Fermionic cosmologies

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Abstract.

In this work we review if fermionic sources could be responsible for accelerated periods during the evolution of a FRW universe. In a first attempt, besides the fermionic source, a matter constituent would answer for the decelerated periods. The coupled differential equations that emerge from the field equations are integrated numerically. The self-interaction potential of the fermionic field is considered as a function of the scalar and pseudo-scalar invariants. It is shown that the fermionic field could behave like an inflaton field in the early universe, giving place to a transition to a matter dominated (decelerated) period. In a second formulation we turn our attention to analytical results, specifically using the idea of form-invariance transformations. These transformations can be used for obtaining accelerated cosmologies starting with conventional cosmological models. Here we reconsider the scalar field case and extend the discussion to fermionic fields. Finally we investigate the role of a Dirac field in a Brans-Dicke (BD) context. The results show that this source, in combination with the BD scalar, promote a final eternal accelerated era, after a matter dominated period.

1. Introduction

The search for constituents responsible for accelerated periods in the evolution of the universe is a fundamental topic in cosmology. The formulations include usually elements of general relativity, field theory and thermodynamics, analysing the evolution of space-time variables like the scale factor, its acceleration and energy densities. Several candidates has been tested for describing both the inflationary period and the present era: scalar fields, exotic equations of state and cosmological constants[1]. Another possibility is to consider fermionic fields as gravitational sources in an FWR expanding universe. Fermionic sources has been tested using several approaches, with results including exact solutions, anisotropy-to-isotropy scenarios and cyclic cosmologies (see, for example[2, 6]).

In our work the connection between general relativity and the Dirac equation is done via the tetrad formalism; the interactions between the constituents can be modeled through the presence of a non-equilibrium pressure term in the source’s energy-momentum tensor. Besides, it can be considered a self-interaction term for the fermion, in the form of a potential that can assume several forms (included the Nambu-Jona-Lasinio case[7]). While testing fermionic sources as responsible of accelerated periods, different regimes are possible[3, 4]. In a young universe scenario the fermion produces a fast expansion where matter is created till it starts to predominate and the initial accelerated expansion gives place to a decelerated era, which ends when the Dirac field predominates again leading to an accelerated era[3]. In this case the
fermionic field plays the role of the inflaton in the early universe and of dark energy for the old universe, without the need of a cosmological constant term or a scalar field. In the old universe case, an initially matter dominated period gradually turns into a dark (fermionic) energy era when an accelerated regime starts and remains for the rest of the systems evolution.

On the other hand, taking a analytical approach, the idea of form-invariant symmetries has been invoked to extract different evolution regimes from Friedmann-Robertson-Walker cosmologies. This is the case when we obtain the so-called phantom cosmologies [14] from ordinary regimes like universes filled with barotropic fluids. These models generate solutions that contain negative pressure eras, therefore promoting positive accelerated expansions [13].

The manuscript is structured as follows: In section II we make a brief review of the tetrad formalism used to include fermionic and matter fields in a dynamical curved space-time. The field equations for an FRW universe are derived in sections II and III and then we present the analysis of the scenarios in which the fermion answers for accelerated eras and transitions to decelerated periods. In section IV we review the basic ideas behind the form-invariance transformations in cosmology; in section V we focus on the fermionic field case. Finally in section VI we consider a fermionic source in the Brans-Dicke (BD) context, showing that this source combines with the BD scalar to promote a final accelerated regime in a FRW universe; to finish we display our conclusions.

2. Fermionic accelerated regimes

As it is well-known[8, 9, 10] the gauge group of GR does not admit a spinor representation and the tetrad formalism solves the problem[8, 9, 10]. Following the general covariance principle, a connection between the tetrad and the metric tensor $g_{\mu\nu}$ is established, $g_{\mu\gamma} = e_{\mu}^{a}e_{\gamma}^{b}\eta_{ab}$, $a = 0, 1, 2, 3$, where $e_{\mu}^{a}$ denotes the tetrad or vierbein and $\eta_{ab}$ is the Minkowski metric tensor; latin indices refer to the local inertial frame an greek indices to the general system.

Considering that we want to construct a fermionic cosmology we need to describe the behavior of fermions in the presence of a gravitational field, so the next step is to construct an action for this system. The Dirac lagrangian density in Minkowski space-time is

$$L_{D} = \frac{1}{2}[\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi - (\partial_{\mu}\overline{\psi})\gamma^{\mu}\psi] - m\overline{\psi}\psi - V,$$

where the spinors are treated as classically commuting fields[9]. $m$ is the fermionic mass, $\overline{\psi} = \psi^{\dagger}\gamma^{0}$ denotes the adjoint spinor field and the term $V$, which is a function of $\psi$ and $\overline{\psi}$, describes the potential density of self-interaction between fermions. The general covariance principle imposes that the Dirac-Pauli matrices $\gamma^{a}$ must be replaced by their generalized counterparts $\Gamma^{\mu} = e_{a}^{\mu}\gamma^{a}$, whereas the generalized Dirac-Pauli matrices satisfy the extended Clifford algebra, i.e., $\{\Gamma^{\mu}, \Gamma^{\nu}\} = 2g^{\mu\nu}$.

In a second step we need to substitute the ordinary derivatives by their covariant versions $\partial_{\mu}\psi \rightarrow D_{\mu}\psi = \partial_{\mu}\psi - \Omega_{\mu}\psi$, where the spin connection $\Omega_{\mu}$ is given by

$$\Omega_{\mu} = -\frac{1}{4}g_{\mu\nu}[\Gamma^{\nu}_{\sigma\lambda} - e_{b}^{\nu}(\partial_{\sigma}\epsilon_{\lambda}^{b})]\gamma^{\sigma}\gamma^{\lambda},$$

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with $\Gamma^\nu_{\mu \lambda}$ denoting the Christoffel symbols. Hence, the generally covariant Dirac lagrangian becomes

$$L_D = \frac{i}{2} \left[ \bar{\psi} \Gamma^\mu D_\mu \psi - (D_\mu \bar{\psi}) \Gamma^\mu \psi \right] - m \bar{\psi} \psi - V. \quad (3)$$

The field equations are obtained from the total action $S(g, \psi, \bar{\psi}) = \int \sqrt{-g} L_1 \, dx$, where $L_1 = L_g + L_D + L_m$ is the total lagrangian density. $L_g = R/2$, with $R$ denoting the curvature scalar is the free Einstein lagrangian. $L_D$ is the Dirac lagrangian density (3) and $L_m$ is the lagrangian density of the matter field. From the lagrangian density (3), through Euler-Lagrange equations, we obtain the Dirac equations for the spinor field and its adjoint coupled to the gravitational field

$$i \Gamma^\mu D_\mu \psi - m \psi - \frac{dV}{d\psi} = 0, \quad i D_\mu \bar{\psi} \Gamma^\mu + m \bar{\psi} + \frac{dV}{d\psi} = 0. \quad (4)$$

The variation of the total action with respect to the metric tensor leads to Einstein field equations $R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = -T_{\mu \nu}$, where $T_{\mu \nu}$ is the total energy-momentum tensor which is a sum of the energy-momentum tensor of the fermionic field $T_f^{\mu \nu}$ and of the matter field $T_m^{\mu \nu}$, i.e., $T^{\mu \nu} = T^{\mu \nu}_f + T^{\mu \nu}_m$. The symmetric form of the energy-momentum tensor of the fermionic field is given by

$$T_f^{\mu \nu} = \frac{i}{4} \left[ \bar{\psi} \Gamma^\mu D^\nu \psi + \bar{\psi} \Gamma^\nu D^\mu \psi - D^\nu \bar{\psi} \Gamma^\mu \psi - D^\mu \bar{\psi} \Gamma^\nu \psi \right] - g^{\mu \nu} L_D. \quad (5)$$

The Robertson-Walker metric mirrors the homogeneity and isotropy properties of the universe, we have $ds^2 = dt^2 - a(t)^2 (dx^2 + dy^2 + dz^2)$, where $a(t)$ is the cosmic scale factor. The total energy-momentum tensor for an isotropic and homogeneous universe is given by $(T^{\mu \nu}) = \text{diag}(\rho, -p - \varpi, -p - \varpi, -p - \varpi)$, where the total energy density $\rho$ and the total pressure $p$ are given as a sum of the corresponding terms of the fermionic and matter fields, i.e., $\rho = \rho_f + \rho_m$ and $p = p_f + p_m$. Moreover, the quantity $\varpi$ refers to a non-equilibrium pressure which is related to dissipative processes during the evolution of the universe and represents an irreversible process of energy transfer between the matter and the gravitational field [12].

Thanks to the Bianchi identities, the covariant differentiation of Einstein field equations lead to the conservation law of the total energy-momentum tensor $T^{\mu \nu; \nu} = 0$, hence it follows by using the representation (12) the evolution equation for the total energy density:

$$\dot{\rho} + 3H(\rho + p + \varpi) = 0, \quad (6)$$

where the dot refers to a differentiation to time and $H = \dot{a}(t)/a(t)$ denotes the Hubble parameter. Furthermore, from Einstein equations follow the Friedmann and acceleration equations

$$H^2 = \frac{1}{3} \rho, \quad \frac{\dot{a}}{a} = -\frac{1}{6}(\rho + 3p + 3\varpi), \quad (7)$$

respectively. Only two equations from (13) and (14) are linearly independent. For the FRW metric the tetrad reads $e^\mu = \delta^\mu_0, e^\mu_i = \frac{1}{a(t)} \delta^\mu_i$. Also, the Dirac matrices become

$$\Gamma^0 = \gamma^0, \quad \Gamma^i = \frac{1}{a(t)} \gamma^i, \quad \Gamma^5 = -\sqrt{-g} \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 = \gamma^5, \quad (8)$$

from which the spin connection components are obtained, yielding $\Omega_0 = 0, \Omega_i = \frac{1}{2} \dot{a}(t) \gamma^0 \gamma^i$. For a FRW universe the fermionic field is an exclusive function of time, so the Dirac equations (4) become

$$\dot{\psi} + \frac{3}{2} H \psi + m \gamma^0 \psi + r \gamma^0 \frac{dV}{d\psi} = 0. \quad (9)$$
Combining the equations above, the non-zero components of the energy-momentum tensor become
\[
(T_f)^0_0 = m(\bar{\psi}\psi) + V, (T_f)^1_1 = (T_f)^2_2 = (T_f)^3_3 = V - \frac{dV}{d\psi} = \frac{\bar{\psi} dV}{2 d\bar{\psi}},
\]
which are only functions of \(\psi\) and \(\bar{\psi}\). By identifying the components of the energy-momentum tensor of the fermionic field as \((T_f)^\mu_\nu = \text{diag}(\rho_f, -p_f, -p_f, -p_f)\), one can obtain from the equations above[3] the conservation law for the energy density of the fermionic field: \(\dot{\rho}_f + 3H(\rho_f + p_f) = 0\). Hence, the evolution equation for the energy density of the fermionic field decouples from the energy density of the matter field, and we obtain[3]: \(\dot{\rho}_m + 3H(\rho_m + p_m) = -3H\varpi\), where the term \(-3H\varpi\) could be interpreted as the energy density production rate of the matter field (see e.g.[12]). According to the extended thermodynamic theory the non-equilibrium pressure \(\varpi(t)\) obeys an evolution equation, whose linear form reads \(\tau\dot{\varpi} + \varpi = -3\eta H\), where \(\tau\) denotes a characteristic time and \(\eta\) is the so-called bulk viscosity.

3. Cosmological Solutions
In order to analyze cosmological solutions for the system of the previous section, we have to specify the fermionic potential \(V\). According to the Pauli-Fierz theorem \(V\) is an exclusive function of the scalar invariant \((\bar{\psi}\psi)^2\) and of the pseudo-scalar invariant \((i\bar{\psi}\gamma^5\psi)^2\), i.e.,
\[
V = V((\bar{\psi}\psi)^2, (i\bar{\psi}\gamma^5\psi)^2).
\]
Here we are interested in self-interaction potentials of the form
\[
V = \lambda \left[\beta_1(\bar{\psi}\psi)^2 + \beta_2(i\bar{\psi}\gamma^5\psi)^2\right]^n,
\]
where the coupling constant \(\lambda\) is a non-negative quantity and \(n\) is a constant exponent. We shall analyze three cases, namely, (i) \(\beta_1 = 1\) and \(\beta_2 = 0\) where \(V\) is a function only of the scalar invariant; (ii) \(\beta_1 = 0\) and \(\beta_2 = 1\) where \(V\) depends only on the pseudo-scalar invariant and (iii) \(\beta_1 = \beta_2 = 1\) where \(V\) is a combination of the scalar and pseudo-scalar invariants. The Nambu-Jona-Lasinio potential [7] is represented by the last case with \(n = 1\).

The energy density and the pressure of the fermions for the potential \(V\) are given by
\[
\rho_f = m(\bar{\psi}\psi) + \lambda \left[\beta_1(\bar{\psi}\psi)^2 + \beta_2(i\bar{\psi}\gamma^5\psi)^2\right]^n, p_f = (2n - 1)[\rho_f - m(\bar{\psi}\psi)].
\]
The fermions could be classified according to the value of the exponent \(n\). Indeed, for \(n \geq 1/2\) the fermions represent a matter field with positive pressure \((n > 1/2)\) or a pressureless fluid \((n = 1/2)\), whereas for \(n < 1/2\) the pressure of the fermions is negative and they could represent either the inflaton or the dark energy[3].

For massless fermions, the pressure is connected with the energy density by a simple barotropic equation of state \(p_f = (2n - 1)\rho_f\) and it follows from the conservation equation (20) for the fermions that \(\rho_f \propto 1/a^{6n}\). We shall not analyze this case here, since the behavior of the fermionic field does not differ from that of a matter field when \(n > 1/2\) or from that of a bosonic field when \(n < 1/2\). Moreover, for the massive case, we shall deal with only the case where the fermionic field behaves as inflaton or dark energy, i.e., the case where \(n < 1/2\).

For the pressure of the matter field we shall adopt a barotropic equation of state, i.e.,
\[
p_m = w_m p_m\]
with \(0 \leq w_m \leq 1\). Furthermore, the coefficient of bulk viscosity \(\eta\) and the characteristic time \(\tau\) are assumed to be related with the energy density \(\rho\) by \(\eta = \alpha \rho\) and \(\tau = \eta / \rho\), where \(\alpha\) is a constant[12].

The system of field equations we shall investigate in order to find the cosmological solutions are: (a) the acceleration equation \(\frac{\ddot{a}}{a} = -\frac{1}{2} (\dot{\rho}_f + \dot{p}_f + 3p_f + 3p_m + 3\varpi)\); (b) the evolution equation for the energy density of the matter field and the non-eq. pressure \(\dot{\rho}_m + 3H(\rho_m + p_m + \varpi) = 0, \tau\dot{\varpi} + \varpi = -3\eta H\), and c) the Dirac equation. This consists of a system of seven coupled ordinary differential equations for the fields \(a(t), \rho_m(t), \varpi(t), \psi_1(t), \psi_2(t), \psi_3(t)\) and \(\psi_4(t)\) and
in the following subsections we shall find solutions of this system of equations for given initial conditions.

Let us analyze the case that corresponds to the evolution of the early universe, where the fermionic field plays at the beginning the role of an inflaton and the matter is created by an irreversible process through the presence of a non-equilibrium pressure $\varpi$. The initial conditions we have chosen for $t = 0$ are $a(0) = 1, \psi_1(0) = 0.1, \psi_2(0) = 1, \psi_3(0) = 0.3, \psi_4(0) = 0, \rho_m(0) = 0, \varpi(0) = 0$. The last two initial conditions correspond to a vanishing energy density of the matter field and a vanishing non-equilibrium pressure at $t = 0$. The conditions chosen here characterize qualitatively an initial proportion between the constituents in the corresponding era (i.e., in this case we have a predominating fermionic field over the matter, indicating the initial inflationary state). We need to specify now an initial condition for $\dot{a}(0)$. This condition follows from the Friedmann equation, i.e., $\dot{a}(0) = a(0)\sqrt{\frac{\rho_f(0) + \rho_m(0)}{3}}$. Besides, we also have to specify some parameters in order to obtain numerical solutions of the coupled system above, these parameters are: (a) $\lambda, \beta_1, \beta_2$ and $n$ which define the self-interacting potential $V$; (b) $m$ which is related to the mass of the fermionic field; (c) $w_m$ which defines the matter field through its barotropic equation of state $p_m = w_m\rho_m$ and (d) $\alpha$ which is connected with the bulk viscosity term $\eta = \alpha \rho$. We have chosen the following values for these parameters: $\lambda = 0.1, \beta_1 = \beta_2 = 1, n = 0.3, m = 0.01, w_m = 1/3, \alpha = 1.0, and \alpha = 1.2$. Which correspond to a fermionic field with a negative pressure, described by a self-interacting potential that depends on the scalar and pseudo-scalar invariants and a matter field of massless particles that could describe a radiation field. The numerical integration of the system[3] indicates that there exists an accelerated period where the fermionic field dominates and the matter field is created at the expenses of an irreversible process of energy transfer between the matter and the gravitational field. The accelerated period is followed by a decelerated era which is dominated by the matter field. Due to fact that the self-interaction potential tends to a constant value for large values of time, the energy density of the fermionic field overcomes the energy density of the matter field and the universe goes into another accelerated period. It is noteworthy that for the bosonic case where the potentials are exponentials or inverse power-laws, one has to add a constant value – which is similar to introduce a cosmological constant term – in order to return to an accelerated era after the accelerated-decelerated period (see, for example[12]). Here the same self-potential interaction which plays the role of an inflaton field at the beginning plays the role of a cosmological constant for later times and could be identify as dark energy. The numerical integration[3] also indicates that for large values of the coefficient $\alpha$ it follows that: (a) the energy density of fermionic field decays more rapidly causing a larger accelerated period and (b) the energy density of the matter field has a more significant growth and leads to a larger decelerated period. In this first part we have investigated the possibility that a fermionic field – with a self-interacting potential – could be the responsible for accelerated regimes in the evolution of the universe. We have shown that the fermionic field behaves like an inflaton field for the early universe, where the matter field was created by an irreversible process connected with a non-equilibrium pressure. As a final comment to this section we must mention that there is in fact another possible scenario in this model; the fermion playing the role of dark energy[3]. For an old decelerated universe dominated by ordinary matter the fermionic field drives the universe to a final eternal accelerated era[2, 3, 4].

4. Internal symmetry in FRW

When we turn our attention to analytical solutions in cosmological models one interesting idea is the use of symmetries. The issue of form-invariant symmetries has been invoked recently to extract different evolution regimes from Friedmann-Robertson-Walker cosmologies [14]. This is the case when we obtain the so-called phantom cosmologies [13] from ordinary regimes like universes filled with barotropic fluids [14]. These new solutions contain negative
pressure regimes, therefore promoting positive accelerated expansions [13]. The case were the gravitational source is a bosonic field was investigated in [14]. In spatially flat perfect fluid FRW cosmologies, such symmetries can be viewed as a prescription relating the quantities \(a, H, \rho\) and \(p\) in a given initial scenario to quantities \(\bar{a}, \bar{H}, \bar{\rho}\) and \(\bar{p}\) corresponding to a new cosmological model. As a starting point we present a quick review of this internal symmetry with the purpose of investigating it in bosonic and fermionic cases [13, 19, 15]. As is well-known, the Einstein equations for a flat FRW cosmological model with scale factor \(a\), and filled with a perfect fluid with energy density \(\rho\) and pressure \(p\), are

\[
3H^2 = \rho, \quad \dot{\rho} + 3H(\rho + p) = 0, \quad (13)
\]

where \(H = \dot{a}/a\). The correspondent form-invariance transformation is given by [13]

\[
\bar{\rho} = \bar{\rho}(\rho), \quad \bar{H} = \left(\frac{\bar{\rho}}{\rho}\right)^{1/2} H, \quad \bar{\rho} = -\bar{\rho} + \left(\frac{\rho}{\bar{\rho}}\right)^{1/2} (\rho + p) \frac{d\bar{\rho}}{d\rho}, \quad (14)
\]

where \(\bar{\rho} = \bar{\rho}(\rho)\) is an invertible arbitrary function. This set of transformations give rise to the form-invariance symmetry group (FISG)[14, 13, 15]. For the perfect fluid case, \(p = (\gamma - 1)\rho\), the barotrophic indexes of the original and transformed fluid are related by \(\bar{\gamma} = \left(\frac{\rho}{\bar{\rho}}\right)^{3/2} \frac{d\bar{\rho}}{d\rho} \gamma\). For later application it will useful to investigate the transformation rule for the quantities which characterize the geometry and the fluid when the transformation is generated by [13] \(\bar{\rho} = n^2 \rho\), where \(n\) is the constant parameter of the FISG. Hence, for the geometrical quantities and equation of state we get

\[
\bar{H} = nH, \quad \bar{a} = a^n, \quad \bar{w} = \bar{\gamma} - 1 = \frac{\gamma}{n} - 1, \quad \bar{p} = \bar{\rho} = \left(\frac{\gamma}{n} - 1\right) \bar{\rho}, \quad (15)
\]

Let us analyze the equation of state (15) when the seed fluid represents a normal fluid with \(\gamma > 0\). We can infer: (i) when \(0 < n < \gamma\) it follows that \(\bar{w} > 0\) and the cosmological fluid behaves as a normal fluid; (ii) when \(n > \gamma\) it follows that \(-1 < \bar{w} < 0\) and the cosmological fluid behaves as a quintessence constituent; (iii) when \(n < 0\) it follows that \(\bar{w} < -1\) and the cosmological fluid behaves as a phantom constituent[14].

5. The fermionic case

In this section we extend the form-invariance transformation to a fermionic field satisfying the Dirac equation in curved space-time. As required by the equations above we must calculate the energy density and the hydrostatic pressure of the fermionic field. We have [3]

\[
\rho_\psi = m(\bar{\Psi}\Psi) + V, \quad p_\psi = \frac{dV \Psi}{d\Psi} + \frac{\bar{\Psi} dV}{d\bar{\Psi}} - V. \quad (16)
\]

To obtain the transformation properties of \(\rho_\psi\) and \(p_\psi\) we express the potential in terms of the scalar invariant \(X = (\bar{\Psi}\Psi)^2\) and the pseudo-scalar invariant \(Y = (i\bar{\Psi}\gamma^5\Psi)^2\). To this end, we suppose a generic potential \(V = V(X, Y)\), that include, among others, the Nambu-Jona-Lasinio potential [3]. For this kind of potential the hydrostatic pressure becomes

\[
p = 2X \frac{\partial V}{\partial X} + 2Y \frac{\partial V}{\partial Y} - V. \quad (17)
\]

Applying the transformation rules (14) to the energy density (16) and the pressure we get

\[
\bar{m}\sqrt{X} + \bar{V} = n^2(m\sqrt{X} + V), \quad (18)
\]
The Dirac’s equations in a FRW background for the seed and transformed configurations read

\[ 2X \frac{\partial \tilde{V}}{\partial X} + 2Y \frac{\partial \tilde{V}}{\partial Y} + \tilde{m} \sqrt{X} = n \left[ 2X \frac{\partial V}{\partial X} + 2Y \frac{\partial V}{\partial Y} + m \sqrt{X} \right]. \] \tag{19}

We also obtain the following transformation equation\(^\text{[?]}\)

\[ \left( \frac{m}{2\sqrt{X}} + \frac{\partial V}{\partial X} \right) \left[ \tilde{X} \frac{\partial X}{\partial X} + \tilde{Y} \frac{\partial X}{\partial Y} - \frac{X}{n} \right] + \frac{\partial V}{\partial X} \left[ \tilde{X} \frac{\partial Y}{\partial X} + \tilde{Y} \frac{\partial Y}{\partial Y} - \frac{Y}{n} \right] = 0. \]

All transformations satisfying these last conditions represent internal symmetries of the Einstein’s equations with a fermionic source, provide that condition becomes an identity for any potential and particle mass. Then the two square brackets must vanish

\[ \tilde{X} \frac{\partial X}{\partial X} + \tilde{Y} \frac{\partial X}{\partial Y} = \frac{X}{n}, \quad \tilde{X} \frac{\partial Y}{\partial X} + \tilde{Y} \frac{\partial Y}{\partial Y} = \frac{Y}{n}. \] \tag{20}

In addition, by imposing the condition \(\tilde{m}(n = 1, \tilde{X} = X, \tilde{V} = V) = m\), on the transformed fermionic mass, any solution of the coupled system of partial differential equations (20) is a form-invariance symmetry transformation of the Einstein-Dirac dynamics. For the general massive fermionic field whose potential depends only on \(X\), the solutions lead to the general form-invariance symmetry transformation\(^\text{[?]}\)

\[ \tilde{X} = X^n, \tilde{V} = n^2 V + n^2 m(\sqrt{X} - X^{n/2}), \tilde{m} = n^2 m. \] \tag{21}

Following the same methodology of the bosonic scalar field, let us investigate two simple cases, (i) the free massive fermionic particle and (ii) the massless fermionic particle with a polynomial potential. In the free case, the energy density (16) and pressure (17) of the fermionic particle are given by \(\rho = m \sqrt{X}, p = 0, V = 0\). Solving the Einstein equations we obtain

\[ a = a_0 t^{2/3}, \quad X = \frac{16}{9m^2} \left( \frac{a_0}{a} \right)^6, \] \tag{22}

where \(a_0\) is an integration constant. Suppose now that we want to in comparing this cosmological model with other one, this can be done by applying the transformations (14) together with Eq. (22) to \(a(t)\) and the potential, which is dependent only on \(X\). Inserting the seed solution into the generated equations we find the full set of power law solutions along with the potential, energy density and pressure characterizing the transformed fermionic configuration:

\[ \tilde{a} = \tilde{a}_0 (\pm t)^{2n/3}, \quad \tilde{a}_0 = a_0^n, \tilde{X} = \left( \frac{16}{9m^2} \right)^n \left( \frac{a_0}{a} \right)^6, \] \tag{23}

\[ \tilde{V} = \tilde{m} (\tilde{X}^{1/2n} - \tilde{X}^{1/2}), \quad \tilde{m} = n^2 m, \tilde{\rho} = \tilde{m} \tilde{X}^{1/2n}, \quad \tilde{\rho} = \left( \frac{1}{n} - 1 \right) \tilde{\rho}. \] \tag{24}

The Dirac’s equations in a FRW background for the seed and transformed configurations read

\[ \dot{\psi} + \frac{3}{2} H \psi + m \gamma^0 \psi + r \gamma^0 \frac{dV}{d\psi} = 0, \] \tag{25}

along with the corresponding equation for \(\tilde{\psi}\) and \(\bar{\psi}\). In order to the physics of the fermionic particles maintain their own characteristics we assume that the Dirac’s matrices remain invariants under a form invariance symmetry transformation. For \(n = 1\) the latter becomes the spinor field corresponding to the seed solution (22). The Einstein equations for the new configuration are \(3\dot{H}^2 = \tilde{\rho}, -2\dot{H} = \tilde{\rho} + \tilde{p}\); so the new solution (23)-(24) can be associated to a fermionic barotropic fluid whose equation of state is \(\gamma = 1 + \tilde{p}/\tilde{\rho} = 1/n\). When \(n = 1\), identical
transformation, the solution of the Einstein’s equations become the seed solution (22), which can be identified with some kind of fermionic dust matter[15]. Now, we consider the massless case and investigate a spinor field driven by a potential $V = V(X)$ depending on $X$. The imposition of the energy density transformation
\[ \rho = n^2 \rho \]
allow us to apply the transformation rules (14) for $X$ and the potential $V$:
\[ \bar{X} = X^n, \quad \bar{V} = n^2 V. \]  
\[ (26) \]
By considering the case where the potential is given by a power law $V = X^\alpha$, where $\alpha$ is a constant, we choose the seed potential $V = \sqrt{X}$, and from the Einstein equations it follows
\[ a = a_0 t^{2/3}, \quad X = \frac{16}{9} \left( \frac{a_0}{a} \right)^6. \]  
\[ (27) \]
This solution and the corresponding spinor field can be obtained from the above seed solution (22) by making $m = 1$. However, both cases, $m \neq 0$ and $m = 0$, are absolutely different because a form invariance transformation that relates them does not exist, as can be seen from Eq. (27). In fact, for the $m = 0$ case, the solution for the transformed fermionic configuration read
\[ \bar{a} = \bar{a}_0 t^{2n/3}, \quad \bar{a}_0 = a_0^n, \quad \bar{X} = \left( \frac{16}{9} \right)^n \left( \frac{a_0}{a} \right)^6, \quad \bar{V} = n^2 \bar{X}^{1/2n}, \quad \bar{\rho} = n^2 \bar{X}^{\alpha/n}, \quad \bar{p} = \left( \frac{1}{n} - 1 \right) \bar{\rho}. \]  
\[ (28) \]
Thus, comparing the transformed configurations (23,24) with the original ones for $m = 1$, we see that both potentials are different and the dynamics too.

6. Brans-Dicke fermionic cosmology

Einstein’s General Relativity (GR) is the usually invoked theory for describing the evolution of the fundamental space-time variables in cosmological models[8]. As an alternative to GR we have the Brans-Dicke theory of gravitation[8], a scalar-tensor formulation that rules the gravitational phenomena through the interplay between the metric tensor and a scalar field $\varphi$ that controls the intensity of the gravitational constant $G$. On the other hand, cosmological models that include dark energy are strong candidates for explaining the present regime of positive acceleration of the universe[1]. Going in that direction other possibility, as discussed in the previous approaches, is to consider fermionic fields as gravitational sources for those accelerated universes. As was exposed in the previous sections, these authors[3] proposed a cosmological model (based on GR) in a dissipative Universe and showed that in a young universe scenario the fermionic field produces a fast expansion where matter (included via a barotropic equation of state) is created till it starts to predominate and the initial accelerated period gives place to a decelerated era. In this case the fermionic field plays the role of the inflaton in an early period and the role of dark energy for an old universe; without the need of a cosmological constant term or a scalar field[3]. In an exclusive old universe scenario an initially matter dominated period gradually turns into a dark (fermionic) energy period when an accelerated regime rules for the rest of the system evolution[3, 4]. Testing those properties in a universe ruled by Brans-Dicke gravitation would be of interest and this is the main focus of this section.

As we saw in the previous discussions, for a FRW universe the fermionic field becomes an exclusive function of time. Besides the Dirac field, we have considered as gravitational source a matter constituent that is ruled by a barotropic equation of state, namely $\rho_m = b_m p_m$, where $b_m$ is the barotropic coefficient ($0 \leq b_m \leq 1$); then the total energy density $\rho$ is given as a sum of these individual contributions, i.e., $\rho = \rho_D + \rho_m = T_{00}$, where $T_{\mu \nu}$ is the total energy-momentum tensor of sources. The energy density $\rho$ satisfies the conservation law $\dot{\rho} + 3\dot{a}/a (\rho + p) = 0$. Combining the Dirac and Einstein equations one can obtain an independent conservation law
for the energy density of the fermionic field[3]. This implies into a decoupled conservation equation for the energy density of the matter constituent, i.e., \( \rho_m + 3H(\rho_m + p_m) = 0 \). In order to analyze the cosmological solutions of our model we have to define first some sources properties. As in section II we consider a fermionic self-interaction potential, \( V = (\bar{\psi}\psi)^n \), where \( n \) is a constant real number[3]. On the other hand the matter field, is again following a barotropic equation of state[3, 5]. After some algebraic manipulation we can put the model dynamics in the following form:

\[
\frac{2\ddot{a}}{a} + H^2 + \alpha^2\dot{\varphi} + 2\alpha H\dot{\varphi} + \frac{\alpha^2}{2}\varphi^2 = e^{-\alpha}\varphi\left[(2n-1)(\bar{\psi}\psi)^{2n} + p_m\right],
\]

\[
\alpha^2\dot{\varphi} + 3\alpha H\dot{\varphi} = \frac{e^{-\alpha}}{3 + 2\omega}\left[V + \rho_m - 3(2n-1)(\bar{\psi}\psi)^{2n} - 3p_m\right],
\]

\[
\dot{\psi} + \frac{3}{2}H\psi + \gamma_0 \frac{dV}{d\psi} = 0, \quad \dot{\rho}_m + 3H(\rho_m + p_m) = 0
\]

We consider a pressureless matter field (\( p_m = 0 \)), so that one can obtain from conservation that \( \rho_m(t) = \rho_m(0)/a(t)^3[5] \). The equations (29-31) constitute a highly non-linear system of differential equations and we proceed to solve it numerically. We analyze the time evolution of our model choosing first the conditions for \( t = 0 \): \( a(0) = 1, \dot{a}(0) = 1, |\psi(0)| = 0.001, \rho_m(0) = 1, \varphi(0) = 1, \dot{\varphi}(0) = 0.001 \). These conditions characterize qualitatively an initial proportion between the constituents; an era when matter predominates over the fermionic density. Besides that, we have to specify the magnitude of the remaining parameters: we suppose initially that \( \alpha = 1.0, \omega = 4 \times 10^4[5] \) and a value for the potential power \( n, n = 0.2[3] \). These choices are reference values that permit final adjustments to follow several cosmological constraints, like the spectrum of the Brans-Dicke coupling \( \omega[11] \) and the present value of the Brans-Dicke scalar field \( \varphi \). These parameters can be in fact adjusted due to invariance properties of the Brans-Dicke gravitational field equations, by using the following change of variables: \( \alpha \rightarrow \bar{\alpha}, \quad \varphi \rightarrow \bar{\varphi}, \quad \bar{\alpha}\bar{\varphi} = \gamma/\varphi + \alpha \) where \( \exp(-\gamma) = x_0 \) is associated to the asymptotic value of \( \bar{\varphi}_0 = (2\omega + 4)/(2\omega + 3) \). These transformations show how a value of the Brans-Dicke parameter \( \omega \) is linked to the definition of a new Brans-Dicke scalar field \( \bar{\varphi} \). In fact, after numerical integration, it is possible to verify that these features are included in the \( \varphi \) evolution that imply into the evolution of the gravitational "constant" as \( G(t) = (2\omega + 4)/[(2\omega + 3)\bar{\varphi}(t)] \).

The numerical integration of the system shows[5] that initially the universe is expanding with negative acceleration, a period where the matter constituent predominates over the fermionic field. With the evolution of time we have increasing values in energy transference to the fermion (this is happening via the gravitational field \( a(t) \) and the scalar field \( \varphi(t) \), as the equations of motion show). The negative pressure of \( \psi(t) \) helps in fact to promote a final accelerated period indicating that in this model the dark energy role would be played by a combination of the fermionic constituent with the scalar \( \varphi(t) \). Other interesting results appear when we choose the power \( n \) to be in the neighborhood of values \( n \approx 0.33 \). In fact, for a fixed value of \( \varphi(0) \) and increasing values of \( n \) what emerges is a universe that fails to show a final accelerated period, even when the fermionic field still exhibits a negative pressure (\( p_D = (2n - 1)\left[(\bar{\psi}\psi)^2\right]^n \)). On the other hand, for values \( n \leq 0.33 \) we will find a universe that is permanently in accelerated expansion.

Another important remark here is that this general qualitative behavior depends strongly on the initial value of the time derivative of the scalar field \( \varphi(t) \): what we verify is that increasing values of \( \varphi(0) \) promote an earlier entrance on the accelerated period[5].

For the energy densities of the fermionic field \( \rho_D \) and matter \( \rho_m \) the numerical results show[5] that eventually the energy density of the fermionic field overcomes the energy density of the
matter field, although this does not coincide with the instant when the universe goes into an accelerated period (opposed to what occurs in some Einstein gravity based models[12]). Another feature showed by the numerical results is that for larger values of \( n \) it follows that: (a) the energy density of fermionic field grows more slowly causing a larger decelerated period and (b) the energy density of the matter field has a less significant decay and also promotes a larger decelerated period. On the other hand, both densities have decreasing values in time due to the permanent expansion of the universe (this was verified with plots of the scale factor against time[5]). Again, all cases are strongly dependent on the exponent \( n \) of the self-interacting potential and one can obtain different behaviors, that are in tune with the acceleration patterns presented above. Finally we verify that the scalar field \( \varphi \) approaches a final constant value that validates the accord between Brans-Dicke and Einstein gravitation for large \( t \).

7. Conclusion
In this work we have reviewed several scenarios where fermionic field is responsible for accelerated regimes in the evolution of a FRW universe. Through numerical integrations is is possible to verify that the fermion behaves like an inflaton field for the early universe and later on as a dark energy field[3], whereas ordinary matter is created by including a non-equilibrium pressure. On the analytic solutions analysis we have found the symmetry transformations under which the Einstein equations, for a FRW universe filled with bosons or fermions, preserve their form. This group of transformations has been used to obtain power-law solutions from a seed one. Starting from a contracting spatially flat FRW cosmological model we get, after using the dual transformation, a super-accelerated spatially flat FRW cosmological model, i.e., the phantomization of the model[15]. Finally, using the Brans-Dicke theory of gravitation we investigated a hypothetical FRW universe filled with a fermionic field (with a self interaction potential) and a matter constituent ruled by a barotropic equation of state. The results show that the fermionic field (in combination with the Brans-Dicke scalar field \( \varphi(t) \)) can be responsible for a final eternal accelerated era, after an initial matter dominated period.

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