Some interesting features of new massive gravity

Antonio Accioly\textsuperscript{1,2}, José Helayel-Neto\textsuperscript{1}, Esley Scatena\textsuperscript{2}, Jefferson Morais\textsuperscript{1}, Rodrigo Turcati\textsuperscript{1} and Bruno Pereira-Dias\textsuperscript{1}

\textsuperscript{1} Laboratório de Física Experimental (LAFEX), Centro Brasileiro de Pesquisas Físicas (CBPF), Rua Dr Xavier Sigaud 150, Urca, 22290-180 Rio de Janeiro, Brazil

\textsuperscript{2} Instituto de Física Teórica (IFT), São Paulo State University (UNESP), Rua Dr Bento Teobaldo Ferraz 271, Bl. II-Barra Funda, 01140-070 São Paulo, Brazil

E-mail: accioly@cbpf.br, helayel@cbpf.br, scatena@ift.unesp.br, morais@cbpf.br, turcati@cbpf.br and bp dias@cbpf.br

Received 29 July 2011, in final form 2 September 2011
Published 18 October 2011
Online at stacks.iop.org/CQG/28/225008

Abstract
A sketch of a proof that new massive gravity—the massive 3D gravity model proposed by Bergshoeff, Hohm and Townsend (BHT)—is the only unitary system at the tree level that can be constructed by augmenting 3D general relativity through (curvature)\textsuperscript{2}-terms is presented. Two interesting gravitational properties of the BHT model, namely, time dilation and time delay, which have no counterpart in the usual Einstein 3D gravity, are analyzed as well.

PACS number: 04.60.Kz

1. Introduction

For too long, physicists believed that gravity models containing fourth (or higher) derivatives of the metric were doomed to failure by virtue of one detail: they entail unphysical ghost states of negative norm. The pure scalar curvature models, i.e. the fourth-order gravity systems with the Lagrangian $\mathcal{L} = R + \alpha R^2$ and which are tree-level unitary, seemed to be the only exception to this rule. Actually, these systems are conformally equivalent to Einstein gravity with a scalar field [1]. Consequently, despite having fourth derivatives at the metric level, these models are ultimately second order in their scalar–tensor versions. It is, therefore, perfectly understandable that just about two years ago the physical community were absolutely amazed to learn that a particular higher-derivative extension of 3D general relativity—that is ghost-free at the tree level—has been found out by Bergshoeff, Hohm and Townsend (BHT) [2–14].

It was argued that the aforementioned massive 3D gravity model, that is also known as ‘new massive gravity’ (NMG), is both ghost-free and power-counting UV finite in its pure, irreducibly fourth derivative, quadratic curvature limit [15], which, as it was pointed
out by Ahmedov and Aliev [10], violates the standard paradigm of its ‘cousins’ in four dimensions [16].

NMG is defined by the Lagrangian density

\[ \mathcal{L} = \sqrt{g} \left[ \frac{2R}{\kappa^2} + \frac{2}{\kappa^2 m_2^2} \left( R^2_{\mu\nu} - \frac{3}{8} R^2 \right) \right], \]  

(1)

where \( \kappa^2 = 32\pi G \), with \( G \) being the 3D analog of Newton’s constant, and \( m_2(>0) \) is a mass parameter. It is worth noticing that the Lagrangian density given in (1) has a reversed Einstein–Hilbert (EH) term.

A formal proof of the equivalence of the linearized version of the BHT model and the Einstein–Hilbert–Pauli–Fierz gravity was given in [2]; incidentally, this proof was reviewed in [17]. Nevertheless, the physical meaning of this equivalence is somewhat unclear; indeed, the linearized version of the BHT system is background diffeomorphism invariant, while the Pauli–Fierz theory is only invariant under the Killing symmetries of the spacetime (in particular, the 3D Minkowski space), which clearly shows that a better understanding of the symmetries is still lacking [9].

And what about the odd sign change of the EH term previously mentioned? At the linearized level, Deser [15] showed that the EH term breaks the Weyl invariance of the BHT model without the EH term and, consequently, is responsible for giving mass to the graviton. In other words, the higher-derivative terms provide the kinetic energy, whereas the EH term provides the mass in this linearized model, thus explaining the weird sign change of the EH term. It is remarkable that the EH term gives origin to the mass in the linearized version of the BHT system by breaking the Weyl invariance and not the expected diffeomorphism invariance [9].

At this point it would be interesting to ask ourselves about the reason for doing research on massive gravitons. The increased interest in recent years in this subject is motivated, on the one hand, by the discovery of cosmic acceleration, which might be explained in terms of an infrared modification of general relativity that gives the graviton a small mass [18], and on the other, by the conjecture that some theory involving massive gravitons could be the low energy limit of a noncritical string-theory underlying QED [19]. As is often done for so many other gravitational physical issues, it is advisable to consider first the possibilities for massive gravitons in the simpler context of a 3D spacetime [17]. The BHT model is accordingly the ideal arena for such investigations.

Our aim in this paper is twofold.

(i) To provide a ‘proof’ that the BHT gravity is the only tree-level unitary model that can be constructed in 3D by judiciously combining the Ricci scalar \( R \) with the curvature-squared terms \( R^2 \) and \( R^2_{\mu\nu} \).

(ii) To explore some interesting properties of this remarkable model that have no counterpart in the usual Einstein gravity in three-dimensional spacetime.

The paper is organized as follows. In the next section, we present a sketch of a proof that NMG is the unique 3D tree-level unitary model quadratic in curvatures. In section 3, it is shown that unlike what occurs in 3D general relativity, clocks are slowed down in a gravitational field described by the BHT model. This gravitational time dilation is the basis of the gravitational spectral shift. An expression for a NMG-induced time delay is obtained in section 4. Finally, in section 5 we present some comments and observations. Details of technical computations are relegated to appendices.

We employ natural units, \( c = \hbar = 1 \), and our Minkowski metric is diag(+1, −1, −1). Our Ricci tensor is defined by \( R_{\mu\nu} = R^\lambda_{\mu\nu\lambda} \equiv \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \cdots \).
2. Finding a class of tree-level unitary massive 3D gravity models

We start off our analysis by considering the most general three-dimensional theory obtained by augmenting the usual 3D Einstein gravity through the curvature-squared terms. Now, taking into account that in three dimensions both the curvature tensor and the Ricci tensor have the same number of components [20], we come to the conclusion that the Lagrangian density for the theory at hand can be written as

\[ \mathcal{L} = \sqrt{\frac{2\sigma}{k^2}} R + \frac{\alpha}{2} R^2 + \frac{\beta}{2} R_{\mu\nu}^2, \]

where \( \sigma \) is a convenient parameter that can take the values +1 (EH term with the standard sign), −1 (EH term with the ‘wrong sign’), and \( \alpha \) and \( \beta \) are free coefficients. Note that the constants \( \kappa \), \( \alpha \) and \( \beta \) have the mass dimension \([\kappa] = -\frac{1}{2} \) and \([\alpha] = [\beta] = -1\) in fundamental units. Our next step is to place constraints on the parameters of the above Lagrangian by examining afterward the residues of the saturated unitarity in order to get the BHT model. To accomplish this goal, we make use of a standard procedure that converts the task of checking the tree-level unitarity of a given model, which is in general a time-consuming work, into a straightforward algebraic exercise. The prescription consists basically in saturating the propagator with external conserved currents, compatible with the symmetries of the system, and in examining afterward the residues of the saturated propagator (SP) at each simple pole. Let us then compute the propagator for the gravity model in equation (2). To do that, we recall that for small fluctuations around the Minkowski metric \( \eta \), the full metric assumes the form

\[ g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}. \]

Linearizing equation (2) via equation (3) and adding to the result the gauge-fixing Lagrangian density, \( \mathcal{L}_{gf} = \frac{1}{2} \left( \partial_\mu \nu \right)^2 \), where \( \gamma_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h \), that corresponds to the de Donder gauge, we find

\[ \mathcal{L} = \frac{1}{2} h_{\mu\nu} \mathcal{O}^{\mu\nu\alpha\beta} h_{\alpha\beta}, \]

where, in momentum space,

\[ \mathcal{O} = \left[ \frac{\alpha k^2 + \beta k^2 k^4}{4} \right] p^{(2)} + \frac{k^2}{2\Lambda} p^{(1)} \left[ \frac{k^2}{4\Lambda} \right] p^{(0-w)} - \sqrt{\frac{2}{\Lambda}} \quad \right. \]

\[ - \frac{\sqrt{2} k^2}{4\Lambda} \quad \left. p^{(0-w)} \right] + \left[ \frac{k^2}{2\Lambda} - \sigma k^2 + 2\alpha k^2 k^4 + \frac{3}{4} \beta k^2 k^4 \right] p^{(0-z)}. \]

Here \( p^{(2)}, p^{(1)}, p^{(0-w)}, p^{(0-s)}, p^{(0-sw)} \) and \( p^{(0-ws)} \) are the usual three-dimensional Barnes–Rivers operators (see appendix A).

Therefore, the propagator is given by (see appendix A)

\[ \mathcal{O}^{-1} = \frac{2\Lambda}{k^2} p^{(1)} + \frac{1}{k^2(\sigma + \frac{3\alpha k^2}{4})} p^{(2)} + \frac{1}{-\sigma k^2 + 2\alpha k^2 k^4 + \frac{3}{4} \beta k^2 k^4} p^{(0-s)} \]

\[ + \frac{\sqrt{2}}{-\sigma k^2 + 2\alpha k^2 k^4 + \frac{3}{4} \beta k^2 k^4} \left[ p^{(0-rw)} + p^{(0-ws)} \right] \]

\[ + \frac{-4\Lambda \sigma + 2 + 8\Lambda \alpha k^2 k^4 + 3\Lambda \beta k^2 k^4}{-\sigma k^2 + 2\alpha k^2 k^4 + \frac{3}{4} \beta k^2 k^4} p^{(0-w)}. \]

Contracting now the above propagator with conserved currents \( T^{\mu\nu}(k) \) (\( k_\mu T^{\mu\nu} = 0 \)) yields

\[ \text{SP} = \frac{1}{\sigma} \left[ \frac{1}{k^2} \frac{1}{k^2 - m^2} \right] \left[ T^{\mu\nu}_\mu T^{\mu\nu} \frac{1}{2} T^2 \right] + \frac{1}{\sigma} \left[ -\frac{1}{k^2} + \frac{1}{k^2 - m^2} \right] \frac{1}{2} T^2. \]
where \( m_2^2 \equiv -\frac{4\sigma}{\kappa^2} \), \( m_0^2 \equiv \frac{4\alpha + 3\beta + \kappa^2}{\kappa^2} \). Assuming that there are no tachyons in the model, we promptly find the following constraints:

\[
\frac{\sigma}{\beta} < 0, \quad \frac{\sigma}{8\alpha + 3\beta} > 0.
\]

On the other hand, the residues of \( \text{SP} \) at the poles \( k^2 = m_2^2 \), \( k^2 = 0 \), and \( k^2 = m_0^2 \) are, respectively,

\[
\text{Res}(\text{SP})\bigg|_{k^2 = m_2^2} = -\frac{1}{\sigma} \left( T_{\mu\nu} - \frac{1}{2} T \right) \bigg|_{k^2 = m_2^2},
\]

\[
\text{Res}(\text{SP})\bigg|_{k^2 = 0} = \frac{1}{\sigma} (T_{\mu\nu} - T) \bigg|_{k^2 = 0},
\]

\[
\text{Res}(\text{SP})\bigg|_{k^2 = m_0^2} = \frac{1}{2\sigma} (T^2) \bigg|_{k^2 = m_0^2}.
\]

Now, as is well known, the tree-level unitarity of a generic model is ensured if the residue at each simple pole of \( \text{SP} \) is \( \geq 0 \). Keeping in mind that \( (T_{\mu\nu} - \frac{1}{2} T) \bigg|_{k^2 = m_2^2} > 0 \) and \( (T_{\mu\nu} - T) \bigg|_{k^2 = 0} = 0 \) (see appendix B), we arrive at the conclusion that (i) \( \text{Res}(\text{SP})\bigg|_{k^2 = m_2^2} > 0 \) if \( \sigma = -1 \) (which implies \( \beta > 0 \) and \( \alpha < 0 \)), and (ii) \( \text{Res}(\text{SP})\bigg|_{k^2 = 0} = 0 \). Consequently, we need not worry about these poles; the troublesome one is \( k^2 = m_0^2 \) since \( \text{Res}(\text{SP})\bigg|_{k^2 = m_0^2} < 0 \).

A way out of this difficulty is to consider the \( m_0 \to \infty \) limit of the model under discussion, which leads us to conclude that \( \alpha = -\frac{3}{8} \beta \). Accordingly, the class of models defined by the Lagrangian density

\[
\mathcal{L} = \sqrt{g} \left[ -\frac{2R}{\kappa^2} + \frac{\beta}{2} \left( R_{\mu\nu}^2 - \frac{3}{8} R^2 \right) \right],
\]

are ghost-free at the tree level. For the sake of convenience, we replace \( \beta \) with \( \frac{4}{\kappa^2 m_2^2} \), where \( m_2 \) is a mass parameter. The resulting Lagrangian density,

\[
\mathcal{L} = \sqrt{g} \left[ -\frac{2R}{\kappa^2} + \frac{2}{\kappa^2 m_2^2} \left( R_{\mu\nu}^2 - \frac{3}{8} R^2 \right) \right],
\]

is nothing but the BHT model for massive 3D gravity.

It is worth noting that it is not clear at all whether or not the particular ratio between \( \alpha \) and \( \beta \) we have previously found will survive renormalization at a given loop level, even at one loop; in other words, unitarity beyond tree level has to be checked [9]. Most likely, the BHT model is nonrenormalizable since it improves only the spin-2 projections of the propagator but not the spin-0 projection [21].

3. Gravitational time dilation

The Einstein 3D gravity is trivial outside the sources; consequently, no gravitational time dilation or slowing down of clocks can take place in its framework. This can easily be shown in the particular case of a spherically symmetric distribution of mass \( M \) whose linearized spacetime interval is given by

\[
ds^2 = dr^2 - (1 + \lambda)(dr^2 + r^2 d\theta^2),
\]

where \( \lambda = 8GM \ln \frac{r_0}{r} \), with \( r_0 \) being an infrared regulator, and \( r \) and \( \theta \) are the usual polar coordinates.
Introducing now new radial ($r'$) and angular ($\theta'$) coordinates through the change of variables
\[(1 - \lambda)r^2 = (1 - 8GM)r'^2, \quad \theta' = (1 - 4GM)\theta,\]
we obtain, to linear order in $GM$,
\[dx^2 = dr^2 - dr'^2 - r'^2 \, d\theta'^2.\]  
(16)

The geometry around the spherically symmetric distribution is, therefore, locally identical to that of a flat spacetime as it should; however, it is not globally Minkowskian since the angle $\theta'$ varies in the range $0 \leq \theta' < 2\pi (1 - 4GM)$. Accordingly, the three-dimensional metric (16) describes a conical space with a wedge of angular size equal to $8\pi GM$ removed and the two faces of the wedge identified. We thus come to the conclusion that in the framework of Einstein 3D gravity, no gravitational spectral shift occurs due to the presence of the mentioned odd geometrical effect. It is worth noticing that in this context, the nonexistence of a time dilation does not imply that the spacetime is necessarily flat; in other words, the time dilation is not a ‘classical test’ of 3D general relativity. As we will see in the following, the aforementioned bizarre geometrical effect does not take place in NMG. To do that we have to solve beforehand the linearized field equations related to the BHT system.

The field equations concerning the Lagrangian density
\[\mathcal{L} = \sqrt{g} \left[-\frac{2R}{\kappa^2} + \frac{2}{\kappa^2m^2} \left(R_{\mu\nu}^2 - \frac{3}{8}R^2\right) - \mathcal{L}_M\right],\]
where $\mathcal{L}_M$ is the Lagrangian density for matter, are
\[G_{\mu\nu} + \frac{1}{m^2} \left[\frac{1}{2}R_{\rho\sigma}g_{\mu\nu} - \frac{1}{4} \nabla_\mu \nabla_\nu R - 2R_{\mu\rho\lambda\nu}R^{\rho\lambda} - \frac{1}{4} g_{\mu\nu} \Box R + \Box R_{\mu\nu} \right.\]
\[-\frac{3}{16} R_{\rho\sigma}^2 + \frac{3}{4} RR_{\rho\sigma} \left.\right] = \frac{\kappa^2}{4} T_{\mu\nu},\]
(18)
where $T_{\mu\nu}$ is the energy–momentum tensor, and $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} R$ is the Einstein tensor.

The corresponding linearized field equations are given by
\[\left(1 + \frac{\Box}{m^2}\right) - \frac{1}{2} \Box h_{\mu\nu} + \frac{\eta_{\mu\nu} R_{\text{lin}}}{4\kappa^2} \left] + \frac{1}{2} \left(\partial_\mu \Gamma_\nu + \partial_\nu \Gamma_\mu - \partial_\lambda \Gamma_{\mu\nu} \right) \right.\]
\[= \frac{\kappa}{4} \left(\frac{T}{\Box} \eta_{\mu\nu} - T_{\mu\nu}\right).\]
(19)
where $R_{\text{lin}} = \kappa \left[\frac{1}{4} \Box h - \gamma_{\mu\nu}^\mu\nu\right]$, $\Gamma_\mu = \left(1 + \frac{\Box}{m^2}\right) \partial_\mu \gamma_{\rho\nu}^\rho + \frac{\eta_{\rho\nu} R_{\text{lin}}}{4\kappa^2 m^2}$, $\gamma_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu}$. Note that here indices are raised (lowered) using $\eta^{\mu\nu}(\eta_{\mu\nu})$.

Mimicking Teyssandier’s work on 4D higher-derivative gravity [22], it can be shown that it is always possible to choose a coordinate system such that the gauge conditions, $\Gamma_\mu = 0$, on the linearized metric hold. Assuming that these conditions are satisfied, it is straightforward to show that the general solution of (19) is given by
\[h_{\mu\nu} = \psi_{\mu\nu} - h_{\mu\nu}^{(E)},\]
(20)
where $h_{\mu\nu}^{(E)}$ is the solution of the linearized Einstein equation in the de Donder gauge, i.e.
\[\Box h_{\mu\nu}^{(E)} = \frac{\kappa}{2} (T \eta_{\mu\nu} - T_{\mu\nu}), \quad \partial^\nu \gamma_{\mu\nu}^{(E)} = 0,\]
(21)
where $\gamma_{\mu\nu}^{(E)} \equiv h_{\mu\nu}^{(E)} - \frac{1}{2} \eta_{\mu\nu} h^{(E)}$, while $\psi_{\mu\nu}$ satisfies the equation
\[\left(\Box + m^2\right) \psi_{\mu\nu} = -\frac{\kappa}{2} \left(T_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} T\right).\]
(22)
It is worth noticing that in this very special gauge, the equations for $\psi_{\mu\nu}$ and $h^{(E)}_{\mu\nu}$ are totally decoupled. As a result, the general solution to equation (19) reduces to a linear combination of the solutions of the aforementioned equations.

Solving equations (21) and (22) for a point-like particle of mass $M$ located at $r = 0$, we find

$$h_{00} = -\frac{\kappa M}{8\pi} K_0(m_2 r)$$

$$h_{11} = h_{22} = -\frac{\kappa M}{8\pi} \left[ K_0(m_2 r) + 2 \ln \frac{r}{r_0} \right],$$

where $K_0$ is the modified Bessel function of order zero. Note that $K_0(x)$ behaves as $-\ln x$ at the origin and as $x^{-\frac{1}{2}} e^{-x}$ asymptotically. Hence, the spacetime interval reads

$$ds^2 = [1 - 4MGK_0(m_2 r)] dt^2 - 1 + 4GM \left( K_0(m_2 r) + 2 \ln \frac{r}{r_0} \right) \left( dr^2 + r^2 d\theta^2 \right).$$

In the $m_2 \to \infty$ limit, (25) reproduces (14), as expected. The geometry around the point particle is, of course, not locally identical to that of a Minkowski-like spacetime, signaling in this way the possibility of occurrence of gravitational spectral shift. Let us then show that the gravitational time dilation does occur in the BHT model.

Suppose that a signal sent from an emitter at a fixed point $(r_E, \theta_E)$ is received, after traveling along a null geodesic, by a receiver at a fixed point $(r_R, \theta_R)$ (see figure 1). Now, the difference $t_R - t_E$, where $t_E$ is the coordinate time of emission and $t_R$ the coordinate time of reception, is the same for all signs sent—the worldlines of successive signals are nothing but copies of successive signals merely shifted in time. As a result, if the $t$-time

![Figure 1. Spacetime diagram illustrating the worldlines of two successive identical signals.](image-url)
difference between a signal and the next is \(dt_E\) at the departure point, the corresponding \(t\)-time difference at the arrival point is necessarily the same. However, the clock of an observer situated at the point of emission records proper time \((\tau)\) and not coordinate time \((t)\). Accordingly, \(dt_E = \sqrt{1 - 4MGK_0(m_2r_E)}\) \(dt_E\) and similarly \(dt_R = \sqrt{1 - 4MGK_0(m_2r_R)}\) \(dt_R\). Since \(dt_E = dt_R\), we promptly obtain

\[
\frac{dt_R}{dt_E} = \frac{\sqrt{1 - 4MGK_0(m_2r_R)}}{\sqrt{1 - 4MGK_0(m_2r_E)}} \\
\approx 1 - 2MGK_0(m_2r_R) + 2MGK_0(m_2r_E) \\
= 1 + V_R - V_E, \quad (26)
\]

where \(V(r) = \frac{1}{2} h_{00}(r) = -2MGK_0(m_2r)\) is the gravitational potential. This shows that if the clock at \((r_R, \theta_R)\) is at a lower potential than the clock at \((r_E, \theta_E)\), i.e. \(V_R < V_E\), then \(dt_R\) is smaller than \(dt_E\). In other words, the clock that is deeper in the gravitational potential runs slower. Equation (26) is the gravitational time-dilation formula or redshift formula. It is worth noticing that \(dt_R \to dt_E\) as \(m_2 \to \infty\), implying that no gravitational time dilation takes place in the framework of 3D general relativity, which totally agrees with the result we have previously found.

On the other hand, if the emitter is a pulsating atom which in the proper time interval \(\Delta \tau_E\) emits \(n\) pulses, an observer situated at the emitter will assign to the atom a frequency \(v_E \equiv \frac{n}{\Delta \tau_E}\), which, of course, is the proper frequency of the pulsating atom. The observer located at the receiver, in turn, assigns a frequency \(v_R \equiv \frac{n}{\Delta \tau_R}\) to the pulsating atom. Consequently,

\[
\frac{v_R}{v_E} = \frac{\sqrt{1 - 4MGK_0(m_2r_E)}}{\sqrt{1 - 4MGK_0(m_2r_R)}} \\
\approx 1 + 2MG \left[K_0(m_2r_R) - K_0(m_2r_E)\right].
\]

From this, we immediately get the fractional shift

\[
\frac{\Delta v}{v} \equiv \frac{v_R - v_E}{v_E} \approx 2MG \left[K_0(m_2r_R) - K_0(m_2r_E)\right].
\]

Note that since \(K_0(x)\) is a monotonically decreasing function in the range \(0 \leq x < \infty\), \(\frac{\Delta v}{v}\) is positive if \(r_E > r_R\), and negative if \(r_E < r_R\). Consequently, if the emitter is nearer to the massive object than the receiver, the shift is toward the red, but if the receiver is nearer to the massive object, it is toward the blue.

From the preceding considerations, we come to the conclusion that the gravitational spectral shift is indeed a classical test of the BHT model. It can also be viewed, like in 4D general relativity, as a direct test of the curvature of the spacetime.

4. Gravitational time delay

Another interesting effect that can be obtained from the linear approximation of NMG is the time delay suffered by a light signal sent by an observer—situated at a fixed point in space in the gravitational field generated by a massive object—to a small object and reflected back to the observer. The small object is supposed to be located directly between the observer and the huge body (see figure 2). Consider, in this spirit, a light pulse that moves along a straight line connecting the observer and the small object. It is easy to show that the coordinate time for the whole trip (observer \(\to\) small object \(\to\) observer) is given by

\[
\Delta t_G = 2 \int_{r_1}^{r_2} \sqrt{1 + 4MGK_0(m_2r) + 2 \frac{\ln \frac{r_2}{r_1}}{m_2}} \, dr. \quad (27)
\]
Figure 2. Time delay in ‘radar sounding’.

Accordingly, the proper time lapse measured by the observer, whose clock, of course, records proper time, has the form

$$\Delta \tau_G = 2 \sqrt{1 - 4MGK_0(m_2r_2)} \int_{r_1}^{r_2} \frac{1 + 4MGK_0(m_2r) + 2 \ln \frac{r}{r_0}}{1 - 4MGK_0(m_2r)} \, dr. \quad (28)$$

On the other hand, the distance traveled by the light pulse is equal to

$$2 \int_{r_1}^{r_2} \sqrt{1 + 4MGK_0(m_2r) + 2 \ln \frac{r}{r_0}} \, dr.$$

Consequently, on the basis of the classical theory, we should expect a roundtrip time of

$$\Delta \tau_C = 2 \int_{r_1}^{r_2} \sqrt{1 + 2GMK_0(m_2r) + 2 \ln \frac{r}{r_0}} \, dr. \quad (29)$$

From (28) and (29), we arrive to the conclusion that $\Delta \tau_G \neq \Delta \tau_C$. Note that in the $m_2 \to \infty$ limit, $\Delta \tau_G = \Delta \tau_C = 2 \int_{r_1}^{r_2} \sqrt{1 + 8MG \ln \frac{r}{r_0}} \, dr$, which clearly shows that there is no time delay in the framework of 3D general relativity, as expected.

On the other hand, equations (28) and (29) tell us that

$$\Delta \tau_G \approx 2 \int_{r_1}^{r_2} \left[ 1 + 4MGK_0(m_2r) + \ln \frac{r}{r_0} \right] \, dr - 4MG[K_0(m_2r_2) - K_0(m_2r_1)](r_2 - r_1),$$

$$\Delta \tau_C \approx 2 \int_{r_1}^{r_2} \left[ 1 + 2GMK_0(m_2r) + 2 \ln \frac{r}{r_0} \right] \, dr.$$

As a result,

$$\Delta \tau_G - \Delta \tau_C \approx 4MG \left[ \int_{r_1}^{r_2} K_0(m_2r) \, dr - (r_2 - r_1)K_0(m_2r_2) \right]$$

$$= 4MG[K_0(m_2r_0) - K_0(m_2r_1)](r_2 - r_1).$$
where $r_1 < r_0 < r_2$. Hence, we come to the conclusion that there is a NMG-induced time delay

$$
\Delta \tau_G - \Delta \tau_C \approx 4MG \left[ K_0(m_2r_0) - K_0(m_2r_2) \right] (r_2 - r_1).
$$

(30)

5. Final remarks

As is well known, the three-dimensional Einstein gravity without sources is physically vacuous because Einstein and Riemann tensors are equivalent in $D = 3$. In addition, the quantization of the gravity field does not give rise to propagating gravitons since the spacetime metric is locally determined by the sources. Consequently, the description of gravitational phenomena via 3D gravity leads to some bizarre results, such as the following:

- lack of a gravity force in the nonrelativistic limit,
- gravitational deflection independent of the impact parameter,
- complete absence of gravitational time dilation,
- no time delay.

It can be shown that the first two odd phenomena in the above list do not take place in the context of the BHT model [23]. In fact, in the framework of the latter, short-range gravitational forces are exerted on slowly moving particles; besides, the light bending depends on the impact parameter, as it should. On the other hand, the remaining strange phenomena in the aforementioned list, as we have shown, do not occur in the BHT system either. Indeed, both time delay and spectral shift do take place in the context of the NMG. Like in 4D general relativity, gravitational time dilation and gravitational time delay are also tests of the BHT model. It is worth noticing that the basis for these tests is the time-independent solution of the linearized BHT field equations produced by a static spherical mass.

One of the main reasons for studying 3D gravity models is in reality to try out a gravity system with less austere ultraviolet divergences in perturbation theory. Since general relativity in 3D is dynamically trivial, the BHT model, which is tree-level unitary, is an important step in this direction. This kind of research conducted in lower dimensions certainly helps us to gain insight into difficult conceptual issues, which are present and more opaque in the physical (3+1)-dimensional world. Another strong argument in favor of considering massive gravity theories, as we have already commented, is the fact that the present accelerated expansion of the universe could be partially attributed to a graviton mass-like effect.

It is worth mentioning that the triviality of 3D general relativity can also be cured by adding to the EH action in 3D a parity-violating Chern–Simons term. The resulting model is usually known as topological massive gravity (TMG) [24, 25]. Nonetheless, in contrast with TMG, 3D massive gravity has the great advantage of being a parity-preserving theory. On the other hand, since 3D higher-derivative gravity (3DHGD)—which is defined by the Lagrangian density $\mathcal{L}_{3DHGD} = \sqrt{g} \left( \frac{1}{2} \mathcal{R} + \frac{\sigma}{\kappa} R_{\mu\nu}^2 + \frac{\alpha}{\kappa^2} R_{\mu
u}^2 \right)$—is nonunitary at the tree level [26], it would be interesting to verify whether the addition of a nonunitary Chern–Simons term ($\mathcal{L}_{CS} = \frac{\mu}{2\kappa} \varepsilon^{\lambda\mu\nu} \Gamma_{\rho\lambda}^{\mu} \left[ \partial_\mu \Gamma^{\rho\sigma}_{\nu\lambda} + \frac{\kappa}{\sqrt{g}} \Gamma^{\rho\sigma}_{\nu\lambda} \Gamma^{\mu}_{\alpha\beta} \right]$), where $\mu$ is an arbitrary parameter, to this higher-order model would cure the nonunitarity of the former. It can be shown that in order to avoid ghosts and tachyons in the mixed theory ($\mathcal{L} = \mathcal{L}_{3DHGD} + \mathcal{L}_{CS}$), the following constraints on the parameters must hold [27]:

- (spin-2 sector): $\sigma < 0$, $\beta > 0$,
- (spin-0 sector): $\sigma > 0$, $3\beta + 8\alpha > 0$.

3 The massless excitation, like the massless excitation of 3D general relativity, is not a dynamical degree of freedom, i.e. it is nonpropagating.
Therefore, for arbitrary values of the parameters, the model at hand is nonunitary at the
tree level, which clearly shows that the topological Chen–Simons term is not a panacea for
3DHDG’s unitarity problem. Nevertheless, if we prevent the spin-0 mode from propagating
by choosing $3\beta + 8\alpha = 0$, the resulting model is tree-level unitary. It is amazing that the above
condition is exactly the same constraint that appears in the BHT model ($m_0 \to \infty$ limit). We
call attention to the fact that, contrary to a popular belief, the addition of a Chern–Simons
term to a tree-level unitary model is not necessarily a guarantee that the resulting model
will be tree-level unitary [26]. For instance, the addition of a Chern–Simons term ($\mathcal{L}_{CS}$) to
three-dimensional $R + \alpha R^2$ gravity ($\mathcal{L}_{R+\alpha R^2} = (-\frac{2\mathcal{F}}{\kappa^2} + \frac{\alpha R^2}{2}) \sqrt{\mathcal{G}}$), which is tree-level unitary,
spoils the unitarity of the latter [26]. Therefore, in some cases, the coexistence between the
topological Chern–Simons term and 3D higher-derivative gravity theories is conflicting.

To conclude, we remark that recently the nonlinear classical dynamics of the BHT model
was exhaustively investigated by de Rham, Gabadadze, Pirtskhalava, Tolley and Yavin [28],
who found that the theory passed remarkably nontrivial checks at the nonlinear level, such as
the following.

- In the decoupling limit of the theory, the interactions of the helicity-0 modes are described
  by a single cubic term, the so-called cubic Galileon [29].
- The conformal mode of the metric coincides with the helicity-0 mode in the decoupling
  limit.
- The full theory does not lead to any extra degrees of freedom, which suggests that a 3D
  analog of the 4D Boulware–Deser ghost is not present in the BHT system.

Acknowledgments

The authors are very grateful to FAPERJ, CNPq and CAPES (Brazilian agencies) for financial
support.

Note added. B Tekin, whom we thank, has informed us about a Weyl-invariant extension
of NMG whose vacuum breaks Weyl symmetry, which means that around the vacuum, the
first order expansion is just NMG with a fixed Newton’s constant [30]. Hence, the mass of the
graviton comes from the symmetry breaking in complete analogy with the Higgs mechanism.
It is interesting that in AdS backgrounds, the classical field equations break the symmetry and
the scalar field develops a nonzero expectation value, while in flat backgrounds, symmetry is
broken at the two-loop level. We also thank A Sinha for calling our attention to the fact that the
recently proposed Dirac–Born–Infeld extension of NMG emerges naturally as a counterterm
in AdS$_4$ [31].

Appendix A. A prescription for computing the graviton propagator as well as a list of
some identities that greatly facilitate this task

In order to find the propagator related to the Lagrangian density in equation (1), it is very
convenient to work in terms of the Barnes–Rivers operators in the space of symmetric rank-2
tensors. The complete set of three-dimensional operators in momentum space is [32, 33]

$$P_{\mu\nu,\kappa\lambda}^{(2)} = \frac{1}{2} (\theta_{\mu\kappa} \theta_{\nu\lambda} + \theta_{\mu\lambda} \theta_{\nu\kappa} - \theta_{\mu\nu} \theta_{\kappa\lambda}), \quad (A.1)$$

$$P_{\mu\nu,\kappa\lambda}^{(1)} = \frac{1}{2} (\theta_{\mu\kappa} \omega_{\nu\lambda} + \theta_{\mu\lambda} \omega_{\nu\kappa} + \theta_{\nu\kappa} \omega_{\mu\lambda} + \theta_{\nu\lambda} \omega_{\mu\kappa}), \quad (A.2)$$

$$P_{\mu\nu,\kappa\lambda}^{(0-2)} = \frac{1}{2} \theta_{\mu\nu} \theta_{\kappa\lambda},$$
appeal to the following identities:

\[ \text{space, we reproduce (5).} \]

The task of computing the operator \( O_{\theta} \) where \( \theta_{\mu\nu} \) is the external conserved current.

If \( m \) is the mass of a generic spin-2 physical particle related to a given 3D gravity model

\[ \text{From (A.6) and (5) we obtain (6).} \]

To compute the graviton propagator, we need the bilinear part of the Lagrangian density

\[ \text{To compute the graviton propagator, we need the bilinear part of the Lagrangian density (1). With the gauge fixing } \frac{1}{\sqrt{\Lambda}} (\partial_{\mu} Y^{\mu})^2 \text{(de Donder gauge), and going over to momentum space, we reproduce (5).} \]

The multiplicative table for these operators is displayed in Table A1.

### Table A1. Multiplicative table for the Barnes–Rivers operators.

| \( p^{(2)} \) | \( p^{(1)} \) | \( p^{(0-s)} \) | \( p^{(0-w)} \) | \( p^{(0-sw)} \) | \( p^{(0-sw)} \) |
|----------------|----------------|----------------|----------------|----------------|----------------|
| \( p^{(2)} \)  | \( p^{(2)} \)  | 0              | 0              | 0              | 0              |
| \( p^{(1)} \)  | 0              | \( p^{(1)} \)  | 0              | 0              | 0              |
| \( p^{(0-s)} \) | 0              | 0              | \( p^{(0-s)} \) | 0              | 0              |
| \( p^{(0-w)} \) | 0              | 0              | 0              | \( p^{(0-w)} \) | 0              |
| \( p^{(0-sw)} \) | 0              | 0              | 0              | 0              | \( p^{(0-sw)} \) |

\[ p_{\mu\nu,\lambda\kappa}^{(0-w)} = \alpha_{\mu\nu} \alpha_{\lambda\kappa}, \]

\[ p_{\mu\nu,\lambda\kappa}^{(0-sw)} = \frac{1}{\sqrt{2}} \theta_{\mu\nu} \alpha_{\lambda\kappa}, \]

\[ p_{\mu\nu,\lambda\kappa}^{(0-sw)} = \frac{1}{\sqrt{2}} \omega_{\mu\nu} \theta_{\lambda\kappa}, \]

where \( \theta_{\mu\nu} \equiv \eta_{\mu\nu} - \frac{k_{\mu} k_{\nu}}{2} \) and \( \alpha_{\mu\nu} \equiv \frac{k_{\mu} k_{\nu}}{2} \) are, respectively, the usual transverse and longitudinal projection operators. The multiplicative table for these operators is displayed in Table A1.

To compute the graviton propagator, we need the bilinear part of the Lagrangian density (1). With the gauge fixing \( \frac{1}{\sqrt{\Lambda}} (\partial_{\mu} Y^{\mu})^2 \) (de Donder gauge), and going over to momentum space, we reproduce (5). The task of computing the operator \( O \) is greatly facilitated if we appeal to the following identities:

\[ \left[ p^{(2)} + p^{(1)} + p^{(0-s)} + p^{(0-w)} \right]_{\mu\nu,\lambda\kappa} = \frac{1}{2} \left( \eta_{\mu\nu} \eta_{\lambda\kappa} + \eta_{\mu\lambda} \eta_{\nu\kappa} \right), \]

\[ \left[ 2p^{(0-s)} + p^{(0-w)} + \sqrt{2} \left( p^{(0-sw)} + p^{(0-sw)} \right) \right]_{\mu\nu,\lambda\kappa} = \eta_{\mu\nu} \eta_{\lambda\kappa}, \]

\[ \left[ 2p^{(1)} + 4p^{(0-w)} \right]_{\mu\nu,\lambda\kappa} = \frac{1}{k^2} \left( \eta_{\mu\nu} k_{\lambda} k_{\kappa} + \eta_{\mu\lambda} k_{\nu} k_{\kappa} + \eta_{\nu\lambda} k_{\mu} k_{\kappa} + \eta_{\nu\kappa} k_{\mu} k_{\lambda} \right), \]

\[ \left[ 2p^{(0-w)} + \sqrt{2} \left( p^{(0-sw)} + p^{(0-sw)} \right) \right]_{\mu\nu,\lambda\kappa} = \frac{1}{k^2} \left( \eta_{\mu\nu} k_{\lambda} k_{\kappa} + \eta_{\mu\lambda} k_{\nu} k_{\kappa} \right), \]

\[ p_{\mu\nu,\lambda\kappa}^{(0-w)} = \frac{1}{k^4} (k_{\mu} k_{\nu} k_{\lambda} k_{\kappa}). \]

Now, if we write the operator \( O \) in the generic form

\[ O = x_1 p^{(1)} + x_2 p^{(2)} + x_3 p^{(0-s)} + x_4 p^{(0-w)} + x_{sw} p^{(0-sw)} + x_{sw} p^{(0-sw)} \]

and take into account that \( \mathcal{O} \mathcal{O}^{-1} = I \), where \( \mathcal{O}^{-1} \) is the propagator, we promptly find

\[ \mathcal{O}^{-1} = \frac{1}{x_1} p^{(1)} + \frac{1}{x_2} p^{(2)} + \frac{1}{x_3 x_{sw} - x_{sw} x_{sw}} \left[ x_4 p^{(0-s)} + x_{sw} p^{(0-w)} - x_{sw} p^{(0-sw)} - x_{sw} p^{(0-sw)} \right]. \]

(A.6)

From (A.6) and (5) we obtain (6).

Appendix B. A useful result for checking the unitarity of a generic 3D gravity model

**Theorem 1.** If \( m \) is the mass of a generic spin-2 physical particle related to a given 3D gravitational model and \( k \) is the corresponding exchanged momentum, then

\[ (T_{\mu\nu}^2 - \frac{1}{2} T^2) |_{k^2 = m^2} > 0 \quad \text{and} \quad (T_{\mu\nu}^2 - T^2) |_{k^2 = 0} = 0. \]

Here, \( T_{\mu\nu} (= T^{\mu\nu}) \) is the external conserved current.
We begin by remarking that the set of independent vectors in momentum space, \( k^\mu \equiv (k^0, \vec{k}), \bar{k}^\mu \equiv (k^0, -\vec{k}), \epsilon^\mu \equiv (0, \hat{e}) \), where \( \hat{e} \) is a unit vector orthogonal to \( k \), is a suitable basis for expanding any three-vector \( V^\mu (k) \). Using this basis we can write the symmetric current tensor as follows:

\[ T^{\mu \nu} = A k^\mu k^\nu + B \bar{k}^\mu \bar{k}^\nu + C \epsilon^\mu \epsilon^\nu + D k^{(\mu} \bar{k}^{\nu)} + E k^{(\mu} \epsilon^{\nu)} + F \bar{k}^{(\mu} \epsilon^{\nu)}, \]

where \( a^{(\mu} b^{\nu)} = \frac{1}{2} (a^\mu b^\nu + b^\mu a^\nu) \).

The current conservations gives the following constraints on the coefficients \( A, B, D, E \) and \( F \):

\[ Ak^2 + \frac{D}{2} (k_0^2 + k^2) = 0 \quad (B.1) \]
\[ B(k_0^2 + k^2) + \frac{D}{2} k^2 = 0 \quad (B.2) \]
\[ Ek^2 + F (k_0^2 + k^2) = 0. \quad (B.3) \]

From equations (B1) and (B2), we get \( Ak^4 + B(k_0^2 + k^2)^2 \), while equation (B3) implies \( E^2 > F^2 \). On the other hand, saturating the indices of \( T^{\mu \nu} \) with momenta \( k_\mu \), we arrive at a consistent relation for the coefficients \( A, B \) and \( D \):

\[ Ak^4 + B(k_0^2 + k^2)^2 + Dk^2 (k_0^2 + k^2) = 0. \quad (B.4) \]

After a lengthy but otherwise straightforward calculation using the earlier equations, we obtain

\[ T_{\mu \nu}^2 - \frac{1}{2} T^2 = \left[ \frac{k^2 (A - B)}{\sqrt{2}} - \frac{C}{\sqrt{2}} \right]^2 + \frac{k^2}{2} (E^2 - F^2), \quad (B.5) \]

\[ T_{\mu \nu}^2 - T^2 = k^2 \left[ \frac{1}{2} (E^2 - F^2) - 2C(A - B) \right]. \]

Therefore,

\[ (T_{\mu \nu}^2 - \frac{1}{2} T^2)|_{k^2=\text{det}} > 0 \quad \text{and} \quad (T_{\mu \nu}^2 - T^2)|_{k^2=0} = 0. \]

References

[1] Whitt B 1984 Phys. Lett. B 145 176
[2] Bergshoeff E, Hohm O and Townsend P 2009 Phys. Rev. Lett. 102 201301
[3] Bergshoeff E, Hohm O and Townsend P 2009 Phys. Rev. D 79 124042
[4] Andringa R et al 2010 Class. Quantum Grav. 27 025010
[5] Bergshoeff E, Hohm O and Townsend P 2010 Ann. Phys. 325 1118
[6] Bergshoeff E et al 2011 Class. Quantum Grav. 28 015002
[7] Bergshoeff E et al 2010 Class. Quantum Grav. 27 235012
[8] Nakasone M and Oda I 2009 Prog. Theor. Phys. 121 1389
[9] Güllü I and Tekin B 2009 Phys. Rev. D 80 064033
[10] Ahmedov H and Aliiev A 2011 Phys. Rev. Lett. 106 021301
[11] Dalmazi D 2009 Phys. Rev. D 80 085008
[12] Dalmazi D and Mendonça E 2009 J. High Energy Phys. JHEP09(2009)011
[13] Helayël-Neto J, Hernaski C, Pereira-Dias B, Vargas-Paredes A and Vasquez-Otoya V 2010 Phys. Rev. D 82 064014
[14] Hernaski C, Vargas-Paredes A and Helayël-Neto J 2009 Phys. Rev. D 80 124012
[15] Deser S 2009 Phys. Rev. Lett. 103 101302
[16] Stelle K 1977 Phys. Rev. D 16 953
[17] Bergshoeff E, Hohm O and Townsend P 2010 J. Phys.: Conf. Ser. 229 012005
[18] Eckhardt D, Pestiaufa J and Fischbach E 2010 New Astron. 15 175
[19] ’t Hooft G 2008 arXiv:0708.3184 [hep-th]
[20] Staruszkiewicz A 1963 Acta Phys. Pol. 24 734
[21] Bergshoeff E, Hohm O and Townsend P 2010 Gravitons in flatland arXiv:1007.4561v2 [hep-th]
[22] Teyssandier P 1989 Class. Quantum Grav. 6 219
[23] Accioly A, Helayël-Neto J, Morais J, Scatena E and Turcati R 2011 Phys. Rev. D 83 104005
[24] Deser S, Jackiw R and Templeton S 1982 Phys. Rev. Lett. 48 975
[25] Deser S, Jackiw R and Templeton S 1982 Ann. Phys., NY 140 372
[26] Deser S, Jackiw R and Templeton S 1988 Ann. Phys., NY 185 406 (erratum)
[27] Accioly A 2003 Phys. Rev. D 67 127502
[28] Hernaski C, Pereira-Dias B and Vargas-Paredes A 2010 Phys. Lett. A 374 3410
[29] de Rham C, Gabadadze G, Pirtshkalava D, Tolley A and Yavin I 2011 J. High Energy Phys. JHEP06(2011)028
[30] Nicolis R, Rattazzi R and Trincherini E 2009 Phys. Rev. D 79 064036
[31] Dengiz S and Tekin B 2011 Phys. Rev. D 84 024033
[32] Jatckar D and Sinha A 2011 Phys. Rev. Lett. 106 171601
[33] Nieuwenhuizen P 1973 Nucl. Phys. B 60 478
[34] Antoniadis I and Tomboulis E 1986 Phys. Rev. D 33 2756