Virtual Monopole Geometry and Confinement

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Preface

Generalizing the geometry of the gauge covariant variables in Yang-Mills theory proposed by Johnson and Haagensen, the 4-d geometry associated with a monopole is defined for SU(2). There are three relevant geometries: \( \text{AdS}_2 \times S^2 \), \( R^2 \times S^2 \) and \( H_+ \times S^2 \), depending on the asymptotic behavior of the torsion. Using this geometry, the Wilson loop average is computed \( \text{à la} \) Nambu-Goto action. In case of \( \text{AdS}_2 \times S^2 \), it satisfies the area law.

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1. Introduction

One of the most intriguing parts of Yang-Mills theory is that the fundamental gauge field variable does not transform covariantly under gauge transformations. This causes construction of the physical Hilbert space rather complicated, if not impossible. This in turn makes difficult to investigate the nonperturbative aspect of the theory in the strong coupling regime where perturbation theory fails. Thus we need to look at the theory from somewhat different point of view.

One of the proposals made by Haagensen and Johnson is to introduce a covariant variable from a geometrical point of view[1]. For example, for SU(2) YM theory a new variable \( u^a_i \) can be defined such that a constraint equation relating \( u^a_i \) to the gauge field \( A^a_i \) is

\[
\epsilon^{ijk} \left( \partial_j u^a_k + \epsilon^{abc} A^b_j u^c_k \right) = 0.
\] (1.1)

For this, the Weyl gauge \( A^a_0 = 0 \) is chosen so that the subscript \( i \) runs over the spatial coordinates only. Then \( u^a_i \) transforms covariantly under gauge transformations. In a subsequent paper, this condition was generalized to allow a small nonvanishing rhs[2]. When \( u^a_i \) are identified as dreibeins of some geometry, this constraint equation is equivalent to nothing but the torsion-free condition. Hence, the following combination can be identified as a metric for some geometry associated with YM theory:

\[
g_{ij} \equiv u^a_i u^a_j. \tag{1.2}
\]

The local Lorentz symmetry is now SU(2). The hope is rewriting the theory in terms of \( u^a_i \) rather than \( A^a_i \) so that the outcome can be manifestly gauge invariant. Although explicit metrics are constructed for instanton or monopole backgrounds[2], the role of such a geometry has not been quite clear so far.

In this paper, we will generalize the above construction including the time component of the metric for the BPS monopole. The BPS monopole satisfies YM equation everywhere, yet reduces to 't Hooft-Polyakov monopole asymptotically so that it can clarify the behavior of the metric better. The BPS monopole is derived usually in the nonabelian Higgs model context, but such a monopole also exists for YM theory without necessarily introducing an extra scalar field. The construction is well known as the (anti)self-dual YM equation can be reduced to the Bogomol'nyi equation[3]. It is shown in this paper that the resulting 4-d geometries are asymptotically AdS\(_2 \times S^2\), \( R^2 \times S^2 \), or \( H_+ \times S^2 \), all with a nonvanishing torsion. In the AdS\(_2 \times S^2\)
case the torsion does not vanish even in the \( r \to \infty \) limit, which distinguishes itself from the other two cases. We propose that these are relevant to the Wilson loop average, as a Nambu-Goto action of some geometry is suspected to be relevant for the Wilson loop average\[^4\]. Then we will show how it can be related to the confinement in terms of the area law of a Wilson loop. In fact, we shall find that the relevant geometry for the area law is AdS\(_2 \times S^2\). Other geometries are related to other phases of YM theory.

## 2. Virtual Monopole Geometry

For our purpose of getting 4-d geometry, we will not fix the Weyl gauge. Then, let us first define an antisymmetric tensor field \( B^a_{\mu \nu} \) such that

\[
B = du + \omega u + u \omega,
\]

where \( B = \frac{1}{2} B^a_{\mu \nu} T^a dx^\mu dx^\nu \), \( u = u^a T^a dx^\mu \), \( \omega = \omega_{\mu}{}^{ab} T^a T^b dx^\mu \) and \( \omega_{\mu}{}^{ab} = -\epsilon^{abc} A^c_{\mu} \). Note that \( B \) transforms covariantly under gauge transformations as \( u \) does. Unlike the 3-d case, here we cannot identify \( u \) as a vierbein and \( \omega \) as a connection because \( a, b \) indices run over only 3-d while \( \mu \) runs over 4-d. Nevertheless, we can still construct

\[
g_{\mu \nu} \equiv u^a_\mu u^a_\nu,
\]

which takes the role of 4-d metric. Note that in this construction \( g_{\mu \nu} \) is not dimensionless because \( u^a_\mu \) is not. However, at this stage, since there is no explicit dimensionful parameter, we will not rescale to a dimensionless one.

The corresponding vierbeins, \( e^A_\mu \), \( A = (0, a) \), such that

\[
g_{\mu \nu} = e^A_\mu e^B_\nu \eta_{AB}, \quad \eta_{AB} = \text{diag}(\pm + + + ),
\]

can be constructed as

\[
e^0_0 \equiv \sqrt{|u^0_0 u^0_0|}, \quad e^a_i \equiv u^a_i, \quad e^0_i = 0 = e^0_0.
\]

Since \( e^0_0 \) is an SU(2) singlet, now the “local Lorentz symmetry” is enlarged to U(1)\( \times \)SU(2). Also the spin connection can be generalized to \( \omega_{\mu}{}^{AB} \) by defining

\[
\omega_{\mu}{}^{0b} = 0.
\]

\[^1\]An analogous conjecture is also used to compute the Wilson loop average in the context of the AdS/CFT correspondence in string theory\[^3\].
Then the torsion tensor in this case is given as usual in terms of $e_\mu^A$ and $\omega_\mu^{AB}$. In this paper, we are mainly interested in the static case so that

$$ T^a_{ij} = B^a_{ij}, \quad T^0_{ij} = 0, \quad T^0_{0i} = -\partial_i \sqrt{|u^a_0 u^a_0|}, \quad T^a_{0i} = \omega^{ab}_0 u^b_i = B^a_{0i} + \partial_i u^a_0 + \omega_i^{ab} u^b_0. \quad (2.6) $$

$T^0_{0i} = -\partial_i e^0_0$ already indicates the torsion does not vanish unless $e_0^0$ is constant.

$B$ can be related to the YM field strength as

$$ dB + \omega B - B\omega = F u - u F. \quad (2.7) $$

We can construct this geometry for the monopole background as follows. The relevant BPS monopole solution of self-dual YM equation is

$$ A^a_0 = \frac{x^a}{r^2} \left( \frac{\mu_m r}{\tanh \mu_m r} - 1 \right), \quad (2.8) $$

$$ A^a_i = \epsilon^a_{ij} \frac{x^j}{r^2} \left( 1 - \frac{\mu_m r}{\sinh \mu_m r} \right), \quad (2.9) $$

where $\mu_m$ is a monopole mass scale. Note that $A^a_0$ takes the role of the scalar field for the usual monopole solution in the Georgi-Glashow model (i.e. $SO(3) \simeq SU(2)$ nonabelian Higgs model).

We choose a metric whose spatial component is conformally flat:

$$ g_{ij} = \mu_m^2 e^{2\kappa \phi} \delta_{ij}, \quad (2.10) $$

where $\kappa$ is some length scale. The actual magnitude of $\kappa$ is quite irrelevant since it can always be rescaled and absorbed into $\phi$. Thus, we could even choose $\kappa = \mu_m^{-1}$. Now we have introduced in $g_{ij}$ an explicit dimensionful parameter coming from the monopole mass scale. Thus the usual dimensionless metric tensor can be obtained easily by rescaling. Anyhow, we need such dimensionful parameters to keep track the dimensionalities of variables. Then

$$ u^a_i = \mu_m e^{\phi/\mu_m} \delta^a_i. \quad (2.11) $$

Demanding the spatial part of the torsion vanishes, we obtain

$$ A^a_i = -\frac{1}{\mu_m} \epsilon^a_{ij} \partial_j \phi. \quad (2.12) $$

One can easily solve this for the 't Hooft-Polyakov monopole as $r \to \infty$ such that

$$ A^a_0 = \epsilon^a_{ij} \frac{x^j}{r^2}, \quad (2.13) $$

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then

\[ \phi = -\mu_m \ln (\mu_m r). \tag{2.14} \]

To compute the time component of the metric, it turns out that we cannot just demand the torsion-free condition for all components of the torsion. To understand the property of the torsion better, we can utilize the full BPS monopole to find out the finite \( r \) behavior. In fact, in general the torsion does not have to vanish for finite \( r \). Here we shall first check out the behavior of \( B \), which in turn will let us know about the torsion. It is reasonable to demand the spatial component of \( B \) vanishes asymptotically so that the energy associated with \( B \) to be finite, but there is no reason to demand the same behavior for the time component of \( B \). In particular, \( B_{0i}^a \) never vanishes for finite \( r \). However, since we do not have any other information for \( B \) at this moment, we will take a compromised position. In the following we shall demand that the spatial part of the torsion vanishes everywhere, but leave the time component arbitrary. In this way, we can still obtain the spatial part of the metric from eq.(2.12).

Eq.(2.12) can be solved for the BPS monopole, eq.(2.9), to obtain

\[ e^{2\kappa \phi} = \frac{1}{\mu_m^2 r^2} \tanh^2 \frac{\mu_m r}{2}. \tag{2.15} \]

The time component of the metric can be derived from \( B_{0i}^a \) as follows. Let

\[ u_0^a \equiv \mu_m^2 x^a f(r), \tag{2.16} \]

then

\[ B_{0i}^a = -\delta_i^a \mu_m^2 f(r) \frac{\mu_m r}{\sinh \mu_m r} - \mu_m^2 x^i x^a \left( \frac{1}{r} f'(r) + \frac{1}{r^2} \left( 1 - \frac{\mu_m r}{\sinh \mu_m r} \right) f(r) \right) + e^{ia_b x^b} \frac{\mu_m r}{r^3} \tanh \frac{\mu_m r}{2} \left( \frac{\mu_m r}{\tanh \mu_m r} - 1 \right). \tag{2.17} \]

If we demand \( B_{0i}^a < \infty \) as \( r \to \infty \), in general

\[ f(r) = c(\mu_m r)^{-n}, \quad (n \geq 0), \tag{2.18} \]

where \( c \) is a constant. \( n \) is chosen to be an integer so that the asymptotic behavior can be consistent with the last term of \( B_{0i}^a \). This choice will also let us avoid later unnecessary branch cuts when projected onto a two-dimensional surface of Wilson loop propagation. We also further demand that \( u_0^a < \infty \) as \( \mu_m \to 0 \). This is because eqs.(2.8)(2.9) are well behaved in this limit. As a result, the only allowed \( n \) are \( n = 0, 1, 2 \).
For $n = 0$, $B_{0i}^a$ does not vanish asymptotically:

$$n = 0 : \quad B_{0i}^a \rightarrow -c\mu_m^2 \frac{x^i x^a}{r^2} + \epsilon^{iab} \mu_m \frac{x^b}{r^2} \rightarrow -c\mu_m^2 \frac{x^i x^a}{r^2}.$$

(2.19)

For $n = 1, 2$,

$$n = 1, 2 : \quad B_{0i}^a \rightarrow \epsilon^{iab} \mu_m \frac{x^b}{r^2} \rightarrow 0.$$

(2.20)

For $n = 2$, although the asymptotic behavior of $B$ is the same as $n = 1$ case, the time component of the resulting metric vanishes as $r \rightarrow \infty$. Note that the $\mu_m$ term behaves the same way as $A_i^a$ asymptotically except the prefactor.

The full metric now reads asymptotically

$$\mu_m^2 ds_{\text{monopole}}^2 = c^2 \mu_m^2 (\mu_m r)^{2(1-n)} dt^2 + \frac{dr^2}{r^2} + d\Omega_2^2, \quad (n = 0, 1, 2).$$

(2.21)

For $n = 0$, it is $\text{AdS}_2 \times S^2$. For $n = 1$, $R^2 \times S^2$. For $n = 2$ the first two terms are the Poincaré metric for the upper half plane so that the asymptotic topology can be identified as $H_+ \times S^2$.

The torsion can be computed accordingly:

$$T_{0i}^a = |c|(n-1)\mu_m^2 \frac{x^i}{r} \left( \frac{1}{(\mu_m r)^n} \right), \quad (2.22)$$

and all other components vanish. Note that in other than $n = 0$ case, the torsion vanishes in the limit $r \rightarrow \infty$. However, for $n = 0$, $T_{0i}^0$ survives even in this limit.

This defines a geometry associated with a monopole in SU(2) YM theory. $c^2 = \pm 1$ is a constant that determines the signature of the geometry, which can be either Euclidean or Lorentzian depending on the value of $u_0^a$. Since this geometry is not that of the spacetime, but associated with the dynamical property of the monopole, we call it “virtual monopole geometry.” The exact form of the metric for finite $r$ cannot be determined at this moment because there is no other information about $B$.

We can now compute the Wilson loop average in this background.

3. Wilson Loop Average

It has long been suspected that the Wilson loop average would take the following form in the leading order:[4]

$$W(C) \sim e^{-S_{NC}(\Sigma_C)},$$

(3.1)
where $\Sigma_C$ is a minimal area surface bounded by $C$. Higher order terms are expected to be related to the extrinsic geometry of $\Sigma_C$[4, 5]. Intuitively, this is a fairly plausible assumption in the following sense. The exponent of the Wilson loop operator can be rewritten as an integration over a flat surface bounded by the contour using the Stokes’ theorem. However, the surface integration is restricted by the derivative of the gauge field. This surface integration is equivalent to a geometric form of surface integration over a curved surface in which the classical part of the gauge field provides the necessary geometrical information. The leading term of this integration is nothing but the Nambu-Goto action because the latter simply computes the surface area. Any explicit indisputable proof of this argument does not exist yet for YM theory, but it will most likely turn out to be true.

In fact, a rough estimation along the line of the argument in the above paragraph shows this is quite reasonable in our case. Using the Stokes’ theorem, we obtain

$$\oint_C dx^\mu A^a_\mu = \int_{D_C} d\sigma^{\mu\nu} b^a_{\mu\nu},$$

where $\sigma^{\mu\nu}$ is a surface element of flat surface $D_C$ bounded by $C$, and

$$b^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu. \quad (3.3)$$

Introducing a surface parameter, we can rewrite

$$\int_{D_C} d\sigma^{\mu\nu} b^a_{\mu\nu} = \int_{\Sigma_C} d^2 \xi \epsilon^{IJ} \partial_I x^\mu \partial_J x^\nu b^a_{\mu\nu}, \quad (3.4)$$

where $\Sigma_C$ is now a curved parameter surface. We can now expand $b$ around the monopole background according to $A^a_\mu = A^{(0)a}_\mu + \delta A^a_\mu$ such that

$$b^a_{\mu\nu} = b^{(0)a}_{\mu\nu} + \delta b^a_{\mu\nu}, \quad (3.5)$$

where

$$b^{(0)a}_{ij} = 2 \epsilon_i^a \frac{1}{r^2} - 2 \epsilon_j^a \frac{x^k x^j}{r^4} + 2 \epsilon_{ik} \frac{x^j x^k}{r^4} + \cdots, \quad (3.6)$$

$$b^{(0)a}_{0i} = \mu_m \left(- \delta_i^a \frac{1}{r} + \frac{x^a x^i}{r^3}\right) + \delta_i^a \frac{1}{r^2} - 2 \frac{x^a x^i}{r^4} + \cdots. \quad (3.7)$$

In the leading order, we obtain the identity\footnote{This is not a gauge invariant identification since it involves a specific choice of a background.}

$$\left(\epsilon^{IJ} \partial_I x^\mu \partial_J x^\nu b^{(0)a}_{\mu\nu}\right)^2 = \det \gamma_{IJ}, \quad (3.8)$$

\footnote{This is not a gauge invariant identification since it involves a specific choice of a background.}
\[ \gamma_{IJ} \equiv g_{\mu\nu} \partial_I x^\mu \partial_J x^\nu \]  
(3.9)

and \( g_{00} = \mu^2_m, \) \( g_{ii} = \frac{1}{r^2}. \) This precisely corresponds to \( n = 1 \) case of the metric we constructed before in eq.(2.21). If we perform the same computation for \( \mu_m \to 0 \) limit, we obtain the \( n = 2 \) case of the metric. \( n = 0 \) case is slightly more complicated because \( b^{(0)a}_{0i} = 0 \) for the 't Hooft-Polyakov monopole limit in the Weyl gauge. However, a careful analysis based on eq.(2.8) in the limit \( \mu_m \to 0 \) first and then \( r \to \infty \) leads to \( b^{(0)a}_{0i} \to \frac{x^a}{r^2}. \) This leads to \( g_{00} = r^2 \) and \( g_{ii} = r^{-2}. \) Properly rescaling by \( t \to \mu^2_m t, \) we can obtain the \( n = 0 \) case of eq.(2.21).

This certainly indicates the Wilson loop average is likely of the form

\[ W(C) = e^{-\int_{\Sigma_C} \sqrt{\gamma} + h.o.} \int D\delta A e^{iS[\delta A]} \text{Tr} e^{\int_{D_C} d\sigma \delta \mu \delta b_{\mu \nu}} \]  
(3.10)

for a proper action of \( \delta A \) derived from the YM action.

Thus, assuming there is no further complication due to the \( \delta A \) part, here we propose the relevant Nambu-Goto action is the one defined by the string of a magnetic flux embedded in the virtual monopole geometry we constructed:

\[ S_{NG}(\Sigma_C) = \frac{1}{2\pi\alpha'} \int_{\Sigma_C} d^2\xi \sqrt{\det|\gamma_{IJ}|}, \]  
(3.11)

where \( g_{\mu\nu} \) is given by eq.(2.21). The rational choice of the string tension is the one associated with the monopole mass scale such that \( \frac{1}{2\pi\alpha'} = \mu^2_m \) can be chosen.

Fixing the worldsheet coordinates as \( \xi^0 = t \) and \( \xi^1 = x^1, \) we can compute the Nambu-Goto action over a rectangle in the \((t, x^1)\)-plane with sides \( T \) and \( R \) to obtain

\[ S_{NG} \sim \mu^2_m 2^{-n} TV(R). \]  
(3.12)

\( V(R) \) is equivalent to the potential energy between quarks due to the nature of the Wilson loop average. Note that different \( n \) values lead to different behaviors of the Wilson loop average, which suggests that this index is a parameter classifying different phases of YM theory. For given \( n = 0, 1, 2, \) the potential energy between quarks behaves for large \( R \) like

\[ V(R) \sim \begin{cases} R, & n = 0, \\ \ln(\mu_m R), & n = 1, \\ -\frac{1}{R}, & n = 2. \end{cases} \]  
(3.13)

The \( n = 0 \) case has the linear potential, hence, it satisfies the area law. The corresponding metric has asymptotic \( \text{AdS}_2 \times S^2 \) topology. Recall that \( n = 0 \) is the only case that the torsion
does not vanish in the $r \to \infty$ limit, indicating that the confinement case has a distinctive geometry compared to other two cases.

The $n = 2$ case is the Coulomb phase.

The third possibility of $n = 1$ is intriguing, but we suspect that this might be the case of the oblique confinement[7]. This is not actually a far fetched identification. The monopole solution we have is in fact dyonic for finite $r$. This is because the nonvanishing leading order of nonabelian electric field $E_i^a$ for large $r$ is $-\frac{ix^a}{r^4}$, although the contribution of the ’t Hooft-Polyakov monopole limit is zero. Based on the argument that condensations of these objects cause confinements, it is clear there are two different possibilities in our case depending on the characteristics in which condensations occur. Since the dyonic behavior is due to a higher order contribution, we could identify $n = 1$ case as the oblique confinement. Note that the logarithmic potential is also confining in the sense the potential increases asymptotically, even though it does meet the area law criterion.

4. Final Remarks

It is shown in this paper that the geometrical structure suggested by Haagensen and Johnson can be generalized to 4-d and it can take an important role to prove confinement in YM theory, provided that the Wilson loop average can be computed in terms of Nambu-Goto action. The result leads to three possible phases of YM theory, presumably, confining, oblique confining and Coulomb phases, although the approach in this paper does not address any dynamical issues of them.

The behavior of different phases are closely related to the property of $B$ field, which is related to the torsion of the 4-d geometry we obtained. Our result applies only for large $r$, but it is possible to know more about the finite $r$ region if further information on $B$ is available. One possible way of incorporating $B$ is to introduce it as the nonabelian version of the Kalb-Ramond field. Then it should be possible that YM theory can be described by $A$ as well as by $(u, B)$. In this sense, one can speculate that $B$ could take the role of the dual field in the nonabelian case[4]. It will be interesting to see if $B$ is relevant in the string formulation of gauge theory.

The result of this paper also suggests that there might be a string theory on AdS$_2 \times S^2$ with a torsion which could be relevant to YM theory or QCD in 4-d in the spirit of [8]. The case of string theory on AdS$_2 \times S^2$ without torsion was investigated in [9].
The most important remaining question in this paper is of course if one can show explicitly that the Wilson loop average in YM theory is related to the Nambu-Goto action of some string theory. In our approach, it is necessary to show that $\delta A$ quantum contribution does not spoil the structure. This will most likely lead to a dual geometrical formulation of YM theory, presumably a string theory in an extrinsic geometry, in which $\delta A_\mu$ is properly translated into $\delta g_{\mu\nu}$. We believe that the geometry provided here is some clue to that.

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