Quad-equations and auto-Bäcklund transformations of NLS-type systems

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Abstract
Treating an integrable quad-equation along with its two generalized symmetries as a compatible system allows one to construct an auto-Bäcklund transformation for solutions of the related NLS-type system. A fixed periodic reduction of the quad-equation yields a quasi-periodic reduction of its generalized symmetries that turns them into differential constraints compatible with the NLS-type system.

Keywords: quad-equation, Bäcklund transformation, superposition formula, NLS-type system
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1. Introduction

Integrable differential–difference equations with one continuous and one discrete variable, subsequently referred to as chains, are known to be closely connected with (systems of) integrable partial differential equations (PDEs). In particular, many integrable chains can be interpreted as Bäcklund transformations of some PDEs [1]. The integrability of a chain assumes the existence of a formal recursion operator and infinitely many commuting flows. This property has been used to classify both integrable chains and PDEs [2]. A pair of commuting flows from the same hierarchy is called compatible. Shabat and Yamilov demonstrated that a pair of compatible chains with some restrictions on their form always yields a system of PDEs through the construction often referred to as elimination of shifts [3]. A by-product of this construction is an invertible auto-transformation of the resulting system of PDEs.

As far as construction of exact solutions is concerned, a more important class of transformations is non-invertible auto-transformations containing an arbitrary parameter (auto-Bäcklund transformations). A direct calculation of such transformations is a tedious task.
The knowledge of other structures associated with integrability, e.g. a Lax pair or Painlevé structure, may significantly simplify the calculation of such transformations [4]. In this paper we show how an auto-Bäcklund transformation can be constructed when a system of PDEs is obtainable through the elimination of shifts from a compatible system of two integrable chains. The necessary ingredient in this construction is that the chains should represent the generalized symmetries of an integrable quad-equation.

To illustrate the idea we consider the integrable chain

$$\partial_t u_{k,l} = \frac{1}{u_{k+1,l} - u_{k-1,l}}; \quad \text{(1)}$$

where $u_{k,l} = u(k, l; x, y)$ is a function that simultaneously depends on discrete and continuous variables: $(k, l) \in \mathbb{Z}^2$, $(t, x) \in \mathbb{C}^2$. Throughout the article the subscripts $k$ and $l$ indicate dependence on discrete variables, while the subscripts $t$ and $x$ indicate partial derivatives. Equation (1) is related to the famous Volterra equation

$$\partial_t w_{k,l} = w_{k,l}(w_{k+1,l} - w_{k-1,l})$$

via the substitution [5]

$$w_{k,l} = \frac{1}{(u_{k+1,l} - u_{k-1,l})(u_{k+2,l} - u_{k,l})}.$$  

The complete classification of the Volterra-type equations can be found in [6].

The simplest commuting flow, i.e. an equation $\partial_t u_{k,l} = G$ of the lowest order that satisfies $\partial_t \partial_t u_{k,l} = \partial_t \partial_t u_{k,l}$, of (1) is given by

$$\partial_t u_{k,l} = \frac{u_{k+2,l} - u_{k-2,l}}{(u_{k+1,l} - u_{k-1,l})^2(u_{k+2,l} - u_{k,l})(u_{k,l} - u_{k-2,l})}. \quad \text{(2)}$$

It can be computed by using the standard tools, such as master symmetry [5] or recursion operator [7]. On the other hand the whole hierarchy of (1) can be represented by a single formula (see formula (9) of [8]). Note that neither of chains (1) or (2) depends on shifts with respect to variable $l$. Nevertheless it is indicated here in order to make possible a connection with a quad-equation (see below).

In order to obtain a system of PDEs satisfied by $u_{k,l}$ and $u_{k+1,l}$, we use (1) and its shifted versions to express variables $u_{k-2,l}$, $u_{k-1,l}$ and $u_{k+2,l}$:

$$u_{k-2,l} = u_{k,l} - \frac{1}{\partial_t u_{k-1,l}}, \quad u_{k-1,l} = u_{k+1,l} - \frac{1}{\partial_t u_{k,l}}, \quad u_{k+2,l} = u_{k,l} + \frac{1}{\partial_t u_{k+1,l}}. \quad \text{(3)}$$

The substitution of (3) into (2) yields the derivative NLS system [9] in the potential form:

$$u_t = u_{xx} + 2u_{x}^2v_x, \quad v_t = -v_{xx} + 2v_{x}^2u_x,$$

where $u_{k,l} = u$, $u_{k+1,l} = v$. The shifts along chain (1)

$$(u_{k+1,l}, u_{k+1,l}) \rightarrow (u_{k+1,l}, u_{k+2,l}), \quad (u_{k-1,l}, u_{k,l}) \rightarrow (u_{k,l}, u_{k+1,l}),$$

can now be interpreted as the auto-transformation of (4)

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} v \\ u + 1/v_x \end{pmatrix} \quad \text{(5)}$$

and its inverse

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} v - 1/u_x \\ u \end{pmatrix} \quad \text{(6)}$$

correspondingly.
It is known that integrable quad-equations possess hierarchies of generalized symmetries (see e.g. [10, 11]). For instance, the hierarchy of equations (1) and (2) is related to the quad-equation
\[
(u_{k,l} - u_{k+1,l+1})(u_{k+1,l} - u_{k,l+1}) - \lambda + \mu = 0,
\]
(7)
where \(\lambda, \mu = \text{const}\). This equation is often referred to as \(H_1\) due to the labeling it received in the classification [12] of equations consistent around the cube. The \(H_1\) equation is also well known in the context of the potential KdV equation where it serves as a superposition formula for solutions related by the auto-Bäcklund transformation [13]. Moreover, equation (7) reduces to pKdV in the continuum limit [14, 15]. This example therefore highlights the link between the classes of NLS and KdV-type equations.

In what follows we are concerned with implications of the mentioned connection between integrable chains, NLS-type systems and quad-equations. We show that it automatically yields an auto-Bäcklund transformation for the related NLS-system. A formula of superposition can then be derived from the assumption of commutativity of the auto-Bäcklund transformations. In general the compatibility of a PDE and a superposition formula needs to be verified separately, and is not always guaranteed. One of the corollaries of the presented construction is that a traveling wave reduction of an integrable quad-equation generates the quasi-periodic closure of the related chains, which turn them into differential constraints compatible with the NLS-type system.

2. Auto-Bäcklund transformations of NLS-type systems

The statement that (1) and (2) are generalized symmetries of (7) implies that the relations
\[
\partial_t F = 0, \quad \partial_x F = 0,
\]
(8)
where \(F\) is the left hand side of (7), are identically satisfied on solutions of the system consisting of (1), (2) and (7). In other words, (8) become identities when partial derivatives are eliminated by using (1) and (2), and mixed shifts by using (7).

Note that due to the covariance \((k, l, \lambda, \mu) \to (l, k, \mu, \lambda)\), equation (7) possesses the generalized symmetries of the form (1) and (2), where \(k\) and \(l\) are interchanged. However, the corresponding system of PDEs will still be the same (potential dNLS). The construction being considered here can be applied to non-symmetrical quad-equations to show that one quad-equation can generate auto-Bäcklund transformations for two different NLS-type systems. However, for the sake of simplicity we will consider only the example of the \(H_1\) equation.

Since equations (1) and (2) do not involve shifts with respect to the variable \(l\), the quantities
\[
p = u_{k,l+1}, \quad q = u_{k+1,l+1}
\]
must satisfy a system of form (4) with \((u, v)\) being replaced by \((p, q)\):
\[
p_t = p_{xx} + 2p^2q_x, \\
q_t = -q_{xx} + 2q^2p_x.
\]
(9)
This observation implies that quad-equation (7) when re-written as
\[
(u - q)(v - p) = \kappa,
\]
(10)
where \(\kappa = \lambda - \mu\), is a part of a certain auto-transformation for the potential dNLS system. Importantly, the constant \(\kappa\) is not present in (9); hence it can play the role of the Bäcklund parameter. Another part of the auto-transformation can be found the following way.
Consider the up- and down-shifted versions of (7):

\[(u_{k+1,l} - u_{k+2,l+1})(u_{k+2,l} - u_{k+1,l+1}) = \kappa, \quad (11)\]

\[(u_{k-1,l} - u_{k,l+1})(u_{k,l} - u_{k-1,l+1}) = \kappa. \quad (12)\]

It follows from (1) that

\[u_{k+2,l} = u + \frac{1}{v_x}, \quad u_{k+2,l+1} = p + \frac{1}{q_x}, \quad u_{k-1,l} = v - \frac{1}{u_x}, \quad u_{k-1,l+1} = q - \frac{1}{p_x}.\]

Substituting these expressions into (11) and (12) we obtain the additional relations

\[\left( v - p - \frac{1}{q_x}\right) \left( u - q + \frac{1}{u_x}\right) = \kappa, \quad (13)\]

\[\left( v - p - \frac{1}{u_x}\right) \left( u - q + \frac{1}{p_x}\right) = \kappa. \quad (14)\]

One can verify that the combination of (10) and (13) implies formula (14). Therefore any combination of two relations from the list of (10), (13) and (14) constitutes an auto-Bäcklund transformation for (4). The analogous transformations for the dNLS system were previously constructed in [16, 17] by using different approaches.

**Superposition formula and construction of solutions**

Now we turn to constructing a superposition formula based on the auto-Bäcklund transformation found previously, i.e. the combination of relations (10) and (13). To this end we look at implications of commutativity of a few transformations (10) which can be schematically represented by the Bianchi diagram:

\[\begin{array}{c}
(m, n) \xrightarrow{\kappa} (r, s) \\
\downarrow \quad \uparrow \\
(\alpha, \nu) \xrightarrow{\kappa} (p, q)
\end{array}\]

The relation (13) is used to obtain the new solution from a seed solution. The diagram yields the following relations

\[(u - q)(v - p) = \kappa, \quad (p - s)(q - r) = \nu, \quad (u - n)(v - m) = v, \quad (m - s)(n - r) = \kappa,\]

which in turn give rise to the possible expressions for \(r\) and \(s\):

\[r = u + \frac{\kappa - v}{p - m}, \quad s = v + \frac{\kappa - v}{q - n} \quad (15)\]

and

\[r = n + q - u, \quad s = v - \frac{v}{u - n} - \frac{\kappa}{u - q}. \quad (16)\]

One can check that the second relation is not compatible with the dNLS system, whereas the first one is! The compatibility is verified by differentiating (15) (or (16)) with respect to the time variable and then making use of the potential dNLS system itself, and also of (10), (13) and (15) (or (16)). Obviously (15) is nothing but the two copies of the standard potential KdV.
superposition formula relating the corresponding components in the Bianchi diagram. Note that (15) is not the only possible form of the superposition formula since $m$ and $p$ could be eliminated from the formula.

By iterating formula (15) we obtain rational expressions in terms of a seed solution and the solution obtained through the dNLS system (9), (10) and (13).

**Example.** If we start with the exponential solution

$$ u = \exp(x - t), \quad v = \exp(-x + t), $$

then it follows that $q$ satisfies the system

$$ q_x = \frac{q(1 - qv)}{\kappa}, \quad q_t = \frac{(1 + \kappa)q^2v - q}{\kappa^2} $$

(17)

while $p$ is given explicitly by

$$ p = v + \frac{\kappa}{q} - \frac{1}{q_x}. $$

Integrating equations (17), we obtain

$$ q = \frac{1 - \kappa}{v + c \exp \left( -\frac{\kappa}{c} \frac{1}{v} \right)}, $$

where $c$ is the constant of integration.

A more intricate solution is then obtained through superposition formula (15). Note that expressions for $m$ and $n$ coincide with $p$ and $q$ correspondingly, where the parameter $\kappa$ is replaced by $\nu$. A common feature of the solutions obtained from the exponential seed solution is that the individual components grow/decay exponentially while their product has the shape of a multi-soliton solution. Such solutions are called dissipations [18]. For instance, for the values of parameters $\kappa = 2, \nu = 1/2, c = 1$ the plot for the product of $r$ and $s$ is

![Plot of product rs](image)

**Remark.** The fact that equation (7) serves two different hierarchies suggests the presence of a common member in the KdV and potential dNLS hierarchies. Indeed, the hierarchy of chains (1) and (2) also contains the ‘negative’ flow

$$ \partial_x u_{k,l} = -\partial_x u_{k+1,l} + (u_{k,l} - u_{k+1,l})^2 + \lambda. $$

(18)
By differentiating (1) and (18) with respect to \( z \) and \( x \) correspondingly and then eliminating shifts from the obtained expressions, we get the hyperbolic system

\[
\begin{align*}
  u_{xz} &= 2(u - v)u_x + 1, \\
  v_{xz} &= -2(u - v)v_x - 1.
\end{align*}
\]

It is not difficult to verify that (19) commutes with the potential dNLS system. On the other hand, the compatibility of chains (1) and (18) can be written as one scalar equation [19]

\[
\begin{align*}
  u_{xxzz} &= \frac{1}{2} u_x^{2} - 2u_x(2u_z - \lambda), \\
  v &= u_{zzz} - 6u_z^2.
\end{align*}
\]

### Reductions

Here we discuss the connections of periodic reductions of quad-equations and quasi-periodic closures of the integrable chains. In fact we could have come to the same construction of auto-Bäcklund transformations by considering the reductions \( u_{k,l} \rightarrow u_{k+\beta,l+\alpha} \), where \( \alpha \) and \( \beta \) are some integers, which induce the periodicity constraint \( u_{k+1} = u_{k-\beta,l+\alpha} \). The simplest reduction of this type is when \( \alpha = 1 \). This reduction, being applied to equation (7), brings it to the form

\[
(uk - uk+\beta)(uk+1 - uk+\beta) = \kappa.
\]

It is important that chains (1) and (2) survive this reduction for an arbitrary \( \beta \) and become the symmetries of (21) upon the substitution \( uk+1 \rightarrow uk+1 \). Moreover, the same procedure of elimination of shifts yields the potential dNLS system with unknowns \( uk = u \) and \( uk+1 = v \).

Since \( \beta \) is arbitrary, the quantities

\[
uk+\beta = p, \quad uk+\beta+1 = q
\]

should be treated as algebraically independent from \( uk \) and \( uk+1 \). Thus equation (21) yields the auto-transformation

\[
(u - q)(v - p) = \kappa
\]

of (4) into itself. Relations (13) and (14) can be derived in exactly the same way as before.

In the case when \( \beta \) is fixed, the quantities \( uk+\beta \) and \( uk+\beta+1 \) can no longer be treated as independent because we can express them in terms of \( uk \) and \( uk+1 \) by using the reduction of (1). As a result we obtain a differential constraint in the form of a dynamical system compatible with the potential dNLS equation. On the other hand, the periodicity constraint transforms the quad-equation into an ordinary difference equation which can be interpreted as a mapping acting in a finite-dimensional space. By construction this mapping will preserve the differential constraint.

**Example.** Consider the case \( \alpha = 1, \beta = 2 \). Equation (7) turns into the ordinary difference equation

\[
(uk - uk+3)(uk+1 - uk+2) = \kappa,
\]

while chain (1) becomes

\[
\frac{1}{\partial_x uk} = \frac{1}{uk+1 - uk-1}.
\]
Writing (23) for \( k = 0, \ldots, 2 \) and eliminating \( u_{-1} \) and \( u_3 \) using (22), we obtain the system

\[
\begin{align*}
\partial_x u_0 &= \left( \frac{u_1 - u_0}{u_2 - u_0} \right) \left( \frac{u_2 - u_1}{u_2 - u_0} \right), \\
\partial_x u_1 &= \frac{1}{u_2 - u_0}, \\
\partial_x u_2 &= \left( \frac{u_2 - u_1}{u_2 - u_0} \right) \left( \frac{u_2 - u_1}{u_2 - u_0} \right),
\end{align*}
\] (24)

where

\[
f = \left( \frac{u_2 - u_1}{u_0 - u_1} \right) \left( u_0 - u_1 \right) + \kappa \left( u_2 - u_0 \right),
\] (25)

which can be interpreted as a differential constraint compatible with the potential dNLS system. In order to verify this, one has to eliminate the \( x \)-derivatives in the two copies \( (u_0, u_1) \) and \( (u_1, u_2) \) of the potential dNLS systems using (24), and check that derivatives \( \partial_t \) and \( \partial_x \) commute. By construction, (24) is invariant with respect to the mapping defined by equation (22):

\[
M : (u_0, u_1, u_2) \rightarrow \left( u_1, u_2, u_0 + \frac{\kappa}{u_2 - u_1} \right).
\] (26)

This implies, in particular, that derivative \( \partial_x \) preserves the integral(s) of mapping \( M \). One can check that \( M \) has only one integral given by (25)—it is also the integral of (24). This integral can be obtained by means of the staircase method [20, 21] (see also [22]).

**Concluding remarks**

Integrable quad-equations provide us with auto-transformations for solutions of some NLS-type systems. Although we used only one example of \( H_1 \)-dNLS equations, the presented construction is not specific to this case. It can be applied to other integrable quad-equations as well—this will be the subject of further research.

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