Chern-Simons pre-quantization over four-manifolds

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Abstract

We introduce a pre-symplectic structure on the space of connections in a $G$-principal bundle over a four-manifold and a Hamiltonian action on it of the group of gauge transformations that are trivial on the boundary. The moment map is given by the square of curvature so that the 0-level set is the space of flat connections. Thus the moduli space of flat connections is endowed with a pre-symplectic structure. In case when the four-manifold is null-cobordant we shall construct, on the moduli space of connections, as well as on that of flat connections, a hermitian line bundle with connection whose curvature is given by the pre-symplectic form. This is the Chern-Simons pre-quantum line bundle. The group of gauge transformations on the boundary of the base manifold acts on the moduli space of flat connections by an infinitesimally symplectic way. When the base manifold is a 4-dimensional disc we show that this action is lifted to the pre-quantum line bundle by its abelian extension. The geometric description of the latter is related to the 4-dimensional Wess-Zumino-Witten model.

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0 Introduction

The moduli space of flat connections on the trivial $SU(2)$ bundle over a surface $\Sigma$ is a compact, finite-dimensional symplectic space [2]. Ramadas, Singer and Weitsman [15] described the Chern-Simons pre-quantization of this moduli space. On the other hand, Donaldson proved that if the surface has a boundary then the moduli space of flat connections is a smooth infinite-dimensional symplectic manifold, and it has a Hamiltonian group action of the gauge transformations on the boundary [6]. In general the moduli space of the $\text{Lie}(G)$-valued flat connections on a connected closed manifold $M$ corresponds bijectively to the conjugate classes of the $G$-representations of the fundamental group $\pi_1(M)$.

In this paper we study the Chern-Simons pre-quantization of the moduli space of connections on a four-manifold $M$ generally with non-empty boundary.

Let $A(M)$ be the space of irreducible connections on $M$ and let $G_0(M)$ be the group of gauge transformations on $M$ that are trivial on the boundary $\partial M$. We shall prove in section 1 that $A(M)$ carries a pre-symplectic structure. The pre-symplectic form is given by

$$\omega_A(a, b) = \frac{1}{8\pi^3} \int_M \text{Tr}[ (ab - ba) F_A ] - \frac{1}{24\pi^3} \int_{\partial M} \text{Tr}[ (ab - ba) A ],$$

(0.1)

for $a, b \in T_A A$. The action of $G_0(M)$ becomes a Hamiltonian action with the moment map given by the square of curvature $F^2$. Hence the 0-level set of the moment map is the space of flat connections $A^\flat(M)$. The 2-form $\omega$ is highly degenerated so as to apply the Marsden-Weinstein symplectic reduction theorem [11]. But we know that $M^\flat(M) = A^\flat(M)/G_0(M)$ is a smooth manifold, and if $\partial M \neq \emptyset$, $M^\flat(M)$ becomes a pre-symplectic space with the pre-symplectic form

$$\omega^\flat_{[A]}(a, b) = -\frac{1}{24\pi^3} \int_{\partial M} \text{Tr}[ (ab - ba) A ],$$

(0.2)

for $[A] \in M^\flat(M)$.

By a pre-quantization of $M^\flat(M)$ we mean a hermitian line bundle $L^\flat(M)$ with connection over $M^\flat(M)$ whose curvature is given by $-i$ times the symplectic form on $M^\flat(M)$, [3] [15]. When we say Chern-Simons pre-quantization it means that the transition functions of the pre-quantum line bundle $L^\flat(M)$ come from the Chern-Simons form. It was pointed out in [15] that, when the base manifold $M$ is 2-dimensional,
the Chern-Simons form provides also the transition functions of a hermitian line bundle with connection over the total moduli space $B(M) = \mathcal{A}(M)/\mathcal{G}_0(M)$ and the pre-quantum line bundle is obtained by restricting it to the moduli space of flat connections $\mathcal{M}^\flat(M)$. In section 3 we extend these results to four-manifolds. We construct a hermitian line bundle with connection $L(M)$ over the moduli space $B(M) = \mathcal{A}(M)/\mathcal{G}_0(M)$ whose transition functions come from 5-dimensional Chern-Simons form.

For that we assume after section 3 that the four-manifold $M$ is a submanifold of a closed four-manifold $\hat{M}$ which is null-cobordant, that is, $\hat{M}$ is the boundary of a 5-dimensional manifold $N$. As a particular case when $M$ is without boundary; $M = \hat{M}$, we have the line bundle $L(\hat{M})$ over $B(\hat{M})$. The 5-dimensional Chern-Simons functional $CS(A)$ of $A \in \mathcal{A}(N)$ gives a section of the pullback line bundle $r^*L(\hat{M})$ by the boundary restriction map $r : B(N) \to B(\hat{M})$. The gradient vector field of this section is

$$\nabla s(A) = \frac{i}{8\pi^3} * F^2_A,$$

where $\nabla$ is the covariant differentiation associated to the pullback of the connection on $L(\hat{M})$.

In section 4 we investigate the pre-quantization of the moduli space $(\mathcal{M}^\flat(M), \omega^\flat)$. The pre-quantum line bundle $L^\flat(M)$ is by definition the restriction of $L(M)$ to $\mathcal{M}^\flat(M)$. For a proper submanifold $M \subset \hat{M}$ it is actually a line bundle with connection whose curvature is $-i\omega^\flat$. On the other hand the line bundle $L^\flat(\hat{M})$ admits a flat connection. $\mathcal{M}^\flat(\hat{M})$ is 0-dimensional if every flat connection is non-degenerate, that is, if the cohomology group $H^1_{\text{ad}}(\hat{M}, ad P) = 0$ for $A \in \mathcal{A}(\hat{M})$, [7]. In this case $L^\flat(\hat{M})$ is a finite dimensional vector space. When $\hat{M}$ is simply connected $\mathcal{M}^\flat(\hat{M})$ becomes one-point and $L^\flat(\hat{M}) \simeq \mathbb{C}$. The trivialization is given by the 4-dimensional reduction of Chern-Simons functional

$$CS_{\hat{M}}(A) = CS(A) = \frac{i}{240\pi^3} \int_N Tr[A^5],$$

where $A$ is a flat extension of $A \in \mathcal{A}(\hat{M})$ to $N$. It is well defined independently of the extension $A$. Here we assumed $\hat{M}$ to be simply connected for the sake of having the flat extension of $A \in \mathcal{M}^\flat(\hat{M})$. It will be important to give the definition of $CS_{\hat{M}}$ also for the case where $\hat{M}$ is not necessarily simply connected.

For a proper submanifold $M \subset \hat{M}$ we shall define the Chern-Simons functional $CS_M$ as a section over $\mathcal{A}(M)$ of the pull back line bun-
dle $p^* \mathcal{L}^0(M^c)$, where $M^c = \bar{M} \setminus \partial M$ and $p : \mathcal{A}^0(M) \longrightarrow \mathcal{M}^0(M^c)$ is a mapping which comes from the extension of flat connections across the boundary $\partial M$. $CS_M$ is a horizontal section of $p^* \mathcal{L}^0(M^c)$. This implies the fact that the mapping $p$ is a Lagrangean immersion.

The group of pointed gauge transformations $\mathcal{G}(M)$ acts on $\mathcal{A}^0(M)$ by infinitesimal symplectic automorphisms. So is the action of $\mathcal{G}(\partial M) = \mathcal{G}(M)/\mathcal{G}_0(M)$ on $\mathcal{M}^0(M)$. We shall discuss in section 5 the lift of this action to the pre-quantum line bundle $\mathcal{L}^0(M)$ for the case when $M$ is the 4-dimensional disc $D$ with boundary $S^3$. Let $\hat{\Omega}$ be the Mickelsson’s abelian extension of $\mathcal{G}(\partial D) \simeq \Omega^3_{\mathbb{C}}(G)$ by the group $\text{Map}(A(S^3), U(1))$, [12]. Then $\hat{\Omega}$ acts on $\mathcal{L}(D)$ equivariantly. Restricted to $\mathcal{L}^0(D)$ the action of $\hat{\Omega}$ is equivariant with the infinitesimal symplectic action of $\Omega^3_{\mathbb{C}}G$ on the base space $\mathcal{M}^0(D)$, and the reduction of $\mathcal{L}^0(D)$ by the action becomes $C^1$. As was discussed in [10, 12] the geometric description of the Mickelsson’s abelian extension is given by the four-dimensional Wess-Zumino-Witten model. These are the analogy of the central extension of the loop group acting on the pre-quantization of the moduli space of flat connections over a surface [11].

1 Preliminaries

1.1 Differential calculations on $\mathcal{A}$

Let $M$ be an oriented Riemannian four-manifold with boundary $\partial M$. Let $G = SU(n)$. The inner product on $G$ is given by $< \xi, \eta > = -\text{Tr}(\xi, \eta)$, $\xi, \eta \in \text{Lie}(G) = su(n)$. With this inner product the dual of $\text{Lie}(G)$ is identified with $\text{Lie}(G)$ itself. Let $\pi : P \longrightarrow M$ be a principal $G$-bundle over $M$ which is given by a system of transition functions in the Sobolev space $L^2_s$ for $s > 2$. We write $\mathcal{A} = \mathcal{A}(M)$ for the space of irreducible $L^2_{s-1}$ connections, which differ from a smooth connection by a $L^2_{s-3}$ section of $T^*_M \otimes \text{adP}$, hence the tangent space of $\mathcal{A}$ at $A \in \mathcal{A}$ is $T_A \mathcal{A} = \Omega^1_{s-1}(M, adP)$. The curvature of $A \in \mathcal{A}$ is

$$F_A = dA + \frac{1}{2}[A \wedge A] \in \Omega^2_{s-2}(M, adP).$$

$\Omega^3_{s-1}(M, adP)$, being the formal dual of $\Omega^1_{s-1}(M, adP) = T_A \mathcal{A}$, is identified with the space of 1-forms on $\mathcal{A}$.

Let $\mathcal{A}(\partial M)$ be the space of irreducible connections on $\partial M$. It is the $L^2_{s-\frac{3}{2}}$ connections that differ from a smooth connection on $\partial M$ by a $L^2_{s-\frac{3}{2}}$
section of $T^*_{\partial M} \otimes \text{ad}P$. The boundary restriction map $r : \mathcal{A}(M) \rightarrow \mathcal{A}(\partial M)$ is surjective.

Here are some differential calculations on $\mathcal{A}$ that we shall cite from [4, 8, 16].

The derivation of a smooth function $G = G(A)$ on $\mathcal{A}$ is defined by the functional variation of $A$:

$$(\partial_A G)a = \lim_{t \rightarrow 0} \frac{G(A + ta) - G(A)}{t}, \quad \text{for } a \in T_A \mathcal{A}.$$ \hfill (1.1)

We have, for example,

$$(\partial_A A)a = a, \quad (\partial_A F_A)a = d_A a.$$  

The second follows from the formula

$$F_{A+a} = F_A + d_A a + a \wedge a.$$  

Similarly the derivation of a vector field on $\mathcal{A}$ or a 1-form on $\mathcal{A}$ is defined as that of a smooth function of $A \in \mathcal{A}$ valued in $\Omega^1(M, \text{ad}P)$, respectively, in $\Omega^3(M, \text{ad}P)$. We have, for a vector field $b$ and a 1-form $\beta$,

$$(\partial_A < \beta, b >)a = < \beta, (\partial_A b)a > + < (\partial_A \beta)a, b >.$$ \hfill (1.2)

The Lie bracket for vector fields on $\mathcal{A}$ is seen to have the expression

$$[a, b] = (\partial_A b)a - (\partial_A a)b.$$ \hfill (1.3)

Let $\tilde{d}$ be the exterior derivative on $\mathcal{A}$. For a function on $\mathcal{A}$, $(\tilde{d}G)_A a = (\partial_A G)a$. From (1.2) and (1.3) we have the following formula for the exterior derivative of a 1-form on $\mathcal{A}$:

$$(\tilde{d} \theta)_A (a, b) = (\partial_A < \theta, b >)a - (\partial_A < \theta, a >)b - < \theta, [a, b] >$$

$$= < (\partial_A \theta)a, b > - < (\partial_A \theta)b, a >.$$ \hfill (1.4)

Likewise, if $\varphi$ is a 2-form on $\mathcal{A}$, then

$$(\tilde{d} \varphi)_A (a, b, c) = (\partial_A \varphi(b, c))a + (\partial_A \varphi(c, a))b + (\partial_A \varphi(a, b))c.$$ \hfill (1.5)
1.2 Moduli space of connections

We write the group of $L^2_s$ gauge transformations by $\mathcal{G}' = \mathcal{G}'(M)$:

$$\mathcal{G}'(M) = \Omega^0_s(M, AdP).$$

(1.6)

$\mathcal{G}'$ acts on $\mathcal{A}$ by

$$g \cdot A = g^{-1}dg + g^{-1}Ag = A + g^{-1}dAg.$$  

(1.7)

By Sobolev lemma one sees that $\mathcal{G}'$ is a Banach Lie group and its action is a smooth map of Banach manifolds.

The group of $L^2_s$ gauge transformations on the boundary $\partial M$ is denoted by $\mathcal{G}'(\partial M)$. We have the restriction map to the boundary:

$$r: \mathcal{G}'(M) \rightarrow \mathcal{G}'(\partial M).$$

Let $\mathcal{G}_0 = \mathcal{G}_0(M)$ be the kernel of the restriction map. It is the group of gauge transformations that are identity on the boundary. $\mathcal{G}_0$ acts freely on $\mathcal{A}$ and $\mathcal{A}/\mathcal{G}_0$ is therefore a smooth infinite dimensional manifold, while the action of $\mathcal{G}'$ is not free.

In the following we choose a fixed point $p_0 \in M$ on the boundary $\partial M$ and deal with the group of gauge transformations that are identity at $p_0$:

$$\mathcal{G} = \mathcal{G}(M) = \{g \in \mathcal{G}'(M); g(p_0) = 1\}.$$

If $\partial M = \phi$, $p_0$ may be any point of $M$. $\mathcal{G}$ act freely on $\mathcal{A}$ and the orbit space $\mathcal{A}/\mathcal{G}$ is a smooth infinite dimensional manifold. We have $\text{Lie}(\mathcal{G}) = \Omega^0_s(M, adP)$.

Correspondingly we have the group $\mathcal{G}(\partial M) = \{g \in \mathcal{G}'(\partial M); g(p_0) = 1\}$, and the restriction map $r: \mathcal{G}(M) \rightarrow \mathcal{G}(\partial M)$ with the kernel $\mathcal{G}_0$. We have

$$\text{Lie}(\mathcal{G}_0) = \{\xi \in \text{Lie}(\mathcal{G}); \xi|\partial M = 0\}.$$  

(1.8)

The derivative of the action of $\mathcal{G}$ at $A \in \mathcal{A}$ is

$$d_A = d + [A \wedge ]: \Omega^0_s(M, adP) \rightarrow \Omega^1_{s-1}(M, adP).$$  

(1.9)

The fundamental vector field on $\mathcal{A}$ corresponding to $\xi \in \text{Lie}(\mathcal{G})$ is given by

$$\xi_A(A) = \frac{d}{dt}|_{t=0} (\exp t\xi) \cdot A = d_A \xi,$$
and the tangent space to the orbit at $A \in \mathcal{A}$ is
\[ T_A(G \cdot A) = \{ d_A \xi ; \xi \in \Omega^0_\mathcal{A}(M, adP) \}. \]

We have two moduli spaces of irreducible connections;
\[ \mathcal{B}(M) = \mathcal{A}/G_0, \quad \mathcal{C}(M) = \mathcal{A}/\mathcal{G}. \]

They are smooth manifold modelled locally on the balls in the Hilbert spaces $\ker d^*_A$, and $\ker d^*_A \cap \ker(\ast|\partial M)$ respectively, in $\Omega^2_{s-1}(M, adP)$.

$\mathcal{C}(M)$ is a $\mathcal{G}/G_0$-principal bundle over $\mathcal{B}(M)$. $\mathcal{C}(M)$ coincides with $\mathcal{B}(M)$ if $M$ has no boundary.

Let $\mathcal{A} = \mathcal{A}(M)$, $\mathcal{G} = \mathcal{G}(M)$, $\mathcal{B} = \mathcal{B}(M)$ and $\mathcal{C} = \mathcal{C}(M)$ be as above. We shall investigate the horizontal subspaces of the fibrations $\mathcal{A} \longrightarrow \mathcal{B}$ and $\mathcal{A} \longrightarrow \mathcal{C}$.

The Stokes formula is stated as follows:
\[ \int_{\partial M} < f, *u > = \int_M < d_A f, *u > - \int_M < f, *d_A^* u >, \]
for $f \in \Omega^0_\mathcal{A}(M, adP)$, $u \in \Omega^1_{s-1}(M, adP)$. If $M$ is a compact manifold without boundary we have the following decomposition:
\[ T_A \mathcal{A} = \{ d_A \xi; \xi \in \operatorname{Lie}(\mathcal{G}) \} \oplus H^0_A, \]
where
\[ H^0_A = \{ a \in \Omega^1_{s-1}(M, adP); d_A^* a = 0 \}. \]

In this case we have
\[ T_A^* \mathcal{B} \simeq H^0_A. \]

When $M$ has the boundary, $d_A \operatorname{Lie}(\mathcal{G}_0)$ is orthogonal to $\ker d^*_A = \{ a \in \Omega^1_{s-1}(M, adP); d_A^* a = 0 \}$, and $d_A \operatorname{Lie}(\mathcal{G})$ is orthogonal to $\{ a \in \ker d_A^*, *a|\partial M = 0 \}$.

Let $\Delta_A$ be the covariant Laplacian defined as the closed extension of $d_A^* d_A$ with the domain of definition $\mathcal{D}_{\Delta_A} = \{ u \in \Omega^0_\mathcal{A}(M, adP); u|\partial M = 0 \}$. Since $A \in \mathcal{A}$ is irreducible $\Delta_A : \mathcal{D}_{\Delta_A} \longrightarrow \Omega^0_{s-2}(M, adP)$ is an isomorphism. Let $G_A = (\Delta_A)^{-1}$ be the Green operator of the Dirichlet problem:
\[
\begin{align*}
\Delta_A u &= f \\
\quad u|\partial M &= 0
\end{align*}
\]
Proposition 1.1. Let $A \in A$.

1. We have the following orthogonal decomposition:

$$T_A A = \{d_A \xi; \xi \in \text{Lie}(G_0)\} \oplus H^0_A,$$  \hspace{1cm} (1.14)

where

$$H^0_A = \{a \in \Omega^1_{s-1}(M, adP); \, d^*_A a = 0\}.$$

2. The $G_0$-principal bundle $\pi : A \longrightarrow B$ has a natural connection defined by the horizontal subspace $H^0_A$, which is given by the connection 1-form $\gamma^0_A = G_A d^*_A$.

3. The curvature form $\mathcal{F}^0$ of the connection 1-form $\gamma^0$ is given by

$$\mathcal{F}^0_A(a, b) = G_A([a, *b]) \quad \text{for} \ a, b \in H^0_A.$$

Corollary 1.2.

$$\mathcal{F}^0_A(a, d_A \xi) = 0 \quad \text{for} \ \xi \in \text{Lie}(G_0).$$  \hspace{1cm} (1.15)

Now we proceed to the fibration $A \longrightarrow C = A/G$.

For a 1-form $v$, let $g = K_A v$ denote the solution of the following boundary value problem:

\[
\begin{cases}
\Delta_A g = 0 \\
*d_A g|\partial M = *v|\partial M.
\end{cases}
\]

Proposition 1.3. Let $A \in A$.

1. We have the orthogonal decomposition:

$$T_A A = \{d_A \xi; \xi \in \text{Lie}(G)\} \oplus H_A,$$  \hspace{1cm} (1.16)

where

$$H_A = \{a \in \Omega^1_{s-1}(M, adP); \, d^*_A a = 0, \text{ and } *a|\partial M = 0\}.$$

2. The $G$-principal bundle $\pi : A \longrightarrow C$ has a natural connection defined by the horizontal subspace $H_A$. 

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Proof
Let \( a \in \Omega^1(M, adP) \) and \( a = d_A \xi + b \) be the decomposition of (1.14), then \( \xi = G_A d_A^* a \) and \( b \in \Omega^1_{s-1}(M, adP), \) \( d_A^* b = 0. \) Put \( \eta = K_A b. \) Then we have the orthogonal decomposition

\[
a = d_A (\xi + \eta) + c,
\]
with \( c \in H_A \) and \( \xi + \eta \in \text{Lie}(G). \) If we write

\[
\gamma_A = \gamma_A^0 + K_A (I - d_A \gamma_A^0), \tag{1.17}
\]
where \( I \) is the identity transformation on \( T_A A, \) then \( \gamma_A \) is a \( \text{Lie}(G) \)–valued 1-form which vanishes on \( H_A \) and \( \gamma_A d_A \xi = \xi, \) that is, \( \gamma_A \) is the connection 1-form.

Let \( g = N_A f \) be the solution of Neuman problem:

\[
\begin{align*}
\Delta^{(n)}_{A} g &= f \\
*d_{A} g|_{\partial M} &= 0 \quad \text{on} \ \partial M.
\end{align*}
\]

Where \( \Delta^{(n)}_{A} \) is the closed extension of \( d_A^* d_A \) with the domain of definition \( D_{\Delta^{(n)}_{A}} = \{ u \in \Omega^0(M, adP) ; * d_A u|_{\partial M} = 0 \}. \)

**Corollary 1.4.** The curvature form of \( \gamma_A \) is given by

\[
\mathcal{F}_{A}(a, b) = N_A (* [a, * b]) \quad \text{for} \ a, b \in H_A. \tag{1.18}
\]

### 1.3 Moduli space of flat connections

The space of flat connections is

\[
\mathcal{A}^\flat(M) = \{ A \in \mathcal{A}(M) ; F_A = 0 \}, \tag{1.19}
\]
which we shall often abbreviate to \( \mathcal{A}^\flat. \) The tangent space of \( \mathcal{A}^\flat \) is given by

\[
T_A \mathcal{A}^\flat = \{ a \in \Omega^1_{s-1}(M, adP) ; d_A a = 0 \}. \tag{1.20}
\]

We shall suppress the Sobolev indices in the following.

The moduli space of flat connections is by definition

\[
\mathcal{M}^\flat = \mathcal{A}^\flat / G_0.
\]
When there is a doubt about which manifold is involved, we shall write \( \mathcal{M}^\flat(M) \) for the orbit space \( \mathcal{M}^\flat = \mathcal{A}^\flat(M)/\mathcal{G}_0(M) \).

We know that \( \mathcal{M}^\flat \) is a smooth infinite-dimensional manifold. In fact, the coordinate mappings are described by the implicit function theorem [8]: For \( A \in \mathcal{A}^\flat \) there is a slice for the \( \mathcal{G}_0 \)-action on \( \mathcal{A}^\flat \) given by the Coulomb gauge condition:

\[
V_A = \{ a \in \Omega^1(M, adP); |a| < \epsilon, d_Aa + a \wedge a = 0, d_A^*a = 0 \}. \tag{1.21}
\]

Let

\[
H^1_A = \{ \Omega^1(M, adP); d_Aa = 0, d_A^*a = 0 \}.
\]

The Kuranishi map is defined by

\[
K_A : \Omega^1(M, adP) \ni \alpha \mapsto K_A(\alpha) = \alpha + d_A^*G_A(\alpha \wedge \alpha) \in \Omega^1(M, adP).
\]

Since the differential of \( K_A \) at \( \alpha = 0 \) becomes the identity transformation on \( \Omega^1(M, adP) \), the implicit function theorem in Banach space yields that \( K_A \) gives an isomorphism on a small neighborhood of 0. Thus we see that the slice \( V_A \) is a neighborhood of \([A]\) that is homeomorphic to the following subset \( H^1_A \):

\[
\{ \beta \in H^1_A; |\beta| < \epsilon, \kappa_A(\beta) = 0 \},
\]

where

\[
\kappa_A(\beta) = (I - G_A \Delta_A)(\alpha \wedge \alpha), \quad \alpha = K_A^{-1}\beta.
\]

Now the moduli space of flat connections modulo the total gauge transformation group \( \mathcal{G} \) is

\[
\mathcal{N}^\flat = \mathcal{A}^\flat/\mathcal{G}.
\tag{1.22}
\]

A slice in a neighborhood of \( A \in \mathcal{A}^\flat \) in this case is

\[
W_A = \{ a \in \Omega^1(M, adP); |a| < \epsilon, d_Aa + a \wedge a = 0, d_A^*a = 0, \text{and } *a|\partial M = 0 \}. \tag{1.23}
\]

The Kuranishi map is defined by

\[
L_A : \Omega^1(M, adP) \ni \alpha \mapsto L_A(\alpha) = \alpha + d_A^*N_A(\alpha \wedge \alpha) \in \Omega^1(M, adP).
\]

The same argument as above yields that there is a neighborhood of \([A]\) in \( \mathcal{N}^\flat \) that is homeomorphic to

\[
\{ \beta \in H^1_A; |\beta| < \epsilon, \lambda_A(\beta) = 0 \},
\]
where
\[ \lambda_A(\beta) = (I - N_A\Delta_A)(\alpha \wedge \alpha), \quad \alpha = L_A^{-1} \beta. \]

The dimension of \( N^\circ \) is finite.

\section{Pre-symplectic structure on \( A \)}

\subsection{Pre-symplectic structure on \( A \) and the action of \( G_0(M) \)}

For each \( A \in A \) we define a skew-symmetric bilinear form on \( T_AA \) by:

\[ \omega_A(a, b) = \omega^0_A(a, b) + \omega'_A(a, b), \quad (2.1) \]

\[ \omega^0_A(a, b) = \frac{1}{8\pi^3} \int_M Tr[(a \wedge b - b \wedge a) \wedge F_A], \quad (2.2) \]

\[ \omega'_A(a, b) = -\frac{1}{24\pi^3} \int_{\partial M} Tr[(a \wedge b - b \wedge a) \wedge A], \quad (2.3) \]

for \( a, b \in T_AA \).

**Proposition 2.1.** \( G_0 \) acts symplectically on \( A \),

\[ g^*\omega = \omega. \]

Note that the action of \( G \) is not even infinitesimally symplectic if \( \partial M \neq \emptyset \).

**Theorem 2.2.** \( (A, \omega) \) is a pre-symplectic space, that is, \( \omega \) is a closed 2-form on \( A \).

**Proof**

In the following we shall abbreviate \( ab \) for the exterior product \( a \wedge b \). Differentiating the 2-form \( \omega^0 \), we have

\[ (d\omega^0)_A(a, b, c) = \partial_A(\omega^0(a, b))(c) + \partial_A(\omega^0(b, c))(a) + \partial_A(\omega^0(c, a))(b), \]

for \( a, b, c \in T_AA \). From the definition we have

\[ \partial_A(\omega^0(a, b))(c) = \frac{1}{8\pi^3} \int_M Tr[(ab - ba)d_A c], \]
hence
\[(\tilde{\omega}^0)_A(a, b, c) = \frac{1}{8\pi^3} \int_M \text{Tr} [(ab - ba)d_A c + (bc - cb)d_A a + (ca - ac)d_A b].\]

Since
\[d(\text{Tr}(ab - ba)c) = \text{Tr} [(ab - ba)d_A c + (bc - cb)d_A a + (ca - ac)d_A b],\]
we have
\[(\tilde{\omega}^0)_A(a, b, c) = \frac{1}{8\pi^3} \int_M d\text{Tr} [(ab - ba)c] = \frac{1}{8\pi^3} \int_{\partial M} \text{Tr}[(ab - ba)c].\]

On the other hand we have
\[(\tilde{\omega}')_A(a, b, c) = 3\partial A(\omega'(a, b))(c) = -\frac{1}{8\pi^3} \int_{\partial M} \text{Tr}[(ab - ba)c].\]

Therefore \(\tilde{\omega} = 0\).

Thus \((A, \omega)\) is a pre-symplectic manifold. \(\square\)

\(B(M)\) is a smooth manifold endowed with the pre-symplectic structure coming from \(\omega\).

Remark 2.1. For \(M\) without boundary the pre-symplectic form \(\omega^0\) and the Hamiltonian action of the group \(G = G_0\) in the next theorem were introduced by Bao and Nair [4]. More generally they gave the pre-symplectic form on \(n\)-dimensional manifolds.

**Theorem 2.3.** The action of \(G_0\) on \(A\) is a Hamiltonian action and the corresponding moment map is given by
\[
\Phi : A \longrightarrow (\text{Lie } G_0)^* = \Omega^1(M, adP) : A \longrightarrow F_A^2.
\]

\[
\langle \Phi(A), \xi \rangle = \Phi^\xi(A) = \frac{1}{8\pi^3} \int_M \text{Tr}(F_A^2 \xi).
\]

**Proof**

The equivariance of \(\Phi : A \longrightarrow (\text{Lie } G_0)^*\) with respect to the \(G_0\)-action on \(A\) and the coadjoint action on \((\text{Lie } G_0)^*\) is evident.

We have
\[
(\partial \Phi^\xi)_A = \frac{1}{8\pi^3} \int_M \text{Tr}[(d_A a \wedge F_A + F_A \wedge d_A a) \xi],
\]
and

\[
(\partial \Phi^\xi)_A a - \omega^0_A(a, d_A \xi) = \frac{1}{8\pi^3} \int_M Tr[(d_A a \wedge F_A + F_A \wedge d_A a)\xi - (F_A a + aF_A)d_A \xi]
\]

\[
= \frac{1}{8\pi^3} \int_M dTr[(aF_A + F_A a)\xi] = \frac{1}{8\pi^3} \int_{\partial M} Tr[(aF_A + F_A a)\xi]
\]

\[
= 0,
\]

since \(\xi = 0\) on \(\partial M\). The following equation is valid on \(\partial M\):

\[
dTr[(Aa - aA)\xi] = Tr[(F_A a + aF_A - Ad_A a - d_A aA + A^2 a + aA^2)\xi]
\]

\[
+ Tr[(Aa - aA)d_A \xi]
\]

\[
= Tr[(ad_A \xi - d_A \xi a)A].
\]

Hence we have

\[
\omega'(a, d_A \xi) = -\frac{1}{24\pi^3} \int_{\partial M} dTr[(Aa - aA)\xi] = 0.
\]

Therefore

\[
(\partial \Phi^\xi)_A a = \omega_A(a, d_A \xi).
\]

The moment map for the action of \(G_0\) on \(A\) is given by

\[
\Phi : A \rightarrow F^2_A
\]

\[
\square
\]

**Proposition 2.4.** We have

\[
\Phi[\xi, \eta](A) = \omega^0_A(d_A \xi, d_A \eta) = \omega_A(d_A \xi, d_A \eta).
\]

(2.5)

**Proof**

For \(\xi, \eta \in \text{Lie } G_0\), we have

\[
dTr[(\xi d_A \eta - d_A \xi \eta)F_A] = 2Tr[d_A \xi d_A \eta F_A - \xi \eta F^2_A + \xi F_A \eta F_A].
\]

This equation and the same one with \(\xi\) and \(\eta\) reversed yield

\[
\int_M Tr(F^2_A[\xi, \eta]) = \int_M Tr[(d_A \xi d_A \eta - d_A \eta d_A \xi)F_A],
\]

and we have the assertion. \(\square\)
The proposition says that the map \( \Phi : \text{Lie} \mathcal{G}_0 \rightarrow C^\infty(\mathcal{A}) \) is a Lie algebra homomorphism if \( C^\infty(\mathcal{A}) \) is endowed with the Poisson bracket coming from \( \omega \). Since \( \omega \) may be degenerate the Poisson structure may not be endowed to \( C^\infty(\mathcal{A}) \). The discussion of Poisson algebra associated to a degenerate two-form is found in [9] and their theory fits well to our case.

The symplectic quotient of \( \mathcal{A} \) by \( \mathcal{G}_0 \) is the moduli space of flat connections \( \mathcal{M}^\flat = \Phi^{-1}(0)/\mathcal{G}_0 = \mathcal{A}^\flat/\mathcal{G}_0 \).

**Theorem 2.5.** Suppose \( \partial \mathcal{M} \neq \emptyset \). Then \( \mathcal{M}^\flat \) is a smooth manifold endowed with a pre-symplectic structure. The pre-symplectic form on \( \mathcal{M}^\flat \) is given by

\[
\omega^\flat_{[A]}(a,b) = \omega'_A(a,b)
\]

for \( a \in \mathcal{M}^\flat \) and \( a, b \in T_a \mathcal{M}^\flat \).

**Proof**

As was explained in 1.3 we know that \( \mathcal{M}^\flat \) is an infinite dimensional smooth manifold. The symplectic structure on \( \mathcal{M}^\flat \) is given by

\[
\omega^\flat_{[A]}(a,b) = \omega'_A(a,b).
\]

Here \( [A] \in \mathcal{M}^\flat \) denotes the \( \mathcal{G}_0 \)-orbit of \( A \in \mathcal{A}^\flat \), and as for a tangent vector \( a \in T_{[A]} \mathcal{M}^\flat \) we take the representative tangent vector to the slice; \( a \in H_A \cap T_A \mathcal{A}^\flat \). \( \omega^\flat \) is well defined because we have \( g \cdot A = A \) on \( \partial \mathcal{M} \) for \( g \in \mathcal{G}_0 \), and \( \omega'_A(a,d_A \xi) = 0 \) for \( \xi \in \text{Lie} \mathcal{G}_0 \) and \( a \in T_A \mathcal{A}^\flat \).

**Example 1.**

For \( \mathcal{M} = S^4 \), the moduli space of flat connections

\[
\mathcal{M}^\flat(S^4) = \mathcal{A}^\flat(S^4)/\mathcal{G}(S^4).
\]

is one-point.

In fact, let \( p_0 \in S^4 \) and let \( A \in \mathcal{A}^\flat(S^4) \). Let \( T^A_\gamma(x) \) denote the parallel transformation by \( A \) along the curve \( \gamma \) joining \( p_0 \) and \( x \). We put \( f_A(x) = T^A_\gamma(x)1 \in \mathcal{G} \). It is independent of the choice of curve \( \gamma \) joining \( p_0 \) and \( x \). Then \( f_A \in \mathcal{G}(S^4) \), and by the definition \( A = df_A \cdot f_A^{-1} \).

In general \( \mathcal{M}^\flat(\mathcal{M}) \) is one-point for an oriented, connected and simply connected compact four-manifold \( \mathcal{M} \).

**Example 2.**
For a disc $D^4 = \{ x \in \mathbb{R}^4; |x| \leq 1 \}$ with boundary $S^3$, we have

$$\mathcal{M}^\flat(D^4) \simeq \Omega^3_0 G.$$ 

Where

$$\Omega^3_0 G = \{ f \in \text{Map}(S^3, G); f(p_0) = 1 \},$$

and $\Omega^3_0 G$ is the connected component of the identity. To prove it, first we note that

$$\mathcal{G}(D^4) \simeq D^4 G = \{ f \in \text{Map}(D^4, G); f(p_0) = 1 \},$$

and

$$\mathcal{G}_0(D^4) \simeq D^4_0 G = \{ f \in D^4 G; f|S^3 = 1 \}.$$ 

Hence $\mathcal{G}/\mathcal{G}_0 \simeq \Omega^3_0 G$. As before we put, for $A \in \mathcal{A}^\flat(D^4)$, $f_A(x) = T^A_\gamma(x)1, x \in D^4$. We have a well defined bijective map from $\mathcal{A}^\flat(D^4)$ to $D^4 G$. In particular, $f_A = g$ for $A = dg g^{-1}$ with $g \in \mathcal{G}(D^4)$ . It holds also that $f_{g \cdot A}(x) = f_A(x)g(x)$ for $g \in \mathcal{G}(D^4)$. Hence we have the isomorphism

$$\mathcal{M}^\flat(D^4) = \mathcal{A}^\flat / \mathcal{G}_0 \simeq D^4 G / D^4_0 G \simeq \Omega^3_0 G.$$ 

### 2.2 The action of $\mathcal{G}(M)$

By the action of the group of gauge transformations $\mathcal{G}$ on $\mathcal{A}^\flat$ we have the orbit space $N^\flat = \mathcal{A}^\flat / \mathcal{G}$. Then we have a fibration $\mathcal{M}^\flat \longrightarrow N^\flat$ with the fiber $\mathcal{G}/\mathcal{G}_0$. We note the fact that any vector which is tangent to the $\mathcal{G}$-orbit through $A \in \mathcal{A}^\flat$ is in $T_A \mathcal{A}^\flat$. Propositions 1.3 yields the following proposition.

**Proposition 2.6.** Let $A \in \mathcal{M}^\flat$.

1. We have the following decomposition

$$T_A \mathcal{M}^\flat = \{d_A \xi; \xi \in \text{Lie}(\mathcal{G}(\partial M))\} \oplus H^\flat_A,$$

where

$$H^\flat_A = \{a \in \Omega^1(M, \text{ad}P); d_A a = d^*_A a = 0, \text{ and } a|\partial M = 0 \}.$$ 

2. The $\mathcal{G}/\mathcal{G}_0$-principal bundle $\pi : \mathcal{M}^\flat \longrightarrow N^\flat$ has a natural connection defined by the horizontal subspace $H^\flat_A$. The connection form is given by $K_A(I - d_A \gamma^0_A)$.
The action of $G$ on $A$ is far from symplectic. But on $A^\oplus$ it is infinitesimally symplectic. In fact, we have $d_A \xi \in T_A A^\oplus$ for $\xi \in \text{Lie}(G)$ and $A \in A^\oplus$. If we denote by $L_\xi$ the Lie derivative by the fundamental vector field corresponding to $\xi \in \text{Lie}(G)$ we have:

$$(L_\xi \omega)_A(a, b) = (\tilde{d} i_{d_A \xi} \omega)_A(a, b) = \partial_A (i_{d_A \xi} \omega_A(b))(a) - \partial_A (i_{d_A \xi} \omega_A(a))(b)$$

$$= -\frac{1}{24\pi^3} \int_{\partial M} \text{Tr}[(b d_A \xi - d_A \xi b)a] + \frac{1}{24\pi^3} \int_{\partial M} \text{Tr}[(a d_A \xi - d_A \xi a)b]$$

$$= -\frac{1}{12\pi^3} \int_{\partial M} \text{Tr}[(ab - ba)d_A \xi] = -\frac{1}{12\pi^3} \int_{\partial M} d\text{Tr}[(ab - ba)A]$$

$$= 0,$$

for $A \in A^\oplus$ and for $a, b \in T_A A^\oplus$.

$G_0$ being a normal subgroup of $G$ the action of the quotient group $G/G_0$ on $(M^\oplus, \omega^\oplus)$ is also infinitesimally symplectic.

**Example** The argument in Example 2 shows also that

$$N^3(D^4) = 1 \text{ point.}$$

The same argument by using parallel transformations along the curves in $S^3$ yields that

$$\Omega^3 G \simeq A^\oplus(S^3).$$

So we have an injective mapping $M^\oplus(D^4) \rightarrow A^\oplus(S^3)$. It corresponds to the embedding $\Omega^3_0 G \rightarrow \Omega^3 G$.

### 3 Line bundle with connection on $B(M)$

#### 3.1 Descent equations

Let $P$ be a $G$-principal bundle over a manifold. The group of gauge transformations on $P$ is denoted by $G$.

Let $\Omega^q$ be the differential $q$-forms on $P$ and let $V^q$ be the vector space of polynomials $\Phi(A)$ of $A \in A$ and its curvature $F_A$ that take values in $\Omega^q$. The group of gauge transformations $G$ acts on $V^q$ by $(g \cdot \Phi)(A) = \Phi(g^{-1} \cdot A)$. We shall investigate the double complex

$$C^{p, q} = C^p(G, V^{q+3}),$$

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that is doubly graded by the chain degree $p$ and the differential form degree $q$. The coboundary operator $\delta : C^p \rightarrow C^{p+1}$ is given by

$$(\delta c^p)(g_1, g_2, \cdots, g_{p+1}) = g_1 \cdot c^p(g_2, \cdots, g_{p+1}) + (-1)^{p+1} c^p(g_1, g_2, \cdots, g_p)$$

$$+ \sum_{k=1}^{p} (-1)^k c^p(g_1, \cdots, g_k, g_{k+1}, g_{k+2}, \cdots, g_{p+1}).$$

The following Proposition is a precise version of the Zumino’s descent equation. Though stated for $C^p(G, V_q^{q+3})$ it holds for general $C^p(G, V_q^{q+n})$. The first equation is nothing but the Zumino’s equation \cite{18}. The author learned the second equation with its transparent proof from Y. Terashima \cite{17}. For $C^p(G, V_q^{q+3})$ stated here the calculations were already appeared in \cite{12, 13}

**Proposition 3.1.** Let $c^{0,3} \in C^{0,3}$ be defined by

$$c^{0,3}(A) = Tr F_A^3, \quad A \in A.$$

Then there is a sequence of cochains $c^{p,q} \in C^{p,q}$, $0 \leq p, q \leq 3$, that satisfies the following relations:

$$dc^{p,3-p} + (-1)^p \delta c^{p-1,3-p+1} = 0 \quad (3.1)$$

$$dc^{p,2-p} + (-1)^{p-1} \delta c^{p-1,3-p} = c^{p,3-p} \quad (3.2)$$

$$c^{p,q} = 0, \quad \text{if } p + q \neq 2, 3$$

Each term is given by the following forms:

$$c^{0,2}(A) = Tr (AF^2 - \frac{1}{2} A^3 F + \frac{1}{10} A^5),$$

$$c^{1,2}(g) = \frac{1}{10} Tr(dg \cdot g^{-1})^5,$$

$$c^{1,1}(g; A) = Tr[-\frac{1}{2} V(AF + FA - A^3) + \frac{1}{4}(VA)^2 + \frac{1}{2} V^2 A],$$

where $V = dg g^{-1},$

$$c^{2,1}(g_1, g_2) = c^{1,1}(g_2; g_1^{-1} dg_1),$$

$$c^{2,0}(g_1, g_2; A) = -Tr[\frac{1}{2}(dg_2 g_1^{-1})(g_1^{-1} dg_1)(g_1^{-1} A g_1) - \frac{1}{2}(dg_2 g_1^{-1})(g_1^{-1} A g_1)(g_1^{-1} dg_1)].$$
Remark 3.1. We write here the analogy for $n = 2$ that is more familiar. In this case we consider the double complex $C^p,q = C^p(G, V^{q+2})$. Put $c^{0,2} = Tr F^2 \in C^{0,2}$, then there is a sequence of cochains $c^{p,q} \in C^{p,q}$, $0 \leq p, q \leq 2$, that satisfies the following relations:

\[
\begin{align*}
d c^{0,1} &= c^{0,2}, \\
d c^{1,0} + \delta c^{0,1} &= c^{1,1}, \\
d c^{2,0} - \delta c^{1,1} &= 0,
\end{align*}
\]

$c^{p,q} = 0$, if $p + q \neq 2, 1$.

Each terms are given as follows.

\[
\begin{align*}
c^{0,1}(A) &= Tr(AF - \frac{1}{3} A^3), \\
c^{1,0}(A; g) &= Tr(dg^{-1} A), \\
c^{1,1}(g) &= \frac{1}{3} Tr(dg^{-1})^3, \\
c^{2,0}(g_1, g_2) &= c^{1,0}(g_1^{-1} dg; g_2).
\end{align*}
\]

We note that the relation $dc^{2,0} - \delta c^{1,1} = 0$ represents the Polyakov-Wiegmann formula appeared in [14], and that the relation $\delta c^{2,0} = 0$ is the cocycle condition for the central extension of the loop group $LG$.

### 3.2 4-dimensional Polyakov-Wiegmann formula

Let $M$ be a connected compact four-manifold that is the boundary of an oriented 5-dimensional manifold $N$; $\partial N = M$. Let $G$ denote the Lie group $SU(n)$ with $n \geq 3$. Then $\pi_4(G) = 1$ and $\pi_5(G) \simeq \mathbb{Z}$, and $g \in G(M)$ has a smooth extension $g \in G(N)$, $g|M = g$, that is unique modulo $\mathbb{Z}$. Let $P$ be a $G$-principal bundle on $N$ and $P$ be its restriction to $M$.

The Chern-Simons form on the principal bundle $P$ is by definition.

\[
c^{0,2}(A) = Tr(AF^2 - \frac{1}{2} A^3 F + \frac{1}{10} A^5), \quad A \in A(N). \tag{3.3}
\]

We have

\[
d c^{0,2}(A) = Tr(F^3), \tag{3.4}
\]

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where $\mathbf{F} = \mathbf{F}_A$. The variation of the Chern-Simons form along the $G(N)$-orbit is described by the equation

$$
c^{0,2}(g \cdot A) - c^{0,2}(A) = d c^{1,1}(g, A) + c^{1,2}(g), \quad g \in G(N). \tag{3.5}
$$

Where, as in Proposition 3.1,

$$
c^{1,1}(g, A) = Tr[-\frac{1}{2}V(AF + FA - A^3) + \frac{1}{4}(VA)^2 + \frac{1}{2}V^3A],
$$

with $V = dg g^{-1}$, and

$$
c^{1,2}(g) = \frac{1}{10} Tr(g^{-1}dg)^5.
$$

The following four-dimensional version of Polyakov-Wiegmann formula is a relation in the Lie group cohomology $C^*(G(M), \mathbb{R})$.

For $A \in A(M)$ and $g \in G(M)$, we put

$$
\Gamma(g; A) = \frac{i}{24\pi^3} \int_M c^{1,1}(g; A) + C_5(g), \tag{3.6}
$$

$$
C_5(g) = \frac{i}{24\pi^3} \int_N c^{1,2}(g)
$$

$$
= \frac{i}{240\pi^3} \int_N Tr(dg \cdot g^{-1})^5. \tag{3.7}
$$

$C_5(g)$ may depend on the extension $g$ of $g$ to $N$, but since $\pi_5(G) = \mathbb{Z}$ the two choices give the same value for $C_5$ modulo $\mathbb{Z}$, and $\exp 2\pi i C_5(g)$ is independent of the extension.

**Lemma 3.2** (Polyakov-Wiegmann). \[10, 12, 14\] For $f, g \in G(M)$ we have

$$
C_5(fg) = C_5(f) + C_5(g) + \gamma(f, g) \mod \mathbb{Z}, \tag{3.8}
$$

where

$$
\gamma(f, g) = \frac{i}{24\pi^3} \int_M c^{2,1}(f, g)
$$

$$
= \frac{i}{48\pi^3} \int_M Tr[(dg^{-1})(f^{-1}df)^3 + \frac{1}{2}(dgg^{-1}f^{-1}df)^2 + (dgg^{-1})^3(f^{-1}df)]. \tag{3.9}
$$
Proof
From (3.2) we have
\[ \delta c^{1,2} + dc^{2,1} = 0. \]
Integration of this equation over \( N \) yields the desired equation. \( \square \)

Lemma 3.3.
\[ \Gamma(fg, A) = \Gamma(g, f \cdot A) + \Gamma(f, A) \] (3.10)

proof
From (3.2) we have
\[ \delta c^{0,2} + dc^{1,1} = c^{1,2}. \]
Hence \( \delta dc^{1,1} = dc^{2,1} \). Integration of the equation over \( N \) and the Polyakov-Wiegmann formula yield the desired result. \( \square \)

3.3 Line bundle with connection over \( \mathcal{B}(M) \)

In this section \( M \) is a connected compact four-manifold that is the boundary of an oriented 5-dimensional manifold \( N; \partial N = M \).

Let \( \mathcal{A} = \mathcal{A}(M), \mathcal{G} = \mathcal{G}(M) \) and \( \mathcal{B} = \mathcal{A}/\mathcal{G} \) as before. The pre-symplectic form \( \omega_A = \omega_A^0 \) on \( \mathcal{A} \) being invariant under the action of \( \mathcal{G} \), \( \mathcal{B} \) has the induced pre-symplectic structure.

We consider the \( U(1) \)-valued function on \( \mathcal{A} \times \mathcal{G} \):
\[ \Theta(g, A) = \exp 2\pi i \Gamma(g, A). \] (3.11)

Lemma 3.3 yield the cocycle condition:
\[ \Theta(g, A)\Theta(h, g \cdot A) = \Theta(gh, A). \] (3.12)

Therefore if we define the action of \( \mathcal{G} \) on \( \mathcal{A} \times \mathbb{C} \) by
\[ (g, (A, c)) \to (g \cdot A, \Theta(g, A)c), \]
we have a complex line bundle:
\[ \mathcal{L} = \mathcal{A} \times \mathbb{C}/\mathcal{G} \to \mathcal{B} = \mathcal{A}/\mathcal{G}. \] (3.13)

\( \Theta \) being \( U(1) \)-valued, it is a hermitian line bundle. The associated \( U(1) \)-principal bundle is given by
\[ \mathcal{P} = \mathcal{A} \times U(1)/\mathcal{G} \xrightarrow{\pi} \mathcal{B}. \] (3.14)
We put
\[ \theta_A(a) = -\frac{i}{24\pi^3} \int_M Tr[(AF + FA - \frac{1}{2} A^3)a], \quad (3.15) \]
here \( a \) represents a tangent vector to \( \mathcal{B} \) and satisfies \( d_A^*a = 0 \).

**Proposition 3.4.**
\[ (\delta \theta_A(a))(g) = \frac{1}{2\pi i} (\tilde{d} \log \Theta(g, A))(a). \quad (3.16) \]
That is, the connection 1-form on \( P \) is given by
\[ \tilde{d}\phi - \theta, \]
\( \phi \) being the angle coordinate on \( U(1) \).

**Proof**
Since \( \tilde{d}C_5(g) = 0 \), we have
\[ \tilde{d}\Gamma(g, A) = \frac{i}{24\pi^3} \int_M Tr[-V(aF + Fa) \]
\[ + \frac{1}{2}(A^2V + AVA + VA^2 + V^2A + AV^2 + +VAV + V^3)a], \quad (3.17) \]
where we put \( V = dg \cdot g^{-1} \) and we used the relation
\[ dTr[(V A a - V a A)] = Tr[V(Ad_a + d_A a A) - V(aF + Fa) \]
\[ + (V^2 A + 2AVA + AV^2)a]. \quad (3.18) \]

On the other hand
\[ (\delta \theta_A(a))(g) = g \cdot \theta_A(a) - \theta_A(a) \]
\[ = -\frac{i}{24\pi^3} \int_M Tr[VF a + FVa - \frac{1}{2}((A + V)^3a - A^3a)], \quad (3.19) \]
which is equal to the right hand side of \( (3.17) \). Therefore
\[ (\delta \theta_A(a))(g) = \tilde{d}\Gamma(g, A)(a) = \frac{1}{2\pi i} (\tilde{d} \log \Theta(g, A))(a). \quad (3.20) \]
Thus \( \tilde{d}\phi - \theta_A \) gives a connection 1-form on \( P \).
**Proposition 3.5.** The curvature of the connection 1-form on $P$ is equal to $\pi^*(-i\omega)$.

**Proof**

$$(\ddbar\theta)_A(a,b) = \langle (\partial_A \theta)a, b \rangle - \langle (\partial_A \theta)b, a \rangle$$

$$= \frac{-i}{24\pi^3} \int_M Tr[2(ab-ba)F - (ab-ba)A^2 - (bd_Aa + d_Aab - d_Aba - ad_Ab)A].$$

But since

$$dTr[(ab-ba)A] = Tr[(bd_Aa + d_Aab - d_Aba - ad_Ab)A] + Tr[(ab-ba)(F+A^2)],$$

we have

$$(\ddbar\theta)_A(a,b) = \frac{-i}{8\pi^3} \int_M Tr[(ab-ba)F] = -i\omega_A(a,b).$$

We have obtained the line bundle with connection $\mathcal{L} \longrightarrow \mathcal{B}$ whose curvature is $-i\omega$, where $\omega$ is the pre-symplectic form induced on $\mathcal{B}$.

### 3.4 The Chern-Simons functional

The manifold $M$ is assumed to be closed as in the previous part. Let $\mathcal{P}_N$ be the pullback of the $U(1)$-principal bundle $\mathcal{P} \longrightarrow \mathcal{B}$ by the boundary restriction map

$$r : \mathcal{B}(N) = \mathcal{A}(N)/\mathcal{G}(N) \longrightarrow \mathcal{B}(M).$$

$$\mathcal{P}_N = r^*\mathcal{P} = \mathcal{A}(N) \times_{r^*\theta} U(1). \quad (3.21)$$

The connection $\theta$ pulls back to give a connection $\theta_N$ on $\mathcal{A}(N) \times U(1)$.

We shall investigate the section of $\mathcal{P}_N$ induced by the Chern-Simons functional.

The *Chern-Simons functional* on $\mathcal{A}(N)$ is defined by

$$\text{CS}(\mathcal{A}) = \frac{i}{24\pi^3} \int_N c^{0,2}(\mathcal{A}). \quad (3.22)$$
The equation (3.5) implies

\[ \text{CS}(g \cdot A) = \text{CS}(A) + \Gamma(g, A), \]
\[ \exp 2\pi i \text{CS}((g \cdot A)) = \Theta(g, A) \exp 2\pi i \text{CS}(A), \quad (3.23) \]

where \( A \) and \( g \) are respectively the restriction of \( A \) and \( g \) to the boundary \( M = \partial N \).

Then the map

\[ s : \mathcal{A}(N) \rightarrow \mathcal{A}(N) \times U(1) \]
given by

\[ s(A) = (A, \exp 2\pi i \text{CS}(A)) \quad (3.24) \]
gives a section of the pullback line bundle \( \mathcal{P}_N \).

**Proposition 3.6.** The gradient vector field of the section \( s \) over \( \mathcal{B}(N) \) is given by

\[ \ast\frac{i}{8\pi^3} F_A^2. \]

**Proof**

Let \( A = rA \in \mathcal{B}(M) \). Let \( a \in T_A\mathcal{B}(N) \) and denote the image of \( a \) by the boundary restriction map \( r \) by \( a = r_a \in T_A\mathcal{B}(M) \). We have

\[ (\tilde{d}\text{CS}(A))a = \frac{i}{24\pi^3} \int_N Tr[F_A^2 a + (F_A A + A F_A - \frac{1}{2} A^3)a]d_A a \]
\[ - \frac{1}{2}(A^2 F_A + A F_A A + F_A A^2 - A^4)a. \]

On the other hand

\[ (\theta_N)_A(a) = \theta_A(a) = -\frac{i}{24\pi^3} \int_M Tr[(A F_A + F_A A - \frac{1}{2} A^3)a] \]
\[ = \frac{i}{24\pi^3} \int_N Tr[-2F_A^2 a - \frac{1}{2}(A^2 F_A + F_A A^2 + A F_A A - A^4)a \]
\[ + (A F_A + F_A A - \frac{1}{2} A^3)d_A a], \]

Therefore we have

\[ (\nabla s)_A a = \frac{1}{2\pi i} \tilde{d}\log s - \theta_N)(a) = (\tilde{d}\text{CS}(A))a - (\theta_N)_A(a) \]
\[ = \frac{i}{24\pi^3} \int_N Tr[3F_A^2 a] = \langle a, \ast\frac{i}{8\pi^3} F_A^2 \rangle_{L^2(N)}. \quad (3.25) \]
The proposition says that flat connections \( A \in \mathcal{A}^0(N) \) are the critical points of the gradient vector field of Chern-Simons functional \( \text{CS} \).

For \( A \in \mathcal{A}^0(N) \), the Chern-Simons functional is

\[
\text{CS}(A) = \frac{i}{240 \pi^3} \int_N \text{Tr}(A^5). \tag{3.26}
\]

If \( M \) is simply connected then any \( A \in \mathcal{A}^0(M) \) has a flat extension \( A \in \mathcal{A}^0(N) \). In fact, there is a smooth function \( f \in \Omega^0(M, \text{AdP}) \) such that \( A = f^{-1} df \), and if we let \( f \) its smooth extension to \( N \), that is assured by \( \pi_4(G) = 1 \), then \( A = f^{-1} df \) gives a flat extension of \( A \) to \( N \).

Under the assumption that \( M \) is simply connected, we put, for \( A \in \mathcal{A}^0(M) \),

\[
\text{CS}_M(A) = \text{CS}(A) = \frac{i}{240 \pi^3} \int_N \text{Tr}(A^5), \tag{3.27}
\]

where \( A \) is the extension of \( A \) to \( N \). It is well defined modulo \( \mathbb{Z} \) independently of the extension.

From (3.23) we have

\[
\exp 2\pi i \text{CS}_M(g \cdot A) = \Theta(g, A) \exp 2\pi i \text{CS}_M(A). \tag{3.28}
\]

It defines over \( \mathcal{M}^0(M) \) a non-vanishing section \([ A, \exp 2\pi i \text{CS}_M(A)] \in \mathcal{L}(M)\), that is, the Chern-Simons functional \( \text{CS}_M \) gives the trivialization of the line bundle \( \mathcal{L}(M) \) restricted on \( \mathcal{M}^0(M) \).

### 3.5 Line bundle with connection over \( B(M) \) for a proper submanifold \( M \subset \hat{M} \)

Let \( \hat{M} \) be a compact four-manifold that is the boundary of a five-manifold \( N \). Let \( M \) be a connected four-dimensional submanifold of \( \hat{M} \) with smooth boundary \( \partial M \). Let \( P \) be a \( G \)-principal bundle over \( M \) and let \( ( B(M), \omega ) \) be the moduli space of connections over \( M \) with the pre-symplectic structure \( \omega \). We shall construct a line bundle with connection over \( B(M) \) whose curvature is \(-i \omega \). Let \( \hat{P} \) be the trivial extension of \( P \) to \( \hat{M} \). We extend any \( g \in \mathcal{G}_0(M) \) across the boundary \( \partial M \) by defining it to be the identity transformation \( 1' \) on \( \hat{P}|(\hat{M} \setminus \hat{M}) = (\hat{M} \setminus \hat{M}) \times G \). Then \( g \vee 1' \in \mathcal{G}(\hat{M}) \). Here the upper prime will indicate that the function is defined on \( \hat{M} \setminus \hat{M} \), for example,
1′ is the constant function \( \hat{M} \setminus \hat{M} \ni x \rightarrow 1'(x) = 1 \in G \), while 1 is the constant function \( M \ni x \rightarrow 1(x) = 1 \in G \). For \( g \in \mathcal{G}(M) \) and \( g' \in \mathcal{G}(\hat{M} \setminus \hat{M}) \) such that \( g|\partial M = g'|\partial M \), we write by \( g \vee g' \) the gauge transformation on \( \hat{M} \) that are obtained by sewing \( g' \) and \( g \).

We put, for \( A \in \mathcal{A}(M) \) and \( g \in \mathcal{G}_0(M) \),

\[
\Gamma_M(g; A) = \frac{i}{24\pi^3} \int_M c^{11}(g; A) + C_5(g \vee 1') \tag{3.29}
\]

\[
\Theta_M(g; A) = \exp 2\pi i \Gamma_M(g; A) \tag{3.30}
\]

We note that \( C_5(g \vee 1') \) is well defined independently of the extension \( g \) of \( g' \) to \( N \). It follows from (3.12) that \( \Theta_M(g, A) \) satisfies the cocycle condition:

\[
\Theta_M(g, A) \, \Theta_M(h, g \cdot A) = \Theta_M(gh, A) \quad \text{for } g, h \in \mathcal{G}_0(M).
\]

So if we define the action of \( \mathcal{G}_0(M) \) on \( \mathcal{A}(M) \times \mathbb{C} \) by

\[
(g, (A, c)) \rightarrow (g \cdot A, \Theta_M(g, A) c),
\]

we have a hermitian line bundle on \( \mathcal{B}(M) \) with the transition function \( \Theta_M(g, A) \):

\[
\mathcal{L}(M) = \mathcal{A}(M) \times \mathbb{C}/\mathcal{G}_0(M) \longrightarrow \mathcal{B}(M). \tag{3.31}
\]

We next give a connection on the line bundle \( \mathcal{L}(M) \) with the curvature \( -i \omega \).

We define the 1-form \( \theta \) on \( \mathcal{A}(M) \) by

\[
\theta_A(a) = -\frac{i}{24\pi^3} \int_M Tr[(AF + FA - \frac{1}{2}A^2)a], \quad \text{for } a \in T_A \mathcal{A}(M). \tag{3.32}
\]

\( \theta \) is a 0-cochain on the Lie group \( \mathcal{G}_0(M) \) taking its value in the space of 1-forms on \( \mathcal{A}(M) \), the coboundary of \( \theta \) becomes

\[
(\delta \theta_A(a))(g) = -\frac{i}{24\pi^3} \int_M Tr[V Fa + FVa - \frac{1}{2}((A + V)^3a - A^3a)]
\]

for \( g \in \mathcal{G}_0(M) \), where \( V = dgg^{-1} \). By the same calculation as in (3.17) we see that the right hand side is equal to

\[
\tilde{d} \Gamma(g, A) = \frac{i}{24\pi^3} \int_M Tr[-V(aF + Fa)
\]

\[+ \frac{1}{2}(A^2V + AVA + VA^2 + V^2A + AV^2 + VAV + V^3)a],
\]

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for $a \in T_A A(M)$, where we used the relation $3.18$ again and the fact that $dg^{-1}|\partial M = 0$ for $g \in G_0(M)$. Therefore

$$
(\delta \theta_A(a))(g) = \tilde{d} \Gamma_M(g, A)(a) = \frac{1}{2\pi i} \langle \tilde{d} \log \Theta_M(g, A)(a) \rangle.
$$

Thus $\tilde{d} \phi - \theta_A$ gives a connection 1-form on the $U(1)$-principal bundle associated to $L(M)$, where $\phi$ is the angle coordinate on the fiber. By the same calculation as in Proposition 3.5 we see that the curvature form becomes $\pi^*(-i\omega)$;

$$
(\tilde{d} \phi_A(a, b) = -\frac{i}{8\pi^3} \int_M Tr[(ab - ba)F] + \frac{i}{24\pi^3} \int_{\partial M} Tr[(ab - ba)A]
$$

$$
= -i \omega_A(a, b).
$$

We have proved the following

**Theorem 3.7.** There exists a hermitian line bundle with connection $L(M) \to B(M)$, whose curvature is equal to $-i\omega$.

## 4 Pre-quantization of the moduli space of flat connections

### 4.1 Pre-quantum line bundle on $\mathcal{M}(M)$

Let $\hat{M}$ be a compact four-manifold that is the boundary of a five-manifold $N$. Let $M$ be a connected open submanifold in $\hat{M}$ with smooth boundary $\partial M$ which may be empty. Let $P = M \times G$ be the trivial bundle. We know from Theorem 2.5 that if $\partial M \neq \emptyset$ the moduli space $\mathcal{M}(M)$ of flat connections on $P$ has the pre-symplectic structure $\omega^P$. If $\partial M = \emptyset$, that is, if $M = \hat{M}$, $\mathcal{M}(\hat{M})$ is a finite dimensional manifold (possibly with singularity). $\mathcal{M}(\hat{M})$ is one-point if $\hat{M}$ is simply connected.

In the previous section we constructed the line bundle with connection

$$
\mathcal{L}(M) \to B(M)
$$

with the curvature $-i\omega$. We restrict it to the moduli space $\mathcal{M}(M)$ and obtain the line bundle with connection

$$
\mathcal{L}^P(M) = \mathcal{A}(M) \times C/G_0(M) \to \mathcal{M}(M). \quad (4.1)
$$
The connection is given by
\[ \theta_A(a) = \frac{i}{48\pi^3} \int_M Tr[A^3 a], \quad (4.2) \]
here \( a \) represents a tangent vector to \( \mathcal{M}^\flat(M) \) and satisfies \( d_A^*a = d_A a = 0 \). We have
\[ (d\theta)_A(a, b) = \begin{cases} -i \omega^\flat_A(a, b), & \text{when } \partial M \neq \emptyset \\ 0, & \text{when } M = \widehat{M} \end{cases} \]
Therefore, if \( M \) is a proper submanifold of \( \widehat{M} \), \( \theta \) is a connection with the curvature given by the symplectic form \( \omega^\flat \). But, for \( \mathcal{L}^\flat(\widehat{M}) \), \( \theta \) is a flat connection.

We have obtained the following theorem.

**Theorem 4.1.**

1. \( \mathcal{L}^\flat(\widehat{M}) \to \mathcal{M}^\flat(\widehat{M}) \) admits a flat connection.

2. For a proper submanifold \( M \) of \( \widehat{M} \), there exists a pre-quantization of the moduli space \( (\mathcal{M}^\flat(M), \omega^\flat) \), that is, there exists a hermitian line bundle with connection \( \mathcal{L}^\flat(M) \to \mathcal{M}^\flat(M) \), whose curvature is equal to the symplectic form \( -i \omega^\flat \).

We call \( \mathcal{L}^\flat(M) \) the pre-quantum line bundle over \( M \).

If \( \widehat{M} \) is simply connected, then \( \mathcal{M}^\flat(\widehat{M}) \) is one-point, and
\[ \mathcal{L}^\flat(\widehat{M}) \simeq C, \quad (4.3) \]
The isomorphism is given by the Chern-Simons functional in \([3.27]\)
\[ \mathcal{L}^\flat(\widehat{M}) \simeq C, \quad (4.4) \]
\[ [A, c \exp 2\pi i \text{CS}_{\widehat{M}}(A)] \to c. \]

**4.2 The Chern-Simons functional on \( \mathcal{M}^\flat(M) \)**

Now we investigate the Chern-Simons functional \( \text{CS}_M \) for the submanifold \( M \subset \widehat{M} \). Hereafter we assume that \( \widehat{M} \) is simply connected. We put \( M^c = \widehat{M} \setminus \hat{M} \), and we assume that every connected component of
$M^c$ are simply connected. We have as in Example 2 of 2.1 the bijective correspondence of $A^b(M^c)$ and $G(M^c) = Map(M^c, G)$ given by the parallel transformations. Therefore

$$M^b(M^c) \simeq G(M^c)/G_0(M^c).$$

In particular, for $A_1$ and $A_2 \in A^b(M^c)$ such that $A_1|\partial M^c = A_2|\partial M^c$ there exists $g \in G_0(M^c)$ such that

$$A_2 = g \cdot A_1.$$

We define the map

$$p : A^b(M) \rightarrow M^b(M^c). \quad (4.5)$$

by

$$p(A) = [A'],$$

where, for $A \in A^b(M)$ we associate a $A' \in A^b(M^c)$ such that $A|\partial M = A'|\partial M^c$. The equivalence class $p(A) = [A']$ is well defined because, for another $A'' \in A^b(M^c)$ with $A''|\partial M^c = A'|\partial M^c$, there is a $g' \in G_0(M^c)$ such that $A'' = g' \cdot A'$.

Let $A \in A^b(M)$. We consider

$$(A', \exp 2\pi i CS_M(A \lor A')) \in A^b(M^c) \times C,$$

for $A' \in A^b(M^c)$ such that $[A'] = p(A)$.

Let $A''$ be another flat connection on $M^c$ with $A''|\partial M^c = A|\partial M$. There is a $g' \in G_0(M^c)$ such that $A'' = g' \cdot A'$. So we have $(1 \lor g') \cdot (A \lor A') = A \lor A''$. It follows from (3.28)

$$\exp 2\pi i CS_M(A \lor A'') = \Theta_{M^c}(g', A') \cdot \exp 2\pi i CS_M(A \lor A').$$

Thus we have a map

$$\exp 2\pi i CS_M : A \rightarrow [A', \exp 2\pi i CS_M(A \lor A')] \in L^b(M^c)_{\pi(A)}, \quad (4.6)$$

that is, we have a section $\exp 2\pi i CS_M$ of $p^*L^b(M^c)$ over $A^b(M)$.

We call $CS_M$ the Chern-Simons functional over $M$.

**Proposition 4.2.** The functional $\exp 2\pi i CS_M$ is a horizontal section over $A^b(M)$ of the line bundle $p^*L^b(M^c)$:

$$\nabla \exp 2\pi i CS_M = 0. \quad (4.7)$$
In fact we have from Proposition 3.6

\[ (\dd exp 2\pi i CS_M)_A(a) = \exp 2\pi i CS_M(A) (\dd CS_M)_A(a) = \theta_A(a) \exp 2\pi i CS_M(A). \]

**Theorem 4.3.** \( p : \mathcal{A}^0(M) \to \mathcal{M}^0(M^c) \) is a Lagrangean immersion.

**Proof**

Let \( \mathcal{P}^0(M^c) \to \mathcal{M}^0(M^c) \) be the \( U(1) \)-principal bundle associated to \( \mathcal{L}^0(M^c) \). The connection form on \( \mathcal{P}^0(M^c) \) is \( \alpha_A = \dd \phi - \theta_A \) with the curvature form \( \dd \alpha = \pi^*(-i\omega) \). We have seen in Proposition 3.6 that

\[ \alpha_A((\exp 2\pi i CS_M)_* a) = 0, \quad a \in T_A\mathcal{A}^0(M). \]

Since \( p = \pi \circ \exp 2\pi i CS_M \) we have

\[ \omega_{\mathcal{A}}(p_* a, p_* b) = 0, \]

for \( a, b \in T_A\mathcal{A}^0(M) \).

\[ \square \]

### 4.3 duality

Now we assume that \( \widehat{M} \), \( M \) and \( M^c \) are all simply connected. In this case \( p : \mathcal{A}^0(M) \to \mathcal{M}^0(M^c) \) yields the isomorphism

\[ \mathcal{M}^0(M) \simeq \mathcal{M}^0(M^c). \quad (4.8) \]

We have hermitian line bundles with connection \( \mathcal{L}^0(M) \) over \( \mathcal{M}^0(M) \) and \( \mathcal{L}^0(M^c) \) over \( \mathcal{M}^0(M^c) \). \( \mathcal{L}^0(M) \) is given by the transition function

\[ \Theta_M(g, A) = \exp 2\pi i \Gamma_M(g, A) \]

for \( g \in G_0(M) \) and \( A \in \mathcal{A}^0(M) \). Here any \( g \in G_0(M) \) is extended from \( M \) to \( \widehat{M} \) by the identity transformation on \( M^c \). The transition function of \( \mathcal{L}^0(M^c) \) is

\[ \Theta_{M^c}(g', A') = \exp 2\pi i \Gamma_{M^c}(g', A') \]

for \( g' \in G_0(M^c) \) and \( A' \in \mathcal{A}^0(M^c) \).

We shall see that the pre-quantum line bundles \( \mathcal{L}^0(M) \) and \( \mathcal{L}^0(M^c) \) are in duality. Here we identified the base spaces \( \mathcal{M}^0(M) \) and \( \mathcal{M}^0(M^c) \) by (4.8).

Now let \( A \in \mathcal{A}^0(M) \) and \( A' \in \mathcal{A}^0(M^c) \) be such that \( A|\partial M = A'|\partial M^c \).

Since

\[ \gamma(F, G) = \Gamma(F^{-1}dF, G) - C_5(G), \]

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we have from (3.29) and the Polyakov-Wiegmann formula that
\[ \Gamma_M(g, A) + \Gamma_{M^c}(g', A') = \Gamma(g \lor g', A \lor A'), \quad \text{mod} \, \mathbb{Z}, \]
for \( g \in G_0(M) \) and \( g' \in G_0(M^c) \). Here we have used the fact \( \gamma(g \lor 1', 1 \lor g') = 0 \).
Therefore
\[ \Theta_M(g, A) \Theta_{M^c}(g', A') = \Theta_{\hat{M}}(g \lor g', A \lor A'). \quad (4.9) \]

By virtue of (4.9) we have a homomorphism of line bundles:
\[ L^0(M) \times L^0(M^c) \longrightarrow L^0(\hat{M}), \quad (4.10) \]
that is given by
\[ ([A', c'], [A, c]) \longrightarrow [A \lor A', c \lor c'], \quad (4.11) \]
where \( c \lor c' \) in the right hand side means that \( c \in \pi^{-1}([A]) \) and \( c' \in \pi^{-1}([A']) \). Composed with the homomorphism \( L^0(\hat{M}) \longrightarrow C, \quad (4.4) \), we have the duality of \( L^0(M) \) and \( L^0(M^c) \):
\[ L^0(M) \times L^0(M^c) \longrightarrow C. \quad (4.12) \]

5 The action of \( \Omega_3 G \) on \( \mathcal{M}^\flat(D) \) and its lift to \( L^3(D) \)

5.1 Abelian extension of the group \( \Omega_3 G \)

Let \( G = SU(n) \) with \( n \geq 3 \). Let \( \Omega_3 G \) be the set of smooth mappings from \( S^3 \) to \( G \) that are based at some point. \( \Omega_3 G \) is not connected but is divided into connected components by the degree. We put
\[ \Omega_3 G = \{ g \in \Omega_3 G; \deg g = 0 \}. \quad (5.1) \]

J. Mickelsson gave an extension of \( \Omega_3^0 G \) by the abelian group \( \text{Map}(A(S^3), U(1)) \), where \( A(S^3) \) is the space of connections on \( S^3 \). In the following we shall explain it after [10, 12, 13].

The oriented 4-dimensional disc with boundary \( S^3 \) is denoted by \( D \). We write \( DG = \text{Map}(D, G) \), the set of smooth mappings from \( D \) to \( G \) based at a \( p_0 \in \partial D = S^3 \). The restriction to \( S^3 \) of a \( f \in DG \) has degree 0: \( f|S^3 \in \Omega_3^0 G \). We let \( D_0 G = \{ f \in DG; f|S^3 = 1 \} \).
We define the action of $D_0 G$ on $DG \times \text{Map}(A(S^3), U(1))$ by

$$h \cdot (f, \lambda) = (fh, \lambda(\cdot)\Theta_D(h, f^{-1} df)), \quad (5.2)$$

for $h \in D_0 G$ and $(f, \lambda) \in DG \times \text{Map}(A(S^3), U(1))$.

We consider the quotient space by this action;

$$\hat{\Omega} G = DG \times \text{Map}(A(S^3), U(1))/D_0 G. \quad (5.3)$$

The equivalence class of $(f, \lambda)$ is denoted by $[f, \lambda]$. The projection $\pi: \hat{\Omega} G \longrightarrow \Omega^3_0 G$ is defined by $\pi([f, \lambda]) = f|S^3$. Then $\hat{\Omega} G$ becomes a principal bundle over $\Omega^3_0 G$ with the structure group $\text{Map}(A(S^3), U(1))$. The transition function is $\chi(f, g) = \Theta_D(f^{-1}g, f^{-1} df)$ for $f, g \in DG$ such that $f|S^3 = g|S^3$. Here the $U(1)$ valued function $\chi(f, g)$ is considered as a constant function in $\text{Map}(A(S^3), U(1))$.

The group structure of $\hat{\Omega} G$ is given by the Mickelsson’s 2-cocycle on $D$ which is defined by the following formula.

$$\gamma_D(f, g; A) = \frac{i}{24\pi^3} \int_D (\delta c^{11})(f, g; A)$$
$$= \frac{i}{24\pi^3} \int_{S^3} c^{20}(f, g; A) + \frac{i}{24\pi^3} \int_D c^{21}(f, g). \quad (5.4)$$

We define the multiplication on $DG \times \text{Map}(A(S^3), U(1))$ by

$$(f, \lambda) \bullet (g, \mu) = (fg, \lambda(\cdot)\mu_f(\cdot) \exp 2\pi i \gamma_D(f, g; \cdot)), \quad (5.5)$$

where

$$\mu_f(A) = \mu ((f|S^3)^{-1}A(f|S^3) + (f|S^3)^{-1}d(f|S^3)).$$

Then $DG \times \text{Map}(A(S^3), U(1))$ is endowed with a group structure. The associative law follows from (3.1) and (3.2);

$$\delta dc^{20} = dc^{30} = 0, \quad \delta c^{21} = 0.$$

From the definition and Lemma 3.2 we can verify that $[f, \lambda] = [g, \mu]$ if and only if there is a $h \in D_0 G$ such that $(g, \mu) = (f, \lambda) \bullet (h, \exp 2\pi i C_5(h \vee 1'))$. Since the set of elements $(h, \exp 2\pi i C_5(h \vee 1'))$ with $h \in D_0 G$ forms a normal subgroup of $DG \times \text{Map}(A(S^3), U(1))$, the group structure descends to the quotient. Thus $\hat{\Omega} G$ is endowed with the group structure. The group $\text{Map}(A(S^3), U(1))$ being embedded as a normal subgroup of $\hat{\Omega} G$, $\hat{\Omega} G$ is an extension of $\Omega^3_0 G$ by the abelian group $\text{Map}(A(S^3), U(1))$. 

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5.2 The action of $\Omega\hat{G}$ on pre-quantum line bundles

Let $P = D \times G$ be the trivial bundle over $D$. The group of gauge transformations $\mathcal{G}(D)$ acts on $\mathcal{A}(D)$. It descends to the actions on $\mathcal{B}(D) = \mathcal{A}(D)/\mathcal{G}_0(D)$ and on $\mathcal{C}(D) = \mathcal{A}(D)/\mathcal{G}(D)$. $\mathcal{B}(D)$ is a $\mathcal{G}/\mathcal{G}_0$-principal bundle over $\mathcal{C}(D)$. Since $\mathcal{G}_0(D) = DG_G$, $G_0(D) = D$ and $DG/DG_0 \cong \Omega^3_0G$, we have the action of $\Omega^3_0G$ on $\mathcal{B}(D)$ and $\mathcal{B}(D)$ is a $\Omega^3_0G$-principal bundle over $\mathcal{C}(D)$.

The action of $\Omega^3_0G$ on $\mathcal{B}(D)$ does not lift to the action on the line bundle $\mathcal{L}(D)$. The abelian extension $\Omega\hat{G}$ is needed to have the lift on $\mathcal{L}(D)$.

For $A \in \mathcal{A}(D)$ and $f \in DG$, we put

$$\beta_D(f, A) = \frac{i}{24\pi^3} \int_D c^{1,1}(f, A). \quad (5.6)$$

Notice that the following two relations hold:

$$\delta \beta_D = \gamma_D, \quad (5.7)$$

$$\Gamma_D(f, A) = \beta_D(f, A) + C_5(f \lor 1'), \quad \text{for } f \in D_0G . \quad (5.8)$$

Let $(f, \lambda) \in DG \times Map(\mathcal{A}(S^3), U(1))$ and $(A, c) \in \mathcal{A}(D) \times C$. The action of $(f, \lambda)$ on $(A, c)$ is defined by

$$(f, \lambda) \cdot (A, c) = (f \cdot A, c\lambda(A|S^3) \exp 2\pi i \beta_D(f, A)). \quad (5.9)$$

It is a right action. By virtue of the relation $\gamma_D = \delta \beta_D$ we can verify

$$(g, \mu) \cdot ((f, \lambda) \cdot (A, c)) = ((f, \lambda) \cdot (g, \mu)) \cdot (A, c),$$

for $(f, \lambda), (g, \mu) \in DG \times Map(\mathcal{A}(S^3), U(1))$ and $(A, c) \in \mathcal{A}(D) \times C$. Hence the action is certainly well defined. In particular we have, for $h \in D_0G$,

$$(h, \exp 2\pi i C_5(h \lor 1')) \cdot (A, c) = (h \cdot A, c \Theta_D(h, A)). \quad (5.10)$$

From the definition of $\mathcal{L}(D)$ and the fact that $[h, \exp 2\pi i C_5(h \lor 1')]$ for $h \in D_0G$ gives the unit element of $\Omega\hat{G}$ we see that the above action descends to the action of $\Omega\hat{G}$ on $\mathcal{L}(D)$. Thus we have proved the following theorem.

**Theorem 5.1.** The line bundle $\mathcal{L}(D)$ carries an action of $\Omega\hat{G}$ that is equivariant with respect to the action of $\Omega^3_0G$ on the base space $\mathcal{B}(D)$. 

The reduction of $L(D)$ by the action of $\hat{\Omega}G$ becomes a line bundle on $C(D)$ that is described as follows.

Let $\mathcal{K}(D) = \mathcal{A}(D) \times C/\hat{\Omega}G,$

and let $\pi : \mathcal{K}(D) \longrightarrow C(D)$ be the projection induced from $\pi : L(D) \longrightarrow B(D).$ Then $\pi : \mathcal{K}(D) \longrightarrow C(D)$ becomes a line bundle with the structure group $Map(\mathcal{A}(S^3), U(1)).$

We saw in section 2.2 that the action of $G(D)$ on $M^\flat(D)$ is infinitesimally symplectic. Hence the action of $\Omega^3_0G$ on $M^\flat(D)$ is also infinitesimally symplectic. This action is lifted to the action of $\hat{\Omega}G$ on the pre-quantum line bundle $L^\flat(D) \longrightarrow M^\flat(D).$ In fact it is given by the restriction to $L^\flat(D)$ of the action $\hat{\Omega}G.$

**Theorem 5.2.** The line bundle $L^\flat(D)$ carries an action of $\hat{\Omega}G$ that is equivariant with respect to the infinitesimally symplectic action of $\Omega^3_0G$ on the base space $M^\flat(D).$ The reduction of $L^\flat(D)$ by $\hat{\Omega}G$ becomes the complex line $C.$

The reduction of $L^\flat(D)$ by the action of $\hat{\Omega}G$ is the restriction of the line bundle $\mathcal{K}(D)$ to $N^\flat(D).$ But $N^\flat(D)$ being one point the restriction becomes

$\mathcal{K}(D)|N^\flat(D) \simeq C.$

**References**

[1] Atiyah, M. F., *Topological quantum field theories*, Publ. Math. Inst. Hautes Etudes Sci. 68(1989),175-186.

[2] Atiyah, M. F. and Bott, R., *Yang-Mills equations over Riemann surfaces*, Phil. Trans. R. Soc. Lond. A. 308(1982), 523-615.

[3] Axelrod, S., Della Pietra S. and Witten, E., *Geometric quantization of Chern-Simons gauge theory*, J. Differential Geometry 33 (1991), 787-902.

[4] Bao, D. and Nair, V. P., *A note on the covariant anomaly as an equivariant momentum mapping*, Commun. Math. Phys. (1985), 101, 437-448.
[5] Babelon, O. and Viallet, C. M., *The riemannian geometry of the configuration space of gauge theories*, Commun. Math. Phys. (1981), 81, 515-525.

[6] Donaldson, S. K., *Boundary value problems for Yang-Mills fields*, J. Geom. Phys. 8(1992) 89-122.

[7] Donaldson, S. K., *Floer homology groups in Yang-Mills theory*, Cambridge University Press (2002).

[8] Donaldson, S. K. and Kronheimer, P. B., *The Geometry of Four-Manifolds*, Oxford Science Publications (1990).

[9] Guillemin, V., Ginzburg, V. and Karshon, Y., *Moment Maps, Cobordisms, and Hamiltonian Group Actions*, A. M. S. (2002).

[10] Kori, T., *Four-dimensional Wess-Zumino-Witten actions*, J. Geom. and Phys. 47 (2003), 235-258.

[11] Meinrenken, E. and Woodward, C., *Hamiltonian loop group actions and Verlinde factorization*, J. Differential Geometry 50 (1998), 417-469.

[12] Mickelsson, J., *Current Algebras and Groups*, (1989), Plenum Press New York.

[13] Mickelsson, J., *Kac-Moody Groups, Topology of the Dirac Determinant Bundle and Fermionization*, Commun. Math. Phys. 110(1987),175-183.

[14] Polyakov, A. M. and Wiegmann, P. B., *Goldstone fields in two dimensions with multivalued actions*, Phys. Lett. B 141, 223-228 (1984).

[15] Ramadas, T. R., Singer, I. M. and Weitsman, J., *Some comments on Chern-Simons gauge theory*, Commun. Math. Phys. 126 (1989), 409-420.

[16] Singer, I. M., *The geometry of the orbit space for non-abelian gauge theories*, Physica Scripta, Vol. 24(1981), 817-820.

[17] Terashima, Y., Private communication.

[18] Zumino, B., *Chiral anomalies and differential geometry*, Les Houches Proc.ed. B. S. Dewit and R. Stora (1983),1291-1332.