The Center for the Elliptic Quantum Group $E_{\tau,\eta}(sl_n)$

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Abstract

We give the center of the elliptic quantum group in general case. Based on the Dynamic Yang-Baxter Relation and the fusion method, we prove that the center commute with all generators of the elliptic quantum group. Then for a kind of assumed form of these generators, we find that the coefficients of these generators form a new type closed algebra. We also give the center for the algebra.

1 Introduction

Recently, many papers have focused on the many-body long-distance integrable dynamical system, such as the Ruijsenaar Schneider model and the Calogero Moser (CM) model[1-3]. They are closely connected with the quantum Hall effect in the condense matter physics and the Seiberg Witten (SW) theory in the field theory, especially for the equations of the spectral curve in the SW theory, namely, the modified eigenvalue equations of the Lax matrices in the above integrable models[4-6]. These Lax matrices are the classical limit of the $L$-matrices which is associated with the interaction-round-a-face (IRF) model of Lie group[7-9] and the modified Yang-Baxter relation (NSF equation)[10-13]. All these $L$-matrices are corresponding to the representation of the elliptic quantum group which was proposed by Felder and Varchenko[11, 12]. The elliptic quantum group is an algebraic structure underlying the elliptic solution of the Yang-Baxter relation in the statistical mechanics and connected with the Knizhnik-Zamolodchikov-Bernard equation on torus. In the theory of the group and algebra, The center of them play an important role. The center is generally defined by the elements which can commute with all generators of the group and algebra. In Ref. [12], Felder and Varchenko obtained the center (also called the quantum determinant) for

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n = 2 case. In first part of this paper, we will discuss the center of the elliptic quantum group for the general case. And in the second part of this paper, assuming that the generators of the elliptic quantum group (namely, the elements of the \( L \)-matrices) take a certain kind of form, we find that their coefficients can construct a closed algebra. To this algebra, we also obtain its center.

### 2 The DYBR and the elliptic quantum group \( E_{\tau, \eta}(sl_n) \)

It is well known that the Boltzmann weight of the \( A_{n-1}^{(1)} \) IRF\([7, 8, 9]\) model can be written as

\[
R(a|z)_{ij}^{ii} = \frac{\sigma(z + w)}{\sigma(w)}, \quad R(a|z)_{ij}^{ij} = \frac{\sigma(z)\sigma(a_{ij} - w)}{\sigma(w)\sigma(a_{ij})} \quad \text{for } i \neq j, \\
R(a|z)_{ij}^{ji} = \frac{\sigma(z + a_{ij})}{\sigma(a_{ij})} \quad \text{for } i \neq j, \quad R(a|z)_{ij}^{ij} = 0 \quad \text{for other cases},
\]

(1)

where \( a \equiv (m_0, m_1, \ldots, m_{n-1}) \) is an \( n \)-vector, and \( a_{ij} = a_i - a_j, a_i = w(m_i - \frac{1}{n}\sum_i m_i + w_i), m_i (i = 0, 1, \ldots, n-1) \) are integers which describe the state of model, while \( \{w, w_i\} \) are generic c-numbers which are the parameters of the model, and \( \sigma(z) \equiv \theta [\frac{z}{\tau}] (z, \tau), \) with

\[
\theta [\frac{a}{b}] (z, \tau) \equiv \sum_{m \in \mathbb{Z}} e^{i\pi (m+a) \tau + 2i\pi (m+a)(z+b)}.
\]

We define an n-dimension vector \( \hat{j} = (0, 0, \ldots, 0, 1, 0, \ldots) \), in which \( j \)th component is 1.

We consider a matrix whose elements are linear operators acting on the quantum space \( V_0 \). The elements of the matrix are denoted by \( L(0|z)^i_j \). The \( R \)-matrix and the \( L \)-matrix can also be depicted by the figure 1 and figure 2, respectively, and they satisfy the Well-known Dynamic Yang-Baxter Relation (DYBR). The DYBR is written as (figure 3)

\[
\sum_{i', j'} R(b|z_1 - z_2)_{ij'}^{i'i''} L(0|z_1)^{i''}_{i'''} j'' L(a + i''|z_2)^{j''}_{j'''} = \sum_{i', j'} L(0|z_2)^{j'}_{j''} L(a + j''|z_1)^{i'}_{i''} R(a|z_1 - z_2)_{ij}^{i'j''},
\]

(2)

where \( b \equiv (m_0^b, m_1^b, \ldots, m_{n-1}^b), \ a \equiv (m_0^a, m_1^a, \ldots, m_{n-1}^a) \), We note that Eq.(2) gives the quadratic relation of the elements of \( L \). If we let \( b = a + h \), the form of the equation will be the same as that given in the Ref.[11, 12].

**Figure 1:** The elements of \( R \)-matrix \( R(a|z_1 - z_2)^{i'}_{ij} \).

**Figure 2:** The element of \( L \)-matrix, \( L(a, h|z)^i_j \equiv L(0|z)^i_j \).
The dynamical Yang–Baxter relation.

The elliptic quantum group $E_{r,q}(sl_n)$ is an algebra generated by the matrix elements of the $L$-matrix. For a given linear space $V$, the generators of the algebra satisfy the following relations

$$R(b|z_1 - z_2)_{ii} L(a|z_1)_{i} L(a + j''|z_2)_{i} = R(a|z_1 - z_2)_{ij} L(a|z_2)_{i} L(a + j''|z_1)_{i}$$

$$R(b|z_1 - z_2)_{ii} L(a|z_1)_{i} L(b + i|z_2)_{i} = R(a|z_1 - z_2)_{ij} L(a|z_2)_{i} L(b + i|z_1)_{i}$$

$$R(b|z_1 - z_2)_{ii} L(a|z_1)_{i} L(a + j''|z_2)_{i} = R(a|z_1 - z_2)_{ij} L(a|z_2)_{i} L(a + j''|z_1)_{i}$$

$$R(b|z_1 - z_2)_{ii} L(a|z_1)_{i} L(b + i|z_2)_{i} = R(a|z_1 - z_2)_{ij} L(a|z_2)_{i} L(b + i|z_1)_{i}$$

$$R(b|z_1 - z_2)_{ii} L(a|z_1)_{i} L(a + j''|z_2)_{i} = R(a|z_1 - z_2)_{ij} L(a|z_2)_{i} L(a + j''|z_1)_{i}$$

$$R(b|z_1 - z_2)_{ii} L(a|z_1)_{i} L(b + i|z_2)_{i} = R(a|z_1 - z_2)_{ij} L(a|z_2)_{i} L(b + i|z_1)_{i}$$

3 The center for the elliptic quantum group

In this section, we will discuss the center of the elliptic quantum group. For a given space $V \otimes V \otimes \cdots \otimes V \equiv V^\otimes n$ with a base $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}$, we define the permutation operator $P$ as

$$P(1_{j_1} 2_{j_2} \cdots n_{j_n}) e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n} = e_{i_{j_1}} \otimes e_{i_{j_2}} \otimes \cdots \otimes e_{i_{j_n}}.$$  

Define

$$P_\pm = \frac{1}{n!} \sum_P (-1)^{\delta_P} P,$$

where $\delta_P = 0$ if $P$ is even permutation, otherwise $\delta_P = 1$. Then, one can easily prove the $P_\pm$ satisfies the property of projection operator

$$P_+ P_- = P_-.$$  

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Let $z = -w$ in the Eq.(1), we have
\[
R(a| - w)_{ii}^{ji} = 0, \quad R(a| - w)_{ij}^{ji} = -\frac{\sigma(a_{ij} - w)}{\sigma(a_{ij})}, \quad R(a| - w)_{ji}^{ji} = \frac{\sigma(a_{ij} - w)}{\sigma(a_{ij})},
\]
where $i \neq j$, the other elements of $R$–matrix are all zeros. So we can easily obtain
\[
R(a| - w)P = -R(a| - w).
\]
Then for the above space $V^\otimes n$, we define a Cherednik operator $A$ as
\[
A = R_{12}(a| - w)R_{13}(a| - 2w) \cdots R_{1N}(a| - (n - 1)w) \\
\times R_{23}(a| - w)R_{24}(a| - 2w) \cdots R_{2N}(a| - (n - 2)w) \\
\times \cdots R_{n-1,n}(a| - w).
\]
Similarly with the Eq.(7), one can check that
\[
AP_- = A.
\]
On the other hand, we can always find an invertible matrix $B$ satisfying $A = BP_-$. Define $L$ in the space $V_0 \otimes V^\otimes n$ as in the figure 4. Taking use of the DYBR repeatedly, we can find a relation between $A$ and $L$, i.e., $AL = L'A$, where $L'$ is defined as figure 5. Therefore, applying the projection operator $P$ on $L$, we have
\[
P_-L = B^{-1}AL = B^{-1}L'A = B^{-1}L'AP_- = P_-LP_-
\]

**Figure 4:** The definition of $L$–matrix  

**Figure 5:** The definition of $L'$–matrix

Under the frame of the theory of interact-round-a-face (IRF), as usual, one can define the tri-spin operator $\varphi_{a,a+\hat{\mu}}^{(j)}(z)$ as
\[
\varphi_{a,a+\hat{\mu}}^{(j)}(z) = \theta \left[ \frac{\frac{1}{2} - \frac{z}{n}}{\frac{1}{2}} \right] (z - nw\bar{a}_\mu, n\tau),
\]
where $a_\mu = m_\mu - \frac{1}{n} \sum_k m_k + \delta_\mu$. It satisfies the face-vertex corresponding relation (figure 8)
\[
\sum_{i,j,i_1,j_1} r(z_1 - z_2)_{i,j}^{i_1,j_1} \varphi_{a,a+\hat{\mu}}^{(i_1)}(z_1) \varphi_{a+\hat{\mu},a+\hat{\nu}}^{(j_1)}(z_2) = \sum_k W \left( \begin{array}{c} a \ a + \hat{k} \\ a + \hat{\mu} \ a + \hat{\mu} + \hat{\nu} \end{array} \right) z_1 - z_2 \varphi_{a+k,a+\hat{\mu}+\hat{\nu}}^{(i)}(z_1) \varphi_{a,a+k}^{(j)}(z_2)
\]
where $r$ is the Belavin’s $Z_n$ symmetric $R$-matrix and $W$ is the Boltzmann weight of the IRF model $A^{(1)}_{n-1}$. Here we note that in this paper, we express the Boltzmann weight as $R$ in stead of $W$ for convenience.
Since the $R$-matrix can be regarded as a special case of $L$-matrix, the Eq.(10) is also held for

\[ X(a|z)_{i_0i_1...i_{n-1}j'} = R(a|z)_{i_0j} R(a + \hat{z}_0|z + w)_{i_1j_1} R(a + \hat{z}_1|z + 2w)_{i_2j_2} \times \cdots R(a + \hat{z}_{n-2}|z + (n - 1)w)_{i_{n-1}j_{n-1}}, \]

namely,

\[ P_-(12...n)X(a|z) = P_-(12...n)X(a|z)P_-(12...n), \]

\[ P_-(12...n)X(a|z)_{i_0i_1...i_{n-1}j'} = n! (P_\cdot X(a|z)_{i_0i_1...i_{n-1}j'})_{01...\cdot(n-1)P_-(12...n)_{i_0i_1...i_{n-1}j'} \times \varphi^{(0)}_{j_0j_1...j_{n-1}}(z + w) \varphi^{(1)}_{a+i_0,j_0} \varphi^{(2)}_{a+i_0+i_1,j_1} \varphi^{(n-1)}_{a+i_0+i_1...i_{n-2},j_{n-1}}(z + (n - 1)w). \]

Obviously, only when $(i_0^0,i_1^1,i_2^0...i_{n-1}^0)$ is the permutation of $(012...\cdot(n-1))$, the right hand side of Eq.(13) is not zero, otherwise, $P_-(012...\cdot(n-1))_{01...\cdot(n-1)} = 0$. This
implies that only when \( j = j' \), the right hand side of Eq.(13) is non-zero. Similar to the Eq. (10), applying \( P_- \) on \( \varphi \), we obtain from the face-vertex correspondence

\[
P_- \varphi \varphi \cdots = B^{-1} A \varphi \varphi \cdots = B^{-1} \varphi \varphi \cdots A'
\]

\[
= B^{-1} \varphi \varphi \cdots A' P_- = B^{-1} A \varphi \varphi \cdots P_- = P_- \varphi \varphi \cdots \varphi P_-,
\]

where \( A' \) is the Cherednik operator constructed by Belavin’s \( R \)-matrix \( r \) which also satisfies \( r(-w)P = -r(-w) \). The left hand side of Eq.(13) gives from the face-vertex correspondence

\[
\text{LHS} = P_- (12 \cdots n)^{i_0 i_1 \cdots i_{n-1}} (k_0)(k_1) \varphi_{a+j, i_0} (z, w) \varphi_{a+j, i_0 + i_1, i_2} (z + 2w) \cdots \varphi_{a+j, i_0 + i_1 + \cdots + i_{n-2}, i_{n-1}, i_0} (z + (n - 1)w) \varphi_{a+j, i_0 + i_1 + \cdots + i_{n-2}, i_{n-1}, i_0} (z + (n - 1)w)
\]

\[
\text{Eq.(14)} \equiv P_- (12 \cdots n)^{i_0 i_1 \cdots i_{n-1}} (k_0)(k_1) \varphi_{a+j, i_0} (z, w) \varphi_{a+j, i_0 + i_1, i_2} (z + 2w) \cdots \varphi_{a+j, i_0 + i_1 + \cdots + i_{n-2}, i_{n-1}, i_0} (z + (n - 1)w)
\]

One can show that in the above relation, we can write \( P_-=r \wedge \cdots \wedge f(z) \delta_{i_1} \cdots \delta_{i_{n-1}} \varphi_{a, j} \). Thus, comparing the Eq.(13) and Eq.(15), we obtain

\[
(\varphi_{a_{i_0} + i_1 + \cdots + i_{n-1}, j'}(z)) = \varphi_{a_{i_0} + i_1 + \cdots + i_{n-1}, j'}(z) = \varphi_{a_{i_0} + i_1 + \cdots + i_{n-1}, j'}(z).
\]

From the property of the face wight, we know that

\[
\varphi_{a_{i_0} + i_1 + \cdots + i_{n-1}, j'}(z) = \varphi_{a_{i_0} + i_1 + \cdots + i_{n-1}, j'}(z) = \varphi_{a_{i_0} + i_1 + \cdots + i_{n-1}, j'}(z).
\]

Therefore, the above equation gives

\[
n! (P_- X(a|z)_{j'} \cdots P_-)_{0 \cdots 1 \cdots 2 \cdots (n-1)} \varphi_{a_{i_0} + i_1 + \cdots + i_{n-1}, j'}(z) = \varphi_{a_{i_0} + i_1 + \cdots + i_{n-1}, j'}(z)
\]

\[
\text{Eq.(14)} = P_- (123 \cdots n)^{i_0 i_1 i_2 \cdots i_{n-1}} (k_0)(k_1) \varphi_{a+j, i_0} (z, w) \varphi_{a+j, i_0 + i_1, i_2} (z + 2w) \cdots \varphi_{a+j, i_0 + i_1 + \cdots + i_{n-2}, i_{n-1}, i_0} (z + (n - 1)w)
\]

By using the definition of the tri-spin operator, we can obtain \( \varphi_{a_{i_0} + i_1 + \cdots + i_{n-1} + \cdots + i_{n-2}, i_{n-1}, j'}(z) = \theta^{(i)} (z + n w a, \mu + (n - 1)w) \), where \( \theta^{(i)}(\cdot) \) is the abbreviation of the \( \theta \) function in the definition of
\[ \langle P_- \varphi(a|z)P_- \rangle \]

\[ \equiv P_-(123 \cdots n)^{\nu_0 \lambda' \tau'}_{a \mu \nu_1 \nu_2 \nu_3 \cdots (n-1)} \varphi^{(0)}_{a \mu \nu}(z) \varphi^{(1)}_{a + \mu \nu}(z + w) \varphi^{(2)}_{a + \mu + \nu, \lambda}(z + 2w) \]

\[ \times \cdots \varphi^{(n-1)}_{a + \mu + \nu + \lambda' \cdots \tau'}(z + (n-1)w) \]

\[ = P_-(1234)^{\nu_0 \lambda' \tau'}_{a \mu \nu_1 \nu_2 \nu_3 \cdots (n-1)} \theta^{(0)}(z + nw\bar{\alpha}_\mu + (n-1)w) \theta^{(1)}(z + nw\bar{\alpha}_\nu + (n-1)w) \]

\[ \times \theta^{(2)}(z + nw\bar{\alpha}_\lambda' + (n-1)w) \cdots \theta^{(n-1)}(z + nw\bar{\alpha}_{\tau'}^{(n-1)} + (n-1)w), \]

where \( a' = a + \mu \), \( a'' = a + \hat{\mu} + \hat{\nu} \), \ldots \( a^{(n-1)} = a + \hat{\mu} + \hat{\nu} + \hat{\lambda} + \cdots \), and \( \hat{\mu} \neq \hat{\nu} \neq \hat{\lambda} \neq \cdots \neq \hat{\tau} \).

Then, we have

\[ \bar{a'}_\nu = m'_\nu - \frac{1}{n} \sum_i m'_i + w_\nu = m'_\nu - \frac{1}{n} (\sum_{l \neq \mu} m_l + m_\mu + 1) + w_\nu \]

\[ = m_\nu - \frac{1}{n} (\sum_{l \neq \mu} m_l + m_\mu + 1) + w_\nu \]

\[ = \bar{a}_\nu - \frac{1}{n}. \]

Similarly, we can also have \( a''_\lambda = \bar{a}_\lambda - \frac{2}{n} \), \ldots \( a''^{(n-1)} = \bar{a}_\tau - \frac{n-1}{n} \). Let \( nz_\mu = z + nw\bar{\alpha}_\mu + (n-1) \), we then can obtain the final result of \( \langle P_- \varphi P_- \rangle \) as

\[ \langle P_- \varphi(a|z)P_- \rangle = \text{Det} \left[ \begin{array}{cccc}
\theta^{(0)}(nz_0) & \theta^{(1)}(nz_0) & \theta^{(2)}(nz_0) & \cdots & \theta^{(n-1)}(nz_0) \\
\theta^{(0)}(nz_1) & \theta^{(1)}(nz_1) & \theta^{(2)}(nz_1) & \cdots & \theta^{(n-1)}(nz_1) \\
\theta^{(0)}(nz_2) & \theta^{(1)}(nz_2) & \theta^{(2)}(nz_2) & \cdots & \theta^{(n-1)}(nz_2) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\theta^{(0)}(nz_{n-1}) & \theta^{(1)}(nz_{n-1}) & \theta^{(2)}(nz_{n-1}) & \cdots & \theta^{(n-1)}(nz_{n-1}) 
\end{array} \right] \]

\[ = c\sigma(\sum_i z_i - \frac{n-1}{2}) \prod_{i < j} \sigma(z_i - z_j) \]

\[ = c\sigma(z + w \sum_j w_j + (n-1)(w - \frac{1}{2})) \prod_{i < j} \sigma(w(\bar{a}_i - \bar{a}_j)) \]

\[ \equiv c\sigma(z + w \sum_j w_j + (n-1)(w - \frac{1}{2})) \Delta(a). \quad (17) \]

So we obtain from the Eq.(16),

\[ \langle P_- X(a|z) j' \rangle j_\mu \equiv (P_- X(a|z) j' \rangle j_\mu)_0^{123 \cdots (n-1)} \equiv \delta_{j_\mu}^{j_\nu} \frac{\Delta(a + \hat{j})}{n! \Delta(a)} f(z). \quad (18) \]

By using the DYBR Eq.(2) repeatedly, one can obtain (\( j \) and \( j' \) are not summed)

\[ P_- X(b|z - u) j' \bigg| L_{a\nu}^b(z) L_{a\nu}^b(u) j'' = L_{a\nu}^b(u) j' \bigg| P_- X(a|z - u) j''. \]

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Considering the properties of $P_-$, the above equation gives

$$
P_-X(b|z-u)j'' P_-L(b|z) P_-L(a|u)j''
= L(b|z)j'' P_-L(b|z) P_-X(a|z-u)j'' P_-
\Rightarrow (P_-X(b|z-u)j'' P_-(a|z) P_-) L(a|u)j'' P_-
= L(b|z)j'' (P_-L(b|z) P_-)(P_-X(a|z-u)j'' P_-) \delta_j''
\Rightarrow (P_-X(b|z-u)j'' P_-(a|z) P_-) L(a|u)j''
= L(b|z)j'' (P_-L(b|z) P_-)(P_-X(a|z-u)j'' P_-).
$$

Let $I(a|z) \equiv P_-L(b|z) P_-$. Substituting the Eq.(17) into the last relation of above relations, we have

$$\frac{\Delta(b+j')}{\Delta(b)} I(b|z) L(a|u)j'' = L(b|z)j'' I(a+j''|z) \frac{\Delta(a+j'')}{\Delta(a)}.
$$

Then, we have

$$\left[ \frac{\Delta(a)}{\Delta(b)} I(b|z) \right] L(a|u)j'' = L(b|z)j'' \left[ \frac{\Delta(a+j'')}{\Delta(b+j')} I(a+j''|z) \right]. \quad (19)
$$

Therefore $(\Delta(a)/\Delta(b)) I(a|z)$ is the center in the meaning given by Felder. By using the definition of $P_-$, we can obtain the result of $I$ as follows,

$$I(a|z) = \frac{1}{n!} \sum_{R} (-1)^{\text{Sign} \, R_0,1 \ldots n-1} \left[ \frac{\Delta(a)}{\Delta(b)} \right] L(b|z) \sum_{\mu_0, \mu_1 \ldots \mu_{n-1}} \left[ \frac{\Delta(a+j'')}{\Delta(b+j')} \right] I(a+j''|z),
$$

and $P$'s are permutations of integers $0, 1, \ldots, n-1$. This agrees with that of Ref.[12] for $n = 2$.

4 The coefficient algebra of the $L$-matrix and its center

For elements of the $L$-matrix which satisfy the Eq.(2), we assume that they take the special forms as follows,

$$L(a|z)j_i = (a_j^0)j_i \sigma(z + \delta_0 + b_i - a_j) F(z), \quad (20)
$$

$$L(a+j'|z)j'_j = (a+j')_j^j \sigma(z + \delta_0 + b'_j - a'_j) F(z), \quad (21)
$$

where $(a_j^0)_i$ and $(a+j')_j^j$ are the coefficient parts of the elements of $L$-matrix, and they are independent of $z$, $F(z)$ is the function of $z$. Substituting the above equations into
the DYBR Eq.(2), we can obtain the following relations (the more detailed derivation is given in Ref.[14])

\[ Y_{i,i'}^{j,j'} - Y_{i,i'}^{j,j'} = 0, \quad i' \neq j' \]  
\[ Y_{i,j}^{j,i'} - Y_{i,j}^{j,i'} = 0, \quad i \neq j \]  
\[ \sigma(w)\sigma(a_{i,j'} + b_{ij})Y_{i,i'}^{j,j'} + \sigma(a_{i,j'})\sigma(b_{ij} - w)Y_{i,i'}^{j,j'} - \sigma(a_{i,j'} + w)\sigma(b_{ij})Y_{i,i'}^{j,j'} = 0, \quad i \neq i', \; j \neq j' \]  

where we have used the notation \( a_{i,j'} = a_i - a_{j'}, \; b_{ij} = b_i - b_j \), and the definition

\[ [\mathfrak{a}]_{i}^{[a]}[\mathfrak{a}+\hat{i}]_{j}^{[a]} \equiv Y_{i,j}^{i',j'} \]  

with

\[ (\mathfrak{a})_{i,j}^{[a]} \times \prod_{l \neq l'} \sigma(a_l - a_{l'}) \equiv [\mathfrak{a}]_{i}^{[a]} \]  

Eqs.(21)-(23) can be regarded as the algebraic relations which are satisfied by the operators in the lattice \( a = \sum_{j=0}^{n-1} m_{ij}^a \hat{a}_j, \; b = \sum_{i=0}^{n-1} m_{ij}^b \hat{b}_i \). We define a new operator

\[ A_{i}^{j'} \equiv [\mathfrak{a}]_{i}^{[a]} \Gamma_{i}^{j'}, \]  

where

\[ \Gamma_{i}^{j'} f(a, b) = f(a + \hat{i}, b + \hat{j}) \Gamma_{i}^{j'} \]  

Namely, we regard the \( a, b \) as operators, \( \Gamma_{i}^{j'} \) is not commutative with the functions of \( a, b \). In this way, we have the following exchange relations of the operators \( \{ A_{i}^{j'} \} \)

\[ (a) \quad A_{i}^{j'} A_{i}^{j''} = A_{i}^{j''} A_{i}^{j'}, \quad i' \neq j' \]  
\[ (b) \quad \sigma(a_{i,j'} + w)\sigma(b_{ij})A_{i}^{j'} A_{i}^{j''} = \sigma(a_{i,j'} + b_{ij})A_{i}^{j''} A_{i}^{j'}, \quad i \neq i', \; j \neq j' \]  
\[ (c) \quad A_{i}^{j'} A_{j}^{j'} = A_{i}^{j''} A_{i}^{j'}, \quad i \neq j \]  

These equations are equivalent relations to the Felder and Varchenko’s elliptic quantum algebra under special condition.

For the algebra discussed above, we can prove that it has a set PBW (Poincare-Birkhoff-Witt) base (the precise proof for the existence of PBW base is given in another paper[14]). In the following part of this paper, we will discuss the center of this algebra.

In the previous section, we have proved that \( (\Delta(a)/\Delta(b))I_{(a)}^{(b)}(z) \) is the center of the elliptic quantum group. Here \( \Delta(b) \equiv \prod_{i<j} \sigma(w(\hat{b}_i - \hat{b}_j)) \), the definition of \( \Delta(a) \) is similar, and \( I_{(a)}^{(b)}(z) \) is given by

\[ I_{(a)}^{(b)}(z) = \frac{1}{n!} \sum_{\nu} (-1)^{\nu} [\text{Sign}_{\nu} \prod_{0=1}^{n-1} \rho_{\nu}] \times L_{(a)}^{(b)}(z)_{\mu_0} L_{(a+\hat{1})}^{(b+\hat{1})} \cdots L_{(a+\hat{n-2})}^{(b+\hat{n-2})} |z + (n-1)w \rangle_{\mu_{n-1}}^{n-1} \]
In our present case, the Eq.(20) can be rewritten as

\[ L(a|b|z) = \sigma (z + \delta + b_i - a_i') A_i'. \]

So the quantum determinant of the present algebra can be written as

\[ I(a|b|z) = \frac{1}{n!} \sum_{P} (-1)^{\text{Sign} P} \sigma (z + \delta + b_{\mu_0} - a_0) \cdots \sigma (z + (n - 1)w + \delta + b_{\mu_{n-1}} - a_{n-1}) A_{\mu_0} A_{\mu_1} \cdots A_{\mu_{n-1}}. \]

(29)

For each \( z \), we can obtain a center element \( \Delta(a)/\Delta(b)I(a|b|z) \) of algebra (28). We can show that there are only \( n \) linearly independent such elements by studying the quasi periodicity of the coefficients[14] of Eq.(29)

\[ \Phi(z)_{\mu_0 \cdots \mu_{n-1}} \equiv \sigma (z + \delta + b_{\mu_0} - a_0) \cdots \sigma (z + (n - 1)w + \delta + b_{\mu_{n-1}} - a_{n-1}). \]

5 Conclusion

The elliptic quantum group is proposed based on the dynamic Yang-Baxter Relation. The generators of the group can be given by the elements of \( L \)-matrix. We know that the center for a group or algebra play the role that it can commute with all elements of the group or algebra. In the first part of this paper, by using the fusion method, we gave the center for the elliptic quantum group \( E_{\tau,\eta}(sl_n) \). After we assumed that the elements of \( L \)-matrix take special forms Eq.(19) and Eq.(20), we obtained a algebra whose algebraic relation can be regarded as a Yang-Baxter relation. Then, Taking use of the results of the first part, we also obtained the center for the algebra. For this new type of algebra, we can also discuss other properties, such as its base. Actually, we have found that it have a set of PBW base. We will discuss it in our future paper[14].

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