On the Origins of the Planck Zero Point Energy in Relativistic Quantum Field Theory

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It is argued that the zero point energy in quantum field theory is a reflection of the particle anti-particle content of the theory. This essential physical content is somewhat disguised in electromagnetic theory wherein the photon is its own anti-particle. To illustrate this point, we consider the case of a charged Boson theory \((\pi^+, \pi^-)\) wherein the particle and anti-particle can be distinguished by the charge \(\pm e\). Starting from the zero point energy, we derive the Boson pair production rate per unit time per unit volume from the vacuum in a uniform external electric field. The result is further generalized for arbitrary spin \(s\).

I. INTRODUCTION

The zero point energy in Boson quantum field theories are a consequence of the Boson and anti-Boson content of the theory. This physical conclusion is masked in electromagnetic theory because the photon is its own anti-particle. If one considers a charged Boson theory, say \((\pi^+, \pi^-)\), the particle and anti-particle can be distinguished in virtue of the electric charge. From the zero point energy, one may derive the known Boson pair production rate per unit time per unit volume from the vacuum in a uniform external electric field. The external electric field does the work required to excite the oscillators.

In Sec.III it is argued that the particle anti-particle content of the original Planck photon number distribution can be employed to explain the zero point energy level \(\hbar \omega /2\). Relating the zero point energy to the particle content of field theory is not so evident for the case of photons because the photon and the anti-photon are the same particle. In Sec. III we consider spin zero Bosons that are charged, e.g. the \((\pi^+, \pi^-)\) system wherein the particle has charge \(+e\) and the anti-particle has charge \(-e\). The application of a uniform electric field excites the charged Boson oscillators ripping \((\pi^+, \pi^-)\) pairs out of the vacuum. The external electric fields can cause the dielectric breakdown of the Boson vacuum. The rate of Boson pair production per unit time per unit volume is also computed here. The case of Fermion pair production is considered in Sec. IV together with a general formula for production of charged particle-antiparticle pairs with arbitrary spin. Concluding remarks are given in Sec. V.

II. PLANCK DISTRIBUTIONS

Planck originally \[1\] discussed the mean number of photons of frequency \(\omega\) in the thermal vacuum

\[\bar{n} = \left[\frac{1}{e^{\hbar \omega /k_B T} - 1}\right]. \quad (1)\]

The mean thermal energy of an electromagnetic oscillator was thereby taken to be

\[\bar{E}(\omega) = \hbar \omega \bar{n} = \frac{\hbar \omega}{e^{\hbar \omega /k_B T} - 1}. \quad (2)\]

Later\[2\] Planck somewhat arbitrarily added the zero-point energy via a second theory, now assuming that absorption should be treated classically, but emission by discrete quanta - a procedure which might seem odd by today’s standards, but this was at the very beginning of quantum theory. Einstein and Stern\[3\] noted that the excess energy over and above the equipartition value obeyed

\[\lim_{T \to \infty} \left[\bar{E}(\omega) + \frac{\hbar \omega}{2} - k_B T\right] = 0 \quad (3)\]

that might theoretically be regarded as slight evidence of a zero temperature energy of \(\hbar \omega /2\). A nice discussion of zero point energy in early quantum theory can be found in the book by Milonni\[4\].

On the other hand, by actually solving the quantum mechanical simple harmonic oscillator one obtains a zero point energy term

\[E_T(\omega) = \hbar \omega \left[\bar{n} + \frac{1}{2}\right] = \left(\frac{\hbar \omega}{2}\right) \coth\left(\frac{\hbar \omega}{2k_B T}\right). \quad (4)\]

Let us note the following

**Theorem:**

\[E_T(\omega) = \frac{1}{2} \left[\bar{E}(\omega) + \bar{E}(\omega)\right]. \quad (5)\]

**Proof:** Eq. (5) is true in virtue of Eqs. (2) and (1). Also note the zero point energy

\[E_0(\omega) \equiv \lim_{T \to 0^+} E_T(\omega) = \left(\frac{\hbar |\omega|}{2}\right). \quad (6)\]

If one expresses this result in terms of the photon creation operator \(a^\dagger\) and destruction operator \(a\) with \([a, a^\dagger] = 1,\)
then the photon number operator \( n = a^\dagger a \) enters into the Hamiltonian via the symmetrized product \((aa^\dagger + a^\dagger a)\) as

\[
\mathcal{H} = \frac{\hbar |\omega|}{2} (a^\dagger a + aa^\dagger) = \hbar |\omega| \left( n + \frac{1}{2} \right).
\]

Eq. (7) leads directly to Eq. (9).

Let us consider the physical meaning of Eq. (5). The positive frequency \( \omega > 0 \) represents a particle moving forward in time, while the negative frequency \( \omega < 0 \) represents an anti-particle moving backward in time. Since the photon is its own anti-particle, the physical meaning of Eqs. (5) and (6) may be somewhat obscured. In order to make the particle content in the zero point energy more evident, let us consider a case wherein the particle and anti-particle are distinct.

Interestingly, the simple argument that both positive and negative signs for \( \omega \) should be treated symmetrically would have been available to Planck, and many others afterwards to obtain the zero point energy from Planck’s first formula by simply symmetrizing it to reflect this, but this seems to have been missed.

### III. CHARGED SPINLESS BOSONS

The energy of a spinless charged scalar Boson in a uniform magnetic field \( B = (0, 0, B) = (0, 0, |B|) \) is given by\[\[8\]
\[
\epsilon_\pm(n, p, B) = \pm c \sqrt{m^2 c^2 + p^2 + (2n + 1)|\hbar eB|/c} ,
\]

wherein \( n \) is the label for the circular Landau orbit, the momentum along the magnetic field axis is \( p = \hbar k \) and \( \kappa = (mc/\hbar) \) is the mass in inverse length units. Thus

\[
\omega(n, k, B) = c \sqrt{\kappa^2 + k^2 + (2n + 1)|eB/\hbar c|} .
\]

The zero point charged Boson oscillator energies per unit volume counting the particle and anti-particle separately in virtue of the different charge \( \pm e \) is determined by

\[
U_0(B) = 2 \times \left[ \left( \frac{eB}{2\pi \hbar c} \right) \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left[ \frac{\hbar \omega(n, k, B)}{2} \right] \right]
\]

(10)

This vacuum energy per unit volume in a magnetic field is clearly divergent so one must regulate and renormalize. After doing both processes, a finite vacuum Boson energy per unit volume in a magnetic field \( U(B) \) arises.

#### A. Gamma Function Regulation

The Gamma function is defined in the \( \Re\{z\} > 0 \) part of the complex plane as

\[
\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} \left( \frac{dt}{t} \right) ,
\]

(11) from which we find for \( a > 0 \) the identity

\[
a^{-z} = \frac{1}{\Gamma(z)} \int_0^{\infty} s^{z-1} e^{-as} \left( \frac{ds}{s} \right) .
\]

(12)

In other regimes, \( \Gamma(z) \) is defined by analytic continuation. This analysis is often assisted by the identity

\[
\Gamma(1 + z) = z\Gamma(z).
\]

(13)

For example, by putting \( z = -(1/2) \) in Eq. (13), one finds

\[
\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \Rightarrow \Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi} .
\]

(14)

Eqs. (12) and (14) lead to a formally divergent integral

\[
\sqrt{a} - \sqrt{b} = \left( \frac{1}{2\sqrt{\pi}} \right) \int_0^{\infty} \left[ e^{-bs} - e^{-as} \right] \left( \frac{ds}{s^{3/2}} \right) .
\]

(15)

For those readers who find it strange in Eq. (15) to put a finite positive quantity equal to a negative infinite quantity, we invite the reader to prove the following Theorem: For any \( a > 0 \) and \( b > 0 \)

\[
\sqrt{a} - \sqrt{b} = \left( \frac{1}{2\sqrt{\pi}} \right) \int_0^{\infty} \left[ e^{-bs} - e^{-as} \right] \left( \frac{ds}{s^{3/2}} \right) .
\]

(16)

Subtractions will made below. Eqs. (10) and (15) yield

\[
U_0(B) = -\left( \frac{eB}{8\pi \hbar c} \right) \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left( \frac{ds}{s^{3/2}} \right) \times \exp\left[ -\kappa^2 s - k^2 s - (2n + 1) \frac{|eB|}{\hbar c} \frac{1}{s} \right] ,
\]

\[
U_0(B) = -\left( \frac{\hbar c}{16\pi^2} \right) \int_0^{\infty} \frac{ds}{s^{3/2}} e^{-\kappa^2 s} \left[ \frac{(eB/s\hbar c)}{\sinh(eB/s\hbar c)} \right] ,
\]

(17)

that is still divergent. Now let us subtract the vacuum zero point oscillations when the magnetic field is zero,

\[
\tilde{U}(B) = U_0(B) - U_0(0) .
\]

(18)

The zero point oscillation energy per unit volume due to vacuum particle anti-particle pairs, say \((\pi^+, \pi^-)\), virtual magnetic moments are thereby

\[
\tilde{U}(B) = \left( \frac{\hbar c}{16\pi^2} \right) \times \int_0^{\infty} \frac{ds}{s^{3/2}} e^{-\kappa^2 s} \left[ 1 - \left( \frac{eB/s\hbar c}{\sinh(eB/s\hbar c)} \right) \right] ,
\]

(19)

which still is divergent but only as a logarithm at small distance squared as \( s \to 0^+ \). Once the divergences are only logarithmic one may pass from regularization to renormalization.
B. Charge Renormalization

In quantum electrodynamics one starts with charges and fields described for the problem at hand by $e_0$ and $B_0$. Both the fields and charges have to be renormalized in such a way that $e_0B_0 = eB$ and thus divergent logarithms are buried. The physical fields are defined so that the normal vacuum magnetic energy density is $|B|^2 / 8\pi$. This can all be realized by a charge renormalization subtraction in Eq.(19). Thus, for scalar Boson fields the vacuum energy density is given by

$$U(B) = \left( \frac{\hbar c}{16\pi^2} \right) \int_0^\infty \frac{ds}{s^3} e^{-s^2} \times \left[ 1 - \frac{(eBs/\hbar c)}{\sinh(eBs/\hbar c)} - \frac{(eBs/\hbar c)^2}{6} \right].$$  (20)

Eq.(20) is both finite and exact for the sum of zero point oscillations of charged Boson spin zero systems. All divergences have been buried. The vacuum Boson magnetization is thereby

$$M = - \left( \frac{\partial U}{\partial B} \right).$$  (21)

Let us now consider what happens in an external electric field.

C. Photon Fields

The electromagnetic fields associated with a photon obey the vacuum Maxwell equations

$$\text{div} E = 0, \quad \text{div} B = 0,$$
$$\text{curl} E = -\frac{1}{c} \left( \frac{\partial B}{\partial t} \right), \quad \text{curl} B = \frac{1}{c} \left( \frac{\partial E}{\partial t} \right).$$  (22)

In terms of the complex vector field

$$F = E + iB,$$
$$\text{div} F = 0, \quad i \left( \frac{\partial F}{\partial t} \right) = c \text{curl} F,$$  (23)

Eq.(23) describes a space-time Schrödinger equation for a photon. The square of the vector wave function,

$$F \cdot F = |E|^2 - |B|^2 + 2iE \cdot B,$$  (24)

determines the Lorentz scalar $|E|^2 - |B|^2$ and Lorentz pseudo-scalar $E \cdot B$. To go from a pure external magnetic field to a pure external electric field one takes $B^2 \rightarrow -E^2$. This allows us to obtain the Boson pair production rate $\Gamma$ per unit time per unit volume.

D. Boson Pair Production

The transition rate per unit time per unit volume in an external electric field may be computed from

$$\Gamma = - \left( \frac{2}{\hbar} \right) 3m U(B \rightarrow -iE).$$  (25)

Eqs.(20) and (25) imply

$$\Gamma = \frac{c}{8\pi^3} \left( \frac{eE}{\hbar c} \right)^2 \times \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^2} \exp \left( -\pi n \frac{m^2c^3}{\hbar E} \right).$$  (26)

A uniform electric field can thereby excite the charged Boson oscillators emitting pairs ($\pi^+, \pi^-$) from the vacuum. The electric field does the work required to break down the vacuum.

IV. CHARGED PARTICLE PATHS

Let us here consider how the zero point energy is expressed in terms of paths forward in time (particle) and backward in time (anti-particle). If a particle over here can go forward in time and then backward in time to over there, then right now the quantum amplitude is partly over here and partly over there which evidently spreads out the zero point ground state wave function. Let us turn the above physical picture into a computation of the charged pair production rate in Eq.(20). Let us at first work in one space and one time (1+1) dimensions. With a small modification, this leads to a correct description in physical three space and one time (3+1) dimensions.

In (1+1) dimensions, the energy-momentum relation reads

$$E^2 - c^2p^2 = (mc^2)^2.$$  (27)

Since energy is force times distance and momentum is force times time, Eq.(27) reads

$$(eEx)^2 - c^2(eEt)^2 = (mc^2)^2,$$  (28)

or in terms of the particle acceleration $a$,

$$a = (eE/m).$$  (29)

Eq.(28) reads

$$x^2 - c^2t^2 = (c^2/a)^2,$$  (30)

that describes classical paths. The particle path forward in time is

$$x_+(t) = \sqrt{c^2t^2 + (c^2/a)^2},$$  (31)

while the anti-particle path backward in time is

$$x_-(t) = -\sqrt{c^2t^2 + (c^2/a)^2}.$$  (32)

Pair production at time zero requires a space-like transition from $x_-(0) = -(c^2/a)$ to $x_+(0) = (c^2/a)$ along the semicircle in Euclidean time $t_\xi$, i.e. Eq.(30) reads in Euclidean time

$$x^2 + c^2t_\xi^2 = (c^2/a)^2.$$  (33)
The arc length of the semicircle is \( s = \pi (c^2 / a) \) giving rise to the Euclidean action
\[
W = mcs = \pi \left( \frac{mc^3}{a} \right) = \pi \left( \frac{m^2 c^3}{eE} \right). \tag{34}
\]
The Boson weight of such pair production processes summed over the number \( k \) of pairs produced is related to the partition function
\[
Z = \sum_{k=0}^{\infty} (-1)^k e^{-kW/k} = \left[ \frac{1}{1 + e^{-W/k}} \right]. \tag{35}
\]

The factor of \(-1\) for each semicircle means a Bose factor of one for each circle. Since the rate of change of momentum is equal to the force, \( dp/dt = eE \), the transition rate per unit time per unit length \( \Gamma_1 \) is given by
\[
\Gamma_1 dt = \left( \frac{dp}{2\pi\hbar} \right) [-\ln Z],
\]
\[
\Gamma_1 = \left( \frac{eE}{2\pi\hbar} \right) \ln \left[ 1 + e^{-W/k} \right],
\]
\[
\Gamma_1 = \left( \frac{eE}{2\pi\hbar} \right) \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-n(\pi m^2 c^3/\hbar eE)}, \tag{36}
\]

By taking the momentum perpendicular to the electric field into account, the (3+1) dimensional result follows from Eq.\(36\),
\[
\Gamma = \left( \frac{eE}{2\pi\hbar} \right) \int \left[ \frac{d^3p}{(2\pi\hbar)^2} \right] \times
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \exp \left( -\pi n \left( \frac{m^2 c^3 + cp^2}{\hbar eE} \right) \right),
\]
\[
\Gamma = \frac{c}{8\pi^3} \left( \frac{eE}{\hbar c} \right)^2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \exp \left( -\pi n \left( \frac{n^2 m^2 c^3}{\hbar eE} \right) \right), \tag{37}
\]
in agreement with Eq.\(26\). It is not difficult to write the transition rate for producing pairs wherein the charged particles have spin \( s \), i.e.

\[
s = 0, 1, 2, 3, \cdots \quad \text{(Bosons)},
\]
\[
s = 1/2, 3/2, 5/2, \cdots \quad \text{(Fermions)}. \tag{38}
\]

The statistical index may be defined as
\[
\eta_s = \exp \left( i\pi (2s + 1) \right),
\]
\[
\eta_s = -1 \quad \text{(Bosons)},
\]
\[
\eta_s = +1 \quad \text{(Fermions)}. \tag{39}
\]

From a quantum field theory viewpoint, the statistical index is related to the commutation or anti-commutation relation between creation and destruction operators
\[
[a, a^\dagger]_{\eta_s} = aa^\dagger + \eta_s a^\dagger a = 1 \tag{40}
\]

For arbitrary spin, Eq.\(67\) may be argued from the factor \(-\eta_s\) for each closed circle loop to be
\[
\Gamma = \frac{(2s + 1)c}{8\pi^3} \left( \frac{eE}{\hbar c} \right)^2 \times \sum_{n=1}^{\infty} \frac{\eta_s^{n+1}}{n^2} \exp \left( -\pi n \left( \frac{n^2 m^2 c^3}{\hbar eE} \right) \right). \tag{41}
\]

Eq.\(41\) has been discussed in the literature.\(\text{[7]}\).

Finally, the Euclidean action \(W\) may be associated with an entropy \(S\) via
\[
\frac{W}{\hbar} = \frac{S}{k_B} = \pi \left( \frac{m^2 c^3}{\hbar eE} \right). \tag{42}
\]

The derivative of the entropy with respect to the rest energy determines the reciprocal temperature
\[
\frac{1}{c^2} \left( \frac{dS}{dm} \right) = \frac{1}{T} \Rightarrow k_BT = \left( \frac{\hbar eE}{2\pi mc} \right). \tag{43}
\]

In terms of the acceleration of the charged Bosons, there exists an effective temperature \(\text{[8]}\),
\[
k_BT = \left( \frac{\hbar a}{2\pi e} \right), \tag{44}
\]
of the environment inducing position fluctuations equivalent to the energy fluctuations in the rest frame of the applied electric field.

\section{V. CONCLUSION}

The central results of this work are not new. For example, the spin zero charged Boson pair production rate in Eq.\(26\) as well as its generalization to the general spin \(s\) charged particle pair production rate in Eq.\(41\) are very well known. However, the derivations, physical pictures and consequences of zero point oscillations are to our knowledge original. The notion of zero point energy in relativistic quantum field theories is made real by the particle and anti-particle content of the theory. In electromagnetic theory, the photon is its own anti-particle which may obscure the interpretation of zero point energy. By considering a generic field theory with charged particles, the differences in charge between the particle and anti-particle makes manifest the physical nature of zero-point oscillations. The vacuum polarization and vacuum magnetization can be computed via this simple physical viewpoint.

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