Liouvillian Solutions of Linear Difference-Differential Equations

Ruyong Feng∗ Michael F. Singer† Min Wu‡

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Abstract

For a field $k$ with an automorphism $\sigma$ and a derivation $\delta$, we introduce the notion of liouvillian solutions of linear difference-differential systems $\{\sigma(Y) = AY, \delta(Y) = BY\}$ over $k$ and characterize the existence of liouvillian solutions in terms of the Galois group of the systems. We will give an algorithm to decide whether such a system has liouvillian solutions when $k = \mathbb{C}(x,t), \sigma(x) = x + 1, \delta = \frac{d}{dt}$ and the size of the system is a prime.

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∗Key Laboratory of Mathematics Mechanization, Institute of Systems Science, AMSS, CAS, Beijing 100190, China, rfeng2@ncsu.edu. The author is supported by NKBRPC 2004CB318000 and NSFC10671200. The work was done during a stay of the first author at Department of Mathematics, North Carolina State University (NCSU). The hospitality at NCSU is gratefully acknowledged.
†North Carolina State University, Department of Mathematics, Box 8205, Raleigh, North Carolina 27695-8205, USA, singer@math.ncsu.edu. This material is based upon work supported by the National Science Foundation under Grant No. CCF-0634123.
‡Shanghai Key Laboratory of Trustworthy Computing, East China Normal University, Shanghai 200062, China, mwu@sei.ecnu.edu.cn, The author is supported in part by NSFC-44012140 and NSFC-90718041.
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1 Introduction

One of the initial and key applications of the Galois theory of linear differential equations is to characterize the solvability of such equations in terms of liouvillian functions, i.e., functions built up iteratively from rational functions using exponentiation, integration and algebraic functions ([10, Appendix 6], [25, Chapters 1.5 and 4]). From the differential Galois theory, a linear differential equation

\[ L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0y = 0, \]

over \( \mathbb{C}(x) \) can be solved in these terms if and only if its Galois group has a solvable subgroup of finite index. This allows us to conclude that if a linear differential equation \( L(y) = 0 \) has a liouvillian solution then it has a solution of the form \( e^{\int f} \) where \( f \) is an algebraic function. This characterization is the foundation of many algorithms that allow one to decide if an equation has such solutions and find them if they exist. This theory and these algorithms have been developed for systems of matrix form \( Y' = AY \) as well as for more general coefficients.

For the case of difference equations, the situation is in many ways not as well developed. A Galois theory of linear difference equations is developed in [26]. Later on in [13], a notion of solving in liouvillian terms is introduced for linear difference equations of the form

\[ L(y) = y(x + n) + a_{n-1}y(x + n - 1) + \cdots + a_0y(x) = 0 \quad \text{with} \quad a_i \in \mathbb{C}(x) \]

and for difference equations in matrix form \( Y(x + 1) = AY(x) \) where \( A \) is a matrix over \( \mathbb{C}(x) \). In [13], a characterization of solving in liouvillian terms is presented in terms of Galois groups and an algorithm is given to decide...
whether a linear difference equation can be solved in liouvillian terms. Remark that in [13], solutions are considered as equivalence classes of sequences of complex numbers \( y = (y(0), y(1), \ldots) \) where two sequences are equivalent if they agree from some point onward. A rational function \( f \) is identified with the sequence of the rational values \( (f(0), f(1), \ldots) \). The addition and multiplication on sequences are defined elementwise. Liouvillian sequences are built up from rational sequences by successively adjoining solutions of the equations of the form \( y(x + 1) = a(x)y(x) \) or \( y(x + 1) - y(x) = b(x) \) and using addition, multiplication and \textit{interlacing} to define new sequences (the interlacing of \( u = (u_0, u_1, \ldots), v = (v_0, v_1, \ldots) \) is \( (u_0, v_0, u_1, v_1, \ldots) \)). Similar to the differential case, a linear difference equation \( L(y) = 0 \) can be solved in terms of liouvillian sequences if and only if its Galois group has a solvable subgroup of finite index. When this is the case, \( L(y) = 0 \) has a solution that is the interlacing of \textit{hypergeometric} sequences, and [13] shows how to decide if this is the case. The paper [13] also gives examples of equations which have no hypergeometric solutions but do have solutions that are interlacings of hypergeometric solutions. Similar results also apply to difference equations in matrix form.

We now consider systems of linear difference-differential equations of the form

\[
Y(x + 1, t) = AY(x, t), \quad \frac{dY(x, t)}{dt} = BY(x, t)
\]

where \( A \) and \( B \) are square matrices over \( \mathbb{C}(x, t) \) and \( A \) is invertible. In [7, 21, 22, 28], some theories and algorithms have been developed on determining reducibility and existence of \textit{hyperexponential solutions} of such systems. However, as in the pure difference case, there are systems which have no hyperexponential solutions but have solutions that are interlacings of hyperexponential solutions. In this paper, we shall use a Galois theory that appears as a special case of the Galois theory developed in [12, Appendix] to characterize \textit{liouvillian solutions of linear difference-differential systems}, and then devise an algorithm to determine whether such a system has liouvillian solutions when the order of the system is prime.

Throughout the paper, we use \( (\cdot)^T \) to denote the transpose of a vector or matrix and \( \det(\cdot) \) to denote the determinant of a square matrix. The symbols \( \mathbb{Z}_{\geq 0} \) and \( \mathbb{Z}_{> 0} \) represent the set of nonnegative integers and the set of positive integers, respectively. Denote by \( 1 \) the identity map on the sets in discussion. For a field \( k \), denote by \( gl_n(k) \) the set of all \( n \times n \) matrices over \( k \) and by \( GL_n(k) \) the set of all \( n \times n \) invertible matrices over \( k \). All difference-differential systems of the form \( \{\sigma(Y) = AY, \delta(Y) = BY\} \) with \( A \in GL_n(k) \)
and $B \in \mathfrak{gl}_n(k)$ that are in discussion in the paper are assumed to be integrable.

The paper is organized as follows. In Section 2, we will first review some Galois theoretic results in [12] and then show that the Galois group of a linear difference-differential system is solvable by finite group if and only if a certain associated system has solutions in a tower built up using generalizations of liouvillian extensions. In Section 3, we show that irreducible systems with liouvillian solutions must be equivalent to systems of particular form (Theorem 3.11) and refine this result for systems of prime order. We propose an algorithm for deciding if linear difference-differential systems of prime order have liouvillian solutions. At last, some examples are illustrated.

We would like to thank Reinhart Shaefke for supplying a simple proof of Lemma 3.25.

2 Galois Theory

2.1 Picard-Vessiot extensions and Galois groups

In [12], a general Galois theory is presented for linear integrable systems of difference-differential equations involving parameters. When there exists no parameters this theory yields a Galois theory of difference-differential systems as above. Let us recall some notation and results in [12].

A $\sigma\delta$-ring $R$ is a commutative ring with unit endowed with an automorphism $\sigma$ and a derivation $\delta$ satisfying $\sigma\delta = \delta\sigma$. $R$ is called a $\sigma\delta$-field when $R$ is a field. An element $c$ of $R$ is called a constant if $\sigma(c) = c$ and $\delta(c) = 0$, i.e., it is a constant with respect both $\sigma$ and $\delta$. The set of constants of $R$, denoted by $R^{\sigma\delta}$, is a subring, and it is a subfield if $R$ is a field.

In this section, unless specified otherwise, we always let $k$ be a $\sigma\delta$-field of characteristic zero and with an algebraically closed field of constants.

Consider a system of the form

$$\sigma(Y) = AY, \quad \delta(Y) = BY$$

where $A \in \text{GL}_n(k)$, $B \in \mathfrak{gl}_n(k)$ and $Y$ is a vector of unknowns of size $n$. The integer $n$ is called the order of the system (1). A $\sigma\delta$-ring $R$ is called a $\sigma\delta$-Picard-Vessiot extension, or a $\sigma\delta$-PV extension for short, of $k$ for the system (1) if it satisfies the following conditions
(i) $R$ is a simple $\sigma\delta$-ring;

(ii) there exists $Z \in \text{GL}_n(R)$ such that $\sigma(Z) = AZ$ and $\delta(Z) = BZ$;

(iii) $R = k[Z, \frac{1}{\det(Z)}]$, that is, $R$ is generated by entries of $Z$ and the inverse of the determinant of $Z$.

Note that if the system \((\text{I})\) has a $\sigma\delta$-PV extension, the commutativity of $\sigma$ and $\delta$ implies

$$\sigma(B) = \delta(A)A^{-1} + ABA^{-1},$$

which is called the integrability conditions for the system \((\text{I})\). Conversely, if the system \((\text{I})\) satisfies the above integrability conditions and the constants of $k$ are algebraically closed, it is shown in [7] and [12, Appendix] that $\sigma\delta$-PV extensions for \((\text{I})\) exist and are unique up to $\sigma\delta-k$ isomorphisms.

The following notation will be used throughout the paper.

**Notation 2.1** Let $A$ be a square matrix over a $\sigma\delta$-ring. For a positive integer $m$, denote $A_m = \sigma^{m-1}(A) \cdots \sigma(A)A$. For a linear algebraic group $G$, $G^0$ represents the identity component of $G$.

**Lemma 2.2** [Lemma 6.8 in [12]] Let $k$ be a $\sigma\delta$-field and $R$ a simple $\sigma\delta$-ring, finitely generated over $k$ as a $\sigma\delta$-ring. Then there are idempotents $e_0, \ldots, e_{s-1}$ in $R$ such that

(i) $R = e_0R \oplus \cdots \oplus e_{s-1}R$;

(ii) $\sigma$ permutes the set $\{e_0R, \ldots, e_{s-1}R\}$. Moreover, $\sigma^s$ leaves each $e_iR$ invariant;

(iii) each $e_iR$ is a domain and a simple $\sigma^s\delta$-ring.

The following lemma is an analogue to Lemma 1.26 in [25].

**Lemma 2.3** Let $k$ be $\sigma\delta$-field, $R$ be a $\sigma\delta$-PV extension for the system

$$\sigma(Y) = AY, \quad \delta(Y) = BY$$

with $A \in \text{GL}_n(k)$ and $B \in \text{gl}_n(k)$

and $e_0, e_1, \ldots, e_{s-1}$ be as in Lemma 2.2. Then each $e_iR$ is a $\sigma^s\delta$-PV extension of $k$ for the system $\{\sigma^s(Y) = A_iY, \delta(Y) = BY\}$. 

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Proof. Let \( R \) be a \( \sigma \delta \)-PV extension for \( \{\sigma(Y) = AY, \delta(Y) = BY\} \) and \( F \) be a fundamental matrix over \( R \) for the system. By Lemma 2.2, each \( e_i R \) is a simple \( \sigma \delta \)-ring. Clearly, \( e_i F \) are the solutions of \( \{\sigma^s(Y) = A_s Y, \delta(Y) = BY\} \) since \( \sigma^s(e_i) = e_i \) and \( \delta(e_i) = 0 \) for \( i = 0, \ldots, s-1 \). Assume that \( e_i \det(F) = 0 \) for some \( i \). Then

\[
\sigma^j(e_i \det(F)) = e_{i+j \mod s} \det(A_j) \det(F) = 0
\]

and thus \( e_{i+j \mod s} \det(F) = 0 \) for \( j = 1, \ldots, s-1 \). Therefore

\[
\det(F) = (e_0 + \cdots + e_{s-1}) \det(F) = 0,
\]

a contradiction. So \( e_i F \) is a fundamental matrix for \( \{\sigma^s(Y) = A_s Y, \delta(Y) = BY\} \) for each \( i \). Moreover, \( e_i R = k[e_i F, \frac{1}{e_i \det(F)}] \) for each \( i \). This completes the proof. \( \square \)

Corollary 2.4 Let \( d \geq 1 \) be a divisor of \( s \). Suppose that there exist idempotents \( e_0, \ldots, e_{s-1} \) in \( R \) such that \( R = e_0 R \oplus e_1 R \oplus \cdots \oplus e_{s-1} R \). Then for \( i = 0, \ldots, s-1 \), the subring \( \bigoplus_{j=0}^{s-1} e_{i+jd} R \) of \( R \) is a \( \sigma^d \delta \)-PV extension of \( k \) for the system

\[
\sigma^d(Y) = A_d Y, \quad \delta(Y) = BY.
\]

Here we use a cyclic notation for the indices \( \{0, \ldots, s-1\} \).

Proof. The proof is similar to that of Lemma 2.3. \( \square \)

Definition 2.5 Let \( A \in \text{GL}_n(k) \), \( B \in \text{gl}_n(k) \) and \( Y \) be a vector of unknowns of size \( n \). Let \( R \) be a \( \sigma \delta \)-PV extension for \( \{\sigma(Y) = AY, \delta(Y) = BY\} \). The group consisting of all \( \sigma \delta \)-automorphisms of \( R \) is called the \( \sigma \delta \)-Galois group for the system and denoted \( \text{Gal}(R/k) \).

Denote by \( \text{Gal}(e_0 R/k) \) the \( \sigma^s \delta \)-Galois group for \( \{\sigma^s(Y) = A_s Y, \delta(Y) = BY\} \). Without loss of generality, we assume that \( \sigma(e_i) = e_{i+1 \mod s} \). Construct a map \( \Gamma \) from \( \text{Gal}(e_0 R/k) \) to \( \text{Gal}(R/k) \) as follows. Let \( \varphi \in \text{Gal}(e_0 R/k) \). For any \( r = r_0 + r_1 + \cdots + r_{s-1} \in R \) with \( r_j \in e_j R \) for \( j = 0, \ldots, s-1 \), define

\[
\Gamma(\varphi)(r) = \sum_{j=0}^{s-1} \sigma^j \varphi \sigma^{-j}(r_j).
\]

Let \( \phi \in \text{Gal}(R/k) \). Clearly, \( \phi \) permutes the \( e_i \)'s by the proof of Lemma 6.8 in [12]. Define a map \( \Delta : \text{Gal}(R/k) \to \mathbb{Z}/s\mathbb{Z} \) to be \( \Delta(\phi) = i \) if \( \phi(e_0) = e_i \). We then have the following
Lemma 2.6 Let $R$ be a $\sigma\delta$-PV extension for $\{\sigma(Y) = AY, \delta(Y) = BY\}$ where $A \in \text{GL}_n(k)$ and $B \in \text{gl}_n(k)$. Let $\Gamma$ and $\Delta$ be stated above. Then the map $\Gamma$ is well-defined, i.e., $\varphi \in \text{Gal}(e_0R/k)$ implies $\Gamma(\varphi) \in \text{Gal}(R/k)$. Moreover, the sequence of group homomorphisms

$$0 \rightarrow \text{Gal}(e_0R/k) \xrightarrow{\Gamma} \text{Gal}(R/k) \xrightarrow{\Delta} \mathbb{Z}/s\mathbb{Z} \rightarrow 0$$

is exact.

Proof. The proof is similar to that of Corollary 1.17 in [25]. □

Lemma 2.7 Suppose that $k$ has no proper algebraic $\sigma\delta$-field extension and $R$ is a $\sigma\delta$-PV extension for $\{\sigma(Y) = AY, \delta(Y) = BY\}$ where $A \in \text{GL}_n(k)$ and $B \in \text{gl}_n(k)$. Then $\text{Gal}(R/k)\langle 0 = \text{Gal}(e_0R/k)$.

Proof. Let $\hat{k}$ be the algebraic closure of $k$ in the quotient field of $e_0R$. Then $\text{Gal}(e_0R/k)\langle 0 = \text{Gal}(e_0R/\hat{k})$. Since $k$ has no proper algebraic $\sigma\delta$-field extension, we have $\hat{k} = k$ and therefore $\text{Gal}(e_0R/k) = \text{Gal}(e_0R/k)\langle 0$. From Lemma 2.6 it follows that $\text{Gal}(e_0R/k)$ is a closed subgroup of $\text{Gal}(R/k)$ of finite index. The proposition in [17, p.53] then implies the lemma. □

From [12], we know that a $\sigma\delta$-PV extension $R$ over $k$ is the coordinate ring of a $\text{Gal}(R/k)$-torsor over $k$. Let $E$ be an algebraically closed differential field with a derivation $\delta$. Clearly, $E(x)$ becomes a $\sigma\delta$-field endowed with the extended derivation $\delta$ such that $\delta(x) = 0$ and with an automorphism $\sigma$ on $E(x)$ given by $\sigma|_E = 1$ and $\sigma(x) = x + 1$. For such a field $E(x)$, we will get an analogue of Proposition 1.20 in [25]. Before we state the result, let us look at the following

Lemma 2.8 Let $k$ be a differential field with a derivation $\delta$, $S$ a differential ring extension of $k$ and $I$ a differential radical ideal of $S$. Suppose that $S$ is Noetherian as an algebraic ring and that $I$ has the minimal prime ideal decomposition $\cap_{i=1}^t P_i$ as an algebraic ideal. Then $P_i$ are differential ideals for $i = 1, \ldots, t$.

Proof. Let $f_1 \in P_1$ and select $f_i \in P_i \setminus P_1$ for $i = 2, \ldots, t$. Then $f = f_1f_2 \cdots f_t \in I$. By taking a derivation on both sides, we have

$$\delta(f) = \delta(f_1)f_2 \cdots f_t + f_1\delta(f_2) \cdots f_t + \cdots + f_1f_2 \cdots \delta(f_t) \in I,$$

which implies $\delta(f_1)f_2 \cdots f_t \in P_1$. So $\delta(f_1) \in P_1$ and $P_1$ is a differential ideal. The proofs for other $P_i$'s are similar. □
Now let $S$ be a finitely generated $\sigma\delta$-ring over $k$ and $I$ a radical $\sigma\delta$-ideal of $S$. Suppose that $S$ is Noetherian as an algebraic ring and $I = \cap_{i=1}^{s}P_i$ is the minimal prime decomposition of $I$ as an algebraic ideal. Since $S$ is Noetherian, we have $\sigma(I) = I$, which implies that $\sigma$ permutes the $P_i$’s. From Lemma 2.8 each $P_i$ is a differential ideal. Therefore if $\{P_i\}_{i \in J}$ with $J$ a subset of $\{1, \ldots, s\}$ is left invariant under the action of $\sigma$, then $\cap_{i \in J}P_i$ is a $\sigma\delta$-ideal. We then have the following result. We will use the following notation: if $V$ is a variety defined over a ring $k_0$ and $k_1$ is a ring containing $k_0$, we denote by $V(k_1)$ the points of $V$ with coordinates in $k_1$.

**Proposition 2.9** Let $\tilde{k} = E(x)$ be as in the paragraph preceding Lemma 2.8, $R$ a $\sigma\delta$-PV extension of $k$ for the system $\{\sigma(Y) = AY, \delta(Y) = BY\}$ where $A \in \text{GL}_n(\tilde{k})$ and $B \in \mathfrak{gl}_n(\tilde{k})$, and $G = \text{Gal}(R/\tilde{k})$. Then the corresponding $G$-torsor $Z$ has a point which is rational over $\tilde{k}$ and $Z(\tilde{k})$ and $G(\tilde{k})$ are isomorphic. Moreover, $G/G^0$ is cyclic.

**Proof.** The notation and proof will follow that of Proposition 1.20 in [25]. Let $Z_0, \ldots, Z_{t-1}$ be the $\tilde{k}$-components of $Z$. By Lemma 2.8, the defining ideals $P_i$ of $Z_i$ are differential ideals. As in the proof of Proposition 1.20 in [25], there exists $B \in Z_0(\tilde{k})$ such that $Z_0 = BG^0_k$ and $Z = BG_k$ where $G_k$ denotes the variety $G$ over $\tilde{k}$. Since $Z(\tilde{k})$ is $\tau$-invariant, we have

$$BG_k = \tau(BG_k) = A^{-1}\sigma(B)G_k,$$

which implies $B^{-1}A^{-1}\sigma(B) \in G(\tilde{k})$. There exists $N \in G(\tilde{k}^{\sigma\delta})$ such that

$$B^{-1}A^{-1}\sigma(B) \in G^0(\tilde{k})N.$$

Let $H$ be the group generated by $G^0$ and $N$. One sees that $\tau(BH_k) = BH_k$ and therefore the defining ideal $\tilde{I}$ of $BH_k$ is $\sigma$-invariant. Since the set $BH_\tilde{k}$ is the union of some of the $Z_i$, $\tilde{I}$ is of the form $\cap_{i \in J}P_i$ with $J$ a subset of $\{0, 1, \ldots, t-1\}$. Hence $\tilde{I}$ is a $\sigma\delta$-ideal because each $P_i$ is a differential ideal. Since the defining ideal $I$ of $Z$ is a maximal $\sigma\delta$-ideal, it follows that $\tilde{I} = I$ and so $H = G$. □

From $B^{-1}A^{-1}\sigma(B) \in G^0(\tilde{k})N$ in the proof of Proposition 2.9 we conclude that $N$ is a generator of the cyclic group $G/G^0$.

**Definition 2.10** Let $k$ be a $\sigma\delta$-field and $R$ be a $\sigma\delta$-PV extension of $k$ for the system

$$\sigma(Y) = AY, \quad \delta(Y) = BY$$

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with \( A \in \text{GL}_n(k) \) and \( B \in \text{gl}_n(k) \), and \( V \subset \mathbb{R}^n \) be the solution space of the system. The system is said to be irreducible over \( k \) if \( V \) has no nontrivial \( \text{Gal}(\mathbb{R}/k) \)-invariant subspaces.

In a manner similar to the purely differential case [26, p. 56], one can show that a system \( \{ \sigma(Y) = AY, \delta(Y) = BY \} \) with \( A \in \text{GL}_n(k) \) and \( B \in \text{gl}_n(k) \) is reducible over \( k \) if and only if there exists \( U \in \text{GL}_n(k) \) such that a change of variables \( Z = UY \) yields \( \sigma(Z) = \tilde{A}Z, \delta(Z) = \tilde{B}Z \) with

\[
\tilde{A} = \begin{pmatrix} A_1 & 0 \\ A_2 & A_3 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} B_1 & 0 \\ B_2 & B_3 \end{pmatrix}.
\]

Note that \( \tilde{A} \) and \( \tilde{B} \) are again \( n \times n \) matrices over \( k \).

We end this section with a concrete way of realizing a \( \sigma\delta \)-Picard-Vessiot extensions for linear difference-differential systems over fields of particular form. We proceed in a manner similar to that of [26, Chapter 3].

Let \( K \) be a differential field with a derivation \( \delta \). Denote by \( S_K \) the set of all sequences of the form \( a = (a_0, a_1, \ldots) \) with \( a(i) = a_i \in K \). Define an equivalence relation on \( S_K \) as follows: any two sequences \( a \) and \( b \) are equivalent if there exists \( N \in \mathbb{Z}_{>0} \) such that \( a(n) = b(n) \) for all \( n > N \). Denote by \( \overline{S}_K \) the set of equivalence classes of \( S_K \) modulo the equivalence relation. One sees that \( \overline{S}_K \) forms a differential ring with the addition, multiplication and a derivation \( \delta \) defined on \( \overline{S}_K \) coordinatewise. Clearly, the map \( \sigma \) given by \( \sigma((a_0, a_1, \ldots)) = (a_1, a_2, \ldots) \) is an automorphism of \( \overline{S}_K \) that commutes with the derivation \( \delta \). In addition, any element \( e \in K \) is identified with \((e, e, \ldots)\).

So we can regard \( K \) as a (differential) subfield of \( \overline{S}_K \).

Let \( E \) be an algebraically closed differential field with a derivation \( \delta \). Construct an automorphism \( \sigma \) on \( E(x) \) given by \( \sigma|_E = 1 \) and \( \sigma(x) = x + 1 \) and extend \( \delta \) to be a derivation \( \delta \) on \( E(x) \) such that \( \delta(x) = 0 \). Assume that \( K \) is a differential field extension of \( E \) with an extended derivation \( \delta \). The map \( E(x) \to \overline{S}_K \) given by \( f \mapsto (0, \ldots, 0, f(N), f(N+1), \ldots) \), where \( N \) is a non-negative integer such that \( f \) has no poles at integers \( \geq N \), induces a \( \sigma\delta \)-embedding of \( E(x) \) into \( \overline{S}_K \). Consequently, we may identify any matrix \( M \) over \( E(x) \) with a sequence of matrices \((0, \ldots, 0, M(N), M(N+1), \ldots) \) where \( M(i) \) means the evaluation of the entries of \( M \) at \( x = i \). So we have the following

**Proposition 2.11** Let \( E \subset K \), \( E(x) \), \( \overline{S}_K \) be as above and let \( \bar{k} = E(x) \). Assume that \( E \) and \( K \) have the same algebraically closed field of constants
as differential fields. Let \( R \) be a \( \sigma\delta\)-PV extension for the system

\[
\sigma(Y) = AY, \quad \delta(Y) = BY
\]

where \( A \in \text{GL}_n(\overline{k}) \) and \( B \in \mathfrak{gl}_n(\overline{k}) \). Let \( N \in \mathbb{Z}_{>0} \) such that \( A(m) \) and \( B(m) \) are defined and \( \det(A(m)) \neq 0 \) for all \( m \geq N \) and assume that \( \delta(Y) = B(N)Y \) has a fundamental matrix \( \overline{Z} \in \text{GL}_n(K) \). Then there exists a \( \sigma\delta\)-\( \overline{k} \)-monomorphism of \( R \) into \( S_K \). Moreover, the entries of any solution of \( \{ \sigma(Y) = AY, \delta(Y) = BY \} \) in \( S_K^n \) lies in the image of \( R \) in \( S_K \).

**Proof.** Let \( R = \overline{k}[Y, \frac{1}{\det Y}]/I \) be the \( \sigma\delta\)-PV ring extension for the system \( \{ \sigma(Y) = AY, \delta(Y) = BY \} \) and let \( G \) be its Galois group. From Proposition 2.3.9, the corresponding torsor has a point \( P \) with coordinates in \( \overline{k} \). This implies that if we introduce a new matrix of variables \( X \) and let \( Y = PX \), then \( R = \overline{k}[X, \frac{1}{\det X}]/J \) where \( J \) is the defining ideal of \( G \). Furthermore, \( \sigma(X) = AX \) and \( \delta(X) = BX \) where \( A = \sigma(P)^{-1}AP \in \mathfrak{g}(\overline{k}) \) and \( B = P^{-1}BP \) and assume that \( \det(Y) = 0 \) for any \( m \geq N \). This implies that if we introduce a new matrix of variables \( X \) and let \( Y = PX \), then \( R = \overline{k}[X, \frac{1}{\det X}]/J \) where \( J \) is the defining ideal of \( G \). Moreover, \( \sigma(X) = AX \) and \( \delta(X) = BX \) where \( A = \sigma(P)^{-1}AP \in \mathfrak{g}(\overline{k}) \) and \( B = P^{-1}BP \) and \( \det(Y) = 0 \) for any \( m \geq N \).

Define recursively a sequence of matrices \( Z_m \in \text{GL}_n(K) \) for \( m \geq N \):

\[
Z_N = \overline{Z} \quad \text{and} \quad Z_{m+1} = A(m)Z_m \quad \text{for any} \quad m \geq N.
\]

The integrability condition on \( A \) and \( B \) implies that \( Z_m \) satisfies \( \delta(Y) = B(m)Y \) for any \( m \geq N \) and so \( Z = (\ldots, Z_N, Z_{N+1}, \ldots) \) is a fundamental matrix in \( \text{GL}_n(S_K) \) of \( \{ \sigma(Y) = AY, \delta(Y) = BY \} \).

Remark that the key to proving the proposition is to show that \( Z \) generates a \( \sigma\delta\)-Picard-Vessiot extension. Unfortunately, we do not see a direct way to show this and our proof is a little circuitous. Clearly, \( U := P^{-1}Z \) satisfies that \( \sigma(U) = AU \) and \( \delta(U) = BU \). Then \( \delta(U(N')) = B(N')U(N') \) for a sufficiently large \( N' \) and therefore \( K \) contains a (differential) Picard-Vessiot extension of \( E \) for the equation \( \delta(U) = BU \). Since \( B(N') \in \mathfrak{g}(\overline{k}) \), Proposition 1.3.1 (and its proof) in [26] together with the uniqueness of Picard-Vessiot extensions imply that there exists \( \nabla \in \mathfrak{g}(K) \) such that \( \delta(\nabla) = \nabla \). Define \( V \in S_K \) by \( V(N') = \nabla \) and \( V(m+1) = A(m)V(m) \) for \( m \geq N' \). This implies that the map from \( R = \overline{k}[X, \frac{1}{\det X}]/J \) to \( S_K \) given by \( X \mapsto V \) is a \( \sigma\delta\overline{k} \)-homomorphism. Since \( I \) is a maximal \( \sigma\delta \)-ideal, this map must be injective, and so is the desired embedding from \( R \) into \( S_K \).

Let \( W \in S_K^n \) be a solution of \( \{ \sigma(Y) = AY, \delta(Y) = BY \} \). For a sufficiently large \( M \), \( W(M) \) is defined and is a solution of \( \delta(Y) = B(M)Y \)
and \( W(m + 1) = A(m)W(m) \) for \( m \geq M \). Therefore \( W(M) = V(M)D \) for some constant vector \( D \) and thus \( W = VD \in R^n \). It follows that \( Z = PU \) also generates a \( \sigma\delta \)-Picard-Vessiot extension, as claimed in the above. □

**Remarks 2.12** If \( K \) is a maximal Picard-Vessiot extension of \( E \) with the same constants (Zorn's lemma guarantees that such fields exist), then the hypothesis on the existence of \( \overline{Z} \) in Proposition 2.11 is always satisfied. Therefore for such a field \( K \), \( S_K \) contains a \( \sigma\delta \)-Picard-Vessiot ring for any system \( \{\sigma(Y) = AY, \delta(Y) = BY\} \).

### 2.2 Liouvillian solutions

The Galois theory for linear differential equations is stated in terms of differential integral domains and fields [26, Chapter 1] and both theory and algorithms for finding liouvillian solutions are well developed [26, Chapters 1.5, 4.1 - 4.4]. The main result is that the associated Picard-Vessiot extension lies in a tower of fields built up by successively adjoining, exponentials, integrals and algebraics if and only if the associated Galois group has a solvable identity component. For linear difference equations, the Galois theory is stated in terms of reduced rings and total rings of fractions. A general theory of liouvillian solutions has not been developed in the difference case. However, a case has been investigated in [13] where the coefficient field is of the form \( C(x) \) with a shift operator \( \sigma : x \mapsto x + 1 \) and \( \sigma|_C = 1 \). In this situation, solutions of linear difference equations are identified with sequences whose entries are in \( C \). One says that a linear difference equation is solvable in terms of liouvillian sequences if it has a full set of solutions in a ring of sequences built up by successively adjoining to \( C \) sequences representing indefinite sums, indefinite products and interlacings of previously defined sequences. The main result is that a linear difference equation is solvable in terms of liouvillian sequences if and only if its Galois group has a solvable identity component.

In this paper we will combine the approaches for differential and difference cases to investigate the solvability of systems of mixed linear difference-differential equations over \( E(x) \) where \( E \) will always be an algebraically closed differential field unless specified otherwise, \( \sigma(x) = x + 1 \) and \( \sigma|_E = 1 \).

In this section, we will give a characterization of Galois groups for mixed difference-differential systems to have solvable identity component in terms of liouvillian towers over an arbitrary \( \sigma\delta \)-field \( k \) with algebraically closed constants. Then we will define a notion of liouvillian sequences and show
that having a full set of solutions of this type implies that the Galois group has solvable identity component (Proposition 2.23). In a later result (Proposition 3.7), we will show the converse is true as well.

Liouvillian extensions for $\sigma\delta$-fields are defined in the usual way.

**Definition 2.13** Let $k$ be a $\sigma\delta$-field. A $\sigma\delta$-field extension $K$ of $k$ is said to be liouvillian if there is a chain of $\sigma\delta$-field extensions

$$k = K_0 \subset K_1 \subset \cdots \subset K_m = K$$

such that $k^\sigma = K^\sigma$, i.e., $K$ shares the same set of constants with $k$, and $K_{i+1} = K_i(t_i)$ for $i = 0,\ldots,m-1$ where

1. $t_i$ is algebraic over $K_i$, or
2. $\sigma(t_i) = r_1t_i$ and $\delta(t_i) = r_2t_i$ with $r_1, r_2 \in K_i$ i.e., $t_i$ is hyperexponential over $K_i$, or
3. $\sigma(t_i) - t_i \in K_i$ and $\delta(t_i) \in K_i$.

We now define liouvillian solutions of mixed difference-differential systems. In the sequel, let $k$ be a $\sigma\delta$-field with algebraically closed constants, $R$ be a $\sigma\delta$-PV extension of $k$ for the system $\{\sigma(Y) = AY, \delta(Y) = BY\}$ with $A \in \text{GL}_n(k)$ and $B \in \mathfrak{gl}_n(k)$. Suppose that $R$ has a decomposition

$$R = e_0R \oplus e_1R \oplus \cdots \oplus e_{s-1}R$$

and $F_0$ is the quotient field of $e_0R$. Then $F_0$ is a $\sigma^s\delta$-field.

**Definition 2.14** Let $v = \sum_{i=0}^{s-1} v_i$ be a solution of $\{\sigma(Y) = AY, \delta(Y) = BY\}$ in $R^n$ where $v_i := e_i v \in e_iR^n$. We say that $v$ is liouvillian if the entries of $v_0$ lie in a $\sigma^s\delta$-liouvillian extension of $k$ containing $F_0$. We say that the original system is solvable in liouvillian terms if each solution is liouvillian.

Suppose that $v$ is a solution of $\{\sigma(Y) = AY, \delta(Y) = BY\}$ in $R^n$. From

$$\sigma(v) = Av \quad \text{and} \quad \delta(v) = Bv,$$

it follows that

$$\sigma(v_{s-1}) \oplus \sigma(v_0) \oplus \cdots \oplus \sigma(v_{s-2}) = Av_0 \oplus Av_1 \oplus \cdots \oplus Av_{s-1}$$

$$\delta(v_0) \oplus \delta(v_1) \oplus \cdots \oplus \delta(v_{s-1}) = Bv_0 \oplus Bv_1 \oplus \cdots \oplus Bv_{s-1},$$
which implies that \( \sigma(v_i) = Av_i + 1_{s \text{ mod } s} \) and \( \delta(v_i) = Bv_i \) for \( i = 0, \ldots, s-1 \). Hence \( \sigma^s(v_0) = A_s v_0 \) and \( \delta(v_0) = Bv_0 \), i.e., \( v_0 \) is a solution of the system \( \{ \sigma^s(Y) = A_s Y, \delta(Y) = BY \} \). Conversely, assume that \( v_0 \) is a solution of \( \{ \sigma^s(Y) = A_s Y, \delta(Y) = BY \} \) in \( R/k \). Let \( v_i = A^{-1} \sigma(v_{i-1}) \) for \( i = 1, \ldots, s-1 \) and \( v = v_0 \oplus \cdots \oplus v_{s-1} \). We then have \( \sigma(v) = Av \). From the fact that \( \sigma(B) - \delta(A)A^{-1} = ABA^{-1} \), one can easily check that \( \delta(v) = Bv \). Hence \( v \) is a solution of \( \{ \sigma(Y) = AY, \delta(Y) = BY \} \) in \( R^n \). Moreover we have the following

**Proposition 2.15** The system \( \{ \sigma(Y) = AY, \delta(Y) = BY \} \) is solvable in liouvillian terms if and only if the system \( \{ \sigma^s(Y) = A_s Y, \delta(Y) = BY \} \) is solvable in liouvillian terms.

**Proof.** Let \( V_1, \ldots, V_n \) be a basis of the solution space for the system \( \{ \sigma(Y) = AY, \delta(Y) = BY \} \). Since \( V_i \) is liouvillian, \( V_{i0} = e_0 V_i \) is liouvillian for each \( i \). It then suffices to show that \( V_{i0}, \ldots, V_{n0} \) are linearly independent over \( k^{\sigma^s} \). Assume that there exist \( c_1, \ldots, c_n \in k^{\sigma^s} \), not all zero, such that \( c_1 V_{i0} + \cdots + c_n V_{n0} = 0 \). Letting \( V_{i1} = e_1 V_i \) we have \( V_{i1} = A^{-1} \sigma(V_{i0}) \) for \( i = 1, \ldots, n \). Remark that \( k^{\sigma^s} = k^{\sigma^s} \) since \( k^{\sigma^s} \) is algebraically closed. Therefore

\[
c_1 V_{i1} + \cdots + c_n V_{n1} = A^{-1} \sigma(c_1 V_{i0} + \cdots + c_n V_{n0}) = 0.
\]

Similarly, \( c_1 e_i V_{i1} + \cdots + c_n e_i V_{n1} = 0 \) for each \( i \). Hence \( c_1 V_1 + \cdots + c_n V_n = 0 \), a contradiction.

Conversely, suppose that \( V_{i0}, \ldots, V_{n0} \) is a basis of the solution space for the system \( \{ \sigma^s(Y) = A_s Y, \delta(Y) = BY \} \) and that all the \( V_{i0} \)'s are liouvillian. For \( i = 1, \ldots, n \) and \( k = 1, \ldots, s-1 \), let

\[
V_{ik} = A^{-1} \sigma(V_{i,k-1}) \quad \text{and} \quad V_i = V_{i0} \oplus V_{i1} \oplus \cdots \oplus V_{i,s-1}.
\]

Then each \( V_i \) is a solution of \( \{ \sigma(Y) = AY, \delta(Y) = BY \} \). Clearly, \( V_1, \ldots, V_n \) are linearly independent over \( k^{\sigma^s} \). This concludes the proposition. \( \square \)

**Theorem 2.16** The system \( \{ \sigma(Y) = AY, \delta(Y) = BY \} \) is solvable in liouvillian terms if and only if \( \text{Gal}(R/k)^0 \) is solvable.

**Proof.** By Proposition 2.15 \( \{ \sigma(Y) = AY, \delta(Y) = BY \} \) is solvable in liouvillian terms if and only if the associated system \( \{ \sigma^s(Y) = A_s Y, \delta(Y) = BY \} \) is solvable in liouvillian terms. By Lemma 2.6 \( \text{Gal}(e_0 R/k) \) is a subgroup of \( \text{Gal}(R/k) \) of finite index. Then \( \text{Gal}(R/k)^0 = \text{Gal}(e_0 R/k)^0 \). Hence it suffices to show that the associated system is solvable in liouvillian terms if and
only if $\text{Gal}(e_0 R/k)^0$ is solvable. Note that the PV extension of $k$ for the associated system is a domain. Thus the proof is similar to that in differential case. □

Let $E$ be a differential field with algebraically closed constants and $L$ be a field containing $E$ which satisfies the following conditions:

(i) $L$ is a differential field extension of $E$ having the same field of constants as $E$;

(ii) every element in $L$ is liouvillian over $E$;

(iii) $L$ is maximal with respect to (i) and (ii).

Zorn’s Lemma guarantees that such a field exists. We refer to $L$ as a maximal liouvillian extension of $E$. One can show that any differential liouvillian extension of $E$ having the same field of constants as $E$ can be embedded into a maximal liouvillian extension and that any two maximal liouvillian extensions of $E$ are isomorphic over $E$ as differential fields.

Recall that for two sequences $a$ and $b$, and for a nonnegative integer $m$, $b$ is called the $i$th $m$-interlacing ([13, Definition 3.2]) of $a$ with zeroes if

$$b(mn + i) = a(n) \quad \text{and} \quad b(r) = 0 \quad \text{for any } r \not\equiv i \mod m.$$  

A sequence $a$ is called the $i$th $m$-section of $b$ if $a(mn + i) = b(mn + i)$ and $a(r) = 0$ for $r \not\equiv i \mod m$.

We now turn to a definition of liouvillian sequences.

**Definition 2.17** Let $E$ be a differential field with algebraically closed constants, $\sigma$ be an automorphism on $E(x)$ satisfying $\sigma(x) = x + 1$ and $\sigma|_E = 1$. Let $L$ be a maximal liouvillian extension of $E$. The ring of liouvillian sequences over $E(x)$ is the smallest subring $\mathcal{L}$ of $\mathcal{S}_L$ such that

1. $L(x) \subset \mathcal{L}$;

2. For $a \in \mathcal{S}_L$, $a \in \mathcal{L}$ if and only if $\sigma(a) \in \mathcal{L}$;

3. Supposing $\sigma(b) = ab$ with $a, b \in \mathcal{S}_L$, then $a \in E(x)$ implies $b \in \mathcal{L}$. $b$ is called a hypergeometric sequence over $E(x)$;

4. Supposing $\sigma(b) = a + b$ with $a, b \in \mathcal{S}_L$, then $a \in \mathcal{L}$ implies $b \in \mathcal{L}$;

5. For $a \in \mathcal{S}_L$, $a \in \mathcal{L}$ implies that $b \in \mathcal{L}$, where $b$ is the $i$th $m$-interlacing of $a$ with zeroes for some $m \in \mathbb{Z}_{>0}$ and $0 \leq i \leq m - 1$.  

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Set \( \bar{k} = E(x) \) and let \( \mathcal{L} \) be the ring of Liouvilian sequences over \( \bar{k} \). From the remarks in [13, p. 243], if \( b \in S_L \) belongs to \( \mathcal{L} \) then the \( i \)th \( m \)-section of \( b \) also belongs to \( \mathcal{L} \) for any \( i \) and \( m \). We claim that \( \mathcal{L} \) is a \( \sigma \delta \)-ring. Since \( \mathcal{L} \) can be constructed inductively using (1) - (5) above, it is enough to show the following statements:

1. If \( f \in L(x) \) then \( \delta(f) \in L(x) \); 
2. If \( a \in S_L, a, \delta(a) \in \mathcal{L} \) then \( \delta(\sigma(a)) \in \mathcal{L} \); 
3. Supposing \( \sigma(b) = ab \) with \( a, b \in S_L \), then \( a, \delta(a) \in E(x) \) implies \( \delta(b) \in \mathcal{L} \); 
4. Supposing \( \sigma(b) = a + b \) with \( a, b \in S_L \), then \( a, \delta(a) \in \mathcal{L} \) implies \( \delta(b) \in \mathcal{L} \); 
5. For \( a \in S_L \), \( a, \delta(a) \in \mathcal{L} \) implies that \( \delta(b) \in \mathcal{L} \), where \( b \) is the \( i \)th \( m \)-interlacing of \( a \) with zeroes for some \( m \in \mathbb{Z}_{>0} \) and \( 0 \leq i \leq m - 1 \).

Verifying (1'), (2') and (5') is straightforward. To verify (3'), suppose that \( b \in \mathcal{L} \) with \( \sigma(b) = ab \). Set \( y = \delta(b) \). Then \( \sigma(y) - ay = \delta(a) b \). Since \( a \in \bar{k} \), \( b \) is invertible. Then \( u := \delta(b) / b \) satisfies \( \sigma(u) - u = \delta(a) / a \). Therefore \( u \in \mathcal{L} \) and so \( \delta(b) = ub \in \mathcal{L} \). To verify (4'), suppose that \( b \in \mathcal{L} \) with \( \sigma(b) = a + b \). Then \( \sigma(\delta(b)) = \delta(b) + \delta(a) \) which implies that \( \delta(b) \in \mathcal{L} \).

A vector is said to be hypergeometric over \( \bar{k} \) if it can be written as \( Wh \) where \( W \in \bar{k}^n \) and \( h \) is a hypergeometric sequence over \( \bar{k} \). For any positive integer \( d \), we can construct a solution of \( \{ \sigma^d(Y) = A_d Y, \delta(Y) = BY \} \) by interlacing as indicated below. Consider a set of new systems

\[
S_j : \{ \sigma(Z) = A_d dx + j) Z, \quad \delta(Z) = B(dx + j) Z \}, \quad j = 0, \ldots, d - 1,
\]

where \( A_d(dx + j) \) and \( B(dx + j) \) mean replacing \( x \) by \( dx + j \) in each entry of \( A_d \) and \( B \), respectively. Clearly, these new systems are all integrable. Moreover, we have the following

**Proposition 2.18** Let \( V_j \) be a solution of \( S_j \) in \( S^d_K \) where \( S_K \) is as in Remark 2.12 and \( W_j \) be the \( j \)th \( d \)-interlacing with zeros of \( V_j \) for \( j = 0, \ldots, d - 1 \). Then

\[
W = W_0 + \cdots + W_{d-1}
\]

is a solution of \( \{ \sigma^d(Y) = A_d Y, \delta(Y) = BY \} \).
Proof. It suffices to show that \( W_j \) is a solution of \( \{ \sigma^d(Y) = A_dY, \delta(Y) = BY \} \) for each \( j \). Let \( V_j = (V_j(0), V_j(1), \cdots) \) and \( W_j = (W_j(0), W_j(1), \cdots) \) for \( j = 0, \cdots, d - 1 \). Then

\[
V_j(i + 1) = A_d(di + j)V_j(i), \quad \delta(V_j(i)) = B(di + j)V_j(i)
\]

and therefore by definition of interlacing, for \( i \geq 0 \) and \( \ell = 1, \cdots, d - 1 \),

\[
W_j(di + j + d) = V_j(i + 1) = A_d(di + j)V_j(i) = A_d(di + j)A_d(di + j)V_j(i) = A_d(di + j)W_j(di + j),
\]

\[
W_j(di + j + d + \ell) = 0 = A_d(di + j)W_j(di + j + \ell).
\]

So \( \sigma^d(W_j) = A_dW_j \). It is clear that \( \delta(W_j) = BW_j \). So the proposition holds. \( \square \)

**Definition 2.19** Let \( E \) be an algebraically closed differential field with a derivation \( \delta \). Suppose that \( \delta \) is extended to be a derivation \( \delta \) on \( E(x) \) such that \( \delta(x) = 0 \) and \( \sigma \) is an automorphism on \( E(x) \) such that \( \sigma|_E = 1 \) and \( \sigma(x) = x + 1 \). A system

\[
\sigma(Y) = AY, \quad \delta(Y) = BY
\]

over \( E(x) \) is said to be solvable in terms of liouvillian sequences if the \( \sigma\delta\)-PV extension of this system embeds, over \( E(x) \), into \( \mathcal{L} \), the ring of liouvillian sequences over \( E(x) \).

Clearly, Definition 2.19 generalizes the notion of solvability in liouvillian terms for linear differential equations and that for linear difference equations. In addition, we shall show in Proposition 2.23 that this property is also equivalent to the identity component of the Galois group being solvable. However this property is not equivalent to having a solution in the ring \( S_L \), as shown by the following

**Example 2.20** The Hermite polynomials

\[
H_n(t) = n! \sum_{m=0}^{[n/2]} \frac{(-1)^m (2t)^{n-2m}}{m!(n-2m)!}
\]

satisfy a linear differential equation with respect to \( t \) and a difference equation with respect to \( n \) (cf., [6], Ch. 10.13). In matrix terms, the vector \( Y(n,t) = (H(n,t), H(n+1,t))^T \) satisfies

\[
Y(n + 1, t) = \begin{pmatrix} 0 & 1 \\ -2n & 2t \end{pmatrix} Y(n, t),
\]

\[
\frac{\partial Y(n,t)}{\partial t} = \begin{pmatrix} 2t & -1 \\ 2n & 0 \end{pmatrix} Y(n, t)
\]

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so this system has a solution in $S_L$ and one can use the techniques of Section 3.3 to show that this system have no solutions in $L$.

Concerning total rings of fractions, we have the following

**Lemma 2.21** Let $k$ be a $\sigma\delta$-field and $S$ a $\sigma\delta$-PV extension of $k$. A nonzero element $r \in S$ is a zero divisor in $S$ if and only if there exists $j \in \mathbb{Z}_{>0}$ such that $\prod_{i=0}^{j} \sigma^i(r) = 0$.

**Proof.** Suppose that $\prod_{i=0}^{j} \sigma^i(r) = 0$ for some $j \in \mathbb{Z}_{>0}$ and let $j$ be minimal with respect to this assumption. If $0 = \prod_{i=1}^{j} \sigma^i(r) = \sigma(\prod_{i=0}^{j-1} \sigma^i(r))$, then we would also have $\prod_{i=0}^{j-1} \sigma^i(r) = 0$ contradicting the minimality of $j$. Therefore $r$ is a zero divisor. Now suppose that $r$ is a zero divisor. We write $S = \sum_{i=0}^{j-1} S_i$ where the $S_i$ are domains and $\sigma(S_i) = S_{i+1 \text{ mod } j}$. Then $r = \sum_{i=0}^{j-1} r_i$ with $r_i \in S_i$. Writing $r = (r_0, \ldots, r_{j-1})$, one sees that $r$ is a zero divisor in $S$ if and only if some $r_i$ is zero. Assume $r_0 = 0$. Then

$$
\sigma(r) = (\sigma(r_{j-1}), 0, \ldots), \quad \sigma^2(r) = (\sigma^2(r_{j-2}), \sigma^2(r_{j-1}), 0, \ldots), \quad \ldots
$$

so $0 = \prod_{i=0}^{j-1} \sigma^i(r)$. □

Consequently, we have

**Corollary 2.22** Let $k$ and $S$ be as above and let $\bar{S}$ be a $\sigma$-subring of $S$. An element $r \in \bar{S}$ is a zero divisor in $\bar{S}$ if and only if it is a zero divisor in $S$. Therefore the total ring of fractions of $\bar{S}$ embeds in the total ring of fractions of $S$.

**Proposition 2.23** Let $E(x)$ be as in Definition 2.19. If a system

$$
\sigma(Y) = AY, \quad \delta(Y) = BY
$$

over $E(x)$ is solvable in terms of liouvillian sequences, then the identity component of its Galois group is solvable.

**Proof.** Let $R$, $L$ and $\mathcal{L}$ be the $\sigma\delta$-PV ring, a maximally liouvillian extension of $E$ and the ring of liouvillian sequences for the given system, respectively. Then $R \subset \mathcal{L}$. Consider the following diagram

$$
\begin{array}{c}
R L(x) \\
/ \\
R \\
\end{array}
$$

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where $RL(x)$ is the field generated by $R$ and $L(x)$. We will show that

(i) $RL(x) \subset L$ is a $\sigma\delta$-PV extension of $L(x)$ and its Galois group has a solvable identity component.

(ii) The $\sigma\delta$-Galois group of $RL(x)$ over $L(x)$ is isomorphic to the subgroup $H$ of the Galois group $G$ of $R$ over $E(x)$ that leaves the quotient field of $R \cap L(x)$ fixed. Moreover, $H$ is a closed normal subgroup of $G$.

(iii) $G/H$ is the Galois group of the quotient field of $R \cap L(x)$ over $E(x)$ and has solvable identity component.

Once the above claims are proven, the group $G$ has a solvable identity component since both $H$ and $G/H$ have solvable identity component.

To prove (i), we consider $\{\sigma(Y) = AY, \delta(Y) = BY\}$ as a system over $L(x)$. Since $R \subset L$, then for a sufficiently large $N$ there is a fundamental matrix of $\delta(Y) = B(N)Y$ with entries in $L$. Applying Proposition 2.11, we conclude that $S_L$ contains a $\sigma\delta$-PV extension $T$ of $L(x)$ for the system $\{\sigma(Y) = AY, \delta(Y) = BY\}$ and that $R \subset T$. This implies that $RL(x)$ is a $\sigma\delta$-PV extension of $L(x)$. Proposition 4.1 of [25] implies that $RL(x)$ is also a difference Picard-Vessiot extension for $\sigma(Y) = AY$ and the results of [13] then imply that the difference Galois group has solvable identity component. The $\sigma\delta$-Galois group is a subgroup of this latter group and its identity component is a subgroup of the identity component of the larger group. Therefore it is also solvable.

To prove (ii), let $F_L$ be the total ring of fractions of $RL(x)$ and $F_E$ the total ring of fractions of $R$. Corollary 2.22 implies that we can regard $F_E$ as a subset of $F_L$. The elements of the $\sigma\delta$-Galois group $G$ of $F_L$ over $L(x)$ restrict to automorphisms of $F_E$ over $E(x)$, and this gives a homomorphism of this group into $H$. Clearly the image is closed and the set of elements left fixed by this group is $L(x) \cap F_E$, the quotient field of $L(x) \cap R$. Therefore this image is $H$.

Before proving that $H$ is normal in $G$, we first show that

$$F_E \cap L(x) = (F_E \cap L)(x).$$
Since \( x \in F_E \) we have \( F_E \cap L(x) \supset (F_E \cap L)(x) \). To get the reverse inclusion, let \( f \in F_E \cap L(x) \) and write
\[
f = \frac{a_r x^r + \ldots + a_0}{b_s x^s + \ldots + b_0}, \quad a_i, b_i \in L
\]
where the numerator and denominator are relatively prime. We then have that \( \{x^s f, x^{s-1} f, \ldots, f, x^r, \ldots, 1\} \) are linearly dependent over \( L \). Since \( L \) is the set of \( \sigma \)-invariant elements of \( L(x) \), the cassoratian of these elements must vanish (\cite{8}, p.271). This implies further that these elements are linearly dependent over the \( \sigma \)-invariant elements of the field \( F_E \cap L(x) \). Therefore there exist \( \sigma \)-invariant elements \( \tilde{a}_i, \tilde{b}_j \in F_E \cap L(x) \) for \( i = 0, 1, \ldots, r \) and \( j = 0, 1, \ldots, s \), not all zero, such that
\[
\tilde{a}_r x^r + \ldots + \tilde{a}_0 - (\tilde{b}_s x^s f + \ldots + \tilde{b}_0 f) = 0.
\]
Since \( x \) is transcendental over \( \sigma \)-invariant elements, there exists at least one \( \tilde{b}_i \) which is not zero. Hence
\[
f = \frac{\tilde{a}_r x^r + \ldots \tilde{a}_0}{\tilde{b}_s x^s + \ldots \tilde{b}_0} \in (F_E \cap L)(x).
\]

To show that \( H \) is normal, it now suffices to prove that any \( \sigma \delta \)-automorphism of \( F_E \) over \( k \) leaves the field \( F_E \cap L(x) = (F_E \cap L)(x) \) invariant. Note that \( L \) is the set of \( \sigma \)-invariant elements of \( F_L \) and so \( F_E \cap L \) is the set of \( \sigma \)-invariant elements of \( F_E \). This set is clearly preserved by any \( \sigma \delta \)-automorphism.

To prove (iii), note that since \( H \) is normal, the field \( (F_E \cap L)(x) \) is a \( \sigma \delta \)-PV extension of \( E(x) \). Furthermore, \( (F_E \cap L)(x) \) lies in the liouvillian extension \( L(x) \) of \( E(x) \). Theorem 2.16 implies that its Galois group is solvable. \( \square \)

Later in Proposition 3.7 we will show that the converse is true as well.

### 2.3 From \( e_0 R \) to \( R \)

Throughout this section, let \( E \) be an algebraically closed differential field with a derivation \( \delta \) whose extension on \( E(x) \) satisfies \( \delta(x) = 0 \), and let \( \sigma \) be an automorphism on \( E(x) \) such that \( \sigma|_E = 1 \) and \( \sigma(x) = x + 1 \). Set \( \tilde{k} = E(x) \) and let \( \mathcal{L} \) be the ring of liouvillian sequences over \( \tilde{k} \). In this section, we shall show how to construct a solution in \( \mathcal{L}^n \) of \( \{\sigma(Y) = AY, \delta(Y) = BY\} \) from a solution in \( \mathcal{L}^n \) of its associated system \( \{\sigma^d(Y) = A_dY, \delta(Y) = BY\} \). Moreover, we shall prove that if the associated system is equivalent over \( \tilde{k} \) to a
diagonal form, so is the original system under the assumption that $A$ is of particular form.

Let $V \in \mathcal{L}^n$ be a nonzero solution of $\{\sigma^d(Y) = A_d Y, \delta(Y) = BY\}$ and $N \in \mathbb{Z}_{>0}$ be such that $V(N) \neq 0, A(j)$ and $B(j)$ are well defined and $\det(A(j)) \neq 0$ for $j \geq N$. We define a vector $W$ in the following way:

$$W(N) = V(N) \quad \text{and} \quad W(j + 1) = A(j)W(j) \quad \text{for } j \geq N.$$  

Since $\delta(V(N)) = B(N)V(N)$, the integrability condition on $\sigma$ and $\delta$ implies that $W$ is a nonzero solution of $\{\sigma(Y) = AY, \delta(Y) = BY\}$. The proposition below says that $W$ is also in $\mathcal{L}^n$.

**Proposition 2.24** Let $W$ be as above. Then $W \in \mathcal{L}^n$. 

**Proof.** Let $N = d\ell + m$ where $\ell, m \in \mathbb{Z}_{>0}, 0 < m < \ell$ and $V_0$ be the $m$th $d$-section of $V$. Then $V_0$ is a solution of $\{\sigma^d(Y) = A_d Y, \delta(Y) = BY\}$ and $V_0 \in \mathcal{L}^n$. Let

$$V_i(j) = A(j)^{-1}V_{i-1}(j+1) \quad \text{for } j \geq N \text{ and } i = 1, \ldots, d - 1,$$

and $U = V_0 + V_1 + \cdots + V_{d-1}$. Then $U \in \mathcal{L}^n$. We shall prove that $W = U$. Note that for $j > N$,

$$V_i(j) = A(j)^{-1}A(j+1)^{-1} \cdots A(j+i-1)^{-1}V_0(j+i).$$

In particular, $V_{d-1}(j+1) = A(j)V_0(j)$. Then $V_i(N) = 0$ for $i = 1, \ldots, d - 1$. Therefore $W(N) = V_0(N) + V_1(N) + \cdots + V_{d-1}(N) = U(N)$ and

$$U(j + 1) = V_0(j + 1) + V_1(j + 1) + \cdots + V_{d-1}(j + 1) = A(j)V_1(j) + A(j)V_2(j) + \cdots + A(j)V_0(j) = A(j)U(j)$$

for $j \geq N$. Hence $W = U \in \mathcal{L}^n$. \square

**Lemma 2.25** Assume that $w \in \tilde{k}$ satisfies

$$\sigma^s(w) = \sigma^{s-1}(b) \cdots \sigma(b)bw$$

where $s \in \mathbb{Z}_{>0}, b \in \tilde{k} \setminus \{0\}$ and $b = x^\nu + b_1x^{\nu-1} + \cdots$ with $\nu \in \mathbb{Z}$ and $b_i \in E$. Then $\sigma(w) = bw$. 

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Proof. We have
\[ \sigma^s \left( \frac{\sigma(w)}{b} \right) = \frac{1}{\sigma^s(b)} \sigma^{s-1}(b) \cdots \sigma(b)bw = \sigma^{s-1}(b) \cdots \sigma(b) \frac{\sigma(w)}{b}. \]
Since \( \frac{\sigma(w)}{b} \in \tilde{k}, \frac{\sigma(w)}{b} = cw \) for some \( c \in \tilde{k} \sigma^s = \tilde{k}^\sigma \). Note that \( w \) and \( b \) are rational functions in \( x \). Expanding \( w \) and \( b \) as Laurent series at \( x = \infty \). By comparing the coefficients, we get \( c = 1 \), so \( \sigma(w) = bw \).

\[ \square \]

Theorem 2.26 Let \( A = \text{diag}(a_1, \ldots, a_n) \) where \( a_i \in \tilde{k} \setminus \{0\} \) and \( a_i = cx^{\nu_i} + a_{i1}x^{\nu_i-1} + \cdots \) with \( \nu_i \in \mathbb{Z} \) and \( c, a_{ij} \in E \). Assume that the system \( \{ \sigma^d(Y) = A_dY, \delta(Y) = BY \} \) is equivalent over \( \tilde{k} \) to
\[ \sigma^d(Y) = A_dY, \ \delta(Y) = \text{diag}(b_1, \ldots, b_n)Y \]
where \( b_i \in \tilde{k} \) for \( i = 1, \ldots, n \). Then \( \{ \sigma(Y) = AY, \delta(Y) = BY \} \) is equivalent over \( \tilde{k} \) to
\[ \sigma(Y) = AY, \ \delta(Y) = \text{diag}(b_1, \ldots, b_n)Y. \]

Proof. From the assumption, there exists \( G \in \text{GL}_n(\tilde{k}) \) such that
\[ \sigma^d(G)A_d = A_dG, \ G^{-1}BG - G^{-1}\delta(G) = \text{diag}(b_1, \ldots, b_n). \]
It then suffices to prove that \( \sigma(G)A = AG \). Let \( G = (g_{ij})_{n \times n} \). Then
\[ \begin{cases} 
\sigma^d(g_{ii}) - g_{ii} = 0, & i = 1, \ldots, n; \\
\sigma^d(g_{ij}) = \sigma^{d-1} \left( \frac{a_{ij}}{a_j} \right) \cdots \sigma \left( \frac{a_{ij}}{a_j} \right) \frac{a_{ij}}{a_j} g_{ij}, & 1 \leq i \neq j \leq n.
\end{cases} \]
By Lemma 2.25 \( \sigma(g_{ij}) = \frac{a_{ij}}{a_j} g_{ij} \) for all \( i, j = 1, \ldots, n \). This implies that \( \sigma(G)A = AG. \)

\[ \square \]

2.4 From \( \mathbb{C}(x,t) \) to \( \overline{\mathbb{C}(t)}(x) \)

In the following sections, we always let \( k_0 \) be the \( \sigma\delta \)-field \( \mathbb{C}(t,x) \) with an automorphism \( \sigma : x \mapsto x+1 \) and a derivation \( \delta = \frac{d}{dt} \) and let \( k \) be its extension field \( \overline{\mathbb{C}(t)}(x) \). Consider a system of difference-differential equations
\[ \sigma(Y) = AY, \ \delta(Y) = BY \]
over \( k_0 \) where \( A \in \text{GL}_n(k_0) \) and \( B \in \text{gl}_n(k_0) \). We shall analyze this system by focusing on its difference part \( \sigma(Y) = AY \) and use techniques from the theory of difference equations. In this latter theory, one assumes that
the fixed field of $\sigma$, that is, the $\sigma$-constants, are algebraically closed. For this reason we will need to consider properties of $\sigma(Y) = AY$ over $k$ as well as over $k_0$. We shall first show that the $\sigma\delta$-Galois group of this system over $k$ can be identified with a normal subgroup of the Galois group of the same system over $k_0$, and then conclude some results on orders of the factors. For example, if the above system is irreducible over $k_0$, it is possible that the system is reducible over $k$. In this case, we will prove that the factors of the above system over $k$ have the same order. A similar result is well known for differential equations—if one makes a normal algebraic extension of the base field then the differential Galois group over this new field is a normal subgroup of the differential Galois group over the original field and an irreducible equation factors into factors of equal order. In the mixed difference-differential case or even the difference case, the fact that Picard-Vessiot extensions may contain zero divisors introduces some small complication.

We start with some lemmas. Let $R$ and $R_0$ be the $\sigma\delta$-PV extensions of $k$ and $k_0$ for the system \{\(\sigma(Y) = AY, \delta(Y) = BY\)\} respectively.

**Lemma 2.27**

(i) There is a $k_0$-monomorphism of $\sigma\delta$-rings from $R_0$ to $R$.

(ii) Identify $R_0$ with a subring of $R$ as in the first assertion. Suppose that $R_0 = f_0 R_0 \oplus \cdots \oplus f_{d-1} R_0$ where $f_i R_0$ is a domain, $f_i^2 = f_i$ and $\sigma(f_i) = f_{i+1 \mod d}$, and that $R = e_0 R \oplus \cdots \oplus e_{s-1} R$ is a similar decomposition of $R$. Then $s = md$ for some $m \in \mathbb{Z}_{>0}$. Moreover, after a possible renumbering of the $f_i$, we have

$$f_i = e_i + e_{i+d} + \cdots + e_{i+(m-1)d} \quad \text{for } i = 0, \ldots, d-1.$$ 

**Proof.** (i) Clearly, the ring $R_0 \otimes_{k_0} k$ becomes a $\sigma\delta$-ring endowed with the actions $\delta(r \otimes h) = \delta(r) \otimes h + r \otimes \delta(h)$ and $\sigma(r \otimes h) = \sigma(r) \otimes \sigma(h)$ for any $r \in R_0$ and $h \in k$. Since $k_0$ is a field, the two canonical embeddings

$$R_0 \to R_0 \otimes_{k_0} k \quad \text{and} \quad k \to 1 \otimes_{k_0} k \subset R_0 \otimes_{k_0} k$$

are both injective, and clearly are homomorphisms of $\sigma\delta$-rings. Let $M$ be a maximal $\sigma\delta$-ideal of $R_0 \otimes_{k_0} k$ and consider the ring $(R_0 \otimes_{k_0} k)/M$. Because $R_0$ and $k$ are both simple $\sigma\delta$-rings, the above embeddings factor through to $(R_0 \otimes_{k_0} k)/M$ and are still injective. Note that $(R_0 \otimes_{k_0} k)/M$ is a $\sigma\delta$-PV extension of $k$ for \{\(\sigma(Y) = AY, \delta(Y) = BY\)\}. So by uniqueness, we may write $R = (R_0 \otimes_{k_0} k)/M$. Assume that $R_0 = k_0[\overline{Z}, \frac{1}{\det(Z)}]$ where $\overline{Z}$ is a
fundamental matrix of the system. Let $\bar{Z} = Z \mod M$. One sees that $\bar{Z}$ is still a fundamental matrix of the system and that $\det(\bar{Z}) \neq 0$. Hence

$$R = \left( k_0 \left[ Z, \frac{1}{\det(Z)} \right] \otimes_{k_0} k \right) / M = k \left[ \bar{Z}, \frac{1}{\det(\bar{Z})} \right]$$

is a $\sigma\delta$-PV ring for the given system over $k$ and $R_0$ can be embedded into $R$.

(ii) Write $f_0 = \sum_{j=0}^{s-1} a_j e_j$ with $a_j \in e_j R$. Squaring both sides yields $f_0 = \sum_{j=0}^{s-1} a_j^2 e_j$, thus $a_j^2 e_j = a_j e_j$. Since $e_j R$ is a domain, $a_j$ is either $e_j$ or 0 for each $j$. The same holds for other $f_i$'s. Then for any $i = 0, \ldots, d - 1$, there is a subset $T_i \subset \{0, \ldots, s - 1\}$ such that $f_i = \sum_{j \in T_i} e_j$. Assume that $T_{i_0} \cap T_{i_1}$ is not empty for two different $i_0$ and $i_1$. Let $l \in T_{i_0} \cap T_{i_1}$. Since $\sum_{i=0}^{d-1} f_i = \sum_{j=0}^{s-1} e_j = 1$,

$$0 = \sum_{i=0}^{d-1} f_i - \sum_{j=0}^{s-1} e_j = \sum_{i \in T_i} e_i - \sum_{j=0}^{s-1} e_j = pe_l + H$$

where $p > 0$ and $H$ is the sum of all the $e_q$'s with $q \neq l$. Multiplying both sides of the above equality by $e_l$, we get $pe_l = 0$, a contradiction. Hence the $T_i$'s form a partition of $\{0, \ldots, s - 1\}$. Since $\sigma(f_i) = f_{i+1 \mod d}$ and $\sigma(e_j) = e_{j+1 \mod s}$, one sees that the sets $T_i$ have the same size and that a renumbering yields the conclusion. □

According to Lemma 2.27, we will always consider $R_0$ as a subring of $R$ and assume $R = k[\bar{Z}, \frac{1}{\det(\bar{Z})}]$ in the sequel. In particular, we can view $R_0 = k_0[\bar{Z}, \frac{1}{\det(\bar{Z})}]$.

**Lemma 2.28** Let $\gamma : \text{Gal}(R/k) \to \text{Gal}(R_0/k_0)$ be a map given by $\gamma(\phi) = \phi|_{R_0}$ for any $\phi \in \text{Gal}(R/k)$. Then $\gamma$ is a monomorphism. Moreover, we can view the identity component of $\text{Gal}(R/k)$ as a subgroup of that of $\text{Gal}(R_0/k_0)$.

**Proof.** Assume that $R = k[\bar{Z}, \frac{1}{\det(\bar{Z})}]$. Let $\phi \in \text{Gal}(R/k)$. If $\phi(\bar{Z}) = \bar{Z}[\phi]Z$ for some $[\phi]Z \in \text{GL}_n(\mathbb{C})$, then $\gamma(\phi)(\bar{Z}) = \bar{Z}[\phi]Z$. Hence $\gamma(\phi)$ is an automorphism of $R_0$ over $k_0$, that is, $\gamma(\phi) \in \text{Gal}(R_0/k_0)$. Note that $\det(\bar{Z}) \neq 0$. If $\gamma(\phi) = 1$, then $[\phi]Z = I_n$, which implies that $\phi = 1$. So $\gamma$ is an injective homomorphism. Therefore, we can view $\text{Gal}(R/k)$ as a subgroup of $\text{Gal}(R_0/k_0)$. Since $\gamma$ is continuous in the Zariski topology, $\gamma(\text{Gal}(R/k)^0)$ is in $\text{Gal}(R_0/k_0)^0$. So the lemma holds. □

**Lemma 2.29** $\text{Gal}(R/k)^0 = \text{Gal}(R_0/k_0)^0$.

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Proof. Let \( G = \text{Gal}(R/k) \) and \( G_0 = \text{Gal}(R_0/k_0) \). From [12], the \( \sigma \delta \)-PV ring \( R \) (resp. \( R_0 \)) is the coordinate ring of a \( G \)-torsor (resp. \( G_0 \)-torsor). From [20] p. 40, (3)], the Krull dimension of \( R \) (resp. \( R_0 \)) equals the Krull dimension of \( G \) (resp. \( G_0 \)). Since all the components of a linear algebraic group are isomorphic as varieties, one sees that the Krull dimension of \( G \) (resp. \( G_0 \)) equals the Krull dimension of \( G_0 \) (resp. \( G_0 \)). Since \( R \) is generated over \( R_0 \) by the elements of \( k \), by Proposition 2.2 and Corollary 2.3 in [20, p. 44], \( R \) is an integral ring extension of \( R_0 \). By [20, Corollary 2.13], the Krull dimension of \( R \) equals that of \( R_0 \). Hence the Krull dimension of \( G_0 \) equals that of \( G_0 \). Since both \( G_0 \) and \( G_0 \) are connected and \( G_0 \subset G_0 \), by the proposition in [17, p. 25] we have \( G_0 = G_0 \). \( \square \)

From Lemma 2.28, \( \text{Gal}(R/k) \) can be viewed as a subgroup of \( \text{Gal}(R_0/k_0) \). In the following, we prove that \( \text{Gal}(R/k) \) is a normal subgroup of \( \text{Gal}(R_0/k_0) \). Let \( \mathcal{F}_0 \) and \( \mathcal{F} \) be the total ring of fractions of \( R_0 \) and \( R \) respectively. Note that \( \text{Gal}(R/k) = \text{Gal}(\mathcal{F}/k) \) and \( \text{Gal}(R_0/k_0) = \text{Gal}(\mathcal{F}_0/k_0) \). Corollary 2.22 allows us to assume that \( \mathcal{F}_0 \subset \mathcal{F} \).

**Lemma 2.30** Let \( u \in k \) be of degree \( m \) over \( k_0 \) and let \( u = u_1, u_2, \ldots, u_m \) be its conjugates. Then there exist \( M \in \text{GL}_m(k_0) \) and \( N \in \text{gl}_m(k_0) \) such that

\[
Z = \begin{pmatrix}
1 & 1 & \ldots & 1 \\
1 & u_2 & \ldots & u_m \\
\vdots & \vdots & \ddots & \vdots \\
1 & u_1^{-1} & u_2^{-1} & \ldots & u_m^{-1}
\end{pmatrix}
\]

satisfies

\[\sigma(Z) = MZ \quad \text{and} \quad \delta(Z) = NZ.\]

Proof. We claim that any field automorphism \( \tau \) of \( k \) over \( k_0 \) is a \( \sigma \delta \)-field automorphism. Indeed, since \( k \) is an algebraic extension of \( k_0 \) so any automorphism \( \tau \) is automatically a \( \delta \)-field automorphism. One sees that \( \tau \) is a \( \sigma \)-field automorphism by noting that for any \( f \in \mathbb{C}(t)(x) \), \( \tau \) acts on the coefficients of powers of \( x \) while \( \sigma \) acts only on \( x \). For any \( g \) in the automorphism group of \( k \) over \( k_0 \), we have \( g(Z) = Z[g] \) where \([g]\) is a permutation matrix. Since \( g \) is also an automorphism of \( \sigma \delta \)-fields, both \( M = \sigma(Z)Z^{-1} \) and \( N = \delta(Z)Z^{-1} \) are left invariant by \( g \) and therefore must have entries in \( k_0 \). \( \square \)

We now proceed to prove the main result of this section.

**Proposition 2.31** \( \text{Gal}(R/k) \) is a normal subgroup of \( \text{Gal}(R_0/k_0) \).
Proof. Consider the following diagram
\[
\begin{array}{ccc}
\mathcal{F} & \rightarrow & \mathcal{F}_0/k \\
\downarrow & & \downarrow \\
\mathcal{F}_0/k_0 & \rightarrow & \mathcal{F}_0 \cap k/k_0
\end{array}
\]

First, we claim that the map \(Gal(\mathcal{F}/k) \to Gal(\mathcal{F}_0/\mathcal{F}_0 \cap k)\) that sends \(h \in Gal(\mathcal{F}/k)\) to its restriction \(h|_{\mathcal{F}_0}\) on \(\mathcal{F}_0\) is an isomorphism. Any automorphism of \(\mathcal{F}\) over \(k\) is determined by its action on a fundamental matrix of \(\{\sigma(Y) = AY, \delta(Y) = BY\}\) and its restriction on \(\mathcal{F}_0\) is determined in the same way. This implies that the restricted map is injective. To see that it is surjective, note that its image is closed and has \(\mathcal{F}_0 \cap k\) as a fixed field. Therefore, the Galois theory implies that the restricted map must be \(Gal(\mathcal{F}_0/\mathcal{F}_0 \cap k)\).

We now show that \(ghg^{-1}\) is in \(Gal(\mathcal{F}_0/\mathcal{F}_0 \cap k)\) for any \(h \in Gal(\mathcal{F}_0/\mathcal{F}_0 \cap k)\) and \(g \in Gal(\mathcal{F}_0/k_0)\). It suffices to show that \(g\) leaves \(\mathcal{F}_0 \cap k\) invariant, which will imply that \(ghg^{-1}(u) = u\) for any \(u \in \mathcal{F}_0 \cap k\) and so \(ghg^{-1} \in Gal(\mathcal{F}_0/\mathcal{F}_0 \cap k)\). Now let \(u \in \mathcal{F}_0 \cap k\) be of degree \(m\) over \(k_0\). From Lemma \ref{2.30} \(U = (1, u, u^2, \ldots, u^{m-1})^T\) satisfies some difference-differential system over \(k_0\). Therefore the vector \(g(U)\) satisfies the same system and so must be a \(\mathbb{C}\)-linear combination of the columns of \(Z\). In particular, we have \(g(u) \in k\). Therefore \(g\) leaves \(\mathcal{F}_0 \cap k\) invariant. This completes the proof. \(\square\)

Theorem 2.32. If \(\{\sigma(Y) = AY, \delta(Y) = BY\}\) is irreducible over \(k_0\), then it is equivalent over \(\hat{k}_0 := F_0 \cap k\) to the system
\[
\sigma(Y) = \text{diag}(A_1, A_2, \ldots, A_d)Y, \quad \delta(Y) = \text{diag}(B_1, B_2, \ldots, B_d)Y
\]
where \(A_i \in \text{GL}_\ell(\hat{k}_0), B_i \in \text{gl}_\ell(\hat{k}_0)\) and \(\ell = \frac{n}{d}\). Moreover, the system \(\{\sigma(Y) = A_i Y, \delta(Y) = B_i Y\}\) is irreducible over \(k\) for \(i = 1, \ldots, d\), and there exists \(g_i \in \text{Gal}(R_0/k_0)\) such that \(g_i(A_1) = A_i\) and \(g_i(B_1) = B_i\).

Proof. By Proposition 2.31, \(Gal(R/k)\) is isomorphic to \(Gal(\mathcal{F}_0/\hat{k}_0)\) and then \(Gal(\mathcal{F}_0/\hat{k}_0)\) is a normal subgroup of \(Gal(R_0/k_0)\). Let \(V\) be the solution space in \(R_0^n\) of \(\{\sigma(Y) = AY, \delta(Y) = BY\}\). Then Clifford’s Theorem \(\ref{9}\) p.25, Theorem 2.2] tells us that \(V\) can be decomposed into \(V = V_1 \oplus V_2 \oplus \cdots \oplus V_d\) where the \(V_i\) are minimal \(Gal(\mathcal{F}_0/\hat{k}_0)\)-invariant subspaces of \(V\) and, for
each $i$, there exists $g_i \in \text{Gal}(R_0/k_0)$ such that $V_i = g_i(V_1)$. Furthermore, $g \in \text{Gal}(R_0/k_0)$ permutes the $V_i$. Let $Z_1$ be an $n \times \ell$ matrix whose columns are the solutions in $V_1$ of the original system. Then $Z_1$ has the full rank. Then for each $i$, the columns of $Z_i = g_i(Z_1)$ are the solutions in $V_i$ of the original system and $Z_i$ has the full rank too. Let $Z = (Z_1, \ldots, Z_d)$. Then $Z$ is a fundamental matrix of the original system. By Lemma 1 in [11], there exists $\ell \times n$ matrix $P_1$ of the rank $\ell$ with entries from $\hat{k}_0$ such that $P_1V_i = 0$ for $i = 2, \ldots, d$. Since $g \in \text{Gal}(R_0/k_0)$ permutes the $V_i$, 

$$\{g_i(V_2), \ldots, g_i(V_d)\} = \{V_1, \ldots, V_{i-1}, V_{i+1}, \ldots, V_d\}.$$ 

Let $P_1 = g_i(P_1)$. Then $P_1V_j = 0$ for $j \neq i$. Therefore, $P := (P_1^T, \ldots, P_d^T)^T \in \mathfrak{gl}_n(\hat{k}_0)$ satisfies that 

$$PZ = \text{diag}(U_1, U_2, \ldots, U_d) \tag{2}$$ 

with $U_i \in \mathfrak{gl}_l(F_0)$. Moreover we have $U_i = g_i(U_1)$ for each $i$. We now prove that $\det(P) \neq 0$. Assume the contrary that $\det(P) = 0$. Then $w^T P = 0$ for some nonzero $w \in \hat{k}_0^n$. Therefore there exists $w_i \in \hat{k}_0^\ell$ for $1 \leq i \leq d$ such that $w_1^TP_1 + \cdots + w_d^TP_d = 0$. Since the $P_i$ have full rank, there exists at least one $i$ such that $w_i^TP_1 \neq 0$. Without loss of generality, assume that $w_1^TP_1 \neq 0$. From $w_1^TP_1 = -(w_2^TP_2 + \cdots + w_d^TP_d)$ and $P_1Z_1 = 0$ for $i = 2, \ldots, d$, we have $w_1^TP_1Z = 0$. Since $\det(Z) \neq 0$, $w_1^TP_1 = 0$, a contradiction. Therefore $\det(P) \neq 0$. Let $\ell = \frac{n}{d}$. From (2), $\{\sigma(Y) = AY, \delta(Y) = BY\}$ is equivalent over $\hat{k}_0$ to 

$$\sigma(Y) = \text{diag}(A_1, \ldots, A_d)Y, \quad \delta(Y) = \text{diag}(B_1, \ldots, B_d)Y$$ 

where $A_i \in \text{GL}_l(\hat{k}_0)$ and $B_i \in \mathfrak{gl}_l(\hat{k}_0)$. Furthermore, $U_i$ is a fundamental matrix of the system $\{\sigma(Z) = A_iZ, \delta(Z) = B_iZ\}$. Since $U_i = g_i(U_1)$, we have that $A_i = g_i(A_1)$ and $B_i = g_i(B_1)$ for each $i$. From the minimality of $V_i$, the system $\{\sigma(Z) = A_iZ, \delta(Z) = B_iZ\}$ is irreducible over $k$. \hfill $\Box$

**Corollary 2.33** Let $A_i$ and $B_i$ be as in Theorem 2.32 for $i = 1, \ldots, d$. Then for each $i$, the Galois group of $\{\sigma(Z) = A_iZ, \delta(Z) = B_iZ\}$ over $k$ is solvable by finite if and only if the Galois group of $\{\sigma(Y) = AY, \delta(Y) = BY\}$ over $k_0$ is solvable by finite.

### 3 Systems

Throughout this section, let $k_0$ be the field $\mathbb{C}(t, x)$ with an automorphism $\sigma : x \mapsto x + 1$ and a derivation $\delta = \frac{d}{dt}$ and let $k$ the extension field $\mathbb{C}(\hat{t})(x)$. In
In this section, we will first prove that if a system \( \{ \sigma(Y) = AY, \delta(Y) = BY \} \) with \( A \in \text{GL}_n(k_0) \) and \( B \in \text{gl}_n(k_0) \) is irreducible over \( k_0 \) and its Galois group over \( k_0 \) is solvable by finite then there exists \( \ell \in \mathbb{Z}_{>0} \) with \( \ell | n \) such that the solution space of \( \{ \sigma^\ell(Y) = A_\ell Y, \delta(Y) = BY \} \) has a basis each of whose members is the interlacing of hypergeometric solutions over \( k \). We will then refine this result to show that \( \{ \sigma^\ell(Y) = A_\ell Y, \delta(Y) = BY \} \) is equivalent over \( k \) to a special form. Based on this special form, we will describe a decision procedure to find its solutions.

### 3.1 Systems with Liouvillian Sequences as Solutions

By Theorem 2.32, if a system \( \{ \sigma(Y) = AY, \delta(Y) = BY \} \) with \( A \in \text{GL}_n(k_0) \) and \( B \in \text{gl}_n(k_0) \) is irreducible over \( k_0 \), then it can be decomposed into factors that are irreducible over \( k \) and if the Galois group of the system over \( k_0 \) is solvable by finite then the Galois groups of these factors over \( k \) are also solvable by finite. Hence it is enough to consider factors of the original system over \( k \).

**Proposition 3.1** Suppose that \( \{ \sigma(Y) = AY, \delta(Y) = BY \} \) with \( A \in \text{GL}_\ell(k) \) and \( B \in \text{gl}_\ell(k) \) is an irreducible system over \( k \) and that its Galois group over \( k \) is solvable by finite. Then the system is equivalent over \( k \) to

\[
\sigma(Y) = \bar{A}Y, \quad \delta(Y) = \bar{B}Y
\]

where \( \bar{B} \in \text{gl}_\ell(k) \) and

\[
\bar{A} = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
a & 0 & 0 & \cdots & 0
\end{pmatrix} \in \text{GL}_\ell(k)
\]

with \( a \in k \).

**Proof.** The proof is similar to those of Lemma 4.1 and Theorem 5.1 in [13]. \( \square \)

**Remark 3.2** From the proof of Lemma 4.1 in [13], we know that \( \ell \) divides \( |\text{Gal}(\mathcal{R}/k)/\text{Gal}(\mathcal{R}/k)^0| \) because \( \{ \sigma(Y) = AY, \delta(Y) = BY \} \) is irreducible over \( k \). From the proof of Theorem 5.1 in [13], \( \text{Gal}(\mathcal{R}/k)^0 \) is diagonalizable.

As a consequence of Proposition 3.1, we have the following
Corollary 3.3 If a system \(\{\sigma(Y) = \mathcal{A}Y, \delta(Y) = \mathcal{B}Y\}\) with \(\mathcal{A} \in \text{GL}_\ell(k)\) and \(\mathcal{B} \in \text{gl}_\ell(k)\) is irreducible over \(k\) and its Galois group over \(k\) is solvable by finite, then \(\{\sigma^\ell(Y) = \mathcal{A}_\ell Y, \delta(Y) = \mathcal{B}Y\}\) is equivalent over \(k\) to
\[
\sigma^\ell(Y) = \mathcal{D}Y, \quad \delta(Y) = \bar{\mathcal{B}}Y
\]
where \(\bar{\mathcal{B}} \in \text{gl}_\ell(k)\) and \(\mathcal{D} = \text{diag}(a, \sigma(a), \ldots, \sigma^{\ell-1}(a))\) with \(a\) as indicated in Proposition 3.1.

Next, we shall prove further that \(\{\sigma^\ell(Y) = \mathcal{A}_\ell Y, \delta(Y) = \mathcal{B}Y\}\) is equivalent over \(k\) to a system of diagonal form. Note that equivalent systems have the same Picard-Vessiot extension.

Proposition 3.4 If a system \(\{\sigma(Y) = \mathcal{A}Y, \delta(Y) = \mathcal{B}Y\}\) with \(\mathcal{A} \in \text{GL}_\ell(k)\) and \(\mathcal{B} \in \text{gl}_\ell(k)\) is irreducible over \(k\) and its Galois group over \(k\) is solvable by finite, then \(\{\sigma^\ell(Y) = \mathcal{A}_\ell Y, \delta(Y) = \mathcal{B}Y\}\) is equivalent over \(k\) to \(\sigma^\ell(Y) = \mathcal{D}Y, \delta(Y) = \text{diag}(\bar{b}_1, \ldots, \bar{b}_\ell)Y\)
with \(\mathcal{D} = \text{diag}(a, \sigma(a), \ldots, \sigma^{\ell-1}(a))\)
with \(a\) as in Proposition 3.1. Let \(R = \bar{e}_0R \oplus \bar{e}_1R \oplus \cdots \oplus \bar{e}_\ell R\) be the decomposition of \(R\). Then \(\text{Gal}(R/k)^0 = \text{Gal}(\bar{e}_0R/k)\) by Lemma 2.7 and \(\text{Gal}(\bar{e}_0R/k)\) is diagonalizable by Remark 3.2. From Lemma 2.3 it follows that \(\bar{e}_0R\) is a \(\sigma^\ell \delta\)-PV extension of \(k\) for the system
\[
\sigma^\ell(Y) = \sigma^{\ell-1}(\mathcal{D}) \cdots \mathcal{D}Y, \quad \delta(Y) = \bar{B}Y.
\]  
Let \(\mathcal{D} = \sigma^{\ell-1}(\mathcal{D}) \cdots \sigma(\mathcal{D})\mathcal{D} = \text{diag}(\bar{d}_1, \ldots, \bar{d}_\ell)\). By Lemma 2.2, \(\bar{e}_0R\) is a domain. As in the differential case, we can show that \(\mathcal{D}\) is equivalent over \(k\) to the system
\[
\sigma^\ell(Y) = \text{diag}(\bar{a}_1, \ldots, \bar{a}_\ell)Y, \quad \delta(Y) = \text{diag}(\bar{b}_1, \ldots, \bar{b}_\ell)Y
\]  
where \(\bar{a}_i, \bar{b}_i \in k\). Then there exists \(G = (g_{ij}) \in \text{GL}_\ell(k)\) such that
\[
\sigma^\ell(G)\text{diag}(\bar{a}_1, \ldots, \bar{a}_\ell) = \mathcal{D}G,
\]
which implies $\sigma^\ell(g_{ij})\bar{a}_j = g_{ij}\bar{d}_j$. Since $\det(G) \neq 0$, there is a permutation $i_1, \ldots, i_\ell$ of $\{1, 2, \ldots, \ell\}$ such that $g_{i_1}g_{i_2}\cdots g_{i_\ell} \neq 0$. Hence

$$\bar{d}_j = \frac{\sigma^\ell(g_{ji})}{g_{ji}}\bar{a}_i \quad \text{for} \quad j = 1, \ldots, \ell.$$ 

Let $P$ be a multiplication of some permutation matrices such that

$$P^{-1}\text{diag}(\bar{a}_1, \ldots, \bar{a}_\ell)P = \text{diag}(\bar{a}_{i_1}, \ldots, \bar{a}_{i_\ell})$$

and let $T = P\text{diag}(1/g_{i_1}, \ldots, 1/g_{i_\ell})$. Under the transformation $Y \rightarrow TY$, the system (4), and therefore (3), is equivalent over $k$ to

$$\sigma^\ell(Y) = \tilde{D}Y, \quad \delta(Y) = \text{diag}(b_1, \ldots, b_\ell)Y$$

where $b_i \in k$ for $i = 1, \ldots, \ell$. Proposition 2.26 implies that the system $\{\sigma^\ell(Y) = A\ell Y, \delta(Y) = BY\}$ is equivalent over $k$ to

$$\sigma^\ell(Y) = \tilde{D}Y, \quad \delta(Y) = \text{diag}(b_1, \ldots, b_\ell)Y.$$ 

This concludes the proposition. □

Theorem 2.32 together with Proposition 3.4 leads to the following

**Proposition 3.5** If the system $\{\sigma(Y) = AY, \delta(Y) = BY\}$ with $A \in \text{GL}_n(k_0)$ and $B \in \text{gl}_n(k_0)$ is irreducible over $k_0$ and its Galois group over $k_0$ is solvable by finite, then there exists $\ell \in \mathbb{Z}_{>0}$ with $\ell|n$ such that the solution space of $\{\sigma^\ell(Y) = A\ell Y, \delta(Y) = BY\}$ has a basis consisting of the interlacing of hypergeometric solutions over $k$.

**PROOF.** By Theorem 2.32 and Proposition 3.4, there is $\ell \in \mathbb{Z}_{>0}$ with $\ell|n$ such that $\{\sigma^\ell(Y) = A\ell Y, \delta(Y) = BY\}$ is equivalent over $k$ to a system of diagonal form. Since the solution space of the latter system has a basis consisting of the interlacing of hypergeometric solutions over $k$, so does the solution space of $\{\sigma^\ell(Y) = A\ell Y, \delta(Y) = BY\}$. □

**Corollary 3.6** Let $\mathcal{L}$ be the ring of Liouvillian sequences over $k$. Assume that $\{\sigma(Y) = AY, \delta(Y) = BY\}$ with $A \in \text{GL}_n(k_0)$ and $B \in \text{gl}_n(k_0)$ is irreducible over $k_0$ and its Galois group over $k_0$ is solvable by finite. Then the solution space of the system has a basis with entries in $\mathcal{L}$.

**PROOF.** Proposition 3.5 implies that there is $\ell \in \mathbb{Z}_{>0}$ such that the solution space of $\{\sigma^\ell(Y) = A\ell Y, \delta(Y) = BY\}$ has a basis with entries in $\mathcal{L}$. The corollary then follows from Proposition 2.24. □
Let us turn to a general case where a difference-differential system may be reducible over the base field. If the Galois group over the base field of the given system is solvable by finite, then the Galois group over the base field of each factor is of the same type. The method in [13] together with the results in [7] implies the following

**Proposition 3.7** If the Galois group for \( \{ \sigma(Y) = AY, \delta(Y) = BY \} \) with \( A \in \text{GL}_n(k_0) \) and \( B \in \text{gl}_n(k_0) \) is solvable by finite, then the solution space of the system has a basis with entries in \( L \).

**Proof.** By induction, we only need to prove the proposition for the case where the given system has two irreducible factors over \( k_0 \). In this case, the given system is equivalent over \( k_0 \) to \( \sigma(Y) = A_1Y, \delta(Y) = B_1Y \) where the systems \( \{ \sigma(Y) = A_iY, \delta(Y) = B_iY \} \) for \( i = 1, 2 \) are both irreducible over \( k_0 \). Let \( d_i \) be the order of \( A_i \) for \( i = 1, 2 \). By Corollary 3.6 each system \( \{ \sigma(Y) = A_iY, \delta(Y) = B_iY \} \) has a fundamental matrix \( U_i \in \text{GL}_{d_i}(L) \). From the proof of Theorem 3 in [7], Proposition 2.11 and Remark 2.12 it follows that the original system has a \( \sigma \delta \)-PV extension \( \mathcal{R} \) of \( k_0 \) which contains entries of the \( U_i \)'s and, moreover, has a fundamental matrix over \( S_K \) of the form

\[
\begin{pmatrix}
U_1 & 0 \\
V & U_2
\end{pmatrix}.
\]

So \( \sigma(V) = A_1U_1 + A_2V. \) Let \( V = U_2W. \) Then \( \sigma(W) = W + \sigma(U_2)^{-1}A_1U_1. \) Since \( U_i \in \text{GL}_{d_i}(L) \), the entries of \( W \) are in \( L \) and so are the entries of \( V. \) □

### 3.2 Normal Forms

Assume that a system \( \{ \sigma(Y) = AY, \delta(Y) = BY \} \) where \( A \in \text{GL}_n(k_0) \) and \( B \in \text{gl}_n(k_0) \) is irreducible over \( k_0 \) and that its Galois group over \( k_0 \) is solvable by finite. Theorem 2.32 and Proposition 3.4 imply that there exists \( \ell \in \mathbb{Z}_{>0} \) with \( \ell | n \) such that \( \{ \sigma^\ell(Y) = A\ell Y, \delta(Y) = BY \} \) is equivalent over \( k \) to a system of diagonal form. In this section, we will show further that the above system is equivalent over \( k \) to a more special form when the order \( n \) of the original system is prime.
3.2.1 Normal Forms for General Systems

Let us first review some notions and properties concerning rational solutions of difference equations.

**Definition 3.8 (cf. [12], Definition 6.1)** Let \( f = \frac{P}{Q} \) with \( P, Q \in \mathbb{C}(t)[x] \) and \( \text{gcd}(P, Q) = 1 \).

1. The dispersion of \( Q \), denoted by \( \text{disp}(Q) \)
   \[ \alpha \in \mathbb{C}(t) \]

2. The polar dispersion of \( f \) is the dispersion of \( Q \) and denoted \( \text{pdisp}(f) \).

3. \( f \) is said to be standard with respect to \( \sigma^m \), with \( m \in \mathbb{Z}_{>0} \), if \( \text{disp}(P \cdot Q) < m \).

As in [12], we have the following

**Lemma 3.9** Assume that \( f \in k\setminus\{0\}, a \in \mathbb{C}(t) \setminus \{0\} \) and \( m \in \mathbb{Z}_{>0} \).

1. There exist \( \tilde{f}, \tilde{g} \in k \setminus \{0\} \) such that \( f = \frac{\sigma^m(\delta)\tilde{f}}{\tilde{g}} \) where \( \tilde{f} \) is standard with respect to \( \sigma^m \).

2. If \( f \) has a pole, then \( \text{pdisp}(\sigma^m(f) - af) \geq m \).

**Proof.** The proof is similar to that of Lemma 6.2 in [12]. \( \square \)

**Proposition 3.10** Let \( 0 \neq a, b \in k \) satisfy \( \sigma^m(b) - b = \frac{\delta(a)}{a} \) where \( m \in \mathbb{Z}_{>0} \). Then

\[ a = \frac{\sigma^m(f)}{f} \alpha(x)\beta(t) \quad \text{and} \quad b = \frac{\delta(f)}{f} + \frac{\delta(\beta(t))}{m\beta(t)} x + c \]

where \( f \in k, c, \beta(t) \in \mathbb{C}(t), \) and \( \alpha(x) \in \mathbb{C}(x) \) is standard with respect to \( \sigma^m \).

**Proof.** Let \( a = \frac{\sigma^m(f)}{f} \hat{a} \) with \( \hat{a} \) standard with respect to \( \sigma^m \) and \( \hat{b} = b - \delta(f) \).

Then \( \sigma^m(\hat{b}) - \hat{b} = \frac{\delta(\hat{a})}{\hat{a}} \). View \( \hat{a} \) and \( \hat{b} \) as rational functions in \( x \). Then \( \text{pdisp}(\hat{b}) < m \). If \( \frac{\delta(\hat{a})}{\hat{a}} \notin \mathbb{C}(t) \), then \( \frac{\delta(\hat{a})}{\hat{a}} \) has a pole and so does \( \hat{b} \).

By Lemma 3.9, \( \text{pdisp}(\sigma^m(\hat{b}) - \hat{b}) \geq m \), a contradiction. Hence \( \frac{\delta(\hat{a})}{\hat{a}} = w(t) \in \mathbb{C}(t) \), which means that \( \hat{a} = \alpha(x)e^{\int w(t)dt} \). Since \( \hat{a} \in k \), \( \hat{a} \) is of the form \( \alpha(x)\beta(t) \) where \( \alpha(x) \in \mathbb{C}(x) \) and \( \beta(t) \in \mathbb{C}(t) \). Then \( \hat{b} \in \mathbb{C}(t)[x] \).

Suppose that \( \hat{b} = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_0 \) where \( c_i \in \mathbb{C}(t) \) and \( c_n \neq 0 \). Then

\[ \sigma^m(\hat{b}) - \hat{b} = nm c_n x^{n-1} + \cdots + \frac{\delta(\hat{a})}{\hat{a}} = \frac{\delta(\beta(t))}{\beta(t)} x + c_0. \] 

So \( n = 1 \) and \( \hat{b} = \frac{\delta(\beta(t))}{\beta(t)} x + c_0. \) \( \square \)
Theorem 3.11 If \( \{\sigma(Y) = AY, \delta(Y) = BY\} \) with \( A \in \text{GL}_n(k_0) \) and \( B \in \text{gl}_n(k_0) \) is irreducible over \( k_0 \) and its Galois group over \( k_0 \) is solvable by finite, then there exists \( \ell \in \mathbb{Z}_{>0} \) with \( \ell | n \) such that the system

\[
\sigma^\ell(Y) = A\ell Y, \quad \delta(Y) = BY
\]

is equivalent over \( k \) to

\[
\begin{aligned}
\sigma^\ell(Y) &= \text{diag}(\Lambda(x)\beta_1(t), \Lambda(x)\beta_2(t), \cdots, \Lambda(x)\beta_m(t))Y, \\
\delta(Y) &= \text{diag} \left( \frac{\delta(\beta_1(t))}{\ell\beta_1(t)}xI_\ell + C_1, \cdots, \frac{\delta(\beta_m(t))}{\ell\beta_m(t)}xI_\ell + C_m \right)Y
\end{aligned}
\]

(5)

where \( \Lambda(x) = \text{diag}(\alpha(x), \cdots, \alpha(x + \ell - 1)) \), \( C_1 = \text{diag}(c_1, \cdots, c_\ell) \) and \( m\ell = n \). Moreover, \( \alpha(x) \in \mathbb{C}(x) \) is standard with respect to \( \sigma^\ell \), \( \beta_i(t), c_i \in \mathbb{C}(t) \), and there exists \( g_i \) in the Galois group of the original system over \( k_0 \) such that \( \beta_i(t) = g_i(\beta_1(t)) \) and \( C_1 = g_i(C_1) \).

**Proof.** By Theorem 2.32 it suffices to prove the theorem for a factor of the given system. Let \( \{\sigma(Y) = AY, \delta(Y) = BY\} \) be such a factor with \( A \in \text{GL}_\ell(k) \) and \( B \in \text{gl}_\ell(k) \). By Proposition 3.11 \( \{\sigma^{\ell}(Y) = A\ell Y, \delta(Y) = BY\} \) is equivalent over \( k \) to

\[
\sigma^\ell(Y) = DY, \quad \delta(Y) = \text{diag}(b_1, \cdots, b_\ell)Y
\]

where \( D \) is as in Corollary 3.3 and \( b_i \in k \) for \( i = 1, \cdots, \ell \). Since \( \sigma^\ell \) and \( \delta \) commute, we have \( \sigma^\ell(b_1) - b_1 = \frac{\delta(\alpha)}{a} \). By Proposition 3.10 we have

\[
a = \frac{\sigma^\ell(f)}{\sigma^{\ell-1}(f)}\alpha(x)\beta_1(t) \quad \text{and} \quad b_1 = \frac{\delta(f)}{f} + \frac{\delta(\beta_1(t))}{\ell\beta_1(t)}x + c_1
\]

where \( \alpha(x) \in \mathbb{C}(x) \) is standard with respect to \( \sigma^\ell \), \( c_1, \beta_1(t) \in \mathbb{C}(t) \) and \( f \in k \). Then for \( i = 1, \cdots, \ell \),

\[
\sigma^{i-1}(a) = \frac{\sigma(\sigma^{i-1}(f))}{\sigma^{i-1}(f)}\alpha(x+i-1)\beta_1(t) \quad \text{and} \quad b_i = \frac{\delta(\sigma^{i-1}(f))}{\sigma^{i-1}(f)} + \frac{\delta(\beta_1(t))}{\ell\beta_1(t)}x + c_i.
\]

Let \( F = \text{diag}(f, \sigma(f), \cdots, \sigma^{\ell-1}(f)) \). Then the system

\[
\sigma^\ell(Y) = DY, \quad \delta(Y) = \text{diag}(b_1, \cdots, b_\ell)Y
\]

is equivalent over \( k \) to

\[
\sigma^\ell(Y) = \Lambda(x)\beta_1(t)Y, \quad \delta(Y) = \left( \frac{\delta(\beta_1(t))}{\ell\beta_1(t)}xI_\ell + C_1 \right)Y
\]

under the transformation \( Y \rightarrow FY \) where \( \Lambda(x) = \text{diag}(\alpha(x), \cdots, \alpha(x + \ell - 1)) \) and \( C_1 = \text{diag}(c_1, \cdots, c_\ell) \). □
3.2.2 Normal Forms for Systems of Prime Order

If a difference-differential system is of prime order \( n \), then the integer \( \ell \) in Theorem 3.11 equals either 1 or \( n \). For the case where the system is reducible over \( k \), we can refine Theorem 3.11 further in the following

**Proposition 3.12** Assume that \( n \) is a prime number. Suppose that the system \( \{ \sigma(Y) = AY, \delta(Y) = BY \} \) with \( A \in GL_n(k_0) \) and \( B \in g\ell_n(k_0) \) is irreducible over \( k_0 \) and reducible over \( k \) and that its Galois group of is solvable by finite. Then the system is equivalent over \( k \) to

\[
\begin{align*}
\sigma(Y) &= \alpha(x) \text{diag}(\beta_1(t), \beta_2(t), \cdots, \beta_n(t)) Y, \\
\delta(Y) &= \text{diag} \left( \frac{\delta(\beta_1(t))}{\beta_1(t)} x + c_1, \cdots, \frac{\delta(\beta_n(t))}{\beta_n(t)} x + c_n \right) Y
\end{align*}
\]

where \( \alpha(x) \in C(x) \) is standard with respect to \( \sigma \), \( \beta_i(t) = g_i(\beta_1(t)) \in \overline{C(t)} \) and \( c_i = g_i(c_1) \in \overline{C(t)} \) for some \( g_i \) in the Galois group of the original system over \( k_0 \).

Before discussing the other case where a difference-differential system is reducible over \( k \), let us look at the following

**Lemma 3.13** Assume that \( \sigma(Y) = AY \) with \( A \in GL_n(k_0) \) is equivalent over \( k \) to \( \sigma(Y) = \bar{A}Y \) where

\[
\bar{A} = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
\beta(t) \alpha(x) & 0 & 0 & \cdots & 0
\end{pmatrix}
\]

with \( \alpha(x) \in C(x) \) and \( \beta(t) \in \overline{C(t)} \). Then \( \beta(t) \in C(t) \).

**Proof.** There exists \( G \in GL_n(k) \) such that \( \sigma(G) \bar{A} G^{-1} = A \). Then

\[
\det(\sigma(G)) \det(\bar{A}) \det(G^{-1}) = \det(A).
\]

Since \( \det(\sigma(G)) = \sigma(\det(G)) \) and \( \det(G^{-1}) = \frac{1}{\det(G)} \), we have

\[
(-1)^{n-1} \beta(t) \alpha(x) \frac{\sigma(\det(G))}{\det(G)} = \det(A).
\]

Expand the rational functions in \( x \) in the above equation as series at \( x = \infty \). Since \( \frac{\sigma(\det(G))}{\det(G)} = 1 + \frac{1}{2} Q \) where \( Q \in \overline{C(t)}[[\frac{1}{x}]] \), one sees that \( \beta(t) \in C(t) \). \( \square \)
Proposition 3.14 Let $A, \tilde{A} \in \text{GL}_n(k_0)$. If $\sigma(Y) = AY$ and $\sigma(Y) = \tilde{A}Y$ are equivalent over $k$ then they are equivalent over $k_0$.

Proof. Suppose that there exists $G \in \text{GL}_n(k)$ such that $\sigma(G)A = \tilde{A}G$. Then there exists $\gamma(t) \in \mathbb{C}(t)$ such that $G \in \text{GL}_n(k_0(\gamma(t)))$. Let $m = [k_0(\gamma(t)) : k_0]$. Since $1, \gamma(t), \cdots, \gamma(t)^{m-1}$ is a basis of $k_0(\gamma(t))$ over $k_0$, we can write
\[ G = G_0 + G_1\gamma(t) + \cdots + G_{m-1}\gamma(t)^{m-1} \]
where $G_i \in \mathfrak{gl}_n(k_0)$. From $\sigma(G)A = \tilde{A}G$, it follows that $\sigma(G_i)A = \tilde{A}G_i$ for $i = 0, \cdots, m-1$. Let $\lambda$ be a parameter satisfying $\sigma(\lambda) = \lambda$ and let $H(\lambda) = \sum_{i=0}^{m-1}\lambda^iG_i$. Therefore, $\sigma(H(\lambda))A = \tilde{A}H(\lambda)$. Since $\det(G) = \det(H(\gamma(t))) \neq 0$, $\det(H(\lambda))$ is a nonzero polynomial with coefficients in $k_0$. Hence there exists $c \in \mathbb{C}(t)$ such that $\det(H(c)) \neq 0$. So $\sigma(H(c))A = \tilde{A}H(c)$ and $H(c) \in \text{GL}_n(k_0)$. □

We now turn to the case where a difference-differential system over $k_0$ is irreducible over $k$.

Proposition 3.15 Suppose that $\{\sigma(Y) = AY, \delta(Y) = BY\}$ with $A \in \text{GL}_n(k_0)$ and $B \in \mathfrak{gl}_n(k_0)$ is irreducible over $k$ and that its Galois group over $k_0$ is solvable by finite. Then the system is equivalent over $k_0$ to
\[ \sigma(Y) = \tilde{A}Y, \quad \delta(Y) = \tilde{B}Y \]
where $\tilde{B} \in \mathfrak{gl}_n(k_0)$ and
\[ \tilde{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \beta(t)\alpha(x) & 0 & 0 & \cdots & 0 \end{pmatrix} \in \text{GL}_n(k_0) \]
with $\alpha(x) \in \mathbb{C}(x)$ standard with respect to $\sigma^n$ and $\beta(t) \in \mathbb{C}(t)$. Moreover, $\frac{\alpha(x+1)}{\alpha(x)} \neq \frac{\sigma^n(b)}{b}$ for any $b \in \mathbb{C}(x)$.

Proof. By Proposition 3.11 the given system is equivalent over $k$ to the system $\{\sigma(Y) = \tilde{A}Y, \delta(Y) = \tilde{B}Y\}$ where $\tilde{B} \in \mathfrak{gl}_n(k)$ and
\[ \tilde{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a & 0 & 0 & \cdots & 0 \end{pmatrix} \in \text{GL}_n(k) \]
for some \( a \in k \). Since \( \sigma \) and \( \delta \) commute, we have \( \sigma(B)\tilde{A} = \delta(\tilde{A}) + \tilde{A}B \).

Let \( \tilde{B} = (\tilde{b}_{ij})_{n \times n} \) where \( \tilde{b}_{ij} \in k \). Then

\[
\sigma(\tilde{b}_{nn}) - \tilde{b}_{11} = \frac{\delta(a)}{a}, \quad \tilde{b}_{nn} = \sigma(\tilde{b}_{n-1,n-1}), \quad \cdots, \quad \tilde{b}_{22} = \sigma(\tilde{b}_{11}).
\]

Hence \( \sigma^n(\tilde{b}_{11}) - \tilde{b}_{11} = \frac{\delta(a)}{a} \). By Proposition 3.10 we have \( a = \frac{\sigma^n(f)}{f} \alpha(x) \beta(t) \) with \( f \in k \), \( \alpha(x) \in \mathbb{C}(x) \) standard with respect to \( \sigma^n \) and \( \beta(t) \in \overline{\mathbb{C}(t)} \).

Then \( \{\sigma(Y) = \tilde{A}Y, \delta(Y) = \tilde{B}Y\} \) is equivalent over \( k \) to \( \{\sigma(Y) = \tilde{A}Y, \delta(Y) = \tilde{B}Y\} \) under the transformation \( Y \rightarrow \text{diag}(f, \sigma(f), \cdots, \sigma^{n-1}(f))Y \), where \( \tilde{B} \in \text{gl}_n(k_0) \) and

\[
\tilde{A} = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
\beta(t)\alpha(x) & 0 & 0 & \cdots & 0
\end{pmatrix}
\]

with \( \alpha(x) \in \mathbb{C}(x) \) and \( \beta(t) \in \overline{\mathbb{C}(t)} \). By Lemma 3.13 and Proposition 3.14 the original system is equivalent over \( k_0 \) to \( \{\sigma(Y) = \tilde{A}Y, \delta(Y) = \tilde{B}Y\} \) with \( \beta(t) \in \mathbb{C}(t) \). Assume that \( \frac{\alpha(x+1)}{\alpha(x)} = \frac{\sigma^n(b)}{b} \) for some \( b \in \mathbb{C}(x) \) and let \( u = \sigma^{-1}(b) \cdots \sigma(b)b \). We have \( \frac{\alpha(x+1)}{\alpha(x)} = \frac{\alpha(x)}{u(x)} \) thus \( \alpha(x) = cu(x) \) for some constant \( c \) with respect to \( \sigma \). Therefore \( c \in \mathbb{C} \) since \( \alpha(x) \) and \( u(x) \) are both in \( \mathbb{C}(x) \). Let \( P \in \text{GL}_n(\overline{\mathbb{C}(t)}) \) be such that

\[
P^{-1} \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
c\beta(t) & 0 & 0 & \cdots & 0
\end{pmatrix} = \text{diag}(\tilde{\beta}_1(t), \tilde{\beta}_2(t), \cdots, \tilde{\beta}_n(t))
\]

where the \( \tilde{\beta}_i(t) \)'s are the roots of \( Y^n - c\beta(t) \). Let

\[
F = \text{diag}(1, b, \sigma(b)b, \cdots, \sigma^{n-2}(b) \cdots \sigma(b)b)b)
\]

and \( \tilde{B} = F^{-1}\tilde{B}F - F^{-1}\delta(F) \). Then \( \{\sigma(Y) = \tilde{A}Y, \delta(Y) = \tilde{B}Y\} \) is equivalent over \( k \) to

\[
\sigma(Y) = b \cdot \text{diag}(\tilde{\beta}_1(t), \tilde{\beta}_2(t), \cdots, \tilde{\beta}_n(t))Y, \quad \delta(Y) = \tilde{B}Y
\]

under the transformation \( Y \rightarrow FY \). Assume \( \tilde{B} = (\tilde{b}_{ij})_{n \times n} \). Since \( \sigma \) and \( \delta \) commute,

\[
\sigma(\tilde{b}_{ij}) - \frac{\tilde{\beta}_j(t)}{\beta_j(t)}\tilde{b}_{ij} = 0,
\]

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for all \( i, j \) with \( 1 \leq i \neq j \leq n \). Hence \( \bar{b}_{ij} = 0 \) if \( i \neq j \). In other word, \( \bar{B} \) is of diagonal form. This contradicts to the irreducibility over \( k \) of the original system. \( \square \)

**Lemma 3.16** Let \( a \in k_0 \setminus \{0\} \), \( n \) be a positive integer and \( m > 0 \) be the least integer such that \( \frac{\sigma^m(a)}{a} = \frac{\sigma^n(b)}{b} \) for some \( b \in k_0 \). Then \( m|n \).

**Proof.** Suppose that \( \frac{\sigma^m(a)}{a} = \frac{\sigma^n(b)}{b} \) with \( b \in k_0 \). Then for each \( \ell > 0 \),

\[
\frac{\sigma^{\ell m}(a)}{a} = \frac{\sigma^{\ell n}(c_\ell)}{c_\ell}
\]

with \( c_\ell \in k_0 \).

Let \( n = \ell_1 m + \ell_2 \) where \( 0 \leq \ell_2 \leq m - 1 \). Then

\[
\sigma^{\ell_2}(a) = \sigma^{\ell_1 m + \ell_2}(a) \sigma^{\ell_2}(a) = \frac{\sigma^n(a)}{a} c
\]

for some \( c \in k_0 \). Hence \( \ell_2 = 0 \) and so \( m|n \). \( \square \)

**Proposition 3.17** Assume that \( n \) is a prime number, the system

\[
\sigma(Y) = A Y, \quad \delta(Y) = B Y
\]

with \( A \in \text{GL}_n(k_0) \) and \( B \in \text{gl}_n(k_0) \) is irreducible over \( k \) and its Galois group over \( k_0 \) is solvable by finite. Then \( \{ \sigma^n(Y) = A_n Y, \delta(Y) = B Y \} \) is equivalent over \( k_0 \) to

\[
\begin{align*}
\sigma^n(Y) &= \beta(t) \text{diag}(\alpha(x), \ldots, \alpha(x+n-1))Y \\
\delta(Y) &= \left( \frac{\delta(\beta(t))}{m\beta(t)} x I_n + \text{diag}(\bar{b}_1, \ldots, \bar{b}_n) \right) Y
\end{align*}
\]

where \( \alpha(x) \) and \( \beta(t) \) are as in Proposition 3.15 and \( \bar{b}_i \in \mathbb{C}(t) \) for \( i = 1, \ldots, n \).

**Proof.** By Proposition 3.15 \( \{ \sigma^n(Y) = A_n Y, \delta(Y) = B Y \} \) is equivalent over \( k_0 \) to the system

\[
\sigma^n(Y) = \beta(t) \cdot \text{diag}(\alpha(x), \ldots, \alpha(x+n-1))Y, \quad \delta(Y) = B Y
\]

with \( \alpha(x) \) and \( \beta(t) \) as in Proposition 3.15 and \( \bar{B} \in \text{gl}_n(k_0) \). Let \( \bar{B}=(\bar{b}_{ij})_{n \times n} \). From \( \sigma^n \delta = \delta \sigma^n \), we have

\[
\begin{align*}
\sigma^n(\bar{b}_{ii}) - \bar{b}_{ii} &= \frac{\delta(\beta(t))}{\beta(t)}, \quad i = 1, \ldots, n, \\
\sigma^n(\bar{b}_{ij}) - \frac{\alpha(x+i)}{\alpha(x+j)} \bar{b}_{ij} &= 0, \quad 1 \leq i \neq j \leq n.
\end{align*}
\]

Hence \( \bar{b}_{ii} = \frac{\delta(\beta(t))}{m\beta(t)} x + \bar{b}_i \) with \( \bar{b}_i \in \mathbb{C}(t) \). Note that \( n \) is prime and \( \alpha(x+i) \neq \alpha(x) \) for any \( b \in \mathbb{C}(x) \). Then by Lemma 3.16 \( \frac{\alpha(x+i)}{\alpha(x)} \neq \frac{\sigma^n(b)}{b} \) for any \( 1 \leq i \leq n-1 \) and \( b \in \mathbb{C}(x) \). Hence \( \bar{b}_{ij} = 0 \) for \( i \neq j \). This concludes the proposition. \( \square \)
3.3 A Decision Procedure for Systems of Prime Order

Consider a system \( \{\sigma(Y) = AY, \delta(Y) = BY\} \) over \( k_0 \). Assume that the order \( n \) is prime, the system is irreducible over \( k_0 \) and its Galois group is solvable by finite (or, equivalently, the system has Liouvillian solutions). By Proposition 3.12 and Proposition 3.17 either the original system has hypergeometric solutions over \( k_0 \) or the system \( \{\sigma^n(Y) = AY, \delta(Y) = BY\} \) has solutions which are the interlacing of hypergeometric solutions over \( k_0 \). In this section, we will give a decision procedure to find solutions of systems of both forms when the order \( n \) is prime. Our procedure relies on the following three facts in the ordinary cases:

(A1) we can compute all rational solutions in \( k_0^n \) of an ordinary difference equation \( \sigma(Y) = AY \) where \( A \in GL_n(k) \); (1, 2, 3, 14);

(A2) we can compute all hypergeometric solutions over \( \mathbb{C}(x) \) of an ordinary difference equation \( \sigma(Y) = AY \) where \( A \in GL_n(\mathbb{C}(x)) \) (13, 21, 22, 23, 24, 15);

(A3) we can compute all hyperexponential solutions over \( \mathbb{C}(t) \) of an ordinary differential equation \( \delta(Y) = BY \) where \( B \in GL_n(\mathbb{C}(t)) \) (19, 21, 22, 27, 16).

In the following subsections, we will reduce the problem of finding solutions of \( \{\sigma(Y) = AY, \delta(Y) = BY\} \) or of \( \{\sigma^n(Y) = AY, \delta(Y) = BY\} \) to that in the ordinary cases as indicated above. We have two case distinctions according to the reducibility of \( \{\sigma(Y) = AY, \delta(Y) = BY\} \) over \( k \).

3.3.1 The Decision Procedure for the Reducible Case

Assume that \( \{\sigma(Y) = AY, \delta(Y) = BY\} \) is reducible over \( k \). Proposition 3.12 implies that this system has hypergeometric solutions of the form \( W_i h_i \) for \( i = 1, \ldots, n \), where \( W_i \in k_0^n \) and \( h_i \) satisfies

\[
\sigma(h_i) = \alpha(x)\beta_i(t)h_i \quad \text{and} \quad \delta(h_i) = \left( \frac{\delta(\beta_i(t))}{\beta_i(t)}x + c_i \right) h_i
\]

with \( \alpha(x) \in \mathbb{C}(x) \) standard with respect to \( \sigma \), \( \beta_i(t) = g_i(\beta_1(t)) \in \mathbb{C}(t) \) and \( c_i = g_i(c_1) \in \mathbb{C}(t) \) for some \( g_i \) in the Galois group of the original system over \( k_0 \). Substituting each \( W_i h \) into the original system, we get

\[
\sigma(W_i) = \frac{A}{\alpha(x)\beta_i(t)}W_i \quad \text{and} \quad \delta(W_i) = \left( B - \frac{\delta(\beta_i(t))}{\beta_i(t)}x - c_i \right) W_i. \tag{7}
\]
So, to compute hypergeometric solutions of \( \{ \sigma(Y) = AY, \delta(Y) = BY \} \) it suffices to find \( \alpha(x), \beta_i(t), c_i \) and \( W_i \) satisfying (7).

**Remark 3.18** The equalities (7) still hold when replacing \( \alpha(x) \) by \( \frac{\sigma(g)\alpha(x)}{g} \) and \( W_i \) by \( \frac{W_i}{g} \) for \( g \in \mathbb{C}(x) \). So in the sequel, we will compute a suitable \( \frac{\sigma(g)\alpha(x)}{g} \) instead of \( \alpha(x) \).

**Computing** \( \alpha(x) \): By Proposition 3.12 there exists \( G \in \text{GL}_n(k) \) such that

\[
\sigma(\det(G)) \frac{\alpha(x)^n}{\det(G)} \prod_{i=1}^n \beta_i(t) = \det(A).
\]

Without loss of generality, we assume that the numerator and denominator of \( \alpha(x) \) are monic. Expanding the functions in the above equality as series at \( x = \infty \), one can compute \( \prod_{i=1}^n \beta_i(t) \) from the series expansion of \( \det(A) \) at \( x = \infty \). Let \( \tilde{a} = \frac{\det(A)}{\prod_{i=1}^n \beta_i(t)} \). Rewrite \( \tilde{a} = \frac{\sigma(b)}{b} \tilde{a} \) where \( b, \tilde{a} \in k_0 \) and \( \tilde{a} \) is standard with respect to \( \sigma \). Then

\[
\tilde{a} = \frac{\sigma(g)}{g} \alpha(x)^n \quad \text{for some } g \in k_0.
\]

From Proposition 3.12 \( \alpha(x) \) is standard with respect to \( \sigma \) and so is \( \alpha(x)^n \). Proposition 3.20 below shows that \( \frac{\sigma(g)}{g} \in \mathbb{C}(x) \) and thus \( \tilde{a} \in \mathbb{C}(x) \). Moreover, \( \tilde{a} \) has the form \( \left( \frac{\sigma(g)}{g} \alpha(x) \right)^n \) for some \( g \in \mathbb{C}(x) \). To prove Proposition 3.20 let us introduce a notation used in [25, Section 2.1].

**Definition 3.19** A divisor \( D \) on \( \mathbb{P}^1(\mathbb{C}(t)) \) is defined to be a finite formal expression \( \sum n_p[p] \) with \( p \in \mathbb{P}^1(\mathbb{C}(t)) \) and \( n_p \in \mathbb{Z} \). The support of a divisor \( D \), denoted \( \text{supp}(D) \), is the finite set of all \( p \) with \( n_p \neq 0 \). Let \( p \in \text{supp}(D) \). The \( \mathbb{Z} \)-orbit \( E \) of \( p \) in \( \text{supp}(D) \) is defined to be

\[
E(p, \text{supp}(D)) = \{ p + i \mid i \in \mathbb{Z} \text{ and } p + i \in \text{supp}(D) \}.
\]

As usual, the divisor \( \text{div}(f) \) of a rational function \( f \in k \setminus \{0\} \) is given by \( \text{div}(f) = \sum \text{ord}_p(f)[p] \), where \( \text{ord}_p(f) \) denotes the order of \( f \) at the point \( p \). It is clear that \( \text{div}(fg) = \text{div}(f) + \text{div}(g) \). Moreover, if \( p \) is in \( \text{supp}(\text{div}(f)) \) but not in \( \text{supp}(\text{div}(fg)) \), then \( p \in \text{supp}(\text{div}(f)) \cap \text{supp}(\text{div}(g)) \). By Definition 3.19 if \( f \in k \setminus \{0\} \) is standard with respect to \( \sigma \), then \( E(p, \text{supp}(\text{div}(f))) = \{ p \} \) for each \( p \in \text{supp}(\text{div}(f)) \).
Proposition 3.20 Assume that \( f, g \in k \setminus \{0\} \) and \( f \) is standard with respect to \( \sigma \). If \( \sigma(g)g^{-1}f \) is standard with respect to \( \sigma \), then

\[
\frac{\sigma(g)}{g} = \prod_i (x + k_i - c_i)^{m_i} \quad (x - c_i)^{m_i}
\]

with \( k_i \in \mathbb{Z}, m_i \in \mathbb{Z}_{>0}, c_i \in \mathbb{C}(t) \) and \( \text{disp}(\prod_i (x - c_i)) = 0 \). Moreover, for each \( i \), either \( \text{ord}_{c_i}(f) = m_i \) or \( \text{ord}_{c_i}(f) = -m_i \).

Proof. Let \( H = \sigma(g)g^{-1}f \), \( S_1 = \text{supp}(\text{div}(f)) \), \( S_2 = \text{supp}(\text{div}(\sigma(g)g^{-1})) \) and \( S_3 = \text{supp}(\text{div}(H)) \). By Lemma 2.1 in [25],

\[
\sum_{q \in E(p, S_2)} \text{ord}_q \left( \frac{\sigma(g)}{g} \right) = 0 \quad \text{for each } p \in S_2.
\]

Then \( |E(p, S_2)| \geq 2 \) for each \( p \in \text{supp}(S_2) \). Since \( H \) and \( f \) are standard, \( |E(p, S_2) \cap S_3| \leq 1 \) and \( |E(p, S_2) \cap S_1| \leq 1 \)

thus \( |E(p, S_2) \cap (S_1 \cup S_3)| \leq 2 \). From \( S_2 \subseteq S_1 \cup S_3 \), we have \( |E(p, S_2)| \leq 2 \). Hence for each \( p \in S_2 \),

\[
|E(p, S_2)| = 2, \quad |E(p, S_2) \cap S_1| = 1 \quad \text{and } |E(p, S_2) \cap S_3| = 1.
\]

From \( |E(p, S_2)| = 2 \) and \( |E(p, S_2) \cap S_3| = 1 \), either \( \text{ord}_p(\sigma(g)g^{-1}) = -\text{ord}_p(f) \) or \( \text{ord}_{p+j}(\sigma(g)g^{-1}) = -\text{ord}_{p+j}(f) \) with \( p + j_0 \in E(p, S_2) \). The proposition holds. \( \square \)

Let \( g \) be as in (8). Since \( \alpha(x) \in \mathbb{C}(x) \), \( g \) can be chosen in \( \mathbb{C}(x) \) according to Proposition 3.20. Then \( \bar{a} \in \mathbb{C}(x) \). Moreover,

\[
\frac{\sigma(g)}{g} = \prod_i (x + k_i - c_i)^{m_i} \quad (x - c_i)^{m_i}
\]

where \( m_i \) has the form \( \bar{m}_i n \) for some \( \bar{m}_i \in \mathbb{Z}_{>0} \) since \( m_i \) is either \( \text{ord}_{c_i}(\alpha(x)^n) \) or \( -\text{ord}_{c_i-k_i}(\alpha(x)^n) \). Let \( \bar{g} = \prod_i \prod_{j=0}^{k_i-1} (x + j - c_i)^{\bar{m}_i} \). Then

\[
\left( \frac{\sigma(\bar{g})}{\bar{g}} \right)^n = \frac{\sigma(g)}{g} \quad \text{and} \quad \bar{a} = \left( \frac{\sigma(\bar{g})}{\bar{g}} \alpha(x) \right)^n.
\]

Note that the numerator and the denominator of \( \alpha(x) \) are monic, so we can compute \( \frac{\sigma(\bar{g})}{\bar{g}} \alpha(x) \) from \( \bar{a} \).
Example 3.21 Consider the integrable system

\[ \sigma(Y) = AY, \quad \delta(Y) = BY \]

where

\[ A = \begin{pmatrix} \frac{(t^2+1)(2t^2-1-x)}{t^2-x-1} & -\frac{x^2+1}{t^2-x-1} \\ \frac{(x^2+1)(t^4+t^2-x^2-2x)}{t^2-x-1} & \frac{(x^2+1)(2t^2-x)}{t^2-x-1} \end{pmatrix}, \]

\[ B = \begin{pmatrix} -\frac{2xt^3-t^4-x^3+2t^2+2xt^2+x}{(t^2-x)(t^2+1)} & -\frac{t^2}{(t^2-x)(t^2+1)} \\ -\frac{t^2^2-t^4-x^2+2t^2-x^2+2t^3-x^2+1}{(t^2-x)(t^2+1)} & \frac{-x^2t^2-x^2+2t^2-x^2+t^2+1}{(t^2-x)(t^2+1)} \end{pmatrix}. \]

We have

\[ \det(A) = -\frac{(x^2+1)^2(t^4-t^2x+t^2-x)}{t^2-x-1} = -(t^2+1)x^4+(t^2+1)x^3+\cdots. \]

Thus \( \beta_1(t)\beta_2(t) = -(t^2+1). \) Let \( \tilde{a} = \frac{\det(A)}{t^2+1} \) and write

\[ \tilde{a} = \frac{t^2-x}{t^2-(x+1)}(x^2+1)^2. \]

Then \( \alpha(x) = x^2+1. \)

Computing \( \beta_i(t) \): We first prove the following

Lemma 3.22 Either \( \beta_1(t) = \cdots = \beta_n(t) \in \mathbb{C}(t) \) or \( \beta_1(t), \ldots, \beta_n(t) \) are the conjugate roots of an irreducible polynomial of degree \( n \) with coefficients in \( \mathbb{C}(t) \).

Proof. Let \( R_0 \) be a \( \sigma\delta \)-PV extension of \( k_0 \) for \( \{\sigma(Y) = AY, \delta(Y) = BY\} \) and \( P = \prod_{i=1}^n (X - \beta_i(t)) \). From the proof of Theorem 2.32 one sees that \( \text{Gal}(R_0/k_0) \) permutes the \( \beta_i(t) \). Furthermore, the orbits of the \( \beta_i(t) \) under this group action all have the same size. Therefore, \( P \) is a polynomial with coefficients in \( \mathbb{C}(t) \). Since \( n \) is prime, either \( P \) is irreducible or all the factors of \( P \) in \( \mathbb{C}(t)[X] \) are of degree one. This concludes the lemma.

The following two notions can be found in [1] [5].

Definition 3.23 Let \( H = (h_{ij})_{n \times n} \in \text{GL}_n(k_0) \). The order of \( H \) at \( \infty \) is defined as

\[ \text{ord}_\infty(H) = \min \{\text{ord}_\infty(h_{ij})\} \]

where \( \text{ord}_\infty(h_{ij}) \) is the order of \( h_{ij} \) at \( \infty \).
We rewrite $H$ into the form

$$H = \left(\frac{1}{x}\right)^{\text{ord}_\infty(H)} \left(H_0 + H_1 \frac{1}{x} + \cdots\right)$$

where $H_i \in \mathfrak{gl}_n(\mathbb{C}(t))$ and $H_0 \neq 0$.

**Definition 3.24** The rational number

$$m(H) = -\text{ord}_\infty(H) + \frac{\text{rank}(H_0)}{n}$$

is called the first Moser order of $H$. And

$$\mu(H) = \min\{m(\sigma(G)HG^{-1})|G \in \text{GL}_n(k)\}$$

is called the Moser invariant of $H$. A matrix $H$ is said to be irreducible if $m(H) = \mu(H)$, otherwise it is called reducible.

Given $H \in \text{GL}_n(k_0)$, one can use the algorithm in [4, 5] to compute $G \in \text{GL}_n(k_0)$ such that $\tilde{H} := \sigma(G)HG^{-1}$ is irreducible. So we can assume that $A\alpha(x)$ is irreducible where $A$ and $\alpha(x)$ are as in (7). Under this assumption, we will show that $A\alpha(x) = \tilde{A}_0 + \tilde{A}_1 \frac{1}{x} + \cdots$

with $\tilde{A}_i \in \mathfrak{gl}_n(\mathbb{C}(t))$ for each $i$ and that all the $\beta_i(t)$'s are eigenvalues of $\tilde{A}_0$. The following lemma can be deduced from the results of Barkatou in [5]. We will present a self contained proof due to Reinhart Shaefke.

**Lemma 3.25** Let $G \in \text{GL}_n(k)$ and assume that $\text{ord}_\infty(\sigma(G^{-1})G) = 0$. Then all the eigenvalues of $\sigma(G^{-1})G_{x=\infty}$ are 1.

**Proof.** Let $H = \sigma(G^{-1})G$. Then $H = H_0 + H_1 \frac{1}{x} + \cdots$ with $H_i \in \mathfrak{gl}_n(\mathbb{C}(t))$ and $H_0 \neq 0$. We now show that $H_0 - I_n$ is nilpotent. For a positive integer $m$, consider a map $L_m : \mathfrak{gl}_n(k) \to \mathfrak{gl}_n(k)$ given by $U \mapsto \sigma(U) - \sigma^m(H)U$ for any $U \in \mathfrak{gl}_n(k)$. Set $P_m = L_m \circ L_{m-1} \circ \cdots \circ L_0(I_n)$ where $\circ$ denotes the composition of maps. Then $P_m|_{x=\infty} = (I_n - H_0)^{m+1}$. On the other hand, $L_m(\sigma^m(G^{-1})V) = \sigma^{m+1}(G^{-1})\Delta(V)$ where $\Delta = \sigma - 1$ is a difference operator and $V \in \mathfrak{gl}_n(k)$. Hence $P_m = \sigma^{m+1}(G^{-1})\Delta^{m+1}(G)$. Note that when $m$ increases, $\text{ord}_\infty(\Delta^{m+1}(G))$ increases but $\text{ord}_\infty(\sigma^{m+1}(G^{-1}))$ is invariant. Then for a sufficiently large $m$, $P_m|_{x=\infty} = 0$. This concludes the lemma. □
Now we can prove the following

**Proposition 3.26** \( \text{ord}_\infty \left( \frac{A}{\alpha(x)} \right) = 0 \) and \( \beta_1(t), \ldots, \beta_n(t) \) are eigenvalues of \( \frac{A}{\alpha(x)} \big|_{x=\infty} \).

**Proof.** By Proposition 3.12, there exists \( G \in \text{GL}_n(k) \) such that
\[
\sigma(G) \frac{A}{\alpha(x)} G^{-1} = \text{diag}(\beta_1(t), \ldots, \beta_n(t)).
\]
This implies that \( \text{ord}_\infty \left( \det \left( \frac{A}{\alpha(x)} \right) \right) = 0 \) and \( m \left( \frac{A}{\alpha(x)} \right) = \mu \left( \frac{A}{\alpha(x)} \right) \leq 1 \). By the property of orders,
\[
\text{ord}_\infty \left( \frac{A}{\alpha(x)} \right) \leq \frac{1}{n} \text{ord}_\infty \left( \det \left( \frac{A}{\alpha(x)} \right) \right) = 0.
\]
Since \( m \left( \frac{A}{\alpha(x)} \right) \leq 1 \), \( \text{ord}_\infty \left( \frac{A}{\alpha(x)} \right) = 0 \) by the definition of the first Moser orders. Therefore,
\[
\frac{A}{\alpha(x)} = \tilde{A}_0 + \tilde{A}_1 \frac{1}{x} + \ldots
\]
where \( \tilde{A}_i \in \mathfrak{gl}_n(\mathbb{C}(t)) \) and \( \tilde{A}_0 \neq 0 \). From (7), \( \sigma(Y) = \frac{A}{\alpha(x)} \delta(t) Y \) has a rational solution \( W_i \) in \( k^n \). Suppose that
\[
W_i = \left( \frac{1}{x} \right)^{\text{ord}_\infty(W_i)} \left( W_{i0} + \frac{1}{x} W_{i1} + \cdots \right)
\]
where \( W_{ij} \in \mathfrak{gl}_n(\mathbb{C}(t)) \) and \( W_{i0} \neq 0 \). Then \( W_{i0} = \tilde{A}_0 \frac{1}{\beta_i(t)} W_{i0} \). Since \( W_{i0} \neq 0 \), we have \( \det \left( I_n - \tilde{A}_0 \frac{\tilde{A}_0}{\beta_i(t)} \right) = 0 \). Hence all the \( \beta_i(t) \) are the eigenvalues of \( \tilde{A}_0 \). If the \( \beta_i(t) \) are the conjugate roots of some irreducible polynomial with degree \( n \), then they are clearly eigenvalues of \( \tilde{A}_0 \). Thus by Lemma 3.22 we only need to consider the case \( \beta_1(t) = \cdots = \beta_n(t) \in \mathbb{C}(x) \). In this case, \( \frac{A}{\alpha(x)} = \beta_1(t) \sigma(G^{-1})G \). Since \( \text{ord}_\infty \left( \frac{A}{\alpha(x)} \right) = 0 \), we have \( \text{ord}_\infty(\sigma(G^{-1})G) = 0 \). By Lemma 3.25 all the eigenvalues of \( \sigma(G^{-1})G \big|_{x=\infty} \) equal 1. Hence all the eigenvalues of \( \tilde{A}_0 \) equal \( \beta_1(t) \).

**Example 3.27 (Continued)** Let \( \tilde{A} = \frac{A}{x^2 + 1} \). From the process in (3), we can find an irreducible matrix \( \tilde{A} \) which is equivalent to \( A \) where
\[
\tilde{A} = \begin{pmatrix}
\frac{(x+1)(t^2 + 1 - x)}{(t^2 - x - 1)x} & \frac{x+1}{t^2 - x - 1} \\
\frac{t^4 + t^2 - x^2 - x}{(t^2 - x - 1)x} & \frac{t(t^2 - x)}{t^2 - x - 1}
\end{pmatrix}.
\]
Write \( \tilde{A} = \tilde{A}_0 + \tilde{A}_1 + \cdots \) where
\[
\tilde{A}_0 = \begin{pmatrix} -t & 1 \\ 1 & t \end{pmatrix} \quad \text{and} \quad \tilde{A}_1 = \begin{pmatrix} t^2 & t \\ t & -t \end{pmatrix}.
\]
The eigenvalues of \( \tilde{A}_0 \) are \( \pm \sqrt{t^2 + 1} \). So \( \beta_1(t) = \sqrt{t^2 + 1} \) and \( \beta_2(t) = -\sqrt{t^2 + 1} \).

Computing \( c_i \) and \( W_i \): Let \( \Lambda(t) = \text{diag}(\beta_1(t), \ldots, \beta_n(t)) \). From \( (A_1) \) we can find a matrix \( G \in \text{GL}_n(k) \) such that \( \sigma(G) \alpha(x) \Lambda(t) = AG \). Let \( \tilde{B} = G^{-1}BG - G^{-1} \delta(G) \). Then \( \tilde{B} \in \text{gl}_n(k) \) and the system \( \{ \sigma(Y) = AY, \delta(Y) = BY \} \) is equivalent over \( k \) to
\[
\sigma(Y) = \alpha(x) \Lambda(t) Y, \quad \delta(Y) = \tilde{B}Y.
\] (9)

Note that \( G \) may not be the required transformation matrix in Proposition 6.12, so \( \tilde{B} \) may not be of diagonal form. Since \( \sigma \delta = \delta \sigma \), the same argument as in the proof of Proposition 6.15 implies the following conclusions:

(i) If \( \beta_i(t) \neq \beta_j(t) \) for all \( i, j \) with \( 1 \leq i \neq j \leq n \), then
\[
\tilde{B} = \text{diag} \left( \frac{\delta(\beta_1(t))}{\beta_1(t)} x + c_1, \ldots, \frac{\delta(\beta_n(t))}{\beta_n(t)} x + c_n \right)
\]
with \( c_i \in \mathbb{C}(t) \);

(ii) If \( \beta_1(t) = \cdots = \beta_n(t) \in \mathbb{C}(t) \), then by Proposition 6.14, \( G \) can be chosen in \( \text{GL}_n(k_0) \). Thus
\[
\tilde{B} = \hat{B} + \frac{\delta(\beta_1(t))}{\beta_1(t)} x I_n
\]
with \( \hat{B} \in \text{gl}_n(\mathbb{C}(t)) \).

In the case (i), we obtain the \( c_i \)'s, and the \( W_i \)'s are just the columns of \( G \). For the case (ii), since \( \delta(Y) = \tilde{B}Y \) is equivalent over \( k \) to \( \delta(Y) = \hat{B}Y \), there exists \( \hat{G} \in \text{GL}_n(k) \) such that
\[
\left\{ \begin{array}{l}
\sigma(\hat{G}) = \hat{G}, \\
\delta(\hat{G}) + \hat{G} \left( \frac{\delta(\beta_1(t))}{\beta_1(t)} x I_n + \text{diag}(c_1, \cdots, c_n) \right) = \left( \hat{B} + \frac{\delta(\beta_1(t))}{\beta_1(t)} x I_n \right) \hat{G}.
\end{array} \right.
\]
Hence \( \hat{G} \in \text{GL}_n(\mathbb{C}(t)) \) and \( \delta(Y) = \hat{B}Y \) is equivalent over \( \mathbb{C}(t) \) to
\[
\delta(Y) = \text{diag}(c_1, \cdots, c_n) Y.
\]
Solving the system \( \delta(Y) = \hat{B}Y \) by \( (A_3) \), we get the \( c_i \)'s and the \( W_i \) are just the columns of \( \hat{G}G \).
Example 3.28 (Continued) Let \( \Lambda(t) = \text{diag}(\sqrt{t^2 + 1}, -\sqrt{t^2 + 1}) \). From \((A_1)\), we can obtain \( G \in \text{GL}_2(k) \) such that \( \sigma(G)(x^2 + 1)\Lambda(t) = AG \) where

\[
G = \begin{pmatrix}
\frac{t-\sqrt{t^2+1}}{2(t^2-x)} & \frac{t+\sqrt{t^2+1}}{2(t^2-x)} \\
\frac{-x+t\sqrt{t^2+1}}{2(t^2-x)} & \frac{-x+t\sqrt{t^2+1}}{2(t^2-x)}
\end{pmatrix}.
\]

Then

\[
\tilde{B} = G^{-1}BG - G^{-1}\delta(G)
\]

\[
= \begin{pmatrix}
\frac{x^t}{t^2+1} + \sqrt{t^2 + 1} + 1 & 0 \\
0 & \frac{x^t}{t^2+1} - \sqrt{t^2 + 1} + 1
\end{pmatrix}.
\]

Hence \( c_1 = \sqrt{t^2 + 1} + 1, c_2 = -\sqrt{t^2 + 1} + 1 \) and \( W_i \) is the \( i \)-th column of \( G \) for \( i = 1, 2 \). Furthermore, a basis of the solution space is

\[
h(\sqrt{t^2 + 1})^e f \sqrt{t^2+1}dt, h(-\sqrt{t^2 + 1})^e f \sqrt{t^2+1}dt
\]

where \( h \) satisfies that \( \sigma(h) = (x^2 + 1)h \) and \( \delta(h) = 0 \).

### 3.3.2 The Decision Procedure for the Irreducible Case

Assume that \( \{\sigma(Y) = AY, \delta(Y) = BY\} \) with \( A \in \text{GL}_n(k_0) \) and \( B \in \text{gl}_n(k_0) \) is an irreducible system over \( k \) and its Galois group over \( k_0 \) is solvable by finite. By Proposition 3.17, the system \( \{\sigma^n(Y) = A_n Y, \delta(Y) = BY\} \) has solutions of the form \( W_i h_i \) for \( i = 1, \cdots, n \), where \( W_i \in k_0^n \) and \( h_i \) satisfies

\[
\sigma^n(h_i) = \alpha(x + i - 1)\beta(h_i), \quad \delta(h_i) = \left( \frac{\delta(\beta(t))}{n\beta(t)} x + \hat{b}_i \right) h_i
\]

with \( \alpha(x), \beta(t) \) and \( \hat{b}_i \) as in Proposition 3.17. Substituting \( Y = W_i h_i \) into \( \{\sigma^n(Y) = A_n Y, \delta(Y) = BY\} \), we have

\[
\sigma^n(W_i) = \frac{A_n}{\beta(t)\alpha(x + i - 1)} W_i \quad \text{and} \quad \delta(W_i) = \left( B - \frac{\delta(\beta(t))}{n\beta(t)} x - \hat{b}_i \right) W_i.
\]

(10)

To compute \( W_i h_i \), it suffices to compute \( \alpha(x), \beta(t), W_i \) and \( \hat{b}_i \) which satisfy (10). Without loss of generality, we assume that the numerator and denominator of \( \alpha(x) \) are monic. By Proposition 3.15, there exists \( G \in \text{GL}_n(k_0) \) such that

\[
\frac{\sigma(\det(G))}{\det(G)} (-1)^{n-1} \alpha(x) \beta(t) = \det(A).
\]
Expanding \( \det(A) \) as a series in \( \frac{1}{x} \), we get that \((-1)^{n-1}\beta(t)\) is the leading coefficient of the series. Hence we can obtain \( \beta(t) \) from \( \det(A) \). In this case, we cannot find \( \alpha(x) \) by the method used in Section 3.3.1. However, we can reduce this problem to working with difference equations over \( \mathbb{C}(x) \).

By Proposition 3.17, there exists \( G \in \text{GL}_n(k_0) \) (the same as that in Proposition 3.15) such that

\[
\sigma^n(G) \cdot \text{diag}(\alpha(x), \cdots, \alpha(x + n - 1)) = \frac{A_n}{\beta(t)} G.
\]

Assume that \( t = p \) is not a pole of the entries of \( \frac{A_n}{\beta(t)} \) and such that \( \det\left(\frac{A_n}{\beta(t)}|_{t=p}\right) \neq 0 \). Let \( \frac{A_n}{\beta(t)} = \tilde{A}_0 + (t-p)\tilde{A}_1 + \cdots \) where \( \tilde{A}_i \in \text{gl}_n(\mathbb{C}(x)) \). We will show that \( \alpha(x) \) can be found by examining the hypergeometric solutions of \( \sigma^n(Y) = \tilde{A}_0 Y \). This will follow from the next proposition.

**Proposition 3.29** Some factor of \( \sigma^n(Y) = \tilde{A}_0 Y \) is equivalent over \( \mathbb{C}(x) \) to some factor of \( \sigma^n(Y) = \text{diag}(\alpha(x), \alpha(x + 1), \cdots, \alpha(x + n - 1))Y \).

**Proof.** Let \( G \) be as above and let \( \Psi(x) = \text{diag}(\alpha(x), \cdots, \alpha(x + n - 1)) \). We may multiply \( G \) by a power of \( t-p \) and assume that \( G = \tilde{G}_0 + (t-p)\tilde{G}_1 + \cdots \) where \( \tilde{G}_0 \neq 0 \) and \( \tilde{G}_i \in \text{gl}_n(\mathbb{C}(x)) \). Then

\[
\sigma^n(\tilde{G}_0 + (t-p)\tilde{G}_1 + \cdots)\Psi(x) = (\tilde{A}_0 + \cdots)(\tilde{G}_0 + (t-p)\tilde{G}_1 + \cdots).
\]

Therefore \( \sigma^n(\tilde{G}_0)\Psi(x) = \tilde{A}_0 \tilde{G}_0 \). Let \( r = \text{rank}(\tilde{G}_0) \). Then \( r > 0 \) because \( \tilde{G}_0 \neq 0 \). There exist \( P \in \text{GL}_n(\mathbb{C}(x)) \) and \( Q \) which is a product of some permutation matrices such that

\[
\tilde{G} = P\tilde{G}_0 Q = \begin{pmatrix} 0 & 0 \\ \tilde{G}_{21} & \tilde{G}_{22} \end{pmatrix}
\]

where \( \tilde{G}_{22} \in \text{GL}_r(\mathbb{C}(x)) \). Then

\[
\sigma^n(\tilde{G})\text{diag}(\alpha(x + k_1), \cdots, \alpha(x + k_n)) = \sigma^n(P)\tilde{A}_0 P^{-1} \tilde{G}
\]

(11)

where \( k_1, \cdots, k_n \) are a permutation of \( \{0, 1, \cdots, n-1\} \). Now let

\[
\tilde{A} = \sigma^n(P)\tilde{A}_0 P^{-1} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix} \text{ where } \tilde{A}_{22} \in \text{gl}_r(\mathbb{C}(x)),
\]

and \( D_2 = \text{diag}(\alpha(x + k_{n-r+1}), \cdots, \alpha(x + k_n)) \). From (11), we have \( \tilde{A}_{12} \tilde{G}_{22} = 0 \) and \( \sigma^n(\tilde{G}_{22})D_2 = \tilde{A}_{22} \tilde{G}_{22} \). Since \( \tilde{G}_{22} \in \text{GL}_r(\mathbb{C}(x)) \), we have \( \tilde{A}_{12} = 0 \). Therefore \( \sigma^n(Z) = \tilde{A}_{22} Z \) is a factor of \( \sigma^n(Y) = \tilde{A}_0 Y \), which is equivalent over \( \mathbb{C}(x) \) to \( \sigma^n(Z) = D_2 Z \). □

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Remark 3.30 For almost all of \( p \in \mathbb{C} \), \( \sigma^n(Y) = \left. \frac{A_n}{\beta(t)} \right|_{t=p} Y \) is equivalent over \( \mathbb{C}(x) \) to

\[
\sigma^n(Y) = \text{diag}(\alpha(x), \alpha(x+1), \ldots, \alpha(x+n-1))Y.
\]

since \( G|_{t=p} \) is invertible.

The same argument as in Remark 3.18 implies that it is enough to compute \( \frac{\sigma^n(g)\alpha(x)}{g} \) for some suitable \( g \in \mathbb{C}(x) \) instead of \( \alpha(x) \). We can use Proposition 3.29 to find \( \frac{\sigma^n(g)\alpha(x+k)}{g} \) with \( k \in \mathbb{Z} \) and \( g \in \mathbb{C}(x) \) as follows.

From Theorem 3 in [7], if \((z_1, \ldots, z_r)^T\) is a solution of \( \sigma^n(Z) = A_22Z \), then \((0, \ldots, 0, z_1, \ldots, z_r)^T\) is a solution of \( \sigma^n(Y) = AY \). So \( \sigma^n(Y) = \tilde{A}Y \) has at least \( r \) solutions \( \tilde{W}_1h_1, \ldots, \tilde{W}_rh_r \), where \( \tilde{W}_i \in \mathbb{C}(x)^n \) and \( \tilde{h}_i \) satisfies \( \sigma^n(h_i) = \alpha(x+k_{n-r+1})\tilde{h}_i \). By (A2), we can find all hypergeometric solutions of \( \sigma(Z) = \tilde{A}_0(nx)Z \) where \( \tilde{A}_0(nx) \) means replacing \( x \) by \( nx \) in \( \tilde{A}_0 \). Then by interlacing, we can find all solutions of \( \sigma^n(Y) = \tilde{A}_0Y \) of the form \( \tilde{W}_jh_j \) where \( \tilde{W}_j \in \mathbb{C}(x)^n \) and \( \tilde{h}_j \) satisfies \( \sigma^n(h_j) = \tilde{a}_jh_j \) for some \( \tilde{a}_j \in \mathbb{C}(x) \). Then there exists \( \tilde{h}_{jo} \) such that \( \tilde{h}_{jo} = g\tilde{h}_1 \) for some \( g \in \mathbb{C}(x) \) and

\[
\hat{\alpha}(x+k_{n-r+1}) = \frac{\sigma^n(h_{jo})}{\tilde{h}_{jo}} = \frac{\sigma^n(g)}{g} \alpha(x+k_{n-r+1}).
\]

After finding \( \hat{\alpha}(x+k_{n-r+1}) \), we can compute a matrix \( \hat{G} \in \text{GL}_n(k_0) \) in a finite number of steps by (A1), such that

\[
\sigma^n(\hat{G}^{-1})A_n\hat{G} = \beta(t)\text{diag}(\hat{\alpha}(x), \ldots, \hat{\alpha}(x+n-1)).
\]

Let \( B = \hat{G}^{-1}B\hat{G} - \hat{G}^{-1}\delta(\hat{G}) \). Then we get a new system

\[
\sigma^n(Y) = \beta(t)\text{diag}(\hat{\alpha}(x), \ldots, \hat{\alpha}(x+n-1))Y, \quad \delta(Y) = BY
\]

which is equivalent to the original one under the transformation \( Y \to \hat{G}^{-1}Y \). Since \( \sigma^n \) and \( \delta \) commute and \( \frac{\alpha(x)}{\alpha(x+1)} \neq \frac{\alpha(b)}{\alpha(b+1)} \) for any \( b \in \mathbb{C}(x) \), the same argument as in the proof of Proposition 3.17 implies that \( B \) is of diagonal form, that is

\[
B = \text{diag} \left( \frac{\delta(\beta(t))}{n\beta(t)}x + \hat{b}_1, \ldots, \frac{\delta(\beta(t))}{n\beta(t)}x + \hat{b}_n \right).
\]

We then get the \( \hat{b}_i \), and the \( W_i \) are just the \( i \)-th columns of \( \hat{G} \).
Example 3.31 Consider an integrable system:

\[ \sigma(Y) = AY, \quad \delta(Y) = BY \]

where

\[
A = \begin{pmatrix}
\frac{x^3 t^4 + 2x^2 t^4 + x t^4 - x - 1}{t^2 + x + 1} & \frac{t^2 (x^4 + 2x^3 + tx^2 + 1)}{t^2 + x + 1} & \frac{t (t - x - 1)}{t^2 + x + 1} \\
-t (x^2 t^3 + xt^4 - 1) & \frac{-t (t^3 x^3 + t^2 x^2 - 1)}{t^2 + x + 1} & \frac{-t (1 + t)}{t^2 + x + 1} \\
\frac{t^6 x^2 + t^6 x + x + 1}{t (t^2 + x + 1)} & \frac{t (t^3 x^3 + t^3 x^2 - 1)}{t^2 + x + 1} & \frac{-t - x - 1}{t^2 + x + 1}
\end{pmatrix},
\]

\[
B = \begin{pmatrix}
\frac{t^4 (t^2 x^2 + 2x^2 + x^2 + x)}{t (t^2 + x)} & \frac{x (-t^2 + t^3 - 1)}{t^2 + x} & \frac{x t^3 (-1 + t)}{t^2 + x} \\
\frac{-t^2 + t^4 + 1}{t^2 + x} & \frac{2t^2 x^2 + x^2 + 5 t^2}{t (t^2 + x)} & \frac{t^3 (1 + t)}{t^2 + x} \\
\frac{-t^2 + t^4 + 1}{t^2 + x} & \frac{x (-t^2 + t^3 - 1)}{t (t^2 + x)} & \frac{x^2 + x t^3 + t^2 x^2}{t (t^2 + x)}
\end{pmatrix}.
\]

We have

\[
\det(A) = \frac{x t^3 (t^2 x + t^2 + x^2 + x)}{x + 1 + t^2} = \frac{(x + 1)(t^2 + x)}{x(t^2 + x + 1)} x^2 t^3.
\]

By (8), if the Galois group over \( k_0 \) of the given system is solvable by finite, then this system have no hypergeometric solutions over \( k \). Therefore we consider the system

\[ \sigma^3(Y) = A_3 Y, \quad \delta(Y) = BY \]

where

\[
A_3 = \begin{pmatrix}
\frac{t^3 (t^2 x^2 + t^2 x + 21 x + x^3 + 8 x^2 + 18)}{t^2 + x + 3} & \frac{-t^4 (x + 1)(5 x + 6)}{t^2 + x + 3} & \frac{-2 t^4 (x + 2)(x + 3)}{t^2 + x + 3} \\
\frac{-2 t^4 (2 x + 3)}{t^2 + x + 3} & \frac{(x + 1)t^3 x^2 + x^2 + 2 t^2}{t^2 + x + 3} & \frac{-2 t^4 (x + 2)}{t^2 + x + 3} \\
\frac{2 t^4 (2 x + 3)}{t^2 + x + 3} & \frac{(x + 1)t^3 (x + 5)}{t^2 + x + 3} & \frac{(x + 2)(x + 3)(x + 1)}{t^2 + x + 3}
\end{pmatrix}.
\]

We can compute \( \beta(t) = t^3 \) from \( \det(A) \). Let \( \tilde{A} = \frac{A}{t^3} \). Then

\[
\tilde{A}|_{t=0} = \begin{pmatrix}
(x + 2)(x + 3) & 0 & 0 \\
0 & \frac{(x + 1)x^2}{x + 3} & 0 \\
0 & \frac{(x + 1)(5 x + 6)}{x + 3} & (x + 1)(x + 2)
\end{pmatrix}.
\]
By \((A_2)\), all hypergeometric solutions of \(\sigma^3(Y) = \tilde{A}_1|_{t=0}Y\) are
\[
9^\frac{3}{5} \Gamma\left(\frac{\varepsilon + 2}{3}\right) \Gamma\left(\frac{x + 3}{3}\right) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad 9^\frac{3}{5} \Gamma\left(\frac{\varepsilon + 1}{3}\right) \Gamma\left(\frac{\varepsilon + 2}{3}\right) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad 9^\frac{3}{5} \Gamma\left(\frac{\varepsilon}{3}\right) \Gamma\left(\frac{\varepsilon + 1}{3}\right) \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]
where \(\Gamma(x)\) satisfies \(\Gamma(x + 1) = x\Gamma(x)\). By \((A_1)\), we can compute a rational solution of \(\sigma^3(Y) = \frac{A_3}{x(x+1)^2}Y\). Moreover, we can compute a matrix \(G \in \text{GL}_3(\mathbb{C}(x,t))\) such that
\(\sigma^3(G)\text{diag}(x(x+1)t^3, (x+1)(x+2)t^3, (x+2)(x+3)t^3) = A_3G\)
where
\[
G = \begin{pmatrix}
\frac{t}{t^2+x} & -\frac{x}{t^2+x} & \frac{x}{t^2+x} \\
\frac{t^2+x}{t^2} & \frac{t}{t^2} & -\frac{x}{t^2}
\end{pmatrix}.
\]
Let \(\tilde{B} = G^{-1}BG - G^{-1}\delta(G)\). Then \(\tilde{B} = \text{diag}\left(\frac{x}{t}, \frac{x^2}{t}, \frac{x^3}{t}\right)\). Hence a basis of solution space of \(\{\sigma^3(Y) = A_3Y, \delta(Y) = BY\}\) is
\[
V_1(x) := 9^\frac{3}{5} \Gamma\left(\frac{\varepsilon}{3}\right) \Gamma\left(\frac{x + 1}{3}\right) t^x e^{\frac{t}{t}} \begin{pmatrix} t \\ \frac{t^2+x}{t^2} \\ \frac{x}{t^2} \end{pmatrix},
\]
\[
V_2(x) := 9^\frac{3}{5} \Gamma\left(\frac{x+1}{3}\right) \Gamma\left(\frac{x+2}{3}\right) t^x e^{\frac{x}{t}} \begin{pmatrix} x \\ \frac{x}{t^2+x} \\ \frac{t^2+x}{t^2} \end{pmatrix},
\]
\[
V_3(x) := 9^\frac{3}{5} \Gamma\left(\frac{x+2}{3}\right) \Gamma\left(\frac{x+3}{3}\right) t^x e^{\frac{x}{t}} \begin{pmatrix} \frac{x}{t^2} \\ \frac{x}{t^2} \\ \frac{t^2+x}{t^2} \end{pmatrix}.
\]
Clearly, \(V_i(1) \neq 0\) for \(i = 1, 2, 3\), and \(A(j)\) and \(B(j)\) are well defined and \(\text{det}(A(j)) \neq 0\) for \(j \geq 1\). By the results in Section 2.3, we get a basis of the solution space of the original system:
\[
W_1 = 9^\frac{3}{5} \Gamma\left(\frac{x}{3}\right) \Gamma\left(\frac{x}{3}\right) t^x e^{\frac{t}{t}} \begin{pmatrix}
0, \frac{t}{t^2+x}, \frac{4t^3}{t^2+2}, -\frac{6t^3}{t^2+3}, \cdots \\
0, \frac{1}{t^2+x}, -\frac{2t^4}{t^2+2}, \frac{2t^4}{t^2+3}, \cdots \\
0, -\frac{1}{t^2+x}, \frac{2t^4}{t^2+2}, \frac{6t^2}{t^2+3}, \cdots
\end{pmatrix}.
\]
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\[ W_2 = 9^{3/2} \Gamma(\frac{2}{3}) \Gamma(1) \text{te}^{\frac{-3}{2}} \begin{pmatrix} (0, -\frac{1}{t^2+1}, \frac{t}{t^2+2}, \frac{18t^3}{t^2+3}, \cdots) \\ (0, \frac{1}{t^2+1}, \frac{t}{t^2+2}, -\frac{6t^4}{t^2+3}, \cdots) \\ (0, \frac{1}{t(t^2+1)}, -\frac{1}{t^2+2}, \frac{6t^4}{t^2+3}, \cdots) \end{pmatrix} \]

and

\[ W_3 = 9^{1/2} \Gamma(1) \Gamma(\frac{4}{3}) \text{te}^{\frac{-4}{3}} \begin{pmatrix} (0, 0, -\frac{2}{t^2+2}, \frac{t}{t^2+3}, \cdots) \\ (0, \frac{t}{t^2+1}, \frac{1}{t^2+2}, \frac{1}{t^2+3}, \cdots) \\ (0, \frac{1}{t^2+1}, \frac{2}{t(t^2+2)}, -\frac{1}{t^2+3}, \cdots) \end{pmatrix} \]

Note that all the \( W_i \) are liouvillian.

### 3.4 Summary

Consider two systems

\[ \sigma(Y) = AY, \quad \delta(Y) = BY \quad (12) \]

and

\[ \sigma^n(Y) = A_n Y, \quad \delta(Y) = BY \quad (13) \]

where \( A \in \text{GL}_n(k_0), B \in \mathfrak{gl}_n(k_0) \) and \( n \) is a prime number. Assume that \((12)\) is irreducible over \( k_0 \). From the results in Sections 3.3.1 and 3.3.2, if \((12)\) has a liouvillian solution over \( k_0 \), then either the solution space of \((12)\) has a basis consisting of hypergeometric solutions over \( k_0 \) or the solution space of \((13)\) has a basis each of whose members is the interlacing of hypergeometric vectors over \( k_0 \). Let us summarize the previous decision procedure as follows.

**Decision Procedure 1** Compute a fundamental matrix of \((12)\) whose entries are hypergeometric over \( k_0 \) if it exists.

(a) Write \( \det(A) = \sigma(g) \alpha \) where \( g, \alpha \in k_0 \) and \( \alpha \) is standard with respect to \( \sigma \). If \( a \neq \alpha(x)^n \beta(t) \) for any \( \alpha(x) \in \mathbb{C}(x) \) and \( \beta(t) \in \mathbb{C}(t) \), then by the results in Section 6.1, **exit** \((12)\) has no required fundamental matrix.

(b) Assume that \( a = \alpha(x)^n \beta(t) \) for some \( \alpha(x) \in \mathbb{C}(x) \) and \( \beta(t) \in \mathbb{C}(t) \).

By the algorithms in [4, 5], compute an irreducible matrix \( \tilde{A} \) such that \( \tilde{A} = \sigma(\tilde{G}) \tilde{G}^{-1} \) for some \( \tilde{G} \in \text{GL}_n(k_0) \). If \( \text{ord}_\infty(\tilde{A}) \neq 0 \), then by Proposition 3.26 **exit** \((12)\) has no required fundamental matrix. Otherwise, let \( \tilde{A}_0 = \tilde{A}|_{x=\infty} \) and \( \beta_1(t), \cdots, \beta_n(t) \) be the eigenvalues of \( \tilde{A}_0 \).

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(c) Goto Step $(d_1)$ if the $\beta_i(t)$ are conjugate and goto Step $(d_2)$ if $\beta_1(t) = \cdots = \beta_n(t) \in \mathbb{C}(t)$. In other cases, by Lemma 3.22 and Proposition 3.26 exit [12] has no required fundamental matrix.

$(d_1)$ If $(A_1)$ yields no rational solutions, then exit [12] has no required fundamental matrix. Otherwise, suppose that we find $G \in \text{GL}_n(k)$ such that

$$\sigma(G)\alpha(x)\text{diag}(\beta_1(t), \cdots, \beta_n(t)) = AG.$$ 

Then $\hat{B} := G^{-1}BG - G^{-1}\delta(G)$ is of diagonal form. Compute a fundamental matrix $H$ of

$$\sigma(Y) = \alpha(x)\text{diag}(\beta_1(t), \cdots, \beta_n(t))Y, \quad \delta(Y) = \hat{B}Y.$$ 

Return \([GH\text{ is a required fundamental matrix of (12)}].\)

$(d_2)$ If we can compute a matrix $G \in \text{GL}_n(k_0)$ such that $\sigma(G)\alpha(x)\beta_1(t) = AG$ then let

$$\hat{B} := G^{-1}BG - G^{-1}\frac{\delta(\beta_1(t))}{\beta_1(t)}xI_n \in \mathfrak{gl}_n(\mathbb{C}(t)),$$

else exit [12] has no required fundamental matrix]. If we can find a fundamental matrix $H$ of $\delta(Y) = \hat{B}Y$ whose entries are hyperexponential over $\mathbb{C}(t)$, then return \([GH\beta_1(t)x^h\text{ is a required fundamental matrix of (12)}]\) where $h$ satisfies $\sigma(h) = \alpha(x)h$ and $\delta(h) = 0$. Otherwise, exit [12] has no required fundamental matrix.

**Decision Procedure 2** Compute a fundamental matrix of (13) whose entries are the interlacing of hypergeometric vectors over $k_0$ if it exists.

(a) If $\det(A) \neq (-1)^{n-1}\frac{\sigma(\beta)}{g}\alpha(x)\beta(t)$ holds for any $g \in k$, $\beta(t) \in \mathbb{C}(t)$ and $\alpha(x) \in \mathbb{C}(x)$ that is standard with respect to $\sigma^n$, then exit [13] has no required fundamental matrix.

(b) Expand $\det(A)$ as a series at $x = \infty$

$$\det(A) = (-1)^{n-1}\beta(t)x^m + \beta_1(t)x^{m-1} + \cdots$$

where $\beta(t), \beta_i(t) \in \mathbb{C}(t)$ and $m \in \mathbb{Z}$. Suppose that $x = p$ is not a pole of the entries of $\frac{A}{\beta(t)}$ and that $\det(\hat{A}_0) \neq 0$ where $\hat{A}_0 = \frac{A}{\beta(t)}|_{x=p}$. Use (A2) to find all hypergeometric solutions of $\sigma(Z) = \hat{A}_0(nx)Z$. By interlacing, we get all solutions of $\sigma^n(Y) = \hat{A}_0Y$ of the form $W_ih_i$. Denote these solutions by $W_1h_1, \cdots, W_dh_d$ where $W_i \in \mathbb{C}(x)^n$ and $h_i$
satisfies $\sigma^n(h_i) = \tilde{a}_ih_i$ for some $\tilde{a}_i \in \mathbb{C}(x)$. If there is $i_0 \in \{1, \ldots, d\}$ such that $\sigma^n(Y) = \frac{h_{i_0}A}{\sigma^n(h_{i_0})}\beta(t)Y$ has a rational solution in $k_0^n$, then let $\lambda(x) = \frac{\sigma^n(h_{i_0})}{h_{i_0}}$, else exit [13] has no required fundamental matrix. Let $j_0$ be the least integer such that $\sigma^n(Y) = A\lambda(x+j_0)\beta(t)Y$ has a rational solution in $k_0^n$. If we can compute $G \in \text{GL}_n(k_0)$ such that 

$$
\sigma(Y)\beta(t)\text{diag}(\lambda(x+j_0), \ldots, \lambda(x+j_0+n-1)) = AG,
$$

then let $\tilde{B} = G^{-1}BG - G^{-1}\delta(G)$. So $\tilde{B}$ is of diagonal form and by the same process as in Step (d1) of Decision Procedure 1, we can compute a required fundamental matrix of [13]. Otherwise, by the results in Section 3.3.2 exit [13] has no required fundamental matrix.

We can decide whether [12] has liouvillian solutions or not as follows. If we can compute hypergeometric solutions over $k$ of [12] by Decision Procedure 1, then we are done. Otherwise, consider the system [13]. If we can compute liouvillian solutions over $k_0$ of [13] by Decision Procedure 2, then by the results in Section 2.3 we can compute liouvillian solutions over $k_0$ of [12] and we are done. Otherwise [12] has no liouvillian solutions.

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