Small coupling limit and multiple solutions to the Dirichlet Problem for Yang Mills connections in 4 dimensions - Part I

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Abstract

In this paper (Part I) and its sequels (Part II and Part III), we analyze the structure of the space of solutions to the $\epsilon$-Dirichlet problem for the Yang-Mills equations on the 4-dimensional disk, for small values of the coupling constant $\epsilon$. These are in one-to-one correspondence with solutions to the Dirichlet problem for the Yang Mills equations, for small boundary data $\epsilon A_0$. We prove the existence of multiple solutions, and, in particular, non minimal ones, and establish a Morse Theory for this non-compact variational problem. In part I, we describe the problem, state the main theorems and do the first part of the proof. This consists in transforming the problem into a finite dimensional problem, by seeking solutions that are approximated by the connected sum of a minimal solution with an instanton, plus a correction term due to the boundary. An auxiliary equation is introduced that allows us to solve the problem orthogonally to the tangent space to the space of approximate solutions. In Part II, the finite dimensional problem is solved via the Ljusternik-Schirelman theory, and the existence proofs are completed. In Part III, we prove that the space of gauge equivalence classes of Sobolev connections with prescribed boundary value is a smooth manifold, as well as some technical lemmas essential to the proofs of Part I. The methods employed still work when $B^4$ is replaced by a general compact manifold with boundary, and $SU(2)$ is replaced by any compact Lie group.

1 Introduction and statement of the main results

A solution to the Yang Mills equations is a critical point for the Yang Mills functional defined on the space of connections. These equations are particularly interesting in 4 dimensions, since in this case, the Yang Mills equations are not only invariant under the infinite dimensional automorphism group of the bundle, namely, the gauge group, but are also invariant under the group of conformal transformations over the base manifold. Since the latter is non-compact, the associated variational problem is non-compact (i.e., it never satisfies the Palais-Smale condition) even when quotiented out by the automorphism group of the bundle. Finding critical points and establishing a Morse theory for such non-compact variational problems is one of the most challenging problems in nonlinear functional analysis. See [2], [4], [21] and references therein for an interesting list of non-compact variational problems and their applications. For the existence of solutions to the Yang Mills equations on closed manifolds, not necessarily action-minimizing, see [3], [16], [18], [19], [20], [22], [23], [24], [25].

We establish a relation between the small coupling limit problem and the problem of existence of multiple critical points for the Yang Mills functional with small Dirichlet boundary conditions

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on a compact 4-dimensional manifold with boundary. We thoroughly analyze the simplest case, namely, the $SU(2)$-Yang Mills problem on the 4-dimensional disk. Our approach is based on a perturbation method as developed by the first author in [9], [10], [11] for the $H$-surface equations. See also [1] and references therein for applications of perturbation methods to popular non-compact variational problems, and, in particular, to Yamabe-like equations.

In order to describe the problem, we need to establish first some basic notation. We denote by $B^4$ the open unit disk in $\mathbb{R}^4$. For $\epsilon > 0$, we define the Lie algebra $(\mathfrak{su}(2), [\cdot, \cdot]_{\epsilon}) := (\mathfrak{su}(2), \epsilon [\cdot, \cdot])$, where $[\cdot, \cdot]$ is the ordinary Lie bracket on $\mathfrak{su}(2)$. Precisely, $\mathfrak{su}(2)_{\epsilon} = \mathfrak{su}(2)$ as vector spaces, but with Lie bracket $[X, Y]_{\epsilon} := \epsilon [X, Y]$ for $X, Y \in \mathfrak{su}(2)$. The map $\phi_{\epsilon} : \mathfrak{su}(2)_{\epsilon} \to \mathfrak{su}(2)$, defined by $\phi_{\epsilon}(X) := \epsilon X$ is a Lie algebra isomorphism (namely $\phi_{\epsilon}[X, Y]_{\epsilon} = \epsilon^2 [X, Y] = [\phi_{\epsilon} X, \phi_{\epsilon} Y]$), thus induces a Lie group isomorphism $\Phi_{\epsilon} : SU(2)_{\epsilon} \to SU(2)$. In the context of $SU(2)_{\epsilon}$-principal bundles, the covariant differentiation associated to a connection $A$ is locally $\mathcal{D}_{A_{\epsilon}} := d + [\cdot, \cdot]_{\epsilon} := d + \epsilon [\cdot, \cdot]$, where $\epsilon$ is understood as the coupling constant that appears in much of the physics literature (cf., for example, [17], or Ch. 15 of [26]).

Let now $A_0$ be a smooth connection on an $SU(2)_{\epsilon}$-principal bundle over $\partial B^4$. Since such bundles are trivial, $A_0$ lives on the trivial bundle, that is $A_0 \in C^\infty(T^*\partial B^4 \otimes \mathfrak{su}(2))$. For $A \in L^2(T^*B^4 \otimes \mathfrak{su}_{\epsilon}(2))$, the $SU(2)_{\epsilon}$-Yang Mills functional is given by

$$\mathcal{M}_{\epsilon}(A) = \int_{B^4} |F_{A_{\epsilon}}|^2 \, dx,$$

where $F_{A_{\epsilon}} = dA + \frac{1}{2}[A, A]_{\epsilon} := dA + \frac{\epsilon}{2}[A, A]$ is the curvature of the connection $A$, and the $SU(2)_{\epsilon}$-Yang Mills Dirichlet problem in exam, obtained via a variational method, is given by

$$\begin{align*}
(D_{\epsilon}) \quad \left\{
\begin{array}{ll}
\mathcal{D}_{A_{\epsilon}} \cdot F_{A_{\epsilon}} = 0 & \text{in } B^4 \\
\iota^* A \sim A_0 & \text{at } \partial B^4.
\end{array}
\right.
\end{align*}$$

Here, $\iota : \partial B^4 \to \overline{B^4}$ is the inclusion, $\iota^* A \sim A_0$ on $\partial B^4$ means that $\iota^* A$ is gauge equivalent to $A_0$ over $\partial B^4$ via a gauge transformation that extends smoothly to the interior, $d_{A_{\epsilon}}^{\ast} := \ast d \ast - \ast [\cdot, \cdot]_{\epsilon} := \ast d \ast + \epsilon \ast [\cdot, \cdot]$, where $\ast$ is the Hodge star operator with respect to the flat metric on $\mathbb{R}^4$. Notice that, since $\mathfrak{su}(2)_{\epsilon} = \mathfrak{su}(2)$ as sets, $A_0$ may be regarded as an $\mathfrak{su}(2)$-valued 1-form on $\partial B^4$, i.e. $A_0 \in C^\infty(T^*\partial B^4 \otimes \mathfrak{su}(2))$. This is a canonical identification between $\mathfrak{su}(2)_{\epsilon}$-valued 1-forms and $\mathfrak{su}(2)$-valued 1-forms. A different identification is given by $\phi_{\epsilon}$. We will use the former identification throughout this paper, unless we explicitly write otherwise.

By the direct method of the calculus of variations, Marini [15] obtained the first solution to $(D_{\epsilon})$, that is, an absolute Yang Mills action minimizing solution, which we shall call small solution and will denote by $\underline{A}_{\epsilon}$. (Note that the small solution is, in general, not unique, so we choose one of these solutions for each $\epsilon > 0$). Moreover, it is known (cf. [12]) that the space of connections with boundary value $A_0$, denoted by $\mathcal{A}(A_0)$, has infinitely many connected components indexed by $\mathbb{Z}$: $\mathcal{A}(A_0) = \bigsqcup_{k=-\infty}^{\infty} \mathcal{A}_k(A_0)$, where $\mathcal{A}_k(A_0) = \{ A \in \mathcal{A}(A_0) : c_2(A) = k \}$, with $c_2(A) = \frac{\epsilon^2}{8\pi^2} \int_{B^4} \text{Tr}(F_{\epsilon} \wedge F_{\epsilon}) - \frac{\epsilon^2}{8\pi^2} \int_{\partial B^4} \text{Tr}(\underline{A}_{\epsilon} \wedge F_{\epsilon})$ (the relative 2nd Chern number with respect to $\underline{A}_{\epsilon}$, where $\underline{A}_{\epsilon}$ is a fixed absolute minimizer). In [12], the problem of finding a minimum
in each component \( A_k(A_0) \) is thoroughly solved, yielding many so-called *large solutions*. In particular, it is proved that, for non-flat boundary values \( A_0 \), there exists a minimum at least in one of the components \( A_{\pm 1}(A_0) \). Since all the solutions known to the Dirichlet problem for Yang Mills are minima (minimizers for the action restricted to the connected components \( A_k(A_0) \)), it is left open for investigation the interesting problem whether there exist non-minimal solutions and, in general, whether the solution found in \([12]\) is unique in each component. (Notice that the results in \([15, 12]\) cited above, proven for the \( su(2) \)-Yang Mills functional, that is for the standard Dirichlet problem \( (D_1) \) \((\epsilon = 1)\), automatically extend to \( \mathcal{M}_\epsilon(A) \). This problem can also be related to the quantization of Yang Mills theory.

Since the uniqueness result for flat boundary values has been established in \([8]\), we henceforth assume that \( A_0 \) is non-flat, and investigate the existence of non-minimal solutions and, more in general, seek non-uniqueness results in \( A_{+1}(A_0) \), (or, by the same arguments, in \( A_{-1}(A_0) \)), since we know that an absolute minimum exists in at least one of these components. By our method, we find multiple solutions and non-minimal ones, as stated in Theorems 1-3, in \( A_{+1}(A_0) \) for small values of the coupling constant \( \epsilon > 0 \)[1]. It is important to point out that, by the argumentation in §2.2, the isomorphism \( \phi_0 \) establishes a correspondence between solutions to \( (D_0) \) and solutions to \( (D_1) \) (the standard Dirichlet problem) with boundary value \( \epsilon A_0 \).

In order to state our main theorems, we need to introduce further notation used throughout this paper. We denote by \( \mathbb{H} \) the algebra of quaternions, i.e., \( \mathbb{H} \) is the associative algebra over \( \mathbb{R} \) generated by \( i, j, k \), which satisfy \( i^2 = j^2 = k^2 = -1, \ ij = -ji = k, \ jk = -kj = i \) and \( ki = -ik = j \). Thus, \( x \in \mathbb{H} \) is written as \( x = x^0 + x^1 i + x^2 j + x^3 k \), \( x_i \in \mathbb{R} \) \((1 \leq i \leq 4)\). The real and the imaginary components of \( x \) are \( \text{Re} \ x := x^0 \) and \( \text{Im} \ x := x^1 i + x^2 j + x^3 k \), respectively. The inner product of \( x, y \in \mathbb{H} \) is \((x, y) := \text{Re} \ (x \overline{y})\), where \( \overline{y} = y^0 - y^1 i - y^2 j - y^3 k \). The Lie algebra, \( \text{Im} \ \mathbb{H} \), of imaginary quaternions, with Lie bracket \([x, y] := xy - yx\), is isomorphic to \( su(2) \), and the Lie group \( Sp(1) \) of unit quaternions is isomorphic to \( SU(2) \). An isomorphism between \( \text{Im} \ \mathbb{H} \) and \( su(2) \) is given explicitly by \( \text{Im} \ \mathbb{H} \ni x^1 i + x^2 j + x^3 k \mapsto \left( \begin{array}{c} x^1 i \\ x^2 + x^3 i \\ -x^1 i \\ \end{array} \right) \in su(2) \). We endow \( su(2) \) with the inner product \( (X, Y) = -\text{Tr}(XY) \), which translates in terms of quaternions into \( (X, Y) = 2(x, y) \), where \( X, Y \in su(2) \) correspond to \( x, y \in \text{Im} \ \mathbb{H} \) via the above isomorphism. The pointwise inner product on \( su(2) \)-valued forms on \( B^4 \) is defined via the inner product on \( su(2) \) and the standard metric on \( B^4 \). In the following, for any \( \text{Im} \ \mathbb{H} \)-valued q-form \( \omega \), we denote by \( \omega_1, \omega_2, \omega_3 \) its real-valued components, i.e., \( \omega = \omega_1 i + \omega_2 j + \omega_3 k \), where the q-forms \( \omega_1, \omega_2, \omega_3 \) are real-valued. For \( p \in B^4 \), we define \( h_p \) as the \( \text{Im} \ \mathbb{H} \)-valued 1-form which is the unique solution of

\[
\begin{cases}
\Delta h_p = 0 & \text{in } B^4 \\
h_p = \text{Im} \left( \frac{\overline{x} - y}{|x - p|^4} \right) & \text{at } \partial B^4.
\end{cases}
\]

Here, all the components of \( h_p \) (not only the tangential ones) are prescribed at the boundary,

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1 One may also investigate these issues in different connected components, i.e. for \( k \neq \pm 1 \), and in particular, in the connected component \( A_0(A_0) \), where an absolute minimum \( \Delta \) is already known to exist. The techniques used in this paper extend to all the cases (cf. Conjecture 8.1 in \([13]\)). Nevertheless, these non-uniqueness problems via the perturbation approach are technically harder for \( k \neq \pm 1 \).
so the one-form $h_p$ is harmonic component-wise, with assigned Dirichlet boundary data. This one-form, as well as the function $F(p)$ and the matrix $M(A_0,p)$ defined below play a crucial role in this paper. We define the function

$$F(p) := \int_{B^4} |(dh_p)^{-}|^2 \, dx, \quad p \in B^4,$$

and, for a given boundary value $A_0$, and $p \in B^4$, we define the $3 \times 3$-matrix

$$M(A_0,p) := (m_{ij}(A_0,p)), \quad 1 \leq i, j \leq 3,$$

with

$$m_{ij}(A_0,p) := \int_{B^4} ((dA_0_{0,i})^-, (dh_p)^-) \, dx, \quad 1 \leq i, j \leq 3,$$

where $A_0$ is a solution to

$$(\mathcal{D}_0) \quad \begin{cases} d^* dA = 0 & \text{in } B^4 \\ \iota^* A \sim A_0 & \text{on } \partial B^4, \end{cases}$$

and, for a given 2-form $\omega$, we denote by $\omega^-$ the anti-self dual component of $\omega$ (that is $\omega^- := (\omega - *\omega)/2$).

**Remark 1.1** Note that $dA_0$ is uniquely determined by (the gauge equivalence class of) $A_0$, since it satisfies the system of equations $d^2 A_0 = 0$, $d^*dA_0 = 0$, $\iota^*dA_0 = d_\tau A_0$, which has a unique solution by relative Hodge theory. Thus the matrix above is a well defined matrix-valued function of $A_0$ and $p$.

We denote by $\mu_1(A_0, p) \geq \mu_2(A_0, p) \geq \mu_3(A_0, p) \geq 0$ the eigenvalues of the non-negative symmetric matrix $M(A_0, p)^T M(A_0, p)$, and state Theorems 1-3 for solutions in $A_{+1}(A_0)$. (Analogous results would hold for solutions in $A_{-1}(A_0)$ by the same arguments, or by simply reversing the orientation of $B^4$).

**Theorem 1** Let us define the function $G_1^\pm(p) := \frac{\sqrt{\mu_1(p)} + \sqrt{\mu_2(p)} \pm \sqrt{\mu_3(p)}}{F(p)}$, $p \in B^4$, and assume that $p_0 \in B^4$ satisfies either of the following hypotheses (1),(2):

(1) $\det M(A_0, p_0) > 0$ and $p_0$ is an isolated local maximum point of $G_1^+(p)$;

(2) $\det M(A_0, p_0) < 0$ and $p_0$ is an isolated local maximum point of $G_1^-(p)$.

Then, there exists $\epsilon_0 > 0$ and a family of connections $\{A_\epsilon\}$ indexed by $\epsilon \in (0, \epsilon_0]$ with the following properties: $A_\epsilon$ is a solution to $\mathcal{D}_\epsilon$ in $A_{+1}(A_0)$; $\epsilon^2 |F_{A_\epsilon}|^2 \, dx \to 8\pi^2 \delta_{p_0}$ as $\epsilon \to 0$ in the sense of measures (i.e. $\epsilon F_{A_\epsilon}$ concentrates at $p_0$ as $\epsilon \to 0$).

**Remark 1.2** Note that the $A_\epsilon$’s above do not satisfy the Yang Mills equations, but the $SU(2)_{\epsilon}$-Yang Mills equations (cf. $\mathcal{D}_\epsilon$). Nonetheless the thesis of Theorem 1 can be restated by means of the isomorphism $\Phi_\epsilon$ as follows:

$$\langle \langle \text{There exists } \epsilon_0 > 0 \text{ and a family of Yang Mills connections } \{A(\epsilon)\}, \text{ indexed by } \epsilon \in (0, \epsilon_0], \text{ with the following properties: } A(\epsilon) \text{ is a solution to the Dirichlet problem } \mathcal{D}_1 \text{ with boundary value } \epsilon A_0 $$
in \( A_{+1}(\varepsilon A_0) \): \(|F_{A(\varepsilon)}|^2 \, dx \to 8\pi^2 \delta_{p_0} \) as \( \varepsilon \to 0 \) in the sense of measures (i.e. the \( F_{A(\varepsilon)} \) concentrates at \( p_0 \) as \( \varepsilon \to 0 \)).

Note that there holds the relation \( A(\varepsilon) = \varepsilon A_\epsilon \) (cf. also §2.2). The theses of Theorems 2, 3 can be restated in similar fashion in terms of the \( A(\varepsilon) \).

In Lemma 2.1 (1) of [13] we show that \( F(\varepsilon) > 0 \) in \( B^4 \). Also, one of the basic properties of \( F(\varepsilon) \), namely dist\((p, \partial B^4)\)\( F(\varepsilon) \to C \) for some constant \( C > 0 \) as dist\((p, \partial B^4) \to 0 \) (cf. Lemma 2.1 (2) of [13]), implies that the maximum of \( G^\pm \) is always attained at some point in \( B^4 \) (provided that \( G^\pm \neq 0 \)). Notice that the solution obtained in [12] corresponds to the global maximum of \( G^\pm \).

Here, we point out that a different sign convention is used in [12]: in the main theorem of [12], “self dual” should be replaced by “anti-self dual” and viceversa, and the glued connection in that proof is in \( A_{-1}(A_0) \) (not in \( A_{+1}(A_0) \)), accordingly to our current convention.

**Theorem 2** Let us define the functions \( G^\pm_0(p) := (\sqrt{\mu_1(p)} - \sqrt{\mu_2(p)^2 + \mu_3(p)})^2 \); \( G^-_3(p) := (\sqrt{\mu_1(p)} - \sqrt{\mu_2(p)^2 + \mu_3(p)})^2 \); \( G^0_1(p) := (\sqrt{\mu_2(p)^2 + \mu_2(p)^2})^2 \); \( G^0_2(p) := (\sqrt{\mu_2(p)^2 - \mu_3(p)})^2 \).

Assume that \( p_0 \in B^4 \) satisfies one of the following conditions (1)-(a), (b), (2)-(a), (b), (c), (3)-(a), (b):

1. \( \det M(A_0, p_0) > 0 \) and
   - (a) \( p_0 \) is a non-degenerate critical point of \( G^+_1(p) \), or
   - (b) \( \sqrt{\mu_1(A_0, p_0)} > \sqrt{\mu_2(A_0, p_0)} + \sqrt{\mu_3(A_0, p_0)} \) and \( p_0 \) is a non-degenerate critical point of \( G^+_2(p) \);

2. \( \det M(A_0, p_0) < 0 \) and
   - (a) \( \mu_2(A_0, p_0) > \mu_3(A_0, p_0) \) and \( p_0 \) is a non-degenerate critical point of \( G^-_1(p) \), or
   - (b) \( \mu_1(A_0, p_0) > \mu_2(A_0, p_0) > \mu_3(A_0, p_0) \) and \( p_0 \) is a non-degenerate critical point of \( G^-_2(p) \), or
   - (c) \( \mu_1(A_0, p_0) > \mu_2(A_0, p_0) \), \( \sqrt{\mu_1(A_0, p_0)} < \sqrt{\mu_2(A_0, p_0)} + \sqrt{\mu_3(A_0, p_0)} \) and \( p_0 \) is a non-degenerate critical point of \( G^-_3(p) \);

3. \( \det M(A_0, p_0) = 0 \) and
   - (a) \( \mu_2(a_0, p_0) > 0 \) and \( p_0 \) is a non-degenerate critical point of \( G^0_1(p) \), or
   - (b) \( \mu_1(A_0, p_0) > \mu_2(A_0, p_0) > 0 \) and \( p_0 \) is a non-degenerate critical point of \( G^0_2(p) \).

Then, there exists \( \varepsilon_0 > 0 \) and a family of connections \( \{ A_\varepsilon \} \) indexed by \( \varepsilon \in (0, \varepsilon_0] \) with the following properties: \( A_\varepsilon \) is a solution to \((\mathcal{D}_\varepsilon)\) in \( A_{+1}(A_0) \); \( \varepsilon^2 |F_{A_\varepsilon}|^2 \, dx \to 8\pi^2 \delta_{p_0} \) as \( \varepsilon \to 0 \) in the sense of measures (i.e. \( \varepsilon F_{A_\varepsilon} \) concentrates at \( p_0 \) as \( \varepsilon \to 0 \)).

One of the main properties of \( F(p) \), namely \( \nabla F(p) \sim \text{dist}(p, \partial B^4)^{-5} p/|p| \), as dist\((p, \partial B^4) \to 0 \) (cf. Lemma 2.1 (3) in [13]), implies that, if the hypotheses of the theorem above are satisfied,
the derivatives of the $G^\pm_i$ s, $G^0_i$ s do not vanish at $\partial B^4$. For each $G^\pm_i$, $G^0_i$, there always exists at least one critical point in $B^4$, a maximum indeed. However, these critical points may be degenerate.

The next theorem holds without assuming such non-degeneracy.

**Theorem 3** Assume that there exists $p_0 \in B^4$ such that one of the following holds:

1. $\det M(A_0, p_0) > 0, \mu_1(A_0, p_0) > \mu_2(A_0, p_0) > \mu_3(A_0, p_0)$ and $\sqrt{\mu_1(A_0, p_0)} > \sqrt{\mu_2(A_0, p_0)} + \sqrt{\mu_3(A_0, p_0)}$;

2. $\det M(A_0, p_0) < 0$ and $\mu_1(A_0, p_0) > \mu_2(A_0, p_0) > \mu_3(A_0, p_0)$;

3. $\det M(A_0, p_0) = 0$ and $\mu_1(A_0, p_0) > \mu_2(A_0, p_0) > 0$.

Then, for all sufficiently small $\epsilon > 0$, there exist at least two distinct solutions to $(D_\epsilon)$ in $A_{\epsilon+1}(A_0)$. Furthermore, the following alternative holds: there exists at least one non-minimizing solution, or there exist infinitely many minimizing solutions. In the hypotheses (2), if in addition $\sqrt{\mu_1(A_0, p_0)} < \sqrt{\mu_2(A_0, p_0)} + \sqrt{\mu_3(A_0, p_0)}$, then there exist at least three distinct solutions, of which at least two non-minimizing, or there exist infinitely many minimizing solutions to $(D_\epsilon)$ in $A_{\epsilon+1}(A_0)$.

Let us point out to the reader’s attention some important results obtained in [13], regarding the hypotheses of Theorems 1-3. Precisely, we construct a family of boundary values yielding matrices $M(A_0, p_0)$ which realize each of the cases in Theorem 3 for any given point $p_0 \in B^4$ (cf. Proposition 8.1 in [13]). We also show that for any boundary value $A_0$, there exists an arbitrarily small perturbation $\tilde{A}_0$ of $A_0$ such that $\det M(\tilde{A}_0, p_0) \neq 0$ and $\mu_1(\tilde{A}_0, p_0) > \mu_2(\tilde{A}_0, p_0) > \mu_3(\tilde{A}_0, p_0)$ (cf. Proposition 8.2 in [13]).

In §2 of the present paper, we describe the asymptotic profile of small and large solutions as $\epsilon \to 0$, thus giving a heuristic explanation of our method. In §3.1 we construct the spaces of approximate solutions and introduce the technical notation used in the estimates that follow. In §3.2 we obtain the asymptotic expansion of the $su(2)_\epsilon$-Yang Mills functional evaluated on the approximate solutions, and in §3.3 – §3.4 we estimate the Hessian and the remainder. In §3.5 we introduce and estimate the modified Hessian. In §3.6 we define the auxiliary equation and solve it.

### 2 Asymptotic profile of small and large solutions as $\epsilon \to 0$

In this section, we analyze the asymptotic behavior as $\epsilon \to 0$ of the family $\{A_\epsilon\}$ of small solutions to the Dirichlet problems $(D_\epsilon)$ defined in §1 (i.e. absolute minimizers for the $SU(2)_\epsilon$-Yang Mills functionals). This is a crucial ingredient in the proofs of Theorems 1-3. We also describe the asymptotic profile as $\epsilon \to 0$ of the family of large solutions $\{\overline{A}_\epsilon\} \subset A_{\epsilon+1}(A_0)$ (or $A_{\epsilon-1}(A_0)$) obtained in [12], in order to give a heuristic argumentation that motivates the procedure we employ to construct approximate solutions to $(D_\epsilon)$.
2.1 Asymptotic profile of small solutions

Let \((D_0)\) be the Dirichlet problem defined in §1. The following Proposition holds for the family of \(\{A_\epsilon\}\), small solutions to \((D_\epsilon)\).

**Proposition 2.1** There exists a solution \(A_\epsilon\) to \((D_0)\) such that \(A_\epsilon \to A_0\) in \(C^\infty(B^4)\), as \(\epsilon \to 0\), in a suitable gauge. More precisely, for any \(k \geq 1\) there exists \(C_k \geq 0\) such that \(\|A_\epsilon - A_0\|_{C^k(B^4)} \leq C_k \epsilon\), for small \(\epsilon > 0\), in a suitable gauge.

**Proof.** We define 1-forms \(\omega_\epsilon := A_\epsilon - A_0\). These satisfy the following boundary value problems:

\[
\begin{align*}
    & \{ \begin{array}{l}
        d^*d\omega_\epsilon + \epsilon \{ \frac{\partial}{\partial \epsilon}[A_\epsilon, A_\epsilon] + \ast [A_\epsilon, \ast dA_\epsilon] + \ast \xi [A_\epsilon, \ast [A_\epsilon, A_\epsilon]] \} = 0 \quad \text{on } B^4 \\
        i^*\omega_\epsilon = 0 \quad \text{on } \partial B^4.
    \end{array} \}
\end{align*}
\]  

(2.1)

One needs the following lemma.

**Lemma 2.1** There exist constants \(B_0, B_1, C, \epsilon_0\) such that

\[
\begin{align*}
    & \|A_\epsilon\|_{L^2(B^4)} \leq C \|F_{A_\epsilon}\|_{L^2(B^4)} \leq B_0, \\
    & \|A_\epsilon\|_{L^p(B^4)} \leq C \|F_{A_\epsilon}\|_{L^p(B^4)}, \\
    & \|\omega_\epsilon\|_{L^2(B^4)} \leq \epsilon B_1,
\end{align*}
\]

for \(2 \leq p < 4\), for all \(\epsilon\) such that \(0 \leq \epsilon \leq \epsilon_0\), where \(L^p\) is the Sobolev space of functions with \(L^p\)-integrable partial derivatives up to order \(k\) (in the sense of distributions).

**Proof.** We denote by \(YM_\epsilon(A)\) the Yang Mills functional on \(su(2)_\epsilon\)-connections \(A\), i.e., explicitly, in local coordinates:

\[
YM_\epsilon(A) := \int_{B^4} |F_{A_\epsilon}| := \int_{B^4} \left| dA + \frac{1}{2}[A, A_\epsilon] \right|^2 = \int_{B^4} \left| dA + \frac{\epsilon}{2}[A, A] \right|^2. \tag{2.5}
\]

We first show that \(YM_\epsilon(A_\epsilon)\) is uniformly bounded for \(\epsilon\) sufficiently small. In fact, \(YM_\epsilon(A_\epsilon) = m_\epsilon := \min_{A \in \mathcal{A}} YM_\epsilon(A)\), where \(\mathcal{A} := \{\text{smooth } su(2)\text{-connections : } i^*A = A_0\}\). Thus,

\[
YM_\epsilon(A_\epsilon) \leq YM_\epsilon(A_0) = \|dA_0\|_{L^2(B^4)} + \epsilon \langle dA_0, [A_0, A_0] \rangle + \frac{\epsilon^2}{4} \|dA_0, A_0\|_{L^2(B^4)}^2
\]

\[
= YM_0(A_0) + \epsilon \langle dA_0, [A_0, A_0] \rangle + \frac{\epsilon^2}{4} \|dA_0, A_0\|_{L^2(B^4)}^2 \leq 2m_0, \tag{2.7}
\]

for \(\epsilon\) sufficiently small, where \(m_0 := YM_0(A_0)\).

Now, let us recall that the absolute Yang Mills minimizer for the Dirichlet problem found in [15] is in the good gauge, i.e., it satisfies \(d^*A = 0\) on \(B^4\) and the boundary condition \(d^*_\tau A_\tau = 0\) at \(\partial B^4\), where \(\tau\) represents tangential directions. These conditions yield the estimate

\[
\|A_\epsilon\|_{L^p(B^4)} \leq h\|dA_\epsilon\|_{L^p(B^4)}, \tag{2.8}
\]

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for $\epsilon \geq 0$, and $2 \leq p < 4$, with a constant $h$ depending only on dimension (not on $\epsilon$). For $\epsilon = 0$, this becomes $\epsilon \left\| A_0 \right\|_{L^p(B^4)} \leq h \left\| F A_0 \right\|_{L^p(B^4)}$, i.e., (2.3). This estimates also holds on local charts $U$ of type one and type two (boundary and interior neighborhoods, respectively, cf. [15]).

This yields

\[ \left\| A_0 \right\|_{L^2(U)} \leq h \left\| F A_0 \right\|_{L^2(U)} + hC' \epsilon \left\| A_0 \right\|_{L^4(U)} \left\| A_0 \right\|_{L^4(U)}. \tag{2.9} \]

Let us now consider covers $C_j$ of charts $U$ of radius $\rho_j$, with $\rho_j \to 0$ as $j \to \infty$, satisfying the following conditions:

(1) there exists $k$, independent of $j$, such that any $k + 1$ charts have empty intersection (in particular each cover $C_j$ is finite);

(2) $\forall \epsilon > 0$ given, there exists $j = j(\epsilon)$ such that $hC' \epsilon \left\| A_0 \right\|_{L^4(U)} < \frac{1}{2}$, for every $U \in C_j$.

This can be achieved using the compactness of $\overline{B}^4$ and using charts of type one and type two. Then, by (2.9),

\[ \left\| A_0 \right\|_{L^2(U)} \leq 2h \left\| F A_0 \right\|_{L^2(U)}, \forall U \in C_j. \tag{2.10} \]

Thus on $B^4$,

\[ \left\| A_0 \right\|_{L^2(B^4)} \leq 2hk \left\| F A_0 \right\|_{L^2(B^4)}, \]

with $h, k$ independent of $\epsilon$. This, together with (2.7) yields (2.2) with $C = 2hk$ and $B_0 = 2Cm_0$.

For $\epsilon > 0$ and general $p \in [2, 4)$, (2.8) yields

\[ \epsilon \left\| A_0 \right\|_{L^p(B^4)} \leq h \left\| F A_0 \right\|_{L^p(B^4)} + hC' \epsilon \left\| A_0 \right\|_{L^4(B^4)} \left\| A_0 \right\|_{L^4(B^4)}, \]

for $q = \frac{4p}{4 - p}$. Thus, applying (2.2), there exists $\epsilon_0 > 0$ such that (2.3) holds in the “good gauge”, for some constant $C$ depending only on dimension, for $2 \leq p < 4$ and for $0 \leq \epsilon \leq \epsilon_0$.

In the same gauge, for $\omega_\epsilon := A_\epsilon - A_0$, one also obtains

\[ \left\| \omega_\epsilon \right\|_{L^2(B^4)} \leq h \left\| d \omega_\epsilon \right\|_{L^2(B^4)}. \tag{2.11} \]

Then, system (2.1) and integration by parts yield

\[ \left\| d \omega_\epsilon \right\|_{L^2(B^4)}^2 = - \int_{B^4} \omega_\epsilon \wedge * d \omega_\epsilon + \int_{\partial B^4} \iota^* (\omega_\epsilon \wedge * d \omega_\epsilon) = \epsilon \int_{B^4} \omega_\epsilon \wedge * \left( \frac{d}{2} a \iota_\epsilon, A_\epsilon \right) + \left( A_\epsilon, * F A_0 \right). \tag{2.12} \]

Sobolev embeddings and Hölder inequalities give

\[ \left\| d \omega_\epsilon \right\|_{L^2(B^4)}^2 \leq \epsilon C' \left\| A_\epsilon \right\|_{L^2(B^4)}^2 \left\| F A_0 \right\|_{L^2(B^4)}^2. \]

Thus,

\[ \left\| \omega_\epsilon \right\|_{L^2(B^4)} \leq h \left\| d \omega_\epsilon \right\|_{L^2(B^4)} \leq \sqrt{\epsilon} B', \tag{2.13} \]

for some constant $B'$, if $\epsilon$ is sufficiently small. Using estimate (2.13) into (2.12) and estimating again, one obtains (2.4), for some constant $B_1$, if $\epsilon$ is sufficiently small.

This completes the proof of the lemma. \qed

To prove the proposition, we cover $B^4$ with coordinate patches $U_1 := \{ x \in B^4; |x|^2 < \delta \}$ of “type one”, in the interior, and $U_2 := \{ (x', x^4); x' \in \partial B^4; x^4 \geq 0; |x|^2 < \delta \}$ of “type two”
near the boundary. Here, the functions $x^4 \to (x', x^4)$ describe unit speed geodesics orthogonal to $\partial B^4$. This way, the metric $g_{ij}$ satisfies $g_{i4}(x', 0) = 0$ and $g_{44} = 1$ in a neighborhood of the boundary. Doubling $B^4$ via reflection across the boundary, yields a Lipschitz-bounded metric on the resulting manifold. We lift this action trivially to the bundle. We show that $\|\omega_\epsilon\|_{C^k(U_2)} \leq C_k \epsilon$ holds for small $\epsilon > 0$, in a suitable gauge, all the way up to $\partial B^4$, on neighborhoods of type two. (We omit the proof for neighborhoods of type one).

Locally, on $U_2$, system (2.1) and the good gauge theorem for boundary neighborhoods (cf. [15]) yield

\[
\begin{align*}
\left\{ \begin{array}{ll}
 d^*d\omega_\epsilon + \epsilon\{ \frac{d^*}{2}[A_\epsilon, \Lambda] + *\Lambda, *dA_\epsilon \} + \frac{\epsilon}{2}[\Lambda, *\Lambda] \} = 0 & \text{on } U_2 \\
 d^*F \omega_\epsilon = 0 & \text{on } U_2 \\
 d^*F(\omega_\epsilon)_x = 0 & \text{on } \{x^4 = 0\} \\
i^*\omega_\epsilon = 0 & \text{on } \{x^4 = 0\}
\end{array} \right.
\]

(2.14)

where $\tau$ denotes tangential components and $*_F$ is the flat Hodge operator. This becomes

\[
\begin{align*}
\left\{ \begin{array}{ll}
 \Lambda \omega_\epsilon := \Delta_F \omega_\epsilon + \mathcal{E} \omega_\epsilon + \epsilon R_\epsilon \omega_\epsilon = \epsilon G(A_\epsilon, dA_\epsilon) & \text{on } U_2 \\
 d^*F \omega_\epsilon = 0 & \text{on } U_2 \\
 d^*_F(\omega_\epsilon)_x = 0 & \text{on } \{x^4 = 0\} \\
i^*\omega_\epsilon = 0 & \text{on } \{x^4 = 0\}
\end{array} \right.
\]

(2.15)

where $\mathcal{E} = *_F d(*_F d)$ contains only first order derivatives of the metric and can be made small by dilations, $R_\epsilon(\cdot) = *_F \{ \frac{d^*_F}{2}[\cdot, A_\epsilon] + *\Lambda, *dA_\epsilon \} + \frac{\epsilon}{2}[\Lambda, *\Lambda] \} = \text{lower order, and } G(A_\epsilon, dA_\epsilon) = - *_F \{ \frac{d^*_F}{2}[A_\epsilon, A_\epsilon] + [A_\epsilon, *dA_\epsilon] + \frac{\epsilon}{2}[A_\epsilon, *\Lambda] \} \} \text{ is uniformly bounded in } L^2 \text{ for small } \epsilon$ by the previous lemma.

Following the procedure in [15], we reflect $U_2$ across the boundary $\{x^4 = 0\}$ and work on the doubled neighborhood $\tilde{U}$, after doubling all the operators above via the formula $r^* \tilde{\Lambda} = \tilde{\Lambda} r^*$. More in detail,

\[\tilde{\Lambda}(\omega)(x) = \Lambda(\omega|_{U^+})(x) + r^* \Lambda(r^*(\omega|_{U^-}))(x),\]

(here $U^\pm = \{x^4 > 0(<0)\}$), for all 1-forms $\omega$. An easy computation shows that the double $\tilde{\Lambda}_F$ is $\Delta_F$ on $\tilde{U}$.

Moreover, $\tilde{\mathcal{E}}$ and $\tilde{R}_\epsilon$ are small operators from $L^p_{0,1}(T^*\tilde{U} \otimes \mathfrak{su}(2))$ to $L^p_{-1}(T^*\tilde{U} \otimes \mathfrak{su}(2))$ and, also, from $L^p_{0,1}(T^*\tilde{U} \otimes \mathfrak{su}(2))$ to $L^p(T^*\tilde{U} \otimes \mathfrak{su}(2))$ for $p > 1$, where $L^p_{0,1}(T^*\tilde{U} \otimes \mathfrak{su}(2))$ is the completion of $C^\infty(T^*\tilde{U} \otimes \mathfrak{su}(2))$ with respect to the $L^1_{1,-1}$-norm, see also §3.3.

We take care of the boundary conditions on $\partial \tilde{U}$ by introducing a smooth appropriate cut-off function $\phi$. System (2.15) then becomes

\[
\begin{align*}
\left\{ \begin{array}{ll}
 \Delta_F (\phi \tilde{\omega}_\epsilon) + \mathcal{E}(\phi \tilde{\omega}_\epsilon) + \epsilon R_\epsilon(\phi \tilde{\omega}_\epsilon) = \epsilon(\tilde{\phi} G(A_\epsilon, dA_\epsilon) + \frac{1}{\epsilon} T_{\phi} \tilde{\omega}_\epsilon) := \epsilon \alpha_\epsilon & \text{on } \tilde{U} \\
 \phi(\tilde{\omega}_\epsilon)_x = 0 & \text{on } \partial \tilde{U}
\end{array} \right.
\]

(2.16)

where $\tilde{\omega}$ and $\tilde{G}$ are odd extensions of $\omega$ and $G$ (i.e., $r^* \tilde{\omega} = -\tilde{\omega}$, and $r^* \tilde{G} = -\tilde{G}$), and $T_{\phi} \tilde{\omega}_\epsilon = \Lambda(\phi \tilde{\omega}_\epsilon) - \phi \Lambda(\tilde{\omega}_\epsilon)$ contains only first order derivatives of $\tilde{\omega}_\epsilon$. With this definition, $\tilde{G}$ is uniformly bounded in $L^2$, thus in $L^p_{-1}$ with $p \leq 4$, and so is $\frac{1}{\epsilon} T_{\phi} \tilde{\omega}_\epsilon$ (by the estimate (2.4)). So $\alpha_\epsilon$ is uniformly bounded in $L^p_{-1}$ with $p \leq 4$ for $0 \leq \epsilon \leq \epsilon_0$. 

It is well known that the system

\[
\begin{align*}
\Delta \omega &= \gamma \quad \text{on } \|x\| \leq \delta \\
\omega &= 0 \quad \text{on } \|x\| = \delta ,
\end{align*}
\]

with \( \gamma \in L_{-1}^1 \) and \( \omega \in L_1^1 \), admits a unique solution in \( L_1^p \). Let \( S \) be the solution operator (bounded). Applying \( S \) to \((2.16)\) one obtains

\[
I(\phi \tilde{\omega}_\varepsilon) + S[\mathcal{E}(\phi \tilde{\omega}_\varepsilon) + \varepsilon R_\varepsilon(\phi \tilde{\omega}_\varepsilon)] = \varepsilon S(\alpha_\varepsilon).
\]

Thus, since \( \mathcal{E} + \varepsilon R_\varepsilon : L_{0,1}^p \rightarrow L_{-1}^p \) is small, we can invert \( I + S(\mathcal{E} + \varepsilon R_\varepsilon) \) and obtain

\[
\phi \tilde{\omega}_\varepsilon = \varepsilon [I + S(\mathcal{E} + \varepsilon R_\varepsilon)]^{-1} S \alpha_\varepsilon.
\]

Thus

\[
\|\phi \tilde{\omega}_\varepsilon\|_{L_1^p(U)} \leq \varepsilon \|I + S(\mathcal{E} + \varepsilon R_\varepsilon)^{-1} S\|_{L_{-1}^p(U)} \leq C'_\varepsilon,
\]

yielding \( \varepsilon^{-1}\omega_\varepsilon \) uniformly bounded in \( L_1^p(U_2) \), on a smaller neighborhood \( U_2 \), all the way up to the boundary \( \{x^4 = 0\} \). Iterating the procedure above, recalling that \( A_\varepsilon = A_0 + \omega_\varepsilon \) and estimate \((2.4)\), one obtains a system similar to \((2.16)\), but simpler (this time there is no need for the operator \( R_\varepsilon \)), with the right hand side uniformly bounded in \( L_p^p \) for any \( p < 4 \), yielding finally

\[
\|\omega_\varepsilon\|_{C^q_0} \leq C \|\omega_\varepsilon\|_{L_p^p(U_2)} \leq C C'_\varepsilon := C_0 \varepsilon ,
\]

for some \( q > 4 \), on a smaller neighborhood \( U_2 \), all the way up to and including the boundary \( \{x^4 = 0\} \). To show the analogous result for \( \nabla \omega_\varepsilon \), we take first tangential derivatives in \((2.15)\) and proceed with the doubling procedure above (using that \( \partial_j \omega_\varepsilon = 0 \) at \( \{x^4 = 0\} \), for \( j = 1, 2, 3 \), thus the one forms \( \partial_j \omega \) are continuous). For normal components \( \partial_4 \omega_\varepsilon \), one uses the relations between tangential and normal derivatives given by the good gauge and the field equations. Iterating this procedure, after some calculation, one obtains

\[
\|A_\varepsilon - A_0\|_{C^k(B^4)} := \|\omega_\varepsilon\|_{C^k(B^4)} \leq C_k \varepsilon \text{ for small } \varepsilon > 0, \text{ in a suitable gauge}.
\]

2.2 Asymptotic profile of large solutions

Here we give a heuristic argumentation to motivate our approach to the existence of new solutions to \((\mathcal{D}_\varepsilon)\). We start by observing that the Lie algebra isomorphism \( \phi_\varepsilon \) (and the Lie group isomorphism \( \Phi_\varepsilon \)) transform the \( SU(2)_2 \)-Yang Mills Dirichlet problem \((\mathcal{D}_\varepsilon)\) for \( A \), into the \( SU(2)_2 \)-Yang Mills Dirichlet problem for \( \phi_\varepsilon(A) := \varepsilon A \)

\[
(\mathcal{D}(\varepsilon)) \quad \begin{cases} 
\partial_4^* A = 0 \\
\varepsilon^* A \sim \phi_\varepsilon(A) := \varepsilon A_0 \\
\text{in } B^4 \text{ on } \partial B^4 ,
\end{cases}
\]

where \( d_A^* = d \star + \star [A, \star] \), and \( F_A = dA + \frac{1}{2}[A, A] \).

Let us set \( \underline{A}(\varepsilon) := \phi_\varepsilon(A) \) and \( \overline{A}(\varepsilon) := \phi_\varepsilon(\overline{A}) \), where \( A_0 \) is the absolute minimizing solution to \((\mathcal{D}_\varepsilon)\), and \( \overline{A}_\varepsilon \) is the absolute minimizer restricted to the class \( A_{+1}(A_0) \) (or \( A_{-1}(A_0) \)). Then, \( A(\varepsilon) \) is an absolute minimizing solution to \((\mathcal{D}(\varepsilon))\), and \( \overline{A}(\varepsilon) \) is a large solution to \((\mathcal{D}(\varepsilon))\),
i.e., it minimizes the Yang Mills functional in $\mathcal{A}_{+1}(\epsilon A_0)$ (or $\mathcal{A}_{-1}(\epsilon A_0)$). Passing to the limit $\epsilon \to 0$ formally in $(\mathcal{D}(\epsilon))$, one obtains the Dirichlet problem for Yang Mills connections with the trivial boundary value. It is known (cf. [8]) that this only admits flat solutions, therefore $\mathcal{A}(\epsilon)$ cannot converge strongly since $\mathcal{A}(\epsilon) \in \mathcal{A}_{+1}(\epsilon A_0)$ for $\epsilon > 0$. Indeed, following the proof in [12], one may argue that $\int_{B^4} |F_{\mathcal{A}(\epsilon)}|^2 dx \to 8\pi^2$ as $\epsilon \to 0$, and $|F_{\mathcal{A}(\epsilon)}|^2 dx \to 8\pi^2 \delta_p dx$ as a Radon measure for some $p \in \mathbb{B}^4$. It follows that, in a suitable gauge, one has asymptotically $\mathcal{A}(\epsilon) \approx \mathcal{A}(\epsilon) \#(\pm 1$-instanton on $S^4$), where $\#$ denotes the connected sum. In terms of $\mathcal{A}_\epsilon$, one has asymptotically $\mathcal{A}_\epsilon \approx \mathcal{A}_\epsilon \#^1(\pm 1$-instanton on $S^4$).

**Remark 2.1** The argumentation above would require a little extra work to be made rigorous. In fact, the spaces $\mathcal{A}_{\pm 1}(A_0)$ do depend on $\mathcal{A}_\epsilon$, thus on $\epsilon$. However, in this paper we construct solutions to $(\mathcal{D}_\epsilon)$ for ‘fixed’ small positive $\epsilon$ and we are not concerned with this issue, nor with the issue of constructing paths of solutions parameterized by $\epsilon$.

### 3. Reduction to a finite dimensional problem

In this section we construct approximate solutions to $(\mathcal{D}_\epsilon)$ (for small $\epsilon$) via a gluing technique, and study the asymptotic expansion of the $SU(2)_s$-Yang Mills functional, its gradient and its Hessian. The approximate solutions depend on a finite-dimensional parameter (cf. §3.1). The space tangent to the space of approximate solutions is a good approximation for the kernel of the Hessian, thus it constitutes the obstruction to the direct application of the implicit function theorem. We follow the standard procedure for this type of problems, consisting of first solving the Yang Mills equation orthogonally to the kernel of the Hessian by means of the auxiliary equation introduced (and solved) in §3.6. Thus, the problem is transformed into a finite dimensional problem (cf. in particular Lemma 3.9 and Proposition 3.2).

We focus on solutions that create a 1-bubble in the limit as $\epsilon \to 0$.

#### 3.1 The space of approximate solutions and introduction of the notation

Motivated by the discussion in §2.2, we seek approximate solutions to $(\mathcal{D}_\epsilon)$ in $\mathcal{A}_{+1}(A_0)$ of the form: $A_\epsilon = A_\epsilon \#^1(1$-bubble) + $a$, where $a \in C^\infty(T^*B^4 \otimes \text{Ad}(P(p, g, \lambda)))$ is small and satisfies $a = 0$ on $\partial B^4$. (The bundle $P(p, g, \lambda)$ will be defined soon). In this section, we introduce all the technical notation used to prove Theorems 1-3.

We start by describing the main part of the solution $A_\epsilon \#^1(1$-bubble).

For $\lambda > 0$, $p \in \mathbb{R}^4$, the 1-instanton solution $I_{\lambda, p}$ with center at $p$ and scale $\lambda$ to the Yang Mills equation on $\mathbb{R}^4$ is defined by

$$I_{\lambda, p} = \begin{cases} 
I_{\lambda, p}^2(x) = \text{Im} \frac{\lambda^2 (x - p) dx}{\lambda^2 + |x - p|^2} & \text{in } U_2(p) = \mathbb{R}^4 \setminus \{p\} \\
I_{\lambda, p}^1(x) = \text{Im} \frac{\lambda^2 (x - p) dx}{\lambda^2 + |x - p|^2} & \text{in } U_1(p) = \{x \in \mathbb{R}^4 : |x - p| < 1\} = B^4(p),
\end{cases}$$

where $x = (x^0, x^1, x^2, x^3) \in \mathbb{R}^4$ is identified with the quaternion $x = x^0 + x^1 i + x^2 j + x^3 k \in \mathbb{H}$ and the transition map is $g_{12, p}(x) = \frac{x - p}{|x - p|}$ for $x \in U_1(p) \cap U_2(p)$ (notice that the gluing relation $I_{\lambda, p}^2 = g_{12, p}^{-1} I_{\lambda, p}^1 g_{12, p}$ holds in $U_1(p) \cap U_2(p)$).
For \( p \in B^4, \lambda > 0 \), we define the 1-form \( h_{\lambda,p} := (h_{\lambda,p})_j \, dx^j \in C^\infty(T^*B^4 \otimes \mathfrak{su}(2)) \), the components of which solve the Dirichlet problems
\[
\begin{cases}
\Delta (h_{\lambda,p})_j = 0 & \text{in } B^4, \\
(h_{\lambda,p})_j = (I^2_{\lambda,p})_j & \text{at } \partial B^4,
\end{cases}
\]
and set \( PI^2_{\lambda,p} := I^2_{\lambda,p} - h_{\lambda,p} \), the projection of \( I^2_{\lambda,p} \) on the Sobolev space \( L^2_{0,1}(T^*B^4 \otimes \mathfrak{su}(2)) := \{ a \in L^2_1(T^*B^4 \otimes \mathfrak{su}(2)) : a = 0 \text{ on } \partial B^4 \}. \)

We also define a cut-off function \( \beta(x) = \beta(|x|) \in C^\infty_0(\mathbb{R}^4) \), such that \( \beta = 1 \) for \( |x| \leq 1 \), \( \beta(x) = 0 \) for \( |x| \geq 2 \) and \( 0 \leq \beta(x) \leq 1 \), and, for \( \lambda > 0, p \in \mathbb{R}^4 \), we define \( \beta_{\lambda,p}(x) := \beta(\lambda^{-1}(x-p)) \).

For \( d_0 \) and \( \lambda_0 \) small fixed numbers satisfying \( 0 < 2\lambda_0 < d_0 < 1 \), we consider the set of parameters \( \tilde{\mathcal{P}}(d_0, \lambda_0) := B^4_{1-d_0} \times SU(2) \times (0, \lambda_0) \).

For \( q := (p, g, \lambda) \in \tilde{\mathcal{P}}(d_0, \lambda_0) \), we define the connections \( A(q) \) as
\[
A(q) = \begin{cases}
(1 - \beta_{\lambda,p})A_1 + \frac{1}{\epsilon} \beta_{\lambda/4,p} \, g \, I^2_{\lambda,p} \, g^{-1} + \frac{1}{\epsilon}(1 - \beta_{\lambda/4,p}) \, g \, PI^2_{\lambda,p} \, g^{-1} & \text{in } B^4 \setminus \{p\}, \\
\frac{1}{\epsilon} g I^1_{\lambda,p} g^{-1} & \text{in } B^4_{\lambda/4}(p),
\end{cases}
\]
obtained by gluing the 1-instanton to \( A_1 \). These connections live on the bundles \( P(q) \), defined by the data
\[
\left( B^4_{\lambda/4}(p), B^4 \setminus \{p\}, g \, g_{12,p} \, g^{-1} \right).
\]

Notice that the relative 2nd Chern number of \( P(q) \) with respect to \( A_1 \) is 1, thus \( A(q) \in \mathcal{A}_{+1}(A_0) \).

We observe that the effective parameter space is
\[
\mathcal{P}(d_0, \lambda_0) := \tilde{\mathcal{P}}(d_0, \lambda_0)/\{\pm 1\} = B^4_{1-d_0} \times SO(3) \times (0, \lambda_0),
\]
since \( P(p, -g, \lambda) = P(p, g, \lambda) \) and \( A(p, -g, \lambda) = A(p, g, \lambda) \), therefore from now on we quotient out with respect to this action and redefine \( q := (p, [g], \lambda) \), where \([g] \in SO(3) = SU(2)/\{\pm 1\}\).

We also observe that the bundles \( P(q) \), for \( q \in \mathcal{P}(d_0, \lambda_0) \) are all isomorphic, so we fix \( q_0 := (p_0, [g_0], \lambda_0) \in \mathcal{P}(d_0, \lambda_0) \) and apply the convention that everything is pulled back to \( P(p_0, [g_0], \lambda_0) \), via the bundle isomorphisms \( \varphi(q) : P(q_0) \xrightarrow{\sim} P(q) \).

We define the map \( \mathcal{G} : \tilde{\mathcal{P}}(d_0, \lambda_0) \to \mathcal{A}_{+1}(A_0) \), \( q \mapsto A(q) \), and the space
\[
\mathcal{N}(d_0, \lambda_0) := \mathcal{G}(\tilde{\mathcal{P}}(d_0, \lambda_0)),
\]
as the space of approximate solutions to \( (\mathcal{D}_{\epsilon}) \) in \( \mathcal{A}_{+1}(A_0) \).

Let \( \mathcal{A}(q) := \text{Aut } P(q) \) be the space of smooth gauge transformations of \( P(q) \), that is, the automorphism group of \( P(q) \), and \( \mathcal{G}_{k+1}^P := L^p_{k+1}(\text{Aut } P(q)) \) the space of \( L^p_{k+1} \)-gauge transformations. For \( A_0 \in C^\infty(T^*\partial B^4 \otimes \mathfrak{su}(2)) \) and \( q \in \tilde{\mathcal{P}}(d_0, \lambda_0) \), we define the following spaces of connections:
\[
\mathcal{A}(A_0; q) = \{ A : A \text{ is a smooth connection on } P(q) \text{ such that } \iota^* A \sim A_0 \text{ on } \partial B^4 \},
\]
where \( \iota^* A \sim A_0 \) on \( \partial B^4 \) means that \( \iota^* A \) is gauge equivalent to \( A_0 \) over \( \partial B^4 \), via a gauge transformation which extends smoothly to \( B^4 \):

\[
\mathcal{A}_k^p(A_0; q) = \{ A : A \text{ is a connection of class } L_k^p \text{ on } P(q) \text{ such that } \iota^* A \sim A_0 \text{ on } \partial B^4 \},
\]

where this time \( \iota^* A \sim A \) on \( \partial B^4 \) means that \( \iota^* A \) is gauge equivalent to \( A_0 \) over \( \partial B^4 \), via an \( L_{k+1-1/p}^p \)-gauge transformation on \( P(q)|_{\partial B^4} \) which admits an \( L_{k+1}^p \) extension to \( B^4 \). We henceforth assume that \((k+1)p \geq 4\).

The spaces \( \mathcal{A}(A_0; q) \) and \( \mathcal{A}_k^p(A_0; q) \) have connected components labeled by the integers \( \mathbb{Z} \) (cf. \([12]\)):

\[
\mathcal{A}(A_0; q) = \bigsqcup_{j \in \mathbb{Z}} \mathcal{A}_j(A_0; q), \quad \text{with } \mathcal{A}_j(A_0; q) = \{ A \in \mathcal{A}(A_0; q) : c_2(A) = j \},
\]

\[
\mathcal{A}_k^p(A_0; q) = \bigsqcup_{j \in \mathbb{Z}} \mathcal{A}_k^p,j(A_0; q), \quad \text{with } \mathcal{A}_k^p,j(A_0; q) = \{ A \in \mathcal{A}_k^p(A_0; q) : c_2(A) = j \},
\]

where \( c_2(A) := \frac{2}{\pi^2} \int_{B^4} \text{Tr}(F_A^\epsilon \wedge F_A^\epsilon) - \frac{2}{8\pi^2} \int_{B^4} \text{Tr}(F_{A^\epsilon}^\epsilon \wedge F_{A^\epsilon}^\epsilon) \) is the relative 2nd Chern class of \( A \) with respect to \( \mathcal{A}_j \).

Since the groups \( \mathcal{G}(q) \), \( \mathcal{G}_{k+1}^p(q) \), respectively, act on \( \mathcal{A}(A_0; q) \), \( \mathcal{A}_k^p(A_0; q) \), preserving these connected components, we consider the quotient spaces

\[
\mathcal{B}(A_0; q) := \mathcal{A}(A_0; q)/\mathcal{G}(q), \quad \mathcal{B}_j(A_0; q) := \mathcal{A}_j(A_0; q)/\mathcal{G}(q),
\]

\[
\mathcal{B}_k^p(A_0; q) := \mathcal{A}_k^p(A_0; q)/\mathcal{G}_{k+1}^p(q), \quad \mathcal{B}_k^p,j(A_0; q) := \mathcal{A}_k^p,j(A_0; q)/\mathcal{G}_{k+1}^p(q),
\]

and denote by \([A]\) the class of \( A \) in \( \mathcal{B}(A_0; q) \), or in \( \mathcal{B}_k^p(A_0; q) \).

We will also make use of the subgroups \( \mathcal{G}^*(q) \) of \( \mathcal{G}(q) \), and \( \mathcal{G}_{k+1}^{*,p}(q) \) of \( \mathcal{G}_{k+1}^p(q) \), consisting of all gauge transformations \( g \) such that \( g((1, 0, 0, 0)) = 1 \), and of corresponding subspaces \( \mathcal{A}^*(A_0; q) \) of \( \mathcal{A}(A_0; q) \), and \( \mathcal{A}_k^*,p(A_0; q) \) of \( \mathcal{A}_k^p(A_0; q) \), consisting of all connections satisfying \( \iota^* A \sim A_0 \) on \( \partial B^4 \) via a gauge transformation \( g \) with \( g((1, 0, 0, 0)) = 1 \) extendible to \( B^4 \). The subspaces \( \mathcal{A}_j^*(A_0; q) \) and \( \mathcal{A}_k^*,p,j(A_0; q) \) are defined analogously. The groups \( \mathcal{G}^*(q) \), \( \mathcal{G}_{k+1}^{*,p}(q) \), act freely on \( \mathcal{A}^*(A_0; q) \), \( \mathcal{A}_k^*,p(A_0; q) \), respectively, and the corresponding quotients \( \mathcal{B}^*(A_0; q) \), \( \mathcal{B}_k^*,p(A_0; q) \), are proved to be differentiable manifolds provided that \((k+1)p > 4\) (cf. §1.1 in \([14]\)).

The \( \text{YM}_t \)-action (cf. \([11]\)), descends to the quotients \( \mathcal{B}(A_0; q) \), \( \mathcal{B}^*(A_0; q) \), \( \mathcal{B}_k^p(A_0; q) \), \( \mathcal{B}_k^*,p(A_0; q) \), thus solutions to \( (D_e) \) are critical points of \( \text{YM}_t \) on \( \mathcal{B}(A_0; q) \), \( \mathcal{B}^*(A_0; q) \), \( \mathcal{B}_k^p(A_0; q) \), \( \mathcal{B}_k^*,p(A_0; q) \).

As remarked previously, the bundles \( P(q) \) are all isomorphic to \( P(q_0) \) for any fixed \( q_0 \in \mathcal{P}(d_0, \lambda_0) \), thus, from now on, we may omit the indication of \( q_0 \) (or \( q \)) from our notation and simply write \( P, \mathcal{A}(A_0), \mathcal{A}^*(A_0), \mathcal{A}_k^p(A_0), \mathcal{A}_k^*,p(A_0), \mathcal{G}, \mathcal{G}^*, \mathcal{G}_k, \mathcal{G}_{k+1}, \mathcal{B}^*(A_0) \) and \( \mathcal{B}_k^*,p(A_0) \) for the corresponding objects.

### 3.2 Asymptotic expansion of \( \text{YM}_t(A(q)) \)

In this and in the next sections, we show that the connections \( A(q) \) introduced in §3.1 are indeed good approximate solutions to the Dirichlet problem \( (D_e) \). For this, we need to consider the
In the hypotheses above, for Proposition 3.1, the following asymptotic expansion:

\[ J = 2 \lambda^2 \int_{B^4} \left| \frac{d}{dx} \right|^2 dx + 4 \lambda^2 \int_{B^4} (dh_p)^{-1} d x - 4 \lambda^2 \int_{B^4} \left( (F_{A_0}^0)^{-1}, g(dh_p)^{-1} \right) dx + r_1(q), \]

where \( r_1(q) = O(\epsilon^3) \) as \( \epsilon \to 0 \).

Moreover, for \( \mathcal{F}_\epsilon \) defined by

\[ \mathcal{F}_\epsilon(q) := 2 \lambda^4 \int_{B^4} (dh_p)^{-1} d x - 4 \lambda^2 \int_{B^4} \left( (F_{A_0}^0)^{-1}, g(dh_p)^{-1} \right) dx, \]

one has

\[ J'(q) = \mathcal{F}_\epsilon'(q) + r_2(q), \]

where \( J'(q) \) and \( \mathcal{F}_\epsilon'(q) \) are the derivatives of \( J \) and \( \mathcal{F}_\epsilon \) with respect to the variable \( q \), and

\[ r_2(q) \left( \frac{\partial}{\partial \eta_i} \right) = O(\epsilon^{5/2}) \quad (1 \leq i \leq 4), \quad r_2(q)(\xi_i g) = O(\epsilon^3 | \log \epsilon |) \quad (1 \leq i \leq 3), \]

\[ r_2(q) \left( \frac{\partial}{\partial \lambda} \right) = O(\epsilon^{5/2} | \log \epsilon |), \]

for \( q \in \mathcal{P}(d_0, \lambda_0; D_1, D_2; \epsilon) \) and small \( \epsilon > 0 \).

In order to prove the proposition above, we need an estimate for the one-form \( h_{\lambda, p} \) defined in \$3.1. We recall that \( h_{\lambda, p} \) and \( h_p \) are harmonic componentwise on \( B^4 \), \( h_{\lambda, p} = \text{Im} \frac{\lambda^2 \frac{dx}{|p|^2}}{|x - p|^2 (\lambda^2 + |x - p|^2)} \) at \( \partial B^4 \), and \( h_p = \text{Im} \frac{\frac{dx}{|p|^2}}{|x - p|^2} \) at \( \partial B^4 \). We have the following:
Lemma 3.1 Let $d_0, \lambda \in (0, 1)$. For any $k \geq 1$, there exists a constant $C_{k, d_0}$ depending only on $k$ and $d_0$ such that
\[
\|h_{\lambda, p} - \lambda^2 h_p\|_{C^k(B^4)} \leq C_{k, d_0} \lambda^4
\]
for any $p \in B_{1-d_0}$.

Proof: For $x \in \partial B^4$, we have
\[
h_{\lambda, p} = \text{Im} \left( \frac{\lambda^2 x - p \, dx}{|x-p|^4} - \frac{\lambda^4 x - p \, dx}{|x-p|^4(\lambda^2 + |x-p|^2)} \right). \tag{3.11}
\]
Let us define
\[
r_{\lambda, p}(x) = -\frac{\lambda^4 x - p \, dx}{|x-p|^4(\lambda^2 + |x-p|^2)}, \ x \neq p.
\]
It is easy to see that there exists a constant $C$ depending only on $k$ and $d_0$ such that $|\nabla^k r_{\lambda, p}(x)| \leq C \lambda^4$ for any $p \in B_{1-d_0}$, $x \in B^4 \setminus B_{1-d_0/2}$, and $\lambda \in (0, 1)$.

Let $\varphi$ be a smooth cut-off function such that $\varphi(x) = 0$ in $B_{1-d_0/2}$, $\varphi(x) = 1$ on $\mathbb{R}^4 \setminus B_{1-d_0/4}$ and $|\nabla \varphi(x)| \leq 8d_0^{-1}$. Then there exists another constant $C$ depending only on $k$ and $d_0$ such that
\[
|\nabla^k (\varphi(x) r_{\lambda, p}(x))| \leq C \lambda^4 \tag{3.12}
\]
for $p \in B_{1-d_0}$, $x \in B^4$ and $\lambda \in (0, 1)$. Since $h_{\lambda, p} - \lambda^2 h_p$ is a harmonic function on $B^4$ with the same boundary value as $\varphi(x) r_{\lambda, p}(x)$ at $\partial B^4$, the assertion of the lemma follows from (3.12) and elliptic estimates (c.f. [7]). \qed

Proof of Proposition 3.1 Throughout these estimates, the expression $Q(q) \lesssim f(\epsilon)$ for a quantity $Q(q)$ depending on $q \in \mathcal{P}(d_0, \lambda_0; D_1, D_2; \epsilon)$ and a function $f(\epsilon)$ means that there exists a constant $C = C(d_0; \lambda_0; D_1, D_2)$ such that $Q(q) \leq C f(\epsilon)$ for all $q \in \mathcal{P}(d_0, \lambda_0; D_1, D_2; \epsilon)$ and small negative $\epsilon$. We write $Q(q) \simeq f(\epsilon)$ if the inequality holds both ways.

We first prove the asymptotic expansion (3.8). Due to the definition of $A(q)$, we decompose the domain $B^4$ into four subdomains: $B^4 \setminus B_{2\lambda}(p)$, $B_{2\lambda}(p) \setminus B_{\lambda/2}(p)$, $B_{\lambda/2}(p) \setminus B_{\lambda/4}(p)$ and $B_{\lambda/4}(p)$.

(1) Estimate of $\int_{B^4 \setminus B_{2\lambda}(p)} |F_{\lambda}(q)|^2 \, dx$

On $B^4 \setminus B_{2\lambda}(p)$, $A(q) = A_\epsilon + \frac{\epsilon}{2} g P_{\lambda, p}^2 g^{-1}$ and $F_{\lambda}(q)^\epsilon = F_{A_\epsilon}^\epsilon + F_{g P_{\lambda, p}^2 g^{-1}}^\epsilon + [A_\epsilon, g P_{\lambda, p}^2 g^{-1}]$.

We recall that $F_{A_\epsilon}^\epsilon = dA_\epsilon + \frac{1}{2}[A_\epsilon, A_\epsilon]$, $F_{g P_{\lambda, p}^2 g^{-1}}^\epsilon = d(g P_{\lambda, p}^2 g^{-1})^\epsilon + \frac{1}{2}[g P_{\lambda, p}^2 g^{-1}, g P_{\lambda, p}^2 g^{-1}]^\epsilon$.

Thus,
\[
\epsilon^2 \int_{B^4 \setminus B_{2\lambda}(p)} |F_{\lambda}(q)|^2 \, dx = \epsilon^2 \int_{B^4} |F_{A_\epsilon}|^2 \, dx + \int_{B_{2\lambda}(p)} |F_{g P_{\lambda, p}^2 g^{-1}}|^2 \, dx + \int_{B_{2\lambda}(p)} \|A_\epsilon, g P_{\lambda, p}^2 g^{-1}\|^2 \, dx
\]
\[
+ 2\epsilon \int_{B^4 \setminus B_{2\lambda}(p)} (F_{A_\epsilon}^\epsilon, F_{g P_{\lambda, p}^2 g^{-1}}^\epsilon) \, dx + 2\epsilon^2 \int_{B_{2\lambda}(p)} (F_{A_\epsilon}^\epsilon, [A_\epsilon, g P_{\lambda, p}^2 g^{-1}]) \, dx
\]
\[
+ 2\epsilon \int_{B_{2\lambda}(p)} (F_{g P_{\lambda, p}^2 g^{-1}}, [A_\epsilon, g P_{\lambda, p}^2 g^{-1}]) \, dx := E_1 + \cdots + E_6. \tag{3.13}
\]

We now estimate all terms $E_2 - E_6$ in (3.13).
• **Estimate of $E_2$:**

On any domain $D \subset B \setminus \{p\}$ one has

$$
\int_D |F_{P^2_{\lambda,p}}|^2 \, dx = \int_D |F_{I^2_{\lambda,p}}|^2 \, dx + \int_D |d_{I^2_{\lambda,p}} \, h_{\lambda,p}|^2 \, dx - 2 \int_D (F_{I^2_{\lambda,p}}, d_{I^2_{\lambda,p}} \, h_{\lambda,p}) \, dx
$$

$$
+ \int_D (d_{I^2_{\lambda,p}} \, h_{\lambda,p}, h_{\lambda,p}) \, dx + \frac{1}{4} \int_D |h_{\lambda,p}, h_{\lambda,p}|^2 \, dx - \int_D (d_{I^2_{\lambda,p}} \, h_{\lambda,p}, h_{\lambda,p}) \, dx
$$

recalling that $PI^2_{\lambda,p} = I^2_{\lambda,p} - h_{\lambda,p}$, $F_{I^2_{\lambda,p}} = dI^2_{\lambda,p} + \frac{1}{2} [I^2_{\lambda,p}, I^2_{\lambda,p}]$ and $d_{I^2_{\lambda,p}} \, h_{\lambda,p} = dh_{\lambda,p} + [I^2_{\lambda,p}, h_{\lambda,p}]$.

By Lemma [5.1] ($||h_{\lambda,p}||_{\infty} \lesssim \lambda^2 ||h_p||_{\infty}$, and $||dh_{\lambda,p}||_{\infty} \lesssim \lambda^2 ||\nabla h_p||_{\infty}$),

$$
\int_D |h_{\lambda,p}, h_{\lambda,p}|^2 \, dx \lesssim \int_D |h_{\lambda,p}|^4 \, dx \lesssim \lambda^8 \lesssim \epsilon^4,
$$

$$
\int_D (d_{I^2_{\lambda,p}} \, h_{\lambda,p}, h_{\lambda,p}) \, dx \lesssim \int_D |h_{\lambda,p}|^2 \, dx + \int_D |I^2_{\lambda,p}|^3 \, dx \lesssim \lambda^6 \lesssim \epsilon^3.
$$

Thus, for any $D \subset B \setminus \{p\}$, one has

$$
\int_D |F_{P^2_{\lambda,p}}|^2 \, dx = \int_D |F_{I^2_{\lambda,p}}|^2 \, dx + \int_D |d_{I^2_{\lambda,p}} \, h_{\lambda,p}|^2 \, dx
$$

$$
- 2 \int_D (F_{I^2_{\lambda,p}}, d_{I^2_{\lambda,p}} \, h_{\lambda,p}) \, dx + \int_D (d_{I^2_{\lambda,p}} \, h_{\lambda,p}, h_{\lambda,p}) \, dx + O(\epsilon^3) \quad (3.14)
$$

• **Estimates of $E_3 - E_5$:**

By Proposition [2.4] ($||A_\epsilon|| \lesssim ||A_\lambda||_{\infty}$, $||F_{A_\epsilon}|| \lesssim ||F_{A_\lambda}||_{\infty}$), and by Lemma [5.1]

$$
E_3 := \epsilon^2 \int_{B \setminus B_{2\lambda}(p)} ||A_\epsilon, g(I^2_{\lambda,p} - h_{\lambda,p})g^{-1}||^2 \, dx \lesssim \epsilon^2 \int_{B \setminus B_{2\lambda}(p)} |I^2_{\lambda,p}|^2 \, dx \lesssim \epsilon^2 \lambda^2 \lesssim \epsilon^3;
$$

$$
E_4 := 2\epsilon \int_{B \setminus B_{2\lambda}(p)} (F_{A_\epsilon}, gF_{I^2_{\lambda,p}}g^{-1}) \, dx - 2\epsilon \int_{B \setminus B_{2\lambda}(p)} (F_{A_\epsilon}, gd_{I^2_{\lambda,p}} \, h_{\lambda,p}g^{-1}) \, dx + O(\epsilon^3);
$$

$$
E_5 := 2\epsilon^2 \int_{B \setminus B_{2\lambda}(p)} (F_{A_\epsilon}, |A_\epsilon, g(I^2_{\lambda,p} - h_{\lambda,p})g^{-1}|) \, dx \lesssim \epsilon^2 \int_{B \setminus B_{2\lambda}(p)} |I^2_{\lambda,p}| \, dx \lesssim \epsilon^2 \lambda^2 \lesssim \epsilon^3.
$$

• **Estimate of $E_6$:**

$$
E_6 = 2\epsilon \int_{B \setminus B_{2\lambda}(p)} (g(F_{I^2_{\lambda,p}} - d_{I^2_{\lambda,p}} \, h_{\lambda,p} + \frac{1}{2}[h_{\lambda,p}, h_{\lambda,p}])g^{-1}, [A_\epsilon, g(I^2_{\lambda,p} - h_{\lambda,p})g^{-1}]) \, dx
$$

$$
= 2\epsilon \int_{B \setminus B_{2\lambda}(p)} (gF_{I^2_{\lambda,p}}g^{-1}, [A_\epsilon, gP^2_{I_{\lambda,p}}g^{-1}]) \, dx + O(\epsilon^3),
$$

since

$$
\epsilon \int_{B \setminus B_{2\lambda}(p)} |A_\epsilon| \, d_{I^2_{\lambda,p}} \, h_{\lambda,p} \, |I^2_{\lambda,p}| \, dx \lesssim \int_{B \setminus B_{2\lambda}(p)} \left( |dh_{\lambda,p}| |I^2_{\lambda,p}| + |I^2_{\lambda,p}|^2 |h_{\lambda,p}| \right) \, dx \lesssim \epsilon \lambda^4 \sim \epsilon^3
$$

$$
\int_{B \setminus B_{2\lambda}(p)} |A_\epsilon| \, d_{I^2_{\lambda,p}} \, h_{\lambda,p} \, |h_{\lambda,p}| \, dx \lesssim \int_{B \setminus B_{2\lambda}(p)} \left( |dh_{\lambda,p}| |h_{\lambda,p}| + |I^2_{\lambda,p}|^2 |h_{\lambda,p}| \right) \, dx \lesssim \epsilon \lambda^4 \sim \epsilon^3,
$$

$$
\int_{B \setminus B_{2\lambda}(p)} (|A_\epsilon|^2 |I^2_{\lambda,p}|^2 + |h_{\lambda,p}|^3) \, dx \lesssim \epsilon \lambda^6 \sim \epsilon^4.
$$
And, adding up $E_1 + \cdots + E_6$,

$$
\frac{\epsilon^2}{2} \int_{B \setminus B_{2\lambda}(p)} |F_A(q)|^2 \, dx = \frac{\epsilon^2}{2} \int_{B \setminus B_{2\lambda}(p)} |F_A(\lambda, p)|^2 \, dx + \int_{B \setminus B_{2\lambda}(p)} |F_{I^2_{\lambda, p}}|^2 \, dx
$$

$$
+ \int_{B \setminus B_{2\lambda}(p)} |d_{I^2_{\lambda, p}} h_{\lambda, p}|^2 \, dx - 2 \int_{B \setminus B_{2\lambda}(p)} (F_{I^2_{\lambda, p}} - d_{I^2_{\lambda, p}} h_{\lambda, p}) \, dx + \int_{B \setminus B_{2\lambda}(p)} (F_{I^2_{\lambda, p}}) \, dx
$$

$$
+ 2\epsilon \int_{B \setminus B_{2\lambda}(p)} (F_{A^*}, g F_{I^2_{\lambda, p}} g^{-1}) \, dx - 2\epsilon \int_{B \setminus B_{2\lambda}(p)} (F_{A^*}, g d_{I^2_{\lambda, p}} h_{\lambda, p} g^{-1}) \, dx
$$

$$
+ 2\epsilon \int_{B \setminus B_{2\lambda}(p)} (g F_{I^2_{\lambda, p}} g^{-1}, [A_x, g P I^2_{\lambda, p} g^{-1}]) \, dx + O(\epsilon^3).
$$

(3.15)

Quite analogously, we obtain the following estimate:

$$
\frac{\epsilon^2}{2} \int_{B \setminus B_{2\lambda}(p)} \text{Tr}(F_A(q) \wedge F_A(q)^*) = \frac{\epsilon^2}{2} \int_{B \setminus B_{2\lambda}(p)} \text{Tr}(F_A(\lambda, p) \wedge F_A(\lambda, p)^*) + \int_{B \setminus B_{2\lambda}(p)} \text{Tr}(F_{I^2_{\lambda, p}} \wedge F_{I^2_{\lambda, p}}) +
$$

$$
+ \int_{B \setminus B_{2\lambda}(p)} \text{Tr}(d_{I^2_{\lambda, p}} h_{\lambda, p} \wedge d_{I^2_{\lambda, p}} h_{\lambda, p}) - 2 \int_{B \setminus B_{2\lambda}(p)} \text{Tr}(F_{I^2_{\lambda, p}} \wedge d_{I^2_{\lambda, p}} h_{\lambda, p}) + \int_{B \setminus B_{2\lambda}(p)} \text{Tr}(F_{I^2_{\lambda, p}} \wedge h_{\lambda, p}, h_{\lambda, p})
$$

$$
+ 2\epsilon \int_{B \setminus B_{2\lambda}(p)} \text{Tr}(F_{A^*} \wedge g F_{I^2_{\lambda, p}} g^{-1}) - 2\epsilon \int_{B \setminus B_{2\lambda}(p)} (F_{A^*} \wedge g d_{I^2_{\lambda, p}} h_{\lambda, p} g^{-1})
$$

$$
+ 2\epsilon \int_{B \setminus B_{2\lambda}(p)} (g F_{I^2_{\lambda, p}} g^{-1}, [A_x, g P I^2_{\lambda, p} g^{-1}]) + O(\epsilon^3).
$$

(3.16)

Summing (3.10) to (3.15), and using $*F_{I^2_{\lambda, p}} = F_{I^2_{\lambda, p}}^*$, finally yields

$$
\frac{\epsilon^2}{2} \int_{B \setminus B_{2\lambda}(p)} |F_A(q)|^2 \, dx + \frac{\epsilon^2}{2} \int_{B \setminus B_{2\lambda}(p)} \text{Tr}(F_A(q) \wedge F_A(q)^*) = \frac{\epsilon^2}{2} \int_{B \setminus B_{2\lambda}(p)} |F_A(\lambda, p)|^2 \, dx
$$

$$
+ \frac{\epsilon^2}{2} \int_{B \setminus B_{2\lambda}(p)} \text{Tr}(F_A(\lambda, p) \wedge F_A(\lambda, p)^*) + 2 \int_{B \setminus B_{2\lambda}(p)} |d_{I^2_{\lambda, p}} h_{\lambda, p} - \lambda, p)|^2 \, dx - 4\epsilon \int_{B \setminus B_{2\lambda}(p)} ((F_{A^*} \wedge, g d_{I^2_{\lambda, p}} h_{\lambda, p} g^{-1}) \, dx + O(\epsilon^3).
$$

(3.17)

(2) **Estimate of** $\frac{\epsilon^2}{2} \int_{B(\lambda/2)(p)} |F_A(q)|^2 \, dx$.

On $B_{2\lambda}(p) \setminus B_{\lambda/2}(p)$, $A(q) = (1 - \beta_{\lambda, p}) A_x + \frac{\epsilon}{2} g P I^2_{\lambda, p} g^{-1}$ and

$$
\frac{\epsilon^2}{2} \int_{B_{2\lambda}(p) \setminus B_{\lambda/2}(p)} |F_A(q)|^2 \, dx = \frac{\epsilon^2}{2} \int_{B_{2\lambda}(p) \setminus B_{\lambda/2}(p)} |F_{(1 - \beta_{\lambda, p}) A_x}|^2 \, dx
$$

$$
+ \int_{B_{2\lambda}(p) \setminus B_{\lambda/2}(p)} |P_{I^2_{\lambda, p}}|^2 \, dx + \int_{B_{2\lambda}(p) \setminus B_{\lambda/2}(p)} \left| [(1 - \beta_{\lambda, p}) A_x, g P I^2_{\lambda, p} g^{-1}] \right|^2 \, dx
$$

$$
+ 2\epsilon \int_{B_{2\lambda}(p) \setminus B_{\lambda/2}(p)} (F_{(1 - \beta_{\lambda, p}) A_x} \wedge, g P I^2_{\lambda, p} g^{-1}) dx + 2\epsilon \int_{B_{2\lambda}(p) \setminus B_{\lambda/2}(p)} (F_{(1 - \beta_{\lambda, p}) A_x} \wedge, [(1 - \beta_{\lambda, p}) A_x, g P I^2_{\lambda, p} g^{-1}]) dx
$$

$$
+ 2\epsilon \int_{B_{2\lambda}(p) \setminus B_{\lambda/2}(p)} (g P I^2_{\lambda, p} g^{-1}, [(1 - \beta_{\lambda, p}) A_x, g P I^2_{\lambda, p} g^{-1}]) dx := E_1 + \cdots + E_6
$$

(3.18)

**Estimates of** $E_1 – E_6$:

Since $F_{(1 - \beta_{\lambda, p}) A_x} = (1 - \beta_{\lambda, p}) F_{A_x} - d\beta_{\lambda, p} \wedge A_x - \frac{\epsilon}{2} \beta_{\lambda, p} (1 - \beta_{\lambda, p}) |A_x|$, we find

$$
E_1 := \frac{\epsilon^2}{2} \int_{B_{2\lambda}(p) \setminus B_{\lambda/2}(p)} |F_{(1 - \beta_{\lambda, p}) A_x}|^2 \, dx \leq \epsilon^3.
$$
The estimate of $E_2$ is already done (cf. (3.14) with $D = B_{2\lambda}(p) \setminus B_{\lambda/2}(p)$).

By Lemma 3.1 and Proposition 2.1,

$$|E_3| \lesssim \epsilon^2 |A_p|_\infty \int_{B_{2\lambda}(p) \setminus B_{\lambda/2}(p)} (|I_{\lambda,p}^2|^2 + |h_{\lambda,p}|^2) \, dx \lesssim \epsilon\lambda^2 \lesssim \epsilon^3;$$

$$E_4 = \epsilon \int_{B_{2\lambda}(p) \setminus B_{\lambda/2}(p)} (F_{(1-\beta_{\lambda,p})A_p}, gF_{I_{\lambda,p}^2}g^{-1}) \, dx + O(\epsilon^3);$$

$$|E_5| \lesssim \epsilon^2 \lambda^{-1} \int_{B_{2\lambda}(p) \setminus B_{\lambda/2}(p)} |I_{\lambda,p}^2| \, dx + \epsilon^2 \lambda^{-1} \int_{B_{2\lambda}(p) \setminus B_{\lambda/2}(p)} |h_{\lambda,p}| \, dx \lesssim \epsilon^2 \lambda^2 \lesssim \epsilon^3;$$

$$E_6 = \epsilon \int_{B_{2\lambda}(p) \setminus B_{\lambda/2}(p)} (gF_{I_{\lambda,p}^2}g^{-1}, [1 - \beta_{\lambda,p}A_p, gPI_{\lambda,p}^2g^{-1}]) \, dx + O(\epsilon^3),$$

since

$$\int_{B_{2\lambda}(p) \setminus B_{\lambda/2}(p)} |A_p| (|d_{I_{\lambda,p}^2}^2 h_{\lambda,p}| + |d_{I_{\lambda,p}^2} h_{\lambda,p}| + |I_{\lambda,p}^2| |h_{\lambda,p}|^2 + |h_{\lambda,p}|^3) \, dx \leq \epsilon (\lambda^4 + \lambda^4 + \lambda^7 + \lambda^{10}) \lesssim \epsilon^3.$$

Adding up $E_1 + \cdots + E_6$, we obtain

$$\epsilon^2 \int_{B_{2\lambda}(p) \setminus B_{\lambda/2}(p)} |F_{A(q)}\epsilon|^2 \, dx = \epsilon^2 \int_{B_{2\lambda}(p) \setminus B_{\lambda/2}(p)} |F_{I_{\lambda,p}^2}|^2 \, dx + \epsilon^2 \int_{B_{2\lambda}(p) \setminus B_{\lambda/2}(p)} |d_{I_{\lambda,p}^2} h_{\lambda,p}|^2 \, dx$$

$$- 2 \int_{B_{2\lambda}(p) \setminus B_{\lambda/2}(p)} (F_{I_{\lambda,p}^2}, d_{I_{\lambda,p}^2} h_{\lambda,p}) \, dx + 2\epsilon \int_{B_{2\lambda}(p) \setminus B_{\lambda/2}(p)} (F_{(1-\beta_{\lambda,p})A_p}, gF_{I_{\lambda,p}^2}g^{-1}) \, dx$$

$$+ 2\epsilon \int_{B_{2\lambda}(p) \setminus B_{\lambda/2}(p)} (gF_{I_{\lambda,p}^2}g^{-1}, [1 - \beta_{\lambda,p}A_p, gPI_{\lambda,p}^2g^{-1}]) \, dx + O(\epsilon^3).$$ (3.19)

With the same calculation, one obtains the analogous formula for trace. Adding up the two and using self duality (as done earlier to obtain (3.17)), finally yields

$$\epsilon^2 \int_{B_{2\lambda}(p) \setminus B_{\lambda/2}(p)} |F_{A(q)}\epsilon|^2 \, dx + e^{2} \int_{B_{2\lambda}(p) \setminus B_{\lambda/2}(p)} \text{Tr} (F_{A(q)}\epsilon \wedge F_{A(q)}\epsilon) = 2 \int_{B_{2\lambda}(p) \setminus B_{\lambda/2}(p)} |d_{I_{\lambda,p}^2} h_{\lambda,p}|^2 \, dx + O(\epsilon^3).$$ (3.20)

(3) **Estimate of** $\int_{B_{\lambda/2}(p) \setminus B_{\lambda/4}(p)} |F_{A(q)}\epsilon|^2 \, dx$.

On $B_{\lambda/2}(p) \setminus B_{\lambda/4}(p)$, $A(q) = \frac{1}{2} gPI_{\lambda,p}^2g^{-1} + \frac{1}{2} \beta_{\lambda/4,p}g h_{\lambda,p}g^{-1}$ and

$$\epsilon^2 \int_{B_{\lambda/2}(p) \setminus B_{\lambda/4}(p)} |F_{A(q)}\epsilon|^2 \, dx = \epsilon^2 \int_{B_{\lambda/2}(p) \setminus B_{\lambda/4}(p)} |PF_{I_{\lambda,p}^2}|^2 \, dx$$

$$+ \int_{B_{\lambda/2}(p) \setminus B_{\lambda/4}(p)} |d_{PF_{I_{\lambda,p}^2}} (\beta_{\lambda/4,p} h_{\lambda,p})|^2 \, dx + \frac{1}{4} \int_{B_{\lambda/2}(p) \setminus B_{\lambda/4}(p)} |[\beta_{\lambda/4,p} h_{\lambda,p}, \beta_{\lambda/4,p} h_{\lambda,p}]|^2 \, dx$$

$$+ 2 \int_{B_{\lambda/2}(p) \setminus B_{\lambda/4}(p)} (PF_{I_{\lambda,p}^2}, d_{I_{\lambda,p}^2} \beta_{\lambda/4,p} h_{\lambda,p}) \, dx + \int_{B_{\lambda/2}(p) \setminus B_{\lambda/4}(p)} (PF_{I_{\lambda,p}^2}, [\beta_{\lambda/4,p} h_{\lambda,p}, \beta_{\lambda/4,p} h_{\lambda,p}]) \, dx$$

$$+ \int_{B_{\lambda/2}(p) \setminus B_{\lambda/4}(p)} (d_{I_{\lambda,p}^2} (\beta_{\lambda/4,p} h_{\lambda,p}), [\beta_{\lambda/4,p} h_{\lambda,p}, \beta_{\lambda/4,p} h_{\lambda,p}]) \, dx := F_1 + \cdots + F_6.$$ (3.21)
• Estimates of $F_1$–$F_6$:

The term $F_1$ has already been estimated in (3.14). Moreover,

\[ |F_2| \lesssim \int_{B_{\lambda/2}(p) \setminus B_{\lambda/4}(p)} (|d\lambda, p|^2 + |I_{\lambda,p}^2|^2 + |dh_{\lambda,p}|^2 |h_{\lambda,p}|^4) \, dx \lesssim \lambda^8 + \lambda^6 + \lambda^8 \lesssim \epsilon^3; \]

\[ |F_3| \lesssim \int_{B_{\lambda/2}(p) \setminus B_{\lambda/4}(p)} |h_{\lambda,p}|^4 \, dx \lesssim \lambda^{12} \sim \epsilon^6; \]

\[ F_4 = \int_{B_{\lambda/2}(p) \setminus B_{\lambda/4}(p)} (F_{I_{\lambda,p}^2}, d_{P_\lambda}^2 (\beta_{\lambda/4,p} h_{\lambda,p})) \, dx + O(\epsilon^3), \]

since

\[ \left| \int_{B_{\lambda/2}(p) \setminus B_{\lambda/4}(p)} (-2d_{I_{\lambda,p}^2} h_{\lambda,p} + \frac{1}{2} [h_{\lambda,p}, h_{\lambda,p}], d_{P_\lambda}^2 (\beta_{\lambda/4,p} h_{\lambda,p}) \right| \, dx \]

\[ \lesssim \int_{B_{\lambda/2}(p) \setminus B_{\lambda/4}(p)} (|d\lambda, p|^2 + |dh_{\lambda,p}||I_{\lambda,p}^2| + |h_{\lambda,p}| + |dh_{\lambda,p}| |h_{\lambda,p}|^2 + |I_{\lambda,p}^2| |h_{\lambda,p}|^2 + |I_{\lambda,p}^2| |h_{\lambda,p}|^4) \, dx \lesssim \epsilon^3; \]

\[ F_5 = \int_{B_{\lambda/2}(p) \setminus B_{\lambda/4}(p)} (F_{I_{\lambda,p}^2}, [\beta_{\lambda/4,p} h_{\lambda,p}, \beta_{\lambda/4,p} h_{\lambda,p}]) \, dx + O(\epsilon^2); \]

since

\[ \left| \int_{B_{\lambda/2}(p) \setminus B_{\lambda/4}(p)} (-d_{I_{\lambda,p}^2} h_{\lambda,p} + \frac{1}{4} [h_{\lambda,p}, h_{\lambda,p}], [\beta_{\lambda/4,p} h_{\lambda,p}, \beta_{\lambda/4,p} h_{\lambda,p}] \right| \]

\[ \lesssim \int_{B_{\lambda/2}(p) \setminus B_{\lambda/4}(p)} (|dI_{\lambda,p}^2| |h_{\lambda,p}|^2 + |I_{\lambda,p}^2| |h_{\lambda,p}|^3 + |h_{\lambda,p}|^4) \, dx \lesssim \epsilon^2; \]

\[ |F_6| \lesssim \int_{B_{\lambda/2}(p) \setminus B_{\lambda/4}(p)} (|dh_{\lambda,p}| |h_{\lambda,p}|^2 + |I_{\lambda,p}^2| |h_{\lambda,p}|^3 + |h_{\lambda,p}|^4) \, dx \lesssim \lambda^{10} + \lambda^9 + \lambda^{12} \lesssim \epsilon^2. \]

Adding up $F_1 + \cdots + F_6$ yields

\[ \epsilon^2 \int_{B_{\lambda/2}(p) \setminus B_{\lambda/4}(p)} |F_1 \leq p|^2 \, dx = \int_{B_{\lambda/2}(p) \setminus B_{\lambda/4}(p)} |F_{I_{\lambda,p}^2}|^2 \, dx + \int_{B_{\lambda/2}(p) \setminus B_{\lambda/4}(p)} |d_{I_{\lambda,p}^2} h_{\lambda,p}|^2 \, dx \]

\[ + 2 \int_{B_{\lambda/2}(p) \setminus B_{\lambda/4}(p)} (F_{I_{\lambda,p}^2}, -d_{I_{\lambda,p}^2} h_{\lambda,p} + F_{\beta_{\lambda/4,p} h_{\lambda,p}} + |I_{\lambda,p}^2, \beta_{\lambda/4,p} h_{\lambda,p}|) \, dx + O(\epsilon^3). \quad (3.22) \]

The analogous formula for trace added to the above gives
\[ \epsilon^2 \int_{B_{\lambda/4}(p) \setminus B_{\lambda/4}(p)} |F_{A(q)}^\epsilon|^2 \, dx + \epsilon^2 \int_{B_{\lambda/2}(p) \setminus B_{\lambda/4}(p)} \text{Tr} \left( F_{A(q)}^\epsilon \wedge F_{A(q)}^\epsilon \right) = 2 \int_{B_{\lambda/2}(p) \setminus B_{\lambda/4}(p)} |d_{I_{\lambda,p}^2} h_{\lambda,p}^-|^2 \, dx + O(\epsilon^3). \] (3.23)

(4) **Estimate of** \( \int_{B_{\lambda/4}(p)} |F_{A(q)}^\epsilon|^2 \, dx \).

On \( B_{\lambda/4}(p) \), we have \( A(q) = \frac{1}{\epsilon} g I_{\lambda,p}^1 g^{-1} \) and

\[ \epsilon^2 \int_{B_{\lambda/4}(p)} |F_{A(q)}^\epsilon|^2 \, dx = \int_{B_{\lambda/4}(p)} |F_{I_{\lambda,p}^1}^\epsilon|^2 \, dx. \] (3.24)

Thus, combining the above with the likewise formula for trace,

\[ \epsilon^2 \int_{B_{\lambda/4}(p)} |F_{A(q)}^\epsilon|^2 \, dx + \epsilon^2 \int_{B_{\lambda/4}(p)} \text{Tr} \left( F_{A(q)}^\epsilon \wedge F_{A(q)}^\epsilon \right) = 0. \] (3.25)

We are now ready to sum the contributions from the four sub-domains, i.e. (3.17), (3.20), (3.23), (3.25), and obtain

\[
\epsilon^2 \int_B |F_{A(q)}^\epsilon|^2 \, dx = -\epsilon^2 \int_B \text{Tr}(F_{A(q)}^\epsilon \wedge F_{A(q)}^\epsilon) + \epsilon^2 \int_B \text{Tr}(F_{\Delta}^\epsilon \wedge F_{\Delta}^\epsilon) + \epsilon^2 \int_B |F_{\Delta}^\epsilon|^2 \, dx \\
+ 2 \int_B |d_{I_{\lambda,p}^2} h_{\lambda,p}^-|^2 \, dx - 4\epsilon \int_B ((F_{\Delta}^\epsilon)^-, gd_{I_{\lambda,p}^2} h_{\lambda,p} g^{-1}) \, dx + O(\epsilon^3) \\
= 8\pi^2 + \epsilon^2 \int_B |F_{\Delta}^\epsilon|^2 \, dx + 2\lambda^4 \int_B |d_{I_{\lambda,p}^2} h_{\lambda,p}^-|^2 \, dx - 4\epsilon \int_B ((F_{\Delta}^\epsilon)^-, gd_{I_{\lambda,p}^2} h_{\lambda,p} g^{-1}) \, dx + O(\epsilon^3) \\
= 8\pi^2 + \epsilon^2 \int_B |F_{\Delta}^\epsilon|^2 \, dx + 2\lambda^4 \int_B |d_{I_{\lambda,p}^2} h_{\lambda,p}^-|^2 \, dx - 4\epsilon \int_B ((F_{\Delta}^0)^-, gd_{I_{\lambda,p}^2} h_{\lambda,p} g^{-1}) \, dx + O(\epsilon^3),
\] (3.26)

where for the first equality we have used the estimates

\[
\left| \epsilon^2 \int_{B_{2\lambda}} \text{Tr}(F_{\Delta}^\epsilon \wedge F_{\Delta}^\epsilon) \right| \lesssim \epsilon^2 \int_{B_{2\lambda}} |F_{\Delta}^\epsilon|^2 \, dx \lesssim \epsilon^2 \lambda^4 \lesssim \epsilon^4, \\
\left| \epsilon \int_{B_{2\lambda}(p)} ((F_{\Delta}^\epsilon)^-, gd_{I_{\lambda,p}^2} h_{\lambda,p} g^{-1}) \, dx \right| \lesssim \epsilon \int_{B_{2\lambda}(p)} (|dh_{\lambda,p}| + |I_{\lambda,p}^2||h_{\lambda,p}|) \, dx \lesssim \epsilon \lambda^6 \lesssim \epsilon^4, \\
\int_{B_{\lambda/4}(p)} |d_{I_{\lambda,p}^2} h_{\lambda,p}^-|^2 \, dx \lesssim \lambda^6 \lesssim \epsilon^3,
\]

and for the second equality we have used the topological constraint

\[ \epsilon^2 \int_{B^4} \text{Tr}(F_{A(q)}^\epsilon \wedge F_{A(q)}^\epsilon) - \epsilon^2 \int_{B^4} \text{Tr}(F_{\Delta}^\epsilon \wedge F_{\Delta}^\epsilon) = 8\pi^2, \] (3.27)

and the estimate

\[ \left| \epsilon \int_{B_{2\lambda}(p)} ((F_{\Delta}^\epsilon)^- - (F_{\Delta}^0)^-, gd_{I_{\lambda,p}^2} h_{\lambda,p} g^{-1}) \, dx \right| \lesssim \epsilon^2 \lambda^6 \lesssim \epsilon^5. \]

Thus, (3.8) holds.

The asymptotic expansion of the derivative of \( J_\epsilon(q) \) is computed similarly and we omit the calculation. \( \square \)
3.3 Estimate of $\|\nabla \mathcal{M}_\varepsilon(A(q))\|_{A(q);1,2,*}$

For connections $A$ on the bundle $P := P(q)$ and one-forms $a \in C^\infty(T^*T^1 \otimes \text{Ad}(P))$, we define the $L^p_k$-norm $\|a\|_{A,k,p}$ by

$$\|a\|_{A,k,p} := \sum_{j=1}^k \|\nabla A^j\|^2 \cdot \|a\|_p,$$

where $\nabla A^j = \nabla + \varepsilon[A, \cdot]$, $(\nabla A^j)^j = \nabla A^j \cdots \nabla A^1$ (j-times) and $\|\cdot\|_p$ is the $L^p$-norm on $B^4$. We denote by $L^p_k(T^*B^4 \otimes \text{Ad}(P))$ the completion of $C^\infty_0(T^*B^4 \otimes \text{Ad}(P))$ with respect to the norm above, and define the spaces

$$L^p_{0,k}(T^*B^4 \otimes \text{Ad}(P)) := L^p_k(T^*B^4 \otimes \text{Ad}(P)) \cap L^p_{0,1}(T^*B^4 \otimes \text{Ad}(P)),$$

$$L^p_{0^*,k}(T^*B^4 \otimes \text{Ad}(P)) = \{\alpha \in L^p_k(T^*B^4 \otimes \text{Ad}(P)) : \iota^*\alpha = 0\}.$$

Note that these are independent of the choice of the connection $A$.

Let now the spaces $A^*(A_0)$, $\mathcal{B}^*(A_0)$ and their Sobolev correspondents be defined as in §3.1. The results in [14] allow us to identify the tangent bundle $T\mathcal{B}^*_{k+1}(A_0) \to \mathcal{B}^*_{k+1}(A_0)$ with a sub-bundle of $A^*_{k+1}(A_0) \times \mathcal{G}^*_k \times L^p_{0^*,k}(T^*B^4 \otimes \text{Ad}(P)) \to \mathcal{B}^*_{k+1}(A_0)$, defined as

$$\mathcal{S}^p_{k+1}(A_0) := \mathcal{S}^p_{k+1}(A_0)/\mathcal{G}^*_k \to \mathcal{B}^*_{k+1}(A_0),$$

where

$$\mathcal{S}^p_{k+1}(A_0) = \{(A, \alpha) \in A^*_{k+1}(A_0) \times L^p_{0^*,k}(T^*B^4 \otimes \text{Ad}(P)) : d_A^\varepsilon\alpha = 0\}$$

and $\mathcal{G}^*_k$ acts diagonally on $\mathcal{S}^p_{k+1}(A_0)$.

The correspondence $A^*_{k+1}(A_0) \times L^p_{0^*,k}(T^*B^4 \otimes \text{Ad}(P)) \ni (A, a) \mapsto \|a\|_{A,p,k}$ is $\mathcal{G}^*_k$-invariant, therefore it descends to the quotient.

We are interested in the case $k = 1$ and $p > 2$, thus we define the gradient of the functional $\mathcal{M}_\varepsilon$ on $\mathcal{B}^*_{1,+}(A_0)$ as

$$\nabla \mathcal{M}_\varepsilon(A)(a) = 2 \int_{B^4} (F_A^\varepsilon, d_A^\varepsilon a),$$

for $(A, a) \in T\mathcal{B}^*_{1,+}(A_0) = \mathcal{S}^p_{1,+}(A_0)$.

Since (3.29) is $\mathcal{G}^*_2$-invariant, that is, $\nabla \mathcal{M}_\varepsilon(g \cdot A)(g \cdot a) = \nabla \mathcal{M}_\varepsilon(A)(a)$ for $g \in \mathcal{G}^*_2$, the functional $\nabla \mathcal{M}_\varepsilon$ descends to $\mathcal{S}_{1,+}(A_0)$.

Observe that a connection $A$ is a solution to $(\mathcal{D}_\varepsilon)$ if and only if $\nabla \mathcal{M}_\varepsilon(A) = 0$ on $\mathcal{S}^p_{1,+}(A_0)$.

We define the dual $L^2_1$-norm of $\nabla \mathcal{M}_\varepsilon$ as

$$\|\nabla \mathcal{M}_\varepsilon(A)\|_{A;1,2,*} := \sup\{\nabla \mathcal{M}_\varepsilon(A)(a) : a \in L^2_{0,1}(T^*B^4 \otimes \text{Ad}(P)), \|a\|_{A;1,2} \leq 1\},$$

or equivalently, by the $\mathcal{G}^*_2$-invariance, as

$$\|\nabla \mathcal{M}_\varepsilon(A)\|_{A;1,2,*} = \sup\{\nabla \mathcal{M}_\varepsilon(A)(a) : a \in L^2_{0,1}(T^*B^4 \otimes \text{Ad}(P))$$

such that $d_A^\varepsilon a = 0$ and $\|a\|_{A;1,2} \leq 1\}.$
Lemma 3.2 For $q \in \mathcal{P}(d_0, \lambda_0; D_1, D_2; \epsilon)$, there exists a constant $C > 0$ depending only on $d_0, \lambda_0, D_1$ and $D_2$ such that the following holds for all small $\epsilon > 0$:

$$
\|\nabla \mathcal{M}_\epsilon(A(q))\|_{A(q);1,2,*} \leq C\epsilon^{1/2}.
$$

Proof: Using (3.30), we estimate $\nabla \mathcal{M}_\epsilon(A(q))(\alpha)$ for $\alpha \in L^2_{0,1}(T^*B^4 \otimes \text{Ad}(P))$ with $d^*_{A(q)}\epsilon \alpha = 0$ and $\|\alpha\|_{A(q);1,2} \leq 1$, with $A(q)$ represented in the same gauge as in (3.33). We have

$$
\frac{1}{2} \nabla \mathcal{M}_\epsilon(A(q))(\alpha) = \int_{B^4} (F_{A(q)}\epsilon, d_{A(q)}\epsilon \alpha) \, dx = \int_{B^4} (d^*_{A(q)}\epsilon F_{A(q)}\epsilon, \alpha) \, dx
$$

$$
= \int_{B^4 \setminus B_{2\lambda}(p)} (d^*_{A(q)}\epsilon F_{A(q)}\epsilon, \alpha) \, dx + \int_{B_{2\lambda}(p) \setminus B_{\lambda/2}(p)} (d^*_{A(q)}\epsilon F_{A(q)}\epsilon, \alpha) \, dx
$$

$$
+ \int_{B_{\lambda/2}(p) \setminus B_{\lambda/4}(p)} (d^*_{A(q)}\epsilon F_{A(q)}\epsilon, \alpha) \, dx + \int_{B_{\lambda/4}(p)} (d^*_{A(q)}\epsilon F_{A(q)}\epsilon, \alpha) \, dx. \quad (3.31)
$$

The last term in (3.31) vanishes since $A(q)$ is Yang Mills on $B_{\lambda/4}(p)$. We now proceed estimating the remaining three terms.

**Estimate on $B^4 \setminus B_{2\lambda}(p)$:** On $B^4 \setminus B_{2\lambda}(p)$, we write $A(q) = \mathcal{A} + 1/2 gP_{\lambda,p}^1 g^{-1} := A_1 + A_2$ and

$$
d^*_{A(q)}\epsilon F_{A(q)}\epsilon = d^*_{A_1} \epsilon F_{A_1}\epsilon + d^*_{A_2} \epsilon F_{A_2}\epsilon + \epsilon \ast d \ast [A_1, A_2] + \epsilon \ast [A_1, *F_{A_2}\epsilon] + \epsilon \ast [A_2, *F_{A_1}\epsilon]
$$

$$
+ \epsilon^2 \ast [A_1, *[A_1, A_2]] + \epsilon^2 \ast [A_2, *[A_1, A_2]]. \quad (3.32)
$$

Since $A_1 = \mathcal{A}$ is Yang Mills, we have $d^*_{A_1} \epsilon F_{A_1}\epsilon = 0$. Also, since $F_{A_2}\epsilon = 1/2 g(F_{I_{\lambda,p}^2} - d_{I_{\lambda,p}^2} h_{\lambda,p} + 1/2[h_{\lambda,p}, h_{\lambda,p}])g^{-1}$ and $I_{\lambda,p}^2$ is Yang Mills, we have

$$
d^*_{A_2} \epsilon F_{A_2}\epsilon = \epsilon g d^*_{I_{\lambda,p}^2} \epsilon d_{I_{\lambda,p}^2} h_{\lambda,p} g^{-1} - 1/2 \epsilon g d^*_{I_{\lambda,p}^2} \epsilon [h_{\lambda,p}, h_{\lambda,p}] g^{-1} - \epsilon g [h_{\lambda,p}, F_{I_{\lambda,p}^2}] g^{-1}
$$

$$
+ 1/2 \epsilon g [h_{\lambda,p}, *d_{I_{\lambda,p}^2} h_{\lambda,p}] g^{-1} - 1/2 \epsilon g [h_{\lambda,p}, *[h_{\lambda,p}, h_{\lambda,p}]] g^{-1}. \quad (3.33)
$$

For the first term in (3.33) we compute explicitly

$$
d^*_{I_{\lambda,p}^2} \epsilon d_{I_{\lambda,p}^2} h_{\lambda,p} = d^* d h_{\lambda,p} + d^* [I_{\lambda,p}^2, h_{\lambda,p}] + *[I_{\lambda,p}^2, *d h_{\lambda,p}] + *[I_{\lambda,p}^2, *d h_{\lambda,p}]
$$

$$
= -dd^* h_{\lambda,p} + *[I_{\lambda,p}^2, *d h_{\lambda,p}] + d^* [I_{\lambda,p}^2, h_{\lambda,p}] + *[I_{\lambda,p}^2, *d h_{\lambda,p}], \quad (3.34)
$$

where we have used the harmonicity of $h_{\lambda,p}$ (i.e., $d^* d h_{\lambda,p} + dd^* h_{\lambda,p} = 0$). Since $d^*_{A(q)}\epsilon \alpha = 0$, we have $d^* \alpha = -\epsilon \ast [A(q), \epsilon \alpha]$ and

$$
\left| \int_{B^4 \setminus B_{2\lambda}(p)} (-dd^* h_{\lambda,p}, \alpha) \, dx \right| \leq \left| \int_{B^4} (d^* h_{\lambda,p}, d^* \alpha) \, dx \right| + \left| \int_{\partial B_{2\lambda}(p)} (d^* h_{\lambda,p}, \alpha) \, dx \right|
$$

$$
\leq \int_{B^4 \setminus B_{2\lambda}(p)} \epsilon |\nabla h_{\lambda,p}| |A(q)| |\alpha| \, dx + \int_{\partial B_{2\lambda}(p)} |\alpha| \, dx
$$

$$
\leq \lambda^2 \int_{B^4 \setminus B_{\lambda}(p)} (\epsilon + |I_{\lambda,p}^2|) |\alpha| \, dx + \lambda^4 \left( \int_{\partial B_{2\lambda}(p)} |\alpha|^3 \, dx \right)^{1/3}
$$

$$
\leq \lambda^2 \epsilon |\alpha|_{A(q);1,2} + \lambda^2 \left( \int_{B^4 \setminus B_{2\lambda}(p)} |I_{\lambda,p}^2|^{4/3} \, dx \right)^{3/4} \left( \int_{B^4 \setminus B_{2\lambda}(p)} |\alpha|^4 \, dx \right)^{1/4}
$$

$$
+ \lambda^4 |\alpha|_{A(q);1,2} \leq \epsilon ^2 |\log \epsilon|^{3/4} |\alpha|_{A(q);1,2}, \quad (3.35)
$$

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where we have integrated by parts in the first line, used Lemma 3.4 in the second line, Hölder’s inequality in the third line, and Sobolev and trace inequalities in the last line ($L^2 \subset L^4$ in dimension 4, $\|\alpha\|_{L^2(\partial B_{2\lambda}(p))} \leq C\|\alpha\|_{L^4(\partial B_{2\lambda}(p))} \leq C\|\alpha\|_{L^4(B_{2\lambda}(p))} \leq C\|\alpha\|_{A(\alpha):1,2}$).

The remaining terms in (3.34) are also estimated using Hölder and Sobolev inequalities and Lemma 3.1 as follows:

$$\left| \int_{B^4 \setminus B_{2\lambda}(p)} (\pi[\mathcal{J}^p_{\lambda,p}, \pi h_{\lambda,p}], \alpha) \, dx \right| \lesssim \lambda^2 \left( \int_{B^4 \setminus B_{2\lambda}(p)} |\nabla \mathcal{I}^p_{\lambda,p}|^{4/3} \, dx \right)^{3/4} \left( \int_{B^4 \setminus B_{2\lambda}(p)} |\alpha|^4 \, dx \right)^{1/4} \lesssim \epsilon^{3/2} \|\alpha\|_{A(\alpha):1,2}. \tag{3.36}$$

$$\left| \int_{B^4 \setminus B_{2\lambda}(p)} (\pi[\mathcal{J}^p_{\lambda,p}, \pi h_{\lambda,p}], \alpha) \, dx \right| \lesssim \lambda^2 \left( \int_{B^4 \setminus B_{2\lambda}(p)} |\nabla \mathcal{I}^p_{\lambda,p}|^{4/3} \, dx \right)^{3/4} + \left( \int_{B^4 \setminus B_{2\lambda}(p)} |\mathcal{I}^p_{\lambda,p}|^{4/3} \, dx \right)^{3/4} \left( \int_{B^4 \setminus B_{2\lambda}(p)} |\alpha|^4 \, dx \right)^{1/4} \lesssim \lambda^2 \|\alpha\|_{A(\alpha):1,2} + \lambda^4 |\log \lambda|^{3/4} \|\alpha\|_{A(\alpha):1,2} \lesssim \epsilon^{3/2} \|\alpha\|_{A(\alpha):1,2}. \tag{3.37}$$

$$\left| \int_{B^4 \setminus B_{2\lambda}(p)} (\pi[\mathcal{J}^p_{\lambda,p}, \pi h_{\lambda,p}], \alpha) \, dx \right| \lesssim \lambda^2 \left( \int_{B^4 \setminus B_{2\lambda}(p)} |\mathcal{I}^p_{\lambda,p}|^{4/3} \, dx \right)^{3/4} \left( \int_{B^4 \setminus B_{2\lambda}(p)} |\alpha|^4 \, dx \right)^{1/4} \lesssim \lambda^4 |\log \lambda| \|\alpha\|_{A(\alpha):1,2} \lesssim \epsilon^{1/2} \|\alpha\|_{A(\alpha):1,2}. \tag{3.38}$$

The remaining terms in (3.33) are estimated by Lemma 3.1 as

$$\left| \int_{B^4 \setminus B_{2\lambda}(p)} (\pi \mathcal{J}^p_{\lambda,p}, \pi h_{\lambda,p}) g^{-1} \, dx \right| \lesssim \lambda^4 \int_{B^4 \setminus B_{2\lambda}(p)} |\alpha| \, dx + \lambda^4 \int_{B^4 \setminus B_{2\lambda}(p)} |\mathcal{I}^p_{\lambda,p}| |\alpha| \, dx \lesssim \lambda^4 \|\alpha\|_{A(\alpha):1,2} + \lambda^4 \left( \int_{B^4 \setminus B_{2\lambda}(p)} |\mathcal{I}^p_{\lambda,p}|^{4/3} \, dx \right)^{3/4} \left( \int_{B^4 \setminus B_{2\lambda}(p)} |\alpha|^4 \, dx \right)^{1/4} \lesssim \lambda^4 \|\alpha\|_{A(\alpha):1,2} + \lambda^6 |\log \lambda|^{3/4} \|\alpha\|_{A(\alpha):1,2} \lesssim \epsilon^2 \|\alpha\|_{A(\alpha):1,2}. \tag{3.39}$$

$$\left| \int_{B^4 \setminus B_{2\lambda}(p)} (\pi \mathcal{J}^p_{\lambda,p}, \pi F_{\alpha}) g^{-1} \, dx \right| \lesssim \lambda^2 \int_{B^4 \setminus B_{2\lambda}(p)} |\mathcal{I}^p_{\lambda,p}| |\alpha| \, dx \lesssim \lambda^2 \left( \int_{B^4 \setminus B_{2\lambda}(p)} |\mathcal{I}^p_{\lambda,p}|^{4/3} \, dx \right)^{3/4} \left( \int_{B^4 \setminus B_{2\lambda}(p)} |\alpha|^4 \, dx \right)^{1/4} \lesssim \lambda^3 \|\alpha\|_{A(\alpha):1,2} \lesssim \epsilon^{3/2} \|\alpha\|_{A(\alpha):1,2}. \tag{3.40}$$
From (3.35) – (3.42), we obtain
\[
\left| \int_{B^i \setminus B^{2 \lambda}(p)} (g * [h_{\lambda,p}^*, [h_{\lambda,p}^*, h_{\lambda,p}^*]], \alpha) \, dx \right| \lesssim \lambda^6 \| \alpha \|_{A(1,2)} \lesssim \epsilon^3 \| \alpha \|_{A(1,2)}. \tag{3.42}
\]

The remaining terms of (3.32) are estimated as follows:
\[
\left| \int_{B^i \setminus B^{2 \lambda}(p)} (d^*_{A_2^*} F_{A_2^*}, \alpha) \, dx \right| \lesssim \epsilon^{1/2} \| \alpha \|_{A(1,2)}. \tag{3.43}
\]

From (3.35) – (3.42), we obtain
\[
\left| \int_{B^i \setminus B^{2 \lambda}(p)} (d^*_{A_2^*} F_{A_2^*}, \alpha) \, dx \right| \lesssim \epsilon^{1/2} \| \alpha \|_{A(1,2)}.
\]
Combining (3.43)–(3.48), we obtain
\[ \left| \int_{B_1 \setminus B_2} (d_{A(q)}^* F_{A(q)}^\epsilon, \alpha) \right| \lesssim \epsilon^{1/2} \| \alpha \|_{A(q):1,2}. \] (3.49)

**Estimate on** $B_2 \setminus B_1$: On $B_2 \setminus B_1$, we write $A(q) = (1 - \beta_{\lambda,p})A_2 + \frac{1}{\epsilon} gP_{\lambda,p} g^{-1} =: A_1 + A_2$, and make use again of the expansion (3.32).

Since $F_{A_1} = -d\beta_{\lambda,p} \wedge A + (1 - \beta_{\lambda,p})dA_2 + \frac{1}{2} (1 - \beta_{\lambda,p}^2) [A, A_2]$, we easily see that $|d_{A_1}^* F_{A_1}^\epsilon| \lesssim \lambda^{-2}$, and
\[ \left| \int_{B_2 \setminus B_1} (d_{A_1}^* F_{A_1}^\epsilon, \alpha) \right| \lesssim \lambda^{-2} \int_{B_2 \setminus B_1} |\alpha| \, dx \lesssim \lambda \left( \int_{B_2 \setminus B_1} |\alpha|^4 \, dx \right)^{1/4} \lesssim \epsilon^{1/2} \| \alpha \|_{A(q):1,2} . \]

The term $d_{A_2}^* F_{A_2}^\epsilon$ is given by the formula (3.33), thus we may proceed as we did earlier and obtain
\[ \left| \int_{B_2 \setminus B_1} (d_{A_2}^* F_{A_2}^\epsilon, \alpha) \right| \lesssim \epsilon^{1/2} \| \alpha \|_{A(q):1,2} . \]

The estimates of all the remaining terms in (3.32) are quite similar and they yield
\[ \left| \int_{B_2 \setminus B_1} (d_{A(q)}^* F_{A(q)}^\epsilon, \alpha) \right| \lesssim \epsilon^{1/2} \| \alpha \|_{A(q):1,2}. \] (3.50)

**Estimate on** $B_1 \setminus A_2$: On $B_1 \setminus A_2$, we write $A(q) = \frac{1}{\epsilon} \frac{d}{d\beta_{\lambda,p}} + \frac{1}{\epsilon} (\beta_{\lambda,p} - 1) g h_{\lambda,p} g^{-1} =: A_2 + A_3$, and use the expansion
\[
\begin{align*}
\left| d_{A_2}^* F_{A(q)}^\epsilon \right| &= d_{A_2}^* F_{A_2}^\epsilon + \left| d_{A_3}^* F_{A_3}^\epsilon \right| + \epsilon \ast d \ast [A_2, A_3] + \epsilon \ast [A_2, \ast F_{A_3}^\epsilon] + \epsilon \ast [A_3, \ast F_{A_2}^\epsilon] \\
&\quad + \epsilon^2 \ast [A_2, \ast [A_2, A_3]] + \epsilon^2 \ast [A_3, \ast [A_2, A_3]],
\end{align*}
\] (3.51)

where the first term above vanishes, since $d_{A_2}^* F_{A_2}^\epsilon = 0$.

For the second term, we have $F_{A_3}^\epsilon = \frac{1}{\epsilon} d\beta_{\lambda,p} \wedge g h_{\lambda,p} g^{-1} + \frac{1}{\epsilon} (\beta_{\lambda,p} - 1) g h_{\lambda,p} g^{-1} + \frac{1}{2\epsilon} (\beta_{\lambda,p}^2 - 1) g [h_{\lambda,p}, h_{\lambda,p}] g^{-1}$, and we can easily see from Lemma 3.1 that $|d_{A_3}^* F_{A_3}^\epsilon| \lesssim 1$. Thus we have
\[ \left| \int_{B_1 \setminus A_2} (d_{A_2}^* F_{A_2}^\epsilon, \alpha) \right| \lesssim \epsilon^{-1} \int_{B_1 \setminus A_2} |\alpha| \, dx \lesssim \epsilon^{-1} \lambda^3 \left( \int_{B_1 \setminus A_2} |\alpha|^4 \, dx \right)^{1/4} \lesssim \epsilon^{1/2} \| \alpha \|_{A(q):1,2}. \] (3.52)

The remaining terms of (3.51) are estimated as follows:
\[ \begin{align*}
\left| \int_{B_1 \setminus A_2} (\epsilon \ast d \ast [A_2, A_3], \alpha) \right| &\lesssim \epsilon^{-1} \int_{B_1 \setminus A_2} (|\nabla I_{\lambda,p}^2 \epsilon^2 + |I_{\lambda,p}^2 \epsilon^2|) \, dx \\
&\lesssim \epsilon^{-1} \lambda^2 \left( \int_{B_1 \setminus A_2} |\nabla I_{\lambda,p}^2 \epsilon^4 |^{3/4} \, dx \right)^{3/4} \| \alpha \|_{A(q):1,2} + \epsilon^{-1} \lambda \left( \int_{B_1 \setminus A_2} |I_{\lambda,p}^2 \epsilon^4 |^{3/4} \, dx \right)^{3/4} \| \alpha \|_{A(q):1,2} \\
&\lesssim \epsilon^{-1} \lambda^3 \| \alpha \|_{A(q):1,2} + \epsilon^{-1} \lambda^3 \| \alpha \|_{A(q):1,2} \lesssim \epsilon^{1/2} \| \alpha \|_{A(q):1,2} ;
\end{align*}
\] (3.53)
Combining (3.52)–(3.57), we obtain
\[
\left| \int_{B_{\lambda/2}(p) \setminus B_{\lambda/4}(p)} (\epsilon * [A_2, *F_{A_3}]^\epsilon, \alpha) \, dx \right| \lesssim \epsilon^{-1} \lambda \left( \int_{B_{\lambda/2}(p) \setminus B_{\lambda/4}(p)} |I_{\lambda,p}^2|^4/3 \, dx \right)^{3/4} \|\alpha\|_{A(q);1,2} \lesssim \epsilon^{1/2} \|\alpha\|_{A(q);1,2}; \tag{3.54}
\]
\[
\left| \int_{B_{\lambda/2}(p) \setminus B_{\lambda/4}(p)} (\epsilon * [A_3, *F_{A_2}]^\epsilon, \alpha) \, dx \right| \lesssim \epsilon^{-1} \lambda^2 \left( \int_{B_{\lambda/2}(p) \setminus B_{\lambda/4}(p)} |F_{I_{\lambda,p}}|^4/3 \, dx \right)^{3/4} \|\alpha\|_{A(q);1,2} \lesssim \epsilon^{1/2} \|\alpha\|_{A(q);1,2}; \tag{3.55}
\]
\[
\left| \int_{B_{\lambda/2}(p) \setminus B_{\lambda/4}(p)} (\epsilon^2 * [A_2, [A_2, A_3]], \alpha) \, dx \right| \lesssim \epsilon^{-1} \lambda^2 \left( \int_{B_{\lambda/2}(p) \setminus B_{\lambda/4}(p)} |F_{I_{\lambda,p}}|^8/3 \, dx \right)^{3/4} \|\alpha\|_{A(q);1,2} \lesssim \epsilon^{1/2} \|\alpha\|_{A(q);1,2}; \tag{3.56}
\]
\[
\left| \int_{B_{\lambda/2}(p) \setminus B_{\lambda/4}(p)} (\epsilon^2 * [A_3, [A_2, A_3]], \alpha) \, dx \right| \lesssim \epsilon^{-1} \lambda^4 \left( \int_{B_{\lambda/2}(p) \setminus B_{\lambda/4}(p)} |I_{\lambda,p}^2|^4/3 \, dx \right)^{3/4} \|\alpha\|_{A(q);1,2} \lesssim \epsilon^{2} \|\alpha\|_{A(q);1,2}. \tag{3.57}
\]
Combining (3.52)–(3.57), we obtain
\[
\left| \int_{B_{\lambda/2}(p) \setminus B_{\lambda/4}(p)} (d_{A(q)}^\epsilon F_{A(q)}^\epsilon, \alpha) \, dx \right| \lesssim \epsilon^{1/2} \|\alpha\|_{A(q);1,2}. \tag{3.58}
\]

**Estimate on** $B_{\lambda/4}(p)$:**

On $B_{\lambda/4}(p)$, $A(q) = \frac{1}{\epsilon} g I_{\lambda,p}^{-1}$ and $d_{A(q)}^\epsilon F_{A(q)}^\epsilon = 0$. We thus have
\[
\left| \int_{B_{\lambda/4}(p)} (d_{A(q)}^\epsilon F_{A(q)}^\epsilon, \alpha) \, dx \right| = 0. \tag{3.59}
\]

From (3.51), (3.50), (3.58), (3.59), we finally obtain
\[
\left| \int_{B^4} (d_{A(q)}^\epsilon F_{A(q)}^\epsilon, \alpha) \right| \lesssim \epsilon^{1/2} \|\alpha\|_{A(q);1,2}.
\]

This completes the proof. \[\square\]
3.4 Estimate of the remainder $R(q; a)$

Let $A \in A(A_0; q)$ and $a \in L^2_0(T^*B^4 \otimes \text{Ad}(P))$. In this section, we estimate the dual norm of the remainder $R(q; a)$, defined via the formula

$$ R(q; a) := \nabla \mathcal{M}(A(q) + a) - \nabla \mathcal{M}(A(q)) - \nabla^2 \mathcal{M}(A(q)) a, $$

where the Hessian of $\mathcal{M}$, denoted by $\nabla^2 \mathcal{M}(A)$, is given by

$$ \langle \nabla^2 \mathcal{M}(A)(a), b \rangle := 2 \int_{B^4} (d_A^* a, d_A^* b) + 2 \int_{B^4} (F_A^*, \varepsilon[a, b]) \text{ for all } a, b \in L^2_0(T^*B^4 \otimes \text{Ad}(P)), $$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between $L^2_0(T^*B^4 \otimes \text{Ad}(P))$ and its dual.

**Lemma 3.3** For $q \in \mathcal{P}(d_0, \lambda_0; D_1, D_2; \varepsilon)$, $a \in L^2_0(T^*B^4 \otimes \text{Ad}(P))$, one has

$$ \|R(q; a)\|_{A(q);1,2,*} \leq C\varepsilon(\|a\|_{A(q);1,2}^2 + \|a\|_{A(q);1,2}^3), $$

where $C > 0$ is a constant depending only on $d_0, \lambda_0, D_1, D_2$.

**Proof:** For $a, b \in L^2_0(T^*B^4 \otimes \text{Ad}(P))$, one has

$$ \nabla \mathcal{M}(A(q) + a)(b) = 2 \int_{B^4} (F_A^*(a^*) + d_A^*a^*b) = 2 \int_{B^4} (F_A^* + d_A^*a + \varepsilon \frac{a^*}{2}[a, b], d_A^*b + \varepsilon[a, b]) dx $$

$$ = \nabla \mathcal{M}(A(q))(b) + \langle \nabla^2 \mathcal{M}(A(q))a, b \rangle + \langle R(q; a), b \rangle, $$

where

$$ \langle R(q; a), b \rangle = 2 \varepsilon \int_{B^4} (d_A^*a^*, [a, b]) dx + \varepsilon \int_{B^4} ([a, b], d_A^*a^*b) dx + \varepsilon^2 \int_{B^4} ([a, b], [a, b]) dx. $$

Therefore, by the Hölder’s inequality

$$ |\langle R(q; a), b \rangle| \leq C\varepsilon\|d_A^*a^*\|_2\|a\|_4\|b\|_4 + C\varepsilon\|a\|_4^2\|d_A^*a^*b\|_2 + C\varepsilon^2\|a\|_4^3\|b\|_4 $$

(3.62)

Applying the Weitzenböck formula, $\Delta_A^* := d_A^*d_A + d_A^*d_A = \nabla_A^*\nabla_A^* + \varepsilon\{F_A^*, \cdot\}$ (cf. [5], [6]) (here, we only need to know that $\{\cdot, \cdot\}$ is a bilinear form) and the Sobolev inequality $\|a\|_4 \leq C\|a\|_{A;1,2}$, we obtain

$$ \|d_A^*a^*\|_2^2 \leq \|\nabla A(q)^*a\|_2^2 + \varepsilon\|F_A^*a, a\|_2 \leq \|\nabla A(q)^*a\|_2^2 + \varepsilon\|F_A^*a\|_2^2 \|a\|_4^2 \leq C\|a\|_{A(q);1,2}^2. $$

(3.63)

The assertion of the lemma now follows from (3.62), (3.63).

3.5 Estimate of the modified Hessian

For $q \in \mathcal{P}(d_0, \lambda_0; D_1, D_2; \varepsilon)$ and $A \in A_{+1}(A_0)$, the modified Hessian, $\mathcal{H}_A$, is defined as a bilinear form on $L^2_0(T^*B^4 \otimes \text{Ad}(P))$ as follows: for $a, b \in L^2_0(T^*B^4 \otimes \text{Ad}(P))$,

$$ \mathcal{H}_A(a, b) := \frac{1}{2}\langle \nabla^2 \mathcal{M}(A)a, b \rangle + \langle d_A^*a, d_A^*b \rangle_{L^2(B^4)}. $$

(3.64)

With this definition, $\mathcal{H}_A$ is continuous. The following positivity result holds for the modified Hessian:
Lemma 3.4 For \( q \in \mathcal{P}(d_0, \lambda_0; D_1, D_2; \epsilon) \), there exists a constant \( C \) depending only on \( d_0, \lambda_0, D_1, D_2 \) such that for small \( \epsilon > 0 \) and \( a \in L^2_{1,0}(T^*B^4 \otimes \text{Ad}(P)) \cap T_{\mathcal{A}(q)}\mathcal{N}(d_0, \lambda_0)^\perp \), there holds

\[
\mathcal{H}_{\mathcal{A}(q)}(a, a) \geq C\| a \|^2_{\mathcal{A}(q); 1, 2}.
\]

Here, \( L^2_{1,0}(T^*B^4 \otimes \text{Ad}(P)) \cap T_{\mathcal{A}(q)}\mathcal{N}(d_0, \lambda_0)^\perp \) is the orthogonal complement of \( T_{\mathcal{A}(q)}\mathcal{N}(d_0, \lambda_0) \) in \( L^2_{1,0}(T^*B^4 \otimes \text{Ad}(P)) \).

To prove the lemma above, one needs to introduce further notation and some auxiliary lemmas.

We define the following family of SU(2)-Yang Mills connections on \( \mathbb{R}^4 \): for \( q \in \mathcal{P}(d_0, \lambda_0) \),

\[
\tilde{A}(q) = \begin{cases} \frac{1}{2} g I_{\frac{1}{2}p} g^{-1} & \text{in } \mathbb{R}^4 \setminus \{p\} \\ \frac{1}{2} g I_{\frac{1}{2}p} g^{-1} & \text{in } B_{\lambda/4}(p). \end{cases}
\]  

(3.65)

These connections (absolute minimizers for the Yang Mills functional) live on the bundles \( \tilde{P}(q) \), defined by the data

\[
\left\{ B_{\lambda/4}(p), \mathbb{R}^4 \setminus \{p\}, \{gg_{12,p}g^{-1}\} \right\}.
\]  

(3.66)

Note that these bundles are extensions to \( \mathbb{R}^4 \) of the bundles \( P(q) \) defined in §3.1 \( (\tilde{P}(q)|_{B^4} = P(q)) \) and that \( \tilde{A}(q) = A(q) \) on \( B_{\lambda/4}(p) \). Similarly to what has been done in §3.1, we set

\[
\tilde{\mathcal{N}}(d_0, \lambda_0) := \{ \tilde{A}(q) : q \in \mathcal{P}(d_0, \lambda_0) \},
\]  

(3.67)

and apply the convention that everything is pulled back to the bundle \( \tilde{P}(q_0) \) by the bundle isomorphisms \( \tilde{\varphi}(q) : \tilde{P}(q_0) \cong \tilde{P}(q) \). To the bundles and connections just defined over \( \mathbb{R}^4 \), correspond bundles and connections on \( S^4 \) (by pull back under the stereographic projection from the north pole), which we denote by \( P(q)|_{S^4} \) and \( A(q)|_{S^4} \), respectively.

To prove Lemma 3.4 we describe the tangent space of \( \mathcal{N}(d_0, \lambda_0) \) at \( \tilde{A}(q) \) by

\[
T_{\tilde{A}(q)}\mathcal{N}(d_0, \lambda_0) := \text{span} \left\{ \frac{\partial A}{\partial p_i}(q), \frac{\partial A}{\partial \xi_j}(q), \frac{\partial A}{\partial \lambda}(q) \right\}_{1 \leq i \leq 4, 1 \leq j \leq 3} \subset L^2_1(T^*B^4 \otimes \text{Ad}(P)).
\]  

(3.68)

This is a finite dimensional (at most 8 dimensional) space. Indeed, for small \( \epsilon > 0 \), it is exactly 8-dimensional (cf. Lemma 3.2 in [14]).

Likewise, the tangent space of \( \tilde{\mathcal{N}}(d_0, \lambda_0) \) at \( \tilde{A}(q) \) is described by

\[
T_{\tilde{A}(q)}\tilde{\mathcal{N}}(d_0, \lambda_0) := \text{span} \left\{ \frac{\partial}{\partial p_i} \tilde{A}(q), \frac{\partial}{\partial \xi_j}[g] \tilde{A}(q), \frac{\partial}{\partial \lambda} \tilde{A}(q) \right\}_{1 \leq i \leq 4, 1 \leq j \leq 3} \subset L^2_1(T^*\mathbb{R}^4 \otimes \text{Ad}(\tilde{P})).
\]  

(3.69)

For \( q := (p, [g], \lambda) \in \mathcal{P}(d_0, \lambda_0; D_1, D_2; \epsilon) \), we introduce the function space

\[
L^2_{1,\tilde{A}(q)}(T^*\mathbb{R}^4 \otimes \text{Ad}(\tilde{P})) = \{ a \in L^2_{1,\text{loc}}(T^*\mathbb{R}^4 \otimes \text{Ad}(\tilde{P})) : \int_{\mathbb{R}^4} |\nabla_{\tilde{A}(q)} \alpha|^2 + \frac{|a|^2}{(1 + |x|^2)^2} \, dx < +\infty \}.
\]

Thus \( a \in L^2_{1,\tilde{A}(q)}(T^*\mathbb{R}^4 \otimes \text{Ad}(\tilde{P})) \) if and only if its pull-back \( \pi^*a \) is in \( L^2_1(T^*S^4) \), where \( \pi : S^4 \to \mathbb{R}^4 \cup \{ \infty \} \) is the stereographic projection from the north pole. For technical reasons, we define the weighted inner product on \( L^2_{1,\tilde{A}(q)}(T^*\mathbb{R}^4 \otimes \text{Ad}(\tilde{P})) \) by

\[
(\alpha, \beta)_{\tilde{A}(q); 1, 2; \mathbb{R}^4} := \int_{\mathbb{R}^4} (\nabla_{\tilde{A}(q)} \alpha, \nabla_{\tilde{A}(q)} \beta) + w(x)(\alpha, \beta) \, dx,
\]  

(3.70)
where \( w(x) = 1 \) for \( |x| \leq 1 \), and \( w(x) = 1/(1+|x|^2) \) for \( |x| > 1 \).

In the following proofs, we denote by \( T_{\tilde{A}(q)} \tilde{N}(d_0, \lambda_0) \) the orthogonal complement of \( T_{\tilde{A}(q)} \tilde{N}(d_0, \lambda_0) \) in \( L^2_{1;\tilde{A}(q)}(T^*\mathbb{R}^4 \otimes \text{Ad}(\tilde{P})) \) and define

\[
\tilde{A}^p_{k+1} := \{ A \in \tilde{P} : \pi^* A \text{ is an } L^p_k\text{-connection on } P(q)_{S^4} \},
\]

where \( \pi \) is the stereographic projection. We denote by \( \tilde{G}^P_{k+1} \) the group of gauge transformations on \( \tilde{P} \) which come from \( L^p_{k+1}\text{-gauge transformations on the bundle } P(q)_{S^4} \), and we define

\[
\tilde{G}^P_{k+1} := \tilde{A}^p_{k+1}/\tilde{G}^P_{k+1}.
\]

In order to prove Lemma 3.4 we need the following lemma:

**Lemma 3.5** For \( q \in P(d_0, \lambda_0; D_1, D_2; \epsilon) \), there exists \( C > 0 \) such that for \( \epsilon = 0 \) one has

\[
\int_{\mathbb{R}^4} |d_{\tilde{A}(q)} a|^2 dx + \int_{\mathbb{R}^4} (F_{\tilde{A}(q)}, [a, a]) dx + \int_{\mathbb{R}^4} |d_{\tilde{A}(q)}^* a|^2 dx \geq C\left( \|\nabla_{\tilde{A}(q)} a\|_{L^2(\mathbb{R}^4)} + \|a\|_{L^2(\mathbb{R}^4)} \right).
\]

**Proof:** We only prove the assertion for the case \( \epsilon = 1 \). The general case follows by the Lie algebra isomorphism \( \phi_\epsilon : \mathfrak{su}(2) \to \mathfrak{su}(2) \).

Recall that the instanton \( \tilde{A}(q) \) is action minimizing, therefore the Hessian of \( \mathcal{Y}M \) at \( \tilde{A}(q) \) is non-negative, i.e.,

\[
\int_{\mathbb{R}^4} |d_{\tilde{A}(q)} a|^2 dx + \int_{\mathbb{R}^4} (F_{\tilde{A}(q)}, [a, a]) dx \geq 0,
\]

for all \( a \in L^2_{1;\tilde{A}(q)}(T^*\mathbb{R}^4 \otimes \text{Ad}(\tilde{P})) \).

We claim:

**Claim 3.1** For \( a \in T_{\tilde{A}(q)} \tilde{N}(d_0, \lambda_0) \) with \( a \neq 0 \),

\[
\int_{\mathbb{R}^4} |d_{\tilde{A}(q)} a|^2 dx + \int_{\mathbb{R}^4} (F_{\tilde{A}(q)}, [a, a]) dx + \int_{\mathbb{R}^4} |d_{\tilde{A}(q)}^* a|^2 dx > 0.
\]

**Proof of Claim 3.1** We already know that this is non-negative by (3.73). By contradiction, assume that there exists \( a \in T_{\tilde{A}(q)} \tilde{N}(d_0, \lambda_0) \) with \( a \neq 0 \) such that

\[
\int_{\mathbb{R}^4} |d_{\tilde{A}(q)} a|^2 dx + \int_{\mathbb{R}^4} (F_{\tilde{A}(q)}, [a, a]) dx + \int_{\mathbb{R}^4} |d_{\tilde{A}(q)}^* a|^2 dx = 0.
\]

From (3.73) and (3.74), it follows that \( d_{\tilde{A}(q)}^* a = 0 \) and

\[
\int_{\mathbb{R}^4} |d_{\tilde{A}(q)} a|^2 dx + \int_{\mathbb{R}^4} (F_{\tilde{A}(q)}, [a, a]) dx = 0.
\]

It follows that \( a \) minimizes the quadratic functional \( \alpha \mapsto \int_{\mathbb{R}^4} |d_{\tilde{A}(q)} \alpha|^2 dx + \int_{\mathbb{R}^4} (F_{\tilde{A}(q)}, [\alpha, \alpha]) dx \) in \( L^2_{1;\tilde{A}(q)}(T^*\mathbb{R}^4 \otimes \text{Ad}(\tilde{P})) \), thus its first variation computed at \( a \) is zero, that is, for all \( \varphi \in L^2_{1;\tilde{A}(q)}(T^*\mathbb{R}^4 \otimes \text{Ad}(\tilde{P})) \),

\[
\frac{d}{d \epsilon} \bigg|_{\epsilon=0} \int_{\mathbb{R}^4} |d_{\tilde{A}(q)} (a + \epsilon \varphi)|^2 dx + \int_{\mathbb{R}^4} (F_{\tilde{A}(q)}, [a + \epsilon \varphi, a + \epsilon \varphi]) dx + \int_{\mathbb{R}^4} |d_{\tilde{A}(q)}^* (a + \epsilon \varphi)|^2 dx
\]

\[
= 2 \int_{\mathbb{R}^4} (d_{\tilde{A}(q)} a, d_{\tilde{A}(q)} \varphi) dx + \int_{\mathbb{R}^4} (F_{\tilde{A}(q)}, [\varphi, a]) dx + 2 \int_{\mathbb{R}^4} (d_{\tilde{A}(q)}^* a, d_{\tilde{A}(q)}^* \varphi) dx = 0,
\]

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or, equivalently (since \( d^*_{\tilde{A}(q)} a = 0 \)),
\[
\nabla^2 \mathcal{Y} M(\tilde{A}(q))(a) = 0.
\] (3.75)

It follows from the elliptic regularity theory that \( a \in C^\infty(T^*\mathbb{R}^4 \otimes \text{Ad}(\tilde{P})) \cap L^2_{1;\tilde{A}(q)}(T^*\mathbb{R}^4 \otimes \text{Ad}(\tilde{P})) \).

It is well-known that the set of all solutions of (3.75), which satisfy \( d^*_{\tilde{A}(q)} a = 0 \) constitute the tangent space of the 1-instanton moduli space \( \mathcal{M}_{+1}(S^4) \) over \( \mathbb{R}^4 \cup \{\infty\} = S^4 \). One has \( T_{\tilde{A}(q)} \mathcal{M}_{+1}(S^4) \subset T_{\tilde{A}(q)} \mathcal{B}_{1;\tilde{A}(q)} := \{ a \in L^2_{1;\tilde{A}(q)}(T^*\mathbb{R}^4 \otimes \text{Ad}(\tilde{P})) : d^*_{\tilde{A}(q)} a = 0 \} \). Since \( T_{\tilde{A}(q)} \tilde{N}(d_0, \lambda_0) \) coincides with the tangent space of \( \mathcal{M}_{+1}(S^4) \) at \( \tilde{A}(q) \) (up to infinitesimal gauge transformations), this contradicts the fact that \( a \) is orthogonal to the tangent space of \( \tilde{N}(d_0, \lambda_0) \) at \( \tilde{A}(q) \). This completes the proof of Claim 3.1.

\[ \square \]

Completion of the proof of Lemma 3.5

By contradiction, assume that there exists a sequence \( \{a_n\} \subset T_{\tilde{A}(q)} \tilde{N}(d_0, \lambda) \) such that
\[
\int_{\mathbb{R}^4} |\nabla_{\tilde{A}(q)} a_n|^2 \, dx + \int_{\mathbb{R}^4} w(x)|a_n|^2 \, dx = 1 \quad \forall n,
\] (3.76)
and
\[
\int_{\mathbb{R}^4} |d_{\tilde{A}(q)} a_n|^2 \, dx + \int_{\mathbb{R}^4} (F_{\tilde{A}(q)}, [a_n, a_n]) \, dx + \int_{\mathbb{R}^4} |d^*_{\tilde{A}(q)} a_n|^2 \, dx \to 0, \text{ as } n \to \infty.
\] (3.77)

By (3.76), passing to a subsequence, we may assume that \( a_n \rightharpoonup a \) weakly in \( L^2_{1;\tilde{A}(q)}(T^*\mathbb{R}^4 \otimes \text{Ad}(\tilde{P})) \) as \( n \to \infty \), for some \( a \in L^2_{1;\tilde{A}(q)}(T^*\mathbb{R}^4 \otimes \text{Ad}(\tilde{P})) \). The first and the third integrals above are clearly lower semi-continuous with respect to the weak convergence in \( L^2_{1;\tilde{A}(q)}(T^*\mathbb{R}^4 \otimes \text{Ad}(\tilde{P})) \).

We assert that the second integral \( a \mapsto \int_{\mathbb{R}^4} (F_{\tilde{A}(q)}, [a, a]) \) is continuous with respect to the weak convergence in \( L^2_{1;\tilde{A}(q)}(T^*\mathbb{R}^4 \otimes \text{Ad}(\tilde{P})) \). To see this, we write \( a_n = a + b_n \) with \( b_n \rightharpoonup 0 \) weakly in \( L^2_{1;\tilde{A}(q)}(T^*\mathbb{R}^4 \otimes \text{Ad}(\tilde{P})) \). We have
\[
\int_{\mathbb{R}^4} (F_{\tilde{A}(q)}, [a_n, a_n]) \, dx = \int_{\mathbb{R}^4} (F_{\tilde{A}(q)}, [a_n, b_n]) \, dx + 2 \int_{\mathbb{R}^4} (F_{\tilde{A}(q)}, [b_n, a]) \, dx + \int_{\mathbb{R}^4} (F_{\tilde{A}(q)}, [a, a]) \, dx.
\] (3.78)

By the Sobolev embedding, we have (modulo passing to a subsequence) \( b_n \to 0 \) in \( L^p_{\text{loc}}(\mathbb{R}^4) \) for any \( p < 4 \). Fixing an arbitrary \( R > 0 \), we have
\[
\int_{|x| \leq R} (F_{\tilde{A}(q)}, [b_n, b_n]) \, dx = \int_{|x| \leq R} (F_{\tilde{A}(q)}, [b_n, b_n]) \, dx + \int_{|x| > R} (F_{\tilde{A}(q)}, [b_n, b_n]) \, dx.
\]

Both integrals on the right hand side go to 0 as \( n \to \infty \) (by Hölder’s inequality, the second one is bounded above by \( C(\int_{|x| > R} |F_{\tilde{A}(q)}|^2 \, dx)^{1/2} \|b_n\|_{1;\tilde{A}(q)}^2 \), which goes to 0 as \( R \to \infty \)). Thus the first integral in (3.78) goes to 0 as \( n \to \infty \). With similar arguments, one shows that the second integral in (3.78) also goes to zero and the assertion is proved.

Inequality (3.73) and (3.77) then imply
\[
\int_{\mathbb{R}^4} |d_{\tilde{A}(q)} a|^2 \, dx + \int_{\mathbb{R}^4} (F_{\tilde{A}(q)}, [a, a]) + \int_{\mathbb{R}^4} |d^*_{\tilde{A}(q)} a|^2 \, dx = 0,
\]
and, finally, since \( a \in T_{\tilde{A}(q)}N(d_0, \lambda_0)^{-1} \), applying Claim 3.11 one obtains that \( a = 0 \).

We then have, by (3.77),

\[
\int_{\mathbb{R}^4} |d_{\tilde{A}(q)} a_n|^2 \, dx + \int_{\mathbb{R}^4} |d_{\tilde{A}(q)}^* a_n|^2 \, dx \to 0 \quad \text{for } n \to \infty,
\]

and by the Weitzenböck formula,

\[
\int_{\mathbb{R}^4} |d_{\tilde{A}(q)} a_n|^2 \, dx + \int_{\mathbb{R}^4} |d_{\tilde{A}(q)}^* a_n|^2 \, dx = \int_{\mathbb{R}^4} |\nabla_{\tilde{A}(q)} a_n|^2 \, dx + (\{F_{\tilde{A}(q)}, a_n\}, a_n) \, dx
\]

\[
= \int_{\mathbb{R}^4} |\nabla_{\tilde{A}(q)} a_n|^2 \, dx + o(1) \quad (n \to \infty).
\]

Combining (3.79), (3.80), we obtain \( \int_{\mathbb{R}^4} |\nabla_{\tilde{A}(q)} a_n|^2 \, dx \to 0 \) as \( n \to \infty \), and \( \int_{\mathbb{R}^4} w(x)|a_n|^2 \, dx \to 0 \), by the Sobolev embedding. This contradicts (3.76) and completes the proof of Lemma 3.5. □

To prove Lemma 3.6 we also need to estimate the difference between the bilinear forms \( \mathcal{K}_A(q) \) and \( \mathcal{K}_{\tilde{A}(q)} \), where \( \mathcal{K}_A(q)(a, b) := \frac{1}{2}(\nabla^2 \mathcal{M}_q(\tilde{A}(q))a, b) + (d_{\tilde{A}(q)}^* a, d_{\tilde{A}(q)}^* b)_{L^2(\mathbb{R}^4)} \) for \( a, b \in L^2_{1,\tilde{A}(q)}(T^* B^4 \otimes \text{Ad}(\hat{P})) \). This is the content of the following lemma.

**Lemma 3.6** Let \( q \in \mathcal{P}(d_0, \lambda_0; D_1, D_2; \epsilon) \). The bilinear forms \( \mathcal{K}_A(q) \) and \( \mathcal{K}_{\tilde{A}(q)} \) on \( L^2_{1,0}(T^* B^4 \otimes \text{Ad}(P)) \times L^2_{1,0}(T^* B^4 \otimes \text{Ad}(P)) \) satisfy

\[
||\mathcal{K}_A(q) - \mathcal{K}_{\tilde{A}(q)}||_{A(q);1,2,*} \lesssim \epsilon.
\]

(Here, \( (\cdot, \cdot)_2 \) denotes the \( L^2 \)-inner product).

**Proof:** Set \( b = b(q) = A(q) - \tilde{A}(q) \). For \( \alpha, \beta \in L^2_{1,0}(T^* B^4 \otimes \text{Ad}(P)) \), a simple computation yields

\[
(\mathcal{K}_A(q) - \mathcal{K}_{\tilde{A}(q)})(\alpha, \beta) = \epsilon(d_{\tilde{A}(q)}^* \alpha, [b, \beta])_2 + \epsilon([b, \alpha], d_{\tilde{A}(q)}^* \beta)_2 + \epsilon(d_{\tilde{A}(q)}b, [\alpha, \beta])_2 + \epsilon^2([b, \beta], [\alpha, \beta])_2 - \epsilon(d_{\tilde{A}(q)}^* \alpha, \ast [b, \beta])_2 - \epsilon(\ast [b, \alpha], d_{\tilde{A}(q)}^* \beta)_2 + 2\epsilon^2([b, \alpha], [b, \beta])_2.
\]

Since \( b = b(q) = (1 - \beta_{\lambda,p})A_\epsilon + \frac{1}{2}(\beta_{\lambda/4,p} - 1)gh_{\lambda,p}g^{-1} \) satisfies \( \epsilon \|b\|_\infty \lesssim \epsilon \), integrating by parts the third addend at the right hand side of (3.81), one obtains

\[
\|\mathcal{K}_A(q) - \mathcal{K}_{\tilde{A}(q)}(\alpha, \beta)\| \lesssim \epsilon \|\alpha\|_{A(q);1,2} \|\beta\|_{A(q);1,2}
\]

This completes the proof of Lemma 3.6. □

For the next steps, we use the orthonormal basis \( \langle a_1(q), a_2(q), \ldots, a_8(q) \rangle \) of \( T_{A(q)}N(d_0, \lambda_0) \) given by \( a_i(q) = A_{q_i}(q) \), where the vector fields \( q_i \) (1 ≤ \( i \) ≤ 8) on \( \mathcal{P}(d_0, \lambda_0) \) are defined in [14] and \( A_{q_i}(q) \) denotes the directional derivative of \( A(q) \) in the direction \( q_i(q) \), and the basis \( \langle \tilde{a}_1(q), \tilde{a}_2(q), \ldots, \tilde{a}_8(q) \rangle \) of \( T_{\tilde{A}(q)}N(d_0, \lambda_0) \), defined via \( \tilde{a}_i(q) := \tilde{A}_{q_i}(q) \). We also need the orthonormal basis \( \langle \tilde{a}_1(q), \ldots, \tilde{a}_8(q) \rangle \) of \( T_{\tilde{A}(q)}N(d_0, \lambda_0) \) constructed via the Gram-Schmidt’s orthogonalization procedure applied to \( \tilde{a}_1(q), \ldots, \tilde{a}_8(q) \) (for details, see [14]). One needs the following technical lemmas, proved in [14].
Lemma 3.7 For $q \in P(d_0, \lambda_0; D_1, D_2; \epsilon)$,
\[
\|a_i(q) - \tilde{a}_i(q)\|_{A(q);1,2;B^i} \lesssim \epsilon^{3/2} \quad (1 \leq i \leq 4),
\]
\[
\|a_i(q) - \tilde{a}_i(q)\|_{A(q);1,2;B^i} \lesssim \epsilon \quad (5 \leq i \leq 8).
\]

Lemma 3.8 For $q \in P(d_0, \lambda_0; D_1, D_2; \epsilon)$,
\[
\|(\nabla^2 \mathcal{M}_u(A(q)) - \nabla^2 \mathcal{M}_u(\tilde{A}(q)))a_i(q)\|_{A(q);1,2,*} \lesssim \epsilon^{3/2} \quad (1 \leq i \leq 8),
\]
\[
\|(d_{A(q)}^\epsilon d_{\tilde{A}(q)}^\epsilon - d_{A(q)}^\epsilon d_{\tilde{A}(q)}^\epsilon)a_i(q)\|_{A(q);1,2,*} \lesssim \epsilon^{3/2} \quad (1 \leq i \leq 8).
\]

Lemma 3.4 and the results in [11] yield the estimate
\[
\|\tilde{a}_i(q) - a_i(q)\|_{A(q);1,2} \leq \|\tilde{a}_i(q) - \tilde{a}_i(q)\|_{A(q);1,2} + \|\tilde{a}_i(q) - a_i(q)\|_{A(q);1,2} \lesssim \epsilon \quad (1 \leq i \leq 8). \quad (3.82)
\]

In order to prove Lemma 3.4, we need to define the “topological projections”
\[
Q : L^2_{1,0}((T^*B^4 \otimes Ad(P)) \to L^2_{1,0}((T^*B^4 \otimes Ad(P)) \cap T_{A(q)}N(d_0, \lambda_0)^\perp, \quad (3.83)
\]
\[
\tilde{Q} : L^2_{1,\tilde{A}(q)}((T^*\tilde{B}^4 \otimes Ad(\tilde{P})) \to L^2_{1,\tilde{A}(q)}((T^*\tilde{B}^4 \otimes Ad(\tilde{P})) \cap T_{\tilde{A}(q)}\tilde{N}(d_0, \lambda_0)^\perp. \quad (3.84)
\]
(Recall that the orthogonal complements $T_{A(q)}N(d_0, \lambda_0)^\perp$, $T_{\tilde{A}(q)}\tilde{N}(d_0, \lambda_0)^\perp$ are calculated with respect to the inner product $(\cdot, \cdot)_{A(q);1,2;B^i}$ and $(\cdot, \cdot)_{\tilde{A}(q);1,2;\tilde{B}^i}$, respectively).

Proof of Lemma 3.4 Let $a \in L^2_{1,0}((T^*B^4 \otimes Ad(P))$. By extending $a$ trivially to $\mathbb{R}^4 \setminus B^4$ by 0, we may also regard it as an element of $L^2_{1,\tilde{A}(q)}((T^*\tilde{B}^4 \otimes Ad(\tilde{P}))$ and we define the components
\[
a^\top := Pa, \quad \text{with } P := \text{Id} - Q, \quad a^\perp := Qa, \quad (3.85)
\]
\[
a^\top := \tilde{P}a, \quad \text{with } \tilde{P} := \text{Id} - \tilde{Q}, \quad a^\perp := \tilde{Q}a. \quad (3.86)
\]

We now prepare to estimate $\mathcal{H}_{A(q)}(a, a)$. We have
\[
\mathcal{H}_{A(q)}(a, a) = \mathcal{H}_{A(q)}(a^\top + a^\perp, a^\top + a^\perp) = \mathcal{H}_{A(q)}(a^\top, a^\top) + 2\mathcal{H}_{A(q)}(a^\top, a^\perp) + \mathcal{H}_{A(q)}(a^\perp, a^\perp). \quad (3.87)
\]
The last term of (3.87) can be estimated by Lemma 3.6 as follows:
\[
\mathcal{H}_{A(q)}(a^\perp, a^\perp) = \mathcal{H}_{\tilde{A}(q)}(a^\perp, a^\perp) + (\mathcal{H}_{A(q)} - \mathcal{H}_{\tilde{A}(q)})(a^\perp, a^\perp) \geq \mathcal{H}_{\tilde{A}(q)}(a^\perp, a^\perp) - C\epsilon\|a^\perp\|_{A(q);1,2}^2. \quad (3.88)
\]
To estimate the first term of (3.87), we write $a^\perp = (a^\perp)^\top + (a^\perp)^\perp$. The first addend expressed in terms of the orthonormal basis $\langle \tilde{a}_1(q), \ldots, \tilde{a}_8(q) \rangle$ of $T_{A(q)}\tilde{N}(d_0, \lambda_0)$ is given by
\[
(a^\perp)^\top = \sum_{i=1}^8 (a^\perp, \tilde{a}_i(q))_{A(q);1,2;\tilde{R}^4}\tilde{a}_i(q).
\]
These components satisfy
\[
(a^\perp, \tilde{a}_i(q))_{A(q);1,2;\tilde{R}^4} = (a^\perp, \tilde{a}_i(q))_{A(q);1,2;B^4} + ((a^\perp, \tilde{a}_i(q))_{A(q);1,2;\tilde{R}^4} - (a^\perp, \tilde{a}_i(q))_{A(q);1,2;B^4})
\]
\[
= (a^\perp, \tilde{a}_i(q) - a_i(q))_{A(q);1,2;B^4} + ((a^\perp, \tilde{a}_i(q))_{A(q);1,2;\tilde{R}^4} - (a^\perp, \tilde{a}_i(q))_{A(q);1,2;B^4}). \quad (3.89)
\]

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Here, we have (recall that supp $a^\perp \subset B^4$):

\[
(a^\perp, \hat{a}_i(q))_{\hat{A}(q); 1,2;\mathbb{R}^4} - (a^\perp, \hat{a}_i(q))_{\hat{A}(q); 1,2;\mathbb{R}^4} = \int_{B^4} (\nabla_{\hat{A}(q)} \epsilon a^\perp, \nabla_{\hat{A}(q)} \epsilon \hat{a}_i(q)) - \int_{B^4} (\nabla_{\hat{A}(q)} \epsilon a^\perp, \nabla_{\hat{A}(q)} \epsilon \hat{a}_i(q)) = \epsilon \int_{B^4} (\nabla_{\hat{A}(q)} \epsilon a^\perp, [b(q), \hat{a}_i(q)]) + \epsilon \int_{B^4} ([b(q), a^\perp], \nabla_{\hat{A}(q)} \epsilon \hat{a}_i(q))
\]

\[
+ \epsilon^2 \int_{B^4} ([b(q), a^\perp], [b(q), \hat{a}_i(q)]) \lesssim \epsilon \|a^\perp\|_{\hat{A}(q); 1,2;\mathbb{R}^4}, \quad (3.90)
\]

where $b(q) = \hat{A}(q) - A(q)$ and we have used $\epsilon \|b\|_{\infty} \lesssim \epsilon$.

By (3.82), (3.89), (3.90), one obtains

\[
\| (a^\perp, \hat{a}_i(q))_{\hat{A}(q); 1,2;\mathbb{R}^4} \| \lesssim \epsilon \|a^\perp\|_{\hat{A}(q); 1,2;\mathbb{R}^4}, \quad 1 \leq i \leq 8,
\]

(3.91)

and therefore,

\[
\| (a^\perp)^{\perp} \|_{\hat{A}(q); 1,2;\mathbb{R}^4} \lesssim \epsilon \|a^\perp\|_{\hat{A}(q); 1,2;\mathbb{R}^4}.
\]

(3.92)

We thus obtain

\[
\mathcal{H}_{\hat{A}(q)}(a^\perp, a^\perp) = \mathcal{H}_{\hat{A}(q)}(\epsilon a^\perp, \epsilon a^\perp) + 2\mathcal{H}_{\hat{A}(q)}((a^\perp)^{\perp}, (a^\perp)^{\perp}) + \mathcal{H}_{\hat{A}(q)}((a^\perp)^{\perp}, (a^\perp)^{\perp})
\]

\[
\geq \mathcal{H}_{\hat{A}(q)}((a^\perp)^{\perp}, (a^\perp)^{\perp}) - C\| (a^\perp)^{\perp} \|_{\hat{A}(q); 1,2;\mathbb{R}^4} \| (a^\perp)^{\perp} \|_{\hat{A}(q); 1,2;\mathbb{R}^4} - C\epsilon \| (a^\perp)^{\perp} \|_{\hat{A}(q); 1,2;\mathbb{R}^4}^2
\]

\[
\geq C\| (a^\perp)^{\perp} \|_{\hat{A}(q); 1,2;\mathbb{R}^4}^2 - C\epsilon \| (a^\perp)^{\perp} \|_{\hat{A}(q); 1,2;\mathbb{R}^4} \| a^\perp \|_{\hat{A}(q); 1,2;\mathbb{R}^4} - C\epsilon \| a^\perp \|_{\hat{A}(q); 1,2;\mathbb{R}^4}^2,
\]

(3.93)

where we used Lemma 3.5 to estimate $\mathcal{H}_{\hat{A}(q)}((a^\perp)^{\perp}, (a^\perp)^{\perp})$.

From $(a^\perp)^{\perp} = a^\perp - (a^\perp)^{\parallel}$, estimate (3.92) and $\|a^\perp\|_{\hat{A}(q); 1,2;\mathbb{R}^4} \approx \|a^\perp\|_{A(q); 1,2;\mathbb{R}^4}$ (since supp $a^\perp \subset B^4$), one obtains

\[
C^{-1}\|a^\perp\|_{A(q); 1,2;\mathbb{R}^4} \leq \| (a^\perp)^{\parallel} \|_{\hat{A}(q); 1,2;\mathbb{R}^4} \leq C\|a^\perp\|_{A(q); 1,2;\mathbb{R}^4}.
\]

(3.94)

Finally, combining (3.88), (3.93), (3.94),

\[
\mathcal{H}_{A(q)}(a^\perp, a^\perp) \geq C\|a^\perp\|_{\hat{A}(q); 1,2;\mathbb{R}^4}^2
\]

(3.95)

for all small $\epsilon > 0$, which is ‘almost’ the assertion of Lemma 3.74. It is left to prove that the constant $C$ can be taken independent of $q \in \mathcal{P}(d_0, \lambda_0; D_1, D_2; \epsilon)$. To this purpose, we first observe that $\mathcal{H}_A(a, a)$ and $\|\nabla_A a\|_{2;\mathbb{R}^4}$ are conformally invariant. Hence, the inequality $\mathcal{H}_A((a^\perp)^{\parallel}, (a^\perp)^{\parallel}) \geq C\|\nabla_A (a^\perp)^{\parallel}\|_{2;\mathbb{R}^4}^2$ holds for some $C > 0$ independent of $q$. By the Poincaré inequality, we have $\|a\|_2 \leq C\|\nabla a\|_2 \leq C\|\nabla_a (a^\perp)^{\parallel}\|_2$ for $a \in L^2_{1,0}(T^*B^4 \otimes \text{Ad}(P))$ and some $C > 0$ independent of $q$. Thus

\[
\|\nabla_A (a^\perp)^{\parallel}\|_{2;\mathbb{R}^4} \geq \|\nabla_A (a^\perp)^{\parallel}\|_{2;\mathbb{R}^4} - \| (a^\perp)^{\parallel} \|_{\hat{A}(q); 1,2;\mathbb{R}^4}
\]

\[
\geq C\|a^\perp\|_{A(q); 1,2;\mathbb{R}^4} - C\epsilon\|a^\perp\|_{A(q); 1,2;\mathbb{R}^4} - C\epsilon\|a^\perp\|_{A(q); 1,2;\mathbb{R}^4}
\]

\[
\geq C\|a^\perp\|_{A(q); 1,2;\mathbb{R}^4}.
\]
for some $C > 0$ independent of $q$. The assertion follows.

Note that (3.95) yields the estimate

$$H_{A(q)}(a, a) \geq C\|a^\perp\|^2_{A(q); 1, 2, B^4} - C\|a^\perp\|^2_{A(q); 1, 2, B^4},$$  \hspace{1cm} (3.96)

which is used in the next section.

\[\square\]

### 3.6 The auxiliary equation

In this section we solve the equation in $T_{A(q)}N(d_0, \lambda_0)^\perp$ (i.e., essentially orthogonally to the kernel of the Hessian of the $\epsilon$-Yang Mills functional, which is the obstruction to the direct application of the implicit function theorem). Thus we introduce the following auxiliary equation, associated to the Yang Mills equation $\nabla YM_\epsilon(A(q) + a) = 0$:

$$Q\left(\frac{1}{2}\nabla YM_\epsilon(A(q) + a) + d_{A(q)+a}^\epsilon d_{A(q)+a}^\epsilon a\right) = 0,$$  \hspace{1cm} (3.97)

where $Q$ is the topological projection defined in (3.83). We shall solve (3.97) for $a \in L^p_{1,0}(T^*B^4 \otimes \text{Ad}(P)) \cap T_{A(q)}N(d_0, \lambda_0)^\perp$.

For $2 < p < 4$, we define the following duality paring:

$$L^p_{1,0}(T^*B^4 \otimes \text{Ad}(P)) \times L^{p'}_{1,0}(T^*B^4 \otimes \text{Ad}(P)) \rightarrow \mathbb{R}$$

$$(a, b) \rightarrow \langle a, b \rangle := \int_{B^4} (\nabla A(q))^\epsilon a, \nabla A(q))^\epsilon b \, dx + \int_{B^4} (a, b) \, dx,$$

for $a \in L^p_{1,0}(T^*B^4 \otimes \text{Ad}(P))$, $b \in L^{p'}_{1,0}(T^*B^4 \otimes \text{Ad}(P))$, with $1/p + 1/p' = 1$.

For $a, b \in C_0^\infty(T^*B^4 \otimes \text{Ad}(P))$, we define

$$\langle \frac{1}{2}\nabla YM_\epsilon(A(q) + a) + d_{A(q)+a}^\epsilon d_{A(q)+a}^\epsilon a, b \rangle$$

$$= \int_{B^4} (F_{A(q)+a})^\epsilon d_{A(q)+a}^\epsilon b \, dx + \int_{B^4} (d_{A(q)+a}^\epsilon a, d_{A(q)+a}^\epsilon b) \, dx$$

$$= \int_{B^4} (F_{A(q)})^\epsilon d_{A(q)}^\epsilon a + \frac{\epsilon}{2}[a, a], d_{A(q)}^\epsilon b + \epsilon[a, b]) \, dx + \int_{B^4} (d_{A(q)}^\epsilon a + \epsilon [a, a], d_{A(q)}^\epsilon b + \epsilon [a, b]) \, dx.$$

By the Sobolev embeddings $L^p(B^4) \subset L^{4p/(4-p)}(B^4)$, $L^{p'}(B^4) \subset L^{4p/(3p-4)}(B^4)$, one obtains

$$\langle \frac{1}{2}\nabla YM_\epsilon(A(q) + a) + d_{A(q)+a}^\epsilon d_{A(q)+a}^\epsilon a, b \rangle$$

$$\leq (\|F_{A(q)}^\epsilon\| + C\|a\|_{A(q); 1, p} + C\|a\|^2_{A(q); 1, p} + C\|a\|^3_{A(q); 1, p}) \|b\|_{A(q); 1, p'},$$  \hspace{1cm} (3.98)

for some constant $C > 0$ independent of $a$ and $b$. It follows that

$$\sup \left\{ \langle \frac{1}{2}\nabla YM_\epsilon(A(q) + a) + d_{A(q)+a}^\epsilon d_{A(q)+a}^\epsilon a, b \rangle : \|b\|_{A(q); 1, p'} \leq 1 \right\} < \infty,$$

for $a \in L^p_{1,0}(T^*B^4 \otimes \text{Ad}(P))$, thus $\frac{1}{2}\nabla YM_\epsilon(A(q) + a) + d_{A(q)+a}^\epsilon d_{A(q)+a}^\epsilon a \in L^p_{1,0}(T^*B^4 \otimes \text{Ad}(P))$.

We obtain the following existence lemma. The proof is essentially an application of the contraction mapping principle together with uniform estimates of the Hessian as given in Lemma 3.4.
Lemma 3.9 There exist $2 < p_0 < 4$ and $\epsilon_0 > 0$ such that for all $2 < p < p_0$, $0 < \epsilon < \epsilon_0$ and $q \in \mathcal{P}(d_0, \lambda_0; D_1, D_2; \epsilon)$, there exists $\delta > 0$ such that the auxiliary equation (3.97) has a unique solution $a = a(q) \in L^p_{1,0}(T^* B^4 \otimes \text{Ad}(P)) \cap T_{A(q)} N(d_0, \lambda_0)^\perp$ which satisfies $\|a(q)\|_{A(q):1,p} < \delta$ and

$$\|a(q)\|_{A(q):1,2,B^4} \leq C\|\nabla y M_\epsilon(A(q))\|_{A(q):1,2,*}$$  (3.99)

for some $C > 0$ depending only on $d_0, \lambda_0, D_1$ and $D_2$.

Proof: For $q \in \mathcal{P}(d_0, \lambda_0; D_1, D_2; \epsilon)$, we define the functional

$$F : L^p_{1,0}(T^* B^4 \otimes \text{Ad}(P)) \cap T_{A(q)} N(d_0, \lambda_0)^\perp \to L^p_{1,0}(T^* B^4 \otimes \text{Ad}(P)) \cap T_{A(q)} N(d_0, \lambda_0)^\perp,$$

by

$$F(a) := Q \left( \frac{1}{2} \nabla^2 y M_\epsilon(A(q) + a) + d_{A(q)}^* d_{A(q)}^* a \right).$$

We show that

$$F'(0) = Q \left( \frac{1}{2} \nabla^2 y M_\epsilon(A(q)) + d_{A(q)}^* d_{A(q)}^* a \right) : L^p_{1,0}(T^* B^4 \otimes \text{Ad}(P)) \cap T_{A(q)} N(d_0, \lambda_0)^\perp \to L^p_{1,0}(T^* \otimes \text{Ad}(P)) \cap T_{A(q)} N(d_0, \lambda_0)^\perp$$

is an isomorphism. Moreover, for our purpose, we shall give an estimate of the inverse norm for each $q \in \mathcal{P}(d_0, \lambda_0; D_1, D_2; \epsilon)$ which depends only on $d_0, D_1, D_1$ and $\epsilon$. To prove the invertibility, suppose that $F'(0) a = 0$ for $a \in L^p_{1,0}(T^* B^4 \otimes \text{Ad}(P)) \cap T_{A(q)} N(d_0, \lambda_0)^\perp$. We then have

$$0 = \langle F'(0) a, a \rangle = \frac{1}{2} \langle \nabla^2 y M_\epsilon(A(q)) a, a \rangle + \langle d_{A(q)}^* d_{A(q)}^* a, d_{A(q)}^* d_{A(q)}^* a \rangle \geq C\|a\|_{A(q):1,2,B^4}^2,$$

where the last inequality comes from Lemma 3.4. Therefore, $a = 0$ and $F'(0)$ is one to one.

To show that $F'(0)$ is onto, one needs to solve the equation

$$F'(0) a = Q \left( \frac{1}{2} \nabla^2 y M_\epsilon(A(q)) a + d_{A(q)}^* d_{A(q)}^* a \right) = b$$

for arbitrary $b \in L^p_{1,0}(T^* \otimes \text{Ad}(P)) \cap T_{A(q)} N(d_0, \lambda_0)^\perp$. This is equivalent to

$$\langle Q \left( \frac{1}{2} \nabla^2 y M_\epsilon(A(q)) a + d_{A(q)}^* d_{A(q)}^* a \right), \varphi \rangle = \langle b, \varphi \rangle := (b, \varphi)_{A(q):1,2,B^4}$$  (3.100)

for all $\varphi \in L^p_{1,0}(T^* B^4 \otimes \text{Ad}(P))$.

By a density argument, it is sufficient to show the existence of $a \in L^p_{1,0}(T^* B^4 \otimes \text{Ad}(P)) \cap T_{A(q)} N(d_0, \lambda_0)^\perp$ such that (3.100) holds for $\varphi \in L^2_{1,0}(T^* \otimes \text{Ad}(P))$. Since (3.100) is always satisfied for $\varphi \in T_{A(q)} N(d_0, \lambda_0)$, we may also assume that $\varphi \in L^2_{1,0}(T^* B^4 \otimes \text{Ad}(P)) \cap T_{A(q)} N(d_0, \lambda_0)^\perp$. Solving (3.100) is equivalent to finding a critical point for the functional $a \mapsto \frac{1}{2} \langle \nabla^2 y M_\epsilon(A(q)) a, a \rangle + \frac{1}{2} \langle d_{A(q)}^* d_{A(q)}^* a, a \rangle - \langle b, a \rangle$ in $L^p_{1,0}(T^* B^4 \otimes \text{Ad}(P)) \cap T_{A(q)} N(d_0, \lambda_0)^\perp$. By Lemma 3.3, there is a unique critical point $a \in L^2_{1,0}(T^* B^4 \otimes \text{Ad}(P)) \cap T_{A(q)} N(d_0, \lambda_0)^\perp$, which also satisfies the estimate

$$\|a\|_{A(q):1,2,B^4} \leq C\|b\|_{A(q):1,2,B^4} \leq C\|b\|_{A(q):1,p;B^4}$$  (3.101)
We show that this solution is actually in $L^0_{1,0}(T^* B^4 \otimes \text{Ad}(P))$. By the Weitzenböck formula, (3.100) can be written as

$$\langle \nabla_{A(q)}^\epsilon a, \nabla_{A(q)}^\epsilon \varphi \rangle_{2,B^4} = -\epsilon \langle F_{A(q)}^\epsilon, [a, \varphi] \rangle_{2,B^4} + \epsilon \langle \{ F_{A(q)}^\epsilon, \varphi \}, a \rangle_{2,B^4} + \langle b, \varphi \rangle.$$ 

Hölder’s inequality and Sobolev embeddings yield

$$|\langle \nabla_{A(q)}^\epsilon a, \nabla_{A(q)}^\epsilon \varphi \rangle_{2,B^4}| \leq C \| \epsilon F_{A(q)}^\epsilon \|_{p,B^4} \| a \|_{A(q):1,2,B^4} \| \varphi \|_{A(q):1,p',B^4} + \| b \|_{A(q):1,p,B^4} \| \varphi \|_{A(q):1,p',B^4},$$

where $C > 0$ depends only on $d_0, \lambda_0, D_1$ and $D_2$.

On the other hand, by Hölder’s inequality and Sobolev embedding,

$$|\langle a, \varphi \rangle_{2,B^4}| \leq C \| a \|_{A(q):1,2,B^4} \| \varphi \|_{A(q):1,p',B^4}$$

for $2 < p \leq 4$, where $C > 0$ is an absolute constant.

From (3.102), (3.103), it follows that

$$\int_{B^4} \langle \nabla_{A(q)}^\epsilon a, \nabla_{A(q)}^\epsilon \varphi \rangle + \langle a, \varphi \rangle \, dx \leq C \left(1 + \| \epsilon F_{A(q)}^\epsilon \|_{p,B^4}\right) (\| a \|_{A(q):1,2,B^4} + \| b \|_{A(q):1,p,B^4}) \| \varphi \|_{A(q):1,p',B^4},$$

thus (by (3.101)), $(\nabla_{A(q)}^\epsilon \nabla_{A(q)}^\epsilon + 1)a \in L^p_{-1}(T^* B^4 \otimes \text{Ad}(P))$ with $\| (\nabla_{A(q)}^\epsilon \nabla_{A(q)}^\epsilon + 1)a \|_{A(q):-1,p,B^4} \leq C(1 + \| \epsilon F_{A(q)}^\epsilon \|_{p,B^4})$ for some constant $C > 0$ depending only on $d_0, D_1$ and $D_2$, it finally follows that $\| a \|_{A(q):1,p',B^4} \leq C(1 + \| \epsilon F_{A(q)}^\epsilon \|_{p,B^4})$ for some constant $C > 0$ depending only on $d_0, \lambda_0, D_1, D_2$. This completes the proof of the invertibility of $F'(0)$ and the estimate of the norm of $F'(0)^{-1}$ for $q \in \mathcal{P}(d_0, \lambda_0; D_1, D_2; \epsilon)$.

From this, the existence of $a = a(q) \in L^p_{1,0}(T^* B^4 \otimes \text{Ad}(P)) \cap T_{A(q)}N(d_0, \lambda_0)^\perp$ satisfying (3.97) follows directly from the contraction mapping theorem. In fact, $F(a) = 0$ can be written as

$$QF'(0)a = -Q\left(\frac{1}{2} \nabla \mathcal{M}_\epsilon(A(q))\right) - Q\left(\frac{1}{2} R(q; a) + R_2(q; a)\right),$$

where $R_2(q; a) = d_{A(q) + \epsilon}^* d_{A(q) + \epsilon}^* a - d_{A(q)}^* d_{A(q)}^* a$, and, since $QF'(0) : L^p_{1,0}(T^* B^4 \otimes \text{Ad}(P)) \cap T_{A(q)}N(d_0, \lambda_0)^\perp \to L^p_{1,0}(T^* B^4 \otimes \text{Ad}(P)) \cap T_{A(q)}N(d_0, \lambda_0)^\perp$ admits an inverse, (3.105) is equivalent to

$$a = T(q; a) := -(QF'(0))^{-1} Q\left(\frac{1}{2} \nabla \mathcal{M}_\epsilon(A(q))\right) - (QF'(0))^{-1} Q\left(\frac{1}{2} R(q; a) + R_2(q; a)\right),$$

where $T(q; \cdot)$ is a contraction from the ball of radius $\delta$ in $L^p_{1,0}(T^* B^4 \otimes \text{Ad}(P)) \cap T_{A(q)}N(d_0, \lambda_0)^\perp$ into itself, if $p > 2$ is close to 2 and $\delta$ is sufficiently small. To show this, for $b \in L^p_{1,0}(T^* B^4 \otimes \text{Ad}(P))$ we write

$$\left\langle \frac{1}{2} R(q; a_1) + R_2(q; a_1) - \frac{1}{2} R(q; a_2) - R_2(q; a_2), b \right\rangle =$$

$$\epsilon \int_{B^4} \left[ (d_{A(q)}^\epsilon a_1, [a_1, b]) - (d_{A(q)}^\epsilon a_2, [a_2, b]) \right] \, dx + \epsilon \int_{B^4} \left[ (d_{A(q)}^\epsilon a_1, d_{A(q)}^\epsilon b) - ([a_1, a_2], d_{A(q)}^\epsilon b) \right] \, dx +$$

$$\epsilon^2 \int_{B^4} \left[ ([a_1, a_1], [a_1, b]) - ([a_2, a_2], [a_2, b]) \right] \, dx + \epsilon \int_{B^4} \left( [d_{A(q)}^\epsilon a_1, *a_1] - [d_{A(q)}^\epsilon a_2, *a_2] \right) \, dx +$$

$$\epsilon \int_{B^4} \left( [a_1, *a_1], d_{A(q)}^\epsilon b \right) - ([a_2, *a_2], d_{A(q)}^\epsilon b) \, dx + \epsilon^2 \int_{B^4} \left( ([a_1, *a_1], [a_1, b]) - ([a_2, *a_2], [a_2, b]) \right) \, dx.$$
By the Sobolev embeddings $L^p_1 \subset L^{4p/(4-p)}$, $L^{p'}_1 \subset L^{4p/(3p-4)}$, and Hölder’s inequality, one obtains

$$\left| \frac{1}{2} R(q; a_1) + R_2(q; a_1) - \frac{1}{2} R(q; a_2) - R_2(q; a_2), b \right| \leq C \epsilon (\|a_1\|_{1,p} + \|a_2\|_{1,p} + \epsilon \|a_1\|_{1,p}^2 + \epsilon \|a_2\|_{1,p}^2) \|a_1 - a_2\|_{1,p} \|b\|_{1,p'}$$

for some $C > 0$ independent of $a_1$, $a_2$ and $\epsilon$. Combining the estimate of the norm of $F'(0)^{-1}$ as given above, the operator $T(q; \cdot)$ satisfies

$$\|T(q; a_1) - T(q; a_2)\|_{1,p} \leq C (1 + \epsilon^2 p^{-1}) (\|a_1\|_{1,p} + \|a_2\|_{1,p} + \epsilon \|a_1\|_{1,p}^2 + \epsilon \|a_2\|_{1,p}^2) \|a_1 - a_2\|_{1,p}$$

and

$$\|T(q; a)\|_{1,p} \leq C (1 + \epsilon^2 p^{-1}) (\|\nabla Y M_\epsilon (A(q))\|_{1,p} + \epsilon \|a\|_{1,p}^2 + \epsilon^2 \|a\|_{1,p}^3),$$

where we have the estimate of the form $\|\nabla Y M_\epsilon (A(q))\|_{1,p} \leq Ce^{\alpha(p)}$ as $\epsilon \to 0$, and $\alpha(p) \to 1/2$ as $p \to 2$ (c.f. Lemma 3.2 and its proof). It follows that there exists $\delta > 0$, $2 < p_0 < 4$ and $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$ and $2 < p < p_0$, $T(q; \cdot)$ is a contraction of the ball of radius $\delta$ in $L^p_1(T^* B^4 \otimes \text{Ad}(P)) \cap T_0(q) N(d_0, \lambda_0)$. By the contraction mapping theorem, for $0 < \epsilon < \epsilon_0$ there exists a unique solution of (3.100) in the ball of radius $\delta$. Note that $a(q)$ depends smoothly on $q$, by the implicit function theorem.

To prove the estimate (3.99), we observe that the auxiliary equation (3.97) yields

$$\left\{ \frac{1}{2} \nabla Y M_\epsilon (A(q) + a), a \right\} + (d^*_{A(q)} + a, d^*_{A(q)} + a)_{2;B^4} = 0, \quad (3.107)$$

where

$$(d^*_{A(q)} + a, d^*_{A(q)} + a)_{2;B^4} = (d^*_{A(q)} + a, d^*_{A(q)} + a)_{2;B^4} + 2\epsilon (d^*_{A(q)} + a, *[a, *a])_{2;B^4} + \epsilon^2 ([a, *a], [a, *a])_{2;B^4} \geq (d^*_{A(q)} + a, d^*_{A(q)} + a)_{2;B^4} - C \epsilon (\|a\|_{A(q);1,2,B^4}^3 + \epsilon \|a\|_{A(q);1,2,B^4}^4). \quad (3.108)$$

Combining (3.107), (3.108), Lemma 3.3 and Lemma 3.4 we finally obtain

$$C \|a\|_{A(q);1,2,B^4}^2 \leq - \langle \nabla Y M_\epsilon (A(q)), a \rangle - \langle R(q; a), a \rangle + C \epsilon (\|a\|_{A(q);1,2,B^4}^3 + \epsilon \|a\|_{A(q);1,2,B^4}^4) \leq \|\nabla Y M_\epsilon (A(q))\|_{A(q);1,2,B^4} \|a\|_{A(q);1,2,B^4} + C \epsilon (\|a\|_{A(q);1,2,B^4}^3 + \epsilon \|a\|_{A(q);1,2,B^4}^4),$$

thus the required estimate (3.99), since $\|a\|_{A(q);1,2,B^4} \leq C \|a\|_{A(q);1,p,B^4}$ is small.

This completes the proof. \hfill \Box

### 3.7 Estimates for $\|a_{q,i}(q)\|_{A(q);1,2,B^4}$, $1 \leq i \leq 8$

Let $a = a(q)$ be as in Lemma 3.3. We now estimate the directional derivatives of $a(q)$ in the direction $q_i$, denoted by $a_{q,i}(q)$. These estimates are needed to prove Proposition 3.2 which allows us to regard the problem of finding multiple solutions to $(D_c)$ as a finite dimensional problem, and are also needed in [13] where the latter is solved.
Lemma 3.10 The following estimates hold:
\[ \|a_{q_i}(q)\|_{A(q);1,2,B^4} \lesssim \varepsilon^{3/2} \quad \text{for } 1 \leq i \leq 4, \]  \hspace{1cm} (3.109)
\[ \|a_{q_i}(q)\|_{A(q);1,2,B^4} \lesssim \varepsilon \quad \text{for } 5 \leq i \leq 8. \]  \hspace{1cm} (3.110)

Proof: To prove this lemma, we write \( a_{q_i}(q) = a_{q_i}(q)^T + a_{q_i}(q)^\perp \), where \( a_{q_i}(q)^T = P a_{q_i}(q) \) and \( a_{q_i}(q)^\perp = Q a_{q_i}(q) \) are defined in the course of the proof of Lemma 3.2 (cf. 3.85), and estimate these components separately.

Estimate of \( \|a_{q_i}(q)^T\|_{A(q);1,2,B^4} \): For this estimate we need the following lemma, proved in [14].

Lemma 3.11 Let \( a_i(q) \) be the element of the orthonormal basis constructed in Lemma 3.2 in [14]. The following estimates hold:
\[ \|a_{i q_j}(q)^\perp\|_{A(q);1,2,B^4} \lesssim \varepsilon \] for \( 1 \leq i, j \leq 8 \), where \( a_{i q_j}(q) \) denotes the directional derivative of \( a_i(q) \) in the direction \( q_j \).

Since \( a(q) \in L^2_{1,0}(T^*B^4 \otimes \text{Ad}(P)) \cap T_{A(q)}\mathbb{N}(d_0, \lambda_0)^\perp \), one has \( \langle a(q), a_i(q) \rangle_{A(q);1,2,B^4} = 0 \) for \( 1 \leq i \leq 8 \). Differentiating this with respect to \( q_j \), one obtains
\[
\begin{align*}
(a_{q_j}(q), a_i(q))_{A(q);1,2,B^4} + (a(q), a_{i q_j}(q))_{A(q);1,2,B^4} + \epsilon (|a_j(q), a(q)|, A_{A(q)})^T a_i(q)_{2,B^4} + \epsilon (\nabla_{A(q})^T a(q), [a_j(q), a_i(q)])_{1,B^4} = 0. \tag{3.112}
\end{align*}
\]

By Lemma 3.2 (3.99), and Lemma 3.11 the second term of (3.112) is estimated as
\[ |(a(q), a_{i q_j}(q))_{A(q);1,2,B^4}| \leq \|a(q)\|_{A(q);1,2,B^4}\|a_{i q_j}(q)^\perp\|_{A(q);1,2,B^4} \lesssim \varepsilon^{3/2}; \]  \hspace{1cm} (3.113)
while the third and fourth terms are estimated as
\[ \epsilon |(a_j(q), a(q)|, A_{A(q)})^T a_i(q)_{2,B^4} | \leq C \epsilon \|a_j(q)\|_{4,B^4} \|a(q)\|_{4,B^4} \|\nabla_{A(q})^T a_i(q)\|_{2,B^4} \leq C \epsilon \|a_j(q)\|_{A(q);1,2,B^4} \|a(q)\|_{A(q);1,2,B^4} \|a_i(q)\|_{A(q);1,2,B^4} \lesssim \varepsilon^{3/2} \]  \hspace{1cm} (3.114)
and
\[ \epsilon |(\nabla_{A(q})^T a(q), [a_j(q), a_i(q)])_{2,B^4}| \lesssim \varepsilon^{3/2}. \]  \hspace{1cm} (3.115)

From (3.112)–(3.115), it follows
\[ |(a_{q_j}(q), a_i(q))_{A(q);1,2,B^4}| \lesssim \varepsilon^{3/2} \quad \text{for } 1 \leq i \leq 8 \]  \hspace{1cm} (3.116)
which yields finally
\[ \|a_{q_j}(q)^T\|_{A(q);1,2,B^4} \lesssim \varepsilon^{3/2} \quad \text{for } 1 \leq j \leq 8. \]  \hspace{1cm} (3.117)

Estimate of \( \|a_{q_i}(q)^\perp\|_{A(q);1,2,B^4} \): To estimate \( \|a_{q_i}(q)^\perp\|_{A(q);1,2,B^4} \), recall that \( a(q) \) satisfies the auxiliary equation (3.97), thus there exist \( c_i(q) \in \mathbb{R} \) \( (1 \leq i \leq 8) \) such that the following holds:
\[ \frac{1}{2} \nabla y M C (A(q) + a(q)) + d_{A(q) + a(q)} e d_{A(q) + a(q)} e a(q) = \sum_{i=1}^{8} c_i(q) a_i(q). \]  \hspace{1cm} (3.118)

From now on, we simply write \( A = A(q), a = a(q), a_i = a_i(q) \) and \( c_i = c_i(q) \).

We now estimate \( |c_i| \) for \( 1 \leq i \leq 8 \).
Lemma 3.12 We have $|c_i(q)| \lesssim \epsilon^{1/2}$ for $1 \leq i \leq 8$.

Proof of Lemma 3.12 By (3.118), (3.99) and Lemmas 3.2, 3.3

$$|c_i| = \left| \frac{1}{2} \nabla y M_\epsilon(A + a, a_i) + (d^*_{A+a} \epsilon a, d^*_{A+a} \epsilon a_i)_{2;B^4} \right|$$

$$\leq C(\|\nabla y M_\epsilon(A)\|_{A;1,2} + \|\nabla^2 y M_\epsilon(A)\|_{A;1,2} a \|A\|_{A;1,2;B^4} + \epsilon^2 \|a\|^2_{A;1,2;B^4}$$

$$\leq C(\|\nabla y M_\epsilon(A)\|_{A;1,2} + \|\nabla^2 y M_\epsilon(A)\|_{A;1,2} a \|A\|_{A;1,2;B^4}) \lesssim \epsilon^{1/2}.$$  

(3.119)

This completes the proof of Lemma 3.12 $\square$

By differentiating (3.118) in the direction of $q$, and taking the pairing with $a_{q_i}$, one obtains

$$\langle \frac{1}{2} \nabla^2 y M_\epsilon(A + a)(a_i + a_{q_i}), a_{q_i} \rangle + \langle d^*_{A+a} \epsilon a_{q_i}, a_{q_i} \rangle$$

$$+ \epsilon \langle [a_i + a_{q_i}, *a], a_{q_i} \rangle = \sum_{j=1}^8 c_j(a_j q_i, a_{q_i})_{A;1,2;B^4},$$  

(3.120)

thus, by expanding the left hand side of (3.120),

$$\mathcal{H}_A(a_{q_i}, a_{q_i}^\perp) + \frac{1}{2} \langle \nabla^2 y M_\epsilon(A) a_i, a_{q_i}^\perp \rangle + \epsilon (d^*_{A+a} \epsilon a_{q_i}, [a, *a_{q_i}])_{2;B^4}$$

$$+ \epsilon (d^*_{A+a} \epsilon a_{q_i}, [a, *a_{q_i}])_{2;B^4} + \frac{1}{2} \langle (\nabla^2 y M_\epsilon(A) - \nabla^2 y M_\epsilon(A))(a_i + a_{q_i}), a_{q_i} \rangle$$

$$+ \epsilon ([a_i + a_{q_i}, d^*_{A+a} \epsilon a], a_{q_i}^\perp)_{2;B^4} + \epsilon ([a_i + a_{q_i}, *a], d^*_{A+a} \epsilon a_{q_i})(a_i + a_{q_i}, a_{q_i}^\perp)_{2;B^4} = \sum_{j=1}^8 c_j(a_j q_i, a_{q_i})_{A;1,2;B^4}.$$  

(3.121)

By (3.96), there holds

$$\mathcal{H}_A(a_{q_i}, a_{q_i}^\perp) = \mathcal{H}_A(a_{q_i}, q_i) - \mathcal{H}_A(a_{q_i}, a_{q_i}^\perp) \geq C \|a_{q_i}^\perp\|^2_{A;1,2;B^4} - C \|a_{q_i}^\perp\|^2_{A;1,2;B^4} - \mathcal{H}_A(a_{q_i}, a_{q_i}^\perp).$$  

(3.122)

Combining the above with (3.121), one obtains

$$C \|a_{q_i}^\perp\|^2_{A;1,2;B^4} \lesssim \mathcal{H}_A(a_{q_i}, a_{q_i}^\perp) + C \|a_{q_i}^\perp\|^2_{A;1,2;B^4} + \frac{1}{2} \langle (\nabla^2 y M_\epsilon(A) a_i, a_{q_i}^\perp) \rangle$$

$$+ \epsilon (d^*_{A+a} \epsilon a_{q_i}, [a, *a_{q_i}])_{2;B^4} + \epsilon ([a_i + a_{q_i}, d^*_{A+a} \epsilon a_{q_i}])_{2;B^4}$$

$$+ \epsilon ([a_i + a_{q_i}, *a], d^*_{A+a} \epsilon a_{q_i})(a_i + a_{q_i}, a_{q_i}^\perp)_{2;B^4} + \sum_{j=1}^8 c_j(a_j q_i, a_{q_i})_{A;1,2;B^4}.$$  

(3.123)

We now estimate each term in (3.123).

Applying (3.117), we have

$$|\mathcal{H}_A(a_{q_i}, a_{q_i}^\perp)| \lesssim \|a_i\|_{A;1,2;B^4} \|a_{q_i}^\perp\|_{A;1,2;B^4} \lesssim \epsilon^{3/2} \|a_{q_i}^\perp\|_{A;1,2;B^4} + \epsilon^3.$$  

(3.124)
The third term in (3.123) is estimated as (we write $\tilde{A} = \tilde{A}(q)$)

$$|\langle \nabla^2 y M_\epsilon (A) a_i, a_{q_i}^\perp \rangle| \leq |\langle \nabla^2 y M_\epsilon (A) - \nabla^2 y M_\epsilon (\tilde{A}) a_i, a_{q_i}^\perp \rangle| + |\langle \nabla^2 y M_\epsilon (\tilde{A}) a_i, a_{q_i}^\perp \rangle|$$

$$\leq |\langle \nabla^2 y M_\epsilon (A) - \nabla^2 y M_\epsilon (\tilde{A}) a_i, a_{q_i}^\perp \rangle|_{A;1,2,*} + |\langle \nabla^2 y M_\epsilon (\tilde{A}) (a_i - \tilde{a}_i), a_{q_i}^\perp \rangle|_{A;1,2,B^4}$$

$$\lesssim \epsilon^{3/2} \|a_{q_i}^\perp\|_{A;1,2,B^4} + \|a_i - \tilde{a}_i\|_{A;1,2,B^4} \|a_{q_i}^\perp\|_{A;1,2,B^4},$$

(3.125)

where we have used $\nabla^2 y M_\epsilon (\tilde{A}) \tilde{a}_i = 0$ and Lemma 3.8.

By (3.99), (3.117) and Lemma 3.2, the fourth term is estimated as

$$\epsilon |(d_A^\epsilon a_{q_i}, [a, a_{q_i}^\perp])_{2,B^4}| \lesssim \epsilon \|a_{q_i}\|_{A;1,2,B^4} \|a\|_{A;1,2,B^4} \|a_{q_i}^\perp\|_{A;1,2,B^4}$$

$$\lesssim \epsilon^{3/2} \|a_{q_i}\|_{A;1,2,B^4} \|a_{q_i}^\perp\|_{A;1,2,B^4} \lesssim \epsilon^3 \|a_{q_i}^\perp\|_{A;1,2,B^4} + \epsilon^{3/2} \|a_{q_i}^\perp\|_{A;1,2,B^4}^2.$$  

(3.126)

The fifth and the sixth terms are estimated similarly:

$$\epsilon |([a, *a_{q_i}], d_A^\epsilon a_{q_i})_{2,B^4}| \lesssim \epsilon^3 \|a_{q_i}^\perp\|_{A;1,2,B^4} + \epsilon^{3/2} \|a_{q_i}^\perp\|_{A;1,2,B^4}^2$$

(3.127)

and

$$\epsilon^2 |([a, *a_{q_i}], [a, *a_{q_i}])_{2,B^4}| \lesssim \epsilon^{9/2} \|a_{q_i}^\perp\|_{A;1,2,B^4} + \epsilon^3 \|a_{q_i}^\perp\|_{A;1,2,B^4}^2.$$  

(3.128)

To estimate the seventh term, we first observe that

$$\frac{1}{2} \langle \nabla^2 y M_\epsilon (A + a) - \nabla^2 y M_\epsilon (A), \alpha, \beta \rangle \leq \int_{B^4} \langle d_A^\epsilon \alpha, [a, \beta] \rangle + \epsilon \int_{B^4} \langle [a, \alpha], d_A^\epsilon \beta \rangle$$

$$+ \epsilon^2 \int_{B^4} \langle [a, \alpha], [a, \beta] \rangle + \epsilon \int_{B^4} \langle d_A^\epsilon a, [a, \beta] \rangle + \frac{\epsilon^2}{2} \int_{B^4} \langle [a, a], [\alpha, \beta] \rangle$$

$$\lesssim \epsilon \|a\|_{A;1,2,B^4} \|\beta\|_{A;1,2,B^4} \|a\|_{A;1,2,B^4} + \epsilon^2 \|\alpha\|_{A;1,2,B^4} \|\alpha\|_{A;1,2,B^4} \|a\|_{A;1,2,B^4}^2$$

$$\lesssim \epsilon^{3/2} \|a\|_{A;1,2,B^4} \|\beta\|_{A;1,2,B^4}$$

(3.129)

and

$$\|\nabla^2 y M_\epsilon (A + a) - \nabla^2 y M_\epsilon (A)\|_{A;1,2,*} \lesssim \epsilon^{3/2}.$$  

(3.130)

Thus,

$$|\langle \nabla^2 y M_\epsilon (A + a) - \nabla^2 y M_\epsilon (A), a_i + a_{q_i} \rangle| \lesssim \epsilon^{3/2} (1 + \|a_{q_i}\|_{A;1,2,B^4}) \|a_{q_i}^\perp\|_{A;1,2,B^4}$$

$$\lesssim \epsilon^{3/2} \|a_{q_i}^\perp\|_{A;1,2,B^4} + \epsilon^{3/2} \|a_{q_i}^\perp\|_{A;1,2,B^4}^2.$$  

(3.131)

The eighth and the ninth terms are estimated similarly:

$$\epsilon |([a_i + a_{q_i}, d_{A+a}^\epsilon a_{q_i})_{2,B^4}| \lesssim \epsilon^{3/2} \|a_{q_i}^\perp\|_{A;1,2,B^4} + \epsilon^{3/2} \|a_{q_i}^\perp\|_{A;1,2,B^4}^2$$

(3.132)

and

$$\epsilon |([a_i + a_{q_i}, a_{q_i}^\perp]_{2,B^4}| \lesssim \epsilon^{3/2} \|a_{q_i}^\perp\|_{A;1,2,B^4} + \epsilon^{3/2} \|a_{q_i}^\perp\|_{A;1,2,B^4}^2.$$  

(3.133)

Finally, by Lemma 3.12 and (3.111), the last term is estimated as

$$\sum_{j=1}^8 |c_j| \langle a_{j,q_i}, a_{q_i}^\perp\rangle_{A;1,2,B^4} = \sum_{j=1}^8 |c_j| \langle a_{j,q_i}, a_{q_i}^\perp\rangle_{A;1,2,B^4} \lesssim \epsilon^{3/2} \|a_{q_i}^\perp\|_{A;1,2,B^4}.$$  

(3.134)
From (3.123)–(3.134), we obtain
\[ \|a_{\perp_{q_i}}\|_{A;1;2;B^4} \lesssim \epsilon^{3/2} \|a_{\perp_{q_i}}\|_{A;1;2;B^4} + \epsilon^3 + \|a_i - \tilde{a}_i\|_{A;1;2;B^4} \lesssim \epsilon^{3/2} \|a_{\perp_{q_i}}\|_{A;1;2;B^4} . \] (3.135)

Thus, for small \( \epsilon > 0 \),
\[ \|a_{\perp_{q_i}}\|_{A;1;2;B^4} \lesssim \epsilon^{3/2} + \|a_i - \tilde{a}_i\|_{A;1;2;B^4} . \] (3.136)

From (3.136) and Lemma 3.7, one obtains the estimates
\[ \|a_{\perp_{q_i}}\|_{A;1;2;B^4} \lesssim \epsilon^{3/2} \quad \text{for } 1 \leq i \leq 4 \] (3.137)
and
\[ \|a_{\perp_{q_i}}\|_{A;1;2;B^4} \lesssim \epsilon \quad \text{for } 5 \leq i \leq 8. \] (3.138)

Finally, by combining (3.117), (3.137), (3.138), we complete the proof of Lemma 3.10

3.8 Natural constraints

In this section, we prove that the manifold \( \{A(q) + a(q) : q \in \mathcal{P}(d_0, \lambda_0; D_1, D_2; \epsilon)\} \) is a natural constraint for \( \mathcal{YM}_\epsilon \) if \( \epsilon > 0 \) is small, more precisely, we prove the following proposition which allows us to transform the \( \epsilon \)-Dirichlet problem for the Yang Mills functional into a finite dimensional problem.

**Proposition 3.2** There exists \( \epsilon_0 > 0 \) such that for \( 0 < \epsilon < \epsilon_0 \), \( q \in \mathcal{P}(d_0, \lambda_0; D_1, D_2; \epsilon) \) is a critical point for the function
\[ J_\epsilon(q) := \epsilon^2 \mathcal{YM}_\epsilon(A(q) + a(q)) \]
on \( \mathcal{P}(d_0, \lambda_0; D_1, D_2; \epsilon) \) if and only if \( A(q) + a(q) \) is a Yang Mills connection, where \( a(q) \) is given by Lemma 3.7.

**Proof:** It is obvious that if \( A(q) + a(q) \) is Yang Mills, then \( q \) is a critical point for \( J_\epsilon \). Assume on the other hand that \( q \in \mathcal{P}(d_0, \lambda_0; D_1, D_2; \epsilon) \) is a critical point for \( J_\epsilon \). Then,
\[ 0 = \langle J'_\epsilon(q), q_j \rangle = \epsilon^2 \nabla \mathcal{YM}_\epsilon(A(q) + a(q)) (a_i + a_{q_i}(q)) \quad \text{for } 1 \leq j \leq 8. \] (3.139)

By (3.118), we have \( \frac{1}{2} \nabla \mathcal{YM}_\epsilon(A(q) + a(q)) = \sum_{i=1}^8 c_i(q) a_i \) on \( T_{A(q) + a(q)} \mathcal{YM}_{k+1}(A_0) \). Thus, by (3.139) and Lemma 3.10
\[ 0 = \sum_{i=1}^8 c_i(q) (a_i(q), a_j(q) + a_{q_j}(q))_{A(q) + a(q);1;2;B^4} \]
\[ = \sum_{i=1}^8 c_i(q) \left( (a_i(q), a_j(q) + a_{q_j}(q))_{A(q);1;2;B^4} + (\nabla_{A(q)} a_i(q), c[A(q), a_j(q) + a_{q_j}(q)])_{2;B^4} \right. \]
\[ + (c[a(q), a_i(q)], \nabla_{A(q)} a_j(q) + a_{q_j}(q))_{2;B^4} + (\epsilon[a(q), a_i(q)], c[a(q), a_j(q)])_{2;B^4} \right) \]
\[ = \sum_{i=1}^8 c_i(q) \delta_{ij} + o(|c(q)|) = c_j(q) + o(|c(q)|) \quad \text{for } 1 \leq j \leq 8. \] (3.140)
This implies $c_j(q) = 0$ for $1 \leq j \leq 8$, if $\epsilon > 0$ is small, and $\nabla_y M_\epsilon(A(q) + a(q)) = 0$ in $T_{A(q) + a(q)} \mathbb{R}^{b}_{k+1}(A_0)$, thus $A(q) + a(q)$ is Yang Mills.

This completes the proof. □

References

[1] A. Ambrosetti, A. Malchiodi: *Perturbation Methods and Semilinear Elliptic Problems on $\mathbb{R}^n$*. Progress in Math. 240, Birkhäuser, Basel-Boston-Berlin, 2006.

[2] T. Aubin: *Some nonlinear problems in Riemannian Geometry*. Springer Monographs in Mathematics. Springer-Verlag Berlin Heidelberg New York.

[3] G. Bor: *Yang Mills fields which are not self dual*. Commun. Math. Phys. 145 (1992), 393-410.

[4] H. Brezis: *Some variational problems with lack of compactness*. Proc. Symp. Pure Math. 45 (1986) 165–201.

[5] S. K. Donaldson, P. B. Kronheimer: *The geometry of four-manifolds*. Oxford University Press, Oxford (1990).

[6] D. S. Freed, K. Uhlenbeck: *Instantons and 4-manifolds*, 2nd edition. Springer-Verlag, New York-Berlin-Heidelberg (1991).

[7] D. Gilberg, N. Trudinger: *Elliptic Partial Differential Equations of Second Order*. 2nd ed. Springer-Verlag, New York (1983).

[8] T. Isobe: *Non-existence and uniqueness results for boundary value problems for Yang Mills connections*. Proc. Amer. Math. Soc. 125 (1997), 1737-1744.

[9] T. Isobe: *Classification of blow-up points and multiplicity of solutions for $H$-systems*. Comm. Partial Differential Equations 25 (2000), 1259–1325.

[10] T. Isobe: *On the asymptotic analysis of $H$-systems, II: The constructions of large solutions*. Adv. Diff. Eq. 6 (2001), 641–700.

[11] T. Isobe: *Multiple solutions for the Dirichlet problem for $H$-systems with small $H$*. Comm. Contemporary Math. 6 (2004), 579–600.

[12] T. Isobe, A. Marini: *On topologically distinct solutions of the Dirichlet problem for Yang Mills connections*. Car. Var. 5 (1997), 345–358.

[13] T. Isobe, A. Marini: *Small coupling limit and multiple solutions to the Dirichlet Problem for Yang Mills connections in 4 dimensions - Part II.*

[14] T. Isobe, A. Marini: *Small coupling limit and multiple solutions to the Dirichlet Problem for Yang Mills connections in 4 dimensions - Part III.*
[15] A. Marini: *Dirichlet and Neumann boundary problems for Yang Mills connections*. Comm. Pure and Appl. Math. **45** (1992), 1015–1050.

[16] T. H. Parker: *A Morse theory for equivariant Yang Mills*. Duke Math. **66** (1992), 337-355.

[17] L. H. Ryder: *Quantum field theory*. Cambridge University Press, Cambridge (1996).

[18] L. Sadun: *A symmetric family of Yang Mills fields*. Commun. Math. Phys. **163** (1994), 257–291.

[19] L. Sadun, J. Segert: *Non-self dual Yang Mills connections with quadrupole symmetry*. Commun. Math. Phys. **145** (1992), 363–391.

[20] L. M. Sibner, J. R. Sibner, K. Uhlenbeck: *Solutions to Yang Mills equations which are not self dual*. Proc. Nat. Acad. Sci. USA **86** (1989), 8610–8613.

[21] M. Struwe: *Variational Methods. Applications to nonlinear partial differential equations and Hamiltonian systems*, 4th edn. Springer, Berlin, Heidelberg, New York (2008).

[22] C. H. Taubes: *Self dual Yang Mills connections over non-self dual 4-manifolds*. J. Differ. Geom. **17**, 139–170 (1982).

[23] C. H. Taubes: *Self dual connections on manifolds with indefinite intersection matrix*. J. Differ. Geom. **19**, 517–560 (1984).

[24] C. H. Taubes: *A framework for Morse theory for the Yang Mills functional*. Invent. Math. **94** (1988), 327–402.

[25] H-Y. Wang: *The existence of non-minimal solutions to the Yang Mills equation with group SU(2) on S^2 x S^2 and S^1 x S^3*. J. Differential Geom. **34** (1991), 701-767.

[26] S. Weinberg: *The quantum theory of fields*. Vol. 2. Cambridge University Press, Cambridge (1996).