Q-hairs and no-hair theorem for charged black holes

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The no-hair theorem by Mayo and Bekenstein states that there exists no non-extremal static and spherical charged black hole endowed with hair in the form of a charged scalar field with a self-interaction potential. In our recent work [Phys. Lett. B 739 (2020) (2014) 302], we showed that the effect of a scalar mass term is important at an asymptotic infinity, which was omitted to prove the no-hair theorem. In this paper, we demonstrate that there actually exists static and spherical charged scalar hair, dubbed as Q-hair, around charged black holes, by taking into account the backreaction to the metric and gauge field. We also discuss that Q-cloud, which is constructed without the backreaction around a Reissner-Nordström black hole, is a good approximation to Q-hair under a certain limit.

Introduction.– Direct observations of gravitational waves [1] and the black-hole shadow [2] have opened up a new era in black hole (BH) physics. As a unique candidate of the strong gravity regime, a better understanding of BH will be inevitable for a deeper understanding of gravity. Moreover, developments from both theoretical and experimental sides might provide a clue to a long-standing question about a unified description of general relativity and quantum mechanics.

Central to our understanding of BH nature relies on the no-hair theorem [3–5]. In early stage [6–8], it precludes the static black hole with a scalar hair. Later on, however, black hole solutions with the scalar hair have been found, which includes the BHs with Skyrmion [9,10] Yang-Mills [11–13], and Dilaton hairs [14]. For the recent review of the no-hair theorem, see Ref. [15].

In a recent work [16], we reconsidered the novel no-hair theorem [17] for spherically symmetric charged black holes. That theorem is concluded by Mayo and Bekenstein and states as follows [17]:

There exists no non-extremal static and spherical charged black hole endowed with hair in the form of a charged scalar field, whether minimally or nonminimally coupled to gravity, and with a regular positive semidefinite self-interaction potential.

As we argued in Ref. [16], however, the above statement is not correctly concluded because they omitted a scalar mass term at an asymptotic infinity in the equation of motion of the scalar field.

In this paper, we construct the numerical examples of spherical charged black hole with charged scalar hair, which are consistent with our argument. After the detailed explanation to our disproof, we demonstrate the numerical scalar hair solutions, dubbed as Q-hairs, in a polynomial scalar potential. Under certain limits, we also show that the scalar hairs are in accord with the cloud solutions [16] obtained under the Reissner-Nordstrom BH background.

Note added: While completing our work, we became aware of an independent work by Carlos A. R. Herdeiro and Eugen Radu [22], who also demonstrate counter examples to the Mayo Bekenstein no-hair theorem.

Equations of motion.– We focus on a theory with a U(1) gauge field $A_{\mu}$ and a charged scalar field $\psi$ which minimally couples to the gravity. In this paper, we use the same notation as in Ref. [17] with $\xi = 0$ for simplicity.

The action and the Lagrangian density are given by

$$S_{SM} = \int \sqrt{-g} d^4x (\mathcal{L}_M + \mathcal{L}_G),$$

$$\mathcal{L}_M = -\frac{1}{2} \left( (D^\alpha \psi)^* D_\alpha \psi + V(\psi, \psi^*) + \frac{1}{8\pi} F_{\alpha\beta} F^{\alpha\beta} \right),$$

$$\mathcal{L}_G = -\frac{1}{16\pi G} R,$$

where

$$D_\alpha = \partial_\alpha - iq A_\alpha, \quad F_{\alpha\beta} = A_{\beta,\alpha} - A_{\alpha,\beta},$$

are the covariant derivative and the field strength of U(1) gauge interaction, respectively. The four-current density $j_\alpha$ of electric charge is given by

$$j_\alpha = q \text{Im} [\psi^* D_\alpha \psi],$$

1 In the flat spacetime, the attractive scalar self-interaction allows to form a non-topological soliton, known as Q-ball [18,21].
and the Maxwell equation is $F^{\alpha\beta} = 4\pi j^\alpha$.

We are interested in static solutions to the Einstein equation, $R^2 - R/2 \delta^a_b = 8\pi GT^a_b$, with a charged black hole located at the center of the coordinate in an asymptotically flat spacetime. The metric is then written as

$$ds^2 = -e^\nu(dt)^2 + \rho^2(dr^2 + \sin^2 \theta d\phi^2)\ ,$$

where $\nu$ and $\lambda$ are functions of $r$ and $O(r^{-1})$ as $r \to \infty$. We define $r_H$ by the radius of the event horizon at the surface of the BH, where $e^{-\lambda(r_H)} = 0$. We focus on the case of non-extremal black hole, in which $e^\nu, e^{-\lambda} = O(r-r_H)$ for $r \to r_H$.

Let us specify the gauge fixing of U(1) gauge symmetry. In a spherically-symmetric static spacetime, $F_{\ell r}$ is the only non-vanishing component for the field strength, which implies that only $A_\ell$ and $A_r$ are non-vanishing components. We can make $A_\ell = 0$ by a gauge transformation of $A_\alpha \to A_\alpha + \Lambda_\alpha$ with $\Lambda = -\int A_\ell dr$. The time component of the gauge field $A_t$ must be the form of $f(r) + g(t)$ so that $F_{\ell r}$ is stationary. Then we can use a residual gauge transformation $\Lambda = -\int g(t)dt$ to make $A_t$ static. The scalar field must be in a form of $\psi = a(r)e^{ib(r) - i\omega t}$ with a real constant $\omega$ since otherwise the current and charge density depend on time. A further gauge transformation with $\Lambda = \omega t/q$ makes $\psi = a(r)e^{ib(r)}$ and $A_t \to A_t + \omega/q$. The conservation of charge implies that $b(r)$ is independent of $r$. Otherwise charge lead out continually to infinity. In summary, the non-vanishing components for the fields are $A_t(r)$ and $\psi = a(r)$.

The field equations and the Einstein equations are written as

$$a_{r r} + \frac{1}{2} (\frac{4}{r} + \nu - \lambda) a_r - (\hat{V} - q^2 e^{-\nu} A_\ell^2) e^\lambda a = 0\ ,$$

$$A_{t r} + \frac{1}{2} (\frac{4}{r} - \nu - \lambda) A_{r t} - 4q^2 a^2 e^\lambda A_t = 0\ ,$$

$$e^{-\lambda} \left(\frac{1}{r^2} - \frac{\lambda}{r}\right) A_{t r} - \frac{1}{r^2} = 8\pi GT^t_r\ ,$$

$$e^{-\lambda} \left(\frac{\nu}{r^2} + \frac{1}{r}\right) - \frac{1}{r^2} = 8\pi GT^r_r\ ,$$

where $(t, t)$ and $(r, r)$ components of the energy-momentum tensor are given by

$$8\pi T^t_t = 4\pi \left(-e^{-\lambda} a^2_{r r} - e^{-\nu} q^2 A_\ell^2 a^2 - \hat{V}\right) - e^{-\nu - \lambda} A_{t r}^2\ ,$$

$$8\pi T^r_r = 4\pi \left(e^{-\lambda} a^2_{r r} + e^{-\nu} q^2 A_\ell^2 a^2 - \hat{V}\right) - e^{-\nu - \lambda} A_{t r}^2\ ,$$

where $\hat{V} \equiv \partial V/\partial a^2$.

We define $Q(r)$ by the electric charge enclosed by the sphere of radius $r$:

$$Q(r) \equiv Q_{\text{BH}} + Q_\psi(r),$$

where $Q_{\text{BH}}$ is the charge of the BH and

$$Q_\psi(r) = \int \sqrt{-g} d^3 x \rho^0\ ,$$

$$= -4\pi q^2 \int_{r_H}^r dr' r'^2 e^{(\nu - \lambda)/2} a^2(r) A_t(r)\ ,$$

is the electric charge of $\psi$ enclosed by the sphere of radius $r$. Then we find

$$e^{-(\nu + \lambda)/2} A_{t r} = -\frac{Q(r)}{r^2},$$

from the Gauss’s law.

No hair theorem by Mayo and Bekenstein.— Before we are going to dispute the no-hair theorem, let us briefly review the proof by Mayo and Bekenstein [17]. Noting that $e^{-(\nu + \lambda)/2} \approx \text{const.}$ near the event horizon, Eq. (10) implies that $A_{t r}$ is regular at $r = r_H$. The asymptotic form of the fields and the metric near the event horizon $r = r_H$ are then written as

$$A_t = c_0 - c_1 (r - r_H)\ ,$$

$$e^\nu = c_2 (r - r_H)\ ,$$

$$e^\lambda = \frac{c_3}{r - r_H}\ ,$$

where $c_i$ denote positive finite constants. Note that $c_1$ is determined by Eq. (16).

Now, suppose that the gauge field does not vanish at an asymptotic infinity, $A_t(\infty) \neq 0$. Then a must vanish asymptotically to satisfy the Maxwell equation Eq. (7). Since we consider an asymptotically flat spacetime, we require $V(a) \to 0$ and $a \to 0$ for $r \to \infty$. Then one may think that Eq. (7) reduces to

$$a_{r r} + \frac{2}{r} a_r + q^2 A_t(\infty)^2 a = 0 \quad \text{for} \quad r \to \infty.$$

As we show in the next section, this is correct only if we omit the scalar mass term. For a moment, we assume the above equation, following Ref. [17]. The solution to the equation has the form of

$$a \sim \frac{1}{r} \sin (q A_t(\infty) r + \chi) \quad \text{for} \quad r \to \infty,$$

where $\chi$ is a constant. In this case, the electric charge density of the scalar field, $\sqrt{g} e^{i j}$, is given by

$$\sqrt{g} e^{j}(r) \sim -4\pi q^2 A_t(\infty) \sin^2 (q A_t(\infty) r + \chi) \quad \text{for} \quad r \to \infty,$$

Then the total electric charge diverges, which means that the assumption of $A_t(\infty) \neq 0$ does not lead to a physical solution. One thus concludes that

$$A_t(\infty) = 0 \ .$$

2 After the gauge transformation, $A_t$ in this paper corresponds to $g(r)/q (\equiv A_0 + \omega/q)$ in Ref. [19].
Once $A_t(\infty) = 0$ is obtained, we can show that $A_t(r)$ is a monotonic function. Suppose that $A_t(r)$ has an extremal at a certain radius $r_*$. Then Eq. (13) implies that $A_t'(r_*)$ and $A_t(r_*)$ has the same sign and hence the extremal is a minimum for $A_t(r_*) > 0$ and is a maximum for $A_t(r_*) < 0$. This can not be consistent with $A_t(\infty) = 0$, so that we conclude that $A_t(r)$ is a monotonic function. Since the overall sign of $A_t$ can be changed by changing the sign of $q$ without loss of generality, we can set $A_t(r) > 0$. In this notation, $A_t(r)$ is a monotonically decreasing function. In particular, $A_t(r_H) (\equiv c_0)$ must be nonzero and positive.

Given these properties, the second term in Eq. (8) is finite for $r \to r_H$ from Eq. (17)-Eq. (19). In the third term, $A_t(r_H)$ is finite and $e^\lambda$ diverges as $1/(r - r_H)$ from Eq. (19), so that $a$ must behave as

$$a = 0 \quad \text{for} \quad r = r_H. \quad (24)$$

On the other hand, Eq. (7) can be approximated to be

$$a,rr + \frac{1}{r - r_H}a,r + \frac{q^2c_0^2}{2(r - r_H)^2}a = 0 \quad \text{for} \quad r \to r_H. \quad (25)$$

The solution to this equation is given by

$$a = B \sin \left[ q^2c_0^{-1/2} \ln \left( \frac{r - r_H}{D} \right) \right], \quad (26)$$

where $B$ and $D$ are arbitrary constants. This solution is, however, inconsistent with the condition in Eq. (24) because there is no choice of constants to satisfy $a \to 0$ for $r \to r_H$. This means that there is no solution for the equations of motion and one may thus conclude the no-hair theorem for a spherically-symmetric static black hole. However, as we briefly noted below Eq. (20), the above argument is correct only if we omit the scalar mass term in Eq. (20).

Incompleteness of the no hair theorem.-- Now we show that the no-hair theorem is not applicable to the case in which the complex scalar field has non-zero mass $\mu$. In the next section, we explicitly show our numerical solutions of scalar hair for a polynomial potential.

In the above proof, we cannot deduce the condition of Eq. (17) if the mass of the scalar field is non-negligible. Even if $V(\psi)$ and $\psi$ are asymptotic to 0 for $r \to \infty$, we must include the mass term because it is in the same order with the last term in Eq. (20). Indeed, the asymptotic scalar equation in Eq. (20) is modified to

$$a,rr + \frac{2}{r}a,r - (\mu^2 - q^2A_t(\infty)^2)a = 0 \quad \text{for} \quad r \to \infty, \quad (27)$$

where $\mu^2$ is the mass squared for the scalar field. If the parenthesis is positive, the solution is given by

$$a \sim \frac{1}{r} e^{\sqrt{\mu^2 - q^2A_t(\infty)^2} r}, \quad (28)$$

and the total electric charge is finite. Therefore, there may be a consistent solution even for $A_t(\infty) \neq 0$ and Eq. (23) is not necessarily true. In particular, there is no reason that we cannot take $A_t(r_H) = 0$ and $a(r_H) \neq 0$.

In fact, there is a consistent solution if we take

$$A_t = O(r - r_H) \quad \text{for} \quad r \to r_H. \quad (29)$$

Then the third term in Eq. (8) can be finite even if $a$ is finite for $r \to r_H$. From Eq. (7), one can check that $a'$ at $r = r_H$ is finite and is given by

$$a'(r_H) = \frac{V_a}{r_H} \left( \frac{1}{r_H^2} + 4\pi G \left( -V - \frac{Q^2}{4\pi r_H^4} \right) \right). \quad (30)$$

Here we used $e^{-\nu}A_t^2 \to 0$ for $r \to r_H$.

Note that $A_t \to 0$ for $r \to r_H$ is the condition that is used to find a static solution of Q-cloud in Ref. [16]. It is also known to be at the threshold for superradiance [23, 24], which is also the case for scalar hairs around Kerr BH [24, 34]. In the next part, we search numerical solutions for the scalar hairs that have the above asymptotic forms.

Numerical solutions.-- Since the effect of scalar hair is expected to be negligible near the surface of the event horizon, we expect that the metric near the BH surface should be written in the form of the Reissner-Nordstrom BH with a nonzero vacuum energy such as

$$e^{-\lambda} \approx 1 - \frac{2GM_{BH}}{r} + \frac{GQ^2_{BH}}{r^2} - \frac{8\pi GAr^2}{3}, \quad \text{for} \quad r \approx r_H, \quad (31)$$

where $\Lambda = V(\psi(r_H))/2$. One can check that $\lambda, r(r_H)$ derived from Eq. (31) is consistent with the one calculated from Eq. (9). Using Eqs. (33) and (35), we then obtain

$$M_{BH} = \frac{r_H}{2G} + \frac{Q^2_{BH}}{2r_H} - \frac{4\pi r_H^2}{3}\Lambda. \quad (32)$$

If $r_H$ is given, Eq. (32) can be regarded as the definition of the BH mass $M_{BH}$. Conversely, one may specify $M_{BH}$ and determine $r_H$ from Eq. (32).

For the purpose of numerical simulation, it is convenient to define $E(r)$ by

$$e^{-\lambda} \equiv 1 - \frac{2GE(r)}{r}. \quad (33)$$

Then it satisfies

$$\partial_r E(r) = -4\pi r^2 T_r^t, \quad (34)$$

$$E(r_H) = \frac{r_H}{2G}. \quad (35)$$

We note that a boundary condition of $e^{-\lambda} \to 1$ for $r \to \infty$ is manifestly satisfied in Eq. (33). The function $E(r)$ is
The scalar potential is taken to be the following polynomial:

$$V = \mu^2 a^2 - \frac{\lambda}{2} a^4 + \frac{\alpha}{4} a^6,$$

where \( \mu \) is the scalar mass at the potential minimum, \( \lambda \) and \( \alpha \) are couplings. We take \( \lambda = 1 \) and \( \alpha = \lambda^2/(3\mu^2) \) as an example.

First, suppose that \((1 - M_{BH}/E_{tot}) \ll 1 \) and \((1 - Q_{BH}/Q_{tot}) \ll 1 \), namely \( Q_{BH} \gg Q_{\psi}(\infty) \) and \( M_{BH} \gg E_{A}(\infty) + E_{\psi}(\infty) \), where \( E_{tot} \equiv E(\infty) \) and \( Q_{tot} \equiv Q(\infty) \). In this case we can neglect the right-hand side of Eqs. (9) and (10) and the solution to those equations are just given by the one for the Reissner-Nordström BH: \( \nu = -\lambda = \ln(r^2 - 2M_{BH}r + Q_{BH}^2/r^2) \). If the third term of Eq. (8) is negligible, the solution of the gauge field is then given by \( A_t(r_H) = 0 \). The only non-trivial equation is Eq. (7), which can be numerically solved by the shooting method. This has been done in Ref. [16] and there actually exist solutions of scalar hair. We note that these limits can be realized by \( \mu/M_{pl} \to 0 \) with \( c_Q, c_M, c_q \) fixed, where \( G \equiv 1/M_{pl}^2 \) and

$$Q_{BH} \equiv c_Q M_{BH}/M_{pl}, \quad M_{BH} \equiv c_M M_{pl}^2/\mu, \quad q \equiv c_q \mu/M_{pl}. \quad (40)$$

As we have confirmed that there are solutions to Eq. (7) in this limit, we can start from a small \( \mu/M_{pl} \) and increase it to find numerical solutions of Q-hair in the full equations.

We show three examples of Q-hair in Fig. 1 where we take \( c_M = 1.5 \) and \( c_Q = q_q = 0.8 \). The solid lines represent \( \psi(r)/\mu \) while the dashed lines represent \( qA_t(r_H)/\mu \) for the case of \( \mu/M_{pl} = 0.02 \) (blue), 0.005 (orange), and 0.001 (green). We also show the Q-cloud solution around a Reissner-Nordström BH with the same parameters as red dotted lines. We cannot distinguish between the green lines and the red dotted lines as they are completely overlapped. This means that the Q-cloud solution is a good approximation for \( \mu/M_{pl} \ll 0.001 \) in this case.

In Fig. 2 each dot represents a parameter at which there exists Q-hair solution. We fix \( \mu/M_{pl} \) as 0.02 (blue dots), 0.005 (orange squares), and 0.001 (green diamonds) with \( q_q = 0.8 \). We take \( c_M \) and \( c_Q \) randomly.
within (0, 10) and (0, 1), respectively, and no solution is found for $\frac{\mu}{M_{\text{pl}}} \gtrsim 9$. We can see that Q-hairs with smaller $\frac{\mu}{M_{\text{pl}}}$ have smaller $(1 - \frac{M_{\text{BH}}}{E_{\text{tot}}})$ and $(1 - \frac{Q_{\text{BH}}}{Q_{\text{tot}}})$. This implies that the backreaction to the metric is negligible for a small $\frac{\mu}{M_{\text{pl}}}$.

The red line in Fig. 2 is a one-parameter solution for $\frac{\mu}{M_{\text{pl}}}$ with $c_M = 1.5$, $c_Q = 0.8$, and $c_q = 0.8$. It is bounded above as no solution is found for $\frac{\mu}{M_{\text{pl}}} \gtrsim 0.023$ in this case. The three black crosses represent the parameters corresponding to the three examples used in Fig. 1 namely for $\frac{\mu}{M_{\text{pl}}} = 0.02, 0.005$, and 0.001. As we expect from Eqs. (15), (37), and (38), both $(1 - \frac{M_{\text{BH}}}{E_{\text{tot}}})$ and $(1 - \frac{Q_{\text{BH}}}{Q_{\text{tot}}})$ are almost proportional to $(\frac{\mu}{M_{\text{pl}}})^2$ for a small $\frac{\mu}{M_{\text{pl}}}$. We conclude that the Q-cloud solutions are good approximation in the limit of $\frac{\mu}{M_{\text{pl}}} \to 0$.

**Discussion.**—We have constructed the numerical examples of the scalar hair solutions of spherically symmetric charged black holes in full equations of motion, taking into account the backreaction to the metric and gauge field. Concretely, the solutions have been investigated for a polynomial potential. In accordance with our argument of the previous paper, we indeed have found the scalar hair, which are counterexamples to the no-hair theorem of the spherically symmetric charged black holes. Moreover, we have also found that some scalar hair solutions match well with the solutions obtained without including the backreaction to the metric and gauge field, particularly in the limit of $M_{\text{pl}} \to \infty$. This is phenomenologically important because, e.g., a typical grand-unified scale is three orders of magnitude smaller than the Planck scale, which is small enough for the backreaction to the metric to be negligible.

Although the stability of Q-cloud against small perturbations as well as non-perturbative process is justified in Ref. [16], that of Q-hair has not been explored yet. We need to investigate the behavior of small perturbations on top of the Q-hair to ensure its stability. This would be an interesting direction to future work.

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