The evolving many-nucleon theory of nuclear rotations

David J. Rowe

1Department of Physics, University of Toronto, Toronto, ON M5S 1A7, Canada

(Dated: Sept., 2017)

The many approaches that have been pursued in seeking an understanding of nuclear rotational dynamics are reviewed and reassessed with a view to their development in the light of recent progress and the research tools that are now available. A motivation for this review is the widespread observation of nuclear shape coexistence and sequences of rotational states in all regions of the nuclear periodic table combined with the recognition that the study of the rotational dynamics of quantum fluids has led to significant advances in the quantum theory of many-boson systems. Recent experimental investigations of the rotational dynamics of a low-temperature \(^6\text{Li}\) gas indicate that its slow rotational flows are likewise the irrotational flows of a superfluid. In this context, the dynamics of rotating nuclei are of fundamental interest because the nucleus is a unique zero-temperature finite many-fermion quantum system. A promising approach is provided by algebraic mean-field theory which, as its name suggests, is a combination of algebraic and mean-field methods. Static mean-field theories play a central role in many-body theory by defining optimal independent-particle and independent quasi-particle basis states for the quantum mechanics of many-fermion systems. Their time-dependent extensions also lead, in the small-amplitude random-phase approximation, to the quantisation of the classical normal-mode vibrations of many-fermion systems about their static equilibrium states. This review shows that mean-field methods become significantly more powerful when combined with algebraic methods and an appropriate coupling scheme for the nuclear shell model.

PACS numbers: 21.60.Ev, 21.60.Fw, 21.60.Cs, 24.10.Cn

Contents

I. Introduction
II. The cranking model
III. Mean-field theory as an interface between classical and quantum mechanics
   A. Quantisation and dequantisation of a Heisenberg-Weyl algebra
   B. Mean-field theory as a classical representation of a Lie algebra
   C. Classical normal-mode vibrations and the random phase approximation
   D. The quantum-mechanical Random Phase Approximation
IV. The emergence of rotational states in mean-field theory
   A. Broken symmetry and rotational states of nuclei
   B. Angular-momentum projection
V. Kinetic energy considerations
VI. Many-nucleon algebraic models
   A. First attempts
   B. The Elliott SU(3) model
   C. The Ui Rot(3) model
   D. The Weaver-Biedenharn-Cusson GCM(3) model
   E. The symplectic Sp(3,\(\mathbb{R}\)) model
VII. Applications of the symplectic model
   A. Unitary irreps of the Sp(3,\(\mathbb{R}\)) Lie algebra
   B. Calculations with schematic algebraic interactions
   C. Sp(1,\(\mathbb{R}\)) and Sp(2,\(\mathbb{R}\)) submodels
   D. Applications with interacting-nucleon Hamiltonians
   E. Emergence of Sp(3,\(\mathbb{R}\)) symmetry in no-core shell model (NCSM) calculations
I. INTRODUCTION

Following the discovery of the nucleus [1, 2] and the development of the Bohr-Sommerfeld model [3, 4] of the atom, the modern era of nuclear physics was initiated by the observations of patterns in the properties of nuclei which indicated the emergence of simple structures where chaotic behaviour had been expected. Nobel prizes appropriately rewarded the early model interpretations of these properties, one of which [5–7] led to the nuclear shell model [8–11] and another [12–14] to the collective model and the Bohr-Mottelson-Nilsson unified model [15–18]. Reviews of the many developments that followed can be found in three recent edited volumes [19–21].

A comparison of the dynamics of a rotating nucleus with that of a macroscopic quantum superfluid proves to be instructive. For a quantum fluid, such as liquid helium below a critical temperature, it can happen that, after reaching an equilibrium steady state in a slowly rotating non-symmetrical container, the circulation of the fluid’s current flow vanishes and the flow becomes irrotational. Recent experimental investigations [22] also find that the slow rotational dynamics of a low-temperature $^6$Li gas are likewise the irrotational flows of a superfluid. At higher angular velocities, it can become energetically favourable for a rotating quantum fluid to lower its kinetic energy at the expense of increasing its potential energy by the creation of quantised vortices. At temperatures above the critical temperature, the viscosity of the fluid ceases to be zero and the equilibrium rotations become those of a normal fluid with rigid-body moments of inertia and perturbations due to the Coriolis and centrifugal forces.

Experimental observations show that the moments of inertia required to describe the observed rotational states of well-deformed nuclei are typically 4-5 times those for irrotational flow and approximately half those of a rigid body [23]. Thus, given that nuclei are unique zero-temperature finite many-fermion quantum systems, it is of fundamental interest to understand the nature of their rotational dynamics. The objective of this review is to identify realistic but practical methods for describing the dynamics of non-symmetrical nuclear rotations in deformed nuclei and the emergent phenomenon of shape coexistence [24–26] in terms of many-nucleon quantum mechanics. It starts with an assessment of what has been achieved and proceeds by considering promising strategies for further progress in light of the powerful resources that are now available.

It is useful to begin by considering the differences between translations and rotations of a nucleus. A fundamental difference is that, whereas the Hamiltonian of an isolated nucleus is invariant under both translations and rotations, there is no rotational analog of Galilean invariance; rotational motions are coupled to other degrees of freedom by inertial Coriolis and centrifugal forces. One might nevertheless expect that perturbations by the inertial forces of the intrinsic structure of a stable and well-deformed rotational nucleus could be negligible in a state of small angular

---

VIII. Algebraic mean-field (AMF) theory
   A. AMF theory as a generalisation of HF theory
   B. More general AMF theories, co-adjoint orbits, and geometric quantisation
   C. The symplectic model as a unified model

IX. Low-energy collective states of symplectic model irreps
   A. Angular-momentum-projected rotational states of an axially symmetric $\text{Sp}(3, \mathbb{R})$ irrep
   B. The rotational states of a triaxial $\text{Sp}(3, \mathbb{R})$ irrep
   C. Relationship of the AMF approximations to the $\text{Sp}(1, \mathbb{R})$ and $\text{Sp}(2, \mathbb{R})$ submodels
   D. Generator-coordinate theory for single $\text{Sp}(3, \mathbb{R})$ many-nucleon irreps

X. The many-nucleon Hilbert space as a sum of collective model subspaces
   A. Coupling schemes for nuclear Hilbert spaces
   B. Energy-ordered symplectic-model subspaces
   C. Generator-coordinate theory for mixed $\text{Sp}(3, \mathbb{R})$ many-nucleon irreps

XI. Concluding remarks
   Acknowledgments
   A. $\text{SU}(1,1)$ scale transformations of an $\text{Sp}(3, \mathbb{R})$ lowest-weight state
   B. Vector-coherent-state (VCS) representations of $\text{SU}(3)$ in an $\text{SO}(3)$ basis
   C. The $K$ quantum number in the $\text{SU}(3)$ model
   References
momentum. This would be the case for an adiabatically rotating nucleus in classical mechanics. However, in quantum mechanics, the angular momentum is quantised and the smallest non-zero value for an even-mass nucleus might already be too large for a rotational state of a well-deformed nucleus to be considered adiabatic.

A possible interpretation of nuclear rotations is that the low-energy states of a rotational band already contain quantised vortices which increase in number continuously with increasing angular momentum with the result that the kinetic energy of the rotations decreases and the potential energy, associated with the creation of vortices, increases such that the two components of the energy combine to give the near \( L(L + 1) \) energy-levels observed in rotational nuclei.

A first interpretation of nuclear rotations, in terms of many-nucleon quantum mechanics, was given by Elliott’s SU(3) model [10] which, with an effective charge and an effective interaction, has many of the observable properties of a rotor model. This is now understood from the observation that the SU(3) Lie algebra is the projected image of a rigid-rotor algebra onto a shell-model subspace and, as a result, retains many of the properties of a genuine rotor model as an effective shell-model of nuclear rotations, as discussed in Section VI. Thus, for example, whereas the rotational states of a rigid rotor model [27] all have identical potential energies, which implies that its rotational energies are purely kinetic, the states of an SU(3) model are exactly the opposite; they have identical kinetic energies and their energies are purely potential. A realistic theory of nuclear rotations evidently lies between the rigid rotor and SU(3) limits.

In pursuing practical approaches to many-nucleon quantum mechanics, it is profitable to begin with an expression of the Hilbert space of a nucleus as an ordered sum of subspaces, each of which is the space for an irrep (irreducible representation) of some dynamical group of unitary transformations. Such a decomposition and classification of many-nucleon states is said to define a coupling scheme. Expressing the infinite-dimensional many-nucleon Hilbert space of a nucleus as an ordered sum of subspaces is necessary, in practical calculations, in order that it may be restricted to a finite-dimensional subspace in a meaningful way. An ideal ordering is such that calculations of low-energy eigenstates of a given Hamiltonian can be implemented in a sequence of spaces of increasing dimensions until convergence of the results is achieved to some acceptable level of accuracy. However, it is important to be aware, as illustrated dramatically in Sect. VII B, that states of much lower energy could still occur if a differently ordered basis were chosen. This characteristic of an infinite-dimensional Hilbert space undoubtedly underlies the phenomenon of shape coexistence [20] in which states show up in the low-energy spectra of nuclei that, in a standard shell-model coupling scheme, are expected to lie high in energy. Thus, it is appropriate to choose a coupling scheme for a microscopic calculation that is based on the unitary irreps of the Lie algebra of a dynamical model which to some level of approximation describes the properties of the low-energy states of interest. A given nuclear Hilbert space can have many coupling schemes [11, 28, 33]. In accord with the above philosophy, we focus here on schemes associated with algebraic models aimed at understanding the emergence of collective properties in nuclei.

This review starts with outlines of the several approaches to questions concerning the deformations of nuclei and the dynamics of their collective motions. These questions have been addressed primarily in two ways: one based on mean-field theory and the other on algebraic methods. Mean-field approaches are natural because the highly successful phenomenological unified model is based on the correlation of the many-nucleon dynamics within the common mean field in which the nucleons move. On the other hand, an algebraic approach is appropriate because the quantum mechanics of any system is fundamentally an algebraic model [31, 53]. Moreover, the standard model of a nucleus, given by non-relativistic many-nucleon quantum mechanics, is a particularly simple algebraic model with a uniquely defined Hilbert space, namely that of a totally antisymmetric unitary irrep of the Lie group of all one-body unitary transformations. This is significant because of the wealth of elegant and highly developed mathematical tools available for the study of such systems. It implies, for example, that the Hilbert space of an \( A \)-fermion nucleus is spanned by Slater determinants made up of \( A \) single-fermion wave functions. As a result, algebraic and symmetry-based methods provide relationships between the many different models and the possibility of exploiting their various insights. A primary objective of this review is to expose the advantages of combining algebraic and mean-field methods. Also discussed, will be the relationships this provides between the classical and quantum mechanical treatments of a dynamical system.

It must be acknowledged at the outset that the many historical developments mentioned in this review are a subset of those that merit recognition. Those referenced are representative of the many that have been influential in the evolution of the models and microscopic theory of nuclear collective structure outlined in this review. For example, important developments that are not discussed in this review are the significant density-functional methods [30, 38] which are aimed primarily at obtaining reliable characterisations of nuclear properties in terms of their sizes, density distributions, binding energies, etc., for the many practical purposes in which such information is needed. Density-functional theory clearly has a vested interest in the foundations of mean field theory [39] and has some overlaps with the considerations of this review. Also omitted are the IBM (Interacting Boson Model) [40] methods which effectively illustrate the power of algebraic methods and to which some of what is discussed in this review applies.

A particularly significant omission in this review is the important extension of the mean-field theories based on
independent-particle approximations to the independent quasi-particle theories, based on the BCS theory \[41\] of superconductivity, which were brought into nuclear physics by Bohr, Mottelson and Pines \[42\], Belyaev \[43\], Baranger \[44\], and others. Algebraic mean-field (AMF) methods can no doubt be extended to include the symmetry-breaking correlations brought about by pairing interactions. However, an alternative and promising approach, within the framework of the proposed AMF theory, is suggested which exploits the so-called quasi-dynamical symmetries that are observed in situations in which the dominant dynamics are adiabatic \[45–49\] and which avoids violating nucleon-number conservation.

More complete reviews of most of the separate topics that feature in this paper can be found in the literature; see, for example, Refs. \[19–21, 50, 51\] and some standard texts \[52–56\].

II. THE CRANKING MODEL

An early model of nuclear rotational dynamics was the Inglis cranking model \[57, 58\] for deriving moments of inertia. The basic cranking model considers a wave function for a deformed nucleus in a body-fixed frame that is rotating about a fixed axis with a small angular velocity $\omega$, and seeks to determine the rotational component of its energy which it equates with \[1\] $\frac{1}{2} J^2 \omega^2$, where $J$ is the moment of inertia for rotation about the chosen axis. This model has been remarkably influential and has contributed to subsequent developments in mean-field theory, e.g., the Thouless-Valatin model (see Sect. IV).

If $\hat{H}_0$ is an independent-particle model Hamiltonian for the intrinsic states of a deformed rotational nucleus, as given, for example, by the Nilsson model \[17\], then the wave function for the nucleus, when rotating with some small angular velocity $\omega$, is assumed in the cranking model to be a solution of the time-dependent Schrödinger equation

$$\hat{H}_0|\phi_\omega(t)\rangle = i\hbar \frac{\partial}{\partial t}|\phi_\omega(t)\rangle. \tag{1}$$

However, if $|\phi_\omega(t)\rangle$ is transformed to the rotating frame of reference

$$|\phi_\omega(t)\rangle \rightarrow |\tilde{\phi}_\omega(t)\rangle = e^{-i\omega t \hat{J}_3}|\phi_\omega(t)\rangle, \tag{2}$$

where $\hat{J}_3$ is the 3-component of the angular-momentum operator, it becomes a stationary state of the form

$$|\tilde{\phi}_\omega(t)\rangle = e^{-\frac{i}{\hbar}E(\omega)t}|\tilde{\phi}_\omega\rangle \tag{3}$$

It follows that the state $|\tilde{\phi}_\omega\rangle$ is a solution of the time-independent Schrödinger equation

$$(\hat{H}_0 - \hbar \omega \hat{J}_3)|\tilde{\phi}_\omega\rangle = E(\omega)|\tilde{\phi}_\omega\rangle \tag{4}$$

and is given, for an arbitrarily small value of $\omega$, to first-order in perturbation theory by

$$|\tilde{\phi}_\omega\rangle = |0\rangle + \hbar \omega \sum_\nu |\nu\rangle \frac{|\nu\rangle \langle \nu| \hat{J}_3 |0\rangle}{E_\nu - E_0}, \tag{5}$$

where $|0\rangle = |\tilde{\phi}_0\rangle$ is the ground state of $\hat{H}_0$ and $|\nu\rangle$ is an excited state of excitation energy $E_\nu - E_0$. The corresponding moment of inertia of the cranking model

$$\mathcal{J}_3 = 2\hbar^2 \sum_\nu \frac{|\langle \nu| \hat{J}_3 |0\rangle|^2}{E_\nu - E_0}, \tag{6}$$

is then obtained by equating the energy increase with the rotational energy in the equation

$$\langle \tilde{\phi}_\omega|\hat{H}_0|\tilde{\phi}_\omega\rangle = E_0 + \frac{1}{2} \mathcal{J}_3 \omega^2. \tag{7}$$

In a simple application to an even-even nucleus \[59\], $\hat{H}_0$ was taken to be the triaxial harmonic-oscillator Hamiltonian

$$\hat{H}_0 = \frac{1}{2} \sum_{ni} \hbar \omega_i (b_n^\dagger b_n + b_n b_n^\dagger), \tag{8}$$
in which \( n \) is summed over the nucleon number, \( i \) is summed over the three coordinate axes, and \( b_{ni}^\dagger \) and \( b_{ni} \) are harmonic oscillator raising and lowering operators that obey the boson commutation relations

\[
[b_{ni}, b_{mj}^\dagger] = \delta_{m,n} \delta_{i,j}.
\] (9)

The spins of the nucleons were considered to be coupled to zero, so that \( \hat{J}_3 \) could be replaced by the orbital angular momentum

\[
\hbar \hat{L}_3 = \sum_n (\hat{x}_{n1} \hat{p}_{n2} - \hat{x}_{n2} \hat{p}_{n1}),
\] (10)

and expressed in terms of the raising and lowering operators by the standard relationships

\[
\hat{x}_{ni} = \frac{1}{\sqrt{2a_i}} (b_{ni}^\dagger + b_{ni}), \quad \hat{p}_{ni} = i\hbar \frac{a_i}{\sqrt{2}} (b_{ni}^\dagger - b_{ni}),
\] (11)

with \( a_i = \sqrt{M\omega_i / \hbar} \). The cranking model then gives the moment of inertia

\[
\mathcal{I}_3 = \frac{\hbar^2}{2\omega_1 \omega_2} \left[ \frac{\omega_2 - \omega_1}{\omega_2 + \omega_1} (\sigma_1 + \sigma_2) + \frac{\omega_2 + \omega_1}{\omega_2 - \omega_1} (\sigma_1 - \sigma_2) \right],
\] (12)

with

\[
\sigma_i = \frac{1}{2} \sum_n \langle 0 | b_{ni}^\dagger b_{ni} + b_{ni} b_{ni}^\dagger | 0 \rangle,
\] (13)

This result was also derived without the use of perturbation theory by Valatin [60]. It is remarkable, as noted by Bohr and Mottelson [53, 59], because if \( \sigma_1 = \sigma_2 = \sigma_3 \) but the values of \( \omega_i \) are not necessarily equal, the cranking-model moments of inertia are precisely those of an irrotational-flow model. And, if \( \sigma_1, \sigma_2, \) and \( \sigma_3 \) are not all equal, but the harmonic oscillator frequencies satisfy the shape-consistency condition (cf. Section IV),

\[
\sigma_1 \omega_1 = \sigma_2 \omega_2 = \sigma_3 \omega_3,
\] (14)

then the moments of inertia are those of a rigid-body.

Two other results are worth noting. The first is when \( \sigma_1 > \sigma_2 \) and \( \sigma_2 = \sigma_3 \) or \( \sigma_1 = \sigma_2 \) and \( \sigma_2 > \sigma_3 \) and the shape-consistency condition (14) is satisfied. The nucleus then has an axis of symmetry and its wave function is invariant under rotations about this symmetry axis. It then has no collective degree of freedom corresponding to rotations about this symmetry axis, and its cranking model moment of inertia for rotations about axes orthogonal to its symmetry axis is that of a rigid rotor. The second is that the cranking-model moments of inertia are generally consistent with the expectations of the Bohr-Mottelson unified model [10]. As a quantum liquid-drop model, the moments of inertia of the Bohr model in deformed rotational states have the irrotational-flow values of a quantum fluid. However, in the unified model, the closed-shell-core component of a nucleus, when polarised to a non-spherical shape by extra-core nucleons, was expected to contribute an irrotational flow component to the moments of inertia while the extra-core nucleons were expected to make a much larger contribution to the total moments of inertia; in the Inglis model [52], the extra-core nucleons bring the combined moment of inertia up to that of a rigid body.

In fact, the rigid-body moments of inertia of the simple cranking model are much larger than those observed which lie midway between irrotational and rigid-body values. It is also known that closer agreement with observed moments of inertia can be obtained if more realistic rotational wave functions are used in the cranking model. Thus, Belyaev [13] obtained an expression which gave much closer agreement with experiment by including the nucleon spins and replacing the independent-particle intrinsic state of the Inglis model with an independent quasi-particle state. He thereby included the effects of pairing correlations and, as expected from the BCS theory of superconductivity [41], obtained cranking model moments of inertia closer to those of a superfluid. Other extensions have also been made to the cranking model; cf., for example, Refs. [61–66]. Particularly influential were the self-consistent field methods of Thouless and Valatin [67, 68], which are discussed briefly in Section IV.

A possible interpretation of the extended cranking model results is that, with the inclusion of pairing interactions for example, the ratio \( \omega_2/\omega_3 \) in the cranking model formula (12) is effectively changed from the shape-consistent value of mean-field theory without pairing, such that Eqn. (12) does give the experimentally determined moments of inertia. The moments of inertia of the cranking model then have an interpretation [60] as those of a rotating Riemann ellipsoid in which the rotations are linear combinations of irrotational and rigid flows.

Such an interpretation of nuclear moments of inertia in terms of rigid and irrotational current flows implicitly assumes that rotational energies are kinetic energies. This would appear to be highly reasonable given that the
underlying concept of a rotational model is that its energies are kinetic energies with potential energy contributions only arising, for example, due to centrifugal stretching and Coriolis perturbations. It is nevertheless important to ascertain the extent to which the so-called rotational energies arising in model calculations are consistent with this presumption. It is also remarkable that the cranking model leads to such simple and elegant results for what is essentially a complex problem; it is known, for example, that pure rigid flow is only attainable in an unphysical limit in the quantum mechanics of a many-fermion system [71–73].

III. MEAN-FIELD THEORY AS AN INTERFACE BETWEEN CLASSICAL AND QUANTUM MECHANICS

From an algebraic perspective, mean-field theory is a coherent-state theory and, as such, it can be understood in terms of both classical and quantum mechanics. It was characterised as a semi-classical theory long ago as a way of explaining the identical equations obtained in Time-Dependent Hartree-Fock theory [74–76], and in the Random Phase Approximation [76–78]. As reviewed in this section, small-amplitude TDHF theory has a quantal interpretation that is equivalent to that of the RPA [79, 80] and, more generally [81], it provides an interface between classical and quantum mechanics. These properties are important because, as shown in Section VIII, standard Hartree-Fock mean-field theory can be extended to a large class of algebraic models to produce powerful new results.

It is first useful to recall the prototype coherent-state relationships between the classical and quantal representations of the Heisenberg-Weyl algebra [82].

A. Quantisation and dequantisation of a Heisenberg-Weyl algebra

The Dirac quantisation [34, 35, 83] of the dynamics of a particle is achieved by mapping the position and momentum coordinates, \( \{ x_i, p_i ; i = 1, 2, 3 \} \), of its classical phase space to operators \( \{ \hat{x}_i, \hat{p}_i ; i = 1, 2, 3 \} \) on a Hilbert space of square-integrable wave functions of the nucleon coordinates such that

\[
\hat{x}_i \psi(x) = x_i \psi(x), \quad \hat{p}_i \psi(x) = -i\hbar \frac{\partial}{\partial x_i} \psi(x).
\]

This corresponds to constructing a unitary irrep of the Heisenberg-Weyl algebra in which the coordinates satisfy the commutation relations

\[
[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{i,j} \hat{1}, \quad [\hat{x}_i, \hat{x}_j] = [\hat{p}_i, \hat{p}_j] = 0,
\]

where \( \hat{1} \) is the identity operator.

Conversely, classical mechanics is regained from quantum mechanics by coherent-state methods [84–87]. Let \( |0\rangle \) denote a minimum-uncertainty state for the position and momentum observables of the particle given by the ground state of a harmonic-oscillator Hamiltonian \( \hat{H}_{\text{HO}} = \sum_i (\alpha_i \hat{x}_i^2 + \beta_i \hat{p}_i^2) \). Then

\[
\langle 0 | \hat{x}_i | 0 \rangle = \langle 0 | \hat{p}_i | 0 \rangle = 0, \quad \text{for all } i,
\]

and it follows that the coherent states

\[
|x, p\rangle = \exp \left[ \frac{i}{\hbar} \sum_i \left( p_i \hat{x}_i - x_i \hat{p}_i \right) \right] |0\rangle,
\]

with real values of \( x_i \) and \( p_i \), have the property that

\[
\langle x, p | \hat{x}_i | x, p \rangle = x_i, \quad \langle x, p | \hat{p}_i | x, p \rangle = p_i.
\]

Other classical observable are similarly defined as functions of the \( x_i, p_i \) phase-space coordinates by their quantum mechanical expectation values. For example, if \( \hat{H} \) is the quantum mechanical Hamiltonian, the corresponding classical Hamiltonian is given by

\[
\mathcal{H}(x, p) = \langle x, p | \hat{H} | x, p \rangle.
\]

From the identities

\[
\frac{\partial}{\partial x_i} |x, p\rangle = -i\hbar \hat{p}_i |x, p\rangle,
\]

\[
\frac{\partial}{\partial p_i} |x, p\rangle = i\hbar \hat{x}_i |x, p\rangle,
\]

\[
\langle x, p | \hat{x}_i | x, p \rangle = x_i.
\]

\[
\langle x, p | \hat{p}_i | x, p \rangle = p_i.
\]
it then follows that

\[
\frac{\partial \mathcal{H}}{\partial x_i} = -\frac{i}{\hbar} \langle x, p | [\hat{H}, \hat{p}_i] | x, p \rangle, \\
\frac{\partial \mathcal{H}}{\partial p_i} = \frac{i}{\hbar} \langle x, p | [\hat{H}, \hat{x}_i] | x, p \rangle,
\]

(22)

The time-dependent Schrödinger equation

\[
i\hbar \frac{\partial x_i}{\partial t} = \langle x, p | [\hat{x}_i, \hat{H}] | x, p \rangle, \\
i\hbar \frac{\partial p_i}{\partial t} = \langle x, p | [\hat{p}_i, \hat{H}] | x, p \rangle,
\]

(23)

then leads to the classical equations of motion

\[
\frac{\partial x_i}{\partial t} = \frac{\partial \mathcal{H}}{\partial p_i}, \quad \frac{\partial p_i}{\partial t} = -\frac{\partial \mathcal{H}}{\partial x_i},
\]

(24)

These relationships reveal an embedding of the classical phase space for a system of spin-free particles as a smooth manifold in the Hilbert space of its quantum mechanics in which a classical point with coordinates \(x = \{x_n\}\) and \(p = \{p_n\}\) is identified with a quantum-mechanical coherent state \(|x,p\rangle\).

For a system of particles with intrinsic spins, the relationship between its classical and its quantum mechanics would appear to be much more complicated. In fact, for nucleons, which are spin-half fermions, it is actually simple because the Hilbert space of a many-fermion system contains only fully anti-symmetric states; this restricts them to a single irrep and imposes a huge restriction on the possible many-fermion states.

### B. Mean-field theory as a classical representation of a Lie algebra

A remarkable observation is that a coherent-state theory analogous to that used above to obtain an interface between the classical and quantum mechanics of the Heisenberg-Weyl Lie algebra applies, with some generalisation, to any Lie algebra whose irreps have lowest- (and/or highest-) weight states. The best-known example is given by Hartree-Fock mean-field theory.

The Hartree-Fock theory of a many-fermion system starts from the observation that its Hilbert space is that of a fully anti-symmetric unitary irrep of a Lie algebra of one-body operators. In the language of second quantisation, this irrep is expressed in terms of creation and annihilation operators, \(a_\mu^\dagger\) and \(a^\mu\), of single-nucleon states as linear combinations of the one-body operators

\[
\hat{C}_\mu^\nu = a_\mu^\dagger a^\nu,
\]

(25)

for which the commutation relations

\[
[\hat{C}_\mu^\nu, \hat{C}_{\mu'}^{\nu'}] = \delta_{\mu \mu'} \hat{C}_{\nu \nu'} - \delta_{\nu \nu'} \hat{C}_{\nu \nu'},
\]

(26)

are obtained from the fermion anti-commutation relations

\[
\{a_\mu^\dagger, a_\nu^\dagger\} = a_\mu^\dagger a_\nu^\dagger + a_\nu^\dagger a_\mu^\dagger = \delta_{\mu \nu}, \quad \{a^\mu, a^\nu\} = \{a_\mu^\dagger, a_\nu^\dagger\} = 0.
\]

(27)

Following an Einstein convention, creation operators \(\{a_\nu^\dagger\}\), with lower indices, are components of a covariant tensor and annihilation operators \(\{a^\nu\}\), with upper indices, are components of a contravariant tensor. They are defined such that a scalar product of these operators is obtained by summing over common upper and lower indices; e.g., if the indices are SU(2) angular-momentum quantum numbers \(\nu = jm\), for a given value of \(j\), a scalar number operator is defined by the sum

\[
\hat{N}_j = \sum_m a_j^\dagger j m a_j^m.
\]

(28)

The standard coupling of covariant SU(2) tensors of the same angular momentum to form a scalar, is given by

\[
[a_j^\dagger \otimes a_j]_0 = \sum_m (j, -m, j, m|0, 0) a_j^\dagger j m a_j, -m = \sum_m \frac{(-1)^{j+m}}{\sqrt{2j+1}} a_j^\dagger j m a_j, -m.
\]

(29)
Thus, SU(2) tensors with upper and lower indices can be related by defining

$$a_{jm} \equiv a^{jm} = (-1)^{j+m} a_{j-m}.$$  \hspace{1cm} (30)

With this definition, it then follows that

$$\hat{N}_j = \sqrt{2j+1} \left[ a_j \otimes a_j \right]_0,$$  \hspace{1cm} (31)

and that pairs of covariant fermion operators satisfy the anti-commutation relationships

$$\{a_{jm}, a_{jm}'\} \equiv \{a^{jm}, a^{jm}'\} = \delta_{j}^j \delta_{m}^{m'}.$$  \hspace{1cm} (32)

It is important to note, however, that the relationship between the covariant and contravariant components of a tensor depends on the nature of the tensor. It is given here for the standard labelling of SU(2) and SO(3) tensors. For Cartesian tensors, which have a positive-definite metric, there is no need to make a distinction; i.e., in terms of an orthonormal basis, $x^i \equiv x_i$ and $x^2 = \sum_i x_i^2$.

The Hilbert space of a nucleus is now determined by standard Lie algebra methods in terms of a lowest-weight state and sets of raising and lowering operators, defined as follows. For an $A$-nucleon nucleus, a lowest-weight state for an irrep of the Lie algebra of one-body operators is a state whose wave function is a Slater determinant of the wave functions of occupied single-nucleon states, as explained below. These states might be selected, for example, on the basis of independent-particle shell model considerations. However, the choice is, in principle, arbitrary. For any given choice, raising and lowering operators are defined as follows.

Let indices $h$ (for hole) and $p$ (for particle) label single-particle states that are, respectively, occupied and unoccupied in a lowest-weight state $\phi$. The operators

$$\hat{\eta}_h^p = a^h_j a^p_{j} \quad \text{and} \quad \hat{\eta}_h^p = a^p_{j} a^h_{j},$$  \hspace{1cm} (33)

are then, respectively, particle-hole creation and annihilation operators and the lowest-weight state is a so-called particle-hole vacuum state. Starting from such a state, a basis for the many-nucleon Hilbert space of a nucleus is obtained by exciting the vacuum state to many particle-hole states generated by the actions on it of the particle-hole creation operators as raising operators.

It is now apparent that any Slater-determinant state is a particle-hole vacuum state with respect to a corresponding definition of particle and hole operators and can serve as a lowest-weight state for the construction of the quantum-mechanical Hilbert space of a nucleus. Moreover, any normalised Slater determinant is related to any other such Slater determinant by a unitary transformation of its occupied single-particle states. Thus, the Slater determinants of Hartree-Fock theory are said to lie on an orbit of the group of unitary one-body transformations and, by definition, coherent states of this group \[80-82\]. As a result \[83\], the Slater determinants of Hartree-Fock theory form a manifold of lowest-weight states that is in one-to-one correspondence with a classical phase space on which a classical Hamiltonian dynamics is defined in parallel with that for the Heisenberg algebra. Such properties of Lie group orbits are well known in mathematics \[84\] and feature in the theory of geometric quantisation \[85\,86\].

The identification of Hartree-Fock mean-field states (normalised Slater determinants) with points on a classical phase space is obtained by first defining local position and momentum operators

$$\hat{x}_{ph} = \frac{1}{\sqrt{2}} (\hat{\eta}_h^p + \hat{\eta}_p^h), \quad \hat{p}_{ph} = \frac{i\hbar}{\sqrt{2}} (\hat{\eta}_h^p - \hat{\eta}_p^h),$$  \hspace{1cm} (34)

for a neighbourhood of an arbitrary particle-hole vacuum state $|\phi\rangle$. These operators satisfy the equations

$$\langle \phi | [\hat{x}_{ph}, \hat{p}_{ph'}] |\phi\rangle = i\hbar \delta_{p,p'} \delta_{h,h'},$$  \hspace{1cm} (35)

and can be used to define mean-field states with local phase space coordinates $(x_{ph}, p_{ph})$ coordinates defined by

$$|\phi(x, p)\rangle = \exp \left[ \frac{i}{\hbar} \sum_{ph} \left( p_{ph} \hat{x}_{ph} - x_{ph} \hat{p}_{ph} \right) \right] |\phi\rangle.$$  \hspace{1cm} (36)

Such $(x_{ph}, p_{ph})$ coordinates can be defined for a finite neighbourhood of any point on the manifold of lowest-weight states \[87\]. This is always possible for a classical phase space for which a neighbourhood of any point can be put into one-to-one correspondence with a neighbourhood of a flat Euclidian space. However, a single global $(x, p)$ coordinate chart, for which the equations \[45\] hold at every point, would not be possible on a phase space which, for example,
had the topology of a sphere. This now a problem, however, because there is no requirement that a classical phase space should be Euclidean. The important observation is that for a given Hamiltonian $\hat{H}$, the classical equations of motion
\begin{equation}
\frac{dx_{ph}}{dt} = \frac{\partial \mathcal{H}}{\partial p_{ph}}, \quad \frac{dp_{ph}}{dt} = -\frac{\partial \mathcal{H}}{\partial x_{ph}},
\end{equation}
are determined to apply precisely at any point of a smooth classical phase-space. This follows, as for the Heisenberg-Weyl Lie algebra, because any point of the space can be considered as the origin of a local coordinate chart for a neighbourhood of that point. Thus, the infinitesimal generators $\hat{x}_{ph}$ and $\hat{p}_{ph}$ can be chosen at any point to satisfy the Eqsns. exactly.

The Hartree-Fock $|\phi_{0}\rangle$ state is now selected as the independent-particle state on the manifold of lowest-weight states for which the energy $\langle \phi_{0}|\hat{H}|\phi_{0}\rangle$ is a minimum and which therefore satisfies the equations
\begin{equation}
\langle \phi_{0}|[\hat{H}, a_{\mu}^{\dagger} a_{\nu}]|\phi_{0}\rangle = 0, \quad \text{for all } p \text{ and } h.
\end{equation}
Thus, if $(x, p)$ coordinates are defined about such a state $|\phi_{0}\rangle$ as a point of phase space, the classical Hamiltonian $\mathcal{H}$ satisfies the equations
\begin{equation}
\left(\frac{\partial \mathcal{H}}{\partial x_{ph}}\right)_{x=p=0} = \left(\frac{\partial \mathcal{H}}{\partial p_{ph}}\right)_{x=p=0} = 0, \quad \text{for all } p \text{ and } h.
\end{equation}
The standard Hartree-Fock procedure for identifying such a minimal energy independent-particle state is to start from some reasonable first guess $|\phi\rangle$ and define an independent-particle Hamiltonian
\begin{equation}
\hat{h} = \sum_{\mu, \nu} \langle \phi|\{a_{\mu}^{\dagger}, [\hat{H}, a_{\nu}^{\dagger}]\}|\phi\rangle a_{\mu}^{\dagger} a_{\nu}.
\end{equation}
Diagonalisation of this Hamiltonian then gives a new lower-energy state as its lowest-energy eigenstate $|\phi_{0}\rangle$. This process is iterated until it converges to a state $|\phi_{0}\rangle$ that satisfies the variational equation.

It is now observed that Hartree-Fock theory is much more than a means to determine an approximation for the ground state of a quantum-mechanical system. It provides an explicit map of the quantum mechanics of a many-fermion system to a corresponding classical mechanics in which the Hartree-Fock Hamiltonian
\begin{equation}
\hat{h}(x, p) = \sum_{\mu, \nu} \langle \phi(x, p)|\{a_{\mu}^{\dagger}, [\hat{H}, a_{\nu}^{\dagger}]\}|\phi(x, p)\rangle a_{\mu}^{\dagger} a_{\nu},
\end{equation}
defined in terms of local $(x, p)$ coordinates at any point of a phase space, determines the classical dynamics at that point. Thus, a TDHF dynamics is defined by the classical equations of motion on this phase space, without the need of an adiabatic approximation as has often been considered necessary for such purposes. An adiabatic approximation may nevertheless be useful to restrict the large-amplitude mean-field dynamics to the valley floor of the classical phase space, as defined in [97]; cf. also [96, 97]. Such large-amplitude TDHF dynamics can be relevant, for example, to the study of heavy-ion reactions and nuclear fission.

The relationship between the classical and quantum representations of the Lie algebra of one-body operators of a many-fermion system was shown in nuclear physics [98–100] and used by many to explore the topography of the classical space of mean-field energies in terms of valleys, fall lines, peaks, ridges and passes. Applications of some of these methods to the theory of nuclear structure were reviewed in terms of generator-coordinate methods by Reinhard and Goeke [100]. Other methods are currently being developed by Matsuyanagi and colleagues [96, 97]. It should be remembered, however, in considering the topographical maps obtained in mean-field theory, that distance scales on the map will generally vary from point-to-point and can give a distorted view of the situation as shown in a rigorous coordinate-free treatment [95, 107]. Distance scales and their directional dependence, defined by the metric on the multi-dimensional mean-field manifold, can be determined at a point by a so-called local harmonic-oscillator treatment, corresponding to constrained RPA solutions [100, 102]. One result is that the path which follows the bottom of a valley will only, in general, be identical to a fall line, i.e., a line of steepest descent [107], if neither is curved. Such distortions are familiar in topographical maps of landscapes on the curved surface of Earth.
C. Classical normal-mode vibrations and the random phase approximation

It will be taken as understood that the nuclear Hamiltonian $\hat{H}$ is rotationally invariant. For the purposes of this section, it will also be understood that the minimum-energy HF state is uniquely defined. This being the case, implies that it also rotationally invariant and consequently it is a state of zero angular momentum. For, if it were not, it would be one of a set of minimum-energy states generated by its rotations. This more general case, is of special interest and is considered in depth in Section IV.

The dual classical/quantal interpretation of TDHF theory relates the classical small-amplitude normal-mode vibrations of a many-fermion system about its lowest-energy equilibrium state to the quantum mechanics of the elementary excitations of a nucleus as given in the RPA (random-phase approximation).

The small-amplitude normal-mode vibrations of a classical system about an equilibrium state are obtained by expanding the Hamiltonian $H$ to quadratic terms in the amplitudes of the position and momentum coordinates which take zero values when the system is at equilibrium. For position and momentum coordinates $\{x_\alpha\}$ and $\{p_\alpha\}$ that are canonical at the equilibrium point, the Hamiltonian is then of the form

$$\mathcal{H}(x,p) = E_0 + \sum_{\alpha,\beta} \frac{1}{2} B_{\alpha\beta} p_\alpha p_\beta + \sum_{\alpha,\beta} \frac{1}{2} C_{\alpha\beta} x_\alpha x_\beta + \cdots ,$$

where $E_0$ is the equilibrium energy of the lowest-energy state. The Hamilton equations of motion (37) then give harmonic small-amplitude vibrational solutions with

$$\frac{dx_\alpha}{dt} = \sum_\beta B_{\alpha\beta} p_\beta, \quad \frac{dp_\alpha}{dt} = -\sum_\beta C_{\alpha\beta} x_\beta .$$

The corresponding mean-field equations are obtained as follows. With the notation that $m$ and $n$ label particle states and $i$ and $j$ label hole states, a small-amplitude time-dependent mean-field vibrational state is expressed as

$$|\phi(t)\rangle = e^{-iE_0 t} e^{i\hat{X}(t)} |\phi_0\rangle ,$$

where $\epsilon$ is a small parameter and $\hat{X}(t)$ is the skew-Hermitian operator

$$\hat{X}(t) = \sum_{mi} \left[ X_{mi}(t) a_m^\dagger a_i - X^*_m(t) a_i^\dagger a_m \right] .$$

From the identities

$$\langle \phi(t) | a_i^\dagger a^m | \phi(t) \rangle = \epsilon X_{mi}(t) + O(\epsilon^3) ,$$

$$i\hbar \frac{d}{dt} \langle \phi(t) | a_i^\dagger a^m | \phi(t) \rangle = \langle \phi(t) | [a_i^\dagger a^m, \hat{H}] | \phi(t) \rangle = \epsilon \langle \phi_0 | [a_i^\dagger a^m, \hat{H}] , \hat{X}(t) | \phi_0 \rangle + O(\epsilon^3) ,$$

it follows that

$$i\hbar \frac{d}{dt} X_{mi}(t) = \langle \phi_0 | [[a_i^\dagger a^m, \hat{H}] , \hat{X}(t)] | \phi_0 \rangle + O(\epsilon^2) .$$

Thus, for harmonic small-amplitude normal-mode vibrations about the Hartree-Fock minimum-energy state, for which

$$X_{mi}(t) = Y_{mi} e^{-i\omega t} + Z^*_m e^{i\omega t} ,$$

Eqn. (50) separates into a set of coupled eigenvalue equations:

$$\sum_{nj} [A_{mnj} Y_{nj} + B_{mnj} Z_{nj}] = \hbar \omega Y_{mi} ,$$

$$\sum_{nj} [A_{mnj} Z^*_nj + B_{mnj} Y^*_nj] = -\hbar \omega Z^*_mi ,$$

where $A$ is a Hermitian matrix and $B$ is a symmetric matrix with components

$$A_{mnj} = \langle \phi | [a_i^\dagger a^m, [\hat{H}, a_j^\dagger a^n]] | \phi \rangle$$

$$B_{mnj} = -\langle \phi | [a_i^\dagger a^m, [\hat{H}, a_j^\dagger a^n]] | \phi \rangle .$$
After taking the complex conjugate of the second equation, these equations are expressed in the matrix form

$$
\begin{pmatrix}
A & B \\
B^* & A^*
\end{pmatrix}
\begin{pmatrix}
Y \\
Z
\end{pmatrix} = \hbar\omega
\begin{pmatrix}
Y \\
-Z
\end{pmatrix}.
$$

(53)

Thus, the solutions of this matrix equation for the \{Y_{mi}\} and \{Z_{mi}\} coefficients and the corresponding values of \(\omega\) define the classical small-amplitude normal-modes of a nucleus and their vibrational frequencies.

D. The quantum-mechanical Random Phase Approximation

The RPA, regarded as a quantum-mechanical coherent-state version of classical normal-mode vibrations, is now obtained \cite{74,80} by identifying the small-amplitude TDHF vibrational wave functions with harmonic-oscillator coherent states. This is achieved by recognising that the operators

$$
O^\downarrow_\lambda := \sum_{mi} (Y_{mi}(\lambda)a^\dagger_m a^i - Z_{mi}(\lambda)a^i a^\dagger_m),
$$

$$
O^\uparrow_\lambda := \sum_{mi} (Y^*_{mi}(\lambda)a^i a^\dagger_m - Z^*_{mi}(\lambda)a^\dagger_m a^i),
$$

(54)

for which the \(Y_{mi}(\lambda)\) and \(Z_{mi}(\lambda)\) coefficients are solutions of Eqn. (51) with \(\omega_\lambda = \omega\), satisfy the equations

$$
\langle \phi | [\hat{X}, [\hat{H}, O^\downarrow_\lambda]]|\phi\rangle = \hbar\omega_\lambda \langle \phi | [\hat{X}, O^\downarrow_\lambda]|\phi\rangle,
$$

$$
\langle \phi | [\hat{X}, [\hat{H}, O^\uparrow_\lambda]]|\phi\rangle = -\hbar\omega_\lambda \langle \phi | [\hat{X}, O^\uparrow_\lambda]|\phi\rangle,
$$

(55)

for any one-body operator \(\hat{X}\) and can be normalised to satisfy the orthogonality relationships

$$
\langle \phi | O^\kappa_\lambda, O^\downarrow_\lambda |\phi\rangle = \delta^\kappa_\lambda,
$$

$$
\langle \phi | O^\kappa_\lambda, O^\uparrow_\lambda |\phi\rangle = \langle \phi | [O^\downarrow_\lambda, O^\uparrow_\lambda]|\phi\rangle = 0.
$$

(56)

The operators \(O^\downarrow_\lambda\) and \(O^\uparrow_\lambda\) are then interpreted in the RPA as excitation and de-excitation operators of one-phonon vibrational excitations of the nuclear ground state \(|0\rangle\). Thus, the ground state \(|0\rangle\) of the nucleus is implicitly defined in the RPA as a state that is annihilated by the \(O^\downarrow_\lambda\) lowering operators, i.e., \(O^\downarrow_\lambda |0\rangle = 0\), for all \(\lambda\), and states of excitation energy \(E_\lambda - E_0 = \hbar\omega_\lambda\) are given by \(|\lambda\rangle = O^\downarrow_\lambda |0\rangle\).

A significant result is that Eqns. (54) - (56) imply the presence of vibrational correlations in the quantum mechanical ground state \(|0\rangle\). Such correlations are implied when the lowering operators \(\{O^\downarrow_\lambda\}\), which should annihilate the ground state, contain non-zero \(Z^*_{mi}\) terms. However, the properties of the vibrational excited states of the RPA are obtained algebraically without the need to derive explicit expression for either the correlated ground state or its excited states. In particular, matrix elements of a one-body transition operator,

$$
\hat{Q} = \sum_{\mu\nu} Q_{\mu\nu} a^\dagger_{\mu} a^\nu,
$$

(57)

between the ground and excited one-phonon states,

$$
|\lambda\rangle := O^\downarrow_\lambda |0\rangle, \quad \forall O^\downarrow_\lambda \text{ for which } \omega_\lambda > 0,
$$

(58)

are evaluated from the expression

$$
\langle 0 | \hat{Q} | \lambda \rangle = \langle 0 | [\hat{Q}, O^\downarrow_\lambda]|0\rangle,
$$

(59)

and are given, within the harmonic-oscillator approximation, by

$$
\langle 0 | \hat{Q} | \lambda \rangle = \langle \phi | [\hat{Q}, O^\downarrow_\lambda]|\phi\rangle = \sum_{mi} (Y_{mi}(\lambda)Q_{im} + Z_{mi}(\lambda)Q_{mi}).
$$

(60)

The above double-commutator equations-of-motion approach, developed in Refs. 52,108, provides the simplest and now standard expression of the RPA, as reviewed, for example, in the book of Ring and Schuck 54.
IV. THE EMERGENCE OF ROTATIONAL STATES IN MEAN-FIELD THEORY

If the minimum-energy Hartree-Fock state for an even-even nucleus is rotationally invariant, it is a state of zero angular momentum and is considered to be an approximation to the ground state of the nucleus. Moreover, if the nucleus under consideration has a low-energy excited state of some angular momentum $J$ that decays strongly, relative to a single-particle transition, by an electromagnetic $J$-pole transition to the ground state, it will naturally be expected that such a state can be interpreted as a one-phonon collective vibrational state with an excitation operator given to a good approximation by an RPA calculation. The low-energy $3^+$ excited states of $^{16}$O and $^{40}$Ca at 6.13 MeV and 3.73 MeV, respectively, are good examples of this and have been successfully treated as such in Refs. $^{109}$–$^{111}$. Numerous other examples are referenced, for example, in $^{52}$ and $^{54}$. Thus, the TDHF-RPA theory is understood to provide a good first-order many-nucleon description of one-phonon vibrational excitations of spherical nuclei. However, rotationally invariant spherical Hartree-Fock minimum-energy states are relatively uncommon. They occur for doubly closed-shell nuclei. But, even then, it is frequently observed, consistent with experimental observations $^{20}$, $^{112}$, that there are strongly deformed Hartree-Fock solutions with only a little more energy.

A. Broken symmetry and rotational states of nuclei

Most frequently, it transpires that the lowest-energy mean-field state for a rotationally invariant nuclear Hamiltonian, is neither rotationally invariant nor a state of good angular momentum. Its coupling to rotated lowest-energy mean-field states is then far from negligible, but is hidden in mean-field theory because such states have energy expectation values that remain unchanged under rotations. Consequently, the TDHF equations have zero-frequency (Nambu-Goldstone $^{113}$–$^{114}$) normal-mode solutions corresponding to rotations of the broken-symmetry mean-field state for which there are no restoring forces. Such a broken-symmetry state was interpreted by Peierls and Yoccoz $^{115}$ as a semi-classical state of a rotor model with zero angular momentum and a specific orientation. This led to the Thouless-Valatin $^{67}$–$^{68}$ generalisation of the Inglis cranking model in which the Hartree-Fock mean-field equations were solved for a Hamiltonian $\hat{H} - \hbar \omega \hat{J}_1$. An expression for the moment of inertia $\mathcal{I}_1$ was then determined from the energy increase

$$\frac{1}{2} \mathcal{I}_1 \omega^2 = \Delta E. \label{eq:61}$$

Unlike the Inglis model, the Thouless-Valatin procedure, takes full account of the self-consistent adjustment of the mean-field Hamiltonian for a rotating state of the nucleus. It can similarly be extended to include pairing interactions.

B. Angular-momentum projection

A precise expression of the Peierls-Yoccoz $^{115}$ observation, within the framework of the generator-coordinate theory of Hill, Wheeler and Griffin $^{116}$–$^{117}$, is that when the lowest-energy mean-field state is not rotationally invariant a set of low-energy states of the nucleus with good angular-momentum quantum numbers can be obtained as linear combinations of the equal-energy states generated by its rotations. This leads to a microscopic many-nucleon model of nuclear rotations in which the nuclear Hamiltonian is diagonalised in the Hilbert space spanned by these rotated states. A basis of angular-momentum states generated in this way is defined by angular-momentum projection methods. Many applications of these methods have recently been reviewed by Sun $^{118}$. Here we briefly review, with some adjustment, the elegant approach of Lee and Cusson $^{119}$–$^{120}$ as it applies to doubly even nuclei.

A Slater determinant, considered as an intrinsic state for a system of rotational states, can be expressed as a sum of states

$$| \Phi \rangle = \sum_{JK} n_{JK} | \phi_{KJK} \rangle \equiv \sum_{JK} n_{JK} | \phi_{KJK} \rangle \label{eq:62}$$

of angular momentum $J$ and component of angular momentum $K$ relative to a so-called body-fixed axes. For a given intrinsic state $| \phi \rangle$, the objective is then to identify the states $| \phi_{KJ} \rangle$ and the corresponding sets of states $\{ | \phi_{KJM} \rangle, M = -J, \ldots, J \}$ generated by rotating them. In standard mean-field theory, the state $| \Phi \rangle$ is the minimum-energy Slater determinant. More generally, it could be determined separately for each angular-momentum state by variation-after-projection methods $^{120}$. Other possibilities, in which $| \Phi \rangle$ could be one of several optimally chosen intrinsic states for the irreps of a microscopic collective model, are discussed in Sections $^{X \text{D}}$ and $^{X \text{C}}$.

Body-fixed axes, for an intrinsic state $| \Phi \rangle$, are appropriately chosen to be principal axes of its quadrupole mass tensor. However, principal axes of the quadrupole mass tensor are only defined to within the subgroup of rotations.
that leave the quadrupole mass tensor invariant. This subgroup is the so-called vierergruppe group $D_2$; i.e., the group of rotations through multiples of angle $\pi$ about each of the principle axes. A significant characteristic of Hartree-Fock mean-field theory is that, although it frequently breaks the rotational invariance of a nuclear Hamiltonian, it commonly retains its $D_2$ invariance \[12\]. It may be noted that $D_2$ invariance is conserved in the asymmetric-top models of nuclear and molecular rotations and emerges in Elliott’s SU(3) model \[11\] and Ui’s rotor model \[27\] from their algebraic structures. When $D_2$ symmetry is conserved, the intrinsic state belongs to a one-dimensional irrep of the group $D_\infty$ which includes all rotations about the symmetry axis plus rotations through multiples of angle $\pi$ about axes perpendicular to the symmetry axis. It then has one-dimensional irreps with a single basis state given by

$$|\phi_{KJM}\rangle = \sum_{JK} n_{JK} |\phi_{KJK}\rangle D_{MK}^j(\Omega),$$

where

$$D_{MK}^j(\Omega) = \langle KJM | \hat{R}(\Omega) | KJK \rangle$$

is a standard rotation matrix. Thus, the basis states $\{|\phi_{KJM}\rangle\}$ for the rotational model Hilbert space with good angular-momentum quantum numbers are obtained in the form

$$|\phi_{KJM}\rangle = \frac{1}{n_{JK}} \frac{2J+1}{8\pi^2} \int \hat{R}(\Omega) |\Phi\rangle D_{MK}^{J^*}(\Omega) d\Omega$$

with the norm factors

$$|n_{JK}|^2 = \frac{2J+1}{8\pi^2} \int D_{MK}^{J^*}(\Omega) \langle \Phi | \hat{R}(\Omega) |\Phi\rangle d\Omega.$$
If $|A\rangle$ and $|B\rangle$ are the $N$-fermion Slater determinants

$$|A\rangle = a_1^\dagger a_2^\dagger \cdots a_N^\dagger |−\rangle,$$

$$|B\rangle = b_1^\dagger b_2^\dagger \cdots a_N^\dagger |−\rangle,$$

where $|−\rangle$ is the bare-fermion vacuum state and $\{a_1^\dagger\}$ and $\{b_1^\dagger\}$ are each sets of orthogonal single-nucleon creation operators, their overlap is the determinant

$$\langle B|A \rangle = D = \text{det}(d),$$

in which $d$ is the $N \times N$ matrix whose elements,

$$d_{\mu\nu} = \langle b^\nu a_{\mu}^\dagger \rangle = \langle −|b^\nu a_{\mu}^\dagger |−\rangle,$$

are the overlaps of single-particle states; recall that $b^\nu$ is the Hermitian adjoint of $b_\nu^\dagger$.

If $\{c_\alpha^\dagger, c^\alpha\}$ is another set of single-fermion creation and annihilation operators, then the matrix element

$$\langle B|c_\beta^\dagger c^\alpha |A \rangle = \langle −|b^N \cdots b^1 b^1 c_\beta^\dagger c_\alpha^\dagger a_1^\dagger \cdots a_N^\dagger |−\rangle$$

is expressed as the sum of terms obtained by replacing the single-particle overlaps $\langle b^\nu a_{\mu}^\dagger \rangle$ in $\text{det}(d)$, one at a time, by $\langle b^\nu c_\beta^\dagger c^\alpha a_{\mu}^\dagger \rangle$. The result is that

$$\langle B|c_\beta^\dagger c^\alpha |A \rangle = \sum_{\mu \nu} \langle b^\nu c_\beta^\dagger c^\alpha a_{\mu}^\dagger \rangle M_{\mu\nu}^\nu = \sum_{\mu \nu} \langle b^\nu c_\beta^\dagger \rangle M_{\mu\nu}^\nu \langle c^\alpha a_{\mu}^\dagger \rangle,$$

where $M_{\mu\nu}^\nu$ is the cofactor of the element $d_{\mu\nu} = \langle b^\nu a_{\mu}^\dagger \rangle$ in the determinant $D = \text{det}(d)$. The matrix $M$ with elements $M_{\mu\nu}^\nu$, known as the adjugate of the matrix $d$, is simply related to the inverse of the matrix $d$ by Cramer’s rule

$$d^{-1} = \frac{1}{D} M.$$

It follows that $M = Dd^{-1}$ and that

$$\langle B|c_\beta^\dagger c^\alpha |A \rangle = D \sum_{\mu \nu} \langle b^\nu c_\beta^\dagger \rangle (d^{-1})_{\nu\mu}^\nu \langle c^\alpha a_{\mu}^\dagger \rangle,$$

as determined by Cusson and Lee.

Two-body matrix elements can be derived in a similar way. A matrix element

$$X_{\alpha \beta}^{\gamma \delta} = \langle B|c_\alpha^\dagger c_\beta^\dagger c^\gamma c^\delta |A \rangle = \langle b^N \cdots b^1 c^\gamma c^\delta c_\alpha^\dagger a_1^\dagger \cdots a_N^\dagger |−\rangle$$

is expressed as a sum of terms obtained by replacing the two-particle overlaps

$$f_{\mu \nu}^{\nu \nu'} = \langle b^\nu b^\nu' a_{\mu}^\dagger a_{\nu'}^\dagger \rangle = \langle b^\nu a_{\mu}^\dagger \rangle \langle b^\nu' a_{\nu'}^\dagger \rangle - \langle b^\nu a_{\mu}^\dagger \rangle \langle b^\nu' a_{\nu'}^\dagger \rangle,$$

in $\text{det}(d)$, one pair at a time, with $B_{\alpha \beta}^{\nu \nu'} A_{\mu \nu}^{\gamma \delta}$, where

$$B_{\alpha \beta}^{\nu \nu'} = \langle b^\nu b^\nu' c_\alpha^\dagger c_\beta^\dagger \rangle, \quad A_{\mu \nu}^{\gamma \delta} = \langle c^\gamma c^\delta a_{\mu}^\dagger a_{\nu}^\dagger \rangle.$$

Then, if $M_{\mu \nu}^{\nu \nu'}$ is the cofactor of $f_{\mu \nu}^{\nu \nu'}$ in the determinant $\text{det}(d)$, so that

$$\text{det}(d) = \sum_{\mu', \nu', \nu} f_{\mu \nu}^{\nu \nu'} M_{\mu \nu}^{\nu \nu'},$$

the two-particle matrix element $X_{\alpha \beta}^{\gamma \delta}$ is given by

$$X_{\alpha \beta}^{\gamma \delta} = \sum_{\mu', \nu', \nu} B_{\alpha \beta}^{\nu \nu'} M_{\mu \nu}^{\mu \nu'} A_{\mu' \nu'}^{\gamma \delta}.$$

Several other approaches have been reviewed by Sun. Most commonly, an approximation devised by Kamlah has been used in practical calculations.
V. KINETIC ENERGY CONSIDERATIONS

Numerous studies have been made of the decomposition of the many-nucleon kinetic energy into its collective and intrinsic components [27, 71, 126–135]. These studies were motivated by the expectation that the rotational energies of strongly deformed nuclei should be predominantly kinetic energies, with corrections due to centrifugal and Coriolis perturbations, and the recognition that, in quantum mechanics, the kinetic energy of a many-nucleon nucleus is proportional to the Laplacian operator on its Hilbert space. The decomposition to emerge from these many studies was summarised in a simple but precise expression of the Laplacian on functions of the 3A many-nucleon coordinates [136] and gave the nuclear kinetic energy as a sum of three terms

\[ \hat{T} = \hat{T}_{\text{cm}} + \hat{T}_{\text{coll}} + \hat{T}_{\text{intr}} ; \] (81)

a centre-of-mass kinetic energy \( \hat{T}_{\text{cm}} \), a collective kinetic energy \( \hat{T}_{\text{coll}} \), and an intrinsic kinetic energy \( \hat{T}_{\text{intr}} \).

The collective component \( \hat{T}_{\text{coll}} \) of this decomposition is expressed in terms of three kinds of momentum and corresponding mass parameters that are simple functions of the nuclear monopole and quadrupole moments. Each of the three momenta has three components and together they are the infinitesimal generators of a group GL(3, \( \mathbb{R} \)) of general-linear transformations. Three components are those of the orbital angular momentum of the nucleus which are infinitesimal generators of rigid-body rotations. A second three are infinitesimal generators of vortex rotations defined as circulations of the nuclear matter in the intrinsic frame of the nucleus that leave its quadrupole moments invariant. The third three components are infinitesimal generators of quadrupole deformation, likewise relative to the intrinsic axes, for which the quadrupole moments remain diagonal. Thus, as discussed in the following section, the above decomposition contributed to the emergence of an algebraic theory of nuclear collective states [137] with quadrupole vibration, rigid rotation and vorticity degrees of freedom. The irreps of this model [134, 138] were shown to have both standard and vortex angular-momentum quantum numbers.

VI. MANY-NUCLEON ALGEBRAIC MODELS

As mentioned in the introduction, the simplicity of many-nucleon quantum mechanics, relative to what it might have been, follows from the fact that it is an algebraic model with a Hilbert space given for each nucleus by a fully anti-symmetric tensor product of single-nucleon spaces. As a result, the Hilbert space of a nucleus has a basis of fully antisymmetric independent-particle states; this makes it possible, to exploit the facility of large computers to work with huge bases of many-fermion states defined in a binary occupation-number representation [139, 140]. A nuclear Hilbert space is also a product of two subspaces: a subspace of centre-of-mass states and a complementary subspace of states with no centre-of-mass degrees of freedom. Another important property is that the nuclear Hilbert space is invariant and irreducible under all one-body unitary transformations of its basis states. Consequently, essentially all algebraic models of nuclei, that have dynamical groups expressed in terms of one-body unitary transformations, have irreps on subspaces of many-nucleon Hilbert spaces.

The most useful algebraic models define physically meaningful basis states for the many-nucleon Hilbert space of a nucleus that are classified by the quantum numbers of their irreps and those of their submodels. A desirable property of an algebraic model of a nucleus is also that it has unitary irreps which leave the centre-of-mass states of the nucleus invariant. Such models define coupling schemes for more complete calculations in spaces spanned by a number of its irreps, and thereby provide physical interpretations of the results of such calculations.

A. First attempts

The search for an algebraic collective model as a submodel of many-nucleon quantum mechanics was initiated in 1955 by Tomonaga [141] in two dimensions and extended by Miyazima and Tamura [142] to three. These searches identified quadrupole-moment operators as infinitesimal generators of irrotational flows and prepared the way for subsequent developments. However, they did not succeed in identifying a closed subalgebra of the many-nucleon algebra of one-body operators for an algebraic model of nuclear collective states.

B. The Elliott SU(3) model

The many-nucleon algebraic model of nuclear rotational dynamics, given by Elliott’s SU(3) model [10], has been particularly influential in demonstrating the value of a relevant coupling scheme for identifying collective subdynamics.
of the nuclear shell model. It also provided an alternative to the shell-model coupling schemes of Flowers and Edmonds\cite{143,144} and showed how states with rotational properties could emerge within the framework of the nuclear shell model. However, it was not the algebraic model of nuclear rotational states that was being sought. For, although the SU(3) Lie algebra contains angular momentum and quadrupole moment operators expressed in terms of nucleon coordinates and momenta, its quadrupole moments are not the physical quadrupole moments of nuclei; they are only their spherical harmonic-oscillator energy-conserving components. Nevertheless, the SU(3) model provides an important effective shell model of rotations in light nuclei.

In retrospect, the remarkable successes of the SU(3) model in obtaining shell-model states with properties close to those of a rotor model can be attributed to the fact that an SU(3) model Hilbert space is the projected image of a map from a rigid-rotor model space onto that of a single spherical harmonic-oscillator shell. This follows from the observation that the SU(3) quadrupole moments are the restrictions of physical quadrupole moments to the space of a single spherical harmonic-oscillator shell. Thus, it fits into the Lee-Suzuki construction\cite{145} of an effective shell model of nuclear rotations. However, as an effective shell model of a rotor, the SU(3) model is unable to give any information about the dynamics of physical nuclear rotations. It has nevertheless proved to be an important sub-model of the desired algebraic collective model that was being sought. Most significantly, it provides a coupling scheme for large shell model calculations\cite{146,147} in spaces that include many SU(3) irreps.

C. The Ui Rot(3) model

An algebraic model similar to the SU(3) model, but which contains the physical quadrupole moments in addition to the angular momentum operators, was introduced by Ui\cite{27}. It is a genuine rotor model and is useful because its representation theory provides a rigorous and systematic procedure for constructing a rigid-rotor model with intrinsic symmetries. It was also a vital step towards the objective of understanding nuclear rotational dynamics. In particular, it defines an algebraic rotor model for which Elliott’s SU(3) model is its projection onto an algebra that leaves a spherical harmonic-oscillator Hamiltonian invariant. Its limitation is that its irreps contain eigenstates of the quadrupole-moment operators which are delta-functions and can only be realised, in the many-nucleon Hilbert space, as non-normalisable limits.

D. The Weaver-Biedenharn-Cusson GCM(3) model

A next major step was the construction by Weaver, Biedenharn and Cusson\cite{149,150} of an algebraic collective model with both rotational and vibrational degrees of freedom. The unitary irreps of this model can be seen in retrospect as many-nucleon versions of an algebraic expression\cite{56,151,152} of the Bohr model\cite{13} and its extension to include quantised vorticity.

First observe that a many-nucleon version of the Bohr model starts with the replacement of its surface shape coordinates \(\{\alpha_n\}\) by microscopic Cartesian quadrupole moments

\[
Q_{ij} = \sum_{n=1}^{A} x_{ni} x_{nj}, \quad i, j = 1, 2, 3, \quad n = 1, \ldots, A. \tag{82}
\]

(The original CM(3) model of Weaver et al. did not include the monopole moment. However, its inclusion is natural and avoids the necessity of assuming the nuclear fluid to be incompressible.) Time derivatives of the quadrupole moments and corresponding momentum observables are given by

\[
\dot{Q}_{ij} = \frac{dQ_{ij}}{dt} = \sum_{n} (\dot{x}_{ni} x_{nj} + x_{ni} \dot{x}_{nj}), \tag{83}
\]

\[
P_{ij} = M \dot{Q}_{ij} = \sum_{n} (p_{ni} x_{nj} + x_{ni} p_{nj}), \tag{84}
\]

where \(M\) is the nucleon mass. An appropriate quantisation of the Bohr model with these observables is then obtained in the standard way by replacing the nucleon coordinates, \(x_{ni}\) and \(p_{ni}\), by operators \(\hat{x}_{ni}\) and \(\hat{p}_{ni}\) with commutation relations \([\hat{x}_{ni}, \hat{p}_{mj}] = i\hbar \delta_{nj} \delta_{m,n}\), to give quantal shape and momentum observables

\[
\hat{Q}_{ij} := \sum_{n} \hat{x}_{ni} \hat{x}_{nj}, \quad \hat{P}_{ij} = \sum_{n} (\hat{p}_{ni} \hat{x}_{nj} + \hat{x}_{ni} \hat{p}_{nj}), \quad \tag{85}
\]
which satisfy commutation relations

\[ [\hat{Q}_{ij}, \hat{P}_{kl}] = i\hbar (\delta_{il}\hat{Q}_{jk} + \delta_{jk}\hat{Q}_{il} + \delta_{jl}\hat{Q}_{ki} + \delta_{ki}\hat{Q}_{jl}) \].

Together with the angular-momentum operators

\[ \hat{L}_{ij} = \sum_n (\hat{x}_{ni}\hat{p}_{nj} - \hat{x}_{nj}\hat{p}_{ni}) \],

these operators span the so-called GCM(3) Lie algebra of a generalised collective model.

An interesting result that emerges is that whereas the standard Bohr model [13] and its later algebraic collective model version [152, 154] have single unitary irreps, the many-nucleon GCM(3) model has many. They were derived by Rosensteel [138] and again by Weaver et al. [134] and shown to be characterised by quantised vorticity. Thus, only the zero-vorticity irrep corresponds to the irrotational-flow Bohr model.

In addition to being a microscopic version of an extended Bohr model and, to a large extent the Bohr-Mottelson [16, 53] unified model, the GCM(3) model has the desirable characteristic of containing all the physical observables, i.e., quadrupole moments, standard angular momentum, vortex spin, and infinitesimal generators of deformation, that appear in the expression of the collective component \( T_{coll} \) of the many-nucleon kinetic energy. It is also related, as considered by Rosensteel [154, 157], to the Riemann model of rotating ellipsoids [70] with linear combinations of rigid and irrotational flows and a mathematical structure in terms of Yang-Mills theory, as given by Rosensteel and Sparks [154, 158].

A problem with the GCM(3) model is that it is difficult to use in a calculation of nuclear properties with a many-nucleon Hamiltonian. This is because bases for its unitary irreps are not easily constructed in terms of many-nucleon states. Thus, it does not lead in a practical way, to the construction of a coupling scheme for the many-nucleon Hilbert space. A more serious concern, is that the kinetic-energy operator and results from subsequent developments indicate that the vortex spin of the GCM(3) model is not conserved in the rotational states of nuclei.

### E. The symplectic Sp(3, \( \mathbb{R} \)) model

A resolution of the problems with the GCM(3) model is obtained [159, 160] by simply extending it to a symplectic Sp(3, \( \mathbb{R} \)) model which includes the full many-nucleon kinetic energy \( \hat{T} \) in its Lie algebra of observables. The Lie algebra of collective observables is thereby extended to the Lie algebra of all bilinear combinations of the nucleon position and momentum coordinates

\[ \hat{Q}_{ij} = \sum_{n=1}^{A} \hat{x}_{ni}\hat{x}_{nj}, \quad \hat{P}_{ij} = \sum_{n=1}^{A} (\hat{x}_{ni}\hat{p}_{nj} + \hat{p}_{ni}\hat{x}_{nj}), \]  
\[ \hbar \hat{L}_{ij} = \sum_{n=1}^{A} (\hat{x}_{ni}\hat{p}_{nj} - \hat{x}_{nj}\hat{p}_{ni}), \quad \hat{K}_{ij} = \sum_{n=1}^{A} \hat{p}_{ni}\hat{p}_{nj}, \]

that are symmetric with respect to nucleon permutations.

This Sp(3, \( \mathbb{R} \)) Lie algebra (sometimes referred to as Sp(6, \( \mathbb{R} \))) is the smallest Lie algebra that contains both the nuclear quadrupole moments and the many-nucleon kinetic energy. It nevertheless contains all the algebras of the above models as subalgebras. In particular, it contains the U(3) Lie algebra of the Elliott model as a subalgebra and has the valuable property that it defines a coupling scheme in a U(3) \( \supseteq \) SU(3) basis for the many-nucleon Hilbert space in a straightforward way. The Sp(3, \( \mathbb{R} \)) Lie algebra, like that of U(3) can be augmented to include the U(4) supermultiplet spin-isospin algebra with which it commutes. It can also be augmented to an Sp(6, \( \mathbb{R} \)) (alias Sp(12, \( \mathbb{R} \))) Lie algebra so that it can more readily describe independent collective motions of the neutrons and protons; cf., Ref. [161] and references therein.

The Sp(3, \( \mathbb{R} \)) symplectic group is of fundamental importance for many reasons. It can be defined, for example, as the group of linear canonical transformations that leave invariant the commutation relations

\[ [\hat{x}_{ni}, \hat{p}_{mj}] = i\delta_{n,m,\delta_{i,j}} \]

of the Heisenberg algebra. Moreover, whereas the U(3) group of the Elliott model is a symmetry group of a three-dimensional spherical harmonic oscillator, the group Sp(3, \( \mathbb{R} \)), which contains U(3) and SU(3) as subgroups, is a dynamical group that leaves invariant the Hilbert space of a general (not necessarily spherical) three-dimensional harmonic oscillator of a given parity. The Sp(\( n, \mathbb{R} \)) symplectic groups for any integer \( n \) have been well studied both
in mathematics and physics. In particular, the \( \text{Sp}(n, \mathbb{R}) \) irreps, on spaces of \( n \)-dimensional harmonic-oscillators, have been constructed \[162, 164\] and the \( \text{U}(n) \) irreps contained in an \( \text{Sp}(n, \mathbb{R}) \) irrep have been determined \[163\]. The \( \text{GCM}(n) \) irreps contained in an \( \text{Sp}(n, \mathbb{R}) \) irrep have also been determined \[166\]; it is shown, for example, that the range of vorticities in an \( \text{Sp}(3, \mathbb{R}) \) irrep is equal to the range of angular momenta in its lowest-grade \( \text{SU}(3) \) irrep.

Because the elements of the \( \text{Sp}(3, \mathbb{R}) \) Lie algebra are symmetric with respect to nucleon permutations, they conserve the so-called space symmetry of nuclear states. And because they are bilinear combinations of the nucleon position and momentum coordinates, they are expressible as sums of their centre-of-mass and relative components, without any coupling terms, by the simple substitutions

\[
\hat{x}_{ni} = \hat{x}'_{ni} + \hat{X}_i, \quad \hat{p}_{ni} = \hat{p}'_{ni} + \frac{1}{A}\hat{P}_i, \tag{91}
\]

where \( \hat{X}_i = \frac{1}{\sqrt{A}} \sum_n \hat{x}_{ni} \) and \( \hat{P}_i = \sum_n \hat{p}_{ni} \) are centre-of-mass coordinates for the nucleus. With this substitution, the quadrupole moments, for example, are the sums

\[
\hat{Q}_{ij} = \hat{Q}_{ij(\text{rel})} + \hat{Q}_{ij(\text{cm})}. \tag{92}
\]

In practical applications, the spurious centre-of-mass components of these operators are generally removed. However, in the interest of simplicity, this technicality is ignored in the following. In applications to heavy nuclei it is, in any event, a relatively minor correction due largely to the fact that the presence of the spurious contribution of the centre-of-mass degrees of freedom does not change the Lie algebra commutation relations in any way.

\section{VII. Applications of the Symplectic Model}

The symplectic model is an algebraic model of the rotations and collective monopole-quadrupole vibrations of nuclei which has the additional vital property that its representations are well-defined on the many-nucleon Hilbert spaces of nuclei. As an algebraic model, it can be used phenomenologically with a Hamiltonian expressed in terms of its Lie algebra of observables and an irrep that best fits the low-energy properties of a nucleus. This provides insights into the microscopic structures of collective states. At a more fundamental level, use can be made of the fact that the symplectic model subspaces of the many-nucleon Hilbert space assists in the identification of the relevant Sp(3, \( \mathbb{R} \)) irreps for describing the low-energy states of nuclei and for defining a coupling scheme that can be used, for example, to explore the extent to which symplectic-model irreps in nuclei are mixed in multi-shell model calculations.

Huge multi-shell model calculations with realistic interactions have been possible for some time in a \( \text{U}(3) \times \text{SU}(2) \) \( T \)-coupled basis in light nuclei \[147, 167–169\] with strong implications for their Sp(3, \( \mathbb{R} \)) content. Recent developments \[170\] now enable such calculations to be carried out in an \( \text{Sp}(3, \mathbb{R}) \times \text{SU}(2) \) \( T \)-basis. However, and not surprisingly, there are limitations to what is feasible with the available computational resources in heavy nuclei. We therefore seek ways to circumvent these limitations. This preliminary section focuses on the properties of single irreps of the symplectic model that are learned from the calculations that have been done.

\subsection{A. Unitary irreps of the \( \text{Sp}(3, \mathbb{R}) \) Lie algebra}

The irreps of Sp(3, \( \mathbb{R} \)) of relevance to nuclear physics are defined in many-nucleon spherical harmonic-oscillator bases by expressing its Lie algebra in terms of harmonic-oscillator raising and lowering operators for which the nucleon coordinates are given by

\[
\hat{x}_{ni} = \frac{1}{\sqrt{2}a}(c^\dagger_{ni} + c_{ni}), \quad \hat{p}_{ni} = i\hbar\frac{a}{\sqrt{2}}(c^\dagger_{ni} - c_{ni}), \tag{93}
\]

where \( a = \sqrt{M\omega/\hbar} \) is a unit of inverse length. This leads to the expansions

\[
\begin{align*}
\hat{Q}_{ij} &= \frac{1}{2a^2}(2\hat{Q}_{ij} + \hat{A}_{ij} + \hat{B}_{ij}), \\
\hat{K}_{ij} &= \frac{1}{2a^2}(2\hat{Q}_{ij} - \hat{A}_{ij} - \hat{B}_{ij}), \\
\hat{P}_{ij} &= i\hbar(\hat{A}_{ij} - \hat{B}_{ij}), \quad \hat{L}_{ij} = -i(\hat{C}_{ij} - \hat{C}_{ij}),
\end{align*} \tag{94}
\]
follows that a basis of shell-model states for an $\text{Sp}(3, \mathbb{R})$ irrep denoted by $\langle \sigma_1, \sigma_2, \sigma_3 \rangle$ is simultaneously a highest-weight state of a U(3) subalgebra and the giant monopole/quadrupole raising and lowering operators $\{ \hat{A}_{ij} \}$ and $\{ \hat{B}_{ij} \}$. The elements of the U(3) subalgebra commute with the spherical harmonic-oscillator Hamiltonian

$$\hat{H}_0 = \hbar \omega_0 \sum_n (c_n^\dagger c_n + \frac{1}{2} \delta_{i,j}),$$

and the giant-resonance raising and lowering operators satisfy the commutation relations

$$[\hat{H}_0, \hat{A}_{ij}] = 2 \hbar \omega \hat{A}_{ij}, \quad [\hat{H}_0, \hat{B}_{ij}] = -2 \hbar \omega \hat{B}_{ij}.$$  

It follows that an irreducible $\text{Sp}(3, \mathbb{R})$ representation is uniquely defined by a so-called lowest-grade U(3) irrep, the states of which are annihilated by the giant-resonance lowering operators $\{ \hat{B}_{ij} \}$. The U(3) highest-weight state $|\sigma\rangle$ of this lowest-grade U(3) irrep is then the state that satisfies the equations

$$\hat{C}_{ij}|\sigma\rangle = 0, \quad i < j, \quad \hat{C}_{ii}|\sigma\rangle = \sigma_i|\sigma\rangle, \quad i = 1, 2, 3,$$

$$\hat{B}_{ij}|\sigma\rangle = 0, \quad i, j = 1, 2, 3,$$

and is a state of U(3) weight $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ given by the triple of integers or half-odd integers ordered such that

$$\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq 0.$$  

Because the state $|\sigma\rangle$ is annihilated by the $\text{Sp}(3, \mathbb{R})$ lowering operators in Eqn. 97, it is also conveniently regarded as the lowest-weight state for the $\text{Sp}(3, \mathbb{R})$ irrep to which it belongs. Thus, $|\sigma\rangle$ is simultaneously a highest-weight state for a U(3) irrep denoted by $\{ \sigma_1, \sigma_2, \sigma_3 \}$ and a lowest-weight state for an $\text{Sp}(3, \mathbb{R})$ irrep denoted by $\langle \sigma_1, \sigma_2, \sigma_3 \rangle$. It follows that a basis of shell-model states for an $\text{Sp}(3, \mathbb{R})$ irrep is constructed by adding to its lowest-grade U(3) states the infinite set of multiple giant-resonance excitations generated by the $\{ \hat{A}_{ij} \}$ raising operators.

An efficient VCS (vector coherent state) algorithm for the construction of the states and matrix elements of an $\text{Sp}(3, \mathbb{R})$ irrep \textcolor{red}{162} \textcolor{red}{163} is described in some detail in a recent review \textcolor{red}{171}. Computer codes for its implementation, have been developed by Rosensteel \textcolor{red}{163}, Bahri \textcolor{red}{172} and McCoy \textcolor{red}{171}. The VCS algorithm makes use of the readily available Clebsch-Gordan and other coefficients for coupling and recoupling states of SU(3) representations \textcolor{red}{172} \textcolor{red}{172} and the fact that the raising and lowering operators $\hat{A}_{ij}$ and $\hat{B}_{ij}$ are simply related to boson creation and annihilation operators as in the Dyson representation \textcolor{red}{170} of SU(2) operators.

B. Calculations with schematic algebraic interactions

Many early symplectic model calculations were reviewed in 1985 \textcolor{red}{50}. A recent calculation of Bahri \textcolor{red}{172} shows that, with a simple Hamiltonian

$$\hat{H}(\epsilon) = \hat{H}_0 + \chi \left( \hat{Q}_2 \cdot \hat{Q}_2 + \frac{\epsilon}{\hat{Q}_2 \cdot \hat{Q}_2} \right),$$

expressed in terms of the $\text{Sp}(3, \mathbb{R})$ Lie algebra in which $\hat{Q}_2$ is the nuclear quadrupole tensor, it is possible to obtain close-to-converged solutions for the low-energy rotational states of $^{166}\text{Er}$. To within corrections that could be due to overly-suppressed centrifugal effects by the potential-energy term of $\hat{H}(\epsilon)$, remarkably good fits were obtained with this Hamiltonian for the relative energies of the lowest rotational states of this nucleus, up to angular momentum

\[
\hat{A}_{ij} = \hat{A}_{ji} = \sum_n c_n^\dagger c_n, \quad \hat{B}_{ij} = \hat{B}_{ji} = \sum_n c_n c_n, \\
\hat{C}_{ij} = \sum_n (c_n^\dagger c_n + \frac{1}{2} \delta_{i,j}), \quad \hat{Q}_{ij} = \frac{1}{2} (\hat{C}_{ij} + \hat{C}_{ji}),
\]
\( J = 16 \), and for the E2 transitions between them. For these calculations, a symplectic irrep \( \langle \sigma \rangle \) was chosen such that the quadrupole moments and E2 transition data could be fitted without the use of an effective charge. A significant result of this calculation, shown by the wave functions in Fig. 1, is that the U(3) irreps contributing to the rotational states that emerged from the calculation were from spherical harmonic-oscillator shells of energies ranging from \( 8 \hbar \omega \) to \( 20 \hbar \omega \) above that of the lowest available for this \( ^{166}\text{Er} \) nucleus. This is a clear indication that it is unreasonable to attempt a conventional shell model description of such rotational states in heavy deformed nuclei. The Bahri model provides a simple and practical prescription for describing the rotational states of axially symmetric nuclei. It has two parameters, \( \epsilon \) and \( \chi \). The parameter \( \epsilon \) defines the deformation at which the potential is a minimum and the strength \( \chi \) of the interaction can be adjusted to ensure that its eigenstates have the same deformation and therefore give the corresponding E2 transitions between its eigenstates.

A notable result is that the amplitudes of the U(3) basis states, displayed in the figure for three angular-momentum states, are essentially identical. An interpretation of this result is that, whereas the rotational states of a single U(3) irrep can be obtained by angular-momentum projection from its highest-weight states, the rotational states of the Sp(3, \( \mathbb{R} \)) irrep of the model can similarly be projected, to a high level of accuracy, from a single intrinsic state given by a linear combination of the U(3) highest-weight states in a manner similar to that of a quasi-dynamical symmetry \[45\] \[48\]. This interpretation is derived in the following from an algebraic mean-field perspective. Thus, although the lowest-energy SU(3)-model components of the calculated states, which are those of the SU(3) \((78,0)\) irrep at a harmonic-oscillator shell model excitation energy of \( 8 \hbar \omega \), are shown in the figure to be exceedingly small, it can nevertheless be understood why the SU(3) model can work as well as it does as an effective model, with a suitable effective interaction and effective charge. However, it must be emphasised that the lowest U(3) irrep that contributes to the results shown belongs to a spherical harmonic-oscillator shell that is 8 shells above those that would be considered in a standard spherical shell-model calculation. Thus, it is an example of shape co-existence.

A limitation of the Hamiltonian (101) is that it applies only to axially symmetric representations and it makes no allowance for the possible centrifugal stretching of a rotating nucleus with increasing angular momentum which would appear to be in evidence in the experimental spectrum of \( ^{166}\text{Er} \). However, the latter deficiency could be remedied to some extent by allowing the deformation parameter to be a function of the angular momentum.

Another model with an algebraic interaction, that is closer to one with a conventional two-nucleon interaction and applies to triaxial as well as axially symmetric nuclei, has a Hamiltonian of the form

\[
\hat{H}(\gamma) = \hat{H}_0 + \frac{\chi}{2\gamma} \left(e^{-\gamma \hat{Q}_z \hat{Q}_z} - 1\right),
\]

which, for an axially symmetric irrep, can be diagonalised by an adjustment of Bahri’s code. It was proposed and applied by Dreyfuss et al. \[148\] to the rotational states based on the Hoyle state of \( ^{12}\text{C} \) and by Tobin et al. \[177\].

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Amplitudes for the wave functions of the lowest-energy \( L = 0, 6, \) and 10, Sp(3, \( \mathbb{R} \)) model states of \( ^{166}\text{Er} \) in a U(3) basis. The calculation included all basis states of the irreducible Sp(3, \( \mathbb{R} \)) representation \( \langle 826.5(78,0) \rangle \) of spherical harmonic-oscillator energy \( \leq 12 \hbar \omega \) above and including that of its lowest U(3) states. Particularly important is the observation that the lowest U(3) states for this representation are already at \( 8 \hbar \omega \), in spherical harmonic-oscillator units of energy, above the lowest available in spherical harmonic-oscillator shells. Note that the + or − signs of the amplitudes are of no significance because they are defined by the arbitrary signs of the basis states. (The figure is adapted from one of Bahri \[172\].)}
\end{figure}
to the states of a few \textit{sd}-shell nuclei. The parameter \( \chi \) of this Hamiltonian was fixed by a self-consistency argument and the parameter \( \gamma \) was adjustable. Good agreement for the rotational band of states based on the Hoyle state was then obtained with an \( \text{Sp}(3, \mathbb{R}) \) irrep having a \((12,0)\) SU(3) lowest weight of spherical harmonic-oscillator energy \( 4\hbar \omega \) above that of the lowest available for the \(^{12}\text{C}\) nucleus and with wave functions containing components from spherical harmonic-oscillator shells of excitation energies up to \( 16\hbar \omega \).

The decomposition of the Hoyle state, calculated in terms of spherical harmonic-oscillator states and shown in figure 2(b), is similar to that for the rare-earth rotational states shown in figure [1]. Both figures show the lowest-grade U(3) states of the symplectic irrep to be small components of the total. This is in contrast to that shown for the much less deformed ground state of \(^{12}\text{C}\) shown in figure 2(a).

Another significant observation is that the Hoyle state is observed and calculated to be at an energy \( \sim 8 \) MeV which is very much less than the \( 4\hbar \omega \) spherical harmonic-oscillator energy of the lowest-weight state that this \( \text{Sp}(3, \mathbb{R}) \) calculation employed.

**C. \text{Sp}(1, \mathbb{R}) \text{ and Sp}(2, \mathbb{R}) \text{ submodels}**

In parallel with the development of the \( \text{Sp}(3, \mathbb{R}) \) model, a so-called \( \text{Sp}(1, \mathbb{R}) \) model was introduced by Arickx [178] founded on the observation (subsequently confirmed in large multi-shell model calculations in U(3) bases [168]) that the excited states which couple most strongly to those of a U(3) irrep of highest weight \( (\sigma_1, \sigma_2, \sigma_3) \) are those of U(3) irreps with highest weights

\[
(\sigma_1 + 2\nu_1, \sigma_2, \sigma_3), \quad \nu_1 = 1, 2, 3, \ldots
\]  

(103)

Arickx also recognised that the highest-weight states for these U(3) irreps, are a basis for a unitary irrep of a group \( \text{Sp}(1, \mathbb{R}) \) that is isomorphic to SU(1,1) and referred to by him as \( \text{Sp}(2, \mathbb{R}) \). [Note that some physicists denote the real non-compact symplectic group \( \text{Sp}(n, \mathbb{R}) \) of rank \( n \) by \( \text{Sp}(2n, \mathbb{R}) \) because its defining matrix representation is \( 2n \) dimensional.]

The next most strongly-coupled states in an \( \text{Sp}(3, \mathbb{R}) \) irrep were subsequently observed by Peterson and Hecht [179] to be those of the U(3) irreps

\[
\{ \sigma_1 + m, \sigma_2 + n, \sigma_3 \}, \quad m + n = 0, 2, 4, \ldots
\]  

(104)

with \( \sigma_1 + m \geq \sigma_2 + n \). It was also recognised that this subset of states of an \( \text{Sp}(3, \mathbb{R}) \) irrep \( \langle \sigma_1, \sigma_2, \sigma_3 \rangle \), which have weights with a common value of \( \sigma_3 \), span an \( \text{Sp}(2, \mathbb{R}) \) irrep.

**D. Applications with interacting-nucleon Hamiltonians**

Early attempts, pursued by Filippov [180,181], Vassanji [182,185] and colleagues, at diagonalising a conventional many-nucleon Hamiltonian within the space of an \( \text{Sp}(3, \mathbb{R}) \) irrep, were based on generator coordinate methods. Such an approach was used by Elliott and Harvey [10,186] for calculations in the space of an SU(3) irrep. The latter made use of the fact that the Hilbert space of a U(3) irrep is spanned by the states generated by SO(3) rotations of its
highest-weight state $|\sigma\rangle \equiv |\sigma_1, \sigma_2, \sigma_3\rangle$. Expressions for SU(3) basis states were then obtained as integrals over the rotated highest-weight state with the rotational angles serving as generator coordinates. It is similarly observed, in Section 10 that the Hilbert space of an Sp(3, $R$) irrep is spanned by the states generated by GL(3, $R$) transformations of the same state $|\sigma\rangle$, which is both a highest-weight state for a U(3) irrep and a lowest-weight state for an Sp(3, $R$) irrep. This approach was discontinued, but now because of its close relationship with the algebraic mean-field theory given below, it appears to provide the solution not only for handling the problem of a single Sp(3, $R$) irrep but also that of mixed Sp(3, $R$) irreps. Thus, it is revisited in Sections XD and XC.

A different strategy, initiated by Escher and Draayer [187] for the explicit derivation of many-nucleon wave functions for symplectic model states, was to start with an expression of the Sp(3, $R$) Lie algebra in terms of nucleon creation and annihilation operators. This potentially promising approach also does not appear to have been pursued, possibly because of the successes of the symmetry-adapted no-core shell model (SA-NCSM). However, it may yet prove to have important applications.

E. Emergence of Sp(3, $R$) symmetry in no-core shell model (NCSM) calculations

The most sophisticated and most successful approach to date in understanding the emergence of collective rotational states within the framework of many-nucleon quantum mechanics, is that of the symmetry-adapted no-core shell model (SA-NCSM) of the LSU research group of Draayer, Dytrych, Launey and colleagues [168, 169]; see also Refs. 146, 167, 188, 190. This model was introduced to expose the physical content of the NCSM calculations of Navratil et al. [191], by expressing them in a symmetry-adapted basis.

The NCSM makes use of the massive parallel computing resources of supercomputers and a simple $M$-scheme basis [133, 141] to enable huge shell-model calculations to be performed in light nuclei with realistic interactions derived from quantum chromodynamics [190] and nucleon-nucleon interaction data [197]. These calculations are of fundamental importance in establishing the foundations of the many-nucleon quantum theory of nuclei with such interactions. The SA-NCSM calculations are similar, in principal, but are carried out in bases of states classified by U(3) ⊃ SO(3) and intrinsic neutron and proton spins. This was possible because of previous developments in the use of SU(3) and its tensor properties [173, 174, 198].

SA-NCSM calculations have been carried out in complete shell-model spaces of up to six spherical harmonic-oscillator shells for the light nuclei so far considered. A remarkable result is that the eigenstates that emerge contain significant contributions from only a tiny fraction of the huge number of U(3) irreps, with corresponding neutron and proton spin wave functions, that were included in the calculations. Even more remarkable is the finding that most of the U(3) states that make up the low-energy eigenstates of a nucleus in these calculations can be identified with those of a single Sp(3, $R$) irrep at the 60–80% level, while the remaining components belong to just one or two other Sp(3, $R$) irreps. For doubly even nuclei, the dominant components are Sp(3, $R$) states coupled to states of zero intrinsic neutron and proton spins. Thus, the LSU results indicate the emergence of Sp(3, $R$) as a dominant dynamical symmetry in calculations that were not prejudiced by the assumption that this should be the case; i.e., they were carried out in a U(3)-coupled basis purely to facilitate an interpretation of the results. The results for light nuclei are especially remarkable because the symplectic model [159, 160] was expected to be most successful as a microscopic collective model for heavy rotational nuclei. Thus, its relevance to rotational states in light nuclei was not expected to be anywhere near as dominant as it has proved to be.

VIII. ALGEBRAIC MEAN-FIELD (AMF) THEORY

At present it is not feasible to carry out realistic SA-NCSM calculations in heavy deformed nuclei. The major obstacle, as Figure [1] illustrates, is that the rotational states of heavy nuclei have expansions in a standard shell-model basis of states with spherical harmonic-oscillator energies ranging between $\sim 6\hbar\omega$ and $\sim 26\hbar\omega$, in spherical harmonic-oscillator shell-model units above those that would normally be considered in a shell-model calculation. Thus, for anything like a meaningful shell-model calculation of the usual type, the dimension of the active valence-shell space would need to be orders of magnitude larger than could conceivably be handled in the foreseeable future. However, there is cause for optimism because, it is now possible to do shell model calculations in bases of states restricted to just a few Sp(3, $R$) irreps [170]. This will surely initiate a new era in the many-nucleon theory of heavy nuclei, for which we need to be prepared.

The first challenge in attempting to apply the symplectic model to the low-lying rotational states of heavy deformed nuclei, is to select the appropriate Sp(3, $R$) irreps. To be in a position to make a good choice from among the many available for a given nucleus, one should first consider the characteristic properties of an Sp(3, $R$) irrep and how it relates to the observable properties of a nucleus. This is a primary objective of AMF theory.
When the HF and TDHF mean-field theories are expressed algebraically in terms of equations of motion as in Section [108, 199], as in Section [111] it becomes evident that, along with their HFB counterparts, these theories are special cases of a more general AMF theory [81, 89, 200]. This section shows that, as an extension of HF theory, AMF theory likewise provides an interface between the classical and the quantum representations of a variety of algebraic systems and gives an explicit realisation of the Dirac-Weyl theory of quantisation [35, 82], in which the classical Poisson bracket realisation of a finite Lie algebra is quantised by construction of its unitary irreps. Conversely, it gives a realisation of classical mechanics as constrained quantum mechanics. Many AMF properties and applications have been developed by Rosensteel and colleagues [69, 201–204]. However, for present purposes, it is noted that, although AMF theory can be applied to any algebraic model, its application is most useful for a model that has irreps defined by lowest-weight (and/or highest-weight) states. The quantisation of its classical mean-field representation is then achieved by standard algebraic methods.

The following will show that the relationships between AMF theory and the representation of a Lie algebra with a lowest- (and/or highest-) weight state makes it possible to take advantage of the complementary contributions of two major approaches to nuclear structure theory: mean-field theory and algebraic modelling. Mean-field theory [205] is well known to be of fundamental importance in establishing the foundations of many-body theory and the nuclear shell model [206]. The power of algebraic methods in exposing the dynamical content of a system is likewise recognised; e.g., in Elliott’s SU(3) model [10], Kerman’s quasi-spin model [207], the Lipkin model [208], the Interacting Boson Models [209], the algebraic version of the Bohr Model [152], and the other models discussed in Section [71]. Such models are valuable for showing the many ways in which collective structures can emerge in nuclei and in providing solvable models that serve as ways to test the validity of various approximation schemes, such as the RPA. Algebraic models, that define coupling schemes [8, 10, 11, 143] for the many-nucleon Hilbert space, are especially important. However, apart from those of Rosensteel and colleagues, the applications of mean-field methods to algebraic models has, to date, received little attention in physics.

### A. AMF theory as a generalisation of HF theory

The AMF generalisation of HF theory is valuable because of its close relationship with the structure theory of semi-simple Lie algebras [81]. HF theory, is an application to the fully anti-symmetric unitary irrep of the Lie algebra of one-body operators, which has a lowest-weight state which is a Slater determinant of so-called occupied single-particle states. A particle-hole state is created by transferring a nucleon from an occupied state of a Slater determinant to an unoccupied state. Thus, any Slater determinant is a particle-hole vacuum state in that it is annihilated by every one-body operator that would annihilate a nucleon in an unoccupied single-particle state and recreate it in an already-occupied single-particle state. The significance of this observation is that any Slater determinant of single-nucleon states can serve as the lowest-weight state for the unique fully anti-symmetric unitary irrep, for a particular nucleus, of the Lie algebra of one-body operators. Moreover, for any choice of lowest-weight state there are corresponding particle-hole creation and annihilation operators that serve as raising and lowering operators.

The important property, shown in Section [111] is that the HF manifold of all Slater determinants for a given nucleus is a classical phase space which spans the Hilbert space of its quantisation. It is also notable that any Slater determinant of occupied single-nucleon states can be transformed into any other by a unitary transformation of the occupied single-nucleon states. This means that the set of Slater determinants for a nucleus, that form its HF classical phase space, can be generated by the transformations of any single determinant by the elements of the group of one-body unitary transformations. Thus, the HF manifold, given by this classical phase space, is said to be an orbit of the group of one-body unitary transformations.

It is now straightforward to show that parallel relationships apply to any algebraic model with irreps having lowest-weight (and/or highest-weight) states and that, for such irreps, there is a corresponding AMF theory. Suppose that a state \( |\phi\rangle \) is a lowest-weight state for a unitary irrep \( \hat{T} \) of a Lie group \( G \) and that the operators \( \{\eta^\alpha\} \) and their Hermitian adjoints \( \{\eta^\alpha\} \) are, respectively, raising and lowering operators for this irrep. It follows immediately from the observation that the state \( |\phi\rangle \) is annihilated by the \( \{\eta^\alpha\} \) operators that any state of the orbit

\[
|\phi(g)\rangle = \hat{T}(g)|\phi\rangle, \quad \text{for some } g \in G,
\]

is a lowest-weight state that is annihilated by the correspondingly transformed lowering operators

\[
\eta^\alpha(g) = \hat{T}(g)\eta^\alpha\hat{T}^{-1}(g).
\]

In other words, any state \( |\phi(g)\rangle \) of a \( G \) orbit of a particular lowest-weight for a unitary irrep is also a lowest-weight state for that irrep.
In parallel with HF theory, AMF theory now makes it possible to select the minimum-energy lowest-weight state for an algebraic model with a given Hamiltonian and determine its normal-mode vibrational states. The minimum-energy lowest-weight state is interpreted classically as the equilibrium state of the model and, in quantum mechanics, as the closest approximation, among the lowest-weight states, to the ground state of the model. The time-dependent normal-mode solutions of the classical equations of motion in AMF theory likewise have an interpretation in terms of the random-phase approximation. The results that emerge resemble those obtained for a system of coupled harmonic oscillators in standard coherent state theory. Explicit examples will be given in the following for the symplectic model.

The recognition that the manifold of all Slater determinants can be identified with a classical phase space of a nucleus, leads to many insightful results as shown in Section III A. For example: it takes account of the antisymmetry properties of the nucleons; it has a classical Hamiltonian defined as a function on this phase space given by the expectation values of the quantum mechanical Hamiltonian; and it leads naturally to the Nambu-Goldstone interpretation of a broken symmetry in mean-field theory (113, 114). The latter follows because, if a minimum-energy mean-field state is not invariant with respect to SO(3) rotations, it represents a classical equilibrium state which, not surprisingly, has many orientations corresponding to the energy-degenerate set of mean-field states that lie on an SO(3) orbit.

### B. More general AMF theories, co-adjoint orbits, and geometric quantisation

The properties of Lie group orbits in the Hilbert spaces of their unitary irreps have been studied widely in mathematics (99, 210), as reviewed in Ref. 211. In the theory of geometric quantisation (91, 92, 212), which relates closely to AMF theory, the orbits are identified with coadjoint orbits, which are orbits of a Lie group in the dual space of linear operators on its Lie algebra.

In physical terms, an element of a coadjoint orbit is a density operator which characterises a state of an algebraic model by the expectation values that it gives for the elements of the model’s Lie algebra of observables, i.e., for any state $|\psi\rangle$ in a unitary irrep of an algebraic model there is a density operator $\hat{\rho}_\psi$ defined by a map in which

$$ |\psi\rangle \rightarrow \hat{\rho}_\psi, \quad \hat{\rho}_\psi(X) = \langle \psi | \hat{X} | \psi \rangle, \quad (107) $$

for all $X$ in the Lie algebra of the model. For example, the dual of a Lie algebra of one-body operators, is a set of one-body density operators. Thus, mapping a mean-field state to a point on a coadjoint orbit corresponds precisely to what is done in the density-matrix formulation of mean-field theory. However, one should be aware that, in general, the map of Equation (107) is not invertable; i.e., there may be a multiple set of states with the same density. The exceptional cases are then of special importance. The states of HF theory are exceptions in that a Slater determinant is uniquely defined (to within a phase factor) by a one-body density matrix. The essential property of such exceptional orbits of a group is that they consist of lowest-weight states that are uniquely defined by the expectation values in these states of the elements of the Lie algebra of the group. Thus, we restrict consideration, in the following, to such group orbits. These special orbits have the advantage of being simply quantised by the standard construction of the unitary irrep of a Lie group with a given lowest- (highest-) weight state and a corresponding system of raising and lowering operators. They are the coadjoint orbits that are said to be ‘integral’.

AMF theories, based on general coadjoint orbits of a group, have been considered by Rosensteel and colleagues (89, 201, 204). However, because the map $|\psi\rangle \rightarrow \hat{\rho}_\psi$, defined by Eqn. (107) may be many to one, the greater generality is obtained at a price. For example, the expectation value of an operator, such as a Hamiltonian that is not an element of the Lie algebra, is only defined at a point of a coadjoint orbit as the average of its values for the states that map to that point.

It will be shown in the following sections that the application of AMF methods to the irreps of an algebraic collective model of the nucleus, that defines a coupling scheme for the many-nucleon Hilbert space, leads to valuable techniques for the study of nuclear collective properties.

### C. The symplectic model as a unified model

A valuable property of a minimum-energy lowest-weight state for a model is its clear algebraic significance. It has all the properties of a lowest-weight state for an irrep of a Lie algebra and the extra property of minimising the expectation value of a model Hamiltonian.

A minimal energy lowest-weight state for an Sp(3, R) irrep is, to within an arbitrary rotation, an eigenstate of a generally triaxial harmonic-oscillator Hamiltonian and can be chosen as a state for which its quadrupole moments are diagonal in a Cartesian basis. In addition, being a minimal-energy lowest-weight state, means that it has zero first-order coupling, by the Hamiltonian, to any other states of the irrep with the exception of those generated by
its rotations. Thus, a AMF minimum-energy lowest-weight state provides a well-defined distinction between the vibrational and rotational degrees of freedom of an Sp(3, \mathbb{R}) irrep. This is a property that is characteristic of an intrinsic state of the Bohr-Mottelson unified model \cite{53}; in fact, the AMF expression of the symplectic model can be seen as a many-nucleon realisation of a Bohr-Mottelson unified model \cite{53}.

In applications of the symplectic model to the lowest-energy rotational states of heavy doubly even nuclei, the dominant symplectic model irreps are invariably those of maximal space symmetry. Consequently, the states of these irreps have vanishing intrinsic spins and minimum isospin, i.e., \( S = 0 \) and \( T = \frac{1}{2}(N - Z) \). For the purposes of this review, we therefore focus primarily on such Sp(3, \mathbb{R}) irreps.

It follows from Lie algebra theory that the full Hilbert space of an Sp(3, \mathbb{R}) irrep is spanned by the states generated by rotations of both its minimum-energy lowest-weight state \( |\sigma, \omega\rangle \) and its multiple vibrational excitations. As a result, the set of classical equilibrium states of a deformed nucleus is generated by the rotations of a lowest-weight state and a corresponding set of quantum-mechanical rotational states is obtained, in a semi-classical approximation, by angular-momentum projection from the minimum-energy lowest-weight state (cf. Section \textbf{IV B}). In a microscopic version of the unified model, the intrinsic one-phonon vibrational states are likewise determined by solution of the RPA equations with a minimum-energy lowest-weight state considered as the uncorrelated vacuum state.

For a generic irrep with \( \sigma_1 > \sigma_2 > \sigma_3 \), the minimum-energy lowest-weight state \( |\sigma, \omega\rangle \) is an eigenstate of the three operators

\[
\zeta_{\!ii}^\dagger = \sum_n b_n^\dagger b_n^{\prime i}, \quad i = 1, 2, 3, \tag{108}
\]

of a Cartan subalgebra. Thus, a complete basis for the Sp(3, \mathbb{R}) irrep is generated by the repeated action of the raising operators

\[
\zeta_{ij}^\dagger = \sum_n b_n^\dagger b_{n_{ij}}^{\prime j}, \quad i < j, \tag{109}
\]

\[
\eta_{ij}^\dagger = \sum_n b_n^\dagger b_{n_{ij}}^{\prime j}, \quad i, j = 1, 2, 3, \tag{110}
\]

on the lowest-weight state \( |\sigma, \omega\rangle \).

From among the linear combinations of this set of 9 raising operators and their Hermitian-adjoint lowering operators, there are three components of angular momentum \( \{ \hat{L}_k, k = 1, 2, 3 \} \) relative to the axes of the lowest-weight state \( |\sigma, \omega\rangle \) and three angular-momentum boost operators \( \{ \hat{\Theta}^j, j = 1, 2, 3 \} \) that satisfy the equations

\[
\langle \sigma, \omega | [\hat{\Theta}^j, \hat{L}_k] | \sigma, \omega \rangle = i \delta^j_k. \tag{111}
\]

Also, in parallel with the Hartree-Fock RPA theory outlined in Section \textbf{III D} there are six pairs of excitation and de-excitation operators \( \{ O^\lambda_k, O^\lambda_k \} \) of vibrational states that are linear combinations of the raising and lowering operators that satisfy the equations

\[
\langle \sigma, \omega | [\hat{X}, [\hat{H}, O^\lambda_k]] | \sigma, \omega \rangle = \hbar \omega_\lambda \langle \sigma, \omega | [\hat{X}, O^\lambda_k] | \sigma, \omega \rangle, \tag{112}
\]

\[
\langle \sigma, \omega | [\hat{X}, [\hat{H}, O^\lambda_k]] | \phi \rangle = -\hbar \omega_\lambda \langle \sigma, \omega | [\hat{X}, O^\lambda_k] | \sigma, \omega, \omega \rangle, \tag{113}
\]

for every \( \hat{X} \) in the Sp(3, \mathbb{R}) Lie algebra, and the orthogonality relationships

\[
\langle \sigma, \omega | [O^\lambda_k, \hat{L}_k] | \sigma, \omega \rangle = \langle \sigma, \omega | [O^\lambda_k, \hat{X}] | \sigma, \omega \rangle = 0, \tag{114}
\]

\[
\langle \sigma, \omega | [O^\lambda_k, \hat{\Theta}^j] | \sigma, \omega \rangle = \langle \sigma, \omega | [O^\lambda_k, \hat{\Theta}^j] | \sigma, \omega \rangle = 0.
\]

The operators \( \{ O^\lambda_k \} \) are interpreted as one-phonon vibrational excitation and de-excitation operators, respectively. However, in doing so there is no intention to suggest that their repeated applications generate sequences of harmonic vibrational states. It should also be recognised that the vibrational states, excited by these \( \{ O^\lambda_k \} \) operators, which satisfy Eqn. (114), are by construction intrinsic vibrational states.

For \( \sigma_1 > \sigma_2 > \sigma_3 \), the spectrum of states obtained from the above equations will consist of zero-phonon rotational states of a triaxial nucleus and six sets of rotational states corresponding to one-phonon intrinsic vibrational excitations. However, if \( \sigma_1 > \sigma_2 = \sigma_3 \), the spectrum will contain a \( K = 0 \) ground-state rotational band, plus two \( K = 0 \) bands and single \( K = 1 \) and \( K = 2 \) rotational bands corresponding to one-phonon vibrational excitations. These
expectations are indicated by the observation that for an SU(3) irrep \((\lambda \mu)\) the tensor product \((\lambda \mu) \otimes (2 0)\) is a sum of six irreps whereas, for an axially symmetric \((\lambda 0)\) irrep

\[
(\lambda, 0) \otimes (2, 0) = (\lambda + 2, 0) \oplus (\lambda + 1, 1) \oplus (\lambda 2).
\]  

(115)

In the \(\mu = 0\) case, one of the \(K = 0\) bands is a monopole vibrational band, and the other \(K = 1\) and 2 bands are quadrupole vibrational bands. Thus, if the \(K = 0\) and \(K = 2\) vibrational bands were to emerge at low energies, they could be interpreted as beta and gamma vibrational bands. However, in the microscopic unified model, corresponding to un-mixed symplectic model irreps, they are the giant-resonance excitations of a deformed nucleus \([213]\) and are not expected to lie low in energy. A mixing of the rotational states arising from different symplectic model irreps could nevertheless give rise to states having some of the properties of those of low-lying beta- and gamma-vibrational bands.

For both triaxial and axially symmetric irreps, the Inglis cranking model predicts rigid-body moments of inertia for the zero-phonon rotational states generated by rotations of the shape-consistent state \(|\sigma, \omega\rangle\), as shown in Section II, and the Thouless-Valatin prescription for the moments of inertia gives

\[
\langle \sigma, \omega | \{ \hat{\Theta}^k | \hat{\Theta}^k \} | \sigma, \omega \rangle = \hbar^2 \mathcal{I}^k.
\]  

(116)

However, more reliable angular-momentum projection techniques \([120, 214, 215]\), including those developed for wave functions in SU(3)-coupled basis \([174, 175]\), can be used, as in HF theory, first to expand the lowest-weight state as a sum

\[
|\sigma, \omega\rangle = \sum_{KL} n_{KL} |KLK\rangle,
\]  

(117)

and then to project out an orthonormal basis of angular momentum states \(|KLM\rangle\) in term of which a nuclear Hamiltonian can then be diagonalised to obtain bands of rotational states

\[
|\alpha LM\rangle = \sum_{K} \alpha_K |KLM\rangle
\]  

(118)

and their energies. One can then also evaluate the separate contributions to these energies coming from the kinetic energy and interaction components of the Hamiltonian

\[
\hat{H} = \hat{T} + \hat{V},
\]  

(119)

and determine the extent to which the potential energies of symplectic-model rotational states take a constant value for adiabatically small values of the rotational angular momentum.

**IX. LOW-ENERGY COLLECTIVE STATES OF SYMPLECTIC MODEL IRREPS**

It is important to recognise that an application of AMF theory is not necessarily an approximation. It could simply be a first step in the construction of an irrep of a semi-simple Lie algebra. The choice of a minimum-energy lowest-weight is nevertheless a useful starting point for good approximations such as given by angular-momentum projection and RPA methods. A higher-order approximation for the low-energy rotational states known as ‘variation after projection’ \([120]\), is to determine the optimal lowest-weight for each value of the projected angular momentum independently; this enables account to be taken of the changing structure of the intrinsic state with increasing angular momentum.

Methods that make use of the algebraic properties of Sp(3, \(\mathbb{R}\)) irreps are of particular interest, especially when they give results that are valid for any choice of Hamiltonian. It is shown, for example, that angular-momentum projection can be achieved explicitly and analytically for a shape-consistent lowest-weight state of an axially symmetric Sp(3, \(\mathbb{R}\)) irrep. This makes it possible to calculate the properties of the projected states of an Sp(3, \(\mathbb{R}\)) irrep as determined by the matrix elements between these states of any observable in the Sp(3, \(\mathbb{R}\)) Lie algebra, such as quadrupole matrix elements, and E2 transition rates. As an example, the calculated expectation values of the many-nucleon kinetic energies of axially symmetric Sp(3, \(\mathbb{R}\)) states have been calculated. Similar algebraic projection methods are possible, in principle, for the generic triaxial Sp(3, \(\mathbb{R}\)) irreps, but have yet to be developed. However, the first steps of projecting a minimum-energy lowest-weight state onto a U(3) basis are considered in this section.

Also discussed are approximations based on the restriction of an Sp(3, \(\mathbb{R}\)) calculation to subsets of basis states and on a generator-coordinate approach. These approaches, are shown to be related to the AMF methods and provide intermediate steps in assessing the accuracy of the AMF-RPA, as a microscopic unified model, in comparison with full Sp(3, \(\mathbb{R}\)) model calculations.
A. Angular-momentum-projected rotational states of an axially symmetric \( \text{Sp}(3, \mathbb{R}) \) irrep

Having determined a minimum-energy lowest-weight state for an \( \text{Sp}(3, \mathbb{R}) \) irrep, angular-momentum projection methods, such as developed for HF theory (see Section \[ \text{II} \]), can be used to determine its low-energy rotational states. Likewise, standard RPA methods can be used to determine its one-phonon intrinsic vibrational states. In fact, explicit expressions have already been derived \[ (121) \] for the multiplicity-free angular-momentum states projected from a lowest-weight state \( |\sigma, \omega\rangle \) of an axially symmetric \( \text{Sp}(3, \mathbb{R}) \) irrep \( (\sigma) = \{\sigma_1, \sigma_2, \sigma_2\} \) which is an eigenstate of energy \( \sigma_1 \hbar \omega_1 + 2 \sigma_2 \hbar \omega_2 \) of an axially symmetric harmonic oscillator. This derivation, starts with the expression of the state \( |\sigma, \omega\rangle \) as an \( \text{SU}(1,1) \) scale transformation

\[
|\sigma, \omega\rangle = \hat{S}(\epsilon)|\sigma_1, \sigma_2, \sigma_2, \omega_2\rangle \tag{120}
\]

of an eigenstate \( |\sigma_1, \sigma_2, \sigma_2, \omega_2\rangle \) of energy \( (\sigma_1 + 2 \sigma_2) \hbar \omega_2 \) of a spherical harmonic oscillator of frequency \( \omega_2 \), and determines that

\[
\hat{S}(\epsilon)|\sigma_1, \sigma_2, \sigma_2, \omega_2\rangle = \sum_{\nu \geq 0} C_\nu(\sigma_1, \epsilon)|\sigma_1 + 2 \nu, \sigma_2, \sigma_2, \omega_2\rangle, \tag{121}
\]

with

\[
C_\nu(\sigma_1, \epsilon) = \frac{1}{(\cosh \epsilon)^{\sigma_1}} (\tanh \epsilon)^\nu \sqrt{\frac{(\sigma_1 + \nu - 1)!}{(\sigma - 1)! \nu!}} \tag{122}
\]

and \( \epsilon^2 = \omega_2/\omega_1 \). This expression of a scale transformation is derived in Appendix \[ A \] This gives the state \( |\sigma, \omega\rangle \) as a sum,

\[
|\sigma, \omega\rangle = \sum_{\nu \geq 0} C_\nu(\sigma_1, \epsilon)|N_\nu(\lambda + 2 \nu, 0), \omega_2\rangle, \tag{123}
\]

of \( \text{U}(3) \supset \text{SU}(3) \) highest-weight states with \( N_\nu = \sigma_1 + 2 \sigma_2 + 2 \nu \) and \( \lambda = \sigma_1 - \sigma_2 \). (Note that a single scale transformation is sufficient, for an axially symmetric irrep, if the frequency \( \omega_2 \) is chosen to give the required volume of the shape-consistent lowest-weight state.)

Use is then made of the expansion \[ (121) \]

\[
|N(\lambda, 0), \omega_2\rangle = \sum_{L \geq 0} a^L_\lambda |N(\lambda, 0)L0, \omega_2\rangle \tag{124}
\]

with

\[
a^L_\lambda = \frac{1}{L} (1 + (-1)^{\lambda + L}) \left[ \frac{(2L + 1)! \lambda !}{(\lambda - L)!!(\lambda + L + 1)!!} \right]^{\frac{1}{2}}, \tag{125}
\]

to obtain the expansion

\[
|\sigma, \omega\rangle = \sum_L N_L(\epsilon)|\epsilon L0\rangle, \tag{126}
\]

with

\[
N_L(\epsilon)|\epsilon L0\rangle = \sum_{\nu \geq 0} C_\nu(\sigma_1, \epsilon)a^{\lambda + 2 \nu}_L|N_\nu(\lambda + 2 \nu, 0)L0\rangle. \tag{127}
\]

Then, with the \( N_L(\epsilon) \) coefficients chosen to be real and the observation that, with

\[
|N_L(\epsilon)|^2 = \sum_{\nu \geq 0} |C_\nu(\sigma_1, \epsilon)|^2 |a^{\lambda + 2 \nu}_L|^2, \tag{128}
\]

they satisfy the identity \( \sum_L |N_L(\epsilon)|^2 \), it follows that

\[
|\epsilon L0\rangle = \sum_{\nu \geq 0} f_{\nu L}(\epsilon)|N_\nu(\lambda + 2 \nu, 0)L0\rangle, \tag{129}
\]
FIG. 3: Amplitudes for the wave functions of the lowest $^{166}$Er states of angular momentum $L = 0$, 10, and 20 projected from a shape-consistent lowest-weight state (as defined in the text) for an Sp(3, $\mathbb{R}$) irrep $\langle N(\lambda, 0) \rangle = \langle 836.5(78, 0) \rangle$. (The figure is adapted from one of Ref. [215] for which $N_0$ was given the value $N_0 = 826.5$.)

with $f_{\nu L}(\epsilon) = C_\nu(\sigma_1, \epsilon) a_{L+2\nu}/N_L(\epsilon)$.

These amplitudes are shown in Fig. 3 for the $L = 0$, 10 and 20 states of the ground-state rotational band of $^{168}$Er, for the shape-consistent lowest-weight state of the Sp(3, $\mathbb{R}$) irrep $\langle N(\lambda, 0) \rangle = \langle 836.5(78, 0) \rangle$. The irrep $\langle 836.5(78, 0) \rangle$ was chosen to give experimentally observed E2 transition rates [217]. The figure shows that the wave functions for the states of this irrep are spread over $\gtrsim 15$ spherical harmonic-oscillator shells.

From the known representations of the Sp(3, $\mathbb{R}$) Lie algebra in a U(3) $\supset$ SO(3) basis [162] (reviewed in Ref. [171]), it is now straightforward to calculate the matrix elements of any Sp(3, $\mathbb{R}$) observable between the states obtained by angular-momentum-projection from the shape-consistent state. As an example, Table I gives the expectation values of the nuclear kinetic energy of the angular-momentum-projected states, relative to that of the $L = 0$ ground state, for the states of angular momentum ranging from 2 to 16. They show that to within $\sim 2\%$, the expectation values of the nuclear kinetic-energy, relative to that for the $L = 0$ state, are proportional to $L(L+1)$ for states of angular momentum ranging from $L = 2$ to 16. Thus, if it is assumed that the potential energy of a rotor depends only on its intrinsic state and is independent of its rotational angular momentum, then the moment of inertia of the rotor can only depend on its kinetic energy and should be expressible in the form $K.E. = K_0 + \hbar^2 L(L+1)/2J_{\text{KE}}$. This moment of inertia and that obtained from the experimentally observed energy for the $L = 2$ state, have the respective values

$$J_{\text{KE}}/\hbar^2 = 99.5 \text{ MeV}^{-1}, \quad J_{\text{expt}}/\hbar^2 = 37.6 \text{ MeV}^{-1}. \quad (130)$$

For comparison the rigid and irrotational-flow moments of inertia for the minimum-energy lowest-weight state have the values

$$J_{\text{rig}}/\hbar^2 = 84.4 \text{ MeV}^{-1}, \quad J_{\text{irr}}/\hbar^2 = 5.8 \text{ MeV}^{-1}. \quad (131)$$

| $L$ | K.E. | $L(L+1)/K.E.$ |
|-----|------|---------------|
| 2   | 30   | 99.5          |
| 4   | 101  | 99.3          |
| 6   | 212  | 99.1          |
| 8   | 364  | 98.9          |
| 10  | 568  | 98.5          |
| 12  | 795  | 98.1          |
| 14  | 1,076| 97.6          |
| 16  | 1,401| 97.1          |
A conclusion of these results is that the kinetic-energy components of the excitation energy for the $2^+$ state of this $^{168}$Er $Sp(3, \mathbb{R})$ irrep is less than half the experimentally observed value. Obvious interpretations of these results, assuming no mistakes have been made, are that the shape-consistent lowest-weight state for the chosen $Sp(3, \mathbb{R})$ irrep is a poor approximation to the intrinsic state of the ground-state rotational band of $^{166}$Er and/or that a large fraction of rotational energies are potential energies. However, regardless of whether or not the chosen irrep is the most appropriate for this nucleus, which is certainly questionable, the results show that the kinetic-energy component of the rotational energies of a symplectic model irrep is close to that of a rigid rotor. Note that, in Bahri’s symplectic-model calculations, the SU(1,1) transformation

for $^{168}$Er with a Davidson-like potential, good fits were obtained in which the rotational energies were more than half potential energies.

According to conventional wisdom, a large mixing of symplectic model irreps, brought about by pairing interactions, is needed to obtain observed moments of inertia. However, the results of Bahri’s calculations suggest that the potential-energy contributions to rotational energies might be sufficient for this purpose. In any event, it is important to determine the potential and kinetic energy components of the rotational energies of the decoupled collective model states of the symplectic model because, as observed in Section X, the states of different symplectic model irreps are pure uncoupled collective states in the sense that there can be no isoscalar E2 transitions between them. Thus, it is clear that an evaluation of the separate contributions of the kinetic and potential energies to the rotational energies of axially symmetric nuclei is essential for understanding the dynamics of nuclear rotations.

B. The rotational states of a triaxial $Sp(3, \mathbb{R})$ irrep

Angular-momentum projection methods for the low-energy rotational states of a nucleus and RPA methods for its intrinsic excitations, as developed for use in HF theory (see Section XIV), can similarly be applied to the minimum-energy lowest-weight states of triaxial $Sp(3, \mathbb{R})$ irreps of maximal space symmetry. It is also likely that if a minimum-energy lowest-weight state of a triaxial $Sp(3, \mathbb{R})$ irrep is expanded in a $U(3)$ basis, it can be further expanded in an $SO(3)$ basis by precise algebraic methods, as shown above for an axially symmetric irrep and as employed for the calculation of $SU(3)$ Clebsch-Gordan coefficients in an $SO(3)$ basis. Thus, an outline is given in this section of the expansion of a shape-consistent lowest-weight of a triaxial $Sp(3, \mathbb{R})$ irrep in an $U(3)$ basis.

If $\sigma, \omega_3$ is a lowest-weight state of a triaxial $Sp(3, \mathbb{R})$ irrep $\sigma(\omega)$ is an eigenstate of a spherical harmonic-oscillator Hamiltonian of frequency $\omega_3$, then the corresponding shape-consistent lowest-weight state is given, in parallel with that of Eqn. 120, by the $SU(1,1) \times SU(1,1)$ transformation

$$|\sigma, \omega_3\rangle \rightarrow |\sigma, \omega\rangle = \hat{S}_1(\epsilon_1)|\sigma, \omega_3\rangle$$

(132)

to an eigenstate of a triaxial harmonic oscillator with a triple of frequencies $\omega \equiv (\omega_1, \omega_2, \omega_3)$, with

$$e^{2\epsilon_1} = \frac{\omega_3}{\omega_1}, \quad e^{2\epsilon_2} = \frac{\omega_3}{\omega_2}. \quad (133)$$

(Note that a third scale transformation is avoided by setting $\omega_3$ such that the state $|\sigma, \omega_3\rangle$ has the desired volume.)

The scaled state is then the sum of states

$$|\sigma, \omega\rangle = \sum_{\nu_1, \nu_2} C_{\nu_1}(\sigma_1, \epsilon_1)C_{\nu_2}(\sigma_2, \epsilon_2)\langle (2\nu_1, 2\nu_2)|\sigma, \omega_3\rangle,$$

(134)

where

$$\langle (2\nu_1, 2\nu_2)|\sigma, \omega_3\rangle \equiv |(\sigma_1 + 2\nu_1, \sigma_2 + 2\nu_2, \sigma_3), \omega_3\rangle.$$  

(135)

The states $|(2\nu_1, 0)\rangle$ in the expansion of $\hat{S}_1(\epsilon_1)|\sigma, \omega_3\rangle$ have the useful property of being $U(3)$ highest-weight states. They are the states illustrated in Fig. 4 that have $U(3)$ weights on the vertical line above the weight of the $Sp(3, \mathbb{R})$ lowest-weight state $|(0, 0)\rangle \equiv |\sigma, \omega_3\rangle$ and are generated from this lowest-weight state by the $\hat{A}_{11}$ operators of Eqn. 29. It is clear that that these states are all of $U(3)$ highest weight.

The states $|(0, 2\nu_2)\rangle$ in the expansion of $\hat{S}_2(\epsilon_2)|\sigma, \omega_3\rangle$ are similarly generated from the lowest-weight state $|(0, 0)\rangle$ by the $\hat{A}_{22}$ operators. Thus, the states in the expansion of $\hat{S}_1(\epsilon_1)\hat{S}_2(\epsilon_2)|\sigma, \omega_3\rangle$ have $U(3)$ weights defined by all the points in Fig. 4(c) that are connected to the state $|(0, 0)\rangle$ by $\hat{A}_{11}$ and $\hat{A}_{22}$ arrows. However, although a state $|(2\nu_1, 2\nu_2)\rangle$ has well-defined $U(3)$ weight $(\sigma_1 + 2\nu_1, \sigma_2 + 2\nu_2, \sigma_3)$, it is only a state of $U(3)$ highest weight if $\nu_2 = 0$. As a result, the expansion of the shape-consistent lowest-weight state of a triaxial $Sp(3, \mathbb{R})$ irrep as a sum of states of good angular-momentum quantum numbers is not as simple as for an axially symmetric irrep. It can no doubt be determined
FIG. 4: The figure shows weight diagrams: (a) for the SU(3) Lie algebra; (b) for the Sp(3, R) raising operators \( \hat{A}_{11} \) and \( \hat{A}_{22} \) as components of a U(3) tensor of highest weight \((2,0,0)\); and (c) for the U(3) weights of states in the expansion of the scaled lowest-weight state \( \hat{S}_1(\epsilon_1)\hat{S}_2(\epsilon_2)|\sigma, \omega_3\rangle \) in a basis of spherical harmonic-oscillator eigenstates. The weight diagram of figure (c) is two-dimensional in the grey shaded areas in which the third component of the weights takes the constant value \( \sigma_3 \). The state \(|\sigma, \omega_3\rangle\), corresponding to the point \((0,0)\) in the figure, is, by definition, both a lowest-weight state for an Sp(3, R) irrep with \( \sigma = (\sigma_1, \sigma_2, \sigma_3) \) and a highest-weight state for a U(3) irrep. All the states with weights that lie on the vertical line above that of the U(3) highest-weight state \(|\sigma, \omega_3\rangle\) and are connected to this state by \( \hat{A}_{11} \) arrows are basis states for an SU(1,1) \( \times \) SU(1,1) irrep. All the states with weights in the combined light- and dark-grey areas are a basis for an Sp(2, R) irrep. The subset of these states with weights that are connected by \( \hat{A}_{11} \) and \( \hat{A}_{22} \) arrows to the state \(|\sigma, \omega_3\rangle\) are a basis for an SU(1,1) \( \times \) SU(1,1) irrep. As shown in the text, a minimum-energy Sp(3, R) lowest-weight state is a linear combination of the basis states for the above-defined SU(1,1) \( \times \) SU(1,1) irrep.

by use of the transformations between SU(3) states in SU(2) and SO(3) bases \( [198, 216, 218] \). As discussed in the following section, the information gained from the content of Figure 4 when considered in the light of the Sp(1, R) and Sp(2, R) approximations and the general coordinate method, also leads to alternative ways of proceeding to get reliable results.

C. Relationship of the AMF approximations to the Sp(1, R) and Sp(2, R) submodels

Approximate Sp(3, R) calculations, which relate to the AMF-RPA methods, are the so-called Sp(1, R) and Sp(2, R) submodels of Sp(3, R) proposed, respectively, by Arickx \( [178] \) and by Peterson and Hecht \( [179] \), as outlined in Section VII C.

A first observation is that, in truncating the Hilbert space of a full Sp(3, R) model to a subspace, these models both maintain a constant value of \( \sigma_3 \) for every U(3) irrep \( \{\sigma_1, \sigma_2, \sigma_3\} \) in the truncated space. Thus, both models can be improved by choosing the value of \( \omega_3 \) to be that for which the shape-consistent lowest-weight state has the desired volume of essentially incompressible nuclear matter, as opposed to choosing it for the spherical harmonic-oscillator lowest-weight state. With this adjustment, they provide sub-models that are intermediate between those of the AMF-RPA and the full Sp(3, R) model. A comparison of their results for particular Sp(3, R) irreps, for which each of them is able to give converged solutions to an acceptable level of accuracy, could then be very informative. Hopefully, they will show that the simple AMF-RPA model, once programmed, is able to give quick and sufficiently accurate results that could be used to obtain a first understanding of nuclear data in terms of unmixed symplectic model irreps. If this proves to be the case, it then become worthwhile to take it to the next level of approximation given by the generator-coordinate theory of single and mixed Sp(3, R) irreps.
D. Generator-coordinate theory for single $\text{Sp}(3, \mathbb{R})$ many-nucleon irreps

It is known that in the Elliott model, the states of an SU(3) irrep can be expressed as linear combinations of the states obtained by rotations of its highest-weight state. Likewise, the state of an Sp(3, $\mathbb{R}$) irrep can all be expressed as linear combinations of the states obtained by general-linear transformations of its lowest-weight state $[\mathfrak{g}]$. This observation underlies the generator coordinate approach of Filippov, Vassanji and colleagues $[180–183, 219]$, which is now seen to have a potentially powerful relationship with the AMF developments.

From an algebraic perspective, the states of an Sp(3, $\mathbb{R}$) irrep are linear combinations of the states generated by the rotations and deformations of a lowest-weight state. Thus, if we start with a minimum-energy lowest-weight state and augment it to a set of locally deformed states generated by infinitesimal generators of the Sp(3, $\mathbb{R}$) algebra, other than those that generate rotations, the result is a space corresponding to the intrinsic small-amplitude vibrational states of a rotational nucleus; the rotational states are then generated by the relatively adiabatic rotation of all the intrinsic states, as understood in the Bohr-Mottelson unified model.

This perspective leads naturally to an extension of the Sp(3, $\mathbb{R}$) model to a model of mixed Sp(3, $\mathbb{R}$) irreps in which the mixing takes places in the intrinsic frame of an adiabatic rotor in the manner of a quasi-dynamical symmetry as discussed in the following Section, X C.

X. THE MANY-NUCLEON HILBERT SPACE AS A SUM OF COLLECTIVE MODEL SUBSPACES

A significant property of the nuclear shell model $[5–7]$ is that it defines a decomposition of the many-nucleon Hilbert spaces of nuclei into sums of energy-ordered subspaces of spherical harmonic-oscillator eigenstates coupled to spin and isospin states. This energy-ordered decomposition has been used widely to select truncated subspaces for shell-model calculations. It has also been useful for separating nuclei into classes of magic (closed-shell) nuclei, singly-closed shell nuclei, and doubly-open shell nuclei, each of which has characteristic properties. However, in spite of its successes, it is now apparent that the spherical shell model, when truncated to a manageable size, is unable to describe the rotational states of strongly deformed nuclei in anything like a realistic manner; it can do so only with the use of fitted effective interactions and huge effective charges for electric quadrupole (E2) transition operators. The problem is that such effective shell model states have essentially zero overlaps with realistic many-nucleon states in heavy nuclei.

A. Coupling schemes for nuclear Hilbert spaces

As Nilsson model $[17, 220]$, mean-field $[205]$ and symplectic model $[172, 221]$ calculations with schematic interactions (see Fig. 1) testify, the spherical harmonic-oscillator-based decomposition and energy-ordering of the standard shell model is seriously inappropriate for realistic descriptions of the rotational states of heavy deformed nuclei. They show that significant components of the states needed do not begin to occur in standard shell-model spaces of spherical harmonic-oscillator energies of less that $\sim 10\hbar \omega$, in spherical harmonic-oscillator units, above those normally considered. Thus, for a realistic shell-model theory of heavy deformed nuclei, it is necessary to consider decompositions of their many-nucleon Hilbert spaces into more appropriate sequences of subspaces.

An appropriate decomposition is given by a coupling scheme, defined by an algebraic model of nuclear rotations and quadrupole vibrations whose irreps can be ordered in a physically relevant manner. Without prejudice as to the nature of the rotational dynamics of nuclei, the Lie algebra of the model should include the angular momentum, the quadrupole moment operators which define the orientation of a rotor and isoscalar E2 transitions, and the nuclear kinetic energy. It is then inferred, see Section VIII, that the smallest Lie algebra that contains these observables is that of the Sp(3, $\mathbb{R}$) symplectic model which, in addition contains infinitesimal generators of quadrupole deformation. The essential property of the decomposition into irreducible symplectic model subspaces is that, if the whole many-nucleon Hilbert space of a nucleus is expressed as a sum of such subspaces, there is no iso-scalar E2 transitions between the states of these subspaces. A finer decomposition, which distinguishes equivalent subspaces but which retains this property with fewer multiplicities, is given by the irreps of the direct product group Sp(3, $\mathbb{R}$) $\times$ U(4), where U(4) is Wigner’s supermultiplet group which contains, as subgroups, the U(2) spin and isospin groups. Thus, the irreps of the group Sp(3, $\mathbb{R}$) $\times$ U(4) can be said to define ideal collective states for the study of rotational dynamics.

B. Energy-ordered symplectic-model subspaces

It is apparent from Fig. 1 and other symplectic model calculations, that highly deformed rotational states which lie low in energy invariably have their dominant components in much higher spherical harmonic-oscillator shells than the
lowest available to them. Thus, instead of ordering symplectic model irreps by the energies of their lowest spherical harmonic-oscillator components, it is more meaningful to order them by the eigenvalues of the generally triaxial harmonic-oscillator Hamiltonians of which their minimum-energy lowest-weight states are eigenstates. As this section shows, the minimum-energy lowest-weight states of strongly deformed irreps then have much lower energies than those of their spherical harmonic-oscillator counterparts.

Such an energy-ordered sequence of $\text{Sp}(3, \mathbb{R})$ irreps is derived below by mean-field shape-consistency methods. It must be recognised, however, that although the sequence of irreps obtained is much more meaningful than the partially ordered sets (with huge degeneracies) given by the energies of spherical harmonic-oscillator lowest-weight states, it is only intended to provide a list of candidates. A better ordering of irreps would be given by the energies of their lowest-energy eigenstates calculated for a suitable model Hamiltonian restricted to the spaces of single irreps. However, given the huge number of possible irreps, such an approach is only feasible for refining the order of a small set of irreps prepared by other, e.g., shape-consistency, methods. It is also possible to select the most relevant irreps from the shape-consistency list on the basis of their ability to describe experimental observations as in Ref. [217].

To within a rotation, a lowest-weight state for an $\text{Sp}(3, \mathbb{R})$ irrep is an eigenstate $|\sigma, \omega\rangle$ of a generally triaxial harmonic oscillator

$$\hat{H}(\omega) = \sum_{n=1}^{A} \sum_{i=1}^{3} \hbar \omega_i (b_{ni}^\dagger b_{ni} + \frac{1}{2}),$$

(136)

with a triple of frequencies $\omega = (\omega_1, \omega_2, \omega_3)$, for which the harmonic-oscillator raising and lowering $b_{ni}^\dagger$ and $b_{ni}$ operators are defined in terms of the nucleon position and momentum coordinates by

$$\hat{x}_{ni} = \frac{1}{\sqrt{2}} a_i (b_{ni}^\dagger + b_{ni}), \quad \hat{p}_{ni} = i\hbar \frac{a_i}{\sqrt{2}} (b_{ni}^\dagger - b_{ni}),$$

(137)

with generally unequal units of inverse length $a_i = \sqrt{M\omega_i/\hbar}$. The objective is then to determine the lowest-weight state for which the energy $\langle \sigma, \omega | \hat{H} | \sigma, \omega \rangle$ of a chosen many-nucleon nuclear Hamiltonian $\hat{H}$ is minimised.

Based on experience gained from standard Hartree-Fock theory, one can be sure that, for a many-nucleon Hamiltonian with predominantly attractive short-range interactions between the nucleons, the energy $\langle \sigma, \omega | \hat{H} | \sigma, \omega \rangle$ will be minimised when, to a good approximation, the ellipsoid defined by the quadrupole moments of the state $|\sigma, \omega\rangle$ has the same shape as that of the potential-energy component of $\hat{H}(\omega)$. Thus, if the lowest-weight state $|\sigma\rangle$ is an eigenstate of the triaxial harmonic-oscillator Hamiltonian $\hat{H}(\omega)$ with potential energy

$$\hat{V} = \frac{1}{2} \sum_{ni} \omega_i^2 \hat{x}_{ni}^2,$$

(138)

then the single-particle density of the lowest-weight state has expectation values given by

$$\langle \sum_n \hat{x}_{ni}^2 \rangle = \frac{\hbar}{2M\omega_i} \sum_n \langle (b_{ni}^\dagger b_{ni} + b_{ni}b_{ni}^\dagger) \rangle = \frac{\hbar \sigma_i}{M\omega_i},$$

(139)

and defines an $n$-independent ellipsoidal surface given by

$$\sum_i \frac{x_i^2}{\langle \sum_n x_{ni}^2 \rangle} = \sum_i \frac{M}{\hbar \sigma_i \omega_i} \omega_i^2 x_i^2 = \text{const.}$$

(140)

It follows that the density of the lowest-weight state has the same ellipsoidal shape as the potential energy $\hat{V}$ if and only if

$$\sigma_1 \omega_1 = \sigma_2 \omega_2 = \sigma_3 \omega_3.$$  

(141)

This shape-consistency relationship has been used for other purposes by Bohr, Mottelson [53, 59] and others [222].

Equation (141) defines the relative values of $\omega_1$, $\omega_2$ and $\omega_3$ for a given $\text{Sp}(3, \mathbb{R})$ irrep $|\sigma\rangle$. The strength of the interaction component of $\hat{H}$ should also be such that their magnitudes give the volume of the deformed nucleus consistent with that expected for near-incompressible nuclear matter. This then determines the absolute magnitudes of the frequencies. Given that a symplectic-model irrep is defined by its lowest weight, $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, it follows that the set of symplectic model irreps for a nucleus can be ordered by the energies of the corresponding shape-consistent lowest-weight states.
Among a set of $\text{Sp}(3,\mathbb{R})$ irreps with a common value of $N_0 = \sigma_1 + \sigma_2 + \sigma_3$, which are irreps having lowest-weight states of spherical harmonic-oscillator energy $N_0 \hbar \omega_0$, it is a simple matter to identify those of maximal space symmetry and observe that they are the maximally deformed irreps; they have the largest values of the SU(3) Casimir invariant $\lambda^2 + \lambda \mu + \mu^2 + 3(\lambda + \mu)$ and correspondingly the largest values of $2 \lambda + \mu$ or $\lambda + 2 \mu$. The energy expectation values

$$E_{\sigma} = \langle \sigma, \omega | \hat{H}(\omega) | \sigma, \omega \rangle = \sum_1 h \omega_1 \sigma_1 = 3(\sigma_1 \sigma_2 \sigma_3)^{\frac{1}{2}} h \omega_0,$$  \quad \text{with} \quad \omega_0^3 = \omega_1 \omega_2 \omega_3, \quad (142)$$

for the corresponding shape-consistent lowest-weight states for these irreps can then be evaluated for increasing values of $N_0$, starting from the lowest values allowed by the Pauli exclusion principle. Table II shows the lowest energies determined in this way for three doubly-even nuclei, in order of increasing energy, for a range of $N_0$ values. An immediate observation is that the order by increasing values of $E_{\sigma}$ differs markedly from the ordering by increasing values of $N_0$ as would be obtained if the minimum-energy lowest-weight states were eigenstates of spherical harmonic-oscillator Hamiltonians. As expected, it is found that the minimum energies of the shape-consistent lowest-weight states obtained in this way are almost invariably those with $\lambda > \mu$, consistent with the observed dominance of prolate over oblate deformations.

A satisfying property of these simply derived results is that the lowest three irreps obtained for $^{12}\text{C}$ and $^{16}\text{O}$ are precisely those determined to be appropriate for describing the low-energy positive-parity states of these nuclei; cf. [148] for $^{12}\text{C}$ and [224] for $^{16}\text{O}$. It is remarkable that the irreps shown for $^{168}\text{Er}$ have lowest-weight states that range over 16 spherical harmonic oscillator $\hbar \omega_0$ shells but have triaxial harmonic-oscillator energies with a spread of only $\sim 0.5 \hbar \omega_0$. These results are particularly significant because they explain the emergence of shape coexistence in a dramatic way without the use of adjustable parameters. Another characteristic is that the sets of lowest-weight states of nearby energies tend to have significantly different values of $N_0$ and substantially different deformation shapes. As a result, the $\text{Sp}(3,\mathbb{R})$ irreps of similar energies will mix much less than would occur for irreps of similar deformations. This is an indication that $\text{Sp}(3,\mathbb{R})$ is likely to be a considerably better dynamical symmetry than might otherwise have been expected. It also indicates that it is hopeless to give a thought to ever being able to perform realistic many-nucleon shell-model calculations in a conventional spherical harmonic-oscillator basis for heavy rotational nuclei.

The above shape-consistency results suggest that it would be profitable to explore the more reliable ordering of minimum-energy lowest-weight states for realistic Hamiltonians restricted to the triaxial harmonic-oscillator lowest-weight states of each $\text{Sp}(3,\mathbb{R})$ irrep. In identifying the appropriate irrep for the lowest-rotational states of a given nucleus, one should also be aware that the states of lowest angular momentum of a symplectic model irrep are expected to have energies significantly below that of its minimum-energy lowest-weight state and to be lowered most for the most-strongly deformed irreps. For example, although $E_{\sigma}$ for the less-deformed shape-consistent lowest-weight state for the irrep with $N_0 = 824.5$ lies slightly lower in energy than that of the next lowest irrep with $N_0 = 824.5$, it is not surprising to find that the observed low-energy rotational states of $^{168}\text{Er}$ are closer to those of the $N_0 = 824.5$ irrep [217]. This lowering could be estimated from an evaluation of the expectation value of $\hat{L} \cdot \hat{L}$ in the minimum-energy lowest-weight state and the expected moment of inertia for the nucleus.

Note also that the weights $\langle \sigma |$ of $\text{Sp}(3,\mathbb{R})$ irreps needed to describe the observed E2 transitions and quadrupole moments of a number of heavy rotational nuclei, have been estimated by Jarrio et al. [217] and, from a theoretical perspective of what is possible, by Carvalho [222], and are qualitatively in agreement with the above results.
C. Generator-coordinate theory for mixed $\text{Sp}(3, \mathbb{R})$ many-nucleon irreps

The potential for a dramatic reduction in the computational complexity of calculations for microscopic Hamiltonians in spaces of multiple symplectic model irreps by generator coordinate methods, coupled with the insights gained from the AMF perspective, raises expectations for the achievement of accurate results from rapidly converging sequences of calculations in sub-spaces of relatively low dimension. As illustrated in a simple application of the generator coordinate method to the $L = 0$ states of mixed $\text{Sp}(3, \mathbb{R})$ irreps by Carvalho et al. [226], it is profitable to proceed by diagonalising a nuclear Hamiltonian in spaces of states, that are angular-momentum projected from an increasing number of optimally selected lowest-weight states from a set of $\text{Sp}(3, \mathbb{R})$ irreps, until converged results are obtained to the desired level of accuracy for the chosen set of irreps.

Such a procedure could be a solution to the challenge of handling the mixing of states of different $\text{Sp}(3, \mathbb{R}) \times U(4)$ irreps by those of a competing dynamical symmetry. The concept of quasi-dynamical symmetry was introduced in a study of such systems with competing dynamical systems [45] to understand the apparent persistence of the sub-dynamics of a system when it is adiabatic relative to the competition. Such quasi-dynamical symmetries, reviewed in Ref. [48], have been shown to emerge in many examples [46, 47, 49, 227–229].

For a rotor model, quasi-dynamical symmetry is realised as a coherent linear sum of many similar irreps, all of which vary in a similar way with respect to their angular-momentum content so that the result is essentially their average. Thus, it is apparent that, when there are competing dynamical symmetries, the dynamical symmetry of an adiabatic rotational model becomes a quasi-dynamical symmetry and is easily handled. Within the spirit of the Bohr-Mottelson-Nilsson unified model, one has only to replace the shape-consistent lowest weight of a single $\text{Sp}(3, \mathbb{R})$ irrep by an eigenstate of the nuclear Hamiltonian in the space of the shape-consistent lowest-weight states of the several mixed $\text{Sp}(3, \mathbb{R})$ irreps which then becomes the intrinsic state for one of several mixed collective-model representations. However, the possibilities of such a potentially powerful approach to the symmetry mixing of symplectic irreps requires investigation.

XI. CONCLUDING REMARKS

For many years, the Nilsson model [17] has been understood as providing the intrinsic states for the unified model of rotational nuclei. In doing so, it has been interpreted as a phenomenological approximation to HF theory. This perspective is now given a more fundamental foundation in which a Hartree-Fock solution is identified as a minimum-energy lowest-weight state for the unique totally anti-symmetric irreps of the Lie group of one-body unitary transformation of nuclei. There is, however, a problem with the underlying Hartree-Fock theory in that, although it is in principle a complete theory, in the sense that a set of Slater determinants spans the whole Hilbert space of a nucleus and although the Hartree-Fock manifold of Slater determinants has many local minimum-energy states, it is not clear how these local minimum energy states relate to the excited rotational bands of a nucleus. This problem is now resolved by extending Hartree-Fock theory to an algebraic mean-field theory and applying it to the irreps of the symplectic model, each of which has a single minimum-energy lowest-weight state and which together form a complete set of intrinsic states for the collective model states of a nucleus.

The irreps of the symplectic model for a nucleus have many invaluable properties. In particular, there are no isoscalar E2 transitions between the states of its different irreps and all matrix elements of the nuclear kinetic energy and of other elements of the $\text{Sp}(3, \mathbb{R})$ Lie algebra have vanishing values between the states of different irreps. Thus, the irreducible $\text{Sp}(3, \mathbb{R})$ subspaces are completely independent collective-model subspaces of the many-nucleon Hilbert space. Their study is then a profitable first step towards understanding the rotational dynamics of nuclei. At the same time, it is recognised that coherent mixtures of states from the different collective subspaces can result in the emergence of new and unexpected properties. For example, the correlations of nucleon dynamics induced by pairing interactions could no doubt result in more superfluid-like flows, especially in states of small deformation. Thus, in reality one can expect there to be a competition between different dynamical symmetries and the emergence of rotational states characterised by quasi-dynamical symmetries as discussed in Section X C.

Several approximations for the calculation of nuclear rotational properties with many-nucleon Hamiltonians in spaces of single or a few $\text{Sp}(3, \mathbb{R})$ irreps have been outlined in this review, although in most cases further developments of the technical tools for such calculations are needed. Moreover, to assess the accuracy of these approaches, comparisons with precise calculations are required for at least a few $\text{Sp}(3, \mathbb{R})$ irreps of medium-mass nuclei of modest deformation. Fortunately, it appears that such calculations are now becoming possible with the supercomputer program of the LSU group as discussed in Section VII E.

The remarkable successes of the Bohr-Mottelson collective and unified models leave little doubt that nuclei have states with the essential properties of the symplectic models. However, in seeking to understand the dynamics of nuclear rotations many questions arise of which the following are a few that emerge.
1. What is the significance of the cranking model moments of inertia, which in one situation gives moments of inertia corresponding to the kinetic energies of a rigid rotor and, in another, the irrotational-flow moments of inertia of a quantum fluid?

It has traditionally been supposed that an observed moment of inertia is a mass parameter in the expression of the kinetic energy of a rotating nucleus. However, this is clearly inappropriate if the so-called rotational energies are not predominantly kinetic energies. Questions also arise regarding the interpretation of nuclear moments of inertia as being linear combinations of those for irrotational and rigid flows and regarding the magnitude of the inertial (Coriolis and centrifugal) forces, if rotational energies are only fractionally kinetic energies.

2. To what extent are the symplectic model irreps, that are best able to describe the low-energy rotational states of doubly-even nuclei, among the leading irreps of maximal space symmetry given by shape-consistency criteria? How should the criteria be modified for the Sp(3, \mathbb{R}) \times U(4) irreps of other than doubly-even nuclei?

3. Is there a better algorithm, than that of computing the energies of shape-consistent lowest-weight states, for a practical ordering of symplectic irreps and for identifying those that contribute most strongly to the observable lower-energy states of nuclei?

4. What evidence is there for low-energy beta and gamma vibrational bands? And why is the axially symmetric unified model as successful as it is, when the minimum-energy shape-consistent lowest-weight states available to a nucleus, as defined by its maximally decoupled irreducible Sp(3, \mathbb{R}) subspaces, have triaxial shapes much more frequently than axially symmetric shapes?

The existence of beta vibrations has been questioned in recent years \[230, 231\] in the light of experimental observations of the decay properties of possible candidates. One can also question the existence of gamma vibrations. From a symplectic model perspective, it is expected that with no mixing of its irreps, the vibrations of the intrinsic states of rotational nuclei would only have vibrational states at giant-resonance energies. It is also expected that the larger the intrinsic deformation of a nucleus, the more rigid it will be and the less it will mix with other irreps. Of particular note is the observations that, in listing and ordering the symplectic irreps available to a nucleus by their generally triaxial harmonic-oscillator energies, as in Table II, there are relatively few axially symmetric irreps.

More precisely, if the length scales of the shape-consistent lowest-weight state are defined by their Sp(3, \mathbb{R}) quantum numbers \((\sigma_1, \sigma_2, \sigma_3)\), it is found that \(\lambda = \sigma_1 - \sigma_2\) is generally substantially larger than \(\mu = \sigma_2 - \sigma_3\), consistent with the observation that nuclear deformations are more prolate than oblate \[222, 223\]. However, it appears that \(\mu\), which is a measure of axial asymmetry, is rarely zero. Nevertheless, the description of rotational states with an axially symmetric rotor model has been remarkably successful. This suggests that there might be a restoration of axial symmetry about the major axis which could be due to the mixing of states from other Sp(3, \mathbb{R}) irreps, e.g., by pairing interactions. However, the coherent mixing of irreps in the restoration of axial symmetry implies that axially symmetric nuclei can have low-energy gamma vibrational excited states as suggested in Ref. \[232\]. Maybe the standard HF and HFB models \[96, 97, 205\] are already able to address this question!

5. A complementary question is; what is the experimental evidence for rotational states of triaxial nuclei nuclei?

The general triaxial rotor model for an even-even nucleus has mixed \(K = 0, 2, 4, \cdots\) sequences of rotational bands whereas the axially symmetric rotor model predicts a \(K = 0\) band, a one-phonon \(K = 2\) gamma-vibrational band, two-phonon \(K = 0\) and 4 bands, and so on. However, there appears to be little experimental evidence of either two-phonon gamma bands or of \(K \geq 4\) states of a triaxial rotor. Maybe they exist but are not observed. Recall also that there is little unequivocal evidence of multi-phonon excitations of any kind in nuclei. Thus, it is of fundamental interest to explore the extent to which a triaxial rotor model is able to fit the observed data of rotational nuclei \[233\] and to explore the existence or non-existence of \(K = 0\) and \(K = 4\) rotational bands that are strongly coupled to the first excited \(K = 2\) bands of even-even rotational nuclei.

6. To what extent is it possible to determine the goodness of the \(K\) quantum number and hence the mixing of rotational bands in the rotor-model interpretation of nuclear data?

As shown in Appendix \[C\] \(K\) is defined as an integer-valued quantum number for the basis states of the rotor model and, because SU(3) states can be expressed, in a vector coherent-state representation, as linear combinations of rigid rotor-model states, it also has remarkably well-defined essentially integer values for appropriately defined basis states of a U(3) irrep. However, the \(K\) quantum number is conserved only for the states of an axially symmetric rotor. Thus, measurements of the mean values of \(K\) for the rotational states of a nucleus provide a measure of triaxiality.

In addressing these questions, there is much to be gained by comparing applications of the HF, HFB and AMF-symplectic models with one another and experimental data. It must be emphasised, however, that there is no reason
to expect that the AMF-symplectic model calculations, when restricted to a single \( \text{Sp}(3, \mathbb{R}) \) irrep will fit experimental data any better than standard HF and HFB calculations for the same Hamiltonian when applied to a ground-state rotational band. On the contrary, given that the minimum-energy lowest-weight states of symplectic model irreps are most likely to be single Slater determinants when combined with appropriate spin and isospin wave functions, the standard HF and HFB models have the potential, to give better results, as far as the ground-state band of collective states is concerned, and also provide information on the mixing of irreps and some of the questions posed above. Thus, it must be emphasised that the objective of this review is not simply to fit rotational data but to develop tools for understanding the nature of rotational dynamics in nuclei. Thus, comparisons of HF, HFB, AMF and experimental results are expected to provide more insights and answers to important questions than either approach on its own.

The most important contribution of the AMF-symplectic model is that it facilitates the analyses of the dynamics of idealised (that is decoupled) collective states and, hence, a means to interpret experimental data in terms of a mixing of such states. If everything works out in a logically consistent way, we can then be satisfied. However, it is perhaps more likely that unexplained phenomena will emerge which will present new and exciting challenges.

The appendices to this review include brief summaries of some developments of importance in the applications of the symplectic model for which details can be found in the literature.

**Acknowledgments**

The author is pleased to acknowledge many helpful discussions of this review with J. L. Wood and his proof reading of the manuscript.

**Appendix A: SU(1,1) scale transformations of an Sp(3, \( \mathbb{R} \)) lowest-weight state**

A pair of position and momentum operators for a particle satisfies the Heisenberg commutation relations \([\hat{x}, \hat{p}] = i\hbar\) and can be expressed in terms of the raising and lowering operators of a simple harmonic-oscillator Hamiltonian

\[
\hat{h}(\omega_0) = \frac{1}{2M} \hat{p}^2 + \frac{1}{2} M\omega_0^2 \hat{x}^2 \tag{A1}
\]

of frequency \(\omega_0\) with

\[
\hat{x} = \frac{1}{\sqrt{2a}} (c^\dagger + c), \quad \hat{p} = i\hbar \frac{a}{\sqrt{2}} (c^\dagger - c), \tag{A2}
\]

where \(a = \sqrt{M\omega_0/\hbar}\) is a unit of inverse length, and

\[
[c^\dagger, c^\dagger] = [c, c] = 0, \quad [c, c^\dagger] = 1. \tag{A3}
\]

An SU(1,1) scale transformation of these operators, that preserves their Heisenberg commutation relation, is then given by

\[
\hat{x} \rightarrow \hat{S}(\epsilon) \hat{x} \hat{S}(\epsilon^*) = e^{-\epsilon} \hat{x}, \quad \hat{p} \rightarrow \hat{S}(\epsilon) \hat{p} \hat{S}(\epsilon^*) = e^\epsilon \hat{p}, \tag{A4}
\]

with

\[
\hat{S}(\epsilon) = e^{\epsilon (\hat{S}_+ - \hat{S}_-)} \tag{A5}
\]

and

\[
\hat{S}_+ = \frac{1}{2} c^\dagger c, \quad \hat{S}_- = \frac{1}{2} c c, \quad \hat{S}_0 = \frac{1}{4} (c^\dagger c + c c^\dagger). \tag{A6}
\]

Thus, a harmonic-oscillator Hamiltonian with a scaled potential is given by the transformation

\[
\hat{h}(\omega_0) \rightarrow \hat{h}(\omega) = e^{-2\epsilon} \hat{S}(\epsilon) \hat{h}(\omega_0) \hat{S}(\epsilon^*) = \frac{1}{2M} \hat{p}^2 + \frac{1}{2} M\omega^2 \hat{x}^2, \tag{A7}
\]

with \(e^{-2\epsilon} = \omega/\omega_0\).

A parallel expression gives the many-particle Hamiltonian

\[
\hat{h}(\omega) = \frac{1}{2M} \sum_n \hat{p}_n^2 + \frac{1}{2} M\omega^2 \sum_n \hat{x}_n^2 \tag{A8}
\]
as a transformation

$$\hat{h}(\omega) = e^{-2\epsilon} \hat{S}(\epsilon) \hat{h}(\omega_0) \hat{S}(-\epsilon)$$  \hspace{1cm} (A9)$$

of the Hamiltonian $\hat{h}(\omega_0)$, with

$$\hat{S}(\epsilon) = \exp \left[ \frac{1}{2} \epsilon \sum_n (c_n^+ c_n^+ - c_n c_n) \right].$$  \hspace{1cm} (A10)$$

It follows that, if $|0, \omega_0\rangle$ is the ground state of $\hat{h}(\omega_0)$, the ground state of $\hat{h}(\omega)$ is given by

$$|0, \omega\rangle = \hat{S}(\epsilon)|0, \omega_0\rangle.$$  \hspace{1cm} (A11)$$

The $\{C_\nu(\epsilon)\}$ coefficients in the expansion

$$\hat{S}(\epsilon)|\sigma, \omega_0\rangle = \sum_{\nu \geq 0} C_\nu(\epsilon)|\sigma + 2\nu, \omega_0\rangle$$  \hspace{1cm} (A12)$$

can now be derived.

In its fundamental matrix representations, an SU(1,1) group element is a complex matrix of the form

$$g = \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix}, \quad \text{with } |a|^2 - |b|^2 = 1,$$

and its Lie algebra is expressed in terms of the matrices

$$S_+ = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad S_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad S_0 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix},$$  \hspace{1cm} (A14)$$

which satisfy the commutation relations

$$[S_0, S_\pm] = \pm S_\pm, \quad [S_-, S_+] = 2S_0.$$  \hspace{1cm} (A15)$$

An SU(1,1) matrix has a so-called Gauss factorisation

$$\begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix} = \begin{pmatrix} 1/b^a^* & 1/a^* \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & a^* \end{pmatrix} \begin{pmatrix} b^a^* & 1 \\ 0 & 1 \end{pmatrix}.$$  \hspace{1cm} (A16)$$

The Lie algebra element $S(\epsilon) = \exp [\epsilon(S_+ - S_-)] \in SU(1,1)$ is then the matrix

$$S(\epsilon) = \exp \left[ -\epsilon \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = \begin{pmatrix} \cosh \epsilon & -\sinh \epsilon \\ -\sinh \epsilon & \cosh \epsilon \end{pmatrix},$$  \hspace{1cm} (A17)$$

and has the factored form

$$\begin{pmatrix} \cosh \epsilon & -\sinh \epsilon \\ -\sinh \epsilon & \cosh \epsilon \end{pmatrix} = \begin{pmatrix} 1 - \tanh \epsilon & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh \epsilon^{-1} & 0 \\ 0 & \cosh \epsilon \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\tanh \epsilon & 1 \end{pmatrix}.$$  \hspace{1cm} (A18)$$

Hence, with $\gamma(\epsilon)$ defined such that $e^{\gamma(\epsilon)/2} = (\cosh \epsilon)^{-1}$ and

$$\begin{pmatrix} \cosh \epsilon^{-1} & 0 \\ 0 & \cosh \epsilon \end{pmatrix} = \begin{pmatrix} e^{\gamma(\epsilon)/2} & 0 \\ 0 & e^{-\gamma(\epsilon)/2} \end{pmatrix} = e^{\gamma(\epsilon)S_0},$$  \hspace{1cm} (A19)$$

and the observation that

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \exp \left[ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \right],$$  \hspace{1cm} (A20)$$

it follows that $S(\epsilon)$ is represented as an operator

$$\hat{S}(\epsilon) = \exp \left[ (\tanh \epsilon) \hat{S}_+ \right] \exp \left[ \gamma(\epsilon) \hat{S}_0 \right] \exp \left[ -(\tanh \epsilon) \hat{S}_- \right].$$  \hspace{1cm} (A21)$$
It also follows from the expansion

$$\exp[(\tanh \epsilon) \hat{S}_+] = \sum_{\nu} \frac{(\tanh \epsilon)^\nu (\hat{S}_+)^\nu}{\nu!},$$  \hspace{1cm} (A22)$$

that, for an SU(1,1) lowest-weight state \( |\sigma\rangle \) for which \( \hat{S}_-|\sigma\rangle = 0 \) and \( \hat{S}_0|\sigma\rangle = \frac{1}{2}|\sigma\rangle \),

$$\hat{S}(\epsilon)|\sigma\rangle = e^{\sigma \gamma / 2} \exp[(\tanh \epsilon) \hat{S}_+]|\sigma\rangle = 1 (\cosh \epsilon)^\sigma \sum_{\nu} \frac{(\tanh \epsilon)^\nu}{\nu!} \left( \hat{S}_+ \right)^\nu |\sigma\rangle.$$  \hspace{1cm} (A23)$$

Finally, the \( C_\nu(\epsilon) \) coefficients in Eqn. (A12) are obtained from the known expression of a unitary SU(1,1) irrep \[234\], with lowest-weight state \( |\sigma\rangle \) and basis states \( \{|\sigma + 2\nu\rangle, \nu = 0, 1, 2, \ldots \} \), given by

\[
\begin{align*}
\hat{S}_0|\sigma + 2\nu\rangle &= \frac{1}{2}(\sigma + 2\nu)|\sigma + 2\nu\rangle, \\
\hat{S}_+|\sigma + 2\nu\rangle &= \sqrt{(\sigma + \nu)(\nu + 1)}|\sigma + 2\nu + 2\rangle, \\
\hat{S}_-|\sigma + 2\nu\rangle &= \sqrt{(\sigma + \nu - 1)|\sigma + 2\nu - 2\rangle}.
\end{align*}
\]

It follows that

$$\exp[(\tanh \epsilon) \hat{S}_+]|\sigma\rangle = \sum_{\nu} (\tanh \epsilon)^\nu \sqrt{(\nu + 1)! \nu!} |\sigma + 2\nu\rangle$$  \hspace{1cm} (A27)$$

and, from Eqn. (A23), that

$$\hat{S}(\epsilon)|\sigma\rangle = \sum_{\nu \geq 0} C_\nu(\sigma, \epsilon)|\sigma + 2\nu\rangle,$$  \hspace{1cm} (A28)$$

with

$$C_\nu(\sigma, \epsilon) = \frac{1}{(\cosh \epsilon)^\sigma (\tanh \epsilon)^\nu} \sqrt{(\nu + 1)! \nu!}.$$  \hspace{1cm} (A29)$$

**Appendix B: Vector-coherent-state (VCS) representations of SU(3) in an SO(3) basis**

In the notations of Section VII A, the U(3) Lie algebra is expressed as linear combinations of operators

$$\hat{C}_{ij} = \sum_n c_{ni}^\dagger c_{nj}, \quad i, j = 1, 2, 3,$$  \hspace{1cm} (B1)$$

that satisfy the commutation relations

$$[\hat{C}_{ij}, \hat{C}_{kl}] = \delta_{jk} \hat{C}_{il} - \delta_{il} \hat{C}_{jk}.$$  \hspace{1cm} (B2)$$

It is also expressed in terms of the angular-momentum and quadrupole operators defined in a spherical-tensor basis by

\[
\begin{align*}
\hat{L}_0 &= \hat{L}_{23}, & \hat{L}_{\pm 1} &= \mp \frac{1}{\sqrt{2}} (\hat{L}_{31} \pm i \hat{L}_{12}), \\
\hat{Q}_{2,0} &= 2 \hat{Q}_{11} - \hat{Q}_{22} - \hat{Q}_{33}, & \hat{Q}_{2,\pm 1} &= \mp \sqrt{6} (\hat{Q}_{12} \pm i \hat{Q}_{13}), \\
\hat{Q}_{2,\pm 2} &= \sqrt{3} (\hat{Q}_{22} - \hat{Q}_{33} \pm 2i \hat{Q}_{23}),
\end{align*}
\]

in which

$$\hat{L}_{ij} = -i(\hat{C}_{ij} - \hat{C}_{ji}), \quad \hat{Q}_{ij} = \frac{1}{2}(\hat{C}_{ij} + \hat{C}_{ji}).$$  \hspace{1cm} (B4)$$
In a VCS expression of an arbitrary SU(3) representation (λ, μ), in which λ and μ are positive integers, the SU(3) quadrupole operators are represented as operators

$$\hat{\Gamma}(Q_{2\nu}) = (2\lambda + \mu + 3)\hat{Q}_{2\nu} - \frac{1}{2} [\hat{L} \cdot \hat{L}, \hat{Q}_{2\nu}] + \sqrt{6}(\hat{\sigma}_+ \hat{Q}_{2\nu}^2 + \hat{\sigma}_- \hat{Q}_{2\nu}^2),$$  \hspace{1cm} (B5)

that act on linear combinations of rotor-model wave functions of the form

$$\Phi^{(K\lambda \mu)}_{KLM}(\Omega) = \sqrt{\frac{2L+1}{16\pi^2(1+\delta_{K0})}}(\xi_K \hat{\varphi}_{KLM}^L(\Omega) + (-1)^{\lambda+L+K} \xi_{-K} \hat{\varphi}_{KLM}^L(-\Omega)).$$  \hspace{1cm} (B6)

In this representation the operator $\hat{Q}_{2\nu}$ acts multiplicatively, i.e.,

$$\hat{Q}_{2\nu} \Phi(\Omega) = \Phi(\Omega) \Phi(\Omega),$$  \hspace{1cm} (B7)

and $\hat{\sigma}_0, \hat{\sigma}_\pm$ operate on intrinsic-spin states $\{\xi_K, K = -\mu, -\mu + 2, \ldots, \mu\}$ according to the SU(2) equations

$$\hat{\sigma}_0 \xi_K = \frac{1}{2} K \xi_K, \quad \hat{\sigma}_\pm \xi_K = \frac{1}{2} \sqrt{(\mu + K)(\mu + K + 2)} \xi_{K\pm 2},$$  \hspace{1cm} (B8a)

with $K$ taking integer values in the range $-\mu, -\mu + 2, \ldots, \mu$.

In spite of the close similarity between the above VCS representations of SU(3) and the standard irreps of the rigid-rotor model, there is a fundamental difference; for, whereas the rotor-model irreps are of infinite dimension, those of SU(3) are finite. This emerges explicitly in the VCS representation $\hat{\Gamma}(Q_{2\nu})$ of Eqn. (B5) for which the matrix elements of the quadrupole operators between states of increasing angular momentum vanish at some value of $L$ at which a band of states terminates because of the increasingly negative term $-\frac{1}{2} [\hat{L} \cdot \hat{L}, \hat{Q}_{2\nu}]$ in this equation. However, it is apparent that, for values of $2\lambda + \mu + 3 \gg L$ and $\mu \gg K$, the matrix elements of the quadrupole operators of SU(3) approach those of a corresponding rigid-rotor irrep with

$$\hat{\Gamma}(Q_{2\nu}) \rightarrow (2\lambda + \mu + 3)\hat{Q}_{2\nu} + \sqrt{\frac{2}{3}}(\hat{Q}_{2\nu}^2 + \hat{Q}_{2\nu}^2),$$  \hspace{1cm} (B9)

and, therefore, that $K$ should become a good quantum number in the $\lambda/L \rightarrow \infty$ and $\mu/K \rightarrow \infty$ limit.

Appendix C: The $K$ quantum number in the SU(3) model

The presence of the $\frac{1}{2} [\hat{L} \cdot \hat{L}, \hat{Q}_{2\nu}]$ term in the VCS expression of the SU(3) quadrupole operators has the result that the VCS quadrupole operators are not Hermitian relative to the rotor-model inner product for which the wave functions $\{\Phi^{(K\lambda \mu)}_{KLM}(\Omega)\}$ are an orthonormal basis. The adjustment of the inner product to that of a finite-dimensional SU(3) irrep is straightforward, but has the result that $K$ is no longer a precise integer-valued quantum number. Nevertheless, it transpires that an orthonormal SO(3) basis for an SU(3) irrep can be defined as eigenstates of a linear combination of SO(3)-invariant operators that have expectation values of $K$ that take integer values to an extraordinarily high degree of accuracy.

The $K$ quantum number for a rigid-rotor model is a component of angular momentum relative to an axis of an intrinsic body-fixed frame in which the Cartesian quadrupole moments of the nucleus are diagonal. This is a frame in which the $L = 2$ spherical tensor components of the Cartesian quadrupole moments

$$\hat{q}_0 = \sqrt{\frac{5}{16\pi}}(2\hat{Q}_{11} - \hat{Q}_{22} - \hat{Q}_{33}), \quad \hat{q}_\pm = \pm \sqrt{\frac{15}{32\pi}}(\hat{Q}_{12} \pm i\hat{Q}_{13}), \quad \hat{q}_\pm = \sqrt{\frac{15}{32\pi}}(\hat{Q}_{22} - \hat{Q}_{33} \pm 2i\hat{Q}_{23}),$$

can be expressed in the familiar form

$$\hat{q}_0 = \beta \cos \gamma, \quad \hat{q}_\pm = 0, \quad \hat{q}_\pm = \frac{1}{\sqrt{2}} \beta \sin \gamma.$$  \hspace{1cm} (C1)

The $K$ quantum number is then defined in this intrinsic frame as the $\hat{L}_0$ component of the rotor’s angular momentum relative to the $1$-axis. This definition is model dependent. However, it has been determined that, for the states of an arbitrary rigid rotor classified by $KLM$ quantum numbers, the ratios of reduced matrix elements

$$R(L, K) = \frac{\langle K + 2, L \mid (\hat{L} \otimes Q_2 \otimes Q_2^0) \mid K \rangle}{\langle K + 2, L \mid (\hat{L} \otimes Q_2 \otimes L_0) \mid K \rangle} = \frac{\langle K + 2, L \mid \hat{Q}_2 \otimes Q_2^0 \mid K \rangle}{\langle K + 2, L \mid Q_2 \otimes Q_2 \mid K \rangle}$$  \hspace{1cm} (C2)
take the precise value

\[ R(L, K) = \sqrt{\frac{8}{7}} q_0. \]  

This means that rotor-model states with good \( K \) quantum numbers are eigenstates of the SO(3) scalar operator

\[ \hat{Z}^{(\text{rot})} = [\hat{L} \otimes [\hat{Q}_2 \otimes \hat{Q}_2]_2 \otimes \hat{L}]_0 - \sqrt{\frac{8}{7}} q_0 [\hat{L} \otimes \hat{Q}_2 \otimes \hat{L}]_0. \]  

A parallel definition is now adopted for an SU(3) irrep.

For large values of \( 2\lambda + \mu \) and \( L \ll 2\lambda + \mu \), the term \( \frac{1}{2} [\hat{L} \cdot \hat{L}, \hat{\gamma}_\nu^2] \) in Eqn. (B5) is comparatively negligible and, for values of \( K \ll \mu \), the VCS expression \( \Gamma_\nu(Q_2) \) becomes identical to its rotor-model counterpart with intrinsic quadrupole moments

\[ q_0 = 2\lambda + \mu + 3, \quad q_{\pm 1} = 0, \quad q_{\pm 2} = \sqrt{\frac{3}{2}} (\mu + 1). \]  

This is a so-called contraction or macroscopic limit. Thus, a resolution of the multiplicity of SO(3) states of the same angular momentum in an SU(3) irrep that will give states of good \( K \) quantum numbers in this contraction limit is obtained by requiring the SU(3) basis states to be eigenstates of both \( \hat{L} \cdot \hat{L} \) and the linear combination

\[ \hat{Z}^{(\lambda \mu)} = [\hat{L} \otimes [\hat{Q} \otimes \hat{Q}]_2 \otimes \hat{L}]_0 - \sqrt{\frac{8}{7}} (2\lambda + \mu + 3)[\hat{L} \otimes \hat{Q}_2 \otimes \hat{L}]_0 \]  

of the SO(3)-invariant operators. The eigenstates \( \{|\alpha LM\} \) obtained are then expressed in a VCS representation as linear combinations of rotor-model states with good \( K \) quantum numbers

\[ |\alpha LM\rangle = \sum_K |KLM\rangle K_{Ka}. \]

Thus, it is simple to determine the mean values of \( K \) in this orthonormal basis.

Some mean values of \( K \), taken from Ref. [17], are shown in Table III. The remarkable result is how extraordinarily close they are to the corresponding integer rotor-model values, both for even and odd values of \( \mu \), with the exception of the few values of the angular momentum close to those at which the finite-dimensional SU(3) band terminates.
TABLE III: Comparison of the $K$ quantum number and diagonal quadrupole matrix element $\langle \lambda \mu |\hat{Q}_2| \lambda \mu \rangle$ for a state $|\lambda \mu \rangle$ in the rotor-model limit and its mean value in the corresponding SU(3) state is denoted by $\langle K |\hat{Q}_2| K \rangle$. The quadrupole matrix elements are denoted by $\langle \hat{Q}_2 |\hat{Q}_2| \rangle$ for the rotor limit and by $\langle \hat{Q}_2 |\hat{Q}_2| \rangle$ when computed precisely.

| $\lambda \mu$ | $\alpha$ | $L$ | $\langle \hat{Q}_2 |\hat{Q}_2| \rangle_{\alpha L}$ | $\langle \hat{Q}_2 |\hat{Q}_2| \rangle_{\alpha L}^{\text{rot}}$ | $\langle K |\hat{Q}_2| K \rangle_{\alpha L}$ |
|--------------|----------|-----|---------------------------------|---------------------------------|-------------------|
| (70 6)       | 1        | 0   | 0                               | 0                               | 0.000             |
|              | 2        | 2   | 178.089                         | 178.089                         | 2.000             |
|              | 3        | 4   | 318.937                         | 318.937                         | 4.000             |
|              | 4        | 6   | 425.927                         | 425.927                         | 6.000             |
|              | 1        | 30  | -582.098                        | -582.193                        | 0.000             |
|              | 2        | 30  | -574.587                        | -574.609                        | 2.000             |
|              | 3        | 30  | -552.055                        | -552.050                        | 4.000             |
|              | 4        | 30  | -514.500                        | -514.388                        | 6.000             |
| (70 7)       | 1        | 1   | 95.304                          | 95.304                          | 1.000             |
|              | 2        | 2   | 256.174                         | 256.174                         | 3.000             |
|              | 3        | 5   | 377.874                         | 377.874                         | 5.000             |
|              | 4        | 7   | 475.213                         | 475.213                         | 7.000             |
|              | 1        | 31  | -546.077                        | -545.784                        | 1.000             |
|              | 2        | 31  | -579.311                        | -545.784                        | 3.000             |
|              | 3        | 31  | -550.495                        | -550.462                        | 5.000             |
|              | 4        | 31  | -507.272                        | -507.108                        | 7.000             |
| (10 4)       | 1        | 2   | -32.271                         | -32.281                         | 0.000             |
|              | 2        | 2   | -32.271                         | -32.281                         | 2.000             |
|              | 3        | 4   | 57.794                          | 57.811                          | 4.000             |
|              | 1        | 10  | -62.077                         | -63.297                         | 0.003             |
|              | 2        | 10  | -55.305                         | -54.896                         | 2.001             |
|              | 3        | 10  | -34.989                         | -34.177                         | 4.399             |
|              | 1        | 12  | -67.663                         | -39.409                         | 1.012             |
|              | 2        | 12  | -62.458                         | -44.447                         | 3.872             |
|              | 1        | 14  | -72.830                         | -35.837                         | 2.010             |

[23] S. G. Nilsson and O. Prior, Mat. Fys. Medd. Dan. Vid. Selsk. 32, no. 16 (1961).
[24] K. K. Heyde, P. V. Isacker, M. Waroquier, J. L. Wood, and R. A. Meyer, Phys. Rep. 102, 291 (1983).
[25] J. L. Wood, K. Heyde, W. Nazarewicz, M. Huyse, and P. van Duppen, Phys. Reports 215, 101 (1992).
[26] K. Heyde and J. L. Wood, Rev. Mod. Phys. 83, 1467 (2011).
[27] H. U. Hirsch, S. Pittel, and G. Stoitcheva (World Scientific, Singapore, 2004), pp. 165–173, arXiv:1106.1607 [nucl-th].
[221] P. Park, J. Carvalho, M. Vassanji, D. J. Rowe, and G. Rosensteel, Nucl. Phys. A 414, 93 (1984).
[222] B. Castel, D. Rowe, and L. Zamick, Phys. Lett. B 236, 121 (1990).
[223] D. J. Rowe, J. of Phys.G: Nucl. Part. Phys. 43, 024011 (25) (2016).
[224] D. J. Rowe, G. Thiamova, and J. L. Wood, Phys. Rev. Lett. 97, 202501 (2006).
[225] J. Carvalho and D. J. Rowe, Nucl. Phys. A 548, 1 (1992).
[226] M. J. Carvalho, M. Vassanji, and D. J. Rowe, Nucl. Phys. A 465, 265 (1987).
[227] P. Rochford and D. J. Rowe, Phys. Lett. B 210, 5 (1988).
[228] C. Bahri, D. J. Rowe, and W. Wijesundera, Phys. Rev. C 58, 1539 (1998).
[229] P. O. Hess, A. Algora, M. Hunyadi, and J. Cseh, Eur. Phys. J. A 15, 449 (2002).
[230] P. E. Garrett, J. of Phys.G: Nucl. Part. Phys. 27, R1 (2001).
[231] P. E. Garrett and J. L. Wood, J. Phys. G: Nucl. Part. Phys. 37, 064028(17) (2010).
[232] J. Carvalho, R. Le Blanc, M. G. Vassanji, D. J. Rowe, and J. McGrory, Nucl. Phys. A 452, 240 (1986).
[233] J. M. Allmond and J. L. Wood, Phys. Lett. B 767, 226 (2017).
[234] J. Van der Jeugt and R. Jagannathan, J. Math. Phys. 39, 5062 (1998).
[235] D. J. Rowe, R. Le Blanc, and J. Repka, J. Phys. A: Math. Gen. 22, L309 (1989).
[236] D. J. Rowe, J. Phys A: Math. Theor. 45, 244003 (2012).
[237] D. J. Rowe and G. Thiamova, J. Phys A: Math. Theor. 41, 065206(1 (2008).