Instantons in 7-dimensions: contact, Hermitian Yang-Mills and $G_2$

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Abstract

We study a natural contact instanton equation on gauge fields over 7-dimensional Sasakian manifolds, which is closely related both to the transverse Hermitian Yang-Mills (HYM) condition and the $G_2$-instanton equation. We obtain, by Fredholm theory, a finite-dimensional local model for the moduli space of irreducible solutions, following the approach by Baraglia and Hekmati in 5 dimensions [BH16]. We derive cohomological conditions for smoothness, and we express its dimension in terms of the index of a transverse elliptic operator.

As an instance of concrete interest, we specialise to transversely holomorphic Sasakian bundles over 7-dimensional Calabi-Yau links, as studied by Calvo-Andrade, Rodríguez and Sá Earp [CARSE], and we show that in this context the notions of contact instanton, integrable $G_2$-instanton and HYM connection coincide.

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1 Introduction

We describe the local model of the moduli space of solutions to a natural gauge-theoretic equation, on a suitable class of vector bundles $E \to M$ over a Sasakian manifold. In [DK90], Donaldson studied solutions of the instanton equation $\pm F_A = * F_A$ to understand the geometry of 4-manifolds, where $F_A$ is the curvature of a connection $A$ and $*$ is the Hodge star. On manifolds of dimension $d \geq 4$, this equation can be generalised relatively to an appropriate $(d-4)$-form $\sigma$ [DT98, Tia00], by

$$\lambda F_A = *(\sigma \wedge F_A),$$

(1.1)

for eigenvalues $\lambda$ of $*(\sigma \wedge \cdot) : \Omega^2(M) \to \Omega^2(M)$.

We shall be particularly interested in Sasakian 7-manifolds, on which the contact structure $\eta$ induces a natural 3-form $\sigma = \eta \wedge d\eta$ and $\lambda \in \{1, -1\}$ (see Section 2.1). We adopt the approach of Baraglia and Hekmati [BH16], who described the moduli space of contact instantons on contact metric 5-manifolds, providing sufficient conditions for smoothness away from reducibles, and determining its dimension as the index of an elliptic operator transverse to the Reeb foliation. We will see that these results admit precise analogues in the appropriate 7-dimensional context, while some new distinct gauge-theoretic phenomena also occur.

1.1 Three natural notions of instanton

Let $(M^{2n+1}, \eta, \xi)$ denote a contact manifold, with contact form $\eta$ and Reeb vector field $\xi$ [BG08, Bla10]. Then the natural $(2n-3)$-form $\sigma = \eta \wedge (d\eta)^n$ provides an instance of (1.1):

$$\pm F_A = *(\eta \wedge (d\eta)^{n-2} \wedge F_A).$$

(1.2)

Solutions of (1.2) are said to be selfdual contact instantons if $\lambda = 1$ (respectively anti-selfdual for $\lambda = -1$).

When the contact manifold is endowed in addition with a Sasakian structure, namely an integrable transverse complex structure $\Phi$ and a compatible metric $g$, Biswas [BS10] proposes a natural notion of Sasakian holomorphic structure for complex vector bundles $E \to M$ (see Appendix A). We recall that a connection $A$ on a complex vector bundle over a Kähler manifold is said to be Hermitian Yang-Mills (HYM) if

$$\hat{F}_A := (F_A, \omega) = 0 \quad \text{and} \quad F_A^{0,2} = 0.$$

(1.3)

This notion indeed extends to Sasakian bundles, by taking $\omega := d\eta \in \Omega^{1,1}(M)$ as a ‘transverse Kähler form’, and defining HYM connections to be the solutions of (1.3) in that sense. Given a Hermitian bundle metric, the well-known concept of Chern connection also extends, namely as a connection mutually compatible with the holomorphic structure and the Hermitian metric [BS10, Section 3].

An important class of Sasakian manifolds are those endowed with a so-called contact Calabi-Yau (cCY) structure (see Definition 3.8), the Riemannian metrics of which have transverse holonomy $SU(2n+1)$, in the sense of foliations, corresponding to the existence of a global transverse holomorphic volume form $\varepsilon \in \Omega^{n,0}(M)$ [HV15]. Furthermore, when $n = 3$, such cCY 7-manifolds are naturally endowed with a $G_2$-structure defined by the 3-form

$$\varphi := \eta \wedge d\eta + \text{Im}(\varepsilon),$$

(1.4)

which is cocalibrated, in the sense that its Hodge dual $\psi := *_g \varphi$ is closed under the de Rham differential. When a 3-form $\varphi$ on a 7-manifold defines a $G_2$-structure, the instanton condition (1.1) for $\sigma = \varphi$ and $\lambda = 1$ is referred to as the $G_2$-instanton equation. On holomorphic Sasakian bundles over closed cCY 7-manifolds, it has the distinctive feature that integrable solutions are indeed Yang-Mills critical points, even though the $G_2$-structure has torsion [CARSE].

Throughout this paper, we will be concerned with Sasakian 7-manifolds, on which the three above concepts of instanton interrelate. The contact instanton equation (1.2) in this case is determined by the natural 3-form

$$\sigma := \eta \wedge d\eta.$$

(1.5)

The operator $*(\sigma \wedge \cdot)$ splits the space of 2-forms into $\{\pm 1, -2\}$-eigenspaces (see Section 2.1). However, the $(-2)$-eigenspace is unidimensional spanned by $d\eta$ and rather uninteresting, so we will focus on the $(\pm 1)$-eigenspaces, which in some sense still signify the instanton equation (1.1) as an (anti-) selfduality condition. On a contact Calabi-Yau 7-manifold, the three notions of instanton are related in the following ways:
Theorem 1.1. Let $E$ be a holomorphic Sasakian bundle over a 7-dimensional cCY manifold $(M, \eta, \xi, g, \Phi)$ endowed with its natural $G_2$-structure (1.4); then the following hold:

(i) Every solution of the contact instanton equation $\pm F_A = * (\sigma \wedge F_A)$ is also a solution of $*(\varphi \wedge F_A) = \pm F_A$, i.e., every contact instanton is a $G_2$-instanton [Proposition 3.11].

(ii) A Chern connection is a $G_2$-instanton if, and only if, it is a contact instanton [Proposition 3.12].

(iii) A Chern connection is HYM if, and only if, it is a $G_2$-instanton [CARSE, Lemma 21].

In particular, among Chern connections, the three notions are equivalent.

1.2 Local model of the moduli space

Our main result is a complete description of the local deformation theory of 7-dimensional Sasakian contact instantons. Let $E \to M$ be a complex vector bundle with compact, connected, semi-simple structure group $G$, and denote by $g_E$ its adjoint bundle and by $\Omega^k(g_E)$ the $g_E$-valued $k$-forms on $M$. The operator

$$L_{\sigma} := * (\sigma \wedge \cdot) : \Omega^2(g_E) \to \Omega^2(g_E)$$

induces the irreducible splitting (cf. (2.6))

$$\Omega^2(g_E) = \left( \Omega^2_{-1}(g_E) \oplus \Omega^2_{0}(g_E) \oplus \Omega^2_{1}(g_E) \right) \oplus \Omega^2_{-2}(g_E)$$

where $\Omega^2_{-1}(g_E)$, $\Omega^2_{0}(g_E)$ and $\Omega^2_{1}(g_E)$ are the eigenspaces associated to $-1$, $1$ and $-2$, respectively. In Proposition 3.1, we show that Chern connections (compatible with both a Hermitian bundle metric and a Sasakian holomorphic structure) with curvature in $\Omega^2_{-1}(g_E)$, called anti-selfdual contact instantons, are necessarily flat. Hence the meaningful notion in our case is that of selfdual contact instantons, with curvature in $\Omega^2_{0}(g_E)$, i.e., in the kernel of the projection map

$$p : \Omega^2_{-1}(g_E) \to \Omega^2_{0}(g_E) := (\Omega^2_{0}(g_E) \oplus \Omega^2_{1}(g_E)).$$

The Hilbert Lie group $G$ of smooth gauge transformations acts smoothly on the space $A$ of connections on $E$, and the topological quotient $B := A/G$ is a Hausdorff space. We denote by $B^* \subset B$ the open subspace of irreducible connections, and by $M \subset B$ the set of gauge equivalence classes of solutions to the selfdual contact instanton equation:

$$M := \{ [A] \in B \mid p(F_A) = 0 \}$$

and, accordingly, $M^* \subset M$ for its irreducible part. Linearising the selfdual contact instanton condition, in terms of the projection $p$ in (1.7), we introduce:

$$d_\gamma := p \circ d_A : \Omega^1_{-1}(g_E) \to \Omega^2_{0}(g_E).$$

In Section 3, we will see that a local model for the moduli space of selfdual contact instantons $M^*$ is given by the cohomology group $H^1(C) := \ker(d_\gamma) / \im(d_\gamma)$ of the deformation complex [Proposition 3.2]:

$$C^* : 0 \longrightarrow \Omega^0(g_E) \overset{d_A}{\longrightarrow} \Omega^1(g_E) \overset{d_\gamma}{\longrightarrow} \Omega^2_{0}(g_E) \overset{d_\gamma}{\longrightarrow} 0.$$

This complex, however, is not elliptic, and in order to compute the dimension of $H^1(C)$ we resort to an auxiliary construction, studied in Section 4. We introduce the quotient spaces of $k$-forms modulo the Lie algebra ideal $I$ generated by $\Omega^2_{-1}(g_E)$:

$$L^k : \Omega^k(g_E) \to \Omega^k(g_E) / I, \quad k = 0, 1, 2, 3, \quad \text{with} \quad I := \langle \Omega^2_{-1}(g_E) \rangle \subset \langle \Omega^*(g_E), \wedge \rangle.$$

For a natural choice of differentials $D_k$, the $L^k$ spaces fit in a complex [Proposition 4.6]:

$$L^* : 0 \longrightarrow L^0 \overset{D_0}{\longrightarrow} L^1 \overset{D_1}{\longrightarrow} L^2 \overset{D_2}{\longrightarrow} L^3 \longrightarrow 0.$$
We denote by $H^\bullet := H^\bullet(L)$ the cohomology of (1.11), and indeed the first cohomology group $H^1$ is isomorphic to the infinitesimal deformations of $[A] \in M$ as a contact instanton.

Relatively to the Reeb orbits, a $S^1$-invariant differential form $\alpha \in \Omega^\bullet(M)$ is called basic. The graded ring $\Omega^\bullet_B(M)$ of basic forms inherits a natural \textit{basic de Rham differential}

$$d_B := d|_{\Omega^\bullet_B(M)} : \Omega^0_B(M) \rightarrow \Omega^{k+1}_B(M),$$

and the cohomology of $d_B$ is referred to as the \textit{basic de Rham cohomology}. Restricting the differentials $D_k$ in (1.11) to basic forms in $L^k = \Omega^k(M)/\langle \Omega^2 \rangle$, we obtain a basic complex [Proposition 4.6]:

$$L^\bullet_B : 0 \longrightarrow \Omega^0_B(\mathfrak{g}_E) \longrightarrow \Omega^1_B(\mathfrak{g}_E) \longrightarrow (\Omega^2_B)_B(\mathfrak{g}_E) \longrightarrow 0.$$  \hspace{1cm} (1.12)

We denote by $H^\bullet_B := H^\bullet(L_B)$ the corresponding \textit{basic cohomology}. If $A$ is a contact instantant, the \textit{transverse index} of $A$ is defined as the index of the basic complex (1.12):

$$\text{index}_T(A) = \dim(H^0_B) - \dim(H^1_B) + \dim(H^2_B).$$

In particular, when $A$ is irreducible, $\text{index}_T(A) = \dim(H^0_B) - \dim(H^1_B)$. Our main theorem provides a local model for the moduli space $M^*$ in terms of basic cohomology:

\textbf{Theorem 1.2.} Let $E \rightarrow M$ be a $G$-bundle over a closed connected Sasakian 7-manifold $(M, \eta, \xi, g, \Phi)$, with adjoint bundle $\mathfrak{g}_E$, and denote by $M^*$ the moduli space (1.8) of irreducible selfdual contact instantons, i.e. solutions of (1.1) for $\lambda = 1$:

$$F_A = *(\eta \wedge d\eta \wedge F_A).$$

Then:

(i) The tangent space of $M^*$ at $[A]$, i.e. the space of infinitesimal deformations of $[A]$ as a contact instantant, is isomorphic to the finite-dimensional cohomology group $H^1(C) := \frac{\ker(D)}{\text{im}(d)}$ of the complex (1.10), and

$$\dim(T_{[A]}M^*) = \dim(H^1(C)).$$

(ii) The dimension of $M^*$ near $[A]$ can be computed from the cohomology of the basic complex (1.12), which is elliptic transversely to the Reeb foliation, namely there is an isomorphism $H^1 \cong H^1_B$, where $H^1_B$ is the cohomology of (1.12):

$$\dim(T_{[A]}M^*) = \dim(H^2_B) - \text{index}_T(A).$$

(iii) The local model of $M^*$ is cut out as the zero set of an obstruction map (Definition 4.29), which vanishes precisely when $H^2_B = 0$ [Proposition 4.25]. Thus, for an irreducible contact instanton $A$ such that $H^2_B = 0$, $M^*$ is smooth near $A$ with finite dimension $\dim M^* = -\text{index}_T(A)$ [Corollary 4.24].

\textbf{Remark 1.1.} Parts (i) and (ii) in Theorem 1.2 establish somewhat independently that the tangent space near an irreducible contact instanton is finite-dimensional, since it occurs as the first cohomology group in both complexes (1.11) and (1.12). However, in terms of the obstruction theory, we learn something finer from (ii) and (iii). In the context of (i), the moduli space near an acyclic point, i.e. $h^0(C) = h^2(C) = 0$ in (1.10), would be necessarily 0-dimensional, whereas the complex (1.12) in terms of basic cohomology is merely transverse-elliptic, hence the moduli space near an acyclic smooth point, with $h^0_B = h^2_B = 0$, can in principle have nonzero dimension $-\text{index}_T(A)$.

\textbf{Remark 1.2.} For most steps in our argument, it suffices to assume $M$ compact and connected, with possibly nontrivial boundary. However, in Theorem 1.1, (iii), taken from [CARSE, Lemma 21], and in Propositions 3.15 and 4.17, one actually needs $M$ to be closed.

\textbf{Remark 1.3.} For a Sasakian 7-manifold with positive transverse scalar curvature, the second basic cohomology group $H^2_B$ should vanish, and therefore $M^*$ is smooth. This is announced here as Conjecture 4.26, to be expanded in an upcoming version. In particular, $\text{index}_T(A) = -\dim(H^2_B)$, see e.g. part (ii) of Theorem 1.2.

Regarding the various instanton notions related by Theorem 1.1, our main Theorem 1.2 has the following significance:

Corollary 1.4. Let $\mathcal{E} \to M$ be a holomorphic Sasakian bundle (Definition A.5) over a 7-dimensional cCY manifold $(M^7, \eta, \xi, g, \Phi)$, endowed with its natural $G_2$-structure (1.4). Among Chern connections in $\mathcal{A}(\mathcal{E})$, the three notions coincide: contact, HYM and $G_2$-instantons. The complex (1.10) describes locally their moduli space.

Outline: In Section 2.1 we describe a local splitting (2.6) of $\Omega^2(M)$ under the contact structure and the operator $L_\sigma := \ast(\sigma \wedge \cdot)$ (cf. (1.6)). Another natural decomposition of $\Omega^2(g_E)$ comes from the transverse complex structure induced by $\Phi \in \text{End}(TM)$, and both are related by (2.12). Furthermore, the endomorphism $\Phi$ provides a notion of transverse holomorphicity for complex vector bundles over Sasakian manifolds [Appendix A], hence also notions of unitary and integrable connections. We show, in Proposition 3.1, that imposing these conditions on a connection forces its curvature component in $\Omega^2_B(\Theta E)$ to vanish, so we focus our attention on the selfdual contact instanton case.

Parts (i) and (ii) of Theorem 1.1 are proven respectively in Propositions 3.11 and 3.12. The proof of Theorem 1.2 is organised as follows: part (i) is the content of Proposition 3.3, which uses an auxiliary elliptic complex [Proposition 3.7] to establish that this local model has finite dimension; part (ii) is an immediate consequence [Corollary 4.19] of Proposition 4.18; and part (iii) requires a thorough study of the moduli space of the obstruction theory of selfdual contact instantons, under the 5-dimensional paradigm from [BH16], culminating in Proposition 4.25.

Disclaimer: This is a working paper, derived from ongoing research towards the PhD thesis of the first-named author at Unicamp, Brazil. A number of additional results currently in progress, as well as possibly substantial corrections, will appear in ulterior updates (see Afterword on page 31).

2 Preliminaries on Sasakian geometry

We follow the standard references for Sasakian geometry [BG08, Bla10]. A Sasakian structure on a smooth manifold $M^{2n+1}$ is a quadruple $(\eta, \xi, g, \Phi)$ such that $(M, g)$ is a Riemannian manifold, $(M, \eta)$ is a contact manifold with Reeb field $\xi$ and $\Phi \in \text{End}(TM)$ is a transverse complex structure, satisfying the following compatibility relations:

\begin{align*}
(i) \quad & g(\xi, \xi) = 1, & (iv) \quad & \nabla_X^g \xi = -\Phi X, \\
(ii) \quad & \Phi \circ \Phi = -I_{TM} + \eta \otimes \xi, & (v) \quad & (\nabla_X^g \Phi)(Y) = g(X, Y)\xi - \eta(Y)X, \\
(iii) \quad & g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y),
\end{align*}

where $X, Y$ are vector fields on $M$ and $\nabla^g$ is the Levi-Civita connection of $g$. In that case we say $(M, \eta, \xi, g, \Phi)$ is a Sasakian manifold. Hereinafter we will denote $\sigma := \eta \wedge d\eta$.

For a $p$-form $\alpha$ and a vector field $v$ on $M$, the metric $g$ is compatible in the sense that $\ast(v, \alpha) = v^b \wedge \ast \alpha$, where $v^b := g(v, \cdot)$ and $\ast$ is the Hodge operator of $g$. Notice that $\xi^b = \eta$, thus applying the above formula to $v = \xi$ and $\alpha = \ast \beta$ for a $p$-form $\beta$, we obtain

$$
\ast(\xi, \ast \beta) = \xi^b \wedge (\ast \beta) = (-1)^{p(n-p)} \eta \wedge \beta,
$$

or, equivalently,

$$
i_\xi (\ast \beta) = (-1)^p (\eta \wedge \beta), \quad \forall \beta \in \Omega^p(M).
$$

(2.1)

Furthermore, the contact structure induces a natural operator

$$
T := \eta \wedge i_\xi (\cdot) : \Omega^p(M) \to \Omega^p(M),
$$

(2.2)

which is a projection:

$$
T^2(\alpha) = (\eta \wedge i_\xi)(\eta \wedge i_\xi \alpha) = (\eta \wedge i_\xi)(i_\xi \eta \wedge \alpha - i_\xi(\eta \wedge \alpha)) = (\eta \wedge i_\xi)(\alpha) = T(\alpha).
$$

As such, it splits any $\alpha \in \Omega^p(M)$ into horizontal and vertical components:

$$
\alpha = (1 - \eta \wedge i_\xi) \alpha + \eta \wedge i_\xi \alpha = \alpha_H + \alpha_V = i_\xi(\eta \wedge \alpha) + \eta \wedge i_\xi \alpha,
$$

(2.3)
this provides a splitting $\Omega^\bullet(M) = \Omega^\bullet_H(M) \oplus \Omega^\bullet_V(M)$, where $\Omega^\bullet_H(M)$ and $\Omega^\bullet_V(M)$ are the horizontal and vertical parts, respectively.

From now on, unless otherwise stated, we will fix $\dim M = 7$. Equation (2.3) suggests a natural ‘instanton equation’ as follows: consider $\alpha \in \Omega^2(M)$, applying the contraction (2.1) to $d\eta \wedge \alpha \in \Omega^4(M)$, we obtain

$$i_\xi(*(d\eta \wedge \alpha)) = *(\sigma \wedge \alpha), \quad \text{with} \quad \sigma = \eta \wedge d\eta.$$ 

This motivates the introduction of operator $L_\sigma : \Omega^2_H(M) \to \Omega^2_H(M)$ in (1.6). In next section we show that $\pm 1$ are eigenvalues of $L_\sigma$ and that $L_\sigma|_{\Omega^2_H(M)} = 0$, and this extends in the natural way to $\Omega^2(G_E)$. If $\alpha = F_A$ is the curvature of a connection $A$ on a suitable vector bundle $E \to M$, a natural instance of the contact instanton equation is $*(\sigma \wedge F_A) = i_\xi(*(d\eta \wedge F_A)) = \pm F_A$, or, equivalently,

$$* F_A = \pm \sigma \wedge F_A.$$ (2.4)

Notice that $\sigma \wedge (\omega)^3 \neq 0$ is a volume form on $M$, $*(\sigma \wedge \omega) = c_\omega \omega$ where $c_\omega \neq 1$ is a constant, this contrasts with 5-dimensional case in [BH16] and classical 4-dimensional gauge theory in which $\omega$ is selfdual.

### 2.1 Eigenspaces of $L_\sigma = *(\sigma \wedge \cdot)$ from the contact structure

We will now see how equation (2.4) splits into components according to the eigenspaces of $L_\sigma$ defined in (1.6). Let $(x_1, \ldots, x_7)$ be Sasakian Darboux local coordinates on $M$ [Bla10, Theorem. 3.1] such that the contact form $\eta$ is given by

$$\eta = dx^7 - (x_4 dx^1 + x_5 dx^2 + x_6 dx^3).$$

For convenience we denote $\frac{\partial}{\partial x_i} = X_i$, and $dx^{i_1 \cdots i_k} := dx^{i_1} \wedge \cdots \wedge dx^{i_k}$. Then $X_7 = \xi$ is the Reeb vector field, and the transverse symplectic 2-form is expressed by $\omega := d\eta = dx^{14} + dx^{25} + dx^{36}$. In these local coordinates, the projection $T$ defined in (2.2) acts as follows:

$$T(dx^{i_j}) = \eta \wedge i_\xi(dx^{i_j}) = 0, \quad \text{for} \quad 1 \leq i < j \leq 6;$$

$$T(dx^{i_7}) = \eta \wedge i_\xi(dx^{i_7}) = -\eta \wedge dx^i, \quad \text{for} \quad i = 1, \cdots, 6.$$

Therefore, the decomposition (2.3) determines a 15-dimensional horizontal space $\Omega^2_H$ and a 6-dimensional vertical space $\Omega^2_V$:

$$\Omega^2(M) = \operatorname{Span}\{dx^{i_j}| 1 \leq i < j \leq 6\} \oplus \operatorname{Span}\{dx^{i_7}| i = 1 \cdots 6\}. \quad (2.5)$$

In local Darboux coordinates, $L_\sigma$ acts on the horizontal space

$$\Omega^2_H = \operatorname{Span}\{dx^{12}, dx^{13}, dx^{14}, dx^{15}, dx^{16}, dx^{23}, dx^{24}, dx^{25}, dx^{26}, dx^{34}, dx^{35}, dx^{36}, dx^{45}, dx^{46}, dx^{56}\},$$

as follows:

$$L_\sigma(dx^{12}) = dx^{45} \quad L_\sigma(dx^{13}) = dx^{46} \quad L_\sigma(dx^{15}) = dx^{24} \quad L_\sigma(dx^{14}) = -(dx^{36} + dx^{25}) \quad L_\sigma(dx^{45}) = dx^{12};$$

$$L_\sigma(dx^{16}) = dx^{34} \quad L_\sigma(dx^{23}) = dx^{56} \quad L_\sigma(dx^{24}) = dx^{15} \quad L_\sigma(dx^{25}) = -(dx^{36} + dx^{14}) \quad L_\sigma(dx^{46}) = dx^{13};$$

$$L_\sigma(dx^{26}) = dx^{35} \quad L_\sigma(dx^{34}) = dx^{16} \quad L_\sigma(dx^{35}) = dx^{26} \quad L_\sigma(dx^{36}) = -(dx^{25} + dx^{14}) \quad L_\sigma(dx^{56}) = dx^{23}.$$

For upcoming convenience, let us fix the following notation:

$$v := dx^{14} + dx^{25} + dx^{36};$$

$$v_1 := dx^{12} - dx^{45} \quad v_2 := dx^{15} - dx^{24} \quad v_3 := dx^{13} - dx^{46};$$

$$v_4 := dx^{16} - dx^{34} \quad v_5 := dx^{23} - dx^{56} \quad v_6 := dx^{26} - dx^{35};$$

$$w_1 := dx^{12} + dx^{45} \quad w_2 := dx^{15} + dx^{24} \quad w_3 := dx^{13} + dx^{46} \quad w_4 := dx^{16} + dx^{34};$$

$$w_5 := dx^{23} + dx^{56} \quad w_6 := dx^{26} + dx^{35} \quad w_7 := dx^{14} - dx^{36} \quad w_8 := dx^{25} - dx^{36}.$$
By simple inspection we can see that $L_\sigma(v_i) = -v_i$ for $i = 1, \ldots, 6$, that $L_\sigma(w_j) = w_j$ for $j = 1, \ldots, 8$, and that $L_\sigma(v) = -2v$. Hence the operator $L_\sigma$ defined in (1.6) splits $\Omega^2_T(M)$ in eigenspaces associated to eigenvalues $\{-2, -1, 1\}$, respectively, as follows:

$$\Omega^2_{T,1} := \Omega^2_{H,1} = \text{Span}\{v\}, \quad \Omega^2_{T,6} := \Omega^2_{H,6} = \text{Span}\{v_1, \ldots, v_6\}, \quad \Omega^2_8 := \Omega^2_{H,8} = \text{Span}\{w_1, \ldots, w_8\},$$

so the 2-forms decompose irreducibly as

$$\Omega^2(M) = \Omega^2_{1} \oplus \Omega^2_{6} \oplus \Omega^2_8 \oplus \Omega^2_V.$$

Of course, this decomposition extends naturally to $g_E$-valued 2-forms. Let $\langle \cdot, \cdot \rangle_B$ be an invariant metric on $g$, an inner product on $\Omega^*(g_E)$ is defined by:

$$\langle \alpha, \beta \rangle_M = \int_M \langle \alpha \wedge *\beta \rangle_g.$$

Relatively to the Reeb foliation on a Sasakian manifold $(M, \eta, \xi, g, \Phi)$, the usual Hodge star induces a transverse Hodge star operator $*_{T}: \Omega^k_T(M) \rightarrow \Omega^{m-1-k}_T(M)$ by the formula [Ton12, Section 12]

$$*_{T}(\beta) = (-1)^{m-1-k} (\beta \wedge \eta).$$

If $\alpha \in \Omega^*_T(M)$, both operators are compatible, in the sense that

$$* \alpha = *_{T} \alpha \wedge \eta.$$

Therefore the inner product in (2.8) can be rewritten as

$$\langle \alpha, \beta \rangle_M = \int_M \langle \alpha \wedge *_{T} \beta \rangle \wedge \eta.$$

**Lemma 2.1.** The decompositions (2.3) and (2.7) are orthogonal with respect to the above inner product.

**Proof.** For $\alpha \in \Omega^2_{H,1}(M)$ and $\beta \in \Omega^2_{H,6}(M)$ we have

$$\langle \alpha, \beta \rangle = (i_\xi(\eta \wedge \alpha), \beta) = \langle \alpha, \eta \wedge i_\xi \beta \rangle = -a_1(\alpha, \eta \wedge dx^i) = (i_\xi \alpha, \beta) = 0.$$

Moreover, the operator $*(\sigma \wedge \cdot) : \Omega^2_T(M) \rightarrow \Omega^2_T(M)$ is self-adjoint: for $\alpha, \beta \in \Omega^2_T(M)$,

$$\langle \alpha, * (\sigma \wedge \beta) \rangle_M = \int_M \alpha \wedge * (\sigma \wedge \beta) \text{dvol} = \int_M \beta \wedge \sigma \wedge \alpha \text{dvol} = \int_M \beta \wedge *^2 (\sigma \wedge \alpha) \text{dvol} = (\sigma \wedge \alpha, \beta)_M,$$

hence its eigenspaces are orthogonal. 

**2.2 Compatible splitting from the transverse complex structure**

We establish some notation and elementary facts about the complexified tangent bundle, which are largely adapted from [BS10] and reviewed in Appendix A.

The contact structure splits the tangent bundle as $TM = H \oplus N_\xi$, (see (A.1)), where $H = \ker(\eta)$ and $N_\xi$ is the real line bundle spanned by the Reeb field $\xi$. The transverse complex structure $\Phi$ satisfies $(\Phi|_H)^2 = -1$, so the eigenvalues of the complexified operator $\Phi^C$ are $\pm i$, with $i := \sqrt{-1}$. The complexification $H_C := H \otimes \mathbb{C}$ splits as $H_C = H^{1,0} \oplus H^{0,1}$ (see (A.3)), so we obtain a smooth decomposition of direct sum of vector bundles (see (A.4))

$$\Lambda^d(H_C)^* = \bigoplus_{i=0}^{d} (H^{i,d-i})^*.$$

This induces the decomposition of vector bundles (see (A.5))

$$\Omega^d(M) = \left( \bigoplus_{i=0}^{d} \Omega^i_{H} \right) \oplus \eta \otimes \left( \bigoplus_{j=0}^{d-1} \Omega^{d-j-1}_{H}(M) \right),$$

where $\Lambda^d(H_C)^*$ is the dual of $\Lambda^d(H_C)$. The complexified contact structure $\Phi^C$ splits as $\Phi^C = \eta \otimes \Phi^C_M$, and $\Phi^C_M$ is self-adjoint: for $\alpha, \beta \in \Omega^d_T(M)$, we have

$$\langle \alpha, *_{C} (\beta) \rangle_M = \int_M \langle \alpha \wedge *_{C} \beta \rangle \wedge \eta = \int_M \beta \wedge * (\Phi|_H)^2 \sigma \wedge \alpha \rangle_M = (\alpha, *_{T} \beta \wedge \eta)_M.$$
where $\Omega^p_{\pm}(M) := \Gamma(M, \Lambda^p(H_C)^* \otimes \Lambda^q(H_C)^*)$. Now, let us study more closely the space of 2-forms, from the ‘transversely complex’ point of view. In local Darboux coordinates $(x_1, \cdots, x_7)$, we denote the transverse complex coordinates by

\[ z_1 := x_1 + ix_4, \quad z_2 := x_2 + ix_5 \quad \text{and} \quad z_3 := x_3 + ix_6. \tag{2.11} \]

We will denote, as usual, $dz^j := dx^j + i dx^{j+3}$, for $j = 1, 2, 3$, and $dz^j := dx^j - i dx^{j+3}$. In terms of $\{v_i\}_{i=1}^6$ and $\{w_i\}_{i=1}^8$, given in (2.6), the space $\Omega^{2,0}(M)$ is locally spanned by

\[
\begin{align*}
\text{we also compute} & \\
 dz^1 \wedge dz^2 &= (dx^{12} + dx^{45}) + i(dx^{15} + dx^{24}) = v_1 + iv_2 \\
 dz^1 \wedge dz^3 &= (dx^{13} + dx^{46}) + i(dx^{16} + dx^{34}) = v_3 + iv_4 \\
 dz^2 \wedge dz^3 &= (dx^{23} + dx^{56}) + i(dx^{26} + dx^{35}) = v_5 + iv_6,
\end{align*}
\]

Since $\omega := d\eta$ is nowhere vanishing and has type (1, 1) [BS10, Corollary, 3.1], it determines an orthogonal complement $\Omega^{1,1}_\perp(M)$ in

\[
\Omega^{1,1}(M) = (\Omega^0(M) \cdot d\eta) \oplus \Omega^{1,1}_\perp(M),
\]

which is expressed in local Darboux coordinates by

\[
\Omega^{1,1}_\perp(M) = \text{Span}\{\text{Re}(dz^i \wedge \overline{z}^j), -\text{Im}(dz^i \wedge \overline{z}^j), dx^{14} - dx^{36}, dx^{25} - dx^{36} | 1 \leq i < j \leq 3\}
\]

\[ \cong \Omega^2_0. \]

Therefore we obtain two decompositions for the horizontal 2-forms:

\[
\Omega^2_H(M) = \Omega^{2,0}(M) \oplus \Omega^{0,2}(M) \oplus \Omega^0(M) \cdot d\eta \oplus \Omega^{1,1}(M) \tag{2.12}
\]

Note that $\Omega^{0,2}(M)$ is spanned by

\[
\begin{align*}
 dz^1 \wedge dz^2 &= (dx^{12} + dx^{45}) - i(dx^{15} + dx^{24}) = v_1 - iv_2, \\
 dz^1 \wedge dz^3 &= (dx^{13} + dx^{46}) - i(dx^{16} + dx^{34}) = v_3 - iv_4, \\
 dz^2 \wedge dz^3 &= (dx^{23} + dx^{56}) - i(dx^{26} + dx^{35}) = v_5 - iv_6,
\end{align*}
\]

consistently with the fact that $\Omega^{0,2}(M) \cong \overline{\Omega^{2,0}(M)}$. Still by inspection, we have:

\[
\begin{align*}
 w_1 &= \frac{1}{2}(dz^1 \wedge dz^2 - dz^2 \wedge dz^3), \quad w_2 = \frac{1}{2}(dz^1 \wedge dz^2 + dz^2 \wedge dz^3), \quad w_3 = \frac{1}{2}(dz^1 \wedge dz^3 - dz^3 \wedge dz^1), \\
 w_4 &= \frac{1}{2}(dz^1 \wedge dz^3 + dz^3 \wedge dz^1), \quad w_5 = \frac{1}{2}(dz^2 \wedge dz^3 - dz^3 \wedge dz^2), \quad w_6 = \frac{1}{2}(dz^2 \wedge dz^3 + dz^3 \wedge dz^2), \\
 w_7 &= \frac{1}{2}(dz^1 \wedge dz^4 - dz^4 \wedge dz^5), \quad w_8 = \frac{1}{2}(dz^2 \wedge dz^4 - dz^4 \wedge dz^5),
\end{align*}
\]

\[
\omega = \frac{1}{2}(dz^1 \wedge dz^2 + dz^2 \wedge dz^3 + dz^3 \wedge dz^4),
\]

\[
8
\]
\[ v_1 = \frac{1}{2}(dz_1 \wedge dz_2 + d\overline{z}_2 \wedge d\overline{z}_1), \quad v_2 = \frac{1}{2}(d\overline{z}_2 \wedge dz_1 - dz_1 \wedge d\overline{z}_2), \quad v_3 = \frac{1}{2}(dz_1 \wedge dz_3 + d\overline{z}_3 \wedge d\overline{z}_1), \]
\[ v_4 = \frac{1}{2}(d\overline{z}_3 \wedge dz_1 - dz_1 \wedge d\overline{z}_3), \quad v_5 = \frac{1}{2}(dz_2 \wedge dz_3 + d\overline{z}_2 \wedge d\overline{z}_3), \quad v_6 = \frac{1}{2}(d\overline{z}_2 \wedge d\overline{z}_3 - dz_3 \wedge dz_2). \]

From the expressions of \( w_i, i = 1, \cdots, 8 \), and \( v_j, j = 1, \cdots, 6 \), we obtain immediately:

**Lemma 2.2.** Let \( \Omega^2_8 \) and \( \Omega^2_6 \) be the eigenspaces of the operator \( L_\sigma \) from (1.6), associated to the eigenvalues 1 and \(-1\), respectively (see (2.6)); then

(i) The 2-form \( \alpha \) belongs to \( \Omega^2_8 \) if, and only if, \( \alpha \) is of type \((1, 1)\) and orthogonal to \( \omega \).

(ii) The 2-form \( \alpha \) belongs to \( \Omega^2_6 \) if, and only if, \( \alpha = \beta + \overline{\beta} \), for some \( \beta \) of type \((2, 0)\).

We denote by \( g^C \) the complexification of the Lie algebra \( g = \text{Lie}(G) \), so that the complexification of the real vector bundle \( g_E \) coincides with the associated bundle with fibre \( g^C_E \). Let also:

\[ \Omega^k(g^C_E) := \Gamma((\Lambda^k(T^*M)\mathbb{C}) \otimes g^C_E) \quad \text{and} \quad \Omega^{p,q}(g^C_E) := \Gamma((\Lambda^{p,q}(T^*M)\mathbb{C}) \otimes g^C_E). \]

(2.13)

In particular, Lemma 2.2 yields a characterisation of curvature forms in \( \Omega^2_8(g^C_E) \) and \( \Omega^2_6(g^C_E) \). Note that all generators \( \{w_i\}_{i=1}^8 \) of \( \Omega^2_8 \) satisfy the reality condition \( \overline{w_i} = w_i \), hence every \( \alpha \in \Omega^2_8(g^C_E) \) satisfies \( \overline{\alpha} = \alpha \). In summary:

**Proposition 3.3.** Let \( E \rightarrow M \) be a G-bundle with adjoint bundle \( g_E \). A connection \( A \) on \( E \) satisfies the selfdual contact instanton equation *\( F_A = \sigma \wedge F_A \) if, and only if,

\[ F_A \in \Omega^{1,1}(g_E), \quad \hat{F}_A := \langle F_A, \omega \rangle = 0 \quad \text{and} \quad F_A = \sigma. \]

Proof. For a contact instanton \( A \), we know from item (i) of Lemma 2.2 that \( F_A \in \langle \omega \rangle^1 \subset \Omega^{1,1}(g_E) \), i.e. \( \hat{F}_A = 0 \).

The last conclusion follows from the discussion just above, since the generators \( \{w_i\}_{i=1}^8 \) satisfy \( \omega_i = \overline{\omega_i} \). \( \square \)

3. **Gauge theory on 7-dimensional Sasakian manifolds**

In 4-dimensional gauge theory, reversing orientation of the base manifold interchanges selfdual and anti-selfdual connections, thus the two theories are equivalent. On the other hand, in the 5-dimensional contact case, the selfdual and anti-selfdual equations are studied separately [BH16] and some differences appear; for instance, the coboundary map in the long exact sequence in [BH16, Proposition 3.3] is only zero in the selfdual case.

In our contact 7-dimensional case, in view of Lemma 2.2, one could a priori study connections with curvature in \( \Omega^2_8 \) or \( \Omega^2_6 \) separately. However, on a Sasakian manifold \((M^7, \eta, \xi, g, \Phi)\), there is a natural choice favouring the selfdual \((\lambda = 1)\) contact instanton equation in (1.2):

\[ F_A = *(\eta \wedge d\eta \wedge F_A), \]

because the alternative theory is essentially trivial (see Proposition 3.1 below). To see this, let us adopt the point of view of [BS10] and extend Yang-Mills theory to the Sasakian context, as a ‘transversely Kähler’ geometry (Appendix A). On a Kähler manifold \( X \) with symplectic form \( \omega \), a connection \( A \) on a holomorphic vector bundle \( E \rightarrow X \) is called *Hermitian Yang-Mills (HYM)* if

\[ \hat{F}_A := \langle F_A, \omega \rangle = 0 \quad \text{and} \quad F_A^{0,2} = 0. \]

Now, if \( E \rightarrow M \) is a Sasakian holomorphic vector bundle (Definition A.5), the above HYM condition generalises identically, interpreting \( \omega := d\eta \in \Omega^{1,1}(M) \) as the transverse Kähler form. It then follows immediately from Lemma 2.2, (ii):

**Proposition 3.1.** Let \( A \in \mathcal{A}(E) \) be a connection on a Hermitian holomorphic vector bundle \( E \rightarrow M \) over a Sasakian 7-manifold, such that \( F_A \in \Omega^2_8(g_E) \); if \( A \) is simultaneously compatible with the transverse holomorphic structure (i.e. integrable) and the Hermitian structure (i.e. unitary), then its curvature \( F_A \) vanishes (i.e. \( A \) is flat).
3.1 Infinitesimal deformations of contact instantons

The main purpose this Section is to prove part (i) of Theorem 1.2. Namely, the linearisation of the moduli space of selfdual contact instantons \( \mathcal{M}^* \) is given by the cohomology group \( H^1(\mathcal{C}) := \ker(\delta^1) / \text{Im}(\delta^2) \) of the deformation complex

\[
C^1 : 0 \to \Omega^0(\mathfrak{g}_E) \xrightarrow{d_A} \Omega^1(\mathfrak{g}_E) \xrightarrow{d^2} \Omega^2_{1}(\mathfrak{g}_E) \to 0.
\]

The complex \( C^1 \) is manifestly not elliptic, but one can show that \( H^1(\mathcal{C}) \) is finite-dimensional by means of an auxiliary extended complex. We conclude this Section by relating contact instantons and \( G_2 \)-instantons in the 7-dimensional contact Calabi-Yau setting, as outlined in [CARSE].

Under the hypotheses of Theorem 1.2, we will describe the linearisation of the moduli space of contact instantons over a Sasakian 7-manifold, following the approach of [Ito83].

**Proposition 3.2.** An element \( \alpha \in \Omega^1(\mathfrak{g}_E) \) gives an infinitesimal deformation of a selfdual contact instanton \( A \) preserving selfduality if, and only if, \( \alpha \in \ker(d_7) \).

**Proof.** Given a connection \( A \) on the \( G \)-bundle \( E \to M \), the induced covariant exterior derivative \( d_A : \Omega^k(\mathfrak{g}_E) \to \Omega^{k+1}(\mathfrak{g}_E) \) squares to an algebraic curvature operator:

\[
d_A \circ d_A(\beta) = [\beta \wedge F_A], \quad \beta \in \Omega^k(\mathfrak{g}_E).
\]

On the other hand, the transverse complex structure splits \( d_A = \partial^A + \overline{\partial}^A \) into (1, 0) and (0, 1) components, acting on \( \Omega^{p,q}_H(\mathfrak{g}_E) \) (2.13) according to bi-degree decomposition,

\[
\begin{align*}
\Omega^{p,q}_H(\mathfrak{g}_E) &\xrightarrow{\partial^A} \Omega^{p+1,q}_H(\mathfrak{g}_E) \\
&\xleftarrow{\overline{\partial}^A} \Omega^{p,q+1}_H(\mathfrak{g}_E).
\end{align*}
\]

Let \( \alpha \in \Omega^{p,q}_H(\mathfrak{g}_E^C) \), and note that

\[
d_A \circ d_A \alpha = \partial^A \partial^A \alpha + \overline{\partial}^A \overline{\partial}^A \alpha + (\partial^A \overline{\partial}^A + \overline{\partial}^A \partial^A) \alpha.
\]

If \( F_A \) is of type \((1,1)\) and orthogonal to \( \omega \), replacing (3.1) in the above formula and comparing bidegree, we obtain:

\[
\partial^A \partial^A \alpha = 0, \quad \overline{\partial}^A \overline{\partial}^A \alpha = 0, \quad (\partial^A \overline{\partial}^A + \overline{\partial}^A \partial^A) \alpha = [\alpha \wedge F_A].
\]

On the other hand, let \( \{A_t\}_{|t| < 1} \) be a path of connections with \( A_0 = A \) and curvature in \( \Omega^2_H(\mathfrak{g}_E) \). An infinitesimal deformation \( \alpha := \frac{d}{dt} \big|_{t=0} (A_t) \in \Omega^1(\mathfrak{g}_E) \) varies the curvature by

\[
F_{A_t} = F_A + td_A \alpha + t^2 \alpha \wedge \alpha.
\]

Decomposing \( \alpha = \alpha^{(1,0)} + \alpha^{(0,1)} \),

\[
d_A \alpha = (\partial^A + \overline{\partial}^A)(\alpha^{(1,0)} + \alpha^{(0,1)}) = \partial^A \alpha^{(1,0)} + \overline{\partial}^A \alpha^{(0,1)} + (\partial^A \overline{\partial}^A + \overline{\partial}^A \partial^A) \alpha^{(1,0)},
\]

so, by Lemma 2.2 item (i), the infinitesimal deformation \( \alpha \) satisfies:

\[
\partial^A \alpha^{(1,0)} = 0, \quad \overline{\partial}^A \alpha^{(0,1)} = 0, \quad \langle (\partial^A \alpha^{(0,1)} + \overline{\partial}^A \alpha^{(1,0)}), \omega \rangle = 0.
\]

We recall the definition of \( d_7 \) in (1.9), hence the above relations give

\[
d_7(\alpha^{(1,0)} + \alpha^{(0,1)}) = \partial^A \alpha^{(1,0)} + \omega \otimes (\partial^A \alpha^{(0,1)} + \overline{\partial}^A \alpha^{(1,0)}), \omega) + \overline{\partial}^A \alpha^{(0,1)}.
\]
The linearisation of the action $G \times \mathcal{A} \to \mathcal{A}$ at $A \in \mathcal{A}$ is $-d_A : \Omega^2(\mathfrak{g}_E) \to \Omega^1(\mathfrak{g}_E)$, so a natural transverse ‘Coulomb’ slice to the orbit of the $G$-action at $A$ is given by the orthogonal complement $\ker(d_A^*)$ of $\text{Im}(d_A) \subset \Omega^1(\mathfrak{g}_E)$ [DK90, p. 131]. For small $\varepsilon > 0$, we set:

$$T_\varepsilon(A) := \{ \alpha \in \Omega^1(M) \mid d_A^*(\alpha) = 0, \| \alpha \| < \varepsilon \},$$

(3.2)

where the norm $\| \cdot \|$ is given by the Sobolev norms, a neighbourhood of $[A] \in \mathcal{B}$ is described by a quotient of $T_\varepsilon(A)$ [DK90, Eq. (4.2.6)]. By the slice condition described above we can consider the restriction of $d_7$ (1.9) on $\ker(d_A^*)$ as follows:

$$d_7|_{\ker(d_A^*)} : \ker(d_A^*) \subset \Omega^1(\mathfrak{g}_E) \to \Omega^2_{\mathfrak{g} \oplus 1}(\mathfrak{g}_E).$$

**Proposition 3.3.** In the context of Theorem 1.2, the infinitesimal deformations of an irreducible selfdual contact instanton $A$ are described by the complex

$$C^* : \quad 0 \to \Omega^0(\mathfrak{g}_E) \xrightarrow{d_A} \Omega^1(\mathfrak{g}_E) \xrightarrow{d_7} \Omega^2_{\mathfrak{g} \oplus 1}(\mathfrak{g}_E) \to 0.$$  

(3.3)

The tangent space $T_{[A]}M^*$ is isomorphic to the first cohomology group $H^1(C) := \ker(d_7)/\text{Im}(d_A^*)$

**Proof.** Consider a one-parameter family $\{ f_t \} \subset G$ of gauge transformations such that $f_0 = \text{id}$, the derivative $\dot{f} := \frac{df}{dt}|_{t=0}$ is a smooth section of the vector bundle $\mathfrak{g}_E$, i.e., $\dot{f} \in \Omega^0(\mathfrak{g}_E)$. Conversely given a smooth section $f \in \Omega^0(\mathfrak{g}_E)$, we obtain a one-parameter family $\{ f_t := \exp(tf) \} \subset G$ of gauge transformations.

Denote by Lie($G$) the Lie algebra of $G$, and $\mathcal{A} \subset \mathcal{A}$ the space of connections satisfying the SD contact condition and note that the infinitesimal action of Lie($G$) on $\mathcal{A}$ defines for any $f \in \text{Lie}(G)$ a tangent vector $X_f$ on $\mathcal{A}$ given by $X_f(A) = -d_A(f)$. The connection form $B^{\text{hf}}(A)$ satisfies

$$B^{\text{hf}}(A) = df_t \cdot f_t^{-1} + \text{Ad}(f_t)A,$$

since $f_t = 1 + t\dot{f} + o(t)$ follows that:

$$B^{\text{hf}}(A) = A + td_A\dot{f} + o(t),$$

Since $f_t(A)$ is contact instanton, by Proposition 3.2, we have $d_A\dot{f} \in \ker(d_7)$. \qed

Proposition 3.3 shows part (i) of Theorem 1.2. Note that, in contrast to the 4-dimensional case, the complex $C^*$ in (1.10) is not elliptic (compare bundle ranks in (3.3)), so we do not know a priori that $H^1(C)$ is finite-dimensional. In the next section, we show in Proposition 3.7 that $H^1(C)$ is indeed finite-dimensional, but we must resort to an auxiliary construction.

### 3.2 The extended complex of contact instantons

This section is devoted to proving that the first cohomology group $H^1(C)$ in (1.10) is finite-dimensional.

**Lemma 3.4.** Let $\sigma = \eta \wedge d\eta$ the natural 3-form on the Sasakian manifold $(M, \eta, \xi, g, \Phi)$ defined in (1.5); using the exterior product, define the map:

$$L_{*\sigma} : \quad \Omega^2(M) \to \Omega^6(M) \quad \alpha \mapsto \alpha \wedge *\sigma,$$

then $L_{*\sigma}$ satisfies

$$L_{*\sigma}|_{\Omega^2 \oplus \Omega^2_\xi} \equiv 0,$$

and

$$L_{*\sigma}|_{\Omega^2_\xi \oplus \Omega^2} : \Omega^2_\xi \oplus \Omega^2 \to \Omega^6(M).$$

**Proof.** The transverse symplectic 2-form $d\eta = \omega$ is given in local Darboux coordinates [Bla10, Theorem. 3.1] by $d\eta = dx^{14} + dx^{25} + dx^{36}$, so $d\eta^2 = -2(dx^{1245} + dx^{1346} + dx^{2356})$, and therefore $*\sigma$ is given by:

$$* \sigma = -\left( dx^{2356} + dx^{1346} + dx^{1245} \right) = \frac{1}{2}d\eta^2.$$  

(3.5)

Using the decomposition (2.7) of $\Omega^2(M)$ to compute $L_{*\sigma}$ in the basis $\{ w_1 \}_{i=1}^8$ and $\{ v_j \}_{j=1}^6$ given in (2.6), we obtain by a direct computation that $L_{*\sigma}|_{\Omega^2_\xi \oplus \Omega^2_\xi} \equiv 0$ and $L_{*\sigma}|_{\Omega^2_\xi \oplus \Omega^2}$ is an isomorphism. \qed
We are now in position to prove that the first cohomology group $H^1(C)$ in (1.10) is finite-dimensional. From Lemma 3.4 and 3.10, we deduce the following:

**Lemma 3.5.** The operator $d_7 : \Omega^1(g) \to \Omega^2(g) \oplus \Omega^2(g)$ in the complex $C^\bullet$ of (1.10) can be identified with the exterior product map $L_{s \sigma} \circ d_A$ defined in (3.4).

**Proof.** It suffices to consider the maps $L_{s \sigma}$ (3.4), which is an isomorphism when it is restricted to $\Omega^2 \oplus \Omega^2$ (see Lemma 3.4), as summarised in the following diagram:

\[
\begin{array}{c}
\Omega^1(g) \xrightarrow{d_7 := \text{pod}_A} \Omega^2(g) \\
\downarrow{d_A} \quad \downarrow{P} \\
\Omega^2(g) \xrightarrow{L_{s \sigma}} \Omega^6(M) \\
\end{array}
\]

From the definition (1.8) of $M$, we know that this space is described (modulo gauge) near an instanton $A$ as the zero locus of the map

\[
\Psi : \alpha \in T_c(A) \subset A \to \frac{d_A \alpha + \alpha \wedge \alpha}{\Omega^2_{\oplus \mathbb{Q}^1}(g)} ,
\]

where the neighbourhood $T_c(A)$ from (3.2) is transversal to $G$-orbits. Let us check that $\Psi$ is a Fredholm map, so that standard theory provides a finite-dimensional local model for $M$. The linearisation of $\Psi$ at the origin is

\[
D(\Psi)_0 = p \circ d_A = d_7,
\]

and $d_7 |_{\text{ker}(d_A)} : \Omega^1(g) \to \Omega^2_{\oplus \mathbb{Q}^1}(g)$ is shown to be Fredholm via the ‘Euler characteristic’ map

\[
\mathbb{D}_A := d_7 \oplus d_A^* : \Omega^1(g) \to (\Omega^2_{\oplus \mathbb{Q}^1} \oplus \Omega^0)(g)
\]

associated to the complex $C^\bullet$ (1.10). Note that:

\[
H^0(C) \cong \ker(d_A), \quad H^1(C) := \frac{\ker(d_7)}{\text{Im}(d_A)} \cong \ker(\mathbb{D}_A) \quad \text{and} \quad H^2(C) \cong \text{Coker}(d_7).
\]

By Lemma 3.5, we identify $d_7$ with $L_{s \sigma} \circ d_A : \Omega^1(g) \to \Omega^6(g)$, then we can consider the extended complex

\[
D^\bullet : 0 \longrightarrow \Omega^0(g) \xrightarrow{d_A} \Omega^1(g) \xrightarrow{L_{s \sigma} \circ d_A} \Omega^6(g) \xrightarrow{d_A} \Omega^7(g) \longrightarrow 0 .
\]

If (3.8) is elliptic, then $\mathbb{D}_A$ is Fredholm and, in particular, $\ker(\mathbb{D}_A)$ is finite-dimensional. To see that explicitly, we will need the following elementary facts:

**Lemma 3.6.**

(i) **Let $L_{s \sigma} : \Omega^2(M) \to \Omega^6(M)$ the operator defined in (3.4), then $[L_{s \sigma}, d_A] = 0$.**

(ii) **The formal adjoint of $d_7$ defined by (1.9) is given by $d_7^* = * d_7^* : \Omega^6(g) \to \Omega^1(g)$.**

**Proof.** For (i), note from (3.5) that $d(\tau) = d(d_7^2) = 0$, so the assertion is straightforward:

\[
d_A(L_{s \sigma}) = d_A(\tau \wedge \alpha) = (d \tau) \wedge \alpha + \tau \wedge (d_A \alpha) = L_{s \sigma}(d_A \alpha).
\]

For (ii) we use (i) and the identification in Lemma 3.5; for any $\alpha \in \Omega^1(g)$ and $\beta \in \Omega^6(g)$,

\[
\langle d_7 \alpha, \beta \rangle = \langle \tau \wedge d_A \alpha \rangle + \langle \beta \rangle = d_A \alpha \wedge \tau \wedge \beta = d \tau \wedge \alpha \wedge \beta
\]

\[
= \langle d_A \alpha, \tau \wedge (d_A \beta) \rangle
\]

\[
= \langle \alpha, * d_7 \tau \wedge (\beta) \rangle
\]

\[
= \langle \alpha, * d_7 (\beta) \rangle
\]

\[
\square
\]

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Proposition 3.7. If \( A \) is a connection with curvature \( F_A \in \Omega^2_G \), the extended complex (3.8) is elliptic.

Proof. We follow the argument of [SE09, Proposition 1.22]. Fix a non zero section \( \zeta \) of \( \pi: T^* M \setminus \{0\} \to M \), so we have the symbol complex

\[
0 \to \pi^* (\Omega^0(\mathfrak{g}_E)) \xrightarrow{\zeta} \pi^* (\Omega^1(\mathfrak{g}_E)) \xrightarrow{\zeta \wedge \cdot} \pi^* (\Omega^2(\mathfrak{g}_E)) \xrightarrow{\zeta} \pi^* (\Omega^7(\mathfrak{g}_E)) \xrightarrow{\zeta} 0.
\]  

(3.9)

To see the exactness of (3.9) at the middle, take \( \alpha \in \Omega^1(\mathfrak{g}_E) \) such that \( \ast \sigma \wedge \zeta \wedge \alpha = 0 \). We need to show that \( \alpha \) lies in \((\Omega^0(\mathfrak{g}_E)) \zeta \). Note that \( \zeta \wedge \alpha \in \ker(L_{\ast \sigma}) = \Omega^2_G \oplus \Omega^6_G \) (cf. (3.4)), so, by definition of the eigenspaces \( \Omega^2_G \) and \( \Omega^6_G \) (cf. (2.6)), we obtain:

\[
\pm \sigma \wedge \zeta \wedge \alpha = \ast (\zeta \wedge \alpha).
\]  

(3.10)

We show that (3.10) implies that \( \zeta \wedge \alpha = 0 \) and this finish the proof. Let \( \{e^i\}_{i=1}^7 \) be a basis of \( T^*_x M \) with \( e^1 = \zeta \). Then \( \mu \in \Lambda^k T^*_x M \), can be written

\[
\mu = e^1 \wedge \gamma + \beta
\]

where \( \gamma \) and \( \beta \) are products just involving \( e^2, \cdots, e^7 \). Let \( \alpha = \zeta \wedge \gamma + \beta \), \( \eta = \zeta \wedge \gamma' + \beta' \) and \( d\eta = \zeta \wedge \gamma'' + \beta'' \) where \( \gamma, \gamma', \gamma'' \) and \( \beta, \beta', \beta'' \) are products just involving \( e^2, \cdots, e^7 \), hence (3.10) becomes

\[
\eta \wedge d\eta \wedge \zeta \wedge \alpha = (\zeta \wedge \gamma' + \beta') \wedge (\zeta \wedge \gamma'' + \beta'') \wedge \zeta \wedge (\zeta \wedge \gamma + \beta) = \beta' \wedge \beta'' \wedge \zeta \wedge \beta
\]

so \( \beta' \wedge \beta'' \wedge \zeta \wedge \beta = \ast (\zeta \wedge \beta) \) the left-hand side of the above equality involves \( \zeta \), while the right-hand side does not, so \( \zeta \wedge \beta = 0 \). Hence

\[
\zeta \wedge \alpha = \zeta \wedge (\zeta \wedge \gamma + \beta) = \zeta \wedge \beta = 0
\]

as claimed, and so \( \alpha = f \wedge \zeta \). \( \square \)

Together, Propositions 3.3 and 3.7 prove part (i) of Theorem 1.2, i.e., that the space of infinitesimal deformations of \( A \) is finite-dimensional and isomorphic to the first cohomology \( H^1(C) \) of the complex (1.10). In Section 4, we address such deformations from another perspective, showing that this local model is isomorphic to the first cohomology group of the complex (1.11) [cf. Theorem 1.2 item (ii)]. As observed in Remark 1.1, to assume that \( H^2(C) \) vanishes would in general be much too strong, leading to a 0-dimensional local model. Instead, we can show the smoothness of the moduli space under a weaker obstruction theory, in terms of \( H^2_B \) [cf. Proposition 4.25].

3.3 \( G_2 \)-instantons on contact Calabi-Yau manifolds

The Sasakian structure allows us to define and study a moduli space of selfdual contact instantons. We now examine the contact Calabi-Yau case, in which the Sasakian \( \tilde{7} \)-manifold \( (M, \eta, \xi, \Phi) \) is also equipped with a transverse \( SU(3) \)-structure, and hence with a natural \( G_2 \)-structure [HV15], as somewhat of an interpolation between CY-geometry and \( G_2 \)-geometry. We explain the relationship between \( G_2 \)-instantons and contact instantons (see Proposition 3.12) in that context, following the approach of [CARSE].

Definition 3.8. A Sasakian manifold \((M, \eta, \xi, \Phi, \varepsilon)\) is said to be a contact Calabi-Yau manifold (cCY) if \( \varepsilon \) is a nowhere vanishing transverse form of horizontal type \((n,0)\) [cf. (A.1)] such that

\[
\varepsilon \wedge \varepsilon = (-1)^{n(n+1)}i^n\omega^n \quad \text{and} \quad d\varepsilon = 0.
\]

Recall that, for a Calabi-Yau 3-fold \((Z, \omega, \varepsilon)\), the product \( Z \times S^1 \) has a natural torsion-free \( G_2 \)-structure defined by: \( \varphi := dt \wedge \omega + \Im(\varepsilon) \), where \( t \) is the variable in \( S^1 \). The Hodge dual of \( \varphi \) is given by

\[
\psi := \ast \varphi = \frac{1}{2} \omega \wedge \omega + dt \wedge \Re(\varepsilon)
\]

(3.11)

and the induced metric \( g_\varphi = g_Z + dt \otimes dt \) is the Riemannian product metric on \( Z \times S^1 \) with holonomy \( \text{Hol}(g_\varphi) = SU(3) \) properly contained in \( G_2 \). The contact Calabi-Yau structure essentially emulates all of these properties, albeit its \( G_2 \)-structure now has some symmetric torsion. Sasakian manifolds with transverse holonomy \( SU(n) \) are studied by Habib and Vezzoni; a number of facts from [HV15, Section 6.2.1] can be summarised as follows:
Proposition 3.9. Let \((M, \eta, \xi, \Phi, \varepsilon)\) be a cCY 7-manifold. Then \(M\) carries a cocalibrated \(G_2\)-structure defined by
\[
\varphi = \eta \wedge \omega + \text{Im}(\varepsilon) = \sigma + \text{Im}(\varepsilon),
\]
with torsion \(d\varphi = \omega \wedge \omega\) and Hodge dual 4-form \(\psi = *\varphi = \frac{1}{2} \omega \wedge \omega + \eta \wedge \text{Re}(\varepsilon)\), and \(\sigma = \eta \wedge d\eta\) as in (1.5).

Now, considering the Sasakian manifold \((M, \eta, \xi, g, \Phi)\) to be cCY manifold (see Definition 3.8) we have the following

Lemma 3.10. Let \((M, \eta, \xi, g, \Phi)\) be cCY manifold and \(\varepsilon\) be the transverse holomorphic volume form of type \((3,0)\) in Definition 3.8; the operator
\[
L_\varepsilon: \Omega^2(M) \to \Omega^2(M)
\]
\[
\alpha \mapsto * (\alpha \wedge \text{Im}(\varepsilon)),
\]
(3.13)
satisfies \(L_\varepsilon|_{\Omega^2_\varepsilon(M)} : \Omega^2_\varepsilon(M) \to \Omega^2_{\varepsilon}(M)\) is an isomorphism.

Proof. In the local Darboux transverse complex coordinates \(z_i\) defined in (2.11), we have \(\varepsilon = dz^1 \wedge dz^2 \wedge dz^3\). In particular, \(\text{Im}(\varepsilon) = (dx^{234} + dx^{126}) - (dx^{135} + dx^{456})\). Using the decomposition of 2-forms given in (2.7) to compute \(L_\varepsilon\) in the basis given in (2.6) by \(\{w_i\}_{i=1,\ldots,8}\) and \(\{v_j\}_{j=1,\ldots,6}\), we obtain immediately \(L_\varepsilon|_{\Omega^2_\varepsilon(M)} = 0\) and, by inspection,
\[
L_\varepsilon(dx^{17}) = -v_5, \quad L_\varepsilon(dx^{27}) = -v_3, \quad L_\varepsilon(dx^{37}) = -v_1, \quad L_\varepsilon(dx^{47}) = -v_6, \quad L_\varepsilon(dx^{57}) = -v_4, \quad L_\varepsilon(dx^{67}) = -v_2.
\]
So \(L_\varepsilon|_{\Omega^2_\varepsilon(M)} : \Omega^2_\varepsilon(M) \to \Omega^2_{\varepsilon}(M)\) is an isomorphism.

The \(G_2\)-instanton equation on a cCY 7-manifold reads [CARSE, Example 6]:
\[
* ((\eta \wedge \omega + \text{Im}(\varepsilon)) \wedge F_A) = F_A
\]
(3.14)
or, equivalently, \(F_A \wedge \psi = 0\), where \(\psi\) is the dual 4-form defined in (3.11).

Proposition 3.11. On a 7-dimensional cCY manifold \((M, \eta, \Phi, \varepsilon)\) (Proposition 3.9), every selfdual contact instanton solution of (1.2) is a \(G_2\)-instanton solution of (3.14).

Proof. If \(A\) is a selfdual contact instanton, then \(F_A = * (\sigma \wedge F_A) \in \Omega^2_{\varepsilon}(\mathfrak{g}_{E}) \subset \ker L_\varepsilon\), by Lemma 3.10. Therefore
\[
* (\varphi \wedge F_A) = * (\sigma \wedge F_A) + L_\varepsilon(F_A) = F_A.
\]

If the complex vector bundle \(E \to M\) has Sasakian holomorphic structure (see A.5), then, at least among Chern connections (mutually compatible with the holomorphic structure and the Hermitian metric), the sets of solutions of both equations actually coincide:

Proposition 3.12. Let \(E \to M\) be a Sasakian holomorphic vector bundle [cf. Definition A.5] on a cCY 7-manifold with its natural \(G_2\)-structure (3.12). Then a Chern connection \(A\) on \(E\) is a \(G_2\)-instanton if, and only if, \(A\) is a selfdual contact instanton as in (2.4).

Proof. If \(A\) is the Chern connection, then \(F_A \in \Omega^{1,1}(M)\) [cf. Proposition A.2], so taking account of the bi-degree of the transverse holomorphic volume form \(\varepsilon\) (Definition 3.8), it follows that \(F_A \wedge \text{Im}(\varepsilon) = 0\). Therefore
\[
F_A \wedge \varphi = F_A \wedge (\eta \wedge d\eta + \text{Im}(\varepsilon)) = F_A \wedge \sigma + F_A \wedge \text{Im}(\varepsilon)
\]
\[
= F_A \wedge \sigma.
\]
3.4 Yang-Mills theory and Chern-Simons action

We will describe two natural gauge-theoretic action functionals in this context, adapting the approach of [SE14]. Let $E \to M$ a Sasaki bundle (see Definition A.4), the Yang-Mills functional is defined by

$$\text{YM}: A \in \mathcal{A}(E) \mapsto \| F_A \|^2 := \int_M \langle F_A \wedge \ast F_A \rangle_{\mathcal{B}E}.$$  \hspace{1cm} (3.15)

Since the curvature $F_A$ splits orthogonally (cf. (2.3)) into horizontal and vertical parts $F_H = i_\xi(\eta \wedge F_A)$ and $F_V = \eta \wedge i_\xi F_A$ respectively, it has two independent components: $\| F_A \|^2 = \| F_H \|^2 + \| F_V \|^2$. Moreover, the horizontal part further splits orthogonally, by Lemma 2.1:

$$F_H = \frac{1}{2} (F_H^+ \ast (\sigma \wedge F_H)) + \frac{1}{2} (F_H^- \ast (\sigma \wedge F_H)),$$

hence

$$\text{YM}(A) = \| F_H^+ \|^2 + \| F_H^- \|^2 + \| F_V \|^2.$$  \hspace{1cm} (3.16)

Given $A \in \mathcal{A}(E)$, the charge of $A$ is defined by

$$\kappa(A) := \int_M \text{Tr}(F_A^2) \wedge \sigma.$$  \hspace{1cm} (3.17)

**Proposition 3.13.** Let $A \in \mathcal{A}(E)$ be a connection on $E$, the charge (3.17) is determined by the horizontal curvature:

$$\kappa(A) = \| F_H^+ \|^2 - \| F_H^- \|^2.$$  \hspace{1cm} (3.18)

Furthermore,

$$| \kappa(A) | \leq \| F_H^+ \|^2 \leq \text{YM}(A).$$  \hspace{1cm} (3.19)

These bounds are saturated if and only if $A$ is (anti)-selfdual contact instanton.

**Proof.** It suffices to show (3.18), because then (3.19) follows from (3.16). Using the definition of $\kappa(A)$ in (3.17) and the definition of $F_H^+$ given above, we compute:

$$\| F_H^+ \|^2 = \frac{1}{2} \int_M \text{Tr} \left( F_H^+ \wedge \ast (F_H^+ \ast (\sigma \wedge F_H)) \right)$$

$$= \frac{1}{2} \left( \int_M \text{Tr} (F_H^+ \wedge F_H) + \int_M \text{Tr} (F_H^+ \wedge \sigma \wedge F_H) \right)$$

$$= \frac{1}{4} \left( \int_M \text{Tr} ((F_H + \ast (\sigma \wedge F_H)) \wedge F_H) + \int_M \text{Tr} ((F_H + \ast (\sigma \wedge F_H)) \wedge \sigma \wedge F_H) \right)$$

$$= \frac{1}{4} \left( \| F_H \|^2 + \int_M \text{Tr} \left( \ast (\sigma \wedge F_H) \wedge \ast F_H \right) + \int_M \text{Tr} \left( \sigma \wedge F_H^2 \right) + \| \sigma \wedge F_H \|^2 \right)$$

$$= \frac{1}{4} \left( \| F_H \|^2 + 2\kappa(A) + \| \sigma \wedge F_H \|^2 \right).$$

Analogously,

$$\| F_H^- \|^2 = \frac{1}{2} \int_M \text{Tr} \left( F_H^- \wedge \ast (F_H^- \ast (\sigma \wedge F_H)) \right)$$

$$= \frac{1}{4} \left( \| F_H \|^2 - 2\kappa(A) + \| \sigma \wedge F_H \|^2 \right).$$

Therefore, $\kappa(A) = \| F_H^+ \|^2 - \| F_H^- \|^2$. Now using the inequality $|a^2 - b^2| \leq a^2 + b^2$, we obtain the following bounds:

$$| \kappa(A) | \leq \int_M \text{Tr}(F_H \wedge \ast F_H).$$
moreover
\[ |\kappa(A)| \leq \int_M \text{Tr}(F_A \wedge *F_A). \]

Under the hypothesis \( F_V = 0 \) we have that: if \( A \) is selfdual contact instanton, in which case \( F_H^+ = 0 \), follows from (3.18) that the last bound is saturated. \( \square \)

**Definition 3.14.** Fix a reference connection \( A_0 \in \mathcal{A}(E) \). The Chern-Simons action is defined by: \( \text{CS}(A_0) = 0 \) and
\[
\text{CS}(A_0 + \alpha) := \frac{1}{2} \int_M \text{Tr}(dA_0 \wedge \alpha + \frac{2}{3} \alpha \wedge \alpha \wedge \alpha) \wedge \sigma. 
\]

**Lemma 3.15.** On a Sasakian vector bundle \( E \to M \) (Definition A.5) over a closed Sasakian 7-manifold; for any connection \( A' = A + \alpha \), where \( \alpha \in \Omega^1(\mathfrak{g}_E) \), on the underlying Sasakian bundle \( E \), we have:
\[
\kappa(A) = \kappa(A') + \int_M \text{Tr}(F_A \wedge \alpha \wedge d\eta^2). \tag{3.20}
\]

**Proof.** Let \( A \in \mathcal{A}(E) \) and let \( \alpha \in \Omega^1(\mathfrak{g}_E) \) be a variation of \( A \). From standard Chern-Weil theory, we know that:
\[
\text{Tr}(F_{A+\alpha}^2) - \text{Tr}(F_A^2) = d(\text{Tr} \delta)
\]
where \( \delta = \delta(A, \alpha) \in \Omega^1(\mathfrak{g}_E) \) given by \( \delta = F_A \wedge \alpha + \frac{1}{2} d_A \alpha \wedge \alpha + \frac{1}{3} \alpha \wedge \alpha \wedge \alpha. \) Since \( M \) is closed, by Stokes’ theorem we obtain
\[
\kappa(A) := \int_M \text{Tr}(F_A^2) \wedge \sigma = \int_M (\text{Tr}(F_{A+\alpha}^2) - d(\text{Tr} \delta)) \wedge \sigma \\
= \kappa(A') - \left( \int_M d(\text{Tr} \delta \wedge \sigma) - \int_M \text{Tr} \delta \wedge d\sigma \right) \\
= \kappa(A') + \int_M \text{Tr} \delta \wedge d\sigma.
\]

Now, we analyze the term \( \int_M \text{Tr} \delta \wedge d\sigma \) for a transverse 1-form \( \alpha \in \Omega^1(\mathfrak{g}_E) \):
\[
\int_M \text{Tr} \delta \wedge d\sigma = \int_M \text{Tr}(F_A \wedge \alpha + \frac{1}{2} d_A \alpha \wedge \alpha + \frac{1}{3} \alpha \wedge \alpha \wedge \alpha) \wedge d\eta^2 \\
= \int_M \text{Tr}(F_A \wedge \alpha) \wedge d\eta^2 + \frac{1}{2} \int_M \text{Tr}(d_A \alpha \wedge \alpha) \wedge d\eta^2 + \frac{1}{3} \int_M \text{Tr}(\alpha \wedge \alpha \wedge \alpha) \wedge d\eta^2 \\
= \int_M \text{Tr}(F_A \wedge \alpha) \wedge d\eta^2.
\]
The last equality holds since the 7-form \( \text{Tr}(\alpha \wedge \alpha \wedge \alpha) \wedge d\eta^2 \) is basic; the same is true for \( \text{Tr}(d_A \alpha \wedge \alpha) \wedge d\eta^2 \). \( \square \)

**Corollary 3.16.** Among Chern connections, the charge is independent of the Hermitian structure; it is a Sasakian holomorphic topological invariant, denoted by \( \kappa(\mathcal{E}) \).

**Proof.** Fixing a reference Chern connection \( A_0 \in \mathcal{A}(E) \), we know from Proposition A.2 that \( F_{A_0} \wedge d\eta^2 \) has type \( (3,3) \), so the defect term \( \int_M \text{Tr}(F_{A_0}) \wedge d\eta^2 \wedge \alpha \) vanishes, by excess in bi-degree, for any \( \alpha \in \Omega^{1,0}(\mathfrak{g}_E) \). Therefore \( \kappa(A_0) = \kappa(A_0 + \alpha) \). \( \square \)

Now, let \( A \in \mathcal{A}(E) \) be a connection on \( E \), using (3.16) and (3.18) we obtain
\[
\text{YM}(A) = \kappa(A) + 2\|F_H^-\|^2 + \|F_V\|^2 \\
= -\kappa(A) + 2\|F_H^+\|^2 + \|F_V\|^2
\]
It follows that YM attains absolute minimum at selfdual contact instantons, i.e., \( F_H^- = 0 \) and \( F_V = 0 \) or at anti-selfdual contact instantons, i.e., \( F_H^+ = 0 \) and \( F_V = 0 \). Furthermore, from Corollary 3.16 among Chern connections we can replace in the above equalities \( \kappa(A) = \kappa(\mathcal{E}) \) hence, the sign of \( \kappa(\mathcal{E}) \) obstructs the existence of one or the other type of solution.
4 The moduli space of contact instantons in 7-dimensions

We showed in Section 3 that there exists a finite-dimensional local model for the moduli space near an irreducible selfdual contact instanton. In this Section we will show part (ii) of Theorem 1.2, namely, that its dimension can be computed from an associated transverse elliptic complex. Our strategy is inspired by [BH16, Section 3].

4.1 The associated elliptic complex of a contact instanton

Consider $E \to M$ a $G$-bundle on a Sasakian manifold $(M, \eta, \xi, g, \Phi)$ and $g_E$ its adjoint bundle. We recall that $\Omega^k_H(g_E)$ denotes the space of horizontal forms, i.e., $\alpha \in \Omega^k_H(g_E)$ for which $i_\xi \alpha = 0$. We introduce the following operators:

$$
\begin{align*}
\delta_V &: \Omega^k_H(g_E) \to \Omega^k_H(g_E), & d_T &: \Omega^k_H(g_E) \to \Omega^{k+1}_H(g_E)
\end{align*}
$$

Moreover, $\alpha \in \Omega^k_H(g_E)$ is called basic if $i_\xi d_A \alpha = 0$; we denote by $\Omega^k_B(g_E)$ the space of basic forms.

**Remark 4.1.** The operators $d_T$ and $d_V$ are defined in [BH16, Section 3.1] for a 5-dimensional contact manifold. Note that the restriction of the Lie derivative along the Reeb field $L_\xi|_{\Omega^k_B(g_E)} = d_V$, thus $\Omega^k_B(g_E) = \text{ker}(d_V)$. Moreover, we can write $d_T(\alpha) = i_\xi (\eta \wedge d_A \alpha)$, i.e., $d_T \alpha$ is just the horizontal part of $d_A \alpha$. In particular, $d_T$ coincides with the usual covariant exterior differential $d_A$ on basic forms.

**Remark 4.2.** A contact metric manifold $(M, \eta, \xi, g, \Phi)$ such that the Reeb field $\xi$ is Killing is called a $K$-contact manifold. Every Sasakian manifold is a $K$-contact manifold but, at dimensions greater than 3, the $K$-contact condition is weaker than the Sasaki condition. Yet, if the manifold is compact and Einstein both notions indeed coincide [BG01].

**Remark 4.3.** Even though almost all results in this section hold for $(M^7, \eta, \xi, g, \Phi)$ just compact Sasakian, in Proposition 4.17 we must be actually closed.

The graded ring $(\Omega^\bullet(g_E), \wedge)$ has a natural graded Lie algebra structure, given in local coordinates by the bi-linear map

$$
[\cdot, \cdot] : \Omega^p(g_E) \times \Omega^q(g_E) \to \Omega^{p+q} (g_E)
$$

The following properties are immediate to check, for $\Phi \in \Omega^p(g_E)$, $\Psi \in \Omega^q(g_E)$ and $\Theta \in \Omega^r(g_E)$:

$$
\begin{align*}
[[\Phi \wedge \Psi] \wedge \Theta] + (-1)^{pq+r}[[\Psi \wedge \Theta] \wedge \Phi] + (-1)^{pr+qr}[[\Theta \wedge \Phi] \wedge \Psi] &= 0, \\
[\Phi \wedge \Psi] &= (-1)^{pq}[[\Psi \wedge \Phi]].
\end{align*}
$$

The next result is a 7-dimensional adaptation of [BH16, Lemma 3.1]:

**Lemma 4.4.** Let $I \subset \Omega^\bullet(g_E)$ be the algebraic ideal of the graded Lie algebra $(\Omega^\bullet(g_E), \wedge)$ generated by $\Omega^2_B(g_E)$; if the Reeb field $\xi$ is a Killing vector field, then $d_A(I) \subset I$.

**Proof.** Since $I$ lies in the image of the exterior product $\Omega^2_B(g_E) \otimes \Omega^\bullet(g_E) \to \Omega^\bullet(g_E)$, by linearity it suffices to show that $d_A(\alpha \otimes \psi) \in I$, where $\alpha \in \Omega^2_B$ in (2.6) and $\psi \in \Omega^0(g_E)$. Indeed,

$$
\begin{align*}
d_A(\alpha \otimes \psi) &= d_A \alpha \otimes \psi + \alpha \wedge d_A \psi \\
&= \eta \wedge d_V \alpha \otimes \psi + d_T \alpha \otimes \psi + \alpha \wedge d_A \psi,
\end{align*}
$$

and the last two terms clearly lie in $I$. In order to show that $\eta \wedge d_V \alpha \in I$, we check that $d_V \alpha = i_\xi d_A \alpha = L_\xi \alpha$ belongs to $\Omega^2_B$. By the eigenspace decomposition (2.6), if $\alpha \in \Omega^2_B$, then $\alpha = * (\sigma \wedge \alpha)$. Furthermore, since $\xi$ is Killing, the Lie derivative $L_\xi$ commutes with the star Hodge operator, so

$$
\begin{align*}
L_\xi \alpha &= L_\xi * (\sigma \wedge \alpha) \\
&= * (L_\xi \eta \wedge d_\eta \wedge \alpha + \eta \wedge d_\eta d_\eta \wedge \alpha + \alpha \wedge L_\xi \eta) \\
&= * (\sigma \wedge L_\xi \eta).
\end{align*}
$$

We deduce that $L_\xi \alpha$ satisfies the instanton equation (2.4) for $\lambda = 1$, i.e., $L_\xi \alpha \in \Omega^2_B$. \[\square\]
Lemma 4.4 shows that $d_A$ descends to a derivation $D : L^k \to L^{k+1}$ on the quotient

$$L^* = L^*(g_E) := \Omega^* (g_E)/I,$$

which is indeed a complex, since $d^2_A = F_A \in \Omega^2 \subset I$. Therefore

$$(L^*, D) \quad (4.3)$$

is a differential graded Lie algebra.

**Remark 4.5.** The statement of Lemma 4.4 follows analogously if one takes, instead, the ideal $I = \text{Span} \{ \Omega^2_0 \}$, with $L^* = \Omega^*(g_E)/I$. This will have no further bearing in this article, since after all we know that the corresponding instantons in that case are trivial [Proposition 3.1], but, whatever the case, we have explicit characterisations for the spaces $L^k$ as follows.

**Proposition 4.6.** The spaces $L^k$ of complex (4.3), for $k = 0, \ldots, 3$, admit the following decompositions:

(i) For $I = \text{Span} \{ \Omega^2_0 (g_E) \}$,

$$L^0 \cong \Omega^0 (g_E), \quad L^2 \cong \Omega^2_0 (g_E) \oplus \Omega^2_1 (g_E) \oplus \eta \wedge \Omega^1_H (g_E),$$

$$L^1 \cong \Omega^1 (g_E), \quad L^3 \cong \eta \wedge (\Omega^2_0 (g_E) \oplus \Omega^2_1 (g_E)) .$$

(ii) For $I = \text{Span} \{ \Omega^2_0 (g_E) \}$,

$$L^0 \cong \Omega^0 (g_E), \quad L^2 \cong \Omega^2_0 (g_E) \oplus \Omega^2_1 (g_E) \oplus \eta \wedge \Omega^1_H (g_E),$$

$$L^1 \cong \Omega^1 (g_E), \quad L^3 \cong \eta \wedge (\Omega^2_0 (g_E) \oplus \Omega^2_1 (g_E)) .$$

In either case, $L^k = 0$ for $k \geq 4$.

**Proof.** For (i), combining $\Omega^k (g_E) = \Omega^k_H (g_E) \oplus \eta \wedge \Omega^k_H (g_E)$ in (A.5) and the natural decomposition of the space of $g_E$-valued 2-forms $\Omega^2 (g_E)$ induced by (2.7):

$$\Omega^2 (g_E) = \underbrace{\Omega^2_0 (g_E)}_{I_0} \oplus \Omega^2_1 (g_E) \oplus \Omega^2_2 (g_E) \oplus \Omega^2_3 (g_E),$$

the identifications are immediate for $L^0, L^1$ and $L^2$. To show that $\Omega^2_{1j} (g_E) \subset I$, let $\{ z_j \}_{j=1}^3$ (2.11) be the transverse complex coordinates, denote the $\omega$-positive 3-form $\Theta = \Theta_+ + i \Theta_-$, given by the real and imaginary part of the transverse complex volume form $dz^1 \wedge dz^2 \wedge dz^3$. In terms of the real local Darboux coordinates, $\Theta$ is given by

$$\Theta_+ = (dx^{123} + dx^{246}) - (dx^{235} + dx^{156}) \quad \text{and} \quad \Theta_- = (dx^{126} + dx^{234}) - (dx^{456} + dx^{135}).$$

We recall that for every point $x \in M$, $(H_x, \omega = d\eta)$ is a transverse symplectic vector space, and

$$J := \Phi|_H$$

is a complex structure. The group $SU(3)$ acts irreducibly on $\Lambda^5 H^*$ and $H^*$, while $\Lambda^2 H^*$ and $\Lambda^3 H^*$ decompose as follows [BV07, Section 2]:

$$\Lambda^2 H^* = \Lambda^2_1 H^* \oplus \Lambda^2_0 H^* \oplus \Lambda^2_0 H^* ,$$

where

$$\Lambda^2_1 H^* = \mathbb{R} \cdot \omega$$

$$\Lambda^2_0 H^* = \{ \alpha \in \Lambda^2 H^* | J^* \alpha = -\alpha \}$$

$$\Lambda^3 H^* = \{ \alpha \in \Lambda^3 H^* | J^* \alpha = \alpha, \alpha \wedge \omega^2 = 0 \}$$

$$\Lambda^3 H^* = \Lambda^3_{\text{in}} H^* \oplus \Lambda^3_{\text{in}} H^* \oplus \Lambda^3_{\text{in}} H^* \oplus \Lambda^3_{\text{in}} H^* \oplus \Lambda^3_{\text{in}} H^*$$
\[ \Lambda_{31}^3 H^* = \mathbb{R} \cdot \Theta_+ \]
\[ \Lambda_{3m}^3 H^* = \mathbb{R} \cdot \Theta_- = \{ \alpha \in \Lambda^3 H^* | \alpha \wedge \omega = 0, \alpha \wedge \Theta_+ = c \omega^3, \ c \in \mathbb{R} \} \]
\[ \Lambda_{6}^3 H^* = \{ \alpha \wedge \omega | \alpha \in \Lambda^1 H^* \} \]
\[ \Lambda_{12}^3 H^* = \{ \alpha \in \Lambda^3 H^* | \alpha \wedge \omega = 0, \alpha \wedge \Theta_+ = 0, \alpha \wedge \Theta_- = 0 \}. \]

Our goal is to show that all subspaces in the decomposition of \( \Lambda^3 H^* \) lie in \( I \). The following exterior multiplication table (row \( \wedge \) column) is easy to compute:

|  \( \wedge \)  | \( \omega \) | \( \Theta_+ \) | \( \Theta_- \) |
|----------------|-----------|----------------|----------------|
| \( dx^{345} \)  | 0         | 0              | \( dx^{123456} \) |
| \( dx^{156} \)  | 0         | 0              | \( dx^{123456} \) |
| \( dx^{126} \)  | 0         | \( dx^{123456} \) | 0              |
| \( dx^{135} \)  | 0         | \( -dx^{123456} \) | 0              |
| \( dx^{435} \)  | 0         | \( -dx^{123456} \) | 0              |
| \( dx^{234} \)  | 0         | \( dx^{123456} \) | 0              |
| \( dx^{123} \)  | 0         | 0              | \( -dx^{123456} \) |
| \( dx^{246} \)  | 0         | 0              | \( -dx^{123456} \) |

From that we obtain a set of generators for \( \Lambda_{12}^3 H^* \):

\[
\{ dx^{126} + dx^{135}, dx^{126} + dx^{456}, dx^{126} - dx^{234}, dx^{135} - dx^{456} \}
\]

In terms of the \( \{ w_i \}_{i=1}^8 \) in (2.6), this translates into

\[ \Lambda_{12}^3 H^* = \text{Span}\{ w_6 \wedge dx^3, w_1 \wedge dx^5, -w_1 \wedge dx^2, w_3 \wedge dx^5, -w_2 \wedge dx^5, w_5 \wedge dx^4, \]
\[ w_4 \wedge dx^5, w_1 \wedge dx^3, -w_6 \wedge dx^4, w_5 \wedge dx^1, w_2 \wedge dx^6, -w_3 \wedge dx^2 \} \subset I. \]

To see that \( \Lambda_{6}^3 H^* \subset I \), note that \( \omega \wedge dx^i = dx^{14i} + dx^{25i} + dx^{36i} \). Equivalently, in terms of \( w_7 \) and \( w_8 \) in (2.6),

\[ \omega \wedge dx^i = w_7 \wedge dx^i + w_8 \wedge dx^i + 3dx^{36i}. \]

Let us have a closer look at the same form of \( dx^{36i} \). For \( i = 3 \) and \( i = 6 \), clearly \( dx^{36i} = 0 \); for \( i = 1 \) and \( i = 4 \), necessarily \( dx^{36i} = \pm w_7 \wedge dx^i \); for \( i = 2 \) and \( i = 5 \), one has \( dx^{36i} = \pm w_8 \wedge dx^i \), hence in all instances \( \Lambda_{6}^3 H^* = \{ w_7 \wedge dx^i + w_8 \wedge dx^i + 3dx^{36i} \}_{i=1}^6 \subset I. \]

Finally, we observe that:

\[ J^*(X_{i,j}{\Theta}_+) = -J^*(J^2 X_{i,j}{\Theta}_+) = -JX_{i,j}J^*{\Theta}_+ = -JX_{i,j}{\Theta}_- = X_{i,j}{\Theta}_+, \]

whereas

\[ (X_{i,j}{\Theta}_+) \wedge \omega^2 = X_{i,j}(\Theta_+ \wedge \omega^2) + \Theta_+ \wedge (X_{i,j}\omega^2) = \Theta_+ \wedge (X_{i,j}\omega^2) = (X_{i,j}{\Theta}_+ \wedge \omega) = \Theta_+ \wedge \omega X_{i,j}{\Theta}, \]

i.e., \( (X_{i,j}{\Theta}_+) \in \Lambda_{3}^3 H^* \), and thus \( \Lambda_{3}^3 H^* \subset I \). That \( \Lambda_{3}^3 H^* \subset I \) follows analogously.

For (ii), it is still manifest that \( L^0, L^1, L^2 \subset I \). As to \( L^3 \), we can inspect directly the basis elements \( \{ v_i \}_{i=1}^6 \) for \( \Omega^3_6(\mathfrak{g}_E) \) introduced in (2.6). We claim that every element in the Darboux coordinate basis for \( \Omega^3_6(\mathfrak{g}_E) \) is obtained from an element of \( \Omega^3_6(\mathfrak{g}_E) \).

\[
\begin{align*}
& \{ dx^{124} = v_1 \wedge dx^4, \quad dx^{125} = v_1 \wedge dx^5, \quad dx^{134} = v_3 \wedge dx^4, \\
& dx^{136} = v_3 \wedge dx^6, \quad dx^{145} = -v_1 \wedge dx^1, \quad dx^{146} = -v_4 \wedge dx^4, \\
& dx^{235} = v_5 \wedge dx^5, \quad dx^{236} = v_5 \wedge dx^6, \quad dx^{245} = v_2 \wedge dx^5, \\
& dx^{256} = v_6 \wedge dx^5, \quad dx^{346} = v_3 \wedge dx^3, \quad dx^{356} = v_6 \wedge dx^5 \}
\end{align*}
\]

\[ \subset I, \]
as well as
\[
\begin{align*}
-2dx^{156} &= v_2 \wedge dx^6 - v_6 \wedge dx^4 + v_4 \wedge dx^5, \\
2dx^{135} &= v_1 \wedge dx^6 + v_3 \wedge dx^5 - v_5 \wedge dx^1, \\
-2dx^{456} &= v_1 \wedge dx^6 + v_4 \wedge dx^2 + v_5 \wedge dx^4, \\
-2dx^{123} &= v_3 \wedge dx^2 - v_2 \wedge dx^6 - v_5 \wedge dx^1, \\
-2dx^{246} &= v_3 \wedge dx^2 - v_2 \wedge dx^6 - v_5 \wedge dx^1,
\end{align*}
\]
\[\left\{ \begin{array}{c}
\end{array} \right\} \subset \mathcal{I}.
\]
Similarly, one can easily check that \(\Omega^k_H(\mathfrak{g}_E)\), \(\Omega^5_H(\mathfrak{g}_E)\) and \(\Omega^6_H(\mathfrak{g}_E)\) lie in \(\mathcal{I}\), thus \(L^k = 0\) for \(k \geq 4\).

**Lemma 4.7.** The following maps preserve \(I\) (whether \(I\) be generated by \(\Omega^2_0(\mathfrak{g}_E)\) or by \(\Omega^2_0(\mathfrak{g}_E)\))
\[
i_{\xi}: \Omega^*(\mathfrak{g}_E) \rightarrow \Omega^{*+1}(\mathfrak{g}_E), \quad \eta \wedge: \Omega^*(\mathfrak{g}_E) \rightarrow \Omega^{*+1}(\mathfrak{g}_E) \quad \text{and} \quad \omega \wedge: \Omega^*(\mathfrak{g}_E) \rightarrow \Omega^{*+2}(\mathfrak{g}_E),
\]
so these maps descend to the quotient \(L^*\).

**Proof.** If \(\alpha \wedge \beta \in \mathcal{I}\), for some \(\alpha \in \Omega^*(\mathfrak{g}_E)\) and \(\beta \in \Omega^2_0(\mathfrak{g}_E)\),
\[
i_{\xi}(\alpha \wedge \beta) = i_{\xi}\alpha \wedge \beta + (-1)^{\text{deg}(\alpha)} \alpha \wedge i_{\xi}\beta
\]
\[= i_{\xi}\alpha \wedge \beta \in \mathcal{I}.
\]
Therefore the map \(i_{\xi}\) is a well-defined contraction of bi-degree \((0, -1)\) on \(L^*\). We also denote by \(L^k_H\) the kernel of contraction inside of \(L^k\), so that \(L^{k,0} = L^k_H\).

We denote the induced maps on the quotient \(L^*\) as follows:
\[
L_\eta := \eta \wedge: L^*(\mathfrak{g}_E) \rightarrow L^{*+1}(\mathfrak{g}_E) \quad \text{and} \quad L_\omega := \omega \wedge: L^*(\mathfrak{g}_E) \rightarrow L^{*+2}(\mathfrak{g}_E).
\]

In summary, by Lemma 4.6 we have the following

**Proposition 4.8.** If a connection \(A\) has curvature \(F_A \in \Omega^2(\mathfrak{g}_E)\), the associated complex (1.11) is elliptic:

\[
L^*: \quad 0 \rightarrow L^0 \xrightarrow{D_0} L^1 \xrightarrow{D_1} L^2 \xrightarrow{D_2} L^3 \rightarrow 0.
\]

**Proof.** Denote by \(\pi: T^*M \setminus \{0\} \rightarrow M\) the fibre bundle obtained by removal of the zero section in \(T^*M\), and by \(p: \Omega^*(\mathfrak{g}_E) \rightarrow L^*\) the quotient projection, i.e., \(D_k = p \circ d^k_A\). Given \(\zeta \in T^*M \setminus \{0\}\), the first order symbol functor \(\sigma_1\) satisfies:
\[
\sigma_1(p \circ d^k_A) = p(\sigma_1(d^k_A)) = p(\zeta \wedge \cdot), \quad \text{for} \quad k = 0, \ldots, 3.
\]
The fibre \(\pi^*(L^k)\) is isomorphic to \(\Lambda^kT^*xM/I\), where \(I = \langle \Omega^2_0 \rangle\), hence the associated 1-symbol is
\[
\sigma_1(D_k)(x, \zeta) = p(\zeta \wedge \cdot): \Lambda^kT^*xM/I \rightarrow \Lambda^{k+1}T^*xM/I,
\]
and we assert that the associated symbol complex is exact:
\[
\sigma_1(L^*): \quad 0 \rightarrow \Lambda^0T^*xM/I \xrightarrow{p(\zeta \wedge \cdot)} \Lambda^1T^*xM/I \xrightarrow{p(\zeta \wedge \cdot)} \Lambda^2T^*xM/I \xrightarrow{p(\zeta \wedge \cdot)} \Lambda^3T^*xM/I \rightarrow 0.
\]
Using the identifications from Proposition 4.6, the above complex becomes

\[
\Lambda^0(T^*_xM) \xrightarrow{\zeta \wedge} \Lambda^1(T^*_xM) \xrightarrow{p(\zeta \wedge \cdot)} (\Lambda^2_{\mathfrak{g}_1} \oplus \eta \wedge (\Lambda^1_H))(T^*_xM) \xrightarrow{p(\zeta \wedge \cdot)} \eta \wedge (\Lambda^2_{\mathfrak{g}_1}(T^*_xM)).
\]

Exactness at positions \(k = 0, 1, 2\) is shown by the same argument, so we prove explicitly for \(k = 2\) and \(k = 3\).

\(k = 2\): Let \(\alpha \in (\Lambda^2_{\mathfrak{g}_1} \oplus \eta \wedge (\Lambda^1_H))(T^*_xM)\) such that \(\zeta \wedge \alpha \in (\Lambda^2_0)\). Since \(\zeta\) is non zero, this forces \(\zeta \wedge \alpha = 0\), and we know that the de Rham complex is elliptic, so there exists \(\beta \in \Lambda^1(T^*_xM)\) such that \(\zeta \wedge \beta = \alpha\). Hence we have exactness at \((\Lambda^2_{\mathfrak{g}_1} \oplus \eta \wedge (\Lambda^1_H))(T^*_xM)\).

\(k = 3\): This is similar to the \(k = 2\) case, but we need to show that \(\beta \in (\Lambda^2_{\mathfrak{g}_1} \oplus \eta \wedge (\Lambda^1_H))(T^*_xM)\). A priori, \(\beta = \beta_1 + \beta_2\), with
\[
\beta_1 \in (\Lambda^2_{\mathfrak{g}_1} \oplus \eta \wedge (\Lambda^1_H))(T^*_xM) \quad \text{and} \quad \beta_2 \in (\Lambda^2_0 \oplus \Lambda_V)(T^*_xM).
\]
Since \(\beta_2\) projects to zero under \(p\), we have \(\alpha = p(\zeta \wedge \beta_1)\), so (3.9) is exact.

\[\square\]
The above complex in Proposition 4.8 is referred to as the \textit{associated complex} to the selfdual contact instanton \(A\) (Proposition 4.8 can be shown in the same way if \(F_A \in \Omega^2_0(g_E)\)). We denote by \(H^k := H^k(L)\) the cohomology groups of \(L^\bullet \) (1.11). \(H^1\) can be interpreted as the space of infinitesimal deformations of the contact instanton \(A\), so that

\[
h^1 := \dim \mathbb{R}(H^1)
\]

represents the expected dimension of the moduli space.

### 4.2 Deformation theory of selfdual contact instantons

The splitting (A.1) of the tangent space \(TM = H \oplus N_\xi\) given by the Reeb vector field defines a bi-grading on \(\Omega^\bullet (g_E)\) by

\[
\Omega^{k,d}(g_E) := \Gamma(M, \Lambda^k H^* \otimes \Lambda^d N_\xi^* \otimes g_E), \quad \text{for } (k, d) \in \{0, \ldots, 6\} \times \{0, 1\}.
\]

In particular, \(\Omega^{k,0}(g_E) = \Omega^k_H(g_E)\) and \(\Omega^{k,1}(g_E) = \eta \wedge \Omega^k_H(g_E)\), for \(k = 0, 1, 2, 3\).

Since the ideal \(I\) is bi-graded, the bi-grading descends to the quotients \(L^\bullet\) to define components \(L^{k,d}\). In our case of main interest, \(I = \langle \Omega^2_6(g_E) \rangle\) induces the complex (1.11):

\[
\begin{align*}
\Omega^0_H(g_E) & \to \Omega^1_H(g_E) \oplus \eta \wedge \Omega^0_H(g_E) \to \Omega^2_{\eta \oplus 1}(g_E) \oplus \eta \wedge \Omega^1_H(g_E) \to \eta \wedge \Omega^2_{\eta \oplus 1}(g_E) \\
L^{1,0} & \quad L^{0,1} & \quad \quad L^{2,0} & \quad L^{1,1} & \quad \quad L^{2,1}
\end{align*}
\]

Define in the quotient the followings maps

\[
D_V : \quad L^\bullet H \to L^{\bullet+1} H, \quad \text{and} \quad D_T : \quad L^\bullet H \to L^{\bullet+1} H \to D\alpha - \eta \wedge D_V\alpha,
\]

\[(4.10)\]

\[\text{Lemma 4.9. Let } D_T \text{ and } D_V \text{ be the operators defined in (4.10), then}
\]

\[
D_T D_V = D_V D_T
\]

\[(4.11)\]

\[\text{and}
\]

\[
D_T^2 \alpha = -\omega \wedge D_V\alpha \quad \text{for } \alpha \in \Omega^0(g_E).
\]

\[(4.12)\]

\[\text{Proof. From } D^2 = 0, \text{we show the followings identities in fact,}
\]

\[
0 = (D_T + \eta \wedge D_V)^2 = D_T^2 + D_T(\eta \wedge D_V) + \eta \wedge D_V(D_T) + (\eta \wedge D_V)^2
\]

\[
= D_T^2 + \omega \wedge D_V - \eta \wedge D_T D_V + \eta \wedge D_V D_T + \eta \wedge D_V(\eta \wedge D_V)
\]

\[
= D_T^2 + \omega \wedge D_V + \eta \wedge (D_V D_T - D_T D_V) + \eta \wedge (D_V \eta \wedge D_V - \eta \wedge D_V^2)
\]

\[
= D_T^2 + \omega \wedge D_V + \eta \wedge (D_V D_T - D_T D_V),
\]

hence (4.11) and (4.12) follow. \(\square\)

The basic forms in \(L^k\) are denoted by \(L^k_B = \ker(D_V) \subset L^k\) and, by Proposition 4.6, \(\ker(D_V)\) can be identified with \(\Omega^\bullet_B(g_E)\). Since \(D_V\) and \(D_T\) commute (4.11),

\[
D_T(\Omega^\bullet_B(g_E)) \subset \Omega^\bullet_B(g_E).
\]

Moreover, \(D_T\) restricts to \(D_B : L^k_B \to L^{k+1}_B\), so \(D_B\) defines a \textit{basic deformation complex}. Considering \(I = \langle \Omega^2_6(g_E) \rangle\), the associated basic complex (1.12) becomes

\[
\begin{align*}
L^\bullet_B : \quad 0 & \to \Omega^0_B(g_E) \xrightarrow{D_B} \Omega^1_B(g_E) \xrightarrow{D_B} \Omega^2_{\eta \oplus 1}(g_E) \to 0
\end{align*}
\]

Denote by \(H^\bullet_B := H^\bullet(L_B)\) the cohomology of the basic complex \(L^\bullet_B\). This is not an elliptic complex, yet it is elliptic transversely to the Reeb foliation. In particular, its cohomology \(H^\bullet_B\) is finite-dimensional [KA+90, Theorem 3.2.5], and we will see that this complex computes the dimension of the moduli space of contact instantons.
Lemma 4.10. Let $D_T$ and $D_V$ be the operators defined in \((4.10)\), then the formal adjoints of $D_V$ and $D_T$ are given by:

\[
\begin{align*}
D_V^* &= -D_V, \quad (4.13) \\
D_T^* &= -*T \, D_T \ast_T . \quad (4.14)
\end{align*}
\]

Furthermore,

\[
D_V D_T^* = D_T^* D_V. \quad (4.15)
\]

**Proof.** For \((4.13)\) note that, since $\mathcal{L}_\xi$ coincides with $D_V$ on $\Omega^*_H(\mathfrak{g}_E)$, if $\xi$ is a Killing vector then $D_V$ and $*T$ commute, and so

\[
0 = \int_M \mathcal{L}_{\xi}(\alpha, *_T \beta) \land \eta = \int_M (D_V \alpha, *_T \beta) \land \eta + \int_M (\alpha, D_V *_T \beta) \land \eta.
\]

Furthermore, since $D_T D_V = D_V D_T$ we obtain \((4.15)\). To show \((4.14)\), let $\alpha \in \Omega^{k-1}_H(\mathfrak{g}_E)$ and $\beta \in \Omega^k_E(\mathfrak{g}_E)$:

\[
\begin{align*}
(\alpha, (*_T D_T *_T) \beta) &= \int_M \alpha \land *_T^2 (D_T *_T \beta) \land \eta \\
&= \int_M \alpha \land *_T^2 [D(*_T \beta) - \eta \land D_V(*_T \beta)] \land \eta \\
&= \int_M \alpha \land *_T^2 D(*(\eta \land \beta)) \land \eta \\
&= \int_M \alpha \land *_T [i_\xi (*D(*(\eta \land \beta)))] \land \eta = (\alpha, i_\xi D^* \eta(\beta)) \\
&= (\eta^* D_i^* \xi)(\alpha), \beta),
\end{align*}
\]

and the assertion follows by observing that $i_\xi D^* \eta(\alpha) = i_\xi \eta(\alpha) = i_\xi (\omega \land \alpha - \eta \land D\alpha) = -D_T \alpha$. \qed

We now adapt a number of fundamental insights from [BH16, Proposition 3.3]. We begin by introducing the transverse Laplacian on the complex $\mathbf{L}^\bullet$:

**Definition 4.11.** The **Laplacian** of $D$, with respect to the inner product defined in \((2.8)\), is

\[
\Delta := DD^* + D^* D : \mathbf{L}^\bullet \to \mathbf{L}^\bullet, \quad (4.16)
\]

and the **transverse Laplacian** is defined by:

\[
\Delta_T := D_T D_T^* + D_T^* D_T - D_V^2 : \mathbf{L}^\bullet \to \mathbf{L}^\bullet. \quad (4.17)
\]

Clearly $\Delta_T$ and $\Delta$ have the same symbol, so $\Delta_T$ is an elliptic operator. Let us denote the spaces of $\Delta$–harmonic and $\Delta_T$–harmonic forms, respectively, by

\[
\mathcal{H}^k := \ker(\Delta) \subset \mathbf{L}^k \quad \text{and} \quad \mathcal{H}_T^k := \ker(\Delta_T) \subset \mathbf{L}^k.
\]

Since $\langle \Delta_T \alpha, \alpha \rangle = \|D_T \alpha\|^2 + \|D_T^* \alpha\|^2 + \|D_V \alpha\|^2$,

\[
\alpha \in \mathbf{L}^{k,0} \quad \text{is} \quad \Delta_T\text{-harmonic if, and only if,} \quad D_T \alpha = D_T^* \alpha = D_V \alpha = 0. \quad (4.18)
\]

On basic forms, in particular, $\Delta_T$–harmonicity is equivalent to $\Delta$-harmonicity. Note that $\Delta_T$ respects the bi-grading defined in \((4.8)\), hence we can split $\mathcal{H}_T^k$ into components $\mathcal{H}_T^{k,d}$. Since $L_\eta : \mathbf{L}^{k,0} \to \mathbf{L}^{k,1} (4.5)$ is an isomorphism, we know from the outset that

\[
\mathcal{H}_T^{k,0} \cong \mathcal{H}_T^{k,1} = \eta \land (\mathcal{H}_T^{k,0}), \quad \text{for} \quad k \in \{1, \cdots, 6\}. \quad (4.19)
\]

**Lemma 4.12.** There exists an isomorphism $\phi : \mathcal{H}_T^{k,0} \to \mathcal{H}_B^k$, where $\mathcal{H}_T^{k,0}$ is just above defined and $\mathcal{H}_B^k$ is the $k$-th cohomology group of the basic complex \((1.12)\).
Proof. Let the morphism \( \phi: \mathcal{H}^{k,0}_T \to H^k_B \) send \( \alpha \in \mathcal{H}^{k,0}_T \) to its equivalence class \([\alpha] \in H^k_B\). This is well-defined, by \((4.18)\), since \(\mathcal{H}^{k,0}_T \subset \ker(D_V) = \Omega^{k,0}_B(V_E)\) and \(D\) coincides with \(D_T\) on basic forms, so indeed \(\mathcal{H}^{k,0}_T \subset \ker(D_B)\), therefore \(\alpha\) defines an equivalence class in \(H^k_B\).

We first check injectivity of \(\phi\). If \(\phi(\alpha) = [\alpha] = 0 \in H^k_B\), then

\[ \alpha = D_B\beta = D_T\beta + \eta \wedge D_V\beta, \quad \text{with} \quad \beta \in \Omega^{k-1}_B(V_E). \]

\(D_V\beta = 0\) because \(\beta\) is a basic form, hence

\[ \alpha = D_T\beta = D\beta. \]

Applying \(D^*_T\) to \(\alpha\) follows that \(0 = D^*_T\alpha = D^*_T D_T\beta\), now taking inner product with \(\beta\) in the last equality we obtain that \(D_T\beta = 0 = \alpha\).

To check surjectivity, let \(\alpha\) be closed and basic, i.e., \(\alpha \in \Lambda^{k,0}\) and \(D_T\alpha = 0 = D_V\alpha\). Elliptic theory implies that \(\alpha = \beta + \Delta_T\gamma\), where \(\beta \in \ker(\Delta_T)\). From \((4.15)\), we have \(D_V\Delta_T = \Delta_T D_V\), hence

\[ D_V\alpha = 0 = D_V\beta + D_V\Delta_T\gamma = \Delta_T D_V\gamma. \]

It follows that \(D_V\gamma\) is \(\Delta_T\)-harmonic, in particular \(D^*_T\gamma = 0\), and so

\[ 0 = (D_T D^*_T D_T\gamma, D_T\gamma) = \|D^*_T D_T\gamma\|^2, \]

so, \(\alpha = \beta + D_T D^*_T\gamma\). Now, \(D^*_T\gamma\) is a basic form, since \(D_V D^*_T\gamma = D^*_T D_V\gamma = 0\), so in fact \(\alpha = \beta + D_B(D^*_T\gamma)\). This shows that every element in \(H^k_B\) has a \(\Delta_T\)-harmonic representative, thus \(\phi\) is surjective. 

\[ \square \]

**Lemma 4.13.** Consider the vector spaces \(A^k := \mathcal{H}^{k,0}_T \oplus \mathcal{H}^{k-1,0}_T\), together with the differential \(\hat{d}\) defined by:

\[ \hat{d}: \quad A^k \to A^{k+1}, \quad (\alpha, \beta) \mapsto \hat{d}(\alpha, \beta) := (\omega \wedge \beta, 0). \]  

\[ (4.20) \]

The following hold:

(i) \((A^*, \hat{d})\) form a chain complex.

(ii) There is a chain map \(j^*: (A^*, \hat{d}) \to (L^*, D)\) into the complex \((4.3)\), defined by

\[ j^k: (\alpha, \beta) \in A^k \mapsto \alpha + \eta \wedge \beta \in L^k, \quad \text{for} \quad k = 0, 1, 2, 3. \]

\[ (4.21) \]

(iii) \(\chi(A) = \sum_{k=0}^{3} (-1)^k \dim(H^k(A)) = 0.\)

**Proof.**

(i) Clearly \(\hat{d}^2 = 0\), and \(\hat{d}\) is well-defined in view of the isomorphism \(\mathcal{H}^{k,0}_T \cong H^k_B\) from Lemma 4.12:

\[ (\alpha, \beta) \in H^{k,0}_B \oplus H^{k-1,0}_B \quad \mapsto \quad (\omega \wedge \beta, 0) \in H^{k+1,0}_B \oplus H^{k,0}_B \]

\[ \| \| \rightarrow \| \| \]

\[ H^k_B \oplus H^{k-1}_B \quad \mapsto \quad (\omega \wedge \beta, 0) \in H^{k+1}_B \oplus H^k_B \]

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(ii) The maps $j^k$ in (4.21) fit in the following diagram:

\[
    \begin{array}{cccccc}
    \mathcal{H}^0_T \oplus \{0\} & \xrightarrow{\bar{d}^0=0} & \mathcal{H}^1_T \oplus \mathcal{H}^0_T & \xrightarrow{\bar{d}^1} & \mathcal{H}^2_T \oplus \mathcal{H}^1_T & \xrightarrow{\bar{d}^2=0} & \{0\} \oplus \mathcal{H}^2_T \\
    j^0 & & j^1 & & j^2 & & j^3 \\
    L^0 & \xrightarrow{D} & L^1 & \xrightarrow{D} & L^2 & \xrightarrow{D} & L^3.
    \end{array}
\] (4.22)

To see that (4.22) is commutative, the only nontrivial step to check is $Dj^1 = j^2\bar{d}$. Given $(\alpha, \beta) \in \Lambda^1$, by definition, $D\alpha = 0 = D_V\beta$, hence

\[
    D j^1(\alpha, \beta) = D(\alpha + \eta \wedge \beta) = D\alpha + \omega \wedge \beta - \eta \wedge D_V\beta \\
    = \omega \wedge \beta = j^2(\omega \wedge \beta, 0) \\
    = j^2\bar{d}(\alpha, \beta).
\]

(iii) We know, from Lemma 4.12, that each chain $A^k$ in (4.20) is finite-dimensional, for $k = 0, 1, 2, 3$, and

\[
    \chi(A) := \sum_{k=0}^3 (-1)^k \dim(A^k) = \sum_{k=0}^3 (-1)^k \left( \dim(\mathcal{H}^k_T) + \dim(\mathcal{H}^{k-1}_T) \right) \\
    = 0.
\]

The assertion now follows from the rank-nullity theorem.

Lemma 4.14. Let $L^k, k = 1, 2$ be the chain spaces defined in Proposition 4.6. For each $[\alpha] \in H^k$, its harmonic representative can be written as:

\[
    \alpha = \beta + \eta \wedge \gamma, \quad \text{with} \quad D\alpha = D^*\alpha = 0,
\]

such that $D_V\beta = D^*_T\beta = 0$ and $D_V\gamma = D_T\gamma = 0$.

Proof. The harmonic representative of $[\alpha] \in H^k$ has the general form $\alpha = \beta + \eta \wedge \gamma$ by definition of $L^k$ (see Proposition 4.6 item (i)). We know from (4.10) that $D\alpha = D_T\alpha + \eta \wedge D_V\alpha$, so

\[
    0 = D\alpha = D_T\beta + \eta \wedge D_V\beta + \omega \wedge \gamma - \eta \wedge (D_T\gamma + \eta \wedge D_V\gamma) \\
    = D_T\beta + \omega \wedge \gamma + \eta \wedge (D_V\beta - D_T\gamma)
\]

and

\[
    0 = D^*\alpha = D^*_T\beta + (\eta \wedge D_V)^*\beta + D^*_T(\eta \wedge \gamma) + (\eta \wedge D_V)^*(\eta \wedge \gamma) \\
    = D^*_T\beta - D_V(i_\xi(\eta \wedge \gamma)) \\
    = D^*_T\beta - D_V\gamma.
\]

In summary, we obtain the following relations:

\[
    (i) \quad D_T\beta + \omega \wedge \gamma = 0, \quad (ii) \quad D_V\beta - D_T\gamma = 0, \quad (iii) \quad D^*_T\beta - D_V\gamma = 0.
\]

(4.24)

Applying $D_T$ to (iii) and $D_V$ to (ii), and commuting by (4.11), we obtain $D_TD^*_T\beta - D^*_T\beta = 0$. Taking the inner product with $\beta$, we have

\[
    \|D_V\beta\|^2 + \|D^*_T\beta\|^2 = 0,
\]

therefore $D_V\beta = D^*_T\beta = 0$, and so $D_V\gamma = D_T\gamma = 0$.

Proposition 4.15. The chain map $j^* : (A^*, \bar{d}) \to (L^*, D)$ defined in (4.21) is a quasi-isomorphism, i.e., the induced map in cohomology $j^k : H^k(A^*) \to H^k(L^*)$ is an isomorphism, for each $k = 0, 1, 2, 3$. 

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This case is trivial, since $j^0$ is essentially the inclusion map.

$k = 3$: We must show that $H^3 \cong \{0\} \oplus \mathcal{H}_{T}^{2,0}$. From Hodge theory, we have isomorphisms $H^k \cong \mathcal{H}^k$, so we show that $H^3 \cong \mathcal{H}_{T}^{2,0}$. Indeed, $j^3$ is just exterior multiplication by $\eta$, so

$$\mathcal{H}_{T}^{2,0} = \{ \gamma \in \mathcal{L}^{2,0} = \mathcal{L}^2_H \mid \Delta_T \gamma = 0 \} = \{ \gamma \in \Omega_0^2 \oplus \Omega_1^2 : D_T \gamma = D_T^* \gamma = D_V \alpha = 0 \}$$

$$\cong \{ \alpha \in \eta \wedge (\Omega_0^2 \oplus \Omega_1^2) : D_T^* \alpha = D_V \alpha = 0 \}$$

$$= \mathcal{H}^3 \subseteq \mathcal{L}^3.$$

$k = 1$: By (4.23), a class $[\alpha] \in H^1$ has a harmonic representative of the form,

$$\alpha = \beta + \eta \wedge \gamma \in \Omega^1_H(\mathfrak{g}_E) \oplus \Omega^1_H(\mathfrak{g}_E),$$

and we first need to be sure that $\beta, \gamma$ are $\Delta_T$-harmonic. By (4.18) and Lemma 4.14, it only remains to check that $D_T^* \gamma = 0$ and $D_T \beta = 0$. The former holds because $\gamma \in \Omega^1_H(\mathfrak{g}_E) \subset \ker(D_T^*)$. For the latter, take the inner product with $\beta$ after the following computation:

$$0 = D_T^* D_T \beta + D_T^*(\omega \wedge \gamma) = D_T^* D_T \beta - 3 \delta_T (\omega \wedge \gamma)$$

$$= D_T^* D_T \beta - 3 \delta_T (\omega \wedge \gamma) = D_T^* D_T \beta - 3 \delta_T (\omega \wedge \gamma)$$

Now, let us check that $j^1 : H^1(\mathfrak{A}^* \rightarrow H^1(\mathfrak{L}^*)$ is an isomorphism. From relation (i) in (4.24), we have $\omega \wedge \gamma = 0$, and thus $(\beta, \gamma) \in \mathcal{H}_{T}^{0,0} \oplus \mathcal{H}_{T}^{0,0}$ and $\delta(\beta, \gamma) = (\omega, \gamma, 0) = (0, 0)$, i.e., $j^1$ is surjective. For injectivity, suppose that $[j^1(\beta, \gamma)] = [\beta + \eta \wedge \gamma] = 0 \in H^1(\mathfrak{L}^*)$, so $\beta + \eta \wedge \gamma = \Delta \delta$, for some $\delta \in \Omega^0(\mathfrak{g}_E)$, and so

$$0 = \beta - D \delta + \eta \wedge \Delta \delta + \eta \wedge \gamma - \eta \wedge \Delta \delta = \beta - i_\xi (\eta \wedge D \delta) + \eta \wedge (\Delta - D \delta).$$

Comparing types, we have $\beta = D \delta$ and $\gamma = D_V \gamma$, but $\beta$ and $\gamma$ are $\Delta_T$-harmonic, so indeed $(\beta, \gamma) = (0, 0)$.

$k = 2$: From Lemma 4.14, $\alpha = \beta + \eta \wedge \gamma \in \Omega_0^2 \oplus \Omega_1^2 (\mathfrak{g}_E) \oplus \Omega^1_H(\mathfrak{g}_E) = \mathcal{L}^2$, where $D_V \beta = D_T \beta = 0$ and $D \gamma = D_T \gamma = 0$. Note that $\omega \wedge \gamma = 0$, since $\omega \wedge \gamma \in \mathcal{L}_{1,0} = \mathcal{L}^{2,1} \cong \eta \wedge (\Omega_0^2 \oplus \Omega_1^2)(\mathfrak{g}_E)$, so relation (i) in (4.24) implies that $D_T \beta = 0$, i.e., $j^2 : H^2(\mathfrak{A}^*) \rightarrow H^2(\mathfrak{L}^*)$ is injective. For injectivity, recall that the complex $\mathfrak{L}^*, D \in (4.3)$ is elliptic [Proposition 4.8] on a compact odd-dimensional manifold, so

$$0 = \sum (-1)^k \dim(H^k(\mathfrak{L}^*))$$

similarly, $0 = \sum (-1)^k \dim(H^k(\mathfrak{A}^*))$ (see item (iii) of Lemma 4.13). Since $j^k$ is an isomorphism for $k = 0, 1, 3$, we conclude that $\dim(H^2(\mathfrak{L}^*)) = \dim(H^2(\mathfrak{A}^*))$, consequently $j^2$ is also an isomorphism.

Remark 4.16. Lemma 4.12 and Proposition 4.15 can be shown in the same fashion if we adopt the ideal $I = \Span(\Omega_0^2)$, instead of $I = \Span(\Omega_2^0)$.

Proposition 4.17. Assume the Sasakian manifold $M^7$ is closed (see Remark 4.3). Then the map

$$L_\omega : [\alpha] \in H^0_B \mapsto [\alpha \wedge \omega] \in H^2_B,$$

induced in cohomology by $L_\omega : L^k \mapsto L^{k+2}$ [cf. Lemma 4.7], is injective.

Proof. Since $D_\omega = 0$, the exterior product map for $\omega$ is a well-defined map in cohomology. Given $\alpha \in \ker(\omega \wedge \cdot) \subset H^0_B$, clearly $[\alpha \wedge \omega] = 0 \in H^2_B$, so there exists some $\beta \in \Omega^1_B$ such that $\alpha \wedge \omega = D_B \beta$. Then

$$\|\alpha\|^2 = \int_M \langle \alpha \wedge * \alpha \rangle = \int_M \alpha \wedge * \alpha \wedge \frac{\eta \wedge \omega^3}{3!} = \frac{1}{6} \int_M \alpha \omega \wedge * \omega \wedge \eta \wedge \frac{\omega^3}{3!}$$

$$= \frac{1}{6} \int_M D_B \beta \wedge \alpha \omega^2 \wedge \eta.$$
Therefore, it suffices to show that the integrand is exact. Indeed:
\[
d (\langle \beta \wedge \omega^2 \rangle \wedge \eta) = (D_B \beta \wedge \omega^2 + \beta \wedge D_B (\alpha \omega^2)) \wedge \eta + \beta \wedge \omega^2 \wedge \omega
\]
\[
= (D_B \beta \wedge \omega^2) \wedge \eta + (\beta \wedge \omega^2) \wedge \omega
\]
\[
= (D_B \beta \wedge \omega^2) \wedge \eta.
\]

At an instanton, the cohomologies $H^*\omega$ and $H^*_B\omega$ of the complexes (1.11) and (1.12), respectively, fit in a Gysin sequence analogous to [Ton12, Theorem 10.13]:

**Proposition 4.18.** At a selfdual contact instanton, the complex (4.20) induces a long exact sequence in cohomology:

\[
\cdots \longrightarrow H^{k-2}_B(A) \overset{\omega^k}{\longrightarrow} H^k_B(A) \longrightarrow H^k(A) \longrightarrow H^{k-1}_B(A) \overset{\omega^k}{\longrightarrow} H^{k+1}_B(A) \longrightarrow \cdots
\]  

(4.25)

**Proof.** In order to show the exactness of (4.25), we use the isomorphisms $H^*(A) \cong H^*_\omega$ [Proposition 4.15] and $H^*_B \cong H^*_{\omega_B}$ [Lemma 4.12]. We recall that the differential $\hat{d}^k: A^k \to A^{k+1}$ from (4.20) maps $(\alpha, \beta) \in H^{k-0}_T \oplus H^{k-1,0}_T$ to $(\omega \wedge \beta, 0) \in H^{k+1,0}_T \oplus H^{k,0}_T$.

- First note that $H^0(A) = \ker(\hat{d}^0) = H^{0,0}_T \cong H^0_B$, hence we obtain exactness in

\[
0 \longrightarrow H^0_B \longrightarrow H^0 \longrightarrow 0.
\]

Also note that $H^3(A) = H^{2,0}_T \cong H^2_B$, hence we obtain exactness in

\[
0 \longrightarrow H^3 \longrightarrow H^2_B \longrightarrow 0.
\]

It remains to show the exactness of

\[
0 \overset{\omega^k}{\longrightarrow} H^1_B \longrightarrow H^1 \longrightarrow H^0_B \overset{\omega^k}{\longrightarrow} H^2_B \longrightarrow H^2 \longrightarrow H^1_B \overset{\omega^k}{\longrightarrow} 0.
\]

We proceed from left to right.

- From Proposition 4.17, the map $L^*_\omega: H^*_B \to H^*_B$ is injective. Moreover,

\[
H^1(A) = \ker(\hat{d}^1) = \{(\alpha, \beta) \in H^{1,0}_T \oplus H^{0,0}_T | (\omega \wedge \beta, 0) = (0, 0)\} \cong H^{1,0}_T \oplus \{0\} \cong H^1_B,
\]

so mapping $H^1 \to 0$ gives exactness at $H^1_B$, $H^1$ and $H^0_B$.

- The map $H^2_B \to H^2$ is induced by the inclusion $L^*_B \to L^*_B$, and its kernel consists of exact basic forms. By Proposition 4.15, these are identified with the image of $L^*_\omega: H^2_T \to H^2_T$, hence (4.25) is exact at $H^2_B$.

- We assert that the map

\[
(i_\xi)^*: H^2 \to H^1_B,
\]

induced in cohomology by $i_\xi^*: L^2 \to L^2_B$, is surjective. Indeed, if $\alpha \in L^1$ is basic and closed, the form $\eta \wedge \alpha$ belongs to $L^2$, and $D(\eta \wedge \alpha) = \omega \wedge \alpha$ is a basic 3-form, i.e., $\omega \wedge \alpha \in L^{3,0}[\text{cf. (4.9)}]$, so it is zero, and $(i_\xi)^* [\eta \wedge \alpha] = [\alpha]$, as claimed.

- Finally, $H^2(A) = \frac{H^2_T \cap H^*_T}{\operatorname{Im}(d^1)}$, and

\[
\operatorname{Im}(d^1) = \{(\alpha, \beta) \in H^{2,0}_T \oplus H^{1,0}_T | d(\theta, \gamma) = (\alpha, \beta), \text{ for some } (\theta, \gamma) \in H^{1,0}_T \oplus H^{0,0}_T\}
\]

\[
= \{(\alpha, \beta) \in H^{2,0}_T \oplus H^{1,0}_T | (\omega \wedge \gamma, 0) = (\alpha, \beta) \in H^{0,0}_T\}
\]

\[
\cong \omega \wedge (H^{0,0}_T \oplus \{0\}),
\]

so, $H^2 \cong H^2_T \omega(\omega^2_B) \oplus H^1_B \cong H^2_T \omega(\omega^2_B) + H^1_B$.  

The following immediate consequence of Proposition 4.18 proves part (ii) of Theorem 1.2:

**Corollary 4.19.** At a selfdual contact instanton, the inclusion $L^1_B \to L^1$ induces an isomorphism $H^1_B \cong H^1$, thus the expected dimension of the moduli space (1.8) is $h^1_B := \dim H^1_B [\text{cf. (1.12)}]$. 

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4.3 Obstruction and smoothness

In this section we will establish that, if the second cohomology group $H^2$ of the basic complex in (1.12) vanishes, then the moduli space $\mathcal{M}^*$ of irreducible selfdual contact instantons defined in (1.8) has a local structure of a smooth manifold (Corollary 4.24). Our approach follows the general scheme of [DK90, Section 4.2], adapting to 7 dimensions some crucial insights from [BH16, Sections 3.1 & 3.2].

4.3.1 The obstruction map

**Lemma 4.20.** Let $A$ be a selfdual contact instanton and $\alpha \in L^1 \cong \Omega^1(\mathfrak{g}_E)$ [cf. Proposition 4.6]; the connection $A + \alpha$ remains a selfdual contact instanton if, and only if, $\alpha$ is a Maurer-Cartan element of $(L^*, D)$ [cf. (4.2) and (4.3)], i.e.,

$$D\alpha + \frac{1}{2}[\alpha, \alpha] = 0 \in L^2.$$

**Proof.** Given $\alpha \in \Omega^1(\mathfrak{g}_E)$, the curvature of the connection $A + \alpha$ is

$$F_{A+\alpha} - F_A = D\alpha + \frac{1}{2}[\alpha, \alpha] \in \Omega^2(\mathfrak{g}_E).$$

Hence $A+\alpha$ still has curvature in $\Omega^2(\mathfrak{g}_E)$, if, and only if, the Maurer-Cartan torsion $D\alpha + \frac{1}{2}[\alpha, \alpha]$ lies in $I = \langle \Omega^2(\mathfrak{g}_E) \rangle$, and therefore it vanishes in the quotient. □

We introduce the $W^2_m$-sobolev norms on smooth sections $\Gamma(\mathfrak{g}_E)$:

$$\|s\|_m := \left( \sum_{m+1}^k \int_{M} |\nabla^m(s(x))|^2 \, dvol \right)^{\frac{1}{2}}, \quad m \in \mathbb{N},$$

(4.26)

where $\nabla^j s \in \Gamma(\mathfrak{g}_E \otimes T^{0,j}(M))$, and $|\nabla^j s|$ is determined by the metric on $M$ and the fibrewise inner product (2.8) on $\mathfrak{g}_E$.

Let $A_m$ denote the space of $W^2_m$-connections on $E$, and define $\mathcal{G}_{m+1}$ as the topological group of $W^2_{m+1}$-gauge transformations. For $m > 3$, Sobolev multiplication holds [DK90, Appendix II], hence $\mathcal{G}_{m+1}$ has the structure of an infinite-dimensional Lie group modeled on a Hilbert space; its Lie algebra is the space of $W^2_{m+1}$ sections of $\text{End}(\mathfrak{g}_E)$. Moreover, the gauge group $\mathcal{G}_{m+1}$ acts smoothly on $A_m$, so we denote by

$$\mathcal{B}_m = A_m/\mathcal{G}_{m+1}$$

the Hausdorff orbit space with the quotient topology [DK90, Lemma 4.2.4]. Let $\mathcal{M}_m \subset \mathcal{B}_m$ denote the set of moduli of solutions to the selfdual ($\lambda = 1$) contact instanton equation (1.1) and, accordingly, denote by $\mathcal{M}^*_m \subset \mathcal{M}_m$ the stratum of irreducible elements.

We recall that a contact instanton $A \in A$ defines a complex $(L^*, D)$ [cf. (4.3)]. The $L^k$ spaces consist of smooth sections of vector bundles on $M$ (Proposition 4.6), and we denote their $W^2_{m+1}$-completions by $L^*_{m}$. From the formal adjoint of $D$,

$$D^* : L^k_{m} \rightarrow L^k_{m-1},$$

we define the Laplacian $\Delta : L^k_{m} \rightarrow L^k_{m-2}$ as in (4.16). Denote by $G$ its Green operator, i.e., the inverse of $\Delta$ on the orthogonal complement $\ker(\Delta)^{\perp}$, by $H : L^k_{m} \rightarrow \ker(\Delta) \subset L^k_{m}$ the orthogonal projection onto harmonic sections and set

$$\delta := D^* G.$$

**Lemma 4.21.** The map

$$\mathcal{F} : L^1_{m} \rightarrow L^1_{m}, \quad \mathcal{F}(\alpha) := \alpha + \frac{1}{2} \delta[\alpha, \alpha];$$

(4.27)

is invertible near $0 \in L^1_{m}$ (cf. (4.2) for definition of $[\cdot, \cdot]$).
Proof. For \( m > 3 \), \( F \) is a smooth map from the Hilbert space \( L^1_m \) to itself. The linearisation of \( F \) at \( 0 \in L^1_m \) is the identity:

\[
F'(0)(\beta) = (\beta + t\delta[\beta],[\beta])|_{t=0} = \beta,
\]

hence there exists a smooth inverse \( F^{-1} \) near \( 0 \in L^1_m \). Take \( c > 0 \) sufficiently small, such that

\[
U_c = \{ \beta \in L^1_m \mid \|\beta\|_m < c \} \subset H^1 \cap \text{Dom}(F^{-1}). \tag{4.28}
\]

For \( \beta \in U_c \) and \( \alpha := F^{-1}(\beta) \), i.e., \( \beta = \alpha + \frac{1}{2}\delta[\alpha,\alpha] \),

\[
0 = \Delta \beta = \Delta \alpha + \frac{1}{2} \Delta \delta[\alpha,\alpha] = \Delta \alpha + \frac{1}{2} D^*\delta[\alpha,\alpha] - \frac{1}{2} D^*H[\alpha,\alpha]
\]

so, \( \alpha \) is a smooth point, by elliptic regularity.

We define the obstruction map of the deformation complex \( (L^*, D) \) [cf. (4.3)] by:

\[
\Psi : U_c \to H^2 \quad \Psi(\alpha) := H[F^{-1}(\alpha), F^{-1}(\alpha)], \tag{4.29}
\]

where \( U_c \) is defined in (4.28) and \( H^2 = \ker(\Delta) \subset L^2 \).

**Lemma 4.22.** For \( m > 3 \), let \( U_c \) be a neighbourhood of \( 0 \in L^1_m \) as in (4.28), on which the inverse \( \mathcal{F}^{-1} \) of the map (4.27) is defined. Then \( \mathcal{F}^{-1} \) maps \( \Psi^{-1}(\{0\}) \) diffeomorphically to a neighbourhood \( W \) of the set

\[
Z = \{ \alpha \in L^1 \mid \alpha \in \ker(D^2), D\alpha = -\frac{1}{2}[\alpha,\alpha] \}.
\]

**Proof.** Given \( \alpha \in L^1_m \),

\[
D\delta(\alpha) = (\Delta - D^*D)G\alpha = (1 - H)\alpha - D^*G\alpha
\]

i.e.,

\[
D\delta = 1 - H - \delta D. \tag{4.30}
\]

Now, we show that \( \mathcal{F}^{-1}(\Psi^{-1}(0)) \subset Z \). If \( \beta \in \Psi^{-1}(0) \), set \( \alpha = \mathcal{F}^{-1}(\beta) \), and apply \( D^* \) to \( \beta = \alpha + \frac{1}{2}\delta[\alpha,\alpha] \):

\[
0 = D^*\beta = D^*\alpha + \frac{1}{2}(D^*)^2G[\alpha,\alpha]
\]

so, if \( \delta[D\alpha,\alpha] = 0 \), then \( \mathcal{F}^{-1} \) maps \( \Psi^{-1}(0) \) into \( Z \). To see that \( \delta[D\alpha,\alpha] = 0 \), put \( x = \delta[D\alpha,\alpha] \) and note that

\[
\delta[D\alpha,\alpha] = \delta[-\frac{1}{2}\delta[\delta,\delta] + \delta[D\alpha,\alpha],\alpha] = \delta[\delta[D\alpha,\alpha],\alpha],
\]

i.e., \( x = \delta[x,\alpha] \). Now, for each \( m > 3 \), there exists \( K > 0 \) such that

\[
\|x\|_m = \|\delta[x,\alpha]\|_m \leq K\|x\|_m\|\alpha\|_m,
\]

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so we can take $c$ small enough that $\|\alpha\|_m < \frac{1}{4c}$, hence $x = 0$, as claimed.

Conversely, let $\alpha \in Z$ and set $\beta = \mathcal{F}(\alpha)$, as above. Since $D^*\beta = D^*\alpha = 0$,
\[
D\beta = D\alpha + \frac{1}{2} D\delta[\alpha,\alpha] = D\alpha + \frac{1}{2}(1 - H - \delta D)[\alpha,\alpha]
\]
\[
= \frac{1}{2}[\alpha,\alpha] + \frac{1}{2}[\alpha,\alpha] - H[\alpha,\alpha] - \delta[D\alpha,\alpha]
\]
\[
= -H[\alpha,\alpha].
\]
The left-hand side in the above equality is $D$-exact, whereas the right-hand side is harmonic, hence
\[
D\beta = -H[\alpha,\alpha] = \Psi(\beta) = 0.
\]
Therefore $\beta$ is harmonic and lies in $\Psi^{-1}(0)$.

In the context of Lemma 4.22, as shown in [DK90, Section 4.2.3], the tangent model is characterised as follows:

**Proposition 4.23.** Let $[A] \in \mathcal{M}_m$ be a selfdual contact instanton gauge modulus, and denote its isotropy by
\[
\Gamma_A := \text{Aut}(G)/Z(G),
\]
where $Z(G)$ is the centre of $G$. Let $c > 0$ be small enough so that the inverse $\mathcal{F}^{-1}$ is defined on $U_c$ [cf. (4.28)]. Then $\Psi^{-1}(0)/\Gamma_A$ is a neighborhood of $[A]$ in $\mathcal{M}_m$. Furthermore, for $m > 3$, the natural map $\mathcal{M}_{m+1} \to \mathcal{M}_m$ is a homeomorphism, so we may suppress the subscript $m$ in $\mathcal{M}_m$.

**Corollary 4.24.** If $[A] \in \mathcal{M}^*$ is an irreducible selfdual contact instanton such that the obstruction map (4.29) vanishes, then $\mathcal{M}^*$ is a smooth manifold near $[A]$.

### 4.3.2 Cohomological vanishing of obstruction

The following proposition provides necessary conditions for the vanishing of the obstruction map (4.29), and thus the local smoothness of $\mathcal{M}^*$ (1.8).

The transverse Laplacian $\Delta_T$ from (4.17) satisfies, by ellipticity,
\[
L^k = \ker(\Delta_T) \oplus \ker(\Delta_T)^\perp.
\]
Denote by $H_T : L^k \to \ker(\Delta_T)$ the projection onto $\Delta_T$-harmonic sections, by $G_T$ the Green operator of $\Delta_T$, and set $\delta_T := D_T^*G_T$. By the same argument applied to (4.27) in Lemma 4.21, the map
\[
\mathcal{F}_T : \alpha \in L^1_m \mapsto \alpha + \frac{1}{2}\delta_T[\alpha,\alpha] \in L^1_m,
\]
is an isomorphism near $0 \in L^1_m$, for $m > 3$. Taking $c > 0$ small enough, we set
\[
U_c = \{\gamma \in \mathcal{H}_T^1 \mid \|\gamma\|_{L^2_m} < c\}
\]
(4.31)
such that $\mathcal{F}_T^{-1}$ is defined on $U_c$.

**Proposition 4.25.** An irreducible selfdual contact instanton such that $H^2_B = 0$, the obstruction $\Psi$ in (4.29) vanishes.

**Proof.** By (1.11) and Corollary 4.19, infinitesimal contact instanton deformations are parametrised by $H^1 = H^1_B$. Fix $c > 0$ small enough that $\mathcal{F}_T^{-1}$ is defined on $U_c$, as in (4.31). Since $D_T = 0$ on $L^2_{m-1}$, $\Delta_T$ simplifies to $\Delta_T = D_TD_T^* - D_T^2$, and, using the fact that $D_V$ commutes with the Green operator, we obtain
\[
D_T\delta_T = D_TD_T^*G_T = (\Delta_T + D_V^2)G_T
\]
\[
= (1 - H_T) + G_TD_V^2.
\]
Furthermore, the assumption and Lemma 4.12 together imply that $0 = H^2_B \cong H^2_T$, so $H_T$ vanishes on $L^2_m$, and the above equation simplifies to
\[
D_T\delta_T = 1 + G_TD_V^2.
\] (4.32)
For $\gamma \in U_c$, let $\alpha := \mathcal{F}_T^{-1}(\gamma)$, so $\gamma = \alpha + \frac{1}{2} \delta_T[\alpha, \alpha]$. Since $\gamma$ is $\Delta_T$-harmonic, it follows from (4.18) that

$$0 = D_V \gamma = D_T \gamma = D_T^* \gamma.$$ 

Now, since $D_V$ commutes with $D_T$ (4.11) and also with $D_T^*$ and $G_T$, a similar argument to Lemma 4.22 shows that $\delta_T [D_V \alpha, \alpha] = 0$, and so

$$0 = D_V \gamma = D_V \alpha + \frac{1}{2} D_V \delta_T[\alpha, \alpha]$$

$$= D_V \alpha + \frac{1}{2} D_V D_T^* G_T[\alpha, \alpha] = D_V \alpha + \delta_T [D_V \alpha, \alpha]$$

$$= D_V \alpha.$$ 

Thus $D_T \alpha = D \alpha$, and we conclude from (4.32) that

$$0 = D_T \gamma = D_T \alpha + \frac{1}{2} D_T \delta_T[\alpha, \alpha] = D \alpha + \frac{1}{2} (\mathbb{1} + G_T D_T^2)[\alpha, \alpha]$$

$$= D \alpha + \frac{1}{2} [\alpha, \alpha],$$

i.e., $\alpha$ is a Maurer-Cartan element [cf. Lemma 4.20]. Furthermore, since $D_T^2 = -\omega \wedge D_V$ (cf. (4.12)) and $D_V^* = -D_V$ (cf. (4.13)), we have $(D_T^*)^2 = D_V \Lambda(\cdot)$, where $\Lambda$ is the formal adjoint of $L_\omega$ (4.5). This implies

$$0 = D_T^* \gamma = D_T^* \alpha + \frac{1}{2} D_T^* \delta_T[\alpha, \alpha]$$

$$= D_T^* \alpha + \frac{1}{2} (D_T^*)^2 G_T[\alpha, \alpha] = D_T^* \alpha + \frac{1}{2} D_V \Lambda(G_T[\alpha, \alpha])$$

$$= D_T^* \alpha.$$ 

In summary, $\alpha = \mathcal{F}_T^{-1}(\gamma) \in Z = \{ \alpha \in L^1_m | D^* \alpha = 0, \ D \alpha + \frac{1}{2} [\alpha, \alpha] = 0 \}$, and it follows that $\mathcal{F}_T^{-1}$ defines a map from $U_c$ into $Z$. In Lemma 4.22 it was shown that $\mathcal{F}$ maps a neighborhood of $0 \in Z$ to a neighborhood of $\Psi^{-1}(0)$, hence, for $c > 0$ small enough,

$$\Psi \circ \mathcal{F} \circ \mathcal{F}_T^{-1}(\alpha) = \Psi(\mathcal{F} \mathcal{F}_T^{-1}(\alpha)) = H[\mathcal{F}^{-1} \mathcal{F} \mathcal{F}_T^{-1}(\alpha), \mathcal{F}^{-1} \mathcal{F} \mathcal{F}_T^{-1}(\alpha)]$$

$$= H_T[\mathcal{F}^{-1} \mathcal{F}(\alpha), \mathcal{F}_T^{-1}(\alpha)]$$

$$= 0,$$

since the linearisation of $\mathcal{F} \circ \mathcal{F}_T^{-1}$ at 0 is the identity. Therefore $\Psi = 0$ on $U_c$, for sufficiently small $c$. 

If $A$ is a contact instanton, the transverse index of $A$ is defined as the index of the basic complex in (1.12), namely,

$$\text{index}_T(A) = \dim(H^0_B) - \dim(H^1_B) + \dim(H^2_B).$$ 

(4.33)

When $A$ is irreducible we have $H^0_B = \{0\}$. If moreover $H^2_B = \{0\}$, it follows from Proposition 4.25 and Corollary 4.24 that $\mathcal{M}^*$ is locally a smooth manifold of dimension computed by the transverse index (4.33):

$$\dim \mathcal{M}^* = \dim H^1_B = - \text{index}_T(A).$$

Afterword: upcoming developments

We announce a number of ongoing research tracks, to appear in a forthcoming update.

Sufficient conditions for $H_B^2 = 0$

Proposition 4.25 suggests looking for sufficient conditions under which $H_B^2 = 0$. Using Bochner-type methods (see e.g. [Ito83]), we obtain the following result, stated here without proof:

**Conjecture 4.26.** Let $E \to M$ be a Sasakian $G$-bundle [Definition A.5] over a compact Sasakian 7-manifold with positive transverse scalar curvature, then for each irreducible selfdual contact instanton $A$ on $E$ the second basic cohomology group $H_B^2$ vanishes.

A Sasakian manifold $(M, \eta, \xi, g, \Phi)$ is called Sasaki-Einstein if $g$ is an Einstein metric in the usual sense. In this case, the transverse scalar curvature is always positive, so:

**Corollary 4.27.** The moduli space of irreducible selfdual contact instantons over a Sasaki-Einstein manifold is smooth.

Geometry on the moduli space

Arguably the most attractive application of studying the moduli space of contact instantons is the possibility of constructing Donaldson-type invariants for Sasakian 7-manifolds, so we are particularly interested in geometric structures on $M^*$. In [BH16, Section 4.3], we learn that, under suitable assumptions, the moduli space of ASD contact instantons on a Sasakian manifold $(M^*, \Omega, J)$ is Kähler, and moreover hyper-Kähler in the transverse Calabi-Yau case.

The theory in Section 4 should suffice to prove analogous results to [BH16, Propositions 4.7 and 4.9]: in the general Sasakian case, that there exist an integrable complex structure $J$ and a Hermitian form $\Omega$ on $M^*$, such that $(M^*, \Omega, J)$ is Kähler; and in the contact Calabi-Yau case, there exist indeed three independent almost complex structures $J_1, J_2, J_3$, with associated Kähler forms $\Omega_1, \Omega_2$ and $\Omega_3$, such that $(M^*, J_i, \Omega_i)$ is hyper-Kähler.

Computations of transverse index in particular examples

Computing the dimension of moduli space of irreducible contact instantons, with vanishing $H_B^2$, is equivalent to computing the transverse index (4.33). In [BH16, Section 5], this is performed by replacing the foliated complex [BH16, Equation 3.3]

$$0 \longrightarrow \Omega^0_B(g_E) \xrightarrow{D_B} \Omega^1_B(g_E) \xrightarrow{D_B} \Omega^2_B(g_E) \longrightarrow 0$$

with a complex involving a suitable lifted $T$-action on $g_E$ [BH16, Proposition 2.8]. Hence the dimension of $M^*$ is computed from the index of the symbol complex

$$\pi^*(H^* \otimes g_E) \xrightarrow{\sigma(Q)(\cdot)} \pi^* \left( (\Lambda^+ H^* \oplus \mathbb{R}) \otimes g_E \right),$$

associated to the transverse elliptic operator $Q = (D_T^*, D_T)$ [BH16, Equation 5.1]

$$\Omega^0_H(g_E) \xrightarrow{\partial} \Omega^0_H(g_E) \oplus \Omega^+_{H}(g_E).$$

The transverse index can then be computed in several interesting cases (see [BH16, Section 5.2 and 5.3]). We expect the same approach can be applied to the index of the symbol complex

$$\pi^*(H^* \otimes g_E) \xrightarrow{\sigma(D_A)(\cdot)} \pi^* \left( (\Lambda^2_{\oplus} H^* \oplus \mathbb{R}) \otimes g_E \right),$$

associated to the transverse elliptic operator $D_A := d_7 \oplus d_A^*$ defined in (3.7):

$$\Omega^0_H(g_E) \xrightarrow{D_A} \Omega^0_H(g_E) \oplus (\Omega^2_{\oplus} \oplus \Omega^+_{H})(g_E).$$
A Sasakian vector bundles

We gather here some general results and definitions on ‘holomorphic’ vector bundles over Sasakian manifolds. This concept obviously does not make strict sense as in classical complex geometry, but it admits a straightforward adaptation in terms of the transverse complex structure. All results and definitions in this Appendix stem from the original insights in [BS10]. We adopt the usual notation for a \((2n+1)\)-dimensional Sasakian manifold \((M, \eta, \xi, \Phi, g)\). Standard references for Sasakian geometry are [BG08, Bla10].

A.1 Differential forms on Sasakian manifolds

The contact structure induces an orthogonal decomposition of the tangent bundle,

\[
TM = H \oplus \mathbb{R} \cdot \xi =: H \oplus N_\xi,
\]

where \(\ker(\eta) =: H \subset TM\) is the distribution of rank \(2n\) transverse to the Reeb field \(\xi\), and the restriction \(J = \Phi|_H\) defines an almost complex structure on \(H\). We write indistinctly \(\xi \cdot \alpha\) or \(i_\xi(\alpha)\) to denote the interior product by \(\xi\). We denote the complexification of the tangent bundle by \(TM_C := TM \otimes \mathbb{C}\) and also, respectively, \(N_\xi^C\) and \(H_C\).

**Definition A.1.** A differential form \(\alpha \in \Omega^k(M)\) is called transverse if \(i_\xi \alpha = 0\). If moreover \(i_\xi d\alpha = 0\), then \(\alpha\) is said to be basic (i.e., \(S^1\)-invariant).

Let \(\Omega^k_H(M) = \Gamma(M, \Lambda^k H^*)\) denote the space of transverse \(k\)-forms. Let \(\alpha \in \Omega^k_H(M)\) a locally defined, transverse complex form and \(x \in M\). Since \(i_\xi \alpha_x = 0\), the evaluation of \(\alpha_x\) on \(\Lambda^p((TM_C)_x)\) is determined by the values of \(\alpha_x\) on the subbundle \(\Lambda^p((H_C)_x) \subset \Lambda^p((TM_C)_x)\). We denote the complexification of the transverse complex structure \(\Phi\) by

\[
\Phi^C_x := \Phi|_{H_x} \otimes \mathbb{C} : (H_C)_x \rightarrow (H_C)_x.
\]

Since \((\Phi|_{H_x})^2 = -\mathbb{1}\), the eigenvalues of \(\Phi^C_x\) are \(\pm i = \pm \sqrt{-1}\), and the complexified horizontal distribution splits accordingly:

\[
(H_C)_x = H_x^{1,0} \oplus H_x^{0,1}.
\]

For \(p, q \geq 0\) we set

\[
H_x^{p,q} := \Lambda^p(H_x^{1,0})^* \otimes \Lambda^q(H_x^{0,1})^* \subset \Lambda^{p+q}(H_x)^* \otimes \mathbb{R} \mathbb{C},
\]

therefore

\[
\Lambda^d(H_C)_x^* = \bigoplus_{i=0}^{d} (H_x^{i,d-i})^*, \quad 0 \leq d \leq 2n.
\]

A section \(\alpha\) is said to be of type \((a, d-a)\). If the evaluation of \(\alpha\) on \(H_x^{i,d-i}\) is zero for all \(i \neq a\). The decomposition (A.4) gives a decomposition into a direct sum of vector bundles

\[
\Omega^d_{H_C}(M) = \bigoplus_{i=0}^{d} \Omega^{i,d-i}_{H_C}(M),
\]

where \(\Omega^d_{H_C}(M)\) denotes \(\Gamma(M, \Lambda^p(H_C)^* \otimes \Lambda^q(H_C)^*)\). Taking into account the dimension of \(N_\xi\) and \(H\), it follows that

\[
\Lambda^d(TM_C)^* = \Lambda^d((H_C)^* \otimes (N_\xi^C)^*),
\]

\[
= \Lambda^d(H_C)^* \otimes (\Lambda^{d-1}(H_C)^* \otimes \Lambda^1(N_\xi^C)^*) \oplus (\Lambda^{d-2}(H_C)^* \otimes \Lambda^2(N_\xi^C)^*) \oplus \ldots
\]

\[
\cdots \oplus (\Lambda^1(H_C)^* \otimes \Lambda^1(N_\xi^C)^*) \oplus \Lambda^d(N_\xi^C)^* \oplus \ldots
\]

\[
\simeq \Lambda^d(H_C)^* \oplus (\eta \otimes \Lambda^{d-1}(H_C)^*)
\]

hence, combining with the decomposition (A.4),

\[
\Omega^d(M) = \left( \bigoplus_{i=0}^{d} \Omega^{i,d-i}_{H_C}(M) \right) \oplus \left( \eta \otimes \left( \bigoplus_{j=0}^{d-1} \Omega^{i,d-j-1}_{H_C}(M) \right) \right).
\]

**Proposition A.2.** The transverse form \(\omega := d\eta|_H\) is of type \((1,1)\), furthermore \(\omega\) is the fundamental symplectic form on \(H\).
A.2 Partial connections

Consider an integrable subbundle $S \subset TM_C$, i.e. the sections of $S$ are closed under the Lie bracket. Of course we have in mind the particular case in which $S = N^C_\xi \subset TM_C$, but we state the first few definitions in general terms.

**Definition A.3.** Consider a complex vector bundle $E \to M$, a partial connection on $E$ in the direction of $S$ is a smooth operator $D : \Gamma(E) \to \Gamma(\Lambda^1 S^*) \otimes \Gamma(E)$, satisfying the ‘Leibniz rule’

$$D(fs) = fD(s) + q_S(df) \otimes s,$$

where the projection $q_S : (TM_C)^* \to S^*$ is the dual of the inclusion $S \to TM_C$.

Since the distribution $S$ is integrable, smooth sections of $\ker(q_S)$ are closed under the exterior derivation [BS10, Section 3.2], this induces an exterior derivative on the smooth sections of $S^*$:

$$\tilde{d} : \Lambda^1 S^* \to \Lambda^2 S^*.$$

Consider a partial connection $D$ on $S$ and the operator $D_1 : \Gamma(\Lambda^1 S^*) \otimes \Gamma(E) \to \Gamma(\Lambda^2 S^*) \otimes \Gamma(E)$ defined by

$$D_1(\theta \otimes s) = \tilde{d}\theta \otimes s - \theta \wedge D(s).$$

Their composition

$$\Gamma(E) \xrightarrow{D} \Gamma(E) \otimes \Gamma(\Lambda^1 S^*) \xrightarrow{D_1} \Gamma(E) \otimes \Gamma(\Lambda^2 S^*)$$

defines a torsion $D_1 \circ D := K(D) \in \Gamma(\Lambda^2 S^* \otimes \operatorname{End}(E)).$ This section is called the curvature of $D$, we will say that $D$ is a flat connection if $K(D) = 0$. We denote the extended anti-holomorphic $(n+1)$-dimensional foliation

$$\tilde{H}^{0,1} := H^{0,1} \oplus (N^C_\xi) \subset TM \otimes_R \mathbb{C},$$

where $H^{0,1}$ is defined in (A.3), it is shown in [BS10, Lemma 3.2] that the distribution in (A.6) is integrable.

**Definition A.4.** A (complex) Sasakian vector bundle on a Sasakian manifold $(M, g, \xi)$ is a pair $(E, D_0)$. Where $D_0$ is a partial connection in the direction of subbundle $N^C_\xi \subset TM$ and $E \to M$ is a complex vector bundle.

Notice that $N^C_\xi \subset \tilde{H}^{0,1}$, therefore, we can consider partial connections along $\tilde{H}^{0,1}$, which define by restriction a partial connection along $N^C_\xi$. Furthermore, $N^C_\xi$ is a 1-dimensional foliation on $M$, so any partial connection along $N^C_\xi$ is flat. When the context is clear, we denote by $\tilde{\partial}$ a flat partial connection $D$ along $\tilde{H}^{0,1}$ such that $\tilde{\partial}|_{N^C_\xi} = D_0$, and we abbreviate the notation by $E := (E, D_0)$.

**Definition A.5.** A holomorphic Sasakian vector bundle on a Sasakian manifold $(M, g, \xi)$ is a pair $E := (E, \tilde{\partial})$, where $E = (E, D_0)$ is a Sasakian vector bundle and $\tilde{\partial}$ is a flat connection on $E$ along $\tilde{H}^{0,1}$ (A.6).

A.3 Hermitian and holomorphic structures

We define a Hermitian structure on $E$ as a smooth Hermitian structure $h$ on $E$ which is compatible with $D_0$:

$$dh(s_1, s_2)|_{N^C_\xi} = h(D_0(s_1), s_2) + h(s_1, D_0(s_2)) \quad \text{for} \quad s_1, s_2 \in \Gamma(E).$$

A unitary connection on $(E, h)$ is a connection $A$ on $E$ such that $d_A$ satisfies the usual sense.

A connection $A \in \mathcal{A}(E)$ induces a partial connection along $\tilde{H}^{0,1}$ (A.6) given by $D_{\tilde{\partial}A_0} := d_A|_{\tilde{H}^{0,1}}$. If it coincides with $\tilde{\partial}$, then $A$ is called a integrable connection on $E$. Let $\mathcal{A}(E) \subset \mathcal{A}(E)$ denote the subset of integrable connections on $E$. The class of connections mutually compatible with the holomorphic structure and the Hermitian metric is the natural analogue of the concept of Chern connection.

**Proposition A.6.** Let $(E, \tilde{\partial})$ be a holomorphic Sasakian bundle with Hermitian structure, then there exists a unique unitary and integrable Chern connection $A_h$ on $E$ and $F_{A_0} \in \Omega^{1,1}$, moreover the expression

$$\det \left( \mathbb{I}_E + \frac{i}{2\pi} H_{A_0} \right) = \sum_{j=0}^n c_j(E, h)$$

defines closed Chern forms $c_j(E, h) \in \Omega^{1,1}(M)$.
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