Comment on “All quantum observables in a hidden-variable model must commute simultaneously”

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Malley discussed [Phys. Rev. A 69, 022118 (2004)] that all quantum observables in a hidden-variable model for quantum events must commute simultaneously. In this comment, we discuss that Malley’s theorem is indeed valid for the hidden-variable theoretical assumptions, which were introduced by Kochen and Specker. However, we give an example that the local hidden-variable (LHV) model for quantum events preserves noncommutativity of quantum observables. It turns out that Malley’s theorem is not related to the LHV model for quantum events, in general.

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I. INTRODUCTION

Physical observables do not generally commute in the Hilbert space formalism of quantum theory [1, 2]. Recently, Malley discussed [3, 4] that all quantum observables must commute simultaneously if we accept a hidden-variable (HV) model for quantum events.

First, Malley showed that all quantum observables must commute simultaneously under a special set of assumptions valid for some HV model for quantum events. According to Malley’s paper, the special set of assumptions is equivalent to those under which the Kochen-Specker (KS) theorem [5] is derived. And, Malley claimed that these conditions are also equivalent to those under which the Bell inequalities [6] are derived, upon invoking Fine’s paper –proposition (2) in Ref. [7]. Finally, Malley concluded that the experimental violations of the Bell inequalities demonstrate only that quantum observables do not commute.

One can find that the argument by Malley is indeed valid under the special HV theoretical assumptions which were used in order to construct Malley’s theorem. In more detail, the product rule (the KS condition) and the uniqueness feature of Gleason’s theorem imply that all quantum observables commute simultaneously [4].

On the other hand, another type of model for quantum events has been presented [8, 9]. It can be interpreted by a HV model. The HV model for quantum events says that classical random variables are represented by in general noncommutative operators in the Hilbert space formalism of quantum theory. In other words, the HV model for quantum events preserves noncommutativity of quantum observables, even though we accept it.

By reading Bell’s arguments [6] carefully, we can notice the condition under which Bell’s theorem is derived. In fact, the condition is only that quantum correlation functions are reproducible by the classical-random-variables model for quantum events. Further, a classical random variable related to one site must not depend on the (simultaneous) choices of measurement observables on the other site each other. (This condition is related to Bell’s nonlocality.) Namely, we can say that the local hidden-variable (LHV) model for quantum events was constructed only by classical random variables. And they depend on quantum measurement observables (Hermitean operators) for each sites. Hence, one can see that the LHV model for quantum events can coexist with the HV model for quantum events presented in Refs. [8, 9].

Therefore, we may be confused. Do all quantum observables commute simultaneously when we accept the LHV model for quantum events? Do the experimental violations of the Bell inequalities demonstrate only that quantum observables do not commute? In fact, this problem was also discussed in Ref. [10] from different approach (Theorem 7 in Ref. [10]).

We shall investigate the reason why the argument claimed by Malley gives rise to the contradiction against the existence of the HV model presented in Refs. [8, 9]. In what follows, we shall give an example such that the LHV model for quantum events preserves noncommutativity of quantum observables, on using several quantum states.

II. LHV MODEL SUGGESTING NONCOMMUTATIVITY

In what follows, we shall mention the standard LHV model for quantum events. And we shall give a counterexample against Malley’s claim.

Let $L(H)$ be the space of Hermitian operators acting on a finite-dimensional Hilbert space $H$, and $T(H)$ be the space of density operators acting on the Hilbert space $H$. Namely, $T(H) = \{ \rho | \rho \in L(H) \land \rho \geq 0 \land \text{tr}[\rho] = 1 \}$.

Let us consider a classical probability space $(\Omega, \Sigma, M_\rho)$, where $\Omega$ is a nonempty space, $\Sigma$ is a $\sigma$-algebra of subsets of $\Omega$, and $M_\rho$ is a $\sigma$-additive normalized measure on $\Sigma$ such that $M_\rho(\Omega) = 1$. The subscript $\rho$ expresses the following meaning: The probability measure $M_\rho$ is determined uniquely when the state $\rho$ is specified.

Consider bipartite states $\rho$ in $T(H_1 \otimes H_2)$, where $H_k$ represents the Hilbert space with respect to party $k = 1, 2$.  

Then we can define functions \( f_k : v_k, \omega \rightarrow f_k(v_k, \omega) \in \{I(v_k), S(v_k), v_k \in L(H_k), \omega \in \Omega \}. \) Here, \( S(v_k) \) and \( I(v_k) \) are the supremum and the infimum of the spectrum of Hermitian operators \( v_k \), respectively.

The functions \( f_k(v_k, \omega) \) must depend on the choices of \( v_k \)’s on the other site each other. On using the functions \( f_k \), we can define quantum states which admit the LHV model [11]. Namely, a quantum state is said to admit the LHV model if and only if there exist a classical probability space \( (\Omega, \Sigma, \mu) \) and of functions \( f_1, f_2 \), such that

\[
\int_{\Omega} M_{\rho}(d\omega)f_1(v_1, \omega)f_2(v_2, \omega) = tr[\rho v_1 \otimes v_2],
\]

for every Hermitian operator in the following form: \( v_1 \otimes v_2 \). Here, \( v_k \in L(H_k) \). Note that there are several (non-commuting) observables per site (not just one \( v_k \)).

The meaning of Eq. (1) is as follows: All correlation functions \( \text{tr}[\rho v_1 \otimes v_2] \) in the state \( \rho \) are reproducible by the LHV model for quantum events.

Let us consider the Pauli spin-1/2 operators, \( \sigma^k_x, \sigma^k_y, \) and \( \sigma^k_z \). Let us assume the system is in an element of certain set of two-site-1 states. They are bipartite separable states written by

\[
U(\alpha, \beta) = \alpha |+1, +2\rangle \langle +1, +2| + \beta |-1, -2\rangle \langle -1, -2|,
\]

where \( \sigma^k_{\pm} = \pm 1|\pm\rangle \rangle \) and \( \alpha + \beta = 1, \alpha, \beta \geq 0 \).

As is well known, every separable state admits the LHV model [11]. In other words, all correlation functions in those separable states \( U(\alpha, \beta) \) are described with the property that they are reproducible by the LHV model for quantum events. Hence functions \( (f_1, f_2) \) exist. That is, we obtain the following equation:

\[
\int_{\Omega} M_{U(\alpha, \beta)}(d\omega)f_1(v_1, \omega)f_2(v_2, \omega) = \text{tr}[U(\alpha, \beta)v_1 \otimes v_2],
\]

for every observable \( v_1 \otimes v_2 \) and every \( \alpha, \beta \). From Eq. (3), when \( v_k = i[\sigma^k_x, \sigma^k_y] \), we have

\[
\int_{\Omega} M_{U(\alpha, \beta)}(d\omega)f_1(i[\sigma^1_x, \sigma^1_y], \omega)f_2(i[\sigma^2_x, \sigma^2_y], \omega) = \text{tr}[U(\alpha, \beta)i[\sigma^1_x, \sigma^1_y] \otimes i[\sigma^2_x, \sigma^2_y]],
\]

(4)

Please notice that \( i[\sigma^k_x, \sigma^k_y] = -2\sigma^k_z \) are Hermitian operators. On substituting Eq. (2) into Eq. (4) and performing some algebra, we find that

\[
\int_{\Omega} M_{U(\alpha, \beta)}(d\omega)f_1(i[\sigma^1_x, \sigma^1_y], \omega)f_2(i[\sigma^2_x, \sigma^2_y], \omega) = 4(\neq 0),
\]

(5)

in spite of any possible values of \( \alpha \) and of \( \beta \). This implies that there exists an event \( (\Sigma', M_{U(\alpha, \beta)}(\Sigma') \neq 0) \) for a \( \sigma \)-algebra \( \Sigma \) such that

\[
[\sigma^1_x, \sigma^1_y] \neq 0 \land [\sigma^2_x, \sigma^2_y] \neq 0 : \forall \omega \in \Sigma',
\]

(6)

since \( f_k(0, \omega) = 0 \) holds. Here, \( 0 \) represents the null operator. We have assumed that the system is in an element of the set of the states \( U(\alpha, \beta) \). But, the conclusion is independent of the possible values of \( \alpha \) and of \( \beta \). Hence, there exist several quantum events for which the LHV model preserves noncommutativity of quantum observables. Of course, no element of the set of the states \( U(\alpha, \beta) \) says any violation of the Bell inequalities. This fact gives rise to the conflict against Malley’s claim.

The experimental violations of the Bell inequalities demonstrate indeed that quantum observables do not commute (Theorem 7 in Ref. [10]). But, such violations show also the nonexistence of the classical-random-variables model (i.e., the LHV model) for quantum events. Therefore, one can see that such violations show also the nonexistence of the HV model proposed in Refs. [8, 9] (except for any nonlocal HV model even if the model proposed in Refs. [8, 9] could be applicable not only to the LHV model but also to some nonlocal HV model.)

III. SUMMARY AND DISCUSSION

We have pointed out a contradiction. That is, the argumentation presented in Refs. [8, 9] cannot coexist with the argumentation claimed by Malley. And we have given a counterexample against Malley’s claim. Namely, the LHV model for quantum events exists. And it preserves noncommutativity of quantum observables.

From these arguments mentioned above, one can see that Malley’s theorem is indeed true under special assumptions. In more detail, the product rule (the KS condition) and the uniqueness feature of Gleason’s theorem imply that all quantum observables commute simultaneously [4]. In this sense, Malley’s theorem is valid for the KS type of HV model for quantum events. It was introduced by Kochen and Specker. However, Malley’s theorem is not related to the LHV model for quantum events, in general. It was reported by Bell in 1964.

At the end of Malley’s paper, it was discussed about hybrid HV models. And Malley stated that “violations of the Bell inequalities do not constitute a failure of Bell locality and our no-go commutativity result does not extend to a negation of Bell locality”. However, the author thinks that the experimental violations of the Bell inequalities indeed constitute the failure of Bell’s locality discussed in 1964. Malley has explicitly written that the experimental violations of the Bell inequalities demonstrate only that quantum observables do not commute. Hence, it seems that Malley considered that the no-go commutativity result can extend to the negation of original Bell’s locality reported in 1964. But this is not true, because, there exists the explicit LHV model which is compatible with noncommutative observables as we have shown.

It would be worth mentioning that the conclusion discussed in this comment coexists with the explicit difference between the KS theorem and Bell’s theorem in the Hilbert space formalism of quantum theory [13, 14]. This approach is valid only when we are
assumed to be given an arbitrary single state. But, this kind of approach can be seen often in literature [1,2,3,4,5,6,7,8,9,10,11,12,15].

The author suspects that what Theorem 7 in Ref. [10] said is as follows: Quantum observables must commute simultaneously if and only if we introduce a condition. The condition is that a set of quantum observables in the Hilbert space formalism is isomorphic to a set of classical random variables which are defined on a common space. Such a set of classical random variables obeys the classical (commutative) algebraic structure. A similar approach has been seen in von Neumann’s no-hidden-variables theorem [15]. Please notice that the outcome of the set of von Neumann’s assumptions directly tells that the set of all quantum observables cannot be isomorphic to any set obeying the classical algebraic structure. In this mathematical sense, the fact that quantum observables, in general, do not commute (i.e., cannot be isomorphic to any set obeying the classical algebraic structure) is equivalent to von Neumann’s no-hidden-variables theorem. This fact agrees with Malley’s achievements. Finally, we mention that any nonlocal HV model was not taken into account.

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