SMOOTH QUARTIC K3 SURFACES AND CREMONA TRANSFORMATIONS, II

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ABSTRACT. This is a continuation of [Og12], concerning automorphisms of smooth quartic K3 surfaces and birational automorphisms of ambient projective three spaces.

1. INTRODUCTION

Throughout this note we work over \( \mathbb{C} \). This note is a continuation of our previous paper [Og12]. Our main results are Theorems (1.7), (1.8) below. In [Og12], being based on a result of Takahashi ([Ta98], see also Theorem (1.9)), we gave the first negative answer to the following long standing question of Gizatullin ([Do11]):

**Question 1.1.** Let \( S \subset \mathbb{P}^3 \) be a smooth quartic K3 surface and \( g \in \text{Aut}(S) \) as abstract variety of \( X \). Is \( g \) derived from \( \text{Bir}(\mathbb{P}^3) \), the group of birational automorphisms of \( \mathbb{P}^3 \), in the sense that there is \( \tilde{g} \in \text{Bir}(\mathbb{P}^3) \) such that \( \tilde{g}_*S = S \) and \( \tilde{g}|S = g \)?

The counter example in [Og12] is of Picard number 2 and \( g \) is of infinite order. Construction there is lattice theoretic and in this sense implicit. Here and hereafter, a K3 surface is a smooth projective simply connected surface \( S \) with nowhere vanishing global holomorphic 2-form \( \sigma_S \). Recall that \( \text{Aut}(S) \) can be an infinite group and can admit infinitely many different embeddings into \( \mathbb{P}^3 \), up to \( \text{Aut}(\mathbb{P}^3) \), or equivalently, infinitely many different linearly equivalent classes of very ample divisors \( D \) with \( (D^2)_S = 4 \). The Fermat quartic surface is such an example. Keeping these in mind, one can naturally ask more interesting questions now than the original question. For instance:

**Question 1.2.** Can one describe at least one counter example \((S, g)\) of Question (1.1) explicitly in terms of homogeneous coordinates of \( \mathbb{P}^3 \)?

**Question 1.3.** Is there a smooth quartic K3 surface \( S \subset \mathbb{P}^3 \) such that \( \text{Aut}(S) \) is enough complicated but every element of \( \text{Aut}(S) \) is derived from \( \text{Bir}(\mathbb{P}^3) \)?

**Question 1.4.** Is every automorphism of the Fermat quartic surface \( S \subset \mathbb{P}^3 \) derived from \( \text{Bir}(\mathbb{P}^3) \)?

**Question 1.5.** Is every automorphism of finite order of any smooth quartic surface \( S \subset \mathbb{P}^3 \) derived from \( \text{Bir}(\mathbb{P}^3) \)?

**Question 1.6.** Are there a smooth K3 surface \( S \) and \( g \in \text{Aut}(S) \) such that \( g \) is not derived from \( \text{Bir}(\mathbb{P}^3) \) in any embedding \( \Phi : S \to \mathbb{P}^3 \)?
The aim of this note is to give affirmative answers to Questions (1.3), (1.6) in stronger forms as well as an example supporting Question (1.5). Questions (1.2), (1.4) are completely open and Question (1.5) is also mostly open (see however Pr12).

Our main results are the following:

**Theorem 1.7.** There is a smooth quartic K3 surface $S \subset P^3$ of Picard number 3 such that

1. $\text{Aut} (S) \simeq \mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2$, the free product of three cyclic groups of order 2.
2. Every element of $\text{Aut}(S)$ is derived from $\text{Bir}(P^3)$ but no element other than $\text{id}_S$ is derived from $\text{Aut}(P^3)$.
3. $S$ admits infinitely many different embeddings $S \rightarrow P^3$, up to $\text{Aut}(P^3)$, and $\text{Aut}(S)$ is derived from $\text{Bir}(P^3)$ in any embedding $S \rightarrow P^3$.

**Theorem 1.8.** There is a smooth K3 surface $S$ of Picard number 2 such that

1. $\text{Aut}(S) \simeq \mathbb{Z}$.
2. $S$ admits infinitely many different embeddings $S \rightarrow P^3$ up to $\text{Aut}(P^3)$.
3. No element $\text{Aut}(S)$ other than $\text{id}_S$ is derived from $\text{Bir}(P^3)$ in any embeddings of $S$ into $P^3$.

We construct a K3 surface $S$ in Theorem (1.8) by deforming the Fermat quartic K3 surface suitably. This will be done in Section 3 after recalling some preliminary results in Section 2. Our proof is involved but fairly concrete. Note that $S$ admits infinitely many involutions but no automorphism of finite order $\geq 3$ (Theorem (1.7)(1)). Theorem (1.7)(3), in particular, says that all finite order automorphisms of $S$ are derived from $\text{Bir}(P^3)$ but none from $\text{Aut}(P^3)$ except $\text{id}_S$ (cf. Question (1.5)).

As one of relevant important results, it is worth mentioning here that the Kummer surface $\text{Km}(\text{Jac}(C))$ of the Jacobian of a generic curve $C$ of genus 2 is birational to a quartic surface $Q$ with 16 nodes and $\text{Aut}(\text{Km}(\text{Jac}(C)))$ is derived from $\text{Bir}(P^3)$ via $Q$ ([Ke97], [Ko98]).

Unlike Theorem (1.7)(3), Theorem (1.8) gives an affirmative answer to Question (1.6). This is originally asked by Doctor Sergey Galkin to me at Kinosaki conference (2011, October). Theorem (1.8) is proved in Section 4 again being based on the following special case of a more general result of Takahashi ([Ta98, Theorem 2.3, Remark 2.4]):

**Theorem 1.9.** Let $H$ be the hyperplane class of $P^3$ and let $S \subset P^3$ be a smooth quartic surface such that any effective curve $C$ on $S$ with $\text{deg} C := (C.H)_{P^3} < 16$ is of the form $C = T \cdot S$ for some hypersurface $T$ of $P^3$. Then, for any $g \in \text{Bir}(P^3) \setminus \text{Aut}(P^3)$, $K_{P^3} + g_* S$ is ample. In particular, $g_* S \neq S$.

**Notation.** Throughout this note $L_K = L \otimes_{\mathbb{Z}} K$ for $\mathbb{Z}$-module $L$ and $\mathbb{Z}$-algebra $K$. For a K3 surface $S$, $\text{NS}(S)$ is the Néron-Severi lattice. This is naturally isomorphic to $\text{Pic}(S)$ and is a sublattice of $H^2(S, \mathbb{Z})$ with natural intersection form $(\ast, \ast)_S$. The orthogonal complement $T(S) := \text{NS}(S)_{H^2(S, \mathbb{Z})} \perp$ of $\text{NS}(S)$ in $H^2(S, \mathbb{Z})$ is the transcendental lattice. $T(S)$ is the minimal primitive sublattice of $H^2(S, \mathbb{Z})$ such that $H^0(S, \Omega^2_S) = \mathbb{C} \sigma_S \subset T_C$. The dual lattice $\text{NS}(S)^* = \text{NS}(S)^*_{\mathbb{Q}}$ is given by $\{ x \in \text{NS}(S)_{\mathbb{Q}} \mid (x, \text{NS}(S))_S \subset \mathbb{Z} \}$. The dual lattice $T(S)^*$ is defined similarly from $T(S)$. We have $\text{NS}(S) \subset \text{NS}(S)^*$, $T(S) \subset T(S)^*$ and there is a natural isomorphism $\text{NS}(S)^*/\text{NS}(S) \simeq T(S)^*/T(S)$ compatible with the natural action of $\text{Aut}(S)$ ([N179, Proposition 1.6.1]). Similarly $L^*/L \simeq (L^\perp)^*/L^\perp$ for non-degenerate primitive sublattice $L$. This isomorphism is compatible with the natural action of $\text{Aut}(S, L)$,
the subgroup consisting of elements \( g \in \text{Aut}(S) \) such that \( g^*(L) = L \). The positive cone \( P(S) \) is the connected component of \( \{ x \in \text{NS}(S) \mid (x^2)_S > 0 \} \) containing ample classes. The effective nef cone \( \text{Amp}^+(S) \) is the convex cone in \( \text{NS}(S)_{\mathbb{R}} \) generated by the effective nef classes. The nef cone \( \text{Amp}^+(S) \) is the closure of \( \text{Amp}^+(S) \) in the topological vector space \( \text{NS}(S)_{\mathbb{R}} \). Our reference on basic facts on K3 surfaces and their projective models are \[\text{BHPV04, Chapter VIII}\] and \[\text{SD74}\].

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2. Preliminaries.

The following two theorems (Theorems (2.1), (2.2)) explain well why Gizatullin’s question is interesting only for quartic K3 surfaces among smooth hypersurfaces. Theorem (2.1) is due to Matsumura and Monsky (MM63) when \( n \geq 2 \) and Chang (Ch78) when \( n = 1 \).

**Theorem 2.1.** Let \( n \) and \( d \) be positive integers and \( X \) be a smooth hypersurface of degree \( d \) in \( \mathbb{P}^{n+1} \). If \( (n,d) \neq (2,4),(1,3) \), then \( \text{Aut}(X) \) is derived from \( \text{Aut} (\mathbb{P}^{n+1}) \).

When \( (n,d) = (1,3) \), \( X = E \) is a smooth cubic curve in \( \mathbb{P}^2 \). The following theorem should be classical and well known:

**Theorem 2.2.** Let \( E \subset \mathbb{P}^2 \) be a smooth cubic curve. Then \( \text{Aut} (E) \) is derived from \( \text{Bir} (\mathbb{P}^2) \).

Since an argument below will be used in our proof of Theorem (1.7), we sketch the proof. The point is that the same proof works with no modification for an elliptic curve \( E/K \) defined over any field \( K \) of characteristic 0 with \( E(K) \neq \emptyset \).

**Proof.** Choose \( O \in E \) and regard \( O \) as the unit element of the elliptic curve \( E \). Then \( \text{Aut}(E) \) is the semi-direct product of \( E \), the group of translations, and \( \text{Aut}(E,O) \) fixing \( O \). Here \( \text{Aut}(E,O) \) is a finite cyclic group of order \( d \in \{2,4,6\} \). By linear change of the coordinate of \( \mathbb{P}^2 \), we can rewrite the equation of \( E \) in the Weierstrass form:

\[
y^2 = x^3 + px + q .
\]

If \( \text{Aut}(E,O) \) is of order 6 (resp. 4), we can make \( p = 0 \) (resp. \( q = 0 \)). The generator of \( \text{Aut}(E,O) \) is \( (x,y) \mapsto (x,-y) \) when \( d = 2 \), \( (x,y) \mapsto (-x,\sqrt{-1}y) \) when \( d = 4 \) and \( (x,y) \mapsto (\zeta_3 x,-y) \) when \( d = 6 \). Here \( \zeta_3 \) is the primitive third root of unity. So \( \text{Aut}(E,O) \) is derived from \( \text{Aut}(\mathbb{P}^2) \). Let us consider the translation of \( E \) by \( (a,b) \in E \). The classical addition formula says that \( (s',t') = (s,t) + (a,b) \) on \( E \) if and only if

\[
s' = \left( \frac{t-b}{s-a} \right)^2 - s - a , \quad t' = \frac{t-b}{s-a} (s' - a) + b .
\]

Consider the rational map \( \varphi_{(a,b)} : \mathbb{P}^2 \cdots \rightarrow \mathbb{P}^2 \) defined by the same equations above. Then given \( (s',t') \in \mathbb{P}^2 \), the term \( (t-b)/(s-a) \) is uniquely determined from \( (s',t') \) as a rational function of \( (s',t') \), by the second formula. Then \( s \) is uniquely determined by the first
As observed by the intersection matrix below, the sublattice restriction map by $H$ hereafter vector field other than $0$. All the results follow from these three facts.

One can also check that there are only finitely many $g \in \text{Aut}(E)$ of the form $g = \tilde{g}|E$ with $\tilde{g} \in \text{Aut}(\mathbb{P}^2)$. So, it is really necessary to enlarge $\text{Aut}(\mathbb{P}^2)$ to $\text{Bir}(\mathbb{P}^2)$ in Theorem (2.2).

We close this section by recalling the following well known result which will be used in the remaining sections:

**Proposition 2.3.** Let $S$ be a K3 surface. Then:

1. Any embedding $\Phi : S \rightarrow \mathbb{P}^3$ is of the form $\Phi = \Phi_{|H|}$. Here $\Phi_{|H|}$ is the morphism associated with the complete linear system of a very ample divisor $H$ such that $(H^2)_{|S|} = 4$.

2. Under (1), the group $\{g \in \text{Aut}(S) \mid g^*H = H\}$ is the subgroup of $\text{Aut}(S)$ consisting of elements being derived from $\Phi_{|H|}$ and this is a finite group.

3. Let $g \in \text{Aut}(S)$ and $H$ be a very ample line bundle such that $(H^2)_{|S|} = 4$. Then $g^*H = H$ if and only if there is an isomorphism $\tilde{g} : \mathbb{P}^3 \rightarrow \mathbb{P}^3$ between the target spaces such that $\tilde{g} \circ \Phi_{|H|} = \Phi_{|H|} \circ g$.

**Proof.** If $S \subset \mathbb{P}^3$, then $S$ is of degree $4$ and $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) \cong H^0(S, \mathcal{O}_{S}(1))$ under the restriction map by $H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(n)) = 0$. We also recall that $S$ has no global holomorphic vector field other than $0$. All the results follow from these three facts.

3. **Proof of Theorem (1.7).**

Let 

$$S_0 := (x_0^4 - x_1^4 + x_2^4 - x_3^4 = 0) \subset \mathbb{P}^3.$$ 

$S_0$ is isomorphic to the Fermat quartic K3 surface and contains skew lines 

$$L := (x_0 = x_1, \ x_2 = x_3), \ M := (x_0 = -x_1, \ x_2 = -x_3), \ L \cap M = \emptyset.$$

As observed by the intersection matrix below, the sublattice $\mathbb{Z}([H],[L],[M])$, where and hereafter $H$ is the hyperplane class, is primitive in $H^2(S_0, \mathbb{Z})$. Let $S$ be a small generic deformation of $S_0$ inside $\mathbb{P}^3$ such that $L, M \subset S$. Then $S \subset \mathbb{P}^3$ is a smooth quartic K3 surface containing $L$ and $M$. In Hodge theoretical terms, $S$ is a generic small deformation of $S_0$ keeping the classes $[H], [L], [M]$ being $(1,1)$-classes. Thus, such $S$ form a dense subset of a 17 dimensional family, and by the primitivity mentioned above, $\text{NS}(S) = \mathbb{Z}H \oplus \mathbb{Z}L \oplus \mathbb{Z}M$.

The intersection matrix is:

$$
\begin{pmatrix}
(H^2)_{|S|} & (H.L)_{|S|} & (H.M)_{|S|} \\
(L.H)_{|S|} & (L^2)_{|S|} & (L.M)_{|S|} \\
(M.H)_{|S|} & (M.L)_{|S|} & (M^2)_{|S|}
\end{pmatrix} =
\begin{pmatrix}
4 & 1 & 1 \\
1 & -2 & 0 \\
1 & 0 & -2
\end{pmatrix}.
$$

We are going to prove that this $S$ satisfies all the requirements of Theorem (1.7). In what follows, we set:

$$f := H - L, \ e := H - L + M, \ v := -6H + 7L - 3M,$$

$$f' := H - M, \ e' := H - M + L, \ v' := -6H - 3L + 7M,$$

in $\text{NS}(S)$. 


Lemma 3.1. $S$ satisfies:

1. $\text{NS}(S) = \mathbb{Z}f \oplus \mathbb{Z}f \oplus \mathbb{Z}v$.
2. $(e^2)_S = (f^2)_S = 0$, $(e,f)_S = 1$, $(v,f)_S = (v,e)_S = 0$ and $(v^2)_S = -20$.
3. $\text{NS}(S)^*/\text{NS}(S) = \langle v/20 \rangle \cong \mathbb{Z}_{20}$, where $\overline{x} = x \mod \text{NS}(S)$ for $x \in \text{NS}(S)\mathbb{Q}$.

The same are true for $f'$, $e'$, $v'$.

Proof. (2) follows from explicit calculation based on the intersection matrix of $H$, $L$, $M$. We have $\mathbb{Z}\langle e, f, v \rangle \subset \mathbb{Z}\langle H, L, M \rangle$ by definition. Again by explicit calculation, we see that the determinant of the intersection matrix of $H, L, M$ and $e, f, v$ are both $-20$. This implies (1). Since the determinant of NS$(S)$ is $-20$, we have $|\text{NS}(S)^*/\text{NS}(S)| = 20$ by elementary divisor theory. By (2), $v/20 \in \text{NS}(S)^*/\text{NS}(S)$ and it is of order 20. This implies (3). The proof for $f'$, $e'$, $v'$ is identical. \hfill $\square$

Let $*: \text{Aut}(S) \to \text{O}(\text{NS}(S))$ be the natural contravariant group homomorphism.

Lemma 3.2. $S$ satisfies:

1. Let $g \in \text{Aut}(S)$. Then $g^*\sigma_S = \pm \sigma_S$. Moreover $g^*\sigma_S = \sigma_S$ (resp. $g^*\sigma_S = -\sigma_S$) if and only if $g^*|\text{NS}(S)^*/\text{NS}(S) = \text{id}$ (resp. $g^*|\text{NS}(S)^*/\text{NS}(S) = -\text{id}$).
2. The map $*$ is injective.

Proof. Since $S$ is projective and rank $T(S) = 19$ is odd, it follows that $g^*\sigma_S = \pm \sigma_S$ by Nikulin ([Ni80]). Thus $g^*|T(S) = \pm \text{id}$ ([Ni80]). Since there are only two cases above and $T(S)^*/T(S) = \mathbb{Z}_{20}$, it follows that $g^*\sigma_S = \pm \sigma_S$ is equivalent to $g^*|T(S)^*/T(S) = \pm \text{id}$ respectively. This implies (1) via the natural isomorphism explained in Notation. If $g^* = \text{id}$ on NS$(S)$, then $g^*|\text{NS}(S)^*/\text{NS}(S) = \text{id}$. Thus $g^*|T(S) = \text{id}$ by (1). Hence $g^*|H^2(S, \mathbb{Z}) = \text{id}$. The result (2) now follows from the global Torelli theorem for K3 surfaces. \hfill $\square$

Planes $P$ such that $L \subset P \subset \mathbb{P}^3$ form a linear pencil $\{P_t | t \in \mathbb{P}^1\}$. Put $h_t := P_t|S \in |H|$. Then $P_t \cap S = L \cup E_t$ where $E_t \subset P_t \cong \mathbb{P}^2$ is a plane cubic curve such that $(E_t)_S = 0$ and $(E_t, M)_S = 1$. Hence $\{E_t | t \in \mathbb{P}^1\}$ defines an elliptic fibration on $S$ with section $M$: $$\Phi_1 := \Phi_{|H-L|}: S \to \mathbb{P}^1.$$ We regard $M$ as the zero section. We denote by $\text{Aut}(\Phi_1)$ the subgroup of $\text{Aut}(S)$ which preserves the fibration $\Phi_1$, i.e., the set of $g \in \text{Aut}(S)$ for which there is $\overline{g} \in \text{Aut}(\mathbb{P}^1)$ such that $\Phi_1 \circ g = \overline{g} \circ \Phi_1$. Let $\iota_1$ be the inversion of $\Phi_1$ and $\text{MW}(\Phi_1)$ be the Mordell-Weil group of $\Phi_1$, i.e., the group of translations by sections of $\Phi_1$. Then $\text{MW}(\Phi_1) \subset \text{Aut}(\Phi_1)$ and $\iota_1 \in \text{Aut}(\Phi_1)$.

Similarly, we have an elliptic fibration $$\Phi_2 := \Phi_{|H-M|}: S \to \mathbb{P}^1,$$ with section $L$. Regarding $L$ as the zero section of $\Phi_2$, we define $\text{Aut}(\Phi_2)$, $\text{MW}(\Phi_2)$ and the inversion $\iota_2$ of $\Phi_2$ similarly.

Proposition 3.3. $S$ satisfies:

1. $\Phi_1$ has no reducible fiber.
2. The set of global sections of $\Phi_1$ (in $\text{NS}(S)$) is $$\{10n^2f + e + nv | n \in \mathbb{Z}\}.$$
(3) With respect to the basis \( \langle f, e, v \rangle \) of \( \text{NS} (S) \),

\[
\varphi_n^* = \begin{pmatrix} 1 & 10n^2 & 20n \\ 0 & 1 & 0 \\ 0 & n & 1 \end{pmatrix}, \quad \iota_1^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},
\]

where \( \varphi_n^{-1} \in \text{MW} (\Phi_1) \) is the element defined by the translation by \( C_n := 10n^2f + e + nv \). In particular, \( \text{MW} (\Phi_1) = \langle \varphi_1 \rangle \cong \mathbb{Z} \).

The same are true for \( \Phi_2 \) if we replace the basis \( \langle f, e, v \rangle \) of \( \text{NS} (S) \) to the basis \( \langle f', e', v' \rangle \) of \( \text{NS} (S) \).

**Proof.** \( f \) is the class of fiber and \( M \) is the zero section of \( \Phi_1 \). Observe that there is no \( x \in \text{NS} (S) \) such that

\[
(\langle f, f \rangle)_S = 0, \quad (x^2)_S = -2.
\]

In fact, writing \( x = af + be + cv \), where \( a, b, c \in \mathbb{Z} \), and substituting this and \( e = M + f \) into the first two equations, we obtain that \( a = b = 0 \). Hence \( x = cv \). Then the last condition implies \( c = 1/\sqrt{10} \notin \mathbb{Z} \). Thus, there are no \( \mathbb{P}^1 \) in fibers of \( \Phi_1 \) and the result (1) follows.

Let \( C \) be a section of \( \Phi_1 \). Then

\[
C = xf + ye + nv
\]

in \( \text{NS} (S) \) for some \( x, y, n \in \mathbb{Z} \). Since \( f \) is the class of fiber of \( \Phi_1 \), it follows that \( (C, f)_S = 1 \). Hence \( y = 1 \). Substituting into (\( \ast \)), we obtain \( C = xf + e + nv \). Since \( C \cong \mathbb{P}^1 \), it follows that \( (C^2)_S = -2 \), i.e., \( 2x - 2 - 20n^2 = -2 \). Hence \( x = 10n^2 \). Therefore \( C = 10n^2f + e + nv \) for some \( n \in \mathbb{Z} \). Conversely, Let \( c = 10n^2f + e + nv \in \text{NS} (S) \) with \( n \in \mathbb{Z} \). We have \( (c^2)_S = -2 \). Hence either \( c \) or \(-c \) is represented by an effective curve. Since \( (c, f)_S = 1 \), it is \( c \). Set \( [C_0 + A] = c \), where \( C_0 \) and \( A \) are effective curves (possibly 0) such that \( C_0 \) is irreducible and \( (C_0, f)_S \neq 0 \). Since \( (c, f)_S = 1 \) and \( f \) is nef, it follows that \( (C_0, f)_S = 1 \) and \( (A, f)_S = 0 \). Since \( C_0 \) is irreducible, it folows that \( C_0 \) is a section of \( \Phi_1 \). Moreover, \( A \) is in fibers by \( (A, f)_S = 0 \). Since \( \Phi_1 \) has no reducible fiber, it follows that \( A = mf \) in \( \text{NS} (S) \) for some \( m \in \mathbb{Z} \). We calculate that

\[
-2 = (c^2)_S = (C_0^2)_S + 2m(C_0, f)_S = -2 + 2m.
\]

Hence \( m = 0 \). Thus \( A = 0 \) as divisors. Hence \( c = [C_0] \). This proves (2).

Let us show (3). We have \( \iota_1^*f = f \) and \( \iota_1^*e = e \). Hence \( \iota_1^*v = -v \) by \( \iota_1 \neq \text{id}_S \) and Lemma (5.2). We have

\[
\varphi_n^*(f) = f, \quad \varphi_n^*(e) = C_n = 10n^2f + e + nv.
\]

Put \( \varphi_n^*(vf) = xf + ye + zv \) where \( x, y, z \in \mathbb{Z} \). By substituting this into

\[
(\varphi_n^*(f), \varphi_n^*(v))_S = (f, v)_S = 0, \quad (\varphi_n^*(e), \varphi_n^*(v))_S = (e, v)_S = 0, \quad (\varphi_n^*(v^2))_S = (v^2)_S = -20,
\]

we obtain \( \varphi_n^*(v) = \pm(20nf + v) \). Since \( \varphi_n^*\sigma_S = \sigma_S \), it follows that \( \varphi_n^*|_{\text{NS} (S)^*} / \text{NS} (S) = \text{id} \) by Lemma (5.2)(1). Thus \( \varphi_n^*(v) = 20nf + v \). By the explicit form of the matrix and using induction on \( \pm n \), one can see that \( \varphi_n = (\varphi_1)^n \). This proves (3).

The proof for \( \Phi_2 \) is identical. \( \square \)

**Proposition 3.4.** For each \( i = 1, 2 \), we have:

\[
\text{Aut} (\Phi_i) = \text{MW} (\Phi_i) \cdot \langle \iota_i \rangle \cong \mathbb{Z} \cdot \mathbb{Z}_2.
\]
Here and hereafter for groups $A$ and $B$, the group $A \cdot B$ is the semi-direct product in which $A$ is normal.

Proof. We only prove for $i = 1$. Proof for $i = 2$ is identical. We already observed that $\langle \text{MW}(\Phi_1), \iota_1 \rangle \subset \text{Aut}(\Phi_1)$. Let $g \in \text{Aut}(\Phi_1)$. We want to show that $g \in \langle \text{MW}(\Phi_1), \iota_1 \rangle$. We have $g^*(H - L) = H - L$. By composing $\text{MW}(\Phi_1)$, we may and will assume that $g^*M = M$. Hence

$$g^*e = e, \ g^*f = f.$$  

Thus $g^*v = \pm v$. As we observed, $\iota_1^*e = e$, $\iota_1^*f = f$ and $\iota_1^*v = -v$. Thus by composing $\iota_1$ if necessary, we may and will assume that $g^*v = v$. Then $g = \text{id}_S$ by Lemma (3.2)(2). Thus $\text{Aut}(\Phi_1) = \langle \text{MW}(\Phi_1), \iota_1 \rangle$. If $g \in \text{MW}(\Phi_1)$, then $\iota_1 \circ g \circ \iota_1^{-1} \in \text{MW}(\Phi_1)$. In fact, if $g$ is the translation defined by a section $C$, then $\iota_1 \circ g \circ \iota_1^{-1}$ is the translation defined by $\iota_1(C)$. Thus $g \in \text{MW}(\Phi_1) \cdot \langle \iota_1 \rangle$. Since $\iota_1$ is the inversion, we have $\langle \iota_1 \rangle \cong \mathbb{Z}_2$ and $\text{MW}(\Phi_1) \cong \mathbb{Z}$ by Proposition (3.3)(2). This completes the proof.

From now on, we set $\iota_3 := \iota_1 \circ \varphi_1$.

**Corollary 3.5.** With respect to the basis $\langle f, e, v \rangle$ of $\text{NS}(S)$,

$$\iota_3^* = \begin{pmatrix} 1 & 10 & -20 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$  

In particular, $\iota_3$ is also an involution.

**Proof.** Since $\iota_3^* = (\iota_1 \circ \varphi_1)^* = \varphi_1^* \circ \iota_1^*$, the result follows from Proposition (3.3)(3). The fact that $\iota_3$ is an involution also follows from the semi-direct product structure.

**Corollary 3.6.** $\iota_1^*$, $\iota_2^*$ and $\iota_3^*$ are (orthogonal) reflections with respect to the hyperplanes orthogonal to the $-20$-elements, $-6H + 7L - 3M$, $-6H - 3L + 7M$ and $4H - 3L - 3M$ respectively.

**Proof.** The results are clear for $\iota_1, \iota_2$ by their explicit matrix forms. Observe that $\iota_3^*$ is an orthogonal involution with determinant $-1$, being not $-\text{id}$. Hence $\iota_3^*$ is the reflection with respect to the hyperplane orthogonal to a primitive eigenvector, which is unique up to $\pm$, with eigenvalue $-1$ of $\iota_3^*$. Indeed, $4H - 3L - 3M$ is such a vector.

The goal of this section is to prove the following:

**Theorem 3.7.** Let $S \subset \mathbb{P}^3$ be as above. Then:

1. $\text{Aut}(S) = \langle \iota_1, \iota_2, \iota_3 \rangle \cong \mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2$.
2. No element other than $\text{id}_S$ is derived from $\text{Aut}(\mathbb{P}^3)$.
3. Every element of $\text{Aut}(S)$ is derived from $\text{Bir}(\mathbb{P}^3)$.
4. $S$ admits infinitely many different embeddings $S \to \mathbb{P}^3$, up to $\text{Aut}(\mathbb{P}^3)$, and $\text{Aut}(S)$ is derived from $\text{Bir}(\mathbb{P}^3)$ in any embedding $S \to \mathbb{P}^3$.

We prove Theorem (3.7) by deviding into several lemmas.

**Lemma 3.8.** Let $g \in \text{Aut}(S)$. If $g^*H = H$, then $g = \text{id}_S$. In particular, no element of $\text{Aut}(S)$ other than $\text{id}_S$ is derived from $\text{Aut}(\mathbb{P}^3)$. 
Lemma 3.9. The following cone result.

Proof. Consider the following system of equations of $x, y, z \in \mathbb{Z}$:

\[
((xH + yL + zM)^2)_S = -2, \quad ((xH + yL + zM)H)_S = 1.
\]

The solutions are $(x, y, z) = (0, 0, 1)$ and $(0, 1, 0)$. Thus if $g^*H = H$, then either $g^*L = L$ and $g^*M = M$ or $g^*L = M$ and $g^*L = M$. In the first case, $g = \text{id}_S$ by Lemma (3.2)(2).

In the second case, $g^*(-6H + 7L - 3M) = -6H + 7L - 3M$, where $v = -6H + 7L - 3M$. Thus $g^*(v/20) \neq \pm v/20$ in $\text{NS}(S)$, a contradiction to Lemma (3.2)(1). This proves the result. \hfill \square

Lemma 3.10. The set $Q := \{x \in \text{NS}(S) | x \in D, \ (x^2)_S = 4\}$, where $D$ is the cone in Lemma (3.9), is

\[
\{H , \ 2H - L - M , \ 3H - 3L + M , \ 3H - 3M + L \}.
\]

Moreover, among these four elements, only $H$ is very ample.

Proof. Note that $3H - 3L + M = 3f + M$. Thus $|3H - 3L + M| = |3f| + M$. This is because $\dim |3H - 3L + M| = \dim |3f| + M$. By the Reiman-Roch formula, similarly $|3H - 3M + L| = |3(H - M)| + L$. $2H - L - M = (H - L) + (H - M)$ gives a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ given by $\Phi_1 \times \Phi_2$. This proves the last statement. Let $A \in Q$. Since $A \in \text{NS}(S)$, we can write $A = xH + yL + zM$ for some $x, y, z \in \mathbb{Z}$. Since $A \in D$, it follows that $x > 0$. Set

\[
s := \frac{y}{x}, \quad t := \frac{z}{x}.
\]

By symmetry of numerical conditions, we may and will assume that $s \leq t$. Since $A \in D$ with $s \leq t$, it follows that

\[-1 \leq s \leq t \leq \frac{1}{2}, \quad -1 \leq s + t.
\]
Such \((s, t)\) form the quadrangle with four vertices \((1/2, 1/2), (-1, 1/2), (-1, 0), (-1/2, -1/2)\) in the \(st\)-plane.

*Here readers are strongly recommended to draw a picture by oneself.*

On the other hand, by \((A^2)s = 4\), we have
\[
4x^2 - 2y^2 - 2z^2 + 2xy + 2xz = 4.
\]
Substituting \(y = sx\) and \(z = tx\), dividing by \(x^2 \neq 0\) and completing squares, we obtain
\[
(s - \frac{1}{2})^2 + (t - \frac{1}{2})^2 = \frac{5}{2} - \frac{2}{x^2}.
\]
This is the circle with center \((1/2, 1/2)\) in the \(st\)-plane.

Assume that \(x = 1\). Then \(s = y\) and \(t = z\) are integers such that \((y - \frac{1}{2})^2 + (y - \frac{1}{2})^2 = \frac{1}{4}\). Solutions are only \((y, z) = (0, 0)\) and \((1, 1)\). Assume that \(x = 2\). Then \(2s = y\) and \(2t = z\) are integers and satisfy \((y - 1)^2 + (z - 1)^2 = 8\). Partition of 8 into two squares is only \(8 = 4 + 4\). Hence, the solutions with \(s \leq t\) are only \((y, z) = (3, 3), (-1, 3)\) and \((-1, -1)\). But the first two solutions do not satisfy \(t = z/2 \leq 1\). Hence \((y, z) = (-1, -1)\). Assume that \(x = 3\). Then \(3s = y\) and \(3t = z\) are integers and satisfy \((2y - 3)^2 + (2z - 3)^2 = 82\). Here we note that \(2y - 3\) and \(2z - 3\) are odd integers. Partition of 82 into odd squares is only \(82 = 1^2 + 9^2\) up to the order. By taking into account that \(-1 \leq s \leq t \leq 1/2\), i.e., \(-3 \leq y \leq z \leq 3/2\), one can see that the solutions are only \((y, z) = (-3, 1)\). These four solutions give the four elements in the statement.

Assume that \(x \geq 4\). One can compute the intersection points of the circle above with the boundary \(s + t = -1\) of the quadrangle above. In fact, substituting \(t = -1 - s\) into the equation of the circle above and simplifying it, we obtain
\[
s^2 + s = -\frac{1}{x^2}.
\]
Solving this on \(s\) in the range \(-1 \leq s \leq t \leq 1/2\) by using the root formula of quadratic equation, we find that the value of the \(s\)-coordinate at the intersection point is:
\[
-\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{x^2}}.
\]
From this calculation with \(-1 \leq s\), we obtain
\[
-1 \leq s \leq -\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{x^2}}.
\]
If \(s = -1\), then by substituting this into the equation of the circle above and solving it on \(t\) by using the root formula of quadratic equation, we obtain
\[
t = \frac{1}{2} \pm \frac{\sqrt{x^2 - 8}}{2x}.
\]
Since \(t\) is a rational number and \(x\) is an integer, it follows that \(x^2 - 8\) is a square of some integer, i.e., \(x^2 - 4 = a^2\) for some non-negative integer \(a\). This equation is equivalent to \((x - a)(x + a) = 8\). Here \(x + a \geq 4\). Hence \(x + a = 4\) or \(8\) and therefore \(x - a = 2\) or \(1\) respectively. The solutions are \((x, a) = (3, 1)\) and \((9/2, 1/2)\). Since \(x \geq 4\), \(x = 9/2\). But
this is not an integer, a contradiction. Hence $s \neq -1$, i.e., $s > -1$. Since $s$ is a rational number whose denominator divides $x$, it follows from the inequality above for $s$ that
\[ -\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{x^2}} < -1, \]
that is,
\[ \frac{1}{2} - \frac{1}{x} \geq \sqrt{\frac{1}{4} - \frac{1}{x^2}}. \]
In particular, the left hand side is also non-negative. Thus
\[ \frac{1}{2} - \frac{1}{x} \geq \sqrt{\frac{1}{4} - \frac{1}{x^2}}. \]
Expanding the right hand side and transform the right hand side to the left, we obtain:
\[ \frac{2}{x^2} - \frac{1}{x} \geq 0. \]
Multiplying both sides by $x^2 > 0$, we obtain $2 - x \geq 0$. Hence $x \leq 2$. However, this contradicts $x \geq 4$. Hence $x \leq 3$. This completes the proof. \qed

Lemma 3.11. $\text{Aut}(S) = \langle \iota_1, \iota_2, \iota_3 \rangle$.

Proof. We have $\langle \iota_1, \iota_2, \iota_3 \rangle \subset \text{Aut}(S)$. Let $g \in \text{Aut}(S)$. Let $H' := g^*H$. Then $H' \in \text{Amp}(S)$. Hence there is $\varphi \in \langle \iota_1, \iota_2, \iota_3 \rangle$ such that $H'' := \varphi^*H' \in D$ by Lemma (3.10). Since $H$ is in NS$(S)$, very ample and $(H^2)_S = 4$, so is $H''$. Hence $H'' = H$ by Lemma (3.10). That is, $(g \circ \varphi)^*H = H$. Thus $g \circ \varphi = id_S$ by Lemma (3.8). Hence $g \in \langle \iota_1, \iota_2, \iota_3 \rangle$. \qed

Lemma 3.12. $\langle \iota_1, \iota_2, \iota_3 \rangle = \langle \iota_1 \rangle * \langle \iota_2 \rangle * \langle \iota_3 \rangle \simeq \mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2$.

Proof. The positive cone $P(S)$ is divided into the four open domains by the three invariant hyperplanes $L_1, L_2, L_3$ corresponding the reflections $\iota_1, \iota_2, \iota_3$ respectively as in the proof of Lemma (3.9).

Here readers are again strongly recommended to draw a picture by oneselfs.

Let $S_1$ be the one of the two domains of $P(S)$ divided by $L_1$, being in the opposite side of $H$. Similarly we define $S_2$ and $S_3$. Then the intersection of any two of $S_1, S_2, S_3$ is empty. Moreover $i_1^*(S_j) \subset S_i$ whenever $j \neq i$ and $i_1^*(H) \in S_i$. Hence, the result follows. Indeed,
\[ i_{k_1}^*i_{k_2}^* \cdots i_{k_m}^* \neq id \]
for any $k_1 \neq k_2 \neq \cdots \neq k_m$ such that $k_i \in \{1, 2, 3\}$. This is because the image of $H$ by the left hand side is in $S_{k_1}$ but $id(H) = H$ by the right hand side is not. This proves the result. \qed

Lemma 3.13. Every element of $\text{Aut}(S)$ is derived from $\text{Bir}(\mathbb{P}^3)$.

Proof. $\Phi_1$ is the restriction of the linear projection
\[ \Phi_{[P-L]} : \mathbb{P}^3 \cdots \rightarrow \mathbb{P}^1 \]
from $L$. Let $\eta = \text{Spec} \mathbb{C}(\mathbb{P}^1)$. The fiber $\mathbb{P}^2_\eta$ of $\Phi_{[P-L]}$ over $\eta$ is the projective plane defined over $\mathbb{C}(\mathbb{P}^1)$ and the generic fiber $S_\eta$ of $\Phi_1 = \Phi_{[H-L]}$ is a smooth cubic curve in $\mathbb{P}^2_\eta$ with $O \in S_\eta(\mathbb{C}(\mathbb{P}^1))$, corresponding to the section $M$, defined over $\mathbb{C}(\mathbb{P}^1)$. Thus, $\text{MW}(\Phi_1)$ and $\iota_1$ are automorphisms of the smooth cubic curve $S_\eta$ over $\mathbb{C}(\mathbb{P}^1)$. Therefore by Theorem
MW (Φ₁) and ℓ₁ are derived from Bir (P^2) over C(t). Since Bir (P^2) is a subgroup of Bir (P^3), the result follows. The same is true for the inversion ℓ₂ of Φ₂. This proves the result.

Lemma 3.14. S admits infinitely many different embeddings S → P^3 up to Aut (P^3). Moreover, Aut (S) is derived from Bir (P^3) in any embedding Φ : S → P^3.

Proof. Since Aut (S) is an infinite group by Lemma (3.12), it follows that Aut (S)∗ H is an infinite set. Indeed otherwise, the stabilizer subgroup of H would be an infinite group, a contradiction to Proposition (2.3)(2). This proves the first assertion. Let A be a very ample line bundle on S such that (A²)S = 4. Then, by Lemmas (3.9), (3.10), there is g ∈ Aut (S) such that g∗ A = H. Hence the last assertion follows from Proposition (2.3)(3).

This completes the proof of Theorem (3.7). Theorem (1.7) follows from Theorem (3.7).

4. Proof of Theorem (1.8).

Let ℓ be an integer such that ℓ > 5 and S_ℓ be a generic K3 surface such that NS (S_ℓ) = L where

\[ L := Z h₁ + Z h₂ , \ (h_i, h_j)_{S_ℓ} = \begin{pmatrix} 4 & 4 \ell \\ 4 \ell & 4 \end{pmatrix} . \]

Here generic means that g∗ \sigma_S = ± \sigma_S for all g ∈ Aut (S). Since the lattice L is even of signature (1, 1), such K3 surfaces S_ℓ exist ([Mil81 Corollary 2.9]), are projective and form dense subset, in the classical topology, of the 18-dimensional family of the L-polarized K3 surfaces. Here we note that non-generic ones are in the countable union of hyperplanes in the period domain of L-polarized K3 surfaces ([Ni80 Theorem 3.1]).

In this section, we prove the following:

Theorem 4.1. Let S := S_ℓ be as above. Then:

(1) Aut (S) ≃ Z.

(2) The set Q of embeddings Φ : S → P^3, up to Aut (P^3), is an infinite set.

(3) For any (Φ : S → P^3) ∈ Q and for any g ∈ Aut (S) \{ id_S \}, there is no ĝ ∈ Bir (P^3) such that g = ĝ|S with respect to the embedding Φ.

By replacing (h₁, h₂) by (−h₁, −h₂) if necessary, we may and will assume that h₁ is in the positive cone P(S).

Lemma 4.2. S satisfies:

(1) NS (S) represents neither 0 nor ±2, i.e., (d²)S ≠ 0, ±2 for d ∈ NS (S).

(2)

\[ \text{NS (S)}^*/\text{NS (S)} = \left( \frac{h₁}{4}, \frac{h₂ - ℓh₁}{4(ℓ² - 1)} \right) \approx Z_4 ⊕ Z_4(ℓ² - 1) . \]

Proof. For xh₁ + yh₂ ∈ NS (S)R, observe that

\[ ((xh₁ + yh₂)^2)_S = 4x² + 8ℓxy + 4y² = 4((x + ℓy)^2 - (ℓ² - 1)y^2) . \]

The first equality shows that (d²)S is divisible by 4 for any d ∈ NS (S). Hence NS (S) does not represent ±2. Since √(ℓ² - 1) is irrational by ℓ ≥ 2, it follows that (d²)S ≠ 0 for any d ∈ NS (S). This proves (1). The assertion (2) follows from an explicit computation based on
the elementary divisor theorem. Indeed, by performing elementary transformations for the matrix \(((h_i, h_j)_S)\) in \(\mathbb{Z}\), one can transform \(((h_i, h_j)_S)\) to the diagonal matrix \(\text{diag}(4, 4(\ell^2 - 1))\) such that \(4|4(\ell^2 - 1)\). This process also identifies the generators of \(\text{NS}(S)^*/\text{NS}(S)\) as described.

In what follows, we set:

\[
v_1 := (-\ell + \sqrt{\ell^2 - 1})h_1 + h_2, \quad v_2 := h_1 + (-\ell + \sqrt{\ell^2 - 1})h_2.
\]

Note that \(v_1\) and \(v_2\) are both irrational.

**Lemma 4.3.**

\[
\overline{\text{Amp}}(S) = \mathcal{P}(S) = R_{\geq 0}v_1 + R_{\geq 0}v_2.
\]

**Proof.** Since \(h_1\) is in the positive cone, the second equality follows from the second equality in the proof of Lemma (4.2)(1). Since \(\text{NS}(S)\) does not represent \(-2\) by Lemma (4.2)(1), \(S\) contains no \(\mathbf{P}^1\). This implies the first equality. \(\square\)

**Lemma 4.4.** \(S\) satisfies:

1. Let \(g \in \text{Aut}(S)\) such that \(g^*|\text{NS}(S)\) is of finite order. Then \(g\) is of finite order.
2. \(\text{Aut}(S)\) has no element of finite order other than id\(_S\).
3. For each \(g \in \text{Aut}(S)\), there are positive real numbers \(\alpha(g)\), \(\beta(g)\) such that \(g^*v_1 = \alpha(g)v_1\) and \(g^*v_2 = \beta(g)v_2\).
4. \(\text{Aut}(S) = \langle g_0 \rangle \simeq \mathbb{Z}\).

**Proof.** Let \(g \in \text{Aut}(S)\). Then either \(g^*v_1 = \alpha v_1\) and \(g^*v_2 = \beta v_2\) or \(g^*v_1 = \alpha v_2\) and \(g^*v_2 = \beta v_1\) for some real positive numbers \(\alpha\) and \(\beta\).

The assertion (1) follows from \(g^*|T(S) = \pm id\) and the global Torelli theorem for K3 surfaces.

Let us prove (2). Assume that \(g\) is of finite order. Then, in the first case, \(\alpha = \beta = 1\), whence \(g^*|\text{NS}(S) = id\). Then \(g^*|\text{NS}(S)^*/\text{NS}(S) = id\). On the other hand, \(g^*|T(S) = \pm id\), by our genericity assumption. It follows that \(g^*|T(S) = id\). Hence \(g = id\_S\) by the global Torelli theorem for K3 surfaces. In the second case, \((g^2)^*v_1 = \alpha^2 v_1\) and \((g^2)^*v_2 = \beta v_2\). Hence \(g^2 = id\) as we have shown. Assume to the contrary that \(g \neq id\_S\). If \(g^*\sigma_S = \sigma_S\), then \(g\) would have exactly 8 fixed points (\[\text{NS}\] Section 5)). Thus \(\text{tr}(g^*|\text{NS}(S)) = -14\) by the Lefschetz fixed point formula. However, this is impossible, because rank \(\text{NS}(S) = 2\) and \(g^*|\text{NS}(S)\) is of order 2, so that the eigenvalues of \(g^*|\text{NS}(S)\) is in \(\{\pm 1\}\) with multiplicity at most 2. Consider the case \(g^*\sigma_S = -\sigma_S\). Since \(g\) is of finite order, there is an ample class \(h\) such that \(g^*h = h\). We can choose \(h\) to be primitive. Then by \(\langle h \rangle^*/\langle h \rangle \simeq \langle (h^2)^* \rangle^*/\langle (h^2) \rangle^*\) and our case assumption, we would have \(id = -id\) on \(\langle h \rangle^*/\langle h \rangle\). Hence \((h^2)_S = 2\), a contradiction to Lemma (4.2)(1). This proves the assertion (2).

Let us show (3). Assume to the contrary that \(g^*v_1 = \alpha v_2\) and \(g^*v_2 = \beta v_1\). Then, \(g \neq id\_S\) and the characteristic polynomial of \(g^*|\text{NS}(S)\) would be \(t^2 - \alpha \beta \in \mathbb{Z}[t]\). Since \(g^*|\text{NS}(S) \in O(\text{NS}(S))\) and \(\alpha, \beta > 0\), it would follow that \(\alpha \beta = 1\). Hence \(g^*|\text{NS}(S)\) would of finite order, whence so would be \(g\) by (1), a contradiction to (2).

Let us show (4). By Stek (\[\text{S185}\] Section 2), the action \(\text{Aut}(S)^*\) on \(\overline{\text{Amp}^f}(S)\) has a finite rational polyhedral fundamental domain. Since \(\overline{\text{Amp}}(S)\) is irrational, it follows that
$|\text{Aut}(S)| = \infty$. Let $R_{>0}$ be the multiplicative group of positive real numbers. By (3), we have a well-defined group homomorphism

$$\alpha : \text{Aut}(S) \rightarrow R_{>0}, \ g \mapsto \alpha(g),$$

where $g^*v_1 = \alpha(g)v_1$. If $\alpha(g) = 1$, then $\beta(g) = 1$ by $g^*|\text{NS}(S) \in \text{O}(\text{NS}(S))$ and $\alpha, \beta > 0$. Then $g^*|\text{NS}(S) = id$, whence $g = ids$ by (1) and (2). Thus $\alpha$ is injective. Let

$$S := \{\alpha(g) \mid g \in \text{Aut}(S), \alpha(g) > 1\}.$$ 

Since $|\text{Aut}(S)| = \infty$, $\alpha$ is injective, and since $\alpha(g^{-1}) > 1$ if $\alpha(g) < 1$, it follows that $S$ is not empty. Note that $\alpha(g)$ and $\beta(g)$ are zeros of quadratic equation of the form $t^2 - at + 1 = 0$ with $a \in \mathbb{Z}$. By the root formula of quadratic equation, we have $\alpha_{a_1} < \alpha_{a_2}$. Here $\alpha_{a_i}$ is the largest solution of $t^2 - a_it + 1 = 0$ and $a_1$ and $a_2$ are integers such that $3 \leq a_1 < a_2$. It follows that the set $S$ has the minimum, say $\alpha_0 = \alpha(g_0) > 1$. Then for all $h \in \text{Aut}(S)$, we have $h = g_0^m$ for some $m \in \mathbb{Z}$. In fact, we can take an integer $n$ such that $1 \leq \alpha(h)\alpha(g_0)^{-n} < \alpha_0$. Then $\alpha(hg_0^{-n}) = 1$ by definition of $\alpha_0$. Since $\alpha$ is injective, it follows that $h = g_0^m$. Hence $\text{Aut}(S) = \langle g_0 \rangle \simeq \mathbb{Z}$ as claimed. \hfill \ensuremath{\square}

In what follows,

$$\mathcal{H} := \{h \in \text{Ampl}(S) \cap \text{NS}(S) \mid (h^2)_S = 4\},$$

and $g_0$ is a generator of $\text{Aut}(S)$ as in Lemma (4.4) (4).

**Lemma 4.5.** $S$ satisfies:

1. Any $h \in \mathcal{H}$ is very ample.
2. $|\mathcal{H}| = \infty$.

*Proof.* Since $S$ contains no $\mathbb{P}^1$, the complete linear system $|h|$ is free ([SD74, 2.7]). Since there is no $d \in \text{NS}(S)$ with $(d^2)_S \in \{0, \pm -2\}$ by Lemma (4.2) (1), it follows that $\Phi|_d$ is an embedding ([SD74, Theorem 5.2]). This proves (1). Note that $h_1 \in \mathcal{H}$ and $\mathcal{H} \neq \emptyset$. Assume to the contrary that $|\mathcal{H}| < \infty$. Then $\{(g_0^n)^*h_1 \mid n \in \mathbb{Z}\}$ would be a finite set as well. Then there would be $m \in \mathbb{Z} \setminus \{0\}$ such that $(g_0^m)^*h_1 = h_1$. Since $h_1$ is ample, $g_0^m$ would be of finite order by Proposition (2.3) (2), a contradiction to $\langle g_0 \rangle \simeq \mathbb{Z}$ (Lemma (4.4) (4)). Hence $|\mathcal{H}| = \infty$. \hfill \ensuremath{\square}

**Lemma 4.6.** Let $h \in \mathcal{H}$ and $\Phi : S \rightarrow \mathbb{P}^3$ be the embedding defined by the complete linear system $|h|$. We regard $S \subset \mathbb{P}^3$ by this $\Phi$. Let $H$ be the hyperplane class of $\mathbb{P}^3$ and $C$ be an effective curve on $S$ such that $(C.H)_{\mathbb{P}^3} < 16$. Then, there is a hypersurface $T$ in $\mathbb{P}^3$ such that $C = T|S$.

*Proof.* Assuming to the contrary that there would be an effective curve $C \subset S$ such that the class $c := [C]$ is linearly independent to $h = H|S$ in $\text{NS}(S)$, we shall derive a contradiction. Then $N := \langle c, h \rangle$ would be a sublattice of $\text{NS}(S)$ of the same rank 2. In particular, the signature would be of (1, 1). Then

$$|N| := \det \left( \begin{array}{cc} (c^2)_S & (c.h)_S \\ (c.h)_S & (h^2)_S \end{array} \right) = (c.h)^2_S - (c^2)_S \cdot (h^2)_S > 0,$$

and $|N|$ would be divided by $|\text{NS}(S)| := |\det(h_1, h_2)|$. Since $c = [C]$ is an effective class and $C \neq \mathbb{P}^1$ by Lemma (4.2) (1), we have $(c^2)_S \geq 0$. Hence

$$|N| \leq (c.h)^2_S < 16^2.$$
On the other hand, by $\ell > 5$, 
$$|\text{NS}(S)| = 16\ell^2 - 16 > 16 \cdot 17 - 16 = 16^2,$$
and therefore $|N|$ could not be divided by $|\text{NS}(S)|$, a contradiction. \(\square\)

**Lemma 4.7.** Let $h \in \mathcal{H}$ and $\Phi : S \to \mathbb{P}^3$ be the embedding defined by the complete linear system $|h|$. We regard $S \subset \mathbb{P}^3$ by this $\Phi$. Then, no element of $\text{Aut}(S) \setminus \{\text{id}_S\}$ is derived from $\text{Bir} (\mathbb{P}^3)$.

**Proof.** Assuming to the contrary that there would be $g \in \text{Aut}(S) \setminus \{\text{id}_S\}$ such that $g = \tilde{g}|S$ for some $\tilde{g} \in \text{Bir} (\mathbb{P}^3)$, we shall derive a contradiction.

If $\tilde{g} \in \text{Aut} (\mathbb{P}^3)$, then $g^* h = h$. Since $h$ is ample, $g$ would be of finite order by Proposition (2.3) (2), a contradiction to Lemma (4.4) (4).

If $\tilde{g} \in \text{Bir} (\mathbb{P}^3) \setminus \text{Aut} (\mathbb{P}^3)$, then by Lemma (4.6), one can apply Theorem (1.9) for $\tilde{g}$. However, then $g(S) = g_\ast S \neq S$, a contradiction. \(\square\)

This completes the proof of Theorem (4.1). Theorem (1.8) follows from Theorem (4.1).

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