Poisson-Lie sigma models over low dimensional real Poisson-Lie groups

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Abstract

The Poisson-Lie sigma models over nonsemisimple low dimensional real Poisson-Lie groups are investigated. We find two sided models on two, three and some four dimensional Poisson-Lie groups where the Poisson-Lie sigma models over Poissin-Lie groups $G$ and its dual $\tilde{G}$ are topological sigma models or BF gauge models.
1 Introduction

Poisson sigma models originally were obtained from nonlinear extension of classical gauge theory by Ikeda [1]; then independently obtained from generalization of 2d gravity-Yang-Mills systems by Schaller and Strobl [2]. Poisson-sigma model is a 2d topological field theory with Poisson manifold as target space. These models have played a central role and given unified structure in the study of two dimensional gauge theories. They are related to 2d gravity [3], 2d Yang-Mills [4], topological sigma models [5], BF models [6] and G/G WZW models [7] as special cases. Poisson sigma model over Poisson-Lie groups are called Poisson-Lie sigma models; these models over complex simple Lie groups $G$ in the case that $r$ matrix is factorizable and triangular are studied previously in [8],[9] respectively.

Here we study Poisson-Lie sigma models over nonsemisimple low dimensional real Lie groups. Because for most of the Lie algebras of these Lie groups, ad-invariant metrics are nondegenerate; hence here we use other new form of the action for these Poisson-Lie sigma models. The results are different from general results of [8] and [9] for some Lie groups. For example, in [8] it is shown that the models over dual Lie groups $\tilde{G}$ for the triangular $r$ matrices are BF gauge models but these are not true in general for our examples. Furthermore, we find two sided models (models over $G$ and its dual $\tilde{G}$) for the case of bi-$r$-matrix bialgebras; such that the Poisson-Lie sigma models over Lie groups $G$ and its dual $\tilde{G}$ are topological sigma models or BF gauge models.

The paper is organized as follows: in section 2 we review Poisson sigma models and Poisson-Lie sigma models and present new form of the action for the case of Poisson-Lie sigma models. In section 3 after calculation of $r$-matrices and Poisson structures for two dimensional Poisson-Lie groups, we obtain two sided topological sigma models over $G$ and $\tilde{G}$. In section 4 we find topological sigma models and BF gauge models related to three dimensional Lie bialgebras[10]. The models over $(II.i, V)$ are two sided and the models over Poisson-Lie group $III$ with Lie bialgebra $(III, III.ii)$ are topological sigma model and also BF gauge model. Furthermore, we see that there are not two dimensional Yang-Mills and 2d gravity models over three dimensional Poisson-Lie groups. In section 5 we present two examples of four dimensional real Lie bialgebras and find a model which is two sided and models over $G$ and $\tilde{G}$ are BF gauge model and also topological sigma model.

2 Poisson-Lie sigma models

2.1 Review of Poisson sigma models

Poisson sigma model is a two dimensional topological sigma model with the Poisson manifold $M$ as the target space. The fields of the model are scalar bosonic fields $x^\mu : \Sigma \to M$ ($x^a$ are coordinates of $M$ and $\Sigma$ is the world sheet) and the fields $A_\mu$ are 1-forms on $\Sigma$ with values on $T^*M$ so that with coordinates $\xi^a$ on $\Sigma$, $A$ can be written as $A = A_{\alpha \mu} d\xi^\alpha \wedge dx^\mu =$
(A_{\alpha\mu}\partial_\beta x^\mu) d\xi^\alpha \wedge d\xi^\beta. In these coordinates the action of Poisson-sigma model is given by [2]

\[ S_{\text{top}} = \int_{\Sigma} d\xi^\alpha \wedge d\xi^\beta [A_{\alpha\mu}(\xi)x^{\mu}_\beta(\xi) + \frac{1}{2}P^{\mu\nu}(x(\xi))A_{\alpha\mu}(\xi)A_{\beta\nu}(\xi)], \]  

such that with the notation \( A_\mu(\xi) = A_{\alpha\mu}(\xi)d\xi^\alpha \), takes the following form:

\[ S_{\text{top}} = \int_{\Sigma} A_\mu \wedge dx^\mu + \frac{1}{2} P^{\mu\nu} A_\mu \wedge A_\nu. \]  

The different two dimensional topological models such as topological sigma models, BF gauge models, two dimensional gravity, two dimensional Yang-Mills and G/G WZW models are a special examples of Poisson sigma models. For example if the Poisson structures on \( M \) is nondegenerate (i.e when \( M \) is a symplectic manifolds) then by use of equation of motion for \( A_\mu \) we have

\[ A_\mu = \Omega_{\mu\nu} dx^\nu, \]  

where \( \Omega_{\mu\nu} \) is the symplectic structure (inverse of \( P_{\mu\nu} \)). Then in this case the Poisson-sigma model is a Witten’s topological sigma models [5] with the following action:

\[ S = \int_{\Sigma} \Omega_{\mu\nu} dx^\mu \wedge dx^\nu. \]  

On the other hand, if Poisson structures are linear in terms of coordinate \( M \), i.e \( P_{\mu\nu} = f^{\mu\nu}_\lambda x^\lambda \) where \( f^{\mu\nu}_\lambda \) is the structure constant of some Lie algebras \( g \) such that its dual \( g^* \) is identified with \( M[2] \); then the action (1) can be rewritten as the form of a BF gauge theory

\[ S = \int_{\Sigma} x^\lambda (dA_\lambda + \frac{1}{2} f^{\mu\nu}_\lambda A_\mu \wedge A_\nu) = \int_{\Sigma} x^\lambda F_\lambda, \]  

where \( F_\lambda \) is the standard curvature 2-form of the gauge group associated to Lie algebra \( g \). Furthermore, by choosing \( C \) as a quadratic casimir of \( g \) then the following action:

\[ S = S_{\text{BF}} + \int C(x(\xi)), \]  

may be seen to provide the first order formulation of a 2d Yang-Mills theory [2].

To obtain the two dimensional gravity it is enough to choose Poisson structure as a nonlinear Poisson tensor on \( M \) [2]. By choosing

\[ P^{\mu\nu} = \varepsilon^{\mu\nu\lambda} u_\lambda(x), \]

\[ u_a = \eta_{ab} x^b, \quad a, b \in \{1, 2\}, \]

\[ u_3 = V(x^a \eta_{ab} x^b, x^3), \quad \eta_{ab} \equiv \text{diag}(1, \pm 1), \]  

and inserting these in the action (1) we have

\[ S = \int x_c (de^c + \varepsilon_{ab} \eta^{ac} e^b \wedge \omega) + x^3 d\omega + V \varepsilon_{abc} e^a \wedge e^b, \]  

where the first two 1-forms \( A_a \) are interpreted as the zweibein \( e_a \) and the third one \( A_3 \) as the spin connection \( \omega \) of a two dimensional gravity theory. In [11] it’s shown that gauged G/G WZW models for semisimple groups \( G \) are equivalent to a Poisson-sigma model with a topological term (if Gauss decomposition of \( G \) is complete then the topological term is not needed).
2.2 Poisson-Lie sigma models

Now, when the Poisson manifold is a Lie group (i.e \( G \) is a Poisson-Lie group) then the Poisson-Sigma model (2) can be rewritten as the Poisson-Lie sigma model. In [8] the action of this model is written as follow:

\[
S_1 = \int_{\Sigma} (\langle dg g^{-1} \wedge A \rangle - \frac{1}{2} \langle A \wedge (r - Ad r Ad g)A \rangle),
\]

where \( g \in G \), \( A = A_i d\xi^i X_i \) and \( r \in \mathfrak{g} \otimes \mathfrak{g} \) is a classical \( r \) matrix with \( \mathfrak{g} \) as the Lie algebra of \( G \) and \( \{X^i\} \) as a basis of \( \mathfrak{g} \). Note that the above action can be applied for simple or nonsemisimple Lie group \( G \) with ad-invariant symmetric bilinear nondegenerate form \( \langle X_i, X_j \rangle = G_{ij} \) on the Lie algebra \( \mathfrak{g} \). When the metric \( G_{ij} \) of Lie algebra is degenerate then the above action is not good. Here we use the following action instead of the above one\(^1\):

\[
S_2 = \int_{\Sigma} \langle dx^i \wedge A_i - \frac{1}{2} P^{ij} A_i \wedge A_j \rangle,
\]

where \( x^i \) are Lie group parameters with parametrization (e.g)

\[
\forall g \in G, \quad g = e^{x^1 X_1} e^{x^2 X_2} ..., \quad (11)
\]

and \( P^{ij} \) is the Poisson structure on Lie group which for coboundary Poisson-Lie groups it is obtained from Sklyanin bracket as follows:

\[
\{f_1, f_2\} = \sum_{i,j} r^{ij} ((X^L_i f_1) (X^L_j f_2) - (X^R_i f_1) (X^R_j f_2)) \quad \forall f_1, f_2 \in C^\infty(G),
\]

(12)

where in the above relation \( r^{ij} \) is the classical \( r \) matrix i.e \( r : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g} \), that is the solution of classical or modified Yang-Baxter equation and \( X^L_i (X^R_i) \) are left(right) invariant vector fields. Note that the form of the action (9) in this parameterization can be rewritten as:

\[
S = \int_{\Sigma} R^k_m [dx^m \wedge A^i - \frac{1}{4} (P^{mn} R^k_n G_{ij} A^j) \wedge A^i] G_{ik},
\]

(13)

where \( R^k_m \) are vielbiens and inverse to the coefficients of right invariant vector fields. To show the differences between action (9) and (10) it is better to use an example. For Poisson-Lie group with Lie bialgebra structure \( (V \oplus R, V.i \oplus R) \) we have[12](See relations (52-55)):

\[
r = X_1 \wedge X_4, \quad G = \begin{pmatrix}
n & 0 & 0 & m \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
m & 0 & 0 & p
\end{pmatrix}, \quad P_{12} = x_2, \quad P_{34} = -x_3, \quad n, m, p \in \mathbb{R}
\]

(14)

and the form of actions (9) and (10) are obtained as follows:

\[
S_1 = \int_{\Sigma} ndx^1 \wedge A^1 + mdx^1 \wedge A^4 + mdx^4 \wedge A^1 + pdx^4 \wedge A^4,
\]

(15)

\(^1\)Note that the indices in action (2) are geometrical coordinate indices while in action (10) are Lie algebraic indices.
We see that in this example the Poisson structure does not appear in the action (9) i.e (15) (because of the effect of degenerate metric on Lie algebra); and it is not good action for Poisson-Lie sigma model for this case. In the next sections we use action (10) to obtain the Poisson-Lie sigma models over all two, three and some four dimensional Poisson-Lie groups.

3 Models with two dimensional Poisson-Lie groups

The real two dimensional Lie bialgebras are classified in [13] as four classes (Abelian, semi-Abelian and type A and B non-Abelian Lie bialgebras). We find that only type B non-Abelian Lie bialgebras is coboundary. The commutation relations for this Lie bialgebra are as follows:

\[
[X_1, X_2] = X_2, \quad [\tilde{X}^1, \tilde{X}^2] = \tilde{X}^1,
\]

\[
[X_1, \tilde{X}^1] = X_2, \quad [X_1, \tilde{X}^2] = -X_1 - \tilde{X}^2, \quad [X_2, \tilde{X}^2] = \tilde{X}^1.
\]

By use of method [10] we find that it is bi-r matrix bialgebras with \( r \) and \( \tilde{r} \) matrices and we obtain Poisson structure as follows:

\[
r = X_1 \wedge X_2, \quad \tilde{r} = -\tilde{X}^1 \wedge \tilde{X}^2,
\]

\[
P_{12} = \frac{x^1}{1 + x^1}, \quad \tilde{P}^{12} = \frac{\tilde{x}_1}{1 + \tilde{x}_1}.
\]

Now the actions \( S_2 \) and \( \tilde{S}_2 \) for this models have the following forms\(^2\):

\[
S = \int_{\Sigma} \left[ dx^1 \wedge A^1 + dx^2 \wedge A^2 - \frac{x^1}{1 + x^1} A^1 \wedge A^2 \right],
\]

\[
\tilde{S} = \int_{\Sigma} \left[ d\tilde{x}_1 \wedge \tilde{A}_1 + d\tilde{x}_2 \wedge \tilde{A}_2 - \frac{\tilde{x}_1}{1 + \tilde{x}_1} \tilde{A}_1 \wedge \tilde{A}_2 \right].
\]

These actions are topological sigma models, because the Poisson structures are nondegenerate and by use of equation of motions for \( A^i \) i.e

\[
\frac{\partial S}{\partial A^i} = dx^1 - \frac{x^1}{1 + x^1} A^2 = 0,
\]

\[
\frac{\partial S}{\partial A^i} = dx^2 - \frac{x^1}{1 + x^1} A^1 = 0,
\]

and similarly for \( \tilde{A}_i \) we can integrate them and obtain the following actions:

\[
S = - \int \frac{1 + x^1}{x^1} dx^1 \wedge dx^2,
\]

\(^2\)Here we use the same parameterization for \( g^* \) as \( g \) i.e in relation (11) the tilde parameters and generators must be replaced with untilded ones.
\[ \tilde{S} = - \int_{\Sigma} \frac{1 + \tilde{x}_1}{\tilde{x}_1} d\tilde{x}_1 \wedge d\tilde{x}_2. \]  

Therefore Poisson-Lie sigma models on real two dimensional Poisson-Lie groups \( G \) and its dual \( \tilde{G} \) are two sided topological sigma models. Note that there are two singular points \((x^1 = 0, x^1 = -1)\) for symplectic form \( \Omega_{\mu \nu} = \frac{x^1 + 1}{x^1} \) and its dual.

4 Models with three dimensional Poisson-Lie groups

The real three dimensional Lie bialgebras are classified in [14]. Their classical \( r \) matrices and their types (triangular, quasitriangular or factorizable) and also Poisson structures are obtained in [10]. Now by use of those informations one can obtain Poisson-Lie sigma models over these Poisson-Lie groups. We perform this work in the following subsections.

4.1 Topological Sigma Models

We know that because the number of dimension (three) is odd then the Poisson structure of three dimensional real Poisson-Lie groups are degenerate. For this reason one can not obtain topological sigma models over these Poisson-Lie groups. But one can obtain topological sigma models over extremal surfaces (surfaces that specifies from equations of motion) because the Poisson structures on these surfaces (symplectic leaves) are nondegenerate. The three dimensional Lie bialgebras that one can obtain such models over them are \((V, II.i)\), \((III, III.ii)\), \((III, II)\), \((VI, II)\), \((VII, II.i)\) and \((VII, II.ii)\) [10]. In general the Poisson structures for these Lie bialgebras have the following form:

\[ \{x_1, x_2\} = 0, \quad \{x_2, x_3\} = P_{23}, \quad \{x_3, x_1\} = 0. \]  

For these Poisson structures the action \( S_2 \) has the following form:

\[ S = \int_{\Sigma} [A_{1\alpha} \partial_\beta x^1 + A_{2\alpha} \partial_\beta x^2 + A_{3\alpha} \partial_\beta x^3 + P^{23} A_{2\alpha} A_{3\beta}] \varepsilon^{\alpha\beta} d\sigma d\tau, \]  

such that where the equations of motion for \( A_i \) are

\[ \frac{\partial S}{\partial A_{1\alpha}} = \partial_\beta x^1 = 0, \quad \frac{\partial S}{\partial A_{2\alpha}} = \partial_\beta x^2 + P^{23} A_{3\beta} = 0, \quad \frac{\partial S}{\partial A_{3\alpha}} = \partial_\beta x^2 - P^{23} A_{2\beta} = 0. \]  

From these relations we have

\[ x^1 = \text{cte.}, \quad A_{3\beta} = -\frac{1}{P^{23}} \partial_\beta x^2, \quad A_{2\beta} = \frac{1}{P^{23}} \partial_\beta x^3. \]  

Then the general form of the action is

\[ S = \int_{\Sigma} \left[ \frac{1}{P^{23}} \partial_\alpha x^3 \partial_\beta x^2 - \frac{1}{P^{23}} \partial_\alpha x^2 \partial_\beta x^3 - \frac{1}{P^{23}} \partial_\beta x^2 \partial_\alpha x^3 \varepsilon^{\alpha\beta} \right] d\sigma d\tau \]
This model is the topological sigma model over symplectic leaf \( x^1 = cte \). Now, for the above mentioned Lie bialgebras we have the following Poisson structures \([10]\) and actions respectively;

\[(V, II.i) : \{x_1, x_2\} = 0, \quad \{x_2, x_3\} = \frac{1}{2}(e^{2x_1} - 1), \quad \{x_3, x_1\} = 0,\]

\[S = -\int_{\tau} \frac{2}{e^{2x_1} - 1} \partial_{\alpha} x^2 \partial_{\beta} x^3 \varepsilon^{\alpha \beta} d\sigma d\tau \quad (x_1 = cte),\]

\[(III, III.ii) : \{x_1, x_2\} = 0, \quad \{x_2, x_3\} = x_2 + x_3, \quad \{x_3, x_1\} = 0,\]

\[S = -\int_{\tau} \frac{1}{x_2 + x_3} \partial_{\alpha} x^2 \partial_{\beta} x^3 \varepsilon^{\alpha \beta} d\sigma d\tau \quad (x_1 = cte),\]

\[(III, II) : \{x_1, x_2\} = 0, \quad \{x_2, x_3\} = \frac{1}{2}(e^{2x_1} - 1), \quad \{x_3, x_1\} = 0,\]

\[S = -\int_{\tau} \frac{2}{e^{2ax_1} + 1} \partial_{\alpha} x^2 \partial_{\beta} x^3 \varepsilon^{\alpha \beta} d\sigma d\tau \quad (x_1 = cte),\]

\[(VIa, II) : \{x_1, x_2\} = 0, \quad \{x_2, x_3\} = \frac{1}{2a}(e^{2ax_1} + 1), \quad \{x_3, x_1\} = 0,\]

\[S = -\int_{\tau} \frac{2a}{e^{2ax_1} + 1} \partial_{\alpha} x^2 \partial_{\beta} x^3 \varepsilon^{\alpha \beta} d\sigma d\tau \quad (x_1 = cte),\]

\[(VIIa, II.i) : \{x_1, x_2\} = 0, \quad \{x_2, x_3\} = \frac{1}{2a}(e^{2ax_1} - 1), \quad \{x_3, x_1\} = 0,\]

\[S = -\int_{\tau} \frac{2a}{e^{2ax_1} - 1} \partial_{\alpha} x^2 \partial_{\beta} x^3 \varepsilon^{\alpha \beta} d\sigma d\tau \quad (x_1 = cte),\]

\[(VIIa, II.ii) : \{x_1, x_2\} = 0, \quad \{x_2, x_3\} = -\frac{1}{2a}(e^{2ax_1} - 1), \quad \{x_3, x_1\} = 0,\]

\[S = \int_{\tau} \frac{2a}{e^{2ax_1} - 1} \partial_{\alpha} x^2 \partial_{\beta} x^3 \varepsilon^{\alpha \beta} d\sigma d\tau \quad (x_1 = cte).\]

Note that according to \([10]\) all Lie bialgebras except \((V, II.i)\) and \((III, III.ii)\) are triangular Lie bialgebras but \((V, II.i)\) and \((III, III.ii)\) are bi-r matrix bialgebras. In this way the model over \((II.i, V)\) is BF gauge model which we will consider in the following subsections. Furthermore as we see in the next subsection the model over \((III, III.ii)\) is also BF gauge model.

### 4.2 BF Gauge Models

As mentioned in section 2, to obtain BF gauge models, the Poisson structure must be linear in \(x^i\). Among three dimensional real Lie bialgebras only the Poisson structures of Poisson-Lie groups of \((II.i, V)\) and \((III, III.ii)\) have these forms \([10]\). For Lie bialgebra \((II.i, V)\) we have the following Poisson structure for its Poisson-Lie group \([10]\)

\[(II.i, V) : \{x_1, x_2\} = -x_2, \quad \{x_2, x_3\} = 0, \quad \{x_3, x_1\} = x_3.\]
Then the action $S_2$ has the following BF form:

$$S = \int \Sigma x^1 \land dA_1 + x^2 \land dA_2 + x^3 \land dA_3 - x^2 A_1 \land A_2 + x^3 A_3 \land A_1 = \int \Sigma x^\mu (dA_\mu + \frac{1}{2} f^{\lambda\nu}_\mu A_\lambda \land A_\nu), \quad (37)$$

where $f^{\lambda\nu}_\mu$ are the structure constants of dual Lie algebra $V$ with the following commutation relations:

$$[\tilde{X}^1, \tilde{X}^2] = -\tilde{X}^2, \quad [\tilde{X}^2, \tilde{X}^3] = 0, \quad [\tilde{X}^3, \tilde{X}^1] = \tilde{X}^3. \quad (38)$$

Similarly for Lie bialgebra $(III, III.ii)$ we have the following Poisson structure for its Poisson-Lie group [10]

$$(III, III.ii) : \{x_1, x_2\} = 0, \{x_2, x_3\} = x_2 + x_3, \{x_3, x_1\} = 0.$$

Then the action $S_2$ has the following BF form:

$$S = \int \Sigma x^1 \land dA_1 + x^2 \land dA_2 + x^3 \land dA_3 + (x^2 + x^3)A_2 \land A_3 = \int \Sigma x^\mu (dA_\mu + \frac{1}{2} \tilde{f}^{\lambda\nu}_\mu A_\lambda \land A_\nu), \quad (39)$$

where $\tilde{f}^{\lambda\nu}_\mu$ are the structure constants of dual Lie algebra $III.ii$ with the following commutation relations:

$$[\tilde{X}^1, \tilde{X}^2] = 0, \quad [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^2 + \tilde{X}^3, \quad [\tilde{X}^3, \tilde{X}^1] = 0. \quad (40)$$

Note that because the casimir for Lie algebra $V$ and $III.ii$ are not quadratic [15] therefore one can not obtain 2D gravity models over these Poisson-Lie groups.

Furthermore, one can not obtain 2D gravity models over Poisson-Lie group $II.i$ because $u_2(x) = u_3$ and $u_3(x) = -x_2$ (i.e $u_a = \epsilon_{ab} x^b$, $\epsilon_{23} = 1$) and by identifying $A_a = \epsilon_a$ and $A_1 = \omega$ we find the action

$$S = \int [x^c (de_c + e_c \land \omega) + x^1 d\omega], \quad (41)$$

but by use of Cartan structure equation $de_c + e_c \land \omega = 0$ this action is not good.

One can obtain $G/G$ WZW as a Poisson-Lie sigma model over dual groups IX and VIII for Lie bialgebras $(V.i[b, VIII]), (V.ii[b, VIII]), (V.iii, VIII), (V|b, IX)$ with topological term; by use of the following action [8]:

$$S = \int (dg.g^{-1} \land A) - \frac{1}{2} (A \land P^g_\square A), \quad (42)$$

where in the above relation ad-invariant metrics only over Lie algebras IX and VIII are non-degenerate. Note that Lie bialgebras $(V.i[b, VIII]), (V.ii[b, VIII]), (V.iii, VIII), (V|b, IX)$ are non-coboundary [10] and the Poisson structures $P^g_\square$ can be obtained by the method mentioned in [16] and [17].

5 Models with some four dimensional Poisson-Lie groups

The classification of real four dimensional Lie bialgebras is under investigation [12]. Here we give two interesting examples where the Poisson-Lie sigma models over them are two
sided. The real four dimensional Lie algebras are classified in [15]. Note that for the following examples the ad-invariant metrics are nondegenerate, for this reason one can not use the action (9) and must use of the action in the form (10).

a) The commutation relations for Lie bialgebras \((A_{4,1}, A_{4,1}.i)\) are as follows[12]:

\[
\begin{align*}
A_{4,1} & : \quad [X_2, X_4] = X_1, \quad [X_3, X_4] = X_2, \\
A_{4,1}.i & : \quad [\tilde{X}_1, \tilde{X}_2] = -\tilde{X}_3, \quad [\tilde{X}_1, \tilde{X}_3] = -\tilde{X}_4.
\end{align*}
\] (43) (44)

The classical r-matrices and Poisson structures for these Lie bialgebras are obtained as:

\[
r = bX_1 \wedge X_2 + X_1 \wedge X_4 + X_2 \wedge X_3, \quad \tilde{r} = \tilde{X}_1 \wedge \tilde{X}_4 - \tilde{X}_2 \wedge \tilde{X}_3 + l\tilde{X}_3 \wedge \tilde{X}_4,
\] (45)

\[
P_{12} = \frac{1}{2} (x^4)^2 + x^3, \quad P^{34} = \frac{1}{2} (\tilde{x}_1)^2 - \tilde{x}_2.
\] (46)

The action and equations of motions for the Poisson-Lie sigma models over Lie groups \(A_{4,1}\) and its dual \(A_{4,1}.i\) are the following form:

\[
S = \int_{\Sigma} [dx^1 \wedge A^1 + dx^2 \wedge A^2 + dx^3 \wedge A^3 + dx^4 \wedge A^4 - \frac{1}{2} (x^4)^2 + x^3 A^1 \wedge A^2],
\] (47)

\[
\frac{\partial S}{\partial A^1} = dx^1 - \left( \frac{1}{2} (x^4)^2 + x^3 \right) A^2 = 0, \quad \frac{\partial S}{\partial A^2} = dx^2 - \left( \frac{1}{2} (x^4)^2 + x^3 \right) A^1 = 0, \quad \frac{\partial S}{\partial A^3} = dx^3 = 0, \quad \frac{\partial S}{\partial A^4} = dx^4 = 0,
\]

\[
\tilde{S} = \int_{\Sigma} [d\tilde{x}_1 \wedge \tilde{A}_1 + d\tilde{x}_2 \wedge \tilde{A}_2 + d\tilde{x}_3 \wedge \tilde{A}_3 + d\tilde{x}_4 \wedge \tilde{A}_4 - \frac{1}{2} (\tilde{x}_1)^2 - \tilde{x}_2 \tilde{A}_3 \wedge \tilde{A}_4],
\] (48)

\[
\frac{\partial \tilde{S}}{\partial \tilde{A}_1} = d\tilde{x}_1 = 0, \quad \frac{\partial \tilde{S}}{\partial \tilde{A}_2} = d\tilde{x}_2 = 0, \quad \frac{\partial \tilde{S}}{\partial \tilde{A}_3} = d\tilde{x}_3 - \left( \frac{1}{2} (\tilde{x}_1)^2 - \tilde{x}_2 \right) \tilde{A}_4 = 0, \quad \frac{\partial \tilde{S}}{\partial \tilde{A}_4} = d\tilde{x}_4 - \left( \frac{1}{2} (\tilde{x}_1)^2 - \tilde{x}_2 \right) \tilde{A}_3 = 0,
\]

we see that the models are topological sigma models with the following actions:

\[
S = \int_{\Sigma} \frac{1}{2} (x^4)^2 + x^3 dx^1 \wedge dx^2, \quad (x^3 = x^4 = \text{cte.}),
\] (49)

\[
\tilde{S} = \int_{\Sigma} \frac{1}{2} (\tilde{x}_1)^2 - \tilde{x}_2 \tilde{x}_3 \wedge \tilde{x}_4, \quad (\tilde{x}_1 = \tilde{x}_2 = \text{cte.}).
\] (50)

b) The commutation relations for Lie bialgebras \((V \oplus R, V.i \oplus R)\) are as follows[12]:

\[
V \oplus R : \quad [X_1, X_2] = -X_2, \quad [X_1, X_3] = -X_3,
\] (51)

\[
V.i \oplus R : \quad [\tilde{X}_3, \tilde{X}_4] = \tilde{X}_3, \quad [\tilde{X}_2, \tilde{X}_4] = \tilde{X}_2.
\] (52)

The classical r-matrices and Poisson structures for these Lie bialgebras are obtained as:

\[
r = X_1 \wedge X_4, \quad \tilde{r} = -\tilde{X}_1 \wedge \tilde{X}_4,
\] (53)

\[
P_{12} = x^2, \quad P_{34} = -x^3, \quad \tilde{P}^{12} = -\tilde{x}_2.
\] (54)

\[\text{Note that the Poisson structure on the Lie group } V \oplus R \text{ is nondegenerate } (\det P = -(x^2 x^3)^2) \text{ and we have symplectic structure on this Lie group.}\]
The action and equations of motions for the Poisson-Lie sigma models over Lie groups $V \oplus R$ and its dual $V.i \oplus R$ are the following form:

$$ S = \int_{\Sigma} [dx^1 \wedge A^1 + dx^2 \wedge A^2 + dx^3 \wedge A^3 + dx^4 \wedge A^4 - x^2 A^1 \wedge A^2 + x^3 A^3 \wedge A^4], \quad \text{(55)} $$

$$ \frac{\partial S}{\partial A^1} = dx^1 + x^2 A^2 = 0, \quad \frac{\partial S}{\partial A^2} = dx^2 - x^2 A^1 = 0, \quad \frac{\partial S}{\partial A^3} = dx^3 - x^3 A^4 = 0, \quad \frac{\partial S}{\partial A^4} = dx^4 + x^3 A^3 = 0, $$

$$ \tilde{S} = \int_{\Sigma} [d\tilde{x}_1 \wedge \tilde{A}_1 + d\tilde{x}_2 \wedge \tilde{A}_2 + d\tilde{x}_3 \wedge \tilde{A}_3 + d\tilde{x}_4 \wedge \tilde{A}_4 + \tilde{x}_2 \tilde{A}_1 \wedge \tilde{A}_2], \quad \text{(56)} $$

for this case we also see that the models are topological sigma models with the following actions:

$$ S = \int_{\Sigma} \frac{1}{x^2} \, dx^1 \wedge dx^2 - \frac{1}{x^3} \, dx^3 \wedge dx^4, \quad \text{(57)} $$

$$ \tilde{S} = - \int_{\Sigma} \frac{1}{\tilde{x}_2} \, d\tilde{x}_1 \wedge d\tilde{x}_2, \quad (\tilde{x}_3 = \tilde{x}_4 = \text{cte.}). \quad \text{(58)} $$

On the other hand because the Poisson structure for this bialgebra and its dual are linear, one can obtain BF gauge models as follows:

$$ S = \int_{\Sigma} [dx^1 \wedge A^1 + dx^2 \wedge A^2 + dx^3 \wedge A^3 + dx^4 \wedge A^4 - x^2 A^1 \wedge A^2 + x^3 A^3 \wedge A^4] = \int_{\Sigma} x^\mu (dA_\mu + \frac{1}{2} \tilde{f}^\mu_{\nu\lambda} A_\mu \wedge A_\nu), \quad \text{(59)} $$

where $\tilde{f}^\mu_{\nu\lambda}$ is the structure constant of dual Lie algebra $V.i \oplus R$ with the commutation relation in (52) and the dual action is

$$ \tilde{S} = \int_{\Sigma} [d\tilde{x}_1 \wedge \tilde{A}_1 + d\tilde{x}_2 \wedge \tilde{A}_2 + d\tilde{x}_3 \wedge \tilde{A}_3 + d\tilde{x}_4 \wedge \tilde{A}_4 + \tilde{x}_2 \tilde{A}_1 \wedge \tilde{A}_2] = \int_{\Sigma} \tilde{x}^\mu (d\tilde{A}_\mu + \frac{1}{2} f^\mu_{\nu\lambda} \tilde{A}_\mu \wedge \tilde{A}_\nu), \quad \text{(60)} $$

where $f^\mu_{\nu\lambda}$ is the structure constant of Lie algebra $V \oplus R$ with the commutation relation in (51).

6 Conclusion

We have found Poisson-Lie sigma models on two, three and some four dimensional Poisson-Lie groups. Most of these Lie groups are non compact, and ad-invariant symmetric metrics over them are nondegenerate; for this reason one can not apply the action presented in [8] and here we apply other action for construction of Poisson-Lie sigma models over these Lie groups. Models over bi-r-matrices bialgebras are two sided. The results of two sided models are given in the following table (these models all are new.)
Two sided Poisson-Lie sigma models over low dimensional real Lie bialgebras.

| $(g, \tilde{g})$ | Models on $g$ | Models on $\tilde{g}$ |
|----------------|----------------|---------------------|
| typeB : $(A_2, A_2, i)$ | topological sigma model | topological sigma model |
| $(II, i, V)$ | BF gauge model | topological sigma model over symplectic leaf |
| $(A_4, 1, A_4, 1)$ | topological sigma model over symplectic leaf | topological sigma model over symplectic leaf |
| $(V \oplus R, V, i \oplus R)$ | BF gauge model | topological sigma model over symplectic leaf |

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