The extension problem for partial Boolean structures in Quantum Mechanics

Costantino Budroni\textsuperscript{1} and Giovanni Morchio\textsuperscript{1,2}
\textsuperscript{1) Dipartimento di Fisica, Università di Pisa, Italy}
\textsuperscript{2) INFN, Sezione di Pisa, Italy}

Alternative partial Boolean structures, implicit in the discussion of classical representability of sets of quantum mechanical predictions, are characterized, with definite general conclusions on the equivalence of the approaches going back to Bell and Kochen-Specker. An algebraic approach is presented, allowing for a discussion of partial classical extension, amounting to reduction of the “number of contexts”, classical representability arising as a special case. As a result, known techniques are generalized and some of the associated computational difficulties overcome. The implications on the discussion of Boole-Bell inequalities are indicated.

I. INTRODUCTION

The central role of partial classical (Boolean) structures in Quantum Mechanics (QM) has been recognized by many authors, both for their abstract mathematical implications (constraints on truth and probability assignments, Gleason theorem\textsuperscript{[1]}; Kochen-Specker theorem\textsuperscript{[2]}) and for their fundamental role in the analysis of empirical correlations (B\textsuperscript{[3]}, Mermin\textsuperscript{[4]}, Pitowsky\textsuperscript{[5]}).

Most notably, Gleason’s result shows that the partial Boolean structure of QM given by the set of Boolean algebras of projections of a Hilbert space of dimension \( \geq 3 \) forces the corresponding sets of probability measures to be given by quantum states.

Maximal Boolean algebras define contexts, corresponding to the QM notion of jointly measurable observables; more generally, they play the role of maximal sets of observables for which a classical description is given. Independently of QM, they arise in general as a result of compatibility relations (which can be formulated\textsuperscript{[6]} in terms of few properties of sequences of experiments) characterizing sets of observables on which logical operations can be defined.

A basic problem behind the introduction of partial Boolean structures is its necessity, i.e. whether their logical and probabilistic structures are compatible or not with a classical probability theory. More generally, one may ask whether partial classical extensions exist, giving joint probabilistic predictions for certain sets of incompatible observables.

In this language, the classical “hidden variable” problem concerns the existence of a classical representation, identifying partial Boolean structures and measures on them as restrictions of a single probability theory. More generally, the existence of a classical description for sets larger than the given contexts plays an important role in the discussion of the interpretation of QM and gives rise to a more general extension problem.

The object of the present paper is a general analysis of partial classical structures, given by partial Boolean algebras (PBAs) and partial probability theories (PPTs) and of their extension problem. As we shall see, our analysis will cover rather general notions and relations, with substantial implications on the analysis of the interpretation of QM, which seem to have been overlooked.

In Sect.\textsuperscript{[I]} the notions of PBA, PPT and extension are introduced and discussed.

In Sect.\textsuperscript{[II]} alternative PBAs and PPTs are associated to QM observables and predictions. In fact, one may either consider collections of Boolean algebras of commuting projections, or partial Boolean algebras freely generated by yes/no observables and treat Boolean relations as empirical relations, induced by “quotients” associated to experiments.

We will show that such quotients are well defined for PBAs arising in the QM case and lead from free algebras to projection algebras. The distinction between free abstract algebras and concrete projection algebras is essential in order to obtain a unification (Theorem\textsuperscript{[III]} below) of Kochen-Specker-type and Bell-type approaches to the investigation of classical representability.

The aim of rest of the paper is to discuss the problem of the extension of partial Boolean structures. Classical representability is the most studied issue and is usually discussed in terms of Bell-like inequalities\textsuperscript{[5]}. Their violation in QM excludes such a representability in general; on the contrary, as we shall see, non-trivial partial extensions arise automatically in many cases; on one side this gives an interpretation of Bell-like inequalities as conditions for further extensions and allows for a more constructive discussion of their violation; on the other side, such extensions give rise to a simplification of the computation of conditions of classical representability, reducing the problem to the compatibility of automatically provided solutions for certain subproblems.

In Sect.\textsuperscript{[IV]} the extension problem is discussed for all the 3 and 4 observables cases which are relevant for QM.

In Sect.\textsuperscript{[V]} we present general results and techniques for the computation of extensibility conditions in terms of topological properties of compatibility relations.

In Appendix\textsuperscript{[A]} we recall some basic notions and results for Boolean algebras.

In Appendix\textsuperscript{[B]} we collect results that explicitly relate the correlation polytope approach to our algebraic approach.

In Appendix\textsuperscript{[C]} we recall Horn and Tarski’s notion of partial measure which provides, in our framework, an extensibility criterion for a large class of partial probability theories.
II. PARTIAL PROBABILITY THEORIES

We start introducing PBAs and PPTs. The basic notions go back to Kochen and Specker; our approach is more general, since we do not assume a property, indicated in the following as $(K-S)$, which holds for Boolean structures in QM; its role in the extension problem will be discussed below.

A partial Boolean algebra (PBA) is a set $X$ together with a non-empty family $\mathcal{F}$ of Boolean algebras, $\mathcal{F} \equiv \{\mathcal{B}_i\}_{i \in I}$, such that $\bigcup \mathcal{B}_i = X$, that satisfy

\[(P_1)\text{ for every }\mathcal{B}_i,\mathcal{B}_j \in \mathcal{F}, \mathcal{B}_i \cap \mathcal{B}_j \in \mathcal{F}\text{ and the Boolean operations }(\cap_i, \cup_i, ^c_i)\text{, } (\cap_j, \cup_j, ^c_j)\text{ of }\mathcal{B}_i\text{ and }\mathcal{B}_j\text{ coincide on it.}\]

Without loss of generality we can also assume the following

\[(P_2)\text{ for all }\mathcal{B}_i \in \mathcal{F}, \text{ each Boolean subalgebra of }\mathcal{B}_i\text{ belongs to }\mathcal{F}.\]

By $(P_1)$, Boolean operations, when defined, are unique and will be denoted by $(\cap, \cup, ^c)$; we shall denote a partial Boolean algebra by $(X, \{\mathcal{B}_i\}_{i \in I})$, or simply by $\{\mathcal{B}_i\}_{i \in I}$. In the following we shall consider only finite partial Boolean algebras. Their elements will also be called observables.

Given a partial Boolean algebra $(X, \{\mathcal{B}_i\})$, a state is defined as a map $f: X \rightarrow [0, 1]$ such that $f|_{\mathcal{B}_i}$ is a normalized measure on the Boolean algebra $\mathcal{B}_i$ for all $i$. Equivalently, a state is given by a collection of compatible probability measures $\{\mu_i\}$, i.e. measures coinciding on intersections of Boolean algebras, one for each $\mathcal{B}_i$.

A partial probability theory (PPT) is a pair $((X, \{\mathcal{B}_i\}); f)$, where $(X, \{\mathcal{B}_i\})$ is a partial Boolean algebra and $f$ is a state defined on it. Equivalently, a partial probability theory can be denoted with $\{(X, \{\mathcal{B}_i\}); \{\mu_i\}\}$, where $\mu_i = f|_{\mathcal{B}_i}$, or simply by $\{\mathcal{B}_i\}; \{\mu_i\}$.

It can be easily checked that the above properties are satisfied by the set of all orthogonal projections in a Hilbert space of arbitrary dimension, with Boolean operations defined by

\[P \cap Q \equiv PQ, \quad P \cup Q \equiv P + Q - PQ, \quad P^c \equiv 1 - P,\]

for all pairs $P, Q$ of commuting projections. If one considers a finite set of projections, the result of the iteration of the above Boolean operations (on commuting projections) is still a finite set and a partial Boolean algebra.

Moreover, given a set of projections, the corresponding predictions given by a QM state define a PPT on the generated PBA. In fact, given a PBA of projections on a Hilbert space $\mathcal{H}$, by the spectral theorem, a quantum mechanical state $\psi$ defines a state $f_\psi$ on it, given by

\[f_\psi(P) = (\psi, P\psi).\]

The generalization to density matrices is obvious.

We shall name the so obtained PPTs projection algebra partial probability theories. We shall see in Sect. III that they are not the only PPTs that can be associated to QM predictions, other choices being implicit in different approaches to contextuality in QM.

It is interesting to notice that, in QM, PBAs of projections also satisfy the following property

\[(K-S)\text{ if }A_1, \ldots, A_n\text{ are elements of }X\text{ such that any two of them belong to a common algebra }\mathcal{B}_i\text{, then there is a }\mathcal{B}_k \in \mathcal{F}\text{ such that }A_1, \ldots, A_n \in \mathcal{B}_k;\]

which is actually part of the definition of partial Boolean algebra given by Kochen and Specker.

The reason for not assuming $(K-S)$ is that it seems to be only motivated by PBAs arising in QM. In a general theory of measurements, it makes perfectly sense to consider, for instance, three measurements such that every pair can be performed jointly, but it is impossible to perform jointly all the three. Moreover, PPTs arising in such a case are not in general given by a probability on a common Boolean algebra, and therefore property $K-S$ is a real restriction.

Given a PBA $(X, \{\mathcal{B}_i\})$, we shall call a context each maximal, with respect to inclusion, Boolean algebra of $\{\mathcal{B}_i\}$. Moreover, given $A, B \in X$, we shall say that $A$ and $B$ are compatible if they belong to a common context.

Given a subset $\mathcal{G} \subset X$, we shall say that $\mathcal{G}$ generates, or that $\mathcal{G}$ is a set of generators for $(X, \{\mathcal{B}_i\})$, if each maximal Boolean algebra of $\{\mathcal{B}_i\}$ is generated by a subset of $\mathcal{G}$.

Given two partial Boolean algebras $(X, \{\mathcal{B}_i\})$ and $(X', \{\mathcal{B}_j'\})$; we say that a function $\varphi: X \rightarrow X'$ is a homomorphism if for each $\mathcal{B}_i$, the image $\varphi(\mathcal{B}_i)$ belongs to $\{\mathcal{B}_j'\}$ and $\varphi|_{\mathcal{B}_i}$ is a homomorphism of Boolean algebras; moreover, if $\varphi$ is invertible, we say that $\varphi$ is an isomorphism. If $(X', \{\mathcal{B}_j'\})$ is a Boolean algebra (notice that a Boolean algebra is also a PBA) and the homomorphism $\varphi$ is an injection, we say that $\varphi$ is an embedding. Homomorphisms of $(X, \{\mathcal{B}_i\})$ into the Boolean algebra $\{0, 1\}$ define multiplicative states.

In the following, we shall analyze the possibility of extending a partial probability theory to additional algebras, reducing the number of contexts.

We shall say that $(X', \{\mathcal{B}_j'\})$ contains $(X, \{\mathcal{B}_i\})$ if $X \subset X'$ and $\{\mathcal{B}_i\} \subset \{\mathcal{B}_j'\}$.

We shall say that $(X', \{\mathcal{B}_j'\})$ extends $(X, \{\mathcal{B}_i\})$ if $(X', \{\mathcal{B}_j'\})$ contains $(X, \{\mathcal{B}_i\})$ and $X$ generates $(X', \{\mathcal{B}_j'\})$.

Similar notions apply to states. Given two PPTs $\mathcal{C} = ((X, \{\mathcal{B}_i\}); \{\mu_i\})$ and $\mathcal{C}' = ((X', \{\mathcal{B}_j'\}); \{\mu_j'\})$, we shall say that $\mathcal{C}'$ contains $\mathcal{C}$ if $(X', \{\mathcal{B}_j'\})$ contains $(X, \{\mathcal{B}_i\})$ and $\{\mu_i\} \subset \{\mu_j'\}$; we shall say that $\mathcal{C}$ extends $\mathcal{C}'$ if $(X', \{\mathcal{B}_j'\})$ extends $(X, \{\mathcal{B}_i\})$ and $\mathcal{C}'$ contains $\mathcal{C}$.
By classical representation of a PPT $C = ((X, \{\mathcal{B}_i\}); \{\mu_i\})$ we shall mean a Boolean algebra $\mathcal{B}$ and a (normalized) measure $\mu$ such that $(\mathcal{B}; \mu)$ extends $C$.

The fact that a PBA is not embeddable into a Boolean algebra is precisely the original form of the Kochen-Specker theorem. The minimality implicit in the above notion of extension reduces the multiplicity of classical representations in the sense of Kochen and Specker (not requiring that the PBA generates the Boolean algebra); however, a classical representation exists in our sense iff it exists in the K-S sense since clearly a PBA is embeddable in a Boolean algebra iff it can be extended to a Boolean algebra.

If its PBA $\{\mathcal{B}_i\}$ extends to a Boolean algebra $\mathcal{B}$, the existence of a classical representation of a PPT amounts to the extension problem of a function, induced by the corresponding state, defined on a subset of $\mathcal{B}$; the solution of this extension problem (with necessary and sufficient conditions) is then implicit in the work of Horn and Tarski, which is summarized in Appendix C. A PPT $C$ is an empirical quotient of $\{\mathcal{C}_k\}_{k \in K}$ if there exists an equivalence relation $\sim$ on $X = \bigcup_i \mathcal{B}_i$ such that

(i) when restricted to each Boolean algebra $\mathcal{B}_i$, $\sim$ coincides with the equivalence relation induced by the ideal $\mathcal{I}_i \equiv \{A \in \mathcal{B}_i | f_k(A) = 0 \text{ for all } k \in K\}$;

(ii) given $A \in \mathcal{B}_i$ and $B \in \mathcal{B}_j$, with $\mathcal{B}_i$ and $\mathcal{B}_j$ maximal, if $A \sim B$, then there exists $C \in \mathcal{B}_i \cap \mathcal{B}_j$ such that $A \sim C$ (and $B \sim C$ by transitivity);

(iii) the quotient set $X/\sim$ is a partial Boolean algebra isomorphic to the PBA $\tilde{X} = \bigcup_j \mathcal{B}_j$; by (i), this implies that the quotient preserves Boolean operations, namely for all $A, B \in X$, with $A$ and $B$ compatible, it holds $[A] \cap [B] = [A \cap B]$, where $[A]$ denotes the equivalence class of $A$ with respect to $\sim$, and analogous properties hold for $\cup$ and $\wedge$;

(iv) denoted with $\varphi : X/\sim \to \tilde{X}$ the isomorphism in (iii), it holds $f_k(A) = \tilde{f}_k(\varphi([A]))$, for all $k \in K$ and for all $A \in X$.

The above definition clearly applies in the classical case, i.e., when both $X$ and $\tilde{X}$ are Boolean algebras; we shall provide below less trivial examples.

We remark that, unlike the classical case, an equivalence relation on a PPT satisfying (i) and (iv) does not in general give rise to an empirical quotient; a counterexample can be constructed by considering a PPT given by the PBA consisting of three maximal Boolean algebras, generated respectively by the pairs of observables $\{A, B\}$, $\{B, C\}$ and $\{A, C\}$, together with the corresponding subalgebras, and a state $\rho$ that induces in the above Boolean algebras the identification $A \sim B$, $B \sim C$ and $C \sim A^c$.

In fact, if an empirical quotient exists, then by transitivity $A$ is identified with $A^c$ and therefore, by (i), both are identified with $\emptyset$; this contradicts $\tilde{f}(\varphi(\{1\})) = 1$.

The above notion of quotient may look too restrictive; on the contrary, it will turn out that all PPTs with a PBA admitting a complete set of states (see below) can be identified with quotients of PPTs associated to a collection of freely generated Boolean algebras, automatically embeddable into a Boolean algebra. This will imply that all extension problems in QM can be put in the H-T form.

### III. REDUCTION TO HORN-TARSKI PPTS

#### A. Empirical quotients of partial probability theories

The aim of the following discussion is to show how PBAs and PPTs provide a unification of the Kochen-Specker-type and Bell-type approaches to classical representability.

A fundamental role is played by the notion of empirical quotient; we shall briefly discuss it in classical probability theory and then we shall generalize it to PPTs.

Consider a classical probability theory defined by a finite Boolean algebra $\mathcal{B}$ and a probability measure $\mu$. If for two elements $A, B \in \mathcal{B}$ it holds $\mu(A \cap B^c) = \mu(A^c \cap B) = 0$, equivalently $\mu(A) = \mu(B) = \mu(A \cap B)$, it follows that every time $A$ happens also $B$ happens and conversely. In terms of conditional probabilities this can be written as $Pr(A|B) = Pr(B|A) = 1$. Therefore, in the situations described by the measure $\mu$, it makes sense to identify the events $A, B$ and $A \cap B$ with a single event since they “cannot be distinguished by any experiment”.

This procedure induces an equivalence relation $\sim_\mathcal{B}$ on $\mathcal{B}$, given by the ideal $\mathcal{I}_\mathcal{B} = \{A \in \mathcal{B} | \mu(A) = 0\}$, giving rise to the empirical quotient algebra $\mathcal{B} \equiv \mathcal{B}/\sim_\mathcal{B}$, $\mu$ induces a normalized measure $\tilde{\mu}$ on $\mathcal{B}$.

Similar notions, with identical interpretation, apply to the case of a finite Boolean algebra $\mathcal{B}$ and a collection of normalized measures $\{\mu_k\}_{k \in K}$, where $K$ may be any set of indices, through the ideal $\mathcal{I} = \{A \in \mathcal{B} | \mu_k(A) = 0 \text{ for all } k \in K\}$ (any $K$ being admissible since $\mathcal{B}$ is finite).

The extension of the above notions to the case of PPTs is not automatic and requires further conditions.

Given two collections of PPTs $\{\mathcal{C}_k\}_{k \in K} = \{(\{\mathcal{B}_1\}, f_k)\}_{k \in K}$ and $\{\mathcal{C}_k\}_{k \in K} = \{(\{\mathcal{B}_2\}, f_k)\}_{k \in K}$, we shall say that $\{\mathcal{C}_k\}_{k \in K}$ is an empirical quotient of $\{\mathcal{C}_k\}_{k \in K}$ if there exists an equivalence relation $\sim$ on $X = \bigcup_i \mathcal{B}_i$ such that

(i) when restricted to each Boolean algebra $\mathcal{B}_i$, $\sim$ coincides with the equivalence relation induced by the ideal $\mathcal{I}_i \equiv \{A \in \mathcal{B}_i | f_k(A) = 0 \text{ for all } k \in K\}$;

(ii) given $A \in \mathcal{B}_i$ and $B \in \mathcal{B}_j$, with $\mathcal{B}_i$ and $\mathcal{B}_j$ maximal, if $A \sim B$, then there exists $C \in \mathcal{B}_i \cap \mathcal{B}_j$ such that $A \sim C$ (and $B \sim C$ by transitivity);

(iii) the quotient set $X/\sim$ is a partial Boolean algebra isomorphic to the PBA $\tilde{X} = \bigcup_j \mathcal{B}_j$; by (i), this implies that the quotient preserves Boolean operations, namely for all $A, B \in X$, with $A$ and $B$ compatible, it holds $[A] \cap [B] = [A \cap B]$, where $[A]$ denotes the equivalence class of $A$ with respect to $\sim$, and analogous properties hold for $\cup$ and $\wedge$;

(iv) denoted with $\varphi : X/\sim \to \tilde{X}$ the isomorphism in (iii), it holds $f_k(A) = \tilde{f}_k(\varphi([A]))$, for all $k \in K$ and for all $A \in X$.

The above definition clearly applies in the classical case, i.e., when both $X$ and $\tilde{X}$ are Boolean algebras; we shall provide below less trivial examples.

We remark that, unlike the classical case, an equivalence relation on a PPT satisfying (i) and (iv) does not in general give rise to an empirical quotient; a counterexample can be constructed by considering a PPT given by the PBA consisting of three maximal Boolean algebras, generated respectively by the pairs of observables $\{A, B\}$, $\{B, C\}$ and $\{A, C\}$, together with the corresponding subalgebras, and a state $\rho$ that induces in the above Boolean algebras the identification $A \sim B$, $B \sim C$ and $C \sim A^c$.

In fact, if an empirical quotient exists, then by transitivity $A$ is identified with $A^c$ and therefore, by (i), both are identified with $\emptyset$; this contradicts $\tilde{f}(\varphi(\{1\})) = 1$.

The above notion of quotient may look too restrictive; on the contrary, it will turn out that all PPTs with a PBA admitting a complete set of states (see below) can be identified with quotients of PPTs associated to a collection of freely generated Boolean algebras, automatically embeddable into a Boolean algebra. This will imply that all extension problems in QM can be put in the H-T form.

#### B. Classical representations of partial probability theories and of their empirical quotients

An important role is played by the following notions.

Given a PBA $\{\mathcal{B}_i\}_{i \in I}$ and a collection of states $\{f_k\}_{k \in K}$, we shall say that the collection $\{f_k\}_{k \in K}$ is complete with respect to $\{\mathcal{B}_i\}_{i \in I}$ if for all $A \in X = \bigcup_i \mathcal{B}_i$,
with $A \neq 0$ there exists $f_k$ such that $f_k(A) \neq 0$. If, in addition, for all $A \neq B$, with $A, B \in X$, there exists $f_k$ such that $f_k(A) \neq f_k(B)$ then $\{f_k\}_{k \in K}$ is said to be separating for $\{\mathcal{B}_i\}_{i \in I}$.

Notice that for an empirical quotient $\{\tilde{C}_k\}_{k \in K} = \{\{\mathcal{B}_{ij} \in J; f_k\}\}_{k \in K}$, by (i) and (iv), $\{f_k\}_{k \in K}$ is always complete with respect to $\{\mathcal{B}_j\}_{j \in J}$.

The following result relates classical representations of PPTs with embeddings of PBAs associated to empirical quotients.

**Proposition III.1.** Given $\{C_k\}_{k \in K} = \{\{\mathcal{B}_{ij} \in J; f_k\}\}_{k \in K}$ and $\{\tilde{C}_k\}_{k \in K} = \{\{\tilde{B}_{ij} \in J; \tilde{f}_k\}\}_{k \in K}$, with $\{C_k\}_{k \in K}$ an empirical quotient of $\{C_{k0}\}_{k0 \in K}$ if there exists $k_0 \in K$ such that $C_{k0}$ admits a classical representation, then there exists a multiplicative state on $\{\tilde{B}_j\}$, i.e. a homomorphism $\delta_0 : \tilde{X} = \bigcup \tilde{\mathcal{B}}_j \rightarrow \{0,1\}$.

Moreover, if there exists $K' \subset K$ such that $\{f_k\}_{k \in K'}$ is separating for $\{\mathcal{B}_j\}_{j \in J}$ and $C_k$ admits a classical representation for every $k \in K'$, then $\{\tilde{B}_j\}_{j \in J}$ is embeddable into the Boolean algebra $\mathcal{B}$, the power set of a $N$-element set, where $N$ is the number of multiplicative states induced by classical representations of the states $\{f_k\}_{k \in K'}$.

**Proof** Let the Boolean algebra $\mathcal{B}$ together with the normalized measure $\mu$ be a classical representation for $C_{k0}$, then $\mu$ can be written as a convex combination of multiplicative measures (see Lemma A.2 below), namely

$$\mu = \sum_i \lambda_i \delta_i,$$

where the $\delta_i$’s are multiplicative measures and the $\lambda_i$’s are positive numbers that sum up to one. It follows that $\mu(A \cap B^c) = \mu(A^c \cap B) = 0$ for all $A, B \in X$ such that $A \sim B$ and $A$ and $B$ belong to a common Boolean $\mathcal{B}_{i0}$ in $\bigcup \mathcal{B}_j$; therefore $\delta_i(A \cap B^c) = \delta_i(A^c \cap B) = 0$ for each $\delta_i$ that appears in (2). Actually, the same holds even if $A$ and $B$ do not belong to a common maximal algebra of $\{\mathcal{B}_j\}$. In fact, by (ii), there exists an element $C$ in the intersection of the two maximal algebras containing $A$ and $B$ such that $A \sim C \sim B$ and the above statement follows from $A \cap B^c = (A \cap B^c \cap C) \cup (A \cap B^c \cap C^c)$.

It follows that $\delta_i(A) = \delta_i(B)$ for all $A, B \in X$ such that $A \sim B$ and for all $\delta_i$ appearing in (2); therefore each $\delta_i$ induces a well defined $\{0,1\}$-valued function on $\tilde{X}$. To conclude, we shall prove that such functions are homomorphisms when restricted to each algebra of $\{\tilde{B}_j\}$.

This follows from the isomorphism between $X$ and $X/\sim$ and the fact that each $\delta_i$ defines a multiplicative measure on $\mathcal{B}_i/\sim$ for all $\mathcal{B}_i$. In fact, given $A, B \in \mathcal{B}_i$, $[A] \cap [B] = [0]$ implies $\delta_i(A \cap B) = 0$ and therefore $\delta_i(A) + \delta_i(B) = \delta_i(A \cup B)$; each $\delta_i$ defines, therefore, a $\{0,1\}$-valued function on $\mathcal{B}_i/\sim$ which is additive on disjoint elements, i.e. a multiplicative measure on $\mathcal{B}_i/\sim$, which is a homomorphism with the Boolean algebra $\{0,1\}$ (see Lemma A.3).

The proof of the second part follows easily from the first part together with Theorem 0 of Ref. 2. □

**C. Partial probability theories as empirical quotients of free H-T theories**

We now show that any complete set of states on a PBA can be regarded as an empirical quotient of a collection of PPTs on a PBA which is embeddable in a (free) Boolean algebra, i.e. a collection of H-T PPTs.

Consider a collection of PPTs $\{\tilde{C}_k\}_{k \in K}$ such that $\{\tilde{f}_k\}_{k \in K}$ is complete, and take a subset $\tilde{G} = \{\tilde{A}_1, \ldots, \tilde{A}_n\} \subset \tilde{X} = \bigcup \tilde{\mathcal{B}}_j$ of generators of $\{\tilde{B}_j\}_{j \in J}$ satisfying the following property

$$\tilde{G} \text{ given } k \geq 1 \text{ maximal Boolean algebras } \tilde{B}_{i1}, \ldots, \tilde{B}_{in}, \text{ generated respectively by maximal subsets of compatible generators } \tilde{G}_{i1}, \ldots, \tilde{G}_{in} \subset \tilde{G}, \text{ such that } \tilde{B}_{i1} \cap \ldots \cap \tilde{B}_{in} \neq \{0,1\}, \text{ the set } \tilde{G}_{i1} \cap \cdots \cap \tilde{G}_{in} \text{ is not empty and it generates the Boolean algebra } \tilde{B}_{i1} \cap \cdots \cap \tilde{B}_{in};$$

notice that each maximal algebra is generated by a maximal subset of compatible generators and that the above choice is always possible since one can take $\tilde{G} = \tilde{X}$. The role of this property will be clarified below.

Denote with $\{G_i\}$ the collection of subsets of compatible observables of $\tilde{G}$, $G_i = \{\tilde{A}_{s1}, \ldots, \tilde{A}_{sn}\}$. Now consider the PBA $\{B_j\}_{j \in J}$ consisting of Boolean algebras freely generated by subsets $G_i \equiv \{A_{s1}, \ldots, A_{sn}\}$.

We now show how each state $\tilde{f}_k$ induces a state $f_k$ on $\{B_j\}_{j \in J}$. First, notice that, since each state on a PBA is a collection of normalized measures, it is sufficient to define it as measures on maximal Boolean algebras. Each measure on a maximal algebra $\mathcal{B}_j$ of $\{B_j\}_{j \in J}$, generated by a set $G_i = \{A_{s1}, \ldots, A_{sn}\}$, is completely determined by its values on elements of the form $(-1)^{1-\varepsilon_1}A_{s1} \cap \cdots \cap (-1)^{1-\varepsilon_n}A_{sn}$, where $-A \equiv A^c$ and $\varepsilon_i \in \{0,1\}$, since each element of the algebra can be written as a disjoint union of elements of that form (see Lemmas A.2 and A.3). Now, $f_k$ is defined as $f_k((-1)^{1-\varepsilon_1}A_{s1} \cap \cdots \cap (-1)^{1-\varepsilon_n}A_{sn}) \equiv \tilde{f}_k((-1)^{1-\varepsilon_1}A_{s1} \cap \cdots \cap (-1)^{1-\varepsilon_n}A_{sn})$ for all maximal subsets of compatible observables $G_i$ of $\tilde{G}$, and extended as a measure on each maximal algebra. It can be verified that such measures are normalized and they coincide on intersection of Boolean algebras; therefore, they define a state.

In this way, we obtain a collection of PPTs $\{C_k\}_{k \in K} = \{\{B_j\}_{j \in J}; f_k\}_{k \in K}$ such that the initial collection $\{C_k\}_{k \in K} = \{\{B_j\}_{j \in J}; \tilde{f}_k\}_{k \in K}$ is an empirical quotient. The equivalence relation $\sim$ can be, in fact, defined as follows: to each element $A$ of $X$, generated by a subset
of compatible generators $\mathcal{G}_t \subset \mathcal{G}$ there corresponds, via
the correspondence $A_t \mapsto \tilde{A}_t$, a unique element $\tilde{A}$ of $\tilde{X}$, defined as the element generated by $\mathcal{G}_t \subset \mathcal{G}$ by means of
the same operations that generate $A$ from $\mathcal{G}_t$; then an
equivance relation $\sim$ can be defined on $X$ as $A \sim B$ iff
$\tilde{A} = B$.

It can be easily verified that $\sim$ is an equivalence relation
and that it defines an empirical quotient:

(i): it is sufficient to consider each Boolean
algebras $\mathfrak{B}_t$, generated by $\mathcal{G}_t = \{A_{t1}, \ldots, A_{tk}\}$,
and notice that, there, $\sim$ coincides with the
equivalence relation induced by the ideal
$I \equiv \{B \in \mathfrak{B}_t | \bigcap_{l \in H} \sim (1)^{-\varepsilon_l} A_{t1} \cap \ldots \cap (1)^{-\varepsilon_l} A_{tk} = \emptyset\}$
with $H_B \equiv \{\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{0, 1\}^n \mid (1)^{-\varepsilon_l} A_{t1} \cap \ldots \cap (1)^{-\varepsilon_l} A_{tk} \subset B\}$ (see lemma A.3 below); now,
since $\{f_k\}_{k \in K}$ is complete and by construction of
$\{f_k\}_{k \in K}$, $I$ coincides with the set $\{B \in \mathfrak{B}_t | f_k(B) = 0$ for
all $k \in K\}$.

(ii): given $A, B \in X$, belonging respectively to
maximal algebras $\mathfrak{B}_t$, generated by $\mathcal{G}_t$, and $\mathfrak{B}_t'$, generated by
$\mathcal{G}_t'$ with $\mathcal{G}_t \cup \mathcal{G}_t'$ maximal, if $A \sim B$, then there
exists $C \in \mathfrak{B}_t \cap \mathfrak{B}_t'$, which is the Boolean algebra generated
by $\mathcal{G}_t \cap \mathcal{G}_t'$, such that $A \sim C \sim B$. In fact, $A \sim B$
implies, with the same notation as above, $\tilde{A} = \tilde{B}$; therefore
the two maximal algebras generated respectively by $\mathcal{G}_t$
and $\mathcal{G}_t'$ have a non-empty intersection containing $\tilde{A}$,
then, by $(G)$, $\mathcal{G}_t \cap \mathcal{G}_t' \neq \emptyset$ and an element $C$
satisfying the above conditions exists.

(iii): by construction, $X/\sim$ is in a one-to-one correspondence
with $\tilde{X}$; that such a bijection is also an isomorphism follows from the coincidence, within each Boolean
algebra, of $\sim$ with the equivalence relation induced by
the ideal $I$ discussed above.

(iv): it follows by construction of $\{f_k\}_{k \in K}$.

The above partial Boolean algebra $\{\mathfrak{B}_t\}_{t \in T}$ is embeddable into the Boolean algebra freely generated by the set $\mathcal{G}$. The PPTs $\{\tilde{C}_k\}_{k \in K}$ are therefore of the Horn-Tarski type and we shall name $\{\tilde{C}_k\}_{k \in K}$ the collection of free
H-T partial probability theories associated to $\{\tilde{C}_k\}_{k \in K}$ and $\mathcal{G}$.

D. Classical representations and free H-T theories

The following theorem applies the results of Proposition
III.1 to the above construction, allowing to reduce the
discussion of the existence of classical representations
to H-T theories.

**Theorem III.1.** Given a collection of PPTs $\{\tilde{C}_k\}_{k \in K}$ =
$\{(\mathfrak{B}_t)_{t \in T}, f_k\}_{k \in K}$ complete with respect to
$\{\mathfrak{B}_t\}_{t \in T}$, a set of generators $\mathcal{G} = \{\tilde{A}_1, \ldots, \tilde{A}_n\}$
satisfying property (G) and the associated collection of free
H-T PPTs $\{C_k\}_{k \in K}$ = $\{(\mathfrak{B}_t)_{t \in T}, f_k\}_{k \in K}$, then

(a) if, for a given $k \in K$, $\tilde{C}_k$ admits a classical representa-

(b) if there exists $K' \subset K$ such that $\{f_k\}_{k \in K'}$ is separa-
ting for $\{\mathfrak{B}_t\}_{t \in J}$ and $\tilde{C}_k$ admits a classical representa-
tion for all $k \in K'$, then $\tilde{C}_k$ admits a classical representa-
tion for all $k \in K'$.

**Proof** (a) Let the Boolean algebra $\mathfrak{B}$ together
with the normalized measure $\mu_k$ be a classical
representation for $\tilde{C}_k$. By the definition of extension,
the set $\mathcal{G}$ is a set of generators for $\mathfrak{B}$; therefore
the Boolean algebra $\mathfrak{B}$ is isomorphic to the
quotient algebra $\mathfrak{B}/\sim$, where $\mathfrak{B}$ is the Boolean algebra
freely generated by $n$ generators $\{A_1, \ldots, A_n\}$ and
the equivalence relation $\sim$ is that induced by the ideal
$I \equiv \{B \in \mathfrak{B} | \bigcap_{l \in H} \sim (1)^{-\varepsilon_l} A_1 \cap \ldots \cap (1)^{-\varepsilon_l} A_n = \emptyset\}$
with $H_B \equiv \{\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{0, 1\}^n \mid (1)^{-\varepsilon_l} A_1 \cap \ldots \cap (1)^{-\varepsilon_l} A_n \subset B\}$ (see Lemma A.3 below). Then,
denoted with $\sim$ the isomorphism between $\mathfrak{B}/\sim$ and $\mathfrak{B}$, a
measure $\mu_k$ extending the state $f_k$ on $\mathfrak{B}$ can be defined
as $\mu_k(A) \equiv \mu_k(\varphi([A]))$ for all $A \in \mathfrak{B}$, where $[A]$ is the
equivalence class of $A$ with respect to $\sim$. It can be easily verified that $\{\mathfrak{B}, \mu_k\}$ is a classical representation for $\tilde{C}_k$.

(b) Let the free Boolean algebra $\mathfrak{B}$, defined as above,
together with a normalized measure $\mu_k$ be a classical
representation for $\tilde{C}_k$, for all $k \in K'$. By Proposition
III.1, $\{\mathfrak{B}_t\}_{t \in T}$ is embeddable into the Boolean algebra
$2^N$, $N$ as in Proposition III.1, let us denote by $\mathfrak{B}$
the subalgebra of $2^N$ generated by $\mathcal{G}$ and with $S$
the set of all homomorphism $\delta : \tilde{X} \rightarrow \{0, 1\}$ induced by
the normalized measures $\mu_k$, $k \in K'$ (see Proposition
III.1). Such homomorphisms are, by construction (see
Theorem 0 in Ref. [2]), in a one-to-one correspondence
with the multiplicative measures of $2^N$ and can be extended to multiplicative measures on $\mathfrak{B}$ in a way
uniquely determined by the values assumed on the set
of generators $\mathcal{G}$. It follows that each element
$\bigcup_{l \in H} \prod_{l=1}^n (1 - \varepsilon_l) \delta(\tilde{A}_l) + (1 - \varepsilon_l) (1 - \delta(\tilde{A}_l)) = 0$, i.e.
the extension of $\delta$ is zero on such an element, for all
$\delta \in S$. Since the homomorphisms in $S$ are induced by
multiplicative measures associated, eq. (2), to the
normalized measures $\mu_k$, $k \in K'$, it follows that
the ideal $I$ defined as in (a) coincides with the ideal
$\{B \in \mathfrak{B} | \mu_k(B) = 0\}$ for all $k \in K'$. This implies,
as in the proof of Proposition III.1, that $\mu_k$ induces a
normalized measure on $\mathfrak{B}/\sim$, and consequently a
normalized measure $\tilde{\mu}$ on $\mathfrak{B}$, for all $k \in K'$. It can be easily checked that $\{\mathfrak{B}, \tilde{\mu}\}$ is a classical representation for $\tilde{C}_k$ for all $k \in K'$. □
E. Free H-T PPT versus projection algebra PPT in QM

On the basis of the above discussion, it is clear that the projection algebra is not the only possible PBA for the formulation of QM predictions.

In particular, for any given PBA of projections, quantum states generate a complete collection of states on such a PBA; it follows that, for any set of generators satisfying property (G), the construction in Sect III C applies and therefore any collection of QM predictions can be described by a free H-T PPT.

The above results formalize constructions which are often used implicitly in the discussion of the interpretation of QM: consider in fact a finite set of yes/no apparatuses $G = \{A_1, \ldots, A_n\}$, represented as projections $P = \{P_1, \ldots, P_n\}$ in a finite-dimensional Hilbert space $\mathcal{H}$; then for every subset of compatible apparatuses, i.e. commuting projections, in $G$, it makes sense to consider logical combinations obtained by means of logic gates applied to the outcomes in an experiment where they are jointly measured. In this way we obtain a collection of Boolean algebra of observables $\{\mathcal{B}_i\}$, each one freely generated by a subset $G_i \subset G$ of compatible apparatuses; there is no longer a bijection between $\{\mathcal{B}_i\}$ and the partial Boolean algebra generated by $\mathcal{P}$, see eq. (1), but states on the PBA are still given by quantum mechanical states $\psi$ by $f_\psi(A) = (\psi, P_A \psi)$, where $A$ belong to a free Boolean algebra $\mathcal{B}_i$ generated by a subset of compatible apparatuses $G_i \subset G$ and $P_A$ is the projection obtained from the corresponding subset of commuting projection $P_i \subset \mathcal{P}$ by means of the same Boolean operations that generate $A$ from $G_i$ (notice that $P_A$ may be $0$ even if $A \neq 0$).

Notice that the above construction only relies on the notion of compatible apparatuses and observed frequencies. It can be qualified as a Bell-type approach: every attribution of 0 and 1 to a set of observables is assumed to be possible, only a posteriori constrained by experimental information, and the logical structure is that of a free Boolean algebra.

A systematic treatment of such a problem is given by Pitowsky in terms of propositional logic; we shall refer to it as the correlation polytope approach (see Appendix B for an account in terms free Boolean algebras).

The alternative approach based on projection algebras gives rise to results of a rather different form, starting from the K-S theorem, and will be referred to as Kochen-Specker-type approach.

The relation between the two approaches has not been clarified in general and is also confused by the fact that in some cases (e.g. the Bell argument with four measurements) the approaches seem to coincide.

From the above discussion, it is clear the main difference between Kochen-Specker-type and Bell-type approaches resides in which logical relations between observables are assumed.

In fact, the above results imply that the K-S approach is related to the Bell approach by an empirical quotient: by the construction of Sect III C a free H-T PPT is obtained from the projection algebra PPT on the basis of any set $\mathcal{P}$ satisfying property (G), which always exists, as discussed above.

The logical content of such a constructions is that logical relations between compatible observables can be weakened to empirical relations, associated in principle to a collection of experiments or states on a PBA. (Similar distinctions have been introduced, with a different interpretation, by Garola and Solombrino).

The construction in Sect III C Proposition III.1 and Theorem III.1 clarify the relation between Kochen-Specker-type results, presenting a non-embeddable partial Boolean algebra of projections, and Bell-type arguments, giving conditions for the existence of a probability measure reproducing measurable correlations on a free Boolean algebra. The result is that the equivalence of the two viewpoints for the discussion of classical representability in QM, recognized by Cabello (see also Ref. 9) in situations arising in the discussion of the Kochen-Specker theorem, is a very general fact, following from basic logical and probabilistic structures.

In fact, Proposition III.1 implies that a set of predictions that generates a Kochen-Specker-type contradiction, namely the impossibility of a consistent truth assignment (i.e. a homomorphism between projections PBA and $\{0, 1\}$), also generates a Bell-type contradiction for all quantum states in the associated free H-T PPTs, more precisely each quantum state violates at least one Bell inequality (not necessarily the same for all states).

Moreover, as a consequence of Theorem III.1, we obtain that, given a set of apparatuses and a set of quantum states inducing a separating collection of states on their projection PBA, a classical representation of all the corresponding projection algebra PPTs exists if and only if all the corresponding free H-T PPTs, constructed as in Sect. III C, admit a classical representation, independently of the choice of the generators.

It follows that all extension problems arising in QM can be discussed in the framework of free H-T PPTs; the rest of this paper is devoted to the investigation of extensibility conditions in this case.

IV. SYSTEMS OF 3 AND 4 OBSERVABLES

In this section we shall discuss two applications of the criterion of classical representability, presented in Appendix B obtained from the translation of Pitowsky’s correlation polytopes results into the Boolean framework.

The proofs of the following theorems are essentially based on the analysis of Bell-Wigner and Clauser-Horne correlation polytopes made by Pitowsky. Theorem IV.1 shows that for three observables with two compatible pairs a classical probabilistic model which reproduces
observable correlations always exists; it implies that for three quantum mechanical observables a classical probabilistic model always exists for all possible compatibility relations. Theorem [V.2] shows that for four observables with Bell-type compatibility relations a probabilistic model for the four observables exists if and only if there are two models for three observables that coincide on the intersection; a result obtained by Fine in a rather different setting; our approach provides in this case a complete analysis for the case of four quantum mechanical observables.

**Theorem IV.1.** Let \( \mathcal{B} \) be a Boolean algebra freely generated by \( \mathcal{G} = \{ A_1, A_2, A_3, A_4 \} \) and \( \mathcal{B}_{13} \) and \( \mathcal{B}_{23} \) the subalgebras generated respectively by \( \{ A_1, A_3 \} \) and \( \{ A_2, A_3 \} \).

Consider \( f : \mathcal{B}_{13} \cup \mathcal{B}_{23} \rightarrow [0,1] \), such that \( f_{|\mathcal{B}_{13}} \) and \( f_{|\mathcal{B}_{23}} \) are normalized measures on such subalgebras. Then \( f \) is extensible to a normalized measure on the algebra \( \mathcal{B} \).

**Proof** By Lemma [B.2] without loss of generality we can consider \( f : X = \{ A_1, A_2, A_3, A_4, A_1 \cap A_3, A_2 \cap A_3 \} \rightarrow [0,1] \). The vector \( p = (p_1, p_2, p_3, p_{13}, p_{23}) \) is given by \( p_1 = f(A_1) \) and \( p_{ij} = f(A_i \cap A_j) \); since such values come from a measure on \( \mathcal{B}_{13} \) and \( \mathcal{B}_{23} \)

\[
p_{13} \leq \min\{p_1, p_3\} \quad p_{23} \leq \min\{p_2, p_3\};
\]

from

\[
0 \leq f((A_i \cup A_j)^c) = 1 - f(A_i \cup A_j) = 1 - f(A_i) - f(A_j) + f(A_i \cap A_j), \quad \{i, j\} = \{1, 3\}, \{2, 3\}
\]

we obtain

\[
p_1 + p_3 - p_{13} \leq 1, \quad p_2 + p_3 - p_{23} \leq 1.
\]

From Lemma [B.1] and Proposition [B.1] we know that if a normalized measure \( \mu \) extends \( f \) exists, then

\[
\lambda(\varepsilon) \equiv \mu(a_\varepsilon) = \mu((-1)^{1-\varepsilon_1} A_1 \cap (-1)^{1-\varepsilon_2} A_2 \cap (-1)^{1-\varepsilon_3} A_3.)
\]

The coefficients \( \lambda(\varepsilon) \) are obtained from (3) and the property \( (b) \) of the definition of measure (see Appendix A). The convex combination is obtained by means of two coefficients \( \chi \) and \( \eta \) representing the two missing correlations \( \mu(A_1 \cap A_2 \cap A_3) \) and \( \mu(A_1 \cap A_2 \cap A_4) \) (alternatively, one can use \( \mu(A_1 \cap A_2) \) and \( \mu(A_1 \cap A_2 \cap A_3) \), but the inequalities [6] - [9] below become more complicated). The following inequalities are obtained from the non-negativity of the measure in the same way as in (3):

\[
\eta \leq \min\{p_{13}, p_{23}\}; \quad \eta \geq \max\{0, p_{13} + p_{23} - p_3\};
\]

\[
\chi \leq \min\{p_{13} - p_{13}, p_{23} - p_3\}; \quad \chi \geq \max\{0, p_{1} + p_{2} + p_{3} - p_{13} - p_{23} - 1\}. 
\]

Using (3) and (4), one can easily show that each number that appears in \( \min\{\ldots\} \) of (6) is greater or equal to each number that appears in \( \max\{\ldots\} \) of (7), the same for (8) and (9). Therefore, \( 0 - 0 \) define two non-empty intervals where one can choose \( \chi \) and \( \eta \). We can now write explicitly the coefficients \( \lambda(\varepsilon) \)

\[
\lambda(0,0,0) = 1 - (p_1 + p_2 + p_3 - p_{13} - p_{23}) + \chi,
\]

\[
\lambda(1,0,0) = p_1 - p_3 - \chi,
\]

\[
\lambda(0,1,0) = p_2 - p_{23} - \chi,
\]

\[
\lambda(0,1,1) = \eta + p_3 - p_{13} - p_{23},
\]

\[
\lambda(1,1,0) = \chi,
\]

\[
\lambda(1,0,1) = p_{13} - \eta,
\]

\[
\lambda(0,1,1) = p_{23} - \eta,
\]

\[
\lambda(1,1,1) = \eta.
\]

It follows immediately that \( \lambda(\varepsilon) \geq 0 \) for all \( \varepsilon \in \{0,1\}^3 \), and that \( \sum_{\varepsilon \in \{0,1\}^3} \lambda(\varepsilon) = 1 \). To conclude one just has to show, by writing it explicitly, that \( \sum_{\varepsilon \in \{0,1\}^3} \lambda(\varepsilon) u_\varepsilon = p \) and then apply Proposition [B.1] \( \square \).

It follows that, for three observables, there exists a classical representation for any state also in the case in which there is only a pair of compatible observables and in the case of three incompatible observables.

In fact, in the case of three incompatible observables only \( p_1, p_2 \) and \( p_3 \) are given, thus one can add \( p_{13} \) and \( p_{23} \) that satisfy (3) and (4) and then apply the same argument as in the proof of Theorem [IV.1]. The same argument also applies to the case in which there is only a pair of compatible observables. Finally, if property (K-S) is assumed, a classical representation exists also for three pairwise compatible observables.

We can conclude, therefore, that for three quantum mechanical observables a classical probabilistic model which reproduce all observable correlations always exists.

We now discuss the implication of the results for the case of three observables to the analysis of the case of four.

**Theorem IV.2.** Let \( \mathcal{B} \) be a Boolean algebra freely generated by \( \mathcal{G} = \{ A_1, A_2, A_3, A_4, A_5 \} \), and \( \mathcal{B}_{ij}, \mathcal{B}_{ijk} \), be the subalgebras generated respectively by \( \{ A_i, A_j \} \) and \( \{ A_i, A_j, A_k \} \).

Consider \( f : \mathcal{B}_{13} \cup \mathcal{B}_{23} \cup \mathcal{B}_{14} \cup \mathcal{B}_{24} \rightarrow [0,1] \) such that \( f_{|\mathcal{B}_{13}}, f_{|\mathcal{B}_{23}}, f_{|\mathcal{B}_{14}}, f_{|\mathcal{B}_{24}} \) are normalized measures on such subalgebras.

Then \( f \) is extensible to a normalized measure on the algebra \( \mathcal{B} \) if and only if there exist two partial extensions \( f^{123} \) and \( f^{124} \), of \( f_{|\mathcal{B}_{13}}, f_{|\mathcal{B}_{23}}, f_{|\mathcal{B}_{14}}, f_{|\mathcal{B}_{24}} \) on the subalgebras \( \mathcal{B}_{123} \) and \( \mathcal{B}_{124} \), such that \( f^{123} \equiv f^{124} \).

**Proof** One implication is obvious since if a measure that extends \( f \) exists, then the two partial extensions exist and they coincide on the intersection.

For the converse, we note, as in Theorem [IV.1], that we can consider without loss of generality \( X = \{ A_1, A_2, A_3, A_4, A_1 \cap A_3, A_2 \cap A_3, A_1 \cap A_4, A_2 \cap A_4 \} \) and \( f : X \rightarrow [0,1] \) and then apply Proposition [B.1] therefore we construct the vector...
p = (p_1, p_2, p_3, p_4, p_{13}, p_{23}, p_{14}, p_{24}) and find the coefficients λ(ε).

First, we apply Theorem [IV.1] to the subalgebras B_{123} and B_{124} and to f_{124} and f_{123}, obtaining two partial extensions p_{123} and p_{124}, that are normalized measures on the subalgebras B_{123} and B_{124}.

Now we consider the vector
\[ p'(x) = (p_1', p_2', p_3', p_{12}', p_{13}', p_{23}') \]
and
\[ p''(x) = f_{124}(A_i), i = 1, 2, 3 \]
and note that, by Proposition [3], there exist \( u_ε = (ε_1, ε_2, ε_3, ε_4) \) and \( λ'(ε) \) such that
\[ p' = \sum_{ε \in \{0,1\}^3} λ'(ε) u_ε, \quad \sum_{ε \in \{0,1\}^3} λ'(ε) = 1, \quad λ'(ε) \geq 0. \]

The same argument applies to
\[ p'' = (p_1'', p_2'', p_3'', p_{12}'', p_{13}'', p_{23}''), \quad p'' = f_{124}(A_i), i = 1, 2, 4 \]
and we obtain \( λ''(ε) \) such that
\[ p'' = \sum_{ε \in \{0,1\}^3} λ''(ε) u_ε, \quad \sum_{ε \in \{0,1\}^3} λ''(ε) = 1, \quad λ''(ε) \geq 0. \]

Moreover, the above is the only set of compatibility relations consistent with property (K-S) in which classical representability does not follow from Theorem [V.1] below. Theorem [IV.2] together with Theorem [V.1] provide, therefore, a complete analysis for the case of four quantum observables.

In general, the correlation polytope approach to the extension problem is computationally intractable, but the use of PBA's and associated states provides new criteria of classical representability for PPTs. In the next section we shall discuss some results that allow for a simplification of the computation of extensibility conditions in many non-trivial cases. Another extensibility criterion based on Horn and Tarski's notion of partial measure is presented in Appendix C.

V. NEW EXTENSIBILITY CRITERIA

In this section we shall show how classical representability may arise algebraically, i.e. independently of states, in many non-trivial cases and we shall present some techniques that allow for a simplification of the computation of conditions of classical representability.

The following result is given by a generalization of the proof of Theorem [IV.2]

Theorem V.1. Let \( B \) be the Boolean algebra freely generated by \( \{A_1, \ldots, A_n\} \), and let \( B_1 \) and \( B_2 \) be the subalgebras generated respectively by \( \{A_1, \ldots, A_k\} \) and \( \{A_1, \ldots, A_n\} \), with \( 1 \leq i \leq k \leq n \). Let \( μ_1 \) and \( μ_2 \) be two normalized measures on \( B \), such that \( μ_1 \) coincides with \( μ_2 \) on \( B_1 \cap B_2 \). Then a measure \( μ \) which extends \( μ_1 \) and \( μ_2 \) on \( B \) exist.

Proof Using the bijective correspondence between the atoms of the subalgebra \( B_1 \) and the vectors \( ε' = (ε_1, \ldots, ε_k) \in \{0,1\}^k \), given by \( a_{ε'} = (-1)^{1-ε_1}A_1 \cap \ldots \cap (-1)^{1-ε_k}A_k \) (see Lemma A.3 below), we define the function \( f_1 : \{0,1\}^k \to \{0,1\} \) as \( f_1(ε') = μ_1(a_{ε'}) \) for all \( ε' \in \{0,1\}^k \). For all \( A \in B_1 \), \( μ_1(A) = \sum_{ε' \in I_A} f_1(ε') \), where \( I_A \equiv \{ε' \in \{0,1\}^k | a_{ε'} \subset A \} \), therefore \( μ_1(1) = 1 \) implies

\[ \sum_{ε' \in \{0,1\}^k} f_1(ε') = 1 \]

We apply the same procedure to the measure \( μ_2 \), defining \( f_2 : \{0,1\}^n \to \{0,1\} \) as \( f_2(ε'') = μ_2(a_{ε''}) \), where \( ε'' \in \{0,1\}^n \) and \( a_{ε''} \) is an atom of \( B_2 \). Similarly, \( μ_2(1) = 1 \) implies

\[ \sum_{ε'' \in \{0,1\}^n} f_2(ε'') = 1. \]

Moreover, \( μ_1|_{B_{1 \cap B_2}} \equiv μ_2|_{B_{1 \cap B_2}} \) implies

\[ \sum f_1(ε_1, \ldots, ε_{i-1}, ε_i, \ldots, ε_k) = \sum f_2(ε_i, \ldots, ε_k, ε_{k+1}, \ldots, ε_n). \]
where \( \varepsilon_j \in \{0, 1\}, j = 1, \ldots, n \). In fact, since \( \mathfrak{B}_1 \cap \mathfrak{B}_2 \) is the subalgebra generated by \( \{A_1, A_{i_1}, \ldots, A_{k_i}\} \), (13) follows from the two possible ways of writing every atom \( a_\eta \) of \( \mathfrak{B}_1 \cap \mathfrak{B}_2 \), with \( \eta \in \{0, 1\}^{k-i+1} \), as a sum of atoms of \( \mathfrak{B}_1 \) or of \( \mathfrak{B}_2 \).

We define \( f \) on \( \{0, 1\}^n \) as follows

\[
f(\varepsilon_1, \ldots, \varepsilon_n) = \frac{f_1(\varepsilon_1, \ldots, \varepsilon_k, \varepsilon_{k+1}, \ldots, \varepsilon_n)}{\sum_{\varepsilon_{k+1}, \ldots, \varepsilon_n} f_1(\varepsilon_1, \ldots, \varepsilon_k, \varepsilon_{k+1}, \ldots, \varepsilon_n)}
\]

(14)

if the denominator is different from 0, and \( f(\varepsilon) = 0 \) otherwise. Equivalently, by (13), the denominator can be written with \( f_2 \), instead of \( f_1 \), summed over the last \((n-k)\) variables.

A function \( \mu \) on \( \mathfrak{B} \) is induced by \( f \)

\[
\mu(A) = \sum_{\varepsilon \in H_A} f(\varepsilon)
\]

(15)

for all \( A \in \mathfrak{B} \), where \( H_A = \{\varepsilon \in \{0, 1\}^n| a_\varepsilon \subset A\} \) and \( a_\varepsilon \) is an atom of \( \mathfrak{B} \) given by Lemma A.3.

It is obvious that \( \mu \) is non-negative and additive on disjoint elements; normalization \((\mu(1) = 1)\) follows easily from (11) and (12).

To conclude, we need to show that \( \mu \) coincide with \( \mu_1 \) on \( \mathfrak{B}_1 \) and with \( \mu_2 \) on \( \mathfrak{B}_2 \). Given \( A \in \mathfrak{B}_1 \), it can be written either as \( A = \bigcup_{\varepsilon' \in I_A} a_{\varepsilon'} \), where \( \varepsilon' \in \{0, 1\}^k \), \( a_{\varepsilon}' \) is an atom of \( \mathfrak{B}_1 \) and \( I_A \) is defined as above, or else, in terms of the atoms of \( \mathfrak{B}_2 \), as \( A = \bigcup_{\varepsilon \in H_A} a_\varepsilon \). By Lemma A.3 \( H_A = \{(\varepsilon', \varepsilon) \in \{0, 1\}^n| \varepsilon' \in I_A, \varepsilon \in \{0, 1\}^{n-k}\} \) hence, by using also (13), we obtain

\[
\mu(A) = \sum_{\varepsilon \in H_A} f(\varepsilon) = \sum_{\varepsilon' \in I_A} \sum_{\varepsilon_{k+1}, \ldots, \varepsilon_n} f(\varepsilon', \varepsilon) = \sum_{\varepsilon' \in I_A} \sum_{\varepsilon_{k+1}, \ldots, \varepsilon_n} f_1(\varepsilon'_1, \ldots, \varepsilon'_{k-1}, \varepsilon'_k, \varepsilon_{k+1}, \ldots, \varepsilon_n) = \sum_{\varepsilon' \in I_A} f_1(\varepsilon'_1, \ldots, \varepsilon'_{k-1}, \varepsilon'_k) = \mu_1(A).
\]

The proof for \( \mu_2 \) is analogous. \( \Box \)

The above theorem provides an extensibility criterion based only on the partial Boolean structures of observables, i.e. on compatibility relations.

In many other non-trivial cases a classical extension of a PPT is in fact obtainable “for free”, i.e. from partial Boolean structures, without constraints on the states. In the following we describe such cases in terms of compatibility relations, by introducing a compatibility graph representation.

Consider a Boolean algebra \( \mathfrak{B} \) freely generated by a set of observables \( \mathcal{G} = \{A_1, \ldots, A_n\} \) with a function \( f \) defined on a subset \( \bigcup \mathfrak{B}_i \subset \mathfrak{B} \), where each \( \mathfrak{B}_i \) is freely generated by a subset \( \mathcal{G}_i \subset \mathcal{G} \) and \( f_{\mathcal{A}_i} \) is a normalized measure on \( \mathfrak{B}_i \). We shall represent compatibility relations as follows:

- Each node represents a subalgebra \( \mathfrak{B}_i \) on which \( f \) defines a normalized measure, and is depicted as an ellipse where the generators \( \mathcal{G}_i \subset \mathcal{G} \), of the subalgebra are indicated;

- when for two subalgebras, generated respectively by \( \mathcal{G}' \subset \mathcal{G} \) and \( \mathcal{G}'' \subset \mathcal{G} \), \( f \) defines a normalized measure on the subalgebra generated by \( \mathcal{G}' \cup \mathcal{G}'' \), we shall depict an edge connecting the two corresponding nodes.

An example is depicted in fig.1.

![Fig. 1: An example of a compatibility graph representation](image1)

If property \((K-S)\) holds, subgraphs consisting of pairwise connected nodes can be described by a single node. On one side, this allows for a representation using only single observable nodes, on the other, it allows for a restriction to graphs without such completely connected subgraphs. An example is depicted in fig.2.

![Fig. 2: An example of two equivalent graph representations for a PPT where property \((K-S)\) holds](image2)
tree subgraphs substantially simplifies the analysis, as shown by the following examples:

**Example V.1.** Starting from the result of Theorem IV.2 and with the same notation, we discuss explicitly the conditions of extensibility of the state $f$, namely the conditions guaranteeing the existence of two partial extension $f^{123}$ and $f^{124}$ that coincide on $\mathcal{B}_{12}$. By Lemma B.2 it is sufficient (see also proposition [B.1]) that the two measure coincide on the element $A_1 \cap A_2$. It is therefore sufficient to investigate possible attribution to $p_{12}$, by means of the Bell-Wigner polytope (see Pitowsky[19]), i.e. the correlation polytope associated with the subset $X_s = \{A_1, A_2, A_s, A_1 \cap A_s, A_2 \cap A_s, A_1 \cap A_2\}$, with $s = 3, 4$. One obtains two systems of inequalities that can be written as

$$\alpha^{(3)} \leq p_{12} \leq \beta^{(3)}, \quad \alpha^{(4)} \leq p_{12} \leq \beta^{(4)}$$

where $\alpha^{(s)}$ is the maximum of linear combinations $L^{(s)}$, of $p_{1,2}p_{s}, p_{1,s}p_{2,s}$, that appear in inequalities of the form $L^{(s)} \leq p_{12}$, and $\beta^{(s)}$ is the minimum of linear combinations $L^{(s)}$, of $p_{1,2}p_{s}, p_{1,s}p_{2,s}$, that appear in inequalities of the form $p_{12} \leq L^{(s)}$. Therefore, a necessary and sufficient condition for the extensibility of $f$ to a normalized measure on $\mathcal{B}$, i.e. for the classical representability of the corresponding PPT, can be expressed as

$$\max_{s=3,4} \alpha^{(s)} \leq \min_{s=3,4} \beta^{(s)}. \quad (16)$$

By exchanging the role of pairs 1, 2 and 3, 4 one obtains an analogous solution. It is less obvious, however, that (by Theorem IV.2) condition (16) for a consistent attribution of a value to the correlation $p_{12}$ is satisfied if and only if the analogous condition for the correlation $p_{34}$ is satisfied.

As a result, Bell inequalities can be identified as conditions for extensions of states beyond their automatic extension given by algebraic structures. Even if automatic extensions always exist, the violation of Bell inequalities shows the inconsistency of an absolute frequency interpretation.

**Example V.2.** Now consider a PPT as before but with observables $A_2, \ldots, A_n$ added only “on one side”, i.e. only compatible with $A_1, A_2$. More precisely, consider a Boolean algebra $\mathcal{B}$ freely generated by $\{A_1, \ldots, A_n\}$ and a function $f : \bigcup_{i=1,2,j=3,\ldots,n} \mathcal{B}_{ij} \to [0,1]$, such that $f_{|\mathcal{B}_{ij}}$ is a normalized measure on $\mathcal{B}_{ij}$ for $i = 1, 2$, $j = 3, 4, \ldots, n$.

Classical representability is still guaranteed, with the same notation as before, by a condition analogous to (16), namely

$$\max_{s=3,\ldots,n} \alpha^{(s)} \leq \min_{s=3,\ldots,n} \beta^{(s)}, \quad (17)$$

with a large simplification with respect to the general correlation polytope approach; in the present approach, the asymmetry of the problem allows in fact a discussion in terms of the “additional correlations” for the “side with less observables”.

A slightly more complicated example is the following:

**Example V.3.** Let $\mathcal{B}$ be a Boolean algebra freely generated by $\mathcal{G} = \{A_1, \ldots, A_6\}$ and $\mathcal{B}_{ij}$ the sub-algebra generated by $\{A_i, A_j\}$, consider a function $f : \bigcup_{i=1,2,j=4,5,6} \mathcal{B}_{ij} \to [0,1]$, such that $f_{|\mathcal{B}_{ij}}$ is a normalized measure on $\mathcal{B}_{ij}$, with $i = 1, 2, 3$ and $j = 4, 5, 6$. By Theorem V.1 there exist three partial extensions $f^{123s}$ respectively of $f_{|\mathcal{B}_{13}}$, $s = 4, 5, 6$, to normalized measures on $\mathcal{B}_{123s}$, the subalgebra generated $\{A_1, A_2, A_3, A_6\}$. The existence of an extension of $f$ to a normalized measure on $\mathcal{B}$ is equivalent to the condition that the three partial extension $f^{123s}$ can be taken to coincide on the subalgebra $\mathcal{B}_{123}$. Such a condition can be investigated in terms of the values assumed by $\{A_1 \cap A_2, A_1 \cap A_3, A_1 \cap A_6, A_2 \cap A_3, A_2 \cap A_6, A_3 \cap A_6\}$ and by means of the correlation polytopes associated to the sets $X_s = \{A_1, A_2, A_3, A_4, A_1 \cap A_3, A_1 \cap A_6, A_2 \cap A_3, A_2 \cap A_6, A_3 \cap A_6, A_1 \cap A_3 \cap A_6\}$, $s = 4, 5, 6$. Each polytope is generated by 16 vertices in $\mathbb{R}^{11}$, and is described by 48 inequalities, computed by means the double description method[12] with package package[13] (for the application of such methods to QM predictions see Pitowsky and Svozil[15]). However, out of these 48 inequalities only 32 are relevant for our discussion and they can be written as:

Type 1 $\alpha_{1,ij}^{(s)} \leq p_{ij} \leq \beta_{1,ij}^{(s)} \cdot \alpha_{2,ij}^{(s)} \leq p_{23} \leq \beta_{2,ij}^{(s)}$

$\{ij\} = \{12\}, \{13\}, \{23\}$ ,

Type 2 $\alpha_{2,ij}^{(s)} \leq p_{ij} - p_{23} \leq \beta_{2,ij}^{(s)}$

$\{ij\} = \{12\}, \{13\}, \{23\}$ ,

Type 3 $\alpha_{3,ijk}^{(s)} \leq p_{ij} + p_{jk} - p_{23} \leq \beta_{3,ijk}^{(s)}$

$\{ij\} = \{12\}, \{13\}, \{23\}, j \neq k \neq i$ ,

Type 4 $\alpha_{4}^{(s)} \leq p_{12} + p_{13} + p_{23} - p_{123} \leq \beta_{4}^{(s)}$,

where $\alpha_{1,ij}^{(s)}$ is the maximum of of linear combinations $L^{(s)}$, of $p_{1,2}p_{3}, p_{1,s}p_{2,s}, p_{1,s}p_{2,s}$, that appear in inequalities of the form $L^{(s)} \leq p_{ij}$, and $\beta^{(s)}$ is the minimum of of linear combinations $L^{(s)}$, of $p_{1,2}p_{3}, p_{1,s}p_{2,s}, p_{1,s}p_{2,s}$, that appear in inequalities of the form $p_{ij} \leq L^{(s)}$, and so on for each type of inequalities. A necessary and sufficient condition for the extensibility of $f$ is therefore the existence of a solution for the following system of 11 lin-
ear inequalities in the variables $p_{12}, p_{13}, p_{23}, p_{123}$,
\begin{align}
\max_s \alpha_{1,ij}^{(s)} &\leq p_{ij} \leq \min_s \beta_{1,ij}^{(s)}, \quad (18) \\
\{ij\} &= \{12\}, \{13\}, \{23\}; \\
\max_s \alpha_{2,ij}^{(s)} &\leq p_{123} \leq \min_s \beta_{2,ij}^{(s)}, \quad (19) \\
\{ij\} &= \{12\}, \{13\}, \{23\}; \\
\max_s \alpha_{3,ijk}^{(s)} &\leq p_{ij} + p_{jk} - p_{123} \leq \min_s \beta_{3,ijk}^{(s)}, \quad (20) \\
\{ij\} &= \{12\}, \{13\}, \{23\}, j \neq k \neq i; \\
\max_s \alpha_{4}^{(s)} &\leq p_{12} + p_{13} + p_{23} - p_{123} \leq \min_s \beta_{4}^{(s)}. \quad (22)
\end{align}
As in example [V.1] the discussion with 1, 2, 3 substituted with 4, 5, 6 is analogous and the corresponding conditions are equivalent.
Moreover, as in example [V.2] the introduction of additional observables only “on one side” modifies inequalities [18]–[22] only by extending the sets on which maximum and minimum are taken.
In general, our method consists in exploiting the “algebraic” extensions given by Theorem [V.2] conditions of classical representability only arise as consistency (i.e. coincidence on intersections) conditions for putting together partial extensions associated to tree subgraphs, giving rise to a description of the initial compatibility graph as a tree graph on such extended nodes. As pointed out in example [V.2] different strategies are possible, keeping the values of the given set of correlations fixed throughout all partial extensions.

VI. CONCLUSIONS
We have presented an algebraic approach to the extensions of partial probability theories, i.e. partial classical structures, applicable in particular to those arising in QM, that contains in the same logical framework both Bell-type and Kochen-Specker-type approaches.
The above analysis applies in particular to the problem of simulability of quantum algorithms by means of classical (probabilistic) algorithms. It is well known that a partial probability theory can be simulated by a classical theory where additional variables, representing observables in different contexts, are introduced. Given a PPT, associated to a finite set of quantum measurements, it is less obvious whether and how much it can be extended; in other words whether the number of its contexts, i.e., the number of additional classical variables can be reduced.
The analysis of partial extensions of a PPT provides, therefore, an intrinsic measure of its non-classicality. From an information-theoretic viewpoint, the above mentioned additional variables may be interpreted as additional information carried by a quantum system with respect to its classical counterpart.
We stress that because of the well known computational intractability of the correlation polytope approach other approaches should be investigated. Beyond the application of Theorem [V.1] in examples [V.1] [V.2] and [V.3] in appendix C we point out two additional methods: the first one is to investigate the partial order ≤ of definition [C.1] (see Appendix C) since it enters in the definition of partial measure and provides conditions of extendibility; the second one is to investigate properties of interior and exterior measures since they play a fundamental role in Theorem [C.3].
We briefly outline some implications of our approach for the interpretations of QM. It has been shown in example [V.1] that condition (16) is equivalent to the complete (in the sense of Pitowsky16) set of CHSH inequalities; in other words that classical representability is equivalent to the possibility of an attribution of a value to the non-observable correlation $p_{12}$ which is consistent with the observable correlations. This example points out explicitly the role of predictions for non-observable correlations in every attempt to a classical interpretation of quantum mechanical predictions. This allows for an interpretation of the violation of Bell inequalities as a negative answer to the question: given two incompatible observables is it possible to assign a value to their correlation which is consistent with measurable correlations?
It is interesting to notice that this consistency criterion is implicit in the argument by Einstein Podolsky and Rosen. In fact, in their famous paper they discussed the possibility of extending quantum mechanical predictions to include correlations between incompatible observables with the requirement that such added (non-observable) correlations are consistent with the observable correlations; in their case, a value is attributed to the momentum of a particle on the basis of its (perfect anti-) correlation with the momentum of the other particle, giving rise to a joint attribution of definite values to position and momentum of a single particle. We may, therefore, name such consistency conditions as the EPR criterion for extensions of QM predictions.

Appendix A: Boolean Algebras
There exists a vast literature on Boolean algebras that explores deep aspects of the subject and important connections with several branches of mathematics; we briefly recall in this section the elementary results needed for our discussion, involving in particular only finite sets. For more details see Sikorski15 and Givant-Halmos17.
A Boolean algebra is a non-empty set $\mathcal{B}$ in which two binary operations $\cap$ and $\cup$ are defined, called respectively meet and join, and one unary operation $c$ called complement, satisfying certain axioms.
We shall denote by $\emptyset$ the zero element, $\emptyset = A \cap A^c$ for all $A$, and by $1$ the unit, $1 = A \cup A^c$ for all $A$.
Every finite Boolean algebra is isomorphic to the
Boolean algebra $\mathcal{P}(X)$ (the subsets of $X$ with intersection, union and complement) for a certain finite set $X$.

Given a Boolean algebra $\mathcal{B}$, a function $m : \mathcal{B} \rightarrow \mathbb{R}$ is called a measure on $\mathcal{B}$ if it satisfies:

(a) $0 \leq m(A) \leq 1$ for all $A \in \mathcal{B}$, and there exists $A_0 \in \mathcal{B}$ such that $m(A_0) = 0$;

(b) $m(A \cup B) = m(A) + m(B)$, if $A$ and $B$ are two disjoint elements of $\mathcal{B}$.

A measure that satisfy $m(1) = 1$ is called a normalized measure, and a normalized measure such that $m(A) \in \{0, 1\}$ for every $A \in \mathcal{B}$ is called a two-valued measure or a multiplicative measure, since its properties also imply that $m(A \cap B) = m(A)m(B)$, for all $A, B \in \mathcal{B}$.

An element $a \neq \emptyset$ of a Boolean algebra $\mathcal{B}$ is called an atom if, for all $A \in \mathcal{B}$ the inclusion $A \subset a$ implies $A = \emptyset$ or $A = a$. A Boolean algebra $\mathcal{B}$ is called atomic if for every element $A$ in $\mathcal{B}$ exists an atom $a$ such that $a \subset A$.

**Lemma A.1.** Every finite Boolean algebra $\mathcal{B}$ is atomic and, if it has $N$ elements, it has exactly $n$ atoms $\{a_1, \ldots, a_n\}$ such that $N = 2^n$. Moreover, every element $A \in \mathcal{B}$ can be written uniquely as $A = \bigcup_{i \in I} a_i$, where $I \subset \{1, \ldots, n\}$.

**Lemma A.2.** Let $\mathcal{B}$ be a finite Boolean algebra with $n$ atoms $\{a_1, \ldots, a_n\}$; then for every atom $a_i$ there exists a multiplicative measure $\delta_{a_i}$ which is 1 on $a_i$ and 0 on all other atoms. Moreover multiplicative measures on $\mathcal{B}$ are all and only those that are defined in this way.

**Proof.** Given the atom $a_i$, define the function $\delta_{a_i} : \mathcal{B} \rightarrow \{0, 1\}$ as follows: $\delta_{a_i}(A) = 1$ if $a_i \subset A$, and zero otherwise. It follows immediately that $\delta_{a_i}(A) \in \{0, 1\}$, while for the property $\delta_{a_i}(A \cup B) = \delta_{a_i}(A) + \delta_{a_i}(B)$, if $A \cap B = \emptyset$, it is sufficient to check that it holds in the three possible cases: $a_i \subset A \cap B$ and $a_i \subset (A \cup B) \setminus (a_i \subset A \cap B)$ is excluded since $A \cap B = \emptyset$. Therefore, $\delta_{a_i}$ is a two-valued measure.

The result follows from a check that there are no other multiplicative measures. $\square$

Given a Boolean algebra $\mathcal{B}$, a subset $G \subset \mathcal{B}$ is said to be a set of generators for $\mathcal{B}$ if, for all $B \in \mathcal{B}$, $B$ can be represented in the form

$$B = (A_{1,1} \cap \ldots \cap A_{1,r_1}) \cup \ldots \cup (A_{s,1} \cap \ldots \cap A_{s,r_s}), \quad (A1)$$

where for all $m, n$ either $A_{m,n} \in G$ or $A_{m,n}^c \in G$.

A set $G$ of generators of a Boolean algebra $\mathcal{B}$ is said to be free if every mapping from $G$ to an arbitrary Boolean algebra $\mathcal{B}'$ can be extended to a $\mathcal{B}'$-valued homomorphism on $\mathcal{B}$. Moreover, a Boolean algebra is said to be freely generated if simply if it contains a set of free generators.

**Lemma A.3.** Given a free Boolean algebra $\mathcal{B}$ with $n$ free generators $G = \{A_1, \ldots, A_n\}$, it contains $2^n$ atoms $a_e$ which are given by the possible intersections

$$\bigcap_{i=1}^n (-1)^{1-e_i} A_i, \text{ where } -A_i = A_i^c \text{ and } e = (e_1, \ldots, e_n) \in \{0, 1\}^n.$$

Moreover, given a subalgebra $\mathcal{B}_0$ generated by $\{A_1, \ldots, A_k\}$, with $1 < k < n$, every atom of $\mathcal{B}_0$, which can be written as $b_j = (1) \ldots (1) - 1^{1-e_i} A_1 \cap \ldots \cap (-1)^{1-e_k} A_k$, with $e' = (e'_1, \ldots, e'_k) \in \{0, 1\}^k$, can be written in terms of atoms $a_{e(1), \epsilon}$ of $\mathcal{B}_0$, with $\epsilon \in \{0, 1\}^{n-k}$ and $(\epsilon, \bar{\epsilon}) \in \{0, 1\}^n$, as

$$b_j = \bigcup_{\epsilon \in \{0, 1\}^{n-k}} a_{e', \epsilon} \quad (A2)$$

**Proof.** First we notice that a multiplicative measure on $\mathcal{B}$ is a homomorphism between $\mathcal{B}$ and the set $\{0, 1\}$ with Boolean operations defined as $x \cap y = x \land y$, $x \lor y = x \lor y$ and $x = 1 - x$ for all $x, y \in \{0, 1\}$. Therefore, it follows from definition of free Boolean algebra that each map $f : G \rightarrow \{0, 1\}$ can be extended, in a unique way since $G$ is a set of generators, to a multiplicative measure.

All possible $(0, 1) - valued$ maps on $G$, are labeled by $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{0, 1\}^n$, i.e. $f_\epsilon : G \rightarrow \{0, 1\}$, where such a correspondence is given by $f_\epsilon(A_i) = \epsilon_i$. Since each map can be extended in a unique way to a multiplicative measure on $\mathcal{B}$, there is a one-to-one correspondence, by Lemma A.2, between $\epsilon \in \{0, 1\}$ and the atoms of $\mathcal{B}$. Let $m_\epsilon$ be the multiplicative measure that extends $f_\epsilon$. From condition $m_\epsilon(A_i) = \epsilon_i$ we obtain $m_\epsilon((-1) - e_i A_i) = 1$, which implies $m_\epsilon(\bigcap_{i=1}^n (-1)^{1-e_i} A_i) = 1$. It is then easy to verify that for every $\epsilon$ the element $\bigcap_{i=1}^n (-1)^{1-e_i} A_i$ is an atom and that they are all distinct, since they are in the right number for a bijection with multiplicative measures.

It is easy to show that $\bigcap_{i=1}^n (-1)^{1-e_i} A_i \cap \bigcap_{i=1}^n (-1)^{1-e_i} A_i = 0$ if $\epsilon \neq \epsilon'$, in fact, it implies $e_j \neq e_j'\epsilon_i$ for at least one $j$, therefore in the above product $A_j \cap A_j'$ must appear.

Now consider a generic $a_{e'}, \epsilon$, we show that $\forall B \in \mathcal{B}$ either $a_{e'} \subset B$, or $B \cap a_{e'} = \emptyset$. Since $\mathcal{B}$ is a free algebra, for all $B$ we can write $B = \bigcup_{i=1}^k \bigcap_{j \in I_i} (-1)^{1-e_i} A_i$, where $k \in \mathbb{N}$ and $c_i \in \{0, 1\}$ and $I_i \subset \{1, \ldots, n\}$. Now, for fixed $j$, either $c_j = c_j'$, for all $i \in I_j$, and then $a_{e'} \subset (\bigcap_{j \in I_i} (-1)^{1-e_i} A_i)$, or there exists $i$ such that $c_i' \neq c_i$ and then $a_{e'} \cap (\bigcap_{j \in I_i} (-1)^{1-e_i} A_i) = \emptyset$. By repeating this argument for all $j$ we obtain that either $a_{e'}$ is contained in at least one of the $(\bigcap_{j \in I_i} (-1)^{1-e_i} A_i)$, and then in $B$, or it has empty intersection with all of them and therefore $a_{e'} \cap B = \emptyset$. Therefore $a_{e'}$ is an atom.

For the second part, it is sufficient to write explicitly equation (A2) in terms of generators of $\mathcal{B}$ and use iteratively the equation $A = (A \cap B) \cup (A \cap B^c)$. $\square$

**Appendix B: Correlation polytopes**

In this appendix we shall derive extension criteria which follow from the translation of Pitowsky’s correlation polytopes results into the Boolean framework.
First, we briefly outline Pitowsky's result (for pair correlations): given $n$ atomic propositions $a_1, \ldots, a_n$ and positive numbers $p_{1}, \ldots, p_{n}$, and $p_{ij}$, $\{ij\} \in S \subseteq \{\{i\} \mid 1 \leq i < j \leq n\}$ associated to the propositions $a_1, \ldots, a_n$ and some logical conjunction of them $a_i \wedge a_j$, the numbers $p_{i}, p_{ij}$ are interpretable in terms of classical probabilities (i.e. there exists a probability space $(X, \Sigma, \mu)$ and $n$ events $A_1, \ldots, A_n$ such that $p_{i} = \mu(A_i)$, $p_{ij} = \mu(A_i \cap A_j)$) if and only if the vector $(p_{1}, \ldots, p_{n}, p_{ij}, \ldots)$ is a convex combination of the vectors $u_{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n, \ldots, \varepsilon_i \varepsilon_j, \ldots)$, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{0,1\}^n$. The problem can be expressed in terms of free Boolean algebras with the identification (see Givant and Halmos) of atomic propositions with free generators, logical operations with Boolean operations, truth assignments with two-valued measures, and probability assignment with normalized measures. A basic fact is the following:

**Lemma B.1.** Let $\mathfrak{B}$ be a finite Boolean algebra with $n$ atoms $\{a_1, \ldots, a_n\}$; then a function $m: \mathfrak{B} \to [0,1]$ is a normalized measure $\iff m = \sum_{i=1}^{n} \lambda_i \delta_{a_i}$, where $\delta_{a_i}$ is the multiplicative measure which is 1 on $a_i$, and $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$.

**Proof.** Given $A \in \mathfrak{B}$, it can be written as a disjoint union $A = \bigcup_{i \in I} a_i$, $I \subseteq \{1, \ldots, n\}$. Then
\[
 m(A) = \sum_{i \in I} m(a_i) = \sum_{i \in I} m(a_i) \delta_{a_i}(a_i) = \sum_{i=1}^{n} m(a_i) \delta_{a_i}(A).
\] (B1)

Moreover $0 \leq m(a_i) \leq 1$, since $m$ is a normalized measure, and $\sum_{i=1}^{n} m(a_i) = m(\bigcup_{i=1}^{n} a_i) = m(1) = 1$.

The converse is obvious. \(\Box\)

From this lemma we can obtain a criterion of extensibility to normalized measures for functions defined over a subset of a finite Boolean algebra. In fact, if we consider a finite Boolean algebra $\mathfrak{B}$ with $k$ atoms $\{a_1, \ldots, a_k\}$ and a subset $X \subseteq \mathfrak{B}$, then a function $f: X \to [0,1]$ can be extended to normalized measure $\mu$ on $\mathfrak{B}$ if and only if there are $k$ numbers $\lambda_1, \ldots, \lambda_k$, with $\lambda_i \geq 0$ and $\sum_{i=1}^{k} \lambda_i = 1$, such that
\[
f = \sum_{i=1}^{k} \lambda_i \delta_{a_i | X}
\] (B2)

and $\mu$ is given by:
\[
\mu = \sum_{i=1}^{k} \lambda_i \delta_{a_i}
\] (B3)

Therefore assignments of values in $[0,1]$ to elements of a subset of a Boolean algebra have a probabilistic interpretation if and only if such values are given by a convex combination of two-valued measures. This construction is closely related to the notion of correlation polytopes, as shown by the following

**Proposition B.1.** Let $\mathfrak{B}$ be a Boolean algebra freely generated by $\mathcal{G} = \{A_1, \ldots, A_n\}$ and $X \subseteq \mathfrak{B}$ with $X = \{A_1, \ldots, A_n\}$, $A_i \cap A_j \subseteq \mathcal{G}$, $i \neq j$. Then $X = \mathcal{G} \cup S_2 \cup \ldots S_m$, $m \leq n$, where elements of $S_l$ are the intersections of $l$ distinct generators, but not necessarily all of those possible, i.e. $|S_l| \leq \binom{n}{l}$. Now consider $f: X \to [0,1]$ and define the vector
\[
p = (p_1, \ldots, p_n, \ldots, p_{ij}, \ldots, p_{i_1 \ldots i_m}, \ldots) \in \mathbb{R}^{|X|}
\]
which has as components the values assumed by $f$ on $X$, namely
\[
p_i = f(A_i), \quad i = 1, \ldots, n
\]
\[
p_{ij} = f(A_i \cap A_j), \quad A_i \cap A_j \subseteq S_2
\]
\[
odd
\]
\[
p_{i_1 \ldots i_m} = f(A_{i_1} \cap \ldots A_{i_m}), \quad A_{i_1} \cap \ldots A_{i_m} \subseteq S_m
\]

For every $\varepsilon \in \{0,1\}^n$ define the vector $u_{\varepsilon} \in \{0,1\}^{|X|}$ given by
\[
u_{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n, \ldots, \varepsilon_i \varepsilon_j, \ldots, \varepsilon_{i_1} \varepsilon_{i_2} \ldots \varepsilon_{i_m}, \ldots)
\]
i.e. for every component $p_{i_1 \ldots i_k}$ of $p$ there is a corresponding component of $u_{\varepsilon}$ given by $\varepsilon_{i_1} \ldots \varepsilon_{i_k}$.

Then $f$ can be extended to a normalized measure on $\mathfrak{B}$ if and only if there are $2^n$ numbers $\lambda(\varepsilon)$, $\varepsilon \in \{0,1\}^n$, such that:
\[
p = \sum_{\varepsilon \in \{0,1\}^n} \lambda(\varepsilon) u_{\varepsilon}
\]

with $\lambda(\varepsilon) \geq 0$ for all $\varepsilon \in \{0,1\}^n$, and
\[
\sum_{\varepsilon \in \{0,1\}^n} \lambda(\varepsilon) = 1
\]

**Proof.** Given a free Boolean algebra with $n$ free generators, there are (see Lemma A.3) $2^n$ atoms in a one-to-one correspondence with $\varepsilon \in \{0,1\}^n$. The extensibility condition (B2) can therefore be written, with $a_{\varepsilon}$ defined as in Lemma A.3
\[
f = \sum_{\varepsilon \in \{0,1\}^n} \lambda(\varepsilon) \delta_{a_{\varepsilon} | X}
\] (B4)

Now it suffices to verify such a condition for every element of $X$, which is equivalent to verify that the vector $p$ is a convex combination of the vectors $u_{\varepsilon}$. In fact, $\delta_{a_{\varepsilon}}(A_i) = \varepsilon_i$ (see the proof of Lemma A.3) and $\delta_{a_{\varepsilon}}(A_{i_1} \cap \ldots A_{i_k}) = \varepsilon_{i_1} \ldots \varepsilon_{i_k}$, therefore:
We denote the correlation polytope associated with a subset $X \subset \mathcal{B}$ of the above mentioned form as $C(G, S_2, \ldots, S_m)$.

A helpful fact about the characterization of measures is the following

**Lemma B.2.** Let $\mathcal{B}$ be a Boolean algebra freely generated by $\{A_1, \ldots, A_n\}$, let $\mu$ be a measure on $\mathcal{B}$ and let $X \subset \mathcal{B}$ the set of all possible intersections between the free generators of $\mathcal{B}$, i.e. $X = \{A_1, \ldots, A_n, A_1 \cap A_2, \ldots, A_1 \cap A_2 \cap \ldots \cap A_n, \ldots, A_1 \cap \ldots \cap A_n\}$. Then the measure $\mu$ is uniquely defined by the values it assumes on the set $X$.

**Proof** Since $\mathcal{B}$ is a finite algebra, it is atomic. Therefore every element $B \in \mathcal{B}$ can be written in a unique way as a disjoint union of atoms, $B = \bigcup_{\varepsilon \in H_B} a_{\varepsilon}$, where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{0, 1\}^n$, $a_\varepsilon = \bigcap_i (-1)^{1-\varepsilon_i} A_i$ is the atom given by the bijection of Lemmas [A3, Appendix C] and $H_B \equiv \{ \varepsilon \in \{0, 1\}^n | a_\varepsilon \subset B \}$. It follows that $\mu(B) = \sum_{\varepsilon \in H_B} \mu(a_\varepsilon)$ since atoms are disjoint, therefore the measure is completely defined by the values assumed on atoms.

Now, it suffices to prove that the measure of every atom is uniquely determined by the values assumed by $\mu$ on the elements of $X$. But this follows from the fact that every atom can be written in the form $a_\varepsilon = \bigcap_i (-1)^{1-\varepsilon_i} A_i$ and the property of measures

$$
\mu(A) = \mu(A \cap B) + \mu(A \cap B^c) \quad \text{for all } A, B \in \mathcal{B}.
$$

In fact, consider an atom $a_\varepsilon = (-1)^{1-\varepsilon_1} A_1 \cap \ldots \cap (-1)^{1-\varepsilon_n} A_n$; if $\varepsilon_i = 1$ for all $i$, then $a_\varepsilon \in X$ so there is nothing to show. Assume, therefore, that there is $i_0$ such that $\varepsilon_{i_0} = 0$; thus, for [B5, Appendix C], $\mu(a_\varepsilon)$ can be written in terms of elements of $X$ as $\mu(a_\varepsilon) = \mu(\bigcap_{i \neq i_0} A_i) - \mu(\bigcap_i A_i)$.

If there is more than one component of $\varepsilon$ equal to 1, this process can be iterated until $\mu(a_\varepsilon)$ is written in terms of elements of $X$. □

The above general results provide a criterion for the extensibility of functions defined over a subset of a free Boolean algebra to normalized measures on the entire algebra, and therefore a criterion of classical representability for PPTs. A simple application is given by the following result:

**Proposition B.2.** Let $\mathcal{B}$ be the Boolean algebra freely generated by $\{A_1, \ldots, A_n\}$, let $\mathcal{B}_1$ and $\mathcal{B}_2$ be the subalgebras generated respectively by $\{A_1, \ldots, A_k\}$ and $\{A_{k+1}, \ldots, A_n\}$ with $1 \leq k < n$, and let $\mu_1$ and $\mu_2$ be two normalized measure respectively on $\mathcal{B}_1$ and $\mathcal{B}_2$. Then there exists a normalized measure $\mu$ on $\mathcal{B}$ such that $\mu_{\mid_{\mathcal{B}_1}} \equiv \mu_1$ and $\mu_{\mid_{\mathcal{B}_2}} \equiv \mu_2$.

**Proof** Every atom $a_\varepsilon$, $\varepsilon \in \{0, 1\}^n$ (see Lemma [A.3, Appendix C]) of $\mathcal{B}$ can be written uniquely as $a_\varepsilon = a_{\varepsilon'} \cap a_{\varepsilon''}$, where $a_{\varepsilon'}$ is an atom of $\mathcal{B}_1$, $\varepsilon' \in \{0, 1\}^k$ and $a_{\varepsilon''}$ is an atom of $\mathcal{B}_2$, $\varepsilon'' \in \{0, 1\}^{n-k}$. Then $\mu$ can be defined as $\mu(a_\varepsilon) = \mu_1(a_{\varepsilon'}) \mu_2(a_{\varepsilon''})$ for all atoms $a_\varepsilon \in \mathcal{B}$. It is obviously a measure (non-negative and additive on disjoint elements); conditions $\mu_{\mid_{\mathcal{B}_1}} \equiv \mu_1$, $i = 1, 2$ and normalization condition follow easily from (A2) (see Lemma [A.3, Appendix C] and normalization conditions for $\mu_1$ and $\mu_2$. □

This provides a direct elementary construction of a classical extension for a PPT where Boolean subalgebras of compatible observables have only trivial (i.e. $\{0, 1\}$) intersections; this is the case, e.g., of a (finite) collection of spin measurements on a spin 1/2 particle discussed by Bell[13] and Kochen and Specker[2].

**Appendix C: Horn-Tarski partial measures**

The notion of partial measure was introduced by Horn and Tarski[14] to analyze the possibility of extending a measure defined on a subalgebra of a Boolean algebra, or even a function defined on an arbitrary subset of the algebra, to a measure on the entire algebra. Even if such a notion was not related to QM by the authors, the reduction of the extension problem to the case of H-T PPTs transforms the H-T results into fundamental general criteria of classical representability.

For the reader’s convenience we give therefore a brief summary of Horn and Tarsi’s results.

**Definition C.1.** Let $A_0, \ldots, A_{m-1}$ and $B_0, \ldots, B_{n-1}$ be
elements of a Boolean algebra $\mathcal{B}$, we say that
\[ \langle A_0, \ldots, A_{m-1} \rangle \leq \langle B_0, \ldots, B_{n-1} \rangle \quad \text{(C1)} \]
if
\[ \bigcup_{r \in S_{k,m}} \bigcap_{i \leq k} A_r, \subset \bigcup_{r \in S_{k,n}} \bigcap_{i \leq k} B_r, \quad \text{(C2)} \]
for all $k < m$, where $S_{k,n}$ is the set of all sequences of natural numbers $r = (r_0, \ldots, r_k)$ with $0 \leq r_0 < \cdots < r_k < n$.

**Definition C.2.** A real function $f$ defined over a subset $S$ of a Boolean algebra $\mathcal{B}$ is a partial measure if the following conditions are satisfied

(i) $f(x) \geq 0$ for all $x \in S$.

(ii) If $A_0, \ldots, A_{m-1}$ and $B_0, \ldots, B_{n-1}$ are elements of $S$ and
\[ \langle A_0, \ldots, A_{m-1} \rangle \leq \langle B_0, \ldots, B_{n-1} \rangle, \]
then
\[ \sum_{i<n} f(A_i) \leq \sum_{j<p} f(B_j). \]

(iii) $1 \in S$ and $f(1) = 1$.

An obvious consequence of Definition C.2 is that if $1 \in S \subset T$ and $f$ is a partial measure on $T$, then $f$ is a partial measure on $S$. Another important property is shown by the following

**Theorem C.1.** Let $S \subset \mathcal{B}$ be a subalgebra of $\mathcal{B}$ (in particular $S = \mathcal{B}$); then a function $f$ on $S$ is a partial measure on $S$ if and only if $f$ is a measure on $S$.

As shown by Theorem C.3, a fundamental role is played by the following notions

**Definition C.3.** Let $S \subset \mathcal{B}$, and $f$ be a partial measure on $S$ and $x \in \mathcal{B}$. We define the exterior measure of $x$ with respect to $f$, and we write $f_e(x)$, as the greatest lower bound of numbers $\xi$ of the form
\[ \xi = \frac{1}{m} \left[ \sum_{i<n} f(A_i) - \sum_{j<p} f(B_j) \right] \quad \text{(C3)} \]
where $A_i, B_j \in S$ for $i < n, j < p$ and where
\[ \langle B_0, \ldots, B_{p-1}, x_0, \ldots, x_{m-1} \rangle \leq \langle A_0, \ldots, A_{n-1} \rangle. \quad \text{(C4)} \]
with $x_i = x$ for all $i < m$.

Similarly, we define the interior measure of $x$ with respect to $f$, and we write $f_i(x)$, as the least upper bound of numbers $\xi$ of the form (C3), where
\[ \langle A_0, \ldots, A_{n-1} \rangle \leq \langle B_0, \ldots, B_{p-1}, x_0, \ldots, x_{m-1} \rangle. \quad \text{(C5)} \]
with $x_i = x$ for all $i < m$.

**Theorem C.2.** If $f$ is a (partial) measure on a subalgebra $\mathcal{B}_0 \subset \mathcal{B}$, then$f_i(x) = \sup \{ f(y) | y \in \mathcal{B}_0, y \subset x \}$ and $f_e(x) = \inf \{ f(y) | y \in \mathcal{B}_0, x \subset y \}$.

Horn and Tarski’s main results are the following

**Theorem C.3.** Let $f$ be a partial measure on $S \subset \mathcal{B}$, $x \in \mathcal{B}$ and $g$ a function on $S \cup \{ x \}$ that coincide with $f$ on $S$; then $g$ is a partial measure on $S \cup \{ x \}$ if and only if
\[ f_i(x) \leq g(x) \leq f_e(x). \quad \text{(C6)} \]

**Theorem C.4.** Let $f$ be a partial measure on $S$ and $S \subset T \subset \mathcal{B}$; then a partial measure $g$ on $T$ that coincide with $f$ on $S$ exists.

**Theorem C.5.** Let $f$ be a partial measure on a subset $S$ of a Boolean algebra $\mathcal{B}$; then a measure $\mu$ on $\mathcal{B}$ that coincide with $f$ on $S$ exists.

We remark that Horn and Tarski’s notion of partial measure has nothing to do with partial Boolean algebras, in the Kochen-Specker or in our version, nor with partial probability theories; even if H-T partial measures are defined only on subsets, they satisfy conditions which imply their extensibility to measures on the entire algebra. Our approach is rather based on a comparison of the two notions, with the result of transforming the H-T conditions for H-T partial measures into extensibility conditions for PPTs, in the case of PBAs which are embeddable in a Boolean algebra.

Since the condition in Theorem C.5 is obviously also necessary, the extensibility criterion provided by partial measures is equivalent to the correlation polytope criterion. In particular, this implies that inequalities derived from the correlation polytope criterion (Proposition B.1) are equivalent to those derived from condition (ii) of Definition C.2.

It should be also remarked that, while Pitowsky’s correlation polytopes implicitly assume a free Boolean structure, Horn and Tarski’s extensibility criterion also applies to non-free Boolean algebras and to more general subsets with respect to those considered in Proposition B.1 in principle, therefore, the H-T criterion also allows for a direct analysis of extensions of states on a PBA in the framework of projection algebra PPTs.

---

1. A.M. Gleason, J. Math. and Mech. 6, 885 (1957)
2. S. Kochen and E.P. Specker, J. Math. and Mech. 17, 59 (1967)
3. J.S. Bell, Physics 1, 195 (1964)
4. N.D. Mermin, Am. J. Phys. 66, 753 (1998), see also arXiv:quant-ph/9801057
5. I. Pitowsky, Quantum Probability Quantum Logic, (Springer, Berlin, 1989)
6. C. Budroni, Master Thesis, http://etd.adm.unipi.it/theses/available/etd-09302009-171625/ (Università di Pisa, 2009)
7. C. Garola and L. Solombrino, Found. Phys., 26, 1329 (1996)
8. A. Cabello, Phys. Rev. Lett. 101, 210401 (2008), see also arXiv:0808.2456 [quant-ph]
9. P. Badziag, I. Bengtsson, A. Cabello and I. Pitowsky, Phys. Rev. Lett. 103, 050401 (2009), see also arXiv:0809.0430 [quant-ph]
10. A. Fine, Phys. Rev. Lett. 48, 291 (1982)
11. A. Horn and A. Tarski, Trans. Am. Math. Soc. 64, 467 (1948)
15. A. Einstein, B. Podolsky and N. Rosen, Phys. Rev. 47, 777 (1935)
16. R. Sikorski, *Boolean algebras*, (Springer-Verlag, Berlin, 1964)
17. S. Givant and P. Halmos, *Introduction to Boolean Algebras*, (Springer, New York, 2009)
18. J.S. Bell, Rev. Mod. Phys. 38, 447 (1966)