The hidden geometry of the quantum Euclidean space *

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Abstract

We briefly describe how to introduce the basic notions of noncommutative differential geometry on the 3-dim quantum space covariant under the quantum group of rotations $SO_q(3)$.

1 Introduction and preliminaries

It is a rather old idea that the micro-structure of space-time at the Planck level might be better described using a noncommutative geometry. Here

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we consider the formalism of Dubois-Violette, Madore, Masson, Mourad, et al., and apply it to the noncommutative algebra describing the quantum Euclidean space $R^3_q$ [1], namely the quantum space covariant under the quantum group $SO_q(3)$. This involves an interesting cross-fertilization between the noncommutative-geometry formalism with the quantum space and quantum group machinery. We briefly describe the main results of our work [2]. There, we introduced a metric and an ‘almost’ metric-compatible linear connection on the quantum Euclidean space, equipped with its (two) standard $SO_q(3)$-covariant differential calculi; correspondingly, the ‘frame’ or dreibein has been also found. Modulo a conformal factor, which might however be reabsorbed into a formulation of metric compatibility more suitable for the present case, the curvature turns out to be zero, suggesting that the quantum space is flat as in the commutative limit. In a separate paper we shall show that in the same limit the traditional quantum space coordinates go to suitable general (non-cartesian) coordinates. This will allow a cure of some unpleasant features [3] of a naive physical interpretation of the representation theory of $Fun(R^3_q)$.

The preliminaries contained in this section are based especially on the works [4, 5]; for an introduction see Ref. [6]. The starting point is a noncommutative algebra $\mathcal{A}$ which has as commutative limit the algebra of functions on some manifold $M$ and over $\mathcal{A}$ a differential calculus $\{d, \Omega^*(\mathcal{A})\}$ which has as corresponding limit the ordinary de Rham differential calculus; as known [7], $\{d, \Omega^*(\mathcal{A})\}$ is completely determined by the left and right module structure of the $\mathcal{A}$-module of 1-forms $\Omega^1(\mathcal{A})$. By definition a metric is a $\mathcal{A}$-bilinear map

$$g : \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \to \mathcal{A}. \quad (1)$$

$\mathcal{A}$-bilinearity means

$$g(f\xi \otimes \eta) = fg(\xi \otimes \eta), \quad g(\xi \otimes \eta f) = g(\xi \otimes \eta)f,$$

for any $f \in \mathcal{A}$ and $\xi, \eta \in \Omega^1(\mathcal{A})$. This is a definition in the “cotangent space of the deformed manifold”; one could also formulate it in the “tangent space”. In the commutative limit $\mathcal{A}$-bilinearity is equivalent to the very important requirement of locality of $g$ in both arguments at each point $x \in M$:

$$[g(f\xi \otimes \eta)](x) = f(x) \cdot [g(\xi \otimes \eta)](x), \quad [g(\xi \otimes \eta f)](x) = [g(\xi \otimes \eta)](x) f(x). \quad (3)$$

A linear connection is a map (cfr. [8])

$$D : \Omega^1(\mathcal{A}) \to \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \quad (4)$$
together with a “generalized flip” $\sigma$, i.e. a $\mathcal{A}$-bilinear map

$$\sigma : \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \to \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$$

such that $D$ satisfies the left and right Leibniz rules

$$D(f\xi) = df \otimes \xi + fD\xi$$

$$D(\xi f) = \sigma(\xi \otimes df) + (D\xi)f.$$  

Let $\pi$ be the projection

$$\pi : \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \to \Omega^2(\mathcal{A}).$$

The torsion is the map $\Theta = d - \pi \circ D$. The connection $D$ is torsion-free iff

$$\pi \circ (\sigma + 1) = 0.$$  

One can naturally extend $D$ to higher tensor powers, e.g.

$$D_2(\xi \otimes \eta) = D\xi \otimes \eta + \sigma_{12}(\xi \otimes D\eta),$$

where we have introduced the tensor notation $\sigma_{12} = \sigma \otimes 1$. The metric-compatibility condition for $g, D$ reads $g_{23} \circ D_2 = d \circ g$.

The curvature $\text{Curv} : \Omega^1(\mathcal{A}) \to \Omega^2(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$ is defined by

$$\text{Curv} = \pi_{12} \circ D_2 \circ D.$$  

It is always left $\mathcal{A}$-linear, and right $\mathcal{A}$-linear only in certain models; in general, right linearity is guaranteed only in the commutative limit. Therefore in this limit the curvature is local, an essential physical requirement for a reasonable definition of a curvature.

If $\mathcal{A}, \Omega^1(\mathcal{A})$ are $*$-algebras and $d$ is real, $(df)^* = df^*$, $D$ can be made also real if we define [9] the involution on $\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$ by

$$(\xi \otimes \eta)^* = \sigma(\eta \otimes \xi^*)$$

(note that this expression has the correct classical limit), with a $\sigma$ such that the square of $*$ gives the identity. So real structures on the tensor product are in one-to-one correspondence with right Leibniz rules. $D_2$ is real iff $\sigma$ in addition fulfils the braid equation

$$\sigma_{12}\sigma_{23}\sigma_{12} = \sigma_{23}\sigma_{12}\sigma_{23},$$  

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The curvature is real if (12), (13) are satisfied.

Now assume that there exists a frame, i.e. a special basis $\theta^a \in \Omega^1(A)$, $1 \leq a \leq n$, such that

$$[\theta^a, A] = 0$$

and any $\xi \in \Omega^1(A)$ can be uniquely written in the form $\xi_a \theta^a$, with $\xi_a \in A$. This is possible only if the limit manifold $M$ is parallelizable. It has the advantage that for any $f \in A$ the computation of commutator $[\xi, f]$ is reduced to the computation of the commutators $[\xi_a, f]$ in $A$. Assume also that there exist $n$ inner derivations $e_a$,

$$e_a f := [\lambda_a, f]$$

($\lambda_a \in A$), dual to $\theta^a$: $\theta^a(e_b) = \delta^a_b$. Then

$$\theta := -\lambda_a \theta^a$$

is the ‘Dirac operator’ [7] for $d$:

$$df = -[\theta, f].$$

$\theta^a$ is a very convenient basis to work with. For instance, from $A$-bilinearity it immediately follows that the elements

$$g^{ab} := g(\theta^a \otimes \theta^b)$$

lie in the center $Z(A)$ of $A$. We shall be interested in the case that $Z(A) = C$. In the commutative limit the condition $g^{ab} \in C$ characterizes the vielbein or ‘moving frame’ of E. Cartan, which is determined up to a linear transformation; if this condition is fulfilled for any value of the deformation parameter the $\theta^a$ remain uniquely determined up to a linear transformation and are particularly convenient objects to be used to guess a physically sensible formulation of noncommutative-geometric notions.

2 Application of the formalism to the quantum Euclidean space

Take ‘the algebra of functions on the quantum Euclidean space $\mathbb{R}^3_q$ [1] as $A$ and over it one of the two $SO_q(3)$-covariant differential calculi [10]. The
treatment of the other calculus can be done in a completely parallel way, see ref. \[2\]. Here we are interested in the case of a real positive \(q\). We ask if they fit in the previous scheme. We shall denote by \(\|\hat{R}_{kl}\|\) the braid matrix of \(SO_q(3)\), by \(g_{ij} = g^{ij}\) the \(SO_q(3)\)-covariant metric; here and below all indices will take the values \(-, 0, +\). In the commutative limit \(q \to 1\) \(g_{ij} \to \delta_{ij}\). The projector decomposition of \(\hat{R}\) is

\[
\hat{R} = qP_s - q^{-1}P_a + q^{-2}P_t; \quad (19)
\]

\(P_s, P_a, P_t\) are \(SO_q(3)\)-covariant \(q\)-deformations of respectively the symmetric trace-free, antisymmetric and trace projectors. The trace projector is 1-dimensional and is related to \(g_{ij}\) by

\[
P_t^{ij} \propto g_{ij}; \quad (20)
\]

\(\mathcal{A}\) is generated by \(x^-, x^0, x^+\) fulfilling \(P_a x x = 0\), or more explicitly

\[
x^- x^0 = q x^0 x^-; \quad x^+ x^0 = q^{-1} x^0 x^+; \quad [x^+, x^-] = h(x^0)^2. \quad (21)
\]

where we define \(h = \sqrt{q - 1}/\sqrt{q}\). The real structure on \(\mathcal{A}\) is defined by \((x^i)^* = x^j g_{ji}\), or more explicitly

\[
(x^-)^* = \sqrt{q} x^+, \quad (x^0)^* = x^0; \quad (x^+)^* = 1/\sqrt{q} x^- \quad (22)
\]

\(\mathcal{Z}(\mathcal{A})\) is generated by the \(SO_q(3)\)-covariant real element

\[
r^2 := g_{ij} x^i x^j = \sqrt{q} x^+ x^- + (x^0)^2 - 1/\sqrt{q} x^- x^+. \quad (23)
\]

Let \(\xi^i = dx^i\). One \(SO_q(3)\)-covariant calculus, which we shall denoted by \(\{d, \Omega^*(\mathcal{A})\}\), is determined by the commutation relations

\[
x^i \xi^j = q \hat{R}^{ij}_{kl} \xi^k x^l. \quad (24)
\]

Unfortunately neither calculus has a real exterior derivative, and up to now no way was known to make it closed under involution \[11\]; rather, each exterior algebra is mapped into the other under the natural involution. The ‘Dirac operator’ \[17\] corresponding to \(d\) is the \(SO_q(3)\)-invariant element \[12\]

\[
\theta := (q - 1)^{-1} q^2 r^{-2} x^i \xi^j g_{ij}; \quad \text{note that } \theta \text{ is singular in the commutative limit.}
\]

In our work \[2\] we have found the following new results.
1. There exist two torsion-free, ‘almost’ metric-compatible linear connections, given by the formula

\[ D(0)\xi = -\theta \otimes \xi + \sigma_0(\xi \otimes \theta) \] (25)

The two corresponding generalized flips \( \sigma_0 \) are determined by

\[ \sigma_0(\xi^i \otimes \xi^j) =: S^{ij}_{hk} \xi^h \otimes \xi^k; \] (26)

its knowledge allows one to extend by linearity \( \sigma_0 \) to all \( \Omega^1(A) \otimes_A \Omega^1(A) \) in a unique way. \( D(0) \) ‘almost’ metric-compatible means compatible up to a conformal factor with the metric given in the next item; a strict compatibility does not seem possible. Both \( \sigma_0 \) fulfil the braid equation (13) and both \( D(0) \) are \( SO_q(3) \)-invariant.

2. If we extend \( A \) by adding the ‘dilatation’ generator \( \Lambda \)

\[ x^i \Lambda = q \Lambda x^i \] (27)

together with its inverse \( \Lambda^{-1} \) (we shall normalize them so that \( \Lambda^* = \Lambda^{-1} \)) and set \( d\Lambda = 0 \), then up to normalization there exists a unique metric \( g_0 \),

\[ g_0(\xi^i \otimes \xi^j) = g^{ij} r^2 \Lambda^2 \] (28)

(\( g_{ij} \) is the \( SO_q(3) \)-covariant metric matrix), which is compatible with the two \( D(0) \) up to the conformal factors \( q^2, q^{-2} \),

\[ S^{ij}_{hk} g^{kl} S^{mn}_{jl} = q^{\pm 2} g^{im} \delta^j_h, \] (29)

respectively in the cases \( S = (q \hat{R})^{\pm 1} \). A strict compatibility would have required no \( q^{\pm 2} \) at the rhs.

3. Curv=0 for both \( D(0) \).

4. If we further extend \( A \) by adding also the generators \( r \) [the square root of (23)], its inverse \( r^{-1} \) and the inverse \( (x^0)^{-1} \) of \( x^0 \), then there exist a frame \( \theta^a \), \( a = -, 0, +, \) and a dual basis \( e_a \) of inner derivations given by

\[ \theta^a := \Lambda^{-1} \theta^a_i \xi^i \] (30)
with

$$
\|\theta^a_i\| := \begin{pmatrix}
(x^0)^{-1} \sqrt{q(q+1)(rx^0)^{-1}} x^+ & r^{-1} \\
-\sqrt{q(q+1)(r^2x^0)^{-1}} (x^+)^2 & -(q+1)r^{-2}x^+ & r^{-2}x^0
\end{pmatrix}
$$

(31)

$$
\lambda_- = +h^{-1}q\Lambda(x^0)^{-1}x^+, \\
\lambda_0 = -h^{-1}\sqrt{q}\Lambda(x^0)^{-1}r, \\
\lambda_+ = -h^{-1}\Lambda(x^0)^{-1}x^-.
$$

(32)

e_a x^i = q\Lambda e^a_i$, where $\|e^a_i\|$ is (left and right) inverse of the $\mathcal{A}$-valued matrix $\|\theta^a_i\|$. Its elements fulfil the ‘$RTT$-relations’ [4]

$$
\hat{R}^{ij}_{kl} e^k_a e^l_b = e^c_i e^j_d \hat{R}^{cd}_{ab}
$$

(33)

as well as the ‘$gTT$-relations’

$$
g^{ab} e^i_a e^j_b = r^2 g^{ij} \quad \quad g_{ij} e^i_a e^j_b = r^2 g_{ab}.
$$

(34)

In a sense $r^{-1} e^i_a$ are a realization of the generators $T^i_a$ of $SO_q(3)$. As a consequence we find

$$
\mathcal{P}^{ab}_{cd} \theta^c \theta^d = 0 \quad \quad \mathcal{P}^{ab}_{cd} \theta^c \theta^d = 0
$$

(35)

the same commutation relations fulfilled by the $\xi^i$’s. Finally, up to a normalization $g_0(\theta^a \otimes \theta^b) = g^{ab}$.  

5. $\Omega^*(\mathcal{A})$ is closed under the involution defined by

$$
(x^i)^* = x^j g_{ji} \quad \quad (\theta^a)^* = \theta^b g_{ba}
$$

(36)

(the latter acts nonlinearly on the $\xi^i$’s: $(\xi^i)^* = \Lambda^{-2} \xi^j c_{ji}$, with $c_{ji} \in \mathcal{A}$).

The reality structure of these differential calculi is an old but always present problem (see [11]). The solution proposed in item 5 is not fully satisfactory, at least naively. For instance, it does not yield real $d, D$; only the curvature is real, for the simple reason that it vanishes. The involution cannot be consistently extended to $\Omega^*(\mathcal{A}) \otimes \Omega^*(\mathcal{A})$ according to (12). Finally, apparently it has not the correct classical limit.

A more careful analysis is needed at this point, but is out of the scope of the present report (for more details see Ref. 2). It involves the investigation
of the properties of the $\ast$-representations of $\Omega^*(\mathcal{A})$ and seems to suggest a more sophisticated version of the proposal in item 5, in which the opposite properties of the two differential calculi cancel with each other. The problems mentioned above and the fact that the linear connections $D_{(0)}$ are metric-compatible up to conformal factors (or, in other words, are only conformally flat) may be related, in the sense that a satisfactory formulation of the reality properties could eventually yield also a new and satisfactory formulation of metric-compatibility which can be strictly fulfilled. A careful analysis of the commutative limit is also needed in order to propose a reasonable correspondence principle between the ‘new’ theory and classical differential geometry.

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