Quantum function algebras from finite-dimensional Nichols algebras

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Abstract

We describe how to find quantum determinants and antipode formulas from braided vector spaces using the FRT-construction and finite-dimensional Nichols algebras. It improves the construction of quantum function algebras using quantum grassmanian algebras. Given a finite-dimensional Nichols algebra \( B \), our method provides a Hopf algebra \( H \) such that \( B \) is a braided Hopf algebra in the category of \( H \)-comodules. It also serves as source to produce Hopf algebras generated by cosemisimple subcoalgebras, which are of interest for the generalized lifting method. We give several examples, among them quantum function algebras from Fomin-Kirillov algebras associated with the symmetric group on three letters.

Introduction

Let \( k \) be a field and \( V \) a finite-dimensional \( k \)-vector space. A map \( c \in \text{Aut}(V \otimes V) \) is called a braiding if it satisfies the braid equation

\[
(c \otimes \text{id})(\text{id} \otimes c)(c \otimes \text{id}) = (\text{id} \otimes c)(c \otimes \text{id})(\text{id} \otimes c) \quad \text{in} \quad \text{End}(V \otimes V \otimes V).
\]  

(1)

In such a case, the pair \((V, c)\) is called a braided vector space.

Given a braided vector space \((V, c)\), Faddeev, Reshetikhin and Takhtajan \[FRT\] introduced a method, the \textit{FRT-construction} for short, to construct a coquasitriangular bialgebra \( A(c) \) such that \( V \) is an \( A(c) \)-comodule and \( c \) is a morphism of \( A(c) \)-comodules. By the very definition, this bialgebra is universal with such properties. Besides, it turns out that the category \( A(c) \mathcal{M} \) of left \( A(c) \)-comodules is braided monoidal. Notice that if \( V \neq 0 \) then \( A(c) \) is never a Hopf algebra: suppose the contrary and write \( S \) for the antipode. Since \( A(c) = \bigoplus_{n \in \mathbb{N}_0} A(c)^n \) is graded in non-negative degrees and generated by comatrix elements \( t_{ij}^k \),

\[
1 = \epsilon(t_{ij}^k) = \sum_k t_{ij}^k S(t_{ik}^j) \in \bigoplus_{n > 0} A(c)^n,
\]

one gets a contradiction. In the trivial example given by \( \tau = \) the flip map in \( k^n \), the FRT-construction yields the coordinate affine ring \( \mathcal{O}(M_n) \) on \( n \times n \) matrices over \( k \), and one needs to localize the commutative algebra on the determinant in order to obtain the Hopf algebra \( \mathcal{O}(\text{GL}_n) \). In general, the \textit{abelianization} of the FRT-construction gives the bialgebra \( \mathcal{O}(\text{End}(c)) \),

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which is the ring of coordinate functions on the endomorphisms $f$ of $V$ such that $f \otimes f$ commutes with $c$. If one could localize the FRT-construction and get a Hopf algebra $H(c)$, then one would have a surjective map $H(c) \to \mathcal{O}(\text{Aut}(c))$, that is, a quantum group much larger than the “classical” automorphism group of the braiding. In general, by [Sch, Lemma 3.2.9] (see [H] for a review) in case the braiding $c$ is rigid there exists a coquasitriangular Hopf algebra $\mathcal{H}(c)$ associated with $(V, c)$ satisfying a universal property: $V \in \mathcal{H}(c)\mathcal{M}$ with a certain comodule structure map $\lambda$ and if $B$ is a coquasitriangular bialgebra such that $V \in B\mathcal{M}$ with comodule structure $\lambda_B : V \to B \otimes V$, then there exists a coquasitriangular bialgebra map $f : \mathcal{H}(c) \to B$ such that $\lambda_B = (f \otimes \text{id})\lambda$. Furthermore, by [Sch, Lemma 3.2.11], the Hopf algebra $\mathcal{H}(c)$ is generated as algebra by elements $\{t_i^j, u_i^j\}_{1 \leq i, j \leq n}$ satisfying

$$\sum_{k,\ell} c_{i,j}^{k,\ell} t_k^j u_k^\ell = \sum_{k,\ell} t_k^j t_k^\ell c_{k,\ell}, \quad \text{and} \quad \sum_{k=1}^n u_k^i t_k^j = \delta_i^j = \sum_{k=1}^n t_k^i u_k^j.$$  \hspace{1cm} (2)

The coalgebra structure is given by $\Delta(t_i^j) = \sum_{k=1}^n t_k^i \otimes t_k^j$, $\varepsilon(t_i^j) = \delta_i^j$ and $\Delta(u_i^j) = \sum_{k=1}^n u_k^i \otimes u_k^j$, $\varepsilon(u_i^j) = \delta_i^j$. Moreover, one has that $S_{\mathcal{H}(c)}(t_i^j) = u_i^j$ for all $1 \leq i, j \leq n$. Note that, since $\mathcal{H}(c)$ is coquasitriangular, the square of the antipode is an inner automorphism, and as a consequence, the antipode and all its powers are defined on the generators $t_i^j, u_i^j$. The comodule category $\mathcal{H}(c)\mathcal{M}$ is the one generated by $V$ and $V^*$, and in general the map $A(c) \to \mathcal{H}(c)$ needs not to be injective (see example [3.6.1]).

In this paper we consider the following 3-step problem: given a finite-dimensional rigid braided vector space $(V, c)$,

(a) find a “quantum determinant” for the FRT-construction $A(c)$,

(b) prove that the localization $H(c) = A(c)[D^{-1}]$ of $A(c)$ at the quantum determinant is a Hopf algebra,

(c) prove that $H(c) \simeq \mathcal{H}(c)$.

In Subsection 2.4 we introduce a method for finding a quantum determinant associated with a rigid solution of the braid equation. Two of our main results, Theorem 2.19 and Theorem 2.21, give sufficient conditions to ensure the existence, and a concrete way to compute it, of a group-like element $D \in A = A(c)$, such that $D$ is normal in $A$ and, under certain conditions, the localization on $D$ is a Hopf algebra $H(c)$. Moreover, our proof yields an explicit formula for the antipode. Finally, we show in Corollary 2.22 that $H(c)$ is isomorphic to the universal coquasitriangular Hopf algebra $\mathcal{H}(c)$ associated with $(V, c)$. In this way, we obtain a realization of $\mathcal{H}(c)$ as a localization of $A(c)$.

As a classical motivation of this problem one can mention the famous work of Y. Manin [M], see also [M2], where the author introduces two operations $\bullet$ and $\circ$ on quadratic algebras, interpreted as internal tensor products, and proves that the internal end$(A) = A^! \bullet A$ of a quadratic algebra $A$ is always a bialgebra, recovering some remarkable examples such as the quantum function algebra $M_q(2)$. The problem of finding quantum determinant is present in this work, introducing what Manin calls a quantum Grassmannian algebra (qga) in [M], or a Frobenius quantum space (Fqs) in [M2], where a “volume form” plays a crucial role. The definition of a qga, or a Fqs, assures the existence of a group-like element that is the natural candidate for a quantum determinant, but the problem of finding the antipode (or even to prove its existence) remains open.

In [H], Hayashi constructed quantum determinants for multiparametric quantum deformations of $\mathcal{O}(\text{SL}_n)$, $\mathcal{O}(\text{GL}_n)$, $\mathcal{O}(\text{SO}_n)$, $\mathcal{O}(\text{O}_n)$ and $\mathcal{O}(\text{Sp}_{2n})$, inverting all group-like elements in
a given quasitriangular bialgebra, and showing that the ending result is a Hopf algebra. To define the quantum determinants, qga’s are considered for the deformations of the classical examples. The idea of considering quantum exterior algebras (qea) is also present in the work of Fiore [F], where the author defines quantum determinants for the quantum function algebras $\text{SO}_q(N)$, $\text{O}_q(N)$, and $\text{Sp}_q(N)$, which are defined through (a quotient of the FRT-construction, by means of the coaction of these on a volume element. This is where the quantum determinant comes into (co)action. More generally, qea’s and quantum determinants appear in the work of Etingof, Schedler and Soloviev [ESS] as universal objects associated with the exterior algebra when considering set-theoretical (involutive) solutions to QYBE’s. All quantum determinants appearing in this way should be central. Nevertheless, we found an example that this might not be the case, see Subsection 3.2.

Motivated by the results in [ESS], the definition of the qga and the quantum exterior algebras, and properties of the Nichols algebra associated with a rigid solution of the braid equation, in these notes we introduce certain class of graded connected algebras extending Manin’s definition of Fqs, see Definition 2.1, that enable us not only to consider volume elements and prove the existence of a quantum determinant, but also to find an explicit formula for the natural candidate of the antipode in the FRT-construction, localized at the quantum determinant.

These qga’s defined by Hayashi and the quantum exterior algebras considered by Fiore are all quadratic. In general, for a given braiding, there is no quadratic qga, but still there might be a finite-dimensional Nichols algebra associated with it. As a consequence, our method still apply in this case, see example in Subsection 3.5.

Quantum determinants are intensively studied in the literature as the classical problem of defining the determinant of a matrix with non-commutative entries, and because they also give a way to construct new examples of quantum groups, see for example [M2], [KL], [PW], [ER], [CWW], [Z], [JoZ], [KKZ] and references therein. It is worth to mention that in the work on quantum determinants by Etingof and Retakh [ER], the existence of formulas with “quantum minors” is considered. In our approach, the existence and concrete formulas for these “minors” emerge clearly.

Another features of the procedure to find quantum determinants are the following: given a finite-dimensional Nichols algebra $\mathcal{B}$, the method provides a Hopf algebra $H$ such that $\mathcal{B}$ is a braided Hopf algebra in the braided category of left $H$-comodules. It also gives families of Hopf algebras generated by simple subcoalgebras. Finite-dimensional quotients of these kind of Hopf algebras are of interest in the classification program of finite-dimensional complex Hopf algebras by means of the generalized Lifting Method, see for example [AC], [CG].

The paper is organized as follows. In Section 1 we recall the FRT-construction and the definition of the Nichols algebra associated with a braided vector space. In Section 2 we introduce the method for finding quantum determinants and “quantum cofactor formulas”, proving our main results Theorems 2.19, 2.21 and Corollary 2.22. Finally, we illustrate our contribution with several examples, including cases where the determinant is not central, and quantum function algebras from Fomin-Kirillov algebras associated with the symmetric groups on three letters.

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1 Preliminaries: the $A \otimes c$

In this section we give the definitions and basic properties of the FRT-construction and Nichols algebras, and recall known results that are needed for our construction.

Throughout the notes, $\mathbb{k}$ denotes an arbitrary field. We use the standard conventions for Hopf algebras and write $\Delta$, $\varepsilon$ and $S$ for the coproduct, counit and antipode, respectively. We also use Sweedler’s notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$ for the comultiplication. Given a bialgebra $A$, the category of finite-dimensional left $A$-comodules is denoted by $^A \mathcal{M}$. The readers are referred to [Ra] for further details on the basic definitions of Hopf algebras.

1.1 The FRT-construction: $A(c)$

In this subsection, we follow [LR]. Let $(V, c)$ be a finite-dimensional braided vector space and fix $\{x_i\}_{i=1}^n$ a basis of $V$. Write $\{x_i\}_{i=1}^n$ for the basis of $V^*$ dual to $\{x_i\}_{i=1}^n$. Recall that a solution of the braid equation $c$ is rigid, if the map $c^\#: V^* \otimes V \to V \otimes V^*$ given by $c^\#: (f \otimes x) = \sum_{i=1}^n (ev \otimes id \otimes id)(f \otimes c(x \otimes x_i) \otimes x_i^*)$ is invertible.

Let $C = \text{End}(V)^*$ be the coalgebra linearly spanned by the matrix coefficients $\{t_{ij}^k\}_{1 \leq i, j, k \leq n}$. Then, $V$ has a natural left $C$-comodule structure. Note that, as $C \cong M_n(\mathbb{k})^* \cong V \otimes V^*$ these generators are induced by the basis $\{x_i\}_{i=1}^n$, via the correspondence $t_{ij}^k \leftrightarrow x_i \otimes x_j^*$. The coalgebra structure is given by

$$\Delta(t_{ij}^k) = \sum_{k=1}^n t_{ij}^k \otimes t_{kj}^i,$$

$$\varepsilon(t_{ij}^k) = \delta_{ij} \quad \text{for all } 1 \leq i, j \leq n,$$

and $V$ is a (left) $C$-comodule by setting

$$\lambda(x_i) = \sum_{j=1}^n t_{ij}^n \otimes x_j \quad \text{for all } 1 \leq i \leq n.$$

Write $TC$ for the tensor algebra of $C$. Extending as algebra maps the comultiplication and the counit of $C$ to $TC$, the latter becomes a bialgebra and $V \otimes V$ is a (left) $TC$-comodule. In general, a linear map $c : V \otimes V \to V \otimes V$ is not necessarily $TC$-colinear. Actually, if one consider the difference of the two possible compositions in the following diagram, computed in the basis $\{x_i \otimes x_j\}_{i,j}$, one gets

\[
\begin{array}{ccc}
V \otimes V & \xrightarrow{c} & V \otimes V \\
\downarrow \lambda & & \downarrow \lambda \\
\sum_{k \ell} c_{ij}^{k \ell} x_k \otimes x_\ell & & \sum_{k \ell} c_{ij}^{k \ell} x_k \otimes x_\ell
\end{array}
\]

\[
\begin{array}{ccc}
TC \otimes (V \otimes V) & \xrightarrow{id \otimes c} & TC \otimes (V \otimes V) \\
\downarrow \lambda & & \downarrow \lambda \\
\sum_{r \ell} c_{ij}^{k \ell} t_{k \ell}^{r s} x_r \otimes x_s & & \sum_{r \ell} c_{ij}^{k \ell} t_{k \ell}^{r s} x_r \otimes x_s
\end{array}
\]

where the coefficients $c_{ij}^{k \ell}$ are defined by the equality $c(x_i \otimes x_j) = \sum_{k \ell} c_{ij}^{k \ell} x_k \otimes x_\ell$. Hence, one arrives naturally at the following definition:

Definition 1.1. [FRT] The FRT-construction (or universal quantum semigroup) for $(V, c)$ is the $\mathbb{k}$-algebra $A = A(c)$ generated by the elements $\{t_{ij}^k\}_{1 \leq i, j, k \leq n}$, satisfying the following relations:

\[
\sum_{k \ell} c_{ij}^{k \ell} t_{k \ell}^{r s} = \sum_{k \ell} t_{ij}^k c_{ij}^{k \ell} t_{k \ell}^{r s} \quad \forall 1 \leq i, j, r, s \leq n.
\]
It is well-known that $A(c)$ is a bialgebra with comultiplication and counit determined by (3), which satisfies a universal property: the map $\lambda : V \to A(c) \otimes V$ equips $V$ with the structure of a left comodule over $A(c)$ such that the map $c$ becomes a comodule map. If $A$ is another bialgebra coacting on $V$ via a linear map $\lambda'$ such that $c$ is $A$-colinear, then there exists a unique bialgebra morphism $f : A(c) \to A$ such that $\lambda' = (f \otimes \text{id}_V)\lambda$.

**Remark 1.2.** Let $V$ be a finite-dimensional $k$-vector space, $c \in \text{End}(V \otimes V)$ and $A = A(c)$. For $n \geq 2$, the linear map given by $c_k := \text{id}_{V \otimes k-1} \otimes c \otimes \text{id}_{V \otimes n-k-1} : V^\otimes n \to V^\otimes n$ is $A$-colinear. That is, the comodule map $\lambda : V^\otimes n \to A \otimes V^\otimes n$ satisfies that $\lambda c_k = (\text{id}_A \otimes c_k)\lambda$ for all $1 \leq k \leq n - 1$.

**Proof.** This follows from the fact that $c$ is $A$-colinear and the category of $A$-comodules is tensorial.

It is well-known that if $c$ satisfies the braid equation, then $A = A(c)$ is a coquasitriangular bialgebra, that is, there exists a convolution-invertible bilinear map $r : A \times A \to k$ satisfying

- **(CQT1)** $r(ab, c) = r(a, c(1))r(b, c(2))$
- **(CQT2)** $r(a, bc) = r(a, b)r(a, c)$
- **(CQT3)** $r(a(1), b(1))a(2)b(2) = b(1)a(1)r(a(2), b(2))$

This map is uniquely determined by $r(t^k_i, t^j_s) = c^k_{js}$ for all $1 \leq i, j, k, \ell \leq n$.

**Remark 1.3.** (a) The first two conditions say that for any group-like element $D$, the maps $r(D, -)$, $r(-, D) : A \to k$ are algebra maps.

(b) The last condition can be express by the equality $r * m = m^{op} * r$. Moreover, on $a = t^j_s$ and $b = t^s_i$, it reads

$$\sum_{k, \ell} r(t^k_i, t^j_s)t^r_k t^s_\ell = \sum_{k, \ell} t^r_k t^s_\ell r(t^k_i, t^j_s),$$

that is,

$$\sum_{k, \ell} c^r_{kj} t^r_k t^s_\ell = \sum_{k, \ell} t^r_k t^s_\ell c^r_{kj}.$$

Conditions (CQT1) and (CQT2) say that $r$ is determined by the values of $r$ on generators, so it extends to the tensor algebra; (CQT3) says that $r$ descends to $A$.

(c) For a group-like element $D$, (CQT3) gives a commutation rule:

$$r(D, b(1))Db(2) = b(1)Dr(D, b(2))$$

for all $b \in A$.

(d) The category $A \mathcal{M}$ is braided with $c_{M,N} : M \otimes N \to N \otimes M$ given by

$$c(m \otimes n) = r(m(1), n(-1)) n(0) \otimes m(0)$$

for all $M, N \in A \mathcal{M}$.

A stronger result than the commutation rule above is due to Hayashi and holds for any coquasitriangular bialgebra.

**Lemma 1.4.** [H Theorem 2.2] Let $A$ be a coquasitriangular bialgebra. For any group-like element $g \in A$, there is a bialgebra automorphism $\mathcal{J}_g : A \to A$ given by $\mathcal{J}_g(a) = r(a(1), g)a(2)r^{-1}(a(3), g)$ such that $ga = \mathcal{J}_g(a)g$ for all $a \in A$. 

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Example 1.5. Let $X$ be a set and $s : X \times X \to X \times X$ a set-theoretical solution of the braid equation, that is $s$ satisfies

$$(s \times \text{id}_X)(\text{id}_X \times s)(s \times \text{id}_X) = (\text{id}_X \times s)(s \times \text{id}_X)(\text{id}_X \times s).$$

For $x, y, a, b \in X$, let $z, t, u, v \in X$ be such that $(z, t) = s(x, y)$, and $s(u, v) = (a, b)$. Let $V = kX$ be the $k$-vector space linearly spanned by the elements of $X$ and let $c$ be the linearization of $s$. Then, the set of equations for the corresponding FRT-construction on $(V, c)$ is

$$t^b_xt^a_y = t^a_yt^b_x$$

In particular, for the flip solution $\tau(x, y) = (y, x)$ on a finite set $X = \{x_1, \ldots, x_n\}$, we have that $t^b_xt^a_y = t^a_yt^b_x$, in other words, $A(\tau) = O(M_n)$. This is not a Hopf algebra, but if one consider the element in $A$ given by the usual determinant

$$D := \det_n = \sum_{\sigma \in S_n} (-1)^{f(\sigma)}t^1_{\sigma(1)} \cdots t^n_{\sigma(n)}.$$

then the localization on $D$ is the Hopf algebra $A(\tau)[D^{-1}] = O(GL_n)$. We will generalize this construction for nontrivial examples.

Remark 1.6. Note that, since $\Box$ is homogeneous, $A(c) = A(qc)$ for all $0 \neq q \in k$. Also, if $c$ is invertible, then $A(c) = A(c^{-1})$.

### 1.2 Nichols algebras: $\mathcal{B}$

Let $(V, c)$ be a braided vector space. The braid group

$$\mathcal{B}_n = \langle \tau_1, \ldots, \tau_{n-1} | \tau_i\tau_j = \tau_j\tau_i, \tau_i\tau_i\tau_i = \tau_j\tau_i\tau_j, \text{ for } 1 \leq i \leq n - 2 \text{ and } j \neq i \pm 1 \rangle$$

acts on $V^{\otimes n}$ via $\rho_n : \mathcal{B}_n \to \text{GL}(V^{\otimes n})$ with $\rho_n(\tau_i) = c_i = \text{id}_{V^{\otimes i-1}} \otimes c \otimes \text{id}_{V^{\otimes n-i-1}} : V^{\otimes n} \to V^{\otimes n}$. Using the Matsumoto (set-theoretical) section from the symmetric group $S_n$ to $\mathcal{B}_n$:

$$M : S_n \to \mathcal{B}_n, \quad (i, i+1) \mapsto \tau_i, \quad \text{for all } 1 \leq i \leq n - 1,$$

one can define the quantum symmetrizer $QS_n : V^{\otimes n} \to V^{\otimes n}$ by

$$QS_n = \sum_{\sigma \in S_n} \rho_n(M(\sigma)) \in \text{End}(V^{\otimes n}).$$

For example $QS_2 = \text{id} + c$, and

$$QS_3 = \text{id} + c \otimes \text{id} + \text{id} \otimes c + (c \otimes c)(c \otimes \text{id}) + (c \otimes \text{id})(c \otimes \text{id}) + (c \otimes \text{id})(c \otimes c)(c \otimes \text{id}).$$

The Nichols algebra associated with $(V, c)$ is the quotient of the tensor algebra $TV$ by the homogeneous ideal

$$\mathcal{J} = \bigoplus_{n \geq 2} \text{Ker} QS_n,$$

or equivalently, $\mathcal{B}(V, c) := \bigoplus_n \text{Im}(QS_n)$. In particular, $\mathcal{B}(V, c)$ is a graded algebra. Note that $\mathcal{B}^0(V, c) = k$, $\mathcal{B}^1(V, c) = V$ and $\mathcal{B}^2(V, c) = (V \otimes V)/(\text{Ker}(\text{id} + c))$.

There are several equivalent definitions of the Nichols algebra associated with $(V, c)$, each of them particularly useful for different purposes. For more details, see [A].
Proposition 1.7. The Nichols algebra $\mathcal{B}(V, c)$ is an $A(c)$-module algebra.

Proof. By Remark 1.2 we have that $c_k$ is $A(c)$-colinear, which implies that $QS_n$ is an $A(c)$-comodule map. Thus, $\text{Ker} \ QS_n$ is an $A(c)$-subcomodule of $V^\otimes n$ for all $n \geq 2$. Hence, taking the quotient module $\mathcal{J}$ defines an $A(c)$-comodule structure on $\mathcal{B}(V, c) = TV/\mathcal{J}$.

Nichols algebras are a key ingredient in the classification of finite-dimensional pointed Hopf algebras and there is extensive literature covering the problem of finding finite-dimensional Nichols algebras. If the Nichols algebra is finite-dimensional and the braiding is rigid, then special features arise. These properties guide us to make a general construction that motivates the definition of “weakly graded-Frobenius algebra” that is the core of next section.

2 The quantum determinant and the antipode formula

In this section we introduce a method for finding a quantum determinant associated with a rigid solution of the braid equation (and additional assumptions), and prove our main results in Theorems 2.19, 2.21 and Corollary 2.22.

2.1 The quantum determinant

The following definition extends the notion of Frobenius quantum space introduced by Manin in [M2, §8.1]. As in loc. cit., we use it to define quantum determinants, to establish quantum Cramer and Lagrange identities, and to produce categorical dual objects.

Definition 2.1. Let $A$ be a bialgebra and $V \in {}^{A}M$. An $A$-comodule algebra $\mathcal{B}$ is called a weakly graded-Frobenius (WGF) algebra for $A$ and $V$ if the following conditions are satisfied:

(WGF1) $\mathcal{B}$ is an $\mathbb{N}$-graded $A$-comodule algebra, that is $\mathcal{B} = \bigoplus_{n \geq 0} \mathcal{B}^n$, $\lambda(\mathcal{B}^n) \subseteq A \otimes \mathcal{B}^n$, where $\lambda : \mathcal{B} \rightarrow A \otimes \mathcal{B}$ is the structure map, and $\mathcal{B}^n \cdot \mathcal{B}^m \subseteq \mathcal{B}^{n+m}$ for all $n, m \geq 0$;

(WGF2) $\mathcal{B}$ is connected (i.e. $\mathcal{B}^0 = k$) and $\mathcal{B}^1 = V$ as $A$-comodules;

(WGF3) $\dim_k \mathcal{B} < \infty$ and $\dim_k \mathcal{B}^{\text{top}} = 1$, where $\text{top} = \max\{n \in \mathbb{N} : \mathcal{B}^n \neq 0\}$;

(WGF4) the multiplication induces non-degenerate bilinear maps

$$\mathcal{B}^1 \times \mathcal{B}^{\text{top}-1} \rightarrow \mathcal{B}^{\text{top}}, \quad \mathcal{B}^{\text{top}-1} \times \mathcal{B}^1 \rightarrow \mathcal{B}^{\text{top}}.$$

Some remarks are in order:

(i) Let $A, A'$ be bialgebras and let $\mathcal{B}$ a WGF-algebra for $A$ and $V$. If $f : A' \rightarrow A$ is a bialgebra map, then $\mathcal{B}$ is also a WGF-algebra for $A'$.

(ii) Let $A$ be a bialgebra and $(V, c)$ a braided vector space. If $V \in {}^{A}M$ is such that $c$ is $A$-colinear and $\mathcal{B}$ is a WGF-algebra for $A$ and $V$, then the universal property of $A(c)$ determines a unique bialgebra map $f : A(c) \rightarrow A$. Consequently $\mathcal{B}$ is a WGF-algebra for $A(c)$ and $V$. In this case, $\mathcal{B}$ is directly associated with the braided vector space $(V, c)$. For short, we say that $\mathcal{B}$ is a WGF-algebra for $A(c)$.

(iii) A finite-dimensional graded algebra $\mathcal{B} = \bigoplus_{n \geq 0} \mathcal{B}^n$ with $\mathcal{B}^0 = k$ is called graded-Frobenius (GF) if there exists $p \in \mathbb{N}$ such that $\dim \mathcal{B}^p = 1, \mathcal{B}^{p+j} = 0$ for $j > 0$ and the multiplication $\mathcal{B}^1 \times \mathcal{B}^{p-j} \rightarrow \mathcal{B}^p$ is non-degenerate for all $j$ with $0 \leq j \leq p$. For instance any finite-dimensional graded connected Hopf algebra in the category of Yetter-Drinfeld modules over a Hopf algebra $H$ is GF, see [N] and [AG1, §3.2] for more details.
(iv) Let \((V, c)\) be a finite-dimensional rigid braided vector space and let \(\mathcal{B} = \mathcal{B}(V, c)\) be the Nichols algebra associated with it. If \(\dim_k \mathcal{B} < \infty\), then by Proposition \(\text{[17]}\), the very definition of Nichols algebra and (iii) above, it follows that \(\mathcal{B}\) is a GF-algebra and hence a WGF-algebra for \(A(c)\). In this way, the theory of Nichols algebras provides plenty of examples that are not necessarily quadratic, nor \(N\)-homogeneous.

(v) One can easily give examples of WGF-algebras that are not GF by adding to a GF-algebra some elements in intermediate degrees with zero products, but these examples are artificial in the sense that they do not occur naturally from the data \((V, c)\). However, given an algebra \(\mathcal{B}\), it is in general a difficult task to check whether or not it is a Nichols algebra: one should also care about the coalgebra structure, verify that it is generated in degree one and there are no primitive elements of degree bigger than one. But for our purposes, the only property that we need from the algebra \(\mathcal{B}\) is just part of the definition of graded Frobenius, and this is easy to check in examples. For this reason, we decide to extend the notion from GF to WGF, even though the only (no artificial) examples that we have are already GF. As a matter of example, concerning the Foming-Kirillov algebras, for the (known to be) finite dimensional algebras some elements in intermediate degrees with zero products, but these examples are artificial in the sense that they do not occur naturally from the data

\[\xi\]

\(A\) for \(B\), \(N\) necessarily quadratic, nor \(N\)-homogeneous.

For \(D\) \(\in\mathcal{N}\) is given by the usual determinant.

\[\text{Definition generalizes [M2, Gu].}\]

It is known that in both cases the quantum exterior algebras are Nichols algebras, thus this Definition generalizes [M2, Gu].

Fix a braided vector space \((V, c)\) and let \(A = A(c)\) be the bialgebra given by the FRT-construction associated with \((V, c)\). The existence of a weakly graded-Frobenius algebra \(\mathcal{B}\) for \(A\) allows not only to define a quantum determinant for \(A\), but also to give an explicit formula for the antipode. We begin with the definition of a quantum determinant associated with \(\mathcal{B}\). Note that our definition is consistent with quantum (homological) determinants defined previously by other authors, see for example [FK], [MS], [AG2], [Gr] and [GGI].

\[\text{Definition 2.2. Let } \mathcal{B} \text{ be a weakly graded-Frobenius algebra for } A \text{ and write } \mathcal{B}^{\text{top}} = \mathcal{B}\text{ for some } 0 \neq b \in \mathcal{B}. \text{ We call such an element a volume element for } \mathcal{B}. \text{ Since } \mathcal{B}^{\text{top}} \text{ is an } A\text{-subcomodule, we have that the coaction on } b \text{ equals } \lambda(b) = D \otimes b \text{ for some group-like element } D \in A. \text{ We call this element } D \text{ the quantum determinant in } A \text{ associated with } \mathcal{B}.\]

Note that \(D \in G(A)\) is independent of the scalar multiple of \(b\).

\[\text{Example 2.3. Consider the braiding } c = -\tau \text{ on an } n\text{-dimensional space } V. \text{ Then } A(-\tau) = O(M_n) \text{ and } \mathcal{B}(V, c) = \Lambda V \text{ is a left } O(M_n)\text{-comodule algebra. If } \{x_1, \ldots, x_n\} \text{ is a basis of } V, \text{ then one may take } b = x_1 \wedge \cdots \wedge x_n \in \Lambda^n V. \text{ In this case,}\]

\[D = \sum_{\sigma \in S_n} (-1)^{f(\sigma)} t_{\sigma(1)}^{I_1} \cdots t_{\sigma(n)}^{I_n}\]

is given by the usual determinant.

\[\text{Notation 2.4. Let } \{x_1, \ldots, x_n\} \text{ be a basis of } V. \text{ Since by assumption the multiplication } \mathcal{B}^1 \times \mathcal{B}^{\text{top} - 1} \rightarrow \mathcal{B}^{\text{top}} = \mathcal{B}\text{ is non-degenerate, there exists a basis of } \mathcal{B}^{\text{top} - 1}, \text{ say } \{\omega^1, \ldots, \omega^n\} \in \mathcal{B}^{\text{top} - 1}, \text{ such that}\]

\[x_i \omega^j = \delta^j_i b \in \mathcal{B}^{\text{top}}.\]

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For $1 \leq i, j \leq n$, we define the elements $T^j_i \in A$ by the equality

$$\lambda(\omega^j) = \sum_j T^j_i \otimes \omega^j \quad \text{for all } 1 \leq i \leq n.$$ 

It is easy to check that $\Delta(T^j_i) = \sum^n_{k=1} T^j_k \otimes T^j_i$ and $\varepsilon(T^j_i) = \delta^j_i$ for all $1 \leq i, j \leq n$. 

**Example 2.5.** Consider the braiding $c = -\tau$ on $V \otimes V$ as in Example 2.3 above. Then, the elements $w^j = (-1)^{i+j} x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_n$ give a "dual basis" with respect to $\{x_1, \ldots, x_n\}$ and the volume form $b = x_1 \wedge \cdots \wedge x_n$. 

Next we generalize the formula when expanding a determinant by a row using minors: 

**Proposition 2.6.** The following formula holds in $A(c)$: 

$$\sum^n_{k=1} t^k_i T^j_k = \delta^j_i D \quad \text{for all } 1 \leq i, j \leq n. \tag{5}$$ 

**Proof.** Using the fact that $\{x_1\}_{1 \leq i \leq n}$ and $\{w^j\}_{1 \leq j \leq n}$ are dual bases with respect to the multiplication, that is $x_i \omega^j = \delta^j_i b$ for all $1 \leq i, j \leq n$, by the comodule structure on $\mathcal{B}$ we get that 

$$\lambda(\delta^j_i b) = \delta^j_i D \otimes b = \lambda(x_i \omega^j) = \lambda(x_i) \lambda(\omega^j) = \sum_{k, \ell} t^k_i T^j_k \otimes x_k \omega^\ell = \sum_{k, \ell} t^k_i T^j_k \otimes \delta^\ell_k b = \sum_k t^k_i T^j_k \otimes b.$$

$\square$

**Example 2.7.** For $M \in M_n(k)$, let Cof$(M)$ be the $(n \times n)$-matrix whose $(i, j)$-entry is the $ij$-minor. For $c = -\tau$, Proposition 2.6 is nothing else than the well-known fact 

$$M \cdot \text{Cof}(M)^T = \text{det}(M) I \quad \forall M \in M_n(k).$$

### 2.2 Main results

In this subsection we prove our main theorem. We begin by introducing a Hopf algebra associated with the quantum determinant and give some properties of its category of finite-dimensional left comodules. For the rest of this subsection, we fix a finite-dimensional rigid braided vector space $(V, c)$ and assume there exists a weakly graded-Frobenius algebra $\mathcal{B}$ for $A$. We write $D$ for the quantum determinant and $b$ for the volume element. By Lemma 1.4 we know that there exists an automorphism $\mathcal{J} := \mathcal{J}_D \in \text{Aut}(A)$ associated with the quantum determinant $D$ such that $Da = \mathcal{J}(a)D$ for all $a \in A$.

**Definition 2.8.** Let $A$ be a $k$-algebra and $D$ a non-zero element in $A$. We define the localization of $A$ in $D$ as a pair $(H, \iota)$, where $H$ is a $k$-algebra and $\iota : A \to H$ is an algebra map that satisfies the following universal property: for any algebra map $f : A \to B$ such that $f(D)$ is invertible in $B$, there exists a unique algebra map $\bar{f} : H \to B$ such that $\bar{f} \circ \iota = f$; i.e. the following diagram commutes

$$\begin{array}{ccc}
A & \xrightarrow{\iota} & H \\
\downarrow{\forall f \text{ s.t. } f(D) \text{ invertible}} & \nearrow{\exists ! \bar{f}} & B \\
\end{array}$$

We call $\iota : A \to H$ the canonical map. By the universal property above, it follows that the localization, if it exists, is unique up to isomorphism. We denote it as $H = A[D^{-1}]$, if no confusion arises.
Remark 2.9. The localization of a bialgebra with respect to a group-like element $D$ always exists in the following sense: If $A$ is a $k$-algebra one can always consider the polynomial algebra in one indeterminate $k[x_0]$ and the free product $A * k[x_0]$. It has the universal property that given a $k$-algebra map $f : A \to B$ and $b_0 \in B$, then there exists a unique $k$-algebra map $\Phi : A * k[x_0]$ such that $\Phi|_A = f$ and $\Phi(x_0) = b_0$.

Now, given an element $D \in A$, one can consider the two-sided ideal $J$ generated by $x_0D - 1$ and $DX_0 - 1$, and define $A[D^{-1}] := (A * k[x_0])/J$. If $f : A \to B$ is an algebra map such that $f(D) = s_0$ is invertible, then one can consider $s_0^{-1} \in B$ and define $\Phi : A * k[x_0] \to B$ by $\Phi(A) = f$ and $\Phi(x_0) = s_0^{-1}$. This map satisfy $\Phi(x_0D - 1) = 0 = \Phi(DX_0 - 1)$, so, it induces an algebra map on the quotient. In other words, $H := A * k[x_0]/J$ satisfies the universal property.

The only problem that one can face is that maybe $J$ is not a proper ideal. If $J = A * k[x_0]$ (e.g. if $D = 0$ then $J = \langle 1 \rangle$) then $H$ is the zero algebra, and one has $1 = 0$ in $H$.

For a counitary bialgebra $A$ and a nonzero group-like element $D \in A$ one has the advantage that $\varepsilon(D) = 1$ (by counitariness) and $A * k[x_0]$ has a unique counitary bialgebra structure determined by $\Delta_A \otimes_A \Delta_A$, $\Delta(x_0) = x_0 \otimes x_0$, $\varepsilon_A = \varepsilon_A$ and $\varepsilon(x_0) = 1$. One can easily see that $\varepsilon(x_0D - 1) = 0 = \varepsilon(DX_0 - 1)$, so $J$ is included in the kernel of the counit. In particular $(A * k[x_0])/J$ is a non-zero $k$-algebra.

Example 2.10. Let $G = F_2 = F(x, y)$ the free group on two elements $x$ and $y$, $H = k[G]$ the group algebra, and $A \subset H$ the $k$ subalgebra generated by $x$, $x^{-1}$ and $y$. Taking $D = y$, the inclusion $A \to H$ has the universal property of the localization of $A$ in $D$.

In the above example we see that the localization is not necessarily a "calculus of fractions", in the sense that not every element in $H$ is of the form $y^{-n}a$ or $ay^{-n}$ for $a \in A$; that is, $y$ do not satisfy the Ore condition in $k\langle x^{\pm 1}, y \rangle \subset k[F_2]$. Nevertheless, our situation is much simpler:

Remark 2.11. Due to Hayashi’s result (see Lemma 1.4), for a coquasitriangular bialgebra $A$ and a (non-zero) group-like element $D$, there exists an automorphism $\tilde{\lambda}_D : A \to A$ such that $Da = \tilde{\lambda}_D(a)D$ for all $a \in A$. So, the multiplicative set $\{D^n\}_{n \in \mathbb{N}_0}$ satisfies the Ore condition and every element in $A[D^{-1}]$ can be written as $D^{-n}a$ (or $aD^{-n}$) for some $a \in A$, $n \in \mathbb{N}_0$. In particular, the localization $A[D^{-1}]$ as defined above coincides with the Ore-localization corresponding to the multiplicative set $\{D^n\}_{n \in \mathbb{N}_0}$. In particular, $A[D^{-1}]$ is a coquasitriangular bialgebra.

We introduce now the localization of $A(c)$ in the quantum determinant.

Definition 2.12. Let $H(c)$ be the $k$-algebra generated by the elements $\{t_i^j\}_{i,j}$ and $D^{-1}$ satisfying the relations (4) and

$$DD^{-1} = 1 = D^{-1}D.$$  \hfill (6)

It easy to see that $H(c)$ is indeed a localization of $A(c)$ in $D$. For this reason, we write indistinctly $H(c) = A[D^{-1}]$; the canonical map is denoted by $\iota : A(c) \to H(c)$. See Section 3 for examples. The next result follows from [4] Theorem 3.1]. We give its proof for completeness.

Lemma 2.13. $H(c)$ is a coquasitriangular bialgebra.

Proof. Let $A'$ be the algebra generated by the same elements but satisfying only (4) and

$$t_i^j D^{-1} = D^{-1} \tilde{\lambda}(t_i^j)$$  \hfill (7)

for all $1 \leq i, j \leq n$.

Then, $H(c) = A'/J$, where $J$ is the two-sided ideal generated by the relation (5). In particular, we have that $\iota : A \to H(c)$ factorizes through $A'$. Note that since $\tilde{\lambda}$ is a bialgebra map, one has that $aD^{-1} = D^{-1} \tilde{\lambda}(a)$ for all $a \in A$. 

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By defining $\Delta(D^{-1}) = D^{-1} \otimes D^{-1}$ and $\varepsilon(D^{-1}) = 1$, we may endow $A'$ with a bialgebra structure: since $\mathfrak{J}$ is a bialgebra map, one has that
\[
\Delta(aD^{-1} - D^{-1}\mathfrak{J}(a)) = a(1)D^{-1} \otimes a(2)D^{-1} - D^{-1}\mathfrak{J}(a(1)) \otimes D^{-1}\mathfrak{J}(a(2))
\]
\[
= a(1)D^{-1} \otimes a(2)D^{-1} - D^{-1}\mathfrak{J}(a(1)) \odot a(2)D^{-1} + D^{-1}\mathfrak{J}(a(1)) \otimes a(2)D^{-1} - D^{-1}\mathfrak{J}(a(1)) \otimes D^{-1}\mathfrak{J}(a(2))
\]
\[
= (a(1)D^{-1} - D^{-1}\mathfrak{J}(a(1))) \otimes a(2)D^{-1} + D^{-1}\mathfrak{J}(a(1)) \otimes (a(2)D^{-1} - D^{-1}\mathfrak{J}(a(2))),
\]
for all $a \in A$. Thus, $\Delta$ is well-defined on $A'$. Also $\varepsilon$ is well-defined since, by the explicit description of $\mathfrak{J}$ (see Lemma 1.4), we have $\varepsilon(aD^{-1}) = \varepsilon(a) = \varepsilon(D^{-1}\mathfrak{J}(a))$ for all $a \in A$.

To show that $H(c)$ is a bialgebra, it is enough to show that $J$ is also a coideal. This follows by a direct computation since both $D$ and $D^{-1}$ are group-like elements. Finally, the coquasi-triangular structure is defined extending the coquasi-triangular structure on $A$ by $r(D^{-1}, a) = r^{-1}(D, a)$ and $r(a, D^{-1}) = r^{-1}(a, D)$ for all $a \in A$. It is well-defined thanks to (CQT1)-(CQT2).

**Remark 2.14.** In the category $H(c)/M$, the comodule $\mathfrak{B}^{\text{top}}$ is invertible, that is, there exists an $H(c)$-comodule $M$ such that $\mathfrak{B}^{\text{top}} \otimes M \cong \mathfrak{B}^{\text{top}}$, in particular, $\mathfrak{B}^{\text{top}}$ and $M$ are one-dimensional. Indeed, consider the one-dimensional vector space $\mathbb{k}D^{-1}$ with generator $D^{-1}$ and whose left $H(c)$-comodule structure is defined extending the coquasi-triangular structure on $A$ by $r(D^{-1}, a) = r^{-1}(D, a)$ and $r(a, D^{-1}) = r^{-1}(a, D)$ for all $a \in A$. It is well-defined thanks to (CQT1)-(CQT2).

**Definition 2.15.** Let $V^*$ and $^*V$ be the $H(c)$-comodules given by
\[
V^* := \mathfrak{B}^{\text{top}-1} \otimes \mathbb{k}D^{-1}, \quad ^*V := \mathbb{k}D^{-1} \otimes \mathfrak{B}^{\text{top}-1}
\]
Using that the multiplication $m_{\mathfrak{B}}$ of $\mathfrak{B}$ gives non-degenerate colinear maps
\[
V \otimes \mathfrak{B}^{\text{top}-1} \to \mathfrak{B}^{\text{top}} = \mathbb{k}b, \quad \mathfrak{B}^{\text{top}-1} \otimes V \to \mathfrak{B}^{\text{top}} = \mathbb{k}b,
\]
we may define evaluation maps $ev_\ell : V^* \otimes V \to \mathbb{k}$ and $ev_r : V \otimes V^* \to \mathbb{k}$ by the following compositions. For the right evaluation:
\[
\begin{align*}
V \otimes V^* & \xrightarrow{\theta_{V^*}} V \otimes (\mathfrak{B}^{\text{top}-1} \otimes \mathbb{k}D^{-1}) \quad \mathbb{k}b \otimes \mathbb{k}D^{-1} \xrightarrow{\sim} \mathbb{k} \\
x \otimes (w \otimes D^{-1}) & \xrightarrow{\theta_{V^*}} xw \otimes D^{-1} \\
\nonumber
\text{ev}_r(x, w \otimes D^{-1}) b \otimes D^{-1} & \xrightarrow{\sim} \text{ev}_r(x, w \otimes D^{-1})
\end{align*}
\]
and similarly on the left:
\[
\begin{align*}
^*V \otimes V & \xrightarrow{\theta_V} (\mathbb{k}D^{-1} \otimes \mathfrak{B}^{\text{top}-1}) \otimes \mathbb{k}D^{-1} \otimes \mathbb{k}b \xrightarrow{\sim} \mathbb{k} \\
(D^{-1} \otimes w) \otimes x & \xrightarrow{\theta_V} D^{-1} \otimes wx \\
\nonumber
\text{ev}_\ell(D^{-1} \otimes w, x) b \otimes D^{-1} & \xrightarrow{\sim} \text{ev}_\ell(D^{-1} \otimes w, x)
\end{align*}
\]
Observe that everything depends on the choice of the “volume element” $b$. 

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Remark 2.16. By (WGF4) one may define two (possibly different) bases for $\mathfrak{B}^{\text{top}-1}$ which are right and left dual to a given basis $\{x_1, \ldots, x_n\}$ of $V$, say $\{w^1, \ldots, w^n\}$ and $\{w^1, \ldots, w^n\}$, satisfying for all $1 \leq i, j \leq n$ that

\[ x_iw^j = \delta^j_i b, \quad w^j_ix_i = \delta^i_j b. \]

For $V^* = \mathfrak{B}^{\text{top}-1} \otimes \mathbb{k}D^{-1}$ and $^*V = \mathbb{k}D^{-1} \otimes \mathfrak{B}^{\text{top}-1}$ as above, define left and right coevaluation maps $\text{coev}_L : \mathbb{k} \rightarrow V \otimes ^*V$ and $\text{coev}_R : \mathbb{k} \rightarrow *V \otimes V$ by

\[ \text{coev}_L(1) := \sum x_i \otimes (D^{-1} \otimes w^j), \quad \text{coev}_R(1) := \sum (D^{-1} \otimes w^j) \otimes x_i. \]

By a direct computation we obtain the following:

\[ \text{Lemma 2.17.} \quad V^* \text{ and } ^*V \text{ are, respectively, right and left duals of } V \text{ in } H(c)M. \]

Remark 2.18. Using similar arguments as before, one has that $(V^*)^* = \mathbb{k}b \otimes V \otimes \mathbb{k}D^{-1}$ and $^*(^*V) = \mathbb{k}D^{-1} \otimes V \otimes \mathbb{k}b$. In particular, $V^{**} \simeq V \simeq ^*V$.

The next theorem is our first main result. It states that $H(c)$ is indeed a Hopf algebra, provided the canonical map $\iota : A(c) \rightarrow H(c)$ is injective. This is the case when $D$ is not a zero divisor in $A$.

\[ \text{Theorem 2.19.} \quad \text{If the canonical map } \iota : A(c) \rightarrow H(c) \text{ is injective then } H(c)M \text{ is rigid, tensorially generated by } V \text{ and } \mathbb{k}D^{-1}. \text{ As a consequence, } H(c) \text{ is a coquasitriangular Hopf algebra. Moreover, the formula for the antipode is given on generators by } S(D^{-1}) = D, \text{ and for all } 1 \leq i, j \leq n:\]

\[ S(t^j_i) := T^j_iD^{-1}. \]

\[ \text{Proof.} \quad \text{Identify the elements of } A \text{ with their image in } H(c) \text{ under the canonical map. Let } M \in H(c)M \text{ and fix a basis } \{m_1, \ldots, m_k\} \text{ of } M \text{ as vector space. Denote by } h^j_i \in H(c) \text{ the elements such that the coaction on } m_i \text{ is given by } \lambda(m_i) = \sum_j h^j_i \otimes m_j. \text{ If all } h^j_i \text{ belong to the image of } A \text{ under the canonical map, then clearly } M \text{ is an } A\text{-comodule. Since } D \text{ is normal, each } h^j_i \text{ can be written as a polynomial in } D^{-1} \text{ with coefficients in (the image of) } A, \text{ say:}

\[ h^j_i = \sum_{k=0}^{N_{ij}} a^{ij}_k D^{-k} \quad \text{ with } a^{ij} \in A \text{ for all } 1 \leq i, j \leq n. \]

Thus, for all } N \geq N_{ij} \text{ we have } h^j_iD^N \in A. \text{ Taking } N = \max_{i,j}\{N_{ij}\}, \text{ we have that}

\[ \tilde{M} = M \otimes \mathbb{k}b^\otimes N = M \otimes \mathbb{k}b \otimes \cdots \otimes \mathbb{k}b \]

\[ \text{is an } A\text{-comodule: a basis is } \{m_1 \otimes b \otimes b \otimes \cdots \otimes b\}_{1 \leq i \leq n}, \text{ and the } A\text{-comodule structure is}

\[ \lambda(m_1 \otimes b \otimes b \otimes \cdots \otimes b) = \sum_j h^j_iD^N \otimes m_1 \otimes b \otimes b \otimes \cdots \otimes b \in A \otimes \tilde{M} \]

\[ \text{Now using that } \mathbb{k}b \otimes \mathbb{k}D^{-1} \cong \mathbb{k} \text{ we have that } \tilde{M} \otimes \mathbb{k}D^{-1} \cdots \otimes \mathbb{k}D^{-1} \cong M. \text{ That is, } M \text{ is isomorphic to a tensor product of an } A\text{-comodule and the } H(c)\text{-comodule } \mathbb{k}D^{-1}. \text{ Thus } H(c)M \text{ is tensorially generated by } A(c)\text{-comodules and } \mathbb{k}D^{-1}. \text{ Since } A(c)M \text{ is tensorially generated by } V, \text{ the first assertion of the statement follows.} \]
Finally, we prove the formula for the antipode. Since $D$ and $D^{-1}$ are group-like, we have $S(D) = D^{-1}$ and $S(D^{-1}) = D$. For the generators $t^i_j$ we proceed as follows: from Proposition 2.6 we know that $\sum_k t^k_i T^j_k = \delta^j_i D$. So, in $H(c)$ it holds that

$$\sum_k t^k_i T^j_k D^{-1} = \delta^j_i \quad \text{for all } 1 \leq i, j \leq n.$$ 

Now, since $H(c)$ is a Hopf algebra, we must have that $\sum_{i, j} S(t^i_j) t^j_i = \varepsilon(t^i_j) = \delta^i_j$. From one side one gets

$$(*) := \sum_k \sum_{i, j} S(t^i_j) t^k_i T^j_k D^{-1} = \sum_k \delta^k_i T^j_k D^{-1} = T^j_i D^{-1},$$

but, changing the order of the double sum, we obtain

$$(*) = \sum_k \sum_{i, j} S(t^i_j) t^k_i T^j_k D^{-1} = \sum_{i, j} S(t^i_j) \delta^j_i = S(t^i_j).$$

Consequently, $S(t^i_j) = T^j_i D^{-1}$ for all $1 \leq i, j \leq n.$

\[\square\]

**Remark 2.20.** The antipode verifies both $S \ast \text{id} = u\varepsilon$ and $\text{id} \ast S = u\varepsilon$, so we also have

$$\sum_k T^k_i D^{-1} t^j_k = \delta^j_i.$$ 

In particular, if $D$ is central, in addition to $\sum_k t^k_i T^j_k = \delta^j_i D$, one must have $\sum_k T^k_i t^j_k = \delta^j_i D$. In the general case, it holds that

$$\sum_k \bar{\delta}(T^k_i) t^j_k = \delta^j_i D,$$

where $\bar{\delta}$ is as in Lemma 1.4. It would be interesting to have a direct proof of this fact in $A(c)$.

Usually, one does not know a priori if $\iota : A \rightarrow H(c)$ is injective, and there are examples where this map actually have non-zero kernel. Also, it is difficult to check by computer if the element $D$ is a zero divisor or not. We give below a "computer adapted version" of Theorem 2.19 without the assumption of the canonical map $\iota : A \rightarrow H(c)$ being injective.

**Theorem 2.21.** Assume the following equality holds in $A(c)$ for all $1 \leq i, j \leq n$:

$$\sum_{k=1}^n \bar{\delta}(T^k_i) t^j_k = \delta^j_i D. \tag{7}$$

Then $H(c)$ is a coquasitriangular Hopf algebra and the formula for the antipode on generators is given by $S(D^{-1}) = D$, and $S(t^i_j) := T^j_i D^{-1}$ for all $1 \leq i, j \leq n$.

**Proof.** Define an algebra map $\varphi : H(c) \rightarrow H(c)$ by $\varphi(t^i_j) = t^j_i$ and $\varphi(u^i_j) = T^j_i D^{-1}$ for all $1 \leq i, j \leq n$. It is clear that the FRT relations (13) and (2) are the same, also

$$\varphi\left(\sum_k t^i_k u^j_k\right) = \sum_k t^i_k T^j_k D^{-1} = \delta^j_i D D^{-1} = \delta^j_i.$$ 

For the remaining relations, notice that for $a \in H(c)$, we have $D^{-1} \bar{\delta}(a) = a D^{-1}$ and so

$$\varphi\left(\sum_k u^k_i t^j_k\right) = \sum_k T^j_i D^{-1} t^j_k = \sum_k D^{-1} \bar{\delta}(T^j_i) t^j_k = D^{-1} \sum_k \bar{\delta}(T^j_i) t^j_k = D^{-1} \delta^j_i D = \delta^j_i.$$
This implies that $\varphi$ is a well-defined algebra map. Moreover, by \text{[2.4]} it follows that $\varphi$ is indeed a bialgebra map. In particular, $B$ is also a weakly graded-Frobenius algebra for $H(c)$ and there exists a group-like element $D$ on $H(c)$ which is mapped to $D$. Since $H(c)$ is a Hopf algebra, $D^{-1}$ is a group-like element whose image $D^{-1}$ is contained in the image of $\varphi$. Consequently, $f$ is surjective and $H(c)$ is a Hopf algebra. By the universal property of $H(c)$, it follows that $H(c)$ is also coquasitriangular.

The formula for the antipode follows the same lines as in the proof of Theorem \text{[2.19]}.

We end this section with two corollaries. The first one states that actually both Hopf algebra $H(c)$ and $\mathcal{H}(c)$ coincide. The second one is a resume that stresses the results for Nichols algebras.

\textbf{Corollary 2.22.} Assume that \text{[7]} holds in $A(c)$ for all $1 \leq i, j \leq n$. Then $H(c)$ and $\mathcal{H}(c)$ are isomorphic as Hopf algebras.

\textbf{Proof.} By the proof of Theorem \text{[2.21]} we know that there is a surjective Hopf algebra map $\varphi : \mathcal{H}(c) \rightarrow H(c)$ such that $\varphi(D) = D$, $\varphi(\iota^i) = t^i$, and $\varphi(u^i) = T^i D^{-1}$ for all $1 \leq i, j \leq n$, where $D$ is the quantum determinant of $\mathcal{H}(c)$ associated with $B$. To prove that both algebras coincide, we show that $\varphi$ is bijective. Define the algebra map $f : H(c) \rightarrow \mathcal{H}(c)$ by $f(t^i) = t^i$ and $f(D^{-1}) = D^{-1}$. Clearly, it is a well-defined bialgebra map, which satisfies that $u^i = f(T^i D^{-1})$.

Indeed, define the matrices $t, T, \mathcal{H}(T), \iota$ and $u$ by $(t)_{ij} = \iota^i_j$, $(T)_{ij} = T^i_j$, $(\mathcal{H}(T))_{ij} = \mathcal{H}(T)_{ij}$, $(\iota)_{ij} = \iota^i_j$, and $(u)_{ij} = u^i_j$. In particular, by Proposition \text{[2.6]} and our assumptions, we know that $f(t) = t$, $u \cdot t = \iota = t \cdot u$, $t \cdot T = DI$ and $\mathcal{H}(T) \cdot t = DI$.

Thus, $f(T) = (u \cdot t) f(T) = u \cdot (f(t) \cdot f(T)) = u \cdot f(t \cdot T) = u D$.

Namely, $f(T^i_j) = u^i_j D$ and so $u^i_j = f(T^i_j D^{-1})$. Hence, we conclude that the algebra map $f : H(c) \rightarrow \mathcal{H}(c)$ is surjective. Since by definition we have that $f \circ \varphi = \iota$ and $\varphi \circ f = \iota$, the claim is proved.

\textbf{Corollary 2.23.} Let $(V, c)$ be a rigid finite-dimensional braided vector space such that the associated Nichols algebra $\mathcal{B}(V, c)$ is finite-dimensional. If the canonical map $\iota : A(c) \rightarrow H(c)$ is injective or equation \text{[7]} is satisfied, then $\mathcal{B}(V, c)$ is a braided Hopf algebra in $H(c)M$.

\textbf{Proof.} Follows from Proposition \text{[17]} the fact that $\iota : A(c) \rightarrow H(c)$ is a bialgebra map and that $\mathcal{B}(V, c)$ is a braided Hopf algebra in $A(c)M$.

\textbf{Question 2.24.} If the canonical map $\iota : A \rightarrow H(c)$ is injective, one can easily see that the hypothesis of Theorem \text{[2.21]} is superfluous. We do not know if it is actually superfluous in general, or at least superfluous in some situation, e.g. noetherianity.

\section{Formulas for the quantum determinant for set-theoretical solutions}

In this subsection, we give an explicit formula for the quantum determinant in the case where the braided vector space is given by a linearization of a set-theoretical solution of the braid equation. Using the coquasitriangular structure, we also give the commuting relations between the quantum determinant and the generators of the bialgebra.

Let $(X, c)$ be a set-theoretical solution of the braid equation and write

$$c(i, j) = (g_i(j), f_j(i)) \quad \text{for all } i, j \in X,$$
where \( f, g : X \to \text{Fun}(X, X) \). We say that the solution \((X, c)\) is non-degenerate if the images of \( f \) and \( g \) are bijections. Let \( V = kX \) be the vector space linearly spanned by \( X \) and consider the map on \( V \), also written as \( c \), obtained by linearizing \( c \). Then \((V, c)\) is a braided vector space. Set \( \{x_i\}_{i \in X} \) for the linear basis of \( V \). A map \( q : X \times X \to k \) is called a cocycle if the map \( c^q : V \otimes V \to V \otimes V \) given by
\[
c^q(x_i \otimes x_j) = q_{ij} x_{g(j)} \otimes x_{f(j)}
\]
for all \( i, j \in X \), where \( q_{ij} = q(i, j) \), is a solution of the braid equation. It turns out that the braiding \( c^q \) is rigid if and only if \( c \) is non-degenerate, see [AG2, Lemma 5.7].

Assume \( |X| = n \) and let \((c^{k\ell})_{i,j,k,\ell \in X}\) be the \((n^2 \times n^2)\)-matrix given by \( c^q(x_i \otimes x_j) = \sum_{k,\ell} c^{k\ell}_{ij} x_k \otimes x_{\ell} \). By the formula above, it follows that
\[
c^{k\ell}_{ij} = q_{ij} \delta_{k,g(j)} \delta_{\ell,f(j)},
\]
Let \( \mathcal{B}(V, c^q) \) be the Nichols algebra associated with the rigid braided vector space \((V, c^q)\). If \( \dim \mathcal{B}(V, c^q) \) is finite, then \( \mathcal{B}(V, c^q) = \bigoplus_{i=0}^N N^i \) with \( \dim \mathcal{B}^N = 1 \). Assume further that \( \mathcal{B}^N = k b \) with \( b \) a “volume element”. Then for any element \( x_i = b, t_a \) is rigid. Then
\[
\lambda(x_i) = \sum_{a=1}^n t^a_i \otimes x_j \\
\lambda(b) = \lambda(x_j) = \sum_{i=1}^n t^{i_1} \otimes t^{i_2} \otimes \cdots \otimes t^{i_n} \otimes x_i \otimes x_j \otimes \cdots x_N
\]
Since \( \lambda(b) = D \otimes b \), the assertion follows.

Next we give some formulas concerning the quantum determinant and the coquasitriangular structure.

**Proposition 2.25.** Assume \( b = x_{j_1} \cdots x_{j_N} \) with \( x_j \in V \) for all \( 1 \leq s \leq N \). Then
\[
D = \sum_{1 \leq i_1, \ldots, i_N \leq n} \alpha_{i_1, \ldots, i_N} t_{i_1}^{i_2} \cdots t_{i_N}^{i_N}.
\]

**Proof.** Since \( \lambda(x_i) = \sum_{a=1}^n t^a_i \otimes x_j \) for all \( 1 \leq i \leq n \), we have that
\[
\lambda(b) = \lambda(x_{j_1} \cdots x_{j_N}) = \sum_{1 \leq i_1, \ldots, i_N \leq n} t^{i_1}_{j_1} \cdots t^{i_N}_{j_N} \otimes x_{i_1} x_{i_2} \cdots x_{i_N}
\]
Next we give some formulas concerning the quantum determinant and the coquasitriangular structure.

**Proposition 2.26.** Let \( 1 \leq a, b \leq n \) and denote recursively \( a_1 = f_{j_1}(a) \) and \( a_k = f_{j_k}(a_{k-1}) \). Then
\[
r(D, t^b_a) = \delta_{b,a} \alpha_{a}(j_1, a_{a_2}(j_2)) \cdots a_{a_{N-1}}(j_N) q_{j_1} a_{a_2}(j_1) \cdots q_{j_N} a_{a_{N-1}}.
\]

**Proof.** From Proposition 2.25 we get
\[
r(D, t^b_a) = \sum_{1 \leq k \leq n} \alpha_{i_1, \ldots, i_N} r(t^{i_1}_{j_1} t^{i_2}_{j_2} \cdots t^{i_N}_{j_N}, t^b_a)
\]
\[
= \sum_{1 \leq k \leq n} \sum_{1 \leq \ell \leq n} \alpha_{i_1, \ldots, i_N} r(t^{i_1}_{j_1} t^{i_2}_{j_2} \cdots t^{i_N}_{j_N}, t^b_a)
\]
\[
= \sum_{1 \leq k \leq n} \sum_{1 \leq \ell \leq n} \delta_{k,b} \alpha_{a}(j_1) \delta_{a_2, j_1} \cdots \delta_{a_{a_{N-1}}, j_{N-1}} \delta_{a_{a_N}} t^{i_1}_{j_1} t^{i_2}_{j_2} \cdots t^{i_N}_{j_N}
\]
\[
= \alpha_{a}(j_1) \cdots a_{a_{N-1}}(j_N) q_{j_1} a_{a_2}(j_1) \cdots q_{j_N} a_{a_{N-1}} \delta_{b,aN},
\]
15
and the claim follows.

\[ r(D, i_a^b) = \delta_{b,a}q_{a(j_1), g_{a(j_2)}(j_3)}}g_{a(j_N-1)(j_N)}q^N. \]

**Corollary 2.28.** Let \( 1 \leq a, b \leq n \). As before, denote \( a_k = f_{jk}(a_{k-1}) \) with \( a(0) = a \), and set \( b_k = f_{jk}(b_{k-1}) \) with \( b_N = b \). Assume \( q \) is a constant cocycle with \( q_{ij} = q \in k^x \) for all \( 1 \leq i, j \leq n \). Then

\[ \alpha_{g_{a(j_1)}, g_{a(j_2)}}(j_N)D_{a_N}^b = \alpha_{g_{b_0}, g_{b_1}(j_2)}g_{b_N-1}(j_N)D_{a_N}^b. \]

In particular, if \( \alpha_{g_{a(j_1)}, g_{a(j_2)}}(j_N) \neq 0 \), we have that

\[ \tilde{\mathcal{A}}(t_a^b) = \alpha^{-1}_{g_{a(j_1)}, g_{a(j_2)}}g_{b_0}g_{b_1}(j_2)g_{b_N-1}(j_N)D_{a_N}^b. \]

**Proof.** Follows directly from Remark 2.29.(4).

**Remark 2.29.** Let \( (X, \triangleright) \) be a rack and consider the associated set-theoretical solution of the braid equation \( \sigma(x, y) = (xy, yx) \) for all \( x, y \in X \). Set \( V = kX \) and consider the covering group

\[ G := G(X, c) = \langle x \in X \rangle / (g_xg_y = g_{xy}g_x). \]

Let \( q \) be a 2-cocycle and write \( \mathcal{A} \in \operatorname{Aut}(V \otimes V) \) for the braiding on \( V \). Then \( V \) is a left \( kG \)-comodule with structure map \( \delta_V : V \to kG \otimes V \) given by \( \delta_V(x) = g_x \otimes x \) for all \( x \in X \). Moreover, \( \mathcal{A} \) is a \( kG \)-colinear map and, by the universal property of \( A(\mathcal{A}) \), there exists a bialgebra map \( f : A(\mathcal{A}) \to kG \) such that \( (f \otimes \text{id})\lambda_V = \delta_V \). In particular, we have that

\[ g_x \otimes x = \delta_V(x) = (f \otimes \text{id})\lambda_V(x) = (f \otimes \text{id}) \left( \sum_{y \in X} t_{x,y}^y \otimes y \right) = \sum_{y \in X} f(t_{x,y}^y) \otimes y, \]

which implies that \( f(t_{x,y}^y) = \delta_{x,y}g_x \) for all \( x, y \in X \). Clearly, \( f \) is a well-defined bialgebra map. As the FRT relations of \( A(\mathcal{A}) \) are given by \( q_{ij} t_{ki,j} = q_{ij} t_{k,j} t_{i,j}^k \) for all \( i, j, k, \ell \in X \), one might view \( kG \), as the universal Hopf algebra of the algebra generated by the elements \( \{ g_x = t_{x,x} \}_{x \in X} \) satisfying the relations

\[ t_{x,xy}^{xy} = t_{x,y}^x t_{y}^y \quad \text{for all } x, y \in X. \]

It is a bialgebra with the coalgebra structure determined by \( t_{x}^x \) being group-like for all \( x \in X \). Suppose that \( \mathcal{B}(V, \mathcal{A}) \) is finite-dimensional and denote by \( D \in A(\mathcal{A}) \) the associated quantum determinant. As \( D \) satisfies (5), we must have that

\[ \delta_{i,j}(D) = \sum_{k=1}^n f(t_{i,k}^k)f(t_{j}^k) = f(t_{i}^j)f(t_{j}^j) = g_i f(T_{ij}^j) \quad \text{for all } i, j \in X. \]

Since \( f(D), g_i \) are group-like elements in \( kG \), we have that \( f(T_{ij}^j) = 0 \) if \( i \neq j \) and \( f(T_{ij}^j) = g_i^{-1} f(D) \in G \) for all \( i \in X \). From the (quantum) geometrical point of view, one may consider the universal map \( f : H(\mathcal{A}) \to kG \) as the inclusion of the (non-commutative) subgroup of diagonal matrices into the quantum group associated with \( H(\mathcal{A}) \). Compare with examples in Subsection 3.6.
3 Examples

In this section we provide examples and formulas given by our main results. Recall that given a set-theoretical solution of the braid equation \( s : X \times X \to X \times X \), \( s(x, y) = (g_x(y), f_y(x)) \) for all \( x, y \in X \), there is a so-called derived solution that is of rack type \( \tau_s : X \times X \to X \times X \), that is, it is of the form \( \tau_s(x, y) = (y, x \triangleleft_s y) \) for all \( x, y \in X \) (see for instance [AG2, Proposition 5.4] and references therein). Now for every \( n \), one can let act the braided group \( \mathcal{B}_n \), using \( s \) or \( \tau_s \). The derived solution has the remarkable property that there exists a bijection in \( X^n \) intertwining these two possible actions. As a consequence, for any non-zero constant cocycle \( \eta \), the dimensions of the homogeneous components Nichols algebras attached to \((X, q_s)\) and \((X, q\tau_s)\) are the same. A very particular case is when \( s^2 = id \); we also have \( \tau_s^2 = id \) and so necessarily \( \tau_s \) is the flip. Hence, for involutive set theoretical solutions \((X, s)\), one always has that the dimension of \( \mathcal{B}(X, -s) \) is finite, and equal to the dimension of the exterior algebra.

Nevertheless, the FRT-construction on the braiding \( s \) is far from being trivial: Example 3.1 is the smallest example of a non-trivial set-theoretical involutive solution, and Example 3.2 is also coming from an involutive set-theoretical solution whose quantum determinant is not central.

3.1 A non-diagonal type \( 2 \times 2 \) example

Let \( X = \{1, 2\} \), and consider the set-theoretical solution of the braid equation given by \( s : X \times X \to X \times X \) given by

\[
\begin{align*}
  s(1, 2) &= (1, 2), & s(2, 1) &= (2, 1), & s(1, 1) &= (2, 2), & s(2, 2) &= (1, 1).
\end{align*}
\]

Write \( (t^i_{s})_{i,j} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). Then, the FRT relations are

\[
\begin{align*}
  a^2 &= d^2, & ab &= cd, & ba &= dc, & ac &= bd, & ca &= db, & b^2 &= c^2.
\end{align*}
\]

The Nichols algebra \( \mathcal{B} = \mathcal{B}(V, c) \) associated with \( V = kx \oplus ky \) and the linearization of \( c = -s \) is the \( k \)-algebra generated by \( x, y \) with relations

\[
\begin{align*}
  x^2 + y^2 &= 0, & 2xy &= 0 = 2yx.
\end{align*}
\]

If \( \text{char}(k) \neq 2 \), then \( \dim \mathcal{B} \) is finite and \( \mathcal{B} \) has a basis \( \{1, x, y, x^2\} \). The volume element \( b \) can be taken to be \( b = x^2 \). Since \( \lambda(x) = a \otimes x + b \otimes y \), we have that

\[
\lambda(b) = \lambda(x^2) = (a \otimes x + b \otimes y)^2 = a^2 \otimes x^2 + ab \otimes xy + ba \otimes yx + b^2 \otimes y^2 = (a^2 - b^2) \otimes x^2.
\]

Thus, we obtain that \( D := a^2 - b^2 \) is a group-like element. One can check by hand that it is central (hence \( J = id \)) and \( \{\omega_1 = x, \omega_2 = -y\} \) is a “dual basis” with respect to the volume element \( b \). We compute the coaction to get the values of the \( T^i_j \):

\[
\begin{align*}
  \lambda(\omega_1) &= \lambda(x) = a \otimes x + b \otimes y = a \otimes \omega_1 - b \otimes \omega_2, \\
  \lambda(\omega_2) &= \lambda(-y) = -c \otimes x - d \otimes y = -c \otimes \omega_1 + d \otimes \omega_2.
\end{align*}
\]

Actually, it is quite difficult to check whether \( D \) is a zero divisor or not. However, to check the condition of Theorem 2.21 is a very easy task (one can check it directly by hand or use GAP) and conclude that \( H(s) = A[D^{-1}] =: \text{GL}(X, -s) \) is a Hopf algebra. The antipode is given by

\[
\begin{align*}
  S(a) &= aD^{-1}, & S(b) &= -cD^{-1}, & S(c) &= -bD^{-1}, & S(d) &= dD^{-1}.
\end{align*}
\]

Since the quantum determinant \( D \) is central, we may also consider the Hopf algebra \( \text{SL}(X, -s) \) given by \( \text{A}(s)/(D - 1) \). It is the algebra presented by

\[
\text{SL}(X, -s) = k\langle a, b, c, d : a^2 = d^2, ab = cd, ba = dc, ac = bd, ca = db, b^2 = c^2, a^2 - b^2 = 1 \rangle.
\]
3.2 Involutive and non-central example

For $X = \{1, 2, 3\}$, consider the set-theoretical solution of the braid equation given by $s(i, i) = (i, i)$ for $i = 1, 2, 3$, $s(i, j) = (j, i)$ for $i, j = 2, 3$, and

$$
s(1, 2) = (3, 1), \quad s(1, 3) = (2, 1), \quad s(2, 1) = (1, 3), \quad s(3, 1) = (1, 2).
$$

Clearly, this solution is involutive. For $(t^j_i)_{i,j} = \left( \begin{array}{ccc} a & b & c \\ d & e & f \\ g & h & i \end{array} \right)$, the FRT relations reads

$$
c^2 = b^2, \quad g^2 = d^2, \quad h^2 = f^2, \quad i^2 = e^2,
$$

$$
ba = ac, \quad ea = ab, \quad da = ag, \quad db = cg, \quad dc = bg, \quad ea = ai,
$$

$$
eb = ci, \quad ec = bi, \quad eg = di, \quad fa = ah, \quad fb = ch, \quad fc = bh,
$$

$$
fg = dh, \quad fi = eh, \quad ga = ad, \quad gb = cd, \quad gc = bd, \quad gh = fd, \quad gi = ed,
$$

$$
ha = af, \quad hb = cf, \quad hc = bf, \quad hf = gf, \quad hg = df, \quad hi = ef, \quad ia = ae,
$$

$$
ib = ce, \quad ic = be, \quad id = ge, \quad if = he, \quad ig = de, \quad ih = fe.
$$

Let $V = kX$ and write $s$ also for the braiding given by the linearization of $s$. As the set-theoretical solution is involutive, the Nichols algebra $\mathfrak{B}(V, -s)$ is finite-dimensional, its maximal degree is 3. Our construction gives the quantum determinant

$$
D = ae^2 - af^2 + bdf - bed - cde + cfd
$$

It is group-like, normal but not central: $D$ commutes with $a$ but $bD = -Dc, \ cD = -Db, \ dD = -Dg, \ gD = -Dd, \ De = iD, \ Di = eD, \ DF = hD, \ fD = Dh$. On the other hand, these non-commutation relations give us the formula for the automorphism $\mathfrak{J}$:

$$
\mathfrak{J}(a) = a, \ \mathfrak{J}(b) = -c, \ \mathfrak{J}(d) = -g, \ \mathfrak{J}(e) = i, \ \mathfrak{J}(f) = h.
$$

Also, it holds that $\mathfrak{J}^2 = \text{id}$. One can check directly that the hypothesis of Theorem 2.21 holds and conclude that $H(s) =: GL(X, -s)$ is a Hopf algebra. We also have the explicit formula for the antipode:

$$
(S(t^j_i))_{ij} = \left( \begin{array}{ccc} -fh + ci & -ce + bf & ch - bi \\ -fg + dh & -cd + ae & cg - ah \\ eg - di & bd - af & -bg + ai \end{array} \right) D^{-1}.
$$

**Remark 3.1.** The relations defining the FRT-construction in this example are not very enlightening, however, we exhibit them the following reasons: first, to stress the fact that our constructions are very explicit; second, to show that our methods apply to every braiding coming from a set theoretical involutive solution, as it has a finite-dimensional Nichols algebra attached to it. Also, even for very elementary solutions (e.g. a braiding coming from a set theoretical involutive solution on a set with 3 elements!), the Hopf algebras that arise in this way are non-trivial, since the quantum determinants in these cases are not necessarily central. And third, the number of set-theoretical involutive solutions on a finite set $X$ grows really fast with respect to the cardinal of $X$, so, one has a big number of exotic examples.
### 3.3 Fomin-Kirillov algebras

Before introducing quantum determinants for Fomin-Kirillov algebras, we first apply our construction to solutions of the braid equation given by a rack and a cocycle. 

A rack is a pair $(X, \triangleright)$ where $X$ is a non-empty set and $\triangleright : X \times X \to X$ is a map such that $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$ and $x \triangleright y = y$ is bijective for every $x, y, z \in X$. Every rack gives a set-theoretical solution of the braid equation by setting

$$c(i, j) = (i \triangleright j, i) \quad \text{for all } i, j \in X.$$ 

A rack 2-cocycle $q : X \times X \to \mathbb{k}^\times$, $(i, i) \mapsto q_{i, i}$ is a function such that

$$q_{i, j} k q_{j, k} = q_{i \triangleright j \triangleright k} q_{i, k} q_{j, k} \quad \text{for all } i, j, k \in X.$$

Let $(X, \triangleright)$ be a rack with $|X| = n$ and let $q : X \times X \to \mathbb{k}^\times$ be a cocycle. Then, one may define a braiding on the vector space $V = \mathbb{k} X$ by

$$c^q(x_i \otimes x_j) = q_{ij} x_i \triangleright x_j \quad \text{for all } i, j \in X.$$

If we write $c(x_i \otimes x_j) = \sum_{k, \ell=1}^n c_{i,j}^{k, \ell} x_k \otimes x_\ell$, then we have that

$$c_{i,j}^{k, \ell} = q_{ij} \delta_{i,j} \delta_{k, \ell} \quad \text{for all } i, j \in X.$$

In particular, the FRT-relations defining $A(c^q)$ have the following form

$$q_{ij} t_i^{k \triangleright j} t_i^\ell = q_{\ell, j} t_i^{k \triangleright j} t_i^\ell q_{i, k} q_{j, k}^{-1} \quad \text{for all } i, j, k, \ell \in X.$$

Moreover, if we replace $\ell \triangleright 1$ by $k$ by we get

$$[q_{ij} t_i^{k \triangleright j} t_i^\ell = q_{\ell, k} q_{j, k}^{-1} t_i^{k \triangleright j} t_i^\ell].$$

Let $n \in \{3, 4, 5\}$. The Fomin-Kirillov algebras $E_n$ arise as Nichols algebras when one considers the solution of the braid equation associated with the racks given by the conjugacy classes of transpositions in $S_n$ and a constant cocycle, see [MS], [AG2], [GGI] for more details. We describe explicitly the case when $n = 3$ and $q_{ij} = -1$ for all $i, j \in X$.

Let $X = \mathbb{O}_{2}^{S_3}$ be the rack of transpositions in $S_3$ and consider the constant cocycle $q_{ij} = -1$. Let $V = \mathbb{k} \mathbb{O}_{2}^{S_3}$ be the braided vector space associated with them and take the basis $x_1 = x_{(12)}, x_2 = x_{(13)}$ and $x_3 = x_{(23)}$ on $V$. Then $c(x_i \otimes x_j) = -x_i \triangleright j \otimes x_i$ for all $1 \leq i, j \leq 3$. In this case, $A(c^q)$ is generated by the elements $\{t_i^j\}_{1 \leq i, j \leq 3}$ satisfying the relations $t_i^k t_i^j = t_i^j t_i^k$ for all $1 \leq i, j, k \leq 3$. Because the cocycle is constant we may write

$$[t_i^{k \triangleright j} t_i^\ell = t_i^\ell t_i^{k \triangleright j}].$$

The Nichols algebra $\mathfrak{B}(\mathbb{O}_{2}^{S_3}, -1)$ associated with this rack and cocycle is finite-dimensional and it is generated by the elements $x_1, x_2, x_3$ satisfying the relations

$$x_i^2 = 0, \quad x_1 x_2 + x_2 x_3 + x_3 x_1 = 0, \quad x_1 x_3 + x_3 x_2 + x_2 x_1 = 0, \quad \text{for all } 1 \leq i \leq 3.$$ 

for all $1 \leq i \leq 3$. It has dimension 12 and its volume element is in degree $N = 4$. In our case, we may take $\mathfrak{B}^1 = \mathbb{k} x_1 x_2 x_3 x_2$ and the volume element $w = x_1 x_2 x_3 x_2$. In particular, by Remark...
Thus, the quantum determinant is given by

\[ D = a^2c^2 - abdf + a^2f^2 - abgi + b^2d^2 - abgi - abdf + b^2f^2 - abdf + c^2d^2 - abgi + c^2e^2 = c^2e^2 + c^2d^2 + b^2f^2 + b^2d^2 - 3abgi - 3abdf + a^2f^2 + a^2e^2. \]

One can check explicitly by hand using the relations above (or using GAP and non-commutative Gröbner basis) that \( D \) is a central element; in particular, the hypothesis of Theorem 2.21 holds. Thus, \( H(c) = \text{GL}(\mathcal{O}_2^\mathcal{B}, -1) \) is a Hopf algebra. The formula for the antipode follows from considering dual bases in the Nichols algebra and finding the elements \( T_i^j \); this can be done explicitly. For example,

\[ S(a) = ( - fbi + fah - ech + eai ) D^{-1}. \]

As \( D \) is a central group-like element, one may also define the Hopf algebra \( \text{SL}(\mathcal{O}_2^\mathcal{B}, -1) \) given by \( A(c)/\langle D - 1 \rangle \).

We end this example with a question suggested by the referee.

**Question 3.2.** The example above relies heavily on computations. Following our construction, it is possible to describe the quantum function algebras \( H(c) = \text{GL}(\mathcal{O}_2^\mathcal{B}, -1) \) for \( n = 4 \) or \( n = 5 \). However, our methods need computer assistance since the dimension of the Nichols algebra for \( n = 4 \) is 576 and its top degree is 12, while for \( n = 5 \) the dimension of \( \mathcal{B} \) is 8294400 and its top degree is 40. It would be interesting to present \( H(c) = \text{GL}(\mathcal{O}_2^\mathcal{B}, -1) \) in a more conceptual way, since this algebra would give some insight on the Fomin-Kirillov algebras.
3.4 Quantum determinants for quantum planes

In [AJG], the authors consider all solution of the QYBE in dimension 2 and give several examples of finite-dimensional Nichols algebras, arranged in families $\mathcal{R}_{0,i}$, $\mathcal{R}_{1,i}$ ($i = 1, 2, 3, 4$), and $\mathcal{R}_{2,i}$ ($i = 1, 2, 3$). They remark that, up to now, the only case known where one can find a quantum determinant and localize $A(c)$ to obtain a Hopf algebra is $\mathcal{R}_{2,1}$, due to a result of Takeuchi [T]. As our method only has as hypothesis $\text{dim } \mathfrak{B} < \infty$, we can apply it in the other cases (and for certain parameters) to obtain quantum determinants.

For example, let us consider the case $\mathcal{R}_{2,2}$ (with $k^2 = -1$ and $pq = 1$, according to the notation in [AJG]). The braiding associated with this two-dimensional vector space $V_{k,p,q}$ is

$$\left(c(x_i \otimes x_j)\right)_{1 \leq i,j \leq 2} = \begin{pmatrix} -x_1 \otimes x_1 & kq x_2 \otimes x_1 - 2 x_1 \otimes x_2 \\ kp x_1 \otimes x_2 & -x_2 \otimes x_2 \end{pmatrix}.$$ 

The corresponding Nichols algebra is presented as follows

$$\mathfrak{B}(V_{k,p,q}) = T(V_{k,p,q})/(x_1 x_2 - kq x_2 x_1, x_1^2, x_2^2).$$

A PBW-basis is given by $\{1, x_1, x_2, x_1 x_2\}$, $\text{dim } \mathfrak{B}(V_{k,p,q}) = 4$ and one may take the volume element $b = x_1 x_2$. Write $(t^j_i)_{i,j=1,2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then the FRT relations read

$$ab = kp ba, \quad ac = kq ca, \quad bc = q^2 cb, \quad ad - da = kp bc, \quad cd = kp dc, \quad bd = kq db.$$ 

The quantum determinant is given by $D = ad - kp bc$. This element is not central, it verifies $aD = Da$, $dD = Dd$, but $Db = p^2 bD$ and $Dc = q^2 cD$. Hence

$$\mathfrak{J}(a) = a, \quad \mathfrak{J}(d) = d, \quad \mathfrak{J}(b) = p^2 b, \quad \mathfrak{J}(c) = q^2 c.$$ 

The matrices $T$ and $\mathfrak{J}(T)$ are

$$T = (T^j_i) = \begin{pmatrix} d & kqb \\ -kp & a \end{pmatrix}, \quad \mathfrak{J}(T) = (\mathfrak{J}(T^j_i)) = \begin{pmatrix} d & kqb \\ -kp & a \end{pmatrix}.$$ 

One can easily check that $t \cdot T = D \cdot \text{id} = \mathfrak{J}(T) \cdot t$, where $t = (t^j_i) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

3.5 A non-quadratic Nichols algebra

Another feature of [AJG] is the presentation by generators and relations of families of Nichols algebras having quadratic relations. In [X], the author presents another example over a quantum plane considered in loc. cit., but with no quadratic relations. It is a finite-dimensional Nichols algebra with all relations of order bigger than 2. As example, we compute the quantum determinant for the Nichols algebra associated with the quantum plane $V_{4,1}$.

Let $\xi$ be a primitive 6-root of unity and write $\xi = -\omega$, with $\omega$ a primitive 3-root of 1. Let $\mathfrak{B}$ be the $\mathbb{k}$-algebra generated by $x$ and $y$ with relations

$$x^3 = 0, \quad y^3 - x^2 y - yx^2 + xyx = 0, \quad y^2 x + xy^2 - yxy = 0, \quad \xi x^2 y + \xi^5 yx^2 + xyx = 0.$$ 

The volume element is $b = x^2 yxy^2$, and the quantum determinant for $(t^j_i)_{i,j} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, is

$$D = (-\omega + \omega^3) b^2 dbdc + (\omega + \omega^5) b^2 dbdc + (-\omega - \omega^2) b^2 dbdc + (\omega - \omega^2) b^2 dbdc + \omega b^2 dbc^2$$
$$+ \omega^2 b^2 c^2 ad + (\omega^2 + \omega^4) bdbd + \omega bdbd - bdbd - dbd + \omega bdbd + (\omega + \omega^2) dbdbd$$
$$- \omega^2 dbd + abd - abd - \omega^2 abd + \omega^2 abd + a^2 dbd + a^2 dbd.$$
3.6 Two examples of the non-injectiveness of the canonical map \(\iota: A(c) \to H(c)\)

3.6.1 Commutative example

We thank Peter Schauenburg for providing us this example, together with an argument. Nevertheless, we exhibit a different argument to show that the canonical map is not injective. Consider the vector space \(V = \mathbb{k}x \oplus \mathbb{k}y\) with the following braiding

\[
c(x \otimes x) = x \otimes x, \quad c(x \otimes y) = y \otimes x, \quad c(y \otimes x) = x \otimes y, \quad c(y \otimes y) = 2y \otimes y.
\]

The FRT relations of \(A(c)\) are

\[
ba - ab = 0, \quad b^2 = 0, \quad bd = 0, \quad ca - ac = 0, \quad cb - bc = 0,
\]

\[
c^2 = 0, \quad cd = 0, \quad da - ad = 0, \quad db - 2bd = 0, \quad dc - 2cd = 0,
\]

which are equivalent to

\[
ab = ba, \quad ac = ca, \quad ad = da, \quad bc = cb, \quad 0 = bd = db = cd = dc = b^2 = c^2.
\]

Thus, \(A(c) = A = \mathbb{k}[a, b, c, d]/(b^2, c^2, bd, cd)\) is a commutative bialgebra. Assume \(\mathbb{k}\) is algebraically closed of characteristic zero. Then, it is clear that \(A\) cannot inject into a Hopf algebra because it has nilpotent elements. Also, a direct proof (valid on any characteristic) in this particular case can be given as follows: the bialgebra \(A\) is a quotient of \(\mathbb{k}[a, b, c, d] = O(M_2)\), thus \(D := ad - bc\) is a group-like element in \(A\). Notice that \(\Delta V\) is not the Nichols algebra of \((V, qc)\) for any \(q \in \mathbb{k}^\times\), because the former is finite-dimensional and the latter infinite-dimensional, but nevertheless it is a weakly graded-Frobenius algebra for \(A\). Since \(b^2 = 0 = bd\), it follows that \(bD = 0\) and \(cD = 0\). Thus, \(D\) is a zero divisor. If there is a bialgebra map \(f : A \to K\) with \(K\) a Hopf algebra, then \(f(D)\) is invertible and so \(f(b) = 0\).

Also, being \(A\) commutative, we have that \(\iota = \text{id}\), and the left inverse of a matrix with commuting entries is the same as a right inverse, so hypothesis of Theorem \(\square\) are fullfilled. Hence, \(H(c)\) is a Hopf algebra. In this concrete example, we see clearly that \(b\) and \(c\) are killed when inverting \(D\), and we get the isomorphism

\[
H(c) \cong \frac{A}{(b = c = 0)}[[ad - bc]^{-1}] = \mathbb{k}[a, d]((ad)^{-1}) = \mathbb{k}[a^{\pm 1}, d^{\pm 1}],
\]

with \(\Delta(a) = a \otimes a\) and \(\Delta(d) = d \otimes d\), so \(H(c) \cong \mathbb{k}[\mathbb{Z} \times \mathbb{Z}]\).

3.6.2 Quantum linear spaces

We end the paper with another example that the canonical map is not necessarily injective. From our calculations one obtains another proof, under certain hypothesis on the braiding, of the very well-known fact that a Nichols algebra associated with a quantum linear space of dimension \(n\) is realizable as a braided Hopf algebra in the category of \(\mathbb{k}[\mathbb{Z}^n]\)-comodules.

Let \(V\) be a finite-dimensional vector space with basis \(\{x_i\}_{1 \leq i \leq n}\) and consider the following diagonal braiding on it:

\[
c(x_i \otimes x_j) = q_{ij}x_j \otimes x_i \quad \text{for all } 1 \leq i, j \leq n,
\]

where \(q_{ij} \in \mathbb{k}^\times\) satisfy that \(q_{ij}q_{ji} = 1\) for \(i \neq j\) and \(q_{ii}\) are primitive \(N_i\)-roots of 1, with \(N_i \in \mathbb{N}\) and \(N_i > 1\). Then the Nichols algebra \(\mathfrak{B}(V, c)\) has generators \(x_1, \ldots, x_n\) and relations

\[
x_ix_j = q_{ij}x_jx_i, \quad x_i^{N_i} = 0.
\]
In particular, \( \dim \mathfrak{B}(V, c) = \prod_{i=1}^{n} N_i \), and a volume element is given by \( b = x_1^{N_1-1} \cdots x_n^{N_n-1} \). Notice that all \( N_i \) may be different, and not necessarily equal to \( 2 \). That is, this is a family of non-quadratic, non-homogeneous algebras.

Recall that, for a braiding \( c \) of diagonal type, the FRT relations of \( A(c) \) are given by

\[
q_{k\ell}t_{ij}^{k\ell} = q_{ij}t_{ij}^{k\ell} \quad \text{for all } 1 \leq i, j \leq n.
\]

This implies in particular that \( A(c) \) is non-commutative if \( q_{k\ell} \neq q_{ij} \). Besides, it holds that \( q_{k\ell}t_{ij}^{k\ell} = q_{ij}t_{ij}^{k\ell} \) and consequently

\[
t_i^{k\ell} t_j = q_{k\ell}^- q_{ij} t_{ij}^{-1} t_i^{k\ell} = q_{k\ell}^- q_{ij} t_{ij}^{-1} t_i^{k\ell}.
\]

By our assumptions on the braiding, this is nothing else than \( t_i^{k\ell} t_j = t_i^{k\ell} t_j \) if \( i \neq j \) and \( k \neq \ell \). On the other hand, if \( i = j \), we get

\[
t_i^{k\ell} t_i = q_{k\ell}^- q_{ij} q_{ij} t_i^{k\ell}.
\]

Thus, for \( k \neq \ell \) we obtain that \( t_i^{k\ell} t_i = q_{ii}^{2} t_i t_i \). If moreover \( N_i \neq 2 \), it holds that \( t_i^{k\ell} t_i = 0 \). Similarly, we have that

\[
t_i^{k\ell} t_i = q_{k\ell}^{-2} q_{ii} q_{ii}^{2} t_i^{k\ell} \quad \text{for } i \neq j \text{ and } k = \ell,
\]

\[
(t_i^{k\ell})^2 = q_{k\ell}^{-2} q_{ii} q_{ii}^{2} (t_i^{k\ell})^2 \quad \text{for } i = j \text{ and } k = \ell.
\]

From the considerations above we get the following lemma:

**Lemma 3.3.** Under the assumptions above, for \( i \neq j \) we have:

(a) If \( q_{ij}^{-2} q_{ii}^{-1} \neq 1 \) then \( (t_i^{k\ell})^2 = 0 \).

(b) If \( N_k \neq 2 \), then \( t_i^{k\ell} t_k = 0 = t_k^{k\ell} t_k \).

**Corollary 3.4.** Assume that for \( i \neq j \) it holds that \( q_{ij}^{-2} q_{ii}^{-1} \neq 1 \) and \( N_k \neq 2 \) for all \( 1 \leq k \leq n \). Then

(a) The quantum determinant is \( D = (t_1^1)^{N_1-1} (t_2^2)^{N_2-1} \cdots (t_n^n)^{N_n-1} \).

(b) \( t_i^j = 0 \) for all \( i \neq j \), and \( t_i^k \) is group-like for all \( 1 \leq k \leq n \) as elements in \( H(c) \).

(c) \( H(c) \cong \mathbb{K}[t_1^1, (t_2^2)^\pm 1, \ldots, (t_n^n)^\pm 1] \cong \mathbb{K}[\mathbb{Z}^n] \). In particular, the canonical map is not injective if \( q_{k\ell} \neq q_{ij} \) for some \( 1 \leq i, j, k, \ell \leq n \).

**Proof.** (a) The claim follows by a direct computation. Indeed, by Lemma 3.3 for \( 2 \leq \ell \leq N_k - 1 \) one has that

\[
\lambda(x_k^\ell) = \sum_{i_1, \ldots, i_\ell} (t_i^1 \cdots t_i^\ell) \otimes x_{i_1} x_{i_2} \cdots x_{i_\ell} = (t_k^\ell) \otimes x_k^\ell.
\]

In particular, this implies that \( (t_k^\ell) \) is a group-like element for all \( 1 \leq k \leq n \) and \( 2 \leq \ell \leq N_k - 1 \). Hence, \( \lambda(x_1^{N_1-1} \cdots x_n^{N_n-1}) = (t_1^1)^{N_1-1} (t_2^2)^{N_2-1} \cdots (t_n^n)^{N_n-1} \otimes x_1^{N_1-1} \cdots x_n^{N_n-1} \) and the assertion is proved.

(b) Since \( q_{k\ell} t_i^{k\ell} t_j = q_{ij} t_j^{k\ell} t_i \), it follows that \( q_{ij} t_i^{k\ell} t_j = q_{ij} t_j^{k\ell} t_i \), which implies that \( t_i^{k\ell} t_j = t_j^{k\ell} t_i \) for all \( 1 \leq i, j \leq n \). Thus, for all \( 1 \leq k \leq n \) we may write \( D = (t_k^1)^{N_k-1} D' \) for some \( D' \in A(c) \).

Since \( D \) is invertible in \( H(c) \) and \( N_k > 1 \), we have that \( t_k^k \) is a unit in \( H(c) \). On the other hand, as \( t_i^{k\ell} t_j = 0 \) for \( i \neq j \), it follows that \( t_i^{k\ell} t_k = 0 \) for \( i \neq k \), from which follows that \( t_k^k = 0 \) for \( i \neq k \).

The last claim follows from the very definition of the comultiplication in \( H(c) \).

(c) From the considerations above, we obtain

\[
H(c) = \frac{A}{(t_i^j : i \neq j)} [D^{-1}] = k[t_1^1, t_2^2, \ldots, t_n^n][D^{-1}] = k[(t_1^1)^{\pm 1}, (t_2^2)^{\pm 1}, \ldots, (t_n^n)^{\pm 1}] \cong k[\mathbb{Z}^n].
\]
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