Multiplicative Ramanujan coefficients of null-function

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Abstract. The null-function $0(a) \overset{\text{def}}{=} 0 \forall a \in \mathbb{N}$ has classical Ramanujan expansions: $0(a) = \sum_{q=1}^{\infty} (1/q)c_q(a)$ (here $c_q(a)$ is the Ramanujan sum), given by Ramanujan himself (in his 1918 history-making paper), and $0(a) = \sum_{q=1}^{\infty} (1/\varphi(q))c_q(a)$, given by Hardy few years later ($\varphi$ is Euler’s totient function). Both are pointwise converging in any $a \in \mathbb{N}$, but not absolutely convergent. A general $G : \mathbb{N} \to \mathbb{C}$ is called a Ramanujan coefficient, abbrev. R.c., iff (if and only if) $\sum_{q=1}^{\infty} G(q)c_q(a)$ converges in any $a \in \mathbb{N}$; also, for general $F : \mathbb{N} \to \mathbb{C}$ we call $< F >$, the set of its R.c.s, the Ramanujan cloud of $F$. Our Main Theorem in arxiv:1910.14640, for Ramanujan expansions and finite Euler products, applies to give a complete Classification for multiplicative Ramanujan coefficients of $0$. We find that Ramanujan’s coefficient of $0$, $G_{R}(q) := 1/q$, is a normal arithmetic function $G$, i.e., with $G(p) \neq 1$ on all primes $p$; while Hardy’s $G_{H}(q) := 1/\varphi(q)$ is a sporadic $G$, namely $G(p) = 1$ for a (necessarily finite) set of primes, but there’s no prime $p$ with $G(p^K) = 1$ on all integer powers $K \geq 0$ (Hardy’s has $G_{H}(p) = 1$ iff $p = 2$). The $G : \mathbb{N} \to \mathbb{C}$ such that there exists at least one prime $p$ with $G(p^K) = 1$, on all powers $K \geq 0$, are defined to be exotic. So, this definition completes the cases for Ramanujan coefficients of $0$. The exotic ones are a kind of new phenomenon in the “0–cloud” (i.e., $< 0 >$): exotic Ramanujan coefficients represent $0$ only with a convergence hypothesis. The not exotic, apart from the convergence of $\sum_{(q,a)=1} G(q)\mu(q)$ in any $a \in \mathbb{N}$ require, in addition, $\sum_{q=1}^{\infty} G(q)\mu(q) = 0$ for normal $G \in < 0 >$, while sporadic $G \in < 0 >$ need $\sum_{(q,P(G))=1} G(q)\mu(q) = 0$ (where $P(G)$ is the product of all primes $p$ making $G(p) = 1$). We give many examples of Ramanujan coefficients $G \in < 0 >$; we prove, in passing, that the only Ramanujan expansions of $0$ with absolute convergence are the exotic ones; actually, these $G \in < 0 >$ generalize to the “weakly exotic”, which are not necessarily multiplicative.

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1. Introduction

Following Ramanujan [R] we denote by $c_q(a)$ the sum of $q$-th roots of unity given by (henceforth $(h, q)$ is the gcd of $h, q$)

$$c_q(a) \overset{\text{def}}{=} \sum_{\substack{h=1 \atop (h, q)=1}}^{q-1} e^{2\pi i ah/q} = \sum_{\substack{h=1 \atop (h, q)=1}}^{q-1} \cos(2\pi ah/q)$$

and we call it a Ramanujan sum. Then a Ramanujan coefficient is any arithmetic function $G : \mathbb{N} \to \mathbb{C}$ such that

$$\sum_{q=1}^{\infty} G(q)c_q(a)$$
converges pointwise for all \( a \in \mathbb{N} \). If \( F(a) = \sum_{q=1}^{\infty} G(q)c_q(a) \) we also say that the series is a \textit{Ramanujan expansion} (or \textit{Fourier-Ramanujan expansion}) of the arithmetic function \( F : \mathbb{N} \to \mathbb{C} \), or that \( G \) is a Ramanujan coefficient of \( F \).

Ramanujan sums are ubiquitous expressions in number theory with remarkable properties. For example, it was first observed by Hardy that the function \( c_q(a) \) is multiplicative in the variable \( q \), so it is determined by its values at powers of primes \( c_{p^k}(a) \). Other formulae to compute the Ramanujan sums in terms of the Möbius function \( \mu \) and Euler’s totient function \( \varphi \) are

\[
c_q(a) = \sum_{d|(q,a)} \mu \left( \frac{q}{d} \right) d = \mu \left( \frac{q}{(q,a)} \right) \frac{\varphi(q)}{\varphi(q/(q,a))}.
\]

The first expression was found by Kluyver [K] while the second is due to von Sterneck and Hölder [Hö].

For a fixed \( F : \mathbb{N} \to \mathbb{C} \) we denote the set of its Ramanujan coefficients by

\[
< F > \overset{\text{def}}{=} \left\{ G : \mathbb{N} \to \mathbb{C} \mid \forall a \in \mathbb{N}, \sum_{q=1}^{\infty} G(q)c_q(a) = F(a) \right\}.
\]

This set is called the \textit{Ramanujan cloud} of \( F \) or, for brevity, the \( F \)-cloud. The problem of establishing the non-emptiness of \( F \)-clouds, (i.e. proving that an arithmetic function \( F \) can be written as a Ramanujan expansion) has been considered by many authors. One general method to achieve this goal is through mean-value theorems and harmonic analysis. In this sense a fundamental tool which is often exploited is an orthogonality relation for Ramanujan sums discovered by Carmichael [Ca]. For more on the classical and modern theory of Ramanujan sum and Ramanujan expansions, we refer the reader to the surveys [C1], [L1], [L2], [M] and to the monograph [ScSp]. We also mention that there is a theory of Ramanujan expansion for modern theory of Ramanujan sum and Ramanujan expansions, we refer the reader to the surveys [C1], [L1], [L2], [M] and to the monograph [ScSp]. We also mention that there is a theory of Ramanujan expansion for functions of several variables and of “dual Ramanujan expansions” of the form \( \sum_{a=1}^{\infty} F(a)c_q(a) \) [M].

Another interesting problem, which is more related to the results of the present article, it that of the “cardinality” of \( < F > \). It was already known to Ramanujan [R] and Hardy [H] that an arithmetic function may be representable as a Ramanujan expansion in more than one way. Respectively, they found the following two nontrivial expansions of the constant zero function \( 0(a) \overset{\text{def}}{=} 0 \), \( \forall a \in \mathbb{N} \):

\[
\sum_{q=1}^{\infty} \frac{1}{q} c_q(a) = 0, \quad \sum_{q=1}^{\infty} \frac{1}{\varphi(q)} c_q(a) = 0.
\]

An intriguing fact is that the convergence of these series (to zero, for every \( a \in \mathbb{N} \)) turns out to be logically equivalent, in a precise sense, to the Prime Number Theorem. Until recently, the Ramanujan coefficients \( G_R(q) := 1/q \) and \( G_H(q) := 1/\varphi(q) \) corresponding to the two examples above of Hardy and Ramanujan were essentially the only elements of the 0-cloud (together of course with their linear combinations) to be found in the literature. Using the theory of Euler products of Ramanujan expansions the first author was able to produce several new examples of \( G \in < 0 > \) and discovered in particular a whole new class of Ramanujan expansions of 0, which often have a much better convergence rate to zero than the examples of Hardy and Ramanujan given above. These Ramanujan expansions correspond to the class of so-called \textit{exotic} Ramanujan coefficients, described later in this article. Also, since to any Ramanujan coefficient we can add an arbitrary \( G \in < 0 > \) without changing the value of the corresponding Ramanujan expansion, it follows that every non-empty \( F \)-cloud has infinite cardinality, and in fact it has a natural structure of an infinite-dimensional complex affine space [C4].

In this paper we shall provide a classification of the multiplicative Ramanujan coefficients \( G \in < 0 > \) under some mild convergence hypotheses which are similar to those found in work of Lucht [L2, Thm 3.1] and which come out naturally from the theory of finite Euler products of [C3]. The precise statements are given in the next section. With this classification at hand, in sections 5 and 6 of this article we provide many new examples of Ramanujan of the constant function 0. Since the Ramanujan sums \( c_q(a) \) are multiplicative themselves, the decision to classify the multiplicative portion of the 0-cloud is rather natural. At any rate, our results generalise easily to provide also some new examples of non-multiplicative Ramanujan coefficients
Depending on which of these spectra is empty or not: (recall $G \in <0>$, such as the class of weakly exotic Ramanujan coefficients described in section 6.2. By the way, only with the exception of this section and of Lemma 3, our $G : \mathbb{N} \to \mathbb{C}$ will always be multiplicative. Finally, in section 7, we shall discuss criteria of absolute convergence for Ramanujan expansions.

The authors plan to give a more complete account of the non-multiplicative part of the $0$-cloud, in a future publication.

2. Statement of the classification

In order to state and prove our results it is better to introduce some notation. We denote by $P$ the set of prime numbers and by $\mathbb{N}_0 \defeq \mathbb{N} \cup \{0\}$ the set of non-negative integers. For every $a \in \mathbb{N}$ we let $P(a)$ be the set of prime divisors of $a$ (recall $P(1) \defeq \emptyset$) and we let

$$P(a) \defeq \prod_{p \in P(a)} p$$

be its squarefree kernel (also known as the radical of $a$). By extension, for a given $N \subset \mathbb{N}$, we set $P(N) \defeq \{p \in P : p|n, \text{for some } n \in N\}$ and then, whenever this set of primes is finite, we denote by

$$P(N) \defeq \prod_{p \in P(N)} p$$

their product. We also recall that $v_p(a) \defeq \max\{K \in \mathbb{N}_0 : p^K | a\}$ denotes the $p$-adic valuation of a number $a \in \mathbb{N}$ with respect to $p \in \mathbb{P}$. Henceforth, we abbreviate with iff the expression “if and only if”.

Now fix a multiplicative function $G : \mathbb{N} \to \mathbb{C}$. We say that $G$ is multiplicatively trivial (resp. completely multiplicatively trivial) at a prime $p \in \mathbb{P}$ iff $G(p) = 1$ (resp. $G(p^K) = 1$ for all $K \geq 0$). We can then define two kinds of “multiplicative spectra” of $G$ which are the key to our classification:

$$\mathcal{F}(G) \defeq \{p \in \mathbb{P} : G(p) = 1\}, \quad \mathcal{F}_0(G) \defeq \{p \in \mathbb{P} : G(p^K) = 1, \forall K \in \mathbb{N}_0\}.$$  

By multiplicativity, the function $G$ is multiplicatively (resp. completely multiplicatively) trivial at $p$ iff the equality $G(pr) = G(r)$ holds for every $(r,p) = 1$ (resp. for every $r \in \mathbb{N}$). We then may also adopt the following evocative terminology: a prime $p$ is transparent to $G$ iff $p \in \mathcal{F}(G)$, and is invisible to $G$ iff $p \in \mathcal{F}_0(G)$.

We can measure the “transparency” of a prime $p$ by a kind of $p$-adic valuation defined as follows:

$$v_{p,G} \defeq \min\{K \in \mathbb{N}_0 : G(p^{K+1}) \neq 1\},$$

with the convention that $v_{p,G} = \infty$ in case the set on the right-hand side is empty. Thus a prime $p$ is transparent iff $v_{p,G} \geq 1$ and is invisible iff $v_{p,G} = \infty$. Whenever $\mathcal{F}(G)$ is finite, we also define:

$$P(G) \defeq \prod_{p \in \mathcal{F}(G)} p.$$ 

We note (compare Remark 1 in [C3], second version) that

$G : \mathbb{N} \to \mathbb{C}$ is a Ramanujan coefficient $\implies \mathcal{F}(G)$ and $\mathcal{F}_0(G)$ are finite sets.

Since $\mathcal{F}_0(G) \subseteq \mathcal{F}(G)$, we distinguish the following three cases, already exposed in [C3] (second version), depending on which of these spectra is empty or not: (recall $G : \mathbb{N} \to \mathbb{C}$ is multiplicative)

- $G$ is normal $\iff \mathcal{F}(G) = \emptyset$,
- $G$ is sporadic $\iff \mathcal{F}(G) \neq \emptyset$ and $\mathcal{F}_0(G) = \emptyset$,
- $G$ is exotic $\iff \mathcal{F}_0(G) \neq \emptyset$. 


We can now give our Classification theorem of the multiplicative part of $< 0$ > under some technical hypotheses of convergence. Some convergence hypotheses of this sort cannot be removed completely, as we’ll argue in future papers. Recall $\mu$ is the Möbius function [T].

**Theorem 1.** Classification of multiplicative Ramanujan coefficients of $0$.

Let $G : \mathbb{N} \to \mathbb{C}$ be multiplicative. Then one and only one of the following cases happens:

A) $G$ is normal. Then, assuming $\sum_{(r,a) = 1} G(r)\mu(r)$ converges pointwise $\forall a \in \mathbb{N}$, $G$ is a Ramanujan coefficient and

$$G \in < 0 \iff \sum_{q = 1}^{\infty} G(q)\mu(q) = 0;$$

B) $G$ is sporadic. Then, assuming $\sum_{(r,a) = 1} G(r)\mu(r)$ converges pointwise $\forall a \in \mathbb{N}$, $G$ is a Ramanujan coefficient and

$$G \in < 0 \iff \sum_{(q,P(G)) = 1} G(q)\mu(q) = 0;$$

C) $G$ is exotic. Then, in the hypothesis $\exists p_0 \in \mathcal{F}_0(G)$: $\sum_{(q,p_0) = 1} G(q)c_q(a)$ converges pointwise $\forall a \in \mathbb{N}$, $G$ is a Ramanujan coefficient and

$$G \in < 0 .$$

The multiplicative Ramanujan coefficient $G_R(q) = 1/q$ considered in the Introduction is an example of a normal multiplicative function, while the function $G_H(q) = 1/\varphi(q)$ of Hardy is sporadic, as $\mathcal{F}(G_H) = \{2\}$ and $\mathcal{F}_0(G_H) = \emptyset$. We will give many more examples of normal and sporadic $G \in < 0 \gg$ using analytic methods in section 5. We shall demonstrate that the examples of Ramanujan and Hardy are not isolated; in a suitable sense, the main feature that makes them belong to the $0$-cloud is their asymptotic rate of decay to zero along the primes: $G_R(p), G_H(p) \sim 1/p$. We refer to Proposition 1 in section 5 for a precise statement.

An example of exotic $G \in < 0 >$ is given in [C3], and in section 6 we will provide several more examples. The most elementary one is the indicator function of the powers of 2, i.e. the multiplicative function $G_2(q)$ which is equal to 1 if $q = 2^k$ for some $k \in \mathbb{N}$, and is zero otherwise. With a generalization of the concept of exotic multiplicative functions, we are also able in section 6.2 to produce for the first time some examples (in fact, a very large class) of Ramanujan expansions of $0$ whose Ramanujan coefficients are not multiplicative.

Many of the Ramanujan coefficients in the exotic class, including the example $G_2$, give rise to absolutely convergent expansions of $0$ (sometimes even Ramanujan series with finitely many nonzero terms). However we argue in section 6.1 that there exist exotic Ramanujan coefficients whose associated Ramanujan expansion does not converge absolutely to zero. We investigate in more detail the issue of absolute convergence in §7.

The proof of the theorem above will be given in section 4, after having established some elementary technical lemmas in section 3.

In a forthcoming paper we’ll discuss the hypotheses of convergence in our classification theorem, in relation with a general problem of summability of multiplicative functions supported over the squarefree numbers, when coprimality conditions are imposed.

3. Lemmata

We start, at once, with the Lemmas that we’ll use in the Proof of our Classification Theorem.

For multiplicative arithmetic functions $G$ we have following Lemma 1, whose proof is immediate.

**Lemma 1.** Let $G : \mathbb{N} \to \mathbb{C}$ be multiplicative, let $\mathcal{F}$ be any finite, non-empty subset of primes and let $p_1$ be any prime with $p_1 \notin \mathcal{F}$. Then

$$G(p_1) \neq 1, \sum_{(r,\{p_1\} \cup \mathcal{F}) = 1} G(r)\mu(r) \text{ converges and} \sum_{(r,\mathcal{F}) = 1} G(r)\mu(r) = 0 \implies \sum_{(r,\{p_1\} \cup \mathcal{F}) = 1} G(r)\mu(r) = 0.$$
Proof. Simply, for \(x \in \mathbb{N}\) large enough in terms of \(p_1\) (say, \(x > p_1\)), abbreviating \(\mathcal{F}_1 \defeq \{p_1\} \cup \mathcal{F}\),

\[
\sum_{r \leq x \atop (r, \mathcal{F}_1) = 1} G(r) \mu(r) = \sum_{r \leq x \atop (r, \mathcal{F}) = 1} G(r) \mu(r) - \sum_{r \leq x \atop (r, p_1) = 1} G(r) \mu(r) = \sum_{r \leq x \atop (r, \mathcal{F}) = 1} G(r) \mu(r) + G(p_1) \sum_{r \leq x \atop (r, \mathcal{F}_1) = 1} G(r) \mu(r),
\]

whence

\[
\sum_{r \leq x \atop (r, \mathcal{F}) = 1} G(r) \mu(r) = \sum_{r \leq x \atop (r, \mathcal{F}_1) = 1} G(r) \mu(r) - G(p_1) \sum_{r \leq x \atop (r, \mathcal{F}_1) = 1} G(r) \mu(r).
\]

The limit as \(x \to \infty\) vanishes for LHS (Left Hand Side), while for RHS (Right Hand Side) it is \(1 - G(p_1) \neq 0\) times the series over \((r, \mathcal{F}_1) = 1\).

Our second Lemma will be applied for the sporadic arithmetic functions \(G\). Proof comes quickly.

**Lemma 2.** Let \(G : \mathbb{N} \to \mathbb{C}\) be multiplicative. Assume there’s a prime \(p_0\), with \(G(p_0) = 1\). Then

\[
p_0 \not| a \text{ and } \sum_{(r, ap_0) = 1} G(r) \mu(r) \text{ converges } \implies \sum_{(r, a) = 1} G(r) \mu(r) = 0 \implies \sum_{q = 1}^{\infty} G(q) c_q(a) = 0.
\]

**Proof.** The first implication uses \(G(p_0) = 1\) (once fixed a large enough \(x \in \mathbb{N}\), say \(x > p_0\)):

\[
\sum_{r \leq x \atop (r, a) = 1} G(r) \mu(r) = \sum_{r \leq x \atop (r, a) = 1} G(r) \mu(r) + \sum_{r \leq x \atop (r, ap_0) = 1} G(r) \mu(r) = - \sum_{r \leq x / p_0 \atop (r, ap_0) = 1} G(r) \mu(r) + \sum_{r \leq x \atop (r, ap_0) = 1} G(r) \mu(r),
\]

whence passing to the limit as \(x \to \infty\),

\[
\sum_{(r, a) = 1} G(r) \mu(r) = 0,
\]

while the second implication comes from the Proposition in [C3] (second version).

Next Lemma generalizes the exotic case, giving to C) a quicker proof (w.r.t. [C3], second version).

**Lemma 3.** Let \(G : \mathbb{N} \to \mathbb{C}\) be weakly-exotic, i.e., by definition, satisfying

\((*)_{weak}\) \quad \exists p_0 \in \mathbb{P} : G(p_0^K r) = G(r), \quad \forall K \in \mathbb{N}_0, \quad \forall r \in \mathbb{N}, (r, p_0) = 1

and, for at least one “invisible” prime \(p_0\), i.e. satisfying \((*)_{weak}\) above, assume that

\[
\sum_{(r, p_0) = 1} G(r) c_r(a) \text{ converges pointwise } \forall a \in \mathbb{N}.
\]

Then \(G\) is a Ramanujan coefficient and

\[
\sum_{q = 1}^{\infty} G(q) c_q(a) = O(a).
\]

**Proof.** Let’s fix \(a \in \mathbb{N}\) and choose \(Q \in \mathbb{N}\), large enough in terms of \(a\) and \(p_0\) above (say, \(Q > p_0^{v_{p_0}(a)+1}\)):

\[
\sum_{q \leq Q} G(q) c_q(a) = \sum_{K = 0}^{\infty} \sum_{r \leq Q / p_0^K} G(p_0^K r) c_{p_0^K}(a) c_r(a) = \sum_{K = 0}^{v_{p_0}(a)+1} c_{p_0^K}(a) \sum_{r \leq Q / p_0^K} G(r) c_r(a),
\]
because \( c_{p^K}(a) = 0, \forall K > v_{p^0}(a) + 1 \), see Fact 1 before Main Lemma in second version of \([C3]\); whence, passing to the limit over \( Q \to \infty \), we get \( G \) is a Ramanujan coefficient and, since:

\[
\sum_{K=0}^{v_{p^0}(a)+1} c_{p^K}(a) = \sum_{K=0}^{v_{p^0}(a)} \varphi(p^K) - p^{v_{p^0}(a)} = 0,
\]

compare quoted Lemma, we also get \( G \in <0> \).

\[\Box\]

4. Proof of the Classification Theorem

We start with

**Proof of case A).** Follow the proof of Corollary 1 in \([C3]\), second version.

 Thanks to Lemma 3, we prove case C) immediately.

**Proof of case C).** Apply Lemma 3, since \( G \text{ EXOTIC} \Rightarrow G \text{ WEAKLY-EXOTIC} \).

Our last, and longest, proof is when \( G \) is sporadic.

**Proof of case B).**

We quote and prove Corollary 3 of \([C3]\) (second version). For sporadic \( G \), notice:

\[
(r, P(G)) = (r, F(G))
\]

The QED (Quod Erat Demonstrandum = what was to be shown) will indicate the end of a part of the Proof.

**Corollary 3.** Let \( G : \mathbb{N} \to \mathbb{C} \) be sporadic and assume that \( \sum_{(r,a)=1} G(r) \mu(r) \) converges pointwise \( \forall a \in \mathbb{N} \).

Then \( G \) is a Ramanujan coefficient and

\[
\sum_{q=1}^{\infty} G(q)c_q(a) = 0(a) \iff \sum_{(r,F(G))=1} G(r) \mu(r) = 0.
\]

**Remark 1.** The condition of convergence on the series with \( \mu \) immediately implies, from Proposition of \([C3]\) (second version), that \( G \) is a Ramanujan coefficient. \( \Box \)

**Proof.** We start with "\( \Rightarrow \)" , the easiest. Since \( G \) is a non-exotic Ramanujan coefficient, the \( p \)-adic valuation \( v_{p,G} \) of \( G \) is strictly positive for only finitely many primes, and is never equal to \( \infty \). Then the natural number

\[ a_G \overset{def}{=} \prod_{p \in F(G)} p^{v_{p,G}}, \]

is well-defined, and \( v_p(a_G) = v_{p,G} \) for every \( p \in \mathbb{P} \). If we choose \( F = F(G) \) and \( a = a_G \in \mathbb{N} \), in the Main Theorem of \([C3]\) (second version) we get the following formula:

\[ (*)_G \quad 0 = \sum_{q=1}^{\infty} G(q)c_q(a_G) = \prod_{p \in F(G)} \sum_{K=0}^{v_{p,G}} p^K (G(p^K) - G(p^{K+1})) \cdot \sum_{(r,F(G))=1} G(r) \mu(r). \]

By the definition of \( v_{p,G} \) we have \( G(p^K) = 1 \) for every \( K \leq v_{p,G} \) and \( G(p^{v_{p,G}+1}) \neq 1 \). Then, the **finite factor** (i.e., the finite Euler product) in \((*)_G\) becomes

\[
\prod_{p \in F(G)} p^{v_{p,G}} (1 - G(p^{v_{p,G}+1})) = a_G \prod_{p \in F(G)} (1 - G(p^{v_{p,G}+1})) \neq 0,
\]

forcing the **co-finite factor** (the infinite series complementary to the finite factor) in \((*)_G\) to vanish:

\[
\sum_{(r,F(G))=1} G(r) \mu(r) = 0.
\]

QED(\( \Rightarrow \))
The other implication, i.e., "←\n\n\sum_{(r,\mathcal{F}(G))=1} G(r)\mu(r) = 0.

We’ll get the vanishing of Ramanujan expansion (LHS in Corollary) little by little, on all \(a \in \mathbb{N}\).

We do it in three steps.

STEP 1.
From the Proposition in [C3] (second version) we get, choosing \(a \in \mathbb{N}\) with \(P(a) = \mathcal{F}(G)\),
\[
\sum_{q=1}^{\infty} G(q)c_q(a) = \left(\prod_{p | a} \sum_{K=0}^{\infty} G(p^K)c_p^K(a)\right) \cdot \sum_{(r,\mathcal{F}(G))=1} G(r)\mu(r) = 0, \forall a \in \mathbb{N} : \mathbb{P}(a) = \mathcal{F}(G),
\]
whence we get, from both (*) and Lemma 1,
\[
\sum_{q=1}^{\infty} G(q)c_q(a) = 0, \forall a \in \mathbb{N} : \mathbb{P}(a) = \mathcal{F}(G) \cup \{p_1\}.
\]

Iterating this procedure and joining more different primes, applying inductively Lemma 1, we get
\[
\sum_{(r,a)=1} G(r)\mu(r) = 0, \forall a \in \mathbb{N} : a \equiv 0 \pmod{P(G)},
\]
because this LHS depends only on the squarefree kernel \(P(a)\) of \(a\) (and not on \(a\), actually). This last vanishing can be used again in Proposition of [C3] (second version), so WE PROVED:
\[
\sum_{q=1}^{\infty} G(q)c_q(a) = 0, \forall a \in \mathbb{N} : a \equiv 0 \pmod{P(G)}.
\]

STEP 3.
We are left with the task to PROVE:
\[
\sum_{q=1}^{\infty} G(q)c_q(a) = 0, \forall a \in \mathbb{N} : a \not\equiv 0 \pmod{P(G)}.
\]

This is equivalent to proving :
\[
\exists p_0 \in \mathcal{F}(G), p_0 \nmid a \implies \sum_{q=1}^{\infty} G(q)c_q(a) = 0.
\]

Lemma 2 is just what we need to conclude.
5. Examples of Ramanujan expansions of 0 via analytic methods

As we remarked in the introduction, the Ramanujan expansions of 0 given by Ramanujan and Hardy, corresponding respectively to the multiplicative functions \( G_R(q) = 1/q \) and \( G_H(q) = 1/\varphi(q) \), are related in an essential way to the Prime Number Theorem. A close inspection of Hardy’s proof \[H, \text{section 8}\] reveals that this fact does not depend much on the special choice of the multiplicative function \( G = G_R \) or \( G = G_H \), but only on the asymptotic behaviour of \( G \). With this intuition, we are able to exhibit many more examples of Ramanujan expansions of 0.

**Proposition 1.** Let \( G \) be multiplicative, not exotic, such that \( G(q) = O(1/q) \) for \( q \) squarefree and

\[
G(p) = \frac{1}{p} + O(p^{-1-\alpha})
\]

for some \( \alpha > 0 \) as \( p \to \infty \) along the primes. Then \( G \) is a Ramanujan coefficient and \( G \in <0> \).

The proof of this Proposition relies on the following Lemma, which we prove by analytic means.

**Lemma 4.** Let \( \alpha \) and \( G \) be as in the Proposition above. Then

\[
\sum_{q=1}^{\infty} G(q)\mu(q) = 0.
\]

**Proof of Lemma 4.** First we define the Dirichlet series

\[
f(s) := \sum_{q=1}^{\infty} \frac{G(q)\mu(q)}{q^{s-1}},
\]

which is absolutely convergent for \( \text{Re}(s) > 1 \) and is equal to the Euler product

\[
f(s) = \prod_{p \in \mathbb{P}} (1 - G(p)p^{-s+1})
\]
on the same half plane. We recall that the Riemann zeta function is given by the Euler product

\[
\zeta(s) = \prod_{p \in \mathbb{P}} (1 - p^{-s})^{-1}
\]
for \( \text{Re}(s) > 1 \). Then, if we write \( G(p) = (1 + \epsilon(p))p^{-1} \) for some \( \epsilon(p) = O(p^{-\alpha}) \) we get that

\[
f(s)\zeta(s) = \prod_{p \in \mathbb{P}} \frac{1 - (1 - \epsilon(p))p^{-s}}{1 - p^{-s}}.
\]

Since every factor in this product is of the form

\[
1 + \epsilon(p)p^{-s} + O(p^{-2s}) = 1 + O(p^{-\min(\alpha,1/2)})
\]
we have on the half plane \( \text{Re}(s) > 1 \) that

\[
f(s) = h(s) \cdot \frac{1}{\zeta(s)}
\]
for some Dirichlet series \( h(s) \) that extends analytically to the half plane \( \text{Re}(s) > 1 - \min(\alpha, 1/2) \). Moreover by the analytic version of the Prime Number Theorem we know that the inverse of the zeta function extends by continuity to the closed half plane \( \text{Re}(s) \geq 1 \), and so does \( f(s) \). Then, by the Theorem of [N], we get that \( \sum_{q=1}^{\infty} G(q)\mu(q) \) converges and is equal to the value of \( \lim_{s \to 1} f(s) \). Since the function \( h(s) \) is regular at \( s = 1 \) and the zeta function \( \zeta(s) \) has a pole there, we get that this limit is equal to zero.

Proof of Proposition 1. Let \( a \in \mathbb{N} \). We may apply the previous Lemma to the multiplicative function \( G_a = G1_{(q,a)=1} \), which is equal to \( G \) on the numbers coprime to \( a \) and is zero elsewhere. Then we get

\[
\sum_{(q,a)=1} G(q)\mu(q) = 0.
\]

In particular, this series converges for every \( a \). If \( G \) is normal or sporadic, the Classification Theorem applies and we get \( G \in <0> \). □

6. Examples of exotic and weakly-exotic Ramanujan coefficients

It was already observed by the first author in [C3] that there exist exotic Ramanujan coefficients \( G \in <0> \) for which the corresponding Ramanujan series \( \sum_{q=1}^{\infty} G(q)c_q(a) \) converges absolutely to zero for every \( a \in \mathbb{N} \). In fact there are lots of absolutely convergent Ramanujan expansions of this kind, as the following construction shows.

Proposition 2. Let \( p_0 \) be a prime number and let \( G \) be a multiplicative function such that \( G(p_0^k) = 1 \) for all \( k \in \mathbb{N} \) and such that \( \sum_{p \in \mathbb{P}} |G(p)| < \infty \). Then \( G \) is an exotic Ramanujan coefficient and

\[
\sum_{q=1}^{\infty} G(q)c_q(a)
\]

converges absolutely to 0 for all \( a \in \mathbb{N} \).

The proof of this result is postponed to the next section, where we will discuss in more detail the absolutely convergent Ramanujan expansions with multiplicative coefficients. In fact, we will also be able to show that all multiplicative coefficients \( G \in <0> \) with absolutely convergent Ramanujan expansions come from this Proposition.
6.1 A non-absolutely convergent Ramanujan expansion with exotic coefficients

Despite this large class of examples, we would like to remark that not all exotic Ramanujan coefficients give rise to absolutely convergent Ramanujan series. Here is an example. (It’s a small Lemma: we call it “Fact”)

**Fact (Example).** Let \( p_0 \) be a prime number and let \( G_0 \) be a multiplicative function such that \( G_0(p_k^i) = 1 \) for all \( k \in \mathbb{N} \) and \( G_0(p) = p^{-1} \) for all primes \( p \neq p_0 \). Then for all \( a \in \mathbb{N} \) the Ramanujan expansion

\[
R_0(a) := \sum_{q=1}^{\infty} G_0(q)c_q(a)
\]

converges pointwise to zero, but it does not converge absolutely.

*Proof.* The gist of the argument is that for every \( b \in \mathbb{N} \) the following series converges to 0

\[
(PNT)_b \sum_{(q,b)=1} \frac{1}{q} \mu(q) = 0.
\]

This is a consequence of the prime number theorem (PNT) in arithmetic progressions. The rate of convergence to zero of series of this type was examined for instance by Ramaré [Ra]. Taking \( G = G_0, a \not\equiv 0(\text{mod } p_0) \) and \( \sum_{(r,a p_0)=1} G_0(r)\mu(r) = 0 \) from \( (PNT)_{a p_0} \), Lemma 2 gives \( R_0(a) = 0, \forall a \equiv 0(\text{mod } p_0) \); while \( \forall a \equiv 0(\text{mod } p_0) \), from Proposition [C3], second version, \( (PNT)_a \Rightarrow R_0(a) = 0 \), too. The fact that the convergence is not absolute follows from the divergence of \( \sum_{p \in \mathbb{P}} |G_0(p)| \) and by the criterion of absolute convergence in section 7.1. \( \square \)

6.2 Another big class of Ramanujan coefficients in \(< 0>\)

We recall the new definition, of weakly exotic \( G : \mathbb{N} \rightarrow \mathbb{C} \) given in Lemma 3 means simply that \( G \) is satisfying that condition (*) we in Lemma 3: \( G \) exotic and multiplicative \( \Rightarrow \) \( G \) weakly exotic (of course!). We already used this trivial implication in the proof of case C) in the Classification Theorem Proof, see §4.

A highly non-trivial elementary result for the cloud of \( 0 \) is Lemma 3 itself!

Thus the hypothesis “\( G \) multiplicative”, actually, is not necessary for the weakly exotic \( G \).

The Class of weakly exotic \( G : \mathbb{N} \rightarrow \mathbb{C} \) seems to be the widest we know, at the moment, in the cloud of \( 0 \).

7. Absolutely convergent Ramanujan expansions

Since lot of research on Ramanujan expansions has been carried out in the hypothesis of absolute convergence (see for instance the theorems of Wintner [W] and Delange [De] or the discussions in the surveys [C1] [L2]), it is natural to study those Ramanujan expansions of \( 0 \) which are absolutely convergent. The classical examples of Hardy and Ramanujan are known to converge pointwise but not absolutely, while Proposition 2 in the previous section shows that there are many exotic coefficients which imply absolute convergence. In this section we examine this topic in detail, thus completing a discussion in [C3]. We show that all Ramanujan expansions of \( 0 \) with normal and sporadic multiplicative Ramanujan coefficients are necessarily not absolutely convergent. We also show that all absolutely convergent Ramanujan expansions with exotic coefficients are constructed as in Proposition 2 (section 6).
7.1 A criterion for absolute convergence

The problem of absolute convergence for Ramanujan expansions with multiplicative coefficients is easily solved using the theory of finite-cofinite Euler product decomposition developed in [C3]. Indeed, we have the following simple criterion.

**Lemma 5.** Let $G$ be a multiplicative function. Then the following are equivalent:

i) $\sum_{q=1}^{\infty} G(q)c_q(a)$ is absolutely convergent for every $a \in \mathbb{N}$;

ii) $\sum_{q=1}^{\infty} G(q)c_q(a)$ is absolutely convergent for some $a \in \mathbb{N}$;

iii) $\sum_{q=1}^{\infty} G(q)\mu(q)$ is absolutely convergent;

iv) $\sum_{p \in \mathbb{P}} |G(p)| < \infty$.

**Proof.** The implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) are obvious; using $|c_q(a)| \geq \mu^2(q)$, $\forall a, q \in \mathbb{N}$, we get (ii) $\Rightarrow$ (iii). So, now let us assume (iv). The assertion (iii) is equivalent to the summability of $|G|$ over the squarefree numbers. Hence, (iii) follows from (iv), using the absolute summability over the prime numbers thanks to the multiplicativity of $G$ and the inequality $1 + |x| \leq \exp(|x|)$:

$$\sum_{q \text{ squarefree}} |G(q)| = \prod_{p \in \mathbb{P}} (1 + |G(p)|) \leq \exp \left( \sum_{p \in \mathbb{P}} |G(p)| \right) < \infty.$$ 

Now, let us prove (i) and (ii) assuming (iii). For each $a \in \mathbb{N}$ we recall that $c_q(a) = \mu(q)$ if $(q, a) = 1$ and $c_q(a) = 0$ whenever $v_p(q) > v_p(a) + 1$ for some prime number $p$. Together with the multiplicativity of $G$ and of the Ramanujan coefficients, we get the following formula, which is analogous to the finite-cofinite Euler product formula in the Main Lemma of [C3]:

$$\sum_{q=1}^{\infty} |G(q)c_q(a)| = \sum_{d|aP(a)} |G(d)c_d(a)| \sum_{(r,a)=1} |G(r)\mu(r)|$$

where we recall that $P(a)$ is the squarefree kernel of $a$, so that $aP(a) = \prod_{p|a} p^{v_p(a)+1}$. Since the first factor is a finite sum and the second factor is a sum over squarefree numbers, we deduce that the Ramanujan series $\sum_{q=1}^{\infty} G(q)c_q(a)$ is absolutely summable. Since $a \in \mathbb{N}$ is arbitrary, the result follows. \(\square\)

A direct consequence of this criterion of absolute convergence is that the Proposition 2 (section 6) is true and, moreover, it captures the structure of all absolutely convergent Ramanujan expansions with exotic coefficients.

7.2 Only pointwise convergence for normal and sporadic coefficients

We now examine the problem of absolute convergence of non-exotic Ramanujan coefficients in the cloud of $0$. We shall prove that absolute convergence is never achieved in this case:

**Proposition 3.** Let $G \in (-0)$ be a normal or sporadic multiplicative Ramanujan coefficient. Then, for each $a \in \mathbb{N}$, the Ramanujan series

$$\sum_{q=1}^{\infty} G(q)c_q(a),$$

does not converge absolutely (but it converges pointwise to 0).

**Proof.** Let $G \in (-0)$ be a normal or sporadic multiplicative Ramanujan coefficient such that the series

$$\sum_{q=1}^{\infty} G(q)c_q(a)$$

is not absolutely convergent, then there exists a prime number $p$ such that $|G(p)| > 1$. By the multiplicativity of $G$, we have $|G(n)| > 1$ for all $n$ divisible by $p$. But then, for each $a \in \mathbb{N}$, we have $|G(a)c_a(a)| > 1$, and so $\sum_{q=1}^{\infty} |G(q)c_q(a)|$ is not absolutely convergent.

Therefore, the Proposition 3 is proved. \(\square\)
converges absolutely for some \( a \in \mathbb{N} \). We are going to derive a contradiction from this assumption. By the criterion of absolute convergence in the previous section, we have that
\[
\sum_{r \text{ squarefree}} |G(r)| < \infty \quad \text{and} \quad \sum_{p \in \mathcal{P}} |G(p)| < \infty.
\]
We deduce for each \( b \in \mathbb{N} \) that the series
\[
\sum_{(r,b)=1} G(r) \mu(r)
\]
converges and (since \( G \) is multiplicative) is equal to the (also convergent) Euler product
\[
\prod_{\substack{p \text{ prime} \\ (p,b)=1}} (1 - G(p)).
\]
Since \( G \in \langle 0 \rangle \), parts A) and B) of the main Theorem imply that
\[
\sum_{(r,P(G))=1} G(r) \mu(r) = 0,
\]
where \( P(G) = 1 \) in case \( G \) is normal. However we also have (compare Property 2 \([C3]\), second version)
\[
\prod_{\substack{p \text{ prime} \\ (p,P(G))=1}} (1 - G(p)) \neq 0,
\]
because \( \sum_{p \in \mathcal{P}} |G(p)| < \infty \) and because \( G(p) \neq 1 \) for all primes appearing in this Euler product, by definition of \( P(G) \). This is the required contradiction.

We summarize our findings as follows.

**Corollary 4.** Let \( G \) be a multiplicative Ramanujan coefficient in the \( 0 \)-cloud. We have that \( \sum_{q=1}^{\infty} G(q)c_q(a) \) converges absolutely for some (hence all) \( a \in \mathbb{N} \) if, and only if:
\[
\sum_{p \in \mathcal{P}} |G(p)| < \infty \quad \text{AND} \quad \mathcal{F}_0(G) \neq \emptyset.
\]
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