Optimal Rank-1 Hankel Approximation of Matrices: Frobenius Norm, Spectral Norm and Cadzow’s Algorithm

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Abstract. In this paper we derive optimal rank-1 approximations with Hankel or Toeplitz structure with regard to two different matrix norms, the Frobenius norm and the spectral norm. We show that the optimal solutions with respect to these two norms are usually different and only coincide in the trivial case when the singular value decomposition already provides an optimal rank-1 approximation with the desired Hankel or Toeplitz structure. We also show that the often used Cadzow algorithm for structured low-rank approximations always converges to a fixed point in the rank-1 case, however, it usually does not converge to the optimal solution – neither with regard to the Frobenius norm nor the spectral norm.

1 Introduction

Structured low-rank approximations are widely used in many signal processing problems as in system theory, parameter identification and signal analysis, e.g. singular spectral analysis (SSA) [11]. Applications include minimal partial realizations in linear system theory, multi-input, multi-output systems, system identification problems or approximation with finite rate of innovation signals [9] [17] [29]. Low-rank Hankel approximation is closely related to Prony’s method [24], or related modifications [5] [20] [30].

Generally, a low-rank Hankel approximation problem can be written as a non-convex optimization problem. For a given matrix $A \in \mathbb{C}^{M \times N}$ we want to find a Hankel matrix $H_r$ of rank at most $r < \min\{M, N\}$, such that

$$H_r := \arg\min_{H \text{ Hankel} \atop \text{rank } H \leq r} \| A - H \|,$$  \hspace{1cm} (1.1)

where the considered matrix norm is usually taken to be a (weighted) Frobenius norm.

Statement of the problem. In this paper, we want to solve the problem of low-rank Hankel approximation for rank $r = 1$ analytically for the Frobenius norm and the spectral norm. To state the problem precisely, we start with some notations.

For a given matrix $A = (a_{j,k})_{j,k=0}^{M-1,N-1} \in \mathbb{C}^{M \times N}$ we define the Frobenius norm and the spectral norm of $A$ as

$$\| A \|_F := \left( \sum_{j=0}^{M-1} \sum_{k=0}^{N-1} |a_{j,k}|^2 \right)^{1/2}, \quad \| A \|_2 := \max_{\| x \|_2 = 1} \| Ax \|_2 = (\rho(A^*A))^{1/2},$$
where  \( \|x\|_2 := \left( \sum_{j=0}^{N-1} |x_j|^2 \right)^{1/2} \) denotes the Euclidean vector norm,  \( A^* := \overline{A}^T \), and  \( \rho(A^*A) \) is the spectral radius of the positive semi-definite matrix  \( A^*A \), i.e., the largest eigenvalue of  \( A^*A \). Throughout the paper we assume that  \( A^*A \) possesses a single largest eigenvalue  \( \sigma_0^2 = \|A\|_2^2 > \sigma_1^2 \) which is bounded away from the second largest singular value.

Hankel matrices are of the form

\[
H := (h_{k+\ell})_{k,\ell=0}^{M-1,N-1} = \begin{pmatrix} h_0 & h_1 & h_2 & \cdots & h_{N-1} \\ h_1 & h_2 & \cdots & & h_N \\ & h_2 & \cdots & & \\ & & \ddots & \cdots & \\ & & & h_{M-1} & h_M + 1 & \cdots & h_{M+N-2} \end{pmatrix} \in \mathbb{C}^{M \times N}.
\]

In this paper, we are interested in optimal approximations of a given matrix  \( A \in \mathbb{C}^{M \times N} \) by a rank-1 Hankel matrix  \( H_1 \) of the same size, i.e., we want to solve

\[
\min_{H_1 \in \mathbb{C}^{M \times N}} \|A - H_1\|_F^2 \quad \text{or} \quad \min_{H_1 \in \mathbb{C}^{M \times N}} \|A - H_1\|_2^2,
\]

(1.2)

under the restriction that  \( H_1 \) is a Hankel matrix of rank 1. While we will always consider Hankel matrices in this paper, we want to remark that the minimization problems in (1.2) can be rewritten using Toeplitz matrices instead of Hankel matrices. Toeplitz matrices are given by

\[
T = (h_{k-\ell})_{k,\ell=0}^{M-1,N-1},
\]

and it can be simply observed that any Toeplitz matrix  \( T \in \mathbb{C}^{M \times N} \) can be presented as

\[
T = HJ_N,
\]

where  \( H \) is a Hankel matrix and

\[
J_N := \begin{pmatrix} 0 & \ldots & 0 & 1 \\ \vdots & & 1 & 0 \\ 0 & \vdots & & \vdots \\ 1 & 0 & \ldots & 0 \end{pmatrix} \in \mathbb{C}^{N \times N}
\]

(1.3)

denotes the counter identity matrix. The optimal approximation of a given matrix  \( A \in \mathbb{C}^{M \times N} \) by a rank-1 Toeplitz matrix  \( T_1 = H_1J_N \) of the same size, can therefore be transferred to the problem of rank-1 Hankel approximation of  \( AJ_N \),

\[
\min_{T_1 \in \mathbb{C}^{M \times N}} \|A - T_1\|_F^2 = \min_{H_1 \in \mathbb{C}^{M \times N}} \|A - H_1J_N\|_F^2 = \min_{H_1 \in \mathbb{C}^{M \times N}} \|AJ_N - H_1\|_F^2.
\]

This transfer works likewise for the spectral norm.

**Related approaches.** It is well-known that for the Frobenius norm and the spectral norm an optimal (unstructured) low-rank approximation can be found by applying the singular value decomposition. However, the obtained low-rank approximation usually does no longer possess the wanted structure, in our case, Hankel or Toeplitz structure.

The minimization problems in (1.1) and in (1.2) are non-convex. There are different optimization approaches in the literature to tackle this problem. A heuristic approach,
often used in practice because of its simplicity, is Cadzow’s algorithm \[6, 7\]. This method is based on alternating projection.

The problem in (1.1) can also be rewritten as a non-linear structured least squares problem (NSLSP), see e.g. \[8, 10, 14, 15, 16, 18, 19, 27\], or as a nonlinear eigenvalue problem \[5, 20, 30\]. When applying the NSLSP methods one usually assumes that the initial matrix \(A\) itself already is structured (here Hankel or Toeplitz).

Some methods are based on relaxation of the optimization problem using the nuclear norm \[9\], convex envelopes \[2, 12\] or subspace based methods \[17, 28\].

A completely different idea to study the structured low-rank approximation problem arises from the AAK theory \[1\] for optimal low-rank approximation of Hankel operators. The AAK theory shows that infinite Hankel matrices (with certain decay properties of their components) can always be approximated by infinite Hankel matrices of lower rank with optimal error. This means, similarly as in the case of unstructured matrices, the (operator norm) error of the rank-\(r\) approximation is given by the \((r + 1)\)-st largest singular value of the Hankel operator. These optimal infinite low-rank Hankel matrices can also be computed numerically, see \[4, 22\], and have been used to compute adaptive Fourier series with exponential decay for large function classes in \[23\]. Unfortunately, this approach cannot be directly transferred to finite matrices, \[4\].

In this paper we aim at a direct analytic approach for rank-1 Hankel approximation. To the best of our knowledge, there is only one other paper, where an exact optimal structured low-rank approximation has been studied for a weighted Frobenius norm using algebraic geometry, namely \[21\]. This paper did not specify on Hankel or Toeplitz matrices and considers a reformulation of the structured low-rank approximation problem using multivariate polynomials of high order.

**Organization of this paper.** In Section 2 we show that Hankel matrices of rank 1 have a special structure and can be determined by two complex parameters \(c\) and \(z\).

In Section 3 we show how the optimal rank-1 approximation with regard to the Frobenius norm can be obtained. The main theoretical results are already stated in Theorems 3.1 and 3.3 in the complex case. In Theorem 3.3 we show in which case the optimal rank-1 approximation error coincides with the error achieved for unstructured rank-1 approximation. Further, we will present a series of results that simplify the computation of the rank-1 Hankel approximation in the real case.

In Section 4, we solve the rank-1 Hankel approximation problem with regard to the spectral norm. This norm is considerably more difficult to handle than the Frobenius norm. Therefore we restrict ourselves to real symmetric matrices in these considerations. The result also gives rise to a corresponding algorithm. Surprisingly, the optimal rank-1 Hankel approximations for the Frobenius norm and the spectral norm usually differ.

Section 5 is devoted to Cadzow’s algorithm, which is (despite a lot of existing optimization approaches) the most popular method for low-rank Hankel approximation in practice. We give a direct proof, that the Cadzow algorithm always converges to a fixed point in the rank-1 case. However, we observe in our numerical examples that it usually does not converge to the optimal solution – neither with regard to the Frobenius norm nor the spectral norm. It may even fail completely, as Example 5.6 shows.
2 Rank-1 Hankel Matrices

Our goal is to find the optimal Hankel-structured rank-1 approximation of a given matrix $A \in \mathbb{C}^{M \times N}$. Therefore we first look at a characterization of Hankel matrices of rank 1. Let

$$e_N := (0, \ldots, 0, 1)^T \in \mathbb{C}^N \quad \text{and} \quad \tilde{e}_N := (1, 0, \ldots, 0)^T \in \mathbb{C}^N.$$  

(2.1)

Further, for any complex number $z$ and $N \in \mathbb{N}$ we define

$$z_N(z) = z_N := \left(z^k\right)_{k=0}^{N-1} = \left(1, z, z^2, \ldots, z^{N-1}\right)^T.$$  

(2.2)

We use the convention that $z_N(z)$ is abbreviated by $z_N$ if it depends just on $z$ as given in (2.2). Then we have

**Lemma 2.1.** A complex rank-1 matrix $H_1 \in \mathbb{C}^{M \times N}$ with $\min\{M, N\} \geq 2$ has Hankel structure if and only if it is of the form

$$H_1 = cz_M z_N^T = c \left(z^{k+\ell}\right)_{k,\ell=0}^{M-1,N-1} \quad \text{or} \quad H_1 = ce_M e_N^T = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix},$$

where $c \in \mathbb{R} \setminus \{0\}$, $z \in \mathbb{C}$, and $e_N$ as in (2.1).

**Proof.** 1. Obviously, the two matrices $H_1 = cz_M z_N^T$ and $H_1 = ce_M e_N^T$ are rank-1 matrices with Hankel structure. 2. We show that each rank-1 Hankel matrix $H_1 = (h_{k+\ell})_{k,\ell=0}^{M-1,N-1}$ with $h_0 \neq 0$ has the form $H_1 = cz_M z_N^T$ for some $z \in \mathbb{C}$. Since $H_1$ has rank 1, we obtain the representation

$$H_1 = (h_{k+\ell})_{k,\ell=0}^{M-1,N-1} = xy^T$$

for some vectors $x = (x_0, \ldots, x_{M-1})^T \in \mathbb{C}^M$ and $y = (y_0, \ldots, y_{N-1})^T \in \mathbb{C}^N$. The imposed Hankel structure implies the conditions

$$x_k y_\ell = x_m y_n, \quad \text{for } k + \ell = m + n,$$

(2.3)

where $k, m = 0, \ldots, M - 1$ and $\ell, n = 0, \ldots, N - 1$. Since we have assumed that $h_0 = x_0 y_0 \neq 0$, we can define $z := x_1/x_0$. It follows from (2.3) with $k + \ell = 1$, i.e., from $x_0 y_1 = x_1 y_0$, that $z = y_1/y_0$, and thus

$$x_1 = z x_0, \quad y_1 = z y_0.$$  

(2.4)

We show by induction that $x_j = z^j x_0$ for $j = 1, \ldots, M - 1$, and $y_j = z^j y_0$ for $j = 1, \ldots, N - 1$. For $0 < j < M - 1$, we obtain from (2.3) that

$$x_{j+1} y_0 = x_j y_1 = (z^j x_0)(z y_0) = z^{j+1} x_0 y_0.$$  

(2.5)

Since $y_0 \neq 0$, this leads to $x_j = z^j x_0$, for $j = 1, \ldots, M - 1$. Analogously, for $0 < j < N - 1$, we have

$$x_0 y_{j+1} = x_1 y_j = (z x_0)(z^j y_0) = z^{j+1} x_0 y_0.$$  

(2.6)

Since $x_0 \neq 0$, we obtain $y_j = z^j y_0$, for $j = 1, \ldots, N - 1$. Thus, $H_1$ has the desired structure $c z_M z_N^T$ with $z = x_1/x_0$ and $c = x_0 y_0$.

3. If $h_0 = x_0 y_0 = 0$ then either $x_0 = 0$ or $y_0 = 0$. Thus, either the complete first row or the complete first column of $H_1 = xy^T$ contains only zeros. By obeying the Hankel structure and the rank-1 condition, we inductively obtain that then all the entries of $xy^T$ are zero except for the last one $c := x_{M-1}y_{N-1} \neq 0$. Thus $H_1 = c e_M e_N^T$.  

\(\square\)
Remark 2.2. 1. The matrix $H_1 = cz_M z_N^T$ possesses the non-zero singular value

$$|c| \|z_M\|_2 \|z_N\|_2$$

with corresponding left and right singular vectors $z_M$ and $z_N$, respectively.

2. Considering the rank-1 Hankel representation $c z_M z_N^T$ with normalized vectors $\tilde{z}_M = z_M \|z_M\|_2$ and $\tilde{z}_N = z_N \|z_N\|_2$, the special case in Lemma 2.1 can also be understood as the limit case for $z \to \infty$.

3. If we define

$$w_N(z) = w_N := \left(z^{N-1-k}\right)_{k=0}^{N-1} = \left(z^{N-1}, z^{N-2}, \ldots, z, 1\right)^T \in \mathbb{C}^N$$

instead of $z_N(z)$ in (2.2), we can show analogously to Lemma 2.1 that a rank-1 Hankel matrix $H_1 \in \mathbb{C}^{M \times N}$ is of the form

$$H_1 = c w_M w_N^T \quad \text{or} \quad H_1 = c \bar{e}_M e_N^T$$

with $\bar{e}_M := (1, 0, \ldots, 0)^T \in \mathbb{C}^M$. Obviously, we have the connection

$$z_N(z) = J_N w_N(z) = z^{N-1} w_N \left(\frac{1}{z}\right).$$

In particular, for the special case in Lemma 2.1 we have $c \bar{e}_M e_N^T = c w_M(0) w_N(0)^T$.

4. To solve the minimization problems in (1.2), we can restrict ourselves to rank-1 Hankel matrices of the form $c z_M z_N^T$, and instead consider the problem for $A$ and for $J_M A J_N$, since we observe for the Frobenius norm as well as for the spectral norm,

$$\|J_M A J_N - c z_M z_N^T\| = \|A - c J_M z_M z_N^T J_N\| = \|A - c w_M w_N^T\|^2.$$

For the Frobenius norm we even have

$$\min_{c \in \mathbb{C}} \|A - c \bar{e}_M e_N^T\|^2 = \|A - a_{M-1,N-1} e_M e_N^T\|^2 = \|A\|_F^2 - |a_{M-1,N-1}|^2.$$

Therefore, $a_{M-1,N-1} e_M e_N^T$ cannot be the optimal rank-1 Hankel approximation of $A$ if $|a_{0,0}| \geq |a_{M-1,N-1}|$. Hence, we always assume this condition in Section 3. If $|a_{0,0}| \geq |a_{M-1,N-1}|$ is not satisfied, then we can replace $A$ by $J_M A J_N$.

3 Optimal Rank-1 Hankel Approximation in the Frobenius Norm

3.1 Complex Rank-1 Hankel Approximations

First we consider the minimization problem (1.2) in the Frobenius norm. Using the structure of a rank-1 Hankel matrix from Lemma 2.1 and Remark 2.2 we can formulate the minimization problem as

$$\min_{c, z \in \mathbb{C}} \|A - c z_M z_N^T\|_F^2,$$

i.e., we only need to find the two constants $c \in \mathbb{C} \setminus \{0\}$ and $z \in \mathbb{C}$, such that the error $A - c z_M z_N^T$ is minimized in the Frobenius norm.
Theorem 3.1. Let $A = (a_{j,k})_{j,k=0}^{M-1,N-1} \in \mathbb{C}^{M \times N}$ with $M, N \geq 2$ and $|a_{0,0}| \geq |a_{M-1,N-1}|$. Assume that $\text{rank}(A) \geq 1$. Then an optimal rank-1 Hankel approximation $H_1 = c\hat{z}_M \hat{z}_N^T$ of $A$ is determined by

\[ \hat{z} \in \arg\max_{z \in \mathbb{C}} \frac{|z_M^* A z_N^*|}{\|z_M\|_2 \|z_N\|_2}, \quad \hat{c} := \frac{z_M^* A z_N^*}{\|z_M\|_2 \|z_N\|_2}, \quad (3.2) \]

where the vectors $\hat{z}_M$ and $\hat{z}_N$ are defined by $\hat{z}$ via (2.2) and $\hat{z}^* := \tilde{z}^T$.

Proof. Using the definition of the Frobenius norm, we obtain

\[ \|A - c z_M z_N^T\|_F^2 = \sum_{j=0}^{M-1} \sum_{k=0}^{N-1} |a_{j,k} - c z_j^* z_k^*|^2 \]

\[ = \sum_{j=0}^{M-1} \sum_{k=0}^{N-1} |a_{j,k}|^2 + \sum_{j=0}^{M-1} \sum_{k=0}^{N-1} \left(-c a_{j,k} z_j^* z_k^* + |c|^2 |z_j|^2 + |z_k|^2 \right) \]

\[ = \|A\|_F^2 - c \overline{z}_M^T \overline{A} z_N - \overline{c} z_M^* A z_N + |c|^2 (z_M^* z_M)(z_N^* z_N), \quad (3.3) \]

where $z_M^* := z_M^T$. To solve the minimization problem in (3.1), we apply variable projection and consider the first derivatives with respect to $c_1$ and $c_2$, where $c = c_1 + ic_2$ with $c_1, c_2 \in \mathbb{R}$ to obtain the necessary conditions

\[ \frac{\partial}{\partial c_1} \|A - c z_M z_N^T\|_F^2 = -2 \text{Re}(z_M^* A z_N) + 2 c_1 \|z_M\|_2 \|z_N\|_2 = 0, \]

\[ \frac{\partial}{\partial c_2} \|A - c z_M z_N^T\|_F^2 = -2 \text{Im}(z_M^* A z_N) + 2 c_2 \|z_M\|_2 \|z_N\|_2 = 0. \]

These yield the optimal $\hat{c} = \frac{z_M^* A z_N}{\|z_M\|_2 \|z_N\|_2}$, After substituting $\hat{c}$ into (3.3) it remains to solve

\[ \min_{z \in \mathbb{C}} \left( \|A\|_F^2 - 2 \frac{|z_M^* A z_N|^2}{\|z_M\|_2 \|z_N\|_2} + \frac{|z_M^* A z_N|^2}{\|z_M\|_2 \|z_N\|_2} \right) \]

\[ = \min_{z \in \mathbb{C}} \left( \|A\|_F^2 - \frac{|z_M^* A z_N|^2}{\|z_M\|_2 \|z_N\|_2} \right). \]

Therefore,

\[ \hat{z} \in \arg\max_{z \in \mathbb{C}} \frac{|z_M^* A z_N|^2}{\|z_M\|_2 \|z_N\|_2} = \arg\max_{z \in \mathbb{C}} \frac{z_M^* \overline{A} z_N}{\|z_M\|_2 \|z_N\|_2} \]

as claimed. $\square$

Remark 3.2. 1. By Theorem 3.1, the computation of the optimal rank-1 Hankel approximation of the matrix $A$ reduces to the problem of finding a position, where the maximum of the complex rational function $|F(z)|$ with

\[ F(z) := \frac{z_M^* \overline{A} z_N}{\|z_M\|_2 \|z_N\|_2} \]

is attained. According to Theorem 3.1 we obtain $\hat{z} = \arg\max_z |F(z)|$ and $\hat{c} = \frac{F(\hat{z})}{\|z_M\|_2 \|z_N\|_2}$. Since $\|z_M\|_2 \geq 1$ for all $z \in \mathbb{C}$ and all integers $M \geq 2$, the function $F(z)$ has no poles. Moreover, $|F(z)|$ is bounded by $\|A\|_2$, which follows from the proof of the next theorem.
2. For \(|z| \to \infty\), we obtain \(\lim_{|z| \to \infty} |F(z)| = |a_{M-1,N-1}|\). This corresponds to the case that the optimal rank-1 approximation of \(A\) is given by \(a_{M-1,N-1} e_M e_N^*\), see also Remark 2.2.

3. In particular for \(M = N\), \(F(z)\) does not have any poles either. If additionally \(A \in \mathbb{R}^{N \times N}\) is symmetric or \(A \in \mathbb{C}^{N \times N}\) is Hermitian, then \(F(z)\) is a Rayleigh quotient and thus \(\lambda_{\min} \leq F(z) \leq \lambda_{\max}\), where \(\lambda_{\min}\) and \(\lambda_{\max}\) are the smallest and largest eigenvalue of \(A\), respectively, see e.g. [13], p. 176.

4. The value \(\tilde{z}\) in (3.2) may not be unique, i.e., \(\max_{z \in \mathbb{C}} |F(z)|\) may be attained for different values \(\tilde{z}\). In this case, any of these values leads to an optimal Hankel rank-1 approximation. If for example \(A = (a_{j,k})_{j,k=0}^{M-1,N-1} \in \mathbb{C}^{M \times N}\) is itself a Hankel matrix where \(a_{j,k} = 0\) if \(j + k\) is odd, then \(\tilde{z} \in \text{argmax}_{z \in \mathbb{C}} |F(z)|\) implies that also \(\tilde{z} \in \text{argmax}_{z \in \mathbb{C}} |F(z)|\).

We may ask, how well a matrix \(A\) can be approximated by a rank-1 Hankel matrix \(H_1\). More precisely, we ask in which cases the Hankel-structured rank-1 approximation is as good as the unstructured rank-1 approximation. The unstructured low-rank approximation is given by the singular value decomposition according to the Eckart-Young-Mirsky Theorem. Let \(u_0 \in \mathbb{C}^M\) and \(v_0 \in \mathbb{C}^N\) denote the normalized singular vectors corresponding to the largest singular value \(\sigma_0 = \|A\|_2\) of \(A\). Then \(u_0\) and \(v_0\) are determined by the following set of equations

\[
A A^* u_0 = \sigma_0^2 u_0, \quad A^* A v_0 = \sigma_0^2 v_0, \quad u_0 = \frac{1}{\sigma_0} A^* u_0, \quad v_0 = \frac{1}{\sigma_0} A v_0. \tag{3.4}
\]

We show that the optimal approximation error can only be achieved if the singular vectors \(u_0\) and \(v_0\) corresponding to the largest singular value \(\sigma_0\) have the special structure \(z_M/\|z_M\|_2\) and \(z_N/\|z_N\|_2\), respectively, for some \(z \in \mathbb{C}\).

**Theorem 3.3.** The optimal rank-1 Hankel approximation error satisfies

\[
\min_{c,z \in \mathbb{C}} \left\| A - cz_M z_N^T \right\|_F^2 = \|A - \tilde{c} \tilde{z}_M \tilde{z}_N^T\|_F^2 = \|A\|_F^2 - \sigma_0^2 = \|A\|_F^2 - \|A\|_2^2, \tag{3.5}
\]

if and only if the two singular vectors of \(A\) in (3.2) corresponding to the largest singular value \(\sigma_0\) are of the form \(u_0 = \frac{1}{\|z_M\|_2} \tilde{z}_M\) and \(v_0 = \frac{1}{\|z_N\|_2} \tilde{z}_N\), where \(\tilde{z}_M, \tilde{z}_N\) are defined by \(\tilde{z}\) via (2.2), and where \(\tilde{z}\) and \(\tilde{c}\) are given by (3.2).

**Proof.** 1. Considering the singular value decomposition of \(A\) we obtain an optimal (unstructured) rank-1 approximation of \(A\) with respect to the Frobenius norm of the form \(\sigma_0 u_0 v_0^*\) with \(u_0, v_0\) in (3.4). If now \(u_0 = \frac{1}{\|z_M\|_2} \tilde{z}_M\) and \(v_0 = \frac{1}{\|z_N\|_2} \tilde{z}_N\), then it follows with \(\tilde{c}\) from (3.2) that

\[
\tilde{c} \tilde{z}_M \tilde{z}_N^T = \frac{\tilde{z}_M^* A \tilde{z}_N}{\|z_M\|_2^2 \|z_N\|_2^2} \tilde{z}_M \tilde{z}_N^T = (u_0^* A v_0) u_0 v_0^* = \sigma_0 u_0 v_0^*,
\]

i.e., the unstructured and the structured rank-1 approximation coincide.

2. Assume that the structured low-rank approximation \(\tilde{c} \tilde{z}_M \tilde{z}_N^T\) provides the optimal error in (3.5). According to equation (3.3) from the proof of Theorem 3.1 we have

\[
\left\| A - \tilde{c} \tilde{z}_M \tilde{z}_N^T \right\|_F^2 = \|A\|_F^2 - \frac{\|\tilde{z}_M^* A \tilde{z}_N\|_2^2}{\|\tilde{z}_M\|_2^2 \|\tilde{z}_N\|_2^2}\tag{3.6}
\]
and it follows on the one hand
\[ \sigma_0^2 = \|A\|_2^2 = \frac{|\tilde{z}_M^* A \tilde{z}_N|^2}{\|\tilde{z}_M\|_2 \|\tilde{z}_N\|_2}. \]  
\hfill (3.7)

On the other hand, the Theorem of Rayleigh-Ritz (see [13], p. 176) implies
\[ \frac{|\tilde{z}_M^* A \tilde{z}_N|^2}{\|\tilde{z}_M\|_2 \|\tilde{z}_N\|_2} \leq \frac{|\tilde{z}_N^T (A^* \tilde{z}_M)|^2}{\|\tilde{z}_N\|_2^2} = \frac{\|A^* \tilde{z}_M\|_2^2}{\|\tilde{z}_M\|_2^2} \leq \|A\|_2^2. \]

Here, equality at \( (*) \) only holds if \( A^* \tilde{z}_M \) is an eigenvector of \( \tilde{z}_N \tilde{z}_N^T \) to the non-zero eigenvalue \( \|\tilde{z}_N\|_2^2 \). Equality at \( (\circ) \) is achieved if moreover \( \tilde{z}_M \) is an eigenvector of \( AA^* \) to the largest eigenvalue \( \sigma_0^2 = \|A\|_2^2 \). The assertion now follows by comparison with \( \|z\|_2^2 \). \( \square \)

In the remainder of this section, we will derive further properties of the optimal value \( \tilde{z} \) in \( (3.2) \) in order to provide an efficient algorithm to compute \( \tilde{z} \) and \( \tilde{c} \). First we consider the possible range of \( \tilde{z} \). For this purpose, we recall that a rank-1 Hankel matrix \( H_1 \) can also be represented as \( H_1 = c w_N w_N^T \) with \( w_N = w_N(z) \) in \( (2.8) \), where \( z \in \mathbb{C} \). Let \( A \in \mathbb{C}^{M \times N} \) with \( M, N \geq 2 \), \( |a_{0,0}| \geq |a_{M-1,N-1}| \), and \( \text{rank}(A) \geq 1 \).

**Theorem 3.4.** Let \( A \in \mathbb{C}^{M \times N} \) with \( M, N \geq 2 \), \( |a_{0,0}| \geq |a_{M-1,N-1}| \), and \( \text{rank}(A) \geq 1 \). Let
\[ F(z) := \frac{z_M^* A z_N}{\|z_M\|_2 \|z_N\|_2}, \quad F_1(z) := \frac{z_M^* J_M A J_N z_N}{\|z_M\|_2 \|z_N\|_2} \]
with \( J_N \) as in \((1.3)\). Let \( M_0 := \max_{|z| \leq 1} |F(z)| \) and \( M_1 := \max_{|z| \leq 1} |F_1(z)| \). Then the optimal rank-1 Hankel approximation \( H_1 = \tilde{c} \tilde{z}_M \tilde{z}_N^T \) of \( A \) is determined by
\[ \tilde{z} \in \begin{cases} \text{argmax}_{|z| \leq 1} |F(z)| & \text{if } M_0 \geq M_1, \\ \text{argmax}_{|z| \leq 1} |F_1(z)|^{-1} & \text{if } M_1 > M_0, \end{cases} \quad \tilde{c} := \frac{z_M^* A z_N}{\|z_M\|_2 \|z_N\|_2}. \]

**Proof.** We show that \( |F_1(z)| = |F(1/z)| \), then the assertion of the theorem follows from Theorem 3.1. First, we observe that
\[ J_N z_N(z) = w_N(z) = z^{N-1} z_N \left( \frac{1}{z} \right). \]

Thus, we have in particular \( \|z_N\|_2 = \|w_N\|_2 \) and it follows
\[ |F_1(z)| = \frac{|z_M^* J_M A J_N z_N|}{\|z_M\|_2 \|z_N\|_2} = \frac{|w_M^* A w_N|}{\|w_M\|_2 \|w_N\|_2} = \frac{|z|^{N+M-2} |z_M(\frac{1}{z})^* A z_N(\frac{1}{z})|}{|z|^{N+M-2} \|z_M(\frac{1}{z})\|_2 \|z_N(\frac{1}{z})\|_2} = \left| F \left( \frac{1}{z} \right) \right|. \]
\hfill \( \square \)

**Remark 3.5.** Using Theorem 3.4 we can restrict the search for an optimal value \( \tilde{z} \) to the unit disc \( \{ z : |z| \leq 1 \} \) if we consider the two functions \( F(z) \) and \( F_1(z) \).
### 3.2 Real Rank-1 Hankel approximations

In the following we will consider real matrices $A \in \mathbb{R}^{M \times N}$ and restrict the search to real optimal rank-1 Hankel approximations, i.e., we search for real parameters $\tilde{c}$ and $\tilde{z}$. Then we can derive further conditions on $\tilde{z}$ that simplify the computation of the optimal rank-1 Hankel approximation of $A$.

**Theorem 3.6.** Let $A \in \mathbb{R}^{M \times N}$ with $M, N \geq 2$, $|a_{0,0}| \geq |a_{M-1,N-1}|$, and $\text{rank}(A) \geq 1$. If $H_1 = \tilde{c} \tilde{z}_M \tilde{z}_N^T$ is an optimal rank-1 Hankel approximation of $A$, then

$$Q(\tilde{z}) := a'(\tilde{z}) p(\tilde{z}) - a(\tilde{z}) p'(\tilde{z}) = 0,$$

with

$$a(z) := z_M^T A z_N = \sum_{j=0}^{M-1} \sum_{k=0}^{N-1} a_{j,k} z^{j+k},$$

$$p(z) := \|z_M\|_2 \|z_N\|_2 = \left( \sum_{k=0}^{M-1} \sum_{j=0}^{N-1} z^{2k} \right)^{1/2} \left( \sum_{k=0}^{N-1} \sum_{j=0}^{M-1} z^{2k} \right)^{1/2} \geq 1.$$

Here, $a'(z)$ and $p'(z)$ denote the first derivatives of $a(z)$ and $p(z)$, respectively.

**Proof.** According to Theorem 3.3 we obtain $\tilde{z}$ as

$$\tilde{z} \in \arg\max_{z \in \mathbb{R}} |F(z)| \quad \text{with} \quad F(z) = \frac{(z_M^T A z_N)}{\|z_M\|_2 \|z_N\|_2} = \frac{a(z)}{p(z)}.$$

Thus, $F(\tilde{z})$ is an extremal value of $F$, i.e., $F'(\tilde{z}) = 0$. The first derivative of $F$ is given by

$$F'(z) = \frac{a'(z)p(z) - a(z)p'(z)}{p(z)^2}.$$

Since $p(z) \geq 1$ for all $z \in \mathbb{R}$, we obtain for $\tilde{z}$ the necessary condition

$$a'(\tilde{z})p(\tilde{z}) - a(\tilde{z})p'(\tilde{z}) = 0,$$

as was claimed. \hfill $\square$

Looking at the coefficients $a_{\ell}$ of the polynomial

$$a(z) = z_M^T A z_N = \sum_{j=0}^{M-1} \sum_{k=0}^{N-1} a_{j,k} z^{j+k} = \sum_{\ell=0}^{M+N-2} a_{\ell} z^\ell \quad (3.8)$$

in more detail, we can conclude even more.

**Corollary 3.7.** Let $A$ be a real $M \times N$ matrix with $M, N \geq 2$, $|a_{0,0}| \geq |a_{M-1,N-1}|$, and $\text{rank}(A) \geq 1$. Let $a(z)$ be given as in (3.8) and

$$\tilde{z} \in \arg\max_{z \in \mathbb{R}} \frac{(z_M^T A z_N)^2}{\|z_M\|_2^2 \|z_N\|_2^2} = \arg\max_{z \in \mathbb{R}} \left( \frac{a(z)}{p(z)} \right)^2.$$

1. If $a_{\ell} \geq 0$ for $\ell = 0, \ldots, M + N - 2$ and $a_0 \geq a_{M+N-2}$, then there exists $\tilde{z} \geq 0$.
2. If $a_{\ell} \geq 0$ for $\ell$ even and $a_{\ell} \leq 0$ for $\ell$ odd, then there exists $\tilde{z} \leq 0$.  

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Proof. The first assertion follows directly from the observation that $a(z) \geq a(-z)$ for $z \geq 0$ while $p(z) = p(-z)$ is an even function. In the second case we have $a(-z) \geq a(z)$ for all $z \geq 0$, and the assertion follows similarly.

Often, we have the special case that $A$ is a real $N \times N$ matrix with non-negative components. If the coefficients of the corresponding polynomial $a(z)$ in (3.8) are non-negative and monotonically decreasing, then we can easily find $\tilde{z}$ and thereby generate the optimal rank-1 Hankel matrix.

**Theorem 3.8.** Let $A$ be a real $N \times N$ matrix and let $a(z)$ be given as in (3.8) with $a_\ell \geq 0$ for $\ell = 0, \ldots, 2N - 2$. Assume that the two sequences $(a_{2\ell})_{\ell=0}^{N-1}$ and $(a_{2\ell+1})_{\ell=0}^{N-2}$ are monotonically decreasing with $a_0 > a_{2N-2}$ and $a_1 > 0$. Then, there exists $\tilde{z} \in (0, 1)$ with

$$
\tilde{z} \in \operatorname{argmax}_{z \in \mathbb{R}} \frac{(z_N^TAz_N)^2}{\|z_N\|^2} = \operatorname{argmax}_{z \in \mathbb{R}} \left( \frac{a(z)}{p(z)} \right)^2
$$

determining the optimal (real) rank-1 Hankel approximation of $A$. Moreover, $\tilde{z}$ is the only positive zero of $Q(z) = a'(z)p(z) - a(z)p'(z)$. Here $p(z) = z_N^TAz_N = \sum_{\ell=0}^{N-1} z^{2\ell}$.

**Proof.** By Theorem 3.6, the desired value $\tilde{z}$ is a zero of the polynomial $Q(z)$. Corollary 3.7 yields that there exists $\tilde{z} \geq 0$ maximizing the function $\left( \frac{a(z)}{p(z)} \right)^2$. Therefore, it suffices to show that $Q(z)$ possesses only one zero in $[0, \infty)$ which is in $(0, 1)$.

First, we observe that

$$
Q(0) = a'(0)p(0) - a(0)p'(0) = a_1 > 0
$$

since $p(0) = 1$ and $p'(0) = 0$. With $p(1) = N$ and $p'(1) = N(N - 1)$ we obtain that

$$
Q(1) = a'(1)p(1) - a(1)p'(1) = N \left( \sum_{j=0}^{2N-2} j a_j \right) - N(N - 1) \left( \sum_{j=0}^{2N-2} a_j \right)
$$

$$
= N \left( \sum_{j=0}^{2N-2} (j - N + 1) a_j \right) = N \left( \sum_{j=0}^{N-1} j (a_{N-1+j} - a_{N-1-j}) \right) < 0,
$$

since $(a_{2\ell})_{\ell=0}^{N-1}$ and $(a_{2\ell+1})_{\ell=0}^{N-2}$ are monotonically decreasing and $a_0 > a_{2N-2}$. Thus $Q(z)$ possesses at least one zero $\tilde{z}$ in $(0, 1)$.

We show that $Q(z)$ possesses only this one positive root $\tilde{z}$. To this end we will consider $Q(z) \cdot (1 - z^2)^2$. Note that for any $z \in (0, 1)$ we have

$$
p(z) = \frac{1 - z^{2N}}{1 - z^2}, \quad \text{and} \quad p'(z) = \frac{(2N - 2)z^{2N+1} - 2Nz^{2N-1} + 2z}{(1 - z^2)^2}
$$

by the quotient rule. Thus we obtain

$$
R(z) := Q(z) \cdot (1 - z^2)^2 = \left( a'(z)p(z) - a(z)p'(z) \right) \left( 1 - z^2 \right)^2
$$

$$
= \left( \sum_{j=1}^{2N-2} j a_j z^{j-1} \right) \left( 1 - z^{2N} \right) \left( 1 - z^2 \right) - \left( \sum_{j=0}^{2N-2} a_j z^j \right) \left( (2N - 2)z^{2N+1} - 2Nz^{2N-1} + 2z \right)
$$

by the quotient rule. Thus we obtain
\[ = \sum_{j=0}^{2N-2} \left[ j \ a_j z^{j-1} - (j+2) a_j z^{j+1} + (2N-j) a_j z^{2N-1+j} - (2N-2-j) a_j z^{2N+1+j} \right]. \]

An index shift \( j' = j + 2 \) implies

\[ R(z) = \sum_{j=0}^{2N-2} \left[ j a_j z^{j-1} + (2N-j) a_j z^{2N-1+j} \right] - \sum_{j'=2}^{2N} \left[ j' a_{j'-2} z^{j'-1} - (2N-j') a_{j'-2} z^{2N-1+j'} \right] \]

\[ = \left( \sum_{j=2}^{2N-2} j (a_j - a_{j-2}) z^{j-1} \right) + \left[ a_1 - (2N-1) a_{2N-3} z^{2N-2} - 2N a_{2N-2} z^{2N-1} \right] \]

\[ + \left( \sum_{j=2}^{2N-2} (2N-j)(a_j - a_{j-2}) z^{2N-1+j} \right) + \left[ 2N a_0 z^{2N-1} + (2N-1) a_1 z^{2N} - a_{2N-3} z^{4N-2} \right] \]

\[ = a_1 z^0 + \sum_{j=2}^{2N-2} j (a_j - a_{j-2}) z^{j-1} - (2N-1) a_{2N-3} z^{2N-2} + 2N (a_0 - a_{2N-2}) z^{2N-1} \]

\[ + (2N-1) a_1 z^{2N} + \sum_{j=2}^{2N-2} (2N-j)(a_j - a_{j-2}) z^{2N-1+j} - a_{2N-3} z^{4N-2}. \] (3.9)

Thus, the sequence of coefficients of the polynomial \( R(z) = Q(z) \cdot (1 - z^2)^2 \) possesses exactly three changes of sign. In this case, the rule of Descartes implies that \( R(z) \) has either one or three real positive roots (counted according to their multiplicity). Since \((1 - z^2)^2\) already has the positive zero 1 with multiplicity 2, we conclude that \( Q(z) \) has exactly one root in \( \mathbb{R}^+ \). By the preceding considerations, this zero is contained in \((0, 1)\). \( \square \)

**Remark 3.9.** In the special case of Theorem 3.8, we can find \( \tilde{z} \) efficiently by employing a Newton method with starting value \( z_0 = 1 \) to obtain \( \tilde{z} \).

**Corollary 3.10.** Let \( A \) be a real \( N \times N \) matrix and let \( a(z) \) be given as in (3.8) with \( a_\ell \geq 0 \) for \( \ell = 0, \ldots, 2N-2 \). Assume that the two sequences \( (a_{2\ell})_{\ell=0}^{N-1} \) and \( (a_{2\ell+1})_{\ell=0}^{N-2} \) are monotonically increasing with \( a_0 < a_{2N-2} \) and \( a_{2N-3} > 0 \). Then, there exists \( \tilde{z} \in (1, \infty) \) with

\[ \tilde{z} \in \arg\max_{z \in \mathbb{R}} \frac{(z^T A z N)^2}{||z N||_2^2} = \arg\max_{z \in \mathbb{R}} \frac{(a(z))^2}{p(z)} \]

determining the optimal (real) rank-1 Hankel approximation of \( A \). Moreover, \( \tilde{z} \) is the only positive zero of \( Q(z) = a'(z) p(z) - a(z) p'(z) \).

**Proof.** For \( z \in (1, \infty) \), we observe that

\[ \frac{a(1/z)}{p(1/z)} = \frac{a(1/z)}{z^{-2N+2} p(z)} = \frac{\sum_{j=0}^{2N-2} a_j z^{2N-2-j}}{p(z)} = \tilde{a}(z), \]

where the polynomial \( \tilde{a}(z) \) has the coefficients \( \tilde{a}_j = a_{2N-2-j}, \ j = 0, \ldots, 2N-2 \). Thus, the sequences \( (\tilde{a}_{2\ell})_{\ell=0}^{N-1} \) and \( (\tilde{a}_{2\ell+1})_{\ell=0}^{N-2} \) are monotonically decreasing with \( \tilde{a}_1 = a_{2N-3} > 0 \) and \( \tilde{a}_0 > a_{2N-2} \). The assertion now follows from Theorem 3.8 applied to \( \tilde{a}(z) \). \( \square \)

**Corollary 3.11.** Let \( A \) be a real \( N \times N \) matrix and let \( a(z) \) be given as in (3.8). Assume that \( \tilde{z} \) is a value determining the optimal (real) rank-1 Hankel approximation of \( A \). Further, let \( Q(z) \) be given as in Theorem 3.8.
1. If \((a_\ell)_{\ell=0}^{N-1}\) is a non-negative monotonically decreasing sequence with \(a_0 > a_{2N-2} \geq 0\) and \((\tilde{a}_\ell)_{\ell=0}^{N-2}\) is a non-positive monotonically decreasing sequence with \(\tilde{a}_1 < 0\), then there exists \(\tilde{z} \in (-1,0)\) that maximizes \(\left(\frac{a(z)}{p(z)}\right)^2\), and moreover, it is the only negative zero of \(Q(z)\).

2. If \((a_\ell)_{\ell=0}^{N-1}\) is a non-positive monotonically decreasing sequence with \(0 \geq a_0 > a_{2N-2}\) and \((\tilde{a}_\ell)_{\ell=0}^{N-2}\) is a non-negative monotonically increasing sequence with \(a_{2N-3} > 0\), then there exists \(\tilde{z} \in (-\infty,-1)\) that maximizes \(\left(\frac{a(z)}{p(z)}\right)^2\), and moreover, it is the only negative zero of \(Q(z)\).

**Proof.** 1. We consider the first assertion. Corollary 3.7 implies that we can find \(\tilde{z} \leq 0\) that maximizes \(\left(\frac{a(z)}{p(z)}\right)^2\). We apply Theorem 3.8 and show that \(Q(z)\) possesses only one negative zero, which is in \((-1,0)\). Let \(\tilde{a}_{2\ell+1} := -a_{2\ell+1}\) for \(\ell = 0, \ldots, N-2\). Then \((\tilde{a}_{2\ell+1})_{\ell=0}^{N-2}\) is a non-negative, monotonically decreasing sequence with \(\tilde{a}_1 > 0\). We can write

\[
a(z) = \alpha(z) + \beta(z) \quad \text{with} \quad \alpha(z) = \sum_{\ell=0}^{N-1} a_{2\ell} z^{2\ell}, \quad \beta(z) = \sum_{\ell=0}^{N-2} a_{2\ell+1} z^{2\ell+1}
\]

and

\[
\tilde{a}(z) = \alpha(z) - \beta(z) = \sum_{\ell=0}^{N-1} a_{2\ell} z^{2\ell} + \sum_{\ell=0}^{N-2} \tilde{a}_{2\ell+1} z^{2\ell+1}.
\]

Further, let \(\tilde{Q}(z) := \tilde{a}'(z)p(z) - \tilde{a}(z)p'(z)\). By Theorem 3.8, \(\tilde{Q}(z)\) possesses only one positive zero and this zero is in \((0,1)\). For the polynomial \(Q(z)\) it follows in this case

\[
Q(z) = a'(z)p(z) - a(z)p'(z) = (a_0'(z) + a_1'(z))p(z) - (a_0(z) + a_1(z))p'(z)
\]

\[
= (a_0'(-z) + a_1'(-z))p(-z) - (a_0(-z) + a_1(-z))(-p'(-z))
\]

\[
= -a(-z)p(-z) + \tilde{a}(-z)p'(-z) = -\tilde{Q}(-z).
\]

Thus \(Q(z)\) possesses exactly one zero in \((-\infty,0)\), and this zero is in \((-1,0)\).

2. The second assertion follows similarly from Corollary 3.7 and Corollary 3.10.

**Remark 3.12.** For Hankel matrices, the decay condition of Theorem 3.8 is satisfied if the sequence \((\ell + 1)a_\ell\) is non-negative and monotonically decreasing, i.e. \((a_\ell)\) decays faster than \((\ell+1)^{-1}\). The reconstruction of real (usually even exponentially) decreasing sequences by very short exponential sums occurs for example in analyzing large fluorescence lifetime imaging (FLIM), see e.g. [31].

Finally, we want to answer the following question: Given a real matrix \(A\), can we restrict the search for the optimal parameters \(c\) and \(z\) to real numbers, or can we achieve better results by allowing complex parameters? The following example shows that indeed complex parameters may provide better approximations.

**Example 3.13.** We want to find an optimal rank-1 Hankel approximation for

\[
A = \begin{pmatrix}
1 & -\frac{1}{2} & -1 \\
-\frac{1}{2} & -1 & -\frac{1}{2} \\
-1 & -\frac{1}{2} & 1
\end{pmatrix}.
\]
This matrix has the eigenvalues 2.0, -1.366025 and 0.366025 and the Frobenius norm \( \|A\|_F = 2.449490 \). Using Theorem 3.6 we find two solutions for the optimal real parameters \((\tilde{z}, \tilde{c}) = (-0.129135, 1.045778)\) and \((\tilde{z}, \tilde{c}) = (-7.743849, 0.000291)\). The obtained Frobenius norm of the error is \( \|A - \tilde{c}\tilde{z}\tilde{z}^T\|_F = 2.206570 \) (for both solutions).

If we allow \(\tilde{c}\) and \(\tilde{z}\) to be complex, we obtain with \((\tilde{z}, \tilde{c}) = (i, \frac{\sqrt{2}}{3})\) as well as with \((\tilde{z}, \tilde{c}) = (-i, \frac{\sqrt{2}}{3})\) the smaller error \( \|A - \tilde{c}\tilde{z}\tilde{z}^T\|_F = \frac{\sqrt{201}}{9} = 1.795055 \).

**Remark 3.14.** For non-negative matrices \(A\) it can be shown that the optimal rank-1 Hankel approximation is always real.

### 4 Optimal Rank-1 Hankel Approximation in the Spectral Norm

We consider the minimization problem \([152]\) in the spectral norm. The structure of the rank-1 Hankel matrix in Lemma 2.1 and Remark 2.2 imply that we need to solve the minimization problem

\[
\min_{c, z \in \mathbb{C}} \|A - czMz_N^T\|_2^2. \tag{4.1}
\]

This problem is much more difficult to solve than the minimization in the Frobenius norm. Therefore, we restrict our considerations to real symmetric matrices \(A \in \mathbb{R}^{N \times N}\) and show how to obtain the real optimal rank-1 Hankel approximation \(H_1\) in this case.

We will use the following notation: Let \(\lambda_0, \ldots, \lambda_{N-1}\) denote the eigenvalues of \(A\) which are ordered by modulus \(|\lambda_0| > |\lambda_1| \geq |\lambda_2| \ldots \geq |\lambda_{N-1}|\), and assume that \(|\lambda_0|\) occurs with multiplicity 1. Further we assume that \(\lambda_0 = \|A\|_2 > 0\). The values \(|\lambda_j|\) coincide with the singular values of \(A\). With \(\{v_0, \ldots, v_{N-1}\}\) we denote the orthonormal basis of eigenvectors in \(\mathbb{R}^N\) satisfying

\[A v_j = \lambda_j v_j, \quad j = 0, \ldots, N - 1.\]

Since we only consider square matrices, we simplify the notation and write

\[z := z_N = \left(1, z, \ldots, z^{N-1}\right)^T, \tag{4.2}\]

for a structured vector which is determined by \(z \in \mathbb{R}\). Further, for \(z \in \mathbb{R}\) and \(\lambda^2 \in (\lambda_1^2, \lambda_0^2)\) let

\[f(z, \lambda^2) := \frac{1}{z^T z} \sum_{j=0}^{N-1} (v_j^T z)^2 = \frac{z^T (A^2 - \lambda^2 I)^{-1} z}{z^T z}. \tag{4.3}\]

The second equality in (4.3) follows with \(z = \sum_{j=0}^{N-1} (v_j^T z) v_j\) from the observations

\[z^T (A^2 - \lambda^2 I)^{-1} z = \sum_{j=0}^{N-1} (v_j^T z)^2 \left(v_j^T (A^2 - \lambda^2 I)^{-1} v_j\right)\]

and \((\lambda^2_j - \lambda^2)\)^{-1} = \(v_j^T (A^2 - \lambda^2 I)^{-1} v_j\). The function \(f(z, \lambda^2)\) is well defined as long as \(\lambda^2\) is not an eigenvalue of \(A^2\). Moreover, \(f(z, \lambda^2)\) is differentiable, and \(\lim_{\lambda^2 \to \lambda^2_j} f(z, \lambda^2)\) is bounded if and only if \(v_j^T z = 0\). Similarly, \(f(z, \lambda^1_j) := \lim_{\lambda^2 \to \lambda^1_j} f(z, \lambda^2)\) is bounded if and only if \(v_j^T z = 0\) for all \(v_j\) corresponding to eigenvalues \(\lambda_j\) with \(\lambda^1_j = \lambda^2_j\). Further, for any fixed \(\lambda^2 \in (\lambda_1^2, \lambda_0^2)\), the function \(f(z, \lambda^2)\) is bounded from above and from below by

\[
\min \left\{(\lambda^2_j - \lambda^2_{N-1})^{-1}, (\lambda_0^2 - \lambda^2)^{-1}\right\} \leq f(z, \lambda^2) \leq \max \left\{(\lambda_0^2 - \lambda^2)^{-1}, (\lambda^2 - \lambda^2_{N-1})^{-1}\right\},
\]
since it can be seen as a Rayleigh quotient for the matrix $(A^2 - \lambda^2 I)^{-1}$. For fixed $z$, $f(z, \lambda^2)$ is strictly monotonically increasing.

The main theorem of this section contains two parts. The first part generalizes the result from [3] and states exact conditions ensuring that the rank-1 Hankel approximation achieves the same error as the unstructured rank-1 approximation. This error is given by $|\lambda_1|$ and achieved e.g. by truncated SVD. The second part of the theorem states conditions that enable us to derive an algorithm to compute the exact optimal rank-1 Hankel approximation also in the case that the error $|\lambda_1|$ cannot be achieved.

In the course of this section we will consider the case $\lambda^2 \in [\lambda^2_1, \lambda^2_0)$. If $\lambda^2 = \lambda^2_1$ then we will find that necessarily $v_j^T z = 0$ whenever $\lambda^2_j = \lambda^2_1 = \lambda^2$. In order to keep $f$ in (4.3) well-defined without any additional notation, from now on we always use the convention $\frac{0}{0} = 0$. To remind the reader that such terms may occur in the sum, we will use the notation $\sum'$ instead of $\sum$.

Theorem 4.1. Let $A = (a_{j,k})_{j,k=0}^{N-1,N-1} \in \mathbb{R}^{N \times N}$ be symmetric with $N \geq 2$. Assume that $\text{rank}(A) > 1$ and $\lambda_0 = \|A\|_2 > |\lambda_1|$. Let $H_1 = \tilde{c} \tilde{z} \tilde{z}^T$ be an optimal rank-1 Hankel approximation of $A$ with regard to the spectral norm.

(1) The optimal error bound $\|A - H_1\|_2 = \|A - \tilde{c} \tilde{z} \tilde{z}^T\|_2 = \lambda^2_2$ is achieved if and only if there exists $\tilde{z} \in \mathbb{R}$ such that the vector $\tilde{z}$ in (4.2) satisfies

$$v_j^T \tilde{z} = 0, \text{ for all } \lambda_j \text{ with } |\lambda_j| = |\lambda_1| \quad \text{and} \quad f(\tilde{z}, \lambda_2^2) = \frac{1}{\|\tilde{z}\|_2^2} \sum_{j=0}^{N-1} \frac{(v_j^T \tilde{z})^2}{\lambda_j^2 - \lambda_1^2} \geq 0, \quad (4.4)$$

and if $\tilde{c}$ is chosen such that

$$\sum_{j=0}^{N-1} \frac{(v_j^T \tilde{z})^2}{\lambda_j + |\lambda_1|} \leq \frac{1}{\tilde{c}} \leq \sum_{j=0}^{N-1} \frac{(v_j^T \tilde{z})^2}{\lambda_j - |\lambda_1|}. \quad (4.5)$$

(2) If there is no $\tilde{z}$ satisfying (4.3), then the optimal rank-1 Hankel approximation of $A$ possesses the error

$$\tilde{\lambda} := \|A - H_1\|_2 = \|A - \tilde{c} \tilde{z} \tilde{z}^T\|_2 \in (|\lambda_1|, \lambda_0),$$

where $\tilde{\lambda}$ is the minimal number in $(|\lambda_1|, \lambda_0)$ satisfying the relation

$$\max_{z \in \mathbb{R}} f(z, \tilde{\lambda}^2) = 0, \quad (4.6)$$

and we have $\tilde{z} \in \arg\max_{z \in \mathbb{R}} f(z, \tilde{\lambda}^2)$. Further,

$$\tilde{c} := \left( \sum_{k=0}^{N-1} \frac{(v_k^T \tilde{z})^2}{\lambda_k - \tilde{\lambda}} \right)^{-1} = \left( \tilde{z}^T (A - \tilde{\lambda} I)^{-1} \tilde{z} \right)^{-1} > 0. \quad (4.7)$$

To prove this theorem we need some preliminary observations. We start with the following lemma, which can also be found in [3].
Lemma 4.2. Let $D = \text{diag}(d_0, \ldots, d_{N-1}) \in \mathbb{R}^{N \times N}$, $b = (b_0, \ldots, b_{N-1})^T \in \mathbb{R}^N$, and $c \in \mathbb{R}$. Then the matrix $B := D + cbb^T$ has the determinant

$$\det(B) = \det(D) + c \sum_{j=0}^{N-1} b_j^2 \left( \prod_{k=0, k \neq j}^{N-1} d_k \right).$$

If $D$ is invertible, we have

$$\det(B) = \det(D) \left( 1 + c \sum_{j=0}^{N-1} b_j^2 \right).$$

Proof. We employ the rule for computing determinants of block matrices, see [26],

$$\det \begin{pmatrix} D & -b \\ cb^T & 1 \end{pmatrix} = \det(1 \cdot D + cbb^T) = \det(B),$$

and use an expansion of the determinant with respect to the last column. \qed

Further we can show

Lemma 4.3. Let $N \geq 2$, $\Lambda = \text{diag}(\lambda_0, \ldots, \lambda_{N-1})$ with $\lambda_0 > |\lambda_1| \geq |\lambda_2| \geq \ldots, |\lambda_{N-1}| \geq 0$, $c > 0$ and $\mu = (\mu_0, \ldots, \mu_{N-1})^T \in \mathbb{R}^N$ with $\mu_0 \neq 0$. Further let $|\lambda_1| \leq \lambda < \lambda_0$. Then $M_1(\lambda) := (\lambda I - \Lambda) + c \mu \mu^T$ is positive semidefinite if and only if

$$\sum_{j=0}^{N-1} \frac{\mu_j^2}{\lambda_j - \lambda} \geq \frac{1}{c}, \quad \text{(4.8)}$$

where $\mu_j = 0$ in the case where $j$ corresponds to $\lambda_j = |\lambda_j| = \lambda$. Moreover, if $\lambda > |\lambda_1|$ and the inequality in (4.8) is strict, then $M_1(\lambda)$ is even positive definite.

Proof. According to [25], $M_1(\lambda)$ is positive semidefinite if and only if all its principal minors, i.e., the determinants of all possible $r \times r$ principal submatrices of $M_1(\lambda)$ for $r = 1, \ldots, N$, are non-negative.

We observe that $M_1(\lambda)$ as well as all its principal submatrices are of the form $B$ in Lemma 4.2. Let $J \subseteq \{0, 1, \ldots, N-1\}$ be a subset with cardinality $|J| = r$, $1 \leq r \leq N$. Denote by $(M_1(\lambda))_J$ the principal submatrix obtained by restricting $M_1(\lambda)$ to the rows and columns with indices in $J$. Applying Lemma 4.2 we need to distinguish two cases: First, $\lambda_j < \lambda$ for all $\lambda_j$ with $j \neq 0$, i.e., $\lambda > |\lambda_1|$, and second, $\lambda = \lambda_j = |\lambda_1|$ for some $j$ (note that $|\lambda_1|$ can come with higher multiplicity). With $\Lambda_J := \text{diag}(\lambda_j)_{j \in J}$, $\mu_J := (\mu_j)_{j \in J}$, and with the $r \times r$ identity matrix $I_r$, we obtain the principal minors of $M_1(\lambda)$ as

$$\det(M_1(\lambda))_J = \begin{cases} \det(\lambda I_r - \Lambda_J) \cdot \left( 1 + c \sum_{j \in J} \frac{\mu_j^2}{\lambda - \lambda_j} \right) & \text{if } \lambda_j \neq \lambda \ \forall j \in J, \quad \text{(4.9a)} \\
 c \sum_{j \in J} \mu_j^2 \left( \prod_{k \in J, k \neq j} (\lambda - \lambda_k) \right) & \text{if } \exists j \in J: \lambda_j = \lambda, \quad \text{(4.9b)} \end{cases}$$

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for all possible subsets $J$, where for $r = N$ we have $\det(M_1(\lambda))_J = \det M_1(\lambda)$. Observe that the second case can only occur if $\lambda_j = \lambda = |\lambda_1|$ for some $j \in J$.

For $\lambda > |\lambda_1|$ we conclude:

If $0 \notin J$, then $\det(\lambda I_r - A_J) > 0$ since $\lambda - \lambda_j > 0$ for $j = 1, \ldots, N - 1$. Furthermore, all terms $\frac{\mu_j^2}{\lambda - \lambda_j}$ in the sum are non-negative, such that the condition $\det(M_1(\lambda))_J > 0$ holds. If $0 \in J$, then $\det(\lambda I_r - A_J) < 0$ since $\lambda - \lambda_0 < 0$. Thus, the condition $\det(M_1(\lambda))_J \geq 0$ is satisfied, if we have $1 + c \sum_{j \in J} \frac{\mu_j^2}{\lambda - \lambda_j} \leq 0$. This condition is always satisfied, if

$$
\left(1 + c \sum_{j=0}^{N-1} \frac{\mu_j^2}{\lambda - \lambda_j}\right) \leq 0,
$$

(4.10)

since all terms $\frac{\mu_j^2}{\lambda - \lambda_j}$, $j = 1, \ldots, N - 1$, are non-negative. A strict inequality in (4.10) even leads to $\det(M_1(\lambda))_J > 0$.

For $\lambda = |\lambda_1|$ we conclude:

Taking the subset $J = \{0, j\}$ for any $j$ with $\lambda_j = |\lambda_1| = \lambda$, it follows from (4.9b) that $\det(M_1(\lambda))_J = c \mu_j^2 (\lambda - \lambda_0) = c \mu_j^2 (\lambda_j - \lambda_0)$.

Since $c > 0$ and $\lambda - \lambda_0 < 0$, the condition $\det(M_1(\lambda))_J \geq 0$ can thus only be satisfied if $\mu_j = 0$. We conclude that

$$
\mu_j = 0 \quad \text{for all indices } j \text{ with } \lambda_j = |\lambda_1| = \lambda.
$$

(4.11)

It follows easily from (4.9b) that (4.11) is sufficient to ensure $\det(M_1(\lambda))_J \geq 0$ for all subsets $J$ containing at least one index $j$ with $\lambda_j = \lambda$.

Further, for all subsets $J$ with $0 \notin J$ and $\lambda_j \neq \lambda$ for all $j \in J$, we derive with the same argument as before that $\det(M_1(\lambda))_J > 0$.

Finally, for all subsets $J$ with $0 \in J$ and $\lambda_j \neq \lambda$ for all $j \in J$, the condition $\det(M_1(\lambda))_J \geq 0$ is satisfied if

$$
\left(1 + c \sum_{j=0}^{N-1} \frac{\mu_j^2}{\lambda - \lambda_j}\right) \leq 0
$$

(4.12)

while (4.11) holds.

All together, the matrix $M_1(\lambda)$ is positive semidefinite if and only if we have

$$
0 < \frac{1}{c} \leq \sum_{j=0}^{N-1} \frac{\mu_j^2}{\lambda_j - \lambda},
$$

(4.13)

and if additionally (4.11) holds in the case $\lambda = \lambda_j = |\lambda_1|$. If $\lambda > |\lambda_1|$ and $\frac{1}{c} < \sum_{j=0}^{N-1} \frac{\mu_j^2}{\lambda_j - \lambda}$, we even have $\det(M_1(\lambda))_J > 0$ for all $J$, and thus $M_1(\lambda)$ is positive definite.

Lemma 4.4. Let $N \geq 2$, $\Lambda = \text{diag}(\lambda_0, \ldots, \lambda_{N-1})$ with $\lambda_0 > |\lambda_1| \geq |\lambda_2| \geq \ldots, |\lambda_{N-1}| \geq 0$, $c > 0$ and $\mu = (\mu_0, \ldots, \mu_{N-1})^T \in \mathbb{R}^N$ with $\mu_0 \neq 0$. Further let $|\lambda_1| \leq \lambda < \lambda_0$. Then

$$
M_2(\lambda) := (\lambda I + \Lambda) - c \mu \mu^T
$$

is positive semidefinite if and only if

$$
\sum_{j=0}^{N-1} \frac{\mu_j^2}{\lambda_j + \lambda} \leq \frac{1}{c}.
$$

(4.14)

where $\mu_j = 0$ in the case where $j$ corresponds to $\lambda_j = |\lambda_1| = \lambda$. Moreover, if $\lambda > |\lambda_1|$ and the inequality in (4.14) is strict, then $M_2(\lambda)$ is positive definite.
Proof. We use a similar notation for the submatrices of $\mathbf{M}_2(\lambda)$ as in Lemma 4.3. First we consider $\lambda > |\lambda_1|$. To ensure that $\mathbf{M}_2(\lambda)$ is positive semidefinite, we find the condition

$$\det(\mathbf{M}_2(\lambda))_J = \det(\lambda \mathbf{I}_r + \mathbf{A}_J) \left(1 - c \sum_{j \in J} \frac{\mu_j^2}{\lambda + \lambda_j}\right) \geq 0$$

for all possible subsets $J \subset \{0, \ldots, N-1\}$, where for $|J| = r = N$ we have $\det(\mathbf{M}_2(\lambda))_J = \det \mathbf{M}_2(\lambda)$. This time, we have $\det(\lambda \mathbf{I}_r + \mathbf{A}_J) > 0$ for all subsets $J$. Since $c > 0$ and all terms $\frac{\mu_j^2}{\lambda + \lambda_j}$, $j = 0, \ldots, N-1$, are positive, these conditions are satisfied if and only if

$$\left(1 - c \sum_{j=0}^{N-1} \frac{\mu_j^2}{\lambda + \lambda_j}\right) \geq 0. \quad (4.15)$$

Now we consider $\lambda = |\lambda_1|$. For $j \in J$ with $-\lambda_j = \lambda = |\lambda_1|$, we find from Lemma 4.2

$$-c \sum_{j \in J} \mu_j^2 \left(\prod_{k \in J, k \neq j} \lambda + \lambda_k\right) \geq 0. \quad (4.16)$$

In particular, taking $J = \{0, j\}$ for some $j$ with $-\lambda_j = |\lambda_1| = \lambda$, (4.16) can only be satisfied if $\mu_j = 0$. For $J$ not containing any $j$ with $-\lambda_j = |\lambda_1| = \lambda$ we find similarly to (4.12)

$$\left(1 - c \sum_{j=0}^{N-1} \frac{\mu_j^2}{\lambda + \lambda_j}\right) \geq 0. \quad (4.17)$$

Together, it follows that $\mathbf{M}_2(\lambda)$ is positive semidefinite if and only if

$$\sum_{j=0}^{N-1} \frac{\mu_j^2}{\lambda_j + |\lambda_1|} \leq \frac{1}{c}, \quad (4.18)$$

where $\mu_j = 0$ for $-\lambda_j = |\lambda_1| = \lambda$. For $|\lambda_1| < \lambda$ and strict inequality in $4.18$, we observe that $\mathbf{M}_2(\lambda)$ is even positive definite.

Proof of Theorem 4.7. Throughout this proof, let

$$\hat{\lambda} := \|\mathbf{A} - \hat{c} \tilde{z} \tilde{z}^T\|_2 \quad (4.19)$$

denote the optimal rank-1 approximation error, i.e., the parameters $\tilde{z}$, $\hat{c}$ generate an optimal rank-1 Hankel approximation of $\mathbf{A}$.

1. We reformulate (4.19) in order to apply Lemma 4.2 – 4.3. By (4.12) it follows that $\hat{\lambda}$ or $-\hat{\lambda}$ is an eigenvalue of the symmetric matrix $\mathbf{A} - \hat{c} \tilde{z} \tilde{z}^T$. Using the basis transform with the orthogonal matrix $\mathbf{V} = (\mathbf{v}_0 \ldots \mathbf{v}_{N-1})$ of eigenvectors of $\mathbf{A}$, we have $\mathbf{A} = \mathbf{V}^T \mathbf{A} \mathbf{V} = \text{diag}(\lambda_0, \ldots, \lambda_{N-1})$ with the eigenvalues of $\mathbf{A}$ ordered by modulus, $\lambda_0 > |\lambda_1| \geq \ldots \geq |\lambda_{N-1}| \geq 0$. Further, let $\mu := \mathbf{V}^T \tilde{z} = (\mu_0, \ldots, \mu_{N-1})^T$. In other words,

$$\tilde{z} = \mathbf{V} \mu = \sum_{j=0}^{N-1} \mu_j \mathbf{v}_j, \quad \text{with} \quad \mu_j = \mathbf{v}_j^T \tilde{z}, \quad j = 0, \ldots, N-1,$$
is the representation of $\tilde{z}$ in the orthogonal basis $\{v_0, v_1, \ldots, v_{N-1}\}$. Formula (4.19) is now equivalent to

$$\tilde{\lambda} = \|A - \tilde{c} \tilde{z} \tilde{z}^T\|_2 = \|V^T A V - \tilde{c} V^T \tilde{z} \tilde{z}^T V\|_2 = \|A - \tilde{c} \mu \mu^T\|_2.$$ 

Therefore, (4.19) holds if and only if the two symmetric matrices

$$M_1(\tilde{\lambda}) := \tilde{\lambda} I - A + \tilde{c} \mu \mu^T, \quad M_2(\tilde{\lambda}) := \tilde{\lambda} I + A - \tilde{c} \mu \mu^T$$

(4.20)

are positive semidefinite, and at least one of the two matrices possesses the eigenvalue 0.

2. We show upper and lower bounds for the optimal approximation error $\tilde{\lambda}$. By the Eckart-Young-Mirsky theorem it follows that $\tilde{\lambda} \geq |\lambda_1|$. We prove that the optimal error $\tilde{\lambda}$ satisfies $\tilde{\lambda} < \lambda_0$ by showing that we can always find values $\tilde{z}$ and $\tilde{c}$ such that $M_1(\lambda_0)$ and $M_2(\lambda_0)$ are even positive definite. This implies that $\lambda_0$ cannot be the optimal error: Obviously, the diagonal matrix $\lambda_0 I - A$ is positive semidefinite. Therefore, the matrix $M_1(\lambda_0)$ is positive definite for $\tilde{c} > 0$ and any $\mu$ with first component $\mu_0 \neq 0$, i.e., for any value $\tilde{z} \in \mathbb{R}$ with $\mu_0 = v_0^T \tilde{z} \neq 0$. Furthermore, $\lambda_0 I + A$ is positive definite with smallest possible singular value $\lambda_0 - |\lambda_1| > 0$. Thus, $M_2(\lambda_0)$ is positive definite for any $\tilde{z} \in \mathbb{R}$ as long as $\tilde{c} < \frac{\lambda_0 - |\lambda_1|}{|\mu_0|^2} = \frac{\lambda_0 - |\lambda_1|}{|\mu_0|^2}$. Hence, the optimal error is bounded from above and below by $|\lambda_1| \leq \tilde{\lambda} < \lambda_0$.

The optimal parameter $\tilde{z}$ necessarily satisfies $\mu_0 = v_0^T \tilde{z} \neq 0$, since otherwise we would find $(A - \tilde{c} \tilde{z} \tilde{z}^T) v_0 = A v_0 = \lambda_0 v_0$ contradicting $\tilde{\lambda} < \lambda_0$. Moreover, we obtain the necessary condition $\tilde{c} > 0$ since for $\tilde{c} \leq 0$ we would add a positive semidefinite matrix to $A$ thereby enlarging the spectral norm,

$$\|A - \tilde{c} \tilde{z} \tilde{z}^T\|_2 = \max_{\|v\|_2 = 1} |v^T (A - \tilde{c} \tilde{z} \tilde{z}^T) v| 
\geq v_0^T A v_0 - \tilde{c} (v_0^T \tilde{z})^2 = \lambda_0 + |\tilde{c}| (v_0^T \tilde{z})^2 \geq \lambda_0.$$

3. Now we derive necessary and sufficient conditions for the optimal parameters $\tilde{c} > 0$, $\tilde{z} \in \mathbb{R}$ and $\tilde{\lambda} \in [|\lambda_1|, \lambda_0]$ by inspecting the matrices $M_1(\tilde{\lambda})$ and $M_2(\tilde{\lambda})$. Thereby we prove part (1) of Theorem 4.1. From Lemma 4.3 and Lemma 4.4 it follows that $M_1(\tilde{\lambda})$ and $M_2(\tilde{\lambda})$ are both positive semidefinite if and only if

$$\sum_{j=0}^{N-1} \frac{c^2}{\lambda_j + \tilde{\lambda}} \leq \frac{1}{\tilde{c}} \leq \sum_{j=0}^{N-1} \frac{c^2}{\lambda_j - \tilde{\lambda}},$$

(4.21)

and if in case of $\tilde{\lambda} = |\lambda_1|$ moreover $\mu_j = v_j^T \tilde{z} = 0$ for all $\lambda_j$ with $|\lambda_j| = \tilde{\lambda}$. Obviously, a parameter $\tilde{c}$ satisfying (4.21) only exists, if

$$\sum_{j=0}^{N-1} \frac{c^2}{\lambda_j - \tilde{\lambda}} - \sum_{j=0}^{N-1} \frac{c^2}{\lambda_j + \tilde{\lambda}} = 2\tilde{\lambda} \sum_{j=0}^{N-1} \frac{c^2}{\lambda_j^2 - \tilde{\lambda}^2} \geq 0.$$ 

W.l.o.g. we can assume that $\tilde{\lambda} \neq 0$ since otherwise the matrix $A$ would be a Hankel matrix of rank 1 already. Thus, for the function $f(z, \lambda^2)$ from (4.3) it follows that $f(\tilde{z}, \tilde{\lambda}^2) \geq 0$, and we conclude (4.3) and (4.4) for $\tilde{\lambda} = |\lambda_1|$.

4. Finally we prove part (2) of Theorem 4.1. So we assume that (4.4) is not satisfied for any $\tilde{z} \in \mathbb{R}$, i.e., $\tilde{\lambda} > |\lambda_1|$. Inspecting the two sums in (4.21), we observe that the left sum
increases for decreasing $\hat{\lambda}$ while the right sum decreases with decreasing $\hat{\lambda}$. Thus, (4.21) implies the equalities
\[
\sum_{j=0}^{N-1} \frac{\mu_j^2}{\lambda_j + \lambda} = \frac{1}{c} = \sum_{j=0}^{N-1} \frac{\mu_j^2}{\lambda_j - \lambda},
\]
(4.22)
for the minimal error $\hat{\lambda}$. Otherwise, we could find a parameter $\tilde{c}$ such that the two inequalities in (4.21) are strict. But then, Lemma 4.3 and Lemma 4.4 yield that the two matrices $M_1(\hat{\lambda})$ and $M_2(\hat{\lambda})$ are even positive definite, and we could find some $|\lambda_1| \leq \lambda < \hat{\lambda}$ such that $M_1(\lambda)$ and $M_2(\lambda)$ are still positive semidefinite. This would contradict our assumption (4.19).

Relation (4.22) directly implies that $\det M_1(\hat{\lambda}) = \det M_2(\hat{\lambda}) = 0$, or equivalently, that $\hat{\lambda}$ as well as $-\lambda$ are eigenvalues of $A - \tilde{c}\hat{z}\hat{z}^T$. Assertion (4.7) now follows from (4.22). Further, we conclude
\[
\sum_{j=0}^{N-1} \frac{\mu_j^2}{\lambda_j + \lambda} - \sum_{j=0}^{N-1} \frac{\mu_j^2}{\lambda_j - \lambda} = 2\hat{\lambda} \sum_{j=0}^{N-1} \frac{\mu_j^2}{\lambda_j^2 - \lambda^2} = 0.
\]
Since $\hat{\lambda} > |\lambda_1| \geq 0$, this shows that $f(\hat{z}, \hat{\lambda}^2) = 0$.

Finally, we consider $\|\hat{z}\|^2 f(z, \hat{\lambda}^2)$ as a polynomial in $z$ for the fixed optimal error $\hat{\lambda}$ and show that $f(z, \hat{\lambda}^2) \leq 0$ for all $z \in \mathbb{R}$. Assume to the contrary that there is some $z$ with $f(z, \hat{\lambda}^2) > 0$. With the same arguments as before, we then obtain a range for the choice of $\tilde{c}$. But then $\tilde{c}$ can be taken such that the two matrices $M_1(\lambda)$ and $M_2(\lambda)$ are even positive definite. In that case $\hat{\lambda}$ is no longer the optimal error, contradicting our assumption. Thus we have shown (4.6).

**Remark 4.5.** 1. The conditions (4.4) in Theorem 4.1 are particularly satisfied if $\hat{z}$ is of the form $\hat{z} = \|\hat{z}\|_2 v_0$, where $v_0$ is the eigenvector corresponding to the largest eigenvalue $\lambda_0$ of $A$. In this case (4.4) simplifies to
\[
(\langle v_0^T \hat{z} \rangle^2) = \frac{\|\hat{z}\|^2_2}{\lambda_0^2 - \lambda_1^2} \geq 0,
\]
since $v_j^T \hat{z} = 0$ for $j = 1, \ldots, N - 1$.

2. Theorem 4.1 provides a solution to the rank-1 Hankel approximation problem with respect to the spectral norm. However, the solution parameters $(\hat{z}, \tilde{c})$ need not be unique. In the case when (4.4) is satisfied and the optimal error $\|A - \tilde{c}\hat{z}\hat{z}^T\|_2 = \lambda_1^2$ is attained, there are several possible choices for $\tilde{c}$ if the inequality in (4.5) is strict. But even in the case when (4.4) cannot be satisfied and $\tilde{c}$ is determined uniquely by (4.7), it may occur that $\hat{z} \in \operatorname{argmax}_{z \in \mathbb{R}} f(z, \hat{\lambda}^2)$ is not unique, as can be seen in Example 5.6.

The conditions shown in Theorem 4.1 can be used to provide an algorithm for computing the optimal rank-1 Hankel approximation numerically. First, we can verify whether (4.4) can be satisfied. For this purpose, we compute all real roots of the polynomial $v_1(z) := v_1^T \hat{z}$ of degree $N - 1$. Then, for each $z$ satisfying $v_1(z) = 0$ we check, whether $0 \leq f(z, \lambda_1^2) < \infty$. If this is the case for some zero $z$ of $v_1(z)$, then we set $\hat{z} := z$ and determine $\tilde{c}$ by (4.5).

If there is no zero of $v_1(z)$ satisfying (4.4), then we have to employ the relations (4.6) and (4.7) in Theorem 4.1 to determine $\hat{z}$ and $\tilde{c}$.
For fixed $\lambda^2 \in (\lambda_1^2, \lambda_0^2)$ define $f_\lambda(z) := f(z, \lambda^2)$. Since for fixed $z$, $f(z, \lambda^2)$ is strictly monotonically increasing in $\lambda^2$, formula (1.6) implies:

- If $\max_z f_\lambda(z) > 0$, then the optimal error $\hat{\lambda}$ in (1.19) satisfies $\hat{\lambda}^2 < \lambda^2$.
- If $\max_z f_\lambda(z) < 0$, then the optimal error satisfies $\hat{\lambda}^2 > \lambda^2$.
- If $\max_z f_\lambda(z) = 0$, then the optimal error satisfies $\hat{\lambda}^2 = \lambda^2$ and the rank-1 Hankel approximation is generated by this zero $\tilde{z} \in \text{argmax}_z f_\lambda(z)$ and $\tilde{c}$ from (4.7).

To find a simple range, where we have to search for the maximum of $f_\lambda$, we can again apply an observation similar to that used in Theorem 3.4 for the Frobenius norm. Let

$$f_\lambda^{(1)}(z) := \frac{1}{\|z\|_2^2} \sum_{j=0}^{N-1} \frac{(v_j^T J_N z)^2}{\lambda_j^2 - \lambda^2},$$

where $J_N$ denotes the counter identity in (1.3). Then we observe for $z \neq 0$ because $z^{N-1} z(1/z) = (z^{N-1}, \ldots, z, 1)^T = J_N z = J_N z$ that

$$f_\lambda \left( \frac{1}{z} \right) = \frac{1}{\|z(1/z)\|_2^2} \sum_{j=0}^{N-1} \frac{(v_j^T z(1/z))^2}{\lambda_j^2 - \lambda^2} = \frac{1}{\|z(z)\|_2^2} \sum_{j=0}^{N-1} \frac{(v_j^T J_N(z))^2}{\lambda_j^2 - \lambda^2} = f_\lambda^{(1)}(z).$$

Thus, we only have to search for the maximum of $f_\lambda(z)$ in the interval $[-1, 1]$ and for the maximum of $f_\lambda^{(1)}(z)$ in $(-1, 1)$. We obtain the following algorithm to compute the optimal rank-1 Hankel approximation of $A$ with respect to the spectral norm as well as the corresponding error.

**Algorithm 4.6** (Computation of optimal rank-1 Hankel approximation w.r.t. spectral norm).

**Input:** symmetric matrix $A \in \mathbb{R}^{N \times N}$ with single largest singular value $\lambda_0 > 0$, threshold $\epsilon > 0$.

1. Compute the SVD of $A$ to obtain the singular values $\lambda_0 > |\lambda_1| \geq \ldots \geq |\lambda_{N-1}| \geq 0$ and the normalized eigenvectors $v_0, \ldots, v_{N-1}$ such that $V = (v_0 \ldots v_{N-1})$ is an orthogonal matrix.
2. Compute the set $\Sigma$ of joint real zeros of the polynomials $v_j(z) := v_j^T z$ of degree $N-1$ corresponding to eigenvalues $\lambda_j$ with $|\lambda_j| = |\lambda_1|$ and with $z$ as in (1.2). For each $z \in \Sigma$ compute

$$\|z\|_2^2 f_\lambda(z) = \sum_{j=0}^{N-1} \frac{(v_j^T z)^2}{\lambda_j^2 - \lambda_1^2}.$$

If one value $z \in \Sigma$ satisfies $f_\lambda(z) \geq 0$ then set

$$\tilde{\lambda} := |\lambda_1|, \quad \tilde{z} := z, \quad \tilde{c} := \left( \sum_{j=0}^{N-1} \frac{(v_j^T z)^2}{\lambda_j - |\lambda_1|} \right)^{-1}.$$

3. If $\Sigma = \emptyset$ or if there is no $z \in \Sigma$ satisfying $f_\lambda(z) \geq 0$ then apply the following bisection iteration:

- Set $a := |\lambda_1|$ and $b := \lambda_0$. While $b - a > \epsilon$ iterate:
  a) Compute $x := \frac{a + b}{2}$. Find the maximal value $W$ of
    $$\|z\|_2^2 f_x(z) = \sum_{j=0}^{N-1} \frac{(v_j^T z)^2}{\lambda_j^2 - x^2} \quad \text{for } z \in [-1, 1]$$
    and
    $$\|z\|_2^2 f_x(z) = \sum_{j=0}^{N-1} \frac{(v_j^T z)^2}{\lambda_j^2 - x^2} \quad \text{for } z \in [-1, 1].$$

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and of
\[ \|z\|_F^2 f^{(1)}_x (z) = \sum_{j=0}^{N-1} \frac{(v_j^T J_N z)^2}{\lambda_j^2 - x^2} \quad \text{for } z \in (-1, 1). \]

b) If \( W = 0 \), then we have found the optimal solution, go to 4.
If \( W > 0 \), then \( b := x \), else \( a := x \).

4. Set \( \hat{\lambda} := a \). If \( f^{(1)}_{\hat{\lambda}} \) possesses a zero \( z \) in \([-1, 1]\) then \( \hat{z} := z \), otherwise, if \( f^{(1)}_{\hat{\lambda}} \) possesses a zero \( z \) in \((-1, 1)\) then set \( \hat{z} := 1/z \). Compute
\[ \hat{c} := \left( \sum_{j=0}^{N-1} \frac{(v_j^T \hat{z})^2}{\lambda_j^2 - \hat{\lambda}} \right)^{-1}. \]

**Output:** \( \hat{z}, \hat{c} \) generating an optimal rank-1 Hankel approximation of \( A \) with respect to the spectral norm, error \( \hat{\lambda} = \|A - \hat{c} \hat{z} \hat{z}^T\|_2 \).

**Remark 4.7.** Obviously, the optimal rank-1 Hankel approximation depends on the distribution of all eigenvalues of \( A \) as well as on the structure of the eigenvectors of \( A \).
In particular the optimal parameters \( \hat{z} \) and \( \hat{c} \) generating the optimal rank-1 Hankel approximation of \( A \) with regard to the spectral norm usually do not coincide with those parameters found for the Frobenius norm.

**Example 4.8.** We consider the Hankel matrix
\[ A := \begin{pmatrix} 3 & 2 & 1 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 1 & 2 & 5 \\ 1 & 2 & 5 & 2 \end{pmatrix} \]
with the eigenvalues (rounded to 6 digits)
\[ \lambda_0 = 8.421093, \lambda_1 = -3.155074, \lambda_2 = 3.009151, \lambda_3 = -0.275170. \]
With Theorem 4.1 for the optimal rank-1 Hankel approximation with regard to the Frobenius norm, we obtain the parameters
\[ \hat{z} = 1.225640, \quad \hat{c} = 1.020343, \quad (4.23) \]
and the error \( \|A - \hat{c} \hat{z} \hat{z}^T\|_F = 4.568510 \). The spectral norm of the obtained matrix is \( \|A - \hat{c} \hat{z} \hat{z}^T\|_2 = 3.208509 \).

Now we consider the rank-1 Hankel approximation with regard to the spectral norm. In this example, the polynomial \( v_1(z) := v_1^T z \) possesses three real zeros at \( z_1 = -0.391861, z_2 = 0.193813, \) and \( z_3 = 1.126551 \). At these points, we find
\[ f(z_1, \lambda_1^2) = -0.455125, \quad f(z_2, \lambda_2^2) = -0.808914, \quad f(z_3, \lambda_3^2) = -0.002521. \]
Therefore, we cannot achieve the error \( |\lambda_1| = 3.155074 \). Algorithm 4.6 provides the optimal parameters
\[ \hat{z} = 1.143122, \quad \hat{c} = 1.595173, \]
and we obtain the error \( \|A - \hat{c} \hat{z} \hat{z}^T\|_2 = 3.159482 \). At the same time, for these parameters we get the Frobenius norm \( \|A - \hat{c} \hat{z} \hat{z}^T\|_F = 4.932743 \).

For comparison, the Cadzow algorithm (considered in the next section) provides, after 15 iterations the parameters \( z = 1.252213 \) and \( c = 0.936695 \) and achieves the error norms \( \|A - \hat{c} \hat{z} \hat{z}^T\|_2 = 3.239722 \) and \( \|A - \hat{c} \hat{z} \hat{z}^T\|_F = 4.574811. \)
Remark 4.9. The AAK theory for infinite Hankel matrices tells us, that the optimal parameter $\tilde{z}$ should be a zero of the Laurent polynomial obtained from the (infinite) eigenvector corresponding to the second singular value $\sigma_1$, see e.g. [4, 22]. Transferred to our case of finite matrices, we have to inspect all zeros of the polynomial $v_1(z) = v_1^T z$. This is exactly, what we are doing already, when we want to check, whether the error known from the unstructured case can be achieved, see Algorithm 5.1 step 2. As we have seen in the example above, non of the zeros of $v_1(z)$ provides the optimal parameter, but $z_3 = 1.126551$ is close to $\tilde{z}$ in (4.23). We refer to [4] for further error estimates.

5 Rank-1 Hankel approximation using the Cadzow Algorithm

Finally, in this section we will consider the Cadzow algorithm. We will show, that the Cadzow iteration for rank-1 Hankel approximation always converges to a fixed point. However, we will also see that the obtained result is usually not optimal with regard to the Frobenius norm or the spectral norm.

Without loss of generality we assume that $M \leq N$. For a matrix $A = (a_{j,k})_{j,k}^{M-1,N-1}$ let $P(A)$ be the Hankel matrix given by

$$ P(A) := (h_{k,\ell})_{k,\ell=0}^{M-1,N-1} \in \mathbb{C}^{M \times N} $$

with

$$ h_\ell := \begin{cases} \frac{1}{\ell+1} \sum_{r=0}^{\ell} a_{r,\ell-r} & \text{for } \ell = 0, \ldots, M-1, \\ \frac{1}{M} \sum_{r=0}^{M-1} a_{r,\ell-r} & \text{for } \ell = M, \ldots, N-1, \\ \frac{1}{M+N-1-\ell} \sum_{r=\ell+1-N}^{M-1} a_{r,\ell-r} & \text{for } \ell = N, \ldots, N+M-2. \end{cases} $$

Then the Cadzow algorithm can be stated as follows.

**Algorithm 5.1** (Cadzow algorithm for rank-1 Hankel approximation).

**Input:** $A \in \mathbb{C}^{M \times N}$ with rank $A \geq 1$ and single largest singular value.

1. Compute the largest singular value $\sigma_0$ of $A$ and the corresponding normalized singular vectors $u_0, v_0$, such that

$$ A_0 := \sigma_0 u_0 v_0^* $$

is the best (unstructured) rank-1 approximation of $A$.

2. For $j = 1, 2, \ldots$ do
   a) $\tilde{A}_j := P(A_{j-1})$
   b) Compute the optimal (unstructured) rank-1 approximation of $\tilde{A}_j$,

$$ A_j := \sigma_j u_j v_j^*, $$

where $\sigma_j$ is the largest singular value of $\tilde{A}_j$ with normalized singular vectors $u_j, v_j$.

**Output:** $u := \lim_{j \to \infty} u_j$, $v := \lim_{j \to \infty} v_j$, $\sigma := \lim_{j \to \infty} \sigma_j$. 

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As we will see, this algorithm can be understood as an alternating projection algorithm. In case of convergence, we usually obtain a rank-1 Hankel approximation \( \sigma u v^* \) of \( A \). We will show convergence of Algorithm 5.1 to a fixed point. To analyse the convergence properties of Algorithm 5.1, we start with the following lemmata:

**Lemma 5.2.** Let \( M, N \in \mathbb{Z} \) with \( 2 \leq M \leq N \). For \( a = (a_0, \ldots, a_{M-1})^T \in \mathbb{C}^M \) and \( b = (b_0, \ldots, b_{N-1})^T \in \mathbb{C}^N \) let \( P(ab^*) \) be the Hankel matrix obtained by counter diagonal averaging, i.e.,

\[
P(ab^*):= (h_{k+\ell})_{k,\ell=0}^{M-1,N-1}
\]

with

\[
h_\ell := \begin{cases} \frac{1}{\ell+1} \sum_{r=0}^\ell a_r \overline{b}_{\ell-r} & \text{for } \ell = 0, \ldots, M-1, \\ \frac{1}{M-N-1-\ell} \sum_{r=M+1-N}^{M-1} a_r \overline{b}_{\ell-r} & \text{for } \ell = M, \ldots, N-1, \\ \frac{1}{M+N-1-\ell} \sum_{r=\ell+1-N}^{M-1} a_r \overline{b}_{\ell-r} & \text{for } \ell = N, \ldots, N+M-2. \end{cases} \tag{5.1}
\]

Then

\[
\|P(ab^*)\|_2 \leq \|P(ab^*)\|_F \leq \|ab^*\|_F = \|ab^*\|_2 = \|a\|_2 \|b\|_2,
\]

and the equality \( \|P(ab^*)\|_F = \|ab^*\|_F \) holds, if and only if there exists \( z \in \mathbb{C} \) such that \( a = z_M = (1, \ldots, z^{M-1})^T \) and \( b = z_N = (1, \ldots, z^{N-1})^T \) or \( a = e_M = (0, \ldots, 0, 1)^T \) and \( b = e_N = (0, \ldots, 0, 1)^T \).

**Proof.** First note that the Frobenius norm can be written as \( \|ab^*\|^2_F = \sum_{k=0}^{M-1} \sigma_k^2 \), where \( \sigma_k \) denote the non-zero singular values of \( ab^* \). But since \( ab^* \) has rank 1 it only possesses one non-zero singular value and hence we have \( \|ab^*\|_F = \|ab^*\|_2 = \|a\|_2 \|b\|_2 \), for the last equality, see Remark 2.2.

We obtain by definition

\[
\|P(ab^*)\|^2_2 \leq \|P(ab^*)\|^2_F,
\]

\[
= \sum_{\ell=0}^{M-1} (\ell+1) |h_\ell|^2 + \sum_{\ell=0}^{N-1} M |h_\ell|^2 + \sum_{\ell=N}^{N+M-2} (N+M-1-\ell) |h_\ell|^2
\]

\[
= \sum_{\ell=0}^{M-1} \frac{1}{\ell+1} \left( \sum_{r=0}^\ell a_r \overline{b}_{\ell-r} \right)^2 + \sum_{\ell=M}^{N-1} \frac{M}{\ell} \left( \sum_{r=0}^{\ell-1} a_r \overline{b}_{\ell-r} \right)^2
\]

\[
+ \sum_{\ell=N}^{N+M-2} \frac{1}{M+N-1-\ell} \left( \sum_{r=\ell+1-N}^{M-1} a_r \overline{b}_{\ell-r} \right)^2
\]

\[
\leq \sum_{\ell=0}^{M-1} \frac{1}{\ell+1} \sum_{r=0}^\ell |a_r \overline{b}_{\ell-r}|^2 + \sum_{\ell=M}^{N-1} \frac{1}{\ell} \sum_{r=0}^{\ell-1} |a_r \overline{b}_{\ell-r}|^2 + \sum_{\ell=N}^{N+M-2} \frac{1}{M+N-1-\ell} \sum_{r=\ell+1-N}^{M-1} |a_r \overline{b}_{\ell-r}|^2
\]

\[
= \|ab^*\|^2_F = \|ab^*\|^2_2 = \|a\|^2_2 \|b\|^2_2,
\]

where we have used that for any integer \( L > 0 \) and any \( c = (c_0, \ldots, c_{L-1})^T \in \mathbb{C}^L \),

\[
\sum_{k=0}^{L-1} |c_k|^2 \leq L \sum_{k=0}^{L-1} |c_k|^2.
\]

Here, equality only holds, if for any two indices \( k, \ell \) we have \( 2 \text{Re}(c_k \overline{c_\ell}) = |c_k|^2 + |c_\ell|^2 \), i.e., if \( c_k = c_\ell \), thus only, if \( c \) is a constant vector of the form \( c = c (1, \ldots, 1)^T \) with some
Lemma 5.3 implies that, restricted to the set of rank-1 matrices, $P$ is an orthogonal projector to the Hankel matrices with regard to the spectral norm as well as to the Frobenius norm, i.e., $P^2 = P$ and $P$ has norm 1. For general matrices $A$, this is still true with respect to the Frobenius norm since $P$ is a linear operator. Taking the singular value decomposition $A = \sum_{k=0}^{M-1} \sigma_k u_k v_k^*$ with normalized singular vectors $u_k$, $v_k$ to the singular values $\sigma_k$ of $A$, we have

$$\|P(A)\|_F^2 = \left\| P \left( \sum_{k=0}^{M-1} \sigma_k u_k v_k^* \right) \right\|_F^2 = \left\| \sum_{k=0}^{M-1} \sigma_k P(u_k v_k^*) \right\|_F^2 \leq \sum_{k=0}^{M-1} \|\sigma_k P(u_k v_k^*)\|_F^2 \leq \sum_{k=0}^{M-1} \sigma_k^2 \|u_k v_k^*\|_2^2 = \sum_{k=0}^{M-1} \sigma_k^2 = \|A\|_F^2.$$  

Since the map from $\tilde{A}_j$ onto its optimal rank-1 approximation $A_j = \sigma_j u_j v_j^*$ in Algorithm 5.1 also is an orthogonal projection, the Cadzow algorithm is indeed an alternating projection algorithm.

In the next lemma, we need the following concept: For vectors $a = (a_0, \ldots, a_{M-1})^T \in \mathbb{C}^M$ and $b = (b_0, \ldots, b_{N-1})^T \in \mathbb{C}^N$ we define the convolution vector $a \star \tilde{b} \in \mathbb{C}^{M+N-1}$ with components

$$(a \star \tilde{b})_\ell = \sum_{j} a_j \bar{b}_{\ell - j}, \quad \ell = 0, \ldots, M + N - 2,$$

where we set $a_j = 0$ for $j \in \mathbb{Z} \setminus \{0, \ldots, M - 1\}$ and $b_j = 0$ for $j \in \mathbb{Z} \setminus \{0, \ldots, N - 1\}$.

Then we have

**Lemma 5.3.** For $M, N \in \mathbb{Z}$, $2 \leq M \leq N$, vectors $a, c \in \mathbb{C}^M$, and vectors $b, d \in \mathbb{C}^N$ we have

$$c^* P(a b^*) d = (c \star \tilde{d})^* D^{-1} (c \star \tilde{b}) = (a \star \tilde{b})^* D^{-1} (c \star \tilde{d}) = c^* P(c d^*) b, \quad (5.2)$$

with $D := \text{diag}(1, 2, \ldots, M - 1, M, \ldots, M, M - 1, \ldots, 2, 1) \in \mathbb{R}^{(M+N-1) \times (M+N-1)}$, where the component $M$ occurs $(N - M + 1)$ times in $D$.

**Proof.** First, we observe that $P(a b^*) = (h_{k+\ell})_{k,\ell=0}^{M-1,N-1}$ with

$$h := (h_0, \ldots, h_{M+N-2})^T = D^{-1} (a \star \tilde{b})$$

as a direct consequence of (5.1). We set $c_j = 0$ for $j \in \mathbb{Z} \setminus \{0, \ldots, M - 1\}$ and $d_j = 0$ for $j \in \mathbb{Z} \setminus \{0, \ldots, N - 1\}$. Then we find

$$c^* P(a b^*) d = c^* (h_{k+\ell})_{k,\ell=0}^{M-1,N-1} d$$

$$= \sum_{\ell=0}^{N-1} \sum_{k=0}^{M-1} h_{k+\ell} \tau_k d_\ell = \sum_{r=0}^{M+N-2} h_r \sum_{k \in \mathbb{Z}} \tau_k d_{r-k}$$

where $\tau_k = \sum_{\ell=0}^{\min(k,M-k)} h_{\ell}$. Therefore, in the above computation, $\|P(a b^*)\|_F^2 = \|a b^*\|_F^2$ is satisfied, if and only if for each $\ell \in \{0, \ldots, N - 1\}$, we have $a_{\ell} b_{\ell-r} = c_{\ell} \in \mathbb{C}$ for $r = \max(0, \ell + 1 - N), \ldots, \min(\ell, M - 1)$, or in other words, if $a b^*$ is already a Hankel matrix, such that $a b^* = P(a b^*)$. By Lemma 2.1, this implies the assertion. \qed
The Cadzow iteration in Algorithm [5.1] implies the following iteration formulas for \( j = 1, 2, \ldots, \):

\[
\begin{align*}
\mathbf{u}_j &:= \arg\max_{\|\mathbf{u}\|_2=1} \mathbf{u}^* \mathbf{A}_j \mathbf{A}_j^{\dagger} \mathbf{u} = \arg\max_{\|\mathbf{u}\|_2=1} \mathbf{u}^* P(\mathbf{u}_{j-1} \mathbf{v}_{j-1}^*) P(\mathbf{u}_{j-1} \mathbf{v}_{j-1}^*)^* \mathbf{u}, \\
\mathbf{v}_j &:= \arg\max_{\|\mathbf{v}\|_2=1} \mathbf{v}^* \mathbf{A}_j^* \mathbf{A}_j \mathbf{v} = \arg\max_{\|\mathbf{v}\|_2=1} \mathbf{v}^* P(\mathbf{u}_{j-1} \mathbf{v}_{j-1}^*)^* P(\mathbf{u}_{j-1} \mathbf{v}_{j-1}^*) \mathbf{v}, \\
\sigma_j^2 &:= \max_{\|\mathbf{v}\|_2=1} \mathbf{v}^* \mathbf{A}_j^* \mathbf{A}_j \mathbf{v} = \max_{\|\mathbf{v}\|_2=1} \mathbf{v}^* \left( \sigma_{j-1}^2 P(\mathbf{u}_{j-1} \mathbf{v}_{j-1}^*)^* P(\mathbf{u}_{j-1} \mathbf{v}_{j-1}^*) \right) \mathbf{v}.
\end{align*}
\]

These iterations can also be written jointly as

\[
(\mathbf{u}_j, \mathbf{v}_j) = \arg\max_{\|\mathbf{u}\|_2=1, \|\mathbf{v}\|_2=1} \|\mathbf{u}^* P(\mathbf{u}_{j-1} \mathbf{v}_{j-1}^*) \mathbf{v}\|, \quad \sigma_j = \sigma_{j-1} \max_{\|\mathbf{u}\|_2=1, \|\mathbf{v}\|_2=1} \|\mathbf{u}^* P(\mathbf{u}_{j-1} \mathbf{v}_{j-1}^*) \mathbf{v}\|.
\]

Observe that \( |\mathbf{u}_j^* P(\mathbf{u}_{j-1} \mathbf{v}_{j-1}^*) \mathbf{v}_j| = |\mathbf{u}_j^* P(\mathbf{u}_{j-1} \mathbf{v}_{j-1}^*) \mathbf{v}_j| > 0\), since \( \mathbf{u}_j \) and \( \mathbf{v}_j \) are the left and right singular vectors of \( P(\mathbf{u}_{j-1} \mathbf{v}_{j-1}^*) \) to its largest singular value \( \frac{\sigma_j}{\sigma_{j-1}} \). We can now prove convergence of the Cadzow algorithm [5.1].

**Theorem 5.4** (Convergence of Cadzow's algorithm for rank-1 Hankel approximation). Let \( \mathbf{A} \in \mathbb{C}^{M \times N} \) with \( 2 \leq M \leq N \) and \( \text{rank}(\mathbf{A}) \geq 1 \). Then the sequences \( (\mathbf{u}_j)_{j=0}^{\infty}, (\mathbf{v}_j)_{j=0}^{\infty} \) and \( (\sigma_j)_{j=0}^{\infty} \) in the Cadzow iterations (5.3) – (5.5) converge.

If \( \sigma = \lim_{j \to \infty} \sigma_j > 0 \), then the Cadzow algorithm [5.1] provides an approximation \( \mathbf{A} \mathbf{u} \mathbf{v}^* \) of Hankel structure, i.e., there exists \( z \in \mathbb{C} \) such that

\[
\mathbf{u} := \lim_{j \to \infty} \mathbf{u}_j = \frac{1}{\|\mathbf{z}_M\|_2} \mathbf{z}_M \quad \text{and} \quad \mathbf{v} := \lim_{j \to \infty} \mathbf{v}_j = \frac{1}{\|\mathbf{z}_N\|_2} \mathbf{z}_N
\]

with \( \mathbf{z}_M \) and \( \mathbf{z}_N \) as in (2.2), or \( \mathbf{u} = \mathbf{e}_M, \mathbf{v} = \mathbf{e}_N \).

If \( \sigma = \lim_{j \to \infty} \sigma_j = 0 \), then Cadzow’s algorithm converges to the zero matrix, while the matrix \( \mathbf{u} \mathbf{v}^* \) generated by \( \mathbf{u} := \lim_{j \to \infty} \mathbf{u}_j \) and \( \mathbf{v} := \lim_{j \to \infty} \mathbf{v}_j \) does not have Hankel structure.

**Proof.** 1. If the normalized singular vectors \( \mathbf{u}_0 \) and \( \mathbf{v}_0 \) of \( \mathbf{A} \) to the largest singular value \( \sigma_0 \) are already of the form \( \mathbf{u}_0 = \frac{1}{\|\mathbf{z}_M\|_2} \mathbf{z}_M, \mathbf{v}_0 = \frac{1}{\|\mathbf{z}_N\|_2} \mathbf{z}_N \) with \( \mathbf{z}_M \) and \( \mathbf{z}_N \) as in (2.2), or \( \mathbf{u}_0 = \mathbf{e}_M, \mathbf{v}_0 = \mathbf{e}_N \), then the optimal rank-1 approximation of \( \mathbf{A} \) has already Hankel structure, i.e., by definition of \( P \), we have

\[
P(\sigma_0 \mathbf{u}_0 \mathbf{v}_0^*) = \sigma_0 \mathbf{u}_0 \mathbf{v}_0^*.
\]

and the algorithm immediately stops, since we find constant sequences \( (\mathbf{u}_j)_{j=0}^{\infty}, (\mathbf{v}_j)_{j=0}^{\infty} \) and \( (\sigma_j)_{j=0}^{\infty} \)
2. Assume now that \( u_0 \) and \( v_0 \) neither satisfy the condition \( u_0 = \frac{1}{\|z_N\|} z_M \), \( v_0 = \frac{1}{\|z_N\|} z_N \) for some \( z \in \mathbb{C} \), nor \( u_0 = e_M, v_0 = e_N \). Then, by Lemma 5.2, we find for the largest singular value of \( \tilde{A}_1 = P(\sigma_0 u_0 v_0^*) \)

\[
\sigma_1 = \|P(\sigma_0 u_0 v_0^*)\|_2 < \sigma_0 \|u_0 v_0^*\|_2 = \sigma_0.
\]

For any \( j \geq 1 \) we obtain analogously

\[
\sigma_{j+1} = \|P(\sigma_j u_j v_j^*)\|_2 < \sigma_j \|u_j v_j^*\|_2 = \sigma_j,
\]

and this inequality is indeed strict as long as \( u_j \) and \( v_j \) do not have the wanted structure \( u_j = \frac{1}{\|z_M\|} z_M, v_j = \frac{1}{\|z_N\|} z_N \) for some \( z \in \mathbb{C} \) or \( u_j = e_M, v_j = e_N \). Thus, the sequence of singular values \( \sigma_j \) decreases monotonically. Since \( \sigma_j \geq 0 \) for all \( j \), convergence follows, and we write \( \sigma := \lim_{j \to \infty} \sigma_j \).

3. Since the matrix \( D \) given in Lemma 5.3 is positive definite, so is \( D^{-1} \). Therefore \( D^{-1} \) induces a scalar product and a corresponding norm,

\[
\langle a, b \rangle_{D^{-1}} := a^* D^{-1} b, \quad \|a\|_{D^{-1}}^2 := \langle a, a \rangle_{D^{-1}}, \quad a, b \in \mathbb{C}^{M+N-1}.
\]

Then it follows with Lemma 5.3 that

\[
0 \leq \|(u_j \star v_j) - (u_{j-1} \star v_{j-1})\|_{D^{-1}}^2
= \|(u_j \star v_j)\|_{D^{-1}}^2 + \|(u_{j-1} \star v_{j-1})\|_{D^{-1}}^2 - 2 \Re \left( (u_j \star v_j)^* D^{-1} (u_{j-1} \star v_{j-1}) \right)
= u_j^* P(u_j v_j^*) v_j + u_{j-1}^* P(u_{j-1} v_{j-1}^*) v_{j-1} - 2 \Re \left( u_j^* P(u_{j-1} v_{j-1}^*) v_j \right).
\]

But from definition (5.7) it follows that \( u_j^* P(u_{j-1} v_{j-1}^*) v_j = \frac{1}{\sigma_j} \frac{1}{\sigma_{j-1}} \) is real. Further, for all \( j > 0 \) we conclude from (5.6) that

\[
u_{j-1}^* P(u_{j-1} v_{j-1}^*) v_{j-1} \leq \max_{\|u\|_2 = 1, \|v\|_2 = 1} |u^* P(u_{j-1} v_{j-1}^*) v| = u_j^* P(u_{j-1} v_{j-1}^*) v_j.
\]

Together with (5.7), we obtain

\[
u_j^* P(u_j v_j^*) v_j \geq u_{j-1}^* P(u_{j-1} v_{j-1}^*) v_{j-1}.
\]

In other words, the sequence \( \| (u_j \star v_j) \|_{D^{-1}}^2 \) is monotonically increasing. At the same time \( \| (u_j \star v_j) \|_{D^{-1}}^2 = u_j^* P(u_j v_j^*) v_j \) is bounded by 1, since \( \|P(u_j v_j^*)\|_2 \leq \|u_j\|_2 \|v_j\|_2 = 1 \) by Lemma 5.2. Therefore we conclude that the sequence \( (u_j \star v_j) \) converges to a limit vector \( u \star v \in \mathbb{C}^{M+N-1} \). This limit vector defines

\[
P(u v^*) = \left( (D^{-1}(u \star v))^M \right)_{k=0, \ell=0}^{M-1, N-1}
\]

and we obtain the two vectors \( u \) and \( v \) via (5.3) and (5.4), i.e., we find \( u = \lim_{j \to \infty} u_j \) as the normalized singular vector of \( P(\sigma_0 u_0 v_0^*) \) to its largest singular value, and \( v = \lim_{j \to \infty} v_j \) as the normalized singular vector of \( P(\sigma_0 u_0 v_0^*) \) to its largest singular value.

4. We now distinguish two cases.
First, assume that \( u^* P(u v^*) v = \lim_{j \to \infty} (u_j^* P(u_j v_j^*) v_j) = 1. \) In particular, it follows that \( \|P(u v^*)\|_2 = 1 = \|uv\|_2 \), and by Lemma 5.2 this directly implies that
\[ u = \lim_{j \to \infty} u_j = \frac{1}{\|z_M\|^2} z_M \text{ and } v = \lim_{j \to \infty} v_j = \frac{1}{\|z_N\|^2} z_N, \text{ for some } z \in \mathbb{C}, \text{ or that} \\
u = e_M \text{ and } v = e_N. \text{ Thus, the Cadzow algorithm converges to a rank-1 matrix } c z_M z_N^T \]
with \( c = \frac{\sigma}{\|z_M\|^2 \|z_N\|^2} \), where the existence of \( \sigma \) had been shown in the second part of the proof. Obviously, we always have have \( \sigma \leq \sigma_0 \).

In the second case, assume that \( u_j^* P(u_j v_j^*) v = \lim_{j \to \infty} (u_j^* P(u_j v_j^*) v_j) = L < 1 \). Since \( (u_j^* P(u_j v_j^*) v_j)_{j=0}^\infty \) has been monotonically increasing, it follows for all \( j \)
\[ \sigma_{j+1} = \sigma_j u_j^* P(u_{j-1} v_{j-1}^*) v_j \leq \sigma_j L, \]
and thus \( \lim_{j \to \infty} \sigma_j \leq \sigma_0 \lim_{j \to \infty} L^j = 0 \). In this case, Cadzow’s algorithm converges to the zero matrix. Moreover, Lemma \[7\] implies that the limit vectors \( u \) and \( v \) cannot be of the wanted structure to generate a rank-1 Hankel matrix. \( \square \)

**Remark 5.5.** During the Cadzow iteration it may happen that the largest singular value of \( \tilde{A}_j \) is not unique, even if the original matrix \( A \) has a single largest singular value. This means that the obtained approximation is not necessarily unique.

**Example 5.6.** We show in a special example that Cadzow’s algorithm for rank-1 Hankel approximation may indeed converge to the zero matrix. We consider the matrix
\[ A := \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1/2 & 0 \\ 1/2 & 0 & 1 \end{pmatrix} \]
with eigenvalues \( \frac{3}{2}, \frac{1}{2}, \frac{1}{2} \). The singular vector to the largest singular value \( \frac{3}{2} \) is of the form
\[ u_0 = v_0 = \frac{1}{\sqrt{2}} (1, 0, 1)^T. \]
Thus we find
\[ P(u_0 v_0^*) = \frac{1}{2} \begin{pmatrix} 1 & 0 & 2/3 \\ 0 & 2/3 & 0 \\ 2/3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 & 1/3 \\ 0 & 1/3 & 0 \\ 1/3 & 0 & 1/2 \end{pmatrix}. \]
Now, \( u_1 = v_1 = \frac{1}{\sqrt{2}} (1, 0, 1)^T \) is the singular vector of \( P(u_0 v_0^*) \) to the largest singular value \( 5/6 \). Further iterations yield
\[ u_j = v_j = \frac{1}{\sqrt{2}} (1, 0, 1)^T, \quad \sigma_j = \frac{3}{2} \cdot \left( \frac{5}{6} \right)^j. \]
Obviously, \( (u_j)_{j=0}^\infty \) and \( (v_j)_{j=0}^\infty \) are constant sequences with limit vector \( \frac{1}{\sqrt{2}} (1, 0, 1)^T \), and \( \lim_{j \to \infty} \sigma_j = 0 \). In other words, the Cadzow algorithm fails to converge to a rank-1 matrix.

For comparison, Theorem \[3.1\] provides the optimal rank-1 Hankel approximation with regard to the Frobenius norm \( \| z \tilde{z} z^T \| \) with
\[ \tilde{z} = \arg \max_{z \in \mathbb{C}} \left( \frac{z^T A z}{z^T z} \right)^2 = \arg \max_{z \in \mathbb{C}} \frac{1 + \frac{3}{2} z^2 + z^4}{1 + z^2 + z^4} = \arg \max_{z \in \mathbb{C}} \frac{z^2}{1 + z^2 + z^4}. \]
We obtain the two solutions \( \tilde{z} = 1 \) and \( \tilde{z} = -1 \). For both, \( z = 1 \) and \( z = -1 \), we find \( \tilde{c} = \frac{A \tilde{z}}{(\tilde{z}^T \tilde{z})^2} = \frac{7}{18} \). Thus, we get indeed two optimal solutions, namely
\[ \frac{7}{18} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \frac{7}{18} \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}. \]
Both solutions possess the error

\[ \| A - \tilde{c} \tilde{z} \tilde{z}^T \|_F = \left\| \frac{1}{18} \begin{pmatrix} 11 & \pm 7 & 2 \\ 2 & \pm 7 & 11 \end{pmatrix} \right\|_F = \frac{\sqrt{130}}{18} = 1.178511. \]

The spectral norm for this error matrix is \( \| A - \tilde{c} \tilde{z} \tilde{z}^T \|_2 = 1.045820. \)

Finally, let us consider the optimal rank-1 Hankel approximation of \( A \) with respect to the spectral norm. We observe that the eigenvectors of \( A \) corresponding to \( \frac{3}{2}, \frac{1}{2}, \frac{1}{2} \) are

\[ v_0 = \frac{1}{\sqrt{2}} (1, 0, 1)^T, \quad v_1 = (0, 1, 0)^T, \quad v_2 = \frac{1}{\sqrt{2}} (1, 0, -1)^T. \]

The optimal error \( \tilde{\lambda} \) is in the interval \([\frac{1}{2}, \frac{3}{2}]\). Since \( v_1(z) = v_1^T \tilde{z} \) and \( v_2(z) = v_2^T \tilde{z} \) have no common zeros, we obtain \( \Sigma = \emptyset \) in Algorithm 4.6. We need to find \( \tilde{\lambda}^2 \) and \( \tilde{z} \), such that \( f(\tilde{z}, \tilde{\lambda}^2) \) satisfies (4.6), i.e., \( \max_{z \in \mathbb{R}} f_{\lambda}(z) = 0 \) and \( \tilde{z} \in \text{argmax}_{z \in \mathbb{R}} f_{\lambda}(z) \). We obtain

\[ \| \tilde{z} \|_2^2 f_{\lambda}(z) = \frac{(v_0^T \tilde{z})^2}{2 - \lambda^2} + \frac{(v_1^T \tilde{z})^2}{2 - \lambda^2} + \frac{(v_2^T \tilde{z})^2}{2 - \lambda^2} = \frac{1}{2} + z^2 + \frac{z^4}{2 - \lambda^2} + \frac{z^2}{2 - \lambda^2} = \frac{(\lambda^2 - \frac{5}{4})}{2 (\frac{3}{2} - \lambda^2)} \left( (\frac{5}{4} - \frac{\lambda^2}{2}) z^4 + (\frac{1}{4} - \frac{\lambda^2}{2}) z^2 + (\frac{5}{4} - \frac{\lambda^2}{2}) \right), \]

where we assume in the last line that \( \lambda^2 \neq \frac{5}{4}. \) A direct inspection of \( f_{\lambda}(z) \) provides that \( \max_z f_{\lambda}(z) = 0 \) if and only if

\[ 1 - \left( \frac{\lambda^2 - \frac{1}{4}}{2 (\frac{3}{2} - \lambda^2)} \right)^2 = 0, \]

i.e., if \( \tilde{\lambda}^2 = \frac{11}{12}. \) We thus obtain from (4.6) and (4.7)

\[ \tilde{z}^2 = \frac{\tilde{\lambda}^2 - \frac{1}{4}}{2 (\frac{3}{2} - \lambda^2)} = \frac{\frac{11}{12} - \frac{1}{4}}{2 (\frac{3}{2} - \frac{11}{12})} = 1, \quad \tilde{c} = \left( \frac{2}{\frac{3}{2} - \sqrt{\frac{11}{12}}} + \frac{1}{\frac{1}{2} - \sqrt{\frac{11}{12}}} + 0 \right)^{-1} = \frac{2}{3}, \]

and therefore again the two solutions \( \tilde{z} = 1 \) and \( \tilde{z} = -1. \) For the obtained error matrix we have

\[ \left\| A - \frac{2}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \right\|_2 = \left\| \frac{1}{6} \begin{pmatrix} 2 & \pm 4 & -1 \\ \pm 4 & -1 & \pm 4 \\ -1 & \pm 4 & 2 \end{pmatrix} \right\|_2 = \sqrt{\frac{11}{12}} = 0.957427, \]

while for the Frobenius norm we get \( \| A - \tilde{c} \tilde{z} \tilde{z}^T \|_F = 1.443376. \) By construction, the error matrix \( A - \tilde{c} \tilde{z} \tilde{z}^T \) possesses the eigenvalues \( \sqrt{\frac{11}{12}}, -\sqrt{\frac{11}{12}}, \) and \( \frac{1}{2}. \)
Example 5.7. Finally, we consider Example 5 in [10]. Given the matrix

\[ \mathbf{A}_a = \begin{pmatrix} a & 1 & a & 1 & a \\ 1 & a & 1 & a & 1 \end{pmatrix}^T, \]

we obtain for \( a = 0 \) with the Cadzow algorithm and with the optimal Frobenius approximation in Section 3, respectively,

\[ \mathbf{H}_{\text{Cadzow}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}^T, \]

\[ \mathbf{H}_{\text{Frob}} = \begin{pmatrix} 0.467001 & 0.488054 & 0.510057 & 0.533051 & 0.557083 \\ 0.467001 & 0.488054 & 0.510057 & 0.533051 & 0.557083 \end{pmatrix}^T. \]

For the Frobenius norm, we find two optimal solutions, \((\hat{z}, \hat{c}) = (1.045082, 0.446855)\), and \((\hat{z}, \hat{c}) = (-1.045082, -0.446855)\), producing the same optimal error. We obtain

\[ \| \mathbf{A} - \hat{c} \hat{z} \hat{z}^T \|_F = 1.577594, \quad \| \mathbf{A} - \mathbf{H}_{\text{Cadzow}} \|_F = 2. \]

Note that the two algorithms HSVD and HTLS studied for comparison in [10], completely fail in this case.

For \( a = 2 \) we get

\[ \mathbf{H}_{\text{Cadzow}} = \begin{pmatrix} 1.5629 & 1.5369 & 1.5113 & 1.4861 & 1.4614 \\ 1.5369 & 1.5113 & 1.4861 & 1.4614 & 1.4370 \end{pmatrix}^T, \]

\[ \mathbf{H}_{\text{Frob}} = \begin{pmatrix} 1.556291 & 1.533373 & 1.510793 & 1.488545 & 1.466626 \\ 1.533373 & 1.510793 & 1.488545 & 1.466626 & 1.445028 \end{pmatrix}^T. \]

For the Frobenius norm, we have the solution parameters \((\hat{z}, \hat{c}) = (0.985274, 1.556291)\). We obtain

\[ \| \mathbf{A} - \hat{c} \hat{z} \hat{z}^T \|_F = 1.577618, \quad \| \mathbf{A} - \mathbf{H}_{\text{Cadzow}} \|_F = 1.577681. \]

While for \( a = 0 \), Cadzow’s algorithms provides a solution error which is significantly larger than the optimal error, we get for \( a = 2 \) an error which is almost optimal.

Conclusion and Outlook

In Section 2 we showed that a rank-1 Hankel matrix \( \mathbf{H}_1 \) is always of the form \( \mathbf{H}_1 = c \mathbf{z}_M \mathbf{z}_N^T \) or \( \mathbf{H}_1 = c \mathbf{e}_M \mathbf{e}_N^T \) with \( \mathbf{z}_N \) and \( \mathbf{e}_N \) defined in (2.2) and (2.1). This observation enabled us to analytically solve

\[
\min_{\mathbf{H}_1 \in \mathbb{C}^{M \times N}} \| \mathbf{A} - \mathbf{H}_1 \|_F^2 \quad \text{and} \quad \min_{\mathbf{H}_1 \in \mathbb{C}^{M \times N}} \| \mathbf{A} - \mathbf{H}_1 \|_2^2.
\]

In the case of the Frobenius norm our results apply to general matrices \( \mathbf{A} \in \mathbb{C}^{M \times N} \). For the spectral norm we considered real symmetric matrices. Our theoretical results gave rise to algorithms to compute the optimal rank-1 Hankel approximations for the Frobenius and spectral norm. In particular, the optimal solutions for the two norms usually differ. This is in contrast to well-known results for unstructured optimal low-rank approximations.
We showed that the well-known Cadzow algorithm applied for rank-1 Hankel approximation always converges to a fixed point. However, it can happen that the algorithm converges to the zero matrix. Even if Cadzow’s method converges to a rank-1 Hankel matrix it usually does not converge to the optimal solution, neither with respect to the Frobenius norm nor with respect to the spectral norm. We conjecture that the fixed point reached by the Cadzow algorithm coincides with the optimal rank-1 Hankel approximation with respect to the Frobenius or spectral norm only in the trivial case, if the unstructured rank-1 approximation obtained by the singular value decomposition already has the wanted Hankel structure. In this case, Cadzow’s algorithm stops already after one iteration step.

A natural extension of our results would be to ask for analytic solutions to the approximation problem for Hankel matrices with rank $r > 1$. However, due to an increasing number of special cases regarding the structure of higher-rank Hankel matrices, this problem is much more difficult to solve. For the Frobenius norm, we will consider applying our algorithm iteratively in order to get a Hankel approximation of higher rank and study the obtained results in comparison to other numerical methods for low-rank Hankel approximation.

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