We study the occurrence and nature of naked singularities for a dust model with non-zero cosmological constant in \((n + 2)\)-dimensional Szekeres space-times (which possess no Killing vectors) for \(n \geq 2\). We find that central shell-focusing singularities may be locally naked in higher dimensions but depend sensitively on the choice of initial data. In fact, the nature of the initial density determines the possibility of naked singularity in space-times with more than five dimensions. The results are similar to the collapse in spherically symmetric Tolman-Bondi-Lemaître space-times.

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I. INTRODUCTION

An extensive study [1-6] of gravitational collapse has been carried out of Tolman-Bondi-Lemaître (TBL) spherically symmetric space-times containing irrotational dust. Due to simplifications introduced by the spherical symmetry several generalizations of this model have been considered. A general conclusion from these studies is that a central curvature singularity forms but its local or global visibility depends on the initial data. Also the study of higher-dimensional spherical collapse reveals the interesting feature that visibility of singularity is impossible in space-times with more than five dimensions with proper choice of regular initial data [7-9].

By contrast, there is very little progress in studying non-spherical collapse. The basic difficulty is the ambiguity of horizon formation in non-spherical geometries and the influence of gravitational radiation. Schoen and Yau [10] proposed a sufficient criterion for the formation of trapped surfaces in an arbitrary space-time but it fails to say anything about the conditions which lead to the formation of naked singularities. This problem has been restated by Thorne [11] in the form of a conjecture known as the hoop conjecture which states that “horizon form when and only when a gravitational mass \(M\) gets compacted into a region whose circumference in every direction is \(C \lesssim 4\pi M\)”.

Subsequently, there were attempts namely numerical simulations of prolate and oblate collapse [12], gravitational radiation emission in aspherical collapse [13], analytical studies of prolate collapsing spheroids [14] and others [15,16] to prove or disprove the conjecture. Interestingly all of them either confirmed or failed to refute the conjecture.

The quasi-spherical dust collapse models, given by Szekeres metric [17] were analyzed by Szekeres himself [18], Joshi and Krolak [19], Deshingkar, Joshi and Jhingan [20] and extensively by Goncalves [21]. The solutions for dust and a non-zero cosmological constant were found by Barrow and Stein-Schabes [26]. In this work, we study the gravitational collapse in the recently generalized \((n + 2)\)-dimensional Szekeres metric [22]. As in four-dimensional space-time, this higher dimensional model does not admit any Killing vector and the description quasi-spherical arises because it has invariant family of spherical...
hypersurfaces. The paper is organized as follows: In section II we derive the basic equations and regularity conditions. In section III we investigate the formation and the nature of central singularity. We study the formation of an apparent horizon due to collapse in section IV. The nature and the strength of the singularity is investigated by an analysis of the geodesics in sections V and VI respectively. Finally the paper ends with a short discussion.

II. BASIC EQUATIONS AND REGULARITY CONDITIONS

Recently, dust solutions have been obtained for \((n + 2)\)-dimensional Szekeres’ space-time metric for which the line element is [22]

\[
ds^2 = dt^2 - e^{2\alpha}dr^2 - e^{2\beta}\sum_{i=1}^{n} dx_i^2
\]

where \(\alpha\) and \(\beta\) are functions of all the \((n + 2)\) space-time coordinates. Under the assumption that \(\beta’ (= \frac{\partial \beta}{\partial r}) \neq 0\), the explicit form of the metric coefficients are [22]

\[
e^\beta = R(t, r) e^{\nu(r, x_1, \ldots, x_n)}
\]

and

\[
e^\alpha = \frac{R' + R \nu'}{\sqrt{1 + f(r)}}
\]

where

\[
e^{-\nu} = A(r) \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} B_i(r)x_i + C(r)
\]

and \(R\) satisfied the differential equation

\[
\ddot{R}^2 = f(r) + \frac{F(r)}{R^{n-1}} + \frac{2\Lambda}{n(n + 1)} R^2.
\]

Here \(\Lambda\) is cosmological constant, \(f(r)\) and \(F(r)\) are arbitrary functions of \(r\) alone; and the other arbitrary functions, namely \(A(r), B_i(r)\) and \(C(r)\), are restricted by the algebraic relation [22]

\[
\sum_{i=1}^{n} B_i^2 - 4AC = -1
\]

The \(r\)-dependence of these arbitrary functions \(A, B_i\) and \(C\) plays an important role in characterizing the geometry of the \((n + 1)\)-dimensional space. In particular, if we choose
\[ A(r) = C(r) = \frac{1}{2} \] and \( B_i(r) = 0 \ (\forall \ i = 1, 2, \ldots, n) \) then using the transformation [22]

\[
\begin{align*}
x_1 &= \sin \theta_n \sin \theta_{n-1} \ldots \sin \theta_2 \cot \frac{\theta_1}{2} \\
x_2 &= \cos \theta_n \sin \theta_{n-1} \ldots \sin \theta_2 \cot \frac{\theta_1}{2} \\
x_3 &= \cos \theta_{n-1} \sin \theta_{n-2} \ldots \sin \theta_2 \cot \frac{\theta_1}{2} \\
&\vdots \\
x_{n-1} &= \cos \theta_3 \sin \theta_2 \cot \frac{\theta_1}{2} \\
x_n &= \cos \theta_2 \cot \frac{\theta_1}{2}
\end{align*}
\]

the space-time metric (1) reduces to the spherically symmetric TBL form:

\[
ds^2 = dt^2 - \frac{R'^2}{1 + f(r)} \, dr^2 - R^2 d\Omega^2.
\] (7)

In the subsequent discussion we shall restrict ourselves to the quasi-spherical space-time which is characterized by the \( r \) dependence of the function \( \nu \) (i.e., \( \nu' \neq 0 \)). From the Einstein field equations we have an expression for energy density for the dust, as

\[
\rho(t, r, x_1, \ldots, x_n) = \frac{n}{2} \frac{F' + (n + 1)F\nu'}{R^n(R' + R\nu')}.
\] (8)

Thus a singularity will occur when either (i) \( R = 0 \) i.e., \( \beta = -\infty \) or (ii) \( \alpha = -\infty \). Using the standard terminology for spherical collapse, the first case corresponds to a shell-focusing singularity, while the second case gives rise to a shell-crossing singularity. As in a TBL space-time, the shell-crossing singularities are gravitationally weak, and we shall not consider them any further here. Hence, we shall restrict ourselves to the situation with \( \alpha > -\infty \).

Suppose that \( t = t_i \) is the initial hypersurface from which the collapse develops. For the initial data we assume that \( R(t_i, r) \) is a monotonically increasing function of \( r \). So, without any loss of generality, we can label the dust shells by the choice \( R(t_i, r) = r \). As a result, the expression for the initial density distribution is given by

\[
\rho_i(r, x_1, \ldots, x_n) = \rho(t_i, r, x_1, \ldots, x_n) = \frac{n}{2} \frac{F' + (n + 1)F\nu'}{r^n(1 + r\nu')}
\] (9)

If we started the collapse from a regular initial hypersurface the function \( \rho_i \) must be non-singular (and also positive for a physically realistic model). Furthermore, in order for the space-time to be locally flat near \( r = 0 \), we must have \( f(r) \to 0 \) as \( r \to 0 \). Then, from equation (5), the boundedness of \( \dot{R}^2 \) as \( r \to 0 \) demands that \( F(r) \sim O(r^m) \) where \( m \geq n - 1 \). But, for small \( r \), \( \rho_i(r) \approx \frac{\nu''}{2r^n} \) and consequently, for regular \( \rho_i(r) \) near \( r = 0 \), we must have \( F(r) \sim O(r^{n+1}) \). Thus, starting with a regular initial hypersurface, we can express \( F(r) \) and \( \rho_i(r) \) by power series near \( r = 0 \) as

\[
F(r) = \sum_{j=0}^{\infty} F_j \, r^{n+1+j}.
\] (10)

and

\[
\rho_i(r) = \sum_{j=0}^{\infty} \rho_j \, r^j.
\] (11)
As \( \nu' \) appears in the expression for the density as well as in the metric coefficient, we can write

\[
\nu'(r) = \sum_{j=-1}^{\infty} \nu_j r^j
\]  

(12)

where \( \nu_{-1} > -1 \).

Now, using these series expansions in equation (9) we have the following relations between the coefficients,

\[
\rho_0 = \frac{n(n+1)}{2} F_0, \quad \rho_1 = \frac{n}{2} \left( n + 1 + \frac{1}{1 + \nu_{-1}} \right) F_1,
\]

\[
\rho_2 = \frac{n}{2} \left[ \left( n + 1 + \frac{2}{1 + \nu_{-1}} \right) F_2 - \frac{F_1 \nu_0}{(1 + \nu_{-1})^2} \right],
\]

\[
\rho_3 = \frac{n}{2} \left[ \left( n + 1 + \frac{3}{1 + \nu_{-1}} \right) F_3 - \frac{2 F_2 \nu_0}{(1 + \nu_{-1})^2} - \frac{(1 + \nu_{-1}) \nu_1 - \nu_2^2}{(1 + \nu_{-1})^3} F_1 \right]
\]  

(13)

and so on.

Finally, if we assume that the density gradient is negative and falls off rapidly to zero near the centre then we must have \( \rho_1 = 0 \) and \( \rho_2 < 0 \). Consequently, we have the restrictions that \( F_1 = 0 \) and \( F_2 < 0 \).

III. FORMATION OF SINGULARITY AND ITS NATURE

In order to form a singularity from the gravitational collapse of dust we first require that all portions of the dust cloud are collapsing i.e., \( \dot{R} \leq 0 \). Let us define \( t_{sf}(r) \) and \( t_{sc}(r) \) as the time for shell-focusing and shell-crossing singularities to occur occurring at radial coordinate \( r \). This gives the relations [18]

\[
R(t_{sf}, r) = 0
\]  

(14)

and

\[
R'(t_{sc}, r) + R(t_{sc}, r) \nu'(r, x_1, x_2, ..., x_n) = 0.
\]  

(15)

Note that \( 't_{sc}' \) may also depend on \( x_1, x_2, ..., x_n \).

As mentioned earlier, the shell-crossing singularity is not of much physical interest so we shall just consider the shell-focusing singularity for the following two cases:

(i) Marginally bound case : \( f(r) = 0 \) with \( \Lambda \neq 0 \)

In this case equation (5) can easily be integrated to give

\[
t = t_i + \sqrt{\frac{2n}{(n+1)\Lambda}} \left[ \sinh^{-1} \left( \sqrt{\frac{2n}{n(n+1)F(r)}} \right) - \sinh^{-1} \left( \sqrt{\frac{2n\Lambda r^{n+1}}{n(n+1)F(r)}} \right) \right]
\]  

(16)

Also, from equations (14) and (16), we have
\[ t_{sf} = t_i + \sqrt{\frac{2n}{(n+1)\Lambda}} \sinh^{-1}\left(\sqrt{\frac{2\Lambda r^{n+1}}{n(n+1)F(r)}}\right) \]  \hspace{1cm} (17)

(ii) Non-marginally bound case with time symmetry : \( f(r) \neq 0, \: \dot{R}(t_i, r) = 0, \Lambda = 0 \)

Here, the solution of equation (5) gives \( t \) as a function of \( r \):

\[ t = t_i + \frac{2}{(n+1)\sqrt{F}} \left[ \sqrt{\frac{\pi}{\Gamma(b+1)}} \frac{\Gamma(b+1)}{\Gamma(b+1/2)} r^{\frac{b+1}{2}} - R^{\frac{b+1}{2}} F_1\left[\frac{1}{2}, b, b+1, \left(\frac{R}{r}\right)^{n-1}\right] \right] \]  \hspace{1cm} (18)

and so the time of the shell-focusing singularity is given by

\[ t_{sf} = t_i + \frac{2\sqrt{\pi}}{(n+1)\sqrt{F}} \frac{\Gamma(b+1)}{\Gamma(b+1/2)} r^{\frac{b+1}{2}}, \]  \hspace{1cm} (19)

where \( F_1 \) is the usual hypergeometric function with \( b = \frac{1}{2} + \frac{1}{n-1} \). However, for the five-dimensional space-time \( (n = 3) \), \( R \) has the particularly simple form

\[ R^2 = r^2 - \frac{F(r)}{r^2}(t - t_i)^2, \]  \hspace{1cm} (20)

and therefore

\[ t_{sf} = t_i + \frac{r^2}{\sqrt{F(r)}}. \]  \hspace{1cm} (21)

**IV. FORMATION OF TRAPPED SURFACES**

The event horizon of observers at infinity plays an important role in the nature of the singularity. However, due to the complexity of the calculation, we shall consider a trapped surface which is a compact space-time 2-surface whose normals on both sides are future-pointing converging null-geodesic families. In particular, if \( (r = \text{constant}, t = \text{constant}) \) the 2-surface \( S_{r,t} \) is a trapped surface then it, and its entire future development, lie behind the event horizon provided the density falls off fast enough at infinity.

If \( t_{ah} \) is the instant of the formation of apparent horizon then we must have [7,8,18]

\[ 2\Lambda R^{n+1}(t_{ah}, r) - n(n+1)R^{n-1}(t_{ah}, r) + n(n+1)F(r) = 0. \]  \hspace{1cm} (22)

Thus, for the above two cases, the explicit expressions for \( t_{ah} \) are the following:

\[ t = t_i + \sqrt{\frac{2n}{(n+1)\Lambda}} \sinh^{-1}\left(\sqrt{\frac{2\Lambda r^{n+1}}{n(n+1)F(r)}}\right) - \sinh^{-1}\left(\sqrt{\frac{2\Lambda R^{n+1}(t_{ah}, r)}{n(n+1)F(r)}}\right) \]  \hspace{1cm} (23)

for \( f(r) = 0, \Lambda \neq 0 \) and

\[ t_{ah} = t_i + \frac{2}{(n+1)\sqrt{F}} \left[ \sqrt{\frac{\pi}{\Gamma(b+1)}} \frac{\Gamma(b+1)}{\Gamma(b+1/2)} r^{\frac{b+1}{2}} - F^b \right] F_1\left[\frac{1}{2}, b, b+1, \frac{F}{r^{n-1}}\right] \]  \hspace{1cm} (24)
for \( f(r) \neq 0 \), \( \dot{R}(t_i, r) = 0, \Lambda = 0 \) and for \( n = 3 \) we get

\[
t_{ah} = t_i + r \sqrt{\frac{r^2}{F} - 1}.
\]  

(25)

From the expressions for \( t_{sf} \) and \( t_{ah} \), we note that the shell-focusing singularity that appears at \( r > 0 \) is in the future of the apparent horizon in both cases. But since we are interested in the central shell-focusing singularity (at \( r = 0 \)) we require the time of occurrence of central shell-focusing singularity \( t_0 (= t_{sf}(0)) \). From equations (16), (19) and (22), taking the limit as \( r \to 0 \), we have that

\[
t_{ah}(r) - t_0 = -\frac{1}{(n + 1) F_0^{3/2}} \frac{1}{\sqrt{1 + \frac{2\Lambda}{n(n + 1) F_0}}} \left[ F_1 r + \left( F_2 - \frac{(4\Lambda + 3n(n + 1) F_0^2)}{4F_0(2\Lambda + n(n + 1) F_0)} \right) r^2 + \left( F_3 - \frac{(4\Lambda + 3n(n + 1) F_0 F_1)}{4F_0(2\Lambda + n(n + 1) F_0)} \right) r^3 + ... \right]
\]

(26)

for \( f(r) = 0, \Lambda \neq 0 \)

\[
t_{ah}(r) - t_0 = -\frac{\sqrt{\pi}}{(n + 1) F_0^{3/2}} \frac{\Gamma(b + 1)}{\Gamma(b + 1/2)} \left[ F_1 r + \left( F_2 - \frac{3F_1^2}{4F_0} \right) r^2 + \left( F_3 - \frac{9F_1 F_2}{4F_0} \right) r^3 + ... \right]
\]

(27)

for \( f(r) \neq 0, \dot{R}(t_i, r) = 0, \Lambda = 0 \)

for \( n = 3 \) (with \( f(r) \neq 0, \dot{R}(t_i, r) = 0, \Lambda = 0 \))

\[
t_{ah}(r) - t_0 = -\frac{1}{2F_0^{3/2}} \left[ F_1 r + \left( F_2 + F_0^2 - \frac{3F_1^2}{4F_0} \right) r^2 + ... \right]
\]

(28)

Here, \( t_0 \) is the time at which the singularity is formed at \( r = 0 \), and \( t_{ah}(r) \) is the instant at which a trapped surface is formed at a distance \( r \). Therefore, if \( t_{ah}(r) > t_0 \), a trapped surface will form later than the epoch at which any light signal from the singularity can reach an observer. Hence the necessary condition for a naked singularity to form is \( t_{ah}(r) > t_0 \); while that for a black hole to form is \( t_{ah}(r) \geq t_0 \). It is to be noted that this criteria for naked singularity is purely local. Hence, in the present problem it possible to have local naked singularity or a black hole form under the conditions shown in the Table I.
| Marginally bound case: $f(r) = 0, \Lambda \neq 0$ | Naked Singularity | Black Hole |
|---|---|---|
| (i) $\rho_1 < 0, \forall n$ | (i) $\rho_1 = 0, \rho_2 < 0, n = 3$, $F_2 \geq -2F_0^2 \sqrt{1 + \frac{\Lambda}{3F_0}}$ |
| (ii) $\rho_1 = 0, \rho_2 < 0, n = 2$, $F_2 < -2F_0^2 \sqrt{1 + \frac{\Lambda}{3F_0}}$ | (ii) $\rho_1 = 0, \rho_2 < 0, n = 4$, $F_3 \geq -2F_0^{5/2} \sqrt{1 + \frac{\Lambda}{3F_0}}$ |
| (iii) $\rho_1 = 0, \rho_2 < 0, n = 3$, $F_2 < -\frac{8}{3\pi}F_0^2$ | (iii) $\rho_1 = 0, \rho_2 = 0, \rho_3 < 0, n = 2$, $F_3 \geq -\frac{8}{3\pi}F_0^{5/2}$ |
| (iv) $\rho_1 = 0, \rho_2 = 0, \rho_3 < 0, n = 2$, $F_3 < -\frac{8}{3\pi}F_0^2$ | (iv) $\rho_1 = 0, \rho_2 = 0, \rho_3 < 0, n = 3$, $F_3 \geq -\frac{8}{3\pi}F_0^{5/2}$ |
| (v) $\rho_1 = 0, \rho_2 = 0, ..., \rho_j = 0, \rho_{j+1} < 0$, $j \geq 3, \forall n$ | (v) $\rho_1 = 0, \rho_2 = 0, ..., \rho_j = 0, \rho_{j+1} < 0$, $j \geq 3, \forall n$ |

| Non-marginally bound case with time symmetry: $f(r) \neq 0, R(t, r) = 0, \Lambda = 0$ | |
| (i) $\rho_1 < 0, \forall n$ | (i) $\rho_1 = 0, \rho_2 < 0, n = 3$, $F_2 \geq -F_0^2$ |
| (ii) $\rho_1 = 0, \rho_2 < 0, n = 2$, $F_2 < -F_0^2$ | (ii) $\rho_1 = 0, \rho_2 < 0, n = 4$, $F_3 \geq -\frac{8}{3\pi}F_0^{5/2}$ |
| (iii) $\rho_1 = 0, \rho_2 = 0, \rho_3 < 0, n = 2$, $F_3 < -\frac{8}{3\pi}F_0^2$ | (iii) $\rho_1 = 0, \rho_2 = 0, \rho_3 < 0, n = 3$, $F_3 \geq -\frac{8}{3\pi}F_0^{5/2}$ |
| (iv) $\rho_1 = 0, \rho_2 = 0, \rho_3 < 0, n = 2$, $F_3 < -\frac{8}{3\pi}F_0^2$ | (iv) $\rho_1 = 0, \rho_2 = 0, \rho_3 < 0, n = 3$, $F_3 \geq -\frac{8}{3\pi}F_0^{5/2}$ |
| (v) $\rho_1 = 0, \rho_2 = 0, ..., \rho_j = 0, \rho_{j+1} < 0$, $j \geq 3, \forall n$ | (v) $\rho_1 = 0, \rho_2 = 0, ..., \rho_j = 0, \rho_{j+1} < 0$, $j \geq 3, \forall n$ |

However, if we relax the condition on the density gradient (namely the condition that $\rho_1 = 0, \rho_2 < 0$ i.e., $F_1 = 0, F_2 < 0$) then it is possible to have a naked singularity in all dimensions. These results are identical to those obtained in the presence of spherical symmetry. Hence the local nature of singularity is not affected by the geometry of the space-time with respect to whether it is spherical or quasi-spherical. However, we note that the Szekeres space-times retain certain special geometrical properties shared by spherical space-times in all dimensions, in particular they do not allow gravitational radiation to be present in the space-time.

**V. GEODESICS AND THE NATURE OF SINGULARITY**

Here, for simplicity, we shall consider only the marginally bound case (i.e., $f(r) = 0$) with $\Lambda = 0$. Then $R(t, r)$ has the simple explicit solution (choosing the initial time $t_i = 0$):

$$R = \left[ r^{\frac{1}{4\alpha}} - \frac{n + 1}{2} \sqrt{F(r)} \ t \right]^{\frac{n}{4\alpha}}$$

(29)

We now follow the geodesic analysis of Joshi and Dwivedi [23] for TBL model. For simplicity of calculation we introduce the following functions:

$$X = \frac{R}{\alpha}$$
$$\xi = \frac{r F'}{F}$$
$$\eta = \frac{Q'}{Q}$$
$$\zeta = \frac{F}{r^{\alpha(n-1)}}$$
$$Q = e^{-\nu}$$
$$\Theta = \frac{1 - \frac{\mu + 3}{2\alpha(n-1)} + \frac{\mu + 3}{4\alpha(n-1)^2}}{\frac{1}{2\alpha(n-1)} + \frac{\mu + 3}{4\alpha(n-1)^2}}$$

(30)
where the constant $a$ is restricted by $a \geq 1$. In order to determine the nature of the singularity we examine whether it is possible to have outgoing null geodesics which are terminated in the past at the central singularity $r = 0$. Suppose this occurs at singularity time $t = t_0$ at which $R(t_0, 0) = 0$. In order to decide this we start with radial null geodesic, given by

$$\frac{dt}{dr} = e^\alpha = \frac{R'Q - RQ'}{Q}.$$  \hspace{1cm} (31)

Next, we introduce the notation $u = r^a$, so we write

$$\frac{dR}{du} = U(X, u),$$  \hspace{1cm} (32)

where $U(X, u) = \left(1 - \frac{\xi}{X^{n+1}}\right)\frac{H}{a} + \frac{2}{a}\sqrt{\frac{\xi}{X^{n+2}}} \ 	ext{with} \ H = \frac{\xi}{n+1} X + \frac{\eta}{X^{n+2}}$.

We now study the limiting behaviour of the function $X$ as we approach the singularity at $R = 0, u = 0$ along the radial null geodesic identified above. If we denote the limiting value by $X_0$ then

$$X_0 = \lim_{R \to 0} \frac{X}{u} = \lim_{R \to 0} \frac{dR}{du} = \lim_{R \to 0} U(X, u) = U(X_0, 0)$$  \hspace{1cm} (33)

Thus if this polynomial equation in $X_0$ has at least one positive real root then it is possible to have a radial null geodesic outgoing from the central singularity. More explicitly, $X_0$ is the root of the following equation in $X$:

$$\left(a - \frac{\xi_0}{n+1}\right)X^n + \left(\frac{\xi_0}{n+1} - \eta_0\right)\sqrt{\frac{\xi_0}{X^{n+2}}} X^{\frac{n+1}{2}} - \Theta_0 X^{\frac{n-1}{2}} + \sqrt{\xi_0} \Theta_0 = 0$$  \hspace{1cm} (34)

where suffix ‘$o$’ for the variables $\xi, \eta, \zeta$ and $\Theta$ stands for the values of these quantities at $r = 0$. As exact analytic solution is not possible for $X$ so we study the roots by numerical methods. Table II shows the dependence of the nature of the roots on the variation of the parameters involved.

Table II: Positive roots ($X_0$) of the eqn. (34) for different values of the parameters namely $a, \eta_0, \xi_0, \zeta_0, \Theta_0$ and in different dimensions, $n$. 
From the definition of the functions in equation (30) it is clear that the parameters \(a, \xi_0, \zeta_0\) are always positive while \(\eta_0, \Theta_0\) can have positive as well as negative values. From the Table we see that as we decrease the value of \(a\), keeping the other parameters fixed, then the formation of a naked singularity is more probable in higher dimensions. The situation for \(\zeta_0\) is similar. But for \(\xi_0\) and \(\eta_0\) the condition is reversed: black hole formation becomes more probable as we increase the dimension of the space-time if we decrease the value of \(\xi_0\) (or \(\eta_0\)) and keep the other constants fixed. However, no such conclusions can be drawn about the variation of \(\Theta_0\).

### VI. STRENGTH OF THE NAKED SINGULARITY

A singularity is called \textit{gravitationally strong}, or simply \textit{strong}, if it destroys by crushing or tidally stretching to zero volume all objects that fall into it; it is called \textit{weak} if no object that falls into the singularity is destroyed in this way. A precise characterization of Tipler strong [24] singularities has been given by Clarke and Krolak [25], who proposed the strong focusing condition. A sufficient condition for a strong curvature singularity is that, for at least one non-space-like geodesic with affine parameter \(\lambda \to 0\) on approach to the singularity, we must have

\[
\lim_{\lambda \to 0} \lambda^2 R_{ij} K^i K^j > 0
\]

where \(K^i = \frac{dx^i}{d\lambda}\) is the tangent vector to the radial null geodesic.
Our purpose here to investigate the above condition along future-directed radial null geodesics that emanate from the naked singularity. Now equation (35) can be expressed as (using L’Hôpital’s rule)

\[
\lim_{\lambda \to 0} \lambda^2 R_{ij} K^i K^j = \frac{n (\xi_0 - (n+1) \eta_0) (\Theta_0 - (n+1) \eta_0)}{2X_0^2 (N_0 + \eta_0)^{\frac{1}{n+2}}}
\]

(36)

where \(H_0 = H(X_0, 0), N_0 = N(X_0, 0)\).

The singularity is gravitationally strong in the sense of Tipler if

\[
\xi_0 - (n + 1)\eta_0 > \max \left\{ 0, -\frac{(n + 1)\Theta_0}{X_0^{\frac{n+2}{n+1}}} \right\}
\]

or

\[
\xi_0 - (n + 1)\eta_0 < \min \left\{ 0, -\frac{(n + 1)\Theta_0}{X_0^{\frac{n+2}{n+1}}} \right\}
\]

If the above condition is not satisfied for the values of the parameters then

\[
\lim_{\lambda \to 0} \lambda^2 R_{ij} K^i K^j \leq 0 \quad \text{and the singularity may or may not be Tipler strong.}
\]

VII. DISCUSSION AND CONCLUDING REMARKS

In this paper we have studied gravitational collapse in \((n + 2)\)-dimensional space-time using a higher-dimensional generalization of the quasi-spherical Szekeres metrics with non-zero cosmological constant. We have examined the local nature of the central shell-focusing singularity by a comparative study of the time of the formation of trapped surface and the time of formation of the central shell-focusing singularity. If we assume the initial density gradient falls off rapidly and vanishes at \(r = 0\), then naked singularity formation is possible only up to space-time dimension five. However, if we drop the above restriction on the initial density distribution then the Cosmic Censorship Conjecture may be violated in any dimension \((n \geq 2)\). Thus we deduce that the nature of the central singularity depends sensitively in these metrics on the choice of the initial data (particularly on the choice of initial density profile). This is also confirmed by our geodesic study following the approach of Joshi and Dwivedi [23] where we have shown numerically that the nature of singularity depends on the value of the defining parameters at \(r = 0\). Finally, we have examined the strength of the naked singularity using the criterion introduced by Tipler [24]. We found that the naked singularity will be a strong curvature singularity depending on the appropriate choice of the value of the parameters at \(r = 0\). These investigations provide some insights into the phenomenon of gravitational collapse in a situation without any imposed Killing symmetries. However, the collapses are special in other senses which permit exact solutions to be found. In particular, there is an absence of gravitational radiation in these space-times [27]. An investigation of its role is a challenge for future analytic and computational investigations.

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