On the QCD Sum Rule Determination of the Strange Quark Mass

P. Colangelo\textsuperscript{a}, F. De Fazio\textsuperscript{a,b}, G. Nardulli\textsuperscript{a,b}, N. Paver\textsuperscript{c}

\textsuperscript{a} Istituto Nazionale di Fisica Nucleare, Sezione di Bari, Italy
\textsuperscript{b} Dipartimento di Fisica, Università di Bari, Italy
\textsuperscript{c} Dipartimento di Fisica Teorica, Università di Trieste, Italy, and Istituto Nazionale di Fisica Nucleare, Sezione di Trieste, Italy

Abstract

In the QCD Sum Rule determination of $m_s$ using the two-point correlator of divergences of $\Delta S = 1$ vector currents, the final uncertainty on $m_s$ is mainly due to the hadronic spectral function. Using a specific parameterization which fully takes into account the available experimental data on the $K\pi$ ($I = 1/2$, $J^P = 0^+$) system, characterized by the presence of a relevant nonresonant component in addition to the resonant one, we find $\overline{m}_s(1 \text{ GeV}) \geq 120 \text{ MeV}$. In particular, varying only the parameters describing the nonresonant $K\pi$ component and $\Lambda_{\overline{MS}}^{\text{u,d,s}}$ we obtain $\overline{m}_s(1 \text{ GeV}) = 125 - 160 \text{ MeV}$. This result is smaller than analogous ones obtained by using a parameterization in terms of only resonant states. We discuss how to systematically improve the determination of $m_s$ by this method.
Light ‘current’ quark masses have an important role in the theoretical description of low energy hadronic physics, and their actual values are needed as an input to quantitatively predict several relevant effects, such as, e.g., the breaking of chiral and vector flavour symmetries in hadron masses and transition amplitudes, and the size of CP violation in Kaon decays.

Chiral perturbation theory represents a consistent framework where the (scale-independent) ratios of ‘current’ quark masses can be evaluated from experimental data. At the next to leading order in the chiral expansion one obtains \[1, 2\]:

\[
\frac{m_s}{m_d} = 18.9 \pm 0.8 \quad ; \quad \frac{m_u}{m_d} = 0.553 \pm 0.043 \quad . \tag{1}
\]

However, for the determination of the individual light quark masses from the experimental data, one has to adopt some alternative non-perturbative method, such as lattice QCD or QCD sum rules. Once (at least) one of these masses is determined from these methods, the ratios in Eq. (1) give access to the other ones. Here, we shall consider the determination of the strange quark mass \(m_s\) from QCD sum rules. The reason for concentrating on \(m_s\) (rather than on the other masses) is that there is more experimental information concerning the hadronic spectral function on which the QCD sum rule method is based, so that in this case one could expect the final result to be less model-dependent and more reliable.

The present situation concerning the evaluation of \(m_s\) is the following. Lattice QCD calculations, in quenched approximation, give for the running mass:

\[
\overline{m}_s(2 \text{ GeV}) = 128 \pm 18 \text{ MeV} \quad [3] \tag{2}
\]

\[
\overline{m}_s(2 \text{ GeV}) = 100 \pm 21 \pm 10 \text{ MeV} \quad [4] \tag{3}
\]

For \(\Lambda_{MS}^{n_f=3} = 380 \text{ MeV}\) these values correspond to \(\overline{m}_s(1 \text{ GeV}) = 172 \pm 24 \text{ MeV}\), and \(\overline{m}_s(1 \text{ GeV}) = 140 \pm 29 \pm 14 \text{ MeV}\), respectively.

Regarding QCD sum rules, substantial progress has been achieved quite recently by the calculation to \(O(\alpha_s^3)\) of the perturbative part of the two-point correlator of \(\Delta S = 1\) scalar quark currents relevant to the determination of \(m_s\) \[5\]. Accounting for such \(O(\alpha_s^3)\) corrections, the value previously obtained in \[6\] by using the \(O(\alpha_s^2)\) expression of the perturbative contribution to the sum rule:

\[
\overline{m}_s(1 \text{ GeV}) = 189 \pm 32 \text{ MeV} \quad [6] \tag{4}
\]

\[1\] Earlier QCD sum rules estimates, using different versions of the method, can be found, e.g., in refs. \[3, 7, 8\].
has changed to the new value:

$$m_s(1 \text{ GeV}) = 203.5 \pm 20 \text{ MeV} \quad [5] .$$

(5)

The uncertainties in Eqs. (4,5) are related to the dependence of the result on the value of $\Lambda_{\overline{MS}}^{\text{n}f=3}$, which was chosen in the range $280 - 480 \text{ MeV}$, and to the variation of other input parameters in the sum rule analysis. In particular, as concluded in [5], the largely dominant uncertainty is that on the hadronic spectral function, which in both [4] and [5] is assumed to behave as the sum of two $I = 1/2, J^P = 0^+$ ($K\pi$) resonant states.

Following this observation, we would like to discuss a parameterization of the relevant hadronic spectral function which, at least in principle, should be able to fully exploit the current experimental information on the scalar $I = 1/2$ channel. In particular, this parameterization takes into account the fact that, in addition to the resonance component, there exists also a nonresonant continuum. As typical of $s$-wave channels, the nonresonant component persists rather far from the threshold energy range and can interfere with the resonances [11]. Our example shows that the inclusion of the nonresonant contribution may give a significant effect on the sum rule and, also, indicates some directions in order to improve the determination of $m_s$ in this approach.

For convenience, we briefly describe the main points of the QCD sum rule derivation of $m_s$ we are interested in. The basic quantity is the two-point correlator

$$\Psi(q^2) = i \int dx \ e^{iqx} < 0 | T[J(x)J^\dagger(0)] | 0 >$$

(6)

where $J(x) = i(m_s - m_u)\bar{s}(x)u(x)$ is the divergence of the $\Delta S = 1$ vector current $V^\mu(x) = \bar{s}(x)\gamma^\mu u(x)$. The second derivative $\Psi''(q^2) = (\partial^2/(\partial q^2)^2)\Psi(q^2)$ obeys an unsubtracted dispersion relation:

$$\Psi''(q^2) = 2 \int_0^\infty ds \ \rho(s) \frac{1}{(s - q^2 - i\epsilon)^3},$$

(7)

with the spectral function $\rho(s)$ given by

$$\rho(s) = \frac{1}{\pi} \text{Im} \ \Psi(s).$$

(8)

Particularly useful for low-energy phenomenology is the Borel transform $\Psi''(M^2)$ [11], defined by the application of the operator $B(M^2) = \frac{(-1)^n}{n!} \left( \frac{d}{d(-q^2)} \right)^n$ in the limit $n \to \infty$, $-q^2 \to \infty$, $-q^2/n = M^2 = \text{const}$ to Eq. (7). This results into:

$$\Psi''(M^2) = \frac{1}{M^6} \int_0^\infty \! ds \ \rho(s) e^{-s/M^2} .$$

(9)
The main advantage of this transformation is that, due to the presence of the exponential, for moderate values of the Borel parameter $M^2$ (of the order of one to a few GeV$^2$) the r.h.s of Eq. (3) is mostly sensitive to the spectral function $\rho(s)$ in the low energy range, where it can be computed in terms of the available experimental information.

The quark ‘current’ mass $m_s$ enters into the l.h.s of Eq. (7) as a parameter, when the correlation function $\Psi'(Q^2)$ is evaluated in QCD for $Q^2 = -q^2 \gg \Lambda^2_{QCD}$ by means of the Operator Product Expansion. Once $\rho(s)$ is parameterized by a hadronic representation inferred from experimental data, the resulting Eq. (9) represents a $(M^2$-dependent) equation relating $m_s$ to experimental data and known QCD parameters.

To arrive at the final result, one applies the notion of global quark-hadron duality, which essentially consists in identifying $\int_{s_0}^{\infty} ds \rho(s) e^{-s/M^2} \simeq \int_{s_0}^{\infty} ds \rho_{\text{OPE}}(s) e^{-s/M^2}$, since, due to asymptotic freedom, quark and gluon degrees of freedom (rather than hadrons) should dominate above an effective energy threshold $s_0$. Accordingly, Eq. (9) takes the form

$$\Psi''(M^2)|_{\text{OPE}} = \frac{1}{M^6} \int_{s_0}^{\infty} ds \rho(s) e^{-s/M^2} + \frac{1}{M^6} \int_{s_0}^{\infty} ds \rho_{\text{OPE}}(s) e^{-s/M^2} ,$$

and the ultimate numerical determination of $m_s$ will be the one for which this relation is stable in the, as yet undetermined, parameters $M^2$ and $s_0$.

The OPE expression for $\Psi''(Q^2)$ can be given in terms of a perturbative and a non perturbative contribution:

$$\Psi''(Q^2) = \Psi''_P(Q^2) + \Psi''_{NP}(Q^2) .$$

For illustrative purposes, we report here just the leading order expression of $\Psi''_P(Q^2)$:

$$\Psi''_P(Q^2) = \frac{3}{8\pi^2} \frac{(m_s(\mu) - m_u(\mu))^2}{Q^2} \left( 1 + \frac{11}{3} \frac{\alpha_s(\mu)}{\pi} - 2 \frac{\alpha_s(\mu)}{\pi} \log \frac{Q^2}{\mu^2} \right)$$

$$- \frac{6}{8\pi^2} \frac{m_s^2(\mu)(m_s(\mu) - m_u(\mu))^2}{Q^4} \left( 1 + \frac{28}{3} \frac{\alpha_s(\mu)}{\pi} - 4 \frac{\alpha_s(\mu)}{\pi} \log \frac{Q^2}{\mu^2} \right) ,$$

and we refer to [5] for the explicit, lengthy expressions of the order $\alpha_s^2$ and $\alpha_s^3$ contributions. In Eq. (12), $\mu$ is an a priori arbitrary renormalization mass scale. Since $\Psi''(Q^2)$ is related to a physical observable, it obeys the homogeneous renormalization group equation:

$$\mu \frac{d}{d\mu} \Psi''(Q^2) = 0 ,$$

We keep $m_u \neq 0$ only in the coefficient of the perturbative function [12].
and therefore the scale dependence of the renormalized parameters $\alpha_s$ and $m_s$ appearing in the perturbative calculation of $\Psi''(Q^2)$ must cancel against $\log \mu$ factors also appearing in $\Psi''(Q^2)$. The Borel transform of $\Psi''(Q^2)$ in the approximation of Eq. (12) is given by:

$$\Psi_{\mu}''(M^2) = \frac{3}{8\pi^2} \left( \frac{(m_s - \bar{m}_s)(\mu)}{M^2} \right)^2 \left\{ 1 + \frac{11 \alpha_s(\mu)}{3 \pi} - 2 \frac{\alpha_s(\mu)}{\pi} \left( \log \frac{M^2}{\mu^2} + \psi(1) \right) \right\}$$

$$- \frac{6}{8\pi^2} \frac{m_s^2(\mu)(\bar{m}_s(\mu) - \bar{m}_s(\mu))}{M^4} \left\{ 1 + \frac{28 \alpha_s(\mu)}{3 \pi} - 4 \frac{\alpha_s(\mu)}{\pi} \left( \log \frac{M^2}{\mu^2} + \psi(2) \right) \right\} , (14)$$

where $\psi(x)$ is the dilogarithmic function. The choice $\mu = M$ allows to resum the logarithmic terms, transforming the dependence on $\mu$ of the running mass $\bar{m}(\mu)$ and of the running coupling constant $\alpha_s(\mu)$ into a dependence on the Borel parameter $M$.

In an analogous way one can write the non perturbative contribution to $\Psi''(Q^2)$. Referring to [6] for the detailed expression of the operator expansion, we report here only the Borel transformed contribution of $D = 4$ operators:

$$\Psi_{NP}''(M^2) = \frac{m_s - \bar{m}_s}{M^6} \left\{ 2 < m_s \bar{u}u >_0 \left( 1 + \frac{\alpha_s}{\pi} \left( \frac{14}{3} - 2 \psi(1) - 2 \log \frac{M^2}{\mu^2} \right) \right) \right\}$$

$$- \frac{I_G}{9} \left( 1 + \frac{\alpha_s}{\pi} \left( \frac{67}{18} - 2 \psi(1) - 2 \log \frac{M^2}{\mu^2} \right) \right) + I_s \left( 1 + \frac{\alpha_s}{\pi} \left( \frac{37}{9} - 2 \psi(1) - 2 \log \frac{M^2}{\mu^2} \right) \right)$$

$$- \frac{3}{7\pi^2} m_s^4 \left( \frac{\pi}{\alpha_s} + \frac{5}{6} \psi(1) - \frac{15}{4} \log \frac{M^2}{\mu^2} \right) \right\} . (15)$$

$I_s$ and $I_G$ are RG invariant combination given by (for $n_f = 3$):

$$I_s = m_s < \bar{s}s >_0 + \frac{3}{7\pi^2} m_s^4 \left( \frac{\pi}{\alpha_s} - \frac{53}{23} \right)$$

$$I_G = \frac{9}{4} < \frac{\alpha_s}{\pi} G^2 > (1 + \frac{16 \alpha_s}{9 \pi}) + \frac{4 \alpha_s}{\pi} (1 + \frac{91 \alpha_s}{24 \pi}) m_s < \bar{s}s >_0$$

$$+ \frac{3}{4\pi^2} (1 + \frac{4 \alpha_s}{3 \pi}) m_s^4 . (16)$$

The hadronic contribution to the spectral function $\rho(s)$ can be obtained by inserting a set of intermediate states with $J^P = 0^+$ and $I = \frac{1}{2}$ into the correlator (I). The simplest examples are the two-particle states $|K\pi >$, $|K\eta >$, $|K\eta' >$. In particular, the contribution of the $|K\pi >$ intermediate state, which is expected to be the dominant one and whose features are better known from the theoretical as well as the experimental point of view, can be written as:

$$\rho^{(\bar{K}\pi)}(s) = \frac{3}{32\pi^2} \sqrt{(s - s_+)(s - s_-)} |d(s)|^2 \quad (s > s_+)$$

Multiparticle states should be suppressed by phase space.
where \( s_\pm = (M_K \pm M_\pi)^2 \), and \( d(s) \) is the scalar form factor related to the \( K_{\ell 3} \) decay form factors \( f_\pm \):

\[
< \pi^0(p')|s\gamma\mu u|K^+(p)> = \frac{1}{\sqrt{2}}[(p + p')_\mu f_+ + (p - p')_\mu f_-],
\]

and

\[
d(s) = \frac{s}{M_K^2 - M_\pi^2} f_+(s) + \frac{s}{M_K^2 - M_\pi^2} f_-(s) = (M_K^2 - M_\pi^2) f_0(s).
\]

From the theoretical point of view, \( d(s) \) and \( f_0(s) \) can be considered as analytic functions of the complex variable \( s \), with a cut on the real axis starting at the threshold \( s_+ = (M_K + M_\pi)^2 \).

Furthermore, in the range \( 0 \leq s \leq s_- \), which is the physical one for \( K_{\ell 3} \) decay, \( f_0(s) \) admits a linear expansion for small \( s \):

\[
f_0(s) = f_0(0) \left( 1 + \lambda_0 \frac{s}{M_\pi^2} \right)
\]

where, from one-loop chiral perturbation theory \[2, 12\], one has the predictions:

\[
f_0(0) = 0.973; \quad \lambda_0 = 0.017 \pm 0.004.
\]

The theoretical value of the slope \( \lambda_0 \) in Eq. (22) agrees with the experimental result from the high statistics analysis of \( K_{\mu 3} \) decays, which gives the result \( \lambda_0 = 0.019 \pm 0.004 \) \[13\].

The linear extrapolation of Eq. (21) with the theoretical values (22) from the decay region to the threshold \( s_+ \) would imply the prediction \( d(s_+) = 0.30 \pm 0.02 \).

As a final constraint, inspired by the quark counting rules \[15\], we assume the asymptotic \( s \)-behaviour: \( d(s) \sim 1/s \).

Experimental information on the \( K\pi \) system was obtained from the analysis of the reaction \( K^-p \to K^-\pi^+n \) some time ago \[14\]. The partial wave analysis of \( K\pi \to K\pi \) provides evidence, in the \( 0^+, I = 1/2 \) channel of a well-established \( K_0^*(1430) \) resonance, with \( M_R = 1429 \pm 4 \pm 5 \) MeV and \( \Gamma_R = 287 \pm 10 \pm 21 \) MeV, and a signal for a not yet confirmed \( K_0^*(1950) \) state, with \( M_R = 1945 \pm 10 \pm 20 \) MeV and \( \Gamma_R = 201 \pm 34 \pm 79 \) MeV \[15, 17\]. Moreover, a non negligible nonresonant component underlying the \( K_0^*(1430) \) resonance shows up in the measured low energy \( K\pi \) phase shifts.

\[4\] However, this value seems not quite in agreement with the result from \( K_{\mu 3}^+ \) decays \[14\].

\[5\] Actually, a small curvature from higher order terms in the low energy representation of \( f_0 \) is admitted, and in principle might be perceptible at \( s_+ \) which is much larger than the decay endpoint \( s_- \).
In [5, 6], the behaviour of \(|d(s)|^2\) appearing in Eq. (18) is modeled by the sum of two Breit-Wigner forms (with masses and widths as reported previously), normalized at \(s = s_+\) to the theoretical prediction from (21) and (22). To model an alternative parameterization of (18) which includes all the experimental information mentioned above, in particular the existence of the nonresonant component, one can attempt a construction of the form factor \(d(s)\) based on analyticity properties and asymptotic behaviour, with the measured \(K\pi\) phase shifts as an input, consistently with the final state interaction theorem. Such a construction of \(d(s)\) can be realized by assuming the following representation [18]:

\[
\frac{d(s)}{d(0)} = \exp\left[\frac{s}{\pi} \int_{s_+}^{\infty} ds' \frac{\delta(s')}{s'(s' - s - i\epsilon)}\right],
\]

(23)

with \(d(0)\) determined from Eqs. (20), (22) and \(\delta(s)\) the \(K\pi\) \((I = \frac{1}{2}, J^P = 0^+)\) phase shift. As usual in the applications of this representation, we do not consider the possibility of zeroes for \(d(s)\), which would require also a polynomial factor in (23).

The \(K\pi\) phase shift is well known in the range of invariant mass from \(s_+\), to \((1.7\ GeV)^2\); it can be parameterized as the sum of an effective range formula and a resonant phase:

\[
\delta(s) = \delta_{ER}(s) + \delta_{BW}(s)
\]

(24)

with:

\[
\delta_{ER}(s) = \text{arctg} \left[ a q(s)(1 + b q^2(s)) \right]
\]

(25)

\((q(s)\) is the \(K\pi\) CM momentum\) and

\[
\delta_{BW} = \text{arctg}\left[ \frac{M_R \Gamma_R(s)}{M_R^2 - s} \right],
\]

(26)

where \(M_R\) is the mass and \(\Gamma_R(s)\) is the s-dependent width of the \(K^*_0(1430)\) state:

\[
\Gamma_R(s) = \frac{M_R}{\sqrt{s}} \frac{q(s)}{q(M_R^2)} \Gamma_R
\]

(27)

The parameters \(a\) and \(b\) have been fitted, with the result [3]:

\[
a = 2.06\ GeV^{-1}, \quad b = -1.37\ GeV^{-2}.
\]

(28)

As for the region \(s > (1.7\ GeV)^2\), above the \(K\eta'\) threshold, inelastic effects are observed [10, 10, 10, 10]. The inclusion of such effects would require a coupled channel analysis of the \(K\pi\)

– The contribution of the \(K\eta\) state is flavour-\(SU(3)\) suppressed [10].
and $K\eta'$ states, with further contributions to $\rho^{(HAD)}$. Since we do not consider here such additional contributions, strictly speaking the result of our analysis is a lower bound for $m_s$, due to the positivity properties of the spectral function $\rho(s)$. However, the exponential factor in Eq. (8) should suppress the contribution of higher states. This is confirmed by the numerical analysis of ref. [5], where the contribution of the $K^*_0(1950)$ has a very small influence on the result for $m_s$.

As far as $d(s)$ is concerned, the asymptotic $1/s$ behaviour at large $s$ can be obtained from (23) if $\delta(s) \to 180^0$, and in this regard we fix $\delta(s) = 180^0$ for $s > (1.7 \text{ GeV})^2$. This is a delicate assumption from the numerical point of view, which, however, is supported by the available experimental data [10, 16].

We can remark that, using the parameterization (24) of the phase shift, the form factor $d(s)$ in (23) reproduces the slope predicted by chiral perturbation theory, as already observed in [6] and shown in Fig. 1. This feature is not obtained, in the framework of the representation (23) for $d(s)$, if the phase shift is parameterized in terms of the resonant $\delta_{BW}(s)$ phase only. Therefore, in the approach considered here, the inclusion of the nonresonant component is needed on phenomenological grounds. The obtained spectral function is depicted in Fig. 2, where we compare the result from the parameterization (24)-(26) with the case of the pure Breit-Wigner form obtained using $a = b = 0$. The substantial reduction of the resonance peak shows that even a moderate nonresonant contribution in the $K\pi$ phase shift $\delta(s)$ can generate a significant variation of the spectral function via the exponential form of the analytic representation (23).

At this point, we perform the numerical analysis of the sum rule, following the same procedure adopted in [5]. The only slight difference with respect to [5] is that, in the expansion of the $\beta$ function relevant to the perturbative part of the sum rule:

$$\beta(a) = \sum_n \beta_n a^n \quad (a = \frac{\alpha_s}{\pi}) \quad (29)$$

we use the coefficient $\beta_4$ recently computed in the $\overline{MS}$ scheme [20], namely the set of values:

$$\beta_1 = -\frac{9}{2}, \quad \beta_2 = -8, \quad \beta_3 = -\frac{3863}{192}, \quad \beta_4 = -\frac{281198}{4608} - \frac{890}{32} \zeta(3) \quad (30)$$

for $N_c = 3$ and $n_f = 3$; $\zeta$ is the Riemann zeta function. The numerical value for $\beta_4$ differs by a factor of two from that used in [5] obtained by a Padé approximant [21].

The identity of the representation (23) with a Breit-Wigner form in the case of a pure resonance can be shown directly [19].
Moreover, for the analogous expansion of the anomalous dimension \( \gamma(a) \): \( \gamma(a) = \sum_n \gamma_n a^n \), we use the first three computed coefficients (for \( N_c = 3 \), \( n_f = 3 \))

\[
\gamma = \begin{pmatrix} 2 \frac{91}{12} \gamma_3 \frac{8885}{288} - 5\zeta(3) \end{pmatrix},
\]

whereas for \( \gamma_4 \) we use the result of two Padé approximants

\[
\frac{\gamma(a)}{a} = A + Ba + \frac{Qa}{1 + Qa},
\]

(\( A, B, Q \) numerical coefficients) giving \( \gamma_4 = \frac{\gamma_3^2}{\gamma_2} \) \([1, 22]\), and

\[
\frac{\gamma(a)}{a} = \frac{A}{1 + Ra + Sa^2},
\]

giving \( \gamma_4 = 2\gamma_2 \gamma_3 / \gamma_1 - \gamma_2^3 / \gamma_1^2 \). The difference between the two approximants \([32] \) and \([33] \) is about 3\%, and has no practical influence on the final result for \( m_s \)\footnote{As for the parameters in the theoretical side of the sum rule, we use the values: \( \langle \bar{u}u \rangle |_{\mu=1 \ GeV} = (-0.225 \ GeV)^3 \), \( \frac{\tilde{G}}{\pi} = (2 - 6) \times 10^{-2} \ GeV^4 \) and \( \frac{\tilde{s}s}{\bar{u}u} = 0.7 - 1.0 \); in the coefficient of eq. \([12] \) we use \( m_u = m_s/34.2 \) as from Eq. \([1] \). The dependence of the result for \( m_s \) from these nonperturbative contributions is rather weak, and therefore the uncertainty corresponding to the variation of the parameters in the considered ranges is found to be small, of the order of 1 \( MeV \).}

We varied the threshold \( s_0 \) in the range 5 – 7 \( GeV^2 \), the Borel parameter \( M^2 \) in the range 1 – 9 \( GeV^2 \), and we considered the three values for \( \Lambda_{\overline{MS}}^{n_f = 3} \): 280, 380 and 480 \( MeV \). Specifically, the best stability in \( M^2 \) of the resulting \( m_s \) is obtained for \( s_0 = 5.5 \ GeV^2 \) when \( \Lambda_{\overline{MS}}^{n_f = 3} = 380 \ MeV \); for \( s_0 = 5.0 \ GeV^2 \) when \( \Lambda_{\overline{MS}}^{n_f = 3} = 280 \ MeV \) and for \( s_0 = 6 \ GeV^2 \) when \( \Lambda_{\overline{MS}}^{n_f = 3} = 480 \ MeV \).\footnote{We varied the threshold \( s_0 \) in the range 5 – 7 \( GeV^2 \), the Borel parameter \( M^2 \) in the range 1 – 9 \( GeV^2 \), and we considered the three values for \( \Lambda_{\overline{MS}}^{n_f = 3} \): 280, 380 and 480 \( MeV \). Specifically, the best stability in \( M^2 \) of the resulting \( m_s \) is obtained for \( s_0 = 5.5 \ GeV^2 \) when \( \Lambda_{\overline{MS}}^{n_f = 3} = 380 \ MeV \); for \( s_0 = 5.0 \ GeV^2 \) when \( \Lambda_{\overline{MS}}^{n_f = 3} = 280 \ MeV \) and for \( s_0 = 6 \ GeV^2 \) when \( \Lambda_{\overline{MS}}^{n_f = 3} = 480 \ MeV \).}

The stability curves for the running mass \( \overline{m}_s(1 \ GeV) \) are depicted in Fig. 3 where only the dependence on \( \Lambda_{\overline{MS}}^{n_f = 3} \) and \( M^2 \) is displayed. Changing only \( \Lambda_{\overline{MS}}^{n_f = 3} \) the result for \( m_s \) changes in the range 130 – 140 \( MeV \). An additional uncertainty is due to the parameters describing the continuum \( K\pi \) component; changing the parameter \( a \) in the effective range formula \([23] \) by 20\%, and the other parameters in the considered ranges, we have

\[
\overline{m}_s(1 \ GeV) = 125 - 160 \ MeV
\]

\footnote{After this paper was completed, a calculation of \( \gamma_4 \) in the \( \overline{MS} \) scheme appeared in \([23] \). The computed value: \( \gamma_4 = 88.5258 \) differs from the results in \([22, 33] \) by 12\%. This effect has no influence on the result for \( m_s \).}

\footnote{Concerning the possible instanton contribution to the correlator \([6, 24] \), even though a quantitative assessment is difficult, we argue that such contribution is within the uncertainty related to the hadronic spectral function, mainly due to the large energy scale involved in the calculation of the correlator Eq. \([1] \).}
with a lower bound

$$\overline{m}_s(1 \text{ GeV}) \geq 120 \text{ MeV.} \quad (35)$$

The value (34) derives from a parameterization of the hadronic spectral function which has the correct analyticity properties and uses all the available experimental information. This determination can be systematically improved by a dedicated analysis of the scalar \( I = \frac{1}{2} K\pi \) channel using, e.g., the semileptonic decays of the \( \tau \) lepton.

In principle, the result (34) is reflected in the lower bound (35) due to the neglect of higher states; however, we do not expect that the actual determination of \( m_s \) in the theoretical framework considered here can significantly differ from it. The value (34), lower that the result obtained assuming the resonance dominance, shows the significancy of the \( K\pi \) nonresonant continuum. \[10\]

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\[10\] Similar conclusions have also been reached in ref. [25].
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FIGURE CAPTIONS

Fig. 1
Form factor $d(s)$ obtained using the representation Eq. (23) (continuous line). The $\chi$PT prediction Eq. (21), with $\lambda_0 = 1.69 \times 10^{-2}$, is also shown (dashed line).

Fig. 2
The hadronic spectral function $\rho(s)$ obtained including only the resonant $K_0^*(1430)$ state (upper dashed line), or using the representation Eq. (23) (lower lines); the three low-lying curves correspond to a 20% variation of the parameter $a$ in Eq. (25).

Fig. 3
The running mass $\overline{m}_s(\mu)$ at the scale $\mu = 1$ GeV as a function of the Borel parameter $M^2$, for $\Lambda_{\overline{MS}}^{nf=3} = 380$ MeV (continuous line), $\Lambda_{\overline{MS}}^{nf=3} = 280$ MeV (dotted line) and $\Lambda_{\overline{MS}}^{nf=3} = 480$ MeV (dashed line).
fig. 1
fig. 3