Instantaneously complete Chern–Ricci flow and Kähler–Einstein metrics

Shaochuang Huang\(^1\) · Man-Chun Lee\(^2\) · Luen-Fai Tam\(^3\)

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Abstract
In this work, we obtain some existence results of Chern–Ricci Flows and the corresponding Potential Flows on complex manifolds with possibly incomplete initial data. We discuss the behaviour of the solution as \(t \to 0\). These results can be viewed as a generalization of an existence result of Ricci flow by Giesen and Topping for surfaces of hyperbolic type to higher dimensions in certain sense. On the other hand, we also discuss the long time behaviour of the solution and obtain some sufficient conditions for the existence of Kähler-Einstein metric on complete non-compact Hermitian manifolds, which generalizes the work of Lott–Zhang and Tosatti–Weinkove to complete non-compact Hermitian manifolds with possibly unbounded curvature.

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1 Introduction
In this work, we will discuss conditions on the existence of Chern–Ricci Flows and the corresponding Potential Flows on complex manifolds with possibly incomplete initial data.

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✉️ Man-Chun Lee
mcline@math.northwestern.edu
Shaochuang Huang
schuang@mail.tsinghua.edu.cn
Luen-Fai Tam
lftam@math.cuhk.edu.hk

1 Yau Mathematical Sciences Center, Tsinghua University, Beijing, China
2 Department of Mathematics, Northwestern University, 2033 Sheridan Road, Evanston, IL 60208, USA
3 The Institute of Mathematical Sciences and Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong, China
The flows will be described later. We will also discuss conditions on long-time existence and convergence to Kähler–Einstein metrics.

We begin with the definitions of Chern–Ricci flow and the corresponding potential flow. Let \( M^n \) be a complex manifold with complex dimension \( n \). Let \( h \) be a Hermitian metric on \( M \) and let \( \theta_0 \) be the Kähler form of \( h \):

\[
\theta_0 = \sqrt{-1} h_{i\bar{j}} dz^i \wedge d\bar{z}^j
\]

where \( h = h_{i\bar{j}} dz^i \otimes d\bar{z}^j \) in local holomorphic coordinates. In this work, Einstein summation convention is enforced.

In general, suppose \( \omega \) is a real \((1,1)\) form on \( M \), if \( \omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j \) in local holomorphic coordinates then the corresponding Hermitian form \( g \) is given by

\[
g = g_{i\bar{j}} dz^i \otimes d\bar{z}^j.
\]

In case \( \omega \) is only nonnegative, we call \( g \) to be the Hermitian form of \( \omega \) and \( \omega \) is still called the Kähler form of \( g \).

Now if \((M^n, h)\) is a Hermitian manifold with Kähler form \( \theta_0 \), let \( \nabla \) be the Chern connection \( \nabla \) of \( h \) and \( \text{Ric}(h) \) be the Chern–Ricci tensor of \( h \) (or the first Ricci curvature). In holomorphic local coordinates such that \( h = h_{i\bar{j}} dz^i \otimes d\bar{z}^j \), the Chern Ricci form is given by

\[
\text{Ric}(h) = -\sqrt{-1} \partial \bar{\partial} \log \det(h_{i\bar{j}}).
\]

For the basic facts on Chern connection and Chern curvature, we refer readers to [30, section 2], see also [17, Appendix A] for example.

Let \( \omega_0 \) be another nonnegative real \((1,1)\) form on \( M \). Define

\[
\alpha := -\text{Ric}(\theta_0) + e^{-t} (\text{Ric}(\theta_0) + \omega_0)
\]

where \( \text{Ric}(\theta_0) \) is the Chern–Ricci curvature of \( h \). We want to study the following parabolic complex Monge–Ampère equation:

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \log \left( \frac{(\alpha + \sqrt{-1} \partial \bar{\partial} u)^n}{\theta_0^n} \right) - u \quad \text{in } M \times (0, S] \\
u(0) &= 0
\end{aligned}
\]  

(1.2)

so that \( \alpha + \sqrt{-1} \partial \bar{\partial} u > 0 \) for \( t > 0 \). When \( M \) is compact and \( \omega_0 = \theta_0 \) is smooth metric, it was first studied by Gill in [13]. Here we are interested in the case when \( \omega_0 \) is possibly an incomplete metric on a complete non-compact Hermitian manifold \((M, h)\). Following [18], (1.2) will be called the potential flow of the following normalized Chern–Ricci flow:

\[
\begin{aligned}
\frac{\partial}{\partial t} \omega(t) &= -\text{Ric}(\omega(t)) - \omega(t); \\
\omega(0) &= \omega_0.
\end{aligned}
\]

(1.3)

It is easy to see that the normalized Chern–Ricci flow will coincide with the normalized Kähler-Ricci flow if \( \omega_0 \) is Kähler. It is well-known that if \( \omega_0 \) is a Hermitian metric and \( \omega(t) \) is Hermitian and a solution to (1.3) which is smooth up to \( t = 0 \), then

\[
u(t) = e^{-t} \int_0^t e^s \log \left( \frac{(\omega(s))^n}{\theta_0^n} \right) ds.
\]

(1.4)

satisfies (1.2). Moreover, \( u(t) \to 0 \) in \( C^\infty \) norm in any compact set as \( t \to 0 \). On the other hand, if \( u \) is a solution to (1.2) so that \( \alpha + \sqrt{-1} \partial \bar{\partial} u > 0 \) for \( t > 0 \), then

\[
\omega(t) = \alpha + \sqrt{-1} \partial \bar{\partial} u
\]

(1.5)
is a solution to (1.3) on $M \times (0, S]$. However, even if we know $u(t) \to 0$ as $t \to 0$ uniformly on $M$, it is still unclear that $\omega(t) \to \omega_0$ in general.

The first motivation is to study Ricci flows starting from metrics which are possibly incomplete and with unbounded curvature. In complex dimension one, the existence of Ricci flow starting from an arbitrary metric has been studied in details by Giesen and Topping [10–12,29]. In particular, the following was proved in [11]: If a surface admits a complete metric $H$ with constant negative curvature, then any initial data which may be incomplete can be deformed through the normalized Ricci flow for long time and converges to $H$. Moreover, the solution is instantaneously complete for $t > 0$. In higher dimensions, recently it is proved by Ge-Lin-Shen [9] that on a complete non-compact Kähler manifold $(M, h)$ with $\text{Ric}(h) \leq -h$ and bounded curvature, if $\omega_0$ is a Kähler metric, not necessarily complete, but with bounded $C^k$ norm with respect to $h$ for $k \geq 0$, then (1.3) has a long time solution which converges to the unique Kähler–Einstein metric with negative scalar curvature, by solving (1.2). Moreover, the solution is instantaneously complete after it evolves.

Motivated by the above mentioned works, we first study the short time existence of the potential flow and the normalized Chern–Ricci flow. Our first result is the following:

**Theorem 1.1** Let $(M^n, h)$ be a complete non-compact Hermitian manifold with complex dimension $n$. Suppose there is $K > 0$ such that the following hold.

1. There is a proper exhaustion function $\rho(x)$ on $M$ such that
   
   $$|\partial \rho|^2_h + |\sqrt{-1} \partial \bar{\partial} \rho|_h \leq K.$$ 

2. $B K_h \geq -K$;

3. The torsion of $h$, $T_h = \partial \omega_h$ satisfies
   
   $$|T_h|^2_h + |\nabla^h T_h| \leq K.$$ 

Let $\omega_0$ be a nonnegative real $(1,1)$ form with corresponding Hermitian form $g_0$ on $M$ (possibly incomplete or degenerate) such that

(a) $g_0 \leq h$ and

$$|T_{g_0}|^2_h + |\nabla^h T_{g_0}|_h + |\nabla^h g_0|_h \leq K.$$ 

(b) There exist $f \in C^\infty(M) \cap L^\infty(M)$, $\beta > 0$ and $s > 0$ so that

$$-\text{Ric}(\theta_0) + e^{-s} (\omega_0 + \text{Ric}(\theta_0)) + \sqrt{-1} \partial \bar{\partial} f \geq \beta \theta_0.$$ 

Then (1.2) has a solution on $M \times (0, s)$ so that $u(t) \to 0$ as $t \to 0$ uniformly on $M$. Moreover, for any $0 < s_0 < s_1 < s$, $\omega(t) = \alpha + \sqrt{-1} \partial \bar{\partial} u$ is the Kähler form of a complete Hermitian metric which is uniformly equivalent to $h$ on $M \times [s_0, s_1]$. In particular, $g(t)$ is complete for $t > 0$.

Here $B K_h \geq -K$ means that for any unitary frame $\{e_k\}$ of $h$, we have $R(h)_{i\bar{j} j\bar{i}} \geq -K$ for all $i, j$.

**Remark 1.1** It is well-known that when $(M, h)$ is Kähler with bounded curvature, then condition (1) will be satisfied, [23,26]. See also [15,19] for related results under various assumptions.

Condition (b) was used in [17,18,30] with $\omega_0$ replaced by $\theta_0$ and is motivated as pointed out in [18] as follows. If we are considering cohomological class instead, in case that $\omega(t)$ is closed, then (1.3) is:

$$\partial_t [\omega(t)] = -[\text{Ric}(\omega(t)) - [\omega(t)]$$
and so

\[ [\omega(t)] = -(1 - e^{-t})[\text{Ric}(\theta_0)] + e^{-t}[\omega_0]. \]

Condition (b) is used to guarantee that \( \omega(t) > 0 \). In our case \( \omega_0, \theta_0, \omega(t) \) may not be closed and \( \omega_0 \) may degenerate. These may cause some difficulties. Indeed, the result is analogous to running \( \text{Kähler-Ricci flow} \) from a rough initial data. When \( M \) is compact, the potential flow from a rough initial data had already been studied by several authors, see for example \([2,25,28]\) and the references therein.

On the other hand, a solution of (1.2) gives rise to a solution of (1.3) when \( t > 0 \). It is rather delicate to see if the corresponding solution of (1.3) will attain the initial Hermitian form \( \omega_0 \). In this respect, we will prove the following:

**Theorem 1.2** With the same notation and assumptions as in Theorem 1.1. Let \( \omega(t) \) be the solution of (1.3) obtained in the theorem. If in addition \( h \) is Kähler and \( d\omega_0 = 0 \). Let \( U = \{\omega_0 > 0\} \). Then \( \omega(t) \to \omega_0 \) in \( C^\infty(U) \) as \( t \to 0 \), uniformly on compact subsets of \( U \).

We should remark that if in addition \( h \) has bounded curvature, then the theorem follows easily from pseudo-locality. The theorem can be applied to the cases studied in \([9]\) and to the case that \(-\text{Ric}(h) \geq \beta \theta_0\) outside a compact set \( V \) and \( \omega_0 > 0 \) on \( V \) with \( \omega_0 \) and its first covariant derivative are bounded. In particular, when \( \Omega \) is a bounded strictly pseudoconvex domain of another manifold \( M \) with defining function \( \varphi \), then the \( \Omega \) with the metric \( h_{ij} = -\partial_i \partial_j \log(-\varphi) \) will satisfy the above, see \([6, (1.22)]\).

Another motivation here is to study the existence of Kähler–Einstein metric with negative scalar curvature on complex manifolds using geometric flows. In \([1,32]\), Aubin and Yau proved that if \( M \) is a compact Kähler manifold with negative first Chern class \( c_1(M) < 0 \), then it admits a unique Kähler–Einstein metric with negative scalar curvature by studying the elliptic complex Monge–Ampère equation. Later, Cao \([3]\) reproved the above result using the Kähler–Ricci flow by showing that one can deform a suitable initial Kähler metric through normalized Kähler–Ricci flow to the Kähler–Einstein metric. Recently, Tosatti and Weinkove \([30]\) proved that under the same condition that \( c_1(M) < 0 \) on a compact complex manifold, the normalized Chern–Ricci flow (1.3) with an arbitrary Hermitian initial metric also has a long time solution and converges to the Kähler–Einstein metric with negative scalar curvature. In \([6]\), Cheng and Yau proved that if \( M \) is a complete non-compact Kähler manifold with Ricci curvature bounded above by a negative constant, injectivity radius bounded below by a positive constant and curvature tensor with its covariant derivatives are bounded, then \( M \) admits a unique complete Kähler–Einstein metric with negative scalar curvature. In \([4]\), Chau used Kähler–Ricci flow to prove that if \((M, g)\) is a complete non-compact Kähler manifold with bounded curvature and \( \text{Ric}(g) + g = -\sqrt{-1} \partial \bar{\partial} f \) for some smooth bounded function \( f \), then it also admits a complete Kähler–Einstein metric with negative scalar curvature. Later, Lott and Zhang \([18]\) generalized Chau’s result by assuming

\[ -\text{Ric}(g) + \sqrt{-1} \partial \bar{\partial} f \geq \beta g \]

for some smooth function \( f \) with bounded \( k \)th covariant derivatives for each \( k \geq 0 \) and positive constant \( \beta \). In this work, we will generalize the results in \([18,30]\) to complete non-compact Hermitian manifolds with possibly unbounded curvature.

For the long time existence and convergence, we will prove the following:

**Theorem 1.3** Under the assumption of Theorem 1.1, if in addition,

\[ -\text{Ric}(h) + \sqrt{-1} \partial \bar{\partial} f \geq \beta \theta_0 \]
for some \( f \in C^\infty(M) \cap L^\infty(M), \beta > 0. \) Then the solution constructed from Theorem 1.1 is a longtime solution and converges to a unique complete Kähler Einstein metric with negative scalar curvature on \( M. \)

As a consequence, we see that if \( h \) satisfies the conditions in the theorem, then \( M \) supports a complete Kähler–Einstein metric with negative scalar curvature, generalizing the results in [18,30].

The paper is organized as follows: In Sect. 2, we will derive a priori estimates along the potential flow and apply it in Sect. 3 to prove Theorem 1.1. Furthermore, we will study the short time behaviour of the constructed solution. In Sect. 4, we will prove the Theorem 1.3 and discuss longtime behaviour for general Kähler-Ricci flow if the initial data satisfies some extra condition. In Appendix A, we will collect some information about the relation between normalized Chern–Ricci flow and unnormalized one together with some useful differential inequalities. In Appendix B, we will state a maximum principle which will be used in this work.

2 a priori estimates for the potential flow

We will study the short time existence of the potential flow (1.2) with \( \omega_0 \) only being assumed to be nonnegative. We need some a priori estimates for the flow. In this section, we always assume the following:

1. There is a proper exhaustion function \( \rho(x) \) on \( M \) such that
\[
|\partial \rho|^2_h + |\sqrt{-1} \partial \bar{\partial} \rho|_h \leq K.
\]

2. \( B K_h \geq -K. \)

3. The torsion of \( h, T_h = \partial \omega_h \) satisfies
\[
|T_h|^2_h + |\nabla^h \partial T_h| \leq K.
\]

Here \( K \) is some positive constant.

On the other hand, let \( \omega_0 \) be a real \((1,1)\) form with corresponding Hermitian form \( g_0. \) We always assume the following:

a) \( g_0 \leq h \) and
\[
|T_{g_0}|^2_h + |\nabla^h T_{g_0}|_h + |\nabla^h g_0|_h \leq K.
\]

b) There exist \( f \in C^\infty(M) \cap L^\infty(M), \beta > 0 \) and \( s > 0 \) so that
\[
-\text{Ric}(\theta_0) + e^{-s}(\omega_0 + \text{Ric}(\theta_0)) + \sqrt{-1} \partial \bar{\partial} f \geq \beta \theta_0.
\]

Note that if \( g_0 \leq Ch, \) then we can replace \( h \) by \( Ch, \) then (b) is still satisfied with a possibly smaller \( \beta. \)

Since \( g_0 \) can be degenerate, we perturb \( g_0 \) in the following way: Let \( 1 \geq \eta \geq 0 \) be a smooth function on \( \mathbb{R} \) such that \( \eta(s) = 1 \) for \( s \leq 1 \) and \( \eta(s) = 0 \) for \( s \geq 2 \) so that \( |\eta'| + |\eta''| \leq c_1, \) say. For \( \epsilon > 0 \) and \( \rho_0 >> 1, \) let \( \eta_0(x) = \eta(\rho(x)/\rho_0). \) Consider the metric:
\[
\gamma_0 = \gamma_0(\rho_0, \epsilon) = \eta_0 \omega_0 + (1 - \eta_0) \theta_0 + \epsilon \theta_0.
\]

Then

- \( \gamma_0 \) is the Kähler form of a complete Hermitian metric, which is uniformly equivalent to \( h; \)

\( \square \)
• $BK(\gamma_0) \geq -C$ for some $C$ which may depend on $\rho_0$, $\epsilon$;
• The torsion $|T_{\gamma_0}|_{\gamma_0} + |\nabla^\gamma_0 T_{\gamma_0}|_{\gamma_0}$ is uniformly bounded by a constant which may depend on $\rho_0$, $\epsilon$.

We will obtain a short time existence for the potential flow starting with $\gamma_0$:

**Lemma 2.1** (1.2) has a solution $u(t)$ on $M \times [0, s]$ with $\alpha = -\text{Ric}(\theta_0) + e^{-t} (\text{Ric}(\theta_0) + \gamma_0)$ and $\omega(t) = \alpha + \sqrt{-1} \partial \bar{\partial} u$ such that $\omega(t)$ satisfies (1.3) with initial data $\gamma_0$, where $\omega(t)$ is the Kähler form of $g(t)$. Moreover, $g(t)$ is uniformly equivalent to $h$ on $M \times [0, s_1]$ for all $s_1 < s$.

**Proof** By the proof of [17, Theorem 4.1], it is sufficient to prove that for any $0 < s_1 < s$,

$$-\text{Ric}(\gamma_0) + e^{-s_1}(\gamma_0 + \text{Ric}(\gamma_0)) + \sqrt{-1} \partial \bar{\partial} f_1 \geq \beta_1 \gamma_0$$

for some smooth bounded function $f_1$ and some constant $\beta_1 > 0$. To simplify the notations, if $\eta, \zeta$ are real (1,1) forms, we write $\eta \geq \zeta$ if $\eta + \sqrt{-1} \bar{\partial} \bar{\partial} \phi \geq \zeta$ for some smooth and bounded function $\phi$. We compute:

$$-\text{Ric}(\gamma_0) + e^{-s_1}(\gamma_0 + \text{Ric}(\gamma_0)) = - (1 - e^{-s_1})\text{Ric}(\gamma_0) + e^{-s_1} \gamma_0$$

$$\geq - (1 - e^{-s_1})\text{Ric}(\theta_0) + e^{-s_1} \gamma_0$$

$$\geq 1 - e^{-s_1} (\beta \theta_0 - e^{-s} \omega_0) + e^{-s_1} \gamma_0$$

$$\geq 1 - e^{-s_1} \beta \theta_0$$

$$\geq \beta_1 \gamma_0$$

for some $\beta_1 > 0$ because $0 < s_1 < s$ and $\gamma_0 \geq \omega_0$. Here we have used condition (b) above, the fact that $\gamma_0^n = \theta_0^n e^H$ for some smooth bounded function $H$ and the definition of Chern–Ricci curvature. \hfill \Box

Let $\omega(t)$ be the solution in the lemma and let $u(t)$ be the potential as in (1.4). Since we want to prove that (1.2) has a solution $u(t)$ on $M \times (0, s)$ with $\alpha = -\text{Ric}(\theta_0) + e^{-t} (\text{Ric}(\theta_0) + \omega_0)$, in the next section, we need to obtain some uniform estimates of $u$, $\dot{u}$ and $\omega(t)$ which is independent of $\rho_0$ and $\epsilon$. The estimates are more delicate because the initial data $\omega_0$ maybe degenerate. For later applications, we need to obtain estimates on $(0, 1]$ and $[1, s)$ if $s > 1$. Note that for fixed $\rho_0$, $\epsilon$, $u(t)$ is smooth up to $t = 0$. Moreover, $u, \dot{u} =: \frac{\partial}{\partial t} u$ are uniformly bounded on $M \times [0, s_1]$ for all $0 < s_1 < s$.

### 2.1 a priori estimates for $u$ and $\dot{u}$

We first give estimates for upper bound of $u$ and $\dot{u}$.

**Lemma 2.2** There is a constant $C$ depending only on $n$ and $K$ such that

$$u \leq C \min\{t, 1\}, \quad \dot{u} \leq \frac{Ct}{e^t - 1}$$

on $M \times [0, s)$, provided $0 < \epsilon < 1$.

**Proof** The proofs here follow almost verbatim from the Kähler case [27], but we include brief arguments for the reader’s convenience. For notational convenience, we use $\Delta = g^{ij} \partial_i \partial_j$. 

\[ Springer \]
to denote the Chern Laplacian associated to $g$. Since $-\text{Ric}(\theta_0) = \omega(t) - e^{-t}(\text{Ric}(\theta_0) + \gamma_0) - \sqrt{-1}\partial\bar{\partial}u$ by (1.5), we have

\[
\left(\frac{\partial}{\partial t} - \Delta\right)(e^t \dot{u}) = e^t \dot{u} - e^t \text{tr}_\omega \text{Ric}(\theta_0) - e^t \left(\frac{\partial}{\partial t} - \Delta\right)u - ne^t
\]

\[
= e^t \text{tr}_\omega \left(-\text{Ric}(\theta_0) + \sqrt{-1}\partial\bar{\partial}u\right) - ne^t
\]

\[
= e^t \text{tr}_\omega (\omega - e^{-t}(\text{Ric}(\theta_0) + \gamma_0)) - ne^t
\]

\[
= -\text{tr}_\omega(\text{Ric}(\theta_0) + \gamma_0)
\]

\[
= \left(\frac{\partial}{\partial t} - \Delta\right)(\dot{u} + u) + n - \text{tr}_\omega(\gamma_0).
\]

Hence

\[
\left(\frac{\partial}{\partial t} - \Delta\right)(\dot{u} + u + nt - e^t \dot{u}) = \text{tr}_\omega \gamma_0 \geq 0.
\]

At $t = 0$, $\dot{u} + u + nt - e^t \dot{u} = 0$. By maximum principle Lemma B.1, we have

\[
(e^t - 1)\dot{u} \leq nt + u.
\]

Next consider

\[
F = u - At - \kappa \rho
\]

on $M \times [0, s_1]$ for any fixed $s_1 < s$. Here $\kappa > 0$ is a constant. Suppose $\sup_{M \times [0, s_1]} F > 0$, then there exists $(x_0, t_0) \in M \times (0, s_1]$ such that $F \geq F(x_0, t_0)$ on $M \times [0, s_1]$, and at this point,

\[
0 \leq \dot{u} - A = \log\left(\frac{\omega^n(t)}{\theta_0^n}\right) - u - A.
\]

Also, $\sqrt{-1}\partial\bar{\partial}u \leq \kappa \sqrt{-1}\partial\bar{\partial}\rho \leq \kappa K \theta_0$. Hence at $(x_0, t_0)$,

\[
\omega(t) = -\text{Ric}(\theta_0) + e^{-t}(\text{Ric}(\theta_0) + \gamma_0) + \sqrt{-1}\partial\bar{\partial}u
\]

\[
\leq (-1 + e^{-t})\text{Ric}(\theta_0) + e^{-t}\gamma_0 + \kappa K \theta_0
\]

\[
\leq (L + 2 + \kappa K)\theta_0,
\]

here $\text{Ric}(\theta_0) = -L(n, K)\theta_0$. Hence at $(x_0, t_0)$ we have

\[
u \leq n \log(L + 2 + \kappa K) - A
\]

\[
\leq 0
\]

if $A = n \log(L + 2) + 1$ and $\kappa > 0$ is small enough. Hence $F(x_0, t_0) < 0$. This is a contradiction. Hence $F \leq 0$ on $M \times [0, s_1]$ provided $A = A(n, K) = n \log(L + 2) + 1$ and we have

\[
u \leq At
\]

by letting $\kappa \to 0$. Combining this with (2.3), we conclude that

\[
\dot{u} \leq \frac{(A + n)t}{e^t - 1}.
\]

Combining this with (2.4), we conclude that $u \leq C$ for some constant $C$ depending only on $n, K$. Since $s_1$ is arbitrary, we complete the proof of Lemma 2.2. □
Next, we will estimate the lower bound of $u$ and $\dot{u}$.

**Lemma 2.3** (i) $u(x,t) \geq -\frac{C}{1-e^{-s}} t + nt \log(1-e^{-t})$ on $M \times [0,s)$ for some constant $C > 0$ depending only on $n, \beta, K, \|f\|_\infty$.

(ii) For $0 < s_1 \leq 1$ and $s_1 < s$,

$$\dot{u} + u \geq \frac{1}{1-e^{s_1-s}} \left( n \log t - \frac{C}{1-e^{-s}} \right)$$

some constant $C > 0$ depending only on $n, \beta, K, \|f\|_\infty$ on $M \times (0, s_1]$.

(iii) For $0 < s_1 \leq 1$ and $s_1 < s$,

$$\dot{u} + u \geq -C$$

on $M \times [0, s_1]$ for some constant $C > 0$ depending only on $n, \beta, K, \|f\|_\infty, s_1, s$ and $\epsilon$.

(iv) Suppose $s > 1$, then for $1 < s_1 < s$,

$$\dot{u} + u \geq -\frac{C(1 + s_1 e^{s_1-s})}{1-e^{s_1-s}}$$

on $M \times [1, s_1]$ for some constant $C(n, \beta, \|f\|_\infty, K) > 0$.

(v) For $0 < s_1 < s$,

$$u \geq -\frac{C(1 + s_1 e^{s_1-s})}{1-e^{s_1-s}}$$

on $M \times [0, s_1]$ for some constant $C(n, \beta, \|f\|_\infty, K) > 0$.

**Proof** In the following, $C_i$ will denote positive constants depending only on $n, \beta, \|f\|_\infty, K$ and $D_i$ will denote positive constants which may also depend on $\rho_0, \epsilon$ but not on $\kappa$.

To prove (i): Consider

$$F = u(x,t) - \frac{1-e^{-s}}{1-e^{-s}} f(x) + A \cdot t - nt \log(1-e^{-t}) + \kappa \rho(x).$$

Suppose $\inf_{M \times [0, s_1]} F < 0$. Then there exists $(x_0, t_0) \in M \times (0, s_1]$ such that $F \geq F(x_0, t_0)$ on $M \times [0, s_1]$. At this point, we have

$$0 \geq \frac{\partial}{\partial t} F$$

$$\dot{u} + u - \frac{e^{-t}}{1-e^{-s}} f(x) - n \log(1-e^{-t}) - \frac{nt}{e^t - 1}.$$ 

$$= \log\left( \frac{\theta_0^n}{\theta_0^n} \right) \left( -\text{Ric}(\theta_0) + e^{-t}(\text{Ric}(\theta_0) + \gamma_0) + \sqrt{-1} \partial \bar{\partial} u \right) - u + A$$

$$-n \log(1-e^{-t}) - \frac{nt}{e^t - 1} - \frac{e^{-t}}{1-e^{-s}} f$$

$$\geq \log\left( \frac{\theta_0^n}{\theta_0^n} \right) \left( -\text{Ric}(\theta_0) + e^{-t}(\text{Ric}(\theta_0) + \gamma_0) + \frac{1-e^{-t}}{e^t} \sqrt{-1} \partial \bar{\partial} f - \kappa \sqrt{-1} \partial \bar{\partial} \rho \right)$$

$$-C(n, K) - \frac{e^{-t}}{1-e^{-s}} f + A - n \log(1-e^{-t}) - \frac{nt}{e^t - 1},$$
where we have used the fact that \( u \leq C(n, K) \), and \( \sqrt{-1} \partial \bar{\partial} u \geq \frac{1 - e^{-t}}{1 - e^{-s}} \sqrt{-1} \partial \bar{\partial} f - \kappa \sqrt{-1} \partial \bar{\partial} \rho \).

Note that

\[-\text{Ric}(\theta_0) \geq \frac{1}{1 - e^{-s}} \left( \beta \theta_0 - e^{-s} \omega_0 - \sqrt{-1} \partial \bar{\partial} f \right),\]

hence

\[-\text{Ric}(\theta_0) + e^{-t}(\text{Ric}(\theta_0) + \gamma_0) + \frac{1 - e^{-t}}{1 - e^{-s}} \sqrt{-1} \partial \bar{\partial} f - \kappa \sqrt{-1} \partial \bar{\partial} \rho \geq e^{-t} \gamma_0 + \frac{1 - e^{-t}}{1 - e^{-s}} \left( \beta \theta_0 - e^{-s} \omega_0 \right) - \kappa K \theta_0 \]

\[\geq \frac{1}{2} \frac{1 - e^{-t}}{1 - e^{-s}} \beta \theta_0 \]

if \( \kappa \) is small enough. Here we have used the fact that \( 0 < t < s \) and \( \gamma_0 \geq \omega_0 \). Hence at \((x_0, t_0)\),

\[0 \geq n \log(1 - e^{-t}) - C_1 \]

\[-\frac{e^{-t}}{1 - e^{-s}} f + A - n \log(1 - e^{-t}) - \frac{nt}{e^t - 1} \]

\[\geq - \frac{1}{1 - e^{-s}} \|f\|_{\infty} + A - C_2\]

> 0

if \( A = \frac{1}{1 - e^{-t}} \|f\|_{\infty} + C_2 + 1 \). Hence for such \( A, F \geq 0 \) and for all \( \kappa > 0 \) small enough, we conclude that

\[u(x, t) \geq -At + nt \log(1 - e^{-t}).\]

To prove (ii), we have

\[
\left( \frac{\partial}{\partial t} - \Delta \right) (\dot{u} + u) = -\text{tr}_\omega(\text{Ric}(\theta_0)) - n.
\]

On the other hand, by (2.2), we also have

\[
\left( \frac{\partial}{\partial t} - \Delta \right) (e^t \dot{u}) = -\text{tr}_\omega(\text{Ric}(\theta_0) + \gamma_0).
\]

Hence

\[
\left( \frac{\partial}{\partial t} - \Delta \right) ((1 - e^{-s})\dot{u} + u)
\]

\[= \text{tr}_\omega(-\text{Ric}(\theta_0) + e^{-s}(\text{Ric}(\theta_0) + \gamma_0)) - n \]

\[\geq \beta \text{tr}_\omega(\theta_0) - \Delta f - n. \quad (2.5)\]

Let \( F = (1 - e^{-s})\dot{u} + u - f - A \log t + \kappa \rho \), where \( A > 0 \) is a constant to be determined. Since \( \log t \to -\infty \) as \( t \to 0 \), we conclude that for \( 0 < s_1 < s \), if \( \inf_{M \times (0, s_1]} F \leq 0 \), then there is \((x_0, t_0) \in M \times (0, s_1]\) so that \( F(x_0, t_0) = \inf_{M \times [0, s_1]} F \). By (2.5), at \((x_0, t_0)\) we have
\[
0 \geq \left( \frac{\partial}{\partial t} - \Delta \right) F \\
\geq \beta \text{tr}_X(\theta_0) - n - \frac{A}{t} - \kappa D_1 \\
\geq n \beta \exp\left(-\frac{n}{1} (\dot{u} + u)\right) - n - \frac{A}{t} - \kappa D_1 
\]

where \( D_1 > 0 \) is a constant independent of \( \kappa \). Hence at this point,

\[
\dot{u} + u \geq -n \log \left( \frac{1}{n \beta} \left( n + \frac{A}{t} + \kappa D_1 \right) \right).
\]

Hence at \((x_0, t_0)\), noting that \( 0 < t_0 \leq s_1 < s \) and \( s_1 \leq 1 \),

\[
F \geq (1 - e^{t-s})(\dot{u} + u) + e^{t-s}u - f - A \log t \\
\geq -(1 - e^{t-s})n \log \left( \frac{1}{n \beta} \left( n + \frac{A}{t} + \kappa D_1 \right) \right) - \sup_M f - A \log t \\
- \frac{C_3}{1 - e^{-s}} + nt \log(1 - e^{-t}) \\
\geq [(1 - e^{t-s})n - A] \log t - (1 - e^{t-s})n \log \left( \frac{1}{n \beta} (nt + A + \kappa t D_1) \right) \\
- ||f||_\infty - \frac{C_4}{1 - e^{-s}} \\
\geq -n \log \left( \frac{1}{n \beta} (2n + \kappa D_1) \right) - ||f||_\infty - \frac{C_4}{1 - e^{-s}}
\]

if \( A = n \). Here we may assume that \( \beta > 0 \) is small enough so that \( 2/\beta > 1 \). Hence we have

\[
F \geq -n \log \left( \frac{1}{n \beta} (2n + \kappa D_1) \right) - ||f||_\infty - \frac{C_4}{1 - e^{-s}}.
\]

on \( M \times (0, s_1] \). Let \( \kappa \to 0 \), we conclude that

\[
(1 - e^{t-s}) (\dot{u} + u) = (1 - e^{t-s})\dot{u} + u - e^{t-s}u \\
\geq n \log t - \frac{C_5}{1 - e^{-s}}.
\]

where we have used the upper bound of \( u \) in Lemma 2.2. From this (ii) follows because \( t \leq s_1 \).

The proof of (iii) is similar to the proof of (ii) by letting \( A = 0 \). Note that in this case, the infimum of \( F \) may be attained at \( t = 0 \) which depends also on \( \epsilon \).

To prove (iv), let \( F \) as in the proof of (ii) with \( A = 0 \). Suppose \( \inf_{M \times [\frac{1}{2}, s_1]} F = \inf_{M \times [\frac{1}{2}]} F \), then by (i) and (ii), we have

\[
F \geq -C_6.
\]

Suppose \( \inf_{M \times [\frac{1}{2}, s_1]} F < \inf_{M \times [\frac{1}{2}]} F \), then we can find \((x_0, t_0) \in M \times (\frac{1}{2}, s_1) \) such that \( F(x_0, t_0) \) attains the infimum. As in the proof of (ii), at this point,

\[
\dot{u} + u \geq -n \log \left( \frac{1}{n \beta} (n + \kappa D_2) \right).
\]
where \( D_2 > 0 \) is a constant independent of \( \kappa \). Hence as in the proof of (ii),
\[
F(x_0, t_0) \geq (1 - e^{t_0 - s})(\dot{u} + u) + e^{t_0 - s}u - f
\geq -n(1 - e^{t_0 - s}) \log \left( \frac{1}{n\beta} (n + \kappa D_2) \right) - \frac{C_7S_1 e^{s_1 - s}}{1 - e^{-s}} - C_8
\geq -n \log \left( \frac{1}{n\beta} (n + \kappa D_2) \right) - \frac{C_7S_1 e^{s_1 - s}}{1 - e^{-s}} - C_8
\]

because \( t_0 \leq s_1 \), where we have used (i) and we may assume that \( \beta < 1 \). Let \( \kappa \to 0 \), we conclude that on \( M \times \left[ \frac{1}{2}, s_1 \right] \),
\[
(1 - e^{t_0 - s})(\dot{u} + u) + e^{t_0 - s}u - f \geq n \log \beta - \frac{C_7S_1 e^{s_1 - s}}{1 - e^{-s}} - C_8.
\]

By Lemma 2.2, we have
\[
\dot{u} + u \geq -\frac{C_9(1 + s_1 e^{s_1 - s})}{1 - e^{s_1 - s}}
\]
on \( M \times \left[ \frac{1}{2}, s_1 \right] \) for some constant because \( s > 1 \).

Finally, (v) follows from (i), Lemma 2.2 and (iv) by integration.
\( \square \)

### 2.2 a priori estimates for \( \omega(t) \)

Next we will estimate the uniform upper bound of \( g(t) \). Before we do this, we first give uniform estimates for the evolution of the key quantity \( \log \text{tr}_h g(t) \).

Let \( \hat{T} \) and \( T_0 \) be the torsions of \( h, \gamma_0 \) respectively. Note that \( \gamma_0 \) depends on \( \rho_0, \epsilon \). Let \( \hat{\nabla} \) be the Chern connection of \( h \). Recall that \( T_{ijl} = \partial_i g_{jl} - \partial_j g_{il} \) etc.

Let \( \tilde{g} \) be such that \( g(t) = e^{-t} \tilde{g}(e^t - 1) \). Let \( s = e^t - 1 \). Then
\[
-\text{Ric}(\tilde{g}(s)) - g(t) = -\text{Ric}(g(t)) - g(t)
= \frac{\partial}{\partial t} g(t)
= -e^{-t} \tilde{g}(e^t - 1) + \frac{\partial}{\partial s} \tilde{g}(s)
= -g(t) + \frac{\partial}{\partial s} \tilde{g}(s).
\]

So
\[
\frac{\partial}{\partial s} \tilde{g}(s) = -\text{Ric}(\tilde{g}(s))
\]
and \( \tilde{g}(0) = \gamma_0 \).

Let \( \Upsilon(t) = \text{tr}_h g(t) \) and \( \tilde{\Upsilon}(s) = \text{tr}_h \tilde{g}(s) \). By Lemma A.1, we have
\[
\left( \frac{\partial}{\partial s} + \tilde{\Delta} \right) \log \tilde{\Upsilon} = I + II + III
\]
where
\[
I \leq 2\tilde{\Upsilon}^{-2} \text{Re} \left( h^{ij}\tilde{g}^{k\bar{l}}(T_0)_{k\bar{l}i} \hat{\nabla}_{\bar{k}} \Upsilon \right).
II = \tilde{\Upsilon}^{-1} \bar{g}^{ij} h^{k\bar{l}} g_{k\bar{l}} \left( \hat{\nabla}_{\bar{i}} (\tilde{T})_{j\bar{i}} - h_{p\bar{q}} \hat{R}_{ji\bar{p}\bar{q}} \right).
\]
and
\[ III = -\tilde{\gamma}^{-1} g^{ij} h^{kl} \left( \dot{\hat{V}}_l \left( (T_0)_j i k \right) + \dot{\hat{V}}_l \left( (T_0)_{ik j} \right) - \langle \hat{T} \rangle_{j i l} \right) \]

Now
\[ \tilde{\gamma}(s) = e^{\gamma}(t). \]

So
\[ \left( \frac{\partial}{\partial s} - \Delta \right) \log \tilde{\gamma}(s) = e^{-t} \left( \left( \frac{\partial}{\partial t} - \Delta \right) \log \gamma + 1 \right) \]
\[ I \leq 2e^{-2t} \gamma^{-2} \text{Re} \left( h^{ij} g^{k\bar{q}} (T_0)_{k i l} \hat{V}_l \gamma \right) . \]
\[ II = \gamma^{-1} g^{ij} h^{kl} g_{k\bar{q}} \left( \dot{\hat{V}}_l \left( \hat{T} \right)_{j i l} - h^{p\bar{q}} \hat{R}_{i l p j} \right) \]

and
\[ III = -e^{-2t} \gamma^{-1} g^{ij} h^{kl} \left( \dot{\hat{V}}_l \left( (T_0)_j i k \right) + \dot{\hat{V}}_l \left( (T_0)_{ik j} \right) - \langle \hat{T} \rangle_{j i l} \right) \]

Hence
\[ \left( \frac{\partial}{\partial t} - \Delta \right) \log \gamma = I' + II' + III' - 1 \]
\[ (2.6) \]

where
\[ I' \leq 2e^{-t} \gamma^{-2} \text{Re} \left( h^{ij} g^{k\bar{q}} (T_0)_{k i l} \hat{V}_l \gamma \right) . \]
\[ II' = \gamma^{-1} g^{ij} h^{kl} g_{k\bar{q}} \left( \dot{\hat{V}}_l \left( \hat{T} \right)_{j i l} - h^{p\bar{q}} \hat{R}_{i l p j} \right) \]

and
\[ III' = -e^{-t} \gamma^{-1} g^{ij} h^{kl} \left( \dot{\hat{V}}_l \left( (T_0)_j i k \right) + \dot{\hat{V}}_l \left( (T_0)_{ik j} \right) - \langle \hat{T} \rangle_{j i l} \right) \]

Now we want to estimate the terms in the above differential inequality.

*Estimate of II'*

Choose an frame unitary with respect to $h$ so that $g_{i\bar{j}} = \lambda_i \delta_{ij}$. Then
\[ II' = \left( \sum_{l} \lambda_l \right)^{-1} \lambda_l^{-1} \lambda_{kl} \left( \dot{\hat{V}}_l \left( \hat{T} \right)_{kl}^{ik} - \hat{R}_{ikkl} \right) \]
\[ (2.7) \]

*Estimate of III'*

Next, we compute the torsion of $\gamma_0$, $T_0 = T_{\gamma_0}$, where $\gamma_0 = \eta (\rho_0) g_0 + (1 - \eta (\rho_0)) h + \epsilon h$:
\[ (T_0)_{ik\bar{q}} = \partial_i (\gamma_0) k\bar{q} - \partial_k (\gamma_0) i\bar{q} \]
\[ = \eta' \frac{1}{\rho_0} [\rho_i (x) (g_0) k\bar{q} - \rho_k (x) (g_0) i\bar{q}] + \eta [\partial_i (g_0) k\bar{q} - \partial_k (g_0) i\bar{q}] \]
\[ + (1 - \eta + \epsilon) [\partial_i h k\bar{q} - \partial_k h i\bar{q}] - \eta \frac{1}{\rho_0} [\rho_i h k\bar{q} - \rho_k h i\bar{q}]. \]

By the assumptions, all terms above are bounded by $C(n, K)$ for all $\rho_0 \geq 1$ and for all $\epsilon \leq 1$. 

\[ \text{Springer} \]
It remains to control \( \hat{\nabla} \left( (T(\gamma_0))_{ikj} \right) \). We may compute \( \hat{\nabla} \left( (T(\gamma_0))_{ikj} \right) \) directly.

\[
\hat{\nabla} \left( (T(\gamma_0))_{ikj} \right) \\
= \hat{\nabla} \left( \partial_i (\gamma_0)_{kj} - \partial_k (\gamma_0)_{ij} \right) \\
= \hat{\nabla} \left\{ \eta' \frac{1}{\rho_0} [\rho_i (x) (g_0)_{kj} - \rho_k (x) (g_0)_{ij}] + \eta [\partial_i (g_0)_{kj} - \partial_k (g_0)_{ij}] \right\} \\
+ (1 - \eta + \epsilon) [\partial_i h_{kj} - \partial_k h_{ij}] - \eta' \frac{1}{\rho_0} [\rho_i h_{k\bar{q}} - \rho_k h_{i\bar{q}}] \\
= \eta'' \rho _j \frac{1}{\rho_0} [\rho_i (g_0)_{kj} - \rho_k (g_0)_{ij}] + \eta' \left( (g_0)_{kj} - \rho_k (g_0)_{ij} \right) \\
+ \eta' \left( \rho_l \hat{\nabla}_l (g_0)_{kj} - \rho_k \hat{\nabla}_l (g_0)_{ij} \right) + \eta [\partial_i (g_0)_{kj} - \partial_k (g_0)_{ij}] \\
+ \hat{\nabla} (g_0)_{ik\bar{q}} + (1 - \eta + \epsilon) \hat{\nabla} (h)_{ik\bar{q}} - \eta' \frac{1}{\rho_0} \hat{\nabla} (h)_{ik\bar{q}} \\
- \eta' \rho_j \frac{1}{\rho_0} [\rho_l \rho_i h_{k\bar{q}} - \rho_l \rho_k h_{i\bar{q}}].
\]

Since we can control every term of the above equation by \( C(n, K) \). Therefore, \( |\hat{\nabla} \left( (T(\gamma_0))_{ikj} \right)| \leq C(n, K) \).

Therefore, if \( 0 < \epsilon < 1, \rho_0 > 1 \)

\[
III' \leq C(n, K) \cdot e^{-t} \gamma^{-1} \Lambda. \tag{2.8}
\]

where \( \Lambda = \text{tr} g h \).

Now we will prove the uniform upper bound of \( g(t) \).

**Lemma 2.4** (i) For \( 0 < s_1 < s \),

\[
\text{tr}_h g(x, t) \leq \exp \left( \frac{C(E - \log(1 - e^{-s}))}{1 - e^{-t}} \right)
\]

on \( M \times (0, s_1] \) for some constant \( C > 0 \) depending only on \( n, K, \beta, ||f||\infty \) provided such that if \( 0 < \epsilon < 1, \rho_0 > 1 \), where

\[
E = \frac{(1 + s_1 e^{s_1 - s})}{(1 - e^{-s})(1 - e^{s_1 - s})}.
\]

(ii) For \( 0 < s_1 < s \), there is a constant \( C \) depending only on \( n, K, \beta, ||f||\infty, s, s_1 \) and also on \( \epsilon \), but independent of \( \rho_0 \) such that

\[
\text{tr}_h g \leq C
\]

on \( M \times [0, s_1] \).

**Proof** In the following, \( C_i \) will denote constants depending only on \( n, K, \beta \) and \( ||f||\infty \), but not \( \rho_0 \) and \( \epsilon \). \( D_i \) will denote constants which may also depend on \( \epsilon, \rho_0 \), but not \( \kappa \). We always assume \( 0 < \epsilon < 1 < \rho_0 \).

Let \( v(x, t) \geq 1 \) be a smooth bounded function. As before, let \( \Upsilon = \text{tr}_h g \) and \( \Lambda = \text{tr} g h \) and let \( \lambda = 0 \) or 1. For \( \kappa > 0 \), consider the function

\[
F = (1 - \lambda e^{-t}) \log \Upsilon - Av + \frac{1}{v} - \kappa \rho + Bt \log(1 - \lambda e^{-t})
\]
on $M \times [0, s_1]$, where $A, B > 0$ are constants to be chosen. We want to estimate $F$ from above. Let

$$\mathfrak{M} = \sup_{M \times [0, s_1]} F.$$ 

Either (i) $\mathfrak{M} \leq 0$; (ii) $\mathfrak{M} = \sup_{M \times [0]} F$; or (iii) there is $(x_0, t_0)$ with $t_0 > 0$ such that $F(x_0, t_0) = \mathfrak{M}$. If (ii) is true, then

$$\mathfrak{M} \leq C_1(n).$$

(2.9)

because $g(0) = \gamma_0 \leq (1 + \epsilon)h$.

Suppose (iii) is true. If at this point $\Upsilon(x_0, t_0) \leq 1$. Then (2.9) is true with a possibly larger $C_1$. So let us assume that $\Upsilon(x_0, t_0) > 1$. By (2.6), (2.7) and (2.8), at $(x_0, t_0)$ we have:

$$0 \leq \left( \frac{\partial}{\partial t} - \Delta \right) F = (1 - \lambda e^{-t}) \left( \frac{\partial}{\partial t} - \Delta \right) \log \Upsilon + \lambda e^{-t} \log \Upsilon - \left( \frac{1}{v^2} + A \right) \left( \frac{\partial}{\partial t} - \Delta \right) v - \frac{2}{v^3} |\nabla v|^2 + \kappa \Delta \rho + B \left( \log(1 - \lambda e^{-t}) + \frac{\lambda t}{e^t - \lambda} \right)$$

$$\leq (1 - \lambda e^{-t}) C_2 \Lambda \left( 1 + e^{-t} \Upsilon^{-1} \right)$$

$$+ 2(1 - \lambda e^{-t}) e^{-t} \Upsilon^{-2} Re \left( \hat{h}^{ij} g^{k\bar{q}} (T_0)_{k\bar{q}} \hat{\nabla} \Upsilon \hat{\nabla} \Upsilon \right)$$

$$+ \lambda e^{-t} \log \Upsilon - \left( \frac{1}{v^2} + A \right) \left( \frac{\partial}{\partial t} - \Delta \right) v - \frac{2 |\nabla v|^2}{v^3}$$

$$+ B \left( \log(1 - \lambda e^{-t}) + \frac{\lambda t}{e^t - \lambda} \right) + \kappa D_1.$$ 

At $(x_0, t_0)$, we also have:

$$(1 - \lambda e^{-t}) \Upsilon^{-1} \hat{\nabla} \Upsilon - \left( \frac{1}{v^2} + A \right) \hat{\nabla} v - \kappa \hat{\nabla} \rho = 0.$$ 

Hence

$$2(1 - \lambda e^{-t}) e^{-t} \Upsilon^{-2} Re \left( \hat{h}^{ij} g^{k\bar{q}} (T_0)_{k\bar{q}} \hat{\nabla} \Upsilon \hat{\nabla} \Upsilon \right)$$

$$= \frac{2 e^{-t}}{\Upsilon} Re \left( \hat{h}^{ij} g^{k\bar{q}} (T_0)_{k\bar{q}} \left( \frac{1}{v^2} + A \right) \hat{\nabla} v - \kappa \hat{\nabla} \rho \right)$$

$$\leq \frac{1}{v^3} |\nabla v|^2 + \frac{C_3(A + \frac{1}{v^2})^2 \cdot v^3 \Lambda}{\Upsilon^2} + \kappa D_2.$$ 

Using the fact that $\Upsilon(x_0, t_0) > 1$, we have at $(x_0, t_0)$:

Hence

$$0 \leq C_2 (1 - \lambda e^{-t}) \Lambda + \frac{C_3(A + \frac{1}{v^2})^2 \cdot v^3 \Lambda}{\Upsilon^2} + \lambda e^{-t} \log \Upsilon$$

$$- \left( \frac{1}{v^2} + A \right) \left( \frac{\partial}{\partial t} - \Delta \right) v + B \left( \log(1 - \lambda e^{-t}) + \frac{\lambda t}{e^t - \lambda} \right) + \kappa D_3.$$ 

(2.10)

Now let

$$v = u - \frac{1 - e^{-t}}{1 - e^{-s}} f + \frac{C_4(1 + s t e^{t-s})}{(1 - e^{-s})(1 - e^{s-t})}$$
By Lemmas 2.2 and 2.3, we can find $C_4 > 0$ so that $v \geq 1$, and there is $C_5 > 0$ so that

$$v \leq \frac{C_5(1 + s_1 e^{s_1 - s})}{(1 - e^{-s})(1 - e^{s_1 - s})}.$$ 

Let

$$E := \frac{(1 + s_1 e^{s_1 - s})}{(1 - e^{-s})(1 - e^{s_1 - s})}. \quad (2.11)$$

Note that

$$\left(\frac{\partial}{\partial t} - \Delta\right) u = \dot{u} - \Delta u$$

$$\geq \dot{u} - n + \mathrm{tr}_g \left(-(1 - e^{-t}) \mathrm{Ric}(\theta_0) + e^{-t} \gamma_0\right)$$

$$\geq \dot{u} - n + \mathrm{tr}_g \left(\frac{1 - e^{-t}}{1 - e^{-s}} \left(\beta \theta_0 - e^{-s} \omega_0 - \sqrt{-1} \partial \bar{\partial} f\right) + e^{-t} \gamma_0\right)$$

$$\geq \dot{u} + \left[\beta \left(\frac{1 - e^{-t}}{1 - e^{-s}} + e e^{-t}\right) \Lambda - \frac{e^{-t}}{1 - e^{-s}} \Delta f - n\right]$$

$$\geq \dot{u} + \left[\beta \left(\frac{1 - e^{-t}}{1 - e^{-s}} + e e^{-t}\right) \Lambda + \left(\frac{\partial}{\partial t} - \Delta\right) \left(\frac{1 - e^{-t}}{1 - e^{-s}} f\right) - e^{-t} f - n\right]$$

$$\geq \dot{u} + u + \left[\beta \left(\frac{1 - e^{-t}}{1 - e^{-s}} + e e^{-t}\right) \Lambda + \left(\frac{\partial}{\partial t} - \Delta\right) \left(\frac{1 - e^{-t}}{1 - e^{-s}} f\right) - c_6 \frac{C_6}{1 - e^{-s}}\right].$$

because $\gamma_0 \geq \omega_0 + \epsilon \theta_0$ and $t < s$.

On the other hand,

$$-\dot{u} - u = \log \left(\frac{\det h}{\det g}\right) \leq c(n) + n \log \Lambda.$$ 

Hence

$$\left(\frac{\partial}{\partial t} - \Delta\right) v \geq -n \log \Lambda + \left[\beta \left(\frac{1 - e^{-t}}{1 - e^{-s}} + e e^{-t}\right) \Lambda - \frac{C_7}{1 - e^{-s}}\right]. \quad (2.12)$$

On the other hand, in a unitary frame with respect to $h$ so that $g_{ij} = \lambda_i \delta_{ij}$, then

$$\Upsilon = \sum_i \lambda_i$$

$$= \frac{\det g}{\det h} \sum_i (\lambda_1 \ldots \lambda_i \ldots \lambda_n)^{-1} \quad (2.13)$$

$$\leq C_8 \Lambda^{n-1}.$$ 

where we have used the upper bound of $\dot{u} + u = \log \frac{\det g}{\det h}$ in Lemma 2.2. Combining (2.10), (2.12) and (2.13), at $(x_0, t_0)$ we have

$$0 \leq C_2 (1 - \lambda e^{-t}) \Lambda \left(1 + \frac{C_y E^3 (A + 1)^2}{(1 - \lambda e^{-t}) \Upsilon}\right) + \lambda e^{-t} \left(\log C_8 + (n - 1) \log \Lambda\right)$$

$$+ \left(\frac{1}{v^2} + A\right) \left(n \log \Lambda - \left[\beta \left(\frac{1 - e^{-t}}{1 - e^{-s}} + e e^{-t}\right) \Lambda + \frac{C_7}{1 - e^{-s}}\right]\right).$$
\[ + B \left( \log(1 - \lambda e^{-t}) + \frac{\lambda t}{e^t - \lambda} \right) + \kappa D_3 \]
\[ \leq \Lambda \left[ C_2(1 - \lambda e^{-t}) \left( 1 + \frac{C_9 E^3(A + 1)^2}{(1 - \lambda e^{-t}) Y} \right) - \frac{A + 1}{C_5^2 E^2} \left( \frac{\beta(1 - e^{-t})}{1 - e^{-s}} + \epsilon e^{-t} \right) \right] \]
\[ + [n(1 + A) + \lambda(n - 1)] \log \Lambda + \frac{C_{10}(A + 1)}{1 - e^{-s}} \]
\[ + B \left( \log(1 - \lambda e^{-t}) + \frac{\lambda t}{e^t - \lambda} \right) + \kappa D_3 + \lambda \log C_8 \]
\[(2.14)\]

where we have used the fact that \( 1 \leq v \leq C_5 E \).

**Case 1:** Let \( \lambda = 1 \). Suppose at \((x_0, t_0)\),
\[
\frac{C_2 C_9 E^3(A + 1)^2}{(1 - e^{-t}) Y} \geq \frac{1}{2} \frac{1}{C_5^2 E^2} \cdot (A + 1) \cdot \beta \cdot \frac{1}{1 - e^{-s}}
\]

Then
\[
(1 - e^{-t}) Y \leq \frac{2C_2 C_9 C_5^2 E^5(1 - e^{-s})(A + 1)}{\beta} \leq C_{11} E^5(A + 1).
\]

Hence,
\[
(1 - e^{-t}) \log Y \leq (1 - e^{-t}) \log(C_{11} E^5(A + 1)) - (1 - e^{-t}) \log(1 - e^{-t}).
\]

Therefore,
\[
\mathfrak{M} \leq C(1 + \log E) + \log(A + 1). \tag{2.15}
\]

for some \( C(n, \beta, K, \| f \|_\infty) > 0 \). Suppose at \((x_0, t_0)\),
\[
\frac{C_2 C_9 E^3(A + 1)^2}{(1 - e^{-t}) Y} < \frac{1}{2} \frac{1}{C_5^2 E^2} \cdot (A + 1) \cdot \beta \cdot \frac{1}{1 - e^{-s}},
\]

then at \((x_0, t_0)\) we have
\[
0 \leq (1 - e^{-t}) \Lambda \left( C_2 - \frac{1}{2} \frac{1}{C_5^2 E^2} \cdot (A + 1) \cdot \beta \cdot \frac{1}{1 - e^{-s}} \right) + n(A + 2) \log \Lambda \\
+ \frac{C_{10}(A + 1)}{1 - e^{-s}} + B \left( \log(1 - e^{-t}) + \frac{t}{e^t - 1} \right) + \kappa D_3 + \log C_8 \\
= \Lambda \left[ (1 - e^{-t}) \left( C_2 - \frac{1}{2} \frac{1}{C_5^2 E^2} \cdot (A + 1) \cdot \beta \cdot \frac{1}{1 - e^{-s}} \right) \right] \\
+ n(A + 2) \log((1 - e^{-t}) \Lambda) + \frac{C_{10}(A + 1)}{1 - e^{-s}} - n(A + 2) \log(1 - e^{-t}) \\
+ B \left( \log(1 - e^{-t}) + \frac{t}{e^t - 1} \right) + \kappa D_3 + \log C_8 \\
\leq - (1 - e^{-t}) \Lambda + n(A + 2) \log((1 - e^{-t}) \Lambda) + \frac{C_{12} E^2}{1 - e^{-s}},
\]
provided \( A = C_{13} E^2 \) so that

\[
\frac{1}{2} \frac{1}{C_5^2 E^2} \cdot (A + 1) \cdot \beta \cdot \frac{1}{1 - e^{-s}} \geq (C_2 + 1)
\]

and \( B \) is chosen so that \( B = n(A + 2) \) and \( \kappa \) is small enough so that \( \kappa D_2 \leq 1 \). Hence using \( 1 + \frac{1}{2} \log x \leq \sqrt{x}, \forall x > 0 \), we have at \((x_0, t_0)\),

\[
(1 - e^{-t}) \Lambda \leq \frac{C_{14} E^4}{1 - e^{-s}},
\]

and so

\[
\log \Lambda \leq \log \frac{C_{14} E^4}{1 - e^{-s}} - \log(1 - e^{-t}).
\]

By (2.13), we have

\[
(1 - e^{-t}) \log \Upsilon \\
\leq (1 - e^{-t}) (\log C_8 + (n - 1) \log \Lambda) \\
\leq (1 - e^{-t}) \left( \log C_8 + (n - 1) \left( \log \frac{C_{14} E^4}{1 - e^{-s}} - \log(1 - e^{-t}) \right) \right) \\
\leq (n - 1) \log \left( \frac{1}{1 - e^{-s}} \right) + C_{15} (1 + \log E).
\]

(2.16)

Hence \( \mathfrak{M} \leq (n - 1) \log \left( \frac{1}{1 - e^{-s}} \right) + C_{16} (1 + \log E) \). By combining (2.9), (2.15) and using the choice of \( A \), we may let \( \kappa \to 0 \) to conclude that on \( M \times (0, s_1) \),

\[
(1 - e^{-t}) \log \Upsilon \leq (n - 1) \log \left( \frac{1}{1 - e^{-s}} \right) + C_{17} (1 + E).
\]

and hence (i) in the lemma is true. Here we have used the fact that \( E \geq \log E + 1 \).

**Case 2:** Let \( \lambda = 0 \), then (2.14) becomes:

\[
0 \leq \Lambda \left[ C_2 \left( 1 + \frac{C_9 E^3 (A + 1)^2}{\Upsilon} \right) - \frac{1}{C_5^2 E^2} (A + 1) \epsilon e^{-t} \right] \\
+ n(1 + A) \log \Lambda + \frac{C_{10} (A + 1)}{1 - e^{-s}} + \kappa D_3.
\]

We can argue as before to conclude that (ii) is true.

\( \square \)

**Corollary 2.1** For any \( 0 < s_0 < s_1 < s \), there is a constant \( C \) depending only on \( n, K, \beta, ||f||_\infty \) and \( s_0, s_1, s \) but independent of \( \epsilon, \rho_0 \) such that if \( 0 < \epsilon < 1, \rho_0 > 1 \), we have

\[
C^{-1} h \leq g(t) \leq Ch
\]
on \( M \times [s_0, s_1] \). There is also a constant \( \tilde{C}(\epsilon) > 0 \) which may also depend on \( \epsilon \) such that

\[
\tilde{C}^{-1} h \leq g(t) \leq \tilde{C} h
\]
on \( M \times [0, s_1] \).
3 Short time existence for the potential flow and the normalized Chern–Ricci flow

Using the a priori estimates in previous section, we are ready to discuss short time existence for the potential flow and the Chern–Ricci flow. We begin with the short time existence of the potential flow. We have the following:

**Theorem 3.1** Let \((M, h)\) be a complete non-compact Hermitian metric with Kähler form \(\theta_0\). Suppose there is \(K > 0\) such that the following holds.

1. There is a proper exhaustion function \(\rho(x)\) on \(M\) such that
   \[|\partial\rho|^2_h + |\sqrt{-1}\partial\bar{\partial}\rho|_h \leq K.\]
2. \(BK_h \geq -K\);
3. The torsion of \(h\), \(T_h = \partial\omega_h\) satisfies
   \[|T_h|^2_h + |\nabla^h \partial\omega_h| \leq K.\]

Let \(\omega_0\) be a nonnegative real \((1,1)\) form with corresponding Hermitian form \(g_0\) on \(M\) (possibly incomplete or degenerate) such that

1. \(g_0 \leq h\) and
   \[|T_{g_0}|^2_h + |\nabla^h T_{g_0}|_h + |\nabla^h g_0|_h \leq K.\]
2. There exist \(f \in C^\infty(M) \cap L^\infty(M)\), \(\beta > 0\) and \(s > 0\) so that
   \[\alpha(t) = -\mathrm{Ric}(\theta_0) + e^{-t}(\omega_0 + \mathrm{Ric}(\theta_0)) + \sqrt{-1}\partial\bar{\partial}f \geq \beta\theta_0.\]

Then (1.2) has a solution on \(M \times (0, s)\) so that \(u(t) \to 0\) as \(t \to 0\) uniformly on \(M\). Moreover, for any \(0 < s_0 < s_1 < s\), let
   \[\alpha(t) = -\mathrm{Ric}(\theta_0) + e^{-t}(\mathrm{Ric}(\theta_0) + \omega_0)\]
then
   \[\omega(t) = \alpha + \sqrt{-1}\partial\bar{\partial}u\]
is the Kähler form of a complete Hermitian metric which is uniformly equivalent to \(h\) on \(M \times [s_0, s_1]\).

**Proof of Theorem 3.1** For later application, we construct the solution in the following way. Combining the local higher order estimate of Chern–Ricci flow (See [22] for example) with Corollary 2.1 for any \(1 > \epsilon > 0\), using diagonal argument as \(\rho_0 \to \infty\) we obtain a solution \(u_\epsilon(t)\) to (1.2) with initial data \(\omega_0 + \epsilon\theta_0\) on \(M \times [0, s]\) which is smooth up to \(t = 0\), so that the corresponding solution \(g_\epsilon(t)\) of (1.3) has smooth solution on \(M \times [0, s]\) with initial metric \(g_\epsilon(0) = g_0 + \epsilon h\). Moreover, \(g_\epsilon\) is uniformly equivalent to \(h\) on \(M \times [0, s]\) for all \(0 < s_1 < s\) and for any \(0 < s_0 < s_1 < s\), there is a constant \(C > 0\) independent of \(\epsilon\) such that
   \[C^{-1}h \leq g_\epsilon \leq Ch\]
on \(M \times [s_0, s_1]\). Using the local higher order estimate of Chern–Ricci flow [22] again, we can find \(\epsilon_i \to 0\) such that \(u_{\epsilon_i}\) converge locally uniformly on any compact subsets of \(M \times (0, s)\) to a solution \(u\) of (1.2). By Lemmas 2.2, 2.3, we see that \(u(t) \to 0\) as \(t \to 0\) uniformly \(M\). Moreover, for any \(0 < s_0 < s_1 < s\), \(\omega(t) = \alpha + \sqrt{-1}\partial\bar{\partial}u\) is the Kähler form of the solution to (1.3). Also, the corresponding Hermitian metric \(g(t)\) is a complete Hermitian metric which is uniformly equivalent to \(h\) in \(M \times [s_0, s_1]\) for any \(0 < s_0 < s_1 < 1\). \(\square\)

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Next we want to discuss the short time existence of the Chern–Ricci flow. The solution \( \omega(t) \) obtained from the Theorem 3.1 satisfies the normalized Chern–Ricci flow on \( M \times (0, s) \). Hence we concentrate on the discussion of the behaviour of \( \omega(t) \) as \( t \to 0 \) for the solution obtained in Theorem 3.1. In case that \( h \) is Kähler and \( \omega_0 \) is closed, we have the following:

**Theorem 3.2** With the same notation and assumptions as in Theorem 3.1. Let \( \omega(t) \) be the solution of (1.3) obtained in the theorem. If in addition \( h \) is Kähler and \( d\omega_0 = 0 \). Let \( U = \{ \omega_0 > 0 \} \). Then \( \omega(t) \to \omega_0 \) in \( C^\infty(U) \) as \( t \to 0 \), uniformly in compact sets of \( U \).

**Remark 3.1** If in addition \( h \) has bounded curvature, then one can use Shi’s Kähler-Ricci flow [23,24] and the argument in [21] to show that the Kähler-Ricci flow \( g_i(t) \) starting from \( g_0 + \epsilon_i h \) has bounded curvature when \( t > 0 \). The uniform local \( C^k \) estimates will follow from the pseudo-locality theorem [14, Corollary 3.1] and the modified Shi’s local estimate [7, Theorem 14.16].

By Theorem 3.1 we have the following:

**Corollary 3.1** Let \( (M, h) \) be a complete non-compact Kähler manifold with bounded curvature. Let \( \theta_0 \) be the Kähler form of \( h \). Suppose there is a compact set \( V \) such that outside \( V \), 

\[-\text{Ric}(\theta_0) + \sqrt{-1}\partial\bar{\partial} f \geq \beta \theta_0 \]

for some \( \beta > 0 \) for some bounded smooth function \( f \). Then for any closed nonnegative real (1,1) form \( \omega_0 \) such that \( \omega_0 \leq \theta_0 \), \( |\nabla_k \omega_0| \) is bounded, and \( \omega_0 > 0 \) on \( V \), there is \( s > 0 \) such that (1.3) has a solution \( \omega(t) \) on \( M \times (0, s) \) so that \( \omega(t) \) is uniformly equivalent to \( h \) on \( M \times [s_0, s_1] \) for any \( 0 < s_0 < s_1 < s \) and \( \omega(t) \) attains initial data \( \omega_0 \) in the set where \( \omega_0 > 0 \).

**Proof** Let \( s > 0 \), then

\[-(1 - e^{-s})\text{Ric}(\theta_0) + (1 - e^{-s})\sqrt{-1}\partial\bar{\partial} f \geq (1 - e^{-s})\beta \theta_0 \]

outside \( V \). On \( V \),

\[\omega_0 - (1 - e^{-s})\text{Ric}(\theta_0) + (1 - e^{-s})\sqrt{-1}\partial\bar{\partial} f \geq \beta' \theta_0 \]

for some \( \beta' > 0 \), provided \( s \) is small enough. The Corollary then follows from Theorems 3.1 and 3.2.

**Remark 3.2** Suppose \( \Omega \) is a bounded strictly pseudoconvex domain in \( \mathbb{C}^n \) with smooth boundary, then there is a complete Kähler metric with Ricci curvature bounded above by the negative constant near infinity by [6]. Hence Corollary 3.1 can be applied to this case, which has been studied by Ge–Lin–Shen [9].

To prove the Theorem 3.2, suppose \( h \) is Kähler and \( d\omega_0 = 0 \), then solution in Theorem 3.1 is the limit of solutions \( g_i(t) \) of the normalized Kähler-Ricci flow on \( M \times [0, s] \) with initial data \( g_0 + \epsilon_i h \), where \( \epsilon_i \to 0 \). Here we may assume \( s \leq 1 \). By Lemma 2.3 (iii) and Lemma 2.4 (ii), each \( g_i(t) \) is uniformly equivalent to \( h \), the uniform constant here may depend on \( \epsilon_i \). In this section, we will use \( \tilde{g}_i(t) = (t + 1)g_i(\log(t + 1)) \) to denote the unnormalized Kähler-Ricci flow and \( \phi_i \) be the corresponding potential flow to the unnormalized Kähler-Ricci flow \( \tilde{g}_i(t) \), see appendix.

We want to prove the following:

**Lemma 3.1** With the same notation and assumptions as in Theorem 3.2, for any precompact open subset \( \Omega \) of \( U \), there is \( S_1 > 0 \) and \( C > 0 \),

\[C^{-1}h \leq \tilde{g}_i(t) \leq Ch \]

for all \( i \) in \( \Omega \times [0, S_1] \).
Proof of Theorem 3.2} Suppose the lemma is true, then Theorem 3.2 will follow from the local estimates in [21]. □

It remains to prove Lemma 3.1.

**Lemma 3.2** We have \(|\phi_i| \leq C_0\), \(\dot{\phi}_i \leq C_0\) on \(M \times [0, e^s - 1]\) for some positive constant \(C_0\) independent of \(i\).

**Proof** By Lemma 2.2, we have

\[
\log \frac{\omega_0^n(s)}{\theta_0^n} = \dot{u}_i + u_i \leq C.
\]

Here \(C\) is a positive constant independent of \(i\) and \(\tilde{\omega}_i(s)\) is the corresponding normalized flow with the relation

\[
\tilde{g}_i(t) e^{-s} = g_i(s), t = e^s - 1.
\]

Then by the equation \(\dot{\phi}_i = \log \frac{\tilde{\omega}_i(t)}{\theta_0^n}\), we obtain the upper bound on \(\dot{\phi}_i(t)\). The lower bound on \(\phi_i\) follows from Lemma 2.3. □

Before we state the next lemma, we fix some notations. Let \(p \in U\). By scaling, we may assume that there is a holomorphic coordinate neighbourhood of \(p\) which can be identified with \(B_0(2) \subset \mathbb{C}^n\) with \(p\) being the origin and \(B_0(r)\) is the Euclidean ball with radius \(r\). Moreover, \(B_0(2) \Subset U\). We may further assume \(\frac{1}{2} h \leq h_E \leq 4h\) in \(B_0(2)\) where \(h_E\) is the Euclidean metric. Since \(\omega_0 > 0\), there is \(\sigma > 0\) such that \(B_{g_i(0)}(p, 2\sigma) \subset B_0(2)\) and

\[
g_i(0) \geq 4\sigma^2 h
\]

in \(B_0(2)\) for some \(0 < \sigma < 1\). This is because \(g_i(0) = \omega_0 + \epsilon_i h\). Here we use \(h_E\) because we want to use the estimates in [21] explicitly. Let \(\tau = e^s - 1\), where \(s\) is the constant in assumption in Theorem 3.1, and let \(\tilde{\phi}\) be as in the proof of Lemma 3.2. It is easy to see that Lemma 3.1 follows from the following:

**Lemma 3.3** With the same notation and assumptions as in Theorem 3.2 and with the above set up. There exist positive constants \(1 > \gamma_1, \gamma_2 > 0\) with \(\gamma_2 < \tau\) which are independent of \(i\) such that

\[
\gamma_1^{-2} h \geq \tilde{g}_i(t) \geq \gamma_1^2 h
\]

on \(B_{\tilde{g}_i(t)}(p, \sigma), t \in [0, \gamma_2 \alpha^{8(n-1)}]\).

**Proof** The lower bound in lemma will follow from the following:

**Claim:** There are constants \(1 > \gamma_1, \gamma_2 > 0\) independent of \(\alpha > 0\) and \(i\) with \(\gamma_2 < \tau\) such that if \(\tilde{g}_i(t) \geq \alpha^2 h\) on \(B_{\tilde{g}_i(t)}(p, \sigma), t \in [0, \gamma_2 \alpha^{8(n-1)}]\), then \(\tilde{g}_i(t) \geq \gamma_1^2 h\) on \(B_{\tilde{g}_i(t)}(p, \sigma)\) for \(t \in [0, \gamma_2 \alpha^{8(n-1)}]\).

The main point is that \(\gamma_1\) does not depend on \(\alpha\). Suppose the claim is true. Fix \(i\), let \(\alpha \leq \gamma_1\) be the supremum of \(\tilde{\alpha}\) so that \(\tilde{g}_i(t) \geq \tilde{\alpha}^2 h\) on \(B_{\tilde{g}_i(t)}(p, \sigma), t \in [0, \gamma_2 \tilde{\alpha}^{8(n-1)}]\). Since \(\tilde{g}_i(0) \geq \sigma^2 h\) in \(U\), we see that \(\alpha > 0\). Suppose \(\alpha < \gamma_1\). By continuity, there is \(\epsilon > 0\) such that \(\alpha + \epsilon < \gamma_1\). Then \(\gamma_2 \alpha^{8(n-1)} \leq \gamma_2 < \tau\). By the claim, we can conclude that

\[
\tilde{g}_i(t) \geq \gamma_1^2 h \geq (\alpha + \epsilon)^2 h
\]

in \(B_{\tilde{g}_i(t)}(p, \sigma), t \in [0, \gamma_2 \alpha^{8(n-1)}]\). By choosing a possibly smaller \(\epsilon\) and by continuity, the above inequality is also true for \(t \in [0, \gamma_2 (\alpha + \epsilon)^{8(n-1)}]\). This is a contradiction.

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To prove the claim, let $\gamma_1$ and $\gamma_2 > 0$ be two constants to be determined later and are independent of $\alpha$ and $i$. In the following, $C_k$ will denote a positive constant independent of $\alpha$ and $i$. In the following, for simplicity in notation, we suppress the index $i$ and simply write $\tilde{g}_i$ as $g$.

Suppose $\alpha \leq \gamma_1$ is such that

$$g(t) \geq \alpha^2 h$$

in $B_{g(t)}(p, \sigma)$, $t \in [0, \gamma_2 \alpha^{8(n-1)}]$. By Lemma 3.2, $\det (g(t)) / \det (h) \leq C_1$ for some $C_1 > 1$. Hence we have

$$\alpha^2 h \leq g(t) \leq C_1 \alpha^{-2(n-1)} h$$

on $B_{g(t)}(p, \sigma)$, $t \in [0, \gamma_2 \alpha^{8(n-1)}]$ and hence on $B_h(p, C_1^{-1/2} \alpha^{n-1} \sigma) \times [0, \gamma_2 \alpha^{8(n-1)}]$ because $B_h(p, C_1^{-1/2} \alpha^{n-1} \sigma) \subset B_{g(t)}(p, \sigma)$ for $t \in [0, \gamma_2 \alpha^{8(n-1)}]$. This can be seen by considering the maximal $h$-geodesic inside $B_t(p, \sigma)$. Together with the fact that $\frac{1}{4} h \leq h_E \leq 4h$ on $B_0(2)$, we conclude that

$$\frac{\alpha^2 h_E}{1} \leq g(t) \leq \frac{\alpha^{-2} h E}{1}$$

(3.1)

on $B_0(\frac{1}{2 \sqrt{C_1}} \alpha^{n-1} \sigma) \times [0, \gamma_2 \alpha^{8(n-1)}]$, where $\alpha_1 > 0$ is given by

$$\alpha_1^2 = \frac{1}{4C_1} \alpha^{2(n-1)}.$$  

(3.2)

By [21, Theorem 1.1], we conclude that

$$|\text{Rm}(g(t))| \leq \frac{C_2}{\alpha_1^8 t}$$

(3.3)

on $B_0(\frac{\sigma}{2} \alpha_1) \times [0, \gamma_2 \alpha^{8(n-1)}]$. From the proof in [21], the constant $C_2$ depends on an upper bound of the existence time but not its precise value. In particular, it is independent of $\alpha$ here. By (3.1), we conclude that (3.3) is true on $B_{g(t)}(p, \frac{\alpha_1^2}{\sqrt{t}})$, $t \in [0, \gamma_2 \alpha^{8(n-1)}]$.

By [20, Lemma 8.3] (see also [8, Chapter 18, Theorem 18.7]), we have:

$$\left( \frac{\partial}{\partial t} - \Delta \right) \left( d_t(p, x) + C_3 \alpha_1^{-4} t^\frac{1}{2} \right) \geq 0$$

(3.4)

in the sense of barrier (see the definition in Appendix B) outside $B_{g(t)}(p, \alpha_1^4 \sqrt{t})$, provided

$$t^\frac{1}{2} \leq \frac{\sigma}{2} \alpha_1^{-2}.$$  

(3.5)

Let $\xi \geq 0$ be smooth with $\xi = 1$ on $[0, \frac{4}{3}]$ and is zero outside $[0, 2]$, with $\xi' \leq 0$, $|\xi''| \leq 0$. Let $\Phi(x, t) = \xi(\sigma^{-1} \eta(x, t))$ where $\eta(x, t) = d_t(p, x) + C_3 \alpha_1^{-4} t^\frac{1}{2}$. For any $\epsilon > 0$, for $t > 0$ satisfying (3.5), if $d_t(p, x) + C_3 \alpha_1^{-4} t^\frac{1}{2} < \frac{4}{3} \sigma$, then $\Phi(x, t) = 1$ near $x$ and so

$$\left( \frac{\partial}{\partial t} - \Delta \right) \left( \log(\Phi + \epsilon) \right) = 0.$$
If \( d_t(p, x) + C_3 \alpha^{-4}_1 t^{1/2} \geq \frac{4}{3} \sigma \) and \( d_t(p, x) \geq \alpha^{-4}_1 t^{1/2} \), then in the sense of barrier we have:

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \log(\Phi + \epsilon) = \left( \frac{\xi'}{\xi} - 1 \right) \frac{\partial}{\partial t} \eta - \frac{\xi''}{\xi} \sigma^{-2} |\nabla \eta|^2 + \frac{(\xi')^2}{\xi^2} \sigma^{-2} |\nabla \eta|^2 \\
\leq C_4(\Phi + \epsilon)^{-1}.
\]

(3.6)

by the choice of \( \xi \) and (3.4). Hence there exists \( C_5 > 0 \) such that if

\[
t^{1/2} \leq C_5 \alpha^{-4}_1
\]

(3.7)

then \( t \) also satisfies (3.5) and \( C_3 \alpha^{-4}_1 t^{1/2} < \frac{\eta}{4} \). Moreover, \( C_5 \) can be chosen so that either \( d_t(p, x) + C_3 \alpha^{-4}_1 t^{1/2} < \frac{4}{3} \sigma \) or \( d_t(p, x) + C_3 \alpha^{-4}_1 t^{1/2} \geq \frac{4}{3} \sigma \) and \( d_t(p, x) \geq \alpha^{-4}_1 t^{1/2} \). Hence (3.6) is true in the sense of barrier for \( t \in (0, C_5^2 \alpha^{-8}_1] \).

Consider the function

\[
F = \log \text{tr}_h g - L v + m \log(\Phi + \epsilon)
\]

where \( v = (\tau - t) \dot{\phi} + \phi - f + nt, \tau = e^{\epsilon} - 1 \). Here \( L, m > 0 \) are constants to be chosen later which are independent of \( i, \alpha \). Recall that \( v \) satisfies

\[
\left( \frac{\partial}{\partial t} - \Delta \right) v = \text{tr}_g (\omega_0 - \tau \text{Ric}(\theta_0) + \sqrt{-1} \bar{\partial} \bar{\partial} f) \geq \beta \text{tr}_g h.
\]

and

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \log \text{tr}_h g \leq C_6 \text{tr}_g h
\]

by Lemma A.1 with vanishing torsion terms here. Let

\[
L \beta = C_6 + 1 + \tau^{-1}.
\]

(3.8)

Note that by the A.M.-G.M. inequality and the definition of \( \dot{\phi} \), we have

\[
- \dot{\phi} \leq n \log \text{tr}_g h; \quad \log \text{tr}_h g \leq \dot{\phi} + (n - 1) \log \text{tr}_g h.
\]

(3.9)

So

\[
\log \text{tr}_g h \geq \frac{1}{n(\tau L - 1) + (n - 1)} (\log \text{tr}_h g - \tau L \dot{\phi})
\]

Then in the sense of barrier

\[
\left( \frac{\partial}{\partial t} - \Delta \right) F \leq - \text{tr}_g h + mC_4(\Phi + \epsilon)^{-1}
\]

\[
\leq - \exp \left( C_7 (\log \text{tr}_h g - \tau L \dot{\phi}) \right) + mC_4(\Phi + \epsilon)^{-1}
\]

\[
\leq - \exp \left( C_7 F - C_8 - C_7 m \log(\Phi + \epsilon) \right) + mC_4(\Phi + \epsilon)^{-1}
\]

\[
= - (\Phi + \epsilon)^{-1} mC_4 \left[ \exp(C_7 F - C_8 - \log(mC_4)) - 1 \right]
\]

if \( mC_7 = 1 \), where we have used the upper bound of \( \dot{\phi} \) and the bound of \( \phi \) in Lemmas 3.2. So

\[
\left( \frac{\partial}{\partial t} - \Delta \right) (C_7 F - C_8 - \log(mC_4))
\]

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\[ \leq -\frac{mC_4C_7}{\Phi + \epsilon} \left[ \exp(C_7F - C_8 - \log(mC_4)) - 1 \right] \]
\[ \leq 0 \]

in the sense of barrier whenever \( C_7F - C_8 - \log(mC_4) > 0 \). Then by the maximum principle Lemma B.1, we conclude that
\[ C_7F - C_8 - \log(mC_4) \leq \sup_{t=0} \left( C_7F - C_8 - \log(mC_4) \right) . \]

Let \( \epsilon \to 0 \), using the definition of \( \Phi \), the choice of \( C_5 \) and the bound of \( |\phi| \), we conclude that in \( B_{\tilde{g}(t)}(p, \sigma) \),
\[ \log \text{tr}_h g - L(\tau - t)\dot{\phi} \leq C_9 \tag{3.10} \]
provided \( t \in [0, C_5^2\alpha_i^8] \). On the other hand, as in (3.9), we have
\[ \log \text{tr}_h g \leq -\dot{\phi} + (n - 1) \log \text{tr}_h g \]
\[ = (n - 1) \left( \log \text{tr}_h g - L(\tau - t)\dot{\phi} \right) + (n - 1)(L(\tau - t) - 1)\dot{\phi} \]
\[ \leq C_{10} \]
provided
\[ Lt \leq L\tau - 1. \tag{3.11} \]

Here we have used the upper bound of \( \dot{\phi} \) in Lemma 3.2.

Hence there is \( \gamma_1 > 0 \) independent of \( \alpha \) and \( i \) such that if \( t \) satisfies (3.7) and (3.11), then
\[ g_i(t) \geq \gamma_1^2h \]
on \( B_{\tilde{g}(t)}(p, \sigma) \). Let \( \gamma_2 < \tau \) be such that
\[ \gamma_2 = \min\{C_5^2, L^{-1}(L\tau - 1)\} \times (4C_1)^{-4} \]
where \( C_1, C_5 \) are the constants in (3.2) and (3.7) respectively and \( L \) is given by (3.8). If \( t \in [0, \gamma_2\alpha_i^{8(n-1)}] \), then \( t \) will satisfy (3.7). One can see that the claim this true.

By (3.10) and Lemma 3.2, we conclude that
\[ \tilde{g}_i(t) \leq C_{11}h \]
on \( B_{\tilde{g}(t)}(p, \sigma) \) for \( t \in [0, \gamma_2\alpha_i^{8(n-1)}] \). The upper bound in the Lemma follows by choosing a possibly smaller \( \gamma_1 \).

For the case of Chern–Ricci flow, the result is less satisfactory because the property of \( d(x, t) \) does not behave as nice as in the Kähler case. As before, under the assumptions of Theorem 3.1, let \( g(t) \) be the Chern–Ricci flow \( g(t) \) constructed in the theorem. We have the following:

**Proposition 3.1** With the same notation and assumptions as in Theorem 3.1. Suppose \( \text{tr}_h g_0 = o(\rho) \). Then \( g(t) \to g_0 \) as \( t \to 0 \) in \( M \). The convergence is in \( C^\infty \) topology and is uniform in compact subsets of \( M \).

Note that \( g_0 \) may still be complete. But it may not be equivalent to \( h \) and the curvature of \( g_0 \) may be unbounded.

As before, \( g(t) \) is the limit of solutions \( g_i(t) \) of the unnormalized Chern–Ricci flow on \( M \times [0, s) \) with initial data \( g_0 + \epsilon_i h \) with \( \epsilon_i \to 0 \). Here we may assume \( s \leq 1 \). We want to prove the following:
Lemma 3.4 With the same notation and assumptions as in Proposition 3.1 and let $S < \tau := e^\delta - 1$, for any precompact open subset $\Omega$ of $M$, there is $C > 0$,

$$C^{-1} h \leq g_i(t) \leq C g$$

for all $i$ in $\Omega \times [0, S]$.

Suppose the lemma is true, then Proposition 3.1 will follows from the local estimates in [22] for Chern–Ricci flow. To prove the lemma, first we prove the following.

Sublemma 3.1 Suppose

$$\lim \inf_{\rho \to \infty} \rho^{-1} \log \frac{\omega^\rho_0}{\bar{\theta}^\rho_0} \geq 0.$$  

Then for any $\sigma > 0$ (small enough independent of $i$), there is a constant $C > 0$ independent of $i$ such that

$$\dot{\phi}_i \geq -C - \sigma \rho$$

on $M \times [0, S]$.

Proof In the following, we will denote $\phi_i$ simply by $\phi$ and $g_i(t)$ simply by $g(t)$ if there is no confusion arisen. Note that $g(t)$ is uniformly equivalent to $h$. Let $\sigma > 0$.

Let $F = - (\tau - t) \dot{\phi} - \phi + f - nt - \sigma \rho$. By (A.1) and (A.2), for $0 \leq t \leq S$, we have

$$\left( \frac{\partial}{\partial t} - \Delta \right) (-(\tau - t) \dot{\phi} - \phi) = (\tau - t) \text{tr}_g \text{Ric}(\theta_0) + \dot{\phi} - \phi + \text{tr}_g (\sqrt{-1} \partial \bar{\partial} \phi)$$

$$= (\tau - t) \text{tr}_g \text{Ric}(\theta_0) + (n + t \text{tr}_g (\text{Ric}(\theta_0)) - \text{tr}_g (\theta_0))$$

$$= \tau \text{tr}_g \text{Ric}(\theta_0) + n - \text{tr}_g (\theta_0)$$

Hence by the fact that:

$$\omega_0 - \tau \text{Ric}(\theta_0) + \sqrt{-1} \partial \bar{\partial} f \geq \beta \theta_0,$$

we have

$$\left( \frac{\partial}{\partial t} - \Delta \right) F \leq \tau \text{tr}_g \text{Ric}(\theta_0) - \text{tr}_g (\theta_0) - \Delta f + \sigma \Delta \rho$$

$$\leq (-\beta + \sigma C_1) \text{tr}_g (\theta_0)$$

$$< 0$$

for some constant $C_1$ independent of $\sigma$ and $i$ for $\sigma$ with $C_1 \sigma < \beta$. Since $F$ is bounded from above, by the maximum principle Lemma B.1, we conclude that

$$\sup_{M \times [0, S]} F \leq \sup_{M \times [0]} F.$$  

At $t = 0$,

$$F = -\tau \dot{\phi} - \sigma \rho + f.$$  

By the assumption, we conclude that $F \leq C(\sigma)$ at $t = 0$. Hence we have

$$F \leq C(\sigma)$$

on $M \times [0, S]$. Since $\phi$, $f$ are bounded, the sublemma follows. □
Sublemma 3.2 With the same notations as in Sublemma 3.1. Suppose $\text{tr}_{g_0} h = o(\rho)$. Then
$$\text{tr}_h g_i \leq C \exp(C' \rho)$$
on $M \times [0, S]$ for some positive constants $C, C'$ independent of $i$.

Proof We will denote $g_i$ by $g$ again and $\omega_0$ to be the Kähler form of the initial metric $g_i(0) = g_0 + \varepsilon_i h$. Note that

$$\left( \frac{\partial}{\partial t} - \Delta \right) \phi = \dot{\phi} - \Delta \phi$$
$$= \dot{\phi} - (n - \text{tr}_g \omega_0 + t \text{tr}_g (\text{Ric}(\theta_0)))$$
$$\geq \dot{\phi} - n + \text{tr}_g \omega_0 + \frac{t\beta}{\tau} \text{tr}_g h - \frac{t}{\tau} \text{tr}_g \omega_0 - \frac{t}{\tau} \Delta f$$
$$\geq \dot{\phi} - n - \frac{t}{\tau} \Delta f + \left( 1 - \frac{S}{\tau} \right) \text{tr}_g \omega_0.$$

Then we have:

$$\left( \frac{\partial}{\partial t} - \Delta \right) (\phi + nt - \frac{t}{\tau} f) \geq \dot{\phi} + \left( 1 - \frac{S}{\tau} \right) \text{tr}_g \omega_0 - C_0. \quad (3.12)$$

Since $|\phi|$ is bounded by a constant independent of $i$ on $M \times [0, S]$, see Lemma 2.2 and Lemma 2.3, there is a constant $C_1, C_2 > 0$ so that $\xi := \phi + nt - \frac{t}{\tau} f + C_1 \geq 1$ and $\xi \leq C_2$ on $M \times [0, S]$. Here and below $C_j$ will denote positive constants independent of $i$.

Let $\Phi(\xi) = 2 - e^{-\xi}$ for $\xi \in \mathbb{R}$.

Then for $\xi := \phi + nt - \frac{t}{\tau} f + C_1 \geq 1$, we have

$$\begin{cases}
\Phi(\xi) \geq 1 \\
\Phi'(\xi) \geq e^{-C_2} \\
\Phi''(\xi) \leq -e^{-C_2}
\end{cases} \quad (3.13)$$
on $M \times [0, S]$. Next, let $P(\zeta)$ be a positive function on $\mathbb{R}$ so that $P' > 0$. Define

$$F(x, t) = \Phi(\xi) P(\rho).$$

Let $Y = tr_h g$, here $g = g_i$. Let $F \to \infty$ near infinity be a smooth function of $x, t$. Then by Lemma A.1, we have

$$\left( \frac{\partial}{\partial t} - \Delta \right) (\log Y - F) = I + II + III - \left( \frac{\partial}{\partial t} - \Delta \right) F$$

where

$$I \leq 2Y^{-2} \text{Re} \left( h^{ij} g^{k\tilde{l}} (T_0)_{kl} \hat{\nabla}_\xi \nabla \xi \right),$$

$$II = Y^{-1} g^{ij} h^{k\tilde{l}} g_{k\tilde{l}} \left( \hat{\nabla}_i (T)_{ji}^q - h^{pq} \hat{\nabla}_{i\tilde{j}} \right),$$

and

$$III = -Y^{-1} g^{ij} h^{k\tilde{l}} \left( \hat{\nabla}_i \left( (T_0)_{jik} \right) + \hat{\nabla}_i \left( (T_0)_{ikj} \right) - (T_0)_{jik}^q (T_0)_{ikq}^p \right).$$

Let $\Theta = tr_h h$. Suppose $\log Y - F$ attains a positive maximum at $(x_0, t_0)$ with $t_0 > 0$, then at this point,

$$Y^{-1} \hat{\nabla} Y = \hat{\nabla} F,$$
and so

\[
I \leq 2 \gamma^{-2} \text{Re} \left( h_{ij} g^{k\ell} (T_0)_{k \ell} \tilde{\nabla}_q \gamma \right) \\
\leq C \gamma^{-1} \Theta^{\frac{1}{2}} |\nabla F| \\
\leq C' \gamma^{-1} \Theta^{\frac{1}{2}} \left( P|\nabla \xi| + P' \Theta^{\frac{1}{2}} \right) .
\]

because \(|\partial \rho|_h \) is bounded. Here we use the norm with respect to the evolving metric \(g(t)\).

\[
\Pi \leq C \Theta , \\
\text{III} \leq C \gamma^{-1} \Theta .
\]

Here \(C, C'\) are positive constants independent of \(i\). On the other hand,

\[
\left( \frac{\partial}{\partial t} - \Delta \right) F \\
= P \left( \frac{\partial}{\partial t} - \Delta \right) \Phi + 2 \text{Re} \left( g^{ij} \partial_i \Phi \partial_j P \right) + \Phi \left( \frac{\partial}{\partial t} - \Delta \right) P \\
\geq P \left( \Phi' \left( \frac{\partial}{\partial t} - \Delta \right) \xi - \Phi'' |\nabla \xi|^2 \right) - C_4 \Phi' P' \Theta^{\frac{1}{2}} |\nabla \xi| - C_4 \Theta (P' + |P''|) \\
\geq P \Phi' \phi + e^{-C_2} P \left( 1 - \frac{S}{\tau} \right) \text{tr}_g \omega_0 - C_0 P + e^{-C_2} P |\nabla \xi|^2 - \frac{1}{2} e^{-C_2} P |\nabla \xi|^2 \\
- C_5 \left( \frac{(P')^2}{P} \right) \Theta - C_4 \Theta (P' + |P''|) .
\]

Here we have used the fact that \(|\partial \rho|_h, |\partial \bar{\rho}|_h\) are bounded \(\Phi(\xi) \leq 2\) and (3.13).

So at \((x_0, t_0)\),

\[
\left( \frac{\partial}{\partial t} - \Delta \right) (\log \gamma - F) \\
\leq C_3 \left( \gamma^{-1} \Theta^{\frac{1}{2}} \left( P|\nabla \xi| + P' \Theta^{\frac{1}{2}} \right) + \gamma^{-1} \Theta + \Theta \right) \\
- P \Phi' \phi - e^{-C_2} P \left( 1 - \frac{S}{\tau} \right) \text{tr}_g \omega_0 - C_0 P - \frac{1}{2} e^{-C_2} P |\nabla \xi|^2 \\
+ \Theta \left( C_5 \left( \frac{(P')^2}{P} \right) + C_4 (P' + |P''|) \right) \\
\leq - P \Phi' \phi - e^{-C_2} P \left( 1 - \frac{S}{\tau} \right) \text{tr}_g \omega_0 - C_0 P + \left( -\frac{1}{2} e^{-C_2} + \gamma^{-1} \right) P |\nabla \xi|^2 \\
+ C_6 \Theta \left( \gamma^{-1} + 1 + \gamma^{-1} P' + P' + \gamma^{-1} P + \frac{(P')^2}{P} + |P''| \right) .
\]

Now

\[
- \dot{\phi} \leq c(n) \log \Theta .
\]
Suppose $\omega_0 \geq \frac{1}{Q(\rho)} \theta_0$ with $Q > 0$ and suppose $\Upsilon^{-1} \leq \frac{1}{2} e^{-C_2}$ at $(x_0, t_0)$, then at $(x_0, t_0)$, we have

$$
\left( \frac{\partial}{\partial t} - \Delta \right) (\log \Upsilon - F) \leq C_7 P (\log \Theta + 1) + \Theta \left[ -C_8 P Q^{-1} + C_9 \left( 1 + P' + \frac{(P')^2}{P} + |P''| \right) \right].
$$

By the assumption on $\text{tr}_{g_0} h$, for any $\sigma > 0$ there is $\rho_0 > 0$ such that if $\rho \geq \rho_0$, then $\text{tr}_{g_0} h \leq \sigma \rho$. Hence we can find $C = C(\sigma)$ such that $g_0 \geq \frac{1}{\sigma (\rho + C(\sigma))} h$ and $\rho + C(\sigma) \geq 1$ on $M$. Let $Q(\rho) = \sigma (\rho + C(\sigma))$, $P(\rho) = \rho + C(\sigma)$, then above inequality becomes

$$
\left( \frac{\partial}{\partial t} - \Delta \right) (\log \Upsilon - F) \leq C_7 P \log(e \Theta) + \Theta \left( -C_8 \sigma^{-1} + 3C_9 \right)
$$

if we choose $\sigma$ small enough independent of $i$. Since $\log \Upsilon - F \to -\infty$ near infinity and uniform in $t \in [0, S]$, and $\log \Upsilon - F < 0$ at $t = 0$, by maximum principle, either $\log \Upsilon - F \leq 0$ on $M \times [0, S]$ or there is $t_0 > 0, x_0 \in M$ such that $\log \Upsilon - F$ attains a positive maximum at $(x_0, t_0)$. Suppose at this point $\Upsilon^{-1} \geq \frac{1}{2} e^{-C_2}$, then

$$\log \Upsilon - F \leq C_{10}.$$ 

Otherwise, at $(x_0, t_0)$ we have

$$0 \leq C_7 P \log(e \Theta) - \frac{1}{2} C_8 \Theta.$$

Hence we have at this point $\Theta \leq C_{11}$ which implies $\Upsilon \leq C_{12}$ because $\dot{\phi} \leq C$ for some constant independent of $i$. So

$$\log \Upsilon - F \leq \log C_{12}.$$ 

Or

$$\Theta \leq C_{13} P^2.$$ 

This implies $\log \Upsilon \leq C_{14}(1 + \log P)$. Hence

$$\log \Upsilon - F \leq C_{14}.$$ 

From these considerations, we conclude that the sublemma is true. \hfill $\square$

**Proof of Lemma 3.4** The lemma follows from Sublemmas 3.1 and 3.2. \hfill $\square$

### 4 Long time behaviour and convergence

In this section, we will first study the long time behaviour for the solution constructed in Theorem 3.1. Namely, we will show the following theorem:
Theorem 4.1 Under the assumption of Theorem 3.1, if in addition,

\[-\text{Ric}(h) + \sqrt{-1} \partial \bar{\partial} f \geq \beta \theta_0\]

for some \( f \in C^\infty(M) \cap L^\infty(M) \), \( \beta > 0 \). Then the solution constructed from Theorem 3.1 is a longtime solution and converges to a unique complete Kähler Einstein metric with negative scalar curvature on \( M \).

Before we prove Theorem 4.1, let us prove a lower bound of \( \dot{u} \) which will be used in the argument of convergence. Once we have uniform equivalence of metrics, we can obtain a better lower bound of \( \dot{u} \).

Lemma 4.1 Assume the solution constructed from Theorem 3.1 is a longtime solution, then there is a positive constant \( C \) such that

\[ \dot{u} \geq -C e^{-\frac{t}{2}} \]

on \( M \times [2, \infty) \).

Proof Since we do not have upper bound of \( g(t) \) as \( t \to 0 \), we shift the initial time of the flow to \( t = 1 \). Note that

\[
\left( \frac{\partial}{\partial t} - \Delta \right) (e^s \dot{u} - f) = - \text{tr}_g (\text{Ric}(h) + g(1)) + \Delta f \\
\geq - \text{tr}_g g(1) \geq -C_1.
\]

Consider \( Q = e^s \dot{u} - f + (C_1 + 1)t \). Then we can use maximum principle argument as before to obtain \( Q(x, t) \geq \inf_M Q(0) \). Then we have

\[ e^s \dot{u} \geq -C_2 - (C_1 + 1)t \]

which implies

\[ \dot{u} \geq -C_3 e^{-\frac{t}{2}} \]

on \( M \times [1, \infty) \). We shift the time back, we obtain the result. \( \square \)

Proof of Theorem 4.1 The assumption \(-\text{Ric}(h) + \sqrt{-1} \partial \bar{\partial} f \geq \beta \theta_0 \) implies that for all \( s \) large enough,

\[ -\text{Ric}(h) + e^{-s} (\omega_0 + \text{Ric}(h)) + \sqrt{-1} \partial \bar{\partial} \hat{f} \geq \frac{\beta}{2} \theta_0. \]

Here \( \hat{f} = (1 - e^{-s}) f \) is a bounded function on \( M \). By Theorem 3.1 and Lemma 2.4, (1.2) has a smooth solution on \( M \times (0, \infty) \) with \( g(t) \) uniformly equivalent to \( h \) on any \([a, \infty) \subset (0, \infty)\).

Combining the local higher order estimate of Chern–Ricci flow (See [22] for example) and Lemma 4.1, we can conclude that \( u(t) \) converges smoothly and locally to a smooth function \( u_\infty \) as \( t \to \infty \) and \( \log \frac{\omega_\infty}{\omega_0} = u_\infty \). Taking \( \sqrt{-1} \partial \bar{\partial} \) on both sides, we have

\[ -\text{Ric}(g_\infty) + \text{Ric}(h) = \sqrt{-1} \partial \bar{\partial} u_\infty. \]

which implies \(-\text{Ric}(g_\infty) = g_\infty \). Obviously, \( g_\infty \) is Kähler. Uniqueness follows from [31, Theorem 3] (see also Proposition 5.1 in [16]). \( \square \)

Taking \( g_0 = h \) in the theorem, we have
Corollary 4.1 Let \((M, h)\) be a complete Hermitian manifold satisfying the assumptions in Theorem 4.1. Then the Chern–Ricci flow with initial data \(h\) exists on \(M \times [0, \infty)\) and converge uniformly on any compact subsets to the unique complete Kähler–Einstein metric with negative scalar curvature on \(M\).

For Kähler–Ricci flow, we have the following general phenomena related to Theorem 4.1.

Theorem 4.2 Let \((M, h)\) be a smooth complete Hermitian manifold with \(BK(h) \geq -K_0\) and \(|\nabla h|_h \leq K_0\) for some constant \(K_0 \geq 0\). Moreover, assume

\[-\text{Ric}(h) + \sqrt{-1} \partial \bar{\partial} f \geq k h\]

for some constant \(k > 0\) and function \(f \in C^\infty(M) \cap L^\infty(M)\). Suppose \(g(t)\) is a smooth complete solution to the normalized Kähler–Ricci flow on \(M \times [0, +\infty)\) with \(g(0) = g_0\) which satisfies

\[\frac{\det g_0}{\det h} \leq \Lambda\]

and

\[R(g_0) \geq -L\]

for some \(\Lambda, L > 0\). Then \(g(t)\) satisfies

\[C^{-1} h \leq g(t) \leq Ch\]

on \(M \times [1, \infty)\) for some constant \(C = C(n, K_0, k, ||f||_\infty, \Lambda, L) > 0\). In particular, \(g(t)\) converges to the unique smooth complete Kähler–Einstein metric with negative scalar curvature.

Proof We can assume \(k = 1\), otherwise we rescale \(h\). We consider the corresponding unnormalized Kähler–Ricci flow \(\tilde{g}(s) = e^s g(t)\) with \(s = e^t - 1\). Then the corresponding Monge–Ampère equation to the unnormalized Kähler–Ricci flow is:

\[
\begin{cases}
\frac{\partial}{\partial s} \phi = \log \left( \frac{(\omega_0 - s \text{Ric}(\theta_0) + \sqrt{-1} \partial \bar{\partial} \phi)^n}{\theta_0^n} \right) \\
\phi(0) = 0.
\end{cases}
\]

Here \(\theta_0\) is the Kähler form of \(h\). By the assumption \(R(g_0) \geq -L\), Proposition 2.1 in [5] and Lemma 5.1 in [16] with the fact

\[\left( \frac{\partial}{\partial s} - \tilde{\Delta} \right) \tilde{R} \geq \frac{1}{n} \tilde{R}^2,\]

we conclude that \(\tilde{R} := R(\tilde{g}(s)) \geq \max\{-L, -\frac{n}{s}\}\) on \(M \times [0, \infty)\). Note that \(\dot{\phi} = -R(\tilde{g}(s))\), we have on \(M \times [0, 1]\), \(\dot{\phi} \leq C(L, \Lambda)\); on \(M \times [1, \infty)\), \(\dot{\phi} \leq C(L, \Lambda) + n \log s\).
For lower bound of $\dot{\phi}$, we consider $Q = -\dot{\phi} + f$. We compute:

$$
\left( \frac{\partial}{\partial s} - \tilde{\Delta} \right) Q = \left( \frac{\partial}{\partial s} - \tilde{\Delta} \right) \dot{\phi} - \Delta f
$$

$$
= \text{tr}_{\tilde{g}}[\text{Ric}(\theta_0) - \sqrt{-1} \partial \bar{\partial} f]
\leq -\text{tr}_{\tilde{g}} h
\leq -n e^{-\frac{\dot{\phi}}{\bar{\phi}}}
\leq -n e^{\frac{1}{\bar{\phi}}(Q-f)}
\leq -C(n, \|f\|_{\infty}) e^{\frac{Q}{\bar{\phi}}}
\leq -C(n, \|f\|_{\infty}) Q^2,
$$

whenever $Q > 0$.

Then by the same argument as in the proof of Proposition 2.1 in [5], we conclude that $\dot{\phi} \geq -C(n, \lambda, \|f\|_{\infty})$ on $M \times [0, \infty)$. Here $\lambda$ is the lower bound of $\frac{\text{det}\, \tilde{g}_0}{\text{det}\, h}$. However, this estimate is not enough for later applications. We consider $F = -\dot{\phi} + f + n \log \bar{s}$. Then we similarly obtain

$$
\left( \frac{\partial}{\partial s} - \tilde{\Delta} \right) F \leq -C(n, \|f\|_{\infty}) F^2,
$$

whenever $F > 0$. By Lemma 5.1 in [16], we conclude that $F \leq \frac{C(n, \|f\|_{\infty})}{s}$ on $M \times [0, \infty)$. Therefore, we obtain

$$
\dot{\phi} \geq -C(n, \|f\|_{\infty}) + n \log \bar{s}
$$
on $M \times [1, \infty)$.

To sum up, for the bound of $\dot{\phi}$, we have:

(i) On $M \times [0, 1]$, $-C(n, \lambda, \|f\|_{\infty}) \leq \dot{\phi} \leq C(L, \Lambda)$;

(ii) On $M \times [1, \infty)$, $-C(n, \|f\|_{\infty}) + n \log \bar{s} \leq \dot{\phi} \leq C(L, \Lambda) + n \log \bar{s}$.

Then we consider back to the normalized Kähler–Ricci flow $g(t)$. Since

$$
\log \frac{\text{det}\, g(t)}{\text{det}\, h} = -n \log(s + 1) + \frac{\partial}{\partial s} \phi(s),
$$

where $s = e^t - 1$, we obtain:

$$
-C(n, \|f\|_{\infty}) \leq \dot{u}(t) + u(t) \leq C(L, \Lambda)
$$
on $M \times [\log 2, \infty)$. Here $u$ solves (1.2).

Next, we consider $G(x, t) = \log \text{tr}_h g(t) - A(\dot{u}(t) + u(t) + f)$. Here $A$ is a large constant to be chosen. As in Section 1, we have

$$
\left( \frac{\partial}{\partial t} - \Delta \right) \log \text{tr}_h g(t) \leq C(n, K_0) \text{tr}_g(t) h - 1.
$$

Therefore,

$$
\left( \frac{\partial}{\partial t} - \Delta \right) G \leq C(n, K_0) \text{tr}_g(t) h - 1 + An + A(\text{tr}_g \text{Ric}(h) + \text{tr}_g \sqrt{-1} \partial \bar{\partial} f)
\leq (-A + C(n, K_0)) \text{tr}_g(t) h - 1 + An
\leq -\text{tr}_g(t) h + An.
$$
Here we take $A = C(n, K_0) + 1$.

On the other hand,

$$tr_h g(t) \leq \frac{1}{(n-1)!} \cdot (tr_{g(t)} h)^{n-1} \cdot \frac{\det g}{\det h} \leq C(n, L, \Lambda) (tr_{g(t)} h)^{n-1}.$$ 

Then we have

$$\left( \frac{\partial}{\partial t} - \Delta \right) G \leq -C(n, L, \Lambda) (tr_h g(t))^{\frac{1}{n-1}} + C(n, K_0)$$

$$= -C(n, L, \Lambda) e^{\frac{1}{n-1} \log(tr_h g(t))} + C(n, K_0)$$

$$= -C(n, L, \Lambda) e^{\frac{1}{n-1} [G + A(\ddot{u}(t) + u(t))]} + C(n, K_0)$$

$$\leq -C(n, L, \Lambda, ||f||_\infty) e^{\frac{1}{n-1} G} + C(n, K_0)$$

$$\leq -C(n, L, \Lambda, ||f||_\infty) G^2 + C(n, K_0),$$

whenever $G > 0$.

By similar argument as in the proof of Lemma 5.1 in [16], we conclude that $G \leq C(n, L, \Lambda, ||f||_\infty, K_0)$ on $M \times [1, \infty)$. The difference here is that we consider the normalized Kähler–Ricci flow instead of Kähler–Ricci flow. The Perelman’s distance distortion lemma for normalized Kähler–Ricci flow is the following:

$$\left( \frac{\partial}{\partial t} - \Delta \right) dt(x_0, x) \geq -\frac{5(n-1)}{3} r_0^{-1} - dt(x_0, x).$$

We then consider $t \cdot \phi(\frac{1}{A_0} [e^t \cdot dt(x_0, x) + \frac{5(n-1)e^t}{3} r_0^{-1}]) \cdot G(x, t)$, the results follows from the same argument as in the proof of Lemma 5.1 in [16].

This implies

$$g(t) \leq C(n, L, \Lambda, ||f||_\infty, K_0) h$$

on $M \times [1, \infty)$.

For lower bound, combining with $e^{\ddot{u}(t) + u(t)} = \frac{\det g}{\det h}$, we have

$$g(t) \geq C^{-1}(n, L, \Lambda, ||f||_\infty, K_0) h$$

on $M \times [1, \infty)$.

Once we obtain the uniform equivalence of metrics of the normalized Kähler–Ricci flow, the convergence follows from the same argument as in the proof of Theorem 5.1 in [16]. This completes the proof of Theorem 4.2. $\Box$

### Appendix A: Some basic relations

Let $g(t)$ be a solution to the Chern–Ricci flow,

$$\partial_t g = -\text{Ric}(g)$$

and $h$ is another Hermitian metric. Let $\omega(t)$ be the Kähler form of $g(t)$, $\theta_0$ be the Kähler form of $h$. Let
\[ \phi(t) = \int_0^t \log \frac{\omega^n(s)}{\theta_0^n} \, ds. \]

\[ \omega(t) = \omega(0) - t \text{Ric}(\theta_0) + \sqrt{-1} \partial \bar{\partial} \phi. \]  

(A.1)

Let \( \dot{\phi} = \frac{\partial}{\partial t} \phi \). Then

\[ \left( \frac{\partial}{\partial t} - \Delta \right) \dot{\phi} = - \text{tr}_g (\text{Ric}(\theta_0)), \]

(A.2)

where \( \Delta \) is the Chern Laplacian with respect to \( g \).

On the other hand, if \( g \) is as above, the solution \( \tilde{g} \) of the corresponding normalized Chern–Ricci flow with the same initial data

\[ \partial_t \tilde{g} = - \text{Ric}(\tilde{g}) - \tilde{g} \]

is given by

\[ \tilde{g}(x, t) = e^{-t} g(x, e^t - 1). \]

The corresponding potential \( u \) is given by

\[ u(t) = e^{-t} \int_0^t e^s \log \tilde{\omega}^n(s) \frac{\omega^n_0}{\theta_0^n} \, ds \]

where \( \tilde{\omega} \) is the Kähler form of \( \tilde{g} \). Also,

\[ \tilde{\omega}(t) = - \text{Ric}(\theta_0) + e^{-t} (\text{Ric}(\theta_0) + \omega(0)) + \sqrt{-1} \partial \bar{\partial} u. \]  

(A.3)

\[ \left( \frac{\partial}{\partial t} - \tilde{\Delta} \right) (\dot{u} + u) = - \text{tr}_{\tilde{g}} \text{Ric}(\theta_0) - n, \]

(A.4)

where \( \tilde{\Delta} \) is the Chern Laplacian with respect to \( \tilde{g} \).

**Lemma A.1** (See [17,30]) Let \( g(t) \) be a solution to the Chern–Ricci flow and let \( \Upsilon = \text{tr}_h g \), and \( \Theta = \text{tr}_h h \).

\[ \left( \frac{\partial}{\partial t} - \Delta \right) \log \Upsilon = I + II + III \]

where

\[ I \leq 2 \Upsilon^{-2} \text{Re} \left( h^{ij} g^{k\bar{l}} (T_0)_{kij} \hat{\nabla}_q \Upsilon \right). \]

\[ II = \Upsilon^{-1} g^{ij} \hat{h}^{k\bar{l}} g_{k\bar{l}} \left( \hat{\nabla}_i (\hat{T})^q_{ji} - \hat{h}^{p\bar{q}} \hat{R}_{i\bar{p}j} \right) \]

and

\[ III = - \Upsilon^{-1} g^{ij} \hat{h}^{k\bar{l}} \left( \hat{\nabla}_i (\hat{T}_0)_{jik} \right) + \hat{\nabla}_i \left( (T_0)_{ikj} \right) - \left( \hat{T}^q_{ji} (T_0)_{ikj} \right). \]

where \( T_0 \) is the torsion of \( g_0 = g(0) \), \( \hat{T} \) is the torsion of \( h \) and \( \hat{\nabla} \) is the derivative with respect to the Chern connection of \( h \).
Appendix B: A maximum principle

We have the following maximum principle, see [16] for example.

**Lemma B.1** Let \((M^n, h)\) be a complete non-compact Hermitian manifold satisfying condition: There exists a smooth positive real exhaustion function \(\rho\) such that \(|\partial \rho|^2_h + |\sqrt{-1} \partial \bar{\partial} \rho|_h \leq C_1\). Suppose \(g(t)\) is a solution to the Chern–Ricci flow on \(M \times [0, S)\). Assume for any \(0 < S_1 < S\), there is \(C_2 > 0\) such that

\[
C_2^{-1} h \leq g(t)
\]

for \(0 \leq t \leq S_1\). Let \(f\) be a smooth function on \(M \times [0, S)\) which is bounded from above such that

\[
\left( \frac{\partial}{\partial t} - \Delta \right) f \leq 0
\]

on \(\{ f > 0 \}\) in the sense of barrier. Suppose \(f \leq 0\) at \(t = 0\), then \(f \leq 0\) on \(M \times [0, S)\).

We say that

\[
\left( \frac{\partial}{\partial t} - \Delta \right) f \leq \phi
\]

in the sense of barrier means that for fixed \(t_1 > 0\) and \(x_1\), for any \(\epsilon > 0\), there is a smooth function \(\sigma(x)\) near \(x_1\) such that \(\sigma(x_1) = f(x_1, t_1)\), \(\sigma(x) \leq f(x, t_1)\) near \(x_1\), such that \(\sigma\) is \(C^2\) and at \((x_1, t_1)\)

\[
\frac{\partial_{-}}{\partial t} f(x, t) - \Delta \sigma(x) \leq \phi(x) + \epsilon.
\]

Here

\[
\frac{\partial_{-}}{\partial t} f(x, t) = \liminf_{h \to 0^+} \frac{f(x, t) - f(x, t - h)}{h}
\]

for a function \(f(x, t)\).

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