The Weyl law for contractive maps

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Abstract
We find an empirical Weyl law followed by the eigenvalues of contractive maps. An important property is that it is mainly insensitive to the dimension of the corresponding invariant classical set, the strange attractor. The usual explanation for the fractal Weyl law emergence in scattering systems (i.e., having a projective opening) is based on the classical phase space distributions evolved up to the quantum to classical correspondence (Ehrenfest) time. In the contractive case this reasoning fails to describe it. Instead, we conjecture that the support for this behavior is essentially given by the strong non-orthogonality of the eigenvectors of the contractive superoperator. We test the validity of the Weyl law and this conjecture on two paradigmatic systems, the dissipative baker and kicked top maps.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The study of open quantum systems has recently become a very active field [1]. The reasons are many, including the development of quantum information and computation [2, 3], and quantum optics and scattering systems [4, 5]. Particularly in this latter case the fractal Weyl law has been proposed. This law predicts the way in which the long-lived resonances of these systems grow as a function of $\hbar$. The fundamental ingredient is the classical invariant set, which in these kind of systems is the repeller, i.e., the set of trajectories non-escaping in the past and in the future. In fact, this law says that the number of long-lived quasibound states is proportional to $\hbar^{-1+\delta_H}$, where $\delta_H$ is the partial Hausdorff dimension of the repeller [6].

There is a vast literature that has contributed to increased confidence in this conjecture by means of numerical tests conducted on many systems [7]. However, open quantum maps have been the main tool in these studies, as they offer great simplicity in the calculations without losing much generality [8–10]. For them, the fractal Weyl law predicts that the resonances grow as $\hbar^{-d}$, where $d$ is the partial fractal dimension of the repeller. But if the way to open the system is nonprojective, the available literature is very scarce. Recently, in [11] this situation has been analyzed for dissipative quantum operations that can be considered as a phase space
correspondence time $T$ this case shrinks following the associated dissipation) evolved up to the quantum to classical having a projective opening). This is done in terms of an initial classical distribution (that in fractal Weyl law, we first follow the same steps as in the case of scattering systems (i.e., spectra of contractive noise. In order to explain its emergence and discrepancies with the usual contraction leading to dissipative dynamics [12]. In that work a dissipative baker map has been studied, where all classical initial conditions asymptotically fall on a strange attractor. The quantum counterpart has been implemented by means of a noise superoperator written in terms of Kraus operators [13]. The number of long-lived resonances has been found to behave in a rather different way compared to the usual prediction of the fractal Weyl law. In fact, this number grows as a power law in $\hbar$, but the exponent is mainly insensitive to the dimension of the fractal invariant set.

In this work we analyze this behavior in depth. We find an empirical Weyl law for the spectra of contractive noise. In order to explain its emergence and discrepancies with the usual fractal Weyl law, we first follow the same steps as in the case of scattering systems (i.e., having a projective opening). This is done in terms of an initial classical distribution (that in this case shrinks following the associated dissipation) evolved up to the quantum to classical correspondence time $T_{\text{Ehr}}$, the Ehrenfest time. We propose a theoretical expression for this time based on dynamical considerations and confirm its validity by means of the exploration of the classical phase space distributions and the eigenvectors of the contractive superoperator. However, this reasoning does not lead to a satisfactory explanation. We conjecture that the strong non-orthogonality of the right eigenvectors is the main reason behind this behavior. We explore the details and the validity of these results by means of two paradigmatic systems, the dissipative baker and kicked top maps.

This paper is organized as follows: in section 2 we briefly describe the first dissipative model that we have used and give the expression of the Weyl law for the contractive baker map. In section 3, the numerical results are analyzed and possible explanations are explored for the emergence of the here obtained Weyl law supported by studies of the phase space distributions and the properties of eigenvectors. Section 4 is devoted to the study of a second dissipative model, the dissipative kicked top, in order to provide more evidence on the validity of the Weyl law and the conjecture regarding the origin of this behavior. Finally, we give our conclusions in section 5.

2. The Weyl law for the dissipative baker map

As in our previous work [11], we have investigated the spectral behavior of the dissipative baker map, which is defined on the 2-torus $T^2 = [0, 1] \times [0, 1]$ by

$$B(q, p) = \begin{cases} (2q, p/2) & \text{if } 0 \leq q < 1/2 \\ (2q - 1, (p + 1)/2) & \text{if } 1/2 \leq q < 1. \end{cases}$$

(1)

Besides contracting the torus in the $p$ direction by an $\epsilon$ factor, this map stretches the unit square by a factor of 2 in the $q$ direction, squeezes it by the same factor in the $p$ direction, and then stacks the right half onto the left one, leading to the formation of a strange attractor. It is important to note that this map can be visualized as a modification of the one studied in [14].

The first step to quantize it is to impose on any state $|\psi\rangle$ the periodic boundary conditions on the torus, for both the position and momentum representations. Then, we take $|q + 1/\psi\rangle = e^{i2\pi q}\langle q|\psi\rangle$ and $|p + 1/\psi\rangle = e^{i2\pi p}\langle p|\psi\rangle$, with $\chi_q, \chi_p \in [0, 1)$. There is a finite dimension $N = (2\pi \hbar)^{-1}$ for the corresponding Hilbert space and a discrete set of positions and momentum eigenstates, which is given by $|q_j\rangle = |(j + \chi_q)/N\rangle$ ($j = 0, 1, \ldots, N - 1$), and $|p_k\rangle = |(k + \chi_p)/N\rangle$ ($k = 0, 1, \ldots, N - 1$), whose eigenvalues are $q_j, p_k$. A discrete Fourier transform, i.e.

$$\langle p_k|q_j\rangle = \frac{1}{\sqrt{N}} \exp\left(-i\frac{2\pi}{N}(j + \chi_q)(k + \chi_p)\right)$$
Figure 1. The Weyl law for the dissipative baker map: \( f_{\text{long-lived}} \) as a function of \( (\gamma_{\text{cut}} \epsilon)/N \). Results for \( \epsilon = 0.8, 0.7, 0.6 \) and 0.4 are represented by means of up triangles (in red), down triangles (in black), dots (in blue) and squares (in magenta), respectively.

relates these sets. We take the anti-symmetric boundary conditions, this meaning \( \chi_q = \chi_p = 1/2 \). For an even \( N \)-dimensional Hilbert space, the quantum baker map is defined in the momentum representation as [15, 16]

\[
B_N = \begin{pmatrix} G_{N/2} & 0 \\ 0 & G_{N/2} \end{pmatrix} G_N^{-1}. \tag{2}
\]

with \( B_N \), a unitary matrix (closed quantum baker map).

We introduce dissipation by means of a non-unital quantum operation [12] implemented by an \( N^2 \times N^2 \) Kraus superoperator of the form:

\[
M = \sum_{\mu=0}^{N-1} A^\mu \otimes A^\mu \dagger. \tag{3}
\]

Quantum operations act on the density matrix, \( \otimes \) denotes the place where this later must be inserted in order to implement the corresponding quantum operation. Here

\[
A^\mu = \sum_{i=\mu}^{N-1} \sqrt{\frac{i}{i - \mu}} \epsilon^{i-\mu} (1 - \epsilon)^{\mu} |p_{i-\mu}\rangle \langle p_i| \tag{4}
\]

are operators that induce transitions towards the momentum state \( |p_{i=0}\rangle \). The coupling constant \( \epsilon \) has the same value as the dissipation parameter of the corresponding classical map. \( M \) is a trace preserving \( (\sum_{\mu} A^\mu \dagger A^\mu = 1) \) and non-unital \( (\sum_{\mu} A^\mu \dagger A^\mu \neq 1) \) superoperator, which describes a process contracting phase space volume. The complete quantum dissipative dynamics is obtained by composing \( M \) with the unitary map (2),

\[
\mathcal{S} = (B_N \otimes B_N^\dagger) \circ M. \tag{5}
\]

In this work, we have computed the eigenvalue spectrum of superoperator (5) for different values of the contraction parameter \( (\epsilon = 0.8, 0.7, 0.6, 0.4) \) and of the dimension \( (90 \leq N \leq 180) \). For each case we have counted the number of complex eigenvalues \( \lambda \) (with \( |\lambda| = \exp(\frac{\gamma}{N^2}) \)) with a decay rate \( \gamma \) smaller than a given value \( \gamma_{\text{cut}} \). The data are collected in figure 1 which displays the fraction of resonances \( f_{\text{long-lived}} \) as a function of \( \epsilon, N \) and the cut-off value \( \gamma_{\text{cut}} \) (in a wide range \( 2 \leq \gamma_{\text{cut}} \leq 14) \).

By fitting these numerical results, we obtain a remarkably compact and simple expression:

\[
f_{\text{long-lived}}(\epsilon, \gamma_{\text{cut}}, N) = \frac{N_{\gamma<\gamma_{\text{cut}}}}{N^2} = C (\epsilon \gamma_{\text{cut}})^{2/3} (N^2)^{-\nu}. \tag{6}
\]
Table 1. Values of the fitted coefficients $C$ (column two) and $\nu$ (column three) for different values of $\epsilon$ for the dissipative baker map. The fourth column displays the semiclassical prediction $\nu_{sc}$ described in section 3.

| $\epsilon$ | $C$   | $\nu$   | $\nu_{sc}$ |
|------------|-------|---------|-------------|
| 0.8        | 5.3   | 0.72    | 0.24        |
| 0.7        | 4.4   | 0.76    | 0.34        |
| 0.6        | 4.7   | 0.79    | 0.42        |
| 0.4        | 5.3   | 0.85    | 0.57        |

The values of $C$ and $\nu$, for four different values of $\epsilon$, are given in Table 1. In the fourth column we display the semiclassical prediction $\nu_{sc}$, which will be analyzed in section 3.

These findings generalize the ones obtained in [11]. On the one hand they confirm the existence of a power law dependence of $\frac{N_{\gamma<\gamma_{cut}}}{N^2}$ on $N$ with an exponent which, in a meaningful range of validity, is fairly insensitive to the value of the dissipation parameter $\epsilon$. On the other hand, they hint (within a precision of 20%) on a very simple dependence of the prefactor with both $\epsilon$ and the cut-off value $\gamma_{cut}$. We will leave the analysis of this prefactor, which is in general believed to be system-dependent, for future work [17] and concentrate in the following on the scaling of $\frac{N_{\gamma<\gamma_{cut}}}{N^2}$ with $N$. We will seek an expression of $\nu$, in order to determine to which extent this exponent can be related to the underlying classical dynamics. For this, we will follow an approach analogous to the one used in the formulation of the fractal Weyl law for chaotic maps with a projective opening [10] and discuss its limitations in the case of a contractive noise.

3. Classical and quantum support for the eigenvalue statistics

A heuristic formulation of the fractal Weyl law for chaotic maps with a projective opening is based on the assumption that the number of long-lived resonances (associated with the classical repeller) scales as the volume of the evolved initial classical distribution up to the Ehrenfest time, that is, the volume of a finite (Ehrenfest) time repeller [10]. This volume can be calculated by a combination of two exponential laws that relate the probability to reside in the system (non-escaping trajectories) and the quantum to classical correspondence.

In the case of a contractive noise the connection between the long-lived resonances and the structure of the classical invariant also exists. In particular, we have verified in [11] that the Husimi representation of the projector corresponding to the eigenfunctions with slow escape rate concentrates on the phase space region corresponding to the classical strange attractor. It seems then natural to generalize the considerations usually applied to chaotic maps with a projective opening to the contractive case and investigate whether this scheme succeeds in accounting for the Weyl law of (6). Our starting point will be the following relation [18]:

$$f_{\text{long-lived}}(\epsilon, \gamma_{cut}, N) \sim A_{\text{class}}^2$$

between the fraction of long-lived resonances and the volume of the attractor $A_{\text{class}}$ which shrinks exponentially until the Ehrenfest time according to:

$$A_{\text{class}} = \exp\left(-\gamma_{\text{cl}} T_{\text{Ehr}}\right).$$

Note in (7) the square (instead of linear) dependence on $A_{\text{class}}$, which is due to the use of the superoperator formalism to model the contractive noise. The classical decay rate $\gamma_{\text{cl}}$ and the correspondence (Ehrenfest) time $T_{\text{Ehr}}$ are then the two main ingredients of this approach that should be evaluated.

The classical decay rate can be easily calculated by following the time evolution of a uniform distribution in phase space under the action of dissipation. It is straightforward to see that after \( t \) time steps the original distribution will occupy \( 2^t \) fringes in the \( q \) direction, each fringe having a width \( (\frac{\delta}{2})^t \). Hence, the total phase space area occupied by the distribution as a function of time is \( A_{\text{clas}} = e^{-\gamma t} = e^t \), and then the classical decay rate is given by \( \gamma_{cl} = -\ln \epsilon \).

Determination of the Ehrenfest time is a more subtle issue. Understood as the time at which the quantum and the classical descriptions differ, we can start our reasoning following the lines of what is done in the case of area preserving maps. In fact, there are two different ways to conceive this correspondence time. The first is the time \( T_{\text{Ehr1}} \) at which a given initial semiclassical distribution (a coherent state of width \( \sqrt{\hbar} \), for instance) spreads up to the border of the system along the unstable direction (manifold). This time is related to the expansive Lyapunov exponent \( \lambda_1 \), such that \( T_{\text{Ehr1}} \propto \frac{\ln N}{\lambda_1} \). On the other hand, the time \( T_{\text{Ehr2}} \) is the one corresponding to the initial distribution shrinking along the stable direction to a size of the order of the Planck cell \((1/N)\). This time is related to the contractive Lyapunov exponent \( \lambda_2 \), such that \( T_{\text{Ehr2}} \propto \frac{\ln N}{\lambda_2} \). Of course, in the case of an area preserving map \( \lambda_1 + \lambda_2 = 0 \) and both times coincide. However, under a contractive noise, our dissipative map gives \( \lambda_1 = \ln 2 \) while \( \lambda_2 = -\ln 2 \). Hence, we propose the shortest \( T_{\text{Ehr2}} \propto \ln N/\ln 2 \) as the global quantum to classical correspondence time for this map.

In order to verify this assumption, we have numerically estimated the correspondence time. This can be accomplished quite easily by evaluating the overlap \( O_{\text{cl-q}} \) between the finite time classical attractor and the Husimi distribution of a uniform initial state evolved up to the same time. If we exploit the fact that for the baker map the interesting features of the distribution (namely its fractality) are only in the \( p \) coordinate we can notably simplify this task. In fact, we just calculate the norm of the evolved wavefunction, restricted to the region occupied by the classical distribution at any given time. As a result we have obtained figure 2 where these overlaps are shown as a function of the map iterations.

The overlap \( O_{\text{cl-q}} \) decays with time and can be approximated by a Gaussian decay expression of the form

\[
O_{\text{cl-q}} = (1 - A) \exp\left[-(t/T_D)^2\right] + A.
\]

We have fitted the characteristic decay time \( T_D \) and the horizontal shift \( A \) taking into account the short time behavior of \( O_{\text{cl-q}} \). These parameters depend on the number of points used to perform the fit, in figure 2 we show examples of different fits. We have performed an average over the different possibilities in order to obtain a mean characteristic decay time \( \overline{T_D} \), while the standard deviation has been taken as a measure of the corresponding error. In table 2 we display the mean characteristic decay time \( \overline{T_D} \) with its error for different values of \( \epsilon \) and \( N \) in comparison with the Ehrenfest time \( T_{\text{Ehr2}} \).

| \( \epsilon \) | \( N = 100 \) | \( N = 200 \) | \( N = 400 \) |
| --- | --- | --- | --- |
| \( T_{\text{Ehr2}} \) | \( \overline{T_D} \) | \( T_{\text{Ehr2}} \) | \( \overline{T_D} \) | \( T_{\text{Ehr2}} \) | \( \overline{T_D} \) |
| 0.4 | 2.86 | 2.61 ± 0.31 | 3.29 | 3.14 ± 0.41 | 3.73 | 3.95 ± 0.41 |
| 0.6 | 3.82 | 4.06 ± 0.30 | 4.4 | 4.60 ± 0.40 | 4.97 | 4.95 ± 0.50 |
| 0.8 | 5.02 | 4.98 ± 0.41 | 5.78 | 6.02 ± 0.65 | 6.53 | 6.78 ± 0.65 |

Comparison between the Ehrenfest time \( T_{\text{Ehr2}} \) (for the dissipative baker map) and the adjusted characteristic time obtained by fitting data in figure 2 with a Gaussian decay. The first column shows the different values of \( \epsilon \). The second, fourth and sixth columns display the Ehrenfest time \( T_{\text{Ehr2}} \) for \( N = 100, \ 200 \) and 400 respectively. The third, fifth and seventh columns display, for the same values of \( N \), the averaged fitted parameters \( \overline{T_D} \) with their respective errors.
Figure 2. Overlap between the phase space region occupied by the quantum and classical attractors (corresponding to the dissipative baker map) as a function of time $t$ (map iterations). The upper panel corresponds to $\epsilon = 0.8$ and the lower panel to $\epsilon = 0.4$. Results for $N = 100$, 200 and 400 are represented with squares (in black), dots (in blue) and up triangles (in magenta), respectively. Two different fits with (9) have been plotted with full and dashed lines in black, blue and magenta, for $N = 100$, 200 and 400 respectively.

As we see, the averaged characteristic decay time $T_D$ of (9) can be identified with the Ehrenfest time $T_{Ehr2}$. Hence, besides small fluctuations and the lack of precision inherent to the discrete time steps of the map, the results confirm our theoretical prediction.

Inserting the expressions of $\gamma_{cl}$ and $T_{Ehr2}$ in (8) gives $A_{clas} = N^{-\nu_{sc}}$, where $\nu_{sc} = 2 - d$, and $d = 1 + \ln(2)/(\ln(2) - \ln(\epsilon))$ is the fractal dimension of the classical attractor. The values of the semiclassical $\nu_{sc}$ are listed in the fourth column of table 1, showing a dramatic discrepancy with the values obtained by fitting our numerical results with (6). Besides an overall factor of $\sim 2$ between $\nu$ and $\nu_{sc}$, the semiclassical exponent shows a dependence on $\epsilon$ (via the fractal dimension of the attractor) which is absent in the fitted $\nu$ which are practically constant. Then, it becomes clear that the way of reasoning that has provided with a reasonable explanation for the emergence of the usual fractal Weyl law for systems subjected to projective noise can no
longer be applied to contractive dynamics. We are now faced with the question of where this discrepancy comes from.

At the basis of (7) is the assumption that the number of long-lived quantum states can be approximated by the number of Planck cells which fit into the phase space volume of the classical invariant set. This, in turn, supposes that to a good approximation the eigenfunctions supported by this set are non-overlapping. Even though we cannot strictly speak of orthogonality, since the operators describing open systems are not normal, we know that in the case of projective openings the long-lived eigenfunctions are quasi-orthogonal (while the short-lived ones present a high degree of degeneracy). This explains the success of the fractal Weyl law in the projective case. In the case of contractive dynamics we will investigate this point by defining the overlap matrix $P_{ij} = \text{Tr}(R_i^\dagger R_j)$, where $R_i$ are the right eigenstates corresponding to the superoperator $S$ of (5) (this is not to be confused with the biorthogonality of the right and left eigenfunctions of a superoperator, which states that $\text{Tr}(L_i^\dagger R_j) = \delta_{i,j}$). The overlap matrix elements corresponding to the contractive map with $N = 180$ and $\epsilon = 0.4$, $0.6$ and $0.8$ for the 200 longest-lived eigenstates are displayed in panels (a), (b) and (c) of figure 3, respectively. A grayscale is used to represent them, going from white corresponding to value 0 to black corresponding to the maximum values. We observe that the off-diagonal elements are clearly non negligible. Moreover, their value grows with the contractive power of the corresponding map (as $\epsilon$ decreases). For comparison we show in panel (d) the overlap matrix for a projective case, obtained by opening the baker map along two symmetric bands in the $q$-direction, of width $\delta p = 0.1$ and centered at $p = 0$ and $p = N - 1$. In this case, as expected, the matrix is almost diagonal.

The different degrees of non-orthogonality of the long-lived resonances in both models are also reflected in the phase space distribution of these states. In panels (a) and (c) of figure 4 we show the sum up to $\gamma_{\text{cut}}$ of the Husimi representation of the longest-lived right eigenstates:
Figure 4. In panel (a) we show the sum (10) of the Husimi representation corresponding to the first 600 right-right eigenvectors for the dissipative baker map at $\epsilon = 0.6$ for $N = 180$. Panel (b) displays the analogous sum corresponding to the Husimi representation of the Schur eigenvectors. For comparison, in the lower panels (c) and (d) we show the same distributions as in panels (a) and (b) respectively but for the projective opening case of figure 3(d).

$$\sum_{\gamma=0}^{N_{\lambda}} \frac{\langle z|R_\gamma |z\rangle \langle z|R_\gamma^\dagger |z\rangle}{\langle R_\gamma |R_\gamma^\dagger \rangle},$$

(10)

with $\langle z|R_\gamma |z\rangle = \text{Tr}(R_\gamma^\dagger, |z\rangle \langle z|)$ where $|z\rangle$ are coherent states centered at $z = (q, p)$. Panels (b) and (d) display the analogous sum (10) corresponding to the Husimi representation but of the Schur eigenvectors, which constitute the orthogonal basis associated with the eigenvalues $\lambda$ with $|\lambda| > \exp(-\frac{2\gamma_{cut}}{2})$.

In the case of the contractive map (upper line) we observe that the area of phase space occupied by the sum of the Husimi distributions is smaller than the area corresponding to the subspace spanned by the Schur decomposition. This is a clear sign of the non-orthogonality of the eigenstates for these kinds of superoperators [19].

In contrast, the lower panels (c) and (d) show that for the case of a projective opening both distributions look much the same, indicating that the assumption of quasi-orthogonality for the long-lived eigenfunctions is justified.

4. The Weyl law for the dissipative kicked top

As a second example, we have studied the spectrum of the quantum dissipative kicked top [20, 21], which has been extensively used to model super-radiance damping in quantum optics [22].

The unitary dynamics in this case is generated by the Floquet operator:

$$F_J = \exp[-i(k/2J)^2] \exp[-i\beta J_y].$$

(11)
where $J_i$ are the components of the angular momentum $J$, and $k$ and $\beta$ the torsion and rotation parameters, respectively [23]. $J^2 = j(j + 1)$ is conserved and the classical limit corresponding to $j \to \infty$. For sufficiently large values of $k$ and $\beta$ the dynamics is strongly chaotic.

The dissipation propagator $D_\tau$ is obtained from the integration of a master equation for the density matrix $\rho(\tau)$:

$$\frac{d}{d\tau} \rho(\tau) = \frac{1}{J} [[J_-, \rho(\tau)J_+] + [J_- \rho(\tau), J_+]].$$

(12)

This equation is written in terms of a dimensionless parameter $\tau$, which is the time in units of the classical timescale. For a dissipation rate $\Gamma$, $\tau = 2J\Gamma t$ gives the relaxation time between two actions of the unitary operator and thus fixes the strength of the dissipation [22]. The detailed form of $D_\tau$ in the semiclassical limit can be found in [24].

The total dissipative map is given by a $N^2 \times N^2$ (with $N = (2j + 1)$) superoperator of the form:

$$\mathcal{S} = D_\tau \circ (FJ \otimes FJ^\dagger).$$

(13)

We have diagonalized superoperator $\mathcal{S}$ for $k = 4.0$, $\beta = 2.0$ and three values of the dissipation parameter $\tau = 0.5$, $\tau = 0.825$ and $\tau = 1.0$. For these sets of values, the phase space portrait (on the sphere) of the corresponding classical map shows the existence of a strange attractor [25], whose dimension $d$ strongly depends on $\tau$.

Figure 5 displays the fraction of long-lived resonances as a function of $N (81 \leq N \leq 161)$ and $\gamma_{\text{cut}} (2 \leq \gamma_{\text{cut}} \leq 7)$. The numerical data are consistent with a relation:

$$f_{\text{long-lived}}(\tau, \gamma_{\text{cut}}, N) = \frac{N_{\gamma_{\text{cut}}}^\nu}{N^2} = C(\tau) (\gamma_{\text{cut}})^{\nu} (N^2)^{-\nu},$$

(14)

which confirms the power law dependence on $\gamma_{\text{cut}}/N$ previously obtained for the dissipative baker map. In the present case the dependence on the dissipation strength parameter has been absorbed in the prefactor $C(\tau)$ since its scaling is not as clear as in the previous example.

Table 3 shows the values of $C(\tau)$ and $\nu$ for different values of $\tau$. We observe that the strong dependence on $\tau$ of the classical attractor dimension $d$ listed in the last column is not reflected in the variations of the exponent $\nu$, evidencing that the usual explanation for the
Figure 6. Overlap matrices $P_{ij}$ of the first 200 right eigenstates with $N = 161$ (ordered by decreasing modulus of the eigenvalues) corresponding to the dissipative kicked top map for $\tau = 1.0$ and 0.5, (panels (a) and (b), respectively). For this system the projective case is almost diagonal (not shown). Only the upper halves of the matrices are shown.

Figure 7. In panel (a) we show the sum (10) of the Husimi representation corresponding to the first 250 right-right eigenvectors for the dissipative kicked top map at $\tau = 0.5$ for $N = 161$. Panel (b) displays the analogous sum corresponding to the Husimi representation of the Schur eigenvectors.

Table 3. Values of the fitted coefficients $C$ (column two) and $\nu$ (column three) for different values of $\tau$ for the dissipative kicked top map. The fourth column displays the classical fractal dimension $d$ from [22].

| $\tau$  | $C$   | $\nu$ | $d$  |
|---------|-------|-------|------|
| 0.5     | 0.95  | 0.68  | 1.70 |
| 0.825   | 0.44  | 0.63  | 0.45 |
| 1.0     | 0.80  | 0.79  | 0.80 |

The fractal Weyl law also fails in this case. For this reason we do not provide a semiclassical prescription of this exponent. Instead, we verify that the right eigenfunctions of $S$ present a high degree of non-orthogonality which increases with the dissipation strength, as shown in figure 6. This is also reflected in the phase space distribution of these eigenstates. Analogously to what was shown in figure 4 for the dissipative baker map, we now display the sum (10) of the Husimi distributions of the first 250 right eigenstates in figure 7(a) and the same sum corresponding to the Husimi distribution of the Schur eigenvectors in figure 7(b). It is clear that the subspace spanned by the latter covers a region larger than the phase space support of the right eigenstates.
5. Conclusions

We have found an empirical expression of the Weyl law for the spectra of the contractive baker map which has a simple dependence on $\gamma_{\text{cut}}/N$. The same simple behavior has been obtained for the dissipative kicked top map on the sphere. We were not able to explain the emergence of this law by means of the usual line of reasoning applied to the projective case. Very simply put, the idea is counting resonances. This has been traditionally accomplished by partitioning the phase space volume occupied by a finite time classical invariant set (the repeller). In fact, this implies a pseudo orthogonality of the long-lived eigenstates. We could verify that this is indeed the case for the projectively opened baker map, a system that has been paradigmatically used in the fractal Weyl law literature. But when it comes to the dissipative baker map used in this work, we have clearly identified a high degree of non-orthogonality. This is the main reason behind the failure of the usual reasoning for explaining the emergence of the Weyl law. As a result, we think that a new method for counting the long-lived resonance other than just partitioning the corresponding volume in phase space into Planck cells, is the key to understanding the statistical behavior of contractive maps. We have verified these results and conjectures in a second paradigmatic example, the dissipative kicked top. This builds up strong evidence on the robust nature of the behavior we have found for dissipative systems and also on its possible origin.

In the future, we hope to obtain a theoretical explanation for the empirically found Weyl law for dissipative systems. This is a mathematically hard problem in view of the difficulties found in order to demonstrate the fractal Weyl law for projectively opened systems. Moreover, we are also trying to develop a theory for the prefactor and the dependence on the dissipative parameters (in our two examples $\epsilon$ and $\tau$, respectively) and $\gamma_{\text{cut}}$ [17]. But given the strong dependence on the system we think that this will be an even harder problem to solve.

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