Uniqueness of Stable Processes with Drift

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Abstract

Suppose that \(d \geq 1\) and \(\alpha \in (1, 2)\). Let \(Y\) be a rotationally symmetric \(\alpha\)-stable process on \(\mathbb{R}^d\) and \(b\) a \(\mathbb{R}^d\)-valued measurable function on \(\mathbb{R}^d\) belonging to a certain Kato class of \(Y\). We show that \(dX^b_t = dY_t + b(X^b_t)dt\) with \(X^b_0 = x\) has a unique weak solution for every \(x \in \mathbb{R}^d\).

Let \(L^b = -(-\Delta)^{\alpha/2} + b \cdot \nabla\), which is the infinitesimal generator of \(X^b\). Denote by \(C^{\infty}_c(\mathbb{R}^d)\) the space of smooth functions on \(\mathbb{R}^d\) with compact support. We further show that the martingale problem for \((L^b, C^{\infty}_c(\mathbb{R}^d))\) has a unique solution for each initial value \(x \in \mathbb{R}^d\).

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1 Introduction

Throughout this paper, unless otherwise stated, \(d \geq 1\) and \(\alpha \in (1, 2)\). A rotationally symmetric \(\alpha\)-stable process \(Y\) in \(\mathbb{R}^d\) is a Lévy process with characteristic function given by

\[
E[\exp(i\xi \cdot (Y_t - Y_0))] = \exp(-t|\xi|^\alpha), \quad \xi \in \mathbb{R}^d.
\] (1.1)

The infinitesimal generator of \(Y\) is the fractional Laplacian \(\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}\). Here we use “:=” to denote a definition. Denote by \(B(x, r)\) the open ball in \(\mathbb{R}^d\) centered at \(x \in \mathbb{R}^d\) with radius \(r > 0\) and \(dx\) the Lebesgue measure on \(\mathbb{R}^d\).

**Definition 1.1.** For a real-valued function \(f\) on \(\mathbb{R}^d\) and \(r > 0\), define

\[
M^\alpha_f(r) := \sup_{x \in \mathbb{R}^d} \int_{B(x, r)} \frac{|f(y)|}{|x - y|^{d+1-\alpha}} dy.
\] (1.2)

A function \(f\) on \(\mathbb{R}^d\) is said to belong to the Kato class \(\mathcal{K}_{d,\alpha-1}\) if \(\lim_{r \downarrow 0} M^\alpha_f(r) = 0\).

Using Hölder’s inequality, it is easy to see that for every \(p > d/(\alpha-1)\), \(L^\infty(\mathbb{R}^d; dx) + L^p(\mathbb{R}^d; dx) \subset \mathcal{K}_{d,\alpha-1}\). Throughout this paper we will assume \(b = (b_1, \ldots, b_d)\) is a \(\mathbb{R}^d\)-valued function on \(\mathbb{R}^d\) such that \(|b| \in \mathcal{K}_{d,\alpha-1}\).

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In this paper, we are concerned with the existence and uniqueness of weak solutions to following stochastic differential equation (SDE)

$$dX_t^b = dY_t + b(X_t^b)dt, \quad X_0^b = x.$$ (1.3)

A solution of (1.3), if it exists, will be called $\alpha$-stable process with drift $b$. When $Y$ is a Brownian motion (which corresponds to $\alpha = 2$), it is well known that Brownian motion with drift can be obtained from Brownian motion through a change of measure called Girsanov transform. But for symmetric $\alpha$-stable process (where $0 < \alpha < 2$), SDE (1.3) can not be solved by a change of measure. This is because $Y$ is a purely discontious Lévy process and so the effect of a Girsanov transform can only produce a purely discontinuous “drift term”; see [4].

In this paper, we show that (1.3) has a unique weak solution for every initial value $x$. We achieve this by showing that the corresponding martingale problem for SDE (1.3) is well-posed. Define $L^b = \Delta^{\alpha/2} + b \cdot \nabla$. It easy to see by using Ito’s formula that $L^b$ is the infinitesimal generator for solutions of (1.3). Let $D([0, \infty); \mathbb{R}^d)$ be the space of right continuous $\mathbb{R}^d$-valued functions having left limits on $[0, \infty)$, equipped with Skorokhod topology. For $t \geq 0$, denote by $X_t$, the projection coordinate map on $D([0, \infty); \mathbb{R}^d)$. A probability measure $Q$ on the Skorokhod space $D([0, \infty); \mathbb{R}^d)$ is said to be a solution to the martingale problem for $(L^b, C_c^\infty(\mathbb{R}^d))$ with initial value $x \in \mathbb{R}^d$ if $Q(X_0 = x) = 1$ and for every $f \in C_c^\infty(\mathbb{R}^d)$, $\int_0^t |L^b f(X_s)|ds < \infty$ $Q$-a.s. for every $t > 0$ and

$$M_t^f := f(X_t) - f(X_0) - \int_0^t L^b f(X_s)ds$$ (1.4)

is a $Q$-martingale. The martingale problem for $(L^b, C_c^\infty(\mathbb{R}^d))$ with initial value $x \in \mathbb{R}^d$ is said to be well-posed if it has a unique solution. The following is the main result of this paper.

**Theorem 1.2.** For each $x \in \mathbb{R}^d$, SDE (1.3) has a unique weak solution. Moreover, weak solutions with different starting points can all be constructed on the canonical Skorokhod space $D([0, \infty); \mathbb{R}^d)$ and the symmetric $\alpha$-stable process $Y$ in (1.3) can be chosen in such a way that it is the same for all starting point $x \in \mathbb{R}^d$. The law of the unique weak solution to SDE (1.3) is the unique solution to the martingale problem for $(L^b, C_c^\infty(\mathbb{R}^d))$.

The unique weak solutions of (1.3) form a strong Markov process $X^b$. Theorem 1.2 combined with the main result of [3] and [4] readily gives sharp two-sided estimates on the transition density $p^b(t, x, y)$ of $X^b$ as well as on the transition density $p^b_{D}(t, x, y)$ of the subprocess $X^{b, D}$ of $X^b$ killed upon leaving a bounded $C^{1,1}$ open set. We refer the definition of $C^{1,1}$ open set and its $C^{1,1}$ characteristics to [4]. For $x \in D$, let $\delta_D(x)$ denote the Euclidean distance between $x$ and $\partial D$. The diameter of $D$ will be denoted as $\text{diam}(D)$.

**Corollary 1.3.** (i) $X^b$ has a jointly continuous transition density function $p^b(t, x, y)$ with respect to the Lebesgue measure on $\mathbb{R}^d$. Moreover, for every $T > 0$, there is a constant $c_1 > 1$ depending only on $d, \alpha, T$ and on $b$ only through the rate at which $M_{|y|^0}(r)$ goes to zero so that for $(t, x, y) \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d$,

$$c_1^{-1} \left( t^{-d/\alpha} \land \frac{t}{|x - y|^{d + \alpha}} \right) \leq p^b(t, x, y) \leq c_1 \left( t^{-d/\alpha} \land \frac{t}{|x - y|^{d + \alpha}} \right).$$
(ii) Let \( d \geq 2 \) and let \( D \) be a bounded \( C^{1,1} \) open subset of \( \mathbb{R}^d \) with \( C^{1,1} \) characteristics \( (R_0, \Lambda_0) \).

Define
\[
    f_D(t, x, y) = \left( 1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left( t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right).
\]

For each \( T > 0 \), there are constants \( c_2 = c_2(T, R_0, \Lambda_0, d, \alpha, \text{diam}(D), b) \geq 1 \) and \( c_3 = c_3(T, d, \alpha, D, b) \geq 1 \) with the dependence on \( b \) only through the rate at which \( M^\alpha_{|b|}(r) \) goes to zero such that

(a) on \( (0, T] \times D \times D \), \( c_2^{-1} f_D(t, x, y) \leq p_D^b(t, x, y) \leq c_2 f_D(t, x, y) \);

(b) on \( [T, \infty) \times D \times D \),
\[
c_3^{-1} e^{-t \lambda^{b,D}_1} \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2} \leq p_D^b(t, x, y) \leq c_3 e^{-t \lambda^{b,D}_1} \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2},
\]

where \( \lambda^{b,D}_1 := -\sup \text{Re}(\sigma(L^b|_D)) > 0 \). Here \( \sigma(L^b|_D) \) denotes the spectrum of the non-local operator \( L^b \) in \( D \) with zero exterior condition.

Here and in the sequel, for \( a, b \in \mathbb{R}, a \wedge b := \min\{a, b\}, a \vee b := \max\{a, b\}, \) and the meaning of the phrase “depending on \( b \) only via the rate at which \( M^\alpha_{|b|}(r) \) goes to zero” is that the statement is true for any \( \mathbb{R}^d \)-valued function \( \tilde{b} \) on \( \mathbb{R}^d \) with \( M^\alpha_{|\tilde{b}|}(r) \leq M^\alpha_{|b|}(r) \) for all \( r > 0 \).

The existence of martingale solution to \( (L^b, C^\infty_c(\mathbb{R}^d)) \) is established in [4, Theorem 2.5], using a (particular) fundamental solution of \( L^b \) constructed in [3]. One deduces easily from Ito’s formula that the uniqueness of the martingale problem for \( (L^b, C^\infty_c(\mathbb{R}^d)) \) implies the weak uniqueness of SDE \( (\ref{SDE}) \). So the main point of Theorem \( \ref{1.2} \) is on the uniqueness of the martingale problem for \( (L^b, C^\infty_c(\mathbb{R}^d)) \) and the existence of a weak solution to SDE \( (\ref{SDE}) \). The novelty here is that the drift \( b \) is a function in Kato class \( K_{d,\alpha-1} \), which in general is merely measurable and can be unbounded. Thus Picard’s iteration method is not applicable either. Motivated by the approach in [2], we establish the uniqueness of the solutions to the martingale problem for \( (L^b, C^\infty_c(\mathbb{R}^d)) \) by showing that its resolvent is uniquely determined. This uniqueness of the martingale problem for \( (L^b, C^\infty_c(\mathbb{R}^d)) \) in particular gives the uniqueness of the fundamental solution to \( L^b \), which was not addressed in [3]; see Theorem \( \ref{2.2} \) below.

The equivalence between weak solutions to SDE driven by Brownian motion and solutions to martingale problems for elliptic operators is well known. The crucial ingredient in this connection is a martingale representation theorem for Brownian motion. Such a martingale representation theorem is not available for stable processes. Recently, Kurtz [8] studied equivalence between weak solutions to a class of SDEs driven by Poisson random measures and solutions to martingale problems for a class of non-local operators using a non-constructive approach. We point out that one can not deduce the existence of weak solution to SDE \( (\ref{SDE}) \) from the existence of the martingale problem for \( (L^b, C^\infty_c(\mathbb{R}^d)) \) by applying results from [8] because \( L^b f \) is typically unbounded for \( f \in C^\infty_c(\mathbb{R}^d) \). In this paper, we develop a new approach to the weak existence of solutions to SDE \( (\ref{SDE}) \). We believe this new approach is potentially useful to study weak existence for some other SDEs with singular drifts, especially those driven by discontinuous Lévy processes. Our new approach uses the Lévy system of the strong Markov process \( X^b \) obtained from the unique solution to the martingale problem for \( (L^b, C^\infty_c(\mathbb{R}^d)) \) and stochastic calculus to construct a weak solution to SDE \( (\ref{SDE}) \).
Very recently, around the same time as the first version of this paper was completed, Kim and Song [3] studied stable process with singular drift, analogous to Brownian motion with singular drift studied in Bass and Chen [2]. Intuitively speaking, stable process with singular drift studied in [2] corresponding to SDE \( \text{with } b \text{ replacing by suitable measure. However, the existence and uniqueness of the solution in [2] is formulated in a weaker sense, as in [2]. When applying to the Kato function } \) \( b \) case considered in this paper, the results in [2] do not give the existence and uniqueness of weak solutions to SDE \( \text{nor the well-posedness of the martingale problem for } \( \mathcal{L}^b, C^\infty_0(\mathbb{R}^d) \)).

The approach of this paper is quite robust. It can be applied to study some other stochastic models. For example, it can be used to establish, for each \( b \in \mathbb{K}_{d,\alpha-1} \), the well-posedness of martingale problem for \( \text{with } b \text{ corresponding to SDE } \) \( \text{analogous to Brownian motion with singular drift. Intuitively speaking, stable process with singular drift studied in Song [7] studied stable process with singular drift, analogous to Brownian motion with singular drift.}

## 2 Uniqueness of martingale problem

Recall that \( \mathcal{L}^b = \Delta^{\alpha/2} + b \cdot \nabla \). When \( b = 0 \), we simply write \( \mathcal{L}^0 \) as \( \mathcal{L} \); that is, \( \mathcal{L} = \Delta^{\alpha/2} \). Let \( b = (b_1, \cdots, b_d) \) be a \( \mathbb{R}^d \)-valued function on \( \mathbb{R}^d \) with \( |b| \in \mathbb{K}_{d,\alpha-1} \). For simplicity, sometimes we just denote it as \( b \in \mathbb{K}_{d,\alpha-1} \). In this section, we establish the well-posedness of the martingale problem for \( \mathcal{L}^b, C^\infty_0(\mathbb{R}^d) \).

We first recall from Bogdan and Jakubowski [3] the construction of a particular fundamental solution \( q^b(t,x,y) \) for non-local operator \( \mathcal{L}^b \) using a perturbation argument. It is based on the following heuristics: \( q^b(t,x,y) \) of \( \mathcal{L}^b \) can be related to the fundamental solution \( p(t,x,y) \) of \( \mathcal{L} \), which is the transition density of the symmetric stable process \( Y \), by the following Duhamel’s formula:

\[
q^b(t,x,y) = p(t,x,y) + \int_0^t \int_{\mathbb{R}^d} q^b(s,x,z) b(z) \cdot \nabla_z p(t-s,z,y) dz \, ds. \tag{2.1}
\]

Applying the above formula recursively, one expects \( q^b(t,x,y) \) to be expressed as an infinite series in terms of \( p \) and its derivatives. Thus we define \( q^b_0(t,x,y) = p(t,x,y) \) and for \( k \geq 1 \),

\[
q^b_k(t,x,y) := \int_0^t \int_{\mathbb{R}^d} q^b_{k-1}(s,x,z) b(z) \cdot \nabla_z p(t-s,z,y) dz. \tag{2.2}
\]

The following results come from [3, Theorem 1, Lemma 15, Lemma 23] and their proofs.

**Proposition 2.1.** (i) There exist constants \( T_0 > 0 \) and \( c_1 > 1 \) depending only on \( d, \alpha \) and on \( b \) only through the rate at which \( M_{\alpha/2}^0(r) \) goes to zero so that \( \sum_{k=0}^{\infty} q^b_k(t,x,y) \) converges locally uniformly on \( (0,T_0] \times \mathbb{R}^d \times \mathbb{R}^d \) to a jointly continuous positive function \( q^b(t,x,y) \) and that on \( (0,T_0] \times \mathbb{R}^d \times \mathbb{R}^d \),

\[
\frac{1}{c_1} \left( t^{-d/\alpha} \land \frac{t}{|x-y|^{d+\alpha}} \right) \leq q^b(t,x,y) \leq c_1 \left( t^{-d/\alpha} \land \frac{t}{|x-y|^{d+\alpha}} \right). \tag{2.3}
\]

Moreover, \( \int_{\mathbb{R}^d} q^b(t,x,y) dy = 1 \) for every \( t \in (0,T_0] \) and \( x \in \mathbb{R}^d \).
(ii) The function $q^b(t,x,y)$ defined in (i) can be extended uniquely to a jointly continuous positive function on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ so that for all $s, t \in (0,\infty)$ and $x, y \in \mathbb{R}^d$, $\int_{\mathbb{R}^d} q^b(t,x,z)dz = 1$ and

$$q^b(s + t, x, y) = \int_{\mathbb{R}^d} q^b(s, x, z)q^b(t, z, y)dz. \quad (2.4)$$

(iii) Define $T^b_tf(x) := \int_{\mathbb{R}^d} q^b(t,x,y)f(y)dy$. Then for any $f, g \in C^\infty_c(\mathbb{R}^d)$, the space of smooth functions with compact supports,

$$\lim_{t \downarrow 0} \frac{1}{t} \int_{\mathbb{R}^d} (T^b_tf(x) - f(x))g(x)dx = \int_{\mathbb{R}^d} (L^b_f)(x)g(x)dx.$$

Proposition 2.1(iii) indicates $q^b(t,x,y)$ is a fundamental solution of $L^b$ in distributional sense. It is easy to check (see Proposition 2.3) that the operators $\{T^b_t; t \geq 0\}$ determined by $q^b(t,x,y)$ form a Feller semigroup and so there exists a $\mathbb{R}^d$-valued conservative Feller process $\{X_t; t \geq 0, P_x, x \in \mathbb{R}^d\}$ defined on the canonical Skorokhod space $D([0,\infty); \mathbb{R}^d)$ having $q^b(t,x,y)$ as its transition density function. Moreover, it is shown in [4, Theorem 2.5] that $P_x$ is a solution to the martingale problem for $(L^b, C^\infty_c(\mathbb{R}^d))$ with $P_x(X_0 = x) = 1$. However in both [3] and [4], neither the uniqueness of fundamental solution $q^b(t,x,y)$ to $L^b$ nor the uniqueness of the martingale problem for $(L^b, C^\infty_c(\mathbb{R}^d))$ are addressed. The main result of this paper, Theorem 1.2, in particular fills in this missing piece and implies that $q^b(t,x,y)$ is the transition density function of the uniqueness solution $(X_t, P_x, x \in \mathbb{R}^d)$ to the martingale problem for $(L^b, C^\infty_c(\mathbb{R}^d))$.

**Theorem 2.2.** For each $x \in \mathbb{R}^d$, the martingale problem for $(L^b, C^\infty_c(\mathbb{R}^d))$ with initial value $x$ is well-posed. These martingale problem solutions $\{P_x, x \in \mathbb{R}^d\}$ form a strong Markov process $X^b$, which has infinite lifetime and possesses a jointly continuous transition density function $p^b(t,x,y)$ with respect to the Lebesgue measure on $\mathbb{R}^d$. Consequently, $p^b(t,x,y)$ is the same as the kernel $q^b(t,x,y)$ constructed in Proposition 2.1 and enjoys the two-sided estimates (2.3).

Here $X^b_t = X_t$ is the coordinate map defined on $D([0,\infty); \mathbb{R}^d)$ but we use superscript $b$ for emphasis when it is viewed as a Markov process under probability measures $P_x$.

Let $(P_x, x \in \mathbb{R}^d)$ be the probability measures on $D([0,\infty); \mathbb{R}^d)$ obtained from the kernel $q^b(t,x,y)$ in Proposition 2.1. The mathematical expectation taken under $P_x$ will be denoted by $E_x$. As we noted in previous paragraph, for each $x \in \mathbb{R}^d$, $P_x$ solves the martingale problem for $(L^b, C^\infty_c(\mathbb{R}^d))$ with initial value $x$. We will show that $P_x$ is in fact the unique solution. Our approach is motivated by that of Bass and Chen [2, Section 5].

Before the proof of Theorem 2.2 we state two lemmas on the boundedness of the $\lambda$-resolvent operator $R_\lambda$ corresponding to symmetric $\alpha$-stable process $Y$. Denote by $p(t,x,y) = p(t,x-y)$ the transition density function of $Y$. Let $r_\lambda(x) = \int_0^\infty e^{-t\lambda}p(t,x)dt$ and define the resolvent operator $R_\lambda$ by

$$R_\lambda g(x) = \int_{\mathbb{R}^d} r_\lambda(x - y)g(y)dy = \int_{\mathbb{R}^d} r_\lambda(y)g(x - y)dy,$$

for every $g \in C_b(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. Here $C_b(\mathbb{R}^d)$ (resp. $C_0(\mathbb{R}^d)$) denote the space of bounded continuous functions on $\mathbb{R}^d$ (resp. continuous functions on $\mathbb{R}^d$ that vanish at infinity). For $f \in C_0(\mathbb{R}^d)$, define $\|f\|_\infty = \sup_{x \in \mathbb{R}^d} |f(x)|$. Denote by $C_0^\infty(\mathbb{R}^d)$ the space of smooth functions on $\mathbb{R}^d$.
that together with their partial derivatives of any order vanish at infinity. By \[3, \text{Lemma 7}\], for \((\lambda, x) \in (0, \infty) \times \mathbb{R}^d\),
\[
    r_\lambda(x) \asymp \left( \frac{\lambda^{(d-\alpha)/\alpha} \vee |x|^{\alpha-d}}{\lambda^{-2}|x|^{-d-\alpha}} \right),
\]
which can be rewritten as
\[
    r_\lambda(x) \asymp \begin{cases} 
        \frac{1}{|x|^{d-\alpha}} \wedge \frac{\lambda^{-2}}{|x|^{d+\alpha}} & \text{when } d > \alpha, \\
        \lambda^{(d-\alpha)/2} \wedge \frac{\lambda^{-2}}{|x|^{d+\alpha}} & \text{when } d \leq \alpha.
    \end{cases}
\]
Here for any two positive functions \(f\) and \(g\), \(f \asymp g\) means that there is a positive constant \(c \geq 1\) so that \(c^{-1} g \leq f \leq c g\) on their common domain of definition.

**Lemma 2.3.** For every \(\lambda > 0\), \(\nabla R_\lambda\) is a bounded operator on \(C_0(\mathbb{R}^d)\). Moreover, \(R_\lambda f \in C_0^\infty(\mathbb{R}^d)\) for every \(f \in C_0^\infty(\mathbb{R}^d)\).

**Proof.** It is known by \[3, \text{Lemma 9}\] that \(r_\lambda(z)\) is continuously differentiable off the origin and there is a constant \(c_1 > 1\) so that for every \(\lambda > 0\) and \(z \neq 0\),
\[
    c_1^{-1} \left( \frac{1}{|z|^{d+1-\alpha}} \wedge \frac{1}{\lambda^2 |z|^{d+1+\alpha}} \right) \leq |\nabla r_\lambda(z)| \leq c_1 \left( \frac{1}{|z|^{d+1-\alpha}} \wedge \frac{1}{\lambda^2 |z|^{d+1+\alpha}} \right).
\]
It follows that for \(\lambda > 0\), \(f \in C_0(\mathbb{R}^d)\) and \(x \in \mathbb{R}^d\),
\[
    \int_{\mathbb{R}^d} |\nabla r_\lambda(x-y)||f(y)|dy \leq c_1 \|f\|_\infty \int_{\mathbb{R}^d} \left( \frac{1}{|z|^{d+1-\alpha}} \wedge \frac{1}{\lambda^2 |z|^{d+1+\alpha}} \right) dz < \infty.
\]
Thus \(R_\lambda f\) is continuously differentiable and
\[
    \nabla R_\lambda f(x) = \int_{\mathbb{R}^d} \nabla r_\lambda(x-y)f(y)dy = \int_{\mathbb{R}^d} \nabla r_\lambda(y)f(x-y)dy.
\]
Since both \(r_\lambda(y)\) and \(|\nabla r_\lambda(y)|\) are integrable over \(\mathbb{R}^d\) and \(f(x-y)\) converges to 0 as \(|x| \to \infty\), we conclude that both \(R_\lambda f\) and \(\nabla R_\lambda f\) are in \(C_0(\mathbb{R}^d)\) with \(\|R_\lambda f\|_\infty \leq c_2 \|f\|_\infty\) and \(\|\nabla R_\lambda f\|_\infty \leq c_2 \|f\|_\infty\) for some constant \(c_2 > 0\). Similarly, for \(f \in C_0^\infty(\mathbb{R}^d)\), we have
\[
    \partial_{x_1}^{k_1} \cdots \partial_{x_d}^{k_d} R_\lambda f(x) = \int_{\mathbb{R}^d} r_\lambda(y) \partial_{x_1}^{k_1} \cdots \partial_{x_d}^{k_d} f(x-y) dy,
\]
and consequently \(R_\lambda f \in C_0^\infty(\mathbb{R}^d)\). \(\square\)

We may view the function \(b\) as a multiplication operator in the sense that \((bf)(x) = b(x)f(x)\).

**Lemma 2.4.** Let \(b = (b_1, \cdots, b_d) \in \mathbb{R}_{d,\alpha-1}\). There exists \(\lambda_0 > 0\) depending only on \(d\), \(\alpha\) and on \(b\) only via the rate at which \(M_{|b|}^\alpha(r)\) goes to zero such that for every \(\lambda > \lambda_0\) and \(f \in C_0(\mathbb{R}^d)\),
\[
    \|\nabla R_\lambda(bf)\|_\infty \leq \frac{1}{2} \|f\|_\infty.
\]
Proof. It follows from [3, Lemma 11 and Corollary 12] and their proof (with $\beta = 2$ there) that there exists a constant $c_1 > 0$ depending only on $d$ and $\alpha$ such that for every $t > 0$,

$$
\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left( \frac{1}{|x - y|^{d+1-\alpha}} \right) \left( \frac{t^2}{|x - y|^{d+1+\alpha}} \right) |b(y)| dy \leq c_1 M_0^\alpha(t^{1/\alpha}). \tag{2.7}
$$

This together with (2.6) implies that there exists a constant $c_2 > 0$ such that for every $\lambda > 0$

$$
\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\nabla r_\lambda(x - y)| |b(y)| |f(y)| dy \leq c_2 \|f\|_\infty \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left( \frac{1}{|x - y|^{d-\alpha+1}} \right) \left( \frac{\lambda^{-2}}{|x - y|^{d+\alpha+1}} \right) |b(y)| dy \leq c_1 c_2 \|f\|_\infty M_0^\alpha(\lambda^{-1/\alpha}).
$$

It follows that

$$
\|\nabla R_\lambda(bf)\|_\infty \leq c_3 M_0^\alpha(\lambda^{-1/\alpha}) \|f\|_\infty.
$$

Since $M_0^\alpha(\lambda^{-1/\alpha})$ tends to 0 as $\lambda \to \infty$, there exists some $\lambda_0 > 0$ so that $c_3 M_0^\alpha(\lambda^{-1/\alpha}) \leq 1/2$ for every $\lambda > \lambda_0$. This proves the lemma. $\Box$

It is well known that the transition density function $p(t, x, y)$ of the symmetric $\alpha$-stable process $Y$ on $\mathbb{R}^d$ has the two-sided estimates

$$
p(t, x, y) \propto t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}}.
$$

So estimate (2.3) can be restated as there is a constant $C_1 \geq 1$ depending on $b$ only through the rate at which $M_0^\alpha(r)$ goes to zero so that

$$
C_1^{-1} p(t, x, y) \leq b(t, x, y) \leq C_1 p(t, x, y) \quad \text{for every } (t, x, y) \in (0, T_0] \times \mathbb{R}^d \times \mathbb{R}^d.
$$

It follows from (2.3) and the Chapman-Kolmogorov equation (2.4) that there are positive constants $C_2 \geq 1$ and $C_3 > 0$ depending on $b$ only through the rate at which $M_0^\alpha(r)$ goes to zero so that

$$
C_2^{-1} e^{-C_3 t} p(t, x, y) \leq b(t, x, y) \leq C_2 e^{C_3 t} p(t, x, y) \quad \text{for every } t > 0 \text{ and } x, y \in \mathbb{R}^d. \tag{2.8}
$$

Thus for $\lambda > C_3$,

$$
\mathbb{E}_x \left[ \int_0^\infty e^{-\lambda t} |b(X_t)| dt \right] \leq C_2 \int_{\mathbb{R}^d} |b(y)| r_{-\lambda C_3}(x - y) dy.
$$

By [3, Lemma 16], there is a constant $C_4 > C_3$ depending on $b$ only through the rate at which $M_0^\alpha(r)$ goes to zero such that for every $\lambda \geq C_4$,

$$
\sup_{x \in \mathbb{R}^d} R_\lambda |b| = \sup_{x \in \mathbb{R}^d} \mathbb{E}_x \left[ \int_0^\infty e^{-\lambda t} |b(X_t)| dt \right] < \infty. \tag{2.9}
$$

By increasing the value of $\lambda_0$ in Lemma 2.4 if needed, we may and do assume that $\lambda_0 \geq C_4$.

**Theorem 2.5.** For each $x \in \mathbb{R}^d$, $\mathbb{P}_x$ is the unique solution to the martingale problem for $(\mathcal{L}^b, C_c^\infty(\mathbb{R}^d))$ with initial value $x$. 

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Proof. Let $Q$ be any solution to the martingale problem for $(\mathcal{L}^b, C_c^\infty(\mathbb{R}^d))$ with initial value $x$. We will show $Q = \mathbb{P}_x$. We divide its proof into 5 steps.

(i) We show that it suffice to consider the case that

$$\mathbb{E}_Q \left[ \int_0^\infty e^{-\lambda t} |b(X_t)| dt \right] < \infty \quad \text{for every } \lambda > \lambda_0, \tag{2.10}$$

where $\mathbb{E}_Q$ is the mathematical expectation under the probability measure $Q$ and $\lambda_0$ is the constant in Lemma 2.7.

By the definition of martingale problem solution, $\int_0^t b(X_s) \cdot \nabla f(X_s) ds$ is well defined $Q$-a.s. for each $t > 0$, that is, $\int_0^t b(X_s) \cdot \nabla f(X_s) ds < \infty$ $Q$-a.s. for every $t > 0$. Let

$$T_n(f) = \inf \left\{ t > 0 : \int_0^t b(X_s) \cdot \nabla f(X_s) ds \geq n \right\}.$$ 

Then $\{T_n(f), n \geq 1\}$ is an increasing sequence of stopping times such that $\lim_{n \to \infty} T_n(f) = \infty$ $Q$-a.s. with

$$\mathbb{E}_Q \left[ \int_0^{T_n} |b(X_s)| \cdot \nabla f(X_s) ds \right] \leq n. \tag{2.11}$$

Choose a sequence of functions $f_n^{(i)} \in C^\infty_c(\mathbb{R}^d)$ such that $f_n^{(i)}(x) = x_i$ for $x \in B(0, n)$ and $1 \leq i \leq d$. Define

$$S_n = \left( \min_{1 \leq i \leq d} T_n(f_n^{(i)}) \right) \wedge \inf \left\{ t : |X_t| > n \text{ or } |X_{t-}| > n \right\}.$$ 

Then $S_n$ is an increasing sequence of stopping times with $\lim_{n \to \infty} S_n = \infty$. By (2.11),

$$\mathbb{E}_Q \left[ \int_0^{S_n} |b(X_s)| ds \right] \leq \sum_{i=1}^d \mathbb{E}_Q \left[ \int_0^{S_n} |b_i(X_s)| ds \right] \leq \sum_{i=1}^d \mathbb{E}_Q \left[ \int_0^{S_n} |b(X_s)| \cdot \nabla f_n^{(i)}(X_s) ds \right] \leq nd. \tag{2.12}$$

Now we construct a new probability measure $\tilde{Q}$ so that $\tilde{Q}$ is also a solution to the martingale problem for $(\mathcal{L}^b, C_c^\infty(\mathbb{R}^d))$ and that for every $\lambda > \lambda_0$,

$$\mathbb{E}_{\tilde{Q}} \left[ \int_0^\infty e^{-\lambda t} |b(X_t)| dt \right] < \infty.$$ 

Let $\mathcal{F}_t$ be the minimal filtration generated by $\{X_s ; s \leq t\}$. Fix $N \geq 1$. We specify $\tilde{Q}$ by

$$\tilde{Q} \left( B \cap (C \circ \theta_{S_N}) \right) = \mathbb{E}_Q \left[ \mathbb{P}_{X_{S_N}} (C) ; \mathcal{B} \right],$$

for $B \in \mathcal{F}_{S_N}$ and $C \in \mathcal{F}_\infty$. It is easy to see that $\tilde{Q}$ is again a solution to the martingale problem for $(\mathcal{L}^b, C_c^\infty(\mathbb{R}^d))$. Moreover,

$$\mathbb{E}_{\tilde{Q}} \left[ \int_0^\infty e^{-\lambda t} |b(X_t)| dt \right] = \mathbb{E}_Q \left[ \int_0^{S_N} e^{-\lambda t} |b(X_t)| dt \right] + \mathbb{E}_Q \left[ e^{-\lambda S_N} \mathbb{E}_{X_{S_N}} \left[ \int_0^\infty e^{-\lambda t} |b(X_t)| dt \right] \right],$$

which is finite by (2.9) and (2.12). Since $\tilde{Q} = Q$ on $\mathcal{F}_{S_N}$, if we can show $\tilde{Q} = \mathbb{P}_x$ on $\mathcal{F}_{S_N}$, since $N \geq 1$ is arbitrary, this would imply that $Q = \mathbb{P}_x$ on $\mathcal{F}_\infty$. So it suffice to consider the solution $Q$ to the martingale problem satisfying (2.10).
(ii) We next show that for every $g \in C_0^\infty(\mathbb{R}^d)$,

$$
\mathbb{E}_Q \left[ \int_0^\infty e^{-\lambda t} g(X_t) dt \right] = \mathbb{R}_\lambda g(x) + \mathbb{E}_Q \left[ \int_0^\infty e^{-\lambda t} b(X_t) \cdot \nabla R_\lambda g(X_t) dt \right]. \tag{2.13}
$$

By (1.4), $f(X_t)$ is a semimartingale under $Q$ for every $f \in C_c^\infty(\mathbb{R}^d)$. It follows by the Itô’s formula that

$$
e^{-\lambda t} f(X_t) = f(X_0) + \int_0^t e^{-\lambda s} dM_f^s + \int_0^t e^{-\lambda s} \left( \Delta^{\alpha/2} f(X_s) + b(X_s) \cdot \nabla f(X_s) \right) ds - \lambda \int_0^t e^{-\lambda s} f(X_s) ds.
$$

Taking expectation with respect to $Q$, we have

$$
\mathbb{E}_Q[e^{-\lambda t} f(X_t)] = f(x) - \mathbb{E}_Q \left[ \int_0^t e^{-\lambda s} (\lambda f - \Delta^{\alpha/2} f)(X_s) ds \right] + \mathbb{E}_Q \left[ \int_0^t e^{-\lambda s} b(X_s) \cdot \nabla f(X_s) ds \right]. \tag{2.14}
$$

Note that $f$, $\nabla f$ and $\Delta^{\alpha/2} f$ are all bounded. Taking limit $t \to \infty$ in both sides of (2.14) and using the fact (2.10), we obtain

$$
\mathbb{E}_Q \left[ \int_0^\infty e^{-\lambda t} (\lambda f - \Delta^{\alpha/2} f)(X_t) dt \right] = f(x) + \mathbb{E}_Q \left[ \int_0^\infty e^{-\lambda t} b(X_t) \cdot \nabla f(X_t) dt \right]. \tag{2.15}
$$

We want to show that (2.15) holds for all $f \in C_0^\infty(\mathbb{R}^d)$. In fact, for any $f \in C_0^\infty(\mathbb{R}^d)$, there exists a sequence of functions $f_n \in C_c^\infty(\mathbb{R}^d)$ such that $f_n \to f$ in $C_0^\infty(\mathbb{R}^d)$ and in particular, $\|f_n - f\|_\infty \to 0$, $\|\Delta^{\alpha/2} f_n - \Delta^{\alpha/2} f\|_\infty \to 0$. Applying (2.10) again, we have

$$
\lim_{n \to \infty} \mathbb{E}_Q \left[ \int_0^\infty e^{-\lambda t} (\lambda f_n - \Delta^{\alpha/2} f_n)(X_t) dt \right] = \mathbb{E}_Q \left[ \int_0^\infty e^{-\lambda t} (\lambda f - \Delta^{\alpha/2} f)(X_t) dt \right],
$$

and

$$
\lim_{n \to \infty} \mathbb{E}_Q \left[ \int_0^\infty e^{-\lambda t} b(X_t) \cdot \nabla f_n(X_t) dt \right] = \mathbb{E}_Q \left[ \int_0^\infty e^{-\lambda t} b(X_t) \cdot \nabla f(X_t) dt \right].
$$

Thus (2.15) holds for $f \in C_0^\infty(\mathbb{R}^d)$.

By Lemma 2.3 $R_\lambda g \in C_0^\infty(\mathbb{R}^d)$ for $g \in C_0^\infty(\mathbb{R}^d)$. Taking $f = R_\lambda g$ in (2.15) and using the fact $(\lambda - \Delta^{\alpha/2}) R_\lambda g = g$, we obtain (2.13).

(iii) We claim that

$$
\mathbb{E}_Q \left[ \int_0^\infty e^{-\lambda t} \mathbbm{1}_A(X_t) dt \right] = 0 \quad \text{for any } A \subset \mathbb{R}^d \text{ with } m(A) = 0, \tag{2.16}
$$

where $m$ is the Lebesgue measure on $\mathbb{R}^d$.

To see this, suppose $A$ is a bounded subset of $\mathbb{R}^d$ having $m(A) = 0$. Let $\psi_n$ be a sequence of positive functions in $C_c^\infty(\mathbb{R}^d)$ so that $|\psi_n| \leq 2$, $\lim_{n \to \infty} \psi_n = 0$ m.a.e. on $\mathbb{R}^d$ and $\lim_{n \to \infty} \psi_n \geq \mathbbm{1}_A$. It follows from (2.0) and the dominated convergence theorem that $\nabla R_\lambda \psi_n(z) = \int_{\mathbb{R}^d} \nabla r_\lambda(z - y) \psi_n(y) dy$ converges to 0 boundedly as $n \to \infty$. One concludes then from (2.10) and the dominated convergence theorem that

$$
\lim_{n \to \infty} \mathbb{E}_Q \left[ \int_0^\infty e^{-\lambda t} b(X_t) \cdot \nabla R_\lambda \psi_n(X_t) dt \right] = 0.
$$
Applying Fatou’s lemma to (2.13) with \( \psi_n \) in place of \( g \) yields that

\[
\mathbb{E}_Q \left[ \int_0^\infty e^{-\lambda t} 1_A(X_t) dt \right] \leq \liminf_{n \to \infty} \mathbb{E}_Q \left[ \int_0^\infty e^{-\lambda t} \psi_n(X_t) dt \right] = \liminf_{n \to \infty} R_\lambda \psi_n(x) = 0,
\]

where the last equality is due to (2.5) and the dominated convergence theorem. This establishes (2.16) for any bounded and hence for any subset \( A \subset \mathbb{R}^d \) having \( m(A) = 0 \).

(iv) We now show that (2.13) holds for any function \( g \) on \( \mathbb{R}^d \) with \( |g| \leq c|b| \) as well.

Let \( g \) be a function on \( \mathbb{R}^d \) with \( |g| \leq c|b| \) for some \( c > 0 \). Fix \( M > 0 \) and define \( g_M = ((-M) \lor g) \land M \). Let \( \phi \) be a positive smooth function on \( \mathbb{R}^d \) with compact support such that \( \int_{\mathbb{R}^d} \phi(y) dy = 1 \). For \( n \geq 1 \), set \( \phi_n(y) = n^d \phi(ny) \) and \( f_n(z) := e^{-n \phi_n(z-y)} g_M(y) dy \). Then \( f_n \in C_0^\infty(\mathbb{R}^d) \), \( |f_n| \leq M \), and \( f_n \) converges to \( g_M \) almost everywhere on \( \mathbb{R}^d \) as \( n \to \infty \). In view of (2.16) and the bounded convergence theorem, \( \lim_{n \to \infty} R_\lambda f_n(x) = R_\lambda g_M(x) \) and

\[
\lim_{n \to \infty} \mathbb{E}_Q \left[ \int_0^\infty e^{-\lambda t} f_n(X_t) dt \right] = \mathbb{E}_Q \left[ \int_0^\infty e^{-\lambda t} g_M(X_t) dt \right].
\]

On the other hand, by (2.6) and the dominated convergence theorem, \( \nabla R_\lambda f_n \) converges boundedly on \( \mathbb{R}^d \) to \( \nabla R_\lambda g_M \) as \( n \to \infty \). So we deduce from (2.13) with \( f_n \) in place of \( g \) and take \( n \to \infty \) that

\[
\mathbb{E}_Q \left[ \int_0^\infty e^{-\lambda t} g_M(X_t) dt \right] = R_\lambda g_M(x) + \mathbb{E}_Q \left[ \int_0^\infty e^{-\lambda t} b(X_t) \cdot \nabla R_\lambda g_M(X_t) dt \right].
\]  

(2.17)

Clearly by the dominated convergence theorem, (2.5) and (2.10),

\[
\lim_{M \to \infty} \mathbb{E}_Q \left[ \int_0^\infty e^{-\lambda t} g_M(X_t) dt \right] = \mathbb{E}_Q \left[ \int_0^\infty e^{-\lambda t} g(X_t) dt \right] \quad \text{and} \quad \lim_{M \to \infty} R_\lambda g_M(x) = R_\lambda g(x),
\]

while in view of (2.6) and (2.7), \( \nabla R_\lambda g_M(z) = \int_{\mathbb{R}^d} \nabla r_\lambda(z-y) g_M(y) dy \) converges boundedly on \( \mathbb{R}^d \) to \( \nabla R_\lambda g(z) \). Thus by (2.10) and the dominated convergence theorem,

\[
\lim_{M \to \infty} \mathbb{E}_Q \left[ \int_0^\infty e^{-\lambda t} b(X_t) \cdot \nabla R_\lambda g_M(X_t) dt \right] = \mathbb{E}_Q \left[ \int_0^\infty e^{-\lambda t} b(X_t) \cdot \nabla R_\lambda g(X_t) dt \right].
\]

The last two displays together with (2.17) establish the claim that (2.13) holds for any \( g \) with \( |g| \leq c|b| \).

(v) Define a linear functional \( V_\lambda \) by

\[
V_\lambda f = \mathbb{E}_Q \left[ \int_0^\infty e^{-\lambda t} f(X_t) dt \right].
\]

Then (2.13) can be rewritten as

\[
V_\lambda g = R_\lambda g(x) + V_\lambda(BR_\lambda g) \quad \text{for } g \in C_0^\infty(\mathbb{R}^d) \bigcup \{g : |g| \leq c|b| \text{ for some } c > 0\},
\]  

(2.18)

where \( B \) is the operator defined by

\[
B f(x) = b(x) \cdot \nabla f(x).
\]
Fix \( g \in C_c^{\infty}(\mathbb{R}^d) \). It follows from (2.6) that \( |BR_\lambda g| \leq c|b| \) for some constant \( c > 0 \). Applying (2.18) with \( BR_\lambda g \) in place of \( g \) yields

\[
V_\lambda(BR_\lambda g) = R_\lambda(BR_\lambda g)(x) + V_\lambda(BR_\lambda BR_\lambda g).
\] (2.19)

Repeating this procedure, we get that for every \( g \in C_c^{\infty}(\mathbb{R}^d) \) and every integer \( N \geq 1 \),

\[
V_\lambda g = \sum_{k=0}^{N} R_\lambda(BR_\lambda)^k g(x) + V_\lambda(B(R_\lambda B)^N R_\lambda g).
\] (2.20)

It follows from Lemma 2.4 that for \( \lambda > \lambda_0 \),

\[
|V_\lambda(B(R_\lambda B)^N R_\lambda g)| \leq \|\nabla R_\lambda b\|^N \|\nabla R_\lambda g\|_\infty \mathbb{E}_Q \left[ \int_0^\infty e^{-\lambda t}|b(X_t)|dt \right] \\
\leq 2^{-N} \|\nabla R_\lambda g\|_\infty \mathbb{E}_Q \left[ \int_0^\infty e^{-\lambda t}|b(X_t)|dt \right],
\]

which tends to 0 as \( N \to \infty \). Passing \( N \to \infty \) in (2.20) gives

\[
V_\lambda g = \sum_{k=0}^{\infty} R_\lambda(BR_\lambda)^k g(x).
\] (2.21)

Note that \( P_x \) is also a solution to the martingale problem for \((\mathcal{L}^b, C_c^{\infty}(\mathbb{R}^d))\) with initial value \( x \). Then (2.21) also holds with \( Q \) replaced by \( P_x \), that is,

\[
\mathbb{E}_x \left[ \int_0^\infty e^{-\lambda t} g(X_t)dt \right] = \sum_{k=0}^{\infty} R_\lambda(BR_\lambda)^k g(x).
\]

Consequently

\[
\mathbb{E}_Q \left[ \int_0^\infty e^{-\lambda t} g(X_t)dt \right] = \mathbb{E}_x \left[ \int_0^\infty e^{-\lambda t} g(X_t)dt \right]
\] (2.22)

for every \( g \in C_c^{\infty}(\mathbb{R}^d) \) and \( \lambda > \lambda_0 \). By the uniqueness of the Laplace transform, we have \( \mathbb{E}_Q[g(X_t)] = \mathbb{E}_x[g(X_t)] \) for all \( t \), or, the one-dimensional distributions of \( X_t \) under \( Q \) and \( P_x \) are the same. By a standard argument using regular conditional probability (see, e.g., the proof of Theorem VI.3.2 in [1]), one obtains equality of all finite-dimensional distributions and hence \( Q = P_x \). The uniqueness for the martingale problem for \((\mathcal{L}^b, C_c^{\infty}(\mathbb{R}^d))\) is thus proved.

**Proof of Theorem 2.2.** The existence and uniqueness for the martingale problem for \((\mathcal{L}^b, C_c^{\infty}(\mathbb{R}^d))\) is established in Theorem 2.5. By the uniqueness, the remaining assertions then follow from Proposition 2.1.

**3 Stochastic differential equation**

It is known that for any \( \alpha \in (0,2) \) the fractional Laplacian \( \Delta^{\alpha/2} \) can be written in the form

\[
\Delta^{\alpha/2} u(x) = \int_{\mathbb{R}^d} \left( u(x + z) - u(x) - \nabla u(x) \cdot z 1_{\{|z| \leq 1\}} \right) \frac{A(d, -\alpha)}{|z|^{d+\alpha}} dz,
\] (3.1)
where $A(d,-\alpha)$ is a normalizing constant so that

$$\int_{\mathbb{R}^d} \left( e^{i\xi \cdot z} - 1 - i\xi \cdot z \mathbf{1}_{\{|z| \leq 1\}} \right) \frac{A(d,-\alpha)}{|z|^{d+\alpha}} \, dz = -|\xi|^\alpha, \quad \xi \in \mathbb{R}^d. \quad (3.2)$$

In fact, $A(d,-\alpha)$ can be computed explicitly in terms of $\Gamma$-function:

$$A(d,-\alpha) = \alpha^{2\alpha-1} \pi^{-d/2} \Gamma\left(\frac{d+\alpha}{2}\right) \Gamma\left(1 - \frac{\alpha}{2}\right).$$

When $\alpha \in (1,2)$ as is assumed in this paper, the $\mathbf{1}_{\{|z| \leq 1\}}$ term can be dropped from both (3.1) and (3.2). It is also known that the symmetric $\alpha$-stable process $Y$ has Lévy intensity function

$$J(x,y) = A(d,-\alpha)|x-y|^{-(d+\alpha)}. \quad (3.3)$$

The Lévy intensity function gives rise to a Lévy system $(N,H)$ for $X$, where $N(x,dy) = J(x,y)dy$ and $H_t = t$, which describes the jumps of the process $Y$.

Recall that $(X^b, \mathbb{P}_x, x \in \mathbb{R}^d)$ is the unique solution to the martingale problem for $(\mathcal{L}^b, C_c^\infty(\mathbb{R}^d))$ on the canonical Skorokhod space $\Omega := \mathcal{D}([0,\infty); \mathbb{R}^d)$. The following theorem shows that $X^b$ is a weak solution to the SDE (1.3).

**Theorem 3.1.** There exists a process $Z$ defined on $\Omega$ so that all its paths are right continuous and admit left limits (rcll), and that under each $\mathbb{P}_x$, $Z$ is a rotationally symmetric $\alpha$-stable process on $\mathbb{R}^d$ and

$$X^b_t = x + Z_t + \int_0^t b(X^b_s) \, ds, \quad t \geq 0. \quad (3.4)$$

**Proof.** By Theorem 2.2, $X^b$ is the same as the Feller process determined by kernel $q^b(t,x,y)$ in Proposition 2.1. Observe that it follows from (2.9) that $\mathbb{E}_x \left[ \int_0^t |b(X^b_s)| \, ds \right] < \infty$ for every $t > 0$.

Let $\{F_t; t \geq 0\}$ be the minimal augmented filtration generated by $X^b_t$. We know from [4, Theorem 2.6] that $X^b$ has the same Lévy system $(J(x,y)dy,t)$ as that of symmetric $\alpha$-stable process; that is, for any $x \in \mathbb{R}^d$, non-negative measurable function $f$ on $\mathbb{R}^d \times \mathbb{R}^d$ vanishing along the diagonal $\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : x = y\}$, predictable process $\xi_t$ and stopping time $T$ with respect to the filtration $\{F_t; t \geq 0\}$,

$$\mathbb{E}_x \left[ \sum_{s \leq T} \xi_s f(X^b_{s-}, X^b_s) \right] = \mathbb{E}_x \left[ \int_0^T \xi_s \left( \int_{\mathbb{R}^d} f(X^b_s,y)J(X^b_s,y) \, dy \right) \, ds \right]. \quad (3.5)$$

In particular, if $\sum_{s \leq t} f(X^b_{s-}, X^b_s)$ has $\mathbb{P}_x$-integrable variation, then

$$\int_0^t \left( \int_{\mathbb{R}^d} f(X^b_s,y)J(X^b_s,y) \, dy \right) \, ds$$

is its dual predictable projection, that is,

$$\sum_{s \leq t} f(X^b_{s-}, X^b_s) - \int_0^t \left( \int_{\mathbb{R}^d} f(X^b_s,y)J(X^b_s,y) \, dy \right) \, ds \quad (3.6)$$

is a $\mathbb{P}_x$-martingale (cf. [4, Definition 5.21 and Corollary 5.31]).
It follows from (3.5) that \( \sum_{s \in [0,t]} |X^b_s - X^b_{s-}| \mathbb{1}_{\{|X^b_s - X^b_{s-}| \geq 1\}} \) is \( \mathbb{P}_x \)-integrable and so

\[
M_t^{d,1} := \sum_{s \in [0,t]} (X^b_s - X^b_{s-}) \mathbb{1}_{\{|X^b_s - X^b_{s-}| \geq 1\}}
\]

is a \( \mathbb{P}_x \)-martingale. Moreover,

\[
M_t^{d,2} := \lim_{\epsilon \to 0} \sum_{s \in [0,t]} (X^b_s - X^b_{s-}) \mathbb{1}_{\{\epsilon < |X^b_s - X^b_{s-}| < 1\}}
\]

is a purely discontinuous \( \mathbb{P}_x \)-square-integrable martingale with \( M_t^{d,2} - M_t^{d,-2} = (X^b_t - X^b_{t-}) \mathbb{1}_{\{|X^b_t - X^b_{t-}| < 1\}} \).

Define \( Z_t = M_t^{d,1} + M_t^{d,2} \), which is a martingale under each \( \mathbb{P}_x \) with

\[
Z_t - Z_{t-} = X^b_t - X^b_{t-} \quad \text{for } t > 0.
\] (3.7)

Since \( (X^b, \mathbb{P}_x) \) solves the martingale problem, \( X^b \) is a semimartingale. In view of (3.7), it can be uniquely expressed as

\[
X^b_t = X^b_0 + M_t + Z_t + A_t,
\] (3.8)

where \( M = (M^1, \ldots, M^d) \) is a continuous local martingale and \( A \) is a continuous process of finite variation. We will use \((M^i, M^j)\) to denote the quadratic covariation of \( M^i \) and \( M^j \). For \( f \in C^\infty_c(\mathbb{R}^d) \), applying Ito’s formula to (3.8) (cf. [1, Theorem 9.35]) and using the Lévy system for \( X^b \) and (3.1), we have

\[
f(X^b_t) - f(X^b_0)
= \int_0^t \nabla f(X^b_s) d(M_s + Z_s) + \int_0^t \nabla f(X^b_s) dA_s + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X^b_s) d\langle M^i, M^j \rangle_s
+ \sum_{s \leq t} \left( f(X^b_s) - f(X^b_{s-}) - \nabla f(X^b_{s-}) \cdot (X^b_s - X^b_{s-}) \right)
\]

= local martingale + \( \int_0^t \nabla f(X^b_s) dA_s + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X^b_s) d\langle M^i, M^j \rangle_s
+ \int_0^t \int_{\mathbb{R}^d} (f(X^b_s + z) - f(X^b_s) - \nabla f(X^b_s) \cdot z) \frac{A(d, \alpha)}{|z|^{d+\alpha}} \, dz \, ds
\]

= local martingale + \( \int_0^t \nabla f(X^b_s) dA_s + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X^b_s) d\langle M^i, M^j \rangle_s + \int_0^t \Delta^{\alpha/2} f(X^b_s) ds \).

Since \( (X^b_t, \mathbb{P}_x) \) solves the martingale problem for \((\mathcal{L}^b, C^\infty_c(\mathbb{R}^d))\),

\[
M_t^b := f(X^b_t) - f(X^b_0) - \int_0^t \mathcal{L}^b f(X^b_s) ds
\]

is a martingale and so we conclude

\[
\int_0^t b(X^b_s) \cdot \nabla f(X^b_s) ds = \int_0^t \nabla f(X^b_s) dA_s + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X^b_s) d\langle M^i, M^j \rangle_s.
\]
Since the above holds for every \( f \in C^\infty_c(\mathbb{R}^d) \), we must have

\[ A_t = \int_0^t b(X^b_s) \, ds \quad \text{and} \quad \langle M^i, M^j \rangle_t = 0 \text{ for every } 1 \leq i, j \leq d. \]

Hence \( M = 0 \) and so by (3.8), \( X^b_t = X^b_0 + Z_t + \int_0^t b(X^b_s) \, ds \).

It remains to show that \( Z \) is a rotationally symmetric \( \alpha \)-stable process under \( \mathbb{P}_x \). For \( \xi \in \mathbb{R}^d \), applying Ito’s formula for \( f(x) = e^{i \xi \cdot x} \) to martingale \( Z_t \) and using Lévy system formula (3.5), we get

\[
\mathbb{E}_x[e^{i \xi \cdot Z_t}] = 1 + \mathbb{E}_x \left[ \sum_{s \leq t} \left( e^{i \xi \cdot Z_s} - e^{i \xi \cdot Z_{s-}} - e^{i \xi \cdot Z_{s-}} (Z_s - Z_{s-}) \right) \right]
\]

\[
= 1 + \mathbb{E}_x \left[ \sum_{s \leq t} e^{i \xi \cdot Z_{s-}} \left( e^{i \xi \cdot z} - 1 - i \xi \cdot z \right) \frac{A(d_\alpha - \alpha)}{|z|^{d+\alpha}} \, dz \, ds \right]
\]

\[
= 1 + \mathbb{E}_x \left[ \int_0^t \int_{\mathbb{R}^d} e^{i \xi \cdot Z_s} \left( e^{i \xi \cdot z} - 1 - i \xi \cdot z 1_{(|z| \leq 1)} \right) \frac{A(d_\alpha - \alpha)}{|z|^{d+\alpha}} \, dz \, ds \right]
\]

\[
= 1 - |\xi|^a \int_0^t \mathbb{E}_x \left[ e^{i \xi \cdot Z_s} \right] \, ds,
\]

where in the last equality, we used (3.2) and Fubini’s theorem. Set \( \phi(t) = \mathbb{E}_x[e^{i \xi \cdot Z_t}] \). We see from above that \( \phi(t) = 1 - |\xi|^a \int_0^t \phi(s) \, ds \). Differentiate in \( t \), one solves easily that \( \phi(t) = e^{-t|\xi|^a} \). Now by the Markov property of \( X^b \), we have for every \( s, t > 0 \),

\[
\mathbb{E}_x \left[ e^{i \xi \cdot (Z_{t+s} - Z_t)} | \mathcal{F}_t \right] = \mathbb{E}_{X^b_t} \left[ e^{i \xi \cdot Z_s} \right] = e^{-s|\xi|^a}.
\]

This proves that, under each \( \mathbb{P}_x \), \( Z \) is a process having independent stationary increments and its characteristic function is \( e^{-s|\xi|^a} \); that is, \( Z \) is a rotationally symmetric \( \alpha \)-stable process on \( \mathbb{R}^d \).

Now we are ready to complete the proof for the uniqueness of weak solution to the SDE (1.3).

**Proof of Theorem 1.2.** Fix \( x \in \mathbb{R}^d \). Theorem 3.1 gives the existence of a weak solution to SDE (1.3). Using Ito’s formula, every weak solution to (1.3) solves the martingale problem for \( (\mathcal{L}^b, C^\infty_c(\mathbb{R}^d)) \). So the uniqueness of weak solution to (1.3) follows from the uniqueness of the martingale problem for \( (\mathcal{L}^b, C^\infty_c(\mathbb{R}^d)) \), which is established in Theorem 2.2.

**Proof of Corollary 1.3.** By Theorem 2.2 \( \alpha \)-stable process \( X^b \) with drift \( b \) is the Feller process with transition density function \( q^b(t, x, y) \) constructed in Proposition 2.1. The conclusion of the Corollary follows readily from Proposition 2.1 and [4, Theorem 1.3].

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