Gaussian Approximation for High Dimensional Time Series

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Abstract

We consider the problem of approximating sums of high-dimensional stationary time series by Gaussian vectors, using the framework of functional dependence measure. The validity of the Gaussian approximation depends on the sample size $n$, the dimension $p$, the moment condition and the dependence of the underlying processes. We also consider an estimator for long-run covariance matrices and study its convergence properties. Our results allow constructing simultaneous confidence intervals for mean vectors of high-dimensional time series with asymptotically correct coverage probabilities. A Gaussian multiplier bootstrap method is proposed. A simulation study indicates the quality of Gaussian approximation with different $n$, $p$ under different moment and dependence conditions.

1 Introduction

During the past decade, there has been a significant development on high-dimensional data analysis with applications in many fields. In this paper we shall consider simultaneous inference for mean vectors of high-dimensional stationary processes, so that one can perform family-wise multiple testing or construct simultaneous confidence intervals, an important problem in the analysis of spatial-temporal processes. To fix the idea, let $X_i$ be a stationary process in $\mathbb{R}^p$ with mean $\mu = (\mu_1, \ldots, \mu_p)^\top$ and finite second moment in the sense that $\mathbb{E}(X_i^\top X_i) < \infty$. In the scalar case in which $p = 1$ or when $p$ is fixed, under suitable weak dependence conditions, we can have the central limit theorem (CLT)

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \mu) \Rightarrow N(0, \Sigma), \text{ where } \Sigma = \sum_{k=-\infty}^{\infty} \mathbb{E}((X_0 - \mu)(X_k - \mu)^\top).$$

See, for example, Rosenblatt (1956), Ibragimov and Linnik (1971), Wu (2005), Dedecker et al. (2007) and Bradley (2007) among others. In the high dimension case in which $p$ can
also diverge to infinity, Portnoy (1986) showed that the central limit theorem can fail for i.i.d. random vectors if $\sqrt{n} = o(p)$. In this paper we shall consider an alternative form: Gaussian approximation for the largest entry of the sample mean vector $\bar{X}_n = n^{-1} \sum_{i=1}^{n} X_i$. For a vector $v = (v_1, \ldots, v_p)^\top$, let $|v|_\infty = \max_{j \leq p} |v_j|$. Specifically, our primary goal is to establish the Gaussian Approximation (GA) in $\mathbb{R}^p$

$$
\sup_{u \geq 0} |\mathbb{P}(\sqrt{n}|\bar{X}_n - \mu|_\infty \geq u) - \mathbb{P}(|Z|_\infty \geq u)| \to 0,
$$

(2)

where both $n, p \to \infty$. Here the Gaussian vector $Z = (Z_1, \ldots, Z_p)^\top \sim N(0, \Sigma)$. Chernozhukov et al. (2013a) studied the Gaussian approximation for independent random vectors. There has been limited research on high-dimensional inference under dependence. The associated statistical inference becomes considerably more challenging since the autocovariances with all lags should be considered. Zhang and Cheng (2014) extended the Gaussian approximation in Chernozhukov et al. (2013a) to very weakly dependent random vectors which satisfy a uniform geometric moment contraction condition. The latter condition is also adopted in Chen et al. (2015) for self-normalized sums. Chernozhukov et al. (2013b) did a similar extension to strong mixing random vectors. Here we shall establish (7) for a wide class of high-dimensional stationary process under suitable conditions on the magnitudes of $p, n$, and the mild dependence conditions on the process $(X_i)$.

In Section 2 we shall introduce the framework of high-dimensional time series and some concepts about functional and predictive dependence measures that are useful for establishing an asymptotic theory. The main result for Gaussian approximation of the normalized mean vector and the choice of the normalization matrix is established in Section 3. Depending on the moment and the dependence conditions, both high dimension and ultra high dimension cases are discussed.

To perform statistical inference based on (7), one needs to estimate the long-run covariance matrix $\Sigma$. The latter problem has been extensively studied in the scalar case; see Politis et al. (1999), Bühlmann (2002), Lahiri (2003), Alexopoulos and Goldsman (2004), among others. In Section 4 we study the batched-mean estimate of long-run covariance matrices and derive a large deviation result about quadratic forms of stationary processes. The latter tail probabilities inequalities allow dependent and/or non-sub-Gaussian processes under mild conditions, which is expected to be useful in other high-dimensional inference problems for dependent vectors. The consistency of the batched-mean estimate
ensures the validity of the normalized Gaussian multiplier bootstrap method.

We provide in Section 5 some sharp inequalities for tail probabilities for dependent processes in both polynomial tail and exponential tail cases. Part of the proof are relegated to Section 6.

We now introduce some notation. For a random variable $X$ and $q > 0$, we write $X \in L^q$ if $\|X\|_q := (\mathbb{E}|X|^q)^{1/q} < \infty$, and for a vector $v = (v_1, \ldots, v_p)^\top$, let the norm-s length $|v|_s = (\sum_{j=1}^p |v_j|^s)^{1/s}$, $s \geq 1$. Write the $p \times p$ identity matrix as $\text{Id}_p$. For two real numbers, set $x \vee y = \max(x, y)$ and $x \wedge y = \min(x, y)$. For two sequences of positive numbers $(a_n)$ and $(b_n)$, we write $a_n \asymp b_n$ (resp. $a_n \lesssim b_n$ or $a_n \ll b_n$) if there exists some constant $C > 0$ such that $C^{-1} \leq a_n/b_n \leq C$ (resp. $a_n/b_n \leq C$ or $a_n/b_n \to 0$) for all large $n$. We use $C, C_1, C_2, \cdots$ to denote positive constants whose values may differ from place to place. A constant with a symbolic subscript is used to emphasize the dependence of the value on the subscript. Throughout the paper, we assume $p = p_n \to \infty$ as $n \to \infty$.

## 2 High-dimensional Time Series

Let $\varepsilon_i, i \in \mathbb{Z}$, be i.i.d. random variables and $\mathcal{F}_i = (\ldots, \varepsilon_{i-1}, \varepsilon_i)$; let $(X_i)$ be a stationary process taking values in $\mathbb{R}^p$ that assumes the form

$$X_i = (X_{i1}, X_{i2}, \ldots, X_{ip})^\top = G(\mathcal{F}_i),$$

where $G(\cdot) = (g_1(\cdot), \ldots, g_p(\cdot))^\top$ is an $\mathbb{R}^p$-valued measurable function such that $X_i$ is well-defined. In the scalar case with $p = 1$, (3) allows a very general class of stationary processes (cf. Wiener (1958), Rosenblatt (1971), Priestley (1988), Tong (1990), Wu (2005), Tsay (2005), Wu (2011)). It includes linear processes as well as a large class of nonlinear time series models. Within this framework, $(\varepsilon_i)$ can be viewed as independent inputs of a physical system and all the dependences among the outputs $(X_i)$ result from the underlying data-generating mechanism $G(\cdot)$. The function $g_j(\cdot)$, $1 \leq j \leq p$, is the $j$-th coordinate projection of $G(\cdot)$. Unless otherwise specified, assume throughout the paper that $\mathbb{E}X_i = 0$ and $\max_{j \leq p} \|X_{ij}\|_q < \infty$ for some $q \geq 2$. Let $\Gamma(k) = (\gamma_{ij}(k))_{i,j=1}^p = \mathbb{E}(X_iX_i^\top)$ be the autocovariance matrix and recall the long-run covariance matrix

$$\Sigma = (\sigma_{ij})_{i,j=1}^p = \sum_{k=-\infty}^{\infty} \Gamma(k)$$

3
Zhang and Cheng (2014) considered some special cases: the former paper requires that
convenient framework for studying high-dimensional time series. Chen et al. (2013) and
\(\text{which can be interpreted as the uniform and the overall dependence adjusted norms of}
\]
condition max
\(\text{unbounded in}
\]
not depend on
\(\text{that the dependence adjusted norm is equivalent to the classical}
\]

\[
\|X_j\|_{q,\alpha} = \sup_{m \geq 0} (m + 1)^{\alpha} \Delta_{m,q,j}, \quad \alpha \geq 0, \quad \text{where } \Delta_{m,q,j} = \sum_{i=m}^{\infty} \delta_{i,q,j}, \quad m \geq 0.
\]

(5)

Due to the dependence, it may happen that \(\|X_j\|_q < \infty\) while \(\|X_j\|_{q,\alpha} = \infty\). Elementary
calculations show that, if \(X_{ij}, i \in \mathbb{Z}\), are i.i.d., then \(\|X_{ij}\|_q \leq \|X_{ij}\|_{q,0} \leq 2\|X_{ij}\|_q\), suggesting
that the dependence adjusted norm is equivalent to the classical \(L^q\) norm.

To account for high-dimensionality, we define

\[
\Psi_{q,\alpha} = \max_{1 \leq j \leq p} \|X_j\|_{q,\alpha} \quad \text{and} \quad \Upsilon_{q,\alpha} = \left( \sum_{j=1}^{p} \|X_j\|_{q,\alpha}^{q} \right)^{1/q},
\]

which can be interpreted as the uniform and the overall dependence adjusted norms of
\((X_i)_{i \in \mathbb{Z}}\), respectively. The form (3) and its associated dependence measures provide a
convenient framework for studying high-dimensional time series. Chen et al. (2013) and
Zhang and Cheng (2014) considered some special cases: the former paper requires that
\[
\text{max}_{1 \leq j \leq p} \|X_j\|_{q,\alpha} \leq C \quad \text{while the latter imposes the stronger geometric moment contraction}
\]
condition \(\text{max}_{1 \leq j \leq p} \Delta_{m,q,j} \leq C \rho^m\) with \(\rho \in (0, 1)\), and in both cases the constant \(C\) does
not depend on \(p\). Those assumptions can be fairly restrictive. In this paper \(\Psi_{q,\alpha}\) can be
unbounded in \(p\). Additionally, we define the \(L^\infty\) functional dependence measure and its
corresponding dependence adjusted norm for the \(p\)-dimensional stationary process \((X_i)\)

\[
\omega_{i,q} = \|X_i - X_{i,\{0\}}\|_\infty;
\]

\[
\|X_i\|_{\infty,\alpha} = \sup_{m \geq 0} (m + 1)^{\alpha} \Omega_{m,q}, \quad \alpha \geq 0, \quad \text{where } \Omega_{m,q} = \sum_{i=m}^{\infty} \omega_{i,q}, \quad m \geq 0.
\]
Clearly, we have $\Psi_{q,\alpha} \leq \|X\|_{\infty, q, \alpha} \leq \Upsilon_{q, \alpha}$.

### 3 Gaussian Approximations

In this section we shall present main results on Gaussian approximations. Theorem 3.2 concerns the finite polynomial moment case with both weaker and stronger temporal dependence. Consequently the dimension $p$ allowed can be at most a power of $n$. If the underlying process has finite dependence-adjusted sub-exponential norms, Theorem 3.3 asserts that an ultra-high dimension $p$ can be allowed. Theorem 6.4 in Section 6.1 provides a convergence rate of the Gaussian approximation.

Recall (4) for the long-run covariance matrix $\Sigma$. Let $\Sigma_0 = \text{diag}(\Sigma)$ be the diagonal matrix of $\Sigma$, and $D_0 = \text{diag}(\sigma_1^{1/2}, \ldots, \sigma_p^{1/2})$. Assume $\mu = 0$. We consider the following normalized version of (2):

$$\sup_{u \geq 0} |\mathbb{P}(\sqrt{n}D_0^{-1}X_n|\infty \geq u) - \mathbb{P}(|D_0^{-1}Z|\infty \geq u)| \to 0,$$

(7)

**Assumption 3.1.** There exists a constant $c > 0$ such that $\min_{1 \leq j \leq p} \sigma_{jj} \geq c$.

To state Theorem 3.2, we need to define the following quantities: $\Theta_{q,\alpha} = \Upsilon_{q,\alpha} \wedge (\|X\|_{\infty, q, \alpha} \log p)$, $L_1 = (n^{1/q-1/2}(\log p)^{1/2}\Theta_{q,\alpha})^{1/(\alpha-1/2+1/q)}$, $L_2 = (\Psi_{2,\alpha}^{1/2}\Psi_{2,0}(\log p)^{2})^{1/\alpha}$, $W_1 = \Psi_{3,0}^2 + \Psi_{4,0}^2 (\log (pn))^{2}$, $W_2 = \Psi_{2,\alpha}^4 (\log (pn))^{4}$, $W_3 = (n^{-\alpha}(\log (pn))^{3/2}\Theta_{q,\alpha})^{1/(1/2-\alpha-1/\alpha)}$, $N_1 = (n/\log p)^{\alpha/2}/\Theta_{q,\alpha}^q$, $N_2 = n(\log p)^{-2}\Psi_{2,\alpha}^{-2}$, $N_3 = (n^{1/2}(\log p)^{-1/2}\Theta_{q,\alpha}^{1/(1/2-\alpha)})$.

**Theorem 3.2.** Let Assumption 3.1 be satisfied. (i) Assume that $\Theta_{q,\alpha} < \infty$ holds with some $q \geq 4$ and $\alpha > 1/2 - 1/q$ (the weaker dependence case),

$$\Theta_{q,\alpha} n^{1/q-1/2}(\log (pn))^{3/2} \to 0$$

(8)

and

$$\max(L_1, L_2) \max(W_1, W_2) = o(1) \min(N_1, N_2).$$

(9)

Then the Gaussian Approximation (7) holds. (ii) Assume $0 < \alpha < 1/2 - 1/q$ (the stronger dependence case). Then (7) holds if $\Theta_{q,\alpha}(\log p)^{1/2} = o(n^{\alpha})$ and

$$L_2 \max(W_1, W_2, W_3) = o(1) \min(N_2, N_3).$$

(10)
Remark 1. (Optimality of our result on the allowed dimension \( p \)) Assume \( \alpha > 1/2 - 1/q \).
In the special case with \( \Psi_{q,\alpha} \asymp 1 \) and \( \Theta_{q,\alpha} \asymp p^{1/q} \), (8) becomes
\[
p(\log(pn))^{3q/2} = o(n^{q/2-1}),
\]
which by elementary manipulations implies (9), and hence the GA (7). It turns out that condition (11), or equivalently \( p(\log p)^{3q/2} = o(n^{q/2-1}) \), is optimal up to a multiplicative logarithmic term. Consider the special case in which \( X_{ij}, i, j \in \mathbb{Z} \), are i.i.d. symmetric random variables with \( \mathbb{E}(X_{ij}^2) = 1 \) and the tail probability \( \mathbb{P}(X_{ij} \geq u) = u^{-p\ell(u)}, \ u \geq u_0 \), where \( \ell(u) = (\log u)^2 \). By Nagaev (1979), we have the expansion: for \( y \geq \sqrt{n} \),
\[
\mathbb{P}(X_{11} + \ldots + X_{n1} \geq y) \sim ny^{-q\ell(y)} + 1 - \Phi(y/\sqrt{n}). \tag{12}
\]
Let \( M_n = X_{11} + \ldots + X_{n1}, Z = (Z_1, \ldots, Z_p)^\top \sim N(0, \text{Id}_p) \) and assume
\[
n^{q/2-1} = o(p(\log n)^{-2}(\log p)^{-q/2}). \tag{13}
\]
Then the Gaussian approximation (7) does not hold. To see this, let \( u = (2 \log p)^{1/2} \). Then \( p\mathbb{P}(|Z_1| \geq u) \to 0 \), and, by (12) and (13), \( p\mathbb{P}(M_n \geq \sqrt{nu}) \to \infty \). Hence \( \mathbb{P}(\{|M_n| \leq \sqrt{nu}\} \to 0 \) and \( \mathbb{P}(\{|Z_1| \leq u\} \to 1 \), implying that
\[
|\mathbb{P}(\sqrt{n}|X_n| \leq u) - \mathbb{P}(|Z| \leq u)| = |\mathbb{P}(\{|M_n| \leq \sqrt{nu}\}) - \mathbb{P}(\{|Z_1| \leq u\})| = |[1 - 2\mathbb{P}(M_n \geq \sqrt{nu})]^p - \mathbb{P}(|Z_1| \leq u)| \to 1.
\]
Note that (13) is equivalent to \( n^{q/2-1} = o(p(\log p)^{-2-q/2}) \), suggesting that (11) is optimal up to a logarithmic term.

Now suppose there exist \( 0 \leq \kappa_1 \leq \kappa_2 \) such that \( \Psi_{q,\alpha} \asymp p^{\kappa_1} \) and \( \Theta_{q,\alpha} \asymp p^{\kappa_2} \), and \( p^\tau \asymp n \). Elementary but tedious calculations show that, in the weaker dependence case \( \alpha > 1/2 - 1/q \), if
\[
\tau > \max \left\{ \frac{\kappa_2}{1/2 - 1/q}, \frac{2\kappa_1}{\alpha} + 8\kappa_1, \frac{2\kappa_1}{\alpha} + 8\kappa_1 + 2\kappa_2 \right\}, \tag{14}
\]
then conditions in (i) of Theorem 3.2 are satisfied, while for the stronger dependence case with \( 0 < \alpha < 1/2 - 1/q \), a larger sample size \( n \) is required:
\[
\tau > \max \left\{ \frac{\kappa_2}{\alpha}, \frac{2\kappa_1}{\alpha} + 8\kappa_1, (1 - 2\alpha) \left( \frac{2\kappa_1}{\alpha} + 8\kappa_1 \right) + 2\kappa_2 \right\}. \tag{15}
\]
The lower bounds in (14) and (15) are both non-decreasing of $\kappa_1, \kappa_2$ and non-increasing in $q, \alpha$.

Under (11), the allowed dimension $p$ can only be at most a polynomial of $n$. To ensure the validity of GA in the ultra-high dimensional case with $\log p = o(n^c)$ with some $c > 0$, we need to consider the sub-exponential case in which $X_{ij}$ has finite moment with any order. For $\nu \geq 0$ and $\alpha \geq 0$, define the dependence-adjusted sub-exponential norm

$$
\|X_j\|_{\psi, \alpha} = \max_{q \geq 2} \frac{\|X_j\|_{q, \alpha}}{q^\nu} \quad \text{and} \quad \Phi_{\psi, \alpha} = \max_{j \leq p} \|X_j\|_{\psi, \alpha}.
$$

Let $L_3 = \left((\log p)^{1/\beta+1/2\Phi_{\psi, \alpha}}\right)^{1/\alpha} \cdot N_4 = n(\log p)^{-1-2/\beta\Phi_{\psi, 0}^{-2}}$ and $W_4 = (\log(pn))^{3+2/\beta\Phi_{\psi, 0}^2} + (\log(pn))^4$. Here $\beta = 2/(1 + 2\nu)$.

**Theorem 3.3.** Let Assumption 3.1 be satisfied. Assume that $\Phi_{\psi, \alpha} < \infty$ for some $\nu \geq 0$, $\alpha > 0$ and

$$
\max(L_2, L_3) \max(W_1, W_4) = o(N_4), \quad L_2^2 \max(W_1, W_4) = o(n).
$$

Then the Gaussian Approximation (7) holds.

**Proof.** The proof is similar to that of Theorem 3.2, and thus is omitted. \qed

If $\Phi_{\psi, \alpha} \asymp 1$, then the ultra high-dimensional case with $\log p = o(n^c)$ with some $c > 0$ is allowed, where specifically we can let

$$
c = \begin{cases} 
1/(8 + 2/\alpha + 2/\beta), & 2/3 \leq \beta \leq 2 \\
1/[7 + (1/\beta + 1/2)(1/\alpha + 2)], & 1/2 \leq \beta < 2/3 \\
1/[3 + 2/\beta + (1/\beta + 1/2)(1/\alpha + 2)], & 0 < \beta < 1/2
\end{cases}
$$

(17)

### 3.1 Simultaneous Inference of Covariances

Let $X_1, \ldots, X_n$ be i.i.d. $p$-dimensional vectors with mean 0 and covariance matrix $\Gamma = \Gamma_0 = \left(\gamma_{jk}\right)_{j,k=1}^p = \mathbb{E}(X_i X_i^\top)$. We can estimate $\Gamma$ by the sample covariance matrix $\hat{\Gamma} = \left(\hat{\gamma}_{jk}\right)_{j,k=1}^p = n^{-1}\sum_{i=1}^n X_i X_i^\top$. To perform simultaneous inference on $\gamma_{jk}, 1 \leq j, k \leq p$, one needs to derive asymptotic distribution of the maximum deviation $\max_{j,k \leq p} |\hat{\gamma}_{jk} - \gamma_{jk}|$ or the normalized version $\max_{j,k \leq p} |\hat{\gamma}_{jk} - \gamma_{jk}|/\tau_{jk}$; cf Equation (2) in Xiao and Wu (2013). Jiang (2004) established the Gumbel convergence of the maximum deviation assuming that all
entries of $X_i$ are also independent. See Li and Rosalsky (2006) and Liu et al. (2008) for some refined results. Xiao and Wu (2013) considered the extension which allows dependence among entries of $X_i$. However the latter paper requires that the vectors $X_1, \ldots, X_n$ are i.i.d. The problem of further extension to temporally dependent $X_i$ is open. In analyzing electrocorticogram data in the format of multivariate time series, Kramer et al. (2009) proposed to use the maximum cross correlation between time series to identify edges that connect the corresponding nodes in a network, suggesting that an asymptotic theory for maximum deviations of sample covariances is needed.

Our Theorems 3.2 and 3.3 can be applied to the above problem of further extension to temporally dependent process $(X_i)$. Let $(X_i)$ be a mean zero $p$-dimensional stationary process of form (3). To apply Theorems 3.2 and 3.3, one needs to deal with the key issue of computing the functional dependence measure of the $p^2$-dimensional vector $X_i = \text{vec}(X_iX_i^\top - \mathbb{E}(X_iX_i^\top))$. Interestingly, our framework allows a natural and elegant treatment. Let $a = (j,k)$, $j,k \leq p$, and $X_{ia} = X_{ij}X_{ik} - \gamma_a$, where $\gamma_a = \gamma_{jk} = \mathbb{E}(X_{ij}X_{ik})$.

By Hölder’s inequality, the functional dependence of the component process $(X_{ia})_i$

\[
\varphi_{i/2,a} := \left\| X_{ij}X_{ik} - \mathbb{E}(X_{ij}X_{ik}) - X_{ij,(0)}X_{ik,(0)} + \mathbb{E}(X_{ij,(0)}X_{ik,(0)}) \right\|_{q/2}
\]

\[
\leq 2\| X_{ij}X_{ik} - X_{ij,(0)}X_{ik,(0)} \|_{q/2}
\]

\[
\leq 2\| X_{ij}(X_{ik} - X_{ik,(0)}) \|_{q/2} + 2\| (X_{ij} - X_{ij,(0)})X_{ik,(0)} \|_{q/2}
\]

\[
\leq 2\| X_{ij} \|_q \| X_{ik} \|_q \| \gamma_{i,j,k} + 2\| X_{ik} \|_q \| \gamma_{i,k,j} \|
\]

(18)

Hence, we can have an upper bound of the dependence adjusted norm of $(X_{ia})$

\[
\| \mathcal{X}_a \|_{q/2,a} := \sup_{m \geq 0} (m + 1)^a \sum_{i=m}^{\infty} \varphi_{i/2,j,k} \leq 2\| X_{j,0} \|_q \| X_{k,0} \|_q,0 \| X_{j,0} \|_q,0 \|.
\]

Consequently, the uniform and the overall dependence adjusted norms of $X_i$ are

\[
\max_a \| \mathcal{X}_a \|_{q/2,a} \leq 4\| \psi_{q,0} \|_q,0 \| X_{j,0} \|_q,0 \|_q,0 \| X_{j,0} \|_q,0 \|.
\]

(20)

Similarly, the $L^\infty$ dependence adjusted norm for the process $(X_i)$ can be calculated by

\[
\| | \mathcal{X}_i |_{\infty} \|_{q/2,a} \leq 4\| | X_{i,\infty} |_{q,0} \| \| X_{i,\infty} \|_{q,0} \|. \]

(21)
With (18)-(21), conditions in Theorems 3.2 and 3.3 can be formulated accordingly, and under those conditions we can have the following Gaussian Approximation

\[
\sup_{u \geq 0} \left| \mathbb{P}(\sqrt{n} \max_a |\gamma_a - \gamma_a|/\tau_a \geq u) - \mathbb{P}(\max_a |Z_a/\tau_a| \geq u) \right| \to 0, \tag{22}
\]

where \( Z = (Z_a)_a \sim N(0, \Sigma_X) \), \( \Sigma_X \) is the \( p^2 \times p^2 \) long-run covariance matrix of \((X_i)_i\) and \((\tau_a^2)_a\) is the diagonal matrix of \( \Sigma_X \).

4 Estimation of long-run covariance matrix

Given the realization \( X_1, \ldots, X_n \), to apply the Gaussian approximation (7), we need to estimate the long-run covariance matrix \( \Sigma \). Note that \( \Sigma/(2\pi) \) is the value of the spectral density matrix of \((X_i)_i\) at zero frequency. In the one-dimensional case, there is a large literature concerning spectral density estimation; see for example Anderson (1971), Priestley (1981), Brockwell and Davis (1991), Liu and Wu (2010) among others.

In the high-dimensional setting, Chen et al. (2013) studied the regularized estimation of \( \Gamma(0) = \mathbb{E}(X_0X_0^\top) \). Assume \( \mathbb{E}X_i = 0 \). We then consider the batched mean estimate

\[
\hat{\Sigma} = \frac{1}{Mw} \sum_{b=1}^w Y_b Y_b^\top = \frac{1}{Mw} \sum_{b=1}^w \left( \sum_{i \in L_b} X_i \right) \left( \sum_{i \in L_b} X_i \right)^\top. \tag{23}
\]

where the window \( L_b = \{1 + (b - 1)M, \ldots, nM\}, b = 1, \ldots, w \), the window size \( |L_b| = M \to \infty \) and the number of blocks \( w = \lfloor n/M \rfloor \). Theorems 4.1 and 4.2 concern the convergence of the above estimate for processes with finite polynomial and finite sub-exponential dependence adjusted norms, respectively. The convergence rate depends in a subtle way on the temporal dependence characterized by \( \alpha \) (cf. (6)), the uniform and the overall dependence adjusted norms \( \Psi_{q,\alpha} \) and \( \Upsilon_{q,\alpha} \), respectively, the same size \( n \) and the dimension \( p \).

For a random variable \( X \), we define the operator \( \mathbb{E}_0 \) as \( \mathbb{E}_0(X) := X - \mathbb{E}X \).

**Theorem 4.1.** Assume \( \Psi_{q,\alpha} < \infty \) with \( q > 4 \) and \( \alpha > 0 \), and \( M = O(n^\varsigma) \) for some \( 0 < \varsigma < 1 \). Let \( F_\alpha = wM \) (resp. \( wM^{q/2-\alpha q/2} \) or \( w^{q/4-\alpha q/2}M^{q/2-\alpha q/2} \)) for \( \alpha > 1 - 2/q \) (resp. \( 1/2 - 2/q < \alpha < 1 - 2/q \) or \( \alpha < 1/2 - 2/q \)). Then for \( x \geq \sqrt{wM^2\Psi_{q,\alpha}^4} \), we have

\[
\mathbb{P}(n|\text{diag}(\hat{\Sigma}) - \mathbb{E}\text{diag}(\hat{\Sigma})|_\infty \geq x) \lesssim \frac{F_\alpha \Upsilon_{q,\alpha}}{x^{q/2}} + p \exp \left( - \frac{C_{q,\alpha} x^2}{wM^2\Psi_{q,\alpha}^4} \right),
\]
\[ \mathbb{P}(n|\tilde{\Sigma} - \mathbb{E}\tilde{\Sigma}|_{\infty} \geq x) \lesssim \frac{pF_{\alpha} T_{q,\alpha}^q}{x^{q/2}} + p^2 \exp \left( - \frac{C_{q,\alpha} x^2}{w M^2 \Psi_{4,\alpha}} \right) \]  

(24)

for all large \( n \), where the constants in \( \lesssim \) only depend on \( \varsigma, \alpha \) and \( q \).

**Proof.** Fix \( 1 \leq j, k \leq p \); let \( T = \sum_{b=1}^{w} Y_{bj} Y_{bk} \), where \( Y_{bj} = \sum_{i \in L_b} X_{ij} \). For \( \tau \geq 0 \), define \( X_{ij,\tau} = \mathbb{E}(X_{ij}|\varepsilon_{i-\tau}, \ldots, \varepsilon_{i}) \), \( Y_{bj,\tau} = \sum_{i \in L_b} X_{ij,\tau} \) and \( T_{\tau} = \sum_{b=1}^{w} Y_{bj,\tau} Y_{bk,\tau} \). We will first prove

\[ \mathbb{P}(\mathbb{E}_0(T - T_M) | \geq x) \lesssim \begin{cases} x^{-q/2} w M^{q/2 - \alpha q/2} \xi_{q,\alpha}^{q/2} + E_{q,\alpha}(x), & \alpha > 1/2 - 2/q \\ x^{-q/2} w^{q/4 - \alpha q/2} M^{q/2 - \alpha q/2} \xi_{q,\alpha}^{q/2} + E_{q,\alpha}(x), & \alpha < 1/2 - 2/q \end{cases}, \]  

(25)

where the constants in \( \lesssim \) only depend on \( \varsigma, \alpha \) and \( q \), and

\[ \xi_{q,\alpha} = \|X_{j}\|_{q,0} \|X_{k}\|_{q,\alpha} + \|X_{k}\|_{q,0} \|X_{j}\|_{q,\alpha}, \]

\[ E_{q,\alpha}(x) = \exp\{-C_{q,\alpha}(w M^{2-2\alpha} \xi_{\alpha,\alpha}^2)^{-1} x^2\}. \]

Following the argument in the proof of Lemma 5.7, let \( L = \lfloor (\log w)/(\log 2) \rfloor \), \( \varpi_l = 2^l \), \( 1 \leq l < L \), \( \varpi_L = w \) and \( \tau_l = M \varpi_l \) for \( 1 \leq l \leq L \). Let \( \varpi_0 = 1 \) and \( \tau_0 = M \). Write

\[ T - T_M = T - T_{Mw} + \sum_{l=1}^{L} V_{w,l}, \text{ where } V_{w,l} = T_{\tau_l} - T_{\tau_{l-1}}. \]  

(26)

By the argument in Lemma 9 of Xiao and Wu (2012), we have

\[ \|\mathbb{E}_0(T - T_{Mw})\|_{q/2} \leq C_q M \sqrt{w} (\Delta_{0,q,j} \Delta_{Mw+1,q,k} + \Delta_{Mw+1,q,j} \Delta_{0,q,k}) \]

\[ \leq C_q M \sqrt{w(Mw)^{-\alpha} \xi_{q,\alpha}} \]  

(27)

for some constant \( C_q > 0 \). By Markov’s inequality, for \( x > 0 \),

\[ \mathbb{P}(\|\mathbb{E}_0(T - T_{Mw})\| \geq x) \leq \frac{C_q M^{q/2 - \alpha q/2} w^{q/4 - \alpha q/2} \xi_{q,\alpha}^{q/2}}{x^{q/2}}. \]  

(28)

By the same argument for proving (27), we have

\[ \|\mathbb{E}_0(V_{w,l})\|_{q/2} \leq C_q M \sqrt{w \tau_l^{-\alpha} \xi_{q,\alpha}}. \]
Let \( c = q/4 - 1 - \alpha q/2 \), \( \lambda_l = 3l^{-2}\pi^{-2} \) if \( 1 \leq l \leq L/2 \) and \( \lambda_l = 3(L + 1 - l)^{-2}\pi^{-2} \) if \( L/2 < l \leq L \). Then \( \sum_{l=1}^{L} \lambda_l < 1 \). By the Nagaev (1979) inequality, it follows that

\[
\mathbb{P}
\left(
\|\sum_{l=1}^{L} x_l\|_{\infty} \geq x \right)
\leq \sum_{l=1}^{L} \mathbb{P} \left(\|\mathbb{E}_0(x_l)\|_{\infty} \geq \lambda_l \right) 
\leq \sum_{l=1}^{L} \frac{C_1 w l^{-2} \lambda_l^{-\alpha} q/2}{\lambda_l q/2} + 4 \sum_{l=1}^{L} \exp \left( - \frac{C_2 (\lambda_l x)^2}{w M^2 \lambda_l} \right)
\leq \frac{C_3 w M^{-2} l^{-\alpha} q/2}{x q/2} \sum_{l=1}^{L} \lambda_l q/2 + C_4 \sum_{l=1}^{L} E_{q,\alpha}(\lambda_l \omega_l^\alpha x).
\]  
(29)

Elementary calculations show that

\[
\sum_{l=1}^{L} \lambda_l q/2 \leq C_5 \text{ for } c < 0 \text{ and } \sum_{l=1}^{L} \lambda_l q/2 \leq C_6 \omega_L = C_6 w^c \text{ for } c > 0.
\]  
(30)

Furthermore, we can use (57) to obtain

\[
\sum_{l=1}^{L} E_{q,\alpha}(\lambda_l \omega_l^\alpha x) \lesssim E_{q,\alpha}(x).
\]  
(31)

Putting (26), (28), (29), (30) and (31) together, we then have (25).

Now it suffices to consider \( \mathbb{P} \left( \|\mathbb{E}_0(T_M)\|_{\infty} \geq x \right) \). Observe that \( (Y_{b,1}Y_{b,k,M})_b \) is odd and so are \( (Y_{b,1}Y_{b,k,M})_b \) is even. By Corollary 1.7 of Nagaev (1979), for any \( J > 1 \),

\[
\mathbb{P} \left( \|\mathbb{E}_0(T_M)\|_{\infty} \geq x \right) \leq \sum_{b=1}^{w} \mathbb{P} \left( \|\mathbb{E}_0(Y_{b,j,Y_{b,k,M}})\|_{\infty} \geq x/(2J) \right) + 2 \left( \sum_{b=1}^{w} \|\mathbb{E}_0(Y_{b,j,Y_{b,k,M}})\|_{\infty}^{\alpha/2} \right)^{J/2}.
\]

Note that \( \|Y_{b,j,M}\|_{q} \leq C_q \sqrt{M} \|X_{j,0}\|_{q,0} \). Hence for \( 1 \leq b \leq w, 1 \leq j, k \leq p \) and \( q \geq 4 \),

\[
\|\mathbb{E}_0(Y_{b,j,M}Y_{b,k,M})\|_{q/2} \leq 2 \|Y_{b,j,M}Y_{b,k,M}\|_{q/2} \leq 2 \|Y_{b,j,M}\|_{q} \|Y_{b,k,M}\|_{q} \leq C_q M \|X_{j,0}\|_{q,0} \|X_{k,0}\|_{q,0}.
\]

Since

\[
\mathbb{E}|Y_{b,j,M}Y_{b,k,M}| \leq \|Y_{b,j,M}\|_{2} \|Y_{b,k,M}\|_{2} \leq M \|X_{j,0}\|_{q,0} \|X_{k,0}\|_{q,0} \leq \frac{x}{\sqrt{w}},
\]
we have
\[
\mathbb{P}(|\mathbb{E}_0(T_M)| \geq x) \leq \sum_{b=1}^{w} \mathbb{P}(|Y_{b,j,M} Y_{b,k,M}| \geq x/(4J))
\]
\[
+ 2 \left( \frac{wM^{q/2} \|X_j\|_{q,0}^{q/2} \|X_k\|_{q,0}^{q/2}}{Jx^{q/2}} \right)^J + 4 \exp \left(- \frac{C_q x^2}{wM^2 \Psi_{4,0}^2} \right).
\]
Recall that \( M = O(n^\varsigma) \) with \( 0 < \varsigma < 1 \). Let \( J = 1 + (2q - 2)(q - 4)^{-1}(1 - \varsigma)^{-1} \). Since \( x \geq \sqrt{wM}\|X_j\|_{q,0}\|X_k\|_{q,0} \), elementary calculations show that for sufficiently large \( n \) the second term in the above expression is no greater than \( C_J wM \|X_j\|_{q,0}^{q/2} \|X_k\|_{q,0}^{q/2}/x^{q/2} \). As for the first term, we have
\[
\mathbb{P}(|Y_{b,j,M} Y_{b,k,M}| \geq x/(4J)) \leq \mathbb{P}(|Y_{b,j,M}| \geq \sqrt{x/(4J)}) + \mathbb{P}(|Y_{b,k,M}| \geq \sqrt{x/(4J)}).
\]
By Lemma 5.2, for \( \alpha > 1/2 - 1/q \) and \( \alpha < 1/2 - 1/q \), respectively, we have
\[
\mathbb{P}(|Y_{b,j,M}| \geq \sqrt{x}) \leq \begin{cases} 
C_{q,\alpha} x^{-q/2} M \|X_j\|_{q,\alpha}^q + C_{q,\alpha} \exp \left(- \frac{C_{q,\alpha} x^2}{M \|X_j\|_{q,\alpha}^2} \right), \\
C_{q,\alpha} x^{-q/2} M^{q/2 - q/2} \|X_j\|_{q,\alpha}^q + C_{q,\alpha} \exp \left(- \frac{C_{q,\alpha} x^2}{M \|X_j\|_{q,\alpha}^2} \right).
\end{cases}
\]
A similar inequality holds for \( \mathbb{P}(|Y_{b,k,M}| \geq \sqrt{x}) \). Let \( \phi_{q,\alpha} = \|X_j\|_{q,\alpha} + \|X_k\|_{q,\alpha} \). Hence, it follows that for \( \alpha > 1/2 - 1/q \) and \( \alpha < 1/2 - 1/q \) respectively,
\[
\mathbb{P}(|\mathbb{E}_0(T_M)| \geq x) \leq \begin{cases} 
C_{q,\alpha} x^{-q/2} wM \phi_{q,\alpha} + C_{q,\alpha} \exp \left(- \frac{C_{q,\alpha} x^2}{wM^2 \Psi_{4,\alpha}^2} \right), \\
C_{q,\alpha} x^{-q/2} wM^{q/2 - q/2} \phi_{q,\alpha} + C_{q,\alpha} \exp \left(- \frac{C_{q,\alpha} x^2}{wM^2 \Psi_{4,\alpha}^2} \right).
\end{cases}
\] (32)
Combining (25) and (32), and noticing that \( \xi_{q,\alpha} \leq C_q \phi_{q,\alpha} \), it follows that
\[
\mathbb{P}(|\mathbb{E}_0(T)| \geq x) \leq C_{q,\alpha} x^{-q/2} F_{q,\alpha} \phi_{q,\alpha} + C_{q,\alpha} \exp \left(- \frac{C_{q,\alpha} x^2}{wM^2 \Psi_{4,\alpha}^2} \right),
\]
which implies (24) by the Bonferroni inequality by summing over \( j \) and \( k \).

Under stronger moment conditions, we can have an exponential inequality.

**Theorem 4.2.** Assume \( \Phi_{\psi,0} < \infty \) for some \( \nu \geq 0 \). Then for all \( x > 0 \), we have
\[
\mathbb{P}(n|\text{diag}(\hat{\Sigma}) - \mathbb{E}\text{diag}(\hat{\Sigma})|_\infty \geq x) \lesssim p \exp \left(- \frac{x^\gamma}{4e\gamma(\sqrt{wM} \Phi_{\psi,0}^2)^\gamma} \right), \quad (33)
\]
\[
\mathbb{P}(n|\hat{\Sigma} - \mathbb{E}\hat{\Sigma}|_\infty \geq x) \lesssim p^2 \exp \left(- \frac{x^\gamma}{4e\gamma(\sqrt{wM} \Phi_{\psi,0}^2)^\gamma} \right), \quad (34)
\]
where \( \gamma = 1/(1 + 2\nu) \) and the constants in \( \lesssim \) only depend on \( \nu \).
Proof. Let $T = \sum_{b=1}^{w} Y_{bj} Y_{bk}$. By the Burkholder inequality, we have

$$
\|E_0 T\|_{q/2}^2 \leq (q/2 - 1) \sum_{l=-\infty}^{w} \|P^l T\|_{q/2}^2 \leq (q/2 - 1) \sum_{l=-\infty}^{w} \left( \sum_{b=1}^{w} \|P^l Y_{bj} Y_{bk}\|_{q/2} \right)^2
$$

(35)

By Theorem 3 in Wu (2011), $\|Y_{bj}\|_q \leq (q - 1)^{1/2} \sqrt{M} \|X_j\|_{q,0}$. Since $\|Y_{bk} - Y_{bk,(t)}\|_q \leq \sum_{h=1+(b-1),M} \delta_{h-t,q,k}$, we have

$$
\sum_{b=1}^{w} \|P^l Y_{bj} Y_{bk}\|_{q/2} \leq \sum_{b=1}^{w} \|Y_{bj} Y_{bk} - Y_{bj,(t)} Y_{bk,(t)}\|_{q/2}
$$

$$
\leq \sum_{b=1}^{w} (\|Y_{bj}\|_q \|Y_{bk}\|_q + \|Y_{bj} - Y_{bj,(t)}\|_q \|Y_{bk} - Y_{bk,(t)}\|_q)
$$

$$
\leq (q - 1)^{1/2} \sqrt{M} \left( \|X_j\|_{q,0} \sum_{h=1}^{w} \delta_{h-t,q,k} + \|X_k\|_{q,0} \sum_{h=1}^{w} \delta_{h-t,q,k} \right),
$$

which by (35) implies that

$$
\|E_0 T\|_{q/2}^2 \leq (q/2 - 1) \sum_{l=-\infty}^{w} \|P^l T\|_{q/2}^2 \leq (q - 2)(q - 1) w M^2 \|X_j\|_{q,0}^2 \|X_k\|_{q,0}^2.
$$

(36)

Let $R_{jk} = E_0 T / (\sqrt{w}M)$. Similarly as the argument for proving Lemma 5.3, if $\gamma h \geq 2$, it follows that $\|R_{jk}\|_{\gamma h} \leq (2\gamma h - 1)(2\gamma h)^{2\nu} \|X_j\|_{\psi_0,0} \|X_k\|_{\psi_0,0}$. Let $\tau_0 = (2\nu \|X_j\|_{\psi_0,0} \|X_k\|_{\psi_0,0})^{-1}$. Notice that $-2\nu = 1 - 1/\gamma$. Then

$$
\frac{t^h \|R_{jk}\|_{\gamma h}^h}{h!} \leq \frac{t^h (2\gamma h - 1)^\gamma (2\gamma h)^{2\nu} \|X_j\|_{\psi_0,0}^\gamma \|X_k\|_{\psi_0,0}^\gamma}{C_1 (h/e)^h a_h^{-1}}
$$

$$
\leq \frac{a_h t^h (2\gamma h - 1)^\gamma}{C_1 \tau_0 (2\gamma h)^{\gamma h}} \leq \frac{a_h t^h}{C_1 \sqrt{e \tau_0^h}}.
$$

If $\gamma h < 2$, then $\|R_{jk}\|_{\gamma h} \leq \|R_{jk}\|_2 \leq \sqrt{6} \cdot 4^{2\nu} \|X_j\|_{\psi_0,0} \|X_k\|_{\psi_0,0}$. So we have

$$
\mathbb{E}[\exp(t R_{jk}^h)] \leq 1 + \sum_{1 \leq h < 2/\gamma} \frac{t^h (\sqrt{6} \cdot 4^{2\nu} \|X_j\|_{\psi_0,0} \|X_k\|_{\psi_0,0})^h}{h!} + \sum_{h \geq 2/\gamma} \frac{a_h t^h}{C_1 \sqrt{e \tau_0^h}}
$$

$$
\leq 1 + C_\gamma \sum_{h=1}^{\infty} \frac{a_h}{\tau_0^h} \leq 1 + C_\gamma \frac{t/\tau_0}{(1 - t/\tau_0)^{1/2}}.
$$

By choosing $t = \tau_0 / 2$, and applying the Markov inequality and the Bonferroni inequality, (33) and (34) are obtained. \qed
Remark 2. An alternative estimate of $\Sigma$, which also works with unknown mean $\mathbb{E}X_i$, is

$$
\hat{\Sigma} = \frac{1}{wM} \sum_{b=1}^{w} \left( \sum_{i \in L_b} X_i - M\bar{X} \right) \left( \sum_{i \in L_b} X_i - M\bar{X} \right)^\top,
$$

where $\bar{X} = (wM)^{-1} \sum_{i=1}^{wM} X_i$, $w = \lfloor n/M \rfloor$. Then $|\hat{\Sigma} - \hat{\Sigma}|_\infty = M|\bar{X}|_\infty^2$. Applying Lemma 5.2 to $\sum_{i=1}^{wM} X_{ij}$, one can conclude that Theorems 4.1 and 4.2 still hold for $\hat{\Sigma}$ with $\mathbb{E}\hat{\Sigma}$ therein replaced by $\Sigma_M := \sum_{i=-M}^{M} (1 - |i|/M)\Gamma_i$ (which equals to $\mathbb{E}\hat{\Sigma}$ if $\mathbb{E}X_i = 0$).

Corollary 4.3. (i) Under conditions in Theorem 4.1, we have $|\hat{\Sigma} - \Sigma|_\infty = O_p(r_n)$, where

$$
r_n = n^{-1} \max\{p^{2/q}F_{\alpha}^{2/q}\Psi_{q,\alpha}^2, \sqrt{wM}\Psi_{q,\alpha}^2\sqrt{\log p}, \sqrt{wM}\Psi_{q,\alpha}^2 + \Psi_{2,0}\Psi_{2,\alpha}v(M),
$$

where $v(M) = 1/M$ if $\alpha > 1$, $v(M) = \log M/M$ if $\alpha = 1$ and $v(M) = 1/M^\alpha$ if $0 < \alpha < 1$. (ii) Under conditions in Theorem 4.2, we have $|\hat{\Sigma} - \Sigma|_\infty = O_p(r_n)$ with $r_n = n^{-1}\sqrt{wM}\Psi_{q,0}^2(\log p)^{1/\gamma} + \Psi_{2,0}\Psi_{2,\alpha}v(M)$.

The above Corollary easily follows from Theorems 4.1 and 4.2 since the bias $|\Sigma_M - \Sigma|_\infty \lesssim \Psi_{2,0}\Psi_{2,\alpha}v(M)$; see the proof of Lemma 6.3.

For the estimate $\hat{\Sigma}$ in (37), let $\tilde{D}_0 = [\text{diag}(\hat{\Sigma})]^{1/2}$. Let $\tilde{Z} = \hat{\Sigma}^{1/2}\eta$, where $\eta \sim N(0, \text{Id}_p)$ is independent of $(X_i)_i$. Then conditioning on $(X_i)_i$, $\tilde{Z} \sim N(0, \hat{\Sigma})$. Let $0 < \theta < 1$; let $\tilde{\chi}_\theta$ be the conditional $\theta$-quantile of $|\tilde{D}_0^{-1}\tilde{Z}|_\infty$ given $(X_i)_{i=1}^n$. We can use $\tilde{\chi}_\theta$ to estimate the $\theta$-quantile of $|D_0^{-1}(\hat{X}_n - \mu)|_\infty$, thus constructing simultaneous confidence intervals for $\mu = (\mu_1, \ldots, \mu_p)^\top$ as $\hat{\mu}_j \pm \tilde{\chi}_\theta \tilde{\sigma}_{jj}^{1/2}$, $1 \leq j \leq p$. Assume that $r_n = o(1/\log^2 p)$. Then $\pi(|\hat{\Sigma} - \Sigma|_\infty) = o(1)$, and by Lemma 3.1 in Chernozhukov et al. (2013a), the latter simultaneous confidence intervals have the asymptotically correct coverage probability $\theta$. Note that $\tilde{\chi}_\theta$ can be obtained by sample quantile estimates from extensive simulations of $\tilde{Z} = \hat{\Sigma}^{1/2}\eta$.

5 Tail probability inequalities under dependence

Tail probability inequalities play an important role in simultaneous inference. Here we shall provide some Nagaev-type tail probability inequalities. They are of independent interest. Let $\varepsilon_i, \varepsilon'_j, i, j \in \mathbb{Z}$, be i.i.d. random variables. We start with the one-dimensional stationary process $(e_i)_{i=-\infty}^{\infty}$ of the form

$$
e_i = g(\ldots, \varepsilon_{i-1}, \varepsilon_i),
$$

(38)
where \( g \) is a measurable function such that \( e_i \) is well-defined. Recall \( \mathcal{F}_i^j = (\varepsilon_i, \varepsilon_{i+1}, \ldots, \varepsilon_j) \), \( \mathcal{F}^i = (\varepsilon_i, \varepsilon_{i+1}, \ldots) \) and \( \mathcal{F}_i = (\varepsilon_i, \varepsilon_{i+1}, \ldots) \). Let the projection operators \( \mathcal{P}_0 = \mathbb{E}(\cdot | \mathcal{F}_0) - \mathbb{E}(\cdot | \mathcal{F}_1) \) and \( \mathcal{P}_0^0 = \mathbb{E}(\cdot | \mathcal{F}_0^0) - \mathbb{E}(\cdot | \mathcal{F}_1) \). As in (5), define respectively the functional and the predictive dependence measures

\[
\delta_{i,q} = \|e_i - g(\mathcal{F}_i^{i,0})\|_q, \quad \theta_{i,q} = \|\mathcal{P}_0 e_i\|_q, \quad \text{and} \quad \theta'_{i,q} = \|\mathcal{P}_0 e_i\|_q,
\]

where \( \mathcal{F}_i^{i,0} = (\varepsilon_{-1}, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_i) \). Let \( \delta_{i,q} = 0 \) if \( i < 0 \); let \( \Delta_{m,q} = \sum_{i=m}^{\infty} \delta_{i,q}, m \geq 0 \), be the tail dependence measures, and the dependence adjusted norm

\[
\|e_i\|_{q,\alpha} := \sup_{m \geq 0} (m+1)^\alpha \Delta_{m,q}, \text{ for } \alpha \geq 0.
\]

Here \( \delta_{i,q} \) measures the dependence of \( e_i \) on \( \varepsilon_0 \) and \( \Delta_{m,q} \) measures the cumulative impact of \( \varepsilon_0 \) on \( (e_i)_{i \geq m} \). The projections \( (\mathcal{P}_i^{-})_{i \in \mathbb{Z}} \) and \( (\mathcal{P}_i^{i})_{i \in \mathbb{Z}} \) induces martingale differences with respect to \( (\mathcal{F}_i) \) and \( (\mathcal{F}_i^{i}) \), respectively. Both predictive dependence measures provide an evaluation to the effect on the prediction of \( e_i \) when part of the previous inputs is concealed, and they satisfy \( \theta_{i,q} \leq \delta_{i,q} \) and \( \theta'_{i,q} \leq \delta_{i,q} \) in view of Jensen’s inequality.

### 5.1 Inequalities with Finite Polynomial Moments

For \( m \geq 0 \), the \( m \)-dependence approximation of \( e_i \) is denoted by \( e_{i,m} \) where

\[
e_{i,m} = \mathbb{E}(e_i | \varepsilon_{i-m}, \varepsilon_{i-m+1}, \ldots, \varepsilon_i).
\]

Let \( S_n = \sum_{i=1}^n e_i, S_{n,m} = \sum_{i=1}^m e_{i,m} \). With the dependence adjusted norm (39), we are able to provide tail probability inequalities for error bounds when approximating \( (e_i) \) by the \( m \)-dependent process \( (e_{i,m}) \). In lemmas below the constant \( C_{q,\alpha} \) only depends on \( q \) and \( \alpha \) and its values may change from line to line.

**Lemma 5.1.** Assume \( \|e_i\|_{q,\alpha} < \infty \), where \( q > 2 \) and \( \alpha > 0 \). (i) If \( \alpha > 1/2 - 1/q \), then

\[
\mathbb{P}(|S_n - S_{n,m}| \geq x) \leq \frac{C_{q,\alpha} n^{q/2 - 1 - \alpha} x^q}{x^q} + C_{q,\alpha} \exp \left( - \frac{C_{q,\alpha} x^2 m^{2\alpha}}{n \|e_i\|_{q,\alpha}^2} \right)
\]

holds for all \( x > 0 \) and \( 1 \leq m \leq n \). (ii) If \( 0 < \alpha < 1/2 - 1/q \), we have

\[
\mathbb{P}(|S_n - S_{n,m}| \geq x) \leq \frac{C_{q,\alpha} n^{q/2 - 2 - \alpha} x^q}{x^q} + C_{q,\alpha} \exp \left( - \frac{C_{q,\alpha} x^2 m^{2\alpha}}{n \|e_i\|_{q,\alpha}^2} \right).
\]
Proof of Lemma 5.1. It is a special case of Lemma 5.7 for \( p = 1 \).

Lemma 5.2 (cf. Theorem 2 of Wu and Wu (2015)). Assume that \( \| e \|_{q, \alpha} < \infty \), where \( q > 2 \) and \( \alpha > 0 \). (i) If \( \alpha > 1/2 - 1/q \), then there exists some constant \( C_{q, \alpha} \) depending on \( q \) and \( \alpha \) only such that, for \( x > 0 \),

\[
\mathbb{P}(|S_n| \geq x) \leq \frac{C_{q, \alpha} n \| e \|_{q, \alpha}^2}{x^q} + C_{q, \alpha} \exp \left( -\frac{C_{q, \alpha} x^2}{n \| e \|_{2, \alpha}^2} \right).
\]

(ii) If \( 0 < \alpha < 1/2 - 1/q \), we have the following inequality,

\[
\mathbb{P}(|S_n| \geq x) \leq \frac{C_{q, \alpha} n^{q/2 - \alpha q} \| e \|_{q, \alpha}^q}{x^q} + C_{q, \alpha} \exp \left( -\frac{C_{q, \alpha} x^2}{n \| e \|_{2, \alpha}^2} \right).
\]

Remark 3. By Markov’s inequality and Lemma 1 of Liu and Wu (2010), one obtains

\[
\mathbb{P}(|S_n - S_{n,m}| \geq x) \leq \frac{\| S_n - S_{n,m} \|_q^q}{x^q} \leq C_{q, \alpha} n^{q/2 - \alpha q} \| e \|_{q, \alpha}^q.
\]

In comparison, the polynomial tail bounds in (42) and (43) are sharper.

5.2 Inequalities with Finite Exponential Moments

If \( e_i \) satisfies stronger moment condition than the existence of finite \( q \)-th moment, we can have an exponential inequality. We shall assume \( \| e \|_{q, \alpha} < \infty \) for all \( q > 0 \) and some \( \alpha \geq 0 \) and we further assume for some \( \nu \geq 0 \), the dependence adjusted sub-exponential norm

\[
\| e \|_{\psi, \alpha} := \sup_{q \geq 2} q^{-\nu} \| e \|_{q, \alpha} < \infty.
\]

By this definition, if \( e_i \) are i.i.d., \( \| e \|_{\psi, \alpha} \) reduces to the sub-Gaussian norm (\( \nu = 1 \)) or sub-exponential norm (\( \nu = 1/2 \)) of the random variable by the equivalence of \( \| e \|_{q, \alpha} \) and \( \| e_i \|_q \). The parameter \( \nu \) measures how fast \( \| e \|_{q, \alpha} \) increases with \( q \).

Lemma 5.3. Assume (45). Let \( J_n = (S_n - S_{n,m}) / \sqrt{n} \) and \( \beta = 2/(1 + 2\nu) \). Then

\[
h(t) := \sup_{n \in \mathbb{N}} \mathbb{E}[\exp(tJ_n^\beta)] \leq 1 + C_\beta (1 - t/t_0)^{-1/2} t/t_0
\]

holds for \( 0 \leq t < t_0 \) with \( t_0 = m^{\alpha \beta}/(e\beta \| e \|_{\psi, \alpha}^\beta) \). Consequently, letting \( t = t_0/2 \), for \( x > 0 \),

\[
\mathbb{P}(|J_n| \geq x) \leq \exp(-tx^\beta) h(t) \leq C_\beta \exp \left( -\frac{x^\beta m^{\alpha \beta}}{2e\beta \| e \|_{\psi, \alpha}^\beta} \right).
\]
Lemma 5.4 (cf. Theorem 3 of Wu and Wu (2015)). Assume (45) holds for $\alpha = 0$. Let $\beta = 2/(1 + 2\nu)$. Then for $x > 0,$

$$P(|S_n/\sqrt{n}| \geq x) \leq C_\beta \exp \left( -\frac{x^\beta}{2e\beta\|e\|_{\psi,0}^\beta} \right). \quad (47)$$

Proof of Lemma 5.3. Let $Q_{n,l} = \sum_{i=1}^nP_{i-l}X_i$, $l \geq 0$. Then $Q_{n,l}$ is a backward martingale. By Burkholder’s inequality, we have

$$\|Q_{n,l}\|_q^2 \leq (q-1)\sum_{i=1}^n\|P_{i-l}X_i\|_q^2 = (q-1)n(\theta'_{l,q})^2.$$

By $\theta'_{l,q} \leq \delta_{l,q},$ we have $\|J_n\|_q \leq (q-1)^{1/2}\Delta_{m+1,q}$ in view of $\sqrt{n}J_n = \sum_{l=m+1}^\infty Q_{n,l}.$ Write the negative binomial expansion $(1 - s)^{-1/2} = 1 + \sum_{k=1}^\infty a_k s^k$ with $a_k = (2k)!/(2^{2k}(k!)^2)$ for $|s| < 1.$ By Stirling’s formula, we have $a_k \sim (k\pi)^{-1/2}$ as $k \to \infty.$ Hence, there exists absolute constants $C_1, C_2 > 0$ such that for all $k \geq 1,$

$$C_1(k/e)^ka_k^{-1} \leq k! \leq C_2(k/e)^ka_k^{-1}. \quad (48)$$

Under condition (45), if $k\beta \geq 2$, then $\|e\|_{\beta,k,a} \leq \|e\|_{\psi,\alpha}(\beta k)^{\nu}$ and hence

$$\frac{t^k\|J_n\|_k^k}{k!} \leq \frac{t^k(\beta k - 1)^{\beta k/2}\Delta_{m+1,\beta k}}{C_1(k/e)^ka_k^{-1}} \leq \frac{a_k t^k(\beta k - 1)^{\beta k/2}}{C_1 t_0^k(\beta k)^{\beta k/2}} \leq \frac{a_k t^k}{C_1 \sqrt{e} t_0^k}.$$

If $k\beta < 2,$ then $\|J_n\|_{\beta k} \leq \|J_n\|_2 \leq 2^{\nu}m^{-\alpha}\|e\|_{\psi,\alpha}.$ In $e^y = \sum_{k=0}^\infty y^k/k!$, let $y = tJ_n^\beta$, then

$$h(t) \leq 1 + \sum_{1 \leq k < 2/\beta} \frac{t^k(2^{\nu}m^{-\alpha}\|e\|_{\psi,\alpha})^{\beta k}}{k!} + \sum_{k \geq 2/\beta} \frac{a_k t^k}{C_1 \sqrt{e} t_0^k} \leq 1 + C_\beta \sum_{k=1}^\infty \frac{t^k}{t_0^k} \leq 1 + C_\beta \frac{t/t_0}{(1 - t/t_0)^{1/2}};$$

where $C_\beta > 0$ only depends on $\beta.$ So (46) follows by Markov’s inequality. \qed

5.3 Inequalities for High-dimensional Time Series with Finite Polynomial Moments

In this section we shall derive powerful tail probability inequalities for high-dimensional stationary vectors; cf Lemmas 5.7 and 5.8. The proofs require Theorem 4.1 of Pinelis...
(1994), a deep Rosenthal-Burkholder type bound on moments of Banach-spaced martingales. Lemma 5.5 follows from Theorem 4.1 of Pinelis (1994). Lemma 5.6 is a Fuk-Magaev type inequality for the sum of independent random vectors. For a $p$-dimensional vector $v = (v_1, \ldots, v_p)$ recall the $s$-length $|v|_s = (\sum_{j=1}^p |v_j|^s)^{1/s}$, $s \geq 1$.

**Lemma 5.5.** Let $D_i$, $1 \leq i \leq n$, be $p$-dimensional martingale difference vectors with respect to the $\sigma$-field $\mathcal{G}_i$. Let $s > 1$ and $q \geq 2$. Then

$$||D_1 + \ldots + D_n||_q \leq c \left\{ q \sup_i |D_i||_q + \sqrt{q(s-1)} \left\| \sum_{i=1}^n \mathbb{E}(|D_i|^2|G_{i-1}) \right\|_q^{1/2} \right\},$$

where $c$ is an absolute constant.

**Lemma 5.6.** Assume $s > 1$. Let $X_1, \ldots, X_n$ be $p$-dimensional independent random vectors with mean zero such that for some $q > 2$, $||X_i||_q < \infty$, $1 \leq i \leq n$. Let $T_n = \sum_{i=1}^n X_i$ and $\sigma_i = (\|X_{i1}\|_2, \ldots, \|X_{ip}\|_2)$. Then for any $y > 0$,

$$P(||T_n||_s \geq 2E||T_n||_s + y) \leq C_q y^{-q} \sum_{i=1}^n \mathbb{E}|X_i|^q + \exp \left( -\frac{y^2}{3 \sum_{i=1}^n |\sigma_i|^2} \right), \quad (49)$$

where $C_q$ is a positive constant only depending on $q$.

**Proof of Lemma 5.6.** For $s > 1$, we apply Theorem 3.1 of Einmahl and Li (2008) with the Banach space $(\mathbb{R}^p, |\cdot|_s)$ and $\eta = \delta = 1$. The unit ball of the dual of $(\mathbb{R}^p, |\cdot|_s)$ is the set of linear functions $\{u = (u_1, \ldots, u_p) \mapsto \lambda^\top u : \lambda \in \mathbb{R}^p, |\lambda|_a \leq 1\}$ where $1/a + 1/s = 1$. By Minkowski’s and Hölder’s inequalities, we have

$$||\lambda^\top X_i||_2 \leq \sum_{j=1}^p |\lambda_j| \cdot ||X_{ij}||_2 \leq |\lambda|_a |\sigma_i||_s.$$ 

Hence, the $\Lambda_n$ therein is bounded by $\sum_{i=1}^n |\sigma_i|^2$. \qed

Let $X_i$ be a mean zero $p$-dimensional stationary process, and $T_n = \sum_{i=1}^n X_i$, $T_{n,m} = \sum_{i=1}^n X_{i,m}$ where $X_{i,m} = \mathbb{E}(X_i|\varepsilon_{i-m}, \ldots, \varepsilon_i)$. We are interested in bounding the tail probabilities of $P(|T_n - T_{n,m}|_\infty \geq x)$ and $P(|T_n|_\infty \geq x)$ for large $x$. Wrtie $\ell = \ell(p) = 1 \lor \log p$. 

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Lemma 5.7. Assume $|||X|||_{q,\alpha} < \infty$, where $q > 2$ and $\alpha \geq 0$. Also assume $\Psi_{2,\alpha} < \infty$.

(i) If $\alpha > 1/2 - 1/q$, then for $x \geq \sqrt{n\ell\Psi_{2,\alpha}} + n^{1/q}\ell |||X|||_{q,\alpha}m^{-\alpha}$,
\[
\mathbb{P}(|T_n - T_{n,m}|_{\infty} \geq x) \leq \frac{C_{q,\alpha}nm^{q/2-1-\alpha}q^{q/2}|||X|||_{q,\alpha}^q}{x^q} + C_{q,\alpha} \exp\left(-\frac{C_{q,\alpha}x^2m^{2\alpha}}{n\Psi_{2,\alpha}^2}\right) \tag{50}
\]
holds for all $1 \leq m \leq n$. (ii) If $0 < \alpha < 1/2 - 1/q$, the inequality is
\[
\mathbb{P}(|T_n - T_{n,m}|_{\infty} \geq x) \leq \frac{C_{q,\alpha}nq^{2-\alpha}q^{q/2}|||X|||_{q,\alpha}^q}{x^q} + C_{q,\alpha} \exp\left(-\frac{C_{q,\alpha}x^2m^{2\alpha}}{n\Psi_{2,\alpha}^2}\right). \tag{51}
\]

Proof of Lemma 5.7. Let $s = \ell = 1 \lor \log p$. Then $\mathbb{P}(|T_n - T_{n,m}|_{s} \geq x)$ is equivalent to $\mathbb{P}(|T_n - T_{n,m}|_{s} \geq x)$, since for any vector $v = (v_1, \ldots, v_p)^\top$, $|v|_{\infty} \leq |v|_s \leq p^{1/s}|v|_{\infty}$. Let $L = \lfloor (\log n - \log m)/(\log 2) \rfloor$, $\omega_l = 2^l$ if $1 \leq l < L$, $\omega_L = [n/m]$ and $\tau_l = m \cdot \omega_l$ for $1 \leq l < L$, $\tau_0 = m$, $\tau_L = n$. Define $M_{n,l} = T_{n,\tau_l} - T_{n,\tau_{l-1}}$ for $1 \leq l \leq L$ and write
\[
T_n - T_{n,m} = T_n - T_{n,n} + \sum_{l=1}^{L} M_{n,l}. \tag{52}
\]

Notice that $T_n - T_{n,n} = \sum_{j=n}^{\infty} T_{n,j+1} - T_{n,j}$. By Lemma 5.5,
\[
|||T_n - T_{n,n}|||_{s} \leq \sum_{j=n}^{\infty} |||T_{n,j+1} - T_{n,j}|||_{s} \leq \sum_{j=n}^{\infty} C_q(ns)^{1/2}\omega_{j+1,q} = C_q(ns)^{1/2}\Omega_{n+1,q},
\]
where $C_q$ is a constant only depending on $q$. By Markov’s inequality, we have
\[
\mathbb{P}(|T_n - T_{n,n}|_{s} \geq x) \leq \frac{|||T_n - T_{n,n}|_{s}|||_{q}^q}{x^q} \leq C_q(ns)^{q/2}\Omega_{n+1,q}^q. \tag{53}
\]

For each $1 \leq l \leq L$, define
\[
Y_{i,l} = \sum_{k=(i-1)\tau_l+1}^{(i\tau_l)\wedge n} (X_{k,\tau_l} - X_{k,\tau_{l-1}}), \quad \text{for } 1 \leq i \leq \lfloor n/\tau_l \rfloor;
\]
\[
R_{n,l}^{e} = \sum_{i \text{ is even}} Y_{i,l} \text{ and } R_{n,l}^{o} = \sum_{i \text{ is odd}} Y_{i,l}.
\]
Let $c = q/2 - 1 - \alpha q$; let $\lambda_1, \lambda_2, \ldots, \lambda_L$ be a positive sequence such that $\sum_{l=1}^{L} \lambda_l \leq 1$, specifically, $\lambda_l = l^{-2}/(\pi^2/3)$ if $1 \leq l \leq L/2$ and $\lambda_l = (L + 1 - l)^{-2}/(\pi^2/3)$ if $L/2 < l \leq L$. Since $Y_{i,l}$ and $Y_{i',l}$ are independent for $|i - i'| > 1$, by Lemma 5.6, for any $x > 0$,
\[
\mathbb{P}(|R_{n,l}^{e} - 2\mathbb{E}|R_{n,l}^{e}|_{s} \geq \lambda_l x) \leq \frac{C_q}{\lambda_l} \sum_{i \text{ is even}} \frac{\mathbb{E}|Y_{i,l}|_{s}^q}{(\lambda_l x)^q} + \exp\left(-\frac{3}{2} \sum_{i \text{ is even}} |\sigma_{Y_{i,l}}|_{s}^2\right),
\]
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where \( \sigma_{Y_{i,l}} = (\|Y_{i,1}\|_2, \ldots, \|Y_{i,p(l)}\|_2)^\top \). By Lemma 5.5, \( \|Y_{i,l}\|_q \leq C_q(\tau s)^{1/2}\tilde{\omega}_{l,q} \) where 
\[
\tilde{\omega}_{l,q} = \sum_{k=\tau-1}^{\tau} \omega_{k,q} \leq \tilde{\omega}_{l,q} \leq \|X_\infty\|_{q,\alpha}.
\]
For \( 1 \leq j \leq p \), by the Bulkholder inequality, 
\[
\|Y_{i,l}\|_2 \leq \sqrt{\tau} \delta_{l,2,j} \text{ where } \delta_{l,2,j} = \sum_{k=\tau-1}^{\tau} \delta_{k,2,j} \leq \tau^{-\alpha}\|X_j\|_{2,\alpha},
\]
which implies \( \|\sigma_{Y_{i,l}}\|_2 \leq \tau^{1/2}\tau^{-\alpha}\Psi_{2,\alpha} \). So we obtain
\[
\mathbb{P}(\|R_{n,l}\|_s - 2\mathbb{E}|R_{n,l}|_s \geq \lambda_l x) \leq \frac{C_1 s^{q/2}}{x^q} \cdot \frac{\tau^{q/2-1}\tilde{\omega}_{l,q}}{\lambda_l^q} + \exp\left(-\frac{C_2 (\lambda_l x)^2 \tau^{-2\alpha}}{n\Psi_{2,\alpha}^2}\right). \tag{54}
\]
By Lemma 8 of Chernozhukov et al. (2014), for \( s = \log p \vee 1 \),
\[
\mathbb{E}|R_{n,l}|_s \lesssim \sqrt{n s \tau^{-\alpha}\Psi_{2,\alpha} + n^{1/2} s \tilde{\omega}_{l,q}} \lesssim [\sqrt{n s \Psi_{2,\alpha} + n^{1/2} s \|X_\infty\|_{q,\alpha}] m^{-\alpha}\tilde{\omega}_l^{-\alpha}. \tag{55}
\]
Notice that \( \min_{l \geq 0} \lambda_l \tilde{\omega}_l^{-\alpha} > 0 \). Hence, \( \mathbb{E}|R_{n,l}|_s \lesssim \lambda_l x \) and (54) implies
\[
\mathbb{P}(\|R_{n,l}\|_s \geq \lambda_l x) \leq \frac{C_1 s^{q/2}}{x^q} \cdot \frac{\tau^{q/2-1}\tilde{\omega}_{l,q}}{\lambda_l^q} + \exp\left(-\frac{C_2 (\lambda_l x)^2 \tau^{-2\alpha}}{n\Psi_{2,\alpha}^2}\right).
\]
A similar inequality holds for \( R_{n,l}^0 \). Therefore,
\[
\mathbb{P}(\sum_{l=1}^{L} M_{n,l} \geq x) \leq \sum_{l=1}^{L} \mathbb{P}(\|M_{n,l}\|_s \geq \lambda_l x) \leq \sum_{l=1}^{L} \mathbb{P}(\|R_{n,l}\|_s \geq \lambda_l x/2) + \sum_{l=1}^{L} \mathbb{P}(\|R_{n,l}\|^0 \geq \lambda_l x/2) \leq \sum_{l=1}^{L} \frac{C_1 s^{q/2}}{x^q} \cdot \frac{\tau^{q/2-1}\tilde{\omega}_{l,q}}{\lambda_l^q} + 2 \sum_{l=1}^{L} \exp\left(-\frac{C_2 (\lambda_l x)^2 \tau^{-2\alpha}}{n\Psi_{2,\alpha}^2}\right) \tag{56}
\]
By the definition of \( \tilde{\omega}_l \) and \( \lambda_l \) and by some elementary calculation, there exists some constant \( C_6 > 1 \) such that for all \( t \geq 1 \),
\[
\sum_{l=1}^{L} \exp(-C_5 t^2 \lambda_l^2 \tilde{\omega}_l^{2\alpha}) \leq C_6 \exp(-C_5 t \mu), \tag{57}
\]
where \( \mu = \min_{l \geq 1} \lambda_l^2 \tilde{\omega}_l^{2\alpha} > 0 \). If \( c > 0 \), it can be obtained that \( \sum_{l=1}^{L} \tilde{\omega}_l^c / \lambda_l^q \leq C_7 \tilde{\omega}_l^c \leq C_7 m^c / m^c \). If \( c < 0 \), then \( \sum_{l=1}^{L} \tilde{\omega}_l^c / \lambda_l^q \leq C_8 \). Hence, combining (52), (53), (56), (57), Lemma 5.7 follows.

\[\square\]
Lemma 5.8. Assume \( \|X|_\infty\|_{q,\alpha} < \infty \), where \( q > 2 \) and \( \alpha \geq 0 \). Also assume \( \Psi_{2,\alpha} < \infty \).

(i) If \( \alpha > 1/2 - 1/q \), then for \( x \geq \sqrt{n\ell\Psi_{2,\alpha} + n^{1/q}\|X|_\infty\|_{q,\alpha}} \),

\[
\mathbb{P}(|T_n|_\infty \geq x) \leq \frac{C_{q,\alpha}n^{\ell q/2}\|X|_\infty\|_{q,\alpha}^q}{x^q} + C_{q,\alpha}\exp\left(-\frac{C_{q,\alpha}x^2}{n\Psi_{2,\alpha}^2}\right).
\]

(ii) If \( 0 < \alpha < 1/2 - 1/q \), we have the following inequality,

\[
\mathbb{P}(|T_n|_\infty \geq x) \leq \frac{C_{q,\alpha}n^{\ell q/2 - \alpha q/2}\|X|_\infty\|_{q,\alpha}^q}{x^q} + C_{q,\alpha}\exp\left(-\frac{C_{q,\alpha}x^2}{n\Psi_{2,\alpha}^2}\right).
\]

Proof of Lemma 5.8. The proof is similar to that of Lemma 5.7, and thus is omitted.

6 Proofs

6.1 Proof of Theorems 3.2 and 3.3

We shall apply the \( m \)-dependence approximation approach. For \( m \geq 0 \), define

\[
X_{i,m} = (X_{i1,m}, \ldots, X_{ip,m})^\top = \mathbb{E}(X_i|\varepsilon_{i-m}, \varepsilon_{i-m+1}, \ldots, \varepsilon_i).
\]

Write \( T_X = \sum_{i=1}^n X_i \) and \( T_{X,m} = \sum_{i=1}^n X_{i,m} \). For simplicity, suppose \( n = (M + m)w \), where \( M \gg m \) and \( M, m, w \to \infty \) (to be determined) as \( n \to \infty \). We apply the block technique and split the interval \([1, n]\) into alternating large blocks \( L_b = [(b - 1)(M + m) + 1, bM + (b - 1)m] \) and small blocks \( S_b = [bM + (b - 1)m + 1, b(M + m)] \), \( 1 \leq b \leq w \). Let

\[
Y_u = \sum_{i \in L_u} X_i, \quad Y_{b,m} = \sum_{i \in L_b} X_{i,m}, \quad T_Y = \sum_{b=1}^w Y_b, \quad T_{Y,m} = \sum_{b=1}^w Y_{b,m}.
\]

Let \( Z_b, 1 \leq b \leq w \), be i.i.d. \( N(0, MB) \) and \( Z_{b,m} \) be i.i.d. \( N(0, M\tilde{B}) \), where the covariance matrices \( B \) and \( \tilde{B} \) are respectively given by

\[
B = (b_{ij})_{i,j=1}^p = \text{Cov}(Y_b/\sqrt{M}) \quad \text{and} \quad \tilde{B} = (\tilde{b}_{ij})_{i,j=1}^p = \text{Cov}(Y_{b,m}/\sqrt{M}).
\]

Write \( T_{Z,m} = \sum_{b=1}^w Z_{b,m} \) and let \( Z \sim N(0, \Sigma) \).

Lemma 6.1. (i) Assume \( \Theta_{q,\alpha} < \infty \) for some \( q > 2 \) and \( \alpha > 0 \). Then there exists some constant \( C_{q,\alpha} \) such that for \( y > 0 \)

\[
\mathbb{P}(|T_X - T_{Y,m}|_\infty \geq y) \leq f_1^*(y) + f_2^*(y) =: f^*(y)
\]

(62)
where the constant in $\lesssim$ only depends on $q$ and $\alpha$,

$$f_1^*(y) = \begin{cases} y^{-qnm^{\alpha}/(2 - 1 - \alpha)\Theta_{q,\alpha}^q} + p \exp \left( -\frac{C_{q,\alpha}y^2}{n\Psi_{2,\alpha}^2} \right), & \alpha > 1/2 - 1/q \\ y^{-qnm^{\alpha}/2 - \alpha\Theta_{q,\alpha}^q} + p \exp \left( -\frac{C_{q,\alpha}y^2}{n\Psi_{2,\alpha}^2} \right), & \alpha < 1/2 - 1/q \end{cases}$$

(63)

and

$$f_2^*(y) = \begin{cases} y^{-qwm\Theta_{q,\alpha}^q} + p \exp \left( -\frac{C_{q,\alpha}y^2}{m\Psi_{2,\alpha}^2} \right), & \alpha > 1/2 - 1/q \\ y^{-q(wm)^{\alpha/2-\alpha}\Theta_{q,\alpha}^q} + p \exp \left( -\frac{C_{q,\alpha}y^2}{wm\Psi_{2,\alpha}^2} \right), & \alpha < 1/2 - 1/q \end{cases}$$

(64)

(ii) Assume $\Phi_{\psi,\alpha} < \infty$ for some $\nu \geq 0$ and $\alpha > 0$. Let $\beta = 2/(1 + 2\nu)$. Then there exists a constant $C_\beta > 0$ such that for $y > 0$,

$$\mathbb{P}(|T_X - T_{Y,m}|_\infty \geq y) \lesssim f_1^*(y) + f_2^*(y) =: f^*(y),$$

(65)

where the constant in $\lesssim$ only depends on $\beta$ and $\alpha$,

$$f_1^*(y) = p \exp \left\{ -C_\beta \left( \frac{y^{\alpha}}{\sqrt{n}\Phi_{\psi,\alpha}} \right)^\beta \right\}$$

and

$$f_2^*(y) = p \exp \left\{ -C_\beta \left( \frac{y^{\alpha}}{\sqrt{mw\Phi_{\psi,\alpha}}}) \right)^\beta \right\}.$$

Proof. Let $P_1 = \mathbb{P}(|T_X - T_{X,m}|_\infty \geq y/2)$ and $P_2 = \mathbb{P}(|T_{X,m} - T_{Y,m}|_\infty \geq y/2)$. Lemmas 5.1 and 5.7 imply that $P_1 \leq f_1^*(y)$. Write $T_{X,m} - T_{Y,m} = \sum_{i=1}^w \sum_{j \in S_i} X_{i,m}$. By Lemmas 5.2 and 5.8, we also have $P_2 \leq f_2^*(y)$. Hence both cases with $\alpha > 1/2 - 1/q$ and $\alpha < 1/2 - 1/q$ of Lemma 6.1(i) follow in view of $\mathbb{P}(|T_X - T_{Y,m}|_\infty \geq y) \leq P_1 + P_2$.

The exponential moment case (ii) similarly follows from $P_1 \leq f_1^*(y)$ and $P_2 \leq f_2^*(y)$. \qed

Lemma 6.2. Let $D = (d_{ij})_{i,j=1}^w$ be a diagonal matrix. Assume that there exist constants $c > 0, c_2 > c_1 > 0$ such that $c < \min_{1 \leq j \leq p} d_{jj}$ and $c_1 \leq b_{jj}/d_{jj} \leq c_2$ for all $1 \leq j \leq p$. Assume $\Psi_{q,0} < \infty$ for some $q \geq 4$. Then for all $\lambda \in (0, 1),$

$$\sup_{t \in \mathbb{R}} \mathbb{P}(|D^{-1/2}T_{X,m}/\sqrt{n}|_\infty \leq t) - \mathbb{P}(|D^{-1/2}T_{Z,m}/\sqrt{n}|_\infty \leq t) \lesssim w^{-1/8}(\Psi_{3,0}^{1/4} \vee \Psi_{4,0}^{1/2})(\log(pw/\lambda))^{7/8} + w^{-1/2}(\log(pw/\lambda))^{3/2} u_m(\lambda) + \lambda =: h(\lambda, u_m(\lambda)),$$

where the constant in $\lesssim$ depends on $c, c_1, c_2$, and $q$ and $\alpha$ for (i), and $\beta$ for (ii) below, and $u_m(\lambda) \leq u^*_m(\lambda)$ in (i), and $u_m(\lambda) \leq u^\alpha_m(\lambda)$ in (ii).
(i) Assume $\Theta_{q,\alpha} < \infty$ for some $q \geq 4$ and $\alpha > 0$, then

$$u_m^*(\lambda) = \begin{cases} \max\{\Theta_{q,\alpha}(\lambda^{-1}w)^{1/q}M^{1/q-1/2}, \Psi_{2,\alpha}\sqrt{\log(p w/\lambda)}\}, & \alpha > 1/2 - 1/q \\ \max\{\Theta_{q,\alpha}(\lambda^{-1}w)^{1/q}M^{-\alpha}, \Psi_{2,\alpha}\sqrt{\log(p w/\lambda)}\}, & \alpha < 1/2 - 1/q. \end{cases} \tag{66}$$

(ii) Assume $\Phi_{\nu,0} < \infty$ for some $\nu \geq 0$. Then

$$u_m^o(\lambda) = \max\{\Phi_{\nu,0}(\log(p w/\lambda))^{1/3}, \sqrt{\log(p w/\lambda)}\}. \tag{67}$$

Proof. For $1 < l \leq q$, define $R_l = \max_{1 \leq j \leq p} M^{-1/2}\|Y_{b,j,m}\|_{l}$. Since $X_{i,j,m} = \sum_{k=0}^{m} P_{i-k} X_{ij}$, by Burkholder’s inequality (Burkholder (1973)),

$$\|\sum_{i=1}^{M} P_{i-k} X_{ij}\|_{l}^2 \leq C_l \sum_{i=1}^{M} \|P_{i-k} X_{ij}\|_{l}^2 \leq C_l M (\theta_{k,l,j})^2,$$

then we have

$$\|\sum_{i=1}^{M} X_{i,j,m}\|_{l} \leq C_l \sum_{k=0}^{m} \|\sum_{i=1}^{M} P_{i-k} X_{ij}\|_{l} \leq C_l M^{1/2} \Delta_{0,l,j} \tag{68},$$

which implies $R_l \leq C_l \Psi_{l,0}$. For $0 < \lambda < 1$ and the diagonal matrix $D = (d_{ij})_{i,j=1}^p$, define $u_{Y,m}(\lambda)$ as the infimum over all numbers $u > 0$ such that

$$\mathbb{P}(|M^{-1/2}d_{jj}^{-1/2}Y_{b,j,m}| \leq u, 1 \leq b \leq w, 1 \leq j \leq p) \geq 1 - \lambda. \tag{69}$$

Also define $u_{Z,m}(\lambda)$ by the corresponding quantity for the analogue Gaussian case, namely with $Y_{b,m}$ replaced by $Z_{b,m}$ in the above definition. Let $u_m(\lambda) := u_{Y,m}(\lambda) \lor u_{Z,m}(\lambda)$. By Theorem 2.2 of Chernozhukov et al. (2013a), for all $\lambda \in (0, 1)$,

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P}(\|D^{-1/2}T_{Y,m}/\sqrt{n}\|_{\infty} \leq t) - \mathbb{P}(\|D^{-1/2}T_{Z,m}/\sqrt{n}\|_{\infty} \leq t) \right| \leq w^{-1/8}(R_3^{3/4} \lor R_4^{1/2})(\log(pw/\lambda))^{7/8} + w^{-1/2}(\log(pw/\lambda))^{3/2} u_m(\lambda) + \lambda,$$

Now we shall find a bound on the function $u_m(\lambda)$. (i) By Lemmas 5.2 and 5.8, we have

$$\mathbb{P}(|M^{-1/2}d_{jj}^{-1/2}Y_{b,j,m}| > u \text{ for some } b, j) \leq \mathbb{P}(|M^{-1/2}Y_{b,m}|_{\infty} > c^{1/2} u) \leq \begin{cases} C_{q,\alpha} u^{-q} w M^{1-q/2} \Theta_{q,\alpha}^q + C_{q,\alpha}pw \exp\left(-\frac{C_{q,\alpha} u^2}{\Psi_{2,\alpha}^2}\right), & \alpha > 1/2 - 1/q \\ C_{q,\alpha} u^{-q} w M^{-\alpha} \Theta_{q,\alpha}^q + C_{q,\alpha}pw \exp\left(-\frac{C_{q,\alpha} u^2}{\Psi_{2,\alpha}^2}\right), & \alpha < 1/2 - 1/q. \end{cases}$$
This implies $u_{Y,m}(\lambda) \leq C_{q,\alpha} \max\{\Theta_{q,\alpha}(\lambda^{-1}w)^{1/q}M^{1/q-1/2}, \Psi_{2,\alpha}\sqrt{\log(pw/\lambda)}\}$ if $\alpha > 1/2 - 1/q$ and $u_{Y,m}(\lambda) \leq C_{q,\alpha} \max\{\Theta_{q,\alpha}(\lambda^{-1}w)^{1/q}M^{-\alpha}, \Psi_{2,\alpha}\sqrt{\log(pw/\lambda)}\}$ if $\alpha < 1/2 - 1/q$. For $u_{Z,m}(\lambda)$, since $M^{-1/2}Z_{bj,m} \sim N(0, b_{jj})$, we have $E(\exp\{M^{-1}Z_{bj,m}^2/(4\tilde{b}_{jj})\}) \leq C$. Hence

$$\mathbb{P}(|M^{-1/2}d_{jj}^{-1/2}Z_{bj,m}| > u \text{ for some } b, j) \leq \sum_{b=1}^{w} \sum_{j=1}^{p} \mathbb{P}(|M^{-1/2}Z_{bj,m}| > d_{jj}^{1/2} u) \leq C_{pw} \exp(-d_{jj}u^2/(4\tilde{b}_{jj})).$$

With the assumption $c_1 \leq \tilde{b}_{jj}/d_{jj} \leq c_2$, $u_{Z,m}(\lambda) \leq C\sqrt{\log(pw/\lambda)}$.

(ii) By Bonferroni inequality and Lemma 5.4,

$$\mathbb{P}(|M^{-1/2}d_{jj}^{-1/2}Y_{bj,m}| > u \text{ for some } b, j) \leq C_{\beta}pw \exp\left\{-C_{\beta} \frac{u^{\beta}}{\Phi_{\psi,0}}\right\},$$

where $\beta = 2/(1 + 2\nu)$ and $C_{\beta}$ is a constant that depends on $\beta$ only. Combining (69) and (70), it follows that $u_m(\lambda) \leq C_{\beta} \max\{\Phi_{\psi,0}(\log(pw/\lambda))^{1/\beta}, \sqrt{\log(pw/\lambda)}\}$. 

Now we consider the comparison between $Z$ and $T_{Z,m}$. Let $\pi(x) = x^{1/3}(1 \vee \log(p/x))^{2/3}$ for $x > 0$.

**Lemma 6.3.** Assume $\Psi_{2,\alpha} < \infty$ for some $\alpha > 0$. Let $D = (d_{ij})_{i,j=1}^{p}$ be a diagonal matrix such that there exist some constants $0 < C_1 < C_2$ such that $C_1 \leq \sigma_{jj}/d_{jj} \leq C_2$ for all $1 \leq j \leq p$. Then we have

$$\sup_{t \in \mathbb{R}} |\mathbb{P}(|D^{-1/2}T_{Z,m}/\sqrt{n}|_\infty \leq t) - \mathbb{P}(|D^{-1/2}Z|_\infty \leq t)| \lesssim \pi(\max_{1 \leq j \leq p}d_{jj}^{-1}\Psi_{2,\alpha}\Psi_{2,0}(m^{-\alpha} + v(M)) + wm/n),$$

where $v(M)$ is the same as defined in Corollary 4.3.

**Proof.** By the definition of $T_{Z,m}$ and $Z$ and (61),

$$\Sigma_{Z,m} := \text{Cov}(D^{-1/2}T_{Z,m}/\sqrt{n}) = \frac{Mw}{n} D^{-1/2} \tilde{B} D^{-1/2},$$

$$\Sigma_{Z} := \text{Cov}(D^{-1/2}Z) = D^{-1/2} \Sigma D^{-1/2}.$$ 

Let $S_{Mj} = \sum_{i=1}^{M} X_{ij}$ and $S_{Mj,m} = \sum_{i=1}^{M} X_{ij,m}$. By the moment inequality in Wu (2005),

$$\|S_{Mj}\|_2 \leq M^{1/2}\Delta_{0,2,j}, \|S_{Mj,m}\|_2 \leq M^{1/2}\Delta_{0,2,j} \text{ and } \|S_{Mj} - S_{Mj,m}\|_2 \leq M^{1/2}\Delta_{m+1,2,j}.$$ 

Note
that \( b_{jk} = M^{-1}\mathbb{E}(S_{Mj}S_{Mk}) \) and \( \bar{b}_{jk} = M^{-1}\mathbb{E}(S_{Mj,m}S_{Mk,m}) \). Then

\[
|b_{jk} - \bar{b}_{jk}| = \frac{1}{M}|\mathbb{E}(S_{Mj}S_{Mk} - S_{Mj,m}S_{Mk,m})|
\leq \frac{1}{M} (\|S_{Mj}\|_2 \cdot \|S_{Mk} - S_{Mk,m}\|_2 + \|S_{Mk,m}\|_2 \cdot \|S_{Mj} - S_{Mj,m}\|_2)
\leq 2\Psi_{2,0} \Psi_{2,0} m^{-\alpha}.
\]

Recall that \( \sigma_{jk} = \sum_{l=-\infty}^{\infty} \gamma_{jk}(l) \) and

\[
b_{jk} = M^{-1}\mathbb{E}(S_{Mj}S_{Mk}) = M^{-1} \sum_{l=-M}^{M} (M - |l|) \gamma_{jk}(l).
\]

It follows that

\[
\sigma_{jk} - b_{jk} = \sum_{|l| > M} \gamma_{jk}(l) + M^{-1} \sum_{l=-M}^{M} |l| \gamma_{jk}(l).
\]

By \( X_{ij} = \sum_{h=0}^{\infty} \mathcal{P}^{-h}X_{ij} \), we have

\[
|\gamma_{jk}(l)| = \left| \sum_{h=0}^{\infty} \mathbb{E}[(\mathcal{P}^{-h}X_{0j})(\mathcal{P}^{-h}X_{lk})] \right| \leq \sum_{h=0}^{\infty} \left| \mathbb{E}[(\mathcal{P}^{-h}X_{0j})(\mathcal{P}^{-h}X_{lk})] \right| \leq \sum_{h=0}^{\infty} \delta_{h,2,j} \delta_{h+l,2,k}.
\]

Hence, it can be obtained that

\[
\left| \sum_{|l| > M} \gamma_{jk}(l) \right| \leq 2 \sum_{l=M+1}^{\infty} |\gamma_{jk}(l)| \leq 2 \sum_{l=M+1}^{\infty} \sum_{h=0}^{\infty} \delta_{h,2,j} \delta_{h+l,2,k} \leq 2\Delta_{0,2,j} \Delta_{M+1,2,k},
\]

and

\[
\left| \frac{1}{M} \sum_{l=-M}^{M} |l| \gamma_{jk}(l) \right| \leq \frac{2}{M} \sum_{l=1}^{M} \sum_{i=k}^{M} \sum_{h=0}^{\infty} \delta_{h,2,j} \delta_{h+i,2,k} \leq \frac{2}{M} \Delta_{0,2,j} \sum_{l=1}^{M} \Delta_{l,2,k}.
\]

Since \( \Delta_{0,2,j} \leq \Psi_{2,0} \) and \( \Delta_{m,2,j} \leq \Psi_{2,0} m^{-\alpha} \), \( \max_{1 \leq j,k \leq p} |b_{jk} - \sigma_{jk}| \leq \Psi_{2,0} v(M) \). Hence,

\[
|\Sigma_{Z,m} - \Sigma_{Z}\|_\infty \leq \max_{1 \leq j,k \leq p} d_{jj}^{-1}(|\tilde{B} - B|_\infty + |B - \Sigma|_\infty) + (1 - Mw/n)|D^{-1/2}\Sigma D^{-1/2}|_\infty \leq \max_{1 \leq j,k \leq p} d_{jj}^{-1} \Psi_{2,0} \Psi_{2,0} (m^{-\alpha} + v(M)) + C_2wm/n.
\]

By Theorem 2 of Chernozhukov et al. (2014), the result follows. \( \square \)
Theorem 6.4. Let $\Sigma_0$ be the diagonal matrix of the long run covariance matrix $\Sigma$ and $D_0 = \Sigma_0^{1/2}$. Let Assumption 3.1 be satisfied. (i) Assume that $\Theta_{q,\alpha} < \infty$ holds with some $q \geq 4$ and $\alpha > 0$. Then for every $\lambda \in (0,1)$ and $\eta > 0$,

$$
\rho_n := \sup_{t \in \mathbb{R}} |\mathbb{P}(|D_0^{-1}X_1/\sqrt{n}|_\infty \leq t) - \mathbb{P}(|D_0^{-1}Z_1|_\infty \leq t)| \\
\lesssim f^*(\sqrt{n}\eta) + \eta\sqrt{\log p} + h(\lambda, u_m^*(\lambda)) + \pi(\Psi_{2,0}(m^{-\alpha} + v(M)) + \omega n). \quad (71)
$$

(ii) Assume $\Phi_{\psi,\alpha} < \infty$ for some $\psi \geq 0$ and $\alpha > 0$. Then for every $\lambda \in (0,1)$ and $\eta > 0$,

$$
\rho_n := \sup_{t \in \mathbb{R}} |\mathbb{P}(|D_0^{-1}X_1/\sqrt{n}|_\infty \leq t) - \mathbb{P}(|D_0^{-1}Z_1|_\infty \leq t)| \\
\lesssim f^*(\sqrt{n}\eta) + \eta\sqrt{\log p} + h(\lambda, u_m^*(\lambda)) + \pi(\Psi_{2,0}(m^{-\alpha} + v(M)) + \omega n). \quad (72)
$$

Proof. (i) By Lemma 6.2 (i) and Lemma 6.3, we have for every $\lambda \in (0,1)$,

$$
\sup_{t \in \mathbb{R}} |\mathbb{P}(|D_0^{-1}X_1/\sqrt{n}|_\infty \leq t) - \mathbb{P}(|D_0^{-1}Z_1|_\infty \leq t)| \\
\lesssim h(\lambda, u_m^*(\lambda)) + \pi(\Psi_{2,0}(m^{-\alpha} + v(M)) + \omega n). \quad (73)
$$

Observe that each component of the Gaussian vector $D_0^{-1}Z$ has variance 1. By Theorem 3 of Chernozhukov et al. (2014), for every $\eta > 0$,

$$
\sup_{t \in \mathbb{R}} \mathbb{P}(|D_0^{-1}Z_1|_\infty - t| \leq \eta) \lesssim \eta \sqrt{\log p}. \quad (74)
$$

By the triangle inequality, for every $\eta > 0$, we have

$$
\sup_{t \in \mathbb{R}} |\mathbb{P}(|D_0^{-1}X_1/\sqrt{n}|_\infty > t) - \mathbb{P}(|D_0^{-1}Y_1/\sqrt{n}|_\infty > t)| \\
\leq \mathbb{P}(|D_0^{-1}(X_1 - Y_1)|/\sqrt{n}|_\infty > \eta) + \sup_{t \in \mathbb{R}} \mathbb{P}(|D_0^{-1}Y_1/\sqrt{n}|_\infty - t| \leq \eta),
$$

which implies Theorem 6.4 (i) in view of Lemma 6.1 (i), (73) and (74).

(ii) Inequality (72) can be obtained by replacing $f^*$ and $u_m^*$ with $f^*$ and $u_m^*$ in the above proof. 

6.2 Proof of Theorem 3.2

Proof. Recall (62) for $f^*(\cdot)$. By Theorem 6.4, for $\alpha > 1/2 - 1/q$, to have (7), we need

$$
\pi(\Psi_{2,0}(m^{-\alpha} + v(M)) + \omega n) \to 0 \quad (75)
$$
and for some $\eta > 0$ and $\lambda \in (0, 1)$,

$$f^*(\sqrt{n}\eta) + \eta\sqrt{\log p} \to 0, \quad (76)$$

$$h(\lambda, u^*_m(\lambda)) \to 0. \quad (77)$$

Firstly, (75) requires $m \gg L_2$, $wm \ll n/(\log p)^{-2}$, $w \ll n/(\log p)^{-2}(\Psi_{2,\alpha}\Psi_{2,0})^{-1}$ if $\alpha > 1$ and $w \ll n/L_2$ if $0 < \alpha < 1$. Moreover, (76) requires $m \gg (L_1, (\Psi_{2,\alpha}\log p)^{1/\alpha})$ and $wm \ll \min(N_1, N_2)$. And (77) needs (8) and $w \gg \max(W_1, W_2)$. We also need $M \approx n/w \gg m$. Notice that $(\Psi_{2,\alpha}\log p)^{1/\alpha} \approx L_2$, $N_2 \leq n/(\log p)^{-2}$ and $N_2 \leq n/(\log p)^{-2}(\Psi_{2,\alpha}\Psi_{2,0})^{-1}$. If

$$\max(L_1, L_2) \max(W_1, W_2) = o(1) \min(n, N_1, N_2), \quad (78)$$

then we can always choose $m$ and $w$ such that (7) holds. Observe that $N_2 \leq n$, then (78) is reduced to (9).

For $0 < \alpha < 1/2 - 1/q$, the function $f^*$ in (76) is replaced by $f^0$ (cf. (65)), which implies $\Theta_{q,\alpha}(\log p)^{1/2} = o(n^\alpha)$, $m \gg (\Psi_{2,\alpha}\log p)^{1/\alpha}$ and $wm \ll \min(N_2, N_3)$. And $u^*_m$ in (77) is replaced by $u^*_m$, implying $w \gg \max(W_1, W_2, W_3)$. By the similar argument, if (10) is further assumed, then (7) also holds for the case $0 < \alpha < 1/2 - 1/q$.

Remark 4. In the proof of Theorem 3.2, we exclude the case $\alpha = 1$ when $\alpha > 1/2 - 1/q$. If $\alpha = 1$, we need to impose the additional assumption

$$\max(W_1, W_2) = o(n/(L_2 \log n)) \quad (79)$$

to ensure (75). The above condition is very mild since (9) implies $\max(W_1, W_2) = o(n/L_2)$. If $\log n \lesssim (\log p)^2\Psi_{2,\alpha}^2$, which trivially holds in the high-dimensional case $p \approx n^\kappa$ with some $\kappa > 0$, we have $N_2 = O(n/\log n)$ and hence (9) implies (79). Similarly, it is further assumed $\max(W_1, W_4) = o(n/(L_2 \log n))$ in Theorem 3.3 if $\alpha = 1$.

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