SIMILARITY OF HOLOMORPHIC MATRICES ON 1-DIMENSIONAL STEIN SPACES

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Dedicated to the memory of Selim Krein

Abstract. R. Guralnick [Linear Algebra Appl. 99, 85-96 (1988)] proved that two holomorphic matrices on a noncompact connected Riemann surface, which are locally holomorphically similar, are globally holomorphically similar. In the preprints [arXiv:1703.09524] and [arXiv:1703.09530], a generalization of this to arbitrary (possibly, nonsmooth) 1-dimensional Stein spaces was obtained. The present paper contains a revised version of the proof from [arXiv:1703.09524]. The method of this revised proof can be used also in the higher dimensional case, which will be the subject of a forthcoming paper.

1. Introduction

Let $X$ be a (reduced) complex space, e.g., a complex manifold or an analytic subset of a complex manifold. Let $\text{Mat}(n \times n, \mathbb{C})$ be the algebra of complex $n \times n$ matrices, and $\text{GL}(n, \mathbb{C})$ the group of invertible complex $n \times n$ matrices.

Two holomorphic maps $A, B : X \rightarrow \text{Mat}(n \times n, \mathbb{C})$ are called (globally) holomorphically similar on $X$ if there is a holomorphic map $H : X \rightarrow \text{GL}(n, \mathbb{C})$ with $B = H^{-1}AH$ on $X$. They are called locally holomorphically similar at $\xi \in X$ if there is a neighborhood $U$ of $\xi$ such that $A|_U$ and $B|_U$ are holomorphically similar on $U$. Correspondingly, continuous and $C^k$ similarity are defined.

R. Guralnick [13] proved the following theorem.

1.1. Theorem. Suppose $X$ is a noncompact connected Riemann surface. Then any two holomorphic maps $A, B : X \rightarrow \text{Mat}(n \times n, \mathbb{C})$, which are locally holomorphically similar at each point of $X$, are globally holomorphically similar on $X$.

First in [16], and then in [18], the following generalization was obtained.

1.2. Theorem. The claim of Theorem 1.1 remains true if $X$ is an arbitrary 1-dimensional Stein space (for example, a 1-dimensional closed analytic subset of some $\mathbb{C}^N$, or, more general, of a domain of holomorphy in $\mathbb{C}^N$).

Guralnick's proof of Theorem 1.1 consists in proving a theorem for matrices with elements in a Bezout ring (with some extra properties) and then applying this to the ring of holomorphic functions on a noncompact connected Riemann surface. The ring of holomorphic functions on an arbitrary (possibly nonsmooth) 1-dimensional
Stein space need not be Bezout. Therefore, it seems that this proof cannot be used to prove Theorem 1.2 at least not in a straightforward way.

The proof of Theorem 1.2 given in [16] is independent of Guralnick’s proof and is therefore also a new proof of Theorem 1.1. It proceeds as follows. First we use the Oka principle for Oka-pairs of Forster and Ramspott [7] to show that Theorem 1.2 is equivalent to a certain topological statement (see Theorem 3.3 below). Then we prove this topological statement.

In [18], we obtained a shorter proof, using Guralnick’s result in the proof of the topological statement.

The aim of the present paper is to give a revised version of the proof from [16]. The method of this revised version is useful also if $X$ is a higher dimensional.

For example, in a forthcoming paper, we will show that the claim of Theorem 1.1 remains valid if $X$ is a convex domain in $C^2$ (for convex domains in $C^3$, this is not true). With this in view, already in the present paper, sometimes $X$ is allowed to be of arbitrary dimension.

2. Notations and our use of the language of sheaves

$\mathbb{N}$ is the set of natural numbers including 0. $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

If $n, m \in \mathbb{N}^*$, then $\text{Mat}(n \times m, \mathbb{C})$ is the algebra of complex $n \times m$ matrices ($n$ rows, $m$ columns), and $\text{GL}(n, \mathbb{C})$ is the group of invertible complex $n \times n$ matrices.

The unit matrix in $\text{Mat}(n \times n, \mathbb{C})$ will be denoted by $I_n$ or simply by $I$.

If a matrix $\Phi \in \text{Mat}(n \times m, \mathbb{C})$ is interpreted as a linear map from $\mathbb{C}^m$ to $\mathbb{C}^n$, then $\ker \Phi$ denotes the kernel, $\text{im} \Phi$ the image and $\| \Phi \|$ the operator norm (induced by the Euclidean norm) of $\Phi$.

By a complex space we always mean a reduced complex space in the sense of [11], which is the same as an analytic space in the terminology used in [5] and [19].

From now on, in this section, $X$ is a topological space, and $G$ is a topological group (possibly non-abelian).

By $\mathcal{O}^G_X$, or more precisely by $\mathcal{O}^G_X(U)$, we denote the sheaf of continuous $G$-valued maps on $X$, that is, the sheaf which assigns to each nonempty open $U \subseteq X$ the group $\mathcal{O}^G(U)$ of all continuous maps $f : U \to G$. Also $\mathcal{O}^G(\emptyset) := \{1\}$ (1 being the neutral element of $G$).

A subsheaf of $\mathcal{O}^G_X$ is a map $\mathcal{F}$ which assigns to each open $U \subseteq X$ a subgroup $\mathcal{F}(U)$ of $\mathcal{O}^G(U)$ such that:

- If $V \subseteq U$ are nonempty open subsets of $X$, then, for each $f \in \mathcal{F}(U)$, the restriction of $f$ to $V$, $f|_V$, belongs to $\mathcal{F}(V)$.
- If $U \subseteq X$ is open and $f \in \mathcal{O}^G(U)$ is such that, for each $\xi \in U$, there is an open neighborhood $V \subseteq U$ of $\xi$ with $f|_V \in \mathcal{F}(V)$, then $f \in \mathcal{F}(U)$.

The elements of $\mathcal{F}(U)$ are called sections of $\mathcal{F}$ over $U$.

If $\mathcal{F}$ and $\mathcal{G}$ are two subsheaves of $\mathcal{O}^G_X$, then $\mathcal{F}$ is called a subsheaf of $\mathcal{G}$ if $\mathcal{F}(U) \subseteq \mathcal{G}(U)$ for all open $U \subseteq X$.

If $X$ is a complex space and $G$ a complex Lie group, then we denote by $O^G_X$, or simply by $O^G$, the subsheaf of $C^\infty_X$ which assigns to each nonempty open $U \subseteq X$, the group $O^G(U)$ of all holomorphic maps from $U$ to $G$.

Furtheron, in this section, $\mathcal{F}$ denotes a subsheaf of $\mathcal{O}^G_X$.

Let $\mathcal{U} = \{U_i\}_{i \in I}$ an open covering of $X$. 

A family \( f_{ij} \in \mathcal{F}(U_i \cap U_j) \), \( i, j \in I \), is called a \((\mathcal{U}, \mathcal{F})\)-cocycle, or simply a cocycle, if (the group operation of \( G \) being a multiplication)

\[ f_{ij} f_{jk} = f_{ik} \quad \text{on} \quad U_i \cap U_j \cap U_k \quad \text{for all} \quad i, j, k \in I. \]

Note that then always \( f_{ij}^{-1} = f_{ji} \) and \( f_{ii} \equiv 1 \). The set of all \((\mathcal{U}, \mathcal{F})\)-cocycles will be denoted by \( Z^1(\mathcal{U}, \mathcal{F}) \).

Let \( f = \{ f_{ij} \} \in Z^1(\mathcal{U}, \mathcal{F}) \). We say that \( f \) splits (or is trivial) if there exists a family \( h_i \in \mathcal{F}(U_i) \), \( i \in I \), such that

\[ f_{ij} = h_i h_j^{-1} \quad \text{on} \quad U_i \cap U_j \quad \text{for all} \quad i, j \in I. \]

We say that \( f \) is an \( \mathcal{F} \)-cocycle on \( X \), if there exists an open covering \( \mathcal{U} \) of \( X \) with \( f \in Z^1(\mathcal{U}, \mathcal{F}) \). Then \( \mathcal{U} \) is called the covering of \( f \). As usual, we write \( H^1(X, \mathcal{F}) = 0 \) to say that each \( \mathcal{F} \)-cocycle on \( X \) splits, and \( H^1(X, \mathcal{F}) \neq 0 \) to say that there exist non-splitting \( \mathcal{F} \)-cocycles on \( X \).

Now let \( \mathcal{U}' = \{ U_{\alpha}^* \}_{\alpha \in I^*} \) be a second open covering of \( X \), which is a refinement of \( \mathcal{U} \), i.e., there is a map \( \tau : I^* \rightarrow I \) with \( U_{\alpha}^* \subseteq U_{\tau(\alpha)} \) for all \( \alpha \in I^* \). Then we say that a \((\mathcal{U}', \mathcal{F})\)-cocycle \( \{ f_{\alpha \beta}^* \}_{\alpha, \beta \in I^*} \) is induced by a \((\mathcal{U}, \mathcal{F})\)-cocycle \( \{ f_{ij} \}_{i, j \in I} \) if this map \( \tau \) can be chosen so that

\[ f_{\alpha \beta}^* = f_{\tau(\alpha) \tau(\beta)} \quad \text{on} \quad U_{\alpha}^* \cap U_{\beta}^* \quad \text{for all} \quad \alpha, \beta \in I^*. \]

We need the following well-known and simple proposition, see [13] p. 41 for “only if” and [5] p. 101 for “if”.

**2.1. Proposition.** Let \( f \in Z^1(\mathcal{U}, \mathcal{F}) \) and \( f^* \in Z^1(\mathcal{U}', \mathcal{F}) \) such that \( f^* \) is induced by \( f \). Then \( f \) splits if and only if \( f^* \) splits.

Let \( Y \) be a nonempty open subset of \( X \).

Then we denote by \( \mathcal{F}|_Y \) the subsheaf of \( \mathcal{O}^*_Y \) defined by \( \mathcal{F}|_Y(U) = \mathcal{F}(U) \) for each open \( U \subseteq Y \). \( \mathcal{F}|_Y \) is called the restriction of \( \mathcal{F} \) to \( Y \).

If \( \mathcal{U} = \{ U_i \}_{i \in I} \) is an open covering of \( X \), then we define \( \mathcal{U} \cap Y = \{ U_i \cap Y \}_{i \in I} \), and, for each \( f = \{ f_{ij} \}_{i, j \in I} \in Z^1(\mathcal{U}, \mathcal{F}) \), we denote by \( f|_Y = \{ (f|_Y)_{ij} \}_{i, j \in I} \) the \((\mathcal{U} \cap Y, \mathcal{F}|_Y)\)-cocycle defined by

\[ (f|_Y)_{ij} = f_{ij}|_{U_i \cap U_j \cap Y} \quad \text{for} \quad i, j \in I. \]

We call \( f|_Y \) the restriction of \( f \) to \( Y \).

**2.2. Remark.** Let \( \mathcal{U} = \{ U_i \}_{i \in I} \) be an open covering of \( X \), and \( f = \{ f_{ij} \}_{i, j \in I} \in Z^1(\mathcal{U}, \mathcal{F}) \). Then \( f|_{U_i} \) splits for each \( i \in I \) with \( U_i \neq \emptyset \).

Indeed, the one-set family \( \{ U_i \} \) is an open covering of \( U_i \) which is a refinement of \( \mathcal{U} \cap U_i \), and there is precisely one \((\{ U_i \}, \mathcal{F})\)-cocycle which is induced by \( f|_{U_i} \), namely \( \{ f_{ii} \} \). Since \( f_{ii} \equiv 1 \), it is trivial that \( \{ f_{ii} \} \) splits. Therefore it follows from Proposition 2.1 that \( f|_{U_i} \) splits.

**3. Topological Criteria for Global Holomorphic Similarity**

**3.1. Definition.** Let \( \Phi \in \text{Mat}(n \times n, \mathbb{C}) \). Then we denote by \( \text{Com} \Phi \) the algebra of all \( \Theta \in \text{Mat}(n \times n, \mathbb{C}) \) with \( \Phi \Theta = \Theta \Phi \), and by \( \text{GCom} \Phi \) we denote the group of

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[1] We use the convention that statements like “\( f = g \) on \( \emptyset \)” or “\( f := g \) on \( \emptyset \)” have to be omitted.
invertible elements of $\text{Com} \Phi$. It is easy to see that

\begin{equation}
\text{GCom} \Phi = \text{GL}(n, \mathbb{C}) \cap \text{Com} \Phi,
\end{equation}

\begin{equation}
\text{Com} (\Gamma^{-1} \Phi \Gamma) = \Gamma^{-1} (\text{Com} \Phi) \Gamma \quad \text{for all } \Gamma \in \text{GL}(n, \mathbb{C}).
\end{equation}

Now let $X$ be a complex space (of arbitrary dimension), and $A : X \to \text{Mat}(n \times n, \mathbb{C})$ a holomorphic map.

We introduce the families

\[
\text{Com} A := \{ \text{Com} A(\zeta) \}_{\zeta \in X} \quad \text{and} \quad \text{GCom} A := \{ \text{GCom} A(\zeta) \}_{\zeta \in X}.
\]

If the dimension of $\text{Com} A(\zeta)$ does not depend on $\zeta$, then it is well-known (and easy to see) that $\text{Com} A$ is a holomorphic vector bundle. But also in this special case, the family of groups $\text{GCom} A$ need not be locally trivial. It is possible that $\text{Com} A$ is a holomorphic vector bundle, but $\text{GCom} A$ is even not locally trivial as a family of topological spaces. For an example, see [18, Sec. 4].

Nevertheless the sheaves of holomorphic and continuous sections of $\text{Com} A$ and $\text{GCom} A$ are well-defined. We denote them by $\mathcal{O}^{\text{Com} A}$, $\mathcal{O}^{\text{GCom} A}$, $\mathcal{O}^{\text{Com} A}$, and $\mathcal{O}^{\text{GCom} A}$, respectively.

We define a subsheaf $\hat{\mathcal{O}}^{\text{Com} A}$ of $\mathcal{C}^{\text{Com} A}$ as follows: if $U$ is a nonempty open subset of $X$, then $\hat{\mathcal{O}}^{\text{Com} A}(U)$ is the subalgebra of all $f \in \mathcal{C}^{\text{Com} A}(U)$ such that, for each $\xi \in U$,

\begin{equation}
\left\{ \begin{array}{l}
\text{there exist a neighborhood } V_\xi \text{ of } \xi \\
\text{and } h_\xi \in \mathcal{O}^{\text{Com} A}(V_\xi) \text{ with } h(\xi) = f(\xi).
\end{array} \right.
\end{equation}

Further, we define a subsheaf $\hat{\mathcal{O}}^{\text{GCom} A}$ of $\mathcal{C}^{\text{GCom} A}$ setting $\hat{\mathcal{O}}^{\text{GCom} A}(U) = \mathcal{C}^{\text{GL}(n, \mathbb{C})}(U) \cap \hat{\mathcal{O}}^{\text{Com} A}(U)$ for each open $U \subseteq X$.

The Oka principle for Oka pairs of Forster and Ramspott [7, Satz 1], yields the following

3.2. Proposition. Let $X$ be a Stein space, and let $A : X \to \text{Mat}(n \times n, \mathbb{C})$ be holomorphic. Then $H^1(X, \mathcal{O}^{\text{GCom} A}) = 0$ if and only if $H^1(X, \hat{\mathcal{O}}^{\text{GCom} A}) = 0$.

Indeed, it is easy to see that, for each nonempty open $U \subseteq X$ we have: If $h \in \mathcal{O}^{\text{Com} A}(U)$, then $e^h \in \mathcal{O}^{\text{GCom} A}(U)$, and, if $H \in \mathcal{O}^{\text{GCom} A}(U)$ with $\sup_{\zeta \in U} \| H(\zeta) - I \| < 1$, then

$$\log H := - \sum_{\mu=1}^{\infty} \frac{(H - I)^\mu}{\mu}$$

belongs to $\mathcal{O}^{\text{Com} A}(U)$. This shows that $\mathcal{O}^{\text{GCom} A}$ is a coherent $\mathcal{O}$-subsheaf of $\mathcal{C}^{\text{GL}(n, \mathbb{C})}$ in the sense of [7, §2], where $\mathcal{O}^{\text{Com} A}$ is the generating sheaf of Lie algebras. Moreover, as observed in [7, §2.3, example b)], the pair $(\mathcal{O}^{\text{GCom} A}, \hat{\mathcal{O}}^{\text{GCom} A})$ is an admissible pair in the sense of [7], which, trivially, satisfies condition (PH) in Satz 1 of [7]. Therefore Proposition 3.3 is one of the statements of that Satz 1.

3.3. Theorem. Let $X$ be a Stein space, and let $A : X \to \text{Mat}(n \times n, \mathbb{C})$ be holomorphic. Then the following are equivalent.

(i) Each holomorphic $B : X \to \text{Mat}(n \times n, \mathbb{C})$, which is locally holomorphically similar to $A$ at each point of $X$, is globally holomorphically similar to $A$ on $X$.

(ii) Each $\mathcal{O}^{\text{GCom} A}$-cocycle on $X$, which splits as a $\mathcal{C}^{\text{GL}(n, \mathbb{C})}$-cocycle, splits also as an $\hat{\mathcal{O}}^{\text{GCom} A}$-cocycle.
Proof. (ii) \(\implies\) (i): Let \(B : X \to \text{Mat}(n \times n, \mathbb{C})\) be holomorphic and locally holomorphically similar to \(A\) at each point of \(X\). Then we can find an open covering \(\{U_i\}_{i \in I}\) of \(X\) and holomorphic maps \(H_i : U_i \to \text{GL}(n, \mathbb{C}), i \in I,\) such that

\[
B = H_i^{-1}AH_i \quad \text{on} \quad U_i.
\]

Hence, for all \(i, j \in I\) with \(U_i \cap U_j \neq \emptyset\), \(AH_iH_j^{-1} = H_iH_j^{-1}A\) on \(U_i \cap U_j\), i.e., \(H_iH_j^{-1} \in O^{\text{GCom}}A(U_i \cap U_j)\). Clearly,

\[
(H_iH_j^{-1})(H_kH_j^{-1}) = H_iH_k^{-1} \quad \text{on} \quad U_i \cap U_j \cap U_k, \quad i, j, k \in I.
\]

Therefore, the family \(\{H_iH_j^{-1}\}_{i,j \in I}\) is a well-defined \(O^{\text{GCom}}A\)-cocycle. It is clear that this cocycle splits as an \(O^{\text{GL}}\text{(n, C)}\)-cocycle. In particular it splits as a \(O^{\text{GL}}\text{(n, C)}\)-cocycle. Since condition (ii) is satisfied, this implies that \(\{h_{ij}\}_{i,j \in I}\) splits as an \(O^{\text{GCom}}A\)-cocycle. By Proposition 3.3 this further implies that \(\{h_{ij}\}_{i,j \in I}\) splits as an \(O^{\text{GCom}}A\)-cocycle, i.e., there is a family \(h_i \in O^{\text{GCom}}A(U_i)\) with \(H_iH_j^{-1} = h_ih_j^{-1}\) on \(U_i \cap U_j\). Therefore \(h_i^{-1}H_i = h_j^{-1}H_j\) on \(U_i \cap U_j\), and we have a well-defined global holomorphic map \(H : X \to \text{GL}(n, \mathbb{C})\) with \(H = h_i^{-1}H_i\) on \(U_i\) for all \(i \in I\). By (3.4) and since \(h_iAh_i^{-1} = A\), we get \(H^{-1}AH = B\) on \(X\).

(i) \(\implies\) (ii): Let an open covering \(\mathcal{U} = \{U_i\}_{i \in I}\) of \(X\) and a \((\mathcal{U}, O^{\text{GCom}}A)\)-cocycle \(f = \{f_{ij}\}_{i,j \in I}\) be given such that \(f\) splits as a \(O^{\text{GL}}(n, \mathbb{C})\)-cocycle. Then, by Grauert’s Oka principle [10, Satz 1] (see also [8, Theorem 5.3.1]), \(f\) even splits as an \(O^{\text{GL}}(n, \mathbb{C})\)-cocycle. This means (we may assume that \(U_i \neq \emptyset\) for all \(i\)) that there exists a family of holomorphic maps \(f_i : U_i \to \text{GL}(n, \mathbb{C})\) such that

\[
f_{ij} = f_if_j^{-1} \quad \text{on} \quad U_i \cap U_j, \quad i, j \in I.
\]

Since \(f_{ij} \in O^{\text{GCom}}A(U_i \cap U_j)\), this implies that

\[
f_if_j^{-1}A = Af_if_j^{-1} \quad \text{and, hence} \quad f^{-1}Af_j = f^{-1}Af_i \quad \text{on} \quad U_i \cap U_j, \quad i, j \in I.
\]

Hence, there is a well-defined holomorphic map \(B : X \to \text{Mat}(n \times n, \mathbb{C})\) with

\[
B = f^{-1}Af_i \quad \text{on} \quad U_i, \quad i \in I.
\]

From its definition it is clear that \(B\) is locally holomorphically similar to \(A\) at each point of \(X\). Since condition (i) is satisfied, it follows that there is a holomorphic map \(T : X \to \text{GL}(n, \mathbb{C})\) such that

\[
B = T^{-1}AT \quad \text{on} \quad X.
\]

Set \(h_i = f_iT^{-1}\) on \(U_i\). By (3.7) and (3.6), then, on each \(U_i\),

\[
h_iA = f_iT^{-1}A = f_iBT^{-1} = f_iT^{-1}Af_iT^{-1} = Af_iT^{-1} = Ah_i,
\]

i.e., \(h_i \in O^{\text{GCom}}A(U_i)\). Moreover, by (3.6),

\[
h_ih_j^{-1} = f_iT^{-1}Tf_j^{-1} = f_iT^{-1}f_j^{-1} = f_{ij} \quad \text{on} \quad U_i \cap U_j.
\]

So, \(f\) splits as an \(O^{\text{GCom}}A\)-cocycle and, above all, as an \(\hat{O}^{\text{GCom}}A\)-cocycle.

\[\square\]

For us, the following two immediate corollaries of Theorem 3.3 are important.

3.4. Corollary. Let \(X\) be a Stein space, and let \(A, B : X \to \text{Mat}(n \times n, \mathbb{C})\) be holomorphic maps, which are locally holomorphically similar at each point of \(X\). If \(H^1(X, \hat{O}^{\text{GCom}}A) = 0\),

then \(A\) and \(B\) are globally holomorphically similar to \(A\) on \(X\).
3.5. **Corollary.** Let $X$ be a Stein space such that $H^1(X, \mathcal{O}^{GL(n, \mathbb{C})}) = 0$, and let $A : X \to \text{Mat}(n \times n, \mathbb{C})$ be a holomorphic map such that

$$H^1(X, \mathcal{O}^{G\text{Com}} A) \neq 0.$$  

Then there exists a holomorphic $B : X \to \text{Mat}(n \times n, \mathbb{C})$, which is locally holomorphically similar to $A$ at each point of $X$, but which is not globally holomorphically similar to $A$ on $X$.

Corollary 3.4 will be used to prove Theorem 1.2. Corollary 3.5 is useful for counterexamples, as, e.g., in [18, Sec. 8].

4. **Bumps on Riemann surfaces**

4.1. **Definition.** Let $X$ be a Riemann surface. Denote by $\Delta$ the closed unit disk centered at the origin in $\mathbb{C}$, and set

$$\Delta_I = \{ u \in \Delta \mid \text{Im} u \leq 0 \text{ and } 1/2 \leq |u| \leq 1 \},$$  

$$\Delta_{II} = \{ u \in \Delta \mid |\text{Im} u| \leq |\text{Re} u| \text{ and } 1/2 \leq |u| \leq 1 \}.$$  

A pair $(B_1, B_2)$ will be called a bump in $X$ if $B_1, B_2$ are compact subsets of $X$ such that either

(4.1) $B_1 \cap B_2 = \emptyset,$

or there exists a neighborhood $U$ of $B_2$ and a $C^\infty$-diffeomorphism, $z$, from $U$ onto a neighborhood of $\Delta$ such that

(4.2) $B_2 \subseteq \{|z| \leq 1\},$

(4.3) $B_1 \cap \{|z| \leq 1\} \subseteq B_2,$

and one of the following two conditions is satisfied.

(4.4) $B_1 \cap \{1/2 \leq |z| \leq 1\} = B_2 \cap \{1/2 \leq |z| \leq 1\} = \{z \in \Delta_I\},$

(4.5) $B_1 \cap \{1/2 \leq |z| \leq 1\} = B_2 \cap \{1/2 \leq |z| \leq 1\} = \{z \in \Delta_{II}\}.$

An $m$-tuple $(B_1, \ldots, B_m)$, $m \geq 2$, of compact subsets of $X$ is called a bump extension in $X$ if, for each $1 \leq \mu \leq m - 1$, $(B_1 \cup \ldots \cup B_\mu, B_{\mu+1})$ is a bump in $X$.

4.2. **Lemma.** Let $X$ be a Riemann surface, and let $\rho : X \to \mathbb{R}$ be a $C^\infty$ function such that, for some real numbers $\alpha < \beta$, the set $\{\rho \leq \beta\}$ is compact and $\rho$ has no critical points on $\{\alpha \leq \rho \leq \beta\}$. Moreover, let $\mathcal{U}$ be an open covering of $X$. Then there exist $B_1, \ldots, B_m$ such that

(i) $\{\rho \leq \alpha\}, B_1, \ldots, B_m$ is a bump extension in $X$;

(ii) $\{\rho \leq \alpha\} \cup B_1 \cup \ldots \cup B_m = \{\rho \leq \beta\};$

(iii) for each $1 \leq \mu \leq m$, $\{\rho \leq \alpha - 1\} \cap B_\mu = \emptyset$ and $B_\mu$ is contained in at least one set of $\mathcal{U}$.

**Proof.** It is sufficient to prove that, for each $\alpha \leq t \leq \beta$, there exists $\varepsilon > 0$ such that, if $t - \varepsilon \leq t_1 \leq t \leq t_2 \leq t + \varepsilon$, then there exist $B_1, \ldots, B_m$ such that

(i) $\{\rho \leq t_1\}, B_1, \ldots, B_m$ is a bump extension in $X$;

(ii) $\{\rho \leq t_1\} \cup B_1 \cup \ldots \cup B_m = \{\rho \leq t_2\};$

(iii) for each $2 \leq \mu \leq m$, $\{\rho \leq \alpha - 1\} \cap B_\mu = \emptyset$ and $B_\mu$ is contained in at least one set of $\mathcal{U}$.

$^2\{|z| \leq 1\} := \{z \in \Delta\} := z^{-1}(\Delta) := \{\zeta \in U \mid |\zeta| \leq 1\}$ etc.
Let $\alpha \leq t \leq \beta$ be given.

Since $\rho$ has no critical points on $\{\rho = t\}$ and $\{\rho = t\}$ is compact, then we can find $\varepsilon > 0$, open subsets $\tilde{U}_1, \ldots, \tilde{U}_m$ of $X$, and $C^\infty$ diffeomorphisms $\tilde{z}_\mu$ from $\tilde{U}_\mu$ onto a neighborhood of $\Delta$ such that

(a) $\{t - \varepsilon \leq \rho \leq t + \varepsilon\} \subseteq \{|\tilde{z}_1| < 1/8\} \cup \ldots \cup \{|\tilde{z}_m| < 1/8\}$;

(b) $\rho = \text{Im} \tilde{z}_\mu + t$ on $\tilde{U}_\mu$ for $1 \leq \mu \leq m$;

(c) for each $1 \leq \mu \leq m$, $\tilde{U}_\mu \cap \{\rho \leq \alpha - 1\} = \emptyset$ and $\tilde{U}_\mu$ is contained in at least one set of $\mathcal{U}$.

By (a) we can choose $C^\infty$-functions $\chi_1, \ldots, \chi_m : X \to [0,1]$ such that

(d) $\text{supp} \chi_\mu \subseteq \{|\tilde{z}_\mu| < 1/4\}$ for $1 \leq \mu \leq m$;

(e) $\sum_{\mu=1}^m \chi_\mu = 1$ on $\{t - \varepsilon \leq \rho \leq t + \varepsilon\}$, and $0 \leq \sum_{\mu=1}^m \chi_\mu \leq 1$ on $X$.

Further, for each $1 \leq \mu \leq m$, take an open set $U_\mu \subseteq \tilde{U}_\mu$, which is relatively compact in $\tilde{U}_\mu$ and such that $\tilde{z}_\mu(U_\mu)$ is still a neighborhood of $\Delta$. Then we can find $\varepsilon > 0$ so small that, for all $v, w \in \mathbb{C}$ with $|v|, |w| \leq 2\varepsilon$, and, for $1 \leq \mu \leq m$,

(f) the function $\bar{z}_\mu + v + w \sum_{\nu=1}^m \chi_\nu$ restricted to $\tilde{U}_\mu$ is a $C^\infty$ diffeomorphism from $U_\mu$ onto a neighborhood of $\Delta$;

(g) $\{|\bar{z}_\mu| \leq 1/4\} \subseteq \{|\tilde{z}_\mu(\zeta) + v + w \sum_{\nu=1}^m \chi_\nu(\zeta)| \leq 1/2\}$.

Moreover, we may assume that

(h) $\varepsilon < \bar{\varepsilon}/4$.

To prove that this $\varepsilon$ has the required property, let $t_1, t_2$ with $t - \varepsilon \leq t_1 \leq t \leq t_2 \leq t + \varepsilon$ be given. Define

- on $U_\mu$: $z_\mu = \bar{z}_\mu + i(t - t_1) - i(t_2 - t_1) \sum_{\nu=1}^m \chi_\nu$ for $1 \leq \mu \leq m$;
- on $X$: $\rho_0 = \rho - t_1$ and $\rho_\mu = \rho - t_1 - (t_2 - t_1) \sum_{\nu=1}^m \chi_\nu$ for $1 \leq \mu \leq m$;
- $B_\mu = \{\rho_\mu \leq 0\} \cap \{|z_\mu| \leq 1\}$ for $1 \leq \mu \leq m$.

Then, by (e) and (h), $\sum_{\nu=1}^m \chi_\nu = 1$ on $\{t - 4\varepsilon \leq \rho \leq t + 4\varepsilon\}$, and $0 \leq \sum_{\nu=1}^m \chi_\nu \leq 1$ everywhere on $X$. Since $0 \leq t_2 - t_1 - 2\varepsilon$ and $0 \leq t - t_1 \leq \varepsilon$, it follows that

$$
\rho_\mu \begin{cases}
= \rho - t_2 & \text{on } \{t - 4\varepsilon \leq \rho \leq t + 4\varepsilon\}, \\
\geq \rho - t_1 - (t_2 - t_1) > 0 & \text{on } \{\rho \geq t + 4\varepsilon\}, \\
\leq \rho - t_1 < 0 & \text{on } \{\rho \leq t - 4\varepsilon\}.
\end{cases}
$$

Therefore

(4.6) $\{\rho_\mu \leq 0\} = \{\rho \leq t_2\}$.

By (g), $\{|\bar{z}_\mu| \leq 1/4\} \subseteq \{|z_\mu| \leq 1/2\}$ for $1 \leq \mu \leq m$. Together with (d), this gives

(4.7) $\rho_\mu = \rho_{\mu - 1}$ on $X \setminus \{|z_\mu| \leq 1/2\}$ for $1 \leq \mu \leq m$.

Let $1 \leq \mu \leq m$. Then, by (4.7),

$$
\{\rho_{\mu - 1} \leq 0\} = \left(\{\rho_{\mu - 1} \leq 0\} \cap \{|z_\mu| \leq 1\}\right) \cup \left(\{\rho_\mu \leq 0\} \cap (X \setminus \{|z_\mu| \leq 1\})\right)
$$

and, further,

$$
\{\rho_{\mu - 1} \leq 0\} \cup B_\mu = \left(\{\rho_{\mu - 1} \leq 0\} \cap \{|z_\mu| \leq 1\}\right) \\
\quad \cup \left(\{\rho_\mu \leq 0\} \cap (X \setminus \{|z_\mu| \leq 1\})\right) \cup \left(\{\rho_\mu \leq 0\} \cap \{|z_\mu| \leq 1\}\right) \\
= \left(\{\rho_{\mu - 1} \leq 0\} \cap \{|z_\mu| \leq 1\}\right) \cup \{\rho_\mu \leq 0\}.
$$
Since $\rho_{\mu-1} \geq \rho_\mu$ and therefore $\{\rho_{\mu-1} \leq 0\} \cap \{|z_\mu| \leq 1\} \subseteq \{\rho_\mu \leq 0\}$, it follows that

$$\{\rho_{\mu-1} \leq 0\} \cup B_\mu = \{\rho_\mu \leq 0\},$$

and, hence,

$$\{\rho_0 \leq 0\} \cup B_1 \cup \ldots \cup B_\mu = \{\rho_\mu \leq 0\}.$$ 

Since $\rho_0 = \rho - t_1$, so we have proved that

(4.8) \quad \{\rho \leq t_1\} \cup B_1 \cup \ldots \cup B_\mu = \{\rho_\mu \leq 0\} \quad \text{for} \quad 1 \leq \mu \leq m.

Since $\rho_1 \geq \rho_0 = \rho - t_1$, we have $\{\rho \leq t_1\} = \{\rho_0 \leq 0\} \subseteq \{\rho_1 \leq 0\}$, which implies by definition of $B_1$ that

(4.9) \quad \{\rho \leq t_1\} \cap \{|z_1| \leq 1\} \subseteq B_1.

For $2 \leq \mu \leq m$, it follows from (4.8) that

$$\left(\{\rho \leq t_1\} \cup B_1 \cup \ldots \cup B_{\mu-1}\right) \cap \{|z_\mu| \leq 1\} = \{\rho_{\mu-1} \leq 0\} \cap \{|z_\mu| \leq 1\}.$$ 

Sine $\rho_{\mu-1} \leq \rho_\mu$, this implies by definition of $B_\mu$ that

(4.10) \quad \left(\{\rho \leq t_1\} \cup B_1 \cup \ldots \cup B_{\mu-1}\right) \cap \{|z_\mu| \leq 1\} \subseteq B_\mu \quad \text{for} \quad 2 \leq \mu \leq m.

Further it follows from (4.7) that

(4.11) \quad \{\rho_{\mu-1} \leq 0\} \cap \{1/2 \leq |z_\mu| \leq 1\} = \{\rho_\mu \leq 0\} \cap \{1/2 \leq |z_\mu| \leq 1\}

= B_\mu \cap \{1/2 \leq |z_\mu| \leq 1\} \quad \text{for} \quad 1 \leq \mu \leq m.

Since $\{\rho_0 \leq 0\} = \{\rho \leq t_1\}$, this implies that

(4.12) \quad \{\rho \leq t_1\} \cap \{1/2 \leq |z_1| \leq 1\} = B_1 \cap \{1/2 \leq |z_1| \leq 1\},

Moreover, by (4.8), from (4.11) we get

(4.13) \quad \left(\{\rho \leq t_1\} \cup B_1 \cup \ldots \cup B_{\mu-1}\right) \cap \{1/2 \leq |z_\mu| \leq 1\}

= B_\mu \cap \{1/2 \leq |z_\mu| \leq 1\} \quad \text{for} \quad 2 \leq \mu \leq m.

By (b), $\text{Im} z_\mu = \rho_\mu|_{U_\mu}$. Therefore $\{\rho_\mu \leq 0\} \cap \{1/2 \leq |z_\mu| \leq 1\} = \{z_\mu \in \Delta_I\}.$

Since $B_\mu \cap \{1/2 \leq |z_\mu| \leq 1\} = \{\rho_\mu \leq 0\} \cap \{1/2 \leq |z_\mu| \leq 1\}$, this means

(4.14) \quad B_\mu \cap \{1/2 \leq |z_\mu| \leq 1\} = \{z_\mu \in \Delta_I\} \quad \text{for} \quad 1 \leq \mu \leq m.

We summarize:

By (f), $z_1$ is a $C^\infty$ diffeomorphism from $U_1$ onto a neighborhood of $\Delta$ and, by definition of $B_1$, we have $B_1 \subseteq \{|z_1| \leq 1\}$. Together with (1.9), (4.12) and (4.14) (for $\mu = 1$), this shows that $\{\rho \leq t_1\}, B_3$ is a bump in $X$ (4.4 is satisfied).

For $2 \leq \mu \leq m$, by (f), $z_\mu$ is a $C^\infty$ diffeomorphism from $U_\mu$ onto a neighborhood of $\Delta$ and, by definition of $B_\mu$, we have $B_\mu \subseteq \{|z_\mu| \leq 1\}$. Together with (4.10), (4.13) and (4.14), this shows that $\{\rho \leq t_1\} \cup B_1 \cup \ldots \cup B_{\mu-1}, B_\mu$ is a bump in $X$.

Therefore (i') holds. (ii') follows from (4.8) and (4.6). (iii') follows from (c). \qed

4.3. Theorem. Let $X$ be noncompact and connected Riemann surface, and let $U$ be an open covering of $X$. Then there exists a sequence $(B_\mu)_{\mu \in \mathbb{N}}$ of compact subsets of $X$ such that

(a) for each $\mu \in \mathbb{N}$, $B_\mu$ is contained in at least one set of $U$;
(b) for each $\mu \in \mathbb{N}^*$, $(B_0 \cup \ldots \cup B_{\mu-1}, B_\mu)$ is a bump in $X$;
(c) $X = \bigcup_{\mu \in \mathbb{N}} B_\mu$. 

(d) for each compact set $\Gamma \subseteq X$, there exists $N(\Gamma) \in \mathbb{N}$ such that $B_\mu \cap \Gamma = \emptyset$ if $\mu \geq N(\Gamma)$.

Proof. Since $X$ is a Stein manifold (see, e.g., [10 Corollary 26.8]), we can find a strictly subharmonic $C^\infty$ function $\rho : X \to \mathbb{R}$ such that, for all $\alpha \in \mathbb{R}$, $\{\rho \leq \alpha\}$ is compact (see, e.g., [13 Theorem 5.1.6]). By Morse theory (see, e.g., [12 §7 and §19, Exercise 19]), we may assume that all critical points of $\rho$ are non-degenerate, and, for each $t \in \mathbb{R}$, at most one critical point of $\rho$ lies on $\{\rho = t\}$. Since $\rho$ is strictly subharmonic (which implies that $\rho$ has no local maxima), then, for each critical value $t$ of $\rho$, there are only two possibilities: either $\rho$ has precisely one critical point on $\{\rho = t\}$, and this is the point of a strong local minimum of $\rho$, or $\rho$ has precisely one critical point on $\{\rho = t\}$, and this is the point of a strong saddle point of $\rho$.

In particular, then there is precisely one point in $X$, $\xi_{\text{min}}$, where $\rho$ assumes its absolute minimum, and this minimum is strong. Therefore, we can find $\epsilon_0 > 0$ such that $\{\rho \leq \rho(\xi_{\text{min}}) + \epsilon_0\}$ is contained in at least one of the sets of $U$, and $\rho$ has no critical points in $\{\rho(\xi_{\text{min}}) < \rho \leq \rho(\xi_{\text{min}}) + \epsilon_0\}$. Set $B_0 = \{\rho(\xi_{\text{min}}) \leq \rho \leq \rho(\xi_{\text{min}}) + \epsilon_0\}$.

If $\xi_{\text{min}}$ is the only critical point of $\rho$, then the proof of the theorem can be completed inductively, applying Lemma 4.13 with $\alpha = \rho(\xi_{\text{min}}) + \epsilon_0 + N$ and $\beta = \rho(\xi_{\text{min}}) + \epsilon_0 + N + 1$ for $N = 0, 1, 2, \ldots$.

If there are further critical points of $\rho$, it remains to complete Lemma 4.12 by the following statement.

(*) Let $\xi$ be a critical point of $\rho$ with $t := \rho(\xi) > \rho(\xi_{\text{min}}) + \epsilon_0$. Then there exists $\epsilon > 0$ such that, for each $0 < \delta \leq \epsilon$, we can find an $m$-tuple $(A_1, \ldots, A_m)$ such that $\{\rho \leq t - \delta\} \cup A_1 \cup \ldots \cup A_m = \{\rho \leq t + \delta\}$, and, for each $1 \leq \mu \leq m$, $A_\mu \cap \{\rho \leq t - \delta - 1\} = \emptyset$ and $A_\mu$ is contained in at least one set of $U$.

Proof of (*) if $\xi$ is the point of a strong local minimum of $\rho$:

Then $\xi$ is an isolated point of $\{\rho \leq t\}$. Therefore we can find an open neighborhood $U$ of $\xi$ and an open neighborhood $V$ of $\{\rho \leq t\} \setminus \{\xi\}$ such that $U \cap V = 0$ and $U$ is contained in at least one set of $U$. Choose $\epsilon > 0$ such that $\rho$ has no critical points on $V \cap \{t - \epsilon \leq \rho \leq t + \epsilon\}$ and

$$\{\rho \leq t + \epsilon\} \subseteq U \cup V.$$  

To prove that this $\epsilon$ has the required property, let $0 < \delta \leq \epsilon$ be given. Then, by Lemma 4.12, we can find $A_1, \ldots, A_{m-1}$ such that $\{\rho \leq t - \delta\} \cup A_1 \cup \ldots \cup A_{m-1} = V \cap \{\rho \leq t + \delta\}$, and, for $1 \leq \mu \leq m - 1$, $A_\mu \cap \{\rho \leq t - \delta - 1\} = \emptyset$ and $A_\mu$ is contained in at least one set of $U$. Since $U \cap V = 0$, it remains to set $A_m = U \cap \{\rho \leq t + \delta\}$.

Proof of (*) if $\xi$ is the point of a strong saddle point of $\rho$:

By a lemma of Morse (see, e.g., [20 Lemma 2.2]), then we can find $R > 0$, an open neighborhood $W$ of $\xi$ and a $C^\infty$ diffeomorphism $w$ from $W$ onto a neighborhood of $\Delta_R := \{\lambda \in \mathbb{C} \mid |\lambda| \leq R\}$ such that $w(\xi) = 0$ and

$$\rho = t + (\text{Re} w)^2 - (\text{Im} w)^2 \quad \text{on} \quad W.$$  

Moreover, we may choose $W$ so small that

$$W \cap \{\rho \leq t - 1\} = \emptyset \quad \text{and} \quad W \text{ is contained in at least one set of } U.$$
Take $\varepsilon > 0$ so small that $\xi$ is the only critical point of $\rho$ on $\{t - \varepsilon \leq \rho \leq t + \varepsilon\}$. To prove that this $\varepsilon$ has the required property, let $0 < \delta \leq \varepsilon$ be given.

Choose $0 < r \leq R/2$ with

$$(4.17) \quad \{|w| \leq r\} \subseteq \{t - \delta < \rho < t + \delta\},$$

and take a $C^\infty$ function $\chi : X \to [0,1]$ with $\chi = 1$ on $\{|z| \leq r/2\}$ and $\chi = 0$ on $X \setminus \{|w| \leq r\}$. By (4.15), then we can find $c > 0$ so small that the functions $\rho_+, \rho_- : X \to \mathbb{R}$ defined by

$$\rho_+ = \rho + c\chi \quad \text{and} \quad \rho_- = \rho - c\chi$$

have the same critical points as $\rho$. Then $\rho_+ = \rho_- = \rho$ on $X \setminus \{|w| \leq r\}$ and $\rho_+ \geq \rho \geq \rho_-$ on $X$. Therefore and by (4.17),

$$(4.18) \quad \{|\rho+ \leq t - \delta\} = \{|\rho \leq t - \delta\},$$

$$(4.19) \quad \{|t - \delta \leq \rho_+ \leq t\} \subseteq \{|t - \delta \leq \rho \leq t\} \subseteq \{|t - \varepsilon \leq \rho \leq t + \varepsilon\},$$

$$(4.20) \quad \{|\rho_+ \leq t - \delta - 1\} = \{|\rho \leq t - \delta - 1\},$$

$$(4.21) \quad \{|\rho_- \leq t + \delta\} = \{|\rho \leq t + \delta\},$$

$$(4.22) \quad \{|t \leq \rho_- \leq t + \delta\} \subseteq \{|t \leq \rho \leq t + \delta\} \subseteq \{|t - \varepsilon \leq \rho \leq t + \varepsilon\},$$

$$(4.23) \quad \{|\rho_- \leq t - 1\} \supseteq \{|\rho \leq t - \delta - 1\}.$$

Since $\rho_+(\xi) = \rho(\xi) + c > t$, we have $\xi \notin \{|t - \delta \leq \rho_+ \leq t\}$. As $\xi$ is the only critical point of $\rho$ in $\{|t - \varepsilon \leq \rho \leq t + \varepsilon\}$ and by (4.19), this implies that $\rho$ has no critical point in $\{t - \delta \leq \rho_+ \leq t\}$. Since $\rho_+$ has the same critical points as $\rho$, this means that $\rho_+$ has no critical point in $\{t - \delta \leq \rho_+ \leq t\}$. Therefore, by Lemma 4.2 by (4.19) and by (4.20), we can find $A_1, \ldots, A_k$ such that

(a) $\{|\rho \leq t - \delta\}, A_1, \ldots, A_k$ is a bump extension in $X$, $\{|\rho \leq t - \delta\} \cup A_1 \cup \ldots \cup A_k = \{|\rho_+ \leq t\}$ and, for each $1 \leq \mu \leq k$, $A_\mu \cap \{|\rho \leq t - \delta - 1\} = \emptyset$ and $A_\mu$ is contained in at least one set of $U$.

Since $\rho_-(\xi) = \rho(\xi) - c < t$, we have $\xi \notin \{|t \leq \rho_- \leq t + \delta\}$. As $\xi$ is the only critical point of $\rho$ in $\{|t - \varepsilon \leq \rho \leq t + \varepsilon\}$ and by (4.22), this implies that $\rho$ has no critical points in $\{t \leq \rho_- \leq t + \delta\}$. Since $\rho_-$ has the same critical points as $\rho$, this means that $\rho_-$ has no critical points in $\{t \leq \rho_- \leq t + \delta\}$. Therefore, by Lemma 4.2 by (4.21) and by (4.23), we can find $A_{k+1}, \ldots, A_m$ such that

(b) $\{|\rho_- \leq t\}, A_{k+1}, \ldots, A_m$ is a bump extension in $X$, $\{|\rho_- \leq t\} \cup A_{k+1} \cup \ldots \cup A_m = \{|\rho \leq t + \delta\}$ and, for each $k+2 \leq \mu \leq m$, $A_\mu \cap \{|\rho \leq t - \delta - 1\} = \emptyset$ and $A_\mu$ is contained in at least one set of $U$.

Set $z = w/2r$ on $W$. Since $r \leq R/2$, then $z$ is a diffeomorphism from $W$ onto a neighborhood of $\Delta$. Since

$$(4.24) \quad \rho_+ = \rho_- = \rho \quad \text{on} \quad X \setminus \{|z| \leq 1/2\} = X \setminus \{|w| \leq r\},$$

it follows from (4.15) that

$$(4.25) \quad \rho_+ = \rho_- = \rho = t + 4r^2 \left( (\text{Re} \, z)^2 - (\text{Im} \, z)^2 \right) \quad \text{on} \quad \{1/2 \leq |z| \leq 1\}.$$

Set $A_{k+1} = \{|\rho_- \leq t\} \cap \{|z| \leq 1\}$. Then

$$(4.26) \quad A_{k+1} \subseteq \{|z| \leq 1\}.$$

Since $\rho_- \leq \rho_+$, we have

$$(4.27) \quad \{|\rho_+ \leq t\} \cap \{|z| \leq 1\} \subseteq A_{k+1}.$$
and
\[
(\{\rho_+ \leq t\} \cup A_{k+1}) \cap \{|z| \leq 1\} = \{\rho_- \leq t\} \cap \{|z| \leq 1\}.
\]
Together with (4.24), the latter yields
\[
(4.28) \quad \{\rho_+ \leq t\} \cup A_{k+1} = \{\rho_- \leq t\}.
\]
From (4.25) it follows that
\[
(4.29) \quad \{\rho_+ \leq t\} \cap \{1/2 \leq |z| \leq 1\} = A_{k+1} \cap \{1/2 \leq |z| \leq 1\} = \{z \in \Delta_U\}.
\]
By (4.26) - (4.29), \(\{\rho_+ \leq t\}, A_{k+1}\) is a bump in \(X\) (condition (4.3) is satisfied) with \(\{\rho_+ \leq t\} \cup A_{k+1} = \{\rho_- \leq t\}\) and such that, by (4.16), \(A_{k+1} \cap \{\rho \leq t-\delta-1\} = \emptyset\) and \(A_{k+1}\) is contained in at least one set of \(\mathcal{U}\). Together with (a) and (b) it follows that the \(m\)-tuple \((A_1, \ldots, A_m)\) has the required properties. \(\square\)

5. Z-ADAPTED PAIRS OF COMPACT SETS ON 1-DIMENSIONAL COMPLEX SPACES

5.1. Definition. Let \(X\) be a 1-dimensional complex space, and let \(Z\) be a discrete and closed subset of \(X\) such that all points of \(X \setminus Z\) are smooth. A pair \((\Gamma_1, \Gamma_2)\) will be called a \(Z\)-adapted pair of compact sets in \(X\) if \(\Gamma_1 \cap \Gamma_2 = \Gamma_0 \cup \Gamma\), where

- \(\Gamma_0 \cap \Gamma = \emptyset\);
- \(\Gamma_0 \subseteq Z\);
- \(\Gamma \cap Z = \emptyset\) and, if \(\Gamma \neq \emptyset\), then \(\Gamma\) consists of a finite number\(^3\) of connected components each of which has a basis of contractible open neighborhoods.

5.2. Lemma. Let \(X\) be a 1-dimensional complex space, and let \(Z\) be a discrete and closed subset of \(X\) such that all points of \(X \setminus Z\) are smooth. Let \(\pi : \tilde{X} \to X\) be the normalization of \(X\) (see, e.g., [19] Ch. VI, §4), and let \((B_1, B_2)\) be a bump in \(X\) (Def. 4.7). Then there exists a \(Z\)-adapted pair of compact sets in \(X\), \((\Gamma_1, \Gamma_2)\), such that

\[
\begin{align*}
(5.1) & \quad \Gamma_1 \subseteq \pi(B_1), \\
(5.2) & \quad \Gamma_2 \subseteq \pi(B_2), \\
(5.3) & \quad \Gamma_1 \cup \Gamma_2 = \pi(B_1 \cup B_2).
\end{align*}
\]

Proof. First let \(B_1 \cap B_2 = \emptyset\). Since all points of \(X \setminus Z\) are smooth and, hence, \(\pi\) is bijective from \(\tilde{X} \setminus \pi^{-1}(Z)\) onto \(X \setminus Z\), then \(\pi(B_1) \cap \pi(B_2) \subseteq Z\). Set \(\Gamma_1 = \pi(B_1)\) and \(\Gamma_2 = \pi(B_2)\). Then \((\Gamma_1, \Gamma_2)\) a \(Z\)-adapted pair of compact sets in \(X\) (we can take \(\Gamma_0 = \pi(B_1) \cap \pi(B_2)\) and \(\Gamma = \emptyset\) in Def. 5.1), which trivially satisfies conditions (5.1)-(5.3).

Now let \(B_1 \cap B_2 \neq \emptyset\). Then (by Definition 5.1) we have a neighborhood \(U\) of \(B_2\) and a diffeomorphic map, \(z\), from \(U\) onto a neighborhood of \(\Delta\) satisfying (4.2) and (4.3), and one of the relations (4.4) or (4.5).

Since \(Z\) is discrete and closed in \(X\), \(\pi^{-1}(Z)\) is discrete and closed in \(\tilde{X}\). Therefore, we can find \(1/2 < r < R < 1\) such that \(\pi^{-1}(Z) \cap \{r \leq |z| \leq R\} = \emptyset\) and, hence,

\[
(5.4) \quad Z \cap \pi(\{r \leq |z| \leq R\}) = \emptyset.
\]

---

\(^3\)In the applications below, this ‘finite number’ will be one or two.
Set
\[ K = B_1 \cap B_2 \cap \{ r \leq |z| \leq R \}, \]
\[ K_1 = B_1 \cap (\bar{X} \setminus \{ |z| < R \}), \]
\[ K_2 = B_2 \cap \{ |z| \leq r \}, \]
\[ \Gamma = \pi(K), \quad \Gamma_1 = \pi(K \cup K_1), \quad \Gamma_2 = \pi(K \cup K_2), \quad \Gamma^0 = \pi(K_1) \cap \pi(K_2). \]

Then
\[ \Gamma_1 \cap \Gamma_2 = \left( \pi(K) \cup \pi(K_1) \right) \cap \left( \pi(K) \cup \pi(K_2) \right) \]
\[ = \pi(K) \cup \left( \pi(K_1) \cap \pi(K) \right) \cup \left( \pi(K_1) \cap \pi(K_2) \right) = \pi(K) \cup \pi(K_1) \cap \pi(K_2) = \Gamma \cup \Gamma^0. \]

Since all points in \( X \setminus Z \) are smooth points of \( X \), \( \pi \) is bijective from \( \bar{X} \setminus \pi^{-1}(Z) \) onto \( X \setminus Z \). As \( K_1 \cap K_2 = \emptyset \), this implies that
\[ \Gamma^0 \subseteq Z. \]

By (5.2), \( \Gamma \cap Z = \emptyset \). Together with (5.3), this gives
\[ \Gamma \cap \Gamma^0 = \emptyset. \]

From (4.4) resp. (4.5) we get
\[ K = \begin{cases} \{ z \in \Delta_I \} & \text{in case of (4.4)}, \\ \{ z \in \Delta_{II} \} & \text{in case of (4.5)}. \end{cases} \]

Since \( z \) is diffeomorphic, this implies that \( K \) consists of a finite number of connected components (one in case (4.4) and two in case (4.5)) each of which has a basis of contractible open neighborhoods. Since, by (5.3), \( \pi \) is biholomorphic from an open neighborhood of \( K \) onto an open neighborhood of \( \Gamma \), this further implies that \( \Gamma \) consists of a finite number of connected components each of which has a basis of contractible open neighborhoods. Together with (4.6) - (4.7), this shows that \( (\Gamma_1, \Gamma_2) \) is a \( Z \)-adapted pair of compact sets in \( X \). Moreover, since \( K \cup K_1 \subseteq B_1 \) and \( K \cup K_2 \subseteq B_2 \), (5.1), (5.2) and “\( \subseteq \)” in (5.3) are clear. Therefore it remains to prove “\( \supseteq \)” in (5.3). For the latter it is sufficient to prove that
\[ \pi((B_1 \cup B_2) \cap \{ |z| \leq r \}) \subseteq \Gamma_2, \]
\[ \pi((B_1 \cup B_2) \cap \{ r \leq |z| \leq R \}) = \Gamma, \]
\[ \pi((B_1 \cup B_2) \cap (\bar{X} \setminus \{ |z| < R \})) \subseteq \Gamma_1. \]

From (4.3) we get \( B \cap \{ |z| \leq r \} \subseteq B_2 \). Hence
\[ (B_1 \cup B_2) \cap \{ |z| \leq r \} \subseteq B_2 \cap \{ |z| \leq r \} = K_2 \subseteq K \cup K_2, \]
which yields (5.8) by definition of \( \Gamma_2 \).

Since, by the first equality in (4.4) resp. (4.5)
\[ (B_1 \cup B_2) \cap \{ r \leq |z| \leq R \} = B_1 \cap B_2 \cap \{ r \leq |z| \leq R \} = K, \]
we get (5.9).

By (4.2), \( B_2 \subseteq \{ |z| \leq 1 \} \) and, hence,
\[ B_2 \cap (\bar{X} \setminus \{ |z| < R \}) = B_2 \cap \{ R \leq |z| \leq 1 \}. \]
Again using the first equality in (4.4) resp. (4.5), this yields
\[ B_2 \cap (\overline{X} \setminus \{|z| < R\}) = B_1 \cap \{ R \leq |z| \leq 1 \} \]
and, further,
\[ B_2 \cap (\overline{X} \setminus \{|z| < R\}) \subseteq B_1 \cap (\overline{X} \setminus \{|z| < R\}). \]
It follows that
\[ (B_1 \cup B_2) \cap (\overline{X} \setminus \{|z| < R\}) \subseteq B_1 \cap (\overline{X} \setminus \{|z| < R\}) = K_1, \]
which implies (6.10). \( \square \)

6. JORDAN STABLE POINTS

In this section, \( X \) is a complex space (of arbitrary dimension), and \( A : X \to \text{Mat}(n \times n, \mathbb{C}) \) is a holomorphic map.

6.1. Definition. A point \( \xi \in X \) will be called Jordan stable for \( A \) if there exists a neighbourhood \( U \) of \( \xi \) such that the following two conditions are satisfied:
(a) there are holomorphic functions \( \lambda_1, \ldots, \lambda_m : U \to \mathbb{C} \) such that, for each \( \zeta \in U \), \( \lambda_1(\zeta), \ldots, \lambda_m(\zeta) \) are the different eigenvalues of \( A(\zeta) \);
(b) there is a holomorphic map \( T : U \to \text{GL}(n\mathbb{C}) \) such that, for all \( \zeta \in U \), \( T(\zeta)^{-1}A(\zeta)T(\zeta) \) is in Jordan normal form.\(^4\)

6.2. Proposition. The points in \( X \) which are not Jordan stable for \( A \) form a nowhere dense analytic subset of \( X \). (If \( X \) is 1-dimensional, this means that this set is discrete and closed in \( X \).)

This proposition can be found in [17, Theorem 5.5]. If \( X \) is 1-dimensional and smooth, it was first proved H. Baumgärtel [1], [2, Kap. 5, §7], [3, 5.7]. If \( X \) is of arbitrary dimension and smooth, H. Baumgärtel [3, 4, S 3.4] obtained the slightly weaker statement that the points in \( X \) which are not Jordan stable for \( A \) are contained in a nowhere dense analytic subset of \( X \).

6.3. Theorem. Let \( \xi \in X \) be Jordan stable for \( A \). Then there exist a neighborhood \( U \) of \( \xi \) and a holomorphic map \( T : U \to \text{GL}(n, \mathbb{C}) \) such that (cp. Def. 6.1)
\[ T(\zeta)^{-1}(\text{Com} A(\zeta))T(\zeta) = \text{Com} A(\xi) \quad \text{for all} \quad \zeta \in U. \]

Proof. Let \( U \), \( \lambda_j \), \( 1 \leq j \leq m \), and \( T \) be as in Definition 6.1. Let \( J(\zeta) := T(\zeta)^{-1}A(\zeta)T(\zeta) \) for \( \zeta \in U \). We may assume that \( U \) is connected. Then, since \( J \) is continuous and, for each \( \zeta \in U \), \( \lambda_1(\zeta), \ldots, \lambda_m(\zeta) \) are the different eigenvalues of \( J(\zeta) \) and \( J(\zeta) \) is in Jordan normal form, after a possible change of the numbering of the \( \lambda_j \), there are integers \( n_1, \ldots, n_m \geq 1 \) such that \( J \) is the block diagonal matrix with the diagonal
\[ \lambda_1 I_{n_1} + M_1, \ldots, \lambda_m I_{n_m} + M_m, \]
where \( M_j \in \text{Mat}(n_j \times n_j, \mathbb{C}) \) is a matrix in Jordan normal form with the only eigenvalue zero.

\(^4\)Equivalently, one could define: \( \xi \) is Jordan stable for \( A \) if and only if there exists a neighborhood \( U \) of \( \xi \) such that the number of different eigenvalues of \( A(\zeta) \) is the same for all \( \zeta \in U \) and, for all integers \( 1 \leq k \leq n \), the number of Jordan blocks in the Jordan normal forms of \( A(\zeta) \) is the same for all \( \zeta \in U \). This was proved by G. P. A. Thijsse [22] (see also [17, Lemma 5.3]).
Let \( \zeta \in U \). Since the eigenvalues \( \lambda_1(\zeta), \ldots, \lambda_m(\zeta) \) are pairwise different, then, as proved in Gantmacher’s book \([9]\) Ch. VIII, §1], a matrix \( \Theta \in \text{Mat}(n \times n, \mathbb{C}) \) belongs to \( \text{Com} J(\zeta) \) if and only if \( \Theta \) is a block diagonal matrix with matrices

\[
Z_1 \in \text{Com} \left( \lambda_1(\zeta)I_{n_1} + M_1 \right), \ldots, Z_m \in \text{Com} \left( \lambda_m(\zeta)I_{n_m} + M_m \right)
\]
on the diagonal. Since, obviously, \( \text{Com} \left( \lambda_j(\zeta)I_{k_j} + M_j \right) = \text{Com} M_j \), this in particular shows that \( \text{Com} J(\zeta) = \text{Com} J(\xi) \). By (6.3) and since \( T \) is contractible, this cocycle splits as a \( \text{GL}(\mathbb{C}) \)-cocycle \([21], \text{Corollary 11.6}\], i.e., there is a holomorphic \( N \)-principal bundle of complex Lie groups, with the characteristic fiber \( \text{GCom} A(\xi) \). Note that \( \text{GCom} A(\xi) \) is connected (as easy to see – cp. \([15]\)Lemma 4.2]), whereas \( N \) need not be connected (cp. Remark 6.4 below).

Theorem 6.3 immediately yields

**6.5. Corollary.** For each open set \( W \subseteq X \) which contains only Jordan stable points of \( A \), we have \( \text{GCom} A(W) = \text{Com} A(W) \).

**6.6. Theorem.** Let \( W \subseteq X \) be an open set which contains only Jordan stable points of \( A \). Further suppose that \( W \) is contractible. Fix some point \( \xi \in W \). Then there exists a continuous map \( T : W \to \text{GL}(n, \mathbb{C}) \) such that

\[
(6.2) \quad T^{-1}(\zeta)(\text{Com} A(\zeta))T(\zeta) = \text{Com} A(\xi) \quad \text{for all } \zeta \in W.
\]

If, moreover, \( W \) is Stein, then this \( T \) can be chosen holomorphically on \( W \).

**Proof.** Since \( W \) is connected, by Theorem 6.3 for each \( \eta \in W \), we can find a neighborhood \( U_{\eta} \subseteq W \) of \( \eta \) and a holomorphic map \( H_{\eta} : U_{\eta} \to \text{GL}(n, \mathbb{C}) \) such that

\[
(6.3) \quad H_{\eta}^{-1}(\zeta)(\text{Com} A(\zeta))H_{\eta}(\zeta) = \text{Com} A(\xi) \quad \text{for all } \zeta \in U_{\eta}.
\]

Let \( N \) the normalizer of \( \text{GCom} A(\xi) \) in \( \text{GL}(n, \mathbb{C}) \), i.e., the complex Lie group of all \( \Phi \in \text{GL}(n, \mathbb{C}) \) with \( \Phi^{-1}(\text{GCom} A(\xi))\Phi = \text{GCom} A(\xi) \). Then, by (6.3),

\[
H_{\eta}^{-1}(\zeta)H_{\tau}(\zeta) \in N \quad \text{for all } \zeta \in U_{\eta} \cap U_{\tau} \quad \text{and all } \eta, \tau \in W,
\]
i.e., the family \( \{H_{\eta}^{-1}H_{\tau}\}_{\eta, \tau} \) is an \( O^N \)-cocycle with the covering \( \{U_{\eta}\}_{\eta \in W} \). Since \( W \) is contractible, this cocycle splits as a \( C^N \)-cocycle \([21]\) Corollary 11.6], i.e., there is a family \( T_{\eta} \in C^N(U_{\eta}) \) such that \( T_{\eta}^{-1}T_{\tau} = H_{\eta}H_{\tau}^{-1} \) on \( U_{\eta} \cap U_{\tau} \). Then

\[
H_{\eta}T_{\eta} = H_{\tau}T_{\tau} \quad \text{on } U_{\eta} \cap U_{\tau}, \quad \eta, \tau \in W.
\]
Hence, there is a well-defined continuous map \( T : X \to \text{GL}(n, \mathbb{C}) \) with

\[
T = H_{\eta}T_{\eta} \quad \text{on } U_{\eta}, \quad \eta \in W.
\]
By (6.3) and since \( T_{\eta}(\zeta) \in N \), this \( T \) satisfies (6.1).

If \( W \) is Stein, then by Grauert’s Oka principle \([10]\) Satz 6] the maps \( T_{\eta} \) can be chosen to be holomorphic, which implies that \( T \) is holomorphic. \( \square \)

**6.7. Corollary.** Under the hypotheses of Theorem 6.6, \( \hat{\text{GCom}} A(W) \), endowed with the topology of uniform convergence on the compact subsets of \( W \), is connected.
Proof. Let $G := \text{GCom} A(\xi)$, and let $C^G(W)$ be also endowed with the topology of uniform convergence on the compact subsets of $W$. Since $W$ is contractible, each element of $C^G(W)$ can be connected by a continuous path in $C^G(W)$ with a constant map. It is easy to see that $G$ is connected [18, Lemma 4.2]. Therefore, this implies that $C^G(W)$ is connected. Since, by (6.1), $C^\text{GCom} A(W)$ and $C^G(W)$ are isomorphic as topological groups, it follows that $C^\text{GCom} A(W)$ is connected. Since $\hat{O} = C^\text{GCom} A(W)$ (Corollary 6.5), this completes the proof. □

6.8. Remark. Assume that $X$ is a domain in the complex plane and all points of $X$ are Jordan stable for $A$. Let $\xi \in X$. If, moreover, $X$ is simply connected (and, hence, contractible), then, by Theorem 6.6, we can find a holomorphic map $T : X \to \text{GL}(n, \mathbb{C})$ such that $T^{-1}(\xi) (\text{Com} A(\xi)) T(\xi) = \text{Com} A(\xi)$ for all $\xi \in X$.

Question: Is this true also if $X$ is not simply connected?

If the normalizer, $N$, of $G \text{Com} A(\xi)$ in $\text{GL}(n, \mathbb{C})$, which appears in the proof of Theorem 6.6, is connected, this is the case. This follows by the same proof, using [10, Satz 7] (see also, [8, Theorem 5.3.1]) (saying that $H^1(Y, O^G) = 0$ for each connected complex Lie group $G$ and each noncompact connected Riemann surface $Y$). However, this is not always the case.

For example, assume that $A(\xi) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then

$$N = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a, d \in \mathbb{C} \setminus \{0\} \right\} \cup \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \mid b, c \in \mathbb{C} \setminus \{0\} \right\},$$

which is not connected.

Proof of (6.4). It is easy to see that

$$\text{GCom} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a, d \in \mathbb{C} \setminus \{0\} \right\}.$$

To prove “$\supset$” in (6.4), we therefore only have to prove that, for all $b, c \in \mathbb{C} \setminus \{0\},$

$$\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \in N.$$

Let $b, c \in \mathbb{C} \setminus \{0\}$ be given. Then, for all $\alpha, \delta \in \mathbb{C} \setminus \{0\},$

$$\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}^{-1} \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} = \begin{pmatrix} 0 & c^{-1} \\ b^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & \alpha b \\ \delta c & 0 \end{pmatrix} = \begin{pmatrix} \delta & 0 \\ 0 & \alpha \end{pmatrix}.$$ 

In view of (6.5), this implies that

$$\text{GCom} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} = \text{GCom} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

which means (6.6), by definition of $N$.

To prove “$\subseteq$” in (6.4), let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in N$ be given. We have to prove that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix},$$

where $\alpha, \delta \in \mathbb{C} \setminus \{0\}$ with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}.$$
which implies
\[(6.8) \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 + \alpha & 0 \\ 0 & 1 + \delta \end{pmatrix} \]
and, hence,
\[(6.9) \quad \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a(1 + \alpha) & b(1 + \delta) \\ c(1 + \alpha) & d(1 + \delta) \end{pmatrix}.\]

By (6.8), \(\begin{pmatrix} 1 + \alpha & 0 \\ 0 & 1 + \delta \end{pmatrix}\) is of rank 1. Therefore

\[
\text{either } 1 + \alpha \neq 0 \text{ and } 1 + \delta = 0, \quad \text{or } 1 + \alpha = 0 \text{ and } 1 + \delta \neq 0.
\]

By (6.9), this proves (6.7). \(\square\)

7. Proof of Theorem 1.2

Here we prove Theorem 1.2. Furtheron, let \(X\) be a 1-dimensional Stein space, and \(A : X \to \text{Mat}(n \times n, \mathbb{C})\) a holomorphic map. By Corollary 3.5, it is sufficient to prove the following

7.1. Theorem. \(H^1(X, \hat{\mathcal{O}}^{\text{GCom}} A) = 0.\)

The proof of Theorem 7.1 will be given at the end of the section. First we need some preparations.

7.2. Definition. Let \(Z\) be a discrete and closed subset of \(X\).

If \(U\) is a nonempty open subset of \(X\), then we define:
\[
\hat{\mathcal{O}}^{\text{Com}} A(U, Z) \quad \text{is the subalgebra of all } f \in \hat{\mathcal{O}}^{\text{Com}} A(U) \text{ such that there exists an open neighborhood } U_Z \text{ of } Z \text{ (depending on } f) \text{ with } f = 0 \text{ on } U_Z \cap U.
\]
\[
\hat{\mathcal{O}}^{\text{GCom}} A(U, Z) \quad \text{is the subgroup of all } f \in \hat{\mathcal{O}}^{\text{GCom}} A(U) \text{ such that there exists an open neighborhood } U_Z \text{ of } Z \text{ (depending on } f) \text{ with } f = 1 \text{ on } U_Z \cap U.
\]

The so defined sheaves on \(X\) will be denoted by \(\hat{\mathcal{O}}^{\text{Com}} A(\cdot, Z)\) and \(\hat{\mathcal{O}}^{\text{GCom}} A(\cdot, Z)\), respectively.

7.3. Definition. Let \(Z\) be a discrete and closed subset of \(X\). An open covering of \(X\), \(\mathcal{U} = \{U_i\}_{i \in I}\), will be called \(Z\)-adapted if
\[(7.1) \quad U_i \cap U_j \cap Z = \emptyset \quad \text{for all } i, j \in I \text{ with } i \neq j.
\]

7.4. Lemma. Let \(Z\) be a discrete and closed subset of \(X\). Then each open covering of \(X\) admits a \(Z\)-adapted refinement.

Proof. Let an open covering \(\mathcal{U} = \{U_i\}_{i \in I}\) of \(X\) be given. Since \(Z\) is discrete and closed in \(X\), we can find a family \(\{U^*_\xi\}_{\xi \in Z}\) such that, for each \(\xi \in Z, U^*_\xi\) is an open neighborhood of \(\xi, \text{ and } U^*_\xi \cap U^*_\eta = \emptyset \text{ for all } \xi, \eta \in Z \text{ with } \xi \neq \eta.\)
Further, we set \(U^*_i = U_i \cap (X \setminus Z)\) for \(i \in I\). Then \(\{U^*_\alpha\}_{\alpha \in I \cup Z}\) (we may assume that \(I \cap Z = \emptyset\)) is a \(Z\)-adapted refinement of \(\mathcal{U}\). \(\square\)

7.5. Remark. To prove Theorem 7.1 it is sufficient to find a discrete and closed subset \(Z\) of \(X\) such that
\[(7.2) \quad H^1(X, \hat{\mathcal{O}}^{\text{GCom}} A(\cdot, Z)) = 0.
\]

Indeed, assume \(Z\) is such a set and \(f\) is an \(\hat{\mathcal{O}}^{\text{GCom}} A\)-cocycle. Let \(\mathcal{U}\) be the covering of \(f\). Then, by Lemma 7.4 we can find a \(Z\)-adapted open covering of \(X, \mathcal{U}^*,\)
which is a refinement of $\mathcal{U}$. Let $f^*$ be a $(\mathcal{U}^*, \hat{\mathcal{O}}_{GCom}^A)$-cocycle, which is induced by $f$. Since $\mathcal{U}^*$ is $Z$-adapted, then $f^*$ can be interpreted as a $(\mathcal{U}^*, \hat{\mathcal{O}}_{GCom}^A(\cdot, Z))$-cocycle, and it follows from (7.2) that $f^*$ splits as an $\hat{\mathcal{O}}_{GCom}^A(\cdot, Z)$-cocycle. As $\hat{\mathcal{O}}_{GCom}^A(\cdot, Z)$ is a subsheaf of $\hat{\mathcal{O}}_{GCom}^A$, this means in particular that $f^*$ splits as an $\hat{\mathcal{O}}_{GCom}^A$-cocycle.

7.6. Lemma. Let $Z$ be a discrete and closed subset of $X$, which contains all points of $X$ which are not Jordan stable for $A$ (by Proposition 6.2, such $Z$ exist), and let $(\Gamma_1, \Gamma_2)$ be a $Z$-adapted pair of compact sets in $X$ (Def. 6.1). Then:

(i) Let $U$ be an open neighborhood of $\Gamma_1 \cap \Gamma_2$ and $f \in \hat{\mathcal{O}}_{GCom}^A(U, Z)$. Then there exist a section $\tilde{f} \in \hat{\mathcal{O}}_{GCom}^A(X, Z)$ and an open neighborhood $V \subseteq U$ of $\Gamma_1 \cap \Gamma_2$ such that $\tilde{f} = f$ on $V$ and $\tilde{f} = I$ on $X \setminus U$.

(ii) Let $U_1$ be an open neighborhood of $\Gamma_1$, $f_1 \in \hat{\mathcal{O}}_{GCom}^A(U_1, Z)$, and let $V$ be a neighborhood of $\Gamma_1 \cap \Gamma_2$. Then there exist an open neighborhood $W$ of $\Gamma_1 \cup \Gamma_2$ and a section $f \in \hat{\mathcal{O}}_{GCom}^A(W, Z)$ such that $f = f_1$ on $(U_1 \cap W) \setminus V$.

(iii) Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of $X$ such that $\Gamma_2$ is contained in at least one set of $\mathcal{U}$. Let $f \in Z^1(\mathcal{U}, \hat{\mathcal{O}}_{GCom}^A(\cdot, Z))$ such that, for some open neighborhood $W_1$ of $\Gamma_1$, $f|_{W_1}$ splits. Then there exist an open neighborhood $W$ of $\Gamma_1 \cup \Gamma_2$ such that also $f|_W$ splits.

Proof. (i) Let $\Gamma^+$ and $\Gamma$ be as in Definition 6.1. If $\Gamma = \emptyset$, i.e., $\Gamma_1 \cap \Gamma_2 \subseteq Z$, then, by definition of $\hat{\mathcal{O}}_{GCom}^A(U, Z)$, there is a neighborhood $V \subseteq U$ of $\Gamma_1 \cap \Gamma_2$ such that $f = I$ on $V$, and we can define $\tilde{f} = I$.

Therefore, we may assume that $\Gamma \neq \emptyset$. Then, shrinking $U$, we can achieve that $U = W^0 \cup W$, where $W^0$ and $W$ are open neighborhoods of $\Gamma^+$ and $\Gamma$, respectively, such that $W^0 \cap W = \emptyset$, $f = I$ on $W^0$, $W$ consists of a finite number of connected components each of which is contractible, and $W \cap Z = \emptyset$. Then all points of $W$ are Jordan stable for $A$, and it follows from Corollary 6.7 that $\hat{\mathcal{O}}_{GCom}^A(W)$ is connected. Therefore, we can find a continuous map $\theta : [0, 1] \times W \to GL(n, \mathbb{C})$ such that

\[ \theta(t, \cdot) \in \hat{\mathcal{O}}_{GCom}^A(W) \quad \text{for all} \quad 0 \leq t \leq 1, \]

\[ \theta(0, \cdot) = f|_W \quad \text{and} \quad \theta(1, \cdot) = I \quad \text{on} \quad W. \]

Choose open neighborhoods $W^0'' \subseteq W^0 \subseteq W$ of $\Gamma$ such that $W^0''$ is relatively compact in $W'$, and $W'$ is relatively compact in $W$. Then we can find a partition of $[0, 1]$, $0 = t_1 < t_2 < \ldots < t_m = 1$, so fine that the maps $g_j \in \hat{\mathcal{O}}_{GCom}^A(W')$, $1 \leq j \leq m - 1$, defined by $g_j(\zeta) := \theta(t_j, \zeta)\theta(t_{j+1}, \zeta)^{-1} - I$ satisfy $\|g_j\| \leq 1/2$ on $W'$. Then $I + g_j \in \hat{\mathcal{O}}_{GCom}^A(W')$ and

\[ f = (I + g_1) \cdot \cdots \cdot (I + g_{m-1}) \quad \text{on} \quad W'. \]

Choose a continuous function $\chi : X \to [0, 1]$ such that $\chi = 1$ in $W^0''$ and $\chi = 0$ on a neighborhood of $X \setminus W'$, and define $\tilde{f} \in \hat{\mathcal{O}}_{GCom}^A(X)$ by

\[ \tilde{f}(\zeta) = \begin{cases} 
(I + \chi(\zeta)g_1(\zeta)) \cdot \cdots \cdot (I + \chi(\zeta)g_{m-1}(\zeta)) & \text{if} \quad \zeta \in W', \\
I & \text{if} \quad \zeta \in X \setminus W'.
\end{cases} \]

Since $Z \cap W' \subseteq Z \cap W = \emptyset$, then $\tilde{f} \in \hat{\mathcal{O}}_{GCom}^A(X, Z)$. Since $W^0 \cap W'' \subseteq W^0 \cap W = \emptyset$ and $f = I$ on $W^0$, it follows from (7.2) that $\tilde{f} = f$ on $W^0 \cup W''$. Set $V = W^0 \cup W''$. 


(ii) Since \( U_1 \cap V \) is a neighborhood of \( \Gamma_1 \cap \Gamma_2 \), by part (i) of the lemma, we can find a section \( \tilde{f} = f_1 \) on \( V' \) and \( \tilde{f} = f \) on \( X \setminus (U_1 \cap V) \). Choose an open neighborhood \( U'_1 \subseteq U_1 \) of \( \Gamma_1 \) and an open neighborhood \( U'_2 \) of \( \Gamma_2 \) such that \( U'_1 \cap U'_2 \subseteq V' \). Further choose an open neighborhood \( U'_2 \) of \( \Gamma_2 \) such that \( \overline{U'_2} \subseteq U_2 \). Then \( W := U'_1 \cup U'_2 \) is an open neighborhood of \( \Gamma_1 \cup \Gamma_2 \). Consider the open sets
\[
W_1 := W \setminus \overline{U'_2} \quad \text{and} \quad W_2 := W \cap U_2.
\]
Since \( \overline{U'_2} \subseteq U_2 \), then \( W = W_1 \cup W_2 \). Moreover, \( W_1 \subseteq U'_1 \), \( W_2 \subseteq U_2 \) and therefore \( W_1 \cap W_2 \subseteq U'_1 \cap U'_2 \subseteq V' \), which implies that \( f_j \tilde{f}^{-1} = f_1 \) on \( W_1 \cap W_2 \). Hence, there is a well-defined section \( f \in \hat{\mathcal{O}}^{\mathrm{GCom}} A(W, Z) \) such that \( f = f_j \tilde{f}^{-1} \) on \( W_1 \) and \( f = f_1 \) on \( W_2 \). As \( \tilde{f} = f \) on \( X \setminus (U_1 \cap V) \subseteq (U_1 \cap W) \setminus V \), we have \( f = f_1 \) on \( (U_1 \cap W) \setminus V \).

(iii) Since \( f|_{W_1} \) splits, we have a family \( f_i \in \hat{\mathcal{O}}^{\mathrm{GCom}} A(U_i \cap W_1, Z) \), \( i \in I \), with
\[
(7.4) \quad f_{ij} = f_i f_j^{-1} \quad \text{on} \quad U_i \cap U_j \cap W_1, \quad i, j \in I.
\]
Moreover, by hypothesis, we can find \( i_0 \in I \) with \( \Gamma_2 \subseteq U_{i_0} \). By (7.4), then
\[
(7.5) \quad g = f_{i_0}^{-1} f_{i_0} \quad \text{on} \quad U_i \cap W_1 \cap U_{i_0}, \quad i \in I.
\]
Therefore, we have a well-defined section \( g \in \hat{\mathcal{O}}^{\mathrm{GCom}} A(W_1 \cap U_{i_0}, Z) \) with
\[
(7.6) \quad \tilde{g} = f_{i_0}^{-1} f_{i_0} \quad \text{on} \quad U_i \cap V, \quad i \in I.
\]
Choose an open neighborhood \( W'_1 \subseteq W_1 \) of \( \Gamma_1 \) and an open neighborhood \( U'_{i_0} \subseteq U_{i_0} \) of \( \Gamma_2 \) such that \( W'_1 \cap U'_{i_0} \subseteq V \), and set \( W = W'_1 \cup U'_{i_0} \). Then \( W \) is an open neighborhood of \( \Gamma_1 \cup \Gamma_2 \), and by (7.5),
\[
(7.7) \quad g = f_{i_0}^{-1} f_{i_0} \quad \text{on} \quad U_i \cap W'_1 \cap U'_{i_0}, \quad i \in I.
\]
Define an open covering \( \mathcal{U}^* = \{ U^*_i \}_{i \in I} \) of \( W \) by
\[
U^*_i = \begin{cases} 
U_i \cap W'_1 & \text{if} \ i \in I \setminus \{ i_0 \}, \\
U'_{i_0} & \text{if} \ i = i_0.
\end{cases}
\]
Then \( \mathcal{U}^* \) is a refinement of \( \mathcal{U} \cap W \), and the cocycle \( f^* := \{ f_{ij} | U^*_i \cap U^*_j \}_{i, j \in I} \) is induced by \( f|_{W} \). Let
\[
f^* := \begin{cases} 
f_{ij} | U^*_i & \text{if} \ i \in I \setminus \{ i_0 \}, \\
\tilde{g}^{-1} | U^*_{i_0} & \text{if} \ i = i_0.
\end{cases}
\]
Then it follows from (7.4) and (7.7) that \( f^*(f^*)^{-1} = f_{ij} \) on \( U^*_i \cap U^*_j \). Hence, \( f^* \) splits, which means, by Proposition 2.3, that \( f|_{W} \) splits.

7.7. Lemma. Assume that \( X \) is irreducible and \( Z \) is a discrete and closed subset of \( X \) which contains all nonsmooth points of \( X \) and all points of \( X \) which are not Jordan-stable for \( A \). Then \( H^1(X, \hat{\mathcal{O}}^{\mathrm{GCom}} A(\ast, Z)) = 0 \).

\(^5\) \( X \) is called irreducible if the set of smooth points of \( X \) is connected, see, e.g., [11] Ch. 9, §1.2 or [19] Ch. V, §4.5, Prop. α].
Proof. Let an \( \hat{O}\text{GC}om^A(\cdot, Z) \)-cocycle \( f \) on \( X \) be given, and let \( \mathcal{U} = \{ U_\alpha \}_{\alpha \in I} \) be the covering of \( f \). Let \( \pi : \tilde{X} \to X \) be the normalization of \( X \) (see, e.g., [19, Ch. VI, §4.4]). Since \( X \) is irreducible, \( \tilde{X} \) is connected (see, e.g., [19, Ch. VI, §4.2]). Since \( X \) is 1-dimensional, by the Puiseux theorem (see, e.g., [19, Ch. VI, §4.1]), \( \tilde{X} \) is a Riemann surfaces. Since \( X \) is Stein and, hence, noncompact, \( \tilde{X} \) is noncompact. Therefore we can apply Theorem 4.3 to \( \tilde{X} \) and the open covering \( \mathcal{U} := \{ \pi^{-1}(U_\alpha) \}_{\alpha \in I} \). This gives a sequence \((B_\mu)_{\mu \in \mathbb{N}}\) of compact subsets of \( \tilde{X} \) such that

(a) for each \( \mu \in \mathbb{N} \), \( B_\mu \) is contained in at least one set of \( \mathcal{U} \);
(b) for each \( \mu \in \mathbb{N}^* \), \( B_0 \cup \ldots \cup B_{\mu-1}, B_\mu \) is a bump in \( \tilde{X} \);
(c) \( \tilde{X} = \bigcup_{\mu \in \mathbb{N}} B_\mu \).
(d) for each compact set \( L \subseteq \tilde{X} \), there exists \( N(L) \in \mathbb{N}^* \) such that \( B_\mu \cap L = \emptyset \) if \( \mu \geq N(L) \).

From (a) we obtain

(a') For each \( \mu \in \mathbb{N} \), \( \pi(B_\mu) \) is contained in at least one set of \( \mathcal{U} \).

Let \( K_m := \pi(B_0 \cup \ldots \cup B_m) \) and let \( \text{Int} \ K_m \) be the interior of \( K_m \), \( m \in \mathbb{N} \).

Statement 1. Let \( m \in \mathbb{N} \). Then there exists an open neighborhood \( W \) of \( K_m \) such that \( f|_W \) splits.

Proof of Statement 1. By (a'), for some \( \alpha_0 \in I \), \( U_{\alpha_0} \) is a neighborhood of \( K_0 \). Since \( f|_{U_{\alpha_0}} \) splits (Remark 2.2), this proves the claim for \( m = 0 \).

Now, proceeding by induction, we assume that, for some \( \ell \in \mathbb{N} \), we already have an open neighborhood \( W \) of \( K_\ell \) such that \( f|_W \) splits. Then, by (b) and Lemma 5.2 we can find a \( Z \)-adapted pair of compact sets in \( X \), \((\Gamma_1, \Gamma_2)\), such that

\[
\begin{align*}
\Gamma_1 & \subseteq K_\ell, \\
\Gamma_2 & \subseteq \pi(B_{\ell+1}), \\
\Gamma_1 \cup \Gamma_2 & = K_{\ell+1}.
\end{align*}
\]

By Lemma 4.2, \( W \) is an open neighborhood of \( \Gamma_1 \). Since \( f|_{\Gamma_1} \) splits and since, by (a') and (a'), \( \Gamma_2 \) is contained in at least one set of \( \mathcal{U} \), from Lemma 7.9 (iii) we get an open neighborhood \( V \) of \( \Gamma_1 \cup \Gamma_2 \) such that \( f|_V \) splits. By (7.10), this is the claim for \( m = \ell + 1 \).

By (c) and (d), and using the fact that all \( \pi^{-1}(K_m) \) are compact (\( \pi \) is proper), we can find a strictly increasing sequence \( m(j) \in \mathbb{N}^* \) such that, for all \( j \in \mathbb{N}^* \),

\[
\begin{align*}
\pi(B_\mu) \cap K_{m(j)} & = \emptyset \quad \text{if} \quad \mu \geq m(j+1), \\
K_{m(j)} & \subseteq \text{Int} \ K_{m(j+1)}.
\end{align*}
\]

Statement 2. Let \( j, \ell \in \mathbb{N}^* \) with \( m(j+1) \leq \ell \), let \( W \) be an open neighborhood of \( K_\ell \) and \( g \in \hat{O}\text{GC}om^A(W, Z) \). Then there exist an open neighborhood \( W' \) of \( K_{\ell+1} \) and \( h \in \hat{O}\text{GC}om^A(W', Z) \) such that

\[
\begin{align*}
h & = g \quad \text{on} \quad K_{m(j)}, \\
\pi(B_{\ell+1}) & \subseteq V, \\
K_{m(j)} \cap V & = \emptyset.
\end{align*}
\]
By (b) and Lemma 5.2 we can find a $Z$-adapted pair of compact sets in $X$, $(\Gamma_1, \Gamma_2)$, such that

\begin{align*}
(7.16) & \quad \Gamma_1 \subseteq K_\ell, \\
(7.17) & \quad \Gamma_2 \subseteq \pi(B_{\ell+1}), \\
(7.18) & \quad \Gamma_1 \cup \Gamma_2 = K_{\ell+1}.
\end{align*}

Then, by (7.16), $W$ is an open neighborhood of $\Gamma_1$, by (7.17) and (a'), $\Gamma_2$ is contained in at least one set of $U$, and, by (7.14) and (7.17), $V$ is a neighborhood of $\Gamma_2$. Therefore, from Lemma 7.3 (ii) we get an open neighborhood $W'$ of $\Gamma_1 \cup \Gamma_2 = K_{\ell+1}$ (cp. (7.18)) and a section $h \in \overline{GCom}^A(W', Z)$ such that

\begin{equation}
(7.19) \quad h = g \text{ on } (W \cap W') \setminus V.
\end{equation}

Since $K_{m(j)} \subseteq K_\ell \subseteq W$ and $K_{m(j)} \subseteq K_{\ell+1} \subseteq W'$, we have $K_{m(j)} \subseteq W \cap W'$, which means, by (7.19), that $K_{m(j)} \subseteq (W \cap W') \setminus V$. Therefore (7.19) proves (7.18).

**Statement 3.** Let $j \in \mathbb{N}^*$, let $W$ be an open neighborhood of $K_{m(j+1)}$ and $g \in \overline{GCom}^A(W, Z)$. Then there exist an open neighborhood $W'$ of $K_{m(j+2)}$ and $h \in \overline{GCom}^A(W, Z)$ such that

\begin{equation}
(7.20) \quad g = h \text{ on } K_{m(j)}.
\end{equation}

**Proof of Statement 3.** Application of Statement 2 for $\ell = 1, 2, \ldots, m(j+2) - 1$.

To prove the lemma, we have to find a family $f = \{f_\alpha\}_{\alpha \in I}$ of sections $f_\alpha \in \overline{GCom}^A(U_\alpha, Z)$ such that

\begin{equation}
(7.21) \quad f_{\alpha\beta} = f_\alpha f^{-1}_\beta \text{ on } U_\alpha \cap U_\beta, \quad \alpha, \beta \in I.
\end{equation}

Since $\bigcup_{j=1}^{\infty} \text{Int} K_{m(j)} = X$, it is sufficient to construct inductively a sequence of families $\{f^{(j)}_\alpha\}_{\alpha \in I}$, $j \in \mathbb{N}^*$, of sections $f^{(j)}_\alpha \in \overline{GCom}^A(\text{Int} K_{m(j)} \cap U_\alpha, Z)$ such that, for each $j \in \mathbb{N}^*$,

- **S(j):** $f^{(j)}_{\alpha\beta} = f^{(j)}_\alpha (f^{(j)}_\beta)^{-1}$ on $\text{Int} K_{m(j)} \cap U_\alpha \cap U_\beta$ for all $\alpha, \beta \in I$, and
- **T(j):** if $j > 2$, then $f^{(j)}_\alpha = f^{(j-1)}_\alpha$ on $K_{m(j-2)} \cap U_\alpha$ for all $\alpha \in I$.

By Statement 1 there exist a neighborhood $W$ of $K(m(1))$ and a family $\{f^{(W)}_\alpha\}_{\alpha \in I}$ of sections $f^{(W)}_\alpha \in \overline{GCom}^A(W \cap U_\alpha, Z)$ such that $f^{(W)}_{\alpha\beta} = f^{(W)}_\alpha (f^{(W)}_\beta)^{-1}$ on $W \cap U_\alpha \cap U_\beta$ for all $\alpha, \beta \in I$. Setting $f^{(1)}_\alpha = f^{(W)}_\alpha|_{\text{Int} K_{m(1)} \cap U_\alpha}$ we get a family $\{f^{(1)}_\alpha\}_{\alpha \in I}$ of sections $f^{(1)}_\alpha \in \overline{GCom}^A(\text{Int} K_{m(1)} \cap U_\alpha, Z)$ satisfying condition $\text{S}(1)$. Condition $\text{T}(1)$ is trivial.

Now we assume that, for some $k \in \mathbb{N}^*$, we already have families $\{f^{(j)}_\alpha\}_{\alpha \in I}$, $1 \leq j \leq k$, of sections $f^{(j)}_\alpha \in \overline{GCom}^A(\text{Int} K_{m(j)} \cap U_\alpha, Z)$ such that conditions $\text{S}(j)$ and $\text{T}(j)$ are satisfied for $1 \leq j \leq k$. We have to find a family $f^{(k+1)}_\alpha \in \overline{GCom}^A(\text{Int} K_{m(k+1)} \cap U_\alpha, Z)$, $\alpha \in I$, such that also conditions $\text{S}(k+1)$ and $\text{T}(k+1)$ are satisfied. Again by Statement 1, we have a family $F_\alpha \in \overline{GCom}^A(\text{Int} K_{m(k+1)} \cap U_\alpha, Z)$, $\alpha \in I$, such that

\begin{equation}
(7.21) \quad f^{(k+1)}_{\alpha\beta} = F_\alpha F^{-1}_\beta \text{ on } \text{Int} K_{m(k+1)} \cap U_\alpha \cap U_\beta, \quad \alpha, \beta \in I.
\end{equation}

If $k = 2$, then the family $f^{(k)}_\alpha := F_\alpha$, $\alpha \in I$, is as required, because $\text{S}(2)$ holds by (7.21), and $\text{T}(2)$ is trivial. Let $k > 2$. Then it follows from (7.21) and condition
that, for all \( \alpha, \beta \in I \), on \( \mathsf{Int} K_{m(k)} \cap U_\alpha \cap U_\beta \) we have
\[
F_\alpha F_\beta^{-1} = f_{\alpha \beta} = f_{\alpha}^{(k)}(f_{\beta}^{(k)})^{-1}
\]
and, therefore, \( (f_{\alpha}^{(k)})^{-1} F_\alpha = (f_{\beta}^{(k)})^{-1} F_\beta \).
Hence, there is a well-defined section \( g \in \tilde{\mathcal{O}}^{G_{\mathrm{Com}} A}(\mathsf{Int} K_{m(k)}, Z) \) with
\[
g = (f_{\alpha}^{(k)})^{-1} F_\alpha \quad \text{on} \quad \mathsf{Int} K_{m(k)} \cap U_\alpha \quad \text{for all} \quad \alpha \in I,
\]
By Statement 3, we can find a section \( h \in \tilde{\mathcal{O}}^{G_{\mathrm{Com}} A}(\mathsf{Int} K_{m(k+1)}, Z) \) with
\[
h = g \quad \text{on} \quad \mathsf{Int} K_{m(k-1)}.
\]
Set \( f_{\alpha}^{(k+1)} = F_\alpha h^{-1} \) on \( \mathsf{Int} K_{m(k+1)} \cap U_\alpha \) for \( \alpha \in I \). Then, by (7.21),
\[
f_{\alpha}^{(k+1)}(f_{\beta}^{(k+1)})^{-1} = F_\alpha h^{-1} h F_\beta^{-1} = F_\alpha F_\beta^{-1} = f_{\alpha \beta} \quad \text{on} \quad \mathsf{Int} K_{m(k+1)} \cap U_\alpha \cap U_\beta,
\]
\( S(k+1) \) is satisfied. From (7.23) and (7.22) it follows that \( f_{\alpha}^{(k+1)} = F_\alpha h^{-1} = F_\alpha g^{-1} = F_\alpha (f_{\alpha}^{(k)})^{-1} f_{\alpha}^{(k)} = f_{\alpha}^{(k)} \) on \( \mathsf{Int} K_{m(k-1)} \), i.e., \( T(k+1) \) is satisfied. \( \square \)

**Proof of Theorem 7.3** Let \( Z \) be the union of the set of nonsmooth points of \( X \) and the set of points in \( X \) which are not Jordan stable for \( A \) (Def. 6.1). By Proposition 6.2, \( Z \) is discrete and closed in \( X \). Therefore, by Remark 7.5 it is sufficient to prove that \( H^1(X, \tilde{\mathcal{O}}^{G_{\mathrm{Com}} A}(\cdot, Z)) = 0 \).

Let an open covering \( \mathcal{U} = \{ U_i \}_{i \in I} \) of \( X \) and a \( (\mathcal{U}, \tilde{\mathcal{O}}^{G_{\mathrm{Com}} A}(\cdot, Z)) \)-cocycle \( f = \{ f_{ij} \}_{j \in I} \) be given. We have to find a family \( f_i \in \tilde{\mathcal{O}}^{G_{\mathrm{Com}} A}(U_i, Z) \), \( i \in I \), such that, for all \( i, j \in I \) and \( \zeta \in U_i \cap U_j \),
\[
f_{ij}(\zeta) = f_i(\zeta)(f_j(\zeta))^{-1}.
\]
Let \( \mathfrak{X} \) be the family of irreducible components of \( X \) (see, e.g., [11, Ch. 9, §2.2] or [19, Ch. IV, §2.9, Theorem 4, and Ch. V, §4.5]). Then, for each \( Y \in \mathfrak{X} \), \( Z \cap Y \) is discrete and closed in \( Y \), contains all nonsmooth points of \( Y \) and all points of \( Y \) which are not Jordan stable for \( A|_Y \). Therefore, by Lemma 7.7 for each \( Y \in \mathfrak{X} \), we can find a family \( f_Y^i \in \tilde{\mathcal{O}}^{G_{\mathrm{Com}} A|_Y}(U_i \cap Y, Z \cap Y) \) such that
\[
f_{ij}^Y = f_i^Y(f_j^Y)^{-1} \quad \text{on} \quad Y \cap U_i \cap U_j, \quad i, j \in I.
\]
Since \( X \setminus Z \) is smooth and, therefore, the family \( \{ Y \setminus Z \}_{Y \in \mathfrak{X}} \) is pairwise disjoint, for each fixed \( i \in I \), the family \( \{ Y \cap (U_i \setminus Z) \}_{Y \in \mathfrak{X}} \) is a pairwise disjoint open covering of \( U_i \setminus Z \). Hence, for each \( i \in I \), there is a well defined map \( f_i : U_i \rightarrow \text{GL}(n, \mathbb{C}) \) such that \( f_i|_{U_i \setminus Z} \in \tilde{\mathcal{O}}^{G_{\mathrm{Com}} A|_Y}(U_i \setminus Z) \) and
\[
f_i(\zeta) = \begin{cases} f_i^Y(\zeta) & \text{if} \quad \zeta \in Y \cap (U_i \setminus Z), \\ I & \text{if} \quad \zeta \in Z. \end{cases}
\]
Since each \( f_i^Y \) is equal to \( I \) in a \( Y \)-neighborhood of \( Z \cap Y \) and since \( \mathfrak{X} \) is locally finite, we see that \( f_i \in \tilde{\mathcal{O}}^{G_{\mathrm{Com}} A|_Y}(U_i, Z) \). Relation (7.24) follows from (7.25) if \( \zeta \in (U_i \cap U_j) \setminus Z \), and from the fact that, \( f_i(\zeta) = f_j(\zeta) = f_{ij}(\zeta) = I \) if \( \zeta \in (U_i \cap U_j) \setminus Z \). This completes the proof of Theorem 7.3 and, hence, the proof of Theorem 1.2.

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