Abstract

The natural generalization of the notion of bundle in quantum geometry is that of bimodule. If the base space has quantum group symmetries one is particularly interested in bimodules covariant (equivariant) under these symmetries. Most attention has so far been focused on the case with maximal symmetry – where the base space is a quantum group and the bimodules are bicovariant. The structure of bicovariant bimodules is well understood through their correspondence with crossed modules.

We investigate the “next best” case – where the base space is a quantum homogeneous space and the bimodules are covariant. We present a structure theorem that resembles the one for bicovariant bimodules. Thus, there is a correspondence between covariant bimodules and a new kind of “crossed” modules which we define. The latter are attached to the pair of quantum groups which defines the quantum homogeneous space.

We apply our structure theorem to differential calculi on quantum homogeneous spaces and discuss a related notion of induced differential calculus.

1 Preliminaries

We start by introducing notation and reviewing some relevant definitions. Thus, coproduct, counit and antipode of a Hopf algebra are denoted \( \Delta, \epsilon, S \) respectively. We use Sweedler’s notation (with implicit summation) \( \Delta h = h_{(1)} \otimes h_{(2)} \) for the coproduct. A similar notation serves for left coactions.

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Let $H$ be a Hopf algebra. We denote the category of left $H$-modules by $\mathcal{M}_H$, the category of left $H$-comodules by $\mathcal{M}_H^*$, and correspondingly for the right hand side versions. Furthermore, for modules that carry several (co)actions which mutually commute we use the obvious notation for the category. (Such modules are also called Hopf modules.) E.g., for left $H$-module right $H$-comodules such that both structures are compatible we would write $\mathcal{M}_H^H$. A module with a compatible comodule structure is also called a covariant module. Compatible left and right (co)module structures are called bi(co)module.

We consider a second type of module which is called crossed module (or Yetter-Drinfeld module). Let $H$ be a Hopf algebra. A right crossed $H$-module $V$ is a right $H$-module and right $H$-comodule such that the following condition holds:

$$v_{(1)} \triangleright h_{(1)} \otimes v_{(2)} h_{(2)} = (v \triangleright h_{(2)})_{(1)} \otimes h_{(1)} (v \triangleright h_{(2)})_{(2)} \quad \forall h \in H, v \in V$$

We denote the category of such modules by $\mathcal{M}_H^H$. There is also a corresponding left handed version.

The structure theorem for bicovariant bimodules (playing the role of bicovariant bundles over a quantum group) can be formulated as follows (in its right handed version). This result is implicit to some extent in [1]. A complete formulation was given in [2].

**Theorem 1.1.** Let $H$ be a Hopf algebra. The categories $\mathcal{M}_H^H$ and $\mathcal{M}_H^H$ are equivalent.

The equivalence is given in one direction by the functor $\mathcal{M}_H^H \rightarrow \mathcal{M}_H^H$ defined by $E \mapsto \mathcal{M}_H^H := \{e \in E : e_{(1)} \otimes e_{(2)} = 1 \otimes e\}$. $H E$ inherits the right $H$-comodule structure from $E$ and is equipped with the new right $H$-module structure $\tilde{e} \tilde{h} := S h_{(1)} \triangleright e \triangleright h_{(2)}$. Conversely, the inverse functor $\mathcal{M}_H^H \rightarrow \mathcal{M}_H^H$ is given by $X \mapsto H \otimes X$. Here, the left module and comodule structure of $H \otimes X$ are the regular ones of $H$ while the right structures are the tensor product ones.

We will be interested in quantum homogeneous spaces defined as follows.

**Definition 1.2.** Let $\pi : P \rightarrow H$ be a surjection of Hopf algebras. Then the left $P$-comodule algebra $B := P^H = \{p \in P : p_{(1)} \otimes \pi(p_{(2)}) = p \otimes 1\}$ is called a quantum homogeneous space.

The triple $(P, B, H)$ is said to satisfy the Hopf-Galois property if the map $\chi : P \otimes_B P \rightarrow P \otimes H$ given by $\chi = (\cdot \otimes \pi) \circ (\text{id} \otimes \Delta)$ is injective (in addition to being surjective).

Note that the Hopf-Galois condition is automatically satisfied if $H$ is cosemisimple (and thus also has invertible antipode). This follows from [3].
(Apply Remark 3.3.(2) to the integral and use Remark 3.3.(1) in Theorem 3.5.)

A bundle structure of prime importance in differential geometry is the (co)tangent bundle. A noncommutative generalization of this notion (together with the exterior derivative of functions) is captured by the notion of differential calculus given as follows.

**Definition 1.3.** Let $B$ be an algebra. A differential calculus $\Omega$ over $B$ is a $B$-bimodule with a linear map $d : B \to \Omega$ such that (a) the Leibniz rule $d(ab) = ad(b) + d(a)b$ is satisfied and (b) the map $B \otimes B \to \Omega : a \otimes b \mapsto adb$ is surjective.

A basic result about differential calculi is the following (see e.g. [1]).

**Proposition 1.4.** Let $B$ be an algebra. The universal differential calculus over $B$ is given by $\tilde{B} := \ker \cdot \subset B \otimes B$ with left and right $B$-module structures given by multiplication of the left respectively right component. The exterior derivative $d : B \to \tilde{B}$ is given by $db = 1 \otimes b - b \otimes 1$. Any differential calculus over $B$ can be identified with a quotient of $\tilde{B}$ by a subbimodule.

If the base space $B$ has extra symmetries as in the case of a quantum homogeneous space or even a quantum group it is natural to demand these symmetries also from the differential calculus (as in the commutative situation). This leads to the obvious notions of covariant or bicovariant differential calculus. Proposition 1.4 remains valid if quotient is understood to mean quotient by a subbimodule which is (bi)covariant.

## 2 Induced Differential Calculi – Motivation

In this section we construct a differential calculus on a quantum homogeneous space from a given one on the symmetry quantum group. In fact, this is nothing but the construction of the cotangent bundle on a homogeneous space from the cotangent bundle on the symmetry group – but formulated in a way that generalizes to the noncommutative case. We use a notation that is intended to remind the reader of the differential geometric origin.

Recall that (by application of the Structure Theorem [1]) a bicovariant differential calculus over a quantum group $\mathcal{C}(G)$ is given by a bicovariant bimodule $\Gamma(T^*G) = \mathcal{C}(G) \otimes T^*_G (\text{classically the space of sections of the cotangent bundle over the Lie group } G)$ [1]. $T^*_G$ is the right crossed $\mathcal{C}(G)$-module of left-invariant 1-forms, which corresponds classically to the cotangent space at the identity of $G$. $T^*_G$ is a quotient $\mathcal{C}(G)^+ / I$ of $\mathcal{C}(G)^+ := \ker \epsilon \subset \mathcal{C}(G)$ as a right crossed $\mathcal{C}(G)$-module via the right regular action and right adjoint coaction. The exterior derivative $d : \mathcal{C}(G) \to \mathcal{C}(G) \otimes T^*_G$ is determined by $f \mapsto f_{(1)} \otimes f_{(2)} - f \otimes 1$. In the classical case $I = (\ker \epsilon)^2$. Then $I$ is the annihilator of $g \subset U(g)$ (with $g$ the Lie algebra of $G$) in the pairing of $\mathcal{C}(G)$ with $U(g)$ and thus $\mathcal{C}(G)^+ / I \cong g^*$. 

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Proposition 2.1 (Induced differentials on homogeneous spaces). Let \( \pi : C(G) \to C(H) \) be a surjection of Hopf algebras with \( C(M) := C(G)^{C(H)} \).
Let \( \Gamma(T^*G) = C(G) \otimes T^*_e G \) be a bicovariant differential calculus on \( C(G) \).
We obtain a corresponding differential calculus on the homogeneous space \( C(M) \) in two steps.
First, we restrict the cotangent space at each point to those forms that are annihilated by the vector fields generated by the right translations of \( H \).
Thus, we define
\[
T^*_e M := C(M) \cap (C(G)^+ / I) = C(M)^+ / (C(M)^+ \cap I)
\]
with \( C(M)^+ := \ker \epsilon \subset C(M) \). While \( T^*_e M \) does not carry a right \( C(G) \) coaction anymore it does inherit from \( T^*_e G \) the induced right coaction of \( C(H) \). Furthermore, it carries a right \( C(M) \) action, the restriction of the right \( C(G) \) action on \( T^*_e G \).
Now, the second step consists in restricting the so formed “bundle over \( G \)” to a “bundle over \( M \)”. This is accomplished by going to the \( C(H) \)-invariant subspace
\[
\Gamma(T^*M) := (C(G) \otimes T^*_e M)^{C(H)}.
\]
This is now a left \( C(G) \)-covariant \( C(M) \)-bimodule. \( d : C(G) \to \Gamma(T^*G) \) descends to a map \( d : C(M) \to \Gamma(T^*M) \). The classical case recovers the usual differential calculus on \( M \).

Proof. The induced right adjoint coaction of \( C(H) \) on \( C(G) \) is closed on the subspace \( C(M) \):
\[
a \mapsto a_{(2)} \otimes \pi(Sa_{(1)}a_{(3)}) = a_{(2)} \otimes \pi(Sa_{(1)}) \in C(M) \otimes C(H)
\]
for \( a \in C(M) \).
That \( d \) descends follows for step 1 from \( \Delta C(M) \subseteq C(G) \otimes C(M) \) and for step 2 from the right \( C(H) \)-invariance of \( \Delta C(M) \):
\[
a_{(1)} \otimes a_{(2)} \mapsto a_{(1)} \otimes a_{(4)} \otimes \pi(a_{(2)}) \pi(Sa_{(3)}a_{(5)}) = a_{(1)} \otimes a_{(2)} \otimes 1
\]
for \( a \in C(M) \).

3 Structure Theorem

The structure found for induced differential calculi in the previous section naturally leads to the question whether this structure is generic. We present here our main result, a structure theorem for covariant bimodules (i.e., covariant bundles) over quantum homogeneous spaces in analogy to Theorem 1.1. This answers the question in the affirmative. Finally, we reapply the structure theorem to differential calculi.
Definition 3.1. Let $P \to H$ be a surjection of Hopf algebras, $B := P^H$. Then a right $B$-module and right $H$-comodule $X$ is called crossed iff
\[ x_{(1)} \triangleleft b \otimes x_{(2)} = (x \triangleleft b_{(2)})_{(1)} \otimes \pi(b_{(1)})(x \triangleleft b_{(2)})_{(2)} \]
for all $x \in X, b \in B$. We denote the category of such objects by $\mathcal{M}^H_B$.

Note that if $H$ has invertible antipode the right $H$-coaction can be converted to a left $H$-coaction $x \mapsto S^{-1} x_{(2)} \otimes x_{(1)}$ and $\mathcal{M}^H_B \cong ^H \mathcal{M}_B$.

Theorem 3.2. Let $P \to H$ be a surjection of Hopf algebras, $B := P^H$. Then, the categories $\mathcal{M}^H_B$ and $^P \mathcal{M}^H_B$ are equivalent.

Proof. For $X \in \mathcal{M}^H_B$ consider the tensor product $P \otimes X$. Equip it with the left (co)module structures of $P$ and the right (co)module structures of the tensor product. One checks that this makes $P \otimes X$ an object in $^P \mathcal{M}^H_B$. For a morphism $f : X \to X'$ in $\mathcal{M}^H_B$ the map $\text{id} \otimes f : P \otimes X \to P \otimes X'$ is a morphism in $^P \mathcal{M}^H_B$. This defines a functor $\mathcal{M}^H_B \to ^P \mathcal{M}^H_B$.

For $Y \in ^P \mathcal{M}^H_B$ consider its left $P$-invariant subspace $^P Y$. We equip it with the new right action of $B$ via $y \triangleright b := S b_{(1)} \triangleright y \triangleleft b_{(2)}$. This makes it with the inherited right coaction of $H$ an object in $\mathcal{M}^H_B$. A morphism $g : Y \to Y'$ in $^P \mathcal{M}^H_B$ induces a morphism $\tilde{g} : ^P Y \to ^P Y'$ in $\mathcal{M}^H_B$ by restriction. This defines a functor $^P \mathcal{M}^H_B \to \mathcal{M}^H_B$.

Finally, we check that the two functors are mutually inverse. While clearly $^P (P \otimes X) \cong X$ the isomorphism $Y \cong P \otimes ^P Y$ is given by $y \mapsto y_{(1)} \otimes S y_{(2)} \triangleright y_{(3)}$ with inverse $p \otimes y \mapsto p \triangleright y$. To check the inverseness for morphisms is straightforward and left to the reader. \qed

Theorem 3.3. Let $P \to H$ be a surjection of Hopf algebras, $B := P^H$. Then, there are functors $\mathcal{F} : ^P \mathcal{M}_B \to ^P \mathcal{M}^H_B$ and $\mathcal{G} : ^P \mathcal{M}^H_B \to ^P \mathcal{M}_B$ such that $\mathcal{G} \circ \mathcal{F}$ is the identity.

If furthermore $H$ has invertible antipode and $(P, B, H)$ is Hopf-Galois, then also $\mathcal{F} \circ \mathcal{G}$ is the identity and the categories are thus equivalent.

Proof. For $E \in ^P \mathcal{M}_B$ consider the tensor product $P \otimes E$. We equip it with the left $P$-comodule structure as a tensor product, the left $P$-module and right $H$-comodule structure of $P$ and the right $B$-module structure of $E$. These structures descend to the quotient $P \otimes_B E$ and make it an object in $^P \mathcal{M}^H_B$. A morphism $h : E \to E'$ in $^P \mathcal{M}_B$ defines a map $\text{id} \otimes h : P \otimes E \to P \otimes E'$ which induces a morphism $h : P \otimes_B E \to P \otimes_B E'$ in $^P \mathcal{M}^H_B$. This defines the functor $\mathcal{F} : ^P \mathcal{M}_B \to ^P \mathcal{M}^H_B$.

Given $Y \in ^P \mathcal{M}^H_B$ consider the right $H$-invariant subspace $Y^H$. The left $P$-comodule and right $B$-module structures descend while the left $P$-module structure only survives as a left $B$-module structure. Thus, $Y^H \in ^P \mathcal{M}_B$. A morphism $g : Y \to Y'$ in $^P \mathcal{M}^H_B$ clearly gives rise to a morphism $\tilde{g} : Y^H \to Y'^H$ in $^P \mathcal{M}_B$ by restriction. This defines the functor $\mathcal{G} : ^P \mathcal{M}^H_B \to ^P \mathcal{M}_B$. 

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Next, we check that $G \circ F = \text{id}$. Starting with $E$ we obtain the corresponding object $F(E) = P \otimes_B E$ and $(G \circ F)(E) = (P \otimes_B E)^H$. But this is $B \otimes_B E$ (as taking invariant subspace and quotient of the tensor product commute by construction) which in turn is canonically isomorphic to $E$. To check $G \circ F = \text{id}$ on morphisms is straightforward and left to the reader.

We assume now further that $H$ has invertible antipode and that $(P, B, H)$ is Hopf-Galois. For $X \in \mathcal{M}^H_B$ consider the map $\chi \otimes \text{id} : P \otimes_B P \otimes X \to P \otimes H \otimes X$. We define a right coaction of $H$ on both sides as the tensor product one on $P \otimes X$ and $H \otimes X$ respectively. (This definition behaves well with respect to the tensor product $\otimes_B$.) It commutes with $\chi \otimes \text{id}$ which thus restricts to a map on the invariant subspaces under this coaction $\tilde{\chi} : P \otimes_B (P \otimes X)^H \to P \otimes (H \otimes X)^H$. As $(P, B, H)$ is Hopf-Galois, $\chi$ is a bijection and so are $\chi \otimes \text{id}$ and $\tilde{\chi}$.

Now the map $(H \otimes X)^H \to X$ given by $\epsilon \otimes \text{id}$ is a bijection since the antipode of $H$ is invertible. Its inverse is given by $x \mapsto S^{-1} x_{(2)} \otimes x_{(1)}$. Thus, we obtain bijections $P \otimes_B (P \otimes X)^H \to P \otimes (H \otimes X)^H \to P \otimes X$. Using Theorem 3.2 this gives rise to a bijection $(F \circ G)(Y) = P \otimes_B Y^H \to Y$ for any $Y \in \mathcal{M}^H_B$. One easily checks that this is an isomorphism with respect to the relevant (co)module structures. Thus, $F \circ G$ is the identity on objects. To check that it is the identity on maps is now straightforward and left to the reader.

**Proposition 3.4.** Let $\pi : P \to H$ be a surjection of Hopf algebras, $B := P^H$. Let $B^+ := \ker \epsilon \subset B$. Then, each left $P$-covariant differential calculus on $B$ corresponds to a crossed submodule $I \subset B^+ \in \mathcal{M}^H_B$ via the right regular action and the coaction $b \mapsto b_{(2)} \otimes \pi(Sb_{(1)})$.

If furthermore $H$ has invertible antipode and $(P, B, H)$ is Hopf-Galois, then the correspondence is one-to-one.

**Proof.** We use the fact that any differential calculus is a quotient of the universal one (Proposition 1.4) and apply the correspondences of the previous theorems. On $B$ the universal calculus is given by the subspace $\ker \epsilon \subset B \otimes B$ with $d : B \to B \otimes B$ defined by $b \mapsto 1 \otimes b - b \otimes 1$. For simplicity we start by considering the whole space $B \otimes B$. It is a left $P$-covariant $B$-bimodule (i.e. an object in $P^B_M_B$) by the tensor product coaction and the left and right regular actions of $B$ on the left and right component respectively. With Theorem 3.3 we obtain $P \otimes_B (B \otimes B) \cong P \otimes B$ as an object in $P^B_M_B$. This in turn corresponds to $P(P \otimes B)$ as an object in $\mathcal{M}^H_B$ according to Theorem 3.2. This in turn we can identify with $B$ via the map $p \otimes b \mapsto \epsilon(p)b$ and its inverse $b \mapsto Sb_{(1)} \otimes b_{(2)}$. The induced module structure on $B$ is the right regular action while the right $H$-comodule structure is $b \mapsto b_{(2)} \otimes \pi(Sb_{(1)})$.

By applying the inverse functors of Theorems 3.3 and 3.2 to this $B$ we obtain an isomorphism $B \otimes B \to (P \otimes B)^H$ in $P^B_M_B$ given by $b \otimes c \mapsto bc_{(1)} \otimes c_{(2)}$ with inverse $p \otimes b \mapsto pSb_{(1)} \otimes b_{(2)}$. 


Now the subspace \( \ker \cdot \subset B \otimes B \) on the left hand side corresponds to \((P \otimes B^+)^H\) on the right hand side. The differential map \( d : B \to (P \otimes B^+)^H \) is \( b \mapsto b^{(1)} \otimes b^{(2)} - b \otimes 1 \).

Since left \( P \)-covariant differential calculi on \( B \) correspond to quotients of \( \ker \cdot \subset B \otimes B \) in \( \mathcal{P}_B \mathcal{M}_B \), by the above correspondence these in turn correspond to quotients (and thus crossed submodules \( I \)) of \( B^+ \) in \( \mathcal{M}^H_B \).

In general each differential calculus corresponds to a certain such crossed submodule (as the composition \( \mathcal{G} \circ \mathcal{F} = \text{id} \) in Theorem 3.3). If the additional condition of invertibility of the antipode of \( H \) and the Hopf-Galois for \((P, B, H)\) is satisfied the converse is also true, giving rise to a one-to-one correspondence (as also \( \mathcal{F} \circ \mathcal{G} = \text{id} \) in Theorem 3.3).

Note that a result similar to Proposition 3.4 for the more general case where \( B \) is a coideal subalgebra of \( P \) was found recently by direct means [4].

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