General radially moving references frames in the black hole background

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Abstract We consider general radially moving frames realized in the background of nonextremal black holes having causal structure similar to that of the Schwarzschild metric. In doing so, we generalize the Lemaître approach, constructing free-falling frames which are built from the reference particles with an arbitrary specific energy \( e_0 \) including \( e_0 < 0 \) and a special case \( e_0 = 0 \). The general formula of 3-velocity of a freely falling particle with the specific energy \( e \) with respect to a frame with \( e_0 \) is presented. We point out the relation between the properties of considered frames near a horizon and the Banados–Silk–West effect of an indefinite growth of energy of particle collisions. Using our radially moving frames, we consider also nonradial motion of test particles including the regions near the horizon and singularity. We also point out the properties of the Lemaître time at horizons depending on the frame and sign of particle energy.

1 Introduction

The Schwarzschild metric \([1]\) is a part of primer of black hole physics and enters all textbooks on gravitation. In spite of this, it still remains a testing area of different approaches, including classes of coordinate transformations. As is known, this metric is singular in the original (so-called curvature, or Schwarzschild) coordinates. There exists completely different methods to remove such a seeming singularity. These approaches can be united in a one picture \([2]\). What is especially interesting is that if the specific energy \( e_0 \) of reference particles (i.e. particle realizing a frame) is included explicitly in the coordinate transformation, the different standard forms can be obtained as different limiting transitions. In this approach, one can recover some well-known metric like the Eddington–Finkelstein ones \([3,4]\). The Lemaître metric can be also included in this scheme \([5,6]\). (Alternatively, one can use a velocity of the local Lorentz transformation instead of \( e_0 \) \([7]\).) Meanwhile, there exists one more aspect connected not only with the frames themselves but with particle dynamics in corresponding background. One may ask, how particle motion looks like depending on \( e_0 \) and particle specific energy \( e \) and relation between them.

In the previous paper \([8]\) we considered such frames that all reference particles have \( e_0 = 1 \) or \( e_0 = 0 \). In the present work, we make the next step and consider a more general situation when \( e_0 \) is arbitrary. In particular, this includes the case of \( e_0 < 0 \). Motivation for such generalization is at least three-fold. (i) In the aforementioned papers, the introduction of \( e_0 \) was made for frames, now we consider particle dynamics. (ii) If we make \( e_0 \) a free parameter, we can trace the relation between reference particles and any other test particles thus establishing connection between the choice of a frame and properties of particle collisions. This is especially actual in the context of high energy particle collisions \([9]\). (iii) We hope that a general approach developed in our work will be useful tool for description of particle motion under the horizon, some concrete examples of which in the Schwarzschild background were discussed in \([10]\) (see also references therein).

In this work, we develop general formalism. In doing so, we suggest simple classification of frames based on their character (contracting or expanding) and the sign of \( e_0 \). The applications of our formalism will be considered in a next paper. Also, we restrict ourselves by static black holes and postpone the generalization to rotating black holes to future works.

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In what follows we deal with the spherically symmetric metrics of the form
\[ ds^2 = -f\,dt^2 + \frac{dr^2}{f} + r^2(d\theta^2 + d\phi^2 \sin^2 \theta). \] (1)

For the Schwarzschild metric, \( f = 1 - \frac{r_+}{r} \) where \( r_+ \) is the radius of the event horizon.

\section{Reference particles and frames}

One of known frames that removes the coordinate singularity on the horizon is the Gullstrand–Painlevé (GP) one \cite{11,12}. In recent years, this frame again attracts attention in different contexts (see \cite{6} and references therein). Quite recently, some modification of the GP frame was suggested in \cite{13}. The generalization of the original GP frame can be obtained after introducing a new time variable via
\[ d\tilde{t} = e_0 dt + \frac{dr}{f} P_0, \] (2)
where by definition
\[ P_0 \equiv \sqrt{e_0^2 - f}. \] (3)

The static time \( t \) is expressed now as
\[ dt = \frac{1}{e_0} \left( d\tilde{t} - \frac{dr}{f} P_0 \right) \] (4)
that gives us the metric in the form
\[ ds^2 = -\frac{f}{e_0^2} d\tilde{t}^2 + 2\frac{d\tilde{t}dr}{e_0} P_0 + \frac{dr^2}{e_0^2} + r^2 d\omega^2. \] (5)

After introducing a new spatial variable \( \rho \) via
\[ d\rho = \frac{dr}{P_0} + d\tilde{t}, \] (6)
the metric can be set to a synchronous form
\[ ds^2 = -d\tilde{t}^2 + \frac{P_0^2}{e_0^2} d\rho^2 + r^2 d\omega^2. \] (7)

In this form it is evident that the coordinate system is formed by particles, free falling with the specific energy \( e_0 \). Indeed, for a radial fall with the the specific energy \( e \) we have the following equations of motion:
\[ \frac{dt}{d\tau} = e, \] (8)
\[ \frac{dr}{d\tau} = -\sqrt{e^2 - f} = -P, \] (9)
where \( \tau \) is the proper time.

Now we can consider the derivative
\[ \frac{d\tilde{t}}{d\tau} = e_0 \frac{dt}{d\tau} + \frac{dr}{d\tau} \frac{P_0}{f} = \frac{e_0 e - P_0 P}{f}, \] (10)
where \( \tilde{t} \) is the \( e_0 \)-synchronous time, and \( \tau \) is the proper time of a particle with the energy \( e \). This entails
\[ \frac{dr}{dt} = -\frac{P f}{e_0 e - P_0 P} \] (11)
and
\[ \frac{d\rho}{d\tau} = \frac{e_0 (e P_0 - e_0 P)}{f} \] (12)
that gives us the rate with which the synchronous spatial coordinate \( \rho \) changes for a free falling particle with the energy \( e \). If \( e = e_0 \), it follows that \( P = P_0 \) and
\[ \frac{d\rho}{d\tau} = 0, \] (13)
as is should be. We see that indeed the particle with the energy \( e_0 \) has synchronous spatial coordinate \( \rho \) constant during a free fall.

We can note also that for \( e_0 = 1 \) the coordinate flow velocity \( dr/d\tilde{t} = -\sqrt{1-f} \) coincides with the velocity of a particle with respect to a stationary frame. For a general case the velocity of the particle having \( e \) with respect to a stationary frame is equal to \( v_{st} = \sqrt{(e^2 - f)/e^2} \) (see eqs. 4.2, 4.4 and 6.5 in \cite{14}), so that \( v_{st} = P/e = (dr/d\tilde{t})/e \).

Let \( e_0 \rightarrow 0 \). We can easily see that there is no smooth limit for the generalized GP metric \( (5) \). However, as it has been pointed out in \cite{6}, we can write a regular form of the synchronous metric if we make a redefinition \( \rho = e_0 \tilde{\rho} \). As now
\[ e_0 d\tilde{\rho} = \frac{dr}{P_0} + d\tilde{t}, \] (14)
we see than in this limit \( r \) is the function of \( \tilde{t} \) only and we get from \( (7) \) a homogeneous metric under the horizon where we set \( f \equiv -g \):
\[ ds^2 = -d\tilde{t}^2 + g(r(\tilde{t}))d\tilde{\rho}^2 + r^2(\tilde{t})d\omega^2. \] (15)

So far, we have considered only positive energy ingoing particles and frames. To include negative \( e \) and outgoing motion, we generalize the transformation to a synchronous frame. One can write
\[ d\tilde{t} = e_0 dt - \frac{dr}{f} \sigma_0 P_0, \] (16)
\[ d\rho = \frac{dr}{P_0} + \delta d\tilde{t} = \frac{dr}{f P_0} [f (1 + \delta \sigma_0) - e_0^2 \sigma_0 \delta] + \delta e_0 dt, \] (17)
Here \( \sigma_0 = \pm 1 \), \( \delta = \pm 1 \) and we assumed that \( \left( \frac{\partial \rho}{\partial \tau} \right)_\delta > 0 \).

We can introduce the function \( \chi \) according to
\[ \chi(r) = \int \frac{dr}{P_0}. \] (18)

Then, for \( \delta = +1 \)
\[ \chi = \rho - \hat{t} \]

and for \( \delta = -1, \)

\[ \chi = \rho + \hat{t}. \]  

Thus for the contracting and expanding frame the dependences \( r(\rho, \hat{t}) \) are different.

The inverse formulas read

\[ dt = \frac{1}{e_0} \left( d\hat{t} + \frac{\sigma_0 P_0}{f} d\rho \right), \]  

\[ d\rho = \frac{d\hat{t}}{P_0} - \delta P_0 d\hat{t}. \]

We want the metric in new coordinates to be diagonal, so the cross-terms with \( d\hat{t} d\rho \) should cancel. This leads to the condition

\[ \delta = -\sigma_0. \]  

Thus we have

\[ d\rho = \frac{d\hat{t} e_0^2}{f P_0} - \sigma_0 e_0 dt. \]

For the contracting systems, \( \frac{d\rho}{dt} < 0 \) along the line with \( \rho = \text{const}, \) so \( \delta = +1 = -\sigma_0. \) For the expanding one, \( \delta = -1, \sigma_0 = +1. \) The synchronous form reads

\[ ds^2 = -d\hat{t}^2 + \frac{P_0^2}{e_0^2} d\rho^2 + r^2 d\omega^2 \]

both for the contracting and expanding frames.

For a general radial motion of a geodesic particle with a specific energy \( e \) the equations of motion give us (8) and

\[ \frac{d\rho}{d\tau} = \sigma P, \]

where \( \sigma = \pm 1 \) depending on the direction of motion.

Correspondingly, we also have from (16) and (24)

\[ \frac{d\hat{t}}{d\tau} = \frac{ee_0 - \sigma_0 \sigma P P_0}{f}, \]  

\[ \frac{dr}{d\tilde{t}} = \frac{P f}{ee_0 \sigma - \sigma_0 P P_0} = \frac{P \sigma f}{Y}, \]

where \( Y = ee_0 - \sigma_0 \sigma P P_0, \)

\[ \frac{d\rho}{d\tau} = \frac{\sigma P}{f P_0 e_0^2} - \sigma_0 e_0 e, \]  

\[ \frac{d\rho}{d\tau} = \frac{ee_0}{f} (\sigma - \sigma_0), \]

\[ \frac{d\hat{t}}{d\tau} = \frac{ee_0^2 - \sigma_0 \sigma P_0^2}{f}. \]

For reference particles, by definition, we require \( \frac{d\rho}{d\tau} = 0, \)

\[ \frac{d\hat{t}}{d\tau} = 1. \]  

This is achieved if \( \sigma = \sigma_0. \) It is worth noting that \( \sigma_0 \) and \( \sigma \) have different meanings. The first one is the parameter of the coordinate transformation (16), the second one is the characteristic of particle motion.

### 3 Classification of frames

Since the energy enters in (25) explicitly only as square, this formula includes, in fact, four different cases depending on signs of the first and second terms in the right hand side of (16). For example, the standard Lemaître system, being a contracting system with \( e_0 = 1, \) has three counterparts: an expanding system with \( e_0 = 1, \) a contracting system with \( e_0 = -1 \) and an expanding system with \( e_0 = -1. \) All four are represented by the same equation (25), with the differences in particular expressions for \( r \) through the coordinates \( \rho \) and \( \hat{t}. \)

This poses the question about the meaning of such Lemaître frames, their GP counterparts and their classification. For example, the condition for the existence of the metric (25) is \( e_0^2 > f. \) In particular, for the Schwarzschild space-time it is defined everywhere if \( |e_0| \geq 1 \) independently of the sign of \( e_0. \) On the other hand, in the Schwarzschild black hole particles can have negative energy only inside a horizon and should be included in a general scheme. (Under the horizon, the energy and momentum mutually interchange their meaning. However, for shortness, we call \( e \) and \( e_0 \) energies in all regions of a space-time.)

A possible answer is that the metric (25) for a negative \( e_0 \) does exist for all \( r, \) but in another parts of the Carter–Penrose diagram for a geodesically complete space-time (see Fig. 1). In what follows, we consider the Schwarzschild space-time or any other space-time with a similar structure. In doing so, we use standard notations for different regions of space-time (see Sec. 31 in [15]): I is “our” region, II - a black hole region, III - a mirror one, IV is a white hole region. Then,

![Fig. 1 The Kruskal diagram for the Schwarzschild metric](image-url)
it is convenient to characterize a frame by two parameters. Below, C means “contracting”, E means “expanding. We also indicate the sign of $e_0$. Then, we have 4 different frames that are listed in Table 1.

Identification of regions follows from several facts. (i) Only particles with $e_0 < 0$ can cross the branch of the horizon between regions III and II. (ii) Particles that cross the branch between regions IV and I enter “our” usual region I ($R^+$, according to the Novikov classification [16]), so they must have $e_0 > 0$. In a similar way, particle in the asymptotically flat mirror region III ($R^-$) must have $e_0 < 0$. Somewhat different classification of frames was suggested in [13].

It is worth paying attention to an important detail not noted in the textbook [17]. The first equation in eq. 102.1 there with a different classification of frames was suggested in [13].

In a similar way, particle in the asymptotically flat mirror region III ($R^-$) must have $e_0 < 0$. Somewhat different classification of frames was suggested in [13].

| Frame   | Regions covered |
|---------|-----------------|
| $C, +$  | I, II           |
| $C, -$  | III, II         |
| $E, +$  | IV, I           |
| $E, -$  | IV, III         |

Outside the horizon. In the $R^+$ region $e > 0$, $e_0 > 0$, $\sigma$ and $\sigma_0$ can have arbitrary signs. In the $R^-$ region, $e < 0$, $e_0 < 0$, $\sigma$ and $\sigma_0$ can have arbitrary signs.

Inside the horizon. In the $T^-$ region $\sigma_0 = +1$, $\sigma = -1$, $e$ and $e_0$ can have arbitrary signs. In the $T^+$ region $\sigma_0 = -1$, $\sigma = +1$, $e$ and $e_0$ can have arbitrary signs. (Depending on their signs, particle move from region IV to I or III.)

In what follows we omit $\sigma_0$ and $\sigma$ for brevity, assuming that $P_0 < 0$ for expanding frames ($\sigma_0 = +1$) and $P_0 > 0$ for contracting frames ($\sigma_0 = -1$). In a similar way, we assume that $P < 0$ for outgoing particles ($\sigma = +1$) and $P > 0$ for ingoing particles ($\sigma = -1$).

Then, in coordinates ($\tilde{t}, r, \theta, \phi$) the four-velocity of a free radially falling (ingoing) particle reads

$$u^\mu = \left(\frac{e_0 \gamma - P_0 \gamma}{f}, - \gamma, 0, 0\right),$$

(37)

$$u_{\mu} = \left(-\frac{e_0}{e_0}, \frac{P_0 \gamma - P e_0}{f e_0}, 0, 0\right).$$

(38)

4 Radial 3-velocity with respect to a free falling frame

It is well known that the 3-velocity with respect to a particular frame characterized by a tetrad field $h_{(i)\mu}$ is given by (see, e.g. [18])

$$v^{(i)} = \frac{h_{(i)\mu} u^\mu}{h_{(0)\mu} u^\mu}.$$ (39)

We choose a radially moving particle with $e = e_0$ as a reference particle. Then, $h_{(0)\mu} = U^\mu$, where $U^\mu$ is its four-velocity. In coordinates $(\tilde{t}, r, 0, 0)$ it follows from (5) and (37) that

$$h_{(0)\mu} = U^\mu = (1, - P_0, 0, 0),$$

(40)

$$h_{(0)\mu} = U_\mu = (-1, 0, 0, 0),$$

(41)

$$h_{(1)\mu} = e_0 (0, 1, 0, 0),$$

(42)

$$h_{(1)\mu} = \frac{1}{e_0} (P_0, 1, 0, 0).$$

(43)

It is worth noting that Eq. (39) can be also rewritten in the form

$$v^{(i)} = \frac{h_{(i)\mu} u^\mu}{\gamma},$$

(44)

where the Lorentz factor of relative motion between a particle and an observer $\gamma = -u^\mu U_{\mu}$. This expression for $\gamma$ in the Schwarzschild background was analyzed in [20].

Using this we can write

$$-h_{(0)\mu} u^\mu = u \gamma,$$

(45)

$$h_{(1)\mu} u^\mu = \frac{P_0 u^\tilde{t} + u^r}{e_0}$$

(46)

which results in
\[ V^{(1)} = \left( P_0 + \frac{dL}{dt} \right) / e_0 = \nu_P, \]  

(47)

where the quantity \( \nu_P \) has the meaning of the peculiar velocity with respect to the particle flow [14].

For the case of \( e_0 = 1 \) we return to \( V^{(1)} = \nu_{st} + dr/dt \). Here, \( \nu_{st} \) is the velocity in the static frame \( \nu_{st} = P_0 / e_0 \) (see, e.g. e 43 in [8]). This means that for \( e_0 = 1 \) the coordinate velocity \( dr/dt \) is naturally decomposed into the flow velocity \( \nu_{st} \) and the physical velocity \( V^{(1)} \) as \( dr/dt = V^{(1)} - \nu_{st} \). This decomposition resembles the Galilean summation law, but is valid independently of how big the velocities are with respect to the speed of light. Since the left hand side of this decomposition is a coordinate (not a physical) velocity, it can be even superluminal. In the general case such a decomposition is valid only for the rate of change of the proper distance \( l \) with the time \( t \) and for the flow velocity with respect to the stationary frame \( \nu_{st} \) since still \( dl/dt = V^{(1)} - \nu_{st} \). (Here, it is implied that the distance is measured along the hypersurface \( t = \text{const} \) in the metric (5)).

For a free particle it follows from (45), (46), (47) and (11) that

\[ V^{(1)} = \frac{P_0 e - P e_0}{e_0 e - P P_0}. \]  

(48)

This is the general formula for radial motion, giving the 3-velocity of the particle with the energy \( e \) with respect to a free falling frame with the energy \( e_0 \). Equations (44), (48) agree with eqs. (9), (31) of [20] and are more general in that they allow negative energy and outward motion for particles and frames as well. Moreover, they can be easily generalized to non-radial motion of a particle provided the frame is still a radial one (see the next section).

It follows from (48) that

\[ P = \frac{e(P_0 - e_0 V^{(1)})}{e_0} = P_0 V^{(1)}. \]  

(49)

Taking the square of (49) and writing \( P^2 = e^2 - e_0^2 + P_0^2 \) where (3), (9) are taken into account, we obtain

\[ e = \gamma(e_0 - P_0 V^{(1)}), \quad \gamma = \frac{1}{\sqrt{1 - (V^{(1)})^2}}. \]  

(50)

When \( e_0 = 1 \), \( P_0 = v = \sqrt{1 - f} \), the known formulas are reproduced. Namely, Eqs. (48), (49) turn into eqs. (55), (56) of [8] and Eq. (50) turns into eq. (6.8) of [14].

It is remarkable that despite the fact that the GP coordinates (5) have no smooth limit for \( e_0 \to 0 \), corresponding formulae for 3-velocities are regular for \( e_0 = 0 \).

5 Motion with angular momentum and horizon asymptotics

In the most general case of a free fall a particle can have nonzero angular momentum \( L \). Now,

\[ \frac{dr}{d\tau} = -P \]  

(51)

with

\[ P = \pm \sqrt{e^2 - f} \left( 1 + \frac{L^2}{r^2} \right). \]  

(52)

\[ L = \frac{L}{m} = u \phi, \]  

so

\[ \frac{d\phi}{d\tau} = \frac{L}{r^2}. \]  

(53)

As is known, in the central field a particle moves within a plane. If we choose this plane to be \( \theta = \pi / 2 \),

\[ h_{(3)\mu} = r(0, 0, 0, 1) \]  

(54)

and

\[ h_{(3)\mu} u^\mu = ru \phi. \]  

(55)

For the reference particle, Eqs. (40), (41) are valid. If we take also into account (27), this gives us

\[ \frac{d\phi}{d\tau} = \frac{L}{r^2(e_0 e - P P_0)}, \]  

(56)

\[ V^{(3)} = \frac{L}{r(e_0 e - P P_0)}. \]  

(57)

As for the radial component, it follows from (51), (52) that we still have

\[ V^{(1)} = \frac{P_0 e - P e_0}{e_0 e - P P_0}. \]  

(58)

If \( e_0 = 1 \), \( P_0 = \sqrt{1 - f} \equiv v \), and we return to eq. (55) of [8].

If \( e_0 = 0 \),

\[ V^{(1)} = -\frac{e}{P}, \]  

(59)

\[ V^{(3)} = \frac{L}{rP}. \]  

(60)

The general formulae for the 3-velocity have several important limits. Now, we consider the behavior at the horizon. First we assume that \( P \) and \( P_0 \) have the same signs (say, positive, for definiteness). Let \( f \to 0 \), then

\[ P = \left| e \right| - \frac{f}{2|e|} \left( 1 + \frac{L^2}{r^2} \right) + \cdots \]  

(61)

\[ P_0 = \left| e_0 \right| - \frac{f}{2|e_0|} + \cdots \]  

(62)
Substituting into the expressions for velocity components we can identify two different cases. If $e$ and $e_0$ have the same sign, then
\[ V^{(1)} \rightarrow \frac{e_0^2 \left( 1 + \frac{e^2}{r_g^2} \right) - e^2}{e_0^2 \left( 1 + \frac{e^2}{r_g^2} \right) + e^2} \]
and
\[ V^{(3)} \rightarrow \frac{2ee_0c}{r_g \left[ e^2 + e_0^2 \left( 1 + \frac{e^2}{r_g^2} \right) \right]}. \]

If $e$ and $e_0$ have different signs, then independently of particular values of the both energies,
\[ |V^{(1)}| \rightarrow 1, \quad (65) \]
\[ V^{(3)} \rightarrow 0. \quad (66) \]

The same limits exist also if any of energies equals zero.

We can also see that if signs of $e_0$ and $e$ coincide, but the direction of motion is different ($P_0$ and $P$ have different signs), the asymptotic (65) and (66) holds as well.

All these cases are summarized in the Table 2.

Here $V_H^{(1)}$ and $V_H^{(3)}$ are given by (63) and (64) respectively. For the cases where the frame and the particle can exist only in non-intersecting zones of the Carter–Penrose diagram, the velocity can not be defined.

As for critical particles or frames, the angular velocity at a horizon always vanishes and the absolute value of radial velocity tends to 1. As for the sign of the radial component, $\text{sign}(V_H^{(1)}) = \text{sign}(e_0/P_0)$ for a critical particle ($e = 0$, $e_0 \neq 0$) and $\text{sign}(V_H^{(1)}) = -\text{sign}(e/P)$ for a critical frame ($e_0 = 0$, $e \neq 0$).

6 The case $V = 1$ and behavior of the Lemaître time

In a black hole space-time the most natural free-falling frames are contracting ones realized by particles with a positive energy. A particular case of $e_0 = 1$ (the Lemaître frame) is the most convenient choice for such a frame, since constant time slices in this case are flat that leads to some simplification. Regularity of the Lemaître frame at the horizon makes the velocity with respect to this frame a meaningful characteristic of particle motion, including the moment of horizon crossing.

To stress this point, let us consider for a moment the velocity with respect to a stationary frame at a horizon crossing. It is known that this velocity is equal to the speed of light. However, the stationary frame at a horizon becomes singular, so the equality $v_{st} = 1$ tells us nothing about a concrete properties of particle motion. On the contrary, the velocity with respect to a frame which can be realized by massive particles contains information about motion itself. We have seen that an ingoing positive energy particle can have at a horizon any velocity from 0 to 1 depending on the specific energy of the particle.

Meanwhile, we identified several cases when the velocity at the horizon crossing with respect to Lemaître frame is equal to the speed of light. In the Schwarzschild space-time this happens for an outgoing particle in the $R$ region and for negative energy particle in the $T$ region of the black hole. Moreover, the property of the velocity to be equal to the speed of light depends only on signs of terms entering (48), so this property holds for any $e_0$-Lemaître frame provided $e_0 > 0$. This deserves some attention.

One can look at these features from somewhat different point of view. The matter under discussion can be considered in the context of the Bañados–Silk–West (BSW) effect [9]. It states that the energy in the center of mass frame $E_{c.m.}$ of two particles colliding near the black hole horizon, under certain conditions can grow unbounded. Kinematically, this means that the relative velocity of two particles tends to the speed of light [19]. Now, the role of one of particles is played by a reference one. For a static particle, $E_{c.m.}$ does grow unbounded but for the frame itself becomes singular. For free falling particles, the case when $e_0 > 0$ and $e > 0$ corresponds to particle collisions of usual (without fine-tuning) particles and cannot lead to the BSW effect [9]. This can be also seen from the results of [20] where it was shown that the relative velocity of particles moving from the $R$ region towards the nonextremal horizon remains separated from the speed of light.

Meanwhile, if $e_0 > 0$ but $e < 0$, this corresponds to head-on collision that does lead to the indefinite growth of $E_{c.m.}$ [21,22]. However, this is connected with a special role of the bifurcation point which will be discussed elsewhere.

| Frame/particle | $e > 0, P > 0$ | $e < 0, P > 0$ | $e > 0, P < 0$ | $e < 0, P < 0$ |
|----------------|---------------|---------------|---------------|---------------|
| $e_0 > 0, P_0 > 0$ | $V_H^{(1)}; V_H^{(3)}$ | $+1; 0$ | $+1; 0$ | – |
| $e_0 < 0, P_0 > 0$ | $-V_H^{(1)}; V_H^{(3)}$ | $-1; 0$ | – | $-1; 0$ |
| $e_0 > 0, P_0 < 0$ | $-V_H^{(1)}; V_H^{(3)}$ | $-1; 0$ | – | $-1; 0$ |
| $e_0 < 0, P_0 < 0$ | – | $+1; 0$ | $+1; 0$ | $V_H^{(1)}; V_H^{(3)}$ |
We discussed the case when the latter scenario is realized inside the horizon in region II near the right branch of the future horizon. The completely similar situation is possible near the left branch. Also, collisions of such a type can occur in region IV where they have meaning of collisions in the background of white holes [23, 24].

We could expect that something should prevent a realization of \( V = 1 \). Otherwise, the energy \( E_{c.m.} \) would become not only unbounded but infinite in the literal sense that is impossible [26]. A possible reason appears to be clear when we consider the Lemaître time needed to reach the horizon. Using (11) it can be written as

\[
\Delta \tilde{t} = \int_{t_0}^{\tau} \frac{dr(e_0 e - P_0 P)}{P f}. \tag{67}
\]

Let us calculate the \( e_0 \)-Lemaitre time needed to reach a horizon. In general, the right hand side of (67) diverges at a horizon. However, if both \( e_0 \) and \( e \) are positive, as well as \( P_0 \) and \( P \), the divergent part in the integral cancels out and the time is finite. From another side, changing sign of entities entering (67) leads to diverging result. For example, changing the sign of \( P \) leads to

\[
\frac{dr}{d\tau} = +v = \sqrt{e^2 - f} \tag{68}
\]

and

\[
\frac{d\tilde{t}}{d\tau} = \frac{e_0 e + P_0 |P|}{f} \tag{69}
\]

instead of (10), so that we have summations of divergent parts in (67).

We can identify three more situations with infinite time \( \tilde{t} \). Indeed, instead of reversing the sign of \( P \) the same effect can be got by reversing the sign of \( e \), the sign of \( e_0 \) or the sign of \( P_0 \).

So that, by inspecting the Lemaître time in mentioned above the two situations, we have seen that the horizon crossing leading to \( V = 1 \) happens for infinite Lemaître time (for the above cases, actually, in infinitely remote past). However, the reference particle or an observer comoving with the frame with respect to which the velocity is measured, reaches the horizon for finite Lemaître time. Therefore, they cannot meet in the same point of space-time, measurement of the velocity (as well as collision) is impossible, so the value \( V = 1 \) and infinite \( E_{c.m.} \) remain unreachable. Moreover, if the velocity of a particle reaches 1 with respect to any other reference system from the Table II, the time of this system needed for the particle to reach horizon diverges, and the above argument is valid as well.

It is worth noting that the velocity of a particle approaches the speed of light just on the boundary where a corresponding frame covers a part of space-time, so its incompleteness reveals itself. For the stationary observer, this is a horizon beyond which a static frame cannot be extended and where the acceleration tends to infinity. For a free-falling frame, the horizon separates different region where motion with a given sign of \( e \) or \( P \) is possible or not.

### 6.1 Special case: critical particles and frames

In this subsection we present notes about critical frames \((e_0 = 0)\) and particles \((e = 0)\) which are curious by themselves. Let us consider the time needed to fall into a singularity from a horizon. It is equal to

\[
\Delta \tilde{t} = \int_{0}^{\tau} \frac{dr(e_0 e - P_0 P)}{P f} = \int_{0}^{\tau} \frac{d e_0 e}{P f} - \int_{0}^{\tau} \frac{d P}{f}. \tag{70}
\]

We can see that (1) If \( e_0 = 0 \),

\[
\Delta \tilde{t} = \int_{0}^{\tau} \frac{d r}{\sqrt{-f}} \tag{71}
\]

does not depend on \( e \) and, correspondingly, on details of the the observer’s motion. This property is evident in the coordinate system obtained from static coordinates via signature change (15), when the spatial and temporal coordinates interchange. Indeed, the metric in such a system, being a particular case of the Kantowski–Sachs cosmological metric depends on time but not on a spatial coordinate, so that a singularity is in future for any observer, separated from the observer by some time which does not depend on the actual position of the observer. What is remarkable, is that the limit \( e_0 \to 0 \) is smooth since the first term in (70), which does depend on details of the observer trajectory, vanishes smoothly for \( e_0 \to 0 \).

(2) If \( e_0 \neq 0 \) but \( e = 0 \) (the critical particle)

\[
\Delta \tilde{t} = \int_{0}^{\tau} \frac{d r}{\sqrt{e_0^2 + g}}. \tag{72}
\]

Near \( g = 0 \) it diverges as \( \Delta \tilde{t} \sim (\ln(r_h - r)) \). This means that for any critical particle the corresponding time \( \tilde{t} \) of fall is infinite in all frames with \( e_0 \neq 0 \).

### 7 Singularity or turning points

Going further, we consider asymptotics near the singularity. For the Schwarzschild metric, it is reachable. We consider the black hole region, assuming that the frame is contracting and a particle is ingoing, so \( P > 0, P_0 > 0 \). Then, \( f \approx \frac{-2M}{r} \to -\infty \). \( P \approx \sqrt{-T L} \), \( P_0 \approx \sqrt{-T} \).

If \( L \neq 0 \),

\[
V^{(3)} \to \text{sign} L, \tag{73}
\]
The velocity properties with respect to a generalized frame realized by ingoing particles with a positive energy \( e_0 \neq 1 \) differs from those for the original frame \((e_0 = 1)\) only quantitatively. If there are no special reasons, the \( e_0 = 1 \) frame is preferable since it gives a number of simplifications. However, we can indicate certain cases when a \( e_0 \neq 1 \) frame can be useful and can be not only of pure academic interest. All of them deal with non-Schwarzschild space-times, and there are the cases when the standard \( e_0 = 1 \) reference system fails to describe a motion [25]. For example, in the RN black hole the \( e_0 = 1 \) frame can not be prolonged through the turning point at \( f = 1 \). So that a motion of a particle with \( e > 1 \), having its turning point at \( f = e^2 \) cannot be fully covered by the Lemaître frame with \( e_0 = 1 \), but can be covered in any frame with \( e_0 > e \). Another example deals with the Schwarzschild–de Sitter metric with \( f = 1 - r_g/r + Q/r^2 \), where \( \Lambda > 0 \) is the cosmological constant. Now the function \( f(r) \) has a maximum value \( f_{\text{max}} < 1 \), and the flow velocity of (would be) Lemaître frame \( v = \sqrt{T - f} \) never crosses zero. This means that such a frame is nowhere contracting. By reversing sign of \( v \) we can get also an everywhere expanding frame, both are unsuitable to describe a black hole in expanding de Sitter Universe. However, taking \( e_0^2 = f_{\text{max}} \) we can construct a smooth frame expanding for large \( r \) and contracting for small \( r \) with the stationary point at the maximum of \( f(r) \) where contracting and expanding branches are glued together (for close but somewhat different approach to this particular metric see the recent paper [13]).

It is not so easy to imagine a situation where expanding frames, or frames realized by negative energy particles could be preferable (apart from such an exotic as a white hole, for which an expanding frame is as natural as contracting frame for a black hole!), nevertheless, the ability of our general formulae to include these frame as well is rather interesting. It is also of interest to apply this formalism to consideration of quantum problems where choice of frame may be important (see, for example, [27, 28]).

We have shown also that non-zero angular momentum of a particle can be included in the general scheme rather easily if the frame is still radially moving.

8 Discussion and conclusions

In the present paper we have considered the formulae for the velocity with respect to the most general free falling radially frame in a spherically symmetric space-time. The formulae can be applied to any radially falling reference system including expanding systems, and those realized by particles with negative energy. The particle moving in corresponding backgrounds can be, in its turn, ingoing or outgoing, having positive or negative energy.

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