Berry-Esséen bounds for the least squares estimator for discretely observed fractional Ornstein-Uhlenbeck processes

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Abstract: Let $\theta > 0$. We consider a one-dimensional fractional Ornstein-Uhlenbeck process defined as $dX_t = -\theta X_t dt + dB_t$, $t \geq 0$, where $B$ is a fractional Brownian motion of Hurst parameter $H \in (\frac{1}{2}, 1)$. We are interested in the problem of estimating the unknown parameter $\theta$. For that purpose, we dispose of a discretized trajectory, observed at $n$ equidistant times $t_i = i\Delta_n, i = 0, \ldots, n$, and $T_n = n\Delta_n$ denotes the length of the ‘observation window’. We assume that $\Delta_n \to 0$ and $T_n \to \infty$ as $n \to \infty$. As an estimator of $\theta$ we choose the least squares estimator (LSE) $\hat{\theta}_n$. The consistency of this estimator is established. Explicit bounds for the Kolmogorov distance, in the case when $H \in (\frac{1}{2}, \frac{3}{4})$, in the central limit theorem for the LSE $\hat{\theta}_n$ are obtained. These results hold without any kind of ergodicity on the process $X$.

Key words: fractional Ornstein-Uhlenbeck processes, discrete-time observation, least squares estimator, Kolmogorov distance, central limit theorem, Malliavin calculus.

1 Introduction

In this paper we consider a fractional Ornstein-Uhlenbeck process $X = (X_t, t \geq 0)$. That is, it solves the linear stochastic differential equation

$$X_0 = x_0; \quad dX_t = -\theta X_t dt + dB_t, \quad t \geq 0,$$

(1.1)

where $x_0 \in \mathbb{R}$, $B = (B_t, t \geq 0)$ is a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$ and $\theta > 0$ is an unknown parameter.

Assume that the process $X$ is observed equidistantly in time with the step size $\Delta_n$: $t_i = i\Delta_n, i = 0, \ldots, n$, and $T_n = n\Delta_n$ denotes the length of the ‘observation window’. The purpose of this paper is to study the least squares estimator (LSE) $\hat{\theta}_n$ of $\theta$ based on the sampling data $X_{t_i}, i = 0, \ldots, n$. The LSE $\hat{\theta}_n$ is obtained as following: $\hat{\theta}_n$ minimizes (formally)

$$\theta \mapsto \sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}} + \theta X_{t_{i-1}} \Delta_n|^2,$$

where $t_i = i\Delta_n, i = 0, \ldots, n$. Thus $\hat{\theta}_n$ is given by

$$\hat{\theta}_n = -\frac{\sum_{i=1}^{n} X_{t_{i-1}}(X_{t_i} - X_{t_{i-1}})}{\Delta_n \sum_{i=1}^{n} X_{t_{i-1}}^2}. \quad (1.2)$$

Also, by using (1.1), we arrive to the following formula:

$$\hat{\theta}_n - \theta = -\frac{\sum_{i=1}^{n} X_{t_{i-1}}U_i}{\Delta_n \sum_{i=1}^{n} X_{t_{i-1}}^2} \quad (1.3)$$

where

$$U_i = X_{t_i} - X_{t_{i-1}} + \theta \Delta_n X_{t_{i-1}}, \quad i = 1, \ldots, n.$$

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The parametric estimation problems for fractional diffusion processes based on continuous-time observations have been studied e.g. in [5, 13, 12] via maximum likelihood method. Recently, the parametric estimation of continuously observed fractional Ornstein-Uhlenbeck process defined in (1.1) is studied in [4, 3], in the case when \( H \in \left( \frac{1}{2}, 1 \right) \), by using the least squares estimators.

From a practical point of view, in parametric inference, it is more realistic and interesting to consider asymptotic estimation for fractional diffusion processes based on discrete observations. There exists a rich literature on the parameter estimation problem for diffusion processes driven by Brownian motions based on discrete observations, we refer to [11] and [12]. In this paper, we focus our discussion on the LSE and the fractional Ornstein-Uhlenbeck process case.

In general, the study of the asymptotic distribution of any estimator is not very useful for practical purposes unless the rate of convergence of its distribution is known. The rate of convergence of the distribution of LSE for some diffusion processes driven by Brownian motions based on discrete time data was studied e.g. in [7]. To the best of our knowledge there is no study of this problem for the distribution of the LSE of the unknown drift parameter in equation (1.1). Our goal in the present paper is to investigate the consistency and the rate of convergence to normality of the LSE \( \hat{\theta}_n \) defined in (1.2).

Recall that, if \( Y, Z \) are two real-valued random variables, then the Kolmogorov distance between the law of \( Y \) and the law of \( Z \) is given by

\[
d_{Kol}(Y, Z) = \sup_{-\infty < z < \infty} \left| P(Y \leq z) - P(Z \leq z) \right|.
\]

Let us now describe the results we prove in this work. In Theorem 3.3 we show that the consistency of \( \hat{\theta}_n \) as \( \Delta_n \rightarrow 0 \) and \( n\Delta_n \rightarrow \infty \) holds true if \( H \in \left( \frac{1}{2}, 1 \right) \). When \( H \in \left( \frac{1}{2}, \frac{3}{4} \right) \) we use the Malliavin calculus, the so-called Stein’s method on Wiener chaos introduced by [8] and the technical Lemmas 3.4 and 3.5 proved respectively by [6] and [2], to derive Berry-Esséen-type bounds in the Kolmogorov distance for the LSE \( \hat{\theta}_n \) (Theorems 3.6 and 3.7).

We proceed as follows. In Section 2 we give the basic tools of Malliavin calculus for the fractional Brownian motion needed throughout the paper. Section 3 contains our main results, concerning the consistency and the rate of convergence of \( \hat{\theta}_n \).

### 2 Preliminaries

In this section we describe some basic facts on the stochastic calculus with respect to a fractional Brownian motion. For more complete presentation on the subject, see [9] and [1].

The fractional Brownian motion \( (B_t, t \geq 0) \) with Hurst parameter \( H \in (0, 1) \), is defined as a centered Gaussian process starting from zero with covariance

\[
R_H(t, s) = E(B_tB_s) = \frac{1}{2} \left( t^{2H} + s^{2H} - |t - s|^{2H} \right).
\]

We assume that \( B \) is defined on a complete probability space \((\Omega, \mathcal{F}, P)\) such that \( \mathcal{F} \) is the sigma-field generated by \( B \). By Kolmogorov’s continuity criterion and the fact

\[
E \left( (B_t - B_s)^2 \right) = |t - s|^{2H}; \quad s, t \geq 0,
\]

we deduce that \( B \) has Hölder continuous paths of any order \( \gamma < H \).

Fix a time interval \([0, T]\). We denote by \( \mathcal{H} \) the canonical Hilbert space associated to the fractional Brownian motion \( B \). That is, \( \mathcal{H} \) is the closure of the linear span \( \mathcal{E} \) generated by the indicator functions \( 1_{[0,t]}, \quad t \in [0, T] \) with respect to the scalar product

\[
(1_{[0,t]}, 1_{[0,s]}) = R_H(t, s).
\]

2
The application $\varphi \in \mathcal{E} \rightarrow B(\varphi)$ is an isometry from $\mathcal{E}$ to the Gaussian space generated by $B$ and it can be extended to $\mathcal{H}$.

If $\mathcal{H} \in \left(\frac{1}{2}, 1\right)$ the elements of $\mathcal{H}$ may not be functions but distributions of negative order (see [10]). Therefore, it is of interest to know significant subspaces of functions contained in it.

Let $|\mathcal{H}|$ be the set of measurable functions $\varphi$ on $[0,T]$ such that

$$\|\varphi\|_{|\mathcal{H}|}^2 := H(2H - 1) \int_0^T \int_0^T |\varphi(u)||\varphi(v)||u - v|^{2H-2} dudv < \infty.$$ 

Note that, if $\varphi, \psi \in |\mathcal{H}|$,

$$E(B(\varphi)B(\psi)) = H(2H - 1) \int_0^T \int_0^T \varphi(u)\psi(v)|u - v|^{2H-2} dudv.$$ 

It follows actually from [10] that the space $|\mathcal{H}|$ is a Banach space for the norm $\|\cdot\|_{|\mathcal{H}|}$ and it is included in $\mathcal{H}$. In fact,

$$L^2([0,T]) \subset L^\infty([0,T]) \subset |\mathcal{H}| \subset H \quad (2.1)$$

Let $C^\infty_c(\mathbb{R}^n, \mathbb{R})$ be the class of infinitely differentiable functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $f$ and all its partial derivatives are bounded. We denote by $S$ the class of smooth cylindrical random variables $F$ of the form

$$F = f(B(\varphi_1),...,B(\varphi_n)), \quad (2.2)$$

where $n \geq 1$, $f \in C^\infty_c(\mathbb{R}^n, \mathbb{R})$ and $\varphi_1, ..., \varphi_n \in \mathcal{H}$.

The derivative operator $D$ of a smooth cylindrical random variable $F$ of the form (2.2) is defined as the $\mathcal{H}$-valued random variable

$$D_tF = \sum_{i=1}^N \frac{\partial f}{\partial x_i} (B(\varphi_1),...,B(\varphi_n))\varphi_i(t).$$

In this way the derivative $DF$ is an element of $L^2(\Omega; \mathcal{H})$. We denote by $D^{1,2}$ the closure of $S$ with respect to the norm defined by

$$\|F\|_{D^{1,2}}^2 = E(\|F\|^2) + E(\|DF\|_{\mathcal{H}}^2).$$

The divergence operator $\delta$ is the adjoint of the derivative operator $D$. Concretely, a random variable $u \in L^2(\Omega; \mathcal{H})$ belongs to the domain of the divergence operator $\text{Dom}\delta$ if

$$E(|DF, u|_{\mathcal{H}}) \leq c_u \|F\|_{L^2(\Omega)}$$

for every $F \in S$. In this case $\delta(u)$ is given by the duality relationship

$$E(F\delta(u)) = E(F, DF, u)_{\mathcal{H}}$$

for any $F \in D^{1,2}$. We will make use of the notation

$$\delta(u) = \int_0^T u_s \delta B_s, \quad u \in \text{Dom}(\delta).$$

In particular, for $h \in \mathcal{H}$, $B(h) = \delta(h) = \int_0^T h_s \delta B_s$.

Assume that $H \in \left(\frac{1}{2}, 1\right)$. If $u \in D^{1,2}(|\mathcal{H}|)$, $u$ belongs to $\text{Dom}\delta$ and we have (see [9] Page 292)

$$E(|\delta(u)|^2) \leq c_H \left(\|E(u)|^2|_{|\mathcal{H}|} + E\left(\|Du\|^2_{|\mathcal{H}| \otimes |\mathcal{H}|}\right)\right),$$

3
where the constant $c_H$ depends only on $H$.

As a consequence, applying (2.1) we obtain that

$$E(|\delta(u)|^2) \leq c_H \left( \|E(u)\|^2_{L^2(0,T)} + E\left( \|Du\|^2_{L^2(0,T)} \right) \right). \quad (2.3)$$

For every $n \geq 1$, let $\mathcal{H}_n$ be the nth Wiener chaos of $B$, that is, the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_n(B(h)), h \in \mathcal{H}, \|h\| = 1\}$ where $\mathcal{H}_n$ is the nth Hermite polynomial. The mapping $I_n(h \otimes n) = n!H_n(B(h))$ provides a linear isometry between the symmetric tensor product $\mathcal{H}^{\otimes n}$ (equipped with the modified norm $\|\cdot\|_{\mathcal{H}^{\otimes n}} = \frac{1}{\sqrt{n}} \|\cdot\|_{\mathcal{H}^{\otimes n}}$) and $\mathcal{H}_n$. For every $f, g \in \mathcal{H}^{\otimes n}$ the following multiplication formula holds

$$E(I_n(f)I_n(g)) = n!(f, g)_{\mathcal{H}^{\otimes n}}.$$ 

Finally, it is well-known that $L^2(\Omega)$ can be decomposed into the infinite orthogonal sum of the spaces $\mathcal{H}_n$. That is, any square integrable random variable $F \in L^2(\Omega)$ admits the following chaotic expansion

$$F = E(F) + \sum_{n=1}^{\infty} I_n(f_n),$$

where the $f_n \in \mathcal{H}^{\otimes n}$ are uniquely determined by $F$.

We will make use of the following theorem proved in [8].

**Theorem 2.1** (Nourdin-Peccati). *Let $F \in D^{1,2}$ has zero mean and $N \sim \mathcal{N}(0,1)$. Then*

$$d_{Kol}(F, N) \leq \sqrt{E \left[ (1-DF, -DL^{-1}F > H^2)^2 \right]} \quad (2.4)$$

*Moreover, if $F = I_q(f)$ with $q \geq 2$ and $f \in \mathcal{H}^{\otimes q}$, then $<DF, -DL^{-1}F > H^2 = \frac{1}{q} \|DF\|^2_{\mathcal{H}}$ and therefore*

$$d_{Kol}(F, N) \leq \sqrt{E \left[ (1- \frac{1}{q} \|DF\|^2_{\mathcal{H}})^2 \right]} . \quad (2.5)$$

Fix $T > 0$. Let $f, g : [0, T] \to \mathbb{R}$ be Hölder continuous functions of orders $\alpha \in (0, 1)$ and $\beta \in (0, 1)$ with $\alpha + \beta > 1$. Young [14] proved that the Riemann-Stieltjes integral (so-called Young integral) $\int_0^T f_s dg_s$ exists. Moreover, if $\alpha = \beta = (\frac{1}{2}, 1)$ and $\phi : \mathbb{R}^2 \to \mathbb{R}$ is a function of class $C^1$, the integrals $\int_0^t \frac{\partial \phi}{\partial f}(f_u, g_u)du$ and $\int_0^t \frac{\partial \phi}{\partial g}(f_u, g_u)dg_u$ exist in the Young sense and the following change of variables formula holds:

$$\phi(f_t, g_t) = \phi(f_0, g_0) + \int_0^t \frac{\partial \phi}{\partial f}(f_u, g_u)du + \int_0^t \frac{\partial \phi}{\partial g}(f_u, g_u)dg_u, \quad 0 \leq t \leq T. \quad (2.6)$$

As a consequence, if $H \in (\frac{1}{2}, 1)$ and $(u_t, t \in [0, T])$ be a process with Hölder paths of order $\alpha \in (1 - H, 1)$, the integral $\int_0^T u_s dB_s$ is well-defined as Young integral. Suppose moreover that for any $t \in [0, T]$, $u_t \in D^{1,2}$, and

$$P\left( \int_0^T \int_0^T |D_s u_t| |t - s|^{2H-2} ds \right) < \infty = 1.$$ 

Then, by [1], $u \in Dom \delta$ and for every $t \in [0, T]$,

$$\int_0^t u_s dB_s = \int_0^t u_s \delta B_s + H(2H - 1) \int_0^t \int_0^t D_s u_r |s - r|^{2H-2} dr ds. \quad (2.7)$$
In particular, when $\varphi$ is a non-random Hölder continuous function of order $\alpha \in (1 - H, 1)$, we obtain
\[
\int_0^T \varphi_s dB_s = \int_0^T \varphi_s \delta B_s = B(\varphi).
\] (2.8)

In addition, for all $\varphi, \psi \in |\mathcal{H}|$,
\[
E \left( \int_0^T \varphi_s d B_s \int_0^T \psi_s d B_s \right) = H(2H - 1) \int_0^T \int_0^T \varphi(u) \psi(v) |u - v|^{2H - 2} dudv.
\] (2.9)

3 Asymptotic behavior of the least squares estimator

Throughout this paper we assume $H \in (\frac{1}{2}, 1)$ and $\theta > 0$. Let us consider the equation (1.1) driven by a fractional Brownian motion $B$ with Hurst parameter $H$ and $\theta$ is the unknown parameter to be estimated for discretely observed $X$. The linear equation (1.1) has the following explicit solution:
\[
X_t = e^{-\theta t} \left( x_0 + \int_0^t e^{\theta u} dB_u \right), \quad t \geq 0,
\] (3.1)

where the integral can be understood either in the Young sense, or in the Skorohod sense, see indeed (2.8).

Let us introduce the following process
\[
\xi_t := \int_0^t e^{\theta u} dB_u, \quad t \geq 0.
\]

We shall use the notation $a_n \preceq b_n$ to means that there exists a positive constant $c(x_0, \theta, H)$ (depending only on $x_0$, $\theta$ and $H$) such that,
\[
\sup_{n \geq 1} |a_n|/|b_n| < c(x_0, \theta, H) < \infty.
\]

We define the following sequence, which will be used throughout,
\[
\alpha_n = H(2H - 1) \int_0^{T_n} \int_0^t e^{-\theta u} u^{2H - 2} dudt, \quad n \geq 0.
\] (3.2)

We shall be using the following lemmas several times.

**Lemma 3.1.** Let $H \in (\frac{1}{2}, 1)$, let $\theta > 0$, and let $\alpha_n$ be the sequence defined by (3.2). Then
\[
\int_0^{T_n} X_s dB_s = \int_0^{T_n} X_s \delta B_s + \alpha_n, \quad n \geq 0,
\] (3.3)

and
\[
\lim_{n \to \infty} \frac{\alpha_n}{T_n} = \theta^{1 - 2H} H \Gamma(2H).
\] (3.4)

In particular, $\alpha_n \preceq T_n$. 

5
Proof. By (2.7), we have
\[ \int_0^{T_n} X_s dB_s = \int_0^{T_n} X_s \delta B_s + H(2H - 1) \int_0^{T_n} \int_0^t D_s X_s |t - s|^{2H - 2} ds dt \]
\[ = \int_0^{T_n} X_s \delta B_s + H(2H - 1) \int_0^{T_n} \int_0^t e^{-\theta(t-s)} (t-s)^{2H-2} ds dt \]
\[ = \int_0^{T_n} X_s \delta B_s + H(2H - 1) \int_0^{T_n} \int_0^t e^{-\theta u} u^{2H-2} du dt \]
\[ = \int_0^{T_n} X_s \delta B_s + \alpha_n. \]

On the other hand,
\[ \lim_{n \to \infty} \frac{\alpha_n}{T_n} = \lim_{n \to \infty} \frac{H(2H - 1)}{T_n} \int_0^T \int_0^t e^{-\theta u} u^{2H-2} du \]
\[ = \frac{H(2H - 1)}{T_n} \int_0^T e^{-\theta u} u^{2H-2} (T_n - u) du \]
\[ = \frac{H(2H - 1)}{T_n} \int_0^T e^{-\theta u} u^{2H-2} du \]
\[ = \theta^{1-2H} \Gamma(2H). \]

Thus, the proof is finished. \( \square \)

**Lemma 3.2.** Assume \( H \in (1/2, 1) \) and \( \theta > 0 \). Then, there exists a constant \( c > 0 \), depending only on \( x_0, \theta \) and \( H \), such that
\[ \sup_{t \geq 0} E(X_t^2) < c \leq \infty. \]

**Proof.** For any \( t > 0 \), we have
\[ E(X_t^2) = H(2H - 1) \int_0^t \int_0^t e^{\theta x} e^{\theta y} |x - y|^{2H-2} dxdy \]
\[ = 2H(2H - 1) \int_0^t dy e^{\theta y} \int_0^y dx e^{\theta x} (y-x)^{2H-2} \]
\[ = 2H(2H - 1) \int_0^t dy e^{2\theta y} \int_0^y dx e^{-\theta z} z^{2H-2} \]
\[ = H(2H - 1) \int_0^t dz e^{-\theta z} z^{2H-2} e^{2\theta y} \int_0^y dy e^{2\theta y} \]
\[ \leq H(2H - 1) \Gamma(2H - 1) e^{2\theta y} \]
\[ = \frac{H \Gamma(2H)}{\theta^{2H}} e^{2\theta y}. \]

By combining (3.1) and (3.5), we obtain that for any \( t > 0 \)
\[ E(X_t^2) \leq 2 \left( x_0^2 + \frac{H \Gamma(2H)}{\theta^{2H}} \right). \]

This proves the claim. \( \square \)
3.1 Consistency of the LSE

The next statement provides consistency of the LSE $\hat{\theta}_n$ of $\theta$.

**Theorem 3.3.** Assume $H \in (1/2, 1)$ and $\theta > 0$. Then, if $\Delta_n \to 0$ and $n\Delta_n \to \infty$ as $n \to \infty$, we have

$$\hat{\theta}_n \to \theta \quad \text{in probability as } n \to \infty. \quad (3.6)$$

**Proof.** From (1.4), we can write

$$\hat{\theta}_n - \theta = \frac{\theta}{\alpha_n} \sum_{i=1}^{n} X_{t_{i-1}} U_i$$

Let $0 < \rho < 1$. We have

$$P \left( \left| \hat{\theta}_n - \theta \right| > \rho \right) = P \left( \left| \frac{\theta}{\alpha_n} \sum_{i=1}^{n} X_{t_{i-1}} U_i \right| > \rho \right) \leq P \left( \left| \frac{\theta}{\alpha_n} \sum_{i=1}^{n} X_{t_{i-1} - 1} \right| > \rho(1 - \rho) \right) + P \left( \left| \frac{\theta}{\alpha_n} \sum_{i=1}^{n} X_{t_{i-1}}^2 - 1 \right| > \rho \right)$$

Let $0 < \rho < 1$. We have

$$j_1(n) = P \left( \left| \frac{\theta}{\alpha_n} \sum_{i=1}^{n} X_{t_{i-1}} U_i \right| \geq \rho(1 - \rho) \right)$$

$$= P \left( \left| \frac{\theta}{\alpha_n} \sum_{i=1}^{n} X_{t_{i-1}} [U_i - (B_{t_i} - B_{t_{i-1}})] \right| + \frac{\theta}{\alpha_n} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (X_{t_{i-1}} - X_t) \delta B_t \right)$$

$$\leq P \left( \frac{\theta}{\alpha_n} \sum_{i=1}^{n} X_{t_{i-1}} [U_i - (B_{t_i} - B_{t_{i-1}})] \right) \geq \frac{1}{\rho}(1 - \rho) \right) + P \left( \frac{\theta}{\alpha_n} \int_{0}^{T_n} X_t \delta B_t \right) \geq \frac{1}{\rho}(1 - \rho) \right) + P \left( \frac{\theta}{\alpha_n} \int_{0}^{T_n} X_t \delta B_t \right) \geq \frac{1}{\rho}(1 - \rho) \right)$$

$$= j_{1,1}(n) + j_{1,2}(n) + j_{1,3}(n).$$

For the term $j_{1,1}(n)$, by using Lemma 3.2 and the fact that for every $t > 0$

$$X_{t_{i-1}} - X_t = (e^{-\theta t_{i-1}} - e^{-\theta t})(x_0 + \xi_{t_{i-1}}) + e^{-\theta t}(\xi_{t_{i-1}} - \xi_t), \quad (3.7)$$

we obtain

$$\sum_{i=1}^{n} E \left[ X_{t_{i-1}} [U_i - (B_{t_i} - B_{t_{i-1}})] \right] = \sum_{i=1}^{n} E \left| X_{t_{i-1}} \int_{t_{i-1}}^{t_i} \theta(X_{t_{i-1}} - X_t) dt \right|$$
Making the change of variables \( s = t - t_{i-1} \), we obtain

\[
\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} [1 - e^{-\theta(t-t_{i-1})}] dt = n \int_0^{\Delta_n} [1 - e^{-\theta s}] ds
\]

\[
= n\Delta_n \int_0^{\Delta_n} [1 - e^{-\theta s}] ds
\]

\[
\leq n\Delta_n^2
\]

where the last estimate comes from the fact that \( \int_0^{\Delta_n} [1 - e^{-\theta s}] ds/\Delta_n^2 \to \theta^2/2 \) as \( \Delta_n \to 0 \) (thanks to l'Hôpital rule). On the other hand, by the change of variables \( u = \frac{t-t_{i-1}}{t-t_{i-1}}, v = \frac{y-t_{i-1}}{t-t_{i-1}} \) and \( s = t-t_{i-1} \)

\[
\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} e^{-\theta t} (E([\xi_t - \xi_{t_{i-1}}])^2)^{1/2} dt
\]

\[
= \sqrt{H(2H-1)} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} dt e^{-\theta t} \left( \int_{t_{i-1}}^{t} dy e^{\theta y} \int_{t_{i-1}}^{t} dx e^{\theta x} |x-y|^{2H-2} \right)^{1/2}
\]

\[
= \sqrt{H(2H-1)} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} dt e^{-\theta(t-t_{i-1})} \left( \int_{t_{i-1}}^{t} dy e^{\theta(y-t_{i-1})} \int_{t_{i-1}}^{t} dx e^{\theta(x-t_{i-1})} |x-y|^{2H-2} \right)^{1/2}
\]

\[
= \sqrt{H(2H-1)} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} dt(t-t_{i-1})^H e^{-\theta(t-t_{i-1})} \left( \int_0^{1} dv e^{\theta(v-t_{i-1})} v \int_0^{1} du e^{\theta(u-t_{i-1})} u |u-v|^{2H-2} \right)^{1/2}
\]

\[
= n\sqrt{H(2H-1)} \int_0^{\Delta_n} ds s^H e^{-\theta s} \left( \int_0^{1} dv e^{\theta s v} e^{\theta s u} |u-v|^{2H-2} \right)^{1/2}
\]

\[
\leq n \int_0^{\Delta_n} s^H ds
\]

\[
\leq n\Delta_n^H. \]

Hence, we obtain that

\[
\sum_{i=1}^{n} E \left| X_{t_{i-1}} [U_i - (B_{t_i} - B_{t-1})] \right| \leq n \left( \Delta_n^2 + \Delta_n^{H+1} \right)
\]
Similarly, we have
\[ j_{1,1}(n) = P \left( \frac{\theta}{\alpha n} \sum_{i=1}^{n} X_{t_{i-1}}(U_i - (B_{t_i} - B_{t_{i-1}})) \right) \geq \frac{1}{3} \rho(1 - \rho) \leq \frac{\Delta_n^H}{\rho(1 - \rho)}. \] (3.10)

For the term \( j_{1,2}(n) \), from (3.9), we have
\[
E \left[ \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (X_{t_{i-1}} - x_t) dB_t \right] \leq E \left[ \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left( (e^{-\theta t_{i-1}} - e^{-\theta t})(x_0 + \xi_{t,i}) \right) dB_t \right]
+ E \left[ \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (e^{-\theta t}(\xi_{t,i} - \xi_t)) dB_t \right].
\]

Using the inequality (2.3), we can write
\[
E \left[ \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left( (e^{-\theta t_{i-1}} - e^{-\theta t})(x_0 + \xi_{t,i}) \right) dB_t \right] = E \left[ \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left( (e^{-\theta t_{i-1}} - e^{-\theta t}) e^{\theta t_{i-1}} \right) 1[t_{i-1}, t_i] dB_t \right]
\leq c_H \left( \int_{t_{i-1}}^{t_i} \sum_{i=1}^{n} \left( e^{-\theta t_{i-1}} - e^{-\theta t} \right) \frac{1}{H} \right) ds dt
\]
\[
= c_H \left( \int_{t_{i-1}}^{t_i} \sum_{i=1}^{n} \left( e^{-\theta t_{i-1}} - e^{-\theta t} \right) \frac{1}{H} \right) ds dt
\]
\[
= c_H \left( \int_{t_{i-1}}^{t_i} \frac{1}{H} \left( 1 - e^{-\theta (t_{i-1})} \right) \right)
\]
\[
\leq n \Delta_n^{H+1}.
\]

because \( \Delta_n \left[ \int_{0}^{\Delta_n} \frac{1}{H} \right] \rightarrow \theta^{1/H}/(1 + 1/H) \) as \( \Delta_n \rightarrow 0 \) (thanks to l'Hôpital rule). Similarly, we have
\[
E \left[ \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} e^{-\theta t}(\xi_{t,i} - \xi_t) dB_t \right] = E \left[ \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} e^{-\theta t}(\xi_{t,i} - \xi_t) 1[t_{i-1}, t_i] dB_t \right]
\leq c_H \left( \int_{t_{i-1}}^{t_i} \sum_{i=1}^{n} e^{-\theta t} 1[t_{i-1}, t_i] dB_t \right) ds dt
\]
\[
\leq n \Delta_n^{H+1}.
\]

which leads to
\[
\frac{\theta}{\alpha n} \sum_{i=1}^{n} E \left[ X_{t_{i-1}}(U_i - (B_{t_i} - B_{t_{i-1}})) \right] \leq \frac{n \Delta_n^{H+1}}{T_n} = \Delta_n^H.
\] (3.9)
\[
\begin{align*}
\mathbb{E} \left[ \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (X_{t_{i-1}} - X_t) \delta B_t \right] & \leq n^H (\Delta_n^{H+1} + \Delta_n^{2H}) \\
& \leq n^H \Delta_n^{H+1}.
\end{align*}
\]

Therefore,
\[
\frac{\theta}{\alpha} \mathbb{E} \left[ \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (X_{t_{i-1}} - X_t) \delta B_t \right] \leq \frac{\Delta_n^H}{n^{1-H}}.
\]

As consequence,
\[
\begin{align*}
\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (X_{t_{i-1}} - X_t) \delta B_t & \geq \frac{1}{3} \rho (1 - \rho) \\
& \leq \frac{\Delta_n^H}{n^{1-H} \rho (1 - \rho)}.
\end{align*}
\]

For the term \( j_{1,3}(n) \), by setting
\[
F_{T_n} = \frac{1}{\sqrt{T_n}} \int_0^{T_n} X_t \delta B_t
\]
we have
\[
\begin{align*}
\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (X_{t_{i-1}} - X_t) \delta B_t & \geq \frac{1}{3} \rho (1 - \rho) \\
& \leq \frac{9 \theta^2 \left[\frac{T_n}{\alpha} \right]^2 E(F_{T_n}^2)}{\rho (1 - \rho)} \\
& \leq \frac{1}{\rho (1 - \rho)} \left[\frac{T_n}{\alpha} \right]^2 T_n
\end{align*}
\]

since from \([4]\),
\[
E(F_{T_n}^2) \rightarrow A(\theta, H), \quad \text{as } n \rightarrow \infty,
\]

Thus,
\[
\begin{align*}
\mathbb{E} \left[ \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (X_{t_{i-1}} - X_t) \delta B_t \right] & \leq n^H (\Delta_n^{H+1} + \Delta_n^{2H}) \\
& \leq n^H \Delta_n^{H+1}.
\end{align*}
\]

Therefore,
\[
\begin{align*}
\frac{\theta}{\alpha} \mathbb{E} \left[ \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (X_{t_{i-1}} - X_t) \delta B_t \right] & \leq \frac{\Delta_n^H}{n^{1-H}}.
\end{align*}
\]

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\[
\begin{align*}
\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (X_{t_{i-1}} - X_t) \delta B_t & \geq \frac{1}{3} \rho (1 - \rho) \\
& \leq \frac{\Delta_n^H}{n^{1-H} \rho (1 - \rho)}.
\end{align*}
\]

For the term \( j_{1,3}(n) \), by setting
\[
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we have
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\begin{align*}
\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (X_{t_{i-1}} - X_t) \delta B_t & \geq \frac{1}{3} \rho (1 - \rho) \\
& \leq \frac{9 \theta^2 \left[\frac{T_n}{\alpha} \right]^2 E(F_{T_n}^2)}{\rho (1 - \rho)} \\
& \leq \frac{1}{\rho (1 - \rho)} \left[\frac{T_n}{\alpha} \right]^2 T_n
\end{align*}
\]

since from \([4]\),
\[
E(F_{T_n}^2) \rightarrow A(\theta, H), \quad \text{as } n \rightarrow \infty,
\]
Finally, by combining (3.10), (3.12) and (3.14), we conclude that
\[ j_1(n) \leq \frac{1}{|\rho(1 - \rho)|^2} \left( \Delta_n^H + \frac{\Delta_n^{H+1}}{n} + \frac{1}{n\Delta_n} \right). \] (3.16)

Consequently, to achieve the proof of Theorem 3.3, it remains to estimate the term \( j_2(n) \). We have
\[
j_2(n) = P \left( \left| \frac{\Delta_n}{\alpha_n} \sum_{i=1}^{n} X_{t_{i-1}}^2 - 1 \right| > \rho \right)
\]
\[
= P \left( \left| \frac{\theta}{\alpha_n} \Delta_n \sum_{i=1}^{n} X_{t_{i-1}}^2 - 1 \right| > \rho \right)
\]
\[
\leq P \left( \left| \frac{\theta}{\alpha_n} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} |X_{t_{i-1}}^2 - X_t^2| dt \right| > \rho/2 \right) + P \left( \frac{\theta}{\alpha_n} \int_0^{T_n} X_t^2 dt - 1 > \rho/2 \right)
\]
\[
=: j_{2.1}(n) + j_{2.2}(n).
\]

We first estimate \( j_{2.1}(n) \). Similar argument applied in (3.8) yields
\[
E \left[ \frac{\theta}{\alpha_n} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} |X_{t_{i-1}}^2 - X_t^2| dt \right] \leq \frac{\theta}{\alpha_n} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} E \left[ |X_{t_{i-1}}^2 - X_t^2| \right] dt
\]
\[
\leq \frac{2\theta \sup_{t>0}(E[|X_t^2|])^{1/2}}{\alpha_n} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left( E \left[ |X_{t_{i-1}}^2 - X_t^2| \right] \right)^{1/2} dt
\]
\[
\leq \frac{n\Delta_n^2 + n\Delta_n^{H+1}}{T_n}
\]
\[
\leq \Delta_n^H
\]
which implies that
\[ j_{2.1}(n) = P \left( \left| \frac{\theta}{\alpha_n} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} |X_{t_{i-1}}^2 - X_t^2| dt \right| > \rho/2 \right) \]
\[
\leq \frac{\Delta_n^H}{\rho} \quad (3.17)
\]

We now study \( j_{2.2}(n) \). Applying the change of variable formula (2.3) leads to
\[ 2 \int_0^{T_n} X_t dB_t = X_{T_n}^2 - x_0^2 + 2\theta \int_0^{T_n} X_t^2 dt. \] (3.18)

Combining (3.3) and (3.18) we obtain
\[ 2\theta \int_0^{T_n} X_t^2 dt - 2\alpha_n = 2 \int_0^{T_n} X_t dB_t - X_{T_n}^2 + x_0^2. \]

Hence
\[ \frac{\theta}{\alpha_n} \int_0^{T_n} X_t^2 dt - 1 = \frac{1}{\alpha_n} \int_0^{T_n} X_t dB_t - \frac{1}{2\alpha_n} (X_{T_n}^2 - x_0^2), \] (3.19)
We deduce from (3.19) and (3.15) together with (3.4) that
\[ j_{2,2}(n) = P\left( \left| \frac{\theta}{\alpha_n} \int_0^{T_n} X_t^2 dt - 1 \right| > \rho/2 \right) \]
\[ \leq P\left( \left| \frac{1}{\alpha_n} \int_0^{T_n} X_t \delta B_t - \frac{1}{2\alpha_n} (X_{T_n}^2 - x_0^2) \right| > \rho/2 \right) \]
\[ \leq P\left( \left| \frac{1}{\alpha_n} \int_0^{T_n} X_t \delta B_t \right| > \rho/8 \right) + P\left( \left| \frac{1}{2\alpha_n} (X_{T_n}^2 - x_0^2) \right| > \rho/8 \right) \]
\[ \leq \frac{T_n^2}{\alpha_n} \frac{E(F_{T_n}^2)}{\rho^2 T_n} + \frac{T_n}{\alpha_n} \frac{E|X_{T_n}^2 - x_0^2|}{\rho T_n} \]
\[ \leq \frac{1}{\rho^2 T_n} \left( \frac{1}{\rho^2 T_n} + 1 \right) \]
\[ \leq \frac{\Delta H_n}{\rho} + \frac{1}{\rho^2 T_n} \]  
\[ (3.21) \]
Therefore, from (3.17) and (3.21), we obtain
\[ j_{2}(n) \leq \frac{\Delta H_n}{\rho} + \frac{1}{\rho^2 T_n} \]  
\[ (3.22) \]
Finally, combining (3.16) and (3.22), the proof of Theorem 3.3 is done.

3.2 Rate of convergence of the LSE

This paragraph is devoted to derive Berry-Esseen-type bounds in the Kolmogorov distance for the LSE \( \hat{\theta}_n \) of \( \theta \). We first recall the following technical lemmas.

**Lemma 3.4** ([6, Page 78]). Let \( f \) and \( g \) be two real-valued random variables with \( g \neq 0 \) P-a.s. Then, for any \( \delta > 0 \),
\[ d_{Kol}(\frac{f}{g}, N) \leq d_{Kol}(f, N) + P(|g - 1| > \delta) + \delta. \]
where \( N \sim N(0,1) \).

**Lemma 3.5** ([2, Page 280]). Let \( Y \) and \( Z \) be two real-valued random variables. Then, for any \( \eta > 0 \),
\[ d_{Kol}(Y + Z, N) \leq d_{Kol}(Z, N) + P(|Y| > \eta) + \frac{\eta}{\sqrt{2\pi}}, \]
where \( N \sim N(0,1) \).

Define
\[ \lambda_n := \frac{\alpha_n}{\theta T_n \sqrt{E(F_{T_n}^2)}} \]  
\[ n \geq 1, \]
where \( \alpha_n \) and \( F_{T_n} \) are defined by (3.2) and (3.13), respectively.

The following result provides explicit bounds for the Kolmogorov distance, in the case when \( H \in \left( \frac{1}{2}, \frac{3}{4} \right) \), between the law of \( \lambda_n \sqrt{T_n} (\hat{\theta}_n - \theta) \) and the standard normal law.
Theorem 3.6. Let \((\delta, \eta) \in (0, 1)^2\). If \(H \in \left(\frac{1}{2}, \frac{3}{4}\right)\) then, for some constant \(c > 0\) depending uniquely on \(x_0, \theta\) and \(H\), we have: for any \(n \geq 1\),

\[
d_{Kol}\left(\lambda_n \sqrt{T_n} (\hat{\theta}_n - \theta), N\right) \\
\leq c \left( \frac{\sqrt{n \Delta_n} \Delta^H_n}{\eta} + (n \Delta_n)^{4H-3} + \eta + \frac{\Delta^H_n}{\delta} + \frac{1}{n \Delta_n \delta^2} + \delta \right).
\]

where \(N \sim N(0, 1)\).

Proof. Fix \((\delta, \eta) \in (0, 1)^2\). From (1.3) and Lemma 3.4, we obtain that

\[
d_{Kol}\left(\lambda_n \sqrt{T_n} (\hat{\theta}_n - \theta), N\right) \\
\leq d_{Kol}\left(\frac{1}{\sqrt{T_n} E(F^2_{T_n})} \sum_{i=1}^{n} X_{t_{i-1}} U_i, N\right) + P \left( \left| \frac{\theta \Delta_n}{\alpha_n} \sum_{i=1}^{n} X_{t_{i-1}}^2 - 1 \right| \geq \delta \right) + \delta \\
:= J_1(n) + J_2(n) + \delta.
\]

We first study the term \(J_1(n)\). Using Lemma 3.5, we obtain

\[
J_1(n) = d_{Kol}\left(\frac{1}{\sqrt{T_n} E(F^2_{T_n})} \sum_{i=1}^{n} X_{t_{i-1}} U_i, N\right) \\
= d_{Kol}\left(\frac{1}{\sqrt{T_n} E(F^2_{T_n})} \sum_{i=1}^{n} X_{t_{i-1}} [U_i - (B_{t_i} - B_{t_{i-1}})] + \frac{1}{\sqrt{T_n} E(F^2_{T_n})} \int_{0}^{T_n} (X_{t_{i-1}} - X_i) \delta B_i \\
+ \frac{1}{\sqrt{T_n} E(F^2_{T_n})} \int_{0}^{T_n} X_i \delta B_i, N\right) \\
\leq P \left( \left| \frac{1}{\sqrt{T_n} E(F^2_{T_n})} \sum_{i=1}^{n} X_{t_{i-1}} [U_i - (B_{t_i} - B_{t_{i-1}})] \right| \geq \frac{\eta}{2} \right) \\
+ P \left( \left| \frac{1}{\sqrt{T_n} E(F^2_{T_n})} \int_{0}^{T_n} (X_{t_{i-1}} - X_i) \delta B_i \right| \geq \frac{\eta}{2} \right) \\
+ d_{Kol}\left(\frac{1}{\sqrt{T_n} E(F^2_{T_n})} \int_{0}^{T_n} X_i \delta B_i, N\right) + \frac{\eta}{\sqrt{2\pi}} \\
:= J_{1,1}(n) + J_{1,2}(n) + J_{1,3}(n) + \frac{\eta}{\sqrt{2\pi}}.
\]

By using (3.9), (3.11) and (3.15) we deduce that

\[
J_{1,1}(n) + J_{1,2}(n) \lesssim \frac{2}{\eta \sqrt{T_n} E(F^2_{T_n})} (n \Delta_n^{H+1} + n^H \Delta_n^{H+1}) \\
\lesssim \frac{\sqrt{n \Delta_n} \Delta_n^H}{\eta}.
\]

(3.23)
To achieve the estimation of $J_1(n)$ it remains to estimate $J_{1,3}(n)$. We have
\[
\frac{1}{\sqrt{T_n}} \int_0^{T_n} X_t \delta B_t = \frac{1}{2\sqrt{T_n}} \int_0^{2\sqrt{T_n}} I_2 \left( e^{-\theta|t-s|} 1_{[0,T_n]}(t, s) \right).
\]

Thus, by using Theorem 2.3 and the fact that $E \left( \|DF_{T_n}\|^2_H \right) = 2E \left( F_{T_n}^2 \right)$, we obtain
\[
J_{1,3}(n) = d_{kol} \left( \frac{1}{\sqrt{T_n}} \int_0^{T_n} X_t \delta B_t, N \right) = d_{kol} \left( \frac{F_{T_n}}{E(F_{T_n})^2} N \right)
\leq E \left[ \left( 1 - \frac{1}{2} \frac{\|F_{T_n}\|^2}{E(F_{T_n})^2} \right) \right]^{2n}
= \frac{1}{2E(F_{T_n})^2} E \left[ \left( \|DF_{T_n}\|^2_H - E\|DF_{T_n}\|^2_H \right)^2 \right].
\]

Moreover, from [H], we have
\[
E \left[ \left( \|DF_{T_n}\|^2_H - E\|DF_{T_n}\|^2_H \right)^2 \right] \leq T_n^{8H-6}.
\]

Thus
\[
J_{1,3}(n) \leq T_n^{4H-3} = (n\Delta_n)^{4H-3}.
\] (3.24)

Consequently, by combining (3.23) and (3.24) we have
\[
J_1(n) \leq \frac{\sqrt{n\Delta_n} \Delta_n^H}{\eta} + (n\Delta_n)^{4H-3} + \eta.
\] (3.25)

Finally, via the same arguments as in the estimation of $j_2(n)$, we obtain the following upper bound of $J_2(n)$:
\[
J_2(n) \leq \frac{\Delta_n^H}{\delta} + \frac{1}{n\Delta_n^2},
\] (3.26)

which completes the proof of Theorem 3.6.

Let us apply now Theorem 3.6 to the particular case $\eta = \sqrt{n\Delta_n^{1+\beta}}$ and $\Delta_n^\eta$, where $0 < \alpha < \beta < H$ which ensure that $n\Delta_n^{1+2\alpha} \to \infty$ and $n\Delta_n^{1+2\beta} \to 0$ as $n \to \infty$.

**Theorem 3.7.** Let $0 < \alpha < \beta < H$ such that $n\Delta_n^{1+2\alpha} \to \infty$ and $n\Delta_n^{1+2\beta} \to 0$ as $n \to \infty$. If $H \in (\frac{1}{2}, \frac{3}{4})$ then, for some constant $c > 0$ depending uniquely on $x_0$, $\theta$ and $H$, we have: for any $n \geq 1$,
\[
d_{kol} \left( \lambda_n \sqrt{T_n}(\hat{\theta}_n - \theta), N \right) \leq c \left( \Delta_n^{H-\beta} + (n\Delta_n)^{4H-3} + \sqrt{n}\Delta_n^{1+2\beta} + \Delta_n^{H-\alpha} + \frac{1}{n\Delta_n^{1+2\alpha}} + \Delta_n^\eta \right).
\]

In particular, as $n \to \infty$
\[
\sqrt{T_n}(\hat{\theta}_n - \theta) \xrightarrow{law} N(0, \sigma^2_H)
\]
where $\sigma^2_H = (4H-1)\theta \left( 1 + \frac{\Gamma(3-4H)\Gamma(4H-1)}{\Gamma(1-2H)\Gamma(2H)} \right)$.  

14
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