On Jordan angles and triangle inequality in Grassmannian

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Abstract

Let $L, M, N$ be $p$-dimensional subspaces in $\mathbb{R}^n$. Let $\varphi_j$ be the angles between $L$ and $M$, let $\psi_j$ be the angles between $M$ and $N$, and $\theta_j$ be the angles between $L$ and $M$. Consider the orbit of the vector $\psi \in \mathbb{R}^p$ with respect to permutations of coordinates and inversions of axes. Let $Z$ be the convex hull of this orbit. Then $\theta \in \varphi + Z$. We discuss similar theorems for other symmetric spaces.

We also obtain formula for geodesic distance on any invariant convex Finsler metrics on classical symmetric space.

1. Grassmannians. Fix positive integers $p \leq q$. Consider the space $\mathbb{R}^{p+q}$ equipped with the standard scalar product. Denote by $Gr_{p,q}$ the set of all $p$-dimensional linear subspaces in $\mathbb{R}^{p+q}$. The orthogonal group $O(p+q)$ acts in $\mathbb{R}^{p+q}$ and hence it acts on $Gr_{p,q}$. Obviously,

$$Gr_{p,q} = O(p+q)/O(p) \times O(q)$$

2. Jordan angles. Let $L, M \in Gr_{p,q}$. Consider an orthonormal basis $e_1, \ldots, e_p \in L$ and an orthonormal basis $f_1, \ldots, f_p \in M$. Consider the matrix

$$\Lambda = \Lambda[L, M]$$

with the matrix elements $<e_i, f_j>$. Denote by

$$\lambda_1 \geq \ldots \geq \lambda_p$$

the singular values of the matrix $\Lambda$. Obviously, the numbers

$$\lambda_j = \lambda_j[L, M]$$

don’t depend on choice of the bases $e_1, \ldots, e_p \in L$ and $f_1, \ldots, f_p \in M$.

Proposition 1. Let $L, M, L', M' \in Gr_{p,q}$. The following conditions are equivalent

1) $\lambda_j[L, M] = \lambda_j[L', M']$ for all $j$
ii) There exists an element $g \in O(p+q)$ such that $gL = L', gM = M'$.

**Proof.** The statement is obvious.

**Proposition 2.** Consider $L, M \in \text{Gr}_{p,q}$. There exist orthonormal bases $e_1, \ldots, e_p \in L$ and $f_1, \ldots, f_p \in M$ such that

$$<e_i, f_j> = 0 \quad \text{if} \quad i \neq j$$
$$<e_j, f_j> = \lambda_j$$

**Proof.** The statement is obvious.

**Proposition 3.**

**a)** \(\lambda_k[L,M] = \max_{P \subset L} \min_{v \in P} \max_{w \in M} <v,w> \)

where the first maximum is given over all \(k\)-dimensional subspaces \(P\) in \(L\).

**b)** \(\lambda_k[L,M] = \min_{Q \subset L} \max_{v \in Q} \max_{w \in M} <v,w> \)

where the minimum is given over all subspaces \(Q\) having codimension \(k\) in \(L\).

**Proof.** The statement is a corollary of the standard minimax characterizations of eigenvalues and singular values, see [10], [2].

**Proposition 4.** Denote by \(\Pi_M\) the orthogonal projector to the subspace \(M\). Then the numbers \(\lambda_j[L,M]\) are the singular values of the operator \(\Pi_M : L \to M\).

**Proof.** The statement is obvious.

**Lemma 5.** Let \(u_1, u_2, \ldots, u_p\) be a (nonorthogonal) basis in \(L\). Let \(v_1, v_2, \ldots, v_p\) be a (nonorthogonal) basis in \(M\). Denote by \(U\) the matrix with the matrix elements \(<u_i, u_j>\), denote by \(V\) the matrix with the matrix elements \(<v_i, v_j>\), denote by \(W\) the matrix with the matrix elements \(<u_i, v_j>\). Then the numbers \(\lambda_j[L,M]\) coincides with eigenvalues of the matrix

$$U^{-1}WV^{-1}W^t$$

(1)

**Proof.** The statement is obvious.

The angles or stationary angles\(^4\) (K.Jordan, 1875) \(\psi_1 \leq \psi_2 \leq \ldots \leq \psi_p\) between the subspaces \(L, M \in \text{Gr}_{p,q}\) are defined by

$$\psi_1 = \Psi_1[L,M] := \arccos \lambda_1, \ldots, \psi_p = \Psi_p[L,M] := \arccos \lambda_p$$

\(^4\)Other terms are complex distance or compound distance. Models of symmetric spaces given in [12] and [13] shows that these invariants for all classical Riemannian symmetric spaces are really angles.
Obviously, $0 \leq \psi_j \leq \pi/2$. We also will use the notation

$$\Psi[L, M] = (\Psi_1[L, M], \ldots, \Psi_p[L, M])$$

**Remark.** For all $j$ we have $\Psi_j[L, M] = \Psi_j[M, L]$.

### 3. The result of the paper.

Denote by $W_p$ (Weyl group) the group of all transformations of $\mathbb{R}^p$ generated by permutations of the coordinates and by the transformations

$$(t_1, \ldots, t_p) \mapsto (\sigma_1 t_1, \ldots, \sigma_p t_p) \quad \text{where} \quad \sigma_j = \pm 1$$

**Theorem A.** Let $\ell(x)$ be a $W_p$-invariant norm in $\mathbb{R}^p$. Then the function

$$d(L, M) := \ell(\Psi_1[L, M], \ldots, \Psi_p[L, M])$$

is an $O(p + q)$-invariant metric on $\text{Gr}_{p,q}$.

**Remark.** The geodesic distance in $\text{Gr}_{p,q}$ associated with the $O(p) \times O(q)$-invariant Riemannian metrics is given by the formula

$$\text{dist}(L, M) = \sqrt{\Psi_1^2[L, M] + \cdots + \Psi_p^2[L, M]}$$

**Theorem B.** Let $L, M, N \in \text{Gr}_{p,q}$. Let $\varphi_j = \Psi_j[L, M], \psi_j = \Psi_j[M, N], \theta_j = \Psi_j[L, N]$ be the angles. Denote by $Z$ the convex hull of the $W_p$-orbit of the vector $(\psi_1, \ldots, \psi_p) \in \mathbb{R}^p$. Denote by $U$ the shift of $Z$ by the vector $(\varphi_1, \ldots, \varphi_p)$.

Then there exists a vector $(\theta_1, \ldots, \theta_p) \in U$ such that the collection of numbers $(\cos \theta_1, \ldots, \cos \theta_p)$ coincides up to permutation with the collection of numbers $(\cos \varphi_1, \ldots, \cos \varphi_p)$.

### 4. Infinitesimal angular structure.

For $L \in \text{Gr}_{p,q}$ we denote by $T_L(\text{Gr}_{p,q})$ the tangent space to $\text{Gr}_{p,q}$ at the point $L$. It is natural to identify elements $\xi \in T_L(\text{Gr}_{p,q})$ with operators $H$ from $L$ to the orthogonal complement $L^\perp$. We denote by

$$\rho_1[L; H] \leq \cdots \leq \rho_p[L; H]$$

the singular values of the operator $H : L \rightarrow L^\perp$.

**Lemma 6.** Let $L, L' \in \text{Gr}_{p,q}, H \in T_L(\text{Gr}_{p,q}) \ , \ H' \in T_{L'}(\text{Gr}_{p,q})$. The following conditions are equivalent

i) $\rho_j[L; H] = \rho_j[L'; H']$ for all $j$

ii) There exists an operator $g \in O(p + q)$ such that $gL = L', gH = H'$.

**Proof.** The statement is obvious.

**Remark.** The $O(p + q)$-invariant Riemannian metric in $\text{Gr}_{p,q}$ is

$$\text{tr} \ H'H = \sum \rho_j^2[L; H]$$
5. Relations between angles and infinitesimal angular structure.
The following statement is obvious.

**Proposition 7.** Let \( M(t) \) be a smooth \( C^\infty \)-curve in \( \text{Gr}_{p,q} \). Then

\[
\lim_{\varepsilon \to +0} \frac{\Psi_j[M(a+\varepsilon), M(a)]}{\varepsilon} = \rho_j[M(a); M'(a)]
\]

where \( \lim_{\varepsilon \to +0} \) denotes the right limit at 0.

6. Infinitesimal variation of angles. Let \( L, M \in \text{Gr}_{p,q} \). Let \( e_j \in L, \ f_j \in M \) be orthonormal bases satisfying the conditions

\[
<e_i, f_j> = 0 \quad \text{if} \quad i \neq j
\]

\[
<e_j, f_j> = \cos \psi_j
\]

Assume \( \psi_j \) be pairwise different.

Consider an orthonormal basis \( r_1, \ldots, r_q \in M^\perp \) such that for all \( j \leq p \) the vectors \( e_j, f_j, r_j \) span 2-dimensional plane and \( f_j \) is situated in the angle between \( e_j \) and \( r_j \).

\[
R^2
\]

\[
M^\perp
\]

\[
f_j
\]

\[
e_j
\]

We have

\[
e_j = f_j \cos \psi_j - r_j \sin \psi_j
\]

Let \( H : M \to M^\perp \) be a tangent vector to \( \text{Gr}_{p,q} \) at the point \( M \). Let \( h_{ij} \) be the matrix elements of \( H \) in the bases \( f_1, \ldots, f_p \) and \( r_1, \ldots, r_q \).

Consider a \( C^\infty \)-smooth curve \( M(\varepsilon) \in \text{Gr}_{p,q} \) such that

\[
M(0) = H; \quad M'(0) = H
\]

**Proposition 8.** Assume \( \psi_0 \neq 0, \psi_p \neq \pi/2, \) and \( \psi_{j+1} \neq \psi_j \) for all \( j \). Then

\[
\frac{d}{d\varepsilon}\Psi_j[L, M(\varepsilon)] \bigg|_{\varepsilon=0} = h_{jj}
\]

**Proof.** Denote by \( f_j(\varepsilon) \) the unique vector in \( M(\varepsilon) \) having the form

\[
f_j(\varepsilon) = f_j + \sum a_{jk}(\varepsilon)r_k
\]
Obviously, 
\[ a_{jk}(\varepsilon) = \varepsilon h_{jk} + O(\varepsilon^2) \]

For all \( i, j \leq p \) we have

\[ < f_i(\varepsilon), f_j(\varepsilon) > = \begin{cases} 
O(\varepsilon^2), & \text{if } i \neq j \\
1 + O(\varepsilon^2), & \text{if } i = j 
\end{cases} \]

and

\[ < e_i, f_j(\varepsilon) > = \begin{cases} 
-\varepsilon h_{ij} \sin \psi_i + O(\varepsilon^2), & \text{if } i \neq j \\
\cos(\psi_j + \varepsilon h_{jj}) + O(\varepsilon^2), & \text{if } i = j 
\end{cases} \]

Now we are ready to write matrix (1) for the subspaces \( L, M(\varepsilon) \)

\[
\begin{pmatrix}
\cos^2(\psi_1 + \varepsilon h_{11}) + O(\varepsilon^2) & O(\varepsilon) & \cdots & O(\varepsilon) \\
O(\varepsilon) & \cos^2(\psi_2 + \varepsilon h_{22}) + O(\varepsilon^2) & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
O(\varepsilon) & O(\varepsilon) & \cdots & \cos^2(\psi_p + \varepsilon h_{pp}) + O(\varepsilon^2) \\
\end{pmatrix}
\]

This implies required statement.

**Remark.** Let us fix \( L \in \text{Gr}_{p,q} \).

a) The set of all \( M \in \text{Gr}_{p,q} \) such that \( \Psi_0[L, M] = 0 \) has codimension \( q-p+1 \).

b) The set of all \( M \in \text{Gr}_{p,q} \) such that \( \Psi_p[L, M] = \pi/2 \) has codimension 1.

c) The set of all \( M \in \text{Gr}_{p,q} \) such that \( \Psi_{j+1}[L, M] = \Psi_j[L, M] \) has codimension 2.

**7. Preliminaries.** A \( p \times p \) matrix \( A \) is called bistochastic if for all \( k, l \)

\[ \sum_i a_{ik} = 1; \quad \sum_j a_{lj} = 1 \]

We say that a real \( p \times p \) matrix \( A \) is quasistochastic if for all \( k, l \)

\[ \sum_i |a_{ik}| \leq 1; \quad \sum_j |a_{lj}| \leq 1 \]

**Proposition 9.** (Birkhoff) The set of all bistochastic matrices is the convex hull of matrices of permutations.

See [10], [2].

**Lemma 10.** The set of all quasistochastic matrices is the convex hull of the group \( W_p \).

**Proof.** It is sufficient to describe extremal points of the set of all quasistochastic matrices.

\[ ^5 \text{i.e. matrices consisting of 0 and 1 and having strictly one 1 in each column and each row.} \]
a) Obviously, for any extremal point $A$

\[
\sum_i |a_{ik}| = 1; \quad \sum_j |a_{ij}| = 1 \quad (3)
\]

b) Let a matrix $A$ satisfies condition (3). Assume the matrix $|a_{ij}|$ be not an extremal point of the set of stochastic matrices. Then $A$ is not an extremal point of the set of quasistochastic matrices.

Hence any extremal point of the set of quasistochastic matrices is an element of $W_p$.

**Example.** Let $U = (u_{ij})$ and $V = (v_{ij})$ be matrices with norm $\leq 1$. Then the matrix $W = (u_{ij}v_{ij})$ is quasistochastic.

**Lemma 11.** Let $A$ be a real $p \times q$ matrix. Let $\lambda = (\lambda_1, \ldots, \lambda_p)$ be its singular values. Then the convex hull of the $W_p$-orbit of $\lambda$ contains the vector $(a_{11}, \ldots, a_{pp})$.

**Proof.** Indeed, the matrix $A$ can be represented in the form

\[
A = U \begin{pmatrix}
\lambda_1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & \lambda_2 & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \lambda_p & 0 & \ldots & 0
\end{pmatrix} V^t; \quad U \in O(p), V \in O(q)
\]

Hence

\[
\begin{pmatrix}
a_{11} \\
a_{22} \\
\vdots \\
a_{pp}
\end{pmatrix} = \begin{pmatrix}
u_{11}v_{11} & u_{12}v_{12} & \ldots & u_{1p}v_{1p} \\
u_{21}v_{21} & u_{22}v_{22} & \ldots & u_{2p}v_{2p} \\
\vdots & \vdots & \ddots & \vdots \\
u_{p1}v_{p1} & u_{p2}v_{p2} & \ldots & u_{pp}v_{pp}
\end{pmatrix} \begin{pmatrix}
\lambda_1 \\
\lambda_2 \\
\vdots \\
\lambda_p
\end{pmatrix}
\]

Then we apply Lemma 10.

**8. Proof of Theorems A–B.** Fix nonnegative numbers

\[a_1 \leq \ldots \leq a_p\]

Fix arbitrary orthonormal basis

\[e_1, e_2, \ldots, e_p, f_1, f_2, \ldots, f_p, r_1, \ldots, r_{q-p} \in \mathbb{R}^{p+q}\]

Consider the subspace $L_a(s) \in \text{Gr}_{p,q}$ spanned by the vectors $v_1(s), \ldots, v_p(s)$ given by the formula

\[v_j(s) = \cos(a_j s)e_j + \sin(a_j s)f_j\]

We obtain a curve $L_a(s)$ in $\text{Gr}_{p,q}$.  

\(^6\)This is a minor variation of Fan Ky theorem, see [10], [2].
We say that a curve $\gamma(t)$ in $\text{Gr}_{p,q}$ is a $H$-curve if in some orthonormal basis it has the form $L_a(s)$. We say that the numbers $a_j$ are the invariants of the $H$-curve $\gamma$.

We say that points $L_a(s)$, $L_a(t)$ on a $H$-curve are sufficiently near if

$$a_p|s-t| \leq \pi/2$$

The following statements are obvious

**Lemma 12.** Consider sufficiently near points $L(s_1)$, $L(s_2)$, $L(s_3)$ on $H$-curve. Assume $s_1 < s_2 < s_3$. Then for all $j$

$$\Psi_j[L(s_1), L(s_2)] + \Psi_j[L(s_2), L(s_3)] = \Psi_j[L(s_1), L(s_3)]$$

**Lemma 13.** Let $L, M \in \text{Gr}_{p,q}$ and $\Psi_p[L, M] < \pi/2$. Then there exists the unique $H$-curve $\gamma(s)$ joining $L$, $M$ such that $L$, $M$ are sufficiently near points on $\gamma(s)$.

Consider points $L, M, N \in \text{Gr}_{p,q}$ having a general position. Denote by $\theta_j$ the angles between $M$ and $N$. Consider the $H$-curve $\gamma(t)$ such that $\gamma(0) = M$, $\gamma(1) = N$ and $M$, $N$ are sufficiently near points of the curve $\gamma(t)$. Then the invariants of the $H$-curve $\gamma(s)$ are $\theta_1, \ldots, \theta_p$. We assume, that for each $s \in [0, 1]$

$$\Psi_0[L, M(s)] \neq 0, \quad \Psi_p[L, M(s)] \neq \pi/2$$

$$\Psi_j[L, M(s)] \neq \Psi_{j+1}[L, M(s)] \quad \text{for all} \quad j$$

Denote by $\mathcal{Z}$ the convex hull of $W_p$-orbit of the vector $\theta$.

By Proposition 8 and Lemma 11, we have

$$\Psi[L, \gamma(1/n)] \in \Psi[L, M] + \left( \frac{1}{n} + O\left( \frac{1}{n^2} \right) \right) \mathcal{Z}, \quad n \to \infty$$

In the same way,

$$\Psi[L, \gamma(2/n)] \in \Psi[L, \gamma(1/n)] + \left( \frac{1}{n} + O\left( \frac{1}{n^2} \right) \right) \mathcal{Z}$$

$$\Psi[L, \gamma(3/n)] \in \Psi[L, \gamma(2/n)] + \left( \frac{1}{n} + O\left( \frac{1}{n^2} \right) \right) \mathcal{Z}$$

$$\cdots \cdots \cdots \cdots \cdots \cdots$$

and $O(1/n^2)$ are uniform in $k/n$. Hence

$$\Psi[L, \gamma(t)] \in \Psi[L, M] + t(1 + O\left( \frac{1}{n} \right)) \mathcal{Z}; \quad n \to \infty$$

and hence

$$\Psi[L, \gamma(t)] \in \Psi[L, M] + t \mathcal{Z}; \quad (7)$$
Assume that there exists the unique value \( \bar{s} \) that doesn't satisfy the conditions (5)–(6). Consider a small \( \delta \). Then for \( t > \bar{s} \) we have

\[
\Psi[L, \gamma(\bar{s} - \delta)] \in \Psi[L, M] + (\bar{s} - \delta)Z
\]

\[
\Psi[L, \gamma(\bar{s} - \delta)] \text{ is close to } \Psi[L, \gamma(\bar{s} + \delta)]
\]

\[
\Psi[L, \gamma(t)] \in \Psi[L, \gamma(\bar{s} + \delta)] + (t - \bar{s} + \delta)Z
\]

and we again obtain (7).

A \( H \)-curve of general position contains a finite number of points \( \bar{s} \) that don’t satisfy conditions (5)–(6). Hence we can repeat our arguments.

This proves Theorem B.

Consider a \( W_p \)-invariant norm \( \ell(\cdot) \) on \( \mathbb{R}^p \). By Lemma 11 for any \( x \in Z \)

\[
\ell(x) \leq \ell(\theta)
\]

and this finishes the proof of Theorem A.

**Corollary 14.** \( H \)-curves are geodesics in any metrics having the form

\[
d(L, M) = \ell(\Psi_1[L, M], \ldots, \Psi_p[L, M])
\]

(8)

**Remark.** If the sphere \( \ell(x) = 1 \) in \( \mathbb{R}^n \) doesn’t contain a segment, then this geodesics is unique.

**Corollary 15.** Consider the Finsler metric on \( \text{Gr}_{p,q} \) given by the formula

\[
F(L, H) = \ell(\rho_1[L, H], \ldots, \rho_1[L, H]); \quad L \in \text{Gr}_{p,q}, H \in T_L
\]

Then the associated geodesic distance is given by the expression (8).

**9. Other symmetric spaces.** In \[12\] it was explained that arbitrary classical compact symmetric space is a Grassmannian in real, complex or quaternionic linear space. This allows to translate literally our results to all classical compact Riemannian symmetric spaces

\[
U(n) \times U(n)/U(n); \quad U(n)/O(n); \quad U(2n)/Sp(n)
\]

\[
U(p + q)/U(p) \times U(q); \quad O(2n)/U(n); \quad O(p + q)/O(p) \times O(q); \quad O(n) \times O(n)/O(n);
\]

\[
Sp(p + q)/Sp(p) \times Sp(q); \quad Sp(n)/U(n); \quad Sp(n) \times Sp(n)/Sp(n)
\]

Three series of the type \( A \) (i.e (9)) slightly differs from others: we have to replace the group \( W_p \) by the symmetric group.

In the same way, all classical Riemannian noncompact symmetric spaces are open domains in Grassmannians (see \[12\]). This allows to extend our results to all classical Riemannian noncompact symmetric spaces

\[
GL(n, \mathbb{C})/U(n); \quad GL(n, \mathbb{R})/U(n); \quad GL(n, \mathbb{H})/U(n)
\]

\[
U(p, q)/U(p) \times U(q); \quad SO^*(2n)/U(n); \quad O(p, q)/O(p) \times O(q); \quad O(n, \mathbb{C})/O(n);
\]

\[
Sp(p, q)/Sp(p) \times Sp(q); \quad Sp(2n, \mathbb{R})/U(n); \quad Sp(n, \mathbb{C})/Sp(n)
\]
10. Some examples. a) The space $GL(n, \mathbb{C})/U(n)$. We realize points of the space as positive definite $n \times n$ complex matrices. The group $GL(n, \mathbb{C})$ acts on this space by the transformations

$$L \mapsto gLg^*, \quad g \in GL(n, \mathbb{C})$$

The angles $\Psi_j[L, M]$ between points $L$ and $M$ are the solutions of the equation

$$\det(L - e^\psi M) = 0$$

Denote by $\Psi[L, M]$ the vector $(\Psi_1[L, M], \ldots, \Psi_n[L, M])$.

Let $L, M, N$ be points of our space. Consider all vectors in $\mathbb{R}^n$ that can be obtained from $\Psi[L, M]$ by permutations of coordinates. Denote by $Z$ their convex hull. Then

$$\Psi[L, N] \in \Psi[L, M] + Z \quad (10)$$

b) Original Lidskii theorem. Consider the space $S$ of hermitian $n \times n$ matrices. This space also is a (nonsemisimple) symmetric space. The group of isometries is the group of transformations

$$X \mapsto UXU^* + A \quad \text{where} \quad U \in U(n), A \in S$$

Let $X, Y \in S$. The analogy of angles are the eigenvalues of $X - Y$. The analogy of Theorem B is the original Lidskii theorem.

The space $S$ can be identified with the tangent space to $GL(n, \mathbb{C})$ at the point 1. For $X, Y \in S$ we define matrices

$$L = 1 + \varepsilon A, \quad M = 1 + \varepsilon B \quad \in GL(n, \mathbb{C})/U(n)$$

where $\varepsilon$ is small. Then the angles between $L$ and $M$ have the form $\varepsilon \lambda_j$, where $\lambda_j$ are the eigenvalues of $X - Y$. Hence the inclusion (10) implies Lidskii theorem.

Lidskii theorem on singular values of sum of two matrices corresponds to the triangle inequality in a tangent space to $U(p + q)/U(p) \times U(q)$ or $U(p, q)/U(p) \times U(q)$

c) The space $Sp(2n, \mathbb{R})/U(n)$. This spaces can be realized as the space of symmetric $n \times n$ complex matrices with norm $< 1$ (see for instance [14,5,1,6.3]). For two points $T, S$ we define the expression

$$\Lambda[T, S] = (1 - TT^*)^{-1/2}(1 - TS)(1 - SS^*)^{-1/2} \quad (11)$$

Let $\lambda_j$ be its singular values. Then the hyperbolic angles between $T$ and $S$ are given by the formula

$$\psi_j = \text{arcosh} \lambda_j \quad (12)$$

---

[7] hyperbolic angles
The analogy of Theorem B is given by the formula (10).

d) *Arazy norms.* Denote by $V_{\text{fin}}$ the space of finite real sequences $x = (x_1, \ldots, x_N, 0, 0, \ldots)$. Consider a norm $\ell$ on a space $V_{\text{fin}}$ satisfying the conditions

- $\ell$ is invariant with respect to permutations of coordinates
- $\ell$ is invariant with respect to the transformations $(x_1, x_2, \ldots) \mapsto (\sigma_1 x_1, \sigma_2 x_2, \ldots)$, where $\sigma_j = \pm 1$
- if $x^{(j)}$ converges to $x$ coordinate-wise and $\ell(x^{(j)})$ converges to $\ell(x)$, then $\ell(x^{(j)} - x)$ converges to 0 (an equivalent formulation: the $\ell$-convergence on the sphere $\ell(x) = 1$ is equivalent to the coordinate-wise convergence).

Let $V_\ell$ be the completion of $V_{\text{fin}}$ with respect to the norm $\ell$.

A compact operator $A$ in a Hilbert space is an element of *Arazy class* (see [1]) $C_\ell$ if the sequence of its singular values is an element of $V_\ell$. Consider a space $B_\ell$ (operator ball) of all compact operators $T$ in the Hilbert space satisfying the conditions

- $T \in C_\ell$
- $\|T\| < 1$, where $T$ denotes the standard norm of an operator in a hilbert space
- $T = T^t$

We define the angles $\Psi_j(T, S)$ in $B_\ell$ by formulas (11)–(12). We define the distance in $B_\ell$ by

$$d_\ell(T, S) = \ell(\Psi_1(T, S), \Psi_2(T, S), \ldots)$$

**Proposition 16.**

a) $d_\ell(T, S)$ is a metric.

b) The space $B_\ell$ is complete with respect to the metric $d_\ell(T, S)$.

The statement a) can be easily obtained from Theorem A by a limit considerations. For a proof of the statement B see [14], 8.6.3.

**11. Some references.**

a) For matrix inequalities see for instance [10], [2].

b) The formula for the distance in a symmetric space associated with the invariant Riemannian metrics was obtained in [3].

c) Our Theorem B for the unitary group $U(n) = U(n) \times U(n)/U(n)$ is Nudelman–Shvartsman theorem [10].

d) Generalization of Fan Ky theorem to arbitrary simple Lie algebras was obtained in [4].

e) Let $G$ be a simple Lie group, $K$ be its maximal compact subgroup and $K \setminus G/K$ be the hypergroup of $K$-biinvariant subsets in $G$ with convolution product. The problem about triangle inequality in Grassmannians is related to a classical problem on structure of the hypergroup $K \setminus G/K$, see for instance [5], [17], [8].

f) Some nonstandard geometries on groups are discussed in [3].

g) Some applications of geometry of angles are contained in [14], 6.3.
h) **Conjecture.** I think that complete triangle inequality for angles coincides with Horn–Klyachko inequalities [7], see also [11] and some comments in [19].

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