Abstract. We define a new class of quantum vertex algebras, based on the Hopf algebra $H_D = \mathbb{C}[D]$ of "infinitesimal translations" generated by $D$. Besides the braiding map describing the obstruction to commutativity of products of vertex operators, $H_D$-quantum vertex algebras have as main new ingredient a "translation map" that describes the obstruction of vertex operators to satisfying translation covariance. The translation map also appears as obstruction to the state-field correspondence being a homomorphism.

We use a bicharacter construction of Borcherds to construct a large class of $H_D$-quantum vertex algebras. One particular example of this construction yields a quantum vertex algebra that contains the quantum vertex operators introduced by Jing in the theory of Hall-Littlewood polynomials.

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1. Introduction

Vertex operators were introduced in the earliest days of string theory and play now an important role of such areas of mathematics as representation theory, algebraic topology and random matrices. Vertex algebras were introduced to axiomatize the properties of vertex operators.

Similarly, quantum vertex operators were discovered in integrable models in statistical mechanics and in connection with theory of symmetric polynomials and the theory of quantum affine algebras. One would like to have theory of quantum vertex algebras to axiomatize the properties of quantum vertex operators. In this paper we introduce and study a class of quantum vertex algebras that produce the quantum vertex operators related to Hall-Littlewood polynomials.

Recall that a vertex operator on a space $V$ is a series $a(z) = \sum_{n \in \mathbb{Z}} a(n)z^{-n-1}$, $a(n) \in \text{End}(V)$, satisfying some extra conditions. We call vertex operators \textit{local} with respect to each other if the commutator is a sum of derivatives of delta distributions:

$$[a(z_1), b(z_2)] = \sum_{n=0}^{N} c_n(z_2) \partial z_2 \delta(z_1, z_2).$$

For \textit{quantum vertex operators} this will not longer be true: one needs a \textit{braiding} map $S_{z_1, z_2} : b(z_2)a(z_1) \rightarrow ba(z_2, z_1)$, where $ba(z_2, z_1)$ is some other $\text{End}(V)$-valued series. Then we should have $S$-\textit{locality}, \cite{EK00}, i.e., we need that the \textit{braided commutator}

$$[a(z), b(z_2)]_S = a(z_1)b(z_2) - ba(z_2, z_1)$$

is a sum of derivatives of delta distributions. One of the goals of this papers is give explicit examples of quantum vertex algebras where one can easily calculate both the quantum vertex operators and their braiding.

There are several proposals for what a quantum vertex algebra should be. There is Borcherds’ theory of $(A, H, S)$-vertex algebras, see \cite{Bor01}, the Etingof-Kazhdan theory of quantum vertex operator algebras, \cite{EK00}, and the Frenkel-Reshet’kin theory of deformed chiral algebras, see \cite{FR97}. (H. Li has developed the Etingof-Kazhdan theory further, see for example \cite{Li06a}, \cite{Li05}.)

The Borcherds theory is based on the observation that products and iterates of vertex operators in vertex algebras are expansions of rational functions in multiple variables. The idea then is to start with these rational objects instead of constructing them after the fact from the vertex operators. Instead of a single vector space $V$
on which the vertex operators act, one has for any integer $n \geq 1$ the space $V(n)$ of "rational vertex operators" in $n$ variables. This is quite beautiful idea, and is easily adapted to include quantum vertex algebras of $(A, H, S)$-type. However, even for classical vertex algebras it seems not known how to include such basic examples as affine vertex algebras in the $(A, H, S)$-framework. In this paper we therefore we prefer to develop a theory that is closer to the usual theory, with a single underlying vector space $V$. We do take, however, from Borcherds' paper the idea of a *bicharacter* as a method to construct examples: we will use bicharacters both to produce the vertex operators and the braiding. (See [Ang06] for more details.)

The Etingof-Kazhdan theory is very close to the classical theory, in fact so close that it is not suitable to describe quantum vertex operators related to symmetric polynomials. Briefly, in the usual theory (and in [EK00]) vertex operators $a(z)$ satisfy translation covariance of the form

$$e^{\gamma D} a(z_1) e^{-\gamma D} = a(z_1 + \gamma),$$

where $D : V \to V$ is the infinitesimal translation operator and we expand in positive powers of $\gamma$. If we introduce notation $a(z)b = Y_z(a \otimes b)$ we can write this, since $\partial_z a(z) = (Da)(z)$, as

$$e^{\gamma D} Y_z = Y_z \circ (e^{\gamma D} \otimes e^{\gamma D}),$$

making clear the similarity of a vertex algebra with a associative ring $R$ with a group action (where $gm(a \otimes b) = m(ga \otimes gb)$ if $m$ is the multiplication in $M$, and $g \in G$, $a, b \in M$, see Appendix A).

It was shown in the thesis [Ang06] that (1.1) can not hold in the case of quantum vertex operators related to symmetric polynomials. Also, not unrelated, the braiding map $S_{z_1, z_2}$ in [EK00] is in fact assumed to be of the form $S_{z_1, z_2} = \tilde{S}_{z_1 - z_2}$, where $\tilde{S}_z$ is a function of a single variable. It is also shown in [Ang06] that this does not holds for symmetric polynomials.

In this paper we introduce the notion of an $H_D$-quantum vertex algebra (where $H_D = \mathbb{C}[D]$ is the Hopf algebra of infinitesimal translations), generalizing [EK00] in various ways. First we need to relax the translations covariance (1.1). We introduce, besides the braiding map $S_{z_1, z_2}^{(\tau)}$, also another map $S_{z_1, z_2}^{(\gamma)}$ on $V \otimes V$ such that we get instead

$$e^{\gamma D} Y_{z_1} \circ S_{z_1, z_2}^{(\gamma)} = Y_{z_1} \circ (e^{\gamma D} \otimes e^{\gamma D}).$$

Both $S_{z_1, z_2}^{(\gamma)}$ and $S_{z_1, z_2}^{(\tau)}$ are rational functions of both $z_1$ and $z_2$, not just of the difference $z_1 - z_2$ as in [EK00]. Another difference is that in [EK00] vertex operators satisfy a braided version of skew-symmetry:

$$(1.2) \quad Y_z \circ S_{z, 0}^{(\tau)}(a \otimes b) = e^{\varepsilon D} Y(b, -z)a.$$

This relation does not make sense for quantum vertex operators coming from symmetric polynomials: the braiding $S_{z_1, z_2}^{(\tau)}$ is in general *singular* for $z_2 = 0$. This motivates us to take as basic building block of the theory not the vertex operator $Y_z$, but the *two-variable* vertex operators $X_{z_1, z_2} : V \otimes V \to V[[z_1, z_2]][z_1^{-1}, (z_1 - z_2)^{-1}][[t]].$ We can define then $Y$ by $Y(a, z)b = X_{z, 0}(a \otimes b)$, but $Y$ does not longer satisfy (1.2). See Corollary 8.2 for the version of skew-symmetry that holds for $H_D$-quantum vertex algebras.
Conversely, if we start with $Y$, we can introduce $X_{z_1,z_2}$ by analytic continuation: we have the expansion

$$i_{z_1,z_2}X_{z_1,z_2}(a \otimes b) = Y(a,z_1)Y(b,z_2)1,$$

(1.3)

where $i_{z,w}$ is the expansion in the region $|z| > |w|$. See Section 7 for details and an alternative definition of $X_{z_1,z_2}$.

Note that for a classical vertex algebra (and also for an Etingof-Kazhdan quantum vertex operator algebra) the translation map $S_{z_1,z_2}$ is the identity, so that in this case

$$X_{z_1,z_2}(a \otimes b) = e^{z_2}Y(a,z_1 - z_2)b \in V[[z_1, z_2]]((z_1 - z_2)^{-1}).$$

In particular in these cases $X_{z_1,z_2}(a \otimes b)$ is not singular for $z_1 = 0$. We consider a more general theory where in $X_{z_1,z_2}(a \otimes b)$ poles in $z_1$ are allowed (and in fact are necessary to be able to treat the quantum vertex operators associated with the Hall-Littlewood polynomials).

In the construction of quantum vertex algebras one or more quantum parameters will appear. They can usually be thought of as describing the deformation away from an ordinary vertex algebra. We should mention that, just as when quantizing universal enveloping algebras, there are two ways of interpreting the quantum parameters in quantum vertex algebras. Either the quantum parameters are independent formal variables or they are complex numbers. The theory of Etingof-Kazhdan follows the first approach, as opposed to the Frenkel-Reshetikhin definition of deformed chiral algebras, which considers the deformation parameter(s) to be complex number(s). In this paper we also follow the first approach: we have an independent variable $t$ and a $D$-quantum vertex algebra $V$ is a (free) module over the ring $\mathbb{C}[t]$ of formal power series in $t$. When putting $t = 1$ one gets in examples generally an ordinary vertex algebra, although we did not require this in our axioms. Note that Li in [Li06a], for instance, studies a form of the Etingof-Kazhdan axioms where the quantum parameter is a complex number.

Maybe the most important difference between our $D$-quantum vertex algebras and classical vertex algebras (and the theory of Etingof-Kazhdan) is the following. We can define a products of states: $a(n)b$, and for fields: $a(z)(n)b(z)$, but is is not longer true that that the state-field correspondence $a \mapsto Y(a,z)$ a homomorphism of products: in general $Y(a(n)b,z) \neq Y(a,z)(n)Y(b,z)$, see Theorem 17.1 for an exact statement.

The outline of the paper is as follows. There are three parts. In the first part we define $H_D$-quantum vertex algebras in section 8 and study their properties in the following sections. We derive in Section 14 and 15 fundamental identities in our quantum vertex algebras: the braided Jacobi identity and the braided Borcherds identity. These are used Sections 16, 17 and 18 to study the $S$-commutator, $(n)$-products of states and of fields and normal ordered products. In Section 19 we derive a weak associativity relation. In the next part of the paper we assume that our underlying vector space $V$ is a commutative and cocommutative Hopf algebra, which allows us to defines bicharacters on $V$ in section 21. Using bicharacters we construct a class $H_D$-quantum vertex algebras in Section 22 and in the rest of this section we explore some of the properties of bicharacter $H_D$-quantum vertex algebras. In last part of the paper, Section 23 and the following sections, we study in detail a single example of a Hopf algebra $V$ with a fixed bicharacter on it. The resulting $H_D$-quantum vertex algebra is a deformation of the familiar lattice
vertex algebra based on the lattice \(L\) with pairing \((m,n) \mapsto mn\). Some of quantum vertex operators in this example were used by Jing, see [Jin91], to study Hall-Littlewood symmetric polynomials. In the Appendix A we describe the “nonsingular” analog of \(H_D\)-quantum vertex algebras: braided algebras with group action. In Appendix B we discuss the construction of braiding maps for \(H_D\)-quantum vertex algebras.

2. The Hopf Algebra \(H_D\)

Let \(H_D = \mathbb{C}[D]\) be the universal enveloping algebra of the 1-dimensional Lie algebra generated by \(D\). \(H_D\) is a Hopf-algebra, with coproduct \(\Delta_{H_D}: D \mapsto D \otimes 1 + 1 \otimes D\), antipode \(S: D \mapsto -D\) and counit \(\epsilon_{H_D}: D \mapsto 0\). \(H_D\) is a fundamental ingredient in the construction of vertex algebras, where it appears as the symmetry algebra of infinitesimal translations in physical space. In this paper the full Hopf algebra structure of \(H_D\) will play only an explicit role when we discuss bicharacter constructions, in the definition of \(H_D\)-quantum vertex algebras in the next section only the algebra structure will be used. However, from Borcherds’ papers [Bor98] and [Bor01] it will be clear that in fact the Hopf algebra \(H_D\) underlies the whole theory of vertex algebras (and their quantum versions).

3. \(H_D\)-Quantum Vertex Algebras

Let \(t\) be a variable. We will use \(t\) to describe quantum deformations, the classical limit corresponding to \(t \to 0\). Let \(k = \mathbb{C}[t]\) and let \(V\) be an \(H_D\)-module and free \(k\)-module. Denote by \(V[[t]]\) the space of (in general infinite) sums

\[
v(t) = \sum_{i=0}^{\infty} v_i t^i, \quad v_i \in V.
\]

In case \(v(t) \in V[[t]]\) has only finitely many nonzero terms we can identify it with an element of \(V\). In the same way will consider spaces such as \(V[[z]][z^{-1}][t]\) consisting of sums

\[
v(z,t) = \sum_{i=0}^{\infty} v_i(z) t^i, \quad v_i \in V[[z]][z^{-1}].
\]

We will also consider rational expressions in multiple variables and their expansions. For instance for a rational function in \(z_1, z_2\) with only possibly poles at \(z_1 = 0, z_2 = 0\) or \(z_1 = z_2 = 0\) we can define expansion maps

\[
i_{z_1, z_2}: \frac{1}{z_1 - z_2} \mapsto \sum_{n \geq 0} z_1^{-n-1} z_2^n, \quad \frac{1}{z_1} \mapsto \frac{1}{z_1}, \quad \frac{1}{z_2} \mapsto \frac{1}{z_2},
\]

\[
i_{z_2, z_1}: \frac{1}{z_2 - z_1} \mapsto -\sum_{n \geq 0} z_2^{-n-1} z_1^n, \quad \frac{1}{z_2} \mapsto \frac{1}{z_2}, \quad \frac{1}{z_1} \mapsto \frac{1}{z_1},
\]

\[
i_{z_2, z_1 - z_2}: \frac{1}{z_2} \mapsto \sum_{n \geq 0} z_2^{-n-1} (z_1 - z_2)^n, \quad \frac{1}{z_2} \mapsto \frac{1}{z_2}, \quad \frac{1}{z_1 - z_2} \mapsto \frac{1}{z_1 - z_2}.
\]

We will write \(i_{z_1, z_2; w, w_1, w_2}\) for \(i_{z_1; w, z_2; w_1, w_2}\), and \(i_{z_1, z_2; w; w_1, w_2}\) for \(i_{z_1, z_2; w_1, w_2\cdot z_1, z_2; w_2}\). We define \(i_{z_1, z_2; \cdots; z_n}\) to be the expansion in the region \(|z_1| > |z_2| > \cdots > |z_n|\).

If \(A \in V \otimes V\) then we define for instance \(A^{23}, A^{13} \in V^\otimes 3\) by \(A^{23} = 1 \otimes A\), and \(A^{13} = a' \otimes 1 \otimes a''\), if \(A = a' \otimes a''\).
Now we are ready to define the central concept of this paper. The definition is rather complicated, and in Appendix A we explain a simpler version of this notion, called a braided ring with symmetry, where the multiplication is nonsingular.

**Definition 3.1.** Let $V$ be a free $k = \mathbb{C}[t]$-module and an $H_D$-module. An $H_D$-quantum vertex algebra structure on $V$ consists of

- $1 \in V$, the vacuum vector.
- A (singular) multiplication map
  $$X_{z_1,z_2} : V \otimes V \rightarrow V[[z_1, z_2]][z_1^{-1}, (z_1 - z_2)^{-1}][[t]].$$
- A braiding map $S^{(\gamma)}$ and a translation map $S^{(\tau)}$ of the form
  $$S^{(\gamma)}_{z_1,z_2} : V \otimes V[[z_1, z_2]][z_1^{-1}, (z_1 - z_2)^{-1}][[t]],$$
  $$S^{(\tau)}_{z_1,z_2} : V \otimes V[[z_1, z_2]][z_1^{-1}, (z_1 - z_2)^{-1}][[t]].$$

These objects satisfy the following axioms:

(Vacuum): For $i = 1, 2$

(3.1) $X_{z_1, z_2}(a \otimes 1) = e^{\gamma_D} a,$  
(3.2) $X_{z_1, z_2}(1 \otimes a) = e^{\gamma_D} a,$  
(3.3) $X_{z_1, z_2}(a \otimes b) = \partial_{z_2} X_{z_1, z_2}(a \otimes b),$  
(3.4) $(1 \otimes e^{\gamma_D}) i_{z_1 - z_2, z_2} S_{z_1, z_2 + \gamma} = S_{z_1, z_2}(1 \otimes e^{\gamma_D}),$
(3.5) $e^{\gamma_D} X_{z_1, z_2} S^{(\gamma)}_{z_1, z_2} = X_{z_1 + \gamma, z_2 + \gamma}.$

(Yang-Baxter):

(3.6) $S_{z_1, z_2}^{12} S_{z_1, z_3}^{13} S_{z_2, z_3}^{23} = S_{z_2, z_3}^{23} S_{z_1, z_3}^{13} S_{z_1, z_2}^{12}.$

(Compatibility with Multiplication):

(3.7) $S_{z_1, z_2}(X_{w_1, w_2} \otimes 1) = (X_{w_1, w_2} \otimes 1) i_{z_1 - z_2, w_1} S_{z_1 + w_1, z_2 + w_1}^{23} S_{z_1, z_2 + w_1}^{23};$
(3.8) $S_{z_1, z_2}(1 \otimes X_{w_1, w_2}) = (1 \otimes X_{w_1, w_2}) i_{z_1 - z_2, w_1} S_{z_1 + w_1, z_2 + w_1}^{12} S_{z_1, z_2 + w_1}^{12};$

(Group Properties):

(3.9) $S^{(\tau)}_{z_1, z_2} \circ \tau \circ S^{(\tau)}_{z_2, z_1} \circ \tau = 1_{V \otimes 2},$
(3.10) $S^{(\gamma_1)}_{z_1, z_2} S^{(\gamma_2)}_{z_1, z_2 + \gamma_1} = S^{(\gamma_1 + \gamma_2)}_{z_1, z_2},$
(3.11) $S^{(\gamma = 0)}_{z_1, z_2} = 1_{V \otimes 2}.$

(Locality): For all $a, b \in V$ and $k \geq 0$ there is $N \geq 0$ such that for all $c \in V$

(3.12) $((z_1 - z_2)^N X_{z_1, 0}(1 \otimes X_{z_2, a})(b \otimes c) \equiv (z_1 - z_2)^N X_{z_2, 0}(1 \otimes X_{z_1, a})(b \otimes a \otimes c) \mod t^k.$

**Remark 3.2.** In the above definition $z_1, z_2, w_1, w_2, \gamma$ are independent commuting variables. In general one should be careful with specializing these variables. For instance, we can evaluate $X_{z_1, z_2}$ at $z_2 = 0$ but not at $z_1 = 0$, in general. For this reason one can not put $\gamma = -z_1$ in (3.3).
Remark 3.3. The vacuum axioms (3.1) for $z_2 = 0$ are
\[ X_{z_1,0}(a \otimes 1) = e^{z_1 D} a, \quad X_{z_1,0}(1 \otimes a) = a. \]
In the literature on vertex algebras the first equation is called the *creation axiom*, and the second the vacuum axiom. In our formalism it seems unnatural to give different names to very similar statements, so we call in (3.1) both vacuum axioms, as they involve the vacuum vector 1.

4. Intermezzo on Expansions

Let $W$ be a vector space and $A(z_1, z_2) \in W((z_1))((z_2))$. It is well known\(^1\) that if there is an $N \geq 0$ such that
\[
A_N = (z_1 - z_2)^N A(z_1, z_2) \in W[[z_1, z_2]][z_1^{-1}, z_2^{-1}],
\]
then $A(z_1, z_2)$ is in the image of the (injective) map $i_{z_1; z_2}$, i.e., there is a (unique) $X(z_1, z_2) \in W[[z_1, z_2]][z_1^{-1}, z_2^{-1}, (z_1 - z_2)^{-1}]$ such that we have the expansion
\[
A(z_1, z_2) = i_{z_1; z_2} X(z_1, z_2).
\]
In fact, we can take $X(z_1, z_2) = (z_1 - z_2)^{-N} A_N$. In this case we have also
\[
(z_1 - z_2)^N A(z_1, z_2) = (z_1 - z_2)^N X(z_1, z_2).
\]
(Note that although $A_N$ depends on $N$, we obtain the same $X$ for all $N$ that make (4.1) true, cf. [L03].)

One way to check (4.1) is by finding $B(z_2, z_1) \in W((z_2))((z_1))$ such that
\[
(z_1 - z_2)^N A(z_1, z_2) = (z_1 - z_2)^N B(z_2, z_1).
\]
Indeed, the LHS shows that (4.1) has at worst a finite order pole in $z_2$ (by assumption on $A(z_1, z_2)$) and the RHS that at worst it has a finite order pole in $z_1$ (by assumption on $B(z_2, z_1)$). This means that (4.4) belongs to $W[[z_1, z_2]][z_1^{-1}, z_2^{-1}]$, as we wanted to show. In this case we have not only that $A$ is the expansion (4.2) of $X$, but also that $B$ is the “opposite” expansion:
\[
B(z_1, z_2) = i_{z_2; z_1} X(z_1, z_2).
\]
There are generalizations to more variables $z_1, z_2, \ldots, z_n$, and to various expansion maps.

We will need slight refinements of these phenomena in case there is a quantum parameter $t$ present. For example:

Lemma 4.1. Let
\[
A(z_1, z_2; t) \in W((z_1))((z_2))[[t]],
\]
and suppose that for all $k \geq 0$ there is an $N \geq 0$ such that
\[
A_N^k \equiv (z_1 - z_2)^N A(z_1, z_2; t) \mod t^k \in W[[z_1, z_2]][z_1^{-1}, z_2^{-1}][[t]]/(t^k).
\]
Then there is a $X(z_1, z_2) \in W[[z_1, z_2]][z_1^{-1}, z_2^{-1}, (z_1 - z_2)^{-1}][[t]]$ such that
\[
i_{z_1; z_2} X(z_1, z_2) = A(z_1, z_2).
\]

\(^1\) See for instance the notion of compatible fields in Definition 7.3 of [Bor98], [Sny], and the reformulation of compatibility in [L03], [L06].
Proof. If (4.5) holds for some $N$ we can define

$$X^k = (z_1 - z_2)^-N A_N^k \in W[[z_1, z_2]][z_1^{-1}, z_2^{-1}, (z_1 - z_2)^{-1}][[t]] / (t^k),$$

and we have

$$i_{z_1, z_2} X^k(z_1, z_2) = A_N^k(z_1, z_2; t).$$

Then the $X^k$s fit together to define a (unique) $X(z_1, z_2; t) \in W[[z_1, z_2]][z_1^{-1}, z_2^{-1}, (z_1 - z_2)^{-1}][[t]]$ such that (4.6) holds. \qed

Note that there need be no uniform $N$ that makes (4.5) true for all $k$; consider for instance the case $A(z_1, z_2; t) = i_{z_1, z_2} e^{t/(z_1 - z_2)}$.

**Lemma 4.2.** If there are

$$A(z_1, z_2; t) \in W((z_1))(\langle z_2 \rangle)[[t]], \quad B(z_2, z_1; t) \in W((z_2))(\langle z_1 \rangle)[[t]]$$

such that there is for all $k \geq 0$ an $N \geq 0$ such that

$$(z_1 - z_2)^N A(z_1, z_2; t) \equiv (z_1 - z_2)^N B(z_2, z_1; t) \mod t^k,$$

then there is $X(z_1, z_2) \in W[[z_1, z_2]][z_1^{-1}, z_2^{-1}, (z_1 - z_2)^{-1}][[t]]$ such that

$$i_{z_1, z_2} X(z_1, z_2) = A(z_1, z_2), \quad i_{z_2, z_1} X(z_1, z_2) = B(z_2, z_1),$$

and

$$(z_1 - z_2)^N A(z_1, z_2) = (z_1 - z_2)^N B(z_2, z_1) = (z_1 - z_2)^N X(z_1, z_2).$$

5. **First Consequences of the Definition**

**Lemma 5.1.**

$$D_1 = 0.$$  

**Proof.** By the vacuum axiom (3.1) for $i = 1, 2$ we have

$$X_{z_1, z_2}(1 \otimes 1) = e^{z_1 D} 1 = e^{z_2 D} 1 \in V[[z_1]] \cap V[[z_2]].$$

This implies $D_1 = 0$. \qed

We emphasize that $X_{z_1, z_2}$ is assumed to be nonsingular in the $z_2$ variable at zero, so that $X_{z_1, 0}$ is defined. (We used this already in the locality axiom, (3.12).)

Define

$$(5.1) \quad X_z : V \to V[[z]], \quad a \mapsto X_z(a) = e^{z D} a.$$

We think of $X_z$ as the “singular multiplication of 1 element of $V$”, which happens to be nonsingular, just as $X_{z_1, z_2}$ is the singular multiplication of 2 elements. Later, in Theorem 10.1 we will define a singular multiplication $X_{z_1, \ldots, z_n}$ of $n$ elements of $V$.

Then we have

$$(5.2) \quad X_z(a) = X_{z, 0}(a \otimes 1),$$

by the vacuum axiom (3.1).

**Lemma 5.2.** For all $a, b \in V$ we have the following expansion:

$$i_{z_1, z_2} X_{z_1, z_2}(a \otimes b) = X_{z_1, 0}(1 \otimes X_{z_2})(a \otimes b).$$
Proof. Since $X_{z_1,z_2}$ is regular at $z_2 = 0$ we have
\[
i_{z_1;z_2}X_{z_1,z_2}(a \otimes b) = e^{z_2\partial_{z_1}}X_{z_1,w} \big|_{w=0} = X_{z_1,0}(a \otimes e^{z_2}\partial_{z_1})b = X_{z_1,0}(1 \otimes X_{z_2})(a \otimes b) \quad \text{by (5.1).}
\]

\[\square\]

Remark 5.3. We derived the expansion of Lemma 5.2 from the covariance axiom (3.3). Conversely, if we know that $X_{z_1,z_2}$ has this expansion we see that $\partial_{z_2}X_{z_1,z_2}(a \otimes b)$ and $X_{z_1,z_2}(a \otimes Db)$ both have the same image under $i_{z_1;z_2}$. So, $i_{z_1;z_2}$ being injective, we can derive the covariance axiom (3.3) from the existence of the expansion in Lemma 5.2

6. Analytic Continuation for $n = 2$

To make contact with the usual notation and terminology in the theory of vertex algebras we introduce some definitions.

Definition 6.1 (Field). Let $V$ be a $k$-module. A field on $V$ is an element of $\mathrm{Hom}(V,V((z))[[t]])$.

So if $a(z)$ is a field, we have for all $b \in V$
\[a(z)b \in V((z))[[t]].\]

Definition 6.2 (Vertex operator). If $V$ is an $H_D$-quantum vertex algebra we define the vertex operator $Y(a, z)$ associated to $a \in V$ by
\[
(6.1) \quad Y(a, z)b = X_{z,0}(a \otimes b),
\]
for $b \in V$. We will also use the notation $Y_z : a \otimes b \mapsto Y(a, z)b$, so that $Y_z = X_{z,0}$.

Note that the vertex operator $a(z) = Y(a, z)$ for an $H_D$-quantum vertex algebra is a field, for all $a \in V$.

We can rewrite Lemma 5.2 as follows:

Corollary 6.3 (Analytic continuation). The singular multiplication $X_{z_1,z_2}(a \otimes b)$ is the analytic continuation of the product of vertex operators $Y(a, z_1)Y(b, z_2)1$, i.e.,
\[
i_{z_1;z_2}X_{z_1,z_2}(a \otimes b) = Y(a, z_1)Y(b, z_2)1.
\]

Remark 6.4. In Theorem 10.1 we construct an $n$-variable version $X_{z_1,z_2,...,z_n}$ of the singular multiplication satisfying
\[
i_{z_1;z_2,...,z_n}X_{z_1,z_2,...,z_n}(a_1 \otimes a_2 \otimes \cdots \otimes a_n) = Y(a_1, z_1)Y(a_2, z_2)\cdots Y(a_n, z_n)1,
\]
i.e., we construct the analytic continuation of arbitrary product of vertex operators.

Remark 6.5. At this point we would like to emphasize that the axioms we use are much weaker than those of Frenkel-Reshetikhin, [FR97]. Indeed, one of their axioms not only requires that the product of (quantum) vertex operators can be analytically continued, but also that the resulting function is meromorphic in the variables. This is not always the case in our $H_D$-quantum vertex algebras. For instance, we allow a singular multiplication $X_{z_1,z_2}(a \otimes b)$ with a singularity of the form $e^{t/(z_1-z_2)}$, but this does not satisfy the Frenkel-Reshetikhin axioms, as there is an essential singularity at $z_1 = z_2$. In our setup the quantum parameter $t$ is an
independent variable (and we always expand in positive powers of \( t \)), whereas in
Frenkel-Reshetikhin \( t \) is a complex number.

7. Alternative Axioms

We have formulated the axioms of an \( H_D \)-quantum vertex algebra in terms of
the rational singular multiplication \( X_{z_1, z_2} \). Traditionally the axioms of a vertex
algebra have been formulated in terms the 1-variable vertex operator \( Y_\gamma \).
Let us briefly indicate how this would work for \( H_D \)-quantum vertex algebras. Our axioms
from Definition 3.1 would change slightly. We start out with assuming the existence
of a map

\[ Y_\gamma : V \otimes V \to V((z))[[t]], \]

instead of the singular multiplication \( X_{z_1, z_2} \). The braiding and translation maps
\( S^{(\tau)} \) and \( S^{(\gamma)} \) are as before. The vertex operator satisfies the following axioms:

1. (vacuum):
\[ Y_\gamma(1 \otimes a) = a, \quad Y_\gamma(a \otimes 1) = e^{zD}a \]

2. \( (H_D\text{-covariance}) \):
\[ i_{z, \gamma}Y(a, z + \gamma)e^{\gamma D}b = i_{z, \gamma}e^{\gamma D}Y_\gamma(\gamma \otimes a \otimes b). \]

3. \( (\text{Compatibility with Multiplication}) \):
\[ S_{z_1, z_2}(Y_w \otimes 1) = (Y_w \otimes 1)i_{z_1, z_2; z_1 - z_2}w S^{23}_{z_1 + w, z_2} S^{13}_{z_1, z_2}, \]
\[ S_{z_1, z_2}(1 \otimes Y_w) = (1 \otimes Y_w)i_{z_1, z_2; z_1 - z_2}w S^{12}_{z_1, z_2} S^{13}_{z_1, z_2}. \]

4. \( (\text{Locality}) \):
For all \( a, b \in V \) and \( k \geq 0 \) there exist \( N \) such that for all \( c \in V \):
\[ (z - w)^N Y(a, z)Y(b, w)c \equiv (z - w)^N Y_w(1 \otimes Y_\gamma)(S_{w, z}(b \otimes a) \otimes c) \mod t^k. \]

Given these axioms we can reconstruct \( X_{z_1, z_2} \).

**Lemma 7.1.** There exists a map
\[ X_{z_1, z_2} : V \otimes V \to V[[z_1, z_2]][z_1^{-1}, (z_1 - z_2)^{-1}][[t]] \]
such that
\[ i_{z_1, z_2}X_{z_1, z_2}(a \otimes b) = Y_\gamma(a, z_1)Y(b, z_2)1. \]

**Proof.** Let \( A(z_1, z_2) = Y(a, z_1)Y(b, z_2)1. \) By definition of the braiding \( S^{(\tau)} \) and the locality \([7.3]\) we have for all \( k \geq 0 \) an \( N \geq 0 \) such that
\[ (z_1 - z_2)^N A(z_1, z_2) \in V[[z_1, z_2]][z_1^{-1}][[t]] \mod t^k, \]
and the Lemma follows from Lemma \([7.1]\). \( \square \)

Thus we can define in the present setup the singular multiplication \( X_{z_1, z_2} \) to be
the analytic continuation of the product \( Y(a, z_1)Y(b, z_2)1. \)

Alternatively, given the fields \( Y(a, z) \) for any \( a \in V \) we can define \( X_{z_1, z_2} \) as follows:

**Definition 7.2.** For any \( a, b \in V \) define
\[ X_{z_1, z_2}(a \otimes b) = e^{z_2D}Y_{z_1 - z_2} S^{(z_2)}_{z_1 - z_2, 0}(a \otimes b). \]

The two definitions are equivalent:
Lemma 7.3. If $X_{z_1,z_2}$ is given by Definition 7.2 then
\[ i_{z_1; z_2} X_{z_1, z_2}(a \otimes b) = Y(a, z_1)e^{z_2 D} b = Y(a, z_1) Y(b, z_2) 1. \]

The proof follows from (7.1) and the vacuum axiom.

To obtain the axioms of Section 2 note that Lemma 7.3 implies the covariance axiom (8.3), by Lemma 5.2 and Remark 5.3. The rest of the axioms follow immediately.

Remark 7.4. We give in Definition 7.2 a direct construction of $X_{z_1, z_2}$ in terms of $Y_{z_1, z_2, z}$ as fundamental ingredient in the theory. Since there are by now hundreds of papers on vertex algebras written in terms of $Y_{z_1, z_2, z}$ that are no longer “commutative”, but rather “braided commutative”, as shown by the next Lemma.

Remark 7.5. In the case of classical vertex algebras, as well as Etingof-Kazhdan (EK) quantum vertex operator algebras or Frenkel-Reshetikhin deformed chiral algebras, the translation map $S^{(r)}_{z_1, z_2}$ is the identity, so that in this case
\[ X_{z_1, z_2}(a \otimes b) = e^{z_2 D} Y(a, z_1 - z_2)b \in V[[z_1, z_2]][(z_1 - z_2)^{-1}]. \]

In particular in these cases we can let $z_1 = 0$ as $X_{z_1, z_2}(a \otimes b)$ is not singular for $z_1 = 0$. That is no longer the case for the examples of vertex operators connected to symmetric polynomials. Therefore we have allowed for singular multiplication maps which are singular in $z_1$ (but not in $z_2$, if we want to be able to define $Y_{z_2}$ fields as above). It is possible to modify the theory further to allow for singularities in both the variables, but we haven’t yet encountered examples which would call for such generalization.

The conclusion of this section is that we can start either with $Y_{z}$ or with $X_{z_1, z_2}$ as fundamental ingredient in the theory. Since there are by now hundreds of papers on vertex algebras written in terms of $Y_{z}$ we have allowed ourselves to emphasize $X_{z_1, z_2}$ in this paper.

8. Braiding and Skewsymmetry

An important fact of the theory of classical vertex algebras is that the singular multiplication maps $X_{z_1, z_2}$ are “commutative”, i.e., we have for any $a, b \in V$
\[ X_{z_1, z_2}(a \otimes b) = X_{z_2, z_1}(b \otimes a). \]
In the case of $H_D$-quantum vertex algebras the singular multiplication maps $X_{z_1, z_2}$ on $V^\otimes 2$ are no longer “commutative”, but rather “braided commutative”, as shown by the next Lemma.

Lemma 8.1 (Braided Symmetry). For any $a, b \in V$
\[ X_{z_1, z_2}(a \otimes b) = X_{z_2, z_1} S^{(r)}_{z_2, z_1}(b \otimes a). \]

Proof. Let $E = X_{z_1, z_2}(a \otimes b), F = X_{z_2, z_1} S^{(r)}_{z_2, z_1}(b \otimes a)$. We have
\[ i_{z_1; z_2} E = X_{z_1, 0}(1 \otimes X_{z_2})(a \otimes b) = X_{z_1, 0}(1 \otimes X_{z_2, 0})(a \otimes b \otimes 1) \quad \text{by Lemma 5.2.} \]
\[ i_{z_2; z_1} F = X_{z_2, 0}(1 \otimes X_{z_1, 0})(i_{z_2; z_1} S^{(r)}_{z_2, z_1}(b \otimes a) \otimes 1). \]
By the locality axiom [3.12] the RHSs of (8.1) and (8.2) are after multiplication by \((z_1 - z_2)^N\) equal modulo \(t^k\). But then there is for all \(k \geq 0\) an \(N \geq 0\) such that for the LHSs we have

\[
(z_1 - z_2)^N E \equiv (z_1 - z_2)^N F \mod t^k.
\]

Since \(E\) and \(F\) both belong to \(V[[z_1, z_2]][z_1^{-1}, z_2^{-1}, (z_1 - z_2)^{-1}][t]\) this implies that they are in fact equal. \(\square\)

**Corollary 8.2 (Skewsymmetry).** For any \(a, b \in V\) we have

\[
e^{z_2 D} Y_{z_1 - z_2} \circ S^{(z_2)}_{z_1 \rightarrow z_2, 0}(a \otimes b) = e^{z_1 D} Y_{z_2 - z_1} \circ S^{(z_1)}_{z_2 \rightarrow z_1, 0} \circ S^{(z)}_{z_2, z_1}(b \otimes a).
\]

The proof follows from Lemma 8.1 and Definition 7.2.

**Remark 8.3.** In the case of EK quantum vertex operator algebras the translation map \(S^{(z)}_{z_1, z_2}\) is the identity, and the braiding map depends on a single variable \(z_1 - z_2\), therefore we can substitute \(z_1 = 0\) and we get the EK braided skewsymmetry relation

\[
e^{zD} Y(a, -z) b = Y \circ S^{(z)}(b \otimes a),
\]

where \(S^{(z)}_z = S^{(0)}_z\).

Note that we cannot substitute \(z_1 = 0\) in general as \(S^{(z)}_{z_1, z_1}\) might be singular at \(z_1 = 0\), see Section 25. The skewsymmetry relation in Corollary 8.2 looks much less appealing than the braided symmetry relation in Lemma 8.1. Many of the properties of \(H_D\)-quantum vertex algebras look more symmetric in terms of the singular maps \(X_{z_1, z_2}\), which was one of the reasons we prefer working with them, rather than the \(Y_z\) fields.

9. **Braiding Maps for \(n > 2\)**

The singular multiplication map \(X_{z_1, z_2}\) on \(V^{\otimes 2}\) is invariant under simultaneous interchange of the variables \(z_1, z_2\) and the factors in \(V^{\otimes 2}\), up to insertion of the two variable braiding map \(S^{(z)}_{z_1, z_2}\), according to the Lemma 8.1. In the next section we will construct for all \(n \geq 1\) a singular multiplication map \(X_{z_1, \ldots, z_n}\) on \(V^{\otimes n}\), see Theorem 10.1. These are invariant under simultaneous permutation of the variables \(z_i\) and the factors in \(V^{\otimes n}\), up to insertion of an \(n\) variable braiding map \(S^{(z_1, \ldots, z_n)}_{z_1, \ldots, z_n}\), see Corollary 10.2. In this section we construct these braiding maps.

Let \(n \geq 2\), \(I_n = \{1, 2, \ldots, n\}\) and let \(S_n\) be permutation group of \(I_n\), i.e., the group of bijections \(f: I_n \to I_n\). Let \(w_i = (ii + 1) \in S_n\) (where \(i = 1, 2, \ldots, n\)) be the simple transposition given on \(j \in I_n\) by

\[
w_i(j) = \begin{cases} 
  j & j \neq i, i + 1 \\
  i + 1 & j = i \\
  i & j = i + 1
\end{cases}
\]

Then \(S_n\) is generated by the \(w_i\), with as only relations

\[
w_i^2 = 1, \quad w_i w_{i+1} w_i = w_{i+1} w_i w_{i+1},
\]

and

\[
w_i w_j = w_j w_i, \quad |i - j| \geq 2.
\]

Then the simple transposition given on \(j \in I_n\) by

\[
w_i(j) = \begin{cases} 
  j & j \neq i, i + 1 \\
  i + 1 & j = i \\
  i & j = i + 1
\end{cases}
\]
Now let $V$ be a free $k$-module, and define a right action for $f \in S_n$ on the $n$-fold tensor product of $V$ by
\[ \sigma_f : V^{\otimes n} \rightarrow V^{\otimes n}, \quad A_n \mapsto a f(1) \otimes a f(2) \otimes \ldots a f(n), \]
where $A_n = a_1 \otimes a_2 \otimes \ldots a_n \in V^{\otimes n}$. Let $\tau : a \otimes b \mapsto b \otimes a \in V^{\otimes 2}$. Then
\[ \sigma_w = i^{n-1} \otimes \tau \otimes 1_{n-i-1}. \]
Here we write $1^k$ for $1_V \otimes 1_V \otimes \ldots \otimes 1_V$, the $k$-fold tensor product of the identity $1_V : V \rightarrow V$. We emphasize that if $f = g w_i$ then $\sigma_f = \sigma_w \sigma_g$.

Let $\text{Rat}_{z_1, z_2, \ldots, z_n}$ be a space of rational functions in $n$ variables. Then $S_n$ acts on the left on $\text{Rat}_{z_1, z_2, \ldots, z_n}$ by permutation the variables: if $f \in S_n$ and $A_{z_1, \ldots, z_n} \in \text{Rat}_{z_1, z_2, \ldots, z_n}$, then we put
\[ f. A_{z_1, \ldots, z_n} = A_{f(z_1), z_2, \ldots, z_n}, \]
where we write $f(z_1, z_2, \ldots, z_n)$ for $f(z_1), z_2, \ldots, z_n$.

Now let $\text{Map}_{z_1, z_2, \ldots, z_n}(V^{\otimes n})$ be the space of linear maps
\[ V^{\otimes n} \rightarrow V^{\otimes n}[z_i^{\pm 1}, (z_i - z_j)^{-1}][[t]], \quad 1 \leq i < j \leq n. \]
We have an action of $S_n$ on $\text{Map}_{z_1, z_2, \ldots, z_n}(V^{\otimes n})$ combining the action of $S_n$ on $V^{\otimes n}$ and on rational functions: if $f \in S_n$ and $A_{z_1, \ldots, z_n} \in \text{Map}_{z_1, z_2, \ldots, z_n}(V^{\otimes n})$ then define
\[ f. A_{z_1, \ldots, z_n} = \sigma_f^{-1} \circ A_{f(z_1), z_2, \ldots, z_n} \circ \sigma_f. \]

Now let $V$ be an $H_D$-quantum vertex algebra. So we get, by definition, in particular a braiding map $S_{z_1, z_2}^{(\tau)} \in \text{Map}_{z_1, z_2}(V^{\otimes 2})$. For simplicity we denote it by $S_{z_1, z_2}$ in this section, as we will not use $S_{z_1, z_2}^{(\tau)}$ here. It satisfies, see Definition \ref{def:braiding_maps}
\begin{align}
(9.3) \quad S_{z_1, z_2} \circ \tau \circ S_{z_2, z_1} \circ \tau &= 1_{V^{\otimes 2}}, \\
(9.4) \quad S_{z_1, z_2}^{12} S_{z_2, z_3}^{23} S_{z_1, z_3}^{13} &= S_{z_2, z_3}^{12} S_{z_1, z_3}^{13} S_{z_1, z_2}^{23}. 
\end{align}
We will to use the braiding matrix $S_{z_1, z_2}$ to define a map $S_n \rightarrow \text{Map}_{z_1, z_2, \ldots, z_n}(V^{\otimes n})$.

**Definition 9.1 (Braiding maps).** Define for each $f \in S_n$ an element $S_f^{\otimes n}$ of $\text{Map}_{z_1, z_2, \ldots, z_n}(V^{\otimes n})$ called the braiding map associated to $f$, by expanding $f$ (in any way) in simple reflections $w_i$ and using
\[ S_{z_1, \ldots, z_n}^{w_i} = i^{n-1} \otimes S_{z_1, z_i+1}^{(\tau)} \otimes i^{n-i-1}, \]
and
\[ S_{z_1, \ldots, z_n}^{g f} = S_{z_1, \ldots, z_n}^{g} \sigma_f \sigma_{g^{-1}}(\sigma_f)^{-1}. \]

The point is that to define $S_f^{\otimes n}$ for $f \in S_n$, we can take any decomposition of $f$ into simple transpositions $w_i$, i.e., this definition is unambiguous. The proof of this statement is discussed in Appendix \ref{appendix:braiding_maps}.

10. **Analytic Continuation for $n > 2$**

If $V$ is an $H_D$-quantum vertex algebra, recall that we have the “singular” multiplications $X_2$ and $X_{z_1, z_2}$ of 1, respectively 2 elements of $V$, see \ref{def:singular_mult} and Definition \ref{def:singular_mult}. We will in this section construct singular multiplications $X_{z_1, \ldots, z_n}$ of $n$ elements of $V$. 

Let \( f_n = w_1w_2\ldots w_{n-1} \) be the \( n \)-cycle (123...\( n \)) and consider the associated braiding matrix \( S_{f_n}^{E_{X_1,\ldots,X_n}} \). We have \( f_n = w_1(23\ldots n) \). Writing \( f_{n-1} = (23\ldots n) \) and \( \sigma_n = \sigma_{f_n} \), we find from (9.5) that

\[
S_{f_n}^{E_{X_1,\ldots,X_n}}, \sigma_n = (1 \otimes S_{f_{n-1}}^{E_{X_2,\ldots,X_n}},z_1,\sigma_{f_{n-1}})(S_{f_{n-1}}^{E_{X_1,\ldots,X_n}}) \otimes 1^{\otimes n-2}.
\]

We will frequently use the abbreviation

\[
p_n = p_n(z_1, z_2, \ldots, z_n) = \prod_{1 \leq i < j \leq n} (z_i - z_j).
\]

**Theorem 10.1 (Analytic Continuation).** Let \( V \) be an \( H_D \)-quantum vertex algebra. There exists for all \( n \geq 2 \) maps

\[
X_{z_1,\ldots,z_n} : V^{\otimes n} \to V[[z_k]][(z_i - z_j)^{-1}][[t]], \quad 1 \leq i < j \leq n, i \leq k \leq n
\]

such that

\[
i_{z_1; z_2,\ldots, z_n} X_{z_1,\ldots, z_n} = X_{z_1,0}(1 \otimes X_{z_2,\ldots, z_n}).
\]

and

\[
X_{z_1, z_2,\ldots, z_n} = X_{z_2,\ldots, z_n, z_1} S_{f_n}^{E_{X_2,\ldots, X_n}}{z_1}, \sigma_n,
\]

where \( S_{f_n}^{E_{X_2,\ldots, X_n}}{z_1} \) is defined in Definition 9.7.

**Proof.** The theorem is true for \( n = 2 \) by Lemma 5.2 and Lemma 8.1. Assume the theorem is true for \( \ell \), \( 2 \leq \ell \leq n_0 \) and let \( n = n_0 + 1 \). The induction hypothesis implies that

\[
i_{z_2; z_3,\ldots, z_n} X_{z_2, z_3,\ldots, z_n} = X_{z_2,0}(1 \otimes X_{z_3,\ldots, z_n}),
\]

so that for all \( k \geq 0 \) there is an \( N \geq 0 \) such that

\[
p_n^{-1} X_{z_2, z_3,\ldots, z_n} \equiv p_n^{-1} X_{z_2,0}(1 \otimes X_{z_3,\ldots, z_n}) \mod t^k.
\]

Also we have

\[
X_{z_1, z_2,\ldots, z_n} = X_{z_2,\ldots, z_n, z_1} S_{f_n}^{E_{X_2,\ldots, X_n}}{z_1}, \sigma_{f_{n-1}}{z_1}.
\]

Consider \( E = X_{z_1,0}(1 \otimes X_{z_2,\ldots, z_n})(A_n) \), \( A_n \in V^{\otimes n} \). This is an element of \( V((z_1))[[z_2, z_3,\ldots, z_n]][z_1^{-1}, \ldots, z_1^{-1}, (z_i - z_j)^{-1}][[t]], 2 \leq i < j \leq n \), and we want to show \( E \) is in the image of the expansion \( i_{z_2; z_3,\ldots, z_n} \). For this it suffices to show that for all \( k \geq 0 \) there is an \( N \geq 0 \) such that \( p_n^N E \) has at worst a finite order pole in \( z_1 \). This is a small calculation: for all \( k \geq 0 \) there is \( N \geq 0 \) such that modulo \( t^k \) we have

\[
p_n^N E = p_n^N X_{z_1,0}(1 \otimes (X_{z_2,0}(1 \otimes X_{z_3,\ldots, z_n}))) \text{ (A_n)} = \text{ by (10.6)}
\]

\[
= p_n^N X_{z_2,0}(1 \otimes (X_{z_1,0}(1 \otimes X_{z_3,\ldots, z_n}))) \text{ (i_{z_2; z_3,\ldots, z_n}) (S_{f_{n-1}}^{E_{X_2,\ldots, X_n}}{	au}) \otimes 1^{\otimes n-2}) \text{ (A_n)} = \text{ by (3.12)}
\]

\[
= p_n^N X_{z_2,0}(1 \otimes (X_{z_1,0}(1 \otimes X_{z_3,\ldots, z_n}))) \text{ (S_{f_{n-1}}^{E_{X_2,\ldots, X_n}}){\tau} \otimes 1^{\otimes n-2}) \text{ (A_n)} = \text{ by (10.5)}
\]

\[
= p_n^N X_{z_2,0}(1 \otimes (X_{z_3,\ldots, z_n, z_1})) \text{ (1 \otimes S_{f_n}^{E_{X_2,\ldots, X_n}}{z_1} \sigma_{f_{n-1}}{z_1}) \times S_{f_{n-1}}^{E_{X_2,\ldots, X_n}}{z_1} \otimes 1^{\otimes n-2}) \text{ (A_n)} = \text{ by (10.6)}
\]

\[
= p_n^N X_{z_2,0}(1 \otimes (X_{z_3,\ldots, z_n, z_1})) S_{f_n}^{E_{X_2,\ldots, X_n}}{z_1} \sigma_n \text{ (A_n)} = \text{ by (10.1)}.
\]

We see from the last expression that \( p_n^N E \) has at worst a finite order pole in \( z_1 \) and hence there is \( X_{z_1, z_2,\ldots, z_n} \) such that (10.3) holds.
Next consider \( F = X_{z_2, z_3, \ldots, z_n, z_1} S^{(r)}_{z_2, z_3, \ldots, z_n, z_1} \sigma_n(A_n) \) and \( G = X_{z_1, \ldots, z_n}(A_n) \).

For all \( k \geq 0 \) there is an \( N \geq 0 \) such that modulo \( t^k \)
\[
p^n F = p^n G.
\]

By what we just proved we have
\[
i_{z_2: z_3, \ldots, z_n, z_1} F = X_{z_2, 0}(1 \otimes X_{z_3, \ldots, z_n, z_1}) i_{z_2: z_3, \ldots, z_n, z_1} S^{(r)}_{z_2, z_3, \ldots, z_n, z_1} \sigma_n(A_n),
\]
and so for all \( k \geq 0 \) there is \( N \geq 0 \) such that modulo \( t^k \) we have
\[
p^n F = p^n X_{z_2, 0}(1 \otimes X_{z_3, \ldots, z_n, z_1}) i_{z_2: z_3, \ldots, z_n, z_1} S^{(r)}_{z_2, z_3, \ldots, z_n, z_1} \sigma_n(A_n) = p^n F = p^n G.
\]

Since \( G, F \) both belong to \( V[[z_i]](z_i - z_j)^{-1}[[t]] \) this forces \( G = F \), i.e., \((10.4)\) holds.

**Corollary 10.2 (Analytic Continuation for products of fields).** For all \( n \geq 2 \) and \( 1 \leq i \leq n - 1 \) we have, if \( A_n = a_1 \otimes a_2 \otimes \ldots \otimes a_n \), the expansion
\[
i_{z_1: z_2, \ldots, z_n} X_{z_1, \ldots, z_n}(A_n) = X_{z_1, 0}(a_1 \otimes X_{z_2, 0}(a_2 \otimes \ldots \ldots (a_{i-1} \otimes X_{z_i, 0}(a_i \otimes X_{z_{i+1}, z_i+2, \ldots, z_n}(a_{i+1} \otimes a_{i+2} \otimes \ldots \otimes a_n)) \ldots)).
\]

In particular the case of \( i = n - 1 \) of the Corollary is (using the notation \((6.1)\))
\[
i_{z_1: z_2, \ldots, z_n} X_{z_1, z_2, \ldots, z_n}(A_n) = Y(a_1, z_1)Y(a_2, z_2) \ldots Y(a_n, z_n).1.
\]

In other words the \( n \)-variable \( X_{z_1, z_2, \ldots, z_n}(A_n) \) is the analytic continuation of the composition of \( n \) vertex operators acting on the vacuum.

### 11. Further Consequences

**Lemma 11.1.** The infinitesimal forms of the \( H_D \)-covariance axioms \((3.2), (3.5)\) are
\[
\begin{align*}
(1 \otimes D + \partial_{z_2})S_{z_1, z_2} &= S_{z_1, z_2} (1 \otimes D), \\
DY(a, z)b &= \partial_2 Y(a, z)b + Y(a, z)Db - Y_z \circ \alpha_{z, 0}(a \otimes b).
\end{align*}
\]
where \( \alpha_{z_1, z_2} \) is defined to be \( \partial_1 S^{(r)}_{z_1, z_2} \), and satisfies the infinitesimal form of the vacuum axioms \((3.2)\)
\[
\alpha_{z_1, z_2}(a \otimes 1) = 0, \quad \alpha_{z_1, z_2}(1 \otimes b) = 0,
\]

**Lemma 11.2.** For all \( a, b \in V \)
\[
X_{z_1, z_2}(Da \otimes b) = \partial_{z_1} X_{z_1, z_2}(a \otimes b),
\]

**Proof.** By the previous Lemma \((11.1)\)
\[
X_{z_1, z_2}(Da \otimes b) = X_{z_1, z_2}(1 \otimes D)(b \otimes a) =
\]
\[
= X_{z_2, z_1} S^{(r)}_{z_2, z_1}(1 \otimes D)(b \otimes a) =
\]
\[
= X_{z_2, z_1}(1 \otimes D + \partial_{z_2})S^{(r)}_{z_2, z_1}(b \otimes a) =
\]
\[
= \left( \partial_{z_2}(X_{z_2, z_1})S^{(r)}_{z_2, z_1} + X_{z_2, z_1} \partial_2 S^{(r)}_{z_2, z_1} \right)(b \otimes a) =
\]
\[
= \partial_{z_2}(X_{z_2, z_1}S^{(r)}_{z_2, z_1})(b \otimes a) =
\]
\[
= \partial_{z_2}X_{z_1, z_2}(a \otimes b) \text{ by Lemma (8.1)}
\]
proving \((11.3)\).
Corollary 11.3. For all $n \geq 2$, $1 \leq i \leq n$ and $A_n = a_1 \otimes a_2 \otimes \ldots \otimes a_n \in V \otimes n$ we have
\[ \partial_i X_{z_1, z_2, \ldots, z_n}(A_n) = X_{z_1, z_2, \ldots, z_n}(a_1 \otimes \ldots \otimes Da_i \otimes \ldots \otimes a_n). \]

Proof. For $n = 2$ and $i = 2$ this is axiom (11.3), and for $i = 1$ this is Lemma 11.2.

Put for $n > 2$
\[ E = \partial_i X_{z_1, z_2, \ldots, z_n}(A_n), \quad F = X_{z_1, z_2, \ldots, z_n}(a_1 \otimes \ldots \otimes Da_i \otimes \ldots \otimes a_n). \]

Since $\partial_i$ commutes with expansions we have
\[ i_{z_1; z_2, \ldots, z_n} E = X_{z_1, 0}(a_1 \otimes X_{z_2, 0}(a_2 \otimes \ldots \ldots (a_{i-1} \otimes \partial_1 X_{z_1, 0}(a_i \otimes X_{z_{i+1}, \ldots, z_n}(a_{i+1} \otimes \ldots \otimes a_n))) \ldots )) \]
\[ = X_{z_1, 0}(a_1 \otimes X_{z_2, 0}(a_2 \otimes \ldots \ldots (a_{i-1} \otimes X_{z_1, 0}(Da_i \otimes X_{z_{i+1}, \ldots, z_n}(a_{i+1} \otimes \ldots \otimes a_n))) \ldots )) \]
\[ = i_{z_1; z_2, \ldots, z_1, z_{i+1}, \ldots, z_n} F. \]

Since both $E$ and $F$ belong to $V[[z_i]][z_i^{-1}, (z_i - z)^{-1}][[t]]$, $1 \leq i < j \leq n$, and have the same expansion they must be equal. \[ \square \]

Lemma 11.4. For all $n \geq 1$ and $A_n = a_1 \otimes a_2 \otimes \ldots \otimes a_n \in V^n$ we have
\[ X_{z_1, z_2, \ldots, z_n, 0}(A_n \otimes 1) = X_{z_1, z_2, \ldots, z_n}(A_n). \]

Proof. For $n = 1$ this is (11.2). Assume that the lemma is true for all $\ell$, $1 \leq \ell \leq n_0$, and let $n = n_0 + 1$. Put $E = X_{z_1, z_2, \ldots, z_n, 0}(A_n \otimes 1)$, $F = X_{z_1, z_2, \ldots, z_n}(A_n)$. By Theorem 10.1 and the induction hypothesis
\[ i_{z_1; z_2, \ldots, z_n} E = X_{z_1, 0}(a_1 \otimes X_{z_2, 0}(a_2 \otimes a_3 \otimes \ldots \otimes a_n \otimes 1)) = \]
\[ = X_{z_1, 0}(a_1 \otimes X_{z_2, \ldots, z_n}(a_2 \otimes a_3 \otimes \ldots \otimes a_n)) = \]
\[ = i_{z_1; z_2, \ldots, z_n} F. \]

Since both $E$ and $F$ belong to $V[[z_i]][z_i^{-1}, (z_i - z)^{-1}][[t]]$, $1 \leq i < j \leq n$, and have the same expansion they must be equal. \[ \square \]

Lemma 11.5. Suppose that $S^{(\gamma)}_{z_1, z_2}$ is the identity map on $V \otimes V$. Then the following is true:
\[ DX_{z_1, z_2} = (\partial_{z_1} + \partial_{z_2})X_{z_1, z_2} = X_{z_1, z_2}(D \otimes 1 + 1 \otimes D), \]
\[ [D, Y(a, z)] = \partial_2 Y(a, z), \]
\[ X_{z_1, z_2} \circ (\partial_{z_1} + \partial_{z_2})S^{(\gamma)}_{z_1, z_2} = 0. \]

Proof. The second property is a direct consequence of Lemma 11.1. The first equality follows from expanding both sides of (3.5) in powers of $\gamma$ and comparing the coefficients in front of $\gamma^1$.

For the last part rewrite (11.4) as
\[ e^{\gamma D}X_{z_1, z_2} = X_{z_1, z_2}\Delta(e^{\gamma D}), \]
where $\Delta$ is the coproduct of $H_D$, so that $\Delta(e^{\gamma D} = e^{\gamma D} \otimes e^{\gamma D})$. Similarly rewrite the $H_D$-covariance axiom (3.3) for the braiding as
\[ (1 \otimes e^{-\gamma D})S_{z_1, z_2} = e^{\gamma(\partial_{z_1} + \partial_{z_2})}S_{z_1, z_2}(1 \otimes e^{-\gamma D}). \]
By differentiating with respect to $z_1$ the axiom (3.9) we obtain a similar equation involving $\partial z_1$ and $e^{-\gamma D} \otimes 1$, and we combine these as

$$\Delta(e^{-\gamma D})S^{(\tau)} = e^{\gamma(\partial z_1 + \partial z_2)}S^{(\tau)}\Delta(e^{-\gamma D}).$$

Now we calculate

$$e^{-\gamma D}X_{z_2, z_1} = e^{-\gamma D}X_{z_1, z_2}S_{z_1, z_2}\tau =$$

$$= X_{z_1, z_2}\Delta(e^{-\gamma D})S_{z_1, z_2}\tau =$$

$$= X_{z_1, z_2}e^{\gamma(\partial z_1 + \partial z_2)}S^{(\tau)}\Delta(e^{-\gamma D}).$$

On the other hand

$$e^{-\gamma D}X_{z_2, z_1} = X_{z_1, z_2}S_{z_1, z_2}\tau\Delta(e^{-\gamma D}).$$

By multiplying by $\Delta(e^{\gamma D})\tau$ on the right we find

$$X_{z_1, z_2}e^{\gamma(\partial z_1 + \partial z_2)}S^{(\tau)} = X_{z_1, z_2}S^{(\tau)},$$

from which (11.6) follows. □

**Remark 11.6.** In the context of the lemma above it is natural to assume that $S^{(\tau)}_{z_1, z_2}$ is a function of just $z_1 - z_2$. In this case V is a quantum vertex operator algebra as defined by Etingof-Kazhdan, see [EK00] (except for the fact that they insist that the braiding is of the form $S^{(\tau)} = 1 + O(t)$).

12. **Braiding and singular multiplication**

We have seen that the $n$-fold singular multiplication has cyclic symmetry: if $f_n$ is the cyclic permutation (123 \ldots n), then

$$(12.1) \quad X_{z_1, \ldots, z_n} = X_{f_n(z_1, \ldots, z_n)}S^{f_n}_{z_1, \ldots, z_n}\sigma_n,$$

see Theorem 10.1. In this section we show that in fact the $n$-fold singular multiplication has arbitrary permutation symmetry: in (12.1) we can replace $f_n$ by any $f \in S_n$.

**Lemma 12.1.** For all $n \geq 2$ we have

$$X_{z_1, z_2, \ldots, z_n} = X_{w_1(z_1, z_2, \ldots, z_n)}S^{w_1}_{z_1, \ldots, z_n}(\tau \otimes 1^{n-2}).$$

**Proof.** Let $E = X_{z_1, z_2, \ldots, z_n}(A_n)$, $F = \bar{X}(\tau)(S^{(\tau)}_{z_2, z_1} \otimes 1^{n-2}) (A_n)$, $A_n \in V^\otimes n$. Then there exist for all $k \geq 0$ an $N \geq 0$ such that modulo $t^k$

$$p_N^N i_{z_1; \ldots; z_n} E = p_N^N X_{z_1, 0}(1 \otimes X_{z_2, 0}(1 \otimes X_{z_3, \ldots, z_n}))(A_n)$$

by Thm 10.1

$$= p_N^N X_{z_2, 0}(1 \otimes X_{z_1, 0}(1 \otimes X_{z_3, \ldots, z_n}))(S^{(\tau)}_{z_2, z_1} \otimes 1^{n-2})(A_n)$$

by (3.12)

$$= p_N^N i_{z_2; \ldots; z_n}F,$$

by Theorem 10.1 again. Since both E and F belong to $V[\{z_i\}, (z_i - z)^{-1}][[t]]$, 1 \leq i < j \leq n, and have the same expansion they must be equal. □

Recall that the first simple transposition $w_1$ and the cyclic permutation $f_n = (123 \ldots n)$ generate $S_n$.

**Corollary 12.2.** If $f \in S_n$ is a permutation of $\{1, 2, \ldots, n\}$ and $\sigma_f(A_n) = a_{f(1)} \otimes a_{f(2)} \otimes \ldots \otimes a_{f(n)}$, then

$$(12.2) \quad X_{z_1, \ldots, z_n} = X_{f(z_1, \ldots, z_n)}S^{f}_{f(z_1, \ldots, z_n)}\sigma_f.$$
Proof. Suppose we have two elements \( f, g \in S_n \) such that (12.2) holds. Then, by (5.3),
\[
X_{fg(z_1, \ldots, z_n)}^S g_{(z_1, \ldots, z_n)} \sigma_{fg} = X_{fg(z_1, \ldots, z_n)}^{S g_{(z_1, \ldots, z_n)} \sigma_{fg}} = X_{fg(z_1, \ldots, z_n)}^{S g_{(z_1, \ldots, z_n)} \sigma_{fg}} = X_{fg(z_1, \ldots, z_n)}^{S g_{(z_1, \ldots, z_n)} \sigma_{fg}} = X_{(z_1, \ldots, z_n)}^{S f_{(z_1, \ldots, z_n)} \sigma_{f}} = f, \quad \text{by Axiom (5.3)}
\]
So if (12.2) holds for \( f \) and for \( g \) it holds for \( fg \). But we know that (12.2) holds for \( f_n \), by Theorem 10.1, and for \( w_1 \), by Lemma 12.1 and these elements generate \( S_n \). So (12.2) holds for all \( f \in S_n \). 

13. Expansions of \( X_{z_1, z_2, 0} \)

We have seen that the expansion of \( X_{z_1, \ldots, z_n} \) (in the region \( |z_1| > |z_2| > \cdots > |z_n| \)) is expressed as a composition of 1-variable vertex operators. In particular, for \( n = 3 \) we get, if \( A = a \otimes b \otimes c \),
\[
i_{z_1:z_2} X_{z_1, z_2, 0}(A) = Y(a, z_1)Y(b, z_2)c,
\]
see (10.7). In this section we find other expansions of \( X_{z_1, z_2, 0} \) that have useful expressions in terms of \( Y_z \).

First we need a variant of the analytic continuation Theorem 10.1.

Lemma 13.1.
\[
X_{z_1, z_2}(1 \otimes X_{w, 0} i_{z_2:z_2} w_{0} g(z_2)) = i_{z_1:z_2, z_2:z_2} w_{0} X_{z_1, z_2, 0}.
\]

Proof.
\[
i_{z_1:z_2} X_{z_1, z_2}(1 \otimes X_{w, 0} i_{z_2:z_2} w_{0} g(z_2)) =
\]
\[
= X_{z_1, 0}(1 \otimes e^{-z_2 D} X_{w, 0} i_{z_2:z_2} w_{0} g(z_2)) = \quad \text{by Lemma 5.2}
\]
\[
= X_{z_1, 0}(1 \otimes i_{z_2:z_2} w_{0} X_{w, 0} f(z_1 - z_2 - w)) = \quad \text{by Axiom (3.3)}
\]
\[
= i_{z_2:z_2} w_{0} X_{z_1, z_2, 0} = \quad \text{by Thm. 10.1}
\]
\[
i_{z_1:z_2} X_{z_1, z_2}(1 \otimes X_{w, 0} i_{z_2:z_2} w_{0} g(z_2)) =
\]
\[
i_{z_1:z_2} i_{z_1:z_2, z_2:z_2} w_{0} f(z_1 - z_2 - w) = i_{z_2:z_2} w_{0} X_{z_1, z_2, 0}.
\]
The Lemma follows then by cancelling \( i_{z_1:z_2} \).

Next we need a variant of the compatibility with multiplication Axiom (5.8).

Lemma 13.2.
\[
S_{z_1, z_2}^{(r)} (1 \otimes X_{w, 0} S_{w, 0}^{(g)}) = (1 \otimes X_{w, 0} S_{w, 0}^{(g)}) i_{z_1:z_2, z_2:z_2} w_{0} X_{z_1, z_2, 0}.
\]

Proof. We need some simple identities. By Axiom (3.4)
\[
X_{w, 0} S_{w, 0}^{(g)} = e^{-\gamma D} X_{w, 0} S_{w, 0}^{(g)}.
\]
By Axiom (3.4)
\[
S_{z_1, z_2}^{(r)} (1 \otimes e^{-\gamma D}) = (1 \otimes e^{-\gamma D}) i_{z_1:z_2, z_2:z_2} w_{0} X_{z_1, z_2, 0}.
\]
Finally, by Axiom (5.8)
\[
S_{z_1, z_2}^{(r)} (1 \otimes X_{w, 0}) = (1 \otimes X_{w, 0}) i_{z_1:z_2, z_2:z_2} w_{0} X_{z_1, z_2, 0}.
\]
Then
\[
S_{z_1,z_2}^{(\tau)}(1 \otimes X_{w,0}S_{w,0}^{(\gamma)}) = S_{z_1,z_2}^{(\tau)}(1 \otimes e^{-\gamma D}X_{w,\gamma}) =
\]
\[
= (1 \otimes e^{-\gamma D})i_{z_1-z_2}S_{z_1,z_2}^{(\tau)}(1 \otimes X_{w,\gamma}) = \text{by (13.2)}
\]
\[
= (1 \otimes e^{-\gamma D})(1 \otimes X_{w,\gamma}) \times
\]
\[
i_{z_1-z_2}S_{z_1-z_2}^{(\tau)}(1 \otimes e^{-\gamma D})i_{z_1-z_2}S_{z_1,z_2}^{(\tau)} = \text{by (13.3)}
\]
\[
= (1 \otimes X_{w,0}S_{w,0}^{(\gamma)}) \times
\]
\[
i_{z_1-z_2}S_{z_1-z_2}^{(\tau)}S_{z_1,z_2}^{(\tau)} = \text{by (13.1)}
\]
since
\[
i_{z_1-z_2}f((z-\gamma)+(w+\gamma)) = i_{z_1-z_2}f(z+w)
\]
\[
i_{z_1-z_2}f((z-\gamma)+\gamma) = f(z).
\]

\[\square\]

**Remark 13.3.** Note that in Lemma 13.2 we establish the equality of two complicated expressions that depend on \(\gamma\) only via the powers \((w+\gamma)^n\). In particular, we can take \(\gamma = z_2\), and the equalities will still hold, although the proof of Lemma 13.2 breaks down in that case, as \(S_{z_1,0}^{(\tau)}\) need not be defined.

**Proposition 13.4.** Let \(V\) be an \(H_D\)-quantum vertex algebra, and \(A = a \otimes b \otimes c \in V^\otimes 3\). Then we have the following expansions:

\[(13.4) \quad i_{z_1,z_2}X_{z_1,z_2,0}(A) = Y(a, z_1)Y(b, z_2)c,\]
\[(13.5) \quad i_{z_2,z_1}X_{z_2,z_1,0}(A) = Y_{z_2}(1 \otimes Y_{z_1})i_{z_2,z_1}S_{z_2,z_1}^{(\tau)}(b \otimes a \otimes c),\]
\[(13.6) \quad i_{z_2,z_1}X_{z_2+z_1,z_2,0}(A) = Y_{z_2}(Y_{z_1} \otimes 1)i_{z_2,z_1}S_{z_2,z_1}^{(\tau)}(a \otimes b \otimes c).\]

*Proof.* (13.4) is (10.7) for \(n = 3\) and \(z_3 = 0\). By Corollary 12.2 (for \(n = 3\) and \(f = w_1\)) we have

\[
X_{z_1,z_2,0}(A) = X_{z_2,z_1,0}(S_{z_2,z_1}^{(\tau)}(b \otimes a \otimes c)).
\]

Expanding this equation by applying \(i_{z_2,z_1}\) and using (13.4) and definition (6.1) gives (13.5).

For the last part, let \(f = (132) = w_2w_1\), so that \(\sigma_f(a \otimes b \otimes c) = c \otimes a \otimes b\). Then

\[
S(f) = S_{z_1,z_2,z_3}^{(\tau)}(f) = S_{z_1,z_2,z_3}^{w_1}(w_1(z_1,z_2,z_3)) = S_{z_1,z_2}^{(\tau)}S_{z_1,z_2}^{(\tau)}(f).
\]

Therefore

\[(13.7) \quad S_{f(z_2+w),z_1}(A) = S_{z_1+z_2+w}^{(\tau)}S_{z_1,z_2}^{(\tau)}(f).\]
Let $E = X_{z_1 z_2} (X_{z_3,0} i_{z_2 z_3} S_{z_3,0}^{(r)} \otimes 1) (A)$. Then

$$E = X_{z_1 z_2} S_{z_1, z_2}^{(r)} (X_{z_3,0} i_{z_2 z_3} S_{z_3,0}^{(r)} \otimes 1) (A) = \quad \text{by Lem. 5.1}$$

$$= X_{z_1 z_2} S_{z_1, z_2}^{(r)} (1 \otimes X_{z_3,0} i_{z_2 z_3} S_{z_3,0}^{(r)}) (c \otimes a \otimes b) = \quad \text{by Lemma 13.2 and Remark 13.3}$$

$$= \sum_{n \geq 1} z_1^n z_2^{n-1}. \quad \text{Thus we have Thm. 10.1}$$

Putting $z_1 = 0$ proves then 13.6.

14. The Braided Jacobi Identity

In one approach to the usual vertex algebras the Jacobi identity for vertex operators is the basic identity, see e.g., [LL04]. In this section we derive the braided analog in our context of $H_D$-quantum vertex algebras.

Introduce some more notation. If $f(z_1, z_2) \in \mathbb{C}[[z_1, z_2]][z_1^{-1}, z_2^{-1}, (z_1 - z_2)^{-1}]$ define the difference of expansions of $f$

$$\delta(f(z_1, z_2)) = (i_{z_1 z_2} - i_{z_2 z_1}) (f(z_1, z_2)).$$

For instance,

$$\delta \left( \frac{1}{z_1 - z_2} \right) = \delta(z_1, z_2) = \sum_{n \geq 1} z_1^n z_2^{-n-1}. \quad \text{This is the usual Dirac Delta Distribution.}$$

Recall that in this paper we are always expanding all expressions in positive powers of $t$. For instance, if we write $\frac{1}{z_1 - z_2}$ we mean $\sum_{n \geq 0} (z_2 - z_1)^n / z_1^{n+1}$. Thus we have, for instance,

$$\delta \left( \frac{1}{z_1 - z_2} \right) = 0.$$

**Lemma 14.1.** For all $f(z_1, z_2, z_3) \in \mathbb{C}[[z_1, z_2, z_3]][z_1^{-1}, z_2^{-1}, z_3^{-1}]$ we have

$$i_{z_1 z_2} (\delta(z_1 - z_2, z_3) f(z_1, z_2, z_1 - z_2)) - i_{z_2 z_1} (\delta(z_1 - z_2, z_3) f(z_1, z_2, z_1 - z_2)) = i_{z_2 z_3} (\delta(z_1, z_2 + z_3) f(z_2 + z_3, z_2, z_3)).$$

**Proof.** See for example Proposition 2.3.26 in [LL04].

**Definition 14.2.** We will write $a(z)$ for the 1-variable vertex operator $Y(a, z)$, the field associated to $a \in V$.

**Theorem 14.3.** (Braided Jacobi Identity) Let $V$ be an $H_D$-quantum vertex algebra. For all $a, b, c \in V$ we have the identity:

$$i_{z_1 z_2} \delta(z_1 - z_2, z_3) a(z_1) b(z_2) c - i_{z_2 z_1} \delta(z_1 - z_2, z_3) Y_{z_2} (1 \otimes Y_{z_1}) S_{z_2, z_1}^{(r)} (b \otimes a \otimes c) = i_{z_2 z_3} \delta(z_1, z_2 + z_3) Y_{z_3} (Y_{z_2} \otimes 1) S_{z_3, 0}^{(r)} (a \otimes b \otimes c)$$
Since this is true for all \( v \), then we get from Lemma 14.1 that the Jacobi identity, cf., [Kac98]. In this section we derive a braided version of the Borcherds identity, instead of the original definition by Borcherds of vertex algebras was given in [Bor86].

Proof. Let \( V^* \) be the dual of \( V \), fix \( v^* \in V^* \) and let \( \langle , \rangle \) be the pairing \( V^* \otimes V \to k = \mathbb{C}[\![t]\!] \). Then for all \( A = a \otimes b \otimes c \in V^{\otimes 3} \) we have

\[
\langle v^*, X_{z_1, z_2, 0}(A) \rangle = \sum_{p \geq 0} \sum_{l, m, n \in \mathbb{Z}} \frac{g_{l, m, n, p}(z_1, z_2)}{z_1^l z_2^m z_2^n} t^p,
\]

for \( g_{l, m, n, p}(z_1, z_2) \in \mathbb{C}[\![z_1, z_2]\!] \). (The sum over \( l, m, n \) is finite, for each \( p \).) Define then

\[
F(z_1, z_2, z_3) = \sum_{p \geq 0} \sum_{l, m, n \in \mathbb{Z}} \frac{g_{l, m, n, p}(z_1, z_2)}{z_3^l z_1^m z_2^n} t^p \in \mathbb{C}[\![z_1, z_2, z_3]\!] [z_1^{-1}, z_2^{-1}, z_3^{-1}][[t]].
\]

Then we have by Corollary 13.3

\[
\begin{align*}
i_{z_1; z_2} F(z_1, z_2, z_1 - z_2) &= \langle v^*, a(z_1)b(z_2)c \rangle, \\
i_{z_2; z_1} F(z_1, z_2, z_1 - z_2) &= \langle v^*, Y_{z_2}(1 \otimes Y_{z_1})i_{z_2; z_1}S^{(12)}_{z_2, z_1}, Y_{z_1} \rangle, \\
i_{z_2; z_3} F(z_2 + z_3, z_2, z_3) &= \langle v^*, Y_{z_2}(Y_{z_3} \otimes 1)i_{z_2; z_3}S^{(12)}_{z_2, z_3}, a \otimes b \otimes c \rangle.
\end{align*}
\]

Then we get from Lemma 14.1 that

\[
\begin{align*}
\langle v^*, i_{z_1; z_2} \delta(z_1 - z_2, z_3) + a(z_1)b(z_2)c \rangle - \langle v^*, i_{z_2; z_1} \delta(z_1 - z_2, z_3) + a(z_1)b(z_2)c \rangle &= \langle v^*, i_{z_2; z_3} \delta(z_1, z_2 + z_3)Y_{z_2}(Y_{z_3} \otimes 1)i_{z_2; z_3}S^{(12)}_{z_2, z_3}, a \otimes b \otimes c \rangle.
\end{align*}
\]

Since this is true for all \( v^* \in V^* \) the Theorem follows.

\[
\]

Remark 14.4. Suppose \( V \) is an \( H_D \)-quantum vertex algebra where \( S_{z_1, z_2}^{(1)} \) and \( S_{z_1, z_2}^{(2)} \) both are the identity map on \( V \otimes V \). Then the fields \( a(z) = Y(a, z) \) satisfy the usual Jacobi identity:

\[
i_{z_1; z_2} \delta(z_1 - z_2, z_3)a(z_1)b(z_2) - i_{z_2; z_1} \delta(z_1 - z_2, z_3)b(z_2)a(z_1) = i_{z_2; z_1} \delta(z_1, z_2 + z_3)Y(Y(a, z_3)b, z_2),
\]

and it follows that \( V \) is an ordinary vertex algebra, cf., [LL04].

15. Braided Borcherds Identity

The original definition by Borcherds of vertex algebras was given in [Bor86]. He took as starting point what later was called the Borcherds identity, instead of the Jacobi identity, cf., [Kac98]. In this section we derive a braided version of the Borcherds identity.

The following lemma is easy to check and well known (at least for \( t = 0 \), see e.g., [??]).

Lemma 15.1. Let \( W \) be a free \( k \)-module and \( f(z, w) \in W[[z, w]][z_1^{-1}, z_2^{-1}, (z_1 - z_2)^{-1}][[t]]. \) Then

\[
\text{Res}_{z_1} \left( \delta(f(z_1, z_2)) \right) = \text{Res}_{z_2} \left( i_{z_2; z_3}f(z_2 + z_3, z_2) \right).
\]
Theorem 15.2. (Braided Borcherds Identity) Let $V$ be an $H_D$-quantum vertex algebra. Let $F \in \mathbb{C}[[z, w]][z^{-1}, w^{-1}, (z - w)^{-1}][[t]]$ and $a, b, c \in V$. Then we have the following identity:

$$\text{Res}_{z_1} \left( Y(a, z_1) Y(b, z_2)c i_{z_1; z_2} F(z_1, z_2) - Y_{z_2}(1 \otimes Y_{z_1}) i_{z_2; z_1} S^{(r), 12}_{z_2, z_1} (b \otimes a \otimes c) F(z_1, z_2) \right) =$$

$$\text{Res}_{z_3} \left( Y_{z_2}(Y_{z_3} \otimes 1) i_{z_2; z_3} \left( S^{(z_3, 12)}_{z_3, 0} (a \otimes b \otimes c) F(z_2 + z_3, z_2) \right) \right).$$

Proof. Take in Lemma 15.1 $f(z_1, z_2) = X_{z_1, z_2, 0}(a \otimes b \otimes c) F(z_1, z_2)$ and use Corollary 13.4 to relate expansions of the following identity:}

$$\text{Res}_{z_1} \left( Y(a, z_1) Y(b, z_2)c i_{z_1; z_2} F(z_1, z_2) - Y_{z_2}(1 \otimes Y_{z_1}) i_{z_2; z_1} S^{(r), 12}_{z_2, z_1} (b \otimes a \otimes c) F(z_1, z_2) \right) =$$

$$\text{Res}_{z_3} \left( Y_{z_2}(Y_{z_3} \otimes 1) i_{z_2; z_3} \left( S^{(z_3, 12)}_{z_3, 0} (a \otimes b \otimes c) F(z_2 + z_3, z_2) \right) \right).$$

\[ \square \]

16. The $S$-Commutator, Locality, and $(n)$-Products of Fields

Definition 16.1. Let $V$ be an $H_D$-quantum vertex algebra, and let $a, b, c \in V$. The $S$-commutator of the fields associated to $a, b$ is

$$[a(z_1), b(z_2)]_S = \delta \left( X_{z_1, z_2, 0}(a \otimes b \otimes c) \right).$$

Here $\delta$ is the difference of expansions, see (14.11). We can write the $S$-commutator using Corollary 13.4 explicitly as

$$[a(z_1), b(z_2)]_S = a(z_1) b(z_2) - Y_{z_2}(1 \otimes Y_{z_1}) i_{z_2; z_1} S^{(r), 12}_{z_2, z_1} (b \otimes a \otimes c).$$

Now the image of $\delta$ is a powerseries in $t$ with coefficients (finite) sums of derivatives of the Dirac distribution (14.2) with coefficients $V$-valued distributions in $z_2$. So we can write the commutator as

$$[a(z_1), b(z_2)]_S = \sum_{k > 0} t^k \left( \sum_n \gamma_{n, k}(z_2) \partial_{z_2}^{(n)} \delta(z_1, z_2) \right) =$$

$$= \sum_{n \geq 0} \gamma_{n}(z_2; t) \partial_{z_2}^{(n)} \delta(z_1, z_2).$$

This implies that for all $k \geq 0$ there is an $N > 0$ such that

$$(z_1 - z_2)^N [a(z_1), b(z_2)]_S \equiv 0 \mod t^k,$$

and we see that the $S$-commutator of $a, b \in V$ is a local distribution mod $t^k$, see [Kac98]. (The $S$-commutator is of course not necessarily itself local.)

Definition 16.2. For all $n \in \mathbb{Z}$ the $(n)$-product of fields associated to $a, b \in V$ is

$$a(z_2)_{(n)} b(z_2) c = \text{Res}_{z_1} \left( \delta \left( X_{z_1, z_2, 0}(a \otimes b \otimes c)(z_1 - z_2)^n \right) \right).$$

This definition allows us to write the $S$-commutator in terms of the $(n)$-product of fields, for $n \geq 0$.

Theorem 16.3. Let $V$ be an $H_D$-quantum vertex algebra. For all $a, b \in V$

$$[a(z_1), b(z_2)]_S = \sum_{n \geq 0} a(z_2)_{(n)} b(z_2) \partial_{z_2}^{(n)} \delta(z_1, z_2).$$
Proof. By the usual calculus of local distributions, see e.g., [Kac98], it follows from (16.1) that
\[ \gamma_n(z_1; t) = \text{Res}_{z_2} \left( [a(z_1), b(z_2)] S(z_1 - z_2)^n \right) = \text{Res}_{z_2} \left( \delta(X_{z_1, z_2, 0}(a \otimes b \otimes -)(z_1 - z_2)^n) \right), \]
by Definition 16.1. Then the Lemma follows from Definition 16.2. \( \square \)

17. \((n)\)-Products of States

We will call an element of \( V \) also a state. We define the \((n)\)-product of states (as opposed to that of fields) in \( V \) in the usual way:
\[ a^{(n)} b = \text{Res}_z (Y(a, z)b z^n), \]
so that
\[ (17.1) \quad Y(a, z) = \sum_{n \in \mathbb{Z}} a^{(n)} z^{-n-1}. \]

We also have
\[ (17.2) \quad a^{(n)} b = 0, \quad n \gg 0. \]

In contrast to the usual vertex algebras the state-field correspondence \( a \mapsto a(z) \) is not quite a homomorphism of the corresponding \((n)\)-products: in general
\[ a^{(n)} b(z) \neq a(z)^{(n)} b(z). \]

Indeed, introduce the generating series \( \mathcal{Y}_F \) of the \((n)\)-products of fields by
\[ \mathcal{Y}_F(a(z), w) = \sum_{n \in \mathbb{Z}} a^{(n)} w^{-n-1}. \]

Then, if the state-field correspondence were a homomorphism we would have
\[ (17.3) \quad \mathcal{Y}_F(a(z), w)b(z) = Y(Y(a, w)b, z). \]

But this in general not true: the translation map \( S_{z_1 z_2}^{(z)} \) is the obstruction to (17.3) being true. More precisely we have the following theorem.

**Theorem 17.1.**
\[ \mathcal{Y}_F(a(z), w)b(z)c = Y_z(Y_w \otimes 1)i_{z;w}S_{w,0}^{(z)}(a \otimes b) \otimes c. \]

**Proof.** By definition of the \((n)\)-product of fields, Lemma 15.1 and Proposition 13.4 we have
\[ \mathcal{Y}_F(a(z), w)b(z)c = \text{Res}_{z_1} \left( \delta(X_{z_1, z_2, 0}(a \otimes b \otimes c)(z_1 - z))^n w^{-n-1} \right) = \text{Res}_{z_3} \left( i_{z_1; z_3}(X_{z_2 + z_3, z_2, 0}(a \otimes b \otimes c) \delta(z_3, w)) = i_{z; w}X_{z + w, z, 0}(a \otimes b \otimes c) \right) = Y_z(Y_w \otimes 1)i_{z; w}S_{w,0}^{(z)}(a \otimes b) \otimes c. \] \( \square \)
Suppose that the translation map \( S_{a,b}^{(z_1)} \) is such that there exists \( N \in \mathbb{Z} \) such that for all \( a, b \in V \)

\[
(17.4) \quad i_{z_2; z_3} S_{z_3,0}^{(z_2)}(a \otimes b) = \sum_{k \geq -N} \left( \sum_{i} a_{i,k} \otimes b_{i,k} \right) s_k(z_2) z_3^k, \quad s_k(z_2) \in \mathbb{C}((z_2)),
\]

where for fixed \( k \) the summation over \( i \) is finite.

Note that in a general \( HD \)-quantum vertex algebra such expansion need not exist. In the main example (see Section 25) this condition is satisfied, however.

**Corollary 17.2.** Assume that \((17.3)\) holds in \( V \). Then for all \( a, b \in V \) and \( n \in \mathbb{Z} \)

\[
a(z)^{(n)} b(z) = \sum_{k \geq -N} \left( \sum_{i} Y((a_{i,k})^{(n)} b_{i,k}, z) \right) s_k(z).
\]

**Proof.** This is the case \( F = (z_1 - z_2)^n \) of the braided Borcherds identity, Theorem 16.2. Indeed, in this case the LHS is just the \((n)\)-product of the fields \( a(z_2) \) and \( b(z_2) \) acting on \( c \), see Definition 16.2 and Corollary 13.4. On the other hand the RHS of the braided Borcherds identity is in this case

\[
\text{Res}_{z_3} \left( Y_{z_2}(Y_{z_3} \otimes 1) \sum_{k \geq -N} \left( \sum_{i} a_{i,k} \otimes b_{i,k} \otimes c \right) s_k(z_2) z_3^{k+n} \right) = \text{Res}_{z_3} \left( Y \left( \sum_{k \geq -N} \left( \sum_{i} (a_{i,k})^{(n)} b_{i,k} \right) z_3^{-m-1}, z_2 \right) c s_k(z_2) z_3^{k+n} \right) = \sum_{k \geq -N} \left( \sum_{i} Y((a_{i,k})^{(n)} b_{i,k}, z) c \right) s_k(z_2).
\]

The proof is concluded by the substitution \( z_2 \mapsto z \). \( \square \)

18. Normal Ordered Products and Operator Product Expansion

We have used the \((n)\)-product (of fields) for \( n \geq 0 \) to calculate the \( S \)-commutator, see Theorem 16.3. The \((n)\)-products for \( n \leq -1 \) are also of course important.

**Definition 18.1.** The normal ordered product of fields \( a(z_1) \) and \( b(z_2) \) is given by

\[
:\!a(z_1)b(z_2): s = \text{Res}_z \left( \delta \left( X_{z,z_2,0}(a \otimes b \otimes c) \frac{1}{z - z_1} \right) \right).
\]

We introduce projections on singular and holomorphic parts of a formal distribution as usual by

\[
\text{Sing}_{z_1}(f(z_1, z_2, \ldots)) = -\text{Res}_z \left( f(z, z_2, \ldots) i_{z; z_1; z} \frac{1}{z - z_1} \right),
\]

\[
\text{Hol}_{z_1}(f(z_1, z_2, \ldots)) = \text{Res}_z \left( f(z, z_2, \ldots) i_{z; z_1} \frac{1}{z - z_1} \right).
\]

In particular, if \( f \) does not depend on \( z_2, \ldots \) we write

\[
f_{\text{Sing}}(z_1) = \text{Sing}_{z_1}(f(z_1)), \quad f_{\text{Hol}}(z_1) = \text{Hol}_{z_1}(a(z_1)).
\]

Then we can rewrite the definition of the normal ordered product as

\[
:\!a(z_1)b(z_2): s = a_{\text{Hol}}(z_1)b(z_2) + \text{Sing}_{z_1} \left( Y_{z_2}(1 \otimes Y_{z_1}) i_{z; z_1; z_2; z_1} S^{(7)}(\tau) b \otimes a \right),
\]
Comparing this with Definition 16.2 we see that
\[ a(z_2)_{(-1)} b(z_2) =: a(z_2) b(z_2) : S, \]
and more generally
\[ a(z_2)_{(-n-1)} b(z_2) =: \partial_z^{(n)} a(z_2) b(z_2) : S. \]
This gives the Operator Product Expansion of fields \( a(z_1), b(z_2) \):
\[ a(z_1) b(z_2) =: a(z_1) b(z_2) : S + \text{Sing}_{z_1} ([a(z_1), b(z_2)]_S) = \]
\[ =: a(z_1) b(z_2) : S + \sum_{n \geq 0} a(z_2)_{(n)} b(z_2) i_{z_1; z_2} \left( \frac{1}{(z_1 - z_2)^{n+1}} \right). \]
Of course, using Corollary 17.2 we can express the operator product expansion in terms of the \((n)\)-product of states, but this seems rather messy.

19. WEAK ASSOCIATIVITY

Two basic ingredients in the usual theory of vertex algebras are locality and associativity. For \( H_D \)-quantum vertex algebras the analog of locality is \( S \)-locality, (16.2). In this section we derive the analog of associativity. It involves the translation map \( S^{(r)}_{z_1, z_2} \).

**Theorem 19.1 (Weak associativity).** Let \( V \) be an \( H_D \)-quantum vertex algebra. For all \( a, b, c \in V \) and for all powers \( t^k \) there is an \( N \geq 0 \) such that
\[ (z_2 + z_3)^N i_{z_2; z_3} a(z_2 + z_3) b(z_2) c \equiv \]
\[ \equiv (z_2 + z_3)^N Y_{z_2} (Y_{z_3} \otimes 1) i_{z_2; z_3} (S^{(r)}_{z_2; z_3}) (a \otimes b) \otimes c \mod t^k. \]

**Proof.** Take \( \text{Res}_{z_1} \) in the braided Jacobi identity of Theorem 14.3 to find
\[ i_{z_2; z_2} a(z_2 + z_3) b(z_2) c - Y_{z_2} (Y_{z_3} \otimes 1) i_{z_2; z_3} (S^{(r)}_{z_2; z_3}) (a \otimes b) \otimes c = \]
\[ = - \text{Res}_{z_1} \left( i_{z_1; z_3} \delta(z_1 - z_2, z_3) Y_{z_2} (1 \otimes Y_{z_1}) (S^{(r)}_{z_2, z_1}) (b \otimes a) \otimes c \right) = \]
\[ = - \text{Res}_{z_1} \left( \sum_{k=0}^{\infty} (-z_1)^k \partial_{z_2}^{(k)} \delta(-z_2, z_3) Y_{z_2} (1 \otimes Y_{z_1}) i_{z_2; z_1} (S^{(r)}_{z_2, z_1}) (b \otimes a) \otimes c \right) . \]
Expanding the RHS observe that the coefficient of each power of \( t \) is after taking the residue a finite sum of \( z_2 \) derivatives of \( \delta(-z_2, z_3) \), hence vanishes if multiplied by a suitable power of \( z_2 + z_3 \). \( \square \)

**Remark 19.2.** For ordinary vertex algebras the power of \( N \) in weak associativity depends only on \( a \) and \( c \), not on \( b \). The above proof in the case of \( H_D \)-quantum vertex algebras does not allow us to conclude the same, because of the appearance of the braiding \( S^{(r)}_{z_2, z_1} (b \otimes a) \).

20. THE \( H_D \)-BIALGEBRA \( V \)

In the rest of the paper we will construct a class of examples of \( H_D \)-quantum vertex algebras, using bicharacters on the underlying space \( V \). To define bicharacters we need to assume that \( V \) has extra structure: we will assume that \( V \) is a commutative and cocommutative \( k \)-bialgebra, or even a Hopf algebra. The coproduct and counit of \( V \) will be denoted by \( \Delta \) and \( \epsilon \). We assume also that \( V \) has a compatible \( H_D \)-action. This means that
21. Bicharacters

Let $W_2$ be the algebra of power series in $t$, with coefficients rational functions in $z_1, z_2$ with poles at $z_1 = 0$, $z_2 = 0$ or $z_1 = z_2$:

\[(21.1) \quad W_2 = C[z_1^{\pm 1}, z_2^{\pm 1}, (z_1 - z_2)^{\pm 1}][[t]].\]

We extract some results from [Bor01] on bicharacters. A $W_2$-valued bicharacter on an $H_D$-bialgebra $V$ is a linear map

\[r_{z_1, z_2} : V \otimes V \to W_2,\]

satisfying

- **(Vacuum)** $r_{z_1, z_2}(a \otimes 1) = r_{z_1, z_2}(1 \otimes a) = \epsilon(a)$, $a \in V$.
- **(Multiplication)** For all $a, b, c \in V$ we have $r_{z_1, z_2}(a \otimes bc) = \sum r_{z_1, z_2}(a' \otimes b) r_{z_1, z_2}(a'' \otimes c)$ and $r_{z_1, z_2}(ab \otimes c) = \sum r_{z_1, z_2}(a \otimes c') r_{z_1, z_2}(b \otimes c'')$.

Here and below we use the notation $\Delta(a) = \sum a' \otimes a''$ for the coproduct of $a \in V$. Often we will also omit the summation symbol, to unclutter the formulas.

In case the bicharacter additionally satisfies

- **($H_D \otimes H_D$-covariance)** $r_{z_1, z_2}(D^k a \otimes D^l b) = \partial_{z_1}^k \partial_{z_2}^l r_{z_1, z_2}(a \otimes b)$, $a, b \in V$,

we call the bicharacter $H_D \otimes H_D$-covariant.

We can multiply bicharacters:

\[(21.2) \quad (r * s)_{z_1, z_2}(a \otimes b) = r_{z_1, z_2}(a' \otimes b') s_{z_1, z_2}(a'' \otimes b'').\]

The unit bicharacter is

\[(21.3) \quad \epsilon_{z_1, z_2}(a \otimes b) = \epsilon(a) \epsilon(b).\]

The collection of bicharacters on an $H_D$-bialgebra forms then a commutative monoid.

In case $V$ is an $H_D$-Hopf algebra, i.e., comes with an antipode compatible with the $H_D$-action, all bicharacters are invertible, with inverse given by

\[r_{z_1, z_2}^{-1}(a \otimes b) = r_{z_1, z_2}(S(a) \otimes b).\]

In this case the set of bicharacters forms an Abelian group.

The transpose of a bicharacter is defined by

\[r_{z_1, z_2}^\tau(a \otimes b) = r_{z_2, z_1}(b \otimes a).\]

The transpose is an involution of the monoid of bicharacters:

\[(21.4) \quad (r * s)_{z_1, z_2}^\tau = (r^\tau * s^\tau)_{z_1, z_2}.\]

If $r$ is an invertible bicharacter with inverse $r^{-1}$ we relate the transpose $r^\tau$ to $r$ by

\[(21.5) \quad r_{z_1, z_2}^\tau = r_{z_1, z_2}^{-1} * R_{z_1, z_2},\]

where

\[R_{z_1, z_2} = r_{z_1, z_2}^{-1} * r_{z_1, z_2}^\tau.\]
We will call $R_{z_1,z_2}$ the braiding bicharacter associated to $r_{z_1,z_2}$. It is the obstruction to $r$ being symmetric: $r = r^{T}$. It will control the braiding in the quantum vertex algebra we are going to construct from $r_{z_1,z_2}$ in Section 22 below. The braiding bicharacter $R_{z_1,z_2}$ is unitary:

$$R_{z_1,z_2}^{r} = R_{z_1,z_2}^{-1}.$$  

(21.6)

Define for a bicharacter $r_{z_1,z_2}$ a shift

$$\gamma_{z_1,z_2} = r_{z_1+z_2}.$$  

(21.7)

The shift $\gamma_{z_1,z_2}$ is again a bicharacter. If $r_{z_1,z_2}$ is $H_D \otimes H_D$-covariant we have the following expansion:

$$i_{z_1,z_2}\gamma_{z_1,z_2} = r_{z_1,z_2} \circ \Delta(e^D).$$

In case the bicharacter is invertible we relate the shift $\gamma$ to $r$ by

$$\gamma_{z_1,z_2} = r^{-1}_{z_1,z_2} \ast R_{z_1,z_2}^{r}, \quad \gamma_{z_1,z_2} = r^{-1}_{z_1,z_2} \ast r_{z_1,z_2}.$$  

(21.8)

We call $R_{z_1,z_2}$ the translation bicharacter associated to $r_{z_1,z_2}$. It is the obstruction to $r$ being shift invariant (i.e., to $r$ being a function just of $z_1 - z_2$).

### 22. $H_D$-Quantum Vertex Algebras from Bicharacters

Suppose now that $V$ is an $H_D$-bialgebra with invertible bicharacter $r_{z_1,z_2}$. In general, a bicharacter on $V$ takes values in $W_2$, see 21.1. For the purpose of the construction of vertex operators we need to make an extra assumption: that $r_{z_1,z_2}$ can be evaluated at $z_2 = 0$. More precisely, we make the following

**Definition 22.1.** A bicharacter $r_{z_1,z_2}$ satisfies the **Vertex Operator Assumption if it is a map**

(VO assumption) \[
V^2 \to \mathbb{C}[z_1^{\pm 1}, z_2, (z_1-z_2)^{-1}][t].
\]

In the sequel we will use $\rho_{z_1,z_2}$ to denote an arbitrary $W_2$-valued bicharacter, and we will write $r_{z_1,z_2}$ for a bicharacter satisfying the VO assumption.

Following the general philosophy of Borcherds, [Bor01], (but not the technical details) we define in this section, given an invertible bicharacter $r_{z_1,z_2}$ satisfying the VO assumption on an $H_D$-quantum vertex algebra structure on $V$. The final result is summarized in Theorem 22.15 below.

We define for any bicharacter $\rho_{z_1,z_2}$ on $V$ a map $S^{\rho_{z_1,z_2}}$ on $V \otimes V$ by

$$S^{\rho_{z_1,z_2}}(a \otimes b) = a' \otimes b' \rho_{z_1,z_2}(a'' \otimes b'').$$

(22.1)

In particular, to a bicharacter $r_{z_1,z_2}$ satisfying the VO assumption with braiding bicharacter $R_{z_1,z_2}$, see (21.6), we associate the map

$$S^{(r)}_{z_1,z_2} = S^{R_{z_1,z_2}}: V \otimes V \to V \otimes V[z_1^{\pm 1}, z_2, (z_1-z_2)^{\pm 1}][t],$$

(22.2)

and associated to the translation bicharacter (21.8) we get a map

$$S^{(\gamma)}_{z_1,z_2} = S^{R_{z_1,z_2}}: V \otimes V \to V \otimes V[z_1^{\pm 1}, z_2, (z_1+\gamma)^{\pm 1}, (z_2+\gamma), (z_1-z_2)^{\pm 1}][t],$$

(22.3)

**Lemma 22.2.**  
1. If $\epsilon$ is the unit bicharacter on $V$, then $S^{\epsilon} = 1_{V \otimes V}$.  
2. If $\rho_{z_1,z_2} \sigma_{z_1,z_2}$ are bicharacters on $V$, then $S^{\rho_{z_1,z_2} \sigma_{z_1,z_2}} = S^{\rho_{z_1,z_2} \circ \sigma_{z_1,z_2}}$.  
3. If $\rho_{z_1,z_2}$ is a bicharacter, then $\tau \circ S^{\rho_{z_1,z_2}} \circ \tau = S^{\rho_{z_1,z_2} \circ \tau}$.  
4. If $\rho_{z_1,z_2}^\gamma$ is a bicharacter, then $\tau \circ S^{\rho_{z_1,z_2}^\gamma} \circ \tau = S^{\rho_{z_1,z_2}^\gamma \circ \tau}$.  
5. If $S^{\rho}$ is a bicharacter, then $\tau \circ S^{\rho} \circ \tau = S^{\rho \circ \tau}$.  
6. If $S^{\rho}$ is a bicharacter, then $\tau \circ S^{\rho} \circ \tau = S^{\rho \circ \tau}$.
Define then, for given invertible bicharacter \( \rho_{z_1,z_2} \) satisfying the VO assumption singular multiplication maps

\[
X_{z_1,z_2} : V^2 \to V \otimes [z_1, z_2][z_1^{-1}, (z_1 - z_2)^{-1}][t].
\]

by

\[
X_{z_1,z_2} = m_2 \circ (e^{z_1D} \otimes e^{z_2D}) \circ S^{z_1,z_2},
\]

where \( m_2 \) is the (nonsingular) multiplications of the (associative) algebra \( V \). More explicitly (dropping here and below the nonsingular multiplication \( m_2 \) on \( V \)):

\[
X_{z_1,z_2}(a \otimes b) = e^{z_1D}a'e^{z_2D}b'r_{z_1,z_2}(a'' \otimes b'').
\]

Lemma 22.3. For any bicharacter \( \rho_{z_1,z_2} \) on \( V \) we have for \( a \in V \)

\[
S^{z_1,z_2}(a \otimes 1) = a \otimes 1,
\]

\[
S^{z_1,z_2}(1 \otimes a) = a \otimes 1.
\]

Proof. Since \( \rho_{z_1,z_2} \) is a bicharacter we have

\[
\rho_{z_1,z_2}(a \otimes 1) = \epsilon(a) = \rho_{z_1,z_2}(1 \otimes a).
\]

In any bialgebra we have \( a'\epsilon(a'') = a \), and the Lemma follows from the definition of \( S^{z_1,z_2} \), see \((22.1)\). \( \square \)

Corollary 22.4. The vacuum axioms \((3.1)\) and \((3.2)\) hold for \( X_{z_1,z_2} \) defined by \((22.4)\) and for \( S^{(\tau)}_{z_1,z_2}, S^{(\gamma)}_{z_1,z_2} \) defined by \((22.2)\) and \((22.3)\).

Lemma 22.5. For any \( H_D \otimes H_D \)-covariant bicharacter \( \rho_{z_1,z_2} \) we have

\[
[S^{z_1,z_2}, 1 \otimes D] = \partial_2 S^{z_1,z_2}, \\
[S^{z_1,z_2}, D \otimes 1] = \partial_1 S^{z_1,z_2}.
\]

Proof. By assumption on \( V \) we have \( \Delta(Db) = Db' \otimes b'' + b' \otimes Db'' \). By assumption on the bicharacter we have \( \rho_{z_1,z_2}(a \otimes Db) = \partial_2 \rho_{z_1,z_2}(a \otimes b) \). Then, for \( a, b \in V \)

\[
S^{z_1,z_2}(a \otimes Db) = a' \otimes Db' \rho_{z_1,z_2}(a'' \otimes b'') + a' \otimes b' \rho_{z_1,z_2}(a'' \otimes Db'') =
\]

\[
(1 \otimes D)S^{z_1,z_2} + \partial_2 S^{z_1,z_2}(a \otimes b),
\]

proving the first part. The second part is similar. \( \square \)

Corollary 22.6. The \( H_D \)-covariance axiom \((3.3)\) holds for \( X_{z_1,z_2} \) defined by \((22.4)\) and the \( H_D \)-covariance axiom \((3.4)\) holds for \( S^{(\tau)}_{z_1,z_2}, S^{(\gamma)}_{z_1,z_2} \) defined by \((22.2)\) and \((22.3)\).

Lemma 22.7. The \( H_D \)-covariance axiom \((3.3)\) holds for \( X_{z_1,z_2} \) defined by \((22.4)\).

Proof. We have

\[
X_{z_1+\gamma,z_2+\gamma}(a \otimes b) = e^{(z_1+\gamma)D}a'e^{(z_2+\gamma)D}b'r_{z_1+\gamma,z_2+\gamma}(a'' \otimes b'') =
\]

\[
e^{\gamma D}(e^{z_1D}a'e^{(z_2D)}b')r_{z_1,z_2}(a''' \otimes b''')R_{z_1,z_2}(a'''' \otimes b''') =
\]

\[
e^{\gamma D}X_{z_1,z_2} \circ S^{(\gamma)}_{z_1,z_2}(a \otimes b).
\]

Lemma 22.8. For any bicharacter \( \rho_{z_1,z_2} \) the map \( S^{z_1,z_2} \) satisfies the Yang-Baxter equation \((3.6)\).
Proof. This follows from the combined cocommutativity and coassociativity identity
\[ \tau^{23} (\Delta \otimes 1) \Delta = (\Delta \otimes 1) \Delta. \]

\[ \square \]

**Corollary 22.9.** The maps \( S^{(r)}_{i1, i2}, S^{(s)}_{i1, i2} \) defined by (22.2) and (22.3) satisfy the Yang-Baxter axiom (3.7).

**Lemma 22.10.** For any bicharacter \( \rho_{i1, i2} \) the map \( S^{\rho_{i1, i2}} \) is compatible with the singular multiplication:

\[
S^{\rho_{i1, i2}}(X_{w_1, w_2} \otimes 1) = (X_{w_1, w_2} \otimes 1) i_{i1, i2} \circ \rho_{i1, i2} \circ S^{\rho_{i1, i2} \Delta} \Delta,
\]

\[
S^{\rho_{i1, i2}}(1 \otimes X_{w_1, w_2}) = (1 \otimes X_{w_1, w_2}) i_{i1, i2} \circ \rho_{i1, i2} \circ S^{\rho_{i1, i2} \Delta} \Delta.
\]

Proof. For any \( a, b, c \in V \) we have

\[
S^{\rho_{i1, i2}}(X_{w_1, w_2} \otimes 1)(a \otimes b \otimes c) = S^{\rho_{i1, i2}}((e^{w_1} a e^{w_2} b') \otimes c) r_{w_1, w_2}(a'' \otimes b'') =
\]

\[
= (e^{w_1} a e^{w_2} b') c \rho_{i1, i2} ((e^{w_1} a e^{w_2} b') \otimes c) r_{w_1, w_2}(a'' \otimes b'') =
\]

\[
= e^{w_1} a'' e^{w_2} b'' c \rho_{i1, i2} (w_1 + w_2) (a'' \otimes c''') r_{w_1, w_2}(a'' \otimes b'') =
\]

\[
i_{i1, i2} \circ \rho_{i1, i2} (X_{w_1, w_2} \otimes 1) S^{\rho_{i1, i2} \Delta} \Delta (a \otimes b \otimes c)
\]

The proof of the other part is similar. \[ \square \]

**Corollary 22.11.** \( S^{(r)}_{i1, i2}, S^{(s)}_{i1, i2} \) defined by (22.2) and (22.3) satisfy the compatibility with multiplication axioms (3.7) and (3.8).

**Corollary 22.12.** \( S^{(r)}_{i1, i2} \) defined by (22.2) satisfies the unitarity axiom (3.9).

Proof. For unitarity, recall that \( S^{(r)}_{i1, i2} = S^{R_{i1, i2}} \), so that by Lemma 22.2 we have \( \tau \circ S^{(r)}_{i1, i2} \circ \tau = S^{R_{i1, i2}} \Delta \Delta \), so that by Lemma 22.2 again and (21.6) we find

\[
S^{(r)}_{i1, i2} \circ \tau \circ S^{(r)}_{i1, i2} \circ \tau = S^{R_{i1, i2} \Delta} \Delta = S^r = 1_{V \otimes V}.
\]

\[ \square \]

**Corollary 22.13.** \( S^{(r)}_{i1, i2} \) defined by (22.3) satisfies the group axioms (3.11) and (3.10).

Proof. Since \( R^{(r) = (0)}_{i1, i2} = e, \) the unit bicharacter, axiom (3.11) follows.

Now

\[
R^{(r) = (0)}_{i1, i2} = R^{(r) = (0)}_{i1, i2} \circ R^{(r) = (0)}_{i1, i2} =
\]

\[
= R^{(r) = (0)}_{i1, i2} \circ R^{(r) = (0)}_{i1, i2} \circ R^{(r) = (0)}_{i1, i2} =
\]

\[
= R^{(r) = (0)}_{i1, i2} \circ R^{(r) = (0)}_{i1, i2} \circ R^{(r) = (0)}_{i1, i2} =
\]

so that axiom (3.10) follows from Lemma 22.2 and definition (22.3). \[ \square \]

**Lemma 22.14.** \( X_{i1, i2} \) and \( S^{(s)}_{i1, i2} \) defined by (22.4) and (22.5) satisfy the locality Axiom (3.12).
Theorem 22.15. \(\) Let
\[
E = e^{z_1 D} e^{z_2 D} b' c' r_{z_1, z_2} (a'' \otimes b'') r_{z_2, 0} (a'''' \otimes c''') r_{z_2, 0} (b''' \otimes c''').
\]
Then
\[
X_{z_1, 0} (1 \otimes X_{z_2, 0}) (A) = e^{z_1 D} a' (e^{z_2 D} b' c') r_{z_1, 0} (a'' \otimes (e^{z_2 D} b' c')') r_{z_2, 0} (b''' \otimes c''') =
\]
\[
e^{z_1 D} a' e^{z_2 D} b' c' r_{z_1, 0} (a'' \otimes e^{z_2 D} b'') r_{z_1, 0} (a'''' \otimes c''') r_{z_2, 0} (b''' \otimes c''') =
\]
\[
e^{z_1 D} a' e^{z_2 D} b' c' i_{z_1, z_2} r_{z_1, z_2} (a'' \otimes b'') r_{z_1, 0} (a'''' \otimes c''') r_{z_2, 0} (b''' \otimes c''') =
\]
\[= i_{z_1, z_2} E.\]
On the other hand
\[
X_{z_2, 0} (1 \otimes X_{z_1, 0}) i_{z_2; z_1} S^{(\tau)} (b \otimes a \otimes c) =
\]
\[
= X_{z_2, 0} (1 \otimes X_{z_1, 0}) (b' \otimes a' \otimes c) i_{z_2; z_1} R^{(\tau)} (b' \otimes a'') =
\]
\[
= X_{z_2, 0} (b' \otimes e^{z_1 D} a' c') r_{z_1, 0} (a'' \otimes c'') i_{z_2; z_1} R^{(\tau)} (b' \otimes a'') =
\]
\[
e^{z_1 D} b' (e^{z_1 D} a'' c') r_{z_1, 0} (b''' \otimes (e^{z_1 D} a' c')') x
\]
\[
\times r_{z_1, 0} (a'''' \otimes c''') i_{z_2; z_1} R^{(\tau)} (b''' \otimes a'') =
\]
\[
e^{z_1 D} b' e^{z_1 D} a' c' i_{z_2; z_1} \left( r_{z_1, z_2} (b''' \otimes a''') R^{(\tau)} (b''' \otimes a''') \right) \times
\]
\[
\times r_{z_2, 0} (b''' \otimes c''') r_{z_1, 0} (a'''' \otimes c''') =
\]
\[
e^{z_1 D} b' e^{z_1 D} a' c' i_{z_2; z_1} \left( r_{z_1, z_2} (a''' \otimes b'''') \right) r_{z_2, 0} (b''' \otimes c''') \times
\]
\[
\times r_{z_1, 0} (a'''' \otimes c'''') =
\]
\[= i_{z_2; z_1} E.\]
Since
\[
(z_1 - z_2)^N i_{z_2; z_1} E = (z_1 - z_2)^N i_{z_2; z_1} E
\]
the locality Axiom (5.12) follows. \(\square\)

The results in this section are summarized in the following theorem.

**Theorem 22.15.** \(\) Let \(V\) be an \(H_D\)-bialgebra with invertible bicharacter \(r_{z_1, z_2}\), satisfying the VO assumption of Definition 22.4. Then the singular multiplications \(X_{z_1, z_2}, X_{z_2, z_1}, z_{1, z_2}\) and maps \(S_{\tau}^{(\tau)}\), \(S_{\tau}^{(\gamma)}\) defined by (22.2), (22.3) and (22.3) give \(V\) the structure of an \(H_D\)-quantum vertex algebra as in Definition 23.7.

### 23. Bicharacters and EK-Quantum Vertex Operator Algebras

Let \(V\) be an \(H_D\)-bialgebra with invertible bicharacter, so that we have on \(V\) by Theorem 22.15 an \(H_D\)-quantum vertex algebra structure. In case the bicharacter satisfies
\[
(\mathcal{D}_{z_1} + \mathcal{D}_{z_2}) r_{z_1, z_2} = 0
\]
the bicharacter is really just a function of \(z_1 - z_2\); \(r_{z_1, z_2}\) takes values in \(\mathbb{C}[(z_1 - z_2)^{\pm 1}][[t]]\). In this case the translation bicharacter \(R_{z_1, z_2}^2\) is the unit bicharacter on \(V\).
In this situation we can evaluate the bicharacter $r_{z_1,z_2}$, the vertex operator $X_{z_1,z_2}$ and the braiding $S_{z_1,z_2}$ both at $z_1 = 0$ and at $z_2 = 0$.

We have in this case $r_{0,z} = r_{-z,0}$ so that

$$X_{0,z}(a \otimes b) = e^{zD} (e^{-zD}a'b') r_{0,z}(a'' \otimes b'') = e^{zD}Y(a,-z)b.$$  

The braided commutativity Lemma [3.1] gives, by putting $z_2 = 0$,

$$Y(a',z)b'R_{z,0}(a'' \otimes b'') = e^{zD}Y(b,-z)a.$$  

We emphasize that in general $H_D$-quantum vertex algebras one does not have a similar braided skew-symmetry, since the braiding $S_{z_1,z_2}$ cannot be evaluated at $z_2 = 0$.

The $H_D$-covariance axiom (3.5) reduces to the familiar formula

$$(23.2) e^{\gamma D}Y(a,z)e^{-\gamma D} = iz\gamma; e^{\gamma D}Y(b,z)e^{-\gamma D}.$$  

Infinitesimally this gives another familiar formula: by differentiating with respect to $\gamma$ we obtain

$$(23.3) [D, Y(a,z)] = \partial_z Y(a,z).$$  

Bicharacters satisfying condition (23.1) give rise to quantum vertex operator algebras in the sense of Etingof-Kazhdan, [EK00]. In case the bicharacter satisfies (23.1) and is also symmetric:

$$r_{z_1,z_2} = r_{z_1,z_2},$$  

we obtain vertex operators of a vertex algebra as is usually defined (see [FLM88], [Kac98]. This is a special case of a more general result of Borcherds, see [Bor01], Theorem 4.2.

The condition (23.1) is not satisfied in the case we are interested in, see section 25.

24. Bicharacter Expansions and $S$-commutator

We continue to assume that $V$ has an $H_D$-quantum vertex algebra structure via a bicharacter $r_{z_1,z_2}$, see Theorem 22.15. In this section we show how an expansion of the bicharacter leads to a closed formula for the $S$-commutator of fields.

Consider the vectorspace $V \otimes W(z)$, where $W(z)$ is some space of functions (or power series) in $z$. Then we get an action of $H_D$ on this vector space by using the coproduct:

$$D^{(k)}(a \otimes f(z)) = \sum_{p+q=k} D^{(p)}a \otimes \partial_z^{(q)}f(z).$$

**Theorem 24.1.** Let $a, b \in V$ and suppose that

$$\delta(r_{z_1,z_2}(a \otimes b)) = \sum_{k \geq 0} d_k(a \otimes b; t)\partial_z^{(k)}\delta(z_1, z_2),$$

where $d_k(a \otimes b; t) \in \mathbb{C}[[z_1^{\pm 1}, z_2^{\pm 1}]][[t]]$. Then we have

$$[a(z_1), b(z_2)]_S = \sum_{k \geq 0} d_k(a' \otimes b'; t) \sum_{p+q=k} Y\left([D^{(p)}a'']b', z_2\right)\partial_z^{(q)}\delta(z_1, z_2).$$
Proof. The RHS of the $S$-commutator of the fields of $a$ and $b$ acting on $\epsilon^{z_1D}a'(\epsilon^{z_2D}b')c'\delta(r_{z_1,z_2}(a'' \otimes b'''))r_{z_1,0}(a''' \otimes e'')r_{z_2,0}(b''' \otimes e''') =

\begin{align*}
&= (\epsilon^{z_2D}b')c' \sum_{k \geq 0} d_k(a'' \otimes b'''; t) \partial^{(k)}_{z_2} (\epsilon^{z_2D} a') r_{z_2,0}(a''' \otimes e'') \delta(z_1, z_2) \times \\
&\quad \times r_{z_2,0}(b''' \otimes e''') = \\
&= (\epsilon^{z_2D}b')c' \sum_{k \geq 0} d_k(a'' \otimes b'''; t) \sum_{p+q+r=k} \epsilon^{z_2D} (D^{(p)} a') \times \\
&\quad \times r_{z_2,0}(D^{(q)} a''') \otimes e''' r_{z_2,0}(b''' \otimes e''') \partial^{(r)}_{z_2} \delta(z_1, z_2) = \\
&= \sum_{k \geq 0} \sum_{p+q+r=k} d_k(a' \otimes b'; t) \epsilon^{z_2D} (D^{(p)} a''') b''' c'(D^{(q)} a''') r_{z_2,0}(b''' \otimes e''') \times \\
&\quad \times \partial^{(r)}_{z_2} \delta(z_1, z_2) = \\
&= \sum_{k \geq 0} \sum_{p+q=r} d_k(a' \otimes b'; t) Y((D^{(p)} a''') b''', z_2) c\partial^{(q)}_{z_2} \delta(z_1, z_2).
\end{align*}

$25. The Main Example$

For the rest of the paper we will study a particular example of an $H_D$-quantum vertex algebra $V$ obtained from a bicharacter as in Theorem 22.15. As an vector space $V$ is the underlying space of the lattice vertex algebra based on the rank 1 lattice $\mathbb{Z}$ with pairing $(m, n) \mapsto mn$, cf. [Kac98], section 5.4.

To define a bicharacter on $V$ we need an $H_D$-bialgebra structure. As $H_D$-bialgebra $V$ is generated by group-like elements $e^\alpha, e^{-\alpha}$, so that

$\Delta(e^{m\alpha}) = e^{m\alpha} \otimes e^m, \quad \epsilon(e^{m\alpha}) = 1, \quad m \in \mathbb{Z}.$

If we write $h = (De^\alpha)e^{-\alpha}$ then $h$ is primitive: we have $\Delta(h) = h \otimes 1 + 1 \otimes h$, $\epsilon(h) = 0$. Then $V = \bigoplus_{m \in \mathbb{Z}} V_m, \quad V_m = k[D^nh]_{n \geq 0} \otimes e^{m\alpha}$.

In fact $V$ is a Hopf algebra, with antipode $S: e^\alpha \mapsto e^{-\alpha}$. We define in this case a bicharacter on $V$ by putting on generators

$25.1 \quad r_{z_1,z_2}(e^{m\alpha} \otimes e^{n\alpha}) = \sigma^{mn}, \quad \sigma = \frac{z_1 - z_2}{1 - z_2/z_1},$

and extend to all of $V$ by using the properties of bicharacters, see [Bor01] for details. Here (and below) we will expand any rational expression in $t$ in positive powers of $t$. Note that $r_{z_1,z_2}$ satisfies the VO assumption of Definition 22.1. So by Theorem 22.15 $V$ has an $H_D$-quantum vertex algebra structure.

The bicharacter $r_{z_1,z_2}$ of this example is implicit in the paper by Jing, [Jin91]. By putting $t = 0$ we obtain a bicharacter $r_{z_1,z_2}^0$ which is implicit in the usual construction of a lattice vertex algebra from the lattice $\mathbb{Z}$ with pairing $(m, n) \mapsto mn$.

We will collect for later reference some values of this bicharacter and of its associated braiding and translation bicharacters. First a simple lemma.
Lemma 25.1. For any bicharacter \( \rho_{z_1, z_2} \) on \( V \) we have, if \( \rho_{z_1, z_2}(e^{m\alpha} \otimes e^{n\alpha}) = \rho^{mn} \),
\[ \rho_{z_1, z_2}(h \otimes e^{m\alpha}) = m\partial_{z_1} \ln(\rho), \quad \rho_{z_1, z_2}(h \otimes h) = \partial_{z_2} \partial_{z_2} \ln(\rho). \]

Lemma 25.2.
\[ r_{z_1, z_2}(h \otimes e^{m\alpha}) = m\left( \frac{1}{z_1 - z_2} - \frac{t z_2/z_1}{z_1 - t z_2} \right), \]
\[ r_{z_1, z_2}(h \otimes h) = \frac{1}{(z_1 - z_2)^2} - \frac{t}{(z_1 - t z_2)^2}. \]

The bicharacter \( r_{z_1, z_2} \) is invertible \((V \text{ being a Hopf algebra})\), with inverse on generators given by
\[ r_{z_1, z_2}^{-1}(e^{m\alpha} \otimes e^{n\alpha}) = \sigma^{-mn}. \]

Lemma 25.3. The braiding bicharacter \( R_{z_1, z_2} \) of \( r_{z_1, z_2} \) is given on the generators by
\[ R_{z_1, z_2}(e^{m\alpha} \otimes e^{n\alpha}) = \Sigma^{mn}, \quad \Sigma = \Sigma_{z_1, z_2} = \frac{1 - t z_2/z_1}{1 - t z_1/z_2}, \]
and we have
\[ R_{z_1, z_2}(h \otimes e^{m\alpha}) = m t \left( \frac{t z_2/z_1}{z_1 - t z_2} + \frac{t}{z_2 - t z_1} \right), \]
and
\[ R_{z_1, z_2}(h \otimes h) = \frac{t}{(z_1 - t z_2)^2} - \frac{t}{(z_2 - t z_1)^2}. \]

Lemma 25.4. The translation bicharacter \( R^\gamma \) of \( r_{z_1, z_2} \) is given on generators by
\[ R^\gamma_{z_1, z_2}(e^{m\alpha} \otimes e^{n\alpha}) = \Pi^{mn}, \quad \Pi = \Pi_{z_1, z_2} = \frac{1 - t z_2/z_1}{1 - t(z_1 + \gamma)}, \]
and we have
\[ R^\gamma_{z_1, z_2}(h \otimes e^{m\alpha}) = \frac{m t z_2/z_1}{z_1 - t z_2} - \frac{m t (z_2 + \gamma)/(z_1 + \gamma)}{(z_1 + \gamma) - t(z_2 + \gamma)}, \]
and
\[ R^\gamma_{z_1, z_2}(h \otimes h) = \frac{t}{(z_1 - t z_2)^2} - \frac{t}{(z_1 + \gamma) - t(z_2 + \gamma))^2}. \]

We will calculate some \((n)\)-products of states and of fields in \( V \) to illustrate what is involved.

First note that \( r_{z,0}(e^{\alpha} \otimes e^{-\alpha}) = \frac{1}{z} \). This implies that
\[ Y(e^{\alpha}, z)e^{-\alpha} = (e^{zD}e^{\alpha})e^{-\alpha}r_{z,0}(e^{\alpha} \otimes e^{-\alpha}) = \frac{1}{z} + h + O(z), \]
so that we have the following products of states.
\[ e^{\alpha}_{(-1)}e^{-\alpha} = h, \quad e^{\alpha}_{(0)}e^{-\alpha} = 1, \quad e^{\alpha}_{(k)}e^{-\alpha} = 0, k > 0. \]

Note that this are the same \((n)\)-products as for the lattice vertex algebra corresponding to the bicharacter \( r_{z_1, z_2}^0 \) (obtained by putting \( t = 0 \)).
Next we want to use Corollary [17.2] to calculate \((n)\)-products of fields. We have by Lemma [25.4] \(R_{z_1,0}^2(e^\alpha \otimes e^{-\alpha}) = 1 - \frac{z_1}{z_1 + z_2}\), so that
\[
i_{z_2; z_3} S_{z_3,0}^{(z_2)}(e^\alpha \otimes e^{-\alpha}) = e^\alpha \otimes e^{-\alpha} i_{z_2; z_3} (1 - \frac{z_3}{z_2 + z_3}) = e^\alpha \otimes e^{-\alpha} \left(1 + t \sum_{k=1}^{\infty} (-1)^k \frac{z_3^k}{z_2^k}\right).
\]
Hence by Corollary [17.2] and [25.8]
\[
e^\alpha(z)(-1)e^{-\alpha}(z) = Y(e^\alpha_{(-1)} e^{-\alpha}, z) - Y(e^\alpha_0 e^{-\alpha}, z) \frac{t}{z} = h(z) - \frac{t}{z}.
\]
Now, see Section [18] \(e^\alpha(z)(-1)e^{-\alpha}(z) = :e^\alpha(z)e^{-\alpha}(z):_S\), and this normal ordered product of fields is not a vertex operator \(Y(a, z)\) for any \(a \in V\), since the action of \(:e^\alpha(z)e^{-\alpha}(z):_S\) on the vacuum is not regular in \(z\), contradicting the vacuum axiom [3.1]. This is in contrast to the situation in the usual vertex algebras.

26. S-Commutators and Commutators

In this section we calculate some \(S\)-commutators of fields by expanding the bicharacter in our main example and express this in terms of commutators, using Theorem [24.1].

We have
\[
\delta\left(r_{z_1, z_2}(e^{m\alpha} \otimes e^{n\alpha})\right) = \begin{cases} 0 & mn \geq 0, \\
(1 - t \frac{z_2}{z_1})^{k+1} \partial^{(k)}(z_1, z_2) & mn = -k - 1 < 0,
\end{cases}
\]
which follows from the definition [25.1]. Then
\[
[e^{m\alpha}(z_1), e^{n\alpha}(z_2)]_S = \begin{cases} 0 & mn \geq 0, \\
(1 - t \frac{z_2}{z_1})^{k+1} \sum Y(v^{p}_{m,n}, z_2) \partial^{(q)}(z_1, z_2) & mn = -k - 1 < 0,
\end{cases}
\]
where \(v^{p}_{m,n} = D^{(p)}(e^{m\alpha}) e^{n\alpha} \in V\) and the sum is over all \(p, q \geq 0\) such that \(p + q = k\). In particular
\[
[e^\alpha(z_1), e^{-\alpha}(z_2)]_S = (1 - t \frac{z_2}{z_1}) \delta(z_1, z_2) = (1 - t) \delta(z_1, z_2).
\]
So
\[
e^\alpha(z)(0) e^{-\alpha}(z) = 1 - t, \quad e^\alpha(z)(k) e^{-\alpha}(z) = 0, k > 0.
\]
In the same way
\[
\delta\left(r_{z_1, z_2}(h \otimes e^{m\alpha})\right) = m \delta(z_1, z_2),
\]
which follows from Lemma [25.2], see also [14.3]. Hence
(26.1) \([h(z_1), e^{m\alpha}(z_2)]_S = me^{m\alpha}(z_2) \delta(z_1, z_2)\).

Finally, using Lemma [25.2] again, we find
\[
\delta\left(r_{z_1, z_2}(h \otimes h)\right) = \partial_{z_2} \delta(z_1, z_2),
\]
so that
(26.2) \([h(z_1), h(z_2)]_S = \partial_{z_2} \delta(z_1, z_2)\).
It is sometimes useful to express the $S$-commutators of fields in terms of the usual commutators. We give some examples.

We have by definition of the $S$-commutator
\[
[h(z_1), e^{\alpha}(z_2)]_S = h(z_1)e^{\alpha}(z_2) - e^{\alpha}(z_2)h(z_1) = [h(z_1), e^{\alpha}(z_2)] - e^{\alpha}(z_2)R_{z_2, z_1}(e^{\alpha} \otimes h) = \]
\[
= [h(z_1), e^{\alpha}(z_2)] - e^{\alpha}(z_2)m \partial z_1 \ln(S_{z_2, z_1}),
\]
where $\Sigma$ is defined in Lemma 25.3. Combining this with (26.1) gives
\[
[h(z_1), e^{\alpha}(z_2)] = me^{\alpha}(z_2)h(z_1) + \partial z_1 \ln(S_{z_2, z_1}).
\]
Now
\[
\text{Res}_{z_1} \left( z_1^n \partial z_1 \ln(S_{z_2, z_1}) \right) = \begin{cases} 
0 & n = 0, \\
-t^n z_2^n & n \neq 0.
\end{cases}
\]
Hence
\[
[h(n), e^{\alpha}(z_2)] = \begin{cases} 
me^{\alpha}(z_2) & n = 0, \\
mz_2^n(1 - t^n)e^{\alpha}(z_2) & n \neq 0.
\end{cases}
\]
Similarly,
\[
[h(z_1), h(z_2)]_S = [h(z_1), h(z_2)] - R_{z_2, z_1}(h \otimes h) = \]
\[
= [h(z_1), h(z_2)] - \frac{t}{(t z_1 - z_2)^2} - \frac{t}{(t z_2 - z_1)^2},
\]
Note that here we see that the ordinary commutator of $h(z)$ with itself is not killed by any power of $z_1 - z_2$, whereas the $S$-commutator is killed by $(z_1 - z_2)^2$, see (26.2).

By (26.2)
\[
[h(z_1), h(z_2)] = \partial z_2 \delta(z_1, z_2) + R_{z_2, z_1}(h \otimes h).
\]
Now
\[
\text{Res}_{z_1} \left( z_1^n R_{z_2, z_1}(h \otimes h) \right) = -nt^n z_2^{n-1},
\]
and we have
\[
[h(m), h(z_2)] = mz_2^{m-1}(1 - t^m)
\]
and
\[
[h(m), h(n)] = (1 - t^m)\delta_{m+n, 0}.
\]
We see therefore that the coefficients of $h(z)$ generate a deformed Heisenberg algebra $\mathcal{H}_t$. As a Lie algebra $\mathcal{H}_t$ is isomorphic to the usual Heisenberg Lie algebra $\mathcal{H} = \mathcal{H}_{t=0}$. In particular the representation theory of $\mathcal{H}_t$ is the same as in the undeformed case. We have a decomposition
\[
V = \bigoplus_{m \in \mathbb{Z}} V_m, \quad V_m = k[D^n h]e^{\alpha},
\]
where each $V_m$ is an irreducible $\mathcal{H}_t$-module, with action given by
\[
h(m) = \begin{cases} 
multiplication by $D^k h/k!$ & m = -k - 1 < 0, \\
\partial \alpha & m = 0, \\
(m(1 - t^m)\partial)^{\frac{\partial}{\partial k(-m - 1)}} & m > 0.
\end{cases}
\]
The case $m = 0$ follows from Cor 17.2 and (26.1).
27. Braided Bosonization

Define

$$\Gamma_+(z) = \exp \left( \sum_{n>0} h_{(-n)} z^n/n \right), \quad \Gamma_-(z) = \exp \left( -\sum_{n>0} h_{(n)} z^{-n}/n \right).$$

By (26.6) we have for $m \neq 0$

$$[h_{(\pm m)}, \Gamma_{\pm}(z)] = \pm z_{\pm m} (1 - t^{|m|}) \Gamma_{\pm}(z), \quad [h_{(\mp m)}, \Gamma_{\pm}(z)] = 0.$$

Then we see that

$$\Sigma_n(z) = \Gamma^-_n(z) e^{n\alpha}(z) \Gamma^+_n(z) e^{n\alpha}$$

commutes with the deformed boson:

$$[h(z_1), \Sigma_n(z_2)] = 0,$$

and by the usual arguments using the representation theory of the deformed Heisenberg algebra (see e.g., [Kac98]) one finds the bosonization formula

$$e^{n\alpha}(z) = \Gamma^+_n(z) \Gamma^-_n(z) e^{n\alpha} z^{n\alpha}.$$

This formula (for $n = \pm 1$) can be found in Jing’s paper, [Jin91], with a slightly different notation.

28. Hall-Littlewood Polynomials

In this section we recall the Macdonald definition of Hall-Littlewood symmetric polynomials ([Mac95]). Also we explain how the bosonized vertex operators described in the previous section (as considered by N. Jing, [Jin91]), serve as generating functions for the Hall-Littlewood polynomials.

Denote by $\Lambda$ the ring of symmetric functions over $\mathbb{C}[[t]]$ in countably many independent variables $x_i$, $i \geq 0$.

Let $\lambda$ be a partition, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k, \ldots)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq \ldots$ Let $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_k + \ldots$.

Denote $z_\lambda = \prod_{i \geq 0} t^{m_i} m_i!$, where $m_i = m_i(\lambda)$ is the number of parts of $\lambda$ equal to $i$.

We call a family $(a_\lambda)$ of elements in a ring indexed by partitions multiplicative if $a_{\lambda} = \prod a_{\lambda_i}$.

For any partition $\alpha$ we use the vector notation $x^\alpha$ for $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_k^{\alpha_k} \ldots$. We will use the basis $(m_\lambda)$ of monomial symmetric functions:

$$(28.1) \quad m_\lambda = \sum \alpha x^\alpha,$$

where the sum is over distinct permutations of $\lambda$, as well as the multiplicative basis generated by the power sums $p_n = \sum_{i \geq 0} x_i^n$, $p_0 = 1$.

Define a scalar product $\langle \ , \ \rangle_t$ on $\Lambda_F$ by putting for the power functions

$$\langle p_\lambda, p_\mu \rangle_t = \delta_{\lambda\mu} z_\lambda v_\lambda,$$

for any partitions $\lambda, \mu$, where the multiplicative family $v_\lambda$ is defined by $v_n = \frac{1}{1-t^n}$.

Define a set of symmetric functions $\{H_\lambda\}$ indexed by partitions by the following...
two (over-determining) conditions:
\[\langle H_\lambda, H_\mu \rangle_t = 0 \quad \text{for} \quad \lambda \neq \mu,\]
\[H_\lambda = m_\lambda + \sum_{\mu < \lambda} u_{\lambda \mu} m_\mu, \quad u_{\lambda \mu} \in \mathbb{C}[t].\]

Here \(\mu < \lambda\) is with respect to the usual partial order on partitions.

It is proved in ([Mac95]) that such symmetric functions \(\{H_\lambda\}\) exist. Denote also by \(Q_\lambda\) the dual of \(H_\lambda\), i.e., \(\langle H_\lambda, Q_\mu \rangle_t = \delta_{\lambda \mu}\). Note that when \(t = 0\) both the \(H_\lambda\) and the \(Q_\lambda\) reduce to the Schur polynomials (Schur polynomials are self dual).

We can view the \(n\)-th power symmetric function \(p_n\) as an operator acting on \(\Lambda_C[[t]]\) by multiplication. Define also for given multiplicative family \((v_\lambda)\) the operators \(p_n^\perp\) by requiring
\[\langle p_n^\perp f, g \rangle_t = \langle f, p_n g \rangle_t,\]
for any \(f, g \in \Lambda_C[[t]]\).

**Lemma 28.1.** The operators \(\{h(n) | n \in \mathbb{Z}\}\) given by \(h(n) = -(1 - t^n)p_n^\perp\), \(h(-n) = (1 - t^n)p_n\) for \(n \in \mathbb{N}\), \(h(0) = 0\) generate a representation of the deformed Heisenberg algebra \(H_t\) on \(\Lambda_C[[t]]\), i.e.,
\[(28.2) [h(m), h(n)] = m(1 - t^{|m|})\delta_{m+n,0}.\]

The proof is based on the undeformed case \((v_n = 1)\), which can be found in [Mac95].

From the fact that the power symmetric functions form a basis of \(\Lambda_C[[t]]\), it follows that \(\Lambda_C[[t]]\) is a highest weight module for \(H_t\), and is thus an irreducible \(H_t\) module. Therefore we have that \(\Lambda_C[[t]]\) is isomorphic as a module and as an algebra to \(V_0\) (\(V_0\) was defined in Section 25). Thus we can identify \((1 - t^n)p_n\) with \(D^{(n-1)}h\) \((n > 0)\).

The following theorem ([Jin91]) explains the connection between the Hall-Littlewood symmetric functions and the vertex operators considered in the previous section:

**Theorem 28.2.** Let \(\tilde{m}\) is a partition of length \(l\), \(\tilde{m} = (m_1, m_2, \ldots, m_l, 0, \ldots)\), and let \(\rho\) be the partition defined by \(\rho = (l, l-1, \ldots, 1, 0, \ldots)\). The constant term of \(Y(D^{(m_1)}e^\alpha, z_1)Y(D^{(m_2)}e^\alpha, z_2) \ldots Y(D^{(m_l)}e^\alpha, z_l)1\) is \(Q_{\tilde{m} - \rho}e^{\alpha}\), where \(Q_{\tilde{m} - \rho}\) is the dual Hall-Littlewood polynomial corresponding to the partition \(\tilde{m} - \rho\).

The proof is straightforward modification of the main theorem in [Jin91] using the properties of the vertex operators.

Thus the vertex operators \(Y(D^{(m)}e^\alpha, z)\) (as described in Section 25) and the coefficients of their products are very important in the theory of the Hall-Littlewood polynomials. This makes them an important example of quantum vertex operators, and they are the main motivation for our definition of \(H_D\)-quantum vertex algebras. The previous definitions of quantum vertex algebras were not general enough to incorporate the Hall-Littlewood vertex operators.

**APPENDIX A. BRAIDED ALGEBRAS WITH SYMMETRY**

A.1. **Introduction.** To motivate the rather complicated definition of an \(H_D\)-quantum vertex algebra in Section 3 we discuss in this Appendix braided algebras (with symmetry). The idea is that a vertex algebra has a singular multiplication, and that it is good to understand the nonsingular case first.
A.2. Commutative Associative Algebras. As a preliminary, note that an efficient way to describe commutative associative unital algebras is as follows. Let $M$ be a vector space and $1 \in M$ a distinguished element, and let
\[ m: M^\otimes 2 \to M \]
be a multiplication for which 1 is the unit:
\[ m(a \otimes 1) = m(1 \otimes a) = a, \quad a \in M. \]

We need some notation. If $a$ is a linear map on $M^\otimes 2$, then $a^{23}$ is the operator on $M^\otimes 3$ acting on the 2nd and 3rd factor (so $a^{23} = 1 \otimes a$). The other superscripts have a similar meaning. Let $\tau: M^\otimes 2 \to M^\otimes 2$ be the flip $a \otimes b \mapsto b \otimes a$. Let $m_3 = m(1 \otimes m): M^\otimes 3 \to M$. Then we impose

\[ (\text{Commutativity/Associativity Axiom}) \quad m_3 = m_3 \tau^{12}, \]

In other words, writing $m_3(a \otimes b \otimes c) = a(bc)$, we require $a(bc) = b(ac)$. Then one easily checks that $(M, m, 1)$ is in fact commutative ($m = m \tau$) and associative ($m(1 \otimes m) = m(m \otimes 1)$).

A.3. Braided Algebras. We are next interested in non commutative algebras where the noncommutativity is controlled by a braiding map.

**Definition A.1.** A braided algebra is a unital algebra $(M, m, 1)$ with a braiding $S: M^\otimes 2 \to M^\otimes 2$ such that

1. (Vacuum Axiom) $S(a \otimes 1) = a \otimes 1$, and $S(1 \otimes a) = 1 \otimes a$.
2. (Braiding Axiom) $m_3 S^{12} = m_3 \tau^{12}$.
3. (Unitarity Axiom) $S \circ \tau \circ S \circ \tau = 1_{M^\otimes 2}$.
4. (Yang-Baxter Axiom) $S^{12} S^{13} S^{23} = S^{23} S^{13} S^{12}$.
5. (Compatibility with Multiplication Axiom) $S m^{12} = m^{12} S^{23} S^{13}$ and $S m^{23} = m^{23} S^{12} S^{13}$.

The Compatibility with Multiplication Axiom allows us to express the braiding involving a product in terms of a product of the braidings of the factors. Also, together with the braiding axiom it gives associativity, as we now proceed to show.

**Lemma A.2 (Braided Commutativity).**
\[ mS = m\tau. \]

**Proof.** Apply the Braiding Axiom to $a \otimes b \otimes 1$, using $m_3(a \otimes b \otimes 1) = m(a \otimes b)$. \qed

**Theorem A.3 (Associativity).** A braided algebra is associative:
\[ m(1 \otimes m) = m(m \otimes 1). \]

**Proof.** Let $A = a \otimes b \otimes c$. Then
\[ mm^{12}(A) = mS\tau m^{12}(A) = mS m^{23}(c \otimes a \otimes b) = m m^{23} S^{12} S^{13} (c \otimes a \otimes b) = m m^{23} S^{23} (a \otimes c \otimes b) = m m^{23} (A) \]

\qed
We used the Compatibility with Multiplication Axiom to derive associativity. If we don’t impose this axiom, we can only derive braided associativity, (also called quasi-associativity cf. [EK00]):

\[
m_{m^{23}} S_{23}^{13} m_{m^{12}} = m m_{m^{12}} S_{12}^{13} m_{m^{23}}.
\]

We have not yet used the unitarity and Yang-Baxter axioms. They are used to describe the behaviour under permutations of the arguments of the \(n\)-fold multiplication \(m^n\): (defined recursively by \(m^n = m(1 \otimes m_{n-1})\)) as we now proceed to explain.

**Lemma A.4.**

\[
m_{m^n} \tau^{i+1} = m_{m^n} S^{i+1}.
\]

*Proof.* We can use associativity to write

\[
m_{m^n} = m_3 \circ (m_{i-1} \otimes m_2 \otimes m_{n-i-1}).
\]

The Lemma follows from Braided Commutativity, Lemma A.2. \(\square\)

**Remark A.5.** Note that if \(i, j\) are not adjacent, then it is in general not true that the transposition \(\tau_{ij}\) does act on \(m_{m^n}\) by multiplication by \(S_{ij}\).

For instance, a simple example of a non-trivial braided algebra is a super commutative algebra \(M = M_0 \oplus M_1\). The braiding is given (for homogeneous elements) by \(S(a \otimes b) = (-1)^{|a||b|} a \otimes b\). It is then clear that the braiding corresponding to the permutation \(\tau_{13}: a \otimes b \otimes c \mapsto c \otimes b \otimes a\) is given by

\[
S \tau_{13} (a \otimes b \otimes c) = S_{12} S_{13} S_{23} (a \otimes b \otimes c) = (-1)^{|a||b|} (-1)^{|a||c|} (-1)^{|b||c|} (a \otimes b \otimes c),
\]

whereas

\[
S_{13} (a \otimes b \otimes c) = (-1)^{|a||c|} a \otimes b \otimes c.
\]

One knows that the symmetric group \(S_n\) is generated by the simple transpositions \(w_i = (ii+1), i = 1, 2, \ldots, n-1\), see Section 9. Then define a map \(S: S_n \rightarrow GL(M \otimes^n)\) by

\[
S(w_i) = 1^{i-1} \otimes 1^* \otimes 1^{n-i-1},
\]

and extend this as an anti-homomorphism:

\[
S(f) = S(w_{i_1}) S(w_{i_{k-1}}) \ldots S(w_{i_1}),
\]

in \(f = w_{i_1} w_{i_2} \ldots w_{i_k} \in S_n\). Then the unitarity and the Yang-Baxter axioms and Lemma A.4 imply

**Theorem A.6.** The braiding map \(S^f : M \otimes^n \rightarrow M \otimes^n\) is independent of the representation of \(\sigma\) in terms of simple reflections. Furthermore

\[
m_n \circ S(f) = m_n.
\]

for all \(f \in S_n\).

This concludes our discussion of braided algebras an sich.
A.4. Braided Algebras with Symmetry. We now assume that we have additionally an action of a group $G$ on the braided algebra $M$. If $g \in G$ we write $\Delta(g) = g \otimes g \in G \otimes G$ for the coproduct of $g$.

**Definition A.7.** Let $(M, m, 1, S)$ be a braided algebra, with a $G$-action on $M$. We call this a braided $G$-algebra in case for each $g \in G$ there is a map

$$S^g : M^{\otimes 2} \rightarrow M^{\otimes 2},$$

such that

- (Vacuum Axiom) $S^g(a \otimes 1) = a \otimes 1$, and $S^g(1 \otimes a) = 1 \otimes a$.
- (G-Symmetry) $g m S^g = m \Delta(g)$.
- (Multiplicativity) $S^{gh} = S^h \circ \Delta(h^{-1}) \circ S^g \circ \Delta(h)$
- (G-Yang-Baxter) $S^{g_1} S^{g_2} S^{g_3} = S^{g_3} S^{g_2} S^{g_1}$.
- (Compatibility with Multiplication Axiom) $S^g m^{12} = m^{12} S^{g, 23} S^{g, 13}$ and $S^g m^{23} = m^{23} S^{g, 12} S^{g, 13}$.

Of course, the simplest case is were $S^g = 1 \otimes 1$ for all $g \in G$. Then the multiplication intertwines the action of $G$ on $M^{\otimes 2}$ and $M$; usually $M$ is then called a module-algebra.

**Lemma A.8.** Define $\Sigma_n$ and $\tilde{\Sigma}_n : M^{n+1} \rightarrow M^{n+1}$ by $\Sigma_n = S^{12} S^{13} \cdots S^{n+1}$, $\tilde{\Sigma}_n = S^{1n+1} \cdots S^{13}$. Then we have compatibility with the higher multiplications:

$$S(1 \otimes m_n) = (1 \otimes m_n) \Sigma_n, \quad S(m_n \otimes 1) = (m_n \otimes 1) \tilde{\Sigma}_n.$$

**Proof.** For $n = 2$ the Lemma is just the compatibility with multiplication axiom. Assume the Lemma is true for $n = k-1$. Then

$$S(1 \otimes m_k) = S(1 \otimes m)(1 \otimes 1 \otimes m_{k-1}) = (1 \otimes m) S^{12} S^{13} (1 \otimes 1 \otimes m_{k-1}) = (1 \otimes m)(1 \otimes 1 \otimes m_{k-1}) S^{12} S^{13} \cdots S^{n+1} m_{n+1}.$$ 

Noting that $\Sigma_n = S^{12} \circ \Sigma_{n-1} \cdots \Sigma^{n+1}$ the first equation of the Lemma follows. The second one is proved similarly. \(\square\)

Now define

$$S^g_n = \Sigma_{n-1} \circ (1 \otimes S^g_{n-1}).$$

**Theorem A.9.** We have $S^g_n = \tilde{\Sigma}_{n-1} \circ (S^g_{n-1} \otimes 1)$ and

$$g m_n S^g_n = m_n \Delta_n(g).$$

**Proof.** ?? \(\square\)

A.5. Bicharacters. Let $M$ be a commutative and cocommutative Hopf algebra. A bicharacters on $M$ is a linear map

$$r : M^{\otimes 2} \rightarrow \mathbb{C},$$

satisfying

- (Vacuum) $r(a \otimes 1) = r(1 \otimes a) = \epsilon(a), a \in M$.
- (Multiplication) For all $a, b, c \in M$ we have $r(a \otimes bc) = \sum r(a' \otimes b)r(a'' \otimes c)$ and $r(ab \otimes c) = \sum r(a \otimes c')r(b \otimes c'')$. 


Here and below we use the notation $\Delta(a) = \sum a' \otimes a''$ for the coproduct for $a \in V$.

Often we will also omit the summation symbol, to unclutter the formulas.

We can multiply bicharacters: if $r, s$ are bicharacters and $a, b \in M$ then

\[(r \ast s)(a \otimes b) = r(a' \otimes b')s(a'' \otimes b'').\]

The unit bicharacter is

\[\epsilon(a \otimes b) = \epsilon(a)\epsilon(b).\]

Since $M$ is a Hopf algebra it comes with an antipode, and all bicharacters are invertible, with inverse given by

\[r^{-1}(a \otimes b) = r(S(a) \otimes b).\]

The set of bicharacters forms an Abelian group.

The transpose of a bicharacter is defined by

\[r^\tau(a \otimes b) = r(b \otimes a).\]

The transpose is an involution of the algebra of bicharacters:

\[(r \ast s)^\tau = (r^\tau \ast s^\tau).\]

If $r$ is an invertible bicharacter with inverse $r^{-1}$ we define another bicharacter

\[R = r^\tau \ast r^{-1},\]

We will call $R$ the *braiding bicharacter* associated to $r$. It will control the braiding in the braided algebra we are going to construct from $r$ below. The braiding bicharacter $R$ is *unitary*:

\[R^\tau = R^{-1}.\]

Also we have

\[r \ast R = r^\tau.\]

For any bicharacter $\rho$ on $M$ we define a map

\[S^{(\rho)} : M^\otimes 2 \to M^\otimes 2, \quad a \otimes b \mapsto a' \otimes b'\rho(a'' \otimes b'').\]

**Lemma A.10.**
1. If $\epsilon$ is the unit bicharacter on $M$, then $S^{(\epsilon)} = 1_{M^\otimes 2}$.
2. If $\rho, \sigma$ are bicharacters on $M$, then $S^{(\rho \circ \sigma)} = S^{(\rho)} \circ S^{(\sigma)}$.
3. If $\rho$ is a bicharacter, then $\tau \circ S^{(\rho)} \circ \tau = S^{(\rho^\tau)}$.

**Lemma A.11.** For all $a \in M$ and bicharacters $\rho$ on $M$ we have
1. *(Vacuum)* $S^{(\rho)}(a \otimes 1) = a \otimes 1$ and $S^{(\rho)}(1 \otimes a) = 1 \otimes a$.
2. *(Yang-Baxter)* $S^{(\rho)}(a')12S^{(\rho)}(1'3)S^{(\rho)}(23) = S^{(\rho)}(23)S^{(\rho)}(13)S^{(\rho)}(12)$.

Now we fix a bicharacter $r$ on $M$, and define a twisting of the multiplication $m$ on $M$:

\[m_r = m \circ S^{(r)} : M^\otimes 2 \to M.\]

**Lemma A.12.** For any bicharacter $\rho$ the map $S^{(\rho)}$ is compatible with the twisted multiplication $m_r$:

\[S^{(\rho)}(m_r \otimes 1) = (m_r \otimes 1)S^{(\rho)}(12)S^{(\rho)}(13), \quad S^{(\rho)}(1 \otimes m_r) = (1 \otimes m_r)S^{(\rho)}(12)S^{(\rho)}(13).\]
Proof. For $a, b, c \in M$ we have
\[
S^{(\rho)}(m_r \otimes 1)(a \otimes b \otimes c) = S^{(\rho)}(a' \otimes c')r(a'' \otimes b'') =
\]
\[
= (a'b')' \otimes c' \rho((a'b')'' \otimes c'')r(a'' \otimes b'') =
\]
\[
= a''b'' \otimes c' \rho(a''' \otimes c''') \rho(b''' \otimes c''''))r(a'' \otimes b'').
\]
Now by coassociativity and cocommutativity of $M$ we have $a'' \otimes a''' \otimes a'' = a'' \otimes a'' \otimes a''$. So that we get
\[
= a''b''r(a'' \otimes b'') \otimes c' \rho(a''' \otimes c''') \rho(b''' \otimes c''''))r(a'' \otimes b'').
\]
\[
= (m_r \otimes 1)S^{(\rho)}(\rho).\]
\[
\]
\[
= S^{(\rho)}(\rho).13(a \otimes b \otimes c).
\]

The proof of the other part is similar. □

Recall the braiding bicharacter $R = r^{-1} * r^\tau$ associated to $r$, and write $S = S^{(R)}$.

Proposition A.13. For any bicharacter $r$ on $M$ the twist $(M, m_r, 1, S)$ is a braided algebra.

Proof. We need to check the axioms in Definition A.1. The vacuum and Yang-Baxter axioms are dealt with in Lemma A.11. For unitarity we have $\tau \circ S \circ \tau = S^{(R)}$ so that by Lemma 22.2 and (A.5)
\[
S \circ \tau \circ S \circ \tau = S^{(R)} \circ S^{(R)} = S^{(R \ast R^\tau)} = S^{(\epsilon)} = 1_{M_{\otimes 2}}.
\]
Compatibility of $S$ with the multiplication $m_r$ is the case $\rho = R$ of Lemma A.12.

Now $m_r$ is braided commutative:
\[
m_rS = m \circ S^{(r)} \circ S^{(R)} = m \circ S^{(r)} \circ S^{(r)} = m \circ \tau \circ S^{(r)} \circ \tau = m_r \circ \tau
\]
by Lemma 22.2 (A.6) and the fact that $m$ is commutative. From compatibility of $S$ with multiplication $m_r$ and the Yang-Baxter equation it follows that $m_r$ is associative. The braiding axiom for $m_r, 3 = m_r(1 \otimes m_r) = m_r(m_r \otimes 1)$ follows from this. □

A.6. Bicharacters and Group Action. Now we assume that we have an action of a group $G$ on the commutative and cocommutative Hopf algebra $M$ compatible with the multiplication and the comultiplication:
\[
gm = m \circ \Delta(g), \quad \Delta(gm) = \Delta_G(g)\Delta(m).
\]

Define for any bicharacter $r$ on $M$ and $g \in G$
\[
r^g = r \circ \Delta(g).
\]
It is easy to check that $r^g$ is again a bicharacter, so that we can write
\[
(A.8) \quad r^g = r * R^g, \quad R^g = r^{-1} * r^g.
\]
Also $R^g$ is then a bicharacter. Define $S^g = S^{(R^g)}$.

Lemma A.14. For all $g \in G$ and bicharacters $r$ on $M$
\[
\]
\[
gm \cdot S^g = m_r \Delta(g).
\]
Proof. By Lemma 22.2 and (A.6)

\[ gm r S^g = gm \circ S^{(r)} \circ S^{(R^g)} = gm \circ S^{(r+R^g)} = \]
\[ = gm \circ S^{(r^g)} = m \circ \Delta(g) \circ S^{(r^g)} = m \circ S^{(r)} \Delta(g). \]

Here we use

(A.9) \[ \Delta(g) S^{(r^g)} = S^{(r)} \Delta(g), \]

which follows from the definition of \( S^{(r)} \), see (A.7). □

Corollary A.15. Let \( r \) be a bicharacter on a commutative and cocommutative Hopf algebra \( M \) with an action of a group \( G \). Then \((M, m_r, 1, S)\) is a braided \( G \)-algebra for the maps

\[ S^g = S^{(R^g)}, \quad g \in G, \]

where \( R^g \) is defined in (A.8).

Proof. We need to check the axioms in Definition A.7. The \( G \)-symmetry axiom is verified in the previous Lemma A.14. The Vacuum Axiom and \( G \)-Yang-Baxter Axiom for \( S^g \) are verified in Lemma A.11 as \( S^g = S^{(R^g)} \) and \( R^g \) is a bicharacter. The compatibility of \( S^g \) with multiplication \( m_r \) is the case \( \rho = R^g \) of Lemma A.12.

For multiplicativity

\[ S^{gh} = S^{(r^{-1}+r^g)} = S^{(r^{-1})} \circ S^{(r^g)} = \]
\[ = S^{(r^{-1})} \circ \Delta(g h)^{-1} \circ S^{(r)} \circ \Delta(g h) = \]
\[ = S^{(r^{-1})} \circ \Delta(h)^{-1} \circ S^{(r)} \circ S^{(r^{-1})} \circ S^{(r^g)} \circ \Delta(h) = \]
\[ = S^{(r^{-1})} \circ S^{(h^{-1})} \circ S^{(r^g)} \circ S^g \circ \Delta(h) = \]
\[ = S^{h} \circ \Delta(h^{-1}) \circ S^g \circ \Delta(h). \]

□

Remark A.16. In a braided \( G \)-algebra we implement the action of \( G \) by a system of maps \( S^g \) satisfying

\[ gm S^g = m \Delta(g). \]

In the bicharacter case of a twisted multiplication \( m_r = m \circ S^{(r)} \) we can also implement the group action by twisting the coproduct on \( G \): we have

\[ gm_r = gm \circ S^{(r)} = m \circ \Delta(g) \circ S^{(r)} = \]
\[ = m \circ S^{(r)} \circ S^{(r^{-1})} \circ \Delta(g) \circ S^{(r)} = \]
\[ = m_r \Delta_r(g), \]

where the twisted coproduct is

\[ \Delta_r(g) = S^{(r^{-1})} \circ \Delta(g) \circ S^{(r)}. \]

The fact that the two approaches are equivalent,

\[ \Delta(g)(S^g)^{-1} = \Delta_r(g), \]

follows from (A.9). It is at this point not clear whether one can replace in an arbitrary braided \( G \)-algebra the maps \( S^g \) by a twist of the coproduct.
APPENDIX B. BRAIDING MAPS

Let $V$ be a free $k$-module and let $\text{Map}_{z_1,z_2,...,z_n}(V^\otimes n)$ be the space of linear maps

$$V^\otimes n \to V^\otimes n[i^\pm 1, (z_i - z_j)^{-1}][i], \quad 1 \leq i < j \leq n.$$ 

Suppose we are given $S_{z_1,z_2} \in \text{Map}_{z_1,z_2}(V^\otimes 2)$ that satisfies

(B.1) \quad $S_{z_1,z_2} \circ \tau \circ S_{z_2,z_1} \circ \tau = 1_{V^\otimes 2},$

(B.2) \quad $S_{z_1,z_2}^{12} S_{z_1,z_2}^{13} S_{z_2,z_3}^{12} = S_{z_1,z_2}^{13} S_{z_1,z_3}^{12} S_{z_2,z_3}^{12}.$

We then define for each $f \in S_n$ an element $S_{z_1,...,z_n}^f \in \text{Map}_{z_1,z_2,...,z_n}(V^\otimes n)$ as follows. First, for $w_i \in S_n$ a simple transposition, define

$$S_{z_1,...,z_n}^{sw_i} = 1^i \otimes S^\tau_{z_i,z_i+1} \otimes 1^{i-1},$$

and extend this to $f \in S_n$ by expanding it in simple transpositions and using

(B.3) \quad $S_{z_1,...,z_n}^{gf} = S_{z_1,...,z_n}^{g} S_{z_1,...,z_n}^{f^{-1}}(1_{z_1,...,z_n})(\sigma_r)^{-1}.$

The problem is that the expansion of $f$ is not unique, because of the relations (B.1) and (B.2) in $S_n$.

To address this problem introduce the free monoid $F_n$ generated by symbols $\tilde{w}_i$, $i = 1, 2, \ldots, n - 1$. In $F_n$ every element $\tilde{f}$ has a unique expression in terms of the $\tilde{w}_i$. Consider the semi-direct product $\text{Map}_{z_1,z_2,...,z_n}(V^\otimes n) \rtimes S_n$: elements of the semi-direct product are pairs $(A_{z_1,...,z_n}, f)$, with product

(B.4) \quad $(A_{z_1,...,z_n}, f). (B_{z_1,...,z_n}, g) = (A_{z_1,...,z_n} \circ \sigma_f^{-1} \circ B_{(z_1,z_2,...,z_n)} \circ \sigma_f, fg).$

We have a homomorphism $F_n \to S_n$, which maps generator $\tilde{w}_i$ to simple transposition $w_i$. Let

$$\phi: F_n \to \text{Map}_{z_1,z_2,...,z_n}(V^\otimes n) \rtimes S_n$$

be given on generators by

$$\phi(\tilde{w}_i) = (S_{z_1,...,z_n}^{w_i}, w_i),$$

and we extend this to all of $F_n$ as an anti-homomorphism of monoids.

We need some more notation. If $\tilde{f} = \tilde{w}_i, \tilde{w}_i, \ldots, \tilde{w}_i \in F_n$, and the corresponding permutation is $f = w_i, w_i, \ldots, w_i \in S_n$, then introduce

$$g_\ell = w_{i_1} w_{i_2} \ldots w_{i_{\ell-1}}, \quad \ell = 1, 2, \ldots, k - 1,$$

and $g_k = 1.$

**Lemma B.1.** Let $\tilde{f} = \tilde{w}_i, \tilde{w}_i, \ldots, \tilde{w}_i$ and $f = w_i, w_i, \ldots, w_i$. Then

$$\phi(\tilde{f}) = (S_{z_1,...,z_n}^{f}, f^{-1} \in \text{Map}_{z_1,z_2,...,z_n}(V^\otimes n) \rtimes S_n),$$

where

$$S_{z_1,...,z_n}^{f} = S^k S^{k-1} \ldots S^1 \sigma_f^{-1}, \quad S^\ell = S_{g_\ell(z_1,z_2,...,z_n)}^{w_i}. $$

Furthermore, for $\tilde{f}, \tilde{g} \in F_n$

(B.5) \quad $S_{z_1,...,z_n}^{fg} = S_{z_1,...,z_n}^{g} S_{z_1,...,z_n}^{f} \sigma_g S_{g^{-1}(z_1,z_2,...,z_n)}^{f} \sigma_f.$
Proof. Using the anti-homomorphism property of \( \phi \) and the multiplication \([B.4]\) we have

\[
\phi(\tilde{f}) = \phi(\tilde{w}_k)\phi(\tilde{w}_{k-1}) \ldots \phi(\tilde{w}_1) = \\
= (S^w_{z_1, \ldots, z_n}, w_k). (S^w_{z_1, \ldots, z_n}, w_{k-1}) \ldots (S^w_{z_1, \ldots, z_n}, w_1) = \\
= (S^w_{z_1, \ldots, z_n} \tau_{iz} S^w_{w_i (z_1, \ldots, z_n)} \tau_{iz}, w_i w_{i+1}) \ldots (S^w_{z_1, \ldots, z_n}, w_1) = \\
= (S^k S^{k-1} \ldots S^1 \sigma_f^{-1}, f^{-1}).
\]

This proves the first part. Then

\[
\phi(\tilde{f}\tilde{g}) = \phi(\tilde{g})\phi(\tilde{f}) = \\
= (S^g_{z_1, \ldots, z_n}, g^{-1}). (S^f_{z_1, \ldots, z_n}, f^{-1}) = (S^g_{z_1, \ldots, z_n} \sigma_g S^f_{g^{-1}(z_1, \ldots, z_n)} \sigma_g^{-1} g^{-1} f^{-1}).
\]

Since \( \sigma_g^{-1} \sigma_f = \sigma_f \) follows. \( \square \)

The observant reader might object to the notation \( S^f_{z_1, \ldots, z_n} \) used in the above Lemma: this map depends a priori on the element \( \tilde{f} \in \mathcal{F}_n \), not just on its image in \( S_n \). The following Lemma justifies the notation.

**Lemma B.2.** The map \( \phi: \mathcal{F}_n \to \text{Map}_{z_1, \ldots, z_n}(V^\otimes n) \rtimes S_n \) factors through the canonical map \( \mathcal{F}_n \to S_n \).

Proof. We need to check that the relations \([B.1]\) and \([B.2]\) (with \( w_i \) replaced by \( \tilde{w}_i \)) belong to the kernel of \( \phi \). But we have

\[
\phi(\tilde{w}_i^2) = \phi(\tilde{w}_i)\phi(\tilde{w}_i) = \\
= (S^w_{z_1, \ldots, z_n}, w_i). (S^w_{z_1, \ldots, z_n}, w_i) = (S^w_{z_1, \ldots, z_n} \tau_i S^w_{w_i (z_1, \ldots, z_n)} \tau_i, w_i w_i) = \\
= (1, 1)
\]

by the definition \([B.3]\) and the property \([B.3]\). Next, by the definition \([B.3]\) and \([B.2]\) we have, if \( |i-j| \geq 2 \),

\[
\phi(\tilde{w}_i \tilde{w}_j) = \phi(\tilde{w}_j)\phi(\tilde{w}_i) = \phi(\tilde{w}_i)\phi(\tilde{w}_j) = \phi(\tilde{w}_j \tilde{w}_i)
\]

Finally

\[
\phi(\tilde{w}_i \tilde{w}_{i+1} \tilde{w}_i) = \phi(\tilde{w}_{i+1} \tilde{w}_i \tilde{w}_{i+1})
\]

follows from the Yang-Baxter equation \([B.4]\). \( \square \)

The conclusion is that Definition \([B.1]\) of \( S^f_{z_1, \ldots, z_n} \) is well defined.

**References**

[Ang06] I. I. Anguelova, *Bicharacter constructions of quantum vertex algebras*, Ph.D. thesis, University of Illinois, Urbana-Champaign, 2006.

[Bor86] Richard E. Borcherds, *Vertex algebras, Kac-Moody algebras, and the Monster*, Proc. Nat. Acad. Sci. U.S.A. **83** (1986), no. 10, 3068–3071. MR MR843307 (87m:17033)

[Bor98] ______, *Vertex algebras, Topological field theory, primitive forms and related topics* (Kyoto, 1996), Progr. Math., vol. 160, Birkhäuser Boston, Boston, MA, 1998, pp. 35–77. MR MR1653021 (99m:17034)

[Bor01] ______, *Quantum vertex algebras*, Taniguchi Conference on Mathematics Nara '98, Adv. Stud. Pure Math., vol. 31, Math. Soc. Japan, Tokyo, 2001, pp. 51–74. MR MR1865087 (2002k:17054)

[EK00] Pavel Etingof and David Kazhdan, *Quantization of Lie bialgebras, V. Quantum vertex operator algebras*, Selecta Math. (N.S.) **6** (2000), no. 1, 105–130. MR 2002i:17022
Igor Frenkel, James Lepowsky, and Arne Meurman, *Vertex operator algebras and the Monster*, Pure and Applied Mathematics, vol. 134, Academic Press Inc., Boston, MA, 1988. MR MR996026 (90h:17026)

Edward Frenkel and Nikolai Reshetikhin, *Towards Deformed Chiral Algebras*, Proceedings of the Quantum Group Symposium at the XXIth International Colloquium on Group Theoretical Methods in Physics, Goslar 1996, 1997, [arXiv:q-alg/9706023](http://arxiv.org/abs/q-alg/9706023).

Nai Huan Jing, *Vertex operators and Hall-Littlewood symmetric functions*, Adv. Math. 87 (1991), no. 2, 226–248. MR MR1112626 (93c:17039)

Victor Kac, *Vertex algebras for beginners*, second ed., University Lecture Series, vol. 10, American Mathematical Society, Providence, RI, 1998. MR MR1651389 (99f:17033)

Haisheng Li, *Axiomatic $G_1$-vertex algebras*, Commun. Contemp. Math. 5 (2003), no. 2, 281–327. MR MR1966260 (2004e:17026)

Haisheng Li, *Nonlocal vertex algebras generated by formal vertex operators*, Selecta Math. (N.S.) 11 (2005), no. 3-4, 349–397. MR MR2215259

Haisheng Li, *Constructing quantum vertex algebras*, Internat. J. Math. 17 (2006), no. 4, 441–476. MR MR2220654

Haisheng Li, *A new construction of vertex algebras and quasi-modules for vertex algebras*, Adv. Math. 202 (2006), no. 1, 232–286. MR MR2218823

James Lepowsky and Haisheng Li, *Introduction to vertex operator algebras and their representations*, Progress in Mathematics, vol. 227, Birkhäuser Boston Inc., Boston, MA, 2004. MR MR2023933 (2004k:17050)

I. G. Macdonald, *Symmetric functions and Hall polynomials*, second ed., Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1995, With contributions by A. Zelevinsky, Oxford Science Publications. MR MR1354144 (96h:05207)

Craig T. Snydal, *Equivalence of Borcherds G-Vertex Algebras and Axiomatic Vertex Algebras*, [arXiv:math.QA/9904104](http://arxiv.org/abs/math.QA/9904104).

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