A Benchmark Calculation for the Nd Scattering with a Model Three-Body Force

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Using the complex energy method, the problem of nucleon-deuteron scattering is solved with a simple three-body force having a separable form. Our results are compared with the results of modern direct two-variable calculations and a good agreement is found. This forms a firm base for other applications of the complex energy method.

§1. Introduction

Faddeev calculations of nucleon-deuteron (Nd) scattering with a three-nucleon force (3NF) have been performed for many years.\textsuperscript{1} Modern nucleon-nucleon potential stem from the chiral effective field theory.\textsuperscript{2} This theory gives us also various 3NFs but for many years numerous calculations were performed with the Tucson-Melbourne 3NF.\textsuperscript{3} There are two alternative formulations leading to the scattering amplitude with 3NF in the momentum space.\textsuperscript{4} One of them was introduced by Glöckle and Brandenburg.\textsuperscript{5} In this scheme the Alt-Grassberger-Sandhas (AGS) equation for the elastic scattering transition operator $U$ is written as\textsuperscript{5}

$$U = P G_0^{-1} \Big[ (1 + P) t_4 + P t_1 G_0 U + (1 + P) t_4 G_0 t_1 G_0 U \Big],$$

(1.1)

where $P$, $t_1$, $G_0$ and $t_4$ are the permutation operator ($P \equiv P_{12} P_{23} + P_{13} P_{23}$), the two-body t-matrix, the free Green’s function and the t-matrix generated by a 3NF, respectively.

The other approach has been used by the Bochum-Kraków group\textsuperscript{1} (BK). Here the equation for $U$ has the following form:

$$U = P G_0^{-1} + (1 + P) V_4^{(1)} (1 + P) + P t_1 G_0 U + (1 + P) V_4^{(1)} (1 + P) G_0 t_1 G_0 U,$$

(1.2)

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where $V_4^{(1)}$ is a part of 3NF which is symmetric under the exchange of nucleons 2 and 3 so the full 3NF, $V_4$, is given as $V_4 = V_4^{(1)}(1 + P)$. In practice, first the auxiliary equation for the three-body operator $T$ is solved,

$$T|\phi\rangle = t_1 P|\phi\rangle + (1 + t_1 G_0) V_4 |\phi\rangle + t_1 G_0 P T|\phi\rangle + (1 + t_1 G_0) V_4 G_0 T|\phi\rangle,$$  \hspace{1cm} (1.3)

with the incoming momentum space state $|\phi\rangle$ composed of the relative nucleon-deuteron motion and the deuteron wave function. Using this $T$, the elastic transition operator $U$ is given as

$$U|\phi\rangle = P G_0^{-1} |\phi\rangle + PT|\phi\rangle + V_4 |\phi\rangle + V_4 G_0 T|\phi\rangle.$$  \hspace{1cm} (1.4)

The consequence of employing Eq. (1.3) is that the $t$-matrix of the 3NF, $t_4$, does not appear in this formulation. The formalism of Eq. (1.1) has already been used with a simple separable 3NF [3]. In that calculation, however, the $t$-matrix of the 3NF was approximated by taking $t_4 = V_4$. It is the aim of the present investigation to show that both schemes, (1.1) and (1.3)-(1.4), yield the same numerical results under a consistent separable approximation for the underlying dynamics.

In Section 2 we show details of our formalism which leads to an analytical $t$-matrix of 3NF and the modified Amado-Lovelace (AL) equation. In Section 3, using the Yamaguchi separable potential and a separable 3NF, we will demonstrate results for the Nd elastic scattering cross section and the neutron-deuteron scattering length. These results will be compared with the results of standard calculations using the general two-variable formulation [1]. In Section 4 we give a short summary.

§2. Modified Amado-Lovelace equation

We first rewrite Eq. (1.1) as [3]

$$\tilde{U} = t_1 G_0 P t_1 + t_1 G_0 (1 + P) t_4 G_0 t_1 + t_1 G_0 P \tilde{U} + t_1 G_0 (1 + P) t_4 G_0 \tilde{U},$$  \hspace{1cm} (2.1)

where $\tilde{U}$ is related to $U$ via $\tilde{U} = t_1 G_0 U G_0 t_1$ and the $t$-matrix of the 3NF $t_4$ is the solution of the Lippmann-Schwinger equation: $t_4 = V_4 + V_4 G_0 t_4$. Next we introduce the so-called separable approximation for the two-body and three-body interactions. We choose first the Yamaguchi formfactor [3] $g_c(p)$, for the pure S-wave two-body potential:

$$v_c(p, p') = -\lambda_c g_c(p) g_c(p') \equiv -\lambda_c \frac{1}{p^2 + \beta^2} \frac{1}{p'^2 + \beta^2},$$  \hspace{1cm} (2.2)

where $p'$ ($p$) is the magnitude of the initial (final) relative momentum within the two-nucleon subsystem and $\lambda_c$ is the coupling coefficient. We choose the same value for the $\beta$ parameter (1.4488 fm$^{-1}$) both for the $^1S_0$ ($c = \phi$) and $^3S_1$ ($c = d$) states. The coupling coefficient $\lambda_d$ for the $^3S_1$ state is taken as $\lambda_d = 8\pi\beta(\alpha + \beta)^2/m$ to get the experimental deuteron binding energy with $\alpha = 0.2316$ fm$^{-1}$, and the other coupling coefficient $\lambda_\phi = 8\pi\beta(\alpha_0 - \beta)^2/m$ for the $^1S_0$ state allows us to reproduce the nucleon-nucleon phase shift parameters with $\alpha_0 = -0.040$ fm$^{-1}$. The Lippmann-Schwinger equation, $t_1 = v + v G_0 t_1$, is explicitly given as:
\[ t_1(p, p'; E) = v(p, p') + \int_0^\infty \frac{v(p, p')t_1(p'', p'; E) \, p'^2 \, dp''}{E - p'^2/m + i\epsilon} \]  

(2.3)

where \( m \) is the nucleon mass (1/\( m = 41.47 \) MeV \( \text{fm}^2 \)). It is well known\(^3\) that the t-matrix \( t_1 \) of a separable potential can be calculated analytically and has also a separable form \((t_1(p, p'; E) = g(p)\tau(E)g(p'))\).

In the case of the 3NF we assume the following S-wave phenomenological separable potential

\[ V_4(p, q, p', q') = \lambda_4 h(p, q)h(p', q') = \lambda_4 \frac{1}{(p^2 + \frac{4}{3}q^2 + \Lambda^2)^2} \frac{1}{(p'^2 + \frac{4}{3}q'^2 + \Lambda^2)^2}, \]  

(2.4)

where \( q \) is the Jacobi momentum for the spectator particle and \( \lambda_4 \) is the 3NF coupling coefficient. The choice of the \( \Lambda \) parameter will be discussed in Section 3. Like the two-body t-matrix, also the 3NF t-matrix \( t_4 \) can be obtained analytically, again in a separable form: \( t_4(p, q, p', q'; E) = h(p, q)\tau_4(E)h(p', q') \). The derivation is presented in the Appendix.

Using these separable formulas for \( t_1 \) and \( t_4 \) leads to a modified AL equation:

\[
X_{c,c'}(q, q_0; E) = Z_{c,c'}(q, q_0; E) + \sum_{c''} \int_0^\infty Z_{c,c''}(q, q''; E)\tau_{c''}(E - \frac{3q''^2}{4m})X_{c'', c'}(q'', q_0; E) \frac{q''^2 \, dq''}{2\pi^2}. \]  

(2.5)

The amplitudes \( X_{c,c'} \) and \( Z_{c,c'} \) are written as

\[
\langle p, q, c|\tilde{U}|p', q', c'\rangle = g_c(p)\tau_c(E - \frac{3q^2}{4m})X_{c,c'}(q, q'; E)\tau_{c'}(E - \frac{3q'^2}{4m})g_{c'}(p'), \]  

(2.6)

\[
\langle p, q, c|t_4G_0Pt_1 + t_4G_0(1 + P)t_4G_0t_1|p', q', c'\rangle \\
= g_c(p)\tau_c(E - \frac{3q^2}{4m})Z_{c,c'}(q, q'; E)\tau_{c'}(E - \frac{3q'^2}{4m})g_{c'}(p') \\
= g_c(p)\tau_c(E - \frac{3q^2}{4m})\{Z_{c,c'}^{(2)}(q, q'; E) + Z_{c,c'}^{(3)}(q, q'; E)\}\tau_{c'}(E - \frac{3q'^2}{4m})g_{c'}(p') \]  

(2.7)

with

\[
Z_{c,c'}^{(2)}(q, q'; E) = \int_{-1}^1 \frac{g_c(p(q, q'))g_{c'}(p'(q', q))}{E - (q^2 + q'^2 + qq')/m + i\epsilon} P_L(x) \, dx \, \Delta_{cc'} \]  

(2.8)

and

\[
Z_{c,c'}^{(3)}(q, q'; E) = \int_0^\infty \frac{g_c(p)h(p, q)}{E - (p^2 + 3q^2/4)/m + i\epsilon} \frac{p^2 \, dp}{2\pi^2} \tau_4(E) \times \int_0^\infty \frac{h(p', q')g_{c'}(p')}{E - (p'^2 + 3q'^2/4)/m + i\epsilon} \frac{p'^2 \, dp'}{2\pi^2} \delta_{cc'} \delta_{L0}(1 + 2\Delta_{cc'}) \]  

(2.9)

where \( P_L(x) \) is the Legendre polynomial. Here \( \Delta_{cc'} \) is the spin and isospin recoupling coefficient defined in the Appendix. We would like to emphasize that the quantities \( Z^{(2)} \) and \( Z^{(3)} \) can be calculated analytically (see the Appendix for details). If the \( Z_{c,c'}^{(3)} \) term is omitted, Eq. (2.5) is brought back to the original AL form.
§3. Cross Section and Scattering Length of Nd elastic scattering

Here we would like to demonstrate some results. The triton binding energy calculated with the Yamaguchi potential \(^8\) is -13.0 MeV. Comparing this number with the experimental result (-8.48 MeV) clearly shows overbinding. This is because the Yamaguchi potential is too simple and furthermore restricted to act only in S-waves. Corresponding numbers obtained with modern potentials show in turn underbinding by about 1 MeV. Since our aim is to check consistency between the numerical results obtained with Eq. (1.1) and Eqs.(1.2)-(1.4), we introduce a simple 3NF as given in Eq. (2.4). That 3NF contains two parameters (\(\lambda_4\) and \(\Lambda\)), which are chosen to reproduce the experimental triton binding energy 8.48 MeV. Figure 1 shows the relation between \(\lambda_4\) and \(\Lambda\) under this condition.

![Fig. 1. The relation between the parameters \(\Lambda\) and \(\lambda_4\) from Eq. (2.4), which is obtained by fixing the three-nucleon binding energy to \(E_b = -8.48\) MeV.](image)

Before introducing 3NF our result for the doublet scattering length is \(2a = -2.17\) fm, which is very different from the experimental data, 0.65±0.04 fm\(^9\). In the case of the quartet scattering length our prediction is \(4a = 6.86\) fm, which is rather close to the data (6.35±0.02 fm)\(^9\).

Figure 2 shows results for \(2a\) when the 3NF is included. Though the doublet scattering length is sensitive to the strength of the 3NF, within the chosen range of the \(\Lambda\) parameter the experimental value can not be reached. However, inclusion of the 3NF clearly improves the situation. On the other hand, the quartet scattering length \(4a\) hardly changes with \(\Lambda\) (not shown).

In the actual calculation we take \(\Lambda = 2\) fm\(^{-1}\) and \(\lambda_4 = 16520\) MeVfm\(^{-2}\). The Nd elastic scattering differential cross section is calculated at \(E_{lab} = 14.1\) MeV, using Eq. (1.1) and then separately with Eqs.(1.2)-(1.4). The Coulomb force is neglected in both our calculations. In order to achieve convergence, all 3N states with total angular momentum up to 31/2 for the both parities have been included. We use 100 \(q\) Gaussian integral points and 20 \(x\) points for the angular integration. The maximum value of the \(q\) momentum is set to \(q_{max} = 20\) fm\(^{-1}\). The smallest \(\epsilon\), which appears in the Green’s functions in the complex energy method\(^7\) is typically 0.01 fm\(^{-1}\).

In Fig. 3 the differential cross section for elastic Nd scattering is demonstrated, comparing calculations with and without the 3NF. The theoretical predictions ob-
tained with the two different schemes agree very well with each other (up to 1 %) but yield only a fair description of the experimental data.

§4. Conclusions

The aim of the present investigation was twofold. First of all we wanted to compare two calculational schemes, which deal with Nd scattering under the inclusion of a 3NF. To this aim we chose a simple separable form of the nucleon-nucleon potential and the 3NF. In this case, in the scheme using Eq. (1.1), some parts of the calculations can be done analytically, which leads to very accurate results. The comparison of these results with the results basing on Eqs. (1.2)–(1.4) provides a very good test for the numerical performance of this second method, which treats directly any nucleon-nucleon and 3N forces. Secondly, since the first numerical framework uses the complex energy method, we could confirm that this method can be successfully used in the few-nucleon calculations. We plan to apply the complex energy method in the three-dimensional treatment of two- and three-nucleon systems.

Fig. 3. The differential cross section of elastic Nd scattering at $E_{\text{lab}} = 14.1 \text{MeV}$. The short-dashed (dotted) line shows the theoretical results of the separable scheme (BK scheme) without 3NF. The solid (long-dashed) line demonstrates the results of the separable scheme (BK scheme) with the inclusion of 3NF. The cross $^+$ ($^x$) marks are for the experimental data from\textsuperscript{10} $^+$ ($^x$).

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Appendix A

Analytical expressions

We show some analytical expressions used in the present investigation. The 3NF of Eq. (2.4) satisfies

$$t_4(p, q, p', q'; E) = V_4(p, q, p', q')$$
\[ \log \left( -p_i^+ p_i^- p_i^0 q_i^0 \right) \]

Assuming the separable form \( \tau_4(p, q, p', q'; E) = h(p, q) \tau_4(E) h(p', q') \) we have

\[ \tau_4(E) = \lambda_4 + \lambda_4 \tau_4(E) I_4(E), \] (A.2)

where the integration \( I_4 \) is performed using the so-called "hyper coordinates" \( (p^2 = p^2 + 3q^2/4, \tan \theta = \sqrt{3q}/(2p)) \),

\[ I_4(E) \equiv \int_0^\infty \int_0^\infty \frac{h(p, q)^2}{E - (p^2 + 3q^2/4)/m + \imath \epsilon} \frac{dp dq}{2\pi^2} = \frac{1}{2} \frac{1}{\sqrt{3} 4\pi^2} \int_0^\infty \frac{\rho^3 dp}{m(E - \rho^2/m + \imath \epsilon)(\rho^2 + A^2)^2} \int_0^{\pi/2} \sin^2 \theta \cos^2 \theta d\theta \]

\[ = \frac{384\sqrt{3}\pi^3 A^2(Em + A^2)^4}{(Em + A^2)(2E^2 m^2 - 5EmA^2 - A^4) + 6A^2 E^2 m^2 (\log(-Em) - \log(A^2))} \] (A.3)

where \( \log(-Em) = \log(Em) - \pi \imath \) in case of \( E > 0 \).

In order to calculate the \( Z_{cc'}^{(3)} \) terms we use

\[ \int_0^\infty \frac{p^2 dp}{2\pi^2} \frac{h(p, q)g(p)}{E - (p^2 + 3q^2/4)/m + \imath \epsilon} = -\frac{m(\beta - ik') + 2A'}{8\pi(\beta - ik')A'(\beta + A')^2(A' - ik')^2} \] (A.4)

with \( k' = \sqrt{mE - 3q^2/4} \) and \( A' = \sqrt{3q^2/4 + A^2} \).

Finally, the recoupling coefficients \( \Delta_{cc'} \) for spin doublet (\( I = \frac{1}{2} \)) and quartet (\( I = \frac{3}{2} \)) states in Eqs. [228]–[249] are

\[ \Delta_{cc'}^{I=\frac{1}{2}} = \left( \begin{array}{cc} \Delta_{dd} & \Delta_{d\phi} \\ \Delta_{d\phi} & \Delta_{\phi\phi} \end{array} \right) \bigg|_{I=1/2} = \left( \begin{array}{cc} \frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{array} \right), \quad \Delta_{cc'}^{I=\frac{3}{2}} = \left( \begin{array}{cc} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{array} \right) \] (A.5)

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