CASTELNUOVO-MUMFORD REGULARITY OF GRAPHS

TÜRKER BIYIKOĞLU, YUSUF CIVAN*

Received May 5, 2015
Revised November 21, 2016
Online First March 27, 2018

We present new combinatorial results on the calculation of (Castelnuovo-Mumford) regularity of graphs. We introduce the notion of a prime graph over a field \( k \), which we define to be a connected graph with \( \text{reg}_k(G - x) < \text{reg}_k(G) \) for any vertex \( x \in V(G) \). We then exhibit some structural properties of prime graphs. This enables us to provide upper bounds to the regularity involving the induced matching number \( \text{im}(G) \). We prove that \( \text{reg}(G) \leq (\Gamma(G) + 1) \text{im}(G) \) holds for any graph \( G \), where \( \Gamma(G) = \max \{|N_G[x]\setminus N_G[y]| : xy \in E(G)\} \) is the maximum privacy degree of \( G \) and \( N_G[x] \) is the closed neighbourhood of \( x \) in \( G \). In the case of claw-free graphs, we verify that this bound can be strengthened by showing that \( \text{reg}(G) \leq 2 \text{im}(G) \). By analysing the effect of Lozin transformations on graphs, we narrow the search for prime graphs into graphs having maximum degree at most three. We show that the regularity of such graphs \( G \) is bounded above by \( 2 \text{im}(G) + 1 \). Moreover, we prove that any non-trivial Lozin operation preserves the primeness of a graph. That enables us to generate many new prime graphs from the existing ones.

We prove that the inequality \( \text{reg}(G/e) \leq \text{reg}(G) \leq \text{reg}(G/e) + 1 \) holds for the contraction of any edge \( e \) of a graph \( G \). This implies that \( \text{reg}(H) \leq \text{reg}(G) \) whenever \( H \) is an edge contraction minor of \( G \).

Finally, we show that there exist connected graphs satisfying \( \text{reg}(G) = n \) and \( \text{im}(G) = k \) for any two integers \( n \geq k \geq 1 \). The proof is based on a result of Januszkiewicz and Świątkowski on the existence of Gromov hyperbolic right angled Coxeter groups of arbitrarily large virtual cohomological dimension, accompanied with Lozin operations. In an opposite direction, we show that if \( G \) is a \( 2K_2 \)-free prime graph, then \( \text{reg}(G) \leq (\delta(G) + 3)/2 \), where \( \delta(G) \) is the minimum degree of \( G \).

* Both authors are supported by TÜBİTAK (grant no: 111T704), and the first author is also supported by ESF EUROCORES TÜBİTAK (grant no: 210T173)
1. Introduction

Computing or finding applicable bounds on the graded Betti numbers, the regularity and the projective dimension of a monomial ideal \( I \) in \( R = \mathbb{k}[x_1, \ldots, x_n] \) has been a central problem in combinatorial commutative algebra [13]. In general, very little is known even in the case where the ideal \( I \) is square-free. For square-free monomial ideals, the main approach to tackle such a problem is topological-combinatorial, that is, the instruments of (hyper)graph theory are the main tools when finding the exact value or establishing tight bounds on these invariants. A common way to associate a combinatorial object to a square-free monomial ideal \( I \) goes by the language of Stanley-Reisner theory. In detail, let \( I \) be minimally generated by square-free monomials \( m_1, \ldots, m_r \), where \( V = \{x_1, \ldots, x_n\} \). If we define \( F_i := \{x_j : x_j|m_i\} \) for any \( 1 \leq i \leq r \), then the family \( \Delta := \{K \subseteq V : F_i \not\subseteq K \text{ for any } i \in [r]\} \) is a simplicial complex on \( V \) whose minimal non-faces exactly correspond to the generators of \( I \). Under such an association, the ideal \( I \) is said to be the Stanley-Reisner ideal of the simplicial complex \( \Delta \), denoted by \( I = I_\Delta \), and the simplicial complex \( \Delta = \Delta_I \) is called the Stanley-Reisner complex of the ideal \( I \). In particular, when each monomial \( m_i \) is quadratic, then the pair \( G_I := (V, \{F_1, \ldots, F_r\}) \) is a (simple) graph on the set \( V \), and under such a correspondence, the ideal \( I \) is called the edge ideal of the graph \( G_I \).

These interrelations can be reversed in the following way. Let \( G = (V, E) \) be a graph on \( V \). The family of independent sets of \( G \) forms a simplicial complex \( \text{Ind}(G) \) on \( V \), the independence complex of \( G \), and the corresponding Stanley-Reisner ideal \( I_G := I_{\text{Ind}(G)} \) is exactly the edge ideal of \( G \).

The resulting one-to-one correspondence between square-free monomial ideals in \( R \) and simplicial complexes on the set \( V \) enables us to state the definitions of the invariants in question by the well-known Hochster’s formula [29, Chapter 2, Corollary 4.9]. For instance, if \( \Delta \) is a simplicial complex on \( V \), then

\[
\text{reg}(\Delta) := \text{reg}(R/I_\Delta) = \max\{j : \tilde{H}_{j-1}(\Delta[S]; \mathbb{k}) \neq 0 \text{ for some } S \subseteq V\},
\]

where \( \Delta[S] := \{F \in \Delta : F \subseteq S\} \) is the induced subcomplex of \( \Delta \) by \( S \), and \( \tilde{H}_*(-; \mathbb{k}) \) denotes the (reduced) singular homology. Since \( \text{reg}(I) = \text{reg}(R/I) + 1 \) for any ideal \( I \) in \( R \), we have \( \text{reg}(I_\Delta) = \text{reg}(\Delta) + 1 \). In a similar vein, we conclude that \( \text{reg}(G) := \text{reg}(\text{Ind}(G)) = \text{reg}(I_G) - 1 \) for any graph \( G \).

Most of the recent work on the graph’s regularity has been devoted to the existence of tight bounds on the regularity via other graph parameters. One parameter that is frequently useful is the induced matching number. Recall that a matching in a graph is a subset of edges no two of which share a vertex. An induced matching is a matching \( M \) if no two vertices belonging to
different edges of $M$ are adjacent. The maximum size of an induced matching of $G$ is known as the induced matching number $\text{im}(G)$ of $G$. By a theorem of Katzman [17], it is already known that the induced matching number $\text{im}(G)$ provides a lower bound for the regularity of a graph $G$, and the characterization of graphs in which the regularity equals to the induced matching number has been the subject of many recent papers [5,14,21,24,31,35]. Observe that for a graph $G$, if we have $\text{reg}(G) = \text{im}(G) = n$ for some $n \geq 1$, then $G$ contains an induced copy of the graph $nK_2$, and $n$ is the greatest integer with this property. On the other hand, the graph $nK_2$ has the minimal order among all graphs satisfying the equality $\text{reg}(G) = \text{im}(G) = n$. Such an observation brings the idea of decomposing a graph into its induced subgraphs for which each subgraph in the decomposition is inclusion-wise minimal with respect to having the given regularity. In other words, we call a connected graph $H$ a prime graph over a field $k$ if $\text{reg}_k(H - x) < \text{reg}_k(H)$ for any vertex $x \in V(H)$. Now, we say that a family $\mathcal{H} = \{H_1, \ldots, H_r\}$ of induced subgraphs of $G$ is a prime decomposition of $G$ over $k$, if each graph $H_i$ is a prime graph over $k$ and $H_1, \ldots, H_r$ are the connected components of an induced subgraph of $G$. We denote by $\mathcal{PD}_k(G)$, the set of prime decompositions of $G$ over $k$.

**Theorem 1.1.** For any graph $G$ and any field $k$, we have

$$\text{reg}_k(G) = \max \left\{ \sum_{i=1}^r \text{reg}_k(H_i) : \{H_1, \ldots, H_r\} \in \mathcal{PD}_k(G) \right\}.$$  

As the regularity is dependent on the characteristic of the coefficient field, so is the notion of primeness. We verify by a computer calculation that an example due to Morey and Villareal (Example 3.6 in [24]) is a prime graph over $\mathbb{Z}_2$, while it is not prime with respect to $\mathbb{Z}_3$ (see Figure 2). We call a graph $G$ as a perfect prime graph if it is a prime graph over any field. The graph $K_2$, the cycles $C_{3k+2}$ and the complement of cycles $\overline{C}_m$ for any $k \geq 1$ and $m \geq 5$ are examples of perfect prime graphs. In addition to these graphs, we show that the Möbius-Kantor graph, which is the generalized Petersen graph $G(8,3)$ is also a (3-regular bipartite) perfect prime graph.

Theorem 1.1 reduces the graph’s regularity computation into finding prime graphs as well as decompositions of a graph into primes. Obviously, finding such decompositions is still a difficult task. We follow two different roads. In one direction, we look for combinatorial conditions on graphs that may effect their primeness. Such observations allow us to provide upper bounds to the regularity of graphs involving the induced matching number.

**Theorem 1.2.** If $G$ is a claw-free graph, then $\text{reg}(G) \leq 2\text{im}(G)$. 

Note that Theorem 1.2 generalizes an earlier result of Nevo [25] on (claw,2K₂)-free graphs, which also answers a question stated implicitly in [35].

We define the maximum privacy degree of a graph \( G \) by

\[
\Gamma(G) = \max\{|N_G[x]\setminus N_G[y]} : xy \in E(G)\},
\]

where \( N_G[x] \) is the closed neighbourhood of \( x \) in \( G \).

**Theorem 1.3.** The inequality \( \text{reg}(G) \leq (\Gamma(G) + 1)\text{im}(G) \) holds for any graph \( G \).

Note that the inequality \( \Gamma(G) + 1 \leq \Delta(G) \) holds for any graph \( G \), where \( \Delta(G) \) is the maximum degree of \( G \). Therefore, we have the upper bound \( \text{reg}(G) \leq \Delta(G)\text{im}(G) \) as a consequence of Theorem 1.3.

The other direction is shaped by a detailed analysis of the impacts of edge contractions and subdivisions, vertex expansions and Lozin operations on graphs to the regularity.

**Theorem 1.4.** The inequality \( \text{reg}(G/e) \leq \text{reg}(G) \leq \text{reg}(G/e) + 1 \) holds for the contraction of any edge \( e \) of \( G \).

One particular implication of Theorem 1.4 is that the inequality \( \text{reg}(H) \leq \text{reg}(G) \) holds whenever \( H \) is a contraction minor of \( G \). On the other hand, it follows that not only the induced primes of \( G \) but also prime graphs that are contraction minors of \( G \) affect the regularity of \( G \).

We verify that the effects of a double edge subdivision on an edge and the contraction of that edge to the regularity are closely related. This fact enables us to describe the effect of a Lozin operation on the regularity. Recall that any Lozin operation [19] works by considering a vertex \( x \) of a graph \( G \) whose (open) neighbourhood is split into two disjoint parts \( N_G(x) = Y_1 \cup Y_2 \), and replacing the vertex \( x \) with a four-path on \( \{y_1,a,b,y_2\} \) together with edges \( uy_i \) for any \( u \in Y_i \) and \( i = 1,2 \) (see Subsection 4.1). One of the interesting results of Lozin’s work is that the induced matching problem remains NP-hard in a narrow subclass of bipartite graphs. We here prove that his operation has a similar effect on the regularity:

**Corollary 1.5.** Let \( G = (V,E) \) be a graph and let \( x \in V \) be given. Then \( \text{reg}(L_x(G)) = \text{reg}(G) + 1 \), where \( L_x(G) \) is a Lozin transform of \( G \) with respect to the vertex \( x \).

We show that any non-trivial Lozin operation preserves the primeness of a graph. This enables us to generate many new prime graphs from the existing ones. In particular, it follows that the determination of prime graphs can be restricted to bipartite prime graphs having sufficiently large girth and having
maximum degree at most three. Moreover, we further strengthen the bound of Theorem 1.3 for graphs of maximum degree at most three by showing that \( \text{reg}(G) \leq 2\text{im}(G) + 1 \) holds for any such graph \( G \). In addition, if such a graph \( G \) is also prime other than \( K_2 \), we verify that it must be 2-connected.

There are various graph parameters that can be used to bound the regularity, and most of them are far from being tight in general. Recall that a well-known result of Kalai-Meshulam [16] states that \( \text{reg}(G) \leq \sum_{i=1}^{k} \text{reg}(H_i) \) holds whenever \( H_1, \ldots, H_k \) are subgraphs of a graph \( G \) satisfying \( E(G) = \bigcup E(H_i) \). This in particular implies that \( \text{reg}(G) \leq \text{cochord}(G) \), where \( \text{cochord}(G) \) is the least number of cochordal subgraphs \( H_1, \ldots, H_k \) of \( G \) such that \( E(G) = \bigcup E(H_i) \) [35, Lemma 1]. Woodroofe has constructed in [35] graphs for which the gap between the regularity and the cochordal cover number could be arbitrarily large. Moreover, while the bound of Theorem 1.3 is sharp, there exist graphs such that the gap between the regularity and the related upper bound could be arbitrarily large (compare to Figure 1). One particular reason for these large gaps is that when a graph \( G \) is not prime, some of its vertices have no effect to its regularity by Theorem 1.1, while any upper bound on \( \text{reg}(G) \) involving other graph parameters such as the induced matching number may be affected by those vertices.

![Figure 1. The graph \( H_n \)](image)

A similar gap occurs for the lower bound \( \text{im}(G) \leq \text{reg}(G) \). Let \( H_n \) be the graph obtained from \((n+1)\) disjoint five cycles by adding an extra vertex and connecting it to exactly one vertex of each five cycle (see Figure 1). Then the graph \( H_n \) is vertex-decomposable [5,18] and the equality \( \text{reg}(H_n) = \text{im}(H_n) + n \) holds for any \( n \geq 1 \).

Even if the gap between the regularity and the induced matching number could be arbitrarily large, this does not guarantee that any pair \((n,k)\) of positive integers can be realized as \((\text{reg}(G), \text{im}(G))\) for some graph \( G \). So, we may ask whether there exists a connected graph \( G(n,k) \) such that \( \text{reg}(G(n,k)) = n \) and \( \text{im}(G(n,k)) = k \) for every pair \((n,k)\) of integers with
n ≥ k ≥ 1 (compare to Question 7.1. (5) in [34]). The case k = 1 is of particular importance, since no example is known when n ≥ 5 [26]. We remark that for n = 4, there is the Coxeter 600-cell $X_{600}$ [28], which is the independence complex of a $2K_2$-free graph $G_{600}$ and the geometric realization of $X_{600}$ is homeomorphic to the 3-dimensional sphere so that $\text{reg}(G_{600}) = 4$. Furthermore, Przytycki and Świątkowski [28] show that no generalized homology sphere of dimension $n ≥ 4$ can be triangulated as the independence complex of a $2K_2$-free graph. The existence of $2K_2$-free graphs of arbitrary large regularity can be deduced from the result of Januszkiewicz and Świątkowski [15], stating\footnote{In fact, the statement of their result is somewhat different than what we express here (see Section 5 for details).} that there exists a $2K_2$-free graph $G_n$ such that its independence complex is an oriented pseudomanifold of dimension $(n - 1)$ for any $n ≥ 1$.

Now, combining Corollary 1.5 and the Januszkiewicz and Świątkowski’s result, we have a complete answer:

**Corollary 1.6.** For any two integers $n ≥ k ≥ 1$, there exists a connected graph $G(n, k)$ satisfying $\text{reg}(G(n, k)) = n$ and $\text{im}(G(n, k)) = k$.

Observe that the graph $G_n$ is a perfect prime, since $\text{Ind}(G_n)$ is an oriented pseudomanifold. It implies that the resulting graph $G(n, k)$ is also a perfect prime when $n > k$ as it is obtained from $G_n$ by successive Lozin operations (see Section 5 for details).

**The organization of the paper:** The conventional background notation and information that may be needed in the sequel are introduced in the Preliminaries section. The prime graphs and prime factorizations are introduced in Section 3, where we also investigate the structural properties of prime graphs, and provide the proofs of stated upper bounds to the regularity involving the induced matching number. Section 4 is devoted to the detailed analysis of the behavior of the regularity under specific combinatorial operations. The proof of Corollary 1.6 is given in Section 5, where we also provide an upper bound on the regularity of $2K_2$-free prime graphs.

### 2. Preliminaries

We first recall some general notions and notations needed throughout the paper, and repeat some of the definitions mentioned in the introduction more formally. We choose to follow the standard terminology from combinatorial commutative algebra [13,24] and topological combinatorics [18]. For any undefined terms, we refer to these references. We remark that most of our
results are independent of the characteristic of the coefficient field $k$, so whenever it is appropriate we drop $k$ from our notation.

**Graphs:** By a (simple) graph $G$, we will mean an undirected finite graph without loops or multiple edges. If $G$ is a graph, $V(G)$ and $E(G)$ (or simply $V$ and $E$) denote its vertex and edge sets. If $U \subset V$, the graph induced on $U$ is written $G[U]$, and in particular, we abbreviate $G[V \setminus U]$ to $G - U$, and write $G - x$ whenever $U = \{x\}$. For a given subset $U \subseteq V$, the graph induced on $U$ is written $G[U]$, and in particular, we abbreviate $G[V \setminus U]$ to $G - U$, and write $G - x$ whenever $U = \{x\}$. For a given subset $U \subseteq V$, the (open) neighbourhood of $U$ is defined by $N_G(U) := \cup_{u \in U} N_G(u)$, where $N_G(u) := \{v \in V : uv \in E\}$, and similarly, $N_G[U] := N_G(U) \cup U$ is the closed neighbourhood of $U$. Furthermore, if $F = \{e_1, \ldots, e_k\}$ is a subset of edges of $G$, we write $N_G[F]$ for the set $N_G[V(F)]$, where $V(F)$ is the set of vertices incident to edges in $F$. The degree of a vertex $x$, the maximum and the minimum degrees of a graph $G$ are denoted by $\deg_G(x)$, $\Delta(G)$ and $\delta(G)$, respectively.

Throughout $K_n$, $P_n$ and $C_k$ will denote the complete, path and cycle graphs on $n \geq 1$ and $k \geq 3$ vertices, respectively. Moreover, we denote by $K_{n,m}$, the complete bipartite for any $n,m \geq 1$. In particular, the graph $K_{1,3}$ is known as the claw graph.

We say that $G$ is $H$-free if no induced subgraph of $G$ is isomorphic to $H$. A graph $G$ is called chordal if it is $C_r$-free for every $r > 3$. Moreover, a graph $G$ is said to be cochordal if its complement $\overline{G}$ is a chordal graph. A split graph is a graph in which the vertices can be partitioned into a clique and an independent set. Note that the family of split graphs constitutes a subclass of cochordal graphs.

**Simplicial complexes:** An (abstract) simplicial complex $\Delta$ on a finite set $V$ is a family of subsets of $V$ satisfying the following properties.

(i) $\{v\} \in \Delta$ for all $v \in V$,

(ii) If $F \in \Delta$ and $H \subset F$, then $H \in \Delta$.

The elements of $\Delta$ are called faces of it; the dimension of a face $F$ is $\dim(F) := |F| - 1$, and the dimension of $\Delta$ is defined to be $\dim(\Delta) := \max\{\dim(F) : F \in \Delta\}$. The 0 and 1-dimensional faces of $\Delta$ are called vertices and edges while maximal faces are called facets.

Although regularity is not a topological invariant, the use of topological methods plays certain roles. In many cases, we will appeal to an induction on the cardinality of the vertex set by a particular choice of a vertex accompanied by two subcomplexes. To be more explicit, if $x$ is a vertex of $\Delta$, then the subcomplexes del$_{\Delta}(x) := \{F \in \Delta : x \notin F\}$ and lk$_{\Delta}(x) := \{R \in \Delta : x \notin R \text{ and } R \cup \{x\} \in \Delta\}$ are called the deletion and link of
$x$ in $\Delta$, respectively. Such an association brings the use of a Mayer-Vietoris sequence of the pair $(\Delta,x)$:

$$\cdots \to \tilde{H}_j(\text{lk}_\Delta(x)) \to \tilde{H}_j(\text{del}_\Delta(x)) \to \tilde{H}_j(\Delta) \to \tilde{H}_{j-1}(\text{lk}_\Delta(x)) \to \cdots \to \tilde{H}_0(\Delta) \to 0.$$  

When considering the complex $\text{Ind}(G)$ of a graph, the deletion and link of a given vertex $x$ correspond to the independence complexes of induced subgraphs, namely that $\text{del}_{\text{Ind}(G)}(x) = \text{Ind}(G-x)$ and $\text{lk}_{\text{Ind}(G)}(x) = \text{Ind}(G-N_G[x])$.

The following provides an inductive bound on the regularity of graphs.

**Corollary 2.1** ([9,24]). Let $G$ be a graph and let $v \in V$ be given. Then

$$\text{reg}(G) \leq \max\{\text{reg}(G-v), \text{reg}(G-N_G[v]) + 1\}.$$  

Moreover, $\text{reg}(G)$ always equals to one of $\text{reg}(G-v)$ or $\text{reg}(G-N_G[v]) + 1$. In particular, the inequality $\text{reg}(G) \leq \text{reg}(G-v) + 1$ holds.

The existence of vertices satisfying certain extra properties is useful when dealing with the homotopy type of simplicial complexes (see [22] for details). Recall that when $(X,A)$ is a CW pair, then $A$ is said to be contractible in $X$, if the inclusion map $A \hookrightarrow X$ is homotopic to a constant map [12, Example 0.14].

**Theorem 2.2.** If $\text{lk}_\Delta(x)$ is contractible in $\text{del}_\Delta(x)$, then $\Delta \simeq \text{del}_\Delta(x) \lor \Sigma(\text{lk}_\Delta(x))$, where $\Sigma X$ denotes the (unreduced) suspension of $X$.

In the case of the independence complexes of graphs, the existence of a (closed or open) dominated vertex may guarantee that the required condition of Theorem 2.2 holds.

**Corollary 2.3** ([11,22]). If $N_G(v) \subseteq N_G(u)$, then there is a homotopy equivalence $\text{Ind}(G) \simeq \text{Ind}(G-v)$. On the other hand, if $N_G[u] \subseteq N_G[v]$, then the homotopy equivalence $\text{Ind}(G) \simeq \text{Ind}(G-v) \lor \Sigma \text{Ind}(G-N_G[v])$ holds.

**Definition 2.4.** An edge $e = uv$ is called an isolating edge of a graph $G$ with respect to a vertex $w$, if $w$ is an isolated vertex of $G-N_G[e]$.

**Theorem 2.5** ([1]). If $\text{Ind}(G-N_G[e])$ is contractible, then the natural inclusion $\text{Ind}(G) \hookrightarrow \text{Ind}(G-e)$ is a homotopy equivalence.
In particular, Theorem 2.5 implies that $\text{Ind}(G) \simeq \text{Ind}(G-e)$ whenever the edge $e$ is isolating. This brings the use of an operation, adding or removing an edge, on a graph without altering its homotopy type. We will follow [1] to write $\text{Add}(x,y;w)$ (respectively $\text{Del}(x,y;w)$) to indicate that we add the edge $e=xy$ to (resp. remove the edge $e=xy$ from) the graph $G$, where $w$ is the corresponding isolated vertex.

**Remark 2.6.** In order to simplify the notation, we note that when we mention the homology, homotopy or a suspension of a graph, we mean that of its independence complex. Thus, when appropriate, we drop $\text{Ind}(-)$ from our notation.

### 3. Prime graphs and Prime Factorizations

As we have already mentioned in Section 1, the notion of primeness brings a new strategy for the calculation of the regularity. We first restate its definition more formally.

**Definition 3.1.** A connected graph $G$ is called a prime graph over a field $\mathbb{k}$, if $\text{reg}_\mathbb{k}(G-x) < \text{reg}_\mathbb{k}(G)$ for any vertex $x \in V(G)$. Furthermore, we call a connected graph $G$ as a perfect prime graph if it is a prime graph over any field.

There is the degenerate case, the null graph $N = (\emptyset, \emptyset)$, where $\text{Ind}(N) = \{\emptyset\}$ in which we count it as the (trivial) perfect prime. This is consistent with the usual conventions that $\tilde{H}_{-1}([\emptyset]; \mathbb{k}) \cong \mathbb{k}$ and $\tilde{H}_p([\emptyset]; \mathbb{k}) \cong 0$ for any $p \neq -1$ in that case.

Note that if $G$ is a prime graph, then $\text{reg}(G) = \text{reg}(G-N_G[x]) + 1$ for any vertex $x \in V$ as a consequence of Corollary 2.1.

Recall that regularity is in general not a topological invariant. However, if the graph is prime, then the determination of its homology suffices for the calculation of its regularity. In other words, the equality $\text{reg}_\mathbb{k}(G) = \min\{i : \tilde{H}_j(G; \mathbb{k}) = 0 \text{ for any } j > i\} + 1$ holds for any prime graph over $\mathbb{k}$.

We verify by a computer calculation [30] that a graph provided by Morey and Villareal (Example 3.6 in [24]) is a prime graph over $\mathbb{Z}_2$, while it is not prime with respect to $\mathbb{Z}_3$ (see Figure 2).

Apart from the already existing examples of perfect prime graphs, our next example shows that there exists a 3-regular bipartite perfect prime graph.
Example 3.2. The bipartite 3-regular Möbius-Kantor graph $G_{MK}$ (see Figure 3), also known as the generalized Petersen graph $G(8,3)$ [23], is a perfect prime graph. Observe first that $G_{MK}$ is a vertex-transitive graph. In order to verify that $G_{MK}$ is a perfect prime, we have computed the induced matching and cochordal cover numbers of the corresponding graphs as follows.
The computation of the cochordal cover number in each case follows easily from Theorem 27 of [5], since the girth of $G_{MK}$ is 6. This in particular implies that $\text{reg}(G - x) = 4$. The homotopy type of $G_{MK} - N_{G_{MK}}[x]$ can be deduced by repeated use of Theorem 2.2 and Corollary 2.3. In fact, if we define $H := G_{MK} - N_{G_{MK}}[1]$, then $H - 3'$ is contractible, hence $H \approx \Sigma(H - N_{H}[3'])$. It turns out that $H \approx S^2 \vee S^3$, which forces $\text{reg}(G_{MK} - N_{G_{MK}}[x]) = 4$. To derive the homotopy type of $G_{MK}$, we apply to Example 4C.2 of [12] together with a homology calculation of $G_{MK}$ in Sage [30]. We have $\tilde{H}_3(G_{MK}; \mathbb{Z}) \cong \mathbb{Z}^4$, $\tilde{H}_4(G_{MK}; \mathbb{Z}) \cong \mathbb{Z}$ and the rest of the groups are trivial. Furthermore, we note that by a result of Engström [11, Theorem 3.13], the independence complex of $G_{MK}$ is simply-connected. It then follows that

$$G_{MK} \cong \bigvee_{i=1}^{4} S^2 \vee S^4,$$

which in particular implies that $\text{reg}(G_{MK}) \geq 5$. Therefore, we have $\text{reg}(G_{MK}) = 5$ by Corollary 2.1, hence, the graph $G_{MK}$ is a perfect prime.

We further note that the graph $G_{MK}$ lacks certain properties. Firstly, since $S = \{2, 3', 6'\}$ is an independent set, and since $G_{MK} - N_{G_{MK}}[S] \cong P_5$ is not well-covered, the full graph $G_{MK}$ is not well-covered. Moreover, it is not vertex-decomposable as it contains no shedding vertex; hence, it is not even sequentially Cohen-Macaulay by [31, Theorem 2.10].

We next introduce the idea of decomposing a graph into its induced primes that will be needed throughout.

**Definition 3.3.** Let $G$ be a graph and let $\mathcal{R} = \{R_1, \ldots, R_r\}$ be a family of induced subgraphs of $G$ such that $|V(R_i)| \geq 2$ for each $1 \leq i \leq r$. Then $\mathcal{R}$ is said to be an *induced decomposition* of $G$ if $R_1, \ldots, R_r$ are the connected components of an induced subgraph of $G$. The set of induced decompositions of a graph $G$ is denoted by $\mathcal{TD}(G)$.

We note that any induced matching of a graph $G$ is a special induced decomposition of $G$.

**Definition 3.4.** Let $\mathcal{R} = \{R_1, \ldots, R_r\}$ be an induced decomposition of a graph $G$. If each $R_i$ is a prime graph, then we call $\mathcal{R}$ as a *prime decomposition* of $G$, and the set of prime decompositions of a graph $G$ is denoted by $\mathcal{PD}(G)$.

Obviously, the set $\mathcal{PD}(G)$ is non-empty for any graph $G$. 

| $H$  | $G_{MK}$ | $G_{MK} - x$ | $G_{MK} - N_{G_{MK}}[x]$ |
|------|----------|---------------|---------------------------|
| $\text{cochord}(H)$ | 6        | 4             | 4                         |
| $\text{im}(H)$      | 4        | 4             | 3                         |
Proof of Theorem 1.1. If $G$ is itself a prime graph, there is nothing to prove. Otherwise there exists a vertex $x \in V$ such that $\text{reg}(G) = \text{reg}(G - x)$. If $G - x$ is a prime graph, then $\{G - x\} \in \mathcal{PD}(G)$ so that the result follows. Otherwise, we have $\text{reg}(G - x) = \max\{\sum_{i=1}^{t} \text{reg}(S_i) : \{S_1, \ldots, S_t\} \in \mathcal{PD}(G - x)\}$ by the induction. However, since $\mathcal{PD}(G - x) \subseteq \mathcal{PD}(G)$ for such a vertex, the claim follows.

Definition 3.5. A prime decomposition $\mathcal{R}$ of a graph $G$ for which the equality of Theorem 1.1 holds is called a prime factorization of $G$, and the set of prime factorizations of $G$ is denoted by $\mathcal{PF}(G)$.

We note that for any prime decomposition $\mathcal{R} = \{R_1, \ldots, R_k\}$ of $G$, the inequality $k \leq \text{im}(G)$ always holds. Moreover, it is also possible that $\sum_{i=1}^{k} \text{im}(R_i) < \text{im}(G)$, even if $\mathcal{R} \in \mathcal{PF}(G)$. For instance, if $H_n$ is the graph in Figure 1, then the family consisting of $(n + 1)$ disjoint copies of $C_5$ is a prime factorization of $H_n$, while $\sum_{i=1}^{n+1} \text{im}(C_5) = n + 1 < n + 2 = \text{im}(H_n)$.

We now look for combinatorial conditions on a graph that may affect its primeness or its induced primes. The proof of the following is immediate from Corollary 2.3.

Proposition 3.6. If $N_G(y) \subseteq N_G(x)$ for vertices $x$ and $y$, then $G$ can not be a prime graph. Similarly, if $N_G[u] \subseteq N_G[v]$ holds in $G$ such that $\deg_G(v) \geq 2$, then $G$ can not be a prime graph.

One of the immediate consequences of Proposition 3.6 is that the equality $\text{reg}(G) = \text{im}(G)$ holds for any graph in a hereditary graph class defined in terms of having a non-isolated (closed or open) dominated vertex, since only prime graph that such a graph in that class can contain is isomorphic to a $K_2$. For instance, the family of such graphs contains all chordal graphs [5,13,14] or distance-hereditary graphs [7]. We also note that distance-hereditary graphs are contained in the class of weakly chordal graphs [7]; hence, the equality $\text{reg}(G) = \text{im}(G)$ for such graphs is already known [35]. Observe that if $H$ is a prime graph for which the equality $\text{reg}(H) = \text{im}(H)$ holds, then $H$ must be isomorphic to $K_2$. This in particular forces that any weakly chordal graph can not contain primes other than a $K_2$.

As we have already mentioned in Section 1, the determination of prime graphs can be restricted to those having maximum degree at most three by the effect of Lozin operations (see Section 4.1). On this direction, we have the following partial result on the structure of such graphs.

Theorem 3.7. If $G$ is a prime graph with $\Delta(G) \leq 3$ and $|G| \geq 3$, then $G$ is 2-connected.
Proof. Suppose that $G$ is a prime graph that is not 2-connected. Then there exists a vertex $v$ such that $G - v$ is disconnected. By the degree constraint, $G - v$ has a connected component $G_0$ such that $v$ has a unique neighbour $w$ in $V(G_0)$. If we set $R := (G - v) - V(G_0)$, it follows that

$$\text{reg}(G) - 1 = \text{reg}(G - v) = \text{reg}(G_0) + \text{reg}(R)$$

$$= \text{reg}(G - N_G[v]) = \text{reg}(G_0 - w) + \text{reg}(R - N_R(v))$$

$$= \text{reg}(G - N_G[w]) = \text{reg}(G_0 - N_G[w]) + \text{reg}(R)$$

by Corollary 2.1. Therefore, we have $\text{reg}(G_0) = \text{reg}(G_0 - N_G[w])$ and $\text{reg}(R) = \text{reg}(R - N_R(v))$. However, this means that $\{K_2, G_0 - N_G[w], R - N_R(v)\}$ is an induced decomposition of $G$, where the graph $K_2$ is induced by the pair $\{v, w\}$. We may replace each subgraph in this family, if necessary, with a prime factorization of itself that creates a prime factorization for $G$ other than itself, a contradiction.

We next present a reduction process as a direct consequence of Corollary 2.1 from which we derive some new upper bounds on the regularity. We call a vertex $v$ of a graph $G$ a prime vertex if $\text{reg}(G - v) < \text{reg}(G)$. Now, given a subset $F \subseteq V$, we repeatedly apply Corollary 2.1 to the vertices of $F$ and associate an integer prime $i$ at each state $i \geq 0$. We start with the graph $G_0 := G$ and set $F_0 := F$ and $\text{prime}_0 := 0$. Pick a vertex $x_i \in F_i$ for some $i \geq 0$. If $x_i$ is a prime vertex of $G_i$, we define $G_{i+1} := G_i - N_{G_i}[x_i]$, $F_{i+1} := F_i - N_{F_i}[x_i]$ and $\text{prime}_{i+1} := \text{prime}_i + 1$. On the other hand, if $x_i$ is not a prime vertex of $G_i$, we then set $G_{i+1} := G_i - x_i$, $F_{i+1} := F_i - x_i$ and $\text{prime}_{i+1} := \text{prime}_i$. The reduction process terminates when $F_k = \emptyset$ for some $k > 0$, in which case we denote by $G_F$ and $\text{prime}_F$, the resulting graph and the count of in how many steps a closed neighbourhood was deleted. Observe that for any subset $F \subseteq V$, the inequality $\text{reg}(G) \leq \text{reg}(G_F) + \text{prime}_F$ holds.

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. We proceed by an induction on the order of $G$. Let $M = \{u_1v_1, \ldots, u_kv_k\}$ be a maximum induced matching of $G$. We choose $F := V(M)$, and apply the reduction process on $G$ with respect to $F$. Assume that $p_{i_1}, \ldots, p_{i_r} \in F$ are the prime vertices iteratively detected along the reduction process for some $r \geq 0$. Firstly, if $r = 0$, that is, there exists no prime vertex at any stage of the reduction process, then $G_F = G - V(M)$. It then follows that $\text{reg}(G) = \text{reg}(G - V(M)) \leq 2\text{im}(G - V(M)) \leq 2\text{im}(G)$, where the first inequality is due to the induction. Thus, we may assume that $r > 0$. We define $e_s := u_{i_s}v_{i_s}$ for each $s \in [r]$ such that $p_{i_s} \in e_s$. We set $T := \bigcup_{i=1}^r N_G[e_i]$, and run the reduction process on $G_F$ with respect to
the set $F' := V(G_F) \cap T$. Observe that if $p_{ij} = u_{ij}$ for some $j \in [r]$, the set $N_G[e_j] \setminus N_G[u_{ij}]$ is a clique by the claw-freeness of $G$. Therefore, the counter is incremented at most once for each $j \in [r]$. It then follows that the counter of this second reduction process can be at most $r$.

Now, if we let $H := (G_F)_{F'}$ be the graph resulting from the reduction process on $G_F$ with respect to $F'$, we conclude that $\text{reg}(G) \leq \text{reg}(H) + 2r$. On the other hand, we note that $\text{im}(H) \leq \text{im}(G) - r$, since the union of any maximum induced matching in $H$ with $\{u_{i_1}v_{i_1}, \ldots, u_{i_r}v_{i_r}\}$ gives an induced matching in $G$. When we apply the induction hypothesis, we obtain that $\text{reg}(H) \leq 2\text{im}(H) \leq 2\text{im}(G) - 2r$; hence, $\text{reg}(G) \leq 2\text{im}(G)$ as claimed.

We remark that the bound of Theorem 1.2 is tight as shown by the graph $C_n$ for any $n \geq 5$. However, we do not know whether a claw-free graph can contain primes other than a $K_2$ or a $C_n$ for any $n \geq 5$.

Before the proof of Theorem 1.3, we note that the maximum privacy degree $\Gamma(G)$ of a graph is monotone decreasing under taking induced subgraphs.

**Proof of Theorem 1.3.** We proceed exactly as in the proof of Theorem 1.2, by choosing a maximum induced matching $M$ and applying the reduction process. If $r = 0$, it then follows that $\text{reg}(G) = \text{reg}(G - V(M)) \leq (\Gamma(G - V(M)) + 1)\text{im}(G - V(M)) \leq (\Gamma(G) + 1)\text{im}(G)$, where the first inequality is due to the induction. When $r > 0$, we again apply the reduction process on $G_F$ with respect to the set $F' := V(G_F) \cap T$. In this case, observe that if $p_{ij} = u_{ij}$ for some $j \in [r]$, the set $N_G[e_j] \cap F'$ can contain at most $\Gamma(G)$ vertices. It then follows that the counter of this second reduction process can be at most $r\Gamma(G)$.

Now, if we let $H := (G_F)_{F'}$ be the graph resulting from the reduction process on $G_F$ with respect to $F'$, we conclude that $\text{reg}(G) \leq \text{reg}(H) + r + r\Gamma(G)$. When we apply the induction hypothesis, we obtain that $\text{reg}(H) \leq (\Gamma(H) + 1)\text{im}(H) \leq (\Gamma(G) + 1)(\text{im}(G) - r)$; hence, $\text{reg}(G) \leq (\Gamma(G) + 1)\text{im}(G)$ as claimed.

We remark that we have $\text{reg}(G) \leq \Delta(G)\text{im}(G)$ by Theorem 1.3. In particular, it follows that $\text{reg}(G) \leq 3\text{im}(G)$ for graphs with $\Delta(G) \leq 3$. However, we may strengthen it further by a detail look at their local structure.

**Theorem 3.8.** Let $G$ be a connected graph with $\Delta(G) \leq 3$. Then, we have $\text{reg}(G) \leq 2\text{im}(G) + 1$. In particular, if $G$ is non 3-regular, then $\text{reg}(G) \leq 2\text{im}(G)$.

**Proof.** Suppose first that $G$ is a connected non 3-regular graph with $\Delta(G) \leq 3$. We proceed by the induction on the order of $G$. We first prove the claim
when $G$ is a prime graph. By Proposition 3.6, if $G \not\cong K_2$, then $G$ has no vertex of degree 1. Let $v \in V$ be vertex of degree 2 with neighbours $a, b$. Once again, by Proposition 3.6, the vertices $a$ and $b$ are not adjacent in $G$. Now we apply Corollary 2.1 at the vertex $b$ to obtain

$$
\text{reg}(G) = \text{reg}(G - N_G[a]) + 1
\leq \max\{\text{reg}((G - N_G[a]) - b), \text{reg}(G - N_G[\{a, b\}]) + 1\} + 1
\leq \text{reg}((G - N_G[a]) - b) + 2,
$$

where the last inequality follows from the fact that $G - N_G[[a, b]]$ is an induced subgraph of $(G - N_G[a]) - b$.

Note that we have $(G - N_G[a]) - b \cong G - N_G[av]$, and the induced matching number of $G - N_G[av]$ is at least 1 less than that of $G$. Thus, we can consider a prime factorization of $G - N_G[av]$ to conclude that

$$
\text{reg}(G) \leq \text{reg}(G - N_G[av]) + 2 \leq 2\text{im}(G - N_G[av]) + 2
\leq 2(\text{im}(G) - 1) + 2 \leq 2\text{im}(G),
$$

where the second inequality is due to the induction.

Assume now that $G$ is not a prime graph, and let $\{G_1, \ldots, G_k\}$ be a prime factorization of $G$. Since $G$ is connected, the graph $G_i$ is non 3-regular with $\Delta(G_i) \leq 3$ for each $i \in [k]$ so that $\text{reg}(G_i) \leq 2\text{im}(G_i)$ by the induction. But then $\text{reg}(G) = \sum_{i=1}^k \text{reg}(G_i) \leq 2(\sum_{i=1}^k \text{im}(G_i)) \leq 2\text{im}(G)$.

Finally, let $G$ be a 3-regular connected graph. If $G$ is not prime, then there exists a vertex $x \in V$ such that $\text{reg}(G) = \text{reg}(G - x)$. However, the connected components of the graph $G - x$ can not be 3-regular, since $G$ is connected; thus, $\text{reg}(G) = \text{reg}(G - x) \leq 2\text{im}(G - x) \leq 2\text{im}(G)$ by the previous case. On the other hand, if $G$ is a prime graph, then $\text{reg}(G) = \text{reg}(G - N_G[x]) + 1$ for any vertex $x \in V$, while each connected component of $G - N_G[x]$ is not 3-regular. Therefore, we conclude that $\text{reg}(G) = \text{reg}(G - N_G[x]) + 1 \leq 2\text{im}(G - N_G[x]) + 1 \leq 2\text{im}(G) + 1$ as claimed.

4. Contractions, expansions, subdivisions and regularity

The purpose of this section is to measure the effect of various operations on graphs to the regularity. Our analysis is divided into two parts. We first prove Theorem 1.4, and then introduce combinatorial operations on graphs under which the regularity is stable. We next show that the regularities of the graphs resulting from an edge contraction and the double edge subdivision are closely related. This latter fact enables us to describe the effect of a Lozin operation on the regularity of a graph in Subsection 4.1.
Let $e = xy$ be an edge of a graph $G$. Then the contraction of $e$ on $G$ is the graph $G/e$ defined by $V(G/e) = (V(G) \setminus \{x,y\}) \cup \{w\}$ and $E(G/e) = E(G - \{x,y\}) \cup \{wz: z \in N_G(x) \cup N_G(y)\}$.

In order to prove Theorem 1.4, we first state the following result.

**Lemma 4.1.** If $e = xy$ is an edge of a graph $G$, then $\text{reg}(G - N_G[e]) \leq \text{reg}(G) - 1$.

**Proof.** Observe that $\{G - N_G[e], K_2\}$ is an induced decomposition of $G$, where $K_2$ corresponds to the edge $e = xy$. Therefore, we have $\text{reg}(G) \geq \text{reg}(G - N_G[e]) + 1$.

**Proof of Theorem 1.4.** The subgraph consisting of all edges incident to $e$ is a split graph, hence cochordal. It follows that $\text{reg}(G) \leq \text{reg}(G - \{x,y\}) + 1$ by the Kalai-Meshulam Theorem [16, Theorem 1.2]. However, since $G - \{x,y\}$ is an induced subgraph of $G/e$, the upper bound follows.

Suppose next that $w_e$ is the vertex of $G/e$ obtained by contracting the edge $e = xy$ in $G$. Observe that $(G/e) - w_e \cong G - \{x,y\}$ and $(G/e) - N_{(G/e)}[w_e] \cong G - N_G[e]$. By Corollary 2.1, there are two cases. Now, if $\text{reg}(G/e) = \text{reg}((G/e) - w_e) = \text{reg}(G - \{x,y\}) \leq \text{reg}(G)$. On the other hand, if $\text{reg}(G/e) = \text{reg}((G/e) - N_{(G/e)}[w_e]) + 1 = \text{reg}(G - N_G[e]) + 1 \leq \text{reg}(G)$ by Lemma 4.1.

We recall that a graph $H$ is said to be a contraction minor of a graph $G$ if it is obtained from $G$ by a series of edge contractions on $G$.

**Corollary 4.2.** If $H$ is a contraction minor of $G$, then $\text{reg}(H) \leq \text{reg}(G)$.

Since we deal with edge contractions both on graphs and simplicial complexes, we next recall the definition of the latter in order to avoid confusion.

The effects of edge contractions on topology was initially studied by Walkup [32] for 3-manifolds, and it was later investigated in the language of independence complexes by Ehrenborg and Hetyei [10] (see also [2,3]).

Let $\Delta$ be a simplicial complex on the vertex set $V = V(\Delta)$, and consider $x,y \in V$ such that $\{x,y\} \in \Delta$ and $w_{xy} \notin V$. The edge contraction $xy \mapsto w_{xy}$ can be defined to be a map $f: V \to (V \setminus \{x,y\}) \cup \{w_{xy}\}$ by

$$f(v) := \begin{cases} v, & \text{if } v \notin \{x,y\}, \\ w_{xy}, & \text{if } v \in \{x,y\}. \end{cases}$$

We then extend $f$ to all simplices $F = \{v_0, v_1, \ldots, v_k\}$ of $\Delta$ by setting

$$f(F) := \{f(v_0), f(v_1), \ldots, f(v_k)\}.$$
The simplicial complex $\Delta_{xy} := \{ f(F) : F \in \Delta \}$ is called the contraction of $\Delta$ with respect to the edge $\{x, y\}$.

Observe that the contraction of an edge may not need to preserve the homotopy type of the complex in general. However, under a suitable restriction on the contracted edge, we can guarantee this to happen. We say that an edge $\{x, y\}$ is a contractible-edge in $\Delta$ if no minimal non-face of $\Delta$ contains it.

**Theorem 4.3** ([10, Theorem 2.4], [2, Theorem 1]). Let $\Delta$ be a simplicial complex and let $\{x, y\} \in \Delta$ be an edge. Then, the simplicial complexes $\Delta$ and $\Delta_{xy}$ are homotopy equivalent provided that the edge $\{x, y\}$ is a contractible-edge.

We note that the required condition in Theorem 4.3 is equivalent to $\text{lk}_\Delta(x) \cap \text{lk}_\Delta(y) = \text{lk}(\{x, y\})$. Furthermore, if $\Delta = \text{Ind}(G)$ for some graph $G$, then the contraction of any edge in $\text{Ind}(G)$ preserves the homotopy type.

We remark that in the rest of this section, we mainly deal with edge contractions on the independence complexes in which we choose to describe the operation on graphs, while their equivalence is ensured by Lemma 4.5.

We need to be sure that after an edge contraction on $\text{Ind}(G)$, the resulting complex is still the independence complex of a graph.

**Definition 4.4.** Let $G = (V, E)$ be a graph. Two non-adjacent vertices $\{x, y\}$ in $G$ is called a genuine-pair (or simply a $g$-pair) if there exist no vertices $u, v \in V \setminus \{x, y\}$ such that $G[\{x, y, u, v\}] \cong 2K_2$. When $\{x, y\}$ is a $g$-pair in $G$, the graph $g(G; xy)$ constructed by $V(g(G; xy)) := (V \setminus \{x, y\}) \cup \{w\}$ and $E(g(G; xy)) := E(G - \{x, y\}) \cup \{uw : u \in N_G(x) \cap N_G(y)\}$ is called the $g$-contraction of $G$ with respect to the pair $\{x, y\}$.

![Figure 4. A g-contraction](image)

The notion of a $g$-pair appears in the work of Lutz and Nevo [20] in dual form. The edge in the clique complex of $G$ corresponding to a $g$-pair is called an admissible edge. Moreover, the proof of the following is given there in this dual setting (see the proof of [20, Corollary 6.2])
Lemma 4.5. If \( \{x,y\} \) is a \( g \)-pair in \( G \), then the simplicial complexes \( \text{Ind}(g(G;xy)) \) and \( \text{Ind}(G)_{xy} \) are isomorphic.

We next introduce an operation that could be considered as the reversal of a \( g \)-contraction.

Let \( G = (V,E) \) be a graph. We say that two disjoint (possibly empty) subsets \( A \) and \( B \) of \( V \) constitute a complete-pairing in \( G \), denoted by \( [A,B] \), if \( ab \in E \) for any \( a \in A \) and \( b \in B \). We note that \( [A,\emptyset] \) or \( [\emptyset,B] \) is always a complete-pairing in \( G \). We say that a complete-pairing \( [A,B] \) is non-trivial provided that the sets \( A \) and \( B \) are both non-empty. Observe that if \( x \) and \( y \) are two nonadjacent vertices of \( G \), then \( \{x,y\} \) is a \( g \)-pair in \( G \) if and only if \( [N_G(x) \setminus N_G(y),N_G(y) \setminus N_G(x)] \) is a complete-pairing in \( G \).

Definition 4.6. Let \( z \) be any vertex of \( G \) and let \( [A_z,B_z] \) be a complete-pairing in \( G - N_G[z] \). Then the \( g \)-expansion \( g(G;z,A_z,B_z) \) of \( G \) with respect to the vertex \( z \) and the pairing \( [A_z,B_z] \) is the graph constructed by

\[
V(g(G;z,A_z,B_z)) := (V \setminus \{z\}) \cup \{x_z, y_z\} \quad \text{and} \quad E(g(G;z,A_z,B_z)) := E(G - z) \cup \{ux_z, uy_z : u \in N_G(z)\} \cup \{ax_z : a \in A_z\} \cup \{by_z : b \in B_z\}.
\]

![Figure 5. A g-expansion with respect to the vertex z and the pairing \( \{\{a\},\{b\}\} \)](image)

When there is no confusion, we abbreviate \( g(G;z,A_z,B_z) \) to \( g(G;z) \), and note that the pair \( \{x_z,y_z\} \) is always a \( g \)-pair in \( g(G;z,A_z,B_z) \). Furthermore, the \( g \)-contraction and \( g \)-expansion operations are inverse to each other. In other words, if \( w \) is the vertex for which the \( g \)-pair \( \{x,y\} \) is contracted, then \( g(g(G;xy);w,A_w,B_w) \) is isomorphic to \( G \), where \( A_w = N_G(x) \setminus N_G(y) \) and \( B_w = N_G(y) \setminus N_G(x) \). Similarly, the graph \( g(g(G;z,A_z,B_z);x_zy_z) \) is isomorphic to \( G \).

Corollary 4.7. Let \( G \) be a graph.

(i) \( \text{Ind}(G) \cong \text{Ind}(g(G;xy)) \) for any \( g \)-pair \( \{x,y\} \) in \( G \).
(ii) \( \text{Ind}(G) \cong \text{Ind}(g(G;z)) \) for any expansion \( g(G;z) \) of \( G \).
hit follows that \( \sim \) followings hold:

\[
\text{a minimal subset satisfying } G \geq \text{clearly isomorphic to have } G \leq S \leq H \text{ subset such that } \sim \text{Proof. We only prove the case } (i) \text{ of } \sim \text{Proof. If } x, y \text{ is a } g \text{-pair in } G \text{ such that } N_G(x) \cap N_G(y) = \emptyset, \text{ then } \text{Ind}(G) \text{ is contractible.} \]

\text{Proof. In such a case, the vertex } w_{xy} \text{ to which the pair } \{x, y\} \text{ is contracted is an isolated vertex of } g(G; xy); \text{ hence, the complex } \text{Ind}(g(G; xy)) \text{ is contractible, so is } \text{Ind}(G) \text{ by Corollary 4.7.} \]

Our next result describes the effect of a \( g \)-contraction (respectively, a \( g \)-expansion) on a graph to its regularity.

\text{Proposition 4.9. Let } \{x, y\} \text{ be a } g \text{-pair in } G, \text{ and let } z \text{ be an non-isolated vertex of } G \text{ such that } [A_z, B_z] \text{ is a complete-pairing in } G - N_G[z]. \text{ Then the followings hold:}

\[
\begin{align*}
(i) & \quad \text{reg}(G) \geq \text{reg}(g(G; xy)) \geq \text{reg}(G) - 1. \\
(ii) & \quad \text{reg}(g(G; z)) \geq \text{reg}(G) \geq \text{reg}(g(G; z)) - 1.
\end{align*}
\]

\text{Proof. We only prove the case } (i), \text{ since the other case follows from that. Now, assume first that } \text{reg}(g(G; xy)) = n, \text{ and let } S \subseteq V(g(G; xy)) \text{ be a subset such that } \tilde{H}_{n-1}(g(G; xy)[S]) \neq 0. \text{ If } w_{xy} \notin S, \text{ then } S \subseteq V(G) \text{ so that } \tilde{H}_{n-1}(G[S]) \neq 0; \text{ hence, } \text{reg}(G) \geq n. \text{ So, suppose that } w_{xy} \in S. \text{ If we let } S^* := (S \setminus \{w_{xy}\}) \cup \{x, y\}, \text{ then } \{x, y\} \text{ is a } g \text{-pair in } G[S^*]. \text{ Therefore, we have } G[S^*] \simeq (g(G[S^*])); xy) \text{ by Corollary 4.7. However, the latter graph is clearly isomorphic to } g(G; xy)[S], \text{ which implies that } \tilde{H}_{n-1}(G[S^*]) \neq 0, \text{ that is, } \text{reg}(G) \geq n.

\text{For the second inequality, suppose that } \text{reg}(G) = m, \text{ and let } R \subseteq V(G) \text{ be a minimal subset satisfying } \tilde{H}_{m-1}(G[R]) \neq 0. \text{ Note first that if } \{x, y\} \subseteq R, \text{ then for the set } R^* := (R \setminus \{x, y\}) \cup \{w_{xy}\}, \text{ we have } \tilde{H}_{m-1}(g(G; xy)[R^*]) \neq 0 \text{ by the fact that } G[R] \simeq g(G; xy)[R^*]. \text{ On the other hand, if } R \cap \{x, y\} = \emptyset, \text{ then we necessarily have } R \subseteq V(g(G; xy)) \text{ so that } \tilde{H}_{m-1}(g(G; xy)[R]) \neq 0. \text{ It follows that } \text{reg}(g(G; xy)) \geq m \text{ in either case.}

\text{Therefore, we are left to verify the case where exactly one of } x \text{ and } y \text{ belongs to the set } R. \text{ Assume without loss of generality that } x \in R, \ y \notin R. \text{ If we consider the associated Mayer-Vietoris exact sequence of the pair } (G[R], x); \]

\[
\cdots \rightarrow \tilde{H}_{m-1}(G[R] - x) \rightarrow \tilde{H}_{m-1}(G[R]) \rightarrow \tilde{H}_{m-2}(G[R] - N_{G[R]}[x]) \rightarrow \cdots,
\]

it follows that \( \tilde{H}_{m-1}(G[R] - x) = 0 \) by the minimality of \( R \) so that \( \tilde{H}_{m-2}(G[R] - N_{G[R]}[x]) \neq 0. \text{ If we set } L := V(G[R] - N_{G[R]}[x]), \text{ the} \]
graph $g(G; xy)[L]$ is isomorphic to $G[R] - N_{G[R]}[x]$. Thus, we have $\tilde{H}_{m-2}(g(G; xy)[L]) \neq 0$ so that $\text{reg}(g(G; xy)) \geq m - 1$ as claimed.

Remark 4.10. We note that the first inequality in (i) of Proposition 4.9 could be strict, that is, there exist graphs with $g$-pairs such that their $g$-contractions may decrease the regularity by one. For instance, if we consider the graph

$$H = (\{a, b, c, d, x, y\}, \{ab, ac, bc, cd, cx, dy\}),$$

the pair $\{x, y\}$ is genuine, while $\text{reg}(H) = 2$ and $\text{reg}(g(H; xy)) = 1$.

We therefore look for a condition on a $g$-pair for which the regularity remains stable under its contraction.

Definition 4.11. We call a $g$-pair $\{x, y\}$ in $G$ as a true-pair (or simply a $t$-pair) if there exists a vertex $u \in N_G(x) \cap N_G(y)$ with $N_G[u] \subseteq N_G[x] \cup N_G[y]$, and a $g$-contraction on a graph with respect to a $t$-pair is called a $t$-contraction of $G$ and denoted by $t(G; xy)$. When $\{x, y\}$ is a $t$-pair, such a vertex $u$ is called a true-neighbour (or simply a $t$-neighbour) of the pair $\{x, y\}$. Similarly, if $z$ is a non-isolated vertex in $G$, we call a complete pairing $[A_z, B_z]$ in $G - N_G[z]$ as a $t$-pairing of $z$, if there exists a vertex $v \in N_G(z)$ with $N_G[v] \subseteq A_z \cup B_z \cup N_G[z]$. A $g$-expansion of a graph $G$ with respect to a $t$-pairing of it is called a $t$-expansion of $G$ and denoted by $t(G; z, A_z, B_z)$.

We note that if $[A_z, B_z]$ is a $t$-pairing of $z$ in $G - N_G[z]$, then $\{x_z, y_z\}$ is a $t$-pair in $t(G; z, A_z, B_z)$ having the vertex $v$ as a $t$-neighbour.

Theorem 4.12. If $\{x, y\}$ is a $t$-pair in $G$, then $\text{reg}(G) = \text{reg}(t(G; xy))$.

Proof. Assume that $\text{reg}(G) = m$, and let $R \subseteq V(G)$ be a minimal subset satisfying $\tilde{H}_{m-1}(G[R]) \neq 0$. In view of Proposition 4.9, we may restrict the proof to the case where such a set $R$ satisfies neither $\{x, y\} \subset R$ nor $R \cap \{x, y\} = \emptyset$. Suppose that $x \in R$ and $y \notin R$. We first claim that $N_G(y) \cap R = \emptyset$. In order to verify that we define $T := R \cup \{y\}$ and consider the Mayer-Vietoris exact sequence of the pair $(G[T], y)$;

$$\cdots \rightarrow \tilde{H}_{m-1}(G[T] - N_{G[T]}[y]) \rightarrow \tilde{H}_{m-1}(G[T] - y) \rightarrow \tilde{H}_{m-1}(G[T]) \rightarrow \cdots,$$

where $\tilde{H}_{m-1}(G[T]) = 0$, since no such set contains both $x$ and $y$. It then follows that $\tilde{H}_{m-1}(G[T] - N_{G[T]}[y]) \neq 0$ by the exactness. However, since $R$ is a minimal set satisfying $\tilde{H}_{m-1}(G[R]) \neq 0$, this implies that $N_{G[T]}[y] = \{y\}$, that is, $N_G(y) \cap R = \emptyset$. 

Now, since \( \{x, y\} \) is a \( t \)-pair, they have a \( t \)-neighbour, say \( u \in N_G(x) \cap N_G(y) \), satisfying \( N_G(u) \subseteq N_G[x] \cup N_G[y] \). By the above claim, we have \( u \notin R \). On the other hand, the minimality of \( R \) also implies that \( \tilde{H}_{m-2}(G[R] - N_{G[R]}[x]) \neq 0 \). Therefore, if we set \( K := V(G[R] - N_{G[R]}[x]) \cup \{u, w_{xy}\} \), it follows that the graph \( t(G; xy)[K] \) is isomorphic to the suspension of \( G[R] - N_{G[R]}[x] \) so that \( \tilde{H}_{m-1}(t(G; xy)[K]) \neq 0 \), that is, \( \text{reg}(t(G; xy)) \geq m \). This completes the proof.

\[\text{Corollary 4.13.} \text{ Let } z \text{ be a non-isolated vertex of a graph } G \text{ such that } [A_z, B_z] \text{ is a } t \text{-pairing in } G - N_G[z], \text{ then } \text{reg}(G) = \text{reg}(t(G; z, A_z, B_z)).\]

\[\text{Proof.} \text{ The claim is the consequence of Theorem 4.12, since } \{x_z, y_z\} \text{ is then a } t \text{-pair in } t(G; z, A_z, B_z).\]

\[\text{Remark 4.14.} \text{ We note that a graph } G \text{ contains no } g \text{-pair if and only if every edge of } G \text{ is contained in an induced } 2K_2 \text{ of } G, \text{ that is, its complement } \overline{G} \text{ is a 4-cycled graph (see [3] for details). Furthermore, a } g \text{-pair in a connected claw-free graph is always a } t \text{-pair. On the other hand, any graph } G \text{ containing a non-isolated vertex admits a } t \text{-expansion (possibly with respect to a half empty } t \text{-pairing).}\]

\[\text{Remark 4.15.} \text{ The operations of } t \text{-contraction and expansion can be used to define an equivalence relation on graphs in which each graph in an equivalence class has the same regularity. On the other hand, one may easily verify that the induced matching number is monotone increasing under these operations, and the maximum of the induced matching numbers in each equivalence class still provides a lower bound to the regularity (see [6, Section 5] for details).}\]

We next verify that the regularities of the graphs resulting from an edge contraction and the double edge subdivision are closely related. This enables us to describe the effect of a Lozin operation on the regularity of a graph in Subsection 4.1. We first recall the definition of a double edge subdivision operation for completeness.

Let \( G = (V, E) \) be a graph, and let \( e = uv \in E \) be an edge. A double edge subdivision on the edge \( e \) corresponds to replacing the edge \( e \) by a path of length 3, that is, we transform the graph \( G \) to the graph \( D(G; e) \), where \( V(D(G; e)) := V \cup \{x, y\} \) and \( E(D(G; e)) = (E \setminus \{uv\}) \cup \{ux, xy, yv\} \).

\[\text{Theorem 4.16.} \text{ If } e = uv \text{ is an edge of a graph } G, \text{ then } \text{reg}(D(G; e)) = \text{reg}(G/e) + 1.\]
Proof. We begin by applying a $t$-expansion in the graph $D := D(G; e)$ to the vertex $u$ with respect to the $t$-pairing $\{v\}, N_D(v) \setminus N_G(u)$ having the $t$-neighbour the vertex $a$ (see Figure 6). Observe that $\{v, a\}$ is a $t$-pair in the graph $t(D; u)$ with a $t$-neighbour $b$. Once we contract this $t$-pair, in the resulting graph $t(t(D; u); va)$, we have that $\{c, y_u\}$ is a $t$-pair having the vertex $w_{va}$ as a $t$-neighbour, where $w_{va}$ is the vertex for which the pair $\{v, a\}$ is contracted. Finally, the $t$-contraction of $\{b, y_u\}$ yields a graph isomorphic to $(G/e) \cup K_2$, where the isolated edge is induced by the vertices $w_{va}$ and $w_{by_u}$. Therefore, we have $\text{reg}(D(G; e)) = \text{reg}((G/e) \cup K_2) = \text{reg}(G/e) + 1$. 

Figure 6. The $t$-expansion and the following two $t$-contractions
4.1. Lozin operations

In the search of algorithmic aspect of the induced matching problem, Lozin describes an operation that increases the induced matching number exactly by one, and proves that a successive applications of his operations transforms a given graph into a graph having maximum degree at most three and arbitrarily large girth [19]. We prove in this subsection$^2$ that his operation has a similar effect on the regularity. In particular, we show that any non-trivial Lozin operation preserves the primeness of a graph that in turn allows us to generate new prime graphs from the existing ones.

We begin with recalling the definition of a Lozin transformation on graphs [19]. Let $G = (V,E)$ be a graph and let $x \in V$ be given. A Lozin transform $L_x(G)$ of $G$ with respect to the vertex $x$ is defined as follows:

(i) partition the neighbourhood $N_G(x)$ of the vertex $x$ into two subsets $Y$ and $Z$ in arbitrary way;
(ii) delete vertex $x$ from the graph together with incident edges;
(iii) add a $P_4 = (\{y,a,b,z\},\{ya,ab,bz\})$ to the rest of the graph;
(iv) connect vertex $y$ of the $P_4$ to each vertex in $Y$, and connect $z$ to each vertex in $Z$.

Observe that any Lozin transform clearly depends on the choice of partition of $N_G(x)$, and when the decomposition $N_G(x) = Y \cup Z$ is of importance, we will write $L_x(G;Y,Z)$ instead of $L_x(G)$. It should be noted that we allow one of the sets $Y$ and $Z$ to be an empty set, in which case, we call the resulting operation as a trivial Lozin operation. Furthermore, if $x$ is an isolated vertex, the corresponding Lozin transform of $G$ with respect to the vertex $x$ is the graph $(G - x) \cup P_4$.

![Figure 7. A Lozin transformation](image)

$^2$ Some of the results of this subsection is appeared in an unpublished manuscript [4].
Lemma 4.17 ([19]). For any graph $G$ and any vertex $x \in V(G)$, the equality $\text{im}(L_x(G)) = \text{im}(G) + 1$ holds. Furthermore, any graph can be transformed by a sequence of Lozin transformations into a bipartite graph of maximum degree three having arbitrary large girth.

We are now ready to prove Corollary 1.5 by the help of Theorem 4.16.

Proof of Corollary 1.5. Denote by $G'$ the graph obtained from $L_x(G)$ by replacing the 4-path on $\{y,a,b,z\}$ by a single edge $e := yz$. Observe that $L_x(G) \cong D(G';e)$ and $G \cong G'/e$ so that the claim follows from Theorem 4.16.

We next verify that any non-trivial Lozin operation preserves the primeness of a graph. We remark that in the trivial case, the primeness of a graph is not preserved under a Lozin transformation. For instance, the graph $L_x(C_8;N_{C_8}(x),\emptyset)$ is not prime for any vertex $x \in V(C_8)$. We first need a technical result.

Lemma 4.18. Let $x,y,z$ be three vertices of a graph $G$ with $xy,yz \in E$. If $\deg_G(x) = 1$ and $\deg_G(y) = 2$, then $\text{reg}(G) = \text{reg}(G - z)$.

Proof. Assume otherwise that $\text{reg}(G - z) < \text{reg}(G) = k$. It then follows that $\text{reg}(G) = \text{reg}(G - N_G[z]) + 1$ by Corollary 2.1. Therefore, there exists a minimal set $S \subseteq V(G - N_G[z])$ such that $\tilde{H}_{k-2}(G[S]) \neq 0$. However, if we set $S^* := S \cup \{x,y\}$, then $G[S^*] \simeq \Sigma(G[S])$; hence $\tilde{H}_{k-1}(G[S^*]) \neq 0$. But then we must have $\text{reg}(G - z) = k$, since $z \notin S^*$, a contradiction.

Theorem 4.19. If $G$ is a prime graph with $\delta(G) \geq 2$, then so is $L_x(G;Y,Z)$ for any vertex $x$ and any partition $N_G(x) = Y \cup Z$ such that $Y,Z \neq \emptyset$.

Proof. We need to show that $\text{reg}(L_x(G) - w) < \text{reg}(L_x(G))$ for any vertex $w$ of $L_x(G)$. If $w$ is a vertex of $G$ other than $x$, then clearly $L_x(G) - w \cong L_x(G - w)$, and the claim follows from Corollary 1.5, since $G$ is prime. It remains to show that the regularity is strictly smaller when $w$ is $y$ or $a$. (The cases $w = b$ or $w = z$ can be treated similarly).

If $w = y$, then Lemma 4.18 gives

$$\text{reg}(L_x(G) - y) = \text{reg}(L_x(G) - y - z) = \text{reg}((G - x) \cup K_2) = \text{reg}(G - x) + 1 = \text{reg}(G),$$

which yields the claim, where $K_2$ corresponds to the edge $e = ab$.

Finally, if $w = a$, then we apply Corollary 2.1 at the vertex $z$. If $\text{reg}(L_x(G) - a) = \text{reg}(L_x(G) - a - z)$, then the claim follows from the symmetry...
between \( y \) and \( z \). Otherwise,

\[
\text{reg}(\mathcal{L}_x(G) - a) = \text{reg}(\mathcal{L}_x(G) - a - N(\mathcal{L}_x(G) - a)[z]) + 1
\]

\[
= \text{reg}(G - Z) + 1 \leq \text{reg}(G),
\]

which yields the claim, since \( Z \neq \emptyset \) and \( G \) is prime.

We close this subsection with the proof of the fact that the homotopy type of a Lozin transform can be deduced from the source graph.

**Lemma 4.20.** Let \( G = (V,E) \) be a graph and let \( x \in V \) be given. Then \( \mathcal{L}_x(G;Y,Z) \simeq \mathcal{L}_x(G;Y',Z') \) for any two distinct decompositions \( \{Y,Z\} \) and \( \{Y',Z'\} \) of \( N_G(x) \).

**Proof.** To prove the claim, it is enough to verify that for a given decomposition \( N_G(x) = Y \cup Z \) and a vertex \( u \in Z \), the graphs \( \mathcal{L}_x(G;Y,Z) \) and \( \mathcal{L}_x(G;Y\cup\{u\},Z\setminus\{u\}) \) have the same homotopy type. However, moving the vertex \( u \) from \( Z \) to \( Y \) corresponds to the sequence of isolating operations \( \text{Add}(u,y;b) \) and \( \text{Del}(u,z;a) \) in \( \mathcal{L}_x(G;Y,Z) \); therefore, the claim follows from Theorem 2.5.

**Theorem 4.21.** Let \( G = (V,E) \) be a graph and let \( x \in V \) be given. Then \( \mathcal{L}_x(G) \simeq \Sigma(G) \).

**Proof.** In view of Lemma 4.20, it is sufficient to show that \( \mathcal{L}_x(G;N_G(x),\emptyset) \simeq \Sigma(G) \). In such a case, we set \( \mathcal{R}_x(G) = \mathcal{L}_x(G;N_G(x),\emptyset) \) and note that \( N_{\mathcal{R}_x(G)}(z) \subseteq N_{\mathcal{R}_x(G)}(a) \) so that \( \mathcal{R}_x(G) \) is homotopy equivalent to \( \mathcal{R}_x(G) - a \), while the latter graph is clearly isomorphic to \( G \cup K_2 \), where \( K_2 \) is induced by the edge \( bz \). It then follows that \( \mathcal{R}_x(G) \simeq \Sigma(G) \) as required.

We remark that Theorem 4.21 generalizes an earlier result of Csorba [8, Theorem 11], which corresponds to a special Lozin transformation, obtained by taking one of the sets in the partition of \( N_G(x) \) as a singleton. In other words, he considers the **triple edge subdivision**, \( \mathcal{L}_x(G;e) := \mathcal{L}_x(G;\{y\},N_G(x)\setminus\{y\}) \) for an edge \( e = xy \) of \( G \), which can be obtained from \( G \) by replacing the edge \( e = xy \) by a path \( x - x_0 - x_1 - x_2 - y \).

**Remark 4.22.** Note that Lozin transformation does not need to preserve the vertex-decomposability of a graph in general. Furthermore, other graph invariants such as the matching number or cochordal cover number may not need to increase exactly by one under a Lozin transformation.
Remark 4.23. Regarding the results of Section 3 and Subsection 4.1, the most prominent question would be the characterization of bipartite (perfect) prime graphs of maximum degree at most three. The investigation of the structural properties that they need to carry would be valuable. Furthermore, it would be interesting to determine whether there exists a small constant $c > 0$ such that $\text{reg}(B) \leq c \text{im}(B)$ holds for any such graph, while our primarily search hints that the right value of such a constant $c$ would be $\frac{3}{2}$ (compare to Theorem 3.8). On the other hand, we suspect that there is a strong connection between the girth of a graph and its regularity, and predict that the inequality $\text{reg}(G) \leq 2\text{im}(G)$ holds whenever $\text{girth}(G) \geq 5$.

5. Regularity of $2K_2$-free graphs

In this section we prove Corollary 1.6. Beyond that, we provide a local analysis on the structure of $2K_2$-free graphs in general.

As we have already mentioned in Section 1, the existence of $2K_2$-free graphs with arbitrary large regularity can be guaranteed from the following result of Januszkiewicz and Świątkowski:

Theorem 5.1 ([15]). For any $n$ there exists a flag simplicial complex containing no empty square which is an oriented pseudomanifold of dimension $(n-1)$.

Some explanations should be in order. In fact, the existence of such complexes ensures the existence of a right angled Coxeter group $(W,S)$ of virtual cohomological dimension $\text{vcd}(W) = n$, such that $W$ does not contain $\mathbb{Z} \oplus \mathbb{Z}$, thus is Gromov hyperbolic. Furthermore, the proof of Theorem 5.1 is inductive, and uses the notion of a simplex of groups. We note that the flag property corresponds to the fact that the resulting simplicial complex is the independence complex of a graph $G_n$, and that the independence complex of $G_n$ has no empty square if and only if $G_n$ is $2K_2$-free. Finally, since the complex is an $(n-1)$-dimensional oriented pseudomanifold, it follows that $\text{reg}(G_n) = n$. In particular, this implies that $G_n$ is connected, since it is $2K_2$-free.

Proof of Corollary 1.6. Let $n$ and $k$ be two positive integers with $n \geq k \geq 1$. If $n = k$, the graph obtained from $nK_2$ by adding an extra vertex and connecting it to exactly one vertex of each edge of $nK_2$ is an example of a graph satisfying the required conditions. So, we may assume that $n > k$. On the other hand, we know from Theorem 5.1 that there exists a graph $G(n,1)$ with $\text{reg}(G(n,1)) = n$ for any $n$ so that we may further assume that
$k > 1$. We let $m := n - k + 1$, and consider a $2K_2$-free graph $G(m, 1)$ such that $\text{reg}(G(m, 1)) = m$. Now, choose an edge $e$ of $G(m, 1)$, and apply $k - 1$ triple edge subdivisions on $e$ consecutively. If we denote the resulting graph by $G(n, k)$, then we have $\text{im}(G(n, k)) = k$ by Lemma 4.17 and $\text{reg}(G(n, k)) = n$ by Corollary 1.5.

We remark that the graph $G_n$ is a perfect prime, since $\text{Ind}(G_n)$ is an oriented pseudomanifold. When $n > k$, it implies that the resulting graph $G(n, k)$ is also a perfect prime by Theorem 4.19, as it is obtained from $G_n$ by successive Lozin operations. Furthermore, we should note that the graph $G(n, k)$ satisfying the required conditions in Corollary 1.6 is not unique. When $k = 1$, Osajda [27] has recently introduced a different construction that produces new examples of such graphs, and in the general case, any distinct choice of a sequence of Lozin operations will produce a different example (see Section 4.1).

We recall that a graph $G$ is said to be a locally homogeneous graph if $G[N_G(x)]$ is isomorphic to a fixed graph $H$ for each $x \in V(G)$ (in such case, $G$ is sometimes called a locally $H$-graph (see [33] for details)). For instance, the underlying graph of the Coxeter 600-cell, that is, the graph $\overline{G}_{600}$ is locally icosahedral graph, and the underlying graph of the icosahedron is locally $C_5$. It is proved in [15] that the links of vertices are isomorphic that in turn implies that the graph $\overline{G}_n$ is locally homogeneous.

We close this section by strengthening the upper bound of Theorem 1.3 for $2K_2$-free graphs. Beforehand, we particularly note that Dao, Huneke and Schweig [9] have two logarithmic upper bounds on the regularity of $2K_2$-free graphs, one is in terms of the maximum degree and the other involves the number of vertices. We refer to Theorems 4.1 and 4.9 in that paper, and note that a graph is $2K_2$-free if and only if its edge ideal is 1-step linear (compare to Corollary 2.9 in [9]).

**Lemma 5.2.** If $e = xy$ is an edge of a $2K_2$-free graph $G$, then $\text{reg}(G - N_G[y]) \leq \frac{1}{2} |N_G[x] \setminus N_G[y]| + 1$.

**Proof.** Observe that the vertex set of the graph $G - N_G[y]$ can be split up into two parts $N_G[x] \setminus N_G[y]$ and $U = V(G) \setminus N_G[e]$. In particular, the set $U = V(G) \setminus N_G[e]$ is an independent set of $G$, since $G$ is a $2K_2$-free graph. Let $M = \{x_1y_1, \ldots, x_ry_r\}$ be a maximum matching of $G[N_G[x] \setminus N_G[y]]$. We then consider split subgraphs $H_i = (K_i, I_i)$, where $K_i$ is the complete graph on $\{x_i, y_i\}$, i.e., a single edge $x_iy_i$, and $I_i$ is the independent set on the set of vertices $(N_{G - N_G[y]}(x_i) \cup N_{G - N_G[y]}(y_i) \setminus \{x_i, y_i\}$ for each $1 \leq i \leq r$. Furthermore, if we set $A := N_G[x] \setminus N_G[y] - V(M)$, then $A$ is an independent set. It follows that $G[\cup A]$ is a $2K_2$-free bipartite graph; hence, it is a cochordal graph.
Observe that the equality $E(G - N_G[y]) = E(H_1) \cup \cdots \cup E(H_r) \cup E(U \cup A)$ holds. Therefore, we have $\text{reg}(G - N_G[y]) \leq \text{cochord}(G - N_G[y]) \leq r + 1$. However, since $r \leq \frac{1}{2}|N_G[x] \setminus N_G[y]|$, the claim follows.

We may now strengthen the bound of Theorem 1.3 for $2K_2$-free graphs by the help of Lemma 5.2.

**Theorem 5.3.** If $G$ is a $2K_2$-free graph, then $\text{reg}(G) \leq \frac{1}{2} \Gamma(G) + 2$.

**Proof.** We proceed by an induction on the order of $G$. Assume that the claim holds for any such graph whose order is less than $|G|$. Suppose that $\Gamma(G) = |N_G[x] \setminus N_G[y]|$ for some edge $e = xy$ in $G$. It then follows from the induction that $\text{reg}(G - y) \leq \frac{1}{2} \Gamma(G - y) + 2 \leq \frac{1}{2} \Gamma(G) + 2$. On the other hand, we have $\text{reg}(G - N_G[y]) \leq \frac{1}{2} \Gamma(G) + 1$ by Lemma 5.2; hence, $\text{reg}(G) \leq \frac{1}{2} \Gamma(G) + 2$ by Corollary 2.1.

**Proposition 5.4.** Let $G$ be a $2K_2$-free graph. If $x$ is a vertex such that $\deg_G(x) < 2 \text{reg}(G) - 3$, then $\text{reg}(G) = \text{reg}(G - N_G[x])$.

**Proof.** We let $\deg_G(x) = d$. If $y \in N_G(x)$, then $|N_G[x] \setminus N_G[y]| \leq d - 1$. Thus, the inequality $\text{reg}(G - N_G[y]) \leq \frac{d + 1}{2} < \text{reg}(G) - 1$ holds by Lemma 5.2. However, this implies that $\text{reg}(G) = \text{reg}(G - y)$ by Corollary 2.1. We may therefore remove any neighbourhood of $x$ without altering the regularity in which case $x$ becomes an isolated vertex, that is, $\text{reg}(G) = \text{reg}(G - N_G[x])$ as claimed.

In fact Proposition 5.4 provides a strict (local) restriction on the minimum degree of $2K_2$-free graphs with a given regularity.

**Corollary 5.5.** If $G$ is a $2K_2$-free graph such that $\text{reg}(G) = k$, then there exists an induced subgraph $H$ of $G$ such that $\text{reg}(G) = \text{reg}(H)$ and $\delta(H) \geq 2k - 3$.

**Proof.** Suppose that $\delta(G) < 2k - 3$, and let $x$ be a vertex of minimum degree in $G$. We have $\text{reg}(G) = \text{reg}(G - N_G[x])$ by Proposition 5.4, since $\deg_G(x) < 2k - 3$. If $\delta(G - N_G[x]) \geq 2k - 3$, we are done. Otherwise, we continue in a similar way, and since $G$ is finite, this process will eventually terminate.

In terms of prime graphs, we may restate Corollary 5.5 as follows.

**Corollary 5.6.** If $G$ is a prime $2K_2$-free graph, then $\text{reg}(G) \leq \frac{\delta(G) + 3}{2}$.
Acknowledgments. We would like to thank Michał Adamaszek for bringing the work of Przytycki and Świątkowski [28] to our attention, Damian Osajda for explaining to us the details of his Basic Construction [27], and Zakir Deniz for his help with the Sage coding. We have received indispensable help from the referees. They pointed out several inconsistencies in the earlier versions of this paper and have contributed substantially to the overall improvement of the presentation. In particular, we are grateful to them for suggesting current short proofs of Theorem 1.4, Corollary 1.5 and Theorem 4.19 which have replaced the original ones.

References

[1] M. Adamaszek: Splittings of independence complexes and the powers of cycles, Journal of Combinatorial Theory Series A 119 (2012), 1031–1047.
[2] D. Attali, A. Lieutier and D. Salinas: Efficient data structure for representing and simplifying simplicial complexes in high dimensions, International Journal of Computational Geometry and Applications 22 (2012), 279–303.
[3] T. Bıyıkolu and Y. Civan: Four-cycled graphs with topological applications, Annals of Combinatorics 16 (2012), 37–56.
[4] T. Bıyıkolu and Y. Civan: Bounding Castelnuovo-Mumford regularity of graphs via Lozin’s operations, unpublished manuscript, available at arXiv:1302.3064, 2013.
[5] T. Bıyıkolu and Y. Civan: Vertex-decomposable graphs, codismantlability, Cohen-Macaulayness, and Castelnuovo-Mumford regularity, Electronic Journal of Combinatorics, 21(1):#P1, 2014.
[6] T. Bıyıkolu and Y. Civan: Castelnuovo-Mumford regularity of graphs, available at arXiv:1503.06018(v1), 43pp, 2015.
[7] A. Brandstädt, V. B. Le and J. P. Spinrad: Graph Classes, A Survey, SIAM Monographs on Discrete Mathematics and Applications, Philadelphia, 1999.
[8] P. Csorba: Subdivision yields Alexander duality on independence complexes, Electronic Journal of Combinatorics, 16(2):#R11, 2009.
[9] H. Dao, C. Huneke and J. Schweig: Bounds on the regularity and projective dimension of ideals associated to graphs, Journal of Algebraic Combinatorics 38 (2013), 37–55.
[10] R. Ehrenborg and G. Hetyei: The topology of the independence complex, European Journal of Combinatorics 27 (2006), 906–923.
[11] A. Engström: Complexes of directed trees and independence complexes, Discrete Mathematics 309 (2009), 3299–3309.
[12] A. Hatcher: Algebraic Topology, Cambridge University Press, New York, 2006.
[13] H. T. Hà: Regularity of squarefree monomial ideals, in: S. M. Cooper and S. Sather-Wagstaff, editors, Connections Between Algebra, Combinatorics, and Geometry, volume 76, 251–276. Springer, Proceedings in Mathematics and Statistics, 2014.
[14] H. T. Hà and A. V. Tuyl: Monomial ideals, edge ideals of hypergraphs, and their graded betti numbers, Journal of Algebraic Combinatorics 27 (2008), 215–245.
[15] T. Januszkiewicz and J. Świątkowski: Hyperbolic Coxeter groups of large dimension, Commentarii Mathematici Helvetici 78 (2003), 555–583.
[16] G. Kalai and R. Meshulam: Intersection of Leray complexes and regularity of monomial ideals, *Journal of Combinatorial Theory Series A* 113 (2006), 1586–1592.

[17] M. Katzman: Characteristic-independence of Betti numbers of graph ideals, *Journal of Combinatorial Theory Series A* 113 (2006), 435–454.

[18] D. Kozlov: *Combinatorial Algebraic Topology*, volume ACM 21, Springer, Berlin, 2008.

[19] V. V. Lozin: On maximum induced matchings in bipartite graphs, *Information Processing Letters* 81 (2002), 7–11.

[20] F. H. Lutz and E. Nevo: Stellar theory for flag complexes, *Mathematica Scandinavica* 118 (2016), 70–82.

[21] M. Mahmoudi, A. Mousivand, M. Crupi, G. Rinaldo, N. Terai and S. Yassemi: Vertex decomposability and regularity of very well-covered graphs, *Journal of Pure and Applied Algebra* 215 (2011), 2473–2480.

[22] M. Marietti and D. Testa: A uniform approach to complexes arising from forests, *Electronic Journal of Combinatorics*, 15:#R101, 2008.

[23] D. Marušić and T. Pisanski: The remarkable generalized Petersen graph $G(8,3)$, *Mathematica Slovaca* 50 (2000), 117–121.

[24] S. Morey and R. H. Villarreal: Edge ideals: algebraic and combinatorial properties, in: C. Francisco, L. C. Klingler, S. Sather-Wagstaff, and J. C. Vassilev, editors, *Progress in Commutative Algebra 1: Combinatorics and Homology*, chapter 3, 85–126. De Gruyter, Berlin, 2012.

[25] E. Nevo: Regularity of edge ideals of $C_4$-free graphs via the topology of the lcm-lattice, *Journal of Combinatorial Theory Series A* 118 (2011), 491–501.

[26] E. Nevo and I. Peeva: $C_4$-free edge ideals, *Journal of Algebraic Combinatorics* 37 (2013), 243–248.

[27] D. Osajda: A construction of hyperbolic Coxeter groups, *Commentarii Mathematici Helvetici* 88 (2013), 353–367.

[28] P. Przytycki and J. Świątkowski: Flag-no-square triangulations and Gromov boundaries in dimension 3, *Groups, Geometry, and Dynamics* 3 (2013), 453–468.

[29] R. P. Stanley: *Combinatorics and Commutative Algebra, Second Edition*, volume 41, Progress in Mathematics, Birkhäuser, Boston, MA, 1996.

[30] W. A. Stein et al: *Sage Mathematics Software*, The Sage Development Team, http://www.sagemath.org, 2014.

[31] A. V. Tuyl: Sequentially Cohen-Macaulay bipartite graphs: vertex decomposability and regularity, *Archiv der Mathematik* 93 (2009), 451–459.

[32] D. W. Walkup: The lower bound conjecture for 3 and 4-manifolds, *Acta Mathematica* 125 (1970), 75–107.

[33] G. Whieldon: A construction of locally homogeneous graphs, *Journal of the London Mathematical Society* 50 (1994), 68–86.

[34] G. Whieldon: Jump sequences of ideals, preprint, available at arXiv:1012.0108v1, 27pp, 2015.

[35] R. Woodroofe: Matchings, coverings, and Castelnuovo-Mumford regularity, *Journal of Commutative Algebra* 6 (2014), 287–304.
