Bosonisation and Duality Symmetry in the Soldering Formalism

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Abstract

We develop a technique that solders the dual aspects of some symmetry. Using this technique it is possible to combine two theories with such symmetries to yield a new effective theory. Some applications in two and three dimensional bosonisation are discussed. In particular, it is shown that two apparently independent three dimensional massive Thirring models with same coupling but opposite mass signatures, in the long wavelength limit, combine by the process of bosonisation and soldering to yield an effective massive Maxwell theory. Similar features also hold for quantum electrodynamics in three dimensions. We also provide a systematic derivation of duality symmetric actions and show that the soldering mechanism leads to a master action which is duality invariant under a bigger set of symmetries than is usually envisaged. The concept of duality swapping is introduced and its implications are analysed. The example of electromagnetic duality is discussed in details.

Keywords: Bosonisation; Duality symmetry; Soldering
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1 Introduction

This paper is devoted to analyse certain features and applications of the soldering mechanism which is a new technique to work with dual manifestations of some symmetry. This technique, which is independent of dimensional considerations, essentially comprises in lifting the gauging of a global symmetry to its local version and exploits certain concepts introduced in a different context by Stone[1] and us[2, 3]. The analysis is intrinsically quantal without having any classical analogue. This is easily explained by the observation that a simple addition of two independent classical Lagrangeans is a trivial operation without leading to anything meaningful or significant.

A particularly interesting application of soldering is in the context of bosonisation. Although bosonisation was initially developed and fully explored in the context of two dimensions[4], more recently it has been extended to higher dimensions[5, 6, 7, 8]. The importance of bosonisation lies in the fact that it includes quantum effects already at the classical level. Consequently, different aspects and manifestations of quantum phenomena may be investigated directly, that would otherwise be highly nontrivial in the fermionic language. Examples of such applications are the computation of the current algebra[7] and the study of screening or confinement in gauge theories[9]. An unexplored aspect of bosonisation is revealed in the following question: given two independent fermionic models which can be bosonised separately, under what circumstances is it possible to represent them by one single effective theory? The answer lies in the symmetries of the problem. Two distinct models displaying dual aspects of some symmetry can be combined by the simultaneous implementation of bosonisation and soldering to yield a completely new theory.

The basic notions and ideas are first introduced in section 2 in the context of two dimensions where bosonisation is known to yield exact results. The starting point is to take two distinct chiral Lagrangeans with opposite chiralities. Using their bosonised expressions, the soldering mechanism fuses, in a precise way, the left and right chiralities. This leads to a general Lagrangean in which the chiral symmetry no longer exists, but it contains two extra parameters manifesting the bosonisation ambiguities. It is shown that different parametrisations lead either to the gauge invariant Schwinger model or the Thirring model.

Whereas the two dimensional analysis lays the foundations, the subsequent three dimensional discussion (section 3) illuminates the full power and utility of the present approach for yielding new equivalences through bosonisation. While the bosonisation in these dimensions is not exact, nevertheless, for massive fermionic models in the
large mass or, equivalently, the long wavelength limit, well defined local expressions are known to exist. Interestingly, these expressions exhibit a self or an anti self dual symmetry that is dictated by the signature of the fermion mass, thereby providing a novel testing ground for our ideas. Indeed, two distinct massive Thirring models with opposite mass signatures are soldered to yield a massive Maxwell theory. This result is vindicated by a direct comparison of the current correlation functions obtained before and after the soldering process. As another instructive application, the fusion of two models describing quantum electrodynamics in three dimensions is considered. Results similar to the corresponding analysis for the massive Thirring models are obtained.

In section 4 we present a systematic derivation of electromagnetic duality symmetric actions by converting the Maxwell action from a second order to a first order form followed by a suitable relabelling of variables which naturally introduces an internal index. It is crucial to note that there are two distinct classes of relabelling characterised by the opposite signatures of the determinant of the $2 \times 2$ orthogonal matrix defined in the internal space. Correspondingly, in this space there are two actions that are manifestly duality symmetric. Interestingly, their equations of motion are just the self and anti-self dual solutions, where the dual field is defined in the internal space. It is also found that in all cases there is one (conventional duality) symmetry transformation which preserves the invariance of these actions but there is another transformation which swaps the actions. We refer to this property as swapping duality. This indicates the possibility, in any dimensions, of combining the two actions to a master action that would contain all the duality symmetries. Indeed this construction is explicitly done by exploiting the ideas of soldering. The soldered master action also has manifest Lorentz or general coordinate invariance. The generators of the symmetry transformations are also obtained. An appendix is included to show briefly how these ideas can be carried over to two dimensions.

Section 5 contains our concluding remarks and observations.

## 2 Bosonisation and soldering in two dimensions

In this section we develop the ideas in the context of two dimensions. Consider, in particular, the following Lagrangeans with opposite chiralities,

$$\mathcal{L}_+ = \bar{\psi}(i\partial + eA)\psi$$
\[ L_- = \bar{\psi} (i \slashed{\partial} + eA^-) \psi \]  

(1)  

where \( P_\pm \) are the projection operators,  

\[ P_\pm = \frac{1 \pm \gamma_5}{2} \]  

(2)  

It is well known that the computation of the fermion determinant, which effectively yields the bosonised expressions, is plagued by regularisation ambiguities since chiral gauge symmetry cannot be preserved\([10]\). Indeed an explicit one loop calculation following Schwinger’s point splitting method\([11]\) yields the following bosonised expressions for the respective actions,  

\[ W_+[\varphi] = \int d^2 x \bar{\psi} (i \slashed{\partial} + eA_+) \psi = \frac{1}{4\pi} \int d^2 x \left( \partial_+ \varphi \partial_- \varphi + 2 e A_+ \partial_- \varphi + a e^2 A_+ A_- \right) \]  

\[ W_-[\rho] = \int d^2 x \bar{\psi} (i \slashed{\partial} + eA_-) \psi = \frac{1}{4\pi} \int d^2 x \left( \partial_+ \rho \partial_- \rho + 2 e A_- \partial_+ \rho + b e^2 A_- A_- \right) \]  

(3)  

where the light cone metric has been invoked for convenience,  

\[ A_\pm = \frac{1}{\sqrt{2}} (A_0 \pm A_1) = A^\mp \; ; \; \partial_\pm = \frac{1}{\sqrt{2}} (\partial_0 \pm \partial_1) = \partial^\pm \]  

(4)  

Note that the regularisation or bosonisation ambiguity is manifested through the arbitrary parameters \( a \) and \( b \). The latter ambiguity, in particular, is clearly understood since by using the normal bosonisation dictionary \( \bar{\psi} i \slashed{\partial} \psi \rightarrow \partial_+ \varphi \partial_- \varphi \) and \( \bar{\psi} \gamma_\mu \psi \rightarrow \frac{1}{\sqrt{2}} \epsilon_{\mu\nu} \partial^\nu \varphi \) (which holds only for a gauge invariant theory), the above expressions with \( a = b = 0 \) are easily reproduced from (1).  

It is crucial to observe that different scalar fields \( \varphi \) and \( \rho \) have been used in the bosonised forms to emphasize the fact that the fermionic fields occurring in the chiral components are uncorrelated. It is the soldering process which will abstract a meaningful combination of these components\([3]\). This process essentially consists in lifting the gauging of a global symmetry to its local version. Consider, therefore, the gauging of the following global symmetry,  

\[ \delta \varphi = \delta \rho = \alpha \]  

\[ \delta A_\pm = 0 \]  

(5)  

The variations in the effective actions \([3]\) are found to be,
\[
\delta W_+[\varphi] = \int d^2x \partial_- \alpha \ J_+ (\varphi) \\
\delta W_-[\rho] = \int d^2x \partial_+ \alpha \ J_- (\rho)
\]

(6)

where the currents are defined as,

\[
J_\pm (\eta) = \frac{1}{2\pi} (\partial_\pm \eta + e A_\pm) ; \ \eta = \varphi, \rho
\]

(7)

The important step now is to introduce the soldering field \( B_\pm \) coupled with the currents so that,

\[
W_\pm^{(1)} [\eta] = W_\pm [\eta] - \int d^2x B_\mp J_\pm (\eta)
\]

(8)

Then it is possible to define a modified action,

\[
W[\varphi, \rho] = W_+^{(1)} [\varphi] + W_-^{(1)} [\rho] + \frac{1}{2\pi} \int d^2x B_+ B_-
\]

(9)

which is invariant under an extended set of transformations that includes (5) together with,

\[
\delta B_\pm = \partial_\pm \alpha
\]

(10)

Observe that the soldering field transforms exactly as a potential. It has served its purpose of fusing the two chiral components. Since it is an auxiliary field, it can be eliminated from the invariant action (9) by using the equations of motion. This will naturally solder the otherwise independent chiral components and justifies its name as a soldering field. The relevant solution is found to be,

\[
B_\pm = 2\pi J_\pm
\]

(11)

Inserting this solution in (9), we obtain,

\[
W[\Phi] = \frac{1}{4\pi} \int d^2x \left\{ \left( \partial_+ \Phi \partial_- \Phi + 2 e A_+ \partial_- \Phi - 2 e A_- \partial_+ \Phi \right) + \left( a + b - 2 \right) e^2 A_+ A_- \right\}
\]

(12)

where,
\[ \Phi = \varphi - \rho \] (13)

As announced, the action is no longer expressed in terms of the different scalars \( \varphi \) and \( \rho \), but only on their difference. This difference is gauge invariant.

Let us digress on the significance of the findings. At the classical fermionic version, the chiral Lagrangeans are completely independent. Bosonising them includes quantum effects, but still there is no correlation. The soldering mechanism exploits the symmetries of the independent actions to precisely combine them to yield a single action. Note that the soldering works with the bosonised expressions. Thus the soldered action obtained in this fashion corresponds to the quantum theory.

We now show that different choices for the parameters \( a \) and \( b \) lead to well known models. To do this consider the variation of (12) under the conventional gauge transformations, \( \delta \varphi = \delta \rho = \alpha \) and \( \delta A_\pm = \partial_\pm \alpha \). It is easy to see that the expression in parenthesis is gauge invariant. Consequently a gauge invariant structure for \( W \) is obtained provided,

\[ a + b - 2 = 0 \] (14)

The effect of soldering, therefore, has been to induce a lift of the initial global symmetry (5) to its local form. By functionally integrating out the \( \Phi \) field from (12), we obtain,

\[ W[A_+, A_-] = -\frac{e^2}{4\pi} \int d^2x \left\{ A_+ \frac{\partial}{\partial A_+} A_+ + A_- \frac{\partial}{\partial A_-} A_- - 2A_+A_- \right\} \] (15)

which is the familiar structure for the gauge invariant action expressed in terms of the potentials. The opposite chiralities of the two independent fermionic theories have been soldered to yield a gauge invariant action.

Some interesting observations are possible concerning the regularisation ambiguity manifested by the parameters \( a \) and \( b \). As shown by us \([13]\), it is possible to uniquely fix these parameters by demanding Bose symmetry \([12]\). In the present case, this symmetry corresponds to the left-right (or \( \pm \)) symmetry in (3), thereby requiring \( a = b \). Together with the condition (14) this implies \( a = b = 1 \). This parametrisation has important consequences if a Maxwell term was included from the beginning to impart dynamics. Then the soldering takes place among two chiral Schwinger models\([13]\) having opposite chiralities to reproduce the usual Schwinger model\([14]\).

Naively it may appear that the soldering of the left and right chiralities to obtain a gauge invariant result is a simple issue since adding the classical Lagrangeans
and \( \bar{\psi} D_\perp \psi \), with identical fermion species, just yields the usual vector Lagrangean \( \bar{\psi} D_\perp \psi \). The quantum considerations are, however, much involved. The chiral determinants, as they occur, cannot be even defined since the kernels map from one chirality to the other so that there is no well defined eigenvalue problem\[13, 12\]. This is circumvented by working with \( \bar{\psi} (i \partial + e A_\perp) \psi \), that satisfy an eigenvalue equation, from which their determinants may be computed. But now a simple addition of the classical Lagrangeans does not reproduce the expected gauge invariant form. At this juncture, the soldering process becomes important. It systematically combined the quantised (bosonised) expressions for the opposite chiral components. Note that different fermionic species were considered so that this soldering does not have any classical analogue, and is strictly a quantum phenomenon. This will become more transparent when the three dimensional case is discussed.

Next, we show how a different choice for the parameters \( a \) and \( b \) in (12) leads to the Thirring model. Indeed it is precisely when the mass term exists (i.e., \( a + b - 2 \neq 0 \)), that (12) represents the Thirring model. Consequently, this parametrisation complements that used previously to obtain the vector gauge invariant structure. It is now easy to see that the term in parentheses in (12) corresponds to \( \bar{\psi} (i \partial + e A_\perp) \psi \) so that integrating out the nondynamical \( A_\mu \) field yields,

\[
\mathcal{L} = \bar{\psi} i \partial \psi - \frac{g}{2} (\bar{\psi} \gamma_\mu \psi)^2 ; \quad g = \frac{4 \pi}{a + b - 2}
\]  

which is just the Lagrangean for the usual Thirring model. It is known \[16\] that this model is meaningful provided the coupling parameter satisfies the condition \( g > -\pi \), so that,

\[
| a + b | > 2
\]  

This condition is the analogue of (14) found earlier. As usual, there is a one parameter arbitrariness. Imposing Bose symmetry implies that both \( a \) and \( b \) are equal and lie in the range

\[
1 < | a | = | b |
\]

This may be compared with the previous case where \( a = b = 1 \) was necessary for getting the gauge invariant structure. Interestingly, the positive range for the parameters in (13) just commences from this value.

Having developed and exploited the concepts of soldering in two dimensions, it is natural to investigate their consequences in three dimensions. The discerning reader may have noticed that it is essential to have dual aspects of a symmetry that can be
soldered to yield new information. In the two dimensional case, this was the left and right chirality. Interestingly, in three dimensions also, we have a similar phenomenon.

3 Bosonisation and Soldering in three dimensions

This section is devoted to an analysis of the soldering process in the massive Thirring model and quantum electrodynamics in three dimensions. We first show that two apparently independent massive Thirring models in the long wavelength limit combine, at the quantum level, into a massive Maxwell theory. This is further vindicated by a direct comparison of the current correlation functions following from the bosonisation identities. These findings are also extended to include three dimensional quantum electrodynamics. The new results and interpretations illuminate a close parallel with the two dimensional discussion.

3.1 The massive Thirring model

In order to effect the soldering, the first step is to consider the bosonisation of the massive Thirring model in three dimensions[6, 7]. This is therefore reviewed briefly. The relevant current correlator generating functional, in the Minkowski metric, is given by,

$$Z[\kappa] = \int D\psi D\bar{\psi} \exp \left( i \int d^3x \left[ \bar{\psi} \left( i \gamma + m \right) \psi - \frac{\lambda^2}{2} j_\mu j^\mu + \lambda j_\mu \kappa^\mu \right] \right)$$  \hspace{1cm} (19)$$

where $j_\mu = \bar{\psi} \gamma_\mu \psi$ is the fermionic current. As usual, the four fermion interaction can be eliminated by introducing an auxiliary field,

$$Z[\kappa] = \int D\psi D\bar{\psi} Df_\mu \exp \left( i \int d^3x \left[ \bar{\psi} \left( i \gamma + m + \lambda (f + \bar{f}) \right) \psi + \frac{1}{2} f_\mu f^\mu \right] \right)$$  \hspace{1cm} (20)$$

Contrary to the two dimensional models, the fermion integration cannot be done exactly. Under certain limiting conditions, however, this integration is possible leading to closed expressions. A particularly effective choice is the large mass limit in which case the fermion determinant yields a local form. Incidentally, any other value of the
mass leads to a nonlocal structure \[3\]. The large mass limit is therefore very special. The leading term in this limit was calculated by various means \[17\] and shown to yield the Chern-Simons three form. Thus the generating functional for the massive Thirring model in the large mass limit is given by,

\[
Z[\kappa] = \int Df_\mu \exp \left( i \int d^3 x \left( \frac{\lambda^2}{8\pi} \frac{m}{|m|} \epsilon^{\mu\nu\lambda} f^\mu \partial^\nu f^\lambda + \frac{1}{2} f_\mu f^\mu + \frac{\lambda^2}{4\pi} \frac{m}{|m|} \epsilon_{\mu\nu\sigma} \kappa^{\mu} \partial^\nu f^\sigma \right) \right)
\]

(21)

where the signature of the topological terms is dictated by the corresponding signature of the fermionic mass term. In obtaining the above result a local counter term has been ignored. Such terms manifest the ambiguity in defining the time ordered product to compute the correlation functions\[18\]. The Lagrangean in the above partition function defines a self dual model introduced earlier \[19\]. The massive Thirring model, in the relevant limit, therefore bosonises to a self dual model. It is useful to clarify the meaning of this self duality. The equation of motion in the absence of sources is given by,

\[
f_\mu = -\frac{\lambda^2}{4\pi} \frac{m}{|m|} \epsilon^{\mu\nu\lambda} \partial^\nu f^\lambda
\]

(22)

from which the following relations may be easily verified,

\[
\partial_\mu f^\mu = 0 \quad (\Box + M^2) f_\mu = 0 \quad ; \quad M = \frac{4\pi}{\lambda^2}
\]

(23)

A field dual to \( f_\mu \) is defined as,

\[
\tilde{f}_\mu = \frac{1}{M} \epsilon_{\mu\nu\lambda} \partial^\nu f^\lambda
\]

(24)

where the mass parameter \( M \) is inserted for dimensional reasons. Repeating the dual operation, we find,

\[
\left( \tilde{f}_\mu \right) = \frac{1}{M} \epsilon_{\mu\lambda\nu} \partial^\nu \tilde{f}^\lambda = f_\mu
\]

(25)

obtained by exploiting \(23\), thereby validating the definition of the dual field. Combining these results with \(22\), we conclude that,

\[
f_\mu = -\frac{m}{|m|} \tilde{f}_\mu
\]

(26)
Hence, depending on the sign of the fermion mass term, the bosonic theory corresponds to a self-dual or an anti self-dual model. Likewise, the Thirring current bosonises to the topological current

$$j_\mu = \frac{\lambda}{4\pi} \frac{m}{m^2} \epsilon_{\mu\nu\rho} \partial^{\nu} f^\rho$$

(27)

The close connection with the two dimensional analysis is now evident. There the starting point was to consider two distinct fermionic theories with opposite chiralities. In the present instance, the analogous thing is to take two independent Thirring models with identical coupling strengths but opposite mass signatures,

$$\mathcal{L}_+ = \bar{\psi} (i\partial + m) \psi - \frac{\lambda^2}{2} \left( \bar{\psi} \gamma_\mu \psi \right)^2$$

$$\mathcal{L}_- = \bar{\xi} (i\partial - m') \xi - \frac{\lambda^2}{2} \left( \bar{\xi} \gamma_\mu \xi \right)^2$$

(28)

Then the bosonised Lagrangeans are, respectively,

$$\mathcal{L}_+ = \frac{1}{2M} \epsilon_{\mu\nu\lambda} f^{\mu} \partial^{\nu} f^{\lambda} + \frac{1}{2} f^\mu f^\mu$$

$$\mathcal{L}_- = -\frac{1}{2M} \epsilon_{\mu\nu\lambda} g^{\mu} \partial^{\nu} g^{\lambda} + \frac{1}{2} g^\mu g^\mu$$

(29)

where $f_\mu$ and $g_\mu$ are the distinct bosonic vector fields. The current bosonization formulae in the two cases are given by

$$j_+^\mu = \bar{\psi} \gamma_\mu \psi = \frac{\lambda}{4\pi} \epsilon_{\mu\nu\rho} \partial^{\nu} f^\rho$$

$$j_-^\mu = \bar{\xi} \gamma_\mu \xi = -\frac{\lambda}{4\pi} \epsilon_{\mu\nu\rho} \partial^{\nu} g^\rho$$

(30)

The stage is now set for soldering. Taking a cue from the two dimensional analysis, let us consider the gauging of the following symmetry,

$$\delta f_\mu = \delta g_\mu = \epsilon_{\mu\rho\sigma} \partial^{\rho} \alpha^\sigma$$

(31)

Under such transformations, the bosonised Lagrangeans change as,

$$\delta \mathcal{L}_\pm = J^\rho_\pm (h_\mu) \partial_\rho \alpha^\sigma ; \ h_\mu = f_\mu, \ g_\mu$$

(32)
where the antisymmetric currents are defined by,

\[ J_{\pm}^{\rho\sigma}(h_{\mu}) = \epsilon^{\mu\rho\sigma} h_{\mu} \pm \frac{1}{M} \epsilon^{\gamma\rho\sigma} \epsilon_{\mu\nu\gamma} \partial^{\mu} h_{\nu} \]  

(33)

It is worthwhile to mention that any other variation of the fields (like \( \delta f_{\mu} = \alpha_{\mu} \)) is inappropriate because changes in the two terms of the Lagrangeans cannot be combined to give a single structure like (33). We now introduce the soldering field coupled with the antisymmetric currents. In the two dimensional case this was a vector. Its natural extension now is the antisymmetric second rank Kalb-Ramond tensor field \( B_{\rho\sigma} \), transforming in the usual way,

\[ \delta B_{\rho\sigma} = \partial_{\rho} \alpha_{\sigma} - \partial_{\sigma} \alpha_{\rho} \]  

(34)

Then it is easy to see that the modified Lagrangeans,

\[ \mathcal{L}^{(1)}_{\pm} = \mathcal{L}_{\pm} - \frac{1}{2} J_{\pm}^{\rho\sigma}(h_{\mu}) B_{\rho\sigma} \]  

(35)

transform as,

\[ \delta \mathcal{L}^{(1)}_{\pm} = -\frac{1}{2} \delta J_{\pm}^{\rho\sigma} B_{\rho\sigma} \]  

(36)

The final modification consists in adding a term to ensure gauge invariance of the soldered Lagrangean. This is achieved by,

\[ \mathcal{L}^{(2)}_{\pm} = \mathcal{L}^{(1)}_{\pm} + \frac{1}{4} B^{\rho\sigma} B_{\rho\sigma} \]  

(37)

A straightforward algebra shows that the following combination,

\[ \mathcal{L}_{S} = \mathcal{L}^{(2)}_{+} + \mathcal{L}^{(2)}_{-} \]

\[ = \mathcal{L}_{+} + \mathcal{L}_{-} - \frac{1}{2} B^{\rho\sigma} \left( J_{\rho\sigma}^{+}(f) + J_{\rho\sigma}^{-}(g) \right) + \frac{1}{2} B^{\rho\sigma} B_{\rho\sigma} \]  

(38)

is invariant under the gauge transformations (34) and (34). The gauging of the symmetry is therefore complete. To return to a description in terms of the original variables, the auxiliary soldering field is eliminated from (38) by using the equation of motion,

\[ B_{\rho\sigma} = \frac{1}{2} \left( J_{\rho\sigma}^{+}(f) + J_{\rho\sigma}^{-}(g) \right) \]  

(39)
Inserting this solution in (38), the final soldered Lagrangean is expressed solely in terms of the currents involving the original fields,

\[
\mathcal{L}_S = \mathcal{L}_+ + \mathcal{L}_- - \frac{1}{8} \left( J^+_{\rho\sigma}(f) + J^-_{\rho\sigma}(g) \right) \left( J^{+\rho\sigma}(f) + J^{-\rho\sigma}(g) \right)
\]

(40)

It is now crucial to note that, by using the explicit structures for the currents, the above Lagrangean is no longer a function of \( f_\mu \) and \( g_\mu \) separately, but only on the combination,

\[
A_\mu = \frac{1}{\sqrt{2}M} (g_\mu - f_\mu)
\]

(41)

with,

\[
\mathcal{L}_S = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{M^2}{2} A_\mu A^\mu
\]

(42)

where,

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu
\]

(43)

is the usual field tensor expressed in terms of the basic entity \( A_\mu \). Our goal has been achieved. The soldering mechanism has precisely fused the self and anti self dual symmetries to yield a massive Maxwell theory which, naturally, lacks this symmetry.

It is now instructive to understand this result by comparing the current correlation functions. The Thirring currents in the two models bosonise to the topological currents (30) in the dual formulation. From a knowledge of the field correlators in the latter case, it is therefore possible to obtain the Thirring current correlators. The field correlators are obtained from the inverse of the kernels occurring in (29),

\[
\langle f_\mu (+k) f_\nu (-k) \rangle = \frac{M^2}{M^2 - k^2} \left( ig_{\mu\nu} + \frac{1}{M} \epsilon_{\mu\nu\rho} k^\rho - \frac{i}{M^2} k_\mu k_\nu \right)
\]

\[
\langle g_\mu (+k) g_\nu (-k) \rangle = \frac{M^2}{M^2 - k^2} \left( ig_{\mu\nu} - \frac{1}{M} \epsilon_{\mu\nu\rho} k^\rho - \frac{i}{M^2} k_\mu k_\nu \right)
\]

(44)

where the expressions are given in the momentum space. Using these in (30), the current correlators are obtained,

\[
\langle j^\mu_\mu (+k) j^\nu_\nu (-k) \rangle = \frac{M}{4\pi (M^2 - k^2)} \left( ik^2 g_{\mu\nu} - ik_\mu k_\nu + \frac{1}{M} \epsilon_{\mu\nu\rho} k^\rho k^2 \right)
\]

\[
\langle j^\mu_\mu (+k) j^\nu_\nu (-k) \rangle = \frac{M}{4\pi (M^2 - k^2)} \left( ik^2 g_{\mu\nu} - ik_\mu k_\nu - \frac{1}{M} \epsilon_{\mu\nu\rho} k^\rho k^2 \right)
\]

(45)
It is now feasible to construct a total current,

\[ j_\mu = j_\mu^+ + j_\mu^- = \frac{\lambda}{4\pi} \epsilon_{\mu\nu\rho}\partial^\nu \left( f^\rho - g^\rho \right) \quad (46) \]

Then the correlation function for this current, in the original self dual formulation, follows from (45) and noting that \( \langle j_\mu^+ j_\nu^- \rangle = 0 \), which is a consequence of the independence of \( f_\mu \) and \( g_\nu \):

\[ \langle j_\mu(+) j_\nu(-) \rangle = \langle j_\mu^+ j_\nu^- \rangle + \langle j_\mu^- j_\nu^+ \rangle = \frac{iM}{2\pi(M^2 - k^2)} \left( k^2 g_{\mu\nu} - k_\mu k_\nu \right) \quad (47) \]

The above equation is easily reproduced from the effective theory. Using (41), it is observed that the bosonization of the composite current (46) is defined in terms of the massive vector field \( A_\mu \),

\[ j_\mu = \bar{\psi} \gamma_\mu \psi + \bar{\xi} \gamma_\mu \xi = -\sqrt{M^2} \epsilon_{\mu\nu\rho} \partial^\nu A^\rho \quad (48) \]

The current correlator is now obtained from the field correlator \( \langle A_\mu A_\nu \rangle \) given by the inverse of the kernel appearing in (42),

\[ \langle A_\mu(+) A_\nu(-) \rangle = \frac{i}{M^2 - k^2} \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{M^2} \right) \quad (49) \]

From (48) and (49) the two point function (47) is reproduced, including the normalization.

We conclude, therefore, that the process of bosonisation and soldering of two massive Thirring models with opposite mass signatures, in the long wavelength limit, yields a massive Maxwell theory. The bosonization of the composite current, obtained by adding the separate contributions from the two models, is given in terms of a topological current (48) of the massive vector theory. These are completely new results which cannot be obtained by a straightforward application of conventional bosonisation techniques. The massive modes in the original Thirring models are manifested in the two modes of (42) so that there is a proper matching in the degrees of freedom. Once again it is reminded that the fermion fields for the models are different so that the analysis has no classical analogue. Indeed if one considered the same fermion species, then a simple addition of the classical Lagrangeans would lead to a Thirring model with a mass given by \( m - m' \). In particular, this difference can be
zero. The bosonised version of such a massless model is known \cite{3, 8} to yield a highly nonlocal theory which has no connection with (42). Classically, therefore, there is no possibility of even understanding, much less, reproducing the effective quantum result. In this sense the application in three dimensions is more dramatic than the corresponding case of two dimensions.

### 3.2 Quantum electrodynamics

An interesting theory in which the preceding ideas may be implemented is quantum electrodynamics, whose current correlator generating functional in an arbitrary covariant gauge is given by,

\[ Z[\kappa] = \int D\bar{\psi}D\psi DA_{\mu} \exp \left\{ i \int d^3x \left( \bar{\psi} (i\partial + m + eA) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\eta}{2} (\partial_{\mu} A^\mu)^2 + e j_{\mu} \kappa^\mu \right) \right\} \]

where \( \eta \) is the gauge fixing parameter and \( j_{\mu} = \bar{\psi} \gamma_{\mu} \psi \) is the current. As before, a one loop computation of the fermionic determinant in the large mass limit yields,

\[ Z[\kappa] = \int DA_{\mu} \exp \left\{ i \int d^3x \left[ \frac{e^2}{8\pi} \frac{m}{m} |\epsilon_{\mu\nu\lambda}| \epsilon_{\mu\nu\lambda} A^\mu \partial^\nu A^\lambda - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right. \right. \]

\[ + \left. \left. \frac{e^2}{4\pi} \frac{m}{m} \epsilon_{\mu\nu\rho} \kappa^{\mu} \partial^\nu A^\rho + \frac{\eta}{2} (\partial_{\mu} A^\mu)^2 \right] \right\} \]

In the absence of sources, this just corresponds to the topologically massive Maxwell-Chern-Simons theory, with the signature of the topological term determined from that of the fermion mass term. The equation of motion,

\[ \partial^\nu F_{\nu\mu} + \frac{e^2}{4\pi} \frac{m}{m} |\epsilon_{\mu\nu\lambda}| \epsilon_{\mu\nu\lambda} \partial^\nu A^\lambda = 0 \]

expressed in terms of the dual tensor,

\[ F_{\mu} = \epsilon_{\mu\nu\lambda} \partial^\nu A^\lambda \]

reveals the self (or anti self) dual property,

\[ F_{\mu} = \frac{4\pi}{e^2} \frac{m}{m} \epsilon_{\mu\nu\lambda} \partial^\nu F^\lambda \]
which is the analogue of \(22\). In this fashion the Maxwell-Chern-Simons theory manifests the well known \([20, 18, 21]\) mapping with the self dual models considered in the previous subsection. The difference is that the self duality in the former, in contrast to the latter, is contained in the dual field \(53\) rather than in the basic field defining the theory. This requires some modifications in the ensuing analysis. Furthermore, the bosonization of the fermionic current is now given by the topological current in the Maxwell-Chern-Simons theory,

\[
j_\mu = \frac{e}{4\pi} \frac{m}{m} \epsilon_{\mu\nu\lambda} \partial^\nu A^\lambda
\]  

(55)

Consider, therefore, two independent models describing quantum electrodynamics with opposite signatures in the mass terms,

\[
\mathcal{L}_+ = \bar{\psi} (i \partial - m + eA) \psi - \frac{1}{4} F^\mu_\nu (A) F^\nu_\mu (A)
\]

\[
\mathcal{L}_- = \bar{\xi} (i \partial - m' + eB) \xi - \frac{1}{4} F^\mu_\nu (B) F^\nu_\mu (B)
\]  

(56)

whose bosonised versions in an appropriate limit are given by,

\[
\mathcal{L}_+ = -\frac{1}{4} F^\mu_\nu (A) + M^2 \epsilon_{\mu\nu\lambda} \partial^\nu A^\lambda ; \quad M = \frac{e^2}{4\pi}
\]

\[
\mathcal{L}_- = -\frac{1}{4} F^\mu_\nu (B) - \frac{1}{2} e_{\mu\nu\lambda} B^\mu \partial^\nu B^\lambda
\]  

(57)

where \(A_\mu\) and \(B_\mu\) are the corresponding potentials. Likewise, the corresponding expressions for the bosonized currents are found from the general structure \(53\),

\[
j^+_\mu = \bar{\psi} \gamma_\mu \psi = \frac{M}{e} \epsilon_{\mu\nu\lambda} \partial^\nu A^\lambda
\]

\[
j^-_\mu = \bar{\xi} \gamma_\mu \xi = -\frac{M}{e} \epsilon_{\mu\nu\lambda} \partial^\nu B^\lambda
\]  

(58)

To proceed with the soldering of the above models, take the symmetry transformation,

\[
\delta A_\mu = \alpha_\mu
\]  

(59)

Such a transformation is spelled out by recalling \(31\) and the observation that now \(53\) simulates the \(f_\mu\) field in the previous case. Under this variation, the Lagrangeans \(57\) change as,

\[
\delta \mathcal{L}_\pm = J^\rho_\pm (P) \partial_\rho \alpha_\sigma ; \quad P = A, B
\]  

(60)
where the antisymmetric currents are defined by,

$$J_{\pm \sigma} (P) = \pm m \epsilon_{\sigma \mu} P_{\mu} - F_{\rho \sigma} (P) \quad (61)$$

Proceeding as before, it is now straightforward to deduce the final Lagrangean that will be gauge invariant. This is given by,

$$\mathcal{L}_S = \hat{\mathcal{L}}_+ + \hat{\mathcal{L}}_- \quad ; \delta \mathcal{L}_S = 0 \quad (62)$$

where the iterated pieces are,

$$\hat{\mathcal{L}}_{\pm} = \mathcal{L}_{\pm} - \frac{1}{2} J_{\pm \sigma} B_{\rho \sigma} - \frac{1}{4} B_{\rho \sigma} B^{\rho \sigma} \quad (63)$$

To obtain the effective soldered Lagrangean, the auxiliary $B_{\rho \sigma}$ field is eliminated by using the equation of motion and the final result is,

$$\mathcal{L}_S = -\frac{1}{4} F_{\mu \nu} (G) F^{\mu \nu} (G) + \frac{M^2}{2} G_{\mu} G^{\mu} \quad (64)$$

written in terms of a single field,

$$G_{\mu} = \frac{1}{\sqrt{2}} (A_{\mu} - B_{\mu}) \quad (65)$$

The Lagrangean (64) governs the dynamics of a massive Maxwell theory.

As before, we now discuss the implications for the current correlation functions. These functions in the original models describing electrodynamics can be obtained from the mapping (58) by exploiting the field correlators found by inverting the kernels occurring in (57). The results, in the momentum space, are

$$\langle j_\mu^+ (k) j_\nu^- (-k) \rangle = \frac{i (M/e)^2}{M^2 - k^2} \left[ k^2 g_{\mu \nu} - k_\mu k_\nu - i M \epsilon_{\mu \nu \rho} k^\rho \right]$$

$$\langle j_\mu^- (k) j_\nu^+ (-k) \rangle = \frac{i (M/e)^2}{M^2 - k^2} \left[ k^2 g_{\mu \nu} - k_\mu k_\nu + i M \epsilon_{\mu \nu \rho} k^\rho \right] \quad (66)$$

Next, defining a composite current,

$$j_\mu = j_\mu^+ + j_\mu^- = \frac{M}{e} \epsilon_{\mu \nu \lambda} \partial^\nu (A^\lambda - B^\lambda) \quad (67)$$
it is simple to obtain the relevant correlator by exploiting the results for \( j_\mu^+ \) and \( j_\mu^- \) from (66),

\[
\langle j_\mu^+(+k) j_\nu^-(+k) \rangle = 2i \left( \frac{M}{e} \right)^2 \frac{1}{M^2 - k^2} (k^2 g_{\mu\nu} - k_\mu k_\nu)
\]

In the bosonized version obtained from the soldering, (67) represents the mapping,

\[
j_\mu = \bar{\psi} \gamma_\mu \psi + \bar{\xi} \gamma_\mu \xi = \sqrt{2} M e \epsilon_{\mu\nu\lambda} \partial_\nu G_\lambda
\]

where \( G_\mu \) is the massive vector field (65) whose dynamics is governed by the Lagrangean (64). In this effective description the result (68) is reproduced from (69) by using the correlator of \( G_\mu \) obtained from (64), which is exactly identical to (49).

Thus the combined effects of bosonisation and soldering of two independent quantum electrodynamical models with appropriate mass signatures yield a massive Maxwell theory. In the self dual version the massive modes are the topological excitations in the Maxwell-Chern-Simons theories. These are combined into the two usual massive modes in the effective massive vector theory. A complete correspondence among the composite current correlation functions in the original models and in their dual bosonised description was also established. The comments made in the concluding part of the last subsection naturally apply also in this instance.

It is interesting to note from the above analysis that in the quadratic approximation in the large mass limit, the massive Thirring model is equivalent to quantum electrodynamics. Furthermore a direct comparison of the mass terms in the effective vector theory reveals the dual nature of this equivalence since the coupling in one case is related to the inverse coupling in the other. In the next section we analyse the consequences of soldering for electromagnetic duality in four dimensions.

4 Electromagnetic duality and soldering

In recent years the old idea [22, 23, 24] of electromagnetic duality has been revived with considerable attention and emphasis [25, 26, 27, 28, 29]. Different directions of research [23, 30, 31, 32] include an abstraction of manifestly covariant forms for duality symmetric actions or an explicit proof of the equivalence of such actions with the original nonduality symmetric actions. In spite of this spate of papers there does not seem to be a simple clear cut way of obtaining duality symmetric actions. We first
discuss such an approach introducing simultaneously the concept of duality swapping. This is followed by analysing the effects of soldering. To show the generality of the ideas the case of a scalar field in two dimensions is also analysed in an appendix.

Let us start with the usual Maxwell Lagrangean,

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \]  \hspace{1cm} (70)

which is expressed in terms of the electric and magnetic fields as

\[ \mathcal{L} = \frac{1}{2} (E^2 - B^2) \]  \hspace{1cm} (71)

where,

\[ E_i = -F_{0i} = -\partial_0 A_i + \partial_i A_0 \]

\[ B_i = \epsilon_{ijk} \partial_j A_k \]  \hspace{1cm} (72)

The following duality transformation,

\[ E \rightarrow \mp B; \quad B \rightarrow \pm E \]  \hspace{1cm} (73)

is known to preserve the invariance of the full set comprising Maxwell’s equations and the Bianchi identities although the Lagrangean changes its sign. To have a duality symmetric Lagrangean, the primary step is to recast (71) in a symmetrised first order form by introducing an auxiliary field,

\[ \mathcal{L} = \frac{1}{2} (P \dot{A} - \dot{P} \cdot A) - \frac{1}{2} P^2 - \frac{1}{2} B^2 + A_0 \nabla \cdot P \]  \hspace{1cm} (74)

A suitable change of variables is now invoked which naturally introduces an internal index. Significantly, there are two possibilities which translate from the old set \((P, A)\) to the new ones \((A_1, A_2)\). It is, however, important to note that the Maxwell theory has a constraint that is implemented by the Lagrange multiplier \(A_0\). The redefined variables are chosen such that this constraint is automatically satisfied,

\[ P \rightarrow B_2; \quad A \rightarrow A_1 \]

\[ P \rightarrow B_1; \quad A \rightarrow A_2 \]  \hspace{1cm} (75)

\[ ^3 \text{Bold face letters denote three vectors.} \]
It is now simple to show that, in terms of the redefined variables, the original Maxwell Lagrangean takes the form,

$$\mathcal{L}_\pm = \frac{1}{2} \left( \pm \dot{A}_\alpha \epsilon_{\alpha\beta} B_\beta - B_\alpha B_\alpha \right)$$

(76)

where $\epsilon_{\alpha\beta}$ is the second rank antisymmetric tensor defined in the internal space with $\epsilon_{12} = 1$. Adding a total derivative that would leave the equations of motion unchanged, this Lagrangean is expressed directly in terms of the electric and magnetic fields,

$$\mathcal{L}_\pm = \frac{1}{2} \left( \pm B_\alpha \epsilon_{\alpha\beta} E_\beta - B_\alpha B_\alpha \right)$$

(77)

Observe that only one of the above structures (namely, $\mathcal{L}_-$) was given earlier in [25]. The presence of two structures leads to some interesting consequences. Let us first introduce the proper and improper rotation matrices parametrised by the angles $\theta$ and $\varphi$, respectively,

$$R^+(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ - \sin \theta & \cos \theta \end{pmatrix}$$

(78)

$$R^-(\varphi) = \begin{pmatrix} \sin \varphi & \cos \varphi \\ \cos \varphi & - \sin \varphi \end{pmatrix}$$

(79)

Both the Lagrangeans are duality symmetric under the proper ($SO(2)$) transformations,

$$E_\alpha \rightarrow R^+_{\alpha\beta} E_\beta$$

$$B_\alpha \rightarrow R^+_{\alpha\beta} B_\beta$$

(80)

Interestingly, under the improper rotations,

$$E_\alpha \rightarrow R^-_{\alpha\beta} E_\beta$$

$$B_\alpha \rightarrow R^-_{\alpha\beta} B_\beta$$

(81)

the Lagrangean $\mathcal{L}_+$ goes over to $\mathcal{L}_-$ and vice versa. We refer to this property as swapping duality. Note that the discretised version of the above equation is obtained by setting $\varphi = 0$,

$$E_\alpha \rightarrow \sigma^\alpha_1 E_\beta$$

$$B_\alpha \rightarrow \sigma^\alpha_1 B_\beta$$

(82)
It is precisely the $\sigma_1$ matrix that reflects the proper into improper rotations,

$$R^+(\theta)\sigma_1 = R^-(\theta)$$  \hspace{1cm} (83)

which illuminates the reason behind the swapping of the Lagrangeans in this example.

The generators of the $SO(2)$ rotations are given by,

$$Q^{(\pm)} = \pm \frac{1}{2} \int d^3x \ A^\alpha \cdot B^\alpha$$  \hspace{1cm} (84)

so that,

$$A_\alpha \rightarrow A'_\alpha = e^{-iQ\theta} A_\alpha e^{iQ\theta}$$  \hspace{1cm} (85)

This can be easily verified by using the basic brackets following from the symplectic structure of the theory,

$$\left[ A^i_\alpha(x), \epsilon^{ijkl} \partial^k A^l_\beta(y) \right] = \pm i \delta^{ij} \epsilon_{\alpha\beta} \delta(x - y)$$  \hspace{1cm} (86)

It is useful to comment on the significance of the above analysis. Since the duality symmetric Lagrangeans have been obtained directly from the Maxwell Lagrangean, it is redundant to show the equivalence of the former expressions with the latter, which is an essential perquisite in other approaches. Furthermore, since classical equations of motion have not been used at any stage, the purported equivalence holds at the quantum level. The need for any explicit demonstration of this fact, which has been the motivation of several recent papers, becomes, in this analysis, superfluous. A related observation is that the usual way of showing the classical equivalence is to use the equations of motion to eliminate one component from (76), thereby leading to the Maxwell Lagrangean in the temporal $A_0 = 0$ gauge. This is not surprising since the change of variables leading from the second to the first order form solved the Gauss law thereby eliminating the multiplier. Finally, note that there are two distinct structures for the duality symmetric Lagrangeans. These must correspond to the opposite aspects of some symmetry, which is next unravelled. By looking at the equations of motion obtained from (76),

$$\dot{A}_\alpha = \pm \epsilon_{\alpha\beta} \nabla \times A_\beta$$  \hspace{1cm} (87)

it is possible to verify that these are just the self and anti-self dual solutions,

$$F^\alpha_{\mu\nu} = \pm \epsilon^{\alpha\beta\gamma} F^\beta_{\mu\nu} \quad ; \quad \ast F^\beta_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\lambda} F^\rho_{\beta}$$  \hspace{1cm} (88)
obtained by setting \( A_0^\alpha = 0 \). As shown in the appendix, in the two dimensional theory the equation of motion naturally assumes a covariant structure. Here, on the other hand, the introduction of \( A_0^\alpha \) is necessary since, as shown earlier, the term involving \( A_0 \) drops out in the construction of the duality invariant Lagrangean. This is because \( A_0 \) is a multiplier enforcing the Gauss constraint. Such a feature distinguishes a gauge theory from the non gauge theory discussed in the two dimensional example. Following our general strategy, the next task is to solder the two Lagrangeans (76).

Consider then the gauging of the following symmetry,

\[
\delta H_\alpha = h_\alpha \quad; \quad H = P, Q
\]  

where \( P \) and \( Q \) denote the basic fields in the Lagrangeans \( \mathcal{L}_+ \) and \( \mathcal{L}_- \), respectively. The Lagrangeans transform as,

\[
\delta \mathcal{L}_\pm = \epsilon_{\alpha\beta} \left( \nabla \times h_\alpha \right) J^\pm_\beta
\]  

with the currents defined by,

\[
J^\pm_\alpha (H) = \left( \mp \dot{H}_\alpha + \epsilon_{\alpha\beta} \nabla \times H_\beta \right)
\]  

Next, the soldering field \( W_\alpha \) is introduced which transforms as,

\[
\delta W_\alpha = -\epsilon_{\alpha\beta} \nabla \times h_\beta
\]  

Following standard steps as outlined previously, the final Lagrangean which is invariant under the complete set of transformations (89) and (92) is obtained,

\[
\mathcal{L} = \mathcal{L}_+(P) + \mathcal{L}_-(Q) - W^\alpha \cdot \left( J^+_\alpha(P) + J^-_\alpha(Q) \right) - W^2_\alpha
\]  

Eliminating the soldering field by using the equations of motion, the effective soldered Lagrangean following from (93) is derived,

\[
\mathcal{L} = \frac{1}{4} \left( \dot{G}_\alpha \cdot \dot{G}_\alpha - \nabla \times G_\alpha \cdot \nabla \times G_\alpha \right)
\]  

where the composite field is given by the combination,

\[
G_\alpha = P_\alpha - Q_\alpha
\]
which is invariant under (89). It is interesting to note that, reintroducing the \( G_0^\alpha \) variable, this is nothing but the Maxwell Lagrangean with a doublet of fields,

\[
\mathcal{L} = -\frac{1}{4} G_\mu^\alpha G_\mu^\alpha \quad ; \quad G_\mu^\alpha = \partial_\mu G^\alpha_\nu - \partial_\nu G^\alpha_\mu
\]  

(96)

It is now possible to show that by reducing (96) to a first order from, we exactly obtain the two types of the duality symmetric Lagrangeans (77). This shows the equivalence of the soldering and reduction (i.e. conversion of a second order Lagrangean to its first order form) processes.

In terms of the original \( P \) and \( Q \) fields a generalised Polyakov-Weigmann like identity \( \frac{3}{3} \) is obtained,

\[
\mathcal{L}(P - Q) = \mathcal{L}(P) + \mathcal{L}(Q) - 2W_{i,\alpha}^+(P)W_{i,\alpha}^-(Q)
\]

\[
W_{i,\alpha}^\pm(H) = \frac{1}{\sqrt{2}} \left( F_{0i}^\alpha(H) \pm \epsilon_{ijk} \epsilon_{\alpha\beta} F_{jk}^\beta(H) \right) \quad ; \quad H = P, Q
\]  

(97)

With respect to the gauge transformations (89), the above identity shows that a contact term is necessary to restore the gauge invariant action from two gauge variant forms. This, it may be recalled, is just the basic content of the Polyakov-Weigmann identity. It is interesting to note that the “mass” term appearing in the above identity is composed of parity preserving pieces \( W_{i,\alpha}^\pm \), thanks to the presence of the compensating \( \epsilon \)-factor from the internal space.

A particularly illuminating way of rewriting the Lagrangean (96) is,

\[
\mathcal{L} = -\frac{1}{8} \left( G_\mu^\alpha + \epsilon_{\alpha\beta\gamma} G_\mu^\beta \right) \left( G_\mu^\beta - \epsilon_{\alpha\beta\gamma} G_\mu^\gamma \right)
\]

\[
= -\frac{1}{8} \left( G_\mu^\alpha + \tilde{G}_\mu^\alpha \right) \left( G_\mu^\alpha - \tilde{G}_\mu^\alpha \right)
\]  

(98)

where, in the second line, the generalised Hodge dual (\( \tilde{G} \)) in the space containing the internal index has been defined in terms of the usual Hodge dual (\( *G \)) to explicitly show the soldering of the self and anti self dual solutions. The above Lagrangean manifestly displays the following duality symmetries,

\[
A_\mu^\alpha \rightarrow R_{\alpha\beta} A_\mu^\beta
\]  

(99)

where, without any loss of generality, we may denote the composite field, of which \( G_\mu^\alpha \) is a function, by \( A \). The generator of the \( SO(2) \) rotations is now given by,

\[
Q = \int dx \, \epsilon^{\alpha\beta} \Pi^\alpha \cdot A^\beta
\]  

(100)
Now observe that the master Lagrangean was obtained from the soldering of two distinct Lagrangeans \((\ref{eq:76})\). The latter were duality symmetric under the proper rotations while the improper ones effected a swapping. The soldered Lagrangean is therefore duality symmetric under the complete set of transformations \((\ref{eq:99})\) implying that it contains a bigger set of duality symmetries than \((\ref{eq:76})\). Significantly, it is also manifestly Lorentz invariant. Furthermore, recall that under the transformations mapping the field to its dual, the original Maxwell equations are invariant but the Lagrangean changes its signature. The corresponding transformation in the \(SO(2)\) space is given by,

\[
G^\alpha_{\mu\nu} \to R^+_{\alpha\beta} G^\beta_{\mu\nu}
\]

which, written in component notation, looks like,

\[
E^\alpha \to \mp \epsilon^{\alpha\beta} B^\beta ; \quad B^\alpha \to \pm \epsilon^{\alpha\beta} E^\beta
\]

The standard duality symmetric Lagrangean fails to manifest this property. However, as may be easily checked, the equations of motion obtained from the master Lagrangean swap with the corresponding Bianchi identity while the Lagrangean flips sign. In this manner the original property of the second order Maxwell Lagrangean is retrieved. Note furthermore that the master Lagrangean possesses the \(\sigma_1\) symmetry (which is just the discretised version of \(R^-\) with \(\varphi = 0\), a feature expected for two dimensional theories. In the appendix, a similar phenomenon is reported where the master action in two dimensions reveals the \(SO(2)\) symmetry usually associated with four dimensional theories.

### 4.1 Coupling to gravity

The effects of coupling to gravity are straightforwardly included by starting from the following Lagrangean,

\[
\mathcal{L} = -\frac{1}{4} \sqrt{-g} g^\mu\nu g^{\alpha\beta} F_{\mu\nu} F_{\alpha\beta}
\]

From our experience in the usual Maxwell theory we know that an eventual change of variables eliminates the Gauss law so that the term involving the multiplier \(A_0\) may be ignored from the outset. Expressing \((\ref{eq:103})\) in terms of its components to separate explicitly the first and second order terms, we find,

\[
\mathcal{L} = \frac{1}{2} \dot{A}_i M^{ij} \dot{A}_j + M^i \dot{A}_i + M
\]
where,

\[
M_{ij} = \sqrt{-g} \left( g^{0i} g^{0j} - g^{ij} g^{00} \right)
\]
\[
M^i = \sqrt{-g} g^{0k} g^{ji} F_{jk}
\]
\[
M = \frac{1}{4} \sqrt{-g} g^{ij} g^{km} F_{im} F_{kj}
\]  

(105)

Now reducing the Lagrangean to its first order form, we obtain,

\[
\mathcal{L} = P^i E_i - \frac{1}{2} P^i M_{ij} P^j - \frac{1}{2} M^i M_{ij} M^j + P^i M_{ij} M^j + M
\]  

(106)

where ̇A_i has been replaced by E_i and M_{ij} is the inverse of M^{ij},

\[
M_{ij} = \frac{-1}{\sqrt{-g} g^{00} g_{ij}}
\]  

(107)

with,

\[
g^{\mu\nu} g_{\alpha\lambda} = \delta_{\alpha}^{\mu}
\]  

(108)

Next, following the Maxwell example, introduce the standard change of variables which solves the Gauss constraint,

\[
E_i \rightarrow E_i^{(1)}
\]
\[
P^i \rightarrow \pm B^{i(2)}
\]  

(109)

the Lagrangean (106) is expressed in the desired form,

\[
\mathcal{L}_\pm = \pm \ E_i^\alpha \epsilon^{\alpha\beta} B^i_\beta + \frac{1}{\sqrt{-g} g^{00} g_{ij}} B^i_\alpha B^j_\beta \\
\pm \frac{g^{0k}}{g^{00}} \epsilon_{ijk} \epsilon^{\alpha\beta} B^i_\alpha B^j_\beta
\]  

(110)

Once again there are two duality symmetric actions corresponding to \( \mathcal{L}_\pm \). The enriched nature of the duality and swapping symmetries under a bigger set of transformations, the constructing of a master Lagrangean from soldering of \( \mathcal{L}_+ \) and \( \mathcal{L}_- \), the corresponding interpretations, all go through exactly as in the flat metric case. Incidentally, the structure for \( \mathcal{L}_- \) only was previously given in [25].
5 Conclusions

The present analysis clearly revealed the possibility of obtaining new results by combining two apparently independent theories into a single effective theory. The essential ingredient was that these theories must possess the dual aspects of the same symmetry. Then, by a systematic application of the soldering technique, it was feasible to abstract a meaningful combination of such models, which can never be obtained by a naive addition of the classical Lagrangeans. Detailed applications of this technique were presented in the context of bosonisation and duality symmetry. Simultaneously, a method for obtaining duality symmetric actions that were soldered was also developed.

The basic notions and ideas were first illustrated in two dimensional bosonisation. Bosonised expressions for distinct chiral Lagrangeans were soldered to reproduce either the usual gauge invariant theory or the Thirring model. Indeed, the soldering mechanism that fused the opposite chiralities clarified several aspects of the ambiguities occurring in bosonising chiral Lagrangeans. It was shown that unless Bose symmetry was imposed as an additional restriction, there is a whole one parameter class of bosonised solutions for the chiral Lagrangeans that can be soldered to yield the vector gauge invariant result. The close connection between Bose symmetry and gauge invariance was thereby established, leading to a unique parametrisation. Similarly, using a different parametrisation, the soldering of the chiral Lagrangeans led to the Thirring model. Once again there was a one parameter ambiguity unless Bose symmetry was imposed. If that was done, there was a specified range of solutions for the chiral Lagrangeans that combined to yield a well defined Thirring model.

The elaboration of our methods was done by considering the massive version of the Thirring model and quantum electrodynamics in three dimensions. By the process of bosonisation such models, in the long wavelength limit, were cast in a form which manifested a self dual symmetry. This was a basic prerequisite for effecting the soldering. It was explicitly shown that two distinct massive Thirring models, with opposite mass signatures, combined to a massive Maxwell theory. The Thirring current correlation functions calculated either in the original self dual formulation or in the effective massive vector theory yielded identical results, showing the consistency of our approach. The application to quantum electrodynamics followed along similar lines.

The present work also revealed a unifying structure behind the construction of the various duality symmetric actions ((76), (110), (128)). The essential ingredient was the conversion of the second order action into a first order form followed by an appro-
appropriate redefinition of variables such that these may be denoted by an internal index. The duality naturally occurred in this internal space. Since the duality symmetric actions were directly derived from the original action the proof of their equivalence becomes superfluous. This is otherwise essential where such a derivation is lacking and recourse is taken to either equations of motion or some hamiltonian analysis.

A notable feature of the analysis was the revelation of a whole class of new symmetries and their interrelations. Different aspects of this feature were elaborated. To be precise, it was shown that there are actually two duality symmetric actions \( \mathcal{L}_\pm \) for the same theory. These actions carry the opposite (self and anti self dual) aspects of some symmetry and their occurrence was essentially tied to the fact that there were two distinct classes in which the renaming of variables was possible, depending on the signature of the determinant specifying the proper or improper rotations. To discuss further the implications of this pair of duality symmetric actions it is best to compare with the existing results. This also serves to put the present work in a proper perspective. It should be mentioned that the analysis for two \(^4\) and four dimensions were generic for \( 4k + 2 \) and \( 4k \) dimensions, respectively.

It is usually observed \(^5\) that the invariance of the actions in different \( D \)-dimensions is preserved by the following groups,

\[
\mathcal{G}_d = Z_2 \ ; \ D = 4k + 2
\]

and,

\[
\mathcal{G}_c = SO(2) \ ; \ D = 4k
\]

which are called the “duality groups”. The \( Z_2 \) group is a discrete group with two elements, the trivial identity and the \( \sigma_1 \) matrix. Observe an important difference since in one case this group is continuous while in the other it is discrete. In our exercise this was easily verified by the pair of duality symmetric actions \( \mathcal{L}_\pm \). The new ingredient was that nontrivial elements of these groups (\( \sigma_1 \) for \( Z_2 \) and \( \epsilon \) for \( SO(2) \)) were also responsible for the swapping \( \mathcal{L}_+ \leftrightarrow \mathcal{L}_- \), but in dimensions different from where they act as elements of duality groups. In other words the “duality swapping matrices” \( \Sigma_s \) are given by,

\[
\Sigma_s = \sigma_1 \ ; \ D = 4k
\]

\[
= \epsilon \ ; \ D = 4k + 2
\]

\(^4\)Note that usual discussions of duality symmetry consider only one of these actions, namely \( \mathcal{L}_- \).

\(^5\)see appendix
A comparison with (111) and (112) shows the reversal of roles of the matrices with regard to the dimensionality of space time.

It was next shown that $\mathcal{L}_\pm$ contained the self and anti-self dual aspects of some symmetry. Consequently, following our ideas of soldering [3], the two Lagrangeans were merged to yield a master Lagrangean $\mathcal{L}_m = \mathcal{L}_+ \oplus \mathcal{L}_-$. The master action, in any dimensions, was manifestly Lorentz or general coordinate invariant and was also duality symmetric under both the groups mentioned above. Moreover the process of soldering lifted the discrete group $\mathbb{Z}_2$ to its continuous version. The duality group for the master action in either dimensionality therefore simplified to,

$$G = O(2) \ ; \ D = 2k + 2$$

Thus, at the level of the master action, the fundamental distinction between the odd and even $N$-forms gets obliterated. It ought to be stated that the lack of usual Chern Simons terms in $D = 4k + 2$ dimensions to act as the generators of duality transformations is compensated by the presence of a similar term in the internal space. Thanks to this it was possible to explicitly construct the symmetry generators for the master action in either two or four dimensions.

We also showed that the master actions in any dimensions, apart from being duality symmetric under the $O(2)$ group, were factored, modulo a normalisation, as a product of the self and anti-self dual solutions,

$$\mathcal{L} = \left(F^\alpha + \tilde{F}^\alpha\right)\left(F^\alpha - \tilde{F}^\alpha\right) \ ; \ D = 2k + 2$$

where the internal index has been explicitly written and the generalised Hodge operation was defined distinctly in $4k$ and $4k + 2$ dimensions. The key ingredient in this construction was to provide a general definition of self duality ($\tilde{F} = F$) that was applicable for either odd or even $N$ forms. Self duality was now defined to include the internal space and was implemented either by the $\sigma_1$ or the $\epsilon$, depending on the dimensionality. This naturally led to the universal structure (113).

Some other aspects of the analysis deserve attention. Specifically, the novel duality symmetric actions obtained in two dimensions revealed the interpolating role between duality and chirality. Furthermore, certain points concerning the interpretation of chirality symmetric action as the analogue of the duality symmetric electromagnetic action in four dimensions were clarified. We also recall that the soldering of actions to obtain a master action was an intrinsically quantum phenomenon that could be expressed in terms of an identity relating two “gauge variant” actions to a “gauge invariant” form. The gauge invariance is with regard to the set of transformations
induced for effecting the soldering and has nothing to do with the conventional gauge transformations. In fact the important thing is that the distinct actions must possess the self and anti self dual aspects of some symmetry which are being soldered. The identities obtained in this way are effectively a generalisation of the usual Polyakov Weigmann identity. We conclude by stressing the practical nature of our approach to either bosonisation or duality which can be extended to other theories.

**Appendix: The Scalar Theory in 1+1 Dimensions**

Here we discuss the effects of soldering in duality symmetry in $1 + 1$ dimensions illuminating the similarities and distinctions from the case of $3 + 1$ dimensions. It is simple to realise that the scalar theory is a very natural example in these dimensions. For instance, there is no photon and the Maxwell theory trivialises so that the electromagnetic field can be identified with a scalar field. Thus all the results presented here can be regarded as equally valid for the “photon” field. Indeed the computations will also be presented in a very suggestive notation which illuminates the Maxwellian nature of the problem.

The Lagrangean for the free massless scalar field is given by,

$$L = \frac{1}{2} \left( \partial_\mu \phi \right)^2$$

and the equation of motion reads,

$$\ddot{\phi} - \phi'' = 0$$

where the dot and the prime denote derivatives with respect to time and space components, respectively. Introduce a change of variables using electromagnetic symbols,

$$E = \dot{\phi} ; \quad B = \phi'$$

Obviously, $E$ and $B$ are not independent but constrained by the identity,

$$E' - \dot{B} = 0$$

In these variables the equation of motion and the Lagrangean are expressed as,

$$\dot{E} - B' = 0$$

$$L = \frac{1}{2} \left( E^2 - B^2 \right)$$
It is now easy to observe that the transformations,
\[ E \rightarrow \pm B \ ; \ B \rightarrow \pm E \]  
(121)
display a duality between the equation of motion and the ‘Bianchi’-like identity (119) but the Lagrangean changes its signature. Note that there is a relative change in the signatures of the duality transformations (73) and (121), arising basically from dimensional considerations. This symmetry corresponds to the improper group of rotations.

To illuminate the close connection with the Maxwell formulation, we introduce covariant and contravariant vectors with a Minkowskian metric \( g_{00} = -g_{11} = 1 \),
\[ F_\mu = \partial_\mu \phi \ ; \ F^\mu = \partial^\mu \phi \]  
(122)
whose components are just the ‘electric’ and ‘magnetic’ fields defined earlier,
\[ F_\mu = (E, B) \ ; \ F^\mu = (E, -B) \]  
(123)
Likewise, with the convention \( \epsilon_{01} = 1 \), the dual field is defined,
\[ *F_\mu = \epsilon_{\mu\nu} \partial^\nu \phi = \epsilon_{\mu\nu} F^\nu \]  
\[ = (-B, -E) \]  
(124)
The equations of motion and the ‘Bianchi’ identity are now expressed by typical electrodynamical relations,
\[ \partial_\mu F^\mu = 0 \]
\[ \partial_\mu *F^\mu = 0 \]  
(125)

To expose a Lagrangean duality symmetry, the basic principle of our approach to convert the original second order form (120) to its first order version and then invoke a relabelling of variables to provide an internal index, is adopted. This is easily achieved by first introducing an auxiliary field,
\[ \mathcal{L} = PE - \frac{1}{2} P^2 - \frac{1}{2} B^2 \]  
(126)
where \( E \) and \( B \) have already been defined. The following renaming of variables corresponding to the proper and improper transformations (see for instance (78) and (79) ) is used,
\[ \phi \rightarrow \phi_1 \]
\[ P \rightarrow \pm \phi'_2 \]  
(127)
where we are just considering the discrete sets of the full symmetry (78) and (79). Then it is possible to recast (126) in the form,

\[
L \rightarrow L_{\pm} = \frac{1}{2} \left[ \pm \phi'_{\alpha} \sigma_1^{\alpha\beta} \phi_{\beta} - \phi_{\alpha}^2 \right]
\]

\[
= \frac{1}{2} \left[ \pm B_\alpha \sigma_1^{\alpha\beta} E_\beta - B_\alpha^2 \right]
\]

(128)

In the second line the Lagrangean is expressed in terms of the electromagnetic variables. This Lagrangean is duality symmetric under the transformations of the basic scalar fields,

\[
\phi_\alpha \rightarrow \sigma_1^{\alpha\beta} \phi_{\beta}
\]

(129)

which, in the notation of \(E\) and \(B\), is given by,

\[
B_\alpha \rightarrow \sigma_1^{\alpha\beta} B_\beta
\]

\[
E_\alpha \rightarrow \sigma_1^{\alpha\beta} E_\beta
\]

(130)

It is quite interesting to observe that, contrary to the electromagnetic duality in 3 + 1 dimensions, the transformation matrix in the \(O(2)\) space is a Pauli matrix, which is the discretised version of improper rotations. This result is in agreement with that found from general algebraic arguments [25, 28] which stated that for \(d = 4k + 2\) dimensions there is a discrete \(\sigma_1\) symmetry. Observe that (130) is a manifestation of the original duality (121) which was also effected by the same operation. It is important to stress that the above symmetry is only implementable at the discrete level. Moreover, since it is not connected to the identity, there is no generator for this transformation.

To complete the picture, we also mention that the following (proper) rotation,

\[
\phi_\alpha \rightarrow \epsilon_{\alpha\beta} \phi_{\beta}
\]

(131)

interchanges the Lagrangeans (128),

\[
L_+ \leftrightarrow L_-
\]

(132)

Thus, except for a rearrangement of the the matrices generating the various transformations, most features of the electromagnetic example are perfectly retained. The crucial point of departure is that now all these transformations are only discrete.
Interestingly, the master action constructed below lifts these symmetries from the discrete to the continuous.

Let us therefore solder the two distinct Lagrangeans to manifestly display the complete symmetries. Before doing this it is instructive to unravel the self and anti-self dual aspects of these Lagrangeans, which are essential to physically understand the soldering process. The equations of motion following from (128), in the language of the basic fields, are given by,

\[ \partial_\mu \phi_\alpha = \mp \sigma^1_{\alpha\beta} \epsilon_{\mu\nu} \partial^\nu \phi_\beta \]  

(133)

provided reasonable boundary conditions are assumed. Note that although the duality symmetric Lagrangean is not manifestly Lorentz covariant, the equations of motion possess this property. In terms of a vector field \( F_\mu^\alpha \) and its dual \( * F_\mu^\alpha \) defined analogously to (123), (124), the equation of motion is rewritten as,

\[ F_\mu^\alpha = \pm \sigma^1_{\alpha\beta} * F_\mu^\beta = \pm \tilde{F}_\mu^\alpha \]  

(134)

where the generalised Hodge dual (\( \tilde{F} \)) has been introduced. Note the difference in the definition of this dual when compared with the corresponding definition (98) in 3 + 1 dimensions which employs the epsilon matrix. This explicitly reveals the self and anti-self dual nature of the solutions in the combined internal and coordinate spaces. The result can be extended to any \( D = 4k + 2 \) dimensions with suitable insertion of indices.

We now solder the two Lagrangeans. This is best done by using the notation of the basic fields of the scalar theory. These Lagrangeans \( \mathcal{L}_+ \) and \( \mathcal{L}_- \) are regarded as functions of the independent scalar fields \( \phi_\alpha \) and \( \rho_\alpha \). Consider the gauging of the following symmetry,

\[ \delta \phi_\alpha = \delta \rho_\alpha = \eta_\alpha \]  

(135)

Following our iterative procedure the final Lagrangean is obtained,

\[ \mathcal{L} = \mathcal{L}_+(\phi) + \mathcal{L}_-(\rho) - B_\alpha \left( J^+_\alpha(\phi) + J^-_\alpha(\rho) \right) - B^2_\alpha \]  

(136)

where the currents are given by,

\[ J^\pm_\alpha(\theta) = \pm \sigma^1_{\alpha\beta} \dot{\theta}_\beta - \theta'_\alpha ; \quad \theta = \phi, \rho \]  

(137)

The above Lagrangean is gauge invariant under the extended transformations including (135) and,

\[ \delta B_\alpha = \eta'_\alpha \]  

(138)
Eliminating the auxiliary $B_\alpha$ field using the equations of motion, the final soldered Lagrangean is obtained from (136),

$$\mathcal{L}(\Phi) = \frac{1}{4} \partial_\mu \Phi_\alpha \partial^\mu \Phi_\alpha$$  \hspace{1cm} (139)

where, expectedly, this is now only a function of the gauge invariant variable,

$$\Phi_\alpha = \phi_\alpha - \rho_\alpha$$  \hspace{1cm} (140)

This master Lagrangean possesses all the symmetries that are expressed by the continuous transformations,

$$\Phi_\alpha \rightarrow R_{\alpha\beta}^\pm(\theta) \Phi_\beta$$  \hspace{1cm} (141)

The generator corresponding to the $SO(2)$ transformations is easily obtained,

$$Q = \int dy \Phi_\alpha \epsilon_{\alpha\beta} \Pi_\beta$$

$$\Phi_\alpha \rightarrow \Phi'_\alpha = e^{-i\theta Q} \Phi_\alpha e^{i\theta Q}$$  \hspace{1cm} (142)

where $\Pi_\alpha$ is the momentum conjugate to $\Phi_\alpha$. Observe that either the original symmetry in $\sigma_1$ or the swapping transformations were only at the discrete level. The process of soldering has lifted these transformations from the discrete to the continuous form. It is equally important to reemphasize that the master action now possesses the $SO(2)$ symmetry which is more commonly associated with four dimensional duality symmetric actions, and not for two dimensional theories. Note that by using the electromagnetic symbols, the Lagrangean can be displayed in a form which manifests the soldering effect of the self and anti self dual symmetries (134),

$$\mathcal{L} = \frac{1}{8} \left( F_\mu^\alpha + \tilde{F}_\mu^\alpha \right) \left( F_\mu^\beta - \tilde{F}_\mu^\beta \right)$$  \hspace{1cm} (143)

where the generalised Hodge dual has been used.

An interesting observation is now made. Recall that the original duality transformation (121) switching equations of motion into Bianchi identities may be rephrased in the internal space by,

$$E_\alpha \rightarrow \mp R_{\alpha\beta}^\pm B_\beta$$

$$B_\alpha \rightarrow \mp R_{\alpha\beta}^\pm E_\beta$$  \hspace{1cm} (144)
which is further written directly in terms of the scalar fields,

\[ \partial_\mu \Phi_\alpha \rightarrow \pm R^\pm_{\alpha\beta} \epsilon_{\mu\nu} \partial^\nu \Phi_\beta \]  

(145)

It is simple to verify that under these transformations even the Hamiltonian for the theories described by the Lagrangeans \( \mathcal{L}_\pm \) (128) are not invariant. However the Hamiltonian following from the master Lagrangean (139) preserves this symmetry. The Lagrangean itself changes its signature. This is the exact analogue of the original situation. A similar phenomenon also occurred in the electromagnetic theory.

It is now straightforward to give a Polyakov-Weigman type identity, that relates the “gauge invariant” Lagrangean with the non gauge invariant structures, by reformulating (139) after a scaling of the fields \((\phi, \rho) \rightarrow \sqrt{2}(\phi, \rho)\),

\[ \mathcal{L}(\Phi) = \mathcal{L}(\phi) + \mathcal{L}(\rho) - 2\partial_\pm \phi_\alpha \partial_\mp \rho_\alpha \]  

(146)

where the light cone variables are defined in (4).

Observe that, as in the electromagnetic example, the gauge invariance is with regard to the transformations introduced for the soldering of the symmetries. Thus, even if the theory does not have a gauge symmetry in the usual sense, the dual symmetries of the theory can simulate the effects of the former. This leads to a Polyakov-Wiegman type identity which has a similar structure to the conventional identity.

It may be useful to highlight some other aspects of duality and soldering which are peculiar to two dimensions, as for instance, the chiral symmetry. The interpretation of this symmetry with regard to duality seems, at least to us, to be a source of some confusion and controversy. As is well known a scalar field in two dimensions can be decomposed into two chiral pieces, described by Floreanini Jackiw (FJ) actions [33]. These actions are sometimes regarded [32] as the two dimensional analogues of the duality symmetric four dimensional electromagnetic actions [25]. Such an interpretation is debatable since the latter have the \(SO(2)\) symmetry (characterised by an internal index \(\alpha\)) which is obviously lacking in the FJ actions. Our analysis, on the other hand, has shown how to incorporate this symmetry in the two dimensional case. Hence we consider the actions defined by (128) to be the true analogue of the duality symmetric electromagnetic actions discussed earlier. Moreover, by solving the equations of motion of the FJ action, it is not possible to recover the second order free scalar Lagrangean, quite in contrast to the electromagnetic theory [25]. Nevertheless, since the FJ actions are just the chiral components of the usual scalar action, these must be soldered to reproduce this result. But if soldering is possible,
such actions must also display the self and anti-self dual aspects of chiral symmetry. This phenomenon is now explored along with the soldering process.

The two FJ actions defined in terms of the independent scalar fields $\phi_+$ and $\phi_-$ are given by,

$$ L_{\pm}^{FJ}(\phi_{\pm}) = \pm \dot{\phi}_{\pm} \phi'_{\pm} - \phi'_{\pm} \phi_{\pm} $$

(147)

whose equations of motion show the self and anti self dual aspects,

$$ \partial_{\mu} \phi_{\pm} = \mp \epsilon_{\mu\nu} \partial_{\nu} \phi_{\pm} $$

(148)

A trivial application of the soldering mechanism leads to,

$$ L(\Phi) = L_{+}^{FJ}(\phi_+) + L_{-}^{FJ}(\phi_-) + \frac{1}{8} \left( J_{+}(\phi_+) + J_{-}(\phi_-) \right)^2 $$

(149)

where the currents $J_{\pm}$ and the composite field $\Phi$ are given by,

$$ J_{\pm} = 2 \left( \pm \dot{\phi}_{\pm} - \phi'_{\pm} \right) $$

$$ \Phi = \phi_+ - \phi_- $$

(150)

Thus the usual scalar action is obtained in terms of the composite field. The previous analysis has, however, shown that each of the Lagrangeans (128) are equivalent to the usual scalar theory. Hence these Lagrangeans contain both chiralities descried by the FJ actions (147). However, in the internal space, $L_{\pm}$ carry the self and anti self dual solutions, respectively. This clearly illuminates the ubiquitous role of chirality versus duality in the two dimensional theories which has been missed in the literature simply because, following conventional analysis in four dimensions [24, 25], only one particular duality symmetric Lagrangean $L_-$ was imagined to exist.

**Coupling to gravity**

It is easy to extend the analysis to include gravity. This is most economically done by using the language of electrodynamics already introduced. The Lagrangean for the scalar field coupled to gravity is given by,

$$ \mathcal{L} = \frac{1}{2} \sqrt{-g} g^{\mu\nu} F_{\mu} F_{\nu} $$

(151)
where $F_\mu$ is defined in (123) and $g = \det g_{\mu\nu}$. Converting the Lagrangean to its first order form, we obtain,

$$\mathcal{L} = P E - \frac{1}{2\sqrt{-g}g^{00}}(P^2 + B^2) + \frac{g^{01}}{g^{00}}PB$$

(152)

where the $E$ and $B$ fields are defined in (118) and $P$ is an auxiliary field. Let us next invoke a change of variables mapping $(E, B) \rightarrow (E_1, B_1)$ by means of the $O(2)$ transformation analogous to (127), and relabel the variable $P$ by $\pm B_2$. Then the Lagrangean (152) assumes the distinct forms,

$$\mathcal{L}_\pm = \frac{1}{2} \left[ \pm B_\alpha \sigma^1_{\alpha\beta} E_\beta - \frac{1}{\sqrt{-g}g^{00}} B^2 \pm \frac{g^{01}}{g^{00}} \sigma^1_{\alpha\beta} B_\alpha B_\beta \right]$$

(153)

which are duality symmetric under the transformations (130). As in the flat metric, there is a swapping between $\mathcal{L}_+$ and $\mathcal{L}_-$ if the transformation matrix is $\epsilon_{\alpha\beta}$. To obtain a duality symmetric action for all transformations it is necessary to construct the master action obtained by soldering the two independent pieces. The dual aspects of the symmetry that will be soldered are revealed by looking at the equations of motion following from (153),

$$\sqrt{-g}F^\alpha_\mu = \mp g_{\mu\nu}\sigma^1_{\alpha\beta} F^{\nu,\beta}$$

(154)

The result of the soldering process, following from our standard techniques, leads to the master Lagrangean,

$$\mathcal{L} = \frac{1}{4} \sqrt{-g}g^{\mu\nu}F^\alpha_\mu F^\alpha_\nu$$

(155)

where $F^\alpha_\mu$ is defined in terms of the composite field given in (140). In the flat space this just reduces to the expression found previously in (139). It may be pointed out that, starting from this master action it is possible, by passing to a first order form, to recover the original pieces.

To conclude, we show how the FJ action now follows trivially by taking any one particular form of the two Lagrangeans, say $\mathcal{L}_+$. To make contact with the conventional expressions quoted in the literature [34], it is useful to revert to the scalar field notation, so that,

$$\mathcal{L}_+ = \frac{1}{2} \left[ \phi_1' \phi_2 + \phi_2' \phi_1 + 2 \frac{g^{01}}{g^{00}} \phi_1' \phi_2 - \frac{1}{g^{00}\sqrt{-g}} \phi_\alpha' \phi_\alpha \right]$$

(156)

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This is diagonalised by the following choice of variables,

\[
\begin{align*}
\phi_1 &= \phi_+ + \phi_- \\
\phi_2 &= \phi_+ - \phi_-
\end{align*}
\]  \hspace{1cm} (157)

leading to,

\[
\mathcal{L}_+ = \mathcal{L}_+^{(+)}(\phi_+, \mathcal{G}_+) + \mathcal{L}_+^{(-)}(\phi_-, \mathcal{G}_-) \hspace{1cm} (158)
\]

with,

\[
\mathcal{L}_+^{(\pm)}(\phi_\pm, \mathcal{G}_\pm) = \pm \dot{\phi}_\pm \phi'_\pm + \mathcal{G}_\pm \phi'_\pm \phi'_\pm
\]

\[
\mathcal{G}_\pm = \frac{1}{g^{00}} \left( - \frac{1}{\sqrt{-g}} \pm g^{01} \right) \hspace{1cm} (159)
\]

These are the usual FJ actions in curved space as given in [34]. Such a structure was suggested by gauging the conformal symmetry of the free scalar field and then confirmed by checking the classical invariance under gauge and affine transformations [34]. Here we have derived this result directly from the action of the scalar field minimally coupled to gravity.
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