Precision prediction of gauge couplings and the profile of a string theory.

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Abstract

We estimate the significance of the prediction for the gauge couplings in the MSSM with an underlying unification. The correlation between the couplings covers only (0.2-2)% of the a priori reasonable region of the parameter space, while the prediction for $\sin^2 \theta_W$ is accurate to 1.3%. Given that agreement with experiment to such precision is unlikely to be fortuitous, we discuss the profile of a string theory capable of preserving this level of accuracy. We argue that models with a low scale of unification involving power law running in the gauge couplings do not. Even theories with a high scale of unification are strongly constrained, requiring the compactification scale of new space dimensions in which states transforming under the Standard Model propagate to be very close to the string cut-off scale. As a result no new space dimensions can be larger than $10^{-14}$ fm.
1 Introduction

The unification of gauge couplings is one of the few believable precision predictions coming from physics beyond the Standard Model. In the Minimal Supersymmetric Standard Model the radiative corrections to the gauge couplings coming from the states of the MSSM causes the SU(3), SU(2) and U(1) gauge couplings to become very nearly equal at the scale \( (1 - 3) \times 10^{16} \text{ GeV} \), provided one adopts the SU(5) normalisation of the U(1) factor. It is usual to use gauge unification to predict the value of the strong coupling given the weak and electromagnetic couplings. Doing this one finds \( \alpha_3(M_Z) = 0.126 \pm 0.01 \) to be compared with the experimental value \( \alpha_3(M_Z) = 0.118 \pm 0.002 \). However, the gauge unification prediction is really much more precise than this comparison suggests. In the MSSM the precision is limited by threshold effects associated with the masses of the supersymmetry partners of Standard Model states affecting the prediction for \( \alpha_3(M_Z) \). In Section 2 we discuss these corrections and demonstrate that, expressed as a correlation between the gauge couplings, the prediction covers only (0.2-2)% of the overall (perturbative) parameter space. Moreover the prediction for \( \sin^2 \theta_W \) is much more precise than for \( \alpha_3 \), being accurate to 1.3%. The fact that experiment and theory agree to this precision seems to us unlikely to be just a happy accident. It is largely due to this circumstantial evidence for unification and the fact that supersymmetry is needed to solve the mass hierarchy problem that there has been so much interest in supersymmetric models and in compactification schemes that preserve a stage of low energy supersymmetry. For the same reason it is of interest to demand that the physics beyond the MSSM should not spoil the prediction.

There are inevitably further threshold corrections coming from new states lying at and above the unification scale. In the case of Grand Unification the states include the heavy gauge bosons and Higgs bosons needed to make up complete GUT multiplets. The threshold corrections associated with these states have been extensively discussed in the context of Grand Unification and the condition that the gauge unification relations should not be significantly affected requires there should be no substantial splitting between the heavy Higgs and gauge boson states. Provided the ratio between these masses is less than a factor of 10 the precision is essentially unchanged in minimal GUT schemes, although the requirement of near equality of Higgs and gauge boson masses becomes much stronger for the case of non-minimal Higgs sectors responsible for breaking the GUT. While at one loop order the unification scale is well defined in a SUSY GUT, at two loop order the scale of unification is not so easy to define. We discuss this point in Section 3 and show that it is necessary to regulate the theory to determine these two loop corrections. In a string theory this regularisation introduces new threshold corrections (associated with the canonical normalisation of the Kähler term for matter fields) and a sensitivity to the string (cut-off) scale. As a result there are new contributions to the gauge couplings in Grand Unified theories proportional to \( \ln(M_s/M_{\text{GUT}}) \) with the string scale, \( M_s \) providing the cut-off scale, which are not usually included in GUT calculations.

In the case of compactified string theories (with or without Grand Unification below the string scale) the threshold corrections can be much larger if the compactification scale is (significantly) different from the string scale. This may be understood (in the effective low energy field theory) as due to Kaluza Klein states associated with gauge non singlets propagating in the extra dimensions and with mass up to the string scale which contribute to the running of the gauge couplings. For low compactification scales the number of such states is very large leading to very large threshold corrections generating power-like gauge coupling running and significant threshold sensitivity of the predictions. String calculations of the threshold effects for the weakly coupled heterotic string have been performed by several groups (for a review of the topic and references see 3). The regularisation of the effective field

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\(^1\) Additional corrections due to the rescaling of the metric (super-Weyl anomaly) exist, with similar implications.
theory is provided at the (heterotic) string level by the geometry of the string. In type I models with compact “in-brane” dimensions one may have a similar (power-like) behaviour due to the associated Kaluza Klein states. Finally, power-like behaviour may also be present in type I models with anisotropic compactification. The string thresholds have been computed in and (string) regularisation is again provided by the geometry of the string. As we shall discuss in Section 3 the expectation is that all these corrections will spoil the MSSM prediction and imply the compactification scale(s) should be very close or equal to the string scale. One important implication of this is that one cannot appeal to non-standard “power law” running to lower the unification scale and one is left only with high scale unification.

The need for a high unification (and string) scale is broadly in agreement with the expectation in the heterotic string. However in the weakly coupled case there is a residual discrepancy of approximately a factor of 20 between the gauge unification scale found in MSSM unification and the string prediction. Further the requirement that the compactification scale should also be close to the string scale apparently prevents large threshold effects from coming to the rescue. However, as we discuss in Section 3.2, in cases of string thresholds with a Wilson line background, it is possible for large Wilson line effects to lower the unification scale to that of the MSSM while keeping the compactification scale close to the string scale. Thus the weakly coupled heterotic string with Wilson line background provides an example of a string model which can preserve the MSSM precision prediction and have an acceptable scale of unification. Another possibility is that the string theory has a stage of Grand Unification below the compactification scale allowing for a gauge unification scale below the compactification scale. As we discuss in Section 4, even in this case, it is necessary that the compactification scale should be close to the string cut-off scale due to the need to keep two-loop corrections under control. An alternative resolution of the discrepancy is that gravity can propagate in more dimensions than matter and gauge states, changing the string prediction for the unification scale. The original idea for this came from the strongly coupled heterotic string in which the gauge unification scale is lowered through the appearance of an 11th dimension. Even in this case the compactification scale of the other compactified dimensions cannot be far from the string scale. Our conclusions are presented in Section 5 where we present a profile of a compactified string theory capable of satisfying all the constraints needed to preserve the accuracy of the predictions for gauge couplings.

2 Precision gauge unification in the MSSM

In the MSSM the prediction of the gauge couplings follows from the GUT or string relation at the unification scale corrected by the renormalisation group determination of radiative corrections involving the MSSM states. By combining the standard model gauge couplings one may eliminate the one-loop dependence on the unification scale and the value of the unified coupling to obtain a relation which depends only on the threshold effects associated with the unknown masses of the supersymmetric states.

\[ \alpha_3^{-1}(M_Z) = \left\{ \frac{15}{7} \sin^2 \theta_W(M_Z) - \frac{3}{7} \right\} \alpha_{em}^{-1}(M_Z) + \frac{1}{2\pi} \ln \frac{T_{eff}}{M_Z} + (two\text{-}\ loop) \]  

(1)

Note that the leading dependence on the SUSY thresholds comes from \( T_{eff} \), an effective supersymmetric scale, while two loop corrections, which have a milder dependence on the scale, through (gauge and matter) wavefunction renormalisation \( \ln Z \sim \ln \alpha_i(scale) \) can be ignored to a first approximation. In

\(^2\text{i.e. the world-sheet torus.} \)

\(^3\text{via cancellation of the quadratic terms between the annulus and Moebius strip contributions.} \)
Figures 1 (a) and (b). Plots of $\alpha_3(M_Z)$ versus $\sin^2 \theta_W$ calculated in the MSSM for two values of the effective supersymmetric threshold, $T_{\text{eff}} = 20$ GeV and $T_{\text{eff}} = 1000$ GeV. The limits correspond to requiring $\alpha_3(M_Z)$ remains in the perturbative domain and unification occurring above the supersymmetry threshold. The area between the two curves provides a measure of the predictivity of the theory. The experimental range of values is also shown.

terms of the individual SUSY masses $T_{\text{eff}}$ is given by [1]

$$T_{\text{eff}} = m_{\tilde{H}} \left( \frac{m_{\tilde{W}}}{m_{\tilde{g}}} \right)^{\frac{28}{19}} \left( \frac{m_{\tilde{H}}}{m_{\tilde{H}}} \right)^{\frac{3}{19}} \left( \frac{m_{\tilde{W}}}{m_{\tilde{H}}} \right)^{\frac{10}{19}} \left( \frac{m_{\tilde{q}}}{m_{\tilde{q}}} \right)^{\frac{9}{19}}$$

provided the particles have mass above $M_Z$. The precision of the relation, eq.(1), between Standard Model couplings is limited by the dependence on $T_{\text{eff}}$. However the uncertainty in $T_{\text{eff}}$ is limited by the fact that the supersymmetry masses are bounded from below because no supersymmetric states have been observed and from above by the requirement that supersymmetry solve the hierarchy problem. From eq.(1) we see the dependence on the squark, slepton and heavy Higgs masses is very small, the main sensitivity being to the Higgsino, Wino and gluino masses.

If all the supersymmetric partner masses are of $O(1 \text{TeV})$ then so is $T_{\text{eff}}$. However in most schemes of supersymmetry breaking radiative corrections split the superpartners. In the case of gravity mediated supersymmetry breaking with the assumption of universal scalar and gaugino masses at the Planck scale the relations between the masses imply $T_{\text{eff}} \simeq m_{\tilde{H}} \left( \alpha_2(M_Z)/\alpha_3(M_Z) \right)^2 \simeq |\mu|/12$. For $\mu$ of order the weak scale $T_{\text{eff}}$ is approximately 20 GeV. From this we see the uncertainty in the SUSY breaking mechanism corresponds to a wide range in $T_{\text{eff}}$. In what follows we shall take $20 \text{GeV} < T_{\text{eff}} < 1 \text{TeV}$ as a reasonable estimate of this uncertainty. Using this one may determine the uncertainty in the strong coupling for given values of the weak and electromagnetic couplings using eq.(1). The result (including two-loop effects) as a function of $\alpha_3$ and $\sin^2 \theta_W$ is plotted in Figures 1(a) and 1(b).

The precision of the prediction is remarkable. A measure of this is given by the area between the two curves in Figure 1 (a). If one assumes that a random model not constrained by unification may

\footnote{Further constraints on this region, coming from small fine-tuning of other observables of the MSSM [1] may further constrain the area of allowed values of $\alpha_3(M_Z)$ vs. $\sin^2 \theta_W$.}
give any value for \( \alpha_3 \) and \( \sin^2 \theta_W \) between 0 and 1 the relative precision is an impressive 0.002! Of course this estimate is sensitive to the measure chosen. Changing to \( \alpha_3 \) and \( \sin \theta_W \) makes very little difference. Changing to \( \alpha_3^{-1} \) and \( \sin^2 \theta_W \) (and restricting the possible range of \( \alpha_3^{-1} \) to be \( 1 < \alpha_3^{-1} < 10 \)) increases the relative precision by a factor of 10. Using this variation as an estimate of the uncertainty associated with the measure we conclude that a reasonable estimate for the precision of the prediction for the correlation between the Standard Model couplings is in the range \((0.2 - 2\%)\). One may also use the result of eq(4) to predict one of the couplings given the other. However, as may be seen from Figure 1(a), the prediction is not equally precise for each coupling. A quantitative estimate of the accuracy of the prediction for \( \alpha_3(M_Z) \) or \( \sin^2 \theta_W(M_Z) \) may be obtained from eq.(3) by taking the derivative with respect to \( T_{eff} \) giving

\[
\alpha_3^{-1}(M_Z) F_{\alpha_3} + \frac{19}{28 \pi} + two - loop
\]

where

\[
F_{\alpha_3} = \frac{d \ln(\alpha_3(M_Z))}{d \ln T_{eff}} ; \quad F_{\theta_W} = \frac{d \ln(\sin^2 \theta_W)}{d \ln T_{eff}}
\]

The quantities \( F_{\alpha_3} \) and \( F_{\theta_W} \) give the fractional threshold sensitivity of \( \alpha_3 \) and \( \sin^2 \theta_W \) to a change in \( \ln T_{eff} \). From this we see that the unification prediction for \( \sin^2 \theta_W(M_Z) \) is more accurate than that for \( \alpha_3(M_Z) \) since, close to the experimental point, we have from eqs.(3) and (4)

\[
\frac{F_{\theta_W}}{F_{\alpha_3}} \approx \frac{1}{7.6}
\]

Given that the error in \( \alpha_3 \) is \( \approx 10\% \) for a change in \( T_{eff} \) from 20 to 1000GeV we see from this equation that the corresponding error in \( \sin^2 \theta_W \) is 1.3%. Thus the prediction for \( \sin^2 \theta_W \) provides a more realistic measure of the precision of the unification prediction. An expansion of \( \sin^2 \theta_W \) (corresponding to \( \alpha_3(M_z) = 0.119 \)) thus gives

\[
\sin^2(\theta_W(M_Z)) \approx 0.2337 - 0.25(\alpha_3(M_Z) - 0.119) \pm 0.0015
\]

This effect may be seen directly in Figure 1(a) since the curve is more steeply varying in the \( \alpha_3 \) direction than in the \( \sin^2 \theta_W \) direction close to the experimental point. The optimal combination of \( \alpha_3 \) and \( \sin^2 \theta_W \) with minimal uncertainty normal to the curve can be determined numerically but is relatively close to \( \sin^2 \theta_W \). The fact that experiment and theory agree to this level of accuracy is impressive and is a major reason why so much attention has been paid to supersymmetric extensions of the Standard Model.

### 3 High-scale threshold sensitivity of the unification prediction

Given this impressive accuracy it is clearly of interest to determine the effect on this prediction coming from threshold effects at the unification scale in realistic string theories. Our hope is that the requirement that the precision should not be significantly degraded will give us information about the underlying string theory. The value of the gauge couplings at \( M_Z \) is of the form

\[
\alpha_i^{-1}(M_Z) = -\delta_i + \alpha^{-1}(\Lambda) + \frac{b_i}{2 \pi} \Delta(\Lambda, \mu_0) + \frac{b_i}{2 \pi} \ln \frac{\Lambda}{M_Z} + \frac{3T_i(G)}{2 \pi} \ln \left[ \frac{\alpha(\Lambda)}{\alpha_i(M_Z)} \right]^{1/3} - \sum_\phi \frac{T_i(R_\phi)}{2 \pi} \ln Z_\phi(\Lambda, M_Z)
\]
where $\Delta$ is the string threshold correction corresponding to the particular string theory considered (hereafter renamed to $\Delta^{H}$, $\Delta^{I}$ to stand for the weakly coupled heterotic and type I string cases respectively). The parameter $\Lambda$ is the unification scale, $\alpha(\Lambda)$ is the unified coupling, $\mu_0$ is the mass of the heavy states, often the compactification scale in string theories and $Z_\phi$ and $\alpha(\Lambda)/\alpha(M_Z)$ are the matter and gauge wavefunction renormalisation coefficients respectively. In eq(7) there are additional effects due to low energy supersymmetric thresholds $\delta_i b_i = -3T_i(G) + \sum_\phi T_i(R_\phi)$ are the one-loop beta function coefficients, while $\bar{t}_i$ depend on the particular compactification scheme - they are non-zero only for the N=2 massive SUSY states and vanish for the N=4 spectrum.

In comparing the threshold effects at the unification scale to the SUSY threshold effects discussed above we will restrict ourselves to a measure of the sensitivity to these scales for either the strong coupling or $\sin^2 \theta_W$ while the other is maintained fixed. In leading order we have from eq(7)

$$\delta \alpha_3^{-1}(M_Z) - \frac{15}{7} \delta \sin^2 \theta_W = \frac{1}{2\pi} \left[ \bar{t}_1 b_{23} b_{12}^3 + \bar{t}_2 b_{31} b_{12}^3 + \bar{t}_3 \right] \delta \Delta(\Lambda, \mu_0)$$

where $b_{ij} = b_i - b_j$, $b_1 = 33/5$, $b_2 = 1$, $b_3 = -3$. The relative sensitivity of $\alpha_3$ (keeping $\sin^2 \theta_W$ fixed) to changes in $\mu_0$ and $T_{eff}$ is, from eqs(3), (4) and (5), given by

$$\mathcal{R} \equiv \left| \frac{\delta \ln(\alpha_3(M_Z))}{\delta \ln(\alpha_3(M_Z))_{\text{MSSM}}} \right| = \left| \frac{d \ln \alpha_3(M_Z)}{d \ln \mu_0} \right| \times \mathcal{F}_{\alpha_3}^{-1}$$

where we have assumed that the predicted value for $\alpha_3(M_Z)$ in the model considered is close to that of the MSSM\footnote{The definition of $T_{eff}$ eq(2) in terms of $\delta_i$ is $\ln T_{eff}/M_Z = -(28\pi/19)(\delta_1 b_{31}/b_{12} + \delta_{23} b_{12} + \delta_3)$, with $b_{ij} = b_i - b_j$.} One obtains the same result for the relative threshold sensitivity if we compute the prediction for $\sin^2 \theta_W(M_Z)$ (with $\alpha_3(M_Z)$ fixed) normalised to $\mathcal{F}_{\theta_W}$. For this reason in our analysis of threshold sensitivity we will focus our attention only on the value of $\mathcal{R}$.

It follows immediately from eq(4) that models involving only complete additional GUT representations (i.e. with $\bar{t}_i = b_i + n$) do not introduce any threshold sensitivity at one loop order. This will be the case if there is a Grand Unification below the compactification scale. It is also true if one can find models for which $\bar{t}_i = \kappa b_i$ as has been suggested \footnote{For a model to be viable one must in first instance predict the right value for $\alpha_3(M_Z)$ and only after would the question of threshold sensitivity be relevant.} to lower the unification scale by power law running. In both these cases, however, there will be significant threshold dependence at two loop order and we will discuss this separately in Section 4.

Returning to the one-loop threshold effects of eq.(3), we will determine the sensitivity for the case of weakly coupled heterotic string with/without Wilson lines and type I'/I models to analyse their relative threshold sensitivity, $\mathcal{R}$. In what follows $\Lambda$, $\mu_0$ stand for the string and compactification scale respectively. To perform this analysis we use the explicit string computations for the string thresholds, $\Delta$ \footnote{The definition of $T_{eff}$ eq(2) in terms of $\delta_i$ is $\ln T_{eff}/M_Z = -(28\pi/19)(\delta_1 b_{31}/b_{12} + \delta_{23} b_{12} + \delta_3)$, with $b_{ij} = b_i - b_j$.}

\subsection*{3.1 Weakly coupled heterotic models}

We first consider the case of the weakly coupled heterotic string with an N=2 sector (without Wilson lines present) in which six of the dimensions are compactified on an orbifold, $T^6/G$. In such models the spectrum splits into N=1, N=2 and N=4 sectors, the latter two associated with a $T^2 \times T^4$ split
of the $T^6$ torus. Due to the supersymmetric non-renormalisation theorem, the N=4 sector does not contribute to the running of the holomorphic couplings. As we shall discuss in Section 4 even in the case of N=4, two-loop corrections to the effective gauge couplings may introduce substantial threshold corrections. The N=1 sector gives the usual running associated with light states but does not contain any moduli dependence. The latter comes entirely from the N=2 sector. For the heterotic string all states are closed string states and at one loop the string world sheet has the topology of the torus $T^2$. For the case of a six-dimensional supersymmetric string vacuum compactified on a two torus $T^2$ the string corrections take the form $[14, 13]$. Here $T \propto R_1 R_2$ and $U \propto R_1/R_2$ where $T$, $U$ are moduli and $R_1$, $R_2$ are the radii associated with $T^2$. We consider the case of a two torus $T^2$ with $T = iT_2$ (the subscript 2 denotes the imaginary part) and $U = iU_2$. Making the dimensions explicit $T_2$ should be replaced by $T_2 \rightarrow T_2^o/(2\alpha') \equiv 2R_2^2/(2\alpha')$. Performing a summation over momentum and winding modes and integrating over the fundamental domain of the torus gives the following result for $\Delta^T$ $[13]$

$$\Delta^T = -\frac{1}{2} \ln \left\{ \frac{8\pi e^{1-\gamma_E}}{3\sqrt{3}} U_2 T_2 |\eta(iU_2)|^4 |\eta(iT_2)|^4 \right\}$$

(11)

$$= -\frac{1}{2} \ln \left\{ 4\pi^2 |\eta(i)|^4 \left( \frac{M_s}{\mu_0} \right)^2 \left| \eta \left[ \frac{i3\sqrt{3}\pi}{2e^{1-\gamma_E}} \left( \frac{M_s}{\mu_0} \right)^2 \right] \right|^4 \right\}$$

(12)

with the choice $U_2 = 1$ in the last equation. $M_s$ is the string scale, $\mu_0 \equiv 1/R$, $\eta(x)$ is the Dedekind eta function and we have replaced $\alpha'$ in terms of $M_s$. For large $T_2$ the eta function is dominated by the leading exponential and so one finds the power law behaviour $\Delta \propto T^2 \propto R^2$ which has a straightforward interpretation as being due to the decompactification associated with $T^2$. This contribution is basically due to the tower of Kaluza Klein states below the string scale and can be understood at the effective field theory level $[4]$ whose result is regularised by the string world-sheet $[10]$. The presence of power-like running, although taking place over a very small region of energy range, can significantly affect the sensitivity of the unification prediction for $\alpha_3(M_Z)$ with respect to changes of the compactification scale (which, being determined by a moduli vev, is not fixed perturbatively and hence is not presently known). To demonstrate this explicitly, consider the variation of the threshold $\Delta^H$ with respect to $T_2$

$$\frac{\delta \Delta^H}{\delta \ln T_2} = -\frac{1}{2} \left\{ 1 + \frac{4}{\ln \eta(iT_2)} \right\}$$

(13)

This gives the following relative threshold sensitivity (eq.13) with respect to the compactification scale $\mu_0 (x \equiv \mu_0/M_s)$

$$R = \frac{14}{19} \left\{ \frac{5}{7} b_1 - \frac{12}{7} b_2 + b_3 \right\} \left\{ 1 - 2x \frac{d}{dx} \ln \eta(i\sigma x^{-2}) \right\}$$

(14)

where $\sigma = 3\sqrt{3}/2e^{1-\gamma_E} - 1/2$. In Figure 2 we plot the second factor in curly brackets (which is equal to $\delta \Delta^H/\delta (\ln x))$ $[14]$. To understand the importance of $\Delta^H$ and the implications it has for the threshold sensitivity of $\alpha_3(M_Z)$, we note that $\Delta^H$ plays a central role in the attempts to bridge the well-known “gap” (of a factor of $\approx 20$) $[12, 3]$ between the heterotic string scale and the MSSM unification scale.

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7. From $[14]$ $M_s = 2 e^{(1-\gamma_E)/2} 3^{-3/4} / \sqrt{2\alpha'}$ where $g_s$ is the string coupling at the unification.
8. Note that $T_2 \approx 5.5 (M_s R)^2$ in $D^R$ scheme, so one can easily have $T_2 \approx 20$ while $R$ is still close to the string length scale, to preserve the weakly coupled regime of the heterotic string. In this section “large” $R$ corresponds to values of $T_2$ in the above range.
9. This takes place essentially between the compactification and the string scale $[4]$. This situation is somewhat similar to other (effective) models $[5]$ where it has been shown that power-like running brings in a significant amount of threshold sensitivity.
It has been found (for a review see [6]) that for generic examples (Z_8 orbifold) this condition requires $\Delta \approx 8.5$ [6, 10] which in turn requires moduli of considerable size, $T_2 \approx 18.7$ corresponding to a small value of $x \approx 0.53$ [14]. As may be seen from Figure 2 this leads to an even larger value for $|d\Delta^H/d(\ln x)|$ and an enhanced relative threshold sensitivity. For the $Z_8$ orbifold [17], we find (using $b_8 = (-9/2, -5/2, -3/2))$ $R = 5.9$ for $x \approx 0.53$. Since $\mathcal{R}$ gives the relative sensitivity of the gauge coupling predictions to the N=2 threshold and the SUSY threshold we see that even for $x = 0.5$ the precision of the gauge coupling prediction is substantially reduced. The requirement that this precision should not be lost constrains $x \approx 1$ (i.e. $\mu_0 = M_{\text{string}}$). However this fails to “bridge the gap”. This example clearly illustrates the general problem that weakly coupled heterotic string models have to reconcile two conflicting constraints, namely the need for a large $\Delta^H$ to solve the scale mismatch between the MSSM unification scale and the heterotic string scale and the need for a small derivative of $\Delta^H$ to avoid a large threshold sensitivity of the gauge couplings. The latter constraint is introduced by the power-law dependence of the thresholds on the compactification scale, and even though such running takes place over a small energy range it still gives significant effects. The problem can be avoided if $x \approx 1$, corresponding to the compactification scale being very close to the string scale. On the other hand the former constraint seems to require a small value for $x$. As we shall discuss this conclusion may be evaded in theories with Wilson line breaking.

These conclusions apply to string theories with an $N = 2$ sector. It is possible to construct string theories in which this sector is absent in which case there is no significant one-loop sensitivity of the unification prediction for the gauge couplings to changes of the high scale/moduli fields. The downside is that then one does not have the large $\Delta^H$ needed to “close the gap”. Even if this problem is solved in another way the $N = 4$ sector necessarily present introduces strong threshold dependence at two

11This values for $T_2$ can still be on the edge of the weakly coupled regime of the 10D heterotic string as its coupling is equal to $\lambda = (\prod_i Re T_i / Re S)^{1/2}$

12This is due almost entirely to the presence of the towers of Kaluza-Klein modes rather than to winding modes, [3].
loop order which provides almost as restrictive a bound on the compactification scale. We shall discuss
this further in Section 4.

3.2 Weakly coupled heterotic models with Wilson lines.

In the case that there are Wilson lines characterised by the moduli $B$ the masses of the heavy states
contributing to the RG flow of the gauge couplings become $B$-dependent. For this class of models
the string calculation of the threshold contribution to the gauge couplings was computed in [18]. The
presence of the Wilson lines may or may not break the symmetry group. In the latter case the threshold
corrections do not seem to be large enough [16, 6] to account for the energy gap between the MSSM
unification scale and that of the weakly coupled heterotic string. The case when Wilson lines break
the symmetry gauge group is more interesting, as one can obtain a symmetry group close to that of
the standard model [19]. We will consider only the class of models with (0,2) compactifications which
have the additional advantage of avoiding the doublet-triplet splitting problem. For the case of the
$Z_8$ orbifold the threshold contribution is then [18] (see also [6])

$$\Delta^H = -\frac{1}{20} \ln \left[ Y^{10} \left( \frac{1}{128} \prod_{k=1}^{10} \theta_k(\Omega) \right)^4 \right]$$  \hspace{1cm} (15)

with $\theta_k$ the ten even, genus two theta functions. As has been shown in [6, 15, 16], in this case one
can obtain a large value of the threshold $\Delta^H$ which may help solve the mismatch between the MSSM
unification scale and that of the heterotic string theory, without the need of large moduli. For example
one obtains $\Delta^H \approx 8.5$ for $T_2 = U_2 = 4.5$ and $B = 1/2$. The derivative of $\Delta^H$ with respect to $T_2$
can easily be taken by using the expansion of (15) up to $O(B^4)$, to give

$$\frac{\delta \Delta^H}{\delta \ln T_2} = -\frac{1}{2} \left\{ \frac{4U_2 T_2}{4U_2 T_2 - B^2} + \frac{24}{5} \frac{d \ln |\eta(iT_2)|}{d \ln T_2} \right\}$$ \hspace{1cm} (16)

$$\approx -\frac{1}{2} \left\{ 1 + \frac{24}{5} \frac{d \ln |\eta(iT_2)|}{d \ln T_2} \right\}$$ \hspace{1cm} (17)

where $dT_2$ stands for a derivative with respect to $T_2$ and where the approximation used above holds for
the particular point in the moduli space giving the right size of the threshold $\Delta^H$. One may observe
that this expression is very close to that of (13), with the difference that one has smaller moduli than
in the case without Wilson lines present. For the $Z_8$ orbifold at the point in moduli space given above
we obtain $R = 1.50$ which is lower than in the case without Wilson lines by a factor of $\approx 4$ for the
same value of the threshold $\Delta^H$.

The overall conclusion one may draw from this example is that the Wilson line background can
give a threshold $\Delta^H$ which is large enough to obtain the correct unification scale with the additional
benefit that its derivative with respect to $T_2$ is not significantly affected, leading to a significantly
lower threshold sensitivity than in the other cases without Wilson lines. Such models can preserve the
precision prediction for gauge coupling and give good agreement between the string unification scale
and the MSSM value.

3.3 Type I'/I models with a N=2 sector.

In this case the corrections to the gauge couplings come from the N=1 (massless) sector and from the
N=2 massive winding (Type I') sector. The massive N=2 threshold corrections to $\alpha_i^{-1}$ at the string

\[\text{with our normalisation for } \Delta^I']
\[
\Delta I = -\frac{1}{2} \sum_k b_{i,k} \ln \left( \sqrt{G_k} \text{Im} U_k M_I^2 |\eta(U_k)|^4 \right)
\]  

(18)

Here \( G_k \) is the metric on the torus \( T^k \), and \( \sqrt{G_1} = R_1 R_2 \) (for a rectangular torus). The behaviour of the thresholds when \( \text{Im} U = R_1/R_2 \) is of order one is logarithmic, \( \Delta_a \sim \ln(R_1 R_2) \) and when \( \text{Im} U = R_1/R_2 \gg 1 \) it is power-like, \( \Delta_a \sim R_1/R_2 \).

The interest in this class of models was initially due to the observation that, in contrast to the heterotic case, the \( \Delta I \)'s have logarithmic running which, even for a low string scale, may continue up to the Planck scale [21] (more precisely up to the first winding mode close to this scale). Despite the absence of power-like terms in (18), such models still have fine-tuning problems and this has been extensively discussed in [20]. The origin of the fine tuning is the difficulty in keeping light the closed string state with vacuum quantum numbers (dual to the first winding mode) which provides the cut-off for the gauge coupling running. Its mass is \( M_I^2/M_P \) and to obtain running to the MSSM gauge unification scale \( M_I \) should be large (\( > 10^{10} \) GeV).

In the following we will determine the relative precision factor \( \mathcal{R} \), induced by the structure of the thresholds (18), assuming that the linear combination of beta functions entering the definition of \( \mathcal{R} \) is of order \( O(1) \). The case this is not true is discussed in Section 4. The variation of the thresholds with respect to \( r_{1,2} = M_I R_{1,2} \) where \( R_{1,2} \) are the compactification scale(s) gives the relative fine tuning measure

\[
\mathcal{R} \approx \frac{d\Delta I}{d(\ln r_1)} = -\frac{1}{2} - 2 \rho \frac{d}{d\rho} \ln \eta(i\rho)
\]  

(19)

where \( \rho = r_1/r_2 \). Quantitatively \( \mathcal{R} \approx 10 \) for \( r_1/r_2 < 1/20 \) or \( r_1/r_2 > 20 \). Since this significantly degrades the precision we are led to the conclusion that anisotropic compactifications are severely limited for reasonable values of the beta coefficients. Models with large radii of compactification are apparently still allowed as it is possible to have \( r_1 \) and \( r_2 \) very large while keeping \( \rho \equiv r_1/r_2 \approx O(1) \). However they suffer from an even more severe problem in that the \( N = 1 \) beta functions have to be proportional to the \( N = 1 \) beta functions if the MSSM predictions are to be preserved (20). We should stress that this sector does not correspond to Kaluza Klein excitations of the Standard Model states and so to obtain negative beta functions it is necessary that it involves a new massive gauge sector which also carries Standard Model gauge quantum numbers. Even with this it is very difficult to construct a spectrum giving beta functions proportional to the Standard Model beta functions because a \( N=2 \) copy of the MSSM representations gives a contribution to the beta functions which is not proportional to the \( N = 1 \) contribution (20). Even more implausible is the possibility that two-loop corrections should be the same as in the MSSM. Thus in this case the good prediction of the MSSM must be viewed as a complete accident as the running above the string scale comes from a spectrum completely unrelated to that of the MSSM.

### 3.4 Type I/I′ models without a \( N=2 \) sector

For models of this class we address a generic type I string inspired model [22] which shares some similarities with the MSSM because it only has logarithmic running, but the spectrum of the light states and the hypercharge normalisation are different. Somewhat unexpectedly, we will show that logarithmic RG evolution is not always a sufficient condition for a low threshold sensitivity of \( \alpha_3(M_Z) \) prediction. This is so because the definition of \( \mathcal{R} \) depends not only on the scale, but also on the

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14We consider in the following only compactification on a single torus, \( T^1 \).
beta function coefficients which in the model to discuss will play a more important role following the non-standard hypercharge normalisation.

The model is derived from D=4, N=1 compact type IIB orientifolds with $D_p$ branes and anti-$D_p$ branes located at different points of the underlying orbifold. It has gravity mediated supersymmetry breaking, full Standard Model gauge group, has no N=2 sector and the UV cut-off scale for the N=1 sector is the string scale. The predictions of the model for the gauge couplings unification were discussed at one loop order in and at two loop level in. It involves logarithmic unification at a low value of the string scale ($\approx 10^{12}$ GeV). Here we address the issue of the relative sensitivity $R$ of the prediction for $\alpha_3(M_Z)$ (or $\sin^2 \theta_W$) to the heavy thresholds compared to the SUSY thresholds. The reason a low unification scale is possible in such models is because the (string) unification relation between gauge couplings is not that given by $SU(5)$ due to a different hypercharge normalisation. To compensate for this it is necessary to extend the MSSM light spectrum to include five further pairs of Higgs doublets (of mass $m$) and three vector-like right handed colour triplets ($d+d^c$). When the latter have a mass equal to $\tilde{m}$ as well, the full two loop RGE equations derived from (7) are given by

$$\alpha^{-1}_i(M_Z) = -\delta_i + \alpha^{-1}_g + \frac{b_i}{2\pi} \ln \left[ \frac{M_s}{M_Z} \right] + \frac{\delta b_i}{2\pi} \ln \left[ \frac{M_s}{\tilde{m}} \right] + \frac{1}{4\pi} \sum_{j=1}^{3} Y_{ij} \ln \left[ \frac{\alpha_g}{\alpha_j(\tilde{m})} \right] + \frac{1}{4\pi} \sum_{j=1}^{3} \frac{b_{ij}}{b_j} \ln \left[ \frac{\alpha_g}{\alpha_j(M_Z)} \right]$$

where

$$Y_{ij} = \delta b_j/(b_j b'_j)(2b_j T_j(G) \delta b_j - b_{ij}), \quad T_j(G) = \{0, 2, 3\}_j, \quad b_{ij} \text{ is the two loop beta function as in the MSSM but with } 3/11 \text{ hypercharge normalisation, } b_j = \{11\xi, 1, -3\}_j, \quad (\xi = 3/11), \quad b'_j = b_j + \delta b_j, \quad \text{with } \delta b_j = \{7\xi, 5, 3\} \text{ to account for the additional five Higgs pairs and three pairs of right handed colour triplets.}$$

The value of $R$ is easily computed at one loop giving

$$R = \frac{14}{19} (2\delta b_1 - 3\delta b_2 + \delta b_3) \approx 6$$

where the definition of $R$, (similar to that of with $\mu_0 = \tilde{m}, M_s$ fixed) takes into account the different hypercharge normalisation of the model. This value for $R$ implies a threshold sensitivity with respect to $M_s/\tilde{m}$ significantly greater than that of the MSSM with respect to $T_{eff}$. This may also be seen in the full two loop analysis which is used in Figure 4(a) and shows (at a global level) that a change of the ratio $M_s/\tilde{m}$ by a factor of 4, to match the uncertainty in $T_{eff}$, gives a much larger variation than that found due to SUSY threshold uncertainty. The ratio of the two areas is $\approx 6$. The problem is even worse if one relaxes the assumption of equal bare masses for the five Higgs pairs and colour triplets.

This example illustrates the problems associated with any model which seeks to reduce the unification scale by going to a non-SU(5) normalisation for the weak hypercharge. While the increased threshold sensitivity and the associated loss of precision in the prediction for gauge couplings is bad enough we think an even more significant indictment of the scheme is that such models require that the precision agreement of the MSSM unification prediction with experiment is just a celestial joke!

4 Two loop limits on the compactification scale

So far we have discussed the one-loop threshold corrections coming from eq. We now turn to the case the linear combination

$$f(b_i, \tilde{b}_i) = \tilde{b}_1 \frac{b_{23}}{b_{12}} + \tilde{b}_2 \frac{b_{31}}{b_{12}} + \tilde{b}_3$$

\footnote{corresponding to a change of $T_{eff}$ between 250 to 1000 GeV.}
is zero and the one-loop corrections to the correlation between $\alpha_3$ and $\sin^2 \theta_W$ vanish. In such a case the heavy threshold correction only comes from two loops (and beyond).

The two loop corrections come from the coefficients $Z_{\phi}(\Lambda, Q)$ in eq.(7). These are the one-loop matter wavefunction renormalisation coefficients above the scale $Q$. It is necessary to regulate the theory in order to determine the $Z_{\phi}(\Lambda, Q)$. In the case of a GUT (effective) theory it is normally assumed that they are unity at the GUT scale (i.e. $\Lambda$ is chosen as the GUT scale). However this is not the correct prescription if the GUT emerges from string compactification for the string regularisation requires that $\Lambda$ be the string cut-off scale, usually the string scale.

The coefficients $Z$ have two contributions. One is due to the usual (gauge and Yukawa) corrections from the $\mathcal{N}=1$ “MSSM-like” massless states which generate the $Z_{\phi}(\Lambda, Q)$ at one loop. These lead to corrections in the gauge couplings at $\mathcal{O}(\alpha^2/(4\pi))$. The second contribution to $Z_{\phi}(\Lambda, Q)$ comes (for $Q > \mu_0$) from towers of $(\mathcal{N}=2,4)$ Kaluza Klein states. Such corrections of order $\mathcal{O}(N\alpha^2/4\pi)$ and will give $\mathcal{R} \approx N\alpha/4\pi$. The constraint that $\mathcal{R}$ should not be large requires $N < 4\pi/\alpha$ (equivalent to the requirement that the theory remain perturbative). While less constraining than the one-loop constraint, this is still very strong due to the rapid increase in the number of Kaluza Klein states, $N \approx (\Lambda/\mu_0)^{\delta}$, $\delta$ being the number of extra dimensions. For $\delta = 2$, $\Lambda < 10\mu_0$ while for $\delta = 6$, $\Lambda < 2.5\mu_0$. Thus, even if one can avoid the large threshold sensitivity (“power-like”) induced at one loop level by the (heterotic) string thresholds $\Delta$ (or at the effective field theory level by towers of Kaluza Klein states) through an (string) construction which sets $f(b_i, \tilde{b}_i) = 0$, the constraint on two-loop contributions still requires
the cut-off (string scale) $\Lambda$ and the compactification scale $\mu_0$ should be very close. In particular string models with a Grand Unified group unbroken below the compactification scale should still have the compactification scale close to the string scale.

5 Summary : A profile of a String Model

In this paper we have considered two issues. The first is the precision of the prediction in the MSSM for low-energy gauge couplings assuming an underlying unification at a high scale. Taking account of the uncertainties due to the unknown masses of the SUSY partners of Standard Model states we found that the predicted range for the gauge couplings compared to the a priori range of possible couplings is remarkably precise, between 2 and 0.2%. This gives a quantitative estimate of how significant is the gauge unification prediction. The prediction for $\sin^2 \theta_W$ itself is also impressive with an accuracy of 1.3%.

This remarkable precision led us to consider the nature of an underlying (string) theory that can maintain this accuracy. Due to the non-decoupling of the contribution of massive states to renormalisable terms in the low energy effective Lagrangian, the requirement that the gauge predictions be undisturbed places strong constraints on the massive sector. We determined the contribution of states transforming non-trivially under the Standard Model gauge group in compactified string models. Requiring that these contributions leave the MSSM predictions intact leads to a constraint on the magnitude of the compactification radius associated with the propagation of such states to limit the number of states in the Kaluza Klein tower. We found that for a wide class of string theories the contribution of heavy states occurs at one-loop order and the radius should be very close to the inverse of the string scale (well within a factor of 2). In the cases the one-loop contributions vanishes the limit is only slightly relaxed.

The implication of this result is quite far-reaching. One immediate one is that unification at a low scale through power law running of the gauge couplings cannot maintain the precision of the MSSM predictions. Indeed, due to the different contributions to the beta functions of massive compared to massless SUSY representations, low scale unification requires a very different multiplet content from the MSSM in order to obtain the same gauge unification prediction. As a result the precision prediction for gauge couplings must be considered a fortuitous accident, something we find hard to accept given the remarkable precision of the prediction. Even if we do accept this, and the N=2 sector happens to give the same beta function as the N=1 MSSM spectrum, power law running introduces a strong dependence on the heavy thresholds so that the precision of the “prediction” is lost. For both these reasons we consider a low scale of unification due to power law running to be unlikely. Models with a low scale of unification due to a non-$SU(5)$ hypercharge normalisation do not require power law running. Nonetheless it turns out that they too have enhanced threshold dependence due to the need for additional light states and again this loses the predictive power of the MSSM.

The profile of our string model which preserves the precision prediction for gauge couplings found in the MSSM therefore requires a large scale of unification with an $SU(5)$ normalisation for the weak hypercharge. Even so, there is still a strong constraint on the compactification scale because the cutoff of the contribution of heavy states is at the string scale or higher. To avoid the same power law running corrections that degraded the predictions in the case of low scale unification, the radius of compactification of those dimensions in which Standard Model gauge non-singlet fields propagate must not be large compared to the cutoff radius. This means the compactified string theory lies far from the Calabi-Yau limit and close to the superconformal limit. This is quite attractive in that many aspects of the effective low-energy field theory, such as Yukawa couplings, are amenable to calculation in the
superconformal limit.

The need for a high scale of unification broadly fits the expectation in the (weakly coupled) heterotic string. However in detail there is a discrepancy between the MSSM value for the unification scale of $3.10^{16}$ GeV and the prediction in the weakly coupled heterotic string, approximately a factor of 20 larger. Three explanations have been suggested.

The first is that heavy threshold effects raises the unification scale to the predicted value. The requirement that the precision of the gauge coupling prediction be maintained severely limits the magnitude of these threshold effects and precludes explanations requiring large radii. However in models with Wilson line breaking it is possible to have very large threshold corrections to the unification scale while keeping the threshold contribution to gauge couplings small. Given that such Wilson line breaking is very often needed to break the underlying gauge symmetry of the heterotic string, this explanation seems very reasonable.

A second possibility is to have a GUT below the string compactification scale, the GUT breaking scale being the unification scale. Even in this case it is still necessary for the compactification scale to be very close to the string scale to keep the two loop contributions from the Kaluza Klein states small. If this theory comes from the weakly coupled heterotic string the compactification and string scales must be close to the Planck scale.

Of course our analysis does not preclude the existence of large new dimensions not associated with the propagation of Standard Model states. A particular example is the strongly coupled heterotic string in which gravity but not the Standard Model states propagate in the eleventh dimension. For the case the eleventh dimension is three orders of magnitude larger than the string length, corresponding to a compactification radius of $O(10^{-14} \text{ fm})$, the gauge unification scale may be reduced to that found in the MSSM. This provides a third way to reconcile the predicted unification scale with that needed in the MSSM. However the size of the extra dimensions is still severely constrained by the need to have a high unification scale, the minimum occurring for just one additional dimension as in the strongly coupled heterotic case. Thus even in the case of large new dimensions in which Standard Model states do not propagate, the new dimension cannot be larger than $10^{-14}$ fm.

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