On a unified double zeta function of Mordell–Tornheim type

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Abstract. We introduce a double zeta function of Mordell–Tornheim type and compute its values at nonpositive integer points. We then discuss a possible generalization of the Kaneko–Zagier conjecture for all integer points.

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1 Introduction

The double zeta function of Mordell–Tornheim type is defined by

$$\zeta_{MT}(s_1, s_2; s_3) = \sum_{m,n \geq 1} m^{-s_1} n^{-s_2} (m+n)^{-s_3},$$

which converges absolutely when $\Re(s_1 + s_3), \Re(s_2 + s_3) > 1$, and $\Re(s_1 + s_2 + s_3) > 2$ (see [16, Thm. 2.2]). The special values of this function at positive integer points are first studied by Tornheim [18] and independently by Mordell [14] for the case $s_1 = s_2 = s_3$, and also rediscovered by Witten [19] in his volume formula for certain moduli spaces related to theoretical physics (see also Zagier’s number-theoretical treatment [20]). Matsumoto [11, Thm. 1] proves that the function $\zeta_{MT}(s_1, s_2; s_3)$ can be analytically continued to the whole $\mathbb{C}^3$-space. Its true singularities are also determined (see also [12, Thm. 6.1]). As an application, it is clarified that nonpositive integers $s_1, s_2, s_3$ are points of indeterminancy, that is, the values of $\zeta_{MT}(s_1, s_2; s_3)$ at nonpositive integer points depend on a limiting process. Explicit formulas for these values in terms of generalized Bernoulli numbers are computed by Komori [7, Thm. 3].

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In this paper, we study special values of the function
\[ \omega_u(s_1, s_2, s_3) = (-1)^{s_1} \zeta_{MT}(s_1, s_2; s_3) + (-1)^{s_2} \zeta_{MT}(s_3, s_1; s_2) + (-1)^{s_3} \zeta_{MT}(s_2, s_3; s_1), \]
where \( s_1, s_2, s_3 \) are complex variables, and we write \((-1)^s = e^{\pi is}\). The function \( \omega_u(s_1, s_2, s_3) \) originates from the previous work by Bachmann et al. \([1]\), who introduced the values of \( \omega_u(s_1, s_2, s_3) \) at positive integer points as a Mordell–Tornheim-type analogue of the symmetric multiple zeta values (see also \([17]\)). One of main results of this paper is an explicit evaluation for coordinatewise limits of \( \omega_u(s_1, s_2, s_3) \) at nonpositive integer points.

**Theorem 1.** For any nonnegative integers \( m_1, m_2, m_3 \in \mathbb{Z}_{\geq 0} \) and \( a, b, c \in \{1, 2, 3\} \) such that \( \{a, b, c\} = \{1, 2, 3\} \), we have
\[
\lim_{\varepsilon_1 \to 0} \lim_{\varepsilon_2 \to 0} \lim_{\varepsilon_3 \to 0} \omega_u(-m_1 + \varepsilon_1, -m_2 + \varepsilon_2, -m_3 + \varepsilon_3) = \begin{cases} 1, & (m_1, m_2, m_3) = (0, 0, 0), \\ 0, & (m_1, m_2, m_3) \neq (0, 0, 0). \end{cases}
\]

The contents of this paper are as follows. To motivate our work, we briefly recall the Kaneko–Zagier conjecture on finite/symmetric multiple zeta values in Section 2. In Section 3, we give a result on true singularities of the function \( \omega_u(s_1, s_2, s_3) \). Section 4 is devoted to proving Theorem 1. Finally, in Section 5, we discuss a possible generalization of the Kaneko–Zagier conjecture for the multiple zeta function of Mordell–Tornheim type.

**2 Background**

This paper mimics the study of the unified multiple zeta function introduced by Komori \([9]\). We briefly review his results to motivate our work.

For each \( r \geq 0 \), we call a tuple \( k = (k_1, \ldots, k_r) \) of positive integers an index and \( k_1 + \cdots + k_r \) its weight. We regard the empty index \( \emptyset \) as the unique index of weight 0 (and \( r = 0 \)). We set \( F(\emptyset) \) to be a unit element for any function \( F \) on indices.

We first recall the Kaneko–Zagier conjecture \([6]\). For an index \( k = (k_1, \ldots, k_r) \) and a positive integer \( n \), define the multiple harmonic sum \( H_n(k) \) by
\[
H_n(k) = \sum_{1 \leq n_1 < \cdots < n_r \leq n} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}.
\]
The empty sum is understood as 0. If the last component \( k_r \) is greater than 1, then the limit at \( n \to \infty \) exists and is called the multiple zeta value, denoted by
\[
\zeta(k) = \sum_{1 \leq n_1 < \cdots < n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}. \tag{2.1}
\]

Kaneko and Zagier introduce two different types of multiple zeta values constructed from the above objects. One of them is the finite multiple zeta value \( \zeta_A(k) \) defined for each index \( k \) by
\[
\zeta_A(k) = (H_{p-1}(k) \mod p)_p
\]
in the \( \mathbb{Q}\)-algebra \( A = (\prod_p \mathbb{Z}/p\mathbb{Z})/(\bigoplus_p \mathbb{Z}/p\mathbb{Z}) \), where \( p \) runs over all primes. The other is the symmetric multiple zeta value \( \zeta_S(k) \) defined for each index \( k \) by
\[
\zeta_S(k) = \sum_{a=0}^{r} (-1)^{k_{a+1}+\cdots+k_r} \zeta^{III}(k_1, \ldots, k_a) \zeta^{III}(k_r, \ldots, k_{a+1}) \mod \pi^2 \mathbb{Z}
\]
in the $\mathbb{Q}$-algebra $\mathbb{Z}/\pi^2\mathbb{Z}$, where we denote by $\mathbb{Z}$ the $\mathbb{Q}$-algebra generated by all multiple zeta values and by $\zeta^{(1)}(k_1, \ldots, k_r) \in \mathbb{Z}$ the shuffle regularized multiple zeta value. Let $\mathbb{I}^+$ be the set of all indices. The main conjecture of Kaneko and Zagier states that for a finite subset $\{a_k \in \mathbb{Q} \mid k \in \mathbb{I}^+\}$ of $\mathbb{Q}$, we have

$$\sum_{k \in \mathbb{I}^+} a_k \zeta_S(k) = 0 \iff \sum_{k \in \mathbb{I}^+} a_k \zeta_A(k) = 0. \quad (2.2)$$

Komori [9] defines the function

$$\zeta_{\mathcal{U}}(s_1, \ldots, s_r) = \sum_{a=0}^{r} (-1)^{s_{a+1} + \cdots + s_r} \zeta(s_1, \ldots, s_a)\zeta(s_r, \ldots, s_{a+1}).$$

It is a function analogue of symmetric multiple zeta values. A crucial feature is that the function $\zeta_{\mathcal{U}}(s_1, \ldots, s_r)$ is entire. Thus we can study the values $\zeta_{\mathcal{U}}(s_1, \ldots, s_r) \in \mathbb{C}$ for any tuple $k$ of integers. These values are conjecturally elements in the polynomial ring $\mathbb{Z}[\pi]$ over $\mathbb{Z}$. Indeed, for all indices $k \in \mathbb{I}^+$, Komori [9, Thm. 1.4] shows that $\zeta_{\mathcal{U}}(k) \in \mathbb{Z}[\pi]$ and also that

$$\zeta_{\mathcal{U}}(k) \equiv \zeta_S(k) \mod \pi i \mathbb{Z}[\pi]. \quad (2.3)$$

Now, assuming the conjecture $\zeta_{\mathcal{U}}(k) \in \mathbb{Z}[\pi]$, the definition of the symmetric multiple zeta values $\zeta_S(k)$ to any tuple $k$ of integers can be extended by

$$\zeta_S(k) = \zeta_{\mathcal{U}}(k) \mod \pi i \mathbb{Z}[\pi].$$

Hereafter, we denote by $\mathbb{I}$ the set of all tuples $(k_1, \ldots, k_r) \in \mathbb{Z}^r$ ($r \geq 0$) of integers. Then a natural question to ask is whether conjecture (2.2) holds for all tuples of integers. Namely, for a finite subset $\{a_k \in \mathbb{Q} \mid k \in \mathbb{I}\}$ of $\mathbb{Q}$, we have

$$\sum_{k \in \mathbb{I}} a_k \zeta_S(k) = 0 \iff \sum_{k \in \mathbb{I}} a_k \zeta_A(k) = 0. \quad (2.4)$$

Note that since there is no convergence issue, the finite multiple zeta values $\zeta_A(k)$ are already defined for all $k \in \mathbb{I}$, and they span the same space with the $\mathbb{Q}$-vector space generated by the set $\{\zeta_A(k) \mid k \in \mathbb{I}^+\}$ (see [5]). The main result of Komori [9, Thm. 1.5] states that conjecture (2.4) holds when we restrict $k$ to tuples of nonpositive integers. A partial evidence for depth 2 case is also given in [9, Prop. 1.6].

In this paper, we study the Kaneko–Zagier conjecture (2.2) for multiple zeta values of Mordell–Tornheim type, which are independently established in [1] and [17]. Our terminology follows [1]. In this analogy, for each $k = (k_1, \ldots, k_r) \in \mathbb{I}^+$, let

$$\omega_n(k) = \sum_{n_1 + \cdots + n_r = n} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}},$$

and define

$$\omega_A(k) = (\omega_p(k) \mod p)_p \in A$$

for $r \neq 1$. We call this the finite multiple omega value for short. Note that the finite multiple omega value is, up to sign, equal to the finite multiple zeta value of Mordell–Tornheim type, which is first studied by Kamano [4]. As a real counterpart of this, we define the symmetric multiple omega value $\omega_S(k)$ for $k = (k_1, \ldots, k_r) \in \mathbb{I}^+$ with $r \neq 1$ by

$$\omega_S(k) = \sum_{a=1}^{r} (-1)^{k_a} \zeta_{MT}(k_1, \ldots, k_{a-1}, \underbrace{k_{a+1}, \ldots, k_r}_{a-1}, k_a) \in \mathbb{Z}/\pi^2\mathbb{Z},$$

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where
\[
\zeta_{MT}(k_1, \ldots, k_{r-1}; k_r) = \sum_{n_1, \ldots, n_{r-1} \geq 1} \frac{1}{n_1^{k_1} \cdots n_{r-1}^{k_{r-1}} (n_1 + \cdots + n_{r-1})^{k_r}}
\]
are the multiple zeta values of Mordell–Tornheim type that can be written as \(\mathbb{Q}\)-linear combinations of multiple zeta values (2.1) (see [2, Thm. 1.1]). A similar object to \(\omega_S(k)\) is introduced by a different method in [17, Def. 4.5] (in the case \(S\) differs in a sign). The main conjecture of [1, Conjecture 1.5] (see also [17, Sect. 4]) is that for a finite subset \(\{a_k \in \mathbb{Q} \mid k \in \mathbb{Z}^+, r \neq 1\}\) of \(\mathbb{Q}\), we have
\[
\sum_{k \in \mathbb{Z}^+} a_k \omega_S(k) = 0 \iff \sum_{k \in \mathbb{Z}^+} a_k \omega_A(k) = 0. \tag{2.5}
\]
It can be seen from [1, Thm. 1.6] that conjecture (2.5) is supported by the Kaneko–Zagier conjecture (2.2).

An ultimate goal of this project is to establish a Mordell–Tornheim analogue of Komori’s conjecture (2.4). For that purpose, we need to find a suitable function whose values at integer points modulo \(\pi^2\) satisfy the same relations with the finite multiple omega values. As a natural candidate, we consider the following function:
\[
\omega_{\mathcal{L}}(s_1, \ldots, s_r) = \sum_{a=1}^{r} (-1)^{s_a} \zeta_{MT}(s_1, \ldots, s_a-1, s_a+1, \ldots, s_r; s_a).
\]
Similarly to (2.3), the congruence \(\omega_{\mathcal{L}}(k) \equiv \omega_S(k) \mod \pi^2\mathbb{Z}\) holds for all \(k \in \mathbb{Z}^+\), since \(\zeta_{MT}(k)\) converges absolutely for any \(k = (k_1, \ldots, k_r) \in \mathbb{Z}^+\) with \(r \neq 1\). As a function, we see from [10, Thm. 1] that the function \(\omega_{\mathcal{L}}(s_1, \ldots, s_r)\) can be meromorphically continued to \(\mathbb{C}^r\) and its possible singularities are on

\[
\begin{align*}
  s_{j_1} + s_{j_2} &= 1 - l \quad (1 \leq j_1 < j_2 \leq r, l \in \mathbb{Z}_{\geq 0}), \\
  s_{j_1} + s_{j_2} + s_{j_3} &= 2 - l \quad (1 \leq j_1 < j_2 < j_3 \leq r, l \in \mathbb{Z}_{\geq 0}), \\
  &\vdots \\
  s_{j_1} + \cdots + s_{j_{r-1}} &= r - 2 - l \quad (1 \leq j_1 < \cdots < j_{r-1} \leq r, l \in \mathbb{Z}_{\geq 0}), \\
  s_1 + \cdots + s_r &= r - 1.
\end{align*}
\]
Altough the function \(\zeta_{\mathcal{L}}(s_1, \ldots, s_r)\) cancels all possible singularities that come from singularities of multiple zeta functions, some of the above singularities of \(\omega_{\mathcal{L}}(s_1, \ldots, s_r)\) will be true. Thus a generalization of conjecture (2.5) via the function \(\omega_{\mathcal{L}}(s_1, \ldots, s_r)\) is not straightforward. In this paper, as a part of the future project, we explicate the situation for the case \(r = 3\) and then discuss a possible generalization of (2.5) to all integer points.

### 3 Singularities

By [11, Thm. 1] the true singularities of the function \(\zeta_{MT}(s_1, s_2; s_3)\) lie on \(s_1 + s_3 = 1 - l, s_2 + s_3 = 1 - l (l \in \mathbb{Z}_{\geq 0})\), and \(s_1 + s_2 + s_3 = 2\). Hence the possible singularities of \(\omega_{\mathcal{L}}(s_1, s_2, s_3)\) lie on the subsets of \(\mathbb{C}^3\) defined by one of the equations
\[
s_a + s_b = 1 - l \quad (1 \leq a < b \leq 3, l \in \mathbb{Z}_{\geq 0}), \quad s_1 + s_2 + s_3 = 2.
\]
We now prove that these are true singularities.

**Theorem 2.** All points on the subsets of \(\mathbb{C}^3\) defined by one of the equations \(s_a + s_b = 1 - l (1 \leq a < b \leq 3, l \in \mathbb{Z}_{\geq 0})\) and \(s_1 + s_2 + s_3 = 2\) are true singularities of the function \(\omega_{\mathcal{L}}(s_1, s_2, s_3)\).
Proof. We first recall the relevant material from [11]. For \( s_1, s_2, s_3 \in \mathbb{C}, M \in \mathbb{Z}_{>0} \) and \( 0 < \eta < 1 \), let

\[
I(s_1, s_2, s_3; M - \eta) = \frac{1}{2\pi i} \int_{(M-\eta)} \Gamma(s_3 + z) \Gamma(-z) \zeta(s_1 + s_3 + z) \zeta(s_2 - z) \, dz,
\]

where the path of integration is the vertical line from \( M - \eta - i\infty \) to \( M - \eta + i\infty \). From [11, Eq. (5.3)] we have

\[
\zeta_{MT}(s_1, s_2; s_3) = \frac{\Gamma(s_2 + s_3 - 1) \Gamma(1 - s_2)}{\Gamma(s_3)} \zeta(s_1 + s_2 + s_3 - 1)
\]

\[
+ \sum_{k=0}^{M-1} \left(-\frac{s_3}{k}\right) \zeta(s_1 + s_3 + k) \zeta(s_2 - k) + \frac{1}{\Gamma(s_3)} I(s_1, s_2, s_3; M - \eta), \tag{3.1}
\]

where \( M \) is a sufficiently large positive integer. We find that the integral \( I(s_1, s_2, s_3; M - \eta) \) is holomorphic on the region

\[
D_M = \{(s_1, s_2, s_3) \in \mathbb{C}^3 \mid \Re s_3 > -M + \eta, \Re(s_1 + s_3) > 1 - M + \eta, \Re s_2 < 1 + M - \eta\},
\]

because in \( D_M \) the poles of the integrand are not on the path of integration. Since \( M \) is arbitrarily large, by (3.1) the function \( \zeta_{MT}(s_1, s_2; s_3) \) can be meromorphically continued to \( \mathbb{C}^3 \). We note that in the case \( s_2 = l \) for a positive integer \( l \), the first and second terms on the right-hand side of (3.1) are singular where \( M > l \), but these singularities cancel each other. Indeed, put \( s_2 = l + \varepsilon \) for a positive integer \( l \). Using the well-known asymptotic formula

\[
\Gamma(s) = \frac{(-1)^{-n}}{(-n)!} \frac{1}{s-n} + \gamma_n + O(|s-n|) \quad (n \in \mathbb{Z}_{\leq 0}), \tag{3.2}
\]

where \( \gamma_n \) is the constant term of the expansion of \( \Gamma(s) \) at \( s = n \in \mathbb{Z}_{\leq 0} \), we have

\[
\zeta_{MT}(s_1, s_2; s_3) = \frac{\Gamma(s_3 + l - 1 + \varepsilon) \Gamma(1 - l - \varepsilon)}{\Gamma(s_3)} \zeta(s_1 + s_3 + l - 1 + \varepsilon)
\]

\[
+ \left(-\frac{s_3}{l-1}\right) \zeta(s_1 + s_3 + l - 1) \zeta(1 + \varepsilon) + O(1)
\]

\[
= \frac{\Gamma(s_3 + l - 1)(-1)^{l-1}}{(l-1)!} \zeta(s_1 + s_3 + l - 1) \frac{1}{\varepsilon} + O(1)
\]

\[
= \frac{-s_3}{l-1} \zeta(s_1 + s_3 + l - 1) - \frac{1}{\varepsilon} + O(1) + O(1).
\]

We now prove that \( s_1 + s_3 = 1 - l \) \((l \in \mathbb{Z}_{\geq 0})\) determines the true singularity of the function \( \omega_{2l}(s_1, s_2, s_3) \). Take \( M \geq l + 1 \). By (3.1) the singular part of \( \zeta_{MT}(s_1, s_2; s_3) \) corresponding to \( s_1 + s_3 = 1 - l \) comes from

\[
\left(-\frac{s_3}{l}\right) \zeta(s_1 + s_3 + l) \zeta(s_2 - l).
\]
Since \( \zeta_{MT}(s_2, s_3; s_1) = \zeta_{MT}(s_3, s_2; s_1) \), the corresponding singular part of \( \omega_U(s_1, s_2, s_3) \) is

\[
(-1)^{s_1} \left( \frac{-s_1}{l} \right) \zeta(s_1 + s_3 + l) \zeta(s_2 - l) + (-1)^{s_3} \left( \frac{-s_3}{l} \right) \zeta(s_1 + s_3 + l) \zeta(s_2 - l)
\]

\[
= \left\{ (-1)^{s_1} \left( \frac{-s_1}{l} \right) + (-1)^{s_3} \left( \frac{-s_3}{l} \right) \right\} \zeta(s_1 + s_3 + l) \zeta(s_2 - l)
\]

\[
= \left( -1^{s_1} - (-1)^{-s_1} \right) \left( \frac{-s_1}{l} \right) \zeta(s_1 + s_3 + l) \zeta(s_2 - l).
\]

Since

\[
\left( (-1)^{s_1} - (-1)^{-s_1} \right) \left( \frac{-s_1}{l} \right) \zeta(s_2 - l) \neq 0,
\]

we see that \( s_1 + s_3 = 1 - l \) determines the true singularity. Note that we have \( \omega_U(s_1, s_2, s_3) = \omega_U(s_σ(1), s_σ(2), s_σ(3)) \) for any permutation \( σ \in S_3 \). Thus \( s_a + s_b = 1 - l \) \((1 \leq a < b \leq 3, l \in \mathbb{Z}_{\geq 0}) \) determine the true singularities.

For the case \( s_1 + s_2 + s_3 = 2 \), by (3.1) the corresponding singular part of \( \omega_U(s_1, s_2, s_3) \) comes from

\[
\zeta(s_1 + s_2 + s_3 - 1) \times \sum_{j=0}^{2} (-1)^{s_{3+j}} \frac{\Gamma(s_{2+j} + s_{3+j} - 1) \Gamma(1 - s_{2+j})}{\Gamma(s_{3+j})},
\]

where we set \( s_{a+j} = s_b \) if \( a + j \equiv b \mod 3 \) for \( b = 1, 2, 3 \) and \( a, j \in \mathbb{Z} \). Hereafter, we abuse the same notation for other triplets of complex numbers or variables. Since \( s_2 + s_3 = 2 - s_1 \), by Euler’s reflection formula

\[
\Gamma(s) \Gamma(1 - s) = \frac{\pi}{\sin \pi s}
\]

we obtain

\[
(-1)^{s_1} \frac{\Gamma(s_2 + s_3 - 1) \Gamma(1 - s_2)}{\Gamma(s_3)}
\]

\[
= (-1)^{s_3} \frac{\Gamma(1 - s_1) \Gamma(1 - s_2)}{\Gamma(s_3)} = \frac{\pi^2}{\Gamma(s_1) \Gamma(s_2) \Gamma(s_3)} \frac{(-1)^{s_3}}{\sin \pi s_1 \sin \pi s_2}
\]

\[
= \frac{\pi^2}{\prod_{a=1}^{3} \Gamma(s_a) \sin \pi s_a} (-1)^{s_3} \sin \pi s_3 = \frac{\pi^2}{\prod_{a=1}^{3} \Gamma(s_a) \sin \pi s_a} \frac{(-1)^{2s_3} - 1}{2i}
\]

Since

\[
\frac{\pi^2}{2i \prod_{a=1}^{3} \Gamma(s_a) \sin \pi s_a} \sum_{j=0}^{2} \left( (-1)^{2s_{3+j}} - 1 \right) \neq 0
\]

in the case \( s_1 + s_2 + s_3 = 2 \), we conclude that \( s_1 + s_2 + s_3 = 2 \) determines the true singularity of \( \omega_U(s_1, s_2, s_3) \), which completes the proof. \( \Box \)

Our proof of Theorem 2 shows that if \( (k_1, k_2, k_3) \in \mathbb{Z}^3 \) is a singular point of the function \( \omega_U(s_1, s_2, s_3) \), then it is a point of indeterminancy.

**Remark.** Desingularizations of the double zeta function of Mordell–Tornheim type are studied in \([3, 15]\).
4 Proof of Theorem 1

In this section, we first study the asymptotic behavior of the function $\zeta_{MT}(s_1, s_2; s_3)$ at $(s_1, s_2, s_3) = (-m_1, -m_2, -m_3)$ for nonnegative integers $m_1, m_2, m_3$ and then give a proof of Theorem 1.

For $m_1, m_2, m_3 \in \mathbb{Z}_{\geq 0}$, let $M = 2 + m_1 + m_3$. Then $(-m_1, -m_2, -m_3) \in D_M$, so the integral $I(s_1, s_2, s_3; M - \eta)$ is analytic at $(-m_1, -m_2, -m_3)$. Hence we have the estimate

$$\frac{1}{\Gamma(s_3)} I(s_1, s_2, s_3; M - \eta) = O(|s_3 + m_3|). \quad (4.1)$$

Note that a similar argument can be found in [13, Sect. 5].

We now give the asymptotics of the function $\zeta_{MT}(s_1, s_2; s_3)$. Let $B_k$ be the $k$th Seki–Bernoulli number defined by

$$\frac{te^t}{e^t - 1} = \sum_{k \geq 0} B_k \frac{t^k}{k!}.$$ 

For nonnegative integers $m_1, m_2, m_3$, let $m = m_1 + m_2 + m_3 + 2$ and define the rational numbers $b(m_1, m_2; m_3)$ and $c(m_1, m_2; m_3)$ by

$$b(m_1, m_2; m_3) = m_1! m_2! m_3! \left( m - 1 \right) \frac{B_m}{m!},$$

$$c(m_1, m_2; m_3) = \sum_{n_1 + n_2 = m_3 \atop n_1, n_2 \geq 0} \binom{m_3}{n_1} \frac{B_{m_1 + n_1 + 1}}{m_1 + n_1 + 1} \frac{B_{m_2 + n_2 + 1}}{m_2 + n_2 + 1}.$$

Lemma 1. For $m_1, m_2, m_3 \in \mathbb{Z}_{\geq 0}$ and $a, b, c \in \{1, 2, 3\}$ such that $\{a, b, c\} = \{1, 2, 3\}$, we have

$$\lim_{\varepsilon_a \to 0} \lim_{\varepsilon_b \to 0} \lim_{\varepsilon_c \to 0} \zeta_{MT}(-m_1 + \varepsilon_1, -m_2 + \varepsilon_2; -m_3 + \varepsilon_3)$$

$$= \lim_{\varepsilon_a \to 0} \lim_{\varepsilon_b \to 0} \lim_{\varepsilon_c \to 0} \left\{ (-1)^{m_2} b(m_2, m_3; m_1) \frac{\varepsilon_3}{\varepsilon_2 + \varepsilon_3} + (-1)^{m_1} b(m_3, m_1; m_2) \frac{\varepsilon_3}{\varepsilon_1 + \varepsilon_3} + c(m_1, m_2, m_3) \right\}.

Proof. Let $m = m_1 + m_2 + m_3 + 2$. For the first term on the right side of (3.1), we use (3.2) and the expansion $\Gamma(s) = (n - 1)! + O(|s - n|)$ at $s = n \in \mathbb{Z}_{\geq 1}$ to obtain

$$\frac{\Gamma(-m_2 - m_3 - 1 + \varepsilon_2 + \varepsilon_3) \Gamma(1 + m_2 - \varepsilon_2)}{\Gamma(-m_3 + \varepsilon_3)} \zeta(1 - m + \varepsilon_1 + \varepsilon_2 + \varepsilon_3)$$

$$= \left\{ (-1)^{m_3} m_3! \varepsilon_3 + O(|\varepsilon_3|^2) \right\} \left\{ (-1)^{m_2 + m_3 + 1} \frac{1}{(m_2 + m_3 + 1)! \varepsilon_2 + \varepsilon_3} + \gamma_{-m_2 - m_3 - 1} + O(|\varepsilon_2 + \varepsilon_3|) \right\}$$

$$\times \left\{ m_2! \varepsilon_2 + O(|\varepsilon_2|) \right\} \left\{ \zeta(1 - m) + O(|\varepsilon_1 + \varepsilon_2 + \varepsilon_3|) \right\}$$

$$= (-1)^{m_2 + 1} m_2! m_3! \frac{\varepsilon_3}{(m_2 + m_3 + 1)!} \frac{\varepsilon_3}{\varepsilon_2 + \varepsilon_3} + O\left( \frac{|\varepsilon_2|}{\varepsilon_2 + \varepsilon_3} \right) + O\left( \frac{|\varepsilon_3| (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)}{\varepsilon_2 + \varepsilon_3} \right)$$

$$+ O(|\varepsilon_3|) + O\left( \frac{|\varepsilon_3|}{\varepsilon_2 + \varepsilon_3} \right)$$

$$= (-1)^{m_2} b(m_2, m_3; m_1) \varepsilon_3 \varepsilon_2 + \varepsilon_3 + O\left( \frac{|\varepsilon_3| (|\varepsilon_1| + |\varepsilon_2| + |\varepsilon_3|)}{|\varepsilon_2 + \varepsilon_3|} \right) + O(|\varepsilon_3|),$$

where for the last equality, we have used $\zeta(1 - m) = -B_m/m \ (m \in \mathbb{Z}_{\geq 1})$. 

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For the second term on the right side of (3.1), let \( M = 2 + m_1 + m_3 \). Since 
\[
\binom{m - \varepsilon}{k} = \begin{cases} \binom{m}{k} + O(|\varepsilon|) & (m \geq k), \\ (-1)^{k-m}m!(k-m-1)! \varepsilon + O(|\varepsilon|^2) & (m < k) 
\end{cases}
\]
for \( k, m \in \mathbb{Z}_{\geq 0} \), we have 
\[
\sum_{k=0}^{M-1} \binom{m_3 - \varepsilon}{k} \zeta(-m_1 - m_3 + k + \varepsilon_1 + \varepsilon_3) \zeta(-m_2 - k + \varepsilon_2) \\
= \sum_{k=0}^{m_1+m_3} \binom{m_3 - \varepsilon}{k} \zeta(-m_1 - m_3 + k + \varepsilon_1 + \varepsilon_3) \zeta(-m_2 - k + \varepsilon_2) \\
+ \binom{m_3 - \varepsilon}{m_1 + m_3 + 1} \zeta(1 + \varepsilon_1 + \varepsilon_3) \zeta(1 - m + \varepsilon_2) \\
= c(m_1, m_2; m_3) + (-1)^{m_1}b(m_3, m_1; m_2) \frac{\varepsilon_3}{\varepsilon_1 + \varepsilon_3} \\
+ O(|\varepsilon_3|) + O(|\varepsilon_1 + \varepsilon_3|) + O(|\varepsilon_2|) + O\left(\frac{\varepsilon_2 \varepsilon_3}{\varepsilon_1 + \varepsilon_3}\right)
\]
where for the last equality, we have also used the Laurent expansion of \( \zeta(s) \) at \( s = 1 \). Thus by (4.1) we obtain the desired result. \( \square \)

To prove Theorem 1, we use generating functions. Let 
\[
B(t_1, t_2, t_3) = \sum_{m_1, m_2, m_3 \geq 0} (-1)^{m_1+m_2}b(m_1, m_2; m_3) \frac{t_1^{m_1} t_2^{m_2} t_3^{m_3}}{m_1! m_2! m_3!}, \\
C(t_1, t_2, t_3) = \sum_{m_1, m_2, m_3 \geq 0} (-1)^{m_3}c(m_1, m_2; m_3) \frac{t_1^{m_1} t_2^{m_2} t_3^{m_3}}{m_1! m_2! m_3!}.
\]

**Lemma 2.** We have 
\[
\sum_{j=0}^{2} \left( B(t_{1+j}, t_{2+j}, t_{3+j}) + C(t_{1+j}, t_{2+j}, t_{3+j}) \right) = 1, 
\]
where we again put \( t_{a+j} = t_b \) if \( a + j \equiv b \mod 3 \) for \( b = 1, 2, 3 \) and \( a, j \in \mathbb{Z} \).

**Proof.** Set 
\[
\beta(t) = \sum_{k \geq 0} \frac{B_{k+1}}{(k+1)!} t^k = \frac{e^t}{e^t - 1} - \frac{1}{t}.
\]
We compute 
\[
B(t_1, t_2, t_3) = \sum_{l, m_3 \geq 0} \left( \sum_{m_1, m_2 = l} t_1^{m_1} t_2^{m_2} \right) (-1)^l \binom{l + m_3 + 1}{m_3} \frac{B_{l+m_3+2}}{(l + m_3 + 2)!} t_3^{m_3} \\
= \sum_{l, m_3 \geq 0} \frac{t_1^{l+1} - t_2^{l+1}}{t_1 - t_2} (-1)^l \binom{l + m_3 + 1}{m_3} \frac{B_{l+m_3+2}}{(l + m_3 + 2)!} t_3^{m_3}
\]
we only need to prove

Since

\[
\sum_{j=0}^{2} (t_{1+1}-t_{2+1})/(t_{3+1}-t_{2+1}) = 0 \quad \text{and} \quad \sum_{j=0}^{2} e^{t_{1+1}-t_{3+1}} + e^{t_{2+1}-t_{3+1}} = 1,
\]

we get the desired result. \( \square \)

We are now in a position to prove Theorem 1.

**Proof of Theorem 1.** By Lemma 1 we have

\[
\begin{align*}
&= 2 \sum_{j=0}^{2} (-1)^{m_{1+j}+m_{2+j}} b(m_{1+j}, m_{2+j}; m_{3+j}) \lim_{\epsilon_{a} \to 0} \lim_{\epsilon_{b} \to 0} \lim_{\epsilon_{c} \to 0} \frac{(-1)^{\epsilon_{1+j}+\epsilon_{1+j} + (-1)^{\epsilon_{2+j}+\epsilon_{2+j}}} \epsilon_{1+j} + \epsilon_{2+j}}{\epsilon_{1+j} + \epsilon_{2+j}} \epsilon_{1+j} + \epsilon_{2+j} \\
&\quad + \sum_{j=0}^{2} (-1)^{m_{3+j}} c(m_{1+j}, m_{2+j}; m_{3+j}).
\end{align*}
\]

Since

\[
\lim_{\epsilon_{a} \to 0} \lim_{\epsilon_{b} \to 0} \lim_{\epsilon_{c} \to 0} \frac{(-1)^{\epsilon_{1+j}+\epsilon_{1+j} + (-1)^{\epsilon_{2+j}+\epsilon_{2+j}}} \epsilon_{1+j} + \epsilon_{2+j}}{\epsilon_{1+j} + \epsilon_{2+j}} = 1,
\]

we only need to prove

\[
\sum_{j=0}^{2} (-1)^{m_{1+j}+m_{2+j}} b(m_{1+j}, m_{2+j}; m_{3+j}) + \sum_{j=0}^{2} (-1)^{m_{3+j}} c(m_{1+j}, m_{2+j}; m_{3+j})
\]

\[
= \begin{cases} 1, & (m_1, m_2, m_3) = (0, 0, 0), \\ 0, & (m_1, m_2, m_3) \neq (0, 0, 0). \end{cases}
\]

Since the left side of (4.3) coincides with the coefficient of \( t_1^m t_2^m t_3^m/(m_1! m_2! m_3!) \) in the left side of (4.2), the desired result follows from Lemma 2. \( \square \)
5 Conclusion

By Theorem 1 we can define the symmetric double omega values \( \omega_S(k_1, k_2, k_3) \) for nonpositive integers \( k_1, k_2, k_3 \in \mathbb{Z}_{\leq 0} \) by

\[
\omega_S(k_1, k_2, k_3) = \lim_{\varepsilon_1 \to 0} \lim_{\varepsilon_2 \to 0} \omega_{\ell}(k_1 + \varepsilon_1, k_2 + \varepsilon_2, k_3 + \varepsilon_3) \mod \pi^2 \mathbb{Z},
\]

which are either 0 or 1. On the other hand, for \( k_1, k_2, k_3 \in \mathbb{Z}_{\leq 0} \), we compute

\[
\omega_A(k_1, k_2, k_3) = \left( \sum_{n_1, n_2 < p \atop n_1, n_2 \geq 1} n_1^{-k_1} n_2^{-k_2} (p - n_1 - n_2)^{-k_3} \mod p \right)_p
\]

\[
= \left( \sum_{m_2=2 \atop m_1=1}^{p-1} m_1^{-k_1} (m_2 - m_1)^{-k_2} (-m_2)^{-k_3} \mod p \right)_p
\]

\[
= (-1)^{-k_2 - k_3} \sum_{l=0}^{-k_2} (-1)^l \binom{-k_2}{l} \zeta_A(k_1 + k_2 + l, k_3 - l)
\]

in \( A \). Hence by [8, Thm. 1.3], for nonpositive integers \( k_1, k_2, k_3 \in \mathbb{Z}_{\leq 0} \), we have

\[
\omega_A(k_1, k_2, k_3) = \begin{cases} (1)_p, & (k_1, k_2, k_3) = (0, 0, 0), \\ (0)_p, & (k_1, k_2, k_3) \neq (0, 0, 0). \end{cases}
\]

Therefore conjecture (2.5) holds when we restrict \( k \) to elements in \( \mathbb{Z}_{\leq 0}^3 \). Moreover, for integers \( k_1, k_2, k_3 \) such that \( k_1 \leq k_2 = 0 \leq k_3 \), as a consequence of [9, Prop. 1.6], we obtain the correspondence between the limit value (5.1) and \( \omega_A(k_1, k_2, k_3) \), because we have \( \omega_{\ell}(k_1, 0, k_3) = (-1)^{k_3} \zeta_{\ell}(k_1, k_3) \) and \( \omega_A(k_1, 0, k_3) = (-1)^{k_3} \zeta_A(k_1, k_3) \).

Now a possible analogue of (2.4) for multiple zeta values of Mordell–Tornheim type is as follows. Suppose that a certain limit \( \lim_{\varepsilon \to 0} \omega_{\ell}(k + \varepsilon) \) exists in \( \mathbb{Z} \) for all \( k \in \mathbb{I} \). Then we can define the symmetric multiple omega values \( \omega_S(k) \in \mathbb{Z}/\pi^2 \mathbb{Z} \) for all \( k \in \mathbb{I} \) by

\[
\omega_S(k) = \lim_{\varepsilon \to 0} \omega_{\ell}(k + \varepsilon) \mod \pi^2 \mathbb{Z}.
\]

We would ask if

\[
\sum_{k \in \mathbb{I}} a_k \omega_S(k) = 0 \iff \sum_{k \in \mathbb{I}} a_k \omega_A(k) = 0
\]

for a finite subset \( \{ a_k \in \mathbb{Q} \mid k \in \mathbb{I}, r \neq 1 \} \) of \( \mathbb{Q} \).

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