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Coefficients of a Comprehensive Subclass of Meromorphic Bi-Univalent Functions Associated with the Faber Polynomial Expansion

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Abstract: In this paper, we introduce a new comprehensive subclass $\Sigma_B(\lambda, \mu, \beta)$ of meromorphic bi-univalent functions in the open unit disk $U$. We also find the upper bounds for the initial Taylor-Maclaurin coefficients $|b_0|, |b_1|$ and $|b_2|$ for functions in this comprehensive subclass. Moreover, we obtain estimates for the general coefficients $|b_n| (n \geq 1)$ for functions in the subclass $\Sigma_B(\lambda, \mu, \beta)$ by making use of the Faber polynomial expansion method. The results presented in this paper would generalize and improve several recent works on the subject.

Keywords: analytic functions; univalent and bi-univalent functions; meromorphic bi-univalent functions; coefficient estimates; Faber polynomial expansion; meromorphic bi-Bazilevič functions of order $\beta$ and type $\mu$; meromorphic bi-starlike functions of order $\beta$

1. Introduction

Let $A$ denote the class of functions $f$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk

$$U = \{z : z \in \mathbb{C} \text{ and } |z| < 1}\}.$$

We also let $S$ be the class of functions $f \in A$ which are univalent in $U$.

It is well known that every function $f \in S$ has an inverse $f^{-1}$, which is defined by

$$f^{-1}(f(z)) = z \quad (z \in U)$$

and

$$f(f^{-1}(w)) = w \quad (|w| < r_0(f); \quad r_0(f) \geq \frac{1}{4}).$$

If $f$ and $f^{-1}$ are univalent in $U$, then $f$ is said to be bi-univalent in $U$. We denote by $\sigma_B$ the class of bi-univalent functions in $U$. For a brief history and interesting examples of functions in the class $\sigma_B$, see the pioneering work [1]. In fact, this widely-cited work...
by Srivastava et al. [1] actually revived the study of analytic and bi-univalent functions in recent years, and it has also led to a flood of papers on the subject by (for example) Srivastava et al. [2–14] and by others [15,16].

In this paper, let \( \Sigma \) be the family of meromorphic univalent functions \( f \) of the following form:

\[
f(z) = z + b_0 + \sum_{n=1}^{\infty} \frac{b_n}{z^n},
\]

which are defined on the domain

\[
\Delta = \{ z : z \in \mathbb{C} \text{ and } 1 < |z| < \infty \}.
\]

Since a function \( f \in \Sigma \) is univalent, it has an inverse \( f^{-1} \) that satisfies the following relationship:

\[
f^{-1}(f(z)) = z \quad (z \in \Delta)
\]

and

\[
f(f^{-1}(w)) = w \quad (M < |w| < \infty; \ M > 0).
\]

Furthermore, the inverse function \( f^{-1} \) has a series expansion of the form [17]:

\[
g(w) = f^{-1}(w) = w + \sum_{n=0}^{\infty} \frac{B_n}{w^n} \quad (M < |w| < \infty).
\]

A function \( f \in \Sigma \) is said to be meromorphic bi-univalent if both \( f \) and \( f^{-1} \) are meromorphic univalent in \( \Delta \). The family of all meromorphic bi-univalent functions in \( \Delta \) of the form (2) is denoted by \( \Sigma_M \). A simple calculation shows that (see also [18,19])

\[
g(w) = f^{-1}(w) = w - b_0 - b_1 w - b_2 + b_0 b_1 \frac{b_2}{w^2} - \cdots.
\]

Moreover, the coefficients of \( g = f^{-1} \) can be given in terms of the Faber polynomial [20] (see also [21–23]) as follows:

\[
g(w) = f^{-1}(w) = w - b_0 - \sum_{n=1}^{\infty} \frac{1}{n} K_{n+1}^n \frac{1}{w^n} \quad (w \in \Delta),
\]

where

\[
K_{n+1}^n = nb_0^{n-1}b_1 + n(n-1)b_0^{n-2}b_2 + \frac{1}{2} n(n-1)(n-2)b_0^{n-3}(b_3 + b_1^2) + \frac{n(n-1)(n-2)(n-3)}{3!} b_0^{n-4}(b_4 + 3b_1b_2) + \sum_{j=5}^{n} b_0^{n-j}V_j
\]

and \( V_j \) (with \( 5 \leq j \leq n \)) is a homogeneous polynomial of degree \( j \) in the variables \( b_1, b_2, \ldots, b_n \).

Estimates on the coefficients of meromorphic univalent functions were widely investigated in the literature. For example, Schiffer [24] obtained the estimate \( |b_2| \leq 2/3 \) for meromorphic univalent functions \( f \in \Sigma \) with \( b_0 = 0 \) and Duren [25] proved that

\[
|b_n| \leq \frac{2}{n+1} \quad (f \in \Sigma; \ b_k = 0; \ 1 \leq k < \frac{n}{2}).
\]

Many researchers introduced and studied subclasses of meromorphic bi-univalent functions (see, for instance, Janani et al. [26], Orhan et al. [27] and others [28–30]).
Recently, Srivastava et al. [31] introduced a new class \( \Sigma_B^* (\lambda, \beta) \) of meromorphic bi-univalent functions and obtained the estimates on the initial Taylor–Maclaurin coefficients \(|b_0|\) and \(|b_1|\) for functions in this class.

**Definition 1** (see [31]). A function \( f \in \Sigma_M \), given by (2), is said to be in the class \( \Sigma_B^* (\lambda, \beta) \) \((\lambda \geq 1; 0 \leq \beta < 1)\), if the following conditions are satisfied:

\[
\Re\left( \frac{z(f'(z))^\lambda}{f(z)} \right) > \beta
\]

and

\[
\Re\left( \frac{w(g'(w))^\lambda}{g(w)} \right) > \beta,
\]

where the function \( g \), given by (3) is the inverse of \( f \) and \( z, w \in \Delta \).

**Theorem 1** (see [31]). Let the function \( f \in \Sigma_M \), given by (2), be in the class \( \Sigma_B^* (\lambda, \beta) \). Then,

\[
|b_0| \leq 2(1 - \beta) \quad \text{and} \quad |b_1| \leq \frac{2(1 - \beta)\sqrt{4\beta^2 - 8\beta + 5}}{1 + \lambda}.
\]

In this paper, we introduce a new comprehensive subclass \( \Sigma_B (\lambda, \mu, \beta) \) of the meromorphic bi-univalent function class \( \Sigma_M \). We also obtain estimates for the initial Taylor–Maclaurin coefficients \( b_0, b_1 \) and \( b_2 \) for functions in this subclass. Furthermore, we find estimates for the general coefficients \( b_n \) \((n \geq 1)\) for functions in this comprehensive subclass \( \Sigma_B (\lambda, \mu, \beta) \) by using the Faber polynomials [20]. Our results for the meromorphic bi-univalent function subclass \( \Sigma_B (\lambda, \mu, \beta) \) would generalize and improve some recent works by Srivastava et al. [31], Hamidi et al. [32] and Jahangiri et al. [33] (see also the recent works [34,35]).

### 2. Preliminary Results

For finding the coefficients of functions belonging to the function class \( \Sigma_B (\lambda, \mu, \beta) \), we need the following lemmas and remarks.

**Lemma 1** (see [21,22]). Let \( f \) be the function given by

\[
f(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots
\]

be a meromorphic univalent function defined on the domain \( \Delta \). Then, for any \( \rho \in \mathbb{R} \), there are polynomials \( K_n^\rho \) such that

\[
\left( \frac{f(z)}{z} \right)^\rho = 1 + \sum_{n=1}^{\infty} K_n^\rho (b_0, b_1, \cdots, b_{n-1}) \frac{z^n}{2^n},
\]

where

\[
K_n^\rho (b_0, b_1, \cdots, b_{n-1}) = \rho b_{n-1} + \frac{\rho (\rho - 1)}{2} D_n^2 + \frac{\rho !}{(\rho - 3)! 3!} D_n^3 + \cdots + \frac{\rho !}{(\rho - n)! n!} D_n^n
\]

and

\[
D_n^k (x_1, x_2, \cdots, x_{n-k+1}) = \sum \frac{k! (\mu_1)_{\mu_1} \cdots (\mu_{n-k+1})_{\mu_{n-k+1}}}{\mu_1! \cdots \mu_{n-k+1}!},
\]
in which the sum is taken over all non-negative integers \( \mu_1, \cdots, \mu_{n-k+1} \) such that
\[
\begin{align*}
\mu_1 + \mu_2 + \cdots + \mu_{n-k+1} &= k \\
\mu_1 + 2\mu_2 + \cdots + (n - k + 1)\mu_{n-k+1} &= n.
\end{align*}
\]

The first three terms of \( K_n^\rho \) are given by
\[
K_1^\rho(b_0) = \rho b_0,
\]
\[
K_2^\rho(b_0, b_1) = \rho b_1 + \frac{\rho(\rho-1)}{2} b_0^2
\]
and
\[
K_3^\rho(b_0, b_1, b_2) = \rho b_2 + \rho(\rho-1)b_0b_1 + \frac{\rho(\rho-1)(\rho-2)}{3!} b_0^3.
\]

**Remark 1.** In the special case when
\[
b_0 = b_1 = \cdots = b_{n-1} = 0,
\]
it is easily seen that
\[
K_i^\rho(b_0, \cdots, b_{i-1}) = 0 \quad (1 \leq i \leq n)
\]
and
\[
K_{n+1}^\rho(b_0, b_1, \cdots, b_n) = \rho b_n.
\]

**Lemma 2** (see [21,22]). Let \( f \) be the function given by
\[
f(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots
\]
be a meromorphic univalent function defined on the domain \( \Delta \). Then, the Faber polynomials \( F_n \) of \( f(z) \) are given by
\[
\frac{zf'(z)}{f(z)} = 1 + \sum_{n=1}^{\infty} \frac{F_n(b_0, b_1, \cdots, b_{n-1})}{z^n},
\]
where \( F_n(b_0, b_1, \cdots, b_{n-1}) \) is a homogeneous polynomial of degree \( n \).

**Remark 2** (see [36]). For any integer \( n \geq 1 \), the polynomials \( F_n(b_0, b_1, \cdots, b_{n-1}) \) are given by
\[
F_n(b_0, b_1, \cdots, b_{n-1}) = \sum_{i_1 + 2i_2 + \cdots + n_i = n} A_{(i_1, i_2, \cdots, i_n)} b_0^{i_0} b_1^{i_1} \cdots b_{n-1}^{i_n},
\]
where
\[
A_{(i_1, i_2, \cdots, i_n)} := (-1)^n + 2 + \cdots + (n+1)! \frac{(i_1 + i_2 + \cdots + i_n - 1)!n}{i_1!i_2! \cdots i_n!}.
\]

The first three terms of \( F_n \) are given by
\[
F_1(b_0) = -b_0,
\]
\[
F_2(b_0, b_1) = b_0^2 - 2b_1
\]
and
\[
F_3(b_0, b_1, b_2) = -b_0^3 + 3b_0b_1 - 3b_2.
\]

**Remark 3.** In the special case when \( b_0 = b_1 = \cdots = b_{n-1} = 0 \), it is readily observed that
\[
F_i(b_0, \cdots, b_{i-1}) = 0 \quad (1 \leq i \leq n)
\]
Proof. By using Lemmas 1 and 2, we have

\[
\left( \frac{zf'(z)}{f(z)} \right)^{\lambda} \left( \frac{f(z)}{z} \right)^{\mu} = 1 + \sum_{n=1}^{\infty} \frac{L_n(b_0, b_1, \ldots, b_{n-1})}{z^n},
\]

where

\[
L_n(b_0, b_1, \ldots, b_{n-1}) = \sum_{i=0}^{n} K_{n-i}^\lambda (F_1, \ldots, F_{n-i}) K_i^\mu (b_0, \ldots, b_{i-1}) \quad (K_0^\lambda = K_0^\mu = 1)
\]

and \(F_n = F_n(b_0, b_1, \ldots, b_{n-1})\) is given by (5).

Remark 4. In the special case when \(b_0 = b_1 = \cdots = b_{n-1} = 0\), we easily find that

\[
L_i(b_0, \ldots, b_{i-1}) = 0 \quad (1 \leq i \leq n)
\]

Proof. By using Lemmas 1 and 2, we have

\[
\left( \frac{zf'(z)}{f(z)} \right)^{\lambda} \left( \frac{f(z)}{z} \right)^{\mu} = \left( 1 + \sum_{m=1}^{\infty} \frac{F_m(b_0, b_1, \ldots, b_{m-1})}{z^m} \right)^{\lambda} \cdot \left( 1 + \sum_{m=1}^{\infty} \frac{K_m^\mu (b_0, b_1, \ldots, b_{m-1})}{z^m} \right).
\]

In addition, by applying Lemma 1 once again, we obtain

\[
\left( \frac{zf'(z)}{f(z)} \right)^{\lambda} \left( \frac{f(z)}{z} \right)^{\mu} = \left( 1 + \sum_{m=1}^{\infty} \frac{K_m^\lambda (F_1, \ldots, F_m)}{z^m} \right)^{\lambda} \cdot \left( 1 + \sum_{m=1}^{\infty} \frac{K_m^\mu (b_0, \ldots, b_{m-1})}{z^m} \right)
\]

\[
= 1 + \sum_{n=1}^{\infty} \sum_{i=0}^{n} K_{n-i}^\lambda (F_1, \ldots, F_{n-i}) K_i^\mu (b_0, \ldots, b_{i-1}) \frac{1}{z^n}
\]

\[
(K_0^\lambda = K_0^\mu = 1).
\]

Our demonstration of Lemma 3 is thus completed. \(\square\)

The first three terms of \(L_n\) are given by

\[
L_1(b_0) = (\mu - \lambda) b_0,
\]

\[
L_2(b_0, b_1) = \frac{\lambda(1 + \lambda - 2\mu) + \mu(\mu - 1)}{2} b_0^2 + (\mu - 2\lambda) b_1
\]

and

\[
L_3(b_0, b_1, b_2) = \left( \frac{\lambda(2 - \mu)(\mu - \lambda)}{2} + \frac{\mu(\mu - 1)(\mu - 2)}{6} - \lambda(\lambda - 1)(\lambda - 2) \right) b_0^3
\]

\[
+ [\lambda(2\lambda + 1) + \mu(\mu - 3\lambda - 1)] b_0 b_1 + (\mu - 3\lambda) b_2.
\]

Remark 4. In the special case when \(b_0 = b_1 = \cdots = b_{n-1} = 0\), we easily find that

\[
L_i(b_0, \ldots, b_{i-1}) = 0 \quad (1 \leq i \leq n)
\]
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Let $f$ be a meromorphic bi-univalent function.

Lemma 2. By putting

- $\lambda = 1$
- $\mu = 0$
- $\beta = 1$

we have subclasses of meromorphic bi-univalent functions. For example, we have the following special cases:

- $\Sigma_b(1, 0, 1)$
- $\Sigma_b(0, 1, 1)$
- $\Sigma_b(1, 1, 1)$

Definition 2. A function $f \in \Sigma_M$, given by (2), is said to be in the class

\[ \Sigma_b(\lambda, \mu, \beta) \quad (\lambda \geq 1; \mu \geq 0; 0 \leq \beta < 1) \]

of meromorphic bi-univalent functions of order $\beta$ and type $\mu$, if the following conditions are satisfied:

- $\Re\left(\left(\frac{zf'(z)}{f(z)}\right)^{\lambda} \left(\frac{f(z)}{z}\right)^{\mu}\right) > \beta$
- $\Re\left(\left(\frac{wg'(w)}{g(w)}\right)^{\lambda} \left(\frac{g(w)}{w}\right)^{\mu}\right) > \beta$

where the function $g$ given by (4), is the inverse of $f$ and $z, w \in \Delta$.

Remark 5. There are several choices of the parameters $\lambda$ and $\mu$ which would provide interesting subclasses of meromorphic bi-univalent functions. For example, we have the following special cases:

- By putting $\lambda = 1$ and $0 \leq \mu < 1$, the class $\Sigma_b(1, \mu, 1)$ reduces to the subclass $B(\beta, \mu)$ of meromorphic bi-Bazilevič functions of order $\beta$ and type $\mu$.
- By putting $\lambda = 1$ and $\mu = 0$, the class $\Sigma_b(1, 0, 1)$ reduces to the class $\Sigma_b(1)$.
- By putting $\mu = \lambda - 1$, the class $\Sigma_b(\lambda, \mu, 1)$ reduces to the class $\Sigma_b^1(\lambda, 1)$ in Definition 1.

Theorem 2. Let $f \in \Sigma_b(\lambda, \mu, \beta)$. If $b_0 = b_1 = \cdots = b_{n-1} = 0$, then

\[ |b_n| \leq \frac{2(1 - \beta)}{|(n+1)\lambda - \mu|} \quad (n \geq 1). \]

Proof. By using Lemma 3 for the meromorphic bi-univalent function $f$ given by

\[ f(z) = z + b_0 + \sum_{n=1}^{\infty} \frac{b_n}{n^\lambda}, \]

we have

\[ \left(\frac{zf'(z)}{f(z)}\right)^{\lambda} \left(\frac{f(z)}{z}\right)^{\mu} = 1 + \sum_{n=0}^{\infty} L_{n+1}(b_0, b_1, \cdots, b_n) \frac{z^{n+1}}{z^{n+1}}. \]
Similarly, for its inverse map \( g \) given by
\[
g(w) = f^{-1}(w) = w + B_0 + \sum_{n=1}^{\infty} \frac{B_n}{w^n}.
\]
we find that
\[
\left( \frac{wg'(w)}{g(w)} \right)^\lambda \left( \frac{g(w)}{w} \right)^\mu = 1 + \sum_{n=0}^{\infty} \frac{L_{n+1}(B_0, B_1, \cdots, B_n)}{w^{n+1}}.
\]
(7)

Furthermore, since \( f \in \Sigma_B(\lambda, \mu, \beta) \), by using Definition 2, there exist two positive real-part functions
\[
c(z) = 1 + \sum_{n=1}^{\infty} c_n z^{-n}
\]
and
\[
d(w) = 1 + \sum_{n=1}^{\infty} d_n w^{-n}
\]
for which
\[\Re(c(z)) > 0 \quad \text{and} \quad \Re(d(w)) > 0 \quad (z, w \in \Delta),\]
such that
\[
\left( \frac{zf'(z)}{f(z)} \right)^\lambda \left( \frac{f(z)}{z} \right)^\mu = 1 + (1 - \beta) \sum_{n=0}^{\infty} K_{n+1}^1(c_1, c_2, \cdots, c_{n+1}) \frac{1}{z^{n+1}}
\]
(8)
and
\[
\left( \frac{wg'(w)}{g(w)} \right)^\lambda \left( \frac{g(w)}{w} \right)^\mu = 1 + (1 - \beta) \sum_{n=0}^{\infty} K_{n+1}^1(d_1, d_2, \cdots, d_{n+1}) \frac{1}{w^{n+1}}
\]
(9)
Upon equating the corresponding coefficients in (6) and (8), we get
\[
L_{n+1}(b_0, b_1, \cdots, b_n) = (1 - \beta) K_{n+1}^1(c_1, c_2, \cdots, c_{n+1}).
\]
(10)
Similarly, from (7) and (9), we obtain
\[
L_{n+1}(B_0, B_1, \cdots, B_n) = (1 - \beta) K_{n+1}^1(d_1, d_2, \cdots, d_{n+1}).
\]
(11)
Now, since \( b_i = 0 \) \((0 \leq i \leq n - 1)\), we have
\[
B_i = 0 \quad (0 \leq i \leq n - 1) \quad \text{and} \quad B_n = -b_n.
\]
Hence, by using Remark 4, Equations (10) and (11) can be rewritten as follows:
\[
(\mu - (n + 1)\lambda)b_n = (1 - \beta)c_{n+1}
\]
(12)
and
\[
-(\mu - (n + 1)\lambda)b_n = (1 - \beta)d_{n+1},
\]
(13)
respectively. Thus, from (12) and (13), we find that
\[
2(\mu - (n + 1)\lambda)b_n = (1 - \beta)(c_{n+1} - d_{n+1}).
\]
Finally, by applying Lemma 4, we get
\[
|b_n| = \frac{(1 - \beta)|c_{n+1} - d_{n+1}|}{2|n + 1|\lambda - \mu} \leq \frac{2(1 - \beta)}{|(n + 1)\lambda - \mu|}
\]
which completes the proof of Theorem 2 \( \square \)
Theorem 3. Let the function $f \in \mathcal{M}$, given by (2), be in the class

$$\Sigma_B(\lambda, \mu, \beta) \quad (\lambda \geq 1; \mu \geq 0; 0 \leq \beta < 1).$$

Then,

$$|b_0| \leq \min \left\{ \frac{2(1-\beta)}{|\mu-\lambda|}, 2 \sqrt{\frac{1-\beta}{|\lambda(1+\lambda-2\mu)+\mu(\mu-1)|}} \right\},$$

and

$$|b_1| \leq \frac{2(1-\beta)}{|\mu-2\lambda|},$$

and

$$|b_2| \leq \frac{2\left[|\lambda(2\lambda+4)+\mu(\mu-3\lambda-2)|+|\lambda(2\lambda+1)+\mu(\mu-3\lambda-1)|\right](1-\beta)}{|(\mu-3\lambda)(\mu-\lambda)|^3} + \frac{8|T(\mu, \lambda)|(1-\beta)^3}{|(\mu-3\lambda)(\mu-\lambda)|^3},$$

where

$$T(\mu, \lambda) = \frac{\lambda(2-\mu)(\mu-\lambda)}{2} + \frac{\mu(\mu-1)(\mu-2) - \lambda(\lambda-1)(\lambda-2)}{6}.$$

Proof. By putting $n = 0, 1, 2$ in (10), we get

$$(\mu-\lambda)b_0 = (1-\beta)c_1, \quad (14)$$

$$\frac{\lambda(1+\lambda-2\mu)+\mu(\mu-1)}{2}b_0^2 + (\mu-2\lambda)b_1 = (1-\beta)c_2 \quad (15)$$

and

$$T(\mu, \lambda)b_0^3 + [\lambda(2\lambda+1)+\mu(\mu-3\lambda-1)]b_0b_1 + (\mu-3\lambda)b_2 = (1-\beta)c_3. \quad (16)$$

Similarly, by putting $n = 0, 1, 2$ in (11), we have

$$-(\mu-\lambda)b_0 = (1-\beta)d_1, \quad (17)$$

$$\frac{\lambda(1+\lambda-2\mu)+\mu(\mu-1)}{2}b_0^2 - (\mu-2\lambda)b_1 = (1-\beta)d_2 \quad (18)$$

and

$$-T(\mu, \lambda)b_0^3 + (\lambda(2\lambda+4)+\mu(\mu-3\lambda-2))b_0b_1 - (\mu-3\lambda)b_2 = (1-\beta)d_3. \quad (19)$$

Clearly, from (14) and (17), we get

$$c_1 = -d_1 \quad (20)$$

and

$$b_0 = \frac{(1-\beta)c_1}{\mu-\lambda}. \quad (21)$$

Adding (15) and (18), we obtain

$$b_0^2 = \frac{(1-\beta)(c_2 + d_2)}{\lambda(1+\lambda-2\mu)+\mu(\mu-1)}. \quad (22)$$
In view of the Equations (21) and (22), by applying Lemma 4, we get
\[
|b_0| \leq \frac{2(1 - \beta)}{|\mu - \lambda|} \quad \text{and} \quad |b_0|^2 \leq \frac{4(1 - \beta)}{|\lambda(1 + \lambda - 2\mu + \mu(\mu - 1))|},
\]
respectively. Thus, we get the desired estimate on the coefficient $|b_0|$.

Next, in order to find the bound on the coefficient $|b_1|$, we subtract (18) from (15). We thus obtain
\[
b_1 = \frac{(1 - \beta)(c_2 - d_2)}{2(\mu - 2\lambda)}.
\]
(23)

Applying Lemma 4 once again, we get
\[
|b_1| \leq \frac{2(1 - \beta)}{|\mu - 2\lambda|}.
\]

Finally, in order to determine the bound on $|b_2|$, we consider the sum of the Equations (16) and (19) with $c_1 = -d_1$. This yields
\[
\lambda b_1 b_1 = \frac{(1 - \beta)(c_3 + d_3)}{\lambda(4\lambda + 5) + \mu(2\mu - 6\lambda - 3)}.
\]
(24)

Subtracting (19) from (16) with $c_1 = -d_1$, we obtain
\[
2(\mu - 3\lambda)b_2 + (\mu - 3\lambda)b_1b_1 + 2T(\mu, \lambda)b_0^2 = (1 - \beta)(c_3 - d_3).
\]
(25)

In addition, by using (21) and (24) in (25), we get
\[
b_2 = \frac{(1 - \beta)(c_3 - d_3)}{2(\mu - 3\lambda)} - \frac{(1 - \beta)(c_3 + d_3)}{2(\lambda(4\lambda + 5) + \mu(2\mu - 6\lambda - 3))} - \frac{T(\mu, \lambda)(1 - \beta)^3c_3^3}{(\mu - 3\lambda)(\mu - \lambda)^3}.
\]

Hence,
\[
b_2 = \frac{[\lambda(2\lambda + 4) + \mu(\mu - 3\lambda - 2)]c_3 - [\lambda(2\lambda + 1) + \mu(\mu - 3\lambda - 1)]d_3}{(\mu - 3\lambda)[\lambda(4\lambda + 5) + \mu(2\mu - 6\lambda - 3)]} (1 - \beta)
\]
\[- \frac{T(\mu, \lambda)(1 - \beta)^3c_3^3}{(\mu - 3\lambda)(\mu - \lambda)^3}.
\]

Thus, by applying Lemma 4 once again, we get
\[
|b_2| \leq \frac{2[|\lambda(2\lambda + 4) + \mu(\mu - 3\lambda - 2)| + |\lambda(2\lambda + 1) + \mu(\mu - 3\lambda - 1)|]}{|(\mu - 3\lambda)[\lambda(4\lambda + 5) + \mu(2\mu - 6\lambda - 3)]|} (1 - \beta)
\]
\[+ \frac{8|T(\mu, \lambda)(1 - \beta)^3c_3^3}{|(\mu - 3\lambda)(\mu - \lambda)^3|}.
\]

This completes the proof of Theorem 3. □

4. A Set of Corollaries and Consequences

By setting $\lambda = 1$ and $0 \leq \mu < 1$ in Theorem 2, we have the following result.

**Corollary 1.** Let the function $f \in \mathcal{M}$, given by (2), be in the subclass $\mathcal{B}(\beta, \mu)$ of meromorphic bi-Bazilevič functions of order $\beta$ and type $\mu$. If
\[
b_0 = b_1 = \cdots = b_{n-1} = 0,
\]
then
\[
|b_n| \leq \frac{2(1 - \beta)}{n + 1 - \mu} \quad (n \geq 1).
\]
Remark 6. The estimate of $|b_n|$, given in Corollary 1, is the same as the corresponding estimate given by Hamidi et al. [38] Corollary 3.3.

By setting $\mu = 0$ in Corollary 1, we have the following result.

Corollary 2. Let the function $f \in M$, given by (2), be in the subclass $\Sigma_0^*(\beta)$ of meromorphic bi-starlike functions of order $\beta$. If

$$b_0 = b_1 = \cdots = b_{n-1} = 0,$$

then

$$|b_n| \leq \frac{2(1-\beta)}{n+1} \quad (n \geq 1).$$

Remark 7. The estimate of $|b_n|$, given in Corollary 2, is the same as the corresponding estimate given by Hamidi et al. [38] Corollary 3.4.

By setting $\mu = \lambda - 1$ in Theorem 2, we have the following result.

Corollary 3. Let the function $f \in M$, given by (2), be in the subclass $\Sigma_{B^*}(\lambda, \beta)$. If

$$b_0 = b_1 = \cdots = b_{n-1} = 0,$$

then

$$|b_n| \leq \frac{2(1-\beta)}{n\lambda + 1} \quad (n \geq 1).$$

Remark 8. Corollary 3 is a generalization of a result presented in Theorem 1, which was proved by Srivastava et al. [31].

By setting $\lambda = 1$ and $0 \leq \mu < 1$ in Theorem 3, we have the following result.

Corollary 4. Let the function $f \in M$, given by (2), be in the subclass $B(\beta, \mu)$ of meromorphic bi-Bazilevič functions of order $\beta$ and type $\mu$. Then,

$$|b_0| \leq \begin{cases} \sqrt{\frac{4(1-\beta)}{(1-\mu)(2-\mu)}} & (0 \leq \beta \leq \frac{1}{2-\mu}), \\ \frac{2(1-\beta)}{1-\mu} & (\frac{1}{2-\mu} \leq \beta < 1), \end{cases}$$

and

$$|b_1| \leq \frac{2(1-\beta)}{2-\mu}.$$

Remark 9. Corollary 4 also contains the estimate of the Taylor–Maclaurin coefficient $|b_2|$ of functions in the subclass $B(\beta, \mu)$ (see [33]).

By setting $\mu = 0$ in Corollary 4, we have the following result.
Corollary 5. Let the function \( f \in \mathcal{M} \), given by (2), be in the subclass \( \Sigma^*_B(\beta) \) of meromorphic bi-starlike functions of order \( \beta \). Then,

\[
|b_0| \leq \begin{cases} 
\sqrt{2(1-\beta)} & (0 \leq \beta \leq \frac{1}{2}) \\
2(1-\beta) & \left( \frac{1}{2} \leq \beta < 1 \right), 
\end{cases}
\]

and

\[
|b_1| \leq 1 - \beta
\]

\[
|b_2| \leq \frac{2(1-\beta)}{3} + \frac{8(1-\beta)^3}{3}.
\]

Remark 10. Corollary 5 not only improves the estimate of the Taylor–Maclaurin coefficient \( |b_0| \), which was given by Hamidi et al. [32] Theorem 2, but it also provides an improvement of the known estimate of the Taylor–Maclaurin coefficient \( |b_2| \) of functions in the subclass \( \Sigma^*_B(\beta) \). Furthermore, the estimate of \( |b_0| \), presented in Corollary 5, is the same as the corresponding estimate given by Hamidi et al. [38] Corollary 3.5.

By setting \( \mu = \lambda - 1 \) in Theorem 3, we have the following result.

Corollary 6. Let the function \( f \in \mathcal{M} \), given by (2), be in the subclass \( \Sigma_B^*(\lambda, \beta) \). Then,

\[
|b_0| \leq \begin{cases} 
\sqrt{2(1-\beta)} & (0 \leq \beta \leq \frac{1}{2}) \\
2(1-\beta) & \left( \frac{1}{2} \leq \beta < 1 \right), 
\end{cases}
\]

and

\[
|b_1| \leq \frac{2(1-\beta)}{\lambda + 1}
\]

and

\[
|b_2| \leq \frac{2(1-\beta)}{2\lambda + 1} + \frac{8(1-\beta)^3}{2\lambda + 1}.
\]

Remark 11. Corollary 6 improves the estimates of the Taylor–Maclaurin coefficients \( |b_0| \) and \( |b_1| \) in Theorem 1 of Srivastava et al. [31]. In fact, it also provides an improvement of the known estimate of the Taylor–Maclaurin coefficient \( |b_2| \) of functions in the subclass \( \Sigma_B^*(\lambda, \beta) \).

Remark 12. In his recently-published survey-cum-expository review article, Srivastava [39] demonstrated how the theories of the basic (or q-) calculus and the fractional q-calculus have significantly encouraged and motivated further developments in Geometric Function Theory of Complex Analysis (see, for example, [8,40–42]). This direction of research is applicable also to the results which we have presented in this article. However, as pointed out by Srivastava [39] (p. 340), any further attempts to easily (and possibly trivially) translate the suggested q-results into the corresponding \((p, q)\)-results (with \(0 < |q| < p \leq 1\)) would obviously be inconsequential because the additional parameter \( p \) is redundant.

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