Average Entropy of a Subsystem

Siddhartha Sen

School of Mathematics, University of Dublin, Trinity College, Dublin, Ireland

(March 28, 2022)

It was recently conjectured by D. Page that if a quantum system of Hilbert space dimension \(mn\) is in a random pure state then the average entropy of a subsystem of dimension \(m\) where \(m \leq n\) is \(S_{mn} = \sum_{k=n+1}^{mn} (1/k) - (m-1)/2n\). In this letter this conjecture is proved.

PACS numbers: 05.30.ch,03.65.-w,05.90.+m

In a recent letter Page considered a system \(AB\) with Hilbert space dimension \(mn\). The system consisted of two subsystems \(A\) and \(B\) of dimensions \(m\) and \(n\) respectively. Page calculated the average

\[ S_{mn} = \langle S_A \rangle \]

of the entropy \(S_A\) over all pure states \(\rho = |\Psi\rangle \langle \Psi|\) of the total system where \(S_A = -\text{Tr} \rho_A \ln \rho_A\) and \(\rho_A\), the density matrix of subsystem \(A\), is obtained by taking the partial trace of the full density matrix \(\rho\) over the other subsystem, that is, \(\rho_A = \text{Tr}_B \rho\).

The average was defined with respect to the unitary invariant Haar measure on the space of unitary vectors \(|\Psi\rangle\) in the \(mn\) dimensional Hilbert space of the total system. The quantity \((\ln m - S_{mn})\) was used to define the average information of the subsystem \(A\). It is a measure of the amount of information regarding the fact that the entire system is a pure state that is contained in the subsystem \(m\). Using earlier work in this area, Page was led to consider the probability distribution of the eigenvalues of \(\rho_A\) for the random pure states \(\rho\) of the entire system.

He used

\[
P(p_1, \ldots, p_m) \, dp_1 \ldots dp_m
= N\delta(1 - \sum_{i=1}^{m} p_i) \prod_{1 \leq i < j \leq m} (p_i - p_j)^2 \prod_{k=1}^{m} p_k^{n-m} dp_k
\]

where \(p_i\) was an eigenvalue of \(\rho_A\) and the normalisation constant for this probability distribution was given only implicitly by the requirement that the total probability integrated to unity. Page then showed that the average

\[
S_{mn} = -\int \left( \sum_{i=1}^{m} p_i \ln p_i \right) P(p_1, \ldots, p_m) \, dp_1 \ldots dp_m
= \psi(mn + 1) - \frac{1}{mn} \int \frac{Q(q_1, \ldots, q_m) \, dq_1 \ldots dq_m}{Q(q_1, \ldots, q_m) \, dq_1 \ldots dq_m}
\]

where \(q_i = r p_i\) for \(i = 1, \ldots, m\), \(r\) is positive, and

\[
\psi(mn + 1) = -C + \sum_{k=1}^{mn} \frac{1}{k},
\]

\(C\) being Euler’s constant, and

\[
Q(q_1, \ldots, q_m) \, dq_1 \ldots dq_m = \prod_{1 \leq i < j \leq m} (q_i - q_j)^2 \prod_{i=1}^{m} e^{-q_i n - m} dq_i.
\]

On the basis of evaluating \(S_{mn}\) for \(m = 2, 3, 4, 5\) using Mathematica, Page conjectured that the exact result for \(S_{mn}\) was

\[
S_{mn} = \frac{1}{k} - \frac{(m-1)}{2n},
\]

but was not able to prove that this was the case. In this letter, we will prove this conjecture.

We first observe that the van der Monde determinant defined by

\[
\Delta(q_1, \ldots, q_m) = \prod_{i \leq i < j \leq m} (q_i - q_j)
\]

may be written

\[
\Delta(q_1, \ldots, q_m) = \begin{vmatrix}
1 & \cdots & 1 \\
q_1 & \cdots & q_m \\
\vdots & \ddots & \vdots \\
q_1^{m-1} & \cdots & q_m^{m-1}
\end{vmatrix}
\]

We next observe that \(\Delta(q_1, \ldots, q_m)\) can be written as

\[
\Delta(q_1, \ldots, q_m) = \begin{vmatrix}
p_0(q_1) & \cdots & p_0(q_m) \\
p_1(q_1) & \cdots & p_1(q_m) \\
\vdots & \ddots & \vdots \\
p_{m-1}(q_1) & \cdots & p_{m-1}(q_m)
\end{vmatrix}
\]

for any set of polynomials \(p_k(q), k = 0, \ldots, m-1\), which have the property, \(p_0(q) = 1\), and

\[
p_k(q) = q^k + C_{k-1} q^{k-1} + \cdots + C_0, \quad k = 1, \ldots, m-1.
\]

This immediately follows from the fact that the value of a determinant does not change if the multiple of any one row is added to a different row.

We now choose polynomials \(p_k^0(q)\) judiciously. We introduce orthogonal polynomials \(p_k^0(q)\) with the properties:

1. \(p_k^0(q) = q^k + C_{k-1} q^{k-1} + \cdots + C_0\), \(p_0^0(q) = 1\).
2. \( \int_0^\infty dq e^{-q} q^p p_k^m(q) p_k^n(q) = h_k^m \delta_{k,k}, \quad \alpha = n - m. \)

Polynomials with these properties are well known. They are the generalised Laguerre polynomials defined by

\[ p_k^m(q) = \frac{e^q}{q^m} \frac{d^k}{dq^k} (e^{-q} q^{k+\alpha}). \]

We also note, for later use, that

\[ p_k^m(q) \sum_{r=0}^k \binom{k}{r} (-1)^r \frac{\Gamma(k+\alpha+1)}{\Gamma(k+\alpha-r+1)} q^{k-r} \quad (2) \]

\[ \int_0^\infty dq e^{-q} q^p p_k^m(q) p_k^n(q) = \Gamma(k+1)\Gamma(k+\alpha+1)\delta_{k,k} \]

\[ \int_0^\infty dq q^{n-1} e^{-q} p_k^m(q) = (1-a+b)k\Gamma(a)(-1)^k \quad (4) \]

recalling that, \((1-a+b)_k = (1-a+b)(1-a+b+1)\ldots(1-a+b+k-1).\) Writing \(\Delta(q_1,\ldots,q_m)\) in terms of \(p_k^m(q)\) as in Eq. and using the orthogonal property of these polynomials it immediately follows that:

\[ S_{mn} = \psi(mn+1) - \frac{1}{mn} \sum_{k=0}^{m-1} \int_0^\infty e^{-q} q^{n-m} (p_k^{m-n}(q))^2 dq. \]

We thus need to evaluate the integral

\[ I_{mn}^k = \int_0^\infty (q \ln q) q^{n-m} (p_k^{m-n}(q))^2 e^{-q} dq. \]

We first introduce

\[ J^k(\alpha) = \int_0^\infty q^\alpha (p_k^m(q))^2 e^{-q} dq. \]

This integral is easily evaluated. We have

\[ J^k(\alpha) = \Gamma(k+1)\Gamma(k+\alpha+2) + k^2\Gamma(k)\Gamma(k+\alpha+1) \quad (5) \]

and we now note that

\[ I_{mn}^k = \left[ \frac{dJ^k(\alpha)}{d\alpha} - 2 \int_0^\infty dq q^{\alpha+1} p_k^{m-n}(q) \frac{dp_k^m}{d\alpha} \right]_{\alpha=n-m}. \]

Evaluating these two terms using Eqs. (2), (3), (5), and (6) we find

\[ S_{mn} = \psi(mn+1) - \frac{1}{mn} \sum_{k=0}^{m-1} [1 + (1+2k+n-m)\psi(k+n+m+1)] \]

\[ + \frac{2}{mn} \sum_{k=0}^{m-1} \sum_{r=0}^k \binom{k}{r} (-1)^{k+r} \frac{\Gamma(k+n-m+1)}{\Gamma(k+n-m-r+1)} \]

\[ \times \left[ \psi(k+n-m+1) - \psi(k+n-m-r+1) \right] \]

\[ \times \frac{(r-k-1)\Gamma(k+n-m-r+2)}{\Gamma(k+1)\Gamma(k+n-m+1)} \quad (6) \]

where we use the fact that \(\psi(z) = \frac{d}{dz} \Gamma(z).\) We now observe that

\[ \psi(mn+1) - \frac{1}{mn} \sum_{k=0}^{m-1} [1 + (1+2k+n-m)\psi(1+k+n-m)] \]

\[ = \frac{1}{mn} \sum_{k=0}^{m-1} \frac{1}{k} + \frac{(m-1)}{2n}. \quad (7) \]

This follows by examining the coefficient of \(\frac{1}{r}\) in

\[ \sum_{k=0}^{m-1} (1+2k+n-m)\psi(1+k+n-m) \]

and writing

\[ \psi(1+k+n-m) = -C + \sum_{r=1}^{k+n-m} \frac{1}{r}. \]

The third expression in Eq. (3) above is

\[ \frac{2}{mn} \sum_{k=0}^{m-1} \sum_{r=0}^k \binom{k}{r} (-1)^{k+r} \frac{\Gamma(k+n-m+1)}{\Gamma(k+n-m-r+1)} \]

\[ \times \left[ \psi(k+n-m+1) - \psi(k+n-m-r+1) \right] \]

\[ \times \frac{(r-k-1)\Gamma(k+n-m-r+2)}{\Gamma(k+1)\Gamma(k+n-m+1)} \]

\[ = \frac{2}{mn} \sum_{k=0}^{m-1} \binom{k}{1} (-1)^{2k+1} \]

\[ = -2 \frac{(m-1)}{2n}. \quad (8) \]

Since \((r-k-1)_k = 0,\) for all \(r \neq 0\) and \(r \neq 1,\) and also \(\psi(k+n-m+1) - \psi(k+n-m-r+1) = 0\) when \(r = 0,\)

we obtain

\[ S_{mn} = \sum_{k=n+1}^{m} \frac{1}{k} - \frac{(m-1)}{2n}. \quad (9) \]

as conjectured by Page.

This work is part of project SC/218/94 supported by Forbairt.

* Electronic address: sen@maths.tcd.ie

[1] D. N. Page, Phys. Rev. Lett. 71, 1291 (1993).
[2] E. Lubkin, J. Math. Phys. 19, 1028 (1978).
[3] S. Lloyd and H. Pagels, Ann. Phys. (N. Y.) 188, 186 (1988).
[4] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, Integrals and Series Vol.2, Gordon and Breach Publishers (1988).