Karhunen-Loève expansion of Random Measures

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Abstract

We present an orthogonal expansion for real regular second-order finite random measures over $\mathbb{R}^d$. Such expansion, which may be seen as a Karhunen-Loève decomposition, consists in a series expansion of deterministic real finite measures weighted by uncorrelated real random variables with summable variances. The convergence of the series is in a mean-square-$\mathcal{M}_B(\mathbb{R}^d)^*$-weak* sense, with $\mathcal{M}_B(\mathbb{R}^d)$ being the space of bounded measurable functions over $\mathbb{R}^d$. This is proven profiting the extra requirement for a regular random measure that its covariance structure is identified with a covariance measure over $\mathbb{R}^d \times \mathbb{R}^d$. We also obtain a series decomposition of the covariance measure which converges in a separately $\mathcal{M}_B(\mathbb{R}^d)^*$-weak*—total-variation sense. We then obtain an analogous result for function-regulated regular random measures.

Keywords Random Measure, Karhunen-Loève Expansion, Covariance Measure

1 Introduction

Karhunen-Loève expansions are an important tool for the analysis of stochastic processes, both in theory and practice. The essential of such such expansions consists on decompositions of the form

$$X = \sum_{n \in \mathbb{N}} X_n f_n,$$

where $X$ is a (real) random variable with values in a (real) separable Hilbert space $E$, $(f_n)_{n \in \mathbb{N}}$ is an orthonormal basis of $E$, and $(X_n)_{n \in \mathbb{N}}$ is a sequence of uncorrelated random variables with summable variances. Such expansion exists if $\mathbb{E}(\|X\|_E^2) < \infty$. The convergence of (1) is
in such case surely and in the sense of the mean-square error $\mathbb{E}(\|X - \sum_{j \leq n} X_j f_j\|_E^2) \xrightarrow{n \to \infty} 0$. Similar expansions with weaker convergence forms are obtained when $X$ is interpreted as a linear application from $E$ to $L^2(\Omega, \mathcal{A}, \mathbb{P})$ whose covariance induces a continuous bi-linear form with finite trace. In such case, the convergence of $\mathbb{E}(\|X(e) - \sum_{j \leq n} X_j(f_j, e)\|_E^2) \xrightarrow{n \to \infty} 0$ for every $e \in E$, being $\langle \cdot, \cdot \rangle_E$ the inner-product in $E$ (see Section 2). The typical example used in most applications is the case of a mean-square continuous stochastic process over a compact interval $(X(t))_{t \in [a, b]}$ which, through the study of its integral against square-integrable test-functions, is interpreted as a process acting linearly over $L^2([a, b])$. We refer to Loève (1978; Ghanem & Spanos, 2003; Wang, 2008; Red-Horse & Ghanem, 2009) for more information on the Karhunen-Loève expansion and its applications.

Extensions of the principle (1) to non-Hilbert spaces has been worked out in many manners. In every case, the vectors $(f_n)_{n \in \mathbb{N}}$ can be seen as members of some topological vector space $V$, not necessarily Hilbert. Although one can always rely upon the Reproducing Kernel Hilbert Space induced by the covariance structure of the process (see Aronszajn (1950) or Ghanem & Spanos (2003, Chapter 2)), which may play the role of the space $E$ in the decomposition (1), it is often interesting to study other possible forms of convergence of the series regarding the topology of $V$ instead of the one of the Hilbert space $E$. The simplest example may be the one of the aforementioned case of a mean-square continuous process over a compact interval $(X(t))_{t \in [a, b]}$, where the series (1) consists (essentially) of continuous functions $(f_n)_{n \in \mathbb{N}}$, and for which, due to Mercer’s Theorem, the convergence is in a uniform—mean-square sense (Loève, 1978, Section 37.5). In the setting of Gaussian processes, Bay & Croix (2017) studied Karhunen-Loève expansions of processes having values in separable Banach spaces, Rajput (1972) studied the case of separable Fréchet spaces, and Peccati & Pycke (2010) studied expansions for processes defined over compact topological groups. In Meidan (1979, Theorem 9) an orthogonal expansion for Generalized Stochastic Processes, that is, random distributions, is presented over compact subsets of $\mathbb{R}^d$. A case which is particularly interesting to our aims is the one studied in Carrizo Vergara (2021), where the studied process $X$ is a tempered random distribution over $\mathbb{R}^d$, conceived as a continuous and linear application from the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ to $L^2(\Omega, \mathcal{A}, \mathbb{P})$. For such $X$, it is there proven that an orthogonal representation (1) holds with $(f_n)_{n \in \mathbb{N}}$ being a sequence of (essentially) linearly independent tempered distributions, and with the convergence inter-
interpreted in a mean-square–\mathcal{F}'(\mathbb{R}^d)\text{-weak*} sense.

In this work we propose a Karhunen-Loève expansion of a particular kind of process \( M \) which is a real second-order finite random measure over \( \mathbb{R}^d \) with a measure covariance structure (here called \textit{regular} random measure). We prove that such process can be decomposed in the form

\[
M = \sum_{n \in \mathbb{N}} X_n \mu_n, \tag{2}
\]

where \((X_n)_{n \in \mathbb{N}}\) is a sequence of uncorrelated random variables whose variances form a convergent series, and \((\mu_n)_{n \in \mathbb{N}}\) is a sequence of real finite measures over \( \mathbb{R}^d \) which are essentially linearly independent. The convergence of (2) is in the sense

\[
\mathbb{E} \left( \left( \langle M, \varphi \rangle - \sum_{j \in n} X_j \langle \mu_j, \varphi \rangle \right)^2 \right) \xrightarrow{n \to \infty} 0, \tag{3}
\]

for every \( \varphi \) measurable and bounded. The construction of the measures \((\mu_n)_{n \in \mathbb{N}}\) is done in a very similar manner to the construction of the expansion for tempered random distributions done in [Carrizo Vergara, 2021]. We apply a regularising anti-derivative operator to the random measure \( M \) in order to obtain a mean-square continuous stochastic process. This process has a traceable Karhunen-Loève expansion with respect to \( L^2(\mathbb{R}^d, \nu) \), being \( \nu \) a conveniently selected (independently of \( M \)) measure. We then differentiate the terms of the expansion in order to obtain the distributions \((\mu_n)_{n \in \mathbb{N}}\) which turn out to be measures (for \( n \) such that \( \sigma_{X_n}^2 := \text{Var}(X_n) > 0 \), which are the important cases). This also guarantees a convergence of the form (3) for \( \varphi \) smooth and compactly supported. The stronger case with \( \varphi \) measurable and bounded is obtained through a convenient application of Lusin’s Theorem. For the covariance measure \( C_M \) of \( M \), which is a positive-semidefinite measure over \( \mathbb{R}^d \times \mathbb{R}^d \), it holds

\[
C_M = \sum_{n \in \mathbb{N}} \sigma_{X_n}^2 \mu_n \otimes \mu_n. \tag{4}
\]

The mode of convergence of this series is in total-variation in one component and with respect to a measurable and bounded test-function in the other component. This result is then extended to the case of function-regulated random measures, that is, the case where there exists a positive locally bounded function \( f \) such that \( \frac{1}{f} M \) is a finite random measure.
To the knowledge of the author, an expansion such as (2) is not known in the generality here proposed. It is known, for instance, that a White Noise can be decomposed using any orthonormal basis of $L^2(\mathbb{R}^d)$ as measures $(\mu_n)_{n\in\mathbb{N}}$ in (2) obtaining a non-traceable expansion. Such expansion can be anyway re-transformed into a traceable one if the measures and random variables are multiplied by convenient positive reciprocal numbers. Similar arguments can be done in the case of orthogonal random measures by using an orthonormal basis of $L^2(\mathbb{R}^d, \mu)$, $\mu$ being the controlling measure (see the end of Section 4 for our definition of orthogonal random measures; we refer to (Passeggeri, 2020; Kingman, 1967) and (Schilling, 2016) as authors using orthogonal random measures in stronger senses: independently scattered or with almost-surely $\sigma-$additivity properties). However, an expansion for a general function-controlled regular random measure is not broadly known. The key extra supposition which allows to obtain the result is that the covariance structure of $M$ is identified to a covariance measure over $\mathbb{R}^d \times \mathbb{R}^d$. In general, the covariance structure of a $L^2(\Omega, \mathcal{A}, \mathbb{P})$-valued measure over $\mathbb{R}^d$ has the structure of a bi-measure, which is not always possible to identify to a measure (Rao, 2012). This extra regularity property allows to use different results on $M$ and its covariance $C_M$, such as the use of the total-variation measure $|C_M|$ and a Stochastic Fubini Theorem. We remark that this does not imply that $M$ is a strictly speaking random measure, in the sense that $M$ does not necessarily take values in a space of measures, nor almost-surely.

In Section 2 we introduce the Karhunen-Loève expansion over Hilbert spaces and we make precise which form of expansion and convergence we will use. In Section 3.2 we introduce random measures. After setting our notation of measure-theoretical objects, we introduce our concept of regular random measures. We present some properties and result regarding those, including a simplified version of a Stochastic Fubini Theorem 3.2. We also present the properties of an anti-derivative regularising operator acting over finite deterministic and random measures. In Section 4 we present the main Theorem 4.1 of expansion of finite regular random measures together with its proof. The expansion of function-regulated random measures is presented in Theorem 4.2 as a corollary of Theorem 4.1. We end in Section 5 with some concluding remarks.

We set up some notation. We denote $1_A$ the indicator function of the set $A$. The symbol $\| \cdot \|_{\infty}$ denotes the supremum norm, and it is used for real functions. $\mathcal{D}(\mathbb{R}^d)$ denotes the space
of (real) smooth compactly supported test-functions over \( \mathbb{R}^d \) typically used in Distribution Theory. The abbreviation LDCT and CSI stand for Lebesgue Dominated Convergence Theorem and Cauchy-Schwarz Inequality, respectively. All the random variables considered in this work are supposed to be defined over the same probability space \((\Omega, \mathcal{A}, \mathbb{P})\) and to have zero mean, without loss of generality. A stochastic process is understood as a family of random variables indexed by an arbitrary non-empty set. We basically work with \textit{second-order properties}, proper of mean-square analysis. Hence, we do not make direct suppositions of the regularity of the trajectories of the processes used. The most part of results we use, particularly the definition of stochastic integrals, are always considering limits in a mean-square sense. We do not precise the laws of the random variables involved (the Gaussian case is a particular one).

\section{Karhunen-Loève Expansion}

We shall use a weak form of Karhunen-Loève expansions. The principle is the following. Let \( E \) be a real separable Hilbert space, with inner-product \((\cdot, \cdot)_E\) and norm \( \| \cdot \|_E \). Let \( X : E \rightarrow L^2(\Omega, \mathcal{A}, \mathbb{P}) \) be a linear and continuous real mapping satisfying that there exists an orthonormal basis \((e_n)_{n \in \mathbb{N}} \subset E\) such that

\begin{equation}
\sum_{n \in \mathbb{N}} \mathbb{E}(|X(e_n)|^2) < \infty. \quad (5)
\end{equation}

If this holds, we say that \( X \) has a \textit{traceable Karhunen-Loève expansion with respect to} \( E \). This because in such case there exist an orthonormal basis of \( E \), say \((f_n)_{n \in \mathbb{N}}\), and a sequence of uncorrelated random variables \((X_n)_{n \in \mathbb{N}}\) with summable variances, such that

\begin{equation}
X(e) = \sum_{n \in \mathbb{N}} X_n(f_n, e)_E, \quad \forall e \in E, \quad (6)
\end{equation}

the convergence of the series being considered in a mean-square sense. The basis \((f_n)_{n \in \mathbb{N}}\) consists of the eigenvectors of the \textit{covariance operator} induced by the covariance structure of \( X \): if we denote \( K_X : E \times E \rightarrow \mathbb{R} \) the covariance Kernel of \( X \), that is

\begin{equation}
K_X(e, f) = \mathbb{E}(X(e)X(f)), \quad (7)
\end{equation}
then $K_X$ is bilinear, positive-semidefinite and continuous (since $X$ is continuous). By Riesz Representation, for every $e \in E$ there exists an element $Q_X(e) \in E$ such that
\[
K_X(e, f) = (Q_X(e), f)_E, \quad \forall f \in E. \tag{8}
\]
The so-induced operator $Q_X : E \to E$ is called the covariance operator of $X$. This operator is linear, continuous, positive-semidefinite, and by (5) it is also trace-class (Reed & Simon, 1980, Theorem VI.18). Hence, it has a spectral decomposition in an orthonormal basis of eigenvectors $(f_n)_{n \in \mathbb{N}}$, with corresponding positive eigenvalues $(\sigma^2_X)_n$ which form a convergent series (Reed & Simon, 1980, Theorem VI.21). The random variables $(X_n)_{n \in \mathbb{N}}$ are given by $X_n := X(f_n)$, for which we have $\text{Cov}(X_n, X_m) = \sigma^2_X \delta_{n,m}$.

We will mainly apply this principle to the following case. Consider $(U(x))_{x \in \mathbb{R}^d}$ a real and mean-square continuous stochastic process over $\mathbb{R}^d$. Consider $C_U(x, y) = \mathbb{E}(U(x)U(y))$ its covariance function, which is continuous over $\mathbb{R}^d \times \mathbb{R}^d$. Let $\nu$ be a positive finite measure over $\mathbb{R}^d$ such that
\[
\int_{\mathbb{R}^d} C_U(x, x) d\nu(x) < \infty. \tag{9}
\]
Using CSI and the positive-semidefiniteness of $C_U$, one proves
\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} |C_U(x, y)||\varphi(x)||\phi(y)| d(\nu \otimes \nu)(x, y) < \infty, \quad \forall \varphi, \phi \in L^2(\mathbb{R}^d, \nu). \tag{10}
\]
It is known (Soong & TT, 1973, Theorem 4.5.2) that this condition allows to properly define stochastic integrals of the form
\[
\int_{\mathbb{R}^d} U(x) \varphi(x) d\nu(x), \quad \forall \varphi \in L^2(\mathbb{R}^d, \nu). \tag{11}
\]
Hence, one can re-define $U$ as a process indexed by functions in the separable Hilbert space $L^2(\mathbb{R}^d, \nu)$, setting
\[
\tilde{U}(\varphi) := \int_{\mathbb{R}^d} U(x) \varphi(x) d\nu(x), \quad \forall \varphi \in L^2(\mathbb{R}^d, \nu). \tag{12}
\]
The so-defined application $\tilde{U} : L^2(\mathbb{R}^d, \nu) \to L^2(\Omega, \mathcal{A}, \mathbb{P})$ is continuous. The covariance operator $Q_{\tilde{U}} : L^2(\mathbb{R}^d, \nu) \to L^2(\mathbb{R}^d, \nu)$ is given by
\[
Q_{\tilde{U}}(\varphi) = \int_{\mathbb{R}^d} C_U(\cdot, y) \varphi(y) d\nu(y), \tag{13}
\]
which by (9) is trace-class \cite{Brislawn}. If then a traceable Karhunen-Loève expansion with respect to $L^2(\mathbb{R}^d, \nu)$:

$$
\int_{\mathbb{R}^d} U(x) \varphi(x) d\nu(x) = \sum_{n \in \mathbb{N}} X_n \langle \nu, \varphi f_n \rangle, \quad \forall \varphi \in L^2(\mathbb{R}^d, \nu),
$$

(14)

with $(f_n)_{n \in \mathbb{N}}$ the orthonormal basis of eigenfunctions of $Q_{\tilde{U}}$ and

$$
X_n = \int_{\mathbb{R}^d} U(x) f_n(x) d\nu(x), \quad \forall n \in \mathbb{N}.
$$

(15)

## 3 Random measures

### 3.1 Reminders on measures over $\mathbb{R}^d$

Let us recall and make precise our notation of measure-theoretical concepts for the Euclidean space. We will always work in the context of the measurable space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, being $\mathcal{B}(\mathbb{R}^d)$ the Borel $\sigma$-algebra. We denote $\mathcal{B}_B(\mathbb{R}^d)$ the $\delta$-ring of bounded Borel subsets of $\mathbb{R}^d$.

By a *measure over* $\mathbb{R}^d$, we mean a *real* application $\mu : \mathcal{B}_B(\mathbb{R}^d) \rightarrow \mathbb{R}$ which is $\sigma$-additive over $\mathcal{B}_B(\mathbb{R}^d)$. This implies that $\mu$ is locally-finite, but $\mu$ may not be defined over the whole Borel $\sigma$-algebra (some authors use the term *pre-measure* for this object). $\mu$ is called positive if it takes only non-negative values. For a real measure over $\mu$ over $\mathbb{R}^d$, its total-variation measure is defined as

$$
|\mu|(A) := \sup \left\{ \sum_{n \in \mathbb{N}} |\mu(E_n)| \right\},
$$

(16)

which is the smallest positive measure such that $|\mu(A)| \leq |\mu|(A)$ for all $A \in \mathcal{B}_B(\mathbb{R}^d)$ \cite[Chapter 6]{Rudin}. This measure always exists and it is locally-finite. If $|\mu|$ can be extended finitely and $\sigma$-additively to the whole Borel $\sigma$-algebra (hence $|\mu|(\mathbb{R}^d) < \infty$), then $\mu$ is said to be finite.

The space of (real) measurable functions over $\mathbb{R}^d$ is denoted $\mathcal{M}(\mathbb{R}^d)$. We denote $\mathcal{M}_B(\mathbb{R}^d)$ the space of bounded measurable functions and $\mathcal{M}_{B,c}(\mathbb{R}^d)$ the space of bounded measurable functions with compact support. A function $\mathcal{M}(\mathbb{R}^d)$ is said to be integrable with respect to $\mu$.}

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if \(|f|\) is Lebesgue integrable with respect to \(|\mu|\) over \(\mathbb{R}^d\). In such case, we use the notation
\[
\langle \mu, f \rangle := \int_{\mathbb{R}^d} f \, d\mu = \int_{\mathbb{R}^d} f(x) \, d\mu(x).
\] (17)

We remark that the total variation measure \(|\mu|\) can also be expressed as
\[
|\mu|(A) = \sup_{\varphi \in \mathcal{M}(\mathbb{R}^d), \|\varphi\|=1} \langle \mu, \varphi \rangle, \quad \forall A \in \mathcal{B}_{\mathbb{R}^d}.
\] (18)

We recall the useful Lusin’s Theorem, applied over the space \(\mathbb{R}^d\) (Folland, 1999, Theorem 7.10):

**Theorem 3.1 (Lusin).** Let \(\mu\) be a finite measure over \(\mathbb{R}^d\). Let \(\psi \in \mathcal{M}(\mathbb{R}^d)\). Then, for every \(\epsilon > 0\) there exists a closed set \(E \subset \mathbb{R}^d\) such that \(\psi\) is continuous over \(E\) and \(|\mu|(E^c) < \epsilon\).

### 3.2 Random measures and properties

**Definition 3.1.** A regular second-order random measure (from now on regular random measure) over \(\mathbb{R}^d\) is a stochastic process indexed by the bounded Borel sets, \((M(A))_{A \in \mathcal{B}(\mathbb{R}^d)} \subset L^2(\Omega, \mathcal{A}, \mathbb{P})\), such that there exists a real measure \(C_M\) over \(\mathbb{R}^d\) such that
\[
\mathbb{E}(M(A)M(B)) = C_M(A \times B), \quad \forall A, B \in \mathcal{B}_{\mathbb{R}^d}.
\] (19)

It is not difficult to prove that for such a process \(M\) we have a \(\sigma\)-additivity property
\[
M \left( \bigcup_{n \in \mathbb{N}} A_n \right) = \sum_{n \in \mathbb{N}} M(A_n),
\] (20)
for every sequence of mutually disjoint sets \((A_n)_{n \in \mathbb{N}} \subset \mathcal{B}_{\mathbb{R}^d}\) such that \(\bigcup_{n \in \mathbb{N}} A_n\) is bounded, the series being taken in a mean-square sense. Hence, \(M\) is a \(L^2(\Omega, \mathcal{A}, \mathbb{P})\)-valued locally finite measure over \(\mathbb{R}^d\). The extra adjective regular is added because of the identification of the covariance structure of \(M\) to the covariance measure \(C_M\). In general, a \(L^2(\Omega, \mathcal{A}, \mathbb{P})\)-valued locally finite measure over \(\mathbb{R}^d\) does not necessarily have a measure over \(\mathbb{R}^d \times \mathbb{R}^d\) describing its covariance structure; the application \((A, B) \mapsto \mathbb{E}(M(A)M(B))\) defines a bi-measure which is not necessarily identified to a measure. See (Rao, 2012, Chapter 2, Example 2) for a counterexample.
It is clear that covariance measures are symmetric in the sense \( C_M(A \times B) = C_M(B \times A) \). It is also possible to verify that \( |C_M| \) is also a symmetric measure. Covariance measures are also positive-semidefinite in the sense
\[
\langle C_M, \varphi \otimes \varphi \rangle \geq 0, \quad \forall \varphi \in \mathcal{M}_{B,c}(\mathbb{R}^d).
\] (21)

Conversely, every symmetric measure over \( \mathbb{R}^d \times \mathbb{R}^d \) satisfying (21) is the covariance measure of a regular random measure (Kolmogorov Extension Theorem).

**Definition 3.2.** A regular random measure \( M \) over \( \mathbb{R}^d \) is said to be finite if its covariance measure \( C_M \) is finite.

When \( M \) is finite, its definition can be extended uniquely, finitely and \( \sigma \)-additively to the whole Borel \( \sigma \)-algebra \( \mathcal{B}(\mathbb{R}^d) \), the random variable \( M(\mathbb{R}^d) \) having finite variance.

If \( \varphi \in \mathcal{M}(\mathbb{R}^d) \) is such that
\[
\langle |C_M|, |\varphi| \otimes |\varphi| \rangle < \infty,
\] (22)
then the stochastic integral
\[
\langle M, \varphi \rangle := \int_{\mathbb{R}^d} \varphi(x)dM(x)
\] (23)
can be defined as a random variable in \( L^2(\Omega, \mathcal{A}, \mathbb{P}) \). This is just an example of the Dunford-Schwartz (or Bochner) integral of \( \varphi \) with respect to the \( L^2(\Omega, \mathcal{A}, \mathbb{P}) \)-valued measure \( M \) (see (Rao, 2012, Chapter 2) for an effective introduction, (Dunford & Schwartz, 1958, Section IV.10) for the details, and (Carrizo Vergara, 2018, Proposition 3.3.1) for the sufficiency of condition (22)). If \( \varphi \) and \( \phi \) satisfy (22), then
\[
\mathbb{E} (\langle M, \varphi \rangle \langle M, \phi \rangle) = \langle C_M, \varphi \otimes \phi \rangle.
\] (24)

Of course, every function in \( \mathcal{M}_{B}(\mathbb{R}^d) \) is integrable in this sense with respect to any finite regular random measure, while every function in \( \mathcal{M}_{B,c}(\mathbb{R}^d) \) is integrable with respect to any regular random measure.

The next Theorem is a simplified form of what we could call a stochastic Fubini Theorem. Its proof is given in Appendix A.
Theorem 3.2 (Stochastic Fubini). Let $M$ be a regular random measure over $\mathbb{R}^d$ with covariance measure $C_M$ and let $\mu$ be a (deterministic) finite measure over $\mathbb{R}^m$. Let $\psi \in \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^m)$ be such that

(i) $\int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^m} |\psi(x, u)| |\psi(y, v)| d(|C_M| \otimes |\mu| \otimes |\mu|)(x, y, u, v) < \infty.$

(ii) The function

$$ (u, v) \mapsto \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x, u) \psi(y, v) dC_M(x, y), $$

which is defined outside a $|\mu| \otimes |\mu|$-null set, is continuous over a set of the form $E \times E$ with $E \in \mathcal{B}(\mathbb{R}^m)$ such that $|\mu|(E^c) = 0$.

Then,

$$ \int_{\mathbb{R}^d} \int_{\mathbb{R}^m} \psi(x, u) d\mu(u) dM(x) = \int_{\mathbb{R}^m} \int_{\mathbb{R}^d} \psi(x, u) dM(x) d\mu(u). $$

As the reader may notice, we use stochastic processes which are random measures in a purely mean-square sense. That means that $M$ is not necessarily a process with measure sample paths, that is, a random variable taking values in a space of measures. Nor is it necessarily true that the process $M$ has almost-surely measure-sample paths, or that it has a modification with measure-sample paths. For those kinds of random measures, the definition of a stochastic integral and stochastic Fubini theorem are relatively simple results to obtain, since we can rely upon the deterministic theory. See the big treaty [Kallenberg, 2017] for more information. As an important counter-example, a Gaussian White Noise over $\mathbb{R}^d$, denoted $W$, which is a regular random measure such that $C_W(A \times B) = |A \cap B|$, $|\cdot|$ standing for the Lebesgue measure, cannot be conceived as a process having a modification with measure-sample paths. See [Dalang et al., 2009, Example 3.16] for such case, [Horowitz, 1986] for the general case of Gaussian measure-valued random variables, and [Kingman, 1967] for the case of orthogonal (independently-scattered) random measures.

3.3 Regularising operator

For non-trivial results exposed in this Section, we give their justifications in Appendix B.
For a deterministic finite measure $\mu$ over $\mathbb{R}^d$, we define the function $O(\mu) : \mathbb{R}^d \to \mathbb{R}$ as

$$O(\mu)(\vec{x}) := \int_0^{\vec{x}} \mu(-\infty, \vec{u}) d\vec{u}, \quad \forall \vec{x} \in \mathbb{R}^d,$$  \hspace{1cm} (27)

where we have used the abbreviated notations

$$\int_0^{\vec{x}} (\cdot) d\vec{u} := \int_0^{x_1} \int_0^{x_2} \ldots \int_0^{x_d} (\cdot) \, du_d \ldots du_2 \, du_1$$  \hspace{1cm} (28)

and

$$(-\infty, \vec{x}] := (-\infty, x_1] \times (-\infty, x_2] \times \ldots \times (-\infty, x_d],$$  \hspace{1cm} (29)

for every $\vec{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$. Since $\mu$ is finite, the function $\vec{u} \mapsto \mu((-\infty, \vec{u}])$ is measurable and bounded by $|\mu|(\mathbb{R}^d)$. The integral (27) is then well-defined, and the resulting function $\vec{x} \mapsto O(\mu)(\vec{x})$ is actually continuous over $\mathbb{R}^d$. The next bound is easy to obtain:

$$|O(\mu)(\vec{x})| \leq |x_1| \ldots |x_d||\mu|(\mathbb{R}^d), \quad \forall \vec{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d.$$  \hspace{1cm} (30)

Finally and more crucially, we have

$$\frac{\partial^{2d} O(\mu)}{\partial x_1^2 \ldots \partial x_d^2} = \mu$$  \hspace{1cm} (31)

in distributional sense over $\mathbb{R}^d$, that is

$$\int_{\mathbb{R}^d} O(\mu)(x) \frac{\partial^{2d} \varphi}{\partial x_1^2 \ldots \partial x_d^2}(x) dx = \int_{\mathbb{R}^d} \varphi(x) d\mu(x), \quad \forall \varphi \in D(\mathbb{R}^d),$$  \hspace{1cm} (32)

which can be proven using Fubini Theorem or through convolution arguments.

Let us now apply $O$ to a finite regular random measure $M$. The application $\vec{u} \mapsto M((-\infty, \vec{u}])$ determines a stochastic process over $\mathbb{R}^d$ which has the particularity of having a bounded covariance and of being mean-square continuous outside a set of null Lebesgue measure. Hence, the stochastic integral

$$O(M)(\vec{x}) := \int_0^{\vec{x}} M((-\infty, \vec{u}]) d\vec{u}$$  \hspace{1cm} (33)

is well-defined through Riemann-alike approximations. This process has covariance function

$$C_{O(M)}(\vec{x}, \vec{y}) = \int_0^{\vec{x}} \int_0^{\vec{y}} C_M((-\infty, \vec{u}] \times (-\infty, \vec{v}]) d\vec{v} d\vec{u},$$  \hspace{1cm} (34)
which is a continuous function over $\mathbb{R}^d \times \mathbb{R}^d$ (it is actually the function $O(C_M)$ interpreting $O$ as an operator acting over finite measures over $\mathbb{R}^d \times \mathbb{R}^d$). Hence $O(M)$ is mean-square continuous. In addition one has the bound

$$|C_{O(M)}(\vec{x}, \vec{y})| \leq |x_1| \ldots |x_d||y_1| \ldots |y_d||C_M|(\mathbb{R}^d \times \mathbb{R}^d), \quad \forall \vec{x}, \vec{y} \in \mathbb{R}^d. \quad (35)$$

Finally, an application of Stochastic Fubini Theorem 3.2 allows to conclude

$$\frac{\partial^{2d} O(M)}{\partial x_1^* \ldots \partial x_d^*} = M \quad (36)$$

in distributional sense over $\mathbb{R}^d$, that is, we have the equality between the stochastic integrals

$$\int_{\mathbb{R}^d} O(M)(x) \frac{\partial^{2d} \varphi}{\partial x_1^* \ldots \partial x_d^*}(x) dx = \int_{\mathbb{R}^d} \varphi(x) dM(x), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^d). \quad (37)$$

4 Expansion of random measures

We give first a result of a Karhunen-Loève expansion for regular finite random measures.

**Theorem 4.1 (Karhunen-Loève expansion of finite random measures).** Let $M$ be a finite regular random measure over $\mathbb{R}^d$. Then, there exists a sequence of pairwise uncorrelated random variables with summable variances $(X_n)_{n \in \mathbb{N}}$, and a sequence of finite measures over $\mathbb{R}^d$, $(\mu_n)_{n \in \mathbb{N}}$ such that

$$\langle M, \varphi \rangle = \sum_{j \in \mathbb{N}} X_j \langle \mu_j, \varphi \rangle, \quad \forall \varphi \in \mathcal{M}_B(\mathbb{R}^d), \quad (38)$$

with the series being considered in a mean-square sense. In addition, if $C_M$ is the covariance measure of $M$, then,

$$C_M = \sum_{j \in \mathbb{N}} \sigma_{X_j}^2 \mu_j \otimes \mu_j, \quad (39)$$

where $\sigma_{X_n}^2 = \text{Var}(X_n)$ for all $n \in \mathbb{N}$. The convergence of the series is in a separately $\mathcal{M}_B(\mathbb{R}^d)^*\text{-weak}^*\text{–total-variation sense}$, that is,

$$\left| \langle C_M, \varphi \otimes (\cdot) \rangle - \sum_{j \leq n} \sigma_{X_j}^2 \langle \mu_j, \varphi \rangle \mu_j \right|(\mathbb{R}^d) \xrightarrow{n \to \infty} 0 \quad \forall \varphi \in \mathcal{M}_B(\mathbb{R}^d). \quad (40)$$
The reader will recognize very similar arguments to the classical proof of the Karhunen-
Loève expansion for mean-square continuous stochastic process over compact intervals, see
for example (Giambartolomei, 2015) for a complete and detailed non-specialist oriented ex-
position. In such case, Dini’s Theorem allows to conclude a uniform-mean-square conver-
gence. Such role is played in our case by Lusin’s Theorem.

Proof: We begin with the construction of the random variables \((X_n)_{n \in \mathbb{N}}\) and the measures
\((\mu_n)_{n \in \mathbb{N}}\). For that, we consider the function
\[
p(\vec{x}) = \prod_{j=1}^{d} (1 + |x_j|^2)^2, \quad \vec{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d,
\]
(41)
which is strictly positive and locally bounded. We also set the finite measure over \(\mathbb{R}^d\)
\[
d\nu(x) = \frac{dx}{p(x)}.
\]
(42)
Now, we consider the stochastic process over \(\mathbb{R}^d\)
\[
U = \mathcal{O}(M).
\]
(43)
We have seen that this process is mean-square continuous and that it has the properties pre-
sented in Section 3.3. From relation (35) we conclude
\[
\int_{\mathbb{R}^d} C_U(x, x) d\nu(x) \leq |C_M|([\mathbb{R}^d \times \mathbb{R}^d]) \left( \int_{\mathbb{R}} \frac{t^2}{(1 + t^2)^2} dt \right)^d < \infty.
\]
(44)
Hence, as stated in Section 2, the process \(U\) has a traceable Karhunen-Loève expansion with
respect to the Hilbert space \(L^2(\mathbb{R}^d, \nu)\), having thus
\[
\int_{\mathbb{R}^d} U(x) \varphi(x) d\nu(x) = \sum_{j \in \mathbb{N}} X_j \langle \nu, f_j \varphi \rangle, \quad \forall \varphi \in L^2(\mathbb{R}^d, \nu),
\]
(45)
being \((f_j)_{j \in \mathbb{N}}\) the orthonormal basis of \(L^2(\mathbb{R}^d, \nu)\) given by the eigenfunctions of the trace-
class covariance operator (13), and the random variables \((X_j)_{j \in \mathbb{N}}\) are given by (15) and they
are uncorrelated with variances \((\sigma^2_{X_n})_{n \in \mathbb{N}}\) such that \(\sum_{n \in \mathbb{N}} \sigma^2_{X_n} < \infty\).

We remark that a function \(f \in L^2(\mathbb{R}^d, \nu)\) determines a distribution over \(\mathbb{R}^d\). This because
\(f\) is equivalent to the locally finite measure \(f \nu\) in the sense that \(\langle f, \varphi \rangle = \langle f \nu, \varphi \rangle\) for every
\( \varphi \in \mathcal{D}(\mathbb{R}^d) \) (the measure \( p\nu \) is the Lebesgue measure). To see that \( fp\nu \) is locally finite, consider any \( A \in \mathcal{B}_B(\mathbb{R}^d) \) and by CSI one has
\[
\int_A |f| \, d\nu \leq \sqrt{\int_A p^2 \, d\nu} \sqrt{\int_A |f|^2 \, d\nu} \leq \sqrt{\int_A p \| f \|_{L^2(\mathbb{R}^d, \nu)}} < \infty, \quad (46)
\]
the last expression being finite due to the local-boundedness of \( p \). For \( \varphi \in \mathcal{D}(\mathbb{R}^d) \), one has \( \varphi p \in L^2(\mathbb{R}^d, \nu) \) hence from (45) we have
\[
\langle U, \varphi \rangle = \int_{\mathbb{R}^d} U(x) \varphi(x) p(x) \, d\nu(x) = \sum_{j \in \mathbb{N}} X_j \langle \nu, f_j \varphi \rangle = \sum_{j \in \mathbb{N}} X_j \langle f_j, \varphi \rangle. \quad (47)
\]
Considering the derivative relation (36), we conclude
\[
\langle M, \varphi \rangle = \left\langle \frac{\partial^{2d} U}{\partial x_1^2 \cdots \partial x_d^2}, \varphi \right\rangle
\]
\[
= \left\langle U, \frac{\partial^{2d} \varphi}{\partial x_1^2 \cdots \partial x_d^2} \right\rangle
\]
\[
= \sum_{j \in \mathbb{N}} X_j \left\langle f_j, \frac{\partial^{2d} \varphi}{\partial x_1^2 \cdots \partial x_d^2} \right\rangle
\]
\[
= \sum_{j \in \mathbb{N}} X_j \left\langle \frac{\partial^{2d} f_j}{\partial x_1^2 \cdots \partial x_d^2}, \varphi \right\rangle, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^d). \quad (48)
\]
This indicates that we must study the distributions \( \frac{\partial^{2d} f_j}{\partial x_1^2 \cdots \partial x_d^2} \). We set \( \sigma^2_{X_j} := \text{Var}(X_j) \) and we consider the cases where \( \sigma^2_{X_j} > 0 \). From the eigenfunction condition and the covariance of \( C_U \) given by (34), we obtain
\[
f_j(\bar{x}) = \frac{1}{\sigma^2_{X_j}} \int_{\mathbb{R}^d} C_U(\bar{x}, \bar{y}) f_j(\bar{y}) \, d\nu(\bar{y})
\]
\[
= \frac{1}{\sigma^2_{X_j}} \int_{\mathbb{R}^d} \int_0^{\bar{x}} \int_0^{\bar{y}} C_M((-\infty, \bar{u}] \times (-\infty, \bar{v}]) d\bar{u}d\bar{v} f_j(\bar{y}) \, d\nu(\bar{y})
\]
\[
= \frac{1}{\sigma^2_{X_j}} \int_0^{\bar{x}} \int_{\mathbb{R}^d} C_M((-\infty, \bar{u}] \times (-\infty, \bar{v}]) d\bar{u} f_j(\bar{y}) \, d\nu(\bar{y}) d\bar{u}, \quad (49)
\]
where we have used the (deterministic) Fubini Theorem. Following this result, we define the application \( \mu_j : \mathcal{B}(\mathbb{R}^d) \to \mathbb{R} \) as
\[
\mu_j(A) := \frac{1}{\sigma^2_{X_j}} \int_{\mathbb{R}^d} \int_0^{\bar{y}} C_M(A \times (-\infty, \bar{v}]) d\bar{u} f_j(\bar{y}) d\bar{y}. \quad (50)
\]
We remark that we can re-write \( \mu_j \) as

\[
\mu_j(A) = \frac{1}{\sigma_{X_j}^2} \int_{\mathbb{R}^d} \mathcal{O}(C_M(A \times \cdot))(\vec{y}) f_j(\vec{y}) d\nu(\vec{y}),
\]

(51)

where \( C_M(A \times \cdot) \) stands for the measure over \( \mathbb{R}^d \) given by \( B \mapsto C_M(A \times B) \). Given the property (30) of the operator \( \mathcal{O} \), the function \( \mathcal{O}(C_M(A \times \cdot)) \) is in \( L^2(\mathbb{R}^d, \nu) \), hence the integral (51) is well-defined. It is clear that \( \mu_j \) is an additive function. In addition, one has

\[
|\mu_j(A)| \leq \frac{1}{\sigma_{X_j}^2} \int_{\mathbb{R}^d} |y_1| \cdots |y_d| |f_j(\vec{y})| d\nu(\vec{y}) |C_M|(A \times \mathbb{R}^d).
\]

(52)

Since \( |C_M| \) is a finite measure, if we take any sequence of Borel sets \( (A_n)_{n \in \mathbb{N}} \) such that \( A_n \searrow \emptyset \), we must have \( |C_M|(A_n \times \mathbb{R}^d) \searrow 0 \) and hence \( |\mu_j(A_n)| \to 0 \). This proves that \( \mu_j \) is a measure over \( \mathbb{R}^d \) and it is also finite since \( C_M \) is finite. In addition, from (49) we have

\[
f_j(\vec{x}) = \int_0^{\vec{x}} \mu_j((\mathbb{R}, \vec{u})) d\vec{u} = \mathcal{O}(\mu_j)(\vec{x}),
\]

(53)

from where we conclude

\[
\frac{\partial^{2d} f_j}{\partial x_1^{a_1} \cdots \partial x_d^{a_d}} = \mu_j.
\]

(54)

For the case where \( \sigma_{X_j}^2 = 0 \), the distribution \( \frac{\partial^{2d} f_j}{\partial x_1^{a_1} \cdots \partial x_d^{a_d}} \) is not necessarily a measure, but such case is not really important since for such \( j \) we have \( X_j = 0 \), so \( \frac{\partial^{2d} f_j}{\partial x_1^{a_1} \cdots \partial x_d^{a_d}} \) does not really intervene in the expansion (38). For simplicity, we will assume in the following that \( \sigma_{X_j}^2 > 0 \) for all \( j \in \mathbb{N} \) (for the case where the sum (38) is finite we have nothing more to prove).

We have hence proven the expansion

\[
\langle M, \varphi \rangle = \sum_{j \in \mathbb{N}} X_j \langle \mu_j, \varphi \rangle, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^d).
\]

(55)

Now, the objective is to extend the result to \( \varphi \in \mathcal{M}_B(\mathbb{R}^d) \). For that, let us define and analyse the sequence of finite random measures

\[
M_n = \sum_{j \leq n} X_j \mu_j, \quad n \in \mathbb{N}.
\]

(56)
Let us study the covariance of $M - M_n$. For that, we consider $A, B \in \mathcal{B}(\mathbb{R}^d)$. Then,

$$
\mathbb{E} \left( (M(A) - M_n(A))(M(B) - M_n(B)) \right) = C_M(A \times B) - \mathbb{E}(M(A)M_n(B)) - \mathbb{E}(M(B)M_n(A)) + \mathbb{E}(M_n(A)M_n(B))
$$

(57)

The covariance $C_M$ is given by

$$
C_M(A \times B) = \mathbb{E}(M_n(A)M_n(B))
$$

$$
= \mathbb{E} \left( \sum_{j \leq n} \sum_{k \leq n} X_j X_k \mu_j(A) \mu_k(B) \right)
$$

(58)

$$
= \sum_{j \leq n} \sigma^2_{X_j} \mu_j(A) \mu_j(B),
$$

where we have used that $\mathbb{E}(X_jX_k) = \sigma^2_{X_j} \delta_{j,k}$. On the other hand,

$$
\mathbb{E}(M(A)M_n(B)) = \mathbb{E} \left( \sum_{j \leq n} M(A)X_j \mu_j(B) \right)
$$

(59)

$$
= \sum_{j \leq n} \mathbb{E}(M(A)X_j) \mu_j(B).
$$

Let us compute $\mathbb{E}(M(A)X_j)$. For that we re-write the expression (15) for $X_j$ as

$$
X_j = \int_{\mathbb{R}^d} \mathcal{O}(M)(\vec{y}) f_j(\vec{y}) d\vec{y} = \int_{\mathbb{R}^d} \int_0^{\vec{y}} \int_{\mathbb{R}^d} 1_{(-\infty, \vec{u})}[\vec{s}] dM(\vec{s}) d\vec{u} f_j(\vec{y}) d\nu(\vec{y}).
$$

(60)

We seek to write this integral as an integral with respect to $M$. For that, we will use Stochastic Fubini Theorem 3.2. Consider for every $\vec{y} \in \mathbb{R}^d$ the function $\theta_\vec{y} : \mathbb{R}^d \to \{-1, 1\}$ such that

$$
\int_{\mathbb{R}^d} \theta_\vec{y}(\vec{u}) \varphi(\vec{u}) d\vec{u} = \int_0^{\vec{y}} \varphi(\vec{u}) d\vec{u}, \quad \forall \varphi \in \mathcal{M}_{\mathcal{B},c}(\mathbb{R}^d)
$$

(61)

We can re-write the integral (60) as

$$
X_j = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1_{(-\infty, \vec{u})}[\vec{s}] dM(\vec{s}) \theta_\vec{y}(\vec{u}) f_j(\vec{y}) d(\ell^d \otimes \nu)(\vec{u}, \vec{y}),
$$

(62)

\footnote{The function $\theta_\vec{y}$ is not necessarily an indicator function. We remind that the integrals of the form (28) are Riemann integrals, for which minus signs appear when changing the integration limits. This must be considered when some components of $\vec{y}$ are negative. It is clear however that $\theta_\vec{y}$ has compact support.}
where we have denoted by $\ell^d$ the Lebesgue measure over $\mathbb{R}^d$. We remark that the measure over $\mathbb{R}^d \times \mathbb{R}^d$ given by $d\mu(\bar{u}, \bar{y}) := \theta_\nu(\bar{u})f_j(\bar{y})d(\ell^d \otimes \nu)(\bar{u}, \bar{y})$ is finite, since by Fubini
\[ \int_{\mathbb{R}^d \times \mathbb{R}^d} |\theta_\nu(\bar{u})f_j(\bar{y})|d(\ell^d \otimes \nu)(\bar{u}, \bar{y}) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\theta_\nu(\bar{u})|d\bar{u}|f_j(\bar{y})|d\nu(\bar{y}) \leq \int_{\mathbb{R}^d} |y_1|\ldots|y_d||f_j(\bar{y})|d\nu(\bar{y}) < \infty. \] (63)

If we consider the function $\psi \in \mathcal{M}(\mathbb{R}^d \times (\mathbb{R}^d \times \mathbb{R}^d))$ given by
\[ \psi(s, (\bar{u}, \bar{y})) = 1_{(-\infty, \bar{u}]}(s), \] (64)
we see that condition (i) holds since the corresponding full integral equals
\[ \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} |C_M|((-\infty, \bar{u}] \times (-\infty, \bar{v}])d|\mu|(\bar{u}, \bar{y})d|\mu|(\bar{v}, \bar{z}), \] (65)
which is finite since $|\mu|$ is finite and the function $(\bar{u}, \bar{v}) \mapsto |C_M|((-\infty, \bar{u}] \times (-\infty, \bar{v}])$ is bounded. In addition, condition (iii) holds since $(\bar{u}, \bar{v}) \mapsto C_M((-\infty, \bar{u}] \times (-\infty, \bar{v}])$ is continuous over a set of the form $E \times E$ such that $\ell^d(E^c) = 0$, see those details in Appendix B.

Stochastic Fubini Theorem can then be applied, and we have thus
\[ X_j = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} 1_{(-\infty, \bar{u}]}(s)\theta_\nu(\bar{u})f_j(\bar{y})d(\ell^d \otimes \nu)(\bar{u}, \bar{y})dM(s) \]
\[ = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1_{(-\infty, \bar{u}]}(s)d\bar{u}f_j(\bar{y})d\nu(\bar{y})dM(s). \] (66)

Using formula (24) and the deterministic Fubini Theorem,
\[ \mathbb{E}(M(A)X_j) = \mathbb{E} \left( \int_{\mathbb{R}^d} 1_A(\bar{t})dM(\bar{t}) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1_{(-\infty, \bar{u}]}(s)d\bar{u}f_j(\bar{y})d\nu(\bar{y})dM(s) \right) \]
\[ = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1_A(\bar{t}) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1_{(-\infty, \bar{u}]}(s)d\bar{u}f_j(\bar{y})d\nu(\bar{y})dC_M(\bar{t}, \bar{s}) \]
\[ = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1_A(\bar{t}) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1_{(-\infty, \bar{u}]}(s)d\bar{u}f_j(\bar{y})d\nu(\bar{y})dC_M(\bar{t}, \bar{s})d\bar{u}f_j(\bar{y})d\nu(\bar{y}) \]
\[ = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} C_M(A \times (-\infty, \bar{u}])d\bar{u}f_j(\bar{y})d\nu(\bar{y}) \]
\[ = \int_{\mathbb{R}^d} \mathcal{O} \left( C_M(A \times \cdot) \right)(\bar{y})f_j(\bar{y})d\nu(\bar{y}) \]
\[ = \sigma_{X_j}^2 \mu_j(A). \] (67)
We conclude

$$\mathbb{E}(M(A)M_n(B)) = \sum_{j \leq n} \sigma_{X_j}^2 \mu_j(A) \mu_j(B) = C_{M_n}(A \times B).$$  \hfill (68)$$

Of course formula (68) also holds when interchanging $A$ and $B$. Using this in (57) we conclude that $C_{M-M_n}$ is the finite measure given by

$$C_{M-M_n}(A \times B) = C_M(A \times B) - \sum_{j \leq n} \sigma_{X_j}^2 \mu_j(A) \mu_j(B),$$

or equivalently,

$$C_{M-M_n} = C_M - C_{M_n} = C_M - \sum_{j \leq n} \sigma_{X_j}^2 \mu_j \otimes \mu_j.$$  \hfill (70)$$

Since $M - M_n$ is finite, this is equivalent to have for every $\varphi, \phi \in \mathcal{M}_B(\mathbb{R}^d)$,

$$\langle C_{M-M_n}, \varphi \otimes \phi \rangle = \langle C_M, \varphi \otimes \phi \rangle - \sum_{j \leq n} \sigma_{X_j}^2 \langle \mu_j, \varphi \rangle \langle \mu_j, \phi \rangle.$$  \hfill (71)$$

$C_{M-M_n}$ is a covariance measure hence it is positive-semidefinite. Thus, $\langle C_{M-M_n}, \varphi \otimes \varphi \rangle \geq 0$ for every $\varphi \in \mathcal{M}_B(\mathbb{R}^d)$. This together with (71) allows to conclude

$$0 \leq \langle C_{M_n}, \varphi \otimes \varphi \rangle \leq \langle C_M, \varphi \otimes \varphi \rangle, \quad \forall n \in \mathbb{N}, \forall \varphi \in \mathcal{M}_B(\mathbb{R}^d).$$  \hfill (72)$$

Then, by CSI,

$$\sum_{j \leq n} \sigma_{X_j}^2 |\langle \mu_j, \varphi \rangle |^2 |\langle \mu_j, \phi \rangle | \leq \left[ \sum_{j \leq n} \sigma_{X_j}^2 |\langle \mu_j, \varphi \rangle |^2 \right] \left[ \sum_{j \leq n} \sigma_{X_j}^2 |\langle \mu_j, \phi \rangle |^2 \right]^{1/2}
\leq \langle C_{M_n}, \varphi \otimes \phi \rangle \langle C_{M_n}, \phi \otimes \phi \rangle
\leq \langle C_M, \varphi \otimes \varphi \rangle \langle C_M, \phi \otimes \phi \rangle < \infty.$$  \hfill (73)$$

This proves that the series $\sum_{j \in \mathbb{N}} \sigma_{X_j}^2 \langle \mu_j, \varphi \rangle \langle \mu_j, \phi \rangle$ is absolutely convergent for every $\varphi, \phi \in \mathcal{M}_B(\mathbb{R}^d)$. We can hence define a function $\Lambda : \mathcal{M}_B(\mathbb{R}^d) \times \mathcal{M}_B(\mathbb{R}^d) \to \mathbb{R}$ as

$$\Lambda(\varphi, \phi) := \sum_{j \in \mathbb{N}} \sigma_{X_j}^2 \langle \mu_j, \varphi \rangle \langle \mu_j, \phi \rangle.$$  \hfill (74)$$

$\Lambda$ is a bilinear positive-semidefinite function. It is clear that in order to complete the proof of Theorem 4.1 we need only to prove

$$\Lambda(\varphi, \phi) = \langle C_M, \varphi \otimes \phi \rangle, \quad \forall \varphi, \phi \in \mathcal{M}_B(\mathbb{R}^d).$$  \hfill (75)$$
Using the positive-semidefiniteness of $C_{M-M_n}$ and CSI, one has that (75) is equivalent to

$$\Lambda(\varphi, \varphi) = \langle C_M, \varphi \otimes \varphi \rangle, \quad \forall \varphi \in \mathcal{M}_B(\mathbb{R}^d).$$

(76)

The result (55) guarantees that (76) holds for $\varphi \in \mathcal{D}(\mathbb{R}^d)$. We will extend it to $\varphi \in \mathcal{M}_B(\mathbb{R}^d)$.

We begin by considering $\varphi$ of the form $\varphi = 1_I$, where $I \subset \mathbb{R}^d$ is a rectangle, that is, $I = I_1 \times \ldots \times I_d$, the set $I_j$ being an interval of $\mathbb{R}$ for every $j$. It is well-known that in such case $\varphi$ can be approximated point-wise by a sequence of functions in $\mathcal{D}(\mathbb{R}^d)$, being them all dominated by $\|\varphi\|_\infty$. We use hence LCDT considering the finite measure $|C_M|$. Let $\epsilon > 0$.

Then, there exists a function $\phi \in \mathcal{D}(\mathbb{R}^d)$ having $\|\phi\|_\infty \leq \|\varphi\|_\infty$, such that

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} |\varphi - \phi| \otimes |\varphi - \phi|d|C_M| < \epsilon^2. \quad \text{(77)}$$

We consider a typical application of triangular inequality in order to obtain

$$|\Lambda(\varphi, \varphi) - \langle C_M, \varphi \otimes \varphi \rangle| \leq \begin{aligned}
&\Lambda(\varphi, \varphi) - \sum_{j \leq n} \sigma_{X_j}^2 |\langle \mu_j, \varphi \rangle|^2 \\
&+ \sum_{j \leq n} \sigma_{X_j}^2 |\langle \mu_j, \varphi \rangle|^2 - \sum_{j \leq n} \sigma_{X_j}^2 |\langle \mu_j, \phi \rangle|^2 \\
&+ \sum_{j \leq n} \sigma_{X_j}^2 |\langle \mu_j, \phi \rangle|^2 - \langle C_M, \phi \otimes \phi \rangle \\
&+ \langle C_M, \phi \otimes \phi \rangle - \langle C_M, \varphi \otimes \varphi \rangle.
\end{aligned} \quad \text{(78)}$$

We will bound each one of the terms $(a), (b), (c), (d)$. Since $\varphi \in \mathcal{M}_B(\mathbb{R}^d)$, by definition of $\Lambda$ (74), the term $(a)$ goes to 0 as $n \to \infty$. The same idea goes for the term $(c)$, since (76) holds for $\phi \in \mathcal{D}(\mathbb{R}^d)$. There exists hence $n_0$ such that if $n \geq n_0$, $\max\{(a), (c)\} < \epsilon$. On the other
hand, one has for (b):

\[
(b) = \left| \sum_{j \in \mathbb{N}} \sigma^2_{X_j} \langle \mu_j \otimes \mu_j, \varphi \otimes \varphi - \phi \otimes \phi \rangle \right|
\]

\[
= \left| \sum_{j \in \mathbb{N}} \sigma^2_{X_j} \langle \mu_j \otimes \mu_j, \varphi \otimes (\varphi - \phi) + (\varphi - \phi) \otimes \phi \rangle \right|
\]

\[
\leq \left| \sum_{j \in \mathbb{N}} \sigma^2_{X_j} \langle \mu_j, \varphi - \phi \rangle \langle \mu_j, \varphi \rangle \right| + \left| \sum_{j \in \mathbb{N}} \sigma^2_{X_j} \langle \mu_j, \varphi \rangle \langle \mu_j, \phi \rangle \right|
\]  

(79)

We use \( \|\varphi\|_\infty \leq \|\varphi\|_\infty \) and the bounds (73) to obtain

\[
(b) \leq \sqrt{\langle C_M, \varphi \otimes \varphi \rangle \langle C_M, (\varphi - \phi) \otimes (\varphi - \phi) \rangle} + \sqrt{\langle C_M, \phi \otimes \phi \rangle \langle C_M, (\varphi - \phi) \otimes (\varphi - \phi) \rangle}
\]

\[
\leq \left( \sqrt{\|C_M|((\mathbb{R}^d \times \mathbb{R}^d)\|\varphi\|_\infty^2} + \sqrt{\|C_M|((\mathbb{R}^d \times \mathbb{R}^d)\|\phi\|_\infty^2} \right) \sqrt{\|C_M|, |\varphi - \phi| \otimes |\varphi - \phi|}
\]

\[
< \sqrt{\epsilon^2} \text{ from (77)}
\]

\[
\leq 2 \sqrt{\|C_M|((\mathbb{R}^d \times \mathbb{R}^d)\|\varphi\|_\infty\epsilon,
\]  

(80)

On the other hand, with similar arguments one has for (d):

\[
(d) = |\langle C_M, \varphi \otimes (\varphi - \phi) \rangle + \langle C_M, (\varphi - \phi) \otimes \phi \rangle|
\]

\[
\leq \sqrt{\langle C_M, \varphi \otimes \varphi \rangle \langle C_M, (\varphi - \phi) \otimes (\varphi - \phi) \rangle} + \sqrt{\langle C_M, \phi \otimes \phi \rangle \langle C_M, (\varphi - \phi) \otimes (\varphi - \phi) \rangle}
\]

\[
\leq 2 \sqrt{\|C_M|((\mathbb{R}^d \times \mathbb{R}^d)\|\varphi\|_\infty}\epsilon.
\]  

(81)

These bounds for (a), (b), (c) and (d) allow to conclude

\[
|\Lambda(\varphi, \varphi) - \langle C_M, \varphi \otimes \varphi \rangle| \leq (2 + 4 \sqrt{\|C_M|((\mathbb{R}^d \times \mathbb{R}^d)\|\varphi\|_\infty})\epsilon.
\]  

(82)

Since \( \epsilon \) was arbitrary, this implies \( \Lambda(\varphi, \varphi) = \langle C_M, \varphi \otimes \varphi \rangle \) for every \( \varphi \) of the form \( \varphi = 1_I \).

By bi-linearity of \( \Lambda \), we can easily extend this result to every \( \varphi \) in the space

\[
\mathcal{E} := \text{span} \{ 1_I \mid I \subset \mathbb{R}^d \text{ rectangle} \}.
\]  

(83)

Now, in order to extend this result to any \( \varphi \in \mathcal{M}_B(\mathbb{R}^d) \), we use Lusin’s Theorem 3.1 applied to the space \( \mathbb{R}^d \) with the finite measure \( |C_M|((\cdot \times \mathbb{R}^d) \). Hence, given the measurable function \( \varphi \) and given \( \epsilon > 0 \), there exists a closed set \( E \subset \mathbb{R}^d \) such that \( \varphi \) is continuous over \( E \) and
such that \( |C_M|(E^c \times \mathbb{R}^d) < \epsilon^2 \). In order to approximate \( \varphi \) over \( E \), we can consider a typical Riemann-alike approximation by the functions

\[
\phi_n := \sum_{j=1}^{n} \varphi(x_j^n) 1_{I_j^n},
\]

where for each \( n \), \( (I_j^n)_{j=1,...,n} \) is a finite collection of rectangles forming a partition of a subset \( K_n \) of \( \mathbb{R}^d \), satisfying that \( K_n \uparrow \mathbb{R}^d \) and \( \max_{j=1,...,n} \text{diam}(I_j^n) \to 0 \) as \( n \to \infty \), and \( x_j^n \in I_j^n \) is an arbitrary tag-point. With those conditions it is not difficult to prove that \( \phi_n(x) \to \varphi(x) \) for every \( x \) at which \( \varphi \) is continuous, hence in particular we have \( \phi_n \to \varphi \) point-wise over \( E \).

In addition, one has \( \|\phi_n\|_\infty \leq \|\varphi\|_\infty \) for every \( n \), and of course \( \phi_n \in \mathcal{E} \). Thus, from LDCT we can always find a function \( \phi \in \mathcal{E} \) with \( \|\phi\|_\infty \leq \|\varphi\|_\infty \) such that

\[
\int_{E \times E} |\varphi - \phi| \otimes |\varphi - \phi| d|C_M| < \epsilon^2.
\]

Using \( \|\varphi - \phi\| \otimes \|\varphi - \phi\| \leq 4\|\varphi\|^2 \) and the symmetry of \( |C_M| \) we obtain

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} |\varphi - \phi| \otimes |\varphi - \phi| d|C_M| = \int_{E \times E} |\varphi - \phi| \otimes |\varphi - \phi| d|C_M| + \int_{E^c \times E} |\varphi - \phi| \otimes |\varphi - \phi| d|C_M| + \int_{E^c \times E^c} |\varphi - \phi| \otimes |\varphi - \phi| d|C_M|
\]

\[
\leq \epsilon^2 + 8\|\varphi\|^2 |C_M|(E^c \times E) + 4\|\varphi\|^2 |C_M|(E^c \times E^c)
\]

\[
\leq \epsilon^2 + 12\|\varphi\|^2 |C_M|(E^c \times \mathbb{R}^d)
\]

\[
\leq (1 + 12\|\varphi\|^2)\epsilon^2.
\]

With this set up, we can study the expression \( \Lambda(\varphi, \varphi) - \langle C_M, \varphi \otimes \varphi \rangle \) for any \( \varphi \in \mathcal{M}_B(\mathbb{R}^d) \) by using the same arguments exposed in (88), using \( \phi \in \mathcal{E} \) constructed as above. Expressions (a) and (c) can be bounded by \( \epsilon \) for some \( n \) by similar arguments. For expression (b) we can follow arguments (80) and use the result (86) and obtain

\[
\left| \sum_{j \leq n} \sigma_{X_j}^2 |\langle \mu_j, \varphi \rangle|^2 - \sum_{j \leq n} \sigma_{X_j}^2 |\langle \mu_j, \phi \rangle|^2 \right| \leq 2 \sqrt{|C_M|(\mathbb{R}^d \times \mathbb{R}^d)} \|\varphi\|_\infty \sqrt{(1 + 12\|\varphi\|^2)} \epsilon.
\]

The same arguments are given to bound (d) as well, starting from (81):

\[
|\langle C_M, \varphi \otimes \varphi \rangle - \langle C_M, \phi \otimes \phi \rangle| \leq 2 \sqrt{|C_M|(\mathbb{R}^d \times \mathbb{R}^d)} \|\varphi\|_\infty \sqrt{(1 + 12\|\varphi\|^2)} \epsilon.
\]
We can thus conclude
\[
|\Lambda(\varphi, \varphi) - \langle C_M, \varphi \otimes \varphi \rangle| \leq (2 + 4 \sqrt{|C_M(\mathbb{R}^d \times \mathbb{R}^d)} \|\varphi\|_\infty \sqrt{1 + 12 \|\varphi\|^2_\infty}) \epsilon. \tag{89}
\]

Since \(\varphi\) and \(\epsilon\) were arbitrary, we finally obtain
\[
\Lambda(\varphi, \varphi) = \langle C_M, \varphi \otimes \varphi \rangle = \sum_{j \in \mathbb{N}} \sigma_j^2 \langle \mu_j, \varphi \rangle^2, \quad \forall \varphi \in \mathcal{M}_B(\mathbb{R}^d). \tag{90}
\]

This proves hence the result (38).

Concerning the expansion of the covariance \(C_M\), it is clear that (39) holds in the sense
\[
\langle C_M, \varphi \otimes \phi \rangle = \sum_{j \in \mathbb{N}} \sigma_j^2 \langle \mu_j, \varphi \rangle \langle \mu_j, \phi \rangle, \quad \forall \varphi, \phi \in \mathcal{M}_B(\mathbb{R}^d). \tag{91}
\]

Let us prove the proposed slightly stronger mode of convergence. Let \(\varphi \in \mathcal{M}_B(\mathbb{R}^d)\). We use expression (18) for the total-variation measure of \(\langle C_M - M_n, \varphi \otimes \cdot \rangle\). We remind expression (70) which implies both \(C_M - M_n = C_M - C_{M_n}\) and that \(\langle C_M - M_n, \phi \otimes \phi \rangle \leq \langle C_M, \phi \otimes \phi \rangle\) for every \(\phi \in \mathcal{M}_B(\mathbb{R}^d)\). We have hence
\[
|\langle C_M - C_{M_n}, \varphi \otimes \cdot \rangle(\mathbb{R}^d)| = |\langle C_M - C_{M_n}, \varphi \otimes \cdot \rangle(\mathbb{R}^d)|
= \sup_{\phi \in \mathcal{M}(\mathbb{R}^d), |\phi| = 1} |\langle C_M - C_{M_n}, \varphi \otimes \phi \rangle|
\leq \sup_{\phi \in \mathcal{M}(\mathbb{R}^d), |\phi| = 1} \sqrt{|\langle C_M - C_{M_n}, \varphi \otimes \varphi \rangle| \langle C_M - C_{M_n}, \phi \otimes \phi \rangle}
\leq \sqrt{|C_M - C_{M_n}, \varphi \otimes \varphi \rangle |C_M|((\mathbb{R}^d \times \mathbb{R}^d) \rightarrow 0, \tag{92}
\]

which proves the desired result. ■

For the case of non-finite regular random measures we will use the following definition.

**Definition 4.1.** A regular random measure \(M\) is said to be regulated by a function, if there exists a strictly positive and locally bounded function \(f \in \mathcal{M}(\mathbb{R}^d)\) such that \(\frac{1}{f \otimes f} C_M\) is a finite measure.

It is not difficult to prove that if \(M\) is regulated by \(f\), then \(\frac{1}{f} M\) is finite. The following result is obtained immediately from Theorem 4.1.
Theorem 4.2. Let $M$ be a regular random measure over $\mathbb{R}^d$ regulated by a function $f$. Then, there exist a sequence of pairwise uncorrelated random variables with summable variances $(X_n)_{n \in \mathbb{N}}$, and a sequence of real measures over $\mathbb{R}^d$, $(\mu_n)_{n \in \mathbb{N}}$, all of them such that $\frac{1}{n^2} |\mu_n|$ is a finite measure, such that

$$\langle M, \varphi \rangle = \sum_{j \in \mathbb{N}} X_j \langle \mu_j, \varphi \rangle, \quad \forall \varphi \in \mathcal{M}_{B,c}(\mathbb{R}^d),$$

with the series being considered in a mean-square sense. For $C_M$ expansion (39) holds, the convergence of the series being in a separately $\mathcal{M}_{B,c}(\mathbb{R}^d)^*\text{-}\text{weak}^*\text{—total-variation-on-compacts}$ sense, that is,

$$\left| \langle C_M, \varphi \otimes (\cdot) \rangle - \sum_{j \leq n} \sigma^2_{X_j} \langle \mu_j, \varphi \rangle \mu_j \right| (K) \xrightarrow{n \to \infty} 0, \quad \forall \varphi \in \mathcal{M}_{B,c}(\mathbb{R}^d) \forall K \subset \mathbb{R}^d \text{ compact}.$$  

Proof: We apply Theorem 4.1 to the finite random measure $\frac{1}{f} M$, obtaining thus

$$\langle \frac{1}{f} M, \phi \rangle = \sum_{j \in \mathbb{N}} X_j \langle \nu_j, \phi \rangle, \quad \phi \in \mathcal{M}_B(\mathbb{R}^d).$$

The measures $\nu_j$ are of course finite. If $\varphi \in \mathcal{M}_{B,c}(\mathbb{R}^d)$, since $f$ is locally bounded then $f \varphi \in \mathcal{M}_{B,c}(\mathbb{R}^d) \subset \mathcal{M}_B(\mathbb{R}^d)$. Since $f$ is positive we have of course

$$\langle M, \varphi \rangle = \langle \frac{1}{f} M, f \varphi \rangle,$$

hence

$$\langle M, \varphi \rangle = \sum_{j \in \mathbb{N}} X_j \langle f \nu_j, \varphi \rangle, \quad \varphi \in \mathcal{M}_{B,c}(\mathbb{R}^d).$$

Setting $\mu_j := f \nu_j$ we obtain then the desired sequence of measures for which $\frac{1}{f} \mu_j = \nu_j$ is finite for every $j$. The convergence of the covariance (94) is obtained following the same arguments used at the end of the proof of Theorem 4.1 ■

While it is not clear if a general regular random measure $M$ can be regulated by a function $f$, it is possible to do so for some important classes of random measures. For instance, tempered random measures which are regulated by a polynomial. Such random measures
determine tempered random distributions over \( \mathbb{R}^d \). This is the case for instance of White Noise and of second-order stationary random measures (that is, with translation invariant covariance structure). Another important class of random measures when Theorem 4.2 can be applied is the case of orthogonal random measures. An orthogonal random measure is a regular random measure \( M \) such that \( C_M(A \times B) = \mu(A \cap B) \) for some positive control measure \( \mu \). Since for every positive (locally-finite) measure \( \mu \) we can find a function \( f \) such that \( f^2 \) regulates \( \mu \) (in an analogous sense to definition 4.1), we can thus find \( f \) such that \( \frac{1}{f} M \) is a finite random measure and thus obtain the result.

5 Concluding remarks

We finish with some remarks about the obtained results.

**Remark 5.1.** The reader may wonder about a direct connection between the expansion in Theorem 4.1 and the Hilbert space approach presented in Section 2. For that, we remark that \( \mathcal{O}(\mu) \) belongs to \( L^2(\mathbb{R}^d, \nu) \) for every finite measure \( \mu \) (\( \nu \) defined as in (42)). If \( E \) is the space of finite measures over \( \mathbb{R}^d \), we can endow \( E \) with the bi-linear positive-semidefinite form

\[
(\mu_1, \mu_2)_E \mapsto (\mathcal{O}(\mu_1), \mathcal{O}(\mu_2))_{L^2(\mathbb{R}^d, \nu)}.
\]  

From the continuity of \( \mathcal{O}(\mu) \) and the right-continuity at every component of \( \vec{u} \mapsto \mu((-\infty, \vec{u}]) \), one can conclude that \( (\mu, \mu)_E = 0 \) if and only if \( \mu = 0 \). The completion of \( E \) with this product will be hence an abstract Hilbert space with respect to which a Karhunen-Loève decomposition as expressed in Section 2 can be stated.

**Remark 5.2.** Results 4.1 and 4.2 can be used to obtain, as corollaries, diverse forms of Karhunen-Loève expansions of not regular stochastic processes over \( \mathbb{R}^d \). For instance, if we have a process of the form \( Z(\vec{x}) = M((-\infty, \vec{x}]) \), with \( M \) being a finite regular random measure, then an immediate application of Theorem 4.1 to the measurable and bounded function \( 1_{(-\infty, \vec{x}]} \) implies a traceable Karhunen-Loève decomposition for \( Z \) of the form

\[
Z = \sum_{n \in \mathbb{N}} X_n g_n,
\]

24
with the functions \( g_n \) being all bounded and right-continuous at every component \( (g_n(\vec{x}) = \mu_j((\infty, \vec{x}])) \). The convergence of the series (99) is mean-square-point-wise. This can be applied to the case of right-continuous with left-limits process, as for example Lévy processes, in dimension \( d = 1 \). Other examples can be given by choosing parameter depending functions \( \varphi \) in equation (38).

**Remark 5.3.** For a general non-finite regular random measure \( M \) a Karhunen-Loève decomposition can be done locally. That is, for every bounded Borel set \( D \), \( M \) has a decomposition of the form (38) for \( \varphi \in \mathcal{M}_B(\mathbb{R}^d) \) null outside \( D \). This because the restriction measure of \( M \) over \( D \) is a finite measure for which Theorem 4.1 holds. In such case the measures \( \mu_j \) and the random variables \( X_j \) depend upon the set \( D \).

**Remark 5.4.** Of course, if the process \( M \) is Gaussian, the random variables in the Karhunen-Loève expansion are independent. It is also possible to prove in that case, that the convergence of the series (38) holds almost-surely.

## Appendices

### A Proof of Stochastic Fubini Theorem

The first issue with Stochastic Fubini theorem 3.2 is the proper definition of the iterated integrals presented in (26). From condition (i), a typical use of deterministic Fubini Theorem implies that the function \( \int_{\mathbb{R}^m} \psi(\cdot, u)d\mu(u) \otimes \int_{\mathbb{R}^m} \psi(\cdot, v)d\mu(v) \) is in \( L^1(\mathbb{R}^d \times \mathbb{R}^d, |C_M|) \). Hence, the integral of \( \int_{\mathbb{R}^m} \psi(\cdot, u)d\mu(u) \) with respect to \( M \) over \( \mathbb{R}^d \), which is the integral at the left side of (26) is a well-defined stochastic integral. Let us look at the one at the right side.

As it is mentioned in the theorem, from condition (i) it follows that the function

\[
C_Z(u, v) := \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x, u)\psi(y, v)dC_M(x, y)
\]

is \( |\mu| \otimes |\mu| \)-almost-everywhere defined over \( \mathbb{R}^m \times \mathbb{R}^m \). It follows that the stochastic process over \( \mathbb{R}^m \)

\[
Z(u) = \int_{\mathbb{R}^d} \psi(x, u)dM(x),
\]

is
is well-defined for every \( u \in \mathbb{R}^m \) outside \( \mu \)-null set \( D \), and its covariance function is \( C_Z \). The extra continuity supposition \( \text{(ii)} \) says that \( C_Z \) is continuous over \( E \times E \), which implies that \( Z \) is mean-square continuous over \( E \) (we may assume \( D \subset E^c \)). In such case, the definition of the stochastic integral

\[
\int_{\mathbb{R}^m} Z(u) d\mu(u)
\]

can be done with a Riemann approximation which we will make precise. We consider for each \( n \in \mathbb{N} \) a finite collection of rectangles \( (I^n_j)_{j=1,...,n} \) forming a partition of a subset \( K_n \) such that \( K_n \nsubseteq \mathbb{R}^m \) and \( \max_{j=1,...,n} \text{diam}(I^n_j) \to 0 \) as \( n \to \infty \). For every \( I^n_j \) such that \( E \cap I^n_j \neq \emptyset \), we set an arbitrary tag-point \( u^n_j \in E \cap I^n_j \). Then, we have

\[
\int_{\mathbb{R}^m} Z(u) d\mu(u) := \lim_{n \to \infty} \sum_{j=1}^{n} Z(u^n_j) \mu(I^n_j \cap E),
\]

where the limit is considered in a mean-square sense. It is known that such limit exists and it is not depending on the selection of the partitions or tag-points when \( \text{Soong & TT, 1973, Theorem 4.5.2} \)

\[
\int_{\mathbb{R}^m \times \mathbb{R}^m} |C_Z| d|\mu| \otimes |\mu| < \infty. \tag{104}
\]

This is guaranteed by condition \( \text{(i)} \). Hence the integral \( 102 \) is well-defined, and it coincides with the integral at the right-side of equation \( 26 \). Of course, we have exploited the fact that \( |\mu|(E^c) = 0 \) in order to implicitly have

\[
\int_{\mathbb{R}^m} Z(u) d\mu(u) = \int_{E} Z(u) d\mu(u), \quad \int_{E^c} Z(u) d\mu(u) = 0. \tag{105}
\]

Let us now consider the variance of the difference between the integrals

\[
a := \mathbb{E} \left( \left| \int_{\mathbb{R}^m} \int_{\mathbb{R}^d} \psi(x, u) dM(x) d\mu(u) - \int_{\mathbb{R}^d} \int_{\mathbb{R}^m} \psi(x, u) d\mu(u) dM(x) \right|^2 \right). \tag{106}
\]

\(^2\)The author is not aware of any result concerning the proper definition of the mean-square stochastic integral \( 102 \) when \( Z \) has a measurable covariance \( C_Z \) with no any other regularity property more than \( 104 \). This is the only reason why the extra continuity condition \( \text{(ii)} \) has been required in the Stochastic Fubini Theorem 3.2. It is expected that less strict conditions may also work. \( \text{Gill, 1987} \) proposes, for instance, sufficient second-order conditions on \( Z \) which make it have a measurable modification, which could be useful in this aim.
Using the limit expression \((103)\) one has

\[
a = \lim_{n \to \infty} \mathbb{E} \left( \sum_{j=1}^{n} \int_{\mathbb{R}^d} \psi(x, u_j^n) dM(x) \mu(I_j^n \cap E) - \int_{\mathbb{R}^d} \int_{\mathbb{R}^m} \psi(y, v) d\mu(v) dM(y) \right)^2
\]

\[
= \lim_{n \to \infty} \sum_{j=1}^{n} \sum_{k=1}^{n} \mathbb{E} \left( \int_{\mathbb{R}^d} \psi(x, u_j^n) dM(x) \int_{\mathbb{R}^d} \psi(y, u_k^n) dM(x) \right) \mu(I_j^n \cap E) \mu(I_k^n \cap E)
\]

\[
- 2 \lim_{n \to \infty} \sum_{j=1}^{n} \int_{\mathbb{R}^d} \psi(x, u_j^n) \int_{\mathbb{R}^m} \psi(y, v) d\mu(v) dC_M(x, y) \mu(I_j^n \cap E)
\]

\[
+ \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^m} \psi(x, u) d\mu(u) \int_{\mathbb{R}^m} \psi(y, v) d\mu(v) dC_M(x, y).
\]

For the expectations we use \((24)\) in order to obtain

\[
a = \lim_{n \to \infty} \sum_{j=1}^{n} \sum_{k=1}^{n} \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x, u_j^n) \psi(y, u_k^n) dC_M(x, y) \mu(I_j^n \cap E) \mu(I_k^n \cap E)
\]

\[
- 2 \lim_{n \to \infty} \sum_{j=1}^{n} \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x, u_j^n) \int_{\mathbb{R}^m} \psi(y, v) d\mu(v) dC_M(x, y) \mu(I_j^n \cap E)
\]

\[
+ \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^m} \psi(x, u) d\mu(u) \int_{\mathbb{R}^m} \psi(y, v) d\mu(v) dC_M(x, y).
\]

Using the continuity of the function \(C_Z\) over the \(|\mu| \otimes |\mu|\)-full set \(E \times E\), we have the Riemann approximation of the integral

\[
\lim_{n \to \infty} \sum_{j=1}^{n} \sum_{k=1}^{n} \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x, u_j^n) \psi(y, u_k^n) dC_M(x, y) \mu(I_j^n \cap E) \mu(I_k^n \cap E)
\]

\[
= \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^m \times \mathbb{R}^m} \psi(x, u) \psi(y, v) dC_M(x, y) d(\mu \otimes \mu)(u, v),
\]

which exists due to condition \((110)\). On the other hand, by deterministic Fubini Theorem,

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x, u) \int_{\mathbb{R}^m} \psi(y, v) d\mu(v) dC_M(x, y) = \int_{\mathbb{R}^m} C_Z(u, v) d\mu(v),
\]

\(|\mu|(u)-almost everywhere. Let us verify that the function \(u \mapsto \int_{\mathbb{R}^m} C_Z(u, v) d\mu(v)\) is continuous over \(E\). For that, we see

\[
\left| \int_{\mathbb{R}^m} C_Z(u, v) - C_Z(w, v) d\mu(v) \right| = \int_E \left| C_Z(u, v) - C_Z(w, v) \right| d\mu(v) \leq \int_E \left| C_Z(u, v) - C_Z(w, v) \right| d|\mu|(v).
\]

27
Since $|\mu|$ is finite and $C_Z$ is continuous over $E \times E$, it is clear that $C_Z$ goes to 0 as $w \to v \in E$, proving hence the continuity over $E$ of $u \mapsto \int_{\mathbb{R}^m} C_Z(u, v) d\mu(v)$. From (i) this function is integrable with respect to $\mu$ and its integral can be approximated by Riemann sums. Hence,

$$
\lim_{n \to \infty} \sum_{j=1}^{n} \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x, u_j^n) \int_{\mathbb{R}^m} \psi(y, v) d\mu(v) dC_M(x, y) \mu(I_j^n \cap E) = \int_{\mathbb{R}^m} \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x, u) \int_{\mathbb{R}^m} \psi(y, v) d\mu(v) dC_M(x, y) d\mu(u).
$$

Using thus the limits (109) and (112), one obtains

$$
a = \int_{\mathbb{R}^m \times \mathbb{R}^m} \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x, u) \psi(y, v) dC_M(x, y) (\mu \otimes \mu)(u, v)
- 2 \int_{\mathbb{R}^m} \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x, u) \int_{\mathbb{R}^m} \psi(y, v) d\mu(v) dC_M(x, y) d\mu(u)
+ \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^m} \psi(x, u) d\mu(u) \int_{\mathbb{R}^m} \psi(y, v) d\mu(v) dC_M(x, y).
$$

But the deterministic Fubini Theorem, guaranteed by condition (i) and the fact that $C_M$ is a measure, allows to conclude the equality between all these deterministic multiple integrals, implying $a = 0$. This proves the equality of the stochastic integrals (26) as random variables in $L^2(\Omega, \mathcal{A}, \mathbb{P})$. ■

### B Properties of the Regularising Operator

The first claim in Section 3.3 concerning the operator $O$ acting over a finite measure $\mu$ is that $O(\mu)$ is a continuous function. This holds because it consists of $d$ iterated Riemann integrals of right-continuous bounded functions. The derivative property (31) is justified by a convolution argument which we make precise for the case $d = 1$. In such case, we may re-write $O(\mu)$ as

$$
O(\mu) = 1_{[0, \infty)} * \left[ (1_{[0, \infty)} * \mu) 1_{[0, \infty)} \right] - 1_{(-\infty, 0)} * \left[ (1_{[0, \infty)} * \mu) 1_{(-\infty, 0)} \right].
$$

where $*$ denotes the convolution operation. Using the properties of the convolution with respect to derivatives, that the Dirac measure $\delta$ (at 0) is its identity element, and that $\frac{d}{dx} 1_{[0, \infty)} =$
\( \delta \), one has
\[
\frac{d}{dx} O(\mu) = \delta \left[ (1_{[0,\infty]} \ast \mu) 1_{[0,\infty]} \right] - (-\delta) \left[ (1_{[0,\infty]} \ast \mu) 1_{(-\infty,0]} \right]
\]
\[
= (1_{[0,\infty]} \ast \mu) 1_{[0,\infty]} + (1_{[0,\infty]} \ast \mu) 1_{(-\infty,0]}
\]
\[= 1_{[0,\infty]} \ast \mu. \tag{115} \]

Deriving again,
\[
\frac{d^2}{dx^2} O(\mu) = \delta \ast \mu = \mu. \tag{116} \]

The case for \( d > 1 \) is analogous though requiring a more tedious notation.

When applying operator \( O \) to a regular finite random measure \( M \), the question about the proper definition of the process \( O(M) \) arises. Let us verify its definition. First, the process
\[
V(\vec{u}) := M((-\infty, \vec{u}]), \quad \vec{u} \in \mathbb{R}^d, \tag{117}
\]
has a covariance function (Eq. (24))
\[
C_V(\vec{u}, \vec{v}) = C_M((-\infty, \vec{u}] \times (-\infty, \vec{v}]). \tag{118}
\]

Consider the function \( \vec{u} \mapsto |C_M|([\mathbb{R}^d \times (-\infty, \vec{u}])] \). This function is increasing and bounded (and càdlàg) in each component of \( \vec{u} \) when the other are fixed. It is known that such kind of functions are continuous almost-everywhere over \( \mathbb{R}^d \). Let \( E \) be the set of continuity points of such function. Let us verify that \( C_V \) is continuous over \( E \times E \). Since \( C_V \) is a covariance function, it is continuous at \( E \times E \) if it is continuous at diagonal points \((\vec{u}, \vec{u}) \in E \times E \) (Soong & TT, 1973, Theorem 4.3.3). For such points we have
\[
|C_V(\vec{u} + \vec{h}, \vec{u} + \vec{k}) - C_V(\vec{u}, \vec{u})| \leq |C_V(\vec{u} + \vec{h}, \vec{u} + \vec{k}) - C_V(\vec{u} + \vec{h}, \vec{u})| + |C_V(\vec{u} + \vec{h}, \vec{u}) - C_V(\vec{u}, \vec{u})| \tag{119}
\]

Now, using (118), we see that differences of the form \( |C_V(\vec{a}, \vec{b}) - C_V(\vec{a}, \vec{c})| \) are actually of the form \( |C_M(A \times B) - C_M(A \times C)| \) for three sets \( A, B, C \). Since \( C_M \) is a finite measure, one has
\[
|C_M(A \times B) - C_M(A \times C)| \leq |C_M|(A \times (B \triangle C)) \leq |C_M|([\mathbb{R}^d \times (B \triangle C)]), \tag{120}
\]

29
where $B \triangle C$ denotes the symmetric difference between the sets $B$ and $C$. In our case, the sets which play the role of $B$ and $C$ are of the form $(-\infty, \vec{u} + \vec{h}]$ and $(-\infty, \vec{u}]$. In such case we have

$$(-\infty, \vec{u} + \vec{h}] \triangle (-\infty, \vec{u}] \subset \overline{B_{|\vec{h}|}}(\vec{u}), \quad (121)$$

where $\overline{B_{|\vec{h}|}}(\vec{u})$ denotes the closed ball of radius $|\vec{h}|$, centred at $\vec{u}$ with respect to the $\infty$-norm over $\mathbb{R}^d$. $|\vec{h}|_\infty$ denotes the $\infty$-norm of the vector $\vec{h} \in \mathbb{R}^d$. We conclude thus

$$|C_V(\vec{u} + \vec{h}, \vec{u} + \vec{k}) - C_V(\vec{u}, \vec{u})| \leq |C_M|(\mathbb{R}^d \times \overline{B_{|\vec{h}|}}(\vec{u})) + |C_M|(\mathbb{R}^d \times \overline{B_{|\vec{h}|}}(\vec{u})) \quad (122)$$

(we have used the symmetry of $C_M$). Since the function $\vec{u} \mapsto |C_M|(\mathbb{R}^d \times (-\infty, \vec{u}])$ is continuous at $\vec{u} \in E$, expression (122) goes to 0 as $\vec{h}, \vec{k} \to 0$, thus $C_V$ is continuous over $E \times E$.

The definition of $O(M)$ involves hence the stochastic integral of a stochastic process with respect to the Lebesgue measure (a Riemann integral actually), the process being continuous almost-everywhere. This integral is hence well-defined, as it has been stated in Appendix A. The expression of the covariance (34) follows from the properties of such integrals.

The derivative relation (36) comes from Stochastic Fubini Theorem. Indeed, if $\varphi \in \mathcal{D}(\mathbb{R}^d)$ then,

$$\int_{\mathbb{R}^d} O(M)(\vec{x}) \frac{\partial^{2d} \varphi}{\partial x_1^2 \cdots \partial x_d^2}(\vec{x}) d\vec{x} = \int_{\mathbb{R}^d} \int_0^\infty \int_{\mathbb{R}^d} 1_{(-\infty, \vec{u}]}(s) dM(s) d\vec{u} \frac{\partial^{2d} \varphi}{\partial x_1^2 \cdots \partial x_d^2}(\vec{x}) d\vec{x}
= \int_{\mathbb{R}^d} \int_0^\infty \int_0^\infty \delta_s((\vec{u}, \vec{u}]) d\vec{u} \frac{\partial^{2d} \varphi}{\partial x_1^2 \cdots \partial x_d^2}(\vec{x}) d\vec{x} dM(s)
= \int_{\mathbb{R}^d} \langle \delta_s, \varphi \rangle dM(s) = \int_{\mathbb{R}^d} \varphi(s) dM(s). \quad (123)$$

Here we have used (32). The arguments which verify that the hypotheses in Stochastic Fubini Theorem 3.2 hold are quite similar to the ones used in the Proof of Theorem 4.1 (see Eq. (60) and Eq. (62)). The details are left to the reader.
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