Modulus of continuity for superlacunar trigonometric series and continuity of Gaussian stationary random processes.

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Abstract

We deduce bilateral interrelations between Fourier coefficients for lacunar trigonometric series and modulus of their continuity. We obtain also as an application some conditions for continuity and discontinuity for Gaussian periodic stationary random centered processes.

Key words and phrases. Lacunar and superlacunar trigonometric series and functions, necessary condition and sufficient one, modulus of continuity, Fourier coefficients, ordinary and complex functions, Gaussian random variable and stationary random processes, convergence, estimation.

1 Introduction. Lacunar and superlacunar trigonometric series.

Let \( \bar{n} = \{n(k)\}, \ k = \pm 1, \pm 2, \pm 3, \ldots \) be a set of integer numbers, positive as well as negative, such that \( n(1) \geq 0, \ n(-1) \leq 0, \)
\[ n(k+1) \geq n(k) + 1, \ k \geq 1; \ n(k-1) \leq n(k) - 1, \ k < 0. \quad (1) \]

Let also \( \vec{c} = \{c(k)\}, \ k \in \vec{n} \) be a numerical absolute summable numerical sequence

\[ \sum_{k \in \vec{n}} |c(k)| < \infty. \quad (2) \]

Define the following \( 2\pi \) periodical continuous function by means of absolute convergent Fourier series

\[ f(t) = f[\vec{n}, \vec{c}](t) \overset{\text{def}}{=} \sum_{k \in \vec{n}} c(k) \exp(i \ t \ n(k)). \ t \in [-\pi, \pi]. \quad (3) \]

Of course,

\[ c(k) = (2\pi)^{-1} \int_{-\pi}^{\pi} \exp(-i \ t \ n(k)) \ f(t) \ dt. \]

Recall that the series of the form (3) are named lacunar, iff

\[ \lim_{k \to \infty} \left[ \frac{n(k+1)}{n(k)} \right] > 1. \quad (4) \]

and

\[ \lim_{k \to \infty} \left[ \frac{|n(-k-1)|}{|n(-k)|} \right] > 1. \quad (5) \]

**Definition 1.1.** The series of the form (3) are named superlacunar, iff

\[ \lim_{k \to \infty} \left[ \frac{n(k+1)}{n(k)} \right] = \infty \quad (6) \]

and in addition

\[ \lim_{k \to \infty} \left[ \frac{|n(-k-1)|}{|n(-k)|} \right] = \infty, \quad (7) \]

see e.g. [3], chapter 5; [17], chapter 3.

Our claim in this report is to establish the interrelations between modulus of continuity of the superlacunar function and behavior of its Fourier coefficients.

We consider as an application the continuity properties of Gaussian periodic stationary random processes.

As an example: the famous Weierstrass function.
2 Bilateral inequalities. Examples.

Recall that the modulus of continuity $\omega[f](\delta)$, $\delta \in [0, 2\pi]$ for the continuous $2\pi$ periodical numerical valued function $f = f(t)$ is defined as follows

$$\omega[f](\delta) \overset{\text{def}}{=} \sup_t \sup_{k:|h| \leq \delta} |f(t + h) - f(t)|.$$  

(8)

It is well-known [17], chapters 2,3 that for the function in particular of the form (3)

$$|c(k)| \leq (4\pi)^{-1} \omega[f](\pi/n(|k|)), \quad |k| \geq 1.$$  

(9)

Conversely, let the function $f = f(t)$, $t \in R$ has the form (3), such that (we recall) $\vec{c} \in l_1$. Let also $t, s \in T$, $|t - s| < h$, $h \in [0, 2\pi]$. We have

$$|f(t) - f(s)| \leq h ||\vec{c}||_{l_1} \sum_{k:|k| \leq N} |k| n(|k|) +$$

$$2 \sum_{k:|k| > N} |c(k)| = h\Sigma_1[c, n](N) + \Sigma_2[c](N),$$

where

$$||\vec{c}||_{l_1} = \sum_k |c(k)| < \infty, \quad \Sigma_1[c, n](N) = ||\vec{c}||_{l_1} \cdot \sum_{k:|k| \leq N} |k| n(|k|),$$

$$\Sigma_2[c](N) = 2 \sum_{k:|k| > N} |c(k)|; \quad N \geq 2.$$  

To summarize common with (9):

**Proposition 2.1.** Denote

$$\Sigma[c, \vec{n}](\delta) \overset{\text{def}}{=} \inf_{N=2,3,4,\ldots} \left[ \delta \cdot \Sigma_1[c, n](N) + \Sigma_2[c](N) \right].$$

We get under formulated above notations and conditions

$$\omega[f](\delta) \leq \Sigma[c, \vec{n}](\delta), \quad \delta \in [0, 2\pi).$$

(10)

**Remark 2.1.** Obviously,

$$\lim_{\delta \to 0^+} \Sigma[c, \vec{n}](\delta) = 0.$$  

Let us consider some examples.
Example 2.1. Let us put

\[ c(k) = k^{-2\Delta}, \quad k \geq 1; \quad c(k) = 0, \quad k \leq 0; \quad n(k) \overset{\text{def}}{=} 2^k, \quad \Delta = \text{const} > 1/2, \]

the lacunar case. On the other words, we consider the following complex valued continuous function

\[ g(t) = g_{\Delta}(t) := \sum_{k=1}^{\infty} k^{-2\Delta} \exp \left( 2^k i \cdot t \right). \tag{11} \]

It follows from (9)

\[ \omega[g](\delta) \geq C_1 | \ln \delta|^{-2\Delta}, \quad 0 < \delta < 1/e, \quad 0 < C_1 < \infty, \]

a lower bound. The correspondent upper bound may be found on the basis of proposition 2.1:

\[ \omega[g](\delta) \leq C_2 | \ln \delta|^{1-2\Delta}, \quad 0 < \delta < 1/e, \quad 0 < C_2 < \infty. \]

Example 2.2. Let us introduce now

\[ c(k) = k^{-2\Delta}, \quad k \geq 1; \quad c(k) = 0, \quad k \leq 0; \quad n(k) \overset{\text{def}}{=} 2^k, \quad \Delta = \text{const} > 1/2, \]

the superlacunar case. We consider the following complex valued continuous function

\[ s(t) = s[\Delta](t) := \sum_{k=1}^{\infty} k^{-2\Delta} \exp \left( i \cdot 2^k \cdot t \right). \tag{12} \]

It follows from (9) as before

\[ \omega[s](\delta) \geq C_3 \left[ | \ln | \ln \delta| \right]^{-2\Delta}, \quad 0 < \delta < e^{-e}, \quad 0 < C_3 < \infty, \]

a lower bound. The correspondent upper bound may be found alike on the basis of proposition 2.1:

\[ \omega[s](\delta) \leq C_4 \left[ | \ln | \ln \delta| \right]^{1-2\Delta}, \quad 0 < \delta < e^{-e}, \quad 0 < C_4 < \infty. \]

3 Gaussian periodic stationary random processes.

Let \( \eta(t), \ t \in T := [-\pi, \pi] \) be centered (mean zero) stochastic continuous real valued separable Gaussian distributed random process (r.p.) having continuous covariation function
\[ R(t, s) := \mathbb{E}\eta(t)\eta(s), \ t, s \in T. \]

The problem of finding the conditions (necessary conditions and sufficient ones) for the continuity with probability one of the r.p. \( \eta(\cdot) \)

\[ \mathbb{P}(\eta(\cdot) \in C(T)) = 1 \] (13)

can be considered as a classic, see e.g. [1], [2], [4], [5], [6], [7], [8], [9] - [14], [15] etc. We mention here the famous result belonging to X.Fernique [5]: if the following integral convergent

\[ \int_0^{\infty} \omega^{1/2}[R(\exp(-x^2/2))] \ dx < \infty, \] (14)

then the relation (13) holds true. So, the condition (14) is sufficient for the continuity a.e. of the Gaussian r.p. \( \eta(t) \). Open question: what happens if the condition (14) is not satisfied?

We will show in this report that if the condition (14) is not satisfied, i.e. when the covariation function of the Gaussian r.p. may be arbitrarily non-smooth (as rough as you like), then the correspondent stationary Gaussian r.p. may be continuous or conversely may be extremely discontinuous, i.e. may be unbounded on every non-empty interval.

We consider only the case of stationary and periodical centered Gaussian r.p.

To be more concrete, introduce the following Gaussian distributed stationary periodical process of the form (super-lacunar series)

\[ \nu(t) = \nu_\Delta(t) \overset{\text{def}}{=} \sum_{|k| \geq 1} |k|^{-\Delta} \kappa_k \exp \left( i \cdot \text{sign}(k) \cdot 2^{2|k|} \right), \] (15)

\[ \Delta = \text{const} > 0, \ t \in \mathbb{R}, \]

and \( \{\kappa_k\} \) are independent centered real valued standard Gaussian distributed random variables: \( \mathbb{E}\kappa_k^2 = 1. \)

The covariation function \( r(t) = \mathbb{E}\xi(t+s)\overline{\xi}(s) \) of the stationary r.p. \( \nu(t) \) has a form (superlacunar series)

\[ r(t) = 2 \sum_{k=1}^{\infty} k^{-2\Delta} \cos \left( 2^{2k} t \right), \ t \in \mathbb{R}. \] (16)

If \( \Delta \leq 1/2 \), then the function \( r = r(t) \) is extremely discontinuous. If \( \Delta \leq 1 \), then the r.p. \( \nu = \nu(t) \) is also extremely discontinuous a.e., [1].
The behavior of the function $r = r(t)$ in the case when $\Delta > 1$ is obtained in the example 2.2. So, it is very rough, despite the correspondent r.p. $\nu(t)$ is continuous.

Moreover, the covariation function $r(t)$ for continuous Gaussian stationary centered random process may be as rough as you like, for instance when

$$r(t) = 2 \sum_{k=1}^{\infty} k^{-2\Delta} \cos(n(k) t), \ t \in R, \ \Delta = \text{const} > 1,$$

where as before $\{n(k)\}$ is very fast increasing integer sequence, see (9).

4 Concluding remarks.

An open question: let $\omega = \omega(\delta), \ \delta \in [0, 1/e]$ be arbitrary non-trivial modulus of continuity. There exists or not a stationary Gaussian random process $\zeta = \zeta(t), \ t \in R$ with continuous sample path for which its covariation function $r = r[\zeta](t)$ obeys a property

$$\omega[r](\delta) \simeq \omega(\delta), \ \delta \in [0, 1/e].$$

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