THE MAXIMAL COMMUTATIVE SUBALGEBRA OF A LEAVITT PATH ALGEBRA

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Abstract. For any unital commutative ring $R$ and for any graph $E$, we identify a maximal commutative subalgebra of the Leavitt path algebra of $E$ with coefficients in $R$. Furthermore, we are able to characterize injectivity of representations which gives a generalization of the Cuntz-Krieger uniqueness theorem, and on the other hand, to generalize and simplify the result about commutative Leavitt path algebras over fields.

1. Introduction

For a commutative unital ring $R$, the Leavitt path algebras are a specific type of path $R$-algebras associated to a graph $E$, modulo some relations. Initially, the Leavitt path algebra $L_K(E)$ was introduced for row-finite graphs (countable graphs such that every vertex emits only a finite number of edges) and for an arbitrary field $K$ in [1] and [4]; later extended the definition to arbitrary graphs in [2]. M. Tomforde in [12] generalizes the construction of Leavitt path algebras by replacing the field $K$ with a commutative unital ring $R$.

Leavitt path algebras are the algebraic version of Cuntz-Krieger graph $C^*$-algebras: a class of algebras intensively investigated by analysts for more than two decades (see [10] for an overview of the subject). On the other hand, they can be considered as natural generalizations of Leavitt algebras $L(1,n)$ of type $(1,n)$, investigated by Leavitt in [8] in order to give examples of algebras not satisfying the IBN property (i.e., whose modules have bases with different cardinality).

Leavitt path algebras of graphs include many well-known algebras such as matrix rings $M_n(R)$ for $n \in \mathbb{N}$, the Laurent polynomial ring $R[x, x^{-1}]$, or the classical Leavitt algebras $L(1,n)$ for $n \geq 2$.

The algebraic and analytic theories share important similarities, but also present some remarkable differences. The relation between these two classes of graph algebras has been mutually beneficial. This is the case once more for the topic discussed in the current paper: the analytic result was given in [9] for the $C^*$-algebras $C^*(E)$, and we give here the algebraic analogue for the Leavitt path algebras $L_R(E)$.

The paper is organized as follows. In Section 2 we give all the background information, together with the definition of $L_R(E)$ and some basic properties. In particular we introduce Cuntz-Krieger $E$-systems, the analog relations of Leavitt path algebras in a general $R$-algebra.

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In Section 3 we introduce the commutative core $M_R(E)$: a maximal commutative subalgebra inside $L_R(E)$. For this purpose we previously introduce a particular representation of the Leavitt path algebra (Proposition 3.8) on a specific $R$-algebra related to the set of all essentially aperiodic trails in the graph, that is, the set of all infinite aperiodic paths, as well as those infinite that are periodic (those that end in a cycle without exits) together with all finite paths whose range is a sink. This allows us to prove Theorem 3.21 where we prove the existence of an algebraic conditional expectation onto the subalgebra $M_R(E)$. With this useful tool, we will give the main result of this section: Theorem 3.22.

Thanks to the existence of $M_R(E)$, in Corollary 4.3 of Section 4 we generalize the result for commutative Leavitt path algebras of row-finite graphs over fields given in [5]. As a by-product, we also analyze the Cohn path algebras $C_R(E)$ over $R$ which are commutative since these algebras can be realized as Leavitt path algebras of some appropriate graphs.

Finally Section 5 is dedicated to the uniqueness theorems for Leavitt path algebras, i.e., those which set conditions on the graph $E$ or on the map $\Phi$ in order to ensure that a representation $\Phi : L_R(E) \to A$ is injective. Analogously to [9] Theorem 3.13] where it is proved a uniqueness theorem in the spirit of Szymanski’s Uniqueness Theorem for graph $C^*$-algebras (see [11] Theorem 1.2), Theorem 5.2 says in particular that a representation of $L_R(E)$ is injective if and only if it is injective on the maximal commutative subalgebra $M_R(E)$. This theorem also generalizes [6] Theorem 3.7 for fields and the Cuntz-Krieger Uniqueness Theorem given in [12] Theorem 6.5] for graphs in which every cycle has an exit, the so-called Condition (L).

2. Preliminaries

A (directed) graph $E = (E^0, E^1, r, s)$ consists of two sets $E^0$ and $E^1$ together with maps $r, s : E^1 \to E^0$. The elements of $E^0$ are called vertices and the elements of $E^1$ edges.

If a vertex $v$ emits no edges, that is, if $s^{-1}(v)$ is empty, then $v$ is called a sink. A vertex $v$ is called a regular vertex if $s^{-1}(v)$ is a finite non-empty set. The set of regular vertices is denoted by $E^0_{\text{reg}}$.

A path $\mu$ in a graph $E$ is a finite sequence of edges $\mu = e_1 \ldots e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \ldots, n - 1$. In this case, $n = l(\mu)$ is the length of $\mu$; we view the elements of $E^0$ as paths of length 0. For any $n \in \mathbb{N}$ the set of paths of length $n$ is denoted by $E^n$. Also, $\text{Path}(E)$ stands for the set of all paths, i.e., $\text{Path}(E) = \bigcup_{n \in \mathbb{N}} E^n$. We denote by $\mu^0$ the set of the vertices of the path $\mu$, that is, the set $\{s(e_1), r(e_1), \ldots, r(e_n)\}$.

A path $\mu = e_1 \ldots e_n$ is closed if $r(e_n) = s(e_1)$, in which case $\mu$ is said to be based at the vertex $s(e_1)$. The closed path $\mu$ is called a cycle if it does not pass through any of its vertices twice, that is, if $s(e_i) \neq s(e_j)$ for every $i \neq j$. A cycle of length one is called a loop. An exit for a path $\mu = e_1 \ldots e_n$ is an edge $e$ such that $s(e) = s(e_i)$ for some $i$ and $e \neq e_i$. We say that $E$ satisfies Condition (L) if every simple closed path in $E$ has an exit, or, equivalently, every cycle in $E$ has an exit.

Given paths $\alpha, \beta$, we say $\alpha \leq \beta$ if $\beta = \alpha \alpha'$, for some path $\alpha'$.

For each $e \in E^1$, we call $e^*$ a ghost edge. We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$.

**Definition 2.1.** Given an arbitrary graph $E$ and a commutative ring with unit $R$, the Leavitt path algebra with coefficients in $R$, $L_R(E)$ is the universal $R$-algebra generated by
a set \( \{ v : v \in E^0 \} \) of pairwise orthogonal idempotents together with a set of variables \( \{ e, e^* : e \in E^1 \} \) which satisfy the following conditions:

1. \( s(e)e = e = er(e) \) for all \( e \in E^1 \).
2. \( r(e)e^* = e^* = e^*s(e) \) for all \( e \in E^1 \).
3. (The “CK-1 relations”) For all \( e, f \in E^1 \), \( e^* e = r(e) \) and \( e^* f = 0 \) if \( e \neq f \).
4. (The “CK-2 relations”) For every regular vertex \( v \in E^0 \),

\[
v = \sum_{\{ e \in E^1, s(e) = v \}} ee^*.
\]

An alternative definition for \( L_R(E) \) can be given using the extended graph \( \hat{E} \). This graph has the same set of vertices \( E^0 \) and same set of edges \( E^1 \) together with the so-called ghost edges \( e^* \) for each \( e \in E^1 \), whose directions are opposite to those of the corresponding \( e \in E^1 \). Thus, \( L_R(E) \) can be defined as the usual path algebra \( R\hat{E} \) with coefficients in \( R \) subject to the Cuntz-Krieger relations (3) and (4) above.

If \( \mu = e_1 \ldots e_n \) is a path in \( E \), we write \( \mu^* \) for the element \( e_n^* \ldots e_1^* \) of \( L_K(E) \). With this notation it can be shown that the Leavitt path algebra \( L_R(E) \) can be viewed as a

\[
L_R(E) = \text{span}_R \{ \alpha \beta^* : \alpha, \beta \in \text{Path}(E) \text{ and } r(\alpha) = r(\beta) \}
\]

and \( rv \neq 0 \) for all \( v \in E \) and all \( r \in R \setminus \{ 0 \} \). (The elements of \( E^0 \) are viewed as paths of length 0, so that this set includes elements of the form \( v \) with \( v \in E^0 \)).

We can define an \( R \)-linear involution \( x \mapsto x^* \) on \( L_R(E) \) as follows: if \( x = \sum_{i=1}^n r_i \alpha_i \beta_i^* \), then \( x^* = \sum_{i=1}^n r_i \beta_i \alpha_i^* \).

If \( E \) is a finite graph, then \( L_R(E) \) is unital with \( \sum_{v \in E^0} v = 1_{L_R(E)} \); otherwise, \( L_R(E) \) is a ring with a set of local units (i.e., a set of elements \( X \) such that for every finite collection \( a_1, \ldots, a_n \in L_R(E) \), there exists \( x \in X \) such that \( a_i x = a_i = xa_i \) consisting of sums of distinct vertices of the graph.

Another useful property of \( L_R(E) \) is that it is a graded algebra. Note that any ring \( R \) may be viewed as a \( \mathbb{Z} \)-ring in the natural way (set \( R_0 = R \) and \( R_n = 0 \) for every \( n \in \mathbb{Z}, n \neq 0 \)). Then \( L_R(E) \) can be decomposed as a direct sum of homogeneous components

\[
L_R(E) = \bigoplus_{n \in \mathbb{Z}} L_R(E)_n
\]

satisfying \( L_R(E)_n L_R(E)_m \subseteq L_R(E)_{n+m} \). Actually,

\[
L_R(E)_n = \text{span}_R \{ pq^* : p, q \in \text{Path}(E), l(p) - l(q) = n \}.
\]

Every element \( x_n \in L_R(E)_n \) is a homogeneous element of degree \( n \).

Let us define the analogous relations given for a Leavitt path algebra when we consider a general \( R \)-algebra with involution.

**Definition 2.2.** Let \( E \) be a graph and \( A \) be an \( R \)-algebra with involution \( * \). A Cuntz-Krieger \( E \)-system in \( A \) is a collection \( \Sigma = (S_{\mu})_{\mu \in E^0 \cup E^1} \subset A \) which satisfies the following relations:

1. For all \( v, w \in E^0 \), \( S_v S_v = S_v \) and \( S_v S_w = 0 \) if \( v \neq w \).
2. \( S_v^* = S_v \) for all \( v \in E^0 \).
3. \( S_{s(e)} S_e = S_e = S_e S_{r(e)} \) for all \( e \in E^1 \).
4. For all \( e, f \in E^1 \), \( S_e^* S_e = S_{r(e)} \) and \( S_e^* S_f = 0 \) if \( e \neq f \).
(5) For every regular vertex \( v \in E^0 \),
\[
S_v = \sum_{\{e \in E^1, s(e) = v\}} S_e S_e^*.
\]

For convenience we are going to extend the Cuntz-Krieger \( E \)-system \( \Sigma \subset \mathcal{A} \) by defining all elements \( S_\mu \in \mathcal{A}, \mu \in \text{Path}(E) \), considering \( S_\mu S_\nu = S_{\mu\nu} \) if \( r(\mu) = s(\nu) \) and \( S_\mu S_\nu = 0 \) otherwise. So we shall abuse the notation and write it as an extended system \( \Sigma = (S_\mu)_{\mu \in \text{Path}(E)} \).

There exists a new version given for paths of the Cuntz-Krieger relation (5) defined above.

**Proposition 2.3.** Let \( v \in E^0 \) and \( k \in \mathbb{N} \). If there are finitely many paths \( \mu \) with \( s(\mu) = v \) and \( l(\mu) \leq k \), then
\[
S_v = \sum_{\{s(\mu) = v, l(\mu) = k\}} S_\mu S_\mu^* + \sum_{\{s(\mu) = v, l(\mu) < k \text{ and } r(\mu) \text{ is a sink}\}} S_\mu S_\mu^*.
\]

**Proof.** The proof follows completely \cite[Proposition 1.3]{9}. We only have to consider that a path \( e_1 e_2 \ldots e_n \) in \cite{9} corresponds to the path \( e_n e_{n-1} \ldots e_1 \) here, that is, the edges in \cite{9} are multiplied so that the action of \( f \) precedes the action of \( e \) in the product \( ef \); contrary to the action we consider here. In particular when \cite{9} refers to a source in our case it exactly corresponds to a sink. \( \square \)

Analogously to \cite[Equation (2.1)]{12} for Leavitt path algebras, we set here the following proposition for future reference.

**Proposition 2.4.** Given a Cuntz-Krieger \( E \)-system \( \Sigma = (S_\mu)_{\mu \in \text{Path}(E)} \) in some \( R \)-algebra \( \mathcal{A} \) with involution, the collection
\[
G(\Sigma) = \{ S_\mu S_\nu^* : \mu, \nu \in \text{Path}(E), r(\mu) = r(\nu) \}
\]
satisfies the following:

(i) given paths \( \mu, \nu \in \text{Path}(E) \) with \( r(\mu) = r(\nu) \), we have \( (S_\mu S_\nu^*)^* = S_\nu S_\mu^* \);

(ii) given four paths \( \alpha, \beta, \mu, \nu \in \text{Path}(E) \) with \( r(\alpha) = r(\beta) \) and \( r(\mu) = r(\nu) \) then
\[
(S_\alpha S_\beta^*)(S_\mu S_\nu^*) = \begin{cases} S_{\alpha\mu'} S_{\nu'}^* & \text{if } \mu = \beta \mu' \text{ for some } \mu' \in \text{Path}(E), \\
S_{\alpha} S_{\nu'\beta'}^* & \text{if } \beta = \mu \beta' \text{ for some } \beta' \in \text{Path}(E), \\
0 & \text{otherwise.}
\end{cases}
\]

**Definition 2.5.** The collection \( G(\Sigma) \) given in Proposition 2.4 will be called the standard Cuntz-Krieger generator set associated with \( \Sigma \). When we refer to the Leavitt path algebra \( L_R(E) \) we will denote it simply by \( G_E \), i.e.,
\[
G_E = \{ \mu \nu^* : \mu, \nu \in \text{Path}(E), r(\mu) = r(\nu) \}.
\]

As we have mentioned before, note that we have \( L_R(E) = \text{span}_R(G_E) \).

To finish this section we introduce a Graded Uniqueness Theorem that follows similarly to the one given in \cite[Theorem 5.3]{12}. We will use this result later.
Theorem 2.6. Let $\Sigma = (S_\mu)_{\mu \in E^0 \cup E^1}$ be a Cuntz-Krieger $E$-system in a $\mathbb{Z}$-graded $R$-algebra $A$ such that for any $\mu \in E^0$, $S_\mu$ is homogenous of degree zero and for any $\mu \in E^1$, $S_\mu$ is homogenous of degree one. Then the following conditions are equivalent:

(i) the associated representation $\Phi : L_R(E) \to A$ is injective;
(ii) $\Phi(rv) \neq 0$ for all $v \in E^0$ and for all $r \in R \setminus \{0\}$.

3. Maximal commutative subalgebra

Definition 3.1. Suppose $\Sigma = (S_\mu)_{\mu \in \text{Path}(E)}$ is a Cuntz-Krieger $E$-system in an $R$-algebra $A$. Consider the subset

$$G^A(\Sigma) = \{S_\mu S_\mu^* : \mu \in \text{Path}(E)\}.$$ 

By Proposition 2.4 it is clear that $G^A(\Sigma)$ is a commutative set of self-adjoint idempotents in $< \Sigma >$, the $R$-subalgebra of $A$ generated by $\Sigma$. We will refer to $G^A(\Sigma)$ as the standard diagonal generator set. The $R$-subalgebra $< G^A(\Sigma) > \subset A$, which will be denoted by $\Delta(\Sigma)$, is called the diagonal algebra associated with $\Sigma$. In particular when we refer to the case of the Leavitt path algebra $L_R(E)$, the diagonal generator set will be denoted by $G^E_\Sigma$ and the $R$-subalgebra generated by it will be denoted by $\Delta(E)$.

Remark 3.2. If $\Sigma = (S_\mu)_{\mu \in \text{Path}(E)}$ is a Cuntz-Krieger $E$-system in an $R$-algebra $A$ and $B$ is another $R$-algebra such that we have a *-homomorphism $\Pi : A \to B$, then $\Pi(\Sigma) = (\Pi(S_\mu))_{\mu \in \text{Path}(E)}$ is a Cuntz-Krieger $E$-system in $B$ and $\Pi$ maps $G^A(\Sigma)$ onto $G^B(\Pi(\Sigma))$ and $\Delta(\Sigma)$ onto $\Delta(\Pi(\Sigma))$. When we specialize to the case of a representation $\Pi : L_R(E) \to A$, this maps $G^E_\Sigma$ onto $G^B(\Sigma)$ and $\Delta(E)$ onto $\Delta(\Sigma)$.

We define all the terminology we need in order to describe the commutative core of a Leavitt path algebra. These definitions are the analog (in our “graph sense”) of those given in [9] Definitions 2.4-2.5.

Definition 3.3. Given a graph $E$, a trail in $E$ is either

(i) a finite path $\tau = e_1 \ldots e_n$ (possibly of length zero) whose range $r(\tau) = r(e_n)$ is a sink, that is, $s^{-1}(r(\tau)) = \emptyset$; or
(ii) an infinite path, that is, an infinite sequence $\tau = e_1 e_2 \ldots$ of edges, such that $r(e_n) = s(e_{n+1})$ for every $n \in \mathbb{N}$.

In some literature about Leavitt path algebras (see for example [2]), the set of all trails in the graph $E$ is denoted by $E^{\leq \infty}$.

Given a trail $\tau$ and some integer $n \geq 0$, we define the head of length $n$ to be the finite path:

$$\tau(n) = \begin{cases} 
  s(\tau) & \text{if } n = 0, \\
  e_1 \ldots e_n & \text{if } n > 0 \text{ and } \tau \text{ is either infinite, or finite with } l(\tau) > n, \\
  \tau & \text{if } n > 0 \text{ and } \tau \text{ is finite, with } l(\tau) \leq n.
\end{cases}$$

Notice that all heads of $\tau$ have the same source, that is, $s(\tau(n)) = s(\tau(0))$, and this special vertex will be denoted simply by $s(\tau)$, and will be referred to as the source of $\tau$. Given a trail $\tau$ and a path $\mu \in \text{Path}(E)$, we write $\mu \leq \tau$ if $\mu = \tau(n)$ for some $n \geq 0$. It is evident that if $\mu \leq \tau$, then removing $\mu$ from $\tau$ still yields a trail.
On the other hand, if we have trail \( \tau \) and a path \( \mu \in \text{Path}(E) \) with \( r(\mu) = s(\tau) \), the obvious concatenation \( \mu \tau \) is again a trail.

**Remark 3.4.** For \( \mu^*, \nu^* \in G_E^\Delta \) we say \( \mu^* \leq \nu^* \) if and only if \( \mu \leq \nu \), that is, \( \nu = \mu \mu' \) for some \( \mu' \in \text{Path}(E) \). Then for any pair \( \mu^*, \nu^* \in G_E^\Delta \) we have either \((\mu^*)(\nu^*) = 0 \) or \( \mu^* \) and \( \nu^* \) are comparable (either \( \mu^* \leq \nu^* \) or \( \nu^* \leq \mu^* \)). Let us see that if we consider the diagonal generator set \( G_E^\Delta \) (whose elements are all non-zero self-adjoint idempotents), the finite (resp. infinite countable) maximal totally ordered subsets of \( G_E^\Delta \) are in one-to-one correspondence with the finite (resp. infinite) trails in \( E \).

Specifically, every maximal totally ordered subset \( \mathcal{P} \subset G_E^\Delta \) can be uniquely presented as \( \mathcal{P} = \{ \tau(n)^* : n \geq 0 \} \) for some trail \( \tau \):

First let \( \tau \) be a finite trail \( \tau = e_1 \ldots e_n \) and \( r(\tau) \) is a sink). We claim that \( \mathcal{P} = \{ \tau(0)^*, \tau(1)^*, \ldots, \tau(n)^* \} \) is a maximal totally ordered subset of \( G_E^\Delta \). Obviously \( \mathcal{P} \) is a totally ordered set with order \( \leq \). Assume that there exists \( \mathcal{P} \subset \mathcal{Q} \) and \( \mathcal{Q} \) is a totally ordered subset of \( G_E^\Delta \); then there exists \( \beta \beta^* \in \mathcal{Q} \setminus \mathcal{P} \) and so \( \mathcal{P} \cup \{ \beta \beta^* \} \) is totally ordered. Then \( \beta \beta^* \leq \tau(n)^* \) or \( \tau(n)^* \leq \beta \beta^* \). If \( \beta \beta^* \leq \tau(n)^* \) then \( \beta \leq \tau(n) \) which is a contradiction since \( \beta \beta^* \notin \mathcal{P} \); on the other hand if \( \tau(n)^* \leq \beta \beta^* \) then \( \tau(n) \leq \beta \) and since \( r(\tau(n)) = r(\tau) \) is a sink then \( \tau(n) = \beta \) which gives again a contradiction since \( \beta \beta^* \notin \mathcal{P} \). Then \( \mathcal{P} \) is maximal totally ordered subset of \( G_E^\Delta \).

Consider now \( \tau \) an infinite trail: \( \tau = e_1 e_2 e_3 \ldots \). Let \( \mathcal{P} = \{ \tau(0)^*, \tau(1)^*, \tau(2)^* \} \) be a maximal totally ordered subset of \( G_E^\Delta \). Take into account that \( \mathcal{P} \) is totally ordered set with order \( \leq \). Suppose there exists a totally ordered subset \( \mathcal{Q} \) of \( G_E^\Delta \) such that \( \mathcal{P} \subset \mathcal{Q} \) and consider \( \beta \beta^* \in \mathcal{Q} \setminus \mathcal{P} \). Then again \( \mathcal{P} \cup \{ \beta \beta^* \} \) is totally ordered. If there exists \( n \) such that \( \beta \beta^* \leq \tau(n)^* \) then \( \beta \leq \tau(n) \) which is a contradiction. Therefore for any \( n \), \( \tau(n)^* \leq \beta \beta^* \) and then \( \tau(n) \leq \beta \) which is impossible since \( \beta \beta^* \in G_E^\Delta \). So \( \mathcal{P} \) is a maximal totally ordered subset of \( G_E^\Delta \).

Conversely let \( \mathcal{P} \) be a finite maximal totally ordered subset of \( G_E^\Delta \). Then by Zorn’s Lemma we can find maximal element in \( \mathcal{P} \). Assume that \( \beta \beta^* \) is a maximal element of \( \mathcal{P} \). Then \( \beta \) is a finite path and since \( \mathcal{P} \) is maximal totally ordered subset of \( G_E^\Delta \) then \( r(\beta) \) is a sink giving that \( \beta \) is a finite trail.

Finally suppose that \( \mathcal{P} \) is a countable infinite subset of \( G_E^\Delta \). Then by Zorn’s Lemma \( \mathcal{P} \) has a minimal element \( e_0 e_0^* \). Also let \( e_1 e_1^* \) the minimal element of \( \mathcal{P} \setminus \{ e_0 e_0^* \} \); \( e_2 e_2^* \) the minimal element of \( \mathcal{P} \setminus \{ e_0 e_0^*, e_1 e_1^* \} \) etc. Proceeding in this way we have that \( e_0 e_1 e_2 \ldots \) is an infinite trail.

Therefore we can say that for any trail there exists a maximal totally ordered subset of \( G_E^\Delta \) and for any finite or infinite countable maximal totally ordered subset of \( G_E^\Delta \) there exists a trail. In the end if \( E \) is countable, then the maximal totally ordered subsets of \( G_E^\Delta \) are in one-to-one correspondence with the trails in \( E \).

**Definition 3.5.** Let \( E \) be a graph.

(A) An infinite trail \( \tau = e_1 e_2 \ldots \) in \( E \) is said to be periodic, if there exist integers \( j, k \geq 1 \), such that \( e_{n+k} = e_n \) for every \( n \geq j \). In this case, it is clear that the path \( \rho = e_j \ldots e_{j+k-1} \) is closed. If we take \( j \) and \( k \) such that \( j + k \) is the smallest possible value which satisfies the condition \( e_{n+k} = e_n \) for every \( n \geq j \) and we consider the paths \( \alpha = e_1 \ldots e_{j-1} \) and \( \lambda = e_j \ldots e_{j+k-1} \), we will refer to the pair
(α, λ) as the seed of τ. Of course α may have length zero. In any case, λ is a
closed path, which will be called the period of τ.

(B) A trail τ is said to be essentially aperiodic if:

(i) τ is finite, or
(ii) τ is periodic and its period is a closed path without exits (which means that
it has to be a cycle without exits), or
(iii) τ is infinite and not periodic.

The trails of the form (i) and (ii) will be called discrete, while the ones of type
(iii) will be called continuous. In a discrete essentially aperiodic trail τ, we refer
to a certain path as the essential head of τ, and denoted by τ(ess), which is defined
accordingly as follows:

(i) if τ is finite, then τ(ess) = τ;
(ii) if τ is periodic, with seed (α, λ) then τ(ess) = α.

Let us denote by Σ_E the set of all essentially aperiodic trails in E.

Remark 3.6. Note that in the Definition 3.5 (A), the choice of j and k such that j + k
is the smallest possible value which satisfies the condition e_{n+k} = e_n for every n ≥ j,
is unique. Assume that there exist j_1, k_1, j_2, k_2 ≥ 1 such that j_1 + k_1 = j_2 + k_2
is the smallest possible value which satisfies the condition e_{m+k_1} = e_m for every m ≥ j_1
and e_{n+k_2} = e_n for every n ≥ j_2. Then for every r ≥ 0, e_{j_1+r} = e_{j_2+r+k_1} = e_{j_2+r+k_2} = e_{j_2+r}
(since j_1 + k_1 = j_2 + k_2). Suppose that j_2 > j_1 (the case j_1 > j_2 is similar). So for
any r ≥ 0, e_{j_1+r} = e_{j_2+r} = e_{j_1+r} and hence for any m ≥ j_1, e_{m+j_2-j_1} = e_m.
By minimality of k_2 + j_2 we have that k_2 = 0 which is a contradiction. Thus j_1 and k_1 are
unique satisfying that j_1 + k_1 is the smallest value with this desired property.

Lemma 3.7. For every vertex v ∈ E^0 there exists at least one essentially aperiodic trail
τ with v = s(τ).

Proof. The proof follows completely [9, Lemma 2.6]. Only we need to take into account
that the action in the graph is changed.

Proposition 3.8. Let E be a graph and R be a commutative ring with unit. If we
consider M the R-module generated by a set of generators \{ξ^n_τ : n ∈ Z, τ ∈ Σ_E\} then
there exist a Z-graded R-algebra \mathcal{E} ⊆ \text{End}_R(M) and a unique Cuntz-Krieger E-system
Σ = (S_α)_{α ∈ \text{Path}(E)} ⊆ \mathcal{E} such that the map S_α : M → M is given by:

$$S_α(ξ^n_τ) = \begin{cases} ξ^{l(α)+n}_α & \text{if } r(α) = s(τ), \\ 0 & \text{otherwise.} \end{cases}$$

Besides, the associated *-representation Π_{ap} : L_R(E) → \mathcal{E}, which will be referred to as the
essentially aperiodic representation, is injective.

Proof. First let us define for each i ∈ Z, M_i as the R-module generated by the collection
\{ξ^i_τ : τ ∈ Σ_E\}. Consider M = Θ_{i∈Z} M_i as Z-graded R-module. Let f ∈ \text{End}_R(M), we
say that f is homogeneous of degree k (k ∈ Z) if f(M_i) ⊆ M_{i+k} for every i. In this case
we set deg(f) = k. Then define \mathcal{E}_i = \{ f \mid f ∈ \text{End}_R(M), \text{deg}(f) = i \} and let \mathcal{E} = Θ_{i∈Z} \mathcal{E}_i
be an R-subalgebra of \text{End}_R(M) which is Z-graded.
Let us check that $\mathcal{E}$ is $\mathbb{Z}$-graded, that is, for each $i, j \in \mathbb{Z}$, $\mathcal{E}_i \subseteq \mathcal{E}_{i+j}$. Suppose $f \in \mathcal{E}_i$ and $g \in \mathcal{E}_j$ and consider $f \circ g$. Take some $t \in \mathbb{Z}$, and then $(f \circ g)(M_t) = f(g(M_t))$. Since $\deg(g) = j$, $g(M_t) \subseteq M_{t+j}$. Then taking into account that $\deg(f) = i$, we have $f(g(M_t)) \subseteq f(M_{t+j}) \subseteq M_{t+j+i}$ which means $f \circ g \in \mathcal{E}_{i+j}$.

Now let $\alpha \in \text{Path}(E)$ and $S_\alpha : M \to M$ be an $R$-homomorphism which is defined on the generators of $M$ as above. Let us prove that $S_\alpha$ is homogeneous of degree $l(\alpha)$, that is, for any $t \in \mathbb{Z}$ we have $S_\alpha(M_t) \subseteq M_{t+l(\alpha)}$. So consider $\tau \in T_E$ and let $\xi_\tau \in M_t$. Then

$$S_\alpha(\xi_\tau) = \begin{cases} \xi_{\alpha \tau}^{l(\alpha)} + t & \text{if } r(\alpha) = s(\tau), \\ 0 & \text{otherwise}. \end{cases}$$

and therefore $S_\alpha(\xi_\tau) \in M_{t+l(\alpha)}$ obtaining the desired statement.

On the other hand, let us define for each $\alpha \in \text{Path}(E)$, $S_\alpha^* = S_{\alpha^*}$, where $\alpha^*$ is the ghost path of $\alpha$:

$$S_\alpha^*(\xi_\tau^n) = \begin{cases} \xi_\beta^{-l(\alpha)} & \text{if } \alpha \leq \tau \text{ with } \tau = \alpha \beta, \\ 0 & \text{otherwise}. \end{cases}$$

A similar argument shows that $S_\alpha^*$ is homogeneous of degree $-l(\alpha)$.

Let us prove that $\Sigma = (S_\alpha)_{\alpha \in \text{Path}(E)}$ is a Cuntz-Krieger $E$-system inside $\mathcal{E}$. Consider $\mu, \nu \in \text{Path}(E)$ and $n \in \mathbb{Z}$, $\tau \in T_E$ then on the one hand

$$S_\mu S_\nu(\xi_\tau^n) = \begin{cases} S_\mu(\xi_\nu^{(\nu) + n}) & \text{if } r(\nu) = s(\tau), \\ 0 & \text{otherwise}. \end{cases}$$

$$= \begin{cases} \xi_{\mu \nu \tau}^{l(\mu) + l(\nu) + n} & \text{if } r(\mu) = s(\nu \tau) \text{ and } r(\nu) = s(\tau), \\ 0 & \text{otherwise}. \end{cases}$$

On the other hand,

$$S_\mu S_\nu(\xi_\tau^n) = \begin{cases} \xi_{\mu \nu \tau}^{l(\mu) + l(\nu) + n} & \text{if } r(\mu \nu) = s(\tau), \\ 0 & \text{otherwise}. \end{cases}$$

So

$$S_\mu S_\nu = \begin{cases} S_{\mu \nu} & \text{if } r(\mu) = s(\nu), \\ 0 & \text{otherwise}. \end{cases}$$

Considering $\mu, \nu$ also as vertices (paths of length zero), conditions (1) and (3) from Definition 2.2 are satisfied. Condition (2) follows from the fact that for any vertex $v$, $S_v^* = S_{v^*} = S_v$. Again for any edges $e, f \in E^1$, $S_e^* S_e = S_e S_e = S_{e^* e} = S_{e^*}$ and if $e \neq f$, $S_e^* S_f = S_{e^* f} = S_{e^*} = 0$ having condition (4). It remains to prove equation (5) from Definition 2.2 consider $v$ a regular vertex then for any $n \in \mathbb{Z}$, and $\tau \in T_E$,

$$\sum_{\{e \in E^1, \, s(e) = v\}} S_e S_e^*(\xi_\tau^n) = \begin{cases} \xi_\tau^n & \text{if } e \leq \tau \text{ for some } e \in s^{-1}(v), \\ 0 & \text{otherwise}. \end{cases}$$
and also

\[ S_v(\xi^n_\tau) = \begin{cases} 
\xi^n_\tau & \text{if } v = s(\tau), \\
0 & \text{otherwise.}
\end{cases} \]

which means \((S_v - \sum_{e \in E^n_1, s(e) = v} S_e S_v^e)(\xi^n_\tau) = 0\) obtaining condition (5).

Now we are in a position to define the representation \(\Pi_{ap} : L_R(E) \to \mathcal{E}\). We know that every \(x \in L_R(E)\) can be written as \(x = \sum_{i=1}^n r_i \alpha_i \beta_i^*\) where \(r_i \in R\), \(\alpha_i, \beta_i \in \text{Path}(E)\) for \(i = 1, \ldots, n\). For every \(\alpha \beta^*\) of the previous form let \(\Pi_{ap}(\alpha \beta^*) = S_{\alpha \beta^*}\) and extend the definition \(R\)-linearly for every \(x \in L_R(E)\). It is a straightforward computation to see that \(\Pi_{ap}\) is a well-defined \(*\)-homomorphism.

To prove that \(\Pi_{ap}\) is \(\mathbb{Z}\)-graded homomorphism, consider \(n \in \mathbb{Z}\) and \(\alpha \beta^* \in L_R(E)_n\), which means \(l(\alpha) - l(\beta) = n\). Then \(\Pi_{ap}(\alpha \beta^*) = S_{\alpha \beta^*} = S_{\alpha} S_{\beta}^* \in \mathcal{E}_{l(\alpha)} \mathcal{E}_{-l(\beta)} \subseteq \mathcal{E}_{l(\alpha) - l(\beta)} = \mathcal{E}_n\). Hence we obtain,

\[ \Pi_{ap}(L_R(E)_n) \subseteq \mathcal{E}_n. \]

Finally, let us see that \(\Pi_{ap}\) is injective. By Lemma 3.7, for every vertex \(v \in E^0\) there exists at least one essentially aperiodic trail \(\tau\) with \(v = s(\tau)\). Then \(S_v(\xi^n_\tau) = \xi^n_\tau\) for every \(n \in \mathbb{Z}\) implying that \(\Pi_{ap}(rv) \neq 0\) for all \(r \in R \setminus \{0\}\). Apply Theorem 2.6 to obtain the injectivity of the aperiodic representation. \(\square\)

**Remark 3.9.** Consider \(M\) the \(R\)-module generated by the set \(\{\xi^n_\tau : n \in \mathbb{Z}, \tau \in \mathfrak{S}_E\}\) according to Proposition 3.8. Given a path \(\alpha \in \text{Path}(E)\), let us define \(P_\alpha \in \mathcal{E}, P_\alpha : M \to M\) such that: for every \(\tau \in \mathfrak{S}_E\) and every \(n \in \mathbb{Z}\),

\[ P_\alpha(\xi^n_\tau) = \begin{cases} 
\xi^n_\tau & \text{if } \alpha \leq \tau, \\
0 & \text{otherwise.}
\end{cases} \]

Take into account that we have \(P_\alpha = S_{\alpha} S_{\alpha}^*\).

Given \(\tau \in \mathfrak{S}_E\) an essentially aperiodic trail, let us define projection \(Q_\tau \in \mathcal{E}, Q_\tau : M \to M\) such that: for every \(\tau' \in \mathfrak{S}_E\) and every \(n \in \mathbb{Z}\),

\[ Q_\tau(\xi^n_{\tau'}) = \begin{cases} 
\xi^n_{\tau'} & \text{if } \tau' = \tau, \\
0 & \text{otherwise.}
\end{cases} \]

Using this notation we can affirm:

(A) Given a trail \(\tau \in \mathfrak{S}_E\), for any \(m \in M\) there exists some integer \(N \geq 0\) such that for every \(n \geq N\),

\[ P_{\tau(n)}(m) = Q_\tau(m). \]

In particular if \(\tau\) is discrete we could take \(N = l(\tau_{(\text{ess})})\) since for any \(m \in M\) and for every \(n \geq l(\tau_{(\text{ess})})\), \(P_{\tau(n)}(m) = P_{\tau_{(\text{ess})}}(m) = Q_\tau(m)\). By the injectivity of \(\Pi_{ap}\) the same holds in \(L_R(E)\): \(\tau(n) \tau_{(n)} = \tau_{(\text{ess})} \tau_{(\text{ess})}\) for all \(n \geq l(\tau_{(\text{ess})})\).

(B) Of course when we restrict the aperiodic representation \(\Pi_{ap}\) to the diagonal \(\Delta(E)\) is injective. Suppose \(\alpha \in \text{Path}(E)\) is such that \(\alpha \alpha^*\) is one of the generators for \(\Delta(E)\) then \(\Pi_{ap}(\alpha \alpha^*) = P_\alpha\), so therefore

\[ Q_\tau \Pi_{ap}(\alpha \alpha^*) = \Pi_{ap}(\alpha \alpha^*) Q_\tau = \begin{cases} 
Q_\tau & \text{if } \alpha \leq \tau, \\
0 & \text{otherwise.}
\end{cases} \]
Then for every $x \in \Delta(E)$ and every $\tau \in \mathfrak{T}_E$ there exists a unique scalar $\varepsilon_\tau(x) \in R$ such that

$$Q_\tau \Pi_{ap}(x) = \Pi_{ap}(x)Q_\tau = \varepsilon_\tau(x)Q_\tau.$$  

(C) It is straightforward that $Q_\tau$ is self-adjoint idempotent and all elements in the collection $(Q_\tau)_{\tau \in \mathfrak{T}_E}$ are mutually orthogonal. Furthermore it satisfies that, for any $m \in M$, there exists a finite number of trails $\{\tau_1, \ldots, \tau_n\} \subset \mathfrak{T}_E$ such that

$$\left(\sum_{i=1}^{n} Q_{\tau_i}(m)\right) = m.$$

**Definition 3.10.** For any infinite discrete essentially aperiodic trail (Definition [3.5](B)(ii)) which is parameterized by the seed $(\alpha, \lambda_\alpha)$ of the trail (that is, $\alpha \in \text{Path}(E)$ is its essential head and $r(\alpha)$ is visited by the cycle without exits $\lambda_\alpha$), the path $\alpha$ will be called a distinguished path. In the case $l(\alpha) = 0$, we will call it a distinguished vertex.

**Remark 3.11.** Consider a distinguished path $\alpha$. This means that if $l(\alpha) = 0$ then $\alpha$ is simply a vertex $v$ and the cycle $\lambda_\alpha$ starts and ends at $v$. In the case $l(\alpha) \geq 1$, suppose $\alpha = e_1 \ldots e_n$, then $\lambda_\alpha$ is a cycle that starts and ends at $r(\alpha) = r(e_n)$ but does not visit any of the vertices $s(e_1), \ldots, s(e_n)$. Of course, for any distinguished path $\alpha$, $r(\alpha)$ is a distinguished vertex.

We are now in the position for describing a commutative subalgebra inside $L_R(E)$. Previously we need the following lemma.

**Lemma 3.12.** A cycle $\lambda$ has no exits if and only if $\lambda \lambda^* = s(\lambda)$.

**Proof.** The result is proved similarly to [9 Lemma 3.2]: following it we only need to use Proposition 2.3 instead, since the action in the graph in [9] is changed as we have commented before; in particular, the source and the range maps in [9] are respectively the range and the source maps here. \qed

**Definition 3.13.** An element $x \in L_R(E)$ is said to be normal if $xx^* = x^*x$.

**Proposition 3.14.** Let $\alpha, \beta \in \text{Path}(E)$ with $r(\alpha) = r(\beta)$. The generator $\alpha \beta^* \in G_E$ is a normal element in $L_R(E)$ if and only if one of the following holds:

1. $\alpha = \beta$;
2. $\beta \leq \alpha$ and $\alpha$ is a distinguished path, i.e. $\alpha = \beta \lambda_\beta$ and $\lambda_\beta$ is a cycle without exits;
3. $\alpha \leq \beta$ and $\beta$ is a distinguished path, i.e. $\beta = \alpha \lambda_\alpha$ and $\lambda_\alpha$ is a cycle without exits.

Also, if we denote by $G_E^M$ the set of all such normal generators, then the $R$-algebra $M_R(E)$ generated by $G_E^M$, $M_R(E) = \langle G_E^M \rangle \subseteq L_R(E)$ is commutative.

**Proof.** The necessary and sufficient condition for normality is proved similarly following the ideas given in [9 Proposition-Definition 3.1]. Anyway we will include here the proof for completeness. Consider first $x = \alpha \beta^*$ satisfying one of the conditions (1), (2) or (3). If $\alpha = \beta$ then $x$ is clearly normal. Suppose $\beta \leq \alpha$ (the case $\alpha \leq \beta$ is done similarly). We have $\alpha = \beta \lambda$ and $\lambda$ is a cycle without exits; by Lemma 3.12 $\lambda \lambda^* = s(\lambda) = r(\beta)$ so

$$xx^* = \beta \lambda \beta^* \beta \lambda^* \beta^* = \beta \lambda \lambda^* \beta^* = \beta \beta^* = \beta \alpha^* \alpha \beta^* = x^*x.$$
For the converse implication suppose now that \( x = \alpha \beta^* \) is a non-zero normal element of \( L_R(E) \). Since \( x \neq 0 \) then \( r(\alpha) = r(\beta) \) and since \( xx^* = x^*x \) then \( s(\alpha) = s(\beta) \). We have \((xx^*)^2 \neq 0\) but \((xx^*)^2 = x^*xx^* \) so \( x^2 \) is non-zero. But having \( x^2 = (\alpha \beta^*)(\alpha \beta^*) \neq 0 \) implies that \( \beta \leq \alpha \) or \( \alpha \leq \beta \) by Proposition \[2.4\]. We can assume that \( \beta \leq \alpha \). Considering all these facts together necessarily \( \alpha = \beta \lambda \) where \( \lambda \) is a closed path. If we suppose that \( \lambda \) has an exit then by Lemma \[3.12\] \( \lambda \lambda^* \neq s(\lambda) = r(\beta) \) and finally
\[
xx^* = \beta \lambda \lambda^* \beta^* \neq \beta^* = x^*x,
\]
which is a contradiction.

In order to prove the commutativity of \( M_R(E) \), we are going to check the following points:

(i) Elements of the form \((1)\) commute: it is clear since \( \alpha = \beta \) then \( \alpha \alpha^* \in G_E^\lambda \).

(ii) Elements of the form \((1)\) commute with elements of the form \((2)\): let \( x = \alpha \alpha^* \) of the form \((1)\) and \( y = \mu \nu^* \) of the form \((2)\). We know that \( \nu \leq \mu \) and \( \mu = \nu \lambda \nu \) where \( \lambda \nu \) is a cycle without exits. Let us see that \( xy = yx \). First notice that since \( s(\mu) = s(\nu) \) we have
\[
xy = 0 \Leftrightarrow s(\alpha) \neq s(\mu) \Leftrightarrow s(\alpha) \neq s(\nu) \Leftrightarrow yx = 0.
\]

By Proposition \[2.4\] the condition \( xy \neq 0 \) (which implies \( yx \neq 0 \)) holds if and only if \( \alpha = \mu \alpha' \) for some \( \alpha' \in \text{Path}(E) \) or \( \mu = \alpha \mu' \) for some \( \mu' \in \text{Path}(E) \). If \( \alpha = \mu \alpha' \) then \( \alpha = \mu \) since \( \lambda \nu \) is a cycle without exits. So in the case \( \alpha = \mu \) then applying Lemma \[3.12\]
\[
xy = \nu \lambda \nu^* \nu \lambda \nu (\nu \lambda \nu)^* = \nu \lambda \nu \nu^* = \alpha \nu^* = xy.
\]
Now assume that \( \mu = \alpha \mu' \) for some \( \mu' \in \text{Path}(E) \), then by Proposition \[2.4\]
\[
xy = \alpha \mu^* \nu^* = \mu \nu^*.
\]
Since \( yx \neq 0 \), then by Proposition \[2.4\] \( \alpha = \nu \alpha'' \) for some \( \alpha'' \in \text{Path}(E) \) or \( \nu = \alpha \nu' \) for some \( \nu' \in \text{Path}(E) \). In case \( \alpha = \nu \alpha'' \), \( xy = \nu \alpha'' \alpha^* \) and since \( \lambda \nu \) is a cycle without exits necessarily \( \alpha'' \in E^0 \) and \( \mu \alpha'' \alpha^* = \mu \nu^* \); so \( xy = \mu \nu^* = yx \). If \( \nu = \alpha \nu' \), then by Proposition \[2.4\] \( xy = \mu (\alpha \nu')^* = \mu \nu^* = xy \). In the end we have \( xy = yx \) as desired.

(iii) Elements of the form \((2)\) commute: consider both \( x = \alpha \beta^* \) and \( y = \mu \nu^* \) of the form \((2)\), that is, \( \beta \leq \alpha \) with \( \alpha = \beta \lambda \beta \) and \( \nu \leq \mu \) with \( \mu = \nu \lambda \nu \) where \( \lambda \beta \) and \( \lambda \nu \) are cycles without exits. Then on the one hand
\[
xy = 0 \Leftrightarrow s(\beta) \neq s(\nu) \Leftrightarrow yx = 0.
\]

By Proposition \[2.4\] \( xy \neq 0 \) (which implies \( yx = 0 \)) if \( \beta \leq \mu \) or \( \mu \leq \beta \) \( (\alpha \leq \nu \) or \( \nu \leq \alpha) \); but except when \( \beta = \mu \) \( (\alpha = \nu) \), these two options are impossible since \( \lambda \beta \) and \( \lambda \nu \) are cycles without exits. In any case then \( xy = yx \).

Analogously to the previous steps it can be proved that:

(ii) \(^*\) elements of the form \((1)\) commute with elements of the form \((3)\);  
(iii) \(^*\) elements of the form \((2)\) commute with elements of the form \((3)\) and  
(iii) \(^*\) elements of the form \((3)\) commute.

Finally we get that \( M_R(E) \) is commutative. \( \Box \)

**Definition 3.15.** The subalgebra \( M_R(E) \) given in Proposition \[3.14\] is called the commutative core of \( L_R(E) \).
**Definition 3.16.** An element of an $R$-algebra $A$ with involution $\ast$ is said to be *positive* if it is a finite sum of elements of the form $x^*x$ for $x \in A$.

The following definition is an important tool in order to prove that $M_R(E)$ is a maximal commutative subalgebra of $L_R(E)$.

**Definition 3.17.** Let $A$ be an $R$-algebra with involution and $B \subseteq A$ an $R$-subalgebra with involution. A linear map $E : A \to B$ is called a *conditional expectation of $A$ onto $B$* if it satisfies the following conditions:

1. $E$ is positive, that is, for every positive element $a$ then $E(a)$ is positive.
2. $E$ is idempotent and $\text{Im}(E) = B$.
3. For every $a \in A$ and $b \in B$, $E(ab) = E(a)b$ since we can apply the involution.

**Remark 3.18.** Condition (iii) above also implies that for every $a \in A$ and $b \in B$, $E(ab) = E(a)b$ since we can apply the involution.

**Definition 3.19.** Let $N$ be an $R$-module, $V$ be a $*$-$R$-subalgebra of $\text{End}_R(N)$ and $F : V \to V$ be a $*$-homomorphism. We say that $F$ is locally positive if for any positive element $T \in V$ and any $n \in N$, $F(T)(n)$ is positive. Note that in this paper when we work with subalgebras of the endomorphism algebra we use the definition of locally positive instead of positive.

For our purposes we need to find an appropriate conditional expectation of the $R$-algebra $E$ defined in Proposition 3.8.

For every $T \in E$ let us define the map $E_{ap} : E \to E$, $T \mapsto E_{ap}(T)$ as follows: by Remark 3.9 (C), for any $m \in M$ there exist $\{\tau_1, \ldots, \tau_n\} \subset \Sigma_E$ such that $(\sum_{i=1}^n Q_{\tau_i})(m) = m$, we define

$$E_{ap}(T)(m) := (\sum_{i=1}^n Q_{\tau_i}TQ_{\tau_i})(m).$$

**Proposition 3.20.** Let $Q = \{Q_\tau\}_{\tau \in \Sigma_E} \subset E$. Then the map $E_{ap}$ is a conditional expectation of $E$ onto $Q'$ where

$$Q' = \{T \in E : TQ_\tau = Q_\tau T \text{ for every } \tau \in \Sigma_E\}.$$  

**Proof.** We need to check the conditions from the definition of conditional expectation above.

1. $E_{ap}$ is positive: consider $T \in E$ positive, that is, $T = \sum_{j=1}^s B_j^*B_j$ where $B_j \in E$ for all $j = 1, \ldots, s$ for some $s \in \mathbb{N}$. Let $m \in M$, then there exist $\{\tau_1, \ldots, \tau_n\} \subset \Sigma_E$ such that $(\sum_{i=1}^n Q_{\tau_i})(m) = m$. We have

$$E_{ap}(T)(m) = E_{ap}(\sum_{j=1}^s B_j^*B_j)(m) = (\sum_{i=1}^n Q_{\tau_i}(\sum_{j=1}^s B_j^*B_j)Q_{\tau_i})(m) =$$

$$= (\sum_{i=1}^n \sum_{j=1}^s Q_{\tau_i}B_j^*B_jQ_{\tau_i})(m) = (\sum_{i=1}^n \sum_{j=1}^s (B_jQ_{\tau_i}^*)(B_jQ_{\tau_i}))(m) =$$

$$= (\sum_{i=1}^n \sum_{j=1}^s (B_jQ_{\tau_i}^*)(B_jQ_{\tau_i}))(m)$$
since by Remark 3.9(C) every $Q_{\tau_i}$ is self-adjoint idempotent.

(ii) $(E_{ap})^2 = E_{ap}$ follows from the fact that the elements in the collection $(Q_\tau)_{\tau \in \mathcal{E}}$ are mutually orthogonal idempotents. Let us see that $\text{Im}(E_{ap}) = \mathcal{Q'}$.

Consider $T \in \text{Im}(E_{ap})$, that is, $T = E_{ap}(T')$ for some $T' \in \mathcal{E}$. For $m \in M$ there exist $\{\tau_1, \ldots, \tau_n\} \subset \mathcal{E}$ such that $(\sum_{i=1}^n Q_{\tau_i})(m) = m$, so we have $T(m) = E_{ap}(T')(m) = (\sum_{i=1}^n Q_{\tau_i}T'Q_{\tau_i})(m)$. Let us check that $T \in \mathcal{Q'}$: consider $Q_\tau$ for some $\tau \in \mathcal{E}$ then

$$(Q_\tau T)(m) = Q_\tau((\sum_{i=1}^n Q_{\tau_i} T' Q_{\tau_i})(m)) = (\sum_{i=1}^n Q_\tau Q_{\tau_i} T' Q_{\tau_i})(m).$$

Since the elements in the collection $(Q_\tau)_{\tau \in \mathcal{E}}$ are mutually orthogonal idempotents,

$$T((Q_\tau)(m)) = (Q_\tau T' Q_\tau)(Q_\tau(m)) = Q_\tau T' Q_\tau(m).$$

Also since $(\sum_{i=1}^n Q_{\tau_i})(m) = m$, then $Q_\tau(m) = 0$ if $\tau \neq \tau_i$ for each $i$. In any case since the collection $(Q_\tau)_{\tau \in \mathcal{E}}$ is mutually orthogonal idempotent,

$$(TQ_\tau)(m) = (Q_\tau T)(m) = \begin{cases} Q_\tau T' Q_\tau(m) & \text{if } \tau = \tau_i \text{ for some } i = 1, \ldots, n \\ 0 & \text{otherwise.} \end{cases}$$

Conversely consider $T \in \mathcal{Q'}$. For $m \in M$ then again there exist $\{\tau_1, \ldots, \tau_n\} \subset \mathcal{E}$ such that $(\sum_{i=1}^n Q_{\tau_i})(m) = m$. So

$$E_{ap}(T)(m) = (\sum_{i=1}^n Q_{\tau_i} T Q_{\tau_i})(m) = (\sum_{i=1}^n TQ_{\tau_i} Q_{\tau_i})(m) = \text{Im}(T_{ap}) \subset \mathcal{E}.$$
Proof. First let us prove the following claims:

(a) $E_{ap}(\Pi_{ap}(x)) = \Pi_{ap}(x)$ for every $x \in MR(E)$.

Consider the collection $\mathcal{P} = \{P_\alpha\}_{\alpha \in \text{Path}(E)} = \{\Pi_{ap}(\alpha \alpha^*)\}_{\alpha \in \text{Path}(E)}$. By Remark 3.9(B), it is clear that $\mathcal{P} \subseteq \mathcal{Q}'$, where

$$\mathcal{Q}' = \{T \in \mathcal{E} : \tau T Q_\tau = Q_\tau T \text{ for every } \tau \in \mathcal{T}_E\}.$$ 

Suppose $T \in \mathcal{E}$ such that $T$ commutes with all $P_\alpha$, $\alpha \in \text{Path}(E)$. In particular for every $\tau \in \mathcal{T}_E$ we have that $TP_{\tau(n)} = P_{\tau(n)} T$ for every $n \geq 0$. From Remark 3.9(A) we get $TQ_\tau = Q_\tau T$ which implies $\mathcal{P} \subseteq \mathcal{Q}'$ where

$$\mathcal{P}' = \{T \in \mathcal{E} : TP_\alpha = P_\alpha T \text{ for every } \alpha \in \text{Path}(E)\}.$$ 

Therefore we have that

$$E_{ap}(T) = T \quad \text{for every } T \in \mathcal{P}'.$$ 

In particular for all $x \in MR(E)$, it follows from Proposition 3.14 that $\Pi_{ap}(x) \in \mathcal{P}'$ so we get

$$E_{ap}(\Pi_{ap}(x)) = \Pi_{ap}(x), \quad \text{for every } x \in MR(E).$$

(b) $E_{ap}(\Pi_{ap}(x)) = 0$ for every $x \in G_E \setminus G_E^M$.

Consider $x = \alpha \beta^* \in G_E \setminus G_E^M$, that is, $r(\alpha) = r(\beta)$. Let $\tau \in \mathcal{T}_E$ then clearly $Q_\tau S_\alpha S_\beta^* Q_\tau \neq 0$ only if $\beta \leq \tau$ and $\alpha \leq \tau$. In any case then we will have that $\alpha \leq \beta$ or $\beta \leq \alpha$ and since $x \notin G_E^M$, $\alpha \neq \beta$. We can suppose the case $\beta \leq \alpha$; then it follows that $s(\alpha) = s(\beta)$ and necessarily $\alpha = \beta \lambda$ for some closed path $\lambda$. Since $\beta \leq \tau$ assume that $\tau = \beta \mu$ for some path $\mu$. The fact that $Q_\tau S_\alpha S_\beta^* Q_\tau \neq 0$ implies that $\beta \lambda \mu = \alpha \mu = \tau = \beta \mu$, and this yields $\lambda \mu = \mu$ which means that $\tau$ is periodic with period $\lambda$. Since $\tau$ is essentially aperiodic then $\lambda$ has no exits. Then $\mu = r(\lambda)$ and so $\lambda \in E^0$. Thus $\alpha = \beta$ which is impossible. Finally we have that then $E_{ap}(\Pi_{ap}(x)) = 0$ for every $x \in G_E \setminus G_E^M$.

So we have that $E_{ap}(\Pi_{ap}(G_E)) \subseteq \Pi_{ap}(G_E)$. Then extending linearly we get that $E_{ap}(\Pi_{ap}(L_R(E))) \subseteq \Pi_{ap}(L_R(E))$. Since $\Pi_{ap}$ is injective by Proposition 3.8 then there exists a unique linear map $E_M : L_R(E) \rightarrow L_R(E)$ such that $\Pi_{ap} \circ E_M = E_{ap} \circ \Pi_{ap}$. Besides, by the injectivity of $\Pi_{ap}$ and by (a) and (b) we have

$$E_M(x) = \begin{cases} x & \text{if } x \in G_E^M, \\ 0 & \text{if } x \in G_E \setminus G_E^M. \end{cases}$$

It then is immediate that $E_M$ is a conditional expectation of $L_R(E)$ onto $MR(E)$.

We can prove now the main result of this section.

Theorem 3.22. Let $E$ be a graph and $R$ a commutative ring with unit. Consider $MR(E) \subseteq LR(E)$. Then

$$MR(E) = \{x \in LR(E) : xd = dx \text{ for every } d \in \Delta(E)\}.$$ 

Furthermore, $MR(E)$ is a maximal commutative $*$-subalgebra of $LR(E)$.
Proof. For the equality $M_R(E) = \{x \in L_R(E) : xd = dx \text{ for every } d \in \Delta(E)\}$ we need to prove the double inclusion. First it is clear that

$$M_R(E) \subseteq \{x \in L_R(E) : xd = dx \text{ for every } d \in \Delta(E)\}$$

since $\Delta(E) \subset M_R(E)$ and $M_R(E)$ is commutative by Proposition 3.14.

For the other containment consider an element $x \in L_R(E)$ such that $xd = dx$ for every $d \in \Delta(E)$. Then for every path $\alpha \in \text{Path}(E)$ we know that $\alpha \alpha^* \in \Delta(E)$ and then

$$\Pi_{ap}(x)P_\alpha = \Pi_{ap}(x)\Pi_{ap}(\alpha \alpha^*) = \Pi_{ap}(x \alpha \alpha^*) = \Pi_{ap}(\alpha \alpha^* x) = P_\alpha \Pi_{ap}(x).$$

Following the proof of Theorem 3.21 we know that $E_{ap}(T) = T$ for every $T \in \mathcal{P}'$ where

$$\mathcal{P}' = \{T \in \mathcal{E} : TP_\alpha = P_\alpha T \text{ for every } \alpha \in \text{Path}(E)\}.$$ 

In particular it follows that $\Pi_{ap}(x) = E_{ap}(\Pi_{ap}(x))$, using Theorem 3.21 $E_{ap}(\Pi_{ap}(x)) = \Pi_{ap}(E_M(x))$ and hence $\Pi_{ap}(x) = \Pi_{ap}(E_M(x))$. Since $\Pi_{ap}$ is injective by Proposition 3.8 we have that $x = E_M(x) \in M_R(E)$ and we are done.

In order to prove that $M_R(E)$ is a maximal commutative subalgebra inside $L_R(E)$, consider $C$ a commutative subalgebra of $L_R(E)$ such that $M_R(E) \subseteq C$. Since we have $\Delta(E) \subseteq M_R(E) \subseteq \mathcal{C}$ then in particular

$$C \subseteq \{x \in L_R(E) : xd = dx \text{ for every } d \in \Delta(E)\} = M_R(E).$$

So in the end necessarily $C = M_R(E)$.

Finally we would like to state here the result concerning the center of a Leavitt path algebra, studied in [5] and [7].

**Corollary 3.23.** The center of the Leavitt path algebra $L_R(E)$, that is

$$Z(L_R(E)) = \{x \in L_R(E) : xy = yx \text{ for every } y \in L_R(E)\}$$

satisfies that $Z(L_R(E)) \subseteq M_R(E)$.

4. Commutativity of Leavitt and Cohn path algebras

Once we have found the commutative core of $L_R(E)$, we are going to use it in order to study the commutative Leavitt path algebras from a different point of view. In particular we will extend the result given in [5] Proposition 2.7. First we need the following proposition.

**Proposition 4.1.** For every $x \in L_R(E)$ and $\tau \in \mathcal{S}_E$. If $\tau$ is discrete then

$$\tau(ess)^* \tau(ess) x \tau(ess)^* \tau(ess) = E_M(x) \tau(ess)^* \tau(ess).$$

**Proof.** Since $\tau$ is discrete then by Remark 3.9(A) $P_{\tau(ess)} = Q_\tau$, and then we have:

$$\Pi_{ap}(\tau(ess)^* \tau(ess) x \tau(ess)^* \tau(ess)) = P_{\tau(ess)} \Pi_{ap}(x) P_{\tau(ess)} = Q_\tau P_{ap}(x) Q_\tau.$$

On the other hand by using Theorem 3.21 and Proposition 3.20

$$\Pi_{ap}(E_M(x) \tau(ess)^* \tau(ess)) = \Pi_{ap}(E_M(x)) Q_\tau = E_{ap} \circ \Pi_{ap}(x) Q_\tau = Q_\tau \Pi_{ap}(x) Q_\tau.$$

So we get $\Pi_{ap}(\tau(ess)^* \tau(ess) x \tau(ess)^* \tau(ess)) = \Pi_{ap}(E_M(x) \tau(ess)^* \tau(ess))$. Finally it is sufficient to apply Proposition 3.8 since $\Pi_{ap}$ is injective then the statement follows. □
Remark 4.2. For a distinguished path \( \alpha \) let \( \omega_\alpha = \alpha \lambda_\alpha \alpha^* \), where \( \lambda_\alpha \) is the cycle without exits that starts and ends at \( s(\alpha) \). We know by Proposition 3.14 that \( \omega_\alpha \in G_E^M \). It so happens that the \( R \)-algebra \( \langle \omega_\alpha \rangle \subseteq M_R(E) \) generated by \( \omega_\alpha \) is unital (with unit \( \alpha \alpha^* \)) and \( \omega_\alpha \) is invertible in \( \langle \omega_\alpha \rangle \):

\[
\omega^*_\alpha \omega_\alpha = \omega_\alpha \omega^*_\alpha = \alpha \alpha^*.
\]

We can define then the powers \( \omega_\alpha^n \) for every \( n \in \mathbb{Z} \): if \( n < 0 \) we let \( \omega_\alpha^n := (\omega^*_\alpha)^{-n} \) and \( \omega^0_\alpha := \alpha \alpha^* \). In this case it follows that there exists a unique *-isomorphism \( \Gamma_\alpha: \langle \omega_\alpha \rangle \to R[x, x^{-1}] \) by simply defining \( \Gamma_\alpha(\alpha \alpha^*) = 1 \), \( \Gamma_\alpha(\omega_\alpha) = x \) and \( \Gamma_\alpha(\omega^*_\alpha) = x^{-1} \).

Notice that we can write \( G_E^M \) as a disjoint union

\[
G_E^M = G_E^\Delta \cup \{ \omega_\alpha^n : \alpha \text{ is a distinguished path, } n \neq 0 \}.
\]

Remark 4.3. Let \( \tau \in \mathcal{F}_E \) be a discrete trail. We can define a corner \( R \)-subalgebra inside \( L_R(E) \) as follows:

\[
M_\tau(E) = \tau_{(\text{ess})} \tau^*_{(\text{ess})} L_R(E) \tau_{(\text{ess})} \tau^*_{(\text{ess})}.
\]

By Proposition 4.1 \( M_\tau(E) \) is contained in \( M_R(E) \) and concretely we have:

(i) if \( \tau \) is finite, then \( M_\tau(E) = \langle \tau_{(\text{ess})} \tau^*_{(\text{ess})} \rangle \) the \( R \)-subalgebra generated by \( \tau_{(\text{ess})} \tau^*_{(\text{ess})} \). In this case we have a natural *-isomorphism \( \langle \tau_{(\text{ess})} \tau^*_{(\text{ess})} \rangle \cong R \), by simply defining \( \tau_{(\text{ess})} \tau^*_{(\text{ess})} \mapsto 1 \).

(ii) if \( \tau \) is infinite, then \( M_\tau(E) = \langle \omega_{\tau_{(\text{ess})}} \rangle \). By Remark 4.2 we know that there exists a *-isomorphism \( \langle \omega_{\tau_{(\text{ess})}} \rangle \cong R[x, x^{-1}] \).

Furthermore from the proof of Proposition 3.14 we have that the collection \( \{ M_\tau(E) \} \), for \( \tau \) discrete, are pairwise orthogonal: if \( \tau_1 \neq \tau_2 \) are discrete then

\[
M_{\tau_1}(E) M_{\tau_2}(E) = \{ 0 \}.
\]

Let us denote \( M_{\text{disc}}(E) \subset M_R(E) \) the \( R \)-algebra generated by \( \bigcup_{\tau \text{ discrete}} M_\tau(E) \), then taking into account all the previous information we have that

\[
M_{\text{disc}}(E) \cong \bigoplus_{\tau \text{ finite}} R \bigoplus \bigoplus_{\tau \text{ infinite}} R[x, x^{-1}],
\]

up to *-isomorphism.

Let us see now the result about commutativity of Leavitt path algebras.

Corollary 4.4. Let \( E \) be a graph and \( R \) a commutative ring with unit. The following conditions are equivalent:

(i) \( L_R(E) \) is commutative; in particular \( L_R(E) = M_R(E) \).

(ii) For the graph \( E \), the maps \( r \) and \( s \) coincide and are injective.

(iii) \( E = \bigsqcup_{i \in I} E_i \) where each subgraph \( E_i \) is an isolated vertex or an isolated loop.

(iv)

\[
L_R(E) \cong \bigoplus_{\tau \in \mathcal{F}_{\text{disc}}} R \bigoplus \bigoplus_{\tau \in \mathcal{F}_{\text{disc}}^{(\text{ess})}} R[x, x^{-1}],
\]

where \( \mathcal{F}_{\text{disc}} \) and \( \mathcal{F}_{\text{disc}}^{(\text{ess})} \) denote respectively the set of all finite and infinite discrete essentially aperiodic trails of \( E \).
Proof. (i)⇒(ii) Consider an edge \( e \in E^1 \). Then since \( L_R(E) = M_R(E) \), we can write by Proposition 3.14, \( e = \sum_{i=1}^{n} r_i \alpha_i \beta_i^* \) where \( \alpha_i \beta_i^* \in G^M_E \) and \( r_i \in R \setminus \{0\} \) for all \( i = 1, \ldots, n \). Since \( \alpha_i \beta_i^* \in G^M_E \) then \( s(\alpha_i) = s(\beta_i) \). But

\[
\begin{align*}
  e = s(e)e &= s(e)\left(\sum_{i=1}^{n} r_i \alpha_i \beta_i^*\right) = \sum_{i=1}^{n} r_i s(e)\alpha_i \beta_i^* \Rightarrow s(e) = s(\alpha_i) \text{ for all } i.
  \\
  e = er(e) &= (\sum_{i=1}^{n} r_i \alpha_i \beta_i^*)r(e) = \sum_{i=1}^{n} r_i \alpha_i \beta_i^*r(e) \Rightarrow r(e) = s(\beta_i) \text{ for all } i.
\end{align*}
\]

So we get that necessarily \( s(e) = r(e) \); then \( r \) and \( s \) coincide.

Let us prove the injectivity. We know that for every \( e \in E^1 \) \( s(e) = r(e) \). This means that \( e \) is a loop. Suppose we have two different loops \( \mu, \nu \) such that \( r(\mu) = r(\nu) \); then \( \mu \nu \neq \nu \mu \) which is a contradiction since \( L_R(E) \) is commutative.

(ii)⇒(iii) It is straightforward.

(iii)⇒(iv) Suppose \( E \) is a collection of isolated vertices and isolated loops. Then it is immediate that \( L_R(E) \) is commutative; note that the product of any two different isolated vertices is zero, the product of an isolated vertex and an isolated loop is zero too, and the same for the product of two different isolated loops. So by Theorem 3.22, \( L_R(E) = M_R(E) \).

On the other hand observe that by hypothesis in our graph \( E \), any infinite path is periodic, or in other words, there are no no continuous trails; so then \( M_R(E) = M_{\text{disc}}(E) \). By Remark 4.3, \( M_{\text{disc}}(E) \cong \left( \bigoplus_{\tau \in \mathcal{T}_{\text{disc}}^f} R \right) \oplus \left( \bigoplus_{\tau \in \mathcal{T}_{\text{disc}}^i} R[x, x^{-1}] \right) \) where \( \mathcal{T}_{\text{disc}}^f \) is the set of all finite discrete essentially aperiodic trails and \( \mathcal{T}_{\text{disc}}^i \) is the set of all infinite discrete essentially aperiodic trails. Finally we get:

\[
L_R(E) \cong \bigoplus_{\tau \in \mathcal{T}_{\text{disc}}^f} R \oplus \bigoplus_{\tau \in \mathcal{T}_{\text{disc}}^i} R[x, x^{-1}].
\]

(iv)⇒(i) It is obvious. \( \square \)

We now study the commutativity of Cohn path algebras taking advantage of all results we have obtained so far.

**Definition 4.5.** Let \( E \) be an arbitrary graph and \( R \) any commutative ring with unit. The *Cohn path algebra of \( E \) with coefficients in \( R \)*, denoted by \( C_R(E) \), is the free associative \( R \)-algebra generated by the set \( E^0 \cup E^1 \cup (E^1)^* \), subject to the relations given in (1), (2), and (3) of Definition 2.1.

In other words, \( C_R(E) \) is the algebra generated by the same symbols as those which generate \( L_R(E) \), but on which we do not impose the (CK-2) relation. It can be proved that for any graph \( E \) the Cohn path algebra \( C_R(E) \) is isomorphic to the Leavitt path algebra \( L_R(F) \) of some graph \( F \), concretely:

**Definition 4.6.** Let \( E \) be an arbitrary graph. Let \( Y := E^0_{\text{reg}} \) and \( Y' = \{v' : v \in Y\} \) be a disjoint copy of \( Y \). For \( v \in Y \) and for each edge \( e \in r_E^{-1}(v) \), we consider a new symbol \( e' \). We define the graph \( F(E) \), as follows:

\[
F(E)^0 = E^0 \cup Y' \text{ and } F(E)^1 = E^1 \cup \{e' : r_E(e) \in Y\}.
\]
For \( e \in E^1 \) we define \( s_{F(E)}(e) = s_E(e) \) and \( r_{F(E)}(e) = r_E(e) \) and define \( s_{F(E)}(e') = s_E(e) \) and \( r_{F(E)}(e') = r_E(e)' \) for the new symbols \( e' \).

Note that the graph \( F(E) \) is built from \( E \) by adding a new vertex to \( E \) corresponding to each element of \( E^0_{\text{reg}} \), and then including new edges to each of these new vertices as appropriate. Observe in particular that each of the new vertices \( v' \in Y' \) is a sink in \( F(E) \), so that \( E^0_{\text{reg}} = F(E)^0_{\text{reg}} \).

As in [3, Theorem 1.5.18] we obtain the desired result about the isomorphism between the Cohn path algebra \( C_R(E) \) and the Leavitt path algebra \( L_R(F(E)) \).

**Theorem 4.7.** ([3, Theorem 1.5.18]) Let \( E \) be an arbitrary graph and \( R \) a commutative ring with unit. Then \( C_R(E) \cong L_R(F(E)) \) where \( F(E) \) is the graph given in Definition 4.6.

In order to analyze the commutative Cohn path algebras we need the following property of the graph \( F(E) \).

**Lemma 4.8.** For every graph \( E \), its corresponding graph \( F(E) \) satisfies Condition \((L)\).

**Proof.** By the way of contradiction suppose we have a graph \( E \) such that \( F(E) \) has a cycle \( c = e_1 \ldots e_n \) without exits. Let be \( v_i = s_{F(E)}(e_i) \) for every \( i = 1, \ldots, n \). Since \( c \) has no exits then each \( v_i \) is a regular vertex in \( F(E) \). Then \( v_1, \ldots, v_n \) are regular vertices in \( E \). Consider \( v_1 \in E^0 \); for the vertex \( v_1 \in E^0_{\text{reg}} \) we have the same \( e_1 \in E^1 \) such that \( s_E(e_1) = v_1 \). Now \( r_E(e_1) = v_2 \) and \( v_2 \) is a sink in \( E \), this means that we have an edge \( e'_1 \) in \( E^1 \) with \( s_{F(E)}(e'_1) = s_E(e_1) = v_1 \) which is a contradiction since \( e'_1 \neq e_1 \). \( \square \)

**Corollary 4.9.** Let \( E \) be a graph and \( R \) a commutative ring with unit. The following conditions are equivalent:

(i) \( C_R(E) \) is commutative.

(ii) \( E = \bigcup_{i \in I} E_i \) where each subgraph \( E_i \) is an isolated vertex.

**Proof.** (i)\( \Rightarrow \)(ii) We know that \( C_R(E) \cong L_R(F(E)) \) by Theorem 4.7. Since \( C_R(E) \) is commutative then \( L_R(F(E)) \) is commutative too. But using Corollary 4.4, this means that \( F(E) = \bigcup_{i \in I} F_i \) where each subgraph \( F_i \) is an isolated vertex or an isolated loop. However by Lemma 4.8 it is not possible to have isolated loops inside \( F(E) \), so necessarily \( F(E) = \bigcup_{i \in I} F_i \) where each subgraph \( F_i \) is an isolated vertex. This means that then \( E = \bigcup_{i \in I} E_i \) where each subgraph \( E_i \) is an isolated vertex.

(ii)\( \Rightarrow \)(i) It is straightforward; note that the product of two different vertices in \( C_R(E) \) is zero. \( \square \)

### 5. General Uniqueness Theorem for Leavitt Path Algebras

In this final section we focus our attention on giving a new Uniqueness Theorem for Leavitt path algebras; concretely we will prove that we only need to check the injectivity of a homomorphism when we restrict to the commutative core \( M_R(E) \). For this purpose first we will write here the so-called Reduction Theorem for Leavitt path algebras but referred to the case over a commutative ring with unit; the proof is followed similarly to that given in [3, Theorem 2.2.11] so we will omit it here.
Theorem 5.1. Let $E$ be an arbitrary graph and $R$ a commutative ring with unit. For any non-zero element $a \in L_R(E)$ there exist $\mu, \nu \in \text{Path}(E)$ such that either:

(i) $0 \neq \mu^*a\nu = rv$, for some $r \in R \setminus \{0\}$ and $v \in E^0$, or

(ii) $0 \neq \mu^*a\nu = p(\lambda)$, where $\lambda$ is a cycle without exits and $p(x)$ is a non-zero polynomial in $R[x, x^{-1}]$.

Theorem 5.2. Let $E$ be a graph and $R$ a commutative ring with unit. Consider $\Phi : L_R(E) \to A$ a ring homomorphism. Then the following conditions are equivalent:

(i) $\Phi$ is injective;

(ii) the restriction of $\Phi$ to $M_R(E)$ is injective;

(iii) both these conditions are satisfied:

(a) $\Phi(rv) \neq 0$, for all $v \in E^0$ and for all $r \in R \setminus \{0\}$;

(b) for every distinguished path $\alpha$ the $*$-algebra $\langle \Phi(\omega_\alpha) \rangle$ generated by $\Phi(\omega_\alpha)$ is $*$-isomorphic to $R[x, x^{-1}]$; that is, $\langle \Phi(\omega_\alpha) \rangle \cong R[x, x^{-1}]$.

Proof. Let us prove the equivalence between the statements.

• (i) $\Rightarrow$ (ii) It is obvious.

• (ii) $\Rightarrow$ (iii) First let us check (a). Consider $rv$, for some $v \in E^0$ and $r \in R \setminus \{0\}$. But $rv = rvv^* \in \Delta(E) \subset M_R(E)$, so it follows immediately.

Now in order to see (b), note that for every distinguished path $\alpha$ and each $i \in \mathbb{Z}$ then $\Phi(\omega^i_\alpha) \neq 0$: this is straightforward since every $\omega_\alpha \in M_R(E)$ by Proposition 3.14.

Then we have a natural $*$-isomorphism $\langle \omega_\alpha \rangle \cong \langle \Phi(\omega_\alpha) \rangle$. We know on the other hand that $\langle \omega_\alpha \rangle \cong R[x, x^{-1}]$ is a $*$-isomorphism by Remark 4.2. It follows therefore that $\langle \Phi(\omega_\alpha) \rangle \cong R[x, x^{-1}]$ as desired.

• (iii) $\Rightarrow$ (i) Suppose $0 \neq a \in L_R(E)$ such that $a \in \text{Ker} \Phi$. It is a well-known fact that $\text{Ker} \Phi$ is an ideal of $L_R(E)$. By Theorem 5.1 there exist $\mu, \nu \in \text{Path}(E)$ and we have two possibilities.

Assume $0 \neq \mu^*a\nu = rv$, for some $r \in R \setminus \{0\}$ and $v \in E^0$. Since $a \in \text{Ker} \Phi \Rightarrow rv \in \text{Ker} \Phi$, which is a contradiction with hypothesis (a). So necessarily $0 \neq \mu^*a\nu = p(\lambda)$, where $\lambda$ is a cycle without exits and $p(x)$ is a non-zero polynomial in $R[x, x^{-1}]$. On the one hand $a \in \text{Ker} \Phi \Rightarrow p(\lambda) \in \text{Ker} \Phi$, so $\Phi(p(\lambda)) = 0$. Let $v \in \lambda^0$, then $v$ is a distinguished vertex and $\lambda = \lambda_v$, so by hypothesis (b)

$$\langle \Phi(\omega_v) \rangle = \langle \Phi(\lambda_v) \rangle = \langle \Phi(\lambda) \rangle \cong R[x, x^{-1}] .$$

Because of this $*$-isomorphism we have $0 = \Phi(p(\lambda)) = p(\Phi(\lambda)) = p(x)$, so $p(x) = 0$, which is a contradiction since $p$ is a non-zero polynomial.

□

As a corollary we can obtain a Cuntz-Krieger Uniqueness Theorem (see [12], Theorem 6.5]). If the graph $E$ satisfies Condition (L) then there are no distinguished paths in $\text{Path}(E)$, so we get immediately:

Corollary 5.3. Let $E$ be a graph satisfying Condition (L) and let $R$ be a commutative ring with unit. Suppose $\Phi : L_R(E) \to A$ is a ring homomorphism. Then the following conditions are equivalent:

(i) $\Phi$ is injective;

(ii) the restriction of $\Phi$ to $M_R(E)$ is injective;
(iii) $\Phi(rv) \neq 0$, for all $v \in E^0$ and for all $r \in R \setminus \{0\}$.

By using the proof of $(ii) \Rightarrow (iii)$ in the proof of Theorem 5.2 we can reformulate Graded Uniqueness Theorem:

**Theorem 5.4.** (Graded Uniqueness Theorem) Let $E$ be a graph and $R$ a commutative ring with unit. If $A$ is a $\mathbb{Z}$-graded ring and $\Phi : L_R(E) \to A$ is a $\mathbb{Z}$-graded ring homomorphism, then the following conditions are equivalent:

(i) $\Phi$ is injective;

(ii) the restriction of $\Phi$ to $M_R(E)$ is injective;

(iii) $\Phi(rv) \neq 0$, for all $v \in E^0$ and for all $r \in R \setminus \{0\}$.

Finally by using Lemma 4.8 and Theorems 4.7, 5.2 we obtain a uniqueness theorem for Cohn path algebras as well.

**Corollary 5.5.** Let $E$ be a graph and $R$ a commutative ring with unit. Consider $\Phi : C_R(E) \to A$ a ring homomorphism. Then the following conditions are equivalent:

(i) $\Phi$ is injective;

(ii) the restriction of $\Phi$ to $M_R(F(E))$ is injective;

(iii) $\Phi(rv) \neq 0$, for all $v \in F(E)^0$ and for all $r \in R \setminus \{0\}$, where $F(E)$ is the graph given in Definition 4.6.

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**References**

[1] G. Abrams, G. Aranda Pino, The Leavitt path algebra of a graph, *J. Algebra* 293 (2) (2005), 319–334.
[2] G. Abrams, G. Aranda Pino, The Leavitt path algebras of arbitrary graphs, *Houston J. Math.* 34 (2) (2008), 423–442.
[3] G. Abrams, P. Ara, M. Siles Molina, Leavitt path algebras. A primer and handbook, *Springer*, to appear.
[4] P. Ara, M.A. Moreno, E. Pardo, Nonstable K-theory for graph algebras, *Algebr. Represent. Theory* 10 (2007), 157–178.
[5] G. Aranda Pino, K. Crow, The center of a Leavitt path algebra, *Rev. Mat. Iberoam.* 27 (2) (2011), 621–644.
[6] G. Aranda Pino, D. Martín Barquero, C. Martín González, M. Siles Molina, Socle theory for Leavitt path algebras of arbitrary graphs, *Rev. Mat. Iberoam.* 26 (2) (2010), 611–638.
[7] M. G. Corrales García, D. Martín Barquero, C. Martín González, M. Siles Molina, J. F. Solanilla Hernández, Extreme cycles. The center of a Leavitt path algebra, preprint (2013).
[8] W. G. Leavitt, Modules without invariant basis number, *Proc. Amer. Math. Soc.* 8 (1957), 322–328.
[9] G. Nagy, S. Reznikoff, Abelian core of graph algebras, *J. London Math. Soc.* **85** (2) (2012), 889–908.
[10] I. Raeburn. *Graph algebras.* CBMS Regional Conference Series in Mathematics 103, Amer. Math. Soc., Providence (2005).
[11] W. Szymanski, General Cuntz-Krieger uniqueness theorem, *Internat. J. Math.* **13** (5) (2002), 549–555.
[12] M. Tomforde, Leavitt path algebras with coefficients in a commutative ring, *J. Pure Appl. Algebra* **215** (2011), 471–484.

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