New LCD MDS codes of non-Reed-Solomon type
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Abstract—Both linear complementary dual (LCD) codes and maximum distance separable (MDS) codes have good algebraic structures, and they have interesting practical applications such as communication systems, data storage, quantum codes, and so on. So far, most of LCD MDS codes have been constructed by employing generalized Reed-Solomon codes. In this paper we construct some classes of new Euclidean LCD MDS codes and Hermitian LCD MDS codes which are not monomially equivalent to Reed-Solomon codes, called LCD MDS codes of non-Reed-Solomon type. Our method is based on the constructions of Beelen et al. (2017) and Roth and Lempel (1989). To the best of our knowledge, this is the first paper on the construction of LCD MDS codes of non-Reed-Solomon type; any LCD MDS code of non-Reed-Solomon type constructed by our method is not monomially equivalent to any LCD code constructed by the method of Carlet et al. (2018).

Index Terms—Linear complementary dual codes, LCD codes, MDS codes, non-Reed-Solomon codes, Reed-Solomon codes

I. INTRODUCTION

The concept of linear complementary dual (LCD) codes was introduced by Massey in 1992 [18], and they have interesting applications in communication systems, cryptography, and data storage. In particular, Carlet et al. [5, 6] found a new application of binary LCD codes in implementations against side-channel and fault injection attacks. Since then, LCD codes have attracted great attention from researchers in coding community. Yang and Massey [25] showed a necessary and sufficient condition for a cyclic code to guarantee LCD property as well. The authors of [14, 13] constructed two families of both LCD and BCH codes.

A maximum distance separable (MDS) code has the greatest error correcting capability when its length and dimension is fixed. MDS codes are extensively used in communications (for example, Reed-Solomon codes are all MDS codes), and they have good applications in minimum storage codes and quantum codes. There are many known constructions for MDS codes; for instance, Generalized Reed-Solomon (GRS) codes [19], based on the equivalent problem of finding $n$-arcs in projective geometry [17], circulant matrices [20], Hankel matrices [21], or extending GRS codes.

Both LCD codes and MDS codes have good algebraic structures, and they have interesting practical applications as mentioned above. Until now, constructions of most of LCD MDS codes have been achieved by using generalized Reed-Solomon codes (GRS codes) since GRS codes are all MDS. We discuss recent research progresses on LCD MDS codes as follows.

(1) Jin [13] and Shi et al. [24] constructed several LCD MDS codes using generalized Reed-Solomon codes with some additional conditions on special polynomials.

(2) Chen and Liu [10] made a different approach to obtain some LCD MDS codes from generalized Reed-Solomon codes, and he extended some results by Jin [13]. Afterwards, Luo et al. [16] and Fang et al. [11] further extended the results of Chen and Liu and investigated Euclidean and Hermitian hulls of MDS codes and their applications in quantum codes.

(3) Beelen and Jin [2] found an explicit construction of several LCD MDS codes in the odd characteristic case using the theory of algebraic function fields.

(4) Sari and Koroglu [22] constructed LCD codes of non-Reed-Solomon type with small dimension or codimension, self-orthogonal codes and generalized Reed-Solomon codes.

(5) Carlet et al. [8] obtained many parameters of Euclidean and Hermitian LCD MDS codes by using some linear codes with small dimension or codimension, self-orthogonal codes and generalized Reed-Solomon codes.

(6) Carlet et al. [9] introduced a general construction of LCD codes from any linear codes. More exactly speaking, if there is an $[n,k,d]$ linear code over $\mathbb{F}_q$ ($q > 3$) (respectively, over $\mathbb{F}_{q^2}$ ($q > 2$)), then there exists an $[n,k,d]$ Euclidean (respectively, Hermitian) LCD code over $\mathbb{F}_q$ (respectively, over $\mathbb{F}_{q^2}$).

In this paper, we construct some classes of new Euclidean LCD MDS codes and Hermitian LCD MDS codes which are not monomially equivalent to Reed-Solomon codes, called LCD MDS codes of non-Reed-Solomon type. To the best of our knowledge, this is the first paper on the construction of LCD MDS codes of non-Reed-Solomon type. In the coding theory, it is an important issue to find all inequivalent codes of the same parameters. We point out that any LCD MDS code of non-Reed-Solomon type constructed by our method is not monomially equivalent to any LCD code constructed by the method of Carlet et al. [9]. In particular, we construct some twisted Reed-Solomon codes or Roth-Lempel codes which are also LCD MDS codes of non-Reed-Solomon type; these codes cannot be constructed by the method in [9]. We also...
present some examples of non-Reed-Solomon LCD MDS codes, which are obtained by using our results and Magma implementation.

Our method is based on the constructions of Beelen et al. [3] and Roth and Lempel [20]. For construction of the LCD MDS codes of non-Reed-Solomon type, first we use some special matrices to form a generator matrix such that the product of the generator matrix and its (conjugate) transpose is as simple as possible. Secondly, we need to add some conditions to make sure that the product is nonsingular. Finally, we use the lifting of the finite field so that these codes also have the MDS property.

This paper is organized as follows. In Section 2, we recall some basic concepts on Euclidean and Hermitian LCD MDS codes and two constructions of these codes. In Section 3, we find some new Euclidean and Hermitian LCD MDS codes, which are not monomially equivalent to generalized Reed-Solomon codes. We finish this paper with a conclusion in Section 4.

II. Preliminaries

Let \( F_q \) be the finite field of order \( q \), where \( q \) is a power of an odd prime. An \([n,k]\) linear code \( C \) over \( F_q \) is a \( k \)-dimensional subspace of \( F_q^n \). The minimum distance \( d \) of a linear code \( C \) is bounded above by the so-called Singleton bound, that is, \( d \leq n-k+1 \). If \( d = n-k+1 \), then the code \( C \) is called a maximum distance separable (MDS) code.

For \( x \in F_q^n \), the conjugate of \( x \) is denoted by \( \overline{x} = x^T \). For a matrix \( A \), we denote by \( A^T \) the transpose of \( A \), and \( \overline{A} \) the matrix of conjugates of \( A \). For a set \( B = \{ x_1, x_2, \ldots, x_l \} \subseteq F_q^n \), we define \( \overline{B} = \{ x_1^T, x_2^T, \ldots, x_l^T \} \).

A. Equivalence of codes

We recall some equivalence notions of codes over the finite field \( F_q \) (see [12] Sections 1.6 and 1.7).

**Definition 2.1:** Let \( C_1 \) and \( C_2 \) be two linear codes of the same length over \( F_q \). Two linear codes \( C_1 \) and \( C_2 \) are permutation equivalent if there is a permutation matrix \( P \) such that \( G_1 \) is a generator matrix of \( C_1 \) if and only if \( G_1 P \) is a generator matrix of \( C_2 \).

Recall that a monomial matrix is a square matrix which has exactly one nonzero entry in each row and each column. A monomial matrix \( M \) can be written either in the form of \( DP \) or the form of \( PD' \), where \( D \) and \( D' \) are diagonal matrices and \( P \) is a permutation matrix.

**Definition 2.2:** Let \( C_1 \) and \( C_2 \) be two linear codes of the same length over \( F_q \), and let \( G_1 \) be a generator matrix of \( C_1 \). Then \( C_1 \) and \( C_2 \) are monomially equivalent if there is a monomial matrix \( M \) such that \( G_1 M \) is a generator matrix of \( C_2 \).

B. Euclidean and Hermitian LCD codes

Given a linear code \( C \) of length \( n \) over \( F_q \), the Euclidean dual code and the Hermitian dual code of \( C \) are defined by

\[ C^\perp = \{ (x_0, \ldots, x_{n-1}) = x \in F_q^n : \langle x, y \rangle_C = \sum_{i=0}^{n-1} x_i y_i = 0 \} \]

and

\[ C^{\perp H} = \{ (x_0, \ldots, x_{n-1}) = x \in F_q^n : \langle x, y \rangle_H = \sum_{i=0}^{n-1} x_i y_i = 0 \} \]

respectively.

A linear code \( C \) over \( F_q \) is called a Euclidean Linear Complementary Dual (Euclidean LCD) code if \( C \cap C^{\perp H} = \{0\} \), and it is called a Hermitian Linear Complementary Dual (Hermitian LCD) code if \( C \cap C^{\perp H} = \{0\} \).

**Lemma 2.3:** [8, Proposition 2] If \( G \) is a generator matrix of an \([n,k]\) linear code \( C \), then \( C \) is a Euclidean (resp. Hermitian) LCD code if and only if the \( k \times k \) matrix \( GG^T \) (resp. \( G^T G \)) is nonsingular.

C. Constructions of MDS codes

In this subsection, we recall some developments on constructions of MDS codes, which include generalized Reed-Solomon codes, twisted Reed-Solomon codes and Roth-Lempel codes as follows. Hereafter, we denote respectively by \( G_1 \) and \( G_2 \) the generator matrix of a twisted Reed-Solomon code and a Roth-Lempel code.

We begin with the well-known generalized Reed-Solomon codes.

**Definition 2.4:** Let \( \alpha_1, \ldots, \alpha_n \) be distinct elements in \( F_q \cup \{\infty\} \) and \( v_1, \ldots, v_n \) be nonzero elements in \( F_q \). For \( 1 \leq k \leq n \), the corresponding generalized Reed-Solomon (GRS) code over \( F_q \) is defined by

\[ \text{GRS}_k(\alpha, v) := \{ (v_1f(\alpha_1), \ldots, v_nf(\alpha_n)) : f(x) \in F_q[x], \deg(f(x)) < k \} \]

where \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in (F_q \cup \{\infty\})^n \) and \( v = (v_1, v_2, \ldots, v_n) \in (F_q)^n \), and the quantity \( f(\infty) \) is defined as the coefficient of \( x^{k-1} \) in the polynomial \( f \).

If \( v_i = 1 \) for every \( i = 1, \ldots, n \), then \( \text{GRS}_k(\alpha, v) \) is called a Reed-Solomon (RS) code. It is well-known that a generalized Reed-Solomon code \( \text{GRS}_k(\alpha, v) \) is an \([n, k, n-k+1]\) MDS code. In fact, \( \text{GRS}_k(\alpha, v) \) has a generator matrix as follows:

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \ldots & \alpha_n & 0 & 0 & \ldots & 0 \\
\alpha_1^{-1} & \alpha_2^{-1} & \ldots & \alpha_n^{-1} & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \ & \vdots & \ddots & \vdots \\
\alpha_1^{-t} & \alpha_2^{-t} & \ldots & \alpha_n^{-t} & 0 & 0 & \ldots & 0 \\
\end{pmatrix}
\]

In 2017, Beelen et al. [3] presented a generalization of Reed-Solomon codes, so-called twisted Reed-Solomon codes.

**Definition 2.5:** Let \( \eta \) be a nonzero element in the finite field \( F_q \). Let \( k, t, h \) be nonnegative integers such that \( 0 \leq h < k \leq q, k < n \), and \( 0 < t \leq n-k \). Let \( \alpha_1, \ldots, \alpha_n \) be distinct elements in \( F_q \cup \{\infty\} \), and we write \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \).

Then the corresponding twisted Reed-Solomon code over \( F_q \) of length \( n \) and dimension \( k \) is given by

\[ \omega(x, t, h, \eta) = (f(\alpha_1), \ldots, f(\alpha_n)) : f(x) = \sum_{i=0}^{h-1} a_i x^i + \eta a_n x^{k-1+1} \in F_q[x] \].

In fact,

\[ G_1 = \begin{pmatrix}
1 & 1 & \ldots & 1 \\
\alpha_1 & \alpha_2 & \ldots & \alpha_n \\
\alpha_1^{h-1} & \alpha_2^{h-1} & \ldots & \alpha_n^{h-1} \\
\alpha_1^{h-t} + \eta \alpha_2^{h-t} + \ldots + \alpha_n^{h-t} & \alpha_1^{h-t+1} + \eta \alpha_2^{h-t+1} + \ldots + \alpha_n^{h-t+1} & \ldots & \alpha_1^{h-1} + \eta \alpha_2^{h-1} + \ldots + \alpha_n^{h-1}
\end{pmatrix} \]
is the generator matrix of the twisted Reed-Solomon code $C_k(\alpha, t, h, \eta)$.

Note that in general, the twisted Reed-Solomon codes are not MDS. Beelen et al.\[3\] obtained some results on the twisted Reed-Solomon codes as follows:

**Lemma 2.6: [3] Theorem 17** Let $F_q \subset F_q$ be a proper subfield and $\alpha_1, \ldots, \alpha_n \in F_q$. If $\eta \in F_q \setminus F_q$, then the twisted Reed-Solomon code $C_k(\alpha, t, h, \eta)$ is MDS.

As indicated in [4] Remark 8, there is a mistake in the proof of [3] Theorem 18. Here we present an exact statement of [3] Theorem 18. To do so, recall from [3] Theorem 1 that any MDS code having a generator matrix of the form $[I_k | A]$, is a GRS if and only if all $3 \times 3$ minors of $A$ are zero, where $A = (A_{ij})$ and $A_{ij} = A_{ij}^{-1}$.

Let $\eta$ be in $F_q^*$ such that the twisted Reed-Solomon code $C_k(\alpha, t, h, \eta)$ is MDS. Let $[I_k | A]$ be a generator matrix of $C_k(\alpha, t, h, \eta)$. Let $M_i = \eta^{(i)}$ for $i = 1, 2, \ldots, l$ be all $3 \times 3$ minors of $A$, where $A = (A_{ij})$ and $A_{ij} = A_{ij}^{-1}$. Here $p_i, q_i$ are polynomials over $F_q$, and $q_i(\eta) \neq 0$.\[3\] Theorem 18]

**Lemma 2.7:** Let $\alpha_1, \ldots, \alpha_n \in F_q$ and $2 < k < n - 2$. Let $H = \{\eta \in F_q^* :$ the twisted Reed-Solomon code $C_k(\alpha, t, h, \eta)$ is MDS$.\}$. Assume that a certain $3 \times 3$ minor $M_i = \eta^{(i)}$ of $A$ defined in right above is nonzero for some $\eta \in H$. Then $C_k(\alpha, t, h, \eta)$ for such $\eta \in H$ is a non-Reed-Solomon code.

**Remark 2.8:** (1) In Lemma 2.7 if $p_i(\eta) = 0$ for any $3 \times 3$ minor of $A$, then $C_k(\alpha, t, h, \eta)$ for $\eta \in H$ is monomially equivalent to an RS code. By [3] Corollary 2, an $[n, k, n - k + 1]$ MDS code with $k < 3$ or $n - k < 3$, is monomially equivalent to an RS code. On the other hand, it was proved in [11] Corollary 9.2 that for a linear MDS code over $F_{p^n}$ with parameters $[p^n + 1, k, (\leq p), p^n - k + 2]$ is an RS code.

(2) To find a twisted MDS code of non-Reed-Solomon type, we need to check the minor assumption of Lemma 2.7.

Combining Lemma 2.6 and Lemma 2.7 we have.

**Lemma 2.9:** [3] Corollary 20] Let $F_q \subset F_q$ and $\alpha_1, \ldots, \alpha_n \in F_q$. Let $2 < k < n - 2$ and $n \leq s$. Assume that the minor condition for $\eta \in F_q \setminus F_q$ of Lemma 2.7 holds. Then $C_k(\alpha, t, h, \eta)$ is MDS but not monomially equivalent to an RS code.

**Remark 2.10:** In the next section, to obtain LCD MDS codes, we first study the LCD property of twisted Reed-Solomon codes, and then we use a suitable vector $\alpha$. Note that Lemma 2.6 shows the existence of twisted Reed-Solomon code, and in general, it is also hard to find an element $\eta \in F_q$ such that $C_k(\alpha, t, h, \eta)$ is a non-Reed-Solomon LCD MDS code even for small lengths.

Roth and Lempel\[20\] found a new construction of MDS codes of non-Reed-Solomon type. A set $S \subset F_q$ of size $m$ is called an $(m, t, \delta)$-set in $F_q$ if there exists an element $\delta \in F_q$ such that no $t$ elements of $S$ sum to $\delta$. We note that if $S$ belongs to some subfield $F_{q^s}$ of $F_q$, then the set $S$ is an $(m, t, \delta)$-set in $F_q$ for each $\delta \in F_q \setminus F_{q^s}$.

**Definition 2.11:** Let $n$ and $k$ be two integers such that $k \geq 3$ and $k + 1 \leq n \leq q$. Let $\alpha_1, \ldots, \alpha_n$ be distinct elements of $F_q$, $\delta \in F_q$, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. Then an $[n + 2, k]$ Roth-Lempel code $RL(\alpha, k, n + 2)$ over $F_q$ is generated by the matrix $G_k = \begin{pmatrix} 1 & 1 & \ldots & 1 & 0 & 0 \\ \alpha_1 & \alpha_2 & \ldots & \alpha_n & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \alpha_{k-2} & \alpha_{k-3} & \ldots & \alpha_{k-2} & 0 & 1 \\ \alpha_{k-1} & \alpha_{k-2} & \ldots & \alpha_{k-1} & 1 & \delta \end{pmatrix}$.

**Lemma 2.12:** [20] An $[n + 2, k]$ Roth-Lempel code with $k \geq 3$ and $k + 1 \leq n \leq q$ over $F_q$ is a non-Reed-Solomon code. Moreover, the Roth-Lempel code is MDS if and only if the set $\{\alpha_1, \ldots, \alpha_n\}$ is an $(n, k - 1, \delta)$-set in $F_q$.

**Remark 2.13:** In the next section, in order to find LCD MDS codes, we first study the LCD property of Roth-Lempel codes. Then we use some set $\{\alpha_1, \ldots, \alpha_n\}$ which is contained in some subfield of $F_q$.

Roth and Lempel\[20\] proved that the generator matrix $G_2$ in (2.2) cannot generate a GRS code by considering a certain form of codewords. That is, $G_2P$ can not generate a GRS code for any permutation matrix $P$. By Definition 2.2, GRS codes are naturally monomially equivalent to RS codes. We claim that Roth-Lempel codes are not monomially equivalent to RS codes. Suppose that a $RL(\alpha, k, n)$ code is monomially equivalent to a RS code. This means that there exists a monomial matrix $M$ such that $G_2M = G'$, where $G'$ is a generator matrix of a RS code. Recall that a monomial matrix $M$ can be written either in the form $DP$ or the form $PD'$, where $D$ and $D'$ are diagonal matrices and $P$ is a permutation matrix.

(1) If $M = DP$, then $G_2M = G_2DP = G'$. Hence, $G_2 = G'P^{-1}D^{-1}$. Note that $G'P^{-1}$ generates some RS codes: that is, $G_2$ generates some GRS codes, which is a contradiction.

(2) If $M = PD'$, then $G_2M = G_2PD' = G'$. Hence, $G_2 = G'D'^{-1}P^{-1}$. We also see that $G_2$ generates some GRS codes; this leads to a contradiction.

In conclusion, Roth-Lempel codes are not monomially equivalent to RS codes. Moreover, in the whole paper, if a code is not monomially equivalent to an RS code, then we call it a code of non-Reed-Solomon type or a non-Reed-Solomon code.

### III. New LCD MDS Codes

In this section, we use constructions in Section 2 to obtain new Euclidean and Hermitian LCD MDS codes. Subsections 3.1 and 3.2 deal with the Euclidean and Hermitian LCD MDS codes, respectively.

#### A. Euclidean LCD MDS codes

Let $\gamma$ be a primitive element of $F_q$ and $k | (q - 1)$. Then $\gamma^{q^{k-1}}$ generates a subgroup of $F_q^*$ of order $k$. Let $\alpha_i = \gamma^{q^{k-1}}$ for $1 \leq i \leq k$. One can easily check that

$$\theta_f = \alpha_1^f + \cdots + \alpha_k^f = \begin{cases} k & \text{if } f \equiv 0 \pmod{k}, \\ 0 & \text{otherwise} \end{cases}.$$ 

Let
Let \( \alpha, t, h, \eta \) be a positive integer with \( k \mid (q - 1) \) and \( h > 0 \), then there exists a \([2k, k]_q\) Euclidean LCD twisted Reed-Solomon code \( C_k(\alpha, t, h, \eta) \) over \( \mathbb{F}_q \). Let \( C_k(\alpha, t, h, \eta) \) be a twisted Reed-Solomon code, we need \( k \neq q - 1 \), and the entries of \( \alpha \) are distinct, which are obvious. From Equation (1), we recall that the generator matrix of the twisted Reed-Solomon code \( C_k(\alpha, t, h, \eta) \) over \( \mathbb{F}_q \). It follows from Lemma (2) that \( C_k(\alpha, t, h, \eta) \) over \( \mathbb{F}_q \) in Definition (2) is Euclidean LCD if and only if \( G_{1}^{T} \) is nonsingular. Let \( \gamma = \sum_{i=1}^{n} \alpha_{i} \) and \( l = k - l + 1 \). Then we compute \( G_{1}^{T} \) in Equation (4) in the top of next page.

Let \( C_{\beta} = A_{\beta} + B_{\beta} \), where \( A_{\beta}, B_{\beta} \) are given in Equation (4) and Equation (7), respectively. By Equation (5), we have \( C_{\beta}C_{\beta}^{T} \) in Equation (8) in the top of next page.

Since every \( \theta_{l} \) for \( l \leq t \leq l + k - 1 \) is zero except exactly one \( \theta_{l} \), we can rewrite (8) as

\[
C_{\beta}C_{\beta}^{T} = \begin{pmatrix}
1 & 1 & \cdots & 1 & 1 \\
1 & \beta_{\alpha_{1}} & \cdots & \beta_{\alpha_{k-1}} & \beta_{\alpha_{k}} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
(\beta_{\alpha_{1}})^{k-1} & (\beta_{\alpha_{2}})^{k-1} & \cdots & (\beta_{\alpha_{k-1}})^{k-1} & (\beta_{\alpha_{k}})^{k-1}
\end{pmatrix}
\]

By Equation (8), the \( *_{\beta} \) and \( \Delta_{\beta} \) are all elements in \( \mathbb{F}_q \), the \( *_{\beta} \) and \( \Delta_{\beta} \) are respectively entries located in the \((i + 1, h + 1)th\), \((h + 1, i + 1)th\) and \((h + 1, h + 1)th\) positions, and the other elements are all zero.

Let \( G_{1} = [C_{1} : C_{\gamma}] \). Then

\[
G_{1}G_{1}^{T} = C_{1}C_{1}^{T} + C_{\gamma}C_{\gamma}^{T}
\]

\[
= \begin{pmatrix}
2k & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & (1 + \gamma)^{k} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 1 + 0 & \cdots & 0 & 0
\end{pmatrix}
\]

By the assumption on \( k \) and \( h > 0 \), we have \( (\gamma^{k} + 1)k \neq 0 \). Hence, we can delete the element \( *_{\theta} + *_{\gamma} \) by some elementary row and column operations of matrices at the same time. Namely, we can find an elementary matrix \( P \) such that

\[
PG_{1}G_{1}^{T}P^{T} = \begin{pmatrix}
2k & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & (1 + \gamma^{k}) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 1 + 0 & \cdots & 0 & 0
\end{pmatrix}
\]

Since \( \det(P) = 1 \) and the element \( \Delta_{1} + \Delta_{\gamma} \) is the entry in the \((h + 1)th \) row and \((h + 1)th \) column, we have \( \det(G_{1}G_{1}^{T}) = 2(1 + \gamma^{k})^{k-1} \neq 0 \) by the definition of determinant. Since the matrix \( G_{1}G_{1}^{T} \) is nonsingular, the code \( C_k(\alpha, t, h, \eta) \) is a Euclidean LCD code. This completes the proof. \( \blacksquare \)

Remark 3.2: For \( G_{1} = [C_{1} : C_{\gamma}] \), it is not guaranteed that \( G_{1}G_{1}^{T} \) is nonsingular when \( k = \frac{4}{2}^{1/2} \); if \( \alpha \) takes all non-zero elements of \( \mathbb{F}_q \) and \( 1 + \gamma^{k} = 0 \), then \( G_{1}G_{1}^{T} \) is singular. However, if \( q = 5 \) then we have \( k = \frac{4}{2}^{1/2} = 2 \) and \( G_{1} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \end{pmatrix} \). Then it is easy to verify that \( C_{2}(\alpha, 1, 1, 1) \) is a Euclidean LCD code for \( \alpha = (1, 2, -2, -1) \).

Example 3.3: Let \( q = 3^{3} = 81 \), \( k = 4 \), and \( \gamma \) be a primitive element of \( \mathbb{F}_q \). Consider a twisted Reed-Solomon code \( C_{2}(\alpha, 1, 3, \eta) \), when \( \alpha = (1, \gamma^{20}, \gamma^{40}, \gamma, \gamma^{20}, \gamma^{40}, \gamma^{20}, \gamma^{40}) \) and \( \eta = \gamma^{i} \in \mathbb{F}_{81} \). Then its generator matrix \( G_{1} \) is given in the top of next page.
Then the result follows from Lemma 2.7.

\[ G_i G_i^T = \begin{pmatrix}
\eta 
\eta 
\eta 
\eta 
\eta 
\eta 
\eta 
\eta 
\eta 
\eta 
\eta 
\eta 
\eta 
\eta 
\end{pmatrix} \]

\[ C_{A} C_{A}^T = \begin{pmatrix}
\gamma & 0 
\gamma & 0 
\gamma & 0 
\gamma & 0 
\gamma & 0 
\gamma & 0 
\gamma & 0 
\gamma & 0 
\end{pmatrix} \]

By Lemma \[5.1\] \( C_{A} (\alpha, 1, 3, \gamma^i) \) is Euclidean LCD for all \( i \). By Magma, it follows that the codes \( C_{A} (\alpha, 1, 3, \eta) \) are MDS with parameters \([8, s]_q\) if and only if \( \eta \) belongs to \( \{ \gamma^j : j = 0, 1, 5, 6, 7, 11, 15, 16, 17, 19, 20, 21, 25, 26, 27, 31, 35, 36, 37, 39, 40, 41, 45, 46, 47, 51, 55, 56, 57, 59, 60, 61, 65, 66, 67, 71, 75, 76, 77, 79\} \). By Magma the code \( C_{A} (\alpha, 1, 3, 1) \) has a generator matrix of the form of \([I_4 | A] \), where

\[
A = \begin{pmatrix}
\gamma^7 & \gamma^{32} & \gamma^{56} & \gamma^{78} \\
\gamma^{31} & \gamma^{21} & \gamma^{64} & \gamma^{44} \\
\gamma^{12} & \gamma^9 & \gamma^{74} & \gamma^{77} \\
\gamma^{60} & \gamma^{29} & \gamma^{52} & \gamma^{79} 
\end{pmatrix}
\]

Then it is easy to check that the \( 3 \times 3 \) minor of the first three rows and columns of \( A \) is equal to \( \gamma^{26} \), which is confirmed by Magma. By Lemma \[2.7\] \( C_{A} (\alpha, 1, 3, 1) \) is an \([8, s]_q\) Euclidean LCD MDS non-Reed-Solomon code.

An effective method for construction of twisted Reed-Solomon codes with MDS property is to use the lifting of the finite field (refer to \[3\]). We note that the Euclidean LCD property of a given code is preserved under the lifting of the finite field. Hence, we obtain the following theorem.

**Theorem 3.4:** Let \( q \) be a power of an odd prime and let \( k \mid (q - 1) \) and \( k \geq 3 \). Let \( \gamma \) be a primitive element of \( F_q \) and \( \gamma_i = \gamma^{2^{j-1}}, 1 \leq i \leq k \). Then there exists an Euclidean LCD Roth-Lempel code \( RL(\alpha, k, n) \) over \( F_q \) with one of the following parameters:

1. \([k + 2, k]_q \) if \( \alpha = (\alpha_1, \ldots, \alpha_k) \);
2. \([k + 3, k]_q \) if \( \gcd(k + 1, q) = 1 \) and \( \alpha = (0, \alpha_1, \ldots, \alpha_k) \);
3. \([2k + 2, 2]_q \) if \( k < 2^{n-1} \) and \( \alpha = (\alpha_1, \ldots, \alpha_k, \alpha_0, \ldots, \alpha_0) \).

**Proof:** By Lemma \[2.3\] the code \( RL(\alpha, k, n) \) over \( F_q \) is Euclidean LCD if and only if \( G_2 G_2^T \) is nonsingular. Let

\[
D = \begin{pmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 1 \\
1 & \delta 
\end{pmatrix}
\]

(1) Let \( G_2 = [A_1 : D] \), where \( A_1 \) is given in Equation \( 4 \). Then

\[
G_2 G_2^T = A_1 A_1^T + D D^T = \begin{pmatrix}
k & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & k \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & k & \cdots & 1 & \delta \\
0 & k & 0 & \cdots & \delta & 1 + \delta^2 
\end{pmatrix}
\]

Then the matrix \( G_2 G_2^T \) is nonsingular; so, \( RL(\alpha, k, k + 2) \) is an Euclidean LCD code.
(2) Let $G_2 = [e_1 : A_1 : D]$, where $e_1 = (1, 0, \ldots, 0)^T$. Then

$$G_2G_2^T = e_1e_1^T + A_1A_1^T + DD^T$$

Then the matrix $G_2G_2^T$ is nonsingular; hence, $RL(\alpha, k, k+3)$ is a Euclidean LCD code.

(3) Note that $k \neq \frac{4q-1}{3}$. Let $G_2 = [A_1 : A_7 : D]$. Then

$$G_2G_2^T = A_1A_1^T + A_7A_7^T + DD^T$$

Then the matrix $G_2G_2^T$ is nonsingular; therefore, $RL(\alpha, k, 2k+2)$ is a Euclidean LCD code. This completes the proof. ■

Theorem 3.6: Let $q$ be a power of an odd prime and $F_q \subset F_q$. Let $k \geq 3$ be an integer with $k \equiv (s-1)$

(1) If $gcd(k+1, s) = 1$, then there exists a $[k, 3, k]$ Euclidean LCD non-Reed-Solomon code.

(2) If $k < \frac{4q-1}{3}$, then there exists a $[2k, 2, k]$ Euclidean LCD non-Reed-Solomon code.

Proof: By the proof of Lemma 3.5, we can construct a Euclidean LCD Roth-Lempel code over $F_q$. Note that we can require that $\alpha_i \in F_q$ for all $i$. By Lemma 2.12, an Roth-Lempel code is a non-Reed-Solomon code, and it is an MDS code if and only if the set $S = \{\alpha_1, \ldots, \alpha_n\}$ forms an $(n, k-1, \delta)$-set in $F_q$; that is, there exists an element $\delta \in F_q$ such that no $k-1$ elements of $S$ sum to $\delta$. Note that we can require that $\alpha_i \in F_q$ for all $i$. Hence, $S \subset F_q$, and we can find some $\delta \in F_q\setminus\alpha_i$ such that $S$ is an $(n, k-1, \delta)$-set in $F_q$. Similar to Lemma 3.5, by the conditions we can find a vector $\alpha \in F_q^n$ such that the code $C_k(\alpha, t, h, \eta)$ is Euclidean LCD and the result follows. ■

The followings are examples of Theorem 3.6.

Example 3.7: (1) Let $q = 3^2 = 9$ and $k = 4$. Let $\gamma$ be a primitive element of $F_9$, and $\delta = \gamma^i$ for some integer $i$ with $0 \leq i \leq 7$. Then the generator matrix of the code $C_2$ is given as follows:

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & \gamma^2 & \gamma^4 & \gamma^6 & 0 \\
0 & 1 & (\gamma^2)^2 & (\gamma^4)^2 & (\gamma^6)^2 & 0 \\
0 & 1 & (\gamma^3)^2 & (\gamma^4)^3 & (\gamma^6)^3 & 1 \\
\end{pmatrix}
$$

By Magma, there is no $i$ such that $C_2$ is a Euclidean LCD MDS non-Reed-Solomon Roth-Lempel code with parameters $(7, 4)_{F_9}$. However, we can make these LCD codes have the MDS property by lifting the finite field $F_9$ which is shown in the following.

(2) Let $q = 3^4$ and $k = 4$. Let $w$ be a primitive element of $F_{81}$ and $\gamma$ a primitive element of $F_9$ with $\gamma = w^{10}$. Choose $\alpha = (0, 1, \gamma^2, \gamma^4, \gamma^6)$ and $\delta = w^i \in F_{81}$. Then the generator matrix of $RL(\alpha, k, n)$ is given as follows:

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & \gamma^2 & \gamma^4 & \gamma^6 & 0 \\
0 & 1 & (\gamma^2)^2 & (\gamma^4)^2 & (\gamma^6)^2 & 0 \\
0 & 1 & (\gamma^3)^2 & (\gamma^4)^3 & (\gamma^6)^3 & 1 \\
\end{pmatrix}
$$

By Theorem 3.6 (1), $RL(\alpha, k, k+3)$ is a Euclidean LCD MDS non-Reed-Solomon code with parameters $(7, 4)_{F_9}$ when $i$ is not divisible by 10.

Remark 3.8: We emphasize that any Euclidean LCD MDS code of non-Reed-Solomon type constructed in Theorems 3.4 and 3.6 is not monomially equivalent to any Euclidean LCD code constructed by the method of Carlet et al. [9].

Now, we briefly justify why they are not monomially equivalent for the case of Theorem 3.4, and the case of Theorem 3.6 can be also justified similarly. According to the result of Carlet et al. [9] Theorem 5.1, assume that there is a $[2k, k]$ linear MDS code over $F_q$ with generator matrix $[I_k \ A]$ satisfying the conditions of [9] Theorem 5.1. Then there is a monomial matrix $M$ such that $[I_k \ A]M$ generates a LCD MDS code $C$ by [9] Theorem 5.1. Now, if we suppose that the code $C$ is monomially equivalent to our LCD MDS code $C_k(\alpha, t, h, \eta)$ of non-Reed-Solomon type with generator matrix $G_1$, then there should exist a monomial matrix $M'$ such that $G_1 = [I_k \ A]M'M$. Then $M'M' = PDPM'$, where $P$ and $P'$ are permutation matrices and $D$ is a diagonal matrix. Therefore, the entries of the first row of $G_1(M')^{-1}M^{-1}$ are nonzero and all-one except two coordinate positions, and the product of all the entries of the first row of $[I_k \ A]$ is zero; this is impossible.

B. Hermitian LCD MDS codes

In this subsection, we consider Hermitian LCD MDS codes over $F_{q^2}$.

Let $\gamma$ be a primitive element of $F_{q^2}$ and $k \mid (q^2 - 1)$. Then $\gamma^{q^2-1}$ generates a subgroup of order $k$ in $F_{q^2}$. Let $\alpha_i = \gamma^{q^2-i}$ with $1 \leq i \leq k$.

For $i, j \in \{0, \ldots, k-1\}$, assume that $a_{ij}(i, j)$ is the entry in the $(i+1)$-th row and $(j+1)$-th column of the matrix $A_{\beta\overline{\beta}}$, where $A_{\beta}\overline{\beta}$ is given in Equation (4). Then

$$
\begin{align*}
\begin{cases}
\beta^{i+j} k & \text{if } i+j \equiv 0 \pmod{k}, \\
0 & \text{otherwise}.
\end{cases}
\end{align*}
$$

Every row of the matrix $A_{\beta\overline{\beta}}$ has exactly one nonzero element, and every column of the matrix $A_{\beta\overline{\beta}}$ has exactly one nonzero element. Hence, the matrix $A_{\beta\overline{\beta}}$ is nonsingular over $F_{q^2}$.

The following lemma plays an important role in proving our main results of this Subsection 3.2. First, we investigate the twisted Reed-Solomon code $C_k(\alpha, t, h, \eta)$ over $F_{q^2}$.
Lemma 3.9: Let $q$ be a power of an odd prime and $k$ be a positive integer with $k \mid (q^2 - 1)$. If there exists an odd prime number $p$ such that $v_p(k) < v_p(q^2 - 1)$ and $h > 0$, then there exists a $(2k, k)_{q^2}$ Hermitian LCD twisted Reed-Solomon code $C_k(\alpha, t, h, \eta)$ over $\mathbb{F}_{q^2}$ for $\alpha = (\alpha_1, \ldots, \alpha_k, \gamma \alpha_1, \ldots, \gamma \alpha_k)$, where $\gamma$ is a primitive element of $\mathbb{F}_{q^2}$, $\alpha_i = \frac{\alpha}{\gamma^{i-1}}, 1 \leq i \leq k$, and $r = 2v_p(q^2 - 1)$.

Proof: The generator matrix $G_1$ of the twisted Reed-Solomon code $C_k(\alpha, t, h, \eta)$ over $\mathbb{F}_{q^2}$ is shown in Equation (11). By Lemma 2.5, $C_k(\alpha, t, h, \eta)$ is Hermitian LCD if and only if $G_1 G_1^T$ is nonsingular. Let $E = \theta_{h+tq} + \eta^q \theta_{h+iq} + \eta^{q+i} \theta_{i+q} + \eta^q \theta_{i+q} + \eta^{q+i} \theta_{i+q}$, then $l = k - 1 + t$. We compute $G_1 G_1^T$ in Equation (9) in the top of next page.

Let $C_\beta = A_\beta + B_\beta$, where $A_\beta$ is given in Equation (9) and $B_\beta$ is given in Equation (11). By Equation (9), we compute $C_\beta G_1 G_1^T$ in Equation (10) in the top of next page.

Note that $\alpha_i q^2 = \alpha_i$. Then $\{\theta_{1}, \theta_{i+q}, \ldots, \theta_{(k-1)+q}\} = \{\theta_{i+q}, \theta_{1+i+q}, \ldots, \theta_{(k-1)+i+q}\}$. Exactly one element in the set $\{\theta_{1}, \theta_{i+q}, \ldots, \theta_{(k-1)+i+q}\}$ has value $k$. Hence,

$$C_\beta G_1 G_1^T = A_\beta A_\beta^T + \left( \begin{array}{cccccc} 0 & \ldots & 0 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & *_\beta & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & \Delta_\beta & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \end{array} \right)$$

where $*_\beta$ and $\Delta_\beta$ are elements belong to $\mathbb{F}_{q^2}$, $*_\beta, *_\beta$, and $\Delta_\beta$ are entries placed in the $1, h+1, \ldots, h+i, h+i+1$ positions, respectively and the other elements are all zero.

Let $h > 0$ and $G_1 = [C_1 : C_\gamma r]$, where $r = 2v_p(q^2 - 1)$. By the condition that there exists an odd prime number $p$ such that $v_p(k) < v_p(q^2 - 1)$, any two columns in $G_1$ are not same. Then

$$G_1 G_1^T = C_1 C_1^T + C_\gamma r C_\gamma r^T = A_1 A_1^T + A_\gamma r A_\gamma r^T + \left( \begin{array}{cccccc} 0 & \ldots & 0 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & *_1 + *_\gamma r & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & \Delta_1 + \Delta_\gamma r & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \end{array} \right).$$

Let $b(i, j)$ be the entry in the $i$-th row and $j$-th column of the matrix $A_1 A_1^T + A_\gamma r A_\gamma r^T$. Then

$$b(i, j) = \begin{cases} (1 + \gamma^{(i+j)q})k & \text{if } i+jq \equiv 0 \pmod{k}, \\ 0 & \text{otherwise}. \end{cases}$$

When $i+jq \equiv 0 \pmod{k}$, assume that $b(i, j) = 0$. We have $\gamma^{(i+j)q} = -1 = \frac{\gamma^q}{1 + \gamma}$. Therefore, we get a contradiction. So, we have $b(i, j) \neq 0$ when $i + jq \equiv 0 \pmod{k}$. Therefore, every row of the matrix $A_1 A_1^T + A_\gamma r A_\gamma r^T$ has a nonzero element and every column of the matrix $A_1 A_1^T + A_\gamma r A_\gamma r^T$ has exactly one nonzero element.

Since the matrix $G_1 G_1^T$ is conjugate symmetric, we can delete the elements $*_1 + *_\gamma r$ and $\Delta_1 + \Delta_\gamma r$ by some elementary row and column operations of matrices at the same time. Therefore, we can find an elementary matrix $P$ such that

$$PG_1 G_1^T P^T = PA_1 A_1^T P^T + PA_\gamma r A_\gamma r^T P^T = \left( \begin{array}{cccccc} 0 & \ldots & 0 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & \Delta_1 + \Delta_\gamma r & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \end{array} \right)$$

Hence, $G_1 G_1^T$ is nonsingular; thus, the code $C_k(\alpha, t, h, \eta)$ over $\mathbb{F}_{q^2}$ is a Hermitian LCD code. This completes the proof.

Example 3.10: Let $q = 11^2 = 121$, $k = 5$, and $\gamma$ be a primitive element of the finite field $\mathbb{F}_{121}$. Consider a twisted Reed-Solomon code $C_2(\alpha, 1, 3, \eta)$ with $\alpha = (1, \gamma^{24}, \gamma^{48}, \gamma^{72}, \gamma^{96}, \gamma^{8}, \gamma^{8}, \gamma^{8}, \gamma^{8}, \gamma^{8}, \gamma^{8}, \gamma^{8}, \gamma^{96})$ and $\eta = \gamma^j \in \mathbb{F}_{121}$. Then its generator matrix $G_2$ is given in the top of next page. By Lemma 3.8, $C_2(\alpha, 1, 3, \eta)$ is Hermitian LCD for all $i$. By Magma, the codes $C_2(\alpha, 1, 3, \eta^i)$ are MDS with parameters $[10, 5, 121]$ when $\eta \in \{\gamma_j : j = 0, 5, 7, 10, 13, 22, 23\}$. By Magma the code $C_2(\alpha, 1, 3, \eta^2)$ has a generator matrix of the form of $[I_5 | A]$, where

$$A = \left( \begin{array}{cccccc} \gamma^{11} & \gamma^{115} & \gamma^6 & \gamma^{45} & 10 \\ \gamma^{73} & \gamma^{19} & \gamma^5 & \gamma^{54} & \gamma^{81} \\ \gamma^{91} & \gamma^{10} & \gamma^{22} & \gamma^{55} & \gamma^{81} \\ \gamma^{40} & \gamma^{94} & \gamma^7 & \gamma^{43} & \gamma^{62} \\ \gamma^{38} & \gamma^{38} & \gamma^{17} & \gamma^{116} & \gamma^{104} \\ \gamma^{56} & \gamma^{56} \end{array} \right).$$

Then it is easy to check that the $3 \times 3$ minor of the first three rows and columns of $A$ is equal to 8, which is confirmed by Magma. By Lemma 2.5, $C_2(\alpha, 1, 3, \gamma^{23})$ is a $[10, 5, 121]$ Hermitian LCD MDS non-Reed-Solomon code.

In a similar way as Theorem 3.4, we obtain the following result.

Theorem 3.11: Let $q$ be a power of an odd prime and $\mathbb{F}_q \subset \mathbb{F}_{q^2}$. Let $k$ be a positive integer such that $k \mid (q - 1)$. Thus, there exists an odd prime number $p$ such that $v_p(k) < v_p(q - 1)$. Let $\alpha = (\alpha_1, \ldots, \alpha_k, \gamma \alpha_1, \ldots, \gamma \alpha_k)$, where $\gamma$ is a primitive element of $\mathbb{F}_q$, $\alpha_i = \frac{\alpha}{\gamma^{i-1}}, 1 \leq i \leq k$, and $r = 2v_p(q - 1)$. Assume that the minor condition for $\eta \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ of Lemma 2.7 holds. Then $C_k(\alpha, t, h, \eta)$ is a $[2k, k]_{q^2}$ Hermitian LCD MDS non-Reed-Solomon code.

Proof: By Lemma 3.9, $C_k(\alpha, t, h, \eta)$ over $\mathbb{F}_q$ is Hermitian LCD. By Lemma 2.5 and $\eta \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$, then the twisted Reed-Solomon code $C_k(\alpha, t, h, \eta)$ over $\mathbb{F}_{q^2}$ is MDS. Then the result follows from Lemma 2.7.
In the following, we consider Roth-Lempel codes.

**Lemma 3.12:** Let \( q \) be a power of an odd prime, and let \( k \mid (q^2 - 1) \) and \( k \geq 3 \). Let \( \gamma \) be a primitive element of \( \mathbb{F}_{q^2} \) and \( \alpha = \gamma^{2k+1} \) for \( 1 \leq i \leq k \). Then there exists a Hermitian LCD Roth-Lempel code \( RL(\alpha, k, n) \) over \( \mathbb{F}_{q^2} \) with one of the following parameters:

1. \([k + 2, k]_q^2\) if \( \alpha = (\alpha_1, \ldots, \alpha_k) \);
2. \([k + 3, k]_q^2\) if \( \gcd(k+1, q) = 1 \) and \( \alpha = (0, \alpha_1, \ldots, \alpha_k) \);
3. \([2k + 2, k]_q^2\) if there exists an odd prime number \( p \) such that \( v_p(k) < v_p(q^2 - 1) \), \( \alpha = (\alpha_1, \ldots, \alpha_k, \gamma^i, \alpha_{k+1}, \ldots, \gamma^i, \alpha_k) \), and \( r = 2^s(q^2 - 1) \).

**Proof:** By Lemma 2.3, the Roth-Lempel code over \( \mathbb{F}_q \) in Definition 2.11 is Hermitian LCD if and only if \( G_2G_2^T \) is nonsingular.

1. Let \( G_2 = [A_1 : D] \), where \( D \) is given in the proof of Lemma 3.5. Then

\[
G_2G_2^T = A_1A_1^T + DD^T
\]

Since the matrix \( A_1A_1^T \) is nonsingular, the matrix \( G_2G_2^T \) is nonsingular and \( RL(\alpha, k, k+2) \) is a Hermitian LCD code.

2. Let \( G_2 = [e_1 : A_1 : D] \), where \( e_1 = (1, 0, \ldots, 0)^T \). Then

\[
G_2G_2^T = e_1e_1^T + A_1A_1^T + DD^T
\]

and the entry of the \((1,1)\)th position of \( A_1A_1^T \) is \( k \), which is nonzero. Then the entry of the \((1,1)\)th position of the matrix \( G_2G_2^T \) is \( k + 1 \), and so the matrix \( G_2G_2^T \) is nonsingular. Therefore, the \( RL(\alpha, k, k+3) \) code is a Hermitian LCD code.

3. By the proof of Lemma 3.5, let \( G_2 = [A_1 : A_1^T : D] \). By the proof of Lemma 3.1, we can take some element \( \delta \in \mathbb{F}_{q^2} \) such that the matrix \( G_2G_2^T \) is nonsingular, and hence the \( RL(\alpha, k, 2k + 2) \) code is a Hermitian LCD code.

This completes the proof.

In a similar way as Theorem 3.6, we have the following theorem on Hermitian LCD codes over \( \mathbb{F}_{q^2} \).

**Theorem 3.13:** Let \( q \) be a power of an odd prime and \( \mathbb{F}_{q^2} \) be the finite field of order \( q^2 \). Let \( \kappa \) be an integer such that \( \kappa \mid (q - 1) \) and \( \kappa \geq 3 \).

1. If \( \gcd(k + 1, q) = 1 \), then there exists a \([k + 3, k]_q^2\) Hermitian LCD MDS non-Reed-Solomon code over \( \mathbb{F}_{q^2} \).

2. If there exists an odd prime number \( p \) such that \( v_p(k) < v_p(q - 1) \), then there exists a \([2k + 2, k]_q^2\) Hermitian LCD MDS non-Reed-Solomon code over \( \mathbb{F}_{q^2} \).

We give the following example.

**Example 3.14:** (1) Let \( q = 5^2 \) and \( k = 6 \). Let \( \gamma \) be a primitive element of the finite field \( \mathbb{F}_{25} \) with \( \alpha = (0, 1, \gamma^4, \gamma^8, \gamma^{12}, \gamma^{16}, \gamma^{20}) \) and \( \delta = \gamma^i \) for some integer \( i \) with \( 0 \leq i \leq 23 \). Then the generator matrix of the code \( C_1 \) is given as follows:

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & \gamma^4 & \gamma^8 & \gamma^{12} & \gamma^{16} & \gamma^{20} & 0 & 0 \\
0 & 1 & (\gamma^4)^2 & (\gamma^8)^2 & (\gamma^{12})^2 & (\gamma^{16})^2 & (\gamma^{20})^2 & 0 & 0 \\
0 & 1 & (\gamma^4)^3 & (\gamma^8)^3 & (\gamma^{12})^3 & (\gamma^{16})^3 & (\gamma^{20})^3 & 0 & 0 \\
0 & 1 & (\gamma^4)^4 & (\gamma^8)^4 & (\gamma^{12})^4 & (\gamma^{16})^4 & (\gamma^{20})^4 & 0 & 1 \\
0 & 1 & (\gamma^4)^5 & (\gamma^8)^5 & (\gamma^{12})^5 & (\gamma^{16})^5 & (\gamma^{20})^5 & 1 & 1 \\
\end{pmatrix}
\]

By Magma, \( C_1 \) is a Hermitian LCD MDS non-Reed-Solomon code over \( \mathbb{F}_{25} \) with parameters \([9, 6]_{25}\) when \( i = 1, 2, 5, 6, 9, 10, 13, 14, 17, 18, 21, 22 \).

(2) Let \( q = 7^2 \) and \( k = 8 \). Let \( \gamma \) be a primitive element of the finite field \( \mathbb{F}_{25} \) with \( \alpha = (0, 1, \gamma^6, \gamma^{12}, \gamma^{18}, \gamma^{24}, \gamma^{30}, \gamma^{36}, \gamma^{42}) \) and \( \delta = \gamma^i \) for some integer \( i \) with \( 0 \leq i \leq 47 \). Then the generator matrix of the code \( C_2 \) is given as follows:

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & \gamma^2 & \gamma^3 & \gamma^{12} & \gamma^{18} & \gamma^{24} & \gamma^{30} & \gamma^{36} & \gamma^{42} \\
0 & 1 & (\gamma^2)^2 & (\gamma^3)^2 & (\gamma^{12})^2 & (\gamma^{18})^2 & (\gamma^{24})^2 & (\gamma^{30})^2 & (\gamma^{36})^2 & (\gamma^{42})^2 \\
0 & 1 & (\gamma^2)^3 & (\gamma^3)^3 & (\gamma^{12})^3 & (\gamma^{18})^3 & (\gamma^{24})^3 & (\gamma^{30})^3 & (\gamma^{36})^3 & (\gamma^{42})^3 \\
0 & 1 & (\gamma^2)^4 & (\gamma^3)^4 & (\gamma^{12})^4 & (\gamma^{18})^4 & (\gamma^{24})^4 & (\gamma^{30})^4 & (\gamma^{36})^4 & (\gamma^{42})^4 \\
0 & 1 & (\gamma^2)^5 & (\gamma^3)^5 & (\gamma^{12})^5 & (\gamma^{18})^5 & (\gamma^{24})^5 & (\gamma^{30})^5 & (\gamma^{36})^5 & (\gamma^{42})^5 \\
\end{pmatrix}
\]
By Magma, $C_2$ is a Hermitian LCD MDS non-Reed-Solomon code over $\mathbb{F}_{10}$ with parameters $[11,8]_{10}$ when $i = 4, 16, 22, 28, 29, 34, 40, 46$.

Remark 3.15: We point out that any Hermitian LCD MDS code of non-Reed-Solomon type constructed in Theorems 3.11 and 3.13 is not monomially equivalent to any Hermitian LCD code constructed by the method of Carlet et al. [9]. This can be justified in a similar way as Remark 3.8 (for the Euclidean case).

IV. CONCLUDING REMARKS

Main contributions of this paper are constructions of some new Euclidean and Hermitian LCD MDS codes of non-Reed-Solomon type. According to the results of Carlet et al. [9], all parameters of Euclidean LCD codes ($q > 3$) and Hermitian LCD codes ($q > 2$) have been completely determined, including the LCD MDS codes. However, in the coding theory, it is an important issue to find all inequivalent codes of the same parameters. We emphasize that any Euclidean (or Hermitian) LCD MDS code of non-Reed-Solomon type constructed by our method is not monomially equivalent to any Euclidean (or Hermitian) LCD code constructed by the method of Carlet et al. in [9]; this is justified in Remarks 3.8 and 3.15. Finally, we provided some examples of non-Reed-Solomon LCD MDS codes.

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