Markovian approximation in foreign exchange markets

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Abstract

In this paper we test the random walk hypothesis on the high frequency dataset of the bid–ask Deutschemark/US dollar exchange rate quotes registered by the interbank Reuters network over the period October 1, 1992 to September 30, 1993.

Then we propose a stochastic model for price variation which is able to describe some important features of the exchange market behavior. Besides the usual correlation analysis we have verified the validity of this model by means of other approaches inspired by information theory.

These techniques are not only severe tests of the approximation but also evidence some aspects of the data series which have a clear financial relevance.

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\textit{Key words:} Exit times, Foreign exchange markets, Markov process

I Introduction

Any financial theory begins introducing a reasonable model for price variation in terms of a suitable stochastic process. In this paper we want to select an asset-pricing model able to describe some features interesting from a financial point of view at weak level, i.e. including only information arising from analysis of historical prices and not including other public (see for example Cusatis et al. (1993), Asquith (1983), Ritter (1991), Loughran and Ritter (1995)) or

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To attain such an aim we analyze the foreign exchange market which presents several advantages compared with other financial markets. First of all it is a very liquid market. This feature is important because every financial market provides a single sample path, the set of the registered quotes of the asset. To be sure that a statistical description makes sense, a reasonable requirement is that this (unique) sequence of quotes is not dominated by single events or single trader’s operations. A natural candidate for such (if possible) description is then a liquid market involving several billions of dollars, daily traded by thousand of actors. In addition the foreign exchange market has no business time limitations. Many market makers have branches worldwide so trading can occur almost continuously. So in the analysis one can avoid to consider problems involved in the opening and closure of a particular market, at least as a first approximation. Finally if one considers the currency exchange, the returns

\[ r_t = \ln \frac{S_{t+1}}{S_t} \]  

are almost symmetrically distributed, where \( S_t \) is the price at the time \( t \) defined as the average between bid and ask prices.

In this paper we investigate the possibility to describe the Deutschemark/US dollar exchange (the most liquid market) in term of a Markov process. We consider a high frequency dataset to have statistical relevance of the results. Our data, made available by Olsen and Associated, contains all worldwide 1,472,241 bid–ask Deutschemark/US dollar exchange rate quotes registered by the inter-bank Reuters network over the period October 1, 1992 to September 30, 1993.

One of the main problems when analyzing financial series is that the quotes are irregularly spaced. In section 2 we briefly describe some different ways to introduce time in finance, and we discuss why we chose the business time, i.e. the time of a transaction is the position in the sequence of the registered quotes.

The history of the efforts in the proposal of proper stochastic processes for price variations is very long. An efficient foreign currency market, i.e. where prices reflect the whole information, suggests that returns are independently distributed. Following Fama (1970) we shall call hereafter “random walk” a financial model where returns are independent variables. Without entering into a detailed review we recall the seminal work of Bachelier (1900) who assumed (and tested) that price variations follow an independent gaussian process. Now it is commonly believed that returns do not behave according to a gaussian. Mandelbrot (1963) has proposed that returns are Levy-stable distributed, still remaining independent random variables. A recent proposal is the truncated Levy distribution model introduced by Mantegna and Stanley (1995) which well fits financial data, even considering them at different time lags.
Because of the financial importance of correlations for arbitrage opportunities (see Pagan (1996) for a review and Baviera et al. (1999) for more recent results) it is essential to introduce a non questionable technique able to answer the question: can “random walk” models correctly describe return variations? In section 2 we show that “random walk” is inadequate to describe even qualitatively some important features of price behavior. We measure the probability distribution of exit time, i.e. the lag to reach a given return amplitude $\Delta$. This analysis not only shows the presence of strong correlations but also leads to a natural measure of time which is intrinsic of the market evolution, i.e. the time in which the market has such a fluctuation. Following Baviera et al. (1999) we call this time $\Delta$-trading time.

In section 3, measuring the time in $\Delta$-trading time, we discuss the validity of a markovian approximation of the market behavior. First, using the quotes, we build a Markov model of order $m$, then, starting from the usual correlation function approach, we consider several techniques to verify the quality of this description.

We also perform an entropic analysis, inspired by the Kolmogorov (1956) $\epsilon$-entropy. This kind of analysis is equivalent to consider a speculator who cares only of market fluctuation of a given size $\Delta$ (see Baviera et al. (1999)). Finally we use other statistical tools to test the validity of such an approximation, namely the mutual information introduced by Shannon (1948) and the Kullback and Leibler (1956) entropy which measures the “discrepancy” between the Markov approximation and the “true” return process.

In section 4 we summarize and discuss the attained results.

II Exit times

One of the main (and unsolved) problems in tick data analysis concerns the irregular spacing of quotes. There are several candidates to measure the time of each transaction.

The first one is obviously the calendar time, i.e. the Greenwich Meridian Time at which the transaction occurs. The trouble with such a choice is rather evident: there are periods of no transaction. The simplest way (see, e.g. Mantegna and Stanley (1995)) to overcome this difficulty is to cut “nights” and “weekends” from the signal, i.e. assuming a zero time lag between the closure and reopening of the market. Of course in a worldwide series is less evident what does it mean “night” or “weekend”, but it is easy to observe that during a day or a week there are lags when no transaction is present. An improvement of the above procedure is to rescale the calendar time with a measure of market activity, i.e. to create a new time scale under which in all lags the same market activity occurs. We do not enter here in the literature on this subject, we briefly mention the procedure in which one measures the market activity
with the average absolute number of quotes per lag (e.g. of 15 minutes) or with the average absolute price change over each lag (see Dacorogna et al. (1993), Fischer et al. (1997)). Let us note that the weight does not change too much the time (roughly it is between 0.5 and 2.5), therefore there is not a big difference with the naive procedure. A similar approach has been introduced by Galluccio et al. (1997) where the time lags are rescaled with a properly defined instantaneous volatility.

A slightly different approach is the business time. One considers all transactions equivalent and the time of the transaction is simply given by its position in the sequence of quotes. In this section we shall adopt the business time; it looks a reasonable choice when facing a worldwide sequence where lags of no transaction are often a consequence of the geographical position on the earth surface of the most important markets. However we have to stress that, at least for some statistical features, there are not qualitatively differences using the business or the calendar time.

In this section we study the distribution of the exit times at a given resolution $\Delta$. This analysis will allow us to show that the “random walk” models cannot be a reasonable description. This technique will help us to understand how to analyze financial time series.

Introducing

$$ r_{t,t_0} \equiv \ln \frac{S_t}{S_{t_0}} , $$

(2)

where $t_0$ is the initial business time and $t > t_0$, we wait until $t_1$ such as :

$$ |r_{t_1,t_0}| \geq \Delta . $$

(3)

Now, starting from $t_1$ with the above procedure we obtain $S_{t_2}$, and so on. In this way we construct the successions of returns and exit times at given $\Delta$ :

$$ \{\rho_1, \rho_2, \ldots, \rho_k, \ldots\} \text{ where } \rho_k \equiv \ln \frac{S_{t_k}}{S_{t_{k-1}}} $$

(4)

and

$$ \{\tau_1, \tau_2, \ldots, \tau_k, \ldots\} \text{ where } \tau_k \equiv t_k - t_{k-1} , $$

(5)

where $|\rho_k| \geq \Delta$ by definition (see eq.(3)), and $\tau_k$ is the time after which we have the $k$-th fluctuation of order $\Delta$ in the price.

In the following we call $k$ the $\Delta$-trading time, i.e. we enumerate only the transactions at which a fluctuation $\Delta$ is reached. Since the distribution of the returns is almost symmetric, the threshold $\Delta$ has been chosen equal for both
positive and negative values.

Fig. 1. Evolution of $r_{t,t_k}$ with $\Delta = 0.01$. The $\Delta$-trading time is zero ($k = 0$) at the first transaction corresponding to 00:00:14 of October 1, 1992 in calendar time, and $\Delta$-trading time is 4 ($k = 4$) at 11:59:28 of October 2, 1992 (9939 business time).

In the figure 1 we show the evolution of the returns $r_{t,t_k}$.

Let us now study $P_\Delta(\tau)$, the probability distribution function (PDF) of exit times (5) for a given size $\Delta$. From the shape of this PDF one can have indication if a stochastic process can be considered a good candidate to model price variation. In figure 2 we show that $P_\Delta(\tau)$ has a different shape for $\Delta$ smaller or larger than the typical transaction cost $\gamma^T$, where we define

$$
\gamma_t \equiv \frac{1}{2} \ln \frac{S_t^{(ask)}}{S_t^{(bid)}} \simeq \frac{S_t^{(ask)} - S_t^{(bid)}}{2S_t^{(bid)}} 
$$

the transaction cost at time $t$, whose distribution has a narrow peak around its typical value $\gamma^T = 2.4 \cdot 10^{-4}$. Note that $P_\Delta(\tau)$ is roughly exponential at small $\Delta$, while for $\Delta$ larger than $\gamma^T$, $P_\Delta(\tau)$ clearly shows a non-exponential shape.

In the insert we present $\langle \tau \rangle$ vs. $\Delta$. We denote with $\langle \cdot \rangle$ the average of a sequence

$$
\langle A \rangle \equiv \frac{1}{L} \sum_{i=1}^{L} A_i 
$$

where $L$ is the size of the sequence. One has a fairly clear scaling law of the average exit time as a function of $\Delta$ for more than three decades in $\langle \tau \rangle$:

$$
\langle \tau \rangle_\Delta \sim \Delta^\alpha \text{ with } \alpha \simeq 2.2. 
$$

for all $\Delta$ greater than the typical transaction costs $\gamma^T$. Let us now compare the PDF from the data analysis with the result one
Fig. 2. Probability distribution function of the exit times $\tau$ for different values of $\Delta$ : $\Delta = 5 \cdot 10^{-5}$ (full line), $\Delta = 2 \cdot 10^{-4}$ (dashed line) and $\Delta = 6 \cdot 10^{-4}$ (dotted line). Those values are respectively smaller, similar to and larger the typical transaction costs. With the dashed-dotted line, we also show the PDF of the exit times for a Wiener process (see eq. (8)) with the $\langle \tau \rangle = 16.85$ equal to that one obtained with $\Delta = 6 \cdot 10^{-4}$. In the insert we show $\langle \tau \rangle$ versus $\Delta$. The line shows the asymptotic behavior (for $\Delta > \gamma T$) of $\langle \tau \rangle \propto \Delta^\alpha$ with $\alpha = 2.2$.

obtains in “random walk” models. In the case of a Wiener process in absence of drift, following Feller (1971) one can find the exact solution for the PDF:

$$P_{\Delta}^{W}(\tau) = \frac{4}{\pi \hat{\tau}} \sum_{n=0}^{\infty} (-1)^n (2n + 1) e^{-\frac{(2n+1)^2}{\hat{\tau}}}.$$  

(8)

where $\hat{\tau} = (8\Delta^2)/(\pi^2 \sigma^2)$ and $\sigma^2$ is the variance of the Wiener process. The main characteristics of this PDF are:

- it is peaked around $\langle \tau \rangle$. The similarity between average and typical value of this distribution allows us to interpret the average exit time as the one we expect to observe in the future with higher probability. This fact is not any more true in the case of a PDF with a power-law tail, and so it is more difficult to give a direct interpretation of the $\langle \tau \rangle$ shown in the insert.
- It decreases exponentially at large $\tau$. It is simple to understand that the exponential decay is true for any “random walk” model. In fact, in absence of correlations one has a finite probability, let us say less than $q$, to exit in a finite time $\hat{\tau}$. Therefore the probability to exit after a time $\tau$ is less than $q^{\tau/\hat{\tau}}$, i.e. it is exponential. In figure 2 we also compare the normalized PDF computed from financial data with the $P_{\Delta}(\tau)$ of a “random walk” with the $\langle \tau \rangle$ equal to the one obtained from the signal with $\Delta = 6 \cdot 10^{-4}$. We recall that $\gamma T \simeq 2.4 \cdot 10^{-4}$. 

6
A model built with independent variables is unable to reproduce $P_\Delta(\tau)$ and it is simple to understand that also processes with short memory, e.g. Markov process, give for the exit times a PDF qualitatively similar (i.e. decaying exponentially at large $\tau$) to that one of a Wiener process.

![Graph](image)

**Fig. 3.** Data collapse of $P_\Delta(\tau/\langle\tau\rangle)$ vs $\tau/\langle\tau\rangle$. We have plotted four rescaled PDFs: $\Delta = 4 \cdot 10^{-4}$, $\Delta = 6 \cdot 10^{-4}$, $\Delta = 8 \cdot 10^{-4}$, and $\Delta = 1 \cdot 10^{-3}$. The full curve indicates the Cauchy distribution.

Figure 3 shows the PDF data collapse, i.e. $P_\Delta(\tau/\langle\tau\rangle)$ vs $\tau/\langle\tau\rangle$ for different $\Delta$, all greater than $\gamma T$.

One observes that the rescaled PDF behave according to a single density function, which is well approximated, a part for large $\tau/\langle\tau\rangle$, by the Cauchy distribution

$$P_\Delta\left(\frac{\tau}{\langle\tau\rangle}\right) \approx \frac{2a}{\pi \left(a^2 + \left(\frac{\tau}{\langle\tau\rangle}\right)^2\right)}.$$  \hspace{1cm} (9)

The distribution $P_\Delta(\tau)$ shows a power-law tail up to an exponential cut (perhaps due to the finite size of the sample analyzed). We note that if $\Delta$ is larger than the typical transaction cost the probability distribution of $\tau$ has a very long tail and so $\langle\tau\rangle_\Delta$ is not the typical value of the sequence (5).

The scaling behavior of $\langle\tau\rangle$ and the form of the distribution $P_\Delta(\tau)$ indicate the presence of strong correlations in the financial signal for $\Delta > \gamma T$. Furthermore the above analysis indicates the $\Delta$-trading time as a natural candidate to measure time. In the financial context this measure has the great advantage to be intrinsic of the market evolution, i.e. it enumerates the times in which the market has such a fluctuation. In the next section we shall show that, measuring the time in such a way, a simple Markov process is a valid description of the market behavior.
III The markovian approximation

In this section we construct a markovian process of order $m$ using a symbolic sequence obtained from the returns $\{\rho_k\}$ at fixed $\Delta$. We define the markovian process building the transition matrix with this symbolic sequence.

Symbolic dynamics is a rather powerful tool to catch the main statistical features of time series. In order to construct a symbolic sequence from the succession $\{\rho_k\}$ we need a \textit{coarse graining} procedure to partition the range data and then we assign a conventional symbol to each element of the partition. We perform the following transformation:

$$ z_k = \begin{cases} 
-1 & \text{if } \rho_k < 0 \\
+1 & \text{if } \rho_k > 0 
\end{cases}. \quad (10) $$

In such a way we may study a discrete stochastic process which reproduces the feature of the original process we want to analyze. The financial meaning of this codification is rather evident: the symbol $-1$ occurs if the stock price decreases of percentage $\Delta$, while if the stock price increases of the same percentage the symbol is $1$. In the following we indicate with $z^{(i)}$ the two possible values of the symbolic sequence.

Let us briefly explain the financial meaning of the return sequence $\{\rho_k\}$ at fixed $\Delta$. A speculator, who modifies his portfolio only when a fluctuation of size $\Delta$ appears in the price sequence, cares only of these returns. Following Baviera et al. (1999) we call \textit{patient} investor such a speculator. He performs automatically a filtering procedure: he rejects all the quotes which do not change at least of a percentage $\Delta$ of the price.

Starting from this sequence we create a Markov process approximating the symbolic sequence. In a Markov process of order $m$ the probability to have the symbol $z_n$ at the step $n$ depends only on the state of the process at the previous $m$ steps $n-1, n-2, \ldots, n-m$. Given a sequence of $m$ symbols $Z_m = \{z^{(j_1)}, z^{(j_2)}, \ldots, z^{(j_m)}\}$, we define $N(Z_m)$ the number of sequences $Z_m$ and $N(Z_m, j)$ the number of times the symbol $z^{(j)}$ comes after the sequence $Z_m$. The transition matrix of the Markov process of order $m$ is:

$$ W_{Z_m,j} \equiv \frac{N(Z_m, j)}{N(Z_m)}, \quad (11) $$

and the probability of the sequence $Z_m$ is:

$$ P(Z_m) \equiv \frac{N(Z_m)}{N_m}, \quad (12) $$
where \( N_m = L - m \) is the total number of possible sequence of length \( m \) including superposition (\( L \) is the size of the sequence). It can be shown (see e.g. Feller (1971)) that the definitions (11) and (12) are coherent in the framework of ergodic Markov processes.

In the case of a Markov chain (i.e. process of order 1) the transition matrix \( W_{i,j} \), i.e. the probability of a transition in one step to the state \( z^{(j)} \) starting from the state \( z^{(i)} \), contains all the relevant information for the process. Naming \( N(i) \) the number of symbol \( z^{(i)} \) and \( N(i,j) \) the number of symbol \( z^{(j)} \) which comes after the symbol \( z^{(i)} \), the transition matrix is:

\[
W_{i,j} = \frac{N(i,j)}{N(i)}.
\] (13)

If the process is ergodic the probability \( P_i \) of the state \( z^{(i)} \) is given by

\[
P_j = \sum_{i=1}^2 P_i W_{i,j} = \lim_{n \to \infty} (W^n)_{i,j} \forall i,
\]

where \( W^n \) is the \( n \)-th power of the matrix \( W \).

In the following we check, using various statistical approaches, if the markovian approximation of order \( m \) mimics properly the price behaviour. First of all we perform an autocorrelation analysis, which is the most common test in financial econometrics. Then we show that the model reproduces properly the Shannon entropy. This quantity, as discovered by Kelly (1956) and shown in Baviera et al. (1999), has a clear financial meaning and it plays a central role in this field.

Finally we test the validity of our approximation using more sophisticated statistical tools of information theory. We compare the mutual information of the signal to the one of the approximation and we measure a “distance” between the symbolic dynamics process and the Markov approximation, with a technique based on the Kullback entropy.

The results we show in the following analysis have been obtained choosing \( \Delta = 4 \cdot 10^{-4} \), but they are totally independent from this choice for \( \Delta > \gamma_T \).

### A Autocorrelation function

A standard approach to verify temporal independence of processes is the measure of autocorrelation function. A first simple similarity test can be based on it: if two processes have the same autocorrelation function they lose memory of their past in a similar way.

Typically one defines *short memory* series if an exponential decay of autocorrelation function occurs, as in Brockwell and Davis (1991).
The autocorrelation function of a random process $x_t$ is

$$ C(n) = \frac{\langle x_{t+n}x_t \rangle - \langle x_t \rangle^2}{\langle x_t^2 \rangle - \langle x_t \rangle^2} $$ \quad (14)$$

where $\langle . \rangle$ indicates the sequence average introduced in equation (6).

The autocorrelation function can be easily computed for a Markov chain described by the transition matrix $W_{i,j}$ and the probability $P_i$ of the state $z^{(i)}$:

$$ C^{(M)}(n) = \frac{\sum_{i,j} z^{(i)}z^{(j)}(W^n)_{i,j} - \left( \sum_i z^{(i)} P_i \right)^2}{\sum_i z^{(i)}^2 P_i - \left( \sum_i z^{(i)} P_i \right)^2} $$ \quad (15)$$

If the Markov chain is ergodic one has

$$ C^{(M)}(n) \sim e^{(\ln |\lambda_2|)n} \quad \text{for large } n, $$ \quad (16)$$

where $\lambda_2$ is the second eigenvalue of the transition matrix (see Feller (1971)).

In figure 4 we compare the autocorrelation functions of the return $\{\rho_k\}$ and

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4.png}
\caption{Autocorrelation function of return sequence $\{\rho_k\}$ (+), Markov sequence of order 1 ($\times$) and Markov sequence of order 3 ($\ast$). The line is the asymptotic analytical result for a Markov process of order 1 (see eq.(16)).}
\end{figure}

of the sequences of the same length generated by Markov processes of various orders. We show also the theoretical value for a Markov chain (see equation (16)). We observe there is a very good agreement between the return sequence and the Markov process inside the statistical error. This error, for the autocorrelation function measured from a finite dataset, is of the order of $O(L/\tau_c)^{-\frac{1}{2}}$ where $\tau_c$ is the correlation time and $L$ is the length of the sequence ($L \simeq 160000$ for $\Delta = 4 \cdot 10^{-4}$). This corresponds to the value of the “plateau”
in the figure 4. The same results are obtained if one computes autocorrelations of the symbolic return sequence \( \{ z_k \} \). This is a consequence of the fact that the probability density function of \( \rho_k \) is peaked around the value \( \pm \Delta \).

**B Entropic analysis**

Let us briefly recall some basic concepts of information theory and discuss the meaning of entropy in financial data analysis. Given a symbolic sequence \( Z_n = \{ z^{(j_1)}, z^{(j_2)}, \ldots, z^{(j_n)} \} \) of length \( n \) with probability \( p(Z_n) \), we define the block entropy \( H_n \) as

\[
H_n \equiv - \sum_{Z_n} p(Z_n) \ln p(Z_n) .
\]  

(17)

The difference

\[
h_n \equiv H_{n+1} - H_n
\]  

(18)

is the average information needed to specify the symbol \( z_{n+1} \) given the previous knowledge of the sequence \( \{ z_1, z_2, \ldots, z_n \} \). The series of \( h_n \) is monotonically not increasing and for an *ergodic* process one has

\[
h = \lim_{n \to \infty} h_n
\]  

(19)

where \( h \) is the Shannon (1948) entropy.

The maximum value of \( h \) is \( \ln(2) \) (this is because we are considering only two-symbols sequence). This value is reached if the process is totally uncorrelated and the symbols have the same probability. We indicate, following Baviera et al. (1999), with *available* information the difference between the maximum entropy and its real value:

\[
I \equiv \ln(2) - h .
\]  

Khinchin (1957) shows that if the stochastic process \( \{ z_1, z_2, \ldots \} \) is markovian of order \( m \) then \( h_n = h \) for \( n \geq m \). We observe that the Markov process of order \( m \) described by equations (11) and (12) has, by definition, the same entropy \( h_n \) of the symbolic sequence for all \( n \) not greater than \( m \). In this way we can build a markov approximation which mimics the originary entropy as we desire.

From figure 5 one observes that the \( h_n \) are consistent with those ones of a Markov process of order 1 (the asymptotic value \( h \) is reached approximatively
in one step). It is therefore natural to conjecture that such a stochastic process is able to mimic price variations. Using Markov processes of order greater than one does not improve the approximation for the entropy value in a appreciable way. The value of $h_n$ is statistically relevant till the length $n$ of the sequence is of the same order of $(1/h) \log(L)$ as shown in Khinchin (1957): this explains the folding of $h_n$ at large $n$.

Why available information is so important in the financial context? Kelly (1956) has shown the link between available information and the optimal growth rate of a capital in some particular investments. A similar idea can be applied to the patient investor, i.e. a speculator who waits to modify his investment till a fluctuation of size $\Delta$ is present. He observes an available information different from zero. Instead, as shown in figure 5, for a random walk the available information is zero.

If the returns $\{\rho_k\}$ at fixed $\Delta$ are ruled by a Markov chain and if one neglects the transaction costs, Baviera et al. (1999) prove that the optimal growth rate of the capital is equal to the available information. The case with transaction costs is considered by Baviera and Vergni (1999).

Inside a markovian description the available information suggests also the order of the process one has to consider. It is useless to include the information coming from one further step in the past, if one does not observe a significant increasing of the available information involved in the operation.

Furthermore if the Markov approximation well describes the available information, it is not so important for a speculator who wants to maximize the grow rate of his capital that the markovian mimicking is no longer good for other quantities. However we shall show in the following that the markovian
approximation not only reproduces the available information but, in addition it is a proper description of the return dynamics itself. This is important in the case one performs a more complex investment for which a detailed model for price variation is essential (see Merton (1990)).

C Mutual information

The mutual information is a measure of the average information one has about an event \( q \) knowing the result of another event \( s \). In our case the events are the values of a process at different time. Following Shannon (1948) we define:

\[
I(n) = -\int \int P(x_t, x_{t+n}) \ln \frac{P(x_t, x_{t+n})}{P(x_t)P(x_{t+n})} dx_t dx_{t+n}
\]  

where \( P(x_t, x_{t+n}) \) is the joint probability of the variables \( x_t \) and \( x_{t+n} \), and \( P(x_t) \) is the probability density function of the \( x_t \).

The main advantage of this technique, compared with autocorrelation function, is that \( I(n) \) is an intrinsic property of the process, i.e. it has the same value if we use \( x_t \) or a function of it. This because \( I(n) \) depends on the probability density function in such a way that the integral (20) is invariant under the change of variable \( x \rightarrow y = f(x) \).

For a Markov chain the mutual information is:

\[
I^{(M)}(n) = -\sum_{i,j} P_i(W^n)_{i,j} \ln \frac{P_i(W^n)_{i,j}}{P_iP_j}
\]  

and, if the Markov chain is ergodic, for large \( n \) one has

\[
I^{(M)}(n) \sim e^{2(\ln |\lambda_2|)n}
\]  

where \( \lambda_2 \) is again the second eigenvalue of the transaction matrix.

In figure 6 we compare the mutual information of the symbolic return series \( \{z_k\} \) with the sequences obtained starting from Markov processes of various orders, and also with the expected theoretical value for a Markov chain (see equation (22)). In the case of the mutual information the statistical error is of the order \( L^{-1} \) (see also Roulston (1999)) for a sequence of length \( L \), since it is computed starting from probability distribution.

Let us stress that the good agreement between the return sequence and the Markov process for both mutual information (figure 6) and correlation function (figure 4) is a clear indicator of the validity of the Markov approximation.
Fig. 6. Mutual information of the symbolic return sequence \( \{ z_k \} \) (+), Markov sequence of order 1 (×) and Markov sequence of order 3 (*). The line is the asymptotic analytical result for a Markov process of order 1 (see eq.(22)).

D Kullback entropy

Given two discrete random variables \( P, Q \) which can assume only \( M \) different value with probabilities \( p_i, q_i \) \( (i = 1 \ldots M) \) respectively, the Kullback entropy of the probability distribution of the variable \( P \) with respect to \( Q \) :

\[
J(P|Q) \equiv \sum_{i=1}^{M} p_i \ln \frac{p_i}{q_i}
\]  

(23)

is a powerful tool to measure the “distance” between the PDF of those variables. In fact, it is shown by Kullback and Leibler (1956) that the function \( J(P|Q) \) is identically zero only if the two random variables have the same probabilities, i.e. \( p_i = q_i \ \forall \ i \); otherwise \( J(P|Q) > 0 \). The \( J(P|Q) \) is not a symmetric function of the two random variable. To define symmetric “distance” between PDF, following Kullback and Leibler (1956) we define the divergence between \( \{ p_i \} \) and \( \{ q_i \} \) as :

\[
K(P|Q) \equiv J(P|Q) + J(Q|P) = \sum_{i=1}^{M} (p_i - q_i) \ln \frac{p_i}{q_i} \equiv \sum_{i=1}^{M} (q_i - p_i) \ln \frac{q_i}{p_i}
\]  

(24)

This function is still positive definite and it is also symmetric.

This is a probabilistic “distance” between random variables, but we are interested to test the similarity of two stochastic processes. To perform such a test we consider a symbolic sequence of length \( n \), \( Z_n = \{ z^{(j_1)}, z^{(j_2)}, \ldots, z^{(j_n)} \} \) obtained from the processes we are interested in, and we compute the “distance”
in the Kullback way for the PDF of such sequences for all \( n \). When \( n \to \infty \) we test the similarity of the processes as a whole, but, as happens for the entropy analysis, we expect for large \( n \) a limitation due to statistical errors.

![Kullback Entropy](image)

**Fig. 7.** \( K_n(P|Q) \) versus \( n \). The symbols indicates the order of the Markov process involved in the similarity test: (+) indicate the first order and (×) indicate the third order. The full line and the dashed line indicates the comparison between the theoretical Markov process with the symbolic return series and the theoretical Markov process with the symbolic Markov sequence (with the same number of elements of the financial data), respectively.

In figure 7 we show \( K_n(P|Q) \) versus \( n \) where \( P \) and \( Q \) are respectively the Markov process and the financial process. The sequence \( Z_n \) for the financial quotes are obtained from the symbolic return sequence \( \{z_k\} \), and its probability is numerically computed as in equation (12). For the Markov process the probabilities of the \( Z_n \) are calculated starting from the transition matrix \( W \).

In the case of a Markov chain we have:

\[
P_n = P_{j_1} \cdot W_{j_1,j_2} \cdot W_{j_2,j_3} \cdots W_{j_{n-1},j_n}
\]

We have also calculated the Kullback entropies between the PDF of the Markov process computed theoretically using the transition matrix, and numerically with a symbolic sequence (of the same length of the financial sequence). This shows the relevance of the statistical error, and gives an indication of the order of \( n \) at which one must stop to have statistically relevant results.

All the Kullback tests are performed using Markov process of order 1 and 3 and is evident from figure 7 how this last process well reproduces the financial series.

For the Kullback entropies the statistical error increases with \( n \). In order to have reasonable statistics we must restrict ourselves to take \( n \leq (1/h) \log(L) \), as in the case of entropy analysis.
At the end of this section we recall that the symbolic return serie \( \{ z_k \} \) is chosen at fixed \( \Delta \), but the results of our analysis, i.e. the good agreement with a Markov process, is strongly independent from the value of \( \Delta \).

### IV Conclusions

In econometric analysis it is obviously relevant to test the validity of “random walk” models because of their strict link with market efficiency. In this paper we have first discarded “random walk” models as proper description of some features of the financial signal and then built a model considering the Deutschemark/US dollar quotes in the period from October 1, 1992 to September 30, 1993.

We have developed a technique, based on the measure of exit times PDF, which allows us to reject the random walk hypothesis. The presence of strong correlations has been observed by means of an “anomalous” scaling of \( \langle \tau \rangle \) and the presence of a power law behavior of the exit times probability density function \( P_\Delta(\tau) \) for \( \Delta > \gamma^T \). This implies the failure of the “random walk” models where \( \langle \tau \rangle \) scales as \( \Delta^2 \) and the \( P_\Delta(\tau) \) tail is exponential.

We want to interpret \( \gamma^T \) as a natural cut-off due to the absence of a profitable trading rule for a patient investor with \( \Delta \) less than \( \gamma^T \); in this case profits are less than costs. We recall that a patient investor cares only of the quotes where it is present a price variation at least of a percentage \( \Delta \).

The main advantage of this approach is that it shows that this class of models gives an inadequate description even at the qualitative level and it suggests a new point of view in financial analysis. Instead of considering an arbitrary measure of time we suggest to limit the analysis only at the times when something relevant from the financial point of view happens. In particular we focus our attention on return fluctuations of size \( \Delta \). This analysis is equivalent to the behavior of a patient speculator.

We show that the returns of such a sequence can be approximated by a Markov model by means of several tests. We have considered the usual autocorrelation approach obtaining a very good agreement inside the statistical errors. In spite of its simplicity the autocorrelation analysis has the disadvantage that gives different results if one considers the random variable \( x_t \) or a function \( f(x_t) \) of it. The mutual information is a generalization of this tool which does not depend from the function \( f \) considered. The agreement observed for the mutual information is surely a severe test of similarity.

A central role in the comparison between the approximation and the “true” signal is surely played by the available information. Following the idea of Kelly, it has been shown by Baviera et al. (1999) that this quantity corresponds, in absence of transaction costs, to the optimal growth rate of the invested capital following a particular trading rule. We show that even a Markov process of
order 1 mimics properly the Deutschemark/US dollar behavior. Finally we have analyzed a “distance” between processes based on the Kullback and Leibler entropy. The advantage of this technique is that one is able to estimate the difference between the processes with a quantity strictly connected with Shannon entropy and then with the available information, i.e. the quantity of interest for a patient speculator. The Markov model we have considered in this paper not only has the advantage to mimic very well the available information of the financial series but also it is a good approximation of the “true” process itself. While the former is the quantity of interest for a speculator who invests directly on the exchange market, the latter is more interesting from both the theoretical and experimental sides to have a deeper insight of the market behavior. This asset-model will allow to reach a better evaluation of risk with the natural consequences on the derivative field.

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