0.1 Space–time variational methods for Maxwell’s equations

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The efficient and accurate numerical solution of the time–dependent Maxwell equations is one of the most challenging tasks, see, e.g., [1]. Besides semi–discretization methods such as the method of lines and Laplace transformation approaches, space–time variational formulations became popular in recent years. Here the variational principle is applied simultaneously in space and time, which later requires the solution of the global linear system of algebraic equations. But this can be done in parallel, and the space–time formulation allows for an adaptive resolution of the solution in space and time simultaneously.

Following previous work [3, 5] on the acoustic wave equation we present two variational formulations for the solution of the electromagnetic wave equation.

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\S 1 A space–time variational formulation for Maxwell’s equations

For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ and a finite time horizon $T > 0$ we define the space–time cylinder $Q := \Omega \times (0, T)$ with the lateral boundary $\Sigma := \partial \Omega \times (0, T)$, and we consider the electromagnetic wave equation

\begin{align*}
\varepsilon(x) \partial_t E(x, t) + \nabla_x \times ((\mu(x))^{-1} \nabla_x \times E(x, t)) &= -\partial_x J(x, t) \quad \text{for } (x, t) \in Q, \\
E(x, t) \times \nu_x &= 0 \quad \text{for } (x, t) \in \Sigma, \\
\varepsilon \partial_x E(x, 0) &= -J(x, 0) + \nabla \times H_0(x) =: \psi(x) \quad \text{for } x \in \Omega, \\
E(x, 0) &= E_0(x) =: \phi(x) \quad \text{for } x \in \Omega,
\end{align*}

where the permittivity $\varepsilon$ and the permeability $\mu$ are assumed to be symmetric and uniform positive definite matrix valued material coefficients, i.e., $\varepsilon, \mu \in [L^\infty(\Omega)]^{3 \times 3}$.

The Sobolev spaces

\begin{align*}
H_{0;0}^{1,\text{curl}}(Q) &:= L^2(0, T; H_0(\text{curl}; \Omega)) \cap H^1(0, T; [L^2(\Omega)]^3), \\
H_{0;0}^{1,0}(Q) &:= L^2(0, T; H_0(\text{curl}; \Omega)) \cap H^1_0(0, T; [L^2(\Omega)]^3), \\
H_{0;0}^{1,1}(Q) &:= L^2(0, T; H_0(\text{curl}; \Omega)) \cap H^1_0(0, T; [L^2(\Omega)]^3),
\end{align*}

are Hilbert spaces for which we define the weighted (half) norms as follows

\begin{align*}
\|u\|_{L^2(\Omega)}^2 &:= \langle \varepsilon u, u \rangle_{L^2(\Omega)}, \\
\|u\|_{H_{\mu;\nu}(\text{curl}; \Omega)}^2 &:= \langle \varepsilon u, u \rangle_{L^2(\Omega)} + \langle \mu^{-1} \nabla_x \times u, \nabla_x \times u \rangle_{L^2(\Omega)}, \\
|u|_{H_{\mu;\nu}(\text{curl}; \Omega)}^2 &:= \langle \partial_t u, \partial_t u \rangle_{L^2(\Omega)} + \langle \mu^{-1} \nabla_x \times u, \nabla_x \times u \rangle_{L^2(\Omega)}.
\end{align*}

When multiplying the electromagnetic wave equation with a test function $v \in H_{0;0}^{1,1}(Q)$, doing integration over the space–time cylinder $Q$, and applying integration by parts both in space and time, this results in a variational formulation to find $E \in H_{0;0}^{1,\text{curl}}(Q)$, $E(x, 0) = \phi(x)$ for $x \in \Omega$, such that

\begin{equation}
-\langle \varepsilon \partial_t E, \partial_t v \rangle_{L^2(Q)} + \langle \mu^{-1} \nabla_x \times E, \nabla_x \times v \rangle_{L^2(Q)} = -\langle \partial_t J, v \rangle_{L^2(Q)} - \langle \varepsilon \psi, v(0) \rangle_{L^2(\Omega)} \tag{1}
\end{equation}

is satisfied for all $v \in H_{0;0}^{1,1}(Q)$.

**Theorem 1.1** (Existence and Uniqueness) Let $\partial_t J \in L^1(0, T; [L^2(\Omega)]^3)$, $\phi \in H_0(\text{curl}; \Omega)$ and $\psi \in [L^2(\Omega)]^3$ be given. Then there exists a unique solution of the variational formulation (1).

In the case $\partial_t J \in [L^2(\Omega)]^3$ the following inequality holds true

\begin{equation}
|E|_{H_{\mu;\nu}(\text{curl}; \Omega)}^2 \leq 4T \|\psi\|_{L^2(\Omega)}^2 + 4T \|\phi\|_{H_{\mu;\nu}(\text{curl}; \Omega)}^2 + 2T^2 \|\partial_t J\|_{L^2(\Omega)}^2.
\end{equation}

The proof of Theorem 1.1 is, as in the case of the acoustic wave equation [2, 3], based on Fourier analysis using an expansion of the solution into the eigenfunctions of the spatial curl curl operator, and considering the system of the resulting ordinary differential equations; for a different approach see [1]. While the result of Theorem 1.1 requires to assume $\partial_t J \in [L^2(\Omega)]^3$, our main interest is in the more general case when $\partial_t J \in [H_{0;0}^{1,1}(\Omega)]'$.\footnote{Corresponding author: email jhauser@math.tugraz.at}
2 A generalized variational formulation

We start with the simple observation, that in the case of the Laplace equation, \( \| - \Delta_x u \|_{H^1_0(\Omega)} = \| \nabla_x u \|_{L^2(\Omega)} \) defines an equivalent norm for \( u \in H^1_0(\Omega) \). While this remains true for the heat equation, i.e., for \( u \in L^2(0, T; H^1_0(\Omega)) \) with \( \partial_t u \in L^2(0, T; H^{-1}(\Omega)) \) and \( u(x, 0) = 0 \) for \( x \in \Omega \) we obtain \( \| \alpha \partial_t u - \Delta_x u \|_{L^2(0, T; H^{-1}(\Omega))} \) as equivalent norm for \( u \) with homogeneous Dirichlet boundary and initial conditions, the equivalent expression \( \| c^{-1} \partial_t u - \Delta_x u \|_{H^1_{0,0}(\Omega)} \), in case of the scalar wave equation does not define a norm in \( H^1_{0,0}(\Omega) \), which also applies to the electromagnetic wave equation. This is mainly due to the missing inclusion of the second initial condition \( \partial_t u \|_{t=0} = 0 \) in the above norm definition. One possible approach, see [5] in case of the acoustic wave equation, is to consider the electromagnetic wave equation in the distributional sense with respect to the extended space–time cylinder \( Q_- := \Omega \times (-\infty, T) \).

For simplicity we only consider homogeneous initial conditions, i.e., \( \phi(x) = \psi(x) = 0 \) for \( x \in \Omega \). Then we define the Hilbert space

\[
\mathcal{H}^{\operatorname{curl},1}(Q) := \left\{ u \big| Q : u \in L^2(Q_-), u|_{\partial \Omega \times (-\infty, 0)} = 0, D_{tt,cc}^Q u \in [H^{\operatorname{curl},1}_{0,0}(Q)]' \right\}
\]

with the norm

\[
\| u \|_{\mathcal{H}^{\operatorname{curl},1}(Q)} := \| u \|_{L^2(Q)} + \| D_{tt,cc}^Q u \|_{[H^{\operatorname{curl},1}_{0,0}(Q)]'}.
\]

For \( u \in H^{\operatorname{curl},1}_{0,0}(Q) \) and \( v \in H^{\operatorname{curl},1}_{0,0}(Q) \) we obtain

\[
\langle D_{tt,cc}^Q u, v \rangle_Q = \langle \varepsilon \partial_t u, \partial_t v \rangle_{L^2(Q)} + \langle \mu^{-1} \nabla_x \times u, \nabla_x \times v \rangle_{L^2(Q)}.
\]

Finally we define \( \mathcal{H}^{\operatorname{curl},1}_{0,0}(Q) = \overline{H^{\operatorname{curl},1}_{0,0}(Q)}^{\| \cdot \|_{\mathcal{H}^{\operatorname{curl},1}(Q)}} \) and we consider the variational formulation to find \( u \in \mathcal{H}^{\operatorname{curl},1}_{0,0}(Q) \) such that

\[
\langle D_{tt,cc}^Q u, v \rangle_Q = \langle f, v \rangle_Q \quad \text{for all } v \in \mathcal{H}^{\operatorname{curl},1}_{0,0}(Q). \tag{2}
\]

**Theorem 2.1** For \( f \in [H^{\operatorname{curl},1}_{0,0}(Q)]' \) there exists a unique solution \( u \in H^{\operatorname{curl},1}_{0,0}(Q) \) of the generalized variational formulation (2), i.e., \( D_{tt,cc}^Q : \mathcal{H}^{\operatorname{curl},1}_{0,0}(Q) \to [H^{\operatorname{curl},1}_{0,0}(Q)]' \) is an isomorphism.

Note that the variational formulation (2) is of Galerkin–Petrov type, with different ansatz and test spaces. The proof of Theorem 2.1 hence relies on a related inf–sup stability condition, which is a direct consequence of the used norm definitions.

3 Concluding remarks

In this note we have introduced new space–time variational formulations for the numerical solution of the electromagnetic wave equation, which are suitable for Galerkin–Bubnov finite element discretizations. For the numerical solution of (1) we expect, as in the case of the acoustic wave equation [3, 5], to have a CFL condition to be satisfied for the mesh widths in space and time, respectively. While in [4] it was possible to derive an unconditional stable stabilized variational formulation in the case of the acoustic wave equation, this is an open problem in the case of the electromagnetic wave equation. A more detailed numerical analysis of the variational formulations (1) and (2) and their related finite element discretizations as well as numerical results will be published elsewhere.

References

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