Dependencies of nonlinear hereditary creep theory for concrete and creep type equations of hardening under plane stress

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Abstract. Concrete creep is of great practical importance and is taken into account in calculation and design of structures. The article discusses various options for describing the theory of creep of concrete, which are divided into three main groups: the theory of an elastic-creeping body (hereditary theory of aging), the theory of elastic heredity (the theory of an visco-elastic body), and the theory of aging. The main advantages and disadvantages of their use are demonstrated depending on various conditions and processes. Collisions in the study of concrete creep are clearly visible when considering expressions for the creep measure, as well as creep and relaxation nuclei. Different scientists propose their own modification of expressions that reflect various features of the material, and which are often very different from each other. The authors of the article obtained the full creep equations of structural materials.

1. Introduction

Let us consider some types of dependences of the nonlinear hereditary theory of concrete creep and hardening type creep equations for the case of a plane stress state.

It is known that creep under two-axial stress conditions has some peculiarities. For example, the experimental data of some authors show that compression in the X direction reduces creep deformations in the Y direction in some cases up to 70%. Since thin-walled elements are usually made of concrete with an increased water-cement ratio, this causes more intense creep. There are various, mutually unrelated, proposals for describing the dependences between the creep measures of concrete with two-dimensional stress state $c_{2x}$ on the one hand, and simple compression $c_x$ on the other. The ratios $c_{2x}/c_x$ are associated with the existing normal stresses and the transverse creep coefficient, a formula that relates the creep measures $c_x$ and $c_{2x}$ is proposed:

$$c_x = \frac{1}{1 + \frac{\sigma_y}{\sigma_x}},$$

where $\sigma_y$, $\sigma_x$ are normal stresses in concrete, and $\sigma_y \leq \sigma_x$.

Static processing of experimental data of I. E. Prokopovich allowed one to establish an approximate dependence of the ratio $c_{2x}/c_x$ on the thickness of a flat disk $h$:
\[
\frac{c_2x}{c_x} = -0.054h^2 + 0.1344h + 0.258,
\]
where \(2 \text{ cm} \leq h \leq 10 \text{ cm} \).

It must be assumed that the values of the \(c_2x/c_x\) ratio are also significantly affected by the stress level and a number of other factors.

Experimental data of the researchers shows that the coefficient of transverse creep deformation of concrete \(v_2(t, \tau)\) is slightly less than the coefficient of transverse elastic-instantaneous deformation \(v_1(\tau)\). However, by the smallness of the variables \(v_1\) and \(v_2\), we can accept the assumptions about their equality:
\[
v_1(\tau) = v_2(t, \tau) = \text{const}. \tag{3}
\]

As demonstrated by I. E. Prokopovich, hypothesis (3) does not lead to large errors (5–6%).

2. Theories of concrete creep
According to a number of the most common features, the variety of variants of the theory of creep of concrete can now be reduced into three different groups: the theory of an elastic-creeping body (hereditary theory of aging), the theory of elastic heredity (theory of an visco-elastic body), and the theory of aging. The most accurately constructed theory of creep should take into account the partial reversibility of creep strains.

The hereditary theory of aging (Maslov-Arutyunyan’s theory of an elastic-creeping body) takes into account the partial reversibility of creep deformations caused by the influence of concrete aging and therefore most correctly describes the process of stress relaxation in concrete structures, provided that the core is well selected in the original integral equation. Mathematically, this is expressed in the fact that the creep measure is taken in the form of:
\[
C(t, \tau) = \left( C_1 + \frac{A_1}{\tau} \right) \left[ 1 - e^{-\gamma(t-\tau)} \right], \tag{4}
\]
where: \(C_1\) is the limit value; \(A_1, \gamma\) is the experimental data.

The theory of elastic heredity does not take into account the presence of an irreversible part of creep deformations and, as a consequence of this, leads to the greatest possible restoration of stresses in the relaxation process. In this case, the creep measure is written as:
\[
C(t, \tau) = C_1 \left[ 1 - e^{-\gamma(t-\tau)} \right]. \tag{5}
\]
\[
C_1 = \text{const}, E(\tau) = E_0 = \text{const}. \tag{6}
\]

Consequently, the theory of elastic heredity is applicable only to old and dried or old and non-drying concrete.

The theory of aging, created specifically for the calculation of concrete and reinforced concrete structures, postulates the complete irreversibility of creep deformation and, consequently, the absence of the process of stress recovery during relaxation. This theory is based on the hypothesis and “parallelism” of curves of simple creep proposed by Whitney C.:
\[
C(t, \tau) = \frac{1}{E(0)} \left[ \varphi(t) - \varphi(\tau) \right], \tag{7}
\]
where \(\varphi(t)\) and \(\varphi(\tau)\) are characteristics of creep; \(E(0)\) is the strain modulus at the initial instant of time.

If, when considering a process of sufficiently long duration, we neglect the features of the development of nonlinear components of creep deformations in the initial section, then we can assume the affine similarity of simple creep curves corresponding to different stress levels \(\eta = \frac{\sigma}{R}\) and present the expressions for the creep measure in the nonlinear region as:
\[ C(\sigma, t, \tau) = f(\sigma)C(t, \sigma), \] (8)
where \( C(t, \tau) \) is the measure of creep in a conditionally linear region;
\( f(\sigma) \) is a nonlinearity function, which is selected on the basis of experimental data depending on the brand of concrete and other factors.

In determining the effect of concrete creep on the stress-strain state of reinforced concrete structures, the hereditary theory of aging is most accurate. In the field of high stresses, one of the most justified equations having a large experimental justification is the nonlinear creep equation. Arutyunyan N. Kh. with elastic instantaneous characteristics, which for the case of a two-axial stress state taking into account equality (3) is written as:

\[ \varepsilon_{11}(t) = \frac{\sigma_{11}(t) - u\sigma_{22}(0)}{E(0)} - \int_{\tau}^{t} \left[ \sigma_{11}(\tau) - u\sigma_{22}(\tau) \right] \left[ \frac{\partial}{\partial \tau} \left[ \frac{1}{E(\tau)} \right] + f \left[ \sigma_{1}(\tau) \frac{dC(\tau, \tau)}{d\tau} \right] \right] d\tau; \]
\[ \varepsilon_{22}(t) = \frac{\sigma_{22}(t) - u\sigma_{11}(t)}{E(t)} - f_{\sigma_{1}} \left[ \sigma_{22}(\tau) - u\sigma_{11}(\tau) \right] \left[ \frac{\partial}{\partial \tau} \left[ \frac{1}{E(\tau)} \right] + f \left[ \sigma_{1}(\tau) \frac{dC(\tau, \tau)}{d\tau} \right] \right] d\tau; \]
\[ \gamma_{12}(t) = \frac{2(1+u)}{E(t)} \tau_{12}(t) - 2(1+u) \int_{\tau}^{t} \left[ \frac{1}{E(t)} \right] \left[ \frac{d}{d\tau} \left[ \frac{1}{E(t)} \right] \right] \left[ \frac{dC(\tau, \tau)}{d\tau} \right] d\tau. \] (9)

If the kernels of an integral equation of type (9) are degenerate, that is, they can be represented by the product of the functions \( \tau \) and \( t \) or the sum of such products, then after twofold differentiation with respect to \( t \) of the left and right sides of (9), it becomes possible to extract a second-order differential equation with variable coefficients:

\[ \dot{\sigma}_{11}(t) - u\dot{\sigma}_{22}(t) + (\sigma_{11} - u\sigma_{22}) \left[ y - \frac{E(t)}{E(t)} \right] + yE(t) \xi_{11}(t) \dot{\sigma}_{11}(t) - u\sigma_{22}) = E(t)(\dot{\varepsilon}_{11} + \gamma\dot{\varepsilon}_{11}); \]
\[ \dot{\sigma}_{22}(t) - u\dot{\sigma}_{11}(t) + (\sigma_{22} - u\sigma_{11}) \left[ y - \frac{E(t)}{E(t)} \right] + yE(t) \xi_{22}(t) \dot{\sigma}_{22}(t) - u\sigma_{11}) = E(t)(\dot{\varepsilon}_{22} + \gamma\dot{\varepsilon}_{22}); \]
\[ \ddot{\tau}_{12} + \ddot{\tau}_{11} + \frac{2(1+u)}{E(t)} \left[ y + \frac{yE(t)}{2(1+u)} \right] \ddot{\tau}_{12} \left[ y + \frac{yE(t)}{2(1+u)} \right] = \frac{E(t)}{2(1+u)} \left[ y + \frac{yE(t)}{2(1+u)} \right]. \] (10)

For a case of old concrete with \( E(t) = E(0) = const \) the dependences (10) are written as:

\[ \dot{\sigma}_{11} - u\dot{\sigma}_{22} - \dot{\varepsilon}_{11} - E(0) = y(\sigma_{11} + C(\infty)E(0)\sigma_{11} - u\sigma_{22})f(\sigma_{i}) - \varepsilon_{11}E(0); \]
\[ \dot{\sigma}_{22} - u\dot{\sigma}_{11} - \dot{\varepsilon}_{22} - E(0) = y(\sigma_{22} + C(\infty)E(0)\sigma_{22} - u\sigma_{11})f(\sigma_{i}) - \varepsilon_{22}E(0) \]
\[ \ddot{\tau}_{12} - \frac{E(0)}{2(1+u)} \left[ y + \frac{yE(t)}{2(1+u)} \right] \ddot{\tau}_{12} \left[ y + \frac{yE(t)}{2(1+u)} \right] = \frac{E(t)}{2(1+u)} \left[ y + \frac{yE(t)}{2(1+u)} \right]. \] (11)

where \( C(\infty) \) is the limit creep measure.

It is generally accepted that the theory of N. Kh. Arutyunyan [2] gives good results for cases of slow reduction or increase in stress in concrete. Such modes are characteristic for the operation of building structures. The disadvantage of this theory is that it does not take into account the softening of nonlinearity with increasing time.

The basic equation of nonlinear creep of the theory of aging, proposed by I. I. Ulitsky, is written for a two-axial stress state as:

\[ \varepsilon_{11}(t) = \frac{\sigma_{11}(t) - u\sigma_{22}(0)}{E(0)} \left[ 1 + f(\sigma_{i})\varphi(t) \right] + \int_{0}^{t} \left[ \frac{d}{dt} \left[ \sigma_{11}(\tau) - u\sigma_{22}(\tau) \right] \right] \left[ \frac{1}{E(\tau)} \right] \left[ \frac{\varphi(\tau) - \varphi(0)}{E(0)} \right] f(\sigma_{i}) \right] d\tau; \]
\[ \varepsilon_{22}(t) = \frac{\sigma_{22}(t) - u\sigma_{11}(0)}{E(0)} \left[ 1 + f(\sigma_{i})\varphi(t) \right] + \int_{0}^{t} \left[ \frac{d}{dt} \left[ \sigma_{22}(\tau) - u\sigma_{11}(\tau) \right] \right] \left[ \frac{1}{E(\tau)} \right] \left[ \frac{\varphi(\tau) - \varphi(0)}{E(0)} \right] f(\sigma_{i}) \right] d\tau; \]
\[ \gamma_{12}(t) = \frac{2(1+u)}{E(0)} \tau_{12}(0) \left[ 1 + f(\sigma_{i})\varphi(t) \right] + \int_{0}^{t} (2(1+u)) \left[ \frac{d}{dt} \left[ \tau_{12}(\tau) \right] \right] \left[ \frac{1}{E(\tau)} \right] \left[ \frac{\varphi(\tau) - \varphi(0)}{E(0)} \right] f(\sigma_{i}) \right] d\tau. \] (12)
The kernel of the integral equation (12) in the theory of aging in accordance with (7) depends only on \( \tau \), which allows, by differentiating the right and left sides of (12) with respect to \( t \), to reduce them to first order differential equations.

The disadvantages of the theory of aging associated with the hypothesis of "parallelism" and the corresponding complete irreversibility of creep deformations make it unsuitable for describing long-term processes in the presence of repeated or variable short-term and long-term effects. Nevertheless, this theory attracts many authors with the simplicity of solutions, which manifests itself most significantly in the nonlinear version. Its application gives quite acceptable results in solving problems of the theory of creep, especially in the case of considering long single exposures.

A special advantage of expressions (4) and (7) is that they allow one to reduce the solution of the basic integral creep equations (9) and (12) to the solution of linear differential equations of the second or first order. This circumstance significantly facilitates the numerical implementation of the tasks to be solved, which will be illustrated in this paper.

3. Creep equations

Experiments show that in a complex stress state the creep of structural materials is determined by shear stresses and proceeds in general according to the laws of “ordinary” plastic deformation. In this regard, the following main points are adopted:

1) a change in body volume is an elastic deformation, that is:

\[
\xi^c = \xi^c_{11} + \xi^c_{22} + \xi^c_{33} = 0, \tag{13}
\]

where \( \xi^c \), \( \xi^c_{11} \), \( \xi^c_{22} \), \( \xi^c_{33} \) are the creep strain rates;

2) the principal stresses of the creep strain rate tensors and stresses coincide;

3) the shapes of stress deviators and creep strain rates coincide;

4) at each point a simple (or close to simple) loading is realized;

5) the intensity of creep shear strain rates \( \eta_{ij}^c \) is a function of the shear stress intensity \( \tau_{ij} \) characteristic for a given material at a given temperature.

This dependence is taken as:

\[
\eta_{ij}^c = f(\tau_{ij})\tau_{ij}, \tag{14}
\]

where the scalar function \( f(\tau_{ij}) \) depends, generally speaking, on scalar parameters associated with the stress and strain state, on time \( t \) and temperature \( T \) \[1\].

The form of the function can be found by considering the case of uniaxial tension and comparing (14) with experimental data.

From the similarity of Mohr using condition (13), we can obtain formulas that relate the components of the creep strain rate tensor and stress tensor:

\[
\xi_{11} = 0.5f(\tau_{ij})(\sigma_{11} - \sigma); \quad \eta_{13} = f(\tau_{ij})\tau_{13}. \tag{15}
\]

where \( \sigma \) is the average pressure.

Adding to (15) the rates of nonlinear elastic strains, we obtain the complete creep equations of structural materials, which for the biaxial stress state can be written as:

\[
\dot{\varepsilon}_{11} = \frac{1}{E}(\sigma_{11} - \nu\sigma_{22}) + \frac{1}{3}f(\tau_{ij}, t)\left(\sigma_{11} - \frac{1}{2}\sigma_{22}\right); \]

\[
\dot{\varepsilon}_{22} = \frac{1}{E}(\sigma_{22} - \nu\sigma_{11}) + \frac{1}{3}f(\tau_{ij}, t)\left(\sigma_{22} - \frac{1}{2}\sigma_{11}\right); \]

\[
\dot{\gamma}_{12} = \frac{2(1+\nu)\tau_{12}}{E} + f(\tau_{ij}, t)\tau_{12}. \tag{16}
\]
For practical calculations, it is convenient to set the function in some analytical form. The inevitable spread of the experimental data makes various analytical approximations of the creep law acceptable.

The most convenient are exponential approximations:

\[
f(\tau_i, t) = \frac{3\varepsilon}{\sigma_i} \exp\left(\frac{\sigma_i}{\sigma_e}\right)
\]  

and power approximations:

\[
f(\tau_i, t) = 3\sigma_i^{n-1}\varepsilon_n\sigma_n^n
\]

where \(\varepsilon_e, \sigma_e, \varepsilon_n, \sigma_n\) are the temperature functions, which are usually set graphically or in the form of tables; 

\(\sigma_i = \sqrt{3\tau_i}\) – the intensity of normal stresses.

Sometimes when using formula (18), the values of the exponent \(n\) turn out to be very large (for example, \(n = 20\)). In these cases, you can use the following approximation:

\[
f(\tau_i, t) = \varepsilon_n \left(\frac{\sigma_i}{\sigma_n}\right)^n \text{ if } \sigma_i > \sigma_n;
\]

\[
f(\tau_i, t) = 0 \text{ if } \sigma_i \leq \sigma_n.
\]

The value can be called the creep limit, although the use of (19) does not mean at all that the existence of a physical creep limit is allowed. The value of exponent \(n\) in (19) turns out to be significantly less than in formula (18).

References
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