Exponential iteration and Borel sets

David S. Lipham

Abstract. We determine the exact Borel class of escaping sets in the exponential family $\exp(z) + a$. We also prove that the sets of non-escaping Julia points for many of these functions are topologically equivalent.

1. Introduction

The escaping set
\[ I(f) = \{ z \in \mathbb{C} : f^n(z) \to \infty \} \]
is one of the most studied objects in complex dynamics [6, 7, 14, 12]. For any entire function $f$, it is easily seen to be an $F_{\sigma \delta}$-subset of the complex plane $\mathbb{C}$. However, it was recently shown by Lasse Rempe [13] that if $f$ is transcendental (with an essential singularity at $\infty$), then $I(f)$ is not $F_\sigma$. Here we will strengthen this result for functions in the exponential class
\[ f_a(z) = e^z + a ; \quad a \in \mathbb{C}. \]

The main result of the paper is the following.

Theorem 1. $I(f_a)$ is not $G_{\delta \sigma}$ for any $a \in \mathbb{C}$.

In other words, $I(f_a)$ cannot be written as a countable union of $G_{\delta}$-subsets of $\mathbb{C}$. Theorem 1 is new even for the plain exponential $f_0(z) = e^z$, and completes the Borel classification of $I(f_a)$.

| \( G_\delta \) [9, Theorem 4.1] | \( G_{\delta \sigma} \) (Theorem 1) | \( G_{\delta \sigma \delta} \) | \ldots |
| \( F_\sigma \) [13, Theorem 1.2] | \( F_{\sigma \delta} \) [13, Section 1] | \( F_{\sigma \delta \sigma} \) | \ldots |

Figure 1. Borel classes of $I(f_a)$ (in green only).

A simple consequence of Theorem 1 is that given any $G_\delta$-set of escaping points, there exists an escaping point whose orbit does not enter the set.

Corollary 2. If $X$ is a $G_\delta$-subset of $\mathbb{C}$ that is contained in $I(f_a)$, then the pre-images $f_a^{-n}[X]$ do not cover $I(f_a)$. Thus there exists $z \in I(f_a)$ such that $f_a^n(z) \notin X$ for all $n = 0, 1, 2, 3, \ldots$.

Next we focus on the $f_a$’s which have attracting or parabolic cycles. The parameters $a$ associated with this class form a large and conjecturally dense subset of $\mathbb{C}$. The conjugacy in [12] established that all escaping sets are mutually homeomorphic in

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this context. Using a localized version of Theorem 1, we will prove the following complementary result wherein \(J(f_a)\) denotes the Julia set of \(f_a\).

**Theorem 3.** If \(f_a\) and \(f_b\) have attracting or parabolic cycles, then \(J(f_a) \setminus I(f_a)\) and \(J(f_b) \setminus I(f_b)\) are topologically equivalent (homeomorphic).

The homeomorphism in Theorem 3 is not necessarily induced by a homeomorphism of the escaping sets, as \(J(f_a)\) and \(J(f_b)\) are often non-homeomorphic. The result also cannot be extended to all \(a \in \mathbb{C}\); the spaces in Theorem 3 are totally disconnected [10, Corollary 10], but there are other parameters (e.g., postsingularly finite) for which \(J(f_a) \setminus I(f_a)\) contains unbounded connected sets [3, Section 2].

**Ideas behind the proofs.** We will prove Theorem 1 by constructing a stratification, or ‘tree’, of first category \(G_\delta\)-subspaces of \(I(f_{-1})\), such that every infinite branch of the tree has an accumulation point in \(I(f_{-1})\). This idea comes from a classical proof that the infinite power of the rationals \(\mathbb{Q}\) is not \(G_\delta\). The stratification sets in that proof are of the form \(\{q_0\} \times \ldots \times \{q_n\} \times \mathbb{Q} \times \mathbb{Q} \times \ldots\), while ours will be defined with rates of escape in mind. Results from [12] will allow us to work inside a relatively simple topological model of \(J(f_{-1})\), and to generalize from \(I(f_{-1})\) to \(I(f_a)\). For attracting and parabolic parameters we will in fact see that \(I(f_a)\) is nowhere \(G_\delta\), and \(J(f_a) \setminus I(f_a)\) nowhere \(F_\omega\). This latter is one of the conditions in a uniqueness theorem of van Engelen [16]. The other conditions involve Baire category and topological dimension, both of which are known for \(J(f_a) \setminus I(f_a)\) [4, 10]. In this way, Theorem 3 will follow from van Engelen’s characterization.

2. Preliminaries

2.1. Dynamics of entire functions. For each positive integer \(n\), the \(n\)-fold composition of an entire function \(f\) is denoted \(f^n\). The orbit of a point \(z \in \mathbb{C}\) is the sequence of iterates \(\{f^n(z)\}_{n=0}^\infty\).

A point \(z\) belongs to the escaping set \(I(f)\) if \(f^n(z) \to \infty\), that is, if the orbit of \(z\) converges to the point at infinity on the Riemann sphere.

The Julia set \(J(f)\) is the set of non-normality for the family of iterates of \(f\). For \(f_a\) this roughly means that \(z \in J(f_a)\) if every neighborhood of \(z\) contains points whose orbits are very different from one-another; for instance, points with periodic orbits and points whose orbits go to \(\infty\). In context of exponential functions it is well-known that the Julia set is equal to the closure of the escaping set; \(J(f_a) = \overline{I(f_a)}\) [7, Corollary 1].

2.2. Attracting and parabolic parameters. The function \(f_a\) has an attracting (or parabolic) cycle if there exists \(z \in \mathbb{C}\) such that the orbit \(\{f^n_a(z)\}_{n=0}^\infty\) is periodic and \(|f'_a(z)| < 1\) (or \(|f'_a(z)| = 1\)). The number \(a\) is then called an attracting (parabolic) parameter. For example, \(a = -2\) is attracting and \(a = -1\) is parabolic.

The Julia set of \(f_{-1}\) is a Cantor bouquet of uncountably many disjoint rays (homeomorphic images of \([0, \infty)\)); see [5, 1]. Each ray belongs to \(I(f_{-1})\) with the possible exception of its finite endpoint [14, Theorem 4.2].

More generally, if \(a\) is any attracting or parabolic parameter then \(I(f_a)\) is a disjoint union of rays and curves (homeomorphic images of \((0, \infty)\)). Different curves may terminate at the same point of \(\mathbb{C}\). In the event that they do, the point at which

\(1\)In many of the papers cited here, the functions \(f_a\) with \(a \in (-\infty, -1]\) are represented in the form \(\lambda e^z\) with \(\lambda \in (0, \frac{1}{e}]\). Note that \(f_a\) is conjugate to \(e^a e^z\) via the translation \(z \mapsto z - a\).
they terminate is non-escaping and thus belongs to $J(f_a) \setminus I(f_a)$; see Figure 2. This pinched Cantor bouquet phenomenon was proved explicitly for attracting parameters in [12], and is explained further in [10, Section 3]. Generalizations to larger classes of transcendental entire functions appear in [2].

2.3. Borel sets. All spaces under consideration are assumed to be separable and metrizable. A subset $X$ of a space $Y$ is said to be an

- $F_{\sigma}$-subset of $Y$ if $X$ is a countable union of closed subsets of $Y$
- $G_{\delta}$-subset of $Y$ if $X$ is a countable intersection of open subsets of $Y$
- $F_{\sigma\delta}$-subset of $Y$ if $X$ is a countable intersection of $F_{\sigma}$-subsets of $Y$
- $G_{\delta\sigma}$-subset of $Y$ if $X$ is a countable union of $G_{\delta}$-subsets of $Y$.

Recall that in metric spaces, $G_{\delta}$-subsets include all closed subsets.

If $Y$ is locally compact and $X$ is an $F_{\sigma}$-subset of $Y$, then $X$ is $\sigma$-compact. And then $X$ is an $F_{\sigma}$-subset of every space into which it is embedded. In this event $X$ is called an absolute $F_{\sigma}$-set.

If $Y$ is completely metrizable and $X$ is a $G_{\delta}$ (respectively, $F_{\sigma\delta}$ or $G_{\delta\sigma}$) subset of $Y$, then $X$ is a $G_{\delta}$ ($F_{\sigma\delta}$ or $G_{\delta\sigma}$) subset of every space $Z$ into which it is embedded. Now $X$ is called an absolute $G_{\delta}$-set (and similarly for $F_{\sigma\delta}$ and $G_{\delta\sigma}$).

The absolute Borel properties described above are intrinsic to the space $X$ and are preserved by homeomorphisms. Indeed, $X$ is an absolute :

- $F_{\sigma}$-set $\iff$ $X$ is $\sigma$-compact
- $G_{\delta}$-set $\iff$ $X$ is completely metrizable [8, Theorem 3.11]
- $F_{\sigma\delta}$-set $\iff$ $X$ has a Sierpiński stratification [15]
- $G_{\delta\sigma}$-set $\iff$ $X$ is $\sigma$-complete (i.e. $X$ is a countable union of completely metrizable subspaces).

We say that a space $X$ is nowhere $G_{\delta\sigma}$ if every absolute $G_{\delta\sigma}$-subset of $X$ has empty interior; likewise for $F_{\sigma\delta}$.

\footnote{This is a consequence of Lavrentiev’s theorem [8, Theorem 3.9] and the fact that $Z$ has a metric completion. See also [11, p.432 Corollary 1 and Remark 1]}

Figure 2. Disjoint curves in $I(f_{2+\pi^2})$ which terminate at non-escaping points. See also [12, Figure 1].
2.4. Baire category. A Borel set $X$ is first category if $X$ can be written as a countable union of nowhere dense subsets, and Baire if $X$ contains a dense completely metrizable subspace [16, Theorem 1.12.2].

The first category property is inherited by open subspaces, and first category spaces are not Baire [8, Theorem 8.4].

2.5. Lower semi-continuity. A function $\varphi : X \to [0, \infty)$ is lower semi-continuous if $\varphi(x_n) \to \varphi(x)$ whenever $x_n \to x$ and $\varphi(x_n) \leq \varphi(x)$ for all $n$. This is equivalent to $\varphi^{-1}(r, \infty)$ being open in $X$ for every $r \geq 0$.

3. A topological model of $\exp(-1)$

We will prove Theorem 1 first for the function $f = f_{-1}$ using a dynamical system $(J(\mathcal{F}), \mathcal{F})$ which models $(J(f), f \upharpoonright J(f))$. The model was introduced in [12] and was used extensively in [3].

3.1. The model. Let $\mathbb{Z}^\omega$ denote the space of integer sequences $\underline{s} = s_0s_1s_2\ldots$.

Define $\mathcal{F} : [0, \infty) \times \mathbb{Z}^\omega \to \mathbb{R} \times \mathbb{Z}^\omega$ by

$$\langle t, \underline{s} \rangle \mapsto (F(t) - |s_1|, \sigma(\underline{s})),$$

where $F(t) = e^t - 1$ and $\sigma(s_0s_1s_2\ldots) = s_1s_2s_3\ldots$ is the shift on $\mathbb{Z}^\omega$. For each $x = \langle t, \underline{s} \rangle \in [0, \infty) \times \mathbb{Z}^\omega$ put $T(x) = t$ and $\omega(x) = \underline{s}$. Let

$$J(\mathcal{F}) = \{ x \in [0, \infty) \times \mathbb{Z}^\omega : T(F^n(x)) \geq 0 \text{ for all } n \geq 0 \};$$

and

$$I(\mathcal{F}) = \{ x \in J(\mathcal{F}) : T(F^n(x)) \to \infty \}.$$

Remark 1. $J(\mathcal{F})$ is closed in $[0, \infty) \times \mathbb{Z}^\omega$ by continuity of each $T \circ F^n$.

Remark 2. In [12, Section 9] it was shown that $\mathcal{F} \upharpoonright J(\mathcal{F})$ is topologically conjugate to $f \upharpoonright J(f)$, meaning that there is a homeomorphism $\varphi : J(\mathcal{F}) \to J(f)$ such that $\varphi \circ \mathcal{F} \upharpoonright J(\mathcal{F}) = f \circ \varphi$. Technically, the conjugacy in [12] was constructed only in the case that $f$ has an attracting cycle, whereas the fixed point of $f_{-1}$ is parabolic. However, $f_{-1}$ is conjugate on its Julia set to other exponentials with attracting fixed points, such as $f_{-2}$, due to more recent results in [2]. See [2, Example 11.1].

3.2.Endpoints of $J(\mathcal{F})$. For each $\underline{s} \in \mathbb{Z}^\omega$ put

$$t_{\underline{s}} = \min \{ t \geq 0 : \langle t, \underline{s} \rangle \in J(\mathcal{F}) \},$$

or $t_{\underline{s}} = \infty$ if there is no such $t$. Observe that

$$J(\mathcal{F}) = \bigcup_{\underline{s} \in \mathbb{Z}^\omega} [t_{\underline{s}}, \infty) \times \{ \underline{s} \},$$

and thus $E(\mathcal{F}) = \{ \langle t_{\underline{s}}, \underline{s} \rangle : t_{\underline{s}} < \infty \}$ consists of the (finite) endpoints of $J(\mathcal{F})$.

Remark 3. In light of Remark 1 and the representation of $J(\mathcal{F})$ above,

$$J(\mathcal{F}) \setminus E(\mathcal{F}) = \bigcup_{n=1}^{\infty} J(\mathcal{F}) + \langle 1/n, 000\ldots \rangle$$

is an $F_\sigma$-subset of $J(\mathcal{F})$. Hence $E(\mathcal{F})$ is a $G_\delta$-subset of $J(\mathcal{F})$ (and of the completely metrizable space $[0, \infty) \times \mathbb{Z}^\omega$). Therefore $E(\mathcal{F})$ is completely metrizable.

Remark 4. The map $\underline{s} \mapsto t_{\underline{s}}$ is lower semi-continuous [12, Observation 3.1].
Remark 5. The set of endpoints is completely invariant under the mapping \( F \). Moreover, for every \( n \geq 0 \) and \( (t_\#, g) \in E(F) \) we have

\[
F^n((t_\#, g)) = (t_{\sigma^n(g)}, \sigma^n(g)) \in E(F).
\]

In particular, if \( x \in E(F) \) then \( T(F^n(x)) = t_{\sigma^n(g(x))} \).

Remark 6. By Montel’s theorem and the conjugacy between \( F \upharpoonright J(F) \) and \( f \upharpoonright J(f) \), the completely \( F \)-invariant sets \( E(F) \), \( I(F) \), and \( E(F) \cap I(F) \) are each dense in \( J(F) \).

3.3. Estimates of \( t^*_\# \). Let \( F^{-1} \) denote the inverse of \( F \). So \( F^{-1}(t) = \ln(t + 1) \). The \( k \)-fold composition of \( F^{-1} \) will be denoted \( F^{-k} \). The following can be verified with elementary calculus.

Proposition 1. \( F^{-k}(t - 1) > F^{-k}(t) - 1 \) for all \( k \geq 1 \) and \( t \in [1, \infty) \).

Now for each \( g \in \mathbb{Z}^\omega \) define

\[
t^*_g = \sup_{k \geq 1} F^{-k}|s_k|.
\]

The next proposition comes from [3, Lemma 3.8] and [3, Observation 3.7].

Proposition 2.

(a) \( t^*_g \leq t_\# \leq t^*_g + 1 \),

(b) \( (t_\#, g) \in E(F) \cap I(F) \) if and only if \( t^*_g < \infty \) and \( t^*_{\sigma^n(g)} \to \infty \), and

(c) if \( |s_0| \leq |s_n| \) for all \( n < \omega \), then \( t^*_{\sigma^0} \leq t^*_\# \) (and likewise for \( t^* \)).

We will also need the following.

Proposition 3. For any positive real number \( R > 0 \) and integer \( n \geq 0 \),

\[
\{ x \in J(F) : t^*_{\sigma^n(g(x))} > R \}
\]

is open in \( J(F) \). Further, if \( y \in E(F) \cap \{ x \in E(F) : t^*_{\sigma^n(g(x))} > R \} \) then

(a) \( t_{\sigma^n(g(y))} \geq R \), and

(b) \( t^*_{\sigma^n(g(y))} \geq R - 1 \).

Proof. Let \( S = \{ g \in \mathbb{Z}^\omega : t^*_g < \infty \} \). Note that \( S \) is equal to \( \{ g \in \mathbb{Z}^\omega : t^*_g < \infty \} \) by Proposition 2(a). The mapping \( g \mapsto t^*_g \) is easily seen to be lower semi-continuous, so \( \mathbb{U} = \{ g \in S : t^*_g > R \} \) is open in \( S \). Thus \( \sigma^{-n}\mathbb{U} = \{ g \in S : \sigma^n(g) \in \mathbb{U} \} \) is open in \( S \).

By continuity of the projection of \( J(F) \) onto \( S \), we conclude that

\[
\{ x \in J(F) : g(x) \in \sigma^{-n}\mathbb{U} \} = \{ x \in J(F) : t^*_{\sigma^n(g(x))} > R \}
\]

is open in \( J(F) \).

By Remark 5, Proposition 2(a) and continuity of \( T \circ F^n \) we have

\[
E(F) \cap \{ x \in E(F) : t^*_{\sigma^n(g(x))} > R \} \subset E(F) \cap \{ x \in E(F) : t_{\sigma^n(g(x))} > R \} \]

\[
= E(F) \cap \{ x \in E(F) : T(F^n(x)) > R \} \]

\[
\subset \{ x \in E(F) : T(F^n(x)) \geq R \} \]

\[
= \{ x \in E(F) : t_{\sigma^n(g(x))} \geq R \} \]

\[
\subset \{ x \in E(F) : t^*_{\sigma^n(g(x))} + 1 \geq R \},
\]

which proves both (a) and (b).
4. Stratifying the escaping endpoints

Let \( \mathbb{N}^{<\omega} \) denote the set of all finite functions \( \alpha : n \to \mathbb{N} \), where \( n < \omega \) and
\[
\text{dom}(\alpha) = n = \{0, \ldots, n-1\}.
\]
We sometimes represent \( \alpha \) as an \( n \)-tuple of integers \( \langle N_0, N_1, \ldots, N_{\text{dom}(\alpha)-1} \rangle \) where \( N_i = \alpha(i) \) for each \( i < \text{dom}(\alpha) \). Given \( \alpha \in \mathbb{N}^{<\omega} \) and \( N \in \mathbb{N} \), the notation \( \alpha \upharpoonright N \) will stand for the extension of \( \alpha \) that has representation \( \langle N_0, N_1, \ldots, N_{\text{dom}(\alpha)-1}, N \rangle \). For example, if \( \alpha = \langle 1, 2, 5 \rangle \) then \( \alpha \upharpoonright 8 = \langle 1, 2, 5, 8 \rangle \).

We recursively define a system \( \langle X_\alpha \rangle \) of subsets of
\[
\tilde{E}(\mathcal{F}) = E(\mathcal{F}) \cap I(\mathcal{F}) = \{ x \in E(\mathcal{F}) : t_{\sigma^a(g(x))} \to \infty \}.
\]
for increasing functions \( \alpha \in \mathbb{N}^{<\omega} \). To begin, let
\[
X_\emptyset = \tilde{E}(\mathcal{F}).
\]
For each \( N \in \mathbb{N} \) define
\[
X_\langle N \rangle = \{ x \in X_\emptyset : t_{\sigma^a(g(x))} > 2 \text{ for all } n \geq N \}.
\]
If \( \alpha = \langle N_0, N_1, \ldots, N_{\text{dom}(\alpha)-1} \rangle \in \mathbb{N}^{<\omega} \) is increasing, \( X_\alpha \) has been defined, and \( N > N_{\text{dom}(\alpha)-1} \), then define
\[
X_{\alpha \upharpoonright N} = \{ x \in X_\alpha : t_{\sigma^a(g(x))} > 3 \text{dom}(\alpha) + 2 \text{ for all } n \geq N \}.
\]

**Observation 1.** Every \( X_\alpha \) is a \( G_\delta \)-subset of \( \tilde{E}(\mathcal{F}) \).

**Proof.** This is an easy consequence of the first part of Proposition 3. \( \square \)

**Observation 2.**
\[
X_\alpha = \bigcup_{N=1}^{\infty} X_{\alpha \upharpoonright N}.
\]

**Proof.** The inclusion \( (\supset) \) is trivial, and \( (\subset) \) holds by Proposition 2(b). \( \square \)

We will now show that every \( X_\alpha \) is first category, as witnessed by the extensions \( X_{\alpha \upharpoonright N} \). In the proof below, the observation
\[
t_{\sigma^a(g(x))} = \sup_{k \geq 1} F^{-k} |s_{n+k}|
\]
will be helpful.

**Theorem 4.** \( X_{\alpha \upharpoonright N} \) is nowhere dense in \( X_\alpha \).

**Proof.** Let \( \langle t_2, \tilde{s} \rangle \in X_{\alpha \upharpoonright N} \). We will show that there is a sequence of points in \( X_\alpha \setminus \overline{X_{\alpha \upharpoonright N}} \) converging to \( \langle t_2, \tilde{s} \rangle \). To that end, for each \( m < \omega \) define \( \tilde{s}^m \) coordinate-wise by setting
\[
\tilde{s}^m_n = \begin{cases} 
  s_n & \text{if } n \leq m \\
  \min\{|s_n|, |F^{n-m}(3 \text{dom}(\alpha))|\} & \text{if } n > m.
\end{cases}
\]
Clearly \( \tilde{s}^m \to \tilde{s} \) and \( |s^m_n| \leq |s_n| \) for every \( n \). So \( t_{\tilde{s}^m} \leq t_\tilde{s} \) by Proposition 2(c). From lower semi-continuity of \( \tilde{s} \mapsto t_\tilde{s} \) we get \( \langle t_\tilde{s^m}, \tilde{s}^m \rangle \to \langle t_\tilde{s}, \tilde{s} \rangle \).

We will now prove that a subsequence of \( \langle t_\tilde{s^m}, \tilde{s}^m \rangle \) is contained in \( X_\alpha \setminus \overline{X_{\alpha \upharpoonright N}} \). This will be established by showing:

For any integer \( M \) there exists \( m \geq M \) such that \( \langle t_\tilde{s^m}, \tilde{s}^m \rangle \in X_\alpha \setminus \overline{X_{\alpha \upharpoonright N}} \).
For each \( i < \text{dom}(\alpha) \) put \( N_i = \alpha(i) \), and let \( N_{\text{dom}(\alpha)} = N \). For each \( n \in [N_i, N_{i+1}) \) there exists \( k_n \geq 1 \) such that
\[
F^{-k_n} |s_{n+k_n}| > 3i + 2.
\]
Now let \( M \) be given; we may assume that \( M > N + \max\{k_n : n < N\} \). For each \( n \in [N, M] \) there exists \( k_n \geq 1 \) such that
\[
F^{-k_n} |s_{n+k_n}| > 3 \text{dom}(\alpha) + 2.
\]
Let \( m = \max\{n + k_n : n \in [N, M]\} \). Clearly \( m \geq M + k_M > M \).

**Claim 1.** \( \langle \delta^m, \gamma^m \rangle \in X_\alpha \)

**Proof of Claim 1.** By the choice of \( m \), for every \( i \leq \text{dom}(\alpha) \) and \( n \in [N_i, m) \) we have guaranteed that \( t_{\alpha}(\delta^m) > 3 \text{dom}(\alpha | i) + 2 \). Additionally, if \( n \geq m \) then \( t_{\alpha}(\delta^m) > 3 \text{dom}(\alpha) - 1 \). This is trivial if \( s_n^m = |s_{n'}| \) for all \( n' > n \). On the other hand, if there exists \( n' > n \) such that \( s_n^m = [F^{n'-m}(3 \text{dom}(\alpha))] \) then by Proposition 1 we get
\[
t_{\alpha}(\delta^m) \geq F^{-(n'-n)}[F^{n'-m}(3 \text{dom}(\alpha))] \geq F^{-(n'-n)}(F^{n'-m}(3 \text{dom}(\alpha)) - 1) > F^{-(n'-n)}(F^{n'-m}(3 \text{dom}(\alpha))) - 1 = F^{n'-m}(3 \text{dom}(\alpha)) - 1 \geq 3 \text{dom}(\alpha) - 1.
\]
This also shows that \( t_{\alpha}(\delta^m) \to \infty \) in the case when
\[
\{n < \omega : s_n^m = [F^{n'-m}(3 \text{dom}(\alpha))]\}
\]
is infinite, due to the fourth line in the inequality above. So in this case \( \langle \delta^m, \gamma^m \rangle \in \tilde{E}(\mathcal{F}) \) by Proposition 2(b). In the other case \( s^m \) is essentially \( \delta \), which clearly belongs to \( \tilde{E}(\mathcal{F}) \). We thus have \( \langle \delta^m, \gamma^m \rangle \in \tilde{E}(\mathcal{F}) \), and conclude that \( \langle \delta^m, \gamma^m \rangle \in X_\alpha \).

**Claim 2.** \( \langle \delta^m, \gamma^m \rangle \notin \overline{X_{\alpha - N}} \)

**Proof of Claim 2.** Since \( m \geq N \), the hypothesis \( \langle \delta^m, \gamma^m \rangle \in X_{\alpha - N} \) would imply \( t_{\alpha}(\delta^m) \geq 3 \text{dom}(\alpha) + 1 \) by Proposition 3(b). But
\[
t_{\alpha}(\delta^m) = \sup_{k \geq 1} F^{-k} |s_{m+k}| \leq \sup_{k \geq 1} F^{-k} [F^k(3 \text{dom}(\alpha))] \leq 3 \text{dom}(\alpha).
\]
Therefore \( \langle \delta^m, \gamma^m \rangle \notin \overline{X_{\alpha - N}} \).

We have shown that each point of \( X_{\alpha - N} \) lies in the closure of \( X_\alpha \setminus \overline{X_{\alpha - N}} \). Therefore \( X_{\alpha - N} \) is nowhere dense in \( X_\alpha \). This concludes the proof of Theorem 4.

**Corollary 5.** Each \( X_\alpha \) is a first category space.

**Proof.** Observation 2 and Theorem 4.

**5. Proof of Theorem 1**

We are now ready for the main results.

**Theorem 6.** \( \tilde{E}(\mathcal{F}) \) is nowhere \( G_{\delta_\sigma} \).
PROOF. Let $d$ be a complete metric for $E(\mathcal{F})$. All closures in the proof will be taken in the space $E(\mathcal{F})$, and diameters will be with respect to $d$.

Let $\mathcal{A} = \{A_n : n < \omega\}$ be a collection of completely metrizable subspaces of $\tilde{E}(f)$. Let $W$ be any non-empty open subset of $E(\mathcal{F})$. We will show that $\mathcal{A}$ does not cover $W \cap \tilde{E}(\mathcal{F})$. This will prove that $W$ is not $\sigma$-complete, and more generally that $E(\mathcal{F})$ is nowhere $G_{\delta\sigma}$.

Recall that any open subset of a first category space is again of first category. Hence the intersection $W \cap X_{\varnothing}$ is first category by Corollary 5. This set is also non-empty by Remark 6. Therefore $A_0$ is not dense in $W \cap X_{\varnothing}$. So $W \cap X_{\varnothing} \setminus \overline{A_0} \neq \emptyset$. Thus there is a non-empty open set $U \subset E(\mathcal{F})$ such that $U \subset W$, $\text{diam}(U) < 1$, and $\overline{U} \cap A_0 = \emptyset$. Let $U_0 = U \cap X_{\varnothing}$.

By Observation 2 there exists $N_0$ such that $X_{(N_0)} \cap U_0 \neq \emptyset$. Since $U_0 \cap X_{(N_0)}$ is first category (Corollary 5), it does not have a dense completely metrizable subspace. By Observation 1, $U_0 \cap X_{(N_0)} \cap A_1$ is a $G_\delta$-subset of $A_1$ and is therefore completely metrizable. So $U_0 \cap X_{(N_0)} \cap A_1 \neq \emptyset$. Hence there is a non-empty relatively open $U_1 \subset U_0 \cap X_{(N_0)}$ such that $\text{diam}(U_1) < 1/2$ and $\overline{U_1} \cap A_1 = \emptyset$. Now choose $N_1 > N_0$ such that $X_{(N_0,N_1)} \cap U_1 \neq \emptyset$.

This process can be continued to get an increasing sequence

$$
\lambda = (N_0, N_1, N_2, \ldots) \in \mathbb{N}^\omega
$$

and non-empty sets $U_0 \supset U_1 \supset U_2 \supset \ldots$ such that $U_n$ is open in $X_{\lambda[n]}$, $\text{diam}(U_n) < \frac{1}{n+1}$, and $\overline{U_n} \cap A_n = \emptyset$. By completeness of the metric space $(E(\mathcal{F}), d)$ there exists

$$
x \in \bigcap_{n=0}^{\infty} \overline{U_n}
$$

Then $x \in \cap_{n=0}^{\infty} X_{\lambda[n]}$, so by Proposition 3(a) $t_{\sigma^n(\varnothing(x))} \to \infty$. We have

$$
x \in \bigcap_{n=0}^{\infty} \overline{U_n} \cap E(\mathcal{F}) \subset W \cap \tilde{E}(\mathcal{F})
$$

and yet $x \notin \bigcup_{n=0}^{\infty} A_n$. Hence $\mathcal{A}$ does not cover $W \cap \tilde{E}(\mathcal{F})$. \hfill \Box

Recall that from Section 3 that $f = f_{-1}$, and $\mathcal{F} \upharpoonright J(\mathcal{F})$ is conjugate to $f \upharpoonright J(f)$.

COROLLARY 7. $I(f)$ is nowhere $G_{\delta\sigma}$.

PROOF. Since $I(f)$ is homeomorphic to $I(\mathcal{F})$, it suffices to show that $I(\mathcal{F})$ is nowhere a $G_{\delta\sigma}$-subset of $[0, \infty) \times \mathbb{Z}^\omega$. This follows from Theorem 6 and the fact that $\tilde{E}(\mathcal{F})$ is a dense $G_{\delta}$-subset of $I(\mathcal{F})$. \hfill \Box

We can now prove Theorem 1 by combining Corollary 6 with the following.

PROPOSITION 4 (cf. [12, Theorem 1.1]). For every $a \in \mathbb{C}$ there exists $R > 0$ and a homeomorphism $\varphi : \mathbb{C} \to \mathbb{C}$ such that $\varphi[A \cap I(f)] = \varphi[A] \cap I(f_a)$, where

$$
A = \{z \in \mathbb{C} : |f^n(z)| \geq R \text{ for all } n \geq 1\}.
$$

Let $a \in \mathbb{C}$ and let $R > 0$ and $\varphi$ be given by Proposition 4. Note that

$$
I(f) = \bigcup_{n=0}^{\infty} f^{-n}[A \cap I(f)].
$$

Since $G_{\delta\sigma}$-sets are preserved by continuous pre-images and countable unions, by Corollary 7 $A \cap I(f)$ is not $G_{\delta\sigma}$. So $\varphi[A \cap I(f)]$ is not $G_{\delta\sigma}$. This is a closed subset of $I(f_a)$
because it is equal to $\varphi[A] \cap I(f_a)$. Therefore $I(f_a)$ cannot be $G_{\delta \sigma}$. This concludes the proof of Theorem 1.

6. Proof of Theorem 3

A space $X$ is zero-dimensional if the topology of $X$ has a basis consisting of clopen subsets of $X$. We can now state van Engelen’s theorem.

**Proposition 5 ([16, Theorem A.2.6]).** Up to homeomorphism, there is only one zero-dimensional Baire space that is $G_{\delta \sigma}$ and nowhere $F_{\sigma \delta}$.

The standard representation of the space in Proposition 5 is $\mathbb{R} \setminus X$, where $X$ is a densely embedded copy of $\mathbb{Q}^\omega$ (the infinite product of the rationals) in $\mathbb{R}$. It can be expressed more concretely as $\mathbb{P}^\omega \setminus (\mathbb{Q} + \pi)^\omega$, where $\mathbb{P}$ denotes the space of irrationals.

**Theorem 8.** If $f_a$ has an attracting or parabolic cycle, then $J(f_a) \setminus I(f_a)$ is a zero-dimensional Baire space that is $G_{\delta \sigma}$ and nowhere $F_{\sigma \delta}$.

**Proof.** Clearly $J(f_a) \setminus I(f_a)$ is $G_{\delta \sigma}$ because its Julia complement $I(f_a)$ is $F_{\sigma \delta}$. It is zero-dimensional by [10, Corollary 10], and Baire because it contains a dense $G_{\delta}$-set in the form of all points whose orbits are dense in the Julia set [4, 9]. Finally, since $I(f_a)$ is dense in $J(f_a)$, an $F_{\sigma \delta}$-neighborhood in $J(f_a) \setminus I(f_a)$ would complement a $G_{\delta \sigma}$-neighborhood in $I(f_a)$. But $I(f_a)$ is nowhere $G_{\delta \sigma}$ by Corollary 7 and the equivalence $I(f_a) \simeq I(f)$ ([12, Theorem 1.2] and [2, Example 11.1]). Therefore $J(f_a) \setminus I(f_a)$ is nowhere $F_{\sigma \delta}$. □

In light of Proposition 5, Theorem 8 implies Theorem 3.

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Department of Mathematics and Data Science, College of Coastal Georgia, Brunswick GA 31520, United States of America

Email address: ds10003@auburn.edu, dlipham@ccga.edu