NOTE

A simple Fourier transform-based reconstruction formula for photoacoustic computed tomography with a circular or spherical measurement geometry

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Received 10 August 2012, in final form 3 October 2012
Published 20 November 2012
Online at stacks.iop.org/PMB/57/N493

Abstract
Photoacoustic computed tomography (PACT), also known as optoacoustic tomography, is an emerging imaging modality that has great potential for a wide range of biomedical imaging applications. In this note, we derive a hybrid reconstruction formula that is mathematically exact and operates on a data function that is expressed in the temporal frequency and spatial domains. This formula explicitly reveals new insights into how the spatial frequency components of the sought-after object function are determined by the temporal frequency components of the data function measured with a circular or spherical measurement geometry in two- and three-dimensional implementations of PACT, respectively. The structure of the reconstruction formula is surprisingly simple compared with existing Fourier-domain reconstruction formulae. It also yields a straightforward numerical implementation that is robust and two orders of magnitude more computationally efficient than filtered backprojection algorithms.

Photoacoustic computed tomography (PACT) is an emerging imaging modality that has great potential for a wide range of biomedical imaging applications (Oraevsky and Karabutov 2003, Wang 2008, Kruger et al 1999). In PACT, biological tissues of interest are illuminated by the use of short laser pulses, which results in the generation of internal acoustic wavefields via the thermoacoustic effect (Xu and Wang 2006, Xu et al 2010). The initial amplitudes of the induced acoustic wavefields are proportional to the spatially variant absorbed optical energy density within the tissues. The propagated acoustic wavefields are subsequently detected by the use of a collection of ultrasonic transducers that are located outside the object. An image reconstruction algorithm is employed to estimate the absorbed optical energy density within the tissue from these data.
A variety of image reconstruction algorithms have been proposed for PACT (Xu et al 2002, Finch et al 2004, 2007, Xu and Wang 2005, Kunyansky 2007, Hristova et al 2008, Kunyansky 2012, Salehin and Abhayapala 2012, Treeby and Cox 2010). While iterative image reconstruction methods hold great value due to their ability to incorporate accurate models of the imaging physics and the instrument response (Paltauf et al 2002, Yuan and Jiang 2007, Zhang et al 2009, Provost and Lesage 2009, Wang et al 2011, Guo et al 2010, Huang et al 2010, Xu et al 2011, Buehler et al 2011, Bu et al 2012, Wang et al 2012), they can lead to long reconstruction times, even when accelerated by use of modern computing hardware such as graphics processing units (Wang et al 2012). This is especially problematic in three-dimensional (3D) implementations of PACT, in which reconstruction times can be excessively long. Almost all experimental studies of PACT to date have employed analytic image reconstruction algorithms. Even if an iterative image reconstruction algorithm is to be employed, it is often useful to employ an analytic reconstruction algorithm to obtain a preliminary image that can initialize the iterative algorithm and thereby accelerate its convergence.

Most analytic reconstruction algorithms for PACT with a spherical measurement aperture and point-like transducers have been formulated in the form of filtered backprojection (FBP) algorithms. These algorithms possess a large computational burden, requiring $O(N^5)$ floating point operations to reconstruct a 3D image of dimension $N^3$. Image reconstruction algorithms based on the time-reversal principle and finite-difference schemes require $O(N^4)$ operations (Burgholzer et al 2007). Fast reconstruction algorithms for spherical measurement apertures that require only $O(N^3 \log N)$ operations have been proposed (Kunyansky 2012, Salehin and Abhayapala 2012). However, numerical implementations of these formulae require computation of special functions and multi-dimensional interpolations operations in Fourier space, which require special care to avoid degradation in reconstructed image accuracy. It is well known that the temporal frequency components of the pressure data recorded on a spherical surface are related to the Fourier components of the sought-after object function (Anastasio et al 2007). However, to date, a simple reconstruction algorithm based on this relationship, i.e. one that does not require series expansions involving special functions or multi-dimensional interpolations, is yet to be developed.

In this note, we derive a novel reconstruction formula for two-dimensional (2D) and 3D PACT employing circular and spherical measurement geometries, respectively. The mathematical forms of the reconstruction formulae are the same in both dimensions and are surprisingly simple compared with existing Fourier-domain reconstruction formulae for spherical and circular measurement geometries. The reconstruction formulae are mathematically exact and describe explicitly how the spatial frequency components of the sought-after object function are determined by the temporal frequency components of the measured pressure data. Their discrete implementations require only discrete Fourier transform, one-dimensional (1D) interpolation and summation operations. A preliminary computer-simulation study is conducted to corroborate the validity of the reconstruction formula.

We consider the canonical PACT imaging model in which the object and surrounding medium are assumed to possess homogeneous and lossless acoustic properties and the object is illuminated by a laser pulse with negligible temporal width. Point-like, unfocused, ultrasonic transducers are assumed. We also assume that the effects of the acousto-electric impulse responses of the transducers have been deconvolved from the measured voltage signals so that the measured data can be interpreted as pressure signals. The 3D problem is addressed where $p(r, t)$ denotes the photoacoustically induced pressure wavefield at location $r \in \mathbb{R}^3$ and time $t \geq 0$. However, the analysis and reconstruction formula that follows remains valid for the 2D...
case. The imaging physics is described by the photoacoustic wave equation (Oraevsky and Karabutov 2003, Wang 2008, Kruger et al 1999):

$$\nabla^2 p(r, t) - \frac{1}{c^2} \frac{\partial^2 p(r, t)}{\partial t^2} = 0, $$

subject to the initial conditions:

$$p(r, t) \bigg|_{t=0} = \frac{\beta c^2}{C_p} A(r); \quad \frac{\partial p(r, t)}{\partial t} \bigg|_{t=0} = 0,$$

where $\nabla^2$ denotes the 3D Laplacian operator and $A(r)$ is the object function to be reconstructed that is contained within the volume $V$. Physically, $A(r)$ represents the distribution of absorbed optical energy density. The constant quantities $\beta$, $c$ and $C_p$ denote the thermal coefficient of volume expansion, speed of sound and the specific heat capacity of the medium at constant pressure.

Let $p(r, t)$ denote the pressure data recorded at location $r_s \in S$ on a spherical surface $S$ of radius $R_S$ that encloses $V$. The continuous form of the imaging model that relates the measurement data to the object function can be expressed as (Cox and Beard 2005)

$$p(r_s, t) = \mathcal{H}A \equiv \frac{\beta c^2}{C_p} \int_{\mathbb{R}^3} dk \hat{A}(k) \cos(ckt) e^{ik \cdot r_s},$$

where $k \in \mathbb{R}^3$ is the spatial frequency vector conjugate to $r$, $k = |k|$ and $\hat{A}(k)$ is the 3D Fourier transform of $A(r)$. We adopt the Fourier transform convention

$$\hat{A}(k) = \mathcal{F}_3 A(r) \equiv \int_{\mathbb{R}^3} dr A(r) e^{-ik \cdot r}$$   \hspace{1cm} (4a)

$$A(r) = \mathcal{F}_3^{-1} \hat{A}(k) \equiv \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} dk \hat{A}(k) e^{ik \cdot r}.$$   \hspace{1cm} (4b)

The imaging model in equation (3) can be interpreted as a mapping $\mathcal{H} : O \to D$ between infinite-dimensional vector spaces that contain the object and data functions. We will define $O$ as the vector space of the bounded and smooth functions that are compactly supported within the volume $V$.

Let the infinite set of functions $\{\gamma_\mu(r)\}$, indexed by $\mu$, represent an orthonormal basis for $O$. The object function $A(r)$ can be represented as

$$A(r) = \int_{\mathbb{R}^3} d\mu \langle A, \gamma_\mu \rangle \gamma_\mu(r),$$

where the inner product in $O$ is defined as

$$\langle A, \gamma_\mu \rangle \equiv \int_{\mathbb{R}^3} dr A(r) \gamma_\mu(r) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} dk \hat{A}(k) \hat{\gamma}_\mu(k),$$

$\hat{\gamma}_\mu(k) = \mathcal{F}_3 \gamma_\mu(r)$, and the quantity on the right-hand side of equation (6) follows from the fact that the Fourier transform is an isometry. A trace identity (see equation (1.7) in Finch et al 2004) for the 3D case and equation (1.16) in Finch et al 2007 for the 2D case) can be employed to relate the inner products in the spaces $O$ and $D$ as

$$\langle A, \gamma_\mu \rangle = \frac{2C_p^2}{R_S \beta c^2} \int_0^\infty dt \int_2^1 dr t p(r_s, t) v_\mu(r_s, t),$$

where

$$v_\mu(r_s, t) = \mathcal{H} \hat{\gamma}_\mu = \frac{\beta c^2}{C_p(2\pi)^3} \int_{\mathbb{R}^3} dk \hat{\gamma}_\mu(k) \cos(ckt) e^{ik \cdot r_s},$$

and the right-hand side of equation (7) defines a scaled version of the inner product in $D$. 


On substitution of equation (8) into equation (7), one obtains
\[
\langle A, \gamma_r \rangle = \frac{1}{(2\pi)^3} \int_\infty d\mathbf{k} \hat{y}(\mathbf{k}) \hat{\gamma}_r(\mathbf{k}), \tag{9}
\]
where
\[
\hat{y}(\mathbf{k}) = \frac{2C_p}{R \beta} \int_S d\mathbf{r} e^{\mathbf{k} \cdot \mathbf{r}} \int_0^\infty dt \ t \ p(\mathbf{r}, t) \cos(ckt). \tag{10}
\]
Comparison of equations (6) and (9) reveals that \(\hat{A}(\mathbf{k}) = \hat{\gamma}(\mathbf{k})\). By evaluating the Fourier cosine transform that is present on the right-hand side of equation (10), a reconstruction formula for determining \(\hat{A}(\mathbf{k})\) can therefore be expressed as
\[
\hat{A}(\mathbf{k}) = \frac{2C_p}{R \beta} \int_S d\mathbf{r} e^{\mathbf{k} \cdot \mathbf{r}} \Re\{\mathcal{F}_1[p(\mathbf{r}, t)]](\mathbf{r}, \omega)|_{\omega = c \mathbf{k}}\}, \tag{11}
\]
where \(\mathcal{F}_1\) denotes the 1D Fourier transform with respect to time \(t\) and \(\Re\) denotes the operation that takes the real part of quantity in the brackets. Subsequently, \(A(\mathbf{r})\) is determined as \(\mathcal{F}_1^{-1}\hat{A}(\mathbf{k})\).

Equation (11) represents a novel reconstruction for PACT and is the key result of this note. Unlike previously proposed Fourier-domain reconstruction formulae (Norton 1980, Kunyansky 2012, Salehin and Abhayapala 2012), equation (11) has a simple form and does not involve series expansions utilizing special functions. The reconstruction formula reveals that the measured data \(p(\mathbf{r}, t)\) determine the 3D Fourier components of \(A(\mathbf{r})\) via a simple process that involves the following four steps: (1) compute the 1D temporal Fourier transform of the modified data function \(t p(\mathbf{r}, t)\); (2) isolate the real-valued component of this quantity corresponding to temporal frequency \(\omega = c \mathbf{k}\); (3) weight this value by the plane-wave \(e^{\mathbf{k} \cdot \mathbf{r}}\); and (4) sum the contributions, formed in this way, corresponding to each measurement location \(\mathbf{r}_d \in S\). This reveals that the components of \(\hat{A}(\mathbf{k})\) residing on a sphere of radius \(k/c\) are determined by the 1D Fourier transform of \(t p(\mathbf{r}, t)\) corresponding to the temporal frequency \(\omega\). In this sense, equation (11) can be interpreted as an implementation of the Fourier shell identity (Anastasio et al 2007). Finally, the form of equation (11) remains unchanged in the 2D case, where \(\mathbf{r}_d, \mathbf{k} \in \mathbb{R}^2\) and \(S\) is a circle that encloses the object.

A discrete implementation of equation (11) possesses low computational complexity and desirable numerical properties. The 1D fast Fourier transform (FFT) can be employed to approximate the action of \(\mathcal{F}_1\) and only a 1D interpolation is required to determine the value of the Fourier transformed data function corresponding to the temporal frequency \(\omega = c \mathbf{k}\), where \(\mathbf{k}\) corresponds to the magnitude of vectors \(\mathbf{k}\) that specify a 3D Cartesian grid. From the values of \(\hat{A}(\mathbf{k})\) determined on this grid, the 3D FFT algorithm can be employed to estimate values of \(A(\mathbf{r})\). If the object is represented on an \(N \times N \times N\) grid and the number of transducer locations and time samples are both \(O(N)\), the computational complexity is limited by the 3D FFT algorithm, i.e. \(O(N^3 \log N)\) in 2D and \(O(N^3 \log N)\) in 3D.

A preliminary computer-simulation study for the 2D case was conducted to corroborate the correctness of the reconstruction formula. The object function \(A(\mathbf{r})\) was taken to be the numerical phantom shown in figure 1(a), which was comprised of a collection of uniform discs that were blurred by a 2D Gaussian kernel whose full-width at half-maximum was 0.3 mm. The phantom was discretized by the use of pixel expansion functions with a pitch of 0.025 mm. The measurement geometry consisted of 256 point-like transducers that were uniformly distributed over a circle of radius 12.8 mm that enclosed the object. The k-wave toolbox (Treeby and Cox 2010) was employed to numerically solve the photoacoustic wave equation and generate simulated pressure signals at each transducer location at a temporal sampling rate of 30 MHz. The simulated pressure data set generated in this way contained...
A Fourier-based reconstruction formula for PACT

Figure 1. The numerical phantom is shown in subfigure (a). Images reconstructed by use of the proposed reconstruction algorithm from noiseless and noisy data are shown in subfigures (b) and (c), respectively. The grayscale window is $[-0.2, 1.2]$.

Figure 2. Profiles corresponding to the central rows of the images shown in figures 1(b) (subfigure(a)) and 1(c) (subfigure(b)). The solid line in subfigure (a), which corresponds to the image reconstructed from noiseless data, almost completely overlaps with the profile through the numerical phantom.

256 $\times$ 2048 data samples. The speed of sound and $\frac{\beta}{c_p}$ were assigned values of 1.5 mm $\mu$s$^{-1}$ and 1000 (arbitrary units), respectively. A noisy data set was produced by the addition of 5% uncorrelated Gaussian noise to the noiseless pressure data.
Images were reconstructed on a uniform 2D grid of spacing 0.1 mm by the use of a discretized form of equation (11) coupled with the 2D inverse FFT algorithm. In order to reconstruct images of dimension $256 \times 256$, samples of $\hat{A}(k)$ were determined on a uniform 2D grid of dimension $512 \times 512$ with a sampling interval of $(0.1 \times 256)^{-1}$ mm$^{-1}$. The samples of the data function $tp(r, t)$ were zero-padded by a factor of 8 prior to estimating its 1D Fourier transform by use of the FFT algorithm. From these data, nearest-neighbor 1D interpolation was employed to determine the values of the term in brackets in equation (11) corresponding to $\omega = ck$ for the sampled locations $k$.

The images reconstructed from the noiseless and noisy data sets are shown in figures 1(b) and (c). Profiles corresponding to the central rows of these images are shown in figure 2. These results confirm that the proposed reconstruction algorithm can reconstruct images with high fidelity from noise-free measurement data. Although a systematic investigation of the noise propagation properties of the proposed algorithm is beyond the scope of this note, figures 1(c) and 2(b) suggest that its performance is robust in the presence of noise. This is to be expected, since all operations involved in the implementation of equation (11) are numerically stable.

In summary, we have derived a Fourier-based reconstruction formula for PACT employing circular and spherical measurement apertures. The formula is mathematically exact and possesses a surprisingly simple form compared with existing Fourier-domain reconstruction formulae. The formula yields a straightforward numerical implementation that is stable and is two orders of magnitude more computationally efficient than 3D FBP algorithms. The proposed formula serves as an alternative to existing fast Fourier-based reconstruction formulae. A systematic comparison of the proposed reconstruction formula with existing formulae by the use of experimental data remains an important topic for future studies.

Acknowledgment

This work was supported in part by NIH awards EB010049 and CA167446.

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A Fourier-based reconstruction formula for PACT

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