Non-Instantaneous Impulsive Boundary Value Problems Containing Caputo Fractional Derivative of a Function with Respect to Another Function and Riemann–Stieltjes Fractional Integral Boundary Conditions

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Abstract: In the present article we study existence and uniqueness results for a new class of boundary value problems consisting by non-instantaneous impulses and Caputo fractional derivative of a function with respect to another function, supplemented with Riemann–Stieltjes fractional integral boundary conditions. The existence of a unique solution is obtained via Banach’s contraction mapping principle, while an existence result is established by using Leray–Schauder nonlinear alternative. Examples illustrating the main results are also constructed.

Keywords: impulsive differential equations; fractional impulsive differential equations; instantaneous impulses; non-instantaneous impulses

1. Introduction and Preliminaries

Fractional calculus is a generalization of classical differentiation and integration to an arbitrary real order. Fractional differential equations has gained much attention in literature because of its applications for description of hereditary properties in many fields, such as physics, mechanics, engineering, game theory, stability and optimal control. With the help of fractional calculus, the natural phenomena and mathematical models can be described more accurately. Many researchers have shown their interests in fractional differential equations, and the theory and applications of the fractional differential equations have been greatly developed. For the basic theory of fractional calculus and fractional differential equations we refer to the monographs [1–8] and references therein.

The theory of impulsive differential equations arise naturally in biology, physics, engineering, and medical fields where at certain moments they change their state rapidly. There are two type of impulses. One is called instantaneous impulses in which the duration of these changes is relatively short, and the other is called non-instantaneous impulses in which an impulsive action, starting abruptly at some points and continue to be active on a finite time interval. Some examples of such processes can be found in physics, biology, population dynamics, ecology, pharmacokineti, and others. For results with instantaneous impulses see, e.g., the monographs [9–14], the papers [15–19], and the references cited therein. Non-instantaneous impulsive differential equation was introduced by Hernández and O’Regan in [20] pointed out that the instantaneous impulses cannot characterize some processes such as evolution processes in pharmacotherapy. Some practical problems involving non-instantaneous impulses within the area of psychology have been reviewed in [21]. For some recent works, on non-instantaneous impulsive fractional differential equations we refer the reader to [22–25] and references therein.
The scope of this investigation is to establish existence results of the new class of boundary value problems consisting by non-instantaneous impulses and Caputo fractional derivative of a function with respect to another function, supplemented with Riemann–Stieltjes fractional integral boundary conditions of the form

\[
\begin{align*}
\alpha_i \left( D_{a_1}^\gamma a \right) x(t) &= f(t, x(t)), & t \in [a_i, t_{i+1}), & i = 0, 1, 2, \ldots, m, \\
x(t) &= \psi_i(t) x(t_i^-), & t \in [t_i, s_i), & i = 1, 2, 3, \ldots, m, \\
\beta_1 x(0) + \beta_2 x(T) &= \sum_{k=0}^{m} \mu_k \int_{t_k}^{t_{k+1}} (s_k \mathcal{I}_R^\alpha x)(u) \, dH_k(u).
\end{align*}
\]

(1)

Here \( s_i D_{a_i}^\gamma \) is the Caputo fractional derivative of order \( \alpha_i \in (0, 1) \), with respect to a function \( g_i \) starting at the point \( s_i \), over the interval \([s_i, t_{i+1})\), \( s_i \mathcal{I}_R^\alpha \) is the Riemann–Liouville fractional integral with respect to the function \( g_i \) on \([s_i, t_{i+1})\) of order \( \gamma_i > 0 \), \( \mu_i \in \mathbb{R} \), the bounded variation function \( H_i \) of the Riemann–Stieltjes on \([s_i, t_{i+1})\) and a function \( f : \bigcup [s_i, t_{i+1}) \to \mathbb{R} \), for \( i = 0, 1, 2, \ldots, m \). (For details on Riemann–Stieltjes integral we refer to [26]). In impulsive interval \([t_i, s_i)\), \( \psi_i, \psi_i, i = 1, 2, 3, \ldots, m \), are given functions. The points

\[ 0 = s_0 < t_1 < s_1 < t_2 < s_2 < \cdots < t_m < s_m < t_{m+1} = T, \]

are fixed in \([0, T]\) and \( \beta_1, \beta_2 \) are known constants. Note that in problem (1), we have \( x(s_i^+) = x(s_i^-) \) and if \( \psi_i(t) \neq 1 \), \( \psi_i(t) \neq 0 \) at \( t_i \) for all \( i = 1, 2, 3, \ldots, m \), then \( x(t_i^+) \neq x(t_i^-) \).

For \( \gamma > 0 \), the Riemann–Liouville fractional integral of an integrable function \( h : [a, b] \to \mathbb{R} \) with respect to another function \( g \in C^1([a, b], \mathbb{R}) \) such that \( g'(t) > 0 \), for all \( t \in [a, b] \) is defined by [2,27,28]

\[
a \mathcal{I}_R^\gamma h(t) = \frac{1}{\Gamma(\gamma)} \int_a^t \frac{g'(s) h(s)}{[g(t) - g(s)]^{1-\gamma}} \, ds,
\]

(2)

where \( \Gamma \) is the gamma function. The Riemann–Liouville type of fractional derivative of a function \( h \), with respect to another function \( g \) on \([a, b]\) is defined as

\[
a^{\alpha} \mathcal{D}_g^\gamma h(t) = \mathcal{D}_g^{\alpha} a \mathcal{I}_R^{n-\alpha} h(t) = \frac{1}{\Gamma(n-\alpha)} D_g^n \int_a^t \frac{g'(s) h(s)}{[g(t) - g(s)]^{1+\alpha-n}} \, ds,
\]

(3)

while the Caputo type is defined by

\[
a^{\alpha} \mathcal{D}_g^\gamma h(t) = a \mathcal{I}^{n-\alpha} \mathcal{D}_g^\gamma h(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{g'(s) D_g^n h(s)}{[g(t) - g(s)]^{1+\alpha-n}} \, ds,
\]

(4)

where \( D_g^n = \mathcal{I}_S^{n-1} \mathcal{D}_S^n \) \( n-1 < \alpha < n \), \( n \) is a positive integer and \( D_S^n \) is defined by

\[
D_S^n = \frac{1}{g'(t)} \frac{d}{dt}.
\]

(5)

There are relations of fractional integral and derivatives of the Riemann–Liouville and Caputo types which will be used in our investigation, see [2], as

\[
a \mathcal{I}_S^\gamma (a^{\alpha} \mathcal{D}_S^\gamma h)(t) = h(t) - \sum_{j=1}^{n} \frac{(g(t) - g(a))^{\gamma-j}}{\Gamma(\gamma-j+1)} \mathcal{D}_S^{\alpha-j} (a^{\alpha-\gamma} h)(a),
\]

(6)

and

\[
a \mathcal{I}_S^\gamma (a \mathcal{D}_S^\gamma h)(t) = h(t) - \sum_{j=0}^{n-1} \frac{(g(t) - g(a))^j}{j!} D_S^\gamma h(a).
\]

(7)
In addition, for $\gamma, \delta > 0$, the relation
\[ d_\delta^\gamma g(t) - g(a) = \frac{\Gamma(\delta + 1)}{\Gamma(\gamma + \delta + 1)}(g(t) - g(a))^{\gamma + \delta}, \] (8)
is applied in the main results ([2]). For some recent results we refer the interesting reader to the papers [29–31].

Note that (2) is reduced to the Riemann–Liouville and Hadamard fractional integrals when $g(t) = t$ and $g(t) = \log t$, respectively, where $\log(\cdot) = \log_e(\cdot)$. The Hadamard and Hadamard–Caputo types fractional derivatives can be obtained by substituting $g(t) = \log t$ in (3) and (4), respectively. Also the Riemann–Liouville and Caputo fractional derivatives are presented by replacing $g(t) = t$ in (3) and (4), respectively. Therefore, the problem (1) generates many types and also mixed types of impulsive fractional differential equations with boundary conditions. There are some papers that have studied either Hadamard or Caputo fractional derivatives containing in noninstantaneous impulsive equations, see [32–34].

The significance of this studying is to mixed different calculus within the system of non-instantaneous impulsive differential equations. For example if putting $m = 1$, $t_1 = 1$, $s_1 = 2$, $t_2 = 3$, $a_0 = a_1 = 1/2$, $g_0(t) = t$ and $g_1(t) = \log_2 t$ in the first two equations of (1), then we obtain
\[
\begin{cases}
\left(\frac{d}{dt}\right)^{1/2} x = f(t, x(t)), & t \in [0, 1), \\
x(t) = \varphi_1(t) + \psi(t)x(1^-), & t \in [1, 2), \\
\left(\frac{d}{dt}\right)^{1/2} x = f(t, x(t)), & t \in [2, 3),
\end{cases}
\]
which is a special case of mixed Riemann–Liouville and Hadamard fractional impulsive system. In addition, if $H_k(t) = g_k(t)$, for all $t \in [s_i, t_{i+1})$, $k = 0, 1, 2, \ldots, m$, then the nonlocal condition in (1), is reduced to
\[
\beta_1 x(0) + \beta_2 x(T) = \sum_{k=0}^m \mu_k \left(\int_{s_k}^{t_{k+1}} g_k(u) - g_k(s_k) du\right)(t_{k+1}).
\]

If $\varphi_i(t) = 0$, $\psi_i(t) = 1$ and $s_i \to t_i$, $i = 1, 2, 3, \ldots, m$, then (1) is reduced to a non impulsive fractional boundary value problem.

In fact, to the best of the authors knowledge, this is the first paper investigating Riemann–Stieltjes integration acting on fractional integral boundary conditions. Existence and uniqueness results are established for the the non-instantaneous impulsive Riemann–Stieltjes fractional integral boundary value problem (1) by using classical fixed point theorems. We make use of Banach’s contraction mapping principle to obtain the uniqueness result, while the Leray–Schauder nonlinear alternative is applied to obtain the existence result. The main results are presented in Section 3. In Section 2 we prove an auxiliary result concerning a linear variant of the problem (1) which is of great importance in the proof of main results. Illustrative examples are also presented.

2. An Auxiliary Result

Let us set some constants which will be used in our proofs.

\[
\Lambda_k = \frac{1}{\Gamma(\gamma_k + 1)} \int_{s_k}^{t_{k+1}} (g_k(u) - g_k(s_k))^{\gamma_k} dH_k(u), \quad k = 1, 2, 3, \ldots, m,
\] (9)

\[
\Lambda^* (i) = \sum_{j=1}^i \left( \prod_{j'=1}^{i-1} \psi_{j+1}(s_{j+1}) \right) \varphi_j(s_j), \quad i = 1, 2, 3, \ldots, m,
\] (10)

\[
\Omega = \beta_1 + \beta_2 \left( \sum_{j=1}^m \psi_j(s_j) \right) - \sum_{k=0}^m \mu_k \left( \prod_{j=1}^m \psi_j(s_j) \right) \Lambda_k.
\] (11)
Lemma 1. Let $\Omega \neq 0$ and $h \in C([0, T], \mathbb{R})$. Then the integral equation equivalent to problem (1) can be written as

$$x(t) = \frac{1}{\Omega} \left( \prod_{j=1}^{i-1} \psi_j(s_j) \right) \left( \sum_{k=0}^{m} \mu_k \Lambda^*(k) \Lambda_k - \beta_2 \Lambda^*(m) \right) + \sum_{k=0}^{m} \mu_k \sum_{j=1}^{k} \left( \prod_{j=1}^{i} \psi_j(s_j) \right) s_{j-1} \int_{s_{j-1}}^{t} f_x(s) \, ds \right) \Lambda_k + \sum_{k=0}^{m} \mu_k \int_{s_k}^{s_{k+1}} s_k \int_{s_k}^{t} f_x(u) \, du \, dH_k(u) - \beta_2 \sum_{j=1}^{m+1} \left( \prod_{j=1}^{i} \psi_j(s_j) \right) s_{j-1} \int_{s_{j-1}}^{t} f_x(s) \, ds \right) + s_{s_{i+1}} f_x(t),$$

for $t \in [s_i, t_{i+1})$, $i = 0, 1, 2, \ldots, m$, and

$$x(t) = \phi_1(t) + \psi_1(t) \left[ \frac{1}{\Omega} \left( \prod_{j=1}^{i-1} \psi_j(s_j) \right) \left( \sum_{k=0}^{m} \mu_k \Lambda^*(k) \Lambda_k \right) + \sum_{k=0}^{m} \mu_k \sum_{j=1}^{k} \left( \prod_{j=1}^{i} \psi_j(s_j) \right) s_{j-1} \int_{s_{j-1}}^{t} f_x(s) \, ds \right) \Lambda_k + \sum_{k=0}^{m} \mu_k \int_{s_k}^{s_{k+1}} s_k \int_{s_k}^{t} f_x(u) \, du \, dH_k(u) - \beta_2 \sum_{j=1}^{m+1} \left( \prod_{j=1}^{i} \psi_j(s_j) \right) s_{j-1} \int_{s_{j-1}}^{t} f_x(s) \, ds \right) + \Lambda^*(i - 1) + \sum_{j=1}^{i} \left( \prod_{j=1}^{i} \psi_j(s_j) \right) s_{j-1} \int_{s_{j-1}}^{t} f_x(s) \, ds \right],$$

for $t \in [t_i, s_i)$, $i = 1, 2, 3, \ldots, m$, where $f_x(t) = f(t, x(t))$.

Proof. For $t \in (s_0, t_1)$, taking the fractional integral with respect to a function $g_0(t)$ of order $\alpha_0 > 0$, from $s_0$ to $t$ in the first equation of (1) and setting $x(0) = A$, we have

$$x(t) = A + s_0 \int_{s_0}^{t} f_x(t) \, dt.$$  

In particular, we get for $t = t_1^+$, that $x(t_1^+) = A + s_0 \int_{s_0}^{t_1} f_x(t) \, dt$.

In the second interval $[t_1, s_1)$, we have from the second equation of (1) as

$$x(t) = \phi_1(t) + \psi_1(t) x(t_1^-) = \phi_1(t) + A \psi_1(t) + \psi_1(t) s_0 \int_{s_0}^{t_1} f_x(t) \, dt,$$

and also $x(s_1) = \phi_1(s_1) + A \psi_1(s_1) + \psi_1(s_1) s_0 \int_{s_0}^{t_1} f_x(t) \, dt$.

In the third interval $[s_1, t_2)$, again taking the Riemann–Liouville fractional integral with respect to a function $g_1(t)$ of order $\alpha_1$, we obtain

$$x(t) = x(s_1) + s_1 \int_{s_1}^{t} f_x(t) \, dt = \phi_1(s_1) + A \psi_1(s_1) + \psi_1(s_1) s_0 \int_{s_0}^{t_1} f_x(t) \, dt + s_1 \int_{s_1}^{t} f_x(t) \, dt,$$

which has particular case as $x(t_2^-) = \phi_1(s_1) + A \psi_1(s_1) + \psi_1(s_1) s_0 \int_{s_0}^{t_1} f_x(t) \, dt + s_1 \int_{s_1}^{t} f_x(t) \, dt$. 


In the fourth interval $[t_2, s_2)$, it follows that

$$x(t) = \varphi_2(t) + \psi_2(t) \left[ \varphi_1(s_1) + A \psi_1(s_1) + \psi_1(s_1) s_0 l_{g_0}^{g_0} f_x(t_1^-) + s_1 l_{g_1}^{g_1} f_x(t_2) \right].$$

By the previous procedure we can find that

$$x(t) = \begin{cases} 
A \left( \prod_{j=1}^i \varphi_j(s_j) \right) + \sum_{j=1}^i \left( \prod_{j=1}^{i-1} \varphi_j(s_j) \right) \varphi_j(s_j) \\
+ \sum_{j=1}^{i-1} \left( \prod_{j=1}^i \varphi_j(s_j) \right) s_{j-1} l_{g_{j-1}}^{g_{j-1}} f_x(t_j^-) + s_i l_{g_i}^{g_i} f_x(t), 
\end{cases}$$

for $t \in [s_i, t_{i+1})$, $i = 0, 1, 2, \ldots, m,$

$$\varphi_i(t) + \psi_i(t) \left[ A \prod_{j=1}^i \varphi_j(s_j) + \sum_{j=1}^{i-1} \left( \prod_{j=1}^{i-1} \varphi_j(s_{j+1}) \right) \varphi_j(s_j) \\
+ \sum_{j=1}^{i-1} \left( \prod_{j=1}^i \varphi_j(s_j) \right) s_{j-1} l_{g_{j-1}}^{g_{j-1}} f_x(t_j^-) \right], \quad t \in [t_i, s_i), i = 1, 2, 3, \ldots, m.$$

By using the mathematical induction, we will claim that the formula (16) holds. Putting $i = 0$ and $i = 1$ in the first and second parts of (16), respectively, we have results in (14) and (15). Assume that the first part of (16) is true for $i = k$, that is, for $t \in [s_k, t_{k+1})$,

$$x(t) = \varphi_{k+1}(t) + \psi_{k+1}(t) x(t_{k+1}^-)$$

$$= \varphi_{k+1}(t) + \psi_{k+1}(t) \left[ A \left( \prod_{j=1}^k \varphi_j(s_j) \right) + \sum_{j=1}^{k-1} \left( \prod_{j=1}^{k-1} \varphi_j(s_{j+1}) \right) \varphi_j(s_j), \\
+ \sum_{j=1}^{k-1} \left( \prod_{j=1}^k \varphi_j(s_j) \right) s_{j-1} l_{g_{j-1}}^{g_{j-1}} f_x(t_j^-) + n_k l_{g_k}^{g_k} f_x(t_{k+1}) \right]$$

$$= \varphi_{k+1}(t) + \psi_{k+1}(t) \left[ A \left( \prod_{j=1}^k \varphi_j(s_j) \right) + \sum_{j=1}^{k-1} \left( \prod_{j=1}^{k-1} \varphi_j(s_{j+1}) \right) \varphi_j(s_j), \\
+ \sum_{j=1}^{k-1} \left( \prod_{j=1}^k \varphi_j(s_j) \right) s_{j-1} l_{g_{j-1}}^{g_{j-1}} f_x(t_j^-) \right],$$

which implies that the second part of (16) holds. Similarly suppose that the second part of (16) is satisfied for $i = k$. Then for $t \in [s_k, t_{k+1})$, we obtain

$$x(t) = \begin{cases} 
\varphi_k(s_k) + \psi_k(s_k) x(s_k) \\
+ \sum_{j=1}^{k-1} \left( \prod_{j=1}^k \varphi_j(s_j) \right) s_{j-1} l_{g_{j-1}}^{g_{j-1}} f_x(t_j^-) + n_k l_{g_k}^{g_k} f_x(t), 
\end{cases}$$

for $t \in [s_k, t_{k+1})$.
which yields

\[
\sum_{k=0}^{m} \mu_k \int_{s_k}^{T} \left( s_k I_{s_k}^{\gamma_k} x(t) \right) dH_k(u) \]

\[
= A \sum_{k=0}^{m} \mu_k \left( \prod_{j=1}^{k} \Psi_j(s_j) \right) \Lambda_k + \sum_{k=0}^{m} \mu_k \Lambda^*(k) \Lambda_k \\
+ \sum_{k=0}^{m} \mu_k \sum_{j=1}^{k} \left( \prod_{j=1}^{k} \Psi_j(s_j) \right) s_{j-1} I_{s_{j-1}}^{\gamma_j-1} f_x(t_j^-) \Lambda_k \\
+ \sum_{k=0}^{m} \mu_k \int_{s_k}^{T} s_k I_{s_k}^{\gamma_k+\gamma_x} f_x(u) dH_k(u). \tag{18}
\]

The condition in (1) with (17) and (18) implies

\[
A = \frac{1}{\Omega} \left\{ \sum_{k=0}^{m} \mu_k \Lambda^*(k) \Lambda_k + \sum_{k=0}^{m} \mu_k \sum_{j=1}^{k} \left( \prod_{j=1}^{k} \Psi_j(s_j) \right) s_{j-1} I_{s_{j-1}}^{\gamma_j-1} f_x(t_j^-) \Lambda_k \\
+ \sum_{k=0}^{m} \mu_k \int_{s_k}^{T} s_k I_{s_k}^{\gamma_k+\gamma_x} f_x(u) dH_k(u) - \beta_2 \Lambda^*(m) \\
- \beta_2 \sum_{j=1}^{m+1} \left( \prod_{j=1}^{m} \Psi_j(s_j) \right) s_{j-1} I_{s_{j-1}}^{\gamma_j-1} f_x(t_j^-) \right\}. \tag{19}
\]
By substituting the constant $A_1$, (19), into (16), the obtained integral Equations (12) and (13) are presented.

Conversely, by taking the operator $s_i D_{S_i}^{h_i}$ over $[s_i, t_i]$, we get $s_i D_{S_i}^{h_i} x(t) = f(t, x(t))$. Putting $t = t_i$ and replacing $i$ by $i - 1$ in (12), then (13) implies $x(t) = \varphi_i(t) + \psi_i(t) x(t_{i-1})$, $t \in [t_i, s_i]$. By direct computation as substituting $t = 0$, $t = T$ and applying the Riemann–Stieltjes fractional integral of order $\gamma_k$ with respect to $g_k$ to the unknown function $x(t)$ in (12) over $[s_k, t_{k+1}]$, then the condition in (1) is satisfied. Therefore the proof is completed. □

3. Existence and Uniqueness Results

Before going to prove our main results, we have to define the space of functions and the operator which are involved to problem (1). Let $f = [0, T]$ be an interval and let $PC(J, \mathbb{R})$ and $PC^1(J, \mathbb{R})$ be the spaces of piecewise continuous function defined by $PC(J, \mathbb{R}) = \{ x : f \to \mathbb{R} \}$ $x(t)$ is continuous everywhere except for some $t_i$ at which $x(t_i^+)$ and $x(t_i^-)$ exist for $i = 1, 2, 3, \ldots, m$} and $PC^1(J, \mathbb{R}) = \{ x \in PC(J, \mathbb{R}) \}$ $x(t)$ is continuous everywhere except for some $t_i$ at which $x'(t_i^+)$ and $x'(t_i^-)$ exist for $i = 1, 2, 3, \ldots, m$. Let $E = PC(J, \mathbb{R}) \cap PC^1(J, \mathbb{R})$. Then $E$ is the Banach space with norm $\| x \| = \sup \{ \| x(t) \|, t \in J \}$.

Now, we define the operator on $E$ by

$$Qx(t) = \begin{cases} \frac{1}{\Omega} \left( \prod_{j=1}^{i} \psi_j(s_j) \right) \left\{ \sum_{k=0}^{m} \mu_k \Lambda^*(k) \Lambda_k - \beta_2 \Lambda^*(m) \right\} + \sum_{k=0}^{m} \mu_k \sum_{j=1}^{k} \left[ \prod_{j=1}^{k} \psi_j(s_j) \right] s_{j-1}^{1/\gamma_{j-1}} f_x(t_j^-) \Lambda_k \\ + \sum_{k=0}^{m} \mu_k \int_{s_k}^{t_{k+1}} s_k^{1/\gamma_k} f_x(u) dH_k(u) \\ - \beta_2 \sum_{j=1}^{m+1} \left[ \prod_{j=1}^{i} \psi_j(s_j) \right] s_{j-1}^{1/\gamma_{j-1}} f_x(t_j^-) \Lambda_k \\ + \Lambda^*(i) + \sum_{j=1}^{i} \left[ \prod_{j=1}^{i} \psi_j(s_j) \right] s_{j-1}^{1/\gamma_{j-1}} f_x(t_j^-) + s_i s_{i+1} f_x(t_i), \end{cases}$$

$t \in [s_i, t_{i+1}]$, $i = 0, 1, 2, \ldots, m$,

$$\phi_i(t) + \psi_i(t) \left\{ \frac{1}{\Omega} \left( \prod_{j=1}^{i-1} \psi_j(s_j) \right) \left\{ \sum_{k=0}^{m} \mu_k \Lambda^*(i) \Lambda_k \right\} + \sum_{k=0}^{m} \mu_k \sum_{j=1}^{k} \left[ \prod_{j=1}^{k} \psi_j(s_j) \right] s_{j-1}^{1/\gamma_{j-1}} f_x(t_j^-) \Lambda_k \\ + \sum_{k=0}^{m} \mu_k \int_{s_k}^{t_{k+1}} s_k^{1/\gamma_k} f_x(u) dH_k(u) - \beta_2 \Lambda^*(m) \right\} \\ - \beta_2 \sum_{j=1}^{m+1} \left[ \prod_{j=1}^{i-1} \psi_j(s_j) \right] s_{j-1}^{1/\gamma_{j-1}} f_x(t_j^-) \Lambda_k \\ + \Lambda^*(i - 1) + \sum_{j=1}^{i-1} \left[ \prod_{j=1}^{i-1} \psi_j(s_j) \right] s_{j-1}^{1/\gamma_{j-1}} f_x(t_j^-) \right\}, \end{cases}$$

$t \in [t_i, s_i]$, $i = 1, 2, 3, \ldots, m$.

Next, by applying the Banach’s contraction mapping principle, and Leray–Schauder’s nonlinear alternative, we derive the existence and uniqueness of solutions to problem (1).

Some constants are set as follows:

$$\Phi_1 = \frac{1}{\Omega} \left( \prod_{j=1}^{m} |\psi_j(s_j)| \right), \quad \Phi_2 = \sum_{k=0}^{m} |\mu_k| |\Lambda^*(k)| |\Lambda_k|.$$
\[ \Phi_3 = \sum_{k=0}^{m} |\mu_k| \left\{ \prod_{j=1}^{k} |\psi_j(s_j)| \right\} \left( \frac{(s_{j-1}(t_j) - s_{j-1}(s_{j-1}))^{\beta_{j-1}}}{\Gamma(s_{j-1} + 1)} \right) |\Lambda_k|, \]

\[ \Phi_4 = \sum_{k=0}^{m} |\mu_k| \int_{s_k}^{r_{k+1}} (g_k(u) - g_k(s_k))^\gamma_k \, dH_k(u), \]

\[ \Phi_5 = \sum_{j=1}^{m+1} \left[ \left( \prod_{j=1}^{m} |\psi_j(s_j)| \right) \left( \frac{(s_{j-1}(t_j) - s_{j-1}(s_{j-1}))^{\beta_{j-1}}}{\Gamma(s_{j-1} + 1)} \right) \right] |\Lambda_j|, \]

\[ \Phi_6 = \Phi_1(\Phi_3 + \Phi_4) + \Phi_5(\beta_2^2 |\Phi_1 + 1). \] (20)

**Theorem 1.** Suppose that the nonlinear function \( f: J \times \mathbb{R} \to \mathbb{R} \) satisfies the condition:

(H1) There exists a constant \( L > 0 \) such that for all \( t \in J \) and \( x, y \in \mathbb{R} \),

\[ |f(t, x) - f(t, y)| \leq L|x - y|. \]

If \( L \Phi_6 < 1 \), where \( \Phi_6 \) is defined by (20), then the non-instantaneous impulsive Riemann–Stieltjes fractional integral boundary value problem (1) has a unique solution on \( J \).

**Proof.** Let \( B_r \) be the subset of \( E \) defined by \( B_r = \{ x \in E : ||x|| \leq r \} \), where a fixed constant \( r \) satisfies

\[ r \geq \frac{\Phi_1 \Phi_2 + |\Lambda^*(m)|(\beta_2 |\Phi_1 + 1) + M \Phi_6}{1 - L \Phi_6}. \] (21)

Now we will prove that \( \mathcal{Q} B_r \subset B_r \). Setting \( M = \sup\{|f(t, 0)|, t \in J\} \), we have, from triangle inequality and (H1), that

\[ |f(t, x)| \leq |f(t, x) - f(t, 0)| + |f(t, 0)| \leq Lr + M. \]

Then we obtain

\[ |Qx(t)| \leq \frac{1}{|\Omega|} \left( \prod_{j=1}^{i} |\psi_j(s_j)| \right) \left\{ \sum_{k=0}^{m} |\mu_k||\Lambda^*(k)||\Lambda_k| + |\beta_2||\Lambda^*(m)| \right\} 
+ \sum_{k=0}^{m} |\mu_k| \left( \prod_{j=1}^{k} |\psi_j(s_j)| \right) s_{j-1}^{\beta_{j-1}} |f_x|(t_j^-) |\Lambda_k| 
+ \sum_{k=0}^{m} |\mu_k| \int_{s_k}^{r_{k+1}} s_k^{\beta_k + \gamma_k} |f_x|(u) \, dH_k(u) 
+ |\beta_2| \sum_{j=1}^{m+1} \left[ \left( \prod_{j=1}^{m} |\psi_j(s_j)| \right) s_{j-1}^{\beta_{j-1}} |f_x|(t_j^-) \right] 
+ |\Lambda^*(i)| + \sum_{j=1}^{i} \left[ \left( \prod_{j=1}^{i} |\psi_j(s_j)| \right) s_{j-1}^{\beta_{j-1}} |f_x|(t_j^-) \right] + s_i^{\beta_i} |f_x|(t) \]

for \( t \in [s_i, t_{i+1}], i = 0, 1, 2, \ldots, m \), and

\[ |Qx(t)| \leq |\phi_1(t)| + |\psi_i(t)| \left( \frac{1}{|\Omega|} \left( \prod_{j=1}^{i} |\psi_j(s_j)| \right) \right) \left\{ \sum_{k=0}^{m} |\mu_k||\Lambda^*(i)||\Lambda_k| \right\} 
+ \sum_{k=0}^{m} |\mu_k| \left( \prod_{j=1}^{k} |\psi_j(s_j)| \right) s_{j-1}^{\beta_{j-1}} |f_x|(t_j^-) |\Lambda_k| 
+ \sum_{k=0}^{m} |\mu_k| \int_{s_k}^{r_{k+1}} s_k^{\beta_k + \gamma_k} |f_x|(u) \, dH_k(u) + |\beta_2||\Lambda^*(m)| 
+ |\beta_2| \sum_{j=1}^{m+1} \left[ \left( \prod_{j=1}^{m} |\psi_j(s_j)| \right) s_{j-1}^{\beta_{j-1}} |f_x|(t_j^-) \right] \]
Thus \( \| Q \| \) for each \( \| \psi_i(s_j) \| \) and \( |\Lambda^*(i)| \) for \( i = 1, 2, 3, \ldots, m \). Then we have

\[
\sup_{t \in [t_i, s_i]} |Qx(t)| \leq \frac{1}{|\Omega|} \left( \prod_{j=1}^{m} |\psi_j(s_j)| \right) \left\{ \sum_{k=0}^{m} |\mu_k| |\Lambda^*(k)| |\Lambda_k| + |\beta_2| |\Lambda^*(m)| + (Lr + M) \sum_{k=0}^{m} |\mu_k| \int_{s_k}^{t_{k+1}} s_k \int_{s_k}^{t_{k+1}} (1) u \, dH_k(u) \right\} \left| s_{j-1}, l_{s_{j-1}}^{\alpha_{j-1}}(1) (t_j^-) \right|
\]

\[
+ (Lr + M) \sum_{k=0}^{m} |\mu_k| \int_{s_k}^{t_{k+1}} s_k \int_{s_k}^{t_{k+1}} (1) u \, dH_k(u)
\]

\[
+ (Lr + M) |\beta_2| \sum_{j=1}^{m+1} \left\{ \left( \prod_{j=1}^{m} |\psi_j(s_j)| \right) s_{j-1}, l_{s_{j-1}}^{\alpha_{j-1}}(1) (t_j^-) \right\}
\]

\[
+ |\Lambda^*(m)| + (Lr + M) \sum_{k=0}^{m} \left\{ \left( \prod_{j=1}^{m} |\psi_j(s_j)| \right) s_{j-1}, l_{s_{j-1}}^{\alpha_{j-1}}(1) (t_j^-) \right\}
\]

\[
+ (Lr + M) s_m l_m(1)(T) = \Phi_1 \Phi_2 + |\Lambda^*(m)| |\beta_2| \Phi_1 + 1) + rL(\Phi_1(\Phi_3 + \Phi_4) + \Phi_5(|\beta_2| \Phi_1 + 1) + M(\Phi_1(\Phi_3 + \Phi_4) + \Phi_5(|\beta_2| \Phi_1 + 1)
\]

Thus \( \| Q \| \leq r \), where \( r \) satisfies (21). Therefore, we conclude that \( QB_r \subset B_r \).

Next we will prove that the operator \( Q \) is a contraction. For any \( x, y \in B \), we have

\[
|Qx(t) - Qy(t)| \leq \frac{1}{|\Omega|} \left( \prod_{j=1}^{m} |\psi_j(s_j)| \right) \left\{ \sum_{k=0}^{m} |\mu_k| \int_{s_k}^{t_{k+1}} s_k \int_{s_k}^{t_{k+1}} (1) u \, dH_k(u) \right\} \left| s_{j-1}, l_{s_{j-1}}^{\alpha_{j-1}}(1) (t_j^-) \right|
\]

\[
+ |\beta_2| \sum_{j=1}^{m+1} \left\{ \left( \prod_{j=1}^{m} |\psi_j(s_j)| \right) s_{j-1}, l_{s_{j-1}}^{\alpha_{j-1}}(1) (t_j^-) \right\}
\]

\[
+ \sum_{j=1}^{m} \left[ \left( \prod_{j=1}^{m} |\psi_j(s_j)| \right) s_{j-1}, l_{s_{j-1}}^{\alpha_{j-1}}(1) (t_j^-) \right] + s_l l_{s_l}(1) (t)
\]

for \( t \in [s_i, t_{i+1}], i = 0, 1, 2, \ldots, m \), and

\[
|Qx(t) - Qy(t)| \leq |\psi_i(t)| + |\psi_i(t)| \left( \frac{1}{|\Omega|} \left( \prod_{j=1}^{m} |\psi_j(s_j)| \right) \left\{ \sum_{k=0}^{m} |\mu_k| \int_{s_k}^{t_{k+1}} s_k \int_{s_k}^{t_{k+1}} (1) u \, dH_k(u) \right\} \left| s_{j-1}, l_{s_{j-1}}^{\alpha_{j-1}}(1) (t_j^-) \right|
\]

\[
+ |\beta_2| \sum_{j=1}^{m+1} \left\{ \left( \prod_{j=1}^{m} |\psi_j(s_j)| \right) s_{j-1}, l_{s_{j-1}}^{\alpha_{j-1}}(1) (t_j^-) \right\}
\]

\[
+ \sum_{j=1}^{m} \left[ \left( \prod_{j=1}^{m} |\psi_j(s_j)| \right) s_{j-1}, l_{s_{j-1}}^{\alpha_{j-1}}(1) (t_j^-) \right] + s_l l_{s_l}(1) (t)
\]

for \( t \in [s_i, t_{i+1}], i = 0, 1, 2, \ldots, m \), and
\[ x_{s_j} \mathcal{I}^\gamma_{s_j} \left| f_x - f_y \right|(t_j) \mathcal{A}_k + \sum_{k=0}^m |\mu_k| \int_{s_k}^{t_k} s_k \mathcal{I}^\gamma_{s_k} \left| f_x - f_y \right|(u) dH_k(u) \]
\[ + \left| \beta_2 \right| \sum_{j=1}^{m+1} \left( \left( \prod_{j=1}^{m} |\psi_j(s_j)| \right) s_{j-1} \mathcal{I}^\gamma_{s_{j-1}} \left| f_x - f_y \right|(t_j) \right) \]
\[ + \sum_{j=1}^{m} \left( \prod_{j=1}^{m} |\psi_j(s_j)| \right) s_{j-1} \mathcal{I}^\gamma_{s_{j-1}} \left| f_x - f_y \right|(t_j) \]

for \( t \in [t_i, s_i], \ i = 1, 2, 3, \ldots, m. \) Consequently

\[ |Qx(t) - Qy(t)| \leq \frac{1}{|\Omega|} \left( \prod_{j=1}^{m} |\psi_j(s_j)| \right) \left[ \mathcal{L} x - y \right] \sum_{k=0}^m |\mu_k| \sum_{j=1}^{k+1} \left( \prod_{j=1}^{m} |\psi_j(s_j)| \right) s_{j-1} \mathcal{I}^\gamma_{s_{j-1}} \left( 1(t_j) \right) |\mathcal{A}_k| \]
\[ + \mathcal{L} x - y \left( \left| \beta_2 \right| \sum_{j=1}^{m+1} \left( \left( \prod_{j=1}^{m} |\psi_j(s_j)| \right) s_{j-1} \mathcal{I}^\gamma_{s_{j-1}} \left( 1(t_j) \right) \right) \right) \]
\[ + \mathcal{L} x - y \left( \sum_{j=1}^{m+1} \left( \left( \prod_{j=1}^{m} |\psi_j(s_j)| \right) s_{j-1} \mathcal{I}^\gamma_{s_{j-1}} \left( 1(t_j) \right) \right) \right) \]
\[ = \mathcal{L} \Phi_6 \| x - y \|, \]

which yields \( \|Qx - Qy\| \leq \mathcal{L} \Phi_6 \| x - y \|. \) As \( \mathcal{L} \Phi_6 < 1, Q \) is a contraction. Therefore, we deduce by Banach’s contraction mapping principle, that \( Q \) has a fixed point which is the solution of the boundary value problem (1). The proof is completed. \( \Box \)

**Remark 1.** If \( \beta_1 \neq 0, \beta_2 = 0, \) then the problem (1) is reduced to the initial and integral values problem. The constants \( \Omega^*, \Phi_6^* \) and \( \Phi_1^* \), given by

\[ \Omega^* = \beta_1 - \sum_{k=0}^m \mu_k \left( \prod_{j=1}^{k} |\psi_j(s_j)| \right) \mathcal{A}_k, \quad \Phi_6^* = \Phi_3^* + \Phi_4^* + \Phi_5^*, \quad \Phi_1^* = \frac{1}{|\Omega^*|} \left( \prod_{j=1}^{m} |\psi_j(s_j)| \right), \]

with conditions \( (H_1) \) and \( \mathcal{L} \Phi_6^* < 1 \) are used to obtain the existence of a unique solution of such a problem on \( J. \)

The following theorem of Leray–Schauder’s nonlinear alternative will be applied to the next result.

**Theorem 2** ([35]). Given \( E \) is a Banach space, and \( B \) is a closed, convex subset of \( E. \) In addition, let \( G \) be an open subset of \( B \) such that \( 0 \in G. \) Suppose that \( Q: G \rightarrow B \) is a continuous, compact (that is, \( Q(G) \) is a relatively compact subset of \( B \)) map. Then either

(i) \( Q \) has a fixed point in \( G, \)

(ii) there is \( x \in \partial G \) (the boundary of \( G \) in \( B \)) and \( \lambda \in (0, 1) \) with \( x = \lambda Q(x). \)

**Theorem 3.** Suppose that \( f: J \times \mathbb{R} \rightarrow \mathcal{R} \) is a continuous function. In addition we assume that:

\( (H_2) \) There exist a continuous nondecreasing function \( \Psi: [0, \infty) \rightarrow (0, \infty) \) and continuous function \( w: J \rightarrow \mathcal{R}^{+} \), such that

\[ |f(t, x)| \leq w(t) \Psi(|x|), \]

for each \( (t, x) \in J \times \mathcal{R}; \)
Then the non-instantaneous impulsive Riemann–Stieltjes fractional integral boundary value problem (1) has at least one solution on $I$.

**Proof.** Let $\rho$ be a radius of a ball $B_{\rho} = \{ x \in E : \| x \| \leq \rho \}$. It is obvious that $B_{\rho}$ is a closed, convex subset of $E$. Now, we will show that the operator $Q$ is fulfilled all conditions of Theorem 2. Firstly the continuity of operator $Q$ is proved by defining a sequence $\{ x_n \}$ which is converse to $x$. Then

$$|Qx_n(t) - Qx(t)| \leq \frac{1}{|\Omega|} \left( \prod_{j=1}^{i} |\psi_j(s_j)| \right) \left\{ \sum_{k=0}^{m} |\mu_k| \sum_{j=1}^{k} \left[ \left( \prod_{j=1}^{k} |\psi_j(s_j)| \right) s_{j-1}^{\alpha_{j-1}} \right] \right\} \Lambda_k$$

$$+ \sum_{k=0}^{m} |\mu_k| \int_{s_k}^{s_{k+1}} \left. s_{j-1}^{\alpha_{j-1}} \right| f_{x_n} - f_x |(t^-) \right| dH_k(u)$$

$$+ |\beta_{2}| \sum_{j=1}^{m+1} \left[ \left( \prod_{j=1}^{m} |\psi_j(s_j)| \right) s_{j-1}^{\alpha_{j-1}} \right] \Lambda_k \rightarrow 0, \quad \text{as} \quad n \rightarrow \infty,$$

for $t \in [s_i, t_{i+1})$, $i = 0, 1, 2, \ldots, m$, and

$$|Qx(t)| \leq |\varphi(t)| + |\psi(t)| \left[ \frac{1}{|\Omega|} \left( \prod_{j=1}^{i} |\psi_j(s_j)| \right) \left\{ \sum_{k=0}^{m} |\mu_k| \sum_{j=1}^{k} \left[ \left( \prod_{j=1}^{k} |\psi_j(s_j)| \right) \right] \right\} \right.$$

$$\left. \times s_{j-1}^{\alpha_{j-1}} \right| f_{x_n} - f_x |(t^-) \right| \Lambda_k + \sum_{k=0}^{m} |\mu_k| \int_{s_k}^{s_{k+1}} \left. s_{j-1}^{\alpha_{j-1}} \right| f_{x_n} - f_x |(u) \right| dH_k(u)$$

$$+ |\beta_{2}| \sum_{j=1}^{m+1} \left[ \left( \prod_{j=1}^{m} |\psi_j(s_j)| \right) s_{j-1}^{\alpha_{j-1}} \right] \Lambda_k \rightarrow 0, \quad \text{as} \quad n \rightarrow \infty,$$

for $t \in [t_i, s_{i+1})$, $i = 1, 2, 3, \ldots, m$. Then $Q$ is continuous.

Next the compactness of the operator $Q$ will be proved. Assume that $x \in B_{\rho}$, then we have

$$|Qx(t)| \leq \frac{1}{|\Omega|} \left( \prod_{j=1}^{i} |\psi_j(s_j)| \right) \left\{ \sum_{k=0}^{m} |\mu_k| |\Lambda^*(k)||\Lambda_k| + |\beta_{2}| |\Lambda^*(m)| \right.$$
\[ + \|w\| \Omega(\rho) |\beta_2| \sum_{j=1}^{m+1} \left[ \left( \prod_{j}^{m} |\psi_j(s_j)| \right) \lambda_{s_j, j-1} \right] \]

\[ + |\Lambda^*(m)| + \|w\| \Omega(\rho) \sum_{j=1}^{m} \left[ \left( \prod_{j}^{m} |\psi_j(s_j)| \right) \lambda_{s_j, j-1} \right] \]

\[ = \Phi_1 \Phi_2 + |\Lambda^*(m)| (|\beta_2| \Phi_1 + 1) + \|w\| \Omega(\rho) \Phi_6 \]

\[ := \Phi_7, \quad (22) \]

which yields \( \|Q\| \leq \Phi_7 \) and then \( QB_p \) is a uniformly bounded set. To prove equicontinuity of \( QB_p \), we let the points \( \theta_1, \theta_2 \in [0, T] \) such that \( \theta_1 < \theta_2 \). Then for any \( x \in B_p \), it follows that

\[ |Q\theta_2 - Q\theta_1| = |s_i \lambda_{s_i} f_{s_i} (\theta_2) - s_i \lambda_{s_i} f_{s_i} (\theta_1)| \leq \|w\| \Omega(\rho) |s_i \lambda_{s_i} (1) (\theta_2) - s_i \lambda_{s_i} (1) (\theta_1)| \]

\[ = \|w\| \Omega(\rho) \left\{ 2(g(\theta_2) - g(\theta_1))^{\alpha_i} + |g(\theta_2) - g(\theta_1)|^{\alpha_i} - (g(\theta_1) - g(\theta_1))^{\alpha_i} \right\} \to 0, \]

as \( \theta_1 \to \theta_2 \) for \( t \in [s_i, s_{i+1}], i = 0, 1, 2, \ldots, m \), and

\[ |Q\theta_2 - Q\theta_1| = |\psi_i(\theta_2) - \psi_i(\theta_1)| + |\psi_i(\theta_2) - \psi_i(\theta_1)| \times \text{const.} \]

\[ \to 0, \quad \text{as} \quad \theta_1 \to \theta_2. \]

for \( t \in [t_i, s_i], i = 1, 2, 3, \ldots, m \). The above two inequalities are convergent to zero independently of \( x \). Then \( QB_p \) is equicontinuous set. Therefore, we deduce that \( QB_p \) is relatively compact which implies by the Arzelá-Ascoli theorem, that the operator \( Q \) is completely continuous.

In the last step, we will illustrate that the condition \( (ii) \) of Theorem 2 dose not hold. Let \( x \) be a solution of problem (1). Now, we consider the operator equation \( x = \lambda Qx \) for any fixed constant \( \lambda \in (0, 1) \). Consequently, from above computation getting (22), we obtain

\[ \|x\| \leq \Phi_1 \Phi_2 + |\Lambda^*(m)| (|\beta_2| \Phi_1 + 1) + \|w\| \Omega(\|x\|) \Phi_6 \leq 1. \]

The hypothesis \((H_3)\) implies that there exists a positive constants \( N \) such that \( \|x\| \neq N \). Define the open subset of \( B_p \) by \( G = \{ x \in B_p : \|x\| < N \} \). It is easy to see that \( Q : \overline{G} \to E \) is continuous and completely continuous. Thus, there is no \( x \in \partial G \) such that \( x = \lambda Qx \) for some \( \lambda \in (0, 1) \). Hence the condition \( (ii) \) of Theorem 2 is not true. Therefore, by the conclusion from Theorem 2 \( (i) \), the operator \( Q \) has a fixed point \( x \in \overline{G} \) which is a solution of the problem (1) on \( J \). This is the end of the proof. \( \square \)

A special case can be obtain by setting \( p(t) \equiv 1 \) and \( \Psi(x) = \kappa_1 x + \kappa_2, \kappa_1 \geq 0, \kappa_2 > 0 \) in Theorem 3.

**Corollary 1.** If

\[ |f(t,x)| \leq \kappa_1 x + \kappa_2, \]

and if \( \kappa_1 \Phi_6 < 1 \), then the non-instantaneous impulsive Riemann–Stieltjes fractional integral boundary value problem (1) has at least one solution on \( J \).

**Remark 2.** In the same way of Remark 1, if \( \beta_1 \neq 0, \beta_2 = 0 \), and conditions \((H_2)-(H_3)\) are fulfilled with

\[ \frac{N}{\Phi_1^* \Phi_2 + |\Lambda^*(m)| + \|w\| \Omega(N) \Phi_6^*} > 1, \]

then the initial and integral values problem has at least one solution on \( J \).
Example 1. Consider the non-instantaneous impulsive Riemann–Stieltjes fractional integral boundary value problem

\[
x(t) = f(t, x(t)), \quad t \in [2i, 2i + 1), \quad i = 0, 1, 2, 3,
\]

\[
x(t) = \frac{1}{2} \log_2(i + t) + \left(\frac{1}{i + \tan^{-1}(t)}\right)x(t^-), \quad t \in [2i - 1, 2i), \quad i = 1, 2, 3,
\]

\[
3 \frac{x(0) + 4}{11} x(7) = \frac{5}{17} \int_0^1 \left(\frac{\varphi^2}{(x - \gamma)}\right) x(u) \, du + \frac{6}{19} \int_2^3 \left(\frac{\varphi^2}{(x - \gamma)}\right) x(u) \, du + \frac{7}{25} \int_4^5 \left(\frac{\varphi^2}{(x - \gamma)}\right) x(u) \, du + \frac{8}{29} \int_6^7 \left(\frac{\varphi^2}{(x - \gamma)}\right) x(u) \, du.
\]

(23)

Here \(a_i = (4i + 5)/(4i + 6)\), \(g_i(t) = e^{\ell}/(e^{\ell} + 4 + i - t)\), for \(t \in [2i, 2i + 1)\), \(i = 0, 1, 2, 3\), \(\varphi_i(t) = (1/2) \log_2(i + t)\), \(\psi_i(t) = 1/(i + \tan^{-1}t)\), \(t \in [2i - 1, 2i)\), \(i = 1, 2, 3\), \(\beta_1 = 3/11\), \(\beta_2 = 4/13\). Since \([2i, 2i + 1) \cup [2i - 1, 2i) \cup \{7\} = [0, 7]\), for \(i = 0, 1, 2, 3\), we put \(T = 7\). Setting \(h_0 = 5/17\), \(h_1 = 6/19\), \(h_2 = 7/23\), \(h_3 = 8/29\), \(H_j(t) = t^2 + it\), \(j = 1, 2, 3, 4\), \(\gamma_0 = 1/4\), \(\gamma_1 = 1/2\), \(\gamma_2 = 3/4\), \(\gamma_3 = 3/2\). Remark that \(g_i(t) > 0\) for all \(t \in [0, 7]\), \(i = 0, 1, 2, 3\). Then from all information, we can compute that \(|\Omega| \approx 0.5181070744\), \(\Phi_1 \approx 0.06251397190\), \(\Phi_2 \approx 0.8574153788\), \(\Phi_3 \approx 0.1639270834\), \(\Phi_4 \approx 0.1706687388\), \(\Phi_5 \approx 0.1889629435\), \(\Phi_6 \approx 0.2135145724\) and \(\Lambda^*(3) \approx 1.376938726\).

(i) Consider a nonlinear function \(f : [0, 7] \times \mathbb{R} \rightarrow \mathbb{R}\) by

\[
f(t,x) = \frac{4}{3} e^{-t} \left(\frac{2x^2 + 3|x|}{1 + |x|}\right) + \frac{2}{3} t + 1.
\]

(24)

It is easy to check that the function \(f(t, x)\) satisfies the Lipchitz condition with \(L = 4\), as \(|f(t, x) - f(t, y)| \leq 4|x - y|\), for all \(t \in [0, 7]\) and \(x, y \in \mathbb{R}\). Since \(L\Phi_6 \approx 0.8540582896 < 1\), by applying the result in Theorem 1, we have that the problem (23), with \(f\) given by (24), has a unique solution on \([0, 7]\).

(ii) Let now a nonlinear function \(f\) defined by

\[
f(t,x) = \frac{1}{t + 2} \left(\frac{x^{16}}{1 + x^{14}} + \frac{2}{3} \sin^2 x + \frac{1}{3} e^{-x^2}\right).
\]

(25)

Note that

\[|f(t,x)| \leq \frac{1}{t + 2} \big(x^2 + 1\big),\]

which satisfies (H2) with \(p(t) = 1/(t + 2)\) and \(\Psi(x) = x^2 + 1\). Accordingly, \(\|p\| = 1/2\) and there exists a constant \(N \in (1.984010360, 7.383031794)\) satisfying the condition (H2) of Theorem 3. Therefore, by applying Theorem 3, we deduce that the problem (23), with \(f\) given by (25), has at least one solution on \([0, 7]\).

(iii) If the term \(x^{16}\) is replaced by \(|x|^{15}\) in (25) then

\[
f(t,x) = \frac{1}{t + 2} \left(\frac{|x|^{15}}{1 + x^{14}} + \frac{2}{3} \sin^2 x + \frac{1}{3} e^{-x^2}\right).
\]

(26)

Hence we get \(|f(t,x)| \leq (1/2)|x| + (1/2)\). Putting \(k_1 = 1/2\) and \(k_2 = 1/2\), it follows that \(k_1\Phi_6 \approx 0.1067572862 < 1\), which implies, by Corollary 1, that the problem (23) with (26) has at least one solution on \([0, 7]\).
4. Conclusions

We have presented the sufficient criteria for the existence and uniqueness of solutions for a non-instantaneous impulsive Riemann–Stieltjes fractional integral boundary value problem. The given boundary value problem is converted into an equivalent fixed point operator equation, which is solved by applying the standard fixed point theorems. We make use of Banach’s contraction mapping principle to obtain the uniqueness result, while the Leray–Schauder nonlinear alternative is applied to obtain the existence result. We have demonstrated the application of the obtained results by constructing examples.

Our problem generates many types and also mixed types of impulsive fractional boundary value problems. For example, our results are reduced to Riemann–Liouville and Hadamard impulsive fractional boundary value problems when \( g(t) = t \) and \( g(t) = \log t \), respectively. Our results are new in the given configuration and contributes to the theory of fractional boundary value problems.

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