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ESTIMATION OF CONDITIONAL EXTREME RISK MEASURES FROM HEAVY-TAILED ELLIPTICAL RANDOM VECTORS

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Abstract. In this work, we focus on some conditional extreme risk measures estimation for elliptical random vectors. In a previous paper, we proposed a methodology to approximate extreme quantiles, based on two extremal parameters. We thus propose some estimators for these parameters, and study their consistency and asymptotic normality in the case of heavy-tailed distributions. Thereafter, from these parameters, we construct extreme conditional quantiles estimators, and give some conditions that ensure consistency and asymptotic normality. Using recent results on the asymptotic relationship between quantiles and other risk measures, we deduce estimators for extreme conditional $L_p$-quantiles and Haezendonck-Goovaerts risk measures. Under similar conditions, consistency and asymptotic normality are provided. In order to test the effectiveness of our estimators, we propose a simulation study. A financial data example is also proposed.

Keywords: Elliptical distribution; Extreme quantiles; Extreme value theory; Haezendonck-Goovaerts risk measures; Heavy-tailed distributions; $L_p$-quantiles.

1. Introduction

In many fields such as finance or actuarial science, quantile, or Value-at-Risk (see Linsmeier and Pearson (2000)) is a recognized tool for risk measurement. In Koenker and Bassett (1978), quantile is seen as minimum of an asymmetric loss function. However, Value-at-Risk, or VaR, has some disadvantages, such as that of not being a coherent measure in the sense of Artzner et al. (1999). These limits have led many authors to use alternative risk measures.

On the basis of Koenker’s approach, Newey and Powell (1987) proposed another measure called expectile, which has since been widely studied (see for example Sobotka and Kneib (2012) or more recently Daouia et al. (2017a)) and applied (Taylor (2008) and Cai and Weng (2016)). Later, Breckling and Chambers (1988) introduced M-quantiles, a family of measures minimizing an asymmetric loss function, and Chen (1996) focused on asymmetric power functions to define $L_p$-quantiles. The cases $p = 1$ and $p = 2$ correspond respectively to the quantile and expectile. Recently, Bernardi et al. (2017) provided some results concerning $L_p$-quantiles for Student distributions, and have shown that closed formula are difficult to obtain in the general case.

In parallel, Artzner et al. (1999) introduced the Tail-Value-at-Risk as an alternative to Value-at-Risk, and this risk measure subsequently had many applications (see e.g. Bargès et al. (2009)). Moreover, TVaR belongs to a larger family of risk measures called Haezendonck-Goovaerts risk measures and introduced in Haezendonck and Goovaerts (1982), Goovaerts et al. (2004) and Tang and Yang (2012). In the same way as $L_p$-quantiles, we do not have an explicit formula in the general case.

However, for a heavy-tailed random variable, Daouia et al. (2017b) proved that $L_p$-quantile and $L_1$-quantile (or quantile) are asymptotically proportional. Then, as proposed in Daouia et al. (2017a), an estimator of a $L_p$-quantile may be deduced from a suitable estimator of the quantile, for extreme levels. In the same spirit, Tang and Yang (2012) provided a similar asymptotic relationship between a subclass of Haezendonck-Goovaerts risk measures and quantiles. Finally, all these risk measures we introduced may be estimated through a quantile estimation in an asymptotic setting.

Extreme quantiles estimation is a very active area of research. In recent years, we can give many examples: Gardes and Girard (2005) focused on Weibull tail distributions, El Methni et al. (2012) proposed a study for heavy and light tailed distributions, Gong et al. (2015) was interested in functions of dependent variables, and de Valk (2016) provided a methodology for high quantiles estimation. The question of extreme conditional quantiles estimation has also been explored in Wang et al. (2012) in a regression framework. However, Maume-Deschamps et al. (2017a) and Maume-Deschamps et al. (2017b) have

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shown that the regression setting may lead to a poor estimation of extreme measures in the case of elliptical distributions. Elliptical distributions, introduced in [Kelker (1970)], aim to generalize the gaussian distribution, i.e. to define symmetric distributions with different properties, such as a heavy tail. This is why elliptical distributions are more and more used in finance (see for example [Owen and Rabinovitch (1983)] or [Xiao and Valdez (2015)].

For all these reasons, we consider, in this paper, an elliptical random vector, and propose to estimate some extreme quantiles (and deduce $L_\alpha$-quantiles and Haezendonck-Goovaerts risk measures) of a component conditionally to the others. In order to improve the conditional quantile estimation, we proposed in [Maume-Deschamps et al. (2017a)] a methodology based on two extremal parameters, and the unconditional quantile. Indeed, if $F^{-1}_\alpha (\alpha)$ and $F^{-1} (\alpha)$ are respectively the conditional and unconditional quantiles with level $\alpha$, a relationship of this form is obtained:

$$F^{-1}_\alpha (\alpha) \sim F^{-1} (\delta(\alpha, \eta, \ell))$$

where $\delta$ is a known function (detailed later) depending on $\alpha$ and two parameters $\eta$ and $\ell$ called extremal parameters. In this paper, we provide estimators for these parameters, and therefore, using order statistics, for extreme conditional quantiles.

The paper is organized as follows. Section 2 provides some definitions and properties of elliptical distributions, including the extremal parameters introduced in [Maume-Deschamps et al. (2017a)]. Section 3 is devoted to extremal parameters estimation, with some consistency and asymptotic normality results. In Section 4 we use the results of Section 3 to introduce some estimators of extreme quantiles, and give consistency and asymptotic normality results. The asymptotic relationships between $L_\alpha$-quantiles and quantiles recalled in Section 5 allow us to give extreme $L_\alpha$-quantiles estimators. The same approach is proposed for extreme Haezendonck-Goovaerts risk measures. In order to analyze the efficiency of our estimators, we propose a simulation study in Section 6 and a real data example in Section 7.

2. Preliminaries

In this section, we first recall some classical results on elliptical distributions. We consider a $d$-dimensional vector $Z$ from an elliptical distribution with parameters $\mu \in \mathbb{R}^d$ and $\Sigma \in \mathbb{R}^{d \times d}$. Then the density of $Z$, if it exists, is given by:

$$f_d \left( \frac{(z-\mu)^T \Sigma^{-1} (z-\mu)}{||\Sigma||^\frac{1}{2}} \right)$$

$$c_d$$ and $g_d$ will respectively be called normalization coefficient and generator of $Z$. [Cambanis et al. (1981)] gives another way to characterize an elliptical distribution, through the following stochastic representation:

$$Z \overset{d}{=} \mu + RAU(d)$$

where $\Lambda \Lambda^T = \Sigma$, $U(d)$ is a $d$-dimensional random vector uniformly distributed on the unit sphere of dimension $d$, and $R$ is a non-negative random variable independent of $U(d)$. $R$ is called radius of $Z$. In the following, the radius must have a particular shape. Indeed, [Huang and Cambanis (1979)] and [Kano (1994)] propose a representation for some particular elliptical distributions:

$$Z \overset{d}{=} \mu + \chi_d \xi A U(d)$$

where $\chi_d$ is the square root of a $\chi^2$ distribution with $d$ degrees of freedom, $\xi$ is a non-negative random variable which does not depend on $d$, and $\chi_d$, $\xi$ and $U(d)$ are mutually independent. In [Kano (1994)], such elliptical distributions are said consistent, have the advantage of being stable by linear combinations (combining Theorem 2.16 of [Fang et al. (1990)] and Theorem 1 in [Kano (1994)]), and allows us to define elliptical random fields (see, e.g., [Opitz (2016)]). In the following, we focus on consistent elliptical distributions, and take the notation

$$R_d = \chi_d \xi$$

For the sake of clarity, we will say that a random variable with stochastic representation (2.3) is $(\xi, d)$-elliptical with parameters $\mu$ and $\Sigma$. Using this terminology, the purpose of the paper is as follows. Let $Z = (X, Y) \in \mathbb{R}^{N+1}$ be a $(\xi, N + 1)$-elliptical random vector with parameters $\mu$ and $\Sigma$, where $X \in \mathbb{R}^N$ and $Y \in \mathbb{R}$. Consistency property of $Z$ implies that $X$ and $Y$ are respectively $(\xi, N)$- and $(\xi, 1)$-elliptical distributions with parameters $\mu_X \in \mathbb{R}^N$, $\Sigma_X \in \mathbb{R}^{N \times N}$ and $\mu_Y \in \mathbb{R}$, $\Sigma_Y \in \mathbb{R}$. We also denote $\Sigma_{X,Y}$ the covariance vector between $X$ and $Y$. The aim is thus to provide a predictor for the quantile of the conditional distribution $Y|X = x$. According to Theorem 7 of [Frahm (2004)], such
The existence of Assumption 1 (Second order regular variations) in Maume-Deschamps et al. (2017a) that such parameters not always exist for all elliptical distribution \( \eta \) some examples of coefficients \( A \) in Maume-Deschamps et al. (2017a), we have given a predictor for conditional quantiles, based solely on \( X \) a good prediction of (2.5) \( \Phi \), we only observe the unconditional distribution of \( X \). This is why, in Maume-Deschamps et al. (2017a), we have given a predictor for conditional quantiles, based solely on the unconditional c.d.f \( \Phi_R(t) = \mathbb{P}(R^* U^{(1)} \leq t) \):

\[
q_{\alpha}(Y|X = x) = \mu_Y|X + \sigma_Y|X \Phi_R^{-1}(\alpha) \tag{2.6}
\]

where \( \alpha \in [0,1], \mu_Y|X = \mu_Y + \Sigma_{XY} \Sigma_X^{-1} (x - \mu_X) \) and \( \sigma_Y^2|X = \Sigma_Y - \Sigma_{XY} \Sigma_X^{-1} \Sigma_{XY} \). Thus, in order to give a good prediction of \( q_{\alpha}(Y|X = x) \), we need to estimate the conditional function \( \Phi_R^{-1} \). Unfortunately, when we have a data set \( X_1, \ldots, X_n \), we only observe the unconditional distribution of \( X \). This is why, in Maume-Deschamps et al. (2017a), we have given a predictor for conditional quantiles, based solely on the unconditional c.d.f \( \Phi_R(t) = \mathbb{P}(R^* U^{(1)} \leq t) \):

\[
q_{\alpha}(Y|X = x) = \mu_Y|X + \sigma_Y|X \left[ \Phi_R^{-1} \left( 1 - \frac{1}{\ell + 2(1-\ell)} \right) \right]^{1/\eta} \tag{2.6}
\]

where the parameters \( \eta \in \mathbb{R} \) and \( 0 < \ell < +\infty \) are such that:

\[
\lim_{\ell \to \infty} \frac{\Phi_R(t)}{\Phi_R(t^\eta)} = \ell, \tag{2.7}
\]

From there, we have proved that \( q_{\alpha}(Y|X = x) \) and \( q_{\alpha}(Y|X = x) \) were asymptotically equivalent as \( \alpha \to 1, \) i.e

\[
\left[ \Phi_R^{-1} \left( 1 - \frac{1}{\ell + 2(1-\ell)} \right) \right]^{1/\eta} \sim_{\alpha \to 1} \Phi_R^{-1}(\alpha) \tag{2.8}
\]

A similar equivalence has been easily deduced for \( \alpha \to 0 \), using the symmetry properties of elliptical distributions. In this paper, we focus on the case \( \alpha \to 1 \), case \( \alpha \to 0 \) being easily deduced. Table I gives some examples of coefficients \( \eta \) and \( \ell \) for classical elliptical distributions.

The first aim, in this paper, is to propose some estimators of \( \eta \). We have shown in Maume-Deschamps et al. (2017a) that such parameters not always exist for all elliptical distribution (see, e.g., Laplace distribution). In the following, we make an assumption on the radius \( R \) that will ensures the existence of \( \eta \) and \( \ell \).

**Assumption 1** (Second order regular variations). We assume that there exist a function \( A \) such that \( A(t) \to 0 \) as \( t \to +\infty \), and

\[
\lim_{t \to +\infty} \frac{\Phi_R^{-1}(1-t)}{A(t)} = \omega_\gamma = \frac{\omega_\rho - 1}{\rho} \tag{2.9}
\]

where \( \gamma > 0 \) and \( \rho \leq 0 \).

This assumption is widespread in literature of extreme quantiles (see, e.g., Daouia et al. (2017a)). A first consequence is that \( \Phi_R \), or equivalently \( F_R \), is attracted to the maximum domain of Pareto-type distributions with tail index \( \gamma \). Furthermore, it entails \( \Phi_R^{-1}(1-t) \sim c_1 t^{\gamma} \), or equivalently \( \Phi_R(t) \sim c_2 t^{\gamma} \) as \( t \to +\infty \) (see de Haan and Ferreira (2006)). As examples, Student and Slash distributions satisfy Assumption 1. The following lemma provides some results concerning asymptotic equivalences.
Lemma 2.1 (Regular variation properties). Under Assumption $\mathcal{A}$ we get the following regular variations properties:

(i) The random variable $\xi$ satisfies

$$F_\xi(t) \overset{t \to +\infty}{\sim} \lambda^{-\frac{1}{\gamma}}, \lambda \in \mathbb{R}$$

(ii) For all $d \in \mathbb{N}^*$, the random variable $R_d = \chi_d \xi$ is attracted to the maximum domain of Pareto-type distribution with tail index $\gamma$, and

$$\hat{F}_\xi(t) \overset{t \to +\infty}{\sim} 2\pi \frac{d + \gamma - 1}{2} \lambda^{-\frac{1}{\gamma}}, \lambda \in \mathbb{R}$$

(iii) For all $\eta > 0$, $d \in \mathbb{N}^*$,

$$\frac{f_{R_d}(t)}{f_{R_1}(t)} \overset{t \to +\infty}{\sim} \frac{\sqrt{\pi} \Gamma \left( \frac{d + \gamma - 1}{2} \right)}{\Gamma \left( \frac{d}{2} \right) \Gamma \left( \frac{1 + \gamma - 1}{2} \right)} t^{(\eta - 1)(\gamma - 1) + 1}$$

In Section 3 we prove the existence of $\eta$ and $\ell$, and propose some estimators. Before that, we need to do a little simplification. Indeed, Equation (2.6) shows that the extreme quantile estimation requires the prior estimation of quantities $\mu_{Y|X}$ and $\sigma_{Y|X}$. These quantities may be easily estimated by the method of moments or fixed-point algorithm (c.f p.66 of Frahm (2004)). In a spatial setting, even if the variable $Y$ is not observed, a stationarity assumption on the random field makes it possible to estimate these values (see Cressie (1988)). Furthermore, the speed of convergence of these methods is higher than those of the estimators we propose in this paper, and therefore do not interfere in the asymptotic results. This is why, in the following, we suppose that $\mu_{Y|X}$, $\sigma_{Y|X}$, and therefore $\mu_X$, $\Sigma_X$ are known. Then, it remains to estimate $\eta$, $\ell$ and $\Phi_{R_1}^{-1}$. Section 3 focuses on $\eta$ and $\ell$, while Section 4 deals with $\Phi_{R_1}^{-1}$.

3. Extremal coefficients estimation

In this section, the aim is to estimate the extremal parameters $\eta$ and $\ell$ conditionaly to the co-variates vector $X = x$. For that purpose, we consider a random sample $X_1, \ldots, X_n$ independent and identically distributed from an $(\xi, N)$-elliptical vector with the same distribution as $X$, and denote $q_X = (x - \mu_X)^T \Sigma_X^{-1} (x - \mu_X)$. The first step is to verify if $\eta$ and $\ell$ exist.

Proposition 3.1 (Existence of extremal parameters). Under Assumption $\mathcal{A}$ parameters $\eta$ and $\ell$ exist, and are expressed:

$$\eta = \frac{\Gamma \left( \frac{N + \gamma - 1}{2} \right)}{\Gamma \left( \frac{N + \gamma - 1}{2} \right)} 1 + \gamma N$$

$$\ell = \frac{\gamma - 1}{\gamma N} \frac{1}{\Gamma \left( \frac{N + \gamma - 1}{2} \right)} \frac{1}{\Gamma \left( \frac{N + \gamma - 1}{2} \right)} \frac{\gamma - 1}{\gamma N} \frac{1}{\Gamma \left( \frac{N + \gamma - 1}{2} \right)}$$

The aim is now to give two suitable estimator $\hat{\eta}$ and $\hat{\ell}$, respectively for $\eta$ and $\ell$.

3.1. Estimation of $\eta$. We notice that coefficient $\eta$ is directly related on the tail index $\gamma$. Then, using a suitable estimator of $\gamma$, we easily deduce $\eta$. There are several estimators widespread in the literature. As examples, Pickands (1975), Schwizer and Steinemabch (1996) or Kratz and Resnick (1996) provide some estimators for $\gamma$. In the following, we use the Hill estimator, introduced in Hill (1975):

$$\hat{\eta}_{k_n} = \frac{1}{k_n} \sum_{i=1}^{k_n} \ln \left( \frac{W[i]}{W[k_{n+1}]} \right)$$

where $W[1] \geq \ldots \geq W[k_{n+1}] \geq \ldots \geq W[n]$ and $k_n = o(n)$ such that $k_n \to +\infty$ as $n \to +\infty$. In this context, the statistic $W$ may be:

- The first (or indifferently any) component of the reduced centered covariate vector $A_X^{-1} (X - \mu_X)$, where $A_X^{-1} \Lambda_X = \Sigma_X$. This approach works well, but we do not use all available data.
- The Mahalanobis norm $\sqrt{(X - \mu_X)^T \Sigma_X^{-1} (X - \mu_X)}$. This approach has the advantage of using all available data.
In the following we will use the one-component approach, since the asymptotic results we give are valid under Assumption 1 applied to the univariate c.d.f \( \Phi_R \). Moreover, numerical comparisons seem show that the second approach does not significantly improve the estimation of the parameters. Main properties of \( \hat{\gamma}_{k_n} \) may be found in de Haan and Resnick (1998). Under second order condition given in Assumption 1, de Haan and Ferreira (2006) proved the following asymptotic normality for \( \hat{\gamma}_{k_n} \).

\[
\sqrt{k_n} (\hat{\gamma}_{k_n} - \gamma) \overset{d}{\to} N\left(0, \frac{\lambda}{1 - \beta^2} \gamma^2 \right)
\]

where \( \lambda = \lim_{n \to +\infty} \sqrt{k_n} A \left( \frac{u}{k_n} \right) \) and \( k_n = o(n) \) such that \( k_n \to +\infty \) as \( n \to +\infty \). Then, using Proposition 3.1 and Equation (3.2), we define the following estimator for \( \eta \).

**Definition 3.1 (Estimator of \( \eta \)).** We define \( \hat{\eta}_{k_n} \) as

\[
\hat{\eta}_{k_n} = \frac{N}{k_n} \sum_{i=1}^{k_n} \ln \left( \frac{W[i]}{W[k_n+1]} \right) + 1
\]

As an affine transformation of Hill estimator, asymptotic normality of \( \hat{\eta}_{k_n} \) is obvious.

**Proposition 3.2 (Asymptotic normality of \( \hat{\eta}_{k_n} \)).** Under Assumption 1, and if \( \lim_{n \to +\infty} \sqrt{k_n} A \left( \frac{u}{k_n} \right) = 0 \), then

\[
\sqrt{k_n} (\hat{\eta}_{k_n} - \eta) \overset{d}{\to} N\left(0, N^2 \gamma^2 \right)
\]

3.2. **Estimation of \( \ell \).** The form of \( \ell \), given in Proposition 3.1, leads to a more complicated estimation. Indeed, \( \ell \) is related on both \( \gamma \) and \( c_N g_N(q_X) \). Our estimator for \( \gamma \) is given in Equation (3.2). Concerning \( c_N g_N(q_X) \), we propose a kernel estimator. Class of kernel estimators, introduced in Parzen (1962), makes it possible to estimate probability densities. Then, the following lemma will be useful for the construction of our estimator. This result comes from p.108 of Johnson (1987).

**Lemma 3.3.** The Mahalanobis distance \( (X - \mu_X)^T \Sigma_X^{-1} (X - \mu_X) \) has density:

\[
\frac{\pi^\frac{N}{2}}{\Gamma \left( \frac{N}{2} \right) } x^{\frac{N}{2} - 1} c_N g_N(x)
\]

Using Lemma 3.3, we introduced a kernel estimator \( \hat{g}_{h_n} \) for \( c_N g_N(q_X) \).

**Definition 3.2 (Generator estimator).** We define \( \hat{g}_{h_n} \) as

\[
\hat{g}_{h_n} = \frac{q_X^{1 - \frac{N}{2}} \Gamma \left( \frac{N}{2} \right) }{\pi^\frac{N}{2} n h_n} \sum_{i=1}^{n} K \left( \frac{q_X - (X_i - \mu_X)^T \Sigma_X^{-1} (X_i - \mu_X) }{h_n} \right)
\]

where the kernel \( K \) fills some conditions given in Parzen (1962) and bandwidth \( h_n \) verifies \( h_n \to 0 \) and \( nh_n \to +\infty \) as \( n \to +\infty \).

Parzen (1962) provided the asymptotic normality for kernel estimators. Furthermore, Lemma 3.3 leads to the following asymptotic normality for \( \hat{g}_{h_n} \).

**Proposition 3.4 (Asymptotic normality of generator estimator).** The following relationship holds:

\[
\sqrt{n h_n} \left( \hat{g}_{h_n} - c_N g_N(q_X) \right) \overset{d}{\to} N\left(0, \frac{q_X^{1 - \frac{N}{2}} \Gamma \left( \frac{N}{2} \right) }{\pi^\frac{N}{2} } c_N g_N(q_X) \int K(u)^2 du \right)
\]

Replacing \( \gamma \) by \( \hat{\gamma} \) and \( c_N g_N(q_X) \) by \( \hat{g}_{h_n} \) in Equation (3.1), we are now able to provide an estimator \( \hat{\ell} \) for \( \ell \), in the following definition. Furthermore, under Assumption 1 we give the asymptotic normality of \( \hat{\ell} \).
Definition 3.3 (Estimator of $\ell$). We define $\hat{\ell}_{k_n,h_n}$ as:

$$\hat{\ell}_{k_n,h_n} = \frac{\Gamma \left( \frac{N+\gamma_k-1}{2} \right)}{\Gamma \left( \frac{\gamma_k+1}{2} \right)} \frac{\gamma_k^{-1} \pi^{-N/2}}{(N+\gamma_k^{-1}) \hat{g}_{k_n}}$$

where $\gamma_k$ and $\hat{g}_{k_n}$ are respectively given in Equations (3.2) and (3.7).

Proposition 3.5. Under Assumption 1 and if $\lim_{n \to +\infty} \sqrt{k_n}A\left(\frac{\gamma}{k_n}\right) = 0$, the following asymptotic relationships hold:

(i) If $nh_n/k_n \to +\infty$, then

$$\sqrt{k_n} \left( \hat{\ell}_{k_n,h_n} - \ell \right) \overset{d}{\to} N(0, V_1(\gamma, cNgN(q_X)))$$

(ii) If $nh_n/k_n \to 0$, then

$$\sqrt{nh_n} \left( \hat{\ell}_{k_n,h_n} - \ell \right) \overset{d}{\to} N(0, V_2(\gamma, cNgN(q_X)))$$

(iii) If $nh_n/k_n \to c \in \mathbb{R}^+$, then

$$\sqrt{k_n} \left( \hat{\ell}_{k_n,h_n} - \ell \right) \overset{d}{\to} N(0, V_1(\gamma, cNgN(q_X)) + cV_2(\gamma, cNgN(q_X)))$$

where ($\Psi$ is the digamma function (see p.258 of Abramovitz et al. (1966)))

$$V_1(\gamma, cNgN(q_X)) = \frac{1}{c^{\gamma_k N(q_X)/2}} \frac{\Gamma \left( \frac{N+\gamma_k-1}{2} \right)}{\Gamma \left( \frac{\gamma_k+1}{2} \right)} \frac{\psi \left( \frac{N+\gamma_k-1}{2} \right) - \psi \left( \frac{\gamma_k+1}{2} \right)}{\Gamma \left( \frac{\gamma_k+1}{2} \right)} - \frac{N}{\left( N+\gamma_k^{-1} \right)^2}$$

$$V_2(\gamma, cNgN(q_X)) = \frac{\Gamma \left( \frac{\gamma}{2} \right)}{c^{\frac{\gamma}{2}}} \frac{\Gamma \left( \frac{N+\gamma_k-1}{2} \right)}{\Gamma \left( \frac{\gamma_k+1}{2} \right)} \text{E}_{NgN(q_X)} \int K(u)^2 \frac{\gamma^{-1} - \frac{2}{\gamma_k}}{(N+\gamma_k^{-1})^2}$$

We have the asymptotic normality for our estimators $\hat{\eta}_{k_n}$ and $\hat{\ell}_{k_n,h_n}$. The next proposition gives the joint distribution according to the asymptotic relations between $k_n$ and $h_n$. The proof derives from delta method.

Proposition 3.6. Under Assumption 1 and if $\sqrt{k_n}A\left(\frac{\gamma}{k_n}\right) \to 0$ as $n \to +\infty$, then the following asymptotic relationships hold:

(i) If $nh_n/k_n \to 0$, then

$$\sqrt{nh_n} \left( \hat{\ell}_{k_n,h_n} - \ell \right) \overset{d}{\to} N(0, V_2(\gamma, cNgN(q_X)))$$

where $V_2(\gamma, cNgN(q_X))$ is given in Equation (3.13).

(ii) If $nh_n/k_n \to +\infty$, then

$$\sqrt{k_n} \left( \hat{\ell}_{k_n,h_n} - \ell \right) \overset{d}{\to} N(0, -N^2 \gamma^2 \sqrt{V_1(\gamma, cNgN(q_X))})$$

where $V_1(\gamma, cNgN(q_X))$ is given in Equation (3.13).

(iii) If $nh_n/k_n \to c \in \mathbb{R}^+$, then

$$\sqrt{k_n} \left( \hat{\ell}_{k_n,h_n} - \ell \right) \overset{d}{\to} N(0, -N^2 \gamma^2 \sqrt{V_1(\gamma, cNgN(q_X))})$$

where $V_1(\gamma, cNgN(q_X))$ and $V_2(\gamma, cNgN(q_X))$ are given in Equation (3.13).

Using the previous results, we propose, in Section 4 some estimators of extreme conditional quantiles based on $\ell_{k_n,h_n}$ and $\hat{\eta}_{k_n}$. 

4. Extreme quantiles estimation

In this section, we propose some estimators of extreme quantiles $\Phi^{-1}_R(\alpha_n)$, for a sequence $\alpha_n \to 1$ as $n \to +\infty$. We recall that $\Phi^{-1}_R(\alpha_n) = q_{\alpha_n}(R^U(1))$ corresponds to the quantile of the random variable $R^U(1)$ with level $\alpha_n$. For that purpose, we divide the study in two cases:

- Intermediate quantiles, i.e. we suppose $n(1-\alpha_n) \to +\infty$. It entails that the estimation of the $\alpha_n$-quantile leads to an interpolation of sample results.
- High quantiles. According to de Haan and Rootzén (1993), we suppose $n(1-\alpha_n) \to 0$, i.e. we need to extrapolate sample results to areas where no data are observed.

4.1. Conditions. The results we provide in this section hold under certain conditions concerning the sequences $k_n$, $n$ and $\alpha$. We recall $k_n \to +\infty$, $k_n = o(n)$, $h_n \to 0$, $nh_n \to +\infty$ as $n \to +\infty$, and define conditions $C$, $C_{int}$ and $C_{high}$ as follows:

- (C): $k_n = o(nh_n)$ and $\sqrt{k_n}A\left(\frac{\alpha}{k_n}\right) \to 0$ as $n \to +\infty$.
- (C$_{int}$): $n(1-\alpha_n) \to +\infty$, $\ln(1-\alpha_n) = o(\sqrt{k_n})$ and $\frac{\sqrt{k_n}}{\ln(1-\alpha_n)} = o\left(\sqrt{n(1-\alpha_n)}\right)$ as $n \to +\infty$.
- (C$_{high}$): $n(1-\alpha_n) \to 0$, $\ln(n(1-\alpha_n)) = o(\sqrt{k_n})$ and $\frac{\ln(1-\alpha_n)}{\ln\left(\frac{\alpha}{k_n}\right)} \to \theta \in [0, +\infty]$ as $n \to +\infty$.

Condition (C) will be common to both approaches, and ensures in a first time that Hill estimator is unbiased, according to Equation (3.3). Moreover, $k_n = o(nh_n)$ means that $\gamma_{k_n}$ (respectively $\eta_{k_n}$) converges to $\gamma$ (respectively $\eta$) faster than $\ell_{k_n}h_n$ to $\ell$. In practice, this condition seems appropriate, because $k_n$ must not be too large for the Hill estimator to be unbiased, and $h_n$ must be tall enough to provide a good estimation of $\ell$.

Conditions (C$_{int}$) and (C$_{high}$) will be used respectively for intermediate and high quantile approaches, and are only related to sequences $k_n$ and $\alpha_n$. In order to clarify these conditions, let us propose a simple example: we choose our sequences in polynomial forms $k_n = n^b$, $0 < b < 1$ and $\alpha_n = 1 - n^{-a}$, $a > 0$. It is straightforward to see that $\ln(1-\alpha_n) = o(\sqrt{k_n})$ and $\ln(n(1-\alpha_n)) = o(k_n)$, $\forall a > 0, 0 < b < 1$. However, $\frac{\sqrt{k_n}}{\ln(1-\alpha_n)} = o\left(\sqrt{n(1-\alpha_n)}\right)$ if and only if $a < 1$, i.e. $n(1-\alpha_n) \to +\infty$ as $n \to +\infty$. Otherwise, if $a > 1$ (or equivalently $n(1-\alpha_n) \to 0$ as $n \to +\infty$), then (C$_{high}$) is filled with a particular $\theta$ given later. We also add stronger conditions (C$^{HG}_{int}$), (C$^{HG}_{high}$), (C$^{Lp}_{int}$) and (C$^{Lp}_{high}$) used later for asymptotic normality results:

- (C$^{HG}_{int}$): (C$_{int}$) holds. In addition, $\sqrt{k_n}(1-\alpha_n) = o(\ln(1-\alpha_n))$, and one of the following statements is fulfilled:
  (i) $\rho > -2\gamma$ and $\lim_{n \to +\infty} \frac{\sqrt{k_n}}{\ln(1-\alpha_n)} (1-\alpha_n)^{-\frac{\gamma}{\rho}} = 0$,
  (ii) $\rho \leq -2\gamma$ and $\lim_{n \to +\infty} \frac{\sqrt{k_n}}{\ln(1-\alpha_n)} (1-\alpha_n)^{\frac{\gamma}{\rho}} = 0$,

- (C$^{HG}_{high}$): (C$_{high}$) holds. In addition, one of the following statements is fulfilled:
  (i) $\rho > -2\gamma$ and $\lim_{n \to +\infty} \frac{\sqrt{k_n}}{\ln(1-\alpha_n)} (1-\alpha_n)^{-\frac{\gamma}{\rho}} = 0$,
  (ii) $\rho \leq -2\gamma$ and $\lim_{n \to +\infty} \frac{\sqrt{k_n}}{\ln(1-\alpha_n)} (1-\alpha_n)^{\frac{\gamma}{\rho}} = 0$,

- (C$^{Lp}_{int}$): (C$_{int}$) holds. In addition, $\sqrt{k_n}(1-\alpha_n) = o(\ln(1-\alpha_n))$, and one of the following statements is fulfilled:
  (i) $\rho > -\gamma$ and $\lim_{n \to +\infty} \frac{\sqrt{k_n}}{\ln(1-\alpha_n)} (1-\alpha_n)^{-\frac{\gamma}{\rho}} = 0$,
  (ii) $\rho \leq -\gamma$ and $\lim_{n \to +\infty} \frac{\sqrt{k_n}}{\ln(1-\alpha_n)} (1-\alpha_n)^{\frac{\gamma}{\rho}} = 0$.

- (C$^{Lp}_{high}$): (C$_{high}$) holds. In addition, one of the following statements is fulfilled:
  (i) $\rho > -\gamma$ and $\lim_{n \to +\infty} \frac{\sqrt{k_n}}{\ln(1-\alpha_n)} (1-\alpha_n)^{-\frac{\gamma}{\rho}} = 0$,
  (ii) $\rho \leq -\gamma$ and $\lim_{n \to +\infty} \frac{\sqrt{k_n}}{\ln(1-\alpha_n)} (1-\alpha_n)^{\frac{\gamma}{\rho}} = 0$.

These conditions, depending on the sequences $k_n$, $\alpha_n$ and extreme value indices $\gamma$ and $\rho$ introduced in Assumption [1], are not involved in the consistency results. However, we need these conditions for asymptotic normality results.
To sum up, among all these conditions, we can deduce the following ordering:

\[
\begin{align*}
(C^{\text{int}}_n) & \Rightarrow (C^{\text{HG}}) \Rightarrow (C) \\
(C^{\text{int}}_p) & \Rightarrow (C^{\text{HG}}) \Rightarrow (C) \\
(C^{\text{high}}_p) & \Rightarrow (C^{\text{HG}}) \Rightarrow (C) \\
(C^{\text{high}}) & \Rightarrow (C)
\end{align*}
\]

4.2. Intermediate quantiles. We consider the case where \(n(1-\alpha_n) \to +\infty\) as \(n \to +\infty\). Based on Theorem 2.4.1 in de Haan and Ferreira [2006], we introduce the following estimator \(\hat{q}_{\alpha_n}(R^*U^{(1)})\) for \(q_{\alpha_n}(R^*U^{(1)}) = \Phi_R^{-1}(\alpha_n)\).

**Definition 4.1** (Intermediate quantile estimator). We define \((\hat{q}_{\alpha_n}(R^*U^{(1)}))_{n \in \mathbb{N}}\) as:

\[
(4.1) \quad \hat{q}_{\alpha_n}(R^*U^{(1)}) = (W_{[n\hat{p}_n+1]})_{\alpha_n}
\]

where \(\hat{p}_n = \left(2 + \hat{\ell}_{\alpha_n, \alpha_n} \left(1 - (1-\alpha_n)^{-1} - 2\right)\right)^{-1}\) and \(W\) is the first (or indifferently any) component of the vector \(\Lambda^X_{\alpha_n}(X - \mu_X)\).

Using some more or less strong assumptions, we can deduce asymptotic normality results. In a first time, we give a result concerning the asymptotic behavior of \(\hat{q}_{\alpha_n}(R^*U^{(1)})\) with respect to \(\Phi_R^{-1}(1-p_n)^{\frac{1}{\gamma}}\). Then, with Equation (2.8), we easily deduce a consistency result for \(\hat{q}_{\alpha_n}(R^*U^{(1)})\).

**Theorem 4.1** (Consistency of intermediate quantile estimator). Let us denote \(p_n = \left(2 + \ell \left((1-\alpha_n)^{-1} - 2\right)\right)^{-1}\) and \(\tilde{p}_n = \left(2 + \hat{\ell}_{\alpha_n, \alpha_n} \left(1 - (1-\alpha_n)^{-1} - 2\right)\right)^{-1}\). Under Assumption [4.1] and conditions (C), (C\text{int}):

\[
(4.2) \quad \frac{\sqrt{k_n}}{\ln (1-\alpha_n)} \left(\hat{q}_{\alpha_n}(R^*U^{(1)}) - 1\right) \sim_{n \to +\infty} \mathcal{N}(0, \frac{N^{2\gamma^4}}{(\gamma N + 1)^4})
\]

And therefore:

\[
(4.3) \quad \frac{\hat{q}_{\alpha_n}(R^*U^{(1)})}{\Phi_R^{-1}(\alpha_n)} \xrightarrow{p} 1
\]

The same asymptotic normality with \(\Phi_R^{-1}(\alpha_n)\) instead of \(\Phi_R^{-1}(1-p_n)^{\frac{1}{\gamma}}\) may be deduced from Proposition [4.1] under the condition

\[
\lim_{n \to +\infty} \frac{\sqrt{k_n}}{\ln (1-\alpha_n)} \ln \left(\frac{\Phi_R^{-1}(1-p_n)^{\frac{1}{\gamma}}}{\Phi_R^{-1}(\alpha_n)}\right) = 0
\]

This condition, which seems quite simple, is difficult to prove in a general context. Indeed, we need a second order expansion of Equation (2.8). But the second order properties of the unconditional quantile \(\Phi_R^{-1}\) given by Assumption [4.1] are not necessarily the same as those of the conditional quantile \(\Phi_R^{-1}\), which makes the study complicated. However, in some simple cases, we are able to solve the problem. We thus give another assumption, stronger than Assumption [4.1]. In the following, we refer to this assumption for results of asymptotic normality.

**Assumption 2.** \(\forall d \in \mathbb{N}^*,\) there exists \(\lambda_1, \lambda_2 \in \mathbb{R}\) such that:

\[
(4.4) \quad c_d g_d(t) = \lambda_1 t^{-\frac{d-1}{\gamma}} \left[1 + \lambda_2 t^{\frac{\rho}{\gamma}} + o \left(t^{\frac{\rho}{\gamma}}\right)\right]
\]

It is obvious that Assumption [4.1] implies Assumption [4.2]. Indeed, according to Hua and Joe [2011], Equation (4.4) is equivalent to say that \(c_2 g_2(t^2)\) is regularly varying of second order with indices \(-1-\gamma^{-1}, \rho/\gamma\) and an auxiliary function proportional to \(t^{\frac{\rho}{\gamma}}\). Then, Proposition 6 in Hua and Joe [2011] entails \(\Phi_R(t)\) is second order regularly varying with \(-\gamma^{-1}, \rho/\gamma\) and the same kind of auxiliary function. Finally, this is equivalent (see de Haan and Ferreira [2006]) to Assumption [4.1] with indicated \(\gamma\) and \(\rho\), and an auxiliary function \(A(t)\) proportional to \(t^{\rho/\gamma}\). As an example, the Student distribution fills Assumption [4.2].

**Proposition 4.2** (Asymptotic normality of intermediate quantile estimator). Assume that Assumption [4.2] and conditions (C), (C\text{int}) hold. Then:

\[
(4.5) \quad \frac{\sqrt{k_n}}{\ln (1-\alpha_n)} \left(\hat{q}_{\alpha_n}(R^*U^{(1)}) - 1\right) \sim_{n \to +\infty} \mathcal{N}(0, \frac{N^{2\gamma^4}}{(\gamma N + 1)^4})
\]
Theorem 4.3 (Consistency of high quantile estimator)

A consistency result that follows immediately is given just below. For the high quantile estimator, we propose a result of asymptotic normality, which we then refine under Assumption 2. The conditions (i) nor (ii) in Proposition 4.1 are filled if $N$ is large. Thus asymptotic normality (4.5) no longer holds. This is explained by the fact that more $N$ is tall, more $\Phi^{-1}_R(\alpha_n)$ (see Equation (2.8)) tends to 1 slowly.

4.3. High quantiles. We now consider $n(1 - \alpha_n) \to 0$ as $n \to +\infty$. In the following definition, we introduce another quantile estimator $\hat{q}_{\alpha_n} (R^* U(1))$ for $\Phi^{-1}_R(\alpha_n)$. The approach, inspired by de Haan and Ferreira (2006) or Weissman (1978), consists in taking an intermediate sequence $k_n$, and of using the regular variations properties.

Definition 4.2 (High quantile estimator). We define $(\hat{q}_{\alpha_n} (R^* U(1)))_{n \in \mathbb{N}}$ as:

$$\hat{\alpha}_n \left( R^* U(1) \right) = \left[ W_{[k_n+1]} \left( \frac{k_n}{n} \left( 2 + \hat{e}_{k_n \alpha_n} \left( \frac{1}{1 - \alpha_n} - 2 \right) \right) \right) \right]^{1/n}$$

We define $\hat{q}_{\alpha_n} (R^* U(1))$ as:

$$\hat{\alpha}_n \left( R^* U(1) \right) = \left[ W_{[k_n+1]} \left( \frac{k_n}{n} \left( 2 + \hat{e}_{k_n \alpha_n} \left( \frac{1}{1 - \alpha_n} - 2 \right) \right) \right) \right]^{1/n}$$

The aim is now to study the asymptotic properties of $\hat{q}_{\alpha_n} (R^* U(1))$. As for the intermediate quantile estimator, we propose a result of asymptotic normality, which we then refine under Assumption 2. The consistency result that follows immediately is given just below.

Theorem 4.3 (Consistency of high quantile estimator). Let us denote $p_n = (2 + \ell \left( (1 - \alpha_n) - 2 \right))^{-1}$ and $\tilde{p}_n = \left( 2 + \hat{e}_{k_n \alpha_n} (1 - \alpha_n) - 2 \right)^{-1}$. Under Assumption 2 and conditions (C), (C(high)):

$$\frac{\hat{q}_{\alpha_n} (R^* U(1))}{\Phi^{-1}_R(\alpha_n)} \sim \mathcal{N} \left( 0, \frac{\gamma^2}{\gamma N + \gamma} \right)$$

And therefore:

$$\frac{\hat{q}_{\alpha_n} (R^* U(1))}{\Phi^{-1}_R(\alpha_n)} \to 1 \text{ as } n \to +\infty$$

We can emphasize that condition (C(high)) is filled in most of the common cases. Indeed, the simple examples to find that do not satisfy (ii) are of the form $\alpha_n = 1 - n^{-1} \ln(n)^{-\kappa}$, $\kappa > 0$ and $k_n = \ln(n)$. But such a choice of sequences would lead to a poor estimation of $\gamma_{k_n}$ and $\eta_{k_n}$, since $k_n \to +\infty$ very slowly, and moreover a poor estimation of the quantile, the level $\alpha_n$ tending to 1 slowly. These sequences are therefore not recommanded in practice. Next corollary gives the value of $\theta$ when sequences $k_n$ and $\alpha_n$ have a polynomial form.

Corollary 4.4. Under Assumption 2 and conditions (C), (C(high)), and taking $k_n = n^b, 0 < b < 1$ and $\alpha_n = 1 - n^{-a}, a > 1$, asymptotic relationship (4.7) holds with $\theta = \frac{a}{a+b-1}$.

As for the intermediate quantile estimator, asymptotic normality (4.7) may be improved under the condition

$$\lim_{n \to +\infty} \frac{\sqrt{k_n}}{n^{1/(1-\alpha_n)}} \ln \left( \frac{k_n}{\pi(1-\alpha_n)} \right) = 0$$

Assumption 2 places us in a framework where it is quite simple to prove it, hence the following result.

Proposition 4.5 (Asymptotic normality of high quantile estimator). Assume that Assumptions 2 and conditions (C), (C(high)) hold. Then:

$$\frac{\sqrt{k_n}}{n \ln \left( \frac{k_n}{\pi(1-\alpha_n)} \right)} \left( \hat{q}_{\alpha_n} (R^* U(1)) - 1 \right) \sim \mathcal{N} \left( 0, \frac{\gamma^2}{\gamma N + \gamma} \right)$$

We can make the same kind of remark as in the previous subsection when $N$ is large. In the following, we give estimators for two other classes of extreme risk measures, based on the estimators given in Equations (4.1) and (4.6). The first one generalizes quantiles.
5. SOME EXTREME RISK MEASURES ESTIMATORS

5.1. $L_p$-quantiles. Let $Z$ be a real random variable. The $L_p$-quantiles of $Z$ with level $\alpha \in [0,1]$ and $p > 0$, denoted $q_{p,\alpha}(Z)$, is solution of the minimization problem (see Chen (1996)).

\[(5.1)\]

\[q_{p,\alpha}(Z) = \arg\min_{z \in \mathbb{R}} E\left[(1 - \alpha)(z - Z)^p + \alpha (Z - z)^p\right] \]

where $Z_+ = Z I_{\{Z > 0\}}$. According to Koenker and Bassett (1978), the case $p = 1$ leads to the quantile $q_{1,\alpha}(Z) = F_Z^{-1}(\alpha)$, where $F_Z$ is the c.d.l of $Z$. The case $p = 2$, formalized in Newey and Powell (1987), leads to more complicated calculations, and admits, with the exception of some particular cases (see, e.g., Koenker (1992)), no general formula. The general case $p > 0$ has seen some recent advances. Indeed, the particular case of Student distributions has, for example, been explored in Bernardi et al. (2017). However, it seems difficult to obtain a general formula. On the other hand, in the case of extreme levels $\alpha$, i.e. when $\alpha$ tends to 1, Daouia et al. (2017b) proved that the following relationship holds, for a heavy-tailed random variable with tail index $\gamma$.

\[(5.2)\]

\[q_{p,\alpha}(Z) \sim_{\alpha \to 1} \left[\beta(p, \gamma^{-1} - p + 1)\right]^{-\frac{1}{\gamma}} \]

where $\beta$ is the beta function. We add that for a Pareto-type distribution with tail index $\gamma$, the $L_p$-quantile exists if and only if the moment of order $p - 1$ exists, i.e. if $\gamma < 1/p$. The expedite case $p = 2$ leads to the result of Bellini et al. (2014). Using this result, we can estimate the conditional $L_p$-quantiles from the quantile estimated in Section 4. For that purpose, we need to know the tail index of the conditional radius $R^*$, given in the following lemma.

**Lemma 5.1.** The conditional distribution $Y|X = x$ is attracted to a maximum domain of Pareto-type distribution with tail index $(\gamma^{-1} + N)^{-1}$, i.e.

\[(5.3)\]

\[\lim_{t \to +\infty} \frac{\Phi_{R^*(\omega t)}}{\Phi_{R^*(t)}} = \omega^{-\frac{1}{\gamma^{-1} + N}} \]

With Lemma 5.1 and Equation (5.2), we define the following estimators for the $L_p$-quantile of the reduced and centered elliptical distribution $\left(Y|X - \mu_Y|X\right)/\sigma_Y|X \overset{d}{=} R^*U^{(1)}$, according to whether $n(1 - \alpha_n)$ tends to 0 or $+\infty$.

**Definition 5.1.** Let $(\alpha_n)_{n \in \mathbb{N}}$ be a sequence such that $\alpha_n \to 1$ as $n \to +\infty$. If either $p \leq N$ or $\gamma < \frac{1}{p - N}$, we define:

\[(5.4)\]

\[
\begin{cases}
\hat{q}_{p,\alpha_n}(R^*U^{(1)}) = \left[\frac{(\gamma_n^{-1} + N)^{-1}}{\beta[p, \gamma_n^{-1} - N + p + 1]}\right]^{-\frac{1}{\gamma_n^{-1} + N}} \hat{q}_{\alpha_n}(R^*U^{(1)}) \\
\check{q}_{p,\alpha_n}(R^*U^{(1)}) = \left[\frac{(\gamma_n^{-1} + N)^{-1}}{\beta[p, \gamma_n^{-1} + N - p + 1]}\right]^{-\frac{1}{\gamma_n^{-1} + N}} \check{q}_{\alpha_n}(R^*U^{(1)})
\end{cases}
\]

where $\gamma_n$, $\hat{q}_{\alpha_n}(R^*U^{(1)})$, and $\check{q}_{\alpha_n}(R^*U^{(1)})$ are respectively given in Equations (3.2), (4.1) and (4.6).

We have proved the convergence in probability of $\hat{q}_{\alpha_n}(R^*U^{(1)})$ and $\check{q}_{\alpha_n}(R^*U^{(1)})$. Furthermore, the convergence in probability of the asymptotic term, and consequently the empirical $L_p$-quantile is not difficult to get, this is why we omit the proof.

**Proposition 5.2 (Consistency of $L_p$-quantile estimators).** Assume that Assumption [4] and condition (C) hold. Under conditions (C_{int}) and (C_{high}) respectively, $q_{p,\alpha_n}(R^*U^{(1)})$ and $\check{q}_{p,\alpha_n}(R^*U^{(1)})$ are consistent, i.e.:

\[(5.5)\]

\[
\begin{cases}
q_{p,\alpha_n}(R^*U^{(1)}) \overset{p}{\to} 1 \\
\check{q}_{p,\alpha_n}(R^*U^{(1)}) \overset{p}{\to} 1
\end{cases}
\]

Using the second order expansion of Equation (5.2) given in Daouia et al. (2017b), and doing some stronger assumptions, we can easily deduce the following asymptotic normality results.
Proposition 5.3 (Asymptotic normality of $L_p$-quantile estimators). Assume that Assumption 2 and condition (C) hold. Under conditions $(C_{int})$ and $(C_{high})$ respectively, and if $p > 1$, then:

\[
\begin{align*}
\lim_{n \to +\infty} \left( \frac{\hat{q}_{p,\alpha_n}(R^*U^{(1)}) - 1}{\sqrt{n}} \right) &\sim \mathcal{N} \left( 0, \frac{\gamma}{(\gamma N + 1)^{\gamma}} \right), \\
\lim_{n \to +\infty} \left( \frac{\hat{q}_{p,\alpha_n}(R^*U^{(1)}) - 1}{\sqrt{n}} \right) &\sim \mathcal{N} \left( 0, \frac{\gamma^2}{(\gamma N + 1)^{\gamma}} + \theta^2 \frac{\gamma^2}{(\gamma N + 1)^{\gamma}} \right).
\end{align*}
\]

An example of $L_2$-quantile, or expectile, is provided in Section 6. The second risk measure we focus on is called Haezendonck-Goovaerts risk measure.

5.2. Haezendonck-Goovaerts risk measures. Let $Z$ be a real random variable, and $\varphi$ a non-negative and convex function with $\varphi(0) = 0$, $\varphi(1) = 1$ and $\varphi(\pm \infty) = \pm \infty$. The Haezendonck-Goovaerts risk measure of $Z$ with level $\alpha \in ]0, 1]$ associated to $\varphi$, is given by the following (see Tang and Yang (2012)):

\[
H_\alpha(Z) = \inf_{Z \in \mathbb{R}} \{ z + H_\alpha(Z, z) \}
\]

where $H_\alpha(Z, z)$ is the unique solution $h$ to the equation:

\[
E \left[ \varphi \left( \frac{(Z - z)_+}{h} \right) \right] = 1 - \alpha
\]

$\varphi$ is called Young function. This family of risk measures has been firstly introduced as Orlicz risk measure in Haezendonck and Goovaerts (1982), then Haezendonck risk measure in Goovaerts et al. (2004), and finally Haezendonck-Goovaerts risk measure in Tang and Yang (2012). The particular case $\varphi(t) = t$ leads to the Tail Value at Risk with level $\alpha$ TVaR$_\alpha(X)$, introduced in Artzner et al. (1999). In Tang and Yang (2012), the authors provided the following result.

Proposition 5.4 (Tang and Yang (2012)). If $Z$ fills Assumption 2 and taking a Young function $\varphi(t) = t^p$, $p \geq 1$, then the following relationship holds:

\[
\frac{H_\alpha(Z)}{q_\alpha(Z)} \xrightarrow{\alpha \to 1} \gamma^{-1} \left( \gamma^{-1} - p \right)^{p^{-1}} \beta \left( \gamma^{-1} - p, p \right)
\]

In particular, taking $p = 1$ leads to TVaR$_\alpha(Z) \sim (1 - \gamma)^{-1} q_\alpha(Z)$ as $\alpha \to 1$. Using Lemma 5.1 extreme quantiles estimators in Definitions 4.1, 4.2 and Proposition 5.4 we can deduce estimators for extreme Haezendonck-Goovaerts risk measure $H_\alpha(R^*U^{(1)})$ (with power Young function $\varphi(t) = t^p$, $p \geq 1$) of the reduced and centered elliptical distribution $(Y|X - \mu Y|X)/\sigma Y|X \overset{d}{=} R^*U^{(1)}$.

Definition 5.2. Let $(\alpha_n)_{n \in \mathbb{N}}$ be a sequence such that $\alpha_n \to 1$ as $n \to +\infty$. If either $p \leq N$ or $\gamma < \frac{1}{p - N}$, we define:

\[
\begin{align*}
\hat{H}_\alpha_n (R^*U^{(1)}) &= \left( \frac{\gamma^{-1} + N}{\gamma^{-1} + N - p} \right)^{p^{-1}} \beta \left( \gamma^{-1} + N - p, p \right) \hat{q}_{\alpha_n}(R^*U^{(1)}) \\
\hat{H}_\alpha_n (R^*U^{(1)}) &= \left( \frac{\gamma^{-1} + N}{\gamma^{-1} + N - p} \right)^{p^{-1}} \beta \left( \gamma^{-1} + N - p, p \right) \hat{q}_{\alpha_n}(R^*U^{(1)})
\end{align*}
\]

The condition $p \leq N$ or $\gamma < \frac{1}{p - N}$ simply ensures the existence of $H_\alpha_n(R^*U^{(1)})$. Using the consistency results given in Propositions 4.1 and 4.3, the consistency of these estimators is immediate. The proof is also omitted from the appendix.

Proposition 5.5 (Consistency of H-G estimators). Assume that Assumption 2 and condition (C) hold. Under conditions $(C_{int})$ and $(C_{high})$ respectively, $\hat{H}_\alpha_n(R^*U^{(1)})$ and $\hat{H}_\alpha_n(R^*U^{(1)})$ are consistent, i.e.:

\[
\begin{align*}
\hat{H}_\alpha_n(R^*U^{(1)}) &\xrightarrow{p} 1 \\
\hat{H}_\alpha_n(R^*U^{(1)}) &\xrightarrow{p} 1
\end{align*}
\]
Proposition 5.6 (Asymptotic normality of H-G estimators). Assume that Assumption (C) hold. Under conditions $(C_{\text{int}})$ and $(C_{\text{high}})$ respectively, we have:

$$
\begin{align*}
\frac{\sqrt{n}}{\ln(1 - \alpha_n)} \left( \frac{R_n^{(1)}(U^{(1)}) - 1}{\sigma_n^{(1)}} \right) \xrightarrow{n \to +\infty} & \mathcal{N} \left( 0, \frac{N^2 \gamma^2}{(\pi N + 1)^2} \right) \\
\frac{\sqrt{n}}{\ln(1 - \alpha_n)} \left( \frac{R_n^{(1)}(U^{(1)}) - 1}{\sigma_n^{(1)}} \right) \xrightarrow{n \to +\infty} & \mathcal{N} \left( 0, \frac{N^2 \gamma^2}{(\pi N + 1)^2} + \theta^2 \frac{N^2 \gamma^2}{(\pi N + 1)^2} \right)
\end{align*}
$$

We propose some examples (with $p = 1$, i.e TVaR) in Sections 6 and 7.

6. Simulation study

In this section, we apply our estimators to a sample of $n = 10,000$ simulations of a 5-dimensional Student vector with $\nu = 1.5$ degrees of freedom, and compare with theoretical results. Indeed, the Student case is the only heavy-tailed elliptical distribution (to our knowledge) where we can obtain closed formula for conditional quantiles. We can notice that the unconditional distribution has tail index 1/3, however, this example, as other heavy-tailed elliptical distributions, does not admit variance, this is why we use the fixed-point algorithm given in Tyler (1987) and p.66 of Frahm (2004). Once this step has been carried out, we suppose that the observed covariates satisfy $q_X = 1$. The next step is to estimate the quantities $\eta$ and $\ell$. For that purpose, we use our estimators $\hat{\eta}_n$ and $\hat{\ell}_n$, respectively introduced in Equations (3.4) and (3.9). These two estimators are related to the Hill estimator $\hat{\gamma}$, and asymptotic results of Section 3 hold only if the data is independent. This is why we do the estimation of $\gamma$ only with the $n$ realizations of the first component of the vector. The chosen kernel $K$ is the gaussian p.d.f.

![Figure 1](image1.png)

**Figure 1.** From left to right: estimators $\hat{\ell}_n$, $\hat{\eta}_n$ and $\hat{\ell}_n$. Theoretical values are in red. The chosen sequences are $k_n = n^{0.7}$, $h_n = n^{-0.2}$ and $\alpha_n = 1 - n^{-1.25}$.

Figure 2 shows the behavior of our estimators $\hat{\eta}_n$ and $\hat{\ell}_n$. In this example, $\hat{\eta}_{10000}$ returns 4.28, where the theoretical value is 13/3 = 4.33, and $\hat{\ell}_{10000,10000}$ gives 8.20, while $\ell$ is equal to 7.85, cf. Table 1. The next step is to estimate the conditional quantiles $\Phi_{R_n}^{-1}(\alpha_n)$, expectiles and TVaRs. Figure 2 provides a high quantiles, expectiles and TVaRs estimation, with $\alpha_n = 1 - n^{-1.25}$.

Theoretical formulas (an algorithm) for conditional quantiles and expectiles may be found in Deschamps and Maume-Deschamps (2017a) and Maume-Deschamps et al. (2017b). Furthermore, using straightforward calculations, formulas for Tail-Value-at-Risk may be obtained.

$$
\begin{align*}
\Phi_{R_n}^{-1}(\alpha) &= q_n \left( R_n^{(1)} \right) = \\
\text{TVaR}_n \left( R_n^{(1)} \right) &= \frac{1}{\alpha} \Gamma \left( \frac{N + \gamma}{\nu} ; \frac{N + \gamma}{\nu} \right) ^{\frac{1}{N - \gamma}} \\
&= \sqrt{\frac{\nu + q_n}{\nu + q_n}} \Phi_{\nu + q_n}^{-1}(\alpha) \left( 1 + \frac{\Phi_{\nu + q_n}^{-1}(\alpha)^2}{\nu + q_n} \right) ^{\frac{1}{N - \gamma}}
\end{align*}
$$

where $\Phi_{\nu}$ is the c.d.f of a Student distribution with $\nu$ degrees of freedom. In order to give an idea of the performance of our estimator, we propose in Figure 3 some box plots representing 1000 estimates (based on a sample size $n = 10,000$) of an extreme quantile with level $\alpha_n = 1 - n^{-1.3}$ according to the chosen
sequences $k_n$ and $h_n$. It seems interesting to specify that all the sequences $k_n$ that we propose are of the form $n^b$. In the Student case, it is possible, using some results of de Haan and Ferreira (2006), to prove that condition $\sqrt{k_n} A(n/k_n) \to 0$ as $n \to +\infty$ is filled if $b < 4/(\nu + 4)$. In this example, $\nu = 1.5$, then $b$ must be less than $8/11$. This is why the fourth plot, taking $k_n = n^{0.8}$, seems more extensive. By taking $b < 8/11$, the estimation variance appears to be decreasing with $b$. Concerning $h_n$, of the form $n^{-a}$, the estimation variance seems to be increasing with $a$. Finally, we would like to compare these results with other estimators already used. The most common and widespread methods for estimating conditional quantiles and expectiles are respectively quantile and expectile regression, introduced in Koenker and Bassett (1978) and Newey and Powell (1987). In Maume-Deschamps et al. (2017a) and Maume-Deschamps et al. (2017b), we have shown that such approach leads to a poor estimation in case of extreme levels. Indeed, in this example, a quantile regression estimator will converge to $\Phi^{-1}_\nu(\alpha_n) = 1530.15$, very far from 7.31, the theoretical result.

Figure 2. Respectively from left to right: empirical vs theoretical quantiles, empirical vs theoretical expectiles and empirical vs theoretical TVaR. The chosen sequences are $k_n = n^{0.7}$, $h_n = n^{-0.2}$ and $\alpha_n = 1 - n^{-1.25}$.

Figure 3. Box plots representing 1000 estimates (based on a sample size $n = 10,000$) of an extreme quantile for different sequences $k_n$ and $h_n$. The chosen sequence $\alpha_n$ equals $1 - n^{-1.3}$. $h_n = n^{-0.2}$ for the 4 first plots, and $k_n = n^{0.7}$ for the others. Furthermore, from left to right: $k_n = n^{0.3}$, $n^{0.5}$, $n^{0.7}$, $n^{0.8}$ and $h_n = n^{-0.1}$, $n^{-0.25}$, $n^{-0.4}$, $n^{-0.5}$. Theoretical value is in red.

In the previous figures, only the first component of the vector is used to estimate the tail index. There is therefore some loss of information. We have suggested in Section 3 another approach. Furthermore, Resnick and Stărică (1995) or Hsing (1991) proved that the Hill estimator may also work with dependent data. Thus it would be possible to improve the estimation of $\hat{\gamma}_{k_n}$ by adding the other components of the vector in Equation (3.2), but in that case the asymptotic results of Propositions 3.2 or 3.3 would not hold anymore.
7. Real data example

As an application, we use the daily market returns (computed from the closing prices) of financial assets from 2006 to 2016, available at [http://stanford.edu/class/ee103/portfolio.html](http://stanford.edu/class/ee103/portfolio.html). We focus on the first four assets, i.e. iShares Core U.S. Aggregate Bond ETF, PowerShares DB Commodity Index Tracking Fund, WisdomTree Europe SmallCap Dividend Fund and SPDR Dow Jones Industrial Average ETF. The reason for focusing solely on the value of these assets could be, for example, that they are the first available every day. The aim would be to anticipate the behavior of other assets on another market. Figure 4 represents the daily return for each day.

![Figure 4. Daily market returns of 4 different assets.](image)

After a brief study of the autocorrelation functions, we consider that the daily returns can be considered as independent. Concerning the shape of the data, histograms of the marginals seem symmetrical. Furthermore, the measured tail index is approximately the same for the 4 marginals. This is why suppose that the data is elliptical. After having estimated $\mu$ and $\Sigma$ by the method of moments, and considering $q_X = 1$, we apply our estimators $\hat{\eta}_{k_n}$ and $\hat{\ell}_{k_n,h_n}$, given in Equations (3.4) and (3.9). We take as sequences $k_n = n^{0.7}$ and $h_n = n^{-0.4}$, and as kernel $K$ the gaussian p.d.f, hence we deduce the asymptotic confidence bounds from Equation (3.14). Figure 5 shows the behavior of our estimators, according to the number of days considered. The study indicates that, with a probability close to 95%, $\ell$ is between 4.88 and 6.24, and $\eta$ between 2.47 and 2.90.

![Figure 5. $\hat{\ell}_{k_n,h_n}$, $\hat{\eta}_{k_n}$ and dotted related 95% confidence bounds. The chosen sequences are $k_n = n^{0.7}$ and $h_n = n^{-0.4}$.](image)

Figure 6 shows the estimated quantiles $\Phi_{\alpha_n}^{-1}$, expectiles and TVaRs, according to their level $\alpha_n$. Then, if we want to provide an estimator for the quantile (or expectile) of another asset, it only remains to estimate the quantities $\mu_{Y|X}$ and $\sigma_{Y|X}$, and use Equation 2.5.

8. Conclusion

In this paper, we propose two estimators $\hat{\ell}_{k_n,h_n}$ and $\hat{\eta}_{k_n}$ respectively for extremal parameters $\ell$ and $\eta$ introduced in Equation (2.6). We have proved their consistency and asymptotic normality according to the asymptotic relationships between the sequences $k_n$ and $h_n$. Using these estimators, we have defined estimators for intermediate and high quantiles, proved their consistency, given their asymptotic normality under stronger conditions, and deduced estimators for extreme $L_p$-quantiles and Haezendonck-Goovaerts risk measures. Consistency and asymptotic normality are also provided for these estimators, under conditions. We have also illustrated with a numerical example the performance of our estimators, and applied them to real data set.

As working perspectives, we intend to propose a method of optimal choice of the sequences $k_n$ and $h_n$. 
Figure 6. Empirical quantiles, expectiles and TVaRs of random variable $R^*U^{(1)}$, estimated using Equations 4.6, 5.4 and 5.10.

which is not discussed in this paper. Furthermore, the shape of $\ell$ and $\eta$ leaving Assumption 1 is a current research topic. More generally, the asymptotic relationships between conditional and unconditional quantile in other maximum domains of attraction, using for example the results of Hashorva (2007), may be developed. However, we need a second-order refinement, as we need a second-order refinement of Equation (2.8) to propose asymptotic normalities 4.2 and 4.5 under weaker assumptions than Assumption 2. Finally, it seems that the ratio of the two terms in Equation (2.8) tends to 1 more and more slowly when the covariate vector size $N$ becomes large. Then, our estimation approach may perform poorly if $N$ is tall. This is why it might be wise to propose another method when the covariate vector size $N$ is large.

9. Appendix

Proof of Lemma 2.1

(i) Since $R_1 \overset{d}{=} \chi_1 \xi$, where $\chi_1$ has a Lebesgue density $\sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}}$. According to Lemma 4.3 in Jessen and Mikosch (2006), $\xi$ satisfies $F_{\xi}(t\omega)/F_{\xi}(t) \to \omega^{-\frac{1}{\gamma}}$ as $t \to +\infty$. Furthermore, Lemma 4.2 in Jessen and Mikosch (2006) entails

$$P(\xi > t) \sim t \to +\infty \frac{\Gamma(\frac{\gamma+1}{2})}{\Gamma(\frac{2}{2})} \lambda t^{-\frac{1}{\gamma}-1},$$

where $\lambda \in \mathbb{R}$ is not related to $d$.

(ii) Using again Lemma 4.2 in Jessen and Mikosch (2006) for $R_d \overset{d}{=} \chi_d \xi$, it comes immediately

$$P(R_d > t) \sim t \to +\infty \frac{\Gamma(\frac{\gamma+1}{2})}{\sqrt{2\pi}} \lambda t^{-\frac{1}{\gamma}-1},$$

where $\lambda' \in \mathbb{R}$ is not related to $d$.

(iii) From (ii), we have, for all $d \in \mathbb{N}$,

$$f_{R_d}(t) \to t \to +\infty \frac{\Gamma(\frac{\gamma+1}{2})}{\sqrt{2\pi}} \lambda' t^{-\frac{1}{\gamma}-1},$$

where $\lambda' \in \mathbb{R}$ is not related to $d$. The result is immediate with this expression. \qed

Proof of Proposition 3.1. The conditional density (Proposition 3 in Maume-Deschamps et al. (2017a)) leads to:

$$\lim_{t \to +\infty} \Phi_{R^*}(t) = \lim_{t \to +\infty} \frac{c_{N+1} g_{N+1}(q_X + t^2)}{c_{N} g_{N}(q_X)} \eta^{-\frac{1}{N}} c_{1} g_{1}(t^2) = \lim_{t \to +\infty} \frac{\Gamma(\frac{N+1}{2}) (q_X + t^2)^{-\frac{1}{2}}}{\pi^{\frac{N}{2}} c_{N} g_{N}(q_X) \eta^{-\frac{1}{N}}} \frac{f_{R_{N+1}}(\sqrt{q_X + t^2})}{f_{R_1}(t^2)}$$

Using Equation (2.12) of Lemma 2.1, it comes

$$\frac{\Phi_{R^*}(t)}{\Phi_{R}(t^2)} \sim t \to +\infty \frac{1}{\pi^{\frac{N}{2}} c_{N} g_{N}(q_X) \eta} \frac{\Gamma(\frac{N+1+\gamma^{-1}}{2})}{\Gamma(\frac{1+\gamma^{-1}}{2})} \lambda'^{-1} \eta^{-1} g_{N+1}(q_X) \eta^{-N}.$$
Obviously, we impose $0 < \ell < +\infty$, then $1 - \eta - N + (\eta - 1)(\gamma^{-1} + 1) = 0$, hence $\eta = N\gamma + 1$. Replacing $\eta$ in the previous equation, $\ell$ is easily deduced

$$\ell = \frac{1}{\pi^{\frac{d}{2}} c_N g_N(q_X)\eta} \Gamma\left(\frac{N+1+\gamma^{-1}}{2}\right)$$

**Proof of Proposition 3.5.** We have seen that $\sqrt{n\hat{h}_n}\left(\hat{\gamma}_n - c_N g_N(q_X)\right)$ tends to a normal distribution. If $k_n = o(\sqrt{n\hat{h}_n})$, it comes $\sqrt{k_n}\left(\hat{\gamma}_n - c_N g_N(q_X)\right) \xrightarrow{n \to +\infty} 0$. Then we get the following asymptotic normality:

$$\sqrt{k_n}\left(\hat{\gamma}_n - \gamma\right) \xrightarrow{n \to +\infty} \mathcal{N}\left(\left(0, 0\right), \left(\frac{\gamma^2}{0} 0\right)\right)$$

Since $\ell = u(\gamma)$, the delta method entails

$$\sqrt{k_n}\left(\hat{\ell}_n, h_n - \ell\right) \xrightarrow{n \to +\infty} \mathcal{N}\left(0, u'(\gamma)^2\gamma^2\right)$$

A quick calculation of $u'$, using Equation (3.1), gives the first result. The second part of the proof is similar. Indeed, if $mh_n = o(k_n)$, then

$$\sqrt{m\hat{h}_n}\left(\hat{\gamma}_n - \gamma\right) \xrightarrow{n \to +\infty} \mathcal{N}\left(\left(0, 0\right), \left(\frac{\gamma^2}{0} 0\right)\right)$$

The delta method completes the proof. Concerning (iii), we have already calculated the asymptotic variances of $\hat{\gamma}_n$ and $\hat{\gamma}_n\gamma$. It remains to calculate the covariance

$$\text{Cov}\left(\sqrt{k_n}\hat{\gamma}_n, \sqrt{k_n}\hat{\gamma}_n\right) = k_n\text{Cov}\left(\frac{1}{k_n} \sum_{i=1}^{k_n} \ln \left(\frac{W[i]}{W[k_n+1]}\right), \frac{1}{nmh_n} \sum_{j=1}^{n} K\left(\frac{q_X - (X_j - \mu_X)^T \Sigma_X^{-1}(X_j - \mu_X)}{h_n}\right)\right)$$

We notice that the covariance is different from 0 if $j \neq i$ and $j \neq k_n + 1$. Furthermore, Lemma 3.2.3 in [de Haan and Ferreira (2006)] proves that $\ln\left(W[i]/W[k_n+1]\right)$ and $W[k_n+1]$ are independent, hence

$$\text{Cov}\left(\sqrt{k_n}\hat{\gamma}_n, \sqrt{k_n}\hat{\gamma}_n\right) \propto \frac{1}{nh_n} \sum_{i=1}^{k_n} \text{Cov}\left(\ln \left(\frac{W[i]}{W[k_n+1]}\right), K\left(\frac{q_X - (X[i] - \mu_X)^T \Sigma_X^{-1}(X[i] - \mu_X)}{h_n}\right)\right)$$

where $X[i]$ is the realization for which $W[i]$ is present. In our context, this is the realization for which the first component is $W[i]$. Cauchy-Schwartz inequality gives

$$\left|\text{Cov}\left(\sqrt{k_n}\hat{\gamma}_n, \sqrt{k_n}\hat{\gamma}_n\right)\right| \leq \kappa \frac{1}{nlh_n} \sum_{i=1}^{k_n} \text{Var}\left[\ln \left(\frac{W[i]}{W[k_n+1]}\right)\right] \text{Var}\left[K\left(\frac{q_X - (X[i] - \mu_X)^T \Sigma_X^{-1}(X[i] - \mu_X)}{h_n}\right)\right]$$

where $\kappa > 0$. According to [de Haan and Ferreira (2006)], $\ln\left(W[i]/W[k_n+1]\right)$ is a standard exponential distribution. Finally, it remains

$$\left|\text{Cov}\left(\sqrt{k_n}\hat{\gamma}_n, \sqrt{k_n}\hat{\gamma}_n\right)\right| \leq \kappa \sqrt{k_n/nh_n} k_n h_n \text{Var}\left[\frac{1}{k_n nh_n} \sum_{i=1}^{k_n} K\left(\frac{q_X - (X[i] - \mu_X)^T \Sigma_X^{-1}(X[i] - \mu_X)}{h_n}\right)\right]$$

[Parzen (1962)] proved that the quantity under the square root was finite. Furthermore, since $nh_n \sim c k_n$, it comes $\sqrt{k_n} = o(n\sqrt{h_n})$. Thus, we obtain the asymptotic normality

$$\sqrt{k_n}\left(\hat{\gamma}_n - \gamma\right) \xrightarrow{n \to +\infty} \mathcal{N}\left(\left(0, 0\right), \left(\frac{\gamma^2}{0} 0\right)\right)$$

The delta method leads to Equation (3.12) \(\Box\)
Proof of Theorem 4.1. In a first time, we can notice \( \hat{p}_n \) is related to \( \hat{\ell}_{k_n, h_n} \). Then, according to Proposition 3.5 (i) entails that we can deal with \( p_n \) instead of \( \hat{p}_n \) in Equation (4.2). Furthermore, we give the decomposition:

\[
\frac{\sqrt{k_n}}{\ln(1-\alpha_n)} \left( \frac{W_{[np_n+1]}}{\Phi_R^{-1}(1-p_n)^{\frac{-1}{c}}} \right)_{x_n} = \frac{\sqrt{k_n}}{\ln(1-\alpha_n)} \left( \frac{W_{[np_n+1]}}{\Phi_R^{-1}(1-p_n)^{\frac{-1}{c}}} - 1 \right) \Phi_R^{-1}(1-p_n)^{\frac{-1}{c}} - 1 + \frac{\sqrt{k_n}}{\ln(1-\alpha_n)} \left( \Phi_R^{-1}(1-p_n)^{\frac{-1}{c}} - 1 \right)
\]

Under Assumption [1] and according to Proposition 3.2 and Theorem 2.4.1 in de Haan and Ferreira (2006) (with (C)), we have:

\[
\left\{ \begin{align*}
\sqrt{k_n} & \left( \frac{1}{\Phi_R^{-1}(1-p_n)} - \frac{1}{\Phi_R^{-1}(1-\alpha_n)} \right) \sim 0 \\
\sqrt{np_n} & \left( \frac{W_{[np_n+1]}}{\Phi_R^{-1}(1-\alpha_n)^{-\frac{1}{c}}} - 1 \right) \sim 0
\end{align*} \right. 
\N(0, \frac{N^2\gamma^2}{(\gamma N+1)^4})
\]

By noticing that \( p_n \) is equivalent to \( \ell^{-1}(1-\alpha_n) \) as \( n \to +\infty \), and using (C), it comes

\[
\frac{\sqrt{k_n}}{\ln(1-\alpha_n)} \left( \frac{W_{[np_n+1]}}{\Phi_R^{-1}(1-\alpha_n)^{\frac{-1}{c}}} - 1 \right) \to 0 \quad \text{as} \quad n \to +\infty.
\]

Furthermore, under Assumption [1] \( \ln(\Phi_R^{-1}(1-\alpha_n)) \) is clearly equivalent to \( -\gamma \ln(p_n) \), or \( -\gamma \ln(1-\alpha_n) \). Then (C) ensures \( \Phi_R^{-1}(1-p_n)^{\frac{-1}{c}} \to 1 \) as \( n \to +\infty \), and therefore the first term of the decomposition tends to 0. It thus remains to calculate the limit of the second term. It is not complicated to notice that

\[
\lim_{n \to +\infty} \left( \frac{\sqrt{k_n}}{\ln(\Phi_R^{-1}(1-\alpha_n))} \left( \Phi_R^{-1}(1-p_n)^{\frac{-1}{c}} - 1 \right) \right) \sim 0 \quad \text{as} \quad n \to +\infty.
\]

Using the equivalence \( \ln(\Phi_R^{-1}(1-\alpha_n)) \sim -\gamma \ln(1-\alpha_n) \), we get the result (4.2). Using asymptotic relationship (2.8), the consistency of the estimator is obvious. \( \square \)

Proof of Proposition 4.2. We recall that density of \( \Phi_R \) is proportional to \( c_{N+1}g_{N+1}(q_X + t^2) \), and, from Assumption [2] there exist \( \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \) such that:

\[
c_{N+1}g_{N+1}(q_X + t^2) = \lambda_1(q_X + t^2)^{-\frac{N+1\gamma-1}{2}} \left[ 1 + \lambda_2(q_X + t^2)^{\frac{\gamma}{2}} + o \left( t^\gamma \right) \right]
\]

The previous expression may be rewritten as follows, where \( \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \):

\[
c_{N+1}g_{N+1}(q_X + t^2) = \lambda_1 t^{-(N+1\gamma-1)} \left[ 1 + \lambda_2(q_X + t^2)^{\frac{\gamma}{2}} + \lambda_3 t^{-2} + o \left( t^{-2} \right) \right], \lambda_1, \lambda_2 \in \mathbb{R}
\]

In order to make the proof more readable, we do not specify the values of constants \( \lambda_i \), because they are not essential. Then, in the case, \( \rho/\gamma \leq -2 \), we get

\[
c_{N+1}g_{N+1}(q_X + t^2) = \lambda_1 t^{-(N+1\gamma-1)} \left[ 1 + \lambda_2 t^{-2} + o \left( t^{-2} \right) \right], \lambda_1, \lambda_2 \in \mathbb{R}
\]

In other terms, \( c_{N+1}g_{N+1}(q_X + t^2) \) is regularly varying of second order with indices \( -N - 1 - \gamma^{-1}, -2 \), and an auxiliary function proportional to \( t^{-2} \). According to Proposition 6 of Hua and Joe (2011), \( \Phi_R(t) = \int_{t}^{t+\infty} c_{N+1}g_{N+1}(q_X + x^2) \, dx \in 2RV_{-N-\gamma-1,-2} \) with an auxiliary function proportional to \( t^{-2} \). Equivalently, there exists \( \lambda_1, \lambda_2 \in \mathbb{R} \) such that

\[
\Phi_R^{-1}(1-\frac{1}{t}) = \lambda_1 t^{\frac{-N\gamma-1}{2}} \left[ 1 + \lambda_2 t^{-\frac{2\gamma}{\gamma N+1}} + o \left( t^{-\frac{2\gamma}{\gamma N+1}} \right) \right]
\]

Since Assumption [1] and Assumption [2] provide \( \Phi_R^{-1}(1-1/t) = \lambda_3 t^{\gamma}(1 + \lambda_4 t^\rho + o(t^\rho)) \), it comes

\[
\frac{\Phi_R^{-1}(1-p_n)^{\frac{1}{c}}}{\Phi_R^{-1}(\alpha_n)} = t^{-\frac{\gamma\gamma}{\gamma N+1}} \left[ \frac{1 + \alpha_4 p_n^{\gamma} + o(p_n^{\gamma})}{1 + \alpha_4 (1-\alpha_n)^{\frac{1}{c}} \gamma N+1} \right] o \left( (1-\alpha_n)^{-\frac{2\gamma}{\gamma N+1}} \right)
\]

for some constants \( \lambda_1, \lambda_2 \in \mathbb{R} \). In that case, we considered \( \rho \leq -2\gamma \), hence \( -\rho > 2\gamma/\gamma N+1 \). We then deduce the following expansion:

\[
\frac{\Phi_R^{-1}(1-p_n)^{\frac{1}{c}}}{\Phi_R^{-1}(\alpha_n)} = t^{-\frac{\gamma\gamma}{\gamma N+1}} \left[ \frac{1 + \lambda(1-\alpha_n)^{\frac{2\gamma}{\gamma N+1}} + o \left( (1-\alpha_n)^{-\frac{2\gamma}{\gamma N+1}} \right) \right]
\]

\[\square\]
for a certain constant $\lambda \in \mathbb{R}$. We can notice that $(1 - \alpha_n)/p_n = 2(1 - \ell)(1 - \alpha_n) + \ell$, and let us now focus on the limit:

$$\lim_{n \to +\infty} \sqrt{n} \ln \left( \frac{\Phi_R^{-1} (1 - p_n)^\frac{1}{2}}{\Phi_R^2 (\alpha_n)} \right) = \frac{\gamma}{\gamma N + 1} \lim_{n \to +\infty} \sqrt{n} \ln \left( \frac{2 - \ell}{\ell} (1 - \alpha_n) + 1 \right) + \lim_{n \to +\infty} \sqrt{n} \ln \left( 1 + \lambda (1 - \alpha_n)^\frac{2}{\gamma N + 1} + o \left( (1 - \alpha_n)^\frac{2}{\gamma N + 1} \right) \right)$$

The first term gives easy to calculate. Indeed, since $\sqrt{k_n}(1 - \alpha_n)/\ln (1 - \alpha_n) \to 0$ as $n \to +\infty$, we deduce

$$\lim_{n \to +\infty} \sqrt{n} \ln \left( \frac{2 - \ell}{\ell} (1 - \alpha_n) + 1 \right) = 2 \frac{1 - \ell}{\ell} \lim_{n \to +\infty} \sqrt{n} \ln (1 - \alpha_n) = 0$$

By a similar calculation, the second term also tends to 0, supposing $\frac{\sqrt{k_n}}{\ln (1 - \alpha_n)} (1 - \alpha_n)^\frac{2}{\gamma N + 1} \to 0$ as $n \to +\infty$. Then, we deduce, using Proposition 4.1:

$$\frac{\sqrt{n} \ln (1 - \alpha_n)}{\ln (1 - \alpha_n)} \left( \frac{\hat{q}_{\alpha_n} (R^* U^{(1)})}{\Phi_R^2 (\alpha_n)} - 1 \right) = \frac{\sqrt{n} \ln (1 - \alpha_n)}{\ln (1 - \alpha_n)} \left( \frac{\hat{q}_{\alpha_n} (R^* U^{(1)})}{\Phi_R^2 (1 - p_n)^\frac{1}{2}} - 1 \right) \Phi_R^{-1} (1 - p_n)^\frac{1}{2} \Phi_R^2 (\alpha_n)$$

$$+ \frac{\sqrt{n} \ln (1 - \alpha_n)}{\ln (1 - \alpha_n)} \left( \Phi_R^{-1} (1 - p_n)^\frac{1}{2} - 1 \right) \sim_{n \to +\infty} N \left( 0, \frac{N^2 \gamma^4}{(\gamma N + 1)^2} \right)$$

Now, let us focus on the case $\rho/\gamma > -2$. The proof is exactly the same, with

$$c_{N+1} g_{N+1} \left( q_X + t^2 \right) = \lambda_1 t^{-(N+1+\gamma^{-1})} \left[ 1 + 2 \gamma t^2 + o \left( t^2 \right) \right], \lambda_1, \lambda_2 \in \mathbb{R}$$

Using the same calculations and doing the further assumption $\lim_{n \to +\infty} \sqrt{n} \ln (1 - \alpha_n)^{-\frac{2}{\gamma N + 1}} = 0$ leads to the result. \( \square \)

**Proof of Theorem 4.3** Firstly, we can notice that

$$\left[ \frac{W_{[k_n+1]} \left( \frac{k_n}{np_n} \right)^{\gamma_{k_n}}}{\Phi_R^{-1} (1 - p_n)^\frac{1}{2}} \right] \rightarrow_{n \to +\infty} \left[ \frac{W_{[k_n+1]} \left( \frac{k_n}{np_n} \right)^{\gamma_{k_n}}}{\Phi_R^{-1} (1 - p_n)^\frac{1}{2}} \right] \frac{1}{\gamma_{k_n}} \left( \frac{p_n}{p_n} \right)^{\gamma_{k_n}} + \left( \frac{p_n}{p_n} \right)^{\gamma_{k_n}} - 1.$$

Since $k_n = o(n h_n)$, we deduce

$$\sqrt{n} \ln \left( \frac{k_n}{np_n} \right) \left[ \frac{W_{[k_n+1]} \left( \frac{k_n}{np_n} \right)^{\gamma_{k_n}}}{\Phi_R^{-1} (1 - p_n)^\frac{1}{2}} \right] - 1 \sim_{n \to +\infty} \sqrt{n} \ln \left( \frac{k_n}{np_n} \right) \left[ \frac{W_{[k_n+1]} \left( \frac{k_n}{np_n} \right)^{\gamma_{k_n}}}{\Phi_R^{-1} (1 - p_n)^\frac{1}{2}} \right] - 1.$$

Furthermore, according to Theorem 4.3.8 in [de Haan and Ferreira 2006], (C) and (C_{high}) lead to

$$\sqrt{n} \ln \left( \frac{k_n}{np_n} \right) \left[ \frac{W_{[k_n+1]} \left( \frac{k_n}{np_n} \right)^{\gamma_{k_n}}}{\Phi_R^{-1} (1 - p_n)^\frac{1}{2}} \right] - 1 \sim_{n \to +\infty} \sqrt{n} \ln \left( \frac{k_n}{np_n} \right) \left( \frac{k_n}{np_n} \right)^{\gamma_{k_n} - \gamma} - 1.$$

From Assumption 1, it is not difficult to prove that $\ln \left( \Phi_R^{-1} (1 - p_n) \right) / \ln (k_n/(np_n))$ is asymptotically equivalent to $\gamma \ln (1 - \alpha_n)/\ln (n(1 - \alpha_n)/k_n)$. Then, if we focus on the second term, it comes, using the limit given in (C_{high}):

$$\sqrt{n} \ln \left( \frac{k_n}{np_n} \right) \left( \frac{k_n}{np_n} \right)^{\gamma_{k_n} - \gamma - 1} \frac{1}{\gamma_{k_n} - \gamma} - 1 \sim_{n \to +\infty} N \left( 0, \frac{\gamma^2}{\gamma N + 1} \right), \left( \frac{\gamma^2}{\gamma N + 1} \right)^3 - \theta \left( \frac{\gamma^2}{\gamma N + 1} \right)^3 - \frac{N^2 \gamma^4}{(\gamma N + 1)^2} \theta^2.$$
Finally,

\[
\frac{\sqrt{k_n}}{\ln \left( \frac{k_n}{n^{1-\alpha_n}} \right)} \left( \frac{W_{[k_n+1]} \left( \frac{k_n}{np_n} \right)^{\gamma_n} \frac{1}{\Phi_{n-1}^{-1} (1-p_n)^{1/\gamma_n}} \right) = \frac{\sqrt{k_n}}{\ln \left( \frac{k_n}{np_n} \right)} \left( \Phi_{R}^{-1} (1-p_n) \frac{1}{\Phi_{n-1}^{-1} (1-p_n)^{1/\gamma_n} - 1} \right)
\]

Supposing

\[
\frac{\sqrt{k_n}}{\ln \left( \frac{k_n}{np_n} \right)} \left( \frac{W_{[k_n+1]} \left( \frac{k_n}{np_n} \right)^{\gamma_n} \frac{1}{\Phi_{n-1}^{-1} (1-p_n)^{1/\gamma_n}} \right) - 1 \Phi_{R}^{-1} (1-p_n)^{1/\gamma_n} - 1 \]

When \( n \to \infty \), this expression is the sum of the following bivariate normal distribution:

\[
\frac{\sqrt{k_n}}{\ln \left( \frac{k_n}{np_n} \right)} \left( \frac{W_{[k_n+1]} \left( \frac{k_n}{np_n} \right)^{\gamma_n} \frac{1}{\Phi_{n-1}^{-1} (1-p_n)^{1/\gamma_n}} \right) \sim n \to +\infty N \left( (0), \left( \frac{\gamma^2}{(\gamma N + 1)^2}, \frac{-\theta \gamma N^3}{(\gamma N + 1)^3}, \frac{-\theta^2 \gamma N^6}{(\gamma N + 1)^4} \right) \right)
\]

To conclude, \( \ln \left( \frac{k_n}{np_n} \right) \sim \ln \left( \frac{k_n}{n^{1-\alpha_n}} \right) \) as \( n \to +\infty \), hence the result. The consistency is immediate.

**Proof of Proposition 4.5** The proof is similar to that of Proposition 4.2. Indeed, we have given, in the case \( \rho^2 / \gamma \leq -2 \):

\[
\frac{\Phi_{R}^{-1} (1-p_n)^{1/\gamma} \gamma_n}{\Phi_{R}^{-1} (\alpha_n)} = \ell \left( 2(1-\ell)(1-\alpha_n) + \ell \right) \left[ 1 + \lambda (1-\alpha_n)^{-\frac{\gamma}{N+1}} + o \left( (1-\alpha_n)^{-\frac{\gamma}{N+1}} \right) \right]
\]

for a certain constant \( \lambda \in \mathbb{R} \). It thus remains to calculate

\[
\lim_{n \to +\infty} \frac{\sqrt{k_n}}{\ln \left( \frac{k_n}{n^{1-\alpha_n}} \right)} \ln \left( \frac{\Phi_{R}^{-1} (1-p_n)^{1/\gamma}}{\Phi_{R}^{-1} (\alpha_n)} \right) = \frac{\gamma}{\gamma N + 1} \lim_{n \to +\infty} \frac{\sqrt{k_n}}{\ln \left( \frac{k_n}{n^{1-\alpha_n}} \right)} \ln \left( \frac{2 - \ell}{\ell} (1-\alpha_n) + 1 \right)
\]

The first term gives is easy to calculate. Indeed, since \( n(1-\alpha_n) \to 0 \) and \( k_n = o(n) \) as \( n \to +\infty \), we deduce

\[
\lim_{n \to +\infty} \frac{\sqrt{k_n}}{\ln \left( \frac{k_n}{n^{1-\alpha_n}} \right)} \ln \left( \frac{2 - \ell}{\ell} (1-\alpha_n) + 1 \right) = \frac{2 - \ell}{\ell} \lim_{n \to +\infty} \frac{\sqrt{k_n}}{\ln \left( \frac{k_n}{n^{1-\alpha_n}} \right)} (1-\alpha_n) = 0
\]

By a similar calculation, the second term also tends to 0, supposing \( \frac{\sqrt{k_n}}{\ln \left( \frac{k_n}{n^{1-\alpha_n}} \right)} (1-\alpha_n)^{-\frac{\gamma}{N+1}} \to 0 \) as \( n \to +\infty \). Then, we deduce, using Proposition 4.3:

\[
\frac{\sqrt{k_n}}{\ln \left( \frac{k_n}{n^{1-\alpha_n}} \right)} \left( \frac{W_{[k_n+1]} \left( \frac{k_n}{np_n} \right)^{\gamma_n} \frac{1}{\Phi_{n-1}^{-1} (1-p_n)^{1/\gamma_n}} \right) = \frac{\sqrt{k_n}}{\ln \left( \frac{k_n}{n^{1-\alpha_n}} \right)} \left( \Phi_{R}^{-1} (1-p_n)^{1/\gamma_n} - 1 \right) \Phi_{R}^{-1} (1-p_n)^{1/\gamma_n} - 1
\]

\[
\frac{\sqrt{k_n}}{\ln \left( \frac{k_n}{n^{1-\alpha_n}} \right)} \left( \frac{W_{[k_n+1]} \left( \frac{k_n}{np_n} \right)^{\gamma_n} \frac{1}{\Phi_{n-1}^{-1} (1-p_n)^{1/\gamma_n}} \right) - 1 \Phi_{R}^{-1} (1-p_n)^{1/\gamma_n} - 1 \]

Now, let us focus on the case \( \rho^2 / \gamma > -2 \). The proof is exactly the same, with

\[
\frac{\Phi_{R}^{-1} (1-p_n)^{1/\gamma}}{\Phi_{R}^{-1} (\alpha_n)} = \ell \left( 2(1-\ell)(1-\alpha_n) + \ell \right) \left[ 1 + \lambda (1-\alpha_n)^{-\frac{\gamma}{N+1}} + o \left( (1-\alpha_n)^{-\frac{\gamma}{N+1}} \right) \right], \lambda \in \mathbb{R}
\]

Using the same calculations and doing the further assumption \( \lim_{n \to +\infty} \frac{\sqrt{k_n}}{\ln \left( \frac{k_n}{n^{1-\alpha_n}} \right)} (1-\alpha_n)^{-\frac{\gamma}{N+1}} = 0 \) leads to the result. □
Proof of Lemma 5.1. The density of $Y|X = x$ is given by

$$c_{N+1}g_{N+1}\left(q_x + (t - \mu_{Y|X})^2\sigma_{Y|X}^2\right) (c_N g_N (q_x))^{-1}$$

where $q_x = (x - \mu_X)^T\Sigma^{-1}_X (x - \mu_X)$. In order to simplify, we consider the case reduced and centered, i.e. $\mu_{Y|X} = 0$ and $\sigma_{Y|X} = 1$. A quick calculation gives

$$\lim_{t \to +\infty} \frac{\Phi_R'(\omega t)}{\Phi_R'(t)} = \omega \lim_{t \to +\infty} \frac{g_{N+1}(q_x + \omega^2 t^2)}{g_{N+1}(q_x + t^2)} = \omega \lim_{t \to +\infty} \frac{(q_x + \omega^2 t^2)^{-\frac{\alpha}{2}}}{(q_x + t^2)^{-\frac{\alpha}{2}}} \frac{f_{R_{N+1}}(\sqrt{q_x + \omega^2 t^2})}{f_{R_{N+1}}(\sqrt{q_x + t^2})}$$

Equation (2.11) leads to

$$\lim_{t \to +\infty} \frac{\Phi_R'(\omega t)}{\Phi_R'(t)} = \omega \omega^{-\frac{\alpha}{2} - 1} = \omega^{-\frac{1}{2} - N} \square$$

Proof of Proposition 5.3. We recall in a first time that condition \((C_{int}^{LR})\) entails \((C_{int}^{LG})\). Furthermore, if we denote

$$\Lambda = \left[\frac{(\gamma^{-1} + N)^{-1}}{\beta_p} + 1\right]^{-1}$$

$$\hat{\Lambda}_{kn} = \left[\frac{(\gamma^{-1} + N)^{-1}}{\beta_{kn}} + 1\right]^{-1}$$

we have the following decomposition:

$$\frac{\sqrt{K_n}}{\ln(1 - \alpha_n)} \left(\frac{\hat{q}_{p,\alpha_n}(R^*U^{(1)})}{\hat{q}_{p,\alpha_n}(R^*U^{(1)}) - 1}\right) = \frac{\sqrt{K_n}}{\ln(1 - \alpha_n)} \left(\frac{\hat{\Lambda}_{kn} - 1}{\Lambda}\right) \frac{\hat{q}_{a_n}(R^*U^{(1)})}{\tilde{q}_{a_n}(R^*U^{(1)})} - 1\right) = \frac{\sqrt{K_n}}{\ln(1 - \alpha_n)} \left(\frac{\tilde{q}_{a_n}(R^*U^{(1)})}{\tilde{q}_{a_n}(R^*U^{(1)}) - 1}\right) \frac{\Lambda_{q_{a_n}}(R^*U^{(1)})}{\tilde{q}_{a_n}(R^*U^{(1)}) - 1}\right)$$

We know that $\hat{\Lambda}_{kn}$, as a function of $\gamma_{kn}$, is asymptotically normal with rate $\sqrt{K_n}$ (see Equation (3.3)). Then, the first term in the sum clearly tends to 0 as $n \to +\infty$. Using Proposition 4.2, the second term tends to the normal distribution given in (4.5). Finally, we have to check that the third term tends to 0. For that purpose, we use the second order expansion given in [Daouia et al. (2017b)]:

$$\frac{\sqrt{K_n}}{\ln(1 - \alpha_n)} \frac{\hat{q}_{p,\alpha_n}(R^*U^{(1)})}{\tilde{q}_{a_n}(R^*U^{(1)})} - 1 = 1 - (\gamma^{-1} + N)^{-1} r(a_n, p) + (\lambda + o(1)) A^*(\frac{1}{1 - \alpha_n})$$

where $r(a_n, p) = \lambda_1 \frac{1}{q_{a_n}(R^*U^{(1)})} (E[R^*U^{(1)}] + o(1)) + \lambda_2 A^*(\frac{1}{1 - \alpha_n}) (1 + o(1))$, $\lambda$, $\lambda_1$, $\lambda_2 \in \mathbb{R}$ are not related to $n$ and $A^*(t)$ is the auxiliary function of $\Phi_R^*(t)$ (1/2). It seems important to prove that the conditional distribution $R^*U^{(1)}$ is regularly varying with tail index $\gamma^{-1} + N > 0$, then $E[R^*U^{(1)}]$ exists and, $R^*U^{(1)}$ being symmetric, equals 0. Then, a sufficient condition for asymptotic normality may be

$$\lim_{n \to +\infty} \frac{\Lambda}{\sqrt{K_n}} = 0$$

We know, using Assumption 2 and the proof of Proposition 4.5, that $q_{a_n}(R^*U^{(1)}) = \Phi_{R^*}^{-1}(a_n)$ is asymptotically proportional to $\left(1 - \alpha_n\right)^{-\gamma^{-1} + N}$, while $A^*(\frac{1}{1 - \alpha_n})$ is asymptotically proportional to $\left(1 - \alpha_n\right)^{-\gamma^{-1} + N}$ if $\rho > 2\gamma$ and $\left(1 - \alpha_n\right)^{-\gamma^{-1} + N}$ otherwise. Finally, it is not difficult to check that (i) or (ii) in $(C_{int}^{LR})$ lead to the nullity of the two limits, and therefore to the third term of the decomposition, hence the result. The proof is exactly the same for the second normality, replacing $\hat{q}_{p,\alpha_n}(R^*U^{(1)})$ by $\tilde{q}_{p,\alpha_n}(R^*U^{(1)})$, $\ln(1 - \alpha_n)$ by $\ln(\frac{\gamma_{kn}}{n(1 - \alpha_n)})$ and using Proposition 4.5 instead of 4.2. \square

Proof of Proposition 5.6. If we denote

$$\Lambda = \left\{\begin{array}{ll}
\frac{(\gamma^{-1} + N)(\gamma^{-1} + N - p)(\gamma^{-1} + N)^{-1} - 1}{\beta^2(\gamma^{-1} + N - p)} & \text{if } a_n > 0
\end{array}\right.$$
we have the following decomposition:

\[
\frac{\sqrt{k}}{\ln(1 - \alpha_n)} \left( \frac{\hat{H}_{\alpha_n}(R^*U(1))}{H_{\alpha_n}(R^*U(1))} - 1 \right) = \frac{\sqrt{k}}{\ln(1 - \alpha_n)} \left( \frac{\hat{\Lambda}_{\alpha_n} - 1}{\hat{\Lambda}_{\alpha_n}} \right) \frac{\hat{\Lambda}_{\alpha_n}(R^*U(1))}{H_{\alpha_n}(R^*U(1))} \times \frac{\hat{\Lambda}_{\alpha_n}(R^*U(1))}{H_{\alpha_n}(R^*U(1))} - 1 \right)
\]

We know that \(\hat{\Lambda}_{\alpha_n}\), as a function of \(\hat{\gamma}_{\alpha_n}\), is asymptotically normal with rate \(\sqrt{k}\) (see Equation (3.3)).

Then, the first term in the sum clearly tends to 0 as \(n \to +\infty\). Using Proposition 4.2, the second term tends to the normal distribution given in (4.5). Finally, we have to check that the third term tends to 0. For that purpose, we use the result of Theorem 4.5 in Mao and Hu (2012), which ensures that there exists \(\lambda \in \mathbb{R}\) such that:

\[
\frac{\hat{H}_{\alpha_n}(R^*U(1))}{\hat{\Lambda}_{\alpha_n}(R^*U(1))} = 1 + \lambda A^* \left( \frac{1}{1 - \alpha_n} \right) (1 + o(1))
\]

where \(A^*\) is the auxiliary function of \(\Phi_{R^*}^{-1}(1 - \frac{1}{t})\). In the proof of Proposition 4.2, we have seen that \(A^*(t)\) was proportional either to \(t^{-\frac{2}{\gamma - \rho}}\) if \(\rho \leq -2\gamma\) or \(t^{\frac{2}{\gamma - \rho}}\) otherwise. Then, statements (i) and (ii) in \((C^H)_{int}\) ensure

\[
\lim_{n \to +\infty} \frac{\sqrt{k}}{\ln(1 - \alpha_n)} \ln \left( \frac{\hat{H}_{\alpha_n}(R^*U(1))}{\hat{\Lambda}_{\alpha_n}(R^*U(1))} \right) = 0
\]

Hence the third term in the sum tends to 0, and the first result of (5.12) is proved. The proof is exactly the same for the second one, with rate \(\frac{\sqrt{k}}{\ln(1 - \alpha_n)}\) instead of \(\frac{\sqrt{k}}{\ln(1 - \alpha_n)}\). Then conditions (i) and (ii) in \((C^H)_{high}\) give the expected result. \(\Box\)
References

Abramowitz, M., Stegun, I. A., et al. (1966). Handbook of mathematical functions. Applied mathematics series, 55(62):39.

Artzner, P., Delbaen, F., Eber, J., and Heath, D. (1999). Coherent measures of risk. Mathematical Finance, 9(3):203–228.

Bargès, M., Cossette, H., and Marceau, E. (2009). TVaR-based capital allocation with copulas. Insurance: Mathematics and Economics, 45(3):348–361.

Bellini, F., Klar, B., Muller, A., and Giani, E. R. (2014). Generalized quantiles as risk measures. Insurance: Mathematics and Economics, 54:41–48.

Bernardi, M., Bigozzi, V., and Petrella, L. (2017). On the Lp-quantiles for the Student t distribution. Statistics & Probability Letters.

Breckling, J. and Chambers, R. (1988). M-quantiles. Biometrika, 75(4).

Cai, J. and Weng, C. (2016). Optimal reinsurance with expectile. Scandinavian Actuarial Journal, 2016(7):624–645.

Cambanis, S., Huang, S., and Simons, G. (1981). On the theory of elliptically contoured distributions. Journal of Multivariate Analysis, 11:368–385.

Chen, Z. (1996). Conditional Lp-quantiles and their application to the testing of symmetry in non-parametric regression. Statistics & Probability Letters, 29(2):107 – 115.

Daouia, A., Girard, S., and Stupfler, G. (2017a). Estimation of tail risk based on Extreme Expectiles. Journal of the Royal Statistical Society: Series B.

Daouia, A., Girard, S., and Stupfler, G. (2017b). Extreme M-quantiles as risk measures: from L1 to Lp optimization. Bernoulli.

de Haan, L. and Ferreira, A. (2006). Extreme value theory: an introduction. Springer Science & Business Media.

de Haan, L. and Resnick, S. (1998). On asymptotic normality of the hill estimator. Communications in Statistics, 14(4):849–866.

de Haan, L. and Rootzén, H. (1993). On the estimation of high quantiles. Journal of Statistical Planning and Inference, 35(1):1–13.

de Valk, C. (2016). Approximation of high quantiles from intermediate quantiles. Extremes, 19:661–686.

El Methni, J., Gardes, L., Girard, S., and Guillou, A. (2012). Estimation of extreme quantiles from heavy and light tailed distributions. Journal of Statistical Planning and Inference, 142(10):2735–2747.

Fang, K.-T., Kotz, S., and Ng, K. W. (1990). Symmetric multivariate and related distributions. Chapman and Hall.

Frahm, G. (2004). Generalized Elliptical Distributions: Theory and Applications. PhD thesis, Universität zu Köln.

Gardes, L. and Girard, S. (2005). Estimating extreme quantiles of weibull tail distributions. Communications in Statistics-Theory and Methods, 34:1065–1080.

Gong, J., Li, Y., Peng, L., and Yao, Q. (2015). Estimation of extreme quantiles for functions of dependent random variables. Journal of the Royal Statistical Society: Series B (Statistical Methodology), 77(5):1001–1024.

Goovaerts, M., Kaas, R., Dhaene, J., and Tang, Q. (2004). Some new classes of consistent risk measures. Insurance: Mathematics and Economics, 34:505–516.

Haezendonck, J. and Goovaerts, M. (1982). A new premium calculation principle based on Orlicz norms. Insurance: Mathematics and Economics, 1:41–53.

Hashorva, E. (2007). Extremes of conditioned elliptical random vectors. Journal of Multivariate Analysis, 98(8):1583–1591.

Hill, B. M. (1975). A simple general approach to inference about the tail of a distribution. The Annals of Statistics, 3.

Hsing, T. (1991). On Tail Index Estimation Using Dependant Data. The Annals of Statistics, 19(3):1547–1569.

Hua, L. and Joe, H. (2011). Second order regular variation and conditional tail expectation of multiple risks. Insurance: Mathematics and Economics, 49:537–546.

Huang, S. T. and Cambanis, S. (1979). Spherically invariant processes: Their nonlinear structure, discrimination, and estimation. Journal of Multivariate Analysis, 9(1):59–83.

Jessen, H. A. and Mikosch, T. (2006). Regularly varying functions. Publications de l’Institut Mathematique, 80(94):171–192.

Johnson, M. (1987). Multivariate Statistical Simulation. Wiley & Sons.
Kano, Y. (1994). Consistency property of elliptical probability density functions. Journal of Multivariate Analysis, 51:139–147.

Kelker, D. (1970). Distribution theory of spherical distributions and a location-scale parameter generalization. Sankhyā: The Indian Journal of Statistics, Series A, 32(4):419–430.

Koenker, R. (1992). When are expectiles percentiles? Econometric Theory, 8:423–424.

Koenker, R. and Bassett, G. J. (1978). Regression quantiles. Econometrica, 46(1):33–50.

Kratz, M. and Resnick, S. I. (1996). The qq-estimator and heavy tails. Stochastic Models, 12(4):699–724.

Linsmeier, T. J. and Pearson, N. D. (2000). Value at risk. Financial Analysts Journal, 56(2):47–67.

Mao, T. and Hu, T. (2012). Second-order properties of the Haezendonck-goovaerts risk measure for extreme risks. Insurance: Mathematics and Economics, 51:333–343.

Maume-Deschamps, V., Rullière, D., and Usseglio-Carleve, A. (2017a). Quantile predictions for elliptical random fields. Journal of Multivariate Analysis, 159:1 – 17.

Maume-Deschamps, V., Rullière, D., and Usseglio-Carleve, A. (2017b). Spatial expectile predictions for elliptical random fields. Methodology and Computing in Applied Probability.

Newey, W. and Powell, J. (1987). Asymmetric least squares estimation and testing. Econometrica, (55):819–847.

Opitz, T. (2016). Modeling asymptotically independent spatial extremes based on Laplace random fields. Spatial Statistics, 16:1–18.

Owen, J. and Rabinovitch, R. (1983). On the class of elliptical distributions and their applications to the theory of portfolio choice. The Journal of Finance, 38(3):745–752.

Parzen, E. (1962). On estimation of a probability density function and mode. Annals of Mathematical Statistics, 33(3):1065–1076.

Pickands, J. (1975). Statistical inference using extreme order statistics. The Annals of Statistics, 3(1):119–131.

Resnick, S. and Stărică, C. (1995). Consistency of Hill’s Estimator for Dependent Data. Journal of Applied Probability, 32(1):139–167.

Schultze, J. and Steinebach, J. (1996). On least squares estimates of an exponential tail coefficient. Statistics & Decisions, 14(3):353–372.

Sobotka, F. and Kneib, T. (2012). Geoadditive expectile regression. Computational Statistics & Data Analysis, 56(4):755–767.

Tang, Q. and Yang, F. (2012). On the Haezendonck-Goovaerts risk measure for extreme risks. Insurance: Mathematics and Economics, 50:217–227.

Taylor, J. W. (2008). Estimating Value at Risk and Expected Shortfall Using Expectiles. Journal of Financial Econometrics, 6(2):231–252.

Tyler, D. (1987). A Distribution-free M-estimator of Multivariate Scatter. The Annals of Statistics, 15(1):234–251.

Wang, H. J., Li, D., and He, X. (2012). Estimation of high conditional quantiles for heavy-tailed distributions. Journal of the American Statistical Association, 107(500):1453–1464.

Weissman, I. (1978). Estimation of parameters and large quantiles based on the $k$ largest observations. Journal of the American Statistical Association, 73(364):812–815.

Xiao, Y. and Valdez, E. (2015). A Black-Litterman asset allocation under Elliptical distributions. Quantitative Finance, 15(3):509–519.