Equivalence of Recurrence Relations for Feynman Integrals with the Same Total Number of External and Loop Momenta

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Abstract

We show that the problem of solving recurrence relations for $L$-loop $(R + 1)$-point Feynman integrals within the method of integration by parts is equivalent to the corresponding problem for $(L + R)$-loop vacuum or $(L + R - 1)$-loop propagator-type integrals. Using this property we solve recurrence relations for two-loop massless vertex diagrams, with arbitrary numerators and integer powers of propagators in the case when two legs are on the light cone, by reducing the problem to the well-known solution of the corresponding recurrence relations for massless three-loop propagator diagrams with specific boundary conditions.

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1 Introduction

The method of integration by parts (IBP) [1, 2] within dimensional regularization [3] is based on integrating by parts in dimensionally regularized Feynman integrals in loop momenta and putting to zero surface terms. Although this procedure has been justified only for off-shell Feynman integrals [4], no examples are known where it breaks down so that it is successfully applied to Feynman integrals on the mass shell, at threshold and to various integrals in the loop momenta that arise in asymptotic expansions of ‘standard’ Feynman integrals in limits of momenta and masses.

The idea of the method is to use equations following from IBP in order to derive recurrence relations that reduce any Feynman integral from a given family to a set of master integrals, e.g. with lowest powers of propagators. If this program is fulfilled then the evaluation of the given family of integrals reduces to algebraical manipulations.

In this paper, we show that different problems of solving IBP relations are in fact equivalent so that one can use known solutions to provide solutions for other families of Feynman integrals. We show that the problem of solving IBP relations for \(L\)-loop \((R + 1)\)-point Feynman integrals is equivalent to the corresponding problem for \((L + R)\)-loop vacuum or \((L + R - 1)\)-loop propagator-type integrals.

We apply this property to reduce solution of the IBP relations for two-loop massless vertex diagrams, with arbitrary numerators and integer powers of propagators in the case when two legs are on the light cone, to the well-known solution [1] for massless three-loop propagator-type diagrams, but with specific boundary conditions.

2 Equivalence of recurrence procedures

Our goal is to prove the equivalence (in some special sense) of the IBP relations for multi-loop Feynman integrals with the same number of independent kinematical invariants. To do this let us present (following [4], with updated notation) the IBP relations in a modified form. We start with \(L\)-loop vacuum Feynman integrals with the maximum possible number of the propagators, \(N = L(L + 1)/2\):

\[
F(n_1, \ldots, n_N; d; m^2) = \int \ldots \int \frac{d^d p_1 \ldots d^d p_L}{D_1^{n_1} \ldots D_N^{n_N}} \equiv (m^2)^{Ld/2 - \Sigma n_i} f(n_1, \ldots, n_N; d),
\]

where \(p_i (i = 1, \ldots, L)\) are loop momenta and \(D_a = A^{ij}_a p_i \cdot p_j - \mu_a m^2 (a = 1, \ldots, N; \text{summation over repeated indices is understood})\). We use dimensional regularization [3] with \(d = 4 - 2\epsilon\). The matrix \(A^{ij}_a\) is supposed to be symmetrical with respect to upper indices. This maximal number \(N\) of the propagators provides the possibility to express any scalar product of the loop momenta as a linear combination of the factors in the denominator. Feynman integrals for specific graphs are obtained for appropriate choices of the \(N \times N\) matrices \(A^{ij}_a\). Diagrams of practical interest
which usually have less number of the propagators can be considered as special cases with non-positive powers of some propagators. If the number of the propagators is greater than \( N \), partial fractioning can be used to deal with integrals with at most \( N \) propagators.

IBP relations are obtained \([1, 2]\) by acting by the operator \((\partial/\partial p_i) \cdot p_k\) on the integrand, putting the integral of the derivative to zero and expressing all the terms resulted from the differentiation through the initial family of the integrals. Starting from (1) we obtain

\[
d\delta_{ik} f(n_1, \ldots, n_N; d) = 2\tilde{A}_a^{kl}(I_a^- + \mu_a)A_b^{kl} I^+_b f(n_1, \ldots, n_N; d),
\]

where \( I_a^-(n, \ldots, n_a, \ldots) \equiv f(n, \ldots, n_a-1, \ldots) \) and \( I_a^+(n, \ldots, n_a, \ldots) \equiv n_a f(n, \ldots, n_a+1, \ldots) \) (no summation in \( n_a \) in the last relation). The elements \( \tilde{A}_a^{kl} \) of the matrix inverse with respect to \( A_a^{kl} \), i.e. with \( \tilde{A}_a^{ij} A_a^{kl} = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/2 \), arise when expressing the scalar products \( p_i \cdot p_j \) in the numerator, which appear as a result of the differentiation, in terms of the denominators \( D_a \):

\[
p_k \cdot p_l = \tilde{A}_a^{kl}(D_a + \mu_a). \tag{3}
\]

Suppose now that the integrals depend in addition on \( R \) external momenta \( p_i \) \((L < i \leq L + R)\) and we are interested in their values at a particular point \( p_i \cdot p_k = \mu_{ik}m^2 \). Without loss of generality we may assume that \( \text{det}(\mu_{ik}) \neq 0 \). Indeed, if this determinant is equal to zero then the momenta \( p_i \) at this point are linear dependent. Let the rank of the matrix \( \mu_{ik} \) be \( R' < R \). This means that we can represent all expressions as functions of only \( R' \) external momenta so that we come back to the (slightly simplified) original problem.

The number of independent propagators (i.e. factors in the denominator) which can be constructed from \( L \) loop and \( R \) (independent) external momenta is \( N_1 = L(L+1)/2 + LR \), and the number of additional (‘external’) invariants is \( N_2 = R(R + 1)/2 \). Let us expand the integrals in formal series (i.e. expand the integrands in the corresponding Taylor series) in the external kinematical invariants \( p_k \cdot p_l \), \( k, l = L + 1, \ldots, L + R \), or, in other words, in the ‘denominator-like’ objects \( D_a, a = N_1 + 1, \ldots, N_1 + N_2 \), depending only on the external momenta:

\[
F(n_1, \ldots, n_N; d; m^2; p_{L+1}, \ldots, p_{L+R}) = \int \ldots \int \frac{d^d p_1 \ldots d^d p_L}{D_{n_1}^{p_1} \ldots D_{n_N}^{p_L}} \sum_{n_{N_1+1} \leq \ldots \leq n_{N_1+N_2} \geq 1} (m^2)^{Ld/2 - \sum n_i + N_2} f(n_1, \ldots, n_{N_1+N_2}; d) \prod_{a=N_1+1}^{N_1+N_2} D_a^{n_a-1}. \tag{4}
\]

In particular, the ‘on-shell’ value at \( p_i \cdot p_k = \mu_{ik}m^2 \) of the integral is represented by the first term \( f(n_1, .., n_{N_1}, 1, .., 1; d) \) in the expansion in \( D_{ik} = p_i \cdot p_k - \mu_{ik}m^2 \), while the other terms in the expansion play an auxiliary role when proving the equivalence.

Acting by \((\partial/\partial p_i) \cdot p_k\), \((i = 1, \ldots, L; k = 1, \ldots, L + R)\) on the integrand in (4) we obtain \( N_1 \) IBP relations exactly in the ‘vacuum’ form (2). To control the
evolution of the $N_2$ ‘external’ indices we need additional $N_2$ relations. We obtain them by acting by $p_k \cdot (\partial/\partial p_i)$, $(i, k = L + 1, \ldots, L + R)$ on (1). These new relations look like (2) with the only exception that they have no term on the left-hand side proportional to the space-time dimension $d$. To transform these additional relations into a manifestly equivalent form we rescale the initial integrals by a power of the determinant of external kinematics invariants normalized on its ‘on-shell’ value $D \equiv \det(p_i \cdot p_k) / \det(\mu_{ik}m^2)$, $L < i, k \leq L + R$:

$$F(n_1, \ldots, n_N; d; \ldots) = D^{(R+1-d)/2} \tilde{F}(n_1, \ldots, n_N; d; \ldots),$$

with $\tilde{F}$ expanded in $\tilde{f}$ by the same equation (4) as in the case of $F$. For pure ‘on-shell’ values, the prefactor in the right-hand side of (5) is equal to one. For higher terms of the expansion in the external invariants, this prefactor is expanded and provides a linear substitution (easily invertible) of coefficients $f$ through $\tilde{f}$. As a result, the coefficients $\tilde{f}(n_1, \ldots, n_{N_1+N_2}; d)$ exactly satisfy the ‘vacuum’ relations (2) with $N = N_1 + N_2$.

Of course, boundary conditions for the recurrence procedures can be different. In particular, when solving the recurrence relations for pure the ‘on-shell’ $L$-loop $(R+1)$-point integrals we can use the combinatorial results for the corresponding $(L+R)$-loop vacuum or $(L + R - 1)$-loop propagator recurrence relations, but with the additional condition that integrals with non-positive values of the ‘external’ indices should be put to zero.

3 Recurrence procedure for massless 2-loop vertex diagrams

Let us apply the equivalence property described in the previous section to the evaluation of the massless two-loop vertex diagrams shown in Fig. 1a,b,c with two legs.

![Figure 1: Three types of two-loop vertex diagrams with general numerators and integer powers of propagators: (a) planar, (b) non-planar and (c) non-Abelian.](image)
on the light cone, \( p_1^2 = p_2^2 = 0 \), general polynomials in the numerator and arbitrary integer indices of the propagators. The diagrams depend homogeneously on \( q^2 = (p_1 + p_2)^2 = -Q^2 \). For example, a general Feynman integral for the planar graph Fig. 1a can be written as

\[
F_p(Q, \epsilon) = \int \int \frac{d^dk d^dl}{(l^2 - 2p_1 \cdot l)^{n_1}(l^2 + 2p_2 \cdot l)^{n_2}} \times \frac{(k \cdot l)^{n_7}}{(k^2 - 2p_1 \cdot k)^{n_3}(k^2 + 2p_2 \cdot k)^{n_4}(k^2 + (k - l)^2)^{n_6}}. \tag{6}
\]

The standard \( i0 \) prescription is implied, i.e. \( k^2 = k^2 + i0 \). We have chosen the irreducible numerator to be the scalar product of the two loop momenta, \( k \cdot l \).

The planar and non-planar integrals with the powers of propagators equal to one and some numerators of low degree were evaluated in [6] in expansion in \( \epsilon \). The planar integral with the powers of propagators equal to one was evaluated by IBP in gamma functions for general \( \epsilon \) in [7]. Using IBP, an algorithm for the evaluation of general planar integrals without numerators has been developed when expanding two-loop vertex diagrams in the Sudakov limit [8]. This solution of the recurrence relations was reproduced in [9]. For planar diagrams with arbitrary numerators, a complete solution was presented in [10]. The non-Abelian diagrams Fig. 1c are non more complicated than the planar ones. The IBP relations for them are simple, and any diagram can be expressed in gamma functions for general values of \( d \). Thus the only non-trivial case is in fact the non-planar diagram of Fig. 1b.

Following the results of Sect. 2 we reduce the evaluation of the diagrams of Fig. 1 to the evaluation of three-loop massless propagator diagrams of Fig. 2a,b,c, with arbitrary numerators and integer powers of propagators. As is well known the latter

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig2.png}
\caption{Three types of three-loop propagator diagrams with general numerators and integer powers of propagators: (a) planar, (b) non-planar and (c) Mercedes.}
\end{figure}

can be reduced, using the algorithm of [1], to a linear combination of master integrals Fig. 3 and recursively one-loop integrals Fig. 4.

Using the statement justified in the previous section we conclude that to solve IBP relations for Fig. 1 we can use recurrence relations for Fig. 2 presented in [1] in an explicitly solved form. The massless external legs should be identified with a pair of lines of the propagator diagrams which are connected to an (arbitrarily fixed)
external vertex. The additional condition formulated above means that, when using the recurrence procedure, one should put to zero integrals with non-positive values of these ‘former external’ indices of the lines.

To make a correspondence between the two given families of the diagrams let us choose, for definiteness, the right vertex in Fig. 2a,b,c and lines that are incident to this vertex so that the vertex diagrams of Fig. 1 will be obtained from the propagator diagrams of Fig. 2 by cutting the two right internal lines and, vice versa, the propagator diagrams will be obtained from the vertex diagrams by adding a new external vertex and a pair of lines which connect them with the two right external vertices in Fig. 1.

Thus, to evaluate a vertex diagram from Fig. 1 with a given set of indices $n_i$, we can use the following prescriptions:

- using the algorithm presented in [1] for the corresponding 3-loop propagator analog express it as a linear combination of the master integrals of Fig. 3 and recursively one-loop diagrams Fig. 4;

- omit the contribution of the recursively one-loop diagrams in the first row of Fig. 4.
• substitute the values of the recursively one-loop diagrams from the second raw in Fig. 4 by the corresponding values of the recursively one-loop vertex-type integrals shown in Fig. 6 if the indices of the right two lines are equal to one. If at least one of the right indices is greater than one, then evaluate the propagator diagram and express the result (written in gamma functions) through the corresponding propagator diagram with both indices equal to one. Finally, perform the above substitution in this result;

• substitute the values of the master integrals in Fig. 3 by the corresponding values of the master integrals shown in Fig. 5.

Figure 5: Master diagrams for 2-loop vertices. All the lines have indices equal to one and the numerator is equal to one.

Figure 6: Recursively one-loop diagrams for 2-loop vertices. Integer indices of lines and numerators are arbitrary.

The correspondence between the two sets of the integrals is obvious: the diagrams from Fig. 5 and 6 are obtained respectively from Fig. 3 and 4 by cutting a pair of the right lines. Note however that the second master vertex integral in Fig. 5 is in fact recursively one-loop and easily evaluated in gamma functions for general $\epsilon \equiv (4-d)/2$ so that the only ‘true’ master vertex integral is the non-planar diagram in Fig. 5 which was evaluated in [6] in expansion in $\epsilon$ up to the finite part:

$$\int \int \frac{d^d k d^d l}{((k+l)^2 - 2p_1(k+l))(k^2 - 2p_1k)(l^2 + 2p_2l)k^2l^2} = \left(1 - \frac{\pi^2}{\epsilon^2} - \frac{83\zeta(3)}{3\epsilon} - \frac{59\pi^4}{120}\right) \left(i\pi^{d/2} \epsilon^{-\gamma_E}\right)^2 \cdot \frac{(Q^2)^{2+2\epsilon}}{\langle Q^2 \rangle^{2+2\epsilon}}, \quad (7)$$
where $\gamma_E$ is the Euler constant.

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