Universality Class of Two-Offspring Branching Annihilating Random Walks

Dexin Zhong

Physics Department, Clarkson University, Potsdam, New York 13699–5820, USA

Daniel ben-Avraham

Physics Department, Clarkson University, Potsdam, New York 13699–5820, USA
and

Physics Department, Bar-Ilan University, Ramat-Gan 52900, Israel

ABSTRACT: We analyze a two-offspring Branching Annihilating Random Walk ($n = 2$ BAW) model, with finite annihilation rate. The finite annihilation rate allows for a dynamical phase transition between a vacuum, absorbing state and a non-empty, active steady state. We find numerically that this transition belongs to the same universality class as BAW’s with an even number of offspring, $n \geq 4$, and that of other models whose dynamic rules conserve the parity of the particles locally. The simplicity of the model is exploited in computer simulations to obtain various critical exponents with a high level of accuracy.

PACS: 64.60.–i 02.50.–r 05.70.Ln, 82.20.Mj

1 e-mail: zhongd@craft.camp.clarkson.edu
2 e-mail: qd00@craft.camp.clarkson.edu
Nonequilibrium phase transitions in dynamical lattice models are the subject of rapidly growing interest [1,2]. Most second-order (i.e., continuous) phase transitions studied to date seem to belong to the same universality class as directed percolation (DP) [3-14]. An early observation of this fact has led Grassberger [15] and Janssen [16] to the conjecture that all continuous phase transitions from a single absorbing state to an active steady state seen in one-component models are in the DP universality class.

The DP conjecture draws support from the field-theoretical description of many of the models in question. In a coarse-grained fashion, this turns out to be identical to Reggeon Field Theory (RFT) [17]. An important achievement of Cardy and Sugar was proving that DP and RFT are in the same universality class [18].

The DP universality class is extremely robust. Examples not only include one-component models with a transition about a single absorbing state (as postulated in the conjecture) with binary interactions [3-6] and higher-order interactions [5], but also models with many absorbing states [7] and multi-component systems [8-11]. Grinstein et al. [8] have presented theoretical arguments for the extension of the DP conjecture to the latter case. In fact, exceptions to DP transitions are rare. In several cases, models that were initially thought to violate the rule were later found to be in the DP class, upon more careful analysis [12-14].

Recently, the study of Branching Annihilating Walks (BAW’s) has revealed a new universality class. The BAW model consists of particles which diffuse on a lattice and annihilate immediately upon encounter. Each particle gives birth to $n$ other offspring at adjacent sites, at some prescribed rate. The situation may be summarized, schematically, by

$$A + A \rightarrow 0,$$

$$A \rightarrow (1 + n)A.$$

As the birth (or branching) rate increases, the observed long time behavior undergoes a phase transition, from a state where the system is empty to an active steady state with a
finite concentration of particles. For odd $n$, the transition is in the same class as DP [13], but a new class emerges for even $n \geq 4$ [19,20]. Because the number of particles modulo 2 is conserved when $n$ is even, this class is referred to as the “parity-conserving universality class” (PC). Other systems in the PC class include Grassberger’s cellular automata models A [21] and B [22], a kinetic Ising model of Menyhárd [23], and the more complex (two-component) interacting monomer-dimer model of Kim and Park [24].

In the original BAW model the annihilation rate is infinite. In this case, the $n = 2$ BAW does not undergo a transition. It always evolves into the vacuum absorbing state, regardless of the branching rate [25]. However, a transition may be observed upon the introduction of a finite reaction rate [23,26]. Below, we report results for the phase transition of the $n = 2$ BAW with finite reaction rate. We find that it is in the PC class, and the simplicity of the model ($n = 2$ offspring, as opposed to even $n \geq 4$) allows us to obtain accurate estimates of various critical exponents.

We define our model on a one-dimensional lattice by the following computer algorithm. A particle is picked at random. It can undergo either nearest-neighbor diffusion (with probability $1 - p$) or it may branch (with probability $p$). In a diffusion attempt, a random direction (left or right) is picked and the particle moves to the target site if it is unoccupied. If the target site is occupied, then annihilation of the incident and target particles occurs with probability $k$. Otherwise, the incident particle remains at its original position. If a branching attempt is selected, then branching to the two nearest-neighbor sites occurs with probability 1 if both neighbors are empty. If one or both neighbors are occupied, branching to both neighbors occurs with probability $k$. In this case, if a new particle is placed on a previously occupied site, then both particles are removed. The probability $k$ controls the annihilation rate ($k = 1$ corresponds to the original model of infinite reaction rate).

Each time a particle is picked, a “time” counter is incremented by $1/N$, where $N$ is the number of currently surviving particles. Thus, in a time step each particle either
branches or diffuses one time, on the average. The ratio of branching rate to diffusion rate is controlled by the parameter $p$. Both $p$ and $k$ are critical fields: a phase transition could be observed upon varying either of the two. We chose to fix the reaction probability at $k = 1/2$, and to study the dynamical phase transition as a function of $p$.

To determine the transition point, we follow the common technique of epidemic analysis [27,20]. Starting our simulations at $t = 0$ with an empty lattice, except two adjacent occupied sites placed at the origin, we let the system evolve. The two-particle seed may grow and spread, but the system may also hit the absorbing state, when all particles annihilate. We measure: (1) $P(t)$, the probability that the system has not entered the absorbing state up to time $t$, (2) $\langle n(t) \rangle$, the mean number of particles at time $t$, averaged over all runs, and (3) $\langle R^2(t) \rangle$, the mean square distance of spreading from the center of the lattice, averaged over only the surviving runs. When the branching rate is critical, $p \to p_c$, these quantities are governed by power-laws in the long time asymptotic limit:

$$P(t) \sim t^{-\delta},$$

$$\langle n(t) \rangle \sim t^{\eta},$$

$$\langle R^2(t) \rangle \sim t^z.$$  \hspace{1cm} (2)

Off criticality, these quantities either grow or decay exponentially fast. As a result, a plot of the local slopes

$$-\delta(t) = \frac{\ln[P(t)/P(t/m)]}{\ln m}$$  \hspace{1cm} (3)

(and similarly for $\eta$ and $z$) as a function of $1/t$ is linear at criticality—and extrapolates to the long time asymptotic value—but shows a telltale curvature, otherwise. In Eq. (3), $m > 1$ is an arbitrary factor which we took to be equal to 5.

Running concurrently on a cluster of about 30 RISC/6000 workstations, we were able to perform extensive computer simulations. For each value of $p$, about $10^8$ runs were performed, up to 50,000 time steps each. The size of the lattice was chosen such that the spreading seed never hit the boundaries, to avoid finite-size effects.
In Fig. (1), we plot the local slope \( \eta(t) \) for various values of \( p \). One clearly sees that the curves for \( p = 0.4960 \) and \( p = 0.04950 \) veer upward, and the curves for \( p = 0.4942 \) and \( p = 0.4920 \) veer downward, while the curve for \( p = 0.4946 \) shows no significant curvature. This, combined with the fact that for the PC universality class \( \eta \) is expected to be zero \([20]\), helps us determine the critical value at \( p_c = 0.4946(2) \). Without relying on the assumption that \( \eta = 0 \), we estimate from our data that \( \eta = 0.000(1) \).

In Figs. (2) and (3), we show analogous results for \( \delta(t) \) and \( z(t) \). The same trend in the curvature of the local slopes is observed, though the plots for \( \eta \) (Fig. 1) allow for a more conclusive determination of the critical point. From the data we conclude that \( \delta = 0.286(2) \) and \( z = 1.147(4) \).

A critical exponent of particular interest is \( \beta \), the order-parameter critical exponent. This is related to the stationary concentration of particles in the steady state, \( \rho \), through

\[
\rho \sim (p - p_c)^\beta, \quad (p_c < p).
\]

To measure \( \beta \), we start a simulation with a fully occupied lattice and let the system evolve until it reaches a steady state. We then average the concentration over a long period of time. In this fashion, we have measured \( \rho \) for various values of \( p \). Each data point was obtained from over 1000 independent runs on \( 10^4 \)-site lattices, averaged over periods of up to \( t = 10^7 \) time steps. In Fig. (4), we show a log-log plot of the steady state concentration as a function of \( p - p_c \). From the slope of the data we estimate \( \beta = 0.922(5) \).

In Table 1, we compare our critical exponents measured for the \( n = 2 \) BAW with finite reaction rate to the exponents measured for the \( n = 4 \) BAW with infinite reaction rate \([20]\). (We anticipate that making the reaction rate finite wont change the nature of the transition in the latter case.) There is excellent agreement between the two sets of measurements, suggesting that our model is in the PC universality class. Similar figures, but with larger numerical uncertainties, have been obtained for the other models in the PC class \([21-24]\). The relative simplicity of the present model allows for efficient data collection. Indeed,
we have achieved a modest improvement in the estimate of \( \beta \), the order-parameter critical exponent.

The new PC universality class remains poorly understood. For example, one-component models in this class (such as BAW’s and Grassberger’s automata) may be treated through a second-quantization formalism, by means of creation-annihilation operators [28]. The equations thus derived may be coarse-grained through standard renormalization group techniques. This results in RFT equations, indicating that these systems should be in the DP universality class. What happens is that crucial parity-conserving fields renormalize away, and do not show in the final analysis.

Is parity conservation solely responsible for the new PC universality class? Undeniably, it is a feature of all the models found to be in the PC class so far, and its neglect in the field-theoretical approach leads to erroneous conclusions. Nevertheless, Park and Park [29] have convincingly proved, at least for the case of the interacting monomer-dimer model, that it is the existence of equivalent groundstates (in a dynamical sense, i.e., states that the system may access with equal probability), and not parity conservation, that gives rise to the new PC class. It would be interesting to see if this can be shown for other models in the PC class.

The BAW model analyzed in this paper is dual to the non-equilibrium Ising kinetic model of Menyhárd. The particles in our model correspond to the “kinks”, or the boundaries between domains of oppositely oriented spins. With only one component, and local dynamic rules involving a small number of particles, the \( n = 2 \) BAW is one of the simplest models in the PC universality class. We hope that this will be exploited in future numerical work, to obtain better estimates of the critical exponents. An intriguing possibility raised by Jensen [20], is that the PC transition may be solved analytically. It would make sense to search for analytical solutions starting with the simplest models, such as the \( n = 2 \) BAW.

We thank Hyunggyu Park for numerous useful discussions.
References

1) G. Nicolis and I. Prigogine, *Self-organization in Nonequilibrium Systems* (Wiley Interscience, New York, 1977).

2) H. Haken, *Synergetics* (Springer-Verlag, New York, 1983).

3) T. E. Harris, *Ann. Prob.* 2, 969 (1974).

4) R. Dickman and M. A. Burschka, *Phys. Lett. A* 127, 132 (1988).

5) R. Dickman, *Phys. Rev. B* 40, 7005 (1989).

6) D. A. Browne and P. Kleban, *Phys. Rev. A* 40, 1615 (1989); T. Aukrust, D. A. Browne, and I. Webman, *Europhys. Lett.* 10, 249 (1989); *Phys. Rev. A* 41, 5294 (1990).

7) I. Jensen and R. Dickman, *Phys. Rev. E* 48, 1710 (1993).

8) G. Grinstein, Z.-W. Lai, and D. A. Browne, *Phys. Rev. A* 40, 4820 (1989).

9) R. M. Ziff, E. Gulari, and Y. Barshad, *Phys. Rev. Lett.* 56, 2553 (1986).

10) I. Jensen, H. C. Fogedby, and R. Dickman, *Phys. Rev. A* 41, 3411 (1990).

11) H. Park, J. Köhler, I.-M. Kim, D. ben-Avraham, and S. Redner, *J. Phys. A* 26, 2071 (1993).

12) R. Bidaux, N. Boccara, and H. Chaté, *Phys. Rev. A* 39, 3094 (1989); I. Jensen, *Phys. Rev. A* 43, 3187 (1991).

13) H. Takayasu and A. Y. Tretyakov, *Phys. Rev. Lett.* 68, 3060 (1992); I. Jensen, *Phys. Rev. E* 47, 1 (1993).

14) J. Köhler and D. ben-Avraham, *J. Phys. A* 24, L621–L627 (1991); D. ben-Avraham and J. Köhler, *J. Stat. Phys.* 65, 839–848 (1991); I. Jensen, *J. Phys. A* 27, L61 (1994).

15) P. Grassberger, *Z. Phys. B* 47, 365 (1982).

16) H. K. Janssen, *Z. Phys. B* 42, 151 (1981).
17) V. N. Gribov, Sov. Phys. JETP 26, 414 (1968); H. D. I. Abarbanel, J. B. Bronzan, R. L. Sugar, and A. R. White, Phys. Rep. 21C, 120 (1975); R. C. Brower, M. A. Furman, and M. Moshe, Phys. Lett. 76B, 213 (1978).

18) J. L. Cardy and R. L. Sugar, J. Phys. A,13, L423 (1980).

19) I. Jensen, J. Phys. A 26, 3921 (1993).

20) I. Jensen, Phys. Rev. E 50, 3623 (1994).

21) P. Grassberger, F. Krause, and T. van der Twer, J. Phys. A 17, L105 (1984).

22) P. Grassberger, J. Phys. A 22, L1103 (1989).

23) N. Menyhárd, J. Phys. A 27, 6139 (1994).

24) M. H. Kim and H. Park, Phys. Rev. Lett. 73, 2579 (1994); H. Park, M. H. Kim, and H. Park, (preprint, to appear in Phys. Rev. E).

25) A. Sudbury, Ann. Prob. 18, 581 (1990)

26) D. ben-Avraham, F. Leyvraz, and S. Redner, Phys. Rev. E 50, 1843 (1994).

27) P. Grassberger and A. de la Torre, Ann. Phys. (NY) 122, 373 (1979).

28) M. Doi, J. Phys. A 9, 1465 and 1479 (1976); P. Grassberger and M. Scheunert, Fortsch. Phys. 28, 547 (1980); L. Peliti, J. Phys. (Paris) 46, 1469 (1985); R. Dickman, J. Stat. Phys. 55, 997 (1989); I. Jensen and R. Dickman, Physica A 203, 175 (1994).

29) H. Park and H. Park, Critical behavior of an absorbing phase transition in an interacting monomer-dimer model, (preprint, to appear in Physica A).
CAPTIONS

Figure 1: Local slope $\eta(t)$ as a function of $1/t$. Shown are curves for $p = 0.4960, 0.4950, 0.4946, 0.4942,$ and $0.4920$ (top to bottom).

Figure 2: Local slope $\delta(t)$ as a function of $1/t$. Shown are curves for $p = 0.4950, 0.4946,$ and $0.4942$ (top to bottom).

Figure 3: Local slope $z(t)$ as a function of $1/t$. Shown are curves for $p = 0.4950, 0.4946,$ and $0.4942$ (top to bottom).

Figure 4: Steady state concentration, $\rho$, as a function of the branching probability, $p - p_c$. The slope of the data points yields the order-parameter critical exponent, $\beta$. 
Table 1

| n  | $\delta$ | $\eta$ | $z$      | $\beta$  |
|----|----------|--------|----------|----------|
| 4  | 0.285(2) | 0.000(1)| 1.141(2) | 0.92(3)  |
| 2  | 0.286(2) | 0.000(1)| 1.147(4) | 0.922(5) |

**Table 1:** Critical exponents for the $n = 4$ BAW (from Ref. 20) and for the $n = 2$ BAW (as measured in the present work).
