GRAPH $W^*$-PROBABILITY SPACES

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Abstract. In this paper, we construct a $W^*$-probability space $(W^*(G), E)$ with amalgamation over a von Neumann algebra $D_G$, where $W^*(G)$ is a graph $W^*$-algebra induced by the countable directed graph $G$. In this structure, we compute the $D_G$-valued moments and cumulants of arbitrary random variables, by using the lattice path models and we characterize the $D_G$-freeness of generators of $W^*(G)$, by the so-called diagram-distinctness on $G$. As examples, we will compute the $D_G$-moments and $D_G$-cumulants of the generating operators $T_{1N}$ of $W^*(G^1_N)$ and $T_N$ of $W^*(C_N)$, where $G^1_N$ is the graph with one vertex and $N$-distinct loop-edges and $C_N$ is the circulant graph with $N$-vertices and $N$-edges. In particular, the generating operator $T_{1N}$ is the free sum of $N$-semicircular elements. This operator $T_{1N}$ is semicircular with its even moments $(2N)^k \cdot c_k$, for all $k \in 2N$, and its second cumulant $2N$, where $c_k$ is the $k$-th Catalan number.

In this paper, we construct the graph $W^*$-probability spaces. The graph $W^*$-probability theory is one of the good example of Speicher's combinatorial free probability theory with amalgamation (See [16]). In this paper, we will observe how to compute the moment and cumulant of an arbitrary random variables in the graph $W^*$-probability space and the freeness on it with respect to the given conditional expectation. In [10], Kribs and Power defined the free semigroupoid algebras and obtained some properties of them. Our work is highly motivated by [10]. Roughly speaking, graph $W^*$-algebras are $W^*$-topology closed version of free semigroupoid algebras. Throughout this paper, let $G$ be a countable directed graph and let $F^+(G)$ be the free semigroupoid of $G$, in the sense of Kribs and Power, i.e., it is a collection of all vertices of the graph $G$ as units and all admissible finite paths, under the admissibility. As a set, the free semigroupoid $F^+(G)$ can be decomposed by

$$F^+(G) = V(G) \cup FP(G),$$

where $V(G)$ is the vertex set of the graph $G$ and $FP(G)$ is the set of all admissible finite paths. Trivially the edge set $E(G)$ of the graph $G$ is properly contained in $FP(G)$, since all edges of the graph can be regarded as finite paths with their length 1. We define a graph $W^*$-algebra of $G$ by

$$W^*(G) \overset{\text{def}}{=} C[\{L_w, L^*_w : w \in F^+(G)\}]^w,$$

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where $L_w$ and $L_w^*$ are creation operators and annihilation operators on the generalized Fock space $H_G = l^2(\mathbb{F}^+ (G))$ induced by the given graph $G$, respectively. Notice that the creation operators induced by vertices are projections and the creation operators induced by finite paths are partial isometries. We can define the $W^*$-subalgebra $D_G$ of $W^*(G)$, which is called the diagonal subalgebra by

$$D_G = \mathbb{C}[\{L_v : v \in V(G)\}]^w.$$

Then each element $a$ in the graph $W^*$-algebra $W^*(G)$ is expressed by

$$a = \sum_{w \in \mathbb{F}(G) : a} p_w L_w^u,$$

where $\mathbb{F}(G : a)$ is a support of the element $a$, as a subset of the free semigroupoid $\mathbb{F}(G)$. The above expression of the random variable $a$ is said to be the Fourier expansion of $a$. Since $\mathbb{F}(G)$ is decomposed by the disjoint subsets $V(G)$ and $FP(G)$, the support $\mathbb{F}(G : a)$ of $a$ is also decomposed by the following disjoint subsets,

$$V(G : a) = \mathbb{F}(G : a) \cap V(G)$$

and

$$FP(G : a) = \mathbb{F}(G : a) \cap FP(G).$$

Thus the operator $a$ can be re-expressed by

$$a = \sum_{v \in V(G : a)} p_v L_v + \sum_{w \in FP(G : a), u_w \in \{1, \ast\}} p_w L_w^{u_w}.$$

Notice that if $V(G : a) \neq \emptyset$, then $\sum_{v \in V(G : a)} p_v L_v$ is contained in the diagonal subalgebra $D_G$. Thus we have the canonical conditional expectation $E : W^*(G) \to D_G$, defined by

$$E (a) = \sum_{v \in V(G : a)} p_v L_v,$$

for all $a = \sum_{w \in \mathbb{F}(G : a), u_w \in \{1, \ast\}} p_w L_w^{u_w}$ in $W^*(G)$. Then the algebraic pair $(W^*(G), E)$ is a $W^*$-probability space with amalgamation over $D_G$ (See [16]). It is easy to check that the conditional expectation $E$ is faithful in the sense that if $E(a^*a) = 0_{D_G}$, for $a \in W^*(G)$, then $a = 0_{D_G}$.

For the fixed operator $a \in W^*(G)$, the support $\mathbb{F}(G : a)$ of the operator $a$ is again decomposed by

$$\mathbb{F}(G : a) = V(G : a) \cup FP_+(G : a) \cup FP_-(G : a),$$
with the decomposition of $FP(G : a)$,

$$FP(G : a) = FP_*(G : a) \cup FP^*_n(G : a),$$

where

$$FP_*(G : a) = \{ w \in FP(G : a) : \text{both } L_w \text{ and } L^*_w \text{ are summands of } a \}$$

and

$$FP^*_n(G : a) = FP(G : a) \setminus FP_*(G : a).$$

The above new expression plays a key role to find the $D_{\Gamma}$-valued moments of the random variable $a$. In fact, the summands $p_v L_v$’s and $p_w L_w + p_w^* L_w^*$, for $v \in V(G : a)$ and $w \in FP_*(G : a)$ act for the computation of $D_{\Gamma}$-valued moments of $a$. By using the above partition of the support of a random variable, we can compute the $D_{\Gamma}$-valued moments and $D_{\Gamma}$-valued cumulants of it via the lattice path model $LP_n$ and the lattice path model $LP^*_n$ satisfying the $*$-axis-property. At a first glance, the computations of $D_{\Gamma}$-valued moments and cumulants look so abstract (See Chapter 3) and hence it looks useless. However, these computations, in particular the computation of $D_{\Gamma}$-valued cumulants, provides us how to figure out the $D_{\Gamma}$-freedom of random variables by making us compute the mixed cumulants. As applications, in the final chapter, we can compute the moment and cumulant of the operator that is the sum of $N$-free semicircular elements with their covariance 2. If $a$ is the operator, then the $n$-th moment of $a$ is

$$\begin{cases} (2N)^{\frac{n}{2}} \cdot c_n^2 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

and the $n$-th cumulant of $a$ is

$$\begin{cases} 2N & \text{if } n = 2 \\ 0 & \text{otherwise} \end{cases}$$

where $c_k = \frac{1}{k+1} \binom{2k}{k}$ is the $k$-th Catalan number.

Based on the $D_{\Gamma}$-cumulant computation, we can characterize the $D_{\Gamma}$-freedom of generators of $W^*(G)$, by the so-called diagram-distinctness on the graph $G$. i.e., the random variables $L_{w_1}$ and $L_{w_2}$ are free over $D_{\Gamma}$ if and only if $w_1$ and $w_2$ are diagram-distinct the sense that $w_1$ and $w_2$ have different diagrams on the graph $G$. Also, we could find the necessary condition for the $D_{\Gamma}$-freedom of two arbitrary random variables $a$ and $b$. i.e., if the supports $FP^*(G : a)$ and $FP^*(G : b)$ are diagram-distinct, in the sense that $w_1$ and $w_2$ are diagram distinct for all pairs $(w_1, w_2) \in FP^*(G : a) \times FP^*(G : b)$, then the random variables $a$ and $b$ are free over $D_{\Gamma}$. 
1. Graph $W^*$-Probability Spaces

Let $G$ be a countable directed graph and let $\mathbb{F}^+(G)$ be the free semigroupoid of $G$, i.e., the set $\mathbb{F}^+(G)$ is the collection of all vertices as units and all admissible finite paths of $G$. Let $w$ be a finite path with its source $s(w) = x$ and its range $r(w) = y$, where $x, y \in V(G)$. Then sometimes we will denote $w$ by $w = xwy$ to express the source and the range of $w$. We can define the graph Hilbert space $H_G$ by the Hilbert space $l^2(\mathbb{F}^+(G))$ generated by the elements in the free semigroupoid $\mathbb{F}^+(G)$, i.e., this Hilbert space has its Hilbert basis $B = \{\xi_w : w \in \mathbb{F}^+(G)\}$. Suppose that $w = e_1...e_k \in FP(G)$ is a finite path with $e_1,...,e_k \in E(G)$. Then we can regard $\xi_w$ as $\xi_{e_1} \otimes ... \otimes \xi_{e_k}$. So, in [10], Kribs and Power called this graph Hilbert space the generalized Fock space. Throughout this paper, we will call $H_G$ the graph Hilbert space to emphasize that this Hilbert space is induced by the graph.

Define the creation operator $L_w$, for $w \in \mathbb{F}^+(G)$, by the multiplication operator by $\xi_w$ on $H_G$. Then the creation operator $L$ on $H_G$ satisfies that

(i) $L_w = L_x L_w L_y$, for $w = xwy$ with $x, y \in V(G)$.

(ii) $L_{w_1} L_{w_2} = \begin{cases} 
L_{w_1 w_2} & \text{if } w_1 w_2 \in \mathbb{F}^+(G) \\
0 & \text{if } w_1 w_2 \notin \mathbb{F}^+(G),
\end{cases}$

for all $w_1, w_2 \in \mathbb{F}^+(G)$.

Now, define the annihilation operator $L_w^*$, for $w \in \mathbb{F}^+(G)$ by

$$
L_w^* \xi_{w'} \overset{\text{def}}{=} \begin{cases} 
\xi_h & \text{if } w' = wh \in \mathbb{F}^+(G) \\
0 & \text{otherwise}.
\end{cases}
$$

The above definition is gotten by the following observation:

$$
< L_w \xi_h, \xi_{wh} > = < \xi_{wh}, \xi_{wh} > = 1 = < \xi_h, \xi_h > = < \xi_h, L_w \xi_{wh} >,
$$

where $<,>$ is the inner product on the graph Hilbert space $H_G$. Of course, in the above formula we need the admissibility of $w$ and $h$ in $\mathbb{F}^+(G)$. However, even though $w$ and $h$ are not admissible (i.e., $wh \notin \mathbb{F}^+(G)$), by the definition of $L_w^*$, we have that

$$
< L_w \xi_h, \xi_h > = < 0, \xi_h > = 0 = < \xi_h, 0 > = < \xi_h, L_w^* \xi_h >.
$$
Notice that the creation operator $L$ and the annihilation operator $L^*$ satisfy that

$$L^*L_w = L_y \quad \text{and} \quad L_wL^*_w = L_x,$$  \hspace{1cm} (1.1)

under the weak topology, where $x, y \in V(G)$. Remark that if we consider the von Neumann algebra $W^*(\{L_w\})$ generated by $L_w$ and $L^*_w$ in $B(H_G)$, then the projections $L_y$ and $L_x$ are Murray-von Neumann equivalent, because there exists a partial isometry $L_w$ satisfying the relation (1.1). Indeed, if $w = xwy$ in $\mathbb{F}^+(G)$, with $x, y \in V(G)$, then under the weak topology we have that

$$L_wL^*_w = L_w \quad \text{and} \quad L^*_wL_wL^*_w = L^*_w,$$  \hspace{1cm} (1.2)

So, the creation operator $L_w$ is a partial isometry in $W^*(\{L_w\})$ in $B(H_G)$.

Assume now that $v \in V(G)$. Then we can regard $v$ as $v = vv$. So,

$$L^*_vL_v = L_v = L_vL^*_v = L^*_v,$$  \hspace{1cm} (1.3)

This relation shows that $L_v$ is a projection in $B(H_G)$ for all $v \in V(G)$.

Define the graph $W^*$-algebra $W^*(G)$ by

$$W^*(G) \overset{\text{def}}{=} \overline{\mathbb{C}[\{L_w, L^*_w : w \in \mathbb{F}^+(G)\}]^w}.$$ 

Then all generators are either partial isometries or projections, by (1.2) and (1.3). So, this graph $W^*$-algebra contains a rich structure, as a von Neumann algebra. (This construction can be the generalization of that of group von Neumann algebra.) Naturally, we can define a von Neumann subalgebra $D_G \subset W^*(G)$ generated by all projections $L_v$, $v \in V(G)$, i.e.

$$D_G \overset{\text{def}}{=} W^*(\{L_v : v \in V(G)\}).$$

We call this subalgebra the diagonal subalgebra of $W^*(G)$. Notice that $D_G = \Delta_G \subset M_{|G|}(\mathbb{C})$, where $\Delta_G$ is the subalgebra of $M_{|G|}(\mathbb{C})$ generated by all diagonal matrices. Also, notice that $1_{D_G} = \sum_{v \in V(G)} L_v = 1_{W^*(G)}$.

If $a \in W^*(G)$ is an operator, then it has the following decomposition which is called the Fourier expansion of $a$:

$$a = \sum_{w \in \mathbb{F}^+(G : a), u_w \in \{1, *\}} p_w L^*_w u_w,$$  \hspace{1cm} (1.4)

where $p_w \in \mathbb{C}$ and $\mathbb{F}^+(G : a)$ is the support of $a$ defined by

$$\mathbb{F}^+(G : a) = \{w \in \mathbb{F}^+(G) : p_w \neq 0\}.$$
Remark that the free semigroupoid $\mathbb{F}^+(G)$ has its partition $\{V(G), FP(G)\}$, as a set. i.e.,

$$\mathbb{F}^+(G) = V(G) \cup FP(G) \quad \text{and} \quad V(G) \cap FP(G) = \emptyset.$$ 

So, the support of $a$ is also partitioned by

$$\mathbb{F}^+(G : a) = V(G : a) \cup FP(G : a),$$

where

$$V(G : a) \overset{\text{def}}{=} V(G) \cap \mathbb{F}^+(G : a)$$

and

$$FP(G : a) \overset{\text{def}}{=} FP(G) \cap \mathbb{F}^+(G : a).$$

So, the above Fourier expansion (1.4) of the random variable $a$ can be re-expressed by

\begin{equation}
(1.5) \quad a = \sum_{v \in V(G:a)} p_v L_v + \sum_{w \in FP(G:a), u_w \in \{1, *\}} p_w L_u^w.
\end{equation}

We can easily see that if $V(G : a) \neq \emptyset$, then $\sum_{v \in V(G:a)} p_v L_v$ is contained in the diagonal subalgebra $D_G$. Also, if $V(G : a) = \emptyset$, then $\sum_{v \in V(G:a)} p_v L_v = 0_{D_G}$. So, we can define the following canonical conditional expectation $E : W^*(G) \to D_G$ by

\begin{equation}
(1.6) \quad E(a) = E \left( \sum_{w \in \mathbb{F}^+(G:a), u_w \in \{1, *\}} p_w L_u^w \right) \overset{\text{def}}{=} \sum_{v \in V(G:a)} p_v L_v,
\end{equation}

for all $a \in W^*(G)$. Indeed, $E$ is a well-determined conditional expectation; it is a bimodule map satisfying that

$$E(d) = d, \quad \text{for all } d \in D_G.$$ 

And

$$E(da'd') = E(d(a_d + a_0)d') = E(da_d'd' + da_0d')$$

$$= E(da_d'd') = da_d'd' = d(E(a))d',$$

for all $d, d' \in D_G$ and $a = a_d + a_0 \in W^*(G)$, where

$$a_d = \sum_{v \in V(G:a)} p_v L_v \quad \text{and} \quad a_0 = \sum_{w \in FP(G:a), u_w \in \{1, *\}} p_w L_u^w.$$ 

Also,
\[ E(a^*) = E((a_d + a_0)^*) = E(a_d^* + a_0^*) = a_d^* = E(a)^*, \]

for all \( a \in W^*(G) \). Here, \( a_d^* = \left( \sum_{v \in \mathcal{V}(G,a)} p_v L_v \right)^* = \sum_{v \in \mathcal{V}(G,a)} p_v L_v \) in \( D_G \).

**Definition 1.1.** Let \( G \) be a countable directed graph and let \( W^*(G) \) be the graph \( W^* \)-algebra induced by \( G \). Let \( E : W^*(G) \to D_G \) be the conditional expectation defined above. Then we say that the algebraic pair \((W^*(G), E)\) is the graph \( W^* \)-probability space over the diagonal subalgebra \( D_G \). By the very definition, it is one of the \( W^* \)-probability space with amalgamation over \( D_G \). All elements in \((W^*(G), E)\) are called \( D_G \)-valued random variables.

We have a graph \( W^* \)-probability space \((W^*(G), E)\) over its diagonal subalgebra \( D_G \). We will define the following free probability data of \( D_G \)-valued random variables.

**Definition 1.2.** Let \( W^*(G) \) be the graph \( W^* \)-algebra induced by \( G \) and let \( a \in W^*(G) \). Define the \( n \)-th \((D_G \text{-valued})\) moment of \( a \) by

\[ E(d_1a d_2a ... d_na), \text{ for all } n \in \mathbb{N}, \]

where \( d_1, ..., d_n \in D_G \). Also, define the \( n \)-th \((D_G \text{-valued})\) cumulant of \( a \) by

\[ k_n(d_1a, d_2a, ..., d_na) = C(n) (d_1a \otimes d_2a \otimes ... \otimes d_na), \]

for all \( n \in \mathbb{N} \), and for \( d_1, ..., d_n \in D_G \), where \( \tilde{C} = (C(n))_{n=1}^{\infty} \in I^c (W^*(G), D_G) \) is the cumulant multiplicative bimodule map induced by the conditional expectation \( E \), in the sense of Speicher. We define the \( n \)-th trivial moment of \( a \) and the \( n \)-th trivial cumulant of \( a \) by

\[ E(a^n) \text{ and } k_n \left( a, a, ..., a \underbrace{\text{\scriptsize \( \otimes \)}}_{\text{n-times}} \right) = C(n) (a \otimes a \otimes ... \otimes a), \]

respectively, for all \( n \in \mathbb{N} \).

To compute the \( D_G \)-valued moments and cumulants of the \( D_G \)-valued random variable \( a \), we need to introduce the following new definition ;

**Definition 1.3.** Let \((W^*(G), E)\) be a graph \( W^* \)-probability space over \( D_G \) and let \( a \in (W^*(G), E) \) be a random variable. Define the subset \( FP_e(G : a) \) in \( FP(G : a) \) by

\[ FP_e(G : a) \overset{\text{def}}{=} \{ w \in F^+(G : a) : \text{ both } L_w \text{ and } L_w^* \text{ are summands of } a \}. \]
And let $FP^c_*(G : a) \overset{\text{def}}{=} FP(G : a) \setminus FP_*(G : a)$.

We already observed that if $a \in (W^*(G), E)$ is a $D_G$-valued random variable, then $a$ has its Fourier expansion $a_d + a_0$, where

$$a_d = \sum_{v \in V(G : a)} p_v L_v$$

and

$$a_0 = \sum_{w \in FP(G : a), u_w \in \{1, *\}} p_w L^*_w.$$

By the previous definition, the set $FP(G : a)$ is partitioned by

$$FP(G : a) = FP_*(G : a) \cup FP^c_*(G : a),$$

for the fixed random variable $a$ in $(W^*(G), E)$. So, the summand $a_0$, in the Fourier expansion of $a = a_d + a_0$, has the following decomposition:

$$a_0 = a_* + a_{(\text{non-*})},$$

where

$$a_* = \sum_{l \in FP_*(G : a)} (p_l L_l + p_l^* L^*_l),$$

and

$$a_{(\text{non-*})} = \sum_{w \in FP^c_*(G : a), u_w \in \{1,*\}} p_w L^*_w,$$

where $p_l$ is the coefficient of $L_l^*$ depending on $l \in FP_*(G : a)$. (There is no special meaning for the complex number $p_l^*$. But we have to keep in mind that $p_l \neq p_l^*$, in general. i.e. $a_* = \sum_{l_1 \in FP_*(G : a)} p_{l_1} L_{l_1} + \sum_{l_2 \in FP_*(G : a)} p_{l_2} L^*_{l_2}$! But for the convenience of using notation, we will use the notation $p_l^*$ for the coefficient of $L_l^*$.) For instance, let $V(G : a) = \{v_1, v_2\}$ and $FP(G : a) = \{w_1, w_2\}$ and let the random variable $a$ in $(W^*(G), E)$ be

$$a = L_{v_1} + L_{v_2} + L^*_{w_1} + L_{w_1} + L^*_{w_2}.$$ 

Then we have that $a_d = L_{v_1} + L_{v_2}$, $a_* = L^*_{w_1} + L_{w_1}$ and $a_{(\text{non-*})} = L^*_{w_2}$. By definition, $a_0 = a_* + a_{(\text{non-*})}$.

2. $D_G$-Moments and $D_G$-Cumulants of Random Variables
Throughout this chapter, let $G$ be a countable directed graph and let $(W^*(G), E)$ be the graph $W^*$-probability space over its diagonal subalgebra $D_G$. In this chapter, we will compute the $D_G$-valued moments and the $D_G$-valued cumulants of arbitrary random variable

$$a = \sum_{w \in \mathbb{F}^+(G), u \in \{1, \ast\}} p_w L_w^{u,w}$$

in the graph $W^*$-probability space $(W^*(G), E)$.

2.1. Lattice Path Model.

Throughout this section, let $G$ be a countable directed graph and let $(W^*(G), E)$ be the graph $W^*$-probability space over its diagonal subalgebra $D_G$. Let $w_1, \ldots, w_n \in \mathbb{F}^+(G)$ and let $L_{w_1} \ldots L_{w_n} \in (W^*(G), E)$ be a $D_G$-valued random variable. In this section, we will define a lattice path model for the random variable $L_{w_1} \ldots L_{w_n}$. Recall that if $w = e_1 \ldots e_k \in FP(G)$ with $e_1, \ldots, e_k \in E(G)$, then we can define the length $|w|$ of $w$ by $k$. i.e.e, the length $|w|$ of $w$ is the cardinality $k$ of the admissible edges $e_1, \ldots, e_k$.

**Definition 2.1.** Let $G$ be a countable directed graph and $\mathbb{F}^+(G)$, the free semi-groupoid. If $w \in \mathbb{F}^+(G)$, then $L_w$ is the corresponding $D_G$-valued random variable in $(W^*(G), E)$. We define the lattice path $l_w$ of $L_w$ and the lattice path $l_w^{-1}$ of $L_w$ by the lattice paths satisfying that:

(i) the lattice path $l_w$ starts from $\ast = (0, 0)$ on the $\mathbb{R}^2$-plane.

(ii) if $w \in V(G)$, then $l_w$ has its end point $(0, 1)$.

(iii) if $w \in E(G)$, then $l_w$ has its end point $(1, 1)$.

(iv) if $w \in E(G)$, then $l_w^{-1}$ has its end point $(-1, -1)$.

(v) if $w \in FP(G)$ with $|w| = k$, then $l_w$ has its end point $(k, k)$.

(vi) if $w \in FP(G)$ with $|w| = k$, then $l_w^{-1}$ has its end point $(-k, -k)$.

Assume that finite paths $w_1, \ldots, w_s$ in $FP(G)$ satisfy that $w_1 \ldots w_s \in FP(G)$.

Define the lattice path $l_{w_1} \ldots l_{w_s}$ by the connected lattice path of the lattice paths $l_{w_1}, \ldots, l_{w_s}$, i.e.e, $l_{w_s}$ starts from $(k_{w_1}, k_{w_2}) \in \mathbb{R}^+$ and ends at $(k_{w_1} + k_{w_2}, k_{w_1} + k_{w_2})$, where $|w_1| = k_{w_1}$ and $|w_2| = k_{w_2}$. Similarly, we can define the lattice path $l_{w_1}^{-1} \ldots l_{w_s}^{-1}$ as the connected path of $l_{w_s}^{-1}, l_{w_{s-1}}^{-1}, \ldots, l_{w_1}^{-1}$.
Definition 2.2. Let $G$ be a countable directed graph and assume that $L_{w_1}, ..., L_{w_n}$ are generators of $(W^*(G), E)$. Then we have the lattice paths $l_{w_1}, ..., l_{w_n}$ of $L_{w_1}, ..., L_{w_n}$, respectively in $\mathbb{R}^2$. Suppose that $L_{w_1}^{u_1} ... L_{w_n}^{u_n} \neq 0_{DG}$ in $(W^*(G), E)$, where $u_{w_1}, ..., u_{w_n} \in \{1, *\}$. Define the lattice path $l_{w_1}^{u_1} ... l_{w_n}^{u_n}$ of nonzero $L_{w_1}^{u_1} ... L_{w_n}^{u_n}$ by the connected lattice path of $l_{w_1}^{*}, ..., l_{w_n}^{*}$, where $t_{w_j} = 1$ if $u_{w_j} = 1$ and $t_{w_j} = -1$ if $u_{w_j} = *$. Assume that $L_{w_1}^{u_1} ... L_{w_n}^{u_n} = 0_{DG}$. Then the empty set $\emptyset$ in $\mathbb{R}^2$ is the lattice path of it. We call it the empty lattice path. By $LP_n$, we will denote the set of all lattice paths of the $D_G$-valued random variables having their forms of $L_{w_1}^{u_1} ... L_{w_n}^{u_n}$, including empty lattice path.

Also, we will define the following important property on the set of all lattice paths:

Definition 2.3. Let $l_{w_1}^{u_1} ... l_{w_n}^{u_n} \neq \emptyset$ be a lattice path of $L_{w_1}^{u_1} ... L_{w_n}^{u_n} \neq 0_{DG}$ in $LP_n$. If the lattice path $l_{w_1}^{u_1} ... l_{w_n}^{u_n}$ starts from * and ends on the *-axis in $\mathbb{R}^2$, then we say that the lattice path $l_{w_1}^{u_1} ... l_{w_n}^{u_n}$ has the *-axis-property. By $LP_n^*$, we will denote the set of all lattice paths having their forms of $l_{w_1}^{u_1} ... l_{w_n}^{u_n}$ which have the *-axis-property. By little abuse of notation, sometimes, we will say that the $D_G$-valued random variable $L_{w_1}^{u_1} ... L_{w_n}^{u_n}$ satisfies the *-axis-property if the lattice path $l_{w_1}^{u_1} ... l_{w_n}^{u_n}$ of it has the *-axis-property.

The following theorem shows that finding $E(L_{w_1}^{u_1} ... L_{w_n}^{u_n})$ is checking the *-axis-property of $L_{w_1}^{u_1} ... L_{w_n}^{u_n}$.

Theorem 2.1. Let $L_{w_1}^{u_1} ... L_{w_n}^{u_n} \in (W^*(G), E)$ be a $D_G$-valued random variable, where $u_{w_1}, ..., u_{w_n} \in \{1, *\}$. Then $E(L_{w_1}^{u_1} ... L_{w_n}^{u_n}) \neq 0_{DG}$ if and only if $L_{w_1}^{u_1} ... L_{w_n}^{u_n}$ has the *-axis-property (i.e., the corresponding lattice path $l_{w_1}^{u_1} ... l_{w_n}^{u_n}$ of $L_{w_1}^{u_1} ... L_{w_n}^{u_n}$ is contained in $LP_n^*$). Notice that $\emptyset \notin LP_n^*$.

Proof. $(\Rightarrow)$ Let $l = l_{w_1}^{u_1} ... l_{w_n}^{u_n} \in LP_n^*$. Suppose that $w_1 = v_{w_1}v_1'$ and $w_n = v_nv_nv_n'$, for $v_1, v_1', v_n, v_n' \in V(G)$. If $l$ is in $LP_n^*$, then

$$\begin{cases}
  v_1 = v_1' & \text{if } u_{w_1} = 1 \text{ and } u_{w_n} = 1 \\
  v_1 = v_n & \text{if } u_{w_1} = 1 \text{ and } u_{w_n} = * \\
  v_1' = v_n' & \text{if } u_{w_1} = * \text{ and } u_{w_n} = 1 \\
  v_1' = v_n & \text{if } u_{w_1} = * \text{ and } u_{w_n} = *.
\end{cases} \tag{2.1.1}$$

By the definition of lattice paths having the *-axis-property and by (2.1.1), if $l_{w_1}^{u_1} ... l_{w_n}^{u_n} \in LP_n^*$, then there exists $v \in V(G)$ such that
Throughout this section, fix a \( L_{\mathbf{w}} \) of (2.1.1), we have that \( L_{\mathbf{w}} = L_v \) trivially, since \( L_{\mathbf{w}} \) is a nonempty lattice path \( l \). By the previous theorem, we can conclude that \( E (L_{\mathbf{w}}) = L_v \). This shows that \( E (L_{\mathbf{w}}) = L_v \). This contradicts our assumption.

\[
(\Rightarrow) \text{ Assume that } E (L_{\mathbf{w}}) \neq 0_{DG}. \text{ This means that there exists } L_v, \text{ with } v \in V(G), \text{ such that }
\]

\[
E (L_{\mathbf{w}}) = L_v.
\]

Equivalently, we have that \( L_{\mathbf{w}} = L_{\mathbf{w}} = L_v \) in \( W^*(G) \). Let \( l = l_{w_1, \ldots, w_n} \in LP_0 \) be the lattice path of the \( DG \)-valued random variable \( L_{\mathbf{w}} = L_{\mathbf{w}} \). By (2.1.3), trivially, \( l \neq 0 \), since \( l \) should be the connected lattice path. Assume that the nonempty lattice path \( l \) is contained in \( LP_0 \). Then, under the same conditions of (2.1.1), we have that

\[
\begin{aligned}
&v_1 \neq v_n' \quad \text{if } u_{w_1} = 1 \text{ and } u_{w_n} = 1 \\
v_1 \neq v_n \quad \text{if } u_{w_1} = 1 \text{ and } u_{w_n} = * \\
\alpha \neq \alpha' \quad \text{if } u_{w_1} = * \text{ and } u_{w_n} = 1 \\
\alpha' \neq v_n \quad \text{if } u_{w_1} = * \text{ and } u_{w_n} = *.
\end{aligned}
\]

Therefore, by (2.1.2), there is no vertex \( v \) satisfying \( L_{\mathbf{w}} = L_{\mathbf{w}} = L_v \). This contradicts our assumption.

By the previous theorem, we can conclude that \( E (L_{\mathbf{w}}) = L_v \), for some \( v \in V(G) \) if and only if the lattice path \( l_{w_1, \ldots, w_n} \) has the \( * \)-axis-property (i.e., \( l_{w_1, \ldots, w_n} \in LP_0^* \)).

2.2. \( DG \)-Valued Moments and Cumulants of Random Variables. Let \( w_1, \ldots, w_n \in \mathbb{F}^+(G) \), \( u_1, \ldots, u_n \in \{1, *\} \) and let \( L_{\mathbf{w}} = L_{\mathbf{w}} \in (W^*(G), E) \) be a \( DG \)-valued random variable. Recall that, in the previous section, we observed that the \( DG \)-valued random variable \( L_{\mathbf{w}} = L_{\mathbf{w}} = L_v \) with \( v \in V(G) \) if and only if the lattice path \( l_{w_1, \ldots, w_n} \) of \( L_{\mathbf{w}} = L_{\mathbf{w}} \) has the \( * \)-axis-property (equivalently, \( l_{w_1, \ldots, w_n} \in LP_0^* \)). Throughout this section, fix a \( DG \)-valued random variable \( a \in (W^*(G), E) \). Then the \( DG \)-valued random variable \( a \) has the following Fourier expansion,
\[ a = \sum_{v \in V(G:a)} p_v L_v + \sum_{l \in FP_1(G:a)} (p_l L_l + p_l L_l) + \sum_{w \in FP_2(G:a), u_w \in \{1,*\}} p_w L_{w}^{u_w}. \]

Let’s observe the new \( D_G \)-valued random variable \( d_1 a d_2 a ... d_n a \in (W^*(G), E) \), where \( d_1, ..., d_n \in D_G \) and \( a \in W^*(G) \) is given. Put

\[ d_j = \sum_{v_j \in V(G:d_j)} q_{v_j} L_{v_j} \in D_G, \text{ for } j = 1, ..., n. \]

Notice that \( V(G : d_j) = \mathbb{F}^+(G : d_j) \), since \( d_j \in D_G \hookrightarrow W^*(G) \). Then

\[
d_1 a d_2 a ... d_n a \\
= \left( \sum_{v_1 \in V(G:d_1)} q_{v_1} L_{v_1} \right) \left( \sum_{w_1 \in \mathbb{F}^+(G:a), u_{w_1} \in \{1,*\}} p_{w_1} L_{w_1}^{u_{w_1}} \right) \\
\ldots \left( \sum_{v_n \in V(G:d_n)} q_{v_n} L_{v_n} \right) \left( \sum_{w_n \in \mathbb{F}^+(G:a), u_{w_n} \in \{1,*\}} p_{w_n} L_{w_n}^{u_{w_n}} \right)
\]

\[
= \sum_{(v_1, ..., v_n) \in \prod_{j=1}^{n} V(G:d_j)} (q_{v_1} \cdots q_{v_n}) \\
\left( L_{v_1} \left( \sum_{w_1 \in \mathbb{F}^+(G:a), u_{w_1} \in \{1,*\}} p_{w_1} L_{w_1}^{u_{w_1}} \right) \right) \\
\ldots \left. \left( L_{v_n} \left( \sum_{w_n \in \mathbb{F}^+(G:a), u_{w_n} \in \{1,*\}} p_{w_n} L_{w_n}^{u_{w_n}} \right) \right) \right)
\]

(1.2.1)

\[
= \sum_{(v_1, ..., v_n) \in \prod_{j=1}^{n} V(G:d_j)} (q_{v_1} \cdots q_{v_n}) \\
\sum_{(w_1, ..., w_n) \in \mathbb{F}^+(G:a)^n, u_{w_j} \in \{1,*\}} (p_{w_1} \cdots p_{w_n}) L_{v_1} L_{w_1}^{u_{w_1}} \cdots L_{v_n} L_{w_n}^{u_{w_n}}.
\]

Now, consider the random variable \( L_{v_1} L_{w_1}^{u_{w_1}} \cdots L_{v_n} L_{w_n}^{u_{w_n}} \) in the formula (1.2.1). Suppose that \( w_j = x_j w_j y_j \), with \( x_j, y_j \in V(G) \), for all \( j = 1, ..., n \). Then

(1.2.2)

\[
L_{v_1} L_{w_1}^{u_{w_1}} \cdots L_{v_n} L_{w_n}^{u_{w_n}} = \delta_{(v_1, x_1, y_1; u_{w_1})} \cdots \delta_{(v_n, x_n, y_n; u_{w_n})} (L_{w_1}^{u_{w_1}} \cdots L_{w_n}^{u_{w_n}}),
\]

where

\[
\delta_{(v_j, x_j, y_j; u_{w_j})} = \begin{cases} 
\delta_{v_j, x_j} & \text{if } u_{w_j} = 1 \\
\delta_{v_j, y_j} & \text{if } u_{w_j} = \ast
\end{cases}
\]
for all $j = 1, \ldots, n$, where $\delta$ in the right-hand side is the Kronecker delta. So, the left-hand side can be understood as a (conditional) Kronecker delta depending on $\{1, *\}$.

By (1.2.1) and (1.2.2), the $n$-th moment of $a$ is

$$E(d_1 a \ldots d_n a) = E\left( \sum_{(v_1, \ldots, v_n) \in \Pi_{j=1}^n V(G:d_j)} (\Pi_{j=1}^n q_{v_j}) \sum_{(w_1, \ldots, w_n) \in \mathcal{F}(G:a)^n, w_j = x_j y_j, u_{w_j} \in \{1, *\}} (\Pi_{j=1}^n p_{w_j}) \left( \prod_{j=1}^n \delta(v_j, x_j, y_j; u_{w_j}) \right) \left( L_{w_1}^{u_{w_1}} \ldots L_{w_n}^{u_{w_n}} \right) \right).$$

Thus to compute the $n$-th moment of $a$, we have to observe $E \left( L_{w_1}^{u_{w_1}} \ldots L_{w_n}^{u_{w_n}} \right)$. In the previous section, we observed that $E \left( L_{w_1}^{u_{w_1}} \ldots L_{w_n}^{u_{w_n}} \right)$ is nonvanishing if and only if $L_{w_1}^{u_{w_1}} \ldots L_{w_n}^{u_{w_n}}$ has the $*$-axis-property.

**Proposition 2.2.** Let $a \in (W^*(G), E)$ be given as above. Then the $n$-th moment of $a$ is

$$E(d_1 a \ldots d_n a) = \sum_{(v_1, \ldots, v_n) \in \Pi_{j=1}^n V(G:d_j)} (\Pi_{j=1}^n q_{v_j}) \sum_{(w_1, \ldots, w_n) \in \mathcal{F}(G:a)^n, u_{w_j} \in \{1, *\}, L_{w_1}^{u_{w_1}} \ldots L_{w_n}^{u_{w_n}} \in \mathcal{L}_n} (\Pi_{j=1}^n p_{w_j}) \left( \prod_{j=1}^n \delta(v_j, x_j, y_j; u_{w_j}) \right) E \left( L_{w_1}^{u_{w_1}} \ldots L_{w_n}^{u_{w_n}} \right).$$

□

From now, rest of this paper, we will compute the $D_G$-valued cumulants of the given $D_G$-valued random variable $a$. Let $w_1, \ldots, w_n \in FP(G)$ be finite paths and $u_1, \ldots, u_n \in \{1, *\}$. Then, by the Möbius inversion, we have

$$E(d_1 a \ldots d_n a) = \sum_{(v_1, \ldots, v_n) \in \Pi_{j=1}^n V(G:d_j)} (\Pi_{j=1}^n q_{v_j}) \sum_{(w_1, \ldots, w_n) \in \mathcal{F}(G:a)^n, u_{w_j} \in \{1, *\}, L_{w_1}^{u_{w_1}} \ldots L_{w_n}^{u_{w_n}} \in \mathcal{L}_n} (\Pi_{j=1}^n p_{w_j}) \left( \prod_{j=1}^n \delta(v_j, x_j, y_j; u_{w_j}) \right) E \left( L_{w_1}^{u_{w_1}} \ldots L_{w_n}^{u_{w_n}} \right).$$

(2.2.1)
\[ k_n \left( L_{w_1}^{u_1}, ..., L_{w_n}^{u_n} \right) = \sum_{\pi \in NC(n)} \hat{E}(\pi) \left( L_{w_1}^{u_1} \otimes ... \otimes L_{w_n}^{u_n} \right) \mu(\pi, 1_n), \]

where \( \hat{E} = (E^{(n)})_{n=1}^{\infty} \) is the moment multiplicative bimodule map induced by the conditional expectation \( E \) (See [16]) and where \( NC(n) \) is the collection of all noncrossing partition over \( \{1, ..., n\} \). Notice that if \( L_{w_1}^{u_1} ... L_{w_n}^{u_n} \) does not have the \(*\)-axis-property, then

\[ E \left( L_{w_1}^{u_1} ... L_{w_n}^{u_n} \right) = 0_{DG}, \]

by Section 2.1. Consider the noncrossing partition \( \pi \in NC(n) \) with its blocks \( V_1, ..., V_k \). Choose one block \( V_j = (j_1, ..., j_k) \in \pi \). Then we have that

\[ (2.2.2) \quad \hat{E}(\pi | V_j) \left( L_{w_1}^{u_1} \otimes ... \otimes L_{w_n}^{u_n} \right) = E \left( L_{w_1}^{u_1} d_{j_1} L_{w_2}^{u_2} ... d_{j_k} L_{w_k}^{u_k} \right), \]

where

\[ d_{j_i} = \begin{cases} 1_{DG} & \text{if there is no inner blocks} \\ L_{v_j} \neq 1_{DG} & \text{if there are inner blocks} \end{cases} \]

between \( j_{i-1} \) and \( j_i \) in \( V_j \)

where \( v_{j_2}, ..., v_{j_k} \in V(G) \). So, again by Section 2.1, \( \hat{E}(\pi | V_j) \left( L_{w_1}^{u_1} \otimes ... \otimes L_{w_n}^{u_n} \right) \) is nonvanishing if and only if \( L_{w_1}^{u_1} d_{j_1} L_{w_2}^{u_2} ... d_{j_k} L_{w_k}^{u_k} \) has the \(*\)-axis-property, for all \( j = 1, ..., n \).

Assume that

\[ \hat{E}(\pi | V_j) \left( L_{w_1}^{u_1} \otimes ... \otimes L_{w_n}^{u_n} \right) = L_{v_j}, \]

and

\[ \hat{E}(\pi | V_i) \left( L_{w_1}^{u_1} \otimes ... \otimes L_{w_n}^{u_n} \right) = L_{v_i}. \]

If \( v_j \neq v_i \), then the partition-dependent \( DG \)-moment satisfies that

\[ \hat{E}(\pi) \left( L_{w_1}^{u_1} \otimes ... \otimes L_{w_n}^{u_n} \right) = 0_{DG}. \]

This says that \( \hat{E}(\pi) \left( L_{w_1}^{u_1} \otimes ... \otimes L_{w_n}^{u_n} \right) \neq 0_{DG} \) if and only if there exists \( v \in V(G) \) such that

\[ \hat{E}(\pi | V_j) \left( L_{w_1}^{u_1} \otimes ... \otimes L_{w_n}^{u_n} \right) = L_v, \]

for all \( j = 1, ..., k \).
Definition 2.4. Let \( NC(n) \) be the set of all noncrossing partition over \( \{1, ..., n\} \) and let \( L_{u_1}^{u_n} \in (W^*(G), E) \) be \( D_G \)-valued random variables, where \( u_1, ..., u_n \in \{1, *\} \). We say that the \( D_G \)-valued random variable \( L_{u_1}^{u_n} \) is \( \pi \)-connected if the \( \pi \)-dependent \( D_G \)-moment of it is nonvanishing, for \( \pi \in NC(n) \). In other words, the random variable \( L_{u_1}^{u_n} \) is \( \pi \)-connected, for \( \pi \in NC(n) \), if

\[
\hat{E}(\pi) \left( L_{u_1}^{u_n} \otimes ... \otimes L_{w_n}^{w_n} \right) \neq 0_{D_G}.
\]

i.e., there exists a vertex \( v \in V(G) \) such that

\[
\hat{E}(\pi) \left( L_{u_1}^{u_n} \otimes ... \otimes L_{w_n}^{w_n} \right) = L_v.
\]

For convenience, we will define the following subset of \( NC(n) \):

Definition 2.5. Let \( NC(n) \) be the set of all noncrossing partitions over \( \{1, ..., n\} \) and fix a \( D_G \)-valued random variable \( L_{u_1}^{u_n} \) in \( (W^*(G), E) \), where \( u_1, ..., u_n \in \{1, *\} \). For the fixed \( D_G \)-valued random variable \( L_{u_1}^{u_n} \), define

\[
C_{u_1, ..., u_n} \overset{\text{def}}{=} \{ \pi \in NC(n) : L_{u_1}^{u_n} \text{ is } \pi \text{-connected} \},
\]

in \( NC(n) \). Let \( \mu \) be the Möbius function in the incidence algebra \( I_2 \). Define the number \( \mu_{u_1, ..., u_n} \), for the fixed \( D_G \)-valued random variable \( L_{u_1}^{u_n} \), by

\[
\mu_{u_1, ..., u_n} \overset{\text{def}}{=} \sum_{\pi \in C_{u_1, ..., u_n}} \mu(\pi, 1_n).
\]

Assume that there exists \( \pi \in NC(n) \) such that \( L_{u_1}^{u_n} = L_v \) is \( \pi \)-connected. Then \( \pi \in C_{u_1, ..., u_n} \) and there exists the maximal partition \( \pi_0 \in C_{u_1, ..., u_n} \), such that \( L_{u_1}^{u_n} = L_v \) is \( \pi_0 \)-connected. (Recall that \( NC(n) \) is a lattice. We can restrict this lattice ordering on \( C_{u_1, ..., u_n} \) and hence it is a POset, again.) Notice that \( 1_n \in C_{u_1, ..., u_n} \). Therefore, the maximal partition in \( C_{u_1, ..., u_n} \) is \( 1_n \). Hence we have that:

Lemma 2.3. Let \( L_{u_1}^{u_n} \in (W^*(G), E) \) be a \( D_G \)-valued random variable having the *-axis-property. Then

\[
E \left( L_{u_1}^{u_n} \right) = \hat{E}(\pi) \left( L_{u_1}^{u_n} \otimes ... \otimes L_{w_n}^{w_n} \right),
\]

for all \( \pi \in C_{u_1, ..., u_n} \).

Proof. By the previous discussion, we can get the result. \( \square \)
By the previous lemmas, we have that

**Theorem 2.4.** Let \( n \in 2\mathbb{N} \) and let \( L_{w_1}^{u_1}, ..., L_{w_n}^{u_n} \in (W^*(G), E) \) be \( D_G \)-valued random variables, where \( w_1, ..., w_n \in FP(G) \) and \( u_j \in \{1, *\}, j = 1, ..., n \). Then

\[
k_n (L_{w_1}^{u_1} ... L_{w_n}^{u_n}) = \mu_{u_1 w_1}^{u_n w_n} \cdot E(L_{w_1}^{u_1} ... L_{w_n}^{u_n})
\]

where \( \mu_{u_1 w_1}^{u_n w_n} = \sum_{\pi \in C_{u_1 w_1}^{u_n w_n}} \mu(\pi, 1_n) \).

**Proof.** We can compute that

\[
k_n (L_{w_1}^{u_1} ... L_{w_n}^{u_n}) = \sum_{\pi \in NC(n)} \hat{E}(\pi) (L_{w_1}^{u_1} \otimes ... \otimes L_{w_n}^{u_n}) \mu(\pi, 1_n)
\]

by the \( \pi \)-connectedness

\[
= \sum_{\pi \in C_{u_1 w_1}^{u_n w_n}} \hat{E}(\pi) (L_{w_1}^{u_1} \otimes ... \otimes L_{w_n}^{u_n}) \mu(\pi, 1_n)
\]

by the previous lemma

\[
= \left( \sum_{\pi \in C_{u_1 w_1}^{u_n w_n}} \mu(\pi, 1_n) \right) \cdot E (L_{w_1}^{u_1} ... L_{w_n}^{u_n}).
\]

Now, we can get the following \( D_G \)-valued cumulants of the random variable \( a \);

**Corollary 2.5.** Let \( n \in \mathbb{N} \) and let \( a = a_d + a_{(\ast)} + a_{(\text{non-}\ast)} \in (W^*(G), E) \) be our \( D_G \)-valued random variable. Then \( k_1 (d_1 a) = d_1 a_d \) and \( k_n (d_1 a, ..., d_n a) = 0_{D_G} \), for all odd \( n \). If \( n \in \mathbb{N} \setminus \{1\} \), then

\[
k_n (d_1 a, ..., d_n a) = \sum_{(v_1, ..., v_n) \in \Pi_{j=1}^n V(G; d_j)} (\Pi_{j=1}^n q_{v_j})
\]

\[
(\Pi_{j=1}^n \delta(v_j, x_j, y_j, u_j)) (\mu_{w_1}^{w_n}, E (L_{w_1}^{u_1} ... L_{w_n}^{u_n})),
\]

where \( w_1, ..., w_n \in FP^*(G; a)^n, w_j = x_j w_j y_j, u_j \in \{1, *\}, v_1, ..., v_n, u_1, ..., u_n, \in L^{P^*}_n \)
where \( d_1, \ldots, d_n \in D_G \) are arbitrary. \( \square \)

We have the following trivial \( D_G \)-valued moments and cumulants of an arbitrary \( D_G \)-valued random variable:

**Corollary 2.6.** Let \( a \in (W^*(G), E) \) be a \( D_G \)-valued random variable and let \( n \in \mathbb{N} \). Then

1. The \( n \)-th trivial \( D_G \)-valued moment of \( a \) is

\[
E(a^n) = \sum_{(w_1, \ldots, w_n) \in \mathcal{F}^+(G:a)^n, \ w_{ij} \in \{1,*\}, \ l_{w_1}^{u_1} \cdots l_{w_n}^{u_n} \in LP_n^*} \left( \prod_{j=1}^n p_{w_j} \right) \cdot E \left( L_{w_1}^{u_1}, \ldots, L_{w_n}^{u_n} \right),
\]

2. The \( n \)-th trivial \( D_G \)-valued cumulant of \( a \) is

\[
k_1(a) = E(a) = a_d
\]

and

\[
k_n \left( \underbrace{a, \ldots, a}_{n \text{-times}} \right) = \sum_{(w_1, \ldots, w_n) \in \mathcal{F}P(G:a)^n, \ w_{ij} \in \{1,*\}, \ l_{w_1}^{u_1} \cdots l_{w_n}^{u_n} \in LP_n^*} \left( \prod_{j=1}^n p_{w_j} \right) \cdot E \left( L_{w_1}^{u_1}, \ldots, L_{w_n}^{u_n} \right),
\]

where \( d_1, \ldots, d_n \in D_G \) are arbitrary. \( \square \)

3. \( D_G \)-Freeness on \((W^*(G), E)\)

Like before, throughout this chapter, let \( G \) be a countable directed graph and \((W^*(G), E)\), the graph \( W^* \)-probability space over its diagonal subalgebra \( D_G \). In this chapter, we will consider the \( D_G \)-valued freeness of given two random variables in \((W^*(G), E)\). We will characterize the \( D_G \)-freeness of \( D_G \)-valued random variables \( L_{w_1} \) and \( L_{w_2} \), where \( w_1 \neq w_2 \in FP(G) \). And then we will observe the \( D_G \)-freeness of arbitrary two \( D_G \)-valued random variables \( a_1 \) and \( a_2 \) in terms of their supports.

Let

\[
a = \sum_{w \in \mathcal{F}^+(G:a), \ w \in \{1,*\}} p_w L_{w}^{u} \quad \& \quad b = \sum_{w' \in \mathcal{F}^+(G:b), \ w' \in \{1,*\}} p_{w'} L_{w'}^{u'}
\]
be fixed $D_G$-valued random variables in $(W^*(G), E)$.

Now, fix $n \in \mathbb{N}$ and let $(a_{i_1}^{\varepsilon_{i_1}}, ..., a_{i_n}^{\varepsilon_{i_n}}) \in \{a, b, a^*, b^*\}^n$, where $\varepsilon_{i_j} \in \{1, *\}$. For convenience, put

$$a_{i_j}^{\varepsilon_{i_j}} = \sum_{w_{i_j} \in \mathbb{F}^+(G:a_{i_k}), u_{i_j} \in \{1, *\}} p^{(j)}_{w_{i_j}} L^{u_{i_j}}_{w_{i_j}}, \text{ for } j = 1, ..., n.$$

Then, by the little modification of Section , we have that:

(3.1)

$$E(d_{i_1} a_{i_1}^{\varepsilon_{i_1}} ... d_{i_n} a_{i_n}^{\varepsilon_{i_n}})$$

$$= \sum_{(v_{i_1}, ..., v_{i_n}) \in \Pi_{k=1}^n V(G:d_{i_k})} \left( \Pi_{k=1}^n q^p_{v_i} \right) \sum_{(w_{i_1}, ..., w_{i_n}) \in \Pi_{k=1}^n \mathbb{F}^+(G:a_{i_k}), w_{i_j} = x_{i_j} w_{i_j} y_{i_j}, u_{i_j} \in \{1, *\}} \left( \Pi_{j=1}^n \delta_{(v_{i_j}, x_{i_j}, y_{i_j}, u_{i_j})} \right) \left( \Pi_{j=1}^n \delta_{(u_{i_{j-1}}, u_{i_j}, u_{i_j})} \right) E(L^{u_{i_1}}_{w_{i_1}} ... L^{u_{i_n}}_{w_{i_n}}).$$

Therefore, we have that

(3.2)

$$k_n (d_{i_1} a_{i_1}^{\varepsilon_{i_1}} ..., d_{i_n} a_{i_n}^{\varepsilon_{i_n}}) = \sum_{(v_{i_1}, ..., v_{i_n}) \in \Pi_{k=1}^n V(G:d_{i_k})} \left( \Pi_{k=1}^n q^p_{v_i} \right) \sum_{(w_{i_1}, ..., w_{i_n}) \in \Pi_{k=1}^n \mathbb{F}P(G:a_{i_k}), w_{i_j} = x_{i_j} w_{i_j} y_{i_j}, u_{i_j} \in \{1, *\}} \left( \Pi_{j=1}^n \delta_{(v_{i_j}, x_{i_j}, y_{i_j}, u_{i_j})} \right) \left( \Pi_{j=1}^n \delta_{(u_{i_{j-1}}, u_{i_j}, u_{i_j})} \right) \left( \mu_{u_{i_1}, ..., u_{i_n}} L^{u_{i_1}}_{w_{i_1}} ... L^{u_{i_n}}_{w_{i_n}} \right),$$

where $\mu_{u_{i_1}, ..., u_{i_n}} = \sum_{\pi \in C_{w_{i_1}, ..., w_{i_n}}} \mu(\pi, 1_n)$ and

$$C_{w_{i_1}, ..., w_{i_n}} = \{\pi \in NC^{(even)}(n) : L^{u_{i_1}}_{w_{i_1}} ... L^{u_{i_n}}_{w_{i_n}} \text{ is } \pi\text{-connected}\}.$$
Proposition 3.1. Let \( a, b \in (W^*(G), E) \) be \( D_G \)-valued random variables, such that \( a \notin W^*(\{b\}, D_G) \), and let \((a_{i_1}^{\varepsilon_{i_1}}, ..., a_{i_n}^{\varepsilon_{i_n}}) \in \{a, b, a^*, b^*\}^n \), for \( n \in \mathbb{N} \setminus \{1\} \), where \( \varepsilon_{i_j} \in \{1, \ast\} \), \( j = 1, ..., n \). Then

\[
(3.3)\]

\[
k_n (d_{i_1}^{a_{i_1}^{\varepsilon_{i_1}}}, ..., d_{i_n}^{a_{i_n}^{\varepsilon_{i_n}}}) = \sum_{(v_1, ..., v_n) \in (\Pi_{j=1}^n V(G) : d_j)} (\Pi_{k=1}^n q_{v_k}) \sum_{(w_{i_1}, ..., w_{i_n}) \in (\Pi_{k=1}^n FP_*(G : a_{i_k}^{\varepsilon_{i_k}}) \cup W^{i_1}_{u_1} \cdots w_n, w_{i_j} = x_{i_j} w_{i_j} y_{i_j})} \left( \Pi_{j=1}^n \delta(v_{ij} x_{ij} y_{ij} u_{ij}) \right) (\mu_{w_{i_1}^{u_{i_1}} \cdots w_{i_n}^{u_{i_n}}} \cdot Pr_{proj}(L_{w_{i_1}^{u_{i_1}}} \cdots L_{w_{i_n}^{u_{i_n}}}))
\]

where \( \mu_n = \sum_{\pi \in C_{w_{i_1}^{u_{i_1}} \cdots w_{i_n}^{u_{i_n}}}} \mu(\pi, 1_n) \) and

\[
W^{i_1}_{u_1} \cdots w_n = \{ w \in FP_*(G : a) \cup FP_*(G : b) : \text{both } L_{w^{u_{i_1}}} \text{ and } L_{w^{u_{i_n}}} \text{ are in } L_{w_{i_1}^{u_{i_1}}} \cdots L_{w_{i_n}^{u_{i_n}}} \}.
\]

□

Corollary 3.2. Let \( x \) and \( y \) be the \( D_G \)-valued random variables in \((W^*(G), E)\). The \( D_G \)-valued random variables \( a \) and \( b \) are free over \( D_G \) in \((W^*(G), E)\) if

\[
FP_*(G : P(x, x^*)) \cap FP_*(G : Q(y, y^*)) = \emptyset
\]

and

\[
W^{P(x, x^*), Q(y, y^*)}_* = \emptyset,
\]

for all \( P, Q \in \mathbb{C}[z_1, z_2] \). □

By using (3.2), we can compute the mixed \( D_G \)-valued cumulants of two \( D_G \)-valued random variables. However, the formula is very abstract. So, we will consider the above formula for fixed two generators of \( W^*(G) \).

Definition 3.1. Let \( G \) be a countable directed graph and \( \mathbb{F}^+(G) \), the free semigroupoid of \( G \) and let \( FP(G) \) be the subset of \( \mathbb{F}^+(G) \) consisting of all finite paths. Define a subset \( \text{loop}(G) \) of \( FP(G) \) containing all loop finite paths or loops. (Remark that, in general, loop finite paths are different from loop-edges. Clearly, all loop-edges are loops in \( FP(G) \).) i.e.,

\[
\text{loop}(G) \overset{\text{def}}{=} \{ l \in FP(G) : l \text{ is a loop} \} \subset FP(G).
\]
Also define the subset $\text{loop}^c(G)$ of $FP(G)$ consisting of all non-loop finite path by

$$\text{loop}^c(G) \overset{\text{def}}{=} FP(G) \setminus \text{loop}(G).$$

Let $l \in \text{loop}(G)$ be a loop finite path. We say that $l$ is a basic loop if there exists no loop $w \in \text{loop}(G)$ such that $l = w^k$, $k \in \mathbb{N} \setminus \{1\}$. Define

$$\text{Loop}(G) \overset{\text{def}}{=} \{ l \in \text{loop}(G) : l \text{ is a basic loop} \} \subset \text{loop}(G).$$

Let $l_1 = w_1^{k_1}$ and $l_2 = w_2^{k_2}$ in $\text{loop}(G)$, where $w_1, w_2 \in \text{Loop}(G)$. We will say that the loops $l_1$ and $l_2$ are diagram-distinct if $w_1 \neq w_2$ in $\text{Loop}(G)$. Otherwise, they are not diagram-distinct.

Now, we will introduce the more general diagram-distinctness of general finite paths:

**Definition 3.2. (Diagram-Distinctness)** We will say that the finite paths $w_1$ and $w_2$ are diagram-distinct if $w_1$ and $w_2$ have different diagrams in the graph $G$. Let $X_1$ and $X_2$ be subsets of $FP(G)$. The subsets $X_1$ and $X_2$ are said to be diagram-distinct if $x_1$ and $x_2$ are diagram-distinct for all pairs $(x_1, x_2) \in X_1 \times X_2$. This diagram-distinctness implies the diagram-distinctness of loops.

Let $H$ be a directed graph with $V(H) = \{v_1, v_2\}$ and $E(H) = \{ e_1 = v_1v_2, e_2 = v_2v_1 \}$. Then $l = e_1e_2$ is a loop in $FP(H)$ (i.e., $l \in \text{loop}(H)$). Moreover, it is a basic loop (i.e., $l \in \text{Loop}(H)$). However, if we have a loop $w = e_1e_2e_1e_2 = l^2$, then it is not a basic loop. i.e.,

$$l^2 \in \text{loop}(H) \setminus \text{Loop}(H).$$

If the graph $G$ contains at least one basic loop $l \in FP(G)$, then we have

$$\{ l^n : n \in \mathbb{N} \} \subset \text{loop}(G) \quad \text{and} \quad \{ l \} \subset \text{Loop}(G).$$

Suppose that $l_1$ and $l_2$ are not diagram-distinct. Then, by definition, there exists $w \in \text{Loop}(G)$ such that $l_1 = w^{k_1}$ and $l_2 = w^{k_2}$, for some $k_1, k_2 \in \mathbb{N}$. On the graph $G$, indeed, $l_1$ and $l_2$ make the same diagram. On the other hand, we can see that if $w_1 \neq w_2 \in \text{loop}^c(G)$, then they are automatically diagram-distinct.

**Lemma 3.3.** Suppose that $w_1 \neq w_2 \in \text{loop}^c(G)$ with $w_1 = v_{11}v_1v_{12}$ and $w_2 = v_{21}v_2v_{22}$. Then $L_{w_1}$ and $L_{w_2}$ are free over $D_G$ in $(W^*(G), E)$.

**Proof.** By definition, $L_{w_1}$ and $L_{w_2}$ are free over $D_G$ if and only if all mixed $D_G$-valued cumulants of $W^*(\{ L_{w_1} \}, D_G)$ and $W^*(\{ L_{w_2} \}, D_G)$ vanish. Equivalently,
all $D_G$-valued cumulants of $P (L_{w_1}, L_{w_1}^*)$ and $Q (L_{w_2}, L_{w_2}^*)$ vanish, for all $P, Q \in \mathbb{C}[z_1, z_2]$. Since $w_1 \neq w_2$ are non-loop edges, we can easily verify that $w_1^{k_1}$ and $w_2^{k_2}$ are not admissible (i.e., $w_1^{k_1} \notin \mathbb{F}^+(G)$ and $w_2^{k_2} \notin \mathbb{F}^+(G)$), for all $k_1, k_2 \in \mathbb{N} \setminus \{1\}$. This shows that

$$L_{w_1}^k = 0_{D_G} = \left(L_{w_j}^k\right)^*, \text{ for } j = 1, 2.$$  

Thus, to show that $L_{w_1}$ and $L_{w_2}$ are free over $D_G$, it suffices to show that all mixed $D_G$-valued cumulants of $P (L_{w_1}, L_{w_1}^*)$ and $Q (L_{w_2}, L_{w_2}^*)$ vanish, for all $P, Q \in \mathbb{C}[z_1, z_2]$ such that

$$P(z_1, z_2) = \alpha_1 z_1 + \alpha_2 z_1 z_2 + \alpha_3 z_2 z_1 + \alpha_4 z_2$$

and

$$Q(z_1, z_2) = \beta_1 z_1 + \beta_2 z_1 z_2 + \beta_3 z_2 z_1 + \beta_4 z_2,$$

where $\alpha, \beta \in \mathbb{C}$. So, for such $P$ and $Q$, we have that

$$P (L_{w_1}, L_{w_1}^*) = \alpha_1 L_{w_1} + \alpha_2 L_{v_{11}} + \alpha_3 L_{v_{12}} + \alpha_4 L_{w_1}^*$$

and

$$Q (L_{w_2}, L_{w_2}^*) = \beta_1 L_{w_2} + \beta_2 L_{v_{21}} + \beta_3 L_{v_{22}} + \beta_4 L_{w_2}^*.$$

Thus, we have that

$$FP_0 (G : P (L_{w_1}, L_{w_1}^*)) \supseteq \{w_1\}, \quad FP_0 (G : Q (L_{w_2}, L_{w_2}^*)) \supseteq \{w_2\}$$

and

$$FP_0^c (G : P (L_{w_1}, L_{w_1}^*)) \supseteq \{w_1\}, \quad FP_0^c (G : Q (L_{w_2}, L_{w_2}^*)) \supseteq \{w_2\}.$$

Remark that if $FP_0 (G : P (L_{w_1}, L_{w_1}^*)) = \{w_1\}$, then $FP_0^c (G : P (L_{w_1}, L_{w_1}^*)) = \emptyset$, and if $FP_0^c (G : P (L_{w_1}, L_{w_1}^*)) = \{w_1\}$, then $FP_0 (G : P (L_{w_1}, L_{w_1}^*)) = \emptyset$. The similar relation holds for $Q (L_{w_2}, L_{w_2}^*)$. So, we have that

$$FP_0 (G : P (L_{w_1}, L_{w_1}^*)) \cap FP_0 (G : Q (L_{w_2}, L_{w_2}^*)) = \emptyset$$

and

$$W_0^c (P (L_{w_1}, L_{w_1}^*), Q (L_{w_2}, L_{w_2}^*)) = \emptyset.$$

Therefore, by the formula (3.4.3), we have the vanishing mixed $D_G$-valued cumulants of $P (L_{w_1}, L_{w_1}^*)$ and $Q (L_{w_2}, L_{w_2}^*)$, for all $n \in \mathbb{N}$ and for all such $P, Q \in \mathbb{C}[z_1, z_2]$. So, we can conclude that $L_{w_1}$ and $L_{w_2}$ are free over $D_G$ in $(W^*(G), E).$  

Now, we will consider the loop case.

**Lemma 3.4.** Let $l_1 \neq l_2 \in \text{Loop}(G)$ be basic loops such that $l_1 = v_1 l_1 v_1$ and $l_2 = v_2 l_2 v_2$, for $v_1, v_2 \in V(G)$ (possibly $v_1 = v_2$). i.e., two basic loops $l_1$ and $l_2$ are
Then the $D_G$-valued random variables $L_{i_1}$ and $L_{i_2}$ are free over $D_G$ in $(W^*(G), E)$. 

Proof. Different from the non-loop case, if $l_1$ and $l_2$ are loops, then $l_1^{k_1}$ and $l_2^{k_2}$ exist in $FP(G)$, for all $k_1, k_2 \in \mathbb{N}$. To show that $L_{i_1}$ and $L_{i_2}$ are free over $D_G$, it suffices to show that all mixed $D_G$-valued cumulants of $P(L_{w_1}, L_{w_1}^*)$ and $Q(L_{w_2}, L_{w_2}^*)$ vanish, for all $P, Q \in \mathbb{C}[z_1, z_2]$, such that

$$P(z_1, z_2) = f_1(z_1) + f_2(z_2) + P_0(z_1, z_2)$$
and
$$Q(z_1, z_2) = g_1(z_1) + g_2(z_2) + Q_0(z_1, z_2),$$

where $f_1, f_2, g_1, g_2 \in \mathbb{C}[z]$ and $P_0, Q_0 \in \mathbb{C}[z_1, z_2]$ such that $P_0$ and $Q_0$ does not contain polynomials only in $z_1$ and $z_2$. So, for such $P$ and $Q$, we have that

$$P(L_{i_1}, L_{i_1}^*) = f_1(L_{i_1}) + f_2(L_{i_1}^*) + P_0(L_{i_1}, L_{i_1}^*)$$
and
$$Q(L_{i_2}, L_{i_2}^*) = g_1(L_{i_2}) + g_2(L_{i_2}^*) + Q_0(L_{i_2}, L_{i_2}^*).$$

Notice that $L_{i_j}^k = L_{i_j}$, for all $k \in \mathbb{N}$, $j = 1, 2$. Also, notice that

$$P_0(L_{i_1}, L_{i_1}^*) = f_1^0(L_{i_1}) + f_2^0(L_{i_1}^*) + \alpha L_{v_1}$$
and
$$Q_0(L_{i_2}, L_{i_2}^*) = g_1^0(L_{i_2}) + g_2^0(L_{i_2}^*) + \beta L_{v_2},$$

where $f_1^0, f_2^0, g_1^0, g_2^0 \in \mathbb{C}[z]$ and $\alpha, \beta \in \mathbb{C}$, by the fact that

$$L_{i_j}^k L_{i_j} = L_{v_j} = L_{i_j} L_{i_j}^k,$$

under the weak-topology. So, finally, we have that

$$P(L_{i_1}, L_{i_1}^*) = f_1(L_{i_1}) + f_2(L_{i_1}^*) + \alpha L_{v_1}$$
and
$$Q(L_{i_2}, L_{i_2}^*) = g_1(L_{i_2}) + g_2(L_{i_2}^*) + \beta L_{v_2},$$

where $f_1, f_2, g_1, g_2 \in \mathbb{C}[z]$ and $\alpha, \beta \in \mathbb{C}$. Thus, we have that

$$FP_*(G : P(L_{i_1}, L_{i_1}^*)) \subseteq \{t_1^k\}_{k=1}^\infty, \quad FP_*(G : Q(L_{i_2}, L_{i_2}^*)) \subseteq \{t_2^k\}_{k=1}^\infty$$
and

$$FP_*(G : P(L_{i_1}, L_{i_1}^*)) \subseteq \{t_1^k\}_{k=1}^\infty, \quad FP_*(G : Q(L_{i_2}, L_{i_2}^*)) \subseteq \{t_2^k\}_{k=1}^\infty.$$
because $l_1$ and $l_2$ are in $\text{Loop}(G)$ (and hence if $l_1 \neq l_2$, then they are diagram-distinct.) And we have that

$$W^*_1\{P(L_{w_1}, L_{w_1}^*), Q(L_{w_2}, L_{w_2}^*)\} = \emptyset.$$  

Therefore, by the formula (3.4.3), we have the vanishing mixed $D_G$-valued cumulants of $P(L_{l_1}, L_{l_1}^*)$ and $Q(L_{l_2}, L_{l_2}^*)$, for all $n \in \mathbb{N}$ and for all $P, Q \in \mathbb{C}[z_1, z_2]$. Since $P$ and $Q$ are arbitrary, we can conclude that $L_{l_1}$ and $L_{l_2}$ are free over $D_G$ in $(W^*(G), E)$. 

Notice that we assumed that the loops $l_1$ and $l_2$ are basic loops in the previous lemma. Since they are distinct basic loops, they are automatically diagram-distinct. Now, assume that $l_1$ and $l_2$ are not diagram-distinct. i.e., there exists a basic loop $w \in \text{Loop}(G)$ such that $l_1 = w^{k_1}$ and $l_2 = w^{k_2}$, for some $k_1, k_2 \in \mathbb{N}$. In other words, the loops $l_1$ and $l_2$ have the same diagram in the graph $G$. Then the $D_G$-valued random variables $L_{l_1}$ and $L_{l_2}$ are not free over $D_G$ in $(W^*(G), E)$. See the next example ;

**Example 3.1.** Let $G_1$ be a directed graph with $V(G_1) = \{v\}$ and $E(G_1) = \{l = \{v, v\}\}$. So, in this case,

$$E(G_1) = \text{Loop}(G_1), \quad FP(G_1) = \text{loop}(G_1),$$

and

$$\text{loop}(G_1) = \{l^k : k \in \mathbb{N}\}.$$  

Thus, even if $w_1 \neq w_2 \in \text{loop}(G_1)$, $w_1$ and $w_2$ are Not diagram-distinct. Take $l^2$ and $l^3$ in $FP(G_1)$. Then the $D_{G_1}$-valued random variable $L_{l^2}$ and $L_{l^3}$ are not free over $D_{G_1}$ in $(W^*(G_1), E)$. Indeed, let’s take $P, Q \in \mathbb{C}[z_1, z_2]$ as

$$P(z_1, z_2) = z_1^3 + z_2^3$$

and

$$Q(z_1, z_2) = z_1^2 + z_2^2.$$  

Then

$$P(L_{l^2}, L_{l^2}^*) = L_{l^2}^3 + L_{l^2}^* = L_{l^6} + L_{l^6}^*$$

and

$$Q(L_{l^3}, L_{l^3}^*) = L_{l^3}^2 + L_{l^3}^* = L_{l^6} + L_{l^6}^*.$$  

Then

$$k_2(P(L_{l^2}, L_{l^2}^*), Q(L_{l^3}, L_{l^3}^*)) = k_2(L_{l^6} + L_{l^6}^*, L_{l^6} + L_{l^6}^*)$$

$$= \mu_{i_{l^6}, i_{l^6}}^* \text{Pr} oj(L_{l^6}, L_{l^6}^*) + \mu_{i_{l^6}, i_{l^6}}^* \text{Pr} oj(L_{l^6}^*, L_{l^6})$$

$$= \mu_{i_{l^6}, i_{l^6}}^* L_{l^6} + \mu_{i_{l^6}, i_{l^6}}^* L_{l^6} = (\mu_{i_{l^6}, i_{l^6}}^* + \mu_{i_{l^6}, i_{l^6}}^*) L_{l^6}$$

$$= 2L_{l^6} \neq 0_{D_{G_1}}.$$
Lemma 3.5. Let \( W \) be loops and assume that \( W_1 = w_1^{k_1} \) and \( W_2 = w_2^{k_2} \), where \( w_1, w_2 \in \text{Loop}(G) \) are basic loops and \( k_1, k_2 \in \mathbb{N} \). If \( w_1 \neq w_2 \), then the \( D_G \)-valued random variables \( W_1, W_2 \) are not free over \( D_G \).\( \square \)

As we have seen before, if two loops \( l_1 \) and \( l_2 \) are not diagram-distinct, then \( D_G \)-valued random variables \( L_1 \) and \( L_2 \) are not free over \( D_G \). However, if \( l_1 \) and \( l_2 \) are diagram-distinct, we can have the following lemma, by the previous lemma:

Lemma 3.6. Let \( l \in \text{loop}(G) \) and \( w \in \text{loop}^\circ(G) \). Then the \( D_G \)-valued random variables \( L_l \) and \( L_w \) are free over \( D_G \) in \((W^*(G), E)\).

Proof. Let \( l \in \text{loop}(G) \) and \( w \in \text{loop}^\circ(G) \) and let \( L_l \) and \( L_w \) be the corresponding \( D_G \)-valued random variables in \((W^*(G), E)\). Then, for all \( P, Q \in \mathbb{C}[z_1, z_2] \), we have that

\[
FP_*(G : P(L_l, L_l^*)) \cap FP_*(G : Q(L_w, L_w^*)) = \emptyset,
\]

since

\[
FP_*(G : P(L_l, L_l^*)) \subseteq \{ t^k : k \in \mathbb{N} \} \subset \text{loop}(G)
\]

and

\[
FP_*(G : Q(L_w, L_w^*)) = \{ w \} \subset \text{loop}^\circ(G).
\]

Also, since \( \text{loop}(G) \cap \text{loop}^\circ(G) = \emptyset \), we can get that

\[
W_* \{ P(L_l, L_l^*), Q(L_w, L_w^*) \} = \emptyset,
\]

for all \( P, Q \in \mathbb{C}[z_1, z_2] \). Therefore, the \( D_G \)-valued random variables \( L_l \) and \( L_w \) are free over \( D_G \) in \((W^*(G), E)\).\( \square \)

Finally, we will observe the following case when we have a loop and a non-loop finite path.

Lemma 3.7. Let \( l \in \text{loop}(G) \) and \( w \in \text{loop}^\circ(G) \). Then the \( D_G \)-valued random variables \( L_l \) and \( L_w \) are free over \( D_G \) in \((W^*(G), E)\).

Theorem 3.7. Let \( w_1, w_2 \in FP(G) \) be finite paths. The \( D_G \)-valued random variables \( L_{w_1} \) and \( L_{w_2} \) in \((W^*(G), E)\) are free over \( D_G \) if and only if \( w_1 \) and \( w_2 \) are diagram-distinct.
Proof. \(\Leftarrow\) Suppose that finite paths \(w_1\) and \(w_2\) are diagram-distinct. Then the \(D_G\)-valued random variables \(L_{w_1}\) and \(L_{w_2}\) are free over \(D_G\), by the previous lemmas.

(\(\Rightarrow\)) Let \(L_{w_1}\) and \(L_{w_2}\) be free over \(D_G\) in \((W^*(G), E)\). Now, assume that \(w_1\) and \(w_2\) are not diagram-distinct. We will observe the following cases:

(Case I) The finite paths \(w_1, w_2 \in loop(G)\). Since they are not diagram-distinct, there exists a basic loop \(l = vlv \in Loop(G)\), with \(v \in V(G)\), such that \(w_1 = l^{k_1}\) and \(w_2 = l^{k_2}\), for some \(k_1, k_2 \in \mathbb{N}\). As we have seen before, \(L_{w_1}\) and \(L_{w_2}\) are not free over \(D_G\) in \((W^*(G), E)\). Indeed, if we let \(k \in \mathbb{N}\) such that \(k_1 | k\) and \(k_2 | k\) with \(k = k_1 n_1 = k_2 n_2\), for \(n_1, n_2 \in \mathbb{N}\), then we can take \(P, Q \in \mathbb{C}[z_1, z_2]\) defined by

\[
P(z_1, z_2) = z_1^{n_1} + z_2^{n_1} \quad \text{and} \quad Q(z_1, z_2) = z_1^{n_2} + z_2^{n_2}.
\]

And then

\[
P(L_{w_1}, L_{w_1}^*) = L_{w_1}^{n_1} + L_{w_1}^{n_1} = L_{l^{k_1}}^{n_1} = L_{l^{k_1}}^* + L_{l^{k_1}}^*
\]

and

\[
Q(L_{w_2}, L_{w_2}^*) = L_{w_2}^{n_2} + L_{w_2}^{n_2} = L_{l^{k_2}}^{n_2} = L_{l^{k_2}}^* + L_{l^{k_2}}^*.
\]

So,

\[
k_2 \left( P(L_{w_1}, L_{w_1}^*), Q(L_{w_2}, L_{w_2}^*) \right) = k_2 \left( L_{l^{k_1}}^* + L_{l^{k_1}}^*, L_{l^{k_2}}^* + L_{l^{k_2}}^* \right)
\]

\[
= \mu_{l^{k_1}}^{1, v} L_v + \mu_{l^{k_1}}^{1, v} L_v
\]

\[
= 2L_v \neq 0_{D_G}.
\]

Therefore, \(P(L_{w_1}, L_{w_1}^*)\) and \(Q(L_{w_2}, L_{w_2}^*)\) are not free over \(D_G\). This shows that \(W^*(\{L_{w_1}\}, D_G)\) and \(W^*(\{L_{w_2}\}, D_G)\) are not free over \(D_G\) in \((W^*(G), E)\) and hence \(L_{w_1}\) and \(L_{w_2}\) are not free over \(D_G\). This contradict our assumption.

(Case II) Suppose that the finite paths \(w_1, w_2\) are non-loop finite paths in \(loop^c(G)\) and assume that they are not diagram-distinct. Since they are not diagram-distinct, they are identically equal. Therefore, they are not free over \(D_G\) in \((W^*(G), E)\).

(Case III) Let \(w_1 \in loop(G)\) and \(w_2 \in loop^c(G)\). They are always diagram-distinct.

Let \(L_{w_1}\) and \(L_{w_2}\) are free over \(D_G\) and assume that \(w_1\) and \(w_2\) are not diagram-distinct. Then \(L_{w_1}\) and \(L_{w_2}\) are not free over \(D_G\), by the Case I, II and III. So, this contradict our assumption. \(\blacksquare\)
The previous theorem characterize the $D_G$-freeness of two partial isometries $L_{w_1}$ and $L_{w_2}$, where $w_1, w_2 \in FP(G)$. This characterization shows us that the diagram-distinctness of finite paths determine the $D_G$-freeness of corresponding creation operators.

Let $a$ and $b$ be the given $D_G$-valued random variables in (3.0). We can get the necessary condition for the $D_G$-freeness of $a$ and $b$, in terms of their supports. Recall that we say that the two subsets $X_1$ and $X_2$ of $FP(G)$ are said to be diagram-distinct if $x_1$ and $x_2$ are diagram-distinct, for all pairs $(x_1, x_2) \in X_1 \times X_2$.

**Theorem 3.8.** Let $a, b \in (W^*(G), E)$ be $D_G$-valued random variables with their supports $FP(G : a)$ and $FP(G : b)$. The $D_G$-valued random variables $a$ and $b$ are free over $D_G$ in $(W^*(G), E)$ if $FP(G : a_1)$ and $FP(G : a_2)$ are diagram-distinct.

**Proof.** For convenience, let’s denote $a$ and $b$ by $a_1$ and $a_2$, respectively. Assume that the supports of $a_1$ and $a_2$, $FP(G : a_1)$ and $FP(G : a_2)$ are diagram-distinct. Then by the previous $D_G$-freeness characterization,

$$\sum_{l \in FP(G : a_1), u_l \in \{1, \ast\}} p^{(1)}_l L_{u_l}^{w_1} \quad \text{and} \quad \sum_{l \in FP(G : a_2), u_l \in \{1, \ast\}} p^{(2)}_l L_{u_l}^{w_2}$$

are free over $D_G$ in $(W^*(G), E)$. Indeed, since $FP(G : a_1)$ and $FP(G : a_2)$ are diagram-distinct, all summands $L_{w_1}$’s of $a_1$ and $L_{w_2}$’s of $a_2$ are free over $D_G$ in $(W^*(G), E)$. Therefore, $a_1$ and $a_2$ are free over $D_G$ in $(W^*(G), E)$. □

4. Examples

In this chapter, as examples, we will compute the trivial $D_G$-valued moments and cumulants of the generating operator $T$ of the graph $W^*$-algebra $W^*(G)$. Let $G$ be a countable directed graph and let $(W^*(G), E)$ be the graph $W^*$-probability space over its diagonal subalgebra. Let $a \in (W^*(G), E)$ be a $D_G$-valued random variable. Recall that the trivial $D_G$-valued $n$-th moments and cumulants of $a$ are defined by

$$E(a^n) \quad \text{and} \quad k_n \left( a, \ldots, a \right)_{n\text{-times}}.$$
In this chapter, we will deal with the following special $D_G$-valued random variable:

**Definition 4.1.** Define an operator $T$ in $W^*(G)$ by

$$ T = \sum_{e \in E(G)} (L_e + L_e^*) . $$

We will call $T$ the generating operator of $W^*(G)$. The self-adjoint operators $L_e + L_e^*$, for $e \in E(G)$, are called the block operators of $T$.

**Example 4.1.** Let $G$ be a one-vertex directed graph with $N$-edges. i.e.,

$$ V(G) = \{v\} \quad \text{and} \quad E(G) = \{e_j = ve_jv : j = 1, ..., N\} . $$

Then the graph $W^*$-algebra $W^*(G)$ satisfies that

$$ W^*(G) = D_G \ast_{D_G} \left( \bigotimes_{j=1}^{N} \left( W^*(\{L_{e_j}\}, D_G) \right) \right) , $$

by Chapter 4. Notice that $D_G = \Delta_1 = \mathbb{C}$. Therefore, the formula (5.2) is rewritten by

$$ W^*(G) = \bigotimes_{j=1}^{N} \left( W^*(\{L_{e_j}\}) \right) , $$

where $\ast$ means the usual (scalar-valued) free product of Voiculescu. Also notice that $1_{D_G} = L_v = 1 \in \mathbb{C}$ and

$$ L_{e_j}^*L_{e_j} = L_v = 1 = L_{e_j}L_{e_j}^* , \quad \text{for all} \quad j = 1, ..., N . $$

This shows that $L_{e_j}$'s are unitary in $W^*(G)$, for all $j = 1, ..., N$. Now, define the generating operator $T = \sum_{j=1}^{N} \left( L_{e_j} + L_{e_j}^* \right)$ of $W^*(G)$. It is easy to see that each block operator $L_{e_j} + L_{e_j}^*$ is semicircular, by Voiculescu, for all $j = 1, ..., N$. (Remember the construction of creation operators $L_{e_j}$'s and see [9].) Futhermore, by Chapter 3, we can get that all blocks $L_{e_j} + L_{e_j}^*$'s are free from each other in the graph $W^*$-probability space $(W^*(G), E)$.

By (5.4), the canonical conditional expectation $E : W^*(G) \to D_G$ is the faithful linear functional. Moreover, by (5.5), this linear functional $E$ is a trace in the sense that $E(ab) = E(ba)$, for all $a, b \in W^*(G)$. From now, to emphasize that $E$ is a trace, we will denote $E$ by $\text{tr}$.

Let’s compute the $n$-th cumulant of $T$ ;

$$ (5.6) $$
\[ k_n(T, ..., T) = k_n \left( \sum_{j=1}^{N} \left( L_{e_j} + L_{e_j}^* \right), ..., \sum_{j=1}^{N} \left( L_{e_j} + L_{e_j}^* \right) \right) \]
\[ = \sum_{j=1}^{N} k_n \left( (L_{e_j} + L_{e_j}^*), ..., (L_{e_j} + L_{e_j}^*) \right), \]

by the mutual freeness of \( \{L_{e_j}, L_{e_j}^*\} \)'s on \((W^*(G), tr)\), for \( j = 1, ..., N \). Observe that

\[ k_n \left( (L_{e_j} + L_{e_j}^*), ..., (L_{e_j} + L_{e_j}^*) \right) \]

\[ = \begin{cases} 
  k_2 \left( (L_{e_j} + L_{e_j}^*), (L_{e_j} + L_{e_j}^*) \right) & \text{if } n = 2 \\
  0 & \text{otherwise,} 
\end{cases} \]

by the semicircularity of \( L_{e_j} + L_{e_j}^* \), for \( j = 1, ..., N \). By (5.7), the formula (5.6) is

\[ k_n(T, ..., T) \]

\[ = \begin{cases} 
  \sum_{j=1}^{N} k_2 \left( L_{e_j} + L_{e_j}^*, L_{e_j} + L_{e_j}^* \right) & \text{if } n = 2 \\
  0 & \text{otherwise} 
\end{cases} \]

Now, observe \( k_2 \left( L_{e_j} + L_{e_j}^*, L_{e_j} + L_{e_j}^* \right) \);

\[ k_2 \left( L_{e_j} + L_{e_j}^*, L_{e_j} + L_{e_j}^* \right) \]
\[ = k_2 \left( L_{e_j}, L_{e_j} \right) + k_2 \left( L_{e_j}, L_{e_j}^* \right) + k_2 \left( L_{e_j}^*, L_{e_j} \right) + k_2 \left( L_{e_j}^*, L_{e_j}^* \right) \]
\[ = 0 + k_2 \left( L_{e_j}, L_{e_j}^* \right) + k_2 \left( L_{e_j}^*, L_{e_j} \right) + 0 \]

by Section 2.1

\[ = tr \left( L_{e_j} L_{e_j}^* \right) + tr \left( L_{e_j}^* L_{e_j} \right) = 2 \cdot tr \left( L_{e_j}^* L_{e_j} \right) \]

since \( tr \) is a trace

\[ = 2 \cdot L_v = 2, \]

for \( j = 1, ..., N \), by Section 2.1 and 2.2. So, we can get that
Now, we can compute the trivial moments of $T$, via the Möbius inversion.

$$tr(T^n) = \sum_{\pi \in NC(n)} \sum_{\pi \in NC_2(n)} (\prod_{V \in \pi} k_{|V|}(a, a))$$

where $k_{|V|}(a, a) = \prod_{V \in \pi} k_{|V|}(a, a)$, for each $\pi \in NC(n)$, by Nica and Speicher (See [1] and [17]).

$$= \sum_{\pi \in NC_2(n)} (\prod_{V \in \pi} 2N) = \sum_{\pi \in NC_2(n)} (2N)^{|\pi|},$$

where $NC_2(n) = \{\pi \in NC(n) : V \in \pi \leftrightarrow |V| = 2\}$ is the collection of all noncrossing pairings

$$= \sum_{\pi \in NC_2(n)} \sum_{\pi \in NC(n)} (2N)^{|\pi|},$$

where $|\pi|$ is the number of blocks in $\pi$. Notice that the above formula (5.10) shows us that the $n$ should be even, because $NC_2(n)$ is nonempty when $n$ is even. Therefore,

$$tr(T^n) = \begin{cases} \sum_{\pi \in NC_2(n)} (2N)^{|\pi|} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Also, notice that if $\pi \in NC_2(n)$, then $|\pi| = \frac{n}{2}$, for all even number $n \in \mathbb{N}$. So,

$$tr(T^n) = \begin{cases} |NC_2(n)| \cdot (2N)^{\frac{n}{2}} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

$$= \begin{cases} (2N)^{\frac{n}{2}} \cdot c_{\frac{n}{2}} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

where $c_k = \frac{1}{k+1} \left( \begin{array}{c} 2k \\ k \end{array} \right)$ is the $k$-th Catalan number, for all $k \in \mathbb{N}$. Remember that
\[ |NC(k)| = |NC_{2k}| = c_k, \text{ for all } k \in \mathbb{N}. \]

Therefore, by (5.9) and (5.12), we can compute the moments and cumulants of the generating operator \( T \) of \( W^*(G, tr) \):

\[
tr(T^n) = \begin{cases} 
(2N)^{\frac{n}{2}} \cdot c^{\frac{n}{2}} & \text{if } n \text{ is even} \\
0 & \text{if } n \text{ is odd},
\end{cases}
\]

and

\[
k_n(T, \ldots, T) = \begin{cases} 
2N & \text{if } n = 2 \\
0 & \text{otherwise}.
\end{cases}
\]

**Example 4.2.** Let \( N \in \mathbb{N} \) and let \( G \) be the circulant graph with

\[
V(G) = \{v_1, \ldots, v_N\}
\]

and

\[
E(G) = \{e_1, \ldots, e_N\}
\]

with

\[ e_j = v_j e_j v_{j+1}, \text{ for } j = 1, \ldots, N - 1, \text{ and } e_N = v_N e_N v_1.\]

Define the generating operator \( T = \sum_{j=1}^{N} \left( L_{e_j} + L_{e_j}^* \right) \) of the graph \( W^* \)-algebra \( W^*(G) \). In this case, we can get the diagonal subalgebra \( D_G \) of \( W^*(G, tr) \), as a von Neumann algebra which is isomorphic to \( \Delta_N \), where \( \Delta_N \) is a subalgebra of the matri-}

cial algebra \( M_N(\mathbb{C}) \). Define the canonical conditional expectation \( E : W^*(G) \rightarrow D_G \).

Then we can compute the trivial \( n \)-th \( D_G \)-valued cumulant of the operator \( T \), by regarding it as a \( D_G \)-valued random variable in the graph \( W^* \)-probability space \( (W^*(G), E) \) over \( D_G = \Delta_N \). Notice that each block \( L_{e_j} + L_{e_j}^* \)'s are free from each other over \( D_G \) in \( (W^*(G), E) \), by the diagram-distinctness of \( e_j \)'s, for \( j = 1, \ldots, N \).

Fix \( n \in \mathbb{N} \). Then

\[
k_n \left( T, \ldots, T \underbrace{\ldots}_{n \text{-times}} \right) = k_n \left( \sum_{j=1}^{N} \left( L_{e_j} + L_{e_j}^* \right), \ldots, \sum_{j=1}^{N} \left( L_{e_j} + L_{e_j}^* \right) \right)
\]

\[ = \sum_{j=1}^{N} k_n \left( L_{e_j} + L_{e_j}^*, \ldots, L_{e_j} + L_{e_j}^* \right) \]

by the mutual \( D_G \)-freeness of \( \{L_{e_j}, L_{e_j}^*\} \)'s, for \( j = 1, \ldots, N \)

\[ (5.13) = \sum_{j=1}^{N} \sum_{(u_1, \ldots, u_n) \in \{1, \ldots, N\}^n} k_n \left( L_{e_j}^{u_1}, \ldots, L_{e_j}^{u_n} \right). \]
Recall that, by Section 2.2, we can get that

\[ (5.14) \quad k_n \left( L_{e_j}^{u_1}, \ldots, L_{e_j}^{u_n} \right) = \mu_{e_j, \ldots, e_j} \cdot \text{Proj} \left( L_{e_j}^{u_1} \ldots L_{e_j}^{u_n} \right), \]

where \( \mu_{e_j, \ldots, e_j} = \sum_{\pi \in C_{e_j, \ldots, e_j}} \mu(\pi, 1_n). \)

Observe that since \( e_j \)'s are non-loop edges, \( e_j^k \notin \mathcal{E}^+(G) \), for all \( k \in \mathbb{N} \setminus \{1\} \), for \( j = 1, \ldots, N \). In other words, such \( e_j^k \) is not admissible. So, if \( (u_1, \ldots, u_n) \) is not alternating, in the sense that \( (u_1, \ldots, u_n) = (1, *, \ldots, 1, *) \) or \( (*, 1, \ldots, *, 1) \), then \( \text{Proj} \left( L_{e_j}^{u_1} \ldots L_{e_j}^{u_n} \right) = 0_{D_G} \). For instance, \( E \left( L_{e_j}^* L_{e_j}^* \right) = 0_{D_G} \) or \( E \left( L_{e_j}^2 L_{e_j}^* L_{e_j}^* \right) = 0_{D_G} \), by Section 2.1. Therefore, the only nonvanishing case is either

\[ k_n \left( L_{e_j}, L_{e_j}^*, \ldots, L_{e_j}, L_{e_j}^* \right) \quad \text{or} \quad k_n \left( L_{e_j}, L_{e_j}, \ldots, L_{e_j}, L_{e_j} \right), \]

where \( n \) is even. Notice that

\[ (5.15) \quad \mu_{e_j, \ldots, e_j}^{1, \ldots, 1} = \mu_{e_j, \ldots, e_j}^{*, 1, \ldots, 1}, \]

because \( C_{e_j, \ldots, e_j}^{1, \ldots, 1} = C_{e_j, \ldots, e_j}^{*, 1, \ldots, 1} \), for all \( j = 1, \ldots, N \). Moreover, since \( C_{e_j, \ldots, e_j}^{1, \ldots, 1} = C_{e_k, \ldots, e_k}^{1, \ldots, 1} \), for all \( j \neq k \) in \( \{1, \ldots, N\} \),

\[ (5.16) \quad \mu_{e_j, \ldots, e_j}^{1, \ldots, 1} = \mu_{e_k, \ldots, e_k}^{1, \ldots, 1}, \]

for all \( j, k \in \{1, \ldots, N\} \). Let’s denote \( \mu_{e_j, \ldots, e_j}^{1, \ldots, 1} \) by \( \mu_n \), for all \( j = 1, \ldots, N \). Then, by (5.14), we have that

\[ (5.17) \quad k_n \left( L_{e_j}^{u_1}, \ldots, L_{e_j}^{u_n} \right) = \begin{cases} \mu_n L_{v_j} & \text{if } (u_1, \ldots, u_n) = (1, *, \ldots, 1, *) \\ \mu_n L_{v_{j+1}} & \text{if } (u_1, \ldots, u_n) = (*, 1, \ldots, *, 1) \\ 0_{D_G} & \text{otherwise,} \end{cases} \]

for all \( j = 1, \ldots, N \), where \( L_{v_{N+1}} \) means \( L_{v_1} \). So, by (5.13) and (5.17), we can get that

\[ k_n (T, \ldots, T) = \sum_{j=1}^{N} \left( k_n \left( L_{e_j}, L_{e_j}^*, \ldots, L_{e_j}, L_{e_j}^* \right) + k_n \left( L_{e_j}^*, L_{e_j}, \ldots, L_{e_j}^*, L_{e_j} \right) \right) \]

\[ = \sum_{j=1}^{N} \left( \mu_n L_{v_j} + \mu_n L_{v_{j+1}} \right) = \sum_{j=1}^{N} \mu_n \left( L_{v_j} + L_{v_{j+1}} \right) \]

where \( L_{v_{N+1}} \) means \( L_{v_1} \), for all \( n \in 2\mathbb{N} \). Therefore,
\[ k_n(T, \ldots, T) = \begin{cases} \sum_{j=1}^{N} \mu_n (L_{v_j} + L_{v_j+1}) & \text{if } n \text{ is even} \\ 0_{DG} & \text{if } n \text{ is odd.} \end{cases} \]

(5.18)

\[ = \begin{cases} 2\mu_n \cdot 1_{DG} & \text{if } n \text{ is even} \\ 0_{DG} & \text{if } n \text{ is odd.} \end{cases} \]

Unfortunately, it is very hard to compute \( \mu_n \), when \( n \to \infty \). But we have to remark that if we have arbitrary graph \( H \) and its graph \( W^* \)-probability space \((W^*(H), F)\) over its diagonal subalgebra \( D_H \) and if \( w \in \text{loop}^*(G) \), then

\[ \mu_{1, \ldots, 1, w} = \mu_n = \mu_{1, \ldots, 1, w} \text{, for all } n \in 2\mathbb{N}. \]

Now, let’s compute the trivial \( n \)-th \( D_G \)-valued moment of \( T \). Notice that since all odd trivial \( D_G \)-valued cumulants of \( T \) vanish, all odd trivial \( D_G \)-valued moments of \( T \) vanish (See [11] and [14]). Thus it suffices to compute the even trivial \( D_G \)-valued moments of \( T \). Assume that \( n \in 2\mathbb{N} \). Then

(5.19)

\[ E(T^n) = \sum_{\pi \in NC_E(n)} k_\pi (T, \ldots, T), \]

where \( k_\pi (T, \ldots, T) \) is the partition-dependent cumulant of \( T \) (See [16]) and

\[ NC_E(n) \overset{def}{=} \{ \pi \in NC(n) : V \in \pi \iff |V| \in 2\mathbb{N} \}. \]

By (5.18), we can get that \( k_n(T, \ldots, T) \) commutes with all elements in \( W^*(G) \), because \( 1_{DG} \) and \( 0_{DG} \) commutes with \( W^*(G) \) and \( 2\mu_n \in \mathbb{C} \), for all \( n \in \mathbb{N} \). So, the formula (5.19) can be reformed by

\[ E(T^n) = \sum_{\pi \in NC_E(n)} \left( \prod_{V \in \pi} k_{|V|}(T, \ldots, T) \right) \]

\[ = \sum_{\pi \in NC_E(n)} \left( \prod_{V \in \pi} 2\mu_{|V|} \cdot 1_{DG} \right) \]

(5.20)

\[ = \left( \sum_{\pi \in NC_E(n)} \left( \prod_{V \in \pi} 2\mu_{|V|} \right) \right) \cdot 1_{DG}, \]

for all \( n \in 2\mathbb{N} \). Therefore, by (5.18) and (5.20), we have that if \( T \) is the generating operator of the graph \( W^* \)-algebra of the circulant graph \( G \) with \( N \)-vertices, then
\[ E(T^n) = \begin{cases} 
\left( \sum_{\pi \in NC_E(n)} \left( \prod_{V \in \pi} 2\mu_{|V|} \right) \right) \cdot 1_{D_G} & \text{if } n \text{ is even} \\
0_{D_G} & \text{if } n \text{ is odd.} 
\end{cases} \]

and

\[ k_n(T, \ldots, T_{\text{n-times}}) = \begin{cases} 
2\mu_n \cdot 1_{D_G} & \text{if } n \text{ is even} \\
0_{D_G} & \text{if } n \text{ is odd.} 
\end{cases} \]

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