4D, $N = 2$ Supersymmetric Off-shell $\sigma$-Models
on the Cotangent Bundles of Kähler Manifolds

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ABSTRACT

We review the construction of 4D $N = 2$ globally supersymmetric off-shell nonlinear sigma models whose target spaces are the cotangent bundles of Kähler manifolds.

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1Supported in part by US NSF Grant PHY-98-02551, the US DOE Grant DE-FG02-94ER-40854, and the Deutsche Forschungsgemeinschaft.

2Contribution to the proceedings of the Buckow-98 meeting of presentation by S. Kuzenko.

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A classical result of SUSY field theory in four spacetime dimensions is that the target spaces of rigid supersymmetric nonlinear sigma models must be Kähler manifolds \(^1\) when \(N = 1\) and hyperkähler manifolds \(^2\) for \(N = 2\). The hyperkähler manifolds form a subspace in the family of Kähler manifolds of complex dimension \(2n\) and possess more restrictive properties as compared to general Kähler ones (three independent parallel complex structures, Ricci-flatness, \(Sp(n)\) holonomy versus \(U(2n)\) in general), see \(^3\) for a review. Remarkably, there exist at least two constructions which generate hyperkähler manifolds of complex dimension \(2n\) from Kähler manifolds of complex dimension \(n\). The first example is provided by the famous c-map \(^4\) in the limit of rigid supersymmetry. One starts with a holomorphic prepotential \(F(Z^i)\), \(i = 1, 2, \ldots, n\), destined to generate the so-called rigid special Kähler geometry \(^5\) whose potential and metric read

\[
G(Z, \bar{Z}) = F_i(Z)\bar{Z}^i + \overline{F_i(Z)Z^i}, \quad g_{ij}(Z, \bar{Z}) = F_{ij}(Z) + \overline{F_{ij}(Z)}.
\]

(1)

Then, it turns out that the following potential

\[
H(Z, \bar{Z}, W, \bar{W}) = G(Z, \bar{Z}) + \frac{1}{2} g^{ij}(Z, \bar{Z})(W_i + \bar{W}_i)(W_j + \bar{W}_j)
\]

(2)

corresponds to a hyperkähler manifold \(^4\). The second example has its origin in the seminal paper of Calabi \(^6\) where the concept of hyperkähler manifolds was introduced. In \(^6\) Calabi showed that the cotangent bundles of complex projective spaces \(\mathbb{C}P^n\) are hyperkähler manifolds. It was long conjectured and recently proved \(^7\) that the holomorphic cotangent bundles of general Kähler manifolds admit hyperkähler structures. Therefore, one can associate an \(N = 2\) supersymmetric sigma model with any Kähler manifold. Such manifestly \(N = 2\) supersymmetric sigma models have been described in our recent paper \(^8\). Below we will present a review of our construction.

Let us start by recalling a general \(N = 1\) rigid supersymmetric sigma model \(^4\). The model is described by chiral superfields \(\Phi^I\) and their conjugates \(\bar{\Phi}^\bar{I}\) whose physical scalar components \(A^I\) and \(\bar{A}^\bar{I}\) parametrize a Kähler manifold \(\mathcal{M}\). The action reads

\[
S[\Phi, \bar{\Phi}] = \int d^8z \ K(\Phi^I, \bar{\Phi}^{\bar{J}})
\]

where \(K(A, \bar{A})\) is the Kähler potential of \(\mathcal{M}\).

An \(N = 2\) supersymmetric extension of (1), in which we are here interested, is given in \(N = 1\) superspace by

\[
S[\Upsilon, \bar{\Upsilon}] = \int d^8z \left[ \frac{1}{2\pi i} \oint \frac{dw}{w} K(\Upsilon^I(w), \bar{\Upsilon}^{\bar{I}}(w)) \right].
\]

(4)
For $\Upsilon(w)$ and $\bar{\Upsilon}(w)$ we have

$$
\Upsilon^I(w) = \sum_{n=0}^{\infty} \Upsilon_n^I(z)w^n = \Phi^I(z) + w\Sigma^I(z) + O(w^2), \quad \bar{\Upsilon}^I(w) = \sum_{n=0}^{\infty} \bar{\Upsilon}_n^I(z)(-w)^{-n} \quad (5)
$$

with $\Phi$ being chiral, $\Sigma$ being complex linear, and the remaining component superfields being unconstrained complex superfields. The superfields $\Phi$ and $\Sigma$ are constrained by

$$
\bar{D}_\alpha \Phi = 0, \quad \bar{D}^2 \Sigma = 0 \quad (6)
$$

and provide two distinct off-shell realizations of $N = 1$ scalar multiplets. The role of the auxiliary superfields $\Upsilon_2, \Upsilon_3, \ldots$, is to ensure a linearly realized $N = 2$ supersymmetry. The expansions in (5) describe “polar” multiplets in the nomenclature of [1].

The $N = 2$ sigma model introduced respects all the geometric features of its $N = 1$ predecessor in (3). The Kähler invariance of (3)

$$
K(\Phi, \bar{\Phi}) \rightarrow K(\Phi, \bar{\Phi}) + \left(\Lambda(\Phi) + \bar{\Lambda}(\bar{\Phi})\right) \quad (7)
$$

turns into

$$
K(\Upsilon, \bar{\Upsilon}) \rightarrow K(\Upsilon, \bar{\Upsilon}) + \left(\Lambda(\Upsilon) + \bar{\Lambda}(\bar{\Upsilon})\right) \quad (8)
$$

for the model (3). A holomorphic reparametrization $A^I \rightarrow f^I(A)$ of the Kähler manifold has the following counterparts

$$
\Phi^I \rightarrow f^I(\Phi), \quad \Upsilon^I(w) \rightarrow f^I(\Upsilon(w)) \quad (9)
$$

in the $N = 1$ and 2 cases, respectively. Therefore, the physical superfields of the $N = 2$ theory

$$
\Upsilon^I(w) \big|_{w=0} = \Phi^I, \quad \frac{d\Upsilon^I(w)}{dw} \big|_{w=0} = \Sigma^I, \quad (10)
$$

should be regarded, respectively, as a coordinate of the Kähler manifold and a tangent vector at point $\Phi$ of the same manifold. That is why the variables $(\Phi^I, \Sigma^J)$ parametrize the tangent bundle $T\mathcal{M}$ of the Kähler manifold $\mathcal{M}$.

The presence of auxiliary superfields $\Upsilon_2, \Upsilon_3, \ldots$, in (5) makes $N = 2$ supersymmetry manifest, but the physical content of the theory is hidden. To describe the theory in terms of the physical superfields $\Phi$ and $\Sigma$ only, all the auxiliary superfields have to be eliminated with the aid of the corresponding algebraic equations of motion

$$
\oint \frac{dw}{w} w^n \frac{\partial K(\Upsilon, \bar{\Upsilon})}{\partial \Upsilon^I} = \oint \frac{dw}{w} w^{-n} \frac{\partial K(\Upsilon, \bar{\Upsilon})}{\partial \bar{\Upsilon}^I} = 0, \quad n \geq 2 \quad (11)
$$
Their solution $\Upsilon_*(w)$ can be found only perturbatively for general Kähler manifolds. Remarkably, there exist numbers of special cases, for instance, the four series of compact Kähler symmetric spaces (see, e.g. [3, 10])

\[
\begin{align*}
\frac{SU(m+n)}{SU(m) \times SU(n) \times U(1)}, & \quad \frac{Sp(n)}{SU(n) \times U(1)}, \quad \frac{SO(2n)}{SU(n) \times U(1)}, \quad \frac{SO(m+2)}{SO(m) \times SO(2)}, \quad m > 2
\end{align*}
\]

for which the equations (11) can be solved exactly, according to the rules given in [8]. The specific feature of the compact Kähler symmetric spaces is that eqs. (11) are equivalent to the holomorphic geodesic equation

\[
\frac{d^2 \Upsilon^I_*(w)}{dw^2} + \Gamma^I_{JK}(\Upsilon_*, \Phi) \frac{d\Upsilon^J_*(w)}{dw} \frac{d\Upsilon^K_*(w)}{dw} = 0
\]

under the initial conditions (10). Here $\Gamma^I_{JK}(\Phi, \bar{\Phi})$ are the Christoffel symbols for the Kähler metric

\[
g^{I\bar{J}}(\Phi, \bar{\Phi}) = \partial_I \partial_{\bar{J}} K(\Phi, \bar{\Phi}).
\]

Upon elimination of the auxiliary superfields, the action (4) takes the form

\[
S[\Upsilon_*, \bar{\Upsilon}_*] = S_{tb}[\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}] = \int d^8 z \left\{ K(\Phi, \bar{\Phi}) - g_{IJ}(\Phi, \bar{\Phi}) \Sigma^I \Sigma^J \right. \]

\[
\left. + \sum_{p=2}^{\infty} R_{I_1 \cdots I_p J_1 \cdots J_p}(\Phi, \bar{\Phi}) \Sigma^{I_1} \cdots \Sigma^{I_p} \Sigma^{J_1} \cdots \Sigma^{J_p} \right\},
\]

where the tensors $R_{I_1 \cdots I_p J_1 \cdots J_p}$ are functions of the Riemann curvature $R_{IJKL}(\Phi, \bar{\Phi})$ and its covariant derivatives. Each term in the action contains equal powers of $\Sigma$ and $\bar{\Sigma}$, since the original model (4) is invariant under rigid $U(1)$ transformations

\[
\Upsilon(w) \rightarrow \Upsilon(e^{i\alpha}w) \quad \Leftrightarrow \quad \Upsilon_n(z) \rightarrow e^{i\alpha} \Upsilon_n(z).
\]

To get a better feel for this construction, let us consider the simple example of $\mathbb{C}P^n = SU(n+1)/U(n)$ in the role of the Kähler manifold $\mathcal{M}$. For $\mathbb{C}P^n$ we have

\[
K(\Phi, \bar{\Phi}) = r^2 \ln \left( 1 + \frac{1}{r^2} \Phi L \bar{\Phi} L \right), \quad g_{IJ}(\Phi, \bar{\Phi}) = \frac{r^2 \delta_{IJ}}{r^2 + \Phi L \bar{\Phi} L} = \frac{r^2 (\bar{\Phi} \Phi J)}{(r^2 + \Phi L \bar{\Phi} L)^2}
\]

with $1/r^2$ being proportional to the curvature of $\mathbb{C}P^n$. Direct calculations lead to

\[
S[\Upsilon_*, \bar{\Upsilon}_*] = \int d^8 z \left\{ K(\Phi, \bar{\Phi}) + r^2 \ln \left( 1 - \frac{1}{r^2} g_{IJ}(\Phi, \bar{\Phi}) \Sigma^I \Sigma^J \right) \right\}.
\]
As is seen, the action is well defined when $|\Sigma|^2 \equiv g_{I\bar{J}} \Sigma^I \bar{\Sigma}^\bar{J} < r^2$. Under this restriction we can represent the second term in (17) by a Taylor series in $|\Sigma|^2$, and the series is nothing but an expansion in powers of the curvature of $CP^n$.

We can turn this to our advantage to obtain some partial information about the tensors $\mathcal{R}_{I_1...I_p,\bar{J}_1...\bar{J}_p}$ that appear in (14). Since the curvature of $CP^n$ is covariantly constant, the expansion of (17) fixes the form of all the terms in $\mathcal{R}_{I_1...I_p,\bar{J}_1...\bar{J}_p}$ that are not dependent of any derivatives of the Kähler manifold curvature. Appealing to universality, we suggest that these tensors in (14) should not be strongly dependent on the choice of a particular Kähler manifold. By this assertion, it follows that all the non-derivative terms in $\mathcal{R}_{I_1...I_p,\bar{J}_1...\bar{J}_p}$ are fixed by the series expansion of the logarithm in (17). Of course, a really comprehensive understanding of this approach requires a completely explicit description of $\mathcal{R}_{I_1...I_p,\bar{J}_1...\bar{J}_p}$ which presently lies beyond our grasp. This a topic for future investigation.

The Lagrangian of the $N = 2$ supersymmetric model (14) cannot yet be identified with a hyperkähler potential. The point is that for $N = 1$ rigid supersymmetric models their Lagrangians coincide with Kähler potentials only if all the dynamical variables are chiral superfields. But our model (14) is described by chiral superfields $\Phi^I$ (parametrizing the base Kähler manifold) and by complex linear ones $\Sigma^I$ (parametrizing the tangent fibers). It remains, however, to fulfil one more step – to dualize the linear superfields $\Sigma^I$ into chiral ones $\Psi_I$, $\bar{D}_\alpha \Psi_I = 0$, via the Legendre transform

$$S_{tb}[\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}] \rightarrow S[\Phi, \bar{\Phi}, U, \bar{U}, \Psi, \bar{\Psi}] = S_{tb}[\Phi, \bar{\Phi}, U, \bar{U}] - \int d^8z \left\{ U^I \Psi_I + \text{c.c.} \right\} \quad (18)$$

with the auxiliary superfields $U^I$ being complex unconstrained. By construction, $\{U^I\}$ is a tangent vector at point $\Phi$ of $\mathcal{M}$, therefore $\{\Psi_I\}$ is a one-form at the same point. Eliminating the auxiliary variables $U^I$ and $\bar{U}^\bar{J}$ in (18), with the aid of their equations of motion, results in a purely chiral sigma model $S_{cb}[\Phi, \bar{\Phi}, \Psi, \bar{\Psi}]$ which is dually equivalent to the $N = 2$ supersymmetric model (14) and is defined on the cotangent bundle $T^*\mathcal{M}$. Therefore, the superfield Lagrangian for $S_{cb}[\Phi, \bar{\Phi}, \Psi, \bar{\Psi}]$ coincides with the hyperkähler potential of $T^*\mathcal{M}$. In particular, if one applies the Legendre transform described to the model (14), one exactly reproduces the hyperkähler potential on the complete cotangent bundle $T^*(CP^n)$ \[\mathcal{B}\] \[\mathcal{L}\], see \[\mathcal{S}\] for more details.

In conclusion, we would like to point out that the c-map hyperkähler potential (2) is generated by a self-coupling of $N = 2$ tensor multiplets. As concerns the hyperkähler structures on the cotangent bundles of Kähler manifolds, the above consideration shows these are generated by self-couplings of $N = 2$ polar multiplets.
Acknowledgements

We like to thank I. Buchbinder, B. de Wit, N. Dragon, F. Gonzalez-Rey, T. Hübsch, E. Ivanov, U. Lindström, A. Lossev D. Lüst, A. Nersessian, B. Ovrut, M. Roček, S. Theisen and M. Vasiliev for enlightening discussions. SMK is grateful to the organizers of Buckow-98 for financial support. The work of SJG was supported in part by NSF Grant PHY-98-02551. The work of SMK was supported in part by Deutsche Forshungsgemeinschaft.

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