Schrödinger Equation on Fractals Curves Imbedding in $R^3$

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Abstract

In this paper we have generalized the quantum mechanics on fractal time-space. The time is changing on Cantor-set like but space is considered as fractal curve like Von-Koch curve. The Feynman path method in quantum mechanics has been suggested on fractal curve. Using $F^\alpha$-calculus and Feynman path method we found the Schrödinger on fractal time-space. The Hamiltonian operator and momentum operator has been derived. Moreover, the continuity equation and the probability density is given in generalized formulation.

Keywords: Feynman path method, Schrödinger on fractal time-space, continuity equation

1 Introduction

Fractal is objects that are very fragmented and irregular at all scales. Their important properties are non-differentiability and having non-integer dimension. Fractal has topological dimension less than Hausdorff-Besicovitch, box-counting, and similarity dimensions. In general, dimension of fractal can be integer or not well-defined dimension [1,7]. Fractional local calculus and nonlocal has applied to model the process with memory and fractal structure [8,14]. The electric and magnetic fields are derived using fractional integrals as an approximation method on fractals [15]. The quantum space-time on the basis of relativity principle and geometrical concept of fractals is introduced [16]. The probability density of quantum wave function with Dirichlet boundary conditions in a D-dimensional spaces has been studied [17]. The fractal concept to quantum physics and the relationships between fractional integral and Feynman path integral method is developed [18,19]. The generalized wave functions is introduced to fractal dimension, a wide class of quantum problems, including the infinite potential well, harmonic oscillator, linear potential, and free particle [20]. Fractal path in quantum mechanics and their contributing in Feynman path integral is investigated [21]. The classical mechanics is derived without the need of the least-action principle using path-integral approach [22]. The calculus on the fractals has been studied in different methods like probabilistic approach method, sequence of discrete Laplacians, measure-theoretical method, time scale calculus [23]. Riemann integration like method has been studied since that is useful and algorithmic [24,29]. Using the calculus on fractals the Newtonian mechanics, Lagrange and Hamilton mechanics, and Maxwell equations has been generalized [30,32]. As a pursue theses research we generalized the quantum mechanics on fractals.

The plan of this paper is as following:
Section 2 we review the fractal calculus. In section 3 we defined the gradient, divergent and Laplacian on fractal space. Section 4 is explained the quantum mechanics on fractals curves. In section 5 we suggested the probability density and continuity equation on the generalized quantum formalism. Finally, section 6 is devoted to conclusion.

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2 A Summary of the calculus on fractal curves

We review the $F^{\alpha}$-calculus on fractal curves [24, 32]. Suppose fractal curve $F \subset R^3$ which is continuously parameterizable i.e there exists a function $w : [a_0, b_0] \rightarrow F \subset R^3$ which is continuous. We also assume $w$ to be invertible. A subdivision $P_{[a,b]}$ of interval $[a, b]$, $a < b$, is a finite set of points $\{a = v_0 < v_1, ... < v_n = b\}$. For $a_0 \leq a < b < b_0$ and appropriate $\alpha$ be chosen, therefore let

$$\gamma^\alpha(F, a, b) = \lim_{\delta \rightarrow 0} \frac{1}{|P|} \inf_{(P_{[a,b]})(|P| \leq \delta)} \sum_{i=0}^{n-1} \frac{|w(v_{i+1}) - w(v_i)|^\alpha}{f(\alpha + 1)},$$

(1)

where $|.|$ denotes the Euclidean norm on $R^3$ and $|P| = \max\{v_{i+1} - v_i; i = 0, ..., n - 1\}$. A $\gamma$-dimension of $F$, which is defined as

$$\dim_\gamma(F) = \inf\{\alpha : \gamma^\alpha(F, a, b) = 0\} = \sup\{\alpha : \gamma^\alpha(F, a, b) = \infty\}.$$

(2)

After this definition $\alpha$ is equal to $\dim_\gamma(F)$. The staircase function $S^\alpha_F : [a_0, b_0] \rightarrow R$ of order $\alpha$ for a set $F$, is defined as

$$S^\alpha_F(v) = \begin{cases} \gamma^\alpha(F, p_0, v) & v \geq p_0 \\ -\gamma^\alpha(F, v, p_0) & v < p_0, \end{cases}$$

(3)

where $a_0 \leq p_0 \leq b_0$ is arbitrary but fixed, and $v \in [a_0, b_0]$. It is monotonic function. The $\theta = w(v)$, denote a point on fractal curve $F$

$$J(\theta) = S^\alpha_F(w^{-1}(\theta)), \quad \theta \in F.$$

(4)

We suppose that fractal curves whose $S^\alpha_F$ is finite and invertible on $[a, b]$. The $F^{\alpha}$-derivative of the bounded function $f : F \rightarrow R$ ($f \in B(F)$) at $\theta \in F$ is defined.

Then the **directional** $F^{\alpha}$-derivative of function $f$ at $\theta \in F$ is defined as

$$w_j \mathcal{D}^\alpha_F f(w(v)) = F - \lim_{\nu \rightarrow 1} \frac{f(w_1(v), w_2(v), ..., w_j(v'), ..., w_i(v)) - f(w(v))}{S^\alpha_F(v') - S^\alpha_F(v)},$$

(5)

where $w_j$ is shows direction of $F^{\alpha}$-derivative, if the limit exists [27].

Let $f \in B(F)$ is an $F$-continuous function on $C(a,b)$ which is the segment $\{w(v) : v \in [a,b]\}$ of $F$. Now let $g : f \rightarrow R$ be define as

$$g(w(v)) = \int_{C(a,v)} f(\theta) d^\alpha_F \theta,$$

(6)

for all $v \in [a,b]$. So that

$$\mathcal{D}^\alpha_F g(\theta) = f(\theta).$$

(7)

Note: Let $\gamma^\alpha(F, a, b)$ be finite and $f(\theta) = 1$, $\theta \in F$ denote the constant function. Then

$$\int_{C(a,b)} f(\theta) d^\alpha_F \theta = \int_{C(a,b)} 1 d^\alpha_F \theta = S^\alpha_F(b) - S^\alpha_F(a) = J(w(b)) - J(w(a)).$$

(8)

Remark: $F^{\alpha}$-derivative and $F^{\alpha}$-integral is a linear operation.

1) Let $f : F \rightarrow R$, $f(\theta) = k \in R$ then $\mathcal{D}^\alpha_F f = 0$.  
2) If $f : F \rightarrow R$ be a $F$-continuous function such that $\mathcal{D}^\alpha_F f = 0$. Then $f = k$ where $k$ on $C(a,b)$. Suppose $f : F \rightarrow R$ be $F^{\alpha}$-differentiable function and $h : F \rightarrow R$ be $F$-continuous such that $h(\theta) = \mathcal{D}^\alpha_F f(\theta)$, so

$$\int_{C(a,b)} h(\theta) d^\alpha_F \theta = f(w(b)) - f(w(a)).$$

(9)

Analogue Taylor series is defined for $h(\theta) \in B(F)$ as

$$f(w(v)) = \sum_{n=0}^{\infty} \frac{(S^\alpha_F(v) - S^\alpha_F(v'))^n}{n!} (\mathcal{D}^\alpha_F)^n f(w(v')),$$

(10)

where $h(\theta)$ is $F^{\alpha}$-differentiable any number of times on $C(a,b)$. That is $(\mathcal{D}^\alpha_F)^n h \in B(F)$, $\forall n > 0$.

3 Gradient, Divergent, Curl and Laplacian on Fractal Curves

In this section we generalized the $F^{\alpha}$-calculus by defining the gradient, divergent, curl and Laplacian on fractal curves imbedding in $R^3$.

3.1 Gradient on fractal curves

Let us consider the $f \in B(F)$ as an $F$-continuous function on $C(a,b) \subset F$ and $w(v, w_i(v)) : R \rightarrow R^3, i = 1, 2, 3$, so the gradient of the $f(w) : F \rightarrow R$ is

$$\nabla^\alpha_F f(w) = \nabla^\alpha_F f(w) \hat{e}_i \quad i = 1, 2, 3,$$

(11)

where the $\hat{e}_i$ is the basis of $R^3$. 

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3.2 Divergent on fractal curves
Let the $f(w(v)) = f_i(w(v)) \hat{e}^i$ $i = 1, 2, 3$, be a vector field on fractal curve. Then we define the divergent of the $f: F \to R^n$ as follows:
$$\nabla^w_f w(v) = \nabla^w f(w(v)), \tag{12}$$
where $f_i(w(v))$ are components of vector field.

3.3 Laplacian on fractal curves
Consider the $w(v, w_0(v)) : R \to R^3$ on the fractal curve, therefore the Laplacian is defined as
$$\Delta^w_f w = (\nabla^w f)^2 f = (\nabla^w D^a f) f(w(v)) \tag{13}$$
where the $\Delta^w_f$ is called Laplacian on fractal curve.

4 Quantum mechanics on fractal curve
The classical mechanics is reformulated in terms of a minimum principle. The Euler-Lagrange equations of motion is derived from the least action. The Feynman paths for a particle in quantum mechanics are fractals with dimension 2 \cite{33}. In this section, we obtain the Schrödinger equation on fractal curves.

4.1 Generalized Feynman path integral method
Feynman method for studying quantum mechanics using classical Lagrangian and action is presented in Ref \cite{34,35}. Now we want to generalized Feynman method using Lagrangian on fractals curves. Consider generalized action as
$$\mathfrak{A}^w_F = \int_{t_1}^{t_2} L^w_F(t, w(v), \hat{t} D^w_F w(v)) d^w_F v \, d^w_F t \quad L^w_F : F \times F \times F \to R. \tag{14}$$
In view of Feynman method, if wave function on fractal in $t_1, w_0(v_1)$ is $\psi^w_F(t_1, w_0(v_1))$. So it gives the total probability amplitude at $t_2$, $w_h(v_2)$ as
$$\psi^w_F(t_2, w_h(v_2)) = \int_{t_1}^{t_2} K^w_F(t_2, w_h(v_2), t_1, w_a(v_1))(\psi^w_F(t_1, w_a(v_1)) d^w_F w(v), \tag{15}$$
where $K^w_F$ is the propagator which is defined as follows:
$$K^w_F(t_2, w_h(v_2), t_1, w_a(v_1)) = \int_{w_a}^{w_h} \exp\left[\frac{i}{\hbar}\mathfrak{A}^w_F\right] D^a_F w(v). \tag{16}$$
Where symbol $D^a_F$ indicates the integration over all fractal paths from $w_a(v_1)$ to $w_h(v_2)$.
Now we derive the Schrödinger equation for a free particle on fractal curve, which is describes the evolution of the wave function from $w_a(v_1)$ to $w_h(v_2)$, when $t_2$ differs from $t_1$ an infinitesimal amount $\epsilon$. Supposing $S^w_F(v_2) = S^w_F(v_1) + \epsilon$, leads to Lagrangian for free particle as
$$L^w_F(t, w(v), \hat{t} D^w_F w(v)) \simeq \frac{m(w(v) - w(v_0))^2}{2(S^w_F(v_2) - S^w_F(v_1))} \tag{17}$$
The generalized action on fractal $\mathfrak{A}^w_F$ is approximately
$$\mathfrak{A}^w_F \sim \epsilon L^w_F = \frac{m(w(v) - w(v_0))^2}{2\epsilon}. \tag{18}$$
As a consequence, we obtain
$$\psi^w_F(t + \epsilon, w(v)) = \int_{-\infty}^{\infty} \frac{1}{A} \exp\left[\frac{i}{\hbar}\frac{m(w(v) - w_0(v_0))^2}{2\epsilon}\right] \psi^w_F(t, \psi_0(v_0)) D^a_F w_0(v_0). \tag{19}$$
Here, because of properties of exponential function only those fractal paths give significant contributions which are very close to $w(v)$. Changing the variable in the integral $\delta = w(v) - w_0(v_0)$ we have $\psi^w_F(t, w_0(v_0)) = \psi^w_F(t, w(v) + \delta)$. Since both $\epsilon$ and $\delta$ are small quantities, so that $\psi^w_F(t, w(v) + \delta)$ and $\psi^w_F(t + \epsilon, w(v))$ can be expanded using Eq. (10). We only keep up to terms of second order of the $\epsilon$ and $\delta$. As a result we get
$$\psi^w_F(t, w(v)) + \epsilon(\hat{t} D^w_F) \psi^w_F(t, w(v)) \simeq \chi^w_F(t) \int_{-\infty}^{\infty} \frac{1}{A} \exp\left[\frac{i}{\hbar}\frac{m\delta^2}{2\epsilon}\right] \left(\psi^w_F(t, w(v)) + \delta(\hat{t} D^w_F) \psi^w_F(t, w(v))\right)$$
$$+ \frac{\delta^2}{2}(\hat{t} D^w_F)^2 \psi^w_F(t, w(v)) d^w_F \delta, \tag{20}$$
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where \( \chi_F(t) \) is the characteristic function for Cantor like sets. The second term in the right hand side vanishes on integration. It follows by equating the leading terms on both sides we obtain

\[
\psi_F^\alpha(t, w(v)) = \int_{-\infty}^{+\infty} \frac{1}{A} \exp\left(\frac{im\delta^2}{2\epsilon}\right) |\psi_F^\alpha(t, w(v))| d\nu \delta.
\] (21)

Also, we arrive at

\[
A = \int_{-\infty}^{+\infty} \frac{1}{A} \exp\left(\frac{im\delta^2}{2\epsilon}\right) |d\nu \delta = \sqrt{\frac{2\pi\hbar\epsilon}{m}}.
\] (22)

Finally, equating the remaining terms, we get Schrödinger equation on fractal curves for free particle as

\[
(i\hbar \mathcal{D}_F^\alpha)\psi_F^\alpha(t, w(v)) = \chi_F(t) \frac{-\hbar^2}{2m} (u \mathcal{D}_F^\alpha)^2 \psi_F^\alpha(t, w(v)).
\] (24)

The Eq. (21) leads to the definition of the Hamiltonian and momentum operator on fractal curves as

\[
\hat{H}_F^\alpha = i\hbar \mathcal{D}_F^\alpha \quad \hat{P}_F^\alpha = -i\hbar \nabla_F^\alpha.
\] (25)

The solution of Eq. (21) can be find using conjugate equation as

\[
\frac{i\hbar}{\partial t}(t, \xi) = \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial \xi^2} \theta(t, \xi) \quad \theta(t, \xi) = \phi[\psi_F^\alpha(t, w(v))].
\] (26)

Since the solution Eq. (25) is

\[
\theta(t, \xi) = (Ae^{i\epsilon \xi} + Be^{-i\epsilon \xi})e^{-i\beta t},
\] (27)

where \( k = \frac{\sqrt{2m\epsilon}}{A} \) and \( \omega = \frac{E}{m} \) are constants. Now by applying \( \phi^{-1} \) we have the solutions as

\[
\psi_F^\alpha(t, w(v)) = (Ae^{iS_F^\alpha(v)} + Be^{-iS_F^\alpha(v)})e^{-i\beta S_F^\alpha(t)}.
\] (28)

It is straight forward to extended to the case of a free particle to the motion involving the potential. In this case the Lagrangian will be \( L_F^\alpha = T_F^\alpha - V_F^\alpha(t, w(v)) \). By substituting the Lagrangian in the Eq. (20) one can derive the Schrödinger equation as

\[
\psi_F^\alpha(t, w(v)) + \epsilon( \mathcal{D}_F^\alpha )\psi_F^\alpha(t, w(v)) \simeq \chi_F(t) \int_{-\infty}^{+\infty} \frac{1}{A} \exp\left(\frac{im\delta^2}{2\epsilon}\right) |1 - \frac{i\epsilon}{\hbar} V_F^\alpha(t, w(v))] (\psi_F^\alpha(t, w(v)) + \delta(w \mathcal{D}_F^\alpha)^2 \psi_F^\alpha(t, w(v))) d\nu \delta.
\] (29)

The same manner we worked out above the Eq. (21) becomes

\[
(i\hbar \mathcal{D}_F^\alpha)\psi_F^\alpha(t, w(v)) = \chi_F(t) \frac{-\hbar^2}{2m} (u \mathcal{D}_F^\alpha)^2 \psi_F^\alpha(t, w(v)) + \chi_F(t) V_F^\alpha(t, w(v)) \psi_F^\alpha(t, w(v))
\] (30)

5 Continuity equation and probability on fractal

It is well known that the continuity equation is an important concept in quantum mechanics. Therefor, the probability density on the fractal for a particle is defined as

\[
\rho_F^\alpha(t, w(v)) = (\star \psi_F^\alpha(t, w(v)) \star \psi_F^\alpha(t, w(v))
\] (31)

The complex conjugate wave function of Eq. (30) is

\[
(-i\hbar \mathcal{D}_F^\alpha)^* \star \psi_F^\alpha(t, w(v)) = \chi_F(t) \frac{-\hbar^2}{2m} (u \mathcal{D}_F^\alpha)^2 \star \psi_F^\alpha(t, w(v)) + \chi_F(t) V_F^\alpha(t, w(v)) \star \psi_F^\alpha(t, w(v))
\] (32)

where \( V_F^\alpha(t, w(v)) = \star V_F^\alpha(t, w(v)) \). Applying this identity is given below

\[
(i\hbar \mathcal{D}_F^\alpha) \star \psi_F^\alpha(t, w(v)) = \mathcal{D}_F^\alpha (\star \psi_F^\alpha(t, w(v)))^* \mathcal{D}_F^\alpha \psi_F^\alpha(t, w(v)) + \psi_F^\alpha(t, w(v)) \mathcal{D}_F^\alpha (\star \psi_F^\alpha(t, w(v))),
\] (33)

and substituting Eq. (30) and Eq. (32), into Eq. (33), yield us

\[
(i\hbar \mathcal{D}_F^\alpha) \rho_F^\alpha(t, w(v)) = \chi_F(t) \frac{\hbar^2}{2m} \psi_F^\alpha(t, w(v))(u \mathcal{D}_F^\alpha)^2 \star \psi_F^\alpha(t, w(v)) - \star \psi_F^\alpha(t, w(v))(u \mathcal{D}_F^\alpha)^2 \psi_F^\alpha(t, w(v)).
\] (34)
As a consequence the definition of a probability current density on fractal curve is

\[
J^\alpha_F(t, w(v)) = \chi_F(t) \frac{\hbar}{2m} \left[ \psi^* \left( t, w(w(v)) \right) \left( \frac{\partial^{\alpha}}{\partial t} \right)^2 \psi \left( t, w(v) \right) - \psi^* \left( t, w(v) \right) \left( \frac{\partial^{\alpha}}{\partial t} \right)^2 \psi \left( t, w(v) \right) \right]
\]

In the following table correspondence between standard quantum mechanics and generalized quantum framework is presented.

| Postulates       | Standard Quantum | Quantum on Fractals |
|------------------|-------------------|---------------------|
| State            | \( \psi(t, x) \)  | \( \psi^\alpha_F(t, w(v)) \) |
| Hamiltonian      | \( i\hbar \frac{\partial}{\partial x} \) | \( i\hbar \left( \frac{\partial^{\alpha}}{\partial t} \right) \) |
| Momentum         | \( -i\hbar \nabla \) | \( -i\hbar \nabla^\alpha_F \) |

6 Conclusion

The calculus on sets, vector space and manifold is used in the classical, quantum mechanics and general relativity respectively. The geometry has important role in this generalization and modeling the physical phenomena. Recently, fractal geometry has been suggested by Mandelbrot. So the calculus on them has been suggest by many researcher but it is still an open problem. In this work we have studied the calculus on fractal curves. Since the path integral in Feynman formulation is fractal so that is motivated us to suggest this generalization. This framework can suggest correct way for obtaining Schrödinger equation from Feynman path quantum mechanics.

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