Random-matrix behavior of quantum nonintegrable many-body systems with Dyson’s three symmetries

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We propose a one-dimensional nonintegrable spin model with local interactions that covers Dyson’s three symmetry classes (classes A, AI, and AII) by varying parameters. We show that the nearest-neighbor spacing distribution in each of these classes agrees with that of random matrices with the corresponding symmetry. We also investigate the ratios between the standard deviations of diagonal and off-diagonal matrix elements, and numerically find that they become universal, depending only on symmetries of the Hamiltonian and an observable, as predicted by random matrix theory. These universal ratios are evaluated from long-time dynamics of small isolated quantum systems.

I. INTRODUCTION

Extensive studies on quantum many-body systems over the past decade have uncovered universal behaviors in such systems upon breaking of integrability. Such quantum nonintegrable many-body systems are conjectured to have the eigenenergy and eigenstate statistics described by random matrices \cite{1–17}, as a generalization of a related conjecture in semiclassical chaotic systems \cite{18–25}. For example, matrix elements of an observable with respect to energy eigenstates of a nonintegrable Hamiltonian are well described by random matrices. This fact is closely related to the eigenstate thermalization hypothesis (ETH) \cite{23, 26–30}, which states that each energy eigenstate behaves thermal for typical observables. The ETH in many-body systems is numerically verified in various nonintegrable systems and considered to lay the cornerstone for universal aspects of statistical mechanics, such as thermalization \cite{1, 8, 30–37}, information scrambling \cite{38} and irreversibility \cite{39, 40}.

Universal properties of quantum many-body systems are classified by antiunitary symmetry that commutes with the Hamiltonian, in particular time-reversal symmetry. Dyson \cite{41} introduced the three fundamental symmetry classes: (i) class A which does not have antiunitary symmetry, (ii) class AI which has antiunitary symmetry $\hat{T}$ that satisfies $\hat{T}^2 = 1$ and $[\hat{T}, \hat{H}] = 0$, and (iii) class AII which has antiunitary symmetry $\hat{T}$ that satisfies $\hat{T}^2 = -1$ and $[\hat{T}, \hat{H}] = 0$. Gaussian random-matrix ensembles in each of these classes are called (i) Gaussian unitary ensemble (GUE), (ii) Gaussian orthogonal ensemble (GOE), and (iii) Gaussian symplectic ensemble (GSE). It is expected that eigenenergy and eigenstate statistics of quantum nonintegrable many-body systems are understood from those of random matrices that belong to the same symmetry class.

A theoretical study of universality in nonintegrable many-body systems requires appropriate models, since too complicated models are often intractable and too simplified models may have extra symmetry and integrability. For example, while the transverse Ising model in one dimension is integrable, additional longitudinal fields to this model makes it nonintegrable \cite{42}. Indeed, the eigenenergy and eigenstate statistics of this two-field Ising model are described by GOE because the model belongs to class AI. This model consists of only local and up to two-body interactions, and has been used to study quantum chaos, thermalization, etc., both numerically \cite{43–46} and experimentally \cite{47}. On the other hand, currently known nonintegrable many-body models with local interactions do not cover all of the three symmetry classes, especially class AII. For studying the effects of symmetry, it is desirable to have models whose symmetry can be controlled by the parameters of the Hamiltonian.

In this paper \cite{48}, we propose a one-dimensional nonintegrable spin model that covers Dyson’s three symmetries (classes A, AI, and AII) by varying parameters. This model is composed of up to two-body local interactions including the Dzyaloshinskii-Moriya (DM) interaction \cite{49, 50} to describe class AII. We study nearest-neighbor spacing distributions of models in these classes and show that they obey those of random matrices with the corresponding symmetry (GUE, GOE, and GSE). We also study the ratios between the standard deviations of diagonal and off-diagonal matrix elements. We show that the ratios become universal, depending only on symmetries of the Hamiltonian and an observable, as predicted by random matrix theory. These universal ratios are evaluated from long-time dynamics of small isolated quantum systems.

The rest of this paper is organized as follows. In Sec. II, we define three nonintegrable models with different symmetries. In Sec. III, we analyze the nearest-neighbor spacing distributions of eigenenergies for these models. In Sec. IV, we investigate the ratios between the standard deviations of diagonal and off-diagonal matrix elements. After introducing the predictions on these ratios using random matrix theory, we numerically show that the predictions hold true for our models and observables with various symmetries. In Sec. V, we analyze...
two types of quench dynamics in our models to demonstrate that the universal ratios are evaluated from long-time dynamics of autocorrelation functions and temporal fluctuations. In Sec. VI, we summarize the main results of this work and discuss some outlooks. In appendices, we detail crossover transitions between models with different symmetries and prove several relations used in the main text.

II. MODEL AND ITS SYMMETRIES

We introduce a one-dimensional spin model with local interactions, which changes its symmetry by varying parameters. Our model consists of the Ising interaction, transverse and longitudinal fields and the DM interaction as follows:
\[
\hat{H} = \hat{H}_I + \hat{H}_F + \hat{H}_{DM},
\]
\[
\hat{H}_I = - \sum_{i=1}^{N-1} J(1 + \epsilon_i) \hat{\sigma}_i^x \hat{\sigma}_{i+1}^x,
\]
\[
\hat{H}_F = - \sum_{i=1}^{N} (h' \hat{\sigma}_i^x + h \hat{\sigma}_i^z),
\]
\[
\hat{H}_{DM} = \sum_{i=1}^{N-1} \hat{D} \cdot (\vec{\sigma}_i \times \vec{\sigma}_{i+1})
\]
\[
= \frac{D}{\sqrt{2}} \sum_{i=1}^{N-1} [(\hat{\sigma}_i^y \hat{\sigma}_{i+1}^x - \hat{\sigma}_i^x \hat{\sigma}_{i+1}^y) + (\hat{\sigma}_i^x \hat{\sigma}_{i+1}^z - \hat{\sigma}_i^z \hat{\sigma}_{i+1}^x)],
\]
where \(N\) denotes the number of spins, \(\hat{D} = D \frac{1}{\sqrt{2}} (\hat{e}_x + \hat{e}_z)\), and we impose the open boundary condition. In addition, \(\epsilon_i\) is a random variable that is uniformly chosen from \([-\epsilon, \epsilon]\) at each site to break the reflection symmetry of sites \(i \leftrightarrow N - i\) [51].

Let us explain each term in Eq. (1). The Ising Hamiltonian \(\hat{H}_I\) has anisotropy in spin space, which makes it less symmetric compared with an isotropic one, e.g., Heisenberg interactions. By adding transverse and longitudinal fields \(\hat{H}_F\), the Hamiltonian \(\hat{H}_I + \hat{H}_F\) loses unitary symmetry that commutes with it. On the other hand, it has one antiunitary symmetry \(\hat{T} = \hat{K} (\hat{T}^2 = 1)\), where \(\hat{K}\) denotes complex conjugation and satisfies
\[
\hat{K} \hat{\sigma}_i^x \hat{K}^{-1} = \hat{\sigma}_i^x, \quad \hat{K} \hat{\sigma}_i^y \hat{K}^{-1} = -\hat{\sigma}_i^y, \quad \hat{K} \hat{\sigma}_i^z \hat{K}^{-1} = \hat{\sigma}_i^z.
\]

Next, to realize systems with antiunitary symmetry satisfying \(\hat{T}^2 = -1\), we note that \(\hat{H}_I\) has time-reversal symmetry
\[
\hat{T} = \hat{T}_0 := \left( \prod_{i=1}^{N} [i \hat{\sigma}_i^y] \right) \hat{K},
\]
which satisfies \(\hat{T}_0^2 = (-1)^N\) and
\[
\hat{T}_0 \hat{\sigma}_i^x \hat{T}_0^{-1} = -\hat{\sigma}_i^x, \quad \hat{T}_0 \hat{\sigma}_i^y \hat{T}_0^{-1} = -\hat{\sigma}_i^y, \quad \hat{T}_0 \hat{\sigma}_i^z \hat{T}_0^{-1} = -\hat{\sigma}_i^z.
\]

To keep this symmetry, we are only allowed to add every-body interactions of spins to \(\hat{H}_I\). To break other symmetries such as \(\hat{K}\), we choose the DM (DM) interaction \(\hat{H}_{DM}\). Indeed, \(\hat{H}_I + \hat{H}_{DM}\) with \(D \neq 0\) only has \(\hat{T}_0\) as symmetry. Note that interactions of the form \(\hat{\sigma}_i^\alpha \hat{\sigma}_{i+1}^\alpha (\alpha = x, y, z)\) do not work because they cannot break symmetries such as \(\hat{K}\). The DM interaction with \(\hat{D} \propto \hat{e}_\alpha (\alpha = x, y, z)\) does not work for the same reason, either.

Since \(\hat{H}_F\) and \(\hat{H}_{DM}\) have different antiunitary symmetry, \(\hat{H}_I + \hat{H}_F + \hat{H}_{DM}\) for nonzero \(h, h'\) and \(D\) is not constrained by any symmetry. Thus, by varying \(h, h'\) and \(D\) we obtain models belonging to different symmetry classes. We focus on the following three models, whose parameters are chosen such that their unwanted symmetries and integrability are broken:

\[
\text{model a :} \quad \hat{H}_a = \hat{H}_I + \hat{H}_F + \hat{H}_{DM} \quad (8)
\]
with \(h = 0.5, h' = -1.05\) and \(D = 0.9\),
\[
\text{model b :} \quad \hat{H}_b = \hat{H}_I + \hat{H}_F \quad (9)
\]
with \(h = 0.5, h' = -1.05\) and \(D = 0\), and
\[
\text{model c :} \quad \hat{H}_c = \hat{H}_I + \hat{H}_{DM} \quad (10)
\]
with \(h = h' = 0\) and \(D = 0.9\). We also assume that \(J = 1, \epsilon = 0.1\) in \(\hat{H}_I\) in all of the models. Model a belongs to class A and model b belongs to class AI. On the other hand, model c belongs to class AI for even \(N\) and class AI for odd \(N\), since \(\hat{T}_0 = (-1)^N\). In Appendix A, we discuss the property of models with intermediate parameters between these three models.

III. NEAREST-LEVEL-SPACING DISTRIBUTIONS

We first analyze nearest-neighbor level-spacing distributions \(P(s)\) [52], which are primary indicators of universality in nonintegrable systems. Figure 1(i) shows that the distribution for model a defined in the previous section obeys GUE,
\[
P_{\text{GUE}}(s) = \frac{32}{\pi^2} s^2 e^{-\frac{4}{\pi} s^2},
\]
reflecting the fact that this model belongs to class A. Figure 1(ii) shows that the distribution for model b obeys GOE,
\[
P_{\text{GOE}}(s) = \frac{\pi}{2} s e^{-\frac{\pi}{4} s^2}
\]
consistent with the fact that this model belongs to class AI. Finally, the distribution for model c varies according to the parity of \(N\) as shown in Fig. 1(iii) and (iv): it obeys GSE,
\[
P_{\text{GSE}}(s) = \frac{\sqrt{2}}{3} s e^{-\frac{2}{3} s^2}
\]
for odd $N$ and GOE for even $N$. Note that the nearest-neighbor level-spacing distribution for odd $N$ (class AII) is calculated by identifying the two degenerate eigenenergies, i.e., the Kramers pairs. These results indicate that our models are in fact sufficiently nonintegrable and that their eigenenergy statistics are described by random matrix theory that respects the corresponding symmetry.

IV. UNIVERSAL RATIO ON MATRIX ELEMENTS

A. Ratio of the standard deviations of diagonal and off-diagonal matrix elements

Another important universality of nonintegrable systems appear in matrix elements of observables $\hat{O}$ for energy eigenstates $|E_\alpha\rangle$ with eigenenergy $E_\alpha$ ($H |E_\alpha\rangle = E_\alpha |E_\alpha\rangle$). To see this, we consider fluctuations of matrix elements within an energy shell $\mathcal{H}_{E_0,\omega_s}$, which is spanned by energy eigenstates whose eigenenergies are within $[E_0 - \omega_s, E_0 + \omega_s]$ for small $\omega_s$. We especially focus on the ratio $r$ between the standard deviation of diagonal matrix elements $\Delta O_d$ and that of off-diagonal matrix elements $\Delta O_{od}$ [13]:

$$r = \frac{\Delta O_d}{\Delta O_{od}}. \tag{14}$$

We can also define

$$r' = \frac{\Delta O_K}{\Delta O_{od}}. \tag{15}$$

for use to analyze class AII, where $\Delta O_K$ is the standard deviation of matrix elements within the Kramers degenerate pair $O_{a\bar{a}} = (E_a|\hat{O}|E_{\bar{a}})$ with $H |E_a\rangle = E_a |E_a\rangle$ and $|E_{\bar{a}}\rangle = \bar{T} |E_a\rangle$ [53].

In the rest of this section, we use random matrix theory to show that these ratios become universal constants in nonintegrable systems, which depend only on the symmetries of the Hamiltonian and observables.

B. Random matrix theory and symmetry

We first determine $r$ from random matrix theory. We start from a conjecture that matrix elements $\langle E_a|\hat{O}|E_{\beta}\rangle = O_{\alpha\beta}$ can be written in the following form [3, 24] in nonintegrable systems with no degeneracy (i.e., for classes A and AI):

$$O_{\alpha\beta} = \mathcal{A}(E)\delta_{\alpha\beta} + \Omega(E)^{-\frac{1}{2}} f(E,\omega)R_{\alpha\beta}, \tag{16}$$

where $E = (E_\alpha + E_{\beta})/2$, $\omega = E_\alpha - E_{\beta}$, $\Omega(E)$ is the density of states of the Hamiltonian, and $\mathcal{A}(E)$ and $f(E,\omega)$ are smooth functions of their arguments. In addition, $R_{\alpha\beta}$ behaves quasi-randomly as a function of $\alpha$ and $\beta$, and its statistical properties (normalized such that the average is zero and the variance is 1) are well described by random matrix theory: $R_{\alpha\beta}$ fluctuates as if energy eigenstates were eigenstates of random Hamiltonians. Since $\Omega(E)$ is exponentially large with respect to the size of the system, the second term in Eq. (16) is exponentially suppressed in the thermodynamic limit, which leads to the ETH. Equation (16) has numerically been verified for few-body observables [3, 4, 7–10, 13], although recent studies indicate that it may also hold for many-body observables [16, 54].

By assuming the ansatz in Eq. (16), we calculate the ratio $r$ between the standard deviation of fluctuations (the second term of Eq. (16) on the right-hand side) for diagonal matrix elements $\Delta O_d$ and that for off-diagonal matrix elements $\Delta O_{od}$. Since $\Omega(E)$ and $f(E,\omega)$ stay almost constant within the energy shell (note that $|E - E_0| < \omega_s$ and $|\omega| < 2\omega_s$), the ansatz leads to

$$r = \frac{\Delta O_d}{\Delta O_{od}} \simeq \frac{\Delta R_d}{\Delta R_{od}}, \tag{17}$$

where $\Delta R_d$ and $\Delta R_{od}$ are the standard deviations of the diagonal and off-diagonal elements of $R_{\alpha\beta}$, respectively. Since the statistics of $R_{\alpha\beta}$ are described by random matrix theory, $r$ becomes universal irrespective of the details of Hamiltonians and observables.

Using random matrix theory, we find that $r$ (and similarly $r'$) is actually determined only by the symmetries of the Hamiltonian and an observable, as shown in Table I. Here, the symmetry of an observable is defined by using the time-reversal symmetry $\bar{T}$ (for classes AI and AII) as follows: the symmetry of $\hat{O}$ is even if $\bar{T} \hat{O} \bar{T}^{-1} = \hat{O}$ and odd if $\bar{T} \hat{O} \bar{T}^{-1} = -\hat{O}$ [53]. As detailed in Appendix B, Table I is obtained under the assumption that the eigenstates behave as if they were eigenstates of random ma-
for model c with odd magnetization. We especially consider local a, b, and c explained in Sec. II for few-body observables. Note that r’ is defined only for class AII. 

| Ratio | \( r_A \) | \( r_{AI} \) | \( r_{AI}^{(\text{even})} \) | \( r_{AI}^{(\text{odd})} \) |
|-------|---------|---------|----------------|----------------|
| even \( \hat{O} \) | 1       | \( \sqrt{2} \) | 1              | 0              |
| odd \( \hat{O} \) | 0       | 1       | \( \sqrt{2} \) |                |

TABLE II: Prediction of random matrices on matrix elements for each model and observable. Note that r’ is defined only for model c with odd \( N \).

| Ratio | \( r_a \) | \( r_b \) | \( r_c \) | \( r_c^{(\text{even})} \) | \( r_c^{(\text{odd})} \) |
|-------|---------|---------|---------|----------------|----------------|
| Symmetry | A | AI | AI | AII | AII |
| \( \hat{O}_c, \hat{O}_l \) (even \( l \)) | 1 | \( \sqrt{2} \) | \( \sqrt{2} \) | 1 | 0 |
| \( \hat{O}_m, \hat{O}_l \) (odd \( l \)) | 1 | \( \sqrt{2} \) | 0 | \( \sqrt{2} \) | |

traces with appropriate symmetry constraints within the energy shell.

Our finding in Table I extends the previous results for classes A (\( r_A = 1 \)) and AI even (\( r_{AI, \text{even}} = \sqrt{2} [8, 13, 56] \)) to class AII. We also note that the symmetry of the observable is crucial for the universality, which has not been discussed in previous literature.

C. Numerical results for few-body observables

We now calculate the ratios \( r \) and \( r’ \) for our models a, b, and c explained in Sec. II for few-body observables with different symmetries. We especially consider local magnetization

\[
\hat{O}_m := \hat{\sigma}_{[N/2]+1}^z
\]

and the neighboring correlation

\[
\hat{O}_c = \hat{\sigma}_{[N/2]+1}^z \hat{\sigma}_{[N/2]+2}^z,
\]

where \([x]\) is the Gauss symbol, which gives the greatest integer that does not exceed \( x \). Since \( \hat{O}_m \) satisfies \( K \hat{O}_m K^{-1} = \hat{O}_m \), it is an even operator for models a and b. On the other hand, since \( T_0 \hat{O}_m T_0^{-1} = -\hat{O}_m \), it is odd for model c. As for \( \hat{O}_c \), it is an even operator for all of the models because \( K \hat{O}_c K^{-1} = \hat{O}_c, T_0 \hat{O}_c T_0^{-1} = \hat{O}_c \). If we assume the prediction of random matrix theory in Table I, the ratios for matrix elements of each observable in three models are predicted as Table II.

Before showing our numerical results, we comment on an important caveat in calculating \( \Delta \hat{O}_d \) to obtain \( r \). While we need to extract the second term on the right-hand side of Eq. (16) to probe the random matrix behavior, the naive standard deviation of diagonal matrix elements involves unwanted effect of the first term especially for few-body observables [54]. To remove this contribution, we consider the standard deviation of modified fluctuations that are defined by the deviations from the linear regression of diagonal matrix elements within the energy shell, not from the naive average (a similar analysis was performed in, e.g., Refs. [4, 57]).

In Fig. 2, we show the energy dependences of \( r \) and \( r’ \) for the models a, b, c, and observables. The figures show that the ratios do not depend on energy especially in the middle of the spectrum. These values agree well with the predictions of random matrix theory in Tables I and II.

FIG. 2: Ratios \( r \) and \( r’ \) of the standard deviations of matrix elements as a function of energy. The results are shown for the cases with (i) \( \hat{O}_m, N = 12 \), (ii) \( \hat{O}_c, N = 12 \), (iii) \( \hat{O}_m, N = 13 \), and (iv) \( \hat{O}_c, N = 13 \). Each graph shows \( r \) for models a (circle), b (square), and c (upward triangle). For \( N = 13 \), we also show \( r’ \) for model c (downward triangle). By analyzing the data from the viewpoint of symmetry, we can see that the conjecture in Tables I and II is valid for these few-body operators especially in the middle of the spectra. We note that some data points are missing for the edges of the spectrum because only few eigenstates exist there. Each data point is obtained from the standard deviation of the modified fluctuations (see the main text) within small energy shell \( \omega_s = (\max_{\alpha} E_{\alpha} - \min_{\alpha} E_{\alpha})/(12N) \) for the Hamiltonian with a single disorder realization.

D. Numerical results for many-body observables

While we have verified the universal ratios for few-body operators above, we obtain similar results for many-body
operators. To demonstrate this, we introduce $l$-body spin correlations defined by [58]
\[
\hat{\sigma}_l := \prod_{n=1}^{l} \hat{\sigma}_n^z \quad (3 \leq l \leq N - 1).
\] (20)

Figure 4 shows that $\bar{\tau}$ and $\bar{\tau}'$ for both even and odd $N$ are universal. In particular, the ratios do not depend on $l$ for models a and b, and only depend on the parity of $l$ for model c. The results agree well with the predictions in Tables I and II.

It is interesting to note that the predictions in Tables I and II seem to agree better with the numerical data in Fig. 4 for larger $l$, i.e., many-body correlations. This can be attributed to the fact that $\mathcal{A}(E)$ and $f(E, \omega)$ in Eq. (16) are less dependent on $E$ and $\omega$ for many-body operators, as discussed in Ref. [54]. In other words, the approximated equality in Eq. (17), which becomes exact for $\omega_s \to 0$, holds better for many-body observables with larger $l$ for finite $\omega_s$.

V. UNIVERSALITY IN QUENCH DYNAMICS IN SMALL SYSTEMS

In this section, we discuss how our universal ratio found in Sec. IV is evaluated from the long-time thermalization dynamics in small isolated systems after a quench. We show that diagonal matrix elements and off-diagonal matrix elements are related to the autocorrelation function and the temporal fluctuation of observables, respectively. To see this, we assume that the energy fluctuation of the initial state $\rho_0$,
\[
\Delta E = \sqrt{\text{Tr}[\rho_0 \hat{H}^2]} - \text{Tr}[\rho_0 \hat{H}^2],
\] (21)
is so small that the variations of $\mathcal{A}(E)$ and $f(E, \omega)$ in Eq. (16) within the energy range $[E - \Delta E, E + \Delta E]$ are much smaller than the fluctuations $\Delta O_d$, $\Delta O_{od}$, and $\Delta O_K$ ($\hat{H}$ is the Hamiltonian after the quench). Since $\Delta E$ typically becomes larger and $\Delta O_d$, $\Delta O_{od}$, and $\Delta O_K$ become smaller as increasing the system size, our discussion is applicable to relatively small systems. Moreover, our theory holds better for many-body correlations with larger $l$, which are in fact measurable quantities in small systems [34, 59]. This is because $\mathcal{A}(E)$ and $f(E, \omega)$ are expected to be less dependent on their arguments for many-body operators, as discussed in Sec. IV. We also assume that the initial state is a pure state $|\psi_0\rangle$, i.e., $\rho_0 = |\psi_0\rangle \langle \psi_0|$.

First, the standard deviation of diagonal matrix elements $\Delta O_d$ is related to the long-time average of the autocorrelation function [21]:
\[
\tilde{S} := \lim_{T,T' \to \infty} \frac{1}{TT'} \int_0^{T} dt \int_0^{T'} dt' \langle \psi_0 | \hat{O}(t + t') \hat{O}(t) | \psi_0 \rangle,
\] (22)
where $\hat{O}(t) := e^{i\hat{H}t} \hat{O} e^{-i\hat{H}t}$. In fact, if we assume non-degeneracy (which corresponds to classes A and AI), we obtain
\[
\tilde{S} = \sum_{\alpha} |c_\alpha|^2 \mathcal{O}_{\alpha\alpha}^2
\] (23)
where $c_\alpha := \langle E_\alpha | \psi_0 \rangle$. Moreover, the long-time average of the expectation value of $\hat{O}$ is
\[
\tilde{\mathcal{O}} := \lim_{T \to \infty} \frac{1}{T} \int_0^{T} dt \langle \psi_0 | \hat{O}(t) | \psi_0 \rangle = \sum_{\alpha} |c_\alpha|^2 \mathcal{O}_{\alpha\alpha}.
\] (24)
Thus, we obtain
\[ \hat{\mathcal{S}} - \mathcal{O}^2 = \sum_{\alpha} \left[ |c_{\alpha}|^2 \left( \mathcal{O}_{\alpha\alpha} - \sum_{\beta} |c_{\beta}|^2 \mathcal{O}_{\beta\beta} \right) \right]^2. \]

By assuming that \( c_{\alpha} \) distributes unbiasedly over \( \alpha \) [60] with a sufficiently small width \( \Delta E \), the right-hand side of Eq. (25) is approximated by the variance of diagonal matrix elements,
\[ \hat{\mathcal{S}} - \mathcal{O}^2 \simeq \Delta \mathcal{O}^2. \]

Next, the standard deviation of off-diagonal matrix elements is evaluated from the long-time average of temporal fluctuations at each time:
\[ \hat{T}^2 := \lim_{T \to \infty} \frac{1}{T} \int_0^T \langle \psi_0 | \hat{\mathcal{O}}(t) | \psi_0 \rangle - \mathcal{O} \rangle^2 dt. \]

By assuming non-degeneracy in gaps of eigenenergies, we obtain [61]
\[ \hat{T}^2 = \sum_{\alpha \neq \beta} |c_{\alpha}|^2 |c_{\beta}|^2 \mathcal{O}_{\alpha\beta}^2. \]

Again, assuming that \( c_{\alpha} \) distributes unbiasedly over \( \alpha \) with a sufficiently small width \( \Delta E \), the right-hand side of Eq. (28) is approximated by the variance of off-diagonal matrix elements,
\[ \hat{T}^2 \simeq \Delta \mathcal{O}^2_{\text{od}}. \]

From these discussions, we obtain
\[ g := \sqrt{\frac{\hat{S} - \mathcal{O}^2}{\hat{T}^2}} \simeq r \]
for classes A and AI. Thus, we expect \( g_{\text{A}} = 1 \), \( g_{\text{A,even}} = \sqrt{2} \), and \( g_{\text{A,odd}} = 0 \). Moreover, as shown in Appendix C, we obtain
\[ g = \sqrt{r^2 + r'^2} \]
for class AII, which suggests \( g_{\text{AII,even}} = 1 \) and \( g_{\text{AII,odd}} = \sqrt{3} \). The relations in Eqs. (30) and (31) suggest that two different fluctuations, i.e., the correlation between two distant times and the fluctuation at each time are proportional to each other with a universal ratio. This universality cannot be described by statistical mechanics because these fluctuations [62] only matter in small isolated systems. Note that this relation is different from the well-known relations about autocorrelation functions, such as the Wiener-Khintchine theorem. In fact, \( \mathcal{T}^2 \) is different from the squared power spectrum.

We demonstrate the validity of Eqs. (30) and (31) by considering two types of the quench of our models a and c. For the first case, called case (I), we consider an initial Hamiltonian in Eq. (1) with \( h = 0.5, h' = -1.05 \) and \( D = 1 \) to model a with \( h = 0.5, h' = -1.05 \) and \( D = 0.9 \). For the second case, called case (II), we consider an initial Hamiltonian in Eq. (1) with \( h = 0.03, h' = -0.063 \) and \( D = 0.9 \) to model c with \( h = h' = 0 \) and \( D = 0.9 \). For both cases, the initial state is chosen to be a highly excited eigenstate of the initial Hamiltonian. Figure 5 shows the factor \( g \) for these quenches by taking the \( l \)-body correlations \( \hat{O}_l \). For both of the quenches and all \( N \) and \( l \), \( g \) agrees well with the predictions of random matrix theory, which is summarized in Table III.

![FIG. 5: Dependence on \( l \) of \( g \) for different system sizes \((N = 10, 11)\) and different quenches (cases (I) and (II)). For all these cases, the results agree well with the predictions of random matrices in Table III. The results show the averages over different initial states, which are energy eigenstates of the initial Hamiltonian taken from the middle one-fourth of the spectrum.](Image)

| Quench Size \( N = \) even | \( N = \) odd | \( N = \) even | \( N = \) odd |
|--------------------------|-----------|-------------|-------------|
| Symmetry \( A \) | \( A \) | \( A \) | \( A \) |
| \( \hat{O}_l \) \( l = \) even | 1 | 1 | \( \sqrt{2} \) | 1 |
| \( \hat{O}_l \) \( l = \) odd | 1 | 1 | 0 | \( \sqrt{3} \) |

VI. CONCLUSIONS AND OUTLOOK

We have introduced a one-dimensional nonintegrable spin model that covers Dyson’s three symmetries by varying the strengths of transverse, longitudinal fields and the DM interaction. We have analyzed nearest-neighbor spacing distributions of the models with different symmetries and shown that they obey those of random matrices with the corresponding symmetry (GUE, GOE, and GSE). We have studied the ratios between the standard deviations of diagonal and off-diagonal matrix elements, which become universal values that depend only on symmetries of the Hamiltonian and the observable. We have
demonstrated the universality by using our nonintegrable models in addition to the predictions by random matrix theory. Finally, we have discussed that these ratios are evaluated from long-time dynamics of small isolated quantum systems.

Our nonintegrable model provides a suitable playground to investigate spectral transitions between Dyson’s three different symmetries. As shown in Appendix A, our model exhibits a crossover transition of nearest-neighbor spacing distributions by varying parameters for a finite system size. It is an interesting future challenge to understand how this transition becomes sharper with increasing the system size towards the thermodynamic limit. While such a question was discussed in random matrices [63], it is still a nontrivial question for nonintegrable many-body systems with local interactions. It is also interesting to investigate how the change of symmetry leads to different universal consequences in our nonintegrable many-body models other than the factor $g$ discussed in our work. We leave these questions for future investigation.

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Appendix A: Spectral crossover transition among different symmetries

In this appendix, we show that the nearest-neighbor spacing distributions exhibit crossover transitions among different symmetries if we change the parameters of our model Hamiltonian in model (1).

Figure 6 shows the nearest-neighbor spacing distributions $P(s)$ for Hamiltonians (1) with $N = 12$ by varying parameters. We fix $J = 1$ and $h' = -2.1h$ and vary $h$ and $D$. For small $h$ and $D$, the system is still approximately integrable and $P(s)$ does not show level repulsions. For $h = 0$ and large $D$, or for $D = 0$ and large $h$, $P(s)$ is close to GOE statistics, since $N$ is even. For large $h$ and $D$, $P(s)$ is close to GUE statistics because all symmetries are broken. We find crossover transitions between these different statistics. In general, transitions are sharper for nonintegrable-nonintegrable ones (such as GOE-GUE transitions) than integrable-nonintegrable ones.

Similarly, Fig. 7 shows $P(s)$ for Hamiltonian (1) with $N = 13$ by varying parameters with $J = 1$ and $h' = -2.1h$. For small $h$ and $D$, the system is still approximately integrable and $P(s)$ does not show level repulsions. For $h = 0$ and large $D$, $P(s)$ is close to GSE statistics, and for $D = 0$ and large $h$, $P(s)$ is close to GOE statistics, since $N$ is odd. For large $h$ and $D$, $P(s)$ is close to GUE statistics because all symmetries are broken. We again find crossover transitions between these different statistics, where transitions are sharper for nonintegrable-nonintegrable ones (such as GOE-GUE and GSE-GUE transitions) than integrable-nonintegrable ones.

Appendix B: Derivations of the universal ratios $r$ and $r'$ by random matrix theory

Here we derive the universal ratios from random matrix theory. While the ratios for classes A (GUE) and AI (GOE) were discussed in Refs. [8, 13], their calculation is not accurate because they ignore non-negligible correlations for random vectors.

1. Class A: calculation by GUE

Let us examine the class A. We calculate the ensemble average of diagonal and off-diagonal matrix elements of an observable $\hat{O}$, which can be diagonalized as

$$\hat{O} = \sum_{i=1}^{d} |a_i\rangle \langle a_i|,$$

where $a_i$ and $|a_i\rangle$ are eigenvalues and eigenstates for the observable and $d$ is the dimension of the matrix. The matrix elements can be written as

$$O_{\alpha\beta} = \sum_{i} a_i U_{\alpha i} U_{i\beta},$$

where $U_{\alpha i} := \langle E_{\alpha} | a_i \rangle$ denotes a transformation function between bases $|E_{\alpha}\rangle$ and $|a_i\rangle$. Let us assume that the Hamiltonian possesses the symmetry of GUE and that an observable is fixed. Then, it is known that $U$ is distributed uniformly with respect to the unitary Haar measure [52]. In this case, we have the following moments of $U$ [64]:

$$U_{\alpha i} U_{\beta j} = \frac{1}{d} \delta_{\alpha \beta},$$

$$|U_{\alpha i}|^2 |U_{\beta j}|^2 = \frac{1 + \delta_{\alpha \beta}}{d(d+1)},$$

$$|U_{\alpha i}|^2 |U_{\beta j}|^2 = \frac{1 + \delta_{ij}}{d(d+1)},$$

$$U_{\alpha i} U_{\beta j} U_{\beta j} U_{\alpha i} = \frac{1}{d(d-1)(d+1)} \quad (\alpha \neq \beta, i \neq j),$$

$$r$$ and $$r'$$ by random matrix theory

Here we derive the universal ratios from random matrix theory. While the ratios for classes A (GUE) and AI (GOE) were discussed in Refs. [8, 13], their calculation is not accurate because they ignore non-negligible correlations for random vectors.
FIG. 6: Nearest-neighbor spacing distributions $P(s)$ for Hamiltonians (1) with $N = 12$, $J = 1$ and $h' = -2.1h$. For small $h$ and $D$, $P(s)$ does not show level repulsions because the system is still approximately integrable. For $h = 0$ and large $D$ or for $D = 0$ and large $h$, $P(s)$ is close to GOE statistics, and for large $h$ and $D$, $P(s)$ is close to GUE. We find crossover transitions between these different statistics.

where $\overline{\cdot}$ denotes the ensemble average (the average with respect to the unitary Haar measure), and $d$ denotes the dimension of the matrices. These lead to the following average and the variance of the matrix elements:

$$\overline{\sigma_{\alpha\beta}} = \frac{\delta_{\alpha\beta}}{d} \sum_i a_i,$$

(B7)

$$\overline{\sigma_{\alpha\alpha}^2} - \overline{\sigma_{\alpha\alpha}}^2 = \sum_{ij} a_i a_j |U_{\alpha i}|^2 |U_{\beta j}|^2 - \left( \frac{1}{d} \sum_i a_i \right)^2$$

$$= \frac{1}{d(d+1)} \sum_i a_i^2 + \frac{1}{d(d+1)} \left( \sum_i a_i \right)^2$$

$$- \left( \frac{1}{d} \sum_i a_i \right)^2$$

$$= \frac{1}{d+1} \left[ \frac{1}{d} \sum_i a_i^2 - \left( \frac{1}{d} \sum_i a_i \right)^2 \right],$$

(B8)
FIG. 7: Nearest-neighbor spacing distributions $P(s)$ for Hamiltonians (1) with $N = 13$, $J = 1$ and $h' = -2.1h$. For small $h$ and $D$, $P(s)$ does not show level repulsions because the system is still approximately integrable. For $h = 0$ and large $D$, $P(s)$ is close to GSE statistics. For $D = 0$ and large $h$, $P(s)$ is close to GOE statistics. For large $h$ and $D$, $P(s)$ is close to GUE statistics. We find crossover transitions between these different statistics.

and

$$|\mathcal{O}_{\alpha\beta}|^2 = \sum_{ij} a_i a_j U_{i\alpha} U_{j\beta} U_{\beta j} U_{\alpha i}$$

$$= \frac{1}{d(d+1)} \sum_i a_i^2 - \frac{1}{d(d-1)(d+1)} \sum_{i\neq j} a_i a_j$$

$$= \frac{d}{(d+1)(d-1)} \left[ \frac{1}{d} \sum_i a_i^2 - \left( \frac{1}{d} \sum_i a_i \right)^2 \right]$$

(B9)

for $\alpha \neq \beta$. We note that in Refs. [8, 13] the term $-\left( \frac{1}{d} \sum_i a_i \right)^2$ is missing in the variance (B8) and (B9) because the authors of Refs. [8, 13] ignore some correlations such as Eq. (B6), which contribute to the lowest-order term after the summation over $i$ and $j$.

From Eqs. (B8) and (B9), we obtain the universal ratio

$$r_A = 1$$

(B10)

for large $d$.

2. Class AI: calculation by GOE

Next, we consider the Hamiltonian that belongs to GOE and the observables that are even under $\hat{T}$. We assume that neither the Hamiltonian nor the observable has a degeneracy. For $\hat{O} = \sum_{i=1}^d a_i |a_i\rangle \langle a_i|$, we can assume that $\hat{T} |a_i\rangle = |a_i\rangle$ and $\hat{T} |E_\alpha\rangle = |E_\alpha\rangle$ without loss of generality [65]. Then the matrix elements can be taken
For a Hamiltonian that belongs to GSE and the observable is odd under $T$, we obtain $r_{\text{GSE,even}} = 1$, which results from the vanishing $O_{\alpha \alpha}$:

$$O_{\alpha \alpha} = \langle \hat{T} | E_\alpha \rangle, \hat{T} \hat{O} \hat{T}^{-1} \hat{T} \langle E_\alpha \rangle^* = \langle E_\alpha \rangle, \langle -\hat{O} | E_\alpha \rangle^* = -O_{\alpha \alpha} = 0.$$  

(B20)

Finally, we consider the case in which the Hamiltonian belongs to GOE and the observable is odd under $T$. In this case, the observable can be written as $\hat{O} = \sum_{\alpha \beta} a_\alpha |a_\alpha\rangle \langle a_\beta| - |\tilde{a}_\alpha\rangle \langle \tilde{a}_\alpha|$, where $|\tilde{a}_\alpha\rangle := \hat{T} |a_\alpha\rangle$ and $\hat{O} |\tilde{a}_\alpha\rangle = -a_\alpha^* |\tilde{a}_\alpha\rangle$. By calculating the higher moments of inner products such as $\langle E_\alpha |a_\alpha\rangle$ and $\langle E_{\alpha'} |a_{\alpha'}\rangle$ using random matrix theory, we obtain the averages and the variances of diagonal and off-diagonal matrix elements as follows [67]:

$$O_{\alpha \beta} = \frac{\delta_{\alpha \beta}}{d} \sum_i a_i,$$

(B21)

$$\langle O_{\alpha \alpha} - \langle O_{\alpha \alpha} \rangle \rangle^2 = \frac{1}{d + 1} \left[ \frac{1}{d} \sum_i a_i^2 - \left( \frac{1}{d} \sum a_i \right)^2 \right],$$

(B22)

$$|O_{\alpha \beta}|^2 = \frac{d}{(d - 2)(d + 1)} \left[ \frac{1}{d} \sum_i a_i^2 - \left( \frac{1}{d} \sum a_i \right)^2 \right],$$

(B23)

for $\alpha \neq \beta$. Thus we obtain

$$r_{\text{GSE,even}} = 1$$

(B24)

if $d$ is sufficiently large. For the ratio concerning the Kramers pair, we obtain $r_{\text{GSE,even}} = 0$, which results from the vanishing $O_{\alpha \alpha}$:

$$O_{\alpha \alpha} = \langle \hat{T} | E_\alpha \rangle, \hat{T} \hat{O} \hat{T}^{-1} \hat{T} \langle E_\alpha \rangle^* = \langle E_\alpha \rangle, \langle -\hat{O} | E_\alpha \rangle^* = -O_{\alpha \alpha} = 0.$$  

(B25)

Here we have used $\hat{T}^2 = -1$.

3. Class AII: calculation by GSE

If the Hamiltonian belongs to GSE, it can be written as $\hat{H} = \sum_{\alpha \beta} E_\alpha |E_\alpha\rangle \langle E_\alpha| + |\tilde{E}_\alpha\rangle \langle \tilde{E}_\alpha|$. Here, we note that $d$ is always even in GSE. Let us first consider that the observable is even under $T$. In this case, the observable can be written as $\hat{O} = \sum_{i=1}^d a_i |a_i\rangle \langle a_i| = \sum_{i=1}^{d/2} a_i |a_i\rangle \langle a_i| + |\tilde{a}_i\rangle \langle \tilde{a}_i|$, where $|\tilde{a}_i\rangle := \hat{T} |a_i\rangle$ and $\hat{O} |\tilde{a}_i\rangle = a_i^* |\tilde{a}_i\rangle$. We assume that the Hamiltonian and the observable have no degeneracies except for Kramers degeneracies. By calculating (higher) moments of $\langle E_{\alpha'} |a_{\alpha'}\rangle$ and $\langle E_\alpha |a_\alpha\rangle$ using random matrix theory, we obtain the averages and the variances of diagonal and off-diagonal matrix elements as follows [67]:

$$O_{\alpha \beta} = \frac{\delta_{\alpha \beta}}{d} \sum_i a_i,$$

(B21)

$$\langle O_{\alpha \alpha} - O_{\alpha \alpha} \rangle^2 = \frac{1}{d + 1} \left[ \frac{1}{d} \sum_i a_i^2 - \left( \frac{1}{d} \sum a_i \right)^2 \right],$$

(B22)

$$|O_{\alpha \beta}|^2 = \frac{d}{(d - 2)(d + 1)} \left[ \frac{1}{d} \sum_i a_i^2 - \left( \frac{1}{d} \sum a_i \right)^2 \right],$$

(B23)

for $\alpha \neq \beta$. Thus we obtain

$$r_{\text{GSE,even}} = 1$$

(B24)

if $d$ is sufficiently large. For the ratio concerning the Kramers pair, we obtain $r_{\text{GSE,even}} = 0$, which results from the vanishing $O_{\alpha \alpha}$:

$$O_{\alpha \beta} = \langle \hat{T} | E_\alpha \rangle, \hat{T} \hat{O} \hat{T}^{-1} \hat{T} \langle E_\alpha \rangle^* = \langle E_\alpha \rangle, \langle -\hat{O} | E_\alpha \rangle^* = -O_{\alpha \alpha} = 0.$$  

(B25)

Here we have used $\hat{T}^2 = -1$. 

Next, if the Hamiltonian belongs to GOE and the observable is odd under $T$, we obtain $r_{\text{GSE,odd}} = 0$ is obtained. This results comes from the vanishing diagonal matrix elements:

$$O_{\alpha \alpha} = \langle \hat{T} | E_\alpha \rangle, \hat{T} \hat{O} \hat{T}^{-1} \hat{T} \langle E_\alpha \rangle^* = \langle E_\alpha \rangle, \langle -\hat{O} | E_\alpha \rangle^* = -O_{\alpha \alpha} = 0.$$  

(B20)
For the variances, we obtain
\[
\langle O_{\alpha \alpha} \rangle = \frac{1}{d+1} \left[ \frac{1}{d} \sum_i a_i^2 \right],
\]
\[
\langle O_{\alpha \beta} \rangle = \frac{1}{d+1} \left[ \frac{1}{d} \sum_i a_i^2 \right], \quad (E_\alpha \neq E_\beta),
\]
\[
\langle O_{\alpha \alpha} \rangle = \frac{2}{d+1} \left[ \frac{1}{d} \sum_i a_i^2 \right].
\]
We then obtain
\[
r_{\text{GSE, odd}} \to 1,
\]
\[
r_{\text{GSE, odd}}' \to \sqrt{2},
\]
if \( d \) is sufficiently large.

### Appendix C: Factor \( g \) in the presence of the Kramers degeneracy

In this appendix, we derive Eq. (31). We first expand the initial state as
\[
|\psi_0 \rangle = \sum_\alpha (c_\alpha |E_\alpha \rangle + c_\bar{\alpha} |\bar{E}_\alpha \rangle) = \sum_\alpha \sqrt{p_\alpha} |\phi_\alpha \rangle,
\]
where
\[
p_\alpha = |c_\alpha|^2 + |c_\bar{\alpha}|^2
\]
and
\[
|\phi_\alpha \rangle = \frac{1}{\sqrt{p_\alpha}} (c_\alpha |E_\alpha \rangle + c_\bar{\alpha} |\bar{E}_\alpha \rangle).
\]
The autocorrelation function can be written as
\[
\langle \psi_0 | \hat{\Delta} (t+t') \hat{\Delta} (t) | \psi_0 \rangle = \sum_{\alpha \beta} \sqrt{p_\alpha p_\beta} e^{iE_\alpha t+t'} e^{-iE_\beta t} \langle \phi_\alpha | \hat{\Delta} e^{-iHt'} \hat{\Delta} | \phi_\beta \rangle
\]
\[
= \sum_{\alpha \beta \gamma} \sqrt{p_\alpha p_\beta} e^{i(E_\alpha - E_\beta) t - i(E_\gamma - E_\alpha) t'} \times \left( \langle \phi_\alpha | \hat{\Delta} | \phi_\gamma \rangle \langle \phi_\gamma | \hat{\Delta} | \phi_\beta \rangle + \langle \phi_\alpha | \hat{\Delta} | \phi_\bar{\gamma} \rangle \langle \phi_\bar{\gamma} | \hat{\Delta} | \phi_\beta \rangle \right),
\]
which leads to
\[
\tilde{S} = \sum_\alpha p_\alpha \left( |\langle \phi_\alpha | \hat{\Delta} | \phi_\alpha \rangle|^2 + |\langle \phi_\alpha | \hat{\Delta} | \phi_\bar{\alpha} \rangle|^2 \right)
\]
by assuming non-degeneracy except for Kramers degeneracy. Here \(|\phi_\alpha \rangle = T |\phi_\alpha \rangle\). Similarly,
\[
\hat{\Theta} = \sum_\alpha p_\alpha \langle \phi_\alpha | \hat{\Theta} | \phi_\alpha \rangle
\]
and then
\[
\tilde{S} - \tilde{\Theta}^2 = \sum_\alpha p_\alpha \left( \langle \phi_\alpha | \hat{\Theta} | \phi_\alpha \rangle - \sum_\beta p_\beta \langle \phi_\beta | \hat{\Theta} | \phi_\beta \rangle \right)^2
\]
\[
+ \sum_\alpha p_\alpha |\langle \phi_\alpha | \hat{\Theta} | \phi_\alpha \rangle|^2.
\]
Since the random-matrix-theory treatment of GSE discussed in Appendix B is applicable to a general rotation in the two-dimensional Kramers pair space [53], we can replace the role of \(|E_\alpha \rangle\) and \(|\bar{E}_\alpha \rangle\) with \(|\phi_\alpha \rangle\) and \(|\bar{\phi}_\alpha \rangle\) in considering the statistics of matrix elements. Thus, the first and second terms on the right-hand side of Eq. (8) are approximated by \(\Delta O^2_{d} \) and \(\Delta O^2_{K} \), respectively, in analogy with the discussion in the main text. We then have
\[
\tilde{S} - \tilde{\Theta}^2 \simeq \Delta O^2_{d} + \Delta O^2_{K}.
\]
Next, for off-diagonal matrix elements we obtain
\[
\tilde{T}^2 = \sum_{\alpha \neq \beta} p_\alpha p_\beta |\langle \phi_\alpha | \hat{\Theta} | \phi_\beta \rangle|^2.
\]
Again, we can safely apply the discussion in Appendix B on \(|E_\alpha \rangle\) and \(|\bar{E}_\alpha \rangle\) to \(|\phi_\alpha \rangle\) and \(|\bar{\phi}_\alpha \rangle\) in considering the statistics of matrix elements. Then, in analogy with the main text, we obtain \(\tilde{T}^2 \simeq \Delta O^2_{od}\). We thus obtain
\[
g \simeq \sqrt{\frac{\Delta O^2_{d} + \Delta O^2_{K}}{\Delta O^2_{od}}} = \sqrt{r^2 + r'^2}
\]
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