Quantum Networks and Symmetries

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Quantum networks are promising tools for the implementation of long-range quantum communication. The characterization of quantum correlations in networks and their usefulness for information processing is therefore central for the progress of the field, but so far only results for small basic network structures or pure quantum states are known. Here we show that symmetries provide a versatile tool for the analysis of correlations in quantum networks. We provide an analytical approach to characterize correlations in large network structures with arbitrary topologies. As examples, we show that entangled quantum states with a bosonic or fermionic symmetry can not be generated in networks; moreover, cluster and graph states are not accessible. Our methods can be used to design certification methods for the functionality of specific links in a network and have implications for the design of future network structures.

Introduction.— A central paradigm for quantum information processing is the notion of quantum networks [1–4]. In an abstract sense, a quantum network consists of quantum systems as nodes on specific locations, where some of the nodes are connected via links. These links correspond to quantum channels, which may be used to send quantum information (e.g., a polarized photon) or where entanglement may be distributed. Crucial building blocks for the links, such as photonic quantum channels between a satellite and ground stations [5–8] or the high-rate distribution of entanglement between nodes [9, 10] have recently been experimentally demonstrated.

For the further progress of the field, it is essential to design methods for the certification and benchmarking of a given network structure or a single specific link within it. Moreover, it is important to analyze the capability of a given network structure to generate global quantum correlations. In principle, any multiparticle state can be generated in a connected network, but this may require complicated protocols using long-lived quantum memories, assisted by coherent operations and classical communication. So, in view of current experimental limitations, the question arises which states can be prepared in the network with moderate effort, e.g., with simple local operations.

This question has attracted some attention, with several lines of research emerging. First, the problem has been considered in the classical setting, such as the analysis of causal structures [11–13] or in the study of hidden variable models, where the hidden variables are not equally distributed between every party [14–19]. Concerning quantum correlations, several initial works appeared in the last year, suggesting slightly different definitions of network entanglement [20–23]. These have been further investigated [24, 25] and methods from the classical realm have been extended to the quantum scenario [26]. Still, the present results are limited to simple networks like the triangle network, noise-free networks, or bounded to small dimensions due to numerical limitations.

In this paper we present an analytical approach to characterize quantum correlations in arbitrary network topologies. Our approach is based on symmetries, which may occur as permutation symmetries or invariance under certain local unitary transformations. Symmetries play an outstanding role in various field of physics [27] and they have already turned out to be useful for various other problems in quantum information theory [28–34]. On a technical level, we combine the inflation technique for quantum networks [13, 22] with estimates known from the study of entropic uncertainty relations [35–37].

Based on our approach, we derive simply testable inequalities in terms of expectation values, which can be used to decide whether a given state may be prepared in a network or not. With this we can prove that large classes of states cannot be prepared in networks using simple communication, for instance all multiparticle graph states with up to twelve vertices with noise, as well as all mixed entangled permutationally symmetric states. This delivers various methods for benchmarking: First, the observation of such states in a network certifies the implementation of advanced network protocols. Second, our results allow to design simple tests for the proper working of a specific link in a given network.

Network entanglement.— To start, let us define the types of correlations that can be prepared in a network. In the simplest scenario Alice, Bob and Charlie aim to prepare a tripartite quantum state using three bipartite source states ϱa, ϱb and ϱc, see Fig. 1. Parties belonging to a same source state are sent to different parties of the network, i.e. A, B or C, such that the global state reads ϱ_{ABC} = ϱ_c ⊗ ϱ_b ⊗ ϱ_a. Note that here the order of the parties on both sides of the equation is different. After receiving the states, each party may still apply a
A triangle quantum network. Three sources $\varrho_a$, $\varrho_b$ and $\varrho_c$ distribute parties to three nodes, Alice, Bob and Charlie ($A$, $B$ and $C$). Alice, Bob and Charlie each end up with a bipartite system $X = X_1X_2$ on which they perform a local channel $\mathcal{E}_X^{(\lambda)}$ ($X = A, B, C$) depending on a classical random variable $\lambda$.

Local operation $\mathcal{E}_X$ (for $X = A, B, C$), in addition these operations may be coordinated by shared randomness. This leads to a global state of the form

$$\varrho = \sum_\lambda p_\lambda \mathcal{E}_A^{(\lambda)} \otimes \mathcal{E}_B^{(\lambda)} \otimes \mathcal{E}_C^{(\lambda)} [\varrho_{ABC}],$$

and the question arises, which three-party states can be written in this form and which cannot?

Some remarks are in order: First, the definition of network states in Eq. (1) can directly be extended to more parties or more advanced sources, e.g. one can consider the case of five parties $A, B, C, D, E$, where some sources distribute four-party states between some of the parties. Second, the scenario considered uses local operations and shared randomness (LOSR) as allowed operations, which is a smaller set than local operations and classical communication (LOCC). In fact, LOCC are much more difficult to implement, but using LOCC and teleportation any tripartite state can be prepared from bipartite sources. On the other hand, the set LOSR is strictly larger than, e.g., the unitary operations considered in [20, 21]. Finally, the discerning reader may have noticed that in Eq. (1) the state $\varrho_{ABC}$ does not depend on the shared random variable $\lambda$, but since the dimension of the source states $\varrho_i$ is not bounded one can always remove a dependency on $\lambda$ by enlarging the dimension [22].

Symmetries.— Symmetry groups can act on quantum states in different ways. First, the elements of a unitary symmetry group may act transitively on the density matrix $\varrho$. That is, $\varrho$ is invariant under transformations like

$$\varrho \mapsto U \varrho U^\dagger = \varrho.$$

If $\varrho = |\psi\rangle\langle\psi|$ is pure, this implies $U|\psi\rangle = e^{i\phi}|\psi\rangle$ and $|\psi\rangle$ is, up to some phase, an eigenstate of some operator. Second, for pure states one can also identify directly a certain subspace of the entire Hilbert space that is equipped with a certain symmetry, e.g. symmetry under exchange of two particles. Denoting by $\Pi$ the projector onto this subspace, the symmetric pure states are defined via

$$\Pi|\psi\rangle = |\psi\rangle$$

and for mixed states one has $\varrho = \Pi\rho\Pi$. Note that if $\varrho = \sum_k p_k |\phi_k\rangle\langle\phi_k|$ has some decomposition into pure states, then each $|\phi_k\rangle = \Pi|\phi_k\rangle$ has to be symmetric, too.

In the following, we consider mainly two types of symmetries. First, we consider multi-qubit states obeying a symmetry as in Eq. (2) where the symmetry operations consist of an Abelian group of tensor products of Pauli matrices. These groups are usually referred to as stabilizers in quantum information theory [38], and they play a central role in the construction of quantum error correcting codes. Pure states obeying such symmetries are also called stabilizer states, or, equivalently, graph states [39]. Second, we consider states with a permutational (or bosonic) symmetry [30, 33], obeying relations as in Eq. (3) with $\Pi$ being the projector onto the symmetric subspace.

GHZ states.— As a warming-up exercise we discuss the Greenberger-Horne-Zeilinger (GHZ) state of three qubits,

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle)$$

in the triangle scenario. This simple case was already the main example in previous works on network correlations [20–22], but it allows us to introduce our concepts and ideas in a simple setting, such that their full generalization is later conceivable.

The GHZ state is an eigenstate of the observables

$$g_1 = X_A X_B X_C, \quad g_2 = 1_A Z_B Z_C, \quad g_3 = Z_A Z_B 1_C.$$

Here and in the following we use the shorthand notation $1_X X_Y Y_Z$. Indeed, these $g_k$ commute and generate the stabilizer $S = \{1, g_1, g_2, g_3, g_1g_2, g_1g_3, g_2g_3, g_1g_2g_3\}$. Clearly, for any $S_i \in S$ we have $S_i|\text{GHZ}\rangle = |\text{GHZ}\rangle$ and so $\langle \text{GHZ}|S_i|\text{GHZ}\rangle = 1$.

As a tool for studying network entanglement, we use the inflation technique [13, 26]. The basic idea is depicted in Fig. (2). If a state $\varrho$ can be prepared in the network scenario, then one can also consider a scenario where each source state is sent two-times to multiple copies of the parties. In this multicity scenario, the source states may, however, also be wired in a different manner. In the simplest case of doubled sources, this may lead to two different states, $\tau$ and $\gamma$. Although $\varrho$, $\tau$ and $\gamma$ are different states, some of their marginals are identical, see Fig. (2) and Appendix A. If one can prove that states $\tau$ and $\gamma$ with the marginal conditions do not exist, then $\varrho$ cannot be prepared in the network.

Let us start by considering the correlation $\langle Z_A Z_B \rangle$ in $\varrho, \tau$ and $\gamma$. The values are equal in all three states,
one can also derive bounds on the maximal GHZ fidelity covariance matrices [23, 24] and classical networks [19],

\[ \langle A \rangle = \langle B \rangle = \langle C \rangle = \gamma, \]

and the same holds for the correlation \( \langle Z_B \rangle \). Note that these should be large, if \( \varrho \) is close to a GHZ state, as \( Z_B \) is an element of the stabilizer. Using the general relation \( \langle Z_B \rangle \geq \langle Z_B \rangle + \langle Z_B \rangle - 1 \) [22] we can use this to estimate \( \langle Z_B \rangle \) in \( \tau \). Due to the marginal conditions, we have \( \langle Z_B \rangle = \langle Z_B \rangle \), implying that this correlation in \( \tau \) must be large, if \( \varrho \) is close to a GHZ state. On the other hand, the correlation \( \langle X_A X_B X_C \rangle \) corresponds also to a stabilizer element and should be large in the state \( \varrho \) as well as in \( \tau \).

The key observation is that the observables \( X_A X_B X_C \) and \( Z_A \) are commuting, moreover, they have only eigenvalues \( \pm 1 \). For this situation, strong constraints on the expectation values are known: If \( M_i \) are pairwise commuting observables with eigenvalues \( \pm 1 \), then \( \sum_i \langle M_i \rangle^2 \leq 1 \) [37]. This fact has already been used to derive entropic uncertainty relations [35, 36] or monogamy relations [40, 41]. For our situation, it directly implies that for \( \tau \) the correlations \( \langle X_A X_B X_C \rangle \) and \( \langle Z_A \rangle \) cannot both be large. Or, expressing everything in terms of the original state \( \varrho \), if \( \langle Z_B \rangle + \langle Z_B \rangle - 1 \geq 0 \) then a condition for preparability of a state in the network is

\[ \langle X_A X_B X_C \rangle^2 + \langle Z_B \rangle + \langle Z_B \rangle - 1 \leq 1. \]  

This is clearly violated by the GHZ state. In fact, if one considers a GHZ state mixed with white noise, \( \varrho = p(GHZ)GHZ) + (1 - p)\), then these states are detected already for \( p = 4/5 \). Note that using the other observables of the stabilizer and permutations of the particles, also other conditions like \( \langle X_B X_C X_D \rangle^2 + \langle Z_A \rangle + \langle Z_B \rangle - 1 \leq 1 \) can be derived.

Using these techniques as well as concepts based on covariance matrices [23, 24] and classical networks [19], one can also derive bounds on the maximal GHZ fidelity achievable by network states. In fact, for network states

\[ F_{GHZ} = \langle GHZ | \varrho | GHZ \rangle \leq \frac{1}{2} \approx 0.7071 \]  

holds, as explained in Appendix B. This is a clear improvement on previous analytical bounds, although it does not improve a fidelity bound obtained by numerical convex optimization [22].

Cluster and graph states. — The core advantage of our approach is the fact that it can directly be generalized to more particles and complicated networks, while the existing numerical and analytical approaches are mostly restricted to the triangle scenario.

Let us start the discussion with the four-qubit cluster state \( |C_4\rangle \). This may be defined as the unique common +1-eigenstate of

\[ g_1 = X_A X_B X_C, \quad g_2 = X_A X_B X_C, \quad g_3 = Z_A X_B X_C, \quad g_4 = X_A X_B X_C. \]  

For later generalization, it is useful to note that the choice of these observables is motivated by a graphical analogy. For the square graph in Fig. 3 one can associate to any vertex a stabilizing operator in the following manner: One takes \( X \) on the vertex \( i \), and \( Z \) on its neighbours, i.e. the vertices connected to \( i \). This delivers the observables in Eq. (8), but it may also be applied to general graphs, leading to the notion of graph states [39].

If a quantum state can be prepared in the square network, then we can consider the third order inflated state \( \xi \) shown in Fig. 3. In the inflation \( \xi \), the three observables \( X_B, X_D, X_C X_B Z_D, \) and \( X_A Y_B Y_D \) anticommute. These observables act on marginals which are identical to those in \( \varrho \). Consequently, for any state that can be prepared in the square network, the relation

\[ \langle X_B X_D \rangle^2 + \langle Z_B X_C Z_D \rangle^2 + \langle X_A Y_B Y_D \rangle \leq 1 \]  

holds. All these observables are also within the stabilizer of the cluster state, so the cluster state violates this
inequality with an lhs equal to three. This proves that cluster states mixed with white noise cannot be prepared in the square network for \( p > 1/\sqrt{3} \approx 0.577 \). Again, with the same strategy different nonlinear witnesses like \((XAXC)^2 + (YAYBZD)ZD)^2 \leq 1\) for network entanglement in the square network can be derived. This follows from the second inflation in Fig. 3, see also the detailed discussion below. Furthermore, it shows that states with a cluster state fidelity of \(F_G > 0.7377\) cannot be prepared in a network. The full discussion is given in Appendix C.

This approach can be generalized to graph states. As already mentioned, starting from a general graph one can define a graph state by stabilizing operators in analogy to Eq. (6). The resulting states play an eminent role in quantum information processing. For instance, the so-called cluster states, which correspond to graphs of quadratic and cubic square lattices, are resource states for measurement-based quantum computation [42] and topological error correction [43, 44].

Applying the presented ideas to general graphs results in the following: If a graph contains a triangle, then under simple and weak conditions an inequality similar to Eq. (6) can be derived. This excludes then the preparability of noisy graph states in any network with bipartite sources only. Note that this is a stronger statement than proving network entanglement only for the network corresponding to the graph, as was considered above for the cluster state.

At first sight, the identification of a specific triangle in the graph may seem a weak condition, but here the entanglement theory of graph states helps: It is well known that certain transformations of the graph, so-called local-completions, change the graph state only by a local unitary transformation [45, 46], so one may apply these to generate the triangle with the required properties. Indeed, this works for all cases we considered (e.g. the full classification up to 12 qubits from Refs. [47, 48]) and we can summarize [49]:

Observation 1. (a) No graph state with up to 12 vertices can be prepared in a bipartite network. (b) If a graph contains a vertex with degree \( d \leq 3\), then it cannot be prepared in any network with bipartite sources. (c) The two- and three-dimensional cluster states cannot be prepared in any network.

In all the cases, it follows that graph states mixed with white noise, \( \rho = p|G\rangle\langle G| + (1-p)1/2^N \) are network entangled for \( p > 4/5 \), independently of the number of qubits. A detailed discussion is given in Appendix C.

Permutational symmetry.— Now we consider multiparticle quantum states of arbitrary dimension that obey a permutational or bosonic symmetry. Mathematically, these states act on the symmetric subspace only, meaning that \( \rho = \Pi^+ \rho \Pi^+ \), where \( \Pi^+ \) is the projector on the symmetric subspace. For example, in the case of three qubits this space is four-dimensional, and spanned by the Dicke states \( |D_0\rangle = |000\rangle, |D_1\rangle = (|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}, |D_2\rangle = (|011\rangle + |101\rangle + |110\rangle)/\sqrt{3}, \) and \( |D_3\rangle = |111\rangle \).

The symmetry has several consequences [30, 33]. First, if one has a decomposition \( \rho = \sum_k p_k |\psi_k\rangle\langle \psi_k| \) into pure states, then all \( |\psi_k\rangle \) have to come from the symmetric space, too. Since pure symmetric states can be either fully separable (like \( |D_0\rangle \)) or genuine multiparticle entangled (like \( |D_1\rangle \)), this implies that mixed symmetric states have also only these two possibilities. That is, if a mixed symmetric state is separable for one bipartition, it must be fully separable.

Second, permutational invariance can also be characterized by the flip operator \( F_{XY} = \sum_{ij} |ij\rangle_X \langle ji|_Y \) on the particles \( X \) and \( Y \). Symmetric multiparticle states obey \( \rho = F_{XY} \rho = \rho F_{XY} \) for any pair of particles, where the second equality directly follows from hermiticity. Conversely, concluding full permutational symmetry from two-particle properties, does only require this relation for pairs such that the \( F_{XY} \) generate the full permutation group. Finally, it is easy to check that if the marginal \( g_{XY} \) of a multiparticle state \( \rho \) obeys \( g_{XY} = F_{XY} g_{XY} \), then the full state \( \rho \) obeys the same relation, too.

Armed with these insights, we can explain the idea for our main result. Consider a three-particle state with bosonic symmetry which can be prepared in a triangle network, the inflation \( \gamma \) from Fig. (2) and the reduced state \( \gamma_{ABC} \) in this inflation. This obeys \( \gamma_{ABC} = F_{XY} \gamma_{ABC} \) for \( XY \) equal to \( AB \) or \( BC \). Since \( F_{AB} F_{BC} F_{AB} = F_{AC} \), this implies that the reduced state \( \tau_{AC} \) obeys \( \tau_{AC} = F_{AC} \tau_{AC} \) and hence also the six-particle state \( \tau \). Moreover, \( \tau \) also obeys similar constraints for other pairs of particles (like \( AB, BC, A'B' \) and \( B'C' \)) and its easy to see that jointly with \( AC' \) these generate the full permutation group. So, \( \tau \) must be fully symmetric. But \( \tau \) is separable with respect to the \( ABC|A'B'C' \) bipartition, so \( \tau \) and hence \( \rho = \tau_{ABC} \) must be fully separable.

The same argument can easily be extended to more complex networks, which are not restricted to use bipartite sources. We can summarize [49]:

Observation 2. Let \( \rho \) be a permutationally symmetric multiparticle state of \( N \) particles. Then, \( \rho \) can be generated by a network with \( (N - 1)\)-partite sources, if and only if it is fully separable.

We add that this Observation can also be extended to the case of fermionic antisymmetry, a detailed discussion is given in the Appendix D.

Certifying network links.— For the technological implementation of quantum networks, it is of utmost importance to design certification methods to test and benchmark different realizations. One of the basic questions is, whether a predefined quantum link works or not. Interestingly, our method allows to derive a simple, yet powerful test for that. Consider a network where the link between two particles is absent or not properly working.
For definiteness, we may consider the square network on the lhs of Fig. 3 and the parties A and C. In the second inflation \( \tau \) we have for the marginals \( \tau_{AC} = \tau_{AC} \). This implies that the observables \( X_A X_C \) and \( Z_A Z_C \) on the original state \( \varrho \) correspond to anticommuting observables on \( \tau \), so we have \( \langle X_A X_C \rangle^2 + \langle Z_A Z_C \rangle^2 \leq 1 \). Using higher-order inflations, one can extend and formulate it for general networks: If a state can be prepared in a network with bipartite sources but without the link \( AC \),

\[
\langle X_A X_C P_{R_i} \rangle^2 + \langle Y_A Y_C P_{R_i} \rangle^2 + \langle Z_A Z_C P_{R_i} \rangle^2 \leq 1. \tag{10}
\]

Here the \( P_{R_i} \) are arbitrary observables on disjoint subsets of the other particles, \( R_i \cap R_j = \emptyset \). If a state was indeed prepared in a real quantum network then violation of this inequality proves that the link \( AC \) is working and distributing entanglement. In Appendix E details are discussed and examples are given, where this test allows to certify the functionality of a link even if the reduced state \( \varrho_{AC} \) is separable.

**Conclusion.**— We have provided an analytical method to analyze correlations arising in quantum networks from few measurements. With this, we have shown that large classes of states with symmetries, namely noisy graph states and permutationally symmetric states cannot be prepared in networks. Moreover, our approach allows to design simple tests for the functionality of a specified link in a network.

Our results open several research lines of interest. First, they are of direct use to analyze quantum correlations in experiments and to show that multiparticle entanglement is needed to generate observed quantum correlations. Second, they are useful for the design of networks in the realistic setting: For instance, we have shown that the generation of graph states from bipartite sources necessarily requires at least some communication between the parties, which may be of relevance for quantum repeater schemes based on graph states that have been designed \([50]\). Moreover, it has been shown that GHZ states provide an advantage for multipartite conference key agreement over bipartite sources \([51]\), which may be directly connected to the fact that their symmetric entanglement is inaccessible in networks. Finally, our result open the door for further studies of entanglement in networks, e.g. using limited communication (first results on this have recently been reported \([52]\)) or restricted quantum memories, which is central for future realizations of a quantum internet.

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**APPENDIX A: NETWORK CORRELATIONS AND THE INFLATION TECHNIQUE**

Before explaining the inflation technique in some detail, it is useful to note some basic observations on the definition of network correlations. Recall from the main text that triangle network states are of the form

\[
\varrho = \sum_\lambda p_\lambda \, \mathcal{E}_A^{(\lambda)} \otimes \mathcal{E}_B^{(\lambda)} \otimes \mathcal{E}_C^{(\lambda)} \left[ \varrho_{ABC} \right], \tag{11}
\]

i.e. there exist source states \( \varrho_a, \varrho_b, \varrho_c \) with \( \varrho_{ABC} = \varrho_a \otimes \varrho_b \otimes \varrho_c \), a shared random variable \( \lambda \) and channels (that is, trace preserving positive maps) \( \mathcal{E}_A^{(\lambda)}, \mathcal{E}_B^{(\lambda)} \) and \( \mathcal{E}_C^{(\lambda)} \) that can be used to generate the state \( \varrho \).

First, we note that in this definition, the state \( \varrho_{ABC} \) does not depend on the classical variable \( \lambda \). This is however, no restriction, as the dimensions of the source states are not bounded. If in Eq. (11) the \( \varrho_{ABC}(\lambda) \) and hence the \( \varrho_a(\lambda), \varrho_b(\lambda), \varrho_c(\lambda) \) depend on \( \lambda \), one can just combine the set of all \( \varrho_a(\lambda) \) to a single \( \varrho_a \) etc. and redefine the maps \( \mathcal{E}_A^{(\lambda)} \) etc. such that they act on the appropriate \( \varrho_a(\lambda) \). This results in a form where \( \varrho_{ABC} \) does not depend on \( \lambda \) anymore, hence the state can be written as in Eq. (11). Note that this has already been observed in \([22]\). For our purpose, the potential dependence of \( \varrho_{ABC} \) on \( \lambda \) has the following consequence: If we wish to compute for a symmetric |\( \psi \rangle \) the maximum fidelity \( \langle \psi | \varrho | \psi \rangle \) over all network states \( \varrho \), then we may assume that \( \varrho \) permutationally symmetric, too. This follows from the simple fact that we can, without decreasing the overlap, symmetrize the state \( \varrho_{ABC} \), and the symmetrized state will still be preparable in the network.

Second, one may restrict the \( \varrho_{ABC} = \varrho_a \otimes \varrho_b \otimes \varrho_c \) further. Indeed it is straightforward to see that \( \varrho_a = |a\rangle \langle a|, \varrho_b = |b\rangle \langle b|, \) and \( \varrho_c = |c\rangle \langle c| \) can be chosen to be pure, as the channels \( \mathcal{E}_A^{(\lambda)} \) etc. are linear.

Third, as the set of network preparable states is by definition convex, one may ask for its extremal points. Formally, these are of the form \( \mathcal{E}_A^{(\lambda)} \otimes \mathcal{E}_B^{(\lambda)} \otimes \mathcal{E}_C^{(\lambda)} \left[ |a\rangle \langle a| \otimes |b\rangle \langle b| \otimes |c\rangle \langle c| \right] \), but can these further be characterized? Clearly, pure biseparable three-particle states, such as \( |\psi\rangle = |\phi\rangle_{AB} \otimes |\eta\rangle_C \) are extremal points. There are however, also mixed states as extremal points, which can be seen as follows: It was shown in Ref. \([21]\) that pure three-qubit states which are genuine multiparticle entangled (that is, not biseparable) cannot be prepared in the triangle network. On the other hand, in Ref. \([22]\) it was shown that there are network states having a GHZ fidelity of 0.5170, which implies that they are genuine multiparticle entangled \([53]\). So, the set defined in Eq. (11) must have some extremal points, which are genuine multiparticle entangled mixed states.
state that is in general not separable and different from similarly, but with a different rewiring, leading to an inflated us with two network states, $E$ and $A$.

Consider two networks with six vertices it is a direct generalization. We introduce it for triangle networks as for arbitrary networks the characterization of quantum networks [22]. We in-
nique [13, 26], which has already proven to be useful for some repermutation, as depicted here. This procedure implies several equalities between the marginals of the original state and some inflations, and its inflations.

After this prelude, let us explain the inflation tech-
nique [13, 26], which has already proven to be useful for the characterization of quantum networks [22]. We introduce it for triangle networks as for arbitrary networks it is a direct generalization.

We start with constructing two inflations of the tri-
gle network. Consider two networks with six vertices and six edges as in Fig. 4. Identical sources are distributed along the lines of same colour, thus two copies of each source are needed per network. In other words, the source $\rho_b$ is distributed between $AC$ and $A'C'$ to generated $\tau$, and between $AC'$ and $A'C$ for $\gamma$ (analogously for $\rho_a$ and $\rho_c$, following Fig. 4). Then, the channels are performed according to the random parameter $\lambda$. Both on primed and non-primed $A$ nodes, the same channel $\mathcal{E}_A^{(\lambda)}$ is applied and similarly for $B$ and $C$. This leaves us with two network states, $\tau$ and $\gamma$. Those operators are physical states, i.e. they have a unit trace and are positive semi-definite. Formally, they can be written as

$$\tau = \sum_\lambda p_\lambda \left( \mathcal{E}_A^{(\lambda)} \otimes \mathcal{E}_B^{(\lambda)} \otimes \mathcal{E}_C^{(\lambda)} [\varrho_{ABC}] \right)$$

and

$$\gamma = \sum_\lambda p_\lambda \mathcal{E}_A^{(\lambda)} \otimes \mathcal{E}_B^{(\lambda)} \otimes \mathcal{E}_C^{(\lambda)} [\varrho_{ABC}]$$

where $\varrho_{ABC}$ and $\varrho_{A'B'C'}$ are physical states, i.e. they have a unit trace and are generated.

Furthermore, from Eq. (12) it is clear that $\tau$ is separable wrt the partition $ABC|A'B'C'$ and we note that $\tau$ and $\gamma$ are permutationally symmetric under the exchange of non primed and primed vertices. Therefore, if, for some given state $\varrho$, it is not possible to find states $\tau$ and $\gamma$ that satisfy those conditions, then $\varrho$ cannot be generated in the considered network.

An interesting point is that the question for the existence of $\tau$ and $\gamma$ with the desired properties can be directly formulated as a semidefinite program (SDP). This can be used to prove that such inflations do not exist, and the corresponding dual program can deliver an witness-like construction that can be used to exclude prepara-
ility of a state in the network. Still, these approaches are memory intensive. For instance, as the authors of Ref. [22] acknowledge, it is difficult to derive tests for tripartite qutrit states in a normal computer.

If $n$ is an integer, then $\varrho$ may be considered, for instance inflations with $3n$ nodes ($n = 3, 4, \ldots$) or simply wired differently than $\tau$ and $\gamma$. As mentioned previously, this technique can also be used for more complicated networks.

APPENDIX B: FIDELITY ESTIMATE FOR THE GHZ STATE

Let us compute a bound on the fidelity of triangle network states to the GHZ state, i.e. compute $F = \max(\langle \psi | \rho_{GHZ} \rangle)$, where the maximum is taken over all states as in Eq. (11). For this maximization is it suf-
ficient to consider the extremal states, which are of the type $\rho_{ITN}^{(\lambda)} = \mathcal{E}_A^{(\lambda)} \otimes \mathcal{E}_B^{(\lambda)} \otimes \mathcal{E}_C^{(\lambda)} [\varrho_{ABC}]$, here $ITN$ stands for the independent triangle network, that is the triangle network without shared randomness.

Therefore, one may use techniques based on covariance matrices [23, 24], which are designed for the ITN.
The covariance matrix (CM) $\Gamma$ of some random variables $x_1, \ldots, x_N$ is the matrix with elements $\Gamma_{ij} = \text{cov}(x_i, x_j) = \langle x_ix_j \rangle - \langle x_i \rangle \langle x_j \rangle$, $i = 1, \ldots, N$.

We can now explain the general idea of the technique in Ref. [24]: If one computes the CM of the outcomes of $Z$-measurements on each qubit of an ITN state, then this matrix has a certain block structure. Checking this block structure can be done by checking the positivity of the comparison matrix $M(\Gamma)$. The comparison matrix obtained by flipping the signs of the off-diagonal elements of $\Gamma$. The condition then reads: For any quantum state in the ITN the comparison matrix of the CM is positive semi-definite. Hence, a negative eigenvalue in the comparison matrix excludes a state of being preparable in the ITN.

Now, if we apply that to states $\varrho(F) = F|\text{GHZ}\rangle\langle\text{GHZ}| + (1 - F)\varrho$ with a fidelity $F$ to the GHZ state, the comparison matrix of the CM reads

$$M(\Gamma) = \begin{bmatrix}
1 - a^2(1 - F)^2 & -(F + d(1 - F) - ab(1 - F)^2) & -(F + e(1 - F) - ac(1 - F)^2) \\
-(F + d(1 - F) - ab(1 - F)^2) & 1 - b^2(1 - F)^2 & -(F + f(1 - F) - bc(1 - F)^2) \\
-(F + e(1 - F) - ac(1 - F)^2) & -(F + f(1 - F) - bc(1 - F)^2) & 1 - c^2(1 - F)^2
\end{bmatrix},$$

where $a = \langle Z11 \rangle_\varrho$, $b = \langle 1Z1 \rangle_\varrho$, $c = \langle 11Z \rangle_\varrho$, $d = \langle ZZ1 \rangle_\varrho$, $e = \langle Z1Z \rangle_\varrho$ and $f = \langle ZZZ \rangle_\varrho$. From the last paragraph, we have that this matrix is positive semi-definite for ITN states, thus $\langle \varrho | M(\Gamma) | \varrho \rangle \geq 0$, for all vectors $|\varrho\rangle$, and in particular for $|\varrho\rangle = (1, 1, 1)/\sqrt{3}$. We notice that $\langle \varrho | M(\Gamma) | \varrho \rangle$ is upper bounded by $4 - 6F + F^2$ and therefore, $0 \leq \langle \varrho | M(\Gamma) | \varrho \rangle \leq 4 - 6F + F^2$ holds for ITN states and we are able to exclude all states $\varrho(F)$ with $F > 3 - \sqrt{3} \simeq 0.7639$ form the triangle network scenario.

By making use of additional constraints or other criteria, we can obtain tighter bound. For any given three compatible dichotomic measurements $M_1, M_2, M_3$, we have [22]

$$p(M_1 = M_2) \geq p(M_1 = M_3) + p(M_2 = M_3) - 1. \quad (18)$$

This implies

$$\langle M_1 M_2 \rangle \geq \langle M_1 M_3 \rangle + \langle M_2 M_3 \rangle - 1. \quad (19)$$

By substituting $M_i$ with $-M_i$, we obtain

$$\langle M_1 M_2 \rangle \geq |\langle M_1 M_3 \rangle + \langle M_2 M_3 \rangle| - 1, \quad (20)$$

$$\langle M_1 M_2 \rangle \leq 1 - |\langle M_1 M_3 \rangle - \langle M_2 M_3 \rangle|. \quad (21)$$

In our case, we have

$$d \geq |a + b| - 1, \quad e \geq |a + c| - 1, \quad f \geq |b + c| - 1. \quad (22)$$

With this extra constraint, $0 \leq \langle \varrho | M(\Gamma) | \varrho \rangle \leq 9 - 12F$ holds for ITN states and we are able to exclude all states $\varrho(F)$ with $F > 3/4 = 0.75$ form the triangle network scenario.

Another criterion for ITN states [19] states that

$$6 \geq (1 + F + d(1 - F))^2 + (1 + F + e(1 - F))^2 + (1 + F + f(1 - F))^2,$$

$$\geq 3(1 + F - (1 - F))^2 = 12F^2, \quad (26)$$

which implies $F \leq 1/\sqrt{2}$. The bound $1/\sqrt{2} \simeq 0.7071$ is slightly worse, but close to the one 0.6803 obtained in Ref. [22] based on advanced numerical computations.

**APPENDIX C: GRAPH AND CLUSTER STATES**

In this Appendix we present our results on graph states and cluster states. This Appendix is structured as follows. We first recall the basic facts about graph and cluster states. Then, we prove the estimate on the fidelity with cluster states for states in the square network (see the main text). Finally, we present the proof and discussion of Observation 1.

**Graph states and the stabilizer formalism**

Graph states [39, 46] are quantum states defined through a graph $G = (V, E)$, i.e. through a set $V$ of
$N$ vertices and a set $E$ containing edges that connect the vertices. The vertices represent the physical systems, qubits. One way of describing these states is through the stabilizer formalism. For that, as introduced in the main text, one first needs to introduce the generator operators $g_i$ of graph states: a graph state $|G\rangle$ is the unique common +1-eigenstate of the set of operators $\{g_i\}$,

$$g_i = X_i \prod_{j \in N_i} Z_j,$$

where $N_i$ is the neighbourhood of the qubit $i$, i.e. the set of all qubits $j \in V$ connected to the qubit $i \in G$. The state $|G\rangle$ can also be described through its stabilizer, which is the set $S = \{S_1, \ldots, S_{2^N}\} = \{\prod_{i=1}^{N} g_i^{x_i} : (x_1, \ldots, x_N) \in \{0, 1\}^N\}$. This means that $S$ contains all possible products of the generators $g_i$, hence $S_i|G\rangle = |G\rangle$.

We note that $\mathbb{I} \in S$. The projector onto the state $|G\rangle$ reads

$$|G\rangle\langle G| = \frac{1}{2^N} \sum_{i=1}^{2^N} S_i.$$  

Defined like that, graph states are a subset of the more general stabilizer states [38, 39, 54]. First, one has to consider an abelian subgroup $S$ of the Pauli group $P_N$ on $N$ qubits that does not contain the operator $-\mathbb{I}$. To that set corresponds a vector space $V_S$ that is said to be stabilized by $S$, i.e. every element of this vector space is stable under the action of any element of $S$. We call stabilizers that lead only to one state full-rank stabilizers, i.e. there is a unique common eigenstate with eigenvalue +1. That state is completely determined by a subset of $N$ elements of $S$. As an example, one may consider the GHZ state, as explained in the main text. Indeed, it is the unique common eigenstate of $XXX$, $1ZZ$ and $ZZ1$. One can show that any stabilizer state is, after a suitable local unitary transformation, equivalent to a graph state.

More precisely, the local unitary transformations that map any stabilizer state to a graph state belong to the so-called local Clifford group $C_1$. The local Clifford group is defined as the normalizer of the single-qubit Pauli group, i.e. $UP_iU^\dagger = P_i$ for all $U \in C_1$. By construction, the stabilizer formalism is preserved under the action of the local Clifford group, and hence, an interesting question is under which conditions two graph states (or two stabilizer states) are equivalent under local Cliffords. For graph states this question has a simple solution in terms of graphical operations that determine their equivalence. Namely, two graph states are equivalent under the action of the local Clifford group if and only if their corresponding graphs are equivalent under a sequence of local complementations [45]. For a given graph $G$ and vertex $i \in V$ the local complement $G'$ of $G$ at the vertex $i$ is constructed in two steps. First, we have to determine the neighborhood $N(i) \subset V$ of the vertex $i$ and then the induced subgraph is inverted, i.e. considering all possible edges in the neighborhood any pre-existing edge is removed and any non-existing edge is added. E.g. having a graph $V = \{1, 2, 3\}$ and $E = \{(1, 2), (2, 3), (3, 1)\}$, a local complementation on vertex 1 results in the graph with edges $E = \{(1, 2), (3, 1)\}$.

**Estimate for the cluster state fidelity**

We aim at computing a bound on the fidelity of square network states to the four-qubit ring cluster state $|C_4\rangle$, i.e. $F = \max \langle C_4|\varrho|C_4\rangle$, were the maximum is taken over all square network states $\varrho$. The fidelity of a state $\varrho$ with the cluster state is given by $F = \frac{1}{16} \sum_{i=0}^{15} \langle S_i \rangle_{\varrho}$, where $\{S_i\}$ is the stabilizer of $|C_4\rangle$, consisting of $S_0 = \mathbb{I}1111$ and further elements given in Table I.

The symmetry of $|C_4\rangle$ implies that one can assume the network state $\varrho$ that maximizes the fidelity to admit the same expectation value on operators from the same column, as denoted in the last row of Table I.

As explained in the main text, the general idea is to notice that some stabilizers of $|C_4\rangle$ anticommute in the appropriate inflation, and then use the fact that anticommuting operators cannot all have large expectation values for a given state. In the $\tau$-inflation of the square network (see Fig. 5), the observable $X_BX_{D'}$ and $Y_BY_DZ_CZ_D$ anticommute, and since $\tau_{BD'} = \varrho_{BD}$ and $\tau_{ABCD} = \varrho_{ABCD}$.
one has
\[ \Xi^2 + \Lambda^2 \leq 1. \] (29)

Secondly, we have Eq. (9) that we reformulate as
\[ \Xi^2 + \Theta^2 + \Sigma^2 \leq 1. \] (30)

At last, we consider the observables \( X_A X_B X_C X_D \) and \( Z_A X_B Z_{A'} X_{D'} \) in the inflation \( \tau \). However, the latter is not a stabilizer of the four-qubit ring cluster state, but we have \( \langle Z_A X_B Z_{A'} X_{D'} \rangle_\tau = \langle Z_A X_B Z_{A'} X_D \rangle_\gamma \). Then, using the fact that for commuting dichotomic measurements, \( \langle M_1 M_2 \rangle \geq \langle M_1 M_2 \rangle + \langle M_2 M_3 \rangle - 1 \) [22], one gets \( \langle Z_A X_B X_D Z_{A'} \rangle_\gamma \geq \langle Z_A X_B Z_C \rangle_\gamma + \langle Z_C X_D Z_{A'} \rangle_\gamma - 1 \). Since \( X_A X_B X_C X_D \) and \( Z_A X_B Z_{A'} X_{D'} \) are anticommuting, from constraints on the marginals, one finally gets
\[ 2\Theta - 1 \leq \sqrt{1 - \Lambda^2}. \] (31)

Analogously,
\[ 2\Sigma - 1 \leq \sqrt{1 - \Omega^2}, \] (32)
\[ 2\Xi - 1 \leq \sqrt{1 - \Sigma^2}, \] (33)
\[ 2\Xi - 1 \leq \sqrt{1 - \Omega^2}. \] (34)

By exploiting all these inequalities as constraints on the maximization of the fidelity, we finally get
\[
F = \frac{1}{16} \left( 1 + 4\Theta + 4\Lambda + 2\Xi - 4\Sigma + \Omega \right) \\
\leq 0.737684,
\] (35)
hence all states with a larger fidelity to the four-qubit ring cluster state cannot be prepared in a square network.

Proof of Observation 1

Here we provide a detailed proof of Observation 1. To do so, we first need to prove the following theorem.

**Theorem 3.** Let \( G(V,E) \) be a graph as in Fig. 6 with three mutually connected vertices \( A, B \) and \( C \) and let
\[
T_{ABC} = N_A \cap N_B \cap N_C, \\
J_{AB} = (N_A \cap N_B) \setminus T_{ABC}, \\
E_A = N_A \setminus (N_B \cup N_C),
\] (36)
(37)
(38)
etc., where \( N_X \) is the neighborhood of \( X \) (\( X = A,B,C \)). Then the graph state \( |G \rangle \) cannot originate from any network with only bipartite sources, if one of the following conditions is satisfied:

1. \( J_{XY} = J_{XZ} = \emptyset \), where \( X,Y,Z \) is a permutation of \( A,B,C \);
2. \( E_X = E_Y = \emptyset \), where \( X \neq Y \in \{ A,B,C \} \);
3. \( E_X = J_{XY} = \emptyset \), where \( X \neq Y \in \{ A,B,C \} \).

**Proof.** We only need to show that the graph state \( |G \rangle \) cannot be generated in the network with the complete graph \( K \) as shown in Fig. 7, where the number of vertices is the same than in \( G \).

We start our discussion with the inflation \( \gamma \) as in Fig. 8. Let us denote
\[
g_A = X_A Z_{N_A}, \quad g_B = X_B Z_{N_B}, \quad g_C = X_C Z_{N_C},
\] (39)
where \( N_A \) is the neighborhood of the vertex \( A \) in the graph \( G \). Then
\[
g_{AB} = Y_A Y_B Z_{R_{AB}}, \quad g_{AC} = Y_A Y_C Z_{R_{AC}}, \quad g_{BC} = Y_B Y_C Z_{R_{BC}},
\] (40)
(41)
(42)

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
Stabilizer elements & \( S_1 = XZ1Z \) & \( S_2 = ZZX1 \) & \( S_3 = 1ZXZ \) & \( S_4 = Z1ZX \) & \( S_5 = YYZZ \) & \( S_6 = YYZY \) & \( S_7 = ZYYZ \) & \( S_8 = ZZYY \) & \( S_9 = XLX1 \) & \( S_{10} = 1XLX \) & \( S_{11} = -XYY1 \) & \( S_{12} = -YXY1 \) & \( S_{13} = -Y1XY \) & \( S_{14} = -XY1Y \) & \( S_{15} = XXXX \) \\
\hline
\end{tabular}
\caption{Elements of the stabilizer of the four-qubit cluster states. The qubit indices \( A,B,C,D \) are suppressed here. See the text for further details.}
\end{table}
Note that here the bipartite links of the network are shown, and not the edges of the graph of the graph state.

By comparing the marginals of the states $\gamma$ and $\varrho$, we have

$$\langle Y_A Y_B Z_{R_{AB}} \rangle_\gamma \geq \langle Y_A Y_B Z_{R_{AB}} \rangle_\varrho + \langle Y_B Y_C Z_{R_{BC}} \rangle_\gamma - 1.$$  \hspace{1cm} (43)

By comparing the marginals of the states $\gamma$ and $\varrho$, we have

$$\langle Y_A Y_B Z_{R_{AB}} \rangle_\gamma = \langle Y_A Y_B Z_{R_{AB}} \rangle_\varrho,$$  \hspace{1cm} (44)
$$\langle Y_B Y_C Z_{R_{BC}} \rangle_\gamma = \langle Y_B Y_C Z_{R_{BC}} \rangle_\varrho,$$  \hspace{1cm} (45)

since $A, C \not\in R_{AB} \cup R_{BC}$. In the following, we will also use the notation

$$\mathcal{R} = R_{AC} = E_A \cup E_C \cup J_{AB} \cup J_{BC}$$  \hspace{1cm} (46)

in order to avoid a plethora of indices.

Then, we consider the inflation $\eta$ shown in Fig. 9. This is constructed as follows. First, one has two disconnected complete graphs, $K$ and $K'$. Then, one takes the subset $\mathcal{R}$ as a subgraph of $K$ and rewires all connections from vertices in $\mathcal{R}$ to $A$ to $A'$. Similarly, one takes the subset $\mathcal{R}'$ as a subgraph of $K'$ and rewires all connections from vertices in $\mathcal{R}'$ to $A'$ to $A$.

By comparing the marginals of $\gamma$ and $\eta$, this time we have

$$\langle Y_A Y_C Z_{\mathcal{R}} \rangle_\gamma = \langle Y_A Y_C Z_{\mathcal{R}} \rangle_\eta.$$  \hspace{1cm} (47)

With this reasoning, we have established that the correlation $\langle Y_A Y_C Z_{\mathcal{R}} \rangle_\eta$ is large in $\eta$, if the original state $\varrho$ is close to the graph state. Now we have to identify another anticommuting observables in $\eta$ with large expectation value in order to arrive at a contradiction.

A natural first candidate is the stabilizing operator

$$g_B = X_B Z_A Z_C Z_{\mathcal{R}}.$$  \hspace{1cm} (48)

with $\mathcal{R} = E_B \cup J_{AB} \cup J_{BC} \cup T_{ABC}$ of the graph state. This, however, cannot always be identified with some observable in the inflation $\eta$. Still, if

$$\mathcal{R} \cap \mathcal{R} = \emptyset \Leftrightarrow J_{AB} = J_{BC} = \emptyset$$  \hspace{1cm} (49)

the observable $g_B$ is not affected by the rewiring in $\eta$, we have $\langle g_B \rangle_\varrho = \langle g_B \rangle_\eta$. Moreover, in $\eta$ the observables $g_B$ and $Y_A^+ Y_C Z_{\mathcal{R}}$ anticommute. So, in this case we have

$$\langle Y_A Y_C Z_{\mathcal{R}} \rangle_\gamma^2 + \langle g_B \rangle_\gamma^2 \leq 1.$$  \hspace{1cm} (50)

and for the original $\varrho$ we arrive at the condition (assuming $\langle Y_A Y_B Z_{R_{AB}} \rangle_\varrho + \langle Y_B Y_C Z_{R_{BC}} \rangle_\varrho - 1 \geq 0$, as in Eq. (6) in the main text)

$$((\langle Y_A Y_B Z_{R_{AB}} \rangle_\varrho + \langle Y_B Y_C Z_{R_{BC}} \rangle_\varrho - 1)^2 + \langle g_B \rangle_\varrho^2 \leq 1$$  \hspace{1cm} (51)

for states that can be prepared in the network.

Furthermore, in the case that $E_A = E_C = 0$ and by making use of the operator

$$g_{AB} g_{BC} = X_A X_B X_C Z_{E_B \cup T_{ABC}},$$  \hspace{1cm} (52)

we arrive at a similar condition on $\varrho$ that is also violated by the graph state $|G\rangle$. 

---

**Fig. 7.** The bipartite network with complete graph $K$. This is the network that should generate the graph state from Fig. 6. Note that here the bipartite links of the network are shown, and not the edges of the graph of the graph state.

**Fig. 8.** The network for the inflation $\gamma$. This is a two-copy inflation, where only the links between $AC$ and $A'C'$ are rewired.

**Fig. 9.** The network for the inflation $\eta$. This is constructed as follows. First, one has two disconnected complete graphs, $K$ and $K'$. Then, one takes the subset $\mathcal{R}$ as a subgraph of $K$ and rewires all connections from vertices in $\mathcal{R}$ to $A$ to $A'$. Similarly, one takes the subset $\mathcal{R}'$ as a subgraph of $K'$ and rewires all connections from vertices in $\mathcal{R}'$ to $A'$ to $A$. 

Where $R_{AB} = E_A \cup E_B \cup J_{AC} \cup J_{BC}$, and analogously for $R_{AC}$ and $R_{BC}$. Since $g_{AC} = (g_{AB})(g_{BC})$, we can apply the usual argument from the GHZ state to conclude that

$$\langle Y_A Y_C Z_{R_{AC}} \rangle_\gamma \geq \langle Y_A Y_B Z_{R_{AB}} \rangle_\gamma + \langle Y_B Y_C Z_{R_{BC}} \rangle_\gamma - 1.$$  \hspace{1cm} (43)
Finally, if $E_A = J_{AB} = \emptyset$, one can make use of the operator
\[ g_A = X_A Z_B Z_C Z_{J_{AC} \cup J_{ABC}}, \] (53)
in order to arrive at a condition on $\varrho$ that is not satisfied by the graph state $|G\rangle$.

By permuting $A, B, C$ in the above argument, we finish our proof.

In the following, we identify some basic situations where the Theorem 3 can be applied. First, we show that the conditions of Theorem 3 are met, if there is one vertex with a small degree.

**Corollary 4.** Let $G$ be a connected graph with no less than three vertices. If its minimal degree is no more than three, the graph state $|G\rangle$ cannot be generated by any network with bipartite sources.

**Proof.** The proof will be done successively for minimal degree one, two and three.

Let $v$ be a vertex whose degree is one and let $w$ be the vertex connected to $v$. Since $G$ is a connected graph with no less than three vertices,
\[ N_w \setminus \{v\} \neq \emptyset, \] (54)
where $N_w$ is the neighbourhood of $w$. If we apply local complementation on the vertex $w$, we obtain a new graph $G'$, where
\[ u \sim v, \quad \forall u \in N_w \setminus \{v\}, \] (55)
where $u \sim v$ means that the vertices $u, v$ are connected.

By setting
\[ B = w, \quad A = v, \quad C = u_0, \] (56)
where $u_0$ is an arbitrary vertex in $N_w \setminus \{v\}$, we see that
\[ N_B \setminus \{A, C\} = T_{ABC} \cup J_{AB}, \]
\[ A \sim B, \quad A \sim C, \quad B \sim C. \] (57)
Hence, $E_B = J_{BC} = \emptyset$, which implies $|G'\rangle$ cannot be from any network with only bipartite sources. Since $|G\rangle$ is equivalent to $|G'\rangle$ up to a local unitary transformation, we come to the same conclusion for $|G\rangle$.

Now, let us consider graphs with minimal degree equal to two, and let $v$ be a vertex with degree two, and $w$ and $u$ be the two vertices connected to $v$. If $w \not\sim u$, we can apply a local complementation on $v$ to connect them. Hence, we can assume $w \sim u$ without loss of generality.

By setting
\[ A = w, \quad B = v, \quad C = u, \] (58)
we have
\[ E_B = J_{AB} = J_{BC} = T_{ABC} = \emptyset, \] (59)

which leads to the desired conclusion.

Lastly, let $v$ be a vertex with degree three and let $w, u$ and $t$ be the three vertices connected to $v$. Since we can apply local complementation on $v$, without loss of generality, we can assume that there are at least two edges among $w, u$ and $t$, more specifically, $w \sim u$ and $w \sim t$. Let us take
\[ A = w, \quad B = v, \quad C = u, \] (60)

hence we see that
\[ E_B = \emptyset, \quad J_{AB} = \emptyset \quad \text{or} \quad J_{BC} = \emptyset. \] (61)
This implies that the graph state $|G\rangle$ with a vertex of degree three cannot be from any network with only bipartite sources.

Under certain conditions, we can also exclude a network structure for graphs with minimal degree four:

**Corollary 5.** Let $G$ be a graph that has a vertex $v$ of degree four. If the induced subgraph on the neighborhood $N_v$ is not a line graph, then $|G\rangle$ cannot be generated in any network with bipartite sources.

**Proof.** As shown in Fig. 10, there are six inequivalent graphs with four vertices up to permutation and complementation. In case (a), we can set $B = v, A = u_1, C = u_2$, then
\[ T_{ABC} = \{u_3, u_4\}, \quad E_B = J_{AB} = J_{BC} = \emptyset. \] (62)
In case (b) and (c), we can set $B = v, A = u_3, C = u_4$, then
\[ E_B = \{u_1, u_2\}, \quad J_{AB} = J_{BC} = T_{ABC} = \emptyset. \] (63)
In case (d) and (e), we can set $B = v, A = u_1, C = u_3$, then
\[ E_B = \{u_2\}, \quad T_{ABC} = \{u_4\}, \quad J_{AB} = J_{BC} = \emptyset. \] (64)
In all the above cases, Theorem 3 implies that the graph state \( |G\rangle \) cannot be from any network with only bipartite sources. In case (f), the neighbourhood \( N_v \) of \( v \) is a line graph whose complementation is also a line graph.

Having established these results, we can discuss graphs with a small number of vertices. Here, previous works have established a classification of all small graphs with respect to equivalence classes under local complementation. In detail, this classification has been achieved for up to seven vertices in Ref. [46], for eight vertices in Ref. [55] and for nine to twelve vertices in Ref. [48]. These required numerical techniques are advanced, as, for instance, for 12 qubits already \( 1 \ 274 \ 068 \) different equivalence classes under local complementation exist. We can use this classification now, and apply our result on it to obtain:

**Theorem 6.** No graph state with up to 12 vertices can originate from a network with only bipartite sources.

_Proof._ Using the tables in Ref. [48] one can directly check that except the graph \( G_{d5} \) in Fig. 11, all graphs with no more than 12 vertices, up to isomorphism and local complementation, satisfy at least one condition in Corollary 4.

For the graph \( G_{d5} \), the minimal degree is no less than 5 whatever local complementation is applied. However, if we set

\[
B = v_1, \quad A = v_4, \quad C = v_5, \tag{65}
\]

then

\[
E_B = \{v_2, v_3\}, \quad T_{ABC} = \{v_6\}. \tag{66}
\]

Thus, \( J_{AB} = J_{BC} = \emptyset \), which implies that the graph state \( |G_{d5}\rangle \) cannot originate from a network with only bipartite sources.

All in all, no graph state with more than 12 vertices can originate from a network with only bipartite sources.

The statements in Observation 1 concerning the two- or three-dimensional cluster states follow directly from Corollary 5 (in the 2D case) or the application of a local complementation and Theorem 3.

### APPENDIX D: PERMUTATIONALLY SYMMETRIC STATES

Before proving our main results, let us give some definitions. As introduced in the main text, we define \( N \)-partite permutationally symmetric (bosonic) states as states that satisfy \( \Pi_{ij}^a \Pi_{ij}^b = \varrho \) for all \( i, j \in \{1, \ldots, N\} \) with \( 2\Pi_{ij}^+ = 1 + F_{ij} \) and \( F_{ij} \) being the flip operator that exchanges parties \( i \) and \( j \). We can also introduce fermionic states that are antisymmetric for a pair of parties \( \{ij\} \), i.e. \( \Pi_{ij} \Theta \Pi_{ij}^+ = \varrho \), with \( 2\Pi_{ij}^+ = 1 + F_{ij} \). For our discussion we need several basic facts. We stress that the following Lemma 7 and 9 are well known [30, 33, 56], while Lemma 8 is a simple technical statement.

**Lemma 7.** Let \( \varrho = \sum_k p_k |\psi_k\rangle \langle \psi_k| \) be a multipartite state and let \( \Pi \) be a projector such that \( \Pi \varrho \Pi = \varrho \). Then \( \Pi |\psi_k\rangle = |\psi_k\rangle \) for all \( k \).

_Proof._ One has

\[
1 = \text{Tr}(\varrho) = \text{Tr}(\Pi \varrho \Pi) = \sum_k p_k \langle \psi_k | \Pi | \psi_k \rangle. \tag{67}
\]

So \( \langle \psi_k | \Pi | \psi_k \rangle = 1 \), and since \( \Pi \) is a projector, \( |\psi_k\rangle = |\psi_k\rangle \) for all \( k \).

This holds in particular for \( \Pi = \Pi_{ij}^\pm \). As a second lemma, we have

**Lemma 8.** If the reduced state on \( AB \) of some state is symmetric or antisymmetric under the exchange of parties \( A \) and \( B \), then the global state also is.

_Proof._ Let \( \varrho = \sum_k p_k |\psi_k\rangle \langle \psi_k| \) be the state of some tripartite system \( ABC \). Let us prove that if \( F_{AB} (\text{Tr}_C(\varrho)) = \pm \text{Tr}_C(\varrho) \), then \( (F_{AB} \otimes 1) \varrho = \pm \varrho \). If one considers the Schmidt decomposition of \( |\psi_k\rangle \) wrt the bipartition \( AB|C \), one has

\[
\varrho = \sum_k p_k \sum_{i,j} s_{k,i} s_{k,j}^* |\phi_{k,i}^{AB}\rangle \langle \phi_{k,j}^{AB}| \otimes |\chi_{k,i}^C\rangle \langle \chi_{k,j}^C|. \tag{68}
\]

From that, \( \varrho_{AB} = \sum_k p_k \sum_i |s_{k,i}|^2 |\phi_{k,i}^{AB}\rangle \langle \phi_{k,i}^{AB}| \) and since it is a permutationally symmetric (respectively antisymmetric) state, from Lemma 7 all states in its decomposition also are and thus \( (F_{AB} \otimes 1) \varrho = \pm \varrho \).

We note that for both those lemma, the converse is trivial. Finally, we have:
Lemma 9. (a) A $N$-partite symmetric state $\varrho$, is either genuinely $N$-partite entangled or fully separable. (b) A $N$-partite antisymmetric state is always $N$-partite entangled.

Proof. Due to Lemma 7 we only need to consider pure states. Let $|\Psi\rangle$ be a $N$-partite (anti)symmetric state that is not $N$-partite entangled, hence it is separable for some bipartition. Without loss of generality, we assume that

$$|\Psi\rangle = |\varphi_{1,...,t}\rangle \otimes |\phi_{t+1,...,N}\rangle.$$  

(69)

Thus, by tracing out the first $t$ parties, we have a pure state. The symmetry of $|\Psi\rangle$ implies that the reduced state is pure after tracing out any $t$ parties. This can only be true if $|\varphi_{1,...,t}\rangle, |\phi_{t+1,...,N}\rangle$ are fully separable.

Besides, denote $|ab...c\rangle$ a normalized fully separable antisymmetric state, we have $|ab...c\rangle = -|ba...c\rangle$. This implies that $-1 = \langle ab...c|ba...c\rangle = |\langle a\mid b\rangle|^2 \geq 0$, hence we arrive at a contradiction. \hfill \Box

We note that the notions of entanglement used in this Lemma are the standard ones for non-symmetric states, as these are the relevant ones for the main text. In principle, for indistinguishable particles one may separate the “formal” entanglement due to the wave function symmetrization from the “physical” entanglement [30].

Now, let us prove the Observation 2 of the main text. For completeness, we restate it here in the full formulation:

Observation 2'. Let $\varrho$ be a permutationally symmetric multiparticle state. Then, $\varrho$ can be generated in a quantum network with $N-1$-partite sources if and only if it is fully separable. If $\varrho$ is not permutationally antisymmetric, then it cannot be generated in a network.

Proof. Let $\varrho$ be a $N$-partite permutationally (anti)symmetric state. Let us assume that it can be generated in a network of $N$ nodes with some at most $(N-1)$-partite sources. Note that any state that can be generated in a network of $N$ nodes with no $N$-partite sources, can also be generated in a network of $N$ nodes with $N$ different $(N-1)$-partite sources. Let us denote by $\zeta_i$ the source used to generate $\varrho$ that distributes parties to all nodes except the $ith$ one.

If we assume that $\varrho$ is a network state, then the inflation $\eta$ build the following way is a physical state: Consider a network of $2N$ nodes $\{A_i, A'_i : i = 1, \ldots, N\}$ and $2N$ sources $\{\xi_k, \xi'_k : k = 1, \ldots, N\}$ that distribute parties to

$$\xi_k : A_1 \ldots A_{k-1} A'_{k+1} \ldots A'_N,$$

$$\xi'_k : A'_1 \ldots A'_{k-1} A_{k+1} \ldots A_N,$$

(70, 71)

where $\xi_k = \xi'_k = \xi_k$ for all $k$. The state $\eta$ is the network state build with these sources and the same channels on the nodes than $\varrho$ (with some shared randomness). From the inflation technique, we know that for the reduced states

$$\eta_{A_i A_{i+1}} = \eta_{A'_i A'_{i+1}} = \varrho_{A_i A_{i+1}}, \quad \forall i \leq N - 1,$$

$$\eta_{A_i A_N} = \eta_{A'_i A_N} = \varrho_{A_i A_N}.$$  

(72, 73)

Since the state $\varrho$ is fully (anti)symmetric, Lemma 8 and Eq. (72) imply that the state $\eta$ is also fully (anti)symmetric.

Now, we consider the inflated state $\tau$, whose sources $\{\omega_k, \omega'_k : k = 1, \ldots, N\}$ distribute states to

$$\omega_k : A_1 A_2 A_3 \ldots A_{k-1} A_{k+1} \ldots A_N,$$

$$\omega'_k : A'_1 A'_2 A'_3 \ldots A'_{k-1} A'_{k+1} \ldots A'_N.$$  

(74)

Again, the local channels and shared randomness are the same than for $\varrho$. This is the two-copy inflation considered several times in the main text. One has

$$\tau_{A_1 \ldots A_N} = \tau_{A'_1 \ldots A'_N} = \varrho.$$  

(75)

Moreover,

$$\tau_{A_i A'_i} = \eta_{A_i A'_i}$$  

(76)

hence $\tau$ is permutationally fully (anti)symmetric under the exchange of all its parties. However, $\tau$ is separable wrt the bipartition $A_1 \ldots A_N | A'_1 \ldots A'_N$. In the fully symmetric case, this means that $\tau$ is fully separable. Therefore $\varrho$ is also fully separable. So, if a network state is permutationally symmetric, it needs to be fully separable. In the fully antisymmetric case, the full separability of $\tau$ contradicts with the assumption that $\tau$ is fully antisymmetric. So, no network state can be permutationally antisymmetric. \hfill \Box

Finally, we note that cyclic symmetric states may be generated in network scenarios, as already pointed out by Ref. [21]. As an example, we consider three Bell pairs $|\Phi^+\rangle$ as sources in the triangle network, and no channels applied. The global $3 \times 4$-partite state is $|\Psi\rangle_{ABC} = |\Phi^+\rangle_{A_2 B_2} |\Phi^+\rangle_{B_2 C_1} |\Phi^+\rangle_{C_1 A_1}$, with $A = A_1 A_2$ and so on. With the appropriate reordering of the parties and by mapping $|ij\rangle_X \mapsto |2i+j\rangle_X$ for $X = A, B, C$, one gets

$$|\Psi\rangle_{ABC} = \frac{1}{2\sqrt{2}}(|000\rangle + |012\rangle + |120\rangle + |201\rangle$$

$$+ |132\rangle + |321\rangle + |213\rangle + |333\rangle),$$  

(77)

which is a symmetric state under cyclic permutations of the $3 \times 4$-dimensional system $ABC$.

**APPENDIX E: CERTIFYING NETWORK LINKS**

Here, we prove the statement made in the main text, which can be formulated as follows:
Observation 10. If a state \( \varrho \) can be prepared in a network with bipartite sources but without the link \( AC \), then
\[
\langle X_A X_C P_{R_1} \rangle^2 + \langle Y_A Y_C P_{R_2} \rangle^2 + \langle Z_A Z_C P_{R_3} \rangle^2 \leq 1. \tag{78}
\]
Here the \( P_{R_i} \) are arbitrary observables on disjoint subsets of the other particles, \( R_i \cap R_j = \emptyset \). If the state \( \varrho \) was indeed prepared in a real quantum network, then violation of this inequality proves that the link \( AC \) is working and distributing entanglement.

Proof. Without loss of generality, we assume that
\[
R_1 = \{ E \}, \quad R_2 = \{ B \}, \quad R_3 = \{ D \}. \tag{79}
\]
Otherwise, we can prove the result similarly. The disconnected nodes \( A \) and \( C \) may be connected via some source with the \( R_i \) or not, but this is not essential. Then, the graph has a structure as the graph in Fig. 12.

We consider a three-copy inflation \( \xi \) of this graph, where the observables in Eq. (78) overlap only in the node \( A \). This inflation is constructed as follows: All links from \( B \) to \( C \) are rewired from \( B \) to \( C' \) and all links from \( D \) to \( C \) are rewired from \( D \) to \( C'' \). This is shown schematically in Fig. 13.

The anticommuting relations imply that
\[
\langle X_A X_C P_{E\xi} \rangle^2 + \langle Y_A Y_C P_{B\xi} \rangle^2 + \langle Z_A Z_C P_{D\xi} \rangle^2 \leq 1. \tag{80}
\]
By comparing the marginals of \( \varrho \) and \( \xi \), we have
\[
\langle X_A X_C P_{E\xi} \rangle = \langle X_A X_C P_{E\varrho} \rangle, \tag{81}
\]
\[
\langle Y_A Y_C P_{B\xi} \rangle = \langle Y_A Y_C P_{B\varrho} \rangle, \tag{82}
\]
\[
\langle Z_A Z_C P_{D\xi} \rangle = \langle Z_A Z_C P_{D\varrho} \rangle. \tag{83}
\]
By substituting the mean values with state \( \xi \) by the ones with \( \varrho \) in Eq. (80), we complete our proof.

Finally, we give a simple example where this criterion can detect the functionality of a link, while a simple concentration on the reduced two-qubit density matrix does not work. Consider the state
\[
\varrho = \frac{1}{2}(|s_1\rangle\langle s_1| + |s_2\rangle\langle s_2|), \tag{84}
\]
where
\[
|s_1\rangle = (|00\rangle_{AC} + |11\rangle_{AC}) \otimes |00\rangle_{BD}, \tag{85}
\]
\[
|s_2\rangle = (|00\rangle_{AC} - |11\rangle_{AC}) \otimes |11\rangle_{BD}. \tag{86}
\]
Here, we want to check whether the link \( AC \) works or not. Since the reduced state on \( AC \) is separable, we cannot use the criteria which acts only on \( AC \). It is easy to verify that
\[
\langle X_A \mathbb{I}_B X_C \mathbb{I}_D \rangle = 0,
\]
\[
-\langle Y_A \mathbb{I}_B Y_C Z_D \rangle = \langle Z_A \mathbb{I}_B Z_C \mathbb{I}_D \rangle = 1, \tag{87}
\]
which violates Eq. (80). Hence, our criteria can detect the link \( AC \) more effectively.

[1] H. J. Kimble, Nature 453, 1023 (2008).
[2] C. Simon, Nat. Phot. 11, 678 (2017).
[3] S. Wehner, D. Elkouss, and R. Hanson, Science 362, 9288 (2018).
[4] J. Biamonte, M. Faccin, and M. De Domenico, Commun. Phys. 2, 53 (2019).
[5] J. Yin, Y. Cao, Y.-H. Li, et al., Science 356, 1140 (2017).
[6] S.-K. Liao, W.-Q. Cai, J. Handsteiner, et al., Phys. Rev. Lett. 120, 030501 (2017).
[7] S.-K. Liao, W.-Q. Cai, W.-Y. Liu, et al., Nature 549, 43 (2018).
[8] J. Yin, Y.-H. Li, S.-K. Liao, M. Yang, Y. Cao, L. Zhang, J.-G. Ren, W.-Q. Cai, W.-Y. Liu, S.-L. Li, et al., Nature 582, 501 (2020).
[9] P. C. Humphreys, N. Kalb, J. P. Morits, R. N. Schouten, R. F. Vermeulen, D. J. Twitchen, M. Markham, and R. Hanson, Nature 558, 268 (2018).
[10] L. J. Stephenson, D. P. Nadlinger, B. C. Nichol, S. An, P. Drmota, T. G. Ballance, K. Thirumalai, J. F. Goodwin, D. M. Lucas, and C. J. Ballance, Phys. Rev. Lett. 124, 110501 (2020).
[11] R. Chaves, L. Luft, T. O. Maciel, D. Gross, D. Janzing, and B. Schölkopf, Proceedings of the 30th Conference on Uncertainty in Artificial Intelligence (2014), p. 112.
[12] R. Chaves, C. Majenz, and D. Gross, Nature Commun. 6, 5766 (2015).
[13] E. Wolfe, R. W. Spekkens, and T. Fritz, J. Causal Inference 7, 20170020 (2019).
[14] C. Branciard, N. Gisin, and S. Pironio, Phys. Rev. Lett. 104, 170401 (2010).
[15] C. Branciard, D. Rosset, N. Gisin, and S. Pironio, Phys. Rev. A 85, 032119 (2012).
[16] T. Fritz, New J. Phys. 14, 103001 (2012).
[17] D. Rosset, C. Branciard, T. J. Barnea, G. Pütz, N. Brunner, and N. Gisin, Phys. Rev. Lett. 116, 010403 (2016).
[18] M.-O. Renou, E. Bäumer, S. Boreiri, N. Brunner, N. Gisin, and S. Beigi, Phys. Rev. Lett. 123, 140401 (2019).
[19] N. Gisin, J. D. Bancal, Y. Cai, et al., Nat. Commun. 11, 2378 (2020).
[20] T. Kraft, S. Designolle, C. Ritz, N. Brunner, O. Gühne, and M. Huber, Phys. Rev. A. 103, L060401 (2021).
[21] M.-X. Luo, Adv. Quantum Technol., 4, 2000123 (2021).
[22] M. Navascués, E. Wolfe, D. Rosset, and A. Poyraz-Kerstjens, Phys. Rev. Lett. 125, 240505 (2020).
[23] J. Åberg, R. Nery, C. Duarte, and R. Chaves, Phys. Rev. Lett. 125, 110505 (2020).
[24] T. Kraft, C. Spee, X.-D. Yu, and O. Gühne, Phys. Rev. A 103, 052405 (2021).
[25] M.-X. Luo, arXiv:2107.05846 (2021).
[26] M. Navascués, E. Wolfe, D. Rosset, and A. Poyraz-Kerstjens, Phys. Rev. Lett. 125, 240505 (2020).
[27] S. Coleman, Aspects of Symmetry, Cambridge University Press, Massachusetts (1985).
[28] R. F. Werner, Phys. Rev. A 40, 4277 (1989).
[29] K. G. H. Vollbrecht and R. F. Werner, Phys. Rev. A 64, 062307 (2001).
[30] K. Eckert, J. Schliemann, D. Bruß, and M. Lewenstein, Ann. Phys. 299, 88 (2002).
[31] C. M. Caves, C. A. Fuchs, and R. Schack, J. Math. Phys. 43, 4537 (2002).
[32] A. C. Doherty, P. A. Parrilo, and F. M. Spedalieri, Phys. Rev. Lett. 88, 187904 (2002).
[33] G. Tóth and O. Gühne, Phys. Rev. Lett. 102, 170503 (2009).
[34] C. Eltschka and J. Siewert, Phys. Rev. Lett. 108, 020502 (2012).
[35] S. Wehner and A. Winter, J. Math. Phys. 49, 062105 (2008).
[36] S. Niekamp, M. Kleinmann, and O. Gühne, J. Math. Phys. 53, 012202 (2012).
[37] G. Tóth and O. Gühne, Phys. Rev. A 72, 022340 (2005).
[38] D. Gotte, PhD Thesis, Caltech (1997).
[39] M. Hein, W. Dür, J. Eisert, R. Raussendorf, M. Van den Nest, and H.-J. Briegel, in Quantum Computers, Algorithms and Chaos, Proceedings of the International School of Physics “Enrico Fermi,” Vol. 162, Varenna, 2005, edited by G. Casati, D. L. Shepelyansky, P. Zoller, and G. Benenti (IOS Press, Amsterdam, 2006). See also arXiv:quant-ph/0602096.
[40] M. C. Tran, R. Ramanathan, M. McKague, D. Kaszlikowski, and T. Paterek, Phys. Rev. A 98, 052325 (2018).
[41] P. Kurzyński, T. Paterek, R. Ramanathan, W. Laskowski, and D. Kaszlikowski, Phys. Rev. Lett. 106, 180402 (2011).
[42] R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. 86, 5188 (2001).
[43] A. Kitaev, Annals Phys. 303, 2 (2003).
[44] X.-C. Yao, T.-X. Wang, H.-Z. Chen, et al., Nature 482, 489–494 (2012).
[45] M. Van den Nest, J. Dehaene, and B. De Moor, Phys. Rev. A 69, 022316 (2004).
[46] M. Hein, J. Eisert, and H. J. Briegel, Phys. Rev. A 69, 062311 (2004).
[47] A. Cabello, L. E. Danielsen, A. J. Lópex-Tarrida, and J. R. Portillo, Phys. Rev. A 83, 042314 (2011).
[48] L. E. Danielsen, Database of Entanglement in Graph States (2011), see also http://www.ii.uib.no/~larsed/entanglement/.
[49] We note that similar statements on genuine multiparticle states or symmetric states have been formulated in Refs. [21], but these are restricted to pure states only and cannot be applied in the presence of noise.
[50] M. Epping, H. Kampermann, and D. Bruß, New J. Phys. 18, 053036 (2016).
[51] M. Epping, H. Kampermann, C. Macchiavello, and D. Bruß, New J. Phys. 19, 093012 (2017).
[52] C. Spee and T. Kraft arXiv:2105.01090 (2021).
[53] O. Gühne and M. Seevinck, New J. Phys. 12, 053002 (2010).
[54] K. Audenaert and M. B Plenio, New J. Phys. 7, 170 (2005).
[55] A. Cabello, A. J. Lópex-Tarrida, P. Moreno, and J.R. Portillo, Phys. Rev. A 80, 012102 (2009).
[56] B. Kraus, PhD Thesis, University of Innsbruck (2003).