A kinetic regime of hydrodynamic fluctuations and long time tails for a Bjorken expansion

Yukinao Akamatsu\textsuperscript{1} 
Aleksas Mazeliauskas\textsuperscript{2} Derek Teaney\textsuperscript{2}

\textsuperscript{1}Osaka University \\
\textsuperscript{2}Stony Brook University

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Hydrodynamics with noise

- Initial state fluctuations
- Thermal fluctuations – for example Landau-Lifshitz

\[ N_{gg}(t, k) \equiv \langle g^i(t, k) g^j(t, k) \rangle = (e + p) T \delta^{ij} \]

1. Conceptually important (required by the FDT)
2. Larger in smaller systems: \( N_{\text{particle}} \sim 10000 \) in the heavy-ions
3. Essential near a critical point

How do thermal fluctuations evolve during a Bjorken expansion?
How do thermal fluctuations change the Bjorken expansion?
Kinetic regime of hydrodynamic fluctuations – a new scale $k_*$

1. For hydrodynamic fluctuations with wavenumber $k$:
   - Equilibration rate $\sim \gamma_\eta k^2$
   - Expansion rate $\sim 1/\tau$ for a Bjorken expansion

2. Compete at a critical scale:
   $$k_* \sim \frac{1}{\sqrt{\gamma_\eta \tau}}$$

3. Derivative expansion controlled by $\epsilon \equiv \gamma_\eta / \tau \ll 1$

We derive an effective description for the kinetic regime $k_*$
1. Langevin equation

\[ \frac{dp}{dt} = -\gamma p + \xi, \quad \langle \xi(t)\xi(t') \rangle = 2TM\gamma \delta(t - t') \]

2. Calculate how \( \langle p^2(t) \rangle \) evolves through Langevin process

\[ \frac{d}{dt} \langle p^2 \rangle = -2\gamma \left[ \langle p^2 \rangle - MT \right] \]

Follow the same steps for hydrodynamics with external forcing
Hydro-kinetic equation: an analogy with Brownian motion

1. Langevin equation

\[ \frac{dp}{dt} = -\gamma p + \xi, \quad \langle \xi(t)\xi(t') \rangle = 2TM\gamma \delta(t - t') \]

2. Calculate how \( \langle p^2(t) \rangle \) evolves through Langevin process

\[ \frac{d}{dt} \langle p^2 \rangle = -2\gamma \left[ \langle p^2 \rangle - MT \right] + \text{external forcing} \]

Follow the same steps for hydrodynamics with external forcing
1. Linearized analysis in Bjorken: \( e = e_0 + \delta e, \vec{g} = (e_0 + p_0)\vec{v} \)

\[
\phi_a(t, \vec{k}) \equiv (c_s \delta e, \vec{g})
\]

2. Hydro-Langevin equation for \( \phi_a(t, \vec{k}) \)

\[
-\dot{\phi}(t, \vec{k}) = i\mathcal{L}\phi + \mathcal{D}\phi + \xi + \mathcal{P}(t)\phi
\]

3. Four eigenmodes of \( \mathcal{L} \): \( \phi_+, \phi_-, \phi_{T1}, \phi_{T2} \)

- left moving sound: \( \lambda_- = -c_s k \)
- right moving sound: \( \lambda_+ = c_s k \)
- transverse modes: \( \lambda_T = 0 \)

The Hydro-Langevin equation in eigen basis is similar to Brownian motion
Hydro-kinetic equation

1. Analyze the square of the eigenmodes – for example

\[ N_{++}(t, \vec{k}) \equiv \langle \phi_+(t, \vec{k}) \phi^*_+(t, \vec{k}) \rangle \]

2. Hydro-kinetic equations for \( N_{++} \)

\[ \dot{N}_{++} = -\frac{4}{3} \gamma \eta k^2 \left[ N_{++} - T_0 (e_0 + p_0) \right] - \frac{1}{\tau} \left[ 2 + c_s^2 + \cos^2 \theta_k \right] N_{++} \]

\( \underbrace{\text{equilibration}}_{\text{external forcing}} \)

3. Neglect off-diagonal components of density matrix

\[ N_{+-} \sim e^{-i(\lambda_+-\lambda_-)t} \sim 0 \]

\( \underbrace{\text{rotating wave approx}}_{\text{hydro fluctuations are driven out of equilibrium at } \vec{k}_*} \)
Solution of the hydro-kinetic equation
Bjorken expansion at late times

$k_*$ is the critical scale: For larger $k \gg k_*$, closer to equilibrium

$$N_{AA} \sim N_{eq} \left[ 1 + \frac{\#}{\gamma \eta \tau k_*^2} + \cdots \right]$$
Evolution of the background

Hydrodynamic equation for a Bjorken expansion:

\[
\frac{d}{d\tau} (\tau T^{\tau\tau}) = -T^{zz}
\]

- Without hydrodynamic fluctuations:

\[
T^{zz} = p_0 - \frac{4 \eta_0}{3 \tau} + (\lambda_1 - \eta \tau \pi) \frac{8}{9 \tau^2}
\]

  - ideal
  - 1st order
  - 2nd order

- Hydrodynamic fluctuations give another contributions

\[
T^{zz}_{\text{fluct}} = (e_0 + p_0) \langle u^z u^z \rangle = \frac{\langle g^z g^z \rangle}{e_0 + p_0}
\]
Nonlinear contributions from $k_*$ to the background

1. Compute the contribution from fluctuation

$$
\langle (g^z (t, \vec{x}))^2 \rangle = \int_0^\Lambda d^3 k \left[ N_{++} \cos^2 \theta + N_{T_2 T_2} \sin^2 \theta \right]
\sim 1 + \#/(\gamma \eta T^2) + \cdots \text{ for } k \gg k_*
$$

- Regularize cubic and linear UV divergences by a cutoff $\Lambda$

2. Renormalize the divergences c.f. Kovtun-Moore-Romatschke (11)

$$
T^{zz} = p_0(\Lambda) + \frac{\Lambda^3 T}{6 \pi^2} - \frac{4}{3 \tau} \left[ \eta_0(\Lambda) + \frac{17 \Lambda T}{120 \pi^2} \frac{e_0 + p_0}{\eta_0} \right] + \text{finite}
\equiv p_{\text{phys}} \quad \eta_{\text{phys}}
$$

The cutoff dependence is absorbed by renormalization of $p_0$ and $\eta_0$
Finite contributions: Long-time tails

Evaluate the finite parts after renormalization

$$T^{zz} = p - \frac{4\eta}{3\tau} + 1.08318 T \left(\frac{1}{4\pi\gamma\eta\tau}\right)^{3/2} + \cdots$$

Simple understanding of the scaling

$$T^{zz}_{\text{fluct}} \sim \frac{1}{2} k_B T \int d^3k \sim T k_*^3 \sim T \left(\frac{1}{\gamma\eta\tau}\right)^{3/2}$$

The finite contribution from $k_*$ gives the long-time tails
Implications for heavy-ion collisions

Plugging in typical values

\[
\frac{T^3}{s} \simeq \frac{1}{13.5}, \quad \frac{\lambda_1 - \eta \tau \pi}{e + p} \simeq -0.8 \left( \frac{\eta}{e + p} \right)^2
\]

Compute \(4\langle T^{zz}\rangle/(e + p)\)

\[
\frac{\eta}{s} = \frac{1}{4\pi} : \quad 1 - 0.092 \left( \frac{4.5}{\tau T} \right) + 0.034 \left( \frac{4.5}{\tau T} \right)^{3/2} - 0.00085 \left( \frac{4.5}{\tau T} \right)^2
\]

\[
\frac{\eta}{s} = \frac{2}{4\pi} : \quad \left\{ \begin{array}{l}
\text{ideal} \\
\text{1st order} \\
\text{1.5th order} \\
\text{2nd order}
\end{array} \right. \quad 1 - 0.185 \left( \frac{4.5}{\tau T} \right) + 0.013 \left( \frac{4.5}{\tau T} \right)^{3/2} - 0.0034 \left( \frac{4.5}{\tau T} \right)^2
\]

Thermal fluctuation is practically larger than 2nd order viscous correction
Comparison to previous diagrammatic calculation in static systems
c.f. Kovtun-Moore-Romatschke (11), Kovtun-Yaffe (03)

Apply a shear perturbation

\[ h_{xy}(t) = h_{xy}e^{-i\omega t} \]

1. Compute how \( N_{++} \) evolves away from equilibrium:

\[ \dot{N}_{++} = -\gamma\eta k^2 [N_{++} - N_{eq}] + \dot{h}(t)N_{++} \]

2. Compute fluctuation contribution to stress in the linear order of \( h \):

\[
\frac{T^{xy}(\omega)}{h^{xy}(\omega)} = \left( p_0 + \frac{\Lambda^3 T}{6\pi^2} \right) - i \left( \eta_0 + \frac{17\Lambda T}{120\pi^2 \gamma\eta} \right) \omega \\
+ (1 + i) \left( \frac{3}{2} \right)^{3/2} \frac{1}{240\pi} + 7T \left( \frac{\omega}{\gamma\eta} \right)^{3/2}
\]

Agrees with the previous diagrammatic calculations
Summary & Outlook

- Hydro-kinetic equation for $k_*$, advantageous in expanding systems
- Universal renormalization of pressure $p_0(\Lambda)$ and viscosity $\eta_0(\Lambda)$
- Background-dependent long-time tails $\propto T^{-3/2}, \omega^{3/2}$
- Alternative way to solve the hydrodynamics with noise

$$d_\mu T^{\mu\nu} = 0, \quad \dot{N}_{AA} = \cdots, \quad T^{\mu\nu} = T^{\mu\nu}_{bkg} + \int_k N_{AA},$$

- Bulk viscosity renormalization for nonconformal fluid

$\zeta' = \zeta_0(\Lambda) + \frac{\Lambda T^2}{2\pi^2} \left[ \left( \frac{1}{3} + \frac{T}{2} \frac{dc_s^2}{dT} - c_s^2 \right)^2 \frac{s}{4\eta} + \left( \frac{1}{3} - c_s^2 \right)^2 \frac{2s}{\eta} \right]$  

- Application to critical dynamics $\zeta = \zeta_0(\Lambda) + \frac{\Lambda T^2}{2\pi^2} \left[ \left( \frac{1}{3} + \frac{T}{2} \frac{dc_s^2}{dT} - c_s^2 \right)^2 \frac{s}{4\eta} + \left( \frac{1}{3} - c_s^2 \right)^2 \frac{2s}{\eta} \right]$
Thank you for your attention!