Holonomic quantum computation with neutral atoms

A. Recati
ECT*, I-38050 Villazzano (TN) and Istituto Nazionale di Fisica della Materia, I-38050 Povo, Italy
2 Institut für Theoretische Physik, Universität Innsbruck, A-6020 Innsbruck, Austria
3 Dipartimento di Fisica, Università di Trento, I-38050 Povo, Italy
4 Institute for Scientific Interchange (ISI) Foundation, I-10133 Torino, Italy
5 Max-Planck Institut für Quantenoptik, Hans-Kopfermann Str. 1, D-85748 Garching, Germany

We propose an all-geometric implementation of quantum computation using neutral atoms in cavity QED. We show how to perform generic single- and two-qubit gates, the latter by encoding a two-atom state onto a single, many-level atom. We compare different strategies to overcome limitations due to cavity imperfections.

I. HOLONOMIC QUANTUM COMPUTATION

We now briefly recall the basics of HQC [8]. In the HQC paradigm information is encoded in an n-fold degenerate eigenspace C of a Hamiltonian \( H(\lambda_0) \) belonging to a d-dimensional parametric family of isodegenerate Hamiltonians \( \{ H(\lambda) \} \). The \( \lambda \)'s represent parameters that are supposed to be controllable in the given experimental situation. The manipulations of the codewords in \( C \) are enacted by driving the control parameters along loops \( \chi \) in an adiabatic fashion. In this way an initial preparation \( |\Psi_0\rangle \in C \) evolves up to an overall dynamical phase, according to the rule \( |\Psi_0\rangle \in C \mapsto U(\chi)|\Psi_0\rangle \) where \( U(\chi)_A = \text{P exp} \int_\chi A \) is the holonomy associated with \( \chi \) by the the \( u(n) \)-valued connection \( A = \sum_{\mu=1}^d A_\mu d\lambda_\mu \).

One finds

\[
A^{\alpha\beta}_\mu = \langle \psi^\alpha(\lambda) | \frac{\partial}{\partial \lambda_\mu} | \psi^\beta(\lambda) \rangle, \tag{1}
\]

where \( \{ |\psi^\alpha(\lambda)\rangle \}_{\alpha=1}^n \) denotes an orthonormal basis of the degenerate eigenspace \( C \). The set of all possible holonomies, obtained by taking all possible \( \chi \)'s, is a subgroup, known as the holonomy group, of the group \( U(n) \) of unitary transformations over \( C \).

When the holonomy group coincides with the whole \( U(n) \) one can perform universal QC over \( C \) by resorting to geometrical means only. This irreducibility condition can be easily stated in terms of the curvature 2-form \( F \) associated with \( A \) by the relation \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu] \) [10]. The number of linearly independent \( F_{\mu\nu} \)'s gives a lower bound to the dimension of the holonomy group [3]. The curvature form \( F \) encodes for the non-trivial geometric features of the global bundle of n-dimensional quantum codes over the manifold of control parameters. Flat bundles, i.e., with \( F = 0 \), have no computational power.

II. DARK STATES IN (N+1)-LEVEL SYSTEM

Following the previous recipe we will give the expression of a parametric Hamiltonian which turns out to be
suitable to achieve holonomic quantum computation. After that we will show how such Hamiltonian can be implemented using neutral atoms in an optical resonator. We can, in this way, give a precise physical meaning to the abstract objects of the HQC paradigm. The idea relies on the concept of the adiabatic passage via dark states. In the simplest case (a.k.a. Λ-system) we have two states (ground states) which are not directly connected, i.e., in the Hamiltonian describing the system there are no terms which couple these two states – considering an atom interacting with the electromagnetic field there is no single-photon transition between the states we are interested in. However they are independently coupled to a third state (excited state) in a tunable way. By changing in an adiabatic way the coupling constants, it is possible to create coherent superpositions of the two ground states and to pass from one to the other without populating the excited state \( |e \rangle \).

Our starting point is the Hamiltonian

\[
H = \sum_{k=1}^{N} \Omega_k |e \rangle \langle g_k | + h.c.,
\]

which can be seen as a generalization of the Λ-system Hamiltonian and represents a system in which \( N \) ground states \( |g_k \rangle \) are coupled to an excited level \( |e \rangle \). The level structure we have in mind is depicted in Fig. 1.

![FIG. 1. Internal level scheme of the atoms inside the cavity, with the coupling needed to realize the Hamiltonian Eq. (2).](image)

The Hamiltonian (2) admits \( N-1 \) dark states \( |D_i \rangle \), i.e., zero-energy eigenstates having no contribution from the excited state. The complex couplings \( \Omega_k \) represent our control parameters. Once they are fixed, the coding space, i.e., the \( (N-1) \)-fold degenerate eigenspace spanned by the dark states, is the orthocomplement of the vector \( N^{-1/2} \sum_{k=1}^{N} \Omega_k |g_k \rangle \) \( (N = \sum_{k=1}^{N} \Omega_k)^2 \) \). By setting \( \Omega_1 = \Omega_2 = \ldots = \Omega_{N-1} = 0 \), we have that the coding space \( C \) is spanned by the first \( N-1 \) ground states. We write the coupling constants in generalized spherical coordinates

\[
\begin{align*}
\Omega_1 &= |\Omega| \sin(\theta_1), \\
\Omega_2 &= |\Omega| e^{-i\phi_2} \cos(\theta_1) \sin(\theta_2), \\
\vdots \\
\Omega_N &= |\Omega| e^{-i\phi_N} \cos(\theta_1) \cdots \cos(\theta_{N-2}) \sin(\theta_{N-1}).
\end{align*}
\]

By explicitly computing the connection form \( A \) and its curvature for this system we have checked that it allows for universal QC over \( C \) for any \( N \). In particular, if \( N = 2^n + n \) qubits can be encoded in \( C \). In the next two subsections we turn on the cases \( N = 3 \) and \( N = 5 \), or equivalently \( n = 1 \) qubit and \( n = 2 \) qubits. We will show how to realize single-qubit rotations – thus, up to a phase factor, any single-qubit gate – and a 2-qubit phase-gate \( e^{i\alpha} R_z(\alpha) \).

### A. (3+1)-level system

One can realize a single-qubit encoding using a system described by the Hamiltonian (3) where \( N = 3 \). As computational basis we choose the first two ground states, i.e., we assign logical values through the identities

\[
|g_1 \rangle = |0 \rangle, \quad |g_2 \rangle = |1 \rangle,
\]

while the third ground state, \( |g_3 \rangle \), plays the role of an auxiliary state, which is necessary to achieve every single-qubit gate. It is well known that any single-qubit gate can be decomposed (up to a phase factor) in the product of three rotations, for instance a rotation about the \( z \) axis, one about the \( y \) axis and one again about the \( z \) axis, i.e., if \( U \) is the single-qubit gate we want to build up, there exist three numbers, \( \alpha, \beta, \gamma \), such that \( U = R_z(\gamma) R_y(\beta) R_z(\alpha) \), the Euler angles. Within our model Hamiltonian a rotation about the \( y \) axis can be obtained by putting the relative phases in Eq. (3) (with \( N = 3 \)) \( \phi_3 = 0 \), and by adiabatically changing the amplitudes \( \theta_1, \theta_2 \). In this case the connection is just \( A = A_\beta d\theta_2 = -i \sin(\theta_1) \sigma_y d\theta_2 \), where \( \sigma_y \) is the \( y \)-Pauli matrix. We obtain the unitary operator (see App. A 1)

\[
R_y(\beta) = \exp(i\beta \sigma_y),
\]

after a cycle \( \chi \) in the \( (\theta_1, \theta_2) \)-submanifold, where the angle \( \beta \) is given by \( \beta = \int_{S(\chi)} \cos(\theta_1) d\theta_1 d\theta_2 \) and \( S(\chi) \) being the surface enclosed by the loop \( \chi \) on the \( \theta_1, \theta_2 \) submanifold. Up to a global phase, a rotation about the \( z \) axis is equivalent to the operator \( \exp(i\alpha|1\rangle\langle 1|) \), which is easily obtained by putting \( \theta_1 = \phi_3 = 0 \) and by adiabatically performing a closed path \( \chi' \) in the submanifold of \( \theta_2, \phi_2 \). The connection is \( A = A_\phi d\phi_2 = -i \sin(\theta_2) \alpha |1\rangle\langle 1| d\phi_2 \).

After a cycle we obtain

\[
\exp(i\alpha|1\rangle\langle 1|) = e^{i\alpha/2} R_z(\alpha),
\]

where \( \alpha = \int_{S(\chi')} \sin(2\theta_2) d\theta_2 d\phi_2 \).
B. (5+1)-level system

We assume two qubits mapped to a four level system. Thus, to implement a two-qubit gate, we need \( N = 5 \). We want to show how to realize a phase gate, which assigns a phase only to one out of the four computational basis states. As the computational basis we choose \( |D_i\rangle \equiv |g_i\rangle \), \( i \leq 4 \), whereby the corresponding coupling constants \( \Omega_k \) are initially set to zero. The logical states can be identified as

\[
|g_1\rangle = |00\rangle, \quad |g_2\rangle = |01\rangle, \quad |g_3\rangle = |10\rangle, \quad |g_4\rangle = |11\rangle, \quad (7)
\]

the fifth state playing the role of an ancilla. Considering a closed path \( \chi'' \) in the two-dimensional submanifold of the parameter space with coordinates \( (\theta_4, \phi_5) \) – the other parameters being kept to zero – the connection is reduced to the simple form: \( A = A_{\phi_5}d\phi_5 = -i\sin^2(\theta_4)|g_4\rangle\langle g_4|d\phi_5 \). This gives rise to the holonomy (see App. A2)

\[
\Gamma_A(\chi'') = \exp(i\alpha|g_4\rangle\langle g_4|) \quad (8)
\]

which, according to Eq. (7), precisely represents a 2-qubit phase gate, with the phase \( \alpha \) given by \( \int_{S(\chi'')} \sin(2\theta_4)d\theta_4d\phi_5 \).

III. PHYSICAL REALIZATION

In the remaining part of the paper we discuss a possible physical realization of the – up to now quite abstract, but also very general – concepts we introduced in the previous sections. Our proposal is based on atoms trapped inside an optical resonator (Fig. 2). The atoms, which represent our qubits, interact individually with laser beams and with a single quantized mode of the optical cavity.

![FIG. 2. Schematic representation of the setup.](image)

The manipulation of the qubits (in terms of single- and 2-qubit gates) involves only the laser beams, while the cavity mode is used to prepare the system every time a 2-qubit gate is required. Indeed, as we will see, we need to encode the information of two qubits in a single many-level atom and this is performed with the aid of the cavity mode, to which all atoms are coupled.

The single-qubit gates are easily implemented if we consider \((3+1)\)-level atoms, i.e., atoms with three ground states coupled by lasers to a single excited state. The Hamiltonian of the single atom can be reduced to Eq. (2) with \( N = 3 \), where the coupling constants \( \Omega_k, k = 1, 2, 3 \) are the Rabi frequencies of the lasers. Therefore for the feasibility of single-qubit gates the addressing of single levels in single atoms is required.

2. 2-qubit gate implementation

Once we have defined qubits, we need a way to couple them in a suitable way to make the computation universal. In our case we are able to implement any 2-qubit transformation in a single atom, having \( N = 5 \) ground (or meta-stable) states which can be coupled by tunable lasers to a single excited one [12]. Thus we have to deal with a system in which the interacting part is described by the Hamiltonian Eq. (4) with \( N = 5 \) and such that we can use the prescription of sec. IIIA. By adiabatically acting on the coupling lasers, which address the single levels of the atom, any 2-qubit operation is achievable.

The complete picture is based on \((5 + 1)\)-level atoms, single-qubit information being stored in the single atoms and single-qubit operations being performed in any single atom as discussed above, using 3 out of the 5 ground states. Let us consider two \((5 + 1)\)-level atoms and let \( |\alpha\rangle_1 \) and \( |\beta\rangle_2 \) be the (logical) states of the first and the second atom, respectively. A 2-qubit gate is performed in three steps:

1. the two-qubit information is stored in the second system by the transfer

\[
|\alpha\rangle_1|\beta\rangle_2 \rightarrow |0\rangle_1|\xi\rangle_2, \quad (9)
\]

where \( \alpha \) (\( \beta \)) represents the first (second) digit of \( \xi \) in binary notation;

2. since we suppose that any \((5 + 1)\)-level atom can be driven by the Hamiltonian Eq. (4), it is possible to obtain any (2-qubit) gate by manipulating the coupling constants, physically the Rabi frequencies \( \Omega_k \). We have shown above how to obtain the phase gate \( U_\alpha = \exp(i\alpha|11\rangle\langle 11|) \). In this case it is sufficient to act only on two of the couplings \( \Omega_k \), the others being turned off;

3. after the holonomic 2-qubit gate operation, the inverse transformation of Eq. (4) is performed, and each qubit is encoded back in one of the two atoms.

To pursue our purpose what is missing is a method to perform the information transfer Eq. (9). In order to be consistent with the holonomic paradigm, the information transfer has to be adiabatically performed.
A. Information Transfer

We will suggest two possible approaches to achieve the information transfer process. They involve, besides the ground states, excited states as well as the single-cavity mode. Thus, such processes will be affected by both spontaneous emission from the excited levels and imperfections of the cavity. The first approach, that from now on we will call the optical scheme, was envisaged in [14]. It is based on an adiabatic transfer that leaves almost unpopulated the cavity mode, thus reducing the influence of spontaneous emission. The second one, proposed here for the first time, is based on an adiabatic transfer that leaves almost unpopulated the excited levels, thus reducing the influence of cavity losses. We will call it the motional scheme. We turn now to study in detail these two different approaches.

Both of them rely on the level scheme shown in Fig. 3. A laser addressing atom \( i \) (\( i = 1, 2 \)) with Rabi frequency \( \Omega'_i(t) \) couples \( |g_{i1}\rangle \) to the excited state \( |e_{1i}\rangle \) and \( |g_{i2}\rangle \) to \( |e_{2i}\rangle \), while \( |g_{i3}\rangle \) and \( |g_{i4}\rangle \) are coupled by the cavity mode to, respectively, \( |e_{1i}\rangle \) and \( |e_{2i}\rangle \).

\[ \begin{align*}
|g_{i1}\rangle & \quad \Delta \quad |e_{2i}\rangle \\
|g_{i2}\rangle & \quad \Delta \quad |e_{1i}\rangle \\
|g_{i3}\rangle & \quad |\phi\rangle \\
|g_{i4}\rangle & \quad |\phi\rangle
\end{align*} \]

FIG. 3. Level scheme and coupling laser lights for the information transfer. The logical values are also shown (see the text).

Note that the information transfer is equivalent to state-swapping between the two atoms, provided the physical encoding is the one shown in Fig. 3, i.e., \( |g_{i3}\rangle = |0\rangle_1 \), \( |g_{i1}\rangle = |1\rangle_1 \) for the first atom and the 2-bit words, which we want to encode in the second atom, are physically represented as \( |g_{i2}\rangle = |0\rangle_2 \), \( |g_{i1}\rangle = |1\rangle_2 \), \( |g_{i4}\rangle = |2\rangle_2 \), \( |g_{i2}\rangle = |3\rangle_2 \). We have, for the logical state:

\[ \begin{align*}
|g_{i1}\rangle |g_{i3}\rangle & \rightarrow |g_{i3}\rangle |g_{i1}\rangle \leftrightarrow |0\rangle_1 |0\rangle_2 \rightarrow |0\rangle_1 |0\rangle_2 \\
|g_{i1}\rangle |g_{i4}\rangle & \rightarrow |g_{i4}\rangle |g_{i1}\rangle \leftrightarrow |0\rangle_1 |1\rangle_2 \rightarrow |0\rangle_1 |1\rangle_2 \\
|g_{i1}\rangle |g_{i3}\rangle & \rightarrow |g_{i3}\rangle |g_{i1}\rangle \leftrightarrow |1\rangle_1 |0\rangle_2 \rightarrow |0\rangle_1 |2\rangle_2 \\
|g_{i1}\rangle |g_{i2}\rangle & \rightarrow |g_{i2}\rangle |g_{i1}\rangle \leftrightarrow |1\rangle_1 |1\rangle_2 \rightarrow |0\rangle_1 |3\rangle_2
\end{align*} \]

Note that we used a different identification of the logical state with respect to sec. [1][8].

To describe the state-transfer process, we will focus on the evolution of one out of the two three-level systems which are contained in each atom, i.e., the one formed by \( |g_{i1}\rangle \), \( |g_{i3}\rangle \), and \( |e_{1i}\rangle \).

The evolution of the system, in presence of decoherence, will be described by a non-Hermitian Hamiltonian in the framework of the quantum-jump approach to dissipative processes [15].

In the optical approach the following single-atom Hamiltonian is considered:

\[
H^i_{\text{opt}} = \hbar[\omega_{e1} |e_{1i}\rangle \langle e_{1i}| + \omega_{g1} |g_{1i}\rangle \langle g_{1i}| + \omega_{g3} |g_{3i}\rangle \langle g_{3i}| + \omega_e b^\dagger b + (\Omega'_i(t)e^{-i\omega t}|e_{1i}\rangle \langle g_{1i}| + gb|e_{1i}\rangle \langle g_{3i}| + h.c.)]
\]

\[ -i\gamma |e_{1i}\rangle \langle e_{1i}| - i\kappa b^\dagger b \]  

where \( \omega_{e1}, \omega_{g1} \) and \( \omega_{g3} \) are the energies of the state \( |e_{1i}\rangle, |g_{1i}\rangle \) and \( |g_{3i}\rangle \), respectively, \( g \) is the dipole-coupling constant between the cavity mode and the atom and \( b \) is the annihilation operator for the cavity mode. As decoherence mechanism we have considered the spontaneous emission \( \gamma \) of the excited levels \( |e_{1i}\rangle \) and the cavity loss rate \( \kappa \). We will see that the transfer process is based on a dark state of the compound system \( \mathcal{H}_{\text{atom1}} \otimes \mathcal{H}_{\text{atom2}} \otimes \mathcal{H}_{\text{cavity}} \) which does not involve any excited states and thus, within the adiabatic approximation, the main dissipative channel is due to the cavity loss rate.

In the motional approach we assume that the atoms are individually trapped in harmonic potentials. The single-atom Hamiltonian contains also the harmonic trapping potential terms (see for instance [18,19]):

\[
H^i_{\text{ext}} = \hbar[\omega_{e1} |e_{1i}\rangle \langle e_{1i}| + \omega_{g3} |g_{3i}\rangle \langle g_{3i}| + \omega_e b^\dagger b + \nu a^\dagger a + (\Omega'_i(t)e^{-i\omega t}|e_{1i}\rangle \langle g_{1i}| + h.c.)]
\]

\[ + g\sin(kx)(b|e_{1i}\rangle \langle g_{3i}| + h.c.)] - i\gamma |e_{1i}\rangle \langle e_{1i}| - i\kappa b^\dagger b \]

where \( a \) (\( a^\dagger \)) is the annihilation (creation) operator for the harmonic motion, while the sine function describes the standing-wave structure of the cavity field, with \( k \) the wave number of the field and \( x = \hbar/(2m
u) \) \( 1/2 (a^\dagger + a) \).

We will see that, under certain conditions, it is possible to obtain an effective Hamiltonian which involves only the cavity mode and the harmonic motion. With such Hamiltonian the transfer is based on a dark state with respect to the cavity and thus one expects that the most important dissipative channel will be the spontaneous decay.

First of all we will show, neglecting any dissipative mechanism, i.e., \( \gamma = \kappa = 0 \), that indeed we can obtain the state-transfer by acting in an adiabatic fashion on the parameters of the Hamiltonians Eq. (10) and Eq. (11). Later on we will carry out in detail the analysis of the effects of the dissipative channels. In the next two subsections we do not consider any decay mechanism.

1. Optical state transfer

In the optical approach the starting point is the single-atom Hamiltonian Eq. (10) (where for the moment...
\( \gamma, \kappa = 0 \). Considering the energy \( h\omega_{gi} \) as the zero of the energy scale and transforming to the reference frame described by the Hermitian operator \( A = \omega_{L}b^\dagger b + \omega_L|e_1\rangle\langle e_1| + (\omega_L - \omega_{gi})|g_3\rangle\langle g_3| \), i.e., \( |\psi\rangle \to e^{itA}|\psi\rangle \), one gets the Hamiltonian

\[
H_{opt}' = \hbar [ - \Delta |e_1\rangle\langle e_1| + (\Omega_I'(t)|e_1\rangle\langle g_1| + gb|e_1\rangle\langle g_3| + h.c.)].
\]

(12)

In writing Eq. (12) we have considered the resonance condition \( \Delta = (\omega_{gi} - \omega_{g_2}) - (\omega_L - \omega_c) = 0 \) and we introduced the detuning \( \Delta = \omega_L - (\omega_{e_1} - \omega_{g_2}) \) between the laser frequency \( \omega_L \) and the transition \( |g_1\rangle \to |e_1\rangle \). The 2-atom Hamiltonian \( H_{opt} = H_{opt}' + H_{opt}'' \) admits the eigenstate

\[
|\Psi\rangle_{opt} \propto \Omega_I'2|g_1\rangle|g_3\rangle|0\rangle_{cav} + \Omega_I'1|g_3\rangle|g_1\rangle|2\rangle_{cav}
\]

\[
- \Omega_I'2|g_2\rangle|g_1\rangle|2\rangle_{cav}.
\]

(13)

Thus by adiabatically changing the Rabi frequencies \( \Omega_I \) and \( \Omega_2 \) applying a “counterintuitive” (11 and reference therein) pulse (whereby the pulse on atom 2 precedes that on atom 1) it is possible to pass from the state \( |g_1\rangle|g_2\rangle|0\rangle_{cav} \) to the state \( |g_1\rangle|g_2\rangle|2\rangle_{cav} \). The state-swapping is thus realized. Note that in such a scheme (see Eq. (13)) during the information transfer the 1-photons state is populated.

2. Motional state transfer

The single-atom Hamiltonian will be in this case the Eq. (11) with \( \gamma = \kappa = 0 \). Considering the Lamb-Dicke limit, i.e., the size of the harmonic trap small compared with the optical wave-length, \( k[h/(2\nu_{0})]^{1/2} \ll 1 \), and writing the Hamiltonian in the reference frame given by the Hermitian operator (the zero-energy being \( \omega_{gi} \)) \( A = \omega_{L}|e_1\rangle\langle e_1| + \omega_{e_1}b^\dagger b + \nu a^\dagger a \) one gets

\[
H_{ext}' = \hbar [ - \Delta |e_1\rangle\langle e_1| + (\Omega_I'(t)|e_1\rangle\langle g_3| + h.c.)
\]

\[
+ \eta (a^\dagger b|e_1\rangle\langle g_3| + a^\dagger a^\dagger e^{-2i\nu t}|e_1\rangle\langle g_3| + h.c.)].
\]

(14)

where \( \eta = k[h/(2\nu_{0})]^{1/2} \) is the Lamb-Dicke parameter. It has been shown in [18,14] that under the conditions \( \Delta \gg |\Omega I'_{i}|, \eta \) a Hamiltonian which does not involve the atomic internal degrees of freedom is obtainable. By adiabatic elimination and using the rotating wave approximation (RWA) we find the effective Hamiltonian (see App. [13])

\[
H_{eff} = \hbar \sum_{i=1}^{2} G_i(t)a_i^\dagger b + h.c.,
\]

(15)

where we introduced the coupling parameters \( G_i = -q_i \Omega I'(t)/\Delta \). The number of excitations \( \sum a_i a_i^\dagger + b^\dagger b \) is a conserved quantity, in particular the zero-excitation eigen-space is spanned by the vacuum state \( |0\rangle_{1}^{cm}|0\rangle_{2}^{cm}|0\rangle_{cav} \) and the 1-excitation eigen-space by the three states \( |1\rangle_{1}^{cm}|0\rangle_{2}^{cm}|0\rangle_{cav}, |0\rangle_{1}^{cm}|1\rangle_{2}^{cm}|0\rangle_{cav}, |0\rangle_{1}^{cm}|0\rangle_{2}^{cm}|1\rangle_{cav} \), where \( |n\rangle_{i}^{cm} \) is the eigenstate of the free external Hamiltonian of the \( i \)th atom, it satisfies \( a_i^\dagger a_i|n\rangle_{i}^{cm} = n|n\rangle_{i}^{cm} \).

Inside the last sub-space there exists the dark state (dark with respect to the cavity mode)

\[
|\Psi\rangle_{ext} \propto \Omega_2'|1\rangle_{2}^{cm}|0\rangle_{1}^{cm}|0\rangle_{cav} - \Omega_1'|0\rangle_{2}^{cm}|1\rangle_{2}^{cm}|0\rangle_{cav}.
\]

(16)

In an analogous way to what described in the previous Section, by adiabatically changing the Rabi frequencies \( \Omega_2'(t) \) it is possible to pass from the state \( |1\rangle_{1}^{cm}|0\rangle_{2}^{cm}|0\rangle_{cav} \) to the state \( |0\rangle_{1}^{cm}|1\rangle_{2}^{cm}|0\rangle_{cav} \) without populating the cavity mode which in the case of a nonzero cavity loss rate \( \kappa \), is a source of decoherence.

We are now ready to describe how the state-swapping can be performed in three steps starting without photons in the cavity and zero-motional-excitation, see Fig. 4: first of all the logical state of qubit 1 is swapped onto the motional state of atom 1 by adiabatic passage via a dark state; then we utilize the dark state (14) to transfer the motional state to the second atom and this is swapped onto the internal state of (the second atom) by the same adiabatic pulse sequence as in first step, ending with a global state without any excitation, i.e., no photons, no quanta of harmonic motion.

FIG. 4. Example of information transfer by using external degrees of freedom: \( |g_1, g_3\rangle \to |g_3, g_1\rangle \), photons, within the validity of Eq. (13), are never present in the cavity; the corresponding logical operation is \( |1\rangle_{1}|0\rangle_{2} \to |0\rangle_{1}|2\rangle_{2} \).

IV. DECOHERENCE EFFECTS

We will now come back to study what are the limitations imposed by the dissipative channels on the transfer process. As stated well before we want to take into account excited level spontaneous emission and cavity loss. Thus the evolution of the system is described by the Eq.
where the presence of the spontaneous emission rate $\gamma$ and the cavity loss rate $\kappa$ give rise to a non-hermitian evolution.

### A. Optical state transfer

The optical state transfer is based on the dark state Eq. (13). First of all we note that it has a non-zero projection onto the 1-photon cavity state and so a lossy cavity will tend to destroy such a state. The dark state is, indeed, not an eigenstate of the Hamiltonian Eq. (10) for $\kappa \neq 0$. In order for the cavity loss rate to be small, the condition $\kappa \int_0^T P_{1-\text{ph}}(t) dt \ll 1$ has to be satisfied, where $P_{1-\text{ph}}(t) = \langle \Psi_{\text{opt}} | b^\dagger b | \Psi_{\text{opt}} \rangle$ is the population of the cavity mode during the adiabatic evolution and $T$ is the time process. Since the integral in the inequality is always smaller than $T \max \{ P_{1-\text{ph}}(t) \}$, inserting into the expression Eq. (13) Gaussian-shaped laser pulses with peak value $\tilde{\Omega}$, time separation $aT$ and variance $\tau T/\sqrt{2}$, one can give the following requirement

$$\kappa T \max \{ P_{1-\text{ph}}(t) \} = \kappa T \left( 1 + \frac{2g^2}{\Omega^2 g^2 a^2 / \tau^2} \right)^{-1} \ll 1. \quad (17)$$

Furthermore, in any real process (finite time, finite energy), the state of the system will precess around the dark state, instead of following it in a perfect adiabatic way. This means that some population reaches the leaky states $| e \rangle_i$. One should calculate the population $P_i$ of such unwanted states during the adiabatic process and impose the condition $\gamma \int_0^T P_i(t) dt \ll 1$. We give here and in what follows some simplified conditions. The condition for adiabatic passage can be stated as

$$\tilde{\Omega} T \gg 1. \quad (18)$$

Here $\hbar \tilde{\Omega}$ is a parameter expressing an estimate of the global value of the difference between the dark-state energy (zero) and the smallest (non-zero) eigenenergy, $E_{\text{min}}(t)$, of the Hamiltonian Eq. (12) during the transfer process. The condition on $\gamma$ reads:

$$\frac{\gamma T}{(\tilde{\Omega} T)^2} \ll 1. \quad (19)$$

The transfer time, $T$, is bounded from below and from above. Let us compare the on-resonance case, i.e., $\Delta = 0$ and the far off-resonance case, i.e., $\Delta \gg | Q'_i |, g$. We evaluate $\tilde{\Omega}$ as a time average of $E_{\text{min}}(t)$ (11). In these regimes we find

$$\tilde{\Omega}_{\Delta = 0} = \left[ g^2 + \frac{1}{2} \left( \Omega'^2 - \sqrt{(\Omega'^2 - \Omega^2)^2 + 4g^4} \right) \right]^{1/2}, \quad (20)$$

$$\tilde{\Omega}_{\Delta \gg} = \frac{1}{\Delta} \left[ g^2 + \frac{1}{2} \left( \Omega'^2 - \sqrt{(\Omega'^2 - \Omega^2)^2 + 4g^4} \right) \right], \quad (21)$$

with $\Omega'^2 = \Omega_1^2 + \Omega_2^2$.

Since we know that the adiabatic passage works better when $a \simeq \tau$ (11), from Eq. (17), we learn that it is favourable to have $g \gg \Omega$. The transfer time is restricted to be:

$$\frac{2g^2}{\kappa \tilde{\Omega}^2} \gg T \gg \frac{\gamma}{2(\omega_{\text{eff}})^2} C. \quad (22)$$

where $C = 1$ for $\Delta = 0$ and $C = 8\Delta^2 / (\Omega_{\text{eff}}^2)$ for large detuning. Provided that the previous inequality is satisfied on both sides by a factor $\alpha$, we obtained that, at least, it must be $\kappa \gamma \leq (g/\alpha)^2$ for $\Delta = 0$ and $\kappa \gamma \leq (g/\alpha)^2 (\Omega / \Delta)^2$ for large $\Delta$. Then it is clear that the optical scheme works better in the on-resonance regime.

### B. Motional state transfer

For the transfer procedure via motional state swap, it is possible, as we have seen, to obtain a Hamiltonian involving just the external degrees of freedom and the cavity mode. This is true also in the presence of decoherence processes, where one obtains the non-Hermitian Hamiltonian

$$H_{\text{eff}, i} = \frac{\eta g Q_i}{\Delta + i\gamma} (a_i b^\dagger + a_i^\dagger b) - i k b^\dagger b - i \frac{\gamma Q_i^2}{\Delta^2 + \gamma^2}. \quad (23)$$

The details of the calculation are given in App. B.2, where also the Hamiltonian without RWA is shown. To obtain Eq. (23) the conditions $\Delta \gg | Q'_i |, \eta g$ are imposed. If $T$ is the process time, the conditions we can give on $\gamma$ are (first order perturbation theory):

$$\gamma T \left( \frac{| Q'_i |}{\Delta} \right)^2, \gamma T \left( \frac{\eta g}{\Delta} \right)^2 \ll 1. \quad (24)$$

The motional scheme is based on a dark-state with respect to the cavity mode (see Eq. (14)). The actual state of the system during the evolution will precess around the dark state and so we will have a certain population $P_{1-\text{ph}}$ of the 1-photon cavity state. In order to give a condition on $\kappa$ we have to estimate such population along the adiabatic process. For Gaussian-shaped pulse we found (see App. C):

$$P_{1-\text{ph}} \leq \frac{g^2}{\tau^2} e^{a^2 / \tau^2} \left( \frac{g^2 \Omega}{\Delta} \right)^T. \quad (25)$$

The condition on $\kappa$ reads: $T \kappa P_{1-\text{ph}} \ll 1$. Thus considering for convenience $m = \max \{ \eta g, \tilde{\Omega} \}$, we see that the process time has to be ($a \simeq \tau$):

$$\frac{1}{\gamma} \left( \frac{\Delta}{m} \right)^2 \gg T \gg \frac{1}{(\tau \eta)^2 g^2} \left( \frac{\Delta}{\Omega} \right)^2. \quad (26)$$
First of all note the inverted role of $\gamma$ and $\kappa$ in the optical and in the motional scheme in restricting the process time. Then, if the inequality is satisfied on both side by a factor $\alpha$ we have that $\kappa \gamma \leq (g/\alpha)^2 (\tau \Omega/m)^2$. We see that, in the motional scheme, we have a much more strict restriction on $\kappa \gamma$ due to the presence of the small Lamb-Dicke parameter.

C. Modified optical scheme

In this section we describe how to enrich the optical scheme to obtain a new scheme, where the role of the decoherence mechanisms is the same as in the motional scheme, without the Lamb-Dicke parameter.

In the case of large detuning, i.e., $\Delta \gg |\Omega'|$, $g$, we can perform an adiabatic elimination of the excited states. Starting from the Hamiltonian Eq. (23), we obtain the following 2-level Hamiltonian (see App. B 1 and [16]):

$$H_{\text{opt}} = \frac{\Omega^2}{\Delta + i\gamma}|g_1\rangle\langle g_1| + \frac{g^2}{\Delta + i\gamma}b^\dagger b|g_3\rangle\langle g_3| \tag{27}$$

The total Hamiltonian $H_{\text{eff}} = H_{\text{opt}} + H_{\text{eff},2}$, when $\kappa = 0$, admits, as expected, the zero-energy eigenstate Eq. (13). If it is possible to compensate the total Stark shift of the ground state $|g_1\rangle_i$ - i.e., the real part of the complex energy of $|g_1\rangle_i$ in Eq. (22) - one obtains a Hamiltonian which has a similar structure to the one obtained for the motional scheme, Eq. (13):

$$\tilde{H}_{\text{eff},i} = -2i\gamma \frac{\Omega^2}{\Delta + i\gamma}|g_1\rangle\langle g_1| + \frac{g^2}{\Delta + i\gamma}b^\dagger b|g_3\rangle\langle g_3| \tag{28}$$

Physically the compensation can be realized by coupling the state $|g_1\rangle$ with another auxiliary state via a laser pulse which has the same Rabi frequency as the one that couples $|g_1\rangle_i$ to $|e_1\rangle_i$, i.e., $\Omega'$, and a detuning $-\Delta$. For simplicity, the auxiliary state is supposed to have the same spontaneous emission rate $\gamma$ as $|e_1\rangle_i$, however. When $\gamma = 0$ (while $\kappa$ can be non-zero), the Hamiltonian Eq. (28) admits the dark state with respect to the cavity mode

$$|\Psi\rangle \propto \Omega'_2|g_1\rangle_1|g_3\rangle_2|0\rangle_{\text{cav}} + \Omega'_1|g_3\rangle_1|g_1\rangle_2|0\rangle_{\text{cav}}. \tag{29}$$

To obtain the requirement on $\kappa$, we estimate the population of the 1-photon cavity state during the adiabatic transfer. Using the same notation we used above we obtained (see Appendix B):

$$P_{1-\text{ph}} \leq \frac{a^2}{\tau^2} e^{a^2/\tau^2} \left( \frac{g\Omega}{\Delta} \right)^2. \tag{30}$$

The condition on $\kappa$ is $T\kappa P_{1-\text{ph}} \ll 1$. The presence of the auxiliary levels impose on $\gamma$ the restriction

$$\gamma T \left( \frac{\Omega'}{\Delta} \right)^2 \ll 1. \tag{31}$$

Thus, for the process time, $T$, we have (as usual $a \simeq \tau$)

$$\frac{1}{\gamma} \left( \frac{\Delta}{\Omega} \right)^2 \gg T \gg \frac{1}{\tau^2 \frac{\kappa}{g} \left( \frac{\Delta}{\Omega} \right)^2}, \tag{32}$$

which is the same condition we gave for the motional scheme, but without the small Lamb-Dicke parameter. For instance the condition on $\gamma \kappa$, obtained as explained before, reads $\gamma \kappa \leq (g/\alpha)^2 \tau^2$.

D. Summary

In summary, using the motional scheme it is possible to reduce the effects of $\kappa$ by increasing the process time, since it is based on the adiabatic transfer via a dark state with respect to the cavity. The process cannot be too long in order to avoid spontaneous emission effects, which are however reduced by choosing a large detuning. On the other hand the optical scheme, based on a dark state with respect to the excited levels, can avoid the effect of the spontaneous emission by increasing the process time. In this case, however, the process must be fast with respect to the inverse of the cavity loss rate. Finally, provided we can use a stabilizing pulse to compensate for the time-dependent energy of the ground state (see also [16]), the optical scheme in the large detuning regime could be able to operate in the same fashion as the motional scheme but faster (basically by a factor of the order of the Lamb-Dicke parameter $\eta$) and then to allow for a larger spontaneous emission rate $\gamma$.

E. Numerical results

We performed numerical simulations for the suggested methods. They confirm what we have stated in the last sections. We show in Fig. 3 and Fig. 4 the transfer fidelity, respectively, for the optical scheme and for the motional one, under the same condition of the parameter $\gamma, \kappa, \Omega$ and $g$. We expressed all the quantities in unity of the coupling, $g$, between the atoms and the cavity mode. We simulated the evolution of the system as driven by Gaussian-shaped pulses. We have followed the evolution of the projections of the state onto the bare states, i.e., $|g_i\rangle$ and $|e_i\rangle$. We are interested in the population of the transfer’s target state Fig. 3.
FIG. 5. Adiabatic evolution of the system. The figure represents the evolution of the projection of the dark-state, for the optical scheme, onto the initial basis states. We see that, essentially, only the projection onto the states involved in the dark-state are populated.

It is this population that it is plotted, as a function of $\kappa$ and $\gamma$, in the Figs. (6) and (7), the time is fixed. In such shown figures we use: $\Omega = 0.05g$, $\Delta = 0$, for the optical scheme, $\Delta = 10g$ and $\eta = 0.1$ for the motional scheme.

As expected from the previously given inequalities the optical transfer is more affected by the presence of the cavity loss, than by the presence of the spontaneous emission, the vice versa is true for the motional scheme. In this parameter regime, the optical scheme works much better by means of both fidelity and transfer time, than the motional one. However is worth to remind the different role played by the two decoherence channels as it has been summarized above. In principle for a bad cavity and a small spontaneous emission rate, one shall have, waiting enough time, a fidelity unreacheable with the optical scheme.
FIG. 7. Population of the “target” state for the motional scheme as a function of the spontaneous decay rate, $\gamma$, for various values of the cavity loss rate, $\kappa$ (up); as a function of the cavity loss rate, $\kappa$, for various values of the spontaneous decay rate, $\gamma$ (down).

V. CONCLUSION

In this paper we have presented an implementation of quantum computation based on holonomic operations, i.e., non-Abelian geometric phases, with trapped neutral atoms. The method is a generalization of a previous proposal exploiting ions [7]. We showed explicitly how to realize a general unitary transformation on $n$ qubits encoded in a $(2^n + 2)$-level atom. We also developed a scalable scheme, where each qubit is stored in one atom. In this case, two-qubit operations are performed in one of the two atoms, which encodes both atoms’ initial logical state after a properly designed transfer process. We discussed in detail two possible procedures for such a state transfer, relying on adiabatic passage via a dark state of the two-atoms-plus-cavity system. The two schemes (the second of whose is originally proposed here) differ in the atomic degrees of freedom involved – respectively, internal or external. We discussed advantages and limitations of both schemes in the presence of decoherence, for different parameter regimes, and we found that the second proposal is more suitable for a situation with a comparatively lossy cavity, as it might be the case, e.g., with atomic micro-traps coupled to surface-mounted microcavities in the context of the so-called Atom Chips [20].

ACKNOWLEDGMENTS

This research has been supported by the Austrian Science Foundation, the Institute for Quantum Information GmbH, the Istituto Trentino di Cultura and the European Commission through contracts ERB-FMRX-CT96-0087, IST-1999-11055 (ACQUIRE) and HPMF-CT-1999-00211.

[1] M.A. Nielsen and I.L. Chuang, Quantum Computation and Quantum information Eds.Cambridge University Press, 2000.
[2] For a review see Geometric Phases in Physics, A. Shapere and F. Wilczek, Eds. World Scientific, 1989.
[3] A. Kitaev, Fault-tolerant computations with Anyons, quant-ph/9707032. J. Preskill, Fault-tolerant quantum computation in Introduction to quantum computation and information, Hoi-Kwong Lo, S. Popescu and T. Spiller Eds., World Scientific, Singapore, 1999.
[4] J. A. Jones et al, Nature 403, 869 (2000).
[5] G. Falci et al, Nature 407, 355 (2000).
[6] P. Zanardi, M. Rasetti, Phys. Lett A 264, 94 (1999); J. Pachos, P. Zanardi and M. Rasetti, Phys. Rev. A 61, 010305 (2000); J. Pachos, S. Chountasis Phys. Rev. A 62, 052318 (2000).
[7] L.M. Duan, J.I. Cirac and P. Zoller, Science 292, 1695 (2001).
[8] J. Pachos and P. Zanardi, Intl. J. Mod. Phys. 15, 1257 (2001).
[9] F. Wilczek and A. Zee, Phys. Rev. Lett. 52, 2111 (1984).
[10] M.Nakahara, Geometry, Topology and Physics (IOP, 1990).
[11] K. Bergmann, H. Theuer and B.W. Shore, Rev. Mod. Phys. 70, 1003 (1998).
[12] As we have shown, in principle, by choosing a system with a sufficient number of levels, we can perform any $n$ qubit gate, but such a scheme is not scalable up to large $n$ – an exponential number of ground states (namely $2^n +1$) would be required.
[13] More constructively, in order to prove universality, for a given holonomic model one can show how to find suitable
loops whose holonomies are set of quantum gates that are known to be universal.

[14] T. Pellizzari, S.A. Gardiner, J.I. Cirac and P. Zoller, Phys. Rev. Lett. 75, 3788 (1995).
[15] C. W. Gardiner, Quantum noise (Springer-Verlag, 1991)
[16] T. Pellizzari, Phys. Rev. Lett. 79, 5242 (1997).
[17] If it were not the case the term $\Delta \psi(g_3)$ would be add to the Hamiltonian $[\Omega]$.
[18] H. Zeng and F. Lin, Phys. Rev. A 50, R3589 (1994).
[19] A.S. Parkins and H.J. Kimble, J. Opt. B 1, 496 (1999).
[20] K. Brugger et al., J. Mod. Opt. 47, 2789 (2000).
[21] A. Galindo and P. Pascual, Quantum mechanics II (Springer-Verlag, 1990).
[22] A. Messiah, mécanique quantique-tome 2 (Dunod, 1972).

APPENDIX A: HOLONOMIES

1. Single-qubit gate

As described in the text any single-qubit gate can be realized by holonomic means if we consider the Hamiltonian Eq. (2) with $N = 3$. We write the coupling parameters using a kind of spherical coordinates:

$$\Omega_1 = |\Omega| \sin(\theta_1),$$
$$\Omega_2 = |\Omega| e^{-i\phi_2} \cos(\theta_1) \sin(\theta_2),$$
$$\Omega_3 = |\Omega| e^{-i\phi_3} \cos(\theta_1) \cos(\theta_2).$$

(A1)

The Hamiltonian admits two zero-eigenvalue eigenvectors $|\psi^1\rangle$ and $|\psi^2\rangle$, which can be written in terms of the ground-states $|g_k\rangle$, $k = 1, 2, 3$ as

$$|\psi^1\rangle = \cos(\theta_1)|g_1\rangle - \sin(\theta_1)[e^{-i\phi_2} \sin(\theta_2)|g_2\rangle + e^{-i\phi_3} \cos(\theta_2)|g_3\rangle]$$
$$|\psi^2\rangle = -e^{-i\phi_2} \cos(\theta_2)|g_2\rangle + e^{-i\phi_3} \sin(\theta_2)|g_3\rangle.$$ 

(A2)

We fixed the two relative phases $\phi_2$ and $\phi_3$ to zero and calculated the connection components $A_{ij}^{\Omega} = \langle \psi^i|\partial/\partial \lambda j|\psi^j\rangle$ where $\{\lambda j\} = \{\theta k, \phi j; k = 1, 2, 3; j = 2, 3\}$. We obtained the connection

$$A = A_{\theta_1} \, d\theta_2 = -i \sin(\theta_1) \sigma y \, d\theta_2,$$

(A3)

where $\sigma y$ is the $y$-Pauli matrix. The related unitary operation is $U(\chi)A = P \exp \int_\chi A \, P \exp \int_\chi A_{\theta_1} \, d\theta_2$, where the integral is along a loop $\chi$ in the sub-manifold $\theta_1, \theta_2$. The line integral can be converted using the Stokes theorem in the surface integral $\int_{S(\chi)} F_{\theta_1,\theta_2} \, d\theta_1 \, d\theta_2$, where

$$F_{\theta_1,\theta_2} = \partial_{\theta_1} A_{\theta_2} - \partial_{\theta_2} A_{\theta_1} = -i \cos(\theta_1) \sigma y$$

(A4)

are the components of the so-called curvature 2-form, and $S(\chi)$ is the surface enclosed by the loop $\chi$ in the $(\theta_1, \theta_2)$-plane. The unitary operator takes the form:

$$U(\chi) = \exp -i \sigma y \int_{S(\chi)} \cos(\theta_1) \, d\theta_1 \, d\theta_2$$

(A5)

with $\sigma y$ the $y$-Pauli matrix, i.e., we have obtained a qubit rotation around the $y$-axis of the Bloch sphere. In the same way one can obtain the gate Eq. (3), fixing $\theta_2 = \phi_3 = 0$. In this case $|\psi^1\rangle = |g_1\rangle$ for any values of the parameters and so, after a cycle in the remaining parameter submanifold, only the state $|g_2\rangle$ acquires a phase. We stated also in the text that by choosing in a suitable way the loop in the manifold $\{\theta k, \phi j; k = 1, 2, 3; j = 2, 3\}$ it is possible to obtain any single-qubit gate. Showing the feasibility of the 2-qubit phase gate is enough to conclude the universality of this approach. We stress once more that in this approach, anyway, instead of thinking how a particular gate is decomposed in single- and two-qubit gates and then to try to realize them, can be much easier from the experimental point of view to search suitable loops (experimentally it means to search for the simplest loops) to perform the quantum gate we want. Indeed we have that, given any unitary operator, $U$, there exists a closed path, in the parameter space such that the holonomy it generates coincide with $U$. 

10
2. Two-qubit gate

Realizing the phase-gate does not require more effort than realizing a single-qubit rotation. In this case the Hamiltonian is given by Eq. (3) with $N = 5$. We write the coupling parameters using a kind of spherical coordinates:

\[
\begin{align*}
\Omega_1 &= |\Omega| \sin(\theta_1), \\
\Omega_2 &= |\Omega| e^{-i\phi_2} \cos(\theta_1) \sin(\theta_2), \\
\Omega_3 &= |\Omega| e^{-i\phi_3} \cos(\theta_1) \cos(\theta_2), \\
\Omega_4 &= |\Omega| e^{-i\phi_4} \cos(\theta_1) \cos(\theta_2) \sin(\theta_3), \\
\Omega_5 &= |\Omega| e^{-i\phi_5} \cos(\theta_1) \cos(\theta_2) \cos(\theta_4). 
\end{align*}
\]

This Hamiltonian admits four zero-eigenvalue eigenvectors $|\psi^i\rangle$ ($i = 1, 2, 3, 4$), which can be written in terms of the ground states $|g_k\rangle$ ($k = 1, 2, 3, 4, 5$) as

\[
\begin{align*}
|\psi^1\rangle &= \cos(\theta_1)|g_1\rangle - \sin(\theta_1)\{e^{-i\phi_2} \sin(\theta_2)|g_2\rangle + \cos(\theta_2)\{e^{-i\phi_3} |g_3\rangle + \cos(\theta_3)\{e^{-i\phi_4} \sin(\theta_4)|g_4\rangle + e^{-i\phi_5} \cos(\theta_4)|g_5\rangle\}\}, \\
|\psi^2\rangle &= e^{-i\phi_2} \cos(\theta_2)|g_2\rangle - \sin(\theta_2)\{e^{-i\phi_3} \sin(\theta_3)|g_3\rangle + \cos(\theta_3)\{e^{-i\phi_4} \sin(\theta_4)|g_4\rangle + e^{-i\phi_5} \cos(\theta_4)|g_5\rangle\}, \\
|\psi^3\rangle &= e^{-i\phi_3} \cos(\theta_3)|g_3\rangle - \sin(\theta_3)\{e^{-i\phi_4} \sin(\theta_4)|g_4\rangle + e^{-i\phi_5} \cos(\theta_4)|g_5\rangle\}, \\
|\psi^4\rangle &= -e^{-i\phi_4} \cos(\theta_4)|g_4\rangle + e^{-i\phi_5} \sin(\theta_4)|g_5\rangle. 
\end{align*}
\]

It is worth noting that, when all the parameters (actually the angles $\theta_i$) are fixed to zero, the previous eigenstates coincide with the 4 ground states $|g_k\rangle$ ($k = 1, 2, 3, 4$): we will consider always paths which start and end at such a point. We fixed to zero the parameters $\theta_1, \theta_2, \theta_3$ -- i.e., the coupling constants $\Omega_1, \Omega_2, \Omega_3$ -- and the relative phases $\phi_2, \phi_3, \phi_4$, thus the connection gets the simple expression

\[
A = A_{\phi_5} d\phi_5 = -i \sin^2 \theta_4 |g_4\rangle \langle g_4| d\phi_5. 
\]

It gives rise to the unitary operator $U(x'') = P \exp \int_{x''} A_{\theta_2} d\theta_2$, which once again can be easily calculated by using the Stokes theorem to convert to a surface integral the line integral. Indeed, by introducing the curvature 2-form $F$, which in this case has just one non-zero component $F_{\phi_3\phi_4} = i \sin(2\theta_4)$, one gets

\[
U(x'') = \exp -i \int_{S(x'')} \sin(2\theta_4) d\theta_4 d\phi_5 |g_4\rangle \langle g_4|. 
\]

The last expression is precisely a phase gate, i.e., an operation that assigns a phase -- equal to the surface-integral -- to one ($|g_4\rangle$) out of four states.

APPENDIX B: EFFECTIVE HAMILTONIANS

1. Three-level atom in a cavity: Effective Hamiltonian

In this section we are going to find out an effective, approximate Hamiltonian, starting from a well-known Hamiltonian in quantum optics. We consider a single three-level atom interacting with a cavity and we want to take into account also the dissipative terms, i.e., as in the previous Section, the spontaneous emission from the excited level $\gamma$ and the cavity loss $\kappa$. Such information can be embodied in the effective Hamiltonian (10) as follows:

\[
H_{opt} = \hbar \{-\Delta + i\gamma\}|e\rangle \langle e| - i\kappa b^\dagger b + (\Omega(t)|e\rangle \langle g_1| + gb|e\rangle \langle g_3| + h.c.\}. 
\]

We put the state vector $|\psi\rangle \in C^3 \otimes H_{cav}$ of the system in the form

\[
|\psi\rangle = \sum_m (c_{e,m}|e, m| + c_{g_1, m}|g_1, m| + c_{g_3, m}|g_3, m + 1) 
\]

and by the Schrödinger equation we found the equation of motion for the coefficients $c_{g_1, m}$, $c_{g_3, m}$ and $c_{e, m}$.
\[ i\dot{c}_{e,m} = - (\Delta + i\gamma + ikm)c_{e,m} + \Omega c_{g1,m} + g\sqrt{m + 1} c_{g3,m}, \]
\[ i\dot{c}_{g1,m} = \Omega c_{e,m} - ikm c_{g1,m}, \]
\[ i\dot{c}_{g3,m} = g\sqrt{m + 1} c_{e,m} - i\kappa (m + 1) c_{g3,m}. \]

From the first equation one gets:
\[ i\dot{c}_{e,m} = \int_0^t e^{i(\Delta + i\gamma + ikm)(t - \tau)} (\Omega c_{g1,m} + g\sqrt{m + 1} c_{g3,m}) d\tau, \]
then substituting the coefficient \(c_{e,m}\) in the two last expressions of the Eq. (B3) with the previous expression, imposing that \(\Omega, g\sqrt{m + 1} \ll \sqrt{\Delta^2 + \gamma^2}\) and neglecting the terms of higher order in \(\Omega/\sqrt{\Delta^2 + \gamma^2}\) and \(g\sqrt{m + 1}/\sqrt{\Delta^2 + \gamma^2}\) one eventually obtains the equations:
\[ i\dot{c}_{g1,m} = \left( \frac{\Omega^2}{\Delta + i\gamma} - ikm \right) c_{g1,m} + \frac{g\Omega}{\Delta + i\gamma} c_{g3,m}, \]
\[ i\dot{c}_{g3,m} = \left( \frac{g^2 (m + 1)}{\Delta + i\gamma} - i\kappa (m + 1) \right) c_{g3,m} + \frac{g\Omega}{\Delta + i\gamma} c_{g1,m}. \]

These equations can be equivalently derived starting from a 2-level system – with internal states \(|g_1\rangle\) and \(|g_3\rangle\) – interacting with the cavity mode by the effective, approximate Hamiltonian (27).

2. Trapped atom in a cavity: Effective external Hamiltonian

Let us write the non-Hermitian effective Hamiltonian, i.e., considering also the dissipative terms, for a harmonically trapped 2-level atom inside a cavity QED:
\[ H = (\omega_0 - i\gamma)|c\rangle \langle c| + (\omega_c - i\kappa)b^\dagger b + \nu a\dagger a + (\Omega |c\rangle \langle g| e^{i\omega_c t} + h.c.) + \eta (a\dagger + a)(gb|c\rangle \langle g| + h.c.). \]

Here, \(\omega_0\) is the energy difference between the ground and the excited atomic level, \(\omega_c\) and \(b\) are respectively the energy and the annihilation operator for the cavity mode, \(\nu\) and \(a\) are the energy and the annihilation operator for the harmonic motion, \(\Omega\) and \(\omega_L\) are the Rabi frequency and the frequency of the laser light, \(\eta\) is the Lamb-Dicke parameter, \(g\) is the dipole cavity-atom coupling constant and \(\gamma\) and \(\kappa\) are respectively the decay rate from the excited state and the cavity loss.

From now on we do not consider in the equation the cavity loss, because its effect on the final Hamiltonian is trivial – indeed at the end it adds the term \(-i\kappa b^\dagger b\): so we re-introduce it just in the final Hamiltonian(s). Making the canonical transformation \(a \to ae^{-i\omega_c t}, b \to be^{-i\omega_c t}\) and writing the state vector \(|\psi\rangle \in \mathbb{C}^2 \otimes \mathcal{H}_{ext} \otimes \mathcal{H}_{cav}\) of the system in the following form:
\[ |\psi\rangle = \sum_{n,m} (c_{e,n,m}|e, n, m - 1\rangle + c_{g,n,m}|g, n, m\rangle), \]
we found the equation of motion for the coefficients \(c_{e,n,m}\) and \(c_{g,n,m}\) (\(g, \Omega\) real):
\[ i\dot{c}_{e,n,m} = (\omega_0 - i\gamma)c_{e,n,m} + \Omega e^{-i\omega_c t} c_{g,n,m-1} + \eta g e^{-i\omega_c t} (\sqrt{nm} c_{g,n-1,m} + \sqrt{(n + 1)m} e^{-i\omega_c t} c_{g,n+1,m}), \]
\[ i\dot{c}_{g,n,m} = \Omega e^{i\omega_c t} c_{e,n,m+1} + \eta g e^{i\omega_c t} (\sqrt{nm} e^{i\omega_c t} c_{e,n-1,m} + \sqrt{(n + 1)m} e^{-i\omega_c t} c_{e,n+1,m}). \]

Integrating the first equation yields
\[ i\dot{c}_{e,n,m} = \int_0^t e^{-i(\omega_0 - i\gamma)(t - \tau)} (\Omega e^{-i\omega_c \tau} c_{g,n,m-1} + \eta g e^{-i\omega_c \tau} (\sqrt{nm} e^{i\omega_c \tau} c_{g,n-1,m} + \sqrt{(n + 1)m} e^{-i\omega_c \tau} c_{g,n+1,m})) d\tau \]
then integrating by parts, considering the resonance condition \(\omega_L + \nu = \omega_c\) and introducing the parameter \(\Delta = \omega_L - \omega_0\) one gets the expression
\[ i\dot{c}_{e,n,m} = e^{-i(\omega_0 - i\gamma)t} \left[ e^{-i(\Delta + i\gamma)t} (\Omega c_{g,n,m-1} + \eta g (\sqrt{nm} c_{g,n-1,m} + \sqrt{(n + 1)m} e^{-2i\omega_c t} c_{g,n+1,m})) \right]_0^t \]
\[- \frac{i}{\Delta + i\gamma} \int_0^t e^{-i(\Delta + i\gamma)\tau} \frac{d}{d\tau} (\Omega c_{g,n,m-1} + \eta g (\sqrt{nm} c_{g,n-1,m} + \sqrt{(n + 1)m} e^{-2i\omega_c t} c_{g,n+1,m})) d\tau \]
If in the previous expression we keep the terms up to the first order in $\Omega/\sqrt{\Delta^2 + \gamma^2}$, $\eta g/\sqrt{\Delta^2 + \gamma^2}$, we obtain

$$c_{e,n,m} = e^{-i\omega L t} \frac{\eta g \Omega}{\Delta + i\gamma} \left( \sqrt{(n+1)c_{g,n+1,m-1}} + \sqrt{nmc_{g,n-1,m+1}} \right).$$

We can now substitute such an expression in the equation for $c_{g,n,m}$ and to first order in $\eta$ we find

$$i\dot{c}_{g,n,m} = \frac{\eta g \Omega}{\Delta + i\gamma} \left( \sqrt{(n+1)c_{g,n+1,m+1}} + \sqrt{nmc_{g,n+1,m-1}} \right).$$

The first term contains a Stark shift, namely $\Delta \Omega^2/(\Delta^2 + \gamma^2)$, and a decoherence part $\gamma \Omega^2/(\Delta^2 + \gamma^2)$; the other terms, while leaving unchanged the internal atomic states, involve the external (harmonic) atomic states and the cavity state. Thus it is possible to write down an effective Hamiltonian for the external atomic degrees of freedom and the cavity QED:

$$H_{\text{eff}} = \frac{\eta g \Omega}{\Delta + i\gamma} \left( ab^\dagger + e^{2i\omega t} a b^\dagger \right) - i \hbar^2 b^\dagger b - i \frac{\gamma \Omega^2}{\Delta^2 + \gamma^2}.$$  \hspace{1cm} (B12)

If the RWA is applicable (i.e., in this case $\nu \gg 1/t$ where $t$ is the time of the process we are interested in) the previous expression takes the form Eq. (C3):

$$H_{\text{eff}}^{\text{RWA}} = \frac{\eta g \Omega}{\Delta + i\gamma} (ab^\dagger + a b^\dagger) - i \hbar^2 b^\dagger b - i \frac{\gamma \Omega^2}{\Delta^2 + \gamma^2}.$$  \hspace{1cm} (B13)

**APPENDIX C: REMARKS ON THE ADIABATIC APPROXIMATION**

The optical scheme after having introduced the pulse to compensate the light shift of the ground state $|g1\rangle$ and the motional scheme rely essentially on the same Hamiltonian. Thus we will study here the generic problem of adiabatic transfer in a system driven by such a parametric Hamiltonian and we will eventually consider the actual expression of the parameters involved in the specific scheme. The starting point is the 3-level coupling Hamiltonian

$$H = D|3\rangle\langle 3| + h(G_1(t)|1\rangle\langle 3| + G_2(t)|2\rangle\langle 3| + h.c.),$$  \hspace{1cm} (C1)

where we have explicitly shown the time dependence of the coupling constants $G_i$, $i = 1, 2$. Note that for the motional scheme $D = 0$. The eigenfrequencies of the Hamiltonian Eq. (C1) are: $E_0 = 0$, $E_\pm = D \pm \sqrt{D^2 + G^2_{\text{eff}}}$, where we defined $G_{\text{eff}} = \sqrt{G_1^2 + G_2^2}$. The respective eigenstates can be written in the following form:

$$|\psi_0(t)\rangle = G_{\text{eff}}^{-1}(G_2|2\rangle - G_1|1\rangle),$$

$$|\psi_{\pm}(t)\rangle = (E_\pm + G^2_{\text{eff}})^{-1/2}(G_1|1\rangle + G_2|2\rangle + E_\pm |3\rangle).$$  \hspace{1cm} (C2)

Throughout the paper we have called the zero-energy eigenstate $|\psi_0\rangle$ the dark state. We prepare the system in the state $|1\rangle$. If at the beginning of the process $G_1/G_2 = 0$, we have that the dark state coincides with such state. Then we suppose that the coupling constant $G_i$ are modified slowly – adiabatically – towards the ratio $G_1/G_2 \to \infty$. The adiabatic theorem [2] tells us that the state of the system proceeds around the instantaneous eigenstate $|\psi_0(t)\rangle$, the asymptotic state being the state $|2\rangle$. The population of the other (two) states, $P_{\pm}$, can be in principle evaluated in the adiabatic approximation by using the expression

$$P_{\pm}(t) = \left| \int_0^1 e^{-i\alpha_{\pm,0}(\tau)} \left\langle \psi_{\pm}(\tau) \left| \frac{d}{d\tau} \psi_0(\tau) \right\rangle d\tau \right|^2,$$  \hspace{1cm} (C3)

where $\alpha_{\pm,0}(t) = \int_0^t (E_{\pm}(t') - E_0(t'))dt'$. An upper limit of the population is [2]
\[
P_{\pm} \leq \left| \frac{\langle \psi_\pm | dH/dt | \psi_0 \rangle}{\hbar (E_\pm - E_0)^2} \right|^2.
\]

Using the various expressions described before, one obtains

\[
\frac{\langle \psi_\pm | dH/dt | \psi_0 \rangle}{\hbar^2 (E_\pm - E_0)^2} = \left( 1 + \frac{D^2}{G_{\text{eff}}^2} \right)^{-1} \left( 1 \pm \frac{D}{\sqrt{D^2 + G_{\text{eff}}^2}} \right)^{-3/2} \frac{\dot{G}_1 G_2 - G_1 \dot{G}_2}{G_{\text{eff}}^3}.
\]

(C5)

Considering \( D = 0 \) and Gaussian-shaped coupling constant \( G_1 = G \exp[-(t/T - a)^2/\tau^2] \) and \( G_2 = G \exp[-(t/T + a)^2/\tau^2] \) in the previous expression we obtained for both states

\[
P_{\pm} = P = \max_t \left\{ \frac{a^2}{2T^2 G_{\text{eff}}^2} \operatorname{sech}^2 \left( \frac{2at}{T \tau^2} \right) e^{t^2 + a^2} \right\}.
\]

(C6)

The r.h.s. of the previous inequality diverges at \( t \to \pm \infty \). We consider only a finite interval – the same we used in the numerical simulation of the transfer process – where anyway the condition on the initial and final value of the ratio \( G_1/G_2 \) are pretty well satisfied. In such an interval the r.h.s. takes its maximum value when \( G_1 = G_2 \), i.e., at the center of the interval (see also the numerical simulation). One obtains

\[
P_{\pm} \leq \frac{a^2}{2T^2 G_{\text{eff}}^2} e^{a^2/\tau^2}.
\]

(C7)

When \( D \neq 0 \) the population for the two unwanted states is different. In the text we are interested in the population of the leaky states to give a condition on the rates \( \gamma \) and \( \kappa \). In the generic system considered here, this amounts (as we will see soon explicitly) to evaluate the population \( P_3 \) of the state \( |3\rangle \). From the expression for the eigenstate of the Hamiltonian we see that

\[
P_3 = \left| \frac{E_+}{\sqrt{G_{\text{eff}}^2 + E_+^2}} \langle \psi_+ | \psi \rangle + \frac{E_-}{\sqrt{G_{\text{eff}}^2 + E_-^2}} \langle \psi_- | \psi \rangle \right|^2,
\]

(C8)

where \( |\psi\rangle \) is the state of the system during the adiabatic evolution. For the optical scheme we have the identification \( |1\rangle = |g_1\rangle |g_3\rangle |2\rangle_\text{cav}, |2\rangle = |g_3\rangle |g_1\rangle |2\rangle_\text{cav} \) and \( |3\rangle = |g_3\rangle |g_3\rangle |1\rangle_\text{cav} \) for the states and \( G = g \Omega/\Delta \) for the coupling constant peak value. In this case \( D = g^2/\Delta \neq 0 \); we have estimated that, in our simulation, the term containing \( D \) in Eq. (\(\text{C7}\)) is of the order of the unity for \( P_- \) and much smaller than 1 for \( P_+ \). Thus an estimate of \( P_3 \) can be given as in the case \( D = 0 \). In such a way, and substituting the expression for \( G \) in Eq. (\(\text{C5}\)), we obtained Eq. (\(\text{C8}\)). For the motional scheme we have the identification \( |1\rangle = |1\rangle |0\rangle |2\rangle_\text{cav}, |2\rangle = |0\rangle |1\rangle |2\rangle_\text{cav} \) and \( |3\rangle = |0\rangle |0\rangle |1\rangle_\text{cav} \) for the states and \( G = g \eta \Omega/\Delta \) for the coupling constant peak value. Note that in this case \( D = 0 \). Substituting \( G \) in Eq. (\(\text{C7}\)) we obtained Eq. (\(\text{C8}\)).