Symmetry Constraints on Direct-Current Josephson Diodes

Da Wang,1,2 Qiang-Hua Wang,1,2 and Congjun Wu3,4,5,6,*

1 National Laboratory of Solid State Microstructures & School of Physics, Nanjing University, Nanjing 210093, China
2 Collaborative Innovation Center of Advanced Microstructures, Nanjing University, Nanjing 210093, China
3 Department of Physics, School of Science, Westlake University, Hangzhou 310024, Zhejiang, China
4 Institute for Theoretical Sciences, Westlake University, Hangzhou 310024, Zhejiang, China
5 Key Laboratory for Quantum Materials of Zhejiang Province, School of Science, Westlake University, Hangzhou 310024, Zhejiang, China
6 Institute of Natural Sciences, Westlake Institute for Advanced Study, Hangzhou 310024, Zhejiang, China

It is necessary to break both time-reversal and parity symmetries to realize a Josephson, or superconducting, diode exhibiting nonreciprocal critical direct-currents (DC). In fact, these conditions are still insufficient. The dependencies of the free energy on the phase difference across the junction and the magnetic field are classified, exhibiting the current-reversion (JR), field-reversion, and field-current reversion conditions, respectively. To exhibit the DC Josephson diode effect, all symmetries satisfying the JR condition need to be broken. The relations of critical currents with respect to the magnetic field are classified into five classes, including three exhibiting the diode effect. These symmetry considerations are applied to concrete examples. Our work reveals that the DC Josephson diode effect is a natural consequence of the JR symmetry breaking, hence, providing a guiding principle to understand or design a DC Josephson diode.

Introduction. The semiconductor diode plays a fundamental role in modern electronics. A Josephson diode effect with nonreciprocal critical supercurrent was firstly proposed by Hu, one of the author, and Dai [1] in junctions between hole- and electron-doped superconductors close to a Mott insulator which works as the depletion region. When the junction is forward-biased by an electric field \( E \), the depletion region shrinks exhibiting an alternating-current (AC) Josephson effect with a large critical current \( I_c \). On the other hand, recently, a nontrivial diode effect has been observed recently in many experiments in the direct-current (DC) Josephson effect, exhibiting nonreciprocal critical current \( |J_{c+}| \neq |J_{c-}| \) \((\pm \text{labels the forward and backward directions})\) [2–18]. These progresses have triggered a great deal of theoretical studies on the supercurrent diode effect in superconductors [19–34].

We focus on the DC Josephson diode effect below. The AC Josephson diode effect is essentially a nonequilibrium phenomenon, while the DC one is an equilibrium property. An important question is to figure out the conditions for such an effect. Researchers have recognized that all of time-reversal \( (T) \) and parity symmetries [including inversion \( (I) \) and mirror reflection \( (M) \)] have to be broken, since any of them protects \( |J_{c+}| = |J_{c-}| \) [20, 21, 26, 28]. However, one may still find various systems with all the above symmetries broken but still do not exhibit the diode effect. One known example is the Josephson junction connecting two different materials (breaking \( I \) and \( M \)) with magnetic impurities in between (breaking \( T \)), where the magnetic scattering could give rise to a \( \pi \)-junction [35] but does not exhibit the diode effect. Other examples will also be given in the context below. Therefore, except the easily distinguished \( T \), \( I \) and \( M \) breaking conditions, there still exist additional constraints to realize a DC Josephson, or superconducting, diode. A unified picture, if exists, is highly desired for future studies in this field.

In this article, we examine what symmetries forbidding the DC Josephson diode effect, and conversely, all such symmetries have to be broken to realize such an effect. According to even-\( \text{or-odd} \) dependence of the free energy with respect to the supercurrent \( J \) and magnetic field \( B \) reflections, there are three relations including current-reversion (JR), field-reversion (BR), and field-current reversion (BJR) ones. Any unitary or anti-unitary symmetry leading to the JR relation forbids the diode effect. The relations of critical current \( J_{c\pm} \) with respect to the magnetic field \( B \) are classified into five classes, including three exhibiting the diode effect. Model Hamiltonians are constructed with the magnetic field and spin-orbit coupling (SOC) to examine the above relations, which reveals symmetry constraints more stringent than the previous time-reversal and parity ones.

Current and field reversion conditions. We consider a Josephson junction with a superconducting phase difference \( \Delta \phi \) across the junction. When the orbital effect of the magnetic field \( B \) needs to be considered, \( \Delta \phi \) should be replaced by the gauge invariant version \( \Delta \tilde{\phi} = \Delta \phi - (2e/\hbar) \int A \cdot d\ell \), where \( A \) is the vector potential to be integrated along the path across the junction. In addition, \( B \) also breaks \( T \) in the spin channel and hence manifests in the free energy \( F \) as well. In the most generic case, the free energy \( F \) could exhibit no symmetry with respect to \( B \) and \( \Delta \tilde{\phi} \). Nevertheless, the symmetries of an experimental system may lead to the
The absence of DC diode effect is satisfied, as shown in Table I.

| DC diode effect | absence of DC diode effect |
|-----------------|---------------------------|
| **BJR (type-I)** | none                      |
| **BR (type-II)** | none                      |
| **none (type-III)** | none                      |
| **JR&BR&BJR** | none                      |
| **JR** | none                      |

| sketches of $J_{c\pm}(B)$ | broken symmetries | satisfied symmetries | broken symmetries | satisfied symmetries |
|-----------------------------|------------------|--------------------|------------------|--------------------|
| $J \parallel \hat{x}$      | $I, M_{x,y,z}, IT, TM_{x,y,z}$, etc. | $T, I, M_{x}, C_{2y,2z}$, $TM_{y,z}, TC_{2x}$, etc. | $I, M_{x,y,z}, C_{2y,2z}, TM_{y,z}, TC_{2x}, IT$ | $I, M_{x,y,z}$, $IT, TM_{y,z}$, etc. |
| $B \parallel \hat{x}$      | $I, M_{x}, C_{2y,2z}, IT, TM_{x}, TC_{2x}, 2y, 2z$ | $T, I, M_{x}, C_{2y,2z}, 2y, 2z, TM_{y,z}, TC_{2x}$, etc. | $I, M_{x}, C_{2y,2z}$, $IT, TM_{y,z}$, etc. | $T, I, C_{2y,2z}$, $IT, TM_{y,z}$, $TC_{2x}, IT$, etc. |
| $J \parallel \hat{y}$      | $I, M_{x}, C_{2y,2z}, IT, TM_{x}, TC_{2x}, 2y, 2z$ | $I, T$, $I, M_{x}$, $C_{2y,2z}$, $IT, TM_{y,z}$ | $I, C_{2y,2z}$, $IT, TM_{x}$, $TC_{2x}$ | $I, C_{2y,2z}$, $IT, TM_{x}, TC_{2x}$, etc. |

**TABLE I.** Usual time-reversal and spatial symmetry constraints for the five classes of DC Josephson junctions. Without lose of generality, we assume the current $J$ along the $\hat{x}$-direction and the magnetic field $B$ along $\hat{x}$- or $\hat{y}$-directions. For each case, we list the symmetries to be all broken, and, to be satisfied with at least one.

The above free energy conditions give rise to different symmetry relations of the supercurrents. The supercurrent is defined as $J = (2e/\hbar)\partial F/\partial \Delta \phi$, and the critical supercurrents $J_{c\pm}$ are maximal values in the forward and backward directions, respectively. The following convention is employed that $J_{c+} > 0$ and $J_{c-} < 0$. Eq. 1 leads to

$$JR: \quad J_{c+}(B) = -J_{c-}(B),$$

which protects the absence of the diode effect. In contrast, Eq. 2 and Eq. 3 give rise to

$$BR: \quad J_{c\pm}(B) = J_{c\pm}(-B),$$
$$BJR: \quad J_{c\pm}(B) = -J_{c\pm}(-B).$$

Both cases, the Josephson diode effect appears if the JR relation is violated.

If any two of the above three conditions are satisfied, then the third one is automatically true. Hence, there exist five types: all conditions, one of the JR, BJR, and BR conditions, and none, are satisfied, as shown in Table I. The former two situations show the absence of the diode effect, while the latter three exhibit it, labeled by type-I, II, III, respectively. For the type-I diode, the curves of $J_{c\pm}$ versus $B$ are central symmetric, i.e., satisfying Eq. 6, which is widely observed in many experiments [2–8, 10–12, 14–18]. As for the type-II diode, the BR condition protects $J_{c\pm}(B) = J_{c\pm}(-B)$, which is reported in the InAs-junctions under a background magnetic field [3] and the NbSe$_2$/Nb$_3$Br$_9$/NbSe$_2$-junctions [13] under zero external magnetic field.

We next analyze concrete symmetries leading to the above free energy conditions. As an example, a unitary symmetry $U$, or, an anti-unitary symmetry $UK$ ($K$ the complex conjugate) leading to the BJR relation can be expressed as

$$U^\dagger H(B, \Delta \phi)U = H(-B, -\Delta \phi + \theta),$$

where $\theta$ is a constant global phase. Eqs. 1, 2, and 3 are termed as current-reversion (JR), magnetic-field-reversion (BR), and magnetic field and current simultaneously reversion (BJR) relations, respectively.

The above free energy conditions give rise to different symmetry relations of the supercurrents. The supercurrent is defined as $J = (2e/\hbar)\partial F/\partial \Delta \phi$, and the critical supercurrents $J_{c\pm}$ are maximal values in the forward and backward directions, respectively. The following convention is employed that $J_{c+} > 0$ and $J_{c-} < 0$. Eq. 1 leads to

$$JR: \quad J_{c+}(B) = -J_{c-}(B),$$

which protects the absence of the diode effect. In contrast, Eq. 2 and Eq. 3 give rise to

$$BR: \quad J_{c\pm}(B) = J_{c\pm}(-B),$$
$$BJR: \quad J_{c\pm}(B) = -J_{c\pm}(-B).$$

In both cases, the Josephson diode effect appears if the JR relation is violated.

If any two of the above three conditions are satisfied, then the third one is automatically true. Hence, there exist five types: all conditions, one of the JR, BJR, and BR conditions, and none, are satisfied, as shown in Table I. The former two situations show the absence of the diode effect, while the latter three exhibit it, labeled by type-I, II, III, respectively.
hand, the curves of $J_{c}\pm$ violate the reflection symme-
tries, indicating the Hamiltonian should break all of the
following symmetries leading to BR and JR conditions,
including inversion $I$, mirror reflections with respect to
the $yz$, $xz$, and $xy$-planes ($M_{i}$ with $i = x, y, z$), and
the combined symmetries of $TI$, $TM_{i}$ ($i = x, y, z$), etc. If the
field and the current are perpendicular, say, $J \parallel \hat{x}$ and
$B \parallel \hat{y}$, similar conclusions can be drawn. Any one of the
following symmetries, $T$, $M_{x}$, $C_{22}$, $TM_{y}$, is sufficient to
lead to the BJR condition. Conversely, all the following
symmetries leading to BR and JR should be broken, $I$, $M_{z}$,
$C_{22}$, $C_{2y}$, $TI$, $TM_{z}$, $TM_{x}$, and $TC_{2y}$, etc.

Similar analysis can be straightforwardly applied to
other situations including the type-II and type-III diode
effect, and also for the other two cases exhibiting no diode
effect, as summarized in Table I. The symmetry patterns
are much richer than previous results in literature, and
they provide a guidance to design DC Josephson, or su-
perconducting, diodes in future studies.

1D model and the DC Josephson diode effect. We
proceed to consider concrete models to verify the above
symmetry conditions of the critical supercurrents. We
first consider a superconducting chain along the $x$-axis
as shown in Fig. 1(a). The model Hamiltonian of the
Bogoliubov-de Gennes (BdG) mean-field theory reads

$$H_{1D} = \sum_{i} \left[ c_{i}^{\dagger} \left( t_{i} + i\lambda \sigma_{z} \right) c_{i+1} + h.c. \right] + \sum_{i} c_{i}^{\dagger} (-\mu - B \cdot \sigma) c_{i} + \left( \Delta c_{i}^{\dagger} c_{i+1}^{\dagger} + \text{h.c.} \right),$$  \hspace{1cm} (9)

where $c_{i}$ is a two-component spinor, $t_{i} = t$ is the nearest
neighbor hopping inside two leads taken as the energy
unit, and $t_{i} = \kappa t$ on the interface bond, $\mu$ is the chemical
potential, $\Delta_{i}$ is the spin-singlet pairing with phases $\pm \frac{\pi}{2}$
on two sides, respectively. The Zeeman field $B$ lies in the
$xz$-plane. The SOC quantization axis lies along the
$z$-axis, with the strength $\lambda_{i} = \lambda$ inside two leads and $\kappa \lambda$ on the interface bond. By calculating the free energy $F$
as a function of $\Delta \phi = \varphi$, which is just the ground state
energy at $T = 0$K for simplicity, the Josephson current is
obtained as $J(\varphi) = (2e/\hbar)\partial_{\varphi} F$, whose maximal/minimal
values by varying $\varphi$ give $J_{\pm}$, respectively.

Such a system breaks the inversion by the SOC term
and breaks time-reversal by the Zeeman term, respec-
tively. Hence, naively one would expect a DC Josephson
diode effect. However, such a diode effect only appears
when all the quantities of $\lambda$, $B_{x}$, $B_{y}$, and $\mu$ are simultane-
ously nonzero. In Fig. 1(b)-(e), the nonreciprocal factor
$Q$, defined as

$$Q = \frac{|J_{c\pm}|}{J_{c\pm}},$$  \hspace{1cm} (10)

is plotted with varying each one of the parameters $B_{x}$,
$B_{y}$, $\Delta$, and $\mu$, while fixing the others at nonzero values.

The above results show that as long as one of the pa-
rameters becomes zero, the DC Josephson diode effect
vanishes because at least one symmetry leads to the JR
condition of Eq. 1. (I) If $\lambda = 0$, the inversion $I$ is a
unitary symmetry satisfying the JR condition, i.e. leaving
the Hamiltonian invariant but reversing the super-
current. (II) If $B_{y} = 0$, the JR relation is satisfied due to
the mirror reflection symmetry $M_{y}$. After switching on
$B_{x}$, the JR condition is violated, and $M_{y}$ reflects $B_{y}$
satisfying the BJ condition, giving the type-I diode effect
in Table I.

The situations of $B_{y} = 0$ or $\mu = 0$ involve new sym-
metries not shown in Table I. (III) For $B_{x} = 0$, we first
define a spin-twist operation $U_{tw}$ as a position-dependent
spin rotation

$$U_{tw} = \prod_{i} U(i),$$  \hspace{1cm} (11)

with $U(i) = e^{i\frac{\pi}{2}(\pm 1)\eta}$ acting on site-$i$ and $\eta = \arctan(\lambda_{i}/t)$. The spin twist leaves the $B_{x}$ term un-
changed, eliminates the SOC term, and transform it into
the hopping term by replacing $t$ with $\sqrt{t^{2} + \lambda^{2}}$ in Eq. 9.
Since the pairing is spin-singlet, it is not changed by this
spin rotation. Then a combined operation $U = U_{tw}U_{tw}^{\dagger}$
leaves the Hamiltonian invariant except switching the current
direction. For nonzero $B_{y}$, the $\pi$-rotation in the spin space
$R_{\pi}(\pi) = e^{i\frac{\pi}{2}\sigma_{z}}$ brings $B_{y}$ to $-B_{x}$ without
reflecting the current, satisfying the BR condition and
hence the diode effect belongs to type-II in Table I. (IV) For $\mu = 0$, the particle-hole transformation

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{(a) Scheme of the superconducting chain along $x$-
direction. The hopping $t$ and SOC strength $\lambda$ are reduced by
a factor $\kappa$ on the interface (blue) bond. The superconducting
phases on the two leads are $\pm \varphi/2$. From (b) to (e), the non-
reciprocal factor $Q$ defined in Eq. 10 is plotted with varying
$B_{x}$, $B_{y}$, and $\lambda$, respectively, while keeping the others fixed
at $B_{z} = B_{x} = \lambda = 0.5$, $\mu = 1$, $\Delta = 0.2$ and $\kappa = 0.4$.}
\end{figure}
Josephson diode effect is observed at zero external
ment on the NbSe$_2$ variant under the current-reflecting operation
π to λ.

To consider a bilayer toy model connected by a narrow junction schematically shown in Fig. 2(a). The tunneling direction is along the z-direction. The BdG Hamiltonian reads

$$H_{2D} = \sum_{n=t,b,k} \left\{ c_{n\uparrow}^\dagger \left[ \epsilon_k \sigma_0 + \lambda_R (k_x \sigma_y - k_y \sigma_x) - B \cdot \sigma \right] c_{nk} \right\}$$

$$+ \left( c_{n\uparrow}^\dagger \Delta_n c_{n\downarrow} + h.c. \right)$$

$$+ \sum_k c_{k\uparrow}(t_\perp + i \lambda_\sigma \sigma_z) c_{k\downarrow} + h.c.,$$

where $t/b$ refers to the top/bottom layer, $k = (k_x, k_y)$ is the in-plane momentum, $t_\perp$ the tunneling matrix element, $\lambda_R$ is the Rashba SOC, and $\lambda_\sigma$ is another SOC coupling to $\sigma_z$. The Zeeman field $B$ is assumed to lie in the xz-plane. The spin-singlet pairings $\Delta_\parallel = \Delta$ and $\Delta_\perp = \Delta \cos \phi$, and then $\Delta_\parallel = \Delta$. The band dispersion is simply chosen as $\epsilon_k = -2t(\cos k_x + \cos k_y) - \mu$ with $t$ taken as the energy unit.

The nonreciprocal factor $Q$ is plotted on the $(B_x, B_z)$ plane as shown in Fig. 2(b). $B_x$ itself cannot lead to a nonzero $Q$ unless a nonzero $B_z$ exists. Again, all the quantities of $B_z$, $\lambda_\parallel$ and $\lambda_\perp$ need to be nonzero for the appearance of the diode effect. (I) If $B_z = 0$, then the combined symmetry $TC_{2z}$ leads to the JR condition and protects $Q = 0$. For nonzero $B_z$, since $TC_{2z}$ reflects $B_z$ and current simultaneously satisfying the BJR condition, the curves of $J_{\pm}$ vs $B_z$ belongs to type-I in Table I. On the other hand, the $B_z$-dependence belongs to type-II in Table I, since $C_{2z}$ brings $B_x$ to $-B_x$ satisfying the BR condition. (II) If $\lambda_\parallel = 0$, the combined symmetry $TM_x$ leads to the JR condition and hence protects $Q = 0$. (III) If $\lambda_\perp = 0$, the symmetry satisfying the JR condition is a little subtle. We first perform a spin twist $U_{tw} = e^{i \frac{\pi}{2} \sigma_y} e^{i \frac{\pi}{2} \sigma_x \cdot \hat{z}}$ with $\eta = \arctan(\lambda_\parallel/t_\perp)$ to eliminate the $\lambda_\parallel$-SOC term, which transforms $B_z \sigma_x$ to $B_z(\cos \frac{\pi}{2} \sigma_x \pm \sin \frac{\pi}{2} \sigma_y) (\pm \text{for top/bottom})$. Then we perform a $\frac{\pi}{2}$-rotation around $\sigma_x$, i.e. $R_x(\frac{\pi}{2}) = e^{i \frac{\pi}{2} \sigma_x}$, to obtain $(B_z \cos \frac{\pi}{2} \sigma_x \pm B_z \sigma_y \mp B_x \sin \frac{\pi}{2} \sigma_x \lambda_\parallel)$, which is invariant under the current-reflecting operation $C_{2z}$. At last, the above operations are inversely applied to recover the original Hamiltonian except $\Delta_\parallel \rightarrow \Delta_\perp$. Put them together, we obtain the combined symmetry $U = U_{tw} R_x(\frac{\pi}{2}) C_{2z} R_x(-\frac{\pi}{2}) U_{tw}^\dagger$ which protects $Q = 0$.

We tentatively compare our results with the experiment on the NbSe$_2$/Nb$_3$Br$_8$/NbSe$_2$-junctions [13]. The DC Josephson diode effect is observed at zero external magnetic field and is suppressed by the in-plane field $B_x$, showing the type-II behavior in Table I. According to the symmetry principle, the time-reversal must already be broken at $B_z = 0$. It is then natural to conjecture the existence of a spontaneous ferromagnetic moment (still labeled by $B_z$ for our convenience) [36, 37]. (This assumption may be at odds with the absence of longitudinal magnetoresistivity hysteresis with the $\hat{z}$-directional external magnetic field [13], which deserves further studies.) Another possibility for the $T$-breaking is the pairing itself breaks time-reversal like in $s + i d$ or $p + ip$ superconductors [25].

In our model, the $\lambda_\parallel$-SOC term is necessary to cause the diode effect. This term requires $M_z$ breaking, which is indeed possible since the interface Nb$_3$Br$_8$ does break $M_z$ [38]. As a comparison, when the interface is replaced by few-layer graphene preserving $M_z$ symmetry, the $\lambda_\parallel$-term is forbidden, leading to absence of the (external) field-free diode effect [13]. In Fig. 2(b), we find $Q$ is evenly suppressed by $B_x$, qualitatively similar to the experiment, but the required field strength (of order $\Delta/\mu_B \sim 10T$) is much larger than the experimental value $\sim 10mT$ [13]. This small field strength indicates that $B_x$ should couple to $J_z$ mainly through the orbital effect rather than the Zeeman effect. In this regard, we choose the vector potential $A_z = yB_z$ entering
into $t_2$ and $\lambda_2$ through the Peierls phase $e^{i(\Phi_x/\Phi_0)(y/L_y)}$ ($\Phi_0$ the magnetic flux quantum and $L_y$ the magnetic unit cell). Within the quasiclassical picture, the supercurrent $J(\phi, \Phi_x, y)$ is integrated over $y$ to obtain $J_x(\phi, \Phi_x)$ and then $J_{x\pm}(\Phi_x)$. In Fig. 2(c) and (d), $J_{x\pm}(\Phi_x)$ and the modified nonreciprocal factor $Q = [J_{c+}(\Phi_x) + J_{c-}(\Phi_x)]/[J_{c+}(0) - J_{c-}(0)]$ are plotted versus $\Phi_x$, displaying a modulated nonreciprocal Fraunhofer pattern. The feature at small field with weak $\Phi_x$-dependence is similar to the experimental results [13].

**Summary.** In summary, we have specified three types of symmetry conditions, i.e., JR, BR and BJR, based on which the relations of $J_{c\pm}$ and $B_{c\pm}$ are classified into five classes, including three exhibiting the DC Josephson diode effect. These symmetry constraints provide a unified picture to understand or design a DC Josephson diode in future studies.

**Acknowledgement.** D.W. thanks H. Wu, X. Xi for helpful discussions and also thanks S.-J. Zhang for early collaborations in this project. This work is supported by National Natural Science Foundation of China through the Grants Nos. 11874205, 11729402, 12174317, and 51971150. D.W. thanks H. Wu, X. Xi for helpful discussions and also thanks S.-J. Zhang for early collaborations in this project. This work is supported by National Natural Science Foundation of China through the Grants Nos. 11874205, 11729402, 12174317, and 51971150.
and I. Mazin, *Phys. Rev. X* **10**, 041003 (2020).

[38] C. M. Pasco, I. El Baggari, E. Bianco, L. F. Kourkoutis, and T. M. McQueen, *ACS Nano* **13**, 9457 (2019).