Lusztig’s $q$-analogue of weight multiplicity and one-dimensional sums for affine root systems

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Abstract

In this paper we complete the proof of the $X = K$ conjecture, that for every family of nonexceptional affine algebras, the graded multiplicities of tensor products of “symmetric power” Kirillov-Reshetikhin modules known as one-dimensional sums, have a large rank stable limit $X$ that has a simple expression (called the $K$-polynomial) as nonnegative integer combination of Kostka-Foulkes polynomials. We consider a subfamily of Lusztig’s $q$-analogues of weight multiplicity which we call stable KL polynomials and denote by $\infty KL$. We give a type-independent proof that $K = \infty KL$. This proves that $X = \infty KL$: the family of stable one-dimensional sums coincides with family of stable KL polynomials. Our result generalizes the theorem of Nakayashiki and Yamada which establishes the above equality in the case of one-dimensional sums of affine type A and the Lusztig $q$-analogue of type A, where both are Kostka-Foulkes polynomials.

1 Introduction

One-dimensional (1-d) sums $X$ are graded tensor product multiplicities for affine Kac-Moody algebras, which arise from two-dimensional solvable lattice models [11] and which may be defined using the combinatorics of affine crystal graphs [5] [6]. The definition of a 1-d sum depends not only on an affine algebra $\hat{\mathfrak{g}}$ but also on a distinguished simple Lie subalgebra $\mathfrak{g}$. This is the same data as giving an affine Dynkin diagram together with a distinguished 0 node whose removal leaves the Dynkin diagram of a simple Lie algebra.

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Within any nonexceptional family of affine algebras, the 1-d sums have a large rank limit that we shall call stable 1-d sums. There are many nonexceptional families of affine algebras and choices of 0 node, but only four distinct kinds of stable 1-d sums [30] [29]. They depend not on the entire affine Dynkin diagram, but only on the neighborhood of the 0 node. The four kinds of stable 1-d sums \( X^\Diamond \) are labeled by the four partitions \( \Diamond \in \{ \emptyset, (1), (2), (1,1) \} \) having at most 2 cells. This labeling is inspired by the branching rules for the restriction of Kirillov-Reshetikhin modules [5] [6] from the affine algebra to its simple Lie subalgebra.

In [30] it was conjectured that every stable 1-d sum \( X \) has a surprisingly simple explicit expression (called the \( K \) polynomial) in terms of those of type \( A \), the latter being the well-known Kostka-Foulkes polynomials. We finish the proof of this \( X = K \) conjecture for tensor products of symmetric powers, supplying the proof in the case \( \Diamond = (1, 1) \); the other cases were already settled in [29]. Unlike the situation in [29], for the case \( \Diamond = (1, 1) \) we cannot use affine crystal theory alone, but must use a novel grading that is not compatible with the affine crystal structure but only with an \( A_{n-1} \) crystal substructure.

On the other hand there is another family of polynomials, Lusztig’s \( q \)-analogue of weight multiplicity for classical Lie algebras [23]. These are instances of affine Kazhdan-Lusztig polynomials. There is a subfamily which we shall call stable KL polynomials and denote by \( \infty \text{KL} \), which are precisely those Lusztig \( q \)-analogues that are indexed by pairs of dominant weights that contain sufficiently large multiples of a certain fundamental weight. They are called stable because if both weights are positively translated by the above fundamental weight, then the polynomial does not change. The stable KL polynomials were studied in [19] [20].

We give a type-independent proof that \( K = \infty \text{KL} \). The stable 1-d sum types \( \emptyset, (1), (2), (1,1) \) correspond to stable KL polynomials of types \( A_{n-1}, B_n, C_n, D_n \) respectively. For type \( A \), \( K = \infty \text{KL} \) holds by definition. For types \( C \) and \( D \) the proof was given in [19] [20]. It was also noticed in [20] that the usual type \( B \) Lusztig \( q \)-analogue does not agree with \( X^{(1)} \), despite the fact that both polynomials have the same value at \( q = 1 \). Here we repair this problem by using a \( q \)-analogue of Kostant’s partition function that uses different powers of \( q \) for different root lengths, which is related to Hecke algebras with unequal parameters.

Combined with the \( X = K \) theorem we obtain \( X = \infty \text{KL} \): the stable 1-d sums are the stable KL polynomials.

In the \( X = \infty \text{KL} \) theorem the correspondence of weights involves a twist \( \mu \mapsto \hat{\mu} \) that also appears in Howe duality [7]. We do not have an adequate explanation for this twist. Our proof reduces to a type \( A_{n-1} \) situation where the twist is given by contragredient duality, but its meaning for the original weights of classical type is less clear to us.

1.1 Classical algebras and crystal graphs

Let \( \mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h} \) be a simple Lie algebra, a Borel subalgebra and a Cartan subalgebra, \( J \) the set of nodes for the Dynkin diagram of \( \mathfrak{g} \), \( \{ \alpha_i \mid i \in J \} \subset \mathfrak{h}^* \) the simple roots, \( R^+ \subset \mathfrak{h}^* \) the set of positive roots with respect to \( \mathfrak{h} \), \( \{ h_i \mid i \in J \} \subset \mathfrak{h} \) the simple coroots, \( \{ \omega_i \mid i \in J \} \subset \mathfrak{h}^* \) the fundamental weights, \( P = \bigoplus_{i \in J} \mathbb{Z} \omega_i \subset \mathfrak{h}^* \) the weight lattice, \( P^+ = \bigoplus_{i \in J} \mathbb{Z}_{\geq 0} \omega_i \) the set of dominant weights, \( W \) the Weyl group, and \( \langle \cdot, \cdot \rangle \) the evaluation pairing of \( \mathfrak{h} \) with \( \mathfrak{h}^* \). Let \( U_q(\mathfrak{g}) \) be the quantized universal enveloping algebra of \( \mathfrak{g} \) [9].

Each finite-dimensional \( U_q(\mathfrak{g}) \)-module has a crystal graph [9]. Let \( \mathcal{C}(\mathfrak{g}) \) be the category of such crystal graphs. Each \( B \in \mathcal{C}(\mathfrak{g}) \) is a directed graph with vertex set also denoted \( B \) and directed edges labeled by the set \( J \). If we remove all edges of \( B \) except those labeled by a fixed \( i \in J \), then
the connected components are directed paths called $i$-strings. Denote by $\tilde{f}_i(b)$ (resp. $\tilde{e}_i(b)$) the next (resp. previous) vertex in the $i$-string through the vertex $b \in B$. If no such vertex exists then the result is defined to be the special symbol $\emptyset$. Define $\tilde{e}_i(\emptyset) = \tilde{f}_i(\emptyset) = \emptyset$. Let $\varepsilon_i(b)$ (resp. $\varphi_i(b)$) be the number of steps to the beginning (resp. end) of the $i$-string through $b$. Each $B \in \mathcal{C}(\mathfrak{g})$ also has a weight function $\text{wt} : B \rightarrow P$. It satisfies

$$\text{wt}(\tilde{f}_i(b)) = \text{wt}(b) - \alpha_i \quad \text{if } \tilde{f}_i(b) \neq \emptyset$$
$$\text{wt}(\tilde{e}_i(b)) = \text{wt}(b) + \alpha_i \quad \text{if } \tilde{e}_i(b) \neq \emptyset$$
$$\langle h_i, \text{wt}(b) \rangle = \varphi_i(b) - \varepsilon_i(b)$$

A vertex $b \in B$ such that $\varepsilon_i(b) = 0$ for all $i \in J$, is called a highest weight vertex of $B$. For $B, B' \in \mathcal{C}(\mathfrak{g})$ a morphism $\psi : B \rightarrow B'$ is a map $\psi : B \cup \{\emptyset\} \rightarrow B' \cup \{\emptyset\}$ such that $\psi(\emptyset) = \emptyset$ and $\psi(\tilde{e}_i(b)) = \tilde{e}_i(\psi(b))$ and $\psi(\tilde{f}_i(b)) = \tilde{f}_i(\psi(b))$ for all $b \in B$ and $i \in J$. An isomorphism is a bijective morphism whose inverse function is also a morphism.

If $V$ and $V'$ are finite-dimensional $U_q(\mathfrak{g})$-modules with crystal graphs $B$ and $B'$, then their direct sum $V \oplus V'$ has crystal graph $B \oplus B'$, which by definition is the disjoint union of $B$ and $B'$.

Each $B \in \mathcal{C}(\mathfrak{g})$ is the disjoint union of connected crystal graphs. For every connected $B \in \mathcal{C}(\mathfrak{g})$, there is a unique $\lambda \in P^+$ such that $B \cong B(\lambda)$ where $B(\lambda)$ is the crystal graph of the irreducible finite-dimensional $U_q(\mathfrak{g})$-module $V(\lambda)$ of highest weight $\lambda$. For $\mathfrak{g}$ of classical type $B(\lambda)$ is described explicitly in [13]. $B(\lambda)$ has a unique highest weight vertex, which is also the unique vertex of weight $\lambda$ in $B(\lambda)$.

The tensor product $V \otimes V'$ has the crystal graph $B \otimes B'$ whose underlying set is the Cartesian product $B \times B'$, whose elements are written $b \otimes b'$ instead of $(b, b')$, with crystal structure given by

$$\tilde{f}_i(b \otimes b') = \begin{cases} \tilde{f}_i(b) \otimes b' & \text{if } \varphi_i(b) > \varepsilon_i(b') \\ b \otimes \tilde{f}_i(b') & \text{if } \varphi_i(b) \leq \varepsilon_i(b') \end{cases}$$
$$\tilde{e}_i(b \otimes b') = \begin{cases} \tilde{e}_i(b) \otimes b' & \text{if } \varphi_i(b) \geq \varepsilon_i(b') \\ b \otimes \tilde{e}_i(b') & \text{if } \varphi_i(b) < \varepsilon_i(b') \end{cases}$$

The following lemma is a straightforward consequence of (1).

**Lemma 1.** Let $B, B' \in \mathcal{C}(\mathfrak{g})$. Then $b \otimes b' \in B \otimes B'$ is a highest weight vertex if and only if $b \in B$ is a highest weight vertex and $\varphi_i(b) \geq \varepsilon_i(b')$ for any $i \in J$.

If $b \in B'$ is a highest weight vertex in $B' \in \mathcal{C}(\mathfrak{g})$, we denote by $B(b)$ the connected component of $b$ in $B'$.

### 1.2 Partitions and dominant weights

From now on let $\mathfrak{g} = \mathfrak{g}_n$ be a classical Lie algebra, one of type $A_{n-1}$, $B_n$, $C_n$, or $D_n$, with Dynkin diagrams labeled as in Figure [1]. We shall also use a nonstandard labeling $D^1_n$. Let $J$ be the vertex set of the Dynkin diagram of $\mathfrak{g}_n$: $J = \{1, 2, \ldots, n-1\}$ for $\mathfrak{g}_n = A_{n-1}$ and $J = \{1, 2, \ldots, n\}$ for $\mathfrak{g}_n = B_n, C_n, D_n$. The diagram $D^1_n$ has vertex set $J' = \{0, 1, \ldots, n-1\}$.

The weight lattice $P$ of $\mathfrak{g}_n$ may be explicitly realized as a sublattice of $(\frac{1}{2}\mathbb{Z})^n$ where $\omega_i = (1^i, 0^{n-i})$ for $1 \leq i \leq n-2$ and also $i = n-1$ for $\mathfrak{g}_n \neq D_n$, $\omega_n^C = (1^n)$, $\omega_n^B = \omega_n^D = (\frac{1}{2}1^n)$, and $\omega_{n-1}^D = (\frac{1}{2}[n-1], -\frac{1}{2})$. For $D^1_n$, $P$ is identified with the sublattice of $(\frac{1}{2}\mathbb{Z})^n$ generated by the fundamental weights $\omega_0^{D^1_n} = ((-\frac{1}{2})^n)$, $\omega_1^{D^1_n} = (\frac{1}{2}, (-\frac{1}{2})^{n-1})$, and $\omega_i^{D^1_n} = (i, (-1)^{n-i})$ for $2 \leq i \leq n-1$. 

3
The sum of fundamental weights (or half-sum of positive roots) \( \rho \) may be given by \( \rho^{A_{n-1}} = (n - 1, n - 2, \ldots, 1, 0) \), \( \rho^{B_n} = \frac{1}{2}(2n - 1, 2n - 3, \ldots, 1) \), \( \rho^{C_n} = (n, n - 1, \ldots, 2, 1) \), and \( \rho^{D_n} = (n - 1, n - 2, \ldots, 1, 0) \).

Let \( \varepsilon_i \) be the \( i \)-th standard basis vector. The simple roots \( \{\alpha_i \mid i \in J\} \subset h^\ast \) are given by \( \alpha_i = \varepsilon_i - \varepsilon_{i+1} \) for \( 1 \leq i \leq n - 1 \), \( \alpha_n^{B_n} = \varepsilon_n \), \( \alpha_n^{C_n} = 2\varepsilon_n \), \( \alpha_n^{D_n} = \varepsilon_{n-1} + \varepsilon_n \), and \( \alpha_0^{D_n} = -\varepsilon_1 - \varepsilon_2 \).

Later we shall make use of the fact that the root system of type \( A_{n-1} \) is the subsystem of \( C_n \) (resp. \( D_n^{\dagger} \)) that is obtained by removing the distinguished zero node \( n \) (resp. 0).

We identify elements of \( \mathbb{Z}^n \subset P \) with weights (excluding the case of \( D_n^{\dagger} \) here). The set \( P^+ \cap \mathbb{Z}^n \) of dominant weights in the sublattice \( \mathbb{Z}^n \) is given by partitions \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n) \in \mathbb{Z}^n \) where \( \lambda_n \geq 0 \) except in type \( D_n \) in which case \( \lambda_{n-1} \geq |\lambda_n| \). If \( \lambda \) is a partition with at most \( m \) parts and \( n \geq m \) then we may view \( \lambda \) as a dominant weight for \( g_n \).

1.3 Affine algebras, KR crystal graphs, and 1-d sums
Let \( \hat{g} \supset \hat{g}' \supset g \) be an affine Kac-Moody algebra \([8]\), its derived subalgebra, and the simple Lie subalgebra whose Dynkin diagram is obtained from the affine Dynkin diagram by removing a distinguished zero node. Let \( U_q(\hat{g}) \supset U_q'(\hat{g}) \supset U_q(g) \) be the corresponding Drinfeld-Jimbo quantized universal enveloping algebras \([9]\).

Let \( U_q'(\hat{g})-\text{Mod} \) be the category of finite-dimensional irreducible \( U_q'(\hat{g}) \)-crystals which possess an affine crystal graph\(^1\). Such affine crystal graphs have directed edges colored by the set \( I = J \cup \{0\} \) and weight function with values in \( P \) (after projection from the weight lattice of \( \hat{g}' \)). It was conjectured in \([5]\) \([6]\) that for every \((r, s) \in J \times \mathbb{Z}_{>0} \) there is a module \( W_s^{(r)} \in U_q'(\hat{g})-\text{Mod} \) called the Kirillov-Reshetikhin (KR) module, with affine crystal graph denoted \( B^{r,s} \). The KR modules are fundamental objects: it is expected that every \( V \in U_q'(\hat{g})-\text{Mod} \) is isomorphic to a tensor product

\(^1\)For finite-dimensional \( U_q'(\hat{g}) \)-modules this is rare. In comparison, the nonzero irreducible integrable highest weight \( U_q'(\hat{g}) \)-modules are all known to have affine crystal graphs, but all these modules are infinite-dimensional.
of KR modules. The KR module $W_s^{(r)}$ has a prescribed $U_q(\mathfrak{g})$-decomposition of the form \cite{5} \cite{6}

$$W_s^{(r)} \cong V(s\omega_r) \oplus \cdots$$

where $\cdots$ indicates a direct sum of irreducibles $V(\mu)$ where $\mu$ is smaller than $s\omega_r$ in a certain sense.

From now on we assume that $\mathfrak{g}$ is a nonexceptional affine algebra. The “symmetric power” KR modules $W_s^{(1)}$ and their affine crystal graphs $B^{1,s}$ were constructed in \cite{12} \cite{10}. Let $\mathcal{C}$ be the category of tensor products of KR crystals of the form $B^{1,s}$. Let

$$d = \begin{cases} 
2 & \text{if the Dynkin diagram of $\hat{\mathfrak{g}}$ has a double bond directed from node 1 to node 0} \\
1 & \text{otherwise.} 
\end{cases} \quad (2)$$

$\mathcal{C}$ has the following remarkable properties.

1. Every $B \in \mathcal{C}$ is the affine crystal graph of an irreducible $U'_q(\hat{\mathfrak{g}})$-module. In particular $B$ is connected as an affine crystal graph.

2. For every $B_1, B_2 \in \mathcal{C}$ there is a unique affine crystal isomorphism $R_{B_1,B_2} : B_1 \otimes B_2 \to B_2 \otimes B_1$ called the combinatorial R-matrix, and a map $\overline{R} = \overline{R}_{B_1,B_2} : B_1 \otimes B_2 \to \frac{1}{d}\mathbb{Z}$ called the local coenergy function, such that the following holds. $b_1 \otimes b_2 \in B_1 \otimes B_2$ and $b_2' \otimes b_1' = R_{B_1,B_2}(b_1 \otimes b_2)$. Then

$$\overline{R}(\bar{e}_i(b_1 \otimes b_2)) = \overline{R}(b_1 \otimes b_2) + \delta_{i0} \frac{d}{\delta_{i0}} \begin{cases} 
1 & \text{if } \varphi_0(b_1) \geq \varepsilon_0(b_2) \text{ and } \varphi_0(b_2') \geq \varepsilon_0(b_1'), \\
-1 & \text{if } \varphi_0(b_1) < \varepsilon_0(b_2) \text{ and } \varphi_0(b_2) < \varepsilon_0(b_1'), \\
0 & \text{otherwise.} 
\end{cases} \quad (3)$$

Thus $\overline{R}$ is a $U'_q(\mathfrak{g})$-equivariant grading of $B_1 \otimes B_2$ (that is, it is constant under directed edges in $J$). Since $B_1 \otimes B_2 \in \mathcal{C}$ is connected as an affine crystal graph, the map $\overline{R}$ is unique up to a global additive constant. If $B_1 = B_2 = B$, we have $R_{B,B} = 1_{B \otimes B}$ by uniqueness and

$$\overline{R}(\bar{e}_i(b_1 \otimes b_2)) = \overline{R}(b_1 \otimes b_2) + \frac{\delta_{i0}}{d} \begin{cases} 
1 & \text{if } \varphi_0(b_1) \geq \varepsilon_0(b_2) \\
-1 & \text{if } \varphi_0(b_1) < \varepsilon_0(b_2). 
\end{cases} \quad (4)$$

The combinatorial $R$-matrices satisfy the Yang-Baxter equation, which asserts the equality of the two ways to compute the isomorphism $B_1 \otimes B_2 \otimes B_3 \to B_3 \otimes B_2 \otimes B_1$ by switching adjacent factors using maps of the form $R_{B_i,B_j}$.

3. Each $B \in \mathcal{C}$ has a $U'_q(\mathfrak{g})$-equivariant grading $\overline{D}_B : B \to \frac{1}{d}\mathbb{Z}$ called the coenergy function for $B$, which is well-defined up to a global additive constant by the following rules \cite{27} \cite{8}. If $B \in \mathcal{C}$ is a KR crystal then the grading $\overline{D}_B$ is prescribed by \cite{5}. \cite{9}. If $B_1, B_2 \in \mathcal{C}$ are such that $\overline{D}_{B_1}$ and $\overline{D}_{B_2}$ have been defined, let $b_1, b_2, b_1', b_2'$ be as in \cite{13}. Then

$$\overline{D}_{B_1 \otimes B_2}(b_1 \otimes b_2) = \overline{R}(b_1 \otimes b_2) + \overline{D}_{B_1}(b_1) + \overline{D}_{B_2}(b_2'). \quad (5)$$

Suppose $B_1, B_2, \ldots, B_m \in \mathcal{C}$. By induction one may prove the following. Let $b_i \in B_i$ for $1 \leq i \leq m$, $B = B_1 \otimes \cdots \otimes B_m$, and $b = b_1 \otimes \cdots \otimes b_m$. Then one may parenthesize the $m$-fold tensor product
in any way, iterate the pairwise construction of \([5]\), and the resulting coenergy function is always given by \([6] [27]\)

\[
\overline{D}_B(b) = \sum_{1 \leq i < j \leq m} \overline{H}(b_i \otimes b_j^{(i+1)}) + \sum_{j=1}^{m} \overline{D}_{B_j}(b_j^{(1)})
\]

where for \(1 \leq i \leq j \leq m\) the vertices \(b_j^{(i)} \in B_j\) are determined by the affine crystal isomorphisms

\[
B_i \otimes \cdots \otimes B_j \otimes B_{j-1} \rightarrow B_j \otimes B_i \otimes B_{i+1} \otimes \cdots \otimes B_{j-1}
\]

\[
b_i \otimes \cdots \otimes b_{j-1} \otimes b_j \rightarrow b_j^{(i)} \otimes b_j^{(i)} \otimes \cdots \otimes b_{j-1}^{(i)}
\]

given by compositions of combinatorial \(R\)-matrices acting at adjacent tensor factors. If each \(B_i\) is the same crystal \(B\) then

\[
\overline{D}_{B^\otimes m}(b_1 \otimes \cdots \otimes b_m) = \sum_{j=1}^{m-1} (m - j) \overline{H}(b_j \otimes b_{j+1}) + m \overline{D}_B(b_1).
\]

We now change notation. Up to now the subscript in \(B_i\) had merely indicated the position of a generic tensor factor. From now on let \(B_s = B^{1:s}\). Let \(B_\mu \in C\) be the affine crystal graph

\[
B_\mu = B_{\mu_1} \otimes B_{\mu_2} \otimes \cdots \otimes B_{\mu_m}
\]

where \(\mu = (\mu_1, \mu_2, \ldots, \mu_m) \in \mathbb{Z}_{>0}^m\).

For \(\lambda \in P^+\) let \(F_\lambda,\mu\) denote the set of highest weight vertices in \(B_\mu\) of weight \(\lambda\). The 1-d sums for \(C\) are the graded tensor product multiplicities defined by

\[
X_{\lambda,\mu}(q) = \sum_{b \in F_\lambda,\mu} q^{D_{B_\mu}(b)}.
\]

The papers \([5] [6]\) introduced another remarkable expression \(M\) called the fermionic formula and conjectured that \(X = M\). The \(M\) formula, a sum of products of \(q\)-binomial coefficients, arises from the Bethe Ansatz and exhibits quasiparticle behavior of interest to physicists. We shall not pursue the \(M\) formula here.

**Remark 2.** To go from the coenergy functions defined here to the energy functions of \([5] [6]\), set \(d = 1\) in the formulae and take the negative. The factor \(d\) is included in our definitions to make the statement of Theorem \([4]\) smoother. For the coenergy to be completely well-defined the local coenergy functions and the coenergy functions for KR crystals must be normalized. This is done in section \([2.6] [4]\) for the cases under consideration in this paper.

### 1.4 Stable 1-d sums

Fix a nonexceptional affine algebra \(\hat{\mathfrak{g}}_n\) together with a distinguished simple Lie subalgebra \(\mathfrak{g}_n\). Let \(P\) be the set of partitions and \(P_m\) those with at most \(m\) parts.

**Lemma 3.** \([30] [2]\) Fix \(m \in \mathbb{Z}_{>0}, \lambda, \mu \in P_m,\) and a nonexceptional family of affine algebras \(\{\hat{\mathfrak{g}}_n\}\). Then for any \(n \geq m\), the 1-d sum \(X_{\lambda,\mu}^{\mathfrak{g}_n}(q)\) is well-defined and gives the same polynomial, called a stable 1-d sum. Moreover, all the families of nonexceptional affine algebras yield only four distinct kinds of stable 1-d sums, one for each of the partitions \(\emptyset, (1), (2), (1,1)\) of size at most 2. They are grouped as follows, based on the attachment of the zero node to the classical Dynkin diagram of \(\mathfrak{g}_n\).

\(^2\)For technical reasons this result was stated for the fermionic formula \(M\) in \([30]\).
• Kind $\emptyset$: $A_{n-1}^{(1)}$: attach 0 to 1 and $n - 1$ by single bonds, in the diagram $A_{n-1}$.

• Kind $(1)$: $D_{n+1}^{(2)}, A_{2n}^{(2)}, B_n^{(1)}$: attach 0 to 1 with a double bond pointing towards 0, in the diagrams $B_n, C_n,$ and $D_n$ respectively.

• Kind $(2)$: $A_{2n}^{(2)}, C_n^{(1)}, A_{2n-1}^{(2)}$: attach 0 to 1 with a double bond pointing towards 1, in the diagrams $B_n, C_n,$ and $D_n$ respectively.

• Kind $(1,1)$: $B_n^{(1)}, A_{2n-1}^{(2)}, D_n^{(1)}$: Attach 0 to the vertex 2 by a single bond, in the diagrams $B_n, C_n,$ and $D_n$ respectively.

For $\lambda, \mu \in P_m$ and $n \geq m$ we define the stable 1-d sums

$$X_\lambda^\diamondsuit_{\mu}(q) = \sum_{\nu \in P_m} c_{\lambda, \nu}^{\mu} K_{\nu, \mu}(q)$$ (11)

with $\diamondsuit \in \{\emptyset, (1), (2), (1,1)\}$ and $\hat{g}_n$ related as above. The stable 1-d sums of type $A_{n-1}^{(1)}$ satisfy

$$X_\lambda^{\emptyset}_{\mu}(q) = K_{\lambda, \mu}(q) := q^{||\mu||} K_{\lambda, \mu}(q^{-1})$$ (12)

where $K_{\lambda, \mu}(q)$ is the Kostka-Foulkes polynomial ($24$, III.6), $K_{\lambda, \mu}(q)$ is the “cocharge” Kostka-Foulkes polynomial, and

$$||\mu|| = \sum_{i=1}^{m} (i - 1)\mu_i.$$ (13)

The equality (12) of type $A_{n-1}^{(1)}$ 1-d sums (of tensor products of symmetric or exterior powers) with the Kostka Foulkes polynomials, was proved by Nakayashiki and Yamada ($25$) using the combinatorial characterization of Kostka-Foulkes polynomials due to Lascoux and Schützenberger ($16$).

1.5 $X = K$

Let $P_\diamondsuit$ be the set of partitions that can be tiled with the partition $\diamondsuit$. Then $P_{\emptyset}$ contains just the empty partition, $P^{(1)} = P$ is the set of all partitions, $P^{(2)}$ consists of the partitions with even rows and $P^{(1,1)}$ the partitions with even columns. Write $P_m^{\diamondsuit} = P_\diamondsuit \cap P_m$. Let

$$|\mu| = \sum_{i=1}^{m} \mu_i \quad \text{for any } \mu \in \mathbb{Z}^m.$$ (14)

The $K$-polynomials ($30$) are defined by

$$K_{\lambda, \mu}(q) = q^{\frac{|\mu|-|\lambda|}{2}} \sum_{\nu \in P_m} K_{\nu, \mu}(q) \sum_{\gamma \in P_m^{\diamondsuit}} c_{\nu, \gamma}^{\lambda, \mu}.$$ (15)

where $c_{\nu, \gamma}^{\lambda, \mu}$ is the Littlewood-Richardson coefficient ($24$, (I.5.2)). They are just (up to degree shift) nonnegative integer combinations of cocharge Kostka-Foulkes polynomials.

Our first main theorem is the $X = K$ theorem, which was conjectured in ($30$).

**Theorem 4.** ($X = K$) Let $\lambda, \mu \in P_m$. Then for every $\diamondsuit \in \{\emptyset, (1), (2), (1,1)\}$,

$$X_{\lambda, \mu}(q) = K_{\lambda, \mu}(q).$$ (16)
The case \( \diamond = \emptyset \) is trivial. The cases \( \diamond = (1) \) and \( \diamond = (2) \) were proved in [29] using the virtual crystal construction of [27], which embeds affine crystals of types \( D_{n+1}^{(2)}, A_{2n}^{(2)}, \) and \( C_{n}^{(1)} \) into those of type \( A_{2n-1}^{(1)} \). We establish the remaining case \( \diamond = (1,1) \) in section 3. Since such an affine crystal embedding into type \( A_{2n}^{(1)} \) does not exist for types \( B_{n}^{(1)}, A_{2n-1}^{(2)}, \) or \( D_{n}^{(1)} \), we must use an entirely different approach to relate the 1-d sums \( \sum_{i}^{(1,1)}(q) \) to those of type \( A_{2n-1}^{(1)} \).

1.6 Lusztig \( q \)-analogues

Let \( \rho = \frac{1}{2} \sum_{\alpha \in R^{+}} \alpha = \sum_{i \in I} \omega_{i} \). Fix a function \( L : R^{+} \rightarrow \mathbb{Z} \) that is constant on the orbits of the Weyl group \( W \). For \( \lambda, \mu \in P^{+} \), define the polynomial

\[
KL_{\lambda,\mu}^{g,L}(q) = \sum_{w \in W} (-1)^{w} \ell(w) [e^{w(\lambda+\rho)-(\mu+\rho)}] \prod_{\alpha \in R^{+}} \frac{1}{1 - q^{L(\alpha)} e^{\alpha}}
\]

where \([e^{\beta}]f\) denotes the coefficient of \( e^{\beta} \) in \( f \in \mathbb{Z}[P] \). It follows from the Weyl character formula that \( KL_{\lambda,\mu}^{g,L}(1) \) is the multiplicity of the weight \( \mu \) in \( V(\lambda) \). When \( L(\alpha) = 1 \) for all \( \alpha \in R^{+} \), \( KL_{\lambda,\mu}^{g,L}(q) = KL_{\lambda,\mu}^{g}(q) \) is Lusztig’s \( q \)-analogue of weight multiplicity [23].

Lusztig’s \( q \)-analogues are certain affine Kazhdan-Lusztig polynomials. Let \( \hat{W} \cong P \times W \) be the extended affine Weyl group, realized as a subgroup of isometries of the weight lattice \( P \). For \( \lambda \in P \) let \( t_{\lambda} \in \hat{W} \) be translation by \( \lambda \). For \( \lambda \in P^{+} \) let \( w_{\lambda} \) be the element of maximal length in the double coset \( Wt_{\lambda}W \subset \hat{W} \). Then

\[
KL_{\lambda,\mu}^{g}(q) = q^{(\ell(w_{\lambda})-\ell(w_{\mu}))/2} P_{w_{\mu},w_{\lambda}}^{\hat{g}}(q^{-1})
\]

where \( P_{x,y}^{\hat{g}}(q) \) is the Kazhdan-Lusztig polynomial [23].

1.7 Stable KL polynomials

We consider a subfamily of Lusztig \( q \)-analogues that first studied in [19] [20]. Let \( \lambda, \mu \in P_{m} \) with \( n \geq m \), regarded as dominant weights for \( g \in \{A_{n-1}, B_{n}, C_{n}, D_{n} \} \). Define the stable KL polynomial

\[
\infty KL_{\lambda,\mu}^{g_{n},L}(q) = \sum_{w \in S_{n}} (-1)^{\ell(w)} [e^{w(\lambda+\rho)-(\mu+\rho)}] \prod_{\alpha \in R^{+}(g_{n})} (1 - q^{L(\alpha)} e^{\alpha})^{-1}.
\]

This is an expression for \( g_{n} \), but the sum runs over the parabolic subgroup of \( W \) generated by \( s_{i} \) for \( 1 \leq i \leq n - 1 \); this is a copy of the symmetric group \( S_{n} = W(A_{n-1}) \), the Weyl group of type \( A_{n-1} \). The following stability phenomenon, which justifies the above nomenclature, was observed in [19] Theorem 5.1.5.

**Proposition 5.** With \( \lambda, \mu \) as above, we have

\[
KL_{\lambda,\mu}^{g_{n},L}(q) = \infty KL_{\lambda,\mu}^{g_{n},L}(q)
\]

provided that \( k \geq (|\lambda| - |\mu|)/2 \). In particular the left hand side of (20) does not change for \( k \) sufficiently large. Moreover for \( g_{n} = A_{n-1} \) every Lusztig \( q \)-analogue is a stable KL polynomial and Kostka-Foulkes polynomial

\[
\infty KL_{\lambda,\mu}^{A_{n-1}}(q) = KL_{\lambda,\mu}^{A_{n-1}}(q) = K_{\lambda,\mu}(q).
\]
Proof. It is well known that $\lambda \geq \mu \lambda$ for all $w \in W$ and $\lambda \in P^+$, where

$$\lambda \geq \mu \quad \text{if and only if} \quad \lambda - \mu \in Q^+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i.$$  \hfill (22)

In particular $w(\lambda + \rho) - (\mu + \rho) \leq \lambda - \mu$. From this it follows that

$$\text{KL}^a_{\lambda,\mu}(q) = 0 \quad \text{unless} \quad \lambda \geq \mu$$  \hfill (23)

for if $\lambda \not\geq \mu$ then every summand is zero. Note that the definition \ref{equality(21)} makes sense for pairs of elements in the set

$$\mathbb{Z}^n_\geq = \{ \lambda \in \mathbb{Z}^n | \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \}$$  \hfill (24)

where the parts of $\lambda$ may be negative. Since every $w \in S_n$ fixes all multiples of $(1^n)$,

$$\text{KL}^a_{\lambda,\mu}(q) = \text{KL}^a_{\lambda,\mu}(q)$$  \hfill (25)

Let us assume that $k \geq (|\lambda| - |\mu|)/2$, recalling the definition \ref{equality(19)}.

In the case $g_n = A_{n-1}$, since $\lambda - \mu \in Q^+$ and $|\alpha_i| = 0$ for all $1 \leq i \leq n - 1$, we have $|\lambda| = |\mu|$. Thus all Lusztig $q$-analogues are stable KL polynomials and \ref{equality(18)} holds since $W(A_{n-1}) = S_n$. The equality \ref{equality(21)} may be taken as the definition of the Kostka-Foulkes polynomials.

Now let $g_n \in \{B_n, C_n, D_n\}$. Suppose that $w \in W \setminus S_n$. It suffices to show that the $w$-th summand in $\text{KL}^a_{\lambda+(k^n),\mu+(k^n)}(q)$ is zero. Suppose not. Then $\beta = w(\lambda + (k^n) + \rho) - (\mu + (k^n) + \rho) \in Q^+$. In particular $|\beta| = (\beta, (1^n)) \geq 0$ since $(1^n)$ is a positive integer multiple of a fundamental weight. Since $w \notin S_n$, $w(k^n)$ must have some coordinate $-k$ so that $|w(k^n) - (k^n)| \leq -2k$. We also have $|w(\rho) - \rho| < 0$ since $\rho$ has nonnegative coordinates and $w(\rho)$ has a strictly negative coordinate. We obtain the contradiction

$$0 \leq |w(\lambda + (k^n) + \rho) - (\mu + (k^n) + \rho)| \leq |w(k^n) - (k^n)| + |w(\rho) - \rho| + |w(\lambda) - \mu| < -2k + |\lambda| - |\mu| \leq 0.$$  \hfill \Box

1.8 $K = \infty$KL

We now state the $K = \infty$KL theorem, which asserts the equality of the $K$-polynomials and the stable KL polynomials. Let $\lambda, \mu \in P_m$ with $n \geq m$. Define the partitions

$$\hat{\lambda} = (M - \lambda_m, \ldots, M - \lambda_1) \quad \text{and} \quad \hat{\mu} = (M - \mu_m, \ldots, M - \mu_1)$$

where $M \geq \max(\lambda_1, \mu_1) + (|\mu| - |\lambda|)/2$.\hfill (26)

Theorem 6. ($K = \infty$KL) We have

$$\text{KL}^{\diamond}_{\lambda,\mu}(q) = q^{(|\mu| + |\mu| - |\lambda|)} \infty \text{KL}^a_{\lambda,\mu}(q^{-1})$$

$$= q^{(|\mu| + |\mu| - |\lambda|)} \text{KL}^a_{\lambda,\mu}(q^{-1})$$  \hfill (27)

where $\diamond = \{\emptyset, (1), (2), (1,1)\}$ respectively correspond to $g_n = \{A_{n-1}, B_n, C_n, D_n\}$ and $L(\alpha) = 1$ for all $\alpha \in R^+$ except for $g_n = B_n$, in which case $L(\alpha) = \frac{1}{2}$ for short roots.
The choice of $M$ guarantees that the KL polynomials $KL_{\lambda,\mu}^{g_n,L}(q)$ are stable; the second equality in (27) holds by Proposition 5. Theorem 6 proved in section 4 by a type-independent argument.

For $\diamondsuit = \emptyset$, $K = \infty$KL reduces to the definition of $K_{\lambda,\mu}(q)$ in (12) due to a duality satisfied by the Kostka-Foulkes polynomials (see [15] or [58]):

$$K_{\lambda,\mu}(q) = K_{\lambda,\mu}^\ast(q).$$

For $\diamondsuit = (2)$ and $\diamondsuit = (1,1)$, $K = KL$ was already proved in [19] [20].

For $\diamondsuit = (1)$, in [20] it was observed that (27) is false if one takes $L(\alpha) = 1$ for all $\alpha \in R^+$, that is, when the right hand side is the usual Lusztig $q$-analogue for type $B_n$.

Combining $X = K$ and $K = \infty$KL (Theorems 4 and 6) we obtain

**Corollary 7.** ($X = \infty$KL Theorem) With notation as in Theorem 6,

$$X_{\lambda,\mu}^\diamondsuit(q) = q^{||\mu|| + ||\mu|| - |\lambda|} KL_{\lambda,\mu}^{g_n,L}(q^{-1})$$

## 2. Explicit description of crystals and energy functions

In this section we give details for the 1-d sums involved in our proof of $X = K$ for $\diamondsuit = (1,1)$.

### 2.1. Notation for KR crystals

Let $\mu \in \mathbb{Z}_{>0}^m$. By abuse of notation, denote by $B^{\mathfrak{g}_n}_{\mu}$ the affine crystal graph given by the tensor product defined in (9) for the affine root systems $\mathfrak{g}_n \in \{A_{n-1}^{(1)}, A_{2n-1}^{(2)}, D_{n}^{(1)}\}$, which have respective simple Lie subalgebras $\mathfrak{g}_n \in \{A_{n-1}, C_n, D_n\}$; here $D_{n}^{(1)}$ is the affine root system $D_{n}^{(1)}$ except that $n$ is regarded as the affine node, so that $D_{n}^{(1)}$ is its classical sub root system. We have [12] [10]

$$B^{\mathfrak{g}}_{\mu} \cong B^{\mathfrak{g}(s\omega_1)}_{\mu} \quad \text{as } \mathfrak{g}\text{-crystals for } \mathfrak{g}_n \in \{A_{n-1}^{(1)}, A_{2n-1}^{(2)}, D_{n}^{(1)}\}. \quad (30)$$

From now on we shall use $D_{n}^{1}$ and $D_{n}^{(1)}$ instead of $D_n$ and $D_n^{(1)}$ unless specifically indicated otherwise.

### 2.2. $A_{n-1}$, $C_n$ and $D_n$ crystals

The crystals of type $A_{n-1}, C_n, D_n^{1}$ are colored directed graphs with colors in the respective sets $J^A = \{1, 2, \ldots, n-1\}$, $J = \{1, 2, \ldots, n\}$, and $J^\dagger = \{0, 1, 2, \ldots, n-1\}$. The $U_q(\mathfrak{g})$-modules associated to the partition $\lambda = (1, 0, \ldots, 0) \in P_n$ are the vector representations $V^{A_{n-1}}(\omega_1)$, $V^{C_n}(\omega_1)$, and $V^{D_n}(\omega_{n-1}^\dagger)$ respectively. Their respective dimensions are $n, 2n, 2n$. We call these crystal graphs $B_{1}^{A_{n-1}}, B_{1}^{C_n}$, and $B_{1}^{D_n}$ respectively. This agrees with the notation of the previous section, except that the crystal graphs pictured above are $\mathfrak{g}$-crystals as opposed to $\mathfrak{g}_n$-crystals. The crystal graphs $B^{\mathfrak{g}}_{\mu}$ are depicted in Figure 2. The notation $B^{\mathfrak{g}_n}_{\mu}$ is consistent with the usual Dynkin labeling $D_n$, but the pictured crystal graph uses the conventions of $D_n^{1}$. We use the set of symbols $\{1, 2, \ldots, n, \overline{1}, \ldots, \overline{n}, \overline{2}, \overline{1}\}$, with weights given by $\text{wt}(i) = \varepsilon_i$ and $\text{wt}(\overline{i}) = -\varepsilon_i$ for $1 \leq i \leq n$. 
We define a partial order $\leq$ on each set $B^g_1$ based on reachability.

\[
\begin{align*}
(B^{A_{n-1}}_1, \leq_{A_{n-1}}) &= \{1 < 2 < \cdots < n\} \\
(B^{C_n}_1, \leq_{C_n}) &= \{1 < 2 < \cdots < n < \overline{n} < \cdots < \overline{2} < \overline{1}\} \\
(B^{D_n}_1, \leq_{D_n}) &= \{\overline{n} < \cdots < \overline{2} < \overline{1} < 2 \cdots < n\}.
\end{align*}
\]

The letters 1 and $\overline{1}$ are not comparable for $\leq_{D_n}$.

For $g_n \in \{A_{n-1}, C_n, D_n\}$ there is an obvious bijection between the vertices of $B^g_1 = (B^g_1)^{\otimes s}$ and the words of length $s$ in the set of symbols $B^g_1$ given by $b = x_1 \otimes x_2 \otimes \cdots \otimes x_l \mapsto w(b) := x_1 x_2 \cdots x_l$. Denote its inverse by $w \mapsto b(w)$.

### 2.3 Tableaux of type $A_{n-1}, C_n, D_n$

Let $g \in \{A_{n-1}, C_n, D_n\}$. Kashiwara and Nakashima \cite{kn} introduced $g$-tableaux, which for $g = A_{n-1}$ are the well-known semistandard tableaux. For $\lambda \in P_n$, the $g$-tableaux of shape $\lambda$ give a natural labeling of the crystal $B^g_1(\lambda)$. They are fillings of the Young diagram associated to $\lambda$ which are semistandard for the orders defined on the set of symbols $B^g_1$ and satisfy additional conditions on their rows and columns detailed in \cite{kn} and \cite{m}. Write $T^g$ for the set of $g$-tableaux.

**Example 8.** $T_1 \in T^{C_4}$ and $T_2 \in T^{D_4}$ where

\[
T_1 = \begin{array}{ccc}
1 & 3 & 3 \\
3 & 3 \\
2 & 2
\end{array} \quad T_2 = \begin{array}{ccc}
4 & 3 & 1 \\
2 & 2 \\
2 & 3
\end{array}.
\]

In the sequel we identify the word $x_1 \cdots x_s$ with the row tableau

\[
L = \begin{array}{ccc}
x_s & \cdots & x_2 & x_1
\end{array}.
\]

**Proposition 9.** \cite{kn} For $g \in \{A_{n-1}, C_n, D_n\}$, the vertices of $B^g_1$ can be labeled by the decreasing words of length $s$ on the set $B^g_1$, that is, by the words $x_1 x_2 \cdots x_s$ with $x_1 \geq^g x_2 \geq^g \cdots \geq^g x_s$. 

\[11\]
In particular a decreasing word of type $D_n$ cannot contain both letters $1$ and $\overline{1}$.

**Remark 10.** Let $g = C_n, D_n$ and $\mu \in \mathbb{Z}_{\geq 0}^m$ with $n \geq m$. The subset of elements in $B^g_{\mu}$ that involve no barred letters, can be identified with $B^g_{\mu - A_{n-1}}$ as $A_{n-1}$-crystals. So we can write $B^g_{\mu - A_{n-1}} \subseteq B^g_{\mu}$.

## 2.4 Plactic monoids and insertion algorithms for types $A_{n-1}, C_n, D_n$

Let $g \in \{A_{n-1}, C_n, D_n\}$. Consider the tensor crystal $G^g = \bigoplus_{l \geq 0} (B^l)^{\otimes l}$. Denote by $\sim^g$ the equivalence relation on the vertices of $G^g$ defined by $b \sim^g b'$ if $b$ and $b'$ are in the same connected component. The quotient set $G^g/\sim^g$ is a monoid [18] [17] 21 which is a quotient of the free monoid on the set $B^g_{\mu}$ by two kinds of plactic relations. The first consists of relations of length 3. They define Lascoux-Schützenberger’s plactic monoid [15]:

\[
\begin{align*}
xyz & \equiv yxz & \text{for } x \leq y < z \text{ and } \\
xzy & \equiv zyx & \text{for } x < y \leq z.
\end{align*}
\]

For $g = C_n$ we obtain

\[
\begin{align*}
R_1^C : & \ yzx \equiv yxz \text{ for } x \leq C y < C z \text{ with } z \neq \overline{x}, \text{ and } xzy \equiv zyx \text{ for } x < C y \leq C z \text{ with } z \neq \overline{x} \\
R_2^C : & \ y(x-1)(x-1) \equiv yx\overline{x} \text{ and } x\overline{y}y \equiv (x-\overline{1})(x-1)y \text{ for } 1 < C x \leq C n \text{ and } x \leq C y \leq C \overline{x}
\end{align*}
\]

and for $g = D_n$ we have:

\[
\begin{align*}
R_1^D : & \ \text{If } x \neq \overline{x}, \ yzx \equiv yxz \text{ for } x \leq D y < D z \text{ and } xzy \equiv zyx \text{ for } x < D y \leq D z \\
R_2^D : & \ \text{If } 1 < D x < D n \text{ and } \overline{x} \leq D y \leq D x, \ y(x+1)(x+1) \equiv y\overline{x}x \text{ and } \overline{x}y \equiv (x+1)(x+1)y \\
R_3^D : & \ \text{If } x \notin \{1, \overline{1}\}, \ \left\{ \begin{array}{l}
x\overline{x} \equiv x \overline{x} \text{ and } \\
x\overline{1} \equiv x \overline{1} \text{ and } \\
x1 \equiv x \overline{1}
\end{array} \right. \\
R_4^D : & \ \left\{ \begin{array}{l}
\overline{11} \equiv 221 \\
1 \overline{11} \equiv 221 \text{ and } \\
11 \overline{11} \equiv 221 \text{ and } \\
1 \overline{11} \equiv 221
\end{array} \right.
\end{align*}
\]

The second kind of relation is called a “contraction relation”. This kind of relation does not preserve the length of words. These relations reflect the crystals isomorphisms $B(1 \cdots p\overline{p}) \simeq B(1 \cdots p - 1)$ with $p \in \{1, \ldots, n\}$ for $g = C_n$ and $B(\overline{p} \cdots q\overline{q}) \simeq B(\overline{p} \cdots q + \overline{1})$ with $q \in \{1, \ldots, n\}$ for $g = D_n$. The reader is referred to [18] and [17] for a complete description of these relations.

For each $g$ there is an insertion scheme that is compatible with the above plactic relations for $g$-tableaux [18] [17]. We shall call this $g$-insertion. $A_{n-1}$-insertion is the well-known Robinson-Schensted insertion algorithm on semistandard tableaux [11]. Consider a word $w = x_1 \cdots x_s$ with $x_i \in B^g_{\mu}$ such that $b(w)$ belongs to a connected component of $G^g$ isomorphic to $B^g(\lambda)$. $g$-insertion computes the unique $g$-tableau $P^g(w)$ occurring in $B^g(\lambda)$ such that $b(w) \sim^g P^g(w)$.

To define $g$-insertion it suffices to describe the insertion denoted $x \xrightarrow{g} P^g_T$ of the letter $x$ into the tableau $T$. This insertion is obtained directly from the plactic relations. For any word $w = x_1 \cdots x_s$, $P^g(w)$ is defined recursively by

\[
P^g(x_1 \cdots x_i) = (x_i \xrightarrow{g} P^g(x_1 \cdots x_{i-1}))
\]

where the $g$-insertion of the empty word results in the empty tableau. For complete details on $g$-insertion see [18] [17].

For $\delta \in \mathbb{Z}_{\geq 0}$ let $F^g_{\delta - A_{n-1}}$ be the set of $A_{n-1}$-highest weight vectors in $B^g_{\delta - A_{n-1}}$. 

12
Lemma 11. For any \( \delta \in \mathbb{Z}_{\geq 0}^m \) with \( m < n \) and \( \nu \in \mathcal{P}_m \), there is a bijection \( \Psi \) from the set of \( A_{n-1} \)-components of \( B_\delta^{A_{n-1}} \), to the set of \( D_n \)-components of \( B_\delta^{D_n} \) whose highest weight is a partition \( \nu \) such that \(|\nu| = |\delta|\). It is defined by

\[
\Psi(B_\delta^{A_{n-1}}(b)) = B_\delta^{D_n}(b)
\]  

(31)

for any \( b \in B_\delta^{A_{n-1}} \) where \( B_\delta^{A_{n-1}}(b) \) is the \( A_{n-1} \)-component of \( b \) in \( B_\delta^{A_{n-1}} \) and \( B_\delta^{D_n}(b) \) is the \( D_n \)-component of \( b \) in \( B_\delta^{D_n} \).

Remark 12. 1. \( T_\delta^{D_{n-1}} \subset T_\delta^{D_n} \).

2. For \( T \in T_\delta^{D_{n-1}} \) and \( b = x_1 \cdots x_s \in B_\delta^{D_n} \), no contraction occurs during the insertions

\[
x_i \xrightarrow{D_n} (x_2 \xrightarrow{D_n} (\cdots (x_1 \xrightarrow{D_n} T))).
\]

3. Let \( b \in B_\delta^{D_n} \). Then \( B_\delta^{D_n}(b) \) is in the image of the map \( \Psi \) of (31) if and only if there is no contraction during the insertion procedure \( D_n(b_1 b_2 \cdots b_m) \).

2.5 Left splitting embeddings

Lemma 13. For any nonnegative integers \( k, l \) and for \( g = A_{n-1}, C_n, D_n \) there is a unique \( g \)-crystal isomorphism

\[
\sigma : B_l \otimes B_k \to B_k \otimes B_l
\]

Proof. For \( g = A_{n-1}, C_n, D_n \) this follows respectively from the uniqueness of the combinatorial R-matrix for \( \hat{g} = A_{n-1}^{(1)}, A_{2n-1}^{(2)}, D_n^{(1)} \), acting on the affine crystal \( B_l \otimes B_k \). \( \square \)

Remark 14. Lemma 13 follows more simply from the fact that in the above cases, \( B_l \otimes B_k \) is multiplicity-free. This is seen from the following list of highest weight vertices of \( B_l \otimes B_k \) computed by Lemma 1.

\[
v_{l,k; a,b}^{A_{n-1}} = 1^l \otimes 2^b 1_{k-b} \quad 0 \leq b \leq \min(k, l)
\]

\[
v_{l,k; a,b}^{C_n} = 1^l \otimes T^a b_{k-a-b} \quad a, b \geq 0, a + b \leq \min(k, l)
\]

\[
v_{l,k; a,b}^{D_n} = T^a \otimes n^{a-b} 1^n b_{k-a-b} \quad a, b \geq 0, a + b \leq \min(k, l).
\]

Remark 15. The crystal isomorphism \( \sigma^g \) can be computed using insertion for \( g \)-tableaux [17] [18] [25] [3] [4].

Let \( g = A_{n-1}, C_n, D_n \). Let \( b = b_1 \otimes \cdots \otimes b_m \in B_\delta \) and \( b_1 = x_1 x_2 \cdots x_{\delta_1} \). The left splitting operation is defined by

\[
S : B_\delta \to B_{\delta'}
\]

\[
b \mapsto x_1 \otimes b_1' \otimes b_2 \otimes \cdots \otimes b_m
\]

(32)

where \( \delta' = (1, \delta_1 - 1, \delta_2, \ldots, \delta_m) \) and \( b_1' = x_2 \cdots x_{\delta_1} \).

Suppose \( \delta_i = 1 \) for some particular \( i \). Write \( b_i = x_i \) where \( x_i \) is a letter. Let

\[
\sigma_i : B_\delta \leftrightarrow B_{\sigma_i(\delta)}
\]

\[
b_1 \otimes \cdots \otimes x_i \otimes b_{i+1} \otimes \cdots \otimes b_m \mapsto b_1 \otimes \cdots \otimes b_i' \otimes x_i' \otimes \cdots \otimes b_m
\]

(33)
where \( \sigma_i(\delta) \) is obtained by switching the \( i \)-th and the \( i+1 \)-th parts of \( \delta \) and \( \sigma(x_i \otimes b_{i+1}) = b'_i \otimes x'_i \) is the crystal isomorphism of Lemma 13.

By composing left splitting and switching operations one may define \( g \)-crystal embeddings \[ S_\delta : B_\delta \to B_{1^{[\delta]}} \] (34)

We define \( S_\delta \) by descending induction on the number of parts of \( \delta \) equal to 1, and then by the minimum index \( k \) such that \( \delta_k > 1 \). \( S_\delta \) is defined to be the identity if every part of \( \delta \) is 1. Otherwise some part of \( \delta \) is greater than 1. Let \( k \) be as above. If \( k = 1 \) then we first split the left tensor factor and induct: \( S_\delta = S_{\delta'} \circ S \) with the above notation for \( S : B_\delta \to B_{\delta'} \). If \( k > 1 \), let \( s = \delta_k \). We move this part closer to the front using the isomorphism \( \psi_s : B_1 \otimes B_s \to B_s \otimes B_1 \) acting at the \((k-1)\)-th and \( k \)-th tensor positions, and then induct. That is, writing \( s_{k-1} \delta \) for \( \delta \) with its \((k-1)\)-th part \( 1 = \delta_{k-1} \) and \( k \)-th part \( s = \delta_k \) interchanged, define \( S_\delta = S_{s_{k-1}} \circ \psi_s \) where it is understood that \( \psi_s \) acts at the \((k-1)\)-th and \( k \)-th tensor factors.

Remark 16. One may take a shortcut: instead of applying a splitting map \( S : B_s \to B_1 \otimes B_{s-1} \) on the leftmost tensor factor, one may split \( S : B_s \to B_{s}^\otimes s \) all at once, chopping a word of length \( s \) in \( B_s \) into its constituent letters.

Example 17. Let \( x^\otimes_p = x \otimes x \otimes \cdots \otimes x \) denote the \( p \)-th tensor power of \( x \).

1. Computing the map \( S_{(l,k)}^C \) on the element \( v_{l,k,a,b}^C \) of Remark 14 the following elements occur as intermediate values.

\[
1^l \otimes T^{2b}1^{k-a-b} \quad 1^l \otimes T^a 2^{k-a-b} \quad 1^l 2^b \otimes T^a 1^{b} \otimes T^a \quad 1^l 2^{b-k} \otimes 1^a \otimes 2^b \otimes T^a
\]

2. For \( b = 1^4 \otimes T21 \otimes T21 \in B_{D(4,3,3)} \) we obtain \( S_{(4,3,3)}^D(b) = 1^5 \otimes 2 \otimes T^2 \otimes 2 \otimes 1 \) in \( B_{10}^D \).

Remark 18. Let \( g = C_n, D_n \). With \( B_n^{A_n-1} \subset B_n^g \) as in Remark 10 the restriction of the map \( S_\delta \) to \( B_n^{A_n-1} \) agrees with the \( A_n-1 \)-crystal embedding \( S_\delta^{A_n-1} \).

Remark 19. One need not split and permute in exactly the order prescribed in the definition of \( S_\delta \). One may compute \( S_\delta \) by applying any sequence of operations going from \( S_\delta \) to \( S_{1_{[\delta]}} \) comprised of splitting the leftmost tensor factor and permuting factors by combinatorial \( R \)-matrices.

### 2.6 Coenergy functions

In this section we follow [11] [25] [6] [27].

From now on, we write \( D \) and \( \overline{D} \) (resp. \( D^A \) and \( \overline{D}^A \)) for the coenergy functions associated to affine crystals of type \( A_{2n-1}^{(2)} \) (resp. \( A_{n-1}^{(1)} \)). By definition, to specify the coenergy function \( D \) for the affine crystal \( B_{\mu}^{C_n} \) of type \( A_{2n-1}^{(2)} \), it suffices to specify \( D_{B_k} \) and \( \overline{D} = \overline{D}_{B_1,B_k} \) for the affine crystals \( B_l \) and \( B_k \) of type \( A_{2n-1}^{(2)} \) for \( l \geq k \), and similarly for \( D^A \).

By (30) for \( \widehat{g}_n = A_{2n-1}^{(2)} \) and \( \widehat{g}_n = A_{n-1}^{(1)} \), \( D_{B_k} \) and \( D^A_{B_k} \) are constant functions. We normalize them to have the value 0:

\[
D_{B_k} = 0 \quad D^A_{B_k} = 0.
\] (35)

To normalize \( \overline{D} \) and \( \overline{D}^A \) on \( B_1 \otimes B_k \) it suffices to specify a single function value. We set

\[
\overline{D}(1^l \otimes 1^k) = 0 \quad \overline{D}^A(1^l \otimes 1^k) = 0.
\] (36)
Remark 20. Let $\overline{H}$ denote either $\overline{H}_{B_1,B_k}$ for $A^{(2)}_{2n-1}$ crystals, or $\overline{H}^A_{B_1,B_k}$ for $A^{(1)}_{n-1}$-crystals. Then $\overline{H}$ can be computed using respectively $C_n$- or $A_{n-1}$-insertion [3] [25]. Given a vertex $b = b_1 \otimes b_2 \in B_1 \otimes B_k$, we have $\overline{H}(b) = k + l - m$ where $m$ is the size of the first row of the tableau $P^g(b)$ obtained by inserting the row $b_2$ into the row tableau $b_1$.

Example 21. For the highest weight vertices $v^{C_n}_{l,k;a,b} \in B^1_l \otimes B^C_k$ in Remark 14 we have

$$P^{C_n}(v_{l,k;a,b}) = \begin{array}{cccc} 1 & \cdots & \cdots & 1 \\ 2 & \cdots & 2 \end{array}$$

which has shape $(l + k - 2a - b, b)$. Thus $\overline{H}(v_{l,k;a,b}) = 2a + b$.

Example 22. For $x, y \in B^C_1$, $\overline{H}_{B_1,B_1}$ is given by

$$\overline{H}(x \otimes y) = \begin{cases} 0 & \text{if } x \geq y \\ 1 & \text{if } x < y \text{ and } (x,y) \neq (1,\bar{1}) \\ 2 & \text{if } (x,y) = (1,\bar{1}). \end{cases}$$

(37)

For $x, y \in B^{A_{n-1}}_1$ $\overline{H}^A_{B_1,B_1}$ is given by

$$\overline{H}^A(x \otimes y) = \begin{cases} 0 & \text{if } x \geq y \\ 1 & \text{otherwise.} \end{cases}$$

(38)

Proposition 23. [26] Let $g \in \{A_{n-1}, C_n, D_n\}$ (so that $g \in \{A^{(1)}_{n-1}, A^{(2)}_{2n-1}, D^{(1)}_n\}$). For $\delta \in \mathbb{Z}^m_+$ the embedding of $g$-crystals $S^g_\delta : B^g_\delta \to B^{1[\delta]}_1$ preserves coenergy, that is, $D^g(b) = \overline{D}^g(S^g_\delta(b))$ for any $b \in B^\delta_\delta$.

Proof. This is proved in [26] for nonexceptional affine root systems, including $D_n \subset D^{(1)}_n$ with the standard Dynkin labeling. The result for $D^1_k \subset D^{(1)}_n$ follows by applying the affine Dynkin automorphism given by $i \mapsto n - i$ and the automorphism of $B^1_{D_n}$ given by $j \mapsto n + 1 - j$ and $\bar{j} \mapsto n + 1 - j$ for $1 \leq j \leq n$.

Example 24. Let us verify Proposition 23 for $\delta = (l,k)$ and the element $v = v^{C_n}_{l,k;a,b}$ of Remark 14. On one hand, $\overline{D}(b) = 2a + b$ by Example 21. By Example 17 $S^C_k(v) \in \mathbb{Z}^{c(a)}_\delta \otimes B^{1[\delta]}_\delta \otimes \overline{I}^{c(a)}$. Computing $\overline{D}(S^C_k(v))$ using (3) and (37) we obtain the answer $2(a + b) + a$, the sum of the positions (counting from the right) of the ascents $1 \otimes 2$ and $2 \otimes 1$ in $S^C_k(v)$.

2.7 One-dimensional sums for the affine crystals $B^C_{\mu}$

For the sake of completeness we prove Lemma 3 for the affine family $\tilde{A}_n = A^{(2)}_{2n-1}$.

Lemma 25. Let $m \leq n$ be nonnegative integers and $\delta \in \mathbb{N}^m$. Then every highest weight vertex $b \in B^C_{\delta}$ contains only letters of the set $\{1, \ldots, m, m-1, m-2, \ldots, \bar{1}\}$. 
Proof. We fix \( n \) and proceed by induction on \( m \). When \( m = 1 \), \( b = 1^{b_1} \) and the property holds. Let \( m \geq 2 \) and suppose the above property holds for \( m - 1 \). Let \( b = b_1 \otimes \cdots \otimes b_m \in B^{C_n}_{b^\delta} \) be a highest weight vertex for \( \delta \in \mathbb{Z}^m_{\geq 0} \). By Lemma 1 \( b^\delta = b_1 \otimes \cdots \otimes b_{m-1} \in B^{C_n}_{\delta} \) is a highest weight vertex with \( \delta^\prime = (\delta_1, \ldots, \delta_{n-1}) \). By induction \( b^\delta \) contains only letters in \( \{1, \ldots, m-1, m-2, \ldots, \bar{T}\} \); in particular \( \varphi_i(b^\delta) = 0 \) for all \( i \in \{m, \ldots, n\} \). If \( b_m \) contains a letter not in the set \( \{1, \ldots, m-1, m-2, \ldots, \bar{T}\} \) then there exists an integer \( i \in \{m, \ldots, n\} \) such that \( \varphi_i(b_m) \geq 1 \). By (1) we obtain \( \varphi_i(b) \geq 1 \) which contradicts the assumption that \( b \) is a highest weight vertex. \( \square \)

Corollary 26. Fix \( m \) and \( \lambda, \mu \in \mathcal{P}_m \). Then for all \( n \geq m \) the one dimensional sum \( X_{\lambda,\mu}^{A^{2n-1}}(q) \) does not depend on \( n \).

Proof. By the previous lemma the set \( F_{\lambda,\mu} \) does not depend on \( n \). The \( C_n \) plactic relations are stable for increasing \( n \), so the computations of combinatorial \( R \)-matrices and the local energy functions that occur in the computation of \( D \), are independent of \( n \). The corollary follows. \( \square \)

Remark 27. This also follows from the corresponding stability result for the fermionic formula \( M \) [30] and the \( X = M \) theorem for symmetric powers [28], but this is massive overkill.

3 Proof of \( X = K \) for \( \diamond = (1, 1) \)

Fix integers \( 0 \leq m < n \).

3.1 Combinatorial description of \( X = K \)

We reformulate the desired \( X = K \) identity [10] for \( \diamond = (1, 1) \). We recall a classical result of Littlewood [22, appendix p. 295] [14, Theorem A1]:

\[
\sum_{\gamma \in \mathcal{P}_m^{(m,1)}} c_{\lambda,\gamma}^\nu = [V^{D_n}(\nu) : V^{A_{n-1}}(\lambda)] \quad \text{for} \ \nu, \lambda \in \mathcal{P}_m \text{ with } n \geq m, \tag{39}
\]

which is the branching multiplicity of \( V^{A_{n-1}}(\lambda) \) in the restriction of \( V^{D_n}(\nu) \) to \( U_q(A_{n-1}) \). Using (39) and Lemma 3 the \( X = K \) formula (16) for \( \diamond = (1, 1) \) may be rewritten as

\[
\overline{X}_{\lambda,\mu}^{A^{2n-1}}(q) = q^{\nu(\lambda)} \sum_{\nu \in \mathcal{P}_m} \overline{K}_{\nu,\mu}(q) [V^{D_n}(\nu) : V^{A_{n-1}}(\lambda)]. \tag{40}
\]

Let \( E_{\lambda,\mu} \) be the set of \( A_{n-1} \)-highest weight vertices of weight \( \lambda \) in \( B_{\mu}^{D_n} \) belonging to the image of the map \( \Psi \) in Lemma 11. Given \( b \in F_{\mu}^{A_{n-1}} \), let \( E_{\lambda,\mu,b} \subset B_{\mu}^{D_n} \) be the set of \( A_{n-1} \)-highest weight vertices of weight \( \lambda \) in the component \( B_{\mu}^{D_n}(b) \subset B_{\mu}^{D_n} \). Then \( E_{\lambda,\mu} = \bigcup_{b \in F_{\mu}^{A_{n-1}}} E_{\lambda,\mu,b} \).

The identity (40) can be rewritten as

\[
\sum_{v \in F_{\lambda,\mu}} q^{D(v)} = q^{\nu(\lambda)} \sum_{b \in F_{\mu}^{A_{n-1}}} q^{D(b)} \sum_{c \in E_{\lambda,\mu,b}} 1, \tag{41}
\]

due to the definition of the 1-d sums \( \overline{X}_{\lambda,\mu}^{A^{2n-1}}(q) \) and \( \overline{K}_{\nu,\mu}(q) = \overline{X}_{\nu,\mu}^{A^{n-1}}(q) \), and Lemma [11]
Remark 28. To prove Theorem 4 for $\Diamond = (1, 1)$ it suffices to exhibit a bijection $\theta : F_{\lambda, \mu} \rightarrow E_{\lambda, \mu}$ such that, for every $v \in F_{\lambda, \mu}$ such that $\theta(v) \in E_{\lambda, \mu}$, $\tilde{D}(v) = \tilde{D}^A(b) + \frac{|\mu| - |\lambda|}{2}$. It also suffices to give a bijection $\theta : F_{\lambda, \mu} \rightarrow E_{\lambda, \mu}$ and a statistic $\tilde{D} : B^{D_n}_\mu \rightarrow \mathbb{Z}$ such that

1. $\tilde{D}$ is constant on $D_n$-components.
2. $\tilde{D}(v) = \tilde{D}(\theta(v)) + \frac{|\mu| - |\lambda|}{2}$.
3. $\tilde{D}(b) = \tilde{D}^A(b)$ for all $b \in F^{A_{n-1}}_\mu$.

Lemma 29. The sets $F_{\lambda, \mu}$ and $E_{\lambda, \mu}$ have the same cardinality.

Proof. It is equivalent to show that (16) holds for $\Diamond = (1, 1)$ at $q = 1$. Let $s^{(1,1)}_{\lambda}$ denote the universal character of Koike and Terada for the orthogonal groups [14] [30]. We have

$$\overline{X}^{(1,1)}_{\lambda, \mu}(1) = [V^{D_n}(\mu_1 \omega_1) \otimes \cdots \otimes V^{D_n}(\mu_m \omega_1) : V^{D_n}(\lambda)]$$

$$= [s^{(1,1)}_{\lambda}]_{\mu_1 \cdots \mu_m}$$

$$= [s^{(1,1)}_{\lambda}] s_{\mu_1} \cdots s_{\mu_m}$$

$$= [s^{(1,1)}_{\lambda}] \sum_{\nu \in P_m} K_{\nu, \mu} s_{\nu}$$

$$= \sum_{\nu \in P_m} K_{\nu, \mu}(1) \sum_{\tau \in P^{(1,1)}_m} c^\nu_{\tau}$$

using the following facts: by definition, the stable 1-d sum of kind (1, 1) is a graded tensor product for the affine root system $D^{(1)}_n$ which has $D_n$ as classical subalgebra with $B_s \cong B(s \omega_1)$; the orthogonal universal characters multiply like irreducible $D_n$ characters in sufficiently large rank [14]; $s^{(1,1)}_{\tau}$ is the Schur function $s_r$ [14]; the multiplicities of products of single row Schur functions are Kostka numbers; the Kostka polynomial at $q = 1$ is the Kostka number; and the Littlewood formula for the coefficient of an orthogonal universal character in a Schur function [14].

3.2 $A_{n-1}$-subcrystals of $B^{C_n}_\mu$ and $B^{D_n}_\mu$

For $s \in \mathbb{Z}_{>0}$ the $C_n$-crystal $B^{C_n}_s$ decomposes into $A_{n-1}$-connected components obtained by deleting the arrows of color $n$. This is given explicitly by

$$B^{C_n}_s \cong \bigoplus_{\alpha, \beta, \varepsilon \geq 0, \alpha + 2\varepsilon + \beta = s} B^A(\pi^{\alpha + \varepsilon} n^\varepsilon 1^\beta)$$

where $B^A(b)$ denotes the $A_{n-1}$-connected component of $b$. For $\alpha, \beta \geq 0$ with $\alpha + \beta = s$ define $v^{C_n}_{\alpha, \beta} = \pi^\alpha 1^\beta \in B^{C_n}_s$. Define the $A_{n-1}$-subcrystal $\tilde{B}^{C_n}_s \subset B^{C_n}_s$ by

$$\tilde{B}^{C_n}_s = \bigoplus_{\alpha + \beta = s} B^A(v^{C_n}_{\alpha, \beta}).$$

(42)

$\tilde{B}^{C_n}_s$ can also be characterized by the property that it is the largest $A_{n-1}$-subcrystal of $B^{C_n}_s$ such that none of its vertices contain the pair $\pi m$. 

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Similarly the $D_n^{[n]}$-crystal $B_{s}^{D_n}$ decomposes as an $A_{n-1}$-crystal as follows. For $\alpha, \beta \geq 0$ such that $\alpha + \beta = s$, let $v_{\alpha, \beta}^{D_n} = 1^{\beta} \pi^{\alpha}$. We have the $A_{n-1}$-decomposition
\[
B_{s}^{D_n} \cong \bigoplus_{\alpha + \beta = s} B^A(v_{\alpha, \beta}^{D_n}).
\] (43)

As $A_{n-1}$ crystals, $B^A(v_{\alpha, \beta}^{C_n}) \cong B^A(v_{\alpha, \beta}^{D_n}) \cong B^A(\beta \omega_1 + \alpha \omega_{n-1})$. Let $\theta_{\alpha, \beta}$ be the $A_{n-1}$-crystal isomorphism
\[
\theta_{\alpha, \beta} : B^A(v_{\alpha, \beta}^{C_n}) \to B^A(v_{\alpha, \beta}^{D_n}).
\]

These maps patch together to define an $A_{n-1}$-crystal isomorphism $\theta_s : \tilde{B}_{s}^{C_n} \to B_{s}^{D_n}$.

**Remark 30.** 1. In fact $B_{s}^{C_n} \cong \bigoplus_{r=0}^{\lfloor \frac{s}{2} \rfloor} B_{s-2r}^{D_n}$ as $A_{n-1}$-crystals. The element $b \in B_{s}^{C_n}$ is sent to the $r$-th summand $B_{s-2r}^{D_n}$ if $r$ is the maximum of the number of letters $n$ and $\pi$ in $b$. In particular, for $s = 1$, $\tilde{B}_{1}^{C_n} \cong B_{1}^{C_n}$ and $\theta_1 : B_{1}^{C_n} \to B_{1}^{D_n}$ is a $A_{n-1}$-crystal isomorphism given by the identity map.

2. The vertices of $B(v_{\alpha, \beta}^{C_n})$ (resp. $B(v_{\alpha, \beta}^{D_n})$) do not contain any pair $(\pi, n)$ (resp. $(1, \overline{1})$).

3. The map $\theta_{\alpha, \beta}$ may be computed applying transformations of the form
\[
\begin{align*}
\varpi p &\mapsto \varpi q &\text{ if } p \neq q \\
\varpi p &\mapsto (p + 1) \varpi &\text{ if } p \neq n.
\end{align*}
\] (44)

Let $\delta \in \mathbb{Z}_{\geq 0}$. Define the $A_{n-1}$-subcrystal $\tilde{B}_{\delta}^{C_n} \subset B_{\delta}^{C_n}$ by $\tilde{B}_{\delta}^{C_n} = \tilde{B}^{C_n}_{\delta_1} \otimes \cdots \otimes \tilde{B}^{C_n}_{\delta_m}$ and the map $\theta_{\delta} : \tilde{B}_{\delta}^{C_n} \to B_{\delta}^{D_n}$ by $\theta_{\delta} = \theta_{\delta_1} \otimes \cdots \otimes \theta_{\delta_m}$.

**Lemma 31.** The map $\theta_{\delta}$ is an isomorphism of $A_{n-1}$-crystals. In particular, for $\delta = (1^m)$, $\tilde{B}_{1}^{C_n} = B_{1}^{C_n}$ and $\theta_1$ is the identity, or equivalently, the restriction to $A_{n-1}$ of the $C_n$- and $D_n$-crystal structure on the set $B_{1}^{C_n} = B_{1}^{D_n}$ is the same.

**Proof.** This follows immediately from Remark 30 (1) and the fact that $\theta_s$ is an isomorphism of $A_{n-1}$-crystals.

**Lemma 32.** The map $\theta_{\delta}$ restricts to the identity on $B_{\delta}^{A_{n-1}}$.

**Proof.** It suffices to show this for one tensor factor. Let $\delta = (s)$. The restriction of $\theta_s$ to $B_{s}^{A_{n-1}}$ is the map $\theta_{0,s} : B^A(v_{0,s}^{C_n}) \to B^A(v_{0,s}^{D_n})$, which is the identity map on $B_{s}^{A_{n-1}}$.

3.3 The bijection $\theta : F \rightarrow E$

**Proposition 33.** For $\lambda \in \mathcal{P}_{m}$, $F_{\lambda,1^m} = E_{\lambda,1^m}$.

**Proof.** By Lemma 29 it suffices to show that $F_{\lambda,1^m} \subset E_{\lambda,1^m}$. Let $b = x_1 \otimes \cdots \otimes x_m \in F_{\lambda,1^m}$. By restriction from $C_n$ to $A_{n-1}$ it follows that $b \in B_{1}^{C_n}$ is a $A_{n-1}$ highest weight vertex of weight $\lambda$. By Lemma 31 $b \in B_{1}^{D_n}$ is an $A_{n-1}$ highest weight vertex of weight $\lambda$.

We must show that the component $B^{D_n}(b)$ is in the image of the map $\Psi$. By Remark 12 it suffices to show that the tableau $P^{D_n}(x_1 \cdots x_m)$ belongs to $T^{D_n}$ and no contraction happens during the corresponding insertion procedure. We proceed by induction on $m$. When $m = 1$, $b = 1$ and
We obtain 

\[ P_{n,n}^{1} P_{m,n}^{1} \]

Lemma 25

tensor factor and swapping adjacent tensor factors using maps of the form given in Lemma 13, which we shall denote by \( \theta \) defined in (32). This is the required compatibility of isomorphism. By Remark 12(3) it suffices to prove that the tableau compatible with these operations.

\[ \text{Tableau} \]

Proposition 33.

\[ \text{Tableau} \]

The required compatibility of \( \theta \) is obtained by Remark 12(2) no contraction occurs during the corresponding insertion procedure. The proof is essentially the same as in Proposition 33.

\[ \text{Tableau} \]

Proposition 34. Let \( \lambda, \mu \in \mathcal{P}_m \). Then \( F_{\lambda,\mu} \subseteq \hat{B}_\mu^C \) and the \( A_{n-1}\)-crystal isomorphism \( \theta_{\mu} : \hat{B}_\mu^C \rightarrow B_{\mu}^{D_{n}} \) restricts to a bijection \( \theta : F_{\lambda,\mu} \rightarrow E_{\lambda,\mu} \).

Proof. Lemma 25 says that the vertices of \( F_{\lambda,\mu} \) do not contain any pair of letters \((\pi, n)\). By the characterization of \( \hat{B}_\mu^C \subseteq B_{\mu}^C \) it follows that \( F_{\lambda,\mu} \subseteq \hat{B}_\mu^C \).

Due to Lemma 29 and the injectivity of \( \theta \), it suffices to show that \( \theta(F_{\lambda,\mu}) \subseteq E_{\lambda,\mu} \). Let \( b = b_1 \cdots b_m \in F_{\lambda,\mu} \). Then \( \theta(b) \) is an \( A_{n-1}\)-highest weight vertex of weight \( \lambda \) since \( \theta \) is an \( A_{n-1}\)-crystal isomorphism. By Remark 12(3) it suffices to prove that the tableau \( P_{n,n}^{1}(b_1 \cdots b_m) \) belongs to \( T^{D_{m}} \) and no contraction occurs during the corresponding insertion procedure. The proof is essentially the same as in Proposition 33.

\[ \text{Tableau} \]

Proposition 35. For \( \lambda \in \mathcal{P}_m \) and \( \delta \in \mathbb{Z}^m_{\geq 0} \) the following diagram commutes:

\[
\begin{array}{ccc}
F_{\lambda,\delta} & \xrightarrow{\theta_{\delta}} & E_{\lambda,\delta} \\
S_{\delta}^{-1} & \downarrow & S_{\delta}^{-1} \\
F_{\lambda,1[\delta]} & \xrightarrow{\theta_{[1,\delta]}} & E_{\lambda,1[\delta]} \\
\end{array}
\]

Proof. By definition \( S_{\delta} \) the map \( S_{\delta} \) can be computed using two operations: splitting the leftmost tensor factor and swapping adjacent tensor factors using maps of the form \( B_{1} \otimes B_{s} \rightarrow B_{s} \otimes B_{1} \) given in Lemma 13 which we shall denote by \( \psi_{s} \). Therefore it suffices to show that the maps \( \theta \) are compatible with these operations.

Let \( b \in B_{\delta\mu}^{C} \) be a highest weight vertex. By Lemma \( b = b_1 \cdots b_m \) with \( b_1 = 1^{\delta_1} \). By direct computation we obtain the equality \( \theta_{\delta'}(S(b)) = S \circ \theta_{\delta}(b) \) where \( S \) and \( \delta' \) are defined in (32). This is the required compatibility of \( \theta \) with splitting.

The required compatibility of \( \theta \) with the swapping operation is given by

\[
\theta_{(s,1)} \circ \psi_{1,s}^{C}(x \otimes v) = \psi_{1,s}^{D} \circ \theta_{(1,s)}(x \otimes v)
\]

(45)

for any vertex \( x \otimes v \) in \( B_{1}^{C} \otimes B_{s}^{C} \) such that there exists \( \delta \in \mathbb{Z}^m_{\geq 0}, b = b_1 \cdots b_m \) a highest weight vertex of \( B_{\delta}^{C} \) and \( 1 \leq i \leq m \) satisfying \( x = b_i \) and \( v = b_{i+1} \). Assume \( b, x, v, i \) satisfy these conditions. We have \( v \in \hat{B}_{\mu}^{C} \) by Proposition 34. Let \( \alpha, \beta \geq 0 \) be such that \( v = B(v^{C}_{\alpha,\beta}) \). By Lemma...
as an $A_{n-1}$-crystal, $B_1^{C_n} \otimes B(v_{\alpha,\beta}^{C_n})$ decomposes into six connected components. Below we give the six associated highest weight vertices and their images under $\psi^C$, computed using $C_n$-insertion.

\begin{align*}
b_1 &= 1 \otimes \bar{\pi}^\alpha 1^\beta \\
b_2 &= 1 \otimes \bar{\pi}^\alpha -1 1^\beta \\
b_3 &= 1 \otimes \bar{\pi}^\alpha 21^\beta -1 \\
b_4 &= \bar{\pi} \otimes \bar{\pi}^\alpha 1^\beta \\
b_5 &= \bar{\pi} \otimes n-1 \bar{\pi}^\alpha -1 1^\beta \\
b_6 &= \bar{\pi} \otimes n-1 \bar{\pi}^\alpha -1 (n-1)1^\beta -1
\end{align*}

\[\psi^C(b_1) = \bar{\pi}^\alpha -1 1^\beta +1 \otimes \bar{\pi}\]

\[\psi^C(b_2) = \bar{\pi}^\alpha -1 1^\beta +1 \otimes \bar{T}\]

\[\psi^C(b_3) = \bar{\pi}^\alpha 1^\beta \otimes 2\]

\[\psi^C(b_4) = \bar{\pi}^\alpha +1 1^\beta -1 \otimes 1\]

\[\psi^C(b_5) = \bar{\pi}^\alpha 1^\beta \otimes n-1\]

\[\psi^C(b_6) = \bar{\pi}^\alpha -1 \bar{\pi}n 1^\beta \otimes \bar{\pi}\]

If $x \otimes v$ belongs to the connected component of $B_1^{C_n} \otimes B(v_{\alpha,\beta}^{C_n})$ with highest weight vertex $b_6$, then $\sigma_i(b)$ (see [22]) contains a pair of letters $(n, \bar{\pi})$ and is a highest weight vertex of $B_\delta^{C_n}$. This contradicts Proposition [5]. Since $\theta, \psi_s^C$ and $\psi_s^D$ are isomorphisms of $A_{n-1}$-crystals, it is enough to establish (45) for the vertices $b_i, i \in \{1, \ldots, 5\}$. For $b_1$ we obtain the commuting diagram

\[
\begin{array}{ccc}
1 \otimes \bar{\pi}^\alpha 1^\beta & \xrightarrow{\theta} & 1 \otimes 1^\beta \bar{\pi}^\alpha \\
\psi^C & \downarrow & \psi^D \\
\bar{\pi}^\alpha -1 1^\beta +1 \otimes \bar{\pi} & \xrightarrow{\theta} & 1^\beta +1 \bar{\pi}^\alpha -1 \otimes \bar{\pi}
\end{array}
\]

where $\psi^D_s$ is computed using $D_n$-insertion and $\theta$ is obtained by (44). For $b_1$ we only need to apply plactic relations that preserve the letters during the $C_n$- and $D_n$-insertion, so that (45) is immediate. The proof is similar for $b_3, b_4$, and $b_5$. It only remains to consider $b_2$. We must apply a plactic relation of type $B_3^D$ to compute $\psi^D_s(b_2)$. This gives the commuting diagram

\[
\begin{array}{ccc}
1 \otimes \bar{\pi}^\alpha -1 1^\beta & \xrightarrow{\theta} & 1 \otimes 1^\beta -1 2 \bar{\pi}^\alpha -1 \\
\psi^C & \downarrow & \psi^D \\
\bar{\pi}^\alpha -1 1^\beta +1 \otimes \bar{T} & \xrightarrow{\theta} & 1^\beta +1 \bar{\pi}^\alpha -1 \otimes \bar{T}
\end{array}
\]

3.4 The map $\theta_\mu$ and the coenergy functions $\overline{D}$ and $\overline{D}$

We shall establish the crucial relation

\[\overline{D}(b) = \frac{|\mu| - |\lambda|}{2} + \overline{D}(\theta(b))\]

for any vertex $b \in F_{\lambda\mu}$ where $\overline{D}$ and $\overline{D}$ are the coenergy functions defined on $B_\mu^{C_n}$ and $B_\mu^{D_n}$, respectively. We first begin with the case $\mu = (1^m)$. Recall that $\overline{D} : B_1^{C_n} \rightarrow \mathbb{Z}$ and $\overline{D} : B_1^{D_n} \rightarrow \mathbb{Z}$.
are then the statistics on vertices \( b = x_1 \otimes \cdots \otimes x_m \) with \( x_i \in B_{1}^{C_n} = B_{1}^{D_n} \) defined by

\[
\overline{D}(b) = \sum_{i=1}^{m-1} (m - i) \overline{H}^i(x_i \otimes x_{i+1})
\]

\[
\overline{D}(b) = \sum_{i=1}^{m} (m - i) \overline{H}(x_i \otimes x_{i+1})
\]

\[
\overline{H}(x \otimes y) = \begin{cases} 
0 & \text{if } x \geq C_n y \\
1 & \text{if } x < C_n y \text{ and } (x, y) \neq (1, 1) \\
2 & \text{if } (x, y) \neq (1, 1).
\end{cases}
\]

\[
\overline{H}(x \otimes y) = \begin{cases} 
0 & \text{if } x \geq D_n y \\
1 & \text{if } x \not\geq D_n y \text{ and } (x, y) \neq (\overline{1}, n) \\
2 & \text{if } (x, y) = (\overline{1}, n).
\end{cases}
\]

In particular \( \overline{H}(1 \otimes 1) = \overline{H}(1 \otimes 1) = 1, \overline{H}(p \otimes q) = 0 \) and \( \overline{H}(q \otimes p) = 1 \) when \( p, q \in \{1, \ldots, n\} \) and \( (p, q) \neq (1, 1) \).

**Remark 36.** The restriction of the map \( \overline{D} \) and \( \overline{D} \) to \( B_{1}^{A_{n-1}} \) is the coenergy function \( D^A \) for \( A_{n-1}^{(1)} \)-affine crystals.

**Proposition 37.**

1. The statistic \( \overline{D} \) is constant on the \( C_n \)-components of \( B_{1}^{C_n} \).

2. The statistic \( \overline{D} \) is constant on the \( D_n^{\uparrow} \)-components of \( B_{1}^{D_n} \).

**Proof.** The two assertions hold since \( \overline{D} \) and \( \overline{D} \) are the coenergy functions on affine crystals for types \( C_n \subset A_{2(n-1)}^{(2)} \) and \( D_n^{\uparrow} \subset D_n^{(1)\uparrow} \) respectively. \( \square \)

For any vertex \( b = x_1 \otimes \cdots \otimes x_m \in B_{1}^{C_n} \) let

\[ Z_b = \{ i \in \{1, \ldots, m-1\} \mid x_i \text{ and } x_{i+1} \text{ are not simultaneously barred or unbarred} \}. \]

Given \( x \otimes y \) in \( B_{1}^{C_n} \), we set

\[
\overline{h}(x \otimes y) = \begin{cases} 
0 & \text{if } x \geq C_n y \\
1 & \text{if } x < C_n y.
\end{cases}
\]

This means that \( \overline{h} \) is the coenergy function on \( B_{1}^{C_n} \) regarded as an affine \( C_n^{(1)} \)-crystal. In particular we have \( \overline{h}(x \otimes y) = \overline{H}(x \otimes y) \) if \( (x, y) \neq (1, 1) \) and \( \overline{h}(1 \otimes 1) = 1 \).

**Lemma 38.** [17] For \( \lambda \in \mathcal{P}_m \) and any \( b \in F_{\lambda,1^m} \) we have

\[
\sum_{i \in Z_b} (m - i)(-1 + 2 \overline{h}(x_i \otimes x_{i+1})) = \frac{m - |\lambda|}{2}.
\]

**Proof.** We have

\[
-1 + 2 \overline{h}(x_i \otimes x_{i+1}) = \begin{cases} 
1 & \text{if } x_i < x_{i+1} \\
-1 & \text{otherwise}.
\end{cases}
\]

Since \( b \) is a highest weight vertex, \( x_1 = 1 \) by Lemma 11 We obtain

\[
\sum_{i \in Z_b} (-1 + 2 \overline{h}(x_i \otimes x_{i+1})) = \begin{cases} 
0 & \text{if } x_m \text{ is unbarred} \\
1 & \text{otherwise}.
\end{cases}
\]

To prove the lemma we proceed by induction on \( m \). When \( m = 1 \) we have \( b = 1, Z_b \) is empty, \( |\lambda| = 1 \) and the lemma holds. Suppose the lemma holds for any highest weight vertex of \( B_{1}^{C_n} \).
Let $b = b' \otimes x_m \in B_{1,m}^{C_n}$ be a highest weight vertex of weight $\lambda$ with $b' = x_1 \otimes \cdots \otimes x_{m-1} \in B_{1,m-1}^{C_n}$, necessarily a highest weight vertex, of weight $\gamma$, say. Let $s$ be the left hand side of (46). We have $Z_b = Z_{b'}$ if $x_{m-1}$ and $x_m$ are simultaneously barred or unbarred and $Z_b = Z_{b'} \cup \{m - 1\}$ otherwise. Considering the various cases and using the induction hypothesis and (47) we have

$$s = \frac{m - 1 - |\gamma|}{2} + \begin{cases} 0 & \text{if } x_m \text{ is unbarred} \\ 1 & \text{otherwise.} \end{cases}$$

When $x_m$ is unbarred $|\gamma| = |\lambda| - 1$ and when $x_m$ is barred, $|\gamma| = |\lambda| + 1$. Either way we obtain $s = \frac{m - |\lambda|}{2}$ as desired. \hfill \Box

**Proposition 39.** For any $b \in F_{\lambda,1,m}$ we have

$$\overline{D}(b) = \overline{D}(b) + \frac{m - |\lambda|}{2}.$$

**Proof.** Write $b = x_1 \otimes \cdots \otimes x_m$ and let $N_b = \{i \in Z_b \mid x_i = 1 \text{ and } x_{i+1} = 1\}$. By Lemma 25 we have $x_i \notin \{\pi, n\}$ for any $i = 1, \ldots, m$. This gives

$$\overline{H}(x_i \otimes x_{i+1}) = \begin{cases} \overline{H}(x_i \otimes x_{i+1}) & \text{if } i \notin Z_b \\ 1 - \overline{H}(x_i \otimes x_{i+1}) & \text{if } i \in Z_b \text{ and } i \notin N_b \\ \overline{H}(x_i \otimes x_{i+1}) - 1 & \text{if } i \in N_b. \end{cases}$$

and using Lemma 38 we obtain

$$\overline{D}(b) = \sum_{i=1}^{m-1} (m - i) \overline{H}(x_i \otimes x_{i+1}) = \sum_{i \notin N_b} (m - i) \overline{H}(x_i \otimes x_{i+1}) + \sum_{i \in N_b} (m - i)2$$

$$= \sum_{i \notin Z_b} (m - i) \overline{H}(x_i \otimes x_{i+1}) + \sum_{i \in Z_b - N_b} (m - i) \overline{H}(x_i \otimes x_{i+1}) + \sum_{i \in N_b} (m - i)2$$

$$= \sum_{i \notin Z_b} (m - i) \overline{H}(x_i \otimes x_{i+1}) + \sum_{i \in Z_b - N_b} (m - i) \overline{H}(x_i \otimes x_{i+1}) + \sum_{i \in N_b} (m - i) +$$

$$\sum_{i \in Z_b - N_b} (m - i) (-1 + 2 \overline{H}(x_i \otimes x_{i+1})) + \sum_{i \in N_b} (m - i)$$

$$= \overline{D}(b) + \sum_{i \in Z_b - N_b} (m - i) (-1 + 2 \overline{H}(x_i \otimes x_{i+1})) + \sum_{i \in N_b} (m - i)$$

$$= \overline{D}(b) + \sum_{i \in Z_b} (m - i) (-1 + 2 \overline{H}(x_i \otimes x_{i+1}))$$

$$= \overline{D}(b) + \frac{m - |\lambda|}{2}.$$

where the last equality follows from Lemma 38. \hfill \Box

By Proposition 23 the splitting $S^D_\mu$ preserves the coenergy function $\overline{D}_\mu$. Thus we have

$$\overline{D}(b) = \overline{D}(S^D_\mu(b))$$

for any $b \in B^{D_n}_\mu$. 

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Proposition 40. Let \( \lambda, \mu \in \mathcal{P}_m \) and \( b \in F_{\lambda, \mu} \).

1. \( \tilde{D} \) is constant on \( D^\dagger_\mu \)-components.

2. We have \( \tilde{D}(b) = \frac{|\mu| - |\lambda|}{2} + \tilde{D}(\theta(b)) \).

3. Let \( b \in F^A_{\mu^{-1}} \). Then \( \tilde{D}(b) = D^A(b) \).

Proof. Assertion (1) follows from the fact that \( \tilde{D} \) is the coenergy function on \( B^\mu_\mu \).

For (2), by Proposition 39 and Proposition 35 we have

\[
\tilde{D}(b) = \tilde{D}(S^C_\mu(b)) = \frac{|\mu| - |\lambda|}{2} + \tilde{D}(S^D_\mu \circ \theta(b)) = \frac{|\mu| - |\lambda|}{2} + \tilde{D}(\theta(b)).
\]

For (3), let \( b \in F^A_{\mu^{-1}} \). We have

\[
\tilde{D}(b) = \tilde{D}(S^D_\mu(b)) = \tilde{D}(S^D_\mu(\theta(b))) = \tilde{D}(S^C_\mu(b)) = \tilde{D}(S^A_\mu(b)) = D^A(b)
\]

by the definition of \( \tilde{D}_\mu \) and the following facts: \( \theta \) is the identity on \( B^A_{\mu^{-1}} \) (Lemma 32), the splitting maps intertwine \( \theta \) (Proposition 35), \( \theta_{1|\mu} \) is the identity (Lemma 31), type C and type A splitting agree on \( B^A_{\mu^{-1}} \) (Remark 18), \( \tilde{D} \) is type A coenergy on \( B^A_{\mu^{-1}} \) (Remark 36 (3)), and that splitting preserves coenergy (Proposition 23).

Corollary 41. Theorem 4 holds for \( \diamond = (1, 1) \).

Proof. In light of Remark 28, the proof is completed by Propositions 34 and 40.

4 Proof of \( K = \infty KL \)

We prove Theorem 6 with a type-independent argument.

For our realization of each of \( g_n \in \{ A_{n-1}, B_n, C_n, D_n \} \), the set of positive roots \( R^+(g_n) \) contain a copy of the positive roots \( R^+(A_{n-1}) \) for type \( A_{n-1} \). The Weyl group \( W \) also contains a copy of the symmetric group, the Weyl group \( S_n = W(A_{n-1}) \) of type \( A_{n-1} \).

We use a number of tricks. The first, presented in section 4.1, is the well-known idea of viewing the Lusztig \( q \)-analogues as coefficients in a generating function which is the graded character of twisted functions on the nullcone. More precisely we use an analogous generating function for the stable KL polynomials. This allows us to apply the second trick in section 4.2 which is the recognition of Littlewood’s formulae inside the above generating function. This requires a special property of the function \( L \) of Theorem 6. Lesser tricks include manipulations of rational \( gl_n \) characters including contragredient duality and symmetries of tensor product multiplicities.
4.1 Another formulation of the stable KL polynomials

We give an alternative form for the definition of the stable KL polynomials \( \text{KL}_{\lambda,\mu}^{g_n,L}(q) \) which is helpful for our proof of \( K = \infty \text{KL} \). Let \( Z[X] \) be the polynomial ring in a set \( X = (x_1, \ldots, x_n) \) of \( n \) variables, interpreted as the group algebra of the weight lattice of \( gl_n \), where \( x_i = e^{x_i} \) is the exponential of the \( i \)-th standard basis vector in the weight lattice \( Z^n \). Let \( Z_r^n \) (see [24]) be the set of dominant weights. Let \( J = \sum_{w \in S_n} (-1)^{\ell(w)} w \) be the antisymmetrization operator on \( Z[X] \) and \( \rho = \rho^{A_{n-1}} = (n-1, n-2, \ldots, 1, 0) \) the half-sum of positive roots.

Let \( E : Z[X] \to Z[X]^{S_n} \) be the Demazure operator for the longest element \( w_0 \) of \( S_n = W(A_{n-1}) \), defined by the bialternant

\[
E(f) = J(x^\rho)^{-1} J(x^\rho f).
\]

It sends dominant monomials to irreducible rational \( gl_n \)-characters

\[
E(x^\lambda) = s_\lambda[X] \quad \text{for dominant weights } \lambda \in Z_r^n
\]

satisfies the \( \rho \)-shifted antisymmetry for monomials

\[
E(x^\beta) = (-1)^w x^{w(\beta + \rho) - \rho} \quad \text{for } \beta \in Z^n
\]

and is linear with respect to symmetric polynomials

\[
E(fg) = Ef(g) \quad \text{for } f \in Z[X]^{S_n}.
\]

The character \( s_\lambda[X] \) is a Schur polynomial up to an integral power of \( x_1x_2 \cdots x_n \). Since \( \{s_\lambda[X] \mid \lambda \in Z_r^n \} \) is a basis of \( Z[X]^{S_n} \) we may define

\[
\sum_{\lambda \in Z_r^n} s_\lambda[X] \text{KL}_{\lambda,\mu}^{g_n,L}(q) = E \left( e^\mu \prod_{\alpha \in R^+(g_n)} \frac{1}{1 - q^{L(\alpha)} e^\alpha} \right)
\]

Multiplying both sides by the Vandermonde \( J(e^\mu) \) and taking coefficients, one sees that the two definitions of stable KL polynomials (19) and (52) agree for any pair \( (\lambda, \mu) \) of elements in \( Z_r^n \).

**Example 42.** For \( n = 2 \), type \( A_{n-1} = A_1 \), and \( \mu = (0, 0) \), the left hand side of (52) reads

\[
s(0,0) + q s(1,-1) + q^2 s(2,-2) + \cdots
\]

so that \( \text{KL}_{(r,-r),(0,0)}^{\infty}(q) = q^r \). Here \( s_{(r,-r)}(x_1, x_2) = (x_1 x_2)^{-r} s_{(2r,0)}(x_1, x_2) \) where \( s_{(2r,0)}(x_1, x_2) = \sum_{a+b=2r} x_1^a x_2^b \) is the Schur polynomial. By (25) and Proposition 5 we have the Kostka-Foulkes polynomial \( K_{(2r,0),(r,r)}(q) = \text{KL}_{(2r,0),(r,r)}^{\infty}(q) = q^r \). This agrees with [16]: there is one tableau of shape \((2r,0)\) and weight \((r,r)\), the single-rowed tableau \(1^r2^r\), and it has charge \( r \).

4.2 The Littlewood formulae

Here is where the choice of function \( L \) enters in. With the function \( L \) as in Theorem 6, we observe that for all \( \alpha \in R^+(g_n) \setminus R^+(A_{n-1}) \), \( L(\alpha) \) is half the length of the root \( \alpha \). This gives the left hand
side of (53) a homogeneity between $q$ and $X$ that allows us to use Littlewood’s formulae [22] [24] to immediately obtain

$$\prod_{\alpha \in R^+(g_n) \setminus R^+(A_{n-1})} \frac{1}{1 - q^{L(\alpha)} e^{\alpha}} = \sum_{\gamma \in P^\diamond} q^{\gamma/2} s_{\gamma}[X]$$

(53)

where $\diamond = \{\varnothing, (1), (2), (1, 1)\}$ for types $A_{n-1}, B_n, C_n, D_n$ respectively. We have

$$\sum_{\lambda \in \mathbb{Z}^n} s_{\lambda}[X] \sim_{\lambda}^{\mu} L_i^{A_{n-1}}(q) = \sum_{\gamma \in P^\diamond} q^{\gamma/2} s_{\gamma}[X] e^{\mu} \prod_{\alpha \in R^+(A_{n-1})} \frac{1}{1 - q^{L(\alpha)} e^{\alpha}} \sum_{\nu \in \mathbb{Z}^n} s_{\nu}[X] \sim_{\nu}^{\mu} L_i^{A_{n-1}}(q)$$

(54)

The first equality holds because (53) is $A_{n-1}$-symmetric, so by (51) it can be pulled out from the operand of $E$ in (52). The second equality holds by (52) for type $A_{n-1}$.

4.3 Rational $gl_n$ characters and the proof completed

Let $c^\lambda_{\gamma \nu}$ be the $gl_n$ tensor product coefficient, the coefficient of the irreducible $gl_n$-character $s_\lambda$ in the product $s_\gamma s_\nu$, for $\lambda, \gamma, \nu \in \mathbb{Z}^n_{\geq}$. Taking the coefficient of $s_\lambda$ in (54) we have

$$\sim_{\lambda}^{\mu} L_i^{A_{n-1}}(q) = \sum_{\nu \in \mathbb{Z}^n} \sim_{\nu}^{\mu} L_i^{A_{n-1}}(q) \sum_{\gamma \in P^\diamond} q^{\gamma/2} c^\lambda_{\gamma \nu}$$

(55)

The last equality holds because $c^\lambda_{\gamma \nu} = 0$ unless $|\lambda| = |\gamma| + |\nu|$ and $\sim_{\nu}^{\mu} L_i^{A_{n-1}}(q) = 0$ unless $|\nu| = |\mu|$. The right hand side of (55) is close to that of (16) but there are some differences; there are a number of appearances of dominant weights, the tensor product multiplicity looks turned around, and the power of $q$ has been shifted and reversed.

We now exploit the fact that we are working with rational $gl_n$-characters. All of the above works for $\mu \in \mathbb{Z}^n_{\geq}$. There is an involutive automorphism on the weight lattice $\mathbb{Z}^n$ given by reversal and negation: $\beta \mapsto \beta^* := -w_0 A_{n-1} \beta$. It permutes the set of positive roots $R^+(A_{n-1})$ and preserves the set $\mathbb{Z}^n_{\geq}$ of dominant weights, sending $\lambda \in \mathbb{Z}^n_{\geq}$ to $\lambda^* \in \mathbb{Z}^n_{\geq}$, which is the highest weight of the contragredient dual of the irreducible $gl_n$-module of highest weight $\lambda$. With $M$ as in (25),

$$\hat{\lambda} = \lambda^* + (M^n)$$
$$\hat{\mu} = \mu^* + (M^n).$$

(56)
We have

\[ \infty KL_{\lambda,\mu}^{g_n, L}(q) = \infty KL_{\lambda^*,\mu^*}^{g_n, L}(q) \]

\[ = q^{(\lambda^*|-\mu^*)}/2 \sum_{\nu} \infty KL_{\nu^*,\mu^*}^{A_n-1}(q) \sum_{\gamma \in P_n^\gamma} c_{\gamma \nu^*}^{\lambda^*} \]

\[ = q^{(\mu|\lambda)}/2 \sum_{\nu \in \mathbb{Z}_{\geq}^n} \infty KL_{\nu^*,\mu}^{A_n-1}(q) \sum_{\gamma \in P_n^\gamma} c_{\gamma \nu^*}^{\lambda^*} \]

\[ = q^{(\mu|\lambda)}/2 \sum_{\nu \in \mathbb{Z}_{\geq}^n} \infty KL_{\nu^*,\mu}^{A_n-1}(q) \sum_{\gamma \in P_n^\gamma} c_{\gamma \nu^*}^{\lambda^*}. \] (57)

The first equality is given by (56) and the invariance property (25). The second holds by applying (55) for the pair \((\lambda^*, \mu^*)\). The third holds since \(\nu \mapsto \nu^*\) is a bijection from \(\mathbb{Z}_{\geq}^n\) to itself. The last equality holds by the symmetry of rational \(gl_n\) tensor product multiplicities

\[ c_{\gamma \nu^*}^{\lambda^*} = \langle s^*_\lambda, s_\gamma s_\mu^* \rangle = \langle s_\nu^*, s_\lambda s_\gamma \rangle = c_{\lambda\gamma}^\nu \]

and the duality

\[ \infty KL_{\lambda^*,\mu}^{A_n-1}(q) = \infty KL_{\lambda,\mu}^{A_n-1}(q) \] (58)

which follows, for example, from applying the automorphism \(e^\beta \mapsto e^{3\beta}\) to the definition (19) for \(g_n = A_{n-1}\).

We now restrict ourselves to the original situation where \(\lambda\) and \(\mu\) are assumed to be partitions with at most \(m\) parts where \(m \leq n\). We may assume that \(\nu\) is a partition, because \(\gamma\) and \(\lambda\) are partitions, and in this case, \(c_{\gamma \lambda}^\nu = 0\) if \(\nu \in \mathbb{Z}_{\geq}^n\) is not a partition. We may assume that \(\nu \geq \mu\), for otherwise \(\infty KL_{\nu^*,\mu}^{A_n-1}(q) = 0\); see the proof of Proposition 5. Since \(\mu\) is a partition with at most \(m\) parts, it follows that \(\nu\) is as well. The tensor product \(c_{\gamma \lambda}^\nu\) is now indexed by three partitions. We may assume that \(\gamma\) has at most \(m\) nonzero parts, since \(c_{\gamma \lambda}^\nu = 0\) unless \(\gamma \subset \nu\).

Finally, since \(\nu\) and \(\mu\) are partitions we have \(\infty KL_{\nu^*,\mu^*}^{A_n-1}(q) = KL_{\nu^*,\mu^*}(q)\) by Proposition 5. Therefore (57) becomes

\[ \infty KL_{\lambda,\mu}^{g_n, L}(q) = q^{(|\mu|\lambda)}/2 \sum_{\nu \in \mathbb{P}_m} KL_{\nu^*,\mu}(q) \sum_{\gamma \in P_n^\gamma} c_{\gamma \nu^*}^{\lambda^*}. \]

Theorem 6 follows by replacing \(q\) by \(q^{-1}\), multiplying by \(q^{(|\mu|+|\nu|)}\) and using the definitions (12) and (15) of the cocharge Kostka-Foulkes polynomial \(K_{\nu^*,\mu}(q)\) and the \(K\)-polynomial \(K_{\lambda,\mu}(q)\).

5 Combinatorial question

Since KL polynomials are \(q\)-analogues of weight multiplicities, in light of the \(X = \infty KL\) theorem (Corollary 7) there is a grade-preserving bijection from the set \(F_{\lambda,\mu}\) that indexes the one-dimensional sum \(X\), to the elements of weight \(\mu\) in the crystal graph of the irreducible \(U_q(g_n)\)-module of highest weight \(\lambda\), with some grading on the latter objects. It would interesting to find a natural statistic on the above weight vectors and a natural bijection of this kind.
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