Fluctuation of inverse compressibility for electronic systems with random capacitive matrices

Yshai Avishai\textsuperscript{1,4}, Dani Berend\textsuperscript{2}, and Richard Berkovits\textsuperscript{3}

\textsuperscript{1}Department of Physics, Ben Gurion University of the Negev, Beer-Sheva 84 105, Israel
\textsuperscript{2}Department of Mathematics and Computer Sciences, Ben Gurion University of the Negev, Beer-Sheva 84 105, Israel
\textsuperscript{3}Minerva Center, Department of Physics, Bar Ilan University, Ramat Gan, Israel
\textsuperscript{4}NTT Basic Research Laboratories, 3-1 Morinosato, Wakamiya, Atsugi-shi Kanagawa-ken, Japan

This article is concerned with statistics of addition spectra of certain many body systems of identical particles. In the first part, the pertinent system consists of $N$ identical particles distributed among $K < N$ independent sub-systems, such that the energy of each sub-system is a quadratic function of the number of particles residing on it with random coefficients. On a large scale, the ground state energy $E(N)$ of the whole system grows quadratically with $N$, but in general there is no simple relation such as $E_N = aN + bN^2$. The deviation of $E(N)$ from exact quadratic behavior implies that its second difference (the inverse compressibility) $\chi_N \equiv E(N + 1) - 2E(N) + E(N - 1)$ is a fluctuating quantity. Regarding the numbers $\chi_N$ as values assumed by a certain random variable $\chi$, we obtain a closed-form expression for its distribution $F(\chi)$. Its main feature is that the corresponding density $P(\chi) = \frac{dF(\chi)}{d\chi}$ has a maximum at the point $\chi = 0$. As $K \to \infty$ the density is Poissonian, namely, $P(\chi) \to e^{-\chi}$.

This result serves as a starting point for the second part in which coupling between subsystems is included. More generally, a classical model is suggested in order to study fluctuations of Coulomb blockade peak spacings in large two-dimensional semiconductor quantum dots. It is based on the
electrostatics of several electron islands among which there are random inductive and capacitive couplings. Each island can accommodate electrons on quantum orbitals whose energy depend also on an external magnetic field. In contrast with a single island quantum dot where the spacing distribution between conductance peaks is close to Gaussian, here the distribution has a peak at small spacing value. The fluctuations are mainly due to charging effects. The model can explain the occasional occurrence of couples or even triples of closely spaced Coulomb blockade peaks, as well as the qualitative behavior of peak positions with the applied magnetic field.
I. STATISTICS OF ADDITION SPECTRA OF INDEPENDENT QUANTUM SYSTEMS

A. Motivation

Statistics of spectra is an efficient tool for elucidating properties of various physical systems. So far, most of the effort is focused on the study of energy levels of a system with a fixed number of particles. In this context, one of the central earlier results is that the spectral statistics of many-body systems such as complex nuclei agree with the predictions of random matrix theory \[1,2\]. On the other extreme, it was found that level statistics of a single particle in a chaotic or disordered system also obeys a Wigner-Dyson statistics \[3,4\].

Recently, experiments are designed to get information on the statistics of the addition spectra of electrons in quantum dots \[5\]. The pertinent energy levels \(E(N)\) are the ground state energies of a system consisting of \(N\) electrons residing on a quantum dot, which is coupled capacitively to its environment.

Let us single out two properties of the addition spectra of quantum dots. The first one is that, on a large scale, the energy \(E(N)\) grows quadratically with \(N\), while the second one is a consequence of charge quantization, namely, there is, in general, no simple relation such as \(E(N) = aN + bN^2\). In this context, an appropriate quantity whose statistics is of interest is then the inverse compressibility,

\[
\chi_N \equiv E(N + 1) - 2E(N) + E(N - 1).
\]

(1)

It is the deviation of \(E(N)\) from exact quadratic behavior which makes its second difference \(\chi_N\) non-constant. Indeed, in a recent experiment on large quantum dot \[6\] it was found that the inverse compressibility vanishes for numerous values of electron number \(N\).

In the present section we study the statistics of the addition spectrum of a simple physical system with the two basic properties mentioned above. One example of such a system is motivated by considering the electrostatic energy of large quantum dots (although it should be mentioned that the model is too simple to describe the actual physics, see next section). To be specific, we have in mind a system of \(K\) metallic grains such that the number of electrons on the \(i^{th}\) grain is \(n_i\) \((i = 0, 1, 2, \ldots, K - 1)\) and their sum equals \(N\). The electrostatic energy of the pertinent system is a bilinear form in the numbers \(n_i\) with a \(K \times K\) matrix \(w \equiv \frac{1}{2}C^{-1}\). Here \(C\) is a positive-definite symmetric matrix of capacitance and inductance coefficients. If the metallic grains are very far apart, the matrix \(C\) is nearly diagonal. Thus, we concentrate on the special case \(C = \text{diag}[C_i]\), for which the energy of the system is given by
\[ E(N) = \min \sum_{i=0}^{K-1} \frac{1}{2C_i} n_i^2, \text{ (subject to } \sum_{i=0}^{K-1} n_i = N). \] (2)

The minimum in (2) is taken over all possible partitions \( n_i \) of \( N \).

Another example is the energy of a system composed of \( K \) different harmonic oscillators, among which one distributes \( N \) spinless fermions. If there are \( n_i \) fermions on oscillator \( i \) (whose frequency is \( \omega_i \)), then the energy of this oscillator (up to a constant) is \( E_i = \hbar \omega_i n_i (n_i + 1) \), and hence the ground state energy of the system is

\[ E(N) = \min \sum_{i=0}^{K-1} E_i, \text{ (subject to } \sum_{i=0}^{K-1} n_i = N). \] (3)

Discussion is concentrated on the first example, which is borrowed from the electrostatics of quantum dots (2), and refer to the constants \( C_i \) as capacitors. A remark pertaining to the second example (the system of oscillators (3)) are also presented.

Regarding the numbers \( \chi_N \) of (1) as values assumed by a certain random variable, the distribution of this random variable is the main focus of the present work, which culminates in Theorem 1, where we find a closed-form expression for the distribution.

The problem of elucidating the (addition) spectral statistics of a a many-body system, consisting of several independent sub-systems (whose dependence of \( E \) on \( n_i \) is known), looks deceptively simple. As will be evident shortly, this is not the case, and finding the distribution in question is quite a non-trivial task. Note that, even for a single particle system composed of several independent sub-systems (e.g., a system of a particle in several boxes), the derivation of level statistics requires a large degree of mathematical effort [21].

The rest of this section is therefore devoted to a rigorous derivation of our main results. We offer our apology to the causal reader who might find this part rather mathematically oriented. He or she is advised to start looking at problem 1 below and leave this section after reading the statement of Theorem 1.

B. Formalism

**Definition 1.** Let \((\theta_n)_{n=1}^\infty\) be a sequence of real numbers and \( F \) a distribution function. The sequence \((\theta_n)\) is **asymptotically \( F \)-distributed** if

\[
\frac{|\{1 \leq n \leq M : \theta_n \leq x\}|}{M} \rightarrow F(x)
\]

for every continuity point \( x \) of \( F \) (where \(|S|\) denotes the cardinality of a finite set \( S \)).

An equivalent condition is the following. Denote by \( \delta_t \) the point mass at \( t \), and let \( \mu \) be the probability measure corresponding to the distribution \( F \) (namely, \( \mu(A) = \int 1_A dF(x) \) for
any Borel set $A$). Then $(\theta_n)$ is asymptotically $F$-distributed if
\[
\frac{1}{M} (\delta_{\theta_1} + \delta_{\theta_2} + \ldots + \delta_{\theta_M}) \underset{M \to \infty}{\rightarrow} \mu
\]
(the convergence being in the weak*-topology).

The notion of asymptotic distribution has a stronger version whereby, instead of requiring only that initial pieces of the sequence behave in a certain way, we require this to happen for any large finite portion of the sequence. This leads to

**Definition 2.** In the setup of Definition 1, $(\theta_n)$ is *asymptotically well $F$-distributed* if
\[
\frac{|\{L < n \leq M : \theta_n \leq x\}|}{M - L} \underset{M - L \to \infty}{\rightarrow} F(x)
\]
for every continuity point $x$ of $F$.

Recall that the *density* of a set $A \subseteq \mathbb{N}$ is given by
\[
D(A) = \lim_{M \to \infty} \frac{|A \cap [1, M]|}{M}
\]
if the limits exists. If, moreover, the limit
\[
BD(A) = \lim_{M - L \to \infty} \frac{|A \cap (L, M]|}{M - L}
\]
exists (in which case it is certainly the same as $D(A)$), then it is called the *Banach density* of $A$ (cf. [9, p.72]).

The following lemma is routine.

**Lemma 1.** Let $(\theta_n)_{n=1}^{\infty}$ be a sequence of real numbers. Suppose $\mathbb{N} = \bigcup_{j=1}^{r} A_j$, where the union is disjoint. Let $(\theta^{(j)}_{n})_{n=1}^{\infty}$ be the subsequence of $(\theta_n)$, consisting of those elements $\theta_n$ with $n \in A_j$, $1 \leq j \leq r$.

1. If each $(\theta^{(j)}_{n})$ is asymptotically $F_j$-distributed for some distribution functions $F_j$, $1 \leq j \leq r$, and $D(A_j) = d_j$, $1 \leq j \leq r$, then $(\theta_n)$ is asymptotically $F$-distributed, where $F = \sum_{j=1}^{r} d_j F_j$.

2. If each $(\theta^{(j)}_{n})$ is asymptotically well $F_j$-distributed and $BD(A_j) = d_j$, then $(\theta_n)$ is asymptotically well $F$-distributed.

Obviously, a general sequence on the line does not have to be asymptotically distributed according to some distribution function, but one would expect it of sufficiently “regular” bounded sequences. In our case, one might expect $\chi_N$ to be distributed according to some distribution function corresponding to a measure centered at about $1/C$. However, this is not the case. In fact, the measure in question is supported on a finite interval, and is a
convex combination of an absolutely continuous measure with decreasing density function on some interval \([0, a]\) and the point mass \(\delta_a\) at the right end \(a\) of that interval.

We have defined \(E(N)\) indirectly by means of the following:

**Problem 1.** For each non-negative integer \(N\), find non-negative integers \(n_0, n_1, \ldots, n_{K-1}\), satisfying \(n_0 + n_1 + \ldots + n_{K-1} = N\), for which \(\sum_{i=0}^{K-1} \frac{1}{2^{n_i}} \cdot n_i^2\) is minimal.

It turns out that this problem is intimately related to a second optimization problem. Put \(w_i = \frac{1}{2^{n_i}}, 0 \leq i \leq K - 1\), and let \(\Delta\) denote the set of all positive odd multiples of the numbers \(\frac{1}{2^{n_i}}\):

\[\Delta = \{w_0, 3w_0, 5w_0, \ldots, w_1, 3w_1, 5w_1, \ldots, w_{K-1}, 3w_{K-1}, 5w_{K-1}, \ldots\}\]

Here we treat \(\Delta\) as a multi-set, or a sequence, in the sense that if some elements appear in this representation of \(\Delta\) more than once (which occurs iff some ratio \(w_i/w_j\) is a rational number with odd numerator and denominator), then we consider \(\Delta\) as having several copies of these numbers.

**Problem 2.** For each non-negative integer \(N\), minimize \(\sum_{m=1}^{N} \delta_m\), where \(\delta_1, \delta_2, \ldots, \delta_N\) range over all distinct \(N\)-tuples in \(\Delta\).

Note that, if an element appears several times in \(\Delta\), it is allowed to appear the same number of times in the sum as well.

Let us demonstrate the equivalence of the two problems. Given the sum \(\sum_{i=0}^{K-1} w_i \cdot n_i^2\), we may use the equality \(w_i \cdot n_i^2 = w_i + 3w_i + 5w_i + \ldots + (2n_i - 1)w_i\) to see that any feasible value for the objective function of the first problem is a feasible value for the objective function of the second problem as well. On the other hand, solving Problem 2 is trivial. Namely, one minimizes the sum there simply by taking the \(N\) least elements of the set \(\Delta\). In particular, for each \(i\), the multiples of \(w_i\) present in the optimal solution will be all odd multiples \(w_i, 3w_i, 5w_i, \ldots\) up to some \((2n_i - 1)w_i\). Thus, the optimal solution of Problem 2 yields the optimal solution of Problem 1 also. We note in passing that this discussion shows also that the minimum (for each of the problems) is obtained at a unique point unless \(\Delta\) contains multiple elements. (However, we shall always refer to the optimal solution, even when there may be several.)

A simple consequence of the above is

**Proposition 1.** Let \(n = (n_i)_{i=0}^{K-1}\) be the optimal solution of Problem 1 for some value of \(N\). Then the optimal solution of Problem 1, with \(N + 1\) instead of \(N\), is \(n' = (n'_i)_{i=0}^{K-1}\), where \(n'_j = n_j + 1\) for some \(0 \leq j \leq K - 1\) and \(n'_i = n_i\) for \(i \neq j\).

**Remark.** It is convenient to comment here on the effect a certain change in the original problem would make. One may consider the energies \(E_i\) to be \(w_i n_i (n_i + 1)\) instead of \(w_i n_i^2\). This would change \(\Delta\) to be the set of all even multiples of the \(w_i\)'s. Obviously, this would
leave intact the equivalence of Problems 1 and 2. One can check that this would have also no effect on Theorems 1 and 2 infra.

To formulate our main result we need a few definitions and notations. Real numbers \( \theta_1, \theta_2, \ldots, \theta_r \) are independent over \( \mathbb{Q} \) if, considered as vectors in the vector space \( \mathbb{R} \) over the field \( \mathbb{Q} \), they are linearly independent. Equivalently, this is the case if the equality \( m_1 \theta_1 + m_2 \theta_2 + \ldots + m_r \theta_r = 0 \) for integer \( m_1, m_2, \ldots, m_r \) implies \( m_1 = m_2 = \ldots = m_r = 0 \).

Considering the actual physical system (a collection of metallic grains), it is reasonable to assume that the capacitors \( C_i \) are random, so that generically they are independent over \( \mathbb{Q} \). Without loss of generality we may rearrange the \( K \) capacitors such that \( C_0 = \max_{0 \leq i \leq K-1} C_i \). It is also useful to divide all the capacitors by the largest one, so that the scaled capacitors \( c_i \equiv C_i/C_0 \) with \( 1 = c_0 > c_1, c_2, \ldots, c_{K-1} \) are dimensionless. Finally, set \( s = c_0 + c_1 + \ldots + c_{K-1} \).

Now we formulate our main results.

**Theorem 1.** Suppose \( C_0, C_1, \ldots, C_{K-1} \) are independent over \( \mathbb{Q} \). Then the sequence \( (\chi_N)_{N=1}^\infty \) is asymptotically \( F \)-distributed, where the distribution \( F \) is given by either of the following two representations:

\[
F(x) = \begin{cases} 
0, & x < 0, \\
1 - \frac{1}{s} \sum_{i=0}^{K-1} c_i \prod_{j=0}^{K-1} \left(1 - \frac{x}{2w_j}\right), & 0 \leq x < 2w_0, \\
1, & 2w_0 \leq x,
\end{cases}
\]

or

\[
F(x) = \begin{cases} 
0, & x < 0, \\
1 - \frac{1}{s} \sum_{S \subseteq \{1, \ldots, K-1\}} (|S| + 1) \prod_{i \in S} c_i \prod_{i \notin S} (1 - c_i) \cdot \left(1 - \frac{x}{2w_0}\right)^{|S|}, & 0 \leq x < 2w_0, \\
1, & 2w_0 \leq x.
\end{cases}
\]

It is not immediately obvious from the formulas, but \( F \) has one discontinuity, namely at the point \( 2w_0 \). The reason is that, as the elements of \( \Delta \) are all odd multiples of the \( w_i \)'s, and as \( w_0 \) is the smallest of the \( w_i \)'s, it happens occasionally that there is no odd multiple of \( w_1, \ldots, w_{K-1} \) between two consecutive multiples of \( w_0 \). The size of the atom at \( 2w_0 \) is \( \frac{1}{s} \prod_{i=1}^{K-1} (1 - c_i) \). This is easily explained intuitively. In fact, the “density” of odd multiples of \( w_i \) is \( c_i \) times the same density for multiples of \( w_0 \). Hence the “probability” that an interval of the form \( [(2n - 1)w_0, (2n + 1)w_0) \) does not contain an odd multiple of \( w_i \) is \( 1 - c_i \). Assuming that the “events” of containing different \( w_i \)'s are independent, we conclude that the proportion of multiples of \( w_0 \) in \( \Delta \) whose successors are also such is \( \prod_{i=1}^{K-1} (1 - c_i) \). Since the proportion of multiples of \( w_0 \) in \( \Delta \) is \( \frac{1}{s} \), we arrive at the required expression for the size of the atom.
Now we would like to study the asymptotic of the distances between consecutive elements of $\Delta$ as the number of capacitors grows. Obviously, as this happens, the distances become smaller. More precisely, on the average we have $\frac{1}{2}w_j$ odd multiples of each $w_j$ in each unit interval, and hence we have there $\sum_{j=0}^{K-1} \frac{1}{2w_j} = \frac{s}{2w_0}$ elements of $\Delta$ altogether. Hence the average distance between consecutive elements is $\frac{s}{2w_0}$. To understand the asymptotics of the gaps, it makes sense therefore to normalize them so as to have mean 1. Thus, we multiply the distances by $\frac{s}{2w_0}$, and ask about the asymptotic behavior

**Theorem 2.** Suppose the capacitances $C_0, C_1, \ldots$ are chosen uniformly and independently in $[0, 1]$. For each $K$, let $F_K$ denote the distribution corresponding to the normalized gaps when taking into account the first $K$ capacitors only. Then, with probability 1, the distributions $F_K$ converge to an Exp(1) distribution function.

**Remark.** As will be seen in the proof, we actually use much less to prove Theorem 2 than is required by the conditions of the theorem. Namely, we need the capacitances $C_i$ to be linearly independent over $\mathbb{Q}$, and that they do not form a fast diminishing sequence.

It is worthwhile mentioning that this type of “Poissonian” asymptotic behavior of consecutive gaps is typical. For example, this is the case for uniformly selected numbers in $[0, 1]$, and is conjectured to be the case in other interesting cases as well. (See, for example, [11] and [12] and the references there.)

In the course of the proof, we shall make use of the notion of uniform distribution modulo 1 and a few basic results relating to it. (The reader is referred to Kuipers and Niederreiter [10] for more information.) A sequence $(x_n)_{n=1}^{\infty}$ of real numbers is uniformly distributed modulo 1 if

$$\frac{|\{1 \leq n \leq M : a \leq \{x_n\} < b\}|}{M} \rightarrow b - a, \quad 0 \leq a < b \leq 1,$$

where $\{t\}$ is the fractional part of a real number $t$. In terms of Definition 1, $(x_n)$ is uniformly distributed modulo 1 if and only if the sequence $(\{x_n\})$ of fractional parts is $F$-distributed, where $F$ is the distribution function of the uniform distribution on $[0, 1]$:

$$F(x) = \begin{cases} 
0, & x < 0, \\
x, & 0 \leq x \leq 1, \\
1, & x > 1.
\end{cases}$$

The generalization of the notion of an asymptotically $F$-distributed sequence to that of an asymptotically well $F$-distributed sequence clearly carries over to our case. Instead of requiring only that the dispersion of large initial pieces of the sequence becomes more and
more even, we require this to happen at arbitrary locations. This version is termed well-distribution. Thus, \((x_n)_{n=1}^{\infty}\) is well-distributed modulo 1 if
\[
\frac{|\{L < n \leq M : a \leq \{x_n\} < b\}|}{M - L} \rightarrow b - a, \quad 0 \leq a < b \leq 1.
\]
Both notions have multi-dimensional analogue. A sequence \((x_n)_{n=1}^{\infty}\) in \(\mathbb{R}^s\) is uniformly distributed modulo 1 in \(\mathbb{R}^s\) if
\[
\frac{|\{1 \leq n \leq N : a \leq \{x_n\} < b\}|}{N} \rightarrow \prod_{i=1}^{s}(b_i - a_i), \quad 0 \leq a < b \leq 1,
\]
where inequalities between vectors in \(\mathbb{R}^s\) are to be understood component-wise, \(0 = (0,0,\ldots,0) \in \mathbb{R}^s\), \(a = (a_1,a_2,\ldots,a_s)\), and so forth.

Perhaps the most basic example of a sequence which is uniformly distributed modulo 1 is \((na)_{n=1}^{\infty}\), where \(a\) is an arbitrary irrational. In the multi-dimensional case, the sequence \((na_1,na_2,\ldots,na_s)\) is uniformly distributed modulo 1 in \(\mathbb{R}^s\) if and only if the numbers 1, \(\alpha_1,\alpha_2,\ldots,\alpha_s\) are linearly independent over \(\mathbb{Q}\). Moreover, in this case uniform distribution implies well-distribution (cf. [10, Example 1.6.1, Exercise 1.6.14]).

Given a partition \(N = \bigcup_{j=1}^{l} A_j\) and positive integers \(r_j\), \(j = 1,\ldots,l\), we define the \((r_j)_{j=1}^{l}\)-inflation of the given partition as the partition of \(N\) obtained by inflating each element of each of the sets \(A_j\) into \(r_j\) elements. More precisely, we construct sets \(B_j\), \(j = 1,\ldots,l\), as follows. For a positive integer \(i\), let \(f(i) = j\) if \(i \in A_j\). Given any positive integer \(n\), let \(m\) be defined by \(\sum_{i=1}^{m-1} f(i) < m \leq \sum_{i=1}^{m} f(i)\). Let \(n \in B_j\) if \(m \in A_j\). The following lemma is routine.

**Lemma 2.** In this setup:

1. If \(D(A_j) = d_j, 1 \leq j \leq l\), then \(D(B_j) = \frac{r_j d_j}{\sum_{i=1}^{l} r_id_i}\).
2. If \(BD(A_j) = d_j, 1 \leq j \leq l\), then \(BD(B_j) = \frac{r_j d_j}{\sum_{i=1}^{l} r_id_i}\).

**Proof of Theorem 1.** Between any two consecutive odd multiples of \(w_0\), there is at most one odd multiple of each \(w_j\), \(1 \leq j \leq K - 1\). In fact, one easily verifies that, given a positive integer \(m\), there is an odd multiple of \(w_j\) between \((2m - 1)w_0\) and \((2m + 1)w_0\), namely there exists an integer \(n\) with
\[
(2m - 1)w_0 \leq (2n - 1)w_j < (2m + 1)w_0,
\]
if and only if
\[
mc_j \in \left(\frac{1 - c_j}{2}, \frac{1 + c_j}{2}\right) \pmod{1}.
\]
Moreover, the relative position of \((2n - 1)w_j\) within the interval \([(2m - 1)w_0, (2m + 1)w_0)\) is the same, but in the opposite direction, as that of \(mc_j\) (mod 1) within the interval \([\frac{1-c_j}{2}, \frac{1+c_j}{2}]\), that is

\[(2n - 1)w_j = \alpha \cdot (2m - 1)w_0 + (1 - \alpha) \cdot (2m + 1)w_0, \quad (0 < \alpha \leq 1), \quad (8)\]

if and only if

\[mc_j \equiv (1 - \alpha) \cdot \frac{1-c_j}{2} + \alpha \cdot \frac{1+c_j}{2} \pmod{1}. \quad (9)\]

Next we define a partition of \(N\) as follows. Write the elements of \(\Delta\) in ascending order: \(\Delta = \{\delta_1 < \delta_2 < \delta_3 < \ldots\}\). Given \(n \in N\), let \(S \subseteq \{1, 2, \ldots, K - 1\}\) denote the set of all those \(j\)'s such that the unique interval of the form \([(2m - 1)w_0, (2m + 1)w_0)\) containing \(\delta_n\) contains an odd multiple of \(w_j\). The set of all integers \(n\) giving rise in this way to any set \(S\) is denoted by \(B_S\). Consider the partition \(N = \bigcup_{S \subseteq \{1, 2, \ldots, K - 1\}} B_S\). To prove the theorem using Lemma 1, we have to find the Banach densities of the sets \(B_S\) and the asymptotic distribution of the corresponding subsequences \((\chi_n)_{n \in B_S}\) of \(\chi_n\).

The partition of \(N\) into sets of the form \(B_S\) is obtained as an inflation of a somewhat more straightforward partition. In fact, let \(S\) be any subset of \(\{1, 2, \ldots, K - 1\}\). Denote by \(A_S\) the set of those positive integers \(n\) for which the interval \([(2n - 1)w_0, (2n + 1)w_0)\) contains odd multiples of \(w_j\) for \(j \in S\) and does not contain such multiples of the other \(w_j\)'s. Then \(N = \bigcup_{S \subseteq \{1, 2, \ldots, K - 1\}} A_S\) is a partition, and its \(((|S| + 1)S \subseteq \{1, 2, \ldots, K - 1\})\)-inflation yields the partition \(N = \bigcup_{S \subseteq \{1, 2, \ldots, K - 1\}} B_S\).

In view of the equivalence of (8) and (9), \(A_S\) is the set of those \(n\)'s for which \(nc_j \in \left(\frac{1-c_j}{2}, \frac{1+c_j}{2}\right)\) for \(j \in S\) and \(nc_j \notin \left(\frac{1-c_j}{2}, \frac{1+c_j}{2}\right)\) for \(j \notin S\). By the conditions of the theorem, the numbers \(1, c_1, \ldots, c_{K-1}\) are linearly independent over \(Q\), and hence the sequence \(c = (nc_1, nc_2, \ldots, nc_{K-1})_{n=1}^{\infty}\) is well-distributed modulo 1 in \(R^{K-1}\). This means that

\[D(A_S) = BD(A_S) = \prod_{i \in S} c_i \prod_{i \notin S} (1 - c_i). \quad (10)\]

Denote the right hand side of (10) by \(p_S\). In view of the above and Lemma 2, this implies

\[D(B_S) = BD(B_S) = \frac{(|S| + 1)p_S}{\sum_{T \subseteq \{1, 2, \ldots, K - 1\}}(|T| + 1)p_T}. \quad (11)\]

The denominator on the right hand side can be given a simpler form. In fact, let \(X_i, \ i = 1, 2, \ldots, K - 1\), be independent random variables with \(X_i \sim B(1, c_i)\), and \(X = \sum_{i=1}^{K-1} X_i\). Then:

\[\sum_{T \subseteq \{1, 2, \ldots, K - 1\}}(|T| + 1)p_T = E(X + 1) = 1 + c_1 + \ldots + c_{K-1} = s. \quad (12)\]
Hence:

\[ BD(B_S) = \frac{(|S| + 1)p_S}{s}. \]  

(13)

Let \( S \) be an arbitrary fixed subset of \{1, 2, \ldots, K - 1\}, say \( S = \{1, 2, \ldots, l\} \), where \( 0 \leq l \leq K - 1 \). If \( n \in A_S \), then there exist odd integers \( a_{1n}, a_{2n}, \ldots, a_{ln} \) such that \( a_{jn}w_j \in [(2n - 1)w_0, (2n + 1)w_0) \). Put:

\[ v_n = (a_{1n}w_1, a_{2n}w_2, \ldots, a_{ln}w_l) - (2n - 1)w_0 \cdot (1, 1, \ldots, 1) \in [0, 2w_0]^l, \quad n \in A_S. \]

By the equivalence of (8) and (9), the sequence \((v_n)_{n \in A_S}\) is well-distributed modulo \( 2w_0 \) in \( \mathbb{R}^l \).

Now each \( v_n \) gives rise to \( l + 1 \) terms of \((\chi_n)_{n \in B_S}\), as follows. Let \( v_n^{(1)} \leq v_n^{(2)} \leq \ldots \leq v_n^{(l)} \) be all coordinates of \( v_n \) in ascending order. Set:

\[ u_n = (v_n^{(1)}, v_n^{(2)} - v_n^{(1)}, \ldots, v_n^{(l)} - v_n^{(l-1)}, 2w_0 - v_n^{(l)}), \quad n \in A_S. \]

The sequence \((\chi_n)_{n \in B_S}\) consists of all coordinates of all vectors \( u_n \). Now we use the fact that if \( X_1, X_2, \ldots, X_r \) are independent random variables, distributed \( U(0, h) \), and \( X^{(1)}, X^{(2)}, \ldots, X^{(r)} \) are the corresponding order statistics, then each of the random variables \( X^{(1)}, X^{(2)} - X^{(1)}, \ldots, X^{(r)} - X^{(r-1)} \) has the distribution function defined by \( G(x) = 1 - (x/h)^r \) for \( 0 \leq x \leq h \) (which follows as a special case from [8, p.42, ex.23]). Consequently, for each \( 1 \leq j \leq l + 1 \), the sequence given by the \( j \)th coordinate of all vectors \( u_n, n \in A_S \), is asymptotically well \( G_1 \)-distributed, where \( G_1(x) = 1 - (x/2w_0)^l \) for \( 0 \leq x \leq 2w_0 \). Hence the sequence \((\chi_n)_{n \in B_S}\) is asymptotically well \( G_1 \)-distributed. Combined with (13), it proves (3).

We shall indicate only briefly the proof of (4), which is quite simpler. This time, we split \((\chi_n)\) into a union of subsequences \((\chi_n^{(i)})\), \( 0 \leq i \leq K - 1 \), by putting \( \chi_n \) in the sequence \( \chi_n^{(i)} \) if \( \delta_n \) is a multiple of \( w_i \). Clearly, the proportion of terms of \((\chi_n)\) belonging to \((\chi_n^{(i)})\) is \( c_i/s \). Next, consider the minimal odd multiples of all \( w_j \)'s which are larger than \( \delta_n \). The minimum of these \( K \) numbers is \( \delta_{n+1} \). For each \( j \neq i \), the distance from \( \delta_n \) to the minimal odd multiple of \( w_j \) following \( \delta_n \) is “distributed” \( U(0, 2w_j) \). (For \( i = 0 \) it is also possible that the next term will be again a multiple of \( w_0 \).) The linear independence of the \( C_i \)'s over \( \mathbb{Q} \) implies that these \( K - 1 \) distances are (statistically) independent, so that their minimum is distributed according to the function \( G_2(x) = 1 - \prod_{j \neq i}^{K-1} \left(1 - \frac{x}{2w_j}\right) \) on the interval \([0, 2w_0]\). These considerations can be formalized to prove (4). This completes the proof.

**Remark.** It is possible to shorten the proof by proving directly the equality of the right hand sides of (4) and (5). In fact, it is easy to integrate both forms with respect to \( x \); the equality of the resulting expressions follows easily from the binomial theorem. We have chosen the long way, as it is more instructive.
Proof of Theorem 2. The distribution \( F_K \) is obtained from that in Theorem 1 by stretching by the constant factor \( \frac{s}{2w_0} \). Hence:

\[
F_K(x) = \begin{cases} 
0, & x < 0, \\
1 - \frac{1}{s} \sum_{i=0}^{K-1} c_i \prod_{j=0, j \neq i}^{K-1} \left( 1 - \frac{c_j x}{s} \right), & 0 \leq x < s, \\
1, & s \leq x. 
\end{cases}
\] (14)

Note that some of the values appearing on the right hand side depend on \( K \) implicitly. Namely, since \( w_0 \) is assumed in Theorem 1 to be the least \( w_i \), each time a \( C_i \) is selected which is larger than all the heretofore selected \( C_j \)'s, we have to rearrange the \( C_j \)'s, thus changing \( w_0 \) and the \( c_j \)'s. We have to show that

\[
F_K(x) \xrightarrow{K \to \infty} 1 - e^{-x}, \quad x \geq 0. 
\] (15)

Indeed, fix \( x \geq 0 \). Since

\[
s = c_0 + c_1 + \ldots + c_{K-1} = \frac{C_0 + C_1 + \ldots + C_{K-1}}{C_0} \geq C_0 + C_1 + \ldots + C_{K-1} \geq 0 \] (16)

and the \( C_i \)'s are independent and uniformly distributed in \([0, 1]\), we have

\[
s \xrightarrow{K \to \infty} \infty. \] (17)

Hence, with probability 1, for sufficiently large \( K \) we have

\[
F_K(x) = 1 - \frac{1}{s} \sum_{i=0}^{K-1} c_i \prod_{j=0, j \neq i}^{K-1} \left( 1 - \frac{c_j x}{s} \right). \] (18)

Thus, to prove (14) we need to show that

\[
\frac{1}{s} \sum_{i=0}^{K-1} c_i \prod_{j=0, j \neq i}^{K-1} \left( 1 - \frac{c_j x}{s} \right) \xrightarrow{a.s. K \to \infty} e^{-x}, \quad x \geq 0. \] (19)

Now, on the one hand, using the inequality

\[
1 - t \leq e^{-t}, \quad t \in \mathbb{R},
\]

we have

\[
\prod_{j=0, j \neq i}^{K-1} \left( 1 - \frac{c_j x}{s} \right) \leq e^{-x \sum_{j=0, j \neq i}^{K-1} c_j / x} \leq e^{-x+x/s}, \quad i = 0, 1, \ldots, K-1,
\]

and therefore

\[
\frac{1}{s} \sum_{i=0}^{K-1} c_i \prod_{j=0, j \neq i}^{K-1} \left( 1 - \frac{c_j x}{s} \right) \leq \frac{1}{s} \sum_{i=0}^{K-1} c_i e^{-x+x/s} \xrightarrow{a.s. K \to \infty} e^{-x}. \] (20)
On the other hand, as \( t \to 0 \) we have
\[
e^{-(t + t^2)} = 1 - (t + t^2) + \frac{(t + t^2)^2}{2} + O(t^3) = 1 - t - \frac{t^2}{2} + O(t^3),
\]
so that for all \( t \) in some sufficiently small neighborhood of 0
\[
e^{-(t + t^2)} \leq 1 - t.
\]
Consequently:
\[
\prod_{j=0}^{K-1} \left(1 - \frac{c_j x}{s}\right) \geq e^{-x \sum_{j=0}^{K-1} \frac{c_j}{i} + x^2 \sum_{j=0}^{K-1} \frac{c_j^2}{s^2}} \geq e^{-x - Kx^2/s^2}. \tag{21}
\]
Obviously, with probability 1, \( s \) grows linearly with \( K \), namely for all sufficiently large \( K \) we have \( s \geq aK \) for a suitably chosen \( a > 0 \). (In fact, any \( a < \frac{1}{2} \) will do.) By (21):
\[
\prod_{j=0}^{K-1} \left(1 - \frac{c_j x}{s}\right) \geq e^{-x - Kx^2/s^2} \stackrel{a.s., K \to \infty}{\longrightarrow} e^{-x}. \tag{22}
\]
From (21) and (22) it follows that
\[
\frac{1}{s} \sum_{i=0}^{K-1} c_i \prod_{j=0}^{K-1} \left(1 - \frac{c_j x}{s}\right) \stackrel{a.s., K \to \infty}{\longrightarrow} e^{-x}, \tag{23}
\]
which completes the proof.

A plot of \( P(\chi) \) is given in figure 1 and shows indeed that it has a maximum at \( \chi = 0 \) and a delta function component at the inverse of the largest capacitance. In fact, when \( K \to \infty \), the weight of the delta function shrinks to zero and \( P(\chi) \) approaches the Poisson distribution \( e^{-\chi} \).

II. FLUCTUATION OF COULOMB BLOCKADE PEAK SPACINGS IN LARGE SEMICONDUCTOR QUANTUM DOTS

A. Motivation

Recently, it became apparent that the physics exposed in the addition spectra of quantum dots is rather rich, and hence its investigation is at the focus of both experimental and theoretical studies. The present section concentrates on the distribution of spacings between
Coulomb blockade peaks in large semiconductor quantum dots. Coulomb blockade is evidently one of the hallmarks of mesoscopic physics. The experimental achievement of tracing an addition of a single electron to a quantum dot and the appearance of isolated conductance peaks led to the concept of single electron transistors. After the origin of Coulomb blockade peaks has been elucidated, investigation is directed toward more subtle questions like their heights, widths and spacings. The underlying physics is related to the ground state energy, chemical potential and inverse compressibility of a few electron island coupled capacitively to its environment, as well as fluctuations of these quantities with the number $N$ of electrons on the dot.

As far as the distribution of spacings between adjacent Coulomb blockade peaks is concerned, the question can be stated as follows: According to the simplest picture in which the quantum dot is regarded as a single electron island whose coupling with the leads is through its capacitance $C$, the total potential energy of a quantum dot is $Q^2/2C - V_gQ$ where $V_g$ is the corresponding gate voltage. The conductance peaks occur at those values of $V_g$ for which $CV_g = e(N+1/2)$ where $e$ is the electron charge (henceforth $e = -1$). For this value of $V_g$ the addition of an electron to the dot (which contains $N$ electrons) does not cost any charging energy. The position of the $N^{th}$ Coulomb blockade peak is then a linear function of $N$ and therefore, the spacing should be a constant $1/C$, independent of $N$. Recent experiments [5,13] indicate however that spacing between Coulomb blockade peaks in small quantum dots is in general not constant but, rather, a fluctuating quantity close to Gaussian. The average of its distribution approximately coincides with the constant value mentioned above, but the elucidation of its standard deviation is still under investigation [14,15].

The situation is even less clear if the quantum dot is very large. As indicated in a series of recent experiments, the spacing occasionally vanishes, namely, two peaks (and sometimes even three peaks) coincide. Moreover, the evolution of peak positions and spacings with an applied magnetic field indicates the existence of strong correlations between them [16,18]. These observation motivated numerous theoretical models based on the concept of pair tunneling [19] or that of two-electron bound-states in depleted electron islands [20].

In the present section we examine the scenario according to which a large quantum dot like the one used in the last experiment [18], is in fact, composed of several electron islands which are coupled capacitively among themselves as well as to the leads. (This is a natural extension of the scheme discussed in the first section in which no coupling is present). Electrons are added in such a way that the total potential energy of the dot is minimum. This simple generalization of the single island picture leads to a remarkable change in the spacing distribution from a Gaussian [5,13] centered around a finite average to a one which is large at small spacings. When the coupling between islands is weak, the distribution has
indeed a maximum at zero spacing. This result is short of explaining the perfect overlap of peaks, since it requires a delta function component at zero spacing. Yet, it leads to the occurrence of couples and sometimes triples of closely spaced peaks, similar to the experimental observation. Moreover, the evolution of the peak positions with the magnetic field is qualitatively similar to the experimental one. On the other hand, the present model does not predict a definite periodicity in the bunching of Coulomb blockade peaks with electron number $N$. In the next subsection the model is explained and the results of calculations are presented in the third subsection.

III. FORMALISM

Consider a large isolated two dimensional quantum dot in a perpendicular magnetic field $B$ subject to a gate voltage $V_g$. Unlike the traditional Coulomb blockade picture it might contain several electron islands which can be regarded as metallic objects with inductive couplings among themselves. These are determined by a positive definite symmetric matrix $C$ whose diagonal elements $C_{ii} \equiv C_i > 0$ are the corresponding capacities whereas the non-diagonal elements $C_{ij} = C_{ji} < 0$, $(i \neq j)$ are the corresponding coefficients of induction. The electrostatic energy of such a system can be written as,

$$E_c = \frac{1}{2} \sum_{i,j=1}^{K} p_{ij} N_i N_j - V_g N,$$  \hspace{1cm} (24)

where $N_i$ is the number of electrons on island $i$ (the number of islands $K > 1$ might be around 10), $N = \sum_{i=1}^{K} N_i$ is the total numbers of electrons, and the (symmetric positive-definite) matrix $p \equiv C^{-1}$.

Beside the electrostatic energy it is assumed that electrons in each island occupy single particle quantum states (orbitals) whose energies $\epsilon_{i\alpha}$ ($i = 1, 2, ..K; \alpha = 1, 2, ..$) depend on the confining potential as well as on the magnetic field. The latter is manifested through its orbital effects as well as due to Zeeman splitting (in which case the quantum number $\alpha$ contains also a spin label). The corresponding occupation numbers $n_{i\alpha}$ can be either 0 or 1. The system described above might then be represented by a classical Hamiltonian

$$H = H_c + H_{sp},$$  \hspace{1cm} (25)

where the charging Hamiltonian $H_c$ is just the electrostatic energy \[24\] written in terms of the orbital occupation numbers,

$$H_c = \frac{1}{2} \sum_{i,j=1}^{K} p_{ij} \left( \sum_{\alpha} n_{i\alpha} \right) \left( \sum_{\alpha'} n_{j\alpha'} \right) - V_g \sum_{i=1}^{K} \sum_{\alpha} n_{i\alpha},$$  \hspace{1cm} (26)
and the single particle part of the Hamiltonian, $H_{sp}$ is,

$$H_{sp} = \sum_{i=1}^{K} \sum_{\alpha} \epsilon_{i\alpha} n_{i\alpha}. \quad (27)$$

The precise form of the matrix elements $p_{ij} = [C^{-1}]_{ij}$ as well as the single particle energies $\epsilon_{i\alpha}$ are specified in the next section when we present our results. Despite the fact that the Hamiltonian $H = H_c + H_{sp}$ is classical (and relatively simple) the elucidation of its spectrum for large $N$ and $K$ is virtually hopeless. In order to compute the ground state energy $E(N)$ one has to find the minimum of $H$ on all the possible sets ($N_1, N_2...N_K$) with the constraint $\sum_{i=1}^{K} N_i = N$. Note that the so called “Coulomb Glass” model obtains as a special case when $i$ refers to a lattice site with random energy $\epsilon_{i\alpha} = \epsilon_i$, and a single orbital $N_i = n_i = 0, 1$.

The interaction matrix is then given by $p_{ij} = 1/r_{ij}$ for $i \neq j$ and $p_{ii} = 0$ where $r_{ij}$ is the distance between sites $i$ and $j$.

The position of the $N^{th}$ conductance peak is given by the first difference of the ground state energy, namely, the chemical potential of the (isolated) dot, $\mu \equiv E(N + 1) - E(N)$. The spacing between peaks is determined by the second difference (the inverse compressibility), defined already in equation 1.

The occurrence of close peaks for certain values of electron number $N$ correspond to small values of $\chi_N$ (recall that for a single island quantum dot in which the single particle energies are neglected, the inverse compressibility is a constant $1/C_{11}$). For the more general model described above the spacing distribution will of course fluctuate. In general, some constants appearing in the Hamiltonian $H = H_c + H_{sp}$ are random, (e.g. the elements of the matrix $C$ and the single particle energies $\epsilon_{i\alpha}$), but most experiments are performed on a single quantum dot, so that fluctuations are meant with respect to the electron number $N$. The numbers $\chi(N)$ might then be considered as values assumed by a random variable $\chi$ which has a certain distribution function $P(\chi)$.

IV. RESULTS

We now return to the full Hamiltonian of Eq. (25). It contains the coefficients of capacitance and induction matrix $C_{ij}$ and the single particle energies $\epsilon_{i\alpha}$ as input. In choosing the actual numerical values we use a few guidelines, one of them is to avoid too many independent input data. As will be clear below, these are not fitting parameters but rather, a set of constants which are chosen once for all on general physical grounds. First, for the electrostatic part, recall that the matrix $C$ should be a symmetric positive definite matrix with $C_{ii} > 0$ and $C_{ij} < 0$ for $i \neq j$. Besides, we expect it to be random. We then assume that $C_{ii}$
are random numbers uniformly distributed between 0 and $W$ whereas the non-diagonal elements are uniformly distributed between $-w$ and 0. Evidently, in choosing these constants, one has to keep $w \ll W$ in order to maintain positive definiteness. Actually, it is only the ratio $w/W$ which matters, so one can assume $W = 1$ and use $e^2/W = 1$ as an energy unit, leaving $w$ as a constant reflecting the strength of coupling between islands. Second, for the single particle energies, we consider each electron island $i$ as a two-dimensional potential well $V_i(r) = \frac{1}{2}M\omega_i^2 r^2$, where $M$ is an effective mass. One may then regard $\omega_i^{-1/2}$ as a measure of the radius of the corresponding electron island. Since the capacitance $C_{ii}$ is also proportional to this radius we assume $\omega_i = \gamma C_{ii}^{-2}$, where $\gamma$ is a constant reflecting the relation between charging energies and single particle energies. The single particle energies in each electron island (subject to a perpendicular magnetic field $B$) are then known analytically.

To be more specific recall that in two dimensions there are two quantum numbers for the orbital motion (denoted hereafter as $n, m$) to which we add a spin index $\sigma = \pm 1$. Then, with $\alpha = (n, m, \sigma)$, we have
\[
\epsilon_{i\alpha} = \frac{\hbar}{2}[n\omega_c + m\sqrt{\omega_i^2 + \frac{1}{4}\omega_c^2}] + g\mu_B\sigma B, \tag{28}
\]
where $\omega_c = eB/Mc$ is the cyclotron frequency, $g$ is the g-factor and $\mu_B$ is the Bohr magneton. Since $g$ contains the effective mass it is not known accurately. Its value is constrained on physical grounds (see below). Note that in this scheme, the spacings between single particle energies are deterministic, and do not follow the Wigner surmise. The main cause of fluctuation is then due to the combination of non-random single particle energies and the occurrence of numerous charging energies. Finally, the number of electron islands $K$ is determined by the size of the quantum dot. The four input data of the model are then $K, w, \gamma$ and $g$. Note that the gate voltage $V_g$ does not have an important role here. Indeed, in actual experiments the variation of gate voltage serves to adjust the energies $E(N)$ with the chemical potential of the leads but here the ground state energies are calculated directly.

In order to avoid redundancy we are content with having a single set of these constants which is physically reasonable. In particular, it assures that the charging energy is much larger than single particle level spacings and that the Zeeman splitting is small at moderate magnetic fields. Specifically, we take $w = 0.03$, $\gamma = 0.1$ and $g$ is chosen such that for moderate fields the Zeeman splitting is of the order of the mean level spacing.

With these prescriptions, we can find the ground state energy as a function of the magnetic field $B$ for each electron number $N$. We use just a brute force trial algorithm, and hence cannot treat systems with large number of electron islands. An intriguing question is whether the addition of an electron leads to a redistribution of all electrons among the islands or else, the newly added electron will chose a place such that the energy cost is minimal while all
others remain intact. Our results indicate that in most cases there is no redistribution, but in some cases redistribution does occur although it does not involve an overall re-shuffling. Consider as a representative example a quantum dot with $K = 5$ electron islands. The ground state energies $E(N)$ and the inverse compressibilities $\chi(N)$ (in the absence of a magnetic field) are calculated for electron number $N$ up to 200. The first question we addressed is how the electrons are added among the islands. Figure 2 shows electron numbers $N_i$ in each one of the five islands, $v.s$ the total electron number $N$, in the range $60 < N < 80$. Evidently, the order of curves is according to the value of the capacitance $C_{ii}$. On a larger scale, the numbers $N_i$ grow linearly with $N$ as it should be. Let us then consider the question of redistribution. From figure 2 we see that redistribution occurs only once, as $N$ grows between 67 and 68 (see the vertical line). The addition of electron to the second (or third) island involves also a transfer of an electron from the fourth island to the second (or the third) one. This scenario occurs also in other ranges of $N$ with the same proportion (namely about four percents). Thus, within the present model, redistribution is present although it is rare and minimal.

The next question is related to the fluctuation of the inverse compressibility $\chi(N)$. Figure 3 displays $\chi(N)$ $v.s$ $N$ for $40 < N < 80$. From this figure one notices that the inverse compressibility fluctuates quite strongly. In particular, it becomes small, (compared with its average) although it does not vanish. Note also that there is, at first glance, no traces of any periodic structure. This is also verified on applying its Fourier transform. The distribution $P(\chi)$ is drawn in figure 4. It is indeed very different from the constant $P(\chi) = \delta(\chi - 1/C)$ appropriate for a single island quantum dot with capacitive coupling $C$. A trace of this latter characteristic is the peak near $\chi = 1$. The eventual decrease of the distribution near $\chi = 0$ seems to be related to the fact that, unlike the case of independent systems, the capacitance matrix is non-diagonal. Switching on the coupling here then has an effect similar to a weak “spacing repulsion”, similar to the familiar effect of perturbation in two level systems. Recall, however, that beside the capacitance induction matrix $C$ the total energy is determined also by the single particle energies in each island. The remarkable point is that the distribution is not Gaussian (which is the hallmark of spacing distributions in small quantum dots).

Finally, we check the behavior of the first difference $E(N + 1) - E(N)$ as function of the magnetic field. Recall that this quantity is proportional to the position of the $N^{th}$ Coulomb blockade peak. As a measure of the strength of the magnetic field we use the parameter $\omega_c/\omega_0$ where $\omega_0$ is the harmonic oscillator frequency of the largest island. The positions of the peaks for $39 < N < 48$ are displayed in figure 5. Comparison with the results displayed in figure 2 of Ref. \cite{18} reveals a large degree of qualitative agreement. In particular, it
shows that groups of two (and sometimes even three) electrons can tunnel through the quantum dot at almost the same gate voltage. The oscillations at small magnetic field just mark transitions to lower Landau levels as the magnetic field increases. The phenomena of alternate bunching \((N, N + 1) \rightarrow (N - 1, N)\) is also reproduced in the present picture.

In conclusion, we suggest a classical model in which a large semiconductor quantum dot is viewed as a collection of metallic electron islands with capacitive and inductive coupling among them. The effect of magnetic field is manifested through its orbital as well as its spin effects. The model can explain the occasional occurrence of couples or even triples of closely spaced Coulomb blockade peaks, as well as the qualitative behavior of peak positions with the magnetic field.

ACKNOWLEDGMENTS

his work was supported in part by grants from the Israel Academy of Science and Humanities under the program *Centers for Excellence*, by the Basic Research Foundation and the BSF – Binational Israel-US Foundation. One of us (Y. A) is grateful to R. Ashoori and D. V. Averin for discussion and suggestions.
[1] E. P. Wigner, Ann. Math. 53, 36, (1951); 62, 548 (1955); 65, 203 (1957); 67, 325 (1958).

[2] F. J. Dyson, Jour. Math. Phys. 3, 140 (1962); 3, 157 (1962); 3, 166 (1962).

[3] O. Bohigas, M.J. Giannoni and C. Schmit, Phys. Rev. Lett. 52, 1 (1984).

[4] B. L. Al'tshuler and B. I. Shklovskii, Zh. Eksp. Teor. Fiz. 91, 220 (1986). [Sov. Phys. JETP 64, 127 (1986)].

[5] U. Sivan, R. Berkovits, Y. Aloni, O. Prus, A. Auerbach and G. Ben Yoseph, Phys. Rev. Lett. 77, 1123 (1996); F. Simmel, T. Heinzell, and D. A. Wharam, Europhys. Lett. 38, 123 (1997); S. R. Patel, S. M. Cronenwel, P. R. Stewart, A. G. Huiberg, C. M. Marcus, C. I. Durooz, J. S. Harris, K. C. Kampman and A. C. Gossard, Phys. Rev. Lett. 80, 4522 (1998).

[6] N.B. Zhitenev, R.C. Ashoori, L.N. Pfeiffer and K.W. West, Phys. Rev. Lett. 79 (1997), 2308.

[7] M.V. Berry and M. Tabor, Proc. Roy. Soc. London 356, 375 (1977).  

[8] W. Feller, An Introduction to Probability Theory and Its Applications, Vol. II, 2nd ed., Wiley, New York, 1971.

[9] H. Furstenberg, Recurrence in Ergodic Theory and Combinatorial Number Theory, Princeton University Press, Princeton, New Jersey, 1981.

[10] L. Kuipers and H. Niederreiter, Uniform Distribution of Sequences, Wiley, New York, 1974.

[11] P. Kurlberg and Z. Rudnick, The distribution of spacings between quadratic residues, preprint.

[12] Z. Rudnick and A. Zaharescu, The distribution of spacings between small powers of a primitive root, preprint.

[13] L. P. Kouenhoven, T. H. Osterkamp, M. W. S. Danoesastro, M. Eto, D. G. Austing, T. Honda and S. Tarucha, Science, 278, 1788 (1997).

20
[14] R. Berkovits and B. L. Altshuler, Phys. Rev. B55, 5297 (1997); Ya. M. Blanter, A. D. Mirlin and B. A. Muzikantskii, Phys. Rev. Lett. 78, 2449 (1997); A. A. Koulakov, F. G. Pikus and B. I. Shklovskii, Phys. Rev. 54, 9223 (1997).

[15] D. Orgad and S. Levit, unpublished.

[16] R. C. Ashoori et al., Phys. Rev. Lett. 68, 3088 (1992).

[17] R. C. Ashoori et al., Physica (Amsterdam) 189B, 117 (1993).

[18] N.B. Zhitenev, R.C. Ashoori, L.N. Pfeiffer and K.W. West, Phys. Rev. Lett. 79(1997), 2308.

[19] Y. Wan, G. Ortiz and P. Phillips, Phys. Rev. Lett. 75, 2879 (1995); Phys. Rev. B 55, 5313 (1997); Phys. Rev. Lett. 78(C), 3979 (1998).

[20] M. E. Raikh, L. I. Glazman and L. E. Zhukov, Phys. Rev. Lett. 77, 1354 (1996); Phys. Rev. Lett. 78(C) 3980 (1998).

[21] M. V. Berry and M. Tabor, Proc. Roy. Soc. London 356 375 (1977)

[22] Y. Avishai, D. Berend and R. Berkovits, Journal of Physics A (Math. Gen.), in press (1998).
FIG. 1. Distribution of inverse compressibility $P(\chi)$ for a system containing independent sub-systems each of which has a ground state energy proportional to the square of the number of particles it contains. The graph corresponds to a special case of equation ?? with $K = 5$.

FIG. 2. Distribution of electrons among the five islands as function of the total electron number $N$. The constants $K$, $w$, $\gamma$ and $g$ are as explained in the text. The number $N_i$ of electrons on island $i$ is commensurate with its capacitance $C_{ii}$. In most cases the addition of an electron does not perturb the occupation of other islands. An example of redistribution is marked with a vertical line. An addition of an electron causes a minimal redistribution.

FIG. 3. Inverse compressibility $\chi_N$ as function of electron number $N$ in the dot. The units on the ordinate are the charging energies of the largest island, namely, $e^2/W = 1$ (see text).

FIG. 4. Distribution $P(\chi)$ of level spacings in the dot at zero magnetic field (not normalized). Note the peak at small values of $\chi$.

FIG. 5. Coulomb blockade peak position for electron number $N$ between 39 and 48 as function of the magnetic field. Here $\omega_o$ is the oscillator frequency of the largest island and $\omega_c$ is the cyclotron frequency. The units on the ordinate are energy units as explained previously. They are proportional to the gate voltage appropriate for the corresponding peak.
