MODELS FOR THE IRREDUCIBLE REPRESENTATION OF A HEISENBERG GROUP

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Abstract. In its most general formulation a quantum kinematical system is described by a Heisenberg group; the "configuration space" in this case corresponds to a maximal isotropic subgroup. We study irreducible models for Heisenberg groups based on compact maximal isotropic subgroups. It is shown that if the Heisenberg group is 2-regular, but the subgroup is not, the "vacuum sector" of the irreducible representation exhibits a fermionic structure. This will be the case, for instance, in a quantum mechanical model based on the 2-adic numbers with a suitably chosen isotropic subgroup.

The formulation in terms of Heisenberg groups allows a uniform treatment of \( p \)-adic quantum systems for all primes \( p \), and includes the possibility of treating adelic systems.

1. Introduction

In Hermann Weyl’s formulation of quantum kinematics\(^1\), a system with \( d \) degrees of freedom is described by operators \( U(a) \) and \( V(b) \) on \( L_2(\mathbb{R}^d) \), defined by

\[
(U(a)f)(x) = e^{ia \cdot x} f(x), \quad (V(b)f)(x) = f(x + b), \quad f \in L_2(\mathbb{R}^d)
\]

where 
\[
a = (a_1, \ldots, a_d), \quad b = (b_1, \ldots, b_d) \in \mathbb{R}^d
\]

and

\[
x \cdot y = x_1 y_1 + \cdots + x_d y_d.
\]

If we set

\[
W(a,b) = e^{(i/2)a \cdot b} U(a) V(b) \quad (a, b) \in \mathbb{R}^d \oplus \mathbb{R}^d
\]

then \( (a, b) \mapsto W(a, b) \) is an irreducible projective unitary representation of \( \mathbb{R}^d \times \mathbb{R}^d \) with multiplier \( m \) given by

\[
m((a, b), (a', b')) = e^{(i/2)(a' \cdot b - a \cdot b')}
\]

i.e.,

\[
W(a,b)W(a', b') = e^{(i/2)(a' \cdot b - a \cdot b')} W(a + a', b + b')
\]

The original operators \( U(a) \) and \( V(b) \) are recovered from \( W(a, b) \) through \( U(a) = W(a, 0) \) and \( V(b) = W(0, b) \). Focussing on the exponential of the position operator, \( U(a) \), we notice that it is obtained by restricting \( W \) to the configuration space \( \mathbb{R}^d \times \{0\} \), which is a maximal isotropic subgroup of \( \mathbb{R}^d \times \mathbb{R}^d \) with respect to the symplectic multiplier \( m \).

All of the above makes sense in the context of a locally compact abelian group \( G \) with a symplectic multiplier \( m \), i.e., a Heisenberg group \((G, m)\). So, following Weyl, one can take the position that quantum kinematics, in its most general form, is described by an irreducible projective representation \( W \) of a Heisenberg group \((G, m)\).

In this general setting there is no obvious candidate for the "configuration space", but one can adopt the view that it corresponds to a choice of a maximal isotropic subgroup \( L \) for \( m \). One can then study the "position operator" by restricting the projective representation \( W \) to \( L \), thereby obtaining an ordinary representation \( U \) of \( L \).

\(^1\)All the concepts discussed in this section will be explained in the subsequent sections.
The objective of this article is to analyze the representation $U$ in the case where $m$ has a compact maximal isotropic subgroup. Of course, this situation never occurs in the conventional case $G = \mathbb{R}^d \times \mathbb{R}^d$, but it does occur, for instance, if we choose our phase space to be $G = \mathbb{Q}_p^d \times \mathbb{Q}_p^d$. In the generic case (i.e., both $G$ and $L$ are 2-regular), $U$ decomposes into one-dimensional subrepresentations. However, if $G$ is 2-regular, but $L$ is not, an interesting phenomenon occurs: When considered on the so-called vacuum space of $W$, the lift of $U$ to $L/2$ gives rise to a set of operators exhibiting a fermionic structure (Theorem 12).

The results of this article extend results obtained by Vladimirov, Volovich and Zelenov for the case $G = \mathbb{Q}_p^d \times \mathbb{Q}_p^d$ (see their book "$p$-adic Analysis and Mathematical Physics" [VVZ94, 244–247]). In particular, we obtain a version of their Theorem 3 on page 247 which is valid also for $p = 2$; in fact, this is a particularly interesting case, since this is where the phenomenon of a "fermionic structure" occurs. Our approach also brings out more clearly the mechanisms which are at work here.

The paper is organized as follows: In Section 2 we remind the reader of the basic facts concerning conventional quantum mechanics and Weyl systems. Section 3 contains a discussion on multipliers, bicharacters, isotropic subgroups and models for the unique irreducible representation of a Heisenberg group. Finally, in Section 4, we present and prove our main results.

2. Preliminaries

Quantum kinematics is based on the well known Heisenberg commutation rules (with $\hbar = 1$)

$$[p_j, q_k] = -i\delta_{jk} I \quad 1 \leq j, k \leq d \quad (H_d)$$

where $q_j, p_j$ ($1 \leq j \leq d$) are the position and momentum coordinates of a quantum system with $k$ degrees of freedom. Almost at the same time as these were discovered, Weyl noticed that they are the infinitesimal version of commutation rules between the unitary groups generated by the $q_j, p_k$; these are known as the Weyl commutation rules. Let

$$U(a) = e^{ia \cdot x}, \quad V(b) = e^{ib \cdot p}$$

where

$$a = (a_1, \ldots, a_d), \quad b = (b_1, \ldots, b_d)$$

and

$$x \cdot y = x_1y_1 + \cdots + x_dy_d$$

The Weyl commutation rules are then given by

$$U(a)V(b) = e^{-ia \cdot b}V(b)U(a) \quad a, b \in \mathbb{R} \quad (W_d)$$

A pair of unitary representations $U, V$ of $\mathbb{R}^d$ in a Hilbert space is called a Weyl system if they satisfy $(W_d)$. These two types of commutation rules are formally equivalent and so Weyl took the point of view that $(W_d)$ describes the kinematics of quantum systems whose configuration space is a real affine space. However the concept of a Weyl system is much deeper than being just an equivalent way to formulate quantum kinematics as envisioned by Heisenberg. The point is, as Weyl himself discovered, that although it was originally defined for $\mathbb{R}^d$, the notion of a Weyl system can be formulated in much greater generality. In the first place, the Weyl commutation rules involve just $\mathbb{R}^d$ and the duality of $\mathbb{R}^d$ with itself given by the pairing

$$a, b \mapsto e^{ia \cdot b}$$

and so one can speak of a Weyl system whenever one has a pair of unitary representations $U, V$ of abelian groups $A, B$ with a nondegenerate pairing

$$\langle \cdot, \cdot \rangle : A \times B \to T \quad (T = \{ z \in \mathbb{C} \mid |z| = 1 \})$$

satisfying

$$U(a)V(b) = \langle a, b \rangle^{-1}V(b)U(a) \quad a \in A, b \in B$$

One can then identify $B$ with a subgroup of the dual group $\hat{A}$ of $A$ by means of this pairing and then interpret the pair $(U, V)$ as describing the quantum kinematics of a system whose configuration

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2Although not stated explicitly in [VVZ94], Theorem 3 on page 247 requires $p \neq 2$. 
space is the Abelian group $A$, whose momenta lie in $B$, and which are covariant under the group of translations of $A$. Weyl treated only the cases when $A$ and $B$ are either finite or finite dimensional Lie groups; both of these are included in the above scheme if we stipulate that $A$ and $B$ are to be locally compact, and the resulting theory is completely adequate for the treatment of quantum systems which are finite dimensional; for treating quantum systems with infinitely many degrees of freedom, such as a quantum field, one has to give up the local compactness of $A$ and $B$, and this introduces many technical complications. We will not discuss this case here.

Weyl’s generalization already allows us to describe quantum systems whose configuration space can be very arbitrary, in particular if it is an affine space over a nonarchimedean local field $K$. We can take $A$ to be a finite dimensional vector space $V$ over such a field, $B$ to be its (linear) dual $V^*$, and $(\ ,\ )$ to be the pairing

$$
(a, b) = \chi((a, b))
$$

where $(\ ,\ )$ is the natural $K$-valued pairing of $V$ and $V^*$, and $\chi$ is a basic character of $K$; and then work with Weyl systems for the pair $V, V^*$. This is our basic point of view in this article.

Weyl carried out a second generalization of the concept of a Weyl system. To motivate this we observe that the definition $(W_d)$ emphasizes the configuration space and its dual, and so is really associated to a particular splitting of the phase space. To get a more invariant description one should try to formulate the concept of a Weyl system directly on the phase space. Weyl noticed that the map

$$
(a, b) \mapsto W(a, b) = e^{i\beta(a, b)}U(a)V(b) \quad (a, b) \in P = R^d \oplus R^d
$$

is a projective unitary representation of the phase space $P$, with the property that the phase factors which determine the departure of $W$ from being an ordinary representation, encode the symplectic structure of the phase space in the sense that they are of the form

$$
e^{i\beta(x, y)}$$

where $\beta$ is the natural symplectic form on the phase space. He was able to show that an affine space admits a faithful irreducible projective representation only when it is symplectic, and further that in this case the phase factors of the projective representation are of the above form and hence determined by the symplectic structure. He was thus led to his final and decisive formulation of quantum kinematics as described by a projective unitary representation of an abelian group with a symplectic structure.

Abelian groups with a symplectic structure are called Heisenberg groups. Our point of view can thus be described as follows. Quantum kinematics for systems with finitely many degrees of freedom is described by a projective unitary representation that is canonically associated to a locally compact Heisenberg group. By specializing the Heisenberg group to one defined over a local field or an adele ring we can then retrieve the cases that are of greater interest, namely quantum systems over local fields and adele rings. For Weyl’s work, see [Wey31].

2.1. The Schrödinger model for $(W_d)$ and its uniqueness. The Schrödinger model for $(W_d)$ consists of the Hilbert space $H = L^2(R^d)$ with

$$
(W(a, b)f)(t) = e^{i(t/2)a \cdot b}e^{ia \cdot t}f(t + b)
$$

(Sch)

which is the same as the requirement

$$
(U(a)f)(t) = e^{ia \cdot t}f(t), \quad (V(b)f)(t) = f(t + b)
$$

The system of operators $\{W(a, b)\}$ is irreducible. The central question of quantum kinematics is whether the commutation rules $(W_d)$ can be satisfied only by the Schrödinger model, i.e., whether up to irreducibility and unitary equivalence the Schrödinger model is the unique solution of $(W_d)$. The Heisenberg–Weyl commutation rules clearly form the content of matrix mechanics while the Schrödinger model formulates quantum kinematics in the form that is known as wave mechanics. Weyl, who was the first to formulate the uniqueness question as the uniqueness of the pair $(U, V)$ or $W$, clearly understood that its affirmative solution is what is needed to show the equivalence of

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3See Section 3.2 for an explanation of the factor $e^{i(t/2)a \cdot b}$. 
wave and matrix mechanics. He could not prove the uniqueness except in the special case when \( a \) and \( b \) vary in a finite cyclic group; it was proved a bit later by Stone and Von Neumann for \( \mathbb{R}^d \).

**Theorem 1** (Stone–Von Neumann). Any Weyl system for \( \mathbb{R}^d \oplus \mathbb{R}^d \) is a direct sum of copies of the Schrödinger model.

For Von Neumann’s original proof see [vN31, Sto30]. For a more recent treatment see [Var96].

3. **Projective representations**

We give here a brief discussion of projective representations, multipliers, and Heisenberg groups.

Let \( G \) be a separable locally compact group and \( T \) the multiplicative group of complex numbers of absolute value 1. An \( m \)-representation of \( G \) is a (Borel) map of \( G \) into the unitary group of a Hilbert space such that for some (necessarily Borel) map \( m \), called the multiplier of \( U \), of \( G \times G \) with values in \( T \),

\[
U(x)U(y) = m(x, y)U(xy), \quad U(1) = 1
\]

If we change \( U \) to \( U' = aU \), where \( a \) is a Borel map of \( G \) into \( T \), then \( U' \) is an \( m' \)-representation where

\[
m'(x, y) = m(x, y)\frac{a(x)a(y)}{a(xy)} \quad (*)
\]

The multiplier itself satisfies

\[
m(x, 1) = m(1, x) = 1 \quad (1)
m(xy, z)m(x, y) = m(x, yz)m(y, z) \quad (2)
\]

Any Borel function \( m \) on \( G \times G \) with values in \( T \) satisfying the above relations is called a multiplier for \( G \); two multipliers \( m, m' \) related as in \( (*) \) are called equivalent. If

\[
m(x, y) = \frac{a(x)a(y)}{a(xy)}
\]

\( m \) is called trivial; in this case, if \( U \) is an \( m \)-representation, \( aU \) becomes an ordinary representation. Under pointwise multiplication, the multipliers form a group, the trivial ones form a subgroup, and the quotient group of equivalence classes of multipliers is called the multiplier group (of \( G \)). It is denoted by \( M(G) \).

3.1. **Projective representations viewed as ordinary representations of central extensions.** In the above definitions we have considered no continuity assumptions on multipliers. In most situations the multipliers are at least locally continuous near \((1, 1)\), and for the case we are interested in, namely for Weyl systems and Heisenberg groups, even continuous. For simplicity we assume here that \( m \) is a continuous\(^4\) multiplier. Then

\[
G_m = G \times T
\]

becomes a separable locally compact group (in the product topology) with the multiplication

\[
(x, s)(y, t) = (xy, st \cdot m(x, y))
\]

For an \( m \)-representation \( U \), the map

\[
U_m : (x, t) \mapsto tU(x)
\]

is an ordinary representation of \( G_m \) with \( U_m(1, t) = t1 \). We call any representation of \( G_m \) that restricts to \( t \mapsto t1 \) on the group \( 1 \times T \), a basic representation. So \( U_m \) is a basic representation of \( G_m \), and

\[
U \leftrightarrow U_m
\]

\(^4\) This construction can be carried out also when \( m \) is only assumed to be Borel. It leads to the Weil topology, the unique locally compact group topology on \( G_m \) which generates the product Borel structure on \( G \times T \) (see [Var85] for details of this theory which goes back to [Mac58]).
is a bijective correspondence between \( m \)-representations of \( G \) and basic ordinary representations of \( G_m \). The subgroup \( 1 \times T \) (which we identify with \( T \)) lies in the center of \( G_m \) and \( G_m \) is a central extension of \( G \) by \( T \) described by the exact sequence

\[
1 \to T \to G_m \to G \to 1
\]

3.2. Alternating bicharacters as multipliers. From now on we shall work with a separable locally compact Abelian group \( G \) written additively. We shall take a closer look at the structure of the group of multipliers of \( G \). First we consider bicharacters. A bicharacter of \( G \) is a continuous multiplier. Let \( m \) be any multiplier, continuous or not, and let us set

\[
m^\sim(x, y) = \frac{m(x, y)}{m(y, x)}
\]

Then

\[
m^\sim(x, y)m^\sim(y, x) = 1, \quad m^\sim(x, x) = 1
\]

and it is easily checked that \( m^\sim \) is a homomorphism in each argument; in fact,

\[
m^\sim(x, y + z) = \frac{m(x, y + z)}{m(y + z, x)} = \frac{m(x + y, z)m(x, y)}{m(y, z)m(y + z, x)} = \frac{m(x + y, z)m(x, y)}{m(y, z + x)m(z, x)} = \frac{m(x, y)m(y + x, z)m(x, y)}{m(y, x)m(y, z + x)m(z, x)} = m^\sim(x, y)m^\sim(x, z)
\]

while the homomorphism property in the first argument follows from the antisymmetry. So, as \( m^\sim \) is obviously Borel, it is an alternating bicharacter. The bicharacter \( m^\sim \) arises naturally because of the fact that for any \( m \)-representation \( U \) we have the commutation rule

\[
U(x)U(y)U(x)^{-1}U(y)^{-1} = m^\sim(x, y)1
\]

The following definition will be needed only for the case \( p = 2 \), but we state it for a general (prime number) \( p \):

Definition 1. Let \( p \) be a prime number. An Abelian group \( G \) is said to be \( p \)-divisible (resp. \( p \)-injective) if the map

\[
x \in G \mapsto px \in G
\]

is surjective (resp. injective); if the map is bijective, we say that \( G \) is \( p \)-regular.

As an easy verification shows, a locally compact Abelian group \( G \) is \( p \)-divisible (resp. \( p \)-injective) if and only if its dual group \( \hat{G} \) is \( p \)-injective (resp. \( p \)-divisible). We write \( x \mapsto x/p \) for the inverse map of \( x \mapsto px \) (when it exists); more generally, if \( A \) is a subset of \( G \), we write \( A/p \) or \((1/p)A \) for the inverse image of \( A \) under the map \( x \mapsto px \) of \( G \) into \( G \).
Lemma 2. If $G$ is 2-regular and $\beta$ is a bicharacter of $G$, there is a unique bicharacter $\beta^{1/2}$ such that $(\beta^{1/2})^2 = \beta$.

Proof. Define 

$$\beta^{1/2}(a, b) = \beta(a/2, b/2)^2 \quad a, b \in G$$

Then $\beta^{1/2}$ is a bicharacter and $(\beta^{1/2})^2 = \beta$.

Uniqueness. Assume $\beta_1, \beta_2$ are two bicharacters such that $\beta_1^2 = \beta_2^2$, and set $\gamma = \beta_1/\beta_2$. Then $\gamma^2 = 1$, so $\gamma(x, y) = \pm 1$ for all $x, y \in G$. But since $G$ is 2-regular, $\gamma(x, y) = \gamma(x/2, y)^2 = 1$ for all $x, y$. Thus $\beta_1 = \beta_2$, and uniqueness follows. □

Let $\Lambda^2(G)$ be the multiplicative group of alternating bicharacters. Then the map

$$m \mapsto m^\sim$$

is a homomorphism of the group of multipliers of $G$ into the group $\Lambda^2(G)$. We now have the following important lemma:

Lemma 3.

1. If $m_i$ ($i = 1, 2$) are multipliers, then

$$m_1 \simeq m_2 \iff m_1^\sim = m_2^\sim$$

In particular, a multiplier $m$ is trivial if and only if $m^\sim = 1$, i.e., $m$ is symmetric. If $m$ is in addition continuous, we can find a continuous map $a$ of $G$ into $T$ such that

$$m(x, y) = \frac{a(x + y)}{a(x)a(y)}$$

Indeed, any Borel map $a : G \to T$ such that $m(x, y) = a(x + y)/a(x)a(y)$ is automatically continuous.

2. The map

$$m \mapsto m^\sim$$

is a homomorphism whose kernel is the group of trivial multipliers, and so induces an injection of $M(G)$ into $\Lambda^2(G)$.

3. If $G$ is 2-regular, the map (2) induces an isomorphism between $M(G)$ and $\Lambda^2(G)$. In this case, if $m \in M(G)$ with image $n \in \Lambda^2(G)$, then $n^{1/2}$ is the unique alternating bicharacter in the multiplier class of $m$.

Proof. Indeed, if $m$ is trivial, it is symmetric and $m^\sim = 1$. Conversely, if $m^\sim = 1$, then $m$ is symmetric and so $G_m$ is Abelian. But then the dual of the exact sequence

$$1 \to T \to G_m \to G \to 1$$

is the exact sequence

$$1 \to \hat{G} \to \hat{G}_m \to \hat{Z} \to 1$$

As $\hat{Z}$ is free, this exact sequence splits, and so the first exact sequence also splits, showing that $T$ is a direct summand of $G_m$. Hence there is a continuous homomorphism

$$x \mapsto (x, a(x))$$

of $G$ into $G_m$. Writing out the condition that this is a homomorphism we get

$$m(x, y) = \frac{a(x + y)}{a(x)a(y)}$$

In general $a$ is not continuous as $G_m$ need not have the product topology. But if $m$ is continuous, $G_m$ has the product topology, and so $a$ is continuous. Since any two choices of $a$ related as above to $m$ differ by a character, it follows that any $a$ such that $m(x, y) = a(x + y)/a(x)a(y)$ is necessarily continuous. If $m_1^\sim = m_2^\sim$, then $(m_1/m_2)^\sim = 1$ so that $m_1/m_2 \simeq 1$, or $m_1 \simeq m_2$. This proves (1) and (2).

To prove (3) note that $n^{1/2}$ is alternating and $(n^{1/2})^\sim = (n^{1/2})^2 = n$. The rest follows from Lemma 2.
Example 1. In Section 2 we defined the Weyl map
\[
(a, b) \mapsto W(a, b) = e^{(i/2)a \cdot b} U(a)V(b) \quad (a, b) \in \mathbb{P} = \mathbb{R}^d \oplus \mathbb{R}^d
\]
which is a projective representation with multiplier
\[
m((a, b), (a', b')) = e^{(i/2)(a' \cdot b - a \cdot b')}
\]
A more natural definition would perhaps have been
\[
(a, b) \mapsto W'(a, b) = U(a)V(b) \quad (a, b) \in \mathbb{P} = \mathbb{R}^d \oplus \mathbb{R}^d
\]
which is a projective representation with multiplier
\[
m'((a, b), (a', b')) = e^{a' \cdot b}
\]
However, \(m'\) is neither alternating nor nondegenerate, but it is equivalent to \(m\), which has both of those properties (i.e., \(m\) is symplectic). The function \(c(a, b) = e^{(i/2)a \cdot b}\) implements the equivalence between \(m\) and \(m'\), and \(m\) is the unique alternating bicharacter in the multiplier class of \(m'\), the existence of which was guaranteed by part (3) of the lemma above.

Example 2. Similarly, if \(A\) and \(B\) are two Abelian groups with a nondegenerate pairing
\[
\langle \ , \ \rangle : A \times B \rightarrow T \quad (T = \{ z \in \mathbb{C} \mid |z| = 1 \})
\]
a Weyl system for \((A, B)\) was defined in Section 2 as a pair of unitary representations \((U, V)\) satisfying
\[
U(a)V(b) = \langle a, b \rangle^{-1} V(b)U(a) \quad a \in A, b \in B
\]
The corresponding Weyl operator
\[
W'(a, b) = U(a)V(b) \quad a \in A, b \in B
\]
is a projective representation of \(G = A \times B\) with multiplier
\[
m'((a, b), (a', b')) = \langle a', b \rangle \quad a, a' \in A, b, b' \in B
\]
Again, this multiplier is neither alternating nor nondegenerate. But if \(A\), and hence also \(B\), is 2-regular, \(m'\) is equivalent to the symplectic bicharacter
\[
m((a, b), (a', b')) = \frac{\langle a', b \rangle^{1/2}}{\langle a, b' \rangle^{1/2}} \quad a, a' \in A, b, b' \in B
\]
via the function \(c : (a, b) \in G \mapsto (a, b)^{1/2} \in T\) (here the notation \((a, b)^{1/2}\) refers to the square root of the bicharacter \(\langle a, b \rangle\) as discussed in Lemma 3). The representation
\[
W(a, b) = (a, b)^{1/2} U(a)V(b) \quad (a, b) \in G
\]
is projective with multiplier \(m\); and again, \(m\) is the unique alternating bicharacter in the multiplier class of \(m'\).

3.3. Heisenberg groups. Let \(G\) be abelian and \(m\) a multiplier for \(G\). \(G_m\) is called a Heisenberg group (with respect to \(m\)) and \(m\) a Heisenberg multiplier (for \(G\)) if \(m^\sim\) is a symplectic bicharacter of \(G\). This definition is prompted by the observation that if \(G = \mathbb{R}^d \times \mathbb{R}^d\) and \(m\) is the multiplier of the projective representation \(W\) in the Schrödinger model, then \(m^\sim\) is symplectic. If \(G\) is 2-regular we shall usually replace \(m\) by the unique alternating bicharacter within the class of \(m\) and so assume that \(m\) itself is a symplectic bicharacter. The most general form of the uniqueness of projective unitary representations of abelian groups with Heisenberg multipliers is then the following theorem proved by Mackey [Mac49, Mac58]; see also [vN31, Sto30, Mum91, Var96]:

Theorem 4 (Mackey). If \(m\) is a Heisenberg multiplier for \(G\), every \(m\)-representation of \(G\) is a direct sum of copies of a unique irreducible \(m\)-representation.

Mackey’s theorem is thus the climax of the following succession of ideas of Heisenberg, Weyl, and Mackey himself:
- Weyl system
- projective unitary representation of \( A \oplus \hat{A} \)
- projective unitary representation of abelian group with a Heisenberg multiplier
- basic representation of Heisenberg groups

When \( G = A \times \hat{A} \) and the projective representation is

\[
W(a, a^*) = U(a)V(a^*)
\]

where \( U, V \) are unitary representations of \( A, \hat{A} \) respectively, the unique irreducible representation has the Schrödinger model. In the general case of an arbitrary \( G \) with Heisenberg \( m \) the model is a little more subtle to describe.

We shall not give the proof of this theorem here. The interested reader may consult the above references; for a detailed discussion, see [Var96] where a proof is given for the case \( G = A \times \hat{A} \).

3.4. Isotropic subgroups. Let \( m \) be a bicharacter on \( G \) (we assume \( m \) is Borel, and hence continuous). For each subset \( A \) of \( G \) we define the polar \( A'_m \) of \( A \) with respect to \( m \) by

\[
A'_m = \{ x \in G; m(x, a) = 1 \text{ for all } a \in A \}
\]

We clearly have \( A \subset B \implies B'_m \subset A'_m \) and \( A \subset A''_m \), and thus

\[
A \subset A''_m = A'_m \supseteq \cdots = A''_m \supseteq \cdots = A''_m = A''_m = \cdots,
\]

where \( A''_m \) denotes the \( n \)-th polar. Polars are closed subgroups (since we are assuming \( m \) is continuous). A subgroup \( A \) is said to be isotropic for \( m \) if \( m_{|_{A \times A}} \equiv 1 \), which is the same as saying that \( A \subset A'_m \). Maximal isotropic subgroups exist by Zorn’s lemma. Since the closure of an isotropic subgroup is again isotropic, maximal isotropic subgroups are closed. \( A \) is maximal if and only if \( A = A'_m \).

If \( m \) is alternating, then \( m'(x, y) = \frac{m(x, y)}{m(y, x)} = m^2(x, y) = m(2x, y) = m(x, 2y) \), and so we get the following relation between \( A'_m \) and \( A''_m \):

\[
A''_m = \{ x \in G; m(x, 2a) = 1, \forall a \in A \} = (2A)'_m = (1/2)A'_m,
\]

where, as usual, \( (1/2)A'_m \) denotes the inverse image of \( A'_m \) under the mapping \( x \rightarrow 2x \). In particular, if \( A \) is 2-regular, we have

\[
A''_m = A'_m \quad (A \text{ 2-regular}).
\]

So in this case, maximal isotropy with respect to \( m \) is the same as maximal isotropy with respect to \( m' \).

3.5. Model for the irreducible \( m \)-representation. We shall now discuss the structure of the unique basic irreducible representation of a locally compact Heisenberg group. The point is that although the representation is unique, one can have many different models. Each model brings to the foreground certain aspects of the representation which are particularly transparent in that model.

We proceed by analogy with the Schrödinger model when \( G = \mathbb{R}^d \oplus \mathbb{R}^d \) and try to split \( G \) as a direct sum of a “configuration space” and its dual. This is not always possible in the most general case, but we can approximate to this situation.

Let \( G \) be a separable locally compact abelian group with a Heisenberg multiplier \( m \). Choose a subgroup \( A \) which is maximal isotropic with respect to \( m' \). Since \( m' \) is 1 on \( A \times A \), it follows that \( m \) is symmetric on \( A \times A \) and hence, by Lemma 3,

\[
m(a, b) = \frac{c(a + b)}{c(a)c(b)} \quad a, b \in A
\]

for some Borel map \( c \) of \( A \) into \( T \), which is continuous if \( m \) is; the function \( c \) is unique up to multiplication by a character of \( A \). The irreducible model is the Hilbert space \( \mathcal{H} = \mathcal{H}_{m, A, c} \) of
functions $f$ on $G$ such that

$$f(x + a) = m(x, a)c(a)^{-1}f(x) \quad x \in G, a \in A$$  \hspace{1cm} (i)

$$\int_{G/A} |f([x])|^2d[x] < \infty \quad ([x] = x + A)$$  \hspace{1cm} (ii)

The irreducible $m$–representation is defined by

$$(W_m(y))f(x) = m(x, y)f(x + y)$$

**Remark.** By (i), $|f(x)|$ is constant on the $A$–cosets and so defines a function $|f([x])|$ on $G/A$; it is this function that appears under the integral sign in (ii).

For a proof that this is the irreducible version, see the discussion in [Mum91, Var96].

### 3.5.1. Examples.

Suppose that $G$ is 2–regular, $m$ is a symplectic bicharacter, and that $A$ and $B$ are maximal isotropic subgroups such that $G \simeq A \times B$. Then $m$ induces an isomorphism of $B$ with $\hat{A}$ and hence we are in the setting of a Weyl system for$(A, \hat{A})$. The model described is then the Schrödinger model. In the general case the maximal isotropic subgroup $A$ need not be a direct summand of $G$. This is the case if

$$G = V = \mathbb{R}^d \times \mathbb{R}^d, \quad m = e^{2\pi i b}, \quad A = \mathbb{Z}^d \times \mathbb{Z}^d$$

where $b$ is the skew-symmetric bilinear form with matrix $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. In this case the model leads to the classical theta functions [Mum91]. If

$$G = V = \mathbb{Q}_p^d \times \mathbb{Q}_p^d, \quad m = \chi_p \circ b, \quad A = \mathbb{Z}_p^d \times \mathbb{Z}_p^d$$

where $b$ has the matrix above, then $A$ is maximal isotropic and not a direct summand: it is a compact open subgroup, and $A = 2A$ iff $p \neq 2$. In the general adelic case $G = V \times H$ where $V$ is a finite dimensional symplectic vector space over $\mathbb{R}$, and $H$ is a totally disconnected group with a Heisenberg multiplier $m$ admitting compact open subgroups which are maximal isotropic for $m$–.

### 4. Structure of vacuum sector

In this section we perform a more detailed analysis of the unique irreducible $m$–representation $W$. More specifically, we study the case when the Heisenberg group $(G, m)$ has a maximal isotropic subgroup $A$ which is compact. The restriction of $W$ to $A$ is an ordinary representation, and we analyze the so-called vacuum sector of this restriction. It turns out that the results depend very much on the properties of the subgroup $A$: If $A = 2A$ (in particular, if $A$ is 2-regular), then there is a unique (up to scalars) vacuum vector. On the other hand, if $A \neq 2A$, the vacuum sector has dimension larger than 1; in addition, it has a canonical fermionic structure.

Before discussing the two cases separately, we shall look into the structure of $W$ a little more closely.

#### 4.1. Restriction to a compact maximal isotropic subgroup.

Let $G$ be 2–regular; $m$ a symplectic bicharacter for $G$; $L$ a maximal isotropic subgroup for $m$ which is compact. Then we have an isomorphism of $\hat{L}$ with $G/L$ such that $[y] := y + L$ corresponds to the character $a \mapsto m(a, y)$ of $L$. Since $\hat{L}$ is discrete, $L$ is open in $G$. Let $W$ be an $m$–representation, not necessarily irreducible. Then the restriction of $W$ to $L$ is an ordinary representation and so, writing

$$\mathcal{H}_{[y]} = \{ \psi : W(a)\psi = m(a, y)\psi \forall a \in L \}$$

we have

$$\mathcal{H} = \bigoplus_{[y] \in G/L}\mathcal{H}_{[y]}$$

(4)

Since

$$W(a)W(x)\psi = m(a, 2x)W(x)W(a)\psi = m(a, y + 2x)W(x)\psi \quad \psi \in \mathcal{H}_{[y]}, \quad a \in L$$

we see that

$$W(x) : \mathcal{H}_{[y]} \simeq \mathcal{H}_{[y+2x]} \quad (x \in G)$$

(5)
By (4) and (5) $\mathcal{H}$ may be viewed as a vector bundle on $G/L$ with a $G$-action permuting the fibers $\mathcal{H}_{[y]}$ above the transitive action $x, [y] \mapsto [y + 2x]$ on $G/L$. In particular

$$W|_L \simeq \dim(\mathcal{H}^L)$$

where $\mathcal{H}^L = \mathcal{H}_{[0]}$ is the subspace of vectors fixed by $L$.

**Lemma 5.** $L/2$ is the normalizer (in $G$) of $\mathcal{H}^L$. Moreover, the action of $W|_{L/2}$ on $\mathcal{H}^L$ factors through $(L/2)/(2L)$.

**Proof.** Assume $x \in G$ normalizes $\mathcal{H}^L$, i.e., $W(x)\mathcal{H}^L \subset \mathcal{H}^L$, and let $0 \neq \psi \in \mathcal{H}^L$. Then, for $a \in L$,

$$W(a)W(x)\psi = m(a, 2x)W(x)W(a)\psi = m(a, 2x)W(x)\psi$$

Since $W(x)\psi \in \mathcal{H}^L$, it follows that $m(a, 2x) = 1$. This holding for all $a \in L$ and $\mathcal{H}$ being maximal $m$-isotropic, we get $2x \in L$, i.e., $x \in L/2$. On the other hand, if $x \in L/2$ and $\psi \in \mathcal{H}^L$, then

$$W(a)W(x)\psi = m(a, 2x)W(x)W(a)\psi = W(x)\psi$$

for all $a \in L$; thus $W(x)\psi \in \mathcal{H}^L$, i.e., $x$ normalizes $\mathcal{H}^L$.

As for the second statement, if $x \in L/2$, $a \in 2L$, we have on $\mathcal{H}^L$:

$$W(x + a) = m(x, a)W(x)W(a) = m(x, a)W(x) = m(2x, a/2)W(x) = W(x)$$

\hfill $\square$

**Remark.** It is because of the appearance of $L/2$ in this lemma that our further analysis will depend very much on whether $L = 2L$ or $L \neq 2L$. Also, note that $(L/2)/(2L)$ is a finite group, since it is discrete ($2L$ being open in $L/2$) and compact ($L/2$ being compact). -In the sequel we will need to consider the action of $W(L/2)$ on $\mathcal{H}^L$ several times, and we introduce the notation $W_L$ for this action; i.e., $W_L$ is shorthand for the $m$-representation $W|_{L/2}$ acting on $\mathcal{H}^L$.

**Lemma 6.** For each closed $W_L$-invariant subspace $K \subset \mathcal{H}^L$ let $\mathcal{H}(K)$ be the smallest $W$-invariant, closed subspace in $\mathcal{H}$ containing $K$. Further, let $P : \mathcal{H} \to \mathcal{H}^L$ be the orthogonal projection. Then

$$PH(K) = K$$

If $\mathcal{H}^L$ is the orthogonal direct sum $\bigoplus_i \mathcal{K}_i$ of $W_L$-invariant closed subspaces $\mathcal{K}_i$, then

$$\mathcal{H} = \bigoplus_i \mathcal{H}(\mathcal{K}_i)$$

**Proof.** Let $\mathcal{L}$ be the linear span of the vectors $\{W(x)\psi : x \in G, \psi \in K\}$. As $W(y)W(x)\psi = m(y, x)W(x + y)\psi$, $\mathcal{L}$ is stable under $W(G)$, and so $\overline{\mathcal{L}} = \mathcal{H}(K)$. Further, as $W(x)\psi \in \mathcal{H}_{[2x]}$, we have

$$PW(x)\psi = \begin{cases} 0 & \text{if } [2x] \neq 0 \\ W(x)\psi \in K & \text{if } [2x] = 0 \end{cases}$$

So $P\mathcal{L} \subset K$, and hence $PH(K) \subset K$. As the reverse inclusion is obvious, the first statement follows.

Let $\mathcal{K}_i$, $i = 1, 2$, be two orthogonal $W(L/2)$-invariant closed subspaces of $\mathcal{H}^L$. We claim that $\mathcal{H}(\mathcal{K}_i)$ are orthogonal. If $y_1 \in G$, $\psi_i \in \mathcal{K}_i$ and $[2y_1] \neq [2y_2]$, then $W(y_1)\psi_1$ and $W(y_2)\psi_2$ are orthogonal because they lie respectively in $\mathcal{H}_{[2y_1]}$ and $\mathcal{H}_{[2y_2]}$, which are orthogonal. If $2(y_1 - y_2) \in L$, then we have

$$W(y_2)\psi_2 = W(y_1)\psi_2' = m(y_1, y_2 - y_1)^{-1}W(y_2 - y_1)\psi_2$$

where $\psi'_2 \in \mathcal{K}_2$ (because $y_2 - y_1 \in L/2$), and hence orthogonal to $\psi_1$. But then

$$(W(y_2)\psi_2, W(y_1)\psi_1) = (W(y_1)\psi'_2, W(y_1)\psi_1) = (\psi'_2, \psi_1) = 0$$

Thus $W(y_1)\psi_1$ and $W(y_2)\psi_2$ are orthogonal for all $y_1 \in G$, $\psi_1 \in \mathcal{K}_1$, and so $\mathcal{H}(\mathcal{K}_1) \perp \mathcal{H}(\mathcal{K}_2)$. Since $\mathcal{H}$ is generated by $\mathcal{H}^L$ under $W$, the second statement is now clear. \hfill $\square$

**Lemma 7.** $\mathcal{H}^L$ is an orthogonal direct sum of subspaces $\mathcal{K}_i$ which are irreducible under $W_L$. $W$ is irreducible on $\mathcal{H}$ if and only if $\mathcal{H}^L$ is irreducible under the action of $W_L$. In particular, for irreducible $W$,

$$\dim(\mathcal{H}^L) < \infty$$
Theorem 11. \( m \) with \( d \) for some positive integer \( n \)

Lemma 9. Proof. The first statement follows from the compactness of \( L/2 \) (or, even more so, from the finiteness of \( (L/2)/(2L) \)), as does the third (after proving the second statement).

As for the second statement: If \( H^L \) is reducible under \( W^L \), then, by the previous lemma, \( H \) is reducible under \( W \). Conversely, if \( Q \) is a nontrivial projection operator on \( H \) which commutes with \( W(G) \), then \( H^L = QH^L \oplus (I-Q)H^L \) is a nontrivial splitting of \( H^L \) into \( W^L \)-invariant subspaces (if the splitting were trivial, say \( \Gamma(H^L) = \{0\} \), we would have \( \{0\} = W(G)QH^L = QW(G)H^L = QH^L \), a contradiction). This completes the proof of the second statement and the lemma. \( \square \)

4.2. Vacuum vectors and coherent states when \( L = 2L \). Since now \( L/2 = L \), \( W^L \) acts trivially on \( H^L \), and thus is irreducible if and only if \( \dim(H^L) = 1 \). Lemma 7 gives

Theorem 8. If \( L = 2L \), \( W \) acts irreducibly on \( H \) if and only if \( \dim(H^L) = 1 \)

The state defined by the one–dimensional space \( \tilde{H}^L \) for irreducible \( W \) may be called the vacuum state; the states defined by the one–dimensional spaces \( H_{[\gamma]} \) are called the coherent states; and the decomposition (4) gives the coherent state structure of \( W \).

4.3. Structure of the vacuum space when \( L \neq 2L \). We now assume that \( W \) is an irreducible \( m \)-representation in the Hilbert space \( H \); it follows that \( W^L \) is irreducible (Lemma 7). We write \( m_{L/2} \) for the restriction of the multiplier \( m \) of \( W \) to \( L/2 \times L/2 \).

Lemma 9. The bicharacter \( m_{L/2} \) is the lift of a symplectic bicharacter of \( (L/2)/L \).

Proof. For \( x, x' \in L/2 \), \( a, a' \in L \) we have

\[
m_{L/2}(x + a, x' + a') = m_{L/2}(x, x')m_{L/2}(a, a')m_{L/2}(x, a')m_{L/2}(a, x') = m_{L/2}(x, x')m(2x, a)m(a, 2x')m(a, a')^2 = m_{L/2}(x, x'),
\]

showing the lifting property. Let \( n \) denote the corresponding bicharacter on \( (L/2)/L \). If \( x \in L/2 \) and \( n(x + L, y + L) = 1 \) for all \( y \in L/2 \), then \( m(x, 2y) = n(x + L, y + L) = 1 \) for all \( y \in L/2 \), hence \( x \in L \) by the maximal \( m \)-isotropy of \( L \). So \( n \) is non-degenerate, hence symplectic (since \( (L/2)/L \) is finite).

Lemma 10. There is a projective representation \( W^L_{0} \) of \( (L/2)/L \) on \( H^L \) whose lift to \( L/2 \) is projectively equivalent to \( W^L \). The multiplier \( m_{L/2} \) of \( W^L_{0} \) is characterized up to equivalence by the condition that the lift of \( m_{L/2} \) to \( L/2 \times L/2 \) is equal to \( m_{L/2} \). In particular, \( m_{0,L/2} \) is Heisenberg for the group \( (L/2)/L \).

Proof. If \( x \in L/2 \) and \( a \in L \), we have on \( H^L \)

\[
W(x + a) = m(a, x)W(x)
\]

So the morphism defined by \( W^L \) of \( L/2 \) into the projective group of \( H^L \) factors through \( L \). As \( (L/2)/L \) is finite, it is clear that we can choose a projective representation of \( (L/2)/L \) on \( H^L \) which corresponds to the above morphism of \( (L/2)/L \). Clearly the lift of \( m_{0,L/2} \) must coincide with \( m_{L/2} \). The preceding lemma shows that \( m_{0,L/2} \) is Heisenberg. \( \square \)

For an illustration of Lemma 10, and its proof, see Figure 1.

Theorem 11. The vacuum space \( H^L \) carries the unique irreducible \( m_{0,L/2} \)-representation of \( (L/2)/L \). Moreover

\[
\dim(H^L) = 2^d \quad |(L/2)/L| = 2^{2d}
\]

for some positive integer \( d \).

Proof. The first statement follows at once from the lemmas above. Now \( V_2 := (L/2)/L \) is a finite dimensional vector space over the field \( \mathbb{F}_2 := \mathbb{Z}/2\mathbb{Z} \) (since all elements of \( V_2 \) have order \( 2 \)), and \( m_{0,L/2} \) has values \( \pm 1 \); so we can view it as a symplectic form on \( V_2 \). Thus \( V_2 \) has even dimension \( 2d \) over \( \mathbb{F}_2 \), and hence cardinality \( 2^{2d} \). Let \( A \) be a maximal \( m_{0,L/2} \)-isotropic subspace of \( V_2 \). Then \( A \) has dimension \( d \) and cardinality \( 2^d \). The unique irreducible \( m_{0,L/2} \)-representation of \( V_2 \) can be realized on the Hilbert space \( \ell_2(A) \), which has complex dimension \( 2^d \).

Thus \( \dim(H^L) = \dim(\ell_2(A)) = 2^d \). \( \square \)
We first check that the bilinear form $\langle \cdot, \cdot \rangle$ is a basis, is symplectic. For this it suffices to prove that $m_{0,L/2}(\pi(x), \pi(y)) = W_{0,L/2}^e(\pi(x)) W_{0,L/2}^e(\pi(y))$, giving $m_{0,L/2}^e(\pi(x), \pi(y)) = m_{0,L/2}^e(x, y)$, $x, y \in L/2$.

The following theorem shows that, in the case of $L \neq 2L$, the vacuum space comes equipped with a canonical fermionic structure.

**Theorem 12** (Fermionic structure). For the projective representation $W^e_0$ of $(L/2)/L$ there is a basis $(e_i)_{i=1}^{2d}$ for $(L/2)/L$ such that:

$$W^e_0(e_i)^2 = 1, \quad W^e_0(e_i) W^e_0(e_j) = -W^e_0(e_j) W^e_0(e_i), \quad i \neq j \quad (6)$$

**Remark.** The relations (6) define the basic representation of the Clifford algebra in $2d$ dimensions in terms of $2d$ anticommuting spin observables. This gives the fermionic structure on $\mathcal{H}^L$.

To prove the theorem we need the following lemma:

**Lemma 13.** Let $V$ be a finite dimensional vector space over the field $\mathbb{F}_2$, and let $b$ be a symplectic form on $V$. Then there is a basis $(e_i)$ for $V$ such that

$$b(e_i, e_j) = 1 - \delta_{ij}$$

Moreover, $b = b_1 - b_1'$ where $b_1'$ is the transpose of $b_1$ and $b_1$ is the bilinear form given by

$$b_1(e_i, e_j) = \begin{cases} 0, & i \leq j \\ 1, & i > j \end{cases}$$

**Proof.** We first check that the bilinear form $b_0$ on $V \times V$ given by

$$b_0(f_i, f_j) = 1 - \delta_{ij},$$

where $(f_i)$ is a basis, is symplectic. For this it suffices to prove that $b_0$ is nondegenerate. Suppose $v = \sum_{1 \leq i \leq 2d} a_i f_i$ is orthogonal to all of $V$, and set $s = \sum_{1 \leq i \leq 2d} a_i$. Then

$$b_0(v, f_j) = \sum_{i \neq j} a_i = s - a_j = 0 \quad \forall j,$$

so $a_j = s = 2da_1 = 0$, $\forall j$. Since any two symplectic forms are isomorphic, the first statement is clear. The second is an explicit calculation. \qed

**Proof of Theorem 12.** As noted above, the bicharacter $m^e_{0,L/2}$ gives rise to a unique symplectic form $b$ on $V_2$, the relation between them being $m^e_{0,L/2} = \chi \circ b$, where $\chi$ is the non-trivial character on $\mathbb{F}_2$. With $b_1$ as in the previous lemma we have $m^e_{0,L/2} = (\chi \circ b_1)^e$. Hence $m^e_{0,L/2}$ and $\chi \circ b_1$ are equivalent so that we may assume $m_{0,L/2} = \chi \circ b_1$. If the basis $(e_i)$ is chosen as in the previous lemma, we have

$$W^e_0(e_i)^2 = m_{0,L/2}(e_i, e_i) W^e_0(2e_i) = 1$$

$$W^e_0(e_i) W^e_0(e_j) = m_{0,L/2}(e_i, e_j) W^e_0(e_i) W^e_0(e_j) = \chi(b(e_i, e_j)) W^e_0(e_j) W^e_0(e_i)$$

$$= \chi(1) W^e_0(e_j) W^e_0(e_i) = -W^e_0(e_j) W^e_0(e_i), \quad i \neq j$$

\qed
5. Examples involving $Q_p$

Set

$$G = Q_p^d \times Q_p^d, \quad m(x, y) = \chi_p(x_1 \cdot y_2 - x_2 \cdot y_1)$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are in $G$ and $\chi_p$ is a basic character on $Q_p$. The representation $W$ given by

$$(W(y)f)(s) = \chi_p(2s \cdot y_2 + y_1 \cdot y_2)f(s + y_1), \quad y = (y_1, y_2) \in G, \ s \in Q_p^d, \ f \in L_2(Q_p^d)$$

is an $m$-representation of $G$ on $\mathcal{H} = L_2(Q_p^d)$, and

$$L = Z_p^d \times Z_p^d$$

is compact subgroup of $G$ which is maximal isotropic for $m$. As for the vacuum space $\mathcal{H}^L$ we have $f \in \mathcal{H}^L \iff W(a_1, a_2)f = f$ for all $(a_1, a_2) \in L$. Written out this becomes

$$\chi_p(2s \cdot a_2 + a_1 \cdot a_2)f(s + a_1) = f(s)$$

$$(s + a_1) = \chi_p(2s \cdot a_2)^{-1}f(s), \ s \in Q_p^d, \ (a_1, a_2) \in L$$

Setting $a_1 = 0$ and assuming $f(s) \neq 0$, this gives $\chi_p(2sa_2) = 1$ for all $a_2 \in Z_p^d$, and thus $s \in Z_p^d/2$; i.e., all functions in $\mathcal{H}^L$ have support in $Z_p^d/2$. Further, setting $a_2 = 0$ in (7) and letting $a_1$ be arbitrary, we get

$$f(s + a_1) = f(s), \ s \in Z_p^d/2, \ a_1 \in Z_p^d$$

All in all we have:

$$\mathcal{H}^L = \{ f \in L_2(Z_p^d/2) : f(s + a_1) = f(s), s \in Z_p^d/2, a_1 \in Z_p^d \}$$

5.1. $G = Q_p^d \times Q_p^d$ with $p \neq 2$. In this case $Z_p^d/2 = Z_p^d$ and so $(Z_p^d/2)/Z_p^d$ reduces to a point, i.e., $\mathcal{H}^L \simeq C$, confirming the result of Theorem 8.

5.2. $G = Q_2^d \times Q_2^d$. Since $(Z_2^d/2)/Z_2 \simeq F_2$ (the field of two elements), $\mathcal{H}^L$ is now isomorphic to the space of functions on $F_2^d$. To identify the representation $W^L$ of Lemma 10 with the fermionic properties (6), consider the action $W'$ of $(Z_2^d/2 \times (Z_2^d/2)$ on $\mathcal{H}^L$ defined as $W'(x_1, x_2) = W(x_1, 0)W(0, x_2)$. Written out this becomes

$$(W'(x_1, x_2)f)(s) = \chi_2(2s \cdot x_2 + 2x_1 \cdot x_2)f(s + x_1), \ \ x_1, x_2, s \in Z_2^d/2, \ f \in \mathcal{H}^L$$

$W'$ is an irreducible projective representation with multiplier $m'((x_1, x_2), (y_1, y_2)) = \chi_2(-2y_1 \cdot x_2)$. Since $m''((x_1, x_2), (y_1, y_2)) = \chi_2(2y_1 \cdot x_2 - 2y_1 \cdot y_2) = m'((x_1, x_2), (y_1, y_2)), W'$ and $W'$ are projectively equivalent. $W'$ factors through $Z_2^d \times Z_2^d$, and the corresponding projective representation $W''$ of $F_2^d \times F_2^d$ on $\mathcal{H}^L \simeq l_2(F_2^d)$ is given by:

$$(W''(\pi(x_1), \pi(x_2))f)(\pi(s)) = (W'(x_1, x_2)f)(s) = \chi_2(2s \cdot x_2 + 2x_1 \cdot x_2)f(s + x_1)$$

$$= \chi(\pi(s) \cdot \pi(x_2) + \pi(x_1) \cdot \pi(x_2))$$

where $\pi : Z_2^d/2 \to (Z_2^d/2)/Z_2 = F_2$ is the canonical map, $f = \tilde{f} \circ \pi$, and $\chi$ is the nontrivial character on $F_2$. For the multiplier $m''$ of $W''$ we have:

$$m''((\pi(x_1), \pi(x_2)), (\pi(y_1), \pi(y_2))) = m'((x_1, x_2), (y_1, y_2)) = \chi_2(-2y_1 \cdot x_2) = \chi_2(y_1 \cdot \pi(x_2))$$

$$m''((\pi(x_1), \pi(x_2)), (\pi(y_1), \pi(y_2))) = \chi(\pi(y_1) \cdot \pi(x_2) - \pi(y_2) \cdot \pi(x_1)) = \chi_2(2y_2 \cdot x_1 - 2y_1 \cdot x_2)$$

$W''$ factors through $Z_2^d \times Z_2^d$, and the corresponding projective representation $W'''$ of $F_2^d \times F_2^d$ on $\mathcal{H}^L \simeq l_2(F_2^d)$ is given by:

$$(W'''((a_1, a_2), (b_1, b_2))f(c)) = \chi(c \cdot a_2 + a_1 \cdot a_2)f(c + a_1)$$

$$m'''((a_1, a_2), (b_1, b_2)) = \chi(b_1 \cdot a_2)$$

$$m'''((a_1, a_2), (b_1, b_2)) = \chi(b_1 \cdot a_2 - b_2 \cdot a_1)$$

$$f \in l_2(F_2^d), \ a_i, b_i \in F_2^d, \ i = 1, 2$$

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