h} M'N/3 \right) \leq \exp \left( -\frac{(M'N)^2/2}{N} |2/h| M'N/3 \right) \]
\[ \leq \exp \left( -\frac{(M'N)^2/2}{|2/h| M'N(1/4 + 1/3)} \right) \leq \exp(-3M'N/7) \leq \exp(-MN/7). \]

Then

\[ \mathbb{P} \left( |L_i(s,t)| > A_n \frac{1}{n} \frac{NM^2}{N-1} \right) \leq \mathbb{P} \left( A_n \frac{1}{n} \frac{N}{N-1} J_i(s) J_i(t) > A_n \frac{1}{n} \frac{NM^2}{N-1} \right) \]
\[ = \mathbb{P} \left( J_i(s) J_i(t) > M^2 \right) \leq \mathbb{P} \left( J_i(s) > M \right) + \mathbb{P} \left( J_i(t) > M \right) \leq 2 \exp(-MN/3). \]
Denote \[ \tilde{L}_i(s, t) = L_i(s, t) \mathbf{1}_{(|L_i(s, t)| \leq \lambda_n \frac{N M^2}{N-1})}, \quad C_*(s, t) = \sum_{i=1}^{n} \tilde{L}_i(s, t). \]

Then \[ \sup_{(s, t) \in \chi(\rho)} |\tilde{C}(s, t) - \mathbb{E}\tilde{C}(s, t)| \leq \sup_{(s, t) \in \chi(\rho)} |C_*(s, t) - \mathbb{E}C_*(s, t)| + F_1 + F_2 \] \hspace{1cm} (49)

with
\[
F_1 = \sup_{(s, t) \in \chi(\rho)} \sum_{i=1}^{n} |L_i(s, t)| \mathbf{1}_{(|L_i(s, t)| > \lambda_n \frac{N M^2}{N-1})},
\]
\[
F_2 = \sup_{(s, t) \in \chi(\rho)} \mathbb{E} \sum_{i=1}^{n} |L_i(s, t)| \mathbf{1}_{(|L_i(s, t)| > \lambda_n \frac{N M^2}{N-1})}.
\]

By similar arguments, \( F_1 \) and \( F_2 \) can be bounded by
\[
\mathbb{E}|F_1 + F_2| \leq 2\mathbb{E}|F_1| \leq 2 \sum_{(s, t) \in \chi(\rho)} \sum_{i=1}^{n} \mathbb{E} \left[ |L_i(s, t)| \mathbf{1}_{(|L_i(s, t)| > \lambda_n \frac{N M^2}{N-1})} \right]^{1/2}
\]
\[
\leq 2 \sum_{(s, t) \in \chi(\rho)} \sum_{i=1}^{n} \mathbb{E} \left[ |L_i(s, t)|^2 \mathbb{P} \left( |L_i(s, t)| > \lambda_n \frac{N M^2}{N-1} \right) \right]^{1/2}
\]
\[
\leq \text{Const.} \sum_{(s, t) \in \chi(\rho)} \sum_{i=1}^{n} \left[ \frac{1}{n^2} \left( 1 + \frac{1}{N^2 h^2} \right) \exp(-M N h/3) \right]^{1/2}
\]
\[
\leq n^2 \rho \left( 1 + \frac{1}{N h} \right) \exp(-M N h/6).
\]

Then \[ |\tilde{L}_i(s, t) - \mathbb{E}\tilde{L}_i(s, t)| \leq 2A_n \frac{1}{n} \frac{N M^2}{N-1}, \text{ and} \]
\[
\sum_{i=1}^{n} \mathbb{E} \{ \tilde{L}_i(s, t) - \mathbb{E}\tilde{L}_i(s, t) \}^2 \leq \sum_{i=1}^{n} \mathbb{E} |\tilde{L}_i(s, t)|^2 \leq \sum_{i=1}^{n} \mathbb{E} |L_i(s, t)|^2 \leq \frac{1}{n} \left( 1 + \frac{1}{N^2 h^2} \right).
\]

By Bernstein inequality again, for \( \lambda > 0 \)
\[
\mathbb{P} \left( |C_*(s, t) - \mathbb{E}C_*(s, t)| > \lambda \right) = \mathbb{P} \left( \left| \sum_{i=1}^{n} (\tilde{L}_i(s, t) - \mathbb{E}\tilde{L}_i(s, t)) \right| > \lambda \right)
\]
\[
\leq 2 \exp \left( -\frac{\lambda^2/2}{\Sigma_{i=1}^{n} \mathbb{E} \{ \tilde{L}_i(s, t) - \mathbb{E}\tilde{L}_i(s, t) \}^2 + 2A_n \frac{1}{n} \frac{N M^2}{N-1} \lambda/3} \right)
\]
\[
\leq 2 \exp \left( -\frac{\lambda^2/2}{\text{Const.} \frac{1}{n} \left( 1 + \frac{1}{N^2 h^2} \right) + 2A_n \frac{1}{n} \frac{N M^2}{N-1} \lambda/3} \right).
\]
And a expectation bound for \(|C_s(s,t) - \mathbb{E}C_s(s,t)|\) can be obtained by
\[
\mathbb{E} \left[ \sup_{(s,t) \in \chi} |C_s(s,t) - \mathbb{E}C_s(s,t)| \right] = \int_0^\infty \mathbb{P} \left( \sup_{(s,t) \in \chi} |C_s(s,t) - \mathbb{E}C_s(s,t)| > \lambda \right) d\lambda
\leq \int_0^\infty \sum_{(s,t) \in \chi} \mathbb{P} \left( |C_s(s,t) - \mathbb{E}C_s(s,t)| > \lambda \right) d\lambda
\leq \int_0^\infty \min \left\{ 1, 2n^{2\rho} \exp \left( -\frac{\lambda^2/2}{\text{Const.} \frac{1}{n} \left( 1 + \frac{1}{N^2h^2} \right) + 2A_n \frac{1}{n} M^2 \rho \ln n} \right) \right\} d\lambda
\leq \text{Const.} \left[ \sqrt{\frac{\rho \ln n}{n}} \left( 1 + \frac{1}{Nh} \right) + A_n \frac{1}{n} M^2 \rho \ln n \right] + \text{Const.} n^{-\rho}h^{-3}
\leq \text{Const.} A_n^{1-\alpha} h^{-2} + \text{Const.} n^{2\rho} \left( 1 + \frac{1}{Nh} \right) \exp(-M Nh/6).
\]

Take \(M = 5 + 6(2\rho + 1)\ln Nh\), \(A_n = n^{1/\alpha} |M^2h^2\rho \ln n|^{-1/\alpha}\), \(\rho = 3s + 1\) \((h \geq n^{-s})\) then
\[
\mathbb{E} \left[ \sup_{s, t \in [0, 1]} |\hat{C}(s,t) - \mathbb{E}\hat{C}(s,t)| \right] \leq \text{Const.} \left[ \sqrt{\frac{\rho \ln n}{n}} \left( 1 + \frac{1}{Nh} \right) + \frac{M^2 \rho \ln n}{n} \right]^{1-\frac{1}{\alpha}} h^{-\frac{2}{\alpha}}
+ \text{Const.} n^{-\rho}h^{-3} \leq \text{Const.} \left[ \sqrt{\frac{\ln n}{n}} \left( 1 + \frac{1}{Nh} \right) + \frac{\ln n}{n} \right]^{1-\frac{1}{\alpha}} \left[ 1 + \frac{\ln n}{Nh} \right]^{\frac{2}{\alpha}} h^{-\frac{2}{\alpha}}.
\]

**S.1.2 Proof of Theorem 6**

**PROOF** **OF THEOREM 6.** By equation (5.22) in Hsing and Eubank (2015), \(\hat{\lambda}_j - \lambda_j\) admits the following expansion,
\[
\hat{\lambda}_j - \lambda_j = \langle \Delta \phi_j, \phi_j \rangle + \langle (\hat{\mathcal{P}}_j - \mathcal{P}_j)(\hat{C} - \lambda_j I)(\hat{\mathcal{P}}_j - \mathcal{P}_j)\phi_j, \phi_j \rangle,
\]
(50)
where \( \hat{P}_j = \hat{\phi}_j \otimes \hat{\phi}_j, \hat{P}_j = \phi_j \otimes \hat{\phi}_j \) and \( I \) is the identity transformation on \( L^2[0,1] \). By Lemma 5.1.7 and Taylor expansion of \( \sqrt{1-x} \) at 0,

\[
\|\hat{\phi}_j - \phi_j\|^2 = 2\left\{1 - (1 - \|\hat{P}_j - \hat{P}_j\|^2)^{1/2}\right\} = 2\left\{1 - 1 + \frac{\|\hat{P}_j - \hat{P}_j\|^2}{2} + o(\|\hat{P}_j - \hat{P}_j\|^2)\right\} = \|\hat{P}_j - \hat{P}_j\|^2 + o(\|\hat{P}_j - \hat{P}_j\|^2).
\]

(51)

Combine (50), (51) and Cauchy-Schwarz inequality,

\[
\lambda_j - \lambda_j = (\Delta \phi_j, \phi_j) + O(\|\hat{\phi}_j - \phi_j\|^2) + o(\|\hat{\phi}_j - \phi_j\|^2).
\]

(52)

We first focus on the asymptotic behavior of \( (\Delta \phi_j, \phi_j) \). For the bias term \( \mathbb{E}(\Delta \phi_j, \phi_j) \),

\[
\mathbb{E}(\Delta \phi_j, \phi_j) = \mathbb{E}\left\{\int X_i(u) \frac{1}{h} \int K\left(\frac{u-s}{h}\right) \phi_j(s) ds - \phi_j(u) \right\} = \int C(u,v) \frac{1}{h} \int K\left(\frac{u-s}{h}\right) \phi_j(s) ds - \phi_j(u) \right\} \phi_j(v) dv + 2 \int C(u,v) \left\{\frac{1}{h} \int K\left(\frac{u-s}{h}\right) \phi_j(s) ds - \phi_j(u) \right\} \phi_j(v) dv
\]

(53)

For each \( u \in [h,1-h] \),

\[
\frac{1}{h} \int K\left(\frac{u-s}{h}\right) \phi_j(s) ds - \phi_j(u) = \int_{-1}^{1} K(v) \left\{\phi_j(u-hv) - \phi_j(v)\right\} dv + \phi_j(u)
\]

(54)

\[
= \int_{-1}^{1} K(v) \left\{-hv\phi_j^{(1)}(u) + \frac{h^2v^2}{2} \phi_j^{(2)}(u) - \frac{h^3v^3}{3!} \phi_j^{(3)}(u) + \frac{h^4v^4}{4!} \phi_j^{(4)}(u)\right\} dv
\]

\[
= \frac{\sigma_K^2}{2} \phi_j^{(2)}(u) + o(h^2),
\]

where \( \sigma_K^2 = \int v^2 K(v) dv \) and the last equality holds under condition \( h j^c = o(1) \). For each \( u \in [0,h] \),

\[
\frac{1}{h} \int K\left(\frac{u-s}{h}\right) \phi_j(s) ds - \phi_j(u) = \int_{-1}^{h} K(v) \phi_j(u-hv) dv - \phi_j(v)
\]

(55)

\[
= -h \phi_j^{(1)}(u) \int_{-1}^{h} v K(v) dv + \frac{h^2}{2} \phi_j^{(2)}(u) \int_{-1}^{h} v^2 K(v) dv - \frac{h^3}{3!} \phi_j^{(3)}(u) \int_{-1}^{h} v^3 K(v) dv + o(h^2).
\]
Similarly, for each \( u \in [1 - h, 1] \),

\[
\frac{1}{h} \int K\left(\frac{u-s}{h}\right) \phi_j(s) \, ds - \phi_j(u) = - h \phi_j^{(1)}(u) \int_{u-\frac{1}{h}}^1 vK(v) \, dv + \frac{h^2}{2} \phi_j^{(2)}(u) \int_{u-\frac{1}{h}}^1 v^2 K(v) \, dv
\]

\[
- \frac{h^3}{3!} \phi_j^{(3)}(u) \int_{u-\frac{1}{h}}^1 v^3 K(v) \, dv + o(h^2).
\]

Combine equation (54)–(56), for each \( k \in \mathbb{N}_+ \)

\[
\int_0^1 \left\{ \frac{1}{h} \int K\left(\frac{u-s}{h}\right) \phi_j(s) \, ds - \phi_j(u) \right\} \phi_k(u) \, du
\]

\[
= \int_{-h}^h \frac{\sigma_k^2 h^2}{2} \phi_j^{(2)}(u) \phi_k(u) \, du - h \int_0^h \left\{ \int_{-1}^1 vK(v) \, dv \right\} \phi_j^{(1)}(u) \phi_k(u) \, du
\]

\[
- h \int_{-1-h}^1 \left\{ \int_{u-\frac{1}{h}}^1 vK(v) \, dv \right\} \phi_j^{(1)}(u) \phi_k(u) \, du + o(h^2),
\]

where the last equality is due to

\[
\int_0^h \frac{h^2}{2} \phi_j^{(2)}(u) \int_{-1}^1 v^2 K(v) \, dv \phi_k(u) \, du \lesssim h^3 \int_{-1}^1 v^2 K(v) \, dv \phi_k(u) \, du \lesssim o(h^2)
\]

under \( h j^c = o(1) \). Combine equation (53) and (57), under \( h(\frac{j^c}{2} + \frac{j^a}{2}) = o(1) \), the bias term is derived by

\[
\mathbb{E}(\Delta \phi_j, \phi_j)
\]

\[
\leq \sum_{k=1}^\infty \lambda_k \left[ \int_0^1 \left\{ \frac{1}{h} \int K\left(\frac{u-s}{h}\right) \phi_j(s) \, ds - \phi_j(u) \right\} \phi_k(u) \, du \right]^2 + 2\lambda \int_0^1 \left\{ \frac{1}{h} \int K\left(\frac{u-s}{h}\right) \phi_j(s) \, ds - \phi_j(u) \right\} \phi_j(u) \, du
\]

\[
= 2\lambda \sigma_k^2 h^2 \int_{-h}^{1-h} \phi_j^{(2)}(u) \phi_j(u) \, du + o(j^{-a} h^2).
\]

Next we focus on \( \text{Var}(\Delta \phi_j, \phi_j) \),

\[
\text{Var}(\Delta \phi_j, \phi_j) = \frac{1}{n} \text{Var} \left[ \frac{1}{N(N-1)} \sum_{i_1 \neq i_2} \{ \delta_{d'i_1} T_h \phi_j(T_{d'i_1}) T_h \phi_j(T_{d'i_2}) - \lambda_j \} \right]
\]

\[
= \frac{1}{n} \mathbb{E} \left[ \left( \frac{1}{N(N-1)} \sum_{i_1 \neq i_2} \{ \delta_{d'i_1} T_h \phi_j(T_{d'i_1}) T_h \phi_j(T_{d'i_2}) - \lambda_j \} \right)^2 \right]
\]

\[
- \frac{1}{n} \left( \mathbb{E}(\Delta \phi_j, \phi_j) \right)^2.
\]

33
For the first term in the right hand side of equation (59),

\[
\mathbb{E}\left(\left[\frac{1}{N(N-1)} \sum_{t_1 \neq t_2} \delta_{t_1 t_2} T_h \phi_j(T_{t_1}) T_h \phi_j(T_{t_2}) - \lambda_j \right]^2\right)
\]

\[
= \mathbb{E}\left(\left[\frac{1}{N(N-1)} \sum_{t_1 \neq t_2} \delta_{t_1 t_2} T_h \phi_j(T_{t_1}) T_h \phi_j(T_{t_2}) \right]^2\right) - 2\lambda_j \mathbb{E}\left(\left[\frac{1}{N(N-1)} \sum_{t_1 \neq t_2} \delta_{t_1 t_2} T_h \phi_j(T_{t_1}) T_h \phi_j(T_{t_2}) \right]\right) + \lambda_j^2.
\]

By similar calculation of equation (14),

\[
\mathbb{E}\left[\left[\frac{1}{N(N-1)} \sum_{t_1 \neq t_2} \delta_{t_1 t_2} T_h \phi_j(T_{t_1}) T_h \phi_j(T_{t_2}) \right]^2\right]
\]

\[
= \frac{4! \binom{N}{4} B_1 + 3! \binom{N}{3} B_2 + 2! \binom{N}{2} B_3}{N^2 (N-1)^2}
\]

with

\[
B_1 = \mathbb{E}\left[\left[\int X(u) T_h \phi_j(u) \right]^4\right]
\]

\[
B_2 = 4 \mathbb{E}\left[\left[\int X(u) T_h \phi_j(u) \right]^2 \left[\int \{ X^2(u) + \sigma^2_X \} T_h \phi_j^2(u) \right]\right]
\]

\[
B_3 = 2 \mathbb{E}\left[\left[\int \{ X^2(u) + \sigma^2_X \} T_h \phi_j^3(u) \right]^2\right].
\]

For \( B_1 \),

\[
\left[\int X(u) T_h \phi_j(u) \right]^4 = \xi_j^4 + (X, T_h \phi_j - \phi_j)^4 + 6 \xi_j^2 (X, T_h \phi_j - \phi_j)^2 + 4 \xi_j (X, T_h \phi_j - \phi_j)^3 + 4 \xi_j^3 (X, T_h \phi_j - \phi_j).
\]

Note that \( \mathbb{E}(\xi_j^4) \geq \{ \mathbb{E}(\xi_j^2)\}^2 = \lambda_j^2 \) and

\[
\mathbb{E}( (X, T_h \phi_j - \phi_j)^4 ) \leq \mathbb{E}( \|X\|^4 ) \|T_h \phi_j - \phi_j\|^4 \leq h^8 J^{4c} = o(j^{-2a});
\]

\[
\mathbb{E}(\xi_j^2 (X, T_h \phi_j - \phi_j)^2) \leq \sqrt{\mathbb{E}(\xi_j^4) \mathbb{E}( (X, T_h \phi_j - \phi_j)^4 )} = o(j^{-2a});
\]

\[
\mathbb{E}(\xi_j (X, T_h \phi_j - \phi_j)^3) \leq \sqrt{\mathbb{E}(\xi_j^2) \mathbb{E}( \|X\|^6 ) \mathbb{E}( (X, T_h \phi_j - \phi_j)^6 )} = o(j^{-2a});
\]

\[
\mathbb{E}(\xi_j^3 (X, T_h \phi_j - \phi_j)) \leq \sqrt{\mathbb{E}(\xi_j^4) \mathbb{E}(\xi_j^2 (X, T_h \phi_j - \phi_j)^2)} = o(j^{-2a}).
\]

Thus,

\[
B_1 = \mathbb{E}(\xi_j^4) + o(\mathbb{E}(\xi_j^4)).
\]
For $B_2$, we start with $\mathbb{E}((X,T_h \phi_j)^2 \| X T_h \phi_j \|^2)$,

$$
\mathbb{E}((X,T_h \phi_j)^2 \| X T_h \phi_j \|^2) = \mathbb{E} \{(X, \phi_j) + (X,T_h \phi_j - \phi_j))^2 \| X T_h \phi_j \|^2 \}
$$

$$
= \mathbb{E}(\xi_j^2 \| X T_h \phi_j \|^2) + \mathbb{E}((X,T_h \phi_j - \phi_j)^2 \| X T_h \phi_j \|^2) + 2 \mathbb{E}(\xi_j (X,T_h \phi_j - \phi_j) \| X T_h \phi_j \|^2). 
$$

(64)

For the first term in the right hand side of equation (64),

$$
\mathbb{E}(\xi_j^2 \| X T_h \phi_j \|^2)
$$

$$
= \mathbb{E}(\xi_j^2 \| X \|^2) + \mathbb{E}(\xi_j^2 (X(T_h \phi_j - \phi_j))^2) + 2 \mathbb{E}(\xi_j^2 (X \phi_j, X(T_h \phi_j - \phi_j))).
$$

Under Assumption (N.2),

$$
\mathbb{E}(\xi_j^2 \| X \phi_j \|^2) = \mathbb{E} \left\{ \xi_j^2 \int_1^{\infty} \xi_k^2 \phi_k(u) \phi_j^2(u) du \right\} \geq \text{Const.} \lambda_j
$$

and the remaining terms in the right hand side of equation (64) admit

$$
\mathbb{E}(\xi_j^2 \| X(T_h \phi_j - \phi_j) \|^2) \leq \sqrt{\mathbb{E}(\xi_j^4) \mathbb{E}(\| X \|^4)}\| T_h \phi_j - \phi_j \|^2 = o(\lambda_j);
$$

$$
\mathbb{E}(\xi_j^2 (X \phi_j, X(T_h \phi_j - \phi_j))) \leq \mathbb{E}(\xi_j^2 \| X \phi_j \| \| T_h \phi_j - \phi_j \|) = o(\lambda_j).
$$

Thus,

$$
\mathbb{E}(\xi_j^2 \| X T_h \phi_j \|^2) = \mathbb{E}(\xi_j^2 \| X \phi_j \|^2) + o(\mathbb{E}(\xi_j^2 \| X \phi_j \|^2)).
$$

(65)

For the last two terms in (64),

$$
\mathbb{E}((X,T_h \phi_j - \phi_j)^2 \| X T_h \phi_j \|^2) \leq \mathbb{E}(\| X \|^4)\| T_h \phi_j - \phi_j \|^2 \| T_h \phi_j \|^2
$$

$$
\leq h^4 j^2c = o(j^{-a});
$$

$$
\mathbb{E}(\xi_j (X,T_h \phi_j - \phi_j) \| X T_h \phi_j \|^2) \leq \sqrt{\mathbb{E}(\xi_j^2) \mathbb{E}(\| X \|^6)}\| T_h \phi_j - \phi_j \|\| T_h \phi_j \|^2
$$

$$
\leq j^{-\frac{3}{2}} h^2 j^c = o(j^{-a}).
$$

(66)

Combine (64) and (66),

$$
\mathbb{E}((X,T_h \phi_j)^2 \| X T_h \phi_j \|^2) = \mathbb{E}(\xi_j^2 \| X \phi_j \|^2) + o(\mathbb{E}(\xi_j^2 \| X \phi_j \|^2)).
$$

(67)

By similar arguments there is

$$
\mathbb{E}((X,T_h \phi_j)^2 \| T_h \phi_j \|^2) = \mathbb{E}(\xi_j^2) + o(\mathbb{E}(\xi_j^2)).
$$

Thus,

$$
B_2 = 4 \mathbb{E}(\xi_j^2 (\| X \phi_j \|^2 + \sigma_X^2)) + o(\mathbb{E}(\xi_j^2 (\| X \phi_j \|^2 + \sigma_X^2))).
$$

(68)

For $B_3$, note that

$$
B_3 = 2 \mathbb{E} \left[ \left\{ \int (X^2(u) + \sigma_X^2)T_h \phi_j^2(u) du \right\}^2 \right]
$$

$$
= 2 \mathbb{E} \left[ \| XT_h \phi_j \|^2 + \sigma_X^2 \| T_h \phi_j \|^2 \right]^2
$$

$$
= 2 \mathbb{E} \left( \| XT_h \phi_j \|^2 + \sigma_X^2 \right)^2 + o(1).
$$

(69)
Combine equation (61), (63), (68) and (69),
\[
\mathbb{E} \left( \frac{1}{N(N-1)} \sum_{i_1 \neq i_2} \delta_{i_1i_2} T_h \phi_j(T_{i_1}) T_h \phi_j(T_{i_2}) \right)^2 \\
= \{1 + o(1)\} \left\{ \frac{(N-2)(N-3)}{N(N-1)} \mathbb{E} \xi_j^4 + \frac{4(N-2)}{N(N-1)} \mathbb{E} \xi_j^2 (\|X \phi_j\|^2 + \sigma_X^2) \right\} \\
+ \frac{2}{N(N-1)} \mathbb{E} \left\{ (\|X \phi_j\|^2 + \sigma_X^2)^2 \right\}.
\]  
(70)

For the last two terms in equation (60), by equation (58)
\[
\mathbb{E} \left( \frac{1}{N(N-1)} \sum_{i_1 \neq i_2} \delta_{i_1i_2} T_h \phi_j(T_{i_1}) T_h \phi_j(T_{i_2}) \right) = \lambda_j + o(j^{-a}).
\]
Thus
\[
\frac{1}{n} \mathbb{E} \left( \left( \frac{1}{N(N-1)} \sum_{i_1 \neq i_2} \delta_{i_1i_2} T_h \phi_j(T_{i_1}) T_h \phi_j(T_{i_2}) - \lambda_j \right)^2 \right) = \Sigma'_n + o(\Sigma'_n) 
\]  
(71)

with
\[
\Sigma'_n = \frac{(N-2)(N-3)}{N(N-1)} \mathbb{E} \xi_j^4 - \lambda_j^2 + \frac{4(N-2)}{N(N-1)} \mathbb{E} \xi_j^2 (\|X \phi_j\|^2 + \sigma_X^2) \\
+ \frac{2}{N(N-1)} \mathbb{E} \left\{ (\|X \phi_j\|^2 + \sigma_X^2)^2 \right\}.
\]

The proof is complete by combing equation (51), (58), (71) and \(\Sigma_n^{-1/2} \|\hat{\phi}_j - \phi_j\| = o_p(1)\) under Assumption (M.2).

**S.1.3 Proofs of lemmas**

**Proof Proof of Lemma 7.** Note that \(\mathbb{E} \hat{C} = T_h C T_h\) and by the boundedness of \(T_h\),
\[
\|\mathbb{E} \hat{C} - C\|_{\text{HS}} = \|T_h C T_h - C\|_{\text{HS}} \leq \|T_h (C T_h - C)\|_{\text{HS}} + \|T_h C - C\|_{\text{HS}} \leq \|C T_h - C\|_{\text{HS}} + \|T_h C - C\|_{\text{HS}}.
\]

As
\[
C = \sum_{k=1}^{\infty} \lambda_k \phi_k \otimes \phi_k, \quad CT_h = \sum_{k=1}^{\infty} \lambda_k \phi_k \otimes T_h \phi_k, \quad T_h C = \sum_{k=1}^{\infty} \lambda_k T_h \phi_k \otimes \phi_k.
\]

We have
\[
\|C T_h - C\|_{\text{HS}}^2 = \|T_h C - C\|_{\text{HS}}^2 \leq \sum_{k=1}^{\infty} \lambda_k^2 \|T_h \phi_k - \phi_k\|^2 \leq C \sum_{k=1}^{\infty} \lambda_k^2 [\min(1, h^2 k^c)]^2
\]
\[
\leq C \sum_{k=1}^{\infty} k^{-2a} \min(1, h^4 k^{2c}) \leq C \left\{ \begin{array}{ll}
h^4, & 2a - 2c > 1 \\
h^4 \ln h, & 2a - 2c = 1 \\
h^{2(2a-1)/c}, & 2a - 2c < 1 \end{array} \right. = c_{a,c}.
\]

36
Furthermore, if $\partial^2 C(s, t)/\partial s^2$ is bounded on $[0, 1]^2$ then $\|CT_h - C\|_{HS}^2 = \|T_h C - C\|_{HS}^2 \leq Ch^4$ and thus $\|E\hat{C} - C\|_{HS}^2 \leq Ch^4$ without the assumption on $2a - 2c$.

**Proof of Lemma 8.** Let $\xi_k = (X, \phi_k)$, then $X = \sum_{k=1}^{\infty} \xi_k \phi_k$, $\|X\|^2 = \sum_{k=1}^{\infty} \xi_k^2$. By Minkowski Inequality and assumptions (A.1),

$$
\left( \mathbb{E} \|X\|^4 \right)^{\frac{1}{2}} = \left\{ \mathbb{E} \left( \sum_{k=1}^{\infty} \xi_k^2 \right) \right\}^{\frac{1}{2}} \leq \sum_{k=1}^{\infty} \left( \mathbb{E} \xi_k^4 \right)^{\frac{1}{2}} \leq \sum_{k=1}^{\infty} C \lambda_k < \infty.
$$

Notice that

$$
(X, T_h \phi_k) = (X, \phi_k) + (X, T_h \phi_k - \phi_k) = \xi_k + (X, T_h \phi_k - \phi_k),
$$

$$
|\langle X, T_h \phi_k \rangle| \leq \|\xi_k\| + \|X\| \|T_h \phi_k - \phi_k\| \leq |\xi_k| + Ch^2 k^c \|X\|.
$$

Then for $1 \leq k \leq 2j$ we have

$$
\mathbb{E} \left( |\langle X, T_h \phi_k \rangle|^{4} \right) \leq C \left\{ \mathbb{E} \xi_k^4 + (h^4 k^{2c})^2 \mathbb{E} \|X\|^4 \right\} \leq C \left\{ \lambda_k^2 + (h^4 k^{2c})^2 \right\} = C k^{-2a} \left\{ 1 + (h^4 k^{2c+a})^2 \right\} \leq C k^{-2a}.
$$

We write $X = X_{(l)} + X_{(h)}$, $X_{(l)} = \sum_{k=1}^{j} \xi_k \phi_k$, $X_{(h)} = \sum_{k> j} \xi_k \phi_k$. Then $\|X_{(h)}\|^2 = \sum_{k> j} \xi_k^2$. For $k > j$ we have $(X_{(l)}, \phi_k) = 0$ and

$$
(X, T_h \phi_k) = (T_h X, \phi_k) = (T_h X, \phi_k) - (X_{(l)}, \phi_k) = \langle T_h X - X_{(l)}, \phi_k \rangle.
$$

Thus

$$
\sum_{k>j} |\langle X, T_h \phi_k \rangle|^2 \leq \|T_h X - X_{(l)}\|^2 = \|T_h X_{(l)} + T_h X_{(h)} - X_{(l)}\|^2
$$

$$
\leq (\|T_h X_{(l)} - X_{(l)}\| + \|T_h X_{(h)}\|)^2 \leq (\|T_h X_{(l)} - X_{(l)}\| + \|X_{(h)}\|)^2.
$$

By Minkowski Inequality we have

$$
\left( \mathbb{E} \|X_{(h)}\|_{L^2}^4 \right)^{\frac{1}{2}} = \left\{ \mathbb{E} \left( \sum_{k>j} \xi_k^2 \right) \right\}^{\frac{1}{2}} \leq \sum_{k>j} \left( \mathbb{E} \xi_k^4 \right)^{\frac{1}{2}} \leq \sum_{k>j} C \lambda_k \leq C j^{1-a}.
$$

As $X_{(l)} = \sum_{k=1}^{j} \xi_k \phi_k$ we have $T_h X_{(l)} - X_{(l)} = \sum_{k=1}^{j} \xi_k (T_h \phi_k - \phi_k)$ and

$$
\|T_h X_{(l)} - X_{(l)}\| \leq \sum_{k=1}^{j} |\xi_k| \|T_h \phi_k - \phi_k\| \leq C \sum_{k=1}^{j} |\xi_k| h^2 k^c
$$

$$
\left( \mathbb{E} \|T_h X_{(l)} - X_{(l)}\|^4 \right)^{\frac{1}{4}} \leq C \sum_{k=1}^{j} \left( \mathbb{E} \xi_k^4 \right)^{\frac{1}{4}} h^2 k^c \leq C \sum_{k=1}^{j} \lambda_k^{\frac{1}{4}} h^2 k^c \leq C \sum_{k=1}^{j} h^2 k^c
$$

$$
\leq C h^2 j^{c+1} = C j^{1-a} (h^4 j^{2c+2a})^{1/2} \leq C j^{1-a}.
$$
Summing all these terms up, we conclude that
\[
\mathbb{E}\left( \sum_{k,j}|(X, T_h \phi_k)|^2 \right)^2 \leq \mathbb{E}\left( \|T_h X_{(l)} - X_{(l)}\| + \|X_{(h)}\| \right)^4 \leq j^{-2a}.
\]

**Proof Proof of Lemma 9.** As \((\hat{\lambda}_j - \lambda_k)(\hat{\phi}_j, \phi_k) = \langle \hat{C} \hat{\phi}_j, \phi_k \rangle = \langle \Delta \hat{\phi}_j, \phi_k \rangle = \langle \Delta \hat{\phi}_j, \phi_k \rangle = \langle \Delta(\hat{\phi}_j - \phi_j), \phi_k \rangle, \) by (25), we have
\[
\hat{\phi}_j - \phi_j = \sum_{k \neq j} \frac{\langle \Delta \hat{\phi}_j, \phi_k \rangle}{\lambda_j - \lambda_k} \phi_k + \sum_{k \neq j} \frac{\lambda_k\langle \Delta \hat{\phi}_j - \phi_k, \phi_k \rangle}{\lambda_j - \lambda_k} + \left\{ \int (\hat{\phi}_j - \phi_j) \phi_j \right\} 
\]
\[
:= I_1 + I_2 + I_3.
\]
The last term in equation (72) is bounded by
\[
\|I_3\|_\infty \leq \|\hat{\phi}_j - \phi_j\|_2 \|\phi_j\|_2 \|\phi_j\|_\infty \leq \text{Const.} \|\hat{\phi}_j - \phi_j\|_2.
\]
For \(I_2\), note that
\[
I_2 = \frac{\Delta(\hat{\phi}_j - \phi_j)}{\lambda_j} + \sum_{k \neq j} \frac{\lambda_k\langle \Delta \hat{\phi}_j - \phi_k, \phi_k \rangle}{\lambda_j(\lambda_j - \lambda_k)} \phi_k := I_{2,1} + I_{2,2},
\]
where
\[
\|I_{2,1}\|_\infty = \|\Delta(\hat{\phi}_j - \phi_j)\|_\infty \leq \|\Delta\|_\infty \|\hat{\phi}_j - \phi_j\|_2 / \lambda_j \leq \text{Const.} \|\Delta\|_\infty \|\hat{\phi}_j - \phi_j\|_2 j^a,
\]
and
\[
\|I_{2,2}\|_\infty \leq \sum_{k \neq j} \frac{\lambda_k |\langle \Delta \hat{\phi}_j - \phi_k, \phi_k \rangle|}{\lambda_j |\lambda_j - \lambda_k|} \|\phi_k\|_\infty \leq \text{Const.} \sum_{k \neq j} \frac{\lambda_k |\langle \Delta \hat{\phi}_j - \phi_k, \phi_k \rangle|}{\lambda_j |\lambda_j - \lambda_k|}
\]
\[
\leq \text{Const.} \left( \sum_{k \neq j} |\langle \Delta \hat{\phi}_j - \phi_k, \phi_k \rangle|^2 \right)^{1/2} \left( \sum_{k \neq j} \frac{\lambda_k^2}{\lambda_j^2 |\lambda_j - \lambda_k|} \right)^{1/2} \leq \text{Const.} \|\Delta(\hat{\phi}_j - \phi_j)\|_2 j^{a+1}
\]
\[
\leq \text{Const.} \|\Delta\|_{\text{HS}} \|\hat{\phi}_j - \phi_j\|_2 j^{a+1}.
\]
Next, we will focus on \(I_1\), which is the dominating term, by similar arguments,
\[
I_1 = \sum_{k \neq j \leq 2j} \frac{\langle \Delta \phi_j, \phi_k \rangle}{\lambda_j - \lambda_k} \phi_k + \sum_{k = 2j}^{\infty} \frac{\lambda_k \langle \Delta \phi_j, \phi_k \rangle}{\lambda_j(\lambda_j - \lambda_k)} \phi_k + \sum_{k = 2j}^{\infty} \frac{\langle \Delta \phi_j, \phi_k \rangle}{\lambda_j} \phi_k
\]
\[
:= I_{1,1} + I_{1,2} + I_{1,3}.
\]
For \(I_{1,1}\) and \(I_{1,2}\), by Theorem 1, there are
\[
\|I_{1,1}\|_\infty \leq \sum_{k \neq j \leq 2j} \frac{|\langle \Delta \phi_j, \phi_k \rangle|}{|\lambda_j - \lambda_k|} \|\phi_k\|_\infty \leq \sum_{k \neq j \leq 2j} \frac{|\langle \Delta \phi_j, \phi_k \rangle|}{|\lambda_j - \lambda_k|}
\]
\[
\lesssim \sum_{j/2 < k \leq 2j} k^{a+1} \frac{1}{|k - j|} \left\{ \frac{1}{\sqrt{n}} \left( j^{\frac{a}{2}} k^{-\frac{a}{2}} + \frac{j^{-\frac{a}{2}} + k^{-\frac{a}{2}}}{\sqrt{N}} + \frac{1}{N} \right) + h^2 k^{c-a} \right\}
\]
\[
= O_p \left( \frac{j \log j}{\sqrt{n}} + \frac{1}{\sqrt{N}} j^{\frac{a}{2}+1} \log j + \frac{1}{\sqrt{nN}} j^{a+1} \log j + h^2 j^{c+1} \log j \right)
\]
and
\[ \|I_{1,2}\|_\infty \leq \sum_{k=2j}^{\infty} \frac{\lambda_k}{\Delta \lambda_j} \|\phi_k\|_\infty \leq \left( \sum_{k=2j}^{\infty} \left| (\Delta \phi_j, \phi_k) \right|^2 \right)^{\frac{1}{2}} \left( \sum_{k=2j}^{\infty} \frac{\lambda_k^2}{\lambda_j - \lambda_k} \right)^{\frac{1}{2}} \]
\[ \leq \left( \sum_{k=2j}^{\infty} \left| (\Delta \phi_j, \phi_k) \right|^2 \right)^{\frac{1}{2}} j^{-a+1/2} \sim O_P \left( \frac{1}{\sqrt{n}} \left( j + \frac{\sqrt{\lambda_j} a + j^{a+1} + j^{a+\frac{1}{2}}}{\sqrt{\tilde{N}}} \right) + h^2 j^{-c} \right). \]

In summary, there are
\[ \hat{\phi}_j - \phi_j = I_1 + I_2 + I_3 = I_{1,1} + I_{1,2} + I_{1,3} + I_{2,1} + I_{2,2} + I_3 = \phi_{j,0} + \lambda_j^{-1} \phi_{j,1}, \]
\[ \phi_{j,0} := I_{1,1} + I_{1,2} + I_{1,3} + I_{2,2} + I_3, \quad \phi_{j,1} := \lambda_j I_{1,3} = \sum_{k=2j}^{\infty} (\Delta \phi_j, \phi_k) \phi_k. \]

On the high probability set \( \Omega_u \), \( \|\Delta\|_\infty j^a + \|\Delta\|_{HS} j^{a+1} = O(1) \) then
\[ \|\phi_{j,0}\|_\infty \leq \|I_{1,1}\|_\infty + \|I_{1,2}\|_\infty + \|I_{2,1}\|_\infty + \|I_{2,2}\|_\infty + \|I_3\|_\infty \]
\[ \leq \|I_{1,1}\|_\infty + \|I_{1,2}\|_\infty + \text{Const.} (1 + \|\Delta\|_\infty j^a + \|\Delta\|_{HS} j^{a+1}) \|\hat{\phi}_j - \phi_j\|_2 \]
\[ \leq \|I_{1,1}\|_\infty + \|I_{1,2}\|_\infty + \text{Const.} \|\hat{\phi}_j - \phi_j\|_2 \]
\[ = O_P \left( \frac{j \log j}{\sqrt{n}} + \frac{1}{\sqrt{nN}} \left( j^{\frac{a+1}{2}} \log j + h^{-\frac{1}{2}} j^{a+\frac{1}{2}} \right) \right) \]
\[ + \frac{1}{\sqrt{nN}} j^{a+\frac{1}{2}} \log j + h^2 j^{-c} \log j \]

which completes the proof.

**Proof of Lemma 10.** For the first statement in 10, note that \( P_{cm} = \sum_{k=1}^{m-1} (g, \phi_k) \phi_k \).

\[ \|P_{cm}\|_2^2 = \sum_{k=1}^{m-1} |(g, \phi_k)|^2 \leq \sum_{k=1}^{m-1} \|g\|_2^2 \|\phi_k\|_\infty \leq \sum_{k=1}^{m-1} \text{Const.} \leq m \text{Const.}, \]
\[ \|P_{cm}\|_\infty \leq \sum_{k=1}^{m-1} |(g, \phi_k)| \|\phi_k\|_\infty \leq \sum_{k=1}^{m-1} \|g\|_1 \|\phi_k\|_\infty \leq \sum_{k=1}^{m-1} \text{Const.} \leq m \text{Const.}, \]
\[ \|P_{2cm}\|_1 = \|g - P_{2cm}\|_1 \leq \|g\|_1 + \|P_{2cm}\|_2 \leq 1 + (2 \text{Const.})^{1/2} \leq \text{Const.}^{1/2}. \]

For the second statement, note that
\[ |(C(T_k \phi_j - \phi_j), \phi_k)| = \lambda_k |(T_k \phi_j - \phi_j, \phi_k)| \leq \lambda_k \|T_k \phi_j - \phi_j\|_2 \|\phi_k\|_2 \leq \text{Const.} k^{-a} h^2 j^{c}, \]
and for all \( f \in L^2 \),

\[ \| (T_h C - C) f \|_\infty = \left\| \sum_{k=1}^{\infty} \lambda_k (T_h \phi_k - \phi_k) (f, \phi_k) \right\|_\infty \leq \sum_{k=1}^{\infty} \lambda_k \| T_h \phi_k - \phi_k \|_\infty (f, \phi_k) \]

\[ \leq \text{Const.} \sum_{k=1}^{\infty} k^{-a} \min(1, h^2 k^c) \| f, \phi_k \| \]

\[ \leq \text{Const.} \left[ \sum_{k=1}^{\infty} k^{-2a} \min(1, h^4 k^{2c}) \right]^{1/2} \left[ \sum_{k=1}^{\infty} \| (f, \phi_k) \|_2^2 \right]^{1/2} \leq \text{Const.} j^{1/2-a} \| f \|_2. \]

Recall \( \mathbb{E} \hat{C} = T_h C T_h \) and \( \| g \|_1 = 1 \) for \( g \in \chi_1(\rho) \), we have,

\[ \{ \mathbb{E} \hat{C} \phi_j, P_{2^j g} \} = \{ T_h C T_h \phi_j, P_{2^j g} \} = \{ (T_h C - C)(T_h \phi_j - \phi_j), P_{2^j g} \} \]

\[ + (C(T_h \phi_j - \phi_j), P_{2^j g}) + \{ (T_h C - C)\phi_j, P_{2^j g} \}. \]

For the last three terms in the right hand side of last equation, there are

\[ \| (T_h C - C)\phi_j, P_{2^j g} \| \leq \lambda_j \| T_h \phi_j - \phi_j \|_\infty \| P_{2^j g} \|_1 \]

\[ \leq \text{Const.} \lambda_j h^2 j^{-1/2} \leq \text{Const.} h^2 j^{c+1/2-a}, \]

\[ \| (C(T_h \phi_j - \phi_j), P_{2^j g}) \| = \left\| \sum_{|k|=2^j} (C(T_h \phi_j - \phi_j), \phi_k) (\phi_k, g) \right\| \]

\[ \leq \sum_{|k|=2^j} \| (C(T_h \phi_j - \phi_j), \phi_k) \|_{\phi_k} \| g \|_1 \leq \text{Const.} \sum_{|k|=2^j} k^{-a} h^2 j^c \leq \text{Const.} h^2 j^{c+1-a} \]

and

\[ \| (T_h C - C)(T_h \phi_j - \phi_j), P_{2^j g} \| \leq \| (T_h C - C)(T_h \phi_j - \phi_j) \|_\infty \| P_{2^j g} \|_1 \]

\[ \leq \text{Const.} j^{1/2-a} \| T_h \phi_j - \phi_j \|_2 \| P_{2^j g} \|_1 \leq \text{Const.} j^{1/2-a} h^2 j^{c-1/2} = \text{Const.} h^2 j^{c+1-a}. \]

The proof is complete by summing up the above.

**Proof Proof of Lemma 11.** For \( E_1 \),

\[ E_1 = \sup_{g \in \chi_1(\rho)} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{N(N - 1)} \sum_{1 \leq l_1 \neq l_2 \leq N} \| T_h \phi_j(T_{il_1} T_h P_{2^j g}(T_{il_2}) \|_1 \| (\delta_{il_1 l_2} > A_n) \]

\[ \leq \sup_{g \in \chi_1(\rho)} \frac{1}{n} \sum_{i=1}^{n} \frac{\| T_h \phi_j \|_\infty \| T_h P_{2^j g} \|_\infty}{N(N - 1)} \sum_{1 \leq l_1 \neq l_2 \leq N} \| \delta_{il_1 l_2} \|_1 \| (\delta_{il_1 l_2} > A_n) \]

\[ \leq \sup_{g \in \chi_1(\rho)} \| T_h \phi_j \|_\infty \| T_h P_{2^j g} \|_\infty \frac{Z_2}{A_n^{z_1}} \leq \frac{\text{Const.} Z_2}{h A_n^{z_1}}, \]

40
where
\[
Z_2 := \frac{1}{n} \sum_{i=1}^{n} \frac{1}{N(N-1)} \sum_{1 \leq l_1 \neq l_2 \leq N} |\delta_{l_1l_2}|^a \text{ and } \mathbb{E}Z_2 = \mathbb{E}|\delta_{l_1l_2}|^a \leq \text{Const.}
\]

For $F_1$,
\[
\mathbb{E}|F_1| \leq \sum_{g \in \chi_{1}(\rho)} \mathbb{E}\left[ C^{x \Phi}_{x}^{M}(\phi_j, P_{z_{2j}}g) \right] = \sum_{g \in \chi_{1}(\rho)} \frac{\mathbb{E}\left[ \tilde{A}_i(f, P_{z_{2j}}g) 1_{(|\tilde{A}_i(f, P_{z_{2j}}g)| > A_n N^2 \|f\|_\infty M) \right]}{N(N-1)} \leq \sum_{g \in \chi_{1}(\rho)} \left[ \frac{\mathbb{E}\left[ \tilde{A}_i(\phi_j, P_{z_{2j}}g) \right]}{N(N-1)} \right]^{1/2} \mathbb{E}\left[ (\|X_{T_h, \phi_j}\|_2^2 + \sigma^2 \|T_h P_{z_{2j}}g\|_2^2) (\|X_{T_h, \phi_j}\|_2^2 + \sigma^2 \|T_h \phi_j\|_2^2) \right] \leq \frac{n M_2^{1/2}}{N(N-1)} \exp\left( -\frac{MN}{4M_0} \right).
\]

**Proof of Lemma 12.** Recall that
\[
\mathbb{E}|A_i(\phi_j, P_{z_{2j}}g)|^2 \leq N^4 A_{i1}(\phi_j, P_{z_{2j}}g) + 2N^3 \{ A_{i22}(\phi_j, P_{z_{2j}}g) + A_{i23}(\phi_j, P_{z_{2j}}g) \} + 2N^2 A_{i31}(\phi_j, P_{z_{2j}}g).
\]

with
\[
A_{i1}(\phi_j, P_{z_{2j}}g) = \mathbb{E}\left[ (\|X_{T_h, \phi_j}\|_2^2)^2 \right],
A_{i22}(\phi_j, P_{z_{2j}}g) = \mathbb{E}\left[ (\|X_{T_h, \phi_j}\|_2^2 + \sigma^2 \|T_h P_{z_{2j}}g\|_2^2) \right],
A_{i23}(\phi_j, P_{z_{2j}}g) = \mathbb{E}\left[ (\|X_{T_h, \phi_j}\|_2^2 + \sigma^2 \|T_h \phi_j\|_2^2) (\|X_{T_h, \phi_j}\|_2^2 + \sigma^2 \|T_h P_{z_{2j}}g\|_2^2) \right],
A_{i31}(\phi_j, P_{z_{2j}}g) = \mathbb{E}\left[ (\|X_{T_h, \phi_j}\|_2^2 + \sigma^2 \|T_h \phi_j\|_2^2) (\|X_{T_h, \phi_j}\|_2^2 + \sigma^2 \|T_h P_{z_{2j}}g\|_2^2) \right].
\]

Note that $\mathbb{E}\|X\varphi\|_2^2 \leq \int_0^1 \mathbb{E}\|X(t)\|_2^2 dt \|\varphi\|_2 \leq \text{Const.}\|\varphi\|_2^4$ and $\mathbb{E}\|X\varphi\|_2^2 \leq \text{Const.}\|\varphi\|_2^4$ for all $\varphi \in L^2$, we have $A_{i31}(\phi_j, P_{z_{2j}}g) \leq \text{Const.}\|T_h \phi_j\|_2^2 \|T_h P_{z_{2j}}g\|_2^2$ and $A_{i22}(\phi_j, P_{z_{2j}}g) \leq \text{Const.}\|X_{T_h, \phi_j}\|_2^2 \|T_h P_{z_{2j}}g\|_2^2$. As $\|T_h \phi_j\|_2 \leq \|\phi_j\|_2 \leq 1$, $\|T_h P_{z_{2j}}g\|_2 = \|T_h g - T_h P_{z_{2j}}g\|_2 \leq 4h^{-1/2}g_1 + \|P_{z_{2j}}g\|_2 \leq 4h^{-1/2}g_1 + \text{Const.}(2j)^{1/2} \leq \text{Const.}h^{-1/2}$ for all $g \in \chi_{1}(\rho)$ and $\mathbb{E}\|X_{T_h, \phi_j}\|_2^2 \leq \text{Const.}j^{-2a}$ by Lemma 8,

\[
A_{i31}(\phi_j, P_{z_{2j}}g) \leq \text{Const.}h^{-1}, \quad A_{i22}(\phi_j, P_{z_{2j}}g) \leq \text{Const.}j^{-a}h^{-1}. \quad (75)
\]

Next, we focus on quantity $\mathbb{E}(X_{T_h, P_{z_{2j}}g})^4$ and note that
\[
(X_{T_h, P_{z_{2j}}g}) = (T_h X - P_{z_{2j}} X, g) - (T_h X - P_{z_{2j}} X, P_{z_{2j}} g).
\]

41
For the second term in the right hand side of equation (76), note that $|\langle T_h X - P_{c2j} X, P_{c2j} g \rangle| \leq \|T_h X - P_{c2j} X\|_2 \|P_{c2j} g\|_2 \leq \text{Const.} j^{1/2} \|T_h X - P_{c2j} X\|_2$, and $\mathbb{E} \left[ \|T_h X - P_{c2j} X\|_2^4 \right] \leq \text{Const.} j^{2-2a}$, we have
\[
\mathbb{E} [ \langle T_h X - P_{c2j} X, P_{c2j} g \rangle ]^4 \leq \text{Const.} j^{4-2a}. \tag{77}
\]

For the first term in the right hand side of equation (76), we further have
\[
\langle T_h X - P_{c2j} X, g \rangle = \langle T_h P_{c2j} X - P_{c2j} X, g \rangle + \langle T_h P_{2c2j} X, g \rangle. \tag{78}
\]

Note that
\[
\left( \mathbb{E} \|T_h P_{c2j} X - P_{c2j} X\|_\infty^4 \right)^{\frac{1}{4}} \leq \text{Const.} \left\{ \mathbb{E} \left( \sum_{k=1}^{2j-1} |\xi_k| \|T_h \phi_k - \phi_k\|_\infty \right)^4 \right\}^{\frac{1}{4}} \leq \text{Const.} \left\{ \sum_{k=1}^{2j-1} \left( \mathbb{E} \xi_k^4 \right)^{\frac{1}{4}} \|h^2 k^c\| \right\} \leq \text{Const.} \left( \sum_{k=1}^{2j-1} \lambda_k^2 h^2 k^c \right) \tag{79}
\]
\[
\leq \text{Const.} \sum_{k=1}^{2j-1} h^2 k^c \leq \text{Const.} h^2 j^{c+1} = \text{Const.} j^{1-a} (h^4 j^{2c+2a})^{1/2} \leq \text{Const.} j^{1-a}.
\]

For the second term in equation (78), as $T_h g \geq 0$ and $\|T_h g\|_1 = 1$ for all $g \in \chi_1(\rho)$,
\[
\mathbb{E} \langle T_h P_{2c2j} X, g \rangle^4 = \mathbb{E} \langle P_{2c2j} X, T_h g \rangle^4 \leq \mathbb{E} \langle P_{2c2j} X^4, T_h g \rangle = \int_0^1 \mathbb{E} \|P_{2c2j} X(t)\|^4 T_h g(t) dt \leq \int_0^1 \text{Const.} j^{4-2a} T_h g(t) dt \leq \text{Const.} j^{4-2a}. \tag{80}
\]

Combine equation (76) to (80), we have $\mathbb{E} \langle X, T_h P_{2c2j} g \rangle^4 \leq \text{Const.} j^{4-2a}$. Thus,
\[
A_{111}(\phi_j, P_{2c2j} g) \leq \left( \mathbb{E} \langle X, T_h \phi_j \rangle^4 \mathbb{E} \langle X, T_h P_{2c2j} g \rangle^4 \right)^{\frac{1}{2}} \leq \text{Const.} \left( j^{-2a} j^{4-2a} \right)^{\frac{1}{2}} \leq \text{Const.} j^{2-2a};
\]
\[
A_{123}(\phi_j, P_{2c2j} g) \leq \|T_h \phi_j\|_\infty \mathbb{E} \left[ \left( \langle X, T_h P_{2c2j} g \rangle \right)^2 (\|X\|_2^2 + \sigma^2) \right] \leq \text{Const.} \left( \mathbb{E} \langle X, T_h P_{2c2j} g \rangle^4 \right)^{1/2} \leq \text{Const.} j^{2-a}. \tag{81}
\]

The proof is complete by combing equation (75) and (81) and the arbitrariness of $g$.

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