ELECTRICALLY CHARGED AND NEUTRAL WORMHOLE INSTABILITY IN SCALAR-TENSOR GRAVITY

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We study the stability of static, spherically symmetric, traversable wormholes with or without an electric charge, existing due to conformal continuations in a class of scalar-tensor theories with zero scalar field potential (so that Penney’s or Fisher’s well-known solutions hold in the Einstein conformal frame). Specific examples of such wormholes are those with nonminimally (e.g., conformally) coupled scalar fields. All boundary conditions for scalar and metric perturbations are taken into account. All such wormholes with zero or small electric charge are shown to be unstable under spherically symmetric perturbations. The instability is proved analytically with the aid of the theory of self-adjoint operators in Hilbert space and is confirmed by numerical computations.

1. Introduction

Lorentzian wormholes as hypothetic macroscopic or astrophysical objects are of great interest from the viewpoint of possible causality violation (time machines etc.) [1, 2] and from an observational viewpoint, as specific scatterers of stellar, galactic and quasar radiation [3]. From the viewpoint of gravitation theories, they are striking examples of extremely strong gravitational fields free of singularities.

A search for traversable wormhole solutions to the gravitational field equations with realistic matter has been for long, and is still remaining to be, one of the most intriguing challenges in gravitational studies. One of attractive features of wormholes is their ability to support electric or magnetic "charge without charge" [4] by letting the lines of force thread from one spatial asymptotic to another.

As is widely known, traversable wormholes can only exist with exotic matter sources, more precisely, if the energy-momentum tensor (EMT) of the matter source of gravity violates the local and averaged null energy condition (NEC) \( T_{\mu\nu}k^\mu k^\nu \geq 0 \), \( k^\mu k_\mu = 0 \) [5]. It is known, for instance, that nonlinear electrodynamics with any Lagrangian of the form \( L(F) \), \( F = F_{\mu\nu}F_{\mu\nu} \), coupled to general relativity, cannot produce a static, spherically symmetric wormhole metric [6]. Though, an effective wormhole geometry for electromagnetic wave propagation can appear as a result of the electromagnetic field nonlinearity [7, 8].

Scalar fields are able to provide good examples of matter needed for wormholes: on the one hand, in many particular models they do exhibit exotic properties, on the other, many exact solutions are known for gravity with scalar sources. We will consider some examples of charged wormhole solutions in the presence of massless scalar fields.

Let us begin with the action for a general (Bergmann-Wagoner) class of scalar-tensor theories (STT), where gravity is characterized by the metric \( g_{\mu\nu} \) and the scalar field \( \phi \) in the presence of the electromagnetic field \( F_{\mu\nu} \) as the only matter source:

\[
S = \int d^4x \sqrt{|g|} \{ f(\phi)R[g] + h(\phi)g^{\mu\nu}\phi,\mu\phi,\nu - F_{\mu\nu}F^{\mu\nu} \}.
\]

(1)

Here \( R[g] \) is the scalar curvature, \( g = |\det(g_{\mu\nu})| \), \( f \) and \( h \) are certain functions of \( \phi \), varying from theory to theory. Exact static, spherically symmetric solutions for this system are well known [9, 10], but their qualitative behaviour is rather diverse and depends on the nature of the functions \( f \) and \( h \).

Wormholes form one of the generic classes of solutions in theories where the kinetic term in \( H \) is negative [10] (more precisely, if \( l(\phi) \), defined in \( \Phi \), is negative). A particular case of this kind of wormholes, namely, wormholes with a “ghost” massless minimally coupled scalar field in general relativity [Eq. \( I \), \( f(\phi) \equiv 1 \), \( h(\phi) \equiv -1 \] was considered by H. Ellis [11].

The energy conditions, NEC in particular, are, however, violated as well by “less exotic” sources, such as
the so-called nonminimally coupled scalar fields in general relativity, represented by the action \( I \) with the functions

\[
f(\phi) = 1 - \xi \phi^2, \quad \xi = \text{const}; \quad h(\phi) \equiv 1.
\]  

Scalar-vacuum (with \( F_{\mu\nu} = 0 \)) static, spherically symmetric wormhole solutions were found in such a theory in Ref. [10] (and were recently discussed in Ref. [13]) for conformal coupling, \( \xi = 1/6 \), and in Ref. [12] for any \( \xi > 0 \). The easiness of violating the energy conditions, so evident due to the appearance of wormhole solutions, even made Barceló and Visser discuss a “restricted domain of application of the energy conditions” [12]. This class of wormholes exists in theories with scalar fields possessing a normal kinetic term but which admit a conformal continuation [14]. The latter means that a singularity in the Einstein frame maps to a regular sphere in the Jordan frame, and the latter may be smoothly continued beyond this sphere. In wormhole solutions, the second spatial asymptotic occurs in this new region of the Jordan manifold.

We have recently proved [15] that all these scalar-vacuum wormhole solutions are unstable under spherically symmetric perturbations: it turns out that there exists at least a single mode of growing physically meaningful perturbations. The characteristic time of their growth is of the order of the time needed for a photon to cover a length equal to the wormhole throat radius [15]. (Our earlier results [16, 17], according to which the perturbation growth rate is unlimited in the linear approximation, were obtained without taking into account the smoothness requirement for metric perturbations and therefore need revision.)

The purpose of this paper is to extend these results to charged wormholes. We first discuss the background configurations, namely, charged static, spherically symmetric wormholes which appear in scalar-tensor theories (STT) of gravity in which the effective gravitational constant can change its sign due to conformal continuation [14]. The investigation is, however, restricted to massless fields for which Penney’s well-known solution [9] (or Fisher’s [18] in the case of zero charge) holds in the Einstein frame. As examples, we describe scalar-electrovacuum wormhole solutions of the theory \( I \), \( 2 \).

Then we examine the stability problem, including the behaviour of metric perturbations related to those of the scalar field. A physically meaningful metric perturbation of an initially regular configuration should be regular everywhere. This requirement turns out to impose an additional constraint on the scalar field perturbations, which makes the stability problem quite nontrivial. We prove that, for sufficiently small electric charges, there exists at least a single growing mode of physically meaningful perturbations, i.e., such wormholes are unstable, with roughly the same increment of perturbation growth as for similar neutral wormholes.

Finally, we briefly report the results of numerical studies. They confirm the instability conclusion and show that, as the charge grows, the perturbation increment decreases, indicating wormhole stabilization for larger charges.

2. Charged wormhole solutions

2.1. The general static solution

The general STT action \( I \) is simplified by the well-known conformal mapping [19]

\[
g_{\mu\nu} = \f_{\mu\nu}/|f(\phi)|,
\]

accompanied by the scalar field transformation \( \phi \to \psi \) such that

\[
\frac{d\psi}{d\phi} = \pm \sqrt{|l(\phi)|} f(\phi), \quad l(\phi) \equiv fh + \frac{3}{2} \left( \frac{df}{d\phi} \right)^2.
\]

In terms of \( \f_{\mu\nu} \) and \( \psi \) the action takes the form

\[
S = \int d^4x \sqrt{-g} \{ (\text{sign } f) \left[R\f^2\right]
+ \f^{\mu\nu} \psi_\mu \psi_\nu \text{ sign } l(\phi) - F^{\mu\nu} F_{\mu\nu} \}
\]

(up to a boundary term which does not affect the field equations). Here \( R\f^2 \) is the Ricci scalar obtained from \( \f_{\mu\nu} \), and the indices are raised and lowered using \( \f_{\mu\nu} \). The electromagnetic field Lagrangian is conformally invariant, and \( F_{\mu\nu} \) is not transformed.

The space-time \( M_4[g] \) with the metric \( g_{\mu\nu} \) is referred to as the Jordan conformal frame, generally regarded to be the physical frame in STT; the Einstein conformal frame \( \f_{\mu\nu} \) with the field \( \psi \) then plays an auxiliary role. The action \( I \) corresponds to conventional general relativity if \( f > 0 \), and the normal sign of scalar kinetic energy is obtained for \( l(\phi) > 0 \).

The general static, spherically symmetric solution to the Einstein-Maxwell-scalar equations that follow from \( I \), was first found by Penney [9] and in a more complete form in [20, 21]. Let us write it in the form suggested in [10], restricting ourselves to the “normal” case \( f > 0 \), \( l > 0 \):

\[
ds_E^2 = e^{2\gamma(u)} dt^2 - e^{2\alpha(u)} du^2 - e^{2\beta(u)} d\Omega^2
= \frac{q^{-2} dt^2}{s^2(h, u + u_1)} - \frac{q^2 s^2(h, u + u_1)}{s^2(k, u)} \left[ \frac{du^2}{s^2(k, u)} + d\Omega^2 \right],
\]

\[
\psi(u) = Cu + \psi_1,
\]

\[
F_{10} = -F_{01} = q e^{\alpha+\gamma-2\beta}
= \frac{1}{q s^2(h, u + u_1)},
\]
where the subscript “E” stands for the Einstein frame; \(d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2\) is the linear element on a unit sphere; \(q = q_e\) (the electric charge), \(C\) (the scalar charge), \(h\), \(k\) and \(\psi_1\) are real integration constants. The function \(s(k, u)\) is defined as follows:

\[
s(k, u) = \begin{cases} 
k^{-1} \sinh ku, & k > 0 \\
u, & k = 0 \\
k^{-1} \sin ku, & k < 0. 
\end{cases}
\]  
(9)

Here \(u\) is a convenient radial variable (it is a harmonic coordinate in the Einstein frame, \(\Box u = 0\)). The range of \(u\) is \(0 < u < u_{\text{max}}\), where \(u = 0\) corresponds to spatial infinity, while \(u_{\text{max}}\) may be finite or infinite depending on the constants \(k\), \(h\) and \(u_1\).

The integration constants are related by

\[
2k^2 \sinh k = 2h^2 \sinh h + C^2,
\]  
(10)

\[
s^2(h, u_1) = 1/q^2.
\]  
(11)

The latter condition, preserving some discrete arbitrariness of \(u_1\), provides the natural choice of the time scale \((\mathcal{J}_0 = 1)\) at spatial infinity \((u = 0)\). Without loss of generality we put \(C > 0\) and \(\psi_1 = 0\).

As usual, in addition to the electric field \(F_{01} = -F_{10}\) given by \(\Box\), one can include a radial magnetic field \(F_{02} = -F_{23} = q_m \sin \theta\) where \(q_m\) is the magnetic charge. One should then understand \(q^2\) in \(\Box\), and further on as \(q^2 = q_e^2 + q_m^2\); in \(\Box\) one should replace \(q\) with \(q_e/\sqrt{2}\), in the first line and \(1/q\) with \(q_e/q^2\) in the second line. In what follows, we will bear in mind this opportunity without special mentioning.

The solution contains four essential integration constants: \(k\) or \(h\) and the charges \(q_e\), \(q_m\), and \(C\). The mass \(M\) in the Einstein frame is obtained by comparing the asymptotic of \(\Box\) at small \(u\) with the Schwarzschild metric:

\[
GM = \pm \sqrt{q^2 + h^2} \sinh h
\]  
(12)

where \(G\) is Newton’s gravitational constant. The “±” sign depends on the choice of \(u_1\) among the variants admitted by \(\Box\).

The Reissner-Nordstrom solution of general relativity is a special case obtained herefrom by putting \(C = 0\). Then from \(\Box\) it follows \(h = k\), and the familiar form of the Reissner-Nordstrom metric is recovered after a transition to the curvature coordinates, \(\mathcal{J}_{\theta\theta} = r^2\):

\[
r = \frac{|q| s(k, u + u_1)}{s(k, u)} \Rightarrow e^{2ku} = \frac{r + k - GM}{r - k - GM}. \tag{13}
\]

To obtain another special case \(q = 0\) (the scalar-vacuum solution), one should consider the limit \(q \to 0\) preserving the boundary condition \(\Box\). This is possible for \(k > h \geq 0\) and \(u_1 \to \infty\). The resulting metric is

\[
ds_E^2 = e^{-2hu} dt^2 - \frac{k^2 e^{2hu}}{\sinh^2 (ku)} \left[ \frac{du^2}{\sinh^2 (ku)} + d\Omega^2 \right].
\]  
(14)

The scalar field is determined, as before, from \(\Box\), and the integration constants are related by

\[
2k^2 = 2h^2 + C^2 \tag{15}
\]

It should be noted that in \(\Box\), \(\Box\) the constant \(h\) can have any sign, and for the mass \(M\) we have simply \(GM = h\).

This is the Fisher solution \([18]\) in terms of the harmonic \(u\) coordinate. Its more familiar form, used, in particular, in Refs. [12, 13], corresponds to the coordinate \(r\) connected with \(u\) by \(r = 2k/(1 - e^{-2ku})\), and the metric in terms of \(r\) has the form

\[
ds_E^2 = (1 - 2/k)^a dt^2 - \left[ (1 - 2/k)^{-a} [dr^2 + r^2(1 - 2k/r)d\Omega^2] \right], \tag{16}
\]

with \(a = h/k\). The Schwarzschild solution is then recovered in case \(C = 0\), \(a = 1\).

All the corresponding Jordan-frame solutions for \(l(\phi) > 0\) are obtained from \(\Box\), \(\Box\) using the transformation \(\Box\), \(\Box\).

### 2.2. Continued solution in the Jordan frame

Let us now turn to wormhole solutions for the nonminimal coupling \(\Box\), \(\xi > 0\). The transformation \(\Box\) takes the form

\[
\frac{d\psi}{d\phi} = \frac{\sqrt{1 - \phi^2 (\xi - 6\phi^2)}}{1 - \phi^2}, \tag{17}
\]

where, without loss of generality, we have chosen the plus sign before the square root. We assume that spatial infinity in the Jordan space-time \(\mathcal{M}_J\) corresponds to \(|\phi| < 1/\sqrt{\xi}\), where \(f(\phi) > 0\), so that the gravitational coupling has its normal sign.

Generically, the solution in \(\mathcal{M}_E\) \(\Box\) has a naked singularity at \(u = u_{\text{max}}\), and, though its nature can change due to the transformation to \(g_{\mu\nu}\), it remains to be a singularity in \(\mathcal{M}_J[g]\). An exception is the case when the solution is smoothly continued in \(\mathcal{M}_J[g]\) through the sphere \(S_{\text{trans}}(u = \infty, \phi = 1/\sqrt{\xi})\) which is singular in \(\mathcal{M}_E[g]\) but regular in \(\mathcal{M}_J[g]\). The infinity of the conformal factor \(1/f\) thus compensates the zero of both \(g_{tt}\) and \(g_{\theta\theta}\) simultaneously. Wormhole solutions can only be found in this case. It happens when, in accord with \(\Box\),

\[
k = 2h = 2C/\sqrt{6} > 0, \quad u_1 > 0, \tag{18}
\]

which selects a special subfamily among all solutions. We will restrict our attention to this subfamily. Note that now \(s(k, u) = (2h)^{-1} \sinh(2hu)\), \(s(h, u + u_1) = h^{-1} \sinh(hu + hu_1)\) and \(u_{\text{max}} = \infty\). According to \(\Box\) and \(\Box\), we have \(\psi \to \infty\) as \(\phi \to 1/\sqrt{\xi} - 0\).
Under the condition (15) the solutions with and without charge in \( \mathcal{M}_E \) are conveniently written in isotropic coordinates. Indeed, putting \( y = \tanh(hu) \), we obtain:

\[
 ds^2_E = \frac{(1 - y^2) y^2}{(y + y_1)^2} \left[ dt^2 - h^2 \frac{(y + y_1)^2}{y_1^4} (dy^2 + y^2 d\Omega^2) \right],
\]

where

\[
 y_1 = \tanh(hu_1) = \frac{h}{\sqrt{h^2 + y^2}}.
\]

The vacuum solution is included here as the special case \( q = 0, \ y_1 = 1 \). The range of \( u, \ u \in \mathbb{R}_+ \), is converted into \( y \in (0, 1) \) where \( y = 0 \) corresponds to spatial infinity and \( y = 1 \) to a naked singularity.

To proceed to the Jordan frame, let us integrate Eq. (17). This gives [12]³

\[
 \psi = -\sqrt{3/2} \ln[B(\phi)H(\phi)H^2(\phi)],
\]

where

\[
 B(\phi) = B_0 \frac{\sqrt{1 - \eta^2 \phi} - \sqrt{6} \xi \phi}{\sqrt{1 - \eta^2 \phi^2 + 6 \xi \phi}},
\]

\( B_0 = \text{const} \), while \( H(\phi) \) is different for different \( \xi \):

\[
 0 < \xi < 1/6 : \\
 H(\phi) = \text{exp} \left[ -\frac{\sqrt{1 - \eta^2 \phi}}{\sqrt{6} \xi} \arcsin \sqrt{\eta} \phi \right],
\]

\[
 \xi > 1/6 : \\
 H(\phi) = \left[ \sqrt{\eta} \phi + \sqrt{1 - \eta^2 \phi^2} \right]^{\frac{\sqrt{6} \xi + 1}{\sqrt{6} \xi}},
\]

where \( \eta = \xi(1 - 6 \xi) \), and \( H \equiv 1 \) for \( \xi = 1/6 \). The function \( H(\phi) \) is finite in the whole range of \( \phi \) under consideration.

Eq. (23) is valid for \( \phi < 1/\sqrt{\xi} \), and the Jordan-frame metric \( g_{\mu
u} = \frac{\sigma_{\mu\nu}}{f} \) under the condition (13) can be written in terms of the coordinate \( y \) as follows:

\[
 ds^2_J = \frac{\text{BH}^2}{1 - \xi \phi^2} \left[ \frac{1 + y^2}{(y + y_1)^2} y_1^2 dt^2 - h^2 \frac{(y + y_1)^2}{y_1^4} (dy^2 + y^2 d\Omega^2) \right],
\]

where

\[
 y = \frac{1 - \text{BH}^2}{1 + \text{BH}^2}.
\]

³We have changed the notations as compared with [12], in particular, we have replaced \( \Phi \rightarrow \sqrt{6} \phi, \ H \rightarrow 1/H \) and \( F^2 \rightarrow 1/B \), to avoid imaginary \( F \) at \( \phi > 1/\sqrt{\xi} \).

The metric is thus actually expressed in terms of the scalar field \( \phi \) used as a coordinate. The isotropic coordinate \( y \) conveniently shortens the expression (20) and makes it easy to see that the metric, originally built for \( \phi < \phi_0 (y < 1) \), is smoothly continued across the surface \( S_{\text{trans}} (\phi = \phi_0, \ y = 1) \). Indeed, in a close neighbourhood of \( S_{\text{trans}} \), for \( \phi = (\phi - \delta)/\sqrt{\xi} \) with \( \delta \ll 1 \) one has

\[
 B \approx B_0 \delta/(12\xi), \quad 1 - \xi \phi^2 \approx 2\delta
\]

whence

\[
 \frac{\text{BH}^2}{1 - \xi \phi^2} \bigg|_{y=1} = \frac{B_0}{24\xi} H^2 \bigg|_{\phi = \phi_0}.
\]

It is easily shown that this ratio is not only finite on \( S_{\text{trans}} \) but also smoothly changes across it, so that Eq. (20) comprises an analytic continuation of the metric, obtained from (3)–(8) in case (18) by the transformation (3), (4), beyond \( S_{\text{trans}} \). The coordinate \( y \) covers the whole manifold \( \mathcal{M}_3 [g] \), and it is now possible to study the properties of the system as a whole.

Before doing that, let us note that the new region \( \phi > \phi_0 (y > 1) \) in \( \mathcal{M}_3 \) can also be obtained by the same transformation (3), (4) from a certain Einstein frame.

An essential difference from the previous solution is that, since \( f(\phi) \) is now negative, (18) leads to the Einstein equations with a reversed sign of the electromagnetic energy-momentum tensor. As a result, the solution in this second Einstein-frame manifold⁴ \( \mathcal{M}_E' \) will have the same form (6–8), but with the replacement

\[
 s(h, u + u_1) \rightarrow \phi'(1 - \phi^2) \cosh(h'u + h'u_1),
\]

where \( h' > 0 \), and the relation (10) is replaced by \( 2k'^2 = 2h'^2 + C^2 \) where \( k' > 0 \).

The solution in \( \mathcal{M}_E' \) is also regularized by the factor \( 1/f \) on \( S_{\text{trans}} \), and the integration constants in it satisfy the condition \( k' = 2h' \), similar to (13). Other integration constants are adjusted as well, in particular, the charges \( q_e \) and \( q_m \) are the same on both sides of \( S_{\text{trans}} \), providing the continuity of the electromagnetic field.

### 2.3. Wormhole solutions

Let us begin with the simplest case \( \xi = 1/6 \) (conformal coupling). Then instead of (23)–(25) one can write for \( \phi < \sqrt{6} \)

\[
 \phi = \sqrt{6} \tanh[(\psi + \psi_0)/\sqrt{6}], \quad \psi_0 = \text{const},
\]

where \( \psi = Cu \) and due to (13) \( C = h\sqrt{6} \). The Jordan-frame solution in terms of the isotropic coordinate \( y \)

⁴The prime will designate quantities describing the Einstein frame or \( \phi > 1/\sqrt{\xi} \).
takes the form [10]
\begin{align}
 ds^2 &= \frac{(1+y_0)^2}{1-y_0^2} \left[ \frac{y_0^2 dt^2}{(y+y_1)^2} - h^2 \frac{(y+y_1)^2}{y_0^2 y^4} (dy^2 + y^2 d\Omega^2) \right], \quad (31)
\end{align}
\begin{align}
 \phi &= \sqrt{6} \frac{y + y_0}{1 + y_0}, \quad (32)
\end{align}
where \( y_0 = \tanh(\psi_0/\sqrt{6}) \) and \( y_1 \in (0,1) \); the expressions for \( F_{\mu\nu} \) are evident.

The original Einstein-frame solution corresponds to \( y < 1 \), \( y = 0 \) is spatial infinity while the sphere \( y = 1 \) is \( S_{\text{trans}} \), where the solution \([31],[32]\) is manifestly regular. The region \( y > 1 \) is an analytic continuation of the solution in \( M_J[g] \) to \( \phi > \sqrt{6} \) and corresponds to another Einstein-frame solution described above.

The properties of the solution at \( y > 1 \) depend on the constant \( y_0 \) which characterizes the \( \phi \) field at spatial infinity. Namely, if \( y_0 < 0 \), then the solution has a naked singularity at \( y = -1/y_0 > 1 \). If \( y_0 = 0 \), we obtain a black hole with electromagnetic and scalar charges \([10,21-23]\); introducing \( r = h(y+y_1)/(y_1 y) \), we obtain
\begin{align}
 ds^2 &= (1-m/r)^2 dt^2 - (1-m/r)^{-2} dr^2 - r^2 d\Omega^2,
\phi &= C/(r-m),
\end{align}
where \( m = GM = \sqrt{h^2 + q^2}, C = \sqrt{6}h \). On the horizon, \( r = m \), despite \( \phi \to \infty \), the energy-momentum tensor of the scalar field is finite. This solution (mainly its neutral special case \( q = 0 \)) was repeatedly discussed as an interesting counterexample of the well-known no-hair theorems; its instability under spherically symmetric perturbations has been proved in Ref. [24].

Lastly, if \( y_0 > 0 \), then \( y \) ranges from 0 to \( \infty \), and \( y = \infty \) is another flat spatial infinity. This is the sought-for wormhole solution, parametrized by the four constants \( h, q_0, q_m \) and \( y_0 \). The position and radius of the wormhole neck (minimum of \( r^2 = -y_0 y \)) are given by
\begin{align}
 y_{\text{neck}} = \frac{\sqrt{y_1}}{\sqrt{y_0}},
 r_{\text{neck}} = \frac{h(1 + \sqrt{y_0 y_1})}{y_1 \sqrt{1 - y_0}}. \quad (34)
\end{align}

For \( \xi \neq 1/6 \) the analytical relations are much more complicated, but the qualitative behaviour of the solution can be described rather easily.

In case \( \xi > 1/6 \), for any \( B_0 \), with growing \( \phi \) the quantity \( B^2 H^{-4} \) eventually reaches the value 1, where \( y_0 y_\to \to \infty \), i.e., we arrive at another spatial asymptotic, and it is straightforward to verify that this infinity is flat. In other words, we obtain again a static wormhole.

In case \( \xi < 1/6 \) everything depends on \( B_0 \). If
\begin{align}
 B_0 < B_0^{cr} = \exp(\pi \sqrt{1 - 6\xi}), \quad (35)
\end{align}
the situation is the same as for \( \xi > 1/6 \), i.e., a wormhole. If \( B_0 > B_0^{cr} \), then, while \( y_0 y \) is still finite, \( \phi \) reaches the value \( 1/\sqrt{\eta} = 1/\sqrt{\xi(1 - 6\xi)} \), the location of a curvature singularity \([12]\). So we have a naked singularity instead of a wormhole. Lastly, for \( B_0 = B_0^{cr} \), the maximum value of \( \phi \) is again \( 1/\sqrt{\eta} \), but now it is non-flat spatial infinity.

### 3. Stability analysis

#### 3.1. Problem setting

The present stability analysis repeats the main features of the similar analysis for electrically neutral wormholes \([15]\). We will use the results obtained there. In particular, the Schrödinger form of the equations is defined for functions of the same Hilbert space as for uncharged wormholes. Meanwhile, the boundary value problem now depends on two parameters: the energy-like parameter (actually, the increment of perturbation growth) and the wormhole charge. The Schrödinger operator, if written in a more or less visually graspable form, turns out to be non-self-adjoint, which considerably reduces our ability to study its properties. We shall solve the problem for small charges only, with the aid of perturbation theory for operators’ point spectra. At small charges our operator evidently turns into its analogue for uncharged wormholes.

Let us write down the linearized Einstein equations for perturbations \( \delta \alpha, \delta \beta, \delta \gamma \) (as in \([15]\), we are working in the Einstein picture), taking the metric \([19]\) as the background one:
\begin{align}
 e^{2\gamma} R_1 &= 2(\delta \dot{\beta} - \beta' (\delta \dot{\beta} + \delta \dot{\gamma}) - \gamma' \delta \dot{\beta}) = 0,
 e^{2\alpha} R_2 &= 2\delta \beta' (2\delta \dot{\beta} + \delta \gamma) - 2 e^{2\beta + 2\gamma} \delta \beta + e^{4\delta} \delta \beta - \delta \beta'' = -4 q^2 e^{2\gamma} \delta \beta. \quad (36)
\end{align}

The primes here denote \( \partial/\partial y \). The scalar field perturbation is absent since we have used the gauge \( \delta \phi \equiv 0 \). This gauge is manifestly physical (i.e., the perturbations do not comprise a pure gauge) and is particularly convenient for describing connections between the Jordan and Einstein pictures \([15]\).

As in \([15]\), we carry out variable separation with the exponential \( e^{i\Omega t} \) and arrive at a boundary-value problem with containing the separation constant \( \Omega \) as an eigenvalue. The set of perturbation equations reduces to a single equation for \( \delta \beta(y,t) \) expressing the dynamics of the only existing degree of freedom:
\begin{align}
 \delta \beta'' - \Omega^2 \delta \beta s^4(y) + F(y) \delta \beta' + G(y) \delta \beta = 0, \quad (37)
\end{align}
where \( s, F, G \) are functions of \( y \) obtained from the metric \([19]\):
\begin{align}
 F(y) = -2 \delta \beta''/\beta',
\end{align}
\[ G(y) = -2\beta'' + 2\beta''\gamma'/\beta' + 2e^{2\beta}/2\gamma' - 4q^2e^{2\gamma}\delta\beta, \]
\[ s(y) = e^\beta. \]

The boundary conditions will be discussed a little later.

A few words about choosing \( y \) as a radial coordinate and using it in a whole range covering two different Einstein pictures before and beyond \( S_{\text{trans}} \) (\( y = 1 \)). The point is that if one considered the perturbation problem in each Einstein frame separately (using, e.g., the harmonic coordinate \( u \)), it would be necessary to study two different operators and prove that their spectra coincide, which is, even if it is the case, a hard problem by itself since the method we are using make it possible to find only an upper bound of \( \Omega \) rather than its precise value. Even more important is that the spectrum of our problem, actually posed in the whole Jordan spacetime, may in principle be different from those of the “partial” problems formulated separately in its two regions. So we invoke a coordinate that covers the whole Jordan manifold, and we are dealing with a single operator. The circumstance that the whole metric \( S_{\text{trans}} \) changes its sign at \( S_{\text{trans}} \) is inessential since this metric only plays an auxiliary role.

Let us bring the differential equation (37) to a self-adjoint Sturm-Liouville form:
\[-(p \delta\beta')' + q \delta\beta = -E r \delta\beta, \quad (38)\]
where
\[ p = \frac{y(1+y)(1+y_1+2y_1y)^2(1+2y)^2}{(1+y+y_1+2y_1y)^2+y_1(1+y+y_1)^2+y_1^2(1+y+y_1)^2}; \]
\[ q = \frac{1}{(1+y+y_1+y_1^2)^2} - \frac{1}{(1+y+y_1)^2} \cdot \frac{1}{(1+y+y_1)^2} + \frac{1}{(1+y+y_1)^2}; \]
\[ r = \frac{y(1+y)^2}{(1+y+y_1)^2}, \]
\[ E = -\Omega^2h^2. \]

It can be shown that when the wormhole charge tends to zero (\( y_1 \to 1 \)), this equation reduces to that for an uncharged wormhole [15]. In other words, using the linear operator perturbation theory, one can show that, for small charges, our boundary-value problem has solutions with values of \( E \) close to the ones for neutral wormholes.

Let us formulate the boundary conditions. Since our equation covers the whole manifold, we have two conditions at two spatial asymptotics: \( \delta\beta \to 0 \) at both. At the transition sphere \( S_{\text{trans}} \), the solution has the approximate form
\[ \delta\beta \approx c_1 + c_2 \ln(y - 1). \quad (39) \]

It is now necessary to find out whether or not there are solutions with \( E < 0 \) and \( c_2 = 0 \), vanishing at infinity.

Equation (38) is brought to a Schrödinger form by the substitution
\[ x = \frac{1}{m} \int s^2 \, dy, \]
\[ \delta\beta = \frac{\beta}{s} \exp \left( -\frac{1}{2} \int F \, du \right), \quad (40) \]
The result is
\[ d^2z(x)/dx^2 + [E - V(x)]z(x) = 0. \quad (41) \]
with a potential \( V \) having the asymptotics
\[ V(x) \approx \frac{2}{y_1} x^3 \quad (x \to \pm \infty, \text{ flat asymptotics}), \]
\[ V(x) \approx \bar{V}(x) = -\frac{1}{(4x^2)} - \frac{11 y_1^2(y_1 - 1)}{4(1+y_1)^3} \]
\[ (x \to 0, \quad S_{\text{trans}}). \quad (42) \]
The complete form of the potential is very cumbersome and, on the other hand, unnecessary for the further study.

At the transition sphere, the asymptotic form of the solution is
\[ z \simeq c_1 \sqrt{x}(1 + bx) + c_2 \ln x \sqrt{x}(1 + bx), \quad (43) \]
where
\[ b = -\frac{11 y_1^2(y_1 - 1)}{4 (1+y_1)^3}. \]

The further reasoning is carried out similarly to that of the previous work [15]. Namely, we study the properties of the Schrödinger operator corresponding to Eq. (44) for small values of the charge \( q \), i.e., for small \( b \). We prove that this operator is self-adjoint, and its continuous spectrum covers the whole non-negative semiaxis. Thus an instability, if any, corresponds to discrete “energy” levels. A further analysis, which is rather technically complicated and includes the use of linear operator perturbation theory [25, 26], leads to the conclusion that there exists at least one negative “energy level” \( E \).

We have thus shown that, for small values of the charge parameter \( b \), our boundary-value problem has solutions describing exponentially growing perturbations with increment values close to those for uncharged wormholes.

This instability conclusion is applicable to any STT admitting wormhole solutions due to conformal continuation.

Numerical calculations have shown that the perturbation increment diminishes as the charge grows. For large values of the charge (compared to the wormhole radius in proper units), the numerical methods used become unreliable, but the available results make us suppose that at larger charges the wormholes stabilize with respect to spherically symmetric perturbations.
To conclude, we give an example of a numerical estimate. In the case of conformal coupling, the throat radius is approximately equal to $2\hbar$. The characteristic time of decay, $\tau = 1/\Omega$, is proportional to $\hbar$ (which has the dimension of length):

$$\tau \simeq \frac{\hbar}{\sqrt{0.048}} \simeq 5\hbar.$$ (44)

For a wormhole radius of the order of a typical stellar size $\sim 10^6$ km, the time $\tau$ is a few seconds, slightly greater than the time needed for a light signal to cover the stellar diameter.

Similar estimates can be obtained for other STT characterized by different $f(\phi)$.

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