Quantum evolution of scalar fields
in Robertson-Walker space-time

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Abstract

We study the $\lambda \phi^4$ field theory in a flat Robertson-Walker space-time using
the functional Schrödinger picture. We introduce a simple Gaussian approxi-
mation to analyze the time evolution of pure states and we establish the renor-
malizability of the approximation. We also show that the energy-momentum
tensor in this approximation is finite once we consider the usual mass and
coupling constant renormalizations.
I. INTRODUCTION

Establishing the validity of various cosmological scenarios requires understanding the dynamical evolution of the cosmos in time, e. g., the time evolution of the inflation-driving field \[1\] before, during and after inflation, and the details of symmetry changing phase transitions that may have given rise to cosmic strings \[2\]. The conventional formulation of quantum field theory in terms of causal Green’s functions in the Heisenberg picture is not especially suited to time-dependent problems that make use of an initial condition for specific solution. Green’s functions contain all information needed for determining transition rates, \(S\)-matrix elements, etc., of systems in equilibrium where initial data are superfluous. However, following a system’s time evolution from a definite initial configuration is more efficiently accomplished in a Schrödinger picture description, where the initial data consist of specifying a pure or mixed state.

For bosonic fields, the field-theoretic functional Schrödinger picture \[3\] is a generalization from ordinary quantum mechanics to the infinite number of degrees of freedom that constitute a field. Therefore, it allows the use of the mathematical/physical intuition acquired in quantum mechanics to analyze field-theoretic problems. Notwithstanding, the functional Schrödinger picture is not as widely used in actual calculations as the Green’s function method since renormalization is more easily carried out in the latter framework. However, it has been established renormalizability for of the Schrödinger picture for both static \[4\] and time-dependent \[5\] cases.

In the Schrödinger picture, a pure state is given by a single wave functional \(\Psi(\phi)\) of a \(c\)-number field \(\phi(x)\) at fixed time, while a pure state is described by a functional density matrix \(\rho(\phi_1, \phi_2) = \sum_n p_n \Psi_n(\phi_1) \Psi_n^*(\phi_2)\), where \(\{\Psi_n\}\) is a complete set of wave functionals and \(p_n\) is the probability that the system is in the state \(n\) \[6\]. The time evolution of pure and mixed is governed by the time-dependent Schrödinger equation and the Liouville-von Neumann equation, respectively. In general these equations cannot be solved, except for systems whose Hamiltonian is quadratic, and consequently approximation methods are
Variational approximations were developed in Refs. [6–8] which lead to tractable equations in the cases of Gaussian trial states. The Gaussian approximation leads to self-consistent equations that, unlike perturbation theory, reflect some of the non-linearities of the full quantum theory.

In various inflationary models, the dynamics of the early universe is dictated by the time evolution of an scalar field, usually called inflaton. The equation of motion used in most of the analyses is of the form

\[ \ddot{\varphi} + \frac{3}{a} \dot{\varphi} + V'_{\text{eff}} = 0, \]

where \( V_{\text{eff}} \) is the static, finite-temperature effective potential. However, the static effective potential does not properly take into account effects of non-equilibrium dynamics. Moreover, semiclassical [9] or linearized approximations [10] have been frequently made, and the full non-linearity of an interacting theory is lost. It is conceivable that a more complete analysis may change the picture of the early universe drawn from these approximations [11,12].

In this work we study a self-interacting scalar quantum field model in a flat Robertson-Walker space-time in the functional Schrödinger picture. Our purpose is to obtain a set of quantum dynamical equations describing the evolution of a scalar field as well as the scale factor. In order to obtain a small set of tractable equations, which capture some of the non-linearities of the full problem, we employ a variational method using a Gaussian trial state whose kernels depend on just a few parameters. We analyze the renormalization of the equations of motion for the variational parameters to assure the consistency of them. Moreover, we also study the renormalization of the expectation value of the energy-momentum tensor, which is the source for the semi-classical Einstein equation, showing that it is finite once we take into account the usual mass and coupling constant renormalizations. Therefore, we obtain a set of equations that can be used to address various cosmological questions like the dynamics of chaotic inflation.

This article is organized as follows: In Sec. II we present the functional Schrödinger picture for a scalar field and obtain the Gaussian approximation for the ground state of the
\( \lambda \Phi^4 \) model, which will be used to define our simplified Gaussian approximation. Section III contains the Ansatz that we used in our calculations as well as the study of the renormalization of the equations of motion for the variational parameters. We present the analyses of the renormalization of the energy-momentum tensor in Sec. IV and, in Sec. V we draw our conclusions.

II. FUNCTIONAL SCHRÖDINGER PICTURE

A. Generalities

Here we present the basic facts about the functional Schrödinger picture, and the interested reader can find explicit examples and learn more about it in Refs. [3][6][11]. In the field-theoretic Schrödinger picture, pure states are described by wave functionals \( \Psi(\phi) \) of a c-number \( \phi(x) \) at a fixed time. The inner product is defined by functional integration

\[
\langle \Psi_1 | \Psi_2 \rangle \Rightarrow \int D\phi \, \bar{\Psi}_1(\phi)\Psi_2(\phi) ,
\]

while operators are represented by functional kernels.

\[
|\Omega \rangle \Rightarrow \int D\phi' \, O(\phi, \phi')\Psi(\phi')
\]

We adopt a diagonal kernel \( \Phi(x) \Rightarrow \phi(x)\delta(\phi - \phi') \) for the canonical field operator \( \Phi(x) \) at fixed time, while the canonical commutation relations determine the canonical momentum kernel to be \( \Pi(x) \Rightarrow \frac{1}{i}[\delta/\delta\phi(x)]\delta(\phi - \phi') \). Hence \( \Phi \) acts by multiplication on functionals of \( \phi \) and \( \Pi \) acts by functional differentiation. In this way, the action of any operator constructed from \( \Pi \) and \( \Phi \) is

\[
O(\Pi, \Phi) |\Psi \rangle \Rightarrow O \left[ \frac{1}{i} \delta / \delta \phi, \phi \right] \Psi(\phi) .
\]

The fundamental equation for initial value problems is the time-dependent Schrödinger equation for the time-dependent state functional \( \Psi(\phi; t) \). This equation takes definite form, once a Hamiltonian operator \( H(\Pi, \Phi) \) is specified:
The initial value problem is completely defined once we also supply the initial wave functional.

\[ i \frac{\partial}{\partial t} \Psi(\phi; t) = H \left[ \frac{1}{i} \delta \frac{\delta}{\delta \phi}, \phi \right] \Psi(\phi; t). \]  

(5)

B. Dirac’s Variational Principle

The time-dependent Schrödinger equation (5) cannot be directly integrated, unless the system is described by a quadratic Hamiltonian. Therefore, we shall employ a variational approximation to study non-linear (interacting) systems. Applications of variational principles with restricted variational Ansatz, in the Rayleigh-Ritz manner, result in tractable self-consistent dynamical equations for the parameters used in the Ansatz, which still retain some of the non-linearity of the complete problem.

For pure states, the time-dependent Schrödinger equation can be obtained through Dirac’s variational principle [13]. First we define the effective action Γ as the time integral of the diagonal matrix element of \( i \frac{\partial}{\partial t} - H \):

\[ \Gamma = \int dt \left\langle \Psi \left| \frac{i}{i} \frac{\partial}{\partial t} \right. - H \left| \Psi \right. \right\rangle, \]  

(6)

and then we demand that Γ be stationary against arbitrary variations of \(|\Psi\rangle\) and \(\langle\Psi|\), imposing appropriate boundary conditions.

It is possible to implement Dirac’s principle in two steps in such way that Γ can be associated to the functional generator of the one-particle irreducible Green’s functions with arbitrary energy and momentum [7]. In order to do so, we consider the time integral of an off-diagonal matrix element

\[ \Gamma = \int dt \left\langle \Psi_- \left| \frac{i}{i} \frac{\partial}{\partial t} - H \right| \Psi_+ \right\rangle, \]  

(7)

subject to the constraint that the matrix element of the field \( \Phi(x) \) is held fixed at a prescribed function \( \varphi(x, t) \).
\[ \langle \Psi_- | \Phi(x) | \Psi_+ \rangle = \varphi(x, t) \]  
\[ \langle \Psi_- | \Psi_+ \rangle = 1 \]  
\[ (8) \]

These constraints are supplemented by the boundary condition that these states tend to the ground state of \( H \) for \( t \to \pm \infty \) The physical theory is recovered when we remove the constraints by solving

\[ \frac{\delta \Gamma}{\delta \varphi(x, t)} = 0 . \]  
\[ (10) \]

C. Gaussian Ansatz

Dirac's variational principle can be used to obtain approximations provided we restrict the variation of the trial wave functional \( \Psi \) to a subspace of the full Hilbert space. In this work, we shall use Gaussian trial states, whose most general expression is

\[ \Psi(\phi, t) = N(t) \exp \left\{ i \int \hat{\pi}(x, t) [\phi(x) - \varphi(x, t)] \right\} \times \exp \left\{ -\int_{x,y} [\phi(x) - \varphi(x, t)] \left[ \frac{1}{4} \Omega^{-1}(x, y, t) - i \Sigma(x, y, t) \right] [\phi(y) - \varphi(y, t)] \right\} , \]  
\[ (11) \]

where the variational parameters are \( \varphi, \hat{\pi}, \Omega, \) and \( \Sigma \) and we abbreviated the integral in \( d \) spatial dimensions as \( f_x \equiv \int d^d x \). The physical meaning of the parameters of this wave functional can be inferred from linear and bilinear averages. The linear averages are given by

\[ \langle \Phi(x) \rangle = \varphi(x, t) , \]  
\[ (12) \]
\[ \langle \Pi(x) \rangle = \hat{\pi}(x, t) , \]  
\[ (13) \]

while bilinear averages are

\[ \langle \Phi(x) \Phi(y) \rangle = \varphi(x) \varphi(y) + \Omega(x, y, t) , \]  
\[ (14) \]
\[ \langle \Pi(x) \Pi(y) \rangle = \hat{\pi}(x) \hat{\pi}(y) + \frac{1}{4} \Omega^{-1}(x, y, t) + 4 (\Sigma \Omega \Sigma)(x, y, t) , \]  
\[ (15) \]
\[ \langle \Phi(x) \Pi(y) \rangle = \frac{i}{2} \delta(x - y) + 2 (\Omega \Sigma)(x, y, t) , \]  
\[ (16) \]
where we have used the matrix notation \((\mathcal{O}\mathcal{K})(x, y) = \int_z \mathcal{O}(x, z) \mathcal{K}(z, y)\). Moreover, from the average value of the operator \(i(\partial/\partial t)\) appearing in the effective action \([6]\), we find that the imaginary part of the covariance function \((\Sigma)\) plays the role of a canonical momentum conjugate to the real part \(\Omega\).

\[
\left\langle \frac{i}{\hbar} \frac{\partial}{\partial t} \right\rangle = \int_x \hat{\pi}(x, t) \dot{\varphi}(x, t) + \int_{x, y} \Sigma(x, y, t) \dot{\Omega}(y, x, t)
\]

(D. Gaussian Vacuum in Flat Space-Time)

In order to gain some intuition and also to motivate the simplification of the Gaussian Ansatz, that we shall use in this work, it is interesting to obtain the vacuum state, in the Gaussian approximation, for the \(\lambda\Phi^4\) model in Minkowski space-time. The dynamics of this system is governed by the Hamiltonian

\[
H = \int_x \left\{ \frac{1}{2} \Pi^2 + (\nabla \Phi)^2 + U(\Phi) \right\},
\]

where the potential function \(U(\Phi)\) is

\[
U(\Phi) = \frac{\mu^2}{2} \Phi^2 + \frac{\lambda}{4!} \Phi^4.
\]

Substituting the wave functional \([11]\) into the Dirac’s variational principle leads to

\[
\Gamma(\varphi, \hat{\pi}, \Omega, \Sigma) = \int dt \int_x \left\{ \hat{\pi} \dot{\varphi} - \left( \frac{1}{2} \hat{\pi}^2 + \frac{1}{2} (\nabla \varphi)^2 + U(\varphi) \right) \right\}
+ \hbar \left[ \left( \Sigma \dot{\Omega} \right)(x, x, t) - \frac{1}{8} \Omega^{-1} (x, x, t) - 2 (\Sigma \Omega \Sigma)(x, x, t) \right]
- \frac{1}{2} \left( -\nabla_x^2 \Omega(x, y, t) \right|_{x=y} + U^{(2)}(\varphi) \Omega(x, x, t) \right]
- \frac{\hbar^2}{8} U^{(4)}(\varphi) \Omega(x, x, t) \Omega(x, x, t)
\]

where \(U^{(n)} \equiv d^nU/d\varphi^n\). The terms in the first square bracket are the classical action, while terms in the second (last) square bracket are formally \(O(\hbar)\) \((O(\hbar^2))\) corrections. In fact, the expression \([20]\) contains all powers in \(\hbar\) since the kernel \(\Omega\) must satisfy a self-consistent
equation, as shown below. By varying \( (20) \) with respect the parameters in our \textit{Ansatz} we obtain four variational equations

\[
\frac{\delta \Gamma}{\delta \varphi(x, t)} = 0 \implies \tilde{\pi}(x, t) = \dot{\varphi}(x, t) ,
\]

\[
\frac{\delta \Gamma}{\delta \tilde{\pi}(x, t)} = 0 \implies \dot{\tilde{\pi}} = \left[ \nabla_x^2 - U^{(1)}(\varphi) - \frac{1}{2} U^{(3)}(\varphi) \Omega(x, x, t) \right] \varphi ,
\]

\[
\frac{\delta \Gamma}{\delta \Sigma(x, y, t)} = 0 \implies \dot{\Omega}(x, y, t) = 2\left[ (\Sigma \Omega)(x, y, t) + (\Sigma \Omega)(x, y, t) \right] ,
\]

\[
\frac{\delta \Gamma}{\delta \Omega(x, y, t)} = 0 \implies \dot{\Sigma}(x, y, t) = \frac{1}{8} \Omega^{-2}(x, y, t) - 2 \Sigma^2(x, y, t) - \nabla_x^2 U^{(2)}(\varphi) + \frac{1}{2} U^{(4)}(\varphi) \Omega(x, x, t) \right] \delta(x - y) .
\]

Translation invariance implies that \( \varphi \) is homogeneous and that the kernels can be expressed as a Fourier transformation (FT)

\[
\Omega(x, y, t) = \int_k e^{ik \cdot (x-y)} \Omega(k, t) ,
\]

where the momentum-space integral \( (\int d^d k / (2\pi)^d) \) is denoted by \( \int_k \). Moreover, the kernels for the vacuum state are time-independent and the above equations of motion reduce to

\[
\dot{\pi} = 0 ,
\]

\[
\varphi \left( \mu^2 + \frac{\lambda}{6} \varphi^2 + \frac{\lambda}{2} \int_k \Omega(k) \right) = 0 ,
\]

\[
\Omega(k) \Sigma(k) = 0 ,
\]

\[
\frac{1}{8} \Omega^{-2}(k) - 2 \Sigma^2(k) - \frac{1}{2} \left( k^2 + \mu^2 + \frac{\lambda}{2} \varphi^2 + \frac{\lambda}{2} \int_k \Omega(k) \right) = 0 ,
\]

whose solution is \( \dot{\pi} = \Sigma(k) = 0 \) and

\[
\Omega(k) = \frac{1}{2\sqrt{k^2 + m^2}} ,
\]

with \( m^2 \) satisfying the gap equation

\[
m^2 = \mu^2 + \frac{\lambda}{2} \varphi^2 + \frac{\lambda}{2} \int_k \Omega(k) .
\]

Since this last equation is a self-consistent one for \( m^2 \), some of the non-linearities of the complete problem are retained by the Gaussian approximation.
E. Renormalization of the Effective Potential

At this point it is instructive to study the renormalization of the Gaussian effective potential, which can be obtained from the effective action through

$$\Gamma(\varphi, \hat{\pi}, \Sigma, \Omega)|_{\text{static}} = -V_{\text{eff}}(\varphi, \hat{\pi}, \Sigma, \Omega) \int_x .$$  \hspace{1cm} (32)

Therefore, from Eq. (20) we can see that the Gaussian effective potential is

$$V_{\text{eff}}(\varphi, \hat{\pi}, \Sigma, \Omega) = \frac{1}{2} \hat{\pi}^2 + \frac{\mu^2}{2} \varphi^2 + \frac{\lambda}{4!} \varphi^4 \right. \left. + \frac{1}{8} \Omega^{-1}(x, x) + 2 (\Sigma \Omega \Sigma)(x, x) - \frac{1}{2} \nabla_x^2 \Omega(x, y)|_{x=y} \right. \left. + \frac{1}{2} \left( \mu^2 + \frac{\lambda}{2} \varphi^2 \right) \Omega(x, x) + \frac{\lambda}{8} \Omega(x, x) \Omega(x, x). \right.$$  \hspace{1cm} (33)

The effective potential $V_{\text{eff}}(\varphi)$ is obtained from $V_{\text{eff}}(\varphi, \hat{\pi}, \Sigma, \Omega)$ by minimizing with respect to the parameters $\hat{\pi}$, $\Omega$, and $\Sigma$. This procedure leads to $\hat{\pi} = \Sigma(k) = 0$ and Eqs. (30) and (31), resulting in

$$V_{\text{eff}}(\varphi) = \frac{\mu^2}{2} \varphi^2 + \frac{\lambda}{4!} \varphi^4 + \frac{1}{4} \int_k \sqrt{k^2 + m^2} \left[ \frac{1}{4k^2 + m^2} \right] \left[ \frac{1}{\sqrt{k^2 + m^2}} \right]$$

$$+ \frac{\lambda}{32} \int_{k,k'} \frac{1}{\sqrt{k^2 + m^2} \sqrt{k'^2 + m^2}} .$$  \hspace{1cm} (34)

In the limit $d = 3$, the above integrals are clearly divergent, due the short distance behavior of $\Omega$, and we must renormalize $V_{\text{eff}}$. We regularized these integrals by using dimensional regularization $[14]$ on the spatial dimension $d$, obtaining that the regularized effective potential is

$$V_{\text{eff}}(\varphi) = \frac{\mu^2}{2} \varphi^2 + \frac{\lambda}{4!} \varphi^4 + \frac{1}{(4\pi)^{(d+1)/2}(1-d)} \left( \frac{m^2}{\Lambda^2} \right)^{(d-3)/2} \Gamma \left( \frac{3-d}{2} \right)$$

$$\times \left\{ \frac{2m^4}{(1+d)} - m^4 + \left( \mu^2 + \frac{\lambda}{2} \varphi^2 \right) m^2 \right. \right.$$  \hspace{1cm} (35)
where $\Lambda$ is an arbitrary mass scale. Notice that the term $\lambda \Omega \Omega$ gives rise to a double pole in $d = 3$, while the other divergences are single poles. Since the Gaussian approximation is very similar to the large-$N$ approximation, the effective potential becomes finite by the renormalization prescription [13]

$$\frac{\mu^2}{\lambda} = \frac{\mu_R^2}{\lambda_R},$$  

$$\frac{1}{\lambda} = \frac{1}{\lambda_R} - \frac{2}{(4\pi)^{(d+1)/2}} \frac{1}{(d-1)(3-d)}.$$  

(36)  

(37)

This last relation implies that

$$\lim_{d \to 3} \lambda = -16\pi^2(3 - d) \left\{ 1 + \frac{16\pi^2(3 - d)}{\lambda_R} + O[(3 - d)^2] \right\} \text{ for } \lambda_R \neq 0.$$  

(38)

When the above renormalization prescription is substituted into Eq. (35), it leads to a finite expression for the effective potential

$$V_{\text{eff}}(\varphi) = \frac{m^2}{2} \varphi^2 - \frac{m^4}{64\pi^2} \gamma + \frac{m^4}{64\pi^2} \ln \left( \frac{m^2}{4\pi\Lambda^2} \right) + \frac{\mu^2 m^2}{\lambda_R} - \frac{m^4}{2\lambda_R},$$  

(39)

in the limit $d = 3$, where we used that

$$\left( \frac{m^2}{\Lambda^2} \right)^{(d-3)/2} \Gamma \left( \frac{3-d}{2} \right) \sim \frac{2}{(3-d)} + \gamma - \ln \left( \frac{m^2}{\Lambda^2} \right) + O(d-3),$$  

(40)

with $\gamma$ being the Euler constant. At this point we choose the scale $\Lambda$ to be

$$\Lambda^2 = \frac{\mu_R^2 e^{-(\gamma - 1/2)}}{4\pi},$$  

(41)

which leads to

$$V_{\text{eff}}(\varphi) = \frac{m^2}{2} \varphi^2 + \frac{m^4}{64\pi^2} \left[ \ln \left( \frac{m^2}{\mu_R^2} \right) - \frac{1}{2} \right] - \frac{(m^2 - \mu_R^2)^2}{2\lambda_R} + \frac{\mu_R^4}{2\lambda_R},$$  

(42)

which is the standard result.

The renormalized expression for $m^2$ can be obtained either by substituting the renormalization prescription into Eq. (33), or by minimizing the renormalized $V_{\text{eff}}$ with respect to $m^2$.

$$\frac{\partial V_{\text{eff}}(\varphi)}{\partial m^2} = 0 \Rightarrow m^2 = \mu_R^2 + \frac{\lambda_R}{2} \varphi^2 + \frac{\lambda_R}{32\pi^2} m^2 \ln \left( \frac{m^2}{\mu_R^2} \right).$$  

(43)
III. SCALAR FIELD EQUATIONS OF MOTION IN ROBERTSON-WALKER SPACE-TIME

In this section we obtain the renormalized equations of motion for a self-interacting scalar field using a simplified Gaussian approximation. We consider a flat Robertson-Walker space-time with the line element

$$ds^2 = dt^2 - a^2(t)dx^2,$$  \hspace{1cm} (44)

where $a(t)$ is the scale factor. We assume minimal coupling between gravity and the scalar field and that the scalar field dynamics is governed by the Lagrange density

$$\mathcal{L} = a^d \left[ \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - U(\Phi) \right],$$ \hspace{1cm} (45)

where the potential $U$ is given by Eq. (19). Although we are mainly interested in physical space-time dimensionality, $d = 3$, we consider the theory in $d$ spatial dimensions in order to regularize the theory in later discussions.

The canonical momentum $\Pi$ is defined by

$$\Pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = a^d \dot{\Phi},$$ \hspace{1cm} (46)

and the Hamiltonian density of the system is

$$\mathcal{H} = a^d \left\{ \frac{1}{2} \left[ a^{-2d} \Pi^2 + a^{-2} (\nabla \Phi)^2 \right] + U(\Phi) \right\}.$$ \hspace{1cm} (47)

The use of the general Gaussian Ansatz \([1]\) leads to coupled integro-differential equations for the Fourier modes of the kernels $\Omega$ and $\Sigma$ \([1]\). Therefore, we must solve an infinity (large) number of coupled equations either analytically or numerically when we apply this approximation to study a physical problem. In order to reduce the number of free parameters and equations, we introduce a simplified Gaussian Ansatz, which is obtained by fixing the functional dependence on $k$ of the kernels $\Omega$ and $\Sigma$. In this work, we assume that these kernels, appearing in the Gaussian trial state \([1]\), have a form similar to their static ones, that is
\[
\Omega(k, t) = \frac{a^{1-d}}{2\sqrt{k^2 + a^2\alpha(t)}},
\]
and
\[
\Sigma(k, t) = -\frac{a^{d-1}\beta}{8[k^2 + a^2\alpha(t)]^n},
\]
where \(\alpha\) and \(\beta\) are the variational parameters and \(n\) is conveniently chosen to control the infinities in the approximation. Clearly, the above Ansatz recovers the vacuum solution (30) in the static limit, provided we take \(a = 1\), \(\beta = 0\) and \(\alpha = m^2\).

The effective action (6) evaluated in the Gaussian trial state (11) with the kernels (48) and (49) is
\[
\Gamma = \int dt \int_x \left\{ \pi \dot{\varphi} - a^d \left[ \frac{1}{2} a^{-2d} \pi^2 + \frac{\mu^2}{2} \varphi^2 + \frac{\lambda}{4!} \varphi^4 \right] - \frac{1}{16} (1 - d) H \beta \int_k \frac{1}{(k^2 + a^2\alpha)^{n+1/2}} 
\right.
\]
\[
+ \frac{1}{32} a^2 \beta (2H\alpha + \dot{\alpha}) \int_k \frac{1}{(k^2 + a^2\alpha)^{n+3/2}} - \frac{1}{64} a^2 \beta^2 \int_k \frac{1}{(k^2 + a^2\alpha)^{2n+1/2}} 
\]
\[
- \frac{1}{2} a^{-1} \int_k \sqrt{k^2 + a^2\alpha} + \frac{1}{4} a \alpha \int_k \frac{1}{\sqrt{k^2 + a^2\alpha}} - \frac{1}{4} a \left( \mu^2 + \frac{\lambda}{2} \varphi^2 \right) \int_k \frac{1}{\sqrt{k^2 + a^2\alpha}} 
\]
\[
+ \frac{\lambda}{32} a^{-2-d} \int_{k,k'} \sqrt{k^2 + a^2\alpha} \frac{1}{\sqrt{k'^2 + a^2\alpha}} \right\},
\]
where \(H = \dot{a}/a\) is the Hubble constant. In the limit \(d = 3\) infinities appear in \(\Gamma\), due to the short distance behavior of the kernels \(\Omega\) and \(\Sigma\). However, choosing \(n > 1\) we do not introduce further infinities besides the one appearing in \(V_{\text{eff}}\).

Using dimensional regularization, we can separate the divergent and finite parts of \(\Gamma\) as
\[
\Gamma = \int dt \int_x \frac{a^d}{(4\pi)^{(d+1)/2}(1-d)} \left( \frac{\alpha}{\Lambda^2} \right)^{(d-3)/2} \left[ \frac{(1 - d)}{(1 + d)} \alpha^2 + \left( \mu^2 + \frac{\lambda}{2} \varphi^2 \right) \alpha \right]
\]
\[
+ \frac{\lambda}{2} \frac{a^2}{(4\pi)^{(d+1)/2}(1-d)} \left( \frac{\alpha}{\Lambda^2} \right)^{(d-3)/2} \Gamma \left( \frac{3 - d}{2} \right) \right] + \text{terms finite at } d = 3.
\]
The divergent part of \(\Gamma\) is similar to the divergences encountered in \(V_{\text{eff}}\), see Eq. (35). This allow us to conclude that \(\Gamma\) can be made finite in the limit \(d = 3\) using the renormalization prescription (36)–(37). It is straightforward to verify that \(\Gamma\) becomes finite in the limit \(d = 3\) by this renormalization prescription:
\[ \Gamma = \int dt \int \left\{ \hat{\pi} \dot{\varphi} - \frac{a^{-3}}{2} \hat{\pi}^2 + \frac{1}{8} H \beta I_{n+1/2} + \frac{1}{32} a^2 \beta (2H \alpha + \dot{\alpha}) I_{n+3/2} - \frac{1}{64} a^{-1} \beta^2 I_{2n+1/2} \right. \\
- a^3 \left[ \frac{\alpha}{2} \varphi^2 + \frac{2 \alpha^2}{64 \pi^2} \left( \ln \left( \frac{\alpha}{\mu^2 R} \right) - \frac{1}{2} \right) - \frac{(\alpha - \mu^2 R)^2}{2\lambda R} + \frac{\mu^2 R}{2\lambda R} \right] \right\} , \]  

where we chose the scale \( \Lambda \) as in Eq. (41) and we defined the integrals

\[ I_j \equiv \int k \left( k^2 + a^2 \alpha \right)^j = \frac{1}{2^d \pi^{d/2}} (a^2 \alpha)^{(d-2j)/2} \Gamma(j - d/2) \Gamma(j) . \]  

At this point it is interesting to compare our results with the ones for general Gaussian Ansatz as shown in Ref. [5]. In both approximations the dynamical equations of motion become finite by the vacuum sector renormalization prescription. Moreover, the large \( k \) behavior of the kernels in these two approximations are similar: in the general Gaussian Ansatz it is required that \( \Omega \simeq O(k^{-1}) \) and \( \dot{\Omega} \simeq O(k^{-3}) \), while in our approximation \( \Omega \) exhibits the same high energy behavior, by construction, and \( n > 1 \), that means that the asymptotic \( k \) dependence of \( \Sigma \) is similar in both approximations.

By varying the renormalized effective action \( \Gamma \) with respect to the parameters \( \varphi, \hat{\pi}, \alpha \), and \( \beta \), we obtain four coupled variational equations

\[ \frac{\delta \Gamma}{\delta \varphi} = 0 \Rightarrow \dot{\varphi} = -a^3 \alpha \varphi , \]  

\[ \frac{\delta \Gamma}{\delta \hat{\pi}} = 0 \Rightarrow \dot{\hat{\pi}} = a^3 \hat{\pi} , \]  

\[ \frac{\delta \Gamma}{\delta \beta} = 0 \Rightarrow \dot{\beta} = a^3 \frac{I_{2n+1/2}}{I_{n+3/2}} - 6 \frac{n}{n-1} H \alpha , \]  

\[ \frac{\delta \Gamma}{\delta \alpha} = 0 \Rightarrow \dot{\alpha} = a^{-2} \frac{I_{2n+1/2}}{I_{n+3/2}} \left\{ 2 \frac{(4n+1) (3n-1)}{2n-1} a^2 \beta^2 I_{2n+3/2} \right. \\
- 32a^3 \left[ \frac{\varphi^2}{2} + \frac{\alpha}{32 \pi^2} \ln \left( \frac{\alpha}{\mu^2 R} \right) - \frac{(\alpha - \mu^2 R)^2}{2\lambda R} + \frac{\mu^2 R}{2\lambda R} \right] \} - 2 \frac{5n^2 - n - 1}{n-1} H \beta . \]  

Due to the choice of our trial state, the equations of motion for the parameters \( \varphi \) and \( \hat{\pi} \) are free field ones with a time-dependent mass \( \alpha \), whose dynamics is dictated by the last two equations [17].
IV. RENORMALIZING THE ENERGY-MOMENTUM TENSOR

In order to write the semi-classical Einstein equation we must study the renormalization of the expectation value of the energy-momentum tensor in our trial state. The energy-momentum tensor for the scalar field described by the Lagrange density (45) is

\[ T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta I}{\delta g^{\mu\nu}} = \partial_\mu \Phi \partial_\nu \Phi - g_{\mu\nu} \left[ \frac{1}{2} g^{\alpha\beta} \partial_\alpha \Phi \partial_\beta \Phi - \frac{\mu^2}{2} \phi^2 - \frac{\lambda}{4!} \phi^4 \right] , \]  

(58)

where \( I \) is the action, \( G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \) is the Einstein tensor and the notation \( ; \mu \) denotes the covariant derivative with respect to the space-time index \( \mu \). In the functional Schrödinger picture, we express the energy-momentum tensor operator in terms of the field operator \( \Phi(x) \) and its canonical momentum \( \Pi(x) \) and evaluate the expectation value in a given state.

In the flat Robertson-Walker metric (44), the expectation value of \( T_{\mu\nu} \) in the translationally invariant Gaussian state (11) has the form

\[ \langle T_{00} \rangle = \frac{a^{-2d}}{2} \hat{\pi}^2 + \frac{\mu^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4 + \frac{a^{-2d}}{8} \Omega^{-1}(x, x, t) + 2a^{-d}(\Sigma\Omega\Sigma)(x, x, t) \]
\[ + \frac{1}{2} \left[ -a^{-2d} \nabla_x^2 + \mu^2 + \frac{\lambda}{2} \Omega(x, x, t) \right] \Omega(x, y, t) \bigg|_{x=y} \]
\[ - \frac{\lambda}{8} \Omega(x, x, t)\Omega(x, x, t) , \]  

(59)

\[ \langle T_{ij} \rangle = a^2 \delta_{ij} \left\{ \frac{a^{-2d}}{2} \hat{\pi}^2 - \frac{\mu^2}{2} \phi^2 - \frac{\lambda}{4!} \phi^4 + \frac{a^{-2d}}{8} \Omega^{-1}(x, x, t) + 2a^{-d}(\Sigma\Omega\Sigma)(x, x, t) \right. \]
\[ - \frac{1}{2} \left[ (1 - \frac{2}{d}) a^{-2d} \nabla_x^2 + \mu^2 + \frac{\lambda}{2} \phi^2 + \frac{\lambda}{2} \Omega(x, x, t) \right] \Omega(x, y, t) \bigg|_{x=y} \]
\[ + \frac{\lambda}{8} \Omega(x, x, t)\Omega(x, x, t) \} , \]  

(60)

\[ \langle T_{0i} \rangle = 0 . \]  

(61)

As expected, the energy-momentum tensor matrix element is diagonal and can be expressed in terms of the average energy density \( \langle \epsilon \rangle \) and pressure \( \langle p \rangle \). In four space-time
dimensions, we have
\[
\langle T_{\mu\nu} \rangle = \text{diag} \left( \langle \epsilon \rangle, a^2 \langle p \rangle, a^2 \langle p \rangle, a^2 \langle p \rangle \right).
\]

(62)

Once more we can witness from (59-61) that infinities may appear in the limit \( d = 3 \) because of the short distance behavior of the kernels \( \Omega \) and \( \Sigma \).

Now we discuss how to obtain the finite, renormalized expectation value of \( T_{\mu\nu} \) in our Gaussian Ansatz, given by Eqs. (11), (48), and (49). First of all, we evaluate the dimensionally regularized expression for \( \langle T_{\mu\nu} \rangle \), which is finite and well behaved. A nice feature of this regularization procedure is that it preserves the general covariance of \( \langle T_{\mu\nu} \rangle \). Substituting (48) and (49) into (59-60), we obtain
\[
\langle T_{00} \rangle = a^{-2d} \hat{\pi}^2 + \frac{\mu^2}{2} \varphi^2 + \frac{\lambda}{4!} \varphi^4 + \frac{1}{64} a^{-d-1} \beta^2 I_{2n+1/2}
\]
\[
+ \frac{1}{(4\pi)^{(d+1)/2}(1-d)} \left( \frac{\alpha}{\Lambda^2} \right)^{(d-3)/2} \Gamma \left( \frac{3-d}{2} \right) \left[ \frac{2}{(1+d)} \alpha^2 - \alpha^2 + \left( \mu^2 + \frac{\lambda}{2} \varphi^2 \right) \alpha \right]
\]
\[
+ \frac{\lambda}{2 (4\pi)^{(d+1)/2}(1-d)} \left( \frac{\alpha}{\Lambda^2} \right)^{(d-3)/2} \Gamma \left( \frac{3-d}{2} \right) \right],
\]

(63)

\[
\langle T_{ij} \rangle = a^2 \delta_{ij} \left( a^{-2d} \hat{\pi}^2 + \frac{\mu^2}{2} \varphi^2 + \frac{\lambda}{4!} \varphi^4 + \frac{1}{64} a^{-d-1} \beta^2 I_{2n+1/2}
\]
\[
+ \frac{1}{(4\pi)^{(d+1)/2}(1-d)} \left( \frac{\alpha}{\Lambda^2} \right)^{(d-3)/2} \Gamma \left( \frac{3-d}{2} \right) \left[ \frac{2}{d(1+d)} \alpha^2 + \left( 1 - \frac{2}{d} \right) \alpha^2 \right]
\]
\[
- \left( \mu^2 + \frac{\lambda}{2} \varphi^2 \right) \alpha - \lambda \frac{a^2}{2 (4\pi)^{(d+1)/2}(1-d)} \left( \frac{\alpha}{\Lambda^2} \right)^{(d-3)/2} \Gamma \left( \frac{3-d}{2} \right) \right) \right\}.
\]

(64)

Next, we express the expectation value of the energy-momentum tensor in terms of the renormalized parameters \( \mu_R \) and \( \lambda_R \). In order to do so, we substitute Eqs. (36) and (37) into the last two expressions, resulting in
\[
\langle T_{00} \rangle = \frac{a^{-6}}{2} \hat{\pi}^2 + \frac{1}{64} a^{-4} \beta^2 I_{2n+1/2} + \frac{\alpha}{2} \varphi^2
\]
\[
+ \frac{\alpha^2}{64\pi^2} \left[ \ln \left( \frac{\alpha}{\mu^2 R} \right) - \frac{1}{2} \right] - \frac{(\alpha - \mu^2 R)^2}{2\lambda R} + \frac{\mu^4 R}{2\lambda R},
\]

(65)

\[
\langle T_{ij} \rangle = a^2 \delta_{ij} \left( \frac{a^{-6}}{2} \hat{\pi}^2 + \frac{1}{64} a^{-4} \beta^2 I_{2n+1/2} - \frac{\alpha}{2} \varphi^2
\]
\[
\right)\right.}
\]
− \frac{\alpha^2}{64\pi^2} \left[ \ln \left( \frac{\alpha}{\mu_R^2} \right) - \frac{1}{2} \right] \frac{(\alpha - \mu_R^2)^2}{2\lambda_R} - \frac{\mu_R^4}{2\lambda_R} \right], \quad (66)

where we took the limit \( d \to 3 \) and chose the scale \( \Lambda \) according to Eq. (41).

One important feature of our Gaussian approximation is that the expectation value of \( T_{\mu\nu} \) turns out to be finite once we take into account the mass and coupling constant renormalization, and that \( n > 1 \). Therefore, the presence of interactions, more specifically the term \(-g_{\mu\nu}(\lambda/8)\Omega\Omega\), leads to the cancelation of the infinities which exist in the free scalar field model \([11]\).

Now we discuss the limit \( \lambda_R = 0 \). In this limit, the renormalization prescription tells us that

\[
\lambda = 0 ; \quad \mu_R^2 = \mu^2 ,
\]

and hence we get back the unrenormalized free theory result for \( \langle T_{\mu\nu} \rangle \) in which the divergences reappear from the \( 1/\lambda_R \) terms. In order to have a well behaved free theory limit we must subtract the terms that diverge in the limit \( \lambda_R = 0 \). Moreover, this subtraction may be justified as a renormalization of coupling constants in a generalized Einstein equation provided that the entire subtraction is expressible in terms of covariantly conserved tensors \([11]\).

In the free theory limit, the contribution \(-g_{\mu\nu}(\alpha - \mu_R^2)^2/2\lambda_R \) vanishes since the structure equation of motion for \( \alpha \) leads to \( \alpha = \mu^2 \) for \( \lambda_R = 0 \). Therefore, we define the renormalized expectation value of the energy-momentum tensor as

\[
\langle T_{\mu\nu} \rangle_R \equiv \langle T_{\mu\nu} \rangle - g_{\mu\nu} \frac{\mu_R^4}{2\lambda_R} .
\]

Notice that this subtraction is basically a redefinition of the cosmological constant.

V. CONCLUSIONS AND DISCUSSION

In this work we used Dirac’s variational principle and a simple Gaussian Ansatz to describe the time evolution of a self-interaction scalar field in flat Robertson-Walker spacetime. Unlike ordinary perturbation theory, this approximation reflects some of the non-linear
features of the full quantum field theory which may be important for understanding various physical processes. Our trial wave functional was obtained from a general Gaussian state by choosing the momentum dependence of its kernels in such a way that we can recover the conventional vacuum in the limit that we have a Minkowski space-time. Nevertheless, we should point out that our vacuum state for a Robertson-Walker space-time is orthogonal to the one obtained in Refs. [5,11], where the Gaussian adiabatic vacuum was used, since they differ by terms of order $O(k^{-3})$ for $k \to \infty$.

In principle, the equations for the variational parameters and the semi-classical Einstein equation may be used to study dynamical question about the universe, such as stability of the de Sitter space and the conditions for inflation setting in. In this case, however, we are confined to the chaotic inflation scenario since our Ansatz does not describe correctly the dynamics of the low momentum modes, as well as the Gaussian approximation in higher-dimensional field theory suffers from well-known shortcomings, analogous to the ones that appear in the the large-$N$ approximation.

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