WEIGHTED ESTIMATES OF THE BERGMAN PROJECTION WITH MATRIX WEIGHTS

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Abstract. We establish a weighted inequality for the Bergman projection with matrix weights for a class of pseudoconvex domains. We extend a result of Aleman-Constantin and obtain the following estimate for the weighted norm of $P$:

\[ \|P\|_{L^2(\Omega,W)} \leq C(B_2(W))^2. \]

Here $B_2(W)$ is the Bekollé-Bonami constant for the matrix weight $W$ and $C$ is a constant that is independent of the weight $W$ but depends upon the dimension and the domain.

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1. Introduction

The purpose of this paper is to investigate weighted inequalities with matrix weights for the Bergman projection on certain class of pseudoconvex domains.

In harmonic analysis, weighted inequalities characterize weighted spaces on which singular integral operators remain bounded and describe the norm dependence on the weights. For the case of the scalar-valued function with scalar weights, it is well-known that for $1 < p < \infty$, a Calderon-Zygmund singular integral operator is bounded over the weighted space $L^p(\mathbb{R}^N, W)$ with the weight function $W$ satisfying the Muckenhoupt $A_p$ condition, i.e. the $A_p$ constant $\mathcal{A}_p(W) := \sup_B \langle W \rangle_B^{1/p} \langle W^{-1} \rangle_B^{1/p} < \infty$.

Here the supremum is taken over all Euclidean balls $B$ in $\mathbb{R}^N$ and $\langle W \rangle_B$ denotes the average of $W$ over the set $B$ with respect to the Lebesgue measure. See for example [HMW73].

As to the dependence of the norm of the operator on $\mathcal{A}_p(W)$, extensive studies have been made in recent decades. For $p = 2$, the $A_2$ conjecture speculates that the dependence is linear. The conjecture was solved by Wittwer for the martingale transform [Wit00], by Petermichl-Volberg for the Beurling operator, by Petermichl for the Hilbert transform in [Pet07], and then the general Calderon-Zygmund operators in [Hyt12]. More elementary proofs were also obtained by Lerner [Ler13] and Lacey [Lac17].

One direction to extend weighted theory considers the setting of weighted $L^p$ space of vector-valued functions with matrix-valued weight $W$. Let $\Omega$ be a domain. Let $W$ be a locally integrable function on $\Omega$ with its range in the set of positive-semidefinite $d \times d$ matrices. The weighted space $L^2(\Omega, W)$ is the space of vector-valued measurable functions $f : \Omega \to \mathbb{C}^d$ for which

\[ \|f\|_{L^2(\Omega,W)}^2 := \int_{\Omega} \langle W(z) f(z), f(z) \rangle dV(z) < \infty; \]

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where $\langle \cdot , \cdot \rangle$ is the usual inner product of $\mathbb{C}^d$.

For the vector-valued function setting with matrix weights, Treil and Volberg \cite{TV97a,TV97b} first gave a matrix $A_2$ condition and showed it is a necessary and sufficient for the boundedness of the Hilbert transform \cite{TV97a,TV97b}. Then Nazarov-Treil \cite{NT96} and Volberg \cite{Vol97} separately generalized this result and established the boundedness of classical Calderon-Zygmund operators for matrix $A_p$ weights. Christ-Goldberg \cite{CG01} and Goldberg \cite{Gold03} studied a class of weighted, vector analogues of the Hardy-Littlewood maximal function and used them to establish the boundedness of a class of singular integral operators on $L^p(W)$, where $W$ is a matrix $A_p$ weight. Recently, progress has also been made on norm dependence. Bickel-Wick \cite{BW16} and Isralowitz-Kwon-Pott \cite{IKP17} separately showed the following estimates for a sparse operator $S$:

$$\|S\|_{L^2(W)} \lesssim A_2(W)^{\frac{d}{2}}.$$  

Bickel-Petermichl-Wick \cite{BPW16} obtained weighted $L^2$ estimates for the Hilbert transform with the norm bound $A_2(W)^{\frac{d}{2}} \log(1+A_2(W))$. Nazarov-Petermichl-Treil-Volberg \cite{NPTV17} then improved their result and obtained a better bound $A_2(W)^{\frac{d}{2}}$ for Calderon-Zygmund operators as well as Haar shifts and paraproducts. The sharp bound was established to some particular cases such as sparse operators with a simple sparse family by Nazarov-Petermichl-Treil-Volberg \cite{NPTV17} and the dyadic square function by Hytönen-Petermichl-Volberg \cite{HPV19}. It is worth noting that a crucial ingredient in some of these results is the reverse Hölder inequality for a scalar $A_2$ weight.

Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain. Let $dV$ denote the Lebesgue measure. The Bergman projection $P$ is the orthogonal projection from $L^2(\Omega)$ onto the Bergman space $A^2(\Omega)$, the space of all square-integrable holomorphic functions. Associated with $P$, there is a unique function $K_{\Omega}$ on $\Omega \times \Omega$ such that for any $f \in L^2(\Omega)$:

$$P(f)(z) = \int_{\Omega} K_{\Omega}(z; \bar{w}) f(w) dV(w). \quad (1.1)$$

Let $P^+$ denote the positive Bergman operator defined by:

$$P^+(f)(z) := \int_{\Omega} |K_{\Omega}(z; \bar{w}) f(w)| dV(w). \quad (1.2)$$

In \cite{BB78,Bek82}, Bekollé and Bonami established the analogue of Muckenhoupt $A_p$ condition in the Bergman setting:

**Theorem 1.1** (Bekollé-Bonami). Let $T_z$ denote the Carleson tent over $z$ in $\mathbb{B}_n$ defined as below:

- $T_z := \left\{ w \in \mathbb{B}_n : \left| 1 - \bar{w} \frac{z}{|z|} \right| < 1 - |z| \right\}$ for $z \neq 0$, and
- $T_z := \mathbb{B}_n$ for $z = 0$.

Let the weight $W$ be a positive, locally integrable function on $\mathbb{B}_n$. Let $1 < p < \infty$. Then the following conditions are equivalent:

1. $P : L^p(\mathbb{B}_n, W) \to L^p(\mathbb{B}_n, W)$ is bounded;
2. $P^+ : L^p(\mathbb{B}_n, W) \to L^p(\mathbb{B}_n, W)$ is bounded;
3. The Bekollé-Bonami constant $B_p(W)$ is finite where:

$$B_p(W) := \sup_{z \in \mathbb{B}_n} \left( W^{\frac{1}{T_z}} (W^{-\frac{1}{p-1}})^{\frac{1}{T_z}} \right)^{\frac{1}{p}}.$$
People have made progress on the dependence of the operator norm $\|P\|_{L^p(B_n,W)}$ on $B_p(W)$. In [PR13], Pott and Reguera gave a weighted $L^p$ estimate for the Bergman projection on the upper half plane. Their estimates are in terms of the Bekollé-Bonami constant and the upper bound is sharp. Later, Rahm, Tchoundja, and Wick [RTW17] generalized the results of Pott and Reguera to the unit ball case, and also obtained estimates for the Berezin transform. Sharp weighted norm estimates of the Bergman projection have been obtained [HW20] on the Hartogs triangle and a broad class of pseudoconvex domains [HWW20a, HWW20b, GHK20].

Unlike Muckenhoupt $A_2$ weights, the reverse Hölder inequality is generally not available for Bekollé-Bonami weights, thereby making weighted inequalities for the Bergman projection harder to prove in the matrix weight setting. Nevertheless, Aleman and Constantin [AC12] extended Bekollé and Bonami’s result on the unit disc to operator-valued weights for $p = 2$. They showed that a $B_2$ condition for operator weights determines the boundedness of the Bergman projection in vector-valued $L^2$-spaces with operator-valued weights as opposed to just matrix valued weights (i.e. they consider the action on an infinite dimensional Hilbert space as opposed to just $\mathbb{C}^d$). They also obtained a weighted norm estimate for the projection:

$$\|P\|_{L^2(\Omega,W)} \leq B_2^{5/2}(W).$$

In their approach, they related the norms of analytic functions in weighted Bergman spaces to weighted norms of their derivatives, which relied on various properties of the unit disc $\mathbb{D}$.

In this paper, we consider the weighted estimates of the Bergman projection in the matrix weight case. We follow the settings from [HWW20a, HWW20b] and the domain $\Omega$ we consider belong to one of the following classes:

- a bounded, smooth, pseudoconvex domain of finite type in $\mathbb{C}^2$,
- a bounded, smooth, strictly pseudoconvex domain in $\mathbb{C}^n$,
- a bounded, smooth, convex domain of finite type in $\mathbb{C}^n$, or
- a bounded, smooth, decoupled domain of finite type in $\mathbb{C}^n$.

Such a $\Omega$ is referred to as a “simple domain” by McNeal in [McN03]. We will use the same terminology in this paper.

Let $\Omega$ be a simple domain. The main result can be summarized as follows:

**Theorem 1.2.** Let $W$ be a matrix $B_2$ weight on $\Omega$. Let $B_2(W)$ denote the matrix $B_2$ constant given by

$$B_2(W) := \sup_{B \in \mathcal{B}} \left\| (W)_{B}^{1/2} (W^{-1})_{B}^{1/2} \right\|_{2},$$

where $\| \cdot \|$ denotes the norm of the matrix acting on $\mathbb{C}^d$, $(W)_{B} := \frac{\int_{\Omega} W \, dV}{\int_{\Omega} dV}$, and $\mathcal{B}$ is a collection of tents in $\Omega$ that touch its boundary. Then we have

$$\|P\|_{L^2(\Omega,W)} \leq C(B_2(W))^2.$$

The constant $C$ in the inequality only depends on the domain $\Omega$ and the dimension $d$, but not the weight $W$.

This theorem extends and improves the estimates by Aleman-Constantin [AC12] by lowering the constant power of $B_2(W)$ in the upper bound and showing it works for more domains. See Section 2.2 for the detailed definition of the tents of $\Omega$ and the collection $\mathcal{B}$.

Our approach is motivated by ideas from [AC12, NPTV17, APR19] and can be outlined as follows: Given a matrix $B_2$ weight $W$, we construct finitely many step averaging weights $W_l$ according to dyadic systems $\mathcal{T}_l$ and their sum $\hat{W} = \sum W_l$ with the following three properties:
for the norm of the matrix acting on property (3) above uses a duality argument from [AC12] and relates the weighted norm of $\mathcal{L}^2$ balls on the boundary of a simple domain.  

2.1. smooth boundary balls on simple domains, we show that the Bergman projection belongs to a convex body-valued dyadic operator [NPTV17]. Since the reverse Hölder is available for if for any domain $\Omega$, the scalar weight $\{\mathcal{W}_i, v\}$ satisfies the reverse Hölder inequality (Lemma 4.4),

$$\|P\|_{L^2(W)} \leq \mathcal{B}_2^{1/2}(W)\|P\|_{L^2(\mathcal{W})}$$

(Theorem 4.3).

The construction of the step averaging weight $\mathcal{W}_i$ is inspired by the work of [APR19]. Property (3) above uses a duality argument from [AC12] and relates the weighted norm of $\mathcal{L}^2$ on $L^2(W)$ to the ones on $L^2(\mathcal{W}_i)$. Using the known estimates about the Bergman kernel on simple domains, we show that the Bergman projection belongs to a convex body-valued dyadic operator [NPTV17]. Since the reverse Hölder is available for $\langle \mathcal{W}_i v, v \rangle$, we are able to establish weighted estimates using some scalar dyadic square operators.

After the first draft of this paper was posted, we received a draft by Limani and Pott [LP21] where they considered the Bergman projection on the upper half plane and obtained weighted estimates of the projection with operator-valued weights. Their results sharpen the exponent of 2 in the estimate of Theorem 1.2 to $\frac{4}{3}$.

Our paper is organized as follows: In Section 2, we recall the definitions and known results about the dyadic tent structure of $\Omega$, estimates for the Bergman kernel function. In Section 3, we give the definition of the convex body-valued sparse operator and show the convex body domination for the Bergman projection. In Section 4, we introduce the weight $\mathcal{W}$ and go over its properties. In Section 5, we prove Theorem 1.2. We make several remarks for our results in Section 6.

Throughout the paper, $n$ will be the dimension of the complex Euclidean space $\mathbb{C}^n$ that contains the domain $\Omega$ of vector-valued functions and $d$ will be the dimension of $\mathbb{C}^d$ in which vector-valued functions take their range. Hence the matrix weights will have range in the set of $d \times d$ matrices. We use the notation $\| \cdot \|$ for the norm of operators/functions on function spaces, and use $\| \cdot \|$ for the norm of the matrix acting on $\mathbb{C}^d$ or the length of a vector in $\mathbb{C}^d$. Given functions of several variables $f$ and $g$, we use $f \lesssim g$ to denote that $f \leq Cg$ for a constant $C$. If $f \lesssim g$ and $g \lesssim f$, then we say $f$ is comparable to $g$ and write $f \approx g$.

2. Preliminaries

2.1. Balls on the boundary of a simple domain. Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ with smooth boundary $b\Omega$. Then there is a smooth function $\rho$ satisfying $\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0\}$ and $\rho \neq 0$ on $b\Omega$. We call such a function $\rho$ a defining function of $\Omega$. $\Omega$ is a pseudoconvex domain if for any $p \in b\Omega$, the complex Hessian $(\rho_{ij})$ of the defining function is positive semidefinite on holomorphic tangent space $T_p^{1,0}(b\Omega)$ at $p$. For $m > 0$, a point $p \in b\Omega$ is of finite type in the sense of D’Angelo [DA82] if the maximum order of contact of one-dimensional complex analytic varieties with $\Omega$ at $p$ equals $m$. We say a domain $\Omega$ is of finite type if every boundary point $p$ is of finite type.

When $\Omega$ is a simple domain in $\mathbb{C}^n$, non-isotropic sets can be constructed using a special coordinate system near the boundary of $\Omega$. See results of McNeal [McN94b, McN91, McN03]. Let $p \in b\Omega$ be a point of finite type $m$. For a small neighborhood $U$ of the point $p$, there exists a holomorphic coordinate system $z = (z_1, \ldots, z_n)$ centered at a point $q_0 \in U$ and defined on $U$ and quantities $\tau_1(q, \delta), \tau_2(q, \delta), \ldots, \tau_n(q, \delta)$ for each $q \in U$ such that $\tau_1(q, \delta) = \delta$ and $\delta^{1/2} \leq \tau_j(q, \delta) \leq \delta^{1/m}$ for $j = 2, 3, \ldots, n$.  

$$\tau_1(q, \delta) = \delta \quad \text{and} \quad \delta^{1/2} \leq \tau_j(q, \delta) \leq \delta^{1/m} \quad \text{for} \quad j = 2, 3, \ldots, n. \quad (2.1)$$
Moreover, the polydisc $D(q, \delta)$ defined by:

$$D(q, \delta) = \{ z \in \mathbb{C}^n : |z_j| < \tau_j(q, \delta), j = 1, \ldots, n \}$$

(2.2)

is the largest one centered at $q$ on which the defining function $\rho$ changes by no more than $\delta$ from its value at $q$, i.e. if $z \in D(q, \delta)$, then $|\rho(z) - \rho(q)| \leq \delta$.

The polydisc $D(q, \delta)$ is known to satisfy several “covering properties” [McN94a]:

1. There exists a constant $C > 0$, such that for points $q_1, q_2 \in U \cap \Omega$ with $D(q_1, \delta) \cap D(q_2, \delta) \neq \emptyset$, we have

$$D(q_2, \delta) \subseteq CD(q_1, \delta) \text{ and } D(q_1, \delta) \subseteq CD(q_2, \delta).$$

(2.3)

2. There exists a constant $c > 0$ such that for $q \in U \cap \Omega$ and $\delta > 0$, we have

$$D(q, 2\delta) \subseteq cD(q, \delta).$$

(2.4)

It was also shown in [McN94a] that $D(p, \delta)$ induces a global quasi-metric on $\Omega$. Here we will use it to define a quasi-metric on $b\Omega$.

For $q \in b\Omega$ and $\delta > 0$, we define the non-isotropic ball of radius $\delta$ to be the set

$$B(q, \delta) = D(q, \delta) \cap b\Omega.$$ 

Set containments (2.3), (2.4), and the compactness and smoothness of $b\Omega$ imply the following properties for the balls:

1. There exists a constant $C$ such that for $q_1, q_2 \in U \cap b\Omega$ with $B(q_1, \delta) \cap B(q_2, \delta) \neq \emptyset$, $B(q_2, \delta) \subseteq CB(q_1, \delta)$ and $B(q_1, \delta) \subseteq CB(q_2, \delta)$.

(2.5)

2. Let $\mu$ denote the Lebesgue surface area on $b\Omega$. There exists a constant $c > 0$ such that for $q \in U \cap b\Omega$ and $\delta > 0$, we have

$$B(q, \delta) \subseteq cB(q, \delta/2) \quad \text{and} \quad \mu(B(q, \delta)) \approx \prod_{j=2}^{n} \tau_j^{2}(q, \delta).$$

(2.6)

The balls $B$ induce a quasi-metric on $b\Omega \cap U$. For $q, p \in b\Omega \cap U$, we set $\tilde{d}(q, p) = \inf \{ \delta > 0 : p \in B(q, \delta) \}$. Note that $b\Omega$ is coompact. To extend this quasi-metric $\tilde{d}(\cdot, \cdot)$ to a global quasi-metric $d(\cdot, \cdot)$ defined on $b\Omega \times b\Omega$, one can just patch the local metrics together in an appropriate way. The resulting quasi-metric is not continuous, but satisfies all the relevant properties. We refer the reader to [McN94a] for more details on this matter. Since $d(\cdot, \cdot)$ and $\tilde{d}(\cdot, \cdot)$ are equivalent, we may abuse the notation $B$ for the ball on the boundary induced by $d$. Then (2.3) and (2.6) still hold true for $B$.

2.2. Dyadic tents on $\Omega$ and the matrix $B_2$ constant. The non-isotropic ball $B(p, \delta)$ on the boundary $b\Omega$ induces “tents” in the domain $\Omega$. To define what “tents” are we need the orthogonal projection map near the boundary. Let $\text{dist}(\cdot, \cdot)$ denote the Euclidean distance in $\mathbb{C}^n$. For small $\epsilon > 0$, set

$$N_\epsilon(b\Omega) = \{ w \in \mathbb{C}^n : \text{dist}(w, b\Omega) < \epsilon \}.$$ 

Lemma 2.1. For sufficiently small $\epsilon_0 > 0$, there exists a map $\pi : N_{\epsilon_0}(b\Omega) \to b\Omega$ such that

1. For each point $z \in N_{\epsilon_0}(b\Omega)$ there exists a unique point $\pi(z) \in b\Omega$ such that

$$\text{dist}(z, \pi(z)) = \text{dist}(z, b\Omega).$$

(2.2)

2. For $p \in b\Omega$, the fiber $\pi^{-1}(p) = \{ p - \epsilon n(p) : -\epsilon_0 < \epsilon < \epsilon_0 \}$ where $n(p)$ is the outer unit normal vector of $b\Omega$ at point $p$. 


(3) The map \( \pi \) is smooth on \( N_\epsilon(b\Omega) \).

(4) If the defining function \( \rho \) is the signed distance function to the boundary, the gradient \( \nabla \rho \) satisfies
\[
\nabla \rho(z) = n(\pi(z)) \quad \text{for } z \in N_\epsilon(b\Omega).
\]

A proof of Lemma 2.1 can be found in [BB00].

**Definition 2.2.** Let \( \epsilon_0 \) and \( \pi \) be as in Lemma 2.1. For \( z \in b\Omega \) and sufficiently small \( \delta > 0 \), the “tent” \( B^\#(z, \delta) \) over the ball \( B(z, \delta) \) is defined to be the subset of \( N_\epsilon(b\Omega) \) as follows: When \( \Omega \) is a simple domain in \( \mathbb{C}^n \),
\[
B^\#(z, \delta) := \{ w \in \Omega : \pi(w) \in B(z, \delta), \text{dist}(\pi(w), w) < \delta \}.
\]

For \( \delta \geq 1 \) and any \( z \in b\Omega \), we set \( B^\#(z, \delta) = \Omega \).

For the “tent” \( B^\#(z, \delta) \) to be within \( N_\epsilon(b\Omega) \), the constant \( \delta \) in Definition 2.2 needs to satisfy \( \delta < \epsilon_0 \).

Given a subset \( U \in \mathbb{C}^n \), let \( |U| \) denote the Lebesgue measure of \( U \). By the definitions of the tents \( B^\#(z, \delta) \) we have:
\[
|B^\#(z, \delta)| \approx \delta^2 \prod_{j=2}^{n} \tau_j^2(z, \delta)
\]  
(2.7)
and hence also the “doubling property”:
\[
|B^\#(z, \delta)| \approx |B^\#(z, \delta/2)|.
\]  
(2.8)

Now we are in the position of constructing dyadic systems on \( b\Omega \) and \( \Omega \). Note that the ball \( B(\cdot, \delta) \) on \( b\Omega \) satisfies the “doubling property” as in (2.6). By (2.5), the surface area \( \mu(B(q_1, \delta)) \approx \mu(B(q_2, \delta)) \) for any \( q_1, q_2 \in b\Omega \) satisfying \( d(q_1, q_2) \leq \delta \). Combining these facts yields that the metric \( d(\cdot, \cdot) \) is a doubling metric, i.e. for every \( q \in b\Omega \) and \( \delta > 0 \), the ball \( B(q, \delta) \) can be covered by at most \( M \) balls \( B(x_i, \delta/2) \). Results of Hytönen and Kairema in [HK12] then give the following lemmas:

**Lemma 2.3.** Let \( \delta \) be a positive constant that is sufficiently small and let \( s > 1 \) be a parameter. There exist reference points \( \{p_j^{(k)}\} \) on the boundary \( b\Omega \) and an associated collection of subsets \( Q = \{Q_j^k\} \) of \( b\Omega \) with \( p_j^{(k)} \in Q_j^k \) such that the following properties hold:

1. For each fixed \( k \), \( \{p_j^{(k)}\} \) is a largest set of points on \( b\Omega \) satisfying \( d_1(p_j^{(k)}, p_i^{(k)}) > s^{-k}\delta \) for all \( i, j \). In other words, if \( p \in b\Omega \) is a point that is not in \( \{p_j^{(k)}\} \), then there exists an index \( j_o \) such that \( d_1(p, p_j^{(k)}) \leq s^{-k}\delta \).
2. For each fixed \( k \), \( \bigcup_j Q_j^k = b\Omega \) and \( Q_j^k \cap Q_i^k = \emptyset \) when \( i \neq j \).
3. For \( k < l \) and any \( i, j \), either \( Q_j^k \supseteq Q_i^l \) or \( Q_j^k \cap Q_i^l = \emptyset \).
4. There exist positive constants \( c \) and \( C \) such that for all \( j \) and \( k \),
\[
B(p_j^{(k)}, cs^{-k}\delta) \subseteq Q_j^k \subseteq B(p_j^{(k)}, Cs^{-k}\delta).
\]
5. Each \( Q_j^k \) contains at most \( N \) numbers of \( Q_i^{k+1} \). Here \( N \) does not depend on \( k, j \).

The points \( p_j^{(k)} \) above are dyadic points in \( b\Omega \).

**Lemma 2.4.** Let \( \delta \) and \( \{p_j^{(k)}\} \) be as in Lemma 2.3. There are finitely many collections \( \{Q_i\}_{i=1}^N \) such that the following hold:
(1) Each collection \( Q_i \) is associated to some dyadic points \( \{ z_j^{(k)} \} \) and satisfies all the properties in Lemma 2.3.

(2) For any \( z \in b\Omega \) and small \( r > 0 \), there exist \( Q_{j_1}^k \in Q_{t_1} \) and \( Q_{j_2}^k \in Q_{t_2} \) such that

\[
Q_{j_1}^k \subseteq B(z, r) \subseteq Q_{j_2}^k \quad \text{and} \quad \mu(B(z, r)) = \mu(Q_{j_1}^k) = \mu(Q_{j_2}^k).
\]

Letting the sets \( Q_j^k \) in Lemma 2.3 serve as these bases, we construct dyadic tents in \( \Omega \) as follows:

**Definition 2.5.** Let \( \delta, \{ p_j^{(k)} \} \) and \( Q = \{ Q_j^k \} \) be as in Lemma 2.3. We define the collection \( T = \{ \hat{K}_j^k \} \) of dyadic tents in the domain \( \Omega \) as

\[
\hat{K}_j^k := \{ z \in \Omega : \pi(z) \in Q_j^k \text{ and } \text{dist}(\pi(z), z) < s^{-k}\delta \}.
\]

**Lemma 2.6.** Let \( T = \{ \hat{K}_j^k \} \) be a collection of dyadic tents in Definition 2.5 and let \( \{ Q_i \}_{i=1}^N \) be collections in \( \Omega \). The following statements hold true:

(1) For any \( \hat{K}_j^k \), \( \hat{K}_j^{k+1} \) in \( T \), either \( \hat{K}_j^k \supseteq \hat{K}_j^{k+1} \) or \( \hat{K}_j^k \cap \hat{K}_j^{k+1} = \emptyset \).

(2) For any \( z \in b\Omega \) and small \( r > 0 \), there exist \( Q_{j_1}^k \in Q_{t_1} \) and \( Q_{j_2}^k \in Q_{t_2} \) such that

\[
\hat{K}_{j_1}^k \subseteq B^#(z, r) \subseteq \hat{K}_{j_2}^k \quad \text{and} \quad |B^#(z, r)| \approx |\hat{K}_{j_1}^k| \approx |\hat{K}_{j_2}^k|.
\]

Set \( B := \{ B^#(z, r) \}_{z \in b\Omega, r \leq \delta} \cup \{ \Omega \} \). Recall the matrix \( B_2 \) constant given by

\[
B_2(W) := \sup_{B \in B} \left| \langle W \rangle_B \right|^{1/2} \left| \langle W^{-1} \rangle_B \right|^{1/2},
\]

where \( \cdot \) denotes the norm of the matrix acting on \( \mathbb{C}^d \) and \( \langle W \rangle_B := \int_B W \, dV \). Let \( \{ T_i \}_{i=1}^N \) be collections of dyadic tents induced by collections \( \{ Q_i \}_{i=1}^N \) in Lemma 2.4 respectively. Set \( \mathcal{T} := \{ \hat{K}_j^k : \hat{K}_j^k \in T_t \text{ for some } t \in \{ 1, \ldots, N \} \cup \{ \Omega \} \} \). Then Lemma 2.6 implies that

\[
B_2(W) \approx \sup_{\hat{K}_j^k \in \mathcal{T}} \left| \langle W \rangle_{\hat{K}_j^k} \right|^{1/2} \left| \langle W^{-1} \rangle_{\hat{K}_j^k} \right|^{1/2}.
\]

From now on, we will abuse the notation of \( B_2(W) \) to represent both the supremum in \( B \) and \( \mathcal{T} \).

### 2.3. Dyadic kubes on \( \Omega \).

**Definition 2.7.** For a collection \( T \) of dyadic tents, we define the center \( \alpha_j^{(k)} \) of each tent \( \hat{K}_j^k \) to be the point satisfying

- \( \pi(\alpha_j^{(k)}) = p_j^{(k)} \); and
- \( \text{dist}(p_j^{(k)}, \alpha_j^{(k)}) = \frac{1}{2} \sup_{\pi(p) = p_j^{(k)}} \text{dist}(p, b\Omega) \).

We set \( K^{-1} = \Omega \setminus \bigcup_{j} \hat{K}_j^0 \), and for each point \( \alpha_j^{(k)} \) or its corresponding tent \( \hat{K}_j^k \), we define the “kube” \( K_j^k := \hat{K}_j^k \setminus \bigcup_l \hat{K}_l^{k+1} \), where \( l \) is any index with \( p_l^{(k+1)} \in \hat{K}_j^k \).

By the definition of dyadic kubes, the following lemma for dyadic kubes holds true:

**Lemma 2.8.** Let \( T = \{ \hat{K}_j^k \} \) be the system of tents induced by \( Q \) in Definition 2.3. Let \( K_j^k \) be the kubes of \( \hat{K}_j^k \). Then

(1) \( \pi(K_j^k) = Q_j^k \in Q; \)
(2) $K_j^k$’s are pairwise disjoint and $\cup_{j,k} K_j^k = \Omega$;

(3) the following volume estimates hold:

$$|K_j^k| \approx |\hat{K}_j^k| \approx s^{-2k}\delta^{-2} \prod_{j=2}^n \tau_j^2(p_j^{(k)}),$$

(2.9)

The next lemma is crucial in the proof of Lemma 4.4 which relates the norm of a holomorphic vector-valued function on $L^2(\mathbb{W})$.

**Lemma 2.9.** There exists finitely many collections of $\{T_l^i \}_{i=1}^M$ such that for any $z \in \Omega$, we can find a tent $\hat{K}_j^k \in T_l^i$ for some $l$ such that the following holds:

1. $z \in K_j^k$,

2. If $k \geq 0$, then $K_j^k \supset D(z, cs^{-k})$ for some constant $c > 0$. If $k = -1$, then $K^{-1}$ contains a Euclidean ball centered at $z$ with radius $r \approx 1$.

**Proof.** Let $\{T_l^i \}_{i=1}^N$ be collections of dyadic tents induced by collections $\{Q_l^i \}_{i=1}^N$ in Lemma 2.4 respectively with parameter $\delta$ and reference points $p_j^{(k)}$. Now let $\{T_l^i \}_{i=1}^N$ be collections also induced by $\{Q_l^i \}_{i=1}^N$ but each $\hat{K}_j^k \in T_l^i$ and $\hat{K}_j^{nk} \in T_l^i'$ are defined by $Q_j^k \in Q_l$ as follows:

$$\hat{K}_j^k := \left\{ z \in \Omega : \pi(z) \in Q_j^k \text{ and } \text{dist}(\pi(z), z) < \frac{s+2}{3}\delta s^{-k} \right\},$$

$$\hat{K}_j^{nk} := \left\{ z \in \Omega : \pi(z) \in Q_j^k \text{ and } \text{dist}(\pi(z), z) < \frac{2s+1}{3}\delta s^{-k-1} \right\},$$

i.e. $\hat{K}_j^k \in T_l^i$ and $\hat{K}_j^{nk} \in T_l^i'$ have the same “base” $Q_j^k$ as the tent $\hat{K}_j^k$ in $T_l^i$ but different “height”. Thus $T_l^i$ and $T_l^i'$ still satisfies Lemma 2.6. We call the tents $\hat{K}_j^k$ and $\hat{K}_j^{nk}$ cousins of $K_j^k$ and use $K_j^k$ and $K_j^{nk}$ to denote the dyadic cubes induced by $\hat{K}_j^k$ and $\hat{K}_j^{nk}$ respectively as in Definition 2.7. We claim $\{T_l^i \}_{i=1}^N \cup \{T_l^i' \}_{i=1}^N$ are the desired collections in Lemma 2.9. Hence, $M = 3N$.

When the point $z \in \Omega$ is far away from the boundary $b\Omega$, say $z \in K^{r-1}$. Then dist$(z, \pi(z)) \geq \frac{2s+1}{3}\delta$. Therefore, dist$(z, bK^{-1}) \geq \frac{2s-2}{3}\delta$ and $K^{-1}$ contains a ball centered at $z$ with radius $\frac{2s-2}{3}\delta \approx 1$.

When the point $z \in \Omega \setminus K^{r-1}$. Lemma 2.6 implies that there exists a tent $\hat{K}_j^k \in T_l^i$ with $z \in K_j^k$ and a constant $c_1 > 0$ such that $\pi(\hat{K}_j^k) \supset B(\pi(z), c_1 s^{-k})$. We first show that there exists a constant $c_2$ such that the polydisc $D(z, c_2 s^{-k})$ satisfies $\pi(D(z, c_2 s^{-k})) \subset B(\pi(z), c_1 s^{-k})$. Suppose $z \in D(z, c_2 s^{-k})$. Let $\zeta'$ be the projection of $\zeta$ on the hypersurface $\{w : \rho(w) = \rho(z)\}$. Then we have $d(\zeta', z) \leq c_2 s^{-k}$ and

$$d(\zeta', z) \leq d(\zeta', \zeta) + d(z, \zeta) \leq c_2 s^{-k}.$$ 

Since the parameter $\delta$ in Lemma 2.3 is chosen to be sufficiently small and the gradient $\nabla \rho$ is continuous near $b\Omega$, we may assume that $|\nabla \rho(x) - \nabla \rho(y)| < \epsilon$ for all $y \in b\Omega$ and $x \in B(y, \delta)$. Then, from $\pi(\zeta') = \zeta' - \rho(z)\nabla \rho(\zeta')$ and $\pi(z) = z - \rho(z)\nabla \rho(z)$, we obtain

$$d(\pi(\zeta'), \pi(z)) \leq d(\pi(\zeta'), \zeta' - \rho(z)\nabla \rho(\zeta')) + d(\pi(z), \zeta' - \rho(z)\nabla \rho(z))$$

$$\leq -\epsilon \rho(z) + d(\pi(z), \zeta' - \rho(z)\nabla \rho(z)).$$

By (2.3) and (2.4), there exists a constant $C > 0$ such that $D(z, \delta s^{-k}) \subset CD(\pi(z), \delta s^{-k})$. Thus, $D(z, \delta s^{-k}) - \rho(z)\nabla \rho(z) \subset CD(\pi(z), \delta s^{-k})$. Since the sets $D(z, \delta s^{-k}) - \rho(z)\nabla \rho(z)$ and
\[ CD(\pi(z), \delta s^{-k}) \text{ have the common center } \pi(z), \text{ the containment still holds after applying a dilation to both sets. Therefore, there exists a constant } C_2 \text{ such that} \]
\[
C^{-1}C_2^{-1}D(z, \delta s^{-k}) - \rho(z)\nabla \rho(z) \subseteq C^{-1}D(\pi(z), \delta s^{-k}) \subseteq D(\pi(z), c_1 s^{-k}).
\]

We may choose \( c_2 \) to be small so that
\[
D(z, c_2 s^{-k}) \subseteq C^{-1}C_2^{-1}D(z, \delta s^{-k}).
\]

Then we obtain \( D(z, c_2 s^{-k}) - \rho(z)\nabla \rho(z) \subseteq D(\pi(z), c_1 s^{-k}) \) which also implies that
\[
d(\pi(z), \zeta' - \rho(z)\nabla \rho(z)) \subseteq c_1 s^{-k}.
\]

Hence \( d(\pi(\zeta), \pi(z)) \subseteq c_1 s^{-k} \) and \( \pi(D(z, c_2 s^{-k})) \subseteq B(\pi(z), c_1 s^{-k}) \) for some \( c_2 \).

Now we are ready to show that \( D(z, c s^{-k}) \) is in \( K_j^k, K_j^{nk} \), or \( K_j^{nk} \) for some \( c > 0 \). Since \( Q_j^k \supseteq \pi(D(z, c s^{-k})) \), it suffices to check the distance from \( z \) to the boundary of \( \partial \Omega \). Note that
\[
\{-\rho(\zeta) : \zeta \in K_j^k\} = \left[\delta s^{-k-1}, \delta s^{-k}\right],
\]
\[
\{-\rho(\zeta) : \zeta \in K_j^{nk}\} = \left[\frac{s + 2}{3} \delta s^{-k-1}, \frac{s + 2}{3} \delta s^{-k}\right],
\]
\[
\{-\rho(\zeta) : \zeta \in K_j^{nk}\} = \left[\frac{2s + 1}{3} \delta s^{-k-2}, \frac{2s + 1}{3} \delta s^{-k-1}\right].
\]

If \( z \in K_j^k \backslash K_j^{nk} \), then the distance from \( z \) to the boundary of \( K_j^k \) along the normal direction \( \pm \nabla \rho(z) \) is greater than \( \frac{s - 1}{3} \delta s^{-k-1} \). If \( z \in K_j^k \backslash K_j^{nk} \), the distance from \( z \) to the boundary of \( K_j^{nk} \) along the normal direction \( \pm \nabla \rho(z) \) is greater than \( \frac{s - 1}{3} \delta s^{-k-2} \). Otherwise, \( z \in K_j^{nk} \cap K_j^k \) and the distance from \( z \) to the boundary of \( K_j^k \) along the normal direction \( \pm \nabla \rho(z) \) is greater than \( \frac{s - 1}{3} \delta s^{-k-1} \). Hence by choosing \( c = \min\{c_2, \frac{s - 1}{3} \delta s^{-2}\} \), we have \( D(z, c s^{-k}) \) is contained in \( K_j^k, K_j^{nk} \), or \( K_j^{nk} \).

### 2.4. Estimates for the Bergman kernel

When \( \Omega \) is a simple domain in \( \mathbb{C}^n \), estimates of the Bergman kernel function on \( \Omega \) were obtained in [McN94b, McN91, McN03].

**Theorem 2.10.** Let \( \Omega \) be a simple domain in \( \mathbb{C}^n \). Let \( p \) be a boundary point of \( \Omega \). There exists a neighborhood \( U \) of \( p \) so that for all \( q_1, q_2 \in U \cap \Omega \),
\[
|K_{\Omega}(q_1; \bar{q}_2)| \lesssim t^{-2} \prod_{j=2}^{n} \tau_j(q_1, t)^{-2},
\]
(2.10)

where \( t = |\rho(q_1)| + |\rho(q_2)| + \inf\{\epsilon > 0 : q_2 \in D(q_1, \epsilon)\} \).

We can reformulate Theorem 2.10 as below.

**Theorem 2.11.** Let \( \Omega \) be a simple domain in \( \mathbb{C}^n \). Let \( p \) be a boundary point of \( \Omega \). There exists a neighborhood \( U \) of \( p \) so that for all \( q_1, q_2 \in U \cap \Omega \),
\[
|K_{\Omega}(q_1; \bar{q}_2)| \lesssim |B^\#(\pi(q_1), t)|^{-1},
\]
(2.11)

where \( t = |\rho(q_1)| + |\rho(q_2)| + \inf\{\epsilon > 0 : q_2 \in D(q_1, \epsilon)\} \). Moreover, there exists a constant \( c \) such that \( q_1, q_2 \in B^\#(\pi(q_1), c\delta) \).

Here the estimate (2.11) follows from (2.7). Recall that the polydisc \( D(q, \delta) \) induces a global quasi-metric \( [\text{McN94a}] \) on \( \Omega \). Then a triangle inequality argument using this quasi-metric yields the containment \( q_2 \in B^\#(\pi(q_1), c\delta) \).
3. Convex Body domination

3.1. Convex body average $\langle f \rangle_Q$. For a function $f \in L^1(Q)$ on some set $Q \subseteq \mathbb{C}^n$ with values in $\mathbb{C}^d$, we recall the definition of the convex body average $\langle f \rangle_Q$ in [NPTV17]:

$$\langle f \rangle_Q := \{ \varphi f : \varphi \text{ is any complex valued function on } Q \text{ with } \| \varphi \|_{\infty} \leq 1 \}.$$  

By its definition, the set $\langle f \rangle_Q$ is symmetric, convex, and compact. For a collection of dyadic tents $\{T_l\}_{l=1}^N$, we define the dyadic operator $L$ by

$$L f = \sum_{i=1}^N \sum_{K_j \in T_i} \langle f \rangle_{K_j} 1_{K_j}.$$  

This operator takes a vector-valued function and returns a set-valued function.

**Lemma 3.1** ([NPTV17] Lemma 2.7). Let $f \in L^1(Q, \mathbb{C}^d)$ and let $g(x) \in \langle f \rangle_Q$ a.e. on $Q$. Then there exists a measurable function $H : Q \times Q \to \mathbb{C}$ with $\|H\|_{\infty} \leq |Q|^{-1}$ such that

$$g(z) = \int_Q H(z, w) f(w) dV(w).$$

Thus to estimate $L$, it suffices to estimate all operators of the form

$$f \mapsto \sum_{K_j \in T} \int_{K_j} H_{K_j}(z, w) f(w) dV(w),$$

where $H_{K_j}$ is supported on $K_j \times \hat{K}_j$ and satisfies $\|H_{K_j}\|_{\infty} \leq |\hat{K}_j|^{-1}$.

**Theorem 3.2.** There exists a constant $C$ such that for any $f \in L^1(\Omega, \mathbb{C}^d)$ and $z \in \Omega$,

$$P(f)(z) \in CLf(z).$$

**Proof.** Let $\{T_l\}_{l=1}^N$ be collections of dyadic tents induced by $\{Q_l\}_{l=1}^N$ in Lemma 2.6. If $z \in \Omega \setminus N_{s_0}(b\Omega)$, then $|K_{\Omega}(z; \cdot)| \approx 1$ and

$$P(f)(z) = \int_{\Omega} K_{\Omega}(z; \bar{w}) f(w) dV(w) \in C\langle f \rangle_{\Omega} \subseteq CLf(z),$$

for some constant $C$. If $z \in \Omega \cap N_{s_0}(b\Omega)$, then we can set $k_0 := \max \{ k \in \mathbb{N} : z \in B^#(\pi(z), s^{-k}\delta) \}$. Theorem 2.11 yields that for $l \in \{1, \ldots, k_0-1\}$ and point $w \in B^#(\pi(z), s^{-l}\delta) \setminus B^#(\pi(z), s^{-l+1}\delta)$,

$$|K_{\Omega}(z; \bar{w})| \lesssim |B^#(\pi(z), s^{-l}\delta)|^{-1}.$$  

By Lemma 2.6, we can find $k_0$ tents $\{K_j\}_{j=1}^{k_0}$ from $\{T_l\}_{l=1}^N$ such that for each $j$,

$$B^#(\pi(z), s^{-j}\delta) \subseteq \hat{K}_j \text{ and } |\hat{K}_j| \approx |B^#(\pi(z), s^{-j}\delta)|.$$  

Set

$$U_0 = \Omega \setminus B^#(\pi(z), \delta)$$

$$U_j = B^#(\pi(z), s^{-j}\delta) \setminus B^#(\pi(z), s^{-j-1}\delta) \text{ for } j = 1, \ldots, k_0 - 1.$$  

Then there is a constant $C$ independent of $j$ and $z$ such that

$$\int_{U_j} K_{\Omega}(z; \bar{w}) f(w) dV(w) \in C\langle f \rangle_{\hat{K}_j}.$$
This containment implies
\[ P(f)(z) = \int_{\Omega} K_{\Omega}(z; \bar{w}) f(w)dV(w) \]
\[ = \sum_{j=0}^{k_0-1} \int_{U_j} K_{\Omega}(z; \bar{w}) f(w)dV(w) \in C L f(z) , \tag{3.1} \]
which completes the proof. \qed

4. A step averaging weight \( W \) and its properties

In this section, we introduce the step averaging weight which will play a crucial role in the proof of Theorem 1.2.

Definition 4.1. Let \( W \) be a matrix weight over \( \Omega \). For a collection \( T \) of dyadic tents, we define a step weight \( W \) to be
\[ W(z) := \sum_{K_j^k \in T} \langle W \rangle_{K_j^k} 1_{K_j^k}(z). \]
For collections \( \{T_i\}_{i=1}^N \) of tents that satisfies both Lemmas 2.6 and 2.9, we define \( \hat{W} \) to be
\[ \hat{W}(z) := \sum_{i=1}^N W_i(z) = \sum_{i=1}^N \sum_{K_j^k \in T_i} \langle W \rangle_{K_j^k} 1_{K_j^k}(z). \]
Since the \( K_j^k \)'s are disjoint, it is easy to see that
\[ W^{-1}(z) = \sum_{K_j^k \in T} \langle W \rangle^{-1}_{K_j^k} 1_{K_j^k}(z). \]

Lemma 4.2. If \( W \) is a matrix \( B_2 \) weight, then \( W \) is also a matrix \( B_2 \) weight with
\[ \sup_{K_j^k \in T} \| \langle W \rangle^{1/2}_{K_j^k} \langle W \rangle^{1/2}_{K_j^k} \|^2 \leq B_2(W). \tag{4.1} \]

Proof. By its definition, \( \langle W \rangle^{1/2}_{K_j^k} = \langle W \rangle^{1/2}_{K_j^k} \). Given a unit vector \( v \in \mathbb{C}^d \),
\[ \| \langle W \rangle^{1/2}_{K_j^k} \langle W \rangle^{1/2}_{K_j^k} \|^2 = \| \langle W \rangle^{1/2}_{K_j^k} \langle W^{-1} \rangle^{1/2}_{K_j^k} \langle W \rangle^{1/2}_{K_j^k} v, v \| \]
\[ = |\hat{K}_j^k|^{-1} \sum_{K_l^m \subseteq K_j^k} |K_l^m| \| \langle W \rangle^{1/2}_{K_j^k} \langle W \rangle^{1/2}_{K_j^k} \langle W \rangle^{1/2}_{K_j^k} v, v \| \]
\[ = |\hat{K}_j^k|^{-1} \sum_{K_l^m \subseteq K_j^k} |K_l^m| \| \langle W \rangle^{-1/2}_{K_l^m} \langle W \rangle^{1/2}_{K_j^k} v \|^2 \]
\[ = |\hat{K}_j^k|^{-1} \sum_{K_l^m \subseteq K_j^k} |K_l^m| \| \langle W \rangle^{-1/2}_{K_l^m} \langle W^{-1} \rangle^{-1/2}_{K_l^m} \langle W^{-1} \rangle^{1/2}_{K_l^m} \langle W \rangle^{1/2}_{K_j^k} v \|^2 \]
\[ \leq |\hat{K}_j^k|^{-1} \sum_{K_l^m \subseteq K_j^k} |K_l^m| \| \langle W^{-1} \rangle^{1/2}_{K_l^m} \langle W \rangle^{1/2}_{K_j^k} v \|^2 \| \langle W^{-1} \rangle^{1/2}_{K_l^m} \langle W \rangle^{1/2}_{K_j^k} v \|^2. \tag{4.2} \]
The last inequality above uses the fact that \( \langle W^{-1} \rangle^{1/2}_{K_j^k} \langle W \rangle^{1/2}_{K_j^k} \) is expanding. See \cite{TV97b}, Corollary 3.3. \qed
By the technique of [TV97b, Lemma 3.5], the weight \( (\mathcal{W}^{-1})^{1/2} \mathcal{W}(z) (\mathcal{W}^{-1})^{1/2} \) is also a matrix weight satisfying (4.1). Moreover, for a unit vector \( v \in \mathbb{C}^d \), the scalar function
\[
\omega(v; z) := \left( (\mathcal{W}^{-1})^{1/2}_k \mathcal{W}(z) (\mathcal{W}^{-1})^{1/2}_k v, v \right),
\]
is a scalar weight satisfying
\[
\sup_{K^k_j \in \mathcal{T}} \langle \omega(v; \cdot)^{-1}, \omega(v; \cdot) \rangle_{K^k_j} \leq c_2(W).
\]

Note that \( \omega(v; z) \) is a fixed constant over each kube \( K^k_j \). The next lemma shows the \( \omega(v; z) \) satisfies the reverse Hölder inequality. In the case \( \Omega = \mathbb{D} \), the lemma follows from [APR19, Theorem 1.7].

**Lemma 4.3.** The function \( \omega(v; z) \) satisfies the reverse Hölder inequality over the tent \( \hat{K}^k_j \in \mathcal{T} \), i.e. there exists a constant \( r > 1 \) such that
\[
\langle \omega(v; z)^r \rangle_{\hat{K}^k_j} \leq c_3 \langle \omega(v; z) \rangle_{\hat{K}^k_j}. \tag{4.3}
\]

Moreover, the constant \( r \) above can be chosen such that \( r - 1 \approx c_2^{-1}(W) \).

**Proof.** On the fixed tent \( \hat{K}^k_j \) and a constant \( R > 1 \) to be determined, we apply a corona decomposition of \( \hat{K}^k_j \) as follows:

1. First, we define the set of stopping children \( L(\hat{K}^l_m) \) of a tent \( \hat{K}^l_m \in \mathcal{T} \):
\[
\mathcal{L}(\hat{K}^l_m) := \left\{ \text{maximal tents } \hat{K}^p_k \text{ such that } \frac{\int_{\hat{K}^p_k} \omega(v; z) dV}{|\hat{K}^p_k|} > R \frac{\int_{\hat{K}^l_m} \omega(v; z) dV}{|\hat{K}^l_m|} \right\}. 
\]

2. Set \( \mathcal{L}_1 := \mathcal{L}(\hat{K}^k_j) \) and for \( j \geq 2 \), set \( \mathcal{L}_i := \bigcup_{\hat{K}^l_m \in \mathcal{L}_{i-1}} \mathcal{L}(\hat{K}^l_m) \) and \( \mathcal{L} := \cup_{i \geq 1} \mathcal{L}_i \).

Then we have for \( \hat{K}^l_m \in \mathcal{L}_i \),
\[
\sum_{K^p_k \in \mathcal{L}_m, K^p_k \in \mathcal{L}_{m+1}} |\hat{K}^p_k| \leq R^{-1} |\hat{K}^l_m|. \tag{4.4}
\]

Since tents in \( \mathcal{L}_m \) are maximal, we also have for \( \hat{K}^l_m \in \mathcal{L}_i \),
\[
R \int_{\hat{K}^l_m} \omega(v; z) dV \leq \int_{\hat{K}^l_m} \omega(v; z) dV \leq (cR) \frac{\int_{\hat{K}^l_m} \omega(v; z) dV}{|\hat{K}^l_m|}. \tag{4.5}
\]

Here \( c \) can be chosen to be the supremum of \( |\hat{K}^l_m|/|\hat{K}^{l+1}_m| \) over all tents \( \hat{K}^l_m \) and \( \hat{K}^{l+1}_m \) satisfying \( \hat{K}^{l+1}_m \subset \hat{K}^l_m \). By Lemma 2.8, \( c \) is finite. Recall the maximal operator \( M_{\mathcal{T}} \) given by
\[
M_{\mathcal{T}} f(z) = \sup_{\hat{K}^k_j \in \mathcal{T}} \langle f \rangle_{\hat{K}^k_j} 1_{\hat{K}^k_j}(z).
\]

Then (4.3) implies
\[
M_{\mathcal{T}}(1_{\hat{K}^k_j} \omega(v; z)) \leq \left( 1 + \sum_{\hat{K}^l_m \in \mathcal{L}_i} (cR)^i 1_{\hat{K}^l_m} \right) \frac{\int_{\hat{K}^k_j} \omega(v; z) dV}{|\hat{K}^k_j|}.
\]
Since \( \omega(v; z) \) is a fixed constant over each cube \( K^t_m \), we have for \( z \in K^t_m \),
\[
\omega(v; z) = \frac{\int_{K^t_m} \omega(v; z) dV}{|K^t_m|} \leq \frac{\int_{K^t_m} \omega(v; z) dV}{|\hat{K}^t_m|}.
\]

Thus \( \omega(v; z) \leq M_T(\omega(v; \cdot)1_{\hat{K}^t_j})(z) \). By applying a scalar multiplication to \( \omega(v; z) \), we may assume that \( \langle \omega(v; \cdot) \rangle_{\hat{K}^t_j} = 1 \). Then, for a point \( z \in \hat{K}^t_j \setminus \bigcup_{\hat{K}^t_m \in \mathcal{L}_i} \hat{K}^t_m \), we have
\[
M_T(\omega(v; \cdot)1_{\hat{K}^t_j})(z) \approx \langle \omega(v; \cdot) \rangle_{\hat{K}^t_j} \approx e^i \langle \omega(v; \cdot) \rangle_{\hat{K}^t_j} = e^i,
\]
which also implies
\[
\log(e + \omega(v; z)) \lesssim i.
\]

Since \( \omega(v; z) \) is a scalar \( B_2 \) weight, the proof of \cite[Lemma 7.1.9(b)]{Gra14} implies that
\[
\|M_T\|_{L^2(\omega^{-1}; z)} \lesssim B_2(\omega(v; \cdot)) \lesssim B_2(W).
\]

Then Hölder’s inequality implies
\[
\int_{\hat{K}^t_j} M_T(\omega(v; \cdot)1_{\hat{K}^t_j}) dV = \int_{\hat{K}^t_j} M_T(\omega(v; \cdot)1_{\hat{K}^t_j}) \omega^{-1/2}(v; \cdot) \omega^{1/2}(v; \cdot) dV
\]
\[
\leq \left( \int_{\hat{K}^t_j} M_T^2(\omega(v; \cdot)1_{\hat{K}^t_j}) \omega^{-1}(v; \cdot) dV \right)^{1/2} \left( \int_{\hat{K}^t_j} \omega(v; \cdot) dV \right)^{1/2}
\]
\[
\leq B_2(\omega(v; \cdot)) \left( \int_{\hat{K}^t_j} \omega^2(v; \cdot) \omega^{-1}(v; \cdot) dV \right)^{1/2} \left( \int_{\hat{K}^t_j} \omega(v; \cdot) dV \right)^{1/2}
\]
\[
\leq B_2(W) \int_{\hat{K}^t_j} \omega(v; z) dV(z).
\]

This together with \(4.6\) and \(4.7\) implies,
\[
B_2(W) \int_{\hat{K}^t_j} \omega(v; z) dV(z) \gtrsim \int_{\hat{K}^t_j} M_T(\omega(v; \cdot)1_{\hat{K}^t_j}) dV
\]
\[
\gtrsim \int_{\hat{K}^t_j} \sum_{i \geq 1} \sum_{\hat{K}^t_m \in \mathcal{L}_i} \langle \omega(v; \cdot) \rangle_{\hat{K}^t_m} \cdot 1_{\hat{K}^t_m} dV
\]
\[
= \int_{\hat{K}^t_j} \omega(v; \cdot) \sum_{i \geq 1} \sum_{\hat{K}^t_m \in \mathcal{L}_i} 1_{\hat{K}^t_m \cup \hat{K}^t_m' \setminus \mathcal{L}_i} \hat{K}^t_m + 1 dV
\]
\[
\approx \int_{\hat{K}^t_j} \omega(v; z) \log(e + M_T(\omega(v; \cdot))(z)) dV
\]
\[
\gtrsim \int_{\hat{K}^t_j} \omega(v; z) \log(e + \omega(v; z)) dV.
\]

Then an argument as in the proof of \cite[Theorem 4.1]{DMRO16} implies for \( 0 < \beta < 1 \), there exists a constant \( \alpha \approx c(\beta) \exp\{ -B_2(W) \} \) such that for \( \hat{K}^t_m \subseteq \hat{K}^t_j \) with
\[
\int_{\hat{K}^t_m} \omega(v; z) dV \leq \beta \int_{\hat{K}^t_j} \omega(v; z) dV,
\]
we have
\[
|\hat{K}^t_m| \leq \alpha |\hat{K}^t_j|.
\]
Now we are ready to show (4.3). Choose $\beta = (4c)^{-1}$ where $c$ is the constant in (4.5). Set $R = \alpha^{-1} \approx \exp\{\mathcal{B}_2(W)\}$. Then we have

$$
\langle \omega(v; \cdot)^r \rangle_{\hat{K}^k_j} \lesssim \langle (M_T(\omega(v; \cdot)1_{\hat{K}^k_j})(z))^r \rangle_{\hat{K}^k_j}
$$

\[
\lesssim \left( \frac{\int_{\hat{K}^k_j} \omega(v; z) dV}{|\hat{K}^k_j|^{1+\alpha}} \right)^r \int_{\hat{K}^k_j} \left( 1 + \sum_i \sum_{K^i_{m/l} \in \mathcal{L}_i} (cR)^{i}1_{K^i_{m/l}} \right)^{r-1} \sum_{i \geq 1} c^{i} R^{(r-1)} \sum_{K^i_{m/l} \in \mathcal{L}_i} \int_{K^i_{m/l}} \omega(v; z) dV
\]

By choosing $r$ such that $(cR)^{(r-1)} = 2$, we obtain (4.3). Applying the natural logarithm it yields:

$$(r-1) \ln(cR) = \ln 2.$$ 

Since $R \approx \exp\{\mathcal{B}_2(W)\}$, we obtain $r - 1 \approx \mathcal{B}_2^{-1}(W)$. \hfill \qed

**Lemma 4.4.** For any holomorphic vector-valued function $f$ on $\Omega$,

$$
\|f\|_{L^2(W)} \lesssim \|f\|_{L^2(\hat{W})}.
$$

**Proof.** Recall $K^{r-1}$ and $D(z, cs^{-k})$ from Lemma 2.9. For each $z \in \Omega$, we define a $z$-dependent subset $D^*(z)$ as follows: For $z \in K^{r-1}$, we set $D^*(z)$ to be the maximal Euclidean ball in $K^{-1}$ centered at $z$. For $z \in \Omega \setminus K^{r-1}$, we set $D^*(z)$ to be the polydisc $D(z, cs^{-k})$ from Lemma 2.9. Then for all $z \in \Omega$

$$
\langle \hat{W}(z)f(z), f(z) \rangle \gtrsim \langle (W^{\cdot}(z)f(z), f(z) \rangle.
$$

We claim that there is a constant $c < 1$ such that for all $\zeta \in cD^*(z)$, there exists a polydisc (or ball) $D'(\zeta)$ centered satisfying the following properties: (1) $z \in D'(\zeta)$; (2) $D'(\zeta) \subseteq D^*(z)$; (3) $|D'(\zeta)| \approx |D^*(z)|$. When $D^*(z)$ is a Euclidean ball, this is obvious. When $D^*(z)$ is a polydisc, the claim follows from (2.3) and (2.4). Then Fubini’s theorem implies that

\[
\int_{\Omega} \langle \hat{W}f(z), f(z) \rangle dV(z) \gtrsim \int_{\Omega} |D^*(z)|^{-1} \int_{D'(z)} \langle W(\zeta)f(z), f(z) \rangle dV(\zeta) dV(z)
\]

\[
\gtrsim \int_{\Omega} |D'(\zeta)|^{-1} \int_{D'(\zeta)} \langle W(\zeta)f(z), f(z) \rangle dV(\zeta) dV(z) dV(\zeta). \tag{4.11}
\]

Note that $\langle W(\zeta)f(z), f(z) \rangle$ is subharmonic in $z$, we have

\[
\int_{\Omega} |D'(\zeta)|^{-1} \int_{D'(\zeta)} \langle W(\zeta)f(z), f(z) \rangle dV(\zeta) dV(z) dV(\zeta) \gtrsim \int_{\Omega} \langle W(\zeta)f(\zeta), f(\zeta) \rangle dV(\zeta).
\]

This implies $\|f\|_{L^2(W)} \lesssim \|f\|_{L^2(\hat{W})}$. \hfill \qed
As a consequence of Lemma 1.4, the norm of $P$ on the weighted space $L^2(W)$ can be dominated by the norm of $P$ on the weighted space $L^2(\tilde{W})$.

**Theorem 4.5.** Let $W$ be a matrix $B_2$ weight on $\Omega$. Then $\|P\|_{L^2(W)} \lesssim B_2^{1/2}(W)\|P\|_{L^2(\tilde{W})}$.

**Proof.** By Lemma 4.4 and a duality argument, we have

$$\|P\|_{L^2(W)} \lesssim \|P : L^2(\omega) \to L^2(\tilde{W})\| = \|P : L^2(\tilde{W}^{-1}) \to L^2(W^{-1})\|.$$ 

Set $\tilde{W}_1(z) := \sum_{j=1}^{N} \sum_{\hat{K}_j \in \mathcal{T}_j} (W^{-1})^j 1_{\hat{K}_j}(z)$. Applying Lemma 4.4 again to the weight $W^{-1}$ yields

$$\|P\|_{L^2(W)} \lesssim \|P : L^2(\tilde{W}^{-1}) \to L^2(W^{-1})\| \lesssim \|P : L^2(\tilde{W}^{-1}) \to L^2(\tilde{W}_{1})\|.$$ 

For any $z \in \Omega$, we consider those kubes $K_m$ of $K_j \in \bigcup_{j=1}^{N} T_j$ such that $\pi(K_j) \ni \text{dist}(K_m, b\Omega)$ and $K_m \ni |B^j(\pi(z), r)|$. Then Lemma 2.6 implies that there is a dyadic tent $\hat{K}_j \ni \bigcup_{j=1}^{N} T_j$ such that $\hat{K}_j \ni \mathcal{T}_j \ni |B^j(\pi(z), r)|$. Thus

$$\tilde{W}(z) = \sum_{K_m \in \bigcup_{j=1}^{N} T_j} (W)^j 1_{K_m}(z) \lesssim N(W)^j_{\hat{K}_j}, \quad (4.12)$$

$$\tilde{W}_1(z) = \sum_{K_m \in \bigcup_{j=1}^{N} T_j} (W^{-1})^j 1_{K_m}(z) \lesssim N(W^{-1})^j_{\hat{K}_j}. \quad (4.13)$$

Inequalities (4.12) and (4.13) yield

$$\|\tilde{W}_{1}^{1/2}(z)\tilde{W}^{1/2}(z)\|^2 \lesssim \|W^{-1/2}1_{\hat{K}_j}^j(W)^{1/2}\|^2 \lesssim B_2(W).$$

Hence, we have $\|\tilde{W}_{1}^{1/2}(z)f(z)\| \lesssim B_2^{1/2}(W)\|\tilde{W}^{-1/2}(z)f(z)\|$, which implies

$$\|P\|_{L^2(W)} \lesssim \|P : L^2(\tilde{W}) \to L^2(\tilde{W}_{1})\| \lesssim B_2^{1/2}(W)\|P\|_{L^2(\tilde{W}^{-1})} = B_2^{1/2}(W)\|P\|_{L^2(\tilde{W})}.$$ 

\[ \square \]

5. **Proof of Theorem 1.2**

By Theorem 4.5 and the fact that $\tilde{W} = \sum_{j=1}^{N} W_j$, it suffices to show that

$$\|P\|_{L^2(W)} \lesssim B_2^{3/2}(W),$$

where $W$ is a weight defined as in Definition 1.1.

We begin with the following lemma which can be viewed as a special case of the Carleson Embedding Theorem for $T$. We provide the proof below for the completeness of the paper.

**Lemma 5.1.** Let $T$ be a collection of dyadic tents as in Lemma 2.8. Then for any scalar-valued measurable function $f$ on $\Omega$ and for any $p \in (1, \infty)$,

$$\sum_{\hat{K}_j \in T} \langle f \rangle^p_{\hat{K}_j} \lesssim (p')^p \|f\|_{L^p(\Omega)}.$$ 

**Proof.** Recall the maximal operator $M_T$ given by

$$M_T f(z) = \sup_{\hat{K}_j \in \mathcal{T}} \langle |f| \rangle^p_{\hat{K}_j} 1_{\hat{K}_j}(z).$$
For $p \in (1, \infty)$, a standard argument yields that $\|M_{T}\|_{L^{p}(\Omega)} \leq p'$. Recall also that for each tent $\hat{K}_{j}^{k}$, its induced kube $K_{j}^{k}$ satisfying $|\hat{K}_{j}^{k}| \approx |K_{j}^{k}|$. Then

$$\sum_{K_{j}^{k} \in T} \langle |f| \rangle_{K_{j}^{k}}^{p} |\hat{K}_{j}^{k}| \lesssim \sum_{K_{j}^{k} \in T} \langle |f| \rangle_{K_{j}^{k}}^{p} |K_{j}^{k}| \leq \int_{\Omega} (M_{T}f)^{p}dV \leq (p')^{p} \|f\|_{L^{p}(\Omega)}^{p}. \quad \square$$

By Lemma 3.1 and Theorem 3.2, we need to estimate the norm of the operator

$$\sum_{K_{j}^{k} \in T} \int_{K_{j}^{k}} H_{K_{j}^{k}}(z, w)f(w) dV(w) 1_{K_{j}^{k}}(z),$$

on the weighted space $L^{2}(\mathcal{W})$. Here $H_{K_{j}^{k}}$ satisfying $|H_{K_{j}^{k}}(z, w)| \lesssim |\hat{K}_{j}^{k}|^{-1}$. This is equivalent to estimating the operator

$$Q_{W}(f)(z) = \sum_{K_{j}^{k} \in T} \int_{K_{j}^{k}} H_{K_{j}^{k}}(z, w)W^{1/2}(z)W^{-1/2}(w)f(w) dV(w) 1_{K_{j}^{k}}(z),$$

over the unweighted space $L^{2}(\Omega, \mathbb{C}^{d})$. Set

$$\mathcal{W}_{-1}(z) = \sum_{K_{j}^{k} \in T} \langle W^{-1} \rangle_{K_{j}^{k}} 1_{K_{j}^{k}}(z).$$

For vector-valued functions $f, g \in L^{2}(\Omega, \mathbb{C}^{d})$,

$$|\langle Q_{W}(f), g \rangle| \leq \sum_{K_{j}^{k} \in T} \left| \int_{\Omega} \int_{K_{j}^{k}} H_{K_{j}^{k}}(z, w)W^{-1/2}(w)f(w) dV(w) 1_{K_{j}^{k}}(z), W^{1/2}(z)g(z) dV(z) \right| \leq \sum_{K_{j}^{k} \in T} \left| \int_{\Omega} \int_{K_{j}^{k}} (W^{-1})_{K_{j}^{k}}^{1/2}W^{-1/2}(w)f(w), (W^{-1})_{K_{j}^{k}}^{1/2}W^{1/2}(z)g(z) 1_{K_{j}^{k}}(z) dV(w) dV(z) \right| \leq \sum_{K_{j}^{k} \in T} \left| \int_{\Omega} \int_{K_{j}^{k}} (W^{-1})_{K_{j}^{k}}^{1/2}W^{1/2}(z)g(z)H_{K_{j}^{k}}(z, w) dV(w) dV(z) \right| \leq \sum_{K_{j}^{k} \in T} \left| \int_{\Omega} \int_{K_{j}^{k}} (W^{-1})_{K_{j}^{k}}^{1/2}W^{-1/2}(w)f(w) 1_{K_{j}^{k}}(z) \right| \left| \int_{\Omega} (W^{-1})_{K_{j}^{k}}^{1/2}W^{1/2}(z)g(z)H_{K_{j}^{k}}(z, w) dV(z) dV(w) \right| \leq \sum_{K_{j}^{k} \in T} \left| \int_{\Omega} (W^{-1})_{K_{j}^{k}}^{1/2}W^{-1/2}(w)f(w) 1_{K_{j}^{k}}(z) \right| \left| \int_{\Omega} (W^{-1})_{K_{j}^{k}}^{1/2}W^{1/2}(z)g(z)H_{K_{j}^{k}}(z, w) dV(z) dV(w) \right| \leq \sum_{K_{j}^{k} \in T} \left| \int_{\Omega} (W^{-1})_{K_{j}^{k}}^{1/2}W^{-1/2}(w)f(w) 1_{K_{j}^{k}}(z) \right| \left| \int_{\Omega} (W^{-1})_{K_{j}^{k}}^{1/2}W^{1/2}(z)g(z)H_{K_{j}^{k}}(z, w) dV(z) dV(w) \right| \leq \left( \sum_{K_{j}^{k} \in T} \left| \int_{\Omega} (W^{-1})_{K_{j}^{k}}^{1/2}W^{-1/2}(w)f(w) 1_{K_{j}^{k}}(z) \right|^{2} |K_{j}^{k}| \right)^{1/2} \left( \sum_{K_{j}^{k} \in T} \left| \int_{\Omega} (W^{-1})_{K_{j}^{k}}^{1/2}W^{1/2}(z)g(z)H_{K_{j}^{k}}(z, w) dV(z) dV(w) \right|^{2} |K_{j}^{k}| \right)^{1/2}. \quad (5.1)$$
Set operators $S_{1,W}$ and $S_{2,W}$ to be as follows:

$$S_{1,W}(f)(z) := \sum_{K_j^+ \in \mathcal{T}} \left( \left( \mathcal{W}^{-1/2} \mathcal{W}^{-1/2} f \right)_{K_j^+} \right)^2 1_{K_j^+}(z),$$

$$S_{2,W}(g)(z) := \sum_{K_j^+ \in \mathcal{T}} \left( \left( \mathcal{W}^{-1/2} \mathcal{W}^{1/2} g \right)_{K_j^+} \right)^2 1_{K_j^+}(z).$$

For $S_{2,W}(g)$, Hölder’s inequality implies

$$\left< \left| (\mathcal{W}^{-1/2})_{K_j^+} \mathcal{W}^{1/2} g \right|^{2r} \right>_{K_j^+} \leq \left< \left( (\mathcal{W}^{-1/2})_{K_j^+} \mathcal{W}^{1/2} \right)^{1/2r} \right>_{K_j^+} \left< (g)^{(2r)'} \right>_{K_j^+}^{1/(2r)'}. \tag{5.2}$$

Choosing $r$ as in Lemma 4.3 and applying the lemma, we have

$$\left< \left| (\mathcal{W}^{-1})_{K_j^+}^{1/2} \mathcal{W}^{1/2} W \right|^{2r} \right>_{K_j^+} = |\hat{K}_j^+|^{-1} \int_{\hat{K}_j^+} \left< (\mathcal{W}^{-1/2})_{K_j^+} \mathcal{W}(w) (\mathcal{W}^{-1/2})_{K_j^+} \mathcal{W} \right> dV(w)$$

$$\leq |\hat{K}_j^+|^{-1} \int_{\hat{K}_j^+} \left( \left( (\mathcal{W}^{-1/2})_{K_j^+} \mathcal{W}(w) (\mathcal{W}^{-1/2})_{K_j^+} \mathcal{W} \right) \right)^r dV(w)$$

$$\leq \sum_{k=1}^d |\hat{K}_j^+|^{-1} \int_{\hat{K}_j^+} \left( (\mathcal{W}^{-1/2})_{K_j^+} \mathcal{W}(w) (\mathcal{W}^{-1/2})_{K_j^+} \mathcal{W} \right)^r dV(w)$$

$$\leq \left( (\mathcal{B}_2(W))^r \right). \tag{5.3}$$

Applying this inequality to (5.2) gives

$$S_{2,W}(g)(z) \leq (\mathcal{B}_2(W))^\frac{1}{r} \sum_{K_j^+ \in \mathcal{T}} \left( |g|^{(2r)'} \right)_{K_j^+}^{1/(2r)} 1_{K_j^+}(z),$$

Set $h = \|g\|^{(2r)'}$ and $p = 2/(2r)'$. Then

$$\|S_{2,W}(g)\|_{L^2(\Omega)}^2 \leq \mathcal{B}_2(W) \sum_{K_j^+ \in \mathcal{T}} \langle h \rangle_{K_j^+}^p \|h\|_{L^p(\Omega)}^p = \mathcal{B}_2(W) \langle p' \rangle^p \|g\|_{L^2(\Omega)}^2,$$

where the second inequality above follows from Lemma 5.1. Since $p' = 2 + \frac{1}{\ell} \leq 2 + \frac{1}{\epsilon}$ and $p \leq 1 + \epsilon$, we have $(p')^p \leq 1/\epsilon = (r-1)^{-1}$. From the choice of $r$ in Lemma 4.3

$$(r-1)^{-1} \approx \mathcal{B}_2(W).$$

Therefore,

$$\|S_{2,W}(g)\|_{L^2(\Omega)}^2 \leq \mathcal{B}_2(W) \|g\|_{L^2(\Omega)}^2.$$

We turn to estimate the norm of $S_{1,W}f$. Note that

$$\int_{K_j^+} \left( (\mathcal{W}^{-1/2})_{K_j^+} \mathcal{W}^{-1}(w) (\mathcal{W}^{-1/2})_{K_j^+} \mathcal{W} \right) dV(w) = I_d.$$

Using this fact and going through the same argument as $S_{2,W}g$ yield that

$$\|S_{1,W}f\|_{L^2(\Omega)}^2 \leq \mathcal{B}_2(W) \|f\|_{L^2(\Omega)}^2.$$
Combining these inequalities, we obtain
\[
\langle Q_w f, g \rangle \lesssim (B_2(W))^{3\over 2}\|f\|_{L^2(\Omega)}\|g\|_{L^2(\Omega)},
\]
which shows the desired estimate \(\|P\|_{L^2(\Omega,W)} \leq \|Q_W\|_{L^2(\Omega)} \lesssim (B_2(W))^{3\over 2}\).

6. Remarks

1. By the proof of Theorem 3.2, it’s not hard to see that the positive Bergman operator \(P^+\) also belongs to a convex body valued sparse operator. The same argument in Section 5 then yields the following corollary for the weighted norm of \(P^+\) on the weighted space \(L^2(\tilde{W})\):

**Corollary 6.1.** Let \(\Omega\) be a simple domain. Let \(W\) be a matrix \(B_2\) weight. Let \(\tilde{W}\) be a weight constructed based on \(W\) as in Definition 4.4. Then \(\|P^+\|_{L^2(\tilde{W})} \lesssim B_2^{3/2}(W)\).

2. As pointed out in [HWW20b] for the scalar-valued case, the products of averages of \(W\) and \(W^{-1}\) over the whole domain and over the small tents will have different impacts on the estimate for the weighted norm of the projection \(P\). However, our estimate in Theorem 1.2 is less likely to be sharp. Hence we didn’t consider this difference in here for the simplicity of the argument.

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