LARGE-SCALE QUASI-GEOSTROPHIC MAGNETOHYDRODYNAMICS

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ABSTRACT

We consider the ideal magnetohydrodynamics (MHD) of a shallow fluid layer on a rapidly rotating planet or star. The presence of a background toroidal magnetic field is assumed, and the “shallow water” beta-plane approximation is used. We derive a single equation for the slow large length scale dynamics. The range of validity of this equation fits the MHD of the lighter fluid at the top of Earth’s outer core. The form of this equation is similar to the quasi-geostrophic (Q-G) equation (for usual ocean or atmosphere), but the parameters are essentially different. Our equation also implies the inverse cascade; but contrary to the usual Q-G situation, the energy cascades to smaller length scales, while the enstrophy cascades to the larger scales. We find the Kolmogorov-type spectrum for the inverse cascade. The spectrum indicates the energy accumulation in larger scales. In addition to the energy and enstrophy, the obtained equation possesses an extra (adiabatic-type) invariant. Its presence implies energy accumulation in the 30° sector around zonal direction. With some special energy input, the extra invariant can lead to the accumulation of energy in zonal magnetic field; this happens if the input of the extra invariant is small, while the energy input is considerable.

Key words: Earth – magnetohydrodynamics (MHD) – stars: rotation – turbulence – waves

1. INTRODUCTION

The behavior of various stars and planets crucially depends on magnetohydrodynamics (MHD) of some shallow fluid layer (fluid shell); examples are Sun’s tachocline (e.g., Hughes et al. 2007) or Earth’s ocean of the core (e.g., Braginsky 2007; the layer of a lighter fluid at the top of the outer core). Such situations can be studied using the system of “shallow water” MHD introduced by Gilman (2000).

The system includes five (scalar) evolution equations

\[ \begin{align*}
\mathbf{V}_t + (\mathbf{V} \cdot \nabla) \mathbf{V} + \mathbf{f} \times \mathbf{V} &= -g \nabla H + (\mathbf{B} \cdot \nabla) \mathbf{B}, \\
H_t + \nabla \cdot (H \mathbf{V}) &= 0, \\
\mathbf{B}_t + (\mathbf{V} \cdot \nabla) \mathbf{B} &= (\mathbf{B} \cdot \nabla) \mathbf{V},
\end{align*} \] (1.1a)

subject to the constraint

\[ \nabla \cdot (H \mathbf{B}) = 0. \] (1.1d)

Equations (1.1) are written in the Cartesian geometry of the \((x, y)\) plane tangent to the fluid shell. The fluid velocity \(\mathbf{V}\), the magnetic field \(\mathbf{B}\), and the fluid depth \(H\) are unknown functions of \(x, y, t\). The magnetic field is normalized to have velocity units. In (1.1), the vectors \(\mathbf{V}\) and \(\mathbf{B}\) have only two non-zero components, in the plane \((x, y)\); the momentum equation (1.1a) includes the Coriolis force with \(\mathbf{f} = [0, 0, f(y)]\); 

g is the gravity constant. Throughout the paper, subscripts \(x, y, t\) denote partial derivatives, while superscripts \(x, y\) denote vector components. The system (1.1) presents significant simplification over the full system of three-dimensional MHD; and at present, the system (1.1) is well established (see, e.g., Schecter et al. 2001; Sterk 2001; Dellar 2003; Zaqarashvili et al. 2007; Heng & Spitkovsky 2009; Umurhan 2013; Karelsey et al. 2013; Zeitlin 2013).

The system (1.1) has steady solution

\[ \mathbf{V} = 0, \quad H = H_0, \quad B_x = B_0, \quad B_y = 0; \] (1.2)

it describes a resting fluid layer of uniform thickness \(H_0\), penetrated by the uniform magnetic field \(B_0\). Considering the \(f\)-plane approximation (when \(f(y) \equiv f_0\) constant), Schecter et al. (2001) found waves on the background (1.2); they were of two types:

1. fast waves—“magnetogravity” branch; they reduce to the gravity (or Poincaré) waves (familiar in geophysical fluid dynamics (GFD); e.g., Vallis 2006) if \(B_0 \to 0\);
2. slow waves—“Alfvén” branch; their frequencies vanish if \(B_0 \to 0\).

Zaqarashvili et al. (2007) considered dynamics (1.1) on the (mid latitudinal) beta-plane, \(f(y) = f_0 + \beta y\), and found that the slow “Alfvén” branch further splits into two sub-branches; if the Alfvén speed \(B_0\) is much smaller than the gravity wave speed \(c_g = \sqrt{gH_0}\) and if the typical length scale \(\mathcal{L}\) is sufficiently large,

\[ \mathcal{L} \gg \ell \equiv \sqrt{B_0/\beta}, \] (1.3)

then one sub-branch represents Rossby waves, slightly modified by the magnetic field, and the other sub-branch represents much slower waves.

In the present paper, we show that in a certain regime (Section 2.3.1), the slowest mode (the slow sub-branch of the slow “Alfvén” branch) obeys a closed nonlinear equation (Equation (2.19) below). The form of this equation is similar to the usual quasi-geostrophic (Q-G) equation (from GFD) or to the Hasegawa–Mima equation (from plasma physics), but the parameters are very different (Section 2). The obtained equation describes the dynamics of interacting waves with dispersion law

\[ \Omega_k = \frac{B_0^2}{\beta} \frac{\rho}{1 + \rho^2 k^2}, \] (1.4)

and \(k = (p, q)\) is the wave vector \((k^2 = p^2 + q^2)\). In this form, the function \(\Omega_k\) differs from the usual Rossby wave dispersion law only by a Doppler shift. The nonlinearity in the obtained equation is the same as in the usual Q-G one.

We see in Section 2.4 that the regime of validity for this equation is realized in the Earth case.
Unlike system (1.1), the obtained equation, in addition to the energy, conserves another positive-definite quadratic quantity, which is naturally called *enstrophy*. This leads to the inverse cascade. However, contrary to the usual Q-G situation, the energy follows the direct cascade (toward smaller scales), while the enstrophy follows the inverse cascade (toward larger scales)—Section 3.1.

Concentrating on the inverse cascade, we use dimensional considerations to find the Kolmogorov-type spectrum

$$E_k \sim (\beta Q)^{1/2} k^{-3/2},$$  

(1.5)

where \(Q\) is the enstrophy flux. The energy integral has infrared divergence on this spectrum, indicating (Zakharov 1985) that the energy accumulates at larger scales (similar to the sea wave turbulence)—Section 3.2.

We further show that the obtained equation has the extra, adiabatic-like, invariant (Section 3.3)

$$I = \int \frac{\eta(k)}{\Omega_k} E_k \, d\k,$$  

(1.6)

where \(E_k\) is the energy spectrum, and

$$\eta(k) \equiv \arctan \frac{q + p\sqrt{3}}{\rho k^2} - \arctan \frac{q - p\sqrt{3}}{\rho k^2}. \quad \text{(1.7)}$$

The presence of this invariant implies the concentration of large-scale energy in the 30° sector around zonal direction (Section 3.4) and can even lead to the formation of zonal magnetic field (Section 3.5).

2. EQUATION FOR LARGE SCALE Q-G MHD

Studying dynamics on the background (Equation (1.2)), we assume

$$V = [v^x, v^y], \quad H = H_0 + h, \quad B = [B_0 + b^x, b^y].$$

2.1. Preliminary Considerations: Dominant Balances

We assume the following three dominant balances in Equations (1.1a)–(1.1c).

*Geostrophic balance.* In the momentum equation (1.1a), the Coriolis force and the pressure gradient (due to the fluid height) balance each other, while dominating all other terms; the next biggest term is assumed to be the one with the background magnetic field

$$f v^x \approx -gh_x + B_0 b^y_x, \quad -f v^y \approx -gh_x + B_0 b^y_x. \quad \text{(2.1a)}$$

The estimates showing the dominance of the geostrophic balance for Earth’s ocean of the core are discussed by Braginsky (1998). Let us note, that in many papers, by “geostrophic balance” people mean the balance in the entire outer core, leading to the Taylor–Proudman columns extending through the bulk of the core; in the present paper, only the balance in the shallow top layer is considered.

In the continuity Equation (1.1b) after substitution of the velocity (Equation (2.1a)), the term with gradient \(\beta \equiv f'(y)\) balances the term with background magnetic field \(B_0\)

$$\frac{\beta g}{f^2} h_x + \frac{B_0}{f}(b^y_x - b^y_x) \approx 0.$$

This equation can be integrated in \(x\); assuming that \(h\) and \(b^y_x - b^y_x\) vanish at \(x = \infty\), we find

$$h \approx \frac{f B_0}{g\beta} (b^y_x - b^y_x). \quad \text{(2.1b)}$$

*Wave balance in the induction Equation (1.1c).* The term with time derivative balances the term with background magnetic field

$$b^y_x \approx B_0 v^x_y, \quad b^y_x \approx B_0 v^y_y. \quad \text{(2.1c)}$$

The wave balance (Equation (2.1c)) implies

$$j_t \approx B_0 \xi_x, \quad \text{(2.2)}$$

where \(j \equiv b^y_x - b^y_x\) and \(\xi \equiv v^y_x - v^y_x\). In Equation (2.2), we express \(j\) in terms of \(h\) using Equation (2.1b) and substitute geostrophic velocity (given by Equation (2.1a)) without the magnetic correction into \(\xi\)

$$h_t \approx \frac{B_0^2}{\beta} \Delta h_x. \quad \text{(2.3)}$$

This equation describes linear waves with dispersion relation

$$\omega \approx \frac{B_0^2}{\beta} pk^2 \left[ h \propto e^{i(px+qy-\omega t)} \right]. \quad \text{(2.4)}$$

Considering corrections beyond the dominant balances, we will find the interaction of these waves and see the energy transfer between waves with different wave vectors. We will also find correction to the dispersion relation (2.3), which allows for the presence of the extra invariant (in addition to the energy and enstrophy).

Let us consider characteristic scales

$$\partial/\partial x, \partial/\partial y \sim 1/L, \quad \partial/\partial t \sim 1/T, \quad h \sim H, \quad v \equiv [v^x, v^y] \sim V, \quad b \equiv [b^x, b^y] \sim B. \quad \text{(2.4)}$$

The three balances (Equation (2.1)) imply, respectively, that

$$f V \sim \frac{gH}{L}, \quad H \sim \frac{f B_0 B}{g\beta L^2}, \quad \frac{B}{T} \sim \frac{B_0 V}{L}. \quad \text{(2.5)}$$

2.2. Momentum and Induction Equations

2.2.1. Quasi-geostrophic Velocity

The two \(x, y\) components of the momentum Equation (1.1a) can be written in the form

$$f v^x = -gh_x + B_0 b^y_x + b \cdot \nabla v^y - v^x \cdot \nabla v^y, \quad f v^y = gh_x - B_0 b^y_x - b \cdot \nabla v^x + v^y \cdot \nabla v^x,$$

$$\sim f_0 V \left\{ 1 + \frac{\beta L}{f_0} \left[ 1 + \frac{B}{B_0} + \frac{\ell^4}{L^4} + \frac{B}{B_0} \frac{\ell^4}{L^4} \right] \right\}. \quad \text{(2.6)}$$

Below these equations, we wrote the magnitudes of the corresponding terms (in the right-hand sides), using the characteristic scales (Equation (2.4)); these magnitudes follow from the dominant balances scaling (Equation (2.5)); \(f_0 \equiv f(0)\). For instance, to estimate the last terms in the right-hand sides of Equations (2.6), we note \(V \sim B_0 B / \beta L^2\) and find

$$\frac{V^2}{L} \sim f_0 V / f_0 L \sim \frac{B_0 B}{f_0 L^2}, \quad \frac{B_0}{B} \frac{\ell^4}{L^4},$$

now recall the definition of \(\ell\) in Equation (1.3).
2.2.2. Magnetic Potential

The constraint (Equation (1.1d)) implies the existence of a function $A(x, y, t)$ such that

$$HB^x = -H_0A_y, \quad H B^y = H_0A_x$$

(here, the constant factor $H_0$ is just for normalization). There is a simple evolution equation for the function $A$

$$A_t + v^x A_x + v^y A_y = 0; \quad (2.7)$$

a similar equation was derived by Gilman (1967) for stratified flow; exactly Equation (2.7) is given (without detailed derivation) by Zeitlin (2013).

Let $A_0(y)$ be the part of the potential $A$ that corresponds to the background (Equation (1.2)): $A = A_0 + a$, and

$$(H_0 + h)(B_0 + b^x) = -H_0(A'_0 + a_y), \quad (H_0 + h)b^y = H_0a_x.$$ \hspace{1cm} (2.2)

So, $A'_0 = -B_0$, and

$$b^x = -a_y - \frac{B_0}{H_0}h + \frac{H_0a_y + B_0h}{H_0(H_0 + h)},$$

$$b^y = a_x - \frac{a_y h}{H_0 + h};$$

$$\sim B + \frac{\beta L}{f_0} \frac{\rho^2}{B} + \frac{\beta L}{f_0} B \frac{\rho^2}{B} + \frac{\beta L}{f_0} \frac{\rho^2}{B}. \quad (2.8)$$

Below the two equations, we wrote the magnitudes of the corresponding terms; to find these, one can note that the magnitude of the potential $a$ is $A = BL$, and

$$\frac{H_0}{H_0} \sim \frac{\beta L}{f_0} \frac{B}{L B} \frac{\rho^2}{B} \frac{\rho^2}{B}.$$ \hspace{1cm} (2.9)

According to Equation (2.7),

$$a_x + v^x a_x + v^y a_y = B_0v^y.$$ \hspace{1cm} (2.10)

2.3. Continuity Equation: Approximation

The continuity Equation (1.1b) can be written in the form

$$h_t + \frac{H_0}{f(F_v)^{\gamma} + (F_v)^{\gamma} - \beta v^y] + (h v^y)]_x + (h v^y)]_y = 0,$$ \hspace{1cm} (2.11)

which will be used to find $v^y$.

2.3.1. Small Parameters: Considered Regime

So far, no approximation has been made. Now we find the fluid velocity by perturbations using the small parameters

$$\frac{B}{B_0}, \quad \frac{\beta L}{f_0}, \quad \frac{\rho^2}{L^2}, \quad \frac{\ell^2}{L^2}.$$ \hspace{1cm} (2.12)

which appear in the asymptotics of the right-hand sides in Equations (2.6) and (2.8). To guarantee the smallness of parameters (Equation (2.11)), we assume weak nonlinearity $B \ll B_0$ and consider regime

$$\ell, \rho \ll L \ll R_0,$$ \hspace{1cm} (2.13)

where $R_0$ is the radius of the spherical fluid shell (the system (1.1) describes dynamics in the plane tangent to this shell). In Earth’s case, $R_0$ is the radius of the core–mantle boundary. The possibility to use the beta-plane approximation is due to the right inequality in Equation (2.12); it also means that $\beta L \ll f_0$ (since at mid latitudes, $\beta \sim f_0/R_0$). The left inequality in Equation (2.12) means that the length scale $L$ exceeds both lengths $\ell$ and $\rho$, defined in Equations (1.3) and (1.4).

Let us suppose, for instance, that all four ratios (Equation (2.11)) scale proportional to the same small parameter $\epsilon \rightarrow 0$. Then according to Equation (2.8),

$$b^x = -a_y + B O(\epsilon^2), \quad b^y = a_x + B O(\epsilon^2)$$ \hspace{1cm} (2.14)

and according to Equation (2.6),

$$f v^x = -g h_y + B_0 b^x + B \cdot \nabla b^y + f_0 \nabla O(\epsilon^3),$$ \hspace{1cm} (2.15)

$$f v^y = g h_x - B_0 b^x - B \cdot \nabla b^y + f_0 \nabla O(\epsilon^3).$$ \hspace{1cm} (2.16)

2.3.2. Fluid Velocity

Now from the continuity Equation (2.10)

$$v^y = \frac{B_0}{\beta} \Delta a_x + \frac{1}{\beta^3} [a, \Delta a] + \frac{f}{H_0} h_t + V O(\epsilon^2)$$ \hspace{1cm} (2.17)

we use the Jacobian notation: $[F, G] = F_x G_x - F_x G_y$ for arbitrary functions $F, G, \Delta a$. Then from Equation (2.14a)

$$\frac{f B_0}{\beta} \Delta a_x = g h_x + f_0 V O(\epsilon),$$ \hspace{1cm} (2.18)

which we integrate in $x$ (assuming $\Delta a$ and $h$ both vanish at $x = \infty$)

$$h = \frac{f B_0}{\beta} \Delta a + V O(\epsilon).$$ \hspace{1cm} (2.19)

We substitute Equation (2.16) into Equation (2.15)

$$v^y = \frac{B_0}{\beta} \Delta a_x + \frac{1}{\beta} [a, \Delta a] + \frac{f^2 B_0 \Delta a}{g H_0 \beta^2} + V O(\epsilon^2)$$ \hspace{1cm} (2.20)

and into Equation (2.14a)

$$v^y = -\frac{B_0}{\beta} \Delta a_x + V O(\epsilon).$$ \hspace{1cm} (2.21)

2.3.3. The Equation for Magnetic Potential

Finally, we substitute the fluid velocity (Equations (2.17) and (2.18)) into Equation (2.9); and neglect higher-order terms

$$a - \rho^2 \Delta a_x = \frac{B_0^2}{\beta} \Delta a_x + 2 \ell^2 [a, \Delta a].$$ \hspace{1cm} (2.22)

Note the factor two in Equation (2.22): Half of the nonlinear term comes from the left-hand side of Equation (2.9), and the other half—from expression (2.17).

The form of Equation (2.22) is very similar to the Q-G potential vorticity equation widely used in GFD (e.g., Vallis, 2006, for usual ocean and atmosphere). Indeed, the Galilean transformation

$$a(x, y, t) = \psi(x - u t, y, t) \quad \text{with} \quad u = \frac{B_0^2}{\beta \rho^2}$$
turns Equation (2.19) into

\[ \langle \psi - \rho^2 \Delta \psi \rangle_i = \frac{B_0^2}{\beta \rho^3} \psi_s + 2 \ell^2 \{ \psi, \Delta \psi \}, \]  

(2.20)

which has the form of the Q-G equation. However, the parameters are essentially different: Instead of the Rossby radius of deformation \( \rho_B \equiv c_s / f_0 \), we have the radius \( \rho \) defined in Equation (1.4) (\( \rho \) is actually inversely proportional to \( \rho_B \)); instead of the usual \( \beta \) parameter, we have \( B_0^2 / \beta \rho^3 \). An equation of the form (2.20) is also known as the Hasegawa–Mima equation in plasma physics (e.g., Diamond et al. 2010), but again, the physical content and parameters are essentially different.

2.4. Estimates

2.4.1. Earth

Although the obtained Equation (2.19) can be applied to different situations (for different rapidly rotating planets and stars), it is interesting to estimate the parameters (Equation (2.11)) for the Earth.

The fine stratification structure of the outer core is not clearly known at present (Hirose et al. 2013). We assume a model stratification, when the liquid outer core consists of deep heavier fluid and on top of it (near the core–mantle boundary) a shell of stratification, when the liquid outer core consists of deep heavier fluid and on top of it (near the core–mantle boundary) a shell of light fluid (the ocean of the core; e.g., Braginsky 2007). The upper boundary of the layer is rigid, and the lower boundary is moving.

To make estimates, we take the shallow layer depth \( H_0 \sim 50 \text{ km} \) and its relative density deficiency (negative relative density excess) \( 10^{-4} \), so that the gravity acceleration \( g_0 \approx 10 \text{ m s}^{-2} \) is reduced to the value \( g \approx 10^{-3} \text{ m s}^{-2} \). This gives the gravity wave speed \( c_g \approx 7 \text{ m s}^{-1} \).

At latitudes about \( 45^\circ \), the Coriolis parameter has value \( f_0 \approx 10^{-4} \text{ s}^{-1} \), and \( \beta \approx 3 \times 10^{-11} \text{ m}^{-1} \text{ s}^{-1} \). We assume the background toroidal magnetic field corresponding to the Alfvén speed \( B_0 \approx 0.3 \text{ m} \). Then \( \ell \approx 100 \text{ km} \) and \( \rho \approx 150 \text{ km} \). Since \( R_0 \approx 3475 \text{ km} \), the condition (2.12) can be well satisfied; for instance, if \( L \approx 400 \text{ km} \), the small parameters (Equation (2.11)) take the values

\[ \frac{\beta L}{f_0} \sim \frac{\rho^2}{\ell^2} \sim 0.1, \quad \frac{c_g}{\ell^2} \sim 0.06; \]

the ratio \( B/\rho \) can be small as well, provided the nonlinearity level is low.

2.4.2. Sun

For the solar tachocline \( f_0 \approx 4 \times 10^{-6} \text{ s}^{-1} \) and \( \beta \approx 8 \times 10^{-15} \text{ m}^{-1} \text{ s}^{-1} \) (at latitudes about \( 45^\circ \)). The tachocline consists of two layers (lower \textit{radiative} and upper \textit{overshoot}) with significantly different parameters.

To evaluate condition (2.12) for both layers, we note that at mid latitudes, \( \beta \sim f_0 / R_0 \). Then

\[ \rho \sim \frac{B_0}{c_g} R_0, \]

and so the condition (2.12) requires the Alfvén speed \( B_0 \) to be much smaller than the gravity speed \( c_g \).

The \textit{overshoot layer} is only slightly stratified; it is prone to many instabilities (Gilman & Cally 2007) and has a strong magnetic field \( \sim 10 T \) corresponding to the Alfvén velocity \( B_0 \sim 600 \text{ m s}^{-1} \). At the same time, the overshoot layer has rather small reduced gravity and small gravity speed \( c_g \lesssim B_0 \) (Schechter et al. (2001) give specific estimates.) Besides, the length \( l = \sqrt{\frac{B_0^2}{\beta}} \approx 3 \times 10^8 \text{ m} \) turns out of the same magnitude as the tachocline radius \( R_0 \approx 5 \times 10^8 \text{ m} \). Thus, the condition (2.12) fails. This is related to the fact that the geostrophic balance for the Sun, i.e., heliostrophic balance, is marginal (Gilman & Dikpati 2014).

The \textit{radiative layer} has stronger stratification and, correspondingly, bigger \( c_g \). The magnetic field in this layer is not clearly known, but one could expect that the field decreases from the overshoot layer toward the radiative interior, becoming smaller by a few orders of magnitude at the lowest levels of the tachocline. So, the lengths \( \ell \) and \( \rho \) turn out much smaller, and the condition (2.12) could be satisfied. Then the large-scale \textit{“shallow water”} MHD of tachocline’s radiative layer could be reduced to a single equation; but the latter should differ from Equation (2.19)—see Section 3.6.

3. ENERGY DISTRIBUTION

3.1. Energy and Enstrophy: Cascades

It is well known in GFD (see, e.g., Vallis 2006) that the Q-G equation (2.20) conserves two positive-definite quadratic integrals: One is the energy

\[ E^\text{GFD} = \int [\psi^2 + (\rho \nabla \psi)^2] \, dx \, dy, \]

and the other is the enstrophy

\[ F^\text{GFD} = \int [(\nabla \psi)^2 + (\rho \Delta \psi)^2] \, dx \, dy. \]

The presence of the second conserved integral (in addition to the energy) implies the inverse energy cascade (toward larger scales), while the enstrophy follows the direct cascade (toward smaller scales).

Equation (2.19) differs from the Q-G equation only by a Doppler shift, and correspondingly, also conserves two integrals and also implies two cascades. However, the energy now is

\[ E = \int [(\nabla a)^2 + (\rho \Delta a)^2] \, dx \, dy, \]

(similar to the GFD enstrophy \( F^\text{GFD} \)); the integral (3.1a) is indeed the energy, because it corresponds to the energy of the original \textit{“shallow water”} MHD (Equation (1.1)). We can call the other conserved integral the “enstrophy”

\[ F = \int [a^2 + (\rho \nabla a)^2] \, dx \, dy, \]

and we can say—comparing (2.19) to the usual Q-G equation—that the energy and the enstrophy switch.

We will be interested in cascades in the Fourier space, and so, consider Equation (2.19) in the Fourier representation. To shorten notations, for any wave vector \( \mathbf{k} \), we keep only its label \( j \); e.g., for the Fourier transform

\[ a_\mathbf{k}(t) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int a(x, y, t) e^{i(\mathbf{k} \cdot \mathbf{x})} \, dx \, dy \quad [\mathbf{k} = (p, q)], \]

we have \( a_{\mathbf{k}_j} \equiv a_j, (j = 1, 2, 3) \); also, \(-j\) stands for \(-\mathbf{k}_j\). In the Fourier representation, the evolution of a Fourier harmonic with some wave vector, say \( \mathbf{k}_1 \), is determined by the interaction
with all other Fourier harmonics, and Equation (2.19) takes the form
\[ \dot{a}_1 = -\Omega_1 a_1 + \int W_{-1,2,3} a_2 a_3 \, dk_2 \, dk_3, \]  
(3.2)
where the dispersion law \( \Omega_k \) is given in Equation (1.4), and the coupling kernel is
\[ W_{1,2,3} \equiv W(k_1, k_2, k_3) = \ell^2 \frac{k_3^2 - k_2^2}{1 + \rho^2 k_4^2} (p_2 q_3 - p_3 q_2) \delta(k_1 + k_2 + k_3). \]  
(3.3)
Equation (3.2) conserves the energy and enstrophy
\[ E = \int k^2 (1 + \rho^2 k^2) |a_k|^2 \, dk, \]  
(3.4a)
\[ F = \int (1 + \rho^2 k^2) |a_k|^2 \, dk, \]  
(3.4b)
corresponding to the integrals (3.1a) and (21b). If \( E_k \) is the energy spectrum, then \( F_k = E_k / k^2 \) is the enstrophy spectrum. This implies the direct cascade of energy and the inverse cascade of enstrophy (contrary to the usual Q-G situation).

### 3.2. Kolmogorov-type Spectrum for the Inverse Cascade

Let us use dimensional considerations to find the turbulence spectrum for the inverse cascade in the dynamics (Equation (2.19)). Dimensional considerations alone are insufficient, since Equation (2.19) has dimensional parameters. However, we can supplement dimensional considerations by other arguments in the following two ways.

#### 3.2.1. Rescaling the Dynamical Equation

First of all, let us note that the term with \( \rho \) in Equation (2.19) is small in the considered regime (Equation (2.12)), and so, can be neglected for dimensional considerations. Now, we re-scale the dependent variable \( \tilde{a} = \ell^2 a \), so that Equation (2.19) is reduced to the equation with only one dimensional parameter
\[ \tilde{a}_t = \frac{\ell^3}{\beta} \tilde{a}^\lambda + 2 [\tilde{a}, \tilde{\Delta} \tilde{a}]; \]
herewith the dimensions of space and time are unaffected. The corresponding energy spectrum \( \tilde{E}_k = \ell^4 E_k \), and the corresponding enstrophy flux \( \tilde{Q} = \ell^4 Q \). Now we assume
\[ \tilde{E}_k = \left( \frac{B_0}{\beta} \right)^\lambda \tilde{Q}^\mu k^\nu \]
with undetermined exponents \( \lambda, \mu, \nu \). The turbulence spectrum evolves due to the third-order cumulant, which in weakly nonlinear situations has magnitude proportional to the product of two turbulence spectra. Therefore, \( \tilde{Q} \propto \tilde{E}^2 \), and \( \mu = 1/2 \). We have the following dimensions:
- the energy spectrum \( E_k \propto m^3/s^2 \),
- the enstrophy spectrum \( F_k \propto m^5/s^2 \),
- the energy flux \( P \propto m^2/s^3 \),
- the enstrophy flux \( Q \propto m^4/s^3 \).

So, the dimensions of \( \tilde{E}_k \) and \( \tilde{Q} \) are \( m^7/s^2 \) and \( m^8/s^3 \) respectively. Now using dimensional considerations, we determine \( \lambda = 1/2, \nu = -3/2 \) and, returning to the original variables, find the Kolmogorov-type spectrum (Equation (1.5)). Interestingly, the spectrum (Equation (1.5)) is independent of \( B_0 \).

#### 3.2.2. Using the Scaling of the Wave Kinetic Equation

We will arrive at the same spectrum (Equation (1.5)) when dimensional considerations are supplemented by the scaling implied by the wave kinetic equation (similar to the dimensional estimates made by Zakharov 1985).

Using the well studied (Kenyon 1964; Longuet-Higgins & Gill 1967; Reznik 1984; Monin & Piterbarg 1987) kinetic equation for the Q-G turbulence (of Rossby waves) we have (switching the energy and enstrophy spectra) the wave kinetic equation for our Equation (2.19),
\[ \frac{dF_k}{dt} = \int W_{123} (W_{123} F_2 F_3 + W_{231} F_1 + W_{312} F_1 F_2) \]
\[ \times \delta(k_1 + k_2 + k_3) \delta(\Omega_1 + \Omega_2 + \Omega_3) \, dk_2 \, dk_3, \]
(3.5)
We do not care here about numerical factors (like \( 2\pi \) in front of the integral), since we will only use this equation for dimensional estimates. As well, we can neglect \( \rho^2 k^2 \ll 1 \) (due to the condition (2.12)) in the denominator of the integral, and in \( \Omega_k \). The inverse cascade corresponds to the enstrophy flux \( Q \); the quantity \( -Q \) is the enstrophy flux out of large sphere \( |k| \leq K \) of big radius \( K \to \infty \), and so, \( -Q \) is the integral of \( dF/d\Omega \) over this sphere. We also assume that the turbulence (corresponding to the spectrum that will be obtained) is local, i.e., the integral in the kinetic equation converges on this spectrum. Then the dimensional considerations suggest
\[ Q \sim W^2 F^2 k^{-2} \left( \frac{B_0^2}{\beta} k^3 \right)^{-1} (dk)^3 \sim \ell^4 \beta B_0^2 F^2 k^3, \]
from which we find
\[ F_k \sim (\beta Q)^{1/2} k^{-9/2} \]
and the energy spectrum (Equation (1.5)).

If we try to integrate the Kolmogorov-type spectrum (Equation (1.5)) to obtain the total energy, we see that the energy integral diverges at small \( k \). Thus, the dimensional considerations indicate (Zakharov 1985) that the large scales contain most of the energy and this suggests the following picture. Most of the energy follows direct cascade and dissipates at large \( k \); but a little fraction of the supplied energy has to transfer to small \( k \) (since enstrophy takes energy); and the energy piles up there. A similar situation takes place for the sea wave turbulence, where there are direct cascade of energy and inverse cascade of wave action, but the large scales contain most of the energy.

#### 3.3. Extra Invariant

A dispersion law \( \omega_k \) is said to be degenerative (Zakharov 1985) if there exists a function \( \phi_k \) such that the equations of resonance triad interactions
\[ k_1 + k_2 + k_3 = 0 \quad \text{and} \quad \omega_1 + \omega_2 + \omega_3 = 0 \]  
(3.5)
imply the equation
\[ \phi_1 + \phi_2 + \phi_3 = 0 \]  
(3.6)
(recall the notational conventions, \( \omega_j \equiv \omega_k, \phi_j \equiv \phi_k \), \( j = 1, 2, 3 \)). The function \( \phi_k \) is supposed to be linearly independent of \( k \) and \( \omega_k \), so that Equation (3.6) is not a mere linear combination of Equation (3.5).
The Rossby wave dispersion law

\[ \omega(k) = \frac{p}{1 + \rho^2k^2} \]  

(3.7)

turned out to be degenerate (Balk et al. 1991; Balk 1991). The dispersion law (Equation (1.4)) is a linear combination of function \( p \) and function \( \eta(k) \), and so, the system (Equation (3.5))—with dispersion law (Equation (3.7))—is equivalent to the system

\[ k_1 + k_2 + k_3 = 0, \quad \Omega_1 + \Omega_2 + \Omega_3 = 0, \]

with \( \Omega_k \) given in Equation (1.4). Therefore, the dispersion law (Equation (1.4)) is also degenerative with the same \( \phi_k \), which is equal to the function \( \eta(k) \), given by the expression (1.7).

It is important that the ratio \( \eta(k)/\Omega_k \) is not “too” singular so that the integral (1.6) is actually a real physical quantity which can be determined in experiments (exact condition is discussed in Balk & Yoshikawa 2009).

Thus, the weakly nonlinear dynamics (Equation (2.19)) possesses the extra invariant (Equation (1.6)), in addition to the energy and the enstrophy.

The conservation of the extra invariant (Equation (1.6)) is adiabatic-like. The quantity \( I \) is conserved approximately over long (nonlinear) time. (The extra invariant and its significance for GFD are reviewed in Balk 2014.)

3.4. Inverse Cascade

We will now show that for the dynamics (Equation (2.19)), the presence of the extra invariant implies the energy accumulation in the sector \( 60^\circ < |\theta| < 90^\circ \) (where \( \theta \) is the polar angle of the wave vector \( k \)).

A similar fact was derived (Balk 2005) for the turbulence of Rossby waves, but the reasoning here is different. This is because the energy cascade has switched direction.

When \( p \to 0 \), the function (1.7) behaves like \( \Omega_k/k^2 \) (up to a constant factor), and so, we linearly combine \( I \) and \( F \) to eliminate the common asymptotics

\[ J = I - \frac{2\sqrt{3}\rho\beta}{B_0^2} F = \int \varphi(k) E_k \, dk, \quad \text{where} \]

\[ \varphi(k) = \frac{\eta(k)}{\Omega_k} - \frac{2\sqrt{3}\rho\beta}{B_0^2} k^2. \]  

(3.8)

Obviously, the integral \( J \) is also an invariant of dynamics (Equation (2.19)), just like the integral \( I \); from now on, we will deal with \( J \) instead of \( I \). \( \varphi(k) = O(p^2) \), \( p \to 0 \), even though each of the two expressions in the difference (Equation (3.8)) does not approach zero as \( p \to 0 \). Figure 1 shows the function \( \varphi(k) \).

Asymptotically for \( k \to 0 \) (up to a constant factor \( \beta/B_0^2 \)),

\[ \varphi(k) \sim \begin{cases} \pi/pk^2, & |\theta| < 60^\circ, \\ 8\sqrt{3}p^2/(q^2 - 3p^2k^2), & 60^\circ < |\theta| < 90^\circ; \end{cases} \]  

(3.9)

so, \( \varphi \sim k^{-3} \) in the sector \( |\theta| < 60^\circ \), and \( \varphi \sim \rho k^{-2} \) in the sector \( 60^\circ < |\theta| < 90^\circ \). Expression (3.9) gives a simple approximation for function (3.8), away from the lines \( q = \pm\sqrt{3}p \).

Let us consider a simple model situation, when the energy is generated at some scale \( k_0 \) at a rate \( \mathcal{E}_0 \) and dissipated at some scales \( k_1 \) and \( k_2 \) at rates \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \), respectively. We assume \( k_1 \ll k_0 \ll k_2 \), so that both direct and inverse cascades are realized. The conservation of the energy and the enstrophy gives

\[ \mathcal{E}_0 = \mathcal{E}_1 + \mathcal{E}_2, \quad \frac{1}{k_0^2} \mathcal{E}_0 = \frac{1}{k_1^2} \mathcal{E}_1 + \frac{1}{k_2^2} \mathcal{E}_2, \]

from which we have

\[ \mathcal{E}_1 = \frac{1}{k_0^2} - \frac{1}{k_1^2} - \frac{1}{k_2^2} \mathcal{E}_0 \approx \left( \frac{k_1}{k_0} \right)^2 \mathcal{E}_0 \ll \mathcal{E}_0, \]  

(3.10)

so that indeed a small fraction of the generated energy is transferred toward the origin (most of the energy follows the direct cascade toward large \( k \).
We will now arrive at a contradiction if the energy $\mathcal{E}_1$ ends up in the sector $60^\circ < |\theta| < 90^\circ$. Indeed, the dissipation of energy $\mathcal{E}_1$ in this sector is accompanied by dissipation of the extra invariant $k_1^{-3} \mathcal{E}_1$; the latter cannot exceed the total amount of the extra invariant generated at scale $k_0$: $k_0^{-3} \mathcal{E}_0$ (if some part of $\mathcal{E}_0$ was generated in the sector $60^\circ < |\theta| < 90^\circ$, then the generated amount of the extra invariant is even smaller). Thus, $k_1^{-3} \mathcal{E}_1 < k_0^{-3} \mathcal{E}_0$, which upon substitution of $\mathcal{E}_1$ in terms of $\mathcal{E}_0$ from Equation (3.10) leads to contradiction $k_1 > k_0$. The contradiction still remains if only a significant part (not the entire amount) of $\mathcal{E}_1$ ends up in the sector $60^\circ < |\theta| < 90^\circ$, but a longer inertial interval might be required.

3.5. Zonal Magnetic Field

We will now see that for some energy input at the scale $k_0$, the region of energy accumulation is more narrow. The energy accumulates in the vicinity of the $q$ axis. Such accumulation means the emergence of zonal magnetic field (of alternating directions: parallel and anti-parallel to the background magnetic field), rather than the emergence of zonal flow (alternating zonal jets). This is because the function $a$ in Equation (2.19) is the magnetic potential, rather than the stream function.

Suppose that the energy input at the scale $k_0$ consists of the energy source in the sector $60^\circ < |\theta| < 90^\circ$, and the energy sink in the sector $|\theta| < 60^\circ$ (see Figure 2), in such a way that the input of the extra invariant is small, while the energy input is significant. Such input is possible because the extra invariant is anisotropic.

To illustrate this situation, let us continue with the above model situation, assuming that the energy is generated at some $k_*$ at a rate $\mathcal{E}_*$ (in the sector $60^\circ < |\theta| < 90^\circ$), but dissipated at some $k_-$ at a rate $\mathcal{E}_-$ (in the sector $|\theta| < 60^\circ$). When $|k_*| \sim |k_-| \sim k_0$, the dissipation at scales $k_1$ and $k_2$ is still present; see Figure 2. The energy input is $\mathcal{E}_0 = \mathcal{E}_* - \mathcal{E}_-$, and the extra invariant input is $\psi_* \mathcal{E}_* - \psi_- \mathcal{E}_-$. The function $\psi(k)$ has much bigger values in the sector $|\theta| < 60^\circ$ than in the sector $60^\circ < |\theta| < 90^\circ$. So, if $\mathcal{E}_* \gg \mathcal{E}_-$, the energy input will be significant, while the extra invariant input can be small:

$$\frac{\psi_* \mathcal{E}_* - \psi_- \mathcal{E}_-}{\psi_* \mathcal{E}_* + \psi_- \mathcal{E}_-} \ll 1.$$

When the energy (and the enstrophy) dissipate at the scale $k_1$, only a small amount of the extra invariant $J$ can be dissipated (less than the total supplied amount of $J$ at scale $k_0$). This is only possible if the dissipation occurs near the $q$ axis (where the function $\psi(k)$ has small magnitude). Thus, the accumulated energy must squeeze near the $q$ axis, which corresponds to zonal magnetic field.

Zonal fields can severely restrict the turbulent transport. This is similar to the situation in fusion plasmas (e.g., Fujisawa 2009).

3.6. Remarks on the Solar Tachocline

Spiegel & Zahn (1992) suggest that the tachocline remains thin because of sufficiently strong horizontal turbulent transport in the radiative zone (this is reviewed, e.g., by Miesch 2005; Priest 2014). However, drawing the analogy with the well known horizontal dynamics of Earth’s atmosphere and oceans, Gough & McIntyre (1998) note that the horizontal transport can be essentially inefficient; e.g., the interaction of Rossby waves can lead to the formation of zonal flow, which significantly limits the horizontal mixing. Section 3.5 seems to give one more reason for smaller horizontal transport, in addition to the purely hydrodynamic reasons (arising according to the above-mentioned analogy). We will now see that this is not so in the solar regime.

Indeed, when deriving Equation (2.19), we assumed that all small parameters (Equation (2.11)) scale in the same way, proportional to the same small parameter $\epsilon \to 0$. This implies that the length scale $L$ should significantly exceed not only $\ell$ and $\rho$ but also the Rossby radius of deformation:

$$\rho_R \equiv \frac{c_s}{f_0} = \frac{\ell^2}{\rho} \Rightarrow \frac{\rho_R^2}{L^2} = \frac{\ell^4/L^4}{\rho^2/L^2} \sim \epsilon. \quad (3.11)$$

In Sun’s case, $\rho_R$ turns out to be greater than (or at least comparable with) the tachocline radius $R_0$, so that the condition (3.11) cannot be satisfied.

It seems possible, in the case of tachocline’s radiative layer, to reduce the large-scale “shallow water” MHD to a single equation; but the scaling of the small parameters (Equation (2.11)) should be different. For instance, the scaling

$$\frac{\rho_R^2}{L^2} \sim \frac{\ell^4}{L^4} \sim \epsilon \quad (3.12)$$

would allow $\rho_R$ to be larger than $L$.

The scaling (Equation (3.12)) brings into consideration the terms with time derivatives in Equation (2.6). Then the dispersion law would be more general

$$\tilde{\Omega}_k = \frac{B_0^2}{\beta} \frac{p k^2}{1 + \rho^2 k^2 + \ell^4 k^4}. \quad (3.13)$$

It is interesting that in both limits of long and short waves, this dispersion law is degenerate and admits an extra invariant. Indeed,

$$\tilde{\Omega}_k^{\text{long}} = \frac{B_0^2}{\beta} \frac{p k^2}{1 + \rho^2 k^2}; \quad \tilde{\Omega}_k^{\text{short}} = \frac{B_0^2}{\beta} \frac{p}{\rho^2 + \ell^4 k^2}$$

are both Rossby dispersion laws (Doppler shifted in the first case). However, it is unlikely that the general dispersion law (Equation (3.13)) is degenerative.

The extra conservation takes place in the generalized situation only if the $\ell$ term is dominated by the $\rho$ term:

$$\frac{\ell^4}{L^4} \ll \frac{\rho^2}{L^2}$$

(so that the approximate conservation of the extra invariant remains within the same bound). For the solar tachocline, the opposite limiting condition holds. Then the extra invariant is absent, and the arguments of Sections 3.4 and 3.5 no longer apply.

Tobias et al. (2007) performed direct numerical simulations of the two-dimensional MHD on the $\beta$ plane. They came to the conclusion that “in the absence of magnetic fields, nonlinear interactions of (the usual) Rossby waves lead to the formation of strong mean zonal flows; but the addition of even a very weak toroidal field suppresses the generation of mean flows.” A similar conclusion was reached by Tobias et al. (2011) for the MHD on a rotating spherical surface. We should note that there is no contradiction between their conclusion and the present paper: they consider the two-dimensional MHD, when $\rho = 0$.

The derivation of a single equation for tachocline’s radiative layer merits further investigation and will be published elsewhere, along with the proof that the dispersion law (3.13) is non-degenerate.
4. CONCLUSION

We have derived a single equation (Equation (2.19)) for the slow large length scale MHD of a shallow fluid layer on the beta-plane of a rapidly rotating planet or star. This equation (after Doppler shift) has the form of the Q-G equation (familiar in GFD) or the Hasegawa–Mima equation (familiar in plasma physics), but the physical content and the parameters are very different; see Section 2. The validity regime of this equation fits Earth’s ocean of the core, a layer of lighter fluid at the core–mantle boundary; see Section 2.4.

The derived equation implies two cascades: the direct energy cascade and the inverse enstrophy cascade (contrary to the situation in the usual Q-G equation); see Section 3.1. Using the dimensional considerations, we have found the Kolmogorov-type spectrum for the inverse cascade. This spectrum indicates that larger scales contain most of the energy (Section 3.2) in agreement with experimental observations; this indicates that larger scale contains most of the energy. This spectrum is feasible, since the extra invariant is essentially anisotropic.

Equation (2.19) possesses an extra invariant (in addition to the energy and enstrophy); see Section 3.3. Its presence implies the energy accumulation in the sector $60^\circ < |\theta| < 90^\circ$; see Section 3.4. The extra invariant can also imply the emergence of zonal magnetic field if the energy input includes the energy source in the sector $60^\circ < |\theta| < 90^\circ$ and its sink in the sector $|\theta| < 60^\circ$; see Section 3.5.

Zonal flows take place in magnetized plasmas; they improve the particles and heat confinement in tokamaks (e.g., Fujisawa 2009). Unlike Earth’s outer core, in plasmas, we have some control over the forcing and dissipation. The similarity of Equation (2.19) to the Hasegawa–Mima equation suggests a way to generate zonal flows in plasmas. To set up a strong zonal flow, one needs to supplement the energy generation (at scale $k_0$) by its dissipation in the sector $|\theta| < 60^\circ$ (see Figure 2); one should supply significant energy amount while supplying small amount (as little as possible) of the extra invariant. Such input is feasible, since the extra invariant is essentially anisotropic. It is interesting that the inclusion of the dissipation, far away from the $q$ axis, can lead to very focused zonal jets, when the energy is accumulated very tightly around the $q$ axis (which corresponds to zonal flow). It is unclear how small the input of the extra invariant can be achieved, since we only control increment/decrement, but not the actual generation/dissipation.

I plan numerical simulations of the Hasegawa–Mima equation and similar Equation (2.19) to investigate how the inclusion of dissipation into the energy input affects the formation of zonal flows and magnetic fields.

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