Abstract. This paper presents new generators for the center of the universal enveloping algebra of the symplectic Lie algebra. These generators are expressed in terms of the column-permanent, and it is easy to calculate their eigenvalues on irreducible representations. We can regard these generators as the counterpart of central elements of the universal enveloping algebra of the orthogonal Lie algebra given in terms of the column-determinant by A. Wachi. The earliest prototype of all these central elements is the Capelli determinants in the universal enveloping algebra of the general linear Lie algebra.

Introduction. In this paper we give new generators for the center of the universal enveloping algebra of the symplectic Lie algebra $\mathfrak{sp}_N$. These generators $D_k(u)$ are expressed in terms of the “column-permanent,” and similar to the Capelli determinants, i.e., well-known central elements of the universal enveloping algebra of the general linear Lie algebra $\mathfrak{gl}_N$. As the key of the Capelli identity, these Capelli determinants are used to analyze the representations of $\mathfrak{gl}_N$ acting via the polarization operators (see [Ca1], [Ca2], [HU]). One of the remarkable properties of the Capelli determinants is that we can easily calculate their eigenvalues on irreducible representations. It is also easy to calculate the eigenvalues of our central elements $D_k(u)$.

On the other hand, it is not so obvious that $D_k(u)$ is actually central in the universal enveloping algebra. This fact can be proved as follows. In addition to $D_k(u)$, we consider another central element $D'_k(u)$ expressed in terms of the “symmetrized permanent.” We can easily check that this $D'_k(u)$ is central, but its eigenvalue is difficult to calculate. In spite of this difference, these $D_k(u)$ and $D'_k(u)$ are actually equal. We will prove this
coincidence to show the centrality of $D_k(u)$. Then, at the same time, we can also see the eigenvalue of $D_k(u)$.

More directly, our central elements are regarded as the counterpart of the central elements of the universal enveloping algebra of the orthogonal Lie algebra $\mathfrak{o}_N$ recently given by A. Wachi [W] in terms of the “column-determinant.” The discussion between $D(u)$ and $D'(u)$ above can be applied to Wachi’s elements ([I4]).

Let us explain the main result precisely. Let $J \in \text{Mat}_N(\mathbb{C})$ be a non-degenerate alternating matrix of size $N$. We can realize the symplectic Lie group as the isometry group with respect to the bilinear form determined by $J$: $\text{Sp}(J) = \{ g \in GL_N \mid {}^t g J g = J \}$.

The corresponding Lie algebra is expressed as $\mathfrak{sp}(J) = \{ Z \in \mathfrak{gl}_N \mid {}^t Z J + J Z = 0 \}$.

As generators of this $\mathfrak{sp}(J)$, we can take $F_{ij}^{\mathfrak{sp}(J)} = E_{ij} - J^{-1}E_{ji}J$, where $E_{ij}$ is the standard basis of $\mathfrak{gl}_N$. We introduce the $N \times N$ matrix $F^{\mathfrak{sp}(J)}$ whose $(i,j)$th entry is this generator: $F^{\mathfrak{sp}(J)} = (F_{ij}^{\mathfrak{sp}(J)})_{1 \leq i,j \leq N}$. We regard this matrix as an element of $\text{Mat}_N(U(\mathfrak{sp}(J)))$.

In the representation theory, the case

$$J = J_0 = \begin{pmatrix}
1 & & & \\
& 1 & & \\
& & \ddots & \\
& & & -1
\end{pmatrix}$$

is important. Indeed, we can take a triangular decomposition of $\mathfrak{sp}(J_0)$ simply as follows:

$$\mathfrak{sp}(J_0) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

Here $\mathfrak{n}^-$, $\mathfrak{h}$, and $\mathfrak{n}^+$ are the subalgebra of $\mathfrak{sp}(J_0)$ spanned by the elements $F_{ij}^{\mathfrak{sp}(J_0)}$ such that $i > j$, $i = j$, and $i < j$ respectively. Namely, the entries in the lower triangular part, in the diagonal part, and in the upper triangular part of the matrix $F^{\mathfrak{sp}(J_0)}$ belong to $\mathfrak{n}^-$, $\mathfrak{h}$, and $\mathfrak{n}^+$ respectively. We call this $\mathfrak{sp}(J_0)$ be the “split realization” of the symplectic Lie algebras.

The following is the main theorem of this paper:

**Theorem A.** The following element is central in $U(\mathfrak{sp}(J_0))$ for any $u \in \mathbb{C}$:

$$D_k(u) = \sum_{1 \leq \alpha_1 \leq \ldots \leq \alpha_k \leq N} \frac{1}{\alpha!} \text{per}(\tilde{F}_{\alpha}^{\mathfrak{sp}(J_0)} + u \mathbf{1}_\alpha) - \mathbf{1}_\alpha \text{diag}(k/2 - 1, k/2 - 2, \ldots, -k/2).$$
The notation is as follows. First, the symbol “per” means the “column-permanent.” Namely, for an $N \times N$ matrix $Z = (Z_{ij})$, we put

$$\text{per} Z = \sum_{\sigma \in S_N} Z_{\sigma(1)1}Z_{\sigma(2)2} \cdots Z_{\sigma(N)N},$$

even if the entries $Z_{ij}$ are non-commutative. Secondly, $\tilde{F}^{sp}(J_0)$ means the matrix

$$\tilde{F}^{sp}(J_0) = F^{sp}(J_0) - \text{diag}(0, \ldots, 0, 1, \ldots, 1).$$

Here the numbers of 0’s and 1’s are equal to $N/2$. Thirdly, $\mathbf{1}$ means the unit matrix. Moreover, for a matrix $Z = (Z_{ij})$ and a non-decreasing sequence $\alpha = (\alpha_1, \ldots, \alpha_k)$, we denote the matrix $((Z_{\alpha_1, \alpha_j}))_{1 \leq i,j \leq k}$ by $Z_\alpha$. Finally, we put $\alpha! = m_1! \cdots m_N!$, where $m_1, \ldots, m_N$ are the multiplicities of $\alpha = (\alpha_1, \ldots, \alpha_k)$:

$$\alpha = (\alpha_1, \ldots, \alpha_k) = (m_1 \underbrace{1, \ldots, 1}_{m_1}, m_2 \underbrace{2, \ldots, 2}_{m_2}, \ldots, m_N \underbrace{N, \ldots, N}_{m_N}).$$

This central element $D_k(u)$ is remarkable, because we can easily calculate its eigenvalue on irreducible representations of $\mathfrak{sp}(J_0)$ (Theorem 4.4). However, the centrality of $D_k(u)$ (namely Theorem A) is not so obvious.

To prove Theorem A, we consider another central element of $U(\mathfrak{sp}(J_0))$:

$$D'_k(u) = \text{per}_k(F + u \mathbf{1}; \frac{k}{2} - 1, \frac{k}{2} - 2, \ldots, -\frac{k}{2} + 1, 0)$$

Here the symbol “per$_k$” means the “symmetrized permanent.” Namely we put

$$\text{per}_k(Z; a_1, \ldots, a_k) = \frac{1}{k!} \sum_{1 \leq \alpha_1 \leq \cdots \leq \alpha_k \leq N} \sum_{\sigma, \sigma' \in S_N} \frac{1}{\alpha!} Z_{\alpha_{\sigma(1)} \alpha_{\sigma'(1)}(a_1)} \cdots Z_{\alpha_{\sigma(k)} \alpha_{\sigma'(k)}(a_k)}$$

with $Z_{ij}(a) = Z_{ij} + \delta_{ij}a$. From an invariance of this “per$_k$” (Proposition 1.9), the centrality of $D'_k(u)$ is almost obvious. However, its eigenvalue is difficult to calculate.

These $D_k(u)$ and $D'_k(u)$ are actually equal:

**Theorem B.** We have $D_k(u) = D'_k(u)$.

The centrality of $D_k(u)$ (namely Theorem A) and the eigenvalue of $D'_k(u)$ are both immediate from this Theorem B. Namely, proving Theorem B directly, we can settle these two problems at the same time.

Our elements $D_k(u)$ can be regarded as the counterpart of the central elements of $U(\mathfrak{o}_N)$ recently given by A. Wachi [W]. Wachi’s elements are also expressed in two different ways. The first expression $C_k(u)$ is given in terms of the “column-determinant,” and we can easily calculate its eigenvalue under this expression. On the other hand, the second expression $C'_k(u)$ is given in terms of the “symmetrized determinant,” and the centrality is almost obvious under this expression. Proving the coincidence $C_k(u) =$
$C'_k(u)$ directly, we can settle the following two problems: (i) the centrality of $C_k(u)$, and
(ii) the calculation of the eigenvalue of $C'_k(u)$. See Section 3 and [I4] for the details.

We note some central elements related to these elements. First, Wachi’s element $C_k(u)$ is equal to the central elements of $U(\mathfrak{so}_N)$ given in terms of the Sklyanin determinant in [M] (see also [MNO]). This coincidence is seen by comparing their eigenvalues. Moreover, $C_k(0)$ and $D_k(0)$ are equal to the central elements defined by eigenvalues in [MN]. For these elements, Capelli type identities are given.

The symmetrized permanent was also introduced to give Capelli type identities for reductive dual pairs. See [I2] and [I3] for these Capelli type identities in terms of the symmetrized determinant and the symmetrized permanent. These identities are closely related to $C_k(u)$ and $D_k(u)$.

The author is grateful to Professors Tôru Umeda and Akihito Wachi for the fruitful discussions.

1. Capelli type elements for the general linear Lie algebras. In this section, we recall the Capelli determinant, a famous central element of $U(\mathfrak{gl}_N)$ essentially given in [Ca1]. We also recall its generalization in terms of minors given in [Ca2] and its analogue in terms of permanents due to M. Nazarov [N]. These are the prototypes of the main objects of this paper and Wachi’s elements.

1.1. First let us recall the Capelli determinant. Let $E_{ij}$ be the standard basis of $\mathfrak{gl}_N$, and consider the matrix $E = (E_{ij})_{1 \leq i,j \leq N}$ in $\text{Mat}_N(U(\mathfrak{gl}_N))$. The following “Capelli determinant” in $U(\mathfrak{gl}_N)$ is well known as the key of the Capelli identity ([Ca1], [H], [U1]):

$$C^{\mathfrak{gl}_N}(u) = \det(E + u1 + \text{diag}(N-1, N-2, \ldots, 0)).$$

Here the symbol “det” means the “column-determinant.” Namely, for $N \times N$ matrix $Z = (Z_{ij})$, we put

$$\det Z = \sum_{\sigma \in S_N} \text{sgn}(\sigma)Z_{\sigma(1)1}Z_{\sigma(2)2} \cdots Z_{\sigma(N)N}.$$ 

Here each $Z_{ij}$ is an element of a (non-commutative) associative $\mathbb{C}$-algebra $A$. This $C^{\mathfrak{gl}_N}_{\det}(u)$ is known to be central:

**Theorem 1.1.** The element $C^{\mathfrak{gl}_N}(u)$ is central in $U(\mathfrak{gl}_N)$ for any $u \in \mathbb{C}$.

The eigenvalue of this Capelli determinant on irreducible representations is easily calculated:

**Theorem 1.2.** For the irreducible representation $\pi^{\mathfrak{gl}_N}_\lambda$ of $\mathfrak{gl}_N$ determined by the partition $\lambda = (\lambda_1, \ldots, \lambda_N)$, we have

$$\pi^{\mathfrak{gl}_N}_\lambda(C^{\mathfrak{gl}_N}(u)) = (u + l_1) \cdots (u + l_N).$$

Here we put $l_i = \lambda_i + N - i$. 

This is immediate from the definition of the column-determinant and the triangular decomposition

\[ \mathfrak{gl}_N = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+. \]

Here \( \mathfrak{n}^- \), \( \mathfrak{h} \), and \( \mathfrak{n}^+ \) are the subalgebras of \( \mathfrak{gl}_N \) spanned by the elements \( E_{ij} \) such that \( i > j \), \( i = j \), and \( i < j \) respectively. Namely the entries in the lower triangular part, in the diagonal part, and in the upper triangular part of \( E \) belong to \( \mathfrak{n}^- \), \( \mathfrak{h} \), and \( \mathfrak{n}^+ \) respectively. Considering the action of \( \mathfrak{C}^{\mathfrak{gl}_N}(u) \) to the highest weight vector, we can easily check Theorem 1.2.

We can rewrite this Capelli determinant in terms of the “symmetrized determinant” as follows:

**Theorem 1.3.** We have

\[ \det(E + u1 + \text{diag}(N - 1, N - 2, \ldots, 0)) = \text{Det}(E + u1; N - 1, N - 2, \ldots, 0). \]

Here the symbol “\( \text{Det} \)” means the “symmetrized determinant.” Namely, for an \( N \times N \) matrix \( Z = (Z_{ij}) \), we put

\[ \text{Det} Z = \frac{1}{N!} \sum_{\sigma, \sigma' \in S_N} \text{sgn}(\sigma) \text{sgn}(\sigma') Z_{\sigma(1)}\sigma'(1) Z_{\sigma(2)}\sigma'(2) \cdots Z_{\sigma(N)}\sigma'(N). \]

Moreover, for \( N \) parameters \( a_1, \ldots, a_N \in \mathbb{C} \), we put

\[ \text{Det}(Z; a_1, \ldots, a_N) = \frac{1}{N!} \sum_{\sigma, \sigma' \in S_N} \text{sgn}(\sigma) \text{sgn}(\sigma') Z_{\sigma(1)}\sigma'(1)(a_1) Z_{\sigma(2)}\sigma'(2)(a_2) \cdots Z_{\sigma(N)}\sigma'(N)(a_N) \]

with \( Z_{ij}(a) = Z_{ij} + \delta_{ij}a \). It is obvious that \( \text{Det} Z \) is equal to the usual determinant, if the entries are commutative. This non-commutative determinant “\( \text{Det} \)” is useful to construct central elements in \( \mathfrak{U}(\mathfrak{gl}_N) \). Indeed, we have the following.

**Proposition 1.4.** For any \( a_1, \ldots, a_N \in \mathbb{C} \), the determinant

\[ \text{Det}(E; a_1, \ldots, a_N) \]

is invariant under the adjoint action of \( \text{GL}_N(\mathbb{C}) \), and hence this is central in \( \mathfrak{U}(\mathfrak{gl}_N) \).

This is immediate from the following two lemmas:

**Lemma 1.5.** The symmetrized determinant is invariant under the conjugation by \( g \in \text{GL}_N(\mathbb{C}) \):

\[ \text{Det}(gZg^{-1}; a_1, \ldots, a_N) = \text{Det}(Z; a_1, \ldots, a_N). \]

Here \( Z \) is an arbitrary \( N \times N \) matrix whose entries are elements of an associative \( \mathbb{C} \)-algebra \( \mathcal{A} \).
Lemma 1.6. The matrix $E$ satisfies the relation
\[ \text{Ad}(g)E = {}^t g \cdot E \cdot g^{-1} \]
for any $g \in GL_N(\mathbb{C})$. Here $\text{Ad}(g)E$ means the matrix $(\text{Ad}(g)E_{ij})_{1 \leq i, j \leq N}$.

Lemma 1.6 can be checked by a direct calculation. Lemma 1.5 is also easy from the expression of “Det” in the framework of the exterior calculus. See [I1] for the details (cf. Section 2 of this paper).

For convenience, we consider the symbol $\natural_k = (k-1, k-2, \ldots, 0)$. Then both sides of Theorem 1.3 can be expressed simply as
\[ C^{\mathfrak{gl}_N}(u) = \text{det}(E + u1 + \text{diag} \natural_k), \quad C'^{\mathfrak{gl}_N}(u) = \text{Det}(E + u1 ; \natural_k). \]

These two expressions play contrast roles. Indeed, it is not so easy to calculate the eigenvalue of $C'^{\mathfrak{gl}_N}(u)$ directly, but the centrality of $C'^{\mathfrak{gl}_N}(u)$ is immediate from Proposition 1.4, because $\text{Det}(E + u1 ; \natural_k) = \text{Det}(E ; u1_N + \natural_k)$. Here $u1_N + \natural_k$ means the linear combination of the two vectors $1_N = (1, \ldots, 1)$ and $\natural_k$ in $\mathbb{C}^N$. Namely we put $u1_N + \natural_k = (u + N - 1, u + N - 2, \ldots, u)$.

Using Theorem 1.3, we can settle the following two problems at the same time: (i) the centrality of $C^{\mathfrak{gl}_N}(u)$, and (ii) the calculation of the eigenvalue of $C'^{\mathfrak{gl}_N}(u)$. Indeed, as seen above, the eigenvalue of $C^{\mathfrak{gl}_N}(u)$ and the centrality of $C'^{\mathfrak{gl}_N}(u)$ are almost obvious.

The proof of Theorem 1.3 will be given in Section 2.

1.2. Next we recall some generalizations of the Capelli determinant. First, we put
\[ C^{\mathfrak{gl}_N}_k(u) = \sum_{1 \leq \alpha_1 < \cdots < \alpha_k \leq N} \text{det}(E_{\alpha} + u1 + \text{diag} \natural_k). \]

Here we denote by $Z_{\alpha}$ the submatrix $(Z_{\alpha_i, \alpha_j})_{1 \leq i, j \leq k}$ of the matrix $Z = (Z_{ij})$. Obviously we have $C^{\mathfrak{gl}_N}_N(u) = C^{\mathfrak{gl}_N}(u)$. This element $C^{\mathfrak{gl}_N}_k(u)$ is also central in $U(\mathfrak{gl}_N)$ for any $u \in \mathbb{C}$, and known by the name of the “Capelli elements of degree $k$.”

Moreover we consider the element
\[ D^{\mathfrak{gl}_N}_k(u) = \sum_{1 \leq \alpha_1 \leq \cdots \leq \alpha_k \leq N} \frac{1}{\alpha!} \text{per}(E_{\alpha} + u1_{\alpha} - 1_{\alpha} \text{diag} \natural_k) \]
due to Nazarov [N]. Here the symbol “per” means the “column-permanent.” Namely, for any $N \times N$ matrix $Z = (Z_{ij})$, we put
\[ \text{per} Z = \sum_{\sigma \in \mathfrak{S}_N} Z_{\sigma(1)1} \cdots Z_{\sigma(N)N}. \]

Moreover, we put $\alpha! = m_1! \cdots m_N!$, where $m_1, \ldots, m_N$ are the multiplicities of $\alpha = (\alpha_1, \ldots, \alpha_k)$:
\[ \alpha = (\alpha_1, \ldots, \alpha_k) = (m_1, 1, \ldots, 1, 2, \ldots, 2, \ldots, m_2, \ldots, m_N, \ldots, N). \]

Note that $Z_{\alpha} = (Z_{\alpha_i, \alpha_j})_{1 \leq i, j \leq k}$ is not a submatrix of $Z$ in general, because $\alpha$ has some multiplicities. This $D^{\mathfrak{gl}_N}_k(u)$ is also central in $U(\mathfrak{gl}_N)$ for any $u \in \mathbb{C}$.

We can easily calculate the eigenvalues of these elements $C^{\mathfrak{gl}_N}_k(u)$ and $D^{\mathfrak{gl}_N}_k(u)$. The proof is almost the same as that of Theorem 1.2.
Proposition 1.7. For the irreducible representation \( \pi = \pi^{\mathfrak{gl}_N}_\lambda \) of \( \mathfrak{gl}_N \) determined by the partition \( \lambda = (\lambda_1, \ldots, \lambda_N) \), we have

\[
\pi(C^\mathfrak{gl}_N^\dag(u)) = \sum_{1 \leq \alpha_1 < \cdots < \alpha_k \leq N} (u + \lambda_{\alpha_1} + k - 1)(u + \lambda_{\alpha_2} + k - 2) \cdots (u + \lambda_{\alpha_k}),
\]

\[
\pi(D^\mathfrak{gl}_N^\dag(u)) = \sum_{1 \leq \alpha_1 < \cdots < \alpha_k \leq N} (u + \lambda_{\alpha_1} - k + 1)(u + \lambda_{\alpha_2} - k + 2) \cdots (u + \lambda_{\alpha_k}).
\]

We can rewrite these elements in terms of the “symmetrized determinant” and the “symmetrized permanent”:

Theorem 1.8. We have

\[
\sum_{1 \leq \alpha_1 < \cdots < \alpha_k \leq N} \det(E_{\alpha} + u\mathbf{1} + \text{diag} \Delta_k) = \text{Det}_k(E + u\mathbf{1}; \Delta_k),
\]

\[
\sum_{1 \leq \alpha_1 \leq \cdots \leq \alpha_k \leq N} \frac{1}{\alpha!} \text{Per}(E_{\alpha} + u\mathbf{1}_{\alpha} - \mathbf{1}_{\alpha} \text{diag} \Delta_k) = \text{Per}_k(E + u\mathbf{1}; -\Delta_k).
\]

Here \( \text{Det}_k \) and \( \text{Per}_k \) are defined as follows. First we put

\[
\text{Per} Z = \frac{1}{N!} \sum_{\sigma, \sigma' \in \mathfrak{S}_N} Z_{\sigma(1)\sigma'(1)} \cdots Z_{\sigma(k)\sigma'(k)}.
\]

Noting this, we put

\[
\text{Det}_k(Z) = \sum_{1 \leq \alpha_1 < \cdots < \alpha_k \leq N} \text{Det} Z_{\alpha}
\]

\[
= \sum_{1 \leq \alpha_1 < \cdots < \alpha_k \leq N} \frac{1}{k!} \sum_{\sigma, \sigma' \in \mathfrak{S}_k} \text{sgn}(\sigma) \text{sgn}(\sigma') Z_{\alpha_1(1)\alpha_1'(1)} \cdots Z_{\alpha_k(1)\alpha_k'(1)}.
\]

\[
\text{Per}_k(Z) = \sum_{1 \leq \alpha_1 \leq \cdots \leq \alpha_k \leq N} \frac{1}{\alpha!} \text{Per} Z_{\alpha}
\]

\[
= \sum_{1 \leq \alpha_1 \leq \cdots \leq \alpha_k \leq N} \frac{1}{\alpha! k!} \sum_{\sigma, \sigma' \in \mathfrak{S}_k} Z_{\alpha_1(1)\alpha_1'(1)} \cdots Z_{\alpha_k(1)\alpha_k'(1)}.
\]

Moreover, for \( k \) parameters \( a_1, \ldots, a_k \in \mathbb{C} \), we put

\[
\text{Det}_k(Z; a_1, \ldots, a_k)
\]

\[
= \sum_{1 \leq \alpha_1 < \cdots < \alpha_k \leq N} \frac{1}{k!} \sum_{\sigma, \sigma' \in \mathfrak{S}_k} \text{sgn}(\sigma) \text{sgn}(\sigma') Z_{\alpha_1(1)\alpha_1'(1)}(a_1) \cdots Z_{\alpha_k(1)\alpha_k'(1)}(a_k),
\]

\[
\text{Per}_k(Z; a_1, \ldots, a_k)
\]

\[
= \sum_{1 \leq \alpha_1 \leq \cdots \leq \alpha_k \leq N} \frac{1}{\alpha! k!} \sum_{\sigma, \sigma' \in \mathfrak{S}_k} Z_{\alpha_1(1)\alpha_1'(1)}(a_1) \cdots Z_{\alpha_k(1)\alpha_k'(1)}(a_k).
\]

These \( \text{Det}_k \) and \( \text{Per}_k \) are invariant under the conjugations:
Proposition 1.9. For $g \in GL_N(\mathbb{C})$, we have
\[
\begin{align*}
\text{Det}_k(gZg^{-1}; a_1, \ldots, a_k) &= \text{Det}_k(Z; a_1, \ldots, a_k), \\
\text{Per}_k(gZg^{-1}; a_1, \ldots, a_k) &= \text{Per}_k(Z; a_1, \ldots, a_k).
\end{align*}
\]

Hence, combining this with Lemma 1.6, we have the following.

Proposition 1.10. For arbitrary $a_1, \ldots, a_k \in \mathbb{C}$, the two elements
\[
\text{Det}_k(E; a_1, \ldots, a_k), \quad \text{Per}_k(E; a_1, \ldots, a_k)
\]
are invariant under the adjoint action of $GL_N(\mathbb{C})$, and hence central in $U(\mathfrak{gl}_N)$.

Let us denote by $C'_k(\mathfrak{gl}_N)(u)$ and $D'_k(\mathfrak{gl}_N)(u)$ the right hand sides of Theorem 1.8:
\[
\begin{align*}
C'_k(\mathfrak{gl}_N)(u) &= \text{Det}_k(E + u1; \zeta_k) = \text{Det}_k(E; u1_N + \zeta_k), \\
D'_k(\mathfrak{gl}_N)(u) &= \text{Per}_k(E + u1; -\zeta_k) = \text{Per}_k(E; u1_N - \zeta_k).
\end{align*}
\]

These are obviously central in $U(\mathfrak{gl}_N)$ for any $u \in \mathbb{C}$. However it is not so easy to calculate their eigenvalues directly.

Theorem 1.8 settles the following two problems at the same time: (i) the centralities of $C'_k(\mathfrak{gl}_N)(u)$ and $D'_k(\mathfrak{gl}_N)(u)$, and (ii) the calculation of the eigenvalues of $C'_k(\mathfrak{gl}_N)(u)$ and $D'_k(\mathfrak{gl}_N)(u)$.

2. The proof in the case of the general linear Lie algebras. In this section, we express determinants and permanents in the framework of the exterior algebra and the symmetric tensor algebra. These calculations are the prototypes of the proof of the main theorem.

2.1. First, we recall the proof of Theorem 1.3 given in [I1]. We can express the column-determinant in the framework of the exterior algebra as follows. Let $e_1, \ldots, e_N$ be $N$ anti-commuting formal variables, which generate the exterior algebra $\Lambda_N = \Lambda(\mathbb{C}^N)$. Put $\eta_{ij}(u) = \sum_{i=1}^{N} e_i E_{ij}(u)$ as an element in the extended algebra $\Lambda_N \otimes U(\mathfrak{gl}_N)$ in which the two subalgebras $\Lambda_N$ and $U(\mathfrak{gl}_N)$ commute with each other. Then, by a direct calculation, we have the following equality in $\Lambda_N \otimes U(\mathfrak{gl}_N)$:
\[
\eta_1(a_1)\eta_2(a_2)\cdots\eta_N(a_N) = e_1 e_2 \cdots e_N \det(E + \text{diag}(a_1, a_2, \ldots, a_N)).
\]

The symmetrized determinant is expressed similarly by doubling the anti-commuting variables. Let $e_1, \ldots, e_N, e_1^*, \ldots, e_N^*$ be $2N$ anti-commuting formal variables, which generate the exterior algebra $\Lambda_{2N} = \Lambda(\mathbb{C}^N \oplus \mathbb{C}^N)$. We put $\Xi(u) = \sum_{i,j=1}^{2N} e_i e_j^* E_{ij}(u)$ in $\Lambda_{2N} \otimes U(\mathfrak{gl}_N)$. Then, by a direct calculation, we have
\[
\Xi(a_1)\Xi(a_2)\cdots\Xi(a_{2N}) = N! e_1 e_1^* \cdots e_N e_N^* \text{Det}(E; a_1, a_2, \ldots, a_N).
\]
Now we can prove Theorem 1.3 using the commutation relation
\[(2.3) \quad \eta_i(a + 1)\eta_j(a) + \eta_j(a + 1)\eta_i(a) = 0.\]

This relation itself is easy from the relation \([E_{ij}, E_{kl}] = \delta_{kj}E_{il} - \delta_{il}E_{kj}\).

**Proof of Theorem 1.3.** Since \(\Xi(u) = \sum_{i=1}^{N} \eta_i(u)e_i^*\), we have
\[
\Xi(u + N - 1)\Xi(u + N - 2)\cdots\Xi(u) = \sum_{1 \leq i_1, \ldots, i_N \leq N} \eta_{i_1}(u + N - 1)e_{i_1}^* \eta_{i_2}(u + N - 2)e_{i_2}^* \cdots \eta_{i_N}(u)e_{i_N}^*.
\]
Here the indices \(i_1, \ldots, i_N\) can be regarded as a permutation of \(1, \ldots, N\), because \(e_i^*\)’s are anti-commuting. Moreover the factors \(\eta_i(a)\) can be reordered by using the commutation relation (2.3). Thus we have
\[
\Xi(u + N - 1)\Xi(u + N - 2)\cdots\Xi(u) = (-)^{\frac{N(N-1)}{2}} \sum_{\sigma \in \mathfrak{S}_N} \eta_{\sigma(1)}(u + N - 1)\eta_{\sigma(2)}(u + N - 2)\cdots\eta_{\sigma(N)}(u)e_{\sigma(1)}^*e_{\sigma(2)}^*\cdots e_{\sigma(N)}^*.
\]
Comparing this equality with (2.1) and (2.2), we reach to the assertion. \(\Box\)

**Remark.** From (2.2), we can see that \(\text{Det}(E; a_1, \ldots, a_N)\) does not depend on the order of the parameters \(a_1, \ldots, a_N\), because \(\Xi(a_1), \ldots, \Xi(a_N)\) commute with each other. Indeed \(\Xi(u)\) can be expressed as \(\Xi(u) = \Xi(0) + u\tau\) with \(\tau = \sum e_i e_i^*\), and this \(\tau\) is central in \(\Lambda_{2N} \otimes U(\mathfrak{gl}_N)\).

**2.2.** Next we go to the proof of Theorem 1.8. We only prove the second relation here, because the proof of the first one is almost the same.

We start with the expressions of our permanents in the framework of the symmetric tensor algebra. Let \(Z = (Z_{ij})\) be an \(N \times N\) matrix whose entries are elements of a (non-commutative) associative \(\mathbb{C}\)-algebra \(\mathcal{A}\). Let \(e_1, \ldots, e_N\) be \(N\) commutative formal variables, which generate the symmetric tensor algebra \(S_N = S(\mathbb{C}^N)\). We put \(\eta_j = \sum_{i=1}^{N} e_i Z_{ij}\) as an element in the extended algebra \(S_N \otimes \mathcal{A}\) in which the two subalgebras \(S_N\) and \(\mathcal{A}\) commute with each other. Then, by a direct calculation, we have the relation
\[(2.4) \quad \eta_{\beta} = \sum_{1 \leq \alpha_1 \leq \cdots \leq \alpha_k \leq N} \frac{1}{\alpha!} e_\alpha \text{per}(Z_{\alpha\beta}).\]
for 1 ≤ β_1 ≤ · · · ≤ β_k ≤ N. Here η_β and e_α denote η_{β_1} · · · η_{β_k} and e_{α_1} · · · e_{α_k} respectively, and Z_{αβ} means the matrix Z_{αβ} = (Z_{α_iβ_j})_{1 ≤ i, j ≤ k}. Moreover, by putting η_j(u) = \sum_{i=1}^{N} e_i Z_{ij}(u), this relation is generalized to

\[
(2.5) \quad η_{β_1}(a_1) · · · η_{β_k}(a_k) = \sum_{1 ≤ α_1 \leq · · · ≤ α_k ≤ N} \frac{1}{α!} e_α \text{per}(Z_{αβ} + 1_{αβ} \text{diag}(a_1, \ldots, a_k)).
\]

The symmetrized permanent is similarly expressed by doubling the commutative variables. Let e_1, . . ., e_N, e_1^*, . . ., e_N^* be 2N commutative formal variables, which generate the symmetric tensor algebra S_{2N} = S(\mathbb{C}^N \oplus \mathbb{C}^N). We put Ξ = \sum_{i,j=1}^{N} e_i e_j^* Z_{ij} in S_{2N} ⊗ \mathcal{A}. Then, we have the relation

\[
(2.6) \quad Ξ^{(k)} = \sum_{1 ≤ α_1 \leq · · · ≤ α_k ≤ N} \frac{1}{α!β!} e_α e_β^* \text{Per}(Z_{αβ}).
\]

Here x^{(k)} means the divided power: x^{(k)} = \frac{1}{k!} x^k.

It is convenient to consider the bilinear form \langle · | · \rangle on S_{2N} defined by the formula

\[
\langle e_α e_β^* \mid e_α' e_β'^* \rangle = δ_α α' δ_β β' α! β!.
\]

This is known as the “Fischer inner product.” Using this, we can rewrite (2.4) and (2.5) to get

\[
\text{per}(Z_{αβ}) = \langle η_β \mid e_α \rangle, \quad \text{per}(Z_{αβ} + 1_{αβ} \text{diag}(a_1, \ldots, a_k)) = \langle η_{β_1}(a_1) · · · η_{β_k}(a_k) \mid e_α \rangle.
\]

Similarly (2.6) can be rewritten to get

\[
\text{Per}(Z_{αβ}) = \langle Ξ^{(k)} \mid e_α e_β^* \rangle.
\]

Moreover, putting Ξ(u) = \sum_{i,j=1}^{N} e_i e_j^* Z_{ij}(u) and τ = \sum_{i=1}^{N} e_i e_i^*, we can express Per_k as

\[
\text{Per}_k(Z) = \langle Ξ^{(k)} \mid τ^{(k)} \rangle, \quad \text{Per}_k(Z \ ; a_1, \ldots, a_k) = \langle \frac{1}{k!} Ξ(a_1) · · · Ξ(a_k) \mid τ^{(k)} \rangle.
\]

These are immediate by noting the relation

\[
τ^{(k)} = \sum_{1 ≤ α_1 ≤ · · · ≤ α_k ≤ N} \frac{1}{α!} e_α e_α^*.
\]

We have a similar expression for the column-permanent:

\[
\text{per}(Z_α + u 1_α + 1_α \text{diag}(a_1, \ldots, a_k)) = \langle η_α^+(u + a_1) · · · η_α^+(u + a_k) \mid τ^{(k)} \rangle.
\]
Here we put \( \eta_j^+(u) = \eta_j(u) e_j^+ \).

Let us write these expressions simply as

\[
(2.7) \quad \operatorname{Per}_k(Z) = \langle \Xi^{(k)} \rangle, \quad \operatorname{Per}_k(Z; a_1, \ldots, a_k) = \langle \frac{1}{k!} \Xi(a_1) \cdots \Xi(a_k) \rangle,
\]

\[
(2.8) \quad \operatorname{per}(Z_\alpha + u 1_\alpha + 1_\alpha \text{ diag}(a_1, \ldots, a_k)) = \langle \eta_{\alpha_1}^+(u + a_1) \cdots \eta_{\alpha_k}^+(u + a_k) \rangle.
\]

Here we put

\[
\langle \varphi \rangle = \sum_{k=0}^{\infty} \langle \varphi | \tau^{(k)} \rangle
\]

for \( \varphi \in S_{2N} \). Note here that the sum is actually finite.

**Remark.** We can see that \( \operatorname{Per}_k(Z; a_1, \ldots, a_k) \) does not depend on the order the parameters \( a_1, \ldots, a_k \), because \( \Xi(a_1), \ldots, \Xi(a_N) \) commute with each other. Indeed, we can express \( \Xi(u) \) as \( \Xi(0) + u \tau \), and \( \tau \) is central in \( S_{2N} \otimes A \). Similarly, \( \operatorname{Det}_k(Z; a_1, \ldots, a_k) \) does not depend on the order the parameters.

Using these expressions, we can prove the second relation of Theorem 1.8 as follows:

**Proof of the second relation of Theorem 1.8.** We put \( \eta_i(u) = \sum_{j=1}^{N} e_j E_{ij}(u) \). The commutation relation

\[
\eta_i(a) \eta_j(a + 1) - \eta_j(a) \eta_i(a + 1) = 0
\]

is easy from the relation \([E_{ij}, E_{kl}] = \delta_{kj} E_{il} - \delta_{il} E_{kj}\). In particular, \( \eta_i^+(u) = \eta_i(u) e_i^+ \) satisfies the relation

\[
(2.9) \quad \eta_i^+(u) \eta_j^+(a + 1) - \eta_j^+(a) \eta_i^+(a + 1) = 0.
\]

Since \( \Xi(u) = \sum_{i,j=1}^{N} e_i e_j^+ E_{ij}(u) \) is written as \( \Xi(u) = \sum_{i=1}^{N} \tilde{\eta}_i(u) \), we have

\[
\Xi(u - k + 1) \Xi(u - k + 2) \cdots \Xi(u) = \sum_{1 \leq i_1, \ldots, i_k \leq N} \eta_{i_1}^+(u - k + 1) \eta_{i_2}^+(u - k + 2) \cdots \eta_{i_k}^+(u).
\]

The factors \( \eta_i^+(a) \) can be reordered by using the commutation relation (2.9). Thus we have

\[
\Xi(u - k + 1) \Xi(u - k + 2) \cdots \Xi(u) = \sum_{1 \leq \alpha_1 \leq \ldots \leq \alpha_k \leq N} \frac{k!}{\alpha_1! \cdots \alpha_k!} \eta_{\alpha_1}^+(u - k + 1) \eta_{\alpha_2}^+(u - k + 2) \cdots \eta_{\alpha_k}^+(u).
\]

Comparing this equality with (2.7) and (2.8), we reach to the assertion. \( \square \)

Next, let us prove Proposition 1.9. This is an application of the following lemma, an elementary fact for the “Fischer inner product” \( \langle \cdot | \cdot \rangle$: 

\[
\text{...}
\]
Lemma 2.1. Consider the standard action of $g \in GL_N$ on the vector space $\mathbb{C}^N \oplus \mathbb{C}^N$, which is naturally extended to an automorphism of $S^2_N$. In this situation, we have

$$\langle \varphi \mid \varphi' \rangle = \langle g(\varphi) \mid t^g^{-1}(\varphi') \rangle$$

for $\varphi, \varphi' \in S^2_N$.

Proof of the second relation of Proposition 1.9. For $g \in GL_N$, we put

$$h_g = \text{diag}(g, t^g) = \left(\begin{array}{cc} g & 0 \\ 0 & t^g^{-1} \end{array} \right) \in GL_{2N},$$

and consider its natural action on $S^2_N \otimes A$. By direct calculations, we have the relations

$$h_g(\Xi_Z(u)) = \Xi_{gZg^{-1}}(u) \text{ and } h_g^{-1}(\tau) = \tau \text{ for } \Xi_Z(u) = \sum_{i,j=1}^N e_ie_j^*Z_{ij}(u) \text{ and } \tau = \sum_{i=1}^N e_i^*e_i^*.$$

Since $h_g$ and $h_g^{-1}$ are automorphisms of $S^2_N \otimes A$, we have

$$\langle \Xi_{gZg^{-1}}(u_1) \cdots \Xi_{gZg^{-1}}(u_k) \mid \tau^{(k)} \rangle = \langle h_g(\Xi_Z(u_1) \cdots h_g \Xi_Z(u_k)) \mid h_g^{-1}(\tau)^{(k)} \rangle = \langle \Xi_Z(u_1) \cdots \Xi_Z(u_k) \mid \tau^{(k)} \rangle.$$

Here, we used Lemma 2.1 for the second equality. By (2.7) this implies our assertion. □

The relations for determinants in Theorem 1.8 and Proposition 1.9 can be proved similarly by considering the exterior algebra instead of the symmetric tensor algebra (see [I3]).

3. The case of the orthogonal Lie algebras. Before going to the main result in the case of the symplectic Lie algebra $\mathfrak{sp}_N$, we recall the case of the orthogonal Lie algebra $\mathfrak{o}_N$. In this case, two analogues of the Capelli determinant are known. One was given by R. Howe and T. Umeda [HU], and the other was recently given by A. Wachi [W].

3.1. First we see the general realization of $\mathfrak{o}_N$. Let $S \in \text{Mat}_N(\mathbb{C})$ be a nondegenerate symmetric matrix of size $N$. We can realize the orthogonal Lie group as the isometry group with respect to the bilinear form determined by $S$:

$$O(S) = \{g \in GL_N \mid t^gSg = S\}.$$

The corresponding Lie algebra is expressed as

$$\mathfrak{o}(S) = \{Z \in \mathfrak{gl}_N \mid t^ZS + SZ = 0\}.$$

As generators of this $\mathfrak{o}(S)$, we can take $F_{ij}^{\mathfrak{o}(S)} = E_{ij} - S^{-1}E_{ji}S$, where $E_{ij}$ is the standard basis of $\mathfrak{gl}_N$. We consider the $N \times N$ matrix $F^{\mathfrak{o}(S)} = (F_{ij}^{\mathfrak{o}(S)})_{1 \leq i, j \leq N}$ whose $(i, j)$th entry is this generator $F_{ij}^{\mathfrak{o}(S)}$. By a direct calculation, this $F^{\mathfrak{o}(S)}$ satisfies the following relation:
Lemma 3.1. For any $g \in O(S)$, we have
\[ \text{Ad}(g)F^\sigma(S) = t^g \cdot F^\sigma(S) \cdot t^g. \]

Here $\text{Ad}(g)F^\sigma(S)$ means the matrix $(\text{Ad}(g)F^\sigma(S))_{ij}$.1 \leq i,j \leq N$.

Combining this with Proposition 1.9, we have the following.

Proposition 3.2. The two elements
\[ \text{Det}_k(F^\sigma(S); a_1, \ldots, a_k), \quad \text{Per}_k(F^\sigma(S); a_1, \ldots, a_k) \]
are invariant under the adjoint action of $O(S)$, and in particular these are central in $U(o(S))$.

Thus, as in the case of $\mathfrak{gl}_N$, the symmetrized determinant and the symmetrized permanent are useful to construct central elements of $U(o(S))$. On the other hand, unfortunately, it seems not easy to construct central elements of $U(o(S))$ similarly using the column-determinant or the column-permanent at least for general $S$.

However, for some special $S$ ($S = 1$ and $S = S_0 = (\delta_{i,N+1-j})_{1 \leq i,j \leq N}$), we have analogues of the Capelli determinant expressed in terms of the column-determinant.

3.2. First, let us consider the case that $S$ is equal to the unit matrix $1$. Namely we consider the Lie algebra consisting of all alternating matrices:
\[ \mathfrak{o}(1) = \{Z \in \mathfrak{gl}_N \mid Z + t^Z = 0\}. \]

In this case, R. Howe and T. Umeda gave an analogue of the Capelli determinant in terms of the column-determinant:

Theorem 3.3 ([HU]). The following element is central in $U(o(1))$ for any $u \in \mathbb{C}$:
\[ C^{\mathfrak{o}(1)}(u) = \text{det}(F^{\mathfrak{o}(1)} + u1 + \text{diag} \mathbb{Z}_N). \]

As in the case of $\mathfrak{gl}_N$, we can rewrite this in terms of the symmetrized determinant:

Theorem 3.4 ([IU]). We have
\[ \text{det}(F^{\mathfrak{o}(1)} + u1 + \text{diag} \mathbb{Z}_N) = \text{Det}(F^{\mathfrak{o}(1)N} + u1; \mathbb{Z}_N). \]

Theorem 3.3 is immediate from this Theorem 3.4. Indeed, by Proposition 3.2,
\[ C^{\mathfrak{o}(1)}(u) = \text{Det}(F^{\mathfrak{o}(1)} + u1; \mathbb{Z}_N) = \text{Det}(F^{\mathfrak{o}(1)}; u1N + \mathbb{Z}_N) \]
is central in $U(o(1))$ for any $u \in \mathbb{C}$.
Remarks. (1) As in the case of $\mathfrak{gl}_N$, we have the following generalization of $C^{\mathfrak{so}(1)}(u)$:

$$C^{\mathfrak{so}(1)}_k(u) = \sum_{1 \leq \alpha_1 < \cdots < \alpha_k \leq N} \det(F^{\mathfrak{so}(1)}_\alpha + u1 + \text{diag} \tilde{z}_k).$$

This can be rewritten in terms of “$\text{Det}_k$” as

$$\sum_{1 \leq \alpha_1 < \cdots < \alpha_k \leq N} \det(F^{\mathfrak{so}(1)}_\alpha + u1 + \text{diag} \tilde{z}_k) = \text{Det}_k(F^{\mathfrak{so}(1)_N} + u1; \tilde{z}_k).$$

(2) These elements are quite similar to the Capelli elements $C^{\mathfrak{gl}_N}_k(u)$. However, it is not easy to calculate their eigenvalues. Indeed, for this realization $\mathfrak{o}(1)$, we can not take its triangular decomposition so simply as (1.1).

3.3. Next, we see the case $S = S_0 = (\delta_{i,N+1-j})$, namely we consider the split realization of the orthogonal Lie algebra:

$$\mathfrak{o}(S_0) = \{Z = (Z_{ij}) \in \mathfrak{gl}_N \mid Z_{ij} + Z_{N+1-j,N+1-i} = 0\}.$$  

A central element of $U(\mathfrak{o}(S_0))$ was recently given in terms of the column-determinant:

**Theorem 3.5 ([W]).** The element

$$C^{\mathfrak{o}(S_0)}(u) = \det(F^{\mathfrak{o}(S_0)} + u1 + \text{diag} \tilde{z}_N)$$

is central in $U(\mathfrak{o}(S_0))$ for any $u \in \mathbb{C}$. Here $\tilde{z}_N$ is the following sequence of length $N$:

$$\tilde{z}_N = \left\{ \begin{array}{ll}
(\frac{N}{2} - 1, \frac{N}{2} - 2, \ldots, 0, 0, \ldots, -\frac{N}{2} + 1), & N: \text{even}, \\
(\frac{N}{2} - 1, \frac{N}{2} - 2, \ldots, -\frac{1}{2}, 0, -\frac{1}{2}, \ldots, -\frac{N}{2} + 1), & N: \text{odd}.
\end{array} \right.$$  

The proof of this theorem is not so easy (Wachi showed the commutativity with the generators of $\mathfrak{o}(S_0)$ by employing the exterior calculus). On the other hand, we can easily calculate its eigenvalue:

**Theorem 3.6 ([W]).** Let $\pi^{\mathfrak{o}(S_0)}_\lambda$ be the irreducible representation of $\mathfrak{o}(S_0)$ determined by the partition $\lambda = (\lambda_1, \ldots, \lambda_{[N/2]})$, where $[N/2]$ means the greatest integer not exceeding $N/2$. Then we have

$$\pi^{\mathfrak{o}(S_0)}_\lambda(C^{\mathfrak{o}(S_0)}(u)) = \left\{ \begin{array}{ll}
(u^2 - l_1^2)(u^2 - l_2^2) \cdots (u^2 - l_{N/2}^2), & N: \text{even}, \\
u(u^2 - l_1^2)(u^2 - l_2^2) \cdots (u^2 - l_{N/2}^2), & N: \text{odd}.
\end{array} \right.$$

Here we put $l_i = \lambda_i + N/2 - i$.

The proof is almost the same as that of Theorem 1.2. Namely this is easy from the definition of the column-determinant and the triangular decomposition

$$\mathfrak{o}(S_0) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$
Here $n^-$, $h$, and $n^+$ are the subalgebras of $\mathfrak{o}(S_0)$ spanned by the elements $F_{ij}(S_0)$ such that $i > j$, $i = j$, and $i < j$ respectively. Namely, the entries in the lower triangular part, in the diagonal part, and in the upper triangular part of the matrix $F^\mathfrak{o}(S_0)$ belong to $n^-$, $h$, and $n^+$ respectively.

We can also rewrite $C^\mathfrak{o}(S_0)(u)$ in terms of the symmetrized determinant. To see this we put
\[ C^\prime\mathfrak{o}(S_0)(u) = \text{Det}(F^\mathfrak{o}(S_0) + u1; \tilde{2}_N) = \text{Det}(F^\mathfrak{o}(S_0); u1_N + \tilde{2}_N). \]

We can easily check that this $C^\prime\mathfrak{o}(S_0)(u)$ is central in $U(\mathfrak{o}(S_0))$ for any $u \in \mathbb{C}$. On the other hand, it is not so easy to calculate its eigenvalue. However this was given through a hard and complicated calculation:

**Theorem 3.7 ([I1]).** We have
\[
\pi^\mathfrak{o}(S_0)(C^\prime\mathfrak{o}(S_0)(u)) = \begin{cases} 
(u^2 - l_1^2)(u^2 - l_2^2) \cdots (u^2 - l_{N/2}^2), & N: \text{even}, \\
u(u^2 - l_1^2)(u^2 - l_2^2) \cdots (u^2 - l_{[N/2]}^2), & N: \text{odd}.
\end{cases}
\]

Comparing this with Theorem 3.6, we have $C^\mathfrak{o}(S_0)(u) = C^\prime\mathfrak{o}(S_0)(u)$ (recall that any central element in the universal enveloping algebras of semisimple Lie algebras is determined by its eigenvalue):

**Theorem 3.8 ([W]).** We have
\[
\text{det}(F^\mathfrak{o}(S_0) + u1 + \text{diag} \tilde{2}_N) = \text{Det}(F^\mathfrak{o}(S_0) + u1; \tilde{2}_N).
\]

This equality was first shown by A. Wachi in this way. Namely this proof depends on the two non-trivial results Theorems 3.5 and 3.7.

However, we can also prove Theorem 3.8 directly not using Theorems 3.5 and 3.7 (see [I4]; this is similar to the proof of the main theorem in this paper, but easier). Conversely, Theorems 3.5 and 3.7 follow from this Theorem 3.8 immediately.

### 3.4. These results can be generalized in terms of minors:

**Theorem 3.9 ([W]).** The following element is central in $U(\mathfrak{o}(S_0))$ for any $u \in \mathbb{C}$:
\[
C^\mathfrak{o}(S_0)(u) = \sum_{1 \leq \alpha_1 < \cdots < \alpha_k \leq N} \text{det}(\tilde{F}^\mathfrak{o}(S_0) + u1 + \text{diag}(k/2 - 1, k/2 - 2, \ldots, -k/2)).
\]

Here we put
\[
\tilde{F}^\mathfrak{o}(S_0) = \begin{cases} 
F^\mathfrak{o}(S_0) + \text{diag}(0, \ldots, 0, 1, \ldots, 1), & N: \text{even}, \\
F^\mathfrak{o}(S_0) + \text{diag}(0, \ldots, 0, \frac{k}{2}, 1, \ldots, 1), & N: \text{odd},
\end{cases}
\]
where the numbers of 0’s and 1’s are equal to $[N/2]$.

This central element can be rewritten in terms of the symmetrized determinant:
Theorem 3.10 ([W]). We have
\[ \sum_{1 \leq \alpha_1 < \cdots < \alpha_k \leq N} \det(\tilde{F}_\alpha^{(S_0)} + u \mathbf{1} + \text{diag}(k, -k/2, \ldots, -k/2)) = \det_k(F^{(S_0)} + u \mathbf{1} ; \tilde{\tau}_k). \]

These can be deduced from Theorem 3.8. See [W] for the details.

Remarks. (1) We can also express \( C_k^{(S_0)}(u) \) as
\[ C_k^{(S_0)}(u) = \sum_{1 \leq \alpha_1 < \cdots < \alpha_k \leq N} \det(\tilde{F}_\alpha^{(S_0)} + u \mathbf{1} + \text{diag}(k, -k/2, \ldots, -k/2 + 1)). \]
Here \( \tilde{F}_\alpha^{(S_0)} \) is defined by
\[ \tilde{F}_\alpha^{(S_0)} = \begin{cases} F^{(S_0)} - \text{diag}(1, \ldots, 1, 0, \ldots, 0), & N: \text{even}, \\ F^{(S_0)} - \text{diag}(1, \ldots, 1, \frac{1}{2}, 0, \ldots, 0), & N: \text{odd}, \end{cases} \]
where the numbers of 1’s and 0’s are equal to \( \lfloor N/2 \rfloor \).

(2) The element \( C_k^{(S_0)}(u) \) is also equal to the central element given in [M] in terms of the Sklyanin determinant. This is seen by comparing their eigenvalues. See [M], [MN], [MNO], [IU], [II], [W] for the details.

(3) The following relation holds for general \( S \) [IU]:
\[ \text{Det}_{2k}(F^{(S)} ; \tilde{\tau}_{2k}) = \sum_{1 \leq \alpha_1 < \cdots < \alpha_{2k} \leq N} \text{Pf}(F^{(S)} S)_\alpha \text{Pf}(S^{-1}F^{(S)})_\alpha. \]
Here we define the Pfaffian \( \text{Pf} Z \) for an alternating matrix \( Z = (Z_{ij}) \) of size \( 2k \) by
\[ \text{Pf} Z = \frac{1}{2^k k!} \sum_{\sigma \in S_{2k}} \text{sgn}(\sigma) Z_{\sigma(1)\sigma(2)} Z_{\sigma(3)\sigma(4)} \cdots Z_{\sigma(2k-1)\sigma(2k)}. \]

4. The case of the symplectic Lie algebras. In this section, we introduce the main object of this paper, namely an analogue of the Capelli determinant for the symplectic Lie algebra \( \mathfrak{sp}_N \). We can regard this as the direct counterpart of the element \( C_k^{(S_0)}(u) \) due to A. Wachi, but this element is given in terms of the column-permanent not in terms of the column-determinant.

4.1. First we see the general realization of \( \mathfrak{sp}_N \). Let \( J \in \text{Mat}_N(\mathbb{C}) \) be a non-degenerate alternating matrix of size \( N \) (hence \( N \) must be even; let us put \( n = N/2 \)). We can realize the symplectic Lie group as the isometry group with respect to the bilinear form determined by \( J \):
\[ \mathbb{S}p(J) = \{ g \in GL_N \mid {}^t g J g = J \}. \]
The corresponding Lie algebra is expressed as
\[ \mathfrak{sp}(J) = \{ Z \in \mathfrak{gl}_N \mid {}^t Z J + J Z = 0 \}. \]
As generators of this \( \mathfrak{sp}(J) \), we can take \( F_{ij}^{\mathfrak{sp}(J)} = E_{ij} - J^{-1}E_{ji} J \). We consider the \( N \times N \) matrix \( F^{\mathfrak{sp}(S)} = (F_{ij}^{\mathfrak{sp}(J)})_{1 \leq i, j \leq N} \) whose \((i, j)\)th entry is this generator \( F_{ij}^{\mathfrak{sp}(J)} \).
By a direct calculation, this \( F^{\mathfrak{sp}(S)} \) satisfies the following relation:
Lemma 4.1. For any \( g \in Sp(J) \), we have
\[
\text{Ad}(g) F^{\alpha(J)} = t^g \cdot F^{\alpha(J)} \cdot t^{-1}.
\]
Here \( \text{Ad}(g) F^{\alpha(J)} \) means the matrix \( (\text{Ad}(g) F^{\alpha(J)})_{ij} \) for \( 1 \leq i,j \leq N \).

Combining this with Proposition 1.9, we have the following proposition:

Proposition 4.2. The two elements
\[
\text{Det}_k(F^{\alpha(J)}; a_1, \ldots, a_k), \quad \text{Per}_k(F^{\alpha(J)}; a_1, \ldots, a_k)
\]
are invariant under the adjoint action of \( Sp(J) \), and in particular this is central in \( U(\mathfrak{sp}(J)) \).

Thus, the symmetrized determinant and the symmetrized permanent are useful to construct central elements of \( U(\mathfrak{sp}(J)) \) as in the case of \( \mathfrak{gl}_N \). However, unfortunately, the column-determinant and the column-permanent do not seem so useful for this purpose at least for general \( J \).

4.2. Let us consider the split realization of the symplectic Lie algebra. Namely we consider the case
\[
J = J_0 = \begin{pmatrix}
1 & & & \\
& \ddots & & \\
& & -1 & \\
-1 & & & 1
\end{pmatrix}.
\]

It is convenient to introduce the symbols
\[
i' = N + 1 - i, \quad \varepsilon(i) = \begin{cases} -1, & 1 \leq i \leq n, \\ +1, & n + 1 \leq i \leq N, \end{cases}
\]
so that \( J_0 = (\varepsilon(j) \delta_{ij'})_{1 \leq i,j \leq N} \) and \( F^{\alpha(J_0)}_{ij} = E_{ij} - \varepsilon(i) \varepsilon(j) E_{ji'} \). Moreover the commutation relation of \( F^{\alpha(J_0)}_{ij} \) is given by
\[
[F^{\alpha(J_0)}_{ij}, F^{\alpha(J_0)}_{kl}] = F^{\alpha(J_0)}_{il} \delta_{kj} - F^{\alpha(J_0)}_{kj} \delta_{il}
+ \varepsilon(k) \varepsilon(l) F^{\alpha(J_0)}_{ij'} \delta_{kj'} + \varepsilon(i) \varepsilon(j) F^{\alpha(J_0)}_{ji'} \delta_{j'i}.
\]

This realization \( \mathfrak{sp}(J_0) \) is important in the representation theory. Indeed, we can take a triangular decomposition of \( \mathfrak{sp}(J_0) \) simply as follows:
\[
\mathfrak{sp}(J_0) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.
\]

Here \( \mathfrak{n}^- \), \( \mathfrak{h} \), and \( \mathfrak{n}^+ \) are the subalgebra of \( \mathfrak{sp}(J_0) \) spanned by the elements \( F^{\alpha(J_0)}_{ij} \) such that \( i > j \), \( i = j \), and \( i < j \) respectively. We call this \( \mathfrak{sp}(J_0) \) be the “split realization” of the symplectic Lie algebra.
The main object of this paper is the following element of $U(\mathfrak{sp}(J_0))$:

$$D_{k}^{\text{sp}(J_0)}(u) = \sum_{1 \leq \alpha_1 \leq \cdots \leq \alpha_k \leq N} \frac{1}{\alpha!} \text{per}(\tilde{F}_{\alpha}^{\text{sp}(J_0)} + u1_{\alpha} - 1_{\alpha} \text{diag}(\frac{k}{2} - 1, \frac{k}{2} - 2, \ldots, -\frac{k}{2})).$$

Here $\tilde{F}_{\alpha}^{\text{sp}(J_0)}$ means the matrix

$$\tilde{F}_{\alpha}^{\text{sp}(J_0)} = F_{\alpha}^{\text{sp}(J_0)} - \text{diag}(0, \ldots, 0, 1, \ldots, 1),$$

where the numbers of 0’s and 1’s are equal to $n$.

**Theorem 4.3.** The element $D_{k}^{\text{sp}(J_0)}(u)$ is central in $U(\mathfrak{sp}(J_0))$ for any $u \in \mathbb{C}$.

The eigenvalue of $D_{k}^{\text{sp}(J_0)}(u)$ on the irreducible representations of $\mathfrak{sp}(J_0)$ can be calculated easily by noting the triangular decomposition (4.2):

**Theorem 4.4.** For the representation $\pi_{\lambda}^{\text{sp}(J_0)}$ of $\mathfrak{sp}(J_0)$ determined by the partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, we have

$$\pi_{\lambda}^{\text{sp}(J_0)}(D_{k}^{\text{sp}(J_0)}(u)) = \sum_{l=0}^{k} \sum_{\substack{n \leq \alpha_1 \leq \cdots \leq \alpha_l \leq n \leq \alpha_{l+1} \leq \cdots \leq \alpha_k \leq N}} (u + \lambda_{\alpha_1} - \frac{k}{2} + l)(u + \lambda_{\alpha_2} - \frac{k}{2} + 2) \cdots (u + \lambda_{\alpha_l} - \frac{k}{2} + l) \cdot (u - \lambda_{\alpha_{l+1}} - \frac{k}{2} + l + 1) \cdots (u - \lambda_{\alpha_k} + \frac{k}{2} - 1).$$

To prove Theorem 4.3, we additionally consider the following element:

$$D_{k}^{\text{sp}(J_0)}(u) = \text{Per}_k(F_{\alpha}^{\text{sp}(J_0)} + u1_{\alpha} ; \tilde{e}_k) = \text{Per}_k(F_{\alpha}^{\text{sp}(J_0)} ; u1_N + \tilde{e}_k).$$

It is obvious from Proposition 4.2 that this $D_{k}^{\text{sp}(J_0)}(u)$ is central in $U(\mathfrak{sp}(J_0))$ for any $u \in \mathbb{C}$. However it is not so easy to calculate the eigenvalue of $D_{k}^{\text{sp}(J_0)}(u)$ directly.

Actually these two elements $D_{k}^{\text{sp}(J_0)}(u)$ and $D_{k}^{\text{sp}(J_0)}(u)$ are equal:

**Theorem 4.5.** We have $D_{k}^{\text{sp}(J_0)}(u) = D_{k}^{\text{sp}(J_0)}(u)$, namely

$$\sum_{1 \leq \alpha_1 \leq \cdots \leq \alpha_k \leq N} \frac{1}{\alpha!} \text{per}(\tilde{F}_{\alpha}^{\text{sp}(J_0)} + u1_{\alpha} - 1_{\alpha} \text{diag}(\frac{k}{2} - 1, \frac{k}{2} - 2, \ldots, -\frac{k}{2})) = \text{Per}_k(F_{\alpha}^{\text{sp}(J_0)} + u1_{\alpha} ; \tilde{e}_k).$$

Theorem 4.3 is immediate from this. Moreover, using Theorems 4.4 and 4.5, we can easily see the eigenvalue of $D_{k}^{\text{sp}(J_0)}(u)$. Thus, as in the case of $\mathfrak{o}(S_0)$, this Theorem 4.5 settles two problems at the same time.
The remainder of this paper is devoted to the proof of Theorem 4.5.

Remarks. (1) We can also express $D_{k}^{\text{sp}(J_{0})}(u)$ as

$$D_{k}^{\text{sp}(J_{0})}(u) = \sum_{1 \leq \alpha_{1} \leq \cdots \leq \alpha_{k} \leq N} \frac{1}{\alpha!} \text{Per}(\tilde{F}_{\alpha}^{\text{sp}(J_{0})} + u1_{\alpha} - 1_{\alpha} \text{diag}(\frac{k}{2}, \frac{k}{2}, \ldots, \frac{k}{2} + 1)).$$

Here we put $\tilde{F}_{\alpha}^{\text{sp}(J_{0})} = F_{\alpha}^{\text{sp}(J_{0})} + \text{diag}(1, \ldots, 1, 0, \ldots, 0)$.

(2) Considering the generating function of $D_{k}^{\text{sp}(J_{0})}(u)$, we can rewrite Theorem 4.4 more simply (see [I3]). Moreover, the right hand side of Theorem 4.4 can be regarded as the complete symmetric polynomials associated to a kind of factorial power. See [I5] for the details.

(3) From Theorem 4.4, we see that $D_{k}^{\text{sp}(J_{0})}(0)$ is equal to $D_{k}$ defined in [MN]. Moreover, as the counterpart of (3.1), we have the relation

$$\text{Per}_{2k}(F_{\alpha}^{J} \cdot Z_{2k}) = \sum_{1 \leq \alpha_{1} \leq \cdots \leq \alpha_{2k} \leq N} \frac{1}{\alpha!} \text{Hf}(F_{\alpha}^{J}) \text{Hf}(J^{-1} F_{\alpha}^{J}).$$

for general $J$. Here we define the Hafnian $\text{Hf}Z$ for a symmetric matrix $Z = (Z_{ij})$ of size $2k$ by

$$\text{Hf}Z = \frac{1}{2^{k}k!} \sum_{\sigma \in S_{2k}} Z_{\sigma(1)\sigma(2)}Z_{\sigma(3)\sigma(4)} \cdots Z_{\sigma(2k-1)\sigma(2k)}.$$

This is deduced from Theorem 5.1 in [MN].

5. Proof of the main theorem. Let us show Theorem 4.5 using the symmetric tensor algebra. This proof is similar to that of Theorem 1.8, but more complicated. Namely we need the variable transformation method developed in [IU], [I2], [I3].

Hereafter, we omit the superscript $\text{sp}(J_{0})$. Namely we denote $F_{\text{sp}(J_{0})}$, $F_{ij}^{\text{sp}(J_{0})}$, $D_{k}^{\text{sp}(J_{0})}(u)$, and $D_{k}^{\text{sp}(J_{0})}(u)$ simply by $F$, $F_{ij}$, $D_{k}(u)$, and $D_{k}'(u)$ respectively.

5.1. First, let us express both sides of Theorem 4.5 using the symmetric tensor algebra $S_{2N} = S(\mathbb{C}^{N} \oplus \mathbb{C}^{N})$. Let $e_{1}, \ldots, e_{N}, e^{*}_{1}, \ldots, e^{*}_{N}$ be the standard generators of $S_{2N}$.

In the extended algebra $S_{2N} \otimes U(\mathfrak{sp}(J_{0}))$, we put

$$\mathcal{E} = \sum_{i,j=1}^{N} e_{i}e^{*}_{j}F_{ij}, \quad \mathcal{E}(u) = \sum_{i,j=1}^{N} e_{i}e^{*}_{j}F_{ij}(u), \quad \tau = \sum_{i=1}^{N} e_{i}e^{*}_{i},$$

so that $\mathcal{E}(u) = \mathcal{E} + u\tau$. Then, by (2.7), we can express $D_{k}'(u)$ as

$$D_{k}'(u) = \text{Per}_{k}(F + u1_{1} + \cdots + \frac{k}{2} - 1, 0)$$

$$= \frac{1}{k!} \langle \mathcal{E}(u - \frac{k}{2} + 1)\mathcal{E}(u - \frac{k}{2} + 2) \cdots \mathcal{E}(u + \frac{k}{2} - 1) \cdot \mathcal{E}(u) \rangle$$

$$= \frac{1}{k!} \langle \mathcal{E}^{k-1}(u - \frac{k}{2} + 1) \cdot \mathcal{E}(u) \rangle.$$
Here $\Xi^k(u)$ means the “rising factorial power”

$$\Xi^k(u) = \Xi(u)\Xi(u+1)\cdots\Xi(u+k-1).$$

Let us express $D_k(u)$ similarly. We put $\eta_j(u) = \sum_{i=1}^N e_i f_{ij}(u)$ and $\tilde{\eta}_j(u) = \eta_j(u)e_1^*$, so that $\sum_{j=1}^N \eta_j^*(u) = \Xi(u)$. Moreover, we put $\tilde{\eta}_j(u) = \sum_{i=1}^N e_i \tilde{f}_{ij}(u)$ and $\tilde{\eta}_j^*(u) = \tilde{\eta}_j(u)e_1^*$. Then $\eta_j^*(u)$ and $\tilde{\eta}_j^*(u)$ are related by

$$(5.2) \quad \tilde{\eta}_j^*(u) = \begin{cases} 
\eta_i^*(u), & 1 \leq i \leq n, \\
\eta_i^*(u-1), & n+1 \leq i \leq N.
\end{cases}$$

In this notation, we have

$$\text{per}(\tilde{F}_\alpha + u \mathbf{1}_1 - \mathbf{1}_1 \text{diag}(\frac{k}{2} - 1, \frac{k}{2} - 2, \ldots, -\frac{k}{2}))$$

$$= \langle \tilde{\eta}_{\alpha_1}^*(u - \frac{k}{2} + 1) \tilde{\eta}_{\alpha_2}^*(u - \frac{k}{2} + 2) \cdots \tilde{\eta}_{\alpha_k}^*(u + \frac{k}{2}) \rangle$$

by (2.8). Thus we can express $D_k(u)$ as

$$(5.3) \quad D_k(u) = \sum_{1 \leq \alpha_1 \leq \cdots \leq \alpha_k \leq N} \frac{1}{\alpha!} \langle \tilde{\eta}_{\alpha_1}^*(u - \frac{k}{2} + 1) \tilde{\eta}_{\alpha_2}^*(u - \frac{k}{2} + 2) \cdots \tilde{\eta}_{\alpha_k}^*(u + \frac{k}{2}) \rangle.$$

Remark. Recall that $\Xi(u)$ and $\Xi(w)$ are commutative for any $u, w \in \mathbb{C}$, because $\tau$ is central.

5.2. By (5.1) and (5.3), our goal $D_k(u) = D'_k(u)$ can be expressed as

$$\sum_{1 \leq \alpha_1 \leq \cdots \leq \alpha_k \leq N} \frac{k!}{\alpha!} \langle \tilde{\eta}_{\alpha_1}^*(u - \frac{k}{2} + 1) \tilde{\eta}_{\alpha_2}^*(u - \frac{k}{2} + 2) \cdots \tilde{\eta}_{\alpha_k}^*(u + \frac{k}{2}) \rangle = \langle \Xi^{k-1}(u - \frac{k}{2} + 1) \cdot \Xi(u) \rangle.$$

Replacing $u$ by $u + \frac{k}{2} - 1$, we can rewrite this simply as

$$\sum_{1 \leq \alpha_1 \leq \cdots \leq \alpha_k \leq N} \frac{k!}{\alpha!} \langle \tilde{\eta}_{\alpha_1}^*(u) \tilde{\eta}_{\alpha_2}^*(u + 1) \cdots \tilde{\eta}_{\alpha_k}^*(u + k - 1) \rangle = \langle \Xi^{k-1}(u) \cdot \Xi(u + \frac{k}{2} - 1) \rangle.$$

Let us prove Theorem 4.5 in this form. Namely, we hereafter aim the following relation:

Lemma 5.1. We have

$$\langle W_k(u) \rangle = \langle W'_k(u) \rangle.$$

Here we put

$$W_k(u) = \sum_{1 \leq \alpha_1 \leq \cdots \leq \alpha_k \leq N} \frac{k!}{\alpha!} \tilde{\eta}_{\alpha_1}^*(u) \tilde{\eta}_{\alpha_2}^*(u + 1) \cdots \tilde{\eta}_{\alpha_k}^*(u + k - 1),$$

$$W'_k(u) = \Xi^{k-1}(u) \cdot \Xi(u + \frac{k}{2} - 1) = \Xi^k(u) - \frac{k}{2} \Xi^{k-1}(u) \tau.$$
5.3. In Sections 5.3–5.5, we will study the relation between $W_k(u)$ and the factorial powers of $\Xi(u)$. First, $W_k(u)$ is expressed in terms of $\eta^\dagger_i(u)$ as

$$W_k(u) = \sum_{l=0}^{k} \sum_{1 \leq \alpha_1 \leq \cdots \leq \alpha_l \leq n} \sum_{n+1 \leq \alpha_{l+1} \leq \cdots \leq \alpha_k \leq N} \frac{k!}{\alpha_i! \alpha_j!} \eta^\dagger_{\alpha_1}(u) \eta^\dagger_{\alpha_2}(u) \cdots \eta^\dagger_{\alpha_l}(u + l - 1) \cdot \eta^\dagger_{\alpha_{l+1}}(u + l) \cdots \eta^\dagger_{\alpha_k}(u + k - 2).$$

Note that $\eta_i(u)$ and $\eta^\dagger_i(u)$ satisfy the following commutation relation. This is deduced from (4.1) by a direct calculation.

**Lemma 5.2.** We have

$$\eta_j(u)\eta^\dagger(u + 1) - \eta_i(u)\eta^\dagger_i(u + 1) = \Theta J^j_0 = \varepsilon(j)\Theta\delta^j_i,$$

$$\eta_j(u)\eta^\dagger_i(u + 1) - \eta_i(u)\eta^\dagger_i(u + 1) = \Theta J^j_0 e_i^* e_i^* = \varepsilon(j)\Theta\delta^j_i e_i^* e_i^*.$$

Here $J^j_0$ means the $(i, j)$th entry of the matrix $J^{-1}_0$, and $\Theta$ is defined by

$$\Theta = \sum_{a, b=1}^{N} \varepsilon(b) e_a e_b F_{ab}. $$

**Remark.** For any $u \in C$, we have

$$\Theta = \sum_{b=1}^{N} \varepsilon(b) \eta_b(u) e_b.$$

**Corollary 5.3.** When $1 \leq i, j \leq n$ or $n + 1 \leq i, j \leq N$, we have

$$\eta^\dagger_i(u)\eta^\dagger_j(u + 1) = \eta^\dagger_j(u)\eta^\dagger_i(u + 1).$$

Noting this relation, we consider the two elements

$$\Xi^{-}(u) = \sum_{j=1}^{n} \eta^\dagger_j(u) = \sum_{i=1}^{N} \sum_{j=1}^{n} e_i e_j^* F_{ij}(u),$$

$$\Xi^{+}(u) = \sum_{j=n+1}^{N} \eta^\dagger_j(u) = \sum_{i=1}^{N} \sum_{j=n+1}^{N} e_i e_j^* F_{ij}(u).$$

Then we have $\Xi^{-}(u) + \Xi^{+}(u) = \Xi(u)$. Moreover we put

$$\Xi^{-}_k(u) = \Xi^{-}(u)\Xi^{-}(u + 1) \cdots \Xi^{-}(u + k - 1),$$

$$\Xi^{+}_k(u) = \Xi^{+}(u)\Xi^{+}(u + 1) \cdots \Xi^{+}(u + k - 1).$$
By Corollary 5.3, these factorial powers can be expanded as

\[
\Xi_k^-(u) = \sum_{1 \leq \alpha_1 \leq \cdots \leq \alpha_k \leq n} \frac{k!}{\alpha_1! \alpha_2! \cdots \alpha_k!} \eta_{\alpha_1}^\dagger(u) \eta_{\alpha_2}^\dagger(u + 1) \cdots \eta_{\alpha_k}^\dagger(u + k - 1),
\]

\[
\Xi_k^+(u) = \sum_{n + 1 \leq \beta_1 \leq \cdots \leq \beta_k \leq N} \frac{k!}{\beta_1! \beta_2! \cdots \beta_k!} \eta_{\beta_1}^\dagger(u) \eta_{\beta_2}^\dagger(u + 1) \cdots \eta_{\beta_k}^\dagger(u + k - 1).
\]

Thus we can rewrite (5.4) simply as

\[
W_k(u) = \sum_{l \geq 0} \binom{k}{l} \Xi_l^-(u) \Xi_{k-l}^+(u + l - 1).
\]

5.4. Let us consider an analogue of \(W_k(u)\):

\[
V_k(u) = \sum_{l \geq 0} \binom{k}{l} \Xi_l^-(u) \Xi_{k-l}^+(u + l).
\]

This \(V_k(u)\) is related to \(W_k(u)\) as follows:

**Lemma 5.4.** We have

\[
V_k(u) = W_k(u) + kV_{k-1}(u)\tau_+.
\]

Here we define \(\tau_-\) and \(\tau_+\) by

\[
\tau_- = \sum_{i=1}^n e_ie_i^* = \sum_{i=n+1}^N e_ie_i^*,
\]

so that \(\tau = \tau_- + \tau_+\), \(\Xi_-(u) = \Xi_- (0) + u\tau_-\), and \(\Xi_+(u) = \Xi_+(0) + u\tau_+\).

**Remark.** These \(\tau_-\) and \(\tau_+\) are obviously central. Hence \(\Xi_-(u)\) and \(\Xi_- (w)\) commute with one another for any \(u, w \in \mathbb{C}\). Similarly \(\Xi_+(u)\) and \(\Xi_+(w)\) are commutative.

**Proof of Lemma 5.4.** Since \(\Xi_+(u) = \Xi_+(0) + u\tau_+\), we have

\[
\Xi_k^-(u) - \Xi_k^+(u - 1) = k \Xi_{k-1}^+(u)\tau_+.
\]

Hence, we have

\[
V_k(u) - W_k(u) = \sum_{l \geq 0} \binom{k}{l} \Xi_l^+(u) \cdot \{ \Xi_{k-l}^+(u + l) - \Xi_{k-l}^-(u + l - 1) \}
\]

\[
= \sum_{l \geq 0} \binom{k}{l} \Xi_l^+(u) \cdot (k - l) \Xi_{k-l-1}^+(u + l) \tau_+
\]

\[
= k \sum_{l \geq 0} \binom{k - 1}{l} \Xi_l^+(u) \Xi_{k-l-1}^+(u + l) \tau_+
\]

\[
= kV_{k-1}(u)\tau_+.
\]

5.5. Let us study the relation between \(V_k(u)\) and the factorial powers of \(\Xi(u)\). First, the following commutation relations are easy from Lemma 5.2:
Lemma 5.5. We have
\[ \Xi_+(u-1)\Xi_-(u) - \Xi_-(u-1)\Xi_+(u) = \Theta \rho^*. \]
Here we put \( \rho^* = \sum_{i=1}^n e_i^* e_i^*. \)

Lemma 5.6. We have
\[ \eta_j(u)\Theta = \Theta \eta_j(u+2), \]
and in particular
\[ \Xi(u)\Theta = \Theta \Xi(u+2), \quad \Xi_-(u)\Theta = \Theta \Xi_-(u+2), \quad \Xi_+(u)\Theta = \Theta \Xi_+(u+2). \]

Note that \( \rho^* \) is central in \( S_{2N} \otimes U(\mathfrak{sp}(J_0)) \). Thus, the following relation is obtained from Lemmas 5.5 and 5.6 by a simple calculation.

Lemma 5.7. We have
\[ \Xi_+(u-1)\Xi_-(u) - \Xi_-(u-1)\Xi_+(u) = \Theta \rho^*. \]

Moreover, we have the following relation:

Lemma 5.8. We have
\[ V_k(u)\Xi(u+k) - V_{k+1}(u) = kV_{k-1}(u)\Theta \rho^*. \]

Proof of Lemma 5.8. First, we have
\[
V_k(u)\Xi(u+k) - V_{k+1}(u) = \sum_{l \geq 0} \binom{k}{l} \Xi_l(u)\Xi_{k-l}(u+l)\Xi(u+k) - \sum_{l \geq 0} \binom{k}{l} \Xi_{k-l}(u)\Xi_{k-l}(u+l+1)
\]
\[
= \sum_{l \geq 0} \binom{k}{l} \Xi_l(u) \cdot \{\Xi_{k-l}(u+l)\Xi(u+k) - \Xi(u+l)\Xi_{k-l}(u+l+1)\}
\]
\[
= \sum_{l \geq 0} \binom{k}{l} \Xi_l(u) \cdot (k-l)\Xi_{k-l-1}(u+l)\Theta \rho^*
\]
\[
= \sum_{l \geq 0} k\binom{k-1}{l} \Xi_l(u)\Xi_{k-l-1}(u+l)\Theta \rho^*
\]
\[
= kV_{k-1}(u)\Theta \rho^*.
\]

Here we used Lemma 5.7 for the third equality. Moreover, we have
\[ V_k(u)\Xi_+(u+k) + \Xi_-(u)V_k(u+1) \]
Proof of Lemma 5.9. We have

$$\Xi_m(u) = \sum_{l \geq 0} R_l^m V_{m-2l}(u) \Theta^l \rho^s l.$$  

Here we put

$$R_l^k = \binom{k}{2l} (2l-1)!!$$

with \((2l-1)!! = (2l-1)(2l-3) \cdots 1\) (we put \((-1)!! = 1\) when \(l = 0\)).

Using this, we can show the following expansion:

**Lemma 5.9.** We have

$$\Xi_k(u) = \sum_{l \geq 0} R_l^k V_{k-2l}(u) \Theta^l \rho^s l.$$  

Here we put

$$R_l^k = \binom{k}{2l} (2l-1)!!$$

with \((2l-1)!! = (2l-1)(2l-3) \cdots 1\) (we put \((-1)!! = 1\) when \(l = 0\)).

**Proof of Lemma 5.9.** This can be proved by induction on \(k\). First, the case \(k = 0\) is easy. Next, by assuming the case \(k = m\), the case \(k = m + 1\) is deduced as follows:

$$\Xi_{m+1}(u) = \Xi_m(u) \Xi(u + m)$$

$$= \sum_{l \geq 0} R_l^m V_{m-2l}(u) \Theta^l \rho^s l \Xi(u + m)$$

$$= \sum_{l \geq 0} R_l^m V_{m-2l}(u) \Theta^l \rho^s l$$

$$= \sum_{l \geq 0} R_l^m V_{m-2l+1}(u) \Theta^l \rho^s l + \sum_{l \geq 0} (m-2l) R_l^m V_{m-2l-1}(u) \Theta^l+1 \rho^s l+1$$

$$= \sum_{l \geq 0} R_l^m V_{m-2l+1}(u) \Theta^l \rho^s l + \sum_{l \geq 1} (m-2l + 2) R_l^m V_{m-2l+1}(u) \Theta^l \rho^s l+1$$

$$= \sum_{l \geq 0} R_l^{m+1} V_{m-2l+1}(u) \Theta^l \rho^s l.$$
Here we used Lemma 5.8 for the fourth equality. To show the last equality, we also used the relations
\[ R_{0}^{m+1} = R_{0}^{m} = 1, \quad R_{i}^{m+1} = R_{i}^{m} + (m - 2l + 2)R_{i-1}^{m}. \]
These relations themselves are immediate from the definition of \( R_{i}^{k} \). □

The coefficient \( R_{i}^{k} \) also appears in the expansions
\[ u^k = \sum_{l \geq 0} (-1)^{l} R_{i}^{k} u^{k-l}, \quad u^\overline{k} = \sum_{l \geq 0} R_{i}^{k+l-1} u^{k-l}. \]

Here \( u^k \) and \( u^\overline{k} \) mean the two factorial powers
\[ u^k = u(u + 1)(u + 2) \cdots (u + k - 1), \quad u^\overline{k} = u(u + 2)(u + 4) \cdots (u + 2k - 2). \]

In particular, we have
\[ \sum_{l \geq 0} (-1)^{l} R_{i}^{k} R_{m-l}^{k-2l} = \delta_{m,0}. \]

By noting this, the following is immediate from Lemma 5.9:

**Lemma 5.10.** We have
\[ V_{k}(u) = \sum_{l \geq 0} (-1)^{l} R_{i}^{k} \Xi^{k-2l}(u) \Theta^{l} \rho^{*l}. \]

5.6. Next we consider the following relations:

**Lemma 5.11.** We have
\[ k \langle \Xi^{k-1}(u) \Theta^{l} \rho^{*l} \tau \rangle + l \langle \Xi^{\overline{k}}(u) \Theta^{l-1} \rho^{*l-1} \tau \omega \rangle = 0. \]

**Lemma 5.12.** We have
\[ k \langle W'_{k-1}(u) \Theta^{l} \rho^{*l} \rangle = l \langle \Xi^{\overline{k}}(u) \Theta^{l-1} \rho^{*l-1} \omega \rangle. \]

Here \( \omega \) is the central element defined by
\[ \omega = \sum_{i=1}^{N} \varepsilon(i) e_i e_i^* = -\tau_- + \tau_. \]

Moreover \( \Xi^{\overline{k}}(u) \) means the factorial power
\[ \Xi^{\overline{k}}(u) = \Xi(u) \Xi(u + 2) \Xi(u + 4) \cdots \Xi(u + 2k - 2). \]
To prove Lemma 5.11, we use a variable transformation and Lemma 2.1. We put
\[ g = \left( \begin{array}{cc} a & bJ_0 \\ cJ_0^{-1} & d \end{array} \right) \in \text{GL}_{2N}, \]
so that
\[ t_g^{-1} = \frac{1}{ad - bc} \left( \begin{array}{cc} d & -cJ_0^{-1} \\ -bJ_0 & a \end{array} \right). \]
These \( g \) and \( t_g^{-1} \) naturally act on \( \mathbb{C}^{2N} \), the space spanned by the formal variables \( e_1, \ldots, e_N, e_1^*, \ldots, e_N^* \). Let us consider their extended actions on \( S_{2N} = S(\mathbb{C}^{2N}) \) and moreover on \( S_{2N} \otimes U(\text{sp}(J_0)) \) as automorphisms. Then, we have the relations
\[
\begin{align*}
g(\tau) &= (ad - bc)\tau, & g(\omega) &= (ad + bc)\omega + 2ab\rho + 2cd\rho^*, \\
g(\rho) &= a^2\rho + c^2\rho^* + ac\omega, & g(\rho^*) &= b^2\rho + d^2\rho^* + bd\omega, \\
g(\Xi) &= (ad + bc)\Xi + ab\Theta + cd\Theta^*, & g(\Theta) &= a^2\Theta + c^2\Theta^* + 2ac\Xi, \\
g(\Theta^*) &= b^2\Theta + d^2\Theta^* + 2bd\Xi.
\end{align*}
\]
(5.6)

Here we define \( \Theta^* \) and \( \rho \) by
\[
\Theta^* = -\sum_{i,j=1}^{N} \varepsilon(i)e_i^*e_j^*F_{ij}, \quad \rho = -\sum_{i=1}^{n} e_i^*e_i.
\]
Moreover we have
\[ t_g^{-1}(\tau) = (ad - bc)^{-1}\tau. \]

Remark. To show (5.6), it is convenient to consider the “row vectors” \( e = (e_1, \ldots, e_N) \) and \( e^* = (e_1^*, \ldots, e_N^*) \), so that
\[
\begin{align*}
\tau &= e^*e^*, & \Xi &= eF^*e^*, & \Theta &= eFJ_0^t e, & \Theta^* &= e^{*t}J_0^{-1}F^*e^*, \\
\omega &= eK_0^t e^*, & \rho &= \frac{1}{2}eK_0J_0^t e, & \rho^* &= \frac{1}{2}e^{*t}J_0^{-1}K_0^t e^*.
\end{align*}
\]
Here \( K_0 \) means the matrix \( K_0 = \text{diag}(-1, \ldots, -1, 1, \ldots, 1) \).

Proof of Lemma 5.11. Let us suppose that \( a = b = d = 1 \) and \( c = 0 \). Then (5.6) is rewritten as
\[
\begin{align*}
g(\tau) &= \tau, & g(\omega) &= \omega + 2\rho, & g(\rho) &= \rho, & g(\rho^*) &= \rho + \rho^* + \omega, \\
g(\Xi) &= \Xi + \Theta, & g(\Theta) &= \Theta, & g(\Theta^*) &= \Theta + \Theta^* + 2\Xi.
\end{align*}
\]
In particular we have
\[
\begin{align*}
g(\Xi(u)) &= \Xi(u) + \Theta, & g(\Theta) &= \Theta, & g(\omega - 2\rho^*) &= -(\omega + 2\rho^*), & g(\tau) &= \tau.
\end{align*}
\]
Using Lemma 5.6, we can show a kind of binomial expansion:
\[
g(\Xi^k(u)) = (\Xi(u) + \Theta)(\Xi(u + 2) + \Theta) \cdots (\Xi(u + 2k - 2) + \Theta)
\]
\[
= \sum_{s \geq 0} \binom{k}{s} \Xi(u)\Xi(u + 2) \cdots \Xi(u + 2s - 2) \cdot \Theta^{k-s}
\]
\[
= \sum_{s \geq 0} \binom{k}{s} \Xi^s(u) \cdot \Theta^{k-s}.
\]
Moreover, expanding the equality \(g((\omega - 2\rho^*)^l) = (-)^l(\omega + 2\rho^*)^l\), we have
\[
g\left(\sum_{r \geq 0} \binom{l}{r} \omega^r (-2\rho^*)^{l-r}\right) = (-)^l \sum_{r \geq 0} \binom{l}{r} \omega^r (2\rho^*)^{l-r}.
\]
We also have \(g(\Theta^{l-1}) = \Theta^{l-1}\) and \(g(\tau^m) = \tau^m\). Multiplying these equalities, we have
\[
g\left(\Xi^k(u)\Theta^{l-1} \sum_{r \geq 0} \binom{l}{r} \omega^r (-2\rho^*)^{l-r}\tau^m\right)
\]
\[
= (-)^l \sum_{s \geq 0} \binom{k}{s} \Xi^s(u)\Theta^{k-s+l-1} \sum_{r \geq 0} \binom{l}{r} \omega^r (2\rho^*)^{l-r}\tau^m.
\]
Take the inner product with \(\tau^{(k+2l-1)}\), and apply Lemma 2.1. Then, since \(\tau^m = \tau\), we have
\[
(5.7) \quad \sum_{r \geq 0} \binom{l}{r} \langle \Xi^k(u)\Theta^{l-1} \omega^r (-2\rho^*)^{l-r}\tau^m \rangle
\]
\[
= \sum_{s \geq 0} \sum_{r \geq 0} (-)^l \binom{k}{s} \binom{l}{r} \langle \Xi^s(u)\Theta^{k+s+l-1} \omega^r (2\rho^*)^{l-r}\tau^m \rangle.
\]
Note that \(\langle \Xi^b(u)\Theta^b \omega^c \rho^d \tau^e \rangle\) is equal to zero, unless \(b \neq d\) (compare the order of \(c_1, \ldots, c_N\) with the order of \(e_1^*, \ldots, e_N^*\)). Hence, the left hand side of (5.7) is equal to zero, unless \(r = 1\). Similarly, the right hand side is equal to zero, unless \(s = k\), \(r = 1\) or \(s = k - 1\), \(r = 0\). Thus we have
\[
l\langle \Xi^k(u)\Theta^{l-1} \omega(-2\rho^*)^{l-1}\tau^m \rangle
\]
\[
= (-)^l \langle \Xi^k(u)\Theta^{l-1} \omega(2\rho^*)^{l-1}\tau^m \rangle + (-)^l k \langle \Xi^{k-1}(u)\Theta^l(2\rho^*)^l\tau^m \rangle.
\]
Simplifying this equality, we have
\[
k\langle \Xi^{k-1}(u)\Theta^l \rho^s \tau^m \rangle + l\langle \Xi^k(u)\Theta^{l-1} \rho^s(l-1)\tau^m \rangle = 0.
\]
Proof of Lemma 5.12. By the expansion (5.5), we have

\[ (5.8) \quad \Xi^k(u) = \sum_{r \geq 0} (-)^r R_r^k \Xi^{k-r}(u) \tau^r. \]

Moreover, we have

\[ kW'_{k-1}(u) = k(\Xi^{k-1}(u) - \frac{k-1}{2} \Xi^{k-2}(u) \tau) \]
\[ = \sum_{r \geq 0} (-)^r k R_r^{k-1} \Xi^{k-1-r}(u) \tau^r - \frac{k(k-1)}{2} \sum_{r \geq 0} (-)^r R_r^{k-2} \Xi^{k-2-r}(u) \tau^{r+1}. \]

Since \( R_r^k = 0 \) for \( r < 0 \), this is equal to

\[ \sum_{r \geq 0} (-)^r k R_r^{k-1} \Xi^{k-1-r}(u) \tau^r - \frac{k(k-1)}{2} \sum_{r \geq 0} (-)^r R_r^{k-2} \Xi^{k-1-r}(u) \tau^r \]
\[ = \sum_{r \geq 0} (-)^r (k R_r^{k-1} + \frac{k(k-1)}{2} R_r^{k-2}) \Xi^{k-1-r}(u) \tau^r \]
\[ = \sum_{r \geq 0} (-)^r (k - r) R_r^k \Xi^{k-1-r}(u) \tau^r. \]

Here we used the relation

\[ k R_r^{k-1} + \frac{k(k-1)}{2} R_r^{k-2} = (k - 2r) R_r^k + r R_r^k = (k - r) R_r^k. \]

Comparing this with (5.8) and applying Lemma 5.11, we have the assertion. \( \square \)

5.7. Combining Lemmas 5.4 and 5.9, we can write \( W_k(u) \) as follows. Using \( k R_l^{k-1} = (k - 2l) R_l^k \), we have

\[ (5.9) \quad W_k(u) = V_k(u) - k V_{k-1}(u) \cdot \tau_+ \]
\[ = \sum_{l \geq 0} (-)^l R_l^k \Xi^{k-2l}(u) \Theta^l \rho^* l \tau_+ - k \sum_{l \geq 0} (-)^l R_l^{k-1} \Xi^{k-2l-1}(u) \Theta^l \rho^* l \tau_+ \]
\[ = \sum_{l \geq 0} (-)^l R_l^k \Xi^{k-2l}(u) \Theta^l \rho^* l - \sum_{l \geq 0} (-)^l (k - 2l) R_l^k \Xi^{k-2l-1}(u) \Theta^l \rho^* l \tau_+ \]
\[ = \sum_{l \geq 0} (-)^l R_l^k \Xi^{k-2l}(u) \Theta^l \rho^* l - \frac{1}{2} \sum_{l \geq 0} (-)^l (k - 2l) R_l^k \Xi^{k-2l-1}(u) \Theta^l \rho^* l \omega \]
\[ - \frac{1}{2} \sum_{l \geq 0} (-)^l (k - 2l) R_l^k \Xi^{k-2l-1}(u) \Theta^l \rho^* l \tau \]
\[
\sum_{l \geq 0} (-)^l R^k_{l+1} W'_{k-2l-2}(u) \Theta^{l+1} \rho^* \omega - \frac{1}{2} \sum_{l \geq 0} (-)^l (k-2l) R^k_l \Xi^{k-2l-1}(u) \Theta^l \rho^* \omega
\]

\[
= W'_k(u) + \sum_{l \geq 0} (-)^{l+1} R^k_{l+1} W'_{k-2l-2}(u) \Theta^{l+1} \rho^* \omega + \frac{1}{2} \sum_{l \geq 0} (-)^l (k-2l) R^k_l \Xi^{k-2l-1}(u) \Theta^l \rho^* \omega.
\]

By Lemma 5.12, we have

\[
(k-2l-1) \langle W'_{k-2l-2}(u) \Theta^{l+1} \rho^* \omega \rangle = (l+1) \langle \Xi^{k-2l-1}(u) \Theta^l \rho^* \omega \rangle.
\]

Multiplying this by \(\frac{1}{k-2l-1} R^k_{l+1} = \frac{k-2l}{2l+2} R^k_l\), we have

\[
R^k_{l+1} \langle W'_{k-2l-2}(u) \Theta^{l+1} \rho^* \omega \rangle = (k-2l) R^k_l \langle \Xi^{k-2l-1}(u) \Theta^l \rho^* \omega \rangle.
\]

Combining this with (5.9), we have \(\langle W_k(u) \rangle = \langle W'_k(u) \rangle\), namely Theorem 5.1. This finishes the proof of the main theorem.

Remark. This proof of Theorem 4.5 is essentially more difficult than that of Theorem 3.8. In [14], we only used simple commutation relations in \(\mathfrak{A}_{2N} \otimes U(\mathfrak{o}(S_0))\) to prove Theorem 3.8. In contrast to this, to prove Theorem 4.5, we needed a variable transformation and Lemma 2.1 as seen in Section 5.6.

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