Research Article

Further Results on the \((p, k)\) – Analogue of Hypergeometric Functions Associated with Fractional Calculus Operators

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In a previous article, first and last researchers introduced an extension of the hypergeometric functions which is called “\((p, k)\)-extended hypergeometric functions.” Motivated by this work, here, we derive several novel properties for these functions, including integral representations, derivative formula, k-Beta transform, Laplace and inverse Laplace transforms, and operators of fractional calculus. Relevant connections of some of the discussed results here with those presented in earlier references are outlined.

1. Introduction

Recently, various extensions of the hypergeometric functions have been presented and investigated (see, for example, [1–7]).

In particular, Diaz and Pariguan [8] introduced interesting generalizations of the gamma, beta, Pochhammer, and hypergeometric functions as follows.

\[
(w)_{mk} = \frac{\Gamma_k(w + mk)}{\Gamma_k(w)} = \begin{cases} w(w+k)\ldots(w+(m-1)k), & m \in \mathbb{N}, w \in \mathbb{C}, \\
1, & m = 0, k \in \mathbb{R}^+, w \in \mathbb{C}\setminus\{0\}.
\end{cases}
\]

The relation between the \(\Gamma_k(w)\) and the usual gamma function \(\Gamma(w)\) (see, e.g., [9]) follows easily that

\[
\Gamma_k(w) = k^{-(w/k)}\Gamma\left(\frac{w}{k}\right), \quad \text{or} \Gamma(r) = k^{1-r}\Gamma_k(kr).
\]

Definition 1. For \(k \in \mathbb{R}^+\), the k-gamma function \(\Gamma_k(w)\) is defined by

\[
\Gamma_k(w) = \int_0^{\infty} z^{w-1}e^{-w^{1/k}z}dz, \quad w \in \mathbb{C}\setminus k\mathbb{Z}^-.
\]

We note that \(\Gamma_k(w) \longrightarrow \Gamma(w)\), for \(k \longrightarrow 1\), where \(\Gamma(w)\) is the classical Euler’s gamma function and \((w)_{mk}\) is the k-Pochhammer symbol given by

\[
\Gamma_k(w) = \int_0^{\infty} z^{w-1}e^{-w/z}dz, \quad w \in \mathbb{C}\setminus \{0\}.
\]
Definition 2. The $k$-beta function $B^k(z, w)$ is defined by

\[
B^k(z, w) = \frac{1}{k} \int_0^1 u^{(z/k)-1} (1-u)^{(w/k)-1} \, du, \quad (k \in \mathbb{R}^+, \min\{\text{Re}(z), \text{Re}(w)\} > 0),
\]

\[
(k \in \mathbb{R}^+, z, w \in \mathbb{C}\setminus\mathbb{Z}_0).
\]

Clearly, the case $k = 1$ in (4) reduces to the known beta function $\mathcal{B}(z, w)$, and the relation between the $k$-beta function $B^k(z, w)$ and the original beta function $\mathcal{B}(z, w)$ is

\[
B^k(z, w) = \frac{1}{k} \mathcal{B}\left(\frac{z}{k}, \frac{w}{k}\right).
\]

Definition 3. Let $k \in \mathbb{R}^+$ and $\theta_1, \theta_2, \theta_3 \in \mathbb{C}$ and $\theta_4 \in \mathbb{C}\setminus\mathbb{Z}_0$; then, $k$-hypergeometric function is defined in the form

\[
_{2}F_{1}^{(k)}\left[ \begin{array}{c} \theta_1, \theta_2 \\ \theta_3 \end{array} ; w \right] = \sum_{s=0}^{\infty} \frac{(\theta_1)_s(k)}{(\theta_3)_s(k)} \frac{w^s}{s!}, \quad |w| < \frac{1}{k}
\]

(6)

is the usual Pochhammer symbol (or the shifted factorial) and $\Gamma(.)$ is standard gamma function.

Since that period, many different outcomes concerning the $k$-analogue of hypergeometric function and related functions have been contributed by many researchers, for instance, Agarwal et al. [10], Mondal and Akel [11], Nisar et al. [12], Singh et al. [13], Li and Dong [14], Yilmaz et al. [15], Yilmazer and Ali [16], and Akdemir et al. [17].

Very recently, Abdalla and Hidan [18] introduced and investigated the following $(p, k)$-extend hypergeometric function:

\[
_{2}F_{1}^{(p, k)}\left[ \begin{array}{c} \theta_1, \theta_2 \\ \theta_3 \end{array} ; \xi \right] = \sum_{s=0}^{\infty} \frac{(\theta_1)_s(k)(\theta_2)_s(k)}{(\theta_3)_s(k)} \frac{\xi^s}{(p s)!},
\]

(9)

which is an entire function for $p > 1$, where $k \in \mathbb{R}^+$ and $\theta_1, \theta_2, \xi \in \mathbb{C}$ and $\theta_3 \in \mathbb{C}\setminus\mathbb{Z}_0$, and $(\theta)_s(k)$ is the $k$-Pochhammer symbol defined in (2).

They studied several its properties such as the $k$-Euler and Laplace-type integral representations, differentiation type formulae, contiguous function relations, and differential equations (cf. [18]).

Remark 1. From important special cases of $_{2}F_{1}^{(p, k)}$ are equations (6) and (7). Furthermore, when $k \rightarrow 1$, we obtained the $p$-extend hypergeometric function in the following form (see Chapter 3 in [19]):

\[
\sum_{s=0}^{\infty} \frac{(\theta_1)_s(k)}{(\theta_2)_s(k)} \frac{\xi^s}{(p s)!},
\]

(10)

which is an entire function for $p > 1$.

Motivated by some of these aforesaid studies of the $(p, k)$-extended hypergeometric function $_{2}F_{1}^{(p, k)}$ defined by (9), in this manuscript, we investigate certain integral representations, a derivative formula, $k$-Beta transform, Laplace and inverse Laplace transforms, and fractional calculus operators of the $(p, k)$-extended hypergeometric function which may be useful for carrying out further research studies to make more other developments and extensions of this field. In addition, some interesting special cases of our main results are also indicated.

2. Integral Representations and Derivative Formula

Starting, we establish the following theorems in terms of the $k$-integral representations of the $(p, k)$-extended hypergeometric functions as follows.

Theorem 1. The following integral representation for $_{2}F_{1}^{(p, k)}$ in (9) holds true:

\[
\sum_{s=0}^{\infty} \frac{(\theta_1)_s(k)}{(\theta_2)_s(k)} \frac{\xi^s}{(p s)!},
\]

(10)
Proof. Inserting series (9) in the LHS of (11) and by using
relation (2), we obtain
\[
\mathcal{I}^k = \frac{\Gamma_k^{(k)}}{\Gamma^{(k)}(\theta_1, \theta_2)} \int_0^\eta u^{(\theta_1, \theta_2) - 1} (\eta - u)^{(\delta/k) - 1} \cdot
\]
\[
\times \sum_{m=0}^\infty \frac{(\theta_1)_{m,k} (\theta_2)_{m,k} (u \xi)^m}{(m)_{k}!} du
\]
\[
= \sum_{m=0}^\infty \frac{(\theta_1)_{m,k} (\theta_2 + m k) \Gamma_k^{(k)} (\theta_3 + \delta) (\xi)^m}{\Gamma_k^{(k)} (\theta_3 + m k) \Gamma_k^{(k)} (\delta)} \cdot (pm)! \cdot
\]
\[
\times \int_0^\eta u^{(\theta_1, \theta_2) - 1} (\eta - u)^{(\delta/k) - 1} du.
\]
Letting \( u = \eta v \) and according to the k-beta function (4), we find that
\[
\mathcal{I}^k = \sum_{m=0}^\infty \frac{(\theta_1)_{m,k} \Gamma_k^{(k)} (\theta_3 + m k) \Gamma_k^{(k)} (\theta_3 + \delta)}{\Gamma_k^{(k)} (\theta_3 + m k) \Gamma_k^{(k)} (\delta)} \cdot (pm)! \cdot
\]
\[
\times \int_0^1 v^{(\theta_1, \theta_2) m - 1} (1 - v)^{(\delta/k) - 1} dv
\]
\[
= \sum_{m=0}^\infty \frac{(\theta_1)_{m,k} \Gamma_k^{(k)} (\theta_3 + m k) \Gamma_k^{(k)} (\theta_3 + \delta)}{\Gamma_k^{(k)} (\theta_3 + m k) \Gamma_k^{(k)} (\delta)} \cdot (pm)! \cdot \eta^{((\theta_1, \theta_2)/k) - 1}
\]
\[
k B^k (\theta_3 + m k, \delta).
\]
(13)

The above equation gives the RHS of (11). We thus obtain result (11) in Theorem 1. \quad \square

**Theorem 2.** The following integral representation for \( \mathcal{I}_1^{(p,k)} \) in (9) holds true:
\[
\int_0^\infty \exp \left( -\frac{1}{4k} \frac{\xi^2}{u} \right) \frac{1}{\Gamma((\theta_3 + k)/2)} \mathcal{I}_1^{(p,k)} \left( \theta_1, \theta_2, \xi \right) \frac{d\xi}{u}
\]
\[
= \frac{\Gamma_k^{(k)}}{\Gamma^{(k)}(\theta_1, \theta_2)} \int_0^\eta u^{(\theta_1, \theta_2) - 1} \cdot
\]
\[
\times \sum_{m=0}^\infty \frac{(\theta_1)_{m,k} (\theta_2)_{m,k} (u \xi)^m}{(m)_{k}!} du
\]
\[
\left[ \frac{1}{\Gamma((\theta_3 + k)/2)} \mathcal{I}_1^{(p,k)} \left( \theta_1, \theta_2, \xi \right) \right] d\xi,
\]
(14)

Proof. For convenience, let the left-hand side of (14) be denoted by \( T \). Applying the series expression of (9) to \( T \), we observe that
\[
T = \int_0^\infty \exp \left( -\frac{1}{4k} \frac{\xi^2}{u} \right) \sum_{m=0}^\infty \frac{(\theta_1)_{m,k} (\theta_2)_{m,k} \xi^{2m+\theta_3 k}}{(pm)!} \cdot
\]
\[
\times \int_0^\eta u^{(\theta_1, \theta_2) - 1} \cdot
\]
\[
\frac{1}{\Gamma((\theta_3 + k)/2)} \mathcal{I}_1^{(p,k)} \left( \theta_1, \theta_2, \xi \right) \frac{d\xi}{u}
\]
\[
= \frac{1}{\Gamma_k^{(k)}} \sum_{m=0}^\infty \frac{(\theta_1)_{m,k} \Gamma_k^{(k)} (\theta_2 + m k) \xi^{2m+\theta_3 k}}{(pm)!} \cdot
\]
\[
\times \int_0^\eta u^{(\theta_1, \theta_2) - 1} \cdot
\]
\[
\frac{1}{\Gamma((\theta_3 + k)/2)} \mathcal{I}_1^{(p,k)} \left( \theta_1, \theta_2, \xi \right) \frac{d\xi}{u}
\]
(15)

Setting \( \xi^2 = 4^{(1/k)} u v \), we obtain
\[
T = \frac{\Gamma_k^{(k)}}{\Gamma^{(k)}(\theta_1, \theta_2)} \sum_{m=0}^\infty \frac{(\theta_1)_{m,k} \Gamma_k^{(k)} (\theta_2 + m k)}{(pm)!} \cdot
\]
\[
\int_0^\eta \exp \left( -\frac{v}{k} \right) \left[ \frac{\sqrt{u v}}{2^{1/2}} \right] \mathcal{I}_1^{(p,k)} \left( \theta_1, \theta_2, \xi \right) \frac{d\xi}{u}
\]
\[
= \frac{1}{\Gamma_k^{(k)}} \sum_{m=0}^\infty \frac{(\theta_1)_{m,k} \Gamma_k^{(k)} (\theta_2 + m k)}{(pm)!} \cdot
\]
\[
\times \int_0^\eta \exp \left( -\frac{v}{k} \right) \left[ \frac{\sqrt{u v}}{2^{1/2}} \right] \mathcal{I}_1^{(p,k)} \left( \theta_1, \theta_2, \xi \right) \frac{d\xi}{u}
\]
(16)

Making an appeal to the following duplication k-gamma formula (cf. [8]),
\[
\Gamma_k^{(k)}(2w) = \frac{k}{2 \pi} 2^{1/2} \Gamma_k^{(k)}(w + k/2).
\]
In the above series and after a simplification, we obtain
\[
T = \frac{\Gamma^k(\theta_3)}{\Gamma^k(\theta_2) \Gamma^k(\theta_3 - \theta_2)} \sum_{m=0}^{\infty} \frac{(\theta_1)_{m,k} \Gamma^k(\theta_2 + mk)}{(pm)!} \frac{1}{2 \left(\theta_3 + 2mk + k\right)} \frac{1}{2 \left(\theta_3 + 2mk - k\right)} \int_{0}^{1} \frac{v^{\theta_3/2}(1-v)^{\theta_2/2}}{v^{\theta_3/2}(1-v)^{\theta_2/2}} dv.
\]
We thus arrive at the desired result (14) asserted by Theorem 2.

**Theorem 3.** The following integral representation for \(2\mathbf{S}_1^{(p,k)}\) in (9) holds true:

\[
S = \frac{\Gamma^k(\theta_3)}{\Gamma^k(\theta_2) \Gamma^k(\theta_3 - \theta_2)} \sum_{m=0}^{\infty} \frac{(\theta_1)_{m,k} \Gamma^k(\theta_2 + mk)}{(pm)!} \frac{1}{2 \left(\theta_3 + 2mk + k\right)} \frac{1}{2 \left(\theta_3 + 2mk - k\right)} \int_{0}^{1} \frac{v^{\theta_3/2}(1-v)^{\theta_2/2}}{v^{\theta_3/2}(1-v)^{\theta_2/2}} dv.
\]

**Proof.** Taking left-hand side of equation (19) by S and \(v = (u/(u+1))\), we have

\[
S = \frac{\Gamma^k(\theta_3)}{\Gamma^k(\theta_2) \Gamma^k(\theta_3 - \theta_2)} \sum_{m=0}^{\infty} \frac{(\theta_1)_{m,k} \Gamma^k(\theta_2 + mk)}{(pm)!} \frac{1}{2 \left(\theta_3 + 2mk + k\right)} \frac{1}{2 \left(\theta_3 + 2mk - k\right)} \int_{0}^{1} \frac{v^{\theta_3/2}(1-v)^{\theta_2/2}}{v^{\theta_3/2}(1-v)^{\theta_2/2}} dv.
\]

Changing the order of integration and summation in (20) and by using relation (4), we obtain

\[
S = \frac{\Gamma^k(\theta_3)}{\Gamma^k(\theta_2) \Gamma^k(\theta_3 - \theta_2)} \sum_{m=0}^{\infty} \frac{(\theta_1)_{m,k} \Gamma^k(\theta_2 + mk)}{(pm)!} \frac{1}{2 \left(\theta_3 + 2mk + k\right)} \frac{1}{2 \left(\theta_3 + 2mk - k\right)} \int_{0}^{1} \frac{v^{\theta_3/2}(1-v)^{\theta_2/2}}{v^{\theta_3/2}(1-v)^{\theta_2/2}} dv.
\]

We thus get the required integral formula (19).

Similarly, we can easily obtain the following result without proof.

**Theorem 4.** The following integral representation for \(2\mathbf{S}_1^{(p,k)}\) in (9) holds true:
\[ \theta_3 \int_0^\xi \omega^{(\theta_3/k)_1} \left[ 1 + \frac{2}{k} (\xi - \omega) \right] \mathcal{S}_1^{(p,k)} \left[ \frac{\theta_1, \theta_2}{\theta_3} : \omega \right] d\omega \]

\[ - \int_0^\xi \omega^{(\theta_3/k)_2} \mathcal{S}_1^{(p,k)} \left[ \frac{\theta_1, \theta_2}{\theta_3 + k} : \omega \right] d\omega, \]

\[ = k\xi^{(\theta_3/k)_2} \mathcal{S}_1^{(p,k)} \left[ \frac{\theta_1, \theta_2}{\theta_3 + k} : \xi \right], \]

\[ (\theta_1, \theta_2, \omega, \xi \in \mathbb{C}, \theta_3 \in \mathbb{C} \setminus \mathbb{Z}^0, \text{Re}(\theta_1) > 0, \text{Re}(\theta_2) > 0, \theta_3 > 0, |\omega| < 1, k \in \mathbb{R}^+ \text{ and } p \in \mathbb{N}), \]

(22)

Remark 2. At \( p = 1 \) in Theorems 1–4, we obtain integral formulae of the \( k \)-analogue of hypergeometric functions defined in (6).

Remark 3. The substitution \( k = 1 \) in Theorems 1–4 leads to the integral representations of the \( p \)-extended hypergeometric functions defined in (10).

Remark 4. If we take \( p = 1 \) and \( k = 1 \) in the abovementioned theorems, we obtain the corresponding results for the classical hypergeometric functions (see, e.g., [20]).

**Theorem 5.** The following derivative formula holds true:

\[ \frac{d^m}{d\xi^m} \left( \xi^{(\theta_3/k)_k} \mathcal{S}_1^{(p,k)} \left[ \frac{\theta_1, \theta_2}{\theta_3 + k} : \xi \right] \right) = \xi^{(\theta_3/k)_k} \mathcal{S}_1^{(p,k)} \left[ \frac{\theta_1, \theta_2}{\theta_3 + (m-1)k} : \xi \right], \]

(23)

\[ (\theta_1, \theta_2, \xi \in \mathbb{C}, \theta_3 \in \mathbb{C} \setminus \mathbb{Z}^0, \text{Re}(\theta_1) > 0, \text{Re}(\theta_2) > 0, \text{Re}(\theta_3) > 0, |\xi| < 1, k \in \mathbb{R}^+ \text{ and } p \in \mathbb{N}). \]

Therefore, the general result (23) can now be easily derived by using the principle of mathematical induction on \( m \in \mathbb{N}_0 \).

Remark 5. \( p = 1 \) in (23) leads naturally to the differentiation formula for \( _2F_1^k \) in (6).

Remark 6. \( k = 1 \) in (23) leads naturally to the differentiation formula for \( _2S_1^{(p,1)} \) in (10).

Remark 7. For \( p = 1 \) and \( k = 1 \) in (23), we obtain the corresponding result for the classical hypergeometric functions (see, e.g., [20]).

### 3. Integral Transforms

In this section, we prove three theorems, which exhibit the connection between the integral transforms such as \( k \)-Beta transform and Laplace and inverse Laplace transforms for \( _2S_1^{(p,k)} \) given in (8).

**Theorem 6.** The \( k \)-Beta transform for \( _2S_1^{(p,k)} \) in (8) is given in the following form:

\[ \mathcal{B}_k \left[ _2S_1^{(p,k)} \left[ \frac{\theta_1, \theta_2}{\theta_3} : \xi, \omega \right] : \theta_1, \theta_2 \right] = \mathcal{B}^k \left( \theta_1, \theta_2 \right) _2S_1^{(p,k)} \left[ \frac{\theta_1, \theta_2}{\theta_3} : \eta \right], \]

(26)

\[ (\theta_1, \theta_2, \xi, \eta, \omega, \xi \in \mathbb{C}, \theta_3 \in \mathbb{C} \setminus \mathbb{Z}^0, \text{Re}(\theta_1) > 0, \text{Re}(\theta_2) > 0, \text{Re}(\theta_3) > 0, |\eta| < 1, k \in \mathbb{R}^+ \text{ and } p \in \mathbb{N}). \]
where the k-Beta transform of $\Phi(\xi)$ is defined as
\[
\mathbb{B}_k[\Phi(\xi); \delta_1, \delta_2] = \frac{1}{k} \int_0^1 \xi^{(\delta_1/k)-1} (1 - \xi)^{\delta_2/k - 1} \Phi(\xi) \, d\xi,
\]
\[\text{ (Re}(\delta_1) > 0, \text{ Re}(\delta_2) > 0, k \in \mathbb{R}^+).\]

Proof. By invoking definition (27) and applying (9) to the k-Beta transform of (26), we obtain

\[
\frac{1}{k} \int_0^1 w^{(\delta_1/k)-1} (1 - w)^{(\delta_2/k)-1} 2\mathcal{D}_1[p,k]^{\theta_1 + \delta, \theta_2}{\theta_3}{\xi \eta} \, dw = \frac{1}{k} \int_0^1 \xi^{(\delta_1/k)-1} (1 - \xi)^{(\delta_2/k)-1} \sum_{m=0}^{\infty} \frac{(\theta_1 + \delta)_m (\theta_2)_{m,k} (\xi \eta)^m}{(\theta_3)_{m,k} (pm)!} \, d\xi
\]

which, upon using (4) and (9), yields our desired result (26) in Theorem 6. \qed

\[\text{Theorem 7. The following Laplace transform and inverse Laplace transform formulae hold, respectively:}
\]

\[
\mathcal{L}\left[2\mathcal{D}_1[p,k]^{\theta_1, \theta_2} w ; \xi \right] = \frac{1}{w} \mathcal{L}\left[2\mathcal{D}_1[p,k]^{\theta_1, \theta_2, k} \xi / kw \right],
\]

\[\left(\theta_1, \theta_2, w, \xi \in \mathbb{C}, \theta_3 \in \mathbb{C} \setminus \mathbb{Z}_0, \text{Re}(\theta_1) > 0, \text{Re}(\theta_2) > 0, \text{Re}(w) > 0, \frac{\xi}{kw} < 1, k \in \mathbb{R}^+ \text{and } p \in \mathbb{N}\right),
\]

\[\mathcal{L}^{-1}\left[w^{-\theta} 2\mathcal{D}_1[p,k]^{\theta_1, \theta_2} w ; \xi \right] = \frac{\xi^{u-1}}{\Gamma^k(\theta_1)} 2\mathcal{D}_1[p,k]^{\theta_2} \xi^k
\]

\[\left(\theta_1, \theta_2, \xi, w \in \mathbb{C}, \theta_3 \in \mathbb{C} \setminus \mathbb{Z}_0, \text{Re}(\theta_1) > 0, \text{Re}(\theta_2) > 0, \text{Re}(\theta_3) > 0, |\xi w^k| < 1, k \in \mathbb{R}^+ \text{and } p \in \mathbb{N}\right).
\]

Proof. To prove (30), we use the power series expansion (9), and applying definition (32), we observe that

\[\text{By change the order of integration and summation with setting } wz = (w/k) \text{ and then applying the } k\text{-gamma function (1) to the last integral, we obtain}
\]

\[\mathcal{L}\left[2\mathcal{D}_1[p,k]^{\theta_1, \theta_2} w ; \xi \right] = \int_0^{\infty} e^{-uz} \sum_{m=0}^{\infty} \frac{(\theta_1)_{m,k} (\theta_2)_{m} (\xi z)^m}{(\theta_3)_{m,k} (pm)!} \, dz
\]

which leads to the desired result (30). Now, in order to demonstrate (31), making use of (8) in the left-hand side of (31), we find that
After a simplification, we get the required result in (31).

Remark 8. The special cases of (30) and (31) when \( p = 1 \) are easily seen to reduce to the known integral transforms of the \( k \)-hypergeometric functions in (6). Also, \( k = 1 \), in (30) and (31), leads to the integral transforms for \( k \)-hypergeometric functions in (7) (see, e.g., [20]).

4. Fractional Calculus Approach

The \( k \)-Riemann–Liouville fractional integral is using \( k \)-gamma function defined in [22]

\[
I^\nu_k \Phi(\tau)(\xi) = \frac{1}{k\Gamma^k(\nu)} \int_0^\xi \Phi(\tau)(\xi - \tau)^{(\nu/k) - 1} d\tau, \quad \nu, \, k \in \mathbb{R}^+.
\]

Also, the \( k \)-Riemann–Liouville fractional derivative of order \( \nu \) introduced in [23, 24] by

\[
\mathcal{D}^\nu_k \Phi(\xi) = D(1^{(1-\nu)}_k \Phi(\xi)), \quad D = \frac{d}{d\xi}
\]

Nowadays, various studies and extensions of \( k \)-fractional calculus operators were presented by several researchers (see, for example, [22–28]).

Here, we consider the \( k \)-fractional differentiation of the \( (p, k) \)-extend hypergeometric functions using define the extended Riemann–Liouville \( k \)-fractional derivative with the parameters \( \delta, \mu, \nu \in \mathbb{C} \).

\[
\mathcal{D}^\nu_k \{ f(\xi) \} = \left\{ I^{\delta(\nu - 1)}_k \frac{d^n}{d\xi^n} \left( I^{(1-\delta)(n-\nu)}_k f(\xi) \right) \right\} f(\xi).
\]

Remark 9. Some important special cases of this definition for some particular choice of the parameters \( \delta, \mu, \nu \), and \( k \) are enumerated below:

1. Putting \( k = 1 \), we produce the generalized fractional derivative is defined in [26] in the form

\[
\mathcal{D}^\nu \{ f(\xi) \} = \left\{ I^{\delta(\nu - 1)} \frac{d^n}{d\xi^n} \left( I^{(1-\delta)(n-\nu)} f(\xi) \right) \right\} f(\xi).
\]

2. If \( \nu = \mu \) and \( n = 1 \), then we obtain the \( k \)-fractional Hilfer derivative of order \( \nu \) is defined [25] as

\[
\mathcal{D}^\nu_k \{ f(\xi) \} = \left\{ I^{\delta(\nu - 1)} \frac{d}{d\xi} \left( I^{(1-\delta)(n-\nu)} f(\xi) \right) \right\} f(\xi).
\]

3. Taking \( \delta = 0, \mu = \nu \), and \( n = 1 \) in (39), we reduce relation (38).

We need the following lemma.

Lemma 1 (see [27]). The following formula holds true:

\[
\mathcal{D}^\nu_k \{ f(\xi) \} = I^{\delta(\nu - 1)}_k \frac{\Gamma^k(\lambda)}{\Gamma^k(\lambda - nk + 1)} \frac{\Gamma^k(\lambda - n - \nu + 1)}{\Gamma^k(\lambda - nk + 1 - (n - 1)(1-\delta)(n-\nu))} f((\nu)(n-1)+\delta(n-\nu+1)/k)-(n+1)),
\]

\[((n-1) < \Re(\mu), \Re(\nu) \leq n, \Re(\delta) \leq 1, \Re(\xi) > 0, k \in \mathbb{R}^+, n \in \mathbb{N})\].
Proof. From (39), we have
\[ D^\mu_0 \{ \xi^{(\lambda/k)-1} \} = \left\{ I^\mu_k \{ (1-\delta)(n-\mu) \} \right\} \xi^{(\lambda/k)-1}. \]  
(43)

\[ (1^{(1-\delta)(n-\mu)}) \left\{ \xi^{(\lambda/k)-1} \right\} = \frac{1}{kl^k((1-\delta)(n-\mu))} \int_0^\xi (\xi - \tau)^{(1-\delta)(n-\mu)-1} \tau^{(\lambda/k)-1} d\tau, \]  
(44)

Substituting \( \tau = w\xi \) in (44), we obtain
\[ (1^{(1-\delta)(n-\mu)}) \left\{ \xi^{(\lambda/k)-1} \right\} = \frac{1}{kl^k((1-\delta)(n-\mu))} \int_0^\xi (\xi - w)^{(1-\delta)(n-\mu)-1} (1-w)^{(1-\delta)(n-\mu)-1} d\xi. \]  
(45)

According to (4) into the above equation, we obtain
\[ (1^{(1-\delta)(n-\mu)}) \left\{ \xi^{(\lambda/k)-1} \right\} = k^{-\delta} \Gamma^k(\lambda) \Gamma^k(\lambda + ((1-\delta)(n-\mu))), \]  
(46)

Direct computation using differentiating equation (46) with respect to \( \xi \) gives
\[ \frac{d}{d\xi} \left\{ (1^{(1-\delta)(n-\mu)}) \left\{ \xi^{(\lambda/k)-1} \right\} \right\} = k^{-\delta} \Gamma(\lambda) \right\{ \xi^{((1-\delta)(n-\mu)+\lambda)/k-2}. \]  
(47)

Consider the following k-fractional integral operator:
\[ D^\mu_0 \{ \xi^{(\lambda/k)-1} \} = \left\{ I^\mu_k \{ (1-\delta)(n-\mu) \} \right\} \xi^{(\lambda/k)-1}. \]  
(43)

Recursive application of this procedure finally gives
\[ \frac{d^n}{d\xi^n} \left\{ (1^{(1-\delta)(n-\mu)}) \left\{ \xi^{(\lambda/k)-1} \right\} \right\} = \frac{k^{-\delta} \Gamma(\lambda)}{\Gamma^k(\lambda - nk + ((1-\delta)(n-\mu)))} \xi^{((1-\delta)(n-\mu)+\lambda)/k-(n+1)}. \]  
(48)

Now, applying the k-Riemann fractional integral in (37) with the order \( \delta(n-\nu) \) to the above equation, we arrive the desired result in (42). 

\[ \square \]

Theorem 8. The following relation holds true:

\[ D^\mu_0 \{ \xi^{(\lambda/k)-1} \} \leq \sigma r, \theta_2, \theta_3, \frac{\theta_1, \theta_2, \lambda}{2} \left\{ \sigma r, \theta_2, \theta_3, \lambda - \mu + n(1-k) + \delta(\mu-\nu) \right\} \left( \begin{array}{l} \theta_1, \theta_2, \tau, \sigma, \sigma r, \theta_2, \theta_3, \lambda - \mu + n(1-k) + \delta(\mu-\nu) \end{array} \right), \]  
(49)
Proof. According to (39) and by invoking (9) into the left-hand side of equation (49), we find that

\[
Y = \sum_{k=0}^{\infty} \frac{\left(\theta_1\right)_{m,k} \left(\theta_2\right)_{m,k} \left(\sigma \varphi\right)^m}{\left(\theta_3\right)_{m,k} \left(pm\right)!} = \sum_{m=0}^{\infty} \frac{\left(\theta_1\right)_{m,k} \left(\theta_2\right)_{m,k} \sigma^m}{\left(\theta_3\right)_{m,k} \left(pm\right)!} \sum_{k=0}^{\infty} \left[ \Gamma_{\left(l+k\right)-1} \left(\sigma \varphi\right)^m \right].
\]

Applying Lemma 1 and after simple computations lead to

\[
Y = \frac{1}{k^n} \sum_{m=0}^{\infty} \frac{\left(\theta_1\right)_{m,k} \left(\theta_2\right)_{m,k} \sigma^m}{\left(\theta_3\right)_{m,k} \left(pm\right)!} \frac{\Gamma_k \left(\lambda + nk\right)}{\Gamma_k \left(\lambda + nk + (1 - \delta) (n - \mu) + \delta (n - \nu)\right)}
\]

\[
= \frac{1}{k^n} \frac{\Gamma_k \left(\lambda\right)}{\Gamma_k \left(\lambda - nk + (1 - \delta) (n - \mu) + \delta (n - \nu)\right)} \frac{1}{\Gamma_k \left(\lambda + nk + (1 - \delta) (n - \mu) + \delta (n - \nu)\right)}
\]

\[
\times \sum_{m=0}^{\infty} \frac{\left(\theta_1\right)_{m,k} \left(\theta_2\right)_{m,k} \left(\lambda - nk + (1 - \delta) (n - \mu) + \delta (n - \nu)\right)}{\left(\theta_3\right)_{m,k} \left(pm\right)!} \Gamma_k \left(\lambda + nk\right)
\]

We thus get the required result in (49).

\[\square\]

5. Concluding Remarks

As a matter of fact, this manuscript is a continuation of the recent author’s paper [18], where we have introduced the \((p,k)\)-analogues of hypergeometric functions. The researchers did not state the properties of integral transforms as well as the fractional calculus operators in the mentioned work, which are the main objective of the current work.

Also, we have detected that, by setting \(p \rightarrow 1\), the various outcomes considered in this manuscript reduce to the corresponding outcomes in [22–25, 27]. Also, for \(k \rightarrow 1\), we get many interesting new outcomes for the \(p\)-extended hypergeometric functions. Furthermore, if we take both \(k \rightarrow 1\) and \(p \rightarrow 1\), then the obtained results reduce to the results analogous to the classical hypergeometric functions.

More importantly, all the results reported here are expected to find some applications in control theory and to the solutions of fractional-order systems in many works (see, e.g., [29–34]).

Data Availability

No data were used to support the study.

Conflicts of Interest

This work does not have any conflicts of interest.

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