A Law of Large Numbers for an Interacting Particle System with Confining Potential

Matteo Ortisi

April 2, 2018

Abstract

In this paper we consider an interacting particle system modeled as a system of $N$ stochastic differential equations driven by Brownian motions with a drift term including a confining potential acting on each particle, and an interaction potential modeling the interaction among all the particles of the system. The limiting behavior as the size $N$ grows to infinity is achieved as a law of large numbers for the empirical process associated with the interacting particle system.

Introduction

We consider a system of $N \in \mathbb{N} - \{0\}$ particles. From the Lagrangian point of view, the system is described by $N$ random variable, $X^k_N(t) \in \mathbb{R}^d$, $t \geq 0$, $k = 1, \ldots, N$, so that $\{X^k_N(t), t \in \mathbb{R}_+\}$ is a stochastic process in the state space $(\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})$, on a common probability space $(\Omega, \mathcal{F}, P)$. $X^k_N$ may describe the state of the $k$-th particle, e.g. its position. We consider the case of continuous time evolution, i.e. the time evolution is described by a system of stochastic differential equations (EDSs) with additive noise

\[
\begin{align*}
    dX^k_N(t) &= \left[ f(X^k_N(t)) + F_N \left[ X_N(t) \right] \left( X^k_N(t) \right) \right] dt \\
    &+ \sigma_N dW^k(t), \quad k = 1, \ldots, N.
\end{align*}
\]  

(1)

In equation (1) the process $\{W^k, k = 1, \ldots\}$ is a family of independent standard Wiener processes, $f : \mathbb{R}^d_+ \to \mathbb{R}$, and the functional $F_N$ is defined on $\mathcal{M}_P(\mathbb{R}^d)$, the space of all probability measures on $\mathbb{R}^d$, and depends on the empirical measure

\[
X_N(t) = \frac{1}{N} \sum_{k=1}^N \epsilon_{X^k_N(t)} \in \mathcal{M}_P(\mathbb{R}^d).
\]  

(2)
By the empirical measure (2) we describe the system by an Eulerian approach: the collective behavior of the discrete (in the number of particles) system, may be given in terms of the spatial distribution of particles at time $t$.

Correspondingly, the measure valued process

$$X_N: t \in \mathbb{R}_+ \rightarrow X_N(t) = \frac{1}{N} \sum_{k=1}^{N} \epsilon X^k_N(t)$$

is called the empirical process of the system for a population size $N$. The trajectories are random elements of $C([0,T],\mathcal{M}_P(\mathbb{R}^d))$, so that the distributions $\mathcal{L}(X_N)$ of those processes can be considered as elements of $\mathcal{M}_P(C([0,T],\mathcal{M}_P(\mathbb{R}^d)))$.

Equation (1) might describe a system of $N$ individuals whose movement is due to a stochastic individual component coupled with an interaction term, and an (individual) advection term.

**The individual dynamics** The advection term $f : \mathbb{R}^d_+ \rightarrow \mathbb{R}$ may describe the individual dynamics of a particle, which may depend on external information. Indeed we consider the following form for $f$

$$f(x) = -\gamma_1 \nabla U(x),$$

where $\gamma_1 \in \mathbb{R}_+$, and the potential $U : \mathbb{R}^d \rightarrow \mathbb{R}_+$ is a non negative smooth even function. From the modelling point of you the transport term (1) mean to be “confining” potential: there are some external information coming from the environment which attracts the particle along the flow due to $U$.

**The interaction dynamics** $F_N$ it depends on the relative location of the specific individual $X^k_N(t)$ with respect to the other individuals, via on the empirical measure of the whole system. The interaction we consider is due to different phenomena: aggregation and repulsion. These two different forces compete but act at different scales.

Aggregation act at macroscale and is modelled by a McKean-Vlasov interaction kernel

$$G_a : \mathbb{R}^d \rightarrow \mathbb{R}_+.$$

The interaction of the particle located in $X^k_N(t)$ at time $t$ with the others is described by a “generalized” gradient operator as discussed in [1,4,14] acting on the empirical measure

$$(\nabla G_a * X_N(t)) \left( X^k_N(t) \right).$$

Repulsion acts at mesoscale; the mesoscale is introduced as in [11,14,16] via a rescaling of a referring kernel $V_1$

$$V_N(z) = N^\beta V_1(N^{\beta/d}z), \quad \beta \in (0,1)$$
The repelling force exerted on the $k$-th (out of $N$) single particle located at $X_N^k(t)$ is given by

$$-\sum_{i=1}^{N} N^{\beta-1} \nabla V_i \left( N^{\beta/d} \left( X_N^k(t) - X_N^i(t) \right) \right) = -\left( \nabla V_N * X_N(t) \right)(X_N^k(t))$$

From (7) it is clear how the choice of $\beta$ determines the range and the strength of the influence of neighboring particles; indeed, any particle interacts (repelling) with $O\left(N^{1-\beta}\right)$ other particles in a small volume $O\left(N^{-\beta}\right)$.

From (5) and (7), the advection interaction term $F[X_N]$ is given by

$$F[X_N](x) = \gamma_2 \left( \nabla (G_a - V_N) * X_N(t) \right)(x),$$

with $\gamma_2 \in \mathbb{R}_+$.  

**The stochasticity** The stochastic component in equation (11) may describe both the lack of information we have about the environment or the particle itself and the need of each particle to interact with the others, so that they move randomly with a mean free path $\sigma_N$ (depending on $N$) unless they meet other particles and interact.

By (11), (13), and (15) the system we study is the following

$$dX_N^k(t) = -\left[ \gamma_1 \nabla U(X_N^k(t)) + \gamma_2 \nabla (V_N - \nabla X_N) (X_N^k(t)) \right] dt + \sigma_N dW^k(t), \quad k = 1, \ldots, N. \tag{9}$$

In the case $\gamma_1 = 0$, the advection is due only to the interaction and the system become the following

$$dX_N^k(t) = (\nabla G_a * X_N(t))(X_N^k(t)) - (\nabla V_N * X_N(t))(X_N^k(t)) dt + \sigma_N dW^k(t), \quad k = 1, \ldots, N. \tag{10}$$

In previous papers [4, 11, 14] the authors has focused their attention on the time evolution of the system (10). In particular they have analyzed the convergence of the system as the number of particles $N$ increases to infinity. In [11, 14] a "law of large numbers" is presented while in [4] the authors have studied the existence and uniqueness of the solution of the PDE describing the time evolution of the limit system.

In this work we focus our attention on the system (9) and extend to this case the results obtained in [11, 14].

**Notations and Hypotheses**

For some topological space $S$ we denote by $C^m_b(S, \mathbb{R}^d)$ the space of $m$-times differentiable $\mathbb{R}^d$-valued functions on $S$ with continuous bounded derivatives; $C^m_b(S, \mathbb{R})$ is abbreviated with $C^m_b(S)$ and $C^m_b(\mathbb{R}^d) = C^m_b(S, \mathbb{R}^d)$. $C^m_b(S, \mathbb{R}^d)$ is equipped with the supremum norm. On $\mathbb{R}^d \times \mathbb{R}^d$, $(\cdot, \cdot)$ denotes the usual scalar product.
$\mathcal{M}_p(S)$ is the space of probability measures on $S$. This space is equipped with the usual weak topology. On the space $\mathcal{M}_p(\mathbb{R}^d)$ the weak topology is generated by the complete metric

$$||\mu - \nu||_1 = \sup_{f \in \mathcal{H}_1} (\langle \mu, f \rangle - \langle \nu, f \rangle),$$

where

$$\langle \mu, f \rangle = \int_{\mathbb{R}^d} f(x) \mu(dx) \quad f \in C_b(\mathbb{R}^d),$$

and

$$\mathcal{H}_1 = \left\{ f \in C_b(\mathbb{R}^d) : \sup_{x \in \mathbb{R}^d} |f(x)| \leq 1, \sup_{x, y \in \mathbb{R}^d, x \neq y} \frac{|f(x) - f(y)|}{|x - y|} \leq 1 \right\}.$$ 

The metric $d_1(\mu, \nu) := ||\mu - \nu||_1$ is also well known as bounded Lipschitz metric.

For any $S$-valued random variable $Y$ we denote by $\mathcal{L}(Y) \in \mathcal{M}_p(S)$ its distribution.

For some $T \in (0, \infty)$, $C([0, T], \mathcal{M}_p(\mathbb{R}^d))$ is the space of all continuous functions $f = f(t)$, $0 \leq t \leq T$ from $[0, T]$ to $\mathcal{M}_p(\mathbb{R}^d)$, equipped with the metric

$$\rho(f, g) = \sup_{0 \leq t \leq T} ||f(t) - g(t)||_1.$$

For $f \in L^2(\mathbb{R}^d)$ we denote by

$$\tilde{f}(\lambda) = \lim_{a \to \infty} \left( \frac{1}{2\pi} \right)^{d/2} \int_{\{|x| \leq a\}} e^{i\lambda x} f(x) dx$$

its Fourier transform.

In connection with Fourier transforms we shall use the relations

$$\int_{\mathbb{R}^d} f(x) \tilde{g}(x) dx = \int_{\mathbb{R}^d} \tilde{f}(\lambda) \tilde{g}(\lambda) d\lambda \quad f, g \in L^2(\mathbb{R}^d), \quad (11)$$

$$\tilde{f} \ast \tilde{g}(\lambda) = (2\pi)^{d/2} \tilde{f}(\lambda) \tilde{g}(\lambda) \quad f, g \in L^2(\mathbb{R}^d), \quad (12)$$

$$\tilde{\nabla} f(\lambda) = -i\lambda \tilde{f}(\lambda) \quad f \in W^1_2(\mathbb{R}^d); \quad (13)$$

where

$$W^1_2(\mathbb{R}^d) = \{ f \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} (1 + |\lambda|^2) |\tilde{f}(\lambda)|^2 d\lambda = ||f||^2_2 + ||\nabla f||^2_2 < \infty \}.$$ 

Positive constants throughout the thesis are denoted by $c_1, c_2, \ldots$; if a constant depends on a quantity $k$, we denote it with $c(k)$.

By defining

$$g_N(x, t) = (V_N \ast X_N(t))(x), \quad (14)$$

$$\nabla g_N(x, t) = (\nabla V_N \ast X_N(t))(x), \quad (15)$$

$$h_N(x, t) = (W_N \ast X_N(t))(x), \quad (16)$$

where

$$W_N(x) = \chi_N^d W(\chi_N x) \quad (17)$$
and $W_1$ is a symmetric probability density defined on $\mathbb{R}^d$, that is

$$W_1(x) = W_1(-x),$$

(18)
equation (9) becomes

$$dX_k^N(t) = -\gamma_1 \nabla U(X_k^N(t)) + \gamma_2 \left[ (\nabla G_a * X_N(t))(X_k^N(t)) - \nabla g_N(X_k^N(t), t) \right] dt$$

$$+ \sigma_N dW^k(t), \quad k = 1, \ldots, N.$$ We consider the further assumptions:

$$V_1(x) = (W_1 * W_1)(x) = \int_{\mathbb{R}^d} W_1(x-y)W_1(dy),$$

(19)and

$$W_1 \in W^1_2(\mathbb{R}^d) \quad \text{i.e.} \quad \int_{\mathbb{R}^d} (1+|\lambda|^2)|\tilde{W}_1(\lambda)|^2 d\lambda < \infty.$$ (20)

About the initial condition we suppose that

$$\sup_{N \in \mathbb{N}} \mathbb{E} \left[ \int_{\mathbb{R}^d} |X_N(0)|^2 dx \right] < \infty,$$

(21)$$\sup_{N \in \mathbb{N}} \mathbb{E} \left[ \int_{\mathbb{R}^d} |h_N(x,0)|^2 dx \right] = \sup_{N \in \mathbb{N}} \mathbb{E} \left[ ||h_N(\cdot,0)||^2_2 \right] < \infty,$$

(22)$$\forall N \geq 1, \quad X_N(0) = (X_N^1(0), \ldots, X_N^N(0)) \text{ is independent of } W^k, k = 1, \ldots, N.$$ (23)
The assumptions about the interaction potential are the following:

$$G_a \in C^2_b(\mathbb{R}^d, \mathbb{R}^+),$$

(24)is a symmetric function on $\mathbb{R}^d$ supposed to be independent on $N$, $$V_N \text{ is supposed to be of the form } V_N(x) = \chi_N^d V_1(\chi_N x), \quad V_1 \in C^2_b(\mathbb{R}^d, \mathbb{R}^+),$$

(25)is a symmetric probability density on $\mathbb{R}^d$,

$$\chi_N = N^{\beta/d}, \quad \beta \in (0,1).$$

(26)It is clear that

$$\lim_{N \to +\infty} V_N = \delta_0,$$

where $\delta_0$ is the Dirac delta function.

$$\forall N \geq 1, \quad \mathbb{E}\{|X_N(0)|^2\} < +\infty.$$ (27)
Let the confining potential be such that

$$U \in C^1_b(\mathbb{R}^d, \mathbb{R}^+) \cap C^2(\mathbb{R}^d, \mathbb{R}^+).$$

(28)Then we consider the following possible assumptions on the parameter $\beta$:
A Law of Large Numbers

In this section we derive a law of large numbers for the measure valued process \( X_N \) defined by (2) and (3), in the case of boundedness properties of the confining gradient \( U \). In particular by following the approaches proposed in \[12\] and \[15\], we prove the existence of the limit measure for the sequence \( \{L(X_N)\}_{N \in \mathbb{N}} \) of distributions of \( \{X_N\}_{N \in \mathbb{N}} \).

We consider both the unviscous case, i.e. \( \lim_{N \to \infty} \sigma_N = 0 \), and the viscous case, i.e. \( \lim_{N \to \infty} \sigma_N = \sigma_\infty > 0 \).

The procedure may be divided into three steps:

i) relative compactness of the sequence \( L(X_N), N \in \mathbb{N} \), which corresponds to an existence result of the limit \( L(X) \);

ii) regularity of the possible limits: we show that the possible limits \( \{X(t), t \in [0, T]\} \) are absolutely continuous with respect to the Lebesgue measure for almost all \( t \in [0, T] \) \( \mathbb{P} - \text{a.s.} \);

iii) identification of the dynamics of the limit process: all possible limits are shown to be solution of a certain deterministic equation that we assume to have a unique solution.

In the case \( \lim_{N \to \infty} \sigma_N = 0 \) we guess the limit dynamics and show that it is the weak limit of \( \{X_N(t), t \in [0, T]\} \).

Relative Compactness

The first step toward proving a law of large number for a measure-valued process is to obtain a relative compactness result for the sequence of empirical measure’s distribution laws \( \{L(X_N)\}_{N \in \mathbb{N}} \) associated to the system of stochastic differential equations.

**Theorem 0.1.** If either \[22\] or \[23\] holds, under conditions \[14\]-\[23\], the sequence \( \{L(X_N)\}_{N \in \mathbb{N}} \) of distributions of the processes \( \{X_N(t), 0 \leq t \leq T\} \) associated to the system of stochastic differential equations \[3\] is relatively compact in the space \( \mathcal{M}_\mathbb{P}(C([0, T], \mathcal{M}_\mathbb{P}(\mathbb{R}^d))) \).

We consider first some preliminary results regarding the martingale properties of some processes. Up to now we suppose that all the hypotheses of Theorem \[0.1\] are satisfied. We remark that all the results are valid also in the case \( \gamma_1 = 0 \).
For the sake of simplifying the notations, in the following calculations we set $\gamma_1 = \gamma_2 = 1$.

Let

$$A_N(t) = \int_0^t \langle X_N(s), 2(\nabla g_N(\cdot, u)^2 - \nabla g_N(\cdot, u)(\nabla U(\cdot) + (\nabla G_a \ast X_N(u))(\cdot)) - \nabla U(\cdot) + (\nabla G_a \ast X_N(u))(\cdot)^2) + \sigma^2_N ||\nabla h_N(\cdot, u)||_2^2 du. \quad (31)$$

**Lemma 0.1.** The process

$$M_N(t) = ||h_N(\cdot, t)||_2^2 + A_N(t) - \int_0^t \langle X_N(u), 2| - \nabla U(\cdot) + (\nabla G_a \ast X_N(u))(\cdot)|^2 \rangle du - c_1 \sigma^2_N t N^{\beta(d+2)/d-1}$$

is a martingale.

**Proof.**

Because of (24), (18), (20) and (28), by applying Itô’s formula to

$$||h_N(\cdot, t)||_2^2 = \langle X_N(t), g_N(\cdot, t) \rangle = \frac{1}{N^2} \sum_{k,l=1}^N V_N(X_N^k(t) - X_N^l(t)),$$

one obtains

$$\mathbb{E} [||h_N(\cdot, t)||_2^2 | F_s] = ||h_N(\cdot, s)||_2^2 + \mathbb{E} \left[ \frac{2}{N} \int_s^t \sum_{m=1}^N \langle X_N(u), \nabla G_a(\cdot - X_N^m(u)) \cdot \nabla g_N(\cdot, u) \rangle du \right.$$

$$+ \frac{2}{N} \int_s^t \sum_{m=1}^N \langle X_N(u), -\nabla U(\cdot) \cdot \nabla g_N(\cdot, u) \rangle du$$

$$- \frac{2}{N} \int_s^t \sum_{m=1}^N \langle X_N(u), (\nabla V_N(\cdot - X_N^m(u))) \cdot \nabla g_N(\cdot, u) \rangle du$$

$$+ \frac{\sigma^2_N}{N^2} \int_s^t \sum_{k,m=1, k \neq m}^N \Delta V_N(X_N^k(u) - X_N^m(u)) du | F_s]$$

$$= ||h_N(\cdot, s)||_2^2$$

$$- \mathbb{E} \left[ \frac{2}{N} \int_s^t \langle X_N(u), |\nabla g_N(\cdot, u)|^2 \rangle du \right.$$

$$- 2 \int_s^t \langle X_N(u), \nabla g_N(\cdot, u) \cdot (-\nabla U(\cdot) + (\nabla G_a \ast X_N(u))(\cdot)) \rangle du$$

$$+ \frac{\sigma^2_N}{N} \int_s^t ||\nabla h_N(\cdot, u)||_2^2 du | F_s] + \frac{\sigma^2_N(t-s)}{N} \Delta V_N(0). \quad (32)$$
By assumption (20),
\[ \Delta V_1(0) = \Delta(W_1 \ast W_1)(0) = (\nabla W_1 \ast \nabla W_1)(0) \]
\[ = \int_{\mathbb{R}^d} \nabla W_1(-y) \nabla W_1(y) dy \]
\[ \leq \left( \int_{\mathbb{R}^d} |\nabla W_1(-y)|^2 dy \right)^{1/2} \left( \int_{\mathbb{R}^d} |\nabla W_1(y)|^2 dy \right)^{1/2} \]
\[ < \infty. \] 
(33)

As a consequence
\[ |\Delta V_N(x)| = N^{\beta} N^{2\beta/d} |\Delta V_1(N^{\beta/d} x)|. \] 
(34)

So
\[ \frac{\sigma^2_N(t-s)}{N} \Delta V_0(0) = c_1 \sigma^2_N(t-s) N^{\beta (d+2)/d - 1}, \] 
(35)

and the thesis follows.

\[ \square \]

**Remark 0.1.** \( A_N(t) \) is not negative; indeed in general
\[ a^2 - ab + b^2 \geq \frac{a^2}{2} - ab + \frac{b^2}{2} = \left( \frac{a}{\sqrt{2}} - \frac{b}{\sqrt{2}} \right)^2 \geq 0. \]

Let now consider a special class of test functions, i.e. positive function \( \phi \in C^2(\mathbb{R}^d) \) such that
\[ \phi(x) = |x| \text{ for } |x| \geq 1 \text{ and } ||\nabla \phi||_\infty + ||\Delta \phi||_\infty < \infty. \] 
(36)

**Lemma 0.2.** For \( 0 \leq s < t \leq T \) and \( \phi \in C^2(\mathbb{R}^d) \) such that (36) holds
\[ \langle X_N(t), \phi \rangle + c_2 A_N(t) + c_3 t \] is a submartingale
(37)

and
\[ \langle X_N(t), \phi \rangle - c_2 A_N(t) - c_3 t \] is a supermartingale,
(38)

with \( c_2, c_3 \in \mathbb{R}_+ \).

**Proof.**

By applying Ito’s Formula to \( \langle X_N(t), \phi \rangle \),
\[ \mathbb{E} \left[ \langle X_N(t), \phi \rangle | \mathcal{F}_s \right] \]
\[ = \langle X_N(s), \phi \rangle \]
\[ + \mathbb{E} \left[ \int_s^t \langle X_N(u), (-\nabla U(\cdot) + (\nabla G_a \ast X_N(u))(\cdot) - \nabla g_N(\cdot, u)) \nabla \phi \right. \]
\[ + \left. \frac{\sigma^2_N}{2} \Delta \phi \rangle du | \mathcal{F}_s \right] \]
\[ \geq \langle X_N(s), \phi \rangle \]
\[ - c_2 \mathbb{E} \left[ \int_s^t \langle X_N(u), -\nabla U(\cdot) + (\nabla G_a \ast X_N(u))(\cdot) - \nabla g_N(\cdot, u) + 1 \rangle du | \mathcal{F}_s \right]. \] 
(39)
Since
\[
0 \leq \langle X_N(u), | - \nabla U(\cdot) + (\nabla G_a \ast X_N(u))(\cdot) - \nabla g_N(\cdot, u)\rangle + 1|^2)
= \langle X_N(u), | - \nabla U(\cdot) + (\nabla G_a \ast X_N(u))(\cdot) - \nabla g_N(\cdot, u)\rangle^2
- 2(-\nabla U(\cdot) + (\nabla G_a \ast X_N(u))(\cdot) - \nabla g_N(\cdot, u) + 1),
\]
we prove confining compactness property for the process
\[
\text{the relative compactness by Ethier-Kurtz (see [8], theorem 8.2). In particular,}
\]
\[
\begin{align*}
& 2\langle X_N(u), -\nabla U(\cdot) + (\nabla G_a \ast X_N(u))(\cdot) - \nabla g_N(\cdot, u)\rangle \\
& \leq \langle X_N(u), | - \nabla U(\cdot) + (\nabla G_a \ast X_N(u))(\cdot) - \nabla g_N(\cdot, u)\rangle^2 + 1,
\end{align*}
\]
and therefore
\[
\begin{align*}
& \langle X_N(u), -\nabla U(\cdot) + (\nabla G_a \ast X_N(u))(\cdot) - \nabla g_N(\cdot, u)\rangle \\
& \leq \langle X_N(u), | - \nabla U(\cdot) + (\nabla G_a \ast X_N(u))(\cdot) - \nabla g_N(\cdot, u)\rangle^2 + 1.
\end{align*}
\]
This implies that (39) is greater than or equal to
\[
\begin{align*}
& \langle X_N(s), \phi \rangle - c_2\mathbb{E}\left[\int_s^t \langle X_N(u), | - \nabla U(\cdot) + (\nabla G_a \ast X_N(u))(\cdot) - \nabla g_N(\cdot, u)\rangle^2 \right. \\
& \left. \ + 2|du|, \mathcal{F}_s \right] \\
& \geq \langle X_N(s), \phi \rangle - c_3\mathbb{E}\left[\int_s^t \langle X_N(u), | - \nabla U(\cdot) + (\nabla G_a \ast X_N(u))(\cdot) - \nabla g_N(\cdot, u)\rangle^2 \right. \\
& \left. \ + 1|du|, \mathcal{F}_s \right] \\
& \geq \langle X_N(s), \phi \rangle - c_3\mathbb{E}\left[\int_s^t \langle X_N(u), 2| - \nabla U(\cdot) + (\nabla G_a \ast X_N(u))(\cdot)\rangle^2 \right. \\
& \left. \ - ( - \nabla U(\cdot) + (\nabla G_a \ast X_N(u))(\cdot))\nabla g_N(\cdot, u) + |\nabla g_N(\cdot, u)|^2) \right) \ + \sigma_N^2 \\|\nabla h_N(\cdot, u)\|^2 \langle du \right. \\
& \left. \ + \int_s^t |du|, \mathcal{F}_s \right] \\
& = \langle X_N(s), \phi \rangle - c_3\mathbb{E}\left[\langle A_N(t) - A_N(s) + t - s|, \mathcal{F}_s \right] \\
& = \langle X_N(s), \phi \rangle - c_3\mathbb{E}\left[\langle A_N(t) + t|, \mathcal{F}_s \right] + c_3A_N(s) + c_3s.
\end{align*}
\]
Hence,
\[
\begin{align*}
& \mathbb{E}\left[\langle X_N(t), \phi \rangle + c_3A_N(t) + c_3t|, \mathcal{F}_s \right] \geq \langle X_N(s), \phi \rangle + c_3A_N(s) + c_3s
\end{align*}
\]
and (37) follows.

In a completely analogous way (with +c_3 instead of -c_3), we obtain the property (38).

Let us define the sequence of stopped processes \(X_{N,k}(t) = X_N(t \wedge \tau_{N,k}^k), 0 \leq t \leq T, N \in \mathbb{N}, k > 0\) fixed, where \(\tau_{N,k}^k\) is defined by
\[
\begin{align*}
\tau_{N,k}^k &= \inf\{t \geq 0 : ||h_N(\cdot, t)||^2 + A_N(t) \\
& \quad - \int_0^t \langle X_N(u), 2| - \nabla U(\cdot) + (\nabla G_a \ast X_N(u))(\cdot)\rangle^2 \rangle du > k\},
\end{align*}
\]
where \(k \in \mathbb{R}^+\).

We consider a slight modification of the more general characterization of the relative compactness by Ethier-Kurtz (see [8], theorem 8.2). In particular, we prove confining compactness property for the process \(\{X_{N,k}(t)\}\) and then the boundedness of small variations of the process.
Proposition 0.1. For any \( \epsilon > 0 \) there exists a compact \( K^k_\epsilon \) in \( (\mathcal{M}_\mathcal{P}(\mathbb{R}^d), \| \cdot \|_1) \) such that

\[
\inf_{N \in \mathbb{N}} \mathbb{P}\{X_{N,k}(t) \in K^k_\epsilon, \forall t \in [0,T]\} \geq 1 - \epsilon.
\]

Proof.

Let \( B^c_\delta = \{ x \in \mathbb{R}^d : |x| > \lambda, \lambda > 1 \} \);

\[
\langle X_{N,k}(t), \phi \rangle = \int_{\mathbb{R}^d} \phi(x)X_{N,k}(t)(dx) \geq \int_{\{x : |x| > \lambda, \lambda > 1 \}} \phi(x)X_{N,k}(t)(dx)
\]

\[
= \int_{\{x : |x| > \lambda, \lambda > 1 \}} |x|X_{N,k}(t)(dx) \geq \lambda \int_{\{x : |x| > \lambda, \lambda > 1 \}} X_{N,k}(t)(dx)
\]

\[
= \lambda \langle X_{N,k}(t), 1_{B^c_\lambda} \rangle.
\]

If \( \langle X_{N,k}(t), 1_{B^c_\lambda} \rangle > \delta \), then \( \langle X_{N,k}(t), \phi \rangle > \lambda \delta \) and therefore

\[
\mathbb{P}\left\{ \sup_{t \leq T} \langle X_{N,k}(t), 1_{B^c_\lambda} \rangle > \delta \right\} \leq \mathbb{P}\left\{ \sup_{t \leq T} \langle X_{N,k}(t), \phi \rangle > \lambda \delta \right\}
\]

\[
\leq \mathbb{P}\left\{ \sup_{t \leq T} \langle X_{N,k}(t), \phi \rangle + c_3 A_N(t \wedge \tau^k_N) + (t \wedge \tau^k_N)c_3 > \lambda \delta \right\}. \tag{42}
\]

By (37) in Lemma 5.2 and by Doob’s Inequality, (42) is less than or equal to

\[
\frac{1}{\lambda \delta} \left( \mathbb{E}\left[ \langle X_{N,k}(T), \phi \rangle \right] + c_3 \mathbb{E}\left[ A_N(T \wedge \tau^k_N) \right] \right)
\]

\[
\leq \frac{1}{\lambda \delta} \left( \mathbb{E}\left[ \langle X_{N,k}(T), \phi \rangle | F_0 \right] \right) + c_3 \mathbb{E}\left[ A_N(T \wedge \tau^k_N) \right] \tag{43}
\]

\[
+ c_3 \mathbb{E}\left[ T \wedge \tau^k_N \right]
\]

\[
\leq \frac{1}{\lambda \delta} \left( \mathbb{E}\left[ \langle X_{N,k}(0), \phi \rangle \right] + c_3 \mathbb{E}[A_N(T \wedge \tau^k_N)| F_0] + c_3 \mathbb{E}[T \wedge \tau^k_N | F_0] - c_3 A_N(0 \wedge \tau^k_N) - (0 \wedge \tau^k_N)c_3 + c_3 A_N(T \wedge \tau^k_N) + (T \wedge \tau^k_N)c_3 \right). \tag{43}
\]

By the definition of \( \tau^k_N \) and since \( \nabla G_a, \nabla U \in C_b(\mathbb{R}^d, \mathbb{R}_+) \),

\[
A_N(T \wedge \tau^k_N) < k - ||h_N(\cdot, t \wedge \tau^k_N)||^2_2
\]

\[
+ \int_0^{T \wedge \tau^k_N} \langle X_N(u), 2 \rangle \cdot \nabla U(\cdot) \cdot \nabla G_a \cdot X_N(u) ||^2 du
\]

\[
< k + \int_0^{T \wedge \tau^k_N} \langle X_N(u), 2 \rangle \cdot \nabla U(\cdot) \cdot \nabla G_a \cdot X_N(u) ||^2 du
\]

\[
< k + c_4 T; \tag{44}
\]

it follows that (43) is less than or equal to

\[
\frac{1}{\lambda \delta} \left( \mathbb{E}\left[ \langle X_{N,k}(0), \phi \rangle + c_3(k + c_4 T) + c_3 T + c_3(k + c_4 T) + c_3 T \right] \right) \leq \frac{c_5(k, T)}{\lambda \delta}. \tag{45}
\]

As a consequence,

\[
\mathbb{P}\left\{ \sup_{t \leq T} \langle X_{N,k}(t), 1_{B^c_\lambda} \rangle > \delta \right\} \leq \frac{c_5(k, T)}{\lambda \delta}. \tag{46}
\]
Let us now take \( \epsilon > 0 \) and two sequences \( \mu_i \) and \( \delta_i \) of positive numbers such that \( \sum_{i=1}^{\infty} \mu_i = \epsilon \) and \( \delta_i \searrow 0 \). Let \( \lambda_i = \frac{c_5(k,T)}{\mu_i \delta_i} \to \infty \). Then (46) yields
\[
\begin{align*}
\mathbb{P}\left\{ \sup_{t \leq T} \langle X_{N,k}(t), 1_{B_{\lambda_i}^c} \rangle > \delta_i, \ \forall i \in \mathbb{N} \right\} &\leq \sum_{i=1}^{\infty} \mathbb{P}\left\{ \sup_{t \leq T} \langle X_{N,k}(t), 1_{B_{\lambda_i}^c} \rangle > \delta_i \right\} \\
&\leq \sum_{i=1}^{\infty} \frac{c_5(k,T)}{\lambda_i \delta_i} = \sum_{i=1}^{\infty} \mu_i = \epsilon.
\end{align*}
\] (47)

By Prohorov’s Theorem, the set
\[
K_{\epsilon}^k = \{ \mu \in \mathcal{M}_\mathbb{P}(\mathbb{R}^d) : \langle \mu, 1_{B_{\lambda_i}^c} \rangle \leq \delta_i, \ \forall i \in \mathbb{N} \}
\]
is compact in \( \mathcal{M}_\mathbb{P}(\mathbb{R}^d) \); since
\[
\mathbb{P}\left\{ \sup_{t \leq T} \langle X_{N,k}(t), 1_{B_{\lambda_i}^c} \rangle > \delta_i, \forall i \in \mathbb{N} \right\} = 1 - \mathbb{P}\left\{ \langle X_{N,k}(t), 1_{B_{\lambda_i}^c} \rangle \leq \delta_i, \forall i \in \mathbb{N}, \forall t \in [0, T] \right\},
\]
by (47), \( \forall \epsilon > 0 \) there exists a compact set \( K_{\epsilon}^k \subset \mathcal{M}_\mathbb{P}(\mathbb{R}^d) \) such that
\[
\inf_{N \in \mathbb{N}} \mathbb{P}\{ X_{N,k}(t) \in K_{\epsilon}^k, \ \forall t \in [0, T] \} \geq 1 - \epsilon.
\]

Next proposition states that for little time variations we have little variations of the processes \( \{X_{N,k}(t)\} \).

**Proposition 0.2.** For any \( 0 < \delta < 1 \), there exists a sequence \( \{\gamma_n^T(\delta)\}_{n \in \mathbb{N}} \) of non negative random variables such that
\[
\mathbb{E} \left[ ||X_{N,k}(t + \delta) - X_{N,k}(t)||_4^4 \right] \leq \mathbb{E} \left[ \gamma_n^T(\delta) \right] \quad 0 \leq t \leq T
\]
and
\[
\lim_{\delta \to 0} \lim_{n \to \infty} \sup \mathbb{E}[\gamma_n^T(\delta)] = 0.
\] (49)

**Proof.**
\[||X_{N,k}(t) - X_{N,k}(s)||_1 = \sup_{f \in \mathcal{H}_1} \frac{1}{N} \sum_{i=1}^{N} (f(X_{N,k}^i(t)) - f(X_{N,k}^i(s))) \]

\[\leq \frac{1}{N} \sum_{i=1}^{N} |X_{N,k}^i(t) - X_{N,k}^i(s)| \]

\[= \frac{1}{N} \sum_{i=1}^{N} |X^i_N(t \wedge \tau^k_N) - X^i_N(s \wedge \tau^k_N)| \]

\[= \frac{1}{N} \sum_{i=1}^{N} \int_{t \wedge \tau^k_N}^{s \wedge \tau^k_N} \frac{\partial U(X^i_N(u)) + (\nabla G_a \ast X_N(u))(X^i_N(u)) - \nabla g_N(X^i_N(u), u)du}{||} \]

\[+ \sigma_N(W^i(t \wedge \tau^k_N) - W^i(s \wedge \tau^k_N)) \]

\[\leq \frac{c_2}{N} \sum_{i=1}^{N} \left( \int_{s \wedge \tau^k_N}^{t \wedge \tau^k_N} du \right) + \frac{1}{N} \sum_{i=1}^{N} \int_{s \wedge \tau^k_N}^{t \wedge \tau^k_N} \left| (\nabla G_a \ast X_N(u))(X^i_N(u)) - \nabla g_N(X^i_N(u), u) \right| du \]

\[+ \sigma_N \sum_{i=1}^{N} \left| W^i(t \wedge \tau^k_N) - W^i(s \wedge \tau^k_N) \right| \]

By the Cauchy-Schwartz and Jensen inequalities,

\[\int_{s \wedge \tau^k_N}^{t \wedge \tau^k_N} \langle X_N(u), ||(\nabla G_a \ast X_N(u))(\cdot) - \nabla g_N(\cdot, u)|| \rangle du \]

\[\leq \left( \int_{s \wedge \tau^k_N}^{t \wedge \tau^k_N} du \right)^{1/2} \left( \int_{s \wedge \tau^k_N}^{t \wedge \tau^k_N} \langle X_N(u), ||(\nabla G_a \ast X_N(u))(\cdot) - \nabla g_N(\cdot, u)|| \rangle^2 du \right)^{1/2} \]

\[\leq \left( \int_{s \wedge \tau^k_N}^{t \wedge \tau^k_N} du \right)^{1/2} \left( \int_{s \wedge \tau^k_N}^{t \wedge \tau^k_N} \langle X_N(u), ||(\nabla G_a \ast X_N(u))(\cdot) - \nabla g_N(\cdot, u)||^2 \rangle du \right)^{1/2} ; \]

moreover, if \( s \leq \tau^k_N \),

\[\int_{s \wedge \tau^k_N}^{t \wedge \tau^k_N} du = \int_{s \wedge \tau^k_N}^{t \wedge \tau^k_N} du \leq \int_{s \wedge \tau^k_N}^{t \wedge \tau^k_N} du = t - s \]

and if \( s > \tau^k_N \),

\[\int_{s \wedge \tau^k_N}^{t \wedge \tau^k_N} du = \int_{s \wedge \tau^k_N}^{\tau^k_N} du = 0 \leq t - s. \]
Therefore, by (52), (53) and (54), (50) is less than or equal to

\[ c_7 |t - s| + (t - s)^{1/2} \left( \int_0^{t \wedge \tau_N^k} \langle X_N(u), |(\nabla G_a \ast X_N(u))| \rangle \; du \right)^{1/2} \]

\[ + \frac{\sigma_N}{N} \sum_{i=1}^N \left| W^i(t \wedge \tau_N^k) - W^i(s \wedge \tau_N^k) \right| \]

\[ \leq c_7 |t - s| + (t - s)^{1/2} \left( \int_0^{t \wedge \tau_N^k} \langle X_N(u), 2|\nabla G_a \ast X_N(u)| \rangle^2 \; du \right) \]

\[ - 2((\nabla G_a \ast X_N(u))|\nabla g_N(\cdot, u)| + 2|\nabla g_N(\cdot, u)|^2) \]

\[ + \sigma_N^2 \| \nabla h_N(\cdot, T \wedge \tau_N^k) \|_2^2 \; du + \| h_N(\cdot, T \wedge \tau_N^k) \|_2^2 \]^{1/2}

\[ + \frac{\sigma_N}{N} \sum_{i=1}^N \left| W^i(t \wedge \tau_N^k) - W^i(s \wedge \tau_N^k) \right| \]

\[ = c_7 |t - s| + (t - s)^{1/2} \left( A_N(T \wedge \tau_N^k) + \| h_N(\cdot, T \wedge \tau_N^k) \|_2^2 \right)^{1/2} \]

\[ + \frac{\sigma_N}{N} \sum_{i=1}^N \left| W^i(t \wedge \tau_N^k) - W^i(s \wedge \tau_N^k) \right| \]

\[ \leq c_7 |t - s| + (t - s)^{1/2} \left( k + \int_0^{T \wedge \tau_N^k} \langle X_N(s), 2|\nabla G_a \ast X_N(u)| \rangle^2 \; ds \right)^{1/2} \]

\[ + \frac{\sigma_N}{N} \sum_{i=1}^N \left| W^i(t \wedge \tau_N^k) - W^i(s \wedge \tau_N^k) \right| \]

\[ \leq c_7 |t - s| + (t - s)^{1/2}(k + c_6 T)^{1/2} + \frac{\sigma_N}{N} \sum_{i=1}^N \left| W^i(t \wedge \tau_N^k) - W^i(s \wedge \tau_N^k) \right|. \]

(55)

It follows that

\[ \| X_{N,k}(t) - X_{N,k}(s) \|_1^t \leq 2^6 \left[ c_7^4 (t - s)^4 + (t - s)^2(k + c_6 T)^2 + \frac{\sigma_N^4}{N^3} (t - s)^2 \right]. \]

(56)

As a consequence, for \( 0 \leq s < t \leq T \),

\[ \mathbb{E}[\| X_{N,k}(t) - X_{N,k}(s) \|_1^t] \leq 2^6 \mathbb{E} \left[ c_7^4 (t - s)^4 + (t - s)^2(k + c_6 T)^2 + \frac{\sigma_N^4}{N^3} (t - s)^2 \right]; \]

in particular, with \( t - s = \delta \) and

\[ \gamma_N^T : \delta \mapsto 2^6 \left[ c_7^4 \delta^4 + \delta^2(k + c_6 T)^2 + \frac{\sigma_N^4}{N^3} \delta^2 \right], \]

we obtain

\[ \mathbb{E} \left[ \| X_{N,k}(t) - X_{N,k}(s) \|_1^t \right] \leq \mathbb{E} \left[ \gamma_N^T(\delta) \right] \]
and
\[ \lim_{\delta \to 0} \sup_{N \in \mathbb{N}} \mathbb{E}[\gamma_N^T(\delta)] = 0. \]

**Proposition 0.3.** \( \mathcal{L}(X_N(\cdot \wedge \tau_N^k)) \) \( \{N \in \mathbb{N}\} \), the sequence of probability laws of the processes \( \{X_N((t \wedge \tau_N^k)), 0 \leq t \leq T\} \) is relatively compact in \( \mathcal{M}_P(C([0, T], \mathcal{M}_P(\mathbb{R}^d))) \).

*Proof.*

It is an obvious consequence of Proposition 0.2 and Theorem 8.6 p.137, in [8].

**Proposition 0.4.** For any \( \tau \) such that \( 0 < \tau < \infty \),

\[ \lim_{k \to \infty} \inf_{N \in \mathbb{N}} \mathbb{P}\{\tau_N^k > \tau\} = 1. \]

*Proof.*

By Lemma 0.1, the process

\[ t \mapsto S_N(t) = ||h_N(\cdot, t)||_2^2 + A_N(t) - \int_0^t \langle X_N(u), 2| - \nabla U(\cdot) + (\nabla G_a * X_N(u))(\cdot)|^2 \rangle du \]

is a submartingale.

By Doob’s inequality

\[
\mathbb{P}\left\{\sup_{t \leq \tau} S_N(t) > k\right\} \leq \frac{1}{k} \mathbb{E}[S_N(\tau)] = \frac{1}{k} \mathbb{E}[M_N(\tau) + \tau \sigma_N^2 N^\beta(d+2)/d-1 c_1]
\]

\[
= \frac{1}{k} \mathbb{E}\left[\mathbb{E}[M_N(\tau) | \mathcal{F}_0] + \tau \sigma_N^2 N^\beta(d+2)/d-1 c_1\right]
\]

\[
= \frac{1}{k} \mathbb{E}[M_N(0) + \tau \sigma_N^2 N^\beta(d+2)/d-1 c_1]
\]

\[
= \frac{1}{k} \left( \mathbb{E}[||h_N(\cdot, 0)||_2^2] + \tau \sigma_N^2 N^\beta(d+2)/d-1 c_1 \right); \tag{57}
\]

since \( \lim_{N \to \infty} \sigma_N = \sigma_{\infty} \geq 0 \), by (22) and (29) or (30), (57) is less than or equal to \( c_8(\tau)/k \), uniformly in \( N \).

The thesis follows.

**Remark 0.2.** Proposition 0.4 implies that \( t \wedge \tau_N^k = t \), for any \( \tau \) such that \( 0 \leq t \leq \tau \).

*Proof of Theorem 0.1.*

At this point Theorem 0.1 simply follows from Propositions 0.3, 0.4 and Remark 0.2.
Theorem 0.1 implies the existence of a subsequence $N_k \in \mathbb{N}$, $N_1 < N_2 < \ldots$, such that the sequence $\{\mathcal{L}(X_{N_k})\}_{k \in \mathbb{N}}$ converges in $\mathcal{M}_p(C([0, T], \mathbb{R}^d))$ to some limit $\mathcal{L}(X)$, which is the distribution of some process $\{X(t), t \in [0, T]\}$, with trajectories in $C([0, T], \mathcal{M}_p(\mathbb{R}^d))$. We discuss the uniqueness of the limit later on. By now we assume the uniqueness, so that $\{N_k\} = \mathbb{N}$.

By Skorokhod's Theorem, we are allowed to assume that $\{X_N(t), t \in [0, T]\}$ converges $\mathbb{P}$-almost surely to $\{X(t), t \in [0, T]\}$ as $N$ grows to infinity. So, we have

$$\lim_{N \to \infty} \sup_{t \leq T} ||X_N(t) - X(t)||_1 = 0 \quad \mathbb{P} \text{-a.s.}$$

(58)

**Absolute continuity of the limit**

Next proposition deals with the regularity properties of the limit measure $X(t)$. We consider the viscous case $\lim_{N \to \infty} \sigma_N = \sigma_\infty > 0$.

**Proposition 0.5.** Suppose that $\lim_{N \to \infty} \sigma_N = \sigma_\infty > 0$. For any $t \geq 0$, the measure $X(t)$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d$ with a density $\rho \in L^2(\mathbb{R}^d, \mathbb{R}_+)$.

**Proof.**

We begin by showing that there exists a positive function $\rho(x, t)$ such that

$$\lim_{N \to \infty} \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d} |h_N(x, t) - \rho(x, t)|^2 \, dx \, dt \right] = 0.$$

(59)

$$\lim_{N, N' \to \infty} \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d} |h_N(x, t) - h_{N'}(x, t)|^2 \, dx \, dt \right]$$

$$= \lim_{N, N' \to \infty} \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d} |\tilde{h}_N(\lambda, t) - \tilde{h}_{N'}(\lambda, t)|^2 \, d\lambda \, dt \right]$$

$$\leq \lim_{N, N' \to \infty} \mathbb{E} \left[ \int_0^T \int_{\{\lambda \leq k\}} |\tilde{h}_N(\lambda, t) - \tilde{h}_{N'}(\lambda, t)|^2 \, d\lambda \, dt \right] + 2 \lim_{N, N' \to \infty} \mathbb{E} \left[ \int_0^T \int_{\{\lambda > k\}} |\tilde{h}_N(\lambda, t)|^2 + |\tilde{h}_{N'}(\lambda, t)|^2 \, d\lambda \, dt \right];$$

(60)

since, by [20], $|\tilde{W}_N(\lambda)|$ is bounded and $\tilde{h}_N(\lambda, t) = \langle X_N(t), e^{i \lambda} \rangle \tilde{W}_N(\lambda)$, expression (60) is less than or equal to

$$\lim_{N, N' \to \infty} \mathbb{E} \left[ \int_0^T \int_{\{\lambda \leq k\}} |\langle X_N(t), e^{i \lambda} \rangle - \langle X_{N'}(t), e^{i \lambda} \rangle|^2 \, d\lambda \, dt \right]$$

$$+ 2 \lim_{N, N' \to \infty} \mathbb{E} \left[ \int_0^T \int_{\{\lambda > k\}} \frac{|\lambda|^2}{k} |\tilde{h}_N(\lambda, t)|^2 + \frac{|\lambda|^2}{k} |\tilde{h}_{N'}(\lambda, t)|^2 \, d\lambda \, dt \right]$$

$$\leq \lim_{N, N' \to \infty} \mathbb{E} \left[ \int_0^T \int_{\{\lambda \leq k\}} |\langle X_N(t), e^{i \lambda} \rangle - \langle X_{N'}(t), e^{i \lambda} \rangle|^2 \, d\lambda \, dt \right]$$

$$+ \frac{2}{k} \lim_{N, N' \to \infty} \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d} |\lambda|^2 |\tilde{h}_N(\lambda, t)|^2 + |\lambda|^2 |\tilde{h}_{N'}(\lambda, t)|^2 \, d\lambda \, dt \right].$$

(61)
Since by (58)
\[ \lim_{N,N' \to \infty} \sup_{t \leq T} \sup_{|\lambda| \leq k} |\langle X_N(t), e^{i\lambda} \rangle - \langle X_{N'}(t), e^{i\lambda} \rangle| = 0, \quad \forall k > 0 \quad \mathbb{P} - a.s., \]
expression (61) is equal to
\[ \frac{2}{k} \lim_{N,N' \to \infty} \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d} |\lambda|^2 |\tilde{h}_N(\lambda, t)|^2 + |\lambda|^2 |\tilde{h}_{N'}(\lambda, t)|^2 d\lambda dt \right]. \quad (62) \]
Now
\[ \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d} |\lambda|^2 |\tilde{h}_N(\lambda, t)|^2 + |\lambda|^2 |\tilde{h}_{N'}(\lambda, t)|^2 d\lambda dt \right] = \mathbb{E} \left[ \int_0^T ||\nabla h_N(\cdot, t)||^2 dt + \int_0^T ||\nabla h_{N'}(\cdot, t)||^2 dt \right]; \quad (63) \]
by (57), with $A_N(t)$ as defined in (31) and $S_N(t)$ defined in Proposition 0.4, we obtain that
\[ \mathbb{E} \left[ ||\nabla h_N(\cdot, t)||^2 \right] \leq \frac{\mathbb{E} [S_N(T)]}{\sigma_N^2} + \frac{cT}{\sigma_N^2} < \infty \quad (64) \]
uniformly in $N$ with $c$ positive constant. As a consequence (63) is finite.

It follows that, for $k$ sufficiently large, (62) can be made smaller than any given $\varepsilon > 0$ and there exists a positive function $\rho(x,t) \in L^2(\mathbb{R}^d, \mathbb{R}_+)$ satisfying equation (59).

Since by (59) and $\lim_{N \to \infty} W_N(\cdot) = \delta_0$ (in the sense of distributions)
\[ \lim_{N \to \infty} \int_{\mathbb{R}^d} f(x) X_N(t)(dx) = \int_{\mathbb{R}^d} f(x) \rho(x,t) dx \quad f \in C_0^0(\mathbb{R}^d \times [0,T]) \quad \mathbb{P} - a.s., \]
we have by (58)
\[ \int_{\mathbb{R}^d} f(x,t) X(t)(dx) = \int_{\mathbb{R}^d} f(x,t) \rho(x,t) dx \quad f \in C_0^0(\mathbb{R}^d \times [0,T]), \mathbb{P} - a.s. \]
Therefore the measure $X(t)$ is absolutely continuous with respect to the Lebesgue measure with density $\rho(x,t)$.

As a consequence of Proposition (0.5)
\[ \lim_{N \to \infty} \langle X_N(t), f(\cdot) \rangle = \langle X(t), f(\cdot) \rangle = \int_{\mathbb{R}^d} f(x) \rho(x,t) dx \quad f \in C_0^0(\mathbb{R}^d), t \in [0,T] \]
(65)

As next point we need the description of the dynamics governing the time evolution of the possible limit process $\{X(t), t \in [0,T]\}$.

**A formal derivation of the continuum models**

In this section, following [12], we characterize the limit behavior, as $N \to \infty$, of the process $X_N$ both in the case $\lim_{N \to \infty} \sigma_N = 0$ and $\lim_{N \to \infty} \sigma_N = \sigma_\infty > 0$. 

\[ \blacksquare \]
By taking into account expression (14) and by using Ito’s formula we get the following weak form of the time evolution of $X_N(t)$:

$$
\langle X_N(t), f(\cdot, t) \rangle = \langle X_N(0), f(\cdot, 0) \rangle + \int_0^t \langle X_N(s), (\nabla G_a \ast X_N(s)) \cdot \nabla f(\cdot, s) \rangle ds \\
- \int_0^t \langle X_N(s), \nabla g_N(\cdot, s) \cdot \nabla f(\cdot, s) \rangle ds \\
- \int_0^t \langle X_N(s), \nabla U(\cdot) \cdot \nabla f(\cdot, s) \rangle ds \\
+ \int_0^t \langle X_N(s), \frac{1}{2} \sigma_N^2 \Delta f(\cdot, s) + \frac{\partial}{\partial s} f(\cdot, s) \rangle ds \\
+ \frac{\sigma_N}{N} \int_0^t \sum_{k=1}^N \nabla f(X_N^k(s), s) dW_k(s), \quad f \in C^2_b([0, T]).
$$

(66)

Last term in (66)

$$
M_N(f, t) := \frac{\sigma_N}{N} \int_0^t \sum_{k=1}^N \nabla f(X_N^k(s), s) dW_k(s)
$$

is a martingale with respect to the natural filtration of the process $\{X_N(t), t \in [0, T]\}$ and the quadratic variation

$$
\lim_{N \to \infty} \mathbb{E} \left[ \sup_{t \leq T} |M_N(f, t)|^2 \right] = 0
$$

(67)

(see [12, 14]). This implies, in both cases, convergence to zero in probability, that is the substantial reason of the deterministic limiting behavior of the process, as $N \to \infty$, since in this limit the evolution equation of the process will not contain the Brownian noise anymore (see [14]).

In order to derive a formal limit for the process $X_N$ also when $\lim_{N \to \infty} \sigma_N = 0$, let us assume that $X(t)$ admits density with respect to the Lebesgue measure also in this case. As a formal consequence of this assumption and (65), we get

$$
\lim_{N \to \infty} g_N(x, t) = \lim_{N \to \infty} (V_N \ast X_N(t))(x) = \rho(x, t), \\
\lim_{N \to \infty} \nabla g_N(x, t) = \nabla \rho(x, t), \\
\lim_{N \to \infty} (\nabla G_a \ast X(t))(x) = (\nabla G_a \ast X(t))(x) \\
= \int \nabla G_a(x - y) \rho(y, t) dy, \quad x \in \mathbb{R}^d, t \in [0, T].
$$

Hence by applying the above limits, from (66) and the hypothesis
\[ \lim_{N \to \infty} \sigma_N = \sigma_\infty \geq 0, \] we get the following equation

\[
\int_{\mathbb{R}^d} f(x,t)\rho(x,t)dx = \int_{\mathbb{R}^d} f(x,0)\rho(x,0)dx \\
+ \int_0^t ds \int_{\mathbb{R}^d} [\nabla (\nabla f(x,s))\rho(x,s) - \nabla f(x,s)\rho(x,s)] dx \\
+ \int_0^t ds \int_{\mathbb{R}^d} \left[ \frac{\partial}{\partial s} f(x,s)\rho(x,s) + \frac{\sigma_\infty^2}{2} \Delta f(x,s)\rho(x,s) \right] dx.
\]

Equation (68) is the weak version of the following equation for the spatial density \( \rho \):

\[
\frac{\partial}{\partial t} \rho(x,t) = \frac{\sigma_\infty^2}{2} \Delta \rho(x,t) + \nabla \cdot (\rho(x,t)\nabla \rho(x,t)) + \nabla \cdot (\rho(x,t)\nabla U(x)) \\
- \nabla \cdot \left[ \rho(x,t)(\nabla G_a * \rho(\cdot, t))(x) \right], \quad x \in \mathbb{R}^d, t \in [0,T],
\]

\[
\rho(x,0) = \rho_0(x), \quad x \in \mathbb{R}^d.
\]

Equation (68) is the weak version of the following equation for the spatial density \( \rho \):

\[
\frac{\partial}{\partial t} \rho(x,t) = \frac{\sigma_\infty^2}{2} \Delta \rho(x,t) + \nabla \cdot (\rho(x,t)\nabla \rho(x,t)) + \nabla \cdot (\rho(x,t)\nabla U(x)) \\
- \nabla \cdot \left[ \rho(x,t)(\nabla G_a * \rho(\cdot, t))(x) \right], \quad x \in \mathbb{R}^d, t \in [0,T],
\]

\[
\rho(x,0) = \rho_0(x), \quad x \in \mathbb{R}^d.
\]

Obviously if \( \sigma_\infty = 0 \) the diffusive term in (68) and (69) vanishes, while if \( \sigma_\infty > 0 \) the dynamics of the density is smoothed by the diffusive term. This is due to the memory of the fluctuations existing when the number of particles \( N \) is finite.

**Main results**

In the present section we present the main results of this chapter, namely two theorems on the convergence of the interacting particle system (9) to the integro-differential equation (69), both for \( \sigma_\infty = 0 \) and \( \sigma_\infty > 0 \).

We begin with the case \( \sigma_\infty = 0 \) (non-viscous case) following the approach proposed in [12] and then we move to the case \( \sigma_\infty > 0 \) (viscous case).

**Non-viscous case**

We are not aware of general results concerning the existence of sufficiently regular solutions \( \rho \) for equation (69); therefore we need the following assumption:

**Assumption 0.1.** For some \( T \in [0,\infty) \) system (69) with \( \sigma_\infty = 0 \) admits a unique, nonnegative solution \( \rho \in C^{(d+2)/d+1.1}([0,T]) \).

About the uniqueness of the solution of equation (69) without confining potential we address to [4].

Let \( \sigma_\infty = 0 \) and suppose that

\[
\hat{\nabla} W_1 \in L^\infty(\mathbb{R}^d),
\]

\[
|W_1(x)| \leq \frac{c}{1 + |x|^{d+2}}, \quad \text{if} \quad |x| \geq 1.
\]
Consider the following assumption for the aggregation kernel $G_a(x)$ and the confining potential $U(x)$:

$$\nabla G_a, \nabla U \in C_b^{((d+2)/d)+1}(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$$ (72)

As far as $\beta$ is concerned, we need to assume that one of the following conditions is satisfied:

$$0 < \beta < \frac{d}{d+2}.$$ (73)

or

$$\frac{d}{d+2} \leq \beta < 1 \quad \text{and} \quad \lim_{N \to +\infty} \sigma_N N^{\beta(d+2)/d-1} = 0.$$ (74)

Under previous hypotheses, we can prove the following theorem:

**Theorem 0.2.** Assume (70)-(74) and Assumption 0.1. If

$$\lim_{N \to \infty} \mathbb{E} \left[ ||h_N(\cdot, 0) - \rho_0(\cdot)||_2^2 \right] = 0,$$

then

$$\lim_{N \to \infty} \mathbb{E} \left[ \sup_{t \leq T} ||h_N(\cdot, t) - \rho(\cdot, t)||_2^2 \right] = 0,$$ (75)

where $\rho$ is the unique solution of (69) with $\sigma_\infty = 0$.

**Corollary 0.1.** Equation (75) implies

$$\lim_{N \to \infty} \langle X_N(t), f \rangle = \langle X(t), f \rangle = \int f(x) \rho(x, t) dx$$

uniformly in $t \in [0, T]$, for any $f \in C_b^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$.

**Proof.**

$$||X_N(t) - \rho(\cdot, t), f|| \leq ||h_N(\cdot, t) - \rho(\cdot, t), f|| + ||X_N(t), |f - f \ast W_N||.$$ From (75) and Lemma 0.3 we obtain our thesis.

Previous corollary state that the empirical measure $X_N(t)$ converges weakly to the density $\rho(\cdot, t)$.

**Proof of Theorem 0.2**

We prove this result by following the same approach used in [12] for proving it in the case of system (9) without confining potential ($\gamma_1 = 0$).

We obtain the convergence of $h_N(\cdot, t)$ to its limit by performing the following steps:

1. we guess the dynamics (69) for the limit $\rho$;

2. we try to control $||h_N(\cdot, t) - \rho(\cdot, t)||_2^2$ in term of its initial value $||h_N(\cdot, 0) - \rho_0(\cdot)||_2^2$ by writing down Ito’s formula for that process and by estimating the different contributions.
We have

$$
\| h_N(\cdot, t) - \rho(\cdot, t) \|^2 = \| h_N(\cdot, t) \|^2 + \| \rho(\cdot, t) \|^2 - 2 \langle \rho(\cdot, t), h_N(\cdot, t) \rangle. 
$$

(76)

From (68) and integration by parts, one gets:

$$
\| \rho(\cdot, t) \|^2 = \| \rho(\cdot, 0) \|^2 
+ \int_0^t ds \int_{\mathbb{R}^d} \left[ (\nabla G_a * \rho(\cdot, s))(x) - \nabla \rho(x, s) - \nabla U(x) \right] \cdot \nabla \rho(x, s) \rho(x, s) dx
+ \int_0^t ds \int_{\mathbb{R}^d} \frac{\partial}{\partial s}(\rho(x, s)) \rho(x, s) dx
= \| \rho(\cdot, 0) \|^2 
- \int_0^t ds \int_{\mathbb{R}^d} \nabla \cdot [\rho(x, s)(\nabla G_a * \rho(\cdot, s))(x)] \rho(x, s) dx
+ \int_0^t ds \int_{\mathbb{R}^d} \nabla \cdot [\nabla \rho(x, s) \rho(x, s)] \rho(x, s) dx
+ \int_0^t ds \int_{\mathbb{R}^d} \nabla \cdot [\nabla U(x) \rho(x, s)] \rho(x, s) dx
+ \int_0^t ds \int_{\mathbb{R}^d} \nabla \cdot [\nabla \rho(x, s) \rho(x, s)] \rho(x, s) \nabla \rho(x, s) \rho(x, s)
- \nabla \cdot [\rho(x, s)(\nabla G_a * \rho(\cdot, s))(x)] \rho(x, s) dx

= \| \rho(\cdot, 0) \|^2 
- 2 \int_0^t ds \int_{\mathbb{R}^d} \nabla \cdot [\rho(x, s)(\nabla G_a * \rho(\cdot, s))(x)] \rho(x, s) dx
+ 2 \int_0^t ds \int_{\mathbb{R}^d} \nabla \cdot [\nabla \rho(x, s) \rho(x, s)] \rho(x, s) dx
+ 2 \int_0^t ds \int_{\mathbb{R}^d} \nabla \cdot [\nabla U(x) \rho(x, s)] \rho(x, s) dx
= \| \rho(\cdot, 0) \|^2 + 2 \int_0^t \langle \rho(\cdot, s), \rho(\cdot, s) * \nabla G_a \rangle \cdot \nabla \rho(\cdot, s) ds
- 2 \int_0^t \langle \rho(\cdot, s), \nabla \rho(\cdot, s) \rangle^2 ds - 2 \int_0^t \langle \rho(\cdot, s), \nabla U(\cdot) \cdot \nabla \rho(\cdot, s) \rangle ds. 
$$

(77)

With the same computations used to obtain expression (32), for the first term
of (76) one obtains
\[
||h_N(\cdot, t)||_2^2 = ||h_N(\cdot, 0)||_2^2 - 2 \int_0^t \langle X_N(u), \nabla g_N(\cdot, s) \rangle du \\
- 2 \int_0^t \langle X_N(u), \nabla g_N(\cdot, u) \cdot \nabla U \rangle du \\
+ 2 \int_0^t \langle X_N(u), \nabla g_N(\cdot, u) \cdot (\nabla G_a * X_N(u)) \rangle du \\
- \sigma_N^2 \int_0^t ||\nabla h_N(\cdot, u)||_2^2 du \\
+ \frac{2\sigma_N^2}{N} \int_0^t \sum_{k=1}^N \nabla (\rho(\cdot, s) * W_N)(X_k^N(s), s) dW^k(s) \\
+ c_1 \sigma_N^2 (t - s) N^{\beta(d+2)/d-1}.
\]

From (66), (69), the symmetry of $W_N$, by Itô’s formula and integration by parts, one obtains
\[
\langle \rho(\cdot, t), h_N(\cdot, t) \rangle = \langle X_N(t), \rho(\cdot, t) * W_N \rangle \\
= \langle \rho(\cdot, 0), h_N(\cdot, 0) \rangle \\
+ \int_0^t \langle X_N(t), [(X_N(s) * \nabla G_a) - \nabla g_N(\cdot, s) - \nabla U(x)] \cdot \nabla (\rho(\cdot, t) * W_N) \rangle ds \\
+ \frac{\sigma_N^2}{2} \int_0^t \langle X_N(s), \Delta \rho(\cdot, s) * W_N \rangle ds + \int_0^t \langle X_N(s), \frac{\partial}{\partial s} \rho(\cdot, s) * W_N \rangle ds \\
+ \frac{\sigma_N^2}{N} \int_0^t \sum_{k=1}^N \nabla (\rho(\cdot, s) * W_N)(X_k^N(s), s) dW^k(s)
\]
\[
= \langle \rho(\cdot, 0), h_N(\cdot, 0) \rangle \\
+ \int_0^t \langle X_N(t), [(X_N(s) * \nabla G_a) - \nabla g_N(\cdot, s) - \nabla U(x)] \cdot \nabla \rho(\cdot, t) * W_N \rangle ds \\
+ \frac{\sigma_N^2}{2} \int_0^t \langle X_N(s), \Delta \rho(\cdot, s) * W_N \rangle ds \\
+ \int_0^t \langle X_N(s), [\nabla \cdot (\rho(\cdot, s) \nabla \rho(\cdot, s))] + \nabla \cdot (\rho(\cdot, s) \nabla U) \\
- \nabla \cdot [\rho(\cdot, s)(\nabla G_a * \rho(\cdot, s))] * W_N \rangle ds + \frac{\sigma_N^2}{N} \int_0^t \sum_{k=1}^N \nabla (\rho(\cdot, s) * W_N)(X_k^N(s), s) dW^k(s)
\]
The constant $c$ Furthermore if $f$, $x$ any Lemma 0.4. Let us suppose Assumption 0.1 and let be (71) and let $W$ Lemma 0.3. In order to get estimates for the terms on the right hand side of (80) we need Estimates for the terms on the right side of (80)

It follows that for (76) one gets the following expression:

$$
||h_N(\cdot, t) - \rho(\cdot, t)||_2^2 = ||h_N(\cdot, 0) - \rho(\cdot, 0)||_2^2 - 2 \int_0^t \left[ (X_N(s), \nabla g_N(\cdot, s) \cdot (\nabla g_N(\cdot, s) - \nabla \rho(\cdot, s) * W_N) \right]
+ \langle \rho(\cdot, s), \nabla \rho(\cdot, s) \cdot [\nabla \rho(\cdot, s) - \nabla h_N(\cdot, s)] \rangle ds
+ 2 \int_0^t \left[ (X_N(s), ((X_N(s) * \nabla G_a) - \nabla U) \nabla g_N(\cdot, s) - \nabla \rho(\cdot, s) * W_N) \right] 
+ \langle \rho(\cdot, s), ((\rho(\cdot, s) * \nabla G_a) - \nabla U) \nabla \rho(\cdot, s) - \nabla h_N(\cdot, s) \rangle ds
- \sigma_N^2 \int_0^t \left[ ||\nabla h_N(\cdot, s)||_2^2 - ||\nabla h_N(\cdot, s), \nabla \rho(\cdot, s) || \right] ds + c_1 \sigma_N^2(t - s) N^{\beta(d+2)/d-1}
+ 2 \sigma_N^2 \int_0^t \sum_{k=1}^N \langle \nabla g_N(\cdot, s) - \nabla \rho(\cdot, s) * W_N)(X_N^k(s)) \rangle dW^k(s). \quad (80)

Estimates for the terms on the right side of (80)

In order to get estimates for the terms on the right hand side of (80) we need the following lemmas:

**Lemma 0.3.** Let $W_1$ be a symmetric density function which satisfies (77) and (71) and let $W_N$ be defined as in (17). Let $f, \nabla f \in C_b^1(\mathbb{R}^d)$. Then for any $x \in \mathbb{R}^d$

$$
|f(x) - (f * W_N)(x)| \leq c_2 \chi_N^{-1} ||\nabla f||_\infty.
$$

Furthermore if $f, \nabla f \in L^2(\mathbb{R}^d)$, then

$$
||f - f * W_N||_2 \leq c_2 \chi_N^{-2} ||\nabla f||_2^2.
$$

The constant $c_2$ is independent of $f$.

**Lemma 0.3** is proved in (17).

**Lemma 0.4.** Let us suppose Assumption 0.1 and let be $L = [(d+2)/2]$ and $v(x) \in C_b^L(\mathbb{R}^d, \mathbb{R}^d)$, $G(x) \in L^1(\mathbb{R}^d, \mathbb{R})$. 

Then we have

\[ |(X_N(s) - \rho(\cdot, s), (\nabla[h_N(\cdot, s) - \rho(\cdot, s)] \ast W_N) \cdot v)| \]
\[ \leq c_3 \left( ||h_N(s) - \rho(\cdot, s)||_2^2 + \chi_N^{-2} + N^{\beta(1-2L/d)} \right), \]

\[ |\langle X_N(s) - \rho(\cdot, s) \rangle G_r (\nabla[h_N(\cdot, s) - \rho(\cdot, s)] \ast W_N) \cdot v| \]
\[ \leq c_4 \left( ||h_N(s) - \rho(\cdot, s)||_2^2 + \chi_N^{-2} + N^{\beta(1-2L/d)} \right). \]

For a proof of this lemma see [12]. We begin by considering the second term on the right hand side of (80); since this term does not depend on the potential \( U \), we can recall a result proved in [12]. Indeed, as showed in [12], by considering \( v = \nabla \rho(\cdot, s) \ast W_N \), from Lemma 0.3 Lemma 0.4 and Assumption 0.1 we have

\[ A := |\langle X_N(s), (\nabla g_N(\cdot, s) \cdot (\nabla g_N(\cdot, s) - \nabla \rho(\cdot, s) \ast W_N)) \]
\[ + \langle \rho(\cdot, s), \nabla \rho(\cdot, s) \cdot [\nabla \rho(\cdot, s) - \nabla h_N(\cdot, s)]\rangle| \]
\[ \leq c_5 ||h_N(\cdot, s) - \rho(\cdot, s)||_2^2 + \chi_N^{-2} + N^{\beta(1-2L/d)}. \]  

(81)

Now let us consider the term in [80] involving the aggregating kernel \( G_a \) and the confining potential \( U \)

\[ B = \langle X_N(s), ((X_N(s) \ast \nabla G_a) - \nabla U) \cdot [\nabla (h_N(\cdot, s) - \rho(\cdot, s)) \ast W_N] \]
\[ - \langle \rho(\cdot, s), ((\rho(\cdot, s) \ast \nabla G_a) - \nabla U) \cdot [-\nabla \rho(\cdot, s) + \nabla h_N(\cdot, s)]\rangle \]
\[ = \langle X_N(s) - \rho(\cdot, s), ((X_N(s) \ast \nabla G_a) - \nabla U) \cdot [\nabla (h_N(\cdot, s) - \rho(\cdot, s)) \ast W_N] \]
\[ + \langle \rho(\cdot, s), ((X_N(s) \ast \nabla G_a) - \nabla U) \cdot [\nabla (h_N(\cdot, s) - \rho(\cdot, s)) \ast W_N] \]
\[ - (\rho(\cdot, s) \ast \nabla G_a) - \nabla U) \cdot [-\nabla \rho(\cdot, s) + \nabla h_N(\cdot, s)]\rangle \]
\[ = B_1 + B_2. \]

(82)

By (72) and Lemma 0.4 with \( v = -\nabla U + X_N(s) \ast \nabla G_a \) we get

\[ |B_1| \leq c_3 \left( ||h_N(s) - \rho(\cdot, s)||_2^2 + \chi_N^{-2} + N^{\beta(1-2L/d)} \right). \]

(83)

On the other hand

\[ B_2 = \langle \rho(\cdot, s), ((X_N(s) \ast \nabla G_a) - \nabla U) \cdot [\nabla (h_N(\cdot, s) - \rho(\cdot, s)) \ast W_N] \]
\[ - (\rho(\cdot, s) \ast \nabla G_a) - \nabla U) \cdot [-\nabla \rho(\cdot, s) + \nabla h_N(\cdot, s)]\rangle \]
\[ = \langle \rho(\cdot, s), ((X_N(s) - \rho(\cdot, s)) \ast \nabla G_a) \cdot [\nabla (h_N(\cdot, s) - \rho(\cdot, s)) \ast W_N] \]
\[ + \langle \rho(\cdot, s), ((\rho(\cdot, s) \ast \nabla G_a) - \nabla U \cdot [\nabla (h_N(\cdot, s) - \rho(\cdot, s)) \ast W_N] \]
\[ - \langle \rho(\cdot, s), ((\rho(\cdot, s) \ast \nabla G_a) - \nabla U) \cdot (h_N(\cdot, s) - \rho(\cdot, s)) \ast W_N \rangle \]
\[ = B_2^1 + B_2^2. \]

(84)
By Assumption 0.1, condition 0.2 and Lemma 0.4, with $G = \nabla G_a$ and $v = \rho - \nabla U$ we get
\[
|B_2^1| \leq c_4 \left( |h_N(\cdot, s) - \rho(\cdot, s)|^2 + \chi_N^{-2} + N^\beta(1 - 2L/d) \right)
\]
and by taking into account also Lemma 0.3
\[
|B_2^2| \leq c_2 \left( |h_N(\cdot, s) - \rho(\cdot, s)|^2 + \chi_N^{-2} \right).
\]
So for the second term $B$ of (80) we get the following estimate:
\[
|B| \leq c_6 \left( |h_N(\cdot, s) - \rho(\cdot, s)|^2 + \chi_N^{-2} + N^\beta(1 - 2L/d) \right).
\]
For the third integrand on the right side of (80) we have
\[
C := -\sigma_N^2 \left[ \frac{||\nabla h_N(\cdot, s)||^2}{2} - \langle \nabla h_N(\cdot, s), \nabla \rho(\cdot, s) \rangle \right]
\]
\[
\leq -\sigma_N^2 \left[ \frac{||\nabla h_N(\cdot, s)||^2}{2} - \langle \nabla h_N(\cdot, s), \nabla \rho(\cdot, s) \rangle \right]
\]
\[
= -\sigma_N^2 \frac{1}{2} ||\nabla h_N(\cdot, s) - \nabla \rho(\cdot, s)||^2 + \frac{\sigma_N^2}{2} ||\nabla \rho(\cdot, s)||^2.
\]
(85)

Let us consider the submartingale term in (80):
\[
M_N(t) := \frac{\sigma_N}{N} \left| \sum_{k=1}^{N} \langle \nabla g_N(\cdot, s) - \nabla \rho(\cdot, s) * W_N)(X_N^k(s))dW^k(s) \right|.
\]
As showed in [12], by Doob’s inequality,
\[
\mathbb{E} \left[ \sup_{t \leq T'} M_N(t) \right]^2 \leq \frac{c_7 \sigma_N d}{N} \mathbb{E} \left[ \int_0^{T'} ||\nabla(h_N(\cdot, s) - \rho(\cdot, s)||^2 ds \right], \quad T' \leq T.
\]
(86)

By collecting all contributions, (80) becomes
\[
||h_N(\cdot, t) - \rho(\cdot, t)||^2 \leq ||h_N(\cdot, 0) - \rho(\cdot, 0)||^2 + c_5 \int_0^t ||h_N(\cdot, s) - \rho(\cdot, s)||^2 ds
\]
\[
+ c_5 t \left[ \chi_N^{-2} + N^\beta(1 - 2L/d) + \sigma_N^2 \frac{\beta(\beta + 1)}{d - 1} + \sigma_N^2 \right]
\]
\[
- \frac{\sigma_N^2}{2} \int_0^t ||\nabla(h_N(\cdot, s) - \rho(\cdot, s)||^2 ds
\]
\[
+ 2\sigma_N \int_0^t \sum_{k=1}^{N} \langle \nabla g_N(\cdot, s) - \nabla \rho(\cdot, s) * W_N)(X_N^k(s))dW^k(s) \right|,
\]
(87)

for $0 \leq t \leq T$.

From (86) and (87)
\[
\mathbb{E} \left[ \sup_{t \leq T'} ||h_N(\cdot, t) - \rho(\cdot, t)||^2 \right] + \frac{\sigma_N^2}{2} \left( 1 - \frac{2 c_7 \chi_N^4}{N} \right) \int_0^{T'} ||\nabla(h_N(\cdot, t) - \rho(\cdot, t))||^2 dt
\]
\[
\leq \mathbb{E} \left[ ||h_N(\cdot, 0) - \rho(\cdot, 0)||^2 \right] + c_5 \int_0^{T'} ||h_N(\cdot, t) - \rho(\cdot, t)||^2 dt
\]
\[
+ c_5 T' \left[ \chi_N^{-2} + N^\beta(1 - 2L/d) + \sigma_N^2 \frac{\beta(\beta + 1)}{d - 1} + \sigma_N^2 \right].
\]
(88)
For $N$ sufficiently large and by applying Gronwall’s inequality we obtain

\[
E \left[ \sup_{t \leq T} ||h_N(\cdot, t) - \rho(\cdot, t)||^2 \right] \\
\leq \left[ E \left[ ||h_N(\cdot, 0) - \rho(\cdot, 0)||^2 \right] + T(\chi_N^2 + N^{\beta(1-2L/d)} + \sigma_N^2 N^{\frac{\beta(d+2)}{d} - 1} + \sigma_N^2) \right] e^{c_T T}.
\]  
(89)

As $N \to \infty$, by (74) and since $\lim_{N \to \infty} \sigma_N^2 N^{\frac{\beta(d+2)}{d} - 1} = 0$, we obtain our thesis.

\[\square\]

Viscous case

Now we move to the case $\sigma_\infty > 0$.

Due to technical difficulties (the presence of the non vanishing term $\sigma_\infty > 0$), in this case we cannot carry out the same proof as for Theorem 0.2. Therefore, by following [15], we try to control directly $E \left[ \langle X(t), f(t) \rangle - \langle \rho(t), f(t) \rangle \right]$, obtaining a result analogous to Corollary 0.1.

About the regularity and uniqueness of the solution of equation (69) we make the following assumption

Assumption 0.2. System (69) with $\sigma_\infty > 0$ admits a unique, nonnegative solution $\rho \in C^{2,1}(\mathbb{R}^d \times [0,T])$.

The requirements of Assumption 0.2 are weaker than those of Assumption 0.1 ($\rho \in C^{2,1}(\mathbb{R}^d \times [0,T])$ instead of $\rho \in C^{((d+2)/d)+1,1}_b(\mathbb{R}^d \times [0,T])$), but we need a further restriction on the function $W_1$ defined by (19): $W_1$ must have compact support.

Theorem 0.3. If

i) $\lim_{N \to \infty} \mathcal{L}(X_N(0)) = \delta_{\mu_0}$ in $\mathcal{M}(\mathcal{M}(\mathbb{R}^d))$,  
(90)

where $\mu_0$ has density $\rho(x,0)$ with respect the Lebesgue measure,

ii) the parameter $\beta$ satisfies condition (30),

iii) $W_1$ defined in (19) has compact support,

then

\[
\lim_{N \to \infty} \langle X_N(t), f(\cdot, t) \rangle = \langle X(t), f(\cdot, t) \rangle = \int_{\mathbb{R}^d} f(x, t) \rho(x, t) dx
\]  
(91)

for any $f \in C^{2,1}_b(\mathbb{R}^d, \mathbb{R}^+)$, where $\rho$ is the unique solution of (69) with $\sigma_\infty > 0$.

Proof.

We have to show that

\[
\sup_{0 \leq t \leq T} ||\langle X(t), f(\cdot, t) \rangle - \int_{\mathbb{R}^d} f(x, t) \rho(x, t) dx ||_1 = 0.
\]
Since (68) is the weak form of (69), it is sufficient to show that for any $\mu \in C^{2,1}_b([0,\infty))$,
\[
E \left[ \left| \langle X(t), f(\cdot, t) \rangle - \langle \mu(0), f(\cdot, 0) \rangle - \int_0^t \langle \rho(\cdot, s), \frac{1}{2} \sigma^2 \Delta f(\cdot, s) + \nabla f(\cdot, s) \rangle + \left[ (\nabla G_a * \rho(\cdot, s))(\cdot) - \nabla U(\cdot) - \nabla \rho(\cdot, s) \right] \cdot \nabla f(\cdot, s) \right\rangle ds \right] = 0.
\]

For fixed $f \in C^{2,1}_b([0,\infty))$
\[
E \left[ \left| \langle X(t), f(\cdot, t) \rangle - \langle X_N(t), f(\cdot, t) \rangle \right| \right] 
\leq E \left[ \left| \langle X(t), f(\cdot, t) \rangle - \langle \mu(0), f(\cdot, 0) \rangle \right| \right] 
+ E \left[ \left| \langle \mu(0), f(\cdot, 0) \rangle - \langle X_N(0), f(\cdot, 0) \rangle \right| \right] 
+ \frac{\sigma^2}{2} E \left[ \left| \left( -\rho(\cdot, s), \Delta f(\cdot, s) \right) + \langle X_N(s), \Delta f(\cdot, s) \rangle \right| ds \right] 
+ E \left[ \right| \langle \rho(\cdot, s), \frac{\partial}{\partial s} f(\cdot, s) \rangle + \langle X_N(s), \frac{\partial}{\partial s} f(\cdot, s) \rangle \right| ds \right] 
+ E \left[ \right| \langle \rho(\cdot, s), \nabla \rho(\cdot, s) \cdot \nabla f(\cdot, s) \rangle - \langle h_N(\cdot, s), \nabla h_N(\cdot, s) \cdot \nabla f(\cdot, s) \rangle \right| ds \right] 
+ E \left[ \right| \langle h_N(\cdot, s), \nabla h_N(\cdot, s) \cdot \nabla f(\cdot, s) \rangle - \langle X_N(s), \nabla g_N \cdot \nabla f(\cdot, s) \rangle \right| ds \right] 
+ E \left[ \right| \left( -\rho(\cdot, s), \left[ (\nabla G_a * \rho(\cdot, s))(\cdot) - \nabla U(\cdot) \right] \cdot \nabla f(\cdot, s) \right) \right| ds \right] 
+ E \left[ \right| \left( X_N(s), \left[ (\nabla G_a + X_N(s))(\cdot) - \nabla U(\cdot) \right] \cdot \nabla f(\cdot, s) \right) \right| ds \right] 
+ E \left[ \right| \left| \left( \frac{\sigma_N}{N} \right) \int_0^t \sum_{k=1}^N \nabla f(X_N^k(s), s) dW_k(s) \right| \right] 
+ E \left[ \right| \langle X_N(t), f(\cdot, t) \rangle - \langle X_N(0), f(\cdot, 0) \rangle \right| \right] 
- \int_0^t \langle X_N(s), \langle \nabla G_a * X_N(s) \rangle \cdot \nabla f(\cdot, s) \rangle ds + \int_0^t \langle X_N(s), \nabla g_N(\cdot, s) \cdot \nabla f(\cdot, s) \rangle ds 
+ \int_0^t \langle X_N(s), \nabla U(\cdot) \cdot \nabla f(\cdot, s) \rangle ds - \int_0^t \langle X_N(s), \frac{1}{2} \sigma^2 \Delta f(\cdot, s) + \nabla f(\cdot, s) \rangle ds 
- \frac{\sigma_N}{N} \int_0^t \sum_{k=1}^N \nabla f(X_N^k(s), s) dW_k(s) \right] 

:= \sum_{i=1}^q I^q_N(t). \tag{92}
\]
\[ I^7_N(t) = \mathbb{E} \left[ \int_0^t |\langle \rho(\cdot, s), \rho(\cdot, s) \Delta f(\cdot, s) \rangle - \langle h_N(\cdot, s), h_N(\cdot, s) \Delta f(\cdot, s) \rangle| ds \right] \]

\[ \leq ||\Delta f||_\infty \int_0^T \mathbb{E} \left[ \int_{\mathbb{R}^d} |h_N(x, t) - \rho(x, t)||h_N(x, t) + \rho(x, t)| dx \right] dt \]

\[ \leq ||\Delta f||_\infty \left( \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d} |h_N(x, t) - \rho(x, t)|^2 dx dt \right] \right)^{1/2} \]

\[ \cdot \left( \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d} |h_N(x, t) + \rho(x, t)|^2 dx dt \right] \right)^{1/2} ; \]

by (57) and (59) we obtain

\[ \lim_{N \to \infty} I^7_N(t) = 0. \] (93)

By the symmetry of \( W_1 \),

\[ I^6_N(t) = \mathbb{E} \left[ \int_0^t \langle X_N(s), W_N * (\nabla h_N(\cdot, s) \cdot \nabla f(\cdot, s)) - (W_N * \nabla h_N(\cdot, s)) \cdot \nabla f(\cdot, s) \rangle ds \right] \]

\[ = \mathbb{E} \left[ \int_0^t \left( \int_{\mathbb{R}^d} X_N(s) dx \right) \int_{\mathbb{R}^d} W_N(x - y) \nabla h_N(y, s) \cdot (\nabla f(y) - \nabla f(x)) dy \right] ds \right]. \] (94)

By the definition of \( W_N \) and since \( W_1 \) has compact support, with \( c = \text{diam}(\text{supp} W_1(\cdot)) \) and \( ||D^2 f||_\infty = \sup_{i,j \leq d} ||\partial^2_{ij}||_\infty \), \( (94) \) is less than or equal to

\[ c \chi_N^{-1} ||D^2 f||_\infty \mathbb{E} \left[ \int_0^t \langle X_N(s) * W_N, |\nabla h_N(\cdot, s)| \rangle ds \right] \]

\[ \leq c \chi_N^{-1} ||D^2 f||_\infty \left( \mathbb{E} \left[ \int_0^T ||h_N(\cdot, s)||^2_2 ds \right] \right)^{1/2} \left( \mathbb{E} \left[ \int_0^T ||\nabla h_N(\cdot, s)||^2_2 ds \right] \right)^{1/2} \]

\[ \leq c \chi_N^{-1} ||D^2 f||_\infty. \]

It follows that

\[ \lim_{N \to \infty} I^6_N(t) = 0. \] (95)

\[ I^5_N(t) = \mathbb{E} \left[ \int_0^t \langle \rho(\cdot, s), \left( (\nabla G_a * \rho(\cdot, s))(\cdot) - \nabla U(\cdot) \right) \cdot \nabla f(\cdot, s) \rangle \right] \]

\[ + \langle X_N(s), \left( (\nabla G_a \ast X_N(s))(\cdot) - \nabla U(\cdot) \right) \cdot \nabla f(\cdot, s) \rangle \]

\[ + \langle X_N(s), \left( (\nabla G_a * \rho(\cdot, s))(\cdot) - \nabla U(\cdot) \right) \cdot \nabla f(\cdot, s) \rangle \]

\[ - \langle X_N(s), \left( (\nabla G_a * \rho(\cdot, s))(\cdot) - \nabla U(\cdot) \right) \cdot \nabla f(\cdot, s) \rangle ds \]

\[ \leq \mathbb{E} \left[ \int_0^t \langle X_N(s) - \rho(s), (\nabla G_a * \rho(\cdot, s))(\cdot) - \nabla U(\cdot) \rangle \cdot \nabla f(\cdot, s) \rangle \right] \]

\[ + \langle X_N(s), (\nabla G_a * \rho(\cdot, s))(\cdot) - (\nabla G_a * X_N(s))(\cdot) \rangle \cdot \nabla f(\cdot, s) \rangle ds \right]. \]
By Assumption 0.2 and (58)

\[
\lim_{N \to \infty} \mathcal{I}_N^7(t) = 0.
\]

As a consequence

\[
\lim_{N \to \infty} \sum_{i=1}^{9} \mathcal{I}_N^i(t) = 0.
\]

Conclusions

In this paper we have studied the asymptotic behavior of system (9) for the size of the population \(N\) growing to infinity, being the time \(t\) fixed, in terms of a law of large numbers for the empirical process \(\{X_N(t), t \in \mathbb{R}_+\}\).

It is also of interest to study the limiting behavior of such a system for fixed \(N\) and time growing to infinity. In [6] the authors investigate conditions for the existence of an invariant measure for system (9), i.e. conditions about the interaction potential and the confining potential such that there exists an invariant measure for the particle positions and, as a consequence, for the empirical process \(\{X_N(t), t \in \mathbb{R}_+\}\).

References

[1] Billingsley P. Convergence of Probability Measures, John Wiley & Sons, NY, 1968.
[2] Billingsley P. Probability and Measure, John Wiley & Sons, New York, 1986.
[3] Bodnar M., Velazquez J.J.L. Derivation of macroscopic equations for individual cell-based models: A formal approach. Math. Meth. Appl. Sci., 28, 1757-1779, 2005.
[4] Burger M., Capasso V., Morale D. On an Aggregation Model with Long and Short Range Interactions, J. Mathematical Biology, (2005) submitted.
[5] Capasso V., Bakstein D. An Introduction to Continuous-Time Stochastic Processes - Theory, Models and Applications to Finance, Biology and Medicine. Birkhäuser, Boston, 2004.
[6] Capasso V., Morale D., Ortisi M. Long Time Behavior of a System of Stochastic Differential Equations Modelling Aggregation. Math Everywhere, Deterministic and Stochastic Modelling in Biomedicine, Economics and Industry. Springer, 2006.
[7] Dawson D.A., Gärtner J. Large deviations, free energy functional and quasi-potential for a mean field model of interacting diffusions. Memoirs of the American Mathematical Society, 78, N.398, 1989.
[8] Ethier S.N., Kurtz, T.G. Markov processes. Characterization and convergence. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons, Inc., N.Y., 1986.

[9] Has’minski, R.Z. Stochastic stability of differential equations. Sijthoff & Noordhoff, Alphen aan den Rijn, The Netherlands and Rockville, Maryland, USA, 1980.

[10] Ikeda N., Watanabe S. Stochastic Differential Equations and Diffusion Processes. North-Holland Mathematical Library, Amsterdam, 1981.

[11] Morale D. Cellular automata and many-particles systems modeling aggregation behaviour among populations, Int. J. Appl. Math. & Comp. Sci. 10, 157-173, 2000.

[12] Morale D., Capasso V., Oelschläger K. A rigorous derivation of a nonlinear integro-differential equation from a system of stochastic differential equations for an aggregation model. Preprint 98-38 (SFB 359) Reaktive Strömugen, Diffusion und Transport, IWR, Universität Heidelberg, Juni 1998.

[13] Morale D., Capasso V., Boi S. Modeling the aggregative behavior of ants of the species Polyergus rufescens. Spatial heterogeneity in ecological models. Nonlinear Anal. Real World Appl., 1, no.1, 163-176, 2000.

[14] Morale D., Capasso V., Oelschläger K. An interacting particle system modelling aggregation behavior: from individuals to populations. J. Mathematical Biology, 50, 49-66, 2005.

[15] Oelschläger K. A law of large numbers for moderately interacting diffusion processes. Z. Wahrscheinlichkeitstheorie verw. Gabiete 69, 279-322 1985.

[16] Oelschläger K. On the derivation of reaction-diffusion equations as limit dynamics of systems of moderately interacting stochastic processes. Prob. Th. Rel. Fields, 82, 565-586, 1989.

[17] Oelschläger K. Large systems of interacting particles and porous medium equation. J. Diff. Eqns., 88, 294-346, 1990.