Scaling in the Neutrino Mass Matrix

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Abstract

In an attempt to uncover any underlying structure in the neutrino mass matrix, we discuss the possibility that the ratios of elements of its Majorana mass matrix are equal. We call this “strong scaling Ansatz” for neutrino masses and study its phenomenological implications. Of three possible independent scale invariant possibilities, only one is allowed by current data, predicting in a novel way the vanishing of $U_{e3}$ and an inverted hierarchy with the lightest neutrino having zero mass. The Ansatz has the additional virtue that it is not affected by renormalization running. We also discuss explicit models in which the scaling Ansatz is realized.

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1 Introduction

Understanding the new physics behind the neutrino observations is one of the main challenges for theory [1] today. While it is widely expected that the extreme smallness of the neutrino masses is most likely related to the seesaw mechanism [2], i.e., the existence of a $B - L$ symmetry broken at very high scale, there is no common consensus on how to understand the detailed mixing pattern. Observation implies a particular structure in the neutrino mass matrix

$$
\begin{pmatrix}
    m_{ee} & m_{e\mu} & m_{e\tau} \\
    m_{e\mu} & m_{\mu\mu} & m_{\mu\tau} \\
    m_{e\tau} & m_{\mu\tau} & m_{\tau\tau}
\end{pmatrix}
\equiv M_\nu = U \mathcal{M}_\nu^{\text{diag}} U^T,
$$

whose diagonalization in the charged lepton mass basis gives the PMNS matrix $U$. There are several distinct approaches:

(i) symmetry approach, where one postulates family symmetries to constrain the mass matrix elements and studies their consequence;

(ii) anarchy approach, where all elements of the neutrino mass matrix $M_\nu$ are allowed to vary at random and the most probable values are then confronted with experiments and

(iii) top-down GUT approach, where one considers a quark-lepton unified grand unified framework such as $SO(10)$ and obtains predictions for neutrino parameters.

In this paper, we consider a different approach which is located somewhere between the anarchy and the symmetry approaches. We consider the Ansatz that the elements of the neutrino mass matrix, in the basis in which the charged lepton mass matrix is diagonal, obey scaling laws. According to this law, ratios of certain elements of $M_{ij}$ are same. We will call this the “strong scaling Ansatz” (SSA) on neutrino masses. An important property of the SSA is that it is not affected by renormalization group extrapolation as we run it down from the seesaw scale to the weak scale. This property is not shared by many other Ansätze used in literature such as $\mu - \tau$ or other symmetries, texture zeros, etc.

The first question to ask now is which matrix elements should one impose the scaling Ansatz to? If we consider scaling for a given matrix element as $m_{\alpha\beta} \simeq m_0 r^{x_\alpha + x_\beta}$, where $m_0$ is flavor independent, then this implies three kinds of strong scaling Ansätze of the form that $\frac{m_{\alpha\beta}}{m_{\alpha\gamma}}$ is independent of the flavor $\alpha_i$. In terms of matrix elements, this means

(A) : \[ \frac{m_{ee}}{m_{e\tau}} = \frac{m_{e\mu}}{m_{\mu\tau}} = \frac{m_{e\tau}}{m_{\tau\tau}} \equiv c. \] (2)

and cyclic variations of these relations, i.e.,

(B) : \[ \frac{m_{ee}}{m_{e\tau}} = \frac{m_{\mu e}}{m_{\mu\tau}} = \frac{m_{\tau e}}{m_{\tau\tau}} \equiv c'. \] (3)
and
\[ (C) : \frac{m_{ee}}{m_{e\mu}} = \frac{m_{\mu e}}{m_{\mu\mu}} = \frac{m_{\tau e}}{m_{\tau\mu}} \equiv c'' . \] (4)

These are the only three independent strong scaling Ansätze for the 3 × 3 case. In the next Section 2, we will consider the phenomenological implications of these strong scaling constraints, showing in particular that only case (A) is allowed by current data.

Clearly, we do not know the fundamental origin of the SSA, but if one looks at the hierarchy of fermion masses, such flavor dependent power laws for masses could easily exist and we only need to make some assumptions about the coefficients. Alternatively, SSA could arise from seesaw physics, i.e., from the structure of the Dirac and heavy Majorana neutrino mass matrices. In any case, we will discuss theoretical scenarios where SSA can emerge in Section 3.

2 Phenomenology of the Strong Scaling Ansatz

In order to explore the implications of SSA, we start by writing down the usual form of the PMNS matrix:
\[ U = \begin{pmatrix} c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta} \\ -c_{23} s_{12} - s_{23} s_{13} c_{12} e^{i\delta} & c_{23} c_{12} - s_{23} s_{13} s_{12} e^{i\delta} & s_{23} c_{13} \\ s_{23} s_{12} - c_{23} s_{13} c_{12} e^{i\delta} & -s_{23} c_{12} - c_{23} s_{13} s_{12} e^{i\delta} & c_{23} c_{13} \end{pmatrix} P , \] (5)

where \( P = \text{diag}(1, e^{i\alpha}, e^{i(\beta + \delta)}) \) contains the Majorana phases. Let us discuss the phenomenology of SSA, starting with case (A) defined in Eq. (2):
\[ \mathcal{M}_\nu = m_0 \begin{pmatrix} A & B & B/c \\ B/c & D/c & D/c^2 \\ B & D & D/c \end{pmatrix} . \] (6)

To discuss the phenomenology of the model, we first note that the mass matrix in Eq. (6) has rank 2 and therefore predicts one vanishing neutrino mass. The eigenvector of Eq. (6) corresponding to the zero eigenvalue is given by \((0, -1/\sqrt{1 + \vert c \vert^2}, c/\sqrt{1 + \vert c \vert^2})^T\). Therefore, the strong scaling condition is only compatible with an inverted mass hierarchy, \( U_{e3} = 0 \) and hence no CP violation in oscillation experiments. Thus, SSA provides another way to understand the vanishing of \( U_{e3} \) and one can use small deviations from strong scaling as a way to understand any possible non-zero \( U_{e3} \) value, see Section 2.3 for more.

Atmospheric neutrino mixing is described by \( \tan^2 \theta_{23} = |1/c|^2 \), and data requires that \( |c| \) is close, but not necessarily equal, to 1. Let us assume for the moment that the parameters are all real. The non-zero masses read
\[ m_{2,1} = \frac{m_0}{2c^2} \left( D + c^2(A + D) \pm w \right), \text{ where } w = \sqrt{4B^2c^2(1+c^2)+(D+c^2(D-A))^2} . \] (7)
The solar neutrino mixing angle is given by
\[
\sin^2 \theta_{12} = \left(1 + \frac{4 B^2 c^2 (1 + c^2)}{(A c^2 - (1 + c^2) D + w)^2}\right)^{-1}, \quad \text{or} \quad \tan 2 \theta_{12} = \frac{2 B \sqrt{1 + c^2}}{(c + 1/c) D - A},
\] (8)
and the effective mass governing neutrinoless double beta decay is \(m_0 |A|\). To have the ratio \(\Delta m^2_\odot / \Delta m^2_A = (m^2_2 - m^2_1) / m^2_2 \simeq 1/25\), typically \(D + c^2 (A + D) \ll w\), or \(A \simeq -2D\), must hold. This is a tuning of parameters shared by many other Ansätze for neutrino mass models. In this case (with \(c = +1\) one needs negative \(A\)) \(\sin^2 \theta_{12} \simeq \frac{1}{2} (1 + A / \sqrt{A^2 + 2 B^2})\) and therefore \(B^2 > A^2\) in order to have \(\sin^2 \theta_{12} < \frac{1}{2}\). This pattern of \(M_\nu\) corresponds to the approximate conservation of the flavor symmetry \(L_e - L_\mu - L_\tau\) [3], and one has \(\langle m \rangle \simeq \sqrt{\Delta m^2_A} \cos 2 \theta_{12} = \sqrt{\Delta m^2_A} (1 - 2 \sin^2 \theta_{12})\). There is also a second, somewhat tuned possibility for the parameters \(A, B, D\), namely \(A^2, D^2 > B^2\), which corresponds to \(\langle m \rangle \simeq \sqrt{\Delta m^2_A}\).

One might compare SSA matrix to the case of exact \(\mu-\tau\) symmetry [4], which also predicts \(U_{e3} = 0\) and in addition \(\cos 2 \theta_{23} = 0\). A mass matrix obeying \(\mu-\tau\) symmetry reads
\[
M_\nu = m_0 \begin{pmatrix}
A & B & B \\
B & D & E \\
B & E & D
\end{pmatrix}.
\] (9)

Hence, \(\mu-\tau\) symmetry with the additional constraint \(E = D\) reproduces SSA when in addition \(|c| = 1\) holds. Breaking of \(\mu-\tau\) symmetry will in general lead to non-zero \(U_{e3}\) and \(\cos 2 \theta_{23}\), together with a correlation of these parameters which depends on the way the symmetry is broken [5]. SSA in turn has in general non-zero \(\cos 2 \theta_{23}\) and breaking of SSA (treated in Section 2.3) will generate non-zero \(U_{e3}\) not directly linked to the deviation from zero \(\cos 2 \theta_{23}\). Thus a key test that distinguishes this model from an approximate \(\mu-\tau\) symmetric model is that here we can have departure from the maximal atmospheric mixing angle even though \(U_{e3} = 0\), whereas in the case of generic breaking of \(\mu-\tau\) symmetry, there is a strong correlation between \(\theta_{23} - \frac{\pi}{4}\) and \(U_{e3}\).

Another comparison should be done with the flavor symmetry \(L_e - L_\mu - L_\tau\), which is the usual symmetry to enforce an inverted hierarchy [3]. The mass matrix reads
\[
M_\nu = m_0 \begin{pmatrix}
0 & A & B \\
A & 0 & 0 \\
B & 0 & 0
\end{pmatrix},
\] (10)
and predicts vanishing \(\Delta m^2_\odot\). In analogy to SSA, \(m_3\) and \(U_{e3}\) are zero and atmospheric neutrino mixing is non-maximal: \(\tan^2 \theta_{23} = |A/B|\). However, solar neutrino mixing is maximal, and one requires large and tuned breaking of the symmetry in order to obtain a sufficiently non-maximal \(\theta_{12}\) [6]. The breaking parameters have to have at least 40% the size of the parameters allowed by the symmetry. In the case of SSA one has naturally large
but non-maximal solar neutrino mixing, as given in Eq. \((8)\), and therefore does not face
the problems of \(L_e - L_\mu - L_\tau\).

It is interesting to elaborate on renormalization aspects of the strong scaling condition. Presumably, SSA will be a property introduced at the scale \(M_X \lesssim M_{\text{GUT}} \approx 2 \cdot 10^{16} \text{ GeV}\) of neutrino mass generation. Its predictions have to be compared to the measurements performed at low scale \(m_Z\). It is well-known that the effect of running from high to low scale can be taken into account by multiplying the \(\alpha \beta\) entry of \(M_\nu\) with \((1 + \delta_\alpha)(1 + \delta_\beta)\), where \(\delta_\alpha = C \frac{m_\alpha^2}{16 \pi^2 v^2} \ln \frac{M_X}{m_Z}\), with \(m_\alpha\) being the charged lepton mass \([7]\). The parameter \(C\) is given by \(3/2\) in the SM and by \(-(1 + \tan^2 \beta)\) in the MSSM. Obviously, with \(\delta_\tau \gg \delta_{e,\mu}\), this leads in the strong scaling condition \([2]\) only to a rescaling of \(c\) with \((1 + \delta_\tau)\), i.e., the three ratios of the mass matrix elements still have the same value, which is however slightly changed. The predictions \(m_3 = 0\) and \(U_{e3} = 0\) are therefore invariant under the renormalization running. The other predictions of SSA are affected: for instance, if SSA at high scale imprints the ratio \(c\) and therefore predicts \(\tan^2 \theta_{23} = 1/c^2\), then at low scale we have \(\tan^2 \theta_{23} = (1 - 2 \delta_\tau)/c^2\). Note that \(\mu - \tau\) symmetry and \(L_e - L_\mu - L_\tau\) are in general affected by renormalization running, hence one generically expects \(U_{e3} \neq 0\) for such models.

### 2.1 CP violating case

The fact that \(U_{e3} = 0\) means that there is no CP violation in oscillation experiments. This can also be seen by working with invariants: a useful measure of CP violation in neutrino oscillation is given by \([8]\)

\[
J_{CP} = \frac{1}{8} \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \cos \theta_{13} \sin \delta = \frac{\text{Im}\{h_{12} h_{23} h_{31}\}}{\Delta m^2_{21} \Delta m^2_{31} \Delta m^2_{32}},
\]

where \(h = \mathcal{M}_\nu \mathcal{M}^\dagger_\nu\).

From Eq. \((6)\) it follows that \(h_{13} = h_{12}/c^*\) and \(h_{23} = c h_{33}\). Since \(h\) is hermitian, one has

\[
\text{Im}\{h_{12} h_{23} h_{31}\} = \text{Im}\{h_{12} h_{23} h_{13}^*\} = \text{Im}\{c^* h_{13} c h_{33} h_{13}^*\} \text{Im}\{|c|^2 |h_{13}|^2 h_{33}\} = 0.
\]

From \(h_{13} = h_{12}/c^*\) and \(h_{23} = c h_{33}\) it also follows that \(h_{12}/h_{13} - h_{32}/h_{33} = 0\). Using the parametrization of the PMNS matrix from Eq. \((5)\), this condition leads for an inverted mass ordering to \([9]\)

\[
|U_{e3}| = \frac{\frac{1}{2} \Delta m^2_{32}}{m^2_3 + \Delta m^2_{31} + \Delta m^2_{32}} \sin 2\theta_{12} \cot \theta_{23} \csc^2 \theta_{12} \cot \theta_{23}.
\]

Since we know that \(m_3 = 0\), it automatically follows again that \(|U_{e3}| = 0\) (and vice versa). Suppose now that all parameters in Eq. \((11)\) are complex, i.e., \(A \rightarrow A e^{i\phi_1}, B \rightarrow B e^{i\phi_2}, D \rightarrow D e^{i\phi_3}\) and \(c \rightarrow c e^{i\phi_4}\), where the redefined \(A, B, D, c\) are real. It is easy to show that...
one can rephase the lepton fields in a way such that only the $e\mu$ and $e\tau$ entries are left with a phase $\phi = \phi_2 - \phi_1/2 - \phi_3/2$:

$$M_\nu = m_0 \begin{pmatrix} A & Be^{i\phi} & B/c e^{i\phi} \\ B e^{i\phi} & D & D/c \\ B/c e^{i\phi} & D/c & D/c^2 \end{pmatrix}. \tag{14}$$

There are thus five relevant parameters $A, B, D, c, \phi$ for the observables $\theta_{12}, \theta_{23}, \Delta m^2_\odot, \Delta m^2_\Lambda$ and $\langle m \rangle$. The Ansatz can therefore completely be reconstructed, in contrast to many other Ansätze. A $23$ rotation with $\tan \theta_{23} = 1/c$ gives

$$M_\nu = m_0 \begin{pmatrix} A & B \sqrt{1 + 1/c^2} e^{i\phi} & 0 \\ B \sqrt{1 + 1/c^2} e^{i\phi} & D (1 + 1/c^2) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{15}$$

We note here that the SSA prediction of $m_3 = 0$ indicates that one of the Majorana phases, in this case $\beta$, is absent. The only observable phase is therefore the Majorana phase $\alpha$. With $A, B, D$ typically of the same order of magnitude, it is rather cumbersome to rephase this simple looking matrix in order to identify the sole surviving low energy phase $\alpha$ and also the two non-vanishing masses. Again, invariants are helpful: the in general three low energy phases $\alpha, \beta, \delta$ correspond to three independent invariants, which can be chosen to be $^{[10]}$

$$I_{\alpha\beta} = \text{Im} \left\{ m_{\alpha\alpha} m_{\beta\beta}^* m_{\alpha\beta}^* m_{\beta\alpha}^* \right\}, \text{ where } (\alpha, \beta) = (e, \mu) \text{ or } (e, \tau) \text{ or } (\mu, \tau).$$

In general, these expressions are rather lengthy, but for $m_3 = 0$ and $U_{e3} = 0$ they simplify to $I_{\mu\tau} = 0$ and $I_{e\mu} = -m_1 m_2 \Delta m^2_\odot c_{12}^4 s_{12}^4 c_{23}^4 \sin 2\alpha$ and $I_{e\tau} = -m_1 m_2 \Delta m^2_\odot c_{12}^4 s_{12}^4 s_{23}^4 \sin 2\alpha$. The single physical low energy phase $\alpha$ corresponds to only one independent invariant. With the rephased mass matrix Eq. (14) we have $I_{\mu\tau} = 0$, $I_{e\mu} = -m_0^3 A B^2 D \sin 2\phi$ and $I_{e\tau} = -m_0^3 A B^2 D c^{-4} \sin 2\phi$. By calculating the ratio $I_{e\mu}/I_{e\tau}$ it follows again that $c^4 = \cot^4 \theta_{23}$. Moreover, we can write

$$m_1 m_2 \Delta m^2_\odot c_{12}^4 s_{12}^4 c_{23}^4 \sin 2\alpha \simeq \Delta m^2_\Lambda \Delta m^2_\odot c_{12}^4 s_{12}^4 c_{23}^4 \sin 2\alpha = m_0^3 A B^2 D \sin 2\phi = \langle m \rangle m_0^2 B^2 D \sin 2\phi, \tag{16}$$

which shows that $\phi$ is closely related to the low energy Majorana phase $\alpha$. This phase $\alpha$ shows up in the effective mass governing neutrinoless double beta decay, which in our case is just $^{[11]}$

$$\langle m \rangle \simeq \sqrt{\Delta m^2_\Lambda} \sqrt{1 - \sin^2 2\theta_{12} \sin^2 \alpha}. \tag{17}$$

We show in Fig. 1 a scatter plot of $\sin^2 \theta_{12}$ against the effective mass for Eq. (6), where all parameters are complex. The oscillation parameters were required to lie in their current $3\sigma$
ranges from Ref. [12]. One can identify basically two bands in this plot, one corresponding to 
\( \langle m \rangle \simeq \sqrt{\Delta m_A^2} \cos 2\theta_{12} = \sqrt{\Delta m_A^2} (1 - 2 \sin^2 \theta_{12}) \), the other to 
\( \langle m \rangle \simeq \sqrt{\Delta m_A^2} \).

Let us compare our discussion with that in the \( \mu-\tau \) symmetric case. With the generic prediction of \( \theta_{13} = 0 \), it follows that there is no Dirac phase. However, in contrast to the SSA case, in general both Majorana phases are present because there are in general three non-vanishing neutrino masses. This extra phase in \( \mu-\tau \) case is hard to detect since as far as neutrinoless double beta decay is concerned, the effective mass will depend only on one phase due to the fact that \( U_{e3} = 0 \). The second phase is in principle observable in processes such as \( \nu_\mu - \bar{\nu}_\mu \) and \( \nu_\tau - \bar{\nu}_\tau \) oscillations [13] or rare decays such as \( K^+ \to \pi^- \mu^+ \mu^- \) [14]. The experimental challenges in order to observe these processes at their predicted rates are however depressingly breathtaking, see the discussion in Ref. [14].

For the case of \( L_e - L_\mu - L_\tau \), there is also no Dirac phase, and in addition only one Majorana phase \( \alpha \), which is fixed by the symmetry to the value \( \pi/2 \). The required (large and tuned) breaking of \( L_e - L_\mu - L_\tau \) will induce non-zero \( m_3 \) and \( \theta_{13} \) and therefore all three possible phases will be present. The phase \( \alpha \) will stay close to \( \pi/2 \) and therefore the effective mass will stay close to 
\( \langle m \rangle \simeq \sqrt{\Delta m_A^2} \cos 2\theta_{12} \), unless the breaking parameters are as large as the parameters allowed by the symmetry. In case of SSA the effective mass can take any value between \( \sqrt{\Delta m_A^2} \cos 2\theta_{12} \) and \( \sqrt{\Delta m_A^2} \), which can be used to distinguish the two scenarios. We stress however again that \( L_e - L_\mu - L_\tau \) requires unnaturally large breaking.

2.2 Cases (B) and (C)

Let us now discuss the cases (B) and (C) given in Eq. (3) and Eq. (4). We note here that the effect of renormalization in case (B) is basically identical to case (A) discussed in the previous Subsection: \( c \) gets rescaled with \( (1 + \delta_r) \). In case (C) there is basically no running, because \( c \) gets rescaled with \( (1 + \delta_\mu) \ll (1 + \delta_r) \). In case (B) the mass matrix has the form:

\[
\mathcal{M}_\nu = m_0 \begin{pmatrix}
A & B & A/c \\
B & D & B/c \\
A/c & B/c & A/c^2
\end{pmatrix}.
\] (18)

Exactly analogous to case (A), in this case also we have a zero eigenvalue with an eigenvector given by \( (-1/\sqrt{1 + |c|^2}, 0, c/\sqrt{1 + |c|^2})^T \), therefore \( U_{\mu 1} = 0 \) or \( U_{\mu 3} = 0 \), depending on whether the hierarchy is normal or inverted. This is however incompatible with observations which show that \( U_{\mu i} \neq 0 \) for all \( i = 1, 2, 3 \). Similar situation holds for case (C) also, which predicts that \( U_{\tau 1} = 0 \) or \( U_{\tau 3} = 0 \). Therefore both these cases (B) and (C) are not phenomenologically viable. A possible way out would be to take correlations from the charged lepton sector into account, which however requires additional input and leads to less predictivity.
2.3 Breaking of Strong Scaling

What happens if we break the SSA conditions? There are three possibilities for this, namely

(A1) : \( \frac{m_{e\mu}}{m_{e\tau}} = \frac{m_{\mu\mu}}{m_{\mu\tau}} = c \), but \( \frac{m_{\tau\mu}}{m_{\tau\tau}} = c (1 + \epsilon) \),

(A2) : \( \frac{m_{e\mu}}{m_{e\tau}} = \frac{m_{\tau\mu}}{m_{\tau\tau}} = c \), but \( \frac{m_{\mu\mu}}{m_{\mu\tau}} = c (1 + \epsilon) \),

(A3) : \( \frac{m_{\mu\mu}}{m_{\mu\tau}} = \frac{m_{\tau\mu}}{m_{\tau\tau}} = c \), but \( \frac{m_{e\mu}}{m_{e\tau}} = c (1 + \epsilon) \).

As a consequence, non-zero \( m_3 \) and \( U_{e3} \) will be generated and the prediction \( \tan^2 \theta_{23} = 1/c^2 \) receives corrections of order \( \epsilon \). It turns out that in cases (A1) and (A2) \(|U_{e3}|\) is roughly \( \epsilon D/2 \) and \( m_3 \) is roughly \( D \epsilon \sqrt{\Delta m_2^2/2} \). In case (A3) one finds \(|U_{e3}| \simeq \epsilon D \) and \( m_3 \) is of order \( \epsilon^2 \sqrt{\Delta m_2^2} \). By allowing \( \epsilon \) to vary between zero and 0.22, we display the behavior of \( m_3 \) and \(|U_{e3}|\) as a function of \( \epsilon \) in Fig. 2. The oscillation parameters were again required to lie in their current 3σ ranges from Ref. [12]. Only for sizable breaking of SSA of order of the Cabibbo angle one can probe such \(|U_{e3}|\) values in next generation experiments. The corrections to the effective mass given in Eq. (17) are of order \( m_3 |U_{e3}|^2 \) and therefore completely negligible.

2.4 Weak Scaling?

One could relax the strong scaling condition and introduce a “weak scaling Ansatz”. This weaker version of SSA corresponds to the case where the equality holds for only a subset of flavors, e.g.,

\[ \frac{m_{e\mu}}{m_{e\tau}} = \frac{m_{\mu\mu}}{m_{\mu\tau}} \equiv c \text{ , or } \frac{m_{e\mu}}{m_{e\tau}} = \frac{m_{\tau\mu}}{m_{\tau\tau}} \equiv c \text{ , or } \frac{m_{\mu\mu}}{m_{\mu\tau}} = \frac{m_{e\mu}}{m_{e\tau}} \equiv c . \]

These relations are also invariant under the usual renormalization group extrapolations. Note that breaking of SSA, as discussed in the previous Subsection, with large \( \epsilon \) corresponds to weak scaling. The third weak scaling condition has sometimes been invoked as a way to understand large atmospheric mixing simultaneously with small \( \Delta m_2^2/\Delta m_2^2 \) [15]. Without additional input, for instance extra symmetries to make certain elements of \( M_\nu \) equal, there is not much predictivity for these cases. Therefore, we do not consider this possibility anymore.

3 Possible Theoretical Origin of Strong Scaling

In this Section, we speculate on the theoretical origin of the strong scaling rule from fundamental principles. We present two gauge models based on discrete family symmetries, which when broken lead to the strong scaling rule.
Scenario I: A gauge model
First we present a gauge model, where a symmetry leads to the strong scaling rule. The model is based on the Standard Model gauge group supplemented by a family symmetry group $D_4 \times Z_2$ as described in Ref. [9], where also the mathematical details of $D_4$ can be found. The leptons are assigned to the following representations of the discrete symmetry group:

\[
\{ L_e, \left( \frac{L_\mu}{L_\tau} \right) \} \sim \{ 1^+_1, 2^+ \}; \quad \{ e_R, \left( \frac{\mu_R}{\tau_R} \right) \} \sim \{ 1^-_1, 2^- \};
\]

\[
\{(N_e), (N_\mu), (N_\tau)\} \sim \{ 1^-_1, 1^+_2, 1^-_2 \},
\]

where the subscripts $+, -$ refer to the transformation under $Z_2$ and the rest are the $D_4$ representations. One requires [9] five Higgs doublets assigned to the group representations:

\[
\phi_1 \sim 1^-_1; \quad \phi_2 \sim 1^+_2; \quad \phi_3 \sim 1^-_4; \quad \left( \begin{array}{c} \phi_4 \\ \phi_5 \end{array} \right) \sim 2^+.
\]

As has been shown in [9], this leads to the neutrino Dirac mass matrix of the form

\[
m_D = \begin{pmatrix} a & 0 & 0 \\ b & d & e \\ 0 & 0 & 0 \end{pmatrix},
\]

a diagonal charged lepton mass matrix with arbitrary elements and also diagonal right-handed neutrino mass matrices with arbitrary elements. Using these, we can calculate the neutrino mass matrix using the type I seesaw formula to get

\[
M_\nu = \begin{pmatrix} a^2 A + b^2 B & b d B & b e B \\ b d B & d^2 B & d e B \\ b e B & d e B & e^2 B \end{pmatrix},
\]

where $A, B$ are the inverse values of the first two right-handed neutrino masses. Again, strong scaling with $c = d/e$ is obeyed in this case.

Scenario II: SSA from spontaneous breaking of $\mu-\tau$ symmetry
Now we present another type I seesaw model in which SSA follows from spontaneous breaking of discrete $\mu-\tau$ symmetry embedded into a $D_4 \times Z_2$ model. Consider an extension of the leptonic sector of the Standard Model with additional Higgs doublets $H_{e,\mu,\tau}$ and $H'_{1,2}$ with transformation properties under $D_4 \times Z_2$ given in Table [1]. The Yukawa couplings in this model can then be written as:

\[
L = \bar{L}_e H_e (h_1 N_e + h_2 N_\mu + h_3 N_\tau) + (\bar{L}_\mu H_\mu + \bar{L}_\tau H_\tau) (h_4 N_e + h_5 N_\mu + h_6 N_\tau) + f_{11} \bar{L}_e H_e e_R + f_{21} (\bar{L}_\mu H_\mu + \bar{L}_\tau H_\tau) e_R + f_{22} (\bar{L}_\mu \mu_R + \bar{L}_\tau \tau_R) H'_1
\]

\[
+ f_{33} (\bar{L}_\mu \mu_R - \bar{L}_\tau \tau_R) H'_2 + \frac{1}{2} \sum_{i,j} M_{ij} N_i^T C^{-1} N_j + h.c.
\]

\[ (25) \]
Table 1: Transformation properties of the particle content of the model in Scenario IV.

| Field       | $D_4 \times Z_2$ quantum number |
|-------------|---------------------------------|
| $L_e, H_e$  | $1^+$                           |
| $L_\mu$     | $2^+$                           |
| $L_\tau$    |                                 |
| $H_\mu$     | $2^+$                           |
| $H_\tau$    |                                 |
| $N_{e,\mu,\tau}$ | $1_1^+$                     |
| $e_R$       | $1^+$                           |
| $\mu_R$     | $2^-$                           |
| $\tau_R$    |                                 |
| $H'_2$      | $1_4^-$                         |
| $H'_1$      | $1_1^-$                         |

Symmetry breaking in general would lead to vevs of the form $\langle H_{e,\mu,\tau} \rangle = v_{e,\mu,\tau}$, all being different from each other. Using the seesaw formula, we can then calculate the light neutrino mass matrix $M_\nu$ and find that it satisfies SSA with $c = \frac{v_\mu}{v_\tau}$. Note that scaling is obeyed independent of the form of $M_R$. The atmospheric neutrino observations imply that $v_\mu \simeq v_\tau$ which is a constraint on the model.

Turning to the charged lepton masses, with the definition $\bar{\psi}_L M_\ell \psi_R$ we can write

$$M_\ell = \begin{pmatrix} f_{11} v_e & 0 & 0 \\ f_{21} v_\mu & f_{22} v'_1 + f_{33} v'_2 & 0 \\ f_{21} v_\tau & 0 & f_{22} v'_1 - f_{33} v'_2 \end{pmatrix}, \quad (26)$$

where $v'_{1,2}$ are the vevs of the $H'_{1,2}$ fields. For $f_{21} = 0$, this can lead to the desired diagonal charged lepton masses with no contribution to $U$. For non-zero $f_{21}$ one would expect non-zero $U_{e3}$ of the order of the 12 mixing angle in the matrix diagonalizing $M_\ell$. If the entries of $M_\ell$ are hierarchical, $(M_\ell)_{33} \simeq m_\tau \gg (M_\ell)_{22} \simeq m_\mu$, then $\theta^e_{12} \sim (M_\ell)_{21}/m_\mu$.

4 Summary

We have introduced a new concept of “strong scaling” to the neutrino mass matrix in which the ratio of certain elements of the neutrino mass matrix is a constant. We find it to be a new possibility to generate zero $U_{e3}$ and also the inverted hierarchy. In contrast to the flavor symmetry $L_e - L_\mu - L_\tau$ one can get correct phenomenology without any breaking of the Ansatz. Renormalization group running from the scale of neutrino mass generation to low scale has basically no effect on scaling, in particular $U_{e3} = 0$ and $m_3 = 0$ remain true for all energy scales. The phenomenology is predictive and testable, and the number of parameters equals the number of observables so that the Ansatz can be fully reconstructed. Observation of a departure of the atmospheric mixing angle from maximal
with tiny $U_{e3}$ would distinguish this model from an approximate $\mu-\tau$ symmetric mass matrix. Observation of a normal hierarchy for neutrinos or quasi-degenerate masses will rule out our Ansatz. We have also presented scenarios for the theoretical realization of the strong scaling Ansatz.

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Note added: After the paper was submitted to the arXiv, it was brought to our attention that there is a model in the literature [16] leading to a mass matrix in our scaling Ansatz (A). It was also noted in Ref. [16] that the results $m_3 = 0$ and $U_{e3} = 0$ are invariant under RGE corrections.

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Figure 1: Scatter plot of $\sin^2 \theta_{12}$ against the effective mass for Eq. (6).

Figure 2: Scatter plots of $|U_{e3}|$ and $m_3$ against $\epsilon$ for broken SSA from Eq. (19).