Partition function of $\mathcal{N} = (2, 2)$ supersymmetric sigma models and Special geometry for the two-moduli non-Fermat Calabi-Yau manifold

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Abstract

We study the new case of the application of the JKLMR conjecture on the connection between the exact partition functions of $\mathcal{N} = (2, 2)$ supersymmetric gauged linear sigma models (GLSM) on $S^2$ and special Kähler geometry on the moduli spaces of Calabi-Yau manifold $Y$. The last ones arise as manifolds of the supersymmetric vacua of the GLSM. We establish this correspondence using the Mirror symmetry in Batyrev's approach. Namely, starting from the two-moduli non-Fermat Calabi-Yau manifold $X$ we construct the dual GLSM with the supersymmetric vacua $Y$, which is the mirror for $X$. Knowing the special geometry on the complex moduli space of $X$ we verify the mirror version of the JKLMR conjecture by explicit computation.

1 Introduction

Superstring theory is considered as a possible approach for unifying the Standard model and Quantum gravity. To obtain $4d$ theory with Spacetime supersymmetry (which is needed for the phenomenological reasons) we have to compactify 6 of 10 dimensions of Superstring theory on Calabi-Yau manifolds $X$ \cite{1,2}. The resulting Lagrangian of the low-energy effective theory is defined by so called Special geometry which appears on the moduli space of CY manifold $X$ \cite{3,4,5}. Indeed the moduli space of Calabi-Yau manifold $X$ is a product of two factors: Moduli space $M_K(X)$ of the Kähler structure deformations and Moduli space of the complex structure deformations $M_C(X)$. Therefore for finding the Effective low energy theory we have to compute the Special Kähler geometry on the both Moduli spaces of CY manifolds.

A new approach for computing the Special geometry on the moduli space of complex structures $M_K(X)$ has been introduced in \cite{6} some time ago. This method is based on the isomorphism between the middle cohomologies on CY and Chiral ring defined by the
polynomial $W_X$ whose zero locus is the CY hypersurface $X$ in the weighted projective space.

On the other hand the conjecture for the explicit expression for the Kähler potential for the moduli space of the Kähler structure deformations $M_K(Y)$ was suggested and checked recently \[7\]. This conjecture (JKLMR conjecture) is the equality

$$e^{-K_Y^K} = Z_Y,$$

where $K_Y^K$ is the Kähler potential of the special geometry on the Kähler moduli spaces of Calabi-Yau manifold $Y$ defined as a hypersurface in the toric variety. The $Z_Y$ is the partition function of some $N = (2,2)$ gauge linear sigma model (GLSM) \[8\] on $S^2$. Partition function $Z_Y$ was computed exactly by the Supersymmetric localization technique in \[9, 10\]. In this case the CY manifold $Y$ coincides with the space of the supersymmetric vacua states of this GLSM. The JKLMR conjecture was proven some time ago in \[11, 12, 13\].

Since the JKLMR conjecture is right, then due to the mirror symmetry, its mirror version should be right \[14, 15, 16\] as well

$$Z_Y = e^{-K_X^X(\psi_l)}.$$

Here $K_X^X(\psi_l)$ is the Kähler potential on $M_C(X)$ - moduli space of complex structures of Calabi-Yau family $X$, which is the mirror partner to $Y$, and $\psi_l$ are the complex structure moduli, the coordinates on $M_C(X)$.

Kähler potential $K_X^X(\psi_l)$ for mirror quintic threefold was computed firstly in \[5\]. Thereafter in \[18, 19, 20, 21\] the special geometry on the moduli space of complex structures $M_C(X)$ has been computed for the wide set of Calabi-Yau manifolds.

The mirror version of JKLMR conjecture \[2\] has been verified \[7, 15, 16\] for a few cases, see also \[17\].

In this work we present a verification of the mirror conjecture \[2\] for the case that belongs to a class of Calabi-Yau manifolds considered by Berglund and Hubsch in \[22\].

The key point of our approach is using the Batyrev’s construction \[23\]. A Calabi-Yau threefold $X$, defined as a hypersurface in a weighted projective space $\mathbb{P}^4_{(k_1, \ldots, k_5)}$ is given by zero locus of quasihomogenous polynomial $W_X$. Exponents of the monomials in the $W_X$ define finite set $\vec{V}_a$, $a = 1, \ldots, N$. Their convex set defines Batyrev’s polytope $\Delta_X$ \[23\]. We use the set of vectors $\vec{V}_a$ for constructing the fan \[24\], which defines a toric variety. Calabi-Yau manifold $Y$, which is the mirror of $X$, is realized as a hypersurface in this toric variety by a homogeneous polynomial $W_Y$. Knowing the fan we find GLSM, its gauge group and the chiral multiplet charges. The last ones appear as a coefficients of linear relations between the vectors of the fan and define a weights of a toric variety.

### 2 Special geometry for the two-moduli non-Fermat Calabi-Yau

In this paper, we consider the non-Fermat type manifold $X$ with two deformations of complex structure \[22\]. The manifold $X$ is defined as a hypersurface in a weighted projective space:

$$\mathbb{P}^4_{(k_1, \ldots, k_5)} = \{(x_1, \ldots, x_5) \mid (x_1, \ldots, x_5) \sim (\lambda x_1, \ldots, \lambda x_5)\}.$$

The main point is that this model is not of the Fermat type, which was considered before in \[10\]. Namely, let Calabi-Yau $X$ is given by equation in $\mathbb{P}^4_{(3,2,2,7,7)}$

$$W_X(x|\psi) = x_1^7 + x_2^4x_4 + x_3^3 + x_4^3x_5 + x_5^3 - \psi_1x_1x_2x_3x_4x_5 + \psi_2x_1^3x_2^3x_3^3 = 0.$$
The degree of the polynomial equals $d = 21$. The phase symmetry group of $W_X$ at $\psi_1 = \psi_2 = 0$ is $\mathbb{Z}_2^2 \times \mathbb{Z}_7$.

We consider a quotient $X/ H$, where $H := (\mathbb{Z}_{21} : 12, 2, 0, 7, 0)$. The Hodge numbers of this 2-parameter family are $h_{1,2}(X) = 2$ and $h_{1,1}(X) = 95$. The basis in the invariant part of Milnor’s ring $R$ consists of monomials $R^Q = \{ e_1, \ldots, e_6 \}$ where $e_1 = 1, e_2 = x_1 x_2 x_3 x_4 x_5, e_3 = x_1^2 x_2 x_3^3, e_4 = e_2^2, e_5 = e_2 e_3 \text{ and } e_6 = e_2^2 e_3$.

Special geometry on the moduli space of complex structures is given by Kähler potential, which was computed in [21] and can be rewritten in a form

$$e^{-K_X^*(\psi_1,\psi_2)} = \sum_{\bar{\mu} = (\nu, \mu)} (-1)^{|\bar{\mu}|} \gamma^3 \left( \frac{\mu}{7} \right) \gamma^2 \left( \frac{\nu}{7} \right) |\sigma_{\bar{\mu}}(\psi_1, \psi_2)|^2,$$

here $|\bar{\mu}| = 3(\nu - 1) + 2(\mu - 1)$.

$$\sigma_{\bar{\mu}}(\psi_1, \psi_2) = \sum_{m,n \in \Sigma_{\bar{\mu}}} (-1)^m \frac{\Gamma^3 \left( \frac{1+n+3m}{7} \right)}{\Gamma^3 \left( \frac{n}{7} \right)} \frac{\Gamma^2 \left( \frac{2+2n-m}{7} \right)}{\Gamma^2 \left( \frac{m}{7} \right)} \psi_1^n \psi_2^m \frac{n!m!}{!},$$

$$\Sigma_{\bar{\mu}} = \{ n, m \in \mathbb{Z}_{\geq 0} \mid 1+n+3m = \nu \text{ (mod) } 7; 2+2n-m = \mu \text{ (mod) } 7 \},$$

$$\bar{\mu} = (\nu, \mu) = (1, 2), (2, 4), (3, 6), (4, 1), (5, 3), (6, 5).$$

3 Gauged Linear Sigma Model

For the first time, this model was considered by Witten in [8]. It was shown in [9,10] that this model can be defined on $S^2$ with preserving $\mathcal{N} = (2, 2)$ supersymmetry. It allows to compute the exact partition function by supersymmetric localization technique [9,10].

Lagrangian of this model [8] is a sum of Yang-Mills Lagrangian $\mathcal{L}_{YM}$, kinetic term $\mathcal{L}_{\text{mat}}$ for matter chiral superfields $\Phi_{al}$, $a = 1, \ldots, N$, Fayet–Iliopoulos term $\mathcal{L}_{FI}$ and the term with superpotential $W_Y(\Phi_{al})$.

We consider the theory with gauge group $G = U(1)^h := \prod_{i=1}^h U(1)_l$ with $h$ gauge vector superfields $(V_1, \ldots, V_h)$. Supersymmetric vacua space of the potential energy for the scalar fields $\phi_a$ of chiral multiplet is a hypersurface $Y$ in a toric variety for suitable values of Fayet–Iliopoulos parameters $r_l$.

The toric varieties themselves form a family depending on parameters $r_l$ and theta angles $\theta_l$. Hypersurfaces $Y$ in each of these toric varieties form a family of the Calabi-Yau manifolds, which depend on the coefficients of the polynomial $W_Y$. The last ones are moduli of the complex structure of $Y$. The polynomial $W_Y$ is invariant with respect to coordinate transformations in the toric variety $\phi_a \rightarrow \lambda Q_{al} \phi_a$. The weights $Q_{al}$ are nothing but charges of $U(1)_l$ action.

Potential energy for the scalar fields in this theory is given by [8]

$$U(\phi) = \sum_{l=1}^h e_l^2 \left( \sum_{a=1}^N Q_{al} |\phi_a|^2 - r_l \right)^2 + \frac{1}{4} \sum_{a=1}^N \left( \frac{\partial W_Y}{\partial \phi_a} \right)^2,$$

Here we denote by $(e_1, \ldots, e_h)$ the coupling constants.

Supersymmetric vacua of the theory is given by minima of the potential modulo gauge symmetry. For $r_l > 0$ it is defined as a manifold

$$Y = \left\{ (\phi_1, \ldots, \phi_N) \mid \sum_{a=1}^N Q_{al} |\phi_a|^2 = r_l, l = 1, \ldots, h, \frac{\partial W_Y}{\partial \phi_a} = 0 \right\} / U(1)^h.$$
An equivalent way \cite{24} to define a manifold (10) is to set $\frac{\partial W_Y}{\partial \phi_a} = 0$ in a toric variety defined as a quotient
\[ C^N / (C^*)^h := (C^N - Z) / (C^*)^h, \]
where $Z$ is some $(C^*)^h$ invariant subset. The charges $Q_{al}$ are weights of the torus action $(C^*)^h$ on $C^N$, $\phi_a \rightarrow \lambda^{Q_{al}} \phi_a$, $a = 1, \ldots, N$, $l = 1, \ldots, h$. That definition is needed for constructing the mirror manifold $Y$ to the initial one $X$ according to the Batyrev construction.

Partition function of GLSM was computed exactly using the Supersymmetric Localization and given by the formula \cite{9, 10}:
\[
Z_Y = \sum_{m_l \in \mathbb{Z}} \prod_{l=1}^{h} e^{-i\theta_{ml}} \int_{C_1} \cdots \int_{C_h} \prod_{l=1}^{h} \frac{d\tau_l}{(2\pi i)} e^{4\pi i \tau_l \sum_{a=1}^{N} \Gamma \left( \frac{q_a/2 + \sum_{l=1}^{h} Q_{al}(\tau_l - \frac{m_l}{2})}{2} \right) \Gamma \left( 1 - \frac{q_a/2 + \sum_{l=1}^{h} Q_{al}(\tau_l + \frac{m_l}{2})}{2} \right)},
\]
where the contours $C_i$ go along the imaginary axis. The parameters $q_a$ denote the R-symmetry charges. Note that partition function (12) does not depend on the coupling constants $\epsilon_i$ and the specific choice of the superpotential $W_Y$.

The conjecture proposed by Jockers et al \cite{7} is that partition function (12) matches with the exponent of the Kähler potential $e^{-K_Y}$ of the Kähler moduli space of Calabi-Yau manifold $Y$, defined as a hypersurface in a toric variety. The last one arises as a manifold of the supersymmetric vacua of the GLSM. \cite{10}. This statement has been checked for a few examples of Calabi-Yau manifolds in \cite{6, 14, 15}. The main problem of verification is the complexity of computing the special Kähler geometry $e^{-K_Y}$.

Since the construction of the mirror symmetry implies the equality
\[ K_Y = K_X, \]
then the conjecture (1) as mentioned above, can be checked in mirror form \cite{14, 15}
\[ e^{-K_C} = Z_Y. \]

We will verify (14) for the case of Calabi-Yau 2-parameter family mentioned above, using the previously computed $K_Y$ in the work \cite{21}.

4 Mirror symmetry

Let us discuss the method for constructing Calabi-Yau threefold $Y$ that is a mirror partner for the $X$, developed in \cite{15}. Consider Calabi-Yau manifold $X$ defined by zero locus of the quasi-homogeneous polynomial in a weighted projective space $\mathbb{P}^4(k_1, \ldots, k_5)$.
\[
W_X(x_1, \ldots, x_5|\psi_1, \ldots, \psi_h) = \sum_{a=1}^{h+5} C_a \prod_{i=1}^{5} x_i^{v_{ai}}, \quad \sum_{i=1}^{5} k_i v_{ai} = d = \sum_{i=1}^{5} k_i. \]

Here $\psi_i$ are the coordinates on the moduli space of complex structures $M_C(X)$. In fact, the equation (15) defines the whole family of Calabi-Yau manifolds, corresponding to the points on the moduli space.

The set of exponents $v_{ai}$ corresponds to the coordinates of vectors $\vec{V}_a \in \mathbb{R}^5$, that is $v_{ai} = (V_a)_i$. These vectors define a fan which defines a toric manifold that contains the mirror Calabi-Yau manifold $Y$. More precisely, they are the edges of this fan. Using
this fact we will build the mirror manifold 

These vectors \( \vec{V}_a \) being vectors in five-dimensional space satisfy the linear relations

\[
\sum_{a=1}^{5+h} Q_{al} \vec{V}_a = 0, \quad l = 1, \ldots, h,
\]  

(16)

where \( Q_{al} \) is a set of integer numbers that we choose such that they form an integral basis of the linear relations between the exponents of the monomials of \( W_X \).

So now, using the data \( Q_{al} \) we can define a toric variety \( CN// (\mathbb{C}^*)^h, N = h + 5 \). Namely, consider the coordinates \((\phi_1, \ldots, \phi_N)\) in \( \mathbb{C}^N \) and define the factorization

\[
(\phi_1, \ldots, \phi_N) \sim (\lambda^{Q_{al}} \phi_1, \ldots, \lambda^{Q_{al}} \phi_N), \quad l = 1, \ldots, h.
\]  

(17)

Then the mirror Calabi-Yau manifold \( Y \) for \( X \) is realized as a hypersurface in the toric variety given by the homogeneous polynomial \( W_Y \) such that

\[
W_Y(\lambda^{Q_{al}} \phi_1, \ldots, \lambda^{Q_{al}} \phi_N) = W_Y(\phi_1, \ldots, \phi_N).
\]  

(18)

5 Verification of the conjecture

Let us proceed to the verification of the mirror version of the JKLMMR conjecture \cite{7}. Namely, we will show by explicit computation the equality \( Z_Y = e^{-K_X} \).

We write the polynomial (4) in a form

\[
W_X(x|\psi) = \sum_{a=1}^{7} C_a \prod_{i=1}^{5} x_i^{v_{ai}}.
\]  

(19)

The exponents \( v_{ai} \) are coordinates of the vectors \( \vec{V}_a \in \mathbb{R}^5 \), that is \( v_{ai} = (\vec{V}_a)_i \). Where vectors \( \vec{V}_a \) are

\[
\vec{V}_1 = \begin{pmatrix} 7 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{V}_2 = \begin{pmatrix} 7 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{V}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{V}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{V}_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{V}_6 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{V}_7 = \begin{pmatrix} 3 \\ 3 \\ 3 \\ 0 \\ 0 \end{pmatrix}.
\]  

(20)

These seven vectors being five-dimensional satisfy two linear relations:

\[
\sum_{a=1}^{7} Q_{al} \vec{V}_a = 0, \quad l = 1, 2,
\]  

(21)

here the \( Q_{al} \) is a set of integer numbers such that (21) defines an integer basis in a space of linear relations between vectors \( \vec{V}_a \).

The convenient choice of the \( Q_{al} \) is:

\[
Q_{a1} = (1, 1, 0, 1, 0, -1, -2), \quad Q_{a2} = (0, 0, 1, 0, 1, -3, 1).
\]  

(22)

Let us construct the connection between the model with manifold \( X \) and Gauged Linear Sigma Model. Following the approach developed in \cite{15} we set \( h = h_{2,1} = 2 \), i.e. consider a theory with a gauge group \( U(1)^2 \) and \( N = 5 + h = 7 \) chiral multiplets with charges \( Q_{al} \) from the (22).
The exact partition function of this model is given by the expression:

\[ Z_Y = \sum_{m_i \in \mathbb{Z}} e^{-i\theta_1 m_1} e^{-i\theta_2 m_2} \int_{C_1} \int_{C_2} \frac{d\tau_1}{(2\pi i)} \frac{d\tau_2}{(2\pi i)} e^{4\pi i \tau_1 \tau_1} e^{4\pi i \tau_2 \tau_2} \times \]

\[ \times \prod_{a=1}^{7} \frac{\Gamma \left( q_a/2 + Q_{a1}(\tau_1 - \frac{m_1}{2}) + Q_{a2}(\tau_2 - \frac{m_2}{2}) \right)}{\Gamma \left( 1 - q_a/2 - Q_{a1}(\tau_1 + \frac{m_1}{2}) - Q_{a2}(\tau_2 + \frac{m_2}{2}) \right)}. \]  \hspace{1cm} (23)

We set the charges of R-symmetry \( q_1 = q_2 = q_4 = 2/7, \) \( q_3 = q_5 = 4/7, \) \( q_6 = q_7 = 0. \)

We introduce a change of coordinates on a vacua space

\[ z_1 = e^{-\frac{2\pi}{7} \left[ (r_1 + 2\theta_1) + 2(r_2 + 2\theta_2) \right]}, \]

\[ z_2 = e^{-\frac{2\pi}{7} \left[ 3(r_1 + 2\theta_1) - (r_2 + 2\theta_2) \right]}. \]  \hspace{1cm} (24)

Then we obtain

\[ Z_Y = \sum_{m_1} \sum_{m_2} \int_{C_1} \int_{C_2} \frac{d\tau_1}{(2\pi i)} \frac{d\tau_2}{(2\pi i)} z_1^{-(\tau_1 - \frac{m_1}{2}) - 3(\tau_2 - \frac{m_2}{2})} z_2^{-(\tau_1 + \frac{m_1}{2}) - 3(\tau_2 + \frac{m_2}{2})} \times \]

\[ \times z_2^{-2(\tau_1 - \frac{m_1}{2}) + (\tau_2 - \frac{m_2}{2})} z_1^{-2(\tau_1 + \frac{m_1}{2}) + (\tau_2 + \frac{m_2}{2})} \times \]

\[ \times \frac{\Gamma^3 \left( 1/7 + (\tau_1 - \frac{m_1}{2}) \right)}{\Gamma^3 \left( 1 - 1/7 - (\tau_1 + \frac{m_1}{2}) \right)} \frac{\Gamma^2 \left( 2/7 + (\tau_2 - \frac{m_2}{2}) \right)}{\Gamma^2 \left( 1 - 2/7 - (\tau_2 + \frac{m_2}{2}) \right)} \times \]

\[ \times \frac{\Gamma \left( -2(\tau_1 - \frac{m_1}{2}) - 3(\tau_2 - \frac{m_2}{2}) \right)}{\Gamma \left( 1 + (\tau_1 + \frac{m_1}{2}) + 3(\tau_2 + \frac{m_2}{2}) \right)} \frac{\Gamma \left( -2(\tau_1 + \frac{m_1}{2}) + (\tau_2 - \frac{m_2}{2}) \right)}{\Gamma \left( 1 + 2(\tau_1 + \frac{m_1}{2}) - (\tau_2 + \frac{m_2}{2}) \right)}. \]  \hspace{1cm} (25)

For \( |z_i| > 1 \) the contours can be deformed to the right half-plane. Then, by Cauchy theorem, the integrals \([25]\) are reduced to the sum of residues at the poles of the gamma function

\[ \left( \tau_1 - \frac{m_1}{2} \right) + 3 \left( \tau_2 - \frac{m_2}{2} \right) = n, \quad 2 \left( \tau_1 - \frac{m_1}{2} \right) - \left( \tau_2 - \frac{m_2}{2} \right) = m, \quad m, n = 0, 1, \ldots \]  \hspace{1cm} (26)

Also denote

\[ \bar{n} = \left( \tau_1 + \frac{m_1}{2} \right) + 3 \left( \tau_2 + \frac{m_2}{2} \right), \quad \bar{m} = 2 \left( \tau_1 + \frac{m_1}{2} \right) - \left( \tau_2 + \frac{m_2}{2} \right), \]  \hspace{1cm} (27)

\[ \left( \tau_1 - \frac{m_1}{2} \right) = \frac{n + 3m}{7}, \quad \left( \tau_2 - \frac{m_2}{2} \right) = \frac{2n - m}{7}, \]

\[ \left( \tau_1 + \frac{m_1}{2} \right) = \frac{\bar{n} + 3\bar{m}}{7}, \quad \left( \tau_2 + \frac{m_2}{2} \right) = \frac{2\bar{n} - \bar{m}}{7}. \]  \hspace{1cm} (28)

When \( 1 + n + 3m = 0 \) (mod 7) and \( 2 + 2n - m = 0 \) (mod 7) the gamma functions in the denominator of the formula \([25]\) have poles, therefore the corresponding terms in the sum vanish. It follows that the sum in \([25]\) effectively goes over the set:

\[ \Sigma_{\bar{m}} = \{ n, m \in \mathbb{Z}_{>0} \mid 1 + n + 3m = \nu \) (mod 7); \( 2 + 2n - m = \mu \) (mod 7) \}, \]  \hspace{1cm} (29)

\[ \bar{\mu} = (\nu, \mu) = (1, 2), (2, 4), (3, 6), (4, 1), (5, 3), (6, 5). \]  \hspace{1cm} (30)

From the relations \( \bar{n} + 3\bar{m} = n + 3m + 7m_1, \) \( 2\bar{n} - \bar{m} = 2n - m + 7m_2 \) we conclude that the numbers \( \bar{n} \) and \( \bar{m} \) belong to the same classes \( \Sigma_{\bar{m}}. \)
Therefore the partition function can be rewritten as

\[
Z_Y = \sum_{\mu=(\nu,\mu),(m,n),(\bar{m},\bar{n}) \in \Sigma_{\mu}} z_1^{-n} z_2^{-\bar{n}} z_1^{-m} z_2^{-\bar{m}} \left( \frac{1+n+3m}{7} \right) \Gamma^2 \left( \frac{2+2n-m}{7} \right) \times \\
\times \frac{(-1)^n(-1)^m}{n!m! \Gamma(1+n) \Gamma(1+m)} = \\
\pi^{-5} \sum_{\mu=(\nu,\mu),(m,n),(\bar{m},\bar{n}) \in \Sigma_{\mu}} (-1)^n(-1)^m z_1^{-n} z_2^{-\bar{n}} z_1^{-m} z_2^{-\bar{m}} \times \\
\times \sin^3 \left( \frac{\pi}{7} \right) \sin^2 \left( \frac{\mu}{7} \right) \times \\
\times \Gamma^3 \left( \frac{1+n+3m}{7} \right) \Gamma^2 \left( \frac{2+2n-m}{7} \right) \Gamma^3 \left( \frac{1+n+3\bar{m}}{7} \right) \Gamma^2 \left( \frac{2+2\bar{n}-m}{7} \right) . (31)
\]

Using the identities:

\[
\pi^{-5} \sin^3 \left( \frac{\pi}{7} \right) \sin^2 \left( \frac{\mu}{7} \right) = \\
\pi^{-5}(-1)^{\bar{n}+\bar{m}}(-1)^{\bar{m}} \sin^3 \left( \frac{\pi \nu}{7} \right) \sin^2 \left( \frac{\pi \mu}{7} \right) = \\
(-1)^{\bar{n}+\bar{m}}(-1)^{\bar{m}} \sin^3 \left( \frac{\pi \nu}{7} \right) \sin^2 \left( \frac{\pi \mu}{7} \right) = \\
\left( \frac{1}{\Gamma^3 \left( \frac{\nu}{7} \right) \Gamma^3 \left( 1-\frac{\nu}{7} \right) \Gamma^2 \left( \frac{\nu}{7} \right) \Gamma^2 \left( 1-\frac{\nu}{7} \right) \Gamma^3 \left( \frac{\mu}{7} \right) \right) = \\
\left( \frac{1}{\Gamma^6 \left( \frac{\nu}{7} \right) \Gamma^4 \left( \frac{\mu}{7} \right) \Gamma^4 \left( \frac{\mu}{7} \right) \Gamma^4 \left( \frac{\mu}{7} \right) \Gamma^4 \left( \frac{\mu}{7} \right) \Gamma^4 \left( \frac{\mu}{7} \right) \Gamma^4 \left( \frac{\mu}{7} \right) \right), (32)
\]

we find:

\[
Z_Y = \sum_{\mu=(\nu,\mu)} (-1)^{\bar{m}} \gamma^3 \left( \frac{\nu}{7} \right) \gamma^2 \left( \frac{\mu}{7} \right) |\sigma_{\mu}(z_1, z_2)|^2, (33)
\]

\[
\sigma_{\mu}(z_1, z_2) = \sum_{m,n \in \Sigma_{\mu}} (-1)^m(-1)^n \Gamma^3 \left( \frac{1+n+3m}{7} \right) \Gamma^2 \left( \frac{2+2n-m}{7} \right) \frac{z_1^{-n} z_2^{-m}}{n!m!} . (34)
\]

The expression for the partition function matches with the formula (5) for $e^{-K_{\mathcal{X}}\bar{Z}}$ from the paper [21]. In order to obtain this equality we must identify moduli of complex structures of the Calabi-Yau $X$ with Kähler moduli of the manifold $Y$ as

\[
\psi_1 = -z_1^{-1}, \quad \psi_2 = z_2^{-1}, (35)
\]

where $z_1, z_2$ are connected with the parameters $r_1, \theta_1$ by the formula (24).

These relations give the mirror map for the considered case.

**Conclusion**

Starting from the model with non-Fermat Calabi-Yau $X$ we have constructed the $\mathcal{N} = (2,2)$ Gauged Linear Sigma Model with the manifold of supersymmetric vacua $Y$, which is the mirror for $X$. Knowing the Special geometry on the moduli space of complex structures on $X$ and using Mirror symmetry in Batyrev’s approach [23] we have checked JKLMR conjecture [7] for this case having obtained the explicit equality $Z_Y = e^{-K_{\mathcal{X}}\bar{Z}}$. We done that for the case of Calabi-Yau $X$ of non-Fermat type which was not considered before in this way.

The formula (25) is also important because it gives an analytic continuation for the Kähler potential $K_{\mathcal{X}}$ on the moduli space of complex structures outside the region of convergence of the series (6).
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