\( \gamma \text{ and } \beta \text{ approximations via general ordered topological spaces} \)

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Abstract

In this paper, we introduce the concepts of \( \gamma \) and \( \beta \) approximations via general ordered topological approximation spaces. Also, increasing (decreasing) \( \gamma \), \( \beta \) boundary, positive and negative regions are given in general ordered topological approximation spaces (GOTAS, for short). Some important properties of them were investigated. From this study, we can say that studying any properties of rough set concepts via GOTAS is a generalization of Pawlak approximation spaces and general approximation spaces.

1. Introduction

Rough set theory was first proposed by Pawlak for dealing with vagueness and granularity in information systems. Various generalizations of Pawlak’s rough set have been made by replacing equivalence relations with kinds of binary relations and many results about generalized rough set with universe being finite were obtained [4, 13,14,15,16,17,18]. An interesting and natural research topic in rough set theory is studying it via topology [5,12]. Neighborhood systems was first applied in generalizing rough sets in 1998 by T. Y. Lin as a generalization of topological connections with rough sets. Lin also introduced the concept of granular computing as a form of topological generalizations [36,37,38,39]. In this paper we give the concept of \( \gamma \), \( \beta \) via topological ordered spaces and studied their properties which may be viewed as a generalization of previous studies in general approximation spaces, as if we take the partially ordered relation as an equal relation we obtain the concepts in general approximation spaces[2].

2. Preliminaries

In this section, we give an account for the basic definitions and preliminaries to be used in the paper.

**Definition 2.1[10].** A subset \( A \) of \( U \), where \((U,\rho)\) is a partially ordered set is said to be increasing (resp. decreasing) if for all \( a \in A \) and \( x \in U \) such that \( a \rho x \) (resp. \( x \rho a \)) imply \( x \in A \).

**Definition 2.2[10].** A triple \((U,\tau,\rho)\) is said to be a topological ordered space, where \((U,\tau)\) is a topological space and \( \rho \) is a partially order relation on \( U \).

**Definition 2.3[11].** Information system is a pair \((U,A)\), where \( U \) is a non-empty finite set of objects and \( A \) is a non-empty finite set of attributes.
Definition 2.4[4]. A non-empty set $U$ equipped with a general relation $R$ which generate a topology $\tau_R$ on $U$ and a partially order relation $\rho$ written as $(U, \tau_R, \rho)$ is said to be general ordered topological approximation space (for short, GOTAS).

Definition 2.5[3]. Let $(U, \tau_R, \rho)$ be a GOTAS and $A \subseteq U$. We define:

1. $\overline{R}_{inc} (A) = A^{inc}$, $A^{inc}$ is the greatest increasing open subset of $A$.
2. $\overline{R}_{dec} (A) = A^{dec}$, $A^{dec}$ is the greatest decreasing open subset of $A$.
3. $\overline{\overline{R}}_{inc} (A) = \overline{\overline{A}}$, $\overline{\overline{A}}$ is the smallest increasing closed superset of $A$.
4. $\overline{\overline{R}}_{dec} (A) = \overline{\overline{A}}$, $\overline{\overline{A}}$ is the smallest decreasing closed superset of $A$.

5. $\alpha^{inc} = \frac{card(\overline{R}_{inc} (A))}{card(\overline{\overline{R}}_{inc} (A))}$ (resp. $\alpha^{dec} = \frac{card(\overline{R}_{dec} (A))}{card(\overline{\overline{R}}_{dec} (A))}$) and $\alpha^{inc}$ (resp. $\alpha^{dec}$) is $R-$ increasing (resp. decreasing) accuracy.

Definition 2.6[4]. Let $(U, \tau_R, \rho)$ be a GOTAS and $A \subseteq U$. We define:

1. $S_{inc} (A) = A \cap \overline{R}_{inc} (R_{inc} (A))$, $S_{inc} (A)$ is called $R-$increasing semi lower.
2. $\overline{S}_{inc} (A) = A \cup \overline{R}_{inc} (R_{inc} (A))$, $\overline{S}_{inc} (A)$ is called $R-$increasing semi upper.
3. $S_{dec} (A) = A \cap \overline{R}_{dec} (R_{dec} (A))$, $S_{dec} (A)$ is called $R-$decreasing semi lower.
4. $\overline{S}_{dec} (A) = A \cup \overline{R}_{dec} (R_{dec} (A))$, $\overline{S}_{dec} (A)$ is called $R-$decreasing semi upper.

$A$ is $R-$ increasing (resp. decreasing) semi exact if $S_{inc} (A) = \overline{S}_{inc} (A)$ (resp. $S_{dec} (A) = \overline{S}_{dec} (A)$), otherwise $A$ is $R-$ increasing (resp. decreasing) semi rough.

Proposition 2.7[1]. Let $(U, \tau_R, \rho)$ be a GOTAS and $A \subseteq U$. Then:

1. $R_{inc} (A) \subseteq \alpha_{inc} (A) \subseteq S_{inc} (A)$ ($R_{dec} (A) \subseteq \alpha_{dec} (A) \subseteq S_{dec} (A)$).
2. $\overline{S}_{inc} (A) \subseteq \overline{\alpha}_{inc} (A) \subseteq \overline{R}_{inc} (A)$ ($\overline{S}_{dec} (A) \subseteq \overline{\alpha}_{dec} (A) \subseteq \overline{R}_{dec} (A)$).

3. New approximations and their properties

In this section, we introduce some definitions and propositions about near approximations, near boundary regions via GOTAS which are essential for a present study.
Definition 3.1. Let \((U, \tau_R, \rho)\) be a GOTAS and \(A \subseteq U\). We define:

1. \(\gamma_{\text{Inc}}(A) = A \cap [\overline{R}_\text{Inc} (R_\text{inc} (A)) \cup R_\text{inc} (\overline{R}_\text{Inc} (A))], \) \(\gamma_{\text{Inc}}(A)\) is called \(R\)–increasing \(\gamma\) lower.

2. \(\gamma_{\overline{\text{Inc}}}(A) = A \cup [\overline{R}_\text{Inc} (R_\text{inc} (A)) \cup R_\text{inc} (\overline{R}_\text{Inc} (A))], \) \(\gamma_{\text{Inc}}(A)\) is called \(R\)–increasing \(\gamma\) upper.

3. \(\gamma_{\text{Dec}}(A) = A \cap [\overline{R}_\text{Dec} (R_\text{dec} (A)) \cup R_\text{dec} (\overline{R}_\text{Dec} (A))], \) \(\gamma_{\text{Dec}}(A)\) is called \(R\)–decreasing \(\gamma\) lower.

4. \(\gamma_{\overline{\text{Dec}}}(A) = A \cup [\overline{R}_\text{Dec} (R_\text{dec} (A)) \cup R_\text{dec} (\overline{R}_\text{Dec} (A))], \) \(\gamma_{\text{Dec}}(A)\) is called \(R\)–decreasing \(\gamma\) upper.

\(\gamma_{\text{Dec}}(A) = \gamma_{\overline{\text{Dec}}}(A)\) otherwise \(A\) is \(R\)–increasing (resp. \(R\)–decreasing) \(\gamma\) exact if \(\gamma_{\text{Inc}}(A) = \gamma_{\overline{\text{Inc}}}(A)\) (resp. \(\gamma_{\text{Dec}}(A) = \gamma_{\overline{\text{Dec}}}(A)\)).

Proposition 3.2. Let \((U, \tau_R, \rho)\) be a GOTAS and \(A, B \subseteq U\). Then

1. \(A \subseteq B \rightarrow \gamma_{\text{Inc}}(A) \subseteq \gamma_{\text{Inc}}(B)\) (\(A \subseteq B \rightarrow \gamma_{\overline{\text{Inc}}}(A) \subseteq \gamma_{\overline{\text{Inc}}}(B)\)).

2. \(\gamma_{\text{Inc}}(A \cap B) \subseteq \gamma_{\text{Inc}}(A) \cap \gamma_{\text{Inc}}(B)\) (\(\gamma_{\overline{\text{Inc}}}(A \cap B) \subseteq \gamma_{\overline{\text{Inc}}}(A) \cap \gamma_{\overline{\text{Inc}}}(B)\)).

3. \(\gamma_{\text{Dec}}(A \cup B) \supseteq \gamma_{\text{Dec}}(A) \cup \gamma_{\text{Dec}}(B)\) (\(\gamma_{\overline{\text{Dec}}}(A \cup B) \supseteq \gamma_{\overline{\text{Dec}}}(A) \cup \gamma_{\overline{\text{Dec}}}(B)\)).

Proof.

1. Omitted.

2. \(\gamma_{\text{Inc}}(A \cap B) = (A \cap B) \cap [\overline{R}_\text{Inc} (R_\text{inc} (A \cap B) \cup R_\text{inc} (\overline{R}_\text{Inc} (A \cap B)))] \)

\(\subseteq (A \cap B) \cap [\overline{R}_\text{Inc} (R_\text{inc} (A \cap B) \cap R_\text{inc} (\overline{R}_\text{Inc} (B)) \cup R_\text{inc} (\overline{R}_\text{Inc} (A) \cap \overline{R}_\text{Inc} (B)))] \)

\(\subseteq (A \cap B) \cup [\overline{R}_\text{Inc} (R_\text{inc} (A) \cap R_\text{inc} (\overline{R}_\text{Inc} (B)) \cup R_\text{inc} (\overline{R}_\text{Inc} (A) \cap \overline{R}_\text{Inc} (B)))] \)

\(\subseteq A \cup [\overline{R}_\text{Inc} (R_\text{inc} (A) \cup R_\text{inc} (\overline{R}_\text{Inc} (B))) \cap B \cup [\overline{R}_\text{Inc} (R_\text{inc} (B) \cup R_\text{inc} (\overline{R}_\text{Inc} (B)))] \)

\(\subseteq \gamma_{\text{Inc}}(A) \cap \gamma_{\text{Inc}}(B).\)

3. \(\gamma_{\text{Dec}}(A \cup B) = (A \cup B) \cup [\overline{R}_\text{Dec} (R_\text{dec} (A \cup B) \cup R_\text{dec} (\overline{R}_\text{Dec} (A \cup B)))] \)

\(\supseteq (A \cup B) \cup [\overline{R}_\text{Dec} (R_\text{dec} (A) \cup R_\text{dec} (\overline{R}_\text{Dec} (B))) \cup R_\text{dec} (\overline{R}_\text{Dec} (A) \cup \overline{R}_\text{Dec} (B)))] \)

\(\supseteq (A \cup B) \cup [\overline{R}_\text{Dec} (R_\text{dec} (A)) \cup R_\text{dec} (\overline{R}_\text{Dec} (B))) \cup R_\text{dec} (\overline{R}_\text{Dec} (A) \cup \overline{R}_\text{Dec} (B)) \)

\(\supseteq A \cup [\overline{R}_\text{Dec} (R_\text{dec} (A)) \cup R_\text{dec} (\overline{R}_\text{Dec} (B))) \cup B \cup [\overline{R}_\text{Dec} (R_\text{dec} (B) \cup R_\text{dec} (\overline{R}_\text{Dec} (B)))] \)
\[ \supseteq \gamma^\text{inc} (A) \cup \gamma^\text{inc} (B). \]

One can prove the case between parentheses.

**Proposition 3.3.** Let \((U, \tau_R, \rho)\) be a GOTAS and \(A, B \subseteq U\). Then

1. \(A \subseteq B \rightarrow \gamma^\text{inc} (A) \subseteq \gamma^\text{inc} (B)\) (\(A \subseteq B \rightarrow \gamma^\text{Dec} (A) \subseteq \gamma^\text{Dec} (B)\)).
2. \(\gamma^\text{inc} (A \cap B) \subseteq \gamma^\text{inc} (A) \cap \gamma^\text{inc} (B)\) (\(\gamma^\text{Dec} (A \cap B) \subseteq \gamma^\text{Dec} (A) \cap \gamma^\text{Dec} (B)\)).
3. \(\gamma^\text{inc} (A \cup B) \supseteq \gamma^\text{inc} (A) \cup \gamma^\text{inc} (B)\) (\(\gamma^\text{Dec} (A \cup B) \supseteq \gamma^\text{Dec} (A) \cup \gamma^\text{Dec} (B)\)).

**Proof.**

(1) Easy.

(2) \(\gamma^\text{inc} (A \cap B) = (A \cap B) \cap [R^\text{inc} (\overline{R}^\text{inc} (A \cap B) \cup R^\text{inc} (\overline{R}^\text{inc} (A \cap B)))] \)
\(\subseteq (A \cap B) \cap [R^\text{inc} (R^\text{inc} (A) \cap R^\text{inc} (B)) \cup R^\text{inc} (\overline{R}^\text{inc} (A) \cap \overline{R}^\text{inc} (B))]) \)
\(\subseteq (A \cap B) \cap [R^\text{inc} (R^\text{inc} (A) \cup R^\text{inc} (\overline{R}^\text{inc} (B))) \cup R^\text{inc} (\overline{R}^\text{inc} (A) \cap \overline{R}^\text{inc} (B))]) \)
\(\subseteq A \cap (R^\text{inc} (R^\text{inc} (A) \cup R^\text{inc} (\overline{R}^\text{inc} (B))) \cap B \cap [R^\text{inc} (R^\text{inc} (B) \cup R^\text{inc} (\overline{R}^\text{inc} (B))]) \)
\(\subseteq \gamma^\text{inc} (A) \cap \gamma^\text{inc} (B). \)

(3) \(\gamma^\text{inc} (A \cup B) = (A \cup B) \cap [R^\text{inc} (R^\text{inc} (A \cup B) \cup R^\text{inc} (\overline{R}^\text{inc} (A \cup B)))] \)
\(\supseteq (A \cup B) \cap [R^\text{inc} (R^\text{inc} (A) \cup R^\text{inc} (B)) \cup R^\text{inc} (\overline{R}^\text{inc} (A) \cup \overline{R}^\text{inc} (B))]) \)
\(\supseteq (A \cup B) \cap [R^\text{inc} (R^\text{inc} (A)) \cup R^\text{inc} (\overline{R}^\text{inc} (B))) \cup R^\text{inc} (\overline{R}^\text{inc} (A) \cup \overline{R}^\text{inc} (B))]) \)
\(\supseteq A \cap [R^\text{inc} (R^\text{inc} (A) \cup R^\text{inc} (\overline{R}^\text{inc} (B))) \cup B \cap [R^\text{inc} (R^\text{inc} (B) \cup R^\text{inc} (\overline{R}^\text{inc} (B))]) \)
\(\supseteq \gamma^\text{inc} (A) \cup \gamma^\text{inc} (B). \)

One can prove the case between parentheses.

**Proposition 3.4.** Let \((U, \tau_R, \rho)\) be a GOTAS and \(A, B \subseteq U\). If \(A\) is \(R\)– increasing (resp. decreasing) exact then \(A\) is \(R\)– increasing (resp. decreasing) \(\gamma\) exact.

**Proof.**

Let \(A\) be \(R\)– increasing exact. Then \(\overline{R}^\text{inc} (A) = R^\text{inc} (A)\), thus \(\gamma^\text{inc} (A) = R^\text{inc} (A)\) and
\[ \gamma^\text{inc}_\text{Inc}(A) = R^\text{inc}_\text{Inc}(A). \] Therefore \( \gamma^\text{inc}_\text{Dec}(A) = \gamma^\text{inc}_\text{Inc}(A). \)

One can prove the case between parentheses.

\( R \) – increasing (resp. decreasing) exact \( \rightarrow \) \( R \) – increasing (resp. decreasing) \( \gamma \) exact

**Proposition 3.5.** Let \((U, \tau_R, \rho)\) be a GOTAS and \(A \subseteq U\). Then

\[
R^\text{inc}_\text{Inc}(A) \subseteq \gamma^\text{inc}_\text{Inc}(A)(R^\text{dec}_\text{Dec}(A) \subseteq \gamma^\text{inc}_\text{Dec}(A)).
\]

**Proof.** Since \( R^\text{inc}_\text{Inc}(A) \subseteq A \) and \( R^\text{inc}_\text{Inc}(A) \subseteq \overline{R}^\text{inc}_\text{Inc}(A) \), then

\[
R^\text{inc}_\text{Inc}(A) \subseteq \overline{R}^\text{inc}_\text{Inc}(A) \cup R^\text{inc}_\text{Inc}(\overline{R}^\text{inc}_\text{Inc}(A)).
\]

Therefore

\[
R^\text{inc}_\text{Inc}(A) \subseteq A \cap \overline{R}^\text{inc}_\text{Inc}(A) \cup R^\text{inc}_\text{Inc}(\overline{R}^\text{inc}_\text{Inc}(A)).
\]

Thus \( R^\text{inc}_\text{Inc}(A) \subseteq \gamma^\text{inc}_\text{Inc}(A) \). One can prove the case between parentheses.

**Proposition 3.6.** Let \((U, \tau_R, \rho)\) be a GOTAS and \(A \subseteq U\). Then

\[
\overline{\gamma}^\text{inc}_\text{Inc}(A) \subseteq \overline{R}^\text{inc}_\text{Inc}(A)(\overline{\gamma}^\text{inc}_\text{Dec}(A) \subseteq \overline{R}^\text{Dec}_\text{Dec}(A)).
\]

**Proof.** Since \( A \subseteq \overline{R}^\text{inc}_\text{Inc}(A) \) and \( R^\text{inc}_\text{Inc}(A) \subseteq A \subseteq \overline{R}^\text{inc}_\text{Inc}(A) \), then

\[
\overline{R}^\text{inc}_\text{Inc}(A) \subseteq A \subseteq \overline{R}^\text{inc}_\text{Inc}(A).
\]

Thus

\[
\overline{R}^\text{inc}_\text{Inc}(A) \cup R^\text{inc}_\text{Inc}(\overline{R}^\text{inc}_\text{Inc}(A)) \subseteq \overline{R}^\text{inc}_\text{Inc}(A).
\]

Therefore \( A \cup \overline{R}^\text{inc}_\text{Inc}(A) \cup R^\text{inc}_\text{Inc}(\overline{R}^\text{inc}_\text{Inc}(A)) \subseteq \overline{R}^\text{inc}_\text{Inc}(A) \). Hence \( \overline{\gamma}^\text{inc}_\text{Inc}(A) \subseteq \overline{R}^\text{inc}_\text{Inc}(A) \).

**Proposition 3.7.** Let \((U, \tau_R, \rho)\) be a GOTAS and \(A \subseteq U\). Then

\[
P^\text{inc}_\text{Inc}(A) \subseteq \gamma^\text{inc}_\text{Inc}(A)(P^\text{dec}_\text{Dec}(A) \subseteq \gamma^\text{inc}_\text{Dec}(A)).
\]

**Proof.** Let \( x \in P^\text{inc}_\text{Inc}(A) = A \cap R^\text{inc}_\text{Inc}(\overline{R}^\text{inc}_\text{Inc}(A)). \) Then \( x \in A \) and \( R^\text{inc}_\text{Inc}(\overline{R}^\text{inc}_\text{Inc}(A)) \). Therefore \( x \in A \) and \( [x \in \overline{R}^\text{inc}_\text{Inc}(\overline{R}^\text{inc}_\text{Inc}(A)) \cup x \in R^\text{inc}_\text{Inc}(\overline{R}^\text{inc}_\text{Inc}(A))]. \) Thus

\[
x \in A \cap [R^\text{inc}_\text{Inc}(\overline{R}^\text{inc}_\text{Inc}(A)) \cup R^\text{inc}_\text{Inc}(\overline{R}^\text{inc}_\text{Inc}(A))]= \gamma^\text{inc}_\text{Inc}(A). \)

Hence \( P^\text{inc}_\text{Inc}(A) \subseteq \gamma^\text{inc}_\text{Inc}(A) \).

One can prove the case between parentheses.

**Proposition 3.8.** Let \((U, \tau_R, \rho)\) be a GOTAS and \(A \subseteq U\). Then

\[
S^\text{inc}_\text{Inc}(A) \subseteq \gamma^\text{inc}_\text{Inc}(A)(S^\text{dec}_\text{Dec}(A) \subseteq \gamma^\text{inc}_\text{Dec}(A)).
\]

**Proof.** Let \( x \in S^\text{inc}_\text{Inc}(A) = A \cap \overline{R}^\text{inc}_\text{Inc}(\overline{R}^\text{inc}_\text{Inc}(A)). \) Then \( x \in A \) and \( \overline{R}^\text{inc}_\text{Inc}(\overline{R}^\text{inc}_\text{Inc}(A)) \). Therefore \( x \in A \) and \( [x \in \overline{R}^\text{inc}_\text{Inc}(\overline{R}^\text{inc}_\text{Inc}(A)) \cup x \in \overline{R}^\text{inc}_\text{Inc}(\overline{R}^\text{inc}_\text{Inc}(A))]. \) Thus
\[ x \in A \setminus \overline{R}^{-\text{inc}}(\overline{R}^{-\text{inc}}(A)) \cup \overline{R}^{-\text{inc}}(\overline{R}^{-\text{inc}}(A)) = \gamma^{-\text{inc}}(A) \]. Hence \( S^{-\text{inc}}(A) \subseteq \gamma^{-\text{inc}}(A) \).

One can prove the case between parentheses.

**Proposition 3.9.** Let \((U, \tau_R, \rho)\) be a GOTAS and \(A \subseteq U\). Then \(\overline{P}^{-\text{inc}}(A) \subseteq \gamma^{-\text{inc}}(A)(\overline{P}^{-\text{dec}}(A) \subseteq \gamma^{-\text{dec}}(A))\).

**Proof.** Let \(x \in \overline{P}^{-\text{inc}}(A) = A \cup \overline{R}^{-\text{inc}}(\overline{R}^{-\text{inc}}(A))\). Then \(x \in A\) and \(\overline{R}^{-\text{inc}}(\overline{R}^{-\text{inc}}(A))\). Therefore \(x \in A \cup [\overline{R}^{-\text{inc}}(\overline{R}^{-\text{inc}}(A)) \cup \overline{R}^{-\text{inc}}(\overline{R}^{-\text{inc}}(A))]\). Thus \(\overline{P}^{-\text{inc}}(A) \subseteq \gamma^{-\text{inc}}(A)\).

**Proposition 3.10.** Let \((U, \tau_R, \rho)\) be a GOTAS and \(A \subseteq U\). Then \(\overline{\beta}^{-\text{inc}}(A) \subseteq \overline{P}^{-\text{inc}}(A)(\overline{\beta}^{-\text{dec}}(A) \subseteq \overline{P}^{-\text{dec}}(A))\).

**Proof.** Omitted.

**Definition 3.11.** Let \((U, \tau_R, \rho)\) be a GOTAS and \(A \subseteq U\). We define:

1. \(\underline{\beta}^{-\text{inc}}(A) = A \cap \overline{R}^{-\text{inc}}(\overline{R}^{-\text{inc}}(\overline{R}^{-\text{inc}}(A)))\), \(\underline{\beta}^{-\text{inc}}(A)\) is called \(R\)–increasing \(\beta\) lower.
2. \(\overline{\beta}^{-\text{inc}}(A) = A \cup \underline{\beta}^{-\text{inc}}(\overline{\overline{R}^{-\text{inc}}(\overline{R}^{-\text{inc}}(A)))\), \(\overline{\beta}^{-\text{inc}}(A)\) is called \(R\)–increasing \(\beta\) upper.
3. \(\underline{\beta}^{-\text{dec}}(A) = A \cap \overline{R}^{-\text{dec}}(\overline{R}^{-\text{dec}}(\overline{R}^{-\text{dec}}(A)))\), \(\underline{\beta}^{-\text{dec}}(A)\) is called \(R\)–decreasing \(\beta\) lower.
4. \(\overline{\beta}^{-\text{dec}}(A) = A \cup \underline{\beta}^{-\text{dec}}(\overline{\overline{R}^{-\text{dec}}(\overline{R}^{-\text{dec}}(A)))\), \(\overline{\beta}^{-\text{dec}}(A)\) is called \(R\)–decreasing \(\beta\) upper.

\(A\) is \(R\)–increasing (decreasing) \(\beta\) exact if \(\beta^{-\text{inc}}(A) = \beta^{-\text{inc}}(A)\) (resp. \(\beta^{-\text{dec}}(A) = \beta^{-\text{dec}}(A)\)), otherwise \(A\) is \(R\)–increasing (decreasing) \(\beta\) rough.

**Proposition 3.12.** Let \((U, \tau_R, \rho)\) be a GOTAS and \(A, B \subseteq U\). Then

1. \(A \subseteq B \rightarrow \overline{\beta}^{-\text{inc}}(A) \subseteq \overline{\beta}^{-\text{inc}}(B)\) (\(A \subseteq B \rightarrow \overline{\beta}^{-\text{dec}}(A) \subseteq \overline{\beta}^{-\text{dec}}(B)\)).
2. \(\overline{\beta}^{-\text{inc}}(A \cap B) \subseteq \overline{\beta}^{-\text{inc}}(A) \cap \overline{\beta}^{-\text{inc}}(B)\) (\(\overline{\beta}^{-\text{dec}}(A \cap B) \subseteq \overline{\beta}^{-\text{dec}}(A) \cap \overline{\beta}^{-\text{dec}}(B)\)).

**Proof.**

1. Omitted.
2. \(\overline{\beta}^{-\text{inc}}(A \cap B) = (A \cap B) \cup \overline{R}^{-\text{inc}}(\overline{R}^{-\text{inc}}(A \cap B)))\)
\[ (A \cap B) \cup R_{\text{inc}}(R_{\text{inc}}(A) \cap R_{\text{inc}}(B)) \]
\[ \subseteq (A \cap B) \cup R_{\text{inc}}(R_{\text{inc}}(A) \cap R_{\text{inc}}(B)) \]
\[ \subseteq (A \cap B) \cup R_{\text{inc}}(R_{\text{inc}}(A)) \cap R_{\text{inc}}(R_{\text{inc}}(B)) \]
\[ \subseteq A \cup R_{\text{inc}}(R_{\text{inc}}(A)) \cap B \cup R_{\text{inc}}(R_{\text{inc}}(B)) \]
\[ \subseteq \beta_{\text{inc}}(A) \cap \beta_{\text{inc}}(B). \]

(3) \[ \beta_{\text{inc}}(A \cup B) = (A \cup B) \cup R_{\text{inc}}(R_{\text{inc}}(A \cup B)) \]
\[ = (A \cup B) \cup R_{\text{inc}}(R_{\text{inc}}(A) \cup R_{\text{inc}}(B)) \]
\[ \supseteq (A \cup B) \cup R_{\text{inc}}(R_{\text{inc}}(A) \cup R_{\text{inc}}(B)) \]
\[ \supseteq (A \cup B) \cup R_{\text{inc}}(R_{\text{inc}}(A) \cup R_{\text{inc}}(R_{\text{inc}}(B))) \]
\[ \supseteq (A \cup R_{\text{inc}}(R_{\text{inc}}(A)) \cup B \cup R_{\text{inc}}(R_{\text{inc}}(B))) \]
\[ \supseteq \beta_{\text{inc}}(A) \cup \beta_{\text{inc}}(B). \]

One can prove the case between parentheses.

**Proposition 3.13.** Let \((U, \tau_{R}, \rho)\) be a GOTAS and \(A, B \subseteq U\). Then

1. \(A \subseteq B \implies \beta_{\text{inc}}(A) \subseteq \beta_{\text{inc}}(B) ( A \subseteq B \implies \beta_{\text{dec}}(A) \subseteq \beta_{\text{dec}}(B) ). \]
2. \(\beta_{\text{inc}}(A \cap B) \subseteq \beta_{\text{inc}}(A) \cap \beta_{\text{inc}}(B) ( \beta_{\text{dec}}(A \cap B) \subseteq \beta_{\text{dec}}(A) \cap \beta_{\text{dec}}(B) ). \]
3. \(\beta_{\text{inc}}(A \cup B) \supseteq \beta_{\text{inc}}(A) \cup \beta_{\text{inc}}(B) ( \gamma_{\text{dec}}(A \cup B) \supseteq \beta_{\text{dec}}(A) \cup \beta_{\text{dec}}(B) ). \)

**Proof.**

1. Easy.

2. \(\beta_{\text{inc}}(A \cap B) = (A \cap B) \cap R_{\text{inc}}(R_{\text{inc}}(A \cap B)) \)
\[ \subseteq (A \cap B) \cap R_{\text{inc}}(R_{\text{inc}}(A) \cap R_{\text{inc}}(B)) \]
\[ \subseteq (A \cap B) \cap R_{\text{inc}}(R_{\text{inc}}(A) \cap R_{\text{inc}}(R_{\text{inc}}(B))) \]
\[ \subseteq A \cap R_{\text{inc}}(R_{\text{inc}}(A) \cap R_{\text{inc}}(R_{\text{inc}}(B))) \]
\[ \subseteq \beta_{\text{inc}}(A) \cap \beta_{\text{inc}}(B). \]
\[(3) \quad \beta_{\text{inc}}(A \cup B) = (A \cup B) \cap \overline{\mathcal{R}^{\text{inc}}}(\overline{\mathcal{R}^{\text{inc}}}(A \cup B)) \]
\[= (A \cup B) \cap \overline{\mathcal{R}^{\text{inc}}}(\overline{\mathcal{R}^{\text{inc}}}(A \cup B)) \]
\[\supseteq (A \cup B) \cap \overline{\mathcal{R}^{\text{inc}}}(\overline{\mathcal{R}^{\text{inc}}}(A) \cup \overline{\mathcal{R}^{\text{inc}}}(B)) \]
\[\supseteq (A \cup B) \cap \overline{\mathcal{R}^{\text{inc}}}(\overline{\mathcal{R}^{\text{inc}}}(A) \cup \overline{\mathcal{R}^{\text{inc}}}(\overline{\mathcal{R}^{\text{inc}}}(B))) \]
\[\supseteq A \cap \overline{\mathcal{R}^{\text{inc}}}(\overline{\mathcal{R}^{\text{inc}}}(A) \cup \overline{\mathcal{R}^{\text{inc}}}(\overline{\mathcal{R}^{\text{inc}}}(B))) \]
\[\supseteq \beta_{\text{inc}}(A) \cup \beta_{\text{inc}}(B). \]

One can prove the case between parentheses.

**Proposition 3.14.** Let \((U, \tau_R, \rho)\) be a GOTAS and \(A, B \subseteq U\). If \(A\) is \(R\)–increasing (resp. decreasing) exact then \(A\) is \(\beta\)–increasing (resp. decreasing) exact.

**Proof.**

Let \(A\) be \(R\)–increasing exact. Then \(\overline{\mathcal{R}^{\text{inc}}}(A) = \mathcal{R}^{\text{inc}}(A)\). Therefore
\[\overline{\beta^{\text{inc}}}(A) = \overline{\mathcal{R}^{\text{inc}}}(A), \quad \beta_{\text{inc}}(A) = \mathcal{R}^{\text{inc}}(A).\] Thus \(\overline{\beta^{\text{inc}}}(A) = \beta_{\text{inc}}(A)\). Hence \(A\) is \(R\)–increasing \(\beta\) exact.

One can prove the case between parentheses.

\(R\)–increasing (resp. decreasing) exact \(\rightarrow\) \(R\)–increasing (resp. decreasing) \(\beta\) exact

**Proposition 3.15.** Let \((U, \tau_R, \rho)\) be a GOTAS and \(A \subseteq U\). Then
\[\mathcal{R}^{\text{inc}}(A) \subseteq \beta_{\text{inc}}(A)(\mathcal{R}^{\text{dec}}(A) \subseteq \beta_{\text{dec}}(A))\]

**Proof.** Since \(\mathcal{R}^{\text{inc}}(A) \subseteq \mathcal{A} \subseteq \overline{\mathcal{R}^{\text{inc}}}(A)\) and \(\mathcal{R}^{\text{inc}}(A) \subseteq \mathcal{R}^{\text{inc}}(\overline{\mathcal{R}^{\text{inc}}}(A))\). Then
\[\mathcal{R}^{\text{inc}}(A) \subseteq \overline{\mathcal{R}^{\text{inc}}}(\overline{\mathcal{R}^{\text{inc}}}(A)) \subseteq \overline{\mathcal{R}^{\text{inc}}}(\overline{\mathcal{R}^{\text{inc}}}(A)).\] Therefore
\[\mathcal{R}^{\text{inc}}(A) \subseteq A \cap \overline{\mathcal{R}^{\text{inc}}}(\overline{\mathcal{R}^{\text{inc}}}(A))\]. Thus \(\mathcal{R}^{\text{inc}}(A) \subseteq \beta_{\text{inc}}(A)\).

One can prove the case between parentheses.

**Proposition 3.16.** Let \((U, \tau_R, \rho)\) be a GOTAS and \(A \subseteq U\). Then
\[\overline{\beta^{\text{inc}}}(A) \subseteq \overline{\mathcal{R}^{\text{inc}}}(A)(\overline{\beta^{\text{dec}}}(A) \subseteq \overline{\mathcal{R}^{\text{dec}}}(A)).\]
Proof. Since $A \subseteq \overline{R}^{\text{inc}}(A)$ and $R^{\text{inc}}(A) \subseteq \overline{R}^{\text{inc}}(A)$. Then $\overline{R}^{\text{inc}}(R^{\text{inc}}(A)) \subseteq \overline{R}^{\text{inc}}(A)$.

Thus $R^{\text{inc}}(R^{\text{inc}}(A)) \subseteq R^{\text{inc}}(\overline{R}^{\text{inc}}(A)) \subseteq \overline{R}^{\text{inc}}(A)$. Therefore $A \cup R^{\text{inc}}(\overline{R}^{\text{inc}}(A)) \subseteq \overline{R}^{\text{inc}}(A)$. Hence $\overline{R}^{\text{inc}}(A) \subseteq \overline{R}^{\text{inc}}(A)$.

Definition 3.17. Let $(U, \tau_R, \rho)$ be a GOTAS and $A \subseteq U$. Then

(1) $B_{j\text{inc}}(A) = j_{\text{inc}}(A) - j_{\text{dec}}(A)$ (resp. $B_{j\text{dec}}(A) = j_{\text{dec}}(A) - j_{\text{inc}}(A)$),

is increasing (resp. decreasing) $j$ boundary region.

(2) $Pos_{j\text{inc}}(A) = j_{\text{inc}}(A)$ (resp. $Pos_{j\text{dec}}(A) = j_{\text{dec}}(A)$),

is increasing (resp. decreasing) $j$ positive region.

(3) $Neg_{j\text{inc}}(A) = U - j_{\text{dec}}(A)$ (resp. $Neg_{j\text{dec}}(A) = U - j_{\text{inc}}(A)$),

is increasing (resp. decreasing) $j$ negative region. Where $j_{\text{inc}}$ the near lower approximations s.t. $j \in \{\beta, \gamma\}$.

Proposition 3.18. Let $(U, \tau_R, \rho)$ be a GOTAS and $A, B \subseteq U$. Then

(1) $Neg_{j\text{inc}}(A \cup B) \subseteq Neg_{j\text{inc}}(A) \cup Neg_{j\text{inc}}(B)$

(2) $Neg_{j\text{inc}}(A \cap B) \supseteq Neg_{j\text{inc}}(A) \cap Neg_{j\text{inc}}(B)$.

Proof.

(1) $Neg_{j\text{inc}}(A \cup B) = U - [(A \cup B) \cup [R^{\text{dec}}(A \cup B) \cup R^{\text{dec}}(A \cup B)]]$

$\subseteq U - [(A \cup B) \cup [R^{\text{dec}}(A \cup B) \cup R^{\text{dec}}(A \cup B)]]$

$\subseteq U - [(A \cup B) \cup [R^{\text{dec}}(A \cup B) \cup R^{\text{dec}}(A \cup B)]]$

$\subseteq U - [(A \cup B) \cup [R^{\text{dec}}(A \cup B) \cup R^{\text{dec}}(A \cup B)]]$

$\subseteq U - [(A \cup B) \cup [R^{\text{dec}}(A \cup B) \cup R^{\text{dec}}(A \cup B)]]$

$\subseteq Neg_{j\text{inc}}(A) \cap Neg_{j\text{inc}}(B).$

(2) $Neg_{j\text{inc}}(A \cap B) = U - [(A \cap B) \cup [R^{\text{dec}}(A \cap B) \cup R^{\text{dec}}(A \cap B)]]$
\[ \Omega U - [(A \cap B) \cup [\overline{R}^{\text{Dec}} (R^{\text{Dec}} (A) \cap R^{\text{Dec}} (B))] \cup R^{\text{Dec}} (A) \cap R^{\text{Dec}} (B))] \]
\[ \subseteq U - [(A \cap B) \cup [\overline{R}^{\text{Dec}} (R^{\text{Dec}} (A) \cap R^{\text{Dec}} (B))] \cup R^{\text{Dec}} (A) \cap R^{\text{Dec}} (B))] \]
\[ \subseteq U - [(A \cap B) \cup [\overline{R}^{\text{Dec}} (R^{\text{Dec}} (A) \cap R^{\text{Dec}} (B)))] \]
\[ \subseteq U - [(A \cap B) \cup [\overline{R}^{\text{Dec}} (R^{\text{Dec}} (A) \cap R^{\text{Dec}} (B)))] \]
\[ \subseteq U - [A \cup R^{\text{Dec}} (R^{\text{Dec}} (A) \cap R^{\text{Dec}} (B))] \]
\[ \subseteq U - [A \cup R^{\text{Dec}} (R^{\text{Dec}} (A) \cap R^{\text{Dec}} (B))] \]
\[ \subseteq U - [A \cup R^{\text{Dec}} (R^{\text{Dec}} (A) \cap R^{\text{Dec}} (B))] \]
\[ \subseteq U - [A \cup R^{\text{Dec}} (R^{\text{Dec}} (A) \cap R^{\text{Dec}} (B))] \]
\[ \subseteq U - [A \cup R^{\text{Dec}} (R^{\text{Dec}} (A) \cap R^{\text{Dec}} (B))] \]
\[ \subseteq U - [A \cup R^{\text{Dec}} (R^{\text{Dec}} (A) \cap R^{\text{Dec}} (B))] \]

One can prove the case between parentheses.

**Proposition 3.19.** Let \((U, \tau_\rho, \rho)\) be a GOTAS and \(A, B \subseteq U\). Then

1. \(\neg_{\beta_{\text{Inc}}} (A \cup B) \subseteq \neg_{\beta_{\text{Inc}}} (A) \cup \neg_{\beta_{\text{Inc}}} (B)\)

2. \(\neg_{\beta_{\text{Dec}}} (A \cap B) \supseteq \neg_{\beta_{\text{Dec}}} (A) \cap \neg_{\beta_{\text{Dec}}} (B)\)

**Proof.**

1. \(\neg_{\beta_{\text{Inc}}} (A \cup B) = U - [(A \cup B) \cup R^{\text{Dec}} (R^{\text{Dec}} (A) \cup R^{\text{Dec}} (B))]\)

2. \(\neg_{\beta_{\text{Dec}}} (A \cap B) = U - [(A \cap B) \cup R^{\text{Dec}} (R^{\text{Dec}} (A) \cap R^{\text{Dec}} (B))]\)
One can prove the case between parentheses.

**Proposition 3.20.** Let \((U, \tau_R, \rho)\) be a GOTAS and \(A \subseteq U\). Then

\[ S_{\text{Inc}}(A) \subseteq \gamma_{\text{Inc}}(A) \subseteq \beta_{\text{Inc}}(A) \subseteq S_{\text{Dec}}(A) \subseteq \beta_{\text{Dec}}(A) \].

**Proof.**

Let \(x \in S_{\text{Inc}}(A)\). Then \(x \in \overline{R}^{\text{Inc}}(R_{\text{Inc}}(A))\). Therefore \(x \in R^{\text{Inc}}(R_{\text{Inc}}(A)) \cup R_{\text{Inc}}(\overline{R}^{\text{Inc}}(A))\).

Thus \(x \in A \cap [R^{\text{Inc}}(R_{\text{Inc}}(A)) \cup R_{\text{Inc}}(\overline{R}^{\text{Inc}}(A))]\) and thus \(x \in \gamma_{\text{Inc}}(A)\).

Hence \(S_{\text{Inc}}(A) \subseteq \gamma_{\text{Inc}}(A)\) (1).

Since \(x \in R_{\text{Inc}}(A)\), then \(x \in \overline{R}^{\text{Inc}}(A)\). Therefore \(x \in R_{\text{Inc}}(\overline{R}^{\text{Inc}}(A))\).

Thus \(x \in \overline{R}^{\text{Inc}}(R_{\text{Inc}}(A))\), and thus \(x \in A \cap \overline{R}^{\text{Inc}}(R_{\text{Inc}}(A))\). Hence

\(x \in \beta_{\text{Inc}}(A)\) (2).

From (1) and (2) we have,

\[ S_{\text{Inc}}(A) \subseteq \gamma_{\text{Inc}}(A) \subseteq \beta_{\text{Inc}}(A) \].

One can prove the case between parentheses.

**Proposition 3.21.** Let \((U, \tau_R, \rho)\) be a GOTAS and \(A \subseteq U\). Then

\[ \beta^{\text{Inc}}(A) \subseteq \gamma^{\text{Inc}}(A) \subseteq S^{\text{Inc}}(A)(\beta^{\text{Dec}}(A) \subseteq \gamma^{\text{Dec}}(A) \subseteq S^{\text{Dec}}(A)) \].

**Proof.**

Let \(x \in \beta^{\text{Inc}}(A)\). Then \(x \in A \cup R_{\text{Inc}}(\overline{R}^{\text{Inc}}(R_{\text{Inc}}(A)))\). Therefore

\(x \in A \) or \(x \in R_{\text{Inc}}(\overline{R}^{\text{Inc}}(R_{\text{Inc}}(A)))\). Thus \(x \in A \) or \(x \in \overline{R}^{\text{Inc}}(R_{\text{Inc}}(A))\). So

\(x \in A \cup R_{\text{Inc}}(\overline{R}^{\text{Inc}}(A))\), and so \(x \in A \cup [R^{\text{Inc}}(R_{\text{Inc}}(A)) \cup R_{\text{Inc}}(\overline{R}^{\text{Inc}}(A))]\). Thus

\(x \in \gamma^{\text{Inc}}(A)\). Hence \(\beta^{\text{Inc}}(A) \subseteq \gamma^{\text{Inc}}(A)\) (1).

Since \(x \in \gamma^{\text{Inc}}(A)\), \(x \in A \) or \(x \in R_{\text{Inc}}(\overline{R}^{\text{Inc}}(A))\), then \(x \in A \cup R_{\text{Inc}}(\overline{R}^{\text{Inc}}(A))\). Therefore

\(x \in S^{\text{Inc}}(A)\) (2).

From (1) and (2) we have,

\[ \beta^{\text{Inc}}(A) \subseteq \gamma^{\text{Inc}}(A) \subseteq S^{\text{Inc}}(A) \].

One can prove the case between parentheses.

**Definition 3.22.** Let \((U, \tau_R, \rho)\) be a GOTAS and \(A\) is a non-empty finite subset of \(U\).

Then the increasing (decreasing) j accuracy of a finite non-empty subset \(A\) of \(U\) is given by:
\[ \eta_{\text{Inc}}(A) = \frac{j_{\text{Inc}}(A)}{j(A)}, \quad j \in \{\beta, \gamma\}. \]

**Proposition 3.23.** Let \((U, \tau_R, \rho)\) be a GOTAS and \(A\) non-empty finite subset of \(U\).

Then we have \(\eta(A) \leq \eta_{\text{Inc}}(A) (\eta(A) \leq \eta_{\text{Dec}}(A))\), for all \(j \in \{\beta, \gamma\}\), where \(\eta(A) = \frac{|R(A)|}{|R(A)|} \).

**Proof.** Omitted.

**Example 3.24.** Let \(U = \{a, b, c, d\}, U / R = \{\{a\}, \{a, b\}, \{c, d\}\}\),
\(\tau_R = \{U, \phi, \{a, b\}, \{c, d\}, \{a\}, \{a, d\}, \{b\}\}\),
\(\tau_R^C = \{U, \phi, \{c, d\}, \{a, b\}, \{b, c, d\}, \{b\}\}\) and
\(\rho = \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, d), (a, d), (a, c), (c, d)\}\).

For \(A = \{a, c\}\), we have:
\[
R_{\text{Dec}}(A) = \{a\}, \quad \overline{R_{\text{Dec}}}(R_{\text{Dec}}(A)) = \{a, b\}, \quad \overline{R_{\text{Dec}}}(A) = U, \quad R_{\text{Dec}}(\overline{R_{\text{Dec}}}(A)) = U.
\]
\[
S_{\text{Dec}}(A) = \{a\}, \quad \overline{S_{\text{Dec}}}(A) = U, \quad B_{S_{\text{Dec}}}(A) = \{b, c, d\}, \quad \text{Neg}_{S_{\text{Inc}}} = \phi.
\]

\[
\gamma_{\text{Dec}}(A) = A \cup U = A, \quad \overline{\gamma_{\text{Dec}}}(A) = A \cup U = U, \quad B_{\gamma_{\text{Dec}}}(A) = \{b, d\}, \quad \text{Neg}_{\gamma_{\text{Inc}}} = \phi.
\]

\[
\beta_{\text{Dec}}(A) = A \cap U = A, \quad \overline{\beta_{\text{Dec}}}(A) = \{a, b, c\}, \quad B_{\beta_{\text{Dec}}}(A) = \{b\}, \quad \text{Neg}_{\beta_{\text{Inc}}} = \{d\}
\]

**Proposition 3.25.** Let \((U, \tau_R, \rho)\) be a GOTAS and \(A \subseteq U\). Then we have
\[
B_{\beta_{\text{Inc}}}(A) \subseteq B_{\gamma_{\text{Inc}}}(A) \subseteq B_{\text{Inc}}(A) \quad (B_{\beta_{\text{Dec}}}(A) \subseteq B_{\gamma_{\text{Dec}}}(A) \subseteq B_{\text{Dec}}(A))
\]

**Proof.** Omitted.

**Remark 3.26.** \(B_{\beta_{\text{Inc}}}(A) \subseteq B_{\gamma_{\text{Inc}}}(A)(B_{\beta_{\text{Dec}}}(A) \subseteq B_{\gamma_{\text{Dec}}}(A))\).

**Remark 3.27.** \(B_{\beta_{\text{Inc}}}(A) \subseteq B_{\gamma_{\text{Inc}}}(A)(B_{\beta_{\text{Dec}}}(A) \subseteq B_{\gamma_{\text{Dec}}}(A))\).

**Proposition 3.28.** Let \((U, \tau_R, \rho)\) be a GOTAS and \(A\) be a non-empty finite subset of \(U\).

Then \(\eta_{\text{Inc}}(A) \leq \eta_{\beta_{\text{Inc}}}(A) \leq \eta_{\gamma_{\text{Inc}}}(A) (\eta_{\text{Dec}}(A) \leq \eta_{\beta_{\text{Dec}}}(A) \leq \eta_{\gamma_{\text{Dec}}}(A))\).

**Proof.** Omitted.

12
Proposition 3.28. Let \((U, \tau_R, \rho)\) be a GOTAS and \(A \subseteq U\). Then
\[
\gamma_{\text{inc}}(A) \subseteq \beta_{\text{inc}}(A) \gamma_{\text{dec}}(A) \subseteq \beta_{\text{dec}}(A)
\]

Proof. Let \(x \in \gamma_{\text{inc}}(A) = A \cap [R_{\text{inc}}^{-1}(R_{\text{inc}}^{-1}(A)) \cup R_{\text{inc}}^{-1}(\overline{R_{\text{inc}}^{-1}}(A))]\). Then
\[
x \in A \quad \text{and} \quad x \in R_{\text{inc}}^{-1}(R_{\text{inc}}^{-1}(A)) \cup R_{\text{inc}}^{-1}(\overline{R_{\text{inc}}^{-1}}(A)). \therefore
\]
\[
x \in A \quad \text{and} \quad [x \in R_{\text{inc}}^{-1}(R_{\text{inc}}^{-1}(A)) \text{ or } x \in R_{\text{inc}}^{-1}(\overline{R_{\text{inc}}^{-1}}(A))]. \therefore x \in A \quad \text{and}
\]
\[
x \in R_{\text{inc}}^{-1}(\overline{R_{\text{inc}}^{-1}}(A)) \text{ and thus } x \in A \quad \text{and} \quad x \in R_{\text{inc}}^{-1}(R_{\text{inc}}^{-1}(A)). \therefore
\]
\[
x \in A \cap R_{\text{inc}}^{-1}(R_{\text{inc}}^{-1}(A)). \therefore \gamma_{\text{inc}}(A) \subseteq \beta_{\text{inc}}(A).
\]

One can prove the case between parentheses.

4. Conclusion

Our results in this paper, became the results about of \(\gamma\), \(\beta\) approximation in [2] in case of \(\rho\) is the equal relation. Also, The new approximation which we give became as Pawlak’s approximation in case of \(\rho\) is the equal relation and \(R\) is the equivalence relation.

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