Evaluation of cutting off entropy functional measure on trajectories of Markov diffusion process and information path functional

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Abstract
The introduced impulse cuts off entropy functional measure (EF) on the Markov diffusion process’ trajectories-solutions of Ito controllable stochastic equation. The process additive functional expressed via the equation functions drift and diffusion defines the EF and allows reducing this functional on trajectories to a regular integral functional. Compared to conventional Shannon entropy measure of a random state, cutting the process on separated states decreases the quantity of process information by the amount, which was concealed in correlation connections between these states, which holds hidden process information. The $n$-dimensional process cutoffs generates a finite information measure, integrated by information path functional (IPF) whose total information approaches the EF measure at $n \rightarrow \infty$, restricting maximal information of the Markov diffusion process. Both delta-function impulse’ cutoff and finite impulses’ discrete impulse deliver equivalent information at each cutoff. The finite restriction limits the impulses’ discrete stepwise actions applied for cutting the regular integral on the functional increments between the cutoffs. The finite impulse step-up action transfers the entropy functional increment to the following impulse, whose step-down action cuts off generated information and step-up action starts an imaginary impulse carrying the entropy increment to next real cut. The step-down cut generates maximal information, while the step-up action delivers minimal information from that impulse cut to the next impulse’s step-down action, both extract maximum of minimal information of this impulse. The sequential impulse cutoffs and minimal entropy transfer between the impulses implement maxmin-minimax principle of converting the process entropy to information. The imaginary impulse with step-up and step-down virtual actions transfers the conjugated entropy fractions during a microprocess, which ends with adjoining the fractions within actual step-down action at the cutoff. Macroprocess as solution of the EF variation problem integrates both imaginary entropy of the impulses microprocess, and the cutoff information of the real impulses, absorbing the IPF collected information in physical process. The EF functional measure accumulates more process information than sum of Shannon’s entropies, counted for all process’ separated states, and provides information measure of Feller’s kernel. The estimate of all extracting information confirms a non-additivity of the EF measured process’ fractions.

Keywords: integral information measure; cutting off the diffusion process; impulse action, Feller kernel, information path functional, finite process information.
Introduction

Conventional information science generally considers an information process, but traditionally uses the probability measure for the random states and Shannon’s entropy measure as the uncertainty function of the states (C.E. Shannon [1], E.T. Jaynes [2], A.N. Kolmogorov [3], others).

Such a measure evaluates information for a sequence of each process’ states and does not measure inner connections between these states through all process.

The problem is to find an integral information measure, evaluating the inner connection and dependencies of the random process’ states to measure the process’ total information.

Using definition of a conditional information entropy and applying it to transitional probability with density measure, defined along the Markov diffusion process, allows measuring the process’ integral information on the process’ trajectories.

Such an entropy functional (EF) measure has an analogy with R. P. Feynman’s path functional [4], which M. Kac [5] has expanded on trajectories of Brownian processes.

We consider a controllable diffusion process, whose control provides a transformation of its transitional probabilities, applying the process’ multiplicative and additive functionals.

Connection of the transitional probability with density, determined through the additive functional, defines the entropy functional measure via this functional.

Estimating inner connections and dependencies of the diffusion process’ states with EF, allows jump-wise control action, cutting off the diffusion process by its de-correlation.

The theory of additive functional is developed by E.B. Dynkin [6,7], I. V. Girsanov [8], A.D. Ventsel [9], R.M. Blumenthal, R.K. Getoor [10], N. Ikeda, S. Watanabe [11,12], Y.V. Prokhorov, Y.A. Rozanov [13], I.I. Gikhman, A.V. Scorochod [14], other authors.

The information measures of a random process were introduced by R.A. Fisher [15], S. Kulback, R.A. Leibler [16], A. Rényi [17], R.L. Stratonovich [18] and some others.

Fisher’s logarithmic measure calculates the covariance matrices associated with maximum-likelihood estimates in statistics.

Rényi’s entropy is a generalization of Shannon entropy by introducing probability of a discrete random variable with various parameter of the probability’s power.

Kullback–Leibler’s divergence between the probabilities of a process’ states distributions is measured by a relative Shannon’s entropy.

R.L. Stratonovich specified Radon-Nikodym’s density for a probability density measure, applying it to Shannon’s entropy of a random process.
Controllable Markov processes with the optimal stopping and estimation methods are studied by E.B. Dynkin [19], I.I.Gihman, A.V. Scorochod [20], N.V. Krylov [21], J. M. Harrison et al [22], F.B.Handson [23], A. Bensoussan, J.L. Lions [24], A. L. Bronstein et al [25], and many others. W. Feller [26, 27] introduced a diffusion operator of “kernel”, which absorbs the diffusion motion on its limited time interval, covering Markov property.

H.P. McKean and H. Tanaka [28] connected the Feller’s kernel with additive functionals of Brownian path and linked these functionals to both an instant of the path’ passage boundary time, limited the kernel, and to the kernel finite energy.

M.Fukushima et al [29] estimated an energy, covering the kernel, and the minimal boundary time.

A.Borodin and V.Gorin [30] studied a correlation function of a discrete time Markov process with the transitional probability measure of Feller’s kernel that connects the kernel’s measure to the correlations.

The paper’s deterministic cut off impulse control “intervenes” in the diffusion process during the minimal boundary instant time of a kernel’s jumping mechanism [29]. This leads to extraction of information hidden in a kernel, which is evaluated through the additive functional at such cut off.

A deterministic “intervention” in a random process, considered by A.A.Yushkevich [31], M.A.H.Dempster and J.J.Ye [32], others, studied mostly a diffusion process with a drift function.

The jump-transition probability under the impulse control, which is described through the additive functional, being defined by both drift and diffusion, allows us studying a more general case.

The paper shows that cutting off the entropy integral information measure on the separated states’ measures decreases the quantity of process information by the amount, which was concealed in the correlation connections between the separating states. This integral functional measure accumulates more process information than sum of the Shannon’s information measures of its separated states. While each cutoff reveals information hidden in the process before the cutoff, the paper information path functional (IPF) integrates all cutoffs information, approaching the process entropy functional at growing the process dimension. In limit at the \( n \to \infty \) dimensional process cutoffs generate the IPF equivalent to the *entropy functional measure*, restricting the process information by this EF.

**The paper organization:**

*Section 1* introduces an entropy functional (EF), defined on Markov diffusion process defined through the process’s additive functional of transitive probabilities, which is determined by the functions drift and diffusion of Ito’s stochastic differential equation. This relational EF probability measure is conditional to Feller kernel, and optimal transition to the kernel minimizes the EF, which allows formulating the problem in this section.
In Section 2, the entropy functional is expressed via the process’ dispersion (correlation) and the averaged controllable drift functions. This allows not only identifying the EF on an observed Markov process by measuring the correlations at applying control functions, but also reducing the EF functional to a regular integral of non-random functions with the integrand defined by an averaged additive functional.

Section 3 studies an impulse delta-action on the regular entropy integral and determines the information contributions at such cutoff actions, evaluates additive and multiplicative functionals, transitional probabilities and the time correlations.

Section 4 introduces a finite restriction on the cutting function that limits the impulses’ discrete and determines an extremal increment of entropy functional between the discrete cutting off the regular integral. The discrete impulse’s step-up action between the cutoff transfers the entropy functional increment to the following impulse’s step-down cut, whose cutoff information delivers this impulse step-up action, which also starts an imaginary impulse carrying the entropy to next real cut by its step down action. The step-down action cuts maximal information, while the impulse step-up action delivers minimal information from the impulse cut to the next impulse’s step-down action; both extract maximum of minimal information of this impulse. The sequential impulse cutoffs and minimal entropy transfer between the impulses implement maxmin-minimax principle of converting the process entropy to information. Estimating the times at the cutoffs, allows finding both the discrete increments of correlations at each information contributions.

Section 5 analyzes structure of Information Path functional (IPF) in $n$-dimensional Markov process under the $n$-cutoff impulses and evaluates it via summary of the cutoff information contributions. The decreasing distance between the finite cutoff intervals with growing process’ dimension at $n \rightarrow \infty$ limits total time of getting summary information contributions, which IPF collects, approaching the process EF that restricts the process finite IPF. The IPF maximum, integrating unlimited number of Bits’ units with finite distances, limits the total information carrying by the process’ Bits. The maximum confirms non-additivity of the EF measured process’ fractions and estimates information, hidden by interstates’ connection of the diffusion process covered by the kernel’s information. The process uncertainty-entropy, measured by the EF is also finite on the finite time of its transition to total process certainty-information, measured by the IPF.

Section 6 describes information dynamic processes initiating by the EF and IPF functionals, which consist of information microprocesses, carrying entropy increments to each cut-off information contributions within the cutoff, and information macroprocess integrating all these microprocesses. A virtual (imaginary) microprocess, transferring the conjugated entropy increments, ends with adjoining the increments in real information microprocess through the cutoff actual step-down action. The microprocess’ equations, follow macroprocess’ equations in Section 7, which solve variation problem for the entropy functional.
1. The entropy functional

Let us have Markov diffusion process $\tilde{x}_t$ with transition probabilities $P(s,\tilde{x},t,B)$, $\sigma$-algebra $\Psi(s,t)$ created by events $\{\tilde{x}(\tau) \in B\}$ at $s \leq \tau \leq t$, and conditional probability distributions $P_{s,x}$ on $\Psi(s,t)$.

A family of the real or complex random values $\varphi'_s = \varphi'_s(\omega)$ depending on $s \leq t$ defines an additive functional of process $\tilde{x}_t = \tilde{x}(t)$ [27, 28], if each $\varphi'_s = \varphi'_s(\omega)$ is measured regarding the related $\sigma$-algebra $\Psi(s,t)$ at any $s \leq \tau \leq t$ with probability 1 at $\varphi'_s = \varphi'_s + \varphi'_t$, and $E_{s,x}[\exp(-\varphi'_s(\omega))] < \infty$, where $E_{s,x}[,]$ are the related mathematical expectations.

Transformation of probability measures [7, p.48]:

$$\tilde{P}_{s,x}(d\omega) = p(\omega)P_{s,x}(d\omega), \quad (1.1)$$

on trajectories of Markov process $(\tilde{x}_t, P_{s,x})$ holds distributions $\tilde{P}_{s,x} = \tilde{P}_{s,x}(A)$ on extensive $\sigma$-algebra $\Psi(s,\infty)$ with density measure:

$$p(\omega) = \frac{\tilde{P}_{s,x}(d\omega)}{P_{s,x}(d\omega)} = \exp\{-\varphi'_s(\omega)\}, \quad (1.2)$$

and transitional probabilities of transformed diffusion process $\tilde{\zeta}_t$ with the additive functional:

$$\tilde{P}(s,\tilde{\zeta},t,B) = \int_{\tilde{x}(t)\in B} \exp\{-\varphi'_s(\omega)\}P_{s,x}(d\omega) \quad (1.3)$$

Applying the definition of a conditional entropy [17] to mathematical expectation of logarithmic probability functional density measure (1.2) for process $\tilde{x}_t$ regarding process $\tilde{\zeta}_t$, we have

$$S(\tilde{x}_t / \tilde{\zeta}_t) = E[-\ln[p(\omega)]] = \int_{\tilde{x}(t)\in B} -\ln[p(\omega)]\tilde{P}_{s,x}(d\omega), \quad (1.4)$$

where $E = E_{s,x,\tilde{x}_t}$ are conditional mathematical expectation, taken along the process trajectories $\tilde{x}_t$ at a varied $(\tilde{x},s)$ (by analogy with M.Kac [5, 194-218].

From (1.2) and (1.3) we get conditional entropy functional expressed via the additive functional on the trajectories of considered diffusion processes:

$$S[\tilde{x}_t / \tilde{\zeta}_t] = E[\varphi'_s(\omega)], \quad (1.5)$$

which is entropy measure of a distance between distributions $\tilde{P}_{s,x,\tilde{\zeta}_t}, P_{s,x}$.

Minimum of this functional, depending on the additive functional measures closeness above distributions:
min $S[\tilde{x}_i / \zeta_i] = S^0$. \hfill (1.5a)

Let diffusion process $\tilde{x}_i$ be a solution of stochastic Ito $n$-dimensional differential Eqs [12]:

$$d \tilde{x}_i = a(t, \tilde{x}_i) dt + \sigma(t, \tilde{x}_i) d \xi, \quad \tilde{x}_s = \eta, \quad t \in [s, T] = \Delta, \quad s \in [0, T] \subset R^1$$ \hfill (1.6)

with standard limitations [19] on drift function $a = a(t, \tilde{x}_i)$, function of diffusion $\sigma = \sigma(t, \tilde{x}_i)$, and Wiener process $\xi = \xi(t, \omega)$, defined on a probability space of elementary random events $\omega \in \Omega$ for variables located in $R^n$.

The additive functional, according Girsanov’s Theorem [8], satisfies the form [13, p.355-360], [14, Theorem 11]:

$$\phi^T_s = 1/2 \int_s^T a(t, \tilde{x})^T (2b(t, \tilde{x}))^{-1} a(t, \tilde{x}) dt + \int_s^T \sigma(t, \tilde{x})^T a(t, \tilde{x}) d \xi(t), \quad 2b(t, \tilde{x}) = \sigma(t, \tilde{x}) \sigma^T(t, \tilde{x}) > 0$$ \hfill (1.7)

for the considered controllable process with its upper limit $T$.

Let the transformed process be

$$\zeta_i = \int_s^t \sigma(v, \xi_v) d \xi_v$$ \hfill (1.8)

having the same diffusion as the initial process, but the zero drift.

Process $\zeta_i$ is a transformed version of process $\tilde{x}_i$, whose transition probability satisfies (1.1), and transformed probability $\tilde{P}_{s,\xi}$ for this process evaluates the Feller kernel measure [27, 29].

Since transformed process $\zeta_i$ (1.8) has the same diffusion matrix but zero drift, the right part of additive functional in (1.5) satisfies

$$E[\int_s^T (\sigma(t, \tilde{x})^{-1} a(t, \tilde{x}) d \xi(t))] = 0.$$ \hfill (1.9)

We come to entropy functional, expressed via parameters of the Ito stochastic equation in the form:

$$S[\tilde{x}_i / \zeta_i] = 1/2 E[\int_s^T a(t, \tilde{x})^T (2b(t, \tilde{x}))^{-1} a(t, \tilde{x}) dt].$$ \hfill (1.10)

Formulas (1.2-1.5), (1.6), (1.7), (1.8) and (1.10) are in [33] with related citations and references.

The entropy functional in forms (1.4, 1.5, 1.10) is an information indicator of distinction between the probability measures of processes $\tilde{x}_i$ and $\zeta_i$; it measures a quantity of information of process $\tilde{x}_i$. 
regarding process $\zeta_t$. For the process’ equivalent measures, this quantity is zero, and it is positive for the process’ equivalent measures.

Since mathematical expectation on the process’ trajectories (1.5) is conditional to probability measure of Feller kernel, it is invariant at non-degenerative Markovian transformations defined through Radon-Nikodym’s probability density measure (1.3) where both $P_{s,t}$ and $P^\ast_{s,t}$ are defined. Thus, integral (1.10) is the entropy measure of Markov process $\bar{x}_t$ while being conditional to the kernel probability measure. Entropy measure (1.4) is conditional to any transformed Markov diffusion process, not necessary satisfying (1.10). Measuring conditional entropy (1.4), (1.10) relatively to diffusion process $\zeta_t$, which models standard perturbations in controllable systems, is practically usable.

Comments. Differential entropy defined on set of random variable $X$ (with probability density $p(\nu)$) as a Shannon form of conditional entropy:

$$H_o = E_X[-\ln p(\nu)] = -\int_X p(\nu)\ln p(\nu)d\nu, \quad (1.11)$$

is not equal to (1.4), and is not invariant under change of variables. Therefore, it cannot be an entropy measure for an arbitrary continuous process, which is unlimited by special requirements [35,pp.224-238]. For the same reason, Shannon entropy (type $H_\nu$) with its density $p(\nu)$ is not founded for continuous process and is not sufficient and applicable for considered random process. •

The problem consists of evaluation information, that curries entropy functional (EF) at transforming above distributions on minimal distance, while moving probability measure of processes $\bar{x}_t$ to probability measure of $\zeta_t$.

The transformation or movement involves an action on the EF trough the additive functional by changing it drift function. Simultaneous cutting both drift functions for each process dimension will instantaneously implement that transformation. Such cutting requires finding the cutoff function acting immediately on the functional drifts.

The requirement satisfies Dirac’s delta function and its discrete form represented through Heaviside’s step-up and step-down functions at small interval. A straight problem solution involves action on the random drift functions depending on the random diffusion process, which complicates solving the problem.

2. The entropy regular integral functional

Let us have a single dimensional Eq. (1.6) with drift function $a = c\bar{x}(t)$ at a given nonrandom function $c = c(t)$ and the diffusion $\sigma = \sigma(t)$.

Then, entropy functional (1.10) acquires form
\[ S[\tilde{x}_t / \xi_t] = \frac{1}{2} \int_0^T E[c^2(t)\tilde{x}^2(t)\sigma^2(t)]dt , \] (2.1)

from which, at \( \sigma(t) \) and nonrandom function \( c(t) \), we get

\[ S[\tilde{x}_t / \xi_t] = \frac{1}{2} \int_0^T \left[c^2(t)\sigma^2(t)E_s[x^2(t)]\right]dt = \frac{1}{2} \int_0^T c^2[2b(t)]^{-1}r_2 dt , \] (2.2)

where for the diffusion process, the following relations hold true:

\[ 2b(t) = \sigma(t)^2 = dr / dt = \dot{r}, E_s[x^2(t)] = r_2 , \] (2.3)

and functional (2.2) is expressed via the process’ nonrandom functions \( A(t,s) = [2b(t)]^{-1}r_2 \) of dispersion \( b(t) \), with initial correlation \( r_2 \), and given \( c^2(t) \).

This allows us to only to identify the entropy functional on an observed Markov process \( \tilde{x}_t = \tilde{x}(t) \) by measuring the above covariation (correlation) functions at applying a positive function \( c^2(t) = u(t) \), but also to represent the functional (1.10) through a regular integral of non-random function

\[ S[\tilde{x}_t / \xi_t] = \frac{1}{2} \int u(t)A(t,s)dt , \] (2.4)

where the integrant is an averaged additive functional (1.7). The \( n \)-dimensional form of functional (2.4) follows directly from related \( n \)-dimensional covariations (2.3), dispersion matrix, and applying \( n \)-dimensional function \( u(t) \).

At given nonrandom function \( u(t) \), (2.4) is regular integral which measures the entropy functional of the Markov process at the probability transformation (1.2) with additive functional (1.7).

3. Impulse action on the entropy functional

Let’s define \( u(t) = u_\tau \) on space \( KC(\Delta, U) \) of a piece-wise continuous step functions \( u_\tau(u'_\tau, u'_s) \) at \( t \in \Delta \) and applied to integrant of (1.10) in form of the difference of step-down \( u'_\tau = u_\tau(t) / \delta_0 \) and step-up \( u'_s = u_\tau(t + \delta_0) / \delta_0 \) functions at fixed interval \( \delta_0 \):

\[ u_\tau^{\delta_0} = [u_\tau(t) - u_\tau(t + \delta_0)] / \delta_0 = \delta_0 u_\tau \] (3.1)

which forms Dirac’s delta-function at

\[ \lim_{\delta_0 \to 0} u_\tau^{\delta_0} = \delta u_\tau . \] (3.2)

Proposition 3.1.

Entropy functional (2.4) under impulse control (3.2) takes the following information values:

(a)-at switching moments \( t = \tau_\delta \):

\[ \text{(a)} \]
\( S[\tilde{x}_i / \zeta_r]_{t=\tau_k} = \frac{1}{2} \text{Nats} \), \hspace{1cm} (3.3) \\
(b) - at switching left locality \( t = \tau_k^{\text{a}} \): \\
\( S[\tilde{x}_i / \zeta_r]_{t=\tau_k} = \frac{1}{4} \text{Nats} \) \hspace{1cm} (3.3a) \\
(c) - and at switching right locality \( t = \tau_k^{\text{a}} \): \\
\( S[\tilde{x}_i / \zeta_r]_{t=\tau_k} = \frac{1}{4} \text{Nats} \). \hspace{1cm} (3.3b) 

**Proof.** Applying delta function \( \delta^2(t, \tau_k) = \delta(t - \tau_k) \) to integral 

\[
\Delta S[\tilde{x}_i / \zeta_r]|_{t=\tau_k^\text{a}} = \frac{1}{2} \int_{\tau_k^\text{a}}^{\tau_k^\text{a}} \delta(t - \tau_k) A(s_k, t) \, dt \hspace{0.5cm}, \quad \tau_k^\text{a} < \tau_k < \tau_k^\text{+a}. 
\] 

(3.4)

determines functions [36,p.678-681]

\[
\Delta S[\tilde{x}_i / \zeta_r]|_{t=\tau_k^\text{a}} = \begin{cases} 
0, & \tau_k < \tau_k^\text{a} \\
\frac{1}{4} A(s_k, \tau_k - 0), & \tau_k = \tau_k^\text{a} \\
\frac{1}{4} A(s_k, \tau_k + 0), & \tau_k = \tau_k^{\text{+a}} \\
\frac{1}{2} A(s_k, \tau_k), & \tau_k^\text{a} < \tau_k < \tau_k^{\text{+a}} 
\end{cases} 
\] 

(3.5)

where, according to (2.2), relation \( A(s, \tau_k) = r(s)[2b(\tau_k)]^{-1} \) at

\[
r(s_k) = \int_{s_k^\text{a}}^{s_k^{\text{+a}}} 2b(t) \, dt = 2b(s_k)(s_k^{\text{+a}} - s_k^\text{a}) = 2b(s_k)o(s_k), o(s_k) = (s_k^{\text{+a}} - s_k^\text{a}), s_k^\text{a} < s_k < s_k^{\text{+a}}, 
\]

(3.5a)

acquires form

\[
1/2 A(s_k, \tau_k) = 1/2r(s_k)[2b(\tau_k)]^{-1} = 1/2(2b(s_k)o(s_k)[2b(\tau_k)]^{-1}). 
\]

(3.5b)

Considering an initial cutoff within interval \( s_k^\text{a} < s_k < s_k^{\text{+a}} \) at moment \( s_k \to \tau_k \), we get

\[
1/2 A(s_k, s_k) = 1/2o(s_k). 
\]

(3.6)

Or such cutoff brings amount of entropy functional \( S[\tilde{x}_i / \zeta_r]_{s_k^\text{a} < s_k < s_k^{\text{+a}}} = 1/2 \text{Nats} \), while on the borders of the interval we get \( S[\tilde{x}_i / \zeta_r]_{s_k^\text{a} = s_k^\text{a}} = 1/4 \text{Nats} \), \( S[\tilde{x}_i / \zeta_r]_{s_k^{\text{+a}} = s_k^{\text{+a}}} = 1/4 \text{Nats} \) accordingly.

Shifting the cutoff time from \( t = s_k \) to \( t = \tau_k \), we get

\[
S[\tilde{x}_i / \zeta_r]_{t=\tau_k} = 1/2 A(s_k, \tau_k) \to S[\tilde{x}_i / \zeta_r]_{t=\tau_k} = 1/2 A(\tau_k, \tau_k) = 1/2o(\tau_k) = 1/2 \text{Nats}, 
\]

(3.6a)

\[
S[\tilde{x}_i / \zeta_r]_{t=\tau_k} = 1/4 \text{Nats} \to S[\tilde{x}_i / \zeta_r]_{t=\tau_k} = 1/4 \text{Nats} \to S[\tilde{x}_i / \zeta_r]_{t=\tau_k} = 1/4 \text{Nats} 
\]

(3.6b)

accordingly.
The results concur with theory of additive and multiplicative functional for Markov diffusion process according to [13,p.358] and [11,p.23-28]. Consequently, the additive functional on $t = \tau_k$ cut is

$$\phi_s^{-} = \begin{cases} 0, t \leq \tau_k^-; \\ \infty, t > \tau_k, \end{cases}$$  \hspace{1cm} (3.7)

while the function $u_k(3.1)$ jump from $t = \tau_k$ to $t = \tau_k^+$ might cut off the diffusion process after moment $\tau_k$ with the related additive functional

$$\phi_s^{+} = \begin{cases} \infty, t > \tau_k; \\ 0, t \leq \tau_k^+. \end{cases}$$  \hspace{1cm} (3.7a)

The additive functional at a vicinity of $t = \tau_k$ under impulse (3.1) acquires form of an impulse function which summarizes (3.7) and (3.7a):

$$\phi_s^{-} + \phi_s^{+} = \delta\phi_s^\tau,$$  \hspace{1cm} (3.8)

The multiplicative functionals, related to (3.7,3.7a), are

$$p_s^{-} = \begin{cases} 0, t \leq \tau_k^-; \\ 1, t > \tau_k, \end{cases}, \hspace{1cm} p_s^{+} = \begin{cases} 1, t > \tau_k; \\ 0, t \leq \tau_k^+. \end{cases}$$  \hspace{1cm} (3.9)

Impulse (3.1) provides an impulse probability density in the form of multiplicative functional

$$\delta p_s^\tau = p_s^{-} p_s^{+},$$  \hspace{1cm} (3.9a)

where $\delta p_s^\tau$ holds $\delta[\tau_k]$-function, which determines the process’ transitional probabilities with

$$\tilde{P}_{s,x}(d\omega) = 0 \text{ at } t \leq \tau_k^- \text{ and } t \leq \tau_k^+ \text{ and } \tilde{P}_{s,x}(d\omega) = P_{s,x}(d\omega) \text{ at } t > \tau_k.$$  \hspace{1cm} (3.9b)

For the cutoff diffusion process, transitional probability (at $t \leq \tau_k^-$ and $t \leq \tau_k^+$) turns to zero, and the process states $\tilde{x}(t \leq \tau_k^-), \tilde{x}(t = \tau_k), \tilde{x}(t \leq \tau_k^+)$ with related jumping probabilities (3.9) become independent, while their shared time correlations are dissolved:

$$r(\tau_k^- \leq t \leq \tau_k^+) = E[\tilde{x}(t \leq \tau_k^-), \tilde{x}(t \leq \tau_k^+)] \to 0.$$  \hspace{1cm} (3.10)

This also follows from (3.5a) at $o(s_k) \to 0$ or $o(\tau_k) \to 0$.

Entropy of additive functional $\delta\phi_s^\tau$ (3.8), which is produced within, or at a border of control impulse (3.1), defines equality

$$E[\phi_s^{-} + \phi_s^{+}] = E[\delta\phi_s^\tau] = \int_{\tau_k^-}^{\tau_k^+} \delta\phi_s^\tau(\omega) P_\delta(d\omega),$$  \hspace{1cm} (3.11)

where $P_\delta(d\omega)$ is a probability evaluation of impulse $\delta\phi_s^\tau$.  

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Taking integral of symmetric $\delta$-function $\delta \phi_\mu$ between the above time intervals, we get

$$E[\delta \phi_\mu] = 1/2 P_\mu(\tau_k) \quad \text{at} \quad \tau_k = \tau_k^\tau = \tau_k^\tau \text{or} \quad \tau_k = \tau_k^\tau \text{or} \quad \tau_k = \tau_k^\tau. \quad (3.12)$$

The impulse, produced by deterministic controls (3.2) for each process dimension, is random with probability at $\tau_k$-locality

$$P_{oc}(\tau_k) = 1, k = 1, \ldots, m. \quad (3.13)$$

This probability holds a jump-diffusion transition probability in (3.12) (according to [23]), which is conserved during the jump.

Entropy functional (1.10), defined through Radon-Nikodym’s probability density measure (1.3), holds all properties of the considered cutoff process, where both $P_{s,s}$ and $\bar{P}_{s,s}$ are defined.

That includes abilities for measuring $\delta$-cut off information and extracting hidden process information not measured by known information measures.

According to the definition of entropy functional (1.4), it is measured in natural $\ln$ where each of its Nat equals $\log_2 e \cong 1.44 \text{bits}$; therefore, it is not using Shannon entropy measure.

4. **Discrete control action on the entropy functional**

Let us consider the increments of functional (1.10) under the action of step-down $u_-(t)$ and step-up $u_+(t+\Delta)$ functions on a discrete interval $\Delta$.

**Limitation on function** $c^2(t) = u(t)$ to be a finite difference $u^\Delta = [u_-(t) - u_+(t+\Delta)]$ on $\Delta = (\tau_k^\tau - s_k^\tau)$.

**Lemma.**

Functions $u_-(t), u_+(t+\Delta)$ satisfy both

$$\lim_{t \to \Delta} [u_-(t) \times u_+(t+\Delta)] = c^2(\Delta) \quad (4.1A)$$

and their difference:

$$\lim_{t \to \Delta} [u_-(t) - u_+(t+\Delta)] = c^2(\Delta) \quad (4.1B)$$

at conditions:

$$\lim_{t \to \tau_k^\tau} [u_+(t+\Delta) - u_-(t)] = u_-(\tau_k^\tau) \times u_+(\tau_k^\tau) = c^2(\tau_k^\tau, \tau_k^\tau), \Delta = \tau_k^\tau - \tau_k^\tau \quad (4.1)$$

when these functions hold

$$u_-(\tau_k^\tau) = -1 \tau_k^\tau \bar{n}, u_+(\tau_k^\tau) = +1 \tau_k^\tau \bar{n}, \quad (4.1a)$$

and each singular instance-jump (4.1a) has high $\bar{n} = 2$ and satisfies $u_+(\tau_k^\tau) = -u_-(\tau_k^\tau). \quad (4.1b)$

Opposite functions $u_+(t), u_-(t+\Delta)$ also satisfy

$$\lim_{t \to \Delta} [u_+(t) \times u_-(t+\Delta)] = c^2(\Delta),$$

and their difference complies with

$$\lim_{t \to \Delta} [u_-(t+\Delta) - u_+(t)] = c^2(\Delta).$$
at conditions
\[
\lim_{t \to \tau_k^-} [u_-(t + \Delta) - u_-(t)] = u_-(\tau_k^-) = c^2(\tau_k^+, \tau_k^-), \Delta = \tau_k^+ - \tau_k^-,
\]  
(4.2)
and
\[
u_-(\tau_k^-) = +1_{\tau_k^-} \bar{u}, u_-(\tau_k^-) = +1_{\tau_k^-} \bar{u}.
\]  
(4.2a)

Function \[u_-(t), u_-(t + \Delta)\] on \[\Delta = (\tau_k^- - s_k^+)\] also satisfy requirements (4.1A) and (4.1B) if
\[
\lim_{t \to \tau_k^-} [u_+(t + \Delta) - u_+(t)] = u_+(s_k^+) \times u_+(\tau_k^-) = c^2(\tau_k^+, s_k^+),
\]  
(4.3)
at \lim_{t \to \tau_k^-} u_-(t) = u_-(\tau_k^-), \lim_{t \to \tau_k^-} (t + \Delta) = u_+(s_k^+), and
\[
u_-(\tau_k^-) = -1_{\tau_k^-} \bar{u}, u_+(s_k^+) = +1_{s_k^+} \bar{u}.
\]  
(4.3a)

These also hold for opposite functions \[u_-(t), u_-(t + \Delta)\] on \[\Delta = (\tau_k^- - s_k^+)\] at
\[
\lim_{t \to \tau_k^+} [u_+(t + \Delta) - u_+(t)] = u_+(s_k^+) \times u_+(\tau_k^-) = c^2(\tau_k^+, s_k^+),
\]  
(4.4)
at \lim_{t \to \tau_k^+} u_-(t) = u_-(\tau_k^-), \lim_{t \to \tau_k^+} (t + \Delta) = u_+(s_k^+), and
\[
u_+(\tau_k^-) = +1_{\tau_k^-} \bar{u}, u_+(s_k^+) = -1_{s_k^+} \bar{u}.
\]  
(4.4a)

At any \[t \to \Delta\], relations (4.2), (4.3), (4.4) not satisfying discrete relations (4.2a, 4.3a, 4.4a) are not true. Proofs are straight forward. •

Corollary 4.1. Conditions 4.1A., 4.B imply that \[c^2(\tau_k^+, s_k^+), c^2(\tau_k^-, \tau_k^+)\] are discrete functions of fixed moments of switching functions (4.1a).

At \[\Delta = \delta\], this leads to discrete delta-function \[\delta^o u_+(3.1)\] which for \[\delta = (\tau_k^- - \tau_k^+)\] holds
\[
\delta^o u_+ = [u_+(\tau_k^-) - u_+(\tau_k^+)] / (\tau_k^+ - \tau_k^-), \quad \text{that at } \Delta = (\tau_k^- - s_k^+) \text{ brings}
\]  
\[
u_+(\tau_k^-) = -1_{\tau_k^-} \bar{u}, u_+(\tau_k^+) = +1_{\tau_k^+} \bar{u},
\]  
(4.2)
while positivity of \[c^2 > 0\] implies
\[
\delta^o u_+ = [u_+(\tau_k^-) - u_+(\tau_k^+)] / (\tau_k^+ - \tau_k^-) > 0.
\]  
(4.3)

Let us find increment of entropy functional \[\Delta S[\bar{x}_i / \bar{\gamma}]_{s_i \tau_k^-}^{s_i \tau_k^+}\] under function
\[
c^2 = \lim_{t \to \tau_k^+} [u_-(t + \Delta) - u_+(t)] / \Delta, \Delta = \tau_k^- - s_k^+, c^2(\tau_k^-, s_k^+)(u_-(\tau_k^-) - u_+(s_k^+)) / (\tau_k^- - s_k^+),
\]  
(4.4)
at \[u_-(\tau_k^-) = -1_{\tau_k^-} \bar{u}, u_+(s_k^+) = +1_{s_k^+} \bar{u} = 2,\text{or at } u_-(\tau_k^-) = \downarrow_{\tau_k^-} \bar{u}, u_+(s_k^+) = \uparrow_{s_k^+} \bar{u}\].

Proposition 4.1. Extremal increment of entropy functional (2.4) under functions (4.4, 4.4a) is
\[
\Delta S[\bar{x}_i / \bar{\gamma}]_{s_i \tau_k^-}^{s_i \tau_k^+} = 1/2(u_-(t) - u_+(s_k^+))(t - s_k^+)^{-1}(s_k^+)^2 = -1/2\bar{u}(1_{s_k^-} - 1_{\tau_k^-})(t - s_k^+)^{-1}(s_k^+)^2,
\]  
and at \[t = \tau_k^-\] it is
\[
\Delta S[\bar{x}_i / \bar{\gamma}]_{s_i \tau_k^-}^{s_i \tau_k^+} = (\uparrow_{s_k^-} - \downarrow_{\tau_k^-})\bar{u}_k(\tau_k^- - s_k^+)^{-1}(s_k^+)^2, \quad \bar{u}_k = \bar{u}_k s_k^+,
\]  
(4.5)
where finite jump, with high \[\bar{u}\], applied during \[1/2o(\tau_k) = \delta_k\], holds the finite size square
\[
\delta_k \bar{u} = \bar{u}_k, \quad \bar{u}_k = |u_k| \times o(\tau_k), |u_k| = 1.
\]  
(4.5a)

Proof. Applying function (4.4) to integral (2.4), leads to
\[
\Delta S[\tilde{x}_{i}/\xi_{j}]^{t \to \tau_{-}\gamma} = 1/2 (\tau_{-}^{\gamma}-s_{k}^{\tau} - s_{k}^{\gamma})^{-1} \int_{s_{k}^{\tau}}^{t \to \tau_{-}\gamma} u_{-}(\tau_{-}) A(t,s_{k}^{\tau}) dt - \int_{s_{k}^{\gamma}}^{t \to \tau_{-}\gamma} u_{+}(s_{k}^{\gamma}) A(t,s_{k}^{\gamma}) dt = (4.6)
\]

\[
1/2(u_{-}(\tau_{-}^{\gamma})-u_{+}(s_{k}^{\gamma}))(\tau_{-}^{\gamma}-s_{k}^{\gamma})^{-1} A(t,s_{k}^{\gamma}) t \to \tau_{-}\gamma, \ t > s_{k}^{\gamma} \tag{4.6a}
\]

The extremal increment brings extremum of function (4.6) which at fixed
\[
u_{-}(\tau_{-}^{\gamma})-u_{+}(s_{k}^{\gamma}) (\tau_{-}^{\gamma}-s_{k}^{\gamma})^{-1} \text{ leads to equations}
\]
\[
\partial A(t,s_{k}^{\gamma}) / \partial t = \dot{A}(t,s_{k}^{\gamma}) t + A(t,s_{k}^{\gamma}) = 0, \ 1/2 u_{-}(\tau_{-}^{\gamma})-u_{+}(s_{k}^{\gamma}) (\tau_{-}^{\gamma}-s_{k}^{\gamma})^{-1} [\dot{A}(t,s_{k}^{\gamma}) t + A(t,s_{k}^{\gamma})] = 0, (4.7)
\]

\[
\text{at } u_{-}(\tau_{-}^{\gamma})-u_{+}(s_{k}^{\gamma}) (\tau_{-}^{\gamma}-s_{k}^{\gamma})^{-1} \neq 0, \tau_{-}^{\gamma} > s_{k}^{\gamma}, \text{ and}
\]

\[
t \partial A(t,s_{k}^{\gamma}) / \partial t + A(t,s_{k}^{\gamma}) / A(t,s_{k}^{\gamma}) + \partial t / t = 0, t \neq 0, A(t,s_{k}^{\gamma}) \neq 0, \partial t / t \neq 0. \tag{4.7a}
\]

From these it follows
\[
\ln A(t,s_{k}^{\gamma}) + \ln C = \ln [A(t,s_{k}^{\gamma}) Ct] = 0, A(t,s_{k}^{\gamma}) Ct = 1,
\]

\[
A(t,s_{k}^{\gamma})^{-1} = C t, A(s_{k}^{\gamma}, t = s_{k}^{\gamma})^{-1} = C s_{k}^{\gamma}, C = A(s_{k}^{\gamma}, s_{k}^{\gamma})^{-1} / s \tag{4.7b}
\]

On a current time interval \( \Delta_{t} = t-s_{k}^{\tau} \), relation (4.7b) determines extremal function
\[
A(s_{k}^{\gamma}, s_{k}^{\gamma}) / A(t,s_{k}^{\gamma}) = t / s_{k}^{\gamma}. \tag{4.8}
\]

Relations (2.2) allow representations
\[
A(s_{k}^{\gamma}, s_{k}^{\gamma}) = r_{k}(s_{k}^{\gamma}) / 2b_{k}(s_{k}^{\gamma}), A(s_{k}^{\gamma}, t) = r_{k}(s_{k}^{\gamma}) / 2b_{k}(t), \tag{4.8a}
\]

whose substitution in (4.8) brings
\[
t = s_{k}^{\gamma} b_{k}(t) / b_{k}(s_{k}^{\gamma}). \tag{4.9}
\]

Comments 4.1. On interval \( \Delta_{t} \), moment \( t = \tau_{-}\gamma \) is coming when relation (4.9)in form
\[
t = s_{k}^{\gamma} \dot{r}(t) / \dot{r}(s_{k}^{\gamma}) \tag{4.9a}
\]

\[
at b(t) = 1/2 \dot{r}(t), b(s_{k}^{\gamma}) = 1/2 \dot{r}(s_{k}^{\gamma}), \tag{4.9b}
\]

can be identified by controlling correlation:
\[
t \equiv s_{k} r_{k}(t) / r_{k}(s_{k}^{\gamma}) \tag{4.9c}
\]
in \( o(t_{k}) \)-locality of applying control \( \tilde{n}_{k} = u_{k} \times o(t_{k}) \), at
\[
r_{k}(t) = r_{k}(t) o(t_{k}), \dot{r}(s_{k}^{\gamma}) = r_{k}(s_{k}^{\gamma}) o(s_{k}^{\gamma}), o(s_{k}^{\gamma}) = o(t_{k}) \tag{4.9d}
\]

Substituting optimal (4.9) in (4.6), we have
\[
\Delta S[\tilde{x}_{i}/\xi_{j}]^{t \to \tau_{-}\gamma} = 1/2 (u_{-}(\tau_{-}^{\gamma}) - u_{+}(s_{k}^{\gamma})(\tau_{-}^{\gamma}-s_{k}^{\gamma})^{-1} A(t,s_{k}^{\gamma})) s_{k}^{\gamma} b_{k}(t) / b_{k}(s_{k}^{\gamma}) \tag{4.10a}
\]

at
\[
2b_{k}(s_{k}^{\gamma}) s_{k}^{\gamma} = r_{k}(s_{k}^{\gamma}). \tag{4.10a}
\]
The last relation implies that control $u_+(s_k^{+\alpha}) = +1_{s_k^{+\alpha}}$ or $u_-(s_k^{-\alpha}) = \uparrow 1_{s_k^{-\alpha}}$, applying at the moment $s_k^{+\alpha}$, follows control $u_-(\tau_k^{\alpha}) = -1_{\tau_k^{\alpha}}$ or $u_-(\tau_k^{-\alpha}) = \downarrow 1_{\tau_k^{-\alpha}}$, which applies when the moment $t = \tau_k^{+\alpha}$, identified by (4.9), comes. Then (4.10) holds:

$$
\Delta S[\tilde{x}, \varphi]_{t_{\tau_k}^{+\alpha}} = 1/2(u_-(t) - u_-(s_k^{+\alpha}))(t - s_k^{+\alpha})^{-1}(s_k^{+\alpha})^2 = -1/2\bar{\mu}(+1_{s_k^{+\alpha}} - 1_{s_k^{-\alpha}})(t - s_k^{+\alpha})^{-1}(s_k^{+\alpha})^2 =
$$

$$
= -\left(\uparrow 1_{s_k^{+\alpha}} - \downarrow 1_{s_k^{-\alpha}}\right)\bar{\mu}_{\tau_k^{+\alpha}}(t - s_k^{+\alpha})^{-1}(s_k^{+\alpha})^2 = \bar{\mu} s_k^{+\alpha} = \bar{\mu} o(s_k). \quad (4.11)
$$

This means starting jump “Yes” $\uparrow 1_{s_k^{+\alpha}}$, applied within time interval $(t - s_k^{+\alpha}) \rightarrow (\tau_k^{+\alpha} - s_k^{+\alpha})$ follows jump “No” $\downarrow 1_{s_k^{-\alpha}}$, while both measure information one Bit on time interval

$$
\delta_k^{+\alpha} = \lim_{t \rightarrow \tau_k^{+\alpha}} (t - s_k^{+\alpha})^{-1}(s_k^{+\alpha})^2 = s_k^{+\alpha} / (\tau_k^{+\alpha} / s_k^{+\alpha} - 1). \quad (4.12)
$$

That is equivalent of one Nat measured by the initial entropy functional (1.10) on $\delta_k^{+\alpha}$. Increment (4.5) during this time interval opposites to contribution (3.5b), which delta-control (3.1) extracts as information.

Sign minus in (3.5) indicates this, assuming that increment of entropy (4.6a) on $\delta_k^{+\alpha}$ is converted to information by the delta-control on $1/2(o(\tau_k) = \delta_k$.

Symbol $\uparrow 1_{s_k^{\alpha}}$ in relation $(\downarrow 1_{s_k^{\alpha}})\bar{\mu} = \uparrow 1_{s_k^{\alpha}} - \downarrow 1_{s_k^{\alpha}}(\bar{\mu})$ (within function (3.5)), corresponds to jump control $u_+(s_k^{+\alpha}) = +1_{s_k^{+\alpha}}$ applied on right border $s_k^{+\alpha}$ of cutoff interval $s_k^{+\alpha} < s_k < s_k^{+\alpha}$, while $\downarrow 1_{s_k^{-\alpha}}$ corresponds to jump control $u_-(\tau_k^{\alpha}) = -1_{\tau_k^{\alpha}}$ applied on left border of cutoff interval $\tau_k^{+\alpha} < \tau_k < \tau_k^{+\alpha}$.

Let us find a discrete analog of the functional increments under discrete delta-function

$$
\delta_k^{+\alpha} u_{\tau_k} = (u_+(\tau_k^{+\alpha}) - u_+(\tau_k^{+\alpha}))((\tau_k^{+\alpha} - \tau_k^{+\alpha})^{-1}
$$

$$
\text{Proposition 4.2. Applying discrete delta function (4.13) we get}
$$

$$
\Delta S[\tilde{x}, \varphi]_{t_{\tau_k}^{+\alpha}} = \begin{cases} 
0, \tau_k < \tau_k^{+\alpha} \\
1/4u_+(\tau_k^{+\alpha})\tau_k^{+\alpha}, \tau_k = \tau_k^{+\alpha}, 1/4 \downarrow 1_{\tau_k^{+\alpha}} \times \tau_k^{+\alpha} \\
1/4u_+(\tau_k^{+\alpha})\tau_k^{+\alpha}, \tau_k = \tau_k^{+\alpha}, 1/4 \uparrow 1_{\tau_k^{+\alpha}} \times \tau_k^{+\alpha} \\
1/2(u_-(\tau_k^{-\alpha}) - u_+(\tau_k^{+\alpha})) (\tau_k^{+\alpha} - \tau_k^{+\alpha})(\tau_k^{+\alpha} - \tau_k^{+\alpha})^{-1}(\tau_k^{+\alpha})^{-2} \tau_k^{+\alpha} < \tau_k < \tau_k^{+\alpha}, 1/2(\downarrow 1_{\tau_k^{+\alpha}} - \uparrow 1_{\tau_k^{+\alpha}})\bar{\mu} \times (\tau_k^{+\alpha} - \tau_k^{+\alpha}) \end{cases}, \quad (4.14)
$$

where

$$
1/2(u_-(\tau_k^{-\alpha}) - u_+(\tau_k^{+\alpha})) = 1/2(u_-(\tau_k^{-\alpha}) - u_+(\tau_k^{+\alpha})) [(2\tau_k^{-\alpha} - (\tau_k^{+\alpha} - \tau_k^{-\alpha}))^{-1}(\tau_k^{-\alpha})^{-2} \tau_k^{-\alpha} < \tau_k < \tau_k^{+\alpha},
$$

and the discrete intervals hold

$$
\delta_k^{+\alpha} = \lim_{\tau_k^{-\alpha} \rightarrow \tau_k^{+\alpha}} [(\tau_k^{+\alpha} - \tau_k^{-\alpha}) - (\tau_k^{+\alpha} - \tau_k^{-\alpha})]^{-1}(\tau_k^{-\alpha})^{-2} = \lim_{\tau_k^{-\alpha} \rightarrow \tau_k^{+\alpha}} [2\tau_k^{-\alpha} - (\tau_k^{+\alpha} + \tau_k^{+\alpha})^{-1}(\tau_k^{-\alpha})^{-2} =
$$

$$
(2\tau_k^{+\alpha} - \tau_k^{-\alpha})^{-1}(\tau_k^{-\alpha})^{-2} = \tau_k^{-\alpha}, \quad (4.15a)
$$

14
\[ \delta^+ = \lim_{\tau_k \to \tau_k^-} \left((\tau_k^+ - \tau_k^-) - (\tau_k^+ - \tau_k^-)\right)^2 = (2\tau_k^+ - \tau_k^-)^2 = \tau_k^+ \], \quad (4.15b) \\
\delta = \delta^+ - \delta^-, \quad \delta = \tau_k^+ - \tau_k^- = 1/2o(\tau_k) \quad (4.15) \\

Using the finite size of the jump, relation \\
\[(\downarrow 1_{t_k} - \uparrow 1_{t_k})\delta_k = 1/2(\downarrow 1_{t_k} - \uparrow 1_{t_k})\tau_k \times (\tau_k^+ - \tau_k^-) = \tau_k \left(\downarrow 1_{t_k} - \uparrow 1_{t_k}\right) \]

determines a Bit on interval \( \delta_k \) which equals to \\
\[S[\tilde{x}_i / \xi_i]_{\delta_k} = 1/2\tau_k Nats \quad (4.17)
\]
that is discrete analog of (3.4b), while the related contributions:
\[S[\tilde{x}_i / \xi_i]_{1/2} = 1/4\tau_k Nats, S[\tilde{x}_i / \xi_i]_{1/4} = 1/4\tau_k Nats \quad (4.17a)\]
are analogous to (3.6a,b). •

Corollaries 4.2

From (2.7) and (1.5, 1.3) it follows that:
4.2a. The step-wise control function \( u_+ = u_+(\tau_k) \), implementing transformation \( \tilde{x}_i(\tau_k^-) \to \xi_\tau(\tau_k) \), converts the entropy functional from its minimum at \( t \leq \tau_k^- \) to maximum at \( \tau_k^+ \to \tau_k \);
4.2b. The step-wise control function \( u_- = u_-(\tau_k) \), implementing transformation \( \xi_\tau(\tau_k) \to \tilde{x}_i(\tau_k^+) \), converts the entropy functional from its maximum at \( t > \tau_k \) to minimum at \( \tau_k \to \tau_k^+ \);
4.2c. The impulse control function implementing transformations \( \tilde{x}_i(\tau_k^-) \to \xi_\tau(\tau_k) \to \tilde{x}_i(\tau_k^+) \), switches the entropy functional from its minimum to maximum and back from maximum to minimum, while the absolute maximum of the entropy functional at a vicinity of \( t = \tau_k \) allows the impulse control to deliver maximal amount of information (4.17) from these transformations, holding principle of extracting maxmin- minmax of the EF measure;
4.2d. Dissolving correlation between the process cutoff points (3.10) leads to losing functional connections at these discrete points, which evaluate Feller’s kernel measure [27, 29].

4.2e. The relation of that measure to additive functional [29] in form (1.7) allows evaluating the kernel’s information by the entropy functional (1.5), (1.10) and (2.4).
4.2f. The jump action on Markov process, associated with “killing its drift”, selects the Feller kernel measure [12, 29, 34], while the cutoff is a source of a kernel information functional measure, estimated by (3.6a,b) and (4.14). •

Proposition 4.3.

A. Extremal increment of the entropy functional, collected on interval \( \tau_k^- \to \tau_k^+ \) from right border \( s_k^- \) of \( s_k^- < s_k < s_k^+ \) to left border \( \tau_k^- \) of applying control \( u_-(\tau_k^-) \), is \\
\[\Delta S[\tilde{x}_i / \xi_i]_{s_k^-}^{\tau_k^-} = \Delta S[\tilde{x}_i / \xi_i]_{s_k^-}^{\tau_k^-} + \Delta S[\tilde{x}_i / \xi_i]_{s_k^-}^{\tau_k^-}, \quad (4.18)\]
where this sum, measured in Nats, holds
\[ \Delta S[k_i / \zeta_i]_{t_k}^{\tau_k^{-} - \tau_k^{+}} = -1/2 \bar{u}(t - s_k^{+})^{-1}(s_k^{+})^2 + 1/4 \bar{u} \tau_k^{-} \]  \hspace{2cm} (4.19)  

at \( s_k^{+} = o(s_k), \tau_k^{-} = o(\tau_k^{+}), o(s_k) = o(\tau_k^{-}), \bar{u} \) (4.19a) 

B. The impulse control converts entropy increment \( \Delta S[k_i / \zeta_i]_{t_k}^{\tau_k^{-} - \tau_k^{+}} \) to equal information contribution \( \Delta I[k_i / \zeta_i]_{t_k} \), while control \( u_s(s_k^{+} \rightarrow \tau_k^{+}) \), starting with \( u_s(s_k^{+}) = +1, \Delta \bar{u} \), prognosis entropy increment \( \Delta S[k_i / \zeta_i]_{t_k}^{\tau_k^{-} - \tau_k^{+}} \rightarrow \Delta S[k_i / \zeta_i]_{t_k}^{\tau_k^{-} - \tau_k^{+}} - \Delta S[k_i / \zeta_i]_{t_k}^{\tau_k^{-} - \tau_k^{+}} \).

The extreme holds maximum, which with starting at \( s_k^{+} \) minimum, implements minimax principle of converting the entropy to the impulse information. The impulse cutoff extracts maximum of minimal information that impulse cuts.

**Proposition 4.4.**

A. At satisfaction of extremal principle, increment \( \Delta S[k_i / \zeta_i]_{t_k}^{\tau_k^{-} - \tau_k^{+}} \) compensates for

\[ \Delta S[k_i / \zeta_i]_{t_k}^{\tau_k^{-} - \tau_k^{+}} = 1/2 \bar{u}(t - s_k^{+})^{-1}(s_k^{+})^2 + 1/4 \bar{u} \tau_k^{-} \]  \hspace{2cm} (4.20a) 

at triple ratio of time:

\[ t \rightarrow \tau_k^{+}, t = \tau_k^{+} / s_k^{+} = 3, \tau_k^{+} = 3s_k^{+} \]  \hspace{2cm} (4.20a)

B. This ratio determines optimal time interval of converting entropy increment \( \Delta S[k_i / \zeta_i]_{t_k}^{\tau_k^{-} - \tau_k^{+}} \) to kernel information contribution \( \Delta I[k_i / \zeta_i]_{t_k} \) for each dimension for the cutoff.

The proofs follow straightforward from (4.12), (4.18).

5. **Information path functional in \( n \)-dimensional Markov process under \( n \)-cutoff impulses**

Both \( \Delta I[k_i / \zeta_i]_{t_k} \Rightarrow \Delta I[k_i / \zeta_i]_{t_k} \) and distance between them \( \tau_k^{-} \) for each \( k \) dimension determine information path functional (IPF) at each cutoff \( \tau_k, \tau_{k+1}, k, k+1, \ldots, n \):

\[ \Delta I[k_i / \zeta_i]_{t_k}, \Delta I[k_i / \zeta_i]_{t_{k+1}} \]  \hspace{2cm} (5.1) 

The IPF unites information contributions taking along \( n \) dimensional Markov process:

\[ I[k_i / \zeta_i]_{t_k}^{\tau_k^{-} - \tau_k^{+}} = \sum_{k=1}^{k=n} \Delta I[k_i / \zeta_i]_{t_k}^{\tau_k^{-} - \tau_k^{+}} \]  \hspace{2cm} (5.2) 

where each dimensional information contribution \( \Delta I[k_i / \zeta_i]_{t_k}^{\tau_k^{-} - \tau_k^{+}} = \Delta I[k_i / \zeta_i]_{t_k} \) follows entropy increment (4.19) on time interval

\[ \tau_k^{-} = 3\tau_k, k, k+1, \ldots, n \]  \hspace{2cm} (5.2a) 

And invariant ratio \( \tau_k^{+} / \tau_k = 3 \) rises from extremal principle for entropy increments (4.20).

**Proposition 5.1.**

A. Distance between the finite cutoff intervals \( \Delta_k = \tau_{k+1} - \tau_k \), measured by ratio to each following intervals \( \tau_k \), decreases in ratio:
\[ \Delta_k / \tau_k = 8, \quad (5.3) \]

with growing dimension \( k, k+1, \ldots, n \),

while the ratio of these discrete intervals decreases in double triplicate number

\[ \tau_k / \tau_{k-1} = 1/9. \quad (5.4) \]

B. Total sum of the descending time distances:

\[ \lim_{n \to \infty} \sum_{k=1}^{n} \Delta_k = T, \quad (5.5) \]

following from (5.3), (5.4) and (5.2a) converges to finite \( T \).

C. Sum of information contributions \( \Delta I[\bar{x}_i / \bar{x}_{i-1}] \) on this \( T \) is converging to both path functional integral and entropy increments of the initial entropy functional:

\[ \sum_{k=1}^{n} \Delta I[\bar{x}_i / \bar{x}_{i-1}] \to \lim_{n \to \infty} n \Delta I[\bar{x}_i / \bar{x}_{i-1}] = I[\bar{x}_i / \bar{x}_{i-1}] = S[\bar{x}_i / \bar{x}_{i-1}], \quad (5.6) \]

while the information contributions within each cutoff on \((\tau_k^+, \tau_k^-)\):

\[ \Delta I[\bar{x}_i / \bar{x}_{i-1}] = 0.5 Nats, \quad \Delta I[\bar{x}_i / \bar{x}_{i-1}] = 0.25 Nats, \quad (5.7) \]

are invariants, independent on decreasing this discrete, and EF (5.6) limits the converging integrals at the finite (5.5).

Proof A. From \( \tau_k^+ - \tau_k^- = \tau_k^-, \tau_k^- - \tau_{k-1}^- = \tau_{k-1}^- \), follow relations

\[ \tau_k / \tau_{k-1} = (\tau_k^+ / \tau_k^- - \tau_k^- / \tau_{k-1}^-)(1 - \tau_k^- / \tau_{k-1}^-)^{-1} = \tau_k^+ / \tau_{k-1}^-, \quad \text{and} \]

\[ \tau_k / \tau_{k-1} = (\tau_k^- / \tau_k^- - 1)(\tau_{k-1}^- / \tau_{k-1}^- - 1)^{-1} \tau_k^- / \tau_{k-1}^-, \quad \text{which lead to} \]

\[ \tau_k^+ / \tau_k^- = (\tau_k^- / \tau_k^- - 1)(\tau_{k-1}^- / \tau_{k-1}^- - 1)^{-1} \tau_k^- / \tau_{k-1}^-, \]

\[ \tau_k^- / \tau_k^- / \tau_{k-1}^- = (\tau_k^- / \tau_k^- - 1)(\tau_{k-1}^- / \tau_{k-1}^- - 1)^{-1} \tau_k^- / \tau_{k-1}^- = \tau_k^+ / \tau_k^- (\tau_{k-1}^- / \tau_{k-1}^- - 1)^{-1} \tau_k^- / \tau_{k-1}^- , \]

which lead to

\[ \tau_k^+ / \tau_k^- = (\tau_k^- / \tau_k^- - 1)(\tau_{k-1}^- / \tau_{k-1}^- - 1)^{-1} \tau_k^- / \tau_{k-1}^- , \]

\[ \tau_k^+ / \tau_k^- = (\tau_k^- / \tau_k^- - 1)(\tau_{k-1}^- / \tau_{k-1}^- - 1)^{-1} \tau_k^- / \tau_{k-1}^- = \tau_k^+ / \tau_k^- (\tau_{k-1}^- / \tau_{k-1}^- - 1)^{-1} \tau_k^- / \tau_{k-1}^- , \]

\[ \tau_k^+ / \tau_k^- = (\tau_k^- / \tau_k^- - 1)(\tau_{k-1}^- / \tau_{k-1}^- - 1)^{-1} \tau_k^- / \tau_{k-1}^- , \]

\[ \tau_k^+ / \tau_k^- = (\tau_k^- / \tau_k^- - 1)(\tau_{k-1}^- / \tau_{k-1}^- - 1)^{-1} \tau_k^- / \tau_{k-1}^- , \]

From \( \tau_k / \tau_{k-1} = (\tau_k^+ / \tau_k^- - \tau_k^- / \tau_{k-1}^-)(1 - \tau_k^- / \tau_{k-1}^-)^{-1} = 3(\tau_k^+ / \tau_k^- - 1)(1 - \tau_k^- / \tau_{k-1}^-)^{-1}, \)

at \( \tau_k^+ / \tau_k^- = \tau_{k-1}^+ / \tau_{k-1}^- \), it follows \( \tau_k / \tau_{k-1} = 3 \tau_{k-1}^+ / \tau_{k-1}^- = 3 \tau_{k-1}^+ / \tau_{k-1}^- \), and from \( \tau_k / \tau_{k-1} = \tau_k^+ / \tau_k^- \),

we have

\[ \tau_k^+ / \tau_k^- = 3 \tau_k^- / \tau_k^- = 3 \tau_k^- / \tau_k^- = 3, \quad (5.8a) \]

and \( \tau_k = 2 \tau_k^- \), \( \tau_k = 2 / 3 \tau_k^- \), \( \tau_k^- = 1 / 3 \tau_k^- \).

Then from

\[ \tau_k / \tau_{k-1} = [(\tau_k^+ / \tau_k^- - 1)/(\tau_k^- / \tau_{k-1}^- - 1)](\tau_k^+ / \tau_{k-1}^-) = \tau_k^- / \tau_k^- / \tau_{k-1}^- / \tau_{k-1}^- = \tau_k^- / \tau_k^- \]

and at (5.8a) we come to

\[ \tau_k / \tau_{k-1} = 1 / 3 \tau_k^- / \tau_{k-1}^- = 1 / 9. \quad (5.9) \]

Proof B follows from definition of total time (5.5), the finite \( \Delta_k \), and invariant ratio (5.4).
Proof C. Since each extremal entropy increment, supplying the following starting cutoff with equal information contribution $1/4u_\omega(\tau_k^{-\omega})\tau_k^{-\omega} = 1/4\text{Nats}$, is invariant, this increment is also invariant. According to (4.14), information contributions
\[
1/2(u_\omega(\tau_k^{\omega}) - u_\omega(\tau_k^{\omega}))(\tau_k^{\omega} - \tau_k^{-\omega}) = 1/2\text{Nats}, \text{ and } 1/4u_\omega(\tau_k^{\omega})\tau_k^{\omega} = 1/4\text{Nats}
\]
(5.9) depend on both the cutoff moments $\tau_k^{\omega} < \tau_k < \tau_k^{-\omega}$, which satisfy invariant relations (5.4), (5.8a,b), and the fixed controls (4.4a), hence holding the invariant values.

Since EF functional $S[\hat{x}_\omega / \zeta^{\omega}]_T$ (1.5) limits growth of path functional in (5.6), the IPF approach the EF functional is achievable during time $T$ (5.5) at unlimited increase of the process dimensions. Because initially upper time $T$ of this integral is undefined, but limited by (5.5), both path’s and entropy integrals are converging and restrictive at the unlimited dimension number.

While each cutoff contribution $0.5\text{Nats}$ delivers $1/4\text{Nats}$ from cutting the random process correlation, which this control conveys to the next cut, the needed following control information is part of the delivered $0.5\text{Nats}$. Thus, each source of the starting cut information $1/4\text{Nats}$ is the previous cutoff $0.5\text{Nats}$, while the current cutoff $0.5\text{Nats}$ provides information for next cut, satisfying the balance of information. Initial starting cutting control information $1/4\text{Nats}$ should provide an external source, for example, an interactive action, which via cutting random process (at $s_{k=0}$), delivers $0.5\text{Nats}$ that not only compensates for that interactive $1/4\text{Nats}$ but also delivers the control information providing the following cut.

The invariant information contributions on the decreasing discrete intervals actually increase each of these contributions if the intervals summary is limited, while the integral summarizes the previous contributions.

Let us find increments of correlation functions within optimal interval $\Delta_k = \tau_k^{-\omega} - s_k^{\omega}$ and on the cutoff time borders $\tau_k^{-\omega}, \tau_k^{\omega}$.

Proposition 5.2.
A. Correlation function on interval $\Delta_k = t - s_k^{\omega}$ for the extremal process holds
\[
r_k^-(t) = 1/2r_k(s_k^{\omega})(t^2 / (s_k^{\omega})^2 + 1)[(t - s_k^{\omega})],
\]
(5.10)
ending with correlation on the cutoff left border $\tau_k^{-\omega}$:
\[
r_k^-(\tau_k^{-\omega}) = 1/2r_k(s_k^{\omega})[(\tau_k^{-\omega} / s_k^{\omega})^2 + 1].
\]
(5.10a)
After the cutoff, correlation function on the next time interval $(\tau_k^{\omega} - \tau_k^{-\omega})$ holds
\[
r_k^+(t) = 1/2r_k(s_k^{\omega})(t^2 / (s_k^{\omega})^2 + 1)[(t - s_k^{\omega})], \quad \tau_k = 2\tau_k^{-\omega}.
\]
(5.10b)
B. Correlation on right border $\tau_k^{\omega}$ of the finite cutoff:
\[ r_k^+(\tau_k^{x_o}) = 1/2 r_k (\tau_k^{x_o}) [(\tau_k^{x_o}/\tau_k^{x_o})^2 + 1] = 5 r_k (\tau_k^{x_o}). \]  (5.11)

C. Difference of these correlations on \( \delta_k = \tau_k^{x_o} - \tau_k^{x_o} = 1/2 o(\tau_k) \):

\[ r_{ko}^+(\tau_k^{x_o}) - r_{ko}^-(\tau_k^{x_o}) = \Delta r_{ko}(\delta_k), \quad \Delta r_{ko}(\delta_k) = 5 r_k^-(\tau_k^{x_o}) - r_k^+(\tau_k^{x_o}) = 4 r_k (\tau_k^{x_o}) \]  (5.12)

and its relative value during such finite cutoff holds:

\[ \Delta r_{ko}(\delta_k)/r_k(\tau_k^{x_o}) = 4, \]  (5.13)

while at the cutoff moment \( \tau_k \):

\[ r_k^+(\tau_k) = 1/2 o(\tau_k) \to 0 \text{ at } o(\tau_k) \to 0. \]  (5.13a)

**Proof A, B, C.** Relation \( t = s_k^{x_o} b_k(t) / b_k(s_k^{x_o}) \), at \( b_k(t) = 1/2 r_k(t) \), determines functions

\[ r_k(t) = 2b_k(s_k^{x_o}) t / s_k^{x_o}, \]  (5.14)

at \( b_k(s_k^{x_o}) s_k^{x_o} = 1/2 r_k(s_k^{x_o}) \).

That gets correlation function on this interval for the extremal process (5.10) and at its end (5.11). After the cutoff, correlation function on the next time interval \( (\tau_k^{x_o} - \tau_k^{x_o}) \) holds (5.11a).

The optimal correlation preceding the current cut on its left border \( \tau_k^{x_o} \):

\[ r_{ko}^-(\tau_k^{x_o}) = 1/2 r_k^-(s_k^{x_o})[3^2 + 1] = 5 r_k^-(s_k^{x_o}), \quad r_{ko}^+(\tau_k^{x_o}) = 1/2 r_k^+(s_k^{x_o})[3^2 + 1] = 5 r_k^+(s_k^{x_o}) \]  (5.15)

growth in five time of the optimal correlation for previous cutoff at \( s_k^{x_o} \).

Correlation on right border \( \tau_k^{x_o} \) of the finite cutoff holds (5.11), which allows finding both difference of these correlations on \( \delta_k = \tau_k^{x_o} - \tau_k^{x_o} = 1/2 o(\tau_k) \) in (5.12) and its relative value during such finite cutoff in (5.13).

**Examples.** Let us find which entropy functional expression meets requirements (4.1A, B) within discrete intervals \( \Delta_t = (t-s) \to o(t) \), particularly on \( \Delta_k = (\tau_k^{x_o} - s_k^{x_o}) \to o(\tau_k^{x_o}) \) under opposite controls \( u_+, u_- \), thereafter satisfying relations

\[ c^2 = |u_+ u_-| = c_+ c_- = \bar{u}^2, \quad c_+ = u_+, c_- = u_- \]  (5.16)

Specifically under \( u_+(s_k^{x_o}) = +1 s_k^{x_o} \bar{u} = \bar{u}(s_k^{x_o}), u_- = -u_+(s_k^{x_o}) = -1 s_k^{x_o} \bar{u} = -\bar{u}(s_k^{x_o}), \)

following (4.19), we have

\[ S_k[\bar{u} \bar{u} (s_k^{x_o})]_{s_k^{x_o}}^{\tau_k^{x_o}} = -1/2 [u_+(s_k^{x_o})](\tau_k^{x_o} - s_k^{x_o})^{-1}(s_k^{x_o})^2 = -1/2 [u_+(s_k^{x_o})(s_k^{x_o})^2 / s_k^{x_o})(3-1)] = -1/4[4u_+(s_k^{x_o}) s_k^{x_o}] \]  (5.17a)

\[ S_k[\bar{u} \bar{u} (s_k^{x_o})]_{s_k^{x_o}}^{\tau_k^{x_o}} = 1/2 [\bar{u}(s_k^{x_o})](\tau_k^{x_o} - s_k^{x_o})^{-1}(s_k^{x_o})^2 = 1/2 [\bar{u}(s_k^{x_o})(s_k^{x_o})^2 / s_k^{x_o})(3-1)] = 1/4[4\bar{u}(s_k^{x_o}) s_k^{x_o}], \]  (5.17a)

at \( S_{k_k}[\bar{u} \bar{u} (s_k^{x_o})]_{s_k^{x_o}}^{\tau_k^{x_o}} = -S_{k_k}[\bar{u} \bar{u} (s_k^{x_o})]_{s_k^{x_o}}^{\tau_k^{x_o}}, \)  (5.17b)

\[ S_{k_k}[\bar{u} \bar{u} (s_k^{x_o})]_{s_k^{x_o}}^{\tau_k^{x_o}} = -S_{k_k}[\bar{u} \bar{u} (s_k^{x_o})]_{s_k^{x_o}}^{\tau_k^{x_o}} = -1/2[\bar{u}(s_k^{x_o}) s_k^{x_o}] = \Delta S_k[\bar{u} \bar{u} (s_k^{x_o})]_{s_k^{x_o}}^{\tau_k^{x_o}}. \]  (5.17c)

Relations
4S_\tau_\kappa / \zeta_\kappa \| s_\kappa \| s_\kappa \| s_\kappa = -\bar{u}(s_\kappa) (s_\kappa) = -2 \times 1 \times 1, \quad 4S_\tau_\kappa / \zeta_\kappa \| s_\kappa \| s_\kappa \| s_\kappa = \bar{u}(s_\kappa) = 2 \times 1 \times 1 \quad (5.17a)

satisfy condition
4S_\tau_\kappa / \zeta_\kappa \| s_\kappa \| s_\kappa \| s_\kappa = -\bar{u}(s_\kappa) \times \bar{u}(s_\kappa) = -(2 \times 1 \times 1) \times (2 \times 1 \times 1) = -4 \times 1 \times 1 \quad (5.17b)

These entropy expressions at any current moment \( t \) within \( \Delta_t = (t - s_\kappa) \) do not comply with (4.1A,B).

The same results hold true for the entropy functional increments under

\begin{align*}
\Delta S[\tilde{x}_\kappa / \zeta_\kappa]_{\tau_\kappa} &= -1 / 2 (u_\kappa(t) - u_\kappa(s_\kappa)) (s_\kappa)^2 \\
\end{align*}

which for \( t \to \tau_\kappa \) holds

\begin{align*}
\Delta S[\tilde{x}_\kappa / \zeta_\kappa]_{\tau_\kappa} &= -1 / 2 (u_\kappa(\tau_\kappa) - u_\kappa(s_\kappa)) (s_\kappa)^2 \\
\end{align*}

and satisfies relations

\begin{align*}
S_\kappa[\tilde{x}_\kappa / \zeta_\kappa]_{\tau_\kappa} &= -S_\kappa[\tilde{x}_\kappa / \zeta_\kappa]_{\tau_\kappa} = \Delta S[\tilde{x}_\kappa / \zeta_\kappa]_{\tau_\kappa} \quad (5.18a) \\
\end{align*}

which determines

\begin{align*}
S_\kappa[\tilde{x}_\kappa / \zeta_\kappa]_{\tau_\kappa} &= -1 / 4 (u_\kappa(\tau_\kappa) - u_\kappa(s_\kappa))(s_\kappa)^2 \\
\end{align*}

We get the entropy expressions for the opposite directional discrete (5.18):

\begin{align*}
S_\kappa[\tilde{x}_\kappa / \zeta_\kappa]_{\tau_\kappa} 4(\tau_\kappa - s_\kappa)(s_\kappa) = (u_\kappa(\tau_\kappa) - u_\kappa(s_\kappa)), \\
\end{align*}

which satisfy (5.18a) at

\begin{align*}
-2(u_\kappa(\tau_\kappa) - u_\kappa(s_\kappa)) = -2(-1 \times \bar{u} - 1 \times \bar{u}) = 2\bar{u}[1 + 1] = 4[1 + 1], \\
\end{align*}

while for each:

\begin{align*}
S_\kappa[\tilde{x}_\kappa / \zeta_\kappa]_{\tau_\kappa} 4(\tau_\kappa - s_\kappa)(s_\kappa) = \bar{u}[1 - 1] \\
\end{align*}

satisfaction of both 4.1A, B:

\begin{align*}
-(u_\kappa(\tau_\kappa) - u_\kappa(s_\kappa)) \times u_\kappa(\tau_\kappa) - u_\kappa(s_\kappa) = -[u_\kappa(\tau_\kappa) - u_\kappa(s_\kappa)]^2, \\
\end{align*}

Requires

\begin{align*}
\bar{u} = -2j, \\
\end{align*}

when \(-[u_\kappa(\tau_\kappa) - u_\kappa(s_\kappa)]^2 = [-2j[-1 - 1]^2]. \quad (5.21a)\)
Simultaneous satisfaction of both 4.1.A, B leads to
\[ \Delta S \left[ \frac{\tau_k}{\zeta} \right]_{\tau_k \rightarrow \tau_k^o} = 2(\tau_k^o - s_k^o)(s_k^o)^{-2} = -2j[\tau_k^o - 1] \quad \text{and} \quad -2j[\tau_k^o - 1] \tau_k \rightarrow -2j[\tau_k^o] - 2j[\tau_k^o] = 4 \ (5.21b) \]

At \( \theta(t) \rightarrow 0 \), these admit an instant existence of both \(-1 - \tau_k + 1\).

It requires imaginary entropy expressions:
\[ S_1 \left[ \frac{\tau_k}{\zeta} \right]_{\tau_k \rightarrow \tau_k^o} = 4(\tau_k^o - s_k^o)(s_k^o)^{-2} = -2j[\tau_k^o - 1] \quad \text{and} \quad -2j[\tau_k^o - 1] \tau_k \rightarrow = -2j[\tau_k^o] \quad \text{for } (5.22a) \]

while
\[ S_2 \left[ \frac{\tau_k}{\zeta} \right]_{\tau_k \rightarrow \tau_k^o} = 4(\tau_k^o - s_k^o)(s_k^o)^{-2} = 2j[\tau_k^o - 1] \tau_k \rightarrow = 2j[\tau_k^o] \quad \text{for } (5.22b) \]

and
\[ S_3 \left[ \frac{\tau_k}{\zeta} \right]_{\tau_k \rightarrow \tau_k^o} = 4(\tau_k^o - s_k^o)(s_k^o)^{-2} - S_2 \left[ \frac{\tau_k}{\zeta} \right]_{\tau_k \rightarrow \tau_k^o} = 4(\tau_k^o - s_k^o)(s_k^o)^{-2} \]

\[ -j2[\tau_k^o - 1] \tau_k \rightarrow = -j2[\tau_k^o] \quad \text{for } (5.23b) \]

Relations (5.3a), (5.20b), as well as (5.21b) and (5.23a) satisfy only at points \( \tau_k^o, s_k^o \).

Between these points, within \( \Delta t = (t - s_k^o) \rightarrow 0 \), the entropy expressions (5.22ab) are imaginary since condition \( \tau_k = -2j \) in (5.23) is not true. Within this interval relations hold:

\[ (S_1 - S_2)^2 = S_1^2 + S_2^2 - 2S_1 S_2, S_1 = -1/2 j S_1^2, S_2 = 1/2 j S_2^2, S_1^2 = -1/4 S_1^2, S_2^2 = -1/4 S_2^2 \]

We have
\[ -2S_1 S_2 = -2(-1/4) j S_1^2 = -1/2 S_1^2, \quad \text{while } S_1^2 + S_2^2 = 1/2 S_1^2 \quad \text{and} \quad (S_1 - S_2)^2 = (-j S_1^2)^2 = -(S_1^2)^2 \]

At fulfillment of 4.1A,B we get relations
\[ S_{1,2}^2 = -1/2 j S_{1,2}, S_{1,2} = -1/4 S_{1,2}^2, S_{1,2} = \pm j S_{1,2} \quad \text{from which also follows } S_{1,2} = 2 j \]

Let us find the entropy increments under control \( u (\tau_k^o), u (\tau_k^o - o(\tau_k^o)) \) near a left border of the cut \( t = \tau_k^o - o(\tau_k^o) \). Applying (5.18a,b) and (5.19a,b), at \( s_k^o \rightarrow t = \tau_k^o - o(\tau_k^o) \), we get
\[ \Delta S(u (\tau_k^o), u (\tau_k^o - o(\tau_k^o))) = -1/2 (u (\tau_k^o) - u (\tau_k^o - o(\tau_k^o))(o(\tau_k^o - o(\tau_k^o)))^2 = \]

\[ -1/2 u (\tau_k^o) (\tau_k^o - o(\tau_k^o)) - u (\tau_k^o - o(\tau_k^o))(\tau_k^o - o(\tau_k^o))(\tau_k^o - o(\tau_k^o))(\tau_k^o - o(\tau_k^o))(\tau_k^o - o(\tau_k^o)) \quad \text{for } (5.24) \]

If entropy measure of these controls:
\[ \Delta S = 1/2 u (\tau_k^o) (\tau_k^o - o(\tau_k^o)) - u (\tau_k^o - o(\tau_k^o))(\tau_k^o - o(\tau_k^o)) \]

and \( (\tau_k^o - o(\tau_k^o)) \) are finite, then entropy increment is infinite:
\[ \Delta S(u (\tau_k^o - o(\tau_k^o))) = \Delta S(u (\tau_k^o - o(\tau_k^o)))(o(\tau_k^o))^{-1} \rightarrow \infty, \quad \text{at } o(\tau_k^o) \rightarrow 0 \]
At $u_-(\tau_k^o) = 2j(-1_{\tau_k^o})$, $u_+(\tau_k^o - o(\tau_k^o)) = 2j(1_{\tau_k^o - o(\tau_k^o)})$, increment is

$$\Delta S[\bar{x}_i / \gamma_i]_{\tau_k^o - o(\tau_k^o)}^{\tau_k^o - o(\tau_k^o)} = -1/2\{2j(-1_{\tau_k^o}) - 2j(1_{\tau_k^o - o(\tau_k^o)})\}(o(\tau_k^o - o(\tau_k^o)) - 1(\tau_k^o - o(\tau_k^o)))^2,$$

$$o(\tau_k^o - o(\tau_k^o)) = \Delta S[\bar{x}_i / \gamma_i]_{\tau_k^o - o(\tau_k^o)}^{\tau_k^o - o(\tau_k^o)} / j(1_{\tau_k^o - o(\tau_k^o)})\}((\tau_k^o - o(\tau_k^o)) - (\tau_k^o) ) .$$

Or imaginary Bit of control $j(1_{\tau_k^o - o(\tau_k^o)})$ applied on interval $(\tau_k^o - o(\tau_k^o))$ compensates for relative interval $1(\tau_k^o - o(\tau_k^o))$.

$$\tau_k^o - o(\tau_k^o) - 1(\tau_k^o - o(\tau_k^o))$$

where entropy increment

$$\Delta S[\bar{x}_i / \gamma_i]_{\tau_k^o - o(\tau_k^o)}^{\tau_k^o - o(\tau_k^o)} / o(\tau_k^o) = -1/2[u_+(\tau_k^o) - o(\tau_k^o)],$$

adjoins in functional increment (4.11) at the moment $t = \tau_k^o$ of applying control $u_+(\tau_k^o)$ with contributing in following IPF (4.13) at the cutoff.

Particularly, at $\tau_k^o - o(\tau_k^o) - 1(\tau_k^o - o(\tau_k^o))$, we have

$$\Delta S[\bar{x}_i / \gamma_i]_{\tau_k^o - o(\tau_k^o)}^{\tau_k^o - o(\tau_k^o)} = -1/2[u_+(\tau_k^o) - o(\tau_k^o)](o(\tau_k^o) - o(\tau_k^o)) = -2j1_{\tau_k^o},$$

$$\Delta S[\bar{x}_i / \gamma_i]_{\tau_k^o - o(\tau_k^o)}^{\tau_k^o - o(\tau_k^o)} / o(\tau_k^o) = -1/2[u_+(\tau_k^o)],$$

adjoints in functional increment (4.11) at the moment $t = \tau_k^o$ of applying control $u_+(\tau_k^o)$ with contributing in following IPF (4.13) at the cutoff.

Thus, infinite influx of entropy functional arises prior the impulse cut:

$$\lim_{o(\tau_k^o) \rightarrow 0} \Delta S[\bar{x}_i / \gamma_i]_{\tau_k^o - o(\tau_k^o)}^{\tau_k^o - o(\tau_k^o)} / o(\tau_k^o) = j\infty .$$

The cutting impulse brings information

$$1/4u_+(\tau_k^o) \tau_k^o, \tau_k^o = \tau_k^o, 1/4 \downarrow 1_{\tau_k^o}, \bar{u} \times \tau_k^o = 1/4 Nats ,$$

which supposed to compensate, or carry the above entropy flow:

$$\Delta S[\bar{x}_i / \gamma_i]_{\tau_k^o \rightarrow \tau_k^o}^{\tau_k^o \rightarrow \tau_k^o} = -1_{\tau_k^o} [j[t \rightarrow \tau_k^o]] / o(\tau_k^o),$$

within $\Delta_t = (t - \tau_k^o)$, at

$$\Delta S[\bar{x}_i / \gamma_i]_{\tau_k^o \rightarrow \tau_k^o}^{\tau_k^o \rightarrow \tau_k^o} = -1_{\tau_k^o} [j(-j)\tau_k^o] - jo(\tau_k^o) = \downarrow 1_{\tau_k^o} / jo(\tau_k^o),$$.  

entropy gap $jo(\tau_k^o)$ is imaginable (at $t \rightarrow \tau_k^o$), as well as time interval $\Delta_t = (t - \tau_k^o)$, compared with $t = \tau_k^o$ when control $u_+(\tau_k^o)$ applies.
Under both controls: \( u_+ = -u_- \), at
\[
S_+ [\tilde{x}_t / \tilde{y}_t]_{\tau_k^{-o}}^{\tau_k^o} = -1/4 u_+ ( t \rightarrow (\tau_k^{-o} - o(\tau_k^{-o})), S_+ [\tilde{x}_t / \tilde{y}_t]_{\tau_k^{-o}}^{\tau_k^o} = 1/4 u_+ ( t \rightarrow (\tau_k^{-o} - o(\tau_k^{-o})),
\]
total entropy functional approaching \( t = \tau_k^{-o} - o(\tau_k^{-o}) \) is
\[
\Delta S[\tilde{x}_t / \tilde{y}_t]_{\tau_k^{-o}}^{\tau_k^o} = S_+ [\tilde{x}_t / \tilde{y}_t]_{\tau_k^{-o}}^{\tau_k^o} - S_+ [\tilde{x}_t / \tilde{y}_t]_{\tau_k^{-o}}^{\tau_k^o} = -1/4 [ t \rightarrow \tau_k^{-o} ] [ u_+ (\tau_k^{-o}) + u_+ (\tau_k^{-o}) ] / o(\tau_k^{-o}). \quad (5.28a)
\]
If difference \( [ u_+ (\tau_k^{-o}) + u_+ (\tau_k^{-o}) ] \) can compensate for \( o(\tau_k^{-o}) \), at
\[
\Delta S[\tilde{x}_t / \tilde{y}_t]_{\tau_k^{-o}}^{\tau_k^o} \rightarrow -1/4 \text{Nats}. \quad (5.29)
\]
The opposite action of functions \( [ u_+ (\tau_k^{-o}) + u_+ (\tau_k^{-o}) ] \) applied during \( \tau_k^{-o} \), represents an interaction on \( o(\tau_k^{-o}) \) of random process with applied control \( u_+ (\tau_k^{-o}) \) at
\[
\Delta S[\tilde{x}_t / \tilde{y}_t]_{\tau_k^{-o}}^{\tau_k^o} \rightarrow -1/4 \text{Nats}. \quad (5.30)
\]
These results are also applied to beginning of this process at opposite \( u_+ (s_k^{+o})s_k^{+o} \) and \( u_+ (s_k^{+o})s_k^{+o} \) which supposed to compensate for \( o(s_k^{+o}) \).
Assuming that \( u_+ (s_k^{+o})s_k^{+o} = 1/4 \text{Nats} \) is part of the cutoff on \( s_k^{+o} \), it supposed to compensates action \( u_+ (s_k^{+o}) \) according to
\[
u_+ (s_k^{+o})s_k^{+o} = -u_+ (s_k^{+o})s_k^{+o} + o(s_k^{+o}). \quad (5.31)
\]
In other way, both opposite actions: \( u_+ (s_k^{+o})s_k^{+o}, u_+ (s_k^{+o})s_k^{+o} \) and their shift \( o(s_k^{+o}) \) are necessary to compensate the random influx from interactive random process.
Thus, sequence of actions on \( \Delta_+ = (t-s) \) generates interactive action of random process on \( \Delta_+ \rightarrow o(t) \) between the applied cutoffs. Action \( -u_+ (\tau_k^{-o}) \) on interval \( o(\tau_k^{-o}) \) might compensate \( u_+ (\tau_k^{-o}) \), applied on \( 1/2 o(\tau_k) = \delta_k \), where according to (5.8b) holds relation:
\[
\delta_k / \tau_k^{-o} = 2 / 3, \tau_k^{-o} = 3 \delta_k / 2. \quad (5.32)
\]
From (5.28b), at
\[
-\delta_k (\tau_k^{-o}) + o(\tau_k^{-o}) = \delta_k \tau_k^{-o}, u_+ (\tau_k^{-o}) - u_+ (\tau_k^{-o}) = -2 o(\tau_k^{-o}) / 3 \delta_k , \text{it follows}
\]
\[-u_+(\tau_k^-) + 2o(\tau_k^-) / 3\delta_k = u_-(\tau_k^-)\].

Assuming that minimal impulse interval is \(\delta_k\), action \(-u_+(\tau_k^-)\), compensating \(o(\tau_k^-)\), should also be applied on \(\delta_k\). Then required difference is

\[[u_-(\tau_k^-) - u_+(\tau_k^-)] = 2/3\], or \(u_+(\tau_k^-) / u_-(\tau_k^-) = 1 - 2/3u_-(\tau_k^-)\).

If \(u_-(\tau_k^-)\) applies on \(\delta_k\): \(u_+(\tau_k^-)\delta_k = 1/4\text{Nats}\), then the required opposite action applied on the same minimal \(\delta_k\) is

\[u_+(\tau_k^-)\delta_k = 1/4\text{Nats} - 2/3\delta_k \text{or } u_+(\tau_k^-)\delta_k < 1/4\text{Nats}\] (5.33)

while the applied action \(u_-(\tau_k^-)\delta_k\) includes difference

\[u_-(\tau_k^-) - u_+(\tau_k^-) = \delta u_-(\tau_k^-) - o(\tau_k^-)\] (5.33a)

This means all applied impulses have the same wide \(\delta_k\) but are opposite and asymmetric on intervals \(\Delta_k = (t - s_k^-)\) and \(\tau_k^+ - \tau_k^-\) approaching in the limit to \(o(t) \to 0\).

In [37(2),p.13] we evaluated \(\delta_k = 0.2\), then \(u_+(\tau_k^-)\delta_k = 1/4\text{Nats} - 0.133\). (5.33b)

6. Information dynamic processes initiating by the EF and IPF functionals

Since the IPF functional becomes finite–converging with the initial EF functional, which had expressed through the additive functional, the EF covers both the cutoff information contributions and the entropy increments between them.

That is why an extremal solution for the EF describes a dynamic process for IPF at \(n \to \infty\).

Such dynamic process could follow from extreme solution of variation problem for IPF which supposes to integrate discrete sequence of the cutoff fractions of the EF.

Such integration of the discrete fractions and solving a classical variation problem for the IPF to find continuous extreme dynamics presents a difficult mathematical task.

Integral (2.4) measures the EF and the IPF maximal limit, but avoids the direct access to Markov random process. Minimum of this integral provides dynamic process which minimizes the distance (1.5a) and dynamically approximates movement \(t_x\) to \(t_\varsigma\).

Initial entropy functional (1.10) presents a potential information functional of the Markov process until the applied impulse control, carrying the cutoff increment (3.5), transforms it to the physical informational path functional (5.1).

Process \(x(t)\) carries the information collected by the maximal IPF at \(o(t) \to 0\), as the IPF information dynamic macroprocess [38], while at \(n \to \infty\), each interval \(\Delta_k = (t - s_k^-) \to o(t)\).

Thus, information collected from the diffusion process by the IPF approaches its source, measured by the EF, when intervals \(\Delta_k \to o(t)\).

Within discrete \(o(t) = \delta_o\), opposite controls \(u_+ u_-\) satisfy relation

\[c^2 = |u_+ u_-| = c_+ c_- = \bar{u}^2, c_+ = u_+, c_- = u_- \quad |u_+ u_-| = \bar{u}^2.\] (6.1)

These controls, applying within \(\Delta_k \to o(t)\), are imaginable, present an opposite discrete complex:

\[u_+ = j\bar{u}, u_- = -j\bar{u}.\] (6.1a)
Relations 4.1A, B satisfy at 
\[ \bar{u} = -2j. \]  
(6.2)

when 
\[ u_+ j\bar{u} (-j\bar{u}) = \bar{u}^2, \]  
(6.2a)
\[ -j\bar{u} (-+j\bar{u}) = -2j\bar{u}, \]  
(6.2b)
\[ \bar{u}^2 = -2j\bar{u}, \bar{u} = -2j. \]  
(6.2c)

The controls are real when 
\[ u_+ = j(-2j) = 2, u_- = -j(-2j) = -2. \]  
(6.3)

At any \( \Delta > 0 \rightarrow o(t) \) and fulfillment of (6.2a) and (6.2b), controls (6.1a) are the discrete complex, while satisfying (6.3) is not required at the end of interval \( \Delta \rightarrow o(t) \).

Within \( \Delta \rightarrow o(t) \), the requirement is (6.1a) at \( \bar{u} = 2 \) as it follows from (4.1b).

Mathematical expectations of Ito’s Eqs 
\[ E[a] = \bar{x}(t) = E[c\bar{x}(t)] = cE[\bar{x}(t)] = c\bar{x}(t) \]  
(6.4)

approximates it by regular differential Eqs 
\[ \bar{x}(t) = c\bar{x}(t), \]  
(6.4a)

whose common solution averages the random movement via dynamic macroprocess \( \bar{x}(t) \):
\[ \bar{x}(t) = \bar{x}(s)\exp(c\bar{x}(s)) = E[\bar{x}(s)]. \]  
(6.5)

Relations (6.1), (6.4a), satisfy two differential equations
\[ \dot{x}_+(t) = j\bar{x}_+(t), \dot{x}_-(t) = -j\bar{x}_-(t) \]  
(6.5a)
describing the processes \( x_+(t), x_-(t) \) under controls (6.1a) on time interval \( \Delta, \Delta \rightarrow o(t) \).

Solutions of (6.5a) takes forms
\[ \ln x_+(t) = Cu_+ t, \ln x_-(t) = Cu_- t, x_+(t) = C \exp(j\bar{u}t), x_-(t) = C \exp(-j\bar{u}t), \]  
\[ C = x_+(s^\circ) = x_-(s^\circ), \]  
\[ x_+(t) = x_+(s^\circ)(\cos\bar{u}t + j\sin\bar{u}t), x_-(t) = x_-(s^\circ)(\cos\bar{u}t - j\sin\bar{u}t), \]  
(6.6)

where moment \( t \) of reaching minimal entropy functional identifies Eqs (4.9a,b).

Thus, process (6.6) is information microprocess on \( o(t) \rightarrow o(\tau^{-\circ}) \) compared to macroprocess (6.5).

The microprocess becomes an inner part of the dynamics process, minimizing distance (1.5a), when its time interval satisfies optimal time distance between the cutoff information at \( \tau^{-\circ} = 3\tau_{n-1} \).

It implies that imaginary time interval triples the cutoff discrete intervals \( o(\tau_{n-1}) \):
\[ o(\tau^{-\circ}) = 3o(\tau_{n-1}). \]  
(6.7)

Finding a dynamic process \( x(t) \) which approximates movement of \( \dot{x}_i \) to \( \zeta_t \) minimizing distance (1.5a) requires solution of variation problem for functional (1.10) which accumulates the transformed movement.

Other way is considering solution of (6.5), starting in time
\[ t = s^\circ b_k(t)/b_k(s^\circ) \]  
(6.8)

which satisfies extremum of entropy functional within each \( \Delta = t - s \).
\[ dx(t) / x(t) = cd t, \ln x(t) = ct, t = s^\ast b_k(t) / b_k(s^\ast), \ln x(t) = cs^\ast b_k(t) / b_k(s^\ast). \]  
(6.8a)

We get a macroprocess, integrating minimal distance of \( \Delta, = t - s \):

\[ x(t) = \exp[cs^\ast b_k(t) / b_k(s^\ast)], x(s^\ast) = \exp(cs^\ast), c = \ln(x(s^\ast)) / s^\ast, x(s^\ast) = \bar{x}(s^\ast), \]

\[ x(t) = \exp[\ln(x(s^\ast))b_k(t) / b_k(s^\ast)] = \exp[\ln(x(s^\ast))t / s], \ln x(t) = \ln(x(s^\ast))t / s, \]  
(6.9)

which at \( t \to T \) approaches

\[ \ln x(T) = \lim[\ln(x(s^\ast))T] / s], x(T) \to x(s^\ast)T / s . \]  
(6.10)

Process \( x(t) \) is the extremal solution of macroprocess \( \bar{x}(t) \) which averages solution of Ito Eq. under the optimal controls' multiple cutoffs of the EF for n-dimensional Markov process within \( \Delta, = (t - s) \to o(t) \).

Process \( x(t) \) carries the EF increments, while the information dynamic macroprocess collects the maximal IPF at \( o(t) \to 0 \), when at \( n \to \infty \), each interval is \( \Delta, = (t - s) \to o(t) \). Then, information collected from the diffusion process by the IPF, approaches the EF entropy functional. Finding \( x(t) \) requires solution of variation problem for the EF.

7. The solution of variation problem for the entropy functional

Applying the variation principle to the entropy functional, we consider an integral functional

\[ S = \int L(t, x, \dot{x}) dt = S[x], \]  
(7.1)

which minimizes the entropy functional (1.10) of the diffusion process in the form

\[ \min_{u \in KC(\Delta, R^a)} S[\bar{x}(u)] = S[x], Q \in KC(\Delta, R^a). \]  
(7.1a)

Specifically, for integral (2.1), it leads to optimal solution of variation problem

\[ \text{extr} S[\bar{x}/ \zeta_j] = \text{extr} 1 / 2 \int_s^T c^2(t)A(t, s) dt, c^2(t) = \dot{x}(t), \text{ at } \dot{x}(t) = A(x + v), A v = u . \]  
(7.1b)

**Proposition 7.1.**

1. An extremal solution of variation problem (7.1a, 7.1) for the entropy functional (1.10), brings the following equations of extremals for vector \( x \) and conjugate vector \( X \) accordingly:

\[ \dot{x} = a^u, a^u = a(u, t, x) (t, x) \in Q, \]  
(7.2)

\[ \dot{X} = -\partial P / \partial x - \partial V / \partial x , \]  
(7.3)

Where

\[ P = (a^u)^T \frac{\partial S}{\partial x} + b^x \frac{\partial^2 S}{\partial x^2}, \]  
(7.4)

\( S(t, x) \) is function of action on extremals (7.2,7.3); \( V(t, x) \) is integrant of the additive functional (1.7), which defines the probability function (1.3).

**Proof.** Using the Jacobi-Hamilton (JH) equations for function of action \( S = S(t, x) \), defined on the extremals \( x(t), (t, x) \in Q \) of functional (7.1), we have

\[ -\frac{\partial S}{\partial t} = H, H = \dot{x}^T X - L, \]  
(7.5)
where $X$ is a conjugate vector for $x$ and $H$ is a Hamiltonian for this functional.

(All derivations here and below have vector form).

From (7.1a) it follows

$$\frac{\partial S}{\partial t} = \frac{\partial S}{\partial t}, \quad \frac{\partial S}{\partial x} = \frac{\partial S}{\partial x},$$

(7.5a)

where for the $JH$ we have $\frac{\partial S}{\partial x} = X, -\frac{\partial S}{\partial t} = H$.

This allows us to join Eqs (7.5), (7.5a) and (1.10) in the form of Kolmogorov’s Eq. for

$$-\frac{\partial S}{\partial t} = (a^*)^T X + b^T \frac{\partial X}{\partial x} + 1/2a^* (2b)^{-1} a^* = -\frac{\partial S}{\partial t} = H,$$

(7.6)

where dynamic Hamiltonian holds $H = V + P$, which includes function $V(t,x)$ and potential function

$$P(t,x) = (a^*)^T X + b^T \frac{\partial X}{\partial x}.$$

(7.7)

Applying Hamilton equation $\frac{\partial H}{\partial X} = \dot{x}$ and $\frac{\partial H}{\partial x} = -\dot{X}$, to (7.6) we get the extremals for vector $x$ and $X$ in the forms (7.2) and (7.3) accordingly.

More details in [38].

Proposition 7.1.

A minimal solution of variation problem (7.1a, 7.1) for the entropy functional (1.10) brings the following equations of extremals for $x$ and $X$ accordingly:

$$\dot{x} = 2bX_o,$$

(7.9)

satisfying condition

$$\min_{x(\tau)} P = P[x(\tau)] = 0.$$

(7.10)

Condition (7.10) is a dynamic constraint, which is imposed on the solutions (7.2), (7.3) at some set of the functional’s field $Q \in KC(\Delta, R^n)$, where the following relations hold:

$$Q^o \subset Q, \quad Q^o = R^n \times \Delta^o, \Delta^o = [0, \tau], \tau = \{\tau_k\}, k = 1, ..., m \text{ for process } x(t)_{\tau_k} = x(\tau),$$

(7.11)

and Hamiltonian

$$H_o = -\frac{\partial S_o}{\partial t}$$

(7.12)

is defined for the function of action $S_o(t,x)$, which on extremals (7.8,7.9) satisfies condition

$$\min(-\frac{\partial S}{\partial t}) = -\frac{\partial S_o}{\partial t}.$$

(7.13)

Hamiltonian (7.6) and Eq. (7.8) determine a second order differential Eq. of extremals:

$$\ddot{x} = \dot{x}[-2b],$$

(7.14)

Proof. Using (7.4) and (7.6), we find the equation for Lagrangian in (7.1) in the form

$$L = -b^\tau \frac{\partial X}{\partial x} - 1/2(2b)^{-1} \dot{X}.$$ 

(7.15)

On extremals $x_i = x(t)(7.2,7.3)$ both functions drift and diffusion (in 1.10) are nonrandom. After their substitution to (7.1) we get the integral functional $\tilde{S}$ on the extremals:
\[ \tilde{S}[x(t)] = \int_{t_1}^{t_2} \frac{1}{2}(a^\tau)^T (2b)^{-1} a^\tau dt , \]  
\text{(7.15a)}

which should satisfy variation conditions (7.1a), or

\[ \tilde{S}[x(t)] = S_o[x(t)] , \]  
\text{(7.15b)}

where both integrals are determined on the same extremals.

From (7.15), (7.15a,b) it follows

\[ L_o = 1/2(a^\tau)^T (2b)^{-1} a^\tau , \quad \text{or} \quad L_o = \dot{x}^T (2b)^{-1} \dot{x} . \]  
\text{(7.16)}

Both expressions for Lagrangian (7.15) and (7.16) coincide on some extremals, where potential (7.7) satisfies condition (7.10) in the form

\[ P_o = P[x(t)] = (a^\tau)^T (2b)^{-1} a^\tau + b^T \frac{\partial X_o}{\partial x} = 0 , \]  
\text{(7.17)}

for Hamiltonian (7.12) and function of action \( S_o(t,x) \) that satisfies (7.13).

From (7.15b) it also follows

\[ E\{\tilde{S}[x(t)]\} = \tilde{S}[x(t)] = S_o[x(t)] . \]  
\text{(7.17a)}

Applying Lagrangian (5.16) to Lagrange’s equation

\[ \frac{\partial L_o}{\partial \dot{x}} = X_o , \]  
\text{(7.17b)}

we get equations for vector

\[ X_o = (2b)^{-1} \dot{x} \]  
\text{(7.17c)}

and extremals (7.8).

Both Lagrangian and Hamiltonian here are information forms of JH solution for the EF.

Lagrangian (7.16) satisfies the principle maximum for functional (7.15), from which also follows (7.17a). Thus, functional (7.1) reaches its minimum on extremals (7.8), while it is maximal on the extremals (7.2,7.3) of (7.6).

This Hamiltonian, at satisfaction of (7.17), reaches its minimum:

\[ \min H = \min[V + P] = 1/2(a^\tau)^T (2b)^{-1} a^\tau = H_o , \]  
\text{(7.18)}

from which it follows (7.19a) at

\[ \min P = P[x(\tau)] = 0 . \]  
\text{(7.19b)}

Function \(-\partial \tilde{S}(t,x) / \partial t\) = \( H \) in (7.6) on extremals (7.2,7.3) reaches a maximum when the constraint (7.10) is not imposed. Both the minimum and maximum are conditional with respect to the constraint imposition.

Variation conditions (7.18), imposing constraint (7.10), selects Hamiltonian

\[ H_o = -\frac{\partial S_o}{\partial t} = 1/2(a^\tau)^T (2b)^{-1} a^\tau \]  
\text{(7.20)}

on the extremals (7.2,7.3) at discrete moments (\( \tau_k \)) (7.11).

The variation principle identifies two Hamiltonians: \( H \) satisfying (7.6) with function of action \( S(t,x) \), and \( H_o \) (7.20), whose function action \( S_o(t,x) \) reaches absolute minimum at moments (\( \tau_k \)) (7.11) of imposing constraint \( P_o = P_o[x(\tau)] \).
Substituting (7.2) and (7.17b) in both (7.16) and (7.20), we get Lagrangian and Hamiltonian on the extremals:

\[ L_o(x, X_o) = \frac{1}{2} \dot{x}^T X_o = H_o. \]  

\[ (7.21) \]

Using \( \dot{X}_o = -\partial H_o / \partial x \), we have \( \dot{X}_o = -\partial H_o / \partial x = -\frac{1}{2} \dot{x}^T \partial X_o / \partial x \),

and from constraint (7.10), we get

\[ \partial X_o / \partial x = -b^T \dot{x} X_o, \text{ and } \partial H_o / \partial x = 1/2 \dot{x}^T b^{-1} \dot{x} X_o = 2H_o X_o, \]

which after substituting (7.17b) leads to extremals (7.9).

Using Eq. for the conjugate vector (7.3), we write constraint (7.10) in form

\[ \frac{\partial X_o}{\partial x} = -2X_o X_o^T, \]  

\[ (7.21a) \]

which follows from (7.7), (7.8) and (7.17c).

By differencing (7.8) we get a second order differential Eqs on the extremals:

\[ \ddot{x} = 2b \dot{X}_o + 2b X_o, \]

which after substituting (7.9) leads to

\[ \ddot{x} = 2X_o [b - 2bH], \text{ or to } (7.14) \]

8. Discussing Secs.1-7 results

The initial entropy functional (1.5, 1.10) presents a potential information functional of the Markov process until the applied impulse control, carrying the cutoff contributions (3.5), transforms it to the informational path functional (5.1). Markov random process is a source of each information contribution, whose entropy increment of cutting random states delivers information, hidden between these states.

The finite restriction on the cutting function determines the discrete impulse’s step-up and step-down controls, acting between the impulse cutoff \( \delta_k = \tau_k^{+o} - \tau_k^{-o} \), which hold the information received on \( \tau_k^{-o} \) and transfers it on \( \tau_k^{+o} \), when starting interval \( \Delta \rightarrow o(t) \) that follows next cutoff, which delivers new hidden process' information and so on.

Cutting entropy on \( \tau_k^{-o} \) produces the equivalent physical information and memorizes it [39, 40].

That’s why information is a physical entity as a difference from entropy which can be virtual.

In a multi-dimensional diffusion process, the step-wise controls, acting on the process all dimensions, sequentially stops and starts the process, evaluating the multiple functional information. Impulses \( \delta u_i \) or discrete \( \delta u_{i_{a}} \) implement transitional transformations (1.2-1.3), initiating the Feller kernels along the process and extracting total kernel information for \( n \)-dimensional process with \( m \) cuts off. This maximal sum measures the interstates information connections held by the process along the trajectories during its time \( (T - s) \), which is hidden by the process correlating states.

This information is not covered by traditional Shannon entropy measure.

The dissolved element of the functional’s correlation matrix at these moments provides independence of the cutting off fractions, leading to orthogonality of the correlation matrix for these cut off fractions.

Intervals between the impulses do not generate information, and are imaginary in terms of getting information.
They are imaginary since no real double controls are applying within these intervals. The minimized increments of entropy functional between the cutoffs allow prediction each following cutoff with maximal conditional probability. A sequence of the functional a priori-posteriori probabilities provides Bayesian entropy measuring a probabilistic causally [40,41] transforming to physical casualty in information macro dynamics. Sum of extracted from the EF information contributions approaches theoretical measure (1.10) at

\[ S_m \rightarrow S[\tilde{x}_t / \zeta_t], \]  

(8.1)

if all local \( m \)-time intervals

\[ t_1 - s = o_1, t_2 - t_1 = o_2, ..., t_m - t_{m-1} = o_m, \] at \( t_m = T \)
satisfy condition

\[ (T-s) = \lim_{m \to \infty} \sum_{t=s}^{t=T} o_m(t). \]  

(8.2)

Realization of (8.1) requires applying the impulse at each instant \((\tilde{x}, s), (\tilde{x}, s + o(s))\) of the conditional mathematical expectation in (1.5,1.10) along the process trajectories.

Sum \( S_{mo} \) of additive fractions of the EF on the finite time intervals \( s, t_1, t_1 + o_1, t_2, ..., t_m + o_m, t_m = T \):

\[ S_{mo} = \Delta S_{io}[\tilde{x}_t / \zeta_t] + \Delta S_{io}[\tilde{x}_t / \zeta_t]_{t+i_00}, ..., + \Delta S_{io}[\tilde{x}_t / \zeta_t]_{t+i_00}, \]  

(8.3)

is less than \( S[\tilde{x}_t / \zeta_t] \), which is defined by the additive functional (1.7).

As a result, the additive principle for a process’ information, measured by the EF, is violated:

\[ S_m < S[\tilde{x}_t / \zeta_t] \]  

(8.4)

If each \( k \)-cutoff might “kill” its process dimension after moment \( \tau_{k \rightarrow o} \), then \( m = n \), and condition (8.1) requires infinite process dimensions.

The EF measure, taken along the process trajectories during time \((T-s)\), limits maximum of total process information, extracting its hidden cutoff information (during the same time), and brings more information than Shannon traditional information measure for multiple states of the process. Otherwise, maximum of the process cutoff information, extracting its total hidden information, approaches the EF information measure according to Sec.6. Or total physical information, collected by IPF in the infinite dimensional Markov diffusion process, is finite. Since rising process dimension up to \( n \to \infty \) increases number of dimensional kernels, the cutting off kernel information grows, and the IPF finally measure all kernels, when the EF is transformed to the IPF kernels. Total process uncertainty-entropy, measured by the EF, is also finite on the finite time of its transition to total process certainty-information, measured by the IPF.

The IPF maximum, integrating unlimited number of Bits’ units with finite distances, limits the total information carrying by the process’ Bits.

The limitation on total time of collecting the cutoff information (at \( n \to \infty \)) decreases the time interval of each cutoff which increases the inclusive quantity of information extracted on this interval. At the same cutting off information for each impulse, density of this information, related to the impulse interval grows, which integrates sum of all previous cutoffs.

The limitation on the cutoff function determines two classes processes for the cutting EF functional: microprocess with entropy increment on each \( o(t) \) as a carrier of information contribution to each \( o(\tau) \) and macroprocess on \((T-s)\) that describes total process of transformation EF to IPF.

The microprocess specifics within each \( o(t) \) are:

- an imaginary time compared to the information cutoff real time on \( o(\tau) \);
- two opposite asymmetrical sources of entropy-information in the considered model, as an interactive reaction from random process, which generate the microprocess; (a microprocess may exist within Markov kernel or outside, independently of both randomness and interactions, and even prior to them).
- the opposite step-wise actions carry two conjugated microprocesses with opposite imaginary entropy increments, which join at $o(\tau^\omega)$ locality, when the joint entropy is transformed to real information by the cutting action;
- at $o(\tau^\omega)$ locality, two opposite imaginary entropy increments correlate by $r[o(\tau^\omega)]$ before the cutting action dissolves the correlation and generates the impulse information contribution on $o(\tau^\omega)$;
- the entropy cut memorizes the cutting information, while the gap within $o(\tau^\omega)$ delivers external influx of entropy, covered by real stepwise action, which carries energy for the cut.

The cut preceding entropy impulse works as Maxwell’s Demon [39], [40, p.60].

The asymmetry of controls compensates the infinite entropy influx.

The asymmetry gap between these controls we had evaluated in [37].

Macroprocess, integrating both imaginary entropy of the $o(t)$-impulses microprocesses and the cutoff information of the $o(\tau)$ real impulses, incorporates the collected entropy-information in physical process.

The EF-IPF transformations and their generated processes provide the Information Path from Randomness and Uncertainty to Information, Thermodynamics, and Intelligence of Observer [40] allowing to describe the processes physical laws [41].

9. The estimation of information hidden by the interstates’ connection of the diffusion process

The evaluated information effect of losing the functional’s bound information at the cutoff moments, according to (3.5), holds amount of 0.5 Nats (~0.772 bits) at each cutoff in the form of $\delta'[\tau_k]$-function, applied $k = m$ times to each of the process dimension $i = 1, ..., n$, with total information $I_c \approx 0.772m \times n$ bits.

Thus, the process functional’s information measures (1.10) encloses $I_c$ bits more, compared to the information measure, applied separately to each of the process $m \times n$ states (during the same time). This result is applicable to a comparative information evaluation of the divided and undivided fractions of an information process, measured by corresponding EF, where each two bits of undivided process’ pair contains 0.772 bits more hidden information, measured by the functional. That means that information process holds more information than any divided number of its fractions, and the considered entropy functional, measured this process, evaluates the quantity of information that connect these fractions. Moreover, by knowing this initially hidden information, one could determine which information is necessary to connect the data, being measured separately, toward composing a unit of some information process.

According to the evaluation of an upper bound entropy per an English character (token) [42], its minimum is estimated by 1.75 bits, with the average amount between 4.66-7 bits per character. The evaluation includes the inner information bound by a character.

At minimal entropy per symbol in 1 bit, a minimal symbol’s bound information is 0.75, which is close to our evaluation at the cutoff.
The method was applied in solidification processes with impulse controls’ automatic system [43], in different artificial and solidification processes with impulse controls’ automatic system [43], in different artificial and cognitive systems [41], which reveal some unidentified phenomena (such as a compulsive appearance centers of crystallization-indicators of generation information at the considered cutoff [43]).

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