Relativistic Causality and Conservation of Energy in Classical Electromagnetic Theory

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(Dated:)

The change of the electromagnetic field in a particular place due to the event of a change in the motion of a charged particle can occur only after the light signal from the event can reach this place. Naive calculations of the electromagnetic energy and the work performed by the electromagnetic fields might lead to paradoxes of apparent non-conservation of energy. A few paradoxes of this type for a simple motion of two charges are presented and resolved in a quantitative way providing deeper insight into various relativistic effects in the classical electromagnetic theory.

I. INTRODUCTION

Starting from Einstein’s work on special relativity, it became clear that classical electromagnetic theory is consistent with relativity, and no true paradoxes can be found. However, several apparent paradoxes have been extensively discussed and these discussions enriched our understanding of the electromagnetic theory. Some of these controversial topics are: hidden momentum, Feynman’s disk, Trouton-Noble Experiment, and the 4/3 factor for the self energy of an electron. Here we present another situation which, analyzed in a naive way, leads to paradoxical conclusions. The paradoxes are relevant to recent discussions of covariance in the electromagnetic theory.

Let us start with the paradox which is simplest to present: its resolution will be shown at the end of the paper.

Paradox I

There are two particles of charge $q$ initially separated by distance $l$. We consider two ways to bring the particles, initially and finally at rest, to a shorter distance $l - x$, see Fig. 1.

(i) We move one particle the distance $x$ toward the other particle. The work required for this is:

$$W^i = U_{NEW} - U_{OLD} = \frac{q^2}{l-x} - \frac{q^2}{l}. \quad (1)$$

(ii) We move both particles toward each other for the distance $x/2$. We do it simultaneously and fast enough such that the motion of each particle ends before the signal about this motion can reach the location of the other particle. In this case, the work should be the sum of the amounts of work performed by external forces exerted on the two particles calculated as if the other particle has not moved:

$$W^{ii} = W_1 + W_2 = 2 \left( \frac{q^2}{l-x} - \frac{q^2}{l} \right). \quad (2)$$

![Fig. 1. Space-time diagram of the motion in the two processes: (i) one particle moves, (ii) two particles move.](image-url)
After the procedure is ended, we obtain the same situation in both cases, but we applied less work when we moved both particles: $W^{ii} < W^i$.

We can get the energy equal to the work $W^i$ back from the system when we reverse the process (i), moving one of the charges to the original distance $l$

We can repeat the cycle of process (ii) followed by resolved it in a quantitative way. In Section VIII we return to the analysis of Paradox I, the solution of which is due to yet another surprising effect.

In Section VII we consider acceleration of particles involved in Paradox I: accelerating a pair of particles from rest. In Section VI we consider remaining part of the processes involved in Paradox I: accelerating a pair of particles from rest. In Section VII we consider acceleration of particles moving in parallel. This setup leads to a paradox the resolution of which is due to yet another surprising effect. In Section VIII we return to the analysis of Paradox I and resolve it in a quantitative way.

II. PARADOX II: CONSERVATION OF ENERGY FOR TWO STOPPING PARTICLES

Consider two particles of charge $q$ and mass $m$ located on the $x$ axis at the separation $l$ and moving in the $x$ direction with a constant velocity $v$. At time $t_1 = 0$ we stop the first particle and at time $t_2 = t$ we stop the second particle: see Fig. 2. The time $t$ is small enough such that signals about the change of the velocity of one particle cannot reach the location of the other, while it is still moving:

$$\frac{-l}{c-v} < t < \frac{l}{c+v}.$$ (4)

We also require that $\tau$, the time of “stopping” a particle, is small: $\tau \ll |t|$.

Let us consider conservation of energy for this process. The initial energy should be equal to the final energy plus the work of the forces which the particles exert on external systems:

$$E_{in} = E_{fin} + W_1 + W_2 + \tilde{W},$$ (5)

where $W_1$ and $W_2$ are the works due to the forces the particles exert during the process of stopping; $\tilde{W}$ is the work performed by the particle moving with velocity $v$ during the time that the other particle is at rest (for $t > 0$, the work $\tilde{W}$ is performed by particle 2 and for $t < 0$, by particle 1). Of course, no work is performed when both particles are at rest, and, also, the net work vanishes during the time when both particles are moving with velocity $v$.

For making relativistic analysis more convenient, we include the rest mass energy. Then, the final energy of the system is

$$E_{fin} = 2mc^2 + \frac{q^2}{l-x},$$ (6)

where $x$ is the change in the distance between the charges:

$$x = vt.$$ (7)

The distance might decrease or increase (negative $t$ and $x$) depending on which particle stopped first.

When a particle moves with constant velocity, the total force exerted on it is zero. Therefore, the force it exerts on an external system is equal to the electromagnetic force the other particle exerts on it. Since, in the laboratory frame, the distance between the particles is $l$, in the Lorentz frame at which the charges are at rest, the distance between them is $\gamma l$ (where $\gamma \equiv 1/\sqrt{1-v^2/c^2}$).

The Lorentz transformation for the force in the $x$ direction between the rest frame and the laboratory frame is $F_x = F'_x$ and, therefore, the forces the particles exert on the external systems are:

$$F_{1x} = -F_{2x} = \frac{q^2}{(\gamma l)^2}.$$ (8)

Thus, the work $\tilde{W}$ is:

$$\tilde{W} = -\frac{q^2 x}{(\gamma l)^2}.$$ (9)

This formula is correct both for $x > 0$, when the work is done by particle 2, and for $x < 0$, when the work is done by particle 1. Thus, the equation of conservation of energy (5) becomes

![Fig. 2. Space-time diagram of the motion of the two particles.](image-url)
\[ E_{in} = 2mc^2 + \frac{q^2}{l-x} + W_1 + W_2 - \frac{q^2x}{(\gamma l)^2}. \] (10)

The initial energy \( E_{in} \) obviously does not depend on \( x \). Due to causality argument, the works \( W_1 \) and \( W_2 \) do not depend on \( x \) either. The equation of energy has only two terms depending on \( x \). Therefore, Eq. \( (10) \) represents a paradox: it must be true for all \( x \) in the interval \( [\frac{1}{c-v}, \frac{i}{c+v}] \) (corresponding to \( \beta \)), but it cannot, since it has only two terms depending on \( x \) which do not balance each other.

### III. Resolution of Paradox II: Interference of Radiation

In Paradox II, according to our naïve calculation, the final energy together with the obtained work had two terms depending on \( x \), the change in the distance between the particles, which do not sum up to a constant. There is cancellation of the \( x \) dependence in the first order of the parameter \( \frac{v}{c} \sim \beta \), therefore, the leading term of the unbalanced \( x \) dependent term is \( \frac{q^2x^2}{T^2} \). Indeed,

\[ \frac{q^2}{l-x} - \frac{q^2x}{(\gamma l)^2} = \frac{q^2}{l} + \frac{q^2x^2}{l^3} + \ldots \] (11)

The process of stopping cannot be arbitrarily slow since it had to be finished before the signal about stopping the other particle can arrive. Moreover, we choose \( \tau \ll |t| \). Therefore, we should expect a significant contribution due to radiation which we have not taken into account. The charges accelerate during the process of stopping and the magnitude of the acceleration is \( a = \frac{e_k}{\tau} \). According to the Larmor formula, the total energy radiated by a single charge during the whole process of stopping is

\[ R_1 = R_2 = \frac{2q^2a^2}{3c^3} = \frac{2q^2v^2}{3c^3\tau}. \] (12)

The \( x \) dependent term which we have to balance is much smaller than \( R_1 \) and \( R_2 \):

\[ \frac{q^2x^2}{l^3} = \frac{q^2(vl)^2}{l^3} \lesssim \frac{q^2v^2}{c^2l} \ll \frac{q^2v^2}{c^3\tau}. \] (13)

However, everything that happens at the close vicinity of the charges cannot depend on \( x \), and, in particular, the radiation which each charge emits does not depend on \( x \), so how can the radiation energy balance the \( x \) dependent terms in the equation of conservation of energy? The effect is due to the interference of radiation. The total radiated energy is

\[ R_{tot} = R_1 + R_2 + R_{int}. \] (14)

The interference term \( R_{int} \) depends on \( x \) and restores the balance. Now we will show this in detail.

The radiation of the stopping charge propagates inside a spherical shell of width \( c\tau \) and the energy flux \( S \) is given by

\[ S = \frac{q^2a^2\sin^2\theta}{4\pi c^3r^2}\hat{r}, \] (15)

where \( r \) is the radius of the shell and \( \theta \) specifies the direction relative to the \( x \) axis in our setup. Since we have two accelerated charges, the radiation field due to the two charges will interfere in the region of the overlap, see Fig. 3. The complete overlap will take place for the angle \( \theta \) defined by

\[ \sin(\theta - \frac{\pi}{2}) = \frac{ct}{l-vt} = \frac{cx}{v(l-x)}. \] (16)

Since the width of the shells is \( c\tau \), the overlap will be nullified beyond the deviation \( \delta\theta \) from the angle \( \theta \) given by

\[ \delta\theta = \frac{c\tau}{(l-x)\sin\theta}, \] (17)

which is obtained from

\[ c\tau = \delta \left[(l-x)\sin(\theta - \frac{\pi}{2})\right] = (l-x)\sin\theta \delta\theta. \] (18)

Due to the interference, the total energy radiated in the direction of the overlap is twice as much as if the two charges were radiating separately. At the interval \([(\theta - \delta\theta), (\theta + \delta\theta)]\) the overlap increases and then decreases linearly. Therefore, the interference term of the radiation energy is

\[ R_{int} = \frac{2q^2a^2\sin^2\theta}{4\pi c^3r^2}\sin\theta r\tau \int_{-\delta\theta}^{\delta\theta} \frac{d\delta\theta}{\delta\theta} \sin \phi \frac{d\phi}{(l-x)} = \frac{q^2c^2\sin^2\theta}{c^3(l-x)}. \] (19)

Fig. 3. Electromagnetic radiation of the two stopping particles. The area of constructive interference of the radiation field is painted in black.
After expressing $\sin^2 \theta = 1 - \sin^2(\theta - \frac{\pi}{2})$, using (10), and making an approximation up to the lowest order in the parameter $\frac{x}{l} \approx \frac{\gamma}{\tilde{\gamma}}$, we obtain:

$$R_{int} = \frac{q^2 v^2}{c^2(l - x)} - \frac{q^2 x^2}{(l - x)^3} \approx \frac{q^2 v^2}{c^2l} - \frac{q^2 x^2}{l^3}. \quad (20)$$

Thus, we have shown that (at least up to a second order in the parameter $\frac{x}{l}$, the precision to which we made our calculations) the equation of conservation of energy which takes into account the electromagnetic radiated energy does not have $x$ dependence. The corrected equation of conservation of energy (which replaces Eq. 5),

$$E_{in} = E_{fin} + W_1 + W_2 + \tilde{W} + R_{tot}, \quad (21)$$

leads, after the approximation, to the expression which does not exhibit $x$ dependence:

$$E_{in} = 2mc^2 + W_1 + W_2 + R_1 + R_2 + \frac{q^2}{l} \left(1 + \frac{v^2}{c^2}\right). \quad (22)$$

Thus we have resolved Paradox II with regard to the unbalanced $x$ dependence. But can we show that the LHS and the RHS of the equation of conservation of energy (22) are equal? We will analyze this in the next section.

The paradox of non-conservation of energy of the system of two charged particles when radiation is neglected has been considered in another example. The calculation of the scattering on the basis of the Coulomb forces yields an energy of the charged particles after the collision that is larger than the initial energy. The advantage of the scattering example is that no external forces have to be taken into account. However, the resolution of this paradox by taking into account radiation has been shown only qualitatively.

### IV. LORENTZ TRANSFORMATION FOR ELECTROMAGNETIC ENERGY

Without knowing the mass and without knowing the local works $W_1$ and $W_2$ it seems that we cannot test (22). However, we can test the consistency of this equation of conservation of energy with single-particle conservation of energy equations. We can write down the conservation of energy for each particle assuming that it performs exactly the same motion (stopping from velocity $v$ during the time $\tau$) in case the other particle is not present. We can argue that the local works $W_i$ and the emitted radiations $R_i$ are the same as in the original example.

This argument is not as strong as the causality argument, i.e. the independence of $x$ of the values of these variables, but it seems that since we can take the time $\tau$ of the local processes very small, the effect of the external field can be neglected. In order to reduce any doubt that this is a valid approach, we will consider, in the next section, a similar situation with a large number of charged particles. It exhibits the same problem, but we will not need this assumption.

Our approach to finding the initial energy is finding the total energy of the two charges in their rest frame $E_0$ and multiplying it by the factor $\gamma$. In the rest frame of the moving particles, the distance between them is $\gamma l$. Therefore, the total initial energy of the particles is:

$$E_{in} = \gamma \left(2mc^2 + \frac{q^2}{\gamma l}\right). \quad (23)$$

Thus, the equation of conservation of energy (22) becomes:

$$2\gamma mc^2 + \frac{q^2}{l} = 2mc^2 + W_1 + W_2 + R_1 + R_2 + \frac{q^2}{l} \left(1 + \frac{v^2}{c^2}\right). \quad (24)$$

We can write two (identical) single-particle equations of conservation of energy:

$$\gamma mc^2 = mc^2 + W_1 + R_1,$$

$$\gamma mc^2 = mc^2 + W_2 + R_2.$$

But when we subtract these equations from the two-particle conservation of energy equation (24) we see that there is inconsistency: the term $\frac{q^2}{l\gamma}$ is unbalanced! The inconsistency does not follow from the approximations we made in deriving (22). The algebraic approximations can be made irrelevant if we consider simultaneous stopping of the charges corresponding to $x = 0$, in which case (22) follows without approximation. We have reached another paradox. There must be another error in our analysis. (In fact, this paradox will appear also for large $|x|$ when there is no interference of radiation; such a case will be considered in the next section.)

The paradox arises from the error which we made in Eq. (23). We have claimed that if the total energy of a system of charges in their rest frame is $E_0$, then its energy in the Lorentz frame in which the system moves with velocity $v$ is

$$E = \gamma E_0. \quad (26)$$

Equation (20) is, of course, correct when the system is an elementary particle. It is also true when the system is a finite stationary isolated body. But it is, in general, not true for a composite system with external forces such as the system of charged particles we consider here.

In order to obtain the correct transformation of the electromagnetic energy from the rest frame to the moving frame, we consider two charges connected by a rigid rod. The energy of the whole system, charges and the rod, does transform according to (29). Therefore, the anomalous term of the transformation of the electromagnetic energy equals to the negative of the anomalous part of the mechanical energy of the rod. The latter is easier to calculate and we will do it now.

In order to calculate the expression for transformation of energy of the rod, we express it as a volume integral of the energy density $u$:

$$E_{in} = \int u dv, \quad (27)$$
and use the Lorentz transformation for the energy density, the 00 component of the energy-stress tensor:

\[ u = \gamma^2 \left( u' + \frac{v}{c^2} S_x - \frac{v^2}{c^2} \sigma'_{xx} \right), \]

where \( S \) is the energy flux and \( \sigma \) is the stress tensor. The transformation of the energy due to the first term leads to the usual expression (26): the energy density is multiplied by \( \gamma^2 \), but due to the Lorentz contraction the volume is multiplied by the factor \( \gamma^{-1} \). The second term does not contribute since the energy flux in the rest frame vanishes. Therefore, the last term contributes the anomalous term. The tension in the rod prevents the rest frame vanishes. Therefore, the last term contributes to the tension of the rod is:

\[ \sigma'_{xx} = \frac{q^2}{s \gamma l^2}, \]

where \( s \) is the cross-section of the rod. Therefore, the contribution to the energy in the laboratory frame due to the tension of the rod is:

\[-\frac{\gamma^2 q^2}{c^2} \int \sigma'_{xx} dv = -\frac{\gamma^2 q^2}{c^2} \sigma_{xx} l s = -\frac{v^2 q^2}{c^2 l} \]

Thus, the correct expression for the initial energy of the electromagnetic field of the two charges (including the anomalous term of the transformation of the electromagnetic energy which equals to the negative of that of the mechanical energy) is:

\[ E_{\text{in}} = \gamma \left[ 2mc^2 + \frac{q^2}{\gamma l} \left( 1 + \frac{v^2}{c^2} \right) \right]. \]

The correction we found for the initial energy of the system of two charges equals exactly the interference term of the radiation energy thus restoring the equation of conservation of energy.

**V. CONSERVATION OF ENERGY FOR \( N \) STOPPING PARTICLES**

In this section we can strengthen the arguments of the previous section by considering a large number of charged particles in a row. In this case, we do not subtract single-particle equations of conservations of energy from the \( N \)-particle equation of conservation of energy. Thus, we do not need the assumption of the previous section that the terms of single-particle equations are equal to the corresponding terms of the \( N \)-particle equation. However, since this section does not describe conceptually new effects, it can be omitted on the first reading.

Consider a chain of a large number \( N \) of identical particles of charge \( q \) and mass \( m \) separated by the distance \( l \) one from the other, all moving with velocity \( v \) on the \( x \) axis. The first particle stops at \( t_1 = 0 \) during a short time \( \tau \). The second particle stops in the same manner at \( t_2 = t \), the latest possible moment such that the information about stopping of the first particle cannot reach it. This corresponds to

\[ x = vt = \frac{vl}{c + v}. \]

The third particle stops in the same way at time \( t_3 = 2t \) just before the information about the stopping of the first two should arrive, the particle \( n \) stops at time \( t_n = (n - 1)t \), etc. until the stopping of the last particle. This is illustrated in Fig. 4.

One difference from the case of two particles is that, due to the particular choice of \( x \), we have no interference of radiation from different charges. The overlap of the radiation fields takes place only on the \( x \) axis where the intensity is zero. Another change from the case of two particles to the case of \( N \) is reflected in the calculations of the initial and final potential energies, the work \( W \) (the work made by particles during motion with constant velocity), and the anomalous energy transformation term: in all these terms appear the factor \( \eta \):

\[ \eta = \sum_{n=1}^{N-1} \sum_{i=1}^{n} \frac{1}{i}. \]

The appearance of the factor \( \eta \) is obvious for potential energy, since \( \frac{\gamma^2}{m} \sum_{i=1}^{n} \frac{1}{i} \) is the energy of bringing the particle number \( n + 1 \) when there are \( n \) particles in the row. Let us show that the same factor appears for \( W \). When all particles move with velocity \( v \), no net work is done on the system and, of course, no net work is done when all particles are at rest. Consider forces between particle \( n \) and all particles \( i \) to the right of it, (i.e., \( i < n \)). The distance the particle \( n \) moves while \( i \) is at rest is \( (n - i)x \), and the force between them is \( \frac{q^2}{(\gamma l(n-i))^2} \). Therefore, the contribution to \( W \) due to the interaction between particle \( n \) and particle \( i \) is

\[ (N-1)y \ldots \ldots \ldots \ldots \ldots \frac{|x|}{l} \frac{|l|}{x} \]

\[ 2t \]

\[ \frac{|l|}{x} \frac{|l|}{x} \]

\[ \begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
N & N-1 & \ldots & 3 & 2 & 1
\end{array} \]

Fig. 4. Space-time diagram of the motion of \( N \) particles.
\[
\frac{q^2x}{(\gamma l)^2(n-i)}.
\]

We obtain that the contribution to the potential energy due to the forces between particle \( n \) and all particles \( i \) such that \( i < n \) is:
\[
\sum_{i=1}^{n-1} \frac{q^2x}{(\gamma l)^2(n-i)} = \frac{q^2x}{(\gamma l)^2} \sum_{i=1}^{n-1} \frac{1}{i}.
\]

The sum of the works made by all particles, starting from the moment when the first particle stops, is:
\[
W = \frac{q^2x}{(\gamma l)^2}\sum_{n=2}^{N} \sum_{i=1}^{n-1} \frac{1}{i} = \eta \frac{q^2x}{(\gamma l)^2}.
\]

Instead of performing a direct calculation of the anomalous transformation term (deviation from (30)), we can make the following observation. The anomalous term can be found by calculation of the contribution due to the tension of the rod (compare with (31)):
\[
-\frac{\gamma^2v^2}{c^2} \int \sigma'_{xx} dv = -\frac{\gamma^2v^2}{c^2} \sum_{n=1}^{N-1} (\sigma'_{xx}) n l s = -\frac{\gamma^2v^2}{c^2} \sum_{n=1}^{N-1} T_n l.
\]

Consider a rod with \( N \) charges separated by distance \( l \) at rest. We will show that \( \sum_{n=1}^{N-1} T_n l \) is equal to the potential energy of the charges. Since the latter is multiplied by \( \eta \) in the transition from 2 to \( N \), the former is multiplied by \( \eta \) too. Potential energy is equal to the work which electromagnetic forces will do in the process of uniform extension of the length of the rod from \( (N-1)l \) to a very large length, which, in turn, is equal to the negative of the mechanical work made by the tension forces of the rod. When the separation between the charges is \( \tilde{l} \), the tension in the parts of the rod can be expressed as
\[
T_n(\tilde{l}) = T_n(l) \frac{\tilde{l}^2}{l^2},
\]

because this tension compensates the Coulomb force proportional to \( \tilde{l}^{-2} \). Therefore, the work (equal to the potential energy) can, indeed, be expressed as
\[
\sum_{n=1}^{N-1} \int_{l}^{\infty} T_n(\tilde{l}) d\tilde{l} = \sum_{n=1}^{N-1} \int_{l}^{\infty} T_n(l) \frac{\tilde{l}^2}{l^2} d\tilde{l} = \sum_{n=1}^{N-1} T_n l.
\]

Now we can write the equation of conservation of energy for \( N \) charges including the anomalous transformation term (the LHS is the modification of (31) and the RHS is the modification of the RHS of (10)):
\[
N\gamma mc^2 + \frac{q^2}{l} \left( 1 + \frac{v^2}{c^2} \right) = Nmc^2 + \eta \frac{q^2}{l-x} + W_1 + W_2 + ... + W_N - \eta \frac{q^2x}{(\gamma l)^2}.
\]

In order to estimate \( \eta \) for large \( N \) we can replace the sums by the integrals (and \( N - 1 \) by \( N \)):
\[
\eta = \sum_{n=1}^{N-1} \int_{1}^{N} \left( \int_{1}^{\infty} \frac{dy}{y} \right) dz \approx \int_{1}^{N} \ln zdz \approx N \ln N.
\]

(We omitted \( N \), since it is negligible relative to \( N \ln N \).

Since for large \( N \) we have \( \eta \gg N \), all terms in (40) which do not have the factor \( \eta \) can be neglected. These include all terms \( W_n \), the amount of work obtained in the process of stopping particle \( n \). Indeed, for small enough \( \tau \), these terms will not have strong dependence on \( N \) and, therefore, their sum will be approximately proportional to \( N \). After substitution of (32), our choice of \( x \), we can see that the equation of energy with the anomalous transformation term is balanced in the leading \( \eta \) proportional terms (and it is not balanced if we use the transformation of energy (29)).

VI. ACCELERATING PARTICLES FROM REST

In order to prepare ourselves for the analysis of Paradox I presented in the Introduction, let us consider a step which has not been discussed yet: the acceleration of the two charged particles from rest to velocity \( v \). Of course, in the frame of reference moving with velocity \( v \), this is deceleration and stopping of particles moving with equal velocities, the process we analyzed above. However, the transformation from one frame to another might be a difficult task and, as in many other examples [1][2][3], an analysis in a different Lorentz frame allows seeing new physical phenomena; in our case it provides yet another possibility to make an error leading to a paradox.

We accelerate the particles in the same way as in the process of simultaneous deceleration discussed in the preceding paragraph, i.e., we perform simultaneous acceleration from rest to velocity \( v \) during time \( \tau \). The radiated energy during the acceleration, then, should be the same as in the process of stopping, see Eq. (20). For simultaneous acceleration \( x = 0 \), so the interference term is
\[
R_{int} = \frac{q^2v^2}{c^2l}.
\]

Initial energy of the particles is
\[
E_{in} = 2mc^2 + \frac{q^2}{l}.
\]

Since the final state of the particles (motion with velocity \( v \) and separation \( l \)) is identical to the initial state of the particles in the previous example, the expression for the final energy \( E_{fin} \) is given by the RHS of (31).

Again, for each particle we can write the equation of conservation of energy for the process when the other particle is not present:
\[
mc^2 = \gamma mc^2 + W_1 + R_1, \quad mc^2 = \gamma mc^2 + W_2 + R_2,
\]
where \( W_1, W_2 \) are defined as work performed by the particles and are negative in this case.

Since the charges start to move together, it seems that the net work is done only during the acceleration period. Therefore, the equation of conservation of energy is:

\[
E_{in} = E_{fin} + W_1 + W_2 + R_1 + R_2 + R_{int}. \tag{45}
\]

Substituting all the terms and subtracting the single-particle equations we obtain:

\[
\frac{q^2}{l} - \frac{q^2}{l} \left( 1 + \frac{v^2}{c^2} \right) + \frac{q^2v^2}{lc^2}. \tag{46}
\]

The final energy is larger than the initial energy: contradiction!

The error we made here is more transparent. It appears in the sentence stating that the only net work of the charges is done during the acceleration period. It is true that in a stationary case (particles keep their motion all the time) the net work of charges moving with constant velocity vanishes. However, at the beginning of the motion, the fields at the vicinity of the charges are different from the Coulomb field of the stationary motion: each particle feels the static field of the other particle (i.e., as if it has not moved) until the signal from the motion of the other particle can arrive.

Let us calculate the contribution to the work due to the forces between the particles. Particle 2 moves in the static field of particle 1 during the time \( t = \frac{l}{v} \) after which it feels the stationary field of particle 1 which is \( \frac{l}{v} = c \). Similarly, particle 1 moves in the static field of particle 2 during the time \( t' = \frac{l}{v} \), after which it feels the stationary field of particle 2 of the same strength but in the opposite direction. After time \( t' \), there is no contribution to the net work due to the forces between the two particles. Until this time, particle 1 covers distance \( x' = vt' \) in the static field of particle 2. Therefore, the contribution to the work from particle 1 is:

\[
\frac{q^2}{l} - \frac{q^2}{l} \left( 1 + \frac{v^2}{c^2} \right) = \frac{vq^2}{cl}. \tag{47}
\]

The work performed by particle 2 until time \( t' \) has two parts. Until time \( t \), it is:

\[
\frac{q^2}{l} - \frac{q^2}{l - x} = -\frac{vq^2}{cl}. \tag{48}
\]

Between time \( t \) and \( t' \) it feels a constant field so the contribution to the work is:

\[
-\frac{q^2}{c^2l^2}v(t' - t) = -\frac{2v^2q^2}{c^2l}. \tag{49}
\]

The contributions (47) and (48) cancel each other, so the net contribution is given by (49). The net work performed by the particles during the time of motion with constant velocity is:

\[
W = -\frac{2v^2q^2}{c^2l}. \tag{50}
\]

This restores the balance in the equation of conservation of energy canceling the unbalanced terms in (40).

We have resolved the contradiction by taking into account the work performed by charged particles during the transition period from static field to stationary field which lasts \( \frac{l}{v} = c \). But what happens if we skip the intermediate stage? We accelerate the particles to velocity \( v \) and shortly after (before the particles finished performing the work (50)) stop them in a similar manner. What is the source of the radiation energy (12) in this case? In fact, since we have two processes with the acceleration, it seems that we are missing even more, twice this amount!

No, this is another error. We need not look for the source of the radiation energy, because the radiation field due to the acceleration and the radiation field due to stopping interfere destructively. (Note a more bizarre example of a destructive interference of radiation field from a moving body.) We are not going to analyze the destructive interference in a quantitative way in this case. The same effect yields the resolution of Paradox I which is demonstrated in a quantitative way in Section VIII.

Before this, in the next section, we present a paradox arising from yet another subtle effect.

VII. ACCELERATING PARTICLES MOVING IN PARALLEL

Let us consider acceleration of two charged particles lined up in the \( y \) axis instead of the \( x \) axis. The particles accelerate simultaneously from rest to velocity \( v \) in the \( x \) direction. The expression for the initial energy is again (13). The final energy, however, is different:

\[
E_{fin} = \gamma \left( 2mc^2 + \frac{q^2}{l} \right). \tag{51}
\]

Indeed, in the rest frame of the moving particles the distance between them is \( l \). In this case the electromagnetic energy is transformed in the usual way (20) because the energy of the composite system of charges and the rod is transformed according to (28) and the energy of the rod with the tension in the \( y \) direction is transformed according to (20). The anomalous behavior of the rod in the previous case followed from the presence of \( \sigma_{xx} \) component of the stress tensor which vanishes in the vertical configuration.

The interference term of the radiation energy is modified too. In the \( x \) axis configuration, the interference is in the \( y - z \) plane and it always has the angle \( \frac{\pi}{2} \) relative to the direction of the acceleration. In the \( y \) axis configuration, the interference is in the \( x - z \) plane with varying angle \( \theta \) relative to the direction of the acceleration. Therefore, the interference is not always in the direction of the maximal intensity. The intensity is proportional to \( \sin^2 \theta \), see (15), and, therefore, averaging on \( \theta \) reduces the interference term relative to the \( x \) configuration (12).
by the factor of 2:

$$R_{\text{int}} = \frac{q^2 v^2}{2c^2 l}. \tag{52}$$

Although during the stationary motion the charges do not exert forces in the direction of motion, in the transition period, when the charges move in the static field, there is a small component of the force in the direction of motion. The particles move in the static field during the time \( t \) which fulfills

$$ct = \sqrt{l^2 + \ell^2 v^2}. \tag{53}$$

Therefore \( ct = \gamma l \). The total work each particle performs is

$$\tilde{W}_1 = \tilde{W}_2 = \frac{q^2}{l} - \frac{q^2}{\gamma l}. \tag{54}$$

The single-particle equations of conservation of energy, of course, remain the same. Thus, putting together all terms of the conservation of energy equation

$$E_{\text{in}} = E_{\text{fin}} + W_1 + W_2 + \tilde{W}_1 + \tilde{W}_2 + R_1 + R_2 + R_{\text{int}}, \tag{55}$$

and subtracting single particle equations \((44)\), we obtain:

$$\frac{q^2}{l} = \frac{\gamma q^2}{l} + 2\ell^2 \left(1 - \frac{1}{\gamma}\right) + \frac{q^2 v^2}{2c^2 l}. \tag{56}$$

Contradiction again: The final energy is larger than the initial energy. Calculating up to the second order in the parameter \( \frac{v^2}{c^2} \) we see three contributions in the units of \( \frac{c^2 q^2}{l} \). The increase in the potential energy contributes \( \frac{1}{2} \), the work of the static fields during the transition period contributes \( 2 \cdot \frac{l}{c^2} = 1 \), and the interference of radiation contributes \( \frac{1}{2} \). All terms together contribute \( \frac{2c^2 q^2}{l} \).

The effect we missed here is, probably, the most subtle one. We have not taken into account the work of the radiation field. The electric field at the point \( r \) (relative to the charge), due to the radiation of the charge \( q \) moving with acceleration \( a \), is

$$\mathbf{E} = \frac{q}{c^2 r} \hat{r} \times (\hat{r} \times a). \tag{57}$$

For our configuration, the field is:

$$\mathbf{E} = -\frac{qa}{c^2 l} \hat{r}. \tag{58}$$

This field exerts force during the time \( \tau \) during which the particle moves distance \( \tau v \). Taking into account that \( a = \frac{v}{c} \) we obtain that the force of the radiation field changes the energy of each particle by

$$W_{\text{rad}} = -\frac{q^2 v^2}{c^2 l}. \tag{59}$$

Both particles lose their energy in this way and, therefore, we lose two units of \( \frac{v^2 q^2}{c^2 l} \) restoring the equation of conservation of energy.

Note that in the \( x \) configuration of charges, the work of the radiation field vanishes because in this case the radiation fields at the locations of the particles vanish.

VIII. RESOLUTION OF THE PARADOX I

Now we have learned all the effects necessary for the resolution of Paradox I. In fact, we have seen a few effects which do not play a role in this case, but understanding them increases our confidence that our explanation is the correct one.

The anomalous transformation of the electromagnetic energy is not relevant because the charges are at rest at the beginning and at the end of the process. The subtle effect of the work performed by the radiation field is not present too, since the radiation fields in the locations of the particles vanish. It is the interference of radiation which we have not taken into account that resolves the paradox. The radiation energy \((12)\) is much larger than the term \((3)\) which we have to compensate, see \((13)\). However, we would like to get quantitative resolution of this paradox showing how the missing term \((3)\) arises from the calculation of the radiation energy.

In order to obtain a quantitative result we specify how we perform the process described in Section I. We assume that we perform it exactly as in all other setups described here: we accelerate particles during small time \( \tau \) until they reach velocity \( v \). In case (i), the particle moves with constant velocity the distance \( x \) when it stops in the same manner as it was accelerated. In case (ii), both particles reach velocity \( v \) (absolute value) and stop at time \( t \) after passing the distance \( x/2 \). The time \( t \) is short enough such that each particle cannot receive a signal during its motion about the motion of the other particle. Thus,

$$\tau \ll t = \frac{x}{2v} < \frac{l}{c + v}. \tag{60}$$

In case (i) the radiation energy is created in the same amount due to the acceleration and due to the stopping of the particle and, therefore, it is twice the amount given by the Larmor formula \((13)\):

$$R^i = \frac{4 q^2 v^2}{3 c^3 \tau}. \tag{61}$$

In case (ii) there are four events of changing velocity of a particle by amount \( v \) and, therefore, there are four spherical shells of radiation field of the width \( \tau c \), see Fig. 5. The radiation energy is four times the Larmor energy \((13)\) with the correction due to the interference. The correction due to the interference has four terms. The interferences are due to radiation emitted during the acceleration of the two particles, stopping of the two particles, acceleration of the first and stopping of the second, and acceleration of the second and stopping of the first. All these terms can be calculated in the same way as we have calculated the interference of radiation energy of two stopping charged particles in Section IV.

Accelerations and decelerations of the particles are performed simultaneously, therefore, the direction of interference is \( \theta = \frac{\pi}{2} \). This is the direction of the maximal
power of the radiation energy, see (15). The range of the angles for which there is the interference is given by (17) and, thus, similarly to derivation of (21), we obtain that the interference term due to simultaneous acceleration is equal to the term due to simultaneous stopping, and it equals

$$-\frac{q^2v^2}{c^2l}.$$ (62)

where the minus sign is because particles accelerate in opposite directions and the second term of (21) does not appear since simultaneity corresponds to $x = 0$ in the notation of Section III.

The interference between acceleration of the first and stopping of the second takes place in the direction $\theta_1$ defined by

$$\sin(\theta_1 - \frac{\pi}{2}) = \frac{ct}{l - vt}. \quad \text{(63)}$$

For calculating this correction we can use (20) again, taking into account that the particle stops after passing the distance $vt = \frac{\pi}{2}$. Therefore, this contribution is:

$$\frac{q^2v^2}{c^2l} - \frac{q^2x^2}{4l^2}. \quad \text{(64)}$$

The contribution to the correction of the radiation energy due to the interference between acceleration of the second and stopping of the first particles takes place in the direction defined by

$$\sin(\theta_2 - \frac{\pi}{2}) = \frac{ct}{l + vt}. \quad \text{(65)}$$

In our case $l \gg vt$, so we can make an approximation $l - vt \approx l + vt \approx l$ and, therefore, we get the same expression again. Summing up all the expressions, we obtain:

$$R^i = \frac{8}{3} \frac{q^2v^2}{c^3\tau} - \frac{2q^2v^2}{c^2l} + 2 \left( \frac{q^2v^2}{c^2l} - \frac{q^2x^2}{4l^2} \right) = \frac{8}{3} \frac{q^2v^2}{c^3\tau} - \frac{q^2x^2}{2l^3}.$$ (66)

Now we are able to analyze the setup of Paradox I taking into account the radiation energy. In case (i) the work performed by the external forces should include the radiation energy (61). Thus, instead of (6), we obtain

$$W^i = U_{NEW} - U_{OLD} + R^i = \frac{q^2}{l - x} - \frac{q^2}{l} + \frac{4q^2v^2}{3c^3\tau}. \quad \text{(67)}$$

In case (ii), following the structure of Paradox I, we have to calculate the work taking into account the causality argument: each particle “does not know” that the other particle moved. Therefore, the work against the field and the radiated energy should be calculated as if the other particle has not moved. The work is twice the amount of work in case (i) with the change of $x \rightarrow \frac{\pi}{2}$. Thus, instead of (2), we obtain

$$W^{ii} = W_1 + W_2 = 2 \left( \frac{q^2}{l - \frac{\pi}{2}} - \frac{q^2}{l} + \frac{4q^2v^2}{3c^3\tau} \right). \quad \text{(68)}$$

Clearly we cannot gain energy from constructing a machine with a cycle of process (ii) and reversed process (i). The work required for reversed process (i) is:

$$W^i = \frac{q^2}{l} - \frac{q^2}{l - x} + \frac{4q^2v^2}{3c^3\tau}. \quad \text{(69)}$$

Thus, the work during the whole cycle is:

$$W_{tot} = W^i + W^{ii} = \frac{q^2}{l} - \frac{q^2}{l - x} + \frac{4q^2v^2}{3c^3\tau} + 2 \left( \frac{q^2}{l - \frac{\pi}{2}} - \frac{q^2}{l} + \frac{4q^2v^2}{3c^3\tau} \right) \approx -\frac{q^2x^2}{2l^3} + \frac{4q^2v^2}{c^3\tau}. \quad \text{(70)}$$

This work is greater than zero, since the radiation term is much larger than the gain in the potential energy; it can be seen explicitly using (60). However, even if we collect the radiation energy, we still cannot gain energy. Indeed, the total radiation energy is:

$$R_{tot} = R^i + R^{ii} = \frac{4q^2v^2}{3c^3\tau} + \frac{8}{3} \frac{q^2v^2}{c^3\tau} - \frac{q^2x^2}{2l^3} = \frac{4q^2v^2}{c^3\tau} - \frac{q^2x^2}{2l^3}. \quad \text{(71)}$$

We obtained exactly the same expression, i.e. our calculations have shown (up to the precision of the order of $\frac{c^2}{\tau^2}$) that during the complete cycle $W_{tot} = R_{tot}$. This completes the analysis of the paradox presented at the beginning of this paper.

Is it a simple task to demonstrate conservation of energy to a higher order in $\frac{c^2}{\tau^2}$? It is not difficult to expand the algebraic expressions we have to a higher order,
but this is not enough. We have used more approximations, in particular, the formulas for the radiation of the charged particles we have used are correct only in the approximation of small acceleration and small velocities. Indeed, Eq. (12) cannot be universally correct, since it says that by reducing $\tau$, the time of stopping the charged particle, we can obtain unlimited amount of radiation energy: clearly we cannot get more energy than the particle has. Thus, higher order calculations of the equation of conservation of energy is an elaborate task which goes beyond the scope of this paper.

In this paper we have analyzed some relativistic features of the classical electromagnetic theory. We demonstrated in a quantitative way the relevance of the several effects to the balance of conservation of energy in a system of charges. These effects are: changing of the electromagnetic field according to relativistic causality constraints, anomalous transformation of the electromagnetic energy, energy radiated by accelerated charges, interference of radiation, and work performed by the radiation field. We believe that presenting the subject in the form of “paradoxes” helps to achieve a deeper understanding. Obtaining quantitative resolutions of presented paradoxical situations helps to reach confidence in applying the equation of conservation of energy for indirect calculations of various effects.

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