Fixed volume discrepancy in the periodic case

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Abstract

The smooth fixed volume discrepancy in the periodic case is studied here. It is proved that the Frolov point sets adjusted to the periodic case have optimal in a certain sense order of decay of the smooth periodic discrepancy. The upper bounds for the $r$-smooth fixed volume periodic discrepancy for these sets are established.

1 Introduction

Discrepancy theory is a classical well established area of research in geometry and numerical integration (see [2], [8], [15], [17]). Recently, in [18], a new phenomenon has been discovered. A typical upper bound for the discrepancy of a good point set of cardinality $m$ is $\leq C(d)m^{-1}(\log m)^{d-1}$ and for the $r$-smooth discrepancy $\leq C(d,r)m^{-r}(\log m)^{d-1}$. These bounds are too rough for functions with small volume of their support. It was proved in [18] that for the Fibonacci point sets ($d = 2$) and the Frolov point sets we can improve the above upper bound to $\leq C(d,r)m^{-r}(\log mV)^{d-1}$, $V \geq c(d,r)/m$, for the functions with the volume of their support equals $V$. We establish a similar phenomenon for the $r$-smooth fixed volume discrepancy in the periodic case.

We begin with a classical definition of discrepancy ("star discrepancy", $L_\infty$-discrepancy) of a point set $T \colon= \xi \colon= \{\xi^{\mu}\}_{\mu=1}^m \subset [0,1)^d$. Let $d \geq 2$ and $[0,1)^d$ be the $d$-dimensional unit cube. For $\mathbf{x}, \mathbf{y} \in [0,1)^d$ with $\mathbf{x} = (x_1, \ldots, x_d)$ and $\mathbf{y} = (y_1, \ldots, y_d)$ we write $\mathbf{x} < \mathbf{y}$ if this inequality holds coordinate-wise. For $\mathbf{x} < \mathbf{y}$ we write $[\mathbf{x}, \mathbf{y})$ for the axis-parallel box $[x_1, y_1) \times \cdots \times [x_d, y_d)$ and define

$$B \colon= \{[\mathbf{x}, \mathbf{y}) : \mathbf{x}, \mathbf{y} \in [0,1)^d, \mathbf{x} < \mathbf{y}\}.$$ 

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Introduce a class of special $d$-variate characteristic functions

$$
\chi^d := \{ \chi_{[0,b]}(x) := \prod_{j=1}^{d} \chi_{[0,b_j]}(x_j), \quad b_j \in [0,1), \quad j = 1, \ldots, d \}
$$

where $\chi_{[a,b]}(x)$ is a univariate characteristic function of the interval $[a, b)$. The classical definition of discrepancy of a set $T$ of points $\{\xi^1, \ldots, \xi^m\} \subset [0,1]^d$ is as follows

$$
D(T, m, d) := \max_{b \in [0,1]^d} \left| \prod_{j=1}^{d} b_j - \frac{1}{m} \sum_{\mu=1}^{m} \chi_{[0,b]}(\xi^\mu) \right|.
$$

It is equivalent within multiplicative constants, which may only depend on $d$, to the following definition

$$
D^1(T) := \sup_{B \in \mathcal{B}} \left| \text{vol}(B) - \frac{1}{m} \sum_{\mu=1}^{m} \chi_B(\xi^\mu) \right|, \quad (1.1)
$$

where for $B = [a, b) \in \mathcal{B}$ we denote $\chi_B(x) := \prod_{j=1}^{d} \chi_{[a_j,b_j]}(x_j)$. Moreover, we consider the following optimized version of $D^1(T)$

$$
D^{1,o}(T) := \inf_{\lambda_1, \ldots, \lambda_m} \sup_{B \in \mathcal{B}} \left| \text{vol}(B) - \sum_{\mu=1}^{m} \lambda_\mu \chi_B(\xi^\mu) \right|, \quad (1.2)
$$

In the definition of $D^1(T)$ and $D^{1,o}(T)$ – the 1-smooth discrepancy – we use as a building block the univariate characteristic function. In numerical integration $L_1$-smoothness of a function plays an important role. A characteristic function of an interval has smoothness 1 in the $L_1$ norm. This is why we call the corresponding discrepancy characteristics the 1-smooth discrepancy. In the definition of $D^2(T)$, $D^{2,o}(T)$, $D^2(T, V)$, and $D^{2,o}(T, V)$ (see below and [18]) we use the hat function $h_{[-u,u)}(x) = |u - x|$ for $|x| \leq u$ and $h_{[-u,u)}(x) = 0$ for $|x| \geq u$ instead of the characteristic function $\chi_{[-u/2,u/2)}(x)$. Function $h_{[-u,u)}(x)$ has smoothness 2 in $L_1$. This fact gives the corresponding name. Note that

$$
h_{[-u,u)}(x) = \chi_{[-u/2,u/2)}(x) \ast \chi_{[-u/2,u/2)}(x),
$$

where

$$
f(x) \ast g(x) := \int_{\mathbb{R}} f(x-y)g(y)dy.
$$
Now, for \( r = 1, 2, 3, \ldots \) we inductively define
\[
h^1(x, u) := \chi_{[-u/2,u/2]}(x), \quad h^2(x, u) := h_{[-u,u]}(x),
\]
\[h^r(x, u) := h^{r-1}(x, u) \ast h^1(x, u), \quad r = 3, 4, \ldots.
\]
Then \( h^r(x, u) \) has smoothness \( r \) in \( L_1 \) and has support \( (-ru/2, ru/2) \). Represent a box \( B \in \mathcal{B} \) in the form
\[
B = \prod_{j=1}^{d} [x_j^0 - ru_j/2, x_j^0 + ru/2)
\]
and define
\[
h^r_B(x) := h^r(x, x^0, u) := \prod_{j=1}^{d} h^r(x_j - x_j^0, u_j).
\]

In \cite{18} we modified definitions (1.1) and (1.2), replacing the characteristic function \( \chi_B \) by a smoother hat function \( h^r_B \).

The \( r \)-smooth discrepancy is now defined as
\[
D^r(T) := \sup_{B \in \mathcal{B}} \left| \int h^r_B(x)dx - \frac{1}{m} \sum_{\mu=1}^{m} h^r_B(\xi^\mu) \right|.
\]
and its optimized version as
\[
D^{r,o}(T) := \inf_{\lambda_1, \ldots, \lambda_m} \sup_{B \in \mathcal{B} : \text{vol}(B) = V} \left| \int h^r_B(x)dx - \sum_{\mu=1}^{m} \lambda_{\mu} h^r_B(\xi^\mu) \right|.
\]
Note that the known concept of \( r \)-discrepancy (see, for instance, \cite{15}, \cite{17}, and Section 4 below) is close to the above concept of \( r \)-smooth discrepancy.

Along with \( D^r(T) \) and \( D^{r,o}(T) \) we consider a more refined quantity – \( r \)-smooth fixed volume discrepancy – defined as follows (see \cite{18})
\[
D^r(T, V) := \sup_{B \in \mathcal{B} : \text{vol}(B) = V} \left| \int h^r_B(x)dx - \frac{1}{m} \sum_{\mu=1}^{m} h^r_B(\xi^\mu) \right|;
\]
\[
D^{r,o}(T, V) := \inf_{\lambda_1, \ldots, \lambda_m} \sup_{B \in \mathcal{B} : \text{vol}(B) = V} \left| \int h^r_B(x)dx - \sum_{\mu=1}^{m} \lambda_{\mu} h^r_B(\xi^\mu) \right|.
\]
Clearly, 

\[ D^r(T) = \sup_{V \in (0,1]} D^r(T, V). \]

In Section 2 of this paper we study a periodic analog of the quantities \( D^r,o(T, V) \) for a set \( T \) generated with a help of the Frolov lattice. We first describe the periodic analogs of the above discrepancy concepts. For a function \( f \in L_1(\mathbb{R}^d) \) with a compact support we define its periodization \( \tilde{f} \) as follows

\[ \tilde{f}(x) := \sum_{m \in \mathbb{Z}^d} f(m + x). \]

Consider \( u \in (0, \frac{1}{2}]^d \). Then for all \( z \in [0,1)^d \) we have

\[ \text{supp}(h^r(x, z, u)) \subset (-r/4, 1 + r/4)^d. \]

Now, for each \( z \in [0,1)^d \) consider a periodization of function \( h^r(x, z, u) \) in \( x \) with period 1 in each variable \( \tilde{h}^r(x, z, u) \). It is convenient for us to use the following abbreviated notation for the product

\[ pr(u) := pr(u, d) := \prod_{j=1}^d u_j. \]

Define the corresponding periodic discrepancy as follows (we only give one modified definition)

\[ \tilde{D}^r,o(T, v) := \]

\[ \inf_{\lambda_1, \ldots, \lambda_m} \sup_{z \in [0,1)^d ; u; pr(u) = v} \left| \int_{[0,1)^d} \tilde{h}^r(x, z, u)dx - \sum_{\mu=1}^m \lambda_\mu \tilde{h}^r(\xi^\mu, z, u) \right|. \quad (1.7) \]

Second we describe the Frolov cubature formulas. We refer the reader for detailed presentation of the theory of the Frolov cubature formulas to [15], [17], [19], and [5]. The following lemma plays a fundamental role in the construction of such point sets (see [15] for its proof).

**Lemma 1.1.** There exists a matrix \( A \) such that the lattice \( L(m) = Am \)

\[ L(m) = \begin{bmatrix} L_1(m) \\ \vdots \\ L_d(m) \end{bmatrix}, \]
where \( \mathbf{m} \) is a (column) vector with integer coordinates, has the following properties

1. \( \prod_{j=1}^{d} L_j(\mathbf{m}) \geq 1 \) for all \( \mathbf{m} \neq \mathbf{0} \);

2. each parallelepiped \( P \) with volume \( |P| \) whose edges are parallel to the coordinate axes contains no more than \( |P| + 1 \) lattice points.

Let \( a > 1 \) and \( A \) be the matrix from Lemma 1.1. We consider the cubature formula

\[
\Phi(a, A)(f) := (a^d | \det A|)^{-1} \sum_{\mathbf{m} \in \mathbb{Z}^d} f \left( \frac{(A^{-1})^T \mathbf{m}}{a} \right)
\]

for \( f \) with compact support.

We call the Frolov point set the following set associated with the matrix \( A \) and parameter \( a \)

\[
\mathcal{F}(a, A) := \left\{ \left( \frac{(A^{-1})^T \mathbf{m}}{a} \right) \right\}_{\mathbf{m} \in \mathbb{Z}^d \cap [0, 1)^d} =: \{\mathbf{z}^n\}_{n=1}^{N}.
\]

Clearly, the number \( N = |\mathcal{F}(a, A)| \) of points of this set does not exceed \( C(A)a^d \). The following results were obtained in [18].

**Theorem 1.1.** Let \( r \geq 2 \). There exists a constant \( c(d, A, r) > 0 \) such that for any \( V \geq V_0 := c(d, A, r)a^{-d} \) we have for all \( B \in \mathcal{B} \), \( \text{vol}(B) = V \),

\[
|\Phi(a, A)(\hat{h}_B^n) - \hat{h}_B^n(0)| \leq C(d, A, r)a^{-rd}(\log(2V/V_0))^{d-1}. \tag{1.8}
\]

**Corollary 1.1.** For \( r \geq 2 \) there exists a constant \( c(d, A, r) > 0 \) such that for any \( V \geq V_0 := c(d, A, r)a^{-d} \) we have

\[
D^{r,a}(\mathcal{F}(a, A), V) \leq C(d, A, r)a^{-rd}(\log(2V/V_0))^{d-1}. \tag{1.9}
\]

In Section 2 we extend Theorem 1.1 and Corollary 1.1 to the periodic case. For that we need to modify the set \( \mathcal{F}(a, A) \) and the cubature formula \( \Phi(a, A) \).

For \( \mathbf{y} \in \mathbb{R}^d \) denote \( \{\mathbf{y}\} := \{y_1, \ldots, y_d\} \), where for \( y \in \mathbb{R} \) notation \( \{y\} \) means the fractional part of \( y \). For given \( a \) and \( A \) denote

\[
\eta := \{\eta^\mu\}_{\mu=1}^{m} := \left\{ \left( \frac{(A^{-1})^T \mathbf{m}}{a} \right) \right\}_{\mathbf{m} \in \mathbb{Z}^d \cap [-1/2, 3/2)^d}
\]

and

\[
\xi := \{\xi^\mu\}_{\mu=1}^{m} := \{\{\eta^\mu\}\}_{\mu=1}^{m}. \tag{1.10}
\]
Clearly, \( m \leq C(A)a^d \). Next, let \( w(t) \) be infinitely differentiable on \( \mathbb{R} \) function with the following properties

\[
\text{supp}(w) \subset (-1/2, 3/2) \quad \text{and} \quad \sum_{k \in \mathbb{Z}} w(t + k) = 1. \tag{1.11}
\]

Denote \( w(x) := \prod_{j=1}^d w(x_j) \). Then for \( f(x) \) defined on \([0, 1)^d\) we consider the cubature formula

\[
\Phi(a, A, w)(f) := \sum_{\mu=1}^m w_\mu f(\xi^\mu), \quad w_\mu := w(\eta^\mu).
\]

In Section 2 we prove the following analogs of Theorem 1.1 and Corollary 1.1.

**Theorem 1.2.** Let \( r \geq 2 \). There exists a constant \( c(d, A, r) > 0 \) such that for any \( v \geq v_0 := c(d, A, r)a^{-d} \) we have for all \( u \in (0, 1/2)^d \), \( pr(u) = v \), and \( z \in [0, 1)^d \)

\[
|\Phi(a, A, w)(\tilde{h}^r(\cdot, z, u)) - \tilde{h}^r(0, z, u)| \leq C(d, A, r, w)a^{-rd}(\log(2/v/v_0))^{d-1}.
\]

**Corollary 1.2.** For \( r \geq 2 \) there exists a constant \( c(d, A, r) > 0 \) such that for any \( v \geq v_0 := c(d, A, r)a^{-d} \) we have for the point set \( \xi \) defined by (1.10)

\[
\check{D}^{r,o}(\xi, v) \leq C(d, A, r)a^{-rd}(\log(2/v/v_0))^{d-1}.
\]

In particular, Theorem 1.2 implies that the \( r \)-smooth periodic discrepancy

\[
\check{D}^{r,o} := \inf_{\lambda_1, \ldots, \lambda_m} \sup_{z \in [0, 1]^d, u \in (0, 1/2)^d} \left| \int_{[0, 1)^d} \tilde{h}^r(x, z, u)dx - \sum_{\mu=1}^m \lambda_\mu \tilde{h}^r(\xi^\mu, z, u) \right| \tag{1.12}
\]

satisfies the bound (for \( r \in \mathbb{N}, r \geq 2 \))

\[
\check{D}^{r,o} \leq C(d, r)m^{-r}(\log m)^{d-1}. \tag{1.13}
\]

In Section 3 we show that the bound (1.13) cannot be improved for a natural class of weights \( \lambda_1, \ldots, \lambda_m \) used in the optimization procedure in the definition of \( \check{D}^{r,o} \), namely, for weights, satisfying

\[
\sum_{\mu=1}^m |\lambda_\mu| \leq B.
\]
2 Point sets based on the Frolov lattice

We prove Theorem 1.2 in this section. Let \( f(x) \) be 1-periodic in each variable function integrable on \( \Omega_d := [0, 1)^d \). Then function \( w(x)f(x) \) is integrable on \( \mathbb{R}^d \) and has a finite support: \( \text{supp}(w f) \subset (-1/2, 3/2)^d \). We note that the idea of applying the Frolov cubature formulas to the product of the form \( w(x)f(x) \), where one function is very smooth and takes care of the support of the product (in our case it is \( w(x) \)) and the other function has a prescribed decay of its Fourier coefficients (in our case it is \( f(x) = h^r(x, z, u) \)), goes back to the very first paper [7] on the Frolov cubature formulas. Further detailed development of this idea was made in [9].

Property (1.11) implies

\[
\int_{\mathbb{R}^d} w(x)f(x)dx = \sum_{k \in \mathbb{Z}^d} \int_{\Omega_d} w(k + x)f(k + x)dx = \sum_{k \in \mathbb{Z}^d} \int_{\Omega_d} w(k + x)f(x)dx
\]

\[
= \int_{\Omega_d} \left( \sum_{k \in \mathbb{Z}^d} w(k + x) \right) f(x)dx = \int_{\Omega_d} f(x)dx. \tag{2.1}
\]

Next, using periodicity of \( f \) we write

\[
\Phi(a, A, w)(f) := \sum_{\mu = 1}^m w_\mu f(\xi^\mu) = \sum_{\mu = 1}^m w(\eta^\mu)f(\eta^\mu) = \Phi(a, A)(wf). \tag{2.2}
\]

Thus, for a 1-periodic function \( f \) we have

\[
\int_{\Omega_d} f(x)dx - \Phi(a, A, w)(f) = \int_{\mathbb{R}^d} w(x)f(x)dx - \Phi(a, A)(wf). \tag{2.3}
\]

We use identity (2.3) for \( f(x) = \tilde{h}^r(x, z, u) \) and estimate the right hand side of (2.3). It is clear that it is sufficient to estimate

\[
\int_{\mathbb{R}^d} w(x)\tilde{h}^r(x, z, u)dx - \Phi(a, A)(w\tilde{h}^r). \tag{2.4}
\]

In the case \( w(x) = 1 \) the above error is bounded in [18]. We follow a similar way and use some technical lemmas from [18]. Denote for \( f \in L_1(\mathbb{R}^d) \)

\[
\mathcal{F}(f)(y) := \hat{f}(y) := \int_{\mathbb{R}^d} f(x)e^{-2\pi i(y,x)}dx.
\]
For a function $f$ with finite support and absolutely convergent series \( \sum_{m \in \mathbb{Z}^d} \hat{f}(aA\mathbf{m}) \) we have for the error of the Frolov cubature formula (see [15])

\[
\Phi(a, A)(f) - \hat{f}(0) = \sum_{m \neq 0} \hat{f}(aA\mathbf{m}).
\] (2.5)

We begin with the following simple univariate lemma.

**Lemma 2.1.** Suppose that $r \in \mathbb{N}$ and $f \in L_1(\mathbb{R})$ satisfies the following conditions

\[
| \text{supp}(f) | \leq C_1 u, \quad |f(x)| \leq C_2 u^{r-1}, \quad \| \Delta_t f \|_1 \leq C_3 |t|^r,
\]

where $\Delta_t f(x) := f(x) - f(x + t)$, $\Delta_t^r := (\Delta_t)^r$. Then,

\[
| \hat{f}(y) | \leq C_4 \min\left( u^r, \frac{1}{|y|^r} \right).
\]

**Proof.** It is easy to see that

\[
\mathcal{F}(f)(y) := \hat{f}(y) = \mathcal{F}\left( \frac{1}{2} \Delta_{\frac{1}{2}|y|^r} f \right)(y).
\]

Iterating the above identity $r$ times we obtain

\[
\mathcal{F}(f)(y) = \mathcal{F}\left( \left( \frac{1}{2} \Delta_{\frac{1}{2}|y|^r} \right)^r f \right)(y).
\]

Using the above representation and our assumptions on $f$, we get

\[
| \mathcal{F}(f)(y) | \leq C_4 \min\left( u^r, \frac{1}{|y|^r} \right).
\]

The lemma is proved.

We return back to estimation of (2.4). We have

\[
w(\mathbf{x}) h^r(\mathbf{x}, \mathbf{z}, \mathbf{u}) = \prod_{j=1}^d w(x_j) h^r(x_j, z_j, u_j).
\]
It is easy to check that \( f(x) := w(x)h^r(x, z, u) \) satisfies conditions of Lemma 2.1. Therefore, for \( f(x) := w(x)h^r(x, z, u) \) by Lemma 2.1 we have

\[
|\hat{f}(y)| \leq C(d, r, w) \prod_{j=1}^{d} \min \left( u_j^{r/2}, \frac{1}{|y_j|^r} \right)
\]

\[
= C(d, r, w) \prod_{j=1}^{d} \left( \frac{u_j}{|y_j|} \right)^{r/2} \min \left( |y_j u_j|^{r/2}, \frac{1}{|y_j u_j|^{r/2}} \right). \tag{2.6}
\]

Consider

\[
\sigma^r(n, u) := \sum_{|s|_1=n} \prod_{j=1}^{d} \min \left( (2^{s_j} u_j)^{r/2}, \frac{1}{(2^{s_j} u_j)^{r/2}} \right), \quad v \in \mathbb{N}_0.
\]

The following lemma was established in [18].

**Lemma 2.2.** Let \( n \in \mathbb{N}_0 \) and \( u \in (0, 1/2]^d \). Then we have the following inequalities.

(I) Under condition \( 2^n pr(u) \geq 1 \) we have

\[
\sigma^r(n, u) \leq C(d) \left( \frac{\log(2^{n+1} pr(u))}{(2^n pr(u))^{r/2}} \right)^{d-1}. \tag{2.7}
\]

(II) Under condition \( 2^n pr(u) \leq 1 \) we have

\[
\sigma^r(n, u) \leq C(d) \left( 2^n pr(u) \right)^{r/2} \left( \log \frac{2}{2^n pr(u)} \right)^{d-1}. \tag{2.8}
\]

For \( s \in \mathbb{N}_0^d \) – the set of vectors with nonnegative integer coordinates, define

\[
\rho(s) := \{ k \in \mathbb{Z}^d : [2^{s_j-1}] \leq |k_j| < 2^{s_j}, \quad j = 1, \ldots, d \}
\]

where \([a]\) denotes the integer part of \( a \).

By (2.5) we have for the error

\[
\delta := \left| \int f(x)dx - \Phi(a, A)(f) \right| \leq \sum_{n=1}^{\infty} \sum_{|s|_1=n} \sum_{m:a Am \in \rho(s)} \left| \hat{f}(a Am) \right|.
\]
Lemma 1.1 implies that if \( n \neq 0 \) is such that \( 2^n < a^d \) then for \( s \) with \( \|s\|_1 = n \) there is no \( m \) such that \( aA_m \in \rho(s) \). Let \( n_0 \in \mathbb{N} \) be the smallest number satisfying \( 2^{n_0} \geq a^d \). Then we have
\[
\delta \leq \sum_{n=n_0}^{\infty} \sum_{\|s\|_1=n} \sum_{m:aA_m \in \rho(s)} |\hat{f}(aA_m)|. \tag{2.9}
\]

Lemma 1.1 implies that for \( n \geq n_0 \) we have
\[
|\rho(s) \cap \{aA_m\}_{m \in \mathbb{Z}^d}| \leq C_1 2^{n-n_0}, \quad \|s\|_1 = n. \tag{2.10}
\]

Using (2.10) we obtain by (2.6) for \( f(x) = w(x)h^r(x, z, u) \)
\[
\delta \leq C(d, r, w) \sum_{n=n_0}^{\infty} \sum_{\|s\|_1=n} 2^{n-n_0}(pr(u)2^{-n})^{r/2} \prod_{j=1}^{d} \min \left( (2^{s_j}u_j)^{r/2}, \frac{1}{(2^{s_j}u_j)^{r/2}} \right).
\]

We now assume that the constant \( c(d, A) \) is such that \( v_0 = 2^{-n_0} \). Then for \( v \geq v_0 \) we have \( 2^n pr(u) \geq 1 \), \( n \geq n_0 \). Using inequality (2.7) of Lemma 2.2 we obtain from here
\[
\delta \leq C(d, r, w)2^{-n_0} \sum_{n=n_0}^{\infty} 2^{-n(r-1)} \left( \log (2^{n+1}pr(u)) \right)^{d-1}
\]
\[
\leq C(d, r, w)2^{-r n_0} \left( \log (2v/v_0) \right)^{d-1} \leq C(d, r, w)a^{-r} \left( \log (2v/v_0) \right)^{d-1}.
\]

### 3 A lower bound for the smooth periodic discrepancy

In this section we prove a lower bound for an analog of the smooth periodic discrepancy \( \tilde{D}^{r,o}(T) \) for any set \( T \) of fixed cardinality. In fact we prove a weaker result. In the definition of optimal smooth periodic discrepancy

\[
\tilde{D}^{r,o}(T) := \inf_{\lambda_1, \ldots, \lambda_m} \sup_{z \in [0,1]^d : u \in (0,1/2]^d} \left| \int_{[0,1]^d} \tilde{h}^r(x, z, u)dx - \sum_{\mu=1}^{m} \lambda_\mu \tilde{h}^r(\xi^\mu, z, u) \right| \tag{3.1}
\]
we allow to optimize over all weights $\lambda_1, \ldots, \lambda_m$. We prove a lower bound under an extra (albeit mild) restriction on the weights. Let $B$ be a positive number and $Q(B, m)$ be the set of cubature formulas

$$\Lambda_m(f, \xi) := \sum_{\mu=1}^m \lambda_\mu f(\xi^\mu), \quad \xi := \{\xi^\mu\}_{\mu=1}^m \subset [0, 1)^d, \quad \lambda_\mu \in \mathbb{R}, \quad \mu = 1, \ldots, m,$$

satisfying the additional condition

$$\sum_{\mu=1}^m |\lambda_\mu| \leq B. \quad (3.2)$$

We obtain the lower estimates for the quantities

$$\tilde{D}_r,B^m := \inf_{\Lambda_m(\cdot, \xi) \in Q(B, m)} \sup_{\xi \in [0,1)^d; \nu \in (0,1/2)^d} \left| \int_{[0,1)^d} \tilde{h}_r(x, z, u) dx - \Lambda(\tilde{h}_r(\cdot, z, u), \xi) \right|.$$

We prove the following relation.

**Theorem 3.1.** Let $r \in \mathbb{N}$ be an even number. Then

$$\tilde{D}_r,B^m \geq C(r, B, d)m^{-r} \log m, \quad C(r, B, d) > 0.$$

**Proof.** Theorem 3.1 is an analog of Theorem 3 from [16] (see also [17]). Our proof follows the ideas from [16]. We use a notation

$$\Lambda(k) := \Lambda_m(e^{i2\pi(k,x)}, \xi) = \sum_{\mu=1}^m \lambda_\mu e^{i2\pi(k,\xi^\mu)}.$$

Let a set $T$ with cardinality $|T| = m$ be given. We specify $\xi := T$ and consider along with the cubature formula $\Lambda(\cdot, \xi)$ the following auxililary cubature formula

$$\Lambda^*(f) := \sum_{\mu=1}^m \lambda_\mu \Lambda_m(f(x - \xi^\mu), \xi).$$

Then

$$\Lambda^*(k) = \Lambda^*(e^{i2\pi(k,x)}) = \sum_{\nu=1}^m \lambda_\nu \sum_{\mu=1}^m \lambda_\mu e^{i2\pi(k,(\xi^\mu - \xi^\nu))} = |\Lambda(k)|^2. \quad (3.3)$$
Suppose that for each \( f(x) := f(x, u) := \tilde{h}^r(x, z, u) \) we have for all \( z \in [0, 1)^d \) and \( u \in (0, \frac{1}{2}]^d \) the bound

\[
|\hat{f}(0, u) - \Lambda_m(f, \xi)| \leq e_m. \tag{3.4}
\]

Integrating over \([0, 1)^d\) with respect to \( z \) we get from here

\[
|\hat{f}(0, u)(1 - \Lambda_m(0))| \leq e_m. \tag{3.5}
\]

In particular, this implies

\[
|1 - \Lambda_m(0)| \leq c(r, d)e_m. \tag{3.6}
\]

Therefore, we have

\[
|\Lambda^*(f) - \Lambda^*(0)\hat{f}(0)| \leq |\Lambda^*(f) - \Lambda_m(0)\hat{f}(0)| + |(\Lambda_m(0) - \Lambda_m(0)^2)\hat{f}(0)|
\]

\[
\leq \left| \sum_{\mu=1}^m \lambda_{\nu}\left(\Lambda_m(f(x - \xi_{\nu}), \xi) - \hat{f}(0)\right) \right| + Be_m \leq 2Be_m. \tag{3.7}
\]

We now need a known result on the lower bound for the weighted sum of \(|\Lambda(k)|^2\) (see [15] and [17]).

**Lemma 3.1.** The following inequality is valid for any \( r > 1 \)

\[
\sum_{k \neq 0} |\Lambda(k)|^2 \nu(k)^{-r} \geq C(r, d)|\Lambda(0)|^2 m^{-r}(\log m)^{d-1},
\]

where \( \bar{k}_j := \max(|k_j|, 1) \) and \( \nu(\bar{k}) := \prod_{j=1}^d \bar{k}_j \).

By (3.3) we get

\[
|\Lambda^*(f) - \Lambda^*(0)\hat{f}(0)| = \left| \sum_{k \neq 0} \Lambda^*(k)\hat{f}(k) \right| = \left| \sum_{k \neq 0} |\Lambda(k)|^2 \hat{f}(k) \right|. \tag{3.8}
\]

Applying (3.8) for \( f(x) = \tilde{h}^r(x, 0, u) \) and using (3.7) we obtain

\[
2Be_m \geq |\Lambda^*(f) - |\Lambda(0)|^2 \hat{f}(0)| = \left| \sum_{k \neq 0} |\Lambda(k)|^2 \hat{h}^r(k, 0, u) \right|. \tag{3.9}
\]
Next
\[ \hat{h}^r(0, 0, u) = u^r, \quad \hat{h}^r(k, 0, u) = \left( \frac{\sin(\pi k u)}{\pi k} \right)^r, \quad k \neq 0, \]
which implies for \( r \) even
\[ \int_0^{1/2} \hat{h}^r(k, 0, u)du \geq c(r)(\bar{k})^{-r}. \]
Integrating the right hand side of (3.9) with respect to \( u \) over \((0, 1/2]^d\) and using Lemma 3.1 we get for \( r \) even
\[ e_m \geq C(r, d)|\Lambda(0)|^{2m^{-r}}(\log m)^{d-1}. \]
It is clear (see, for instance (3.6)) that it must be \(|\Lambda(0)| \geq c(r, d) > 0\). This completes the proof of Theorem 3.1. \( \square \)

4 Discussion

The paper addresses some issues of discrepancy theory. Discrepancy theory is a well established topic with deep elaborate technique and with some open fundamental problems (see, for instance, [2, 8, 15, 17]). One of the most acute open problems is the problem of the right order of decay of the quantity
\[ D(m, d)_{\infty} := \inf_T D(T, m, d)_{\infty}. \]
The upper bound is known (see [2])
\[ D(m, d)_{\infty} \leq C(d)m^{-1}(\log m)^{d-1}. \quad (4.1) \]
In case \( d = 2 \) it is complemented by the lower bound proved by W. Schmidt [13]
\[ D(m, 2)_{\infty} \geq Cm^{-1}\log m. \quad (4.2) \]
In the case \( d \geq 3 \) the problem is still open. The following conjecture has been formulated in [2] as an excruciatingly difficult great open problem.

**Conjecture 4.1.** We have for \( d \geq 3 \)
\[ D(m, d)_{\infty} \geq C(d)m^{-1}(\log m)^{d-1}. \]
This problem is still open. Recently, D. Bilyk and M. Lacey \cite{BilykLacey2013} and D. Bilyk, M. Lacey, and A. Vagharshakyan \cite{Bilyketal2014} proved
\[ D(m, d)_{\infty} \geq C(d) m^{-1} (\log m)^{(d-1)/2+\delta(d)} \]
with some positive $\delta(d)$.

In this paper we introduce a concept of $r$-smooth discrepancy and prove the lower bound for the $r$-smooth periodic discrepancy $\tilde{D}_m^r$ (see Theorem 3.1) for $r$ even numbers. This lower bound does not prove Conjecture 4.1 but it supports it. There is another variant of smooth discrepancy, which shows similar behavior. We discuss it in detail (see \cite{BilykLacey2014} and \cite{Vagharshakyan2015}). In the definition of the $r$-discrepancy instead of the characteristic function (this corresponds to 1-discrepancy) we use the following function

\[ B_r(t, x) := \prod_{j=1}^{d} \left( (r - 1)! \right)^{-1} (t_j - x_j)_{+}^{r-1}, \]

$t, x \in \Omega_d$, $(a)_+ := \max(a, 0)$. Then for point set $\xi := \{\xi^\mu\}_{\mu=1}^m$ of cardinality $m$ and weights $\Lambda := \{\lambda^\mu\}_{\mu=1}^m$ we define the $r$-discrepancy of the pair $(\xi, \Lambda)$ by the formula

\[ D_r(\xi, \Lambda, m, d)_{\infty} := \sup_{t \in (0,1]^d} \left| \sum_{\mu=1}^{m} \lambda^\mu B_r(t, \xi^\mu) - \prod_{j=1}^{d} (t_j^r / r!) \right|. \]

Define

\[ D_r(\xi, m, d)_{\infty} := \inf_{\Lambda} D_r(\xi, \Lambda, m, d)_{\infty}. \]

Then $D_r(\xi, m, d)_{\infty}$ is close in a spirit to the quantity $D^{r,0}(\xi)$ defined in \cite{Vagharshakyan2015}. The following known result (see \cite{BilykLacey2014} and \cite{Vagharshakyan2015}) gives the lower bounds in the case of weights $\Lambda$ satisfying an extra condition (3.2).

**Theorem 4.1.** Let $B$ be a positive number. For any points $\xi^1, \ldots, \xi^m \subset \Omega_d$ and any weights $\Lambda = (\lambda_1, \ldots, \lambda_m)$ satisfying the condition

\[ \sum_{\mu=1}^{m} |\lambda^\mu| \leq B \]

we have for even integers $r$

\[ D_r(\xi, \Lambda, m, d)_{\infty} \geq C(d, B, r) m^{-r} (\log m)^{d-1} \]

with a positive constant $C(d, B, r)$. \hfill 14
Theorem 3.1 is an analog of the above Theorem 4.1.

The concept of fixed volume discrepancy was introduced and studied in [18]. It is an interesting concept by itself and it is closely related to the concept of dispersion. For \( n \geq 1 \) let \( T \) be a set of points in \([0, 1]^d\) of cardinality \(|T| = n\). The volume of the largest empty (from points of \( T \)) axis-parallel box, which can be inscribed in \([0, 1]^d\), is called the dispersion of \( T \):

\[
\text{disp}(T) := \sup_{B \in \mathcal{B} : B \cap T = \emptyset} \text{vol}(B).
\]

An interesting extremal problem is to find (estimate) the minimal dispersion of point sets of fixed cardinality:

\[
\text{disp}^*(n, d) := \inf_{T \subseteq [0,1]^d, |T| = n} \text{disp}(T).
\]

It is known that

\[
\text{disp}^*(n, d) \leq C^*/n. \tag{4.4}
\]

A trivial lower bound \( \text{disp}^*(n, d) \geq (n + 1)^{-1} \) combined with (4.4) shows that the optimal rate of decay of dispersion with respect to cardinality \( n \) of sets is \( 1/n \). Another interesting problem is to find (provide a construction) of sets \( T \) with cardinality \( n \), which have optimal rate of decay of dispersion: \( \text{disp}(T) \leq C(d)/n \). Inequality (4.4) with \( C^*(d) = 2^{d-1} \prod_{i=1}^{d-1} p_i \), where \( p_i \) denotes the \( i \)th prime number, was proved in [6] (see also [11]). The authors of [6] used the Halton-Hammersly set of \( n \) points (see [8]). Inequality (4.4) with \( C^*(d) = 2^{7d+1} \) was proved in [1]. The authors of [1], following G. Larcher, used the \((t, r, d)\)-nets (see [10] and [8] for results on \((t, r, d)\)-nets).

For further recent results on dispersion we refer the reader to papers [20], [12], [14] and references therein. In [18] we proved that the Fibonacci and the Frolov point sets have optimal in the sense of order rate of decay of dispersion. This result was derived from the bounds on the 2-smooth fixed volume discrepancy of the corresponding point sets. In the case of the Frolov point sets it is provided by Corollary 1.1 formulated above.

Acknowledgment. The author would like to thank the Erwin Schrödinger International Institute for Mathematics and Physics (ESI) at the University of Vienna for support. This paper was written, when the author participated in the ESI-Semester "Tractability of High Dimensional Problems and Discrepancy", September 11–October 13, 2017.
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