Signal processing of nonlinear dynamic systems

S V Sarkisov, S Z El-Salim, A V Bondarev 1, A N Korpusov and P A Putilin

Military Institute (engineering) of the Military Academy logistics support named after Army General A. V. Hrulev, 22, Zakharyevskaya, St. Petersburg, 196121, Russian Federation

1 E-mail: Bondarev.aspb@mail.ru

Abstract. The paper considers Hermite polynomials that act as a self-similar basis for the decomposition of functions in phase space. It is shown that the equations of behavior of nonlinear dynamical systems are simplified. It is also noted that the wavelet decomposition over Hermite polynomials reduces the number of approximation coefficients and improves the quality of approximation.

1. Introduction

Signals recorded in real time and depending on a number of physical parameters require the use of such processing methods that allow increasing the resolution of the analysis and the selectivity of identification, as well as the sensitivity and stability characteristic of nonlinear dynamic systems.

Integral transformations are effective for solving a wide range of data analysis tasks. The problems directly related to spectral analysis include the issues of data smoothing and filtering. They consist in constructing another sequence based on the initial experimental dependence, which is close to the original one, taking into account the error expressing the noise component of the measurements.

The development of digital measurement methods involves the use of spectral methods of information processing, which include Fourier transforms, wavelet analysis, correlation transformations and other integral algorithms.

Transformations that can be applied in real time with varying degrees of detail are of great interest for signal processing characteristic of nonlinear dynamic systems. Most spectral transformations, which include the Fourier, Gabor, Hilbert transformations and a number of others, are subject to the uncertainty principle, according to which the higher the density of a function over a selected time interval, the greater its variance in the frequency domain. The advantage of the wavelet transform is that the phase plane is covered with cells of the same area, but of different shapes. This allows you to localize both the low-frequency details of the signal and the high-frequency component of the measured values.

The idea of using wavelets for multiscale analysis is that they are self-similar - the signal is decomposed according to the basis formed by shifts and different-scale prototype functions, that is, practically fractals. Thus, the convolution of a signal with one of the wavelets allows us to identify the characteristic features of the signal in the localization region of this wavelet. The larger the scale of the wavelet, the wider the signal area affects the convolution result [1].

Moreover, the wavelet analysis allows us to study the behavior of fractal functions – that is, those that do not have explicit derivatives at any of their points. Such systems include open systems that are not amenable to classical methods of description. High sensitivity to the setting of initial conditions in
processes occurring in open systems leads to the impossibility of predicting their behavior for a long period of time. The wavelet transformations by successive coarsening of the signal allow us to identify its local features and divide them by intensity. That is, they allow you to determine the dynamics of signal changes depending on the scale and intensity.

Due to the high efficiency of algorithms and resistance to interference, the wavelet transform is a powerful tool in areas where other methods of analysis are traditionally used, for example, the Fourier transform. The possibility of using existing methods for processing measurement results, as well as the characteristic features of the behavior of the wavelet transform in the time-frequency domain, allow us to significantly expand and supplement the capabilities of the analysis of nonlinear systems.

2. Basic functions of the wavelet transform

A distinctive feature of the wavelet analysis is the possibility of using a family of functions that implement various variants of the uncertainty ratio.

The wavelet basis of the space \( L^2(R) \), \( R(-\infty, \infty) \), it is advisable to construct from the finite functions belonging to the same space. The faster these functions tend to zero, the more convenient it is to use them as a transformation basis when analyzing real signals.

Functions that meet the conditions of self-similarity and scaling must have certain characteristics, which include: localization, striving for zero of the average value, limitation and self-similarity [1].

The wavelet transform, in contrast to the Fourier transform, uses a localized basis function, that is, the wavelet-generating function must be localized both in time space and in frequency. The equality of the average value of the wavelet to zero is a necessary condition for its locality: \( \int_{-\infty}^{\infty} \psi(t) dt = 0 \).

It should be noted that the equality of the higher moments to zero allows us to ignore the regular polynomial components of the signal under study and analyze small-scale fluctuations and high-order features: \( \int_{-\infty}^{\infty} t^m \psi(t) dt = 0 \), where \( m \) is the order of the moment. The boundedness of the wavelet function is given by the condition: \( \int_{-\infty}^{\infty} |\psi(t)|^2 dt < \infty \). Localization and limitations for temporary space is defined by the condition that \( |\psi(t)| < (1 + |t|^n)^{-1} \), and for frequency conversion: \( |\psi(t)| < (1 + |k - \omega_b|^n)^{-1} \).

The characteristic feature of the basis of wavelet transform is its self-similarity, that is, all generated by a wavelet tour of this family have the same number of oscillations that the wavelet basis, as obtained through large-scale transformations and shifts: \( WT[f(t)] = \frac{1}{|a|} \int_{-\infty}^{\infty} f(t) \psi\left(\frac{t-b}{a}\right) dt \), where \( a \) is the compression parameter, \( b \) is the shift parameter [2].

Currently, there are generally accepted wavelet bases that make up standard functions that satisfy the characteristic features inherent in wavelet functions. The most well-known ones can be noted: the basis consisting of derivatives of normal distribution functions («Mexican hat»), the Daubechey basis («French hat»), Morlaix, Haar, Shannon, Paul, Meyer, Koifman and a number of other functions and functional [2, 3, 4].

However, for the analysis of nonlinear dynamical systems, which, as a rule, are represented by a large amount of data, the use of the bases accepted today is limited for a number of reasons. First of all, these are the requirements of computational resources for organizing real-time processing. Modification of basic functions is usually limited. To improve the quality of the analysis, it is necessary to use complex-valued functions [4, 5].
3. Modification of hermite polynomials

To expand the possibilities and increase the functionality of using the wavelet transform, it is proposed to use Hermite polynomials of the form as the basis of the decomposition $H_n(x) = (-1)^n e^{-x^2} \frac{d^n e^{-x^2}}{dx^n}$, where the variable $x$ can represent any regular physical parameter - time, frequency, energy, and others. Hermite polynomials have all the features inherent in functions that form a wavelet basis [6]. First of all, Hermite polynomials with a weight function $e^{-x^2}$ are orthogonal on the interval $(-\infty, \infty)$:

$$\int e^{-x^2} H_m(x)H_n(x)dx = \begin{cases} 0, & m \neq n \\ 2^n n! \sqrt{\pi}, & m = n \end{cases}.$$

The sequence of polynomials has the form $H_n(x) = (-1)^n e^{-x^2} \frac{d^n e^{-x^2}}{dx^n}$, and for $n=0$, $H_0(x) = 1$.

The first Hermite polynomials are equal to:

- $H_0(x)$ is a polynomial of degree $n$;
- for even $n$, the polynomial $H_n(x)$ contains only even powers of $x$, for odd $n$, only odd powers of $x$;
- the coefficient at the highest degree of $x$ is $2^n$.

Figure 1 shows the form of Hermite functions of initial (even) orders [6].

Figure 1. Basic Hermite functions of even orders 2, 4, 8.

A recurrent formula is known that connects Hermite polynomials in increasing degree. According to the Leibniz rule for calculating derivatives containing an exponent, in the case of Hermite polynomials we have: $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$. Hence the recurrence relation for the derivatives of regular parameter [3, 5]: $H'_n(x) = 2nH_{n-1}(x)$.

The generating function for Hermite polynomials in its expansion in a Taylor series corresponds to the desired polynomials with the precision of a numeric multiplier that can be algebraically equal [7].

Expanding the function of two variables $F(\varphi, t) = e^{2\varphi t^2}$ in a Taylor series in powers of $t$ will receive

$$\sum_{n=1}^\infty \frac{\varphi_n(x)t^n}{n!},$$

the coefficients $\varphi_n = H_n(x)$, hence, $e^{2\varphi t^2} = \sum_{n=1}^\infty \frac{H_n(x)t^n}{n!}$. 


The integral of the Euler–Poisson allows you to calculate the rate of polinoms Hermite weight \( e^{-x^2} \) which is equal to:

\[
\int_{-\infty}^{\infty} e^{-x^2} H_n^2(x) dx = 2^n n! \sqrt{\pi}.
\]

In physical applications and for performing spectral transformations, it is more convenient to use not the Hermite polynomials themselves, but the functions corresponding to the Hermitian norm:

\[
\psi_n = \frac{H_n(x)}{\| H_n(x) \|}, \quad \psi_n(x) = \frac{e^{-x^2/2}}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x),
\]

which form an orthonormal system:

\[
(\psi_m, \psi_n) = \int_{-\infty}^{\infty} \psi_m(x) \psi_n(x) dx = \begin{cases} 0, & m \neq n \\ 1, & m = n. \end{cases}
\]

Thus, the wavelet basis constructed on the basis of normalized Hermite functions not only has the property of self-similarity (self-similarity), but also is orthogonal, which increases the reliability of spectral transformations of nonlinear dynamical systems with high fractal (correlation) dimension [8].

4. Increasing the detection sensitivity using hermite convolution

Consider a model signal with a Signal-to-Noise ratio = 10. The interference is represented by white (Gaussian) noise that masks the formation of a useful signal (figure 2).

![Figure 2. Model of the useful signal with the parameters "Signal/Noise" = 10.](image)

For comparison, the Fourier transform of the original signal and the convolution of the values of the original time series (figure 2) with the Hermite function of the 2nd order (figure 3) are carried out.

![FFT image (Real x Complex) of the source signal](image)  ![Convolution with the Hermite function of the 2nd order](image)

**Figure 3.** Comparison of transformations.
The model signal (figure 2) is processed by two methods: using the traditional Fourier transform and using the convolution of the original signal with a wavelet formed on the basis of the Hermite function. If the FFT image is not scalable, then Hermite wavelets are subject to compression and scaling operations. Figure 4 shows graphs of the parent wavelets with different scales \([6, 6]\) and \([60, 60]\).

![Figure 4](image)

Figure 4. Examples of mother wavelets at different scales.

Then a signal with a Signal-to-Noise ratio \(= 1\) is processed, that is, this model has a low detectability order, the resolution of useful components by traditional methods is almost impossible. Figure 5 shows the results of processing the original signal by performing an FFT transformation and convolution with the Hermite function of the 2nd order.

![Figure 5](image)

Figure 5. Results of processing a highly noisy model signal.

The advantages of using convolution are obvious: useful components are highlighted and noise is minimized, the nature of which can be both natural and hardware origin. In the case of using convolution as a conversion method, the requirements for the hardware design of analytical equipment are significantly reduced and the sensitivity to the useful component of the measured signal is increased. Indeed, for a model signal with a noise ratio of no more than one, after processing, the ratio of the useful signal and noise increases by 2-3 orders of magnitude \([8, 9]\).

5. The wavelet transform in the basis of hermite functions

Let us consider the analytical signal obtained during the research by a bioactive detecting system, which clearly refers to nonlinear dynamic structures. Figure 6 shows the dependencies of the initial signal on the measurement time and the result of the primary transformations.
After the initial transformations, wavelet images with different resolution are constructed: 1, 5, 10, 30, 60 and 75. High-resolution decompositions with successive zoom increases are shown. Table 1 shows the wavelet expansions with different scales to illustrate the use of the Hermite function of the 2nd order as the parent wavelet.

It can be seen that changing the zoom from 1 and higher affects the detail of the signal under study. Also, the wavelet image obtained by the Hermite basis decomposition allows us to restore the differential equation for a dynamic nonlinear system in which the process under study proceeds, generating the measured signal.

**Table 1.** Wavelet decomposition of the original signal by Hermite functions with different resolution.

| Scale Parameter | Wavelet Image |
|-----------------|---------------|
| 1               | ![Wavelet Image](image1) |
| 5               | ![Wavelet Image](image2) |
| 10              | ![Wavelet Image](image3) |
| 30              | ![Wavelet Image](image4) |
| 60              | ![Wavelet Image](image5) |
| 75              | ![Wavelet Image](image6) |

**Figure 6.** The original and converted test signal.
4. Conclusion

Thus, if there is a wavelet decomposition by Hermite functions of order \( n \) of the form
\[
\psi_n(t) = \frac{H_n(t)}{\|H_n(t)\|} e^{-\frac{t^2}{2}},
\]
then the series \( X(t) \) can be represented by Hermite polynomials \( H_n(t) \) with the weight function \( e^{-\frac{t^2}{2}} \) and the norm \( \|H_n(x)\|^2 = 2^n n! \sqrt{\pi} : X(t) = H_n(t) + e^{\frac{H_n(t+1)}{H_n(t)}} \).

In this case, the dynamic equation of the nonlinear system
\[
d\frac{X(t)}{dt} = F(X, t, \alpha_1, ..., \alpha_n)
\]
will be as follows:
\[
\frac{dX(t)}{dt} = e^{\frac{H_n(t+1)}{H_n(t)}} \frac{dH_n(t)}{dt} + \text{Const}.
\]

Then the trajectories in the phase space are determined by Hermite polynomials of order \( n \), in accordance with the dimension of the wavelet basis.

Introducing the replacement of the variable \( \frac{H_n(t)}{\|H_n(t)\|} = \frac{t^2}{2} \), and expressing the time as
\[
t = \sqrt{\frac{H_n(t+1)}{H_n(t)}} \frac{1}{2},
\]
it is seen that the nonlinearity of the process flow change parameter is characteristic for open dynamical systems [9,10].

The use of Hermite functions as the basis of the spectral decomposition shows the irreversibility of the processes inherent in nonlinear systems with fractal phase trajectories.

Another property of Hermite polynomials should be noted, which are used as a basis for the decomposition of functions in phase coordinates. The wavelet image, forming an orthonormal system of function decomposition, increases the number of compression and shift coefficients, which requires an increase in the power of the computer systems used. If Hermite polynomials are used as a basis, only the degree of the polynomial is subject to change, and the degree can be either even or odd. This greatly simplifies the calculations and improves the quality of the approximation.

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