CLUSTER RANDOM FIELDS AND RANDOM-SHIFT REPRESENTATIONS

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Abstract: This paper investigates random-shift representations of $\alpha$-homogeneous shift-invariant classes of random fields (rf’s) $\mathcal{K}_{\alpha}[Z]$, which were introduced in [1]. Here $Z(t), t \in \mathcal{T}$ is a stochastically continuous $\mathbb{R}^d$-valued rf with $\mathcal{T} = \mathbb{R}^l$ or $\mathcal{T} = \mathbb{Z}^l$. We show that that random-shift representations of interest are obtained by constructing cluster rf’s, which play a crucial role in the study of extremes of stationary regularly varying rf’s. An important implication of those representations is their close relationship with Rosiński (or mixed moving maxima) representations of max-stable rf’s. We show that for a given $\mathcal{K}_{\alpha}[Z]$ different cluster rf’s can be constructed, which is useful for the derivation of new representations of extremal functional indices, Rosiński representations of max-stable rf’s as well as for random-shift representations of shift-invariant tail measures.

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Date: November 22, 2022.
Let \( \mathfrak{W} \) be the class of all stochastically continuous \( \mathbb{R}^d \)-valued rf’s \( V(t), t \in \mathcal{T} \) defined on some complete probability space, with \( \mathcal{T} = \mathbb{R}^l \) or \( \mathcal{T} = \mathbb{Z}^l \) and \( d, l \) two positive integers. In view of [2][Thm 5, p. 169, Thm 1, p. 171], we shall assume without loss of generality that all \( V \in \mathfrak{W} \) are separable and jointly measurable. Hereafter \( Z \in \mathfrak{W} \) is defined on the complete probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) and it satisfies

\[
\mathbb{P}\left\{ \sup_{t \in \mathcal{T}} \|Z(t)\| > 0 \right\} = 1, \quad \mathbb{E}\{\|Z(0)\|^\alpha\} \in (0, \infty), \quad \mathbb{E}\left\{ \sup_{t \in [-a,a] \cap \mathcal{T}} \|Z(t)\|^\alpha \right\} < \infty, \forall \alpha \in (0, \infty),
\]

with \( \alpha > 0 \) and \( \|\cdot\| : \mathbb{R}^d \mapsto [0, \infty) \) a 1-homogeneous continuous map. If \( Z \in \mathfrak{W} \) it might be defined on another complete probability space \( (\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) \).

Write \( \mathcal{H} \) for the class of all maps \( F : D \mapsto \mathbb{R} \) and all maps \( F : D \mapsto [0, \infty] \) which are \( \mathcal{B}([0, \infty]) \)-measurable, where \( D \) is the space of all maps \( f : \mathcal{T} \mapsto \mathbb{R}^d \) equipped with the product \( \sigma \)-field \( \mathcal{B} \). Here \( \mathcal{B}(V) \) stands for the Borel \( \sigma \)-field of a topological space \( V \).

**Definition 1.1.** As in [1], we call \( \mathbb{K}_\alpha[Z] \) an \( \alpha \)-homogeneous class of rf’s with representer \( Z \), if it contains all \( \tilde{Z} \in \mathfrak{W} \) that satisfy (1.1) and further

\[
\mathbb{E}\{\|Z(h)\|^\alpha F(\|Z(h)\|)\} = \mathbb{E}\{\|Z(h)\|^\alpha F(\|Z(\cdot)\|)\}, \quad \forall F \in \mathcal{H}, \forall h \in \mathcal{T}.
\]

\( \mathbb{K}_\alpha[Z] \) is called shift-invariant (and then denoted by \( \mathbb{K}_\alpha[Z] \)) if further

\[
B^h Z \in \mathbb{K}_\alpha[Z], \quad \forall h \in \mathcal{T},
\]

where \( B \) stands for the back-shift operator, i.e., \( B^h Z(\cdot) = Z(\cdot - h), \forall h \in \mathcal{T} \).

Hereafter we simply write \( \mathbb{P}, \mathbb{E} \) instead of \( \tilde{\mathbb{P}}, \tilde{\mathbb{E}} \), respectively. Since \( \|Z\| \) is jointly measurable, then

\[
S(Z) = \int_{\mathcal{T}} \|Z(t)\|^\alpha \lambda(dt)
\]

is a well-defined rv with \( \lambda \) the Lebesgue measure if \( \mathcal{T} = \mathbb{R}^l \), or the counting measure on \( \mathcal{T} \) when the latter is discrete.

**Definition 1.2.** If \( \mathbb{P}\{S(Z) = \infty\} = 1/0 \), then we call \( \mathbb{K}_\alpha[Z] \) purely conservative/dissipative.

The following simple result (we omit its proof) shows that if \( Z \) is stationary, then \( \mathbb{K}_\alpha[Z] \) is shift-invariant and purely conservative.

**Lemma 1.3.** If \( Z \in \mathfrak{W} \) is stationary and satisfies (1.1), then \( \mathbb{K}_\alpha[Z] \) is shift-invariant and \( \mathbb{P}\{S(Z) = \infty\} = 1 \).

Purely dissipative \( \mathbb{K}_\alpha[Z] \)'s do exist as demonstrated by the so-called Brown-Resnick \( \mathbb{K}_\alpha[Z] \) constructed next, see [3] for the definition of Brown-Resnick max-stable rf’s.

**Example 1.4.** (Brown-Resnick \( \alpha \)-homogeneous class) Considering \( \|\cdot\| \) a norm on \( \mathbb{R}^d \) define

\[
Z(t) = (e^{Y_1(t) - \text{aVar}(Y_1(t))/2}, \ldots, e^{Y_d(t) - \text{aVar}(Y_d(t))/2}), \quad 1 \leq i \leq d, t \in \mathcal{T},
\]

where \( Y(t), t \in \mathcal{T} \) is a centered \( \mathbb{R}^d \)-valued Gaussian rf with almost surely continuous sample paths such that \( \gamma_{ij}(s, t) = \text{Var}(Y_i(t) - Y_j(s)), s, t \in \mathcal{T} \) depends only on \( t - s \) for all \( s, t \in \mathcal{T} \) and all \( i, j \leq d \), see [1]. This
Lemma A.1 is well-defined and belongs to $\mathcal{W}$ below. Moreover, it agrees with Theorem 3.1 proves that it generates a purely dissipative shift-invariant class of rf's. The previous example also shows that not every $Z \in \mathcal{W}$ that satisfies (1.1) generates a shift-invariant $\mathcal{K}_\alpha[Z]$. One such instance is $Z$ in (1.3) for which $\gamma_{ij}(s,t)$ depends on both $s,t$ and not only on the difference $t-s$.

A direct approach to construct a purely dissipative $\mathcal{K}_\alpha[Z]$ is discussed in the next example. Hereafter, the $T$-valued rv $N$ with positive probability density function (pdf) $p_N(t) > 0$, $t \in T$ is assume to be independent of $Q$.

Example 1.5. Let $Q \in \mathcal{W}$ be independent of $N$ and set

$$ Z_N(t) = \frac{B^N Q(t)}{|p_N(N)|^{1/\alpha}}, \quad t \in T, $$

which in view of Lemma A.1 is well-defined and belongs to $\mathcal{W}$. Although $Z_N$ is in general non-stationary, Theorem 3.1 proves that it generates a purely dissipative shift-invariant class of rf's $\mathcal{K}_\alpha[Z_N]$ denoted simply below by $C_\alpha(Q)$, provided that $Q$ is a cluster rf, which is defined next.

Definition 1.6. $Q \in \mathcal{W}$ is called a cluster rf if

$$ \mathbb{P}\left\{ \sup_{t \in T} \| Q(t) \| > 0 \right\} = 1, \quad \int_T \mathbb{E} \left\{ \sup_{t \in [-a,a]} \| Q(t-v) \|^{\alpha} \right\} \lambda(\text{dv}) < \infty, \quad \forall a > 0. $$

One particular interesting instance of a cluster rf is $Q$ being the pdf of a non-singular Gaussian df on $\mathbb{R}^d$.

Cluster rf's with càdlàg sample paths and $T = \mathbb{R}$ are discussed in connection with random-shift representations of shift-invariant tail measures in [6]. The case of discrete $T$ is investigated in [7–13]. These rf's are very important for numerous applications which are discussed in [6–8, 12, 14]. We note in passing, that in [8][Def. 5.4.6] cluster rf’s are referred to as conditional spectral tail rf, while [10] refers to them as the anchored tail process due to their definition through an anchoring map.

Definition 1.7. We say that $\mathcal{K}_\alpha[Z]$ has a random-shift representation, if $Z_N$ defined in (1.5) belongs to $\mathcal{K}_\alpha[Z]$.

When $\mathcal{K}_\alpha[Z]$ has a random-shift representation, then it is necessarily purely dissipative and shift-invariant, see Theorem 3.1 below. Moreover, it agrees with $C_\alpha(Q)$ and necessarily $Q$ is a cluster rf.

In the study of regular variation of càdlàg random processes and rf's tail measures play a crucial role, see [6–9, 14, 15]. In view of the applications presented in [6, 8], it is of interest to construct random-shift representations of tail measures, see [6–8, 10]. Such representations for $T = \mathbb{R}$ are derived recently for càdlàg case in [6], whereas [8, 9, 16] discusses $T = \mathbb{Z}^l$. The recent manuscript [17] deals with a more abstract framework.

In this paper, we consider the general case of stochastically continuous rf's. Compared to the case of càdlàg rf's, various technical difficulties arise in our settings for which several functionals are not measurable any more, see [1]. Based on available results for càdlàg and discrete rf’s, natural questions that arise here include:

A) Given a shift-invariant $\mathcal{K}_\alpha[Z]$, under what conditions on $Z$ does $\mathcal{K}_\alpha[Z]$ have a random-shift representation and how to determine the rf $Q$?
B) What is the role of the rv $N$, does its particular choice matter?
C) Can we construct different cluster rf’s $Q$ for the random-shift representation of $\mathcal{K}_\alpha[Z]$ as in the càdlàg case?
D) What is the relation between random-shift representations for a given $\mathcal{K}_\alpha[Z]$ and those discussed in [6, 8, 9] for shift-invariant tail measures?
E) How do cluster rf’s arise in applications and what are some key implications of random-shift representations?

Brief outline of the rest of the paper: We shall present the main notation and definitions in Section 2 which includes also few preliminary results. Section 3 answers the first three questions raised above by discussing first the basic properties of cluster rf’s followed by explicit constructions of random-shift representations for $\mathcal{K}_\alpha[Z]$ based on results and ideas in [1, 6–9]. All the constructions in Section 3 are new if $\|Z(0)\|$ is almost surely positive. Section 4 discusses several applications related to the last two questions above. In Appendix A we present some technical results, whereas Appendix B briefly explores $\alpha$-homogeneous positive measures.

2. Preliminaries

We shall introduce first some classes of maps followed by a brief discussion on spectral tail and tail rf’s. We conclude with some properties of shift-invariant $\mathcal{K}_\alpha[Z]$’s related to universal maps introduced in [1].

2.1. Homogeneous, anchoring and involution maps. Write $\mathcal{H}_\beta, \beta \geq 0$ for the class of measurable maps $F \in \mathcal{H}$ (recall the definition of $\mathcal{H}$ in the Introduction) which are $\beta$-homogeneous, i.e., $F(cf) = c^\beta F(f)$ for all $f \in \mathcal{D}$ and $c > 0$. We say that $H \in \mathcal{H}$ is shift-invariant if $H(B^hf) = H(f)$ for all $f \in \mathcal{D}, h \in \mathcal{T}$.

Throughout this paper $g : \mathbb{R}^l \mapsto [0, \infty)$ is continuous satisfying

$$\sup_{t \in \mathcal{T}} g(t) > 0$$

and $\lambda$ denotes the Lebesgue measure on $\mathbb{R}^l$ or the counting measure on $\mathbb{Z}^l$ if $\mathcal{T} = \mathbb{R}^l$ or $\mathcal{T} = \mathbb{Z}^l$, respectively. As in [1], write $\mathcal{H}_v, v \geq 0$ for the class of maps $F$ determined for given $\Gamma_v \in \mathcal{H}_v$ by

$$(2.1) \quad F(f) = \frac{\Gamma_v(f) \Gamma_\beta(f)}{\mathcal{J}_\beta(f, g)}, \quad \mathcal{J}_\beta(f, g) = \int_{\mathcal{T}} \|f(t)\|^\beta g(t) \lambda(dt), \quad f \in \mathcal{D}, \quad \beta \geq 0,$$

with $\Gamma_\beta : \mathcal{D} \mapsto [0, c], \Gamma_\beta \in \mathcal{H}_\beta$ and $c$ some positive constant, or

$$(2.2) \quad F(f) = \Gamma_v(f) \mathcal{I}(\mathcal{J}_\beta(f, g) \in A), \quad f \in \mathcal{D}, \quad A \in \{\{0\}, \{0, \infty\}, \{\infty\}\}.$$  

For $f \in \mathcal{D}$ such that $\|f\|$ is not $\lambda$-measurable put $F(f) = 0$.

When $\mathcal{T} = \mathbb{R}^l$, the following integral map

$$F_\ell : f \mapsto \int_{\mathcal{T}} \|f(t)\| \lambda(dt), \quad f \in \mathcal{D}$$

is not $\mathcal{D}/\mathcal{B}([0, \infty])$-measurable (we set $F_\ell(f) = 0$ if $f \in \mathcal{D}$ is not Lebesgue measurable).\(^1\)

The anchoring maps are introduced in [10] in order to deal with tail rf’s. Let $\mathcal{L}$ be an additive subgroup of $\mathcal{T}$.

Hereafter 0 denotes also the origin of $\mathbb{R}^l$ or the zero-function on $\mathcal{D}$, depending on the context.

**Definition 2.1.** Let $J : \mathcal{D} \mapsto \mathbb{R}^d \cup \{\infty\}$ be $\mathcal{D}/\mathcal{B}(\mathbb{R}^d \cup \{\infty\})$-measurable.

\(^1F_\ell(Z)$ is well-defined rv as noted in the Introduction although $F_\ell$ is not measurable. This is the case for $F(Z)$ for all $F \in \mathcal{H}_v$.\)
A1) For all \( j \in \mathcal{L}, f \in \mathbb{D} \) we have \( \mathcal{J}(B^j f) - j = \mathcal{J}(f) \);
A2) \( \mathcal{J}(f) = j \in \mathcal{L} \) implies \( \|f(j)\| > \min(1, \|f(0)\|) \) for all \( f \in \mathbb{D} \);
A3) \( \mathcal{J}(f) = j \in \mathcal{L} \) implies \( \|f(j)\| > 0 \) for all \( f \in \mathbb{D} \).

Suppose that \( \mathcal{J} \) satisfies A1). If \( \mathcal{J} \) is 0-homogeneous it is called a shift-involution and if further A3) holds it is called a positive shift-involution. When A2) is valid \( \mathcal{J} \) is referred to as anchoring.

Hereafter \( \prec \) stands for some total order on \( \mathcal{T} \) which is shift-invariant, i.e., \( i \prec j \) implies \( i + k \prec j + k \) for all \( i, j, k \in \mathcal{T} \). We write \( i \preceq j \) if \( i \prec j \) or \( i = j \). Both inf and sup are taken with respect to \( \prec \) order and the infimum of an empty set is equal to \( \infty \). As in [10] define the first exceedance functional \( \mathcal{I}_{\mathcal{L},f} \) by
\[
\mathcal{I}_{\mathcal{L},f}(f) = \inf\{j \in \mathcal{L} : \|f(j)\| > 1\}, \quad f \in \mathbb{D},
\]
where \( \mathcal{I}_{\mathcal{L},f}(f) = \infty \) if there are infinitely many exceedance on \( \{j \in \mathcal{L}, j \prec k_0\} \) for some \( k_0 \in \mathcal{L} \) with all components positive.

For \( f : \mathcal{T} \mapsto \mathbb{R}^d \) satisfying \( \|f(\cdot)\| \) is \( \lambda \)-measurable define for \( \mathcal{L} \subset \mathcal{T} \) such that
\[
\mathcal{S}_\mathcal{L}(f) = \int_\mathcal{L} \|f(t)\|^\alpha \lambda(dt), \quad \mathcal{B}_{\mathcal{L},\tau}(f) = \int_\mathcal{L} \|f(t)\|^\tau \mathbb{I}(\|f(t)\| \geq 1)\lambda(dt), \quad \tau \in \mathbb{R},
\]
with \( \lambda(\mathcal{L}) > 0 \) if \( \mathcal{T} = \mathbb{R}^l \).

If \( \mathcal{L} = \mathcal{T} \) we write simply \( \mathcal{S} \) instead of \( \mathcal{S}_\mathcal{L} \). Define next the infrargsup map
\[
\mathcal{I}_\mathcal{L}(f) = \inf\{j \in \mathcal{L} : \sup_{i \in \mathcal{L}}\|f(i)\|\}, \quad f \in \mathbb{D},
\]
which is a positive shift-involution and anchoring.

A particular choice for \( \mathcal{L} \) will be a discrete subgroup of the additive group \( \mathcal{T} \), also referred to as a lattice. For such a lattice, we can find an \( l \times l \) real matrix \( A \) (called a base matrix) such that \( \mathcal{L} = \{Ax, x \in \mathbb{Z}^l\} \), where \( x \) denotes an \( l \times 1 \) vector. Two base matrices \( A, B \) generate the same lattice if and only if (iff) \( A = BU \), where \( U \) is an \( l \times l \) real matrix with determinant \( \pm 1 \). Denote the fundamental parallelogram of \( \mathcal{L} \) by
\[
P(\mathcal{L}) = \{Ax, x \in [0,1)^l\}.
\]
The volume of the fundamental parallelogram does not depend on the choice of the base matrix and is simply given by
\[
\Delta(\mathcal{L}) = |\det(A)|.
\]
We call \( \mathcal{L} \) a full rank lattice if \( A \) is non-singular.

2.2. Spectral tail and tail rf’s. In the sequel \( \mathcal{K}_\alpha[\mathcal{Z}] \) is a shift-invariant class of rf’s. In view of [1] condition (1.2) is equivalent with
\[
\mathbb{E}\{F(\mathcal{Z})\} = \mathbb{E}\{F(B^h \tilde{\mathcal{Z}})\}, \quad \forall F \in \mathcal{H}_\alpha, \forall h \in \mathcal{T}, \forall \tilde{\mathcal{Z}} \in \mathcal{K}_\alpha[\mathcal{Z}].
\]
Hereafter, we shall suppose for simplicity that \( \mathbb{E}\{\|\mathcal{Z}(0)\|^\alpha\} = 1 \), hence (2.3) yields
\[
\mathbb{E}\{\|\tilde{\mathcal{Z}}(t)\|^\alpha\} = 1, \quad \forall t \in \mathcal{T}, \forall \tilde{\mathcal{Z}} \in \mathcal{K}_\alpha[\mathcal{Z}].
\]
Write \( \Theta \) for the rf \( \mathcal{Z}/\|\mathcal{Z}(0)\| \) under the probability measure
\[
\hat{P}(A) = \frac{1}{\mathbb{E}\{\|\mathcal{Z}(0)\|^\alpha\}}\mathbb{E}\{\|\mathcal{Z}(0)\|^\alpha \mathbb{I}(A)\}, \quad \forall A \in \mathcal{F},
\]
where \( \mathbb{I}(A) \) is the indicator function.
with $\mathbb{I}(A)$ the indicator function of some set $A$ and let hereafter $R$ be an $\alpha$-Pareto (i.e., its survival function is $s^{-\alpha}, s \geq 1$) rv defined on $(\Omega, \mathcal{F}, \hat{P})$ being independent of $\Theta$.

Note that since $(\Omega, \mathcal{F}, P)$ is complete, then also $(\Omega, \mathcal{F}, \hat{P})$ is complete. For notational simplicity we shall write below $P$ instead of $\hat{P}$. Moreover, we have that $\Theta \in \mathcal{W}$.

**Definition 2.2.** We shall call $\Theta$ a spectral tail and $Y(t) = R\Theta(t), t \in \mathcal{T}$ the tail rf of $\mathcal{K}_\alpha[Z]$, respectively.

An important consequence of [1][Thm 3.4] is that (2.3) is equivalent with

$$E\{\|Z(h)\|^{\alpha} F(Z)\} = E\{\|Z(0)\|^{\alpha} F(B^hZ)\} = E\{\|Z(0)\|^{\alpha} E\{F(B^h\Theta)\}, \forall F \in \mathcal{F}_0, \forall h \in \mathcal{T}, \forall Z \in \mathcal{K}_\alpha[Z],$$

which implies for all $x > 0, h \in \mathcal{T}$

$$E\{\|\Theta(h)\|^{\alpha} \Gamma(\Theta)\} = E\{\mathbb{I}(\|\Theta(-h)\| \neq 0) \Gamma(B^h\Theta)\}, \forall \Gamma \in \mathcal{F}_0,$$

$$x^{-\alpha}E\{\Gamma(xB^hY) \mathbb{I}(x\|Y(-h)\| > 1)\} = E\{\Gamma(Y) \mathbb{I}(\|Y(h)\| > x)\}, \forall \Gamma \in \mathcal{F}_0,$$

where $\mathcal{F}_0$ is the class of maps $\Gamma$ defined by

$$\Gamma = F_1F_2, \quad F_1 \in \mathcal{F}_0, F_2 \in \mathcal{H}, \quad v \geq 0.$$

Condition (1.1) for $Z$ is crucial for the properties of $\mathcal{K}_\alpha[Z]$. In view of [1] or directly by Lemma A.4 condition (1.1) is equivalent with

$$E\left\{\frac{1}{\int_{[-a,a] \cap \mathcal{T}} \|\Theta(s-t)\|^{\tau} \mathbb{I}(\|Y(s-t)\| > 1) \lambda(ds)}\right\} < \infty$$

for all $a > 0$ and all $\tau \in \mathbb{R}$ such that

$$\sup_{s \in [-a,a]} E\{\|\Theta(s)\|^{\tau}\} < \infty, \quad \forall a > 0,$$

which is originally shown in [6] for $Z$ with càdlàg sample paths and $l = 1, \tau = 0$.

**Remark 2.3.**

(i) In both (2.7) and (2.8) we interpret $\infty \cdot 0$ and $0/0$ as $0$ and these rules apply hereafter;

(ii) For discrete $\mathcal{T}$ the equivalence of (2.6) and (2.8) is first discovered in [7], see [6, 9, 10, 14, 16, 17] for other extensions and [18] for the initial introduction of (2.7).

Spectral tail and tail rf’s play a crucial role in the asymptotic analysis of time series, see [8] for an excellent monograph treatment and the references therein.

Both argsup map $\mathcal{I}_L$ and the first exceedance map $\mathcal{I}_{L,f_e}$ satisfy certain relationships in connections with $\Theta$ and $Y$, namely (set $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{I}_L$)

$$P\{S_L(\Theta) < \infty\} = P\{S_L(\Theta) < \infty, J_1(\Theta) \in \mathcal{L}\}, \quad P\{B_{L,\tau}(Y) < \infty\} = P\{B_{L,\tau}(Y) < \infty, J_2(Y) \in \mathcal{L}\}$$

for all $\tau \in \mathbb{R}$ and further

$$P\{S_L(Z) < \infty\} = P\{S_L(Z) < \infty, J_3(Z) \in \mathcal{L}\},$$

where $\mathcal{F}_1 = \mathcal{I}_{L,f_e}$.

In the following $\|t\|_\ast = \sum_{i=1}^t |t_i|, t = (t_1, \ldots, t_i) \in \mathbb{R}^l$ and all set inclusions or set equalities are modulo null sets.
Theorem 2.4. Let $J_1$ be a positive shift-involution, $J_2$ an anchoring map that satisfy (2.11) and let $J_3$ be a shift-involution satisfying (2.12) if $L$ is a full rank lattice on $T$. For all $b$ positive, $\tau \in \mathbb{R}$ satisfying (2.10) and $L$ as above or $L = T$

\begin{equation}
(2.13) \quad \{S(Y) < \infty\} = \left\{ \lim_{||t|| \to \infty, t \in L} \|\Theta(t)\| = 0 \right\} = \left\{ \int_{T} \sup_{t \in [-a,a] \cap T} \|\Theta(t + s)\|^{\alpha} \lambda(ds) < \infty \right\}
\end{equation}

\begin{equation}
(2.14) \quad \{J_1(\Theta) \in L\} = \{J_2(Y) \in L\} = \{I_1(\Theta) \in T\} = \{I_2(Y) \in T\}
\end{equation}

\begin{equation}
(2.15) \quad \{S_L(Y) < \infty\} = \{B_{L,\tau}(bY) < \infty\} = \{B_{T,\tau}(bY) < \infty\}
\end{equation}

and

\begin{equation}
(2.16) \quad \{S(\tilde{Z}) < \infty\} = \left\{ \lim_{||t|| \to \infty, t \in L} \|\tilde{Z}(t)\| = 0 \right\} = \left\{ \int_{T} \sup_{t \in [-a,a] \cap T} \|\tilde{Z}(t + s)\|^{\alpha} \lambda(ds) < \infty \right\}
\end{equation}

\begin{equation}
(2.17) \quad \{J_3(\tilde{Z}) \in L\} = \{I_1(\tilde{Z}) \in T\}, \quad \forall \tilde{Z} \in K_\alpha[Z],
\end{equation}

with $I_1$ equal to the infargmax map $I_T$ and $I_2$ equal to the first exceedance map $I_{T,fe}$.

3. Main Results

In the first part of this section we shall discuss some basic properties of cluster rf’s and their relations with purely dissipative shift-invariant $\alpha$-homogeneous classes of rf’s. The second part is dedicated to explicit constructions of cluster rf’s $Q$ that yield a random-shift representations of $K_\alpha[Z]$’s. For simplicity we shall assume (2.4) in the following.

3.1. Shift-invariant classes and cluster rf’s. As shown next $C_\alpha(Q)$ defined in the Introduction (it is equal to $K_\alpha[Z_N]$ with $Z_N$ determined in (1.5)) is shift-invariant, purely dissipative and does not depend on the choice of the rv $N$. Moreover, the next theorem shows that a given cluster rf’s determines a purely dissipative $K_\alpha[Z]$.

Theorem 3.1. Let $K_\alpha[Z]$ be an $\alpha$-homogeneous rf’s and let $Q \in \mathcal{M}$ satisfy

$$P\left(\sup_{t \in T} ||Q(t)|| > 0 \right) = 1.$$ 

If (1.5) holds, then it is valid for all rv’s $N$ with pdf $p_N(t) > 0, t \in T$ being further independent of $Q$. Furthermore, $K_\alpha[Z]$ is shift-invariant and $Q$ is a cluster rf such that $K_\alpha[Z] = C_\alpha(Q)$.

Conversely, if $Q$ is a cluster rf, then $C_\alpha(Q)$ is shift-invariant and for all $\tilde{Z} \in C_\alpha(Q)$ we have $P\{S(\tilde{Z}) \in (0,\infty)\} = 1$.

An interesting relation holds for the expectations of $H(\Theta), H(Z)$ and $H(Q)$, when $H$ is shift-invariant (recall the definition in Section 2.1) and 0-homogeneous, which is stated next.

Proposition 3.2. Let $K_\alpha[Z]$ be an $\alpha$-homogeneous shift-invariant class of rf’s. If $K_\alpha[Z] = C_\alpha(Q)$, then for all shift-invariant $F \in \mathcal{S}_0$

(i) $E\{H(\Theta)\} = 0$;

(ii) $E\{H(\tilde{Z})\} = 0$ for some (and then for all) $\tilde{Z} \in K_\alpha[Z]$;

(iii) $E\{H(Q)\} = 0$
are all equivalent. In particular, we have
\begin{equation}
(3.1) \mathbb{P}\{S(\Theta) \in (0, \infty)\} = 1 \iff \mathbb{P}\{S(\bar{Z}) \in (0, \infty)\} = 1 \iff \mathbb{P}\{S(Q) \in (0, \infty)\} = 1, \quad \forall \bar{Z} \in \mathcal{K}_\alpha[Z].
\end{equation}
Conversely, if \(\mathbb{P}\{S(\Theta) \in (0, \infty)\} = 1\), then \(\mathcal{K}_\alpha[Z] = C_\alpha(Q)\) with \(Q = c^{1/\alpha}\Theta, c = 1/S(\Theta)\).

The shift-invariant class of Brown-Resnick rf's introduced in Example 1.4 satisfies \(\mathbb{P}\{\|Z(0)\| > 0\} = 1\). We present next a characterisation for such \(\mathcal{K}_\alpha[Z]'s.\)

**Proposition 3.3.** Given a shift-invariant \(\mathcal{K}_\alpha[Z]\), the following are equivalent:

(i) \(\|Z(0)\| > 0\) almost surely;

(ii) For some (and then for all) \(\bar{Z} \in \mathcal{K}_\alpha[Z]\) we have \(\|\bar{Z}(t)\| > 0\) almost surely for all \(t \in \mathcal{T}\);

(iii) \(\|\Theta(t)\| > 0\) almost surely for all \(t \in \mathcal{T}\);

(iv) If further \(\mathcal{K}_\alpha[Z] = C_\alpha(Q)\), then there exist rf's \(Q \in \mathcal{W}\) such that \(\|Q(t)\| > 0\) almost surely for all \(t \in \mathcal{T}\).

**Example 3.4.** Let \(Q\) be a cluster rf and let \(m > 0\) be given. Setting \(Q^{(m)}(t) = Q(t)1_{\|t\| \leq m}, t \in \mathcal{T}\) it follows easily that \(Q^{(m)}\) is also a cluster rf. Clearly, \(Q^{(m)}\) does not satisfy Proposition 3.3, Item (iv) even when \(Q\) satisfies it. Moreover, in general \(C_\alpha(Q)\) and \(C_\alpha(Q^{(m)})\) are different, however by construction they are both purely dissipative for all \(m > 0\).

**Example 3.5.** Let \(D^*\) be the set of functions \(f : \mathcal{T} \mapsto \mathbb{R}^d\) such that when \(\mathcal{T} = \mathbb{R}^l\) then it consists of generalised càdlàg functions. Equip \(D^*\) with Skorohod \(J_1\)-topology and denote by \(\mathcal{D}\) the corresponding Borel \(\sigma\)-field. Let further \(Q\) be deterministic cluster rf which belongs to \(D^*\). If \(\mathcal{K}_\alpha[Z] = C_\alpha(Q)\), then the spectral tail rf of \(\mathcal{K}_\alpha[Z]\) has law defined by
\begin{equation}
(3.2) \quad \mathbb{P}\{\Theta \in A\} = \mathbb{E}\{\|Z(0)\|^\alpha1_{\|Z(0)\| \in A}\}, \quad \forall A \in \mathcal{D}.
\end{equation}
In view of Proposition 3.2 it follows that
\[ Q_E = B^E Q / (S(Q))^{1/\alpha} \]
is a cluster rf which is non-deterministic, where the rv \(E\) has pdf \(\|Q(t)\|^\alpha / S(Q), t \in \mathcal{T}\) being further independent of \(Q\). See also \([6, 16]\) for constructions of \(\Theta\) from \(Q\).

**Example 3.6.** Let \(Q_i, i \in \mathbb{N}\) be independent spectral tail rf's and let \(N\) be an \(\mathbb{N}\)-valued rv independent of any other random element. Clearly, \(Q_N\) is a spectral tail rf. A special case is if \(Q_i's\) are pdf's with \(Q_N\) being a mixture pdf. If \(\mathcal{K}_\alpha[Z], i \in \mathbb{N}\) are shift-invariant \(\alpha\)-homogeneous rf's and \(Z_i's\) are independent, then \(\mathcal{K}_\alpha[Z_N]\) is also shift-invariant. Moreover, if \(\mathcal{K}_\alpha[Z_i] = C_\alpha(Q_i)\), then \(\mathcal{K}_\alpha[Z_N] = C_\alpha(Q_N)\).

### 3.2. Constructions of cluster rf's of purely dis dissipative \(\mathcal{K}_\alpha[Z]'s.\)

If \(S\) is a positive rv satisfying \(\mathbb{E}\{S^\alpha\} = 1\) being further independent of a cluster rf \(Q\), then \(Q^* = SQ\) is also a cluster rf and \(C_\alpha(Q^*) = C_\alpha(Q)\). Given a purely dissipative shift-invariant \(\mathcal{K}_\alpha[Z]\), it is of interest to construct cluster rf's \(Q\) such that \(\mathcal{K}_\alpha[Z] = C_\alpha(Q)\). This is not only of theoretical interest, but also important for statistical applications. Indeed, numerous contributions have considered this problem: see \([1, 3, 5, 7, 19]\) for constructions related to max-stable processes and \([6, 8, 9, 14]\) for more general settings and applications.
As we show next, besides random scaling and random shifting, cluster rf's can be directly determined by \( Z, \Theta \) or \( Y \) utilising \( S_\mathcal{L} \) and \( \mathcal{B}_{\mathcal{L},\tau} \). Set below

\[
(3.3) \quad \mathcal{M}_{\mathcal{L}}(Y) = \sup_{t \in \mathcal{L}} \|Y(t)\|
\]

and recall that \( \mathcal{P}(\mathcal{L}) \) denotes the fundamental parallelogram of the lattice \( \mathcal{L} \).

**Theorem 3.7.** Let \( \mathcal{K}_\alpha[Z] \) be shift-invariant and suppose that \( \mathbb{P}\{S(\Theta) \in (0, \infty)\} = 1 \). For all \( \tau \in \mathbb{R} \) as in (2.10) and all full rank lattices \( \mathcal{L} \) such that almost surely

\[
(3.4) \quad \{S_\mathcal{L}(\Theta) \in (0, \infty)\} = \{S_\mathcal{L}(B^t\Theta) > 0\}, \quad \forall t \in \mathcal{P}(\mathcal{L}) \cap \mathcal{T}
\]

or \( \mathcal{L} = \mathcal{T} \), a cluster rf \( Q \) such that \( \mathcal{K}_\alpha[Z] = \mathcal{K}_\alpha(Q) \) can be constructed for all \( b \in [1, \infty) \) as follows:

(i) \( Q = c^{1/\alpha} \Theta \), where \( 1/c = \Delta(\mathcal{L})S_\mathcal{L}(\Theta) \);

(ii) \( Q = c^{1/\alpha}Y(\mathcal{M}_\mathcal{L}(Y) > b) \), where \( c = b^\alpha \|Y(0)\|^\tau / (\Delta(\mathcal{L})\mathcal{M}_\mathcal{L}(Y)\mathcal{B}_{\mathcal{L},\tau}(Y)) \);

(iii) \( Q = c^{1/\alpha}Z \), where \( c = \|Z(0)\|^\alpha / \Delta(\mathcal{L}S_\mathcal{L}(Z)) \);

(iv) \( Q = bY(\tau)1_{\mathcal{M}_\mathcal{L}(Y(\tau)) > b_1}/\mathcal{M}_\mathcal{L}(Y(\tau)), t \in \mathcal{T} \), where \( Y(\tau) \) is the rf \( Y \) under the tilting with respect to \( \|Y(0)\|^\tau / (\Delta(\mathcal{L})\mathcal{B}_{\mathcal{L},\tau}(Y)) \).

It is possible to construct cluster rf's utilising shift-involutions acting on \( Z \), positive shift-involutions acting on the spectral tail rf \( \Theta \) and anchoring maps applied to the tail rf \( Y \). We shall discuss next only the case that \( \mathcal{L} \) is a full rank lattice on \( \mathcal{T} \) considering positive shift-involutions and anchoring maps denoted by \( \mathcal{J}_1 \) and \( \mathcal{J}_2 \), respectively. Further, we shall denote by \( \mathcal{J}_3 \) a shift-involution.

**Theorem 3.8.** Let \( \mathcal{L} \) be a full rank lattice on \( \mathcal{T} \) or \( \mathcal{L} = \mathcal{T} \) and suppose that \( \mathbb{P}\{S(\Theta) \in (0, \infty)\} = 1 \). If \( \mathcal{J}_1, \mathcal{J}_2 \) satisfy (2.11), \( \mathcal{J}_3 \) satisfies (2.12) and further (3.4) holds, then \( \mathcal{K}_\alpha[Z] \) has a random-shift representation with \( Q = c^{1/\alpha}Q_c \), \( c > 0 \) determined as follows:

(i) \( \overline{Q}(t) = \Theta(0)(\mathcal{J}_1(\Theta) = 0) \) and \( c = \mathbb{P}\{\mathcal{J}_1(\Theta) = 0\}/\Delta(\mathcal{L}) > 0 \);

(ii) \( \overline{Q}(t) = Y(t)_{\mathcal{M}_\mathcal{L}(Y)}(\mathcal{J}_2(Y) = 0, \mathcal{M}_\mathcal{L}(Y) > b) \) and \( c = b^\alpha \mathbb{P}\{\mathcal{J}_2(Y) = 0, \mathcal{M}_\mathcal{L}(Y) > b\}/\Delta(\mathcal{L}), b \in [1, \infty) \);

(iii) \( \overline{Q}(t) = Z(t)(\mathcal{J}_3(Z) = 0) \) and \( c = \mathbb{P}\{\mathcal{J}_3(Z) = 0\}/\Delta(\mathcal{L}) > 0 \).

**Remark 3.9.**

i) Condition (3.4) is fulfilled if \( \mathbb{P}\{\|Z(0)\| > 0\} = 1 \), since in view of Proposition 3.3 this implies that \( \mathbb{P}\{\|\Theta(t)\| > 0\} = 1 \) for all \( t \in \mathcal{T} \) and hence \( \mathbb{P}\{S_{\mathcal{L}}(B^t\Theta) > 0\} = 1 \) follows from [20][Thm 2.1, Rem 2.2,iii]);

ii) If \( \mathcal{K}_\alpha[Z] = \mathcal{K}_\alpha(Q) \), then under the assumptions of Theorem 3.7 and Theorem 3.8, for all shift-invariant \( H \in S_\alpha \) applying (2.3) we obtain

\[
(3.5) \quad E\left\{ \frac{H(\Theta)}{\Delta(\mathcal{L})S_{\mathcal{L}}(\Theta)} \right\} = E\left\{ \frac{H(Y)}{\Delta(\mathcal{L})\sup_{t \in \mathcal{L}}\|\mathcal{M}_\mathcal{L}(Y(t))\|^\alpha} | J_2(Y) = 0 \right\} = E\left\{ \|Z(0)\|^\alpha \frac{H(Z)}{S(Z)} \right\} = E\{H(Q)\}.
\]

Recall that \( H \) is shift-invariant means \( H(B^h f) = H(f) \) for all \( f \in \mathcal{D}, h \in \mathcal{T} \).

iii) For the setup of this paper, all the constructions above are new. Related results for the case of càdlàg rf's or \( \mathcal{T} = \mathbb{R}^1 \) can be found in [6–9].
4. Discussions and Applications

Let hereafter $\mathcal{K}_\alpha[Z]$ be shift-invariant and as above suppose that (2.4) holds. Given the fact that tail measures play a crucial role in the analysis of regularly varying time series [6, 8, 10, 21, 22] we shall discuss next the relation between a given $\mathcal{K}_\alpha[Z]$ and shift-invariant tail measures followed by a discussion on tail measures on the space of generalised càdlàg functions. We continue then with the properties of the candidate extremal index showing further that it agrees with the extremal index of an associated max-stable stationary rf’s. A short investigation on the approximations of purely dissipative $\alpha$-homogeneous classes of rf’s is then followed by the properties of spectral cluster rf’s.

4.1. Random-shift representations of tail measures. Consider $D = D(\mathcal{T},\mathbb{R}^d)$ the set of functions $f : \mathcal{T} \mapsto \mathbb{R}^d$ equipped with the product $\sigma$-field $\mathcal{D}$ and let $\mathcal{D}$ be a countable dense subset of $\mathcal{T}$ which is an additive subgroup of $\mathcal{T}$. Let $Z$ be as in the Introduction satisfying (1.1). In view of [2]

\begin{equation}
(4.1) \quad \tilde{\mathcal{D}} = \mathbb{A}_D,
\end{equation}

where $\tilde{\mathcal{D}}$ is the completion of $\mathcal{D}$ with respect to the measure $\mu$ induced by $Z$ on $(D, \mathcal{D})$ and $\mathbb{A}_D$ is the completion of $\mathbb{A}_D = \sigma(\pi(t), t \in D)$ with respect to the measure $\mu_D$ induced by $Z_D$. Here $\pi(t), t \in \mathbb{R}$ is the projection map. It follows that $T(s,f) = sf, s > 0, f \in D$ is $\mathcal{B}((0,\infty)) \times \tilde{\mathcal{D}}/\mathcal{D}$ measurable. Hence we can define the measure $\nu_Z$ on $\tilde{\mathcal{D}}$ by

\begin{equation}
(4.2) \quad \nu_Z[H] = \int_0^\infty \mathbb{E}\{H(zZ)\}\lambda_\alpha(dz), \quad \lambda_\alpha(dz) = az^{\alpha-1}dz
\end{equation}

for all $H : D \mapsto [0,\infty)$ being $\tilde{\mathcal{D}}/\mathcal{B}(\mathbb{R})$ measurable.

Clearly, $\nu_Z(tA) = t^{-\alpha}\nu_Z(A)$ for all $A \in \tilde{\mathcal{D}}$ and hence $\nu_Z$ is $-\alpha$-homogeneous.

An important consequence of (1.1) is that $\nu = \nu_Z$ satisfies

\begin{equation}
(4.3) \quad \nu(A_a) < \infty, \quad \forall a > 0,
\end{equation}

where $A_a = \{ f \in D : \sup_{t \in [-a,a]} \| f(t) \| > 1 \}$. The measure $\nu_Z$ is a tail measure with index $\alpha$ as defined for instance in [14], see also [1, 17]. In view of [1] $\nu_Z$ is shift-invariant, i.e., $\nu_{\Theta^hZ}$ for all $h \in \mathcal{T} \text{ iff } \mathbb{K}_\alpha[Z]$ is shift-invariant.

The choice of $Z$ is not important for the definition of the measure above, i.e., for all $H$ as above

\begin{equation}
(4.4) \quad \nu[Z][H] = \nu[Z][H], \quad \forall Z \in \mathbb{K}_\alpha[Z].
\end{equation}

If $\nu_Z$ is shift-invariant and $\mathbb{P}\{S(\Theta) < \infty\} = 1$, then by Theorem 3.7 we have $\mathbb{K}_\alpha[Z] = C_\alpha(Q)$. Hence utilising (4.4) for all $H : D^* \mapsto [0,\infty]$ being $\tilde{\mathcal{D}}/\mathcal{B}(\mathbb{R})$ measurable

\begin{equation}
(4.5) \quad \nu_Z[H] = \int_{\mathcal{T}} \int_0^\infty \mathbb{E}\{H(zB^tQ)\}\lambda_\alpha(dz)\lambda(dt).
\end{equation}

An immediate consequence of Proposition 3.2 is the following result, presented for special cases in [7][Cor 3.2], [6][Lem 2.8] and [8][Lem 5.4.3, 5.4.7].

**Corollary 4.1.** If $\mathbb{K}_\alpha[Z] = C_\alpha(Q)$, then for all $H : D \mapsto [0,\infty]$ shift-invariant, $0$-homogeneous and $\tilde{\mathcal{D}}/\mathcal{B}(\mathbb{R})$-measurable

\begin{equation}
(4.6) \quad \mathbb{E}\{H(\Theta)\} = 0 \iff \mathbb{E}\{H(Y)\} = 0 \iff \mathbb{E}\{H(Z)\} = 0 \iff \mathbb{E}\{H(Q)\} = 0 \iff \nu[Z][H] = 0.
\end{equation}
If the tail measure \( \nu \) on \( \tilde{\mathcal{G}} \) satisfies one of the conditions in Proposition 4.2, then for a cluster rf \( Q \) as therein we define a new measure \( \nu^* = \nu_Q \) on \( \tilde{\mathcal{G}} \), referred to as the cluster measure of \( \nu \), see [6, 8]. We can rewrite now (4.5) as

\[
\nu[H] = \int_T \int_0^\infty \mathbb{E}\{H(zB^tQ)\} \lambda(\alpha, \pi) dt = \int_T \nu^*[B^tH] \lambda(dt)
\]

for all \( \mathcal{G}(R) \)-measurable maps \( H : D \to [0, \infty] \), which shows that for \( H \) shift-invariant with respect to \( T \)

\[ \nu[H] \in \{0, \infty\}, \]

with \( \nu[H] = 0 \) iff \( \nu^*[H] = 0 \).

As in Example 3.5 let \( D^* \subset D(T, \mathbb{R}^d) \) consists of all maps \( f : T \to \mathbb{R}^d \) such that \( f \) is a generalised càdlàg if \( T = \mathbb{R}^l \). We can equip \( D^* \) with the Skorohod \( J_1 \)-topology and the corresponding Borel \( \sigma \)-field \( \mathcal{G} \), which coincides with \( \tilde{\mathcal{G}} \). If \( Z \) has a.s. càdlàg sample paths, since by assumption \( \mathbb{P}\{\sup_{t \in T} \|Z(t)\| > 0\} = 1 \), then Remark B.2, Item (i) (or Example B.3) implies that \( \nu_Z \) is \( \sigma \)-finite.

The next result extends [6][Thm 2.9] and [8][Thm 5.4.8, Thm 5.4.9], see also [9][Prop. 2.18].

**Proposition 4.2.** Let \( K_\alpha[Z] \) be given with \( Z \) with càdlàg sample paths and let \( \nu = \nu_Z \) be defined as above. The following are equivalent:

(i) \( \nu \) has a shift representation with cluster rf’s \( Q \in \mathcal{M} \);

(ii) one of the events in Theorem 2.4 hold with probability 1;

(iii) \( \nu \) is supported on \( \{f \in D : \mathcal{B}_{\tau, \pi}(bf) < \infty\} \) for some (and then all) \( b > 0 \) and \( \tau \) as in (2.10) (here \( 0\mathbb{Z}^l = \mathbb{R}^l \));

(iv) \( \nu \) is supported on \( \{f \in D : \lim_{\|t\|_\pi \to \infty} \|f(t)\| = 0\} \);

(v) \( \mathbb{P}\{\lim_{\|t\|_\pi \to \infty} \|Z(t)\| = 0\} = 1 \).

4.2. The candidate extremal index. Adopting the terminology of [7, 8] we call \( \nu_L \) defined by

\[
\nu_L = \mathbb{E}\left\{ \frac{1}{\Delta(L)B_{L,0}(Y)} \right\}
\]

the candidate extremal index of \( K_\alpha[Z_L] \), where \( L \) is a lattice on \( T \) or \( L = T \). The finiteness of \( \nu_T \) follows from (2.9). We discuss in this section the case of full rank lattices \( L \). In view of Theorem 2.4 a crucial property of the candidate extremal index is

\[
\partial_L = 0 \iff \partial_T = 0 \iff \mathbb{P}\{S(\Theta) = \infty\} = 1.
\]

The latter condition is equivalent with one of the events in Theorem 2.4 holds with probability zero.

It follows from the proof of Theorem 3.7 that for all \( \tau \) as in (2.10) and all \( b \in [1, \infty) \)

\[
(4.8) \quad \nu_L = b^\alpha \mathbb{E}\left\{ \frac{\|Y(0)\|^{\tau} \Delta(L)B_{L,\tau}(bY)}{\Delta(L)B_{L,\tau}(Y)} > 0 \right\} < \infty.
\]

The case \( b = 1, \tau = 0 \) in (4.8) was derived for the extremal index of a regularly varying time series in [23] under some asymptotic restrictions inspired by [24, 25] where it appears (not explicitly) in relation to Pickands constants, see [26] which uses that representation to derive a lower bound for Pickands constant. We refer to the representation (4.8) of \( \nu_L \) as the Berman representation.
For \( T = \mathbb{Z}, b = 1, \tau = 0 \) it appeared in [16] and for \( T = \mathbb{Z}^l, b = 1, \tau = 0 \) in [27], see also [1, 6] for a less restrictive framework.

From the proof of Theorem 3.7 it follows that

\[
\nu_\mathcal{L} = \mathbb{E}\left\{ \frac{\sup_{t \in \mathcal{L}} \| \Theta(t) \|^\alpha}{\Delta(\mathcal{L}) S_{\mathcal{L}}(\Theta)} \right\}.
\]

The representation (4.9) appears in the settings of [23] and as the extremal index of max-stable stationary rf’s in [19, 28] being essentially motivation from [29], which considered the classical Brown-Resnick stationary max-stable processes. As noted in [30] this representation is already implied from the seminal papers [31, 32] and can be thus referred to as the Samordnitsky representation.

Other representations for \( \nu_\mathcal{L} \) are obtained utilising Lemma A.2. For instance when \( \mathcal{J}_2 \) satisfies (2.11) and \( \mathcal{L} \) is a lattice on \( T \), using further (A.6)

\[
(4.10) \quad \nu_\mathcal{L} = \frac{b^\alpha}{\Delta(\mathcal{L})} \mathbb{P}\{ \mathcal{J}_2(Y) = 0, \mathcal{M}_\mathcal{L}(Y) > b \} = \frac{b^\alpha}{\Delta(\mathcal{L})} \mathbb{P}\left\{ \sup_{0 < t, t \in \mathcal{L}} \| Y(t) \| \leq 1, \mathcal{M}_\mathcal{L}(Y) > b \right\}, \quad b \in [1, \infty).
\]

The second expression above, which follows from the first taking \( \mathcal{J}_2 \) to be the first exceedance map (it satisfies (2.11)), goes back to works of Albin (\( b = 1 \) case) and appears as limiting constant (Pickands constant) of supremum of various processes, see e.g., [33–35].

Note that (4.10) simply implies that when \( \nu_\mathcal{L} > 0 \)

\[
(4.11) \quad \mathcal{M}_\mathcal{L}(Y) |_{\mathcal{J}_2(Y) = 0}
\]

has an \( \alpha \)-Pareto law and this is in particular also the case for \( \mathcal{M}_\mathcal{L}(Y) |_{\sup_{0 < t, t \in \mathcal{L}} \| Y(t) \| \leq 1} \). See [10][Lem 3.7] for the case \( \mathcal{L} = \mathbb{Z}^p = T \) and \( \mathbb{P}\{S(\Theta) < \infty\} = 1 \).

Applying [27][Lem 1] we obtain from the second expression in (4.10)

\[
(4.12) \quad \Delta(\mathcal{L}) \nu_\mathcal{L} = \mathbb{E}\left\{ \sup_{0 \leq t, t \in \mathcal{L}} \| \Theta(t) \|^\alpha - \sup_{0 < t, t \in \mathcal{L}} \| \Theta(t) \|^\alpha \right\} = \mathbb{E}\left\{ \sup_{0 \leq t, t \in \mathcal{L}} \| Z(t) \|^\alpha - \sup_{0 < t, t \in \mathcal{L}} \| Z(t) \|^\alpha \right\},
\]

which is derived for the Brown-Resnick model in [22][Corr 6.3] and initially obtained in [18], see also [8, 10, 13, 28]. If \( \mathcal{K}_\alpha[Z] = \mathcal{C}_\alpha(Q) \), we have in view of Theorem 3.7 and (3.5) the general expression

\[
\nu_\mathcal{L} = \mathbb{E}\left\{ \sup_{t \in \mathcal{L}} \| Q(t) \|^\alpha \right\}
\]

obtained for \( l = 1 \) in [30]. Consequently, different choices for \( Q \) give different representations for \( \nu_\mathcal{L} \). In the special case \( \| Z(0) \| > 0 \) almost surely, in view of Theorem 3.7, for all full rank lattice \( \mathcal{L} \) and \( \tau \) as in Section 2.2

\[
(4.13) \quad \nu_{\mathbb{Z}^l} = \frac{b^\alpha}{\Delta(\mathcal{L})} \mathbb{E}\left\{ \sup_{t \in \mathcal{T}} \| Y(t) \|^\alpha \| Y(0) \|^{\tau, \mathcal{M}_\mathcal{L}(Y) > b} \right\}.
\]

We note in passing that new expressions for \( \nu_\mathcal{L} \) are shown also in the recent contribution [16] for \( T = \mathcal{L} = \mathbb{Z}^l \).

As discussed in [6, 8, 10, 13, 18, 36–39] calculation of the candidate extremal index is closely related to the calculation of extremal index, with few notable exception pointed out in [40].

Note further that for particular cases, representations of extremal indices are obtained in [41–43]. Below we show that \( \nu_\mathcal{L} \) is exactly the extremal index of a corresponding max-stable rf and its calculation can be dealt with within the framework of max-stable rf’s.
4.3. Max-stable rf’s. Set next

\[ Z(t) = \|Z(t)\|, \quad \Theta(t) = \|\Theta(t)\|, \quad \|Y(t)\| = \|Y(t)\|, \quad t \in \mathcal{T}. \]

Since \( \mathcal{K}_\alpha[Z] \) is shift-invariant, we have that \( \mathbb{K}_\alpha[Z] \) is also shift-invariant and its spectral tail rf is \( \Theta \) with the corresponding tail rf \( Y \).

Let \( Z^{(i)}_t \in \mathbb{N} \) be independent copies of \( Z \) and define the max-stable stationary rf \( X(t), t \in \mathcal{T} \) via its de Haan representation (e.g., [5, 44])

\[ X(t) = \max_{i \geq 1} \Gamma_i^{-1/\alpha} Z^{(i)}(t), \quad t \in \mathcal{T}, \]

which has unit Fréchet marginals. Here \( \Gamma_i = \sum_{k=1}^i \nu_k \) with \( \nu_k, k \geq 1 \) unit exponential rv’s being independent of any other random element. The rf \( Z \) is a representer for \( X \). It is of interest to derive a Rosiński representation for \( X \), namely

\[ X(t) = \max_{1 \leq i \leq \infty} P_i Q^{(i)}(t - \tau_i), \]

where \( \sum_{i=1}^\infty \delta_{\tau_i, Q^{(i)}}(\cdot) \) is a PPP on \((0, \infty) \times \mathbb{R}^l \times \mathbb{D}^s\), with mean measure \( \lambda(\cdot) \circ \mu_\alpha(\cdot) \circ \mathbb{P}(\cdot) \) and \( Q^{(i)} \)'s are independent copies of some cluster rf \( Q \), where \( \mu_\alpha(dt) = \cot^{\alpha-1} \mu(dt), c > 0 \) and \( \mu \) the Lebesgue measure on \((0, \infty)\). In our settings this simply means that \( p_N(N)^{-1/\alpha} B^NQ \) is also a representer for \( X \) for all \( N \) a \( \mathcal{T} \)-valued rv independent of \( Q \) with \( \text{pdf} p_N(t) > 0, t \in \mathcal{T} \). A necessary and sufficient condition for such a representation is \( \mathbb{P}\{ S(Z) < \infty \} = 1 \), which is equivalent with one of the events in Theorem 2.4 holds with probability 1. In view of our assumptions we have

\[ \mathbb{P}\left\{ \sup_{t \in \mathcal{T}} Z(t) > 0, S(Z) = 0 \right\} = 0, \]

hence [19][Eq. 6.5] holds. If \( \mathbb{P}\{ S(Z) < \infty \} = 1 \) we can take

\[ Q(t) = \|Q(t)\|, \quad t \in \mathcal{T}, \]

with \( Q \) a cluster rf such that \( \mathcal{K}_\alpha[Z] = \mathcal{C}_\alpha(Q) \). The construction of different \( Q \)'s has been the topic of numerous papers, see e.g., [5–9, 19, 30, 44] and the references therein. Our results imply new constructions when \( \tau \neq 0, b \in [1, \infty) \) or \( \|Z(0)\| > 0 \) almost surely.

**Proposition 4.3.** If one of the events in Theorem 2.4 is 1 with probability 1 and \( \mathcal{L} \) is a full rank lattice on \( \mathcal{T} \), then a stochastically continuous cluster rf \( Q \) that defines a Rosiński representation (4.15) for \( X \) exists and \( c = \nu_T \).

Since for all \( t_i \in \mathcal{T}, x_i \in (0, \infty), i \leq n, n \in \mathbb{N} \) we have (see e.g., [45, 46])

\[ -\ln \mathbb{P}\{ X(t_1) \leq x_1, \ldots, X(t_n) \leq x_n \} = \mathbb{E}\left\{ \max_{1 \leq i \leq n} \frac{1}{x_i} Z^\alpha(t_i) \right\} \]

for all \( z > 0 \) and all full rank lattice \( \mathcal{L} \) or \( \mathcal{L} = \mathcal{T} \) (recall that \( X \) is also taken to be separable)

\[ -\ln \mathbb{P}\left\{ \sup_{t \in [0,n] \cap \mathcal{L}} X(t) \leq zn^{1/\alpha} \right\} = \frac{1}{z^{\alpha} n^{\alpha}} \mathbb{E}\left\{ \sup_{t \in [0,n] \cap \mathcal{L}} \|Z(t)\|^\alpha \right\} = \frac{1}{z^{\alpha}} \mathbb{E}_Z^n(\mathcal{L}). \]

In view of [1][Lem 6.2]

\[ \lim_{n \to \infty} \mathbb{E}_Z^n(\mathcal{L}) = \mathbb{E}_Z^\mathcal{L} = \nu_\mathcal{L} < \infty. \]
Since $\mathbb{B}_Z^\mathcal{C} = 0$ if and only if $\mathbb{P}\{S(Z) = \infty\} = 1$, then
\begin{equation}
\mathbb{B}_Z^\mathcal{C} = \mathbb{P}\{S(\Theta) < \infty\} \lim_{n \to \infty} \mathbb{B}_Z^\mathcal{C}(n),
\end{equation}
with $Z_*$ belonging to the $\alpha$-homogeneous shift-invariant class of rf's with spectral tail rf $\Theta|S(\Theta) < \infty$. Consequently, we can assume without loss of generality that $S(Z) < \infty$ almost surely and hence there exists a spectral rf $Q$ such that $\mathcal{K}_\alpha[Z] = \mathcal{C}_\alpha(Q)$ implying
\begin{equation}
\mathbb{B}_Z^\mathcal{C} = \nu_\mathcal{L} = \mathbb{E}\left\{ \max_{t \in \mathcal{L}} |Q(t)|^\alpha \right\}.
\end{equation}
The case $Z$ has càdlàg sample paths has been considered in [30], see also [6, 8, 14].

**Example 4.4.** Let $\alpha = d = 1$ and $Z(t) = e^{\overline{X}(t)}, \overline{X}(t) = X(t) - \text{Var}(X(t))/2, t \in \mathbb{R}^l$ be as in (1.3) satisfying further (1.4). Let $\mathcal{L} = \delta \mathbb{Z}^l$ with $\delta > 0$ such that $\delta \in \mathbb{N}$ if $\mathcal{T} = \mathbb{Z}^l$ and let $||x|| = |x|, x \in \mathbb{R}$. Since $Z(0) = 1$ almost surely, then the spectral tail rf $\Theta$ can be taken equal to $Z$ and thus $Y(t) = e^{\mathbb{E}Z(t)}$ with $E$ a unit exponential rv independent of $Z$. For this case we can take $\tau \in [0, \infty)$ and hence for all $\theta \geq 0$ and $\Delta(\mathcal{L}) = \delta l$, in view of Theorem 3.7, Item (ii) and Theorem 3.8
\begin{equation}
\nu_\mathcal{T} = \frac{e^\theta}{\delta^l} \mathbb{E}\left[ \sup_{t \in \mathcal{T}} e^{\overline{X}(t)} \mathbb{1}\left( \sup_{t \in \mathcal{L}} \overline{X}(t) + E > \theta \right) \right]
\end{equation}
\begin{equation}
= \frac{e^\theta}{\delta^l} \mathbb{E}\left[ \sup_{t \in \mathcal{T}} e^{\overline{X}(t)} \mathbb{1}\left( \sup_{0 \leq t, t \in \mathcal{L}} \overline{X}(t) + E \leq 0, \sup_{t \in \mathcal{T}} \overline{X}(t) + E > \theta \right) \right]
\end{equation}
\begin{equation}
= \frac{e^\theta}{\delta^l} \mathbb{E}\left[ \mathbb{1}\left( \sup_{t \in \mathcal{T}} \overline{X}(t) + E > \theta \right) \int_{\mathcal{T}} e^{\tau\overline{X}(t)} \mathbb{1}\left( \overline{X}(t) + E > 0 \right) \lambda(dt) \right]
\end{equation}
\begin{equation}
= \frac{1}{\delta^l} \mathbb{E}\left[ \sup_{t \in \mathcal{T}} e^{\overline{X}(t)} \right] \mathbb{1}\left( \sum_{t \in \mathcal{L}} e^{\overline{X}(t)} \right).
\end{equation}
We note that (4.22) is shown for $l = 1$ in [20] and (4.23) in [47].

### 4.4. $m$-approximation

Let below $\mathcal{K}_\alpha[Z]$ be purely dissipative, i.e., $\mathbb{P}\{S(\Theta) < \infty\} = 1$ and thus $\mathcal{K}_\alpha[Z] = \mathcal{C}_\alpha(Q)$ for some cluster rf $Q$. Define $\mathcal{C}_\alpha(Q^{(m)})$ as in Example 3.4 and let
\begin{equation}
Z^{(m)}_N = B^N Q^{(m)}/p_N(N)^{1/\alpha}, \quad Z_N = B^N Q/p_N(N)^{1/\alpha},
\end{equation}
with $m \in (0, \infty), Q^{(m)}(t) = Q(t)\mathbb{1}(||t|| \leq m), t \in \mathcal{T}$ and $N$ a $\mathcal{T}$-valued rv independent of $Q$ with positive pdf $p_N(t) > 0, t \in \mathcal{T}$. The next result shows that the elements of $\mathcal{K}_\alpha[Z]$ can be approximated by those of $\mathcal{C}_\alpha(Q^{(m)})$ as $m \to \infty$, which is in line with the so-called $m$-approximation discussed in [6, 8–10].

**Proposition 4.5.** For all bounded compact sets $K \subset \mathbb{R}^l$
\begin{equation}
\lim_{m \to \infty} \sup_{n > 0} \frac{1}{n} \mathbb{E}\left\{ \sup_{t \in nK \cap \mathcal{T}} \|Z_N(t) - Z^{(m)}_N(t)\|^\alpha \right\} = 0.
\end{equation}
As an application of (4.5), given a full rank lattice $\mathcal{L}$ on $\mathcal{T}$ or $\mathcal{L} = \mathcal{T}$ we obtain by (4.24)
\begin{equation}
\mathbb{B}_Z^\mathcal{C} = \mathbb{B}_Z^{\mathcal{C}N} = \lim_{m \to \infty} \mathbb{B}_Z^{\mathcal{C}N},
\end{equation}
which follows also from (4.19) and the definition of $Q^{(m)}$. 
4.5. Spectral cluster rf’s. Suppose next that $K_\alpha[Z] = C_\alpha(Q)$. In applications, given some map $\Gamma \in \mathcal{H}$, it is of interest to construct another cluster rf $Q_\Gamma$, such that $\mathbb{P}\{\Gamma(Q_\Gamma) = 1\} = 1$. In [13] $Q_\Gamma$ is referred to as a spectral cluster rf’s on $D_\Gamma$, where

$$ D_\Gamma = \{f \in D : \Gamma(f) = 1\}. $$

Note in passing that in the literature, the label “spectral” has the same meaning for the rf $\Theta$, since by definition $\mathbb{P}\{\Gamma(\Theta) = 1\} = 1$, where $\Gamma(f) = \|f(0)\|$, $f \in D$.

We discuss below some interesting maps $\Gamma$ and their spectral cluster rf’s $Q_\Gamma$.

**Lemma 4.6.** Let $K_\alpha[Z] = C_\alpha(Q)$ and let $\Gamma : \mathbb{S}_p \mapsto \mathbb{R}$, $p \geq 0$ be such that $\Gamma(Q)$ is a strictly positive rv with $c = \mathbb{E}\{\Gamma(Q)\} \in (0, \infty)$ for some $\xi \in \mathbb{R}$. If further $\mathbb{P}\{\Gamma(Q/\Gamma^\gamma(Q)) = 1\} = 1$, with $\gamma = \xi/\alpha$, then $Q_\Gamma$ having the same law as $Q/\Gamma^\gamma(Q)$ under

$$ \hat{P}(A) = c^{-1}\mathbb{E}\{[\Gamma(Q)]^{\xi/\alpha}\mathbb{1}(A)\}, \quad \forall A \in \mathcal{F} $$

is a spectral cluster rf on $D_\Gamma$ and moreover $K_\alpha[Z] = C_\alpha(c^{1/\alpha}Q_\Gamma)$.

As in [13], taking $\xi = \alpha/p, p > 0$ and

$$ \Gamma(f) = \int_T \|f(t)\|^p \lambda(dt) $$

define $Q^{(p)} = Q_\Gamma$ for all $p > 0$ such that $\mathbb{E}\{[\Gamma(Q)]^{\alpha/p}\} < \infty$. Define also $Q^{(\infty)} = Q_{\Gamma^*}$ taking

$$ \Gamma^*(f) = \sup_{t \in \partial} \|f(t)\|^\alpha, \quad \xi = 1. $$

Note that $\Gamma(Q^{(p)}) > 0$ almost surely follows from [20][Thm 2.1, Rem 2.2,iii] since $\mathbb{P}\{\Gamma^*(Q^*) > 0\} = 1$. Further (4.19) implies $\mathbb{E}\{\Gamma^*(Q)\} = \vartheta_T < \infty$. In view of (2.3) for all shift-invariant $H \in \mathbb{S}_T$

$$ \mathbb{E}\{H(Q^{(\alpha)})\} = \mathbb{E}\{H(\vartheta_T^{-1/\alpha}Q^{(\infty)})\} = \mathbb{E}\{H(\Theta/S(\Theta))\} = \mathbb{E}\{H(Q)\}. $$

Moreover, from Theorem 3.7, Item (iv) for all $\tau$ as in (2.10)

$$ \mathbb{E}\{H(Q^{(\infty)})\} = \mathbb{E}\{H(Y^{(\tau)}/\sup_{t \in T}\|Y^{(\tau)}(t)\|^\alpha)\}. $$

We note in passing that since [10] defines $Q$ through an anchoring map, the spectral cluster rf’s there are referred to as the anchored spectral tail process. We do not use that terminology since $Q$ can be determined also without referring to anchoring maps.

5. Proofs

**Proof of Theorem 2.4** For all $b > 0$ and $\tau \in \mathbb{R}$ by (2.8)

$$ \int_\mathcal{C}\mathbb{E}\{\mathbb{1}(B_{L,\tau}(Y) = \infty, \|Y(s)\| > b)\} \lambda(ds) = b^{-\alpha}\int_\mathcal{C}\mathbb{E}\{\mathbb{1}(B_{L,\tau}(bY) = \infty, b\|Y(-s)\| > 1)\} \lambda(ds) $$

$$ = b^{-\alpha}\mathbb{E}\{\mathbb{1}(B_{L,\tau}(bY) = \infty)B_{L,\tau}(bY)\}.$$

and hence

$$ \mathbb{P}\{B_{L,\tau}(Y) < \infty\} = 1 \implies \mathbb{P}\{B_{L,\tau}(bY) < \infty\} = 1 $$

and thus from [1][Lem 5.1] modulo null sets

$$ \{B_{L,\tau}(Y) < \infty\} \subset \{B_{L,\tau}(bY) < \infty\}. $$
The reverse inclusion can be shown with the same arguments and hence
\[(5.2) \quad \{B_{L,T}(Y) < \infty\} = \{B_{L,T}(bY) < \infty\}.
\]

Apart from the last equality in (2.14), if $b = 1$ the other claims follow from [1][Lem 5.4, Thm 5.6].

Clearly, almost surely
\[
\lim_{\|t\| \to \infty, t \in T} \|\Theta(t)\| = 0 \subset \{I_2(Y) \in T\},
\]
with $I_2$ the first exceedance map. The latter event implies $\{J_2(Y) \in \mathcal{L}\}$, hence the proof follows. 

**Proof of Theorem 3.1** If for some $\mathcal{T}$-valued rv $N$ with pdf $p_N(t) > 0, t \in \mathcal{T}$ independent of $Q$ we have $Z_N = p_N(N)^{-\alpha}B^N \mathcal{Q} \in \mathbb{K}_\alpha[Z]$, then applying (2.6) and the Fubini-Tonelli Theorem we obtain
\[
\mathbb{E}\{F(B^h\Theta)\} = \mathbb{E}\{\|Z_N(h)\|^\alpha F(Z_N)\}
\]
\[
= \mathbb{E}\left\{\int_{\mathcal{T}} \|B^tQ(h)\|^\alpha F(B^tQ)\lambda(dt)\right\}
\]
\[
= \mathbb{E}\{\|Z_{N^*}(h)\|^\alpha F(Z_{N^*})\}, \quad \forall F \in \mathcal{H}_0,
\]
where the first equality follows since by the assumption $Z_N \in \mathbb{K}_\alpha[Z]$ and $N^*$ is another $\mathcal{T}$-valued rv with positive pdf $p_{N^*}$ being further independent of $Q$. This shows that $Z_{N^*} \in \mathbb{K}_\alpha[Z]$ independent of the choice of $N^*$, which in turn implies that $B^hZ_N$ belongs to $\mathbb{K}_\alpha[Z]$ for all $h \in \mathcal{T}$ and thus $\mathbb{K}_\alpha[Z]$ is shift-invariant.

Further, by (2.4) and the shift-invariance of the Lebesgues measure
\[
1 = \mathbb{E}\{\|Z(0)\|^\alpha\} = \mathbb{E}\left\{\int_{\mathcal{T}} \|Q(t)\|^\alpha \lambda(dt)\right\} = \mathbb{E}\{S(Q)\},
\]
hence $\mathbb{P}\{S(Q) < \infty\} = 1$. In view of [20][Thm 2.1, Rem 2.2,iii]
\[
\mathbb{P}\{\sup_{t \in \mathcal{T}} \|Q(t)\| > 0\} = 1 \implies \mathbb{P}\{S(Q) > 0\} = 1.
\]
Suppose for simplicity in the rest of the proof that $l = 1$ and let $M$ be a positive integer. By (1.1) and (1.5) the Fubini-Tonelli Theorem implies
\[
\infty > \mathbb{E}\left\{\sup_{t \in [0,2M+1] \cap \mathcal{T}} \|Z(t)\|^\alpha\right\} = \mathbb{E}\left\{\int_{\mathcal{T}} \sup_{t \in [0,2M+1] \cap \mathcal{T}} \|Q(t + x)\|^\alpha \lambda(dx)\right\}
\]
\[
= \sum_{i \in \mathbb{Z}} \int_{i}^{i+1} \mathbb{E}\left\{\sup_{t \in [0,2M+1] \cap \mathcal{T}} \|Q(t + x)\|^\alpha\right\} \lambda(dx)
\]
\[
\geq \sum_{i \in \mathbb{Z}} \int_{i}^{i+1} \lambda(dx) \mathbb{E}\left\{\sup_{s \in [i+1+M,i+1+2M] \cap \mathcal{T}} \|Q(s)\|^\alpha\right\}
\]
\[
= \sum_{j \in \mathbb{Z}} \mathbb{E}\left\{\sup_{s \in [j,j+M] \cap \mathcal{T}} \|Q(s)\|^\alpha\right\},
\]

hence (1.6) follows using further (2.3) and thus $Q$ is a cluster rf.

To prove the converse, assume that $Z = Z_N$ is given as above with $Q$ a cluster rf. By (1.6)
\[
\mathbb{P}\{\sup_{t \in \mathcal{T}} \|Z(t)\| > 0\} = \mathbb{P}\{\sup_{t \in \mathcal{T}} \|Q(t)\| > 0\} = 1
\]
for all $\tilde{Z} \in \mathcal{C}_\alpha(Q)$. Further
\[
\mathbb{E}\left\{ \sup_{t \in [-C,C]\cap \mathcal{T}} \|Z(t)\|^\alpha \right\} = \mathbb{E}\left\{ \int_{\mathcal{T}} \sup_{t \in [-C,C]\cap \mathcal{T}} \|Q(t + x)\|^\alpha \lambda(dx) \right\} < \infty, \quad C > 0
\]
and by (2.3) and (1.6) for all $h \in \mathcal{T}, \tilde{Z} \in \mathcal{C}_\alpha(Q)$
\[
(5.3) \quad \mathbb{E}\{\|\tilde{Z}(h)\|^\alpha\} = \mathbb{E}\{S(Q)\} < \infty, \quad \forall h \in \mathcal{T}
\]
implies thus (1.1). If $\mathcal{T} = Z^l$, then (1.1) follows immediately from (5.3). Since $B^h Z_N$ belongs also to $\mathcal{C}_\alpha(Q)$ by the definition, we have that $\mathcal{C}_\alpha(Q)$ is shift-invariant and if $\tilde{Z} \in \mathcal{C}_\alpha(Q)$, then [20][Thm 2.1, Rem 2.2,iii]] implies $\mathbb{P}\{S(\tilde{Z}) > 0\} = 1$. Next, by [1][Lem B.2]
\[
\mathbb{P}\{S(\tilde{Z}) \in (0,\infty)\} = \mathbb{P}\{S(Z) \in (0,\infty)\}
\]
\[
= \int_{\mathcal{T}} \mathbb{P}\{S(B^tQ) \in (0,\infty)\} p_N(t) \lambda(dt)
\]
\[
= \mathbb{P}\{S(Q) \in (0,\infty)\} \int_{\mathcal{T}} p_N(t) \lambda(dt)
\]
\[
= \mathbb{P}\{S(Q) \in (0,\infty)\},
\]
hence using further (1.6) $\mathbb{P}\{S(\tilde{Z}) \in (0,\infty)\} = 1$ establishing the proof.

\section*{Proof of Proposition 3.2}
The first two equivalences follow from [1][Lem B.2]. Clearly, if $\mathbb{E}\{H(\tilde{Z})\} = 0$ for all $\tilde{Z} \in \mathcal{C}_\alpha[Z]$, since $Z_N = B^N Q/[p_N(N)]^{1/\alpha} \in \mathcal{K}_\alpha[Z]$, then by the shift-invariance and 0-homogeneity of $H$
\[
\mathbb{E}\{H(Z_N)\} = \mathbb{E}\{H(Q)\} = 0
\]
follows. If $\mathbb{E}\{H(Q)\} = 0$ holds, then again $\mathbb{E}\{H(Z_N)\} = 0$.

Since a cluster rf $Q$ satisfies $\mathbb{P}\{S(Q) \in (0,\infty)\} = 1$, then (3.1) follows from the stated equivalences. Assuming the converse, if $\mathbb{P}\{S(\Theta) \in (0,\infty)\} = 1$ we have that $Q = c^{1/\alpha} \Theta, c = 1/S(\Theta)$ is well-defined. Applying now (2.7) to $Z_N$ shows that (2.3) holds and thus $Z_N \in \mathcal{K}_\alpha[Z]$ establishing the claim. \hfill \Box

\section*{Proof of Proposition 3.3}
The first three equivalences follow from [1][Cor B.3]. Assume next that $\mathcal{K}_\alpha[Z] = \mathcal{C}_\alpha(Q)$ with $Q \in \mathcal{M}$ a cluster rf. If Item (iv) holds, then since $Z_N$ defined by (1.5) belongs to $\mathcal{K}_\alpha[Z]$, then $Z_N(t) > 0$ almost surely for all $t \in \mathcal{T}$ and hence Item (ii) holds. If Item (iii) holds, taking $Q = c^{1/\alpha} \Theta, c = 1/S(\Theta)$, then clearly for this construction Item (iv) follows. \hfill \Box

\section*{Proof of Theorem 3.7}
It suffices to show that (2.6) holds for $Z_N = p_N(N)^{-1/\alpha} B^N Q$, i.e., for all $F \in \mathcal{F}_0$ and all $h \in \mathcal{T}$
\[
(5.4) \quad \mathbb{E}\{\|Z_N(h)\|^\alpha F(Z_N)\} = \mathbb{E}\{G_h(Q)\} = \mathbb{E}\{F(B^h \Theta)\} = \mathbb{E}\{\|Z(h)\|^\alpha F(Z)\},
\]
with
\[
(5.5) \quad G_h(f) = \int_{\mathcal{T}} \|B^h f(h)\|^\alpha F(B^h f) \lambda(dy), \quad f \in D, \quad F \in \mathcal{F}_0.
\]

\section*{Proof of Item (iii), Item (i):} It is enough to show the proof of Item (i). Since $\mathbb{P}\{S(\Theta) \in (0,\infty)\} = 1$, which by Theorem 2.4 is equivalent with $\mathbb{P}\{S(\Theta) \in (0,\infty)\} = 1$, then the rf $Q(t) = c^{1/\alpha} \Theta$ is well defined with $c = 1/(\Delta(\mathcal{L}) S_\mathcal{L}(\Theta))$. For $F \in \mathcal{F}_0$ and $h \in \mathcal{T}$ we have
\[
(5.6) \quad \mathbb{E}\{G_h(Q)\} = \mathbb{E}\left\{ \frac{G_h(\Theta)}{\Delta(\mathcal{L}) S_\mathcal{L}(\Theta)} \right\}.
\]
Note that when $\mathcal{L} = \mathcal{T}$ we set $\Delta(\mathcal{L}) = 1$. For this case applying (2.7), for all $\Gamma \in \mathcal{F}_0$, $h \in \mathcal{T}$ we obtain
\begin{equation}
(5.7) \quad \mathbb{E}\left\{ \frac{G_h(\Theta)}{S(\Theta)} \right\} = \mathbb{E}\{ F(B^h \Theta) \}.
\end{equation}
Consider therefore the case $\mathcal{L}$ is a full rank lattice of $\mathcal{T} = \mathbb{R}^l$, hence
\begin{equation}
(5.8) \quad \text{span}(\mathcal{L}) = \text{span}(\mathbb{Z}^l) = \mathbb{R}^l,
\end{equation}
where $\text{span}(A)$ denotes the smallest linear subspace of $\mathbb{R}^l$ containing $A \subset \mathbb{R}^l$. It is well-known (see e.g., [48] [Lem 10.7]) that we can tilt $\mathcal{T}$ by the fundamental domain of the full rank lattice $\mathcal{L}$ on $\mathcal{T}$. We take as fundamental domain the fundamental parallelepiped $\mathcal{P}(\mathcal{L}) = \{ A x, x \in [0,1]^l \}$, where $A$ is the $l \times l$ real matrix (called base matrix of $\mathcal{L}$) which is non-singular since $\mathcal{L}$ is a full rank lattice; recall $\mathcal{L} = \{ A x, x \in \mathbb{Z}^l \}$ and
$$
\text{Vol}(\mathcal{P}(\mathcal{L})) = \Delta(\mathcal{L}) = \text{det}(A) > 0.
$$
Consider $\mathcal{T} = \mathbb{R}^l$ which is spanned by $\mathcal{L}$. Hence we have the tilt of $\mathcal{T}$ as
$$
\mathcal{T} = \text{span}(\mathcal{L}) = \cup_{t \in \mathcal{L}} \{ t + \mathcal{P}(\mathcal{L}) \},
$$
where $t + \mathcal{P}(\mathcal{L})$ and $s + \mathcal{P}(\mathcal{L})$ are disjoint for $t \neq s \in \mathbb{R}^l$. By Theorem 2.4 and (3.4)
$$
\mathcal{A}(\Theta) = \{ S(\Theta) \in (0, \infty), S_{\mathcal{L}}(\Theta) \in (0, \infty) \} = \{ S(\Theta) \in (0, \infty) \} = \{ S_{\mathcal{L}}(\Theta) \in (0, \infty) \},
$$
\begin{equation}
(5.9) \quad \mathcal{A}(B^{s+t}\Theta) = \{ S(\Theta) \in (0, \infty), S_{\mathcal{L}}(B^{s+t}\Theta) \in (0, \infty) \} = \mathcal{A}(B^t\Theta) = \{ S(\Theta) \in (0, \infty), S_{\mathcal{L}}(B^t\Theta) > 0 \}
\end{equation}
for all $s \in \mathcal{L}, t \in \mathcal{P}(\mathcal{L}) \cap \mathcal{T}$. Further, (3.4) yields
$$
\mathcal{A}(B^{s+t}\Theta) = \mathcal{A}(B^t\Theta) = \mathcal{A}(\Theta)
$$
aalmost surely. Write hereafter $\mathbb{E}\{ A; B \}$ instead of $\mathbb{E}\{ A I(B) \}$ and take $F \in \mathcal{F}_0$. Using (2.7) for the derivation of the fourth line below, the Fubini-Tonelli theorem and the shift-invariance of $\lambda(dt)$ (recall that $\| \Theta(0) \| = 1$ almost surely and we interpret $0 : 0$ as 0)
\begin{align*}
\mathbb{E}\left\{ \frac{G_h(\Theta)}{S_{\mathcal{L}}(\Theta)} \right\} &= \mathbb{E}\left\{ \frac{G_h(\Theta)}{S_{\mathcal{L}}(\Theta)} ; A(\Theta) \right\} = \mathbb{E}\left\{ \frac{G_h(\Theta) S(\Theta)}{S_{\mathcal{L}}(\Theta) S(\Theta)} ; A(\Theta) \right\} \\
&= \int_{t \in \mathcal{P}(\mathcal{L})} \int_{s \in \mathcal{L}} \mathbb{E}\left\{ \frac{G_h(\Theta)}{S(\Theta)} \right\} \frac{\| \Theta(t) \|^{\alpha}}{S_{\mathcal{L}}(\Theta)} ; A(\Theta) \right\} \lambda(dt) \\
&= \int_{t \in \mathcal{P}(\mathcal{L})} \int_{s \in \mathcal{L}} \mathbb{E}\left\{ \| \Theta(s+t) \|^{\alpha} \frac{G_h(\Theta)}{S(\Theta)} \right\} \frac{\| \Theta(0) \|^{\alpha}}{S_{\mathcal{L}}(\Theta)} ; A(\Theta) \right\} \lambda(dy) \lambda(dt) \\
&= \int_{t \in \mathcal{P}(\mathcal{L})} \int_{s \in \mathcal{L}} \int_{\mathcal{T}} \mathbb{E}\left\{ \| \Theta(s+t) \|^{\alpha} \frac{B^{y \Theta}(h)}{S(\Theta)} \right\} \frac{\| B^{y \Theta}(0) \|^{\alpha}}{S_{\mathcal{L}}(\Theta)} ; A(B^{s+t}\Theta) \right\} \lambda(dy) \lambda(dt) \\
&= \int_{t \in \mathcal{P}(\mathcal{L})} \int_{s \in \mathcal{L}} \int_{\mathcal{T}} \mathbb{E}\left\{ \frac{G_h(\Theta) \| B^t \Theta(-s) \|^{\alpha}}{S_{\mathcal{L}}(B^{s+t}\Theta)} ; A(B^{s+t}\Theta) \right\} \lambda(dy) \lambda(dt) \\
&= \int_{t \in \mathcal{P}(\mathcal{L})} \int_{s \in \mathcal{L}} \mathbb{E}\left\{ \frac{G_h(\Theta) \sum_{s \in \mathcal{L}} \| B^t \Theta(-s) \|^{\alpha}}{S_{\mathcal{L}}(B^{s+t}\Theta)} ; A(B^{s+t}\Theta) \right\} \lambda(dy) \lambda(dt) \\
&= \int_{t \in \mathcal{P}(\mathcal{L})} \int_{s \in \mathcal{L}} \mathbb{E}\left\{ \frac{G_h(\Theta) S_{\mathcal{L}}(B^t \Theta)}{S(\Theta) S_{\mathcal{L}}(B^t \Theta)} ; A(B^t \Theta) \right\} \lambda(dy) \lambda(dt)
\end{align*}
By the Fubini-Tonelli theorem, the shift-invariance of \( \lambda \) choice of \( T \) shows that we have a tiling of \( T \) where the last equality follows from (5.9).

Next, take \( T = \mathbb{Z}^l \) and define the additive quotient group \( T / \mathcal{L} = \{ x + \mathcal{L}, x \in T \} \). In view of (5.8) we have that the order \( m \) of the quotient group is given by (see [49])

\[
m = |T / \mathcal{L}| = |\mathbb{Z}^l \cap \mathcal{P}(\mathcal{L})| = \Delta(\mathcal{L})/\Delta(\mathcal{T}) = \Delta(\mathcal{L}).
\]

This shows that we have a tiling of \( T \) by \( T \cap \mathcal{P}(\mathcal{L}) \). Hence repeating the calculations where the integral is substituted by the sum establishes the claim.

**Proof of Item (ii):** Recall that in our notation \( \mathcal{M}_\mathcal{L}(Y) = \max_{t \in \mathcal{L}} \| Y(t) \| \) and let \( G_h \) be as above. Since \( \mathbb{P}\{S(\Theta) < \infty\} = 1 \), then **Theorem 2.4** and **Lemma A.1** show that \( Q \) is well-defined and belongs to \( \mathfrak{M} \). For this choice of \( Q \) we have

\[
E\{G_h(Q)\} = b^\alpha \mathbb{E}\left\{ \frac{G_h(Y) \| Y(0) \| \tau \mathbb{I}(\mathcal{M}_\mathcal{L}(Y) > b)}{\Delta(\mathcal{L})[\mathcal{M}_\mathcal{L}(Y)]^{\alpha} B_{\mathcal{L}, \tau}(Y)} \right\}.
\]

By **Theorem 2.4** and \( \mathbb{P}\{\mathcal{M}_\mathcal{L}(Y) > 1\} = 1 \) we have modulo null sets for all \( b \geq 1 \) (recall that \( \| Y(0) \| > 1 \) almost surely)

\[
\mathcal{A}(Y) = \{ S_\mathcal{L}(Y) \in (0, \infty) \} = \{ S_\mathcal{L}(Y) \in (0, \infty), B_{\mathcal{L}, \tau}(Y) \in (0, \infty), B_{\mathcal{L}, \tau}(Y/b) \in (0, \infty) \}
\]

By the shift-invariance of \( \lambda(dt) \) and (2.8) applied to obtain the second last equality

\[
E\left\{ \frac{G_h(Y) \| Y(0) \| \tau \mathbb{I}(\mathcal{M}_\mathcal{L}(Y) > b)}{[\mathcal{M}_\mathcal{L}(Y)]^{\alpha} B_{\mathcal{L}, \tau}(Y)} \right\} = b^\alpha \mathbb{E}\left\{ \frac{G_h(Y) \| Y(0) \| \tau \mathbb{I}(\mathcal{M}_\mathcal{L}(Y) > b)}{[\mathcal{M}_\mathcal{L}(Y)]^{\alpha} B_{\mathcal{L}, \tau}(Y)} \right\} = b^\alpha E\left\{ \frac{G_h(Y) \| Y(0) \| \tau \mathbb{I}(\mathcal{M}_\mathcal{L}(Y) > b)}{[\mathcal{M}_\mathcal{L}(Y)]^{\alpha} B_{\mathcal{L}, \tau}(Y)} \right\} \lambda(dt)
\]

holds for all \( F \in \mathfrak{S}_0 \), shift-invariant with respect to \( \mathcal{L} \).
As shown in [1] since \( \mathbb{P}\{A(Y)\} = 1 \), then almost surely
\[
A(Y) = A(Y) \cap \{B_{L,r}(Y/(zM_L(Y))) \in (0, \infty)\} = \mathcal{E}_z
\]
for all \( z \in (0, 1) \) up to a set of \( \mathbb{R} \) with Lebesgue measure zero.

Using again Fubini-Tonelli theorem and (2.8) (recall the definition of \( \lambda_\alpha(dz) = \alpha z^{-\alpha-1}dz \) with \( \mu \) the Lebesgue measure on \( \mathbb{R} \))
\[
E\{G_h(Q)\}
= \int_E \left\{ \frac{G_h(Y)\|Y(0)\|^\tau \|Y(h)\|^\alpha}{B_{L,r}(Y)[M_L(Y)]^\alpha}; A(Y) \right\} \lambda(dh)
\]
\[
= \int_E \left\{ \frac{G_h(Y)\|Y(0)\|^\tau \|Y(h)\| > 1}{B_{L,r}(Y)}; A(Y) \right\} \lambda_\alpha(dz) \lambda(dh)
\]
\[
= \int_E \left\{ \frac{G_h(Y)\|Y(-h)/z\|^\tau \|Y(-h)\| > z}{B_{L,r}(Y/z)}; A(Y) \cap \{B_{L,r}(Y/z) \in (0, \infty)\} \right\} \alpha z^{-\alpha-1}dz \lambda(dh)
\]
\[
= \int_E \left\{ \frac{G_h(Y)\|Y(-h)/z\|^\tau \|Y(-h)\| > z}{B_{L,r}(Y/(zM_L(Y)))}; A(Y) \cap \{B_{L,r}(Y/z) \in (0, \infty)\} \right\} \alpha z^{-\alpha-1}dz \lambda(dh)
\]
\[
= \int_E \left\{ \frac{G_h(Y)\|Y(-h)/z\|^\tau \|Y(-h)\| > z}{B_{L,r}(Y/(zM_L(Y)))}; A(Y) \cap \{B_{L,r}(Y/z) \in (0, \infty)\} \right\} \alpha z^{-\alpha-1}dz \lambda(dh)
\]
\[
= \int_1^\infty \left\{ \frac{G_h(Y)}{B_{L,r}(Y/(zM_L(Y)))}; A(Y) \right\} \alpha z^{-\alpha-1}dz
\]
where 1 in the upper bound of the integral in the last fourth line above is a consequence of \( \mathbb{P}\{M_L(Y) > 1\} = 1 \), the indicator \( I(B_{L,r}(Y/z) \in (0, \infty)) \) appears in the last forth line thanks to Remark 2.3, Item (i), and the last line follows from (5.9). Hence the proof is complete.

Proof of Item (iv): We have that \( \vartheta_L = E\{\|Y(0)\|^\tau /(\Delta(L)B_{L,r}(Y))\} \in (0, \infty) \), which is clear for \( L \) a lattice on \( \mathbb{R}^d \), while for \( L = \mathcal{T} \) it follows from (A.7). Taking \( Y^{(\tau)} \) having the same law as \( Y \) under
\[
\mathbb{P}^*\{A\} = \vartheta_L^{-1}E\{I(A)\|Y(0)\|^\tau /(\Delta(L)B_{L,r}(Y))\}, \quad A \in \mathcal{F}
\]
the claim follows from the calculations in the previous case.

Proof of Theorem 3.8 First note that by the assumption \( \mathbb{P}\{S(\Theta) \in (0, \infty)\} = 1 \). Hence Theorem 2.4 implies \( \mathbb{P}\{S(\Theta) \in (0, \infty)\} = \mathbb{P}\{B_{L,r}(Y) \in (0, \infty)\} = 1 \) and moreover since \( \vartheta_L \in (0, \infty) \) we have from (A.5) and (A.6) that the constant \( c \) is positive in both three cases treated below.

Proof of Item (i): With the notation of Theorem 3.1 using (A.5) we have
\[
E\{G_h(Q)\} = \frac{1}{\Delta(L)} E\{G_h(\Theta); J_1(\Theta) = 0\} = E\left\{ \frac{G_h(\Theta)}{\Delta(L)S_L(\Theta)} \right\},
\]
hence (5.9) implies (5.4) establishing thus the claim.

Proof of Item (ii): Applying (A.6) and then (5.11) we obtain (recall \( M_L(Y) = \sup_{t \in \mathcal{L}} \|Y(t)\| \))
\[
E\{G_h(Q)\} = b^\alpha \frac{1}{\Delta(L)} E\left\{ \frac{G_h(Y)|M_L(Y)| > b}{[M_L(Y)]^\alpha} ; J_2(Y) = 0 \right\}
\]
\[
= b^\alpha \frac{1}{\Delta(L)} E\left\{ \frac{G_h(Y)|M_L(Y)| > b}{[M_L(Y)]^\alpha B_{L,0}(Y)} \right\}
\]
\[
\frac{1}{\Delta(\mathcal{L})} \mathbb{E} \left\{ \frac{G_h(Y)}{[\mathcal{M}_\mathcal{L}(Y)]^\alpha B_{\mathcal{L},0}(Y)} \right\} = \mathbb{E} \left\{ \frac{G_h(\Theta)}{\Delta(\mathcal{L})S_{\mathcal{L}}(\Theta)} \right\},
\]

where the last equality is shown in the proof of Theorem 3.1. Hence again (5.4) is satisfied and thus the claim follows.

**Proof of Item (iii):** The proof follows by applying (A.5). \qed

**Proof of Proposition 4.2** When \( \nu \) is shift-invariant, then by [1][Prop 6.5] \( \mathbb{K}_\alpha[Z] \) is shift-invariant. Define the spectral tail rf \( \Theta \) by (3.2) for all \( A \in \mathcal{D} \), which has a.s. sample paths on \( \mathcal{D}^* \). If \( \mathbb{P}\{S(\Theta) < \infty\} = 1 \), we can define \( Q = \Theta/(S(\Theta))^{1/\alpha} \), which has again a.s. sample paths on \( \mathcal{D}^* \). In view of [50][Lem p. 1276] we can define \( Z, \Theta, Q \) on the same non-atomic complete probability space since \( \mathcal{D}^* \) is a Polish space. The rest of the proof follows from Corollary 4.1, (2.16) and Theorem 3.7, Theorem 3.8. \qed

**Proof of Proposition 4.3** In view of [1][Lem 6.1] \( X \) is stationary and stochastically continuous with locally bounded sample paths. As shown in [5] by [44][Lem 2] \( X \) has a representor \( Z^* \) which is stochastically continuous. In view of [19][Thm 2.6] and (2.6) it follows that \( Z^* \in \mathbb{K}_\alpha[Z] \). This implies that \( \Theta^* \) and \( Y^* \) constructed from \( Z^* \) are both stochastically continuous. \qed

**Proof of Proposition 4.5** Since \( K \) is bounded we have for all \( t_0 \in \mathbb{R}^l \) that \( K - t_0 \in [0,k]^l \) for some positive integer \( k \). Hence by the definition, the shift-invariance of Lebesgue measure and the representations for \( Z_N \) and \( Z_N^{(m)} \) imply for all \( n > 0 \)

\[
\frac{1}{n^l} \mathbb{E} \left\{ \sup_{t \in nK} \|Z_N(t) - Z_N^{(m)}(t)\|^\alpha \right\} \leq \frac{1}{n^l} \int_T \mathbb{E} \left\{ \sup_{s \in n(K - t_0)} \|Q(t + s) - Q^{(m)}(t + s)\|^\alpha \right\} \lambda(ds) \leq \int_T \mathbb{E} \left\{ \sup_{s \in [0,nk]^l} \|Q(t + s)\|^\alpha \mathbb{1}(\|t + s\| > m) \right\} \lambda(ds) \rightarrow 0
\]

as \( m \to \infty \) establishing the proof. \sq

**Proof of Lemma 4.6** It can be easily verified that \( Q_{\Gamma} \) is a spectral rf. The assumption \( \mathbb{P}\{\Gamma(Q/\Gamma(Q)) = 1\} = 1 \) implies \( \mathbb{P}\{\Gamma(Q_{\Gamma}) = 1\} = 1 \) and hence \( Q_{\Gamma} \) is a spectral cluster rf on \( \mathcal{D}_{\Gamma} \). Hence with \( G_h \) given by (5.5), a direct application of the Fubini-Tonelli theorem and the assumption that \( \xi = \alpha \gamma \) implies

\[
\mathbb{E}\{G_h(e^{1/\alpha}Q_{\Gamma})\} = c^{1-\mathbb{E}\left\{[\Gamma(Q)]^{\alpha}\right\}}G_h(e^{1/\alpha}Q/\Gamma(Q))^{\gamma} = \mathbb{E}\{G_h(Q)\} = \mathbb{E}\{F(B^h\Theta)\},
\]

where the last equality follows from (5.4). Consequently, we have \( \mathbb{K}_\alpha[Z] = C_\alpha(e^{1/\alpha}Q_{\Gamma}) \) establishing the proof. \sq

**Appendix A. Technical results**

**Lemma A.1.** Let \( Q \in \mathfrak{M} \) and \( N \) the \( \mathcal{T} \)-valued rv be being independent of \( Q \), then \( B^NQ \in \mathfrak{M} \).

**Proof of Lemma A.1** We show first that \( B^NQ(t) = Q(t - N), t \in \mathcal{T} \) is a rf. If \( N \) is a discrete rv taking only finite values \( t_i, 1 \leq i \leq n \), then for any \( c \in \mathbb{R} \) we have \( \{Q(t - N) < c\} \) is an event since \( \{Q(t - N) < c, N = t_i\} \)
is an event for all $1 \leq i \leq n$. For a general $N$ we can approximate it almost surely by discrete rv’s $N_n, n \in \mathbb{N}$. Hence $B^NQ(t), t \in T$ is the almost sure limit of $B^N\alpha Q(t), t \in T$ as $n \to \infty$, implying that $B^NQ$ is a rf. Since $N$ is independent of $Q$, by dominated convergence theorem, we have that the assumption $Q \in \mathcal{M}$ implies $B^NQ \in \mathcal{M}$ establishing the claim.

Recall that in our notation $E\{A; B\}$ stands for $E\{A 1_B\}$. For a non-empty $\mathcal{V} \subset \mathcal{L}$ and given non-negative weights $w_i, i \in \mathcal{V}$ define

$$A_k(f) = \sum_{i \in \mathcal{V}} w_i 1_{(J_k(f) = i)}, \quad k = 1, 2, 3$$

and note that for $w_i = 1, i \in \mathcal{V}$ we simply have

$$\chi_k(f) = 1_{(J_k(f) \in \mathcal{V})}.$$ 

**Lemma A.2.** Let $\mathcal{V} \subset \mathcal{L}$ be non-empty with $\mathcal{L}$ a lattice on $T$ and let $F \in \mathcal{F}_0, \Gamma = F_1F_2, F_1 \in \mathcal{F}_v, v \geq 0, F_2 \in \mathcal{H}$. If $J_1$ is a positive shift-involutions, $J_2$ is anchoring and $J_3$ is a shift-involutions, respectively, then for all $\tau \in \mathbb{R}$

\begin{align}
(A.1) \quad E\{F(\Theta); \chi_1(\Theta)\} &= E\left\{\sum_{i \in \mathcal{V}} w_i 1_{(B^i\Theta(0))} 1^{\alpha} F(B^i\Theta); J_2(\Theta) = 0\right\}, \\
(A.2) \quad E\{\|Y(0)\|^\Gamma(\mathcal{Y}); \chi_2(\mathcal{Y})\} &= E\left\{\sum_{i \in \mathcal{V}} w_i 1_{(B^i\mathcal{Y}(0))} 1^{\Gamma}\{\|B^i\mathcal{Y}(0)\| > 1\} 1_{\Gamma(\mathcal{Y})}; J_2(\mathcal{Y}) = 0\right\}, \\
(A.3) \quad E\{\|Z(0)\|^\alpha F(Z); \chi_3(Z)\} &= E\left\{\sum_{i \in \mathcal{V}} w_i 1_{(B^i\mathcal{Z}(0))} 1^{\alpha} F(B^i\mathcal{Z}); J_3(Z) = 0\right\}
\end{align}

and if $\mathcal{V}$ is also a lattice on $T$, then

\begin{align}
(A.4) \quad P\{S\mathcal{V}(\mathcal{Y}) = \infty, J_1(\Theta) \in \mathcal{V}\} &= P\{B_{\mathcal{V},\tau}(\mathcal{Y}) = \infty, J_2(\mathcal{Y}) \in \mathcal{V}\} = P\{S\mathcal{V}(\mathcal{Z}) = \infty, J_3(\mathcal{Z}) \in \mathcal{V}\} = 0.
\end{align}

**Remark A.3.**

(i) Note that when $\mathcal{L}$ is a full rank lattice on $T$, then $P\{J_2(\mathcal{Y}) \in \mathcal{L}\} = 1$ is equivalent with one of the events in Theorem 2.4 holds with probability 1. If $\mathcal{V} = \mathcal{L} = T = Z^l, P\{J_2(\mathcal{Y}) \in \mathcal{L}\} = 1$ and $\|\|\|$ is a norm on $\mathbb{R}^d$, then (A.2) reduces to [10][Prop 3.6];

(ii) If $E\{\|Z(0)\|^\alpha\} = 1$ and $J_k, k \leq 3$ are as in Lemma A.2, satisfying (2.11) and (2.12), then for all $F \in \mathcal{F}_\alpha$ shift-invariant with respect to $\mathcal{L}$

\begin{align}
(A.5) \quad E\left\{\frac{F(\Theta)}{S_{\mathcal{L}}(\Theta)}\right\} &= E\{F(\Theta); J_1(\Theta) = 0\} = E\{F(\Theta); J_2(\mathcal{Y}) = 0\} = E\{F(\Theta); J_3(\mathcal{Z}) = 0\},
\end{align}

with $\Theta(t) = \Theta(t)/\sup_{t \in \mathcal{V}}\|\Theta(t)\|$ and for any $\Gamma = F_1, F_2, F_1 \in \mathcal{F}_v, v \geq 0, F_2 \in \mathcal{H}$ shift-invariant with respect to $\mathcal{L}$ and all $\tau \in \mathbb{R}$

\begin{align}
(A.6) \quad E\left\{\frac{\|Y(0)\|^\Gamma F(Y)}{B_{\mathcal{L},\tau}(\mathcal{Y})}\right\} &= E\{F(\mathcal{Y}); J_2(\mathcal{Y}) = 0\},
\end{align}

where we interpret $\infty/\infty$ and $0/0$ as 0. Note in passing that (A.4), and both (A.5), (A.6) imply for $T = Z^l$ the claims of [8][Thm 5.5.3].

The first two identities in (A.5) and (A.6) for $\tau = 0, \mathcal{V} = \mathcal{L} = T = Z^d$ and $\|\|\|$ a norm on $\mathbb{R}^d$ are stated in [8].
Proof of Lemma A.2 Next, if $J_1$ is a positive shift-involution, then using that $\Theta$ is a spectral tail rf by the Fubini-Tonelli theorem, for all $F \in \mathcal{H}_0$

$$E\left\{ \sum_{s \in \mathcal{V}} w_s \|B^s \Theta(0)\|^\alpha F(B^s \Theta); J_1(\Theta) = 0 \right\} = \sum_{s \in \mathcal{V}} w_s E\{\|B^s \Theta(0)\|^\alpha F(B^s \Theta); J_1(\Theta) = 0\}$$

$$= \sum_{s \in \mathcal{V}} w_s E\{F(\Theta)I(J_1(\Theta) = 0)I(\|B^{-s} \Theta(0)\| \neq 0)\}$$

$$= \sum_{s \in \mathcal{V}} w_s E\{F(\Theta)I(J_1(\Theta) = 0)\}$$

$$= \sum_{s \in \mathcal{V}} w_s E\{F(\Theta)\} = \sum_{s \in \mathcal{V}} w_s J_1(\Theta) = s$$

where the second equality follows since by the assumption $I(\|B^{-s} f\| \neq 0) \in \mathcal{H}_0$ and the fact that $\Theta$ is a spectral tail rf, whereas the third equality follows by the assumption that $J_1$ is a positive shift-invariant involution which yields $J_1(\Theta) = k$ implies $\|\Theta(k)\| \neq 0$ almost surely for any $k \in \mathcal{L}$. Hence (A.2) follows.

Applying (2.8), the Fubini-Tonelli theorem implies for all $\Gamma \in \mathcal{H}_0$ and $J_2$ anchoring

$$E\left\{ \sum_{i \in \mathcal{V}} w_i \|B^i Y(0)\|^\top I(\|B^i Y(0)\| > 1)\Gamma(B^i Y); J_2(Y) = 0 \right\}$$

$$= \sum_{i \in \mathcal{V}} w_i E\{I(\|Y(-i)\| > 1)\|Y(-i)\|^\top \Gamma(B^i Y); J_2(Y) = 0 \}$$

$$= \sum_{i \in \mathcal{V}} w_i E\{I(\|Y(i)\| > 1)\|Y(0)\|^\top \Gamma(Y); J_2(B^{-i} Y) = 0 \}$$

$$= \sum_{i \in \mathcal{V}} w_i E\{\|Y(0)\|^\top \Gamma(Y); J_2(B^{-i} Y) = 0 \}$$

$$= \sum_{i \in \mathcal{V}} w_i E\{\|Y(0)\|^\top \Gamma(Y); J_2(Y) = i \}$$

$$= \sum_{i \in \mathcal{V}} w_i E\{\|Y(0)\|^\top \Gamma(Y); J_2(Y) = i \}$$

$$= \sum_{i \in \mathcal{V}} w_i I(J_2(Y) = i)$$

and thus (A.2) follows. Set $\Gamma(Y) = I(B_{\mathcal{V}, r}(Y) = \infty)/R^r$, which is shift-invariant with respect to $\mathcal{V}$ by the assumption that $\mathcal{V}$ is a subgroup of the additive group $\mathcal{T}$. Since $\mathcal{V}$ has infinite number of elements, the above yields $P\{B_{\mathcal{V}, r}(Y) = \infty, J_2(Y) \in \mathcal{V}\} = 0$.

Next, taking $J_3$ to be a shift-involution and thus it is 0-homogeneous, utilising the shift-invariance of $K_\alpha[Z]$, for all $F \in \mathcal{H}_0$ the Fubini-Tonelli theorem implies

$$E\left\{ \sum_{s \in \mathcal{L}} w_s \|B^s Z(0)\|^\alpha F(B^s Z); J_3(Z) = 0 \right\} = \sum_{s \in \mathcal{L}} w_s E\{\|B^s Z(0)\|^\alpha F(B^s Z); J_3(Z) = 0\}$$

$$= \sum_{s \in \mathcal{L}} w_s E\{\|Z(0)\|^\alpha F(Z)I(J_3(B^{-s} Z) = 0)\}$$

$$= \sum_{s \in \mathcal{L}} w_s I(J_3(Z) = s)$$
and therefore there exists \( h \) such that \( \int_{\mathbb{R}} \|\Theta(s-t)\|^\tau \mathbb{I}(\|Y(s-t)\| > 1) \lambda(ds) \) follows.

Borrowing the idea of the proof of [8][Thm 5.5.3], namely using that by the above choice of \( F \)

\[
\mathbb{E}\{S_\mathcal{V}(Z)F(Z); J_3(Z) = 0\} = \mathbb{E}\{F(\Theta); J_3(\Theta) \in \mathcal{V}\}
\]

we obtain

\[
P\{S_\mathcal{C}(Z) = \infty, J_3(Z) \in \mathcal{V}\} = P\{S_\mathcal{C}(\Theta) = \infty, J_3(\Theta) \in \mathcal{V}\} = 0,
\]

hence (A.4) follows. \( \square \)

**Lemma A.4.** If \( \tau \in \mathbb{R} \) satisfies (2.10), then

\[
(A.7) \quad \mathbb{E}\left\{\sup_{t \in K} \|Z(t)\|^\alpha\right\} = \int_K \mathbb{E}\left\{\frac{1}{\int_{\mathbb{R}} \|\Theta(s-t)\|^\tau \mathbb{I}(\|Y(s-t)\| > 1) \lambda(ds)}\right\} \lambda(dt) \in (0, \infty)
\]

for all \( a > 0 \), with \( K = [-a,a]^l \cap \mathcal{T} \) and \( Y = R\Theta \).

**Proof of Lemma A.4** If \( \mathcal{T} = \mathbb{R}^l \), by the assumption on \( \tau \) for all \( a > 0 \) and \( K = [-a,a]^l \)

\[
M \leq \mathbb{E}\{\|\Theta(t)\|^\tau \mathbb{I}(\|Y(t)\| > 1)\} \leq \mathbb{E}\{\|\Theta(t)\|^\tau\} < \infty, \forall t \in \mathcal{T}.
\]

Note in passing that for all \( \tau \in [0,\alpha] \) we have that

\[
\mathbb{E}\{\|\Theta(t)\|^\tau\} \leq \mathbb{E}\{\|\Theta(t)\|^\alpha\} = P\{\|\Theta(-t)\| \neq 0\} \leq 1
\]

and therefore there exists \( \tau \)'s that satisfy (2.10). Consequently, we can find a bounded positive càdlàg function \( p_N(t) > 0, t \in \mathcal{T} \) such that

\[
p_N(t) = 1, \quad t \in K
\]

and

\[
\int_{\mathcal{T}} \mathbb{E}\{\|\Theta(t)\|^\tau \mathbb{I}(\|Y(t)\| > 1)\}p_N(t) \lambda(dt) < \infty.
\]

Since \( \|\Theta(0)\|^\tau \mathbb{I}(\|Y(0)\| > 1) = 1 \) almost surely, in view of [20][Thm 2.1, Rem 2.2,iii] we conclude

\[
\int_{\mathcal{T}} \|\Theta(t)\|^\tau \mathbb{I}(\|Y(t)\| > 1)p_N(t) \lambda(dt) \in (0, \infty)
\]

almost surely. It follows as in [1] that for a \( \mathcal{T} \)-valued rv \( N \) with pdf \( p_N(t) \)

\[
Z_N(t) = \frac{[p_N(N)]^{1/\alpha}}{\left(\max_{h \in \mathbb{R}^l} p_N(h) \|B^NY(h)\|^\alpha \int_{\mathcal{T}} \|B^NY(h)\|^\tau \mathbb{I}(\|B^NY(h)\| > 1)p_N(h) \lambda(dh)\right)^{1/\alpha}}B^NY(t)
\]

belongs to \( \mathcal{K}_\alpha[Z] \) with \( R \) an \( \alpha \)-Pareto rv, and both \( R \) and \( N \) are independent of all other random elements.

We shall assume further that \( p_N(t) = 1 \) for all \( t \in K \). Set below

\[
W_K(Y) = \int_K \|\Theta(t)\|^\tau \mathbb{I}(\|Y(t)\| > 1) \lambda(dt)
\]
and

$$S_h(Y) = \int_T \|B^h \Theta(t)\|^2 \mathbb{I}(\|B^h Y(t)\| > 1) p_N(t) \lambda(dt), \quad M_h = \sup_{t \in T} [p_N(t)]^{1/\beta} \|B^h Y(t)\|.$$

We have that almost surely

$$(A.8) \quad W_K(Y) \in (0, \infty), \quad S_h(Y) \in (0, \infty), \quad M_h \in (0, \infty), \quad \forall h \in T.$$

By the Fubini-Tonelli theorem (set $\lambda_\alpha(dr) = \alpha r^{-\alpha-1} dr$)

$$\mathbb{E} \left\{ \sup_{s \in [-a,a]} \|Z(t)\|^2 \right\} = \mathbb{E} \left\{ \sup_{s \in [-a,a]} \|B^N Y(s)\|^\alpha \right\}^{1/[M_N]^{\alpha} S_N(Y)}$$

$$= \int_0^\infty \mathbb{E} \left\{ p_N(t) \mathbb{I}(r \sup_{s \in K} \|B^N Y(s)\| > 1) \right\} \lambda_\alpha(dr)$$

$$= \int_T \int_0^\infty \mathbb{E} \left\{ p_N(t) \mathbb{I}(r \sup_{s \in K} \|B^t Y(s)\| > 1) \frac{W_K(r B^t Y)}{W_K(r B^t Y)} \right\} \lambda_\alpha(dr) \lambda(dt),$$

where we used that for all $t \in T$ by [20][Thm 2.1, Rem 2.2,iii] almost surely

$$\left\{ r \sup_{s \in K} \|B^t Y(s)\| > 1 \right\} \subset \{W_K(r B^t Y) > 0\}$$

and $\mathbb{P}\{S_t > 0\} = 1$ since $\mathbb{P}\{\|Y(0)\| > 1\} = 1$ together with

$${\mathbb{P}\{W_K(r B^t Y) < \infty\} = 1}.$$ 

For all $s \in (0, 1)$ by [20][Thm 2.1, Rem 2.2,iii] we have

$$\mathbb{P}\{L_{s,z} > 0\} = 1, \quad L_{s,z} = S_z((M_{t,z})^{-1} Y), \quad M_{t,z} = M_{z}/[p_N(t)]^{1/\alpha}$$

since for all $s \in (0, 1), z \in T$ and

$$\sup_{t \in T} \mathbb{I}([p_N(t)]^{1/\beta} \|B^z Y(t)\| > M_{z}s) = \sup_{t \in T} \mathbb{I}(\|B^z Y(t)\| > M_{t,z}s) = 1.$$

Applying the Fubini-Tonelli theorem and (2.8)

$$\int_0^1 \mathbb{E}\{L_{s,z}(L_{s,z} = \infty)\} \lambda_\alpha(ds)$$

$$= \int_T \int_0^1 \mathbb{E}\{[B^z \Theta(t)]^\alpha \mathbb{I}(\|B^z Y(t)\| > M_{z}s)\} \lambda_\alpha(ds) p_N(t) \lambda(dt)$$

$$= \int_T \int_0^\infty \mathbb{E}\{[M_{t,z}]^{\alpha} \|B^z \Theta(t)\|^\alpha \mathbb{I}(\|B^z Y(t)\| > s)\} \lambda_\alpha(ds) \lambda(dt)$$

$$= \int_T \int_0^\infty \mathbb{E}\{[M_{t,z}]^{\alpha} \mathbb{I}(s \mathbb{I}(z-t) > 1) \mathbb{I}(S_t(Y) = \infty)\} \lambda_\alpha(ds) \lambda(dt)$$

$$= 0,$$

where the last equality follows from (A.8). Consequently, for all $s \in (0, 1)$ up to a set with Lebesgue measure equal zero and all $z \in T$

$$(A.9) \quad \mathbb{P}\{L_{s,z} \in (0, \infty)\} = 1.$$
Consequently, we have

\[ \mathbb{E}\left\{ \sup_{s \in [-a,a]} \|Z(t)\|^\alpha \right\} \]

\[ = \int_{z \in K} \int_{t \in T} \int_{0}^{\infty} p_N(t) \mathbb{E}\left\{ \frac{p_N(t) M_{zt}(Y)}{W_K(rB^tY)} \right\} \lambda(\alpha^r) \lambda(dt) \lambda(dz) \]

\[ = \int_{z \in K} \int_{t \in T} \int_{0}^{\infty} \mathbb{E}\left\{ \frac{p_N(t) M_{zt}(Y)}{W_K(B^tY)} \right\} \lambda(\alpha^r) \lambda(dt) \lambda(dz) \]

\[ = \int_{z \in K} \int_{t \in T} \int_{0}^{\infty} \mathbb{E}\left\{ \frac{p_N(t) M_{zt}(Y)}{W_K(B^tY)} \right\} \lambda(\alpha^r) \lambda(dt) \lambda(dz) \]

where in the third equality we used (2.8) and putting 1 in the integral bound above is justified by the fact that for all \( s > 1, z \in K, t \in T \) almost surely

\[ \mathbb{I}([p_N(t)]^{1/\alpha} \|B^tY(t)\| \geq M_t) = \frac{1}{\mathbb{P}[p_N(t)]^{1/\alpha} \|B^tY(t)\| > M_t]} = 0, \]

hence the proof is complete. \( \square \)

**Appendix B. \( \sigma \)-finite homogeneous measures**

Let \( V \) be equipped with a \( \sigma \)-field \( \mathcal{V} \) and let \( (s, t) \mapsto s \cdot t \in V \) be a paring \( s \in \mathbb{R}_+ = (0, \infty) \) and \( t \in V \). We shall assume that \( (V, \cdot, \mathcal{V}, \mathbb{R}_+, \cdot) \) is a measurable cone, i.e.,

E1) \( 1 \cdot t = t \) and \( (s_1 s_2) \cdot t = s_1 (s_2 \cdot t) = s_2 (s_1 \cdot t) \) for all \( s_1, s_2 \in [0, \infty), t \in V \);

E2) \( (s, t) \mapsto s \cdot t \in \mathcal{B}((0, \infty)) \times \mathcal{V} / \mathcal{V} \) measurable.

If \( Z \) is a random element on \( V \) defined on a complete probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) set

\[ \nu_Z(A) = \mathbb{E}\left\{ \int_{0}^{\infty} \mathbb{I}(z \cdot Z \in A) \lambda_\alpha(dz) \right\} = \int_{0}^{\infty} \mathbb{P}\{z \cdot Z \in A\} \lambda_\alpha(dz), \quad A \in \mathcal{V} \]

for some \( \alpha > 0 \) (recall \( \lambda_\alpha(dr) = \alpha^{-\alpha-1} dr \)). By the assumption that \( (s, t) \mapsto s \cdot t \) is jointly measurable, the measure \( \nu_Z \) is the push-forward of the map \( (s, t) \mapsto s \cdot t \) with respect to the product measure \( \mathbb{P} \circ Z^{-1}(dv) \times \lambda_\alpha(dr) \), and thus \( \nu_Z \) is well-defined satisfying further

\[ \nu_Z(s \cdot A) = s^{-\alpha} \nu_Z(A), \quad \forall s > 0, \quad \forall A \in \mathcal{V}. \]

The next result is a variation of [8][Thm B.2.5].

**Proposition B.1.** Suppose that there exist a \( \mathcal{V} / \mathcal{B}([0, \infty]) \)-measurable map \( \psi : V \mapsto [0, \infty] \) such that \( \psi(s \cdot t) = s\psi(t), \forall (s, t) \in (0, \infty) \times V \) and \( F = \{ v \in V : \psi(v) \in \{0, \infty\} \} \neq V \).
(i) If $Z$ is a $V$-valued random element such that $\mathbb{P}\{\psi(Z) = 1\} = 1$ or $\mathbb{P}\{\psi(Z) \in (0, \infty)\} = 1$ and $\mathbb{E}\{(\psi(Z))^{\alpha}\} \in (0, \infty), \alpha > 0$, then $\nu_Z$ is a non-zero $\sigma$-finite measure on $(V, \mathcal{V})$ such that $\nu(F) = 0$ and moreover for all maps $H : V \to \mathbb{R}$ which are $\mathcal{V}/\mathcal{B}(\mathbb{R})$-measurable

\begin{equation}
\nu_Z[H] = \int_V H(x)\nu_Z(dx) = \int_0^\infty \mathbb{E}\{H(z \cdot Z)\}\lambda_\alpha(dz);
\end{equation}

(ii) Conversely, let a positive measure $\nu$ on $\mathcal{V}$ satisfy (B.2) for some $\alpha > 0$ and

$$\nu(\{v \in V : \psi(v) > 1\}) = 1, \quad \nu(v \in V : \psi(v) = 0) = 0.$$ 

Suppose further that $\psi$ is finite and continuous. If further $V$ is a Polish metric space and $\mathcal{V}$ its Borel $\sigma$-field, then there exists a $V$-valued random element $Z$ such that $\mathbb{P}\{\psi(Z) = 1\} = 1$ and $\nu = \nu_Z$.

**Proof of Proposition B.1** Item (i): If $\mathbb{P}\{\psi(Z) \in (0, \infty)\} = 1$ and $c = \mathbb{E}\{(\psi(Z))^{\alpha}\} \in (0, \infty)$, then $\nu_Z = \nu_{Z^*}$, where $Z^*$ is the law of $Z/\psi(Z)$ with respect to

$$\tilde{\mathbb{P}}(A) = c^{-1}\mathbb{E}\{(\psi(Z))^{\alpha}\} \mathbb{1}(A), \quad A \in \mathcal{F}.$$ 

We have $\psi(Z^*) = 1$ almost surely. We consider now the case $\psi(Z) = 1$ almost surely. The assumption that $\psi$ is measurable and 1-homogeneous together with Items E1 and E2 (which guarantee that $\nu_Z$ is a measure on $(V, \mathcal{V})$) imply for all $a > 0$ (set $B_a = \{v \in V : \psi(v) \in (a, \infty)\}$)

$$\nu_Z(B_a) = \int_0^\infty \mathbb{P}\{\psi(Z) \in (a/s, \infty)\}\lambda_\alpha(ds) = \int_0^\infty \mathbb{1}(s > a)\lambda_\alpha(ds) = a^{-\alpha} < \infty,$$

where we used that $\mathbb{P}\{\psi(Z) = 1\} = 1$. By the homogeneity of $\psi$ we have $cF = F$ for all $c \in (0, \infty)$ and by the measurability of $\psi$ we have $F \in \mathcal{V}$. Hence

$$\nu_Z(F) = \int_0^\infty \mathbb{P}\{s \cdot Z \in \psi^{-1}(\{0, \infty\})\}\lambda_\alpha(ds) \leq \int_0^\infty \mathbb{P}\{\psi(Z) \neq 1\}\lambda_\alpha(ds) = 0$$

and further

$$V_F = V \setminus F = \bigcup_{i=1}^\infty \{v \in V : \psi(v) \in (1/i, \infty)\} = \bigcup_{i=1}^\infty B_{1/i}.$$ 

By the measurability of $\psi$ we have that $B_{1/i}$’s belong to $\mathcal{V}$, hence since further $\nu(B_{1/i}) = i^{\alpha} < \infty$, then $\nu_Z$ is $\sigma$-finite and thus (B.3) follows by (B.1) and the Fubini-Tonelli Theorem.

Item (ii): The assumption $\nu(\{v \in V : \psi(v) > 1\}) = \nu(B_1) = 1$ and (B.2) imply $\nu(B_{1/i}) = i^{\alpha}, i > 0$. Hence $\nu(\{v \in V : \psi(v) = \infty\}) = 0$. Using further that $\nu(\{v \in V : \psi(v) = 0\}) = 0$ we have that $\nu$ is $\sigma$-finite. Let $V_* = \{x \in V : \psi(x) = 1\}$ which belongs to $\mathcal{V}$ since $\psi$ is measurable and define the trace $\sigma$-field on $V_*$ by

$$\mathcal{V}_* = \{B \cap V_F : B \in \mathcal{V}\}.$$ 

The measurability of $\psi$ yields that the map $T(x) = (\psi(x), x/\psi(x))$ is $\mathcal{V}/(\mathcal{B}(C_{(0, \infty)}) \times \mathcal{V}_*)$ measurable. By definition, it is further bijective with inverse map $T^{-1}(s, x) = s \cdot x$. Define the probability measure on $\mathcal{V}_*$ by

\begin{equation}
\sigma(B) = \nu(\{x \in V_F : \psi(x) > 1, x/\psi(x) \in B\})/\nu(\{x \in V_F : \psi(x) > 1\}), \quad \forall B \in \mathcal{V}_*\end{equation}

and let $Z$ be a $V_*$-valued random element with law $\sigma$. As in [9][p. 3890], applying the Fubini-Tonelli theorem (the $\sigma$-finiteness of $\nu$ is crucial), by the 1-homogeneity of $\psi$, the measurability of the bijection $T^{-1}$, the property (B.2), $\psi(x) \in (0, \infty)$ iff $x \not\in F$, the definition of $\sigma$, and $\nu(F) = 0$ we obtain

$$\nu(A) = \int_V \mathbb{1}(x \in A)\nu(dx) = \int_{V_F} \mathbb{1}(x \in A)\nu(dx)$$
for all $A \in \mathcal{Y}$ establishing the proof.

\begin{align*}
\int_{V_F} \mathbb{I}(x \in A)\mathbb{I}(0 < \psi(x) < \infty)\frac{[\psi(x)]^{-\alpha}}{[\psi(x)]^{-\alpha}} \nu(dx) \\
= \int_{V_F} \int_0^\infty \mathbb{I}(x \in A)\mathbb{I}(s\psi(x) \in (1, \infty))[\psi(x)]^{-\alpha} \lambda_\alpha(ds) \nu(dx) \\
= \int_0^\infty \int_{V_F} \mathbb{I}(r \cdot x \in A, \psi(x) \in (1, \infty))[\psi(x)]^{-\alpha} \nu(dx) \lambda_\alpha(dr) \\
= \int_0^\infty \int_{V_F} \mathbb{I}((r/\psi(x)) \cdot x \in A, \psi(x) \in (1, \infty)) \nu(dx) \lambda_\alpha(dr) \\
= \int_0^\infty \int_{V_F} \mathbb{I}(r \cdot u \in A) \sigma(du) \lambda_\alpha(dr) \\
= \int_0^\infty \mathbb{P}\{r \cdot Z \in A\} \lambda_\alpha(dr)
\end{align*}

Remark B.2. \begin{enumerate}
\item Suppose that there exists 1-homogeneous $\mathcal{Y}/\mathcal{B}(\mathbb{R})$-measurable maps $\psi_k : V \mapsto [0, \infty], k \in \mathbb{N}$ and $Z$ is a $V$-valued random element defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assume that the following holds

\begin{equation}
\mathbb{E}\{[\psi_1(Z)]^\alpha\} \in (0, \infty), \quad \mathbb{E}\{[\psi_k(Z)]^\alpha\} \in [0, \infty), \quad \forall k \in \mathbb{N}, \quad \mathbb{P}\left\{\sup_{k \in \mathbb{N}} \psi_k(Z) > 0\right\} = 1.
\end{equation}

Hence we can find positive $w_i$'s such that for $\psi(v) = \max_{k \in \mathbb{N}} \psi_k(w_k(v)), v \in V$ we have

$$
\mathbb{E}\{[\psi(Z)]^\alpha\} \in (0, \infty),
$$

which is possible by (B.5). Set $F = \{v \in V : \max_{k \in \mathbb{N}} \psi_k(v) \in \{0, \infty\}\}$ and let $\nu = \nu_Z$ as specified in (B.1). By this choice and (B.5)

$$
\mathbb{P}\{\psi(Z) \in (0, \infty)\} = 1, \quad \nu_Z(F) = 0.
$$

In view of Proposition B.1, Item (i) the measure $\nu_Z$ satisfies (B.2) and is $\sigma$-finite.

\item Under the assumptions of Proposition B.1, Item (ii) if $V$ is a Polish space with $\mathcal{Y}$ its Borel $\sigma$-field, then since $V_*$ is by definition closed, it is also Polish. Hence by [50][Lem p. 1276] the random element $Z$ with law $\sigma$ can be defined on any given non-atomic complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
\end{enumerate}

Example B.3. Consider the space $D^*$ as in Example 3.5 and set $\psi_1(f) = \|f(t_i)\|, t_i \in \mathbb{D}$ for some countable dense set $\mathbb{D}$ of $T$. If $Z$ is a $D^*$-valued random element that satisfies $\mathbb{P}\{\sup_{t \in T} \|Z(t)\| > 0\} = 1$ and $\mathbb{E}\{\|Z(t)\|^\alpha\} \in (0, \infty)$, then (B.5) holds taking $\psi_j(f) = \|f(j)\|, j \in \mathbb{D}, f \in D^*$. For this case Item E2) is satisfied since this mapping is continuous, see [14]. Hence by Remark B.2, Item (i) $\nu_Z$ satisfies (B.2), (B.3) and is $\sigma$-finite. Moreover

$$
\nu_Z\{\{f \in D^* : \sup_{t \in \mathbb{D}} \|f(t)\| = 0\}\} = 0
$$

and if further the 1-homogeneous continuous function $\|\cdot\|$ satisfies $\|x\| = 0$ iff $x = 0$ for all $x \in \mathbb{R}^l$, then $F = \{0\}$, with 0 the zero function implying $\nu_Z(\{0\}) = 0$.

Acknowledgements: Many thanks go to Krzys Dębicki, Dima Korshunov, Rafal Kulik, Thomas Mikosch, Philippe Soulier, Georgy Shevchenko and Dima Zaporozhets for numerous discussions and comments. Support from the SNSF Grant 1200021-196888 is kindly acknowledged.
References

[1] E. Hashorva, “Shift-invariant homogeneous classes of random fields,” arXiv:2111.00792v4, 2021.
[2] I. I. Gihman and A. V. Skorohod, The theory of stochastic processes. I, vol. 210 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin-New York, english ed., 1980. Translated from the Russian by Samuel Kotz.
[3] Z. Kabluchko, M. Schlather, and L. de Haan, “Stationary max-stable fields associated to negative definite functions,” Ann. Probab., vol. 37, pp. 2042–2065, 2009.
[4] B. M. Brown and S. I. Resnick, “Extreme values of independent stochastic processes,” J. Appl. Probab., vol. 14, pp. 732–739, 1977.
[5] C. Dombry and Z. Kabluchko, “Ergodic decompositions of stationary max-stable processes in terms of their spectral functions,” Stochastic Processes and their Applications, vol. 127, no. 6, pp. 1763–1784, 2017.
[6] P. Soulier, “The tail process and tail measure of continuous time regularly varying stochastic processes,” Extremes, vol. 25, no. 1, pp. 107–173, 2022.
[7] H. Planinić and P. Soulier, “The tail process revisited,” Extremes, vol. 21, no. 4, pp. 551–579, 2018.
[8] R. Kulik and P. Soulier, Heavy tailed time series. Cham: Springer, 2020.
[9] C. Dombry, E. Hashorva, and P. Soulier, “Tail measure and spectral tail process of regularly varying time series,” Ann. Appl. Probab., vol. 28, no. 6, pp. 3884–3921, 2018.
[10] B. Basrak and H. Planinić, “Compound Poisson approximation for regularly varying fields with application to sequence alignment,” Bernoulli, vol. 27, no. 2, pp. 1371–1408, 2021.
[11] Y. Cissokho and R. Kulik, “Estimation of cluster functionals for regularly varying time series: sliding blocks estimators,” Electron. J. Stat., vol. 15, no. 1, pp. 2777–2831, 2021.
[12] C. Youssouph and R. Kulik, “Estimation of cluster functionals for regularly varying time series: runs estimators,” arXiv:2109.02164, 2021.
[13] G. Buriticá, T. Mikosch, and O. Wintenberger, “Threshold selection for cluster inference based on large deviation principles,” arXiv:2106.12822, 2021.
[14] M. Bladt, E. Hashorva, and G. Shevchenko, “Tail measures and regular variation,” Electron. J. Probab., vol. 27, pp. Paper No. 64, 43, 2022.
[15] T. Owada and G. Samorodnitsky, “Tail measures of stochastic processes of random fields with regularly varying tails. Technical report,” 2012.
[16] H. Planinić, “Palm theory for extremes of stationary regularly varying time series and random fields,” arXiv:2104.03810, 2021.
[17] G. Last, “Tail processes and tail measures: An approach via Palm calculus,” arXiv:2112.15380, 2021.
[18] B. Basrak and J. Segers, “Regularly varying multivariate time series,” Stochastic Process. Appl., vol. 119, no. 4, pp. 1055–1080, 2009.
[19] E. Hashorva, “Representations of max-stable processes via exponential tilting,” Stochastic Process. Appl., vol. 128, no. 9, pp. 2952–2978, 2018.
[20] K. Bisewski, E. Hashorva, and G. Shevchenko, “The harmonic mean formula for random processes,” arXiv:2106.11707, to appear in Stochastic Analysis, 2021.
[21] J. Segers, Y. Zhao, and T. Meinguet, “Polar decomposition of regularly varying time series in star-shaped metric spaces,” *Extremes*, vol. 20, no. 3, pp. 539–566, 2017.
[22] L. Wu and G. Samorodnitsky, “Regularly varying random fields,” *Stochastic Process. Appl.*, vol. 130, no. 7, pp. 4470–4492, 2020.
[23] E. Hashorva, “New representation for the extremal index of jointly regularly varying stationary time series and random fields,” *Unpublished Manuscript*, 2016.
[24] S. Berman, “Sojourns and extremes of stationary processes,” *Ann. Probab.*, vol. 10, no. 1, pp. 1–46, 1982.
[25] S. Berman, *Sojourns and Extremes of Stochastic Processes*. The Wadsworth & Brooks/Cole Statistics/Probability Series, Pacific Grove, CA: Wadsworth & Brooks/Cole Advanced Books & Software, 1992.
[26] K. Dębicki, Z. Michna, and X. Peng, “Approximation of Sojourn Times of Gaussian Processes,” *Methodol. Comput. Appl. Probab.*, vol. 21, no. 4, pp. 1183–1213, 2019.
[27] E. Hashorva, “On extremal index of max-stable random fields,” *Lith. Math. J.*, vol. 61, no. 2, pp. 217–238, 2021.
[28] K. Dębicki and E. Hashorva, “On extremal index of max-stable stationary processes,” *Probab. Math. Statist.*, vol. 37, no. 2, pp. 299–317, 2017.
[29] A. B. Dieker and B. Yakir, “On asymptotic constants in the theory of extremes for Gaussian processes,” *Bernoulli*, vol. 20, no. 3, pp. 1600–1619, 2014.
[30] K. Dębicki and E. Hashorva, “Approximation of Supremum of Max-Stable Stationary Processes & Pickands Constants,” *J. Theoret. Probab.*, vol. 33, no. 1, pp. 444–464, 2020.
[31] G. Samorodnitsky, “Extreme value theory, ergodic theory and the boundary between short memory and long memory for stationary stable processes,” *Ann. Probab.*, vol. 32, no. 2, pp. 1438–1468, 2004.
[32] G. Samorodnitsky, “Maxima of continuous-time stationary stable processes,” *Adv. in Appl. Probab.*, vol. 36, no. 3, pp. 805–823, 2004.
[33] J. M. P. Albin, “On extremal theory for stationary processes,” *Ann. Probab.*, vol. 18, no. 1, pp. 92–128, 1990.
[34] J. Albin and H. Choi, “A new proof of an old result by Pickands,” *Electron. Commun. Probab.*, vol. 15, pp. 339–345, 2010.
[35] J. M. P. Albin, “Extremes of diffusions over fixed intervals,” *Stochastic Process. Appl.*, vol. 48, no. 2, pp. 211–235, 1993.
[36] B. Basrak, H. Planinic, and P. Soulier, “An invariance principle for sums and record times of regularly varying stationary sequences,” *https://arxiv.org/abs/1609.00687*, 2016.
[37] J. Beirlant, Y. Goegebeur, J. Teugels, and J. Segers, *Statistics of extremes*. Wiley Series in Probability and Statistics, John Wiley & Sons, Ltd., Chichester, 2004.
[38] H. Drees, “Statistical inference on a changing extremal dependence structure,” *arXiv.2201.06389*, 2022.
[39] A. Rønn-Nielsen and M. Stehr, “Extremal clustering and cluster counting for spatial random fields,” 2022.
[40] S. Bai and Y. Wang, “Tail processes for stable-regenerative model,” *arXiv:2110.07499*, 2021.
[41] Z. Kabluchko and Y. Wang, “Limiting distribution for the maximal standardized increment of a random walk,” *Stochastic Process. Appl.*, vol. 124, no. 9, pp. 2824–2867, 2014.
[42] K. Dębicki and K. Tabiś, “Pickands-Piterbarg constants for self-similar Gaussian processes,” *Probab. Math. Statist.*, vol. 40, no. 2, pp. 297–315, 2020.
[43] B. Yakir, *Extremes in random fields*. Wiley Series in Probability and Statistics, John Wiley & Sons, Ltd., Chichester; Higher Education Press, Beijing, 2013. A theory and its applications.

[44] L. de Haan, “A spectral representation for max-stable processes,” *Ann. Probab.*, vol. 12, no. 4, pp. 1194–1204, 1984.

[45] A. B. Dieker and T. Mikosch, “Exact simulation of Brown-Resnick random fields at a finite number of locations,” *Extremes*, vol. 18, pp. 301–314, 2015.

[46] I. Molchanov and K. Stucki, “Stationarity of multivariate particle systems,” *Stochastic Process. Appl.*, vol. 123, no. 6, pp. 2272–2285, 2013.

[47] K. Dębicki, E. Hashorva, and Z. Michna, “On the continuity of Pickands constants,” *J. Appl. Probab.*, vol. 59, no. 1, pp. 187–201, 2022.

[48] R. Takloo-Bighash, *A Pythagorean introduction to number theory*. Undergraduate Texts in Mathematics, Springer, Cham, 2018. Right triangles, sums of squares, and arithmetic.

[49] D. Dadush and O. Regev, *Lattices, Convexity and Algorithms: Fundamental Domains, Lattice Density and Minkowski Theorems*. https://cs.nyu.edu/courses/spring13/CSCI-GA.3033-013/lectures/lecture-3.pdf, 2013.

[50] V. S. Varadarajan, “On a problem in measure-spaces,” *Ann. Math. Statist.*, vol. 29, pp. 1275–1278, 1958.

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