Simplicial principal bundles in parametrized spaces

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Abstract

In this paper, motivated by recent interest in higher gauge theory, we prove that the fiberwise geometric realization functor takes a certain class of simplicial principal bundles in a suitable category of spaces over a fixed space $B$ to fiberwise principal bundles. As an application we show that the fiberwise geometric realization of the universal simplicial principal bundle for a simplicial group $G$ in the category of spaces over $B$ gives rise to a fiberwise principal bundle with structure group $|G|$. An application to classifying theory for fiberwise principal bundles is described.

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1 Introduction

The construction of a classifying space $BG$ for a topological group $G$ is most conveniently done using simplicial techniques, namely one constructs $BG$ as the geometric realization of a certain simplicial space. Similarly the universal principal $G$-bundle $EG \rightarrow BG$ arises as the geometric realization of a certain simplicial
principal bundle. Using a synthesis of techniques from classical simplicial homotopy theory on the one hand and ordinary bundle theory on the other hand, we will prove in this paper (amongst other things) a generalization of this fact: geometric realization sends a certain class of simplicial principal bundles to ordinary principal bundles.

Moreover, we will work in the fiberwise setting with the category of topological spaces replaced by a suitable category \( \mathcal{K}/B \) of spaces over a fixed space \( B \) and geometric realization replaced by fiberwise geometric realization. The main result of the paper (for a precise statement see Theorem 13 in Section 5) can then be summarized as follows.

**Theorem.** The fiberwise geometric realization functor

\[ -|: s\mathcal{K}/B \to \mathcal{K}/B \]

preserves locally trivial bundles \( P \to M \) in \( s\mathcal{K}/B \) under suitable conditions on \( M \) and the bundles \( P_n \to M_n \) in \( \mathcal{K}/B, n \geq 0 \).

Our motivation comes from recent interest in higher principal bundles or gerbes \([2, 3, 18, 22, 33, 35, 50]\). Recall that for a paracompact space \( M \), there is a bijection between \( H^3(M, \mathbb{Z}) \) and the set of equivalence classes of \( S^1 \) bundle gerbes on \( M \). An \( S^1 \) bundle gerbe on \( M \) is, roughly speaking, a principal bundle on \( M \) where the structure group \( S^1 \) is replaced by the groupoid \( S^1 \), viewed as a group object in the category of groupoids, or equivalently (via the nerve construction) as a simplicial group \( NS^1 \). In Section 4 we explain how \( S^1 \) bundle gerbes on \( M \) are naturally viewed as simplicial principal bundles (more precisely as principal twisted cartesian products) for the simplicial topological group \( NS^1 \). From another point of view, \( H^3(M, \mathbb{Z}) \) parametrizes the set of isomorphism classes of principal \( BS^1 \) bundles on \( M \). The process of passing from a simplicial principal bundle for \( NS^1 \) to a principal \( BS^1 \) bundle is exactly the process of geometric realization.

Our interest lies in a generalization of this, namely when the simplicial group \( NS^1 \) is replaced by an arbitrary simplicial topological group \( G \) and we consider simplicial principal bundles with structure group \( G \) on \( M \). The resulting set of equivalence classes is isomorphic to the non-abelian cohomology set \( H^1(M, G) \).

In this case the process of geometric realization produces an ordinary principal \( |G| \) bundle from a simplicial principal \( G \) bundle and therefore gives rise to a map \( H^1(M, G) \to H^1(M, |G|) \). In [45], based on the results of this paper, we prove that this map is an isomorphism provided that \( M \) is paracompact and \( G \) satisfies some mild topological conditions.

In outline then this paper is as follows. We begin in Section 2 with a review of some of the more elementary material from [27] on the category \( \mathcal{K}/B \) of \( k \)-spaces over a fixed compactly generated space \( B \). We review the fiberwise model structure (or \( f \)-model structure) on \( \mathcal{K}/B \), this is in a certain sense the natural generalization of the classical Strøm model structure \([48]\) on topological spaces to the parametrized setting. We study the notion of group object in \( \mathcal{K}/B \) and the corresponding equivariant fiberwise notions of homotopy equivalence, cofibration and fibration.

In Section 3 we put the notion of group object in \( \mathcal{K}/B \) to further use by studying the notion of fiberwise principal bundles, in other words principal bundle objects in \( \mathcal{K}/B \). We review the definition of fiberwise principal bundle from [8] and recall the result from [8] to the effect that every fiberwise principal bundle over a numerable base is an \( f \)-fibration. We finish this section with a result (Proposition 8) stating that certain fiberwise cofibrations \( A \subset M \) pullback along fiberwise principal bundles over \( M \) to equivariant fiberwise cofibrations. This last result is put to later use in the proof of Theorem 13 in Section 5.

In Section 4 we study simplicial principal bundles in \( \mathcal{K}/B \) or more generally in a category \( \mathcal{C} \) admitting finite limits. Our treatment here is heavily influenced by Duskin’s memoir [12]. We explain how the universal simplicial principal bundle for a group \( G \) in \( s\mathcal{C} \) gives rise, via the Artin-Mazur total simplicial object functor \( T: ss\mathcal{C} \to s\mathcal{C} \), to the classical universal bundle \( WG \to \overline{WG} \). We show (Lemma 10) how this perspective on the classical \( \overline{WG} \) construction equips \( WG \) with a canonical structure as a simplicial group, an observation that seems to be missing from the literature. We also give a new treatment of the classifying map for a principal twisted cartesian product and show how the bundle gerbes of Murray [31] have a natural interpretation as principal twisted cartesian products.
In Section 5 we prove the main result of the paper, Theorem [13]. This result says that if \( P \to M \) is a principal bundle in \( s\mathcal{K}/B \) such that \( M \) is suitably cofibrant and each fiberwise bundle \( P_n \to M_n \) is numerable, then the fiberwise geometric realization \( |P| \to |M| \) is a fiberwise principal bundle in \( \mathcal{K}/B \). Our method of proof is an adaptation of the arguments from [26] [39] [11], the main idea being to construct local sections of \( |P| \to |M| \) via a filtration of \(|P| \) by equivariant fiberwise neighborhood deformation retracts.

In Section 6 we study the universal simplicial principal bundle \( WG \to \bar{W}G \) associated to a group object \( G \) in \( s\mathcal{K}/B \). We prove (Proposition [18]) that the fiberwise geometric realization \( |WG| \to |\bar{W}G| \) is a model for the universal principal \( |G| \) bundle in \( \mathcal{K}/B \) provided that \( G \) is good in the sense that every degeneracy map is a fiberwise neighborhood deformation retract. Along the way we give a proof (Proposition [20]) of the folk result that every good simplicial object in a topological model category \( \mathcal{C} \) is automatically proper (i.e. Reedy cofibrant in the Reedy model structure on \( \mathcal{C} \)) provided that certain pullbacks in \( \mathcal{C} \) commute with colimits and cofibrations in \( \mathcal{C} \) are closed under forming unions (i.e. the analog of Lillig’s union theorem on cofibrations [23] holds in \( \mathcal{C} \)). This result is perhaps well known to experts but it seems difficult to find a proof in the literature at the level of generality that we are interested in. In Proposition [22] we adapt the proof of [23] to show Lillig’s union theorem extends to \( \mathcal{K}/B \). In Proposition [23] we give a convenient criterion to recognize when a simplicial group \( G \) in \( \mathcal{K}/B \) is good, namely \( G \) is good provided that each group \( G_n \) is well sectioned, i.e. the identity section \( B \to G \) is an \( f \)-cofibration.

Finally, in Section 7 we show in Theorem [27] that if \( G \) is a well sectioned group in \( \mathcal{K}/B \) then the fiberwise geometric realization of the universal \( G \) bundle \( WG \to W\bar{G} \) in \( s\mathcal{K}/B \) is a classifying space for fiberwise principal \( G \)-bundles over a paracompact base.

2 Parametrized spaces

We let \( \mathcal{K} \) denote the category of \( k \)-spaces [49] and we let \( \mathcal{U} \) denote the full subcategory of compactly generated spaces (i.e. weakly Hausdorff \( k \)-spaces). Let \( B \) be an object of \( \mathcal{U} \). We will be interested in the category \( \mathcal{K}/B \) of spaces over \( B \). Recall from [27] that \( \mathcal{K}/B \) is a topological bicomplete category, in the sense that \( \mathcal{K}/B \) is enriched over \( \mathcal{K} \), the underlying category is complete and cocomplete, and that it is tensored and cotensored over \( \mathcal{K} \). For any space \( K \) and space \( X \) over \( B \) the tensor \( K \otimes X \) is defined to be the space \( K \times X \) in \( \mathcal{K} \), considered as a space over \( B \) via the obvious map \( K \times X \to B \). Similarly, the cotensor \( X^K \) is defined to be the space \( \text{Map}_B(K, X) \) given by the pullback diagram

\[
\begin{array}{ccc}
\text{Map}_B(K, X) & \longrightarrow & \text{Map}(K, X) \\
\downarrow & & \downarrow \\
B & \longrightarrow & \text{Map}(K, B)
\end{array}
\]

in \( \mathcal{K} \), where the map \( B \to \text{Map}(K, B) \) is the adjoint of \( B \times K \to B \). Recall also (see [27]) that \( \mathcal{K}/B \) is cartesian closed under the fiberwise cartesian product \( X \times_B Y \) and the fiberwise mapping space \( \text{Map}_B(X, Y) \) over \( B \). The definition of the fiberwise mapping space \( \text{Map}_B(X, Y) \) is rather subtle and we will not give it here, we instead refer the reader to Definition 1.37 of [27]. Let us note though that \( \text{Map}_B(X, Y) \) is generally not weak Hausdorff even if \( X \) and \( Y \) are, which is one of the main reasons why May and Sigurdsson choose to work with the category \( \mathcal{K}/B \) rather than the category \( \mathcal{U}/B \).

In [27] several model structures on \( \mathcal{K}/B \) are introduced. The model structure on \( \mathcal{K}/B \) that we will be interested in has its origins in the work [39] of Schwänzl and Vogt. In [39] (see also [7] and [27]) the authors consider a topological bicomplete category \( \mathcal{C} \) and define three classes of morphisms: \( h \)-equivalences, \( h \)-fibrations and \( h \)-cofibrations. A morphism \( f: X \to Y \) in \( \mathcal{C} \) is an \( h \)-equivalence if and only if it is a homotopy equivalence, defined in terms of the cylinder objects \( X \otimes I \) and \( Y \otimes I \) where \( I \) denotes the unit interval. A morphism \( f: X \to Y \) is called an \( h \)-fibration if and only if it has the LLP (left lifting property) with respect to all morphisms of the form \( Z \otimes \{0\} \to Z \otimes I \), while \( f \) is called an \( h \)-cofibration if and only if \( X \otimes I \cup_{X \otimes 0} Y \otimes 0 \to Y \otimes I \) has the LLP with respect to all \( h \)-fibrations in \( \mathcal{C} \).
In [36], the $\tilde{h}$-cofibrations are called *strong cofibrations* and the following alternative characterization of them is given: a morphism $f: X \to Y$ is an $\tilde{h}$-cofibration if and only if it has the RLP (right lifting property) with respect to all $h$-fibrations which are also $h$-equivalences — i.e. the $h$-acyclic $h$-fibrations. When $\mathcal{C} = \mathcal{K}$, the class of strong cofibrations equals the class of closed cofibrations. Under suitable hypotheses on $\mathcal{C}$ (see Theorem 4.2 of [7] and Theorem 4.2.12 of [27]) these three classes of morphisms equip $\mathcal{C}$ with the structure of a proper, topological model category. This model structure is sometimes called the $h$-model structure.

Recall that a model category $\mathcal{C}$ is said to be topological if it is a $\mathcal{K}$-model category in the sense of Definition 4.2.18 of [16], for the monoidal model structure on $\mathcal{K}$ given by the above $h$-model structure (observe that this coincides with the classical Strøm model structure [7, 27, 48] on $\mathcal{K}$). Moreover (see [24]) this adjunction extends to a Quillen equivalence when $\mathcal{K}$ is equipped with a certain model structure (the fibrant objects in $\infty\text{Sh}(B)$ are the so-called $\infty$-sheaves or homotopy sheaves on $B$). From this perspective the model structure $(\mathcal{K}/B)_{ij}$ is perhaps more interesting than the $f$-model structure on $\mathcal{K}/B$. The $f$-model structure and the $ij$-model structure on $\mathcal{K}/B$ are related however — in [40] it is proven that the identity functor on $\mathcal{K}/B$ is left Quillen from the $ij$-model structure to the $f$-model structure.

**Theorem 1** (May-Sigurdsson [27]). $\mathcal{K}/B$ has the structure of a proper, topological model category for which

- the weak equivalences are the $f$-equivalences,
- the fibrations are the $f$-fibrations,
- the cofibrations are the $f$-cofibrations.

To be completely explicit, we explain the labels on the three classes of maps in the theorem. A map $g: X \to Y$ in $\mathcal{K}/B$ is called an $f$-equivalence if it is a fiberwise homotopy equivalence. This needs the notion of homotopy over $B$, which is formulated in terms of $X \times_B I$. A map $g: X \to Y$ in $\mathcal{K}/B$ is called an $f$-fibration if it has the fiberwise covering homotopy property, i.e. it has the RLP property with respect to all maps of the form $i_0: Z \to Z \times_B I$ for all $Z \in \mathcal{K}/B$. A map $g: X \to Y$ in $\mathcal{K}/B$ is called an $f$-cofibration, or a strong cofibration if it has the LLP property with respect to all $f$-acyclic $f$-fibrations. There is also the notion of an $f$-cofibration: this is a map $g: X \to Y$ which satisfies the LLP property with respect to all maps of the form $p_0: \text{Map}_B(I, Z) \to Z$ for some $Z \in \mathcal{K}/B$. Every $f$-cofibration $g: X \to Y$ in $\mathcal{K}/B$ is an $f$-cofibration. The converse is not true in general. However May and Sigurdsson prove (see Theorems 4.4.4 and 5.2.8 of [27]) that if $g: X \to Y$ is a closed $f$-cofibration then $g$ is an $f$-cofibration.

Moreover, in analogy with the standard characterization of closed Hurewicz cofibrations in terms of NDR pairs, May and Sigurdsson give a criterion (see Lemma 5.2.4 of [27]) which detects when a closed inclusion $i: A \to X$ in $\mathcal{K}/B$ is an $f$-cofibration. Such an inclusion $i: A \to X$ in $\mathcal{K}/B$ is an $f$-cofibration if and only if $(X, A)$ is a fiberwise NDR pair in the sense that there is a map $u: X \to I$ for which $A = u^{-1}(0)$ and a homotopy $h: X \times_B I \to X$ over $B$ such that $h_0 = id$, $h_t|_A = id_A$ for all $0 \leq t \leq 1$ and $h_1(x) \in A$ whenever $u(x) < 1$.

As we mentioned above, there are several model structures on $\mathcal{K}/B$. The model structure described in Theorem 1 above is the most convenient for us, as it is the natural generalization of the Strøm model structure to parametrized spaces. In [17, 24] a different model structure on $\mathcal{K}/B$ was introduced (following [40]) we denote this model structure by $(\mathcal{K}/B)_{ij}$ which has the property that it models homotopy sheaves on $B$. More precisely, in [24] (see also [40]) it is proven that there is a canonical adjunction

$$| - |: \infty\text{Sh}(B) \rightleftarrows (\mathcal{K}/B)_{ij}: \text{Sing}_B$$

between the category $\infty\text{Sh}(B)$ of simplicial presheaves on $B$ and the category $\mathcal{K}/B$ of spaces over $B$. Moreover (see [24]) this adjunction extends to a Quillen equivalence when $\infty\text{Sh}(B)$ is equipped with a certain model structure (the fibrant objects in $\infty\text{Sh}(B)$ are the so-called $\infty$-sheaves or homotopy sheaves on $B$). From this perspective the model structure $(\mathcal{K}/B)_{ij}$ is perhaps more interesting than the $f$-model structure on $\mathcal{K}/B$. The $f$-model structure and the $ij$-model structure on $\mathcal{K}/B$ are related however — in [40] it is proven that the identity functor on $\mathcal{K}/B$ is left Quillen from the $ij$-model structure to the $f$-model structure.
We shall also be interested in group objects in \( \mathcal{X}/B \), these are mentioned in passing in Remark 1.4.6 of [27] and in more detail in [8]. Such an object consists of a space \( G \) over \( B \) together with a morphism \( G \times_B G \to G \) in \( \mathcal{X}/B \) such that the induced map \( G_b \times G_b \to G_b \) on fibers equips \( G_b \) with the structure of a topological group for all \( b \in B \). For more details we refer to [8].

If \( G \) is a group object in \( \mathcal{X}/B \) then there is a natural notion of a \( G \)-space over \( B \) and a \( G \)-map between \( G \)-spaces over \( B \). A \( G \)-space over \( B \) is a space \( X \) over \( B \) equipped with an action of \( G \), i.e. a map \( X \times_B G \to X \) of spaces over \( B \) making the usual diagrams commute. Thus for every \( b \in B \) the fibers \( X_b \) have a natural structure of a \( G_b \)-space. Similarly, if \( X \) and \( Y \) are \( G \)-spaces over \( B \) then a \( G \)-map \( X \to Y \) over \( B \) consists of a map \( X \to Y \) of spaces over \( B \) such that for all \( b \in B \) the induced map \( X_b \to Y_b \) on fibers is a map of \( G_b \)-spaces. We write \( G.\mathcal{X}/B \) for the subcategory of \( \mathcal{X}/B \) consisting of \( G \)-equivariant objects and \( G \)-maps between them. We have the following lemma.

**Lemma 2.** The category \( G.\mathcal{X}/B \) is a topological bicomplete category.

**Proof.** To construct limits in \( G.\mathcal{X}/B \) one first constructs the corresponding limit in \( \mathcal{X}/B \) and then equips it with the induced \( G \)-action. To construct colimits in \( G.\mathcal{X}/B \) one first constructs the colimit in \( \mathcal{X}/B \) and then one observes that, since \( G \times_B - \) is a left adjoint and therefore preserves colimits, the colimit in \( \mathcal{X}/B \) comes equipped with a natural \( G \)-action. \( G.\mathcal{X}/B \) is naturally enriched over \( \mathcal{X} \); if \( X \) and \( Y \) are objects of \( \mathcal{X}/B \) then the space of morphisms \( G.\mathcal{X}/B(X,Y) \) is given by the equalizer diagram

\[
G.\mathcal{X}/B(X,Y) \to \mathcal{X}/B(X,Y) \rightleftharpoons \mathcal{X}/B(X \times_B G, Y)
\]

in \( \mathcal{X} \), where the two maps are induced by the projection onto \( X \) and the action of \( G \) on \( Y \). If \( X \in G.\mathcal{X}/B \) and \( K \in \mathcal{X} \) then the tensor \( X \otimes K \) is the usual one in \( \mathcal{X}/B \) equipped with the \( G \)-action where \( G \) acts trivially on the \( K \) factor. The cotensor in \( G.\mathcal{X}/B \) is the usual cotensor in \( \mathcal{X}/B \) equipped with an action of \( G \) described as follows. The commutative diagram

\[
\begin{array}{ccc}
G & \to & \text{Map}(K,G) \\
\downarrow & & \downarrow \\
B & \to & \text{Map}(K,B),
\end{array}
\]

where the top horizontal map is the adjoint of \( G \times K \to G \) in \( \mathcal{X} \), shows that there is a natural morphism \( G \to \text{Map}_B(K,G) \) in \( \mathcal{X}/B \). The action of \( G \) on \( \text{Map}_B(K,X) \) is given by the following composite:

\[
\text{Map}_B(K,X) \times_B G \to \text{Map}_B(K,X) \times_B \text{Map}_B(K,G) \cong \text{Map}_B(K \times_B G) \to \text{Map}_B(K,X).
\]

One can check that this gives a \( G \)-action as claimed. To check that we have required adjunction homeomorphisms, observe that we have the following isomorphisms of diagrams in \( \mathcal{X} \):

\[
\begin{align*}
\mathcal{X}/B(X \otimes K,Y) & \cong \mathcal{X}/B(X,Y^K) \cong \mathcal{X}(X,\mathcal{X}/B(X,Y)) \\
\mathcal{X}/B((X \times_B G) \otimes K,Y) & \cong \mathcal{X}/B(X \times_B G,Y^K) \cong \mathcal{X}(K,\mathcal{X}/B(X \times_B G,Y))
\end{align*}
\]

where we have used the fact that we have an isomorphism \( (X \otimes K) \times_B G \cong (X \times_B G) \otimes K \). Therefore, on forming equalizers we get the required natural isomorphisms

\[
G.\mathcal{X}/B(X \otimes K,Y) \cong G.\mathcal{X}/B(X,Y^K) \cong \mathcal{X}(K,G.\mathcal{X}/B(X,Y)),
\]

using the fact that \( \mathcal{X}(K,-) \) preserves equalizers. \( \square \)
Let $G$ continue to denote a group object in $\mathcal{X}/B$. As a topological bicomplete category, in $G\mathcal{X}/B$ there are natural notions of $f$-equivalence, $f$-fibration, $f$-cofibration and $\tilde{f}$-cofibration. Thus a map $g: X \to Y$ in $G\mathcal{X}/B$ is an $f$-cofibration if it has the LLP in $G\mathcal{X}/B$ with respect to $G$-maps of the form $p_0: \text{Map}_B(I,Z) \to Z$ for all $Z$ in $G\mathcal{X}/B$. Similarly, we say that a map $g: X \to Y$ in $G\mathcal{X}/B$ is an $f$-equivalence if it is a fiberwise $G$-homotopy equivalence. A map $g: X \to Y$ in $\mathcal{X}/B$ is an $f$-fibration if it has the RLP in $G\mathcal{X}/B$ with respect to $G$-maps of the form $i_0: Z \to Z \times_B I$ for all $Z$ in $G\mathcal{X}/B$. A map $g: X \to Y$ in $G\mathcal{X}/B$ is an $f$-cofibration if it has the LLP in $G\mathcal{X}/B$ with respect to all $f$-acyclic $f$-fibrations in $G\mathcal{X}/B$.

Just as above, there is a criterion to detect when an inclusion $i: A \to X$ in $G\mathcal{X}/B$ is a $\tilde{f}$-cofibration in $G\mathcal{X}/B$. We have the following result which will play a key role in the proof of Theorem 13.

**Lemma 3.** An inclusion $i: A \to X$ in $G\mathcal{X}/B$ is a $\tilde{f}$-cofibration if and only if $i(A)$ is closed in $X$ and there is a representation of $(X, A)$ as a $G$-fiberwise NDR pair.

Here by a representation of $(X, A)$ as a $G$-fiberwise NDR pair we understand, in analogy with [11], that there is a pair $(u, h)$ of maps with $u: X \to I$ and $h: X \times_B I \to X$ which represent $(X, A)$ as a fiberwise NDR pair and which satisfy $u(xg) = u(x)$ for all $x \in X$ and $g \in G$, and $h(xg, t) = h(x, t)g$ for all $(x, t) \in X \times_B I$ and $g \in G$.

**Proof.** We will explain how to adapt Steps 1–3 in the proof of Theorem 4.4.4 of [27] to our setting. Step 3 adapts in a straightforward way to show that $i(A)$ is closed in $X$: factor the inclusion $i: A \to X$ as $A \to E \to X$ where $E = A \times_B I \cup X \times_B (0,1]$ and where $i_0: A \to E$ is given by $i_0(a) = (a,0)$. Analogous to the corresponding statement in [27], the projection $\pi: E \to X$ is an $f$-acyclic $f$-fibration in $G\mathcal{X}/B$. Therefore there exists a map $\lambda: X \to E$ extending $i$, i.e. $\lambda \circ i = i_0$. Let $\psi: E \to I$ denote the projection onto the second factor and note that $\psi^{-1}(0) = i_0(A)$, so that $i_0(A)$ is closed in $E$. Therefore $\lambda^{-1}i_0(A) = i(A)$ is closed in $X$ (since $\lambda$ is injective). Standard arguments now show that $(X, A)$ has a representation as a $G$-fiberwise NDR pair.

Next we explain how Steps 1 and 2 can be adapted to show that if $(X, A)$ has a representation as a $G$-fiberwise NDR pair then $i: A \to X$ is an $\tilde{f}$-cofibration. The usual argument shows that $X \times_B \{0\} \cup A \times_B I$ is a retract of $X \times_B I$ in $G\mathcal{X}/B$. Hence $i: A \to X$ is a closed $\tilde{f}$-cofibration in $G\mathcal{X}/B$ and so $Mi: X \times_B I \to X$ is the inclusion of a strong deformation retraction (see Lemma 4.2.5 of [27]). The map $u$ in a representation $(u, h)$ of $(X, A)$ as a $G$-fiberwise NDR pair can be used to show that there exists a map $\psi: X \times_B I \to I$ such that $\psi^{-1}(0) = Mi$. The analogue of Theorem 3 of [10] for the category $G\mathcal{X}/B$ then shows that $Mi: X \times_B I \to X$ has the LLP with respect to all $f$-fibrations and hence $i: A \to X$ is an $\tilde{f}$-cofibration.

Finally, let us note (36 Lemma 2.6) that since $\tilde{f}$ cofibrations in $G\mathcal{X}/B$ are defined by a left lifting property, the following is true.

**Lemma 4.** If $X_0 \to X_1 \to \cdots \to X_n \to \cdots$ is a sequence of $\tilde{f}$-cofibrations in $G\mathcal{X}/B$, then $X_n \to X$ is an $f$-cofibration in $G\mathcal{X}/B$ for all $n \geq 0$, where $X = \text{colim} X_n$.

### 3 Fiberwise principal bundles

We shall also be interested in the notion of a principal bundle in $\mathcal{X}/B$ for a group object $G$ in $\mathcal{X}/B$. Such a notion is studied in [8] where the following definition is essentially made.

**Definition 5** ([8]). Let $G$ be a group object in $\mathcal{X}/B$. A fiberwise principal $G$ bundle in $\mathcal{X}/B$ consists of a map $\pi: P \to M$ in $\mathcal{X}/B$ which admits local sections together with an action of $G$ on $P$ which is strongly free in the sense that the diagram

\[
P \times_B G \xrightarrow{p_1} P \xrightarrow{\pi} M
\]
is a pullback in $\mathcal{K}/B$. Here the upper horizontal map is the action of $G$ on $P$ and the left hand vertical map is projection onto the first factor. The condition that $\pi: P \to M$ admits local sections means that for every point of $m$ of $M$ there is an open neighborhood $U_m \subset M$ of $m$ together with a fiberwise map $s: U_m \to P$ which is a section of $\pi$.

Let $P \to M$ be a fiberwise principal $G$ bundle for some group object $G$ in $\mathcal{K}/B$. The canonical map $P/G \to M$ is an isomorphism and hence it follows that $P \to M$ is an effective epimorphism in $\mathcal{K}/B$ — in other words the diagram

$$P \times_M P \cong P \to M$$

is a pullback in $\mathcal{K}/B$.

There is an obvious notion of morphism of fiberwise principal $G$-bundles over $M$, and, just as for ordinary principal bundles, every morphism is an isomorphism. We denote the set of isomorphism classes of fiberwise principal $G$ bundles on $M$ by $H^1(M,G)$.

Every fiberwise principal $G$-bundle $\pi: P \to M$ is a fiberwise fiber bundle in the sense that each point of $M$ has an open neighborhood $U$ such that the restriction of $P$ to $U$ is isomorphic to the trivial fiberwise bundle $U \times B G$. Such a trivial fiberwise bundle is clearly an $f$-fibration in the sense of Theorem 1. When $B$ is a point it is a well known theorem that every fiberwise fiber bundle $E \to M$ is a Hurewicz fibration. There is an obvious extension of this notion to the notion of a numerable fiberwise fiber bundle: a fiberwise fiber bundle is numerable if it is fiberwise locally trivial relative to a numerable open cover of the base space. We have the following theorem from [3].

**Theorem 6 (3).** Let $p: E \to M$ be a map in $\mathcal{K}/B$. Suppose that $p^{-1}V_i \to V_i$ is an $f$-fibration for each open set $V_i$ in a numerable covering $\{V_i\}_{i \in I}$ of $M$. Then $p$ is an $f$-fibration. In particular any fiberwise principal $G$ bundle $\pi: P \to M$ in $\mathcal{K}/B$ over a paracompact base space $M$, or more generally any numerable fiberwise principal $G$ bundle in $\mathcal{K}/B$, is an $f$-fibration.

This theorem has the following important corollary. In the fiberwise context, principal $G$-bundles $P_0$ and $P_1$ on $M$ are said to be fiberwise concordant if there exists a principal $G$-bundle $P$ on $M \times I$ together with fiberwise isomorphisms $P_0 \cong P|_{M \times \{0\}}$ and $P_1 \cong P|_{M \times \{1\}}$. The fiberwise concordance relation is clearly an equivalence relation. When $B$ is a point it is well known that there is a bijection between the set of isomorphism classes of numerable principal $G$ bundles on $M$ and concordance classes of principal $G$ bundles on $M$. In the fiberwise setting there is an analogous bijection.

**Corollary 7.** Let $M$ be a paracompact space in $\mathcal{K}/B$ and let $G$ be a group object in $\mathcal{K}/B$. Then there is a bijection between $H^1(M,G)$ and the set of fiberwise concordance classes of principal $G$-bundles on $M$.

**Proof.** To prove that there is such a bijection one needs to know that fiberwise concordant bundles are isomorphic. For this, it is enough to prove that there is an isomorphism $P \cong P_0 \times_B I$, when $P$ is a fiberwise $G$-bundle on $M \times I$ isomorphic to $P_0$ and $P_1$ when restricted to $M \times \{0\}$ and $M \times \{1\}$ respectively. Consider the bundle $P \times_G (P_0 \times I)$ on $M \times_B I$. There is a section of this bundle over the closed subspace $M \times_B \{0\}$ of $M \times_B I$. We want to know that this section extends to a section defined over $M \times_B I$. Since $P \times_G (P_0 \times I)$ is a fiberwise locally trivial bundle on $M \times_B I$, it is an $f$-acyclic $\tilde{f}$-cofibration. It follows that the set of fiberwise concordance classes of $G$-bundles on $M$ is isomorphic to $H^1(M,G)$.

We shall also need the following result, related to Theorem 12 of [47].

**Proposition 8.** Let $\pi: P \to M$ be a fiberwise principal $G$ bundle in $\mathcal{K}/B$ and suppose that $A \subset M$ is a closed inclusion which is an $f$-cofibration in $\mathcal{K}/B$. Then the closed inclusion $P|_A \subset P$ is an $f$-cofibration in $G\mathcal{K}/B$. 


Proof. The proof of the analogous result in [47] can be adapted to this setting as follows. Choose a representation \((u, h)\) of \((M, A)\) as a fiberwise NDR pair in \(\mathcal{E}/B\). Next observe that in the diagram

\[
\begin{array}{ccc}
P & \xrightarrow{\pi} & P \\
\downarrow{\iota} & & \downarrow{\pi} \\
P \times I & \xrightarrow{h(x \times 1)} & M \\
\end{array}
\]

the indicated lifting \(\bar{h}\) can be found, and moreover can be chosen to be \(G\)-equivariant, in light of the proof of Corollary 7 above. To finish the proof, we need to show that we can choose \(\bar{h}\) so that \(\bar{h}(x, t) = x\) for any \(x \in P|_{A}\). Consider the associated bundle \(Aut_0(P \times I) = (P \times I) \times_G G\) on \(M \times I\), where the right action of \(G\) on itself is conjugation. Note that sections of \(Aut_0(P \times I)\) are bundle automorphisms of \(P \times I\) covering the identity on \(M \times I\). Since \(\pi\bar{h} = h(\pi \times 1)\) and \(\bar{h}\) is equivariant, it follows that \(\bar{h}\) restricts to a section of \(Aut_0(P \times I)\) over \(A \times I \subset M \times I\). Similarly the restriction of \(\bar{h}\) to \(P \times \{0\}\) defines a section of \(Aut(P \times I)\) over \(M \times \{0\}\). Since \(Aut_0(P \times I) \to M \times I\) is a locally trivial, numerable, fiberwise bundle, and \((A \times I) \cup (M \times \{0\}) \subset M \times I\) is a closed \(\bar{f}\)-cofibration, it follows that we can find the indicated lifting in the diagram

\[
\begin{array}{ccc}
(A \times I) \cup (M \times \{0\}) & \xrightarrow{\bar{h}} & Aut_0(P \times I) \\
\downarrow{\bar{h}} & & \downarrow{\bar{h}} \\
M \times I & \xrightarrow{1} & M \times I.
\end{array}
\]

Now define \(\bar{h} = \bar{h}^{-1}\). Then \(\bar{h} : P \times I \to P\) is \(G\)-equivariant and satisfies \(\bar{h} \pi = (\bar{h} \pi \times 1)\). If we set \(\bar{u} = u\pi\) then it is easily checked that \((\bar{u}, \bar{h})\) is a representation of \((P, P|_{A})\) as a \(G\)-equivariant NDR pair. 

4 Simplicial principal bundles and twisted cartesian products

The concept of a twisted Cartesian product is ubiquitous in the context of simplicial sets. In fact this concept makes sense in any category \(s\mathcal{E}\) of simplicial objects in some category \(\mathcal{E}\) admitting finite limits. In [12] Duskin interpreted twisted Cartesian products in \(s\mathcal{E}\) in terms of his notion of ‘pseudo-torsor’ (see Definition 4.5.2 of [12]). We recall this notion here, although we shall use use the terminology ‘torsor’ rather than ‘pseudo-torsor’. Let \(\mathcal{E}\) be a category admitting finite limits. Recall (see [12]) that if \(G\) is a group object in \(\mathcal{E}\) and \(M\) is an object of \(\mathcal{E}\) then a \(G\)-torsor in \(\mathcal{E}\) above \(M\) consists of an object \(P\) of \(\mathcal{E}/M\) equipped with an action \(P \times_M (M \times G) \to P\) of the group object \(M \times G\) in \(\mathcal{E}/M\) on \(P\) satisfying the usual axioms. Further, the diagram

\[
\begin{array}{ccc}
P \times G & \xrightarrow{p_1} & P \\
\downarrow{p_1} & & \downarrow{p_1} \\
P & \xrightarrow{M} & M
\end{array}
\]

is required to be a pullback in \(\mathcal{E}\) (as in Definition 5 above we will say that the \(G\)-action is strongly free). When \(\mathcal{E} = s\mathcal{E}\) is the category of simplicial objects in a category \(\mathcal{E}\) admitting finite limits, we will say that a \(G\)-torsor in \(\mathcal{E}\) is a simplicial \(G\)-torsor in \(\mathcal{E}\).

Since the notion of \(G\)-torsor over \(M\) is formulated in terms of finite limits, it is stable under pullback in the sense that if \(N \to M\) is a map in \(\mathcal{E}\) and \(P \to M\) is a \(G\)-torsor over \(M\), then \(N \times_M P \to N\) is a \(G\)-torsor over \(N\). To see this observe that in the diagram

\[
\begin{array}{ccc}
N \times_M P \times G & \xrightarrow{N \times_M P} & N \times_M P \\
\downarrow{N \times_M P} & & \downarrow{N \times_M P} \\
N \times_M P & \xrightarrow{M} & M
\end{array}
\]
the outer square and the right hand square are both pullbacks, hence the left hand square is also a pullback and $N \times_M P \to N$ is a $G$-torsor over $N$.

There is a canonical torsor in $Gpd(\mathcal{E})$ associated to any group in $\mathcal{E}$ (here $Gpd(\mathcal{E})$ denotes the category of groupoid objects in $\mathcal{E}$). To explain this, recall that if $G$ is a group in $\mathcal{E}$ (which we will sometimes think of as a groupoid in $\mathcal{E}$ in the usual way) and $G$ acts on $P$ then we can form the action groupoid $P//G$ in $\mathcal{E}$ whose object of objects is $P$ and whose object of morphisms is $P \times G$. The source and target morphisms come from the action of $G$ on $P$ and projection onto the first factor respectively. The remainder of the groupoid structure comes from the axioms for the action of $G$ on $P$. Note that any $G$-map $P \to Q$ in $\mathcal{E}$ between objects equipped with a $G$-action induces a map $P//G \to Q//G$ between the corresponding action groupoids.

Observe that $G$ has a natural structure of a $G$-torsor in $\mathcal{E}$ over 1. If we equip 1 with the trivial $G$-action then by the remark above we obtain a canonical map

$$G//G \to 1//G$$

in $Gpd(\mathcal{E})$. If we regard $G$ as a group object in $Gpd(\mathcal{E})$ whose underlying groupoid is the discrete groupoid $G$, then it is easy to see that $G//G$ has a $G$-action (given by right multiplication in the group $G$) and that the diagram

$$\begin{array}{ccc}
G//G \times G & \longrightarrow & G//G \\
\downarrow & & \downarrow \\
G//G & \longrightarrow & 1//G
\end{array}$$

is a pullback in $Gpd(\mathcal{E})$. Hence $G//G$ is an example of a $G$ torsor over $1//G$ — we will return to study this in more detail shortly.

We now describe Duskin’s formulation of the notion of twisted cartesian product. For this, we need one extra piece of terminology. Suppose that $U: \mathcal{E} \to \mathcal{B}$ is a functor, then a $G$-torsor $P$ is said to be a $G$-torsor over $M$ (rel. $U: \mathcal{E} \to \mathcal{B}$) when $P \to M$ is equipped with a section $s: U(M) \to U(P)$.

In Example 4.5.7.2 of [12] Duskin shows how to recover the notion of a principal twisted cartesian product by taking Illusie’s décalage functor $\text{Dec}_0: s\mathcal{E} \to a_c s\mathcal{E}$ [19] as the functor $U$, where $a_c s\mathcal{E}$ denotes the category of contractible augmented simplicial objects in $\mathcal{E}$.

Recall (see for example [12] [19] [14]), that $\text{Dec}_0$ is the functor which shifts degrees up by one so that if $X$ is a simplicial object in $\mathcal{E}$ then $\text{Dec}_0(X)_n = X_{n+1}$ with the last face and degeneracy map at each level forgotten or ‘stripped away’. In other words $\text{Dec}_0$ is the functor induced by restriction along the functor $\sigma_0: \Delta \to \Delta$, where $\Delta$ denotes the augmented simplex category and where $\sigma_0$ is defined by $\sigma_0([n]) = ([n], [0])$, where $\sigma: \Delta \times \Delta \to \Delta$ is ordinal sum, i.e. $\sigma([m], [n]) = [m+n+1]$. Observe that the last face map at every level defines a simplicial map $d_{\text{last}}: \text{Dec}_0 X \to X$ for any simplicial object $X$ in $\mathcal{E}$ which in degree $n$ is given by $d_{n+1}: X_{n+1} \to X_n$.

Here an augmented simplicial object $X$ in $\mathcal{E}$ is said to be contractible if it can be equipped with a contraction, i.e. a family of maps $s_{n+1}: X_n \to X_{n+1}$ which are ‘extra degeneracies’ in the sense that they satisfy all the simplicial identities involving degeneracies (see for example [12]). A morphism in $a_c s\mathcal{E}$ is a morphism of the underlying augmented simplicial objects which respects the extra degeneracies. For more details we refer to [12].

**Definition 9** (Duskin [12]). Let $G$ be a group object in $s\mathcal{E}$. Then a principal twisted cartesian product (PTCP) in $s\mathcal{E}$ with structure group $G$ and base $M$ is defined to be a $G$-torsor $P$ over $M$ (rel. $\text{Dec}_0: s\mathcal{E} \to a_c s\mathcal{E}$) in $s\mathcal{E}$.

Specialized to the case when $\mathcal{E} = \text{Set}$, we see that this data amounts to an action of the simplicial group $G$ on $P$ over $M$ such that $G_n$ acts freely and transitively on $P_n$ for each $n$, and also the existence of a section $\sigma: \text{Dec}_0 M \to \text{Dec}_0 P$. Such a section corresponds exactly to the notion of pseudo-cross section (see for example [28] Definition 18.5).
One needs to exercise a slight degree of restraint in adapting the general theory of PCTPs developed for simplicial sets to arbitrary categories \( \mathcal{C} \) — not all of the theory generalizes readily. Some arguments (most notably Lemma 18.6 of [28]) involve constructions of pseudo-cross sections which ultimately rely on the axiom of choice. That said, all of the theory from [28] that we shall need goes through.

To support this claim, we will now describe some of the classical theory of PTCPs in Duskin’s abstract setting (\( \mathcal{C} \) will continue to denote a category with all finite limits). Let \( G \) be a group in \( s \mathcal{C} \). Recall the canonical \( G \)-torsor \( G//G \rightarrow 1//G \) in \( \text{Gpd}(s \mathcal{C}) \) described earlier. Write \( N: \text{Gpd}(s \mathcal{C}) \rightarrow s s \mathcal{C} \) for the functor which sends any groupoid in \( s \mathcal{C} \) to its nerve. Then \( N(1//G) = NG \), the nerve of the groupoid \( G \) in \( s \mathcal{C} \), and \( N(G) = G \), considered as a constant simplicial object in \( s \mathcal{C} \). Since \( N \) preserves products and pullbacks we see that \( N(1//G) \rightarrow NG \) is a \( G \)-torsor in \( s s \mathcal{C} \).

We would now like a way to pass from torsors in \( s s \mathcal{C} \) to torsors in \( s \mathcal{C} \). To this end, consider the Artin-Mazur total complex functor \( T: s s \mathcal{C} \rightarrow s \mathcal{C} \). This functor was first described in the case where \( \mathcal{C} = \text{Set} \) in [19] and was extended to arbitrary categories \( \mathcal{C} \) with finite limits by Duskin. We will describe the object \( (TX)_n \) of \( n \)-simplices for a bisimplicial object \( X \) in \( \mathcal{C} \), for a more detailed exposition of this functor we refer to [6][11]. We have

\[
(TX)_n \rightarrow \prod_{i=0}^{n} X_{i,n-i} \Rightarrow \prod_{i=0}^{n-1} X_{i,n-i-1}
\]  

where the components of the two maps are defined by the composites

\[
\prod_{i=0}^{n} X_{i,n-i} \xrightarrow{p_i} X_{i,n-i} \xrightarrow{d^v_{i,n-i}} X_{i,n-i-1}
\]

and

\[
\prod_{i=0}^{n} X_{i,n-i} \xrightarrow{p_{i+1}} X_{i+1,n-i-1} \xrightarrow{d^h_{i+1,n-i}} X_{i,n-i-1}.
\]

For our purposes, the most important property of the functor \( T \) is that it is right adjoint to Illusie’s total décalage functor \( \text{Dec} \) [19]. Recall (see [19] for more details) that \( \text{Dec}: s \mathcal{C} \rightarrow s s \mathcal{C} \) is the functor induced by restriction along ordinal sum \( \sigma: \Delta \times \Delta \rightarrow \Delta \). Alternatively, if \( X \) is a simplicial object in \( \mathcal{C} \), then \( \text{Dec}X \) is the simplicial comonadic resolution of \( X \) via the comonad \( \text{Dec} \). As a right adjoint, \( T \) preserves all limits and so, since \( T \) applied to the constant simplicial group \( G \) in \( s \mathcal{C} \) is seen to be equal to the group \( G \) in \( s s \mathcal{C} \) using the formula (3), we see that

\[
TN(G//G) \rightarrow TNG
\]

is a \( G \)-torsor in \( s \mathcal{C} \). We have the following lemma.

**Lemma 10.** The following statements are true:

1. \( TN(G//G) = \text{Dec}_0TNG \) and the map (4) above coincides with the canonical map \( \text{Dec}_0TNG \rightarrow TNG \),

2. \( TN(G//G) \rightarrow TNG \) is a \( G \)-torsor in \( s \mathcal{C} \) rel. \( \text{Dec}_0 \),

3. \( TN(G//G) \) is a group in \( s \mathcal{C} \).

**Proof.** We first prove 1. Regard \( NG \) as the simplicial object in \( s \mathcal{C} \) whose object of \( n \)-simplices in \( s \mathcal{C} \) is the nerve \( NG_n \) of the group \( G_n \) in \( \mathcal{C} \). Likewise regard \( N(G//G) \) as the simplicial object in \( s \mathcal{C} \) whose object of \( n \)-simplices in \( s \mathcal{C} \) is the nerve \( N(G_n//G_n) \) of the action groupoid \( G_n//G_n \) in \( \text{Gpd}(\mathcal{C}) \). Then it is an easy calculation to check that \( \text{Dec}_0NG = N(G//G) \) where \( \text{Dec}_0 \) is the functor \( \text{Dec}_0: s(s \mathcal{C}) \rightarrow a.s(s \mathcal{C}) \). To finish the proof it suffices to show that \( \text{Dec}_0NG = \text{Dec}_0TNG \).

Let \( X \in s(s \mathcal{C}) \) be a simplicial object in \( s \mathcal{C} \) such that the object \( X_0 \) is the terminal object of \( s \mathcal{C} \). We will show that \( T\text{Dec}_0X = \text{Dec}_0TX \) in this case. The object \( (T\text{Dec}_0X)_n \) is defined by the equalizer

\[
(T\text{Dec}_0X)_n \rightarrow \prod_{i=0}^{n} (\text{Dec}_0X)_{i,n-i} \Rightarrow \prod_{i=0}^{n-1} (\text{Dec}_0X)_{i,n-i-1},
\]
or in other words by the equalizer

\[(T \text{Dec}_0 X)_n \to \prod_{i=0}^{n} X_{i+1,n-i} \rightleftarrows \prod_{i=0}^{n-1} X_{i+1,n-i-1}.\]

By hypothesis this is the same as the equalizer

\[(T \text{Dec}_0 X)_n \to \prod_{i=0}^{n+1} X_{i,n+1-i} \rightleftarrows \prod_{i=0}^{n} X_{i,n-i}\]

and hence there is an isomorphism \((T \text{Dec}_0 X)_n \cong (\text{Dec}_0 T X)_n\) for all \(n \geq 0\). Moreover one can show that this isomorphism is compatible with face and degeneracy maps, hence we have an isomorphism \(T \text{Dec}_0 X \cong \text{Dec}_0 T X\) in \(s\mathcal{C}\).

Next we prove 2. We need to show that there is a section of the map \(\text{Dec}_0 T N(G//G) \to \text{Dec}_0 T N G\). In other words (using 1) we need to show that there is a section of \(\text{Dec}_0 \text{Dec}_0 T N G \to \text{Dec}_0 T N G\) in the category \(a_{\mathcal{C}} s\mathcal{C}\) of augmented simplicial objects and coherent maps. We will show that there exists such a section of the canonical map \(d_{\text{last}} : \text{Dec}_0 \text{Dec}_0 X \to \text{Dec}_0 X\) for any simplicial object \(X\).

First observe that in degree \(n\) the canonical map \(d_{\text{last}}\) is given by the last face map \(d_{n+2} : X_{n+2} \to X_{n+1}\). Therefore the last degeneracy map \(s_{n+1} : X_{n+1} \to X_{n+2}\) gives a section for every \(n \geq 0\), and moreover extends to define a simplicial map \(s_{\text{last}} : \text{Dec}_0 \to \text{Dec}_0 \text{Dec}_0 X\). It is easy to check that \(s_{\text{last}}\) is compatible with the contractions on \(\text{Dec}_0 X\) and \(\text{Dec}_0 \text{Dec}_0 X\).

Finally 3 follows from the fact that the functors \(N\) and \(T\) preserve products and the well known fact that \(G//G\) is a group object in \(Gpd(\mathcal{C})\).

Specialized to the case when \(\mathcal{C} = \text{Set}\), the principal twisted cartesian product \(T N (G//G) \to T N G\) constructed in Lemma 11 reduces to the classical principal twisted cartesian product \(WG \to W G\) as described for instance in Corollary 21.8 of [28]. Recall that \(WG\) is the simplicial set with \(WG_0 = 1\) and

\[WG_n = G_{n-1} \times G_{n-2} \times \cdots \times G_0\]

for \(n \geq 1\). The face and degeneracy maps are described in [28]. It is an easy exercise, using the formula (3) for \(T\) given above, to prove the following well-known lemma.

**Lemma 11** (Duskin). Let \(G\) be a group in \(s\text{Set}\). Then there is an equality of simplicial sets \(T N G = W G\).

On the basis of this lemma we will denote (following [6]) \(T N G = W G\) and \(T N (G//G) = W G\). Lemma 11 can then be re-stated as

**Lemma 12.** Let \(\mathcal{C}\) be a category with all finite limits and let \(G\) be a group in \(s\mathcal{C}\). Then there is a canonical PCTP \(WG \to W G\) with structure group \(G\). Moreover \(WG\) is a group in \(s\mathcal{C}\).

As far as we are aware the group structure on \(WG\) does not seem to be widely known — the first place that we know of where this fact is explicitly recorded is in [32], where it is also shown that the endofunctor \(W : \text{Grp}(s\mathcal{C}) \to \text{Grp}(s\mathcal{C})\) is a monad. Hence it gives rise to an augmented cosimplicial simplicial group which is contractible in each (cosimplicial) dimension — in other words, a cosimplicial resolution. This raises the interesting question of what sort of cohomology theory can be defined using the standard definition of cohomology arising from a monad, however this would lead us too far astray to answer here.

When \(G\) is the nerve of a group object in \(\text{Cat}\), Lemma 12 can be rephrased in globular terms, rather than simplicial, and even in terms of crossed modules and generalizations [22]; this was in fact the original motivation for the result in [32].

There is also a very nice abstract approach to the classifying theory of PTCPs, which was most likely known to Duskin, although it does not appear in [12]. Before we can describe this however, we need to review the notion of Čech nerve. Suppose that \(\mathcal{E}\) is a complete category, \(G\) is a group object in \(\mathcal{E}\) and \(P\) is
a $G$-torsor over $M$ in $\mathcal{E}$. Then we can form the Čech nerve $\check{C}(P)$ of the map $P \to M$: this is a simplicial object

\[
\cdots \quad P \times_M P \times_M P \cdots \to P \times_M P \to P
\]

in $\mathcal{E}/M$ where the face and degeneracy maps are given by omission and insertion of identities. In other words, the Čech nerve $\check{C}(P)$ is nothing but the 0-coskeleton $\cosk_0 P$ of the 0-truncated simplicial object $P$ in $\mathcal{E}/M$. From the definition of $G$-torsor over $M$ we see that there is a natural simplicial map

\[
\cosk_0 P \to NG
\]

in $s\mathcal{E}$, where $NG$ denotes the simplicial object which is the usual nerve of the groupoid $G$ in $\mathcal{E}$. In a completely analogous manner, we can form the simplicial object $\cosk_0 P^{[2]}$ in $s\mathcal{E}/P$ and obtain a commutative diagram

\[
\begin{array}{ccc}
\cosk_0 P^{[2]} & \to & N(G//G) \\
\downarrow & & \downarrow \\
\cosk_0 P & \to & NG
\end{array}
\]

Now suppose that $\mathcal{E} = s\mathcal{E}$. If we apply the above discussion to a $G$-torsor $P$ over $M$ (rel. Dec0) in $s\mathcal{E}$, then we obtain a diagram

\[
\begin{array}{ccc}
P & \leftarrow & \cosk_0 P^{[2]} \to N(G//G) \\
\downarrow & & \downarrow \\
M & \leftarrow & \cosk_0 P \to NG
\end{array}
\]

in $s s\mathcal{E}$, where $\cosk_0 P \to M$ denotes the natural augmentation to $M$, regarded as a bisimplicial object constant in the vertical direction (note that $\cosk_0 P$ is also augmented over $M$, regarded as a bisimplicial object constant in the horizontal direction). Likewise $\cosk_0 P^{[2]} \to P$ denotes the natural augmentation to $P$. We have a commutative diagram

\[
\begin{array}{ccc}
Dec_0 P & \to & P_0 \\
\downarrow & & \downarrow \\
Dec_0 M & \to & M_0
\end{array}
\]

and by definition of PCTP, there is a section of $Dec_0 P \to Dec_0 M$. Hence we have a map $Dec_0 M \to P_0$ in $s\mathcal{E}/M$, which we can regard as the 0-truncation of a map $Dec M \to \cosk_0 P$, where $Dec M$ denotes the total décalage of $M$. We can compose the map $Dec M \to \cosk_0 P$ with the canonical map $\cosk_0 P \to NG$ and obtain a map

\[
Dec M \to NG
\]

in $s s\mathcal{E}$. In a completely analogous manner we can use the section of $Dec_0 P^{[2]} \to Dec_0 P$ forming part of the data of the pullback PTCP $P^{[2]} \to P$ and construct a map $Dec P \to N(G//G)$ which fits into a commutative diagram

\[
\begin{array}{ccc}
Dec P & \to & N(G//G) \\
\downarrow & & \downarrow \\
Dec M & \to & NG
\end{array}
\]

By adjointness we have a commutative diagram

\[
\begin{array}{ccc}
P & \to & WG \\
\downarrow & & \downarrow \\
M & \to & WG.
\end{array}
\]
We will now describe simplicial torsors in more detail when $\mathcal{K}$ is the topological bicomplete category $\mathcal{K}/B$ of spaces over $B$. Let $G$ be a group object in $s\mathcal{K}/B$. We will say that a simplicial $G$-torsor $P$ in $\mathcal{K}/B$ above a simplicial object $M$ in $\mathcal{K}/B$ is a simplicial principal bundle in $\mathcal{K}/B$ if the map $P_n \to M_n$ is a fiberwise principal $G_n$ bundle for all $n \geq 0$. In this case we will call $P$ the total space, $M$ the base space and $G$ the structure group.

In particular we have the notion of a PCTP over $M$ in $s\mathcal{K}/B$. Just as in the case for simplicial sets, such a PCTP with structure group $G$ can be described in terms of twisting functions. In this context, a twisting function is a family of maps $t_n : M_n \to G_{n-1}$ in $\mathcal{K}/B$ satisfying the identities $(T)$ on page 71 of [28]. From this data we can form a new simplicial object $P = M \times^1 G$ in $\mathcal{K}/B$, whose object of $n$-simplices is

$$P_n = M_n \times_B G_n$$

and whose face and degeneracy maps are defined by the usual formulas (see (i)–(iii) on page 71 of [28]). Note that the family of maps given in degree $n$ by $\sigma_n : M_n \to P_n$ defined by $\sigma_n(m) = (m, 1)$ is a pseudo-cross section. Every PCTP in $s\mathcal{K}/B$ is given by a twisting function in this way.

In the remainder of this section we will study in some detail an example: we will see that the bundle gerbes of Murray [31] provide interesting examples of twisted cartesian products in the category $s\mathcal{K}$ of simplicial spaces (i.e. when $B$ is a point). Recall from [31] that a bundle gerbe on a space $M$ consists of a map $\pi : Y_0 \to M$ which admits local sections together with a $S^1$ principal bundle $Y_1 \to Y_0^{[2]}$. The $S^1$ bundle $Y_1$ is required to have an associative product, in other words there is an isomorphism of $S^1$ bundles over $Y_0^{[3]}$ which on fibers takes the form

$$(Y_1)_{(y_1,y_2)} \otimes (Y_1)_{(y_2,y_3)} \to (Y_1)_{(y_1,y_3)}$$

for points $y_1, y_2, y_3$ all lying in the same fiber of $\pi$ over $M$ and satisfies the obvious associativity condition over $Y_0^{[4]}$. Associated to each bundle gerbe on $M$ is a characteristic class in $H^3(M, \mathbb{Z})$ which is the obstruction to the bundle gerbe being trivial in a certain sense.

Such a bundle gerbe determines, and is determined by, a PTCP in the category of simplicial spaces. To explain this, we first describe a simplicial space canonically associated to a bundle gerbe $Y_1 \to Y_0^{[2]}$, generalizing the Čech nerve of [5]. Recall, see [31] for details, that there is a section of $Y_1 \to Y_0^{[2]}$ defined over the diagonal $Y_0 \subset Y_0^{[2]}$. This map together with the two obvious maps $Y_1 \to Y_0$ defines a 1-truncated simplicial space $Y$:

$$Y_1 \xrightarrow{d_0} \xrightarrow{d_1} \xrightarrow{s_0} Y_0.$$

We want to enlarge this to a simplicial space. One way to do this is to observe that this 1-truncated simplicial space determines an internal groupoid in $\mathcal{K}$, where composition is defined using the associative product on $Y_1$. The nerve of this internal groupoid then determines a simplicial space.

We will construct an enlargement in a different way, using the following well known canonical procedure for enlarging a truncated simplicial space into a simplicial space: if $s_{\leq n}\mathcal{K}$ denotes the category of $n$-truncated simplicial spaces then there is a functor $\cosk_n : s_{\leq n}\mathcal{K} \to s\mathcal{K}$ (the $n$-coskeleton functor) which is right adjoint to the obvious restriction functor $\tr_n : s\mathcal{K} \to s_{\leq n}\mathcal{K}$. In particular the 1-coskeleton $\cosk_1 Y$ is the simplicial space whose space of $p$-simplices is the subspace of $Y^k$, where $k = p(p + 1)/2$, consisting of all tuples $(y_1, \ldots, y_k)$ such that $d_i(y_j) = d_{j-1}(y_i)$ if $i < j$. Thus

$$(\cosk_1 Y)_2 = (Y_1 \times_{Y_0} Y_1) \times_{Y_0^{[2]}} Y_1,$$

for example. The simplicial space $\cosk_1 Y$ is augmented over $M$, so that we have a simplicial map $\cosk_1 Y \to M$, where $M$ is regarded as a constant simplicial space. In fact $\cosk_1 Y$ is a generalized hypercover of $M$ in the language of [10].

If we denote by $uv$ the image of $u \otimes v$ under the isomorphism $(6)$ above then we can define an element $t(u, v, w) \in S^1$ such that

$$uv = wt(u, v, w)$$

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In this way we get a map $t: (\cosk_1 Y)_2 \to S^1$. This map $t$ is a twisting function for the simplicial topological group $NS^1[1]$ which is the nerve $NS^1$ of the topological groupoid $S^1$ (note that since $S^1$ is abelian, $NS^1$ has a natural structure as a simplicial abelian group). Therefore we have associated a PCTP to the bundle gerbe.

Conversely, if $\cosk_1 Y$ is a generalized hypercover of $M$ (so that the maps $Y_0 \to M$ and $Y_1 \to Y_0^{[2]}$ admit local sections), then any twisting cochain $t_n: (\cosk_1 Y)_n \to (NS^1)_{n-1}$ determines a bundle gerbe on $M$. To see this observe that there is a canonical inclusion $Y_1 \times_{Y_0^{[2]}} Y_1 \subset (\cosk_1 Y)_2$ defined by sending $(u, v)$ in $Y_1 \times_{Y_0^{[2]}} Y_1$ to $(u, v, s_0d_1(u))$ in $(\cosk_1 Y)_2$. Therefore the twisting cochain $t_2: (\cosk_1 Y)_2 \to S^1$ restricts to define a map $f: Y_1 \times_{Y_0^{[2]}} Y_1 \to S^1$. A little calculation shows that $f$ satisfies

$$f(u, v)f(v, w) = f(u, w)$$

for $(u, v, w) \in Y_1 \times_{Y_0^{[2]}} Y_1 \times_{Y_0^{[2]}} Y_1$. Since the map $Y_1 \to Y_0^{[2]}$ admits local sections we can define a principal $S^1$ bundle $L$ on $Y_0^{[2]}$ by forming the quotient of $Y_1 \times S^1$ by the equivalence relation which identifies $(u, \xi) \sim (v, \eta)$ if and only if $\xi = \eta f(u, v)$. It is not hard to show that $L \to Y_0^{[2]}$ has an associative product of the form (6).

So we see that bundle gerbes are more or less the same things as PTPCs. In fact PTPCs provide a means of unifying many different structures in the theory of gerbes and non-abelian cohomology. For instance bundle 2-gerbes [13] can be thought of as PTPCs for a suitable choice of simplicial topological group.

As another example, if $H \to G_0$ is a crossed module of topological groups corresponding to a simplicial group $G$, then a PTPC for $G$ on $M$ is precisely the same thing as a crossed module bundle gerbe on $M$ in the sense of [20].

5 Geometric realization of simplicial principal bundles

As with any topological bicocomplete category, there is a notion of geometric realization for simplicial objects in $\mathcal{K}/B$ (see for example VII.3 of [13]). If $X$ is a simplicial object in $\mathcal{K}/B$, then the fiberwise geometric realization $|X|$ of $X$ is defined by the usual coend formula:

$$|X| = \int_{[n] \in \Delta} X_n \otimes \Delta^n.$$ 

In other words, one regards $X$ as a simplicial object in $\mathcal{K}$ and computes the ordinary geometric realization, and then one equips this with the induced map to $B$. Fiberwise geometric realization gives rise to a functor $|\cdot|: s\mathcal{K}/B \to \mathcal{K}/B$. Since ordinary geometric realization commutes with products and fiber products, fiberwise geometric realization also commutes with products and fiber products in $\mathcal{K}/B$, and moreover is compatible with the topological structure on $\mathcal{K}/B$ in the sense that $|X \otimes K| = |X| \otimes |K|$ for any space $K$ in $\mathcal{K}$. For us, this means that fiberwise geometric realization $|\cdot|$ has an important technical advantage over the corresponding notion of fiberwise ‘fat’ realization $||\cdot||$ — if $G$ is a group in $s\mathcal{K}/B$, then $|G|$ is a group in $\mathcal{K}/B$ whereas $||G||$ has only a partially defined product.

Let $G$ be a group in $s\mathcal{K}/B$ and suppose that $P$ is a simplicial principal bundle in $\mathcal{K}/B$ with base $M$ and structure group $G$. It follows from the discussion above that, since the diagram (2) is a pullback in $s\mathcal{K}/B$, the diagram

$$\begin{array}{ccc}
|P| \times_B |G| & \to & |P| \\
\downarrow \psi & & \downarrow \pi \\
|P| & \to & |M| \\
\end{array}$$

is a pullback in $\mathcal{K}/B$. It follows that $|P|$ is a $|G|$ torsor over $|M|$ in $\mathcal{K}/B$. We want to investigate when the map $|P| \to |M|$ admits local sections, so that $|P|$ is a principal $|G|$ bundle over $|M|$ in $\mathcal{K}/B$. For this, we need a slight digression on the notion of proper simplicial object.
In [25] May isolated a condition on the degeneracy maps of a simplicial space $X$ which ensured that the geometric realization of $X$ had certain nice homotopy theoretic properties. May called a simplicial space $X$ proper if $(X_{n+1}, sX_n)$ is an NDR pair for every $n \geq 0$. Here
\[ sX_n = \bigcup_{i=0}^{n} s_i(X_n) \]
denotes the degenerate part of $X_{n+1}$. In modern terminology one would identify $sX_n$ with the latching object $L_nX$ of $X$ (see (10)) and say that a simplicial space $X$ in $s\mathcal{K}$ is proper if and only if it is Reedy cofibrant for the Reedy model structure on $s\mathcal{K}$ associated to the Strøm model structure on $\mathcal{K}$ (see Theorem 4.4.4 of [27]).

In [28] Segal studied the related notion of a good simplicial space: this was a simplicial space $X$ for which all the degeneracy maps were closed cofibrations. Both the notion of proper and the notion of good make sense in the setting of a bicomplete topological category $\mathcal{C}$. We say (in analogy with Definition 22.4.2 of [27]) that a simplicial object $X$ in $s\mathcal{C}$ is proper if the latching maps $L_nX \to X_n$ are $h$-cofibrations for all $n \geq 0$. Similarly we say that $X$ is good if all of the degeneracy morphisms $s_i: X_n \to X_{n+1}$ are all $h$-cofibrations.

With the notion of a proper simplicial object in $\mathcal{K}/B$ in hand, we can state the main result of our paper.

**Theorem 13.** Let $G$ be a simplicial group object in $\mathcal{K}/B$ and let $M$ be a proper simplicial object in $\mathcal{K}/B$. If $P$ is a simplicial principal bundle over $M$ with structure group $G$ such that each $P_n \to M_n$ is a numerable, fiberwise principal $G_n$ bundle in $\mathcal{K}/B$, then the induced map
\[ |P| \to |M| \]
on fiberwise geometric realizations is a locally trivial fiberwise principal $|G|$ bundle in $\mathcal{K}/B$.

To prove Theorem 13 we will adapt the approach of the papers [26], [30], [41] (which deal with the case where $G$ is a constant simplicial group) to the case where $G$ is an arbitrary simplicial group object in $\mathcal{K}/B$. An important ingredient in [26], [30], [41] is the notion of an equivariant NDR pair, a notion which we have already explained (see Section 3 above) has a straightforward generalization to the parametrized setting.

**Proof of Theorem 13.** Let $n \geq 0$ be an integer. Recall that the $n^{th}$ skeleton $s_k M$ of $M$ comes equipped with a map $s_k M \to M$ and that there are natural maps $s_k M \to s_m M$ whenever $m \leq n$. Recall also that $M = \text{colim}_m s_k M$ and that $s_{k-1} M \to s_k M$ fits into a pushout diagram built out of the $n^{th}$ latching object $L_n M$ of $M$. For more details we refer to Chapter VII of [15].

We use the $n^{th}$ skeletons of $M$ to define a filtration $|P|_0 \subset |P|_1 \subset \cdots \subset |P|_n \subset \cdots \subset |P|$ of $|P|$. The canonical maps $s_k M \to M$ induce by pullback simplicial principal bundles with structure group $G$ on each of the simplicial spaces $s_k M$. Let $|P|_n = |s_k M \times_M P|$. For convenience of notation we will also denote $|s_k M|$ by $|M|_n$. Note no confusion should result from this. Recall that $M = \text{colim}_n s_k M$ and hence $|M| = \text{colim}_n |M|_n$ in $\mathcal{K}/B$. We claim that $P = \text{colim}_n (s_k M \times_M P)$. This is easy to see in the special case that $P$ is trivial. We can reduce the general statement to this special case, since $P$ is a colimit of trivial bundles and colimits commute amongst themselves.

The map $|P| \to |M|$ is a quotient map, since the map $\Pi_{n \geq 0} P_n \otimes \Delta^n \to \Pi_{n \geq 0} M_n \otimes \Delta^n$ is a quotient map, and both of the maps $\Pi_{n \geq 0} P_n \otimes \Delta^n \to |P|$ and $\Pi_{n \geq 0} M_n \otimes \Delta^n \to |M|$ are quotient maps. Since the diagram
\[
\begin{array}{ccc}
|P|_n & \longrightarrow & |P| \\
|\downarrow| & & |\uparrow|
\end{array}
\]
\[
\begin{array}{ccc}
|M|_n & \longrightarrow & |M|
\end{array}
\]
is a pullback, we see that $|P|_n \to |M|_n$ is also a quotient map ($|M|_n \to |M|$ is a closed inclusion, and quotient maps pullback along closed inclusions to quotient maps). In particular $|M|_n$ has the quotient topology induced by the map $|\pi|: |P|_n \to |M|_n$. 

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The main step in our proof is to prove that \((|P|_n, |P|_{n-1})\) is a \(|G|\)-fiberwise NDR pair in \(\mathcal{X}/B\) for all \(n \geq 1\), so that we can apply the method of [26, 30, 41]. As a first step in this direction we have the following lemma.

**Lemma 14.** For every \(n \geq 1\) we have a pushout diagram in \(s\mathcal{X}/B\) of the form

\[
\begin{array}{c}
((M_n \otimes \partial[\Delta[n]]) \cup (L_n M \otimes [\Delta[n]]) \times_M P) \to \sk_{n-1}M \times_M P \\
(M_n \otimes [\Delta[n]]) \times_M P \to \sk_n M \times_M P
\end{array}
\]

(7)

**Proof.** Observe that the canonical map from the pushout to \(\sk_n M \times_M P\) is a continuous bijection in each degree. Therefore it suffices to show that for each \(m \geq 0\) the induced map

\[
((M_n \otimes [\Delta[n]])_m \times_M P)_m \to ((\sk_{n-1}M)_m \times_M P)_m
\]

is a quotient map. The map (8) is the map of fiberwise principal bundles induced by pullback along the quotient map

\[
(M_n \otimes [\Delta[n]])_m \to (\sk_n M)_m.
\]

Therefore to prove the lemma it suffices to establish the following claim: if \(P \to M\) is a fiberwise principal bundle and \(f: N \to M\) is a quotient map in \(\mathcal{X}/B\), then \(f^*P \to P\) is also a quotient map. To see this observe that since \(P\) can be constructed as a quotient of a coproduct of spaces of the form \(U \times_B G\), and \(f^*P\) can be constructed as a quotient of a coproduct of spaces of the form \(f^{-1}U \times_B G\), it suffices to prove that \(f^{-1}U \times_B G \to U \times_B G\) is a quotient map for any open set \(U \subset M\). Since the functor \((-) \times_B G\) preserves colimits this follows from the fact that \(f^{-1}U \to U\) is a quotient map, since \(U \subset M\) is open. \(\square\)

The second step is to show that in the diagram (7) the realization of the left hand vertical map is an \(\bar{f}\)-cofibration in \(|G|\mathcal{X}/B\). For this we will need the hypotheses that each \(P_n \to M_n\) is a numerable principal \(G_n\) bundle, and that \(M\) is proper.

**Lemma 15.** For every \(n \geq 1\), the map

\[
|((M_n \otimes \partial[\Delta[n]]) \cup (L_n M \otimes [\Delta[n]]) \times_M P| \to |M_n \otimes [\Delta[n]] \times_M P|
\]

is an \(\bar{f}\)-cofibration in \(|G|\mathcal{X}/B\) and hence \((|P|_n, |P|_{n-1})\) is a \(|G|\)-fiberwise NDR pair in \(\mathcal{X}/B\) for all \(n \geq 1\).

**Proof.** We have a pullback diagram

\[
\begin{array}{c}
|((M_n \otimes \partial[\Delta[n]]) \cup (L_n M \otimes [\Delta[n]]) \times_M P| \to |M_n \otimes [\Delta[n]] \times_M P| \\
(M_n \otimes [\Delta[n]]) \cup (sM_{n-1} \otimes [\Delta^n]) \to M_n \otimes [\Delta^n].
\end{array}
\]

Since \(M\) is proper, the inclusion \(sM_{n-1} \subset M_n\) is an \(\bar{f}\)-cofibration and standard results show that this induces an \(\bar{f}\)-cofibration \((M_n \otimes [\Delta^n]) \cup (sM_{n-1} \otimes [\Delta^n]) \to M_n \otimes [\Delta^n]\). Therefore the result will follow from Proposition 8 if we can show that

\[
|M_n \otimes [\Delta[n]] \times_M P| \to M_n \otimes [\Delta^n]
\]

is a numerable fiberwise principal \(|G|\) bundle in \(\mathcal{X}/B\). Since \(|P| \to |M|\) is a \(|G|\)-torsor, and torsors pullback along maps, \(|M_n \otimes [\Delta[n]] \times_M P| \to M_n \otimes [\Delta^n]\) is also a \(|G|\)-torsor. Therefore we only need to show that this map
admits local sections relative to a numerable open cover of \( M_n \otimes \Delta^n \). For this, consider the commutative diagram

\[
\begin{array}{ccc}
  P_n \otimes \Delta^n & \to & |P| \\
  \downarrow & & \downarrow \\
  M_n \otimes \Delta^n & \to & |M|
\end{array}
\]

where the horizontal maps are the canonical ones into the colimits defining \(|P|\) and \(|M|\). The map \( P_n \otimes \Delta^n \to |P| \) factors through \(|M_n \otimes \Delta[n] \times_M P|\) and hence \(|M_n \otimes \Delta[n] \times_M P| \to M_n \otimes \Delta^n\) admits local sections relative to a numerable open cover of \( M_n \otimes \Delta^n \) since the principal \( G_n \) bundle \( P_n \otimes \Delta^n \to M_n \otimes \Delta^n \) does by hypothesis.

We now proceed in analogy with the arguments in [26, 30, 41]. Since \(|P|_n, |P|_{n-1}\) is a fiberwise \(|G|\) equivariant NDR pair for every \( n \geq 1 \) and \(|P| = \text{colim}_n \, |P|_n\), we see that \(|P|, |P|_n\) is a fiberwise \(|G|\) equivariant NDR pair for every \( n \geq 0 \) (Lemma 1). For any \( n \geq 0 \) let \( h_n : |P| \times_\Delta I \to |P| \) and \( u_n : |P| \to I \) be a representation of \(|P|, |P|_n\) as a fiberwise \(|G|\) equivariant NDR pair. Define functions \( \hat{\rho}_n : |P| \to I \) for every \( n \geq 1 \) by

\[
\hat{\rho}_n(x) = (1 - u_n(x))h_n(x, 1)).
\]

The functions \( \hat{\rho}_n \) are easily seen to be \(|G|\) invariant and hence descend to functions \( \rho_n : |M| \to I \). Let \( U_n = \hat{\rho}_n^{-1}(0, 1) \) and let \( V_n = \rho_n^{-1}(0, 1) \) so that \( U_n = |\pi|^{-1}V_n \) (and hence \( U_n \) is \(|G|\) invariant). Following [26, 30] let \( r_n : |P| \to |P| \) denote the map \( r_n(x, t) = h_n(x, t, 1) \). Then we have (see [26, 30]) the following chain of inclusions

\[
|P|_n \setminus |P|_{n-1} \subset U_n \subset r_n^{-1}(|P|_n \setminus |P|_{n-1})
\]

Observe that we have a commutative diagram

\[
\begin{array}{ccc}
  U_n & \xrightarrow{r_n} & |P|_n \setminus |P|_{n-1} \\
  \downarrow & & \downarrow |\pi| \\
  V_n & \xrightarrow{\hat{\rho}_n} & |M|_n \setminus |M|_{n-1}
\end{array}
\]

in which the top horizontal maps are \(|G|\)-equivariant. In the previous Lemma 13 we observed that \(|M_n \otimes \Delta[n] \times_M P| \to |M_n \otimes \Delta[n]|\) is a numerable fiberwise principal \(|G|\)-bundle in \( \mathcal{K}/B \) and hence locally trivial. Using local sections of this map, we can find an open cover \( \{V_{n,i}\} \) of \( V_n \) and \(|G|\)-equivariant maps \( \zeta_{n,i} : U_{n,i} \to |G| \), where \( U_{n,i} = |\pi|^{-1}V_{n,i} \). Then we can define \(|G|\) invariant maps \( \delta_{n,i} : U_{n,i} \to U_{n,i} \) by \( \delta_{n,i}(x) = x\zeta_{n,i}(x)^{-1} \). Since \( \delta_{n,i} \) is \(|G|\)-invariant, it descends to define a unique map \( \sigma_{n,i} : V_{n,i} \to U_{n,i} \) so that the diagram

\[
\begin{array}{ccc}
  U_{n,i} & \xrightarrow{\sigma_{n,i}} & U_{n,i} \\
  \downarrow |\pi| & & \downarrow |\pi| \\
  V_{n,i} & \xrightarrow{\sigma_{n,i}} & V_{n,i}
\end{array}
\]

commutes. \( V_{n,i} \) has the quotient topology induced by \(|\pi|\) and hence \( \sigma_{n,i} \) is continuous. Clearly \( \sigma_{n,i} \) is a section of \(|\pi|\). Thus we have proven that there exist trivializations of \(|\pi| : |P| \to |M| \) over the \( V_{n,i} \).

It is not clear to us that one can find a **numerable** open cover of \(|M|\) over which \(|\pi| : |P| \to |M|\) has local sections. Clearly numerable bundles have some advantages over ordinary bundles and so it is of interest to know when the principal bundle constructed in Theorem 13 is numerable. It is possible that, extending remarks in [41], one could find sufficient conditions on \( M \) to ensure that \(|M|\) is paracompact, but we have not investigated this. We do however have the following sufficient condition which is satisfied in the case of interest for us in the next section.
Theorem 16. Let \( G \) be a simplicial group in \( \mathcal{X}/B \) and let \( M \) be a proper simplicial object in \( \mathcal{X}/B \). If \( P \) is a simplicial principal bundle over \( M \) with structure group \( G \) such that \( P_n \to M_n \) is trivial for all \( n \geq 0 \), then the induced map

\[
|P| \to |M|
\]

on fiberwise geometric realizations is a locally trivial, numerable fiberwise principal \( |G| \) bundle.

In particular this theorem applies whenever \( P \to M \) is a PTCP.

Proof. Follow the proof above to obtain the commutative diagram (8). In this case the bundle \( |M_n \times [n]_\times M| \to |M_n \times [n]| \) is trivial, and therefore we can define \( |G| \)-equivariant maps \( \zeta_n: U_n \to |G| \). In exactly the same way as above we can use the maps \( \zeta_n \) to define \( |G| \)-invariant maps \( \sigma_n: U_n \to U_n \) which descend to sections \( \sigma_n: V_n \to U_n \) of \( |\pi| \). The problem now is to show that the open cover \( \{V_n\} \) is numerable. To do this we use the functions \( \rho_n: V_n \to I \) constructed earlier. The collection of functions \( \{\rho_n\} \) may not be locally finite, this can be fixed however using the method of Dold [11]; one defines new functions \( \phi_n: U_n \to I \) with \( \text{supp}(\phi_n) \subseteq U_n \) by

\[
\phi_n(x) = \max \left( 0, \rho_n(x) - n \sum_{i=1}^{n-1} \rho_i(x) \right).
\]

Then one can check as in [11] that the collection of functions \( \{\phi_n\} \) is locally finite. It is now clear how to form a partition of unity from the \( \phi_n \). \( \square \)

6 Good simplicial groups in \( \mathcal{X}/B \)

In this section we will prove, as a corollary of Theorem [13] a generalization of the classical fact that \( |WG| \) is a model for \( BG \), when \( G \) is a well pointed topological group. The analogue of a pointed space in the parametrized context is the notion of an ex-space. An ex-space over \( B \) is a space \( X \) in \( \mathcal{X}/B \) together with a section of the map \( X \to B \). For example any group \( G \) in \( \mathcal{X}/B \) has a natural structure as an ex-space — the identity section gives a canonical section of \( G \to B \). The category of ex-spaces and maps between them is denoted by \( \mathcal{X}_B \). An ex-space \( X \) is said to be well-sectioned if the section \( B \to X \) is an \( \hat{f} \)-cofibration. The following definition describes the analog of a well pointed topological group in the parameterized setting.

Definition 17. Let \( G \) be a group in \( s\mathcal{X}/B \). We will say that the group \( G \) is well sectioned if, for every \( n \geq 0 \), the object \( G_n \) in \( \mathcal{X}/B \) is well sectioned by the identity map.

With this defintion in hand we can state the main result of this section.

Proposition 18. Let \( G \) be a well sectioned simplicial group in \( \mathcal{X}/B \). Then the geometric realization

\[
|WG| \to |WG|\]

of the universal \( G \)-bundle \( WG \to WG \) is a numerable principal \( |G| \) bundle. Moreover \( |WG| \) is a fiberwise contractible group in \( \mathcal{X}/B \) containing \( |G| \) as a closed subgroup.

In [4] it was proven that when \( G \) is a group in simplicial sets then there is a homeomorphism between \( |WG| \) and \( B|G| \). It is well known that there is a simplicial homotopy equivalence \( dNG \to WG \). It is not so clear that this simplicial map induces a homeomorphism on geometric realizations — it would therefore be interesting to obtain an explicit description of the homeomorphism from [4].

We conjecture that if \( G \) is a well sectioned group in \( s\mathcal{X}/B \) then the methods of [4] adapt to give a homeomorphism between \( |WG| \) and \( B|G| \). We have not pursued this in detail since the approach of [4] relies on a result of Steenrod [41] which identifies a recursive definition of the classifying space with the geometric realization \( B|G| \). Hence one would need to first extend this result of Steenrod’s to the parametrized setting, which we have not attempted to do.

Proposition [13] follows immediately from Corollary [10] once one knows that \( WG \) is proper if \( G \) is well sectioned. The remainder of this section is devoted to the proof of this last fact (see Proposition [25]). Our
strategy will be to show firstly that good simplicial objects in $\mathcal{X}/B$ are automatically proper and secondly, that $G$ well sectioned implies that $\overline{W}G$ is good.

We begin by examining the relation between good simplicial objects and proper simplicial objects. The slogan “good implies proper” seems to be folklore in some contexts. This is implicit in the proof of Lemma A–5 of [15] and a proof in the context of simplicial spaces is given in Corollary 2.4 (b) of [13], while a similar statement is made in Remark 22.4.3 of [27]. We will give a proof here in a reasonably general setting under the assumption that some additional axioms hold, one of which is a generalization of Lillig’s union theorem on cofibrations [23].

Recall that a proper simplicial object $X$ in a topological bicomplete category $\mathcal{C}$ is one for which the latching maps $L_nX \to X_n$ are $h$-cofibrations for all $n \geq 0$. We need to examine the notion of latching object in a little more detail. Recall that $L_nX$ is defined to be the colimit

$$L_nX = \colim_{[k] \to [n] \in \Delta_+^o/[n]} X_k$$

where $\Delta_+$ denotes the subcategory of $\Delta$ whose morphisms are the surjections. This colimit can also be calculated as the coequalizer

$$\prod_{1 \leq i < j \leq n - 2} X_{n-2} \rightrightarrows \prod_{1 \leq i \leq n - 1} X_{n-1} \twoheadrightarrow L_nX$$

where the two maps defining the coequalizer arise from the simplicial identity $s_is_{j-1} = s_js_i$ if $i < j$ (see for example V Lemma 1.1 and VII Remark 1.8 of [15]). It is well known that $L_0X = \emptyset$, $L_1X = X_0$ and $L_2X = \bigcup_{n \geq 0} X_n$. We will need a decomposition of the latching object $L_nX$ for any $n \geq 0$.

From Chapter VII of [15] we have the following. Let $\mathcal{O}_n$ be the full subcategory of $[n]/\Delta$ whose objects are the surjections $\phi: [n] \to [m]$, for $m < n$. Ordinal sum (on the right) with $[0]$ defines a functor $\sigma_0: \mathcal{O}_n \to \mathcal{O}_{n+1}$. Consider the image $\sigma_0(\mathcal{O}_n)$. The objects of this category consist of all surjections $\phi: [n+1] \to [m]$ with $0 < m < n + 1$ and with $\phi(n) < \phi(n+1)$. Likewise the morphisms are all morphisms in $[n+1]/\Delta$ of the form $f: [m] \to [k]$ such that $f(m-1) \leq k - 1$.

Next, let $\mathcal{M}(n)$ denote the full subcategory of $\mathcal{O}_{n+1}$ whose objects consist of the surjections $\phi: [n+1] \to [m]$ with $\phi(n) = \phi(n+1)$. Note that $s^n: [n+1] \to [n]$ is an initial object of $\mathcal{M}(n)$. We note the following fact about these subcategories (compare with Lemma VII 1.26 of [15]): the following diagram is a pushout in the category of small categories

$$\begin{array}{ccc}
\mathcal{O}_n & \longrightarrow & \mathcal{M}(n) \\
\sigma_0(\mathcal{O}_n) & \downarrow & \mathcal{O}_{n+1} \\
\end{array}$$

As in [15], if $J$ is a small category and $F: J \to \Delta^{op}$ is a functor we define the generalized latching object $L_JX$ of a simplicial object $X$ to be the colimit

$$L_JX = \colim X$$

where we regard $X$ as a functor $X: J \to \mathcal{C}$ via composition with $F$. Therefore $L_{\mathcal{M}(n)^{op}}X = L_nX$ and $L_{\mathcal{M}(n+1)^{op}}X = X_n$, using the obvious functor $\mathcal{O}_{n+1}^{op} \to \Delta^{op}$. Before we continue, we need to observe the following fact about computing colimits: if $X: D \to \mathcal{C}$ is a diagram in $\mathcal{C}$ and $D$ is a pushout $D = A \cup_B C$ in $\mathcal{C}$, the category of small categories, then (with the obvious meanings) the diagram

$$\begin{array}{ccc}
\colim_B X & \longrightarrow & \colim_A X \\
\downarrow & & \downarrow \\
\colim_C X & \longrightarrow & \colim_D X \\
\end{array}$$

is a pushout in $\mathcal{C}$. Since $\mathcal{O}_{n+1}^{op}$ is the pushout $\mathcal{O}_{n+1}^{op} = \sigma_0(\mathcal{O}_n^{op}) \cup_{\mathcal{O}_n^{op}} \mathcal{M}(n)^{op}$, we can apply this fact to prove the following Lemma.
Lemma 19. The following diagram is a pushout:

\[
\begin{array}{ccc}
L_n X & \rightarrow & X_n \\
\downarrow & & \downarrow \\
L_n \text{Dec} \rightarrow L_{n+1} X
\end{array}
\] (12)

Proof. We need to show that

\[L_{\sigma_0(\sigma_n)} X = L_n \text{Dec} X.\]

This follows by the computation:

\[L_{\sigma_0(\sigma_n)} X = \colim_{[n+1]=[m+1], 0<m<n} X_{m+1} = \colim_{[n]=[m], 0\leq m<n} (\text{Dec} X)_m = L_n \text{Dec} X.\]

Definition 20. Let \(\mathcal{C}\) be a topological bicomplete category. We say that \(\mathcal{C}\) satisfies the Lillig condition if the following is true. Suppose that the following diagram in \(\mathcal{C}\) is a pullback:

\[
\begin{array}{ccc}
A_3 & \rightarrow & A_2 \\
\downarrow & & \downarrow \\
A_1 & \rightarrow & X
\end{array}
\]

and that the morphisms \(A_1 \rightarrow X, A_3 \rightarrow X\) and \(A_2 \rightarrow X\) are \(\tilde{h}\)-cofibrations. Then the canonical map \(A_1 \cup A_3 A_2 \rightarrow X\) is an \(\tilde{h}\)-cofibration.

When \(\mathcal{C} = \mathcal{K}\) this is Lillig’s union theorem on cofibrations [23]. We will prove shortly that a reworking of the proof in [23] shows that the Lillig condition holds when \(\mathcal{C} = \mathcal{K}/B\) — we do not know if this condition holds more generally.

Proposition 21. Let \(\mathcal{C}\) be a topological bicomplete category and let \(X\) be a good simplicial object in \(\mathcal{C}\). Suppose that the following two conditions are satisfied:

1. \(\mathcal{C}\) satisfies the Lillig condition of Definition 20
2. the functor \(s^*_n: \mathcal{C}/X_{n+1} \rightarrow \mathcal{C}/X_n\) commutes with finite colimits for all \(n \geq 0\).

Then \(X\) is proper.

What follows is based upon the proof of Proposition 23.6 of [39].

Proof. Suppose that \(\mathcal{C}\) is a topological bicomplete category satisfying 1 and 2 above. We will prove by induction on \(n\) that for each \(n \geq 0\) the latching maps \(L_n X \rightarrow X_n\) are \(\tilde{h}\)-cofibrations for any good simplicial object \(X\) in \(\mathcal{C}\) such that condition 2 above is satisfied. When \(n = 0\) the map \(L_0 X \rightarrow X_0\) is \(0 \rightarrow X_0\) which is an \(\tilde{h}\)-cofibration since every object of \(\mathcal{C}\) is \(\tilde{h}\)-cofibrant. Assume that \(L_n X \rightarrow X_n\) is an \(\tilde{h}\)-cofibration and that \(X\) satisfies condition 2.

From the diagram (12) we get the following commutative diagram from the universal property of pushouts.
and we need to prove that the dotted morphism is an $h$-cofibration. By hypothesis $s_n: X_n \to X_{n+1}$ is an $h$-cofibration and so our inductive assumption shows that the composition $L_n X \to X_n \to X_{n+1}$ is an $h$-cofibration. Similarly, we can identify the morphism $L_n \text{Dec}_0 X \to X_{n+1}$ with the latching map $L_n \text{Dec}_0 X \to (\text{Dec}_0 X)_n$; clearly $\text{Dec}_0 X$ is good and condition 2 is satisfied, therefore our inductive hypothesis shows that $L_n \text{Dec}_0 X \to (\text{Dec}_0 X)_n$ is an $h$-cofibration. We would like to make use of the fact that $C$ satisfies the Lillig condition and conclude that $L_n X \to X_{n+1}$ is an $h$-cofibration. For this we need to know that the diagram

\[
\begin{array}{ccc}
L_n X & \to & X_n \\
\downarrow & & \downarrow s_n \\
L_n \text{Dec}_0 X & \to & X_{n+1}
\end{array}
\]

is a pullback. To see this we use the formula (11) for the latching object $L_n \text{Dec}_0 X$ as a coequalizer and the fact that $s_n^*$ commutes with finite colimits to write

\[
X_n \times X_{n+1} L_n \text{Dec}_0 X = \text{coeq} \left( \coprod_{0 \leq i < j \leq n-2} X_{n-1} \times X_{n+1} X_n \Rightarrow \coprod_{0 \leq i \leq n} X_n \times X_{n+1} X_n \right)
\]

Now the well known fact (see for example [15]) that the diagrams

\[
\begin{array}{ccc}
X_{n-1} & \overset{s_i}{\to} & X_n \\
\downarrow s_{i-1} & & \downarrow s_i \\
X_n & \overset{s_i}{\to} & X_{n+1}
\end{array}
\]

are pullbacks for $i < j$ shows that the above expression can in turn be written as the coequalizer

\[
\text{coeq} \left( \coprod_{0 \leq i < j \leq n-2} X_{n-2} \Rightarrow \coprod_{0 \leq i \leq n-1} X_{n-1} \right)
\]

defining the latching object $L_n X$. \hfill $\Box$

**Proposition 22.** Let $\mathcal{C} = \mathcal{K}/B$. Then $\mathcal{C}$ satisfies conditions 1 and 2 above. Hence any good simplicial object in $\mathcal{K}/B$ is proper in the above sense.

**Proof.** We deal with condition 2 first: we need to know that the functor $s_n^*: (\mathcal{K}/B)/X_{n+1} \to (\mathcal{K}/B)/X_n$, i.e. restriction along the closed inclusion $s_n: X_n \to X_{n+1}$ preserves finite colimits. In other words, since $(\mathcal{K}/B)/X \cong \mathcal{K}/X$ for any object $X$ in $\mathcal{K}/B$, we have to show that $s_n^*: \mathcal{K}/X_{n+1} \to \mathcal{K}/X_n$ preserves finite colimits.

A colimit in $\mathcal{K}/X_{n+1}$ is constructed as a quotient of a coproduct in $\mathcal{K}$ and then equipped with the canonical map to $X_{n+1}$. Therefore it is sufficient to prove two things: firstly that restriction along $X_n$ preserves coproducts in $\mathcal{K}/X_{n+1}$ and secondly that if $q: Y \to Z$ is a quotient map in $\mathcal{K}/X_{n+1}$ then in the pullback diagram

\[
\begin{array}{ccc}
X_n \times X_{n+1} Y & \to & Y \\
\downarrow & & \downarrow q \\
X_n \times X_{n+1} Z & \to & Z
\end{array}
\]

in $\mathcal{K}$ the map $X_n \times X_{n+1} Y \to X_n \times X_{n+1} Z$ is a quotient map. The first of these things is easy to prove, for the second it is enough to prove that $X_n \times X_{n+1} Z \to Z$ is a closed inclusion, since quotient maps restrict
to quotient maps along closed subspaces. This is clear however, since $s_n: X_n \to X_{n+1}$ is a closed inclusion, and closed inclusions pull back along arbitrary maps to closed inclusions.

We now prove that condition 1 holds, i.e. we prove that the analog of Lillig’s theorem holds in the category $\mathcal{K}/B$. Suppose that

$$
\begin{array}{c}
A_3 \\
A_2 \\
A_1 \longrightarrow \ A
\end{array}
$$

is a pullback diagram in $\mathcal{K}/B$ as in Definition 20 above so that the maps $A_1 \to X$, $A_2 \to X$ and $A_3 \to X$ are $\tilde{f}$-cofibrations. From the pushout-product theorem (see [36]) it follows that

$$A_1 \cup_{A_3} A_3 \otimes I \cup_{A_3} A_2 \to X \otimes I$$

is an $\tilde{f}$-cofibration. This map fits into the commutative diagram

$$
\begin{array}{ccc}
A_1 \cup_{A_3} A_3 \otimes I \cup_{A_3} A_2 & \longrightarrow & A_1 \cup_{A_3} A_2 \\
X \otimes I & \longrightarrow & X
\end{array}
$$

The pushout of (13) along $A_1 \cup_{A_3} A_3 \otimes I \cup_{A_3} A_2 \to A_1 \cup_{A_3} A_2$ can be identified with a map

$$A_1 \cup_{A_3} A_2 \to X \otimes I \cup_{A_3 \otimes I} A_3$$

which is also an $\tilde{f}$-cofibration. Therefore, to prove that $A_1 \cup_{A_3} A_2 \to X$ is an $\tilde{f}$-cofibration it suffices to prove that $A_1 \cup_{A_3} A_2 \to X$ is a retract of $A_1 \cup_{A_3} A_2 \to X \otimes I \cup_{A_3 \otimes I} A_3$. Suppose $(u_1, h_1)$ and $(u_2, h_2)$ are representations of $(X, A_1)$ and $(X, A_2)$ as fiberwise NDR pairs. As in [23] define a map $u: X \to X \otimes I \cup_{A_3 \otimes I} A_3$ by

$$u(x) = \begin{cases}
[x, u_1(x)/(u_1(x) + u_2(x))] & \text{if } x \notin A_3, \\
[x, 0] & \text{if } x \in A_3.
\end{cases}$$

Then it is easy to check that $u(x) = [x, 0]$ if $x \in A_1$ and $u(x) = [x, 1]$ if $x \in A_2$. This map exhibits $A_1 \cup_{A_3} A_2 \to X$ as a retract, as required. We do not know if the analogous result holds more generally; the proof we have given (which is a re-working of Lillig’s original proof) uses crucially the characterization of $\tilde{f}$-cofibrations in terms of fiberwise NDR pairs. We note that the result is false in general if $\tilde{f}$-cofibrations are replaced by $f$-cofibrations.

We would like to find a sufficient condition which ensures that the simplicial object underlying a group in $\mathcal{K}/B$ is good (and hence proper by the above proposition). We will say that a simplicial group $G$ in $\mathcal{K}/B$ is a good simplicial group if the object in $s\mathcal{K}/B$ underlying $G$ is good. The following is a sufficient condition which is relatively easy to check in practice.

**Proposition 23.** Let $G$ be a well sectioned group in $s\mathcal{K}/B$. Then $G$ is a good simplicial group in $\mathcal{K}/B$.

For the proof of this proposition we will need the following lemma.

**Lemma 24.** Suppose that $A_1 \to X$ and $A_2 \to Y$ are $\tilde{f}$-cofibrations in $\mathcal{K}/B$. Then $A_1 \times_B A_2 \to X \times_B Y$ is also an $\tilde{f}$-cofibration.

**Proof.** It is clearly sufficient to prove that if $A \to X$ is an $\tilde{f}$-cofibration and $Y$ is any space over $B$, then $A \times_B Y \to X \times_B Y$ is an $\tilde{f}$-cofibration, in other words it has the LLP with respect to all $f$-acyclic $f$-fibrations $U \to V$. By adjointness, this is equivalent to checking that $A \to X$ has the LLP against all maps of the form $\text{Map}_B(Y, U) \to \text{Map}_B(Y, V)$ where $U \to V$ is an $f$-acyclic $f$-fibration.
By an adjointness argument, the functor $\text{Map}_B(Y, -) : \mathcal{K}/B \to \mathcal{K}/B$ preserves $f$-fibrations. It also preserves fiberwise homotopies: if $g_0, g_1 : X \to Z$ are fiberwise homotopic through a fiberwise homotopy $h : X \times_B I \to Z$, then the maps $\text{Map}_B(Y, g_0)$ and $\text{Map}_B(Y, g_1)$ are fiberwise homotopic through the fiberwise homotopy $h : \text{Map}_B(Y, X) \times_B I \to \text{Map}_B(Y, Z)$ defined as the composite

$$\text{Map}_B(Y, X) \times_B I \to \text{Map}_B(Y, X) \times_B I \xrightarrow{\text{Map}_B(Y, h)} \text{Map}_B(Y, Z), \quad (14)$$

where the first map is the adjoint of the canonical map $Y \times_B \text{Map}_B(Y, X) \times_B I \to X \times_B I$. One can check that $\tilde{h}$ so defined does give such a fiberwise homotopy as claimed. It follows that the functor $\text{Map}_B(Y, -)$ preserves $f$-equivalences, and hence $f$-acyclic $f$-fibrations, which proves the lemma.

Proof of Proposition 23. We need to show that $s_i : G_n \to G_{n+1}$ is an $\bar{f}$-cofibration for all $0 \leq i \leq n$ and all $n \geq 0$. Since $s_i$ is a section of the corresponding face operator $d_i$, we can identify $s_i$ with the map $G_n \to G_n \times_B \ker(d_i)$ which sends $g \mapsto (g, 1)$. Therefore, by the lemma, to prove that $s_i$ is an $\bar{f}$-cofibration it is sufficient to prove that $\ker(d_i)$ is well sectioned.

For this, we observe that $\ker(d_i)$ is a retract of $G_{n+1}$ by the map $G_{n+1} \to \ker(d_i)$ sending $g$ to $g s_i d_i(g)^{-1}$. Therefore the section $B \to \ker(d_i)$ is an $\bar{f}$-cofibration since it is a retract of the map $B \to G_{n+1}$ which is an $\bar{f}$-cofibration by hypothesis.

This result has a partial converse: if $G$ is a good simplicial group in $\mathcal{K}/B$ such that $G_0$ is well sectioned, then $G_n$ is well sectioned for every $n \geq 0$. We have the following proposition.

Proposition 25. If $G$ is a well-sectioned group in $s\mathcal{K}/B$ then $\overline{WG}$ is proper in $s\mathcal{K}/B$.

Proof. To prove this we need to recall the definition of the degeneracy maps of $\overline{WG}$. From [28] we have $s_0(c) = 1 \in G_0$ if $n = 0$ and

$$s_i(g_{n-1}, \ldots, g_0) = \begin{cases} (1, g_{n-1}, \ldots, g_0) & \text{if } i = 0, \\ (s_{i-1}(g_n), \ldots, s_0(g_{n-i}), 1, g_{n-i}, \ldots, g_0) & \text{if } 1 \leq i \leq n \end{cases}$$

if $n \geq 1$. Since each degeneracy map $s_i$ of $G$ is an $\bar{f}$-cofibration, and each $G_n$ is well sectioned, it follows from Lemma 24 that

$$s_{i-1} \times_B \cdots \times_B s_0 \times_B e_{G-n-1} \times_B 1 \times_B \cdots \times_B 1 : \overline{WG}_n \to \overline{WG}_{n+1}$$

is also an $\bar{f}$-cofibration for all $1 \leq i \leq n$ and all $n \geq 1$. The remaining degeneracy maps are also $\bar{f}$-cofibrations and so $\overline{WG}$ is good, and hence proper by Proposition 24.

Recall that fiberwise geometric realization gives a functor $| \cdot | : s\mathcal{K}/B \to \mathcal{K}/B$ which preserves products and fiber products. It follows that $| \cdot |$ sends group objects in $s\mathcal{K}/B$ to group objects in $\mathcal{K}/B$. Hence $|G|$ is a topological group over $B$ whenever $G$ is a simplicial topological group over $B$. As a final result in this section, which we shall need elsewhere, we have the following.

Proposition 26. Let $G$ be a well sectioned group in $s\mathcal{K}/B$. Then $|G|$ is a well sectioned group in $\mathcal{K}/B$.

Proof. Since $G$ is well sectioned, the simplicial object $G$ is proper, and hence the inclusion $|G|_n \subset |G|_{n+1}$ is an $\bar{f}$-cofibration for all $n \geq 0$ (with the notation of the proof of Theorem 13). Therefore the inclusion $|G|_n \subset |G|$ is an $\bar{f}$-cofibration for all $n \geq 0$. Since $|G|_0$ is well sectioned and the composite of two $\bar{f}$-cofibrations is an $\bar{f}$-cofibration, it follows that $|G|$ is well sectioned.
7 Classifying theory for parametrized bundles

In [8] (see pages 37–39) a construction of a universal fiberwise principal $G$-bundle is given, where $G$ is a group object in $\mathcal{K}/B$. This construction is based on the Milnor construction of a universal bundle, using infinite joins. This model of the universal bundle is very useful as it makes almost no assumptions on $G$, however it is also useful to impose a mild restriction on $G$ and build a slightly more economical model with some convenient properties.

In particular, when $G$ is a well sectioned group in $\mathcal{K}/B$, Proposition [8] specializes (with $G$ replaced by the corresponding constant simplicial group object) to give a construction of a numerable fiberwise principal $G$-bundle $EG \to BG$. Unlike the Milnor model of $EG \to BG$ constructed in [8] the bundle $EG \to BG$ has the property that its total space $EG$ is a (fiberwise contractible) group object in $\mathcal{K}/B$. This is useful for some purposes. In this section we will describe the classifying theory for parametrized bundles based on this model from Theorem [19]. Our main result in this section is the following.

**Theorem 27.** Let $M$ be a paracompact space over $B$ and let $G$ be a well sectioned group object in $\mathcal{K}/B$. Then there is a bijection

$$H^1(M,G) \cong [M,BG]_{\mathcal{K}/B}.$$

To prove this Theorem we make use of the fact that $H^1(M,G)$ is isomorphic to the set of fiberwise concordance classes of fiberwise principal $G$ bundles on $M$ (see Corollary [7]). We define a map

$$[M, BG]_{\mathcal{K}/B} \to H^1(M,G)$$

$$[f] \mapsto [f^*EG]$$

(15)

for $f: M \to BG$. It is easy to verify that this map is well defined. To prove that it is a bijection we construct an inverse. For this we need some preparation.

Recall the Čech nerve $\check{C}(P)$ associated to $P$ (see [5]). This is a simplicial object in $\mathcal{K}/B$ which is augmented over $M$ and hence on taking fiberwise geometric realizations we obtain a map

$$|\check{C}(P)| \to M$$

in $\mathcal{K}/B$. The Čech nerve $\check{C}(Y)$ can of course be defined for any map $\pi: Y \to M$. It is a well known fact (essentially going back to [37]) that if $\pi$ admits local sections and $M$ is paracompact then the map $|\check{C}(Y)| \to M$ is a homotopy equivalence (in fact the hypothesis on $M$ can be removed and one can show, at least when $B$ is a point, that the map is a weak homotopy equivalence — see [10]). We have the following Lemma.

**Lemma 28.** If $\pi: Y \to M$ is a map in $\mathcal{K}/B$ which admits local sections and $M$ is paracompact then the canonical map

$$|\check{C}(Y)| \to M$$

is a fiberwise homotopy equivalence.

**Proof.** For convenience of notation let $\pi$ denote the canonical map $|\check{C}(Y)| \to M$. Since $M$ is paracompact we can choose an open cover $U = (U_i)_{i \in I}$ of $M$ such that there exist local sections $s_i: U_i \to Y$ of $\pi$, and such that there exists a partition of unity $(\phi_i)_{i \in I}$ subordinate to $U$.

We define a map $s: M \to |\check{C}(Y)|$ which is a section of $\pi$ as follows. Choose a total ordering on the set $I$. If $m \in M$ let $i_0 \leq i_1 \leq \cdots \leq i_{p(m)}$ be the ordered set of indices $i$ such that $\phi_i(m) \neq 0$. Let

$$s(m) = |\phi_{i_0}(m), \ldots, \phi_{i_{p(m)}}(m), s_{i_0}(m), \ldots, s_{i_{p(m)}}(m)|.$$

Then it is clear that $s$ is the identity on $M$. Define a homotopy $h_t$ between $s\pi$ and the identity on $|\check{C}(Y)|$ by

$$h_t([s_0, \ldots, s_p, y_0, \ldots, y_p]) = [ts_0, \ldots, ts_p, (1-t)\phi_{i_0}(m), \ldots, (1-t)\phi_{i_{p(m)}}(m),$$

$$y_0, \ldots, y_p, s_{i_0}(m), \ldots, s_{i_{p(m)}}(m)].$$

It is easy to check that $h_t$ is well-defined and is a fiberwise homotopy between $s\pi$ and the identity.
With these preparations out of the way, we can return to the problem of defining an inverse for the map 
\([M, BG]_{\mathcal{X}/B} \to H^1(M, G)\). Since \(G\) acts strongly freely on \(P\), there exist maps 

\[ P \times_M P \to G, \ P \times_M P \times_M P \to G \times G, \ldots \text{ etc} \]

which fit together to give a simplicial map \(\hat{C}(P) \to WG\). On taking fiberwise geometric realizations this gives a map

\[ |\hat{C}(P)| \to BG \tag{16} \]

in \(\mathcal{X}/B\). Composing this map with the map \(s: M \to |\hat{C}(P)|\) of Lemma 28 gives a map \(M \to BG\). It is clear that this map respects the concordance equivalence relation (recall that we are identifying \(H^1(M, G)\) with the set of fiberwise concordance classes) to give a map

\[ H^1(M, G) \to [M, BG]_{\mathcal{X}/B}. \tag{17} \]

We need to prove that this map is the inverse of the map \(15\). We first examine the composite \(H^1(M, G) \to [M, BG]_{\mathcal{X}/B} \to H^1(M, G)\). To show that this is the identity we need to show that the pullback of \(EG \to BG\) under the map \(|\hat{C}(P)| \to BG\) is fiberwise isomorphic to \(\pi^*P\) where, as in the proof of Lemma 28 we use \(\pi\) to denote the map \(|\hat{C}(P)| \to M\). It then follows that the pullback of \(EG \to BG\) under the composite map

\[ M \to |\hat{C}(P)| \to BG \] is isomorphic to \(s^*\pi^*P \cong P\). Observe that we have the commutative diagram

\[
\begin{array}{ccc}
P & \xleftarrow{\hat{C}(\pi^*P)} & WG \\
\downarrow & & \downarrow \\
M & \xleftarrow{\hat{C}(P)} & WG
\end{array}
\]

in \(s\mathcal{X}/B\). Each square in this diagram is a pullback and so it follows that on taking fiberwise geometric realizations we have a diagram

\[
\begin{array}{ccc}
P & \xleftarrow{|\hat{C}(\pi^*P)|} & EG \\
\downarrow & & \downarrow \\
M & \xleftarrow{|\hat{C}(P)|} & BG
\end{array}
\]

in which each square is a pullback. From the remark above it follows that the composite \(H^1(M, G) \to [M, BG]_{\mathcal{X}/B} \to H^1(M, G)\) is the identity.

Now we turn our attention to the composite \([M, BG]_{\mathcal{X}/B} \to H^1(M, G) \to [M, BG]_{\mathcal{X}/B}\). To prove that this is the identity it is sufficient to prove the following: in the diagram

\[
\begin{array}{ccc}
|\hat{C}(EG)| & \longrightarrow & BG \\
\downarrow & & \downarrow \\
BG & & BG
\end{array}
\]

the two maps \(|\hat{C}(EG)| \to BG\) are fiberwise homotopic. This is the part of our proof that we believe is new. Here the horizontal map \(|\hat{C}(EG)| \to BG\) is the map \(16\) above. For suppose there exists such a fiberwise homotopy \(|\hat{C}(EG)| \times_B I \to BG\), then for any map \(f: M \to BG\) in \(\mathcal{X}/B\) we obtain a diagram

\[
\begin{array}{ccc}
M & \xrightarrow{s} & |\hat{C}(P)| \\
\downarrow & & \downarrow \\
M \times_B I & \xrightarrow{s \times_B I} & |\hat{C}(P)| \times_B I \\
\downarrow & & \downarrow \\
M & \xrightarrow{s} & |\hat{C}(P)|
\end{array}
\]

\[
\begin{array}{ccc}
& |\hat{C}(P)| \\
& \downarrow \\
& |\hat{C}(EG)|
\end{array}
\]

\[
\begin{array}{ccc}
& |\hat{C}(P)| \times_B I \\
& \downarrow \\
& |\hat{C}(EG)| \times_B I \\
\downarrow & & \downarrow \\
& BG
\end{array}
\]

\[
\begin{array}{ccc}
& |\hat{C}(EG)| \\
& \downarrow \\
& BG
\end{array}
\]

25
in which \( P \) is the fiberwise principal \( G \) bundle induced by \( f \). The composite of the bottom-most rightwards pointing maps in this diagram gives a map \( M \to BG \) which is equal to \( f \). We then obtain by composition a fiberwise homotopy between \( f \) and the composite \( M \to |\tilde{C}(P)| \to BG \).

So we need to prove that there is such a fiberwise homotopy. To do this we proceed as follows. Let \( \bar{W}G_{\Delta[1]} \) denote the usual simplicial path space, so that \( \bar{W}G_{\Delta[1]} \) is the simplicial object in \( \mathcal{X}/B \) whose object of \( n \)-simplices is the generalized matching object \( (\bar{W}G_{\Delta[1]})_n = M_{\Delta[n] \times \Delta[1]} \bar{W}G \) (see VII 1.21 of [15]). Then the evaluation map \( \bar{W}G_{\Delta[1]} \times \Delta[1] \to \bar{W}G \) gives a canonical simplicial homotopy

\[
\begin{array}{ccc}
\bar{W}G_{\Delta[1]} & \to & \bar{W}G \\
\downarrow & & \downarrow \\
\bar{W}G_{\Delta[1]} \times \Delta[1] & \to & \bar{W}G \\
\end{array}
\]

in \( \mathcal{X}/B \), where the two maps \( \bar{W}G_{\Delta[1]} \to \bar{W}G \) are induced by the two inclusions of \( \Delta[0] \) into \( \Delta[1] \). Hence on taking fiberwise geometric realizations we get a fiberwise homotopy

\[
\begin{array}{ccc}
|\bar{W}G_{\Delta[1]}| & \to & |\bar{W}G| \\
\downarrow & & \downarrow \\
|\bar{W}G_{\Delta[1]}| \times_B I & \to & BG \\
\end{array}
\]

We will prove the following proposition.

**Proposition 29.** There is a map \( |\tilde{C}(EG)| \to |\bar{W}G_{\Delta[1]}| \) in \( \mathcal{X}/B \) such that the diagram

\[
\begin{array}{ccc}
|\tilde{C}(EG)| & \to & |\bar{W}G_{\Delta[1]}| \\
\downarrow & & \downarrow \\
BG & \to & BG \\
\end{array}
\]

commutes.

Before we give the proof of this proposition let us show how it implies the existence of a fiberwise homotopy between the two maps in \([13]\) above. Given a map \( |\tilde{C}(EG)| \to |\bar{W}G_{\Delta[1]}| \) as in the proposition, observe that we get a diagram

\[
\begin{array}{ccc}
|\tilde{C}(EG)| & \to & |\bar{W}G_{\Delta[1]}| \\
\downarrow & & \downarrow \\
|\tilde{C}(EG)| \times_B I & \to & |\bar{W}G_{\Delta[1]}| \times \Delta[1] \to BG \\
\end{array}
\]
and hence the required fiberwise homotopy. We now give the proof of Proposition 29.

Proof of Proposition 29. Firstly, observe that there is a canonical simplicial map \( \text{Dec} \overline{WG} \to \check{C}(WG) \), where \( \check{C}(WG) \) denotes the Čech nerve of the map \( WG \to \overline{WG} \). It is a straightforward calculation to check that this map is an isomorphism.

Therefore, using the fact that the fiberwise geometric realization preserves finite limits, the geometric realization \( |d \text{Dec} \overline{WG}| \) is isomorphic to the realization of the simplicial object \( \check{C}(EG) \). Therefore, to prove the proposition it is enough to prove that there is a morphism of simplicial objects \( d \text{Dec}(\overline{WG}) \to WG \Delta[1] \) and that this morphism is compatible with projections to \( \overline{WG} \). It is an easy calculation to see that the object of \( n \)-simplices of \( d \text{Dec}(\overline{WG}) \) is given by

\[
(d \text{Dec}(\overline{WG}))[n] = M_{\Delta[2n+1]} \overline{WG} = (\overline{WG})_{2n+1}.
\]

We will prove that there is a simplicial map

\[
\Delta[n] \times \Delta[1] \to \Delta[2n + 1],
\]

natural in \( [n] \), such that the diagram

\[
\begin{array}{ccc}
\Delta[n] & \to & \Delta[2n+1] \\
\downarrow & & \downarrow \\
\Delta[n] \times \Delta[1] & \to & \Delta[2n+1] \\
\Delta[n] & \downarrow & \\
& & \\
\end{array}
\]

commutes, where the two maps \( \Delta[n] \to \Delta[2n + 1] \) are induced by \( \sigma([n], [-1]) \to \sigma([n], [n]) \) and \( \sigma([-1], [n]) \to \sigma([n], [n]) \) respectively (here \( [-1] \) denotes the empty set). Given the above identification of \( d \text{Dec} \overline{WG} \), this is enough to prove that there is a simplicial map

\[
d \text{Dec} \overline{WG} \to WG \Delta[1],
\]

which is compatible with the projections to \( \overline{WG} \). To prove the existence of the simplicial map above, note that in \( \text{Cat} \) there is a functor \( f: [n] \times [1] \to [2n + 1] \) defined by

\[
f(i, 0) = i, \quad f(i, 1) = n + i + 1.
\]

Note that this functor is natural in \( [n] \). Taking nerves then gives the required simplicial map. \( \square \)

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