q-Mittag-Leffler stability and Lyapunov direct method for differential systems with q-fractional order

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Abstract

In this paper, using the theory of q-fractional calculus, we deal with the q-Mittag-Leffler stability of q-fractional differential systems, and based on it, we analyze the direct Lyapunov method of q-fractional differential systems. Several sufficient criteria are established to guarantee the q-Mittag-Leffler stability and asymptotic stability for the differential systems with q-fractional order.

Keywords: q-fractional calculus; q-Mittag-Leffler functions; q-Mittag-Leffler stability; Lyapunov method

1 Introduction

The development of the theory of q-calculus can be dated back to the early 20th century in order to look for a better description of the phenomena having both discrete and continuous behaviors. The q-analog of fractional integrals and derivatives were first studied by Al-Salam [1–3] and then by Agrawal [4]. Recently, the q-fractional calculus has been payed more attention [5–8] because it serves as a bridge between fractional calculus and q-calculus.

In nonlinear systems, Lyapunov’s direct method provides an effective way to analyze the stability of a system without explicitly solving the differential equations. Motivated by the application of fractional calculus in nonlinear systems Li, Chen, and Podlubny [9, 10] proposed the Mittag-Leffler stability and Lyapunov direct method, and a considerable number results of stability analysis for fractional systems have been reported; see [11–21] and the references therein. However, to our knowledge, the q-Mittag-Leffler stability of q-fractional dynamic systems has not been studied. In this paper, we propose the q-Mittag-Leffler stability and the q-fractional Lyapunov direct method with a hope to enrich the knowledge of the theory of q-fractional calculus. We also present a simple Lyapunov function to get the q-Mittag-Leffler stability for many q-fractional-order systems and show that q-fractional-order dynamical systems also do not have to decay exponentially for the system to be stable in the Lyapunov sense.
2 Preliminaries

2.1 Definitions and properties of $q$-calculus

This section is devoted to recall some essential definitions and properties of $q$-calculus [1–4, 8].

If $q \in \mathbb{R}, 0 < q < 1$, a subset $A$ of $\mathbb{R}$ is called $q$-geometric if $qx \in A$ whenever $x \in A$. If a subset $A$ of $\mathbb{R}$ is $q$-geometric, then it contains all geometric sequences $\{xq^n\}_{n=0}^{\infty}, x \in A$.

Definition 2.1 ([8]) Let $f(x)$ be a real function defined on a $q$-geometric set $A$. The $q$-derivative is defined by

$$D_q f(x) = \frac{f(qx) - f(x)}{(q - 1)x}, \quad x \in A \setminus \{0\},$$

and

$$D_q f(x)|_{x=0} = \lim_{n \to \infty} \frac{f(q^n x) - f(0)}{q^n}.$$  

Setting $q \to 1$, we have $\lim_{q \to 1} D_q f(x) = f'(x)$.

Also, the $q$-integral is given as

$$\int_0^x f(t) d_q t = (1 - q)x \sum_{n=0}^{\infty} q^n f(q^n x), \quad x \in A,$$

and

$$\int_a^b f(t) d_q t = \int_0^a f(t) d_q t - \int_0^b f(t) d_q t, \quad a, b \in A.$$

We present here two basic properties concerning $q$-derivatives.

Property 1 ([7])

$$D_q (f \pm g)(x) = D_q f(x) \pm D_q g(x).$$

Property 2 ([7]) The $q$-Leibniz product rule is given by

$$D_q [g(x)f(x)] = g(qx)D_q f(x) + f(x)D_q g(x),$$

where $D_q$ is the $q$-derivative.

The $q$-analogue of exponent $(s - t)^{(k)}$ is

$$(s - t)^{(0)} = 1, \quad (s - t)^{(k)} = \prod_{j=0}^{k-1} (x - yq^j), \quad k \in \mathbb{N}, x, y \in \mathbb{R}.$$ 

Definition 2.2 ([7]) A $q$-analogue of the Riemann–Liouville fractional integral is defined as

$$I_{q,a}^\alpha f(x) = \int_0^x \frac{(x - qs)^{(\alpha - 1)}}{\Gamma_q(\alpha)} f(s) d_q s, \quad \alpha > 0.$$
If we let $q \to 1$, then the $q$-analogue of Riemann–Liouville fractional integral $\int_{q,a}^q f(x) \to I_q^a f(x)$.

**Definition 2.3** ([6]) The Riemann–Liouville type fractional $q$-derivative of a function $f : (0, \infty) \to R$ is defined by

$$
(D_q^\alpha f)(x) = \begin{cases} 
(I_{q,a}^{-\alpha} f)(x), & \alpha \leq 0, \\
(I_{q,a}^\alpha I_{q,a}^{-\alpha} f)(x), & \alpha > 0,
\end{cases}
$$

where $[\alpha]$ denotes the smallest integer greater than or equal to $\alpha$.

**Definition 2.4** ([6]) The Caputo type fractional $q$-derivative of a function $f : (0, \infty) \to R$ is define by

$$
(CD_q^\alpha f)(x) = \begin{cases} 
(I_{q,a}^{-\alpha} f)(x), & \alpha \leq 0, \\
(I_{q,a}^\alpha I_{q,a}^{-\alpha} f)(x), & \alpha > 0,
\end{cases}
$$

where $[\alpha]$ denotes the smallest integer greater or equal to $\alpha$.

### 2.2 $q$-Mittag-Leffler function

Similar to the Mittag-Leffler function frequently used in the solutions of fractional-order equations, the functions frequently used in the solutions of $q$-fractional-order equations are the $q$-analogues of Mittag-Leffler functions defined as

$$
e_{\alpha, \beta}(z, q) = \sum_{n=0}^{\infty} \frac{z^n q}{\Gamma_q(n\alpha + \beta)} \quad (|z(1-q)^\alpha| < 1)
$$

and

$$
E_{\alpha, \beta}(z, q) = \sum_{n=0}^{\infty} \frac{z^n q}{\Gamma_q(n\alpha + \beta)} \quad (z \in \mathbb{C}),
$$

where $\alpha > 0$ and $\beta \in \mathbb{C}$. When $\beta = 1$, the functions $e_{\alpha, \beta}(z, q)$ and $E_{\alpha, \beta}(z, q)$ are defined by

$$
e_{\alpha, 1}(z, q) = \sum_{n=0}^{\infty} \frac{z^n q}{\Gamma_q(n\alpha + 1)} \quad (|z(1-q)^\alpha| < 1)
$$

and

$$
E_{\alpha, 1}(z, q) = \sum_{n=0}^{\infty} \frac{z^n q}{\Gamma_q(n\alpha + 1)} \quad (z \in \mathbb{C}).
$$

### 2.3 $q$-Laplace transform of fractional $q$-integrals, $q$-derivatives, and $q$-Mittag-Leffler functions

**Theorem 2.5** ([6]) If $f \in L_q^1[0, a]$ and $\Phi(s) = q L_q f(x)$, then

$$
q L_q f(x) = \frac{(1-q)^\alpha}{s^\alpha} \Phi(s) \quad \text{for} \ \alpha > 0.
$$
If \( n - 1 < \alpha \leq n \) and \( L_q^{n-\alpha} f(x) \in C_{\alpha}^1[0,a] \), then let \( \Phi(s) = qL f(x) \). The \( q \)-Laplace transform of the Riemann–Liouville fractional and the Caputo fractional \( q \)-derivatives are given by

\[
qL_s^\alpha C qD_q^\alpha f(x) = \frac{s^\alpha}{(1 - q)^\alpha} \left( \Phi(s) - \sum_{r=0}^{n-1} D_q^r f(0^+) \left( \frac{1 - q}{s^{r+1}} \right) \right)
\]

and

\[
qL_s^\alpha L_q^\alpha f(x) = \frac{s^\alpha}{(1 - q)^\alpha} \Phi(s) - \sum_{m=1}^{n} D_q^{\alpha - m} f(0^+) \frac{s^{m-1}}{(1 - q)^m}.
\]

**Theorem 2.6** ([6]) If \( \| \frac{s}{1 - q} \| > |a|^{\frac{1}{\alpha}} \), then

\[
qL_s^\alpha \left( e^{\beta - 1} e^{\alpha \beta - 1} x(t_0; q) \right) = \frac{1}{1 - q} \left( \frac{s}{1 - q} \right)^{\alpha - \beta}.
\]

Taking \( \beta = 1 \), we have

\[
qL_s^\alpha \left( e_\alpha \alpha (ax; q) \right) = \frac{1}{1 - q} \left( \frac{s}{1 - q} \right)^{\alpha - 1}.
\]

### 3 \( q \)-Mittag-Leffler stability and Lyapunov direct method for differential systems with \( q \)-fractional order

Consider the Caputo fractional nonautonomous system \( q \)-Mittag-Leffler stability of solutions of the following system:

\[
\begin{align*}
C D_q^\alpha x(t) &= f(t, x(t)), \\
x(t_0) &= x_0,
\end{align*}
\]

where \( t \geq t_0, t_0 \in A, A = [t_0, t], 0 < \alpha < 1 \), and \( f : [t_0, t] \times R \to R \) is a function with \( f \in L^1[t_0, t] \). Let \( f(t, 0) = 0 \), for all \( t \in [t_0, t] \), so that system (19) admits the trivial solution.

Now we give some definitions that will be used in studying the \( q \)-Mittag-Leffler stability of (19).

**Definition 3.1** The trivial solution \( x(t) = 0 \) of (19) is said to be asymptotically stable if for all \( \epsilon > 0 \) and \( t_0 \in A \), there exists \( \delta = \delta(t_0, \epsilon) \) such that if \( \| x_0 \| < \delta \) implies that \( \lim_{t \to \infty} \| x(t) \| = 0 \).

**Definition 3.2** (\( q \)-Mittag-Leffler stability) The solution of (19) is said to be \( q \)-Mittag-Leffler stability if

\[
\| x(t) \| \leq \left\{ m \left[ x(t_0) \right] e_{\alpha, \alpha} (-\lambda (t - t_0)^\alpha) \right\}^b,
\]

where \( t_0 \in A \) is the initial time, \( \alpha \in (0, 1) \), \( \lambda \geq 0, b > 0, m(0) = 0, m(x) \geq 0, \) and \( m(x) \) is locally Lipschitz on \( x \in B \subset R \) with Lipschitz constant \( m_0 \). We further assume that \( t_0 = 0 \).
Theorem 3.3  Let $x = 0$ be an equilibrium point for system (19), and let $D \subset \mathbb{R}$ be a domain containing origin. Let $V(t,x(t)) : [0,T] \times D \rightarrow \mathbb{R}$ be a continuously differentiable function and locally Lipschitz with respect to $x$ such that

$$\beta_1 \|x(t)\|^a \leq V(t,x(t)) \leq \beta_2 \|x(t)\|^ab,$$

$$C_0^q D_q^a V(t,x(t)) \leq (-\beta_3) \|x(t)\|^ab,$$

where $t \in [0,T], t > 0, 0 < \alpha < 1,$ and $\beta_1, \beta_2, \beta_3, a,$ and $b$ are arbitrary positive constants. Then $x = 0$ is $q$-Mittag-Leffler stable.

Proof  It follows from equations (19) and (20) that

$$C_0^q D_q^a V(t,x(t)) \leq -\frac{\beta_3}{\beta_2} V(t,x(t)).$$

There exists a nonnegative function $M(t)$ satisfying

$$C_0^q D_q^a V(t,x(t)) + M(t) = -\frac{\beta_3}{\beta_2} V(t,x(t)).$$

Taking the $q$-Laplace transform of (24) gives

$$\frac{s^a}{(1-q)^s} (V(s) - \frac{1}{s} V(0,x(0)) + M(s)) = -\frac{\beta_3}{\beta_2} V(s),$$

where $V(s) =_q L_s[V(t,x(t))]$. It then follows that

$$V(s) = V(0,x(0)) \frac{s^{a-1}}{(1-q)^s} + \frac{M(s)}{s^{a-1}} \frac{\beta_3}{\beta_2} \frac{1}{1-q} \left( \frac{s}{1-q} \right)^a + \frac{\beta_1}{\beta_2} \frac{1}{1-q}$$

It follows from the inverse Laplace transform that the unique solution of (24) is

$$V(t) = V(0,x(0)) e_{a,1} \left(-\frac{\beta_3}{\beta_2} t; q \right) - \int_0^t M(\tau)(t-\tau)^{a-1} e_{a,\alpha} \left( -\frac{\beta_3}{\beta_2} (t-\tau)^a; q \right) d\tau.$$

Since $0 < q < 1, M(t) \geq 0,$ and $e_{a,\alpha}(-\frac{\beta_3}{\beta_2} (t-\tau)^a; q)$ are nonnegative functions, we get

$$V(t) \leq V(0,x(0)) e_{a,1} \left(-\frac{\beta_3}{\beta_2} t; q \right).$$

Substitution of (28) into (21) yields

$$\|x(t)\| \leq \left[ \frac{V(0,x(0))}{\beta_1} e_{a,1} \left(-\frac{\beta_3}{\beta_2} t; q \right) \right]^\frac{1}{a},$$

where $\frac{V(0,x(0))}{\beta_1} > 0$ for $x(0) \neq 0$. 
Let \( m = \frac{V(0,x(0))}{p_l} \geq 0 \). Then we have

\[
\|x(t)\| \leq \left[ me_{\alpha,1}\left( \frac{-\beta_3}{\beta_2} t; q \right) \right]^{\frac{1}{\alpha}},
\]

(30)

where \( m = 0 \) if and only if \( x(0) = 0 \). Because \( V(t,x) \) is locally Lipschitz with respect to \( x \)
and \( V(0,x(0)) = 0 \) if and only if \( x(0) \), it follows that \( m \) is also Lipschitz with respect to \( x(0) \)
and \( m(0) \), which implies the \( q \)-Mittag-Leffler stability.

In [8], an identity relation between the Caputo fractional \( q \)-derivative and the Riemann–Liouville fractional \( q \)-derivative is introduced:

\[
f(t) = \int_0^t D_q^\alpha f(t) - \Gamma_q(1-\alpha) f(t_0).
\]

(31)

where \( \alpha > 0 \) and \( n = [\alpha] + 1 \). When \( 0 < \alpha < 1 \), we have

\[
C_{\alpha,0}^\alpha D_q^\alpha f(t) = \int_0^t D_q^\alpha f(t) - \frac{(t-t_0)^2}{\Gamma_q(1-\alpha)} f(t_0).
\]

(32)

\[\square\]

**Theorem 3.4** If the assumptions in Theorem 3.3 are satisfied except replacing \( C_{\alpha,0}^\alpha D_q^\alpha \) by \( t_0 D_q^\alpha \), then the trivial solution of (19) is \( q \)-Mittag-Leffler stable.

**Proof** From (32) we have

\[
C_{\alpha,0}^\alpha D_q^\alpha V(t,x(t)) = \int_0^t D_q^\alpha V(t,x(t)) - \frac{t_0^\alpha}{\Gamma_q(1-\alpha)} V(0,x(0)) \quad \text{for} \quad t \in [0,T],
\]

(33)

and since \( V(0,x(0)) \geq 0 \) and \( \frac{t_0^\alpha}{\Gamma_q(1-\alpha)} \geq 0 \), we obtain the result.

Furthermore, if we extend the Lyapunov direct method to the case of \( q \)-fractional-order systems, then the asymptotic stability of the corresponding systems can be obtained. The following properties of the \( q \)-Mittag-Leffler function and the class-K functions are applied to analysis of the \( q \)-fractional Lyapunov direct method.

**Remark 3.5** Since

\[
D_q e_{\alpha,1}\left( (-\lambda t; q) \right) = -\lambda t^\alpha e_{\alpha,\alpha-1}(-\lambda t; q),
\]

(34)

where \( t > 0, 0 < \alpha < 1, \lambda > 0 \), the \( q \)-Mittag-Leffler function \( e_{\alpha,1}((-\lambda t)^\alpha; q) \) is decreasing, so the \( q \)-Mittag-Leffler stability implies the asymptotic stability.

### 4 \( q \)-Mittag-Leffler stability of linear systems with \( q \)-fractional order

In this section, we present a new result that allows us to find Lyapunov candidate functions for demonstrating the \( q \)-Mittag-Leffler of many fractional-order systems using the results of the Lyapunov direct method in Theorem 3.3.
Theorem 4.1 Let \( x(t) \in \mathbb{R} \) be defined on a suitable \( q \)-geometric set \( A = [0, a] \), \( D_q x(t) \in C_q[0, q] \) (where \( C_q[0, a] \) is the space of all continuous functions on the interval \( [0, a] \)). Then, for any time \( t > 0, t \in A \),

\[
C_0D_q^\alpha x^2(t) \leq (x(t) + x(tq))^2, \quad 0 < \alpha < 1.
\]

Proof Proving expression (35) is equivalent to proving that

\[
(x(t) + x(tq))^2 \leq (x(t) + x(tq))^2.
\]

Using Definition 2.2 and Definition 2.4, \( x(t) + x(tq) \)\( C_0D_q^\alpha x(t) \) and \( C_0D_q^\alpha x^2(t) \) can be written as

\[
(x(t) + x(tq))^2 = \int_0^t (t - qs)^{-\alpha} D_q x(s) d_q s.
\]

So, the left side of expression (36) can be written as

\[
\int_0^t (t - qs)^{-\alpha} \left[ (x(t) - x(s)) + (x(tq) - x(sq)) \right] D_q x(s) d_q s.
\]

Now, let us define the auxiliary variable \( y(s) = x(t) - x(s) \), which implies that

\[
D_q y^2(s) = (y(s) + y(sq))D_q y(s)
\]

\[
\begin{align*}
&= -\left[ (x(t) - x(s)) + (x(tq) - x(sq)) \right] D_q x(s).
\end{align*}
\]

In this way, expression (39) can be written as

\[
\int_0^t (t - qs)^{-\alpha} d_q y^2(s) = -\int_0^t (t - qs)^{-\alpha} y^2(qs) d_q s.
\]

Since \( x(t) \) is regular at zero, using the rule of \( q \)-integration by parts, expression (41) becomes

\[
\int_0^t (t - qs)^{-\alpha} d_q y^2(s) = y^2(t)(t - qt)^{-\alpha} - \Gamma(1 - \alpha) y^2(0)t^{-\alpha}
\]

\[
- \alpha q \int_0^t (t - qs)^{-\alpha - 1} y^2(qs) d_q s.
\]

Since \( y^2(t) = (x(t) - x(s))^2 = 0 \), it follows that

\[
C_0D_q^\alpha x(t) - C_0D_q^\alpha x^2(t)
\]

\[
= \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - qs)^{-\alpha} \left[ (x(t) - x(s)) + (x(tq) - x(sq)) \right] D_q x(s) d_q s.
\]
\[ -\frac{1}{\Gamma(1-\alpha)} \int_0^t (t - qs)^{-\alpha} \, dq y^2(s) \]
\[ = \frac{1}{\Gamma(1-\alpha)} y^2(0) t^{-\alpha} + \frac{\alpha q}{\Gamma(1-\alpha)} \int_0^t (t - qs)^{-\alpha-1} y^2(s) \, dq(s) \]
\[ \geq 0. \] \tag{43}

This concludes the proof. \square

**Corollary 4.2** For the q-fractional-order system
\[ C_0^\alpha D_q x(t) = f(t, x(t)), \] \tag{44}
where \( \alpha \in (0, 1) \), \( x = 0 \) is the equilibrium point, and \( D_q x(t) \in C_q[0,a], f(t,x(t)) \in S_q^1[0,a] \). If
\[ (x(t) + x(tq))f(t,x(t)) \leq 0, \quad \forall x \in A, \] \tag{45}
then the origin of system (44) is q-Mittag-Leffler stable.

**Proof** Let us propose the following Lyapunov candidate function:
\[ V(t,x(t)) = x^2. \] \tag{46}
Applying Theorem 4.1 results in
\[ C_0^\alpha D_q^\alpha V(t,x(t)) \leq (x(t) + x(tq))C_0^\alpha D_q x(t) \leq (x(t) + x(tq))f(t,x(t)) \leq 0, \] \tag{47}
and thus the origin of system (44) is q-Mittag-Leffler stable. \square

**Proposition 4.3** For the system
\[ C_0^\alpha D_q x(t) = -x(t) - x(tq), \] \tag{48}
where \( 0 < \alpha < 1 \) and \( D_q x(t) \in C_q[0,a], \) the origin of system (44) is q-Mittag-Leffler stable.

**Proof** Let \( V(x(t)) = x^2(t) \). Then
\[ C_0^\alpha D_q^\alpha x^2(t) \leq (x(t) + x(tq))C_0^\alpha D_q x(t) \]
\[ = - (x(t) + x(tq))^2 \leq -\|x(t)\|^2. \] \tag{49}

So we can conclude that the trivial solution of system (48) is asymptotically stable.

Furthermore, from the expression of exact solution for (48) using two \( q \)-analogues of the Mittag-Leffler functions defined by (12) and (13),
\[ x(t) = c_1 e_{(\alpha,1)}(-x,q) + c_2 E_{(\alpha,1)}(-x,q), \] \tag{50}
and the properties of these two functions the asymptotical stability can also be derived. \square
5 Conclusions

In this paper, we studied the stability of systems with $q$-fractional order. We proposed the definition of $q$-Mittag-Leffler stability, presented sufficient criteria of $q$-Mittag-Leffler stability and the $q$-fractional Lyapunov direct method of nonlinear systems with $q$-fractional order. Meanwhile, the $q$-fractional Lyapunov candidate functions for demonstrating the $q$-Mittag-Leffler stability of many $q$-fractional-order systems were discussed. With the rapid development of advanced applied science, we believe that many other study subjects of the $q$-fractional calculus and $q$-fractional dynamical systems will attract more attention of researchers. In our following study, we will still focus on the stability problem of $q$-fractional differential equations in a variety of different forms.

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Competing interests

The authors declare that they have no competing interests.

Authors’ contributions

All authors read and approved the final manuscript.

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