LACK OF GROUND STATE FOR NLS ON BRIDGE-TYPE GRAPHS

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Abstract. We prove the nonexistence of ground states for NLS on bridge-like graphs, i.e. graphs with two halflines and four vertices, of which two at infinity, with Kirchhoff matching conditions. By ground state we mean any minimizer of the energy functional among all functions with the same mass.

1. Introduction

A graph \( \mathcal{G} \) consists of a set \( V \) of points \( v_1, \ldots, v_{N_v} \) called vertices and a set \( A \) of edges \( e_1, \ldots, e_{N_e} \) joining pairs of vertices. Multiple connections between the same couple of vertices (i.e. several edges between the same vertices) and also edges connecting a vertex with itself, called self-loops, are allowed. We assume that the number \( N_v \) of vertices as well as the number \( N_e \) of edges, are finite.

We require \( \mathcal{G} \) to be a metric graph, that is we identify every edge with a real interval, namely

\[
 e_j \mapsto I_j := (0, l_j)
\]

with \( l_j \in (0, +\infty] \).

Notice that a given vertex \( v \) can act both as the left endpoint for an edge and as the right endpoint for another one. It is then meaningful to define on the set \( A \) the functions \( R \) and \( L \), such that \( L(e_j) = v \) if \( v \) is the left endpoint of \( e_j \), and \( R(e_j) = v \) if \( v \) is the right endpoint of \( e_j \).

Owing to the metric structure, it is natural to define functions \( u : \mathcal{G} \rightarrow \mathbb{C} \) as

\[
 u := (u_1, \ldots, u_{N_e}),
\]
where $u_j : I_j \rightarrow \mathbb{C}$ is the restriction of $u$ to the edge $e_j$. The definition of $u$ is made complete by specifying the value of $u$ at any vertex of $G$. We define $L^p$ spaces on $G$ according to the norm

$$
\|u\|_{L^p(G)}^p = \sum_{k=1}^{N_e} \|u_k\|_{L^p(I_k)}^p,
$$

(In the following we will use the shorthand notation $\|u\|_p = \|u\|_{L^p(G)}$ and $\|u_j\|_p = \|u_j\|_{L^p(I_j)}$.)

Analogously, we define the space $H^1(G)$ as the subspace of $L^2(G)$ consisting of functions $u$ such that $u' := (u'_1, \ldots, u'_{N_e})$ is an element of $L^2(G)$ too, and satisfies the continuity condition at vertices, that states that the limit of $u(x)$ as $x$ approaches a vertex $v$ exists and is independent of the particular edge on which $x$ runs.

Owing to the previous definitions, one can introduce the energy functional

$$
E(u, G) = \frac{1}{2}\|u'\|_2^2 - \frac{1}{p}\|u\|_p^p,
$$

defined on any function $u \in H^1(G)$. It is well-known that this functional corresponds to the conserved energy of the equation

$$
i\partial_t u(t) = -\Delta u(t) - |u(t)|^{p-2}u(t),$$

i.e., a nonlinear Schrödinger equation on $G$ with nonlinearity power $p - 1$ (for a general introduction to NLS see [13]). In such equation multiplication and powers are to be understood componentwise (i.e. edge by edge), while the definition of the Laplacian has to be completed by Kirchhoff boundary conditions (see the end of this Section, and [18] for the classification of all self-adjoint vertex conditions).

There is nowadays a huge literature on quantum graphs, i.e. metric graphs with differential or pseudo-differential operators acting on functions defined on it ([21, 7, 8, 15, 19, 20]). However, most of such a literature is concerned with linear systems, while for nonlinear systems the research started more than two decades ago ([3]), got relevant results, but has remained less extensive ([9, 12, 11, 10, 22]). Important
results have been obtained for dispersive estimates, that are often used to link linear and nonlinear evolutions ([4, 5]).

NLS-type equations are currently used to model several systems where the propagation of waves in branched structures is relevant: Bose-Einstein condensates in ramified traps, optical fibers, T-junctions and others. In all applications it proves important to get information on stationary solutions (i.e. the modes of the system) and on their stability. The stationary solutions for which one can typically state a stability result are the ground states of the systems, i.e., the minimizers, possibly under suitable constraints, of the functional \((1.1)\) ([14, 23, 16]). Observe indeed that this functional is not bounded from below, since, for all non-trivial \(u\), \(E(\lambda u) \to -\infty\) as \(\lambda \to +\infty\). However, as soon as the nonlinearity power \(p\) is subcritical, i.e.

\[ 2 < p < 6, \]

the restriction of \(E\) to the manifold of functions \(u\) sharing the same, fixed, value \(\mu\) for the mass, namely the constraint

\[ \|u\|_2^2 = \mu > 0, \]

is bounded from below. Indeed, by the Gagliardo-Nirenberg inequality

\[ \|u\|_p \leq C \|u\|_2^{\frac{1}{p}} \|u\|_{H^1}^{\frac{1}{p} - \frac{1}{2}} \]

that can be easily extended to graphs with a finite number of edges, one has

\[ E(u, G) \geq \frac{1}{2} \|u'\|_2^2 - C \|u'\|_2^{\frac{p}{2} - 1} - C \]

where the mass constraint was taken into account, and lower boundedness immediately follows. So the following questions arise.

(i) Is the infimum of \(E\) on the space \(H^1_\mu = \{ u \in H^1, \|u\|_2^2 = \mu \}\) larger or smaller than the infimum on the line?

(ii) Is the infimum attained?

Question (ii) can be rephrased in a more physical language, as follows: does there exist a ground state?

The answers to (i) and (ii) depend on the nature of \(G\), as the following examples illustrate.
(1) $\mathcal{G} = \mathbb{R}$. The set of minimizers is given by the soliton
\[
\phi_\mu(x) = C\mu^{\frac{2}{p-2}}\text{sech}^{\frac{2}{p-2}}(c\mu^{\frac{2}{p-2}}x),
\]
where $C$ and $c$ are constants depending on $p$ only, and by the orbit of $\phi_\mu$ with respect to translations and multiplication by a phase. Namely, the only minimizers are given by the functions
\[
e^{i\theta}\phi_\mu(\cdot - y), \quad \theta \in [0, 2\pi), \; y \in \mathbb{R}
\]
([14, 16, 17]).

Remark 1.1. By this classical result one immediately has that, if at least one edge of $\mathcal{G}$ is infinite, then $\inf_{H^1_\mu} E(u, \mathcal{G}) \leq E(\phi_\mu, \mathbb{R})$.
Indeed, assuming that first edge is infinite, consider the functions
\[
u_n(x) := (A_n\chi_+(x)\phi_\mu(x - n), 0, \ldots, 0)
\]
where $\chi_+$ is a smooth function, supported on $\mathbb{R}^+$, with $\chi_+(x) = 1$ for all $x > 1$, and $A_n$ are constants such that $\|u_{(n)}\|_2^2 = \mu$.
Then, it is easily seen that $E(u_{(n)}, \mathcal{G})$ converges to $E(\phi_\mu, \mathbb{R})$ as $n$ goes to infinity, so that $\inf_{H^1_\mu} E(u, \mathcal{G}) \leq E(\phi_\mu, \mathbb{R}).$

(2) $\mathcal{G} = \mathbb{R}^+$. In this case the only positive minimizer is given by the “half-soliton”, i.e., by the restriction of $\phi_{2\mu}$ to the positive halfline. Any further minimizer can be obtained by multiplying $\phi_{2\mu}$ by a phase factor.

(3) $\mathcal{G} = \mathcal{S}_{n,\infty}$ with $n \geq 3$, namely the star-graph made up of $n \geq 3$ halflines (in Figure 1 the case $n = 3$ is plotted). In that case
\[
\inf_{u \in H^1_\mu} E(u, \mathcal{S}_{n,\infty}) = E(\phi_\mu, \mathbb{R})
\]
but the infimum is not achieved ([11]).

(4) $\mathcal{G} = \mathcal{B}_3$, i.e. the three-bridge graph portrayed in Figure 2.
This graph is Eulerian, i.e. it can be unfolded into a line, as shown in Figure 3. Correspondingly, every $u \in H^1_\mu(\mathcal{G})$ unfolds into a function $\tilde{u} \in H^1_\mu(\mathbb{R})$ such that $E(u, \mathcal{B}_3) = E(\tilde{u}, \mathbb{R})$. Notice that the Eulerian path on $\mathcal{B}_3$ crosses every vertex three times, so that the unfolded function $\tilde{u}$ must assume three times the
values at vertices. This implies that \( \tilde{u} \) cannot be a soliton, so the infimum cannot be attained.

(5) \( \mathcal{G} = \mathcal{B}_2 \), i.e. the two-bridge graph in Figure 4.
This time, the graph is not Eulerian, so the problem is not immediate to solve. We will show in the next section that the situation is exactly the same as in the previous example: \( \inf_{H_\mu} E(u, \mathcal{B}_2) = E(\phi_\mu, \mathbb{R}) \) but the infimum is not attained. The same holds for any \( 2k \)-bridge, and this is the main result of this note.

(6) \( \mathcal{G} = \mathcal{S}_{2+1} \), i.e. the star-graph consisting of two infinite and one finite edge, displayed in Figure 5. In this case \( \inf_{H_\mu} E(u, \mathcal{S}_{2+1}) < \)
$E(\phi_\mu, \mathbb{R})$ and the infimum is attained, so it is actually a minimum. This result will be proved in the forthcoming paper [2].

![Figure 5](image1)

**Figure 5.** The three-star graph $S_{2+1}$ with two infinite and one finite edge.

(7) The *exceptional graph* $E_3$ displayed in Figure 6. In this case $\inf_{H_\mu} E(u, B_3) = E(\phi_\mu, \mathbb{R})$ and the minimum is attained. Details will be given in [2].

![Figure 6](image2)

**Figure 6.** The exceptional graph $E_3$. Edges connecting the same couple of vertices have the same length.

In this note we treat the case of the $n$-bridge graphs $B_n$, i.e. a graph consisting of two halflines whose origins are connected by $n$ finite edges (not necessarily of the same length).

We prove the following

**Theorem 1.2.** Let $B_n$, $n \geq 2$, be an $n$-bridge graph. Consider the energy functional $E$ defined in (1.1) with $2 < p < 6$. Then,

(a) $\inf_{H_\mu} E(u, B_n) = E(\phi_\mu, \mathbb{R})$.

(b) The infimum is not attained.
This is the first result on the minimization of NLS energy on non-star graphs. A more general result, including cases where the infimum is attained, will be proved in \cite{2}.

In order to illustrate the physical meaning of the absence of the ground state, consider for instance the case of a Bose-Einstein condensate in a ramified trap with two long branches. Under the critical temperature, a macroscopic fraction of the particles of the system is known to collapse in the ground state of the Gross-Pitaevskii functional (i.e. the energy $E$ with $p = 4$). In absence of a ground state, one could imagine the system that follows a minimizing sequence. Of course, an actual trap will always be finite and therefore a ground state will exist. Nevertheless, provided that two branches exhibit a larger lengthscale than the rest of the graph and that some other technical hypotheses are fulfilled, the ground state should not be sensitively different from a soliton escaping along one of the branches.

Before proving Theorem \ref{1.2}, let us comment on the matching conditions at vertices. Even though our nonexistence result holds for bridges only, the argument we give for vertex conditions is general, see also \cite{2}.

Any minimizer is a stationary point for the unconstrained functional

$$\tilde{E}(u) = E(u) + \nu \|u\|_2^2,$$

where $\nu$ is a Lagrange multiplier. Now, since $\tilde{E}$ is differentiable on $H^1(\mathcal{G})$,

$$\nabla \tilde{E}(u) \eta = \Re \int_{\mathcal{G}} \left( \bar{u}' \eta' - |u|^{p-2} \bar{u} \eta + 2 \nu \bar{u} \eta \right) dx$$

$$= \sum_{j=1}^{N_c} \Re \int_{I_j} \left( \bar{u}_j' \eta_j' - |u_j|^{p-2} \bar{u}_j \eta_j + 2 \nu \bar{u}_j \eta_j \right) dx = 0$$

for all $\eta \in H^1(\mathcal{G})$. By standard arguments (integrating by parts and using the Euler–Lagrange equation in each interval), the preceding
identity yields

\[ \mathcal{R} \sum_{j=1}^{N_e} \bar{u}'_j \eta_j \big|_0^l_j = 0. \]

Focusing on vertices instead of edges, this can be equivalently written as

\[ \mathcal{R} \sum_{k=1}^{N_v} \eta(v_k) \left( \sum_{R(e_j)=v_k} \bar{u}'_j(l_j) - \sum_{L(e_j)=v_k} \bar{u}'_j(0) \right) = 0. \]

Finally, by the arbitrariness of \( \eta \), one concludes

\[ \sum_{R(e_j)=v_k} u'_j(l_j) - \sum_{L(e_j)=v_k} u'_j(0) = 0, \quad \text{for all } k, \]

which are the well-known Kirchhoff conditions.

2. PROOF

We start by giving a lemma that compares the contributions of two different edges to the energy and shows how to construct a third edge and a function which, properly inserted in the graph, makes the energy decrease. Theorem 1.2 then follows as an easy consequence.

**Lemma 2.1 (Comparison).** For \( i = 1, 2 \), let \( l_i \) be an element of \((0, +\infty]\) and denote by \( I_i \) the interval \((0, l_i)\).

Given a pair of functions \( u_i \in H^1(I_i) \backslash \{0\} \), there exist an interval \( I = (0, l) \) with \( l \in (0, +\infty] \) and a function \( w \in H^1(I) \), such that

1. \( \|w\|_{L^2(I)}^2 = \|u_1\|_{L^2(I_1)}^2 + \|u_2\|_{L^2(I_2)}^2. \)
2. For either \( i = 1 \) or \( i = 2 \), \( w(0) = u_i(0) \) and \( w(l) = u_i(l_i) \).
3. \( E(w, I) \leq E(u_1, I_1) + E(u_2, I_2). \)

Furthermore,

\[ E(w, I) < E(u_1, I_1) + E(u_2, I_2), \]

unless \( u_1 = u_2 = c \) for some constant \( c \).
\textbf{Proof.} Set

\[
\lambda := \frac{\int_0^{l_2} |u_2(x)|^2 \, dx}{\int_0^{l_1} |u_1(x)|^2 \, dx},
\]

and define \( \tilde{l}_1 := (1+\lambda)l_1 \) if \( l_1 \) is finite, \( l_2 := \frac{1+\lambda}{\lambda} l_2 \) if \( l_2 \) is finite, \( \tilde{l}_i := +\infty \) if \( l_i = +\infty \), and \( \tilde{I}_i := (0, \tilde{l}_i) \). Consider the functions \( \tilde{u}_i : \tilde{I}_i \to \mathbb{R} \) defined by

\[
\tilde{u}_1(x) := u_1 \left( \frac{x}{1+\lambda} \right), \quad \tilde{u}_2(x) := u_2 \left( \frac{\lambda x}{1+\lambda} \right).
\]

An elementary computation gives

\[
\int_0^{\tilde{l}_i} |\tilde{u}_i'(x)|^2 \, dx = \frac{1}{1+\lambda} \int_0^{l_1} |u_1'(x)|^2 \, dx
\]

\[
\int_0^{\tilde{l}_i} |\tilde{u}_i(x)|^q \, dx = (1+\lambda) \int_0^{l_1} |u_1(x)|^q \, dx
\]

\[
\int_0^{\tilde{l}_2} |\tilde{u}_2'(x)|^2 \, dx = \frac{\lambda}{1+\lambda} \int_0^{l_2} |u_2'(x)|^2 \, dx
\]

\[
\int_0^{\tilde{l}_2} |\tilde{u}_2(x)|^q \, dx = \frac{1+\lambda}{\lambda} \int_0^{l_2} |u_2(x)|^q \, dx,
\]

for any \( q > 0 \). Setting \( q = 2 \), and owing to the definition of \( \lambda \), one immediately finds that for both \( i = 1, 2 \)

\[
\|\tilde{u}_i\|_{L^2(\tilde{I}_i)}^2 = \|u_1\|_{L^2(I_1)}^2 + \|u_2\|_{L^2(I_2)}^2
\]

(2.2)

and

\[
\tilde{u}_i(0) = u_i(0), \quad \tilde{u}_i(\tilde{l}_i) = u_i(l_i).
\]

(2.3)
If either \( u_1 \) or \( u_2 \) is nonconstant, then by (2.1) with \( q = p \) one obtains
\[
E(\tilde{u}_1, \tilde{I}_1) + \lambda E(\tilde{u}_2, \tilde{I}_2)
= \frac{1}{2(1 + \lambda)} \int_0^{l_1} |u_1'(x)|^2 \, dx - \frac{1 + \lambda}{p} \int_0^{l_1} |u_1(x)|^p \, dx
+ \frac{\lambda^2}{2(1 + \lambda)} \int_0^{l_2} |u_2'(x)|^2 \, dx - \frac{1 + \lambda}{p} \int_0^{l_2} |u_2(x)|^p \, dx
\]
\[
< (1 + \lambda) \left( E(u_1, I_1) + E(u_2, I_2) \right).
\]

Then, for either \( i = 1 \) or \( i = 2 \) one gets
\[
E(\tilde{u}_i, \tilde{I}_i) < E(u_1, I_1) + E(u_2, I_2).
\]

Denote this index by \( \bar{i} \) and define \( w = u_i, I = \tilde{I}_i \). By (2.2), (2.3), and (2.4), items (1), (2), and (3) with the strict inequality are proved for \( u_1 \) and \( u_2 \) that are not both constant.

Finally, let us suppose that \( u_i \equiv \bar{u}_i \) for both \( i = 1, 2 \), where \( \bar{u}_i \) is a constant. Then, from (2.1) one has
\[
E(\tilde{u}_1, \tilde{I}_1) = (1 + \lambda) E(u_1, I_1), \quad E(\tilde{u}_2, \tilde{I}_2) = \frac{1 + \lambda}{\lambda} E(u_2, I_2)
\]
thus
\[
E(\tilde{u}_1, \tilde{I}_1) + \lambda E(\tilde{u}_2, \tilde{I}_2) = (1 + \lambda) \left( E(u_1, I_1) + E(u_2, I_2) \right).
\]

As a consequence, either \( E(\tilde{u}_1, \tilde{I}_1) < E(u_1, I_1) + E(u_2, I_2) \) or \( E(\tilde{u}_2, \tilde{I}_2) < E(u_1, I_1) + E(u_2, I_2) \) unless \( E(\tilde{u}_1, \tilde{I}_1) = E(\tilde{u}_2, \tilde{I}_2) = E(u_1, I_1) + E(u_1, I_2) \).

By a straightforward computation, one finds that this implies \( \bar{u}_1 = \bar{u}_2 \).

The proof is complete.

Now we are ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** Consider the \( n \)-bridge graph \( \mathcal{B}_n \) and a function \( u \in H^1_\mu(\mathcal{B}_n) \). If \( n \) is odd, then \( \mathcal{B}_n \) is Eulerian, so that the function \( u \) unfolds to a function \( \tilde{u} : \mathbb{R} \to \mathbb{C} \) s.t.
\[
E(u, \mathcal{B}_n) = E(\tilde{u}, \mathbb{R}) \geq E(\phi_\mu, \mathbb{R})
\]
and the last inequality is an identity only if $\tilde{u}(x) = \phi_\mu(x - y)$ for some $y$. But this is not possible, since any value attained by $u$ at a vertex is attained at least $n$ times by $\tilde{u}$. As $n \geq 3$, $\tilde{u}$ cannot be equal to a soliton. As a consequence,

$$E(u, B_n) > E(\phi_\mu, \mathbb{R}).$$

This inequality, together with Remark 1.1, proves Theorem 1.2 when $n$ is odd.

When $n$ is even, let $e_i$, $1 \leq i \leq n$, be the $i$-th edge between the two halflines. As stated in Section 1, an interval $I_i = (0, l_i)$ is associated with the edge $e_i$. Focusing on $e_1, e_2$, by Lemma 2.1 there exist an interval $I := (0, l)$ and a function $w_2 : I \rightarrow \mathbb{C}$ such that $\|w_2\|_2^2 = \|u_1\|_2^2 + \|u_2\|_2^2$, $w_2(0) = u(v_1)$ and $w_2(l) = u(v_2)$, where $v_1$ and $v_2$ are the two vertices corresponding to the origins of the two halflines. Then, the function

$$w = (w_2, u_3, \ldots, u_n, u_{n+1}, u_{n+2}),$$

where $u_{n+1}$ and $u_{n+2}$ are the components of $u$ on the two halflines, is an element of $H^1(\mathcal{B}_{n-1})$. Furthermore, owing to point (3) in Lemma 2.1 again, one gets

$$E(w, \mathcal{B}_{n-1}) = E(w_2, I) + \sum_{j=3}^{n+2} E(u_j, I_j) \leq \sum_{j=1}^{n+2} E(u_j, I_j) = E(u, \mathcal{B}_n).$$

Since $n - 1$ is odd, one concludes

$$E(\phi_\mu, \mathbb{R}) < E(w, \mathcal{B}_{n-1}) \leq E(u, \mathcal{B}_n)$$

and the proof is complete.

\[\square\]

3. Possible extensions and perspectives

The reduction technique described in the preceding section can be extended to treat more general graphs.
For instance, a self-loop attached to an edge can be melted in a single edge, as illustrated in the following lemma.

Lemma 3.1 (Removing self-loops). Let \( l_1 > 0 \) and \( l_2 \in (0, +\infty) \) and denote by \( I_i \) the interval \((0, l_i)\) and by \( I \) the interval \((0, l_1 + l_2)\).

Given a pair of functions \( u_i \in H^1(I_i) \), with \( u_1(0) = u_1(l_1) = u_2(0) \), there exists a function \( w \in H^1(I) \), such that

\[
\begin{align*}
1) & \quad \|w\|_{L^2(I)}^2 = \|u_1\|_{L^2(I_1)}^2 + \|u_2\|_{L^2(I_2)}^2, \\
2) & \quad w(0) = u_1(0) \text{ and } v(l) = u_2(l_2), \\
3) & \quad E(w, I) = E(u_1, I_1) + E(u_2, I_2).
\end{align*}
\]

Proof. It is sufficient to define \( w \) on \( I \) as

\[
w(x) = \begin{cases} 
  u_1(x) & \text{if } x \in (0, l_1) \\
  u_2(x - l_1) & \text{if } x \in (l_1, l_1 + l_2).
\end{cases}
\]

Lemmas 2.1 and 3.1 can be used in order to develop a “haircut” strategy suitable to work on a larger class of graphs. Indeed, consider a graph \( \mathcal{G} \) with \( N_e \) edges and a function \( u \in H^1_{\mu}(\mathcal{G}) \). In several cases one may use Lemma 2.1 or Lemma 3.1 to construct a graph \( \mathcal{G}' \) with \( N_e - 1 \) edges and a function \( w \in H^1_{\mu}(\mathcal{G}') \) such that

\[
E(w, \mathcal{G}') \leq E(u, \mathcal{G}).
\]

This could be the starting point of an inductive procedure aimed at reducing any graph (by removing one edge at a time) to simpler graphs that one is able to handle explicitly. This is exactly described in the forthcoming paper [2].

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