Supplementary Information for
Observation of Weyl Exceptional Rings in Thermal Diffusion

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Supplementary Note 1: Effective non-Hermitian topology in diffusive system

The horizontal and vertical temperature components along x-direction can be calculated with a conduction-advective function. As indicated in (1), these temperatures can be effectively written as the plane-wave solution once the conductive components \( D = \kappa / \rho c \) in Eq. (1) is small enough. Hence, the system is dominated by advections, which can be regarded as effective "oscillations". Thus, the horizontal and vertical temperature components in the x-y and x-z spaces can be written as:

\[
T_{\text{hor}} = A_{\text{hor}} e^{i(k_{\text{hor}}x + \omega_{\text{hor}}t + \phi_{\text{hor}})} + A_{\bar{\text{hor}}} e^{i(k_{\text{hor}}x - \omega_{\text{hor}}t + \phi_{\text{hor}})},
\]

\[
T_{\text{ver}} = A_{\text{ver}} e^{i(k_{\text{ver}}z + \omega_{\text{hor}}t + \phi_{\text{ver}})} + A_{\bar{\text{ver}}} e^{i(k_{\text{ver}}z - \omega_{\text{hor}}t + \phi_{\text{ver}})},
\]

Taking Eq. (S1) into Eq. (1) with the multiplication of \( i \) on the thermal process in the x-z space, we can obtain the effective Hamiltonian of the coupled thermal system:

\[
H = i \left( \begin{pmatrix} m_{\text{hor}} + im_{\text{ver}} \sigma_z + ik_{\text{ver}} v_{\text{ver}} \sigma_y - ik_{\text{hor}} v_{\text{hor}} \sigma_x \end{pmatrix} - \left( D \begin{pmatrix} k_{\text{hor}}^2 + k_{\text{ver}}^2 \end{pmatrix} + m_{\text{hor}} + im_{\text{ver}} \right) I_{2\times2} \right).
\]

In Eq. (S2), \( I_{2\times2} \) is the \( 2 \times 2 \) identity matrix. The term \( D k_{\text{hor}}^2 + D k_{\text{ver}}^2 + m_{\text{hor}} + m_{\text{ver}} \) only leads to the global shift of the eigenvalues, it can be neglected in Eq. (S2) for further analyzing the eigenvectors. To create the periodic condition on the central strip with the length \( L \), we define the effective wavelengths as \( L \) both for the horizontal and vertical advections. Thus, the effective wavenumbers can be characterized as \( k_{\text{hor}} = k_{\text{ver}} = 1/L \). Since Eq. (S2) is a non-Hermitian Hamiltonian, it possesses left and right eigenvectors which can be obtained with the functions of:

\[
\langle \psi^L \vert H(k) \vert \psi^L \rangle = E^L(k) \langle \psi^L \vert \psi^L \rangle, \quad \langle \psi^R \vert H(k) \vert \psi^R \rangle = E^R(k) \langle \psi^R \vert \psi^R \rangle.
\]

\[
\langle \psi^L \rangle(k) = \left( \frac{1}{N_z} \right) \begin{pmatrix} \langle \psi^L \rangle(k) \end{pmatrix} \begin{pmatrix} \psi^L \rangle(k) \end{pmatrix} \right),
\]

\[
\langle \psi^R \rangle(k) = \left( \frac{1}{N_z} \right) \begin{pmatrix} \langle \psi^R \rangle(k) \end{pmatrix} \begin{pmatrix} \psi^R \rangle(k) \end{pmatrix} \right),
\]

where \( N_z \) is the normalization factor:

\[
(N_z)^2 = 2E^L(E^L - k_{\text{hor}} v_{\text{hor}}).
\]

It is noted that the eigenvalues of the current Hamiltonian are complex beyond the simple PT/APT symmetry. This property is quite different from that of the 2D non-Hermitian Hamiltonian for thermal diffusion (1), where anti-parity time (APT) symmetry is exhibited in heat transfer. To investigate the topological transitions of this non-Hermitian Weyl Hamiltonian, one should determine the topological charge first with the Berry connection and Berry curvature. In this paper, we adopt the Berry connection calculated by the left and right eigenvectors as a representation:

\[
A^L_\gamma(k) = i \langle \psi^L \rangle(k) \nabla_k \langle \psi^R \rangle(k).
\]

Based on the above Berry connection, we can obtain the Berry curvature:

\[
\Omega^L_\gamma(k) = \nabla \times A^L_\gamma(k).
\]

Then, the topological charge can be subsequently calculated:

\[
\gamma^L = \frac{1}{2\pi} \oint_S \Omega^L_\gamma(k) \cdot dS.
\]

where \( S \) denotes the integration surface. For determining the integration surface conveniently, we carry out a spatial transformation to choose a sphere integration surface with the radius of \( m_R = \sqrt{(m_{\text{hor}})^2 + (k_{\text{ver}} v_{\text{ver}})^2 + (k_{\text{hor}} v_{\text{hor}})^2} \). The spatial components on the real space can be defined as:
\[ d_x = -m_{e\tau} = -m_e \sin(\theta) \cos(\phi), \]
\[ d_y = -k_{e\tau} \nu_{e\tau} = -m_e \sin(\theta) \sin(\phi), \]
\[ d_z = k_{e\tau} \nu_{e\tau} = m_e \cos(\theta). \]  

(S9)

Taking Eq. (S9) into (S2), the eigenvectors can be rewritten as:
\[
\psi_1^\pm(k) = \frac{1}{N_\pm} \begin{pmatrix} 1 & i \left( m_{\text{ext}} - m_e \sin(\theta) \sin(\phi) \right) - m_e \sin(\theta) \cos(\phi) \\ -i \pm \sqrt{\left( m_{\text{ext}} \right)^2 - \left( m_e \right)^2 + 2im_{\text{ext}} m_e \cos(\theta)} & -m_e \cos(\theta) \end{pmatrix}. \]

(S10)
\[
\psi_3^\pm(k) = \frac{1}{N_\pm} \begin{pmatrix} i \left( m_{\text{ext}} + m_e \sin(\theta) \sin(\phi) \right) - m_e \sin(\theta) \cos(\phi) \\ -i \pm \sqrt{\left( m_{\text{ext}} \right)^2 - \left( m_e \right)^2 + 2im_{\text{ext}} m_e \cos(\theta)} & -m_e \cos(\theta) \end{pmatrix}. \]

(S11)

Further considering Eqs. (S6) and (S7), we can respectively achieve the Berry connection and Berry curvature for the Weyl exceptional ring in such a thermal system. The three components of its Berry connection in the spherical coordinates can be expressed as:
\[
A_k^{\theta \tilde{\theta}} = i \left\langle \psi^\dagger \right| \frac{\partial}{\partial \theta} | \psi^k \rangle = \left( \frac{-2m_{\text{ext}} m_e \sin(\theta) \cos(\phi)}{N_\pm^{\theta \tilde{\theta}}} + i \left( E_\pm^2 - m_e^2 - m_{\text{ext}}^2 + 2E_m \cos(\theta) \right) \right) \frac{\partial N_\pm}{\partial m_e}
\]  
\[
A_k^{\phi \tilde{\phi}} = i \left\langle \psi^\dagger \right| \frac{\partial}{\partial \phi} | \psi^k \rangle = \left( \frac{-2m_{\text{ext}} m_e \sin(\theta) \cos(\phi)}{N_\pm^{\phi \tilde{\phi}}} - i \left( E_\pm^2 - m_e^2 + m_{\text{ext}}^2 - 2E_m \cos(\theta) \right) \right) \frac{\partial N_\pm}{\partial m_e}
\]  
\[
A_k^{\psi \tilde{\psi}} = i \left\langle \psi^\dagger \right| \frac{\partial}{\partial \psi} | \psi^k \rangle = \left( \frac{-2m_{\text{ext}} m_e \sin(\theta) \cos(\phi)}{N_\pm^{\psi \tilde{\psi}}} - i \left( m_{\text{ext}}^2 - m_e^2 + m_{\text{ext}}^2 - 2E_m \cos(\theta) \right) \right) \frac{\partial N_\pm}{\partial m_e}
\]  
\[
+ i \left( N_e m_e \cos(\theta) + N_m m_e \cos(\theta) \right) \frac{\partial E_\pm}{\partial \phi}
\]  
\[
+ \left( \left( N_e m_{\text{ext}} m_e \sin(\theta) \sin(\phi) + N_m m_e \cos^2(\theta) - N_m m_e \right) + i \cdot N_{e\tau} m_{\tau} \sin(\theta) \cos(\phi) \right).
\]

The related Berry curvature can be subsequently calculated:
\[ \Omega_{\phi}^{iR} = \left( \frac{1}{N_s^2 m^2_{\phi} \sin^2(\theta) - 1} \right) \begin{pmatrix} 
abla \theta \left( m^2_{\phi} \sin(\theta) \sin(\phi)^2 \theta - m^2_{\phi} \sin(\theta) \right) + N_s m^2_{\phi} \cos(\theta) + 2 \left( m^2_{\phi} \sin(\theta) \cos^2(\theta) - m^2_{\phi} \sin(\theta) \right) 
abla \theta + \frac{1}{N_s m^2_{\phi} \cos(\theta) + 2 \left( m^2_{\phi} \sin(\theta) \cos^2(\theta) - m^2_{\phi} \sin(\theta) \right)} \frac{\partial N_s}{\partial \theta} + 2 m^2_{\phi} \sin(\theta) \cos(\theta) \cos(\theta) + 2 N_s m^2_{\phi} \sin(\theta) \cos(\theta) \right) 
abla \theta \frac{\partial N_s}{\partial \theta} + i \left( N_s m^2_{\phi} \sin(\theta) \cos(\theta) \right) \right) \] 
\[ \Omega_{\phi}^{iR} = \left( \frac{1}{N_s^2 m^2_{\phi} \sin^2(\theta) - 1} \right) \begin{pmatrix} N_s m^2_{\phi} \cos(\theta) - N_s m^2_{\phi} \cos^2(\theta) - i E_s N_{\phi} \frac{\partial E_s}{\partial \phi} + i \left( E_s^2 N_{\phi} - N_s m^2_{\phi} \cos(\theta) \right) \frac{\partial N_{\phi}}{\partial \phi} + 2 \left( N_s m^2_{\phi} \sin(\theta) \cos(\theta) + N_s m^2_{\phi} \cos^2(\theta) \right) \frac{\partial N_s}{\partial \phi} + 2 \left( N_s m^2_{\phi} \sin(\theta) \cos(\theta) \right) \frac{\partial N_{\phi}}{\partial \phi} \right) 
\[ \Omega_{\phi}^{iR} = \left( \frac{1}{N_s^2 m^2_{\phi} \sin^2(\theta) - 1} \right) \begin{pmatrix} -2 \left( N_s m^2_{\phi} \cos(\theta) \cos(\theta) \right) - i \left( E_s N_{\phi} m^2_{\phi} \sin(\theta) - N_s m^2_{\phi} \cos(\theta) \sin(\phi) \right) \frac{\partial N_{\phi}}{\partial \phi} + N_s m^2_{\phi} \sin(\theta) \frac{\partial E_s}{\partial \phi} - i E_s N_{\phi} \frac{\partial E_s}{\partial \phi} \right) + i \left( E_s N_{\phi} - N_s m^2_{\phi} \cos(\theta) \right) \frac{\partial N_{\phi}}{\partial \phi} \right) 
\[ \Omega_{\phi}^{iR} = \left( \frac{1}{N_s^2 m^2_{\phi} \sin^2(\theta) - 1} \right) \begin{pmatrix} -2 \left( N_s m^2_{\phi} \sin(\theta) \cos(\theta) \right) - 2 E_s N_{\phi} \left( N_s m^2_{\phi} \cos(\theta) - N_s m^2_{\phi} \right) \frac{\partial N_{\phi}}{\partial \phi} + 2 i \left( N_s m^2_{\phi} \sin(\theta) \cos(\theta) \right) \frac{\partial N_{\phi}}{\partial \phi} + 2 i \left( N_s m^2_{\phi} \cos(\theta) \right) \frac{\partial N_{\phi}}{\partial \phi} - E_s \frac{\partial E_s}{\partial \phi} + 2 i \left( N_s m^2_{\phi} \cos(\theta) \right) \frac{\partial E_s}{\partial \phi} \right) \] 

Based on the above Berry curvature, we can further calculate the topological charge of the current system with Eq. (S8). It is noted that the topological charge is determined by the...
integration surface. Here, we consider two typical cases, i.e., scheme 1: the selected integration surface encircles the entire WER ($|m_{hor}| < |m_{ver}|$); and scheme 2: the selected integration surface does not enclose the WER (within or outside the WER). For scheme 1, the topological charge can be directly solved by Eqs. S8, S15, S16, and S17:

$$\gamma_s = \frac{1}{2\pi} \oint_S \Omega_s \cdot \mathbf{k} \cdot dS = \frac{1}{2\pi} \oint_S \left( \Omega^{\alpha \beta}_{\alpha \beta}, \Omega^{\alpha \beta}_{\alpha \beta}, \Omega^{\alpha \beta}_{\alpha \beta} \right) \cdot dS.$$  \hspace{1cm} (S18)

It can be indicated that the Berry curvatures of $\Omega^{\alpha \beta}_{\alpha \beta}$ and $\Omega^{\alpha \beta}_{\alpha \beta}$ lead to zero components when solving Eq. (S18), hence the integration results in the following form:

$$\gamma_s = \pm \frac{1}{2\pi} \int_0^\pi d\theta \int_0^\pi d\varphi \cdot \Omega^{\alpha \beta}_{\alpha \beta} = \pm \frac{1}{2\pi} \int_0^\pi -\pi \cdot \sin(\theta) d\theta = \pm 1.$$  \hspace{1cm} (S19)

The value of Eq. (S19) proves that the topological charge is a quantized integer when the integration surface encircles the entire WER in this thermal system.

For scheme 2 without encircling any WER, we can investigate the topological charge with two types of integration surfaces. The first one is completely outside the WER for arbitrary $|m_{hor}|$, while the second one is within the WER under the condition of $|m_{hor}| > |m_{ver}|$. For the first type, the integration surface is selected in one separative band (upper or lower) which is different from the former one encircling the entire WER around the degenerate bands. Thus, a complete path ranging from 0 to $2\pi$ should be satisfied in $\theta$ direction, which reveals a zero charge. For the second case within the WER, the selected surface is entirely comprised of branch points. Referring to the strategy in (2), we can rewrite the eigenvectors following a single band in the current system:

$$\langle \psi^s_1(k) \rangle = \langle \psi^s_1(k) \rangle \Theta(\mathrm{Re}(d)) + \langle \psi^s_0(k) \rangle \Theta(-\mathrm{Re}(d)).$$  \hspace{1cm} (S20)

$$\langle \psi^s_2(k) \rangle = \langle \psi^s_2(k) \rangle \Theta(\mathrm{Re}(d)) + \langle \psi^s_1(k) \rangle \Theta(-\mathrm{Re}(d)).$$

$$\langle \psi^s_3(k) \rangle = \Theta(\mathrm{Re}(d)) \langle \psi^s_2(k) \rangle + \Theta(-\mathrm{Re}(d)) \langle \psi^s_1(k) \rangle.$$  \hspace{1cm} (S21)

where $\Theta$ denotes the Heaviside function. With the conditional eigenvectors, the Berry connection and curvature can be calculated, and only a zero charge can be subsequently observed under the condition of $|m_{hor}| > |m_{ver}|$. In general, the topological charges remain zero when the integration surface $S$ does not encircle the WER for the proposed system, no matter the selected integration surface is within or outside the WER.

**Supplementary Note 2: Spatiotemporal-modulated velocities and heat exchanges of Cases 1 and 2 and supplementary simulated and experimental profiles.**

Owing to the larger vertical thickness of the central strip $d_{ver}$ and material parameters of the red advective strips, the heat exchanges caused by vertical-advection components could be regarded as an effect of sweeping slab on the vertical surface of the central strip. Thus, a temporal $h_{ver}=0.332 Re^2 pr_{f}^{-3}$ should be considered and indicate a direct proportion to the square root of time-changing $ver$ as a contrast to its ideal value $v/d_{ver}$. It further contributes to the concomitant oscillations of the exothermic and endothermic processes along the synthetic dimensions $m_{ver}$. Besides, the smaller horizontal thickness of the central strip $d_{hor}$ and the effective conductivity of hollow horizontal strips allow fast and concentrated heat transfer through the central strip along the horizontal direction. Hence, we could maintain a quasi-constant heat exchange $m_{hor}$ with the ideal value of $h_{hor}=k/d_{hor}$ in the practical implementations. Under the above conditions, the critical values of $|m_{hor}|$ on the WER is 0.13 s$^{-1}$, while the initial $|m_{ver}|$ is 0.009 s$^{-1}$. The configurations of the employed spatiotemporal-modulated velocities are as follows: Case 1 (encircling the WER based on one pair of EPs (EP1) of $|k_{hor}V_{hor}|^{EP1} = |k_{ver}V_{ver}|^{EP1}$): $k_{hor}V_{hor} = 6\pi |k_{hor}V_{hor}|^{EP1} \sin(2\pi/10-t)$ and $k_{ver}V_{ver} = 6\pi |k_{ver}V_{ver}|^{EP1} \cos(2\pi/10-t)$; and Case 2 (outside the WER): $k_{hor}V_{hor} = 6\pi |k_{hor}V_{hor}|^{EP1} \sin(2\pi/10-t)+8\pi |k_{hor}V_{hor}|^{EP1}$ and $k_{ver}V_{ver} = 6\pi |k_{ver}V_{ver}|^{EP1} \cos(2\pi/10-t)+8\pi |k_{ver}V_{ver}|^{EP1}$, while the vertical heat exchanges oscillate as the function
of $m_{\text{ver}} = \pm \frac{k_{\text{hor}, \text{ver}}}{k_{\text{hor}, \text{ver}}^{\text{EP1}}} \varepsilon \pi$ and the sign is dependent on the locally endothermic or exothermic processes according to the directions local advections and coupled heat flux. Here, Case 1 serves to present the nontrivial transitions around the WER. It is noted that the WER consists of multiple pairs of EPs in different synthetic planes of the parameter space, thus Case 1 is one of the representative schemes (SI Appendix Supplementary Notes 3 and 5). Case 2 aims to demonstrate the trivial transitions in thermal diffusion when the spatiotemporal-modulated route of advections outside the WER.

To further indicate the topologically non-trivial and trivial transitions around the WER in thermal diffusion, the measured surfaces in $x$-$y$ and $x$-$z$ spaces and the supplementary simulated/experimental profiles are shown in Fig. S1. The robust thermal profiles are significant when the WER is completely encircled in Case 1, while moving profiles are obtained if the encircling surface is out of the WER in Case 2.

**Supplementary Note 3: Contrast scheme based on another pair of EPs on the WER**

As indicated in the main content, the WER consists of countless EP pairs in the advective space. To illustrate such a property in non-Hermitian thermal system, we define the EP pair used in the main content as EP1, when the advections meet $|k_{\text{hor,ver}}|_{\text{EP1}} = |k_{\text{ver,ver}}|_{\text{EP1}}$. Then, we further adopt another EP pair as EP2 at the plane $|k_{\text{hor,ver}}|_{\text{EP2}} \leq |k_{\text{ver,ver}}|_{\text{EP2}}$ to build a Contrast scheme 1 to study the WER encircling behaviors. In that case, the advective configurations of Contrast scheme 1 are $k_{\text{hor,ver}} = 6 |k_{\text{hor,ver}}|_{\text{EP1}} \sin(2\pi/10 - t)$ and $k_{\text{ver,ver}} = 6 |k_{\text{ver,ver}}|_{\text{EP1}} \cos(2\pi/10 - t)$. This EP pair is selected in a different $k_{\text{hor,ver}}=k_{\text{ver,ver}}$ plane from the EP1 of the main content (see **Supplementary Note 5**). For simplification, we track the $T_{\text{max}}$ trajectories along $x$-directions in the two measured surfaces in all the following contents. The related behaviors are illustrated in Fig. S2. The observed behaviors also present the robust thermal profiles, thus further revealing the significances of the observed WER and the path independence of the topological transitions in thermal diffusion. These findings indicate the significance of encircling the WER rather than a single pair of EPs.

**Supplementary Note 4: Encircling behaviors within the WER**

Case 2 presents a trivial transition with a zero topological charge, whose integration surface is completely out of the observed WER. There is also another case possessing a zero topological charge when the integration surface is completely within the WER. To further demonstrate such behaviors, we adopt a Contrast scheme 2 with the advective configurations of $k_{\text{hor,ver}} = 0.5 |k_{\text{hor,ver}}|_{\text{EP1}} \sin(2\pi/10 - t)$ and $k_{\text{ver,ver}} = 0.5 |k_{\text{ver,ver}}|_{\text{EP1}} \cos(2\pi/10 - t)$ based on EP1 in the horizontal and vertical advections, and the thermal profiles are shown in Fig. S3. The stationary thermal profiles and the $T_{\text{max}}$ locations can be achieved both in the horizontal and vertical interlayers for Contrast scheme 2. Owing to the smaller advections velocities than the critical values on the WER, i.e., $|k_{\text{hor}}|_{\text{EP}}$ and $|k_{\text{ver}}|_{\text{EP}}$ such stationary thermal profiles are dominated by the internal thermal conduction (attenuation) of the central strip rather than the imposed advections (propagation). This behavior is a distinctive phase transition in diffusion when the system is strongly dominated by dissipation.

**Supplementary Note 5: Different pairs of EPs on the WER**

To illustrate the topological transitions of Case 1 and Contrast scheme 1 around the observed WER, we investigate the behaviors on specific EP pairs in different parameter-planes ($k_{\text{hor,ver}}=k_{\text{ver,ver}}$ plane) of the current systems. Here, we select two schemes with different advective configurations based on Eq. (3), i.e., $|k_{\text{hor}}|_{\text{EP1}} = |k_{\text{ver}}|_{\text{EP1}}$ for scheme S1 and $|k_{\text{hor}}|_{\text{EP1}} = 2 |k_{\text{ver}}|_{\text{EP1}}$ for scheme S2. Considering the structural parameters of the current system indicated in the Method, the EP pairs of the two schemes respectively locate at $k_{\text{hor,ver}} = \pm 0.705m_{\text{hor}}$ in the plane of $\theta = 0.022\pi$, $\varphi = 0.446/6$ (EP1), and $k_{\text{hor,ver}} = 0.446m_{\text{hor}}$ in the plane of $\theta = 0.022\pi$, $\varphi = 0.446/6$ (EP2) for schemes S1 and S2, where $\theta$ and $\varphi$ are same with those of Eq. (S9). As indicated in (1), the thermal profiles would remain stationary when the selected parameters are on the EPs. Thus, the observations of stationary thermal profiles corresponding to different EP pairs are the direct evidence for the existence of WER, since the WER is the set of multiple EP pairs in varied
parameter-planes. The thermal profiles of schemes S1 and S2 are shown in Fig. S4. Both the thermal profiles almost remain stationary, thus further revealing the significant WER in the current thermal system.

Supplementary Note 6: Experimental setups for demonstrating the WER

Based on the model shown in Fig. 2, we have fabricated an experimental sample supporting the orthogonal advections along the heat flux as illustrated in Fig. S5. The two green strips are driven by two conveyor belts to translate in x-y surface, and each conveyor belt is connected to two motors (H-motors, marked by green stars) through a motion controller. The other two red strips are driven by another two conveyor belts to translate in x-z surface (orthogonal direction). These two conveyor belts are driven by motors (V-motors, marked by red stars), since we connect one side of each belt to their motors and the other side bypasses a fixed cylinder. The two motion controllers (H and V) respectively provide the movement instructions for the horizontal and vertical advections of Cases 1 and 2 and the Contrast schemes 1 and 2 during the experiments. The lower insert exhibits the assembled sample (the lateral conveyor belt in x-z surface is removed), where the arrows indicate the initial motion-directions.

Supplementary Note 7: Strategy for realizing two opposite-charge WER and experimental setups for observing the surface-like state

To validate the presence of the surface-like state, we further parallel another hybrid thermal system with two horizontal advections, another central strip, and one vertical advective strip with the initial system. The newly inserted system also consists of the above three parts as illustrated in Figs. 3B and C, i.e., the central medium strip, the green horizontal advective component, and the red vertical advective component. The structural parameters of the central strip are also \( d_{hor} = 5 \text{ mm}, d_{ver} = 10 \text{ mm}, \) and \( L = 100 \text{ mm}. \) Similar to the initial sample, the structural parameters of the advective components can be defined as \( b_{hor} = b_{ver} = 5 \text{ mm} \) and \( L_{hor} = L_{ver} = 3L. \) Besides, the related medium properties are same with those of the initial sample. We further parallel it with one of the vertical strips of the initial system (the upper inset of Fig. 3B) to keep their vertical-advective components in the same space. Therefore, one of the vertical strips of the initial system is shared to simultaneously act as a vertical strip of the inserted one as shown in Fig. 3C. After observing the WER in the initial system, the subsequent system possessing the same initial temperatures with the initial one is imposed. To create another robust WER with an opposite charge in the inserted system, we need to realize an opposite spatiotemporal-modulated strategy either in the horizontal or vertical advection pair of the inserted system. Considering the shared vertical strip, the spatiotemporal-modulated configuration of the vertical advection pair is spontaneously opposite to the initial one. Thus, we only need to keep a reversed velocity in the inserted vertical strip to the shared one as presented in Fig. 3C, while the same horizontal spatiotemporal-modulated advections as the initial system can be retained.

The experimental setups for demonstrating the robust profiles of surface-like state are presented in Fig. S6, while some additional simulated/experimental profiles measured at other moments are illustrated in Fig. S7. Compared with the one shown in Fig. S5A, four conveyor belts are employed to drive the four green strips in x-y surface, while three strips are imposed to actuate three red strips in x-z surface. The upper insert exhibits the assembled sample (the upper conveyor belts in x-y surface are removed), where the arrows indicate the initial motion-directions. The green and red strips denote the motion strips for horizontal (green) and vertical (red) advections, while their driving motors are marked by green and red stars. The supplementary thermal profiles are illustrated in Figs. S7B ~ D, which further verify the significances of the observed surface-like state with robust thermal profiles and process.

Due to the similar changing trends of the surface and interface temperature profiles of the central strip, the thermal images of the WER and surface-like states are captured on corresponding surfaces of the central strip by an IR camera. The locations of IR camera for capturing different surfaces are shown in Figs. S8A and B. For directly observing the temperature distributions on these surfaces at one specific moment, the advective strips and their connected conveyor belts in front of these surfaces are removed and the other motions are also terminated.
during the measurements. When capturing the thermal images at other moments, the entire experiment and the above implementations are repeated.

**Supplementary Note 8: Bulk state in non-Hermitian thermal diffusion**

Similar to the wave systems (2, 3), the surface-like state is observed when the pairs of WERs possessing opposite topological charges exist in the coupled system. The presentation of surface-like state provides a connection between the opposite WERs in the initial and inserted systems and contributes to the robust heat transfer processes. If the pairs of WERs are removed, the analogue of bulk state possessing zero topological charges with moving thermal profiles can be expected. To confirm the bulk state in the current diffusive system, we reset the horizontal and vertical advectons of the initial and inserted systems to make their integration surfaces outside the WERs. For simplification, the horizontal and vertical advectons employed in Case 2 of the main content are used in the initial and inserted systems, i.e., $k_{\text{hor}} = 6 |k_{\text{hor}} V_{\text{hor}}| \sin(2\pi/10-t)+8 |k_{\text{hor}} V_{\text{hor}}| \pi$ and $k_{\text{ver}} = 6 |k_{\text{ver}} V_{\text{ver}}| \pi \cos(2\pi/10-t)+8 |k_{\text{ver}} V_{\text{ver}}| \pi$. The $T_{\text{max}}$ locations are illustrated Figs. 4A − D of the main text, while the simulated/experimental phase changes are illustrated in Fig. S9. It is significant that the thermal profiles of central strips keep moving during the entire heat transfer process. These moving thermal profiles are in accordance with the expectation and can be considered as the bulk state in 3D non-Hermitian thermal diffusion.

**Supplementary Note 9: Effects of the central strip conductivity**

The thermal distributions are highly affected by the thermal conductivity of the central strip, since the terms of $m_{\text{hor}}$ and $m_{\text{ver}}$ are dependent of its value. These terms $m_{\text{hor}}$ and $m_{\text{ver}}$ determines the radius $m_{\text{p}} = \sqrt{(m_{\text{ver}})^2 + (k_{\text{ver}} v_{\text{ver}})^2 + (k_{\text{hor}} v_{\text{hor}})^2}$ of the critical integration surface and velocities for reaching the WER. When it changes, the size of WER and critical integration surface also change accordingly. That is, the enclosing behaviors might change with different thermal conductivities of the central strip under the same advective configurations. The different encircling behaviors further leads to the thermal distributions of stationary and moving thermal profiles, once the integration surface is respectively within/enclosing and outside the WER. Here, two referenced calculations (Schemes S3 and S4) with different conductivities of the central strip are implemented to indicate the related effects. To make fair comparisons, we keep the same velocity configurations ($v_{\text{ver}}$ and $v_{\text{hor}}$) of Case 2 presenting trivial thermal distributions. While the conductivities are respectively $k_{\text{S3}} = 0.05 k_{\text{Case 2}}$ and $k_{\text{S4}} = 20 k_{\text{Case 2}}$. For Scheme S3, the radius of WER and integration surface are smaller than the ones of Case 2 due to the decreased central strip conductivity. In that case, the WER of Scheme S3 is also outside the integration surface and further leads to the moving temperature distributions and changing $T_{\text{max}}$ locations at different moments as shown in Figs. S10 A − D. For Scheme S4, the radius of WER and integration surface are larger than the ones of Case 2, while the enlarged integration surface is completely within such a larger WER at this stage. Thus, the entire thermal progress is dominated by the inherent conductions and reveal the stationary temperature distributions as illustrated in Figs. S10 E − H. In general, the thermal conductivity of central strip would affect the size of WER, while the temperature distributions would change accordingly upon the enclosing behaviors around the newly formed WER.
Fig. S1. Measured surfaces and supplementary simulated/experimental profiles of Cases 1 and 2. (A) The experimental setups. (B) The measured surfaces of the central strip, where the light green and light red surfaces respectively denote the ones in x-y and x-z spaces; (C) and (D) present the supplementary simulated/experimental thermal profiles at specific moments of Cases 1 and 2. The upper and lower inserts respectively illustrate the profiles in the x-y (light green) and x-z (light red) spaces. The white dots indicate the $T_{\text{max}}$ locations at specific moments.
**Fig. S2.** Contrast scheme 1 based on the EP2 at $|k_{hor} V_{hor}|_{EP2} = 2|k_{ver} V_{ver}|_{EP2}$.  
(A) The locations of maximum temperatures observed in the horizontal and vertical measured surfaces along x-directions; (B) The encircling surface of Contrast scheme 1; (C) Simulated/experimental thermal profiles at specific moments in the x-y (horizontal) space. (D) Simulated/experimental thermal profiles at specific moments in the x-z (vertical) space. The measured surfaces are same with the ones in Fig. S1A.
Fig. S3. Contrast scheme 2 within the WER. (A) The $T_{\text{max}}$ locations observed in the horizontal and vertical measured surfaces along $x$-directions; (B) The encircling surface of Contrast scheme 2; (C) Simulated/experimental thermal profiles at specific moments in the $x$-$y$ (horizontal) space. (D) Simulated/experimental thermal profiles at specific moments in the $x$-$z$ (vertical) space. The measured surfaces are same with the ones in Fig. S1A.
Fig. S4. Thermal profiles of schemes S1 and S2 at specific EP pairs in different $k_{\text{hor}}$-$k_{\text{ver}}$ planes on the WER (EP1 and EP2). (A) and (B) are the employed EP pairs respectively in the planes of $|k_{\text{hor}}| = |k_{\text{ver}}|$ (grey plane in A) and $|k_{\text{hor}}| = 2|k_{\text{ver}}|$ (blue plane in B). The red circles denote the observed WER, while the red and blue dots indicate the selected EP pairs. The $T_{\text{max}}$ locations along $x$-directions and related thermal profiles of schemes S1 and S2 are respectively presented in (C) and (D).
Fig. S5. Experimental setups for demonstrating the WER in thermal diffusion. (A) presents a global view of the experimental setups and fabricated sample (right-lower insert). (B) illustrates the inputs and the thermal images of the initially periodic thermal profiles. The periodic thermal profiles are achieved by periodically and alternatively configure the heating and cooling strips on the combined sample as shown in the upper insert. The lower inserts demonstrate the initial thermal profiles (without advective motions) of the entire system, while the valid area involving the central medium, i.e., the IR images shown in Fig. 2 of the main content and Figs. S1 ~ S3 are marked by dashed borders.
Fig. S6. Experimental setups for observing the surface-like state. The experimental setups with four green and three red strips respectively in the x-y and x-z spaces. The upper insert is the combined sample.
Fig. S7. Temperature profiles of surface-like state measured on the central strips. (A) The measured surfaces used in the simulations and experiments for capturing the locations of $T_{\text{max}}$, where the green and red surfaces respectively denote the ones in x-y and x-z spaces. The black and red borders are employed to indicate the initial and inserted systems; (B) presents the simulated (upper insert) and experimental (lower insert) temperature profiles at specific moments of the surface-like state in the x-y (horizontal) surface. (C) and (D) are the temperature profiles in the two lateral sides of x-z (vertical) spaces, while the upper and lower inserts of each subgraph denote their simulated and experimental temperature profiles. Note that, the IR images are captured on the surfaces of the central strip instead of the interfaces, while the locations of $T_{\text{max}}$ are directly measured on the interfaces by the embedded fine-wire thermocouples.
Fig. S8. Locations of IR camera for capturing the thermal profiles on different surfaces and illustrations on the normalized spatial dimensions. (A) and (B) respectively indicate the locations of IR camera for measuring the related behaviors based on the WER and surface-like states. (C) and (D) provide the graphical illustrations of the normalizations on each surface. The origin $O$ of these local coordinates are the initial $T_{\text{max}}$ locations at $0s$ of these measured surfaces.
Fig. S9. Temperature profiles of the thermal bulk state measured on the central strips. (A) Simulated (upper insert) and experimental (lower insert) profiles of the bulk state at specific moments measured in the horizontal surface in the x-y space. (B) and (C) respectively illustrate the simulated (upper insert) and experimental (lower insert) profiles of the two lateral sides in the x-z space. The black and red dashed-borders indicate the initial and inserted systems, while the white dots implies the $T_{\text{max}}$ locations for the two central strips. The white line indicates the $T_{\text{max}}$ location at the initial moment.
Fig. S10. Thermal behaviors of Schemes S3 and S4 with different central strip conductivities under the same advective configurations of Case 2. (A) ~ (D) and (E) ~ (H) respectively denote the thermal behaviors of Schemes S3 and S4 with the conductivities of $\kappa_{S3} = 0.05\kappa_{case 2}$ and $\kappa_{S4} = 20\kappa_{case 2}$. Among them, (A) and (E) present the newly formed WERs and the enclosing situations. (B) and (F) illustrate temperature distributions of the horizontal and vertical surfaces of these two schemes. (C)/(D) and (G)/(H) present the $T_{max}$ locations on the horizontal/vertical surface of Schemes S3 and S4. The moving and stationary profiles are significant in Schemes S3 and S4 respectively with smaller and larger conductivities, since their integration surfaces are respectively outside and within the corresponding WERs.
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