1. Introduction

Green and Lazarsfeld \cite{GL2} have proven the following remarkable theorem:

**Theorem 1.1.** Let $X$ be a smooth complex projective variety. Then the set $S^{pq}(X)$ of line bundles in $\text{Pic}^0(X)$ satisfying $H^q(X, \Omega^p_X \otimes L) \neq 0$ is a union of a finite number of translates of abelian subvarieties.

In this paper, we seek a generalization for higher rank bundles. The analogue of $\text{Pic}^0(X)$ is the moduli space $M = M_{V}(X, n)$ of semistable bundles of a given rank $n$ with trivial Chern classes, and within it a subset $\{ E | H^q(X, \Omega^p_X \otimes E) \neq 0 \}$ can be defined as above. However, $M$ is very far from an abelian variety in general, so it is not immediately clear what the analogous theorem should even say. A clue is provided by the following theorem of Hitchin \cite{H2}:

**Theorem 1.2.** Suppose that $X$ is a smooth projective curve and $M^s \subset M$ the smooth open set of stable bundles. Then there is a morphism from the cotangent bundle $T^*M^s$ to an affine space such that the general fibers are open subsets of abelian varieties.

Our ultimate goal then is to give a common generalization of both theorems. We will establish a result similar to 1.2 for any component of the above cohomology support locus in $M$ when $X$ has arbitrary dimension.

In order to delve deeper into the story, it will be necessary to explain how to compactify Hitchin’s map. For this, we need to make the transition from vector bundles to Higgs bundles, which can be motivated as follows. Suppose that $E$ is a stable bundle corresponding to a smooth point $[E] \in M$. Then a cotangent vector to $[E]$ is just a section $\theta \in H^0(X, \Omega^1_X \otimes \text{End}(E))$. As $[E]$ is a smooth point, there is no obstruction to extending a first order deformation of $E$ to one of second order. The dual condition is $\theta \wedge \theta = 0$. The pair $(E, \theta)$ is an example of a Higgs bundle. Simpson \cite{S2} has shown that the set of isomorphism classes of Higgs bundles of rank $n$ with vanishing rational Chern classes (and subject to a suitable semistability condition weaker than semistability of the underlying vector bundle), can be parameterized by a quasiprojective moduli scheme $M_{Dol}(X, n)$. Furthermore there is a proper morphism $h$, the analogue of Hitchin’s map, from $M_{Dol}(X, n)$ to an affine space (which assigns to $(E, \theta)$ the characteristic polynomial of $\theta$). We can define $\Sigma^k_{m,Dol}(X, n) \subseteq M_{Dol}(X, n)$ as the set of those pairs $(E, \theta)$ such that the appropriate $k$th cohomology group has dimension at least $m$. In section 4, we prove the main result:

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Theorem 1.3. If $X$ is a smooth projective variety, then $\Sigma_{m,Dol}^k(X, n)$ is a Zariski closed subset of $M_{Dol}(X, n)$. If $\tilde{\Sigma}$ is the normalization of an irreducible component of $\Sigma_{m,Dol}^k(X, n)$ (with its reduced subscheme structure) containing a stable point, then the connected components of the general fibers of the pullback of $h$ to $\tilde{\Sigma}$ are abelian varieties.

The case $m = k = 0$ gives an analogue of Hitchin’s result. The above sets can be further subdivided into $(p, q)$ parts, some components of which form partial compactifications of the “cotangent bundles” of the sets $\{E \mid H^q(X, \Omega_X^p \otimes E) \neq 0\}$ considered earlier. Similar results will be proved for these sets.

The analytic space associated to $M_{Dol}(X, n)$ has a second complex structure $M_{B}(X, n)^{an}$ which comes about via a correspondence between Higgs bundles (of the above type) and semisimple local systems $[S1, S2]$. The key observation is that when taken together, these yield a quaternionic structure on this space, and the set $\Sigma_{m,Dol}^k(X, n)$ is compatible with this structure. We use this fact to show that the general fiber of the restriction of $h$ is lagrangian with respect to a suitable symplectic structure, then the theorem follows easily. An important precedent for the use of the quaternionic structure in this context is Deligne’s and Simpson’s $[S3]$ approach to proving theorem $[G1]$. One complication, absent in the rank one case, is the presence of singularities. Recent work of Verbitsky, discussed in the next section, allows us to handle these issues.

The cohomology group of a Higgs bundle has a number of different incarnations. To begin with, it can be defined as the hypercohomology of an explicit complex. It is also (isomorphic to) the cohomology of the associated local system. Both interpretations are needed in order to verify that the cohomology support loci are quaternionic. The first description will also be used establish the invariance of these loci under a natural $C^*$-action. This will imply that any irreducible component contains a complex variation of Hodge structure. The second point of view will be useful for establishing certain homotopy invariance properties for these sets. Finally, we will reinterpret the cohomology group of a Higgs bundle as an Ext group for certain sheaves on the cotangent bundle of $X$. Then using the local to global spectral sequence, we prove a generic vanishing theorem in the spirit of $[GL1]$. This leads to estimates on the codimension of the cohomology support loci.

The final section of this paper contains some explicit examples. So readers may wish to skip to it from time to time. For the most part, schemes over $\mathbb{C}$ will be treated as sets of $\mathbb{C}$-valued points. As usual, the superscript “an” indicates the analytic space associated to a scheme.

2. QUATERNIONIC GEOMETRY

This section is completely expository. It is intended to give a quick introduction to quaternionic geometry, and to some of Verbitsky’s work in particular. A nice discussion of some related ideas and examples can be found in $[H3]$. See also $[H1], [F], [S4], [V1], [V2]$ and references contained therein.

A quaternionic (or hypercomplex) manifold is a $C^\infty$-manifold $X$ with two complex structures $\mathcal{I}$ and $\mathcal{J}$ which induce the same real analytic structure on $X$ and satisfies $\mathcal{I}\mathcal{J} = -\mathcal{J}\mathcal{I}$. Setting $\mathcal{K} = \mathcal{I}\mathcal{J}$ gives an action of the quaternions $\mathbb{H}$ on the tangent bundle. Any quaternionic vector space is naturally a quaternionic manifold. A morphism of quaternionic manifolds is $C^\infty$ map which is holomorphic with respect to $\mathcal{I}, \mathcal{J}$ and $\mathcal{K}$ (it suffices to check holomorphicity with respect to any two).
Lemma 2.1. Let $V_1$ and $V_2$ be two finite dimensional quaternionic vector spaces, and let $U_i \subseteq V_i$ be open neighbourhoods of the origin with their induced quaternionic structures. Any morphism $f : U_1 \to U_2$ satisfying $f(0) = 0$ is the restriction of an $\mathbb{H}$-linear map.

**Proof.** (Deligne, see [S3]). Choose a point $x \in U_1$. After identifying the tangent space at $x$ and $f(x)$ with $V_1$ and $V_2$, the differential gives an $\mathbb{R}$-linear map $df_x : V_1 \to V_2$. By assumption $df_x I = Id df_x$, $df_x J = J df_x$ so in fact $df_x$ is $\mathbb{H}$-linear. Thus $df$ can be viewed as a $C^\infty$ map from $U_1$ to $\text{Hom}_\mathbb{H}(V_1, V_2)$. Differentiating again yields the Hessian, which is a symmetric $\mathbb{R}$-bilinear form $H_x : V_1 \times V_1 \to V_2$. $H_x$ is $\mathbb{H}$-linear in one variable and therefore in both. Now for the punchline:

$$K H_x(\alpha, \beta) = I H_x(J \alpha, \beta) = J H_x(\alpha, I \beta) = -K H_x(\alpha, \beta)$$

Therefore $H_x = 0$ and the lemma follows immediately.

A quaternionic submanifold of a quaternionic manifold is a $C^\infty$ submanifold such that the inclusion is a morphism. More generally a quaternionic subvariety $Y$ of a quaternionic manifold $X$ is a reduced real analytic subvariety whose complexified ideal is locally defined by both $I$ and $J$ holomorphic functions.

Corollary 2.2. Any quaternionic submanifold of a quaternionic vector space $V$ is a translate of a linear subspace. A quaternionic subvariety is a union of submanifolds.

A hyperkähler manifold is a $C^\infty$ manifold with a Riemannian metric $g$ and two anticommuting complex structures $I$ and $J$, such that $g$ is Kähler with respect to both of these structures.

Proposition 2.3. ([V1, 6.5]) If $X$ is hyperkähler then the underlying real analytic structures associated to $I$ and $J$ coincide. Therefore $X$ is a quaternionic manifold.

Theorem 2.4. (Verbitsky [V2]). Let $(X, x)$ be a germ of a hyperkähler manifold. The the germ of quaternionic subvariety $(Y, x)$ is the union of a finite number of germs of quaternionic submanifolds.

Here is an outline of the proof: Let $R$ be the local ring of real analytic functions of $(X, x)$. Given a complex structure $L$ on $X$, let $O_L$ be the local ring of $L$-holomorphic functions. There is a natural inclusion $O_L \subset R \otimes \mathbb{C}$ which splits: the complement is the ideal generated by $L$-antiholomorphic functions vanishing at $x$. Let $\phi$ be the composition of local homomorphisms:

$$O_I \hookrightarrow R \otimes \mathbb{C} \rightarrow O_J \hookrightarrow R \otimes \mathbb{C} \rightarrow O_X$$

A direct calculation shows that the induced endomorphism on the cotangent space $m/m^2$ of $O_I$ is a homothety associated to a scalar $\lambda \in \mathbb{C}^*$ which is not a root of unity. After a change of variables, one can arrange $I$-holomorphic coordinates so that $\phi(x_i) = \lambda x_i$. Let $I$ be the ideal of $Y$ in $O_I$. Then $I(R \otimes \mathbb{C})$ is also generated by $J$-holomorphic functions. Therefore $\phi(I) \subset I$ and this implies that $I$ is homogeneous, and thus $(Y, x)$ is isomorphic to the germ of its tangent cone. The tangent cone is a quaternionic subvariety of a quaternionic vector space, and therefore by 2.2 is a union of manifolds.
Corollary 2.5. The normalization of a quaternionic subvariety of a hyperkähler manifold with respect to \( \mathcal{I} \) coincides with the normalization with respect to \( \mathcal{J} \) and it is smooth. Furthermore it inherits a hyperkähler structure from the ambient manifold.

Given a hyperkähler manifold \( X \), set \( \omega_L(\alpha, \beta) = g(L\alpha, \beta) \) for \( L = \mathcal{I}, \mathcal{J}, \mathcal{K} \). These are just the Kähler forms associated to the complex structures. In particular, they define (real) symplectic structures on \( X \). The form \( \omega_J + \sqrt{-1}\omega_K \) defines an \( \mathcal{I} \)-holomorphic symplectic structure on \( X \).

3. Lagrangian maps

Let \( (X, \omega) \) be a real or holomorphic symplectic manifold. A closed submanifold \( Y \subset X \) is lagrangian if the tangent spaces of \( Y \) are maximal isotropic subspaces of the tangent spaces of \( X \) with respect to the symplectic pairing induced by \( \omega \). A map \( h : X \to B \) (\( C^\infty \) or holomorphic according to the category) to manifold \( B \) will be called lagrangian if its differential has maximal rank, and all its fibers are lagrangian submanifolds. Note that a lagrangian map is the same thing as a completely integrable system. A complete discussion of these notions would take us too far afield, see for example [GS] for further details.

If \( \omega \) is only defined on a dense open subset of \( U' \subseteq X \), a map \( h : X \to B \) will be called generically lagrangian if there is a dense nonsingular open set \( U \subset B \) such that \( h^{-1}(U) \subseteq U' \) and \( h^{-1}(U) \to U \) is lagrangian, the complement of the largest such \( U \) will be called the discriminant.

Lemma 3.1. Let \( X \) be a Kähler manifold with a real \( C^\infty \) symplectic structure \( \omega \) (not necessarily equal to the Kähler form). If \( h : X \to B \) is a proper lagrangian holomorphic map, then the connected components of the fibers are complex tori.

Proof. After Stein factorization, we can assume that the fibers are connected. Standard results in symplectic geometry [GS, page 353] show that any fiber \( F \) is diffeomorphic to a torus. It follows easily that the Albanese map

\[
F \to H^1(F, \mathbb{R})^*/H_1(F, \mathbb{Z}) \cong H^0(F, \Omega^1_F)^*/H_1(F, \mathbb{Z})
\]

is a diffeomorphism and therefore a biholomorphism. Note that the isomorphism \( H^1(F, \mathbb{R}) \cong H^0(F, \Omega^1_F) \) is only place where Kähler condition on \( X \) is used, so the lemma holds under considerably weaker hypotheses.

Let \( (X, g, \mathcal{I}, \mathcal{J}) \) be a hyperkähler manifold. We will usually take \( \mathcal{I} \) as the preferred complex structure with holomorphic symplectic structure given by \( \omega_J + \sqrt{-1}\omega_K \). In particular, a lagrangian map \( h : X \to B \) will be assumed to be \( \mathcal{I} \)-holomorphic and lagrangian with respect to the indicated symplectic structure. Note that such a map is also lagrangian with respect to the real symplectic structure \( \omega_J \), consequently the above lemma applies.

Theorem 3.2. Let \( (X, g_X, \mathcal{I}, \mathcal{J}) \) and \( (Y, g_Y, \mathcal{I}, \mathcal{J}) \) be a hyperkähler manifolds Suppose that \( h : X \to B \) is a lagrangian map. If \( f : Y \to X \) is a finite quaternionic morphism such that \( g_Y = f^*g_X \) (away from critical points). Then \( h \circ f : Y \to B' \) is a generically lagrangian map, where the image \( B' = h(f(Y)) \) is endowed with the reduced analytic structure.
Proof. For any \( x \in X \), \( V_x = \ker dh_x \) is a maximal isotropic subspace of the tangent space \( T_x \) with respect to \( \omega_F \). Thus \( \mathcal{J}V_x \) is orthogonal to \( V_x \) with respect to \( g_X \). Since
\[
\dim \mathcal{J}V_x = \dim V_x = \dim T_x / 2,
\]
\( T_x = V_x \oplus \mathcal{J}V_x \). Let \( y \in Y \) be a general point, then we can identify \( T_y \) with a quaternionic subspace of \( T_{f(y)} \) and \( \ker d(h \circ f)_y \) with \( V_{f(y)} \cap T_y \), and so \( T_y = \ker d(h \circ f)_y \oplus \mathcal{J} \ker d(h \circ f)_y \). Therefore \( \ker d(h \circ f)_y \) is a maximal isotropic subspace. So the general fibers of \( h \circ f \) are lagrangian.

Corollary 3.3. If in the above notation \( h \circ f \) is proper, then the connected components of its general fibers are complex tori.

4. Cohomology support loci for Higgs bundles.

Let \( X \) be a smooth complex projective variety with a fixed ample line bundle \( L \). A Higgs bundle on \( X \) consists of an algebraic vector bundle \( E \) together with a section \( \theta \in H^0(X, \Omega^1_X \otimes \text{End}(E)) \) satisfying \( \theta \wedge \Theta = 0 \). A Higgs bundle \( (E, \theta) \), with \( c_1(E) = 0 \) (in rational cohomology) is called stable if \( c_1(F) \cdot c_1(L)^{\dim X - 1} < 0 \) for any coherent subsheaf \( F \subset E \) satisfying \( rk F < rk E \) and \( \theta(F) \subseteq \Omega^1_X \otimes F \). A Higgs bundle with \( c_1(E) = 0 \) is called polystable if it is a direct sum of stable Higgs bundles with vanishing first Chern class.

It will be convenient to combine the main results of Simpson \([S1, S2]\) into one big theorem:

Theorem 4.1. There is an affine scheme (of finite type over \( \text{Spec}\mathbb{Z} \)) \( M_B(X, n) \) whose complex points parameterize the isomorphism classes of semisimple representation of \( \pi_1(X) \) into \( \text{Gl}_n(\mathbb{C}) \). There is a quasiprojective scheme \( M_{\text{Dol}}(X, n) \), over \( \text{Spec}\mathbb{C} \), whose complex points parameterize polystable rank \( n \) Higgs bundles with vanishing first and second rational Chern classes. There are open subsets \( M_{\text{Dol}}^*(X, n) \subseteq M_{\text{Dol}}(X, n) \) and \( M_{\text{Birr}}^*(X, n) \subseteq M_B(X, n) \) which parameterize stable bundles and irreducible representations respectively. The spaces \( M_{\text{Dol}}(X, n)_{\text{an}} \) and \( M_B(X, n)_{\text{an}} \) are homeomorphic, and \( M_{\text{Dol}}^*(X, n) \) and \( M_{\text{Birr}}^*(X, n) \) correspond under this homeomorphism.

Remark 4.2. 0) \( X \) and \( n \) will be omitted from the notation when it is safe to do so.

1) The word “parameterize” is a bit vague. The correct statement is that these are coarse moduli spaces for the appropriate moduli functors.

2) A semistable Higgs bundle with \( c_1 = 0 \) is an iterated extension of stable Higgs bundles with \( c_1 = 0 \). Two semistable bundles are equivalent if their stable factors coincide (up to isomorphism). Every equivalence class of semistable bundles has a unique polystable representative. Thus \( M_{\text{Dol}} \) parameterizes equivalence classes of semistable bundles. Similarly \( M_B \) parameterizes equivalence classes of arbitrary representations, where two representations are equivalent if they have isomorphic semisimplifications.

3) These moduli spaces may be nonreduced. However we will usually suppress the scheme structure and just treat them as sets of \( \mathbb{C} \)-valued points. For every (poly, semis)stable Higgs bundle or semisimple representation \( V \), let \( [V] \) denote the corresponding point in the moduli space.
A family of Higgs bundles on $X$ parameterized by $T$ is a vector bundle $E$ on $X \times T$, with a section $\Theta$ of $p_X^* \Omega_X \otimes \text{End}(E)$ satisfying $\Theta \wedge \Theta = 0$. $M_{Dal}$ is only a coarse moduli space, so there may not be a universal family of Higgs bundles. However Simpson’s construction gives a bit more. Namely $M_{Dal}(X, n)$ is a quotient, in the sense of geometric invariant theory, of a locally closed subscheme $Q_n$ of an appropriate Quot or Hilbert scheme. $X \times Q_n$ will in fact carry a family of Higgs bundles $(E, \Theta)$ such that its restriction $(E, \Theta)|_q$ to a slice $X \times \{q\}$ is semistable and corresponds to the image of $q$ in $M_{Dal}(X, n)$. Over $M_{Dal}^s(X, n)$ it possible to find cross sections to $Q_n \rightarrow M_{Dal}(X, n)$ locally in the etale topology. The following is an almost immediate consequence:

**Proposition 4.3.** Let $S$ be a set of semistable Higgs bundles on $X$. Suppose that for every $T$ and semistable family of Higgs bundles $(E, \Theta)$ parameterized by $T$, the set $S_T = \{ t \mid (E, \Theta)|_t \in S \}$ is Zariski closed. Then the set $S_M$ of all $m \in M_{Dal}$ which possess a semistable representative in $S$ is Zariski closed.

**Proof.** $S_M$ is constructible because it is the image of $S_Q$ under the canonical map. Given a limit point $s$ of $S_M$, there is an irreducible curve $C$ such that $s \in C$ and $C - \{s\} \subset S_M$. As $Q \rightarrow M_{Dal}$ is surjective, there is an irreducible curve $C' \subset Q$ and finite map $C' \rightarrow C$. One obtains a family of Higgs bundles on $C'$ by restriction. $S_{C'} = C'$ since it is closed and contains the preimage of $C - \{s\}$. Therefore $s \in S_M$ and so $S_M$ is closed.

**Proposition 4.4.** Let $C \subset X$ be a curve obtained as a complete intersection of divisors associated to an $N$th power of $L$, with $N >> 0$. Then the restriction maps $M_B(X, n) \rightarrow M_B(C, n)$ and $M_{Dal}(X, n) \rightarrow M_{Dal}(C, n)$ are injective (on $\mathbb{C}$-valued points) and compatible with the homeomorphisms of the associated analytic spaces.

**Proof.** This well known to the experts, so we will merely indicate the main ideas. The first part follows from the Lefschetz hyperplane theorem \[^{[M]}\]. For the second part, first note that the restriction of a semistable Higgs bundle is semistable thanks to Simpson’s generalization of the Mehta-Ramanathan theorem \[^{[S]}\], so the map is well defined. As for injectivity, choosing $N >> 0$ guarantees that

$$H^1(E^*_1 \otimes E^*_2 \otimes IC) = 0$$

for any pair of Higgs bundles on $X$. Thus any isomorphism of their restrictions can be lifted to map of the Higgs bundles which can be seen to be an isomorphism (using, for example, the fact that polystable bundles are direct sums of simple bundles). The last part is just a restatement of the naturality of the correspondence.

**Remark 4.5.** The images of $M_{Dal}^s(X, n)$ and $M_{Dal}^{ur}(X, n)$ under restriction lie in the corresponding subsets for $C$.

Given a Higgs bundle $(E, \theta)$, define $H^i(E, \theta)$ to be the $i$th hypercohomology of the complex:

$$E \xrightarrow{\theta \wedge} \Omega^1_X \otimes E \xrightarrow{\theta \wedge} \Omega^2_X \otimes E \ldots$$

We define the cohomology support loci as:

$$\Sigma^k_{m, Dal}(X, n) = \{(E, \theta) \in M_{Dal}(X, n) \mid (E, \theta) \text{ polystable and } \dim H^k(E, \theta) \geq m\}$$

We can make an analogous definition for $M_B$. Given a representation of $\rho: \pi_1(X) \rightarrow \text{Gl}_n(\mathbb{C})$, there exists (up to isomorphism) a unique rank $n$ locally constant sheaf,
or local system, with $\rho$ as its monodromy representation. So we can, and will, view $M_B(X, n)$ as a moduli space of local systems on $X$. Let

$$\Sigma_{m,B}^k(X, n) = \{ V \in M_B(X, n) \mid \text{V semisimple and } \dim H^k(X, V) \geq m \}$$

**Proposition 4.6.** $\Sigma_{m,Dol}^k$ is Zariski closed in $M_{Dol}$. $\Sigma_{m,B}^k$ is Zariski closed in $M_B$. $\Sigma_{m,Dol}^k$ and $\Sigma_{m,B}^k$ coincide under the correspondence between $M_{an,Dol}$ and $M_{an,B}$.

**Proof.** Note that if $(E_i, \theta_i)$, $i = 1, 2$ are equivalent semistable bundles with $(E_1, \theta_1)$ polystable, then

$$\dim H^k(E_1, \theta_1) \geq \dim H^k(E_2, \theta_2)$$

by subadditivity of cohomology. Therefore the first statement follows from proposition 4.3 and the semicontinuity theorem for cohomology [EGA III, 7.7.5, 7.7.12]. The second statement is proved in [A2]. The last part follows from [S1, 2.2].

**Theorem 4.7.** (Hitchin [H1]) Let $C$ be a smooth projective curve. Then the stable locus $M_{Dol}^{s,an}(C, n)$ is smooth and carries a natural hyperkähler structure $(\mathcal{G}, \mathcal{I}, \mathcal{J})$. Where $\mathcal{I}$ is the usual complex structure and $\mathcal{J}$ is the structure induced from the identification $M_{Dol}^{s,an} \cong M_{Birr,an}^{s,an}$.

**Remark 4.8.** Hitchin stated this only when $n = 2$, although his proof presumably works for higher rank bundles. In any case, Fujiki [F] has established a much more general result.

Given a rank $n$ Higgs bundle $(E, \theta)$ on $X$, let $\theta^i \in H^0(X, S^i \Omega^1_X)$ be the $i$th (symmetric) power Let

$$h_X(E, \theta) \in S_n(X) = \bigoplus_{i=1}^n H^0(X, S^i \Omega^1_X)$$

be the map $(E, \theta) \mapsto \text{trace}(\theta^i)$. This differs from Simpson’s and Hitchin’s definition in that they use the characteristic polynomial. However there is an algebraic automorphism $\sigma$ of the target space such that $\sigma \circ h$ agrees with their map, so they are essentially the same. Therefore:

**Theorem 4.9.** (Simpson [S2], Hitchin [H2]) $h_X$ gives a proper morphism from $M_{Dol}$ to $S_n(X)$. When $\dim X = 1$, $h_X$ is surjective and generically lagrangian with respect to the hyperkähler structure on $M_{Dol}^{s,an}$.

Suppose that a curve $C \subseteq X$ has been chosen as in [14], then we obtain a commutative diagram:

$$
\begin{array}{ccc}
M_{Dol}(X, n) & \hookrightarrow & M_{Dol}(C, n) \\
h_X \downarrow & & h_C \downarrow \\
S_n(X) & \overset{i}{\rightarrow} & S_n(C)
\end{array}
$$

While $i$ need not be injective, we do have:

**Lemma 4.10.** The map $h_X(M_{Dol}(X, n)) \rightarrow h_C(M_{Dol}(C, n))$ is finite.

**Proof.** This follows from the properness of all the other maps in the diagram.

Consequently, fibers of $h_X$ are components of fibers of $i \circ h_X$. Putting the above results together yields the main theorem:
Theorem 4.11. Let $X$ be a smooth projective variety. If $\tilde{\Sigma}$ is a connected component of the normalization of $\Sigma_{m, Dol}^k$ (with its reduced subscheme structure) which meets $M_{Dol}^n$, and $\tilde{h} : \tilde{\Sigma} \to h(\tilde{\Sigma})$ the natural map. Then $\tilde{h}$ is generically lagrangian. In particular the connected components of its general fibers are abelian varieties of dimension half of that of $\dim \tilde{\Sigma}$.

Proof. Let $C \subseteq X$ be a complete intersection curve of high degree. Then $\Sigma_{m, Dol}^k(X, n)$ is a quaternionic subvariety of $M_{Dol}(C, n)$ by the previous results. Therefore the theorem follows from 3.2. □

Corollary 4.12. Any irreducible component $F$ of every fiber of $\tilde{h}$ satisfies $\dim F = \dim \tilde{\Sigma}/2$.

Proof. By upper semicontinuity of dimensions, it is enough to prove $\dim F \leq \dim \tilde{\Sigma}/2$. By 2.4, $\tilde{\Sigma}$ is compatible with the complex structure coming from $\Sigma_{m, B}^k(X, n)$. With this structure $\tilde{\Sigma}^{an}$ is Stein, therefore $H_i(\tilde{\Sigma}^{an}, \mathbb{Z}) = 0$ for $i > \dim \tilde{\Sigma}$. On the other hand, with the original complex structure $\tilde{\Sigma}$ is quasiprojective, and $F$ is a proper subvariety. Therefore the fundamental class of $F$ defines a nonzero element of $H_{2\dim(F)}(\tilde{\Sigma}^{an}, \mathbb{Z})$, and this forces the inequality. □

Remark 4.13. The theorem can also be used to recover a result of Biswas [8.2] about Poisson commutivity of higher dimensional Hitchin maps.

In order to get a better analogue of the original theorem of Green and Lazarsfeld, we need to break $\Sigma^k_{m, Dol}$ up into $(p, q)$ parts. For a Higgs bundle $(E, \theta)$, let $H^{pq}(E, \theta)$ be the $E^{pq}_{\infty}$ term of the spectral sequence:

$$E^{pq}_{1} = H^q(X, \Omega^p_X \otimes E) \Rightarrow H^{pq}(E, \theta)$$

Then set $S^{pq}_{m}(X, n)$ equal to the closure of

$$\{(E, \theta) \in M_{Dol}(X, n)|\dim H^{pq}(E, \theta) \geq m\}$$

Clearly

$$\Sigma^k_{m, Dol}(X, n) = \bigcup_{m_0 + m_1 + \ldots = m} \bigcap_{p} S^{p-k, p}_{m_p}(X, n)$$

Corollary 4.14. Let $\tilde{S}$ be a connected component of the normalization of $S^{pq}_{m}$ which meets $M_{Dol}^n$. The connected components of general fibers of the restriction of $\tilde{h}$ to $\tilde{S}$ are abelian varieties. All fibers have dimension equal to one half of that of $\tilde{S}$.

This can be deduced from the theorem and the previous corollary using the next lemma:

Lemma 4.15. Let $X$ be a noetherian topological space. Suppose that there are nested closed sets

$$X = X_0 \supseteq X_1 \supseteq \ldots$$

Then any irreducible component of $X_i$ is an irreducible component of some set of the form

$$Y_m = \bigcup_{m_0 + m_1 + \ldots = m} \bigcap_{i} X^i_{m_i}$$
Proof. Suppose that $S$ is an irreducible component of $X^0_{m_0}$. We can assume that $S$ is not contained in $X^0_{m_0+1}$. Similarly, let $m_1, \ldots$ be the largest integers for which $S$ is contained in $X^1_{m_1}, \ldots$. Then it is easily seen to be an irreducible component of $Y_m$, where $m = m_0 + m_1 + \ldots$.

If $(E, \theta)$ is a Higgs bundle, its dual is $(E^*, -\theta)$ where $\theta$ is viewed as section of $\Omega^1_X \otimes \text{End}(E^*) \cong \Omega^1_X \otimes \text{End}(E)$. This is compatible with the duality of local systems. Simpson has already observed a duality theorem holds on cohomology. This can be refined slightly:

**Proposition 4.16.** If $d = \dim X$ then

$$H^{pq}(E, \theta) \cong H^{d-p, d-q}(E^*, -\theta)^*$$

Proof. Set $\omega_X = \Omega^d_X$. A special case of the Grothendieck-Serre duality theorem is that

$$H^{d-i}(\text{Hom}(V^\bullet, \omega_X)) \cong H^i(V^\bullet)^*$$

for any finite complex of locally free sheaves $V^\bullet$. The natural pairing of $\Omega^p_X \otimes \Omega^d-p_X \to \omega_X$ induces an isomorphism of complexes

$$\text{Hom}(\Omega^p_X \otimes E, \omega_X) \cong (\Omega^p_X \otimes E^*, -\theta)[d]$$

upto sign. This respects the the filtrations

$$\text{Hom}(\Omega^p_X \otimes E, \omega_X) \cong (\Omega^p_X \otimes E^*, -\theta)[d]$$

and induces isomorphisms on the associated graded parts. Therefore there is an isomorphism between the associated spectral sequences converging to the hypercohomology groups on the left and right hand sides. The $E_\infty$ terms can be identified with the groups of the proposition.

**Corollary 4.17.** $S^{pq}_m(X, n) = S^{d-p, d-q}_m(X, n)$

5. **Cohomology support loci for vector bundles.**

We will use the same notation as in the previous section. Let $M_V(X, n)$ be the closed subscheme of $M_{\text{Dol}}(X, n)$ parameterizing polystable Higgs bundles of the form $(E, 0)$. Let $M_V^s$ be the open subset of stable bundles. Note that $(E, 0)$ is (poly, semi)stable if and only if $E$ is (poly, semi)stable in the usual sense (with respect to “slope”). So $M_V$ is just the coarse moduli space of semistable vector bundles of rank $n$ with trivial Chern classes. Note that $(E, \theta)$ is (poly, semi)stable if $E$ is. Let $T^*M_V$ (respectively $T^*M_V^s$) be the set of all Higgs bundles $[(E, \theta)]$ such that $E$ is polystable (respectively stable). $M_V$ may have singularities, so the notation $T^*M_V$ is merely suggestive; some justification for it will be given below. There is a morphism $T^*M_V \to M_V$ given by projection.

A vector bundle is determined by a 1-cocycle $g_{ij} \in Z^1(U, GL_n(O_X))$, and a first order deformation to it is described by a cocycle of the form

$$g_{ij} + \epsilon \gamma_{ij} \in Z^1(U, GL_n(O_X[\epsilon]/(\epsilon^2)))$$

$\gamma_{ij}$ defines a cocycle with values in $\text{End}(E)$. So the Zariski tangent space to $[E] \in M_V^s$ can be identified with $H^1(X, \text{End}(E))$. The obstruction to lifting a first order deformation $v \in H^1(X, \text{End}(E))$ to one of second order is given by $[v, v] \in H^2(X, \text{End}(E))$. There are in fact no higher obstructions, so the tangent
Proposition 5.1. This As pointed out in the remarks preceding 4.3, etale local cross sections to $\text{End}(E)$ carry unitary flat connection $\nabla$. Consequently, $\text{End}(E)$ also carries a unitary flat connection. Hodge theory with unitary flat coefficients and the self duality of $\text{End}(E)$ shows that there is a conjugate linear isomorphism

$$H^i(X, \text{End}(E)) \cong H^0(X, \Omega_X^1 \otimes \text{End}(E))$$

preserving the graded Lie brackets. The Higgs fields on $E$ are precisely the conjugates of the vectors in the tangent cone. This implies that they have the same dimension as real algebraic varieties, and therefore as complex algebraic varieties. It is sometimes better to view $T_{[E]} = H^0(X, \Omega_X^1 \otimes \text{End}(E))$ as dual to the tangent space, via the hard Lefschetz pairing

$$< \alpha, \beta > = \int_X \text{trace}(\alpha \cup \beta) \cup L^{\dim X - 1}$$

In an analogous fashion, the tangent space to any stable point $[(E, \theta)] \in M_{\text{Dol}}$ is $H^1(\text{End}(E, \theta))$ where $\text{End}(E, \theta)$ is the Higgs bundle $((\text{End}(E), \text{ad}(\theta)))$ [S2, 10.5]. The tangent cone is defined by the quadratic form associated to the cup product

$$H^1(\text{End}(E, \theta)) \times H^1(\text{End}(E, \theta)) \rightarrow H^2(\text{End}(E, \theta)).$$

Proposition 5.2. $T^*M_V$ is an open subset of $M_{\text{Dol}}$. If $M \subseteq M_V$ is an irreducible component, then $\dim M = \dim(\pi^{-1}M)/2$ and $M$ is an irreducible component of $h^{-1}(0) \cap \pi^{-1}M$.

Proof. As pointed out in the remarks preceding 4.3, etale local cross sections to $Q^s \rightarrow M_{\text{Dol}}^s$ exist. Therefore there is an etale neighbourhood $T \rightarrow M_{\text{Dol}}$ of any stable point $(E_0, \theta_0)$ and family of Higgs bundles $(E_t, \theta_t)$ parameterized by $T$, such that $[(E_0, \theta_0)]$ is precisely the image of $t$. Stability is an open condition [S2, 3.7], thus if $E_0$ were stable, then this would hold in open neighbourhood of $0 \in T$.

Note that $M_V \subseteq h^{-1}(0)$, and the dimension of any component of $h^{-1}(0)$ is half the dimension of an irreducible component of $M_{\text{Dol}}$ containing it by 4.12. Thus it suffices to prove that $\dim M \geq 2\dim(\pi^{-1}M)$. Choose a general point $[E] \in M$. Then consider the terms of low degree for the spectral sequence converging to the cohomology of $\text{End}(E, 0)$:

$$0 \rightarrow H^0(X, \Omega_X^1 \otimes \text{End}(E)) \rightarrow H^1(\text{End}(E, 0)) \rightarrow H^1(X, \text{End}(E)) \rightarrow 0$$

It follows that the tangent cone $C_1$ of $M_{\text{Dol}}$ at $[(E, 0)]$ maps to the tangent cone $C_2$ of $M_V$ at $[E]$, and the fiber over 0 is the cone $T^*_{[E]}$. This implies that the dimension of $C_1$ is less than or equal to $\dim C_2 + \dim T^*_{[E]} = 2\dim C_2$. □

The intersections of $S_m^{pq}$ with $M_V$ and $T^*M$ have a rather concrete description which is very close to the spirit of [GL1]. Let us write $T^*S_m^{pq}$ for $S_m^{pq} \cap T^*M_V$.

Proposition 5.2. If $(E, \theta)$ is polystable, then $[(E, \theta)] \in T^*S_m^{pq}$ if and only if the $p$th cohomology of the complex

$$\ldots \rightarrow H^q(X, \Omega_X^1 \otimes E) \stackrel{0}{\rightarrow} H^q(X, \Omega_X^{p+1} \otimes E) \stackrel{0}{\rightarrow} \ldots$$

has dimension greater than or equal to $m$. In particular $(E, 0) \in S_1^{pq}$ if and only if $H^0(X, \Omega_X^1 \otimes E) \neq 0$.

The proposition is a consequence of the next two lemmas.
Lemma 5.3. If $E$ is polystable, then $H^{pq}(E, \theta)$ is just the $p$th cohomology of the complex

$$
\ldots H^q(X, \Omega^p_X \otimes E) \xrightarrow{\partial \wedge} H^q(X, \Omega^{p+1}_X \otimes E) \xrightarrow{\partial \wedge} \ldots
$$

Proof. This is equivalent to the assertion that the spectral sequence degenerates at $E_2$. By the theorem of Donaldson and Uhlenbeck-Yau, $E$ carries a unitary flat connection $\nabla$. The lemma now follows by applying [A2, III 3.6] to the complex constructed in the proof of [loc.cit, IV 2.1]. (In the notation of that paper, the spectral sequence associated to $\nabla(0)$ coincides with the one above.)

An alternative argument can be given by modifying the proof of [GL1, 3.7] by replacing $\partial$ and $\bar{\partial}$ by the $(1,0)$ and $(0,1)$ parts of $\nabla$.

Lemma 5.4. If $(E_t, \theta_t)$ is a family of Higgs bundles parameterized by a smooth curve $T$ such that $E_0$ is polystable for some $0 \in T$ and $\dim H^{pq}(E_t, \theta_t) \geq m$ for $t \neq 0$ then $\dim H^{pq}(E_0, \theta_0) \geq m$.

Proof. We can assume that $T = \mathrm{Spec}R$. Consider the complex of $R$-modules

$$
\ldots H^q(X, \Omega^p_X \otimes E) \xrightarrow{\partial \wedge} H^q(X, \Omega^{p+1}_X \otimes E) \xrightarrow{\partial \wedge} \ldots
$$

Our assumptions imply that if one tensors this by the residue field of $t \neq 0$, the $p$th cohomology has dimension greater than or equal to $m$ (This is true regardless of whether the spectral sequence degenerates for $t \neq 0$, because at any rate $\dim E_2 \geq \dim E_{\infty}$). Therefore this property persists for $t = 0$, and the lemma follows from the previous one.

The next result gives a useful dimension estimate on the cohomology support loci. It can also be deduced using Green’s and Lazarsfeld’s deformation theory [GL1].

Proposition 5.5. Let $S_V$ be an irreducible component of $S^m_{pq} \cap M_V$, and let $S$ be the irreducible component of $S^m_{pq}$ containing $S_V$. Then for a general point $[E] \in S_V$

$$
\dim S_V = \dim (T^*_e M^*_V) \cap S = \frac{\dim S}{2}
$$

Proof. By 4.12 and 5.1, $\dim S_V = \frac{\dim S}{2}$. The remaining equality follows from $\dim S = \dim S_V + \dim (T^*_e M^*_V) \cap S$.

6. $\mathbb{C}^*$-invariance

In the last section, we made an explicit study of the intersection of the cohomology support loci with $T^*M^*_V$. There are some features of this geometry which extend to the whole space. First, recall:

**Theorem 6.1. (Simpson [S2])** There is an algebraic $\mathbb{C}^*$-action on $M_{Dol}$ given by $t : (E, \theta) \mapsto (E, t\theta)$. For any point $e \in M_{Dol}$ the limit of $te$ as $t \to 0$ exists.

Of particular interest are the fixed points. A Higgs bundle $(E, \theta)$ is called a complex variation of Hodge structure if $E$ admits a grading $\oplus E^p$ such that “Griffiths transversality” $\theta(E^p) \subset E^{p-1}$ holds. Given a complex variation of Hodge structure, let $T$ be the automorphism which acts by $t^{-p}$ on $E^p$ where $t \in \mathbb{C}^*$. Then $T$ induces an isomorphism $(E, \theta) \cong (E, t\theta)$. Therefore $[(E, \theta)]$ is a fixed point. Conversely,
Simpson has shown that all fixed points arise this way, and in fact if the underlying Higgs bundle \((E, \theta)\) is stable then the grading is uniquely determined (up to a shift of indices). Note that if \(\theta = 0\), then we can take \(E^0 = E\).

The previous theorem implies that the \(\mathbb{C}^*\)-action extends to a morphism of reduced schemes \(k^1 \times M_{Dol,red} \to M_{Dol,red}\). The image of \(\{0\} \times M_{Dol}\) is the fixed point set \(F\). Thus we get a morphism \(\pi : M_{Dol,red} \to F_{red}\) by composing

\[M_{Dol,red} \cong \{0\} \times M_{Dol,red} \to F_{red}.\]

\(\pi\) extends the map \(T^*M_V \to M_V\) constructed earlier, and exhibits \(M_{Dol}\) as a family of cones. The cohomology support loci are compatible with this conic al structure:

**Lemma 6.2.** \(\Sigma_{m,Dol}^k\) is invariant under the \(\mathbb{C}^*\)-action.

**Proof.** There is an isomorphism \(H^k(E, \theta) \cong H^k(E, t\theta)\) given on the level of complexes by multiplication by \(t^p\) on \(\Omega^p_X \otimes E\).

As these sets are closed, we obtain:

**Corollary 6.3.** \(\pi(\Sigma_{m,Dol}^k) \subset \Sigma_{m,Dol}^k\), so \(\Sigma_{m,Dol}^k\) is a family of subcones.

**Corollary 6.4.** Any irreducible component of \(\Sigma_{m,Dol}^k\) contains a complex variation of Hodge structure.

Conjectures of Simpson [S1] and Pantev [P] suggest that much more should be true, for example every component should contain an integral variation of Hodge structure. In the rank one case, the \(\mathbb{C}^*\) invariance of the above sets is a powerful constraint. In fact, it leads to a proof of theorem 1.1, [A1, S3].

### 7. Generic vanishing

In this section, we relax the condition on the Higgs bundles. We no longer insist that the Chern classes vanish, or even that they are locally free. Let \(X\) be a smooth projective \(d\)-dimensional variety A Higgs sheaf on \(X\) is a torsion free coherent \(O_X\)-module \(E\) together with morphism \(\theta : E \to \Omega^1_X \otimes E\) satisfying \(\theta \wedge \theta = 0\). There is a useful alternative viewpoint which we now recall. Let \(\pi : T^*X \to X\) be the cotangent bundle. A cotangent vector \(\eta \in T^*_X\) can also be viewed as an element of the fiber of \(\pi^*\Omega^1_X\). This defines a canonical section \(\Theta\) of \(\pi^*\Omega^1_X\). If \(E\) is a torsion free \(O_{T^*X}\)-module, such that \(\text{supp}(E) \to X\) is finite, then \(E = \pi_*\mathcal{E}\) is a torsion free coherent sheaf, and the map

\[E \overset{\theta}{\to} \Omega^1_X \otimes E \cong \pi_*(\mathcal{E} \otimes [O_{T^*X} \to \pi^*\Omega^1_X])\]

defines a Higgs structure on \(E\). In fact, every Higgs sheaf arises this way from a unique \(\mathcal{E}\) [S1] [S2]. The support of \(\mathcal{E}\) is precisely the set of eigenforms for \(\theta\). In other words, \(\text{supp}(\mathcal{E}) \cap T^*_X\) is the set of cotangent vectors \(\eta\) satisfying \(\theta_x(v) = \eta v\) for some nonzero \(v \in E_x\). Define the degeneracy locus of \((E, \theta)\) as \(\text{supp}(\mathcal{E}) \cap X\) where \(X \subset T^*X\) is identified with the zero section. More concretely, if \(E\) is locally free then \(x\) lies in the degeneracy locus if and only if \(\theta_x\) has a zero eigenform or, equivalently, is not injective. Let \(\text{degen}(\theta)\) be the dimension of the degeneracy locus of \((E, \theta)\) if it is nonempty, or \(-1\) otherwise. It will simplify matters to define \(\text{dim} \theta = -1\).
Given a Higgs sheaf \((E, \theta)\), the cohomology \(H^i(E, \theta)\) can be defined as the hypercohomology of the complex
\[
E \xrightarrow{\theta \wedge} \Omega^1_X \otimes E \xrightarrow{\theta \wedge} \Omega^2_X \otimes E \ldots
\]
as before.

**Proposition 7.1.** If \(X \subset T^*X\) is identified with the zero section, then for any Higgs sheaf \((E, \theta)\) and corresponding sheaf \(E\) on \(T^*X\),
\[
H^i(E, \theta) \cong \text{Ext}^i(O_X, E)
\]

**Proof.** The zero locus of \(\Theta\) is precisely \(X\). Therefore \(O_X\) is quasiisomorphic to the Koszul complex
\[
K_\bullet = \ldots \wedge^2 \pi^*T_X \to \pi^*T_X \to O_{T^*X} \to 0
\]
Therefore
\[
\text{Ext}^i(O_X, E) \cong H^i(\text{Hom}(K_\bullet, E)) \cong H^i(\pi_*(K_\bullet \otimes E))
\]
By the projection formula,
\[
\pi_*(K_\bullet \otimes E) = \Omega_X^\bullet \otimes E
\]
and the differentials are easily seen to be given by \(\theta \wedge\).

**Theorem 7.2.** If \((E, \theta)\) is a Higgs sheaf and \(i > \text{degen}(\theta) + d\), then \(H^i(E, \theta) = 0\).

**Proof.** The support of \(\text{Ext}^\bullet(O_X, E)\) lies in the degeneracy locus \(\text{supp}(E) \cap X\). Therefore the first part of the theorem follows from 7.1 and the spectral sequence
\[
H^p(\text{Ext}^q(O_X, E)) \Rightarrow \text{Ext}^{p+q}(O_X, E).
\]
The second statement follows by duality [S1, 2.5].

As an immediate corollary: \(\Sigma^1 \neq M_{Dol}\) if there exists a Higgs bundle \((E, \theta)\) with \(|i - d| > \text{degen}(\theta)\). When \(E\) is stable, this can be interpreted as a generic vanishing theorem:

**Corollary 7.3.** If \([E] \in M^s_V\), and \(\theta\) is a Higgs field on \(E\). Then in any neighbourhood of \([E]\) in \(M^s_V\) there is an \([E']\) such that
\[
H^q(X, \Omega^p_X \otimes E') = 0
\]
for \(|p + q - d| > \text{degen}(\theta)|\).

One gets a very concrete statement by restricting to diagonal Higgs fields.

**Corollary 7.4.** If \(\phi \in H^0(X, \Omega^1_X)\), then for any irreducible component \(M\) of \(M_V(X, n)\), there exists a semistable vector bundle \(E\), with \([E] \in M\) and
\[
H^q(X, \Omega^p_X \otimes E) = 0
\]
for \(|p + q - d| > \text{dim}\{x | \phi_x = 0\}\).

**Proof.** Apply the theorem to \((E, \theta)\) where \([E] \in M\) is a general point and \(\theta = \phi I_E\).

The above arguments can be refined to yield a codimension estimate. Given a Higgs field \(\theta\), clearly \(\text{degen}(\lambda \theta) = \text{degen}(\theta)\) for any \(\lambda \neq 0\). Thus \(\text{degen}\) gives a map, from the projectivization \(\mathbb{P}T^*_X[E]\) to the set of natural numbers, which is easily seen to be upper semicontinuous.
Corollary 7.5. If $S \subseteq \Sigma_1^i \cap M_\nu^r$ is an irreducible component, and $[E] \in S$ a general point. Assume that $M_V$ is smooth at $[E]$, then the codimension of $S$, in the irreducible component of $M_V$ containing it, is greater than $\dim \{ [\theta] \in \mathbb{P} T^*_{[E]} | \deg \theta \geq |i - d| \}$.

Proof. The smoothness assumption implies that $T^*_{[E]}$ is a vector space. By 5.5 the codimension of the projectivized cone $V = \mathbb{P}(T^*_{[E]} \cap \Sigma_1^i) \subset \mathbb{P}T^*_{[E]}$ coincides with the above codimension. By the theorem, $V$ cannot meet $\{ [\theta] \in \mathbb{P} T^*_{[E]} | \deg \theta \geq |i - d| \}$. Thus the corollary is a consequence of Bezout’s theorem.

8. Homotopy invariance

The cohomology support loci are clearly isomorphism invariants. But much more is true, namely the $\Sigma_{m,B}$ depend only on the homotopy type of the space. We will give some variants of this which will be quite useful for the construction of examples.

Proposition 8.1. Suppose that $f : X \to Y$ is a morphism of smooth projective varieties, with connected fibers, such that the induced map on homotopy groups $\pi_i(X^{an}) \to \pi_i(Y^{an})$ is an isomorphism for $i \leq k$ and a surjection for $i = k + 1$ with $k > 0$. Then for all $m$, $f^*(\Sigma_{m,Dol}(Y))$ is contained in $\Sigma_{m,Dol}(X)$ when $i = k + 1$ and equality holds when $i \leq k$.

Proof. It suffices to prove the corresponding statement for $\Sigma_B$. First note that by standard arguments in topology [Sp, pages 99-100] $f^{an}$ is homotopy equivalent to a fibration of topological spaces $f' : X' \to Y'$. The homotopy long exact sequence and the hypothesis implies that the first $k$ homotopy groups of the fiber $F$ vanish. Therefore by Hurwitz theorem $H^j(F, V) = 0$ for $j \leq k$ and V and arbitrary coefficient group. Thus the Leray-Serre spectral sequence yields an injection

$$H^i(Y^{an}, V) \to H^i(X^{an}, f^*V)$$

when $i = k + 1$ and an isomorphism when $i \leq k$.

Corollary 8.2. If $X \subseteq Y$ is general hyperplane section where $\dim Y \geq 3$, then under restriction: $\Sigma_{m,Dol}(Y) \subseteq \Sigma_{m,Dol}(X)$ for $k = \dim X$ and equality holds when $k < \dim X$.

Proof. The hypothesis of the theorem is fulfilled by the Lefschetz hyperplane theorem [M].

Proposition 8.3. If $f : X \to Y$ is a surjective map of smooth projective varieties, then $f^*\Sigma_{m,Dol}(Y) \subseteq \Sigma_{m,Dol}(X)$ for $k \leq \dim Y$.

Proof. Once again, we will work with $\Sigma_B$. We will also omit the superscript “an”. We will show that the map $f^* : H^k(Y, V) \to H^k(X, V)$ is injective for any local system $V$. To begin with, assume that $f$ is generically finite. Then one gets a transfer or Gysin homomorphism $f_! : H^k(X, V) \to H^k(Y, V)$ as the Poincaré dual of $H^{2\dim Y-i}(Y, V^*) \to H^{2\dim Y-i}(X, V^*)$. Arguing as in the case of constant coefficients, $\deg f_! f_*$ splits $f^*$.

In general, choose a complete intersection $Z \subseteq X$ of hyperplane sections, such that $f|_Z$ is finite. The map $H^k(Y, V) \to H^k(Z, f^*V)$ factors through $f^*$. Consequently $f^*$ is again injective.
We extend the notation $\Sigma^k_B(X)$ so as to allow $X$ to be any topological space. Of course, there won’t be an analogue $\Sigma_{Dol}$ in general. Our next task is to prove a Künneth decomposition. Given two spaces $X_1, X_2$, there is a morphism

$$\tau : M_B(X_1, n_1) \times M_B(X_2, n_2) \rightarrow M_B(X_1 \times X_2, n_1n_2)$$

given by $\tau(V_1, V_2) = p^*_1 V_1 \otimes p^*_2 V_2$. Also let

$$\sigma : M_B(X_1, n_1) \times M_B(X_2, n_2) \rightarrow M_B(X, n_1 + n_2)$$

be the morphism $\sigma(V_1, V_2) \rightarrow V_1 \oplus V_2$. When the $X_i$ are smooth projective, similar morphisms can be defined for $M_{Dol}$.

**Proposition 8.4.** $\Sigma^k_{1,B}(X_1 \times X_2 \times \ldots X_N, n)$ is the union of images of products

$$\sigma(\tau(\Sigma^1_{1,B}(X_1, n_1) \times \ldots \Sigma^k_{1,B}(X_N, n_N)) \times M_B(X_1 \times \ldots X_N, n - n_1n_2\ldots n_N))$$

as $k_i$ and $n_i$ range over partitions of $k$ and factorizations of integers no greater than $n$ respectively.

We will prove the proposition when $N = 2$. Let $G_i = \pi_1(X_i)$, then of course $G = G_1 \times G_2$ is the fundamental group of the produce. The classification of irreducible $G$-modules is well known, at least for finite groups.

**Lemma 8.5.** Any irreducible finite dimensional complex representation of $G$ is the form $V_1 \otimes_C V_2$ where $V_i$ are irreducible $G_i$ representations.

**Proof.** Let $V$ be a nonzero irreducible representation of $G$. Then it contains a nonzero irreducible $G_1$-module $V_1$. The vector space $G_2V_1 = \sum_{g \in G_2} gV_1$ is a $G$-module, so it must coincide with $V$. Therefore $V$ is isomorphic to direct sum of copies of $V_1$, so by standard arguments [CR, section 27], $R = \text{End}_{G_1}(V)$ is isomorphic to the $C$-endomorphism ring of an irreducible $R$-submodule $V_2$. $V_2$ is necessarily a $G_2$-module, and the natural map $V_1 \otimes V_2 \rightarrow V$ is easily seen to be an isomorphism. $V_2$ must be irreducible, since otherwise, we could find a smaller submodule $V_2'$, and the $G$-module $V_1 \otimes V_2'$ would violate irreducibility of $V$. \qed

The above proposition is now a consequence of the usual Künneth formula for cohomology. If $X$ is a product of subvarieties then, there is a similar decomposition for $\Sigma_{Dol}^k$.

9. Examples

In this final section, we will give some explicit examples. To simplify notation, we will drop the decorations “an, Dol, B”; it should be clear from context where we are. In addition to the previous results, we will need the following well known fact:

**Lemma 9.1.** The Euler characteristic of a rank $n$ local system $V$ on a variety $X$ is $n$ times the Euler characteristic $e(X)$ of $X$.

**Proof.** There are a number of ways to prove this; perhaps the easiest is choose a finite triangulation of $X$, then observe that each term $S^\bullet(V)$ in the simplicial cochain complex with coefficients in $V$ is (noncanonically) isomorphic to $S^\bullet(C)^\otimes n$. \qed
Example 9.2. Let $C$ be a smooth projective curve of genus $g > 1$. If $V$ is a nontrivial irreducible rank $n$ local system then $H^0(C, V) = 0$ and $H^2(C, V) \cong H^0(C, V^*) = 0$. Therefore $\dim H^1(C, V) = 2n(1-g)$. So that $\Sigma_{2m(1-g)}^1(C, n) = M(C, n)$ because irreducible local systems are dense in $M(C, n)$. If $m > m_0 = n(1-g)$, then $\Sigma_{2m}^1(C, n)$ is locus of semisimple local systems with a trivial summand of rank $m - m_0$. If $X = C_1 \times C_2$ is a product of two curves, then

$$M(X, n) = \bigcup_{a+b=n} M(C_1, a) \times M(C_2, b)$$

and $V \in \Sigma_1^1(X, n)$ if and only if it has a direct summand of the form $p_i^* V_i$ by Example 9.4. Hitchin’s map respects this decomposition. More precisely, the product

$$h_{C_1} \times h_{C_2} : M(C_1, a) \times M(C_2, b) \to S_a(C_1) \times S_b(C_2)$$

factors through the restriction of $h_X$ to the above component, and the map

$$h_X(M(C_1, a) \times M(C_2, b)) \to S_a(C_1) \times S_b(C_2)$$

is a bijection. To see this, let

$$(E, \theta) = (p_1^* E_1 \otimes p_2^* E_2, p_1^* \theta_1 \otimes I_{E_2} + I_{E_1} \otimes p_2^* \theta_2)$$

be a point on this component. Then the projection of $h_X(E, \theta)$ to $S_a(C_1) \times S_b(C_2)$ is $(h_{C_1}(E_1, \theta_1), h_{C_2}(E_2, \theta_2))$. Moreover, the value of $h_X(E, \theta)$ at any point $x = (x_1, x_2) \in X$ is determined by the eigenvalues of $\theta_i(x_i)$ which is, in turn, determined by $h(E_i, \theta_i)(x_i)$.

The generic vanishing theorem, shows that $\Sigma_1^1(X, n)$ is proper whenever the $X$ has 1-form without zeros. When $X$ is an abelian variety, we can say much more.

Example 9.3. Let $A$ be an abelian variety. Then for each $k$, $\Sigma_1^k(A, n)$ is locus of semisimple local systems with a trivial direct summand. To prove this, notice that $A$ is homeomorphic to a product of circles, and by Example 9.4, it suffices to treat the case of a single circle. In this case its clear, as $H^0(S^1, V)$ and $H^1(S^1, V)$ are respectively just the invariants and coinvariants of $V$.

Example 9.4. Let $X \subset A$ be an ample divisors. Then $\Sigma_1^k(X, n)$ has the same description as in the previous example when $k < \dim X$ by Example 9.4. The same holds for $k > \dim X$ by duality. An easy Chern class argument shows that the Euler characteristic of $X$ is nonzero. Consequently $\Sigma_1^{\dim X}(X, n)$ is all of $M(X, n)$.

The last example has an abelian fundamental group. Now we will show that any group $\Gamma$ which can be realized as the fundamental group of a smooth projective variety, is realizable as the fundamental group of smooth projective variety with $\Sigma_m^k(x, n)$ as large as possible, for any $m, n$ and $k \geq 2$. Note that $\Sigma_1$ cannot be changed without altering the fundamental group.

Example 9.5. Let $Y$ be smooth projective variety with fundamental group $\Gamma$. We can arrange that $\dim Y = k + 1$ as follows. First after replacing $Y$ with a product with a projective space, we can assume that $\dim Y \geq k + 1$. Now replace $Y$ by a $k + 1$ dimensional generic complete intersection. Choose a very ample line bundle $O_Y(1)$ on $Y$, and let $X_d$ be the general member of the linear system associated to $O_Y(d)$. Then $\Sigma_m^k(Y) = \Sigma_m^k(X_d) = \Sigma_m^{2k-i}(X_d)$ for $i < k$ by Example 9.4 and duality. Thus these sets are independent of $d$. The Euler characteristic is just the Chern number $c_k(X_d)$, and this can be computed to obtain a polynomial of degree $k$ in $d$. Thus
the Euler characteristic approaches $\pm \infty$ as $d \to \infty$. Therefore after choosing $m$, we can find $d$ large enough so that $\Sigma^k_m(X_d, n) = M(X_d, n)$

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