Small-angle scattering in a marginal Fermi-liquid

E. C. Carter and A. J. Schofield
Theoretical Physics Group, School of Physics and Astronomy,
University of Birmingham, Edgbaston, Birmingham B15 2TT, United Kingdom
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We study the magnetotransport properties of a model of small-angle scattering in a marginal Fermi liquid. Such a model has been proposed by Varma and Abrahams [Phys. Rev. Lett. 86, 4652 (2001)] to account for the anomalous temperature dependence of in-plane magnetotransport properties of the high-$T_c$ cuprates. We study the resistivity, Hall angle and magnetoresistance using both analytical and numerical techniques. We find that small-angle scattering only generates a new temperature dependence for the Hall angle near particle-hole symmetric Fermi surfaces where the conventional Hall term vanishes. The magnetoresistance always shows Kohler’s rule behavior.

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The unusual magnetotransport properties in the normal state of the superconducting cuprates pose a major challenge to existing models of metallic behavior. Experiments on near optimally-doped materials have shown a linear-$T$ dependence of the resistivity, indicative of a marginal Fermi-liquid scattering rate. However, the scattering rate of the Hall component of an electric current exhibits an entirely different temperature dependence from that of the resistivity. Moreover, Kohler’s rule for the magnetoresistance is also violated.

An influential hypothesis was proposed by Anderson to explain these results. He argued that two independent scattering rates are present in the cuprates which govern the resistivity and Hall-current scattering respectively by appearing multiplicatively in the Hall conductivity. This “two-lifetime” scenario has often been used to interpret experimental data. Within this picture, for example, the deviations of Kohler’s rule in the magnetoresistance are accounted for by a “modified Kohler’s rule” whereby the lifetime controlling the Hall angle is the one that also governs the magnetoresistance. There have been many attempts to give a more microscopic explanation of the “two-lifetime” phenomenology, either by invoking a Fermi-liquid picture with anisotropic scattering effects (for example, hot or cold spots on the Fermi surface, or skew scattering), or going beyond a Fermi-liquid description inspired by Anderson’s original motivation.

More recently Varma and Abrahams (VA) have proposed that there is only a single temperature dependent scattering rate in the cuprates but this rate appears in magnetotransport responses in an unconventional fashion. In particular, the Hall angle is given by the square of the marginal scattering lifetime. This idea has received experimental support from very recent infrared optical Hall angle studies. The optical data fits best to a square Lorentzian in exactly the manner that Varma and Abrahams would predict.

Given this experimental support for VA’s hypothesis, we re-examine the microscopic basis they used to derive their form for the Hall conductivity. They considered how transport in a marginal Fermi-liquid is affected by small-angle impurity scattering anisotropically distributed around the Fermi surface. They make an expansion in the small scattering angle and argue that in the Hall conductivity the first term in the expansion dominates the conventional (zeroth order) term. In the longitudinal conductivity the conventional term always dominates.

In this paper, we address a number of key questions raised by their work, for when a perturbative correction dominates the zeroth order term it is usually an indication that the expansion is breaking down. Thus, we must establish first: under what conditions does the first correction dominates the zeroth order term? Secondly, we must check whether the expansion remains controlled under such conditions. Finally we consider the magnetoresistance since, as we will show, it is necessary for higher order terms to dominate the expansion in the magnetoresistance if the deviations from Kohler’s rule seen in experiment are to be accounted for within this model. Our approach combines an analytic expansion in the small-angle scattering parameter together with an extensive numerical investigation including anisotropy in the Fermi velocity and the small angle of scattering.

Our main findings are as follows. We find that for both the longitudinal conductivity and the Hall conductivity the corrections due to small-angle scattering are generically small compared to the leading term. However by tuning the Fermi surface close to particle-hole symmetry (zero average curvature) the conventional term can of course be tuned to zero. So under these conditions the Hall conductance is dominated by the leading correction in a controlled fashion leading to the form suggested originally by Varma and Abrahams. It is here that we emphasize the importance of the magnetoresistance as a key test for any microscopic theory (see for example Ref. [3]). We calculate the magnetoresistance directly and find that no new temperature dependence generated even at particle-hole symmetry.

We begin by describing our model. Varma and Abrahams’ suggestion of anisotropic small-angle scattering processes in a marginal Fermi-liquid was inspired by angle-resolved photoemission spectroscopy (ARPES)
measurements which indicate that the scattering rate of electrons has the following form:

$$\Gamma \sim \tau_M^{-1} + \tau_i^{-1} \cos^2 2\theta,$$  \hspace{1cm} (1)

where there is an isotropic temperature dependent contribution, \(\tau_M^{-1} \sim A + BT\) (the marginal Fermi-liquid inverse lifetime) and also an anisotropic temperature-independent contribution (proportional to \(\tau_i^{-1}\)). The idea of Ref. 1 is that \(\tau_i^{-1}\) is an outcome of small-angle scattering processes (possibly from out-of-plane impurities).

Here, we introduce (but do not restrict ourselves to) an explicit form of the scattering rate which satisfies these criteria:

$$\tau^{-1}(\theta, \theta') = \tau_M^{-1} + \tau_i^{-1} |\cos 2\theta| |\cos 2\theta'| \frac{e^{-(\theta - \theta')^2}}{\theta c}, \hspace{1cm} (2)$$

where \(\theta, \theta'\) measure distance round the Fermi surface [1].

The linearized Boltzmann transport equation is:

$$\sum_{k'} \left( \frac{1}{\tau(k)} + \frac{e}{\hbar} \mathbf{v}_k \times \mathbf{B} \cdot \nabla \mathbf{v}_k \right) \delta_{k,k'} - C(k,k') \right) g(k') \right) g(k') = \sum_{k,k'} A(k,k') g(k') = e \mathbf{E} \cdot \mathbf{v}_k \delta(\epsilon_k - \epsilon_F), \hspace{1cm} (3)$$

where the scattering rate \(1/\tau(k) = \sum_{k'} C(k,k')\) and \(C(k,k') = 2\pi \delta(\epsilon_k - \epsilon_{k'}) \tau^{-1}(\theta, \theta')\). One must solve for \(g(k)\), the deviation from the equilibrium distribution.

Our numerical calculations involve constructing the matrix \(A\) of Eq. 3 for an arbitrary, discretized Fermi surface and the scattering rate in Eq. 2 sums are weighted appropriately by the density of states per unit length along the surface, and \(g(k)\) thus obtained directly.

For our analytic calculations (following VA), the Boltzmann equation is solved by expanding to first order in \(\theta^2\), in addition to the Zener–Jones expansion. The derivatives caused by integrating with a sharply peaked function turn the scattering time into an operator, labeled \(\tau\), and the integral of \(\tau^{-1}(\theta, \theta')\) over \(\theta'\) must yield the ARPES inverse lifetime (Eq. 1). The transport equation (3) is reduced to a relationship which must be solved for \(\tau(\theta)\).

We illustrate the form of our results with an explicit example, within which coefficients can be calculated analytically. For a circular Fermi surface, the equation for \(\tau(\theta)\) is:

$$\tau v = \tau_M v + \frac{\theta^2 \tau_M}{\tau_i} \frac{d}{d\theta} \left[ \cos^2 2\theta \frac{d(\tau v)}{d\theta} \right].$$  \hspace{1cm} (4)

Noting that derivatives of \(\tau v\) only enter at the next order in \(\theta^2\) means we can write \((\tau v)' \sim \tau_M v'\) etc. and so

$$\tau(\theta) = \tau_M + \frac{\theta^2 \tau_M^2}{\tau_i} \left( \cos^2 2\theta \frac{d^2}{d\theta^2} - 2 \sin 4\theta \frac{d}{d\theta} \right). \hspace{1cm} (5)$$

This is equivalent to Eq. 16 of Ref. 1 except the dependence on the original scattering times and \(\theta_c\) is shown.

Conductivities are obtained from expressions such as

$$\sigma^{xy} = \frac{ne^2 eB}{\pi m} \int d\theta v^x \frac{d}{d\theta} \hat{\sigma}^{xy}, \hspace{1cm} (6)$$

and we find

$$\sigma^{xx} \sim \tau_M \left( C_0^{0} - C_0^{2} \frac{\theta^2 \tau_M}{\tau_i} + O(\theta^2) \right), \hspace{1cm} (7)$$

$$\sigma^{xy} \sim \omega_c \tau_M^2 \left( C_{xy}^{0} - C_{xy}^{2} \frac{\theta^2 \tau_M}{\tau_i} + O(\theta^2) \right), \hspace{1cm} (8)$$

$$\Delta \sigma^{xx} \sim -\omega_c \tau_M^3 \left( C_{\Delta xx}^{0} - C_{\Delta xx}^{2} \frac{\theta^2 \tau_M}{\tau_i} + O(\theta^2) \right). \hspace{1cm} (9)$$

This shows the apparent appearance of new \(T\)-dependences at each order in \(\theta^2\). The essence of VA’s argument rests on the value of the numerical coefficients \(C\). To account for the measured resistivity, \(C_{xy}^{1}/C_{xx}^{0}\) must be generally small: small-angle scattering does not dramatically affect the marginal Fermi-liquid scattering rate so \(\rho_{xx} = \sigma_{xx} \sim \tau_M^{-1} \sim T\). However, for the Hall conductivity VA argue that \(C_{xy}^{1}/C_{xx}^{0}\) is large so that the first correction dominates the zeroth order term. This would have a dramatic effect on the inverse Hall angle, since \(\cos \Theta_H = \sigma_{xx}/\sigma_{xy} \sim \tau_M^{-2} \sim T^2\), thus generating an apparently new temperature dependent scattering rate.

For the circular Fermi surface we can calculate these coefficients and find

$$C_{xx,xy,\Delta xx}^{0} = 1; \hspace{0.5cm} C_{xx}^{1} = 1, C_{xy}^{1} = 2, C_{\Delta xx}^{1} = 3. \hspace{1cm} (10)$$

When substituted into the expression for the inverse Hall angle above we see that for this case, to order \(\theta^2\), the inverse Hall angle and the resistivity have an identical temperature dependence and there is no new effect from small-angle scattering.

Clearly then, for small-angle scattering to have any effect, further anisotropy must be included [2]. Also we have assumed that \(\theta_c^2\) is a reasonable expansion parameter but, if the leading correction is to ever dominate the zeroth order term, this must be questionable. To address both these issues we now adopt a numerical approach which makes no assumptions on the size of \(\theta_c\) and can handle arbitrary Fermi surfaces and other anisotropies.

We have solved the model of Eq. 3 and 4 numerically without recourse to an expansion, so any new temperature dependence in the Hall angle can be seen directly. We studied a wide variety of Fermi surfaces and, in particular, the ARPES best-fit parameterizations [3]. We also allowed the small scattering angle itself to vary around the Fermi surface [for example as \(\theta_c \sim 0.1(1 + \cos^2 2\theta)\)]. The results of this study were that we were unable to find any significant new temperature dependence appearing in the Hall angle with the exception of the particle-hole symmetric Fermi surface discussed below.

To make quantitative comparison with the analytic work on the circular Fermi surface above we used the numerics to extract the numerical coefficients of the expansion of Eqs 7, 8, 9. We found that for the Fermi surface...
FIG. 2: Log-log graph of cot Θ_H against temperature (arbitrary units). A T^2 law (dashed line) is possible for the right Fermi surface (as Fig. 1, ε_F = -0.70t) but on altering the Fermi level slightly, the linear-T law (dot-dashed line) quickly returns (ε_F = -0.75t). [Parameters used: θ_e ~ 0.02, τ_1^-1 ~ 10, τ_M^-1 = 0.1 + T.]

FIG. 1: An example of a “particle-hole symmetric” Fermi surface, for which the conventional Hall term vanishes (ε_k = -t(cos k_x + cos k_y + 0.5966 cos k_x cos k_y); solid line: ε_F = -0.70t). Note that the shape does not coincide well with observed cuprate Fermi surfaces. A linear-T Hall angle has returned when the surface is tuned as far as either of the dashed shapes (ε_F = -0.75t, -0.65t).
much different from a $T^{-2}$ law, whether or not the Fermi surface is particle-hole symmetric. We illustrate our findings with a representative plot (Fig. 3), which shows that Kohler’s rule is well obeyed, in contradiction with the experimental measurements.

In conclusion, Varma and Abraham’s suggestion that the inverse Hall angle is measuring the square of a relaxation rate provides an important new perspective on the unusual magneto-transport properties of the cuprates. Their prediction of the form of the optical Hall angle is powerful evidence in favor of their suggestion. In this paper we have sought to analyze whether a model of small-angle scattering in a marginal Fermi-liquid can account for the form of Hall angle that Varma and Abrahams suggest. We have studied such a model both analytically and numerically, including various anisotropies, and have also gone beyond the Hall conductivity to consider magneto-resistance.

We find that in general small-angle scattering does not discriminate significantly differently between resistivity and inverse Hall angle except very close to particle-hole symmetry. At this point however the magneto-resistance continues to be dominated by the scattering rate seen in the resistivity. This conventional Kohler behavior of the magneto-resistance is found in this model independent of proximity to particle-hole symmetry and in contrast to the cuprates which show strong deviations from Kohler’s rule. Thus we conclude that small-angle scattering in a marginal Fermi-liquid is not the origin of the unusual magneto-transport in the cuprates. This problem remains a tantalizing key to deciphering the unconventional normal state properties of the cuprates.

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FIG. 3: Log-log graph of $\rho \Delta \rho$ (solid line) and $\Delta \rho/\Theta_H^2$ (dashed line) against $T$ (parameters as Fig 2 special Fermi surface $\epsilon_F = -0.70$). Kohler’s rule $\Delta \rho/\rho \sim \rho^{-2}$ can be seen to be obeyed ($\rho \Delta \rho \sim T$ independent). Experiment suggests a “modified Kohler’s rule” $(\Delta \rho/(\rho \Theta_H^2) \sim T$ independent) which cannot be reproduced within the model.

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21. The moduli are unimportant as the narrow Gaussian only allows contributions where $|\theta - \theta'| \lesssim \theta_c$.
22. However, this apparent new temperature dependence can be misleading: for example in the case of small-angle impurity scattering which is isotropic around the Fermi surface these terms can be summed to all orders in $\theta_c$ to give a temperature independent addition to the resistivity (i.e. Matthiessen’s rule). Since we will be looking for a large correction at leading order in the analytic work described here this effect will not be important. In the numerical work described later, we effectively do work to all orders in $\theta_c$ so take this effect into account.