On homotopy exact sequences for normal schemes

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Abstract
Let \( f : X \to S \) be a surjective morphism of finite type between connected locally Noetherian normal schemes. We discuss sufficient conditions that the sequence of the étale fundamental groups

\[
\pi_1(X \times_S \overline{\eta}, *) \to \pi_1(X, *) \to \pi_1(S, *) \to 1
\]

is exact, where \( \overline{\eta} \) is a geometric generic point of \( S \) and \( * \) is a geometric point of \( X \times_S \overline{\eta} \). In the present paper, we generalize those in [SGA1], [Ho1], and [Mit]. We show that the conditions we give are also necessary conditions in the case where, for instance, \( S \) is an affine smooth curve over a field of characteristic 0.

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0 Introduction

Let \( f : X \to S \) be a surjective morphism of finite type between connected locally Noetherian normal schemes, \( \overline{\eta} \) a geometric generic point of \( S \), and \( * \)
a geometric point of $X \times_S \eta$. Suppose that the scheme $X \times_S \eta$ is connected. Consider the sequence of the étale fundamental groups

\[ \pi_1(X \times_S \eta, \ast) \to \pi_1(X, \ast) \to \pi_1(S, \ast) \to 1 \]  

(1)

In [SGA1], the following proposition is proved:

**Proposition 0.1.** ([SGA1] Exp.X Corollaire 1.4) Suppose that $f$ is proper and flat with geometrically reduced fibers. Moreover, suppose that $f_*O_X = O_S$. Then the sequence (1) is exact.

Note that the scheme $S$ is not assumed to be normal in [SGA1]. This proposition has been improved by Hoshi [Ho1] and Mitsui [Mit] (cf. Propositions 1.2 and 1.3). They discussed the case where the morphism $f$ has geometrically reduced fibers.

In the present paper, we discuss homotopy exact sequences without this assumption. Our main result is as follows (see Theorem 1.12 for weak conditions):

**Theorem 0.2.** Suppose that the following conditions are satisfied:

- The morphism $f$ is flat or the scheme $S$ is regular.
- Let $s$ be a point of $S$ whose local ring is of dimension 1. Write $\xi_1, \ldots, \xi_n$ for the generic points of the scheme $f^{-1}(s)$, $e_i$ for the multiplicity of $\xi_i$, and $k(\xi_i)$ (resp. $k(s)$) for the residual field of $\xi_i$ (resp. $s$). Then $\gcd(e_1, \ldots, e_n) = 1$ and the algebraic closure of $k(s)$ in $k(\xi_i)$ is separable for some $i$.

Then the sequence (1) is exact.

We cannot drop any assumption of Theorem 1.12 (cf. Section 3, Example 5.1, and Remark 5.2). For instance, we have the following two propositions (see Section 3 for general settings):

**Proposition 0.3.** (cf. Corollary 3.4 and Example 3.5.2) Suppose that the scheme $S$ is the spectrum of a semi-local Dedekind domain which contains $\mathbb{Q}$, and that the scheme $X$ is regular. Then the sequence (1) is exact if and only if the greatest common divisor of the multiplicities of the irreducible components of each closed fiber of $f$ is 1.

**Proposition 0.4.** (cf. Proposition 3.6) Suppose that the scheme $S$ is a smooth curve over a field $k$ of characteristic 0, and the scheme $X$ is regular. Moreover, suppose that the scheme $S$ is not proper rational (cf. Definition 2.5). Then the sequence (1) is exact if and only if the greatest common divisor of the multiplicities of the irreducible components of each closed fiber of $f$ is 1.
We apply the above results to the case where $f : X \to S$ is a morphism from a regular variety to a hyperbolic curve (cf. Definition 2.4). In particular, we prove that a certain morphism is characterized by the property that the kernel of the induced homomorphism between the étale fundamental groups is topologically finitely generated (see Theorem 4.2 for more details):

**Theorem 0.5.** (cf. Theorem 4.2) Suppose that $S$ is a hyperbolic curve over a field of characteristic 0 and the scheme $X$ is regular. The following three conditions are equivalent:

1. The greatest common divisor of the multiplicities of the irreducible components of each closed fiber of $f$ is 1.
2. The sequence (1) is exact.
3. The profinite group $\text{Ker}(\pi_1(X, *) \to \pi_1(S, *))$ is topologically finitely generated.

Note that condition 1 is stated only in the language of schemes and that condition 3 is stated only in the language of topological groups. Such statements are natural in the framework of anabelian geometry (cf. [Tama1], [Moch1], [Ho1]). In anabelian geometry, we attempt to get information of varieties from their étale fundamental groups. In this sense, Theorem 0.5 may be regarded as a group theoretical characterization of a morphism as written in condition 1.

Let us explain the strategies of the proofs of the exactness of the homotopy exact sequences given in [SGA1], [Ho1], [Mit], and the present paper. Let $X' \to X$ be an étale covering space whose pull-back $X' \times_S \overline{\eta} \to X \times_S \overline{\eta}$ has a section. To show that the sequence (1) is exact, we need to construct an étale covering space $S' \to S$ such that the pull-back $X \times_S S'$ is isomorphic to $X'$ over $X$. In [SGA1], the Stein factorization of the morphism $X' \to S$ plays the role of $S'$. In [Ho1], where $f$ is not always proper, the normalization of $S$ in the function field of $X'$ plays the role of $S'$ there. (In the present paper, we need to use the normalization of $S$ in the separable closure of the function field of $S$ in the function field of $X'$). In [Ho1] and [Mit], they replace $X$ by another scheme over $S$ which is faithfully flat with geometrically normal fibers to show that the morphism $S' \to S$ is étale. In our situation, we cannot find such a good scheme. If the scheme $S$ is regular, it suffices to show that the morphism $S' \to S$ is étale over an open subscheme of $S$ whose complement is of codimension $\geq 2$ by the Zariski-Nagata purity theorem. If the scheme $S$ is not regular, we need to assume that the morphism $f$ is flat (cf. Example 3.7.1). In this case, we use Serre’s criterion for normality to compare the morphism $X' \to X$ and the morphism $S' \to S$.

The content of each section is as follows: In Section 1 we give the proof of Theorem 0.2. In Section 2 we discuss properties of Dedekind schemes to
have many tame extensions. In Section 3 we give the proofs of Proposition 0.3 and Proposition 0.4. In Section 4 we give the proof of Theorem 4.2. In Section 5 we discuss property (F). In Section 6 we discuss the homotopy exact sequence for geometrically connected (not necessarily generic) fibers.

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1 Sufficient conditions

In this section, we give the proof of Theorem 0.2 in a generalized setting.

Let \( f : X \to S \) be a surjective morphism of locally finite type between connected locally Noetherian normal separated schemes. We write \( K(X) \) (resp. \( K(S) \)) for the function field of \( X \) (resp. \( S \)). Take a geometric generic point \( \eta \) of \( S \) and write \( X_\eta \) for the scheme \( X \times_S \eta \). Suppose that \( X_\eta \) is connected (and hence irreducible). Take a geometric point \( \overline{x} \) of \( X_\eta \). Then we obtain the following sequence of the \( \acute{e} \)tale fundamental groups:

\[
\pi_1(X_\eta, \overline{x}) \to \pi_1(X, \overline{x}) \to \pi_1(S, \overline{x}) \to 1.
\] (2)

Remark 1.1. 1. Since \( f \) is generically geometrically connected, the homomorphism \( \pi_1(X, \overline{x}) \to \pi_1(S, \overline{x}) \) is surjective by [Ho1] Lemma 1.6.

2. Let \( S' \to S \) be a finite \( \acute{e} \)tale morphism which the morphism \( \eta \to S \) factors through. By Remark 1.1.1, the sequence (2) is exact if and only if the sequence

\[
\pi_1(X_\eta, \overline{x}) \to \pi_1(X \times_S S', \overline{x}) \to \pi_1(S', \overline{x}) \to 1
\]

is exact.

3. The composite homomorphism \( \pi_1(X_\eta, \overline{x}) \to \pi_1(X, \overline{x}) \to \pi_1(S, \overline{x}) \) is trivial.

4. Thus, the sequence (2) is exact if and only if

\[
\text{Im}(\pi_1(X_\eta, \overline{x}) \to \pi_1(X, \overline{x})) \supset \text{Ker}(\pi_1(X, \overline{x}) \to \pi_1(S, \overline{x})).
\]

First, we recall sufficient conditions given by Hoshi and Mitsui which generalize Proposition 0.1.

Proposition 1.2. ([Ho1] Proposition 1.10) Suppose that there exist a connected locally Noetherian normal separated scheme \( Y \) and a morphism \( p : Y \to X \). Moreover, suppose that the following conditions are satisfied:

\[\text{Im}(\pi_1(X_\eta, \overline{x}) \to \pi_1(X, \overline{x})) \supset \text{Ker}(\pi_1(X, \overline{x}) \to \pi_1(S, \overline{x})).\]
• The morphism $p$ is dominant and induces an outer surjection
\[ \pi_1(Y) \to \pi_1(X). \]

• The morphism $f$ is generically geometrically integral.

• The composite morphism $f \circ p$ is of finite type, faithfully flat, geometrically normal, and generically geometrically connected.

Then the sequence (2) is exact.

**Proposition 1.3.** (Mit Theorem 4.22) Suppose that $f$ is flat and geometrically reduced. Moreover, suppose that the sheaf $O_S$ is integrally closed in the sheaf $f_*O_X$. Then the sequence (2) is exact.

Since the schemes $X$ and $S$ are normal, these schemes enjoy the following properties:

**Lemma 1.4.** Let $U$ be a connected locally Noetherian normal scheme. Write $K(U)$ for the function field of $U$.

1. Let $*$ be a geometric point of $\text{Spec} K(U)$. Then the homomorphism
\[ \pi_1(\text{Spec} K(U), *) \to \pi_1(U, *) \]
is surjective.

2. Let $V \to U$ be a connected étale covering space. Write $K(V)$ for the function field of $V$. Let $L$ be an intermediate field of $K(U) \subset K(V)$.

Then the normalization of $U$ in $L$ is an étale covering space of $U$.

**Proof.** Assertion 1 is well-known. Since the $\pi_1(\text{Spec} K(U), *)$-equivariant morphism
\[ (\text{Hom}_U(*, V) \simeq) \text{Hom}_{\text{Spec} K(U)}(*, \text{Spec} K(V)) \to \text{Hom}_{\text{Spec} K(U)}(*, \text{Spec} L) \]
is surjective, there exists a connected étale covering space $W \to U$ such that $W \times_U \text{Spec} K(U)$ is isomorphic to $\text{Spec} L$ over $K(U)$. Since $U$ is normal and the morphism $W \to U$ is étale, $W$ is also normal. Since $W$ is finite over $U$, assertion 2 holds.

We rephrase the exactness of the sequence (2) in terms of étale covering spaces of $X$.

**Proposition 1.5.** The following four conditions are equivalent:

1. $\text{Im}(\pi_1(X_\mathcal{F}, \mathcal{F}) \to \pi_1(X, \mathcal{F})) \supset \text{Ker}(\pi_1(X, \mathcal{F}) \to \pi_1(S, \mathcal{F}))$.

2. Let $C$ be a finite set with a continuous left $\pi_1(X, \mathcal{F})$-action. Suppose that there exists a $\pi_1(X_\mathcal{F}, \mathcal{F})$-orbit of $C$ on which $\pi_1(X_\mathcal{F}, \mathcal{F})$ acts trivially. Then there exist a finite set $D$ with a continuous transitive left $\pi_1(S, \mathcal{F})$-action and a $\pi_1(X, \mathcal{F})$-equivariant isomorphism between $D$ and $C$. 

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3. Let $X'$ be a connected étale covering space of $X$. Suppose that the étale covering space $X' \times_X X' \to X'$ has a section. Then there exist an étale covering space $S' \to S$ and an $X$-isomorphism between $X \times_S S'$ and $X'$.

4. Let $X'$ be a connected étale covering space of $X$. Write $K_{X'/S}$ for the separable closure of $K(S)$ in the function field of $X'$. The normalization $N_{X'/S}$ of $S$ in $K_{X'/S}$ is étale over $S$.

**Proof.** Since the homomorphism $\pi_1(X, x) \to \pi_1(S, x)$ is surjective, condition 1 is equivalent to condition 2. The equivalence of 2 and 3 is clear.

We prove the equivalence of 3 and 4. Let $X'$ be a connected étale covering space of $X$. Write $K(X')$ for the function field of $X'$, $K_{X'/S}$ for the separable closure of $K(S)$ in $K(X')$, and $N_{X'/S} \to S$ for the normalization of $S$ in $K_{X'/S}$.

First, we prove the implication $3 \Rightarrow 4$. The normalization $X_N$ of $X'$ in the composite field $K(X)K_{X'/S}$ in $K(X')$ is an étale covering space of $X$ by Lemma 1.4.2. Moreover, since the morphism $X' \times_X X' \to X'\times_S X'$ has a section, there exist a finite étale covering space $S' \to S$ and an $X$-isomorphism $X_N \simeq X \times_S S'$ by condition 3. Therefore $N_{X'/S}$ is isomorphic to $S'$ over $S$ and thus condition 4 holds.

Next, we prove the implication $4 \Rightarrow 3$. Suppose that the morphism $X' \times_S X_{X'/S} \to X' \times_S X_{X'/S}$ has a section. By condition 4, $N_{X'/S}$ is an étale covering space of $S$. It suffices to show that the induced morphism $\phi : X' \to X \times_S N_{X'/S}$ is an isomorphism. Since $X \times_S N_{X'/S}$ is étale over $X$ and connected, the morphism $\phi$ is finite étale surjective. The number of connected components of $X' \times_S N_{X'/S}$ coincides with the covering degree of $N_{X'/S}$ over $S$. On the other hand, the number of connected components of $X' \times X' = X' \times_S X' \times_S X'$ coincides with the extension degree $[K_{X'/S} : K(S)]$. Therefore, there is a bijection between the set of connected components of $X' \times X'$ and that of $X' \times_S N_{X'/S}$. Since the morphism $X' \times_X X' \to X'$ has a section, we can show that the covering degree of $X'$ over $X \times S N_{X'/S}$ is 1. Thus, condition 3 holds.

**Remark 1.6.** Proposition 1.5 holds if $f$ is dominant (even if $f$ is not surjective).

Recall that we do not assume that the scheme $S$ is regular. Since we cannot use the Zariski-Nagata purity theorem, we show the following technical lemma needed later:

**Lemma 1.7.** Let $S'$ be an integral scheme and $S' \to S$ a quasi-finite dominant morphism.

1. Suppose that $f$ is flat and the extension between the function fields of $S'$ and $S$ is separable. Then the scheme $X \times_S S'$ is integral.
2. Moreover, suppose that the scheme $S'$ is normal and the morphism $S' \to S$ is étale over each point of $S$ whose local ring is of dimension 1. Then the scheme $X \times_S S'$ is normal.

Proof. Write $K(S')$ for the function field of $S'$. Since $f$ is generically geometrically connected and $K(S')$ is separable over $K(S)$, the scheme $X \times_S \text{Spec} K(S')$ is integral. Therefore, assertion 1 follows from flatness of $f$.

By Serre’s criterion for normality, it suffices to show that the scheme $X \times_S S'$ satisfies ($R_1$) and ($S_2$) to prove assertion 2. Any point of the scheme $X \times_S S'$ over a point of $S'$ whose local ring is of dimension $\leq 1$ is normal by the assumption on the morphism $S' \to S$. Since $f$ is flat, the image of any point of the scheme $X \times_S S'$ whose local ring is of dimension 1 is a point of $S'$ whose local ring is of dimension $\leq 1$. Therefore, $X \times_S S'$ satisfies ($R_1$). Since $f$ is flat and $S'$ is normal, any point of the scheme $X \times_S S'$ over a point of $S'$ whose local ring is of dimension $\geq 2$ is of depth $\geq 2$. Therefore, the scheme $X \times_S S'$ satisfies ($S_2$). □

Proposition 1.8. Suppose that $f$ is flat or $S$ is regular. Then the four conditions in Proposition 1.3 are equivalent to the following condition:

5. Let $X'$ be a connected étale covering space of $X$. Write $K_{X'/S}$ for the separable closure of $K(S)$ in the function field of $X'$. The normalization $N_{X'/S}$ of $S$ in $K_{X'/S}$ is étale over each point of $S$ whose local ring is of dimension 1.

Proof. The implication 4 $\Rightarrow$ 5 is clear. We prove the implication 5 $\Rightarrow$ 4. By condition 5, the morphism $N_{X'/S} \to S$ is étale over each point of $S$ whose local ring is of dimension $\leq 1$. If $S$ is regular, the morphism $N_{X'/S} \to S$ is étale by the Zariski-Nagata purity theorem. Hence we may assume that $f$ is flat. Then the scheme $X \times_S N_{X'/S}$ is connected normal by Lemmas 1.7.1 and 1.7.2. Since the scheme $X \times_S N_{X'/S}$ coincides with the normalization of $X$ in the composite field $K(X)K_{X'/S}$ in the function field of $X'$, the morphism $X \times_S N_{X'/S} \to X$ is étale by Lemma 1.4.2. Since $f$ is faithfully flat, the morphism $X \times_S N_{X'/S} \to X$ is also étale. We finish the proof of Proposition 1.8. □

Definition 1.9. Let $\{\iota_i : k \hookrightarrow K_i\}_{i \in I}$ be a set of inclusions of fields. We say that the inclusions $\{\iota_i\}$ satisfy property (F) if the following condition is satisfied:

(F): For any algebraic separable extension $L_i$ of $K_i$ ($i \in I$) and any subfield $l$ of the product ring $\prod_{i \in I} L_i$ which is algebraic over the diagonal subfield $k$ defined by $\iota_i$, the extension $k \subset l$ is separable.

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Remark 1.10. 1. If $K_i$ is geometrically reduced over $k$ for some $i$, the inclusions $\{\iota_i\}$ satisfy property (F). In fact, if $k$ is purely inseparably closed in $K_i$ (i.e., $k^{p^{-\infty}} \cap K_i = k$) for some $i$, the inclusions $\{\iota_i\}$ satisfy property (F).

2. We discuss property (F) in Section 5.

Definition 1.11. We say that $f$ satisfies property (R) if the following condition is satisfied.

(R): Let $s$ be a point of $S$ whose local ring is of dimension 1. Write $\xi_i$ ($i \in I$) for the generic points of the scheme $f^{-1}(s)$, $e_i$ for the multiplicity of $\xi_i$, and $k(\xi_i)$ (resp. $k(s)$) for the residual field of $\xi$ (resp. $s$). Then $\gcd_{i \in I} e_i = 1$ and the inclusions $k(s) \hookrightarrow k(\xi_i)$ ($i \in I$) satisfy property (F).

We prove the main theorem of the present paper (cf. Theorem 0.2).

Theorem 1.12. Suppose that $f$ satisfies property (R) and one of the following conditions is satisfied:

- The morphism $f$ is flat.
- The scheme $S$ is regular.

Then the sequence (2) is exact.

Proof. By Remark 1.14, it suffices to show that

$$\text{Ker}(\pi_1(X,\mathcal{X}) \to \pi_1(S,\mathcal{X})) \subset \text{Im}(\pi_1(X_{\eta,\mathcal{X}}) \to \pi_1(X,\mathcal{X})).$$

Let $X' \to X$ be a finite étale covering space. Write $K_{X'/S}$ for the separable closure of $K(S)$ in the function field of $X'$. By Proposition 1.8 it suffices to show that the normalization $N_{X'/S}$ of $S$ in $K_{X'/S}$ is finite étale over $S$ at each point of $N_{X'/S}$ whose local ring is of dimension 1. Let $n$ be such a point of $N_{X'/S}$. Write $s$ for the image of $n$ in $S$. It suffices to show that the extension of the discrete valuation rings $O_{\xi,s} \subset O_{N_{X'/S},n}$ is unramified. Therefore, the assertion follows from the hypothesis of Theorem 1.12.

Remark 1.13. If the morphism $f$ is not flat and the scheme $S$ is not regular, Theorem 1.12 does not hold in general (cf. Example 3.7.1).

2 Lemmas for Dedekind schemes

In this section, we discuss some properties of Dedekind schemes which we use in Section 3.
2.1 A fundamental lemma

We prove a lemma for Dedekind schemes needed later.

**Lemma 2.1.** Let $R$ be a strictly henselian discrete valuation ring. Write $K$ for the field of fractions of $R$. Let $K'$ be a finite tamely ramified extension field of $K$. Write $R'$ for the normalization of $R$ in $K'$ and $e'$ for the ramification index of the extension $R'/R$. Let $A$ be a discrete valuation ring which dominates $R$ such that the ramification index of the extension $A/R$ is $e$. Suppose that the field of fractions $L$ of $A$ is geometrically connected over $K$ and $e$ is divisible by $e'$. Then the normalization $A'$ of $A \otimes_R R'$ (cf. Lemma 1.7.1) is étale over $A$.

**Proof.** Let $\tilde{A}$ be a strict henselization of $A$. Then $\tilde{A} \otimes_A A'$ is the normalization of $\tilde{A} \otimes_R R'$ (in its total ring of fractions). Therefore, it suffices to show that $\tilde{A} \otimes_A A'$ is the product ring of $e'$ copies of $\tilde{A}$. Let $\varpi$ (resp. $\varpi'$; $\varpi_A$) be a uniformizer of $R$ (resp. $R'$; $A$). There exists a unit $u'$ (resp. $u_A$) of $R'$ (resp. $A$) such that $\varpi = u' (\varpi')^{e'}$ (resp. $\varpi = u_A (\varpi_A)^e$). Since there exists a unit $v'$ (resp. $v_A$) of $R'$ (resp. $A$) which satisfies that $(v')^{e'} = u'$ (resp. $v_A^{e'} = u_A$), we may assume that $(\varpi')^{e'} = \varpi$ (resp. $(\varpi_A)^e = \varpi$). Thus, $R'$ is isomorphic to $R[T]/(T^{e'} - \varpi)$ and $\tilde{A} \otimes_R R'$ is isomorphic to $\tilde{A}[T]/ \prod_{1 \leq i \leq e'} (T - \zeta_i^{e'} (\varpi_A)^{\frac{e}{e'}})$. Here, $\zeta_i^{e'}$ is a primitive $e'$-th root of unity in $\tilde{A}$. Therefore, $\tilde{A} \otimes_A A'$ is isomorphic to $\prod_{1 \leq i \leq e'} \tilde{A}[T]/(T - \zeta_i^{e'} (\varpi_A)^{\frac{e}{e'}})$. We finished the proof of Lemma 2.1.

2.2 Examples of Dedekind schemes

We discuss whether there exists a convenient (cf. Definition 2.2) tame covering space of a given Dedekind scheme.

**Definition 2.2.**

1. Let $S$ be a scheme. We shall say that $S$ is a Dedekind scheme if $S$ is connected, locally Noetherian, normal, and of dimension 1.

2. Let $S$ be a Dedekind scheme. We say that $S$ has property (T) if, for each closed point $s \in S$ and a prime number $l$ which is not divisible by the characteristic of the residual field of $s$, there exist a normal scheme $S'$ and a finite dominant morphism $S' \to S$ which satisfy the following conditions:

- The morphism $S' \to S$ is finite Galois étale over $S \setminus \{s\}$.
- For any closed point $s'$ over $s$, the ramification index of $S' \to S$ at $s'$ (which independent of the choice of $s'$ since $S'$ is Galois over $S$) is $l$.  

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Remark 2.3. The conditions on $S'$ in Definition 2.2.2 is equivalent to the following conditions:

- The ramification index of $S' \to S$ at each point of $S'$ over $s$ is equal to 1 or $l$.
- There exists a point of $S'$ over $s$ where the ramification index of the morphism $S' \to S$ is equal to $l$.
- The morphism $S' \to S$ is finite étale over $S \setminus \{s\}$.

Lemma 2.4. Let $R$ be a semi-local Dedekind domain (hence a principal ideal domain). Then the scheme $S = \text{Spec } R$ satisfies property (T).

Proof. Write $K$ for the field of fractions of $R$, $m_i$ ($1 \leq i \leq n$) for the maximal ideals of $R$, and $p_i$ ($1 \leq i \leq n$) for the characteristic of $R/m_i$. Let $l$ be a prime number which is not divisible by $p_1$. By Chinese Remainder Theorem, we can choose elements $a$ and $b$ of $R$ which satisfy the following conditions:

- $\begin{cases} a \in m_i \ (l \notin m_i) \\ a \equiv 1 \mod m_i \ (l \in m_i) \end{cases}$
- $\begin{cases} b \in m_i \setminus m_1^2 \\ b \equiv 1 \mod m_i \ (m_i \neq m_1) \end{cases}$

Then the extension field of $K$ defined by the polynomial $T^l - aT - b$ satisfies the conditions in Remark 2.3. Therefore, the Dedekind scheme $S$ satisfies property (T).

Definition 2.5. Let $k$ be a field and $C$ a scheme over $k$.

1. We say that $C$ is a smooth curve over $k$ if the structure morphism $C \to \text{Spec } k$ is separated, of finite type, smooth of relative dimension 1, and geometrically connected. Let $C$ be a smooth curve over $k$ and $\overline{C}$ an algebraic closure of $k$. Write $\overline{C}$ for the regular compactification of $C$ over $k$, $g_C$ for the genus of $\overline{C}$, and $r_C$ for the number of closed points of the scheme $(\overline{C} \setminus C) \times_{\text{Spec } k} \text{Spec } \overline{k}$.

2. We say that $C$ is proper rational if $C$ is a smooth curve over $k$ and $g_C = r_C = 0$.

3. We say that $C$ is a hyperbolic curve if $C$ is a smooth curve over $k$, $2g_C + r_C - 2 > 0$, and the reduced closed subscheme $\overline{C} \setminus C$ of $\overline{C}$ is finite étale over $\text{Spec } k$.

Lemma 2.6. Let $k$ be a field and $S$ a smooth curve over $k$. Suppose that $S$ is not proper rational. Then the Dedekind scheme $S$ satisfies property (T).
Proof. We may assume that the field \( k \) is algebraically closed. Take \( s \in S \) and \( l \) as in Definition 2.2.

First, suppose that \( S \) is not proper. Choose a point \( s' \) of \( \overline{S} \setminus S \), where \( \overline{S} \) is the smooth compactification of \( S \) over \( k \). Then there exists a finite dominant morphism \( \overline{S}' \to \overline{S} \) from a proper smooth curve \( \overline{S}' \) over \( k \) to \( \overline{S} \) which is a \( \mathbb{Z}/l\mathbb{Z} \)-étale covering space over \( \overline{S} \setminus \{s, s'\} \) and totally (tamely) ramified over \( s \) and \( s' \). Therefore, \( S \) satisfies property (T).

Next, suppose that \( S \) is proper (and hence the genus of \( S \) is not 0). Then take a nontrivial finite Galois étale covering space \( S' \to S \). Choose two closed points \( s_1 \) and \( s_2 \) of \( S' \) over \( s \). Then there exists a \( \mathbb{Z}/l\mathbb{Z} \)-Galois covering space \( S'' \to S' \) which is finite étale over \( S' \setminus \{s_1, s_2\} \) and totally ramified over \( s_1 \) and \( s_2 \). By Remark 2.3, \( S \) satisfies property (T).

3 Necessary conditions

In this section, we show that the conditions in Theorem 1.12 are necessary for the sequence (2) to be exact in some cases.

Let \( f, X, S, \overline{\pi}, X_\overline{\pi}, \) and \( \overline{\pi} \) be as in Section 1. In this section, we suppose that the morphism \( f \) is generically geometrically reduced (cf. Remark 3.1.1) and the scheme \( X \) is regular (cf. Example 3.7.2). Moreover, suppose that the morphism \( f \) is flat. Since \( f \) is faithfully flat, it follows that \( S \) is also regular.

Remark 3.1. 1. We show that the condition that \( f \) satisfies property (R) is not sufficient for the sequence (2) to be exact. Suppose that \( S \) is a smooth curve over an algebraically closed field \( k \) of positive characteristic and \( f \) satisfies property (R). Note that the sequence of profinite groups (2) is exact by Theorem 1.12. Write \( F : S \to S \) for the absolute Frobenius morphism of \( S \). Then the composite morphism \( F \circ f \) is flat and does not satisfy property (R). For the morphism \( F \circ f \), we can consider a sequence of the étale fundamental groups similar to (2), which is also exact since \( F \) is a universally homeomorphism.

2. Since \( f \) is formally smooth over the generic point of \( S \), there exists a dense open subset of \( S \) such that \( f \) is formally smooth there.

Theorem 3.2. Suppose that there exist a connected normal scheme \( S' \) and a finite dominant morphism \( S' \to S \) which satisfy the following conditions:

- The morphism \( S' \to S \) is étale over the generic point of \( S \).
- Let \( s' \) be a point of \( S' \) whose local ring is of dimension 1. Write \( s \) for the image of \( s' \) in \( S \). Then the extension of discrete valuation rings \( O_{S, s} \subset O_{S', s'} \) is at most tamely ramified with ramification index \( e_{s'} \).
Let $\xi$ be a generic point of the scheme $f^{-1}(s)$ and write $e$ for the multiplicity of $\xi$. Then $e$ is divisible by $e_{s'}$.

Then the normalization $X'$ of the scheme $X \times_S S'$ in its function field is étale over $X$. Moreover, the sequence (2) is not exact.

**Proof.** Note that the scheme $X \times_S S'$ is integral by Lemma 1.7.1. The second assertion follows from the first assertion and Proposition 1.5. To show the first assertion, we may assume that $S(= \text{Spec} \ O_{S,s})$ is the spectrum of a discrete valuation ring by the discussion in the proof of Proposition 1.7. Let $s$ be the closed point of $S$ and $\xi_i (i \in I)$ the generic points of $f^{-1}(s)$. By the Zariski-Nagata purity theorem, it suffices to show that the morphism $X' \to X$ is étale at each $\xi_i$. Write $O_{S,s}^{\text{sh}}$ for the strict henselization of $O_{S,s}$. Since the morphism $\text{Spec} \ O_{S,s}^{\text{sh}} \to S$ is faithfully flat, we may assume that $S = \text{Spec} \ O_{S,s}^{\text{sh}}$. Therefore, we can show Theorem 3.2 by applying Lemma 2.1 to each $O_{X,\xi_i}$.

**Remark 3.3.** We use the same notation in Theorem 3.2. Suppose that the scheme $S$ is quasi-compact (hence Noetherian) and the morphism $f$ is of finite type. Then the set of points $s$ of $S$ satisfying the following properties is finite by Remark 3.1.2:

- $\dim O_{S,s} = 1$.
- There exists a point $s'$ of $S'$ over $s$ satisfying $e_{s'} \neq 1$.

**Corollary 3.4.** Suppose that the following conditions are satisfied:

- $S$ is a Dedekind scheme and satisfies property (T) (cf. Definition 2.2, Lemma 2.4, Lemma 2.6).
- Let $s$ be a closed point of $S$ and $\xi_i (i \in I)$ generic points of the scheme $f^{-1}(s)$. Write $e_i$ for the multiplicity of $\xi_i$, $k(\xi_i)$ (resp. $k(s)$) for the residual field of $\xi$ (resp. $s$), and $p(s)$ for the characteristic of the field $k(s)$. Then $(e_s := \gcd_{i \in I} e_i)$ is not divisible by $p(s)$ and the inclusions $k(s) \hookrightarrow k(\xi_i) (i \in I)$ satisfy property (F).

Then the sequence (2) is exact if and only if $e_s = 1$ for each closed point $s$ of $S$.

**Proof.** Corollary 3.4 follows from Theorem 1.12 and Theorem 3.2.

**Example 3.5.** (cf. Proposition 0.3) We discuss the conditions of Corollary 3.4.

1. Suppose that $S$ is the spectrum of a discrete valuation ring with perfect residual field of characteristic $p$. Then properties (T) and (F) are automatically satisfied (cf. Lemma 2.4). Therefore, we only need to suppose that $e_s$ is not divisible by $p$ to apply Corollary 3.4.
2. Suppose that $S$ is the spectrum of a semi-local Dedekind domain which contains $\mathbb{Q}$. Then all the conditions of Corollary 3.4 are automatically satisfied (cf. Lemma 2.4).

**Proposition 3.6.** (cf. Proposition 0.4) Suppose that $S$ is a smooth curve over a field $k$ of characteristic 0. Moreover, suppose that $S$ is not proper rational. Then the sequence (2) is exact if and only if the greatest common divisor of the multiplicities of the irreducible components of each closed fiber of $f$ is 1.

**Proof.** Proposition 3.6 follows from Lemma 2.6 and Corollary 3.4.

**Example 3.7.** Let $k$ be an algebraically closed field, $C'$ a smooth curve over $k$, and $\sigma$ an automorphism of $C'$ over $k$ of prime order $l (> 2)$. Suppose that $\sigma$ has $n(> 0)$ fixed points $c'_1, \ldots, c'_n$. Write $C' \to C$ for the quotient scheme of $C'$ by $\mathbb{Z}/l \mathbb{Z} = \langle \sigma \rangle$ and $c_i$ for the image of $c'_i$ in $C$ for $1 \leq i \leq n$. Then $\{(c'_i, c'_j) \in C' \times_{\text{Spec } k} C' | 1 \leq i, j \leq n\}$ is the set of fixed points of $C' \times_{\text{Spec } k} C'$ for the action of $\mathbb{Z}/l \mathbb{Z} = \langle (\sigma^2, \sigma) \rangle$. Let $B'$ be the scheme obtained by the blow-ups of $C' \times_{\text{Spec } k} C'$ at all such points. The $\mathbb{Z}/l \mathbb{Z}$-action on $C' \times_{\text{Spec } k} C'$ induces a natural $\mathbb{Z}/l \mathbb{Z}$-action on $B'$. Then the scheme $B'$ has exactly $2n^2$ fixed points. Write $Y'$ for the open subscheme of $B'$ whose complement is the set of the fixed points. Let $Y \to B \to Z$ be the quotient morphisms of the morphisms $Y' \to B' \to C' \times_{\text{Spec } k} C'$ by $\mathbb{Z}/l \mathbb{Z} = \langle (\sigma^2, \sigma) \rangle$. Since $\{(c'_i, c'_j) | 1 \leq i, j \leq n\}$ is the ramified locus of the morphism $C' \times_{\text{Spec } k} C' \to Z$, the scheme $Z$ is not regular by the Zariski-Nagata purity theorem. Note that the morphism $Y' \to Y$ is étale.

1. We show that Theorem 1.12 does not hold in general if the morphism $f$ is not flat and the scheme $S$ is not regular. Consider the case where $f$ is the morphism $Y \to Z$. Note that $f$ is not flat since the dimensions of fibers of $f$ are not constant. Moreover, $S(=Z)$ is not regular. Since $f$ is birational and $Y' \to X(=Y)$ is étale, the normalization of $S(=Z)$ in the separable closure of the function field of $S$ in the function field of $Y'$ coincides with $C' \times_{\text{Spec } k} C'$. Since $C' \times_{\text{Spec } k} C'$ is not étale over $S$, the sequence (2) is not exact by Proposition 1.5.

2. We show that Proposition 3.6 does not hold in general if $X$ is not regular. Suppose that $C' \simeq \text{Spec } k = \text{Spec } k[T]$ and $\sigma$ is induced by the $k$-algebra homomorphism determined by $T \mapsto \zeta_l T$, where $\zeta_l$ is a primitive $l$-th root of unity. Then $n = 1$ and $c'_1 = 0 \in C'$. Note that the second projection $C' \times_{\text{Spec } k} C' \to C'$ is a $\mathbb{Z}/l \mathbb{Z}$-equivariant morphism, which induces a morphism $Z \to C$. Consider the case where $f$ is this morphism $Z \to C$. Then $f^{-1}(c)$ is reduced (resp. irreducible and the multiplicity of its generic point is $l$) if $c \neq c_1$ (resp. $c = c_1$). To see that the sequence (2) is exact, it suffices to show that condition 4
in Proposition 1.5 is satisfied. Let \( X' \) be a connected étale covering
space of \( X (= Z) \). Then the normalization of \( C \) in the function field
of \( X' \) is étale over \( C \setminus \{ c \} \), which is a \( \mathbb{Z}/N\mathbb{Z} \)-Galois étale covering
for some \( N \in \mathbb{N} \) since \( C \setminus \{ c \} \) is isomorphic to \( \mathbb{G}_{m,k} \). By considering
the multiplicities of the fibers of \( f \), we have \( N = 1 \) or \( l \). On the other hand,
since \( X' \) does not factor through \( C' \times_{\text{Spec} k} C' \to X \) by Lemma 1.4.2,
\( l \) does not divide \( N \). Therefore, \( N = 1 \) and condition 4 in Proposition
1.5 is satisfied.

4 An application to morphisms to curves

In this section, we apply Proposition 3.6 to morphisms from smooth varieties
to smooth curves over a field of characteristic 0.

Definition 4.1. ([Ho2] Definition 2.5) We shall write
\[ \mathbb{P}_{\mathcal{J} \to \infty} \]
for the property of a profinite group defined as follows: A profinite group \( G \)
has property \( \mathbb{P}_{\mathcal{J} \to \infty} \) if, for an arbitrary open subgroup \( H \) of \( G \), there exists
a prime number \( l_H \) such that there are no quotient profinite groups of \( H \)
which are free pro-\( l_H \) and not topologically finitely generated.

Let \( k \) be a field of characteristic 0, \( S \) a smooth curve over \( k \), \( X \) a normal,
separated scheme of finite type and geometrically connected over \( k \), and \( f \) a
dominant morphism from \( X \) to \( S \) over \( k \). Write \( N_{X/S} \) for the normalization
of \( S \) in the function field of \( X \) and \( S' \to S \) for the maximal étale subextension
of \( N_{X/S} \to S \). Then we have a natural factorization \( X \to N_{X/S} \to S' \to S \).
Write \( f' \) for the morphism \( X \to S' \) Let \( \eta' \) be a geometric generic point of
\( S' \). Write \( X_{\eta'} \) for the scheme \( X \times_S \eta' \). Take a geometric point \( \overline{x} \) of \( X_{\eta'} \).

Theorem 4.2. (cf. Theorem 0.5) Consider the following conditions:

1. The morphism \( f' \) is surjective and the scheme \( X_{\eta'} \) is connected. Moreover,
   the greatest common divisor of the multiplicities of the irreducible components of each closed fiber
   of \( f' \) is 1.

2. The scheme \( X_{\eta'} \) is connected and the sequence of the étale fundamental
groups
   \[ \pi_1(X_{\eta'}, \overline{x}) \to \pi_1(X, \overline{x}) \to \pi_1(S', \overline{x}) \to 1 \quad (3) \]
is exact.

3. The profinite group \( \text{Ker}(\pi_1(X, \overline{x}) \to \pi_1(S, \overline{x})) \) has property \( \mathbb{P}_{\mathcal{J} \to \infty} \).
Then it holds that $1 \Rightarrow 2 \Rightarrow 3$. If the scheme $S$ is neither a proper rational curve nor an affine line, it holds that $3 \Rightarrow 2$. If the scheme $S$ is not proper rational and $X$ is regular, it holds that $2 \Rightarrow 1$.

**Proof.** The implications between conditions 2 and 3 are results of Hoshi (cf. [Ho2] Theorem 2.8). Note that $X_\overline{\eta}$ is connected if and only if $N_{X/S} = S'$. Then, $1 \Rightarrow 2$ follows from Proposition 3.6.

Suppose that condition 2 is satisfied and the scheme $S$ is not proper rational. Then $S'$ is also not proper rational. Note that, if $f'$ is surjective, condition 1 is satisfied by Proposition 3.6. Hence, to finish the proof of Theorem 4.2 it suffices to show that the morphism $f'$ is surjective. Suppose that $f'$ is not surjective. Since $S'$ satisfies property (T) by Lemma 2.6 there exists a connected étale covering space $X' \rightarrow X$ such that the normalization of $S'$ in the function field of $X'$ is not étale over $S'$. This contradicts the assumption that the sequence (3) is exact by Proposition 1.5 and Remark 1.6.

**Remark 4.3.** By [Ho2] Remark 2.5.1, a topologically finitely generated profinite group satisfies property $P \not\exists \rightarrow \infty$. Suppose that $S$ is a hyperbolic curve over $k$ and $X$ is regular. Since the profinite group $\pi_1(X_\overline{\eta}, \overline{x})$ is topologically finitely generated, the conditions in Theorem 4.2 hold if and only if the profinite group $\text{Ker}(\pi_1(X, \overline{x}) \rightarrow \pi_1(S, \overline{x}))$ is topologically finitely generated.

**Remark 4.4.** If we drop the assumption that the scheme $X$ is regular, the implication $2 \Rightarrow 1$ does not hold in general (cf. Example 3.7.2).

## 5 Appendix 1: property (F)

In this section, we discuss property (F) (cf. Definition 1.9).

### 5.1 Examples

If we drop property (F), Theorem 1.12 does not hold in general.

**Example 5.1.** Let $K$ be a strictly henselian discrete valuation field with imperfect residual field $k$ of characteristic $p > 0$. Write $O_K$ for the valuation ring of $K$. Let $\mathcal{C} \rightarrow \text{Spec} O_K$ be a proper smooth morphism of relative dimension 1 with geometrically connected fibers. Suppose that there exists a $\mathbb{Z}/p\mathbb{Z}$-Galois étale covering space $\mathcal{X} \rightarrow \mathcal{C}$. Choose a generator $\sigma$ of the Galois group $\mathbb{Z}/p\mathbb{Z} \subset \text{Aut}(\mathcal{X})$. Let $K' \supset K$ be a $\mathbb{Z}/p\mathbb{Z}$-Galois extension whose residual extension is purely inseparable of degree $p$. Write $O_{K'}$ (resp. $k'$) for the valuation ring of $K'$ (resp. the residual field of $O_{K'}$). Choose a generator $\tau$ of the Galois group $\mathbb{Z}/p\mathbb{Z} \subset \text{Aut}(\text{Spec} K')$ and consider a $\mathbb{Z}/p\mathbb{Z}$-action on the scheme $\mathcal{X} \times_{\text{Spec} O_K} \text{Spec} O_{K'}$ induced by the automorphism $(\sigma, \tau)$. Then
the second projection \( X \times_{\text{Spec } O_K} \text{Spec } O_{K'} \to \text{Spec } O_{K'} \) is a \( \mathbb{Z}/p\mathbb{Z} \)-equivariant morphism.

\[
\begin{array}{ccc}
X \times_{\text{Spec } O_K} \text{Spec } O_{K'} & \longrightarrow & \text{Spec } O_{K'} \\
\downarrow & & \downarrow \\
X & \longrightarrow & \text{Spec } O_K \\
\end{array}
\]

Write \( \mathcal{Z} \) for the quotient scheme \( (X \times_{\text{Spec } O_K} \text{Spec } O_{K'})/\langle \sigma \times \tau \rangle \). \( \mathcal{Z} \) is a scheme over \( \text{Spec } O_K \) and its special fiber is isomorphic to \( C \times_{\text{Spec } O_K} \text{Spec } k' \) over \( k \). Since the natural morphism \( X \times_{\text{Spec } O_K} \text{Spec } O_{K'} \to \mathcal{Z} \) is finite étale, the scheme \( \mathcal{Z} \) is regular. Note that, in the diagram (4), the left square is Cartesian and the right square is not Cartesian. The normalization of \( \text{Spec } O_K \) in the function field of \( X \times_{\text{Spec } O_K} \text{Spec } O_{K'} \) coincides with \( \text{Spec } O_{K'} \). Therefore, if we consider the case where \( X \to S \) (in Section 1) is \( \mathcal{Z} \to \text{Spec } O_K \), the sequence (2) is not exact by Proposition 1.5. Note that the greatest common divisor of the multiplicities of the irreducible components of the special fibers is 1.

**Remark 5.2.**

1. We do not need to assume that \( C \) is of relative dimension 1 over \( \text{Spec } O_K \) in the discussion given in Example 5.1.
2. If we replace the condition on residual extension of \( K' \supset K \) in Example 5.1 with the condition that the ramification index of the extension \( K' \supset K \) is \( p \), the multiplicity of the (unique) irreducible component of the special fiber is \( p \). Therefore, we need to suppose that the greatest common divisor in property (R) is not divisible by \( p \).

5.2 Generalities on (F)

In this subsection, we discuss generalities on (F). Let \( k \) be a field and \( \iota_i : k \hookrightarrow K_i \ (i \in I) \) inclusions of fields.

**Proposition 5.3.** Write \( k_i \) for the algebraic closure of \( k \) in \( K_i \) and \( \iota'_i : k \hookrightarrow k_i \) for the inclusion induced by \( \iota_i \). Moreover, write \( k_i^{\text{sep}} \) for the (absolute) separable closure of \( k_i \) and \( \iota_i^{\text{sep}} : k \hookrightarrow k_i^{\text{sep}} \) for the inclusion induced by \( \iota'_i \).

The following are equivalent:

1. The inclusions \( \{\iota_i\} \) satisfy property (F).
2. The inclusions \( \{\iota'_i\} \) satisfy property (F).
3. The inclusions \( \{\iota_i^{\text{sep}}\} \) satisfy property (F).

**Proof.** Proposition 5.3 follows from the definition of property (F) and elementary field theory. \( \qed \)
Definition 5.4. We say that the inclusions \( \{ \iota_i \} \) satisfy property (F') if the following condition is satisfied:

(F'): For any subfield \( l \) of the product ring \( \prod_{1 \leq i \leq n} K_i \) which is algebraic over the diagonal subfield \( k \) defined by \( \iota_i \), the extension \( k \subset l \) is separable.

Example 5.5. Property (F) implies property (F'), but property (F') does not imply property (F). Consider the inclusions \( F_p(X_p + Y_p, X_pY_p) \hookrightarrow F_p(X, Y_p) \) and \( F_p(X_p + Y_p, X_pY_p) \hookrightarrow F_p(X + Y, XY) \). These inclusions satisfy property (F'). On the other hand, the field extension \( F_p(X, Y) \subset F_p(X, Y_p) \) is separable and the field \( F_p(X, Y_p) \) contains the field \( F_p(X, Y) \) which is inseparable over the field \( F_p(X_p + Y_p, X_pY_p) \).

Lemma 5.6. Let \( k \subset K' \) be an extension of fields. Write \( k' \) for the algebraic closure of \( k \) in \( K' \), \( k'_s \) (resp. \( k'_n \)) for a(n absolute) separable closure of \( k' \) (resp. an (absolute) algebraic closure of \( k' \)), and \( k_s \) for the separable closure of \( k \) in \( K' \). Moreover, write \( k'_n \) (resp. \( k'_s,n \)) for the normal closure of \( k' \) over \( k \) (resp. the normal closure of \( k'_s \) over \( k \) in \( K' \)). Furthermore, write \( k'_p \) (resp. \( k'_s,p \)) for the field \( k'_s \cap k_p^{-\infty} \) (resp. the field \( k'_s,n \cap k_p^{-\infty} \)). Then the following hold:

1. \( k'_p = k'_s,p \).
2. \( k_s \) and \( k'_s,p \) are linearly disjoint over \( k \) and we have \( k'_s,n = k_s k'_s,p \).

Proof. Note that we have \( k'_s = k'_s k_s \). Since \( k'_n \) is the composite field of the separable closure of \( k \) in \( k'_s \) and \( k'_p \), we have

\[
  k'_s \subset k'_n k_s = k'_p k_s \subset k'_s,n.
\]

Since \( k'_n \) and \( k_s \) are normal over \( k \), we have \( k'_p k_s = k'_s,n \). Since \( k'_p \) and \( k_s \) are linearly independent over \( k \), Lemma 5.6 holds.

Remark 5.7. We obtain an extension field \( k_{i,p} \) of \( k \) for each \( i \in I \) as \( k'_p \) by replacing \( k \subset K' \) in Lemma 5.6 with \( k \subset K_i \). Note that we cannot consider the intersection of \( k_i \) (\( i \in I \)), although we can consider the intersection of \( k_{i,p} \) (\( i \in I \)).

Theorem 5.8. The inclusions \( \{ \iota_i \} \) satisfy property (F) if the intersection of \( k_{i,p} \) (\( i \in I \)) coincides with \( k \).

Proof. This follows from Proposition 5.3 and Lemma 5.6.

Proposition 5.9. Suppose that the degree of extension \( k^p \subset k \) is \( \leq 1 \).

1. Any algebraic extension of \( k \) is a linear disjoint sum of an algebraic separable extension of \( k \) and a purely inseparable extension of \( k \).
2. The inclusions \( \{i \} \) satisfy property (F) if and only if the algebraic closure of \( k \) in each \( k_i \) is separable over \( k \).

**Proof.** Assertion 2 follows from assertion 1. Let \( M \) (resp. \( M_i \); \( M_s \)) be an algebraic extension field of \( K \) (resp. the purely inseparable closure of \( K \) in \( M \); the separable closure of \( K \) in \( M \)). We assume that \( M \) is finite over \( K \) and show that \( [M : M_i] = [M_i : K] \), from which assertion 1 follows. Since we have \( [M^{1/p} : M_s] = [K^{1/p} : K] \leq p \), \( M \) is inseparable over \( K \) if and only if \( K^{1/p} \subset M \). Then \( [M : M_s] = [M_i : K] \) follows from induction on \( [M : M_i] \).

**Example 5.10.** We give some examples of fields \( k \) such that the degree of extension \( k^p \subset k \) is \( \leq 1 \).

1. A perfect field.
2. An extension field of a perfect field with transcendental degree 1.
3. A field of Laurent series over a perfect field.

**6 Appendix 2: geometrically connected fibers**

In this section, we discuss the homotopy exact sequence \(^2\) in the case where the geometric point \( \eta \) is not necessarily over the generic point of \( S \).

Let \( X, S, f, \eta, \) and \( \mathfrak{f} \) be as in Section 1. Consider a geometric (not necessarily generic) point \( \eta' \) of \( S \). Write \( \mathcal{S}^{-}_{\eta'} \) for the strict localization of \( S \) at \( \eta' \) and fix an \( S \)-morphism \( \eta \rightarrow \mathcal{S}^{-}_{\eta'} \).

**Remark 6.1.**

1. If the sequence \(^2\) is exact, the sequence

\[
\pi_1(X \times_S \mathcal{S}^{-}_{\eta'}, \mathfrak{f}) \rightarrow \pi_1(X, \mathfrak{f}) \rightarrow \pi_1(S, \mathfrak{f}) \rightarrow 1 \tag{5}
\]

is exact.

2. Suppose \( f \) satisfies the assumptions other than condition (R) in Theorem \( \text{[1.12]} \) and instead satisfies the following condition (R'):

\((R')\): Let \( s \) be a point of \( S \setminus \text{Im}(\eta) \) whose local ring is of dimension 1. Write \( \xi_i \) (\( i \in I \)) for the generic points of the scheme \( f^{-1}(s) \), \( e_i \) for the multiplicity of \( \xi_i \), and \( k(\xi_i) \) (resp. \( k(s) \)) for the residual field of \( \xi \) (resp. \( s \)). Then \( \gcd_{i \in I} e_i = 1 \) and the inclusions \( k(s) \hookrightarrow k(\xi_i) \) (\( i \in I \)) satisfy property (F).

Then, by using an argument similar to that given in Section \( \text{[1]} \), we can show that the sequence \(^5\) is exact.

3. From Remarks \( \text{[6.11]} \) and \( \text{[6.12]} \), the homomorphism \( \pi_1(X \times_S \eta, \mathfrak{f}) \rightarrow \pi_1(X \times_S \mathcal{S}^{-}_{\eta'}, \mathfrak{f}) \) is not surjective in general.
Take a geometric point $\overline{\eta}$ of $X \times S \overline{\eta}$.

**Remark 6.2.** 1. Suppose that the morphism $X \times S \overline{\eta} \to \tilde{S}_\eta$ is proper and flat. Note that $\text{Spec}(f \times \text{id}_{\tilde{S}_\eta}) \circ O_{X \times S \overline{\eta}}$ is connected and normal and the morphism

$$\pi : \text{Spec}(f \times \text{id}_{\tilde{S}_\eta}) \circ O_{X \times S \overline{\eta}} \to \tilde{S}_\eta$$

is finite. Since $X \times S \overline{\eta}$ is connected, $\pi$ is a universally homeomorphism and therefore the scheme $X \times S \overline{\eta}$ is connected. Then the homomorphism $\iota : \pi_1(X \times S \overline{\eta}, x) \to \pi_1(X \times S \tilde{S}_\eta, x)$ is an isomorphism.

2. $\iota$ in Remark 6.2.1 is neither surjective nor injective in general.

**Corollary 6.3.** Suppose that $f$ is proper and flat (cf. Remark 6.2.1). Then the sequence

$$\pi_1(X \times S \overline{\eta}, x) \to \pi_1(X, x) \to \pi_1(S, x) \to 1 \quad (6)$$

is exact if condition (R') is satisfied.

**Proof.** Corollary 6.3 follows from Remark 6.1.2 and Remark 6.2.1.

Suppose that $X \times S \overline{\eta}$ is connected. We need an ad hoc assumption (cf. the third condition in Proposition 6.4) to make the sequence (6) exact by Remark 6.1.3 and Remark 6.2.2.

**Proposition 6.4.** (cf. [Ho1] Proposition 1.10) Suppose that the following conditions are satisfied:

- The morphism $f$ is flat or the scheme $S$ is regular.
- $f$ satisfies property (R').
- Let $X' \to X$ be a connected finite étale covering space. Write $K_{X'/S}$ for the algebraic separable closure of the function field of $S$ in the function field of $X'$ and $N_{X'/S}$ for the normalization of $S$ in $K_{X'/S}$. For any $S$-morphism $\overline{\eta} \to N_{X'/S}$, the scheme $\overline{\eta} \times N_{X'/S} X'$ is connected.

Then the sequence (6) is exact.

**Proof.** It suffices to show that the implication $4 \Rightarrow 3$ in Proposition 1.5 holds if we replace $\overline{\eta}$ with $\overline{\eta'}$. Let $X'$ be as in condition 3 in Proposition 1.5. Since the number of the connected components of the scheme $X' \times_S X' = \overline{\eta} \times_S X' = (\overline{\eta} \times_S N_{X'/S}) \times_{N_{X'/S}} X'$ coincides with the covering degree of $N_{X'/S} \to S$ by the third condition, the assertion holds.
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