QUASI-ISOMETRY RIGIDITY OF RIGHT-ANGLED ARTIN GROUPS I: THE FINITE OUT CASE

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Abstract. Let $G$ and $G'$ be two right-angled Artin groups (RAAG). We show they are quasi-isometric iff they are isomorphic, under the assumption that $Out(G)$ and $Out(G')$ are finite. If only $Out(G)$ is finite, then $G'$ is quasi-isometric $G$ iff $G'$ is isomorphic to a finite index subgroup of $G$. In this case, we give an algorithm to determine whether $G$ and $G'$ are quasi-isometric by looking at their defining graphs.

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1. Introduction

1.1. Backgrounds and Summary of Results. Given a finite simplicial graph $\Gamma$ with vertex set $\{v_i\}_{i \in I}$, the right-angled Artin group (RAAG) with defining graph $\Gamma$, denoted by $G(\Gamma)$, is given by the following presentation:
\{v_i, \text{ for } i \in I | [v_i, v_j] = 1 \text{ iff } v_i \text{ and } v_j \text{ are joined by an edge}\}

\{v_i\}_{i \in I} \text{ is called a standard generating set for } G(\Gamma).

The class of all RAAGs enjoys a balance between simplicity and complexity. On one hand, RAAGs have many nice geometric, combinatorial and group theoretic properties (see \cite{Cha07} for a summary); on the other hand, this class inherits the full complexity of the collection of finite simplicial graphs, and even a single RAAG could have very complicated subgroups (\cite{BB97}).

In recent years, RAAG has become an important model to understand other unknown groups, either by (virtually) embedding the unknown groups into some RAAGs (see \cite[Section 6]{Wis09}), or by finding embedded copies of RAAGs in the unknown groups (\cite{CLM10, Kob12b, Tay13, KK13a, BKK14}).

In this paper, we will study the (asymptotic) geometry of RAAGs and use this to understand the quasi-isometry classification and rigidity properties of RAAGs. Currently, the following two classes of RAAGs have been classified:

(1) Tree groups by Behrstock and Neumann (\(\Gamma\) is a tree). It is shown in \cite{BN08} that for any two trees \(\Gamma_1\) and \(\Gamma_2\) with diameter \(\geq 3\), \(G(\Gamma_1)\) and \(G(\Gamma_2)\) are quasi-isometric. The higher dimensional analogues of tree groups is studied in \cite{BJN10}.

(2) Atomic groups by Bestvina, Kleiner and Sageev (\(\Gamma\) is connected and does not contain valence one vertices, cycles of length \(<5\) and separating closed stars). It is shown in \cite{BKS08a} that two atomic RAAGs are quasi-isometric iff they are isomorphic.

It is interesting to see these two cases have opposite conclusions. Here the dimension of \(G(\Gamma)\) is defined to be the maximal \(n\) such that \(G(\Gamma)\) contains a \(\mathbb{Z}^n\) subgroup, and it coincides with the cohomological dimension of \(G(\Gamma)\). All atomic groups are 2-dimensional, hence it is natural to ask what should be the proper higher dimensional analogues of atomic groups. This is the starting point of the current paper.

Since we are looking for RAAGs which are as “rigid” as atomic groups, those ones with small quasi-isometry groups would be reasonable candidates. However, \(QI(G(\Gamma))\) is usually very complicated, so we turn to the outer automorphism group \(Out(G(\Gamma))\), which is relatively easy to work with. Moreover, \(QI(G(\Gamma))\) and \(Out(G(\Gamma))\) are loosely related in the way that they both capture the symmetry of \(G(\Gamma)\), one from a geometric viewpoint and one from an algebraic viewpoint.

Now it is natural to ask whether those RAAGs with “small” outer automorphism groups is more rigid or not. Actually, “small” outer automorphism and (quasi-)isometry rigidity results come together in several other cases, for example, Mostow rigidity, mapping class groups (\cite{Ham05, BKMM12}), \(Out(F_n)\) (\cite{FH07}) etc. Our first result is about the quasi-isometry classification for RAAGs with finite outer automorphism group.

**Theorem 1.1.** Pick \(G(\Gamma_1)\) and \(G(\Gamma_2)\) such that \(Out(G(\Gamma_i))\) is finite for \(i = 1, 2\). Then they are quasi-isometric iff they are isomorphic.

The collection of RAAGs with finite outer automorphism group is a reasonably large class. Recall that there is a 1-1 correspondence between finite simplicial graphs and RAAGs (\cite{Dro87}), thus it makes sense to talk about a random RAAG by considering the Erdős–Rényi model for random graph. If the parameters of the model are in the right range, then almost all RAAGs have finite outer automorphism group (\cite{CF12, Day12}).
The class of 2-dimensional RAAGs with finite outer automorphism group is strictly larger than the class of atomic RAAGs, moreover, there are lots of higher dimensional RAAGs with finite outer automorphism group.

Whether $\text{Out}(G(\Gamma))$ is finite or not can be easily read from $\Gamma$. By results in [Ser89, Lau95], $\text{Out}(G(\Gamma))$ is generated by the following four types of elements (we identify the vertex set of $\Gamma$ with a standard generating set of $G(\Gamma)$):

1. Given vertex $v \in \Gamma$, sending $v \to v^{-1}$ and fixing all other vertices.
2. Graph automorphism of $\Gamma$.
3. If $\text{lk}(w) \subset \text{St}(v)$ for vertices $w, v \in \Gamma$, sending $w \to vw$ and fixing all other vertices induces to a group automorphism. It is called a transvection. When $d(v, w) = 1$, it is an adjacent transvection, otherwise it is a non-adjacent transvection.
4. Suppose $\Gamma \setminus \text{St}(v)$ is disconnected. Then one obtains a group automorphism by picking a connected component $C$ and sending $w \to vwv^{-1}$ for vertex $w \in C$. It is called a partial conjugation.

We caution the reader that in this paper, the closed star of a vertex $v$, which we denoted by $\text{St}(v)$, is defined to be the full subgraph spanned by $v$ and vertices adjacent to $v$. This definition is slightly different from the usual one. Similarly, $\text{lk}(v)$ is defined to be the full subgraph spanned by vertices adjacent to $v$. See Section 2.1.

Elements of type (3) and (4) have infinite order in $\text{Out}(G(\Gamma))$ while elements of type (1) and (2) are of finite order. $\text{Out}(G(\Gamma))$ is finite iff $\Gamma$ does not contain any separating closed star, and there does not exist distinct vertices $v, w \in \Gamma$ such that $\text{lk}(w) \subset \text{St}(v)$.

**Theorem 1.2.** Suppose $\text{Out}(G(\Gamma_1))$ is finite. Then the following are equivalent:

1. $G(\Gamma_2)$ is quasi-isometric to $G(\Gamma_1)$.
2. $G(\Gamma_2)$ is isomorphic to a subgroup of finite index in $G(\Gamma_1)$.
3. $\Gamma_2$ is isomorphic to $\Gamma_1$.

Here $\Gamma'$ denotes the extension graph introduced by Kim and Koberda in [KK13b] (see Definition 2.11). Extension graphs can be viewed as “curve graphs” for RAAGs ([KK14]). This analogue carries on to the aspect of quasi-isometry rigidity. Namely, if $G$ is a mapping class group and $q : G' \to G$ is a quasi-isometry, then it is shown in [BKMM12] that $G'$ naturally acts on the curve graph associated to $G$. This is still true if $G$ is a RAAG with some restriction on its outer automorphism group, for example, $\text{Out}(G)$ is finite.

However, in general, there exists a pair of commensurable RAAGs with different extension graphs, see Example [3,28]. There also exists a pair of non-quasi-isometric RAAGs with isomorphic extension graphs.

Theorem 1.2 implies it suffices to study finite index RAAG subgroup (a subgroup which is also a RAAG) of $G(\Gamma_1)$.

Given an arbitrary RAAG $G(\Gamma)$ (not necessarily have finite outer automorphism group) and pick a standard generating set $S$ for $G(\Gamma)$, let $d_S$ be the word metric on $G(\Gamma)$ with respect to $S$. A subset $K \subset G(\Gamma)$ is $S$-convex iff for any three points $x, y \in K$ and $z \in G(\Gamma)$ such that $d_S(x, y) = d_S(x, z) + d_S(z, y)$, we must have $z \in K$. Every finite $S$-convex subset $K$ naturally gives rise to a finite index RAAG subgroup $G \leq G(\Gamma)$ such that $K$ is the fundamental domain of the left action $G \curvearrowright G(\Gamma)$. For example, if $G(\Gamma) = \mathbb{Z} \oplus \mathbb{Z}$ and pick $K$ to be a rectangle of size $n$ by
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If \( m \), then the corresponding subgroup is of form \( n \mathbb{Z} \oplus m \mathbb{Z} \). The detailed construction of this is given in Section 6.1. \( G \) is called a \( S \)-special subgroup of \( G(\Gamma) \). A subgroup of \( G(\Gamma) \) is special iff it is \( S \)-special for some standard generating set \( S \). A similar construction in the case of right-angled Coxeter groups has been done in [Hag08].

Our next result says if \( \text{Out}(G(\Gamma)) \) is finite, then this is the only way to obtain finite index RAAG subgroups.

**Theorem 1.3.** Suppose \( \text{Out}(G(\Gamma)) \) is finite and let \( S \) be a standard generating set for \( G(\Gamma) \). Then all finite index RAAG subgroups are \( S \)-special. Moreover, there is a 1-1 correspondence between non-negative finite \( S \)-convex subsets of \( G(\Gamma) \) based at the identity and finite index RAAG subgroups of \( G(\Gamma) \).

The most simple example is when \( G(\Gamma) = \mathbb{Z} \), we have \( n \mathbb{Z} \leftrightarrow [0, n-1] \).

We need to explain two terms: non-negative and based at the identity. For example, take \( G = n \mathbb{Z} \oplus m \mathbb{Z} \) inside \( \mathbb{Z} \oplus \mathbb{Z} \), then any \( n \) by \( m \) rectangle could be the fundamental domain for the action of \( G \). We naturally require the rectangle to be in the first quadrant and contain the identity, which would give us a unique choice. One could do similar things in all RAAGs and this two terms will be defined precisely in Section 6.

**Corollary 1.4.** If \( \text{Out}(G(\Gamma_1)) \) is finite, then \( G(\Gamma_2) \) is quasi-isometric to \( G(\Gamma_1) \) iff \( G(\Gamma_2) \) is isomorphic to a special subgroup of \( G(\Gamma_1) \).

It turns out that there is an algorithm to enumerate the defining graphs of all special subgroups of a RAAG, so

**Theorem 1.5.** If \( \text{Out}(G(\Gamma)) \) is finite, then \( G(\Gamma') \) is quasi-isometric to \( G(\Gamma) \) iff \( \Gamma' \) can be obtained from \( \Gamma \) by finite many GSEs. In particular, there is an algorithm to determine whether \( G(\Gamma') \) and \( G(\Gamma) \) are quasi-isometric by looking at the graphs \( \Gamma \) and \( \Gamma' \).

GSE is a generalized version of star extension in [BKS08a Example 1.4], see also [KK13b, lemma 50]. It will be defined in Section 6.

Another natural question motivated by Theorem 1.2 is the following:

**Question 1.6.** Let \( G(\Gamma) \) be a RAAG such that \( \text{Out}(G(\Gamma)) \) is finite. And let \( H \) be a finite generated group quasi-isometric to \( G(\Gamma) \). What can we say about \( H \)?

Actually, we have shown that \( H \) admits a cubulation, and the following result will appear in a forthcoming paper:

**Theorem 1.7.** Let \( G(\Gamma) \) and \( H \) be as in Question 1.6. Then the induced quasi-action \( H \acts X(\Gamma) \) is quasi-isometrically conjugate to a geometric action \( H \acts X' \). Here \( X' \) is a \( \text{CAT}(0) \) cube complex which is closely related to \( X(\Gamma) \).

1.2. Comments on the Proof. We start with several notations. The Salvetti complex of \( G(\Gamma) \) is denoted by \( S(\Gamma) \), the universal covering of \( S(\Gamma) \) is denoted by \( X(\Gamma) \), and flat in \( X(\Gamma) \) that covers a standard torus in \( S(\Gamma) \) is called standard flat. See Section 2.4 for precise definition of these terms.

Let \( q : X(\Gamma) \to X(\Gamma') \) be a quasi-isometry. The proof of Theorem 1.1 follows the scheme of proof of the main theorem in [BKS08a]. One can also find similar schemes in [KL97, BKMM12]. There are three steps in [BKS08a]. First they show \( q \) maps top dimensional flat to top dimensional flat up to finite Hausdorff distance. However, the collection of top dimensional flats is too large to be linked.
directly to the combinatorics of RAAG, so the second step is to show quasi-isometry preserve standard flats up to finite Hausdorff distance. The third step is to use this combinatorial information to reconstruct the quasi-isometry such that it actually maps standard flat to standard flat, and the conclusion follows automatically.

In our cases, the first step has been done in [Hua], where we show $q$ still preserves top dimensional flat up to finite Hausdorff distance in higher dimensional case. No assumption on the outer automorphism group is needed for this step.

The second step consists of two parts. First we show $q$ preserves certain top dimensional maximal products up to finite Hausdorff distance. Then one wish to pass to standard flats by intersecting these top dimensional objects. However, in the higher dimensional case, lower dimensional standard flat may not be the intersection of top dimensional objects, and even in the case it is an intersection, one may not be able to read this information directly from the defining graph $\Gamma$. This is different from the 2-dimensional situation in [BKS08a] and brings complications to our proof.

A necessary condition for $q$ to preserve the standard flats is that every elements in $Out(G(\Gamma))$ should do so, which implies there could not be any transvections in $Out(G(\Gamma))$. This condition is also sufficient:

**Theorem 1.8.** Suppose $Out(G(\Gamma))$ is transvection free. Then there exists positive constant $D = D(L, A, |\Gamma|)$ such that for any standard flat $F \subset X(\Gamma)$, there exists a standard flat $F' \subset X(\Gamma')$ such that $d_H(q(F), F') < D$.

$|\Gamma|$ is the number of vertices in $\Gamma$ and $d_H$ denotes the Hausdorff distance.

In step 3, our treatment is different from the one in [BKS08a] and we will rely on a different combinatorial object. First we construct an auxiliary simplicial complex $P(\Gamma)$, which can be viewed as a simplified Tits boundary for $X(\Gamma)$. This complex turns out to coincide with the extension graph introduced in [KK13b].

Denote the Tits boundary of $X(\Gamma)$ by $\partial_T(X(\Gamma))$, and let $T(\Gamma) \subset \partial_T(X(\Gamma))$ be the union of Tits boundaries of standard flats in $X(\Gamma)$. Then $T(\Gamma)$ has a natural simplicial structure. However, $T(\Gamma)$ contains redundant information, this can be seen in the similar situation where the link of the base point of $S(\Gamma)$ looks more complicated than $\Gamma$, but they essentially contain the same information.

This redundancy can be resolved by replacing the spheres in $T(X)$ that arises from standard flats by simplexes of the same dimension. This gives rise to a well defined simplicial complex $P(\Gamma)$, since for any standard flats $F_1$ and $F_2$ with $\partial_T F_1 \cap \partial_T F_2 \neq \emptyset$, there exists a standard flat $F$ such that $\partial_T F = \partial_T F_1 \cap \partial_T F_2$. See Section 4.1 for more properties of $P(\Gamma)$.

By Theorem 1.8 if both $Out(G(\Gamma))$ and $Out(G(\Gamma'))$ are transvection free, then $q$ induces a boundary map $\partial q : P(\Gamma) \to P(\Gamma')$, which is a simplicial isomorphism. Conversely, we can ask if it is possible to reconstruct a map $X(\Gamma) \to X(\Gamma')$ from the boundary map $\partial q$?

More precisely, pick vertex $p \in X(\Gamma)$, let $\{F_i\}_{i=1}^n$ be the collection of maximal standard flats containing $p$. By Theorem 1.8 for each $i$, there exists a unique maximal standard flat $F_i' \subset X(\Gamma')$ such that $d_H(q(F_i), F_i') < \infty$. One may wish to map $p$ to $\cap_{i=1}^n F_i$. Is $\cap_{i=1}^n F_i'$ non-empty?

It turns out that if we also rule out partial conjugations in $Out(G(\Gamma))$, then $\cap_{i=1}^n F_i'$ is exactly a point. This give rises to a well-defined map $\bar{q} : X(\Gamma)^{(0)} \to X(\Gamma')^{(0)}$ which maps vertices in a standard flat to vertices in standard flat. If
Out$(G(\Gamma'))$ is also finite, then we can define an inverse map of $\bar{q}$ and this is enough to deduce Theorem 1.1.

If only Out$(G(\Gamma))$ is assumed to be finite, we can still recover the fact that $\partial q$ is a simplicial isomorphism (this is non-trivial, since Theorem 1.8 does not say for any standard flat $F' \subset X(\Gamma')$, we can find standard flat $F \subset X(\Gamma)$ such that $d_H(q(F), F') < \infty$). Hence we can define $\tilde{q}$ as before. However the inverse of $\tilde{q}$ does not exist in general.

The next step is trying to extend $\tilde{q}$ to a cubical map from $X(\Gamma)$ to $X(\Gamma')$. There are obvious obstructions: though $\tilde{q}$ maps vertices in a standard geodesic to vertices in a standard geodesic, $\tilde{q}$ may not preserve the order of these vertices. A typical example is given in the following picture, where one can permute the green level and the red level in a tree, then the order of vertices in the black line is not preserved.

A remedy is to “flip backwards”. Namely we will pre-compose $\tilde{q}$ with a sequence of permutations of “levels” such that the resulting map restricted to each standard geodesic respects the order. Then we can extend $\tilde{q}$ to a cubical map. This argument relies on the understanding of quasi-isometry flexibility, namely how much room we have to perform this flip process. Here is one formulation which demonstrate this aspect:

**Theorem 1.9.** If Out$(G(\Gamma))$ is finite, then Aut$(P(\Gamma)) \cong$ Isom$(G(\Gamma), d_r)$.

Here $d_r$ denotes the syllable metric, see [KK14, Section 5.2].

Theorem 1.2 - Theorem 1.5 will rely on the cubical map $\tilde{q}$. In particular, $\tilde{q}^{-1}(x)$ ($x \in X(\Gamma')$ is a vertex) is a compact convex subcomplex and this is how we obtain the $S$-convex subset in Theorem 1.3.

1.3. **Organization of the Paper.** Section 2 contains basic notations used in this paper and some background material about CAT$(0)$ cube complex and RAAG. In particular, Section 2.3 collects several technical lemmas about CAT$(0)$ cube complex, one can skip 2.3 on first reading and come back when needed.

In Section 3 we prove Theorem 1.8. Section 3.1 is about stability of top dimensional maximal product subcomplexes under quasi-isometry and Section 3.2 deals with lower dimensional objects. In Section 4 we prove Theorem 1.1. We will construct the extension complex from our viewpoint in Section 4.1 and explain how is this object related to Tits boundary, flat space and contact graph. In Section 4.2, we describe how to reconstruct the quasi-isometry.

Section 5.1 and Section 5.2 are devoted to proving Theorem 1.2. Section 6 is devoted to proving Theorem 1.3, Corollary 1.4 and Theorem 1.5.
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2. Preliminaries

2.1. Notations and Conventions. All graphs in this paper are simplicial.

The flag complex of a graph $\Gamma$ is denoted by $F(\Gamma)$, i.e. $F(\Gamma)$ is a flag complex such that its 1-skeleton is $\Gamma$.

A subcomplex $K'$ in a combinatorial polyhedral complex $K$ is full iff $K'$ contains all the subcomplexes of $K$ which have the same vertex set as $K'$. If $K$ is 1-dimensional, then we also call $K'$ a full subgraph.

We use “$*$” to denote the joint of two simplicial complexes and “$o$” to denote the join of two graphs. Let $K$ be a simplicial complex or a graph. By viewing the 1-skeleton as a metric graph with edge lengths 1, we obtain a metric defined on the 0-skeleton, which we denote by $d$. Let $N \subset K$ be a subcomplex. We define the orthogonal complement of $N$, denoted by $N^\perp$, to be the set $\{v \in K(0) \mid d(v, v) = 1$ for any $v \in V\}$, define $lk(N)$ to be the full subcomplex spanned by $N^\perp$, and define closed star of $N$, denoted by $St(N)$, to be the full subcomplex spanned by $N \cup lk(N)$.

Suppose $L$ is a subcomplex such that $N \subset L \subset K$. We denote the closed star of $N$ in $L$ by $St(v, L)$. If $L$ is a full subcomplex, then $St(N, L) = St(N) \cap L$. We can define $lk(N, L)$ in a similar way. Let $M \subset K$ be an arbitrary subset. We denote the collection of vertices inside $M$ by $\nu(M)$.

We will follow the language of coarse set theory introduced in [MSW04].

Let $(X, d)$ be a metric space. The open ball of radius $r$ centred at $p$ in $X$ will be denoted by $B(p, r)$. Given subsets $A, B \subset X$, the $r$-neighbourhood of a subset $A$ is denoted by $N_r(A)$. The diameter of $A$ is denoted by $\text{diam}(A)$. The Hausdorff distance between $A$ and $B$ is denoted by $d_H(A, B)$. Here is a summary of notations: subsection $\text{CAT}(0)$ space and $\text{CAT}(0)$ cube complex First we recall several basic facts about $\text{CAT}(0)$ space, the standard reference is [BH99].

Let $(X, d)$ be a $\text{CAT}(0)$ space. Pick $x, y \in X$, we denote the unique geodesic segment joining $x$ and $y$ by $\overline{xy}$. For $y, z \in X \setminus \{x\}$, denote the comparison angle between $\overline{xy}$ and $\overline{xz}$ at $x$ by $\angle_x(y, z)$ and the Alexandrov angle by $\angle_x(y, z)$.

The boundary of $X$, denoted by $\partial X$, is the collection of asymptotic classes of geodesic rays. $\partial X$ has an angular metric, which is defined by

$$\angle(\xi_1, \xi_2) = \lim_{t, t' \to \infty} \frac{1}{t} \angle_p(l_1(t), l_2(t'))$$

here $l_1$ and $l_2$ are unit speed geodesic rays emanating from a base point $p$ such that $l_i(\infty) = \xi_i$ for $i = 1, 2$. This metric does not depend on the choice of $p$, and the length metric associated to the angular metric, denoted by $d_T$, is called the Tits metric. The Tits boundary, denoted by $\partial_T X$, is the $\text{CAT}(1)$ space $(\partial X, d_T)$ (see Chapter II.8 and II.9 of [BH99]).
Given two metric spaces \((X_1, d_1)\) and \((X_2, d_2)\), denote the Cartesian product of \(X_1\) and \(X_2\) by \(X_1 \times X_2\), i.e. \(d = \sqrt{d_1^2 + d_2^2}\) on \(X_1 \times X_2\). If \(X_1\) and \(X_2\) are CAT(0), then so is \(X_1 \times X_2\).

A \(n\)-flat in a CAT(0) space \(X\) is the image of an isometric embedding \(\mathbb{E}^n \to X\). Note that any flat is convex in \(X\).

Pick a convex subset \(C \subseteq X\), then \(C\) is also CAT(0). We use \(\pi_C\) to denote the nearest point projection from \(X\) to \(C\). \(\pi_C\) is well-defined and is 1-Lipschitz. Moreover, pick \(x \in X \setminus C\), then \(\angle_{\pi_C(x)}(x, y) \geq \pi/2\) for any \(y \in C\) such that \(y \neq \pi_C(x)\) (see Proposition II.2.4 of [BH99]).

If \(C' \subseteq X\) is another convex set, then \(C'\) is parallel to \(C\) iff \(d(\cdot, C)|_{C'}\) and \(d(\cdot, C')|_C\) are constant functions. In this case, there is a natural isomorphism between \(C \times [0, d(C, C')]\) and the convex hull of \(C\) and \(C'\). We define the parallel set of \(C\), denoted by \(P_C\), to be the union of all convex subsets of \(X\) parallel to \(C\). If \(C\) has geodesic extension property, or more generally, \(C\) is boundary-minimal (Section 3.3 of [CM09]), then \(P_C\) is a convex subset in \(X\). Moreover, \(P_C\) admits a canonical splitting \(P_C = C \times C^\perp\), where \(C^\perp\) is a CAT(0) space.

Now we turn to CAT(0) cube complex. All cube complexes in this paper are assumed to be finite dimensional.

A cube complex \(X\) is obtained by gluing a collection of unit Euclidean cubes isometrically along their faces, see II.7.32 of [BH99] for a precise definition. Then the cube complex has a natural piecewise Euclidean metric. This metric is complete and geodesic since \(X\) is finite dimensional (I.7.19 of [BH99]) and is non-positively curved iff the link of each vertex is a flag complex ([Gro87]). If in addition \(X\) is simply connected, then this metric is CAT(0) and \(X\) is said to be a CAT(0) cube complex. We can put a different metric on the 1-skeleton \(X(1)\) by considering it as a metric graph with all edge lengths 1. This is called the \(l^1\) metric and the natural injection \(X(1) \hookrightarrow X\) is a quasi-isometry (1.7.31 of [BH99] or Lemma 2.2 of [CS11]). In this paper, we will mainly use the CAT(0) metric unless otherwise specified.

A geodesic segment, geodesic ray or geodesic in \(X\) is an isometric embedding of \([a, b], [0, \infty)\) or \(\mathbb{R}\) into \(X\) with respect to the CAT(0) metric. A combinatorial geodesic segment, combinatorial geodesic ray or combinatorial geodesic is a \(l^1\)-isometric embedding of \([a, b], [0, \infty)\) or \(\mathbb{R}\) into \(X(1)\) such that its image is a subcomplex.

Let \(X\) be a CAT(0) cube complex and let \(Y \subseteq X\) be a subcomplex. Then the following are equivalent (see [Hag08]):

1. \(Y\) is convex with respect to the CAT(0) metric.
2. \(Y\) is a full subcomplex and \(Y^{(1)} \subseteq X^{(1)}\) is convex with respect to the \(l^1\) metric.
3. \(Lk(p, Y)\) (the link of \(p\) in \(Y\)) is a full subcomplex of \(Lk(P, X)\) for every vertex \(p \in Y\).

The collection of convex subcomplexes in a CAT(0) cube complex enjoys the following version of Helly’s property ([Ger98]):

**Lemma 2.1.** Let \(X\) be as above and \(\{C_i\}_{i=1}^k\) be a collection of convex subcomplexes. If \(C_i \cap C_j \neq \emptyset\) for any \(1 \leq i \neq j \leq k\), then \(\cap_{i=1}^k C_i \neq \emptyset\).

**Lemma 2.2.** Let \(X_1\) and \(X_2\) be two CAT(0) cube complexes and let \(K \subseteq X_1 \times X_2\) be a convex subcomplex. Then \(K\) admits a splitting \(K = K_1 \times K_2\) where \(K_i\) is a convex subcomplex of \(X_i\) for \(i = 1, 2\).
The lemma is clear when $X_1 \cong [0,1]$, and the general case follows from this special case.

Now we come to the notion of hyperplane, which is the cubical analogue of “track” introduced in [Dun85]. A hyperplane $h$ in a cube complex $X$ is a subset such that

1. $h$ is connected.
2. For each cube $C \subset X$, $h \cap C$ is either empty or a union of mid-cubes of $C$.
3. $h$ is minimal, i.e. if there exists $h' \subset h$ satisfying (1) and (2), then $h = h'$.

Recall that a mid-cube of $C = [0,1]^n$ is a subset of form $f_i^{-1}(1/2)$, where $f_i$ is one of the coordinate functions.

If $X$ is a CAT(0) cube complex, then the following are true (see [Sag95]):

1. Each hyperplane is embedded, i.e. $h \cap C$ is either empty or a mid-cube of $C$;
2. $h$ is a convex subset in $X$ and $h$ with the induced cell structure from $X$ is also a CAT(0) cube complex;
3. $X \setminus h$ has exactly two connected components, they are called halfspaces. The closure of a halfspace is called closed halfspace, which is also convex in $X$.
4. Let $N_h$ be the smallest subcomplex of $X$ that contains $h$. Then $N_h$ is a convex subcomplex of $X$ and there is a natural isometry $i : N_h \to h \times [0,1]$ such that $i(h) = h \times \{1/2\}$. $N_h$ is called the carrier of $h$.
5. For every edge $e \subset X$, there exists a unique hyperplane $h_e$ which intersects $e$ in its midpoint. In this case, we say $h_e$ is the hyperplane dual to $e$ and $e$ is an edge dual to the hyperplane $h_e$.
6. Lemma 2.1 is also true for a collection of hyperplanes.

Now it is easy to see an edge path $\omega \subset X$ is a combinatorial geodesic segment if there does not exist two different edges of $\omega$ such that they are dual to the same hyperplane. Moreover, for two vertices $v, w \in X$, their $l^1$ distance is exactly the number of hyperplanes that separate $v$ from $w$.

Pick an edge $e \subset X$ and let $\pi_e : X \to e \cong [0,1]$ be the CAT(0) projection. Then

1. The hyperplane dual to $e$ is exactly $\pi_e^{-1}(1/2)$.
2. $\pi_e^{-1}(t)$ is convex in $X$ for any $0 \leq t \leq 1$, moreover, if $0 < t < t' < 1$, then $\pi_e^{-1}(t)$ and $\pi_e^{-1}(t')$ are parallel.
3. $N_h$ is the closure of $\pi_e^{-1}(0,1)$. Alternatively, we can describe $N_h$ as the parallel set of $e$.

2.2. Coarse intersection of convex subcomplexes.

**Lemma 2.3** (Lemma 2.10 of [Hua]). Let $X$ be a CAT(0) cube complex of dimension $n$ and $C_1, C_2$ be convex subcomplexes. Put $\Delta = d(C_1, C_2)$. Let $Y_1 = \{y \in C_1 \mid d(y, C_2) = \Delta\}$ and $Y_2 = \{y \in C_2 \mid d(y, C_1) = \Delta\}$. Then:

1. $Y_1$ and $Y_2$ are not empty.
2. $Y_1$ and $Y_2$ are convex; $\pi_{C_1}$ map $Y_2$ isometrically onto $Y_1$ and $\pi_{C_2}$ map $Y_1$ isometrically onto $Y_2$; the convex hull of $Y_1 \cup Y_2$ is isometric to $Y_1 \times [0, \Delta]$.
3. $Y_1$ and $Y_2$ are subcomplexes, and $\pi_{C_2}|_{Y_1}$ is a cubical isomorphism with its inverse given by $\pi_{C_1}|_{Y_2}$.
(4) there exist $A = A(\Delta, n, \epsilon)$ such that if $p_1 \in C_1$, $p_2 \in C_2$ and $d(p_1, Y_1) \geq \epsilon > 0$, $d(p_2, Y_2) \geq \epsilon > 0$, then

$$d(p_1, C_2) \geq \Delta + Ad(p_1, Y_1), d(p_2, C_1) \geq \Delta + Ad(p_2, Y_2)$$

Remark 2.5. Equation (2.4) implies for any $r > 0$, $(C_1 \cap C_2) \subset \gamma Y_i (i = 1, 2)$, here $r' = \min(1,(2r - \Delta)/A) + r$ and $A = A(\Delta, n, 1)$. Moreover, $\partial T C_1 \cap \partial T C_2 = \partial T Y_1 = \partial T Y_2$.

The remark implies $Y_1 \subseteq Y_2 \subseteq (C_1 \cap C_2)$ for $r$ large enough. We use $T(C_1, C_2) = (Y_1, Y_2)$ to describe this situation, where $T$ stands for the word “intersect”. The next lemma gives a combinatorial description of $Y_1$ and $Y_2$.

Lemma 2.6. Let $X, C_1, C_2, Y_1$ and $Y_2$ be as above. Pick an edge $e$ in $Y_1$ (or $Y_2$) and let $h$ be the hyperplane dual to $e$. Then $h \cap C_i \neq \emptyset$ for $i = 1, 2$. Conversely, if a hyperplane $h'$ satisfies $h' \cap C_i \neq \emptyset$ for $i = 1, 2$, then $T(h' \cap C_1, h' \cap C_2) = (h' \cap Y_1, h' \cap Y_2)$ and $h'$ comes from the dual hyperplane of some edge $e'$ in $Y_1$ (or $Y_2$).

Proof. The first part of the Lemma follows from Lemma 2.3. Let $T(h' \cap C_1, h' \cap C_2) = (Y_1', Y_2')$. Pick $x \in Y_1'$ and let $x' = \pi_{h' \cap C_2}(x) \in Y_2'$. Then $\pi_{h' \cap C_1}(x') = x$. Let $N_h' = h' \times [0, 1]$ be the carrier of $h'$. Then $(h' \cap C_i) \times ([1/2 - \epsilon, 1/2 + \epsilon]) = C_i \cap (h' \times ([1/2 - \epsilon, 1/2 + \epsilon]))$ for $i = 1, 2$ and $\epsilon < 1/2$. Thus for any $y \in C_2$, $\angle_{x'}(x, y) \geq \pi/2$, which implies $x' = \pi_{C_2}(x)$. Similarly, $x = \pi_{C_1}(x') = \pi_{C_1} \circ \pi_{C_2}(x)$, hence $x \in Y_1$ and $Y_1' \subset Y_1$. By the same argument, $Y_2' \subset Y_2$, thus $Y_1' = Y_1 \cap h'$ for $i = 1, 2$ and the lemma follows.

Lemma 2.3, Remark 2.5 and Lemma 2.6 can also be applied to $CAT(0)$ rectangle complexes of finite type, whose cells are of form $\Pi_{i=1}^n[0, a_i]$. “Finite type” means there are only finite many isometry types of rectangle cells in the rectangle complex.

Lemma 2.7. Let $X, C_1, C_2, Y_1$ and $Y_2$ be as above. If $h$ is a hyperplane separating $C_1$ from $C_2$, then there exists a convex set $Y \subset h$ such that $Y$ is parallel to $Y_1$ (or $Y_2$).

Proof. Let $M = Y_1 \times [0, \Delta]$ ($\Delta = d(C_1, C_2)$) be the convex hull of $Y_1$ and $Y_2$. We want to prove $M \cap h = Y_1 \times \{t\}$ for some $t \in [0, \Delta]$. It suffices to show for any edge $e \subset Y_1$, $e \times [0, \Delta] \cap h = e \times \{t\}$ for some $t$.

Pick point $x \in e$ and let $\{x\} \times \{t\} = (\{x\} \times [0, \Delta]) \cap h$. Since $e \times \{t\}$ is parallel to $e$, $e \times \{t\}$ sits inside a cube and $e \times \{t\}$ is parallel to an edge of this cube. Thus either $e \times \{t\} \subset h$ or $e \times \{t\}$ is parallel to some edge dual to $h$, but the second case implies that $h$ is dual to $e$ and $h \cap Y_1 \neq \emptyset$, which is impossible, so $e \times \{t\} \subset (e \times [0, \Delta]) \cap h$. Now we are done since $(\{x\} \times [0, \Delta]) \cap h$ is exactly one point for each $x \in e$. \qed

2.3. Right-angled Artin group. Pick a finite simplicial graph $\Gamma$, let $S$ be a standard generating set for $G(\Gamma)$ and label the vertices of $\Gamma$ by elements in $S$. $G(\Gamma)$ has a nice Eilenberg-MacLane space $S(\Gamma)$, called the Salvetti complex (see [CD95, Cha07]). $S(\Gamma)$ is an non-positively curved cube complex. The 2-skeleton of $S(\Gamma)$ is the usual representing complex of $G(\Gamma)$. If the representing complex contains a copy of 2-skeleton of a 3-torus, then we attach a 3-cell to obtain a 3-torus. We can build $S(\Gamma)$ inductively in this manner, and this process will stop after finite many steps. The closure of each k-cell in $S(\Gamma)$ is a k-torus. Torus of this kind are called standard torus. There is a 1-1 correspondence between the
Denote the universal covering of $S(\Gamma)$ by $X(\Gamma)$, which is a $CAT(0)$ cube complex. Our previous labelling of vertices of $\Gamma$ induces a labelling of the standard circles of $S(\Gamma)$, which lifts to a labelling of edges of $X(\Gamma)$. We choose an orientation for each standard circle of $S(\Gamma)$ and this would give us a directed labelling of the edges in $X(\Gamma)$. If we pick a base point $v \in X(\Gamma)$ ($v$ is a vertex), then there is a 1-1 correspondence between words in $G(\Gamma)$ and edge paths in $X(\Gamma)$ which starts at $v$.

Each full subgraph $\Gamma' \subset \Gamma$ gives rise to a subgroup $G(\Gamma') \hookrightarrow G(\Gamma)$, subgroup of this kind is called $S$-standard subgroup and coset of $S$-standard subgroup is called $S$-standard coset (we will omit $S$ when the generating set is clear). There is also an embedding $S(\Gamma') \hookrightarrow S(\Gamma)$ which is locally isometric. Let $p : X(\Gamma) \to S(\Gamma)$ be the covering map. Then each connect component of $p^{-1}(S(\Gamma'))$ is a convex subcomplex isometric to $X(\Gamma')$. We will call these components standard subcomplexes with defining graph $\Gamma'$. A standard $k$-flat is the standard complex which covers a standard $k$-torus in $S(\Gamma)$. When $k = 1$, we also call it a standard geodesic.

We identify $G(\Gamma)$ with the 0-skeleton of $X(\Gamma)$, which gives rise to a left action $G(\Gamma) \curvearrowright X(\Gamma)$. Then for any $h \in G(\Gamma)$, the convex hull of $\{hgv \, | \, g \in G(\Gamma') \}$ ($v$ is a vertex in $X(\Gamma)$) is a standard subcomplex associated with $\Gamma'$. Thus there is a 1-1 correspondence between standard subcomplexes associated with $\Gamma'$ and left cosets of $G(\Gamma')$ in $G(\Gamma)$.

Note that for every edge $e \in X(\Gamma)$, there is a vertex in $\Gamma$ which shares the same label as $e$, and we denote this vertex by $V_e$. If $K \subset X(\Gamma)$ is a subcomplex, we define $V_K$ to be $\{V_e \, | \, e \text{ is an edge in } K\}$ and $\Gamma_K$ to be the full subgraph spanned by $V_K$. In particular, if $K$ is a standard subcomplex, then the defining graph of $K$ is $\Gamma_K$. Pick a vertex $v \in X(\Gamma)$ and a full subgraph $\Gamma' \subset \Gamma$, we denote the unique standard subcomplex with defining graph $\Gamma'$ that contains $v$ by $K(v, \Gamma')$. Every finite simplicial graph $\Gamma$ admits a canonical join decomposition $\Gamma = \Gamma_1 \circ \Gamma_2 \circ \cdots \circ \Gamma_k$, where $\Gamma_1$ is the maximal clique join factor and for $2 \leq i \leq k$, $\Gamma_i$ does not allow any non-trivial join decomposition and is not a point. $\Gamma$ is irreducible iff this join decomposition is trivial. This decomposition induces a product decomposition $X(\Gamma) = \mathbb{E}^n \times \prod_{i=2}^n X(\Gamma_i)$, which is called the De Rahm decomposition of $X(\Gamma)$. This is consistent with the canonical product decomposition of $CAT(0)$ cube complex discussed in Section 2.5 of [CS11].

We turn to the asymptotic geometry of RAAGs. A right-angled Artin group $G(\Gamma)$ is one-ended iff $\Gamma$ is connected. Moreover, the $n$-connectivity at infinity of $G(\Gamma)$ can be read off from $\Gamma$, see [BM01]. In order to classify all RAAGs up to quasi-isometry, it suffices to consider those one-ended RAAGs, this follows from the main results in [PW02]. Moreover, Lemma 3.2 of [PW02] implies the following:

**Lemma 2.8.** If $q : X(\Gamma) \to X(\Gamma')$ is an $(L, A)$-quasi-isometry, then there exists $D = D(L, A) > 0$ such that for any connected component $\Gamma_1 \subset \Gamma$ such that $\Gamma_1$ is not a point and any standard subcomplex $K_1 \subset X(\Gamma)$ with defining graph $\Gamma_1$, there is a unique connected component $\Gamma'_1 \subset \Gamma'$ and a unique standard subcomplex $K'_1 \subset X(\Gamma')$ with defining graph $\Gamma'_1$ such that $d_H(q(K_1), K'_1) < D$.

It is shown in [BCT12] and [ABD+13] that $G(\Gamma)$ has linear divergence iff $\Gamma$ is a join or $\Gamma$ is one point, which implies $\Gamma$ being a join is a quasi-isometry invariant.
Moreover, their results together with Theorem B of [KKL98] implies that the De Rahm decomposition is stable under quasi-isometry:

**Theorem 2.9.** Given $X = X(\Gamma)$ and $X' = X(\Gamma')$, let $X = \mathbb{R}^n \times \prod_{i=1}^k X(\Gamma_i)$ and let $X' = \mathbb{R}^{n'} \times \prod_{j=1}^{k'} X(\Gamma'_j)$ be the corresponding De Rahm decomposition. If $\phi : X \to X'$ is an $(L, A)$-quasi-isometry, then $n = n'$, $k = k'$ and there exist constants $L' = L'(L, A)$, $A' = A'(L, A)$, $D = D(L, A)$ such that after re-indexing the factors in $X'$, we have $(L', A')$-quasi-isometry $\phi_i : X(\Gamma_i) \to X(\Gamma'_i)$ so that $d(\phi_i \circ p, \prod_{i=1}^k \phi_i \circ p) < D$, where $p : X \to \prod_{i=1}^k X(\Gamma_i)$ and $p' : X' \to \prod_{j=1}^{k'} X(\Gamma'_j)$ are the projections.

Thus in order to study the quasi-isometry classification of RAAGs, it suffices to study those RAAGs which are one-ended and irreducible, but this relies on finer quasi-isometry invariant of RAAGs.

Recall that in the case of Gromov hyperbolic spaces, quasi-isometry maps geodesic to geodesic up to finite Hausdorff distance, hence induces a well-defined quasi-isometry invariant of RAAGs. The following is a higher dimensional generalization of Theorem 3.10 of [BKS08b]:

**Theorem 2.10** (Theorem 5.20 of [Hua]). If $\phi : X(\Gamma_1) \to X(\Gamma_2)$ is an $(L, A)$-quasi-isometry, then $\dim(X(\Gamma_1)) = \dim(X(\Gamma_2))$. And there is a constant $D = D(L, A)$ such that for any top-dimensional flat $F_1 \subset X(\Gamma_1)$, there is a unique flat $F_2 \subset X(\Gamma_2)$ such that $d_H(\phi(F_1), F_2) < D$.

For each right-angled Artin group $G(\Gamma)$, there is a simplicial graph $\Gamma^e$, called the extension graph, which is introduced in [KK13b]. Extension graphs can be viewed as “curve graphs” for RAAGs ([KK14]).

**Definition 2.11** (Definition 1 of [KK13b]). The vertex set of $\Gamma^e$ consists of words in $G(\Gamma)$ that are conjugate to elements in $S$ (recall that $S$ is a standard generating set of $G(\Gamma)$), and two vertices are adjacent in $\Gamma^e$ iff the corresponding words commute in $G(\Gamma)$.

The flag complex of the extension graph is called the extension complex.

Since the curve graph captures the combinatorial pattern of how Dehn twist flats intersect in a mapping class group, it plays an important role in the quasi-isometry rigidity of mapping class group ([Ham05] [BKMM12]). Similarly, we will see in Section 4 that the extension graph captures the combinatorial pattern of the coarse intersection of certain collection of flats in a RAAG, and it is a quasi-isometry invariant for certain classes of RAAGs.

3. **Stable subgraph**

We study the behaviour of certain standard subcomplexes under quasi-isometry in this section.

3.1. **Coarse intersection of standard subcomplexes and flats.**

**Lemma 3.1.** Let $\Gamma$ be a finite simplicial graph and let $K_1$, $K_2$ be two standard subcomplexes of $X(\Gamma)$. If $(Y_1, Y_2) = \mathcal{I}(K_1, K_2)$, then $Y_1$ and $Y_2$ are also standard subcomplexes.
Proof. The lemma is clear if $K_1 \cap K_2 \neq \emptyset$. Now we assume $d(K_1, K_2) = c > 0$. Pick a vertex $v_1 \in K_1$, by Lemma 2.3 there exists vertex $v_2 \in K_2$ such that $d(v_1, v_2) = c$. Let $l : [0, c] \to X(\Gamma)$ be the unit speed geodesic from $v_1$ to $v_2$. We can find sequence of cubes $\{B_t\}_{t=1}^{N}$ and $0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = c$ such that each $B_t$ contains $\{l(t) \mid t_{i-1} < t < t_i\}$ as interior points.

Let $V_i = \bigcup_{t=1}^{N} V_{B_t}$ and $V_i = V_{K_i}$ for $i = 1, 2$. Put $V' = V_1 \cap V_2 \cap V_i^\perp$ and let $\Gamma'$ be the full subgraph spanned by $V'$. Let $Y'_1 = K(v_1, \Gamma')$ (if $V'$ is empty, then $Y'_1 = v_1$). We claim $Y'_1 \subset Y'_2$.

Pick an edge $e \subset K_1$ such that $v_1 \in e$ and $V_e \subset V'$. Let $h$ be the hyperplane dual to $e$ and $N_h \cong h \times [0, 1]$ be the carrier of $h$. Since $d(V_e, w) = 1$ for any $w \in V_1$, we can assume $l \subset h \times \{1\} \subset N_h$. By our definition of $V'$, there is an edge $e' \subset K_2$ with $v_2 \in e'$ and $h$ dual to $e'$, thus $e$ and $e'$ cobound an isometrically embedded flat rectangle (one side of the rectangle is $l$), which implies $e \subset Y'_1$. Let $l'$ be the side of the rectangle opposite to $l$. We can define $V_{l'}$ similarly as we define $V_i$, then $V_{l'} = V_i$. Now let $\omega$ be any edge path starting at $v_1$ such that $V_{l'} \subset V'$ for any edge $e' \subset \omega$. Then it follows from the above argument and induction on the combinatorial length of $\omega$ that $\omega \subset Y_1$, thus $Y'_1 \subset Y'_2$.

For the other direction, since $Y_i$ is a convex subcomplex by Lemma 2.3, it suffices to prove every vertex of $Y_i$ belongs to $Y'_i$. By the induction argument as above, we only need to show for edge $e_1$ with $v_1 \in e_1$, if $e_1 \in Y_i$, then $e_1 \in Y'_i$. Lemma 2.3 implies that there exists edge $e_2 \subset Y_2$ such that $e_1$ and $e_2$ cobound an isometrically embedded flat rectangle (one side of the rectangle is $l$). So $l$ is in the carrier of the hyperplane dual to $e_1$. It follows that $V_{e_1} \subset V'$ and $e_1 \in Y'_i$. □

Corollary 3.2. Let $K_1, K_2, Y_1$ and $Y_2$ be as above. Let $h$ be a hyperplane separating $K_1$ and $K_2$ and let $e$ be an edge dual to $h$. Then $V_e \subset V_{Y_1} = V_{Y_2}^\perp$. In particular, pick vertex $v \in \Gamma$, then $v \in V_{Y_1}$ iff

\begin{enumerate}
\item $v \in V_{K_1} \cap V_{K_2}$.
\item For any hyperplane $h'$ separating $K_1$ from $K_2$ and any edge $e'$ dual to $h'$, $d(v, V_{e'}) = 1$.
\end{enumerate}

Proof. Let $V'_{Y_1}$ be a collection of vertices of $\Gamma$ such that $v \in V'_{Y_1}$ iff $v = V_{e'}$ for some edge $e' \subset X(\Gamma)$ satisfying (2). It suffices to prove $V'_{Y_1} = V_i$.

$V'_{Y_1} \subset V_i$ is clear since if a hyperplanes $h$ separates $K_1$ from $K_2$, $l$ intersects $h$ transversally at one point. To see $V_i \subset V'_{Y_1}$, it suffices to show $h \cap K_i = \emptyset$ for $i = 1, 2$, where $h$ is a hyperplane that intersects $l$ transversally. Let $x = l \cap h$. Suppose $h \cap K_1 \neq \emptyset$ and let $x' = \pi_{h \cap K_1}(x)$. Now consider the triangle $\Delta(v_1, x, x')$ (recall that $v_1 = l(0)$), we have $\angle v_1(x, x') \geq \pi/2$ (since $\pi_{K_1}(x) = v_1$), $\angle x'(v_1, x) \geq \pi/2$ (see the proof of Lemma 2.6) and $\angle x(v_1, x') > 0$, which is a contradiction, so $h \cap K_i = \emptyset$, similarly $h \cap K_2 = \emptyset$. □

Remark 3.3. Lemma 3.1 implies that for each pair of standard (left) cosets of $G(\Gamma)$, we can associate another standard coset, which captures the coarse intersection of the pair. Moreover, we can also define a notion of distant between two standard cosets, which takes value on $G(\Gamma)$.

Lemma 3.4. Let $K \subset X(\Gamma)$ be a convex subcomplex and let $\Gamma' = \text{lk}(\Gamma_K)$ (i.e. $\Gamma'$ is the full subgraph spanned by $V_K^\perp$, see Section 2.3). Then $P_{K'}$ is a convex subcomplex and canonically splits as $K \times X(\Gamma')$.

Note that we do not require $K$ to satisfy geodesic extension property.
Proof. Let $\Gamma'' = \Gamma_K$ and let $P_1 = K(v, \Gamma' \circ \Gamma'')$ ($v$ is a vertex in $K$). Then $K \subset P_1$. Let $P'$ be the natural copy of $K \times X(\Gamma')$ inside $P_1$. It is clear $P' \subset P_K$.

Let $K'$ be a convex subset parallel to $K$ and let $\phi : K \to K'$ be the isometry induced by $CAT(0)$ projection onto $K'$. Pick vertex $v \in K$ and let $l$ be the geodesic segment connecting $v$ and $\phi(v)$. We define $V_l$ as in the proof of Lemma 3.1 (note that $\phi(v)$ is not necessarily a vertex). Let $e$ be any edge such that $v \in e \subset K$. Then $e$ is a flat rectangle with $e$, $\phi(e)$ and $l$ as its three sides, thus $l$ is contained in the carrier of the hyperplane dual to $e$ and $V_l \subset V_e^\perp$. Note that if $l'$ is the side opposite to $l$, then $V_{l'} = V_l$. For any given edge $e' \subset K$, we can find an edge path $\omega \subset K$ such that $e$ is the first and $e'$ is the last edge in $\omega$. By induction on the combinatorial length of $w$ and the same argument as above, we can show $V_l \subset V_{e'}^\perp$, thus $V_l \subset V_{e'}^\perp$ and $K' \subset P'$. It follows that $P_K \subset P'$, so $P_K = P'$.

Remark 3.5. The following is a generalization of the above lemma for general $CAT(0)$ cube complex. Let $X$ be a $CAT(0)$ cube complex. A convex set $K \subset X$ is regular iff for any $x \in K$, $\Sigma_x K$ (the space of direction of $x$ in $K$, see Chapter II.3 of BH92) satisfies:

1. $\Sigma_x K$ is a subcomplex of $\Sigma_x X$ with respect to the canonical all-right spherical complex structure on $\Sigma_x X$.

2. There exists $r > 0$ such that $B(x, r) \cap K$ is isometric to the $r$-ball centred at the cone point in the Euclidean cone over $\Sigma_x K$.

If $K \subset X$ is a regular convex subset, the $P_K$ is convex and admits a splitting $P_K \cong K \times N$ where $N$ has an induced cubical structure from $X$ ($N$ is $CAT(0)$).

Lemma 3.6. Let $q : X(\Gamma_1) \to X(\Gamma_2)$ be an $(L, A)$-quasi-isometry and let $F \subset X(\Gamma_1)$ be a subcomplex isometric to $E^k$. Suppose $n = \dim(X(\Gamma_1)) = \dim(X(\Gamma_2))$. If there exist $R_1 > 0$, $R_2 > 0$ and top dimensional flats $F_1$, $F_2$ such that $F \cong F_1 \cap R_1$, $F_2$ and $F \cong F_1 \cap R_2 F_2$ for any $R \geq R_1$, then

1. There exists a constant $D = D(L, A, R_1, R_2, n)$ and subcomplex $F' \subset X(\Gamma_2)$ isometric to $E^k$ such that $q(F) \cong F'$.

2. There exists a constant $D' = D'(L, A)$ such that $q(P_F) \cong D'P_F$.

Proof. By Theorem 2.10 there exist top dimensional flats $F'_1 \subset X(\Gamma_2)$ and $F'_2 \subset X(\Gamma_2)$ such that $q(F_1) \cong F'_1$ for $D_1 = D_1(L, A)$ and $i = 1, 2$. Thus there exists $R_2 = R'_2(L, A, R_1, R_2)$ and $R_3 = R_3(L, A, R_1, R_2) > R_1$ such that $q(F_1 \cap R_1, F_2) \subset F'_1 \cap R_2, F'_2 \subset q(F_1 \cap R_3, F_2)$, this and Remark 2.3 imply

\[q(F_1 \cap R_1, F_2) \cong F'_1 \cap R_2, F'_2 \]

for $D_2 = D_2(n, d(F_1, F_2)) = D_2(L, A, R_1, R_2, n)$.

Let $(Y_1, Y_2) = T(F'_1, F'_2)$. Then there exists $D_3 = D_3(L, A, R_1, R_2, n)$ such that

\[Y_1 \cong F'_1 \cap R_2, F'_2 \]

From (3.7) and (3.8), we have

\[q(F) \cong F_1 \]

for $D_1 = D_4(L, A, R_1, R_2, n)$. By Lemma 2.3, $Y_1$ is a convex subcomplex of $F'_1$, this together with (3.9) imply $F'_1 = F' \times \prod_{i=1}^{k} I_i$ where $F'$ is isometric to $E^k$ and
\[ \{I_i\}_{i=1}^{k'} \text{ are finite intervals. Moreover, by } \text{(3.9), } \text{diam}(\prod_{i=1}^{k'} I_i) \text{ must be bounded in terms of } D_A, L \text{ and } A, \text{ thus (1) follows.} \]

Let \( \{F_{\lambda}\}_{\lambda \in A} \) be the collection of top dimensional flats in \( X(\Gamma_1) \) which are contained in \( P_F \). Lemma 3.4 implies

\[
(3.10) \quad d_H(\cup_{\lambda \in A} F_{\lambda}, P_F) \leq 1.
\]

For \( \lambda \in A \), there exists \( R_{\lambda} > 0 \) such that \( F \subset R_{\lambda} F_{\lambda} \). Let \( F'_{\lambda} \) be the top dimensional flat in \( X(\Gamma_2) \) such that \( q(F_{\lambda}) \cong F'_{\lambda} \). Then by (1), there exists \( R'_{\lambda} > 0 \) such that \( F' \subset R'_{\lambda} (F'_{\lambda}) \). This and Lemma 2.3 imply \( F'_{\lambda} \subset P_{F'} \) for any \( \lambda \in A \). Thus by (3.10), there exists \( D' = D'(L, A) \) such that \( q(P_F) \subset N_{D'}(P_{F'}) \). We can run the same argument for the quasi-isometry inverse \( p: X(\Gamma_2) \rightarrow X(\Gamma_1) \), then (2) follows. \( \square \)

A standard tree product is a standard subcomplex which splits as a product of trees. Actually each tree factor is also a standard subcomplex in this case.

**Lemma 3.11.** Suppose \( q: X(\Gamma) \rightarrow X(\Gamma') \) is a quasi-isometry and suppose \( \dim(X(\Gamma)) = n \). Let \( K = \prod_{i=1}^{n} T_i \) be a top dimensional tree product and its tree factors. Then there exists a standard tree product \( K' \) such that \( q(K) \subset c K' \).

The proof essentially follows [BKS08a, Theorem 4.2].

**Proof.** For \( 1 \leq i \leq n \), let \( V_i = V_{T_i} \in \Gamma \) be the collection of labels of edges in \( T_i \). The case where all \( V_i \) are consist of one point follows from Theorem 2.10. If \( V_i \) contains at least two point for all \( i \), then by Lemma 3.6, for any geodesic \( l \subset T_i \), there exists a subcomplex \( l' \subset X(\Gamma') \) isometric to \( \mathbb{R} \) such that \( q(l) \cong l' \). Since \( l' \) is unique up to parallelism, the collection of labels of edges in \( l' \) does not depend on the choice of \( l' \) and will be denoted by \( V_{q(l)} \). For \( 1 \leq i \leq n \), define \( V_i' = \cup_{l \subset T_i} V_{q(l)} \) where \( l \) varies among all geodesic in \( T_i \).

We claim \( V_i' \subset (V_j')_i \) for \( i \neq j \). To see this, pick geodesic \( l_i \in T_i \) and let \( F = \prod_{i=1}^{n} l_i \). Then there exist top dimensional flat \( F' \) and geodesic lines \( \{l'_i\}_{i=1}^{n} \) \( (l'_i \text{ is a subcomplex}) \) in \( X(\Gamma') \) such that \( q(F) \cong F' \) and \( q(l_i) \cong l'_i \). Since \( l'_i \subset c F' \), by Lemma 2.3, we can assume \( l'_i \) is a subcomplex of \( F' \). Note that \( l'_i \) is not parallel to \( l'_j \) for \( i \neq j \), so \( \{l'_i\}_{i=1}^{n} \) is a mutually orthogonal collection.

Let \( \Gamma'_i = V'_1 \circ V'_2 \circ \cdots \circ V'_n \subset \Gamma' \). Then each \( V_i' \) has to be a discrete full subgraph by our dimension assumption. Let \( \{F_{\lambda}\}_{\lambda \in A} \) be the collection of top dimensional flats in \( K \) and let \( F'_{\lambda} \) be the unique flat such that \( q(F_{\lambda}) \cong F'_{\lambda} \). Note that for arbitrary \( F_{\lambda_1} \) and \( F_{\lambda_2} \), there exists a finite chain in \( \{F_{\lambda}\}_{\lambda \in A} \) which starts with \( F_{\lambda_1} \) and ends with \( F_{\lambda_2} \) such that the intersection of adjacent elements in the chain contains a top dimensional orthant. Thus the collection \( \{F'_{\lambda}\}_{\lambda \in A} \) also have this property. Then \( \cup_{\lambda \in A} F'_{\lambda} \) is contained in some standard subcomplex associated with \( \Gamma'_i \).

It remains to deal with the case where there exists \( i \neq j \) such that \( |V_i| = 1 \) and \( |V_j| \geq 2 \). By applying Lemma 3.6 with \( F = T_i \), we can reduce to lower dimensional case and the theorem follows by induction on dimension. \( \square \)

**Corollary 3.12.** Let \( q: X(\Gamma) \rightarrow X(\Gamma') \) be a quasi-isometry and let \( K \) be a top dimensional maximal standard tree product, i.e. \( K \) is not properly contained in other tree product. Then there exists a standard tree product \( K' \subset X(\Gamma') \) such that \( q(K) \cong K' \).
3.2. **Standard flats of arbitrary dimension.** Up to now, we have only dealt with top dimensional standard subcomplexes. The next goal is to study those which are not necessarily top dimensional. In particular, we are interested in whether the quasi-isometry will preserve standard flats or not, and the answer turns out to be related to the outer automorphism group of $G(\Gamma)$.

One direction is obvious, namely, if $q : X(\Gamma) \to X(\Gamma')$ is a quasi-isometry and the $q$-image of any standard flat in $X(\Gamma)$ is Hausdorff close to a standard flat in $X(\Gamma')$, then $Out(G(\Gamma))$ must be transvection free (i.e. $Out(G(\Gamma))$ does not contain any transvections). The converse is also true. Now we set up several necessary tools to prove the converse.

In this section, $\Gamma$ will always be a finite simplicial graph.

**Definition 3.13.** A subgraph $\Gamma_1 \subset \Gamma$ is **stable in $\Gamma$** iff $\Gamma_1$ is a full subgraph and for any standard subcomplex $K \subset X(\Gamma)$ with $\Gamma_K = \Gamma_1$ and $(L, A)$-quasi-isometry $q : X(\Gamma) \to X(\Gamma')$, there exist $D = D(L, A, \Gamma_1, \Gamma) > 0$ and standard subcomplex $K' \subset X(\Gamma')$ such that $d_H(q(K), K') < D$. For simplicity, we will also say the pair $(\Gamma_1, \Gamma)$ is **stable** in this case. A standard subcomplex $K \subset X(\Gamma)$ is **stable** iff it arises from a stable subgraph of $\Gamma$.

Note that $\Gamma_{K'}$ is stable in $\Gamma'$, so one can obtain quasi-isometry invariants by identifying certain classes of stable subgraphs.

It is immediate from the definition that for finite simplicial graphs $\Gamma_1 \subset \Gamma_2 \subset \Gamma_3$, if $(\Gamma_1, \Gamma_2)$ is stable and $(\Gamma_2, \Gamma_3)$ is stable, then $(\Gamma_1, \Gamma_3)$ is stable. However, it is not necessarily true that if $(\Gamma_1, \Gamma_3)$ and $(\Gamma_2, \Gamma_3)$ are stable, then $(\Gamma_1, \Gamma_2)$ is stable. In the sequel, we will investigate several more properties of stable subgraph. The following lemma is an easy consequence of Lemma 3.1 and Remark 2.5.

**Lemma 3.14.** Suppose $\Gamma_1$ and $\Gamma_2$ are stable in $\Gamma$, then $\Gamma_1 \cap \Gamma_2$ is also stable in $\Gamma$.

The following results follows from Lemma 2.8

**Lemma 3.15.** If $\Gamma_1$ is stable in $\Gamma$, then every connected component of $\Gamma_1$ that contains more than one point is also stable in $\Gamma$.

**Lemma 3.16.** Suppose $\Gamma_1$ is stable in $\Gamma$. Let $V = v(\Gamma_1)$ and let $\Gamma_2$ be the full subgraph spanned by $V$ and $V^\perp$. Then $\Gamma_2$ is also stable in $\Gamma$.

**Proof.** Let $K_2 \subset X(\Gamma)$ be a standard subcomplex with $\Gamma_{K_2} = \Gamma_2$ and let $K_1 \subset K_2$ be any standard subcomplex satisfying $\Gamma_{K_1} = \Gamma_1$. Lemma 3.4 implies $K_2 = P_{K_1} = K_1 \times K_1^\perp$. For vertex $x \in K_1^\perp$, denote $K_x = K_1 \times \{x\}$. Let $q : X(\Gamma) \to X(\Gamma')$ be an $(L, A)$-quasi-isometry. Then there exists standard subcomplex $K_x'$ such that $d_H(q(K_x), K'_x) < D = D(L, A, \Gamma_1, \Gamma)$, thus $K'_x \cong K'_y$ for vertices $x, y \in K_1^\perp$. It follows from Lemma 3.1 that $K_x'$ and $K'_y$ are parallel. Thus $q(P_{K_1}) \subset P_{K_x'}$ for $R = D + L + A$. Moreover, $P_{K_x'}$ is also a standard subcomplex by Lemma 3.4. By considering the quasi-isometry inverse and repeating the previous argument, we know $q(P_{K_1'}) \cong P_{K_x'}$, thus $\Gamma_2$ is also stable in $\Gamma$. \qed

**Lemma 3.17.** Suppose $\Gamma_1$ is stable in $\Gamma$. Pick vertex $v \notin \Gamma_1$, then the full subgraph spanned by $v^\perp \cap \Gamma_1$ is stable in $\Gamma$.

**Proof.** We use $\Gamma_2$ to denote the full subgraph spanned by $v^\perp \cap \Gamma_1$. Let $K_2 \subset X(\Gamma)$ be a standard subcomplex such that $\Gamma_{K_2} = \Gamma_2$ and let $K_1 \subset X(\Gamma)$ be the unique standard subcomplex such that $\Gamma_{K_1} = \Gamma_1$ and $K_2 \subset K_1$. Pick vertex $x \in K_2$ and
let $e \subset X(\Gamma)$ be the edge such that $V_e = v$ and $x \in e$. Suppose $\bar{x}$ is the other end point of $e$, let $\bar{K}_i = K(\bar{x}, \Gamma_i)$ for $i = 1, 2$. Denote the hyperplane dual to $e$ by $h$. Since $v \notin \Gamma_1$, $h \cap \bar{K}_1 = \emptyset$ and $h \cap K_1 = \emptyset$, thus $h$ separates $K_1$ and $\bar{K}_1$ and $d(K_1, \bar{K}_1) = 1$. It follows from Corollary 3.2 that $\mathcal{I}(K_1, \bar{K}_1) = (K_2, \bar{K}_2)$, in particular $K_2 \supseteq \bar{K}_1 \cap \bar{K}_1$ for $D$ depending on $R$ and the dimension of $X(\Gamma)$. Now the lemma follows since $\Gamma_1$ is stable.

The next result is a direct consequence of Corollary 3.12.

**Lemma 3.18.** If $\Gamma_1$ is stable in $\Gamma$, then there exists $\Gamma_2$ which is stable in $\Gamma_1$ such that

1. $\Gamma_2 = \bar{\Gamma}_1 \circ \bar{\Gamma}_2 \circ \cdots \circ \bar{\Gamma}_k$ where $\bar{\Gamma}_i$ is discrete for $1 \leq i \leq k$.
2. $k = \dim(X(\Gamma_1))$.

**Lemma 3.19.** Let $\Gamma$ be a finite simplicial graph such that there does not exist distinct pair of vertices $v, w \in \Gamma$ such that $v^\perp \subset \text{St}(w)$. Then every stable subgraph of $\Gamma$ contains a stable vertex.

**Proof.** Let $\Gamma_1$ be a minimal stable subgraph, i.e. it does not properly contain any stable subgraph of $\Gamma$. It suffices to show $\Gamma_1$ is a point. We argue by contradiction and assume $\Gamma_1$ contains more than one point.

First we claim $\Gamma_1$ can not be discrete. Suppose the contrary is true. Pick vertices $v, w \in \Gamma_1$ and pick vertex $u \in v^\perp \setminus \text{St}(w)$. By Lemma 3.17, $u^\perp \cap \Gamma_1$ is also stable. Note that $v \in u^\perp \cap \Gamma_1$ and $w \notin u^\perp \cap \Gamma_1$, which contradicts the minimality of $\Gamma_1$.

We claim $\Gamma_1$ must be a clique. Since $\Gamma_1$ is not discrete, by Lemma 3.18 we can find a stable subgraph

$$
\Gamma_2 = \bar{\Gamma}_1 \circ \bar{\Gamma}_2 \circ \cdots \circ \bar{\Gamma}_m \subset \Gamma_1
$$

where $\{\bar{\Gamma}_i\}_{i=1}^m$ are discrete full subgraphs and $m \geq 2$. Then $\Gamma_2 = \Gamma_1$. Suppose some $\bar{\Gamma}_i$ contains more than one point, and let $\bar{\Gamma}_3$ be the join of the remaining join factors. Then Theorem 2.2 implies that $\bar{\Gamma}_3$ is stable, contradicting the minimality of $\Gamma_1$. Therefore $\Gamma_1$ is a clique.

Pick distinct vertices $v_1, v_2 \in \Gamma_1$. By our assumption, there exists vertex $w \in v_1^\perp \setminus \text{St}(v_2)$. Since $\Gamma_1$ is a clique, $\Gamma_1 \subset \text{St}(v_2)$, then $w \notin \Gamma_1$. Let $\Gamma_4$ be the full subgraph spanned by $w^\perp \cap \Gamma_1$. Then $\Gamma_4$ is stable by Lemma 3.17. Moreover, $\Gamma_4 \subseteq \bar{\Gamma}_1 (v_2 \notin \Gamma_4)$, which yields a contradiction.

**Lemma 3.21.** Let $\Gamma$ be as in Lemma 3.19 and let $\Gamma_1$ be a stable subgraph of $\Gamma$. Then for any vertex $w \in \Gamma_1$, there exists a stable vertex $v \in \Gamma_1$ such that $d(v, w) \leq 1$.

**Proof.** Denote the combinatorial distance in $\Gamma$ and $\Gamma_1$ by $d$ and $d_1$ respectively. Since $\Gamma_1$ is a full subgraph, $d(x, y) = 1$ iff $d_1(x, y) = 1$ and $d(x, y) \geq 2$ iff $d_1(x, y) \geq 2$ for vertex $x, y \in \Gamma_1$. If $w$ is isolated, we can use the argument in the second paragraph of the proof of Lemma 3.19 to get rid all vertices in $\Gamma_1$ except $w$, which implies $w$ is stable. If $w$ is not isolated, we can assume $\Gamma_1$ is connected by Lemma 3.16.

By Lemma 3.19, there exists a stable vertex $v \in \Gamma_1$. If $d_1(u, w) \leq 1$, then we are done, otherwise let $\omega$ be one of the geodesics in $\Gamma_1$ connecting $u$ and $w$ (which might not be a geodesic in $\Gamma$) and let $\{v_i\}_{i=0}^n$ be the consecutive vertices in $\omega$, here $v_0 = w$, $v_n = u$ and $n = d_1(w, u)$.

Since $u$ is stable, by Lemma 3.16, $\text{St}(u)$ is also stable. Note that $d_1(v_{n-2}, u) = 2$, so $d(v_{n-2}, u) = 2$ and $v_{n-2} \notin \text{St}(u)$. Lemma 3.17 implies $v_{n-2}^\perp \cap \text{St}(u)$ is stable and
by Lemma 3.14, \( v_{i-2}^+ \cap St(u) \cap \Gamma \) is also stable. \( v_{i-2}^+ \cap St(u) \cap \Gamma \neq \emptyset \) since it contains \( v_{i-1} \). Lemma 3.19 implies there is a stable vertex \( u' \in v_{i-2}^+ \cap St(u) \cap \Gamma \) and it is easy to see \( d_1(w,u') = n - 1 \). Now the lemma follows by induction. □

**Lemma 3.22.** Let \( \Gamma \) be as in Lemma 3.19. Then every vertex of \( \Gamma \) is stable.

**Proof.** Let \( \Gamma_w \) be the intersection of all stable subgraph that contains \( w \). By Lemma 3.14, \( \Gamma_w \) is the minimal stable subgraph that contains \( w \). It suffices to prove \( \Gamma_w = w \). We argue by contradiction and denote the vertices in \( \Gamma_w \setminus \{w\} \) by \( \{v_i\}_{i=1}^k \). The minimality of \( \Gamma_w \) implies we can not use Lemma 3.11 to get rid of some \( v_i \) while keep \( w \), thus \( w^+ \setminus St(v_i) \subset \{v_1 \cdots v_{i-1}, v_{i+1} \cdots v_k\} \) for any \( i \), in other words

\[(3.23) \quad w^+ \subset St(v_i) \cup \{v_1 \cdots v_{i-1}, v_{i+1} \cdots v_k\} \]

for \( 1 \leq i \leq k \). Then there does not exists \( i \) such that \( \Gamma_w \subset St(v_i) \), otherwise we would have \( w^+ \subset St(v_i) \) by (3.23).

On the other hand, Lemma 3.21 implies there exists a stable vertex \( u \in \Gamma_w \) with \( d(w,u) = 1 \). Then \( St(u) \) is stable (Lemma 3.16) and \( St(u) \cap \Gamma_w \) is stable (Lemma 3.14). Note that \( w \in St(u) \cap \Gamma_w \), by the minimality of \( \Gamma_w \), \( \Gamma_w \subset St(u) \), which yields a contradiction. □

**Lemma 3.24.** Let \( \Gamma \) be a finite simplicial graph and pick stable subgraphs \( \Gamma_1, \Gamma_2 \) of \( \Gamma \). Let \( \overline{\Gamma} \) be the full subgraph spanned by \( V \) and \( V^\perp \) where \( V = V_{\overline{\Gamma}} \). If \( \Gamma_2 \subset \overline{\Gamma} \), then the full subgraph spanned by \( \Gamma_1 \cup \Gamma_2 \) is stable in \( \Gamma \).

To simplify notation, in the following proof, we will denote \( q(K) \approx K' \) where \( q, K \) and \( K' \) are as in Definition 3.13. We will also assume without loss of generality that \( q(K) \subset K' \).

**Proof.** Let \( q : X(\Gamma) \to X(\Gamma') \) be an \((L,A)\)-quasi-isometry. Suppose \( K_1 \) and \( K \) are standard subcomplexes in \( X(\Gamma) \) such that \( \Gamma_{K_1} = \Gamma_1, \Gamma_K = \Gamma \) and \( \Gamma_1 \subset \Gamma \). Put \( K' \approx q(K), K'_1 \approx q(K_1), K = K_1 \times K_1^\perp \) and \( K' = K'_1 \times K_1^\perp \). The proof of Lemma 3.16 implies there exist quasi-isometry \( q' : K^\perp_1 \to K_1^\perp \) and \( D_1 = D_1(L, A, \Gamma_1, \Gamma) \) such that

\[(3.25) \quad d(q' \circ p_2(x), p_2 \circ q(x)) < D_1 \]

for any \( x \in K \) where \( p_2 : K \to K^\perp_1 \) and \( p_2' : K' \to K_1^\perp \) are projection.

Let \( \Gamma_2 = \Gamma_2 \cap \Gamma_2 \) where \( \Gamma_2 = \Gamma_1 \cap \Gamma_2 \) and let \( K_{22}, K_2 \) be standard subcomplexes such that \( \Gamma_{K_{22}} = \Gamma_{22}, \Gamma_{K_2} = \Gamma_2 \) and \( K_{22} \subset K_2 \subset K \). By (3.25), it suffices to prove there exist standard subcomplex \( K_{22}^\perp \subset K' \) and \( D = D(L, A, \Gamma_1, \Gamma_2, \Gamma) \) such that \( d_H(p_2' \circ q(K_{22}), K_{22}^\perp) < D \). Let \( K_2' \approx q(K_2) \). Then \( K_2' \subset K' \) and \( p_2'(K_2') \) is a standard subcomplex. By (3.25), \( p_2 \circ q(K_{22}) \approx p_2 \circ q(K_2) \approx p_2'(K_2') \), thus we can take \( K_{22}^\perp = p_2'(K_2') \). □

**Remark 3.26.** In general the full subgraph spanned by \( \Gamma_1 \cup \Gamma_2 \) is not necessarily stable even if \( \Gamma_1 \) and \( \Gamma_2 \) are stable, see Remark 3.35.

The next theorem follows from Lemma 3.22 and Lemma 3.24.

**Theorem 3.27.** Given finite simplicial graph \( \Gamma \), the following are equivalent:

(1) \( \text{Out}(G(\Gamma)) \) is transvection free.
For any $(L,A)$-quasi-isometry $q : X(\Gamma) \to X(\Gamma')$, there exists positive constant $D = D(L,A,|\Gamma|)$ ($|\Gamma|$ denotes the number of vertices in $\Gamma$) such that for any standard flat $F \subset X(\Gamma)$, there exists a standard flat $F' \subset X(\Gamma')$ such that $d_H(q(F), F') < D$.

In our argument, the complexity of the combinatorics is bounded by $|\Gamma|^2$ and the dimension of $X(\Gamma)$ is bounded by $|\Gamma|$, thus $D$ only depends on $L, A$ and $|\Gamma|$.

At this point, we have the following natural questions:

(1) In Theorem 3.27 is it true that every standard flat in $X(\Gamma')$ comes from some standard flat in $X(\Gamma)$? A related question could be, is condition (1) in Theorem 3.27 a quasi-isometry invariant for right-angled Artin group?

(2) What can we say about the stable subgraphs of $\Gamma$ if we drop condition (1) in Theorem 3.27?

The following example gives an negative answer to question (1).

**Example 3.28.** Let $\Gamma_1$ be the graph on the left and let $\Gamma_2$ be the one on the right. It is easy to see $Out(G(\Gamma_1))$ is transvection free while $Out(G(\Gamma_1))$ contains non-trivial transvection ($\Gamma_2$ has dead end at vertex $u$). We claim $G(\Gamma_1)$ and $G(\Gamma_2)$ are quasi-isometric.

Let $\Gamma \subset \Gamma_1$ be the pentagon on the left side and let $Y$ be the Salvetti complex of $\Gamma$. Suppose $X_1 = Y \cup Y \cup (S^1 \times [0, 1]) / \sim$, here the two boundary circles of the annulus are identified with two standard circles which are in different copies of $Y$. Then $\pi_1(X_1) = G(\Gamma_1)$. Define homomorphism $h_1 : G(\Gamma) \to \mathbb{Z}/2$ by sending $w$ to the non-trivial element in $\mathbb{Z}/2$ and other generators to the identity element. Let $Y'$ be the 2-sheet covering of $Y$ with respect to $\ker(h_1)$.

Define homomorphism $h_2 : G(\Gamma_1) \to \mathbb{Z}/2$ by sending $w$ and $k$ to the non-trivial element in $\mathbb{Z}/2$ and other generators to the identity element. Let $X$ be the 2-sheet covering of $X_1$ with respect to $\ker(h_2)$. Then $X$ is made of two copies of $Y'$ and two annuli with the boundaries of the annuli identified with the $v$-circles in $Y'$ (each $Y'$ has two $v$-circles which cover the $v$-circle in $Y$), see the picture below.

We claim $X$ is homotopy equivalent to a Salvetti complex. To see this, let $S_w$ be the circle in $X'$ which covers the $w$-circle in $Y$ two times and let $S_z \vee S_v$ be a wedge of the two circles in $Y'$ which covers the wedge of $z$-circle and $v$-circle in $Y$. There is a copy of $S_w \times (S_z \vee S_v)$ inside $Y'$. Let $I$ be a segment in $S_w$ such that its end points are mapped to the base point of $Y$ under the covering. We collapse $I \times (S_z \vee S_v)$ to $\{pt\} \times (S_z \vee S_v)$ inside each copy of $Y'$ in $X$, and collapse one of the annuli in $X$ to a circle by killing the interval factor. Denote the resulting space by $X'$, then $X'$ is homotopy equivalent to $X$ and the un-collapsed annulus in $X$ becomes a torus in $X'$. It is easy to see $X'$ is a Salvetti complex with defining graph $\Gamma_2$. 
Any standard geodesic in \( X(\Gamma_2) \) which comes from vertex \( u \) is not Hausdorff close to a quasi-isometry image of some standard geodesic in \( X(\Gamma_1) \), since \( u \) is not a stable point while every point in \( \Gamma_1 \) is stable.

Here is a generalization of the above example. Suppose \( \Gamma \) is a finite simplicial graph such that there exist vertices \( v_1, v_2 \in \Gamma \) with \( d(v_1, v_2) = 2 \) such that they are separated by \( \text{lk}(v_1) \cap \text{lk}(v_2) \). Define a homomorphism \( h : G(\Gamma) \to \mathbb{Z}/2 \) by sending \( v_1 \) and \( v_2 \) to 1 and killing all other generators. Then \( \ker(h) \) is also a right-angled Artin group by the same argument as before. To find its defining graph, let \( \Gamma \) sending \( v \) to \( 1 \). Define \( C_1 = \text{lk}(v_1) \cap \text{lk}(v_2) \) and suppose \( v_1 \in C_1 \). Define \( \Gamma_1 = C_1 \cup (\text{lk}(v_1) \cap \text{lk}(v_2)) \) and \( \Gamma_2 = (\cup_{i=2}^n C_i) \cup (\text{lk}(v_1) \cap \text{lk}(v_2)) \), then \( \Gamma_1 \) and \( \Gamma_2 \) are full subgraphs of \( \Gamma \); moreover, \( St(v_i) \in C_i \). For \( i = 1, 2 \), let \( \Gamma_i' \) be the graph obtained by gluing two copies of \( \Gamma_i \) along \( St(v_i) \) and let \( \Gamma_3' \) be the join of one point and \( \text{lk}(v_1) \cap \text{lk}(v_2) \). Then the defining graph of \( \ker(h) \) can be obtained by gluing \( \Gamma_1' \), \( \Gamma_2' \) and \( \Gamma_3' \) along \( \text{lk}(v_1) \cap \text{lk}(v_2) \).

Note that we are taking advantage of separating closed stars while constructing the counterexample, if these phenomena are not allowed, we have positive answer to question (1) (see Section 5).

The rest of this section will focus on question (2). \( \Gamma \) will be an arbitrary finite simplicial graph in the rest of this section.

**Lemma 3.29.** Let \( v \in \Gamma \) be a vertex which is not isolated. Then at least one of the following is true:

1. \( v \) is contained in a stable discrete subgraph.
2. \( v \) is contained in a stable clique subgraph.
3. There is a stable discrete subgraph whose vertex set is in \( v^\perp \).
4. There is a stable clique subgraph whose vertex set is in \( v^\perp \).

**Proof.** Since \( v \) is not isolated, we can assume \( \Gamma \) is connected by Lemma 3.15. By Lemma 3.18, we can find a stable subgraph \( \Gamma_1 = \Gamma_1 \circ \Gamma_2 \circ \cdots \circ \Gamma_n \) where \( \{\Gamma_i\}_{i=1}^n \) are discrete full subgraphs and \( n = \dim(X(\Gamma)) \). If \( v \in \Gamma_1 \), by the third paragraph of the proof of Lemma 3.19 we know either (1), (2) or (4) is true.

If \( d(v, \Gamma_1) = 1 \), let \( \Gamma_2 \) be the full subgraph spanned by \( v^\perp \cap \Gamma_1 \). Then \( \Gamma_2 \) is stable by Lemma 3.17. The proof of Lemma 3.19 implies every stable subgraph of \( \Gamma \) contains either a stable discrete subgraph or a stable clique subgraph (this does not depend on the \( v^\perp \), \( St(w) \) assumption), thus either (3) or (4) is true.

If \( d(v, \Gamma_1) \geq 2 \), pick vertex \( u \in \Gamma_1 \) such that \( d(v, u) = d(v, \Gamma_1) = n \) and let \( \omega \) be a geodesic connecting \( v \) and \( u \). Suppose \( \{v_i\}_{i=0}^n \) are the consecutive vertices in
ω such that \( v_0 = v \) and \( v_n = u \). Let \( \Gamma' \) be the full subgraph spanned by \( v_{n-1}^\perp \cap \Gamma \) and let \( \Gamma'' \) be the full subgraph spanned by \( V \) and \( V^\perp \) where \( V = V_{\Gamma'} \). Then \( \Gamma' \) is stable by Lemma 3.17 and \( \Gamma'' \) is stable by Lemma 3.16. Note that \( d(v, x) \geq n \) for any vertex \( x \in V \), so \( d(v, y) \geq n - 1 \) for any vertex \( y \in V^\perp \) and \( d(v, \Gamma'') \geq n - 1 \), but \( v_{n-1} \in \Gamma'' \), hence \( d(v, \Gamma'') = n - 1 \). Now we can induct on \( n \) and reduce to the \( d(v, \Gamma_1) = 1 \) case.

It is interesting to see that if \( \Gamma \) has large diameter, then there are a lot of non-trivial stable subgraphs.

We record the following lemma which is an easy consequence of Theorem 2.9.

**Lemma 3.30.** Suppose \( \Gamma = \Gamma_1 \circ \Gamma_2 \) where \( \Gamma_1 \) is the maximal clique join factor of \( \Gamma \). If \( \Gamma_2 \) is stable in \( \Gamma_2 \), then \( \Gamma_1 \circ \Gamma_2 \) is stable in \( \Gamma \).

**Lemma 3.31.** Pick a vertex \( w \in \Gamma \) and let \( \Gamma_w \) be the intersection of all stable subgraph of \( \Gamma \) that contains \( w \). Define \( W = \{ w' \in \Gamma \mid w^\perp \subset St(w') \} \), then \( \Gamma_w \) is the full subgraph spanned by \( W \).

In other words, \( G(\Gamma_w) \leq G(\Gamma) \) is the minimal standard subgroup containing \( w \) with the property that \( G(\Gamma_w) \) is invariant under any transvection.

**Proof.** By Lemma 3.14, \( \Gamma_w \) is the minimal stable subgraph that contains \( w \). If there exists vertex \( w' \in W \) such that \( w' \notin \Gamma_w \), then sending \( w \to ww' \) and fixing all other vertices would induce an group automorphism, which gives rise to a quasi-isometry from \( X(\Gamma) \) to \( X(\Gamma) \). The existence of such quasi-isometry would contradict the stability of \( \Gamma_w \), thus \( W \subset \Gamma_w \).

Let \( W' \) be the vertex set of \( \Gamma_w \), it remains to prove \( W' \subset W \). Suppose \( W \not\subset W' \) and let \( u \in W' \setminus W \). The minimality of \( \Gamma_w \) implies we can not use Lemma 3.17 to get rid of \( u \) while keep \( w \), thus

\[
(3.32) \quad \emptyset \neq w^\perp \setminus St(u) \subset W' \setminus \{u, w\}.
\]

In particular, \( w \) is not isolated in \( \Gamma_w \) and

\[
(3.33) \quad \Gamma_w \notin St(u).
\]

Now we apply Lemma 3.29 to \( \Gamma_w \) and \( w \), and recall that if a subgraph is stable in \( \Gamma_w \), then it is stable in \( \Gamma \). If case (1) in Lemma 3.29 is true, then we will get a contradiction since \( w \) is not isolated in \( \Gamma_w \). If case (2) is true, then \( \Gamma_w \) sits inside some clique, which is contradictory to (3.33).

If case (3) is true, let \( \Gamma_1 \subset \Gamma_w \) be the corresponding stable discrete subgraph. Let \( V_1 = V_{\Gamma_1} \) and let \( V'_1 = \{ u \in \Gamma_w \mid d(u, v) = 1 \text{ for any } v \in V_1 \} \). Suppose \( \Gamma'_w \) is the full subgraph spanned by \( V_1 \) and \( V'_1 \). Then \( \Gamma'_w \) is stable by Lemma 3.16, hence \( \Gamma'_w = \Gamma_w \). Let \( \Gamma_w = \Gamma_1 \circ \Gamma_2 \circ \cdots \circ \Gamma_k \) be the join decomposition induced by the De Rahm decomposition of \( X(\Gamma_w) \). Then \( k \geq 2 \) and \( u \) does not sit inside the clique factor by (3.33).

If there is no clique factor, then each join factor is stable by Theorem 2.9 and \( w \) is inside one of the join factors, which contradict the minimality of \( \Gamma_w \). If the clique factor exists and \( w \) sits inside the clique factor, by Theorem 2.9, the clique factor is stable and we have the same contradiction as before. If the clique factor exists and \( w \) sits outside the clique factor, this reduces to the next case.

If case (4) is true, let \( \Gamma_2 \subset \Gamma_w \) be the corresponding stable clique subgraph. Let \( V_2 = V_{\Gamma_2} \) and \( V'_2 = \{ v \in \Gamma_w \mid d(u, v) = 1 \text{ for any } v \in V_2 \} \). Suppose \( \Gamma''_w \) is the full subgraph spanned by \( V_2 \) and \( V'_2 \). Then \( \Gamma''_w = \Gamma_w \) as before. Let \( \Gamma_w = \Gamma'_1 \circ \Gamma'_2 \).
where $\Gamma'$ corresponds to the Euclidean De Rahm factor of $X(\Gamma_w)$. Note that $\Gamma_2'$ is non-trivial and $w, u \in \Gamma_2'$ as in the discussion of case (3). \(3.32\) implies $w^\perp \not\subseteq St(u)$ is still true if we take the orthogonal complement of $w$ and the closed star of $u$ in $\Gamma_2'$, in particular, $w$ is not isolated in $\Gamma_2'$. Moreover, $dim(X(\Gamma_2')) < dim(X(\Gamma_w)) \leq dim(X(\Gamma))$.

If $dim(X(\Gamma)) = 2$, we immediately get a contradiction since $\Gamma_2'$ is discrete in this case, and the lemma is proved. If $dim(X(\Gamma)) = n > 2$, by induction we can assume the lemma is true for all lower dimensional graphs, then there exists $\bar{\Gamma}_w$ stable in $\Gamma_2'$ such that $w \in \bar{\Gamma}_w$ and $u \not\in \bar{\Gamma}_w$. By Lemma $3.30$, $\bar{\Gamma}_w \circ \Gamma_1'$ is stable in $\Gamma_w$, hence in $\Gamma$, which contradicts the minimality of $\Gamma_w$. □

**Theorem 3.34.** A clique $\Gamma_1 \subset \Gamma$ is stable iff there does not exist vertices $w \in \Gamma_1$ and $v \in \Gamma \setminus \Gamma_1$ such that $w^\perp \subset St(v)$.

In other words, the clique $\Gamma_1$ is stable iff the corresponding $\mathbb{Z}^n$ subgroup of $G(\Gamma_1)$ is invariant under all transvections.

**Proof.** The only if part can be proved similarly as the previous lemma by choosing a transvection which does not preserve the subgroup $G(\Gamma_1)$. For the converse, let $\{v_i\}_{i=1}^n$ and let $\Gamma_{v_i}$ be the minimal stable subgraph that contains $v_i$ for $1 \leq i \leq n$. By our assumption and Lemma $3.31$, $\Gamma_{v_i} \subset \Gamma_1$. Thus the full subgraph spanned by $\bigcup_{i=1}^n \Gamma_{v_i}$ is stable by Lemma $3.24$, which means $\Gamma_1$ is stable. □

**Remark 3.35.** It is nature to ask what happens if we drop the clique assumption in the above theorem. There turns out to be counterexamples. Let $\Gamma$ be the graph as below and $\Gamma_1 \subset \Gamma$ be the disjoint union of $v$ and $w$. It is easy to check there does not exist $v_1 \in \Gamma_1$ and $v_2 \in \Gamma \setminus \Gamma_1$ such that $v_1^\perp \subset St(v_2)$. Note that $St(u)$ separates $\Gamma$, then we get a partial conjugation that sends $v \rightarrow v$ and $w \rightarrow u^{-1}wu$, which implies $\Gamma_1$ is not stable.

A more interesting example (but of the same nature) is the following. Let $\Gamma_1$ be the graph in the left side as below and $\Gamma_2$ be the graph in the right side. Then $G(\Gamma_1)$ is quasi-isometric to $G(\Gamma_2)$ by the discussion in Section 11 of [BKS08a]. Let $q : X(\Gamma_2) \rightarrow X(\Gamma_2)$ be a quasi-isometry and $K$ be a standard subcomplex in $X(\Gamma_2)$ such that $\Gamma_K$ is a pentagon in $\Gamma_2$. Suppose $q(K)$ is Hausdorff close to a standard subcomplex $K'$ in $X(\Gamma)$. Then $\Gamma_{K'}$ must be a connected proper subgraph of $\Gamma_1$, hence is a tree. But this is impossible by results of [BN08].
The problem in both counterexamples are caused by separating closed stars in the graph. It is natural to ask if there are counterexamples for graphs without separating closed star.

4. Quasi-isometry implies isomorphism

4.1. A boundary map. Let \( q : X(\Gamma) \to X(\Gamma') \) be a quasi-isometry. Usually \( q \) does not induce a well-defined boundary map, see [CK00]. However, Theorem 3.27 implies we still have control on a subset of the Tits boundaries in the case when \( \text{Out}(G(\Gamma)) \) and \( \text{Out}(G(\Gamma')) \) are transvection free. In this section, we will re-organize this piece of information in terms of extension complex.

Recall that we identify the vertex set of \( \Gamma \) with a standard generating set \( S \) of \( G(\Gamma) \), which induces a labelling of the standard circles in the Salvetti complex. By choosing an orientation in each standard circle, we obtain a directed labelling of edges in \( X(\Gamma) \).

Denote the extension complex of \( \Gamma \) by \( \mathcal{P}(\Gamma) \). We give an alternative definition of \( \mathcal{P}(\Gamma) \) here, which is natural for our purposes. The vertices of \( \mathcal{P}(\Gamma) \) are in 1-1 correspondence with the parallel classes of standard geodesics in \( X(\Gamma) \) (two standard flats are in the same parallel class iff they are parallel). Two distinct vertices \( v_1, v_2 \in \mathcal{P}(\Gamma) \) are connected by an edge iff we can find standard geodesic \( l_i \) in the parallel class associated with \( v_i \) \((i = 1, 2)\) such that \( l_1 \) and \( l_2 \) span a standard 2-flat.

The next observation follows from Lemma 3.1 and Lemma 2.3.

**Observation 4.1.** If \( v_1 \neq v_2 \), then \( v_1 \) and \( v_2 \) are joined by an edge iff there exist \( l'_i \) in the parallel class associated with \( v_i \) \((i = 1, 2)\) and \( R > 0 \) such that \( l'_1 \subset N_R(P_{l_2}) \).

\( \mathcal{P}(\Gamma) \) is defined to be the flag complex of its 1-skeleton.

**Lemma 4.2.** \( \mathcal{P}(\Gamma) \) is isomorphic to the extension complex of \( \Gamma \).

**Proof.** To see this, pick vertex \( v \in \mathcal{P}(\Gamma) \) and let \( l \) be a standard geodesic in the parallel class associated with \( v \). We identify \( l \) with \( \mathbb{R} \) in an orientation-preserving way (the orientation in \( l \) is induced by the directed labelling). Recall that \( G(\Gamma) \curvearrowright X(\Gamma) \) by deck transformations, let \( \alpha_v \in G(\Gamma) \) be the element such that \( \alpha_v(l) = l \) and \( \alpha_v(x) = x + 1 \) for any \( x \in l \). It is easy to see \( \alpha_v \) is conjugate to an element in \( S \), thus \( \alpha_v \) gives rise a vertex \( \alpha_v \in \Gamma^c \) by Definition 2.11. \( \alpha_v \) does not depend the choice of \( l \) in the parallel class, so we have a well-defined map from the vertex set of \( \mathcal{P}(\Gamma) \) to the vertex set of \( \Gamma^c \). Moreover, if \( v_1 \) and \( v_2 \) are adjacent, then \( \alpha_{v_1} \) and \( \alpha_{v_2} \) commutes.

Now we define an inverse map. Pick \( \alpha = gsg^{-1} \in \Gamma^c \) \((s \in S)\), then all standard geodesics which are stabilized by \( \alpha \) are in the same parallel class. Let \( v_\alpha \) be the vertex in \( \mathcal{P}(\Gamma) \) associated with this parallel class. For \( i = 1, 2 \), let \( \alpha_i = g_i s_i g_i^{-1} \in \Gamma^c \). By the centralizer theorem of [DS89], \( \alpha_1 \) and \( \alpha_2 \) commutes iff \([s_1, s_2] = 1\) and there exists \( g \in G(\Gamma) \) such that \( \alpha_i = g s_i g^{-1} \), thus \( v_{\alpha_1} \) and \( v_{\alpha_2} \) are adjacent in \( \mathcal{P}(\Gamma) \). \( \square \)

Since every edge in the standard geodesics of the same parallel class has the same label, the labelling of the edges of \( X(\Gamma) \) induces a labelling of the vertices of \( \mathcal{P}(\Gamma) \). Moreover, since \( G(\Gamma) \curvearrowright X(\Gamma) \) by label-preserving cubical isomorphisms, we obtain an induced action \( G(\Gamma) \curvearrowright \mathcal{P}(\Gamma) \) by label-preserving simplicial isomorphisms.

Pick arbitrary vertex \( p \in X(\Gamma) \), one can obtain a simplicial embedding \( i_p : F(\Gamma) \to \mathcal{P}(\Gamma) \) by considering the collection of standard geodesics passing through
$p$ (recall that $F(\Gamma)$ is the flag complex of $\Gamma$). We will denote the image of $i_p$ by $(F(\Gamma))_p$. Since $G(\Gamma)$ acts freely and transitively on the vertex set of $X(\Gamma)$, this induces a label-preserving projection $\pi : P(\Gamma) \to F(\Gamma)$.

Pick $k$-simplex in $P(\Gamma)$ with vertex set $\{v_i\}_{i=1}^k$ and pick standard geodesic $l_i$ in the parallel class associated with $v_i$ for $1 \leq i \leq k$. Since $P_i \cap P_j \neq \emptyset$ for $1 \leq i \neq j \leq k$, by Lemma 2.1 and Corollary 3.2 and Lemma 3.4 there exist standard geodesics $\{l_i\}_{i=1}^k$ satisfying

1. $l_i$ is parallel to $l_i$ for each $i$.
2. The convex hull of $\{l_i\}_{i=1}^k$ is a standard $k$-flat, denoted by $F_k$.
3. $\cap_{i=1}^k P_i = P_{F_k}$.

Thus we have a 1-1 correspondence between the $k$-simplices in $P(\Gamma)$ and parallel classes of standard $k$-flats in $X(\Gamma)$. In particular, maximal simplices in $P(\Gamma)$, namely those simplices which are not properly contained in some larger simplices of $P(\Gamma)$, are corresponding to maximal standard flats in $X(\Gamma)$. For standard flat $F \subset X(\Gamma)$, we denote the simplex in $P(\Gamma)$ associated with the parallel class containing $F$ by $\Delta(F)$.

**Observation 4.3.** Let $\Delta_1$, $\Delta_2$ be two simplices in $P(\Gamma)$ such that $\Delta = \Delta_1 \cap \Delta_2 \neq \emptyset$ and let $F_i \subset X(\Gamma)$ be standard flat such that $\Delta(F_i) = \Delta_i$ for $i = 1, 2$. Set $(F_1', F_2') = I(F_1, F_2)$, then $\Delta(F_1') = \Delta(F_2') = \Delta$.

Define the reduced Tits boundary, $\partial_T(X(\Gamma))$ be the subset of $\partial_T(X(\Gamma))$ which is the union of Tits boundaries of standard flats in $X(\Gamma)$. For standard flat $F \subset X(\Gamma)$, we triangulate $\partial_T F$ into all-right spherical simplices which are the Tits boundaries of orthant subcomplexes in $F$. Pick another standard flat $F' \subset X(\Gamma)$, then $\partial_T F \cap \partial_T F'$ is a subcomplex in both $\partial_T F$ and $\partial_T F'$ by Lemma 3.1 and Remark 2.5. Thus we can endow $\partial_T(X(\Gamma))$ with the structure of an all-right spherical complex. Moreover, we can associate $\partial_T F$ with $\Delta(F) \subset P(\Gamma)$, which induces a surjective simplicial map $s : \partial_T(X(\Gamma)) \to P(\Gamma)$ (s can be defined by induction on dimension). By looking at the inverse image of each simplex in $P(\Gamma)$, it is easy to see that one can construct $\partial_T(X(\Gamma))$ from $P(\Gamma)$ in the following way: starting with a collection of $S^0$ which are in 1-1 correspondence to vertices of $P(\Gamma)$ and forming a joint of $n$ copies of $S^0$ iff the corresponding $n$ vertices in $P(\Gamma)$ span a $(n-1)$-simplex. Hence $\partial_T(X(\Gamma))$ is a flag complex.

Let $K_1 \subset X(\Gamma)$ be a standard subcomplex. We define $\partial_T(K_1)$ to be the union of Tits boundaries of standard flats in $K_1$. Note that $\partial_T(K_1) = \partial_T(X(\Gamma)) \cap \partial_T K_1$. $\partial_T(K_1)$ descends to a subcomplex in $P(\Gamma)$, which will be denoted by $\Delta(K_1)$. Pick another standard subcomplex $K_2 \subset X(\Gamma) and let $(K_1', K_2') = I(K_1, K_2)$. By Remark 2.5 we have $\partial_T K_1' = \partial_T K_2' = \partial_T K_1 \cap \partial_T K_2$, hence $\partial_T K_1' = \partial_T K_2' = \Delta(K_1') = \Delta(K_2') = \Delta(K_1) \cap \Delta(K_2)$, which is a generalization of Observation 4.3.

Now we discuss the relation between $P(\Gamma)$ with several other objects in the literature. One could skip this discussion on first reading and proceed directly to Lemma 4.7.

We can endow $F(\Gamma)$ with the structure of complex of groups, which gives us an alternative definition of $P(\Gamma)$. More specifically, $P(\Gamma) = F(\Gamma) \times G(\Gamma)/\sim$, here $St(v) \times g_1$ and $St(v) \times g_2$ ($v \in F(\Gamma)$ is a vertex) are identified iff there exists integer $m$ such that $g_1^{-1}g_2 = v^m$ (we also view $v$ as one of the generators of $G(\Gamma)$). Hence for $k$-simplex $\Delta^k \subset F(\Gamma)$ with vertex set $\{v_i\}_{i=1}^k$, $St(\Delta^k) \times g_1$ and $St(\Delta^k) \times g_2$ are
identified iff \( g_1^{-1}g_2 \) belongs to the \( \mathbb{Z}^k \) subgroup of \( G(\Gamma) \) generated by \( \{v_i\}_{i=1}^k \). One can compare this with a similar construction for Coxeter group in \( \text{[Dav83]} \).

There is another important object associated with a right-angled Artin group, called the modified Deligne complex in \( \text{[CD95]} \) and the flat space in \( \text{[BKS08a]} \).

**Definition 4.4.** Let \( P(\Gamma) \) be poset of left cosets of standard Abelian subgroup of \( G(\Gamma) \) (include the trivial subgroup) such that the partial order is induced by inclusion of sets. Then the **modified Deligne complex** is defined to be the geometric realization of the derived poset of \( P(\Gamma) \).

Recall that elements in the **derived poset** of a poset \( P \) are totally ordered finite chains in \( P \). It can be viewed as an abstract simplex.

One can put a piecewise Euclidean metric on the modified Deligne complex such that it becomes a \( \text{CAT}(0) \) cube complex (\( \text{[CD95]} \)), moreover, it is hyperbolic when \( \Gamma \) has no cycles of length \( < 5 \) (\( \text{[BKS08a]} \)).

The extension complex \( P'(\Gamma) \) can be viewed as a coarse version of the modified Deligne complex. Let \( A, B \) be two subsets of a metric space. We say \( A \) and \( B \) are **coarsely equivalent** iff \( A = B \), and \( A \) are **coarsely contained** in \( B \) iff \( A \subset_c B \). Let \( P'(\Gamma) \) be the poset whose elements are coarsely equivalent classes of left cosets of non-trivial standard Abelian subgroup of \( G(\Gamma) \), and the partial order is induced by coarse inclusion of sets.

**Observation 4.5.** The poset \( P'(\Gamma) \) is an abstract simplicial complex and it is isomorphic to \( P(\Gamma) \).

Roughly speaking, \( P(\Gamma) \) captures the combinatorial pattern of how standard flats in \( X(\Gamma) \) intersect with each other, and \( P(\Gamma) \) is about how they coarsely intersect with each other, thus \( P(\Gamma) \) contains more information than \( P(\Gamma) \). However, in certain cases, it is possible to recover information about \( P(\Gamma) \) from \( P(\Gamma) \), and this enable us to prove quasi-isometry classification/rigidity results for RAAGs.

We can define the poset \( P'(\Gamma) \) for arbitrary Artin group by considering the collection of coarse equivalent classes of spherical subgroups in an Artin group under coarse inclusion. Then the geometric realization of the derived poset of \( P'(\Gamma) \) would be a natural candidate for the extension complex of an Artin group. It is interesting to ask how much of the results in \( \text{[KK14]} \) can be generalized to this context.

There is a natural surjective simplicial map \( p : C(X(\Gamma)) \rightarrow \Gamma^e \), namely pick vertex \( v \in C(X(\Gamma)) \) and let \( h \) be the corresponding hyperplane. Since all standard geodesics which intersect \( h \) at one point are in the same parallel class, we define \( p(v) \) to be the vertex in \( \Gamma^e \) associated with this parallel class (see Lemma \( \text{[12]} \)). It is clear that if \( v_1, v_2 \in C(X(\Gamma)) \) are adjacent vertices, then \( p(v_1) \) and \( p(v_2) \) are adjacent, so \( p \) extends to a simplicial map. Pick vertex \( w \in \Gamma^e \), then \( p^{-1}(w) \) is the collection of hyperplanes dual to a standard geodesic.
Theorem 4.6 (KK13b, Hag14, HK14). If $\Gamma$ is connected, then $C(X(\Gamma))$, $\mathcal{C}(X(\Gamma))$ and $\mathcal{P}(\Gamma)$ are quasi-isometric to each other, moreover, they are quasi-isometric to a tree.

From this viewpoint, $\mathcal{P}(\Gamma)$ captures both the geometric information of $X(\Gamma)$ (the standard flats) and the combinatorial information (the hyperplanes).

Now we study how the extension complex behaves under quasi-isometry.

**Lemma 4.7.** Given $\Gamma_1$ and $\Gamma_2$ such that $G(\Gamma_i)$ is transvection free for $i = 1, 2$, then any quasi-isometry $q : X(\Gamma_1) \to X(\Gamma_2)$ induces a simplicial isomorphism $q_* : \mathcal{P}(\Gamma_1) \to \mathcal{P}(\Gamma_2)$. If only $G(\Gamma_1)$ is assumed to be transvection free, we still have a simplicial embedding $q_* : \mathcal{P}(\Gamma_1) \to \mathcal{P}(\Gamma_2)$.

**Proof.** By Theorem 3.27, every vertex in $\Gamma_1$ is stable, thus $q$ will send any parallel class of standard geodesics in $X(\Gamma_1)$ to another parallel class of standard geodesics in $X(\Gamma_2)$ up to finite Hausdorff distance. This induces a well-defined map $q_*$ from the 0-skeleton of $\mathcal{P}(\Gamma_1)$ to the 0-skeleton of $\mathcal{P}(\Gamma_2)$. $q_*$ is a bijection by considering the quasi-isometry inverse. Moreover, Observation 4.1 implies two vertices in $\mathcal{P}(\Gamma_1)$ are adjacent iff their images under $q_*$ are adjacent. So we can extend $q_*$ to be a graph isomorphism between the 1-skeleton of $\mathcal{P}(\Gamma_1)$ and the 1-skeleton of $\mathcal{P}(\Gamma_2)$. $\square$

4.2. Reconstruction of the quasi-isometry. We show the boundary map $q_* : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma')$ in Theorem 4.8 induces a well-defined map from $G(\Gamma)$ to $G(\Gamma')$.

**Lemma 4.8.** Let $F_1$ and $F_2$ be two maximal standard flats in $X(\Gamma)$ and let $\Delta_1$, $\Delta_2$ be the corresponding maximal simplexes in $\mathcal{P}(\Gamma)$. If $F_1$ and $F_2$ are separated by a hyperplane $h$, then there exist vertices $v_i \in \Delta_i$ for $i = 1, 2$ and $v \in \mathcal{P}(\Gamma)$ such that $v_1$ and $v_2$ are in different connected component of $\mathcal{P}(\Gamma) \setminus St(v)$.

**Proof.** Let $e$ be an edge dual to $h$ and let $l$ be the standard geodesic that contains $e$. Set $v = \Delta(l) \in \mathcal{P}(\Gamma)$. By Lemma 3.4, $P_l$ is isometric to $h \times \mathbb{E}^1$; thus every standard geodesic parallel to $l$ must have non-trivial intersection with $h$. Since $F_1 \cap h = \emptyset$, $F_1$ can not contain any standard geodesic parallel to $l$, which means $v \notin \Delta_1$. Moreover, $\Delta_1 \not\subseteq St(v)$ since $\Delta_1$ is a maximal simplex. Similarly, $\Delta_2 \not\subseteq St(v)$, thus we can find vertices $v_i \in \Delta_i \setminus St(v)$ for $i = 1, 2$. We claim $v_1$, $v_2$ and $v$ are the vertices we are looking for.

If there is a path $\omega \subset \mathcal{P}(\Gamma) \setminus St(v)$ connecting $v_1$ and $v_2$, we can assume $\omega$ is of the sequence of edges $\{e_i\}_{i=1}^k$ with $v_1 \in e_1$ and $v_2 \in e_k$. For each $e_i$, pick a maximal simplex $\Delta'_i$ that contains $e_i$ and let $F'_i$ to be the maximal standard flat such that $\Delta(F'_i) = \Delta'_i$. Then $v \notin \Delta'_i$ for $1 \leq i \leq k$, hence $F'_i \cap h = \emptyset$.

Set $\Delta_0 = \Delta_1$, $\Delta_{k+1} = \Delta_2$, $F_0 = F_1$ and $F_{k+1} = F_2$. Since $\Delta'_i \cap \Delta'_{i+1}$ contains a vertex in $\omega$, we have

$$\Delta_i \cap \Delta_{i+1} \cap St(v) \neq \emptyset$$

for $0 \leq i \leq k$. Since $F_0$ and $F_{k+1}$ are in different side of $h$, there exists $i_0$ such that $h$ separate $F_{i_0}$ and $F_{i_0+1}$. Let $(F''_{i_0}, F''_{i_0+1}) = \mathcal{I}(F_{i_0}'', F_{i_0+1}'', \Delta'_i \cap \Delta'_{i+1})$. By Observation 4.3 $\Delta(F''_{i_0}) = \Delta(F''_{i_0+1}) = \Delta'_i \cap \Delta'_{i+1}$. However, by Lemma 2.7, there exists a convex subset of $h$ parallel to $F''_{i_0}$, thus $F''_{i_0} \subset h \subset P_l$. It follows from Observation 4.1 that $\Delta'_i \cap \Delta'_{i+1} \subset St(v)$, which contradicts (4.9). $\square$

From now on, we will identify $G(\Gamma)$ with the 0-skeleton of $X(\Gamma)$. Denote the Cayley graph of $G(\Gamma)$ with respect to the standard generating set $S$ by $C(\Gamma)$ and we identify it with the 1-skeleton of $X(\Gamma)$. 


Lemma 4.10. Given $\Gamma_1$ satisfying:

(1) There is no separating closed star in $F(\Gamma_1)$.

(2) $F(\Gamma_1)$ is not contained in a union of two closed stars.

then any simplicial isomorphism $s : \mathcal{P}(\Gamma_1) \to \mathcal{P}(\Gamma_2)$ induces a unique map $s' : G(\Gamma_1) \to G(\Gamma_2)$ such that for any maximal standard flat $F_1 \subset X(\Gamma_1)$, vertices in $F_1$ are mapped by $s'$ to vertices lying in a maximal standard flat $F_2 \subset X(\Gamma_2)$ with $\Delta(F_2) = s'(\Delta(F_1))$.

Proof. Pick vertex $p \in G(\Gamma_1)$, let $\{F_i\}_{i=1}^k$ be the collection of maximal standard flats containing $p$. For $1 \leq i \leq k$, define $\Delta_i = \Delta(F_i)$ and $\Delta'_i = s(\Delta_i)$. Let $F'_i \subset G(\Gamma_2)$ be maximal standard flat such that $\Delta(F'_i) = \Delta'_i$. Let $K_p = (F(\Gamma))_p = \bigcup_{i=1}^k \Delta_i$ (recall that $K_p \cong F(\Gamma_1)$). We claim

\[ \cap_{i=1}^k F'_i \neq \emptyset. \tag{4.11} \]

Suppose the contrary is true. Then by Lemma 2.4, there exist $i_1$ and $i_2$ such that $F'_{i_1} \cap F'_{i_2} = \emptyset$, thus $F'_{i_1}$ and $F'_{i_2}$ are separated by a hyperplane. It follows from Lemma 4.8 that there exist vertices $v' \in \mathcal{P}(\Gamma_2)$, $v'_1 \in \Delta'_{i_1}$ and $v'_2 \in \Delta'_{i_2}$ such that $v'_1$ and $v'_2$ are in different connected components of $\mathcal{P}(\Gamma_2) \setminus St(v')$. Let $v = s^{-1}(v')$, $v_1 = s^{-1}(v'_1)$ and $v_2 = s^{-1}(v'_2)$. Then $K_p \setminus (K_p \cap St(v))$ is disconnected (since $v_1, v_2 \in K_p$ and they are separated by $St(v)$).

If $v \in K_p$, then $K_p$ would contain separating closed star, which yields a contradiction, thus (4.11) is true in this case.

Suppose $v \notin K_p$. Pick a standard geodesic $l$ such that $\Delta(l) = v$ and let $\{h_i\}_{i=1}^n$ be the collection of hyperplanes in $X(\Gamma)$ such that each $h_i$ separates $p$ from $P_1$ (note that $p \notin P_1$). Pick an edge $e_i$ dual to $h_i$ and let $w_i$ be the unique vertex in $K_p$ that has the same label as $e_i$. Let $w_0 \in K_p$ be the unique vertex which has the same label as $v$. We claim

\[ St(v) \cap K_p = \cap_{i=0}^n (St(w_i) \cap K_p). \tag{4.12} \]

For every $u \in K_p$, we let $l_u$ denote the unique standard geodesic such that $\Delta(l_u) = u$ and $p \in l_u$.

Pick $u \in St(v) \cap K_p$. Observation 4.1 implies $I(l_u, P_1) = (l_u, l'_u)$ ($l'_u$ is some standard geodesic in $P_1$). Then for $1 \leq i \leq n$, $h_i$ separates $l_u$ from $K_p$, otherwise $h_i \cap l_u \neq \emptyset$ and Lemma 2.6 implies $h_i \cap P_1 \neq \emptyset$, which is a contradiction. It follows from Corollary 3.2 that $u$ and $w_i$ are adjacent for $0 \leq i \leq n$, thus $u \in \cap_{i=0}^n (St(w_i) \cap K_p)$. Therefore $St(v) \cap K_p \subset \cap_{i=0}^n (St(w_i) \cap K_p)$.

Pick $u \in \cap_{i=0}^n (St(w_i) \cap K_p)$. Suppose the intersection $l_u \cap P_1$ is nonempty, and contains the vertex $z$. Since $v$ and $w_0$ have the same label and $u \in St(w_0)$, it follows that the edge of $l_u$ containing $z$ belongs to the parallel set $P_1$. Then $l_u \subset P_1$, contradicting the fact that $p \notin P_1$. Therefore $l_u \cap P_1 = \emptyset$, and there exists a shortest edge path $\omega$ connecting $l_u$ and $P_1$. Let $\{f_j\}_{j=1}^m$ be the consecutive edges in $\omega$ such that $f_1 \cap l_u \neq \emptyset$ and let $h_j$ be the hyperplane dual to $f_j$. Then $h_j$ separates $l_u$ from $P_1$ (otherwise $\omega$ would not be the shortest edge path), hence separates $p$ from $P_1$. This and $u \in \cap_{i=0}^n (St(w_i) \cap K_p)$ imply that $d(\pi(u), V_{f_j}) \leq 1$ for each $j$. It follows that $\omega$ is contained in the parallel set $P_{f_j}$, and hence the intersection $P_{f_1} \cap P_1$ contains some vertex $z$. Again, since $u \in St(w_0)$, and $w_0$ has the same label as $v$, we find that the standard geodesic $l'_u \subset P_{f_j}$ that is parallel to $l_u$ and passes through $z$, is contained in $P_1$. Therefore $u \in St(v) \cap K_p$. Then (4.12) follows.
By condition (2) of Lemma 4.10, we have
\[(St(w_0) \cap K_p) \cup (\cap_{i=1}^n (St(w_i) \cap K_p)) \subseteq K_p.\]

Then the Mayer-Vietoris sequence for reduced homology imply either \(\cap_{i=1}^n (St(w_i) \cap K_p)\) or \(St(w_0) \cap K_p\) would separate \(K_p\). By induction, there exists \(i_0\) such that \(St(w_{i_0}) \cap K_p\) separates \(K_p\), which is contradictory to condition (1) of Lemma 4.10.

Since \(X(\Gamma_1)\) is Euclidean factor free, \(\cap_{i=1}^k \Delta_i = \emptyset\) and \(\cap_{i=1}^k F_i = p\). It follows that \(\cap_{i=1}^k \Delta_i = \emptyset\) and \(\cap_{i=1}^k F_i\) is exactly one point. We define \(s'\) by sending \(p\) to this point, it is easy to check \(s'\) has the required properties. \(\square\)

Condition (2) in the above lemma is necessary, otherwise we can find a counterexample by taking \(\Gamma_i\) to be 4-gon for \(i = 1, 2\).

**Corollary 4.13.** Suppose \(G(\Gamma_1)\) and \(G(\Gamma_2)\) both satisfy the assumption of Lemma 4.10. Then they are isomorphic iff \(P(\Gamma_1)\) and \(P(\Gamma_2)\) are isomorphic as simplicial complexes.

**Lemma 4.14.** Any right-angled Artin group with finite outer automorphism group satisfies the assumption of Lemma 4.10.

**Proof.** It is clear that \(F(\Gamma)\) should satisfy condition (1) of Lemma 4.10 since no nontrivial partial conjugation is allowed. To see (2), we argue by contradiction and assume \(F(\Gamma) = St(v) \cup St(w)\) for distinct vertices \(v, w \in \Gamma\), since \(v^+ \nsubseteq St(w)\), there exists \(u \in v^+\) such that \(d(u, w) \geq 2\). Pick any edge \(e\) such that \(u \in e\), then \(e \nsubseteq St(w)\), so \(e \subset St(v)\), which implies \(u^+ \subset St(v)\) and \(Out(G(\Gamma))\) is infinite. \(\square\)

By Theorem 4.7, Lemma 4.14 and Corollary 4.13 we have

**Theorem 4.15.** Given graph \(\Gamma_1\) and \(\Gamma_2\) such that \(Out(G(\Gamma_i))\) is finite for \(i = 1, 2\), then \(G(\Gamma_1)\) and \(G(\Gamma_2)\) are quasi-isometric iff they are isomorphic. Moreover, for any \((L, A)\)-quasi-isometry \(q : X(\Gamma_1) \rightarrow X(\Gamma_2)\), there exist an bijection \(q' : G(\Gamma_1) \rightarrow G(\Gamma_2)\) and \(D = D(L, A, |\Gamma_1|)\) such that

1. \(d(q(v), q'(v)) < D\) for any \(v \in G(\Gamma_1)\).
2. For any standard flat \(F_1 \subset X(\Gamma_1)\), there exists a standard flat \(F_2 \subset X(\Gamma_2)\) such that \(q'\) induces a bijection between \(F_1 \cap G(\Gamma_1)\) and \(F_2 \cap G(\Gamma_2)\).

To see (2), note that every vertex in \(v \in \Gamma\) is the intersection of maximal cliques that contain \(v\) (otherwise there exists \(w \neq v\) such that \(v^+ \subset St(w)\)), thus every standard geodesic is the intersection of finite many maximal standard flats.

Denote the word metric on \(G(\Gamma)\) with respect to the standard generators by \(d_w\). Following [KK14], we define the syllable length of a word \(\omega\) to be minimal \(l\) such that \(\omega\) can be written as the product of \(l\) elements of form \(v_i^{k_i}\), where \(v_i\) is a standard generator and \(k_i\) is an integer.

An alternative definition is the following, let \(\{h_i\}_{i=1}^k\) be the be the collection of hyperplanes separating \(\omega\) and the identity element, and for each \(i\), pick a standard geodesic \(l_i\) dual to \(h_i\). Then the syllable length of \(\omega\) is the number of elements in \(\{\Delta(l_i)\}_{i=1}^k\). The syllable length induces a left invariant metric on \(G(\Gamma)\), which will be denoted by \(d_r\).

**Corollary 4.16.** Let \(\Gamma\) be a graph such that \(Out(G(\Gamma))\) is finite and denote the simplicial automorphism group of \(P(\Gamma)\) by \(Aut(P(\Gamma))\). Then
\[
Aut(P(\Gamma)) \cong Isom(G(\Gamma), d_r)
\]
Proof. We have a group homomorphism $h_1 : \text{Aut}(\mathcal{P}(\Gamma)) \to \text{Perm}(\mathcal{P}(\Gamma))$ by Lemma 4.10, where $\text{Perm}(\mathcal{P}(\Gamma))$ is the permutation group of elements in $\mathcal{P}(\Gamma)$. Take $\phi \in \text{Aut}(\mathcal{P}(\Gamma))$, by Lemma 4.14, $\varphi = h_1(\phi)$ and $\varphi^{-1} = h_1(\phi^{-1})$ satisfy the conclusion of Lemma 4.10. Since every standard geodesic is the intersection of finite many maximal standard flats, points in a standard geodesic are mapped to points in a standard geodesic by $\phi$, which implies $d_r(\varphi(v_1), \varphi(v_2)) \leq d_r(v_1, v_2)$ if $d_r(v_1, v_2) \leq 1$. By triangle inequality, we have $d_r(\varphi(v_1), \varphi(v_2)) \leq d_r(v_1, v_2)$ for any $v_1, v_2 \in \mathcal{P}(\Gamma)$ and similarly $d_r(\varphi^{-1}(v_1), \varphi^{-1}(v_2)) \leq d_r(v_1, v_2)$, thus $\varphi \in \text{Isom}(\mathcal{G}(\Gamma), d_r)$ and we have homomorphism $h_1 : \text{Aut}(\mathcal{P}(\Gamma)) \to \text{Isom}(\mathcal{P}(\Gamma), d_r)$.

Now pick $\varphi \in \text{Isom}(\mathcal{G}(\Gamma), d_r)$, let $v_1, v_2, v_3 \in \mathcal{G}(\Gamma)$ such that $d_r(v_1, v_1) = 1$ for $i = 2, 3$. We claim

\begin{equation}
\angle_{v_1}(v_2, v_3) = \pi/2 \Leftrightarrow \angle_{\varphi(v_1)}(\varphi(v_2), \varphi(v_3)) = \pi/2.
\end{equation}

If $\angle_{v_1}(v_2, v_3) = \pi/2$, then we can find $v_4 \in \mathcal{G}(\Gamma)$ such that $\{v_i\}_{i=1}^4$ are the vertices of a flat rectangle in $X(\Gamma)$. Note that $d_r(v_1, v_4) = d_r(v_2, v_3) = 2$ and $d_r(v_4, v_2) = d_r(v_4, v_3) = 1$, so $d_r(\varphi(v_1), \varphi(v_4)) = d_r(\varphi(v_2), \varphi(v_3)) = 2$ and $d_r(\varphi(v_4), \varphi(v_2)) = 1$. Consider the 4-gon formed by $\varphi(v_1)\varphi(v_2), \varphi(v_2)\varphi(v_4), \varphi(v_4)\varphi(v_3)$ and $\varphi(v_3)\varphi(v_1)$, then the angles at the four vertices of the 4-gon are bigger or equal to $\pi/2$, it follows from $\text{CAT}(0)$ geometry that the angles are exactly $\pi/2$ and the 4-gon actually bounds a flat rectangle. Thus one direction of (4.17) is proved, the other direction is similar.

Here is another observation we need: if three points $v_1, v_2, v_3 \in \mathcal{G}(\Gamma)$ satisfies $d_r(v_i, v_j) = 1$ for $1 \leq i, j \leq 3$, then the angle at each vertex of the triangle $\Delta(v_1, v_2, v_3)$ could only be $0$ or $\pi$, thus $\{v_i\}_{i=1}^3$ are inside a standard geodesic. It follows that points in a standard geodesic are mapped by $\varphi$ to points in a standard geodesic. We define $\phi : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$ as follows: for vertex $w \in \mathcal{P}(\Gamma)$, let $l$ be a standard geodesic such that $\Delta(l) = w$. Suppose $l' \subset X(\Gamma)$ is the standard geodesic such that $\phi(l) \subset l'$ and suppose $w' = \Delta(l')$. We define $w' = \phi(w)$. (4.17) implies $w'$ does not depend on the choice of $l$ and $\phi(w_1)$ and $\phi(w_2)$ are adjacent if vertices $w_1, w_2 \in \mathcal{P}(\Gamma)$ are adjacent, thus $\phi$ is a well-defined simplicial map. Note that $\phi^{-1}$ also induces a simplicial map from $\mathcal{P}(\Gamma)$ to itself in a similar way, so $\phi \in \text{Aut}(\mathcal{P}(\Gamma))$.

We define $\phi = h_2(\sigma)$, it is easy to check $h_2 : \text{Isom}(\mathcal{G}(\Gamma), d_r) \to \text{Aut}(\mathcal{P}(\Gamma))$ is a group homomorphism and $h_2 \circ h_1 \circ h_2 = Id$, thus the corollary follows. □

Remark 4.18. If we drop the assumption in the above corollary about $\Gamma$, we can still get a monomorphism $h : \text{Isom}(\mathcal{G}(\Gamma), d_r) \to \text{Aut}(\mathcal{P}(\Gamma))$, moreover, any $\varphi \in \text{Isom}(\mathcal{G}(\Gamma), d_r)$ maps vertices in a standard flat to vertices in a standard flat of the same dimension. Also note that $h$ is surjective iff $\text{Out}(\mathcal{G}(\Gamma))$ is finite.

Remark 4.19. For any finite simplicial graphs $\Gamma_1$ and $\Gamma_2$, $\mathcal{G}(\Gamma_1) \cong \mathcal{G}(\Gamma_2)$ iff $(\mathcal{G}(\Gamma_1), d_r)$ and $(\mathcal{G}(\Gamma_2), d_r)$ are isometric as metric space. This follows from Dro87, Lau95. For the other direction, let $\varphi : (\mathcal{G}(\Gamma_1), d_r) \to (\mathcal{G}(\Gamma_2), d_r)$ be an isometry. Pick $v \in \mathcal{G}(\Gamma_1)$ and let $\{l_i\}_{i=1}^k$ be the collection of standard geodesics passing through $v$. Pick $v_i \in \mathcal{G}(\Gamma_1)$ such that $v_i \in l_i \setminus \{v\}$. Then $d_r(v_i, v_j) = 1$ for $1 \leq i \neq k$ and $d_r(v_i, v_j) = 2$ for $1 \leq i, j \leq k$. So $d_r(\varphi(v_i), \varphi(v_j)) = 1$ for $1 \leq i \neq k$ and $d_r(\varphi(v_i), \varphi(v_j)) = 2$ for $1 \leq i, j \leq k$ and $\angle_{v_i}(v_j, v_3) = \pi/2$ if $\angle_{v_i}(\varphi(v_i), \varphi(v_j)) = \pi/2$ by (4.17). By considering $\varphi^{-1}$, we know $\Gamma_1$ and $\Gamma_2$ are isomorphic.
Suppose Theorem 5.3. $\Gamma$ does not contain any clique join factor, thus isometry. Then $\Gamma_1 \circ \Gamma_2 \circ \Gamma_1 = \Gamma_1 \circ \Gamma_2$ is stable, so $\Gamma_1$ is stable in $\Gamma$ by Theorem 5.3. Pick vertex $v \in \Gamma_2 \setminus \Gamma_1$, then $v \in w^\perp \subset St(u)$ for any vertex $u \in \Gamma_w$ by Lemma 3.31, thus $v \in \Gamma_1 \subset \Gamma_1 \circ \Gamma_2 \circ \Gamma_1$. On the other hand, $w \in \Gamma_2$, so $\Gamma_1 \circ \Gamma_1 \circ \Gamma_2 = \Gamma_1 \circ \Gamma_1$. Since $\Gamma_2$ does not contain any clique factor and $\Gamma_1 \circ \Gamma_2 \circ \Gamma_1 = \Gamma_1 \circ \Gamma_2$ is stable, so $\Gamma_1$ is stable in $\Gamma$ by Theorem 5.3.

Remark 5.2. In the above proof, $\Gamma_1 \circ \Gamma_2$ may be empty, but if $\Gamma_1 \circ \Gamma_2 \neq \emptyset$, then it does not contain any clique join factor, thus $\Gamma_1$ is the maximal clique join factor of $\Gamma_1 \circ \Gamma_2 \circ \Gamma_1$.

The next result answers the question at the end of Example 3.28.

Theorem 5.3. Suppose $Out(G(\Gamma))$ is finite and let $q : X(\Gamma) \to X(\Gamma')$ be a quasi-isometry. Then $q$ induces a simplicial isomorphism $q_* : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma')$, in particular, $Out(G(\Gamma'))$ is transvection free.
In the following proof, we identify $\Gamma$ with the one-skeleton of $F(\Gamma)$. Also recall that there is a label-preserving projection $\pi : \mathcal{P}(\Gamma) \to F(\Gamma)$ and $\pi : \mathcal{P}(\Gamma') \to F(\Gamma')$.

**Proof.** By Lemma 4.7, there is a simplicial embedding $q_* : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma')$. Pick vertex $p \in X(\Gamma)$ and let $\{\Delta_i\}_{i=1}^k$, $\{F_i\}_{i=1}^k$, $\{\Delta'_i\}_{i=1}^k$ and $\{F'_i\}_{i=1}^k$ be as in the proof of Lemma 4.10. We claim

$$\bigcap_{i=1}^k F'_i \neq \emptyset.$$  

Suppose \((5.4)\) is not true. Then there exist $1 \leq i_1 \neq i_2 \leq k$ and hyperplane $h' \subset X(\Gamma)$ such that $h'$ separates $F'_{i_1}$ and $F'_{i_2}$. Let $l'$ be a standard geodesic that intersects $h'$ transversely and suppose $v' = \Delta(l')$. By the discussion in Lemma 4.8 we can find vertices $v_1' \in \Delta'_{i_1}$ and $v_2' \in \Delta'_{i_2}$ such that $v_1'$ and $v_2'$ are separated by $St(v')$. If there exists $i_0$ such that $F'_{i_0} \cap h' \neq \emptyset$, then $v' \in q_*(\mathcal{P}(\Gamma'))$ and we can prove \((5.4)\) as in Lemma 4.10. Now we assume $F'_{i} \cap h' = \emptyset$ for any $i$. Let $w' = \pi(v') \in \Gamma'$ and let $\Gamma_w'$ be the minimal stable subgraph of $\Gamma'$ that contains $w'$.

We apply Lemma 5.1 to $w' \in \Gamma'$, if case (1) is true, let $F'$ be the standard flat in $X(\Gamma')$ such that $l' \subset F'$ and $\Gamma_{F'} = \Gamma_w'$. Since $\Gamma_{F'}$ is stable, $\Delta(F') \subset q_*(\mathcal{P}(\Gamma'))$, in particular, $v' \in q_*(\mathcal{P}(\Gamma'))$ and we can prove \((5.4)\) as in Lemma 4.10.

If case (2) is true, let $\Gamma_1' = lk(w')$ and let $\Gamma_2' = lk(\Gamma'_1)$. Take $K_1'$ and $K'$ to be the standard subcomplexes in $X(\Gamma')$ such that (1) $\Gamma_{K_1'} = \Gamma_1'$ and $\Gamma_{K'} = \Gamma_1' \circ \Gamma_2'$; (2) $l' \subset K'$ and $K_1' \subset K'$. Set $M'_i = \Delta(K'_i)$ and $M' = \Delta(K')$. Let $K_2'$ be a standard subcomplex such that $\Gamma_{K_2'} = \Gamma_2$ and $K' = K'_1 \times K'_2$. It follows that $M' = M'_1 \ast M'_2$ for $M'_2 = \Delta(K'_2)$. By construction, $v' \in M'$ and $lk(v') = M'_1$.

Since $K'$ and $K_1'$ are stable, there exist stable standard subcomplexes $K$ and $K_1$ in $X(\Gamma)$ such that $q(K) \equiv K'$ and $q(K_1) \equiv K_1'$. Moreover, by applying Theorem 2.9 to the quasi-isometry between $K$ and $K'$, there exists standard subcomplex $K_2 \subset K$ such that $K = K_1 \times K_2$ and $K_2$ is quasi-isometric to $K'_2$, thus $\Gamma_{K_2}$ is also disconnected. Let $M_i = \Delta(K_i) \subset \mathcal{P}(\Gamma)$ for $i = 1, 2$ and $M = M_1 \ast M_2 = \Delta(K)$. Then $q_*(M_1) \subset M'_1$ (caution: $q_*(M_1) = M'_1$ is not true in general) and

$$q^{-1}_*(M'_1) = M_1.$$  

To see this, pick simplex $\Delta \subset \mathcal{P}(\Gamma)$ with $q_*(\Delta) \subset M'_1$. Suppose $\Delta = \Delta(F)$ for a stable standard flat $F \subset X(\Gamma)$. Then $q(F) \subset K'$, hence $F \subset C_1$ and $\Delta \subset M_1$.

Let $L = \cup_{i=1}^k \Delta_i$ and $L' = \cup_{i=1}^k \Delta'_i$. By Lemma 4.10 $L' \setminus (St(v') \cap L')$ is disconnected, thus $L \setminus q^{-1}(St(v') \cap L')$ is disconnected. Note that $(St(v') \cap L') \subset M'_1$, hence $q^{-1}(St(v') \cap L') \subset M_1$ by (5.5).

Let $N = \pi(q^{-1}(St(v') \cap L'))$ and let $N_i = \pi(M_i)$ for $i = 1, 2$. Then $N$ separates $F(\Gamma)$, $N \subset N_1$ and $N_2$ is disconnected. Pick vertices $u_1, u_2$ in distinct connected components of $N_2$, then $d(u_1, u_2) \geq 2$ (since $N_2$ is the full subcomplex spanned by $\Gamma_{K_2}$). Since $\pi(M) = N_1 \ast N_2 \subset F(\Gamma)$, $N \subset St(u_1) \setminus \{u_1\}$ for $i = 1, 2$. Let $\{C_j\}_{j=1}^d$ be the connected components of $F(\Gamma) \setminus N$. Then at most one of $C_j$ is contained in $St(u_1)$. If $d \geq 3$, $St(u_1)$ would separate $F(\Gamma)$, contradiction. If $d = 2$, note that for $i = 1, 2$, there must exist $j$ such that $C_j \subset St(u_i)$, otherwise $St(u_i)$ would separate $F(\Gamma)$. Moreover, if $C_j \subset St(u_i)$, then $u_i \in C_j$. So we can assume with out loss of generality that $C_1 \subset St(u_1)$ and $C_2 \subset St(u_2)$, which implies $F(\Gamma) = St(u_1) \cup St(u_2)$, contradiction again (Lemma 4.14). Thus case (2) is impossible and (5.4) is true.

Let $\{F_\lambda\}_{\lambda \in \Lambda}$ be the collection of maximal standard flats in $X(\Gamma)$. Then $X(\Gamma) = \bigcup_{\lambda \in \Lambda} F_\lambda$. For each $\lambda$, let $F'_\lambda$ be the unique maximal standard flat such that $q(F'_\lambda) \equiv
$F'_{\lambda}$. Then

$$X(\Gamma') \equiv \cup_{\lambda \in \Lambda} F'_{\lambda}. \tag{5.6}$$

Pick arbitrary hyperplane $h \subset X(\Gamma')$, we claim $h \cap (\cup_{\lambda \in \Lambda} F'_{\lambda}) \neq \emptyset$, otherwise $\cup_{\lambda \in \Lambda} F'_{\lambda}$ would stay on one side of the hyperplane since it is a connected set by (5.4), and this contradicts (5.6). Pick any standard geodesic $r \subset X(\Gamma')$ and let $h_r$ be a hyperplane dual to $r$. Then there exists $\lambda \in \Lambda$ such that $F'_{\lambda} \cap h_r \neq \emptyset$, it follows that $r \subset F'_{\lambda}$. So $\Delta(r) \in \Delta(F'_{\lambda}) \subset q_2(\mathcal{P}(\Gamma))$, which implies $q_2$ is surjective, hence is a simplicial isomorphism.

5.2. **Coherent ordering and coherent labelling.** We assume $Out(G(\Gamma))$ is finite in this section. If $q : G(\Gamma) \to G(\Gamma')$ be a quasi-isometry, then $G(\Gamma')$ has a quasi-action on $G(\Gamma)$, which induces a group homomorphism:

$$H : G(\Gamma') \to QI(G(\Gamma))$$

On the other hand, since $G(\Gamma)$ acts by isometry on $X(\Gamma)$, we can identify $G(\Gamma)$ as a subgroup of $QI(G(\Gamma))$ (we embed $G(\Gamma)$ into $Isom(G(\Gamma), d_w)$ and embed $Isom(G(\Gamma), d_w)$ into $QI(G(\Gamma))$ by Corollary 4.20). In the light of Mostow rigidity, we could ask the following question:

Does there exists $g \in QI(G(\Gamma))$ such that $q \cdot H(G(\Gamma')) \cdot g^{-1} \subset G(\Gamma)$?

Recall that we have picked an identification between $G(\Gamma)$ and the 0-skeleton of $X(\Gamma)$. The vertices of $F(\Gamma)$ are labelled by elements in a standard generating sets $S$ of $G(\Gamma)$. This induces a $G(\Gamma)$-invariant labelling of vertices in $\mathcal{P}(\Gamma)$ and a $G(\Gamma)$-invariant directed labelling of edges in $X(\Gamma)$.

Let $\{l_\lambda\}_{\lambda \in \Lambda}$ be the collection of standard geodesics in $X(\Gamma)$ and let $V_\lambda = v(l_\lambda)$. A coherent ordering of $G(\Gamma)$ is obtained by assigning a total order $\leq_\lambda$ to $V_\lambda$ for each $\lambda \in \Lambda$ such that if $l_{\lambda_1}$ and $l_{\lambda_2}$ are parallel, then the map $p : V_{\lambda_1} \to V_{\lambda_2}$ induced by parallelism is order preserving. The $G(\Gamma)$-invariant directed labelling of edges in $X(\Gamma)$ induces a unique coherent ordering of $G(\Gamma)$, which will be denoted by $\Omega$.

Recall that for any vertex $v \in X(\Gamma)$, we have a label-preserving simplicial embedding $i_v : F(\Gamma) \to \mathcal{P}(\Gamma)$ by considering the standard geodesics passing through $v$. A coherent labelling of $G(\Gamma)$ is a simplicial map $a : \mathcal{P}(\Gamma) \to F(\Gamma)$ such that $a \circ i_v : F(\Gamma) \to F(\Gamma)$ is a simplicial isomorphism for every vertex $v \in X(\Gamma)$. The label-preserving projection $L : \mathcal{P}(\Gamma) \to F(\Gamma)$ gives rise to a coherent labelling of $G(\Gamma)$.

We have the following alternative characterization of elements in $Isom(G(\Gamma), d_r)$.

**Lemma 5.7.** There is a 1-1 corresponding between elements in $Isom(G(\Gamma), d_r)$ and the following set of information:

1. A point $v \in G(\Gamma)$.
2. A coherent ordering of $G(\Gamma)$.
3. A coherent labelling of $G(\Gamma)$.

**Proof.** Pick $\phi \in Isom(G(\Gamma), d_r)$ and let $\varphi = h(\phi) : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$, where $h$ is the monomorphism in Remark 4.18. Then $\varphi^* L = \varphi \circ L : \mathcal{P}(\Gamma) \to F(\Gamma)$ is a coherent labelling of $G(\Gamma)$. Pick standard geodesic $l_1 \subset X(\Gamma)$, then the parallel set $P_{l_1}$ admits a splitting $P_{l_1} = l_1 \times l_1^\perp$. Since $\phi$ maps vertices in a standard flat bijectively to vertices in a standard flat, there exists a standard geodesic $l_2 \subset X(\Gamma)$ such that $\phi(l_1) = l_2$ and $\phi(P_{l_1}) = P_{l_2}$, moreover, $\phi$ respects the product structure.
exists a unique standard geodesic $L$ for pick word parallel, thus $L$.

Conversely, given point $v \in G(\Gamma)$, a coherent ordering $\Omega'$ and a coherent labelling $L'$, we can construct a map $\phi$ as follows. Set $\phi(e) = v$, for $u \in G(\Gamma)$, pick a word $w_u = a_1a_2 \cdots a_n$ representing $u$, let $u_i$ be the point in $G(\Gamma)$ represented by the word $a_1a_2 \cdots a_i$ for $1 \leq i \leq n$ and let $u_0 = e$. We define $q_i = \phi(a_1a_2 \cdots a_i) \in G(\Gamma)$ inductively as follows: set $q_0 = v$ and suppose $q_{i-1}$ is already defined. Denote the standard geodesic containing $u_{i-1}$ and $u_i$ by $l_i$. Let $v_i = L'((\Delta(l_i)))$ which is a vertex of $\Gamma$ and let $l'_i = K(q_{i-1}, v_i)$. Denote the vertex set of $l_i$ with the order from $\Omega'$ by $(v(l_i), \leq_{\Omega'})$. Suppose $k : (v(l_i), \leq_{\Omega'}) \to (v(l'_i), \leq_{\Omega'})$ is the unique order preserving bijection such that $k(u_{i-1}) = q_{i-1}$, and define $q_i = k(u_i)$.

We claim for any other word $w'_{u}$ representing $u$, $\phi(w_u) = \phi(w_u')$, hence there is a well-defined map $\phi : G(\Gamma) \to G(\Gamma')$. To see this, recall that one can obtain $w_u$ from $w_u'$ by performing the following two basic moves:

1. $w_1aa^{-1}w_2 \to w_1w_2$.
2. $w_1abw_2 \to w_1ba\cdot w_2$ when $a$ and $b$ commute.

It is clear that $\phi(w_1aa^{-1}w_2) = \phi(w_1w_2)$. For the second move, let $u_{i-1}, u_i, u'_{i-1}, u'_{i-1}$ be points in $G(\Gamma)$ represented by $w_1, w_1a, w_1b$ and $w_1ab = w_1ba$ respectively. Define $q_{i-1} = \phi(w_1), q_i = \phi(w_1a), q'_i = \phi(w_1b), q_{i+1} = \phi(w_1ab)$ and $q'_{i+1} = \phi(w_1ba)$. Since $L'$ is a coherent labelling, $\angle_{q_i}(q_{i+1}, q_{i-1}) = \angle_{q'_i}(q_{i+1}, q'_i) = \pi/2$, moreover, the standard geodesic containing $q_i$ and $q_{i+1}$ is parallel to the standard geodesic containing $q_{i-1}$ and $q'_i$. Since $\Omega'$ is a coherent ordering, $d(q_i, q_{i+1}) = d(q'_i, q'_{i+1})$, thus $q_iq_{i+1}$ and $q'_iq'_{i+1}$ are parallel. Similarly, $q_{i-1}q'_i$ and $q'_{i+1}q'_{i+1}$ are parallel, thus $q_{i-1}q'_{i+1}$.

Now we define another map $\phi'$, which serves as the inverse of $\phi$. Set $\phi'(v) = e$ and pick word $w = a_1a_2 \cdots a_n$. Let $r_i$ be the point in $G(\Gamma)$ represented by $va_1a_2 \cdots a_i$ for $1 \leq i \leq n$ and $r_0 = v$. We define $p_i = \phi'(va_1a_2 \cdots a_i)$ inductively as follows: put $p_0 = e$ and suppose $p_{i-1}$ is already defined. Since $L'$ is a coherent labelling, there exists a unique standard geodesic $l_i$ containing $p_{i-1}$ such that $L'((\Delta(l_i)))$ and the edge $r_{i-1}r_i$ share the same label. Let $l'_i$ be the unique standard geodesic containing $r_{i-1}$ and $r_i$ and let $k' : (v(l'_i), \leq_{\Omega'}) \to (v(l_i), \leq_{\Omega'})$ be the unique order preserving bijection such that $k'(r_{i-1}) = p_{i-1}$. Put $p_i = k'(r_i)$. By a similar argument as above, $\phi' : G(\Gamma) \to G(\Gamma')$ is well-defined. It is not hard to deduce the following properties from our construction:

1. $\phi' \circ \phi = \phi \circ \phi' = \text{Id}$.
2. $d_r(\phi(v_1), \phi(v_2)) \leq d_r(v_1,v_2)$ and $d_r(\phi'(v_1), \phi'(v_2)) \leq d_r(v_1,v_2)$ for any vertices $v_1, v_2 \in G(\Gamma)$.
3. If $L' = L$ and $\Omega' = \Omega$, then $\phi$ is a left translation, if in addition $v = e$, then $\phi = \text{Id}$.

It follows from (1) and (2) that $\phi \in \text{Isom}(G(\Gamma), d_r)$. Moreover, $v = \phi(e), L' = \phi^*L$ ($\varphi = h(\phi)$ where $h$ is the monomorphism in Remark 4.18) and $\Omega' = \phi^*\Omega$, thus we have established the required 1-1 correspondence.

Pick finite simplicial graph $\Gamma$ and $\Gamma'$ such that (1) $\text{Out}(G(\Gamma))$ is finite; (2) there exists a simplicial isomorphism $s : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma')$. By Lemma 4.10 $s$ induces a
map \( \phi : G(\Gamma) \to G(\Gamma') \). For every \( g' \in G(\Gamma') \), there is a left translation \( \tilde{\phi}_{g'} : G(\Gamma') \to G(\Gamma') \), which gives rise to a simplicial isomorphism \( \bar{s}_{g'} : P(\Gamma') \to P(\Gamma') \).

Let \( s_{g'} = s^{-1} \circ \bar{s}_{g'} \circ s \). Then \( s_{g'} \) gives rise to a map \( \phi_{g'} \in Isom(G(\Gamma), d_r) \) by Corollary 4.16, moreover, by Lemma 4.10

\[(5.8) \quad \tilde{\phi}_{g'} \circ \phi = \phi \circ \phi_{g'} \]

for any \( g' \in G(\Gamma') \). So \( G(\Gamma') \) acts on \( G(\Gamma) \) and we can define a homomorphism \( \Phi : G(\Gamma') \to Isom(G(\Gamma), d_r) \) by sending \( g' \) to \( \phi_{g'} \). \( \Phi \) is injective since each step in defining \( \Phi \) is injective.

**Lemma 5.9.** In the above setting, there exists an element \( \phi_1 \in Isom(G(\Gamma), d_r) \) such that it conjugates the image of \( \Phi \) to a finite index subgroup of \( G(\Gamma) \).

Here we identify \( G(\Gamma) \) as a subgroup of \( Isom(G(\Gamma), d_r) \) via the left action \( G(\Gamma) \curvearrowright G(\Gamma) \).

**Proof.** Pick a reference point \( q \in \text{Im} \phi \) and let \( K_q = (F(\Gamma'))_q \). Denote the points in \( \phi^{-1}(q) \) by \( \{p_\lambda\}_{\lambda \in \Lambda} \) and let \( K_{p_\lambda} = (F(\Gamma'))_{p_\lambda} \). Since \( \{\phi(K_{p_\lambda})\}_{\lambda \in \Lambda} \) are distinct subcomplexes of \( K_q \), \( \Lambda \) is a finite set.

Let \( L : P(\Gamma) \to F(\Gamma) \) and \( \Omega \) be the coherent labelling and coherent ordering induced by the \( (\Gamma)-\)invariant labelling of \( F(\Gamma) \), \( X(\Gamma) \) and \( P(\Gamma) \). We can obtain a coherent labelling \( L' : P(\Gamma') \to F(\Gamma') \) and a coherent ordering \( \Omega' \) for \( G(\Gamma') \) in a similar fashion. Thus

\[(5.10) \quad (\bar{s}_{g'})^* L' = L' \quad \text{and} \quad (\bar{\phi}_{g'})^* \Omega' = \Omega'. \]

Our goal is to find coherent labelling \( L_1 \) and coherent ordering \( \Omega_1 \) of \( G(\Gamma) \) such that \( (s_{g'})^* L_1 = L_1 \) and \( (\phi_{g'})^* \Omega_1 = \Omega_1 \) for any \( g' \in G(\Gamma') \).

Let \( i_q : F(\Gamma') \to G(\Gamma') \) be the canonical embedding and let

\[L_1 = L \circ s^{-1} \circ i_q \circ L' \circ s\]

be the simplicial map from \( P(\Gamma) \) to \( F(\Gamma) \). Pick arbitrary \( p \in G(\Gamma) \) and let \( i_p : F(\Gamma) \to P(\Gamma) \) be the canonical embedding. We need to show \( L_1 \circ i_p \) is a simplicial isomorphism. Let \( K_p = i_p(F(\Gamma)) \) and let \( g'_1 \in G(\Gamma') \) such that \( g'_1 \circ \phi(p) = q \). Then

\[i_q \circ L'|_{s(K_p)} = \bar{s}_{g'_1}|_{s(K_p)} \quad \text{and} \quad L_1 \circ i_p = L \circ s^{-1} \circ i_q \circ L' \circ s \circ i_p = L \circ s^{-1} \circ \bar{s}_{g'_1} \circ s \circ i_p = L \circ \bar{s}_{g'_1} \circ i_p, \]

which is a simplicial isomorphism by Lemma 4.10. It follows that \( L_1 \) is a coherent labelling, moreover,

\[(s_{g'})^* L_1 = (L \circ s^{-1} \circ a \circ L' \circ s) \circ (s^{-1} \circ \bar{s}_{g'} \circ s) = L \circ s^{-1} \circ a \circ L' \circ \bar{s}_{g'} \circ s = L \circ s^{-1} \circ a \circ L' \circ s = L_1 \]

for any \( g' \in G(\Gamma') \), here the third equality follows from (5.10). So \( L_1 \) is the required coherent labelling.

To simplify notation, we will write \( x <_{\Omega_1} y \) if \( x < y \) under the ordering \( \Omega \). We define \( \Omega_1 \) as follows: let \( p_1, p_2 \in G(\Gamma) \) be two distinct points in a standard geodesic line. If \( \phi(p_1) \neq \phi(p_2) \), then we set \( p_1 <_{\Omega_1} p_2 \) if \( \phi(p_1) <_{\Omega} \phi(p_2) \); if \( \phi(p_1) = \phi(p_2) \), then by (5.8), there exists a unique \( g' \in G(\Gamma') \) such that \( \phi_{g'}(p_1) \in \phi^{-1}(q) \) for \( i = 1, 2 \) and we set \( p_1 <_{\Omega_1} p_2 \) if \( \phi_{g'}(p_1) <_{\Omega} \phi_{g'}(p_2) \). It follows from (5.10), (5.8) and our construction that \( p_1 <_{\Omega_1} p_2 \) if \( \phi_{g'}(p_1) <_{\Omega_1} \phi_{g'}(p_2) \) for any \( p_1, p_2 \) in the same standard geodesic line and any \( g' \in G(\Gamma') \), thus \( (\phi_{g'})^* \Omega_1 = \Omega_1 \).
To verify $\Omega$ is coherent, pick parallel standard geodesic $l_1$ and $l_2$ in $X(\Gamma)$ and pick distinct vertices $p_{11}, p_{12} \in l$. Let $p_{21}, p_{22}$ be the corresponding vertices in $l_2$ via parallelism. We assume $p_{11} <_\Omega p_{12}$, it suffices to prove $p_{21} <_\Omega p_{22}$.

Case 1: If $\phi(p_{11}) \neq \phi(p_{12})$, recall that $l_1$ can be realized as finite intersection of maximal standard flats, so by Lemma 4.10 there exists standard geodesic line $l'_1 \in X(\Gamma')$ such that $\phi(v(l_1)) \subset v(l'_1)$ and $\phi(v(P_{l_1})) \subset v(P_{l'_1})$, moreover, $\phi$ respects the product structure of $P_{l_1}$ and $P_{l'_1}$. Thus $\phi(p_{11})\phi(p_{21})$ and $\phi(p_{21})\phi(p_{22})$ are the opposite sides of a flat rectangle in $X(\Gamma')$. Now $p_{21} <_\Omega p_{22}$ follows since $\Omega'$ is coherent.

Case 2: If $\phi(p_{11}) = \phi(p_{12}) \neq \phi(p_{21})$, we can assume without loss of generality that $\phi(p_{11}) = \phi(p_{12}) = q$ (since $(\phi_q)^*\Omega = \Omega$) and points $p_{11}, p_{21}$ stay in the same standard geodesic. For $i = 1, 2$, let $r_i$ be the standard geodesic passing $p_{1i}$ and $p_{2i}$. Take $r'_i \subset X(\Gamma')$ and $l'_i \subset X(\Gamma')$ to be the standard geodesics such that $\phi(v(r_i)) \subset v(r'_i)$ and $\phi(v(l'_i)) \subset v(l_i)$ respectively. Denote $q' = \phi(p_{11})$, since $\phi$ restricted on $v(P_{l_1})$ respects the product structure, $\phi(p_{21}) = \phi(p_{22}) = q'$ and $r'_i = r'_2$.

Let $\phi_g'$ be the left translation such that $\phi_{g'}(q') = q$. Since $q' \in r'_1$ and $q \in r'_1$, $\phi_g'$ is a translation along $r'_1$ and $\pi_g$ fixes every point in $St(\Delta(r'_1))$, hence $s_g'$ fixes every point in $s^{-1}(St(\Delta(r'_1))) = St(\Delta(r_1))$ and

$$
\phi_g'(r_i) = r_i
$$

for $i = 1, 2$. Let $l_3 = \phi_{g'}(l_2)$. Then $l_3$ is parallel to $l_1$ (or $l_2$). To see this, note that $\Delta(l_1) \subset St(\Delta(r'_1))$, hence $\Delta(l_1)$ is fixed by $s_g'$. Put $p_{3i} = \phi_g'(p_{2i})$ for $i = 1, 2$, then $p_{3i} \in r_i$ by (5.11), hence $p_{31}p_{32}$ and $p_{11}p_{12}$ are the opposite sides of a flat rectangle. Moreover $p_{3i} \in \phi^{-1}(q)$ for $i = 1, 2$ by (5.8), then $p_{31} <_\Omega p_{32}$ since $\Omega$ is coherent and $\Omega = \Omega_1$ while restricted on $\phi^{-1}(q)$. Now the $G(\Gamma')$-invariance of $\Omega_1$ implies $p_{21} <_\Omega p_{22}$.

Case 3: If $\phi(p_{11}) = \phi(p_{12}) = \phi(p_{21})$, we can assume they all equal to $q$. It follows that $\phi(p_{22}) = q$ since $\phi$ respects the product structure while restricted on $v(P_{l_1})$. Thus $p_{21} <_\Omega p_{22}$ by definition.

By Lemma 5.7 there exists $\phi_1 \in Isom(G(\Gamma), d_r)$ such that $\phi_1^*\Omega = \Omega$ and $s_1^*L = L_1 = (s_1 = h(\phi_1))$ where $h$ is the monomorphism in Remark 4.18. Thus

$$
(\phi_1 \circ \phi_{g'} \circ \phi_1^{-1})^*\Omega = (\phi_1^{-1})^* \circ (\phi_{g'})^* \circ (\phi_1^*\Omega) = (\phi_1^{-1})^* \circ (\phi_{g'})^* \Omega_1 = (\phi_1^{-1})^*\Omega_1 = \Omega
$$

for any $g' \in G(\Gamma')$. Similarly, $(s_1 \circ s_{g'} \circ s_1^{-1})^*L = L$ for any $g' \in G(\Gamma')$. Note that $s_1 \circ s_{g'} \circ s_1^{-1} = h(\phi_1 \circ \phi_{g'} \circ \phi_1^{-1})$, thus by Lemma 5.7 $G(\Gamma')$ acts on $G(\Gamma)$ by left translation via $g' \in G(\Gamma') \mapsto \phi_{g'} \circ \phi_1^{-1}$, which induces a monomorphism $G(\Gamma') \to G(\Gamma)$. Moreover, by (5.8) and the fact that $\phi^{-1}(q)$ is finite, this action has finite quotient, thus we can realize $G(\Gamma')$ as a finite index subgroup of $G(\Gamma)$.

The next result is already clear. However, we will give it “proof” to obtain extra insight.

**Theorem 5.12.** Given graph $\Gamma$ and $\Gamma'$ such that $Out(G(\Gamma))$ is finite and $G(\Gamma')$ is quasi-isometric to $G(\Gamma)$, then the induced quasi-action of $G(\Gamma')$ on $G(\Gamma)$ is quasi-isometrically conjugate to an action of $G(\Gamma')$ on $G(\Gamma)$ by left translation.

**Proof.** Let $f : G(\Gamma) \to G(\Gamma')$ be a quasi-isometry. By Corollary 5.6 $f$ induces simplicial isomorphism $s : P(\Gamma) \to P(\Gamma')$, and by Lemma 4.10 $s$ induces a map
\( \phi : G(\Gamma) \rightarrow G(\Gamma') \) such that \( d_w(f(x), \phi(x)) < D \) for any \( x \in G(\Gamma) \), thus \( \phi \) is a quasi-isometry. Let \( \phi_1 \) be the map in Lemma 5.9. It remains to show \( \phi_1 \in QI(G(\Gamma')) \).

Let \( \varphi = \phi \circ \phi_1^{-1} \), it suffices to show \( \varphi \) is a quasi-isometry. We will use the same notation as in the proof of Lemma 5.9.

We claim that if \( F = \cap_{i=1}^{h} F_i \) where each \( F_i \) is a maximal standard flat, then there exists a unique standard flat \( F' \subset G(\Gamma') \) such that \( \phi(v(F)) = v(F') \). To see this, let \( F_i' \) be maximal standard flat in \( X(\Gamma') \) such that \( \Delta(F_i') = s(\Delta(F_i)) \) for \( 1 \leq i \leq h \) and let \( F' = \cap_{i=1}^{h} F_i' \). Then it follows from Lemma 4.10 that \( \phi(v(F)) \subset v(F') \).

Recall that \( G(\Gamma') \) acts on \( G(\Gamma'), P(\Gamma'), G(\Gamma) \) and \( P(\Gamma) \). The stabilizer \( Stab(v(F')) \) fixes \( \Delta(F_i') \) for all \( i \), hence it fixes \( \Delta_i \) for all \( i \) and \( Stab(v(F')) \subset Stab(v(F)) \).

Since \( Stab(v(F')) \) acts on \( v(F') \) transitively, (5.8) implies \( \phi(v(F)) = v(F') \) and \( |\phi^{-1}(y) \cap F| = |\phi^{-1}(y') \cap F| \) for any \( y, y' \in v(F) \). It also follows that \( Stab(v(F)) \subset Stab(v(F')) \), thus \( Stab(v(F')) = Stab(v(F)) \).

Note that the above claim is also true for \( \varphi \) and any standard geodesic satisfies the assumption of the claim. Moreover, \( \varphi \) is surjective since \( \phi_1 \) is surjective by (5.8). Pick standard geodesics \( l \in X(\Gamma) \) and \( l' \in X(\Gamma') \) such that \( v(l') = \varphi(v(l)) \) and we identify \( v(l) \) and \( v(l') \) with \( \mathbb{Z} \) in an order-preserving way, then the above claim and the construction of \( \phi_1 \) imply that \( \varphi|_{v(l)} \) is of form

\[
\varphi(a) = \lfloor a/d \rfloor + r
\]

for some integers \( r \) and \( d (d \geq 1) \). In particular, \( \varphi \) can be extended to a simplicial map from \( C(\Gamma) \) to \( C(\Gamma') \).

Pick a combinatorial geodesic \( \omega \subset C(\Gamma) \) connecting vertices \( x \) and \( y \), we claim that \( \omega' = \phi(\omega) \) is also a geodesic in \( C(\Gamma') \) (it could be a point). Let \( \{v_i\}_{i=0}^n \) be vertices in \( \omega \) such that for \( 0 \leq i \leq n-1 \), \( [v_i, v_{i+1}] \) is a maximal sub-segment of \( \omega \) that is contained in a standard geodesic \( (v_0 = x \text{ and } v_n = y) \). Denote the corresponding standard geodesic by \( l_i \). For \( 0 \leq i \leq n-1 \), let \( l_i' \subset X(\Gamma') \) be the standard geodesic such that \( v(l_i') = \varphi(v(l_i)) \) and \( \omega_i' = \phi([v_i, v_{i+1}]) \). Then \( \omega_i' \) is a (possibly degenerate) segment in \( l_i' \) by (5.13). Since \( \omega \) is a geodesic, none of two geodesics in \( \{l_i\}_{i=0}^{n-1} \) are parallel. Note that \( \varphi \) is induced by a simplicial isomorphism between \( P(\Gamma) \) and \( P(\Gamma') \), then the same property is true for the collection \( \{l_i'\}_{i=0}^{n-1} \), thus no hyperplane in \( X(\Gamma') \) could intersect \( \omega' \) at more than one point and \( \omega' \) is a combinatorial geodesic.

Let \( u_i = \varphi(v_i) \). Then \( d_w(u_i, u_{i+1}) \leq d_w(v_i, v_{i+1}) \) by (5.13), thus

\[
d_w(\varphi(x), \varphi(y)) = \sum_{i=0}^{n-1} d_w(u_i, u_{i+1}) \leq \sum_{i=0}^{n-1} d_w(v_i, v_{i+1}) = d_w(x, y)
\]

for any \( x, y \in G(\Gamma) \).

Pick \( p \in G(\Gamma') \) and let \( k = |\varphi^{-1}(p)| \), then \( k \) does not depend on \( p \) by (5.8). It follows that \( d_w(\varphi(x), \varphi(y)) \geq 1 \) whenever \( d_w(x, y) \geq k+1 \). For arbitrary pair \( x, y \), we can cut \( \omega \) into pieces of length \( k+1 \). Since \( \varphi(\omega) \) is a combinatorial geodesic,

\[
d_w(\varphi(x), \varphi(y)) \geq \frac{d_w(x, y)}{k+1} - 1.
\]

\[ \square \]

**Remark 5.14.** Note that we can extend \( \varphi \) to a cubical map from \( X(\Gamma) \) to \( X(\Gamma') \) such that \( \varphi|_{\Gamma} \) preserves the product structure for any standard geodesic \( l \subset X(\Gamma) \).
which implies that $X(\Gamma')$ can be obtained from $X(\Gamma)$ by collapsing the carries of a certain collection of hyperplanes in $X(\Gamma)$.

The next theorem follows from Corollary 5.6 and Lemma 5.9

**Theorem 5.15.** If $\Gamma$ and $\Gamma'$ are finite simplicial graphs such that $\text{Out}(G(\Gamma))$ is finite, then the following are equivalent:

1. $G(\Gamma')$ is quasi-isometric to $G(\Gamma)$.
2. $\mathcal{P}(\Gamma')$ is isomorphic to $\mathcal{P}(\Gamma)$ as simplicial complexes.
3. $G(\Gamma')$ is isomorphic to a subgroup of finite index in $G(\Gamma)$.

6. THE GEOMETRY OF FINITE INDEX RAAG SUBGROUP

6.1. Construct finite index RAAG subgroup. An right-angled Artin subgroup is a subgroup which is also a right-angled Artin group. In this section, we introduce a natural process to obtain finite index RAAG subgroups of an arbitrary RAAG.

**Lemma 6.1.** Let $X$ be a $\text{CAT}(0)$ cube complex, let $l \subset X$ be a geodesic in the 1-skeleton and let $\{h_i\}_{i \in \mathbb{Z}}$ be consecutive hyperplanes dual to $l$. Then

1. For every edge $e \subset X$, if $e \cap h_i = \emptyset$ for all $i$, then $\pi_i(e)$ is a vertex in $l$, if $e \cap h_i \neq \emptyset$ for some $i$, then $\pi_i(e)$ is an edge in $l$.
2. If $K$ is any subcomplex such that $e \cap h_i = \emptyset$ for all $i$, then $\pi_i(K)$ is a vertex in $l$, moreover, if $K$ stays between $h_i$ and $h_{i+1}$, then $\pi_i(K)$ is the vertex in $l$ that stays between $h_i$ and $h_{i+1}$.
3. For every interval $[a,b] \subset l$, $\pi^{-1}_i([a,b])$ is a convex set in $X$. In particular, if $x \in l$ is a vertex, then $\pi^{-1}_i(x)$ is a convex subcomplex of $X$.
4. If $K$ is a convex subcomplex such that $K \cap l \neq \emptyset$, then $\pi_i(K) = K \cap l$.

Recall that $\pi_l : X \to l$ is the $\text{CAT}(0)$ projection.

**Proof.** Here (1) and (3) follow from the fact that the hyperplane has a carrier, and (2) follows from (1). To see (4), it suffices to show for $i$ such that $h_i \cap l \neq \emptyset$ and $h_i \cap K \neq \emptyset$, we have $e_i \subset K (e_i$ is the edge in $l$ dual to $h_i$). Let $N_{h_i}$ be the carrier of $h_i$. By Lemma 2.2 $d(x, N_{h_i} \cap K) \equiv c$ for any $x \in e_i$. Moreover, $d(x, N_{h_i} \cap K) = d(x, K)$ for $x$ in the interior of $e_i$, so we must have $c = 0$, otherwise the convexity of $d(-, K)$ would imply $K \cap l = \emptyset$. \hfill \qed

**Lemma 6.2.** Let $l \subset X(\Gamma)$ be a standard geodesic. Then there is a canonical map $\pi_{\Delta(l)} : v(\mathcal{P}(\Gamma) \setminus St(\Delta(l))) \to v(l)$ such that if $v_1$ and $v_2$ are in the same connected component of $\mathcal{P}(\Gamma) \setminus St(\Delta(l))$, then $\pi_{\Delta(l)}(v_1) = \pi_{\Delta(l)}(v_2)$.

**Proof.** Let $\pi_l : X(\Gamma) \to l$ be the $\text{CAT}(0)$ projection and $l_1 \subset X(\Gamma)$ be a standard geodesic such that $d(\Delta(l_1), \Delta(l)) \geq 2$. Then $\pi_l(l_1)$ is a vertex in $l$ by Lemma 3.1 and Corollary 5.2. Moreover, we claim $\pi_l(l_1) = \pi_l(l_2)$ if $l_2$ is a standard geodesic parallel to $l_1$. It suffices to prove the case when there is a unique hyperplane $h$ separating $l_1$ from $l_2$. Note that $d(\Delta(l_1), \Delta(l)) \geq 2$ yields $h \cap l = \emptyset$, so $l_1$ and $l_2$ are pinched together by two hyperplanes dual to $l$, then the claim follows from Lemma 6.1. Thus $\pi_l$ induces a well-defined map $\pi_{\Delta(l)} : v(\mathcal{P}(\Gamma) \setminus St(\Delta(l))) \to v(l)$. If $\Delta(l_1)$ and $\Delta(l_2)$ are connected by an edge, then there exists standard geodesic $l'_1$ and $l'_2$ such that $l'_1 \cap l'_2 \neq \emptyset$ and $l'_i$ is parallel to $l_i$ for $i = 1, 2$, thus $\pi_l(l_1) = \pi_l(l'_1) = \pi_l(l'_2) = \pi_l(l_2)$ and $\pi_{\Delta(l)}(\Delta(l_1)) = \pi_{\Delta(l)}(\Delta(l_2))$. \hfill \qed
Pick a standard generating set $S$ of $G(\Gamma)$ and let $C(\Gamma, S)$ be the Cayley graph. We identify $G(\Gamma)$ as a subset of $C(\Gamma, S)$ and attach higher dimensional cubes to $C(\Gamma, S)$ to obtain a $\text{CAT}(0)$ cube complex $X(\Gamma, S)$, which is basically the universal covering of the Salvetti complex. Here we would like to think of $G(\Gamma)$ as a fixed set and $C(\Gamma, S)$, $X(\Gamma, S)$ as a particular way determined by $S$ to connecting points in $G(\Gamma)$, so we write $S$ explicitly. We will choose a $G(\Gamma)$-equivariant orientation for edges in $X(\Gamma, S)$ as before.

A $S$-flat (or a $S$-geodesic) in $G(\Gamma)$ is defined to be the vertex set of a standard flat (or geodesic) in $X(\Gamma, S)$. We define $\mathcal{P}(\Gamma, S)$ as before such that its vertices correspond to coarse equivalence classes of $S$-geodesics.

We define an isometric embedding $I : G(\Gamma) \rightarrow l^1(\nu(\mathcal{P}(\Gamma)))$ which depends on $S$ and the orientations of edges in $X(\Gamma, S)$. Pick standard geodesic $l \in X(\Gamma, S)$ and let $\pi_l : X(\Gamma, S) \rightarrow l$ be the $\text{CAT}(0)$ projection. We identify $\nu(l)$ with $\mathbb{Z}^\Delta(l)$ in an orientation preserving way such that $\pi_l(e) = 0$ ($e$ is the identity element in $G(\Gamma)$), then $\pi_l$ induces a coordinate function $I_{\Delta(l)} : G(\Gamma) \rightarrow \mathbb{Z}^\Delta(l)$. If we change $l$ to a standard geodesic $l_1$ parallel to $l$, then $I_{\Delta(l)}$ and $I_{\Delta(l_1)}$ are identical by Lemma 6.1, thus for every vertex $v \in \mathcal{P}(\Gamma)$, there is a well-defined coordinate function $l_v : G(\Gamma) \rightarrow \mathbb{Z}^v$, which induces $I : G(\Gamma) \rightarrow \mathbb{Z}^{(\nu(\mathcal{P}(\Gamma)))}$.

$I$ is an embedding since every two point in $G(\Gamma)$ is separated by some hyperplane $I(G(\Gamma)) \subset l^1(\nu(\mathcal{P}(\Gamma)))$ since for any $g \in G(\Gamma)$, there are only finite many hyperplanes separating $e$ and $g$. $I$ naturally extends to a map $I : X(\Gamma, S) \rightarrow l^1(\nu(\mathcal{P}(\Gamma)))$ and it maps combinatorial geodesic to geodesic by the argument in Theorem 5.12, thus $I$ is an isometric embedding with respect to the $l^1$ metric in $X(\Gamma, S)$. We say a convex subcomplex $K \subset X(\Gamma, S)$ is non-negative is each point in $K$ has non-negative coordinates (this notion depends on the orientation of edges in $X(\Gamma, S)$). Let $CN(\Gamma, S)$ be the collection of convex, non-negative subcomplexes of $X(\Gamma, S)$ that contain the identity.

For any $K \in CN(\Gamma, S)$, we find a maximal collection of standard geodesics $\{c_i\}_{i=1}^r$ such that $c_i \cap K \neq \emptyset$ for all $i$ and $\Delta(c_i) \neq \Delta(c_j)$ for any $i \neq j$. Let $g_i \in S$ be the label of edges in $c_i$ and let $\alpha_i = \pi_{c_i}(e)$. Put $n_i = |\nu(K \cap c_i)|$ and $v_i = \alpha_i g_i^{-1} \alpha_i^{-1}$. Let $G'$ be the subgroup generated by $\{v_i\}_{i=1}^r$. It follows from the convexity of $K$ that if standard geodesic $c$ is parallel to $c_i$ and $c \cap K \neq \emptyset$, $|\nu(K \cap c_i)| = |\nu(K \cap c)|$. Thus $\{v_i\}_{i=1}^r$ and $G'$ do not depend on the choice of $c_i$.

$G'$ is a finite index subgroup of $G(\Gamma)$. We prove this by showing $G' \cdot v(K) = G(\Gamma)$. Pick word $\alpha \in G(\Gamma)$ and assume $\alpha \in G' \cdot v(K)$ when $d_{e}(\alpha, e) \leq k - 1$. If $d_{e}(\alpha, e) = k$, then there exists $\beta \in G(\Gamma)$ such that $d_{e}(e, \beta) = k - 1$ and $d_{e}(\beta, \alpha) = 1$. Let $\beta = \beta_1 \beta_2$ for $\beta_1 \in G'$ and $\beta_2 \in v(K)$. Then $d_{e}(\beta_2, \beta_1^{-1} \alpha) = 1$. Suppose $c$ is the standard geodesic containing $\beta_2$ and $\beta_1^{-1} \alpha$. Then there exists $i$ such that $c_i$ and $c$ are parallel. Note that $P_c \cap K$ is a convex set in the parallel set $P_c$, hence respects the natural splitting $P_c = c \times c^\perp$, moreover, the left action of $v_i$ translates the factor by $n_i$ units and fixes the other factor. Thus there exists $d \in \mathbb{Z}$ and $\beta_2 \in K \cap c$ such that $v_i^d \beta_2 = \beta_1^{-1} \alpha$, which implies $\alpha = \beta_1 v_i^d \beta_2 \in G' \cdot v(K)$.

$G'$ is a right-angled Artin group. Let $\Gamma'$ be the full subgraph of $\mathcal{P}(\Gamma)$ spanned by points $\{\Delta(c_i)\}_{i=1}^r$. Then there is a natural homomorphism $G(\Gamma') \rightarrow G'$, which is actually an isomorphism. To see this, we follow the strategy in [Kob12a], where the following version ping-pong lemma for right-angled Artin group was used:

**Theorem 6.3** (Theorem 4.1 of [Kob12a]). Let $G = G(\Gamma)$ and let $X$ be a set with a $G$-action. Suppose the following hold:
(1) There exists subsets $X_i \subset X$ for each vertex $v_i$ of $\Gamma$ whose union is properly contained in $X$.

(2) For each nonzero $k \in \mathbb{Z}$ and vertices $v_i, v_j$ joined by en edge, $v_i^k(X_j) \subset X_j$.

(3) For each nonzero $k \in \mathbb{Z}$ and vertices $v_i, v_j$ not joined by en edge, $v_i^k(X_j) \subset X_i$.

(4) There exists $x_0 \in X \setminus \bigcup_{i \in V} X_i$ ($V$ is the vertex set of $\Gamma$) such that for each nonzero $k \in \mathbb{Z}$, $v_i^k(x_0) \in X_i$.

Then the $G$-action is faithful.

Now we will apply the above theorem with $X = X(\Gamma, S)$ and $G = G(\Gamma')$. For $1 \leq i \leq s$, we identify $c_i$ and $\mathbb{R}$ in an orientation preserving way such that $\pi_{c_i}(e)$ corresponds to $0 \in \mathbb{R}$. Define $X_i^+ = \mathbb{R}^{-1}([n_i - 1/2, \infty))$, $X_i^- = \mathbb{R}^{-1}((\infty, -1/2])$ and $X_i = X_i^+ \cup X_i^-$. It clear that $e \notin X_i$ for all $i$, so (1) of Theorem 6.3 is true.

Each $v_i = \alpha_i g_i^{n_i} \alpha_i^{-1}$ translates $c_i$ by $n_i$ units, so (4) is also true with $x_0 = e$.

If $\Delta(c_i)$ and $\Delta(c_j)$ are connected by en edge in $P(\Gamma)$, then $v_i$ stabilizes every hyperplane dual to $v_j$, thus $v_i^k(X_j) = X_j$ and (2) is true. If

$$d(\Delta(c_i), \Delta(c_j)) \geq 2,$$

then $\pi_{c_j}(c_i)$ is a point. Lemma 6.1 and $c_i \cap K \neq \emptyset$ yield that $\pi_{c_j}(c_i) \subset \pi_{c_j}(K) = c_j \cap K = [0, n_j - 1]$, thus

$$c_i \cap X_j = \emptyset,$$

similarly $c_i \cap X_j = \emptyset$. Let $h = \mathbb{R}^{-1}(-1/2)$ be the boundary of $X_j^-$ and let $N_h$ be the carrier of $h$. Then (6.4) implies that $h$ has empty intersection with any hyperplane dual to $c_i$, so is $N_h$. It follows from Lemma 6.1 that $\pi_{c_i}(h) = \pi_{c_i}(N_h) = p$ is a vertex in $c_i$. If $h_1 = \mathbb{R}^{-1}(p - 1/2)$ and $h_2 = \mathbb{R}^{-1}(p + 1/2)$ are two hyperplanes that pinch $p$, then $h \cap h_k = \emptyset$ for $k = 1, 2$. This and (6.5) yield $X_j^- \cap h_k = \emptyset$, hence $\pi_{c_i}(X_j^-) = p$ by Lemma 6.1. Similarly $\pi_{c_i}(X_j^+) = p$, so $p = \pi_{c_i}(X_j) = \pi_{c_i}(c_j) \subset \pi_{c_i}(K) = c_i \cap K = [0, n_i - 1]$. Note that $(\pi_{c_i} \circ v_i^k)(X_j) = (v_i^k \circ \pi_{c_i})(X_j) = v_i^k(p) = p + k n_i$, so $v_i^k(X_j) \subset X_i$ for $k \neq 0$.

Note that there does not exist hyperplane which separates $K$ from $X_i^+$ (or $X_i^-$, $X_j^+$, $X_j^-$). Given all these facts, we must have

$$X_i \cap X_j = \emptyset,$$

here is a picture ($c_i$ and $c_j$ may not intersect):

\[ \text{The above discussion yield a well-defined map } \]

$$\Theta_S : \text{CN}(\Gamma, S) \to \{\text{Finite index RAAG subgroups of } G(\Gamma)\}. $$
6.2. Rigidity of RAAG subgroup. From now on, we will assume $G(\Gamma)$ is a subgroup of finite index in $G(\Gamma)$ and $Out(G(\Gamma))$ is finite. We want to obtain a detailed description of $G(\Gamma')$.

Recall that $Out(G(\Gamma))$ is finite and $Out(G(\Gamma'))$ is transvection free (Corollary 5.6), so any two standard generating sets of $G(\Gamma)$ (or $G(\Gamma')$) differ by a sequence of conjugations or partial conjugations. Then given any two standard generating sets $S$ and $S_1$ for $G(\Gamma)$, there is a canonical way to identify $P(\Gamma, S)$ and $P(\Gamma, S_1)$ (every $S$-geodesic is Hausdorff close to a $S_1$-geodesic). Thus we will write $P(\Gamma)$ and $P(\Gamma')$ and omit the generating set.

**Lemma 6.7.** Let $\phi, s$ be as in the discussion before Lemma 5.9 and let $l$ (or $l'$) be standard geodesic in $X(\Gamma)$ (or $X(\Gamma')$) such that $\phi(v(l)) = v(l')$. Then $\phi \circ \pi_{\Delta(l)} = \pi_{\Delta(l')} \circ s$.

**Proof.** Pick standard geodesics $r \in X(\Gamma)$ and $r' \in X(\Gamma')$ such that $\phi(v(r)) = v(r')$, then $s(\Delta(r)) = \Delta(r')$ by Lemma 4.10 (recall that $r$ is the intersection of maximal flats). Therefore, by the definition of $\pi_{\Delta(l)}$, it suffices to show $\phi \circ \pi_l(x) = \pi_{l'} \circ \phi(x)$ for any vertex $x \in X(\Gamma)$. Let $y$ be a vertex such that $y \notin l$ and let $x = \pi_l(y)$. By Lemma 6.4 we can approximate $\pi_l$ by a combinatorial geodesic $\omega$ in the $l$-skeleton of $\pi^{-1}_l(y)$, then no hyperplane could intersect both $l$ and $\omega$. Let $\{v_i\}_{i=0}^n$ and $\{l_i\}_{i=0}^{n-3}$ be as in the proof of Theorem 5.12. Then $\Delta(l) \neq \Delta(l_i)$ for all $i$. Let $u_i = \phi(v_i)$ and let $l_i'$ be standard geodesic such that $\phi(v(l_i)) = v(l_i')$. Then $\pi_{l_i'}(u_i) = \pi_{l_i'}(u_j)$ for all $1 \leq i, j \leq n$.

The left action $G(\Gamma) \acts G(\Gamma)$ induces $G(\Gamma') \acts G(\Gamma)$ and $G(\Gamma') \acts X(\Gamma, S)$. By choosing a standard generating set $S'$ of $G(\Gamma')$, we have left action $G(\Gamma') \acts X(\Gamma', S')$. For $h \in G(\Gamma')$, we use $\phi_h, \bar{\phi}_h, s_h$ and $\bar{s}_h$ to denote the action of $h$ on $G(\Gamma), G(\Gamma'), P(\Gamma)$ and $P(\Gamma')$ respectively. Pick a $G(\Gamma')$-equivariant quasi-automorphism $q : X(\Gamma, S) \to X(\Gamma', S')$ such that $q|_{G(\Gamma')} = \text{Id}$. By Corollary 5.6 and Lemma 4.10, $q$ induces surjective $G(\Gamma')$-equivariant maps $\phi : G(\Gamma) \to G(\Gamma')$ and $s : P(\Gamma) \to P(\Gamma')$. Note that $\phi$ depends on the choice of generating set $S$ and $S'$, and this flexibility comes from the automorphism group of $G(\Gamma)$ and $G(\Gamma')$. We want to choose a “nice” generating set $S'$ such that $\phi$ behaves like $\varphi$ in Theorem 5.12.

Without loss of generality, we can assume $\phi(e) = e$ ($e$ is the identity element). To see this, assume $\phi(e) = a \neq e$, we claim if we change the generating set from $S'$ to $aS'a^{-1}$, then the resulting $\bar{\phi}$ will satisfy our requirement. By the construction of $\phi$, it suffices to show for any maximal $S'$-flat $F_1'$ such that $a \in F_1'$, there exists a maximal $aS'a^{-1}$-flat $F'$ such that $e \in F'$ and $d_H(F_1', F) < \infty$. Let us assume $F_1' = \{ag^k\}_{k \in \mathbb{Z}}$ for some $g \in S'$. Then $F' = \{(aga^{-1})^k\}_{k \in \mathbb{Z}}$ would satisfy the required condition. We can prove the general case in a similar way.

Pick a standard geodesic $l \subset X(\Gamma, S)$, we want to flip the order of points of $l$ in a $G(\Gamma')$-equivariant way such that (5.13) is true. Choose a order preserving identification of $\phi^{-1}_l(v(p)) \cap v(l)$ for $p \in v(l)$. By the claim in Theorem 5.12 $d$ does not depend on $p$ and $Stab(v(l))$ acts on $v(l)$ in the same way as $d\mathbb{Z}$ acts on $\mathbb{Z}$. We will also write $\chi(l) = d$. If $\overline{l}$ and $l$ are parallel, then $\chi(l) = \chi(\overline{l})$, thus $\chi : P(\Gamma) \to \mathbb{Z}$ is well-defined. Since $\chi(l)$ only depends on how $Stab(v(l))$ acts on $v(l)$, $\chi$ does not depend on the generating set $S'$. However, for any choice of $S'$, $\chi$ descends to $\chi : S' \to \mathbb{Z}$ by the $G(\Gamma')$-equivariance of $\phi$. 


Let $\phi(0) = a$. Then $\text{Stab}(v(l))$ is generated by $aha^{-1}$ for some $h \in S'$. By above discussion, we can assume $a = e$. Let $S' = \{h_\lambda\}_{\lambda \in \Lambda}$. For each $h_\lambda \in S'$, we associated an integer $n_\lambda$ as follows. If $h_\lambda h = hh_\lambda$, we set $n_\lambda = 0$. If $h_\lambda h \neq hh_\lambda$, let $l'_\lambda \subset X(\Gamma', S')$ be the standard geodesic that contains all powers of $h'_\lambda$ and let $b_\lambda = \pi_{\Delta(l')} \circ s^{-1}(\Delta(l'_\lambda)) \circ \pi_{\Delta(l')}$ is the map in Lemma 6.2. Then $n_\lambda$ is defined to be the unique integer such that $b_\lambda + n_\lambda d \in [0, d-1]$. Define $f : S' \to G(\Gamma') \to S''$ by sending $h_\lambda$ to $h^{n_\lambda}h_\lambda$; then $f$ induces an automorphism of $G(\Gamma')$ and $S'' = \{f(h_\lambda)\}_{\lambda \in \Lambda}$ is also a standard generating set. Indeed, if $\Delta(l'_1)$ and $\Delta(l'_2)$ stay in the same connected component of $P(\Gamma') \setminus \text{St}(\Delta(l'))$, then $b_\lambda = b_{\lambda'}$ by Lemma 6.2 hence $n_{\lambda_1} = n_{\lambda_2}$. It follows that $f$ can be realized as a composition of partial conjugations.

For any $i \in [0, d-1]$, there exists $\lambda$ such that $b_\lambda + n_\lambda d = i$. By Lemma 6.7 $\phi(b_\lambda) = e$, hence $\phi(i) = h^{n_\lambda}$. Let $l_i$ be a standard geodesic such that $b_\lambda \in l_i$ and $d(\Delta(l_i), \Delta(l)) \geq 2$. Then there exists $h_\lambda' \in S'$ with $b_\lambda' = b_\lambda$ such that $\phi(v(l_i)) = \{h_\lambda'^k\}_{k \in \mathbb{Z}}$. Then $(\phi_{\lambda'})^{n_\lambda}(v(l_i))$ is a $S'$-geodesic passing through $i$ and $(\phi \circ (\phi_{\lambda'})^{n_\lambda})(v(l_i)) = ((\phi_{\lambda'})^{n_\lambda} \circ \phi)(v(l_i)) = \{h^{n_\lambda}h_\lambda'^k\}_{k \in \mathbb{Z}}$. Note that

$$d_H(h^{n_\lambda}h_\lambda'^k, \{(f(h_{\lambda'}))^k\}_{k \in \mathbb{Z}}) < \infty$$

Now we replace $S'$ by $S''$ in the definition of $\phi$ and denote the new map by $\phi_1$, then $\phi_1(0) = e$ is still true, moreover, Lemma 6.7 imply $\phi_1(i) = e$ for $i \in [0, d-1]$. Since $\phi_1$ and $\phi_{\lambda'}$ satisfies 5.13 for any standard geodesic $l_i$ with $\Delta(l_i) \in \{s_{\lambda}(\Delta(l))\}_{\lambda \in G(\Gamma')}$.

We need to prove the above change of base process does not affect other geodesics in an essential way, namely, let $r \subset X(\Gamma, S)$ be a standard geodesic such that $\Delta(r) \notin \{s_{\lambda}(\Delta(l))\}_{\lambda \in G(\Gamma')}$. Then there exists $h_{\lambda_1}, h_{\lambda_2} \in S'$ such that $\Delta(\phi_{\lambda_1}(r)) = ((\phi_{\lambda_1})^{n_\lambda_1} \circ \phi)(r)$ is still true, moreover, (6.8) and Lemma 6.7 imply (6.9) is immediate. Note that for any $a \in r'_1$ and $b \in r'_2$, we have

$$b = a \cdot (f(h_{\lambda_1}))^{k_1} \cdot h^{n_{\lambda_1} - n_{\lambda_2}} \cdot (f(h_{\lambda_2}))^{k_2}$$

for some $k_1, k_2 \in \mathbb{Z}$, then (6.11) follows from (6.9) and the definition of $\pi_{\Delta(r''')}$. Similarly, one can prove the change of $\phi$ under $S' \to aS'a^{-1}$ also satisfies the statement in the beginning of this paragraph without the restriction on $r$.

Now we can apply the above procedure for finite many times to find appropriate generating set $S'$ such that the corresponding $\phi$ satisfies (5.13) when restricted on any standard geodesic. By the proof of Theorem 5.12 we can extend $\phi$ to a cubical map $\phi : X(\Gamma, S) \to X(\Gamma', S')$ such that combinatorial geodesic in $C(\Gamma, S)$ is mapped to combinatorial geodesic in $C(\Gamma', S')$, thus $\phi^{-1}(e)$ is a combinatorially convex
subcomplex (it is also compact since $\phi^{-1}(e)$ contains finite many vertices). Recall that combinatorial convexity in $l^1$-metric and convexity in CAT(0) metric are the same for subcomplexes in CAT(0) cube complex ([Hag07]), so every finite index subgroup $G(\Gamma') \subset G(\Gamma)$ gives rise to a compact convex subcomplex in $X(\Gamma, S)$.

Next we show this compact convex subcomplex can be assumed to be in $CN(\Gamma, S)$.

For $K \subset G(\Gamma)$, denote the union of all standard geodesics in $X(\Gamma, S)$ that have non-trivial intersection with $K$ by $K^*$. $K$ is $S$-convex iff $K$ is the vertex set of some convex subcomplex in $X(\Gamma, S)$.

Now we return to $\phi$ and assume $\phi^{-1}(y)$ is $S$-convex for any $y \in G(\Gamma')$ and $\phi(e) = e$. In order to make $\phi^{-1}(e)$ non-negative, we proceed as follows. Step 1, let $\{l_i\}_{i=1}^n$ be the collection of standard geodesics passing through $e$ and $A_1 = \{e\}$. Since $v(l_i)$ and $v(l_j)$ are in different $G(\Gamma')$-orbit for $i \neq j$, we can adjust $S'$ as above such that

$$I^{-1}_\Delta(l_i) \cap v(l_i) \subset \phi^{-1}(e)$$

for all $i$. Step 2, let $A_2 = A_1 \cap \phi^{-1}(e)$ and pick vertex $x \in A_2 \setminus A_1$ (if such $x$ does not exist, then our process terminate). Let $l$ be a standard geodesic such that $x \in l$. If $l$ is parallel to some $l_i$, then the $[6.12]$ with $l_i$ replaced by $l$ is automatically true without any modification on $S'$, because both $l$ and $\phi$ respect the product structure of $P_i$. If $l$ is not parallel to any $l_i$, then $I_{\Delta(l)}(x) = 0$. Moreover, $\Delta(l)$ is not in any $G(\Gamma')$-orbits of $\Delta(l_i)$, so we can modify $S'$ as before such that both $[6.12]$ and $I^{-1}_\Delta(l) \cap v(l) \subset \phi^{-1}(e)$ are true. We deal with other standard geodesics passing through $x$ and other points in $A_2 \setminus A_1$ in a similar way. Step 3, we repeat the previous process by induction. Since $|\phi^{-1}(e)|$ is finite and does not change after adjusting $S'$, this procedure must terminate after finitely many steps. Once it terminates, we must have already hit all points in $\phi^{-1}(e)$ since $\phi^{-1}(e)$ stays connected in every step. By construction, the resulting $\phi$ satisfies $\phi^{-1}(e)$ is non-negative and $e \in \phi^{-1}(e)$. Note that the set $\Lambda_i$ actually does not depend on $\phi$ from step $i - 1$, it only depends on $\chi : v(P(\Gamma)) \rightarrow \mathbb{Z}$, thus the above procedure produces a unique set in $G(\Gamma')$ which depends on $S$ and the subgroup $G(\Gamma')$. Then we have a well-defined map

$$\Xi_S : \{\text{Finite index RAAG subgroups of } G(\Gamma)\} \rightarrow CN(\Gamma, S)$$

Now we prove $\Theta_S \circ \Xi_S = \text{Id}$. Let $K = \Xi_S(G(\Gamma'))$ and denote the corresponding map and generating set by $\phi : G(\Gamma) \rightarrow G(\Gamma')$ and $S'$ respectively. We find a maximal collection of standard geodesics $\{c_i\}_{i=1}^n$ such that $c_i \cap K \neq \emptyset$ for all $i$ and $\Delta(c_i) \neq \Delta(c_j)$ for any $i \neq j$. Let $n_i = \chi(c_i)$ and let $g_i \in S$ be the label of edges in $c_i$. Suppose $\alpha_i = \pi_{c_i}(e)$ where $\pi_{c_i} : X(\Gamma, S) \rightarrow c_i$ is the CAT(0) projection. We claim $S' = \{\alpha_i g^n_i \alpha_i^{-1}\}_{i=1}^n$.

Pick $h \in S'$ and let $c_h \subset X(\Gamma, S')$ be the standard geodesic containing $e$ and $h$. Then there exists unique $i$ such that $\phi(h(c_i)) = c_h$. To see this, let $c$ be a standard geodesic in $X(\Gamma, S)$ such that $s(\Delta(c)) = \Delta(c_h)$. Then $\phi(v(c))$ and $c_h$ are parallel and there exists $u \in G(\Gamma')$ which sends $\phi(v(c))$ to $v(c_h)$, thus $\phi \circ \phi_u(v(c)) = v(c_h)$ by [5.5]. We choose $c_i$ to be the geodesic parallel to $\phi_u(v(c))$. For any standard geodesic $c_i'$ parallel to $c_i$, $\phi(c_i')$ is parallel to $c_h$, so $h \in Stab(v(\phi(c_i'))) = Stab(v(c_i'))$. It follows that $\phi_h$ stabilizes the parallel set $P_{c_i}$ and acts by translation along the $c_i$-direction. Note that $(I_{\Delta(c_i)} \circ \phi_h)(x) = I_{\Delta(c_i)}(x) + \chi(c_i)$ for any $x \in v(P_{c_i})$, so $h = \phi_h(e) = \alpha_i g^n_i \alpha_i^{-1}$ and the claim follows.
It remains to show $\Xi_S \circ \Theta_S = \text{Id}$, but this follows from the following more general result:

**Lemma 6.13.** Let $\Gamma$ be an arbitrary finite simplicial graph. Pick a standard generating set $S$ for $G(\Gamma)$ and $K \in \text{CN}(\Gamma, S)$. Let $G(\Gamma') = \Theta_S(K)$ and let $S'$ be the corresponding generating set. Suppose $q : G(\Gamma) \to G(\Gamma')$ is a $G(\Gamma')$-equivariant quasi-isometry such that $q|_{G(\Gamma')} = \text{Id}$. Then

1. $q$ induces a simplicial isomorphism $q_* : \mathcal{P}(\Gamma, S) \to \mathcal{P}(\Gamma', S')$.
2. $q_*$ induces a $G(\Gamma')$-equivariant retraction $r : G(\Gamma) \to G(\Gamma')$ such that $r$ sends every $S$-flat to a $S'$-flat.
3. $r^{-1}(e) = K$. In particular, the vertex set of $K$ is the strict fundamental domain for the left action $G(\Gamma') \curvearrowright G(\Gamma)$.

**Proof.** It suffices to prove the case when $\Gamma$ does not admit a nontrivial join decomposition and $\Gamma$ is not a point.

By the construction of $\Theta_S$, we know the $q$-image of any $S$-flat which intersects $K$ is Hausdorff close to a $S'$-flat which contains the identity. Moreover, if the $S$-flat is maximal, then the corresponding $S'$-flat is unique. Since $G(\Gamma') \cdot v(K) = G(\Gamma)$, so the equivariance of $q$ implies the $q$-image of every $S$-flat is Hausdorff close to a $S'$-flat. Since $q$ is a quasi-isometry, so images of $S$-geodesics are Hausdorff closed to parallel $S'$-geodesics, which induces $q_* : \mathcal{P}(\Gamma, S) \to \mathcal{P}(\Gamma', S')$. $q_*$ is injective since $q$ is a quasi-isometry and $q_*$ is surjective by the $G(\Gamma')$-equivariance.

Pick $x \in G(\Gamma)$, let $\{F_i\}_{i \in I}$ be the collection of maximal $S$-flat and let $F'_i$ be the unique maximal $S'$-flat such that $d_H(q(F_i), F'_i) < \infty$. Note that $\bigcap_{i \in I} F_i = x$ by our assumption on $\Gamma$. So $\bigcap_{i \in I} F'_i$ is either empty or one point. Note that if $x \in K$, then $\bigcap_{i \in I} F'_i = e$. The equivariant of $q_*$ implies for every $x$, $\bigcap_{i \in I} F'_i$ is a point, which is defined to be $r(x)$. It is clear that $v(K) \subset r^{-1}(e)$, but $|G(\Gamma) : G(\Gamma')| \leq |v(K)|$, so $v(K) = r^{-1}(e)$. It follows that $v(K)$ is the strict fundamental domain for the left action of $G(\Gamma')$ and $r$ is the $G(\Gamma')$-equivariant retraction which maps $v(K)$ to $e$.

By the construction of $\Theta_S$, $r$ sends every $S$-flat that intersects $K$ to a $S'$-flat passing through the identity element of $G(\Gamma')$, thus $r$ sends every $S$-flat to a $S'$-flat by the equivariance of $r$. It is easy to see $r$ extends to a cubical map $r : X(\Gamma, S) \to X(\Gamma', S')$ such that $r^{-1}(e) = K$. \qed

**Remark 6.14.** We can generalize some of the above results to infinite convex subcomplex. A convex subcomplex $K \subset X(\Gamma, S)$ is admissible iff for any standard geodesic $l$, the $\text{CAT}(0)$ projection $\pi_l(K)$ is either an interval or the whole $l$ (a ray is not allowed). Let $\{l_\lambda\}_{\lambda \in \Lambda}$ be a maximal collection of standard geodesics such that (1) $l_\lambda \cap K \neq \emptyset$; (2) $l_\lambda$ and $l_\lambda'$ are not parallel for $\lambda \neq \lambda'$; (3) $\pi_{l_\lambda}(K)$ is a finite interval. For each $l_\lambda$, we associated $\alpha_\lambda \in G(\Gamma)$ which translate along $l_\lambda$ with translation length $= 1 + \text{length}(\pi_{l_\lambda}(K))$ as before. Let $G_K$ be the subgroup generated by $S' = \{\alpha_\lambda\}_{\lambda \in \Lambda}$. If $K$ is admissible, we can prove $G_K \cdot v(K) = G(\Gamma)$ as before. Moreover, for any finite subset $S' \subset S'$, the subgroup $G_1$ generated by $S_1'$ is a right-angled Artin group, and $G_1 \hookrightarrow G_K$ is an isometric embedding with respect to the word metric. We can define $S'$-flat as before and view each vertex of $G_K$ as $0$-dimensional $S'$-flat.

$v(K)$ is a strict fundamental domain for the action $G_K \curvearrowright G(\Gamma)$. It suffices to show $\alpha(K) \cap K = \emptyset$ for each nontrivial $\alpha \in G_K$. We can assume there is a
right-angled Artin group $G_1$ such that $\alpha \in G_1 \subset G_K$. Let $\alpha = w_1 w_2 \cdots w_n$ be a canonical form of $\alpha$ (see Section 2.3 of [Cha07]), then

(1) Each $w_i$ belongs to an Abelian standard subgroup of $G_1$.

(2) For each $i$, let $w_i = r_{i,1}^{k_{i,1}} r_{i,2}^{k_{i,2}} \cdots r_{i,n_i}^{k_{i,n_i}}$ ($r_{i,j} \in S'$). Then for each $r_{i+1,j}$ ($1 \leq j \leq n_{i+1}$), there exists $r_{i,j'}$ which does not commute with $r_{i+1,j}$.

We associate each generator $r_{i,j}$ with a subset $X_{i,j} \subset X(\Gamma, S)$ as before and claim there exists $j$ with $1 \leq j \leq n_1$ such that $\alpha(K) \subset X_{1,j}$, then $\alpha(K) \cap K = \emptyset$ follows.

We prove by induction on $n$ and assume $w_2 w_3 \cdots w_n(K) \subset X_{2,j'}$. By (2), there is $r_{1,j}$ such that $r_{1,j}$ and $r_{2,j'}$ does not commute, so $r_{1,j}^{-1}(X_{2,j'}) \subset X_{1,j}$. Moreover, by (1), $r_{1,h}^{-1}(X_{1,j}) = X_{1,j}$ for $h \neq j$, so $\alpha(K) \subset w_1(X_{2,j'}) \subset X_{1,j}$.

Now we can define a $G_K$-equivariant map $r : G(\Gamma) \to G_K$ by sending $v(K)$ to the identity of $G_K$ and we can prove as before that $r$ maps $S$-flat to (possibly lower dimensional or 0-dimensional) $S'$-flat, thus $r$ is 1-Lipschitz with respect to the word metric. Let $i : G_K \hookrightarrow G(\Gamma)$ be the inclusion. Then $r \circ i$ is a left translation, in particular, if $K$ contains the identity, then $r$ is a retraction. It follows that if $S'$ is finite, then $i$ is an quasi-isometric embedding.

Note that a related construction in the case of right-angled Coxeter group had been discussed in [Hag08]. By taking larger and larger convex compact subcomplex of $X(\Gamma, S)$, we know $G(\Gamma)$ is residually finite. Moreover, pick $\beta \in \text{Stab}(K) \subset G(\Gamma)$, by definition of $S'$, $S' = \beta S' \beta^{-1}$, so $\text{Stab}(K)$ normalize $G_K$. Now we have obtained a direct proof of the fact that every word-quasi-convex subgroup of a finite generated right-angled Artin group is separable (Theorem F of [Hag08]) by using the above discussion together with the outline in Section 1.5 of [Hag08].

The following result readily follows from the above discussion:

**Theorem 6.15.** Given a right-angled Artin group $G(\Gamma)$ with $\text{Out}(G(\Gamma))$ finite and a standard generating set $S$ for $G(\Gamma)$, there is a 1-1 correspondence between non-negative convex compact subcomplexes of $X(\Gamma, S)$ that contain the identity and finite index right-angled Artin subgroups of $G(\Gamma)$. In particular, these subgroups are generated by conjugates of powers of elements in $S$.

**Remark 6.16.** If $\text{Out}(G(\Gamma))$ is infinite, then there exist $G(\Gamma_1)$ and its right-angled Artin subgroup $G(\Gamma_2)$ such that $G(\Gamma_2)$ is not isomorphic to any special subgroup of $G(\Gamma_1)$. To see this, let $G(\Gamma_1)$ be a right-angled Artin group such that $\text{Out}(G(\Gamma_1))$ is transvection free. Then Lemma 6.13 and Theorem 3.27 imply every special subgroup of $G(\Gamma_1)$ does not admit non-trivial transvection in its outer automorphism group. Let $\Gamma_1$ and $\Gamma_2$ be the graphs in Example 3.28. Then $G(\Gamma_2)$ is a right-angled Artin subgroup of $G(\Gamma_1)$ and there exists non-trivial transvection in $\text{Out}(G(\Gamma_2))$, thus $G(\Gamma_2)$ is not isometric to any special subgroup of $G(\Gamma_1)$.

**Remark 6.17.** Pick $G(\Gamma)$ such that $\text{Out}(G(\Gamma))$ is finite, then the above theorem can be used to prove certain subgroup of $G(\Gamma)$ is not a right-angled Artin group. For example, pick distinct generators $\{v_i\}_{i=1}^k$ for $G(\Gamma)$ and defining homomorphism $h : G(\Gamma) \to \mathbb{Z}/2$ by sending each $v_i$ to 1 and killing all other generators, then $\text{ker}(h)$ is a right-angled Artin group iff $k = 1$. It is interesting to compare this example with Example 3.28.

**Remark 6.18.** It is shown in Theorem 2 of [KK13b] that if $F(\Gamma')$ embeds into $\mathcal{P}(\Gamma)$ as a full subcomplex, then there exists a monomorphism $G(\Gamma') \hookrightarrow G(\Gamma)$, this result
Suppose $\Delta(l) = l$ for every standard geodesic $l \in X(\Gamma, S)$ with $\Delta(l) = w$. Suppose $M \subset \mathcal{P}(\Gamma, S)$ is a compact full subcomplex and $\Gamma'$ is the 1-skeleton of $M$. Denote the vertex set of $M$ by $\{w_i\}_{i=1}^n$ and let $l_i$ be a standard geodesic with $\Delta(l_i) = w_i$. We identify $l_i$ with $\mathbb{R}$ by the coordinate function $i : G(\Gamma) \rightarrow \mathbb{Z}^{\mathcal{P}(\Gamma, S)}$. For $1 \leq i \leq n$, define $\Delta_i = \{1 \leq j \leq n \mid d(w_i, w_j) \geq 2\}$. If $\Delta_i \neq \emptyset$, let $[a_i, a_i + k_i] \subset \mathbb{Z}$ be the minimal interval such that $\bigcup_{j \in \Delta_i} \pi_2^{-1}([a_i, a_i + k_i]) \subset I_{w_i}^{-1}([a_i, a_i + k_i])$; if $\Delta_i \neq \emptyset$, pick an arbitrary $a_i$ and set $k_i = 0$. Define $X_i = \pi_2^{-1}((\infty, a_i - 1/2)] \cup \pi_2^{-1}([a_i + k_i + 1/2, \infty))$, then by construction, $X_i \cap X_j = \emptyset$ for $i, j$ satisfying $d(w_i, w_j) \geq 2$. Using similar argument as before, we can show the subgroup generated by $S' = \{\alpha_{w_i}^{-1}\}_{i=1}^n$ is a right-angled Artin group with defining graph $\Gamma'$.

By Remark 6.14 the above monomorphism $i : G(\Gamma') \rightarrow G(\Gamma)$ is actually a quasi-isometric embedding. Moreover, it induces a simplicial embedding $i_s : \mathcal{P}(\Gamma', S') \rightarrow \mathcal{P}(\Gamma, S)$ such that the image of $i_s$ is a full subcomplex of $\mathcal{P}(\Gamma)$. It is clear that the image of every $S'$-geodesic under $i$ is Hausdorff close to a $S$-geodesic and the images of two parallel $S'$-geodesics under $i$ are Hausdorff close, which induces $i_s : v(\mathcal{P}(\Gamma', S')) \rightarrow v(\mathcal{P}(\Gamma, S))$. This map is injective since $i$ is quasi-isometric embedding. Pick vertices $w_1, w_2 \in \mathcal{P}(\Gamma', S')$ and let $u_i = i_s(w_i)$. We need to show $d(w_1, w_2) = 1$ if $d(u_1, u_2) = 1$. Let $l_1$ and $l_2$ be $S'$-geodesics with $\Delta(l_1) = w_1$ and let $\alpha_{u_1} \in G(\Gamma')$ be the translation along $l_1$. Note that $i(\alpha_{u_1}) = \alpha_{w_1}$. We argue by contradiction and suppose $d(w_1, w_2) \geq 2$ and $d(u_1, u_2) = 1$. For $j = 1, 2$, let $c_j$ be a $S$-geodesic Hausdorff close to $i(l_j)$. Then $\Delta(\alpha_{u_1}, l_2) \neq \Delta(l_2)$ and $\Delta(\alpha_{u_1} \cdot c_2) = \Delta(c_2) = u_2$. However, $i(\alpha_{u_1}, l_2) = i(\alpha_{w_1}) \cdot i(l_2) = \alpha_{w_1} \cdot i(l_2)$, which implies $d_H(i(\alpha_{w_1}, l_2), \alpha_{u_1} \cdot c_2) < \infty$. Thus $i_s(\Delta(\alpha_{w_1}, l_2)) = \Delta(\alpha_{w_1}, c_2) = u_2$, which contradicts the injectivity of $i_s$.

At this point it is natural to ask the following question:

**Question 6.19.** Let $S$ be a standard generating set of $G(\Gamma)$ and let $S'$ be a finite collection of elements of form $\alpha^k \alpha^{-1}$, here $r \in S$, $k \in \mathbb{Z}$ and $\alpha \in G(\Gamma)$. Suppose $G$ is the subgroup generated by $S'$, is $G$ a right-angled Artin group?

### 6.3. Generalized star extension.

In this section, we are going to find an algorithm to determine whether $G(\Gamma)$ and $G(\Gamma')$ are quasi-isometric or not, given $Out(G(\Gamma))$ is finite.

For convex subcomplex $E \subset X(\Gamma)$, denote the full subcomplex in $\mathcal{P}(\Gamma, S)$ spanned by $\{\Delta(l_{\lambda})\}_{\lambda \in \Lambda}$ by $\tilde{E}$, where $\{l_{\lambda}\}_{\lambda \in \Lambda}$ is the collection of standard geodesics in $X(\Gamma)$ with $l_{\lambda} \cap E \neq \emptyset$.

Now we describe a process to construct a graph $\Gamma'$ from $\Gamma$ such that $G(\Gamma')$ is isomorphic to a special subgroup of $G(\Gamma)$. Let $\Gamma_1 = \Gamma$ and let $K_1$ be one point; we are going to construct a pair $(\Gamma_i, K_i)$ inductively such that

1. $K_i$ is a compact CAT(0) cube complex and there is a cubical embedding $f : K_i \rightarrow X(\Gamma)$ such that $f(K_i)$ is convex in $X(\Gamma)$.
2. $\Gamma_i$ is a finite simplicial graph and there is a simplicial isomorphism $g : F(\Gamma_i) \rightarrow K_i$.

Note that these assumptions are trivially true for $i = 1$. Each edge $e \in K_i$ can be associated with a vertex in $\Gamma_i$, denoted by $v_e$, as follows: let $l_e$ be the standard geodesic in $X(\Gamma)$ that contains $f(e)$ and put $v_e := g^{-1}(\Delta(l_e))$. Each vertex $x \in K_i$ can be associated with a full subcomplex $\Phi(x) \subset F(\Gamma_i)$ defined by $\Phi(x) = g^{-1}(\hat{x})$. 
To define \((\Gamma_{i+1}, K_{i+1})\), pick a vertex \(v \in \Gamma_i\) and let \(\{x_j\}_{j=1}^m\) be the collection of vertices in \(K_i\) such that \(v \in \Phi(x_j)\). Then \(\{\Phi(x_j)\}_{j=1}^k\) are exactly the vertices in \(P_i \cap f(K_i)\), here \(l\) is a standard geodesic such that \(\Delta(l) = g(v)\). Let \(L\) be the convex hull of \(\{x_j\}_{j=1}^m\) in \(K_i\). Then \(e \subseteq L\) for any edge \(e \subseteq K_i\) with \(v_e = v\). \(L\) admits a natural splitting \(L = h \times [0, a]\) such that \(a > 0\) iff there exists an edge \(e \in K_i\) with \(v_e = v\), in this case, \(h\) is isomorphic to the hyperplane dual to \(e\), and for any edge \(e' \in K_i\) with \(v_{e'} = v\), the projection of \(e'\) to the interval factor is an edge. This splitting coincides with the pull-back of the natural splitting of \(f(K_i) \cap P_i\) under \(f\).

Let \(L_i = h \times \{x\} \subseteq L\) and let \(M_i = \cup_{x \in L_i} \Phi(x)\) (\(x\) is a vertex). Define \(F(\Gamma_{i+1})\) to be the simplex obtained by glueing \(F(\Gamma_i)\) and \(M_i\) along \(St(v, M_i)\) and \(K_{i+1}\) to be the \(CAT(0)\) cube complex obtained by glueing \(K_i\) and \(L_i \times [0, 1]\) along \(L_i\). It is easy to see that we can extent \(f\) to a cubical embedding \(f' : K_{i+1} \to X(\Gamma)\) such that \(f'(K_{i+1})\) is convex. This induces an isomorphism \(g' : F(\Gamma_{i+1}) \to K_{i+1}\) which is an extension of \(g\). By construction, each \(G(\Gamma_i)\) is isomorphic to a special subgroup of \(G(\Gamma)\), moreover, the above induction process does not depend on knowing exactly what \(X(\Gamma)\) is, it actually provides a way to construct convex subcomplexes of \(X(\Gamma)\).

The above process of obtaining \((\Gamma_{i+1}, K_{i+1})\) from \((\Gamma_i, K_i)\) is called a generalized star extension (GSE) at \(v\). Note that \(\Gamma_i \subseteq \Gamma_{i+1}\) iff \(St(\pi(g(v))) \subseteq F(\Gamma)\) (\(\pi : \mathcal{P}(\Gamma) \to F(\Gamma)\) is the natural label-preserving projection), in this case we say the GSE is nontrivial. There exists \(v \in \Gamma_i\) such that the GSE at \(v\) is nontrivial iff \(f(\Gamma_i) \subseteq \mathcal{P}(\Gamma)\) iff \(\Gamma\) is not a clique.

Suppose \(G(\Gamma')\) is isomorphic to a special subgroup of \(G(\Gamma)\). Then we can construct \(\Gamma'\) from \(\Gamma\) with using finite many GSEs. To see this, suppose \(G(\Gamma')\) is isomorphic to \(\Theta_S(K)\) for \(K \in CN(\Gamma, S)\). We define a sequence of convex subcomplexes in \(K\) by induction. Let \(K_1\) be the identity element in \(G(\Gamma)\). Suppose \(K_i\) is already defined. If \(K_i = K\), then the induction terminates; if \(K_i \not\subseteq K\), pick an edge \(e_i \subseteq K\) such that \(e_i \cap K_j\) is a vertex and let \(K_{i+1}\) be the convex hull of \(K_i \cup e_i\). Let \(\{K_i\}_{i=1}^n\) be the resulting collection of convex subcomplexes. An alternative way of describing \(K_{i+1}\) is the following. If \(h_i\) is the hyperplane dual to \(e_i\) and \(N_i\) is the carrier of \(h_i\), then \(h_i \cap L_i = \emptyset\) by the convexity of \(K_i\). Let \(M_i\) be the copy of \((K_i \cap N_i) \times [0, 1]\) inside \(N_i\), then \(K_{i+1} = K_i \cup M_i\). Thus we can obtain \((K_{i+1}, K_{i+1})\) from \((K_i, K_i)\) by the process described as above.

The above construction gives rise to an algorithm to detect whether \(G(\Gamma')\) is isomorphic to a special subgroup of \(G(\Gamma)\). If there are \(n\) vertices in \(\Gamma'\), then \(\Gamma'\) can be obtained from \(\Gamma\) by at most \(n\) nontrivial GSEs, so we can enumerate all possibilities and check one by one. In the light of Theorem 6.13 and Theorem 6.13 we have the following result:

**Theorem 6.20.** If \(Out(G(\Gamma))\) is finite, then \(G(\Gamma')\) is quasi-isometric to \(G(\Gamma)\) iff \(\Gamma'\) can be obtained from \(\Gamma\) by finite many GSEs. In particular, there is an algorithm to determine whether \(G(\Gamma')\) and \(G(\Gamma)\) are quasi-isometric.

Note that GSE gives rise to a pair \((\Gamma_i, K_i)\). If one do not care about the associated convex subcomplex \(K_i\), then we have a simpler description of GSE when \(Out(G(\Gamma))\) is finite. Suppose we have already obtained \(F(\Gamma_i)\) together with a finite collection of full subcomplexes \(\{G_\lambda\}_{\lambda \in \Lambda_i}\) such that

1. \(\{G_\lambda\}_{\lambda \in \Lambda_i}\) is a covering of \(F(\Gamma_i)\).
2. Each \(G_\lambda\) is isomorphic to \(F(\Gamma)\).
When \( i = 1 \), we pick the trivial cover of \( \Gamma \) by itself. To construct \( \Gamma_{i+1} \), pick vertex \( v \in F(\Gamma_1) \), let \( \Lambda_v = \{ \lambda \in \Lambda \mid \lambda \in G_x \} \) and let \( \Gamma_v = \cup_{\lambda \in \Lambda_v} C_{\lambda} \). Suppose \( \{ C_{j} \}_{j=1}^{m} \) is the collection of components of \( \Gamma_v \setminus St(v, \Gamma_v) \) and suppose \( C_{j} = C_j \cup St(v, \Gamma_v) \). Then \( F(\Gamma_{i+1}) \) is defined by gluing \( C_1' \) and \( F(\Gamma_i) \) along \( St(v, \Gamma_v) \) and \( \Gamma_{i+1} \) is the 1-skeleton of \( F(\Gamma_{i+1}) \).

We need to show this construction is consistent with the GSE. Assume inductively that there is a \( CAT(0) \) cube complex \( K_i \) such that the two induction assumptions for GSE are satisfied, moreover, \( \{ G_\lambda \}_{\lambda \in \Lambda} \), coincides with \( \{ \Phi(x) \}_{x \in K_i} \) (\( x \) is a vertex). Let \( L = h \times [0,a] \) be as before and let \( L_j = h \times \{ j \} \subset L \). It suffices to show there is a 1-1 correspondence between \( \{ L_j \}_{j=0}^{a} \) and \( \{ C_j' \}_{j=1}^{m} \) such that for each \( j \), there exists unique \( j' \) with \( f(L_j) = g(C_{j'}) \). Pick adjacent vertex \( x_1, x_2 \in f(L_j) \) and let \( w \in \Gamma \) be the label of edge \( \pi_1 \cdot x_2 \). Suppose \( \bar{v} = \pi(g(v)) \). Then \( d(\bar{v}, v) = 1 \). Since \( \text{Out}(G(\Gamma)) \) is finite, \( \bar{w} \notin \Phi \cdot St(\bar{v}) \), then there is a vertex \( \bar{u} \in \bar{w} \) such that \( d(\bar{u}, \bar{v}) = 2 \). The lift of \( \bar{u} \) in \( \tilde{x}_1 \) and \( \tilde{x}_2 \) are the same point, so \( \tilde{x}_1 \cap \tilde{x}_2 \setminus St(g(v)) \) contains a vertex, but \( \tilde{x}_i \setminus St(g(v)) \) is connected for \( i = 1,2 \), so \( \tilde{x}_1 \cap \tilde{x}_2 \setminus St(g(v)) \) is connected. It follows that \( f(L_j) \setminus St(g(v)) \) is connected. Moreover, Lemma 4.8 implies that \( f(L_{j_1}) \setminus St(g(v)) \) and \( f(L_{j_2}) \setminus St(g(v)) \) are in different components of \( \mathcal{P}(\Gamma) \setminus St(g(v)) \) when \( j_1 \neq j_2 \), so there exists a unique \( j' \) such that \( f(L_j) = g(C_{j'}) \).

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