Moduli Space of Cubic Surfaces as Ball Quotient via Hypergeometric Functions

Brent R. Doran

Abstract

We describe hypergeometric functions of Deligne-Mostow type for open subsets of the configuration space of six points on $\mathbb{P}^2$, induced from those for seven points on $\mathbb{P}^1$. The seven point ball quotient example $DM(2^5, 1^2)$ does not appear on Mostow’s original list [9], but does appear on Thurston’s corrected version [12]. We show that $DM(2^5, 1^2)$ is a finite cover of the moduli space of cubic surfaces $\mathcal{M}_C$ endowed with the ball quotient structure $\Gamma_C \backslash \mathbb{B}^4$ of [2]. This answers a question of Allcock [1] about the commensurability of $\Gamma_C$ with the monodromy groups of Deligne-Mostow hypergeometric functions.

1 Introduction

A moduli space $\mathcal{M}$ of algebraic varieties can sometimes be endowed with the structure of a Hermitian locally symmetric space $\Gamma \backslash \mathcal{B}$ via a multi-valued period map with discrete monodromy group $\Gamma$. In classical terminology, the period map (or rather, its single-valued $\Gamma$-invariant inverse) provides a “uniformization” of $\mathcal{M}$ by the symmetric space $\mathcal{B}$. When $\mathcal{B}$ is the complex $n$-dimensional hyperbolic space form, i.e., the “complex $n$-ball” $\mathbb{B}^n$, then the locally symmetric space is called a ball quotient.

Probably the best-known example is the uniformization of the coarse moduli space of cubic (elliptic) curves in $\mathbb{P}^2$ by the upper half plane $\mathbb{H}^1 \cong \mathbb{B}^1$. The ball quotient structure on $\mathcal{M}$ is simply the $j$-line ($\mathbb{P}^1$ with two orbifold points — of order 2 and 3 — and one puncture representing a cusp point). The $j$-function on $\mathbb{H}^1$ is the inverse to the period map and is invariant under the modular group $\Gamma = \text{PSL}_2(\mathbb{Z})$. Here the period map is expressed in terms of classical hypergeometric functions [13].

One might hope for something similar with moduli of cubic surfaces in $\mathbb{P}^3$. Unfortunately cubic surfaces have no non-zero holomorphic 2-forms, and hence no non-trivial periods.

One option is to instead use the periods of naturally associated varieties. The work of Allcock, Carlson, and Toledo [2] uses cubic threefolds that arise as tri-cyclic covers of $\mathbb{P}^3$ branched over a cubic surface. They conclude that the moduli space $\mathcal{M}_C$ of nodal cubic surfaces is a 4-ball quotient $\Gamma_C \backslash \mathbb{B}^4$, and that in fact the GIT compactification (adding semistable points) of $\mathcal{M}_C$ coincides with the Baily-Borel compactification (adding cusp points) of the ball quotient.

Another option is to find a uniformization more directly using hypergeometric functions. In the Deligne-Mostow framework, such functions uniformize certain moduli spaces of $n$ points on $\mathbb{P}^1$ by $\mathbb{B}^{n-3}$. Unfortunately, Allcock [1] showed that $\Gamma_C$ is not equal to any of the Deligne-Mostow monodromy groups, so it is — in that sense — a new example and not of hypergeometric type. However, his proof leaves open whether $\Gamma_C$ is commensurable to a Deligne-Mostow group.

1.1 Motivation, intuition, and the argument in brief

With this in mind, the motivating idea for how to relate $\mathcal{M}_C$ and Deligne-Mostow hypergeometric functions is straightforward.

$\mathcal{M}_C$ is 4-dimensional. The 4-dimensional Deligne-Mostow ball quotients are moduli spaces of 7 points on $\mathbb{P}^1$. All but three of these are descendants (i.e., realized as GIT-stable coincidences of subsets of 12 points, see [5, 7] for details) of a single Deligne-Mostow 9-ball quotient, which we call the Eisenstein ancestral example and denote by $DM(1^{12})$. In particular, their monodromy groups are defined (as automorphisms of lattices) over the Eisenstein ring $\mathcal{E} = \mathbb{Z}[\omega], \omega^3 = 1$.
In \cite{2} it is shown that $\Gamma_C$ is likewise defined over $\mathcal{E}$, and contains the monodromy group for a 6-point Deligne-Mostow example $DM(2^6)$. As the notation suggests, $DM(2^6)$ is the descendant of $DM(1^{12})$ representing six pairs of points sharing the same coordinates. However, only one 7-point Eisenstein descendant contains $DM(2^6)$, namely $DM(2^5, 1^2)$. (Observe that $DM(2^5, 1^2)$ does not appear on the original list of Mostow \cite{9} — descendants are perhaps the easiest way to see how the list misses many examples — although it does appear in Thurston’s corrected list \cite{12}.) This strongly suggests that the ball quotient monodromy group $DM_T(2^5, 1^2)$ and $\Gamma_C$ are naturally related. In Section \ref{sec:1} we give a geometric proof that $DM_T(2^5, 1^2)$ is a finite index subgroup of $\Gamma_C$. The essential argument is summarized below.

A smooth cubic surface is isomorphic to $\mathbb{P}^2$ blown up at 6 generic points. Label these points $m_i$. The task is to associate to these 6 points a set of 7 points on $\mathbb{P}^1$. The first five $m_i$ uniquely determine an irreducible quadric curve $Q$. There are two tangent lines to $Q$ through $m_6$. Thus $m_i, 1 \leq i \leq 5$ together with the two tangent points define 7 points on $Q$. Projection from $m_6$ sends these to 7 points on $\mathbb{P}^1$, where the first 5 are equally weighted (should be considered as unordered points), as are the last 2 (the ordering of the tangent lines is arbitrary). These are the symmetries of $DM(2^5, 1^2)$. The map is not quite invertible: each fiber is a $(\mathbb{Z}/2\mathbb{Z})^4$ orbit.

Nodal cubic surfaces are isomorphic to $\mathbb{P}^2$ blown-up at 6 points on a common irreducible $Q$, with at most pairs of points “colliding” (being infinitely near). The limit in which $m_6$ lies on $Q$ thus corresponds precisely to the class of nodal cubic surfaces, and projection now gives 6 points on $\mathbb{P}^1$ with exactly the structure of $DM(2^6)$.

Nodal cubic surfaces are the only singular cubic surfaces parametrized by $\mathcal{M}_C$. So in fact, up to finite identifications, $DM(2^5, 1^2)$ encodes all the data of $\mathcal{M}_C$.

The main result says more:

\textbf{Theorem 3.} $DM(2^5, 1^2)$ is a finite branched cover of $\mathcal{M}_C$. Furthermore, the Deligne-Mostow ball quotient structure is the same as the Allcock-Carlson-Toledo one induced by the covering map. In particular, $DM_T(2^5, 1^2) \subset \Gamma_C$ as a finite index subgroup.

\section{Eisenstein Ancestral Example}

This section presents necessary background on the hypergeometric functions, monodromy groups, and lattices involved in the main construction.

Deligne and Mostow \cite{3} \cite{9} characterize which moduli spaces of GIT-stable $n$-punctured projective lines are uniformized by a complex ball via hypergeometric functions. The list of
examples they produce, however, has many natural inclusions. One example in particular, which we call the Eisenstein ancestral example, contains almost all of the Deligne-Mostow examples for $n \geq 7$ as natural “descendants” (see \cite{6} for technical details).

A GIT moduli space of $n$ distinct points on $\mathbb{P}^1$ is specified by a choice of ample line bundle on $(\mathbb{P}^1)^n$ that linearizes the action of $SL_2(\mathbb{C})$. This is equivalent data to a list of $n$ positive integers $\mu = (\mu_1, \ldots, \mu_n)$. We denote the GIT-stable moduli space by $DM(\mu)$.

**Remark 1.** Each $DM(\mu)$ is a quasi-projective variety together with an explicit embedding. The projective completion is the GIT compactification obtained by adding the “semi-stable but not stable” points (up to equivalence). All of the $DM(\mu)$ for fixed $n$ are isomorphic on the open subset representing collections of distinct points on $\mathbb{P}^1$.

**Definition 1.** The Eisenstein ancestral example is $DM(1^{12})$, i.e., $DM(1, \ldots, 1)$.

**Remark 2.** Passing from ordered to unordered points by quotienting by $S_{12}$ yields the moduli space of degree 12 binary forms.

Hypergeometric functions on $DM(1^{12})$ are defined as follows. Denote by $\mathbb{P}^1_S$ the 12-punctured projective line $\mathbb{P}^1 \setminus S, S = \{s_1, \ldots, s_{12}\}$. Consider the multi-valued form

$$\omega_{HG}(s_1, \ldots, s_{12}) = \prod_i (z - s_i)^{-\frac{1}{3}} dz$$

on $\mathbb{P}^1_S$ (assuming only for convenience of notation that $s_i \neq \infty, \forall i$). This is the same data as: a rank one complex local system, $l_S \to \mathbb{P}^1_S$, with monodromy about any of the $s_i$ given by multiplication by $e^{\frac{2\pi i}{3}}$. It is easy to see that the dual local system is just the conjugate local system $\overline{l_S}$ (characterized by local monodromies $e^{\frac{-2\pi i}{3}}$).

**Proposition 1 (after \cite{3} \cite{9}).** The local system valued cohomology $H^1(\mathbb{P}^1_S, l_S)$ is 10 dimensional. It admits an anti-Hermitian perfect pairing with compactly supported cohomology, which is defined over the Eisenstein integers $\mathbb{E} = \mathbb{Z}[\omega]$ (for $\omega$ a cube root of unity). The associated Hermitian form is Lorentzian, i.e., has signature $(1,9)$. We call the rank 10 Lorentzian lattice $\Lambda_E$. In the Hodge decomposition $H^{1,0} \oplus H^{0,1}$, the holomorphic subspace is positive definite, and the anti-holomorphic subspace is negative definite, with respect to the Hermitian form.

**Remark 3.** Alternatively, one could use intersection cohomology to get a true intersection pairing on $IH^1(\mathbb{P}^1_S)$.

Now let the points of $S$ vary on $\mathbb{P}^1$. Deligne and Mostow show that what one hopes to be true is in fact true, namely that the $H^1(\mathbb{P}^1_S, l_S)$ are the fibers of a local system $\mathcal{L}$ over the parameter space of $n$ distinct points, $(\mathbb{P}^1)^n \setminus \Delta$, where $\Delta$ is the union of the subdiagonals. The resulting monodromy group $DM_T(1^{12})$ must preserve $\Lambda_E$, hence $DM_T(1^{12}) \subset Aut(\Lambda_E)$.

They also observe that projectivizing the fibers of $\mathcal{L}$ yields a canonical flat projective space bundle. $H^{1,0}(\mathbb{P}^1_S, l_S)$ is one dimensional, so its projective image is a point. By abuse of notation, the projective monodromy group we also denote by $DM_T(1^{12})$, and henceforth take that to be the default meaning of the notation. The natural extension of the group by the permutation group of the weights, here $S_{12}$, is denoted by $DM_T \times S_{12}(1^{12})$.

Choose coordinates on a fiber of the bundle over an arbitrary base point. Use flatness to canonically extend coordinates over the bundle, up to the projective monodromy action. Thus the projectivized holomorphic component of the cohomology defines a multi-valued map, with $DM_T(1^{12})$-monodromy, from the parameter space of 12 distinct points to $\mathbb{B}^9 \subset \mathbb{P}^9$. It is easily seen that this map is invariant with respect to the natural $SL_2(\mathbb{C})$-action on the parameter space, and so descends to the moduli space $DM(1^{12})$. We call this map the hypergeometric function $HG(1^{12})$ on $DM(1^{12})$. They show it extends naturally to the boundary, which corresponds to GIT stable collisions of the 12 points on $\mathbb{P}^1$. This boundary locus is stratified, indexed by partitions of the 12 points into subsets with no more than 5 elements. The image of $HG(1^{12})$, upon extending to the boundary, is all of $\mathbb{B}^9$.

**Theorem 1 (after \cite{3} \cite{9}).** $HG(1^{12})$ defines a complex ball uniformization of $DM(1^{12})$. Specifically, there is an analytic isomorphism $DM(1^{12}) \cong DM_T(1^{12}) \setminus \mathbb{B}^9$. Furthermore, permutations of the coordinates by $S_{12}$ are automorphisms of the lattice.

**Corollary 1 (\cite{3} \cite{9}).** The GIT compactification of $DM(1^{12})$ is isomorphic to the Baily-Borel compactification of $DM_T(1^{12}) \setminus \mathbb{B}^9$.
The closure of a boundary stratum is again a Deligne-Mostow example with \( DM_\ell (\mu) \) defined over \( E \). Its image under \( HG(12^2) \) is a \( DM_\ell (12^2) \)-orbit of a subball of \( \mathbb{B}^9 \). Indeed, the boundary divisor associated to the collision of two points is given by, on the ball quotient side, the Lorentzian sublattice orthogonal to a distinguished vector (see [3] for details). The process is iterated for further collisions. A simple count shows:

**Proposition 2.** There are 6 Deligne-Mostow 4-ball quotient examples that are descendants of \( DM(1^2)^2 \). As \( DM(2^5, 1^2) \) and \( DM(3, 2^3, 1^3) \) are not on Mostow’s original list [2], that list is incomplete.

**Remark 4.** Mostow compiled his list using the correct theory, but made mistakes in his by-hand computations. Thurston later [12] corrected the computation with a computer search, and his list is now believed to be complete (although I am unaware of a direct proof).

Note that each of the Deligne-Mostow monodromy groups is a subgroup of the projective automorphism group of a lattice over \( E \). For 4-ball quotients it turns out there is only one such lattice.

**Theorem 2 ([1]).** Up to isometry, there is a unique Lorentzian rank 5 lattice over \( E \), which we denote by \( \Lambda_5 \).

Denote Eisenstein 7 point Deligne-Mostow monodromy groups by \( DM_\ell, E(\mu) \). The theorem tells us that \( \Gamma_C \) and all the \( DM_\ell, E(\mu) \) are subgroups of the same \( PAut(\Lambda_5) \). It then follows from Theorem 1 that the extension \( DM_{\Sigma, E}(\mu) \) of the Deligne-Mostow monodromy group by \( \Sigma \) (the permutations of the points of equal weight) is likewise a subgroup of \( PAut(\Lambda_5) \).

## 3 Moduli Space of Marked Cubic surfaces and \( DM(2^5, 1^2) \)

Consider the parameter space of 6 points on \( \mathbb{P}^2 \). Let \( \Delta \) denote the divisor representing non-generic configurations, i.e., when all six points lie on a common quadric curve or if any three are collinear (including multiplicities). A smooth cubic surface is the blow up of 6 generic configurations, i.e., when all six points lie on a common quadric curve or if any three are not on Mostow’s original list [9], that list is incomplete.

**Lemma 1 ([2]).** \( \mathcal{M}^{sm}_{C,m}/W(E_6) \cong \mathcal{M}^{sm}_{C} \), where \( \mathcal{M}^{sm}_{C} \) is the moduli space of smooth cubic surfaces. Furthermore, \( W(E_6) \) acts as a subgroup of \( \Gamma_C \).

Basic facts about the geometry of the elementary (3-point) Cremona transforms will be needed.

**Lemma 2.** An elementary Cremona transform, defined by blowing up three points on \( \mathbb{P}^2 \) and then blowing down the proper transforms of the three lines that connect the vertices, has the following properties, up to projective transformations:

- 1. It sends a line through an edge of the triangle to the opposite vertex, and sends any line through a single vertex to itself, albeit non-trivially.
- 2. It preserves any quadric that passes through precisely two of the vertices.
- 3. Lines which don’t meet any of the three vertices are sent to quadrics passing through the three vertices.

**Proof.** The first follows essentially by definition. The last two are easily seen in coordinates. Pick coordinates so that \( m_1 = [1,0,0], m_2 = [0,1,0], \) and \( m_3 = [0,0,1] \). Then the Cremona transformation is \( \psi_{123} : [x,y,z] \mapsto [yz, xz, xy] \). Any nondegenerate quadric passing through two chosen points may be presented, via projective automorphisms, as the Veronese quadric \( y^2 - xz = 0 \) with the marked points \( m_1 \) and \( m_2 \) as above, and with the third vertex sent to \( m_3 \). The Veronese quadric in \( \psi_{123} \)-transformed coordinates is \( Y^2 - XZ = xz(xz - y^2) = 0 \), so indeed the quadric is preserved.
A line that doesn’t intersect \( m_1, m_2, \) or \( m_3 \) may be taken to be \( x + y + z = 0 \). Its image set under \( \psi_{123} \) is \( yz + xz + xy = 0 \), a non-degenerate quadric. Also, \( m_1, m_2, \) and \( m_3 \) are fixed and lie on the image quadric. So all six \( m_i \) are on the image quadric.

In the following Lemma we construct a map, \( \phi_5^0 \), from configurations of 6 generic points on \( \mathbb{P}^2 \) to configurations of 7 points on \( \mathbb{P}^1 \). The two subsequent Lemmas use \( \phi_5^0 \) in order to realize an open subset of \( DM(2^5, 1^2)/(\mathbb{Z}/2\mathbb{Z}) \) as a branched cover of \( \mathcal{M}_{C, sm}^\text{gen} \). This is the central idea governing the whole geometric construction.

**Lemma 3.** There is a morphism

\[
\phi_5^0 : ((\mathbb{P}^2)^6 \setminus \Delta)/\text{SL}_3(\mathbb{C}) \cong \mathcal{M}_{C, sm}^\text{gen} \rightarrow DM_{\mathbb{Z}, \mathcal{E}}(\mu)/(\mathbb{Z}/2\mathbb{Z})
\]

given geometrically by associating 7 points \( p_i \) on a projective line to six generic ordered points \( m_i \) on \( \mathbb{P}^2 \). If \( DM_{\mathbb{Z}, \mathcal{E}}(\mu) \) is taken to be \( DM(2^5, 1^2) \), then the morphism is naturally \( S_5 \)-equivariant. The \( S_5 \) acts as automorphisms of the ball quotient structure on \( DM(2^5, 1^2) \).

**Proof.** Because the \( m_i \) are generic (not a configuration represented in \( \Delta \)), the first five points \( \{m_1, \ldots, m_5\} \) lie on a unique irreducible quadric \( Q \) and \( m_6 \) is not on \( Q \). There are precisely two tangent lines to \( Q \) that pass through \( m_6 \). Label the tangent points \( t_1, t_2 \), for a total of seven labelled points on \( Q \). Since the tangent lines are unordered the labelling of the \( t_i \) is arbitrary. More precisely, there are 5 ordered and 2 unordered points.

Projection from \( m_6 \) determines 7 marked points \( p_i \) on the image \( \mathbb{P}^1 \), by \( m_i \mapsto p_i, 1 \leq i \leq 5 \), and \( t_j \mapsto p_{j+5} \). There is a \( \mathbb{Z}/2\mathbb{Z} \) action interchanging \( p_6 \) and \( p_7 \) since they are unordered. Observe that any one of \( \{p_i, 1 \leq i \leq 5\} \) may equal \( p_6 \) and any one other may equal \( p_7 \), but no other multiplicities are allowed.

The map is by construction compatible with the \( SL_3 \) and \( SL_2 \) actions: Any \( SL_2 \) action on the projective line of course may be realized as the restriction of an action on \( \mathbb{P}^2 \). To see the reverse direction in coordinates, we may use \( SL_3 \) to take \( Q \) to be the Veronese embedding \( V : y^2 = xz \). The associated isotropy subgroup acts as automorphisms of the projective line.

The \( S_5 \)-equivariance follows for the example \( DM(2^5, 1^2) \) because it permutes the first five coordinates on each side. It acts as automorphisms of \( \Delta \) by restriction from Theorem

The map is not one-to-one. Let \( DM_{\mathbb{Z}/2\mathbb{Z}}(2^5, 1^2) \) denote \( DM(2^5, 1^2)/(\mathbb{Z}/2\mathbb{Z}) \), where \( \mathbb{Z}/2\mathbb{Z} \) acts by permuting the two coordinates of weight 1. (Deligne-Mostow descendants inherit the action and so we will use similar notation for them.)

**Lemma 4.** \( \phi_5^0 \) descends to a dominant \( S_5 \)-equivariant injective morphism

\[
\mathcal{M}_{C, sm}^\text{gen} / (\mathbb{Z}/2\mathbb{Z})^4 \rightarrow DM_{\mathbb{Z}/2\mathbb{Z}}(2^5, 1^2),
\]

with image set a union of three strata: \( DM_{\mathbb{Z}/2\mathbb{Z}}^\text{gen}(2^5, 1^2), DM_{\mathbb{Z}/2\mathbb{Z}}^\text{gen}(3, 2^4, 1), \) and \( DM_{\mathbb{Z}/2\mathbb{Z}}^\text{gen}(3^2, 2^3) \). It is an isomorphism onto its image.

**Proof.** In the Veronese presentation above, the pair \( \{p_6, p_7\} \) is canonically identified with the pair of tangent points \( \{[1, 0, 0], [0, 0, 1]\} \). If the remaining \( p_i \) are distinct from \( p_6 \) and \( p_7 \), then \( p_i = [1, x_i, x_i^2], x_i \neq 0, 1 \leq i \leq 5 \), and each one is the image of precisely two points on \( Q \), namely, \( q_1 = [1, x_i, x_i^2] \) and \( q_2 = [1, -x_i, x_i^2] \). Thus any fiber of \( \phi_5^0 \) admits a transitive \( (\mathbb{Z}/2\mathbb{Z})^5 \) action. However, the automorphism of \( \mathbb{P}^2 \) changing the sign of the \( y \) coordinate obviously preserves the quadric \( Q \), the point of projection \([0, 1, 0]\), and the \( p_i \), while simultaneously switching \( q_1 \) and \( q_2 \) for all \( i \). So the effective transitive action on a generic fiber (i.e., over the open subset \( DM_{\mathbb{Z}/2\mathbb{Z}}^\text{gen}(2^5, 1^2) \)) is \( (\mathbb{Z}/2\mathbb{Z})^5 / (\mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^4 \).

If one (or two) of the \( p_i, 1 \leq i \leq 5 \), call it \( p_j \) (and \( p_k \)), coincides with \( p_6 \) or \( p_7 \) (or both), then the description is identical except that \( q_{j1} = q_{j2} \) (and \( q_{k1} = q_{k2} \)). Of course, \( (\mathbb{Z}/2\mathbb{Z})^4 \) still acts transitively, but not freely. The case with a single point of multiplicity 2 is the stratum \( DM_{\mathbb{Z}/2\mathbb{Z}}^\text{gen}(3, 2^4, 1) \) and the case with two such points is the stratum \( DM_{\mathbb{Z}/2\mathbb{Z}}^\text{gen}(3^2, 2^3) \).

Quotienting by the \( (\mathbb{Z}/2\mathbb{Z})^4 \) action makes the inverse well-defined, yielding the isomorphism onto the image. As before, the \( S_5 \) action permutes the \( m_i, 1 \leq i \leq 5 \) on the one hand and the \( p_i, 1 \leq i \leq 5 \) on the other.
The groups \((\mathbb{Z}/2\mathbb{Z})^4\) and \(S_5\) act via Cremona transforms as isomorphisms of cubic surfaces.

Proof. Let \(m_1, \ldots, m_5\) define the nondegenerate quadric \(Q\). Then \(m_6\) lies off \(Q\). The line \(m_1 m_2 m_3 m_4 m_5\) intersects \(Q\) in \(m_i\) and \(n_i\), with \(m_i = n_i\) if and only if \(m_1 m_2 m_3 m_4 m_5\) is a tangent line to \(Q\). Recall that \((\mathbb{Z}/2\mathbb{Z})^4\) acts via \((\mathbb{Z}/2\mathbb{Z})^5\), the \(i\)th factor interchanging \(m_i\) with \(n_i\), up to a \(\mathbb{Z}/2\mathbb{Z}\) projective transformation that switches all the \(m_i\) and \(n_i\) for \(i \neq 6\). Thus it suffices to exhibit Cremona transformations, up to projective equivalence, that switch any four \(m_i\) with \(n_i\) for \(i \neq 6\). Without loss of generality, we show this for \(i = 1, \ldots, 4\).

The Cremona transformation \(\psi_{126}\), composed with a projective change of coordinates whose notation we suppress, preserves \(m_1, m_3,\) and \(m_6\) but switches \(m_3, m_4,\) and \(m_5\) with their respective \(n_i\). This follows from Lemma 2 because \(Q\) passes through \(m_1\) and \(m_2\) and hence is sent to itself, but also lines through \(m_6\) are sent to themselves non-trivially, so it must be the case that \(m_i \leftrightarrow n_i\) for \(i = 3, 4,\) and \(5\). Then \(\psi_{126} \cdot \psi_{126}\) switches \(m_i\) and \(n_i\) for \(i = 1, \ldots, 4\).

\(S_5\) can be understood similarly, but it is easier to observe that it (indeed \(S_6\)) acts by permuting the coordinates and so is obviously an automorphism of a cubic surface and manifestly a subgroup of the \(W(E_6)\) action from Lemma 3.

Let us simplify notation a bit by defining:

\[
DM_{Z/2Z}(2^5, 1^2) = DM_{gen}(2^5, 1^2) \cup DM_{gen}(3, 2^4, 1) \cup DM_{Z/2Z}(2^5, 2^3)
\]

It then follows from Lemma 5 that:

**Corollary 2.** \(DM_{Z/2Z}(2^5, 1^2) \cong M_{gen}(Z/2Z, \mathbb{Z})^4\) is an intermediate branched cover of \(M_{gen}(\mathbb{Z})^4\), as are their respective \(S_5\) quotients.

Let \(U^{sm}\) and \(V^{sm}\) denote the open subsets of \(\mathbb{B}^4\) such that

\[
\Gamma_C \backslash U^{sm} \cong M_C \quad \text{and} \quad DM_{1}(2^5, 1^2) \backslash V^{sm} \cong DM_{Z/2Z}(2^5, 1^2).
\]

Our goal is to show there is a finite branched covering map \(DM_{1}(2^5, 1^2) \backslash \mathbb{B}^4 \to \Gamma_C \backslash \mathbb{B}^4\). Thus we must extend the existing covering map by adding the boundary loci (arrangements of 3-balls) to \(U\) and \(V\) in \(\mathbb{B}^4\).

We can try to extend \(\phi^s\) to strata of \(\Delta\) that match with the boundary strata of \(DM(2^5, 1^2)\). The most important case, which we denote by \(\Delta Q_{gen}\), is when all six \(m_i\) lie on an irreducible quadric \(Q\). Denote by \(\Delta Q\) the partial closure that allows points of multiplicity two.

**Lemma 6.** \(\phi^s\) extends, as an \(S_5\) equivariant map, over \(\Delta Q\). It is an isomorphism onto its image \(DM(2^6) \subset DM_{Z/2Z}(2^5, 1^2)\).

Proof. Projection from \(m_6\) identifies 6 points on \(\mathbb{P}^1\), with \(m_6\) identified with the point at \(\infty\). Thus \(\Delta Q_{gen}\) is mapped to \(DM_{gen}(2^6)\). Pairwise collisions are precisely the stability condition for \(DM(2^6)\) (inherited from that of \(DM(1^{12})\), described in Section 3). So the descendants of \(DM(2^6)\) make up the rest of the image of \(\Delta Q\). The map is clearly invertible. The \(S_5\) equivariance is reordering of coordinates, as before.

It is well-known that the moduli space of singular cubic surfaces with only nodal singularities, \(M_C^{nod}\), is isomorphic to the variety \(DM_C(2^6)\). What is more, the 3-ball quotient structure on \(M_C^{nod}\) induced by restriction from \(M_C \cong \Gamma_C \backslash \mathbb{B}^4\) coincides with the ball quotient structure of \(DM_C(2^6)\) (Section 9). Let \(\mathbb{B}^3_{DM(2^6)}\) denote a 3-subball of \(\mathbb{B}^4\) which covers \(DM(2^6)\). Then, in particular, \(\Gamma_C\) acts on \(\mathbb{B}^3_{DM(2^6)}\) in such a way that the orbit is precisely the arrangement of subballs which make up the complement of \(U^{sm}\) in \(\mathbb{B}^4\). Then as an immediate consequence, we have:

**Corollary 3.** The branched covering extends to

\[
(DM_{1}(2^5, 1^2) \backslash V) / (\mathbb{Z}/2\mathbb{Z}) \to \Gamma_C \backslash \mathbb{B}^4 \cong M_C
\]
To establish our goal, we simply need to show that the $\Gamma_C$ orbit of $B^3_{DM(2^6)}$ in $\mathbb{B}^4$ is in fact the complement of $V^{\mathrm{sm}}$ as well. The subgroup of $\Gamma_C$ that acts non-trivially on cubic surfaces is the factor group associated to $\mathcal{M}_{C,m}^{\mathrm{sm}}$, namely $W(E_6)$ acting via Cremona transformations. Thus it suffices to show that all the remaining strata of $DM_{\mathbf{Z}/2\mathbf{Z}}(2^5,1^2)$ can be obtained as images of subsets of $DM(2^6)$ under Cremona transformations in $W(E_6)$.

We consider another extension of $\phi_7^6$ from $\mathcal{M}_{C,m}^{\mathrm{sm}}$ to handle the other codimension 1 stratum of $DM_{\mathbf{Z}/2\mathbf{Z}}(2^5,1^2)$, namely $DM_{\mathbf{Z}/2\mathbf{Z}}^{\mathrm{gen}}(4,2^3,1^2)$. On $\mathbb{P}^2$, this stratum corresponds to the condition: $m_6$ is collinear with $m_i$ and $m_j$, for some $i \neq j \in 1, \ldots, 5$.

**Lemma 7. Cremona transforms identify $DM_{\mathrm{gen}}^{\mathrm{gen}}(2^6)$ with $DM_{\mathrm{gen}}^{\mathrm{gen}}(4,2^3,1^2)$**

**Proof.** Consider a point in $DM_{\mathrm{gen}}^{\mathrm{gen}}(4,2^3,1^2)$. It represents a configuration where three points, of which $m_6$ is one, are collinear. The result now follows by using the Cremona transformation from the proof of item 3 in Lemma 3.

So, the two extensions of $\phi_7^6$ we have described are identified by the Cremona transform. We now show that same Cremona transform also identifies the boundary of these two divisors.

We check this fact by taking limits in the respective extended domains of $\phi_7^6$. Indeed, it is sufficient to check for the one main case, $DM(4,2^4)$, as is clear from the poset of descendant strata.

The interpretation via the extension of $\phi_7^6$ for $DM(4,2^4) \subset DM(2^6)$ is straightforward: simply collide two points of the six on the quadric $Q$. On the other hand, for $DM(4,2^4) \subset DM_{\mathbf{Z}/2\mathbf{Z}}(4,2^3,1^2)$ one takes the other limit in which there is only one “tangent” point to $Q$, namely when it degenerates into two lines. The weight 4 point on $\mathbb{P}^1$ corresponds to either (I) two distinct points $m_i$ and $m_j$ (i, j $\neq 6$) collinear with $m_6$, or (II) two overlapping points (the cases are equivalent up to the ($\mathbf{Z}/2\mathbf{Z}$)$^4$ action). In each case, the Cremona transform defined by the non-collinear triple results in a quadric with 5 marked points: (I) One of $m_i$ or $m_j$ must also be collinear with a side of the Cremona triangle (i.e., two elements of the triple), and hence is identified with the opposite vertex for a total of five points, one of double weight. Also, all five points must lie on the quadric that is the image of the line defined by $m_6$, $m_i$, and $m_j$. (II) The line connecting $m_6$ and the doubled point is sent to a quadric passing through the three vertices of the Cremona triangle. This proves:

**Theorem 3.** $DM(2^5,1^2)$ is a finite branched cover of $\mathcal{M}_C$. Furthermore, the Deligne-Mostow ball quotient structure is the same as the Allcock-Carlson-Toledo one induced by the covering map. In particular, $DM_{\Gamma}(2^5,1^2) \subset \Gamma_C$ as a finite index subgroup.

**Remark 5.** Alcock shows $\Gamma_C$ is unequal to any $DM_{\Gamma}(\mu)$, and hence that $\Gamma_C$ is in some sense a “new” example. Theorem 3 answers his question [11, Section 8] regarding commensurability.

## 4 Further Remarks

Note that although we used $\phi_7^6$ and two of its extensions to construct associated cubic surfaces and hence the finite cover, a simultaneous extension in $(\mathbb{P}^2)^6 \setminus \Delta$ is not well-defined. For example, the configurations with a point of multiplicity 2 are interpretable in multiple ways in $DM(2^5,1^6)$.

To correctly extend $\phi_7^6$, one needs to extend the domain of definition by “separating” these different interpretations. They are indexed by where the collision “came from”, be it quadric, collinear, or a generic collision.

More formally then, the true extension of $\phi_7^6$ is to the moduli space of marked nodal cubic surfaces compactified by 40 points [10]. There are a total of 36 boundary divisors, interchanged by the action of ($\mathbf{Z}/2\mathbf{Z}$)$^4 \ltimes S_5 \subset W(E_6)$. The 36 are indexed as follows: (a) 1 in which the points all lie on a quadric, (b) 10 such that $m_6$ is collinear with some $m_i$ and $m_j$, (c) 10 such that three of the $m_k$, $k \neq 6$ are collinear, and (d) 15 that represent the “six choose three” pairwise collisions. If the full group $W(E_6)$ is allowed to act, then all the divisors are identified, and the quotient is $\mathcal{M}_C$.

The $S_5$ action identifies the 10 divisors of (b) with a single locus corresponding to $DM(4,2^3,1^2)$. It identifies those of (c) with a single locus corresponding to $DM(2^6)$ (now thought of as parametrizing the projections from $m_6$ of five points on a degenerate quadric).

Rather than write out the full details, for a more formal discussion (one originally motivated by periods of K3 surfaces), one should see the geometric proof in [5]. The authors give
a beautiful interpretation of this cover as the moduli space of cubic surfaces with a marked line. The \((\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5\) group action is interpreted as \(W(D_5) \subset W(E_6)\).

Dolgachev, van Geemen, and Kondô don’t use the locus of 6 points on a quadric in constructing their isomorphism. They use, in effect, the alternate limit where \(Q\) degenerates to a pair of lines while \(m_6\) remains off \(Q\). The singular point is the confluence of the tangent points, and so the image under projection to \(\mathbb{P}^1\) from \(m_6\) has weight 2, yielding \(DM(2^6)\). This is the basis for their list of 19 strata on pp. 15-16.

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