DISCRETE SPECTRUM OF SCHRÖDINGER OPERATORS WITH OSCILLATING DECAYING POTENTIALS

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Abstract. We consider the Schrödinger operator \( H_{\eta W} = -\Delta + \eta W \), self-adjoint in \( L^2(\mathbb{R}^d) \), \( d \geq 1 \). Here \( \eta \) is a non constant almost periodic function, while \( W \) decays slowly and regularly at infinity. We study the asymptotic behaviour of the discrete spectrum of \( H_{\eta W} \) near the origin, and due to the irregular decay of \( \eta W \), we encounter some non semiclassical phenomena. In particular, \( H_{\eta W} \) has less eigenvalues than suggested by the semiclassical intuition.

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1. Introduction

In the present article, we investigate the discrete spectrum of the Schrödinger operator whose potential is a product of an oscillating, e.g. periodic factor \( \eta \), and a factor \( W \) which decays regularly at infinity; by a regular decay we mean that the derivatives of \( W \) decay at infinity faster than \( W \) itself. Note that if \( \eta \) is oscillating and not constant, then the product \( \eta W \) does not decay regularly. Our results show that due to the irregular decay of the potential, the corresponding Schrödinger operator generally has less discrete eigenvalues than suggested by the semiclassical intuition.

For a better understanding of our results, we first recall some known facts concerning the discrete spectrum of the Schrödinger operator

\[
H_V := -\Delta + V,
\]

self-adjoint in \( L^2(\mathbb{R}^d) \), \( d \geq 1 \), with decaying potential \( V : \mathbb{R}^d \to \mathbb{R} \). Throughout the article, we assume that the multiplier by \( V \) is \(-\Delta\)-relatively compact in the sense of the quadratic forms, i.e. the operator \( |V|^{1/2}(-\Delta + 1)^{-1/2} \) is compact in \( L^2(\mathbb{R}^d) \). A simple sufficient condition guaranteeing such a relative compactness is that for each \( \varepsilon > 0 \) we have \( V = V_{1,\varepsilon} + V_{2,\varepsilon} \) where \( V_{1,\varepsilon} \in L^p(\mathbb{R}^d) \) with \( p = 1 \) if \( d = 1 \), \( p > 1 \) if \( d = 2 \), \( p = d/2 \) if \( d \geq 3 \), and \( \|V_{2,\varepsilon}\|_{L^\infty(\mathbb{R}^d)} \leq \varepsilon \). An even simpler sufficient condition is that \( V \in L^p_{\text{loc}}(\mathbb{R}^d) \) and \( \lim_{|x| \to \infty} V(x) = 0 \).

Under our relative compactness condition we can define \( H_V \) as the sum in the sense of the quadratic forms. Then

\[
\sigma_{\text{ess}}(H_V) = \sigma_{\text{ess}}(H_0) = [0, \infty),
\]
so that the possible discrete spectrum of $H_V$ is negative, and it can accumulate only at the origin. In order to address the problem about the finiteness of $\sigma_{\text{disc}}(H_V)$, set

$$N(E; V) := \text{Tr} \mathbf{1}_{(-\infty, E)}(H_V), \quad E \in (-\infty, 0],$$

where $\mathbf{1}_{(-\infty, E)}(H_V)$ is the spectral projection of the operator $H_V$ corresponding to the interval $(-\infty, E)$. Thus, $N(E; V)$ is just the number of the eigenvalues of $H_V$ smaller than $E$, and counted with the multiplicities. We have the following simple

**Proposition 1.1.** Let $d \geq 1$. Assume that $V^{1/2}(-\Delta + 1)^{-1/2}$ is compact in $L^2(\mathbb{R}^d)$, $V$ is measurable, and there exists constants $c \in (0, \infty)$ and $\rho \in (2, \infty)$ such that

$$(1.1) \quad V_-(x) \leq c(1 + |x|)^{-\rho}, \quad x \in \mathbb{R}^d.$$ 

Then

$$(1.2) \quad N(0; V) < \infty,$$

i.e. the discrete spectrum of $H_V$ is finite.

If $d \geq 3$, the result follows from the Cwikel-Lieb-Rozenblum estimate

$$(1.3) \quad N(0; V) \leq c_d \int_{\mathbb{R}^d} V_-(x)^{d/2} \, dx$$

where the constant depends only on $d$ (see e.g. [10 Theorem XIII.12]). If $d = 2$, it is implied, for example, by [3, Eq. (44)], while for $d = 1$ it follows, say, from [10 Problem 22, Chapter XIII]. The following proposition shows that the condition $\rho > 2$ in (1.1) is close to the optimal one.

**Proposition 1.2.** ([10 Theorem XIII.82]) Let $d \geq 1$, $\rho \in (0, 2)$. Assume that there exists constant $C \in (0, \infty)$, $R \in (0, \infty)$ such that

$$(1.4) \quad |V(x)| \leq C(1 + |x|)^{-\rho}, \quad x \in \mathbb{R}^d,$$

$$(1.5) \quad |\nabla V(x)| \leq C(1 + |x|)^{-\rho - 1}, \quad x \in \mathbb{R}^d,$$

and

$$V(x) \leq -C|x|^{-\rho}, \quad x \in \mathbb{R}^d, \quad |x| \geq R.$$

Then we have

$$(1.6) \quad \frac{\tau_d}{(2\pi)^d} \int_{\mathbb{R}^d} (V(x) - E)^{d/2} \, dx \approx |E|^{d(\frac{1}{2} - \frac{1}{\rho})}, \quad E \uparrow 0,$$

where $\tau_d := \frac{\pi^{d/2}}{\Gamma(1 + d/2)}$ is the volume of the unit ball in $\mathbb{R}^d$.

The border-line case $\rho = 2$ is handled in the following

**Proposition 1.3.** ([4]) Let $d \geq 1$. Assume that $V \in L^\infty(\mathbb{R}^d)$ and there exists $L \in \mathbb{R}$ such that $\lim_{|x| \to \infty} |x|^2 V(x) = L$. Then we have

$$(1.7) \quad \lim_{E \uparrow 0} \frac{1}{E} \ln |E|^{-1} N(E; V) = C_\rho(L)$$
where

\[
C_d(L) := \begin{cases} 
\frac{1}{\pi} \left( L + \frac{1}{4} \right)^{1/2} & \text{if } d = 1, \\
\frac{1}{2\pi} \sum_{q=0}^{\infty} \left( L + \frac{(d-2)^2}{4} + \lambda_q \right)^{1/2} & \text{if } d \geq 2,
\end{cases}
\]

and \( \{\lambda_q\}_{q \in \mathbb{Z}_+} \) is the non decreasing sequence of the eigenvalues of the Beltrami-Laplace operator, self-adjoint in \( L^2(S^{d-1}) \). If, moreover, \( L > -\frac{(d-2)^2}{4} \), \( d \geq 1 \), then (1.2) holds true.

**Remark:** The explicit expressions for the eigenvalues of the Beltrami-Laplace operator and their multiplicities are well known; see, for instance, [12, Subsections 22.3-4]. In particular, \( \lambda_0 = 0 \) is a simple eigenvalue for any \( d \geq 2 \).

In this article, we consider the case

\[
V = \eta W
\]

where \( \eta \) is a non constant almost periodic function, while \( W \) decays regularly at infinity; in particular \( W \) satisfies (1.3) and (1.5). Note, however, that in this case the product \( \eta W \) cannot satisfy (1.4). Our main result, Theorem 2.1, describes some non classical phenomena in the asymptotic behaviour of the discrete spectrum of \( H_{\eta W} \), due to the irregular decay of \( \eta W \).

One-dimensional Schrödinger operators with potentials as in (1.9) arose as effective Hamiltonians in [9], where we considered the asymptotic distribution of the discrete spectrum for waveguides with perturbed periodic twisting. Multidimensional operators of this kind were discussed in [5, 11], where the problem about the location of the absolute spectrum of \( H_V \) was studied.

The article is organized as follows. In the next section we formulate Theorem 2.1 and briefly comment on it. Section 3 contains some auxiliary results, while the proof of Theorem 2.1 can be found in Section 4.

## 2. MAIN RESULT

Let us first introduce several definitions needed for the statement of the main result. We will write \( W \in \mathcal{S}_{m,\varrho}(\mathbb{R}^d) \), \( m \in \mathbb{Z}_+ \), \( \varrho \in (0, \infty) \), if \( W \in C^m(\mathbb{R}^d) \), and there exists a constant \( C \in (0, \infty) \) such that

\[
|D^\alpha W(x)| \leq C(1 + |x|)^{-\varrho - |\alpha|}, \quad x \in \mathbb{R}^d,
\]

for each \( \alpha \in \mathbb{Z}_+^d \) with \( 0 \leq |\alpha| \leq m \). If \( W \in \mathcal{S}_{m,\varrho}(\mathbb{R}^d) \) is real valued, we will write, as usual, \( W \in \mathcal{S}_{m,\varrho}(\mathbb{R}^d; \mathbb{R}) \).

Further, set

\[
\eta(x) := \sum_{\ell \in J} \eta_{\ell e^{i\xi_{\ell} \cdot x}}, \quad x \in \mathbb{R}^d,
\]

where:
\[ \mathcal{J} := \{0, 1, \ldots, L\}, \ 1 \leq L \leq \infty; \]
\[ \eta_\ell \in \mathbb{C} \text{ if } \ell \in \mathcal{J}, \ \eta_\ell \neq 0 \text{ if } \ell \in \mathcal{J}_0 := \{1, \ldots, L\}, \ \text{and } \sum_{\ell \in \mathcal{J}} |\eta_\ell| < \infty; \]
\[ \xi_\ell \in \mathbb{R}^d \text{ if } \ell \in \mathcal{J}, \ \xi_\ell \neq \xi_{\ell_0} \text{ if } \ell_1 \neq \ell_2, \ \xi_0 = 0, \ \text{and } r := \inf_{\ell \in \mathcal{J}_0} |\xi_\ell| > 0. \]

Then we write \( \eta \in \mathcal{A}(\mathbb{R}^d) \). Of course, \( \eta \in \mathcal{A}(\mathbb{R}^d) \) implies that \( \eta : \mathbb{R}^d \to \mathbb{C} \) is a (uniformly) continuous almost periodic function whose mean value equals \( \eta_0 \).

For \( u \in \mathcal{S}(\mathbb{R}^d) \), the Schwartz class in \( \mathbb{R}^d \), introduce the Fourier transform

\[ (\mathcal{F}u)(\xi) = \hat{u}(\xi) := (2\pi)^{-1/2} \int_{\mathbb{R}^d} e^{-ix\xi} u(x) dx, \quad \xi \in \mathbb{R}^d. \]

Whenever necessary, the Fourier transform is extended by duality to the dual Schwartz class \( \mathcal{S}'(\mathbb{R}^d) \).

Then, \( \eta \in \mathcal{A}(\mathbb{R}^d) \) implies that

\[ (\text{supp } \hat{\eta}) \setminus \{0\} \cap B_r = \emptyset \]

where \( B_r := \{\xi \in \mathbb{R}^d \mid |\xi| < r\} \). A leading example for \( \eta \in \mathcal{A}(\mathbb{R}^d) \) is a function periodic with respect to a non degenerate lattice of periods in \( \mathbb{R}^d \), which has an absolutely summable series of Fourier coefficients. As above, if \( \eta \in \mathcal{A}(\mathbb{R}^d) \) is real valued, we will write \( \eta \in \mathcal{A}(\mathbb{R}^d; \mathbb{R}) \).

Finally, for \( \rho \in N \) and \( \rho \in (0, \infty) \) set

\[ \nu = \nu(d, \rho) := \begin{cases} \frac{d+2-\rho}{2} & \text{if } \ d = 1, 3, \\ 3 - \rho/2 & \text{if } \ d = 2, \\ \frac{d-\rho}{2} & \text{if } \ d \geq 4. \end{cases} \]

**Theorem 2.1.** Let \( d \geq 1, \ \rho \in (0, 2) \). Suppose that \( \eta \in \mathcal{A}(\mathbb{R}^d; \mathbb{R}) \), and \( W \in S_{2n,d}(\mathbb{R}^d; \mathbb{R}) \) with \( n \in N, \ n > \nu(d, \rho) \).

(i) Let \( \rho \in (0, 2) \). Assume that there exist constants \( C \in (0, \infty) \), and \( R \in (0, \infty) \), such that

\[ W(x) \leq -C|x|^{-\rho}, \quad x \in \mathbb{R}^d, \quad |x| \geq R. \]

If \( \eta_0 > 0 \), then

\[ N(E; \eta W) = \frac{\tau_d}{(2\pi)^d} \int_{\mathbb{R}^d} (\eta_0 W(x) - E)^{d/2} dx(1 + o(1)) \asymp |E|^d(\frac{d-\rho}{2}), \quad E \uparrow 0. \]

If, on the contrary, \( \eta_0 < 0 \), then

\[ N(0; \eta W) < \infty. \]

(ii) Let \( \rho \in (0, 2) \). Assume \( \eta_0 = 0 \). Then we have

\[ N(E; \eta W) = \begin{cases} O\left(|E|^{d(1-\frac{\rho}{2})}\right) & \text{if } \rho \in (0, 1), \\ O(\ln|E|) & \text{if } \rho = 1, \\ O(1) & \text{if } \rho \in (1, 2), \end{cases} \quad E \uparrow 0. \]

(iii) Let \( \rho = 2 \). Suppose that there exists \( L \in \mathbb{R} \) such that

\[ \lim_{|x| \to \infty} |x|^2 W(x) = L. \]
Then we have
\[ \lim_{E \uparrow 0}(|\ln |E||)^{-1}N(E; \eta W) = C_d(\eta_0 L), \]
\( C_d \) being defined in (1.8). If, moreover, \( \eta_0 L > -\frac{(d-2)^2}{4} \), then (2.7) holds true.

Remark: If \( \eta \in L^\infty(\mathbb{R}^d; \mathbb{R}), \ d \geq 1, \) and \( W \in \mathcal{S}_{\rho}(\mathbb{R}^d; \mathbb{R}) \) with \( \rho \in (2, \infty) \), then Proposition 1.1 implies that (2.7) holds true again.

The proof of Theorem 2.1 can be found in Section 4. Let us comment here briefly on the theorem.
One of its curious features is that most of its results are not of semiclassical nature. For example, if \( d \geq 3 \), estimate (2.7) could hold true under assumptions which guarantee the divergence of the integral \( \int_{\mathbb{R}^d}(\eta W)^{d/2}dx \), appearing at the r.h.s. of (1.3). This could happen if, for example, \( W(x) = -(1 + |x|^2)^{-\rho/2}, \ x \in \mathbb{R}^d, \ \rho \in (0, 2) \), and \( \eta \in C^\infty(T^d; \mathbb{R}) \) with \( T^d = \mathbb{R}^d/\mathbb{Z}^d \), such that the mean value of \( \eta \) is negative, but its positive part does not vanish identically. Similarly, relation (2.6) is not semiclassical in the sense of (1.6) since it is not difficult to construct examples of \( W \) and \( \eta \) with \( \eta_0 > 0 \) which satisfy the assumptions of part (i) of Theorem 2.1, such that
\[ \lim_{E \uparrow \eta_0} \frac{\int_{\mathbb{R}^d}(\eta_0 W(x) - E)^{d/2}dx}{\int_{\mathbb{R}^d}(\eta W(x) - E)^{d/2}dx} < 1. \]
A related effect where the main terms of the eigenvalue asymptotics for quantum magnetic Hamiltonians depend only on the mean value of the oscillating magnetic field, can be found in [7, 8].

The methods used in the proof of Theorem 2.1 resemble those developed in [6, 9]. Moreover, in the proof of part (ii), we apply a bound which is inspired by [11]; this bound allowed us to improve slightly asymptotic estimates (2.8).
The assumptions on \( \eta \) and \( W \) in Theorem 2.1 could be essentially relaxed. For example, in part (ii) we could have supposed that \( W \in \mathcal{S}_{1,\rho}(\mathbb{R}^d) \), \( W \) satisfies (2.5), and for all \( \varepsilon > 0 \) we have
\[ \hat{W} \in C^m(\mathbb{R}^d \setminus B_\varepsilon), \quad m = \left\{ \begin{array}{ll} 2 & \text{if } d = 1, 3, \\ 4 & \text{if } d = 2, \\ 0 & \text{if } d \geq 4, \end{array} \right. \]  
(see the proof of Proposition 4.1). Similarly, in part (ii), we could have assumed that \( W \in \mathcal{S}_{1,\rho}(\mathbb{R}^d) \), \( \hat{W} \) satisfies (2.10). Further, we could replace the hypothesis \( \eta \in \mathcal{A}(\mathbb{R}^d; \mathbb{R}) \) by the assumption that \( \eta \) is a fairly more general function which satisfies (2.3). We prefer to use explicit and transparent assumptions allowing us to give a simple and self-contained proof whose main ideas are not hidden by unessential technical details.

3. Auxiliary results
This section contains auxiliary results needed for the proof of Theorem 2.1
Let \( X \) be a separable Hilbert space. We denote by \( S_{\infty}(X) \) the class of linear compact
operators $T : X \to X$. Let $T = T^* \in S_\infty(X)$. For $s > 0$ set
\[ n_\pm(s; T) := \text{Tr} 1_{(s, \infty)}(\pm T). \]
Thus, $n_+(s; T)$ (resp., $n_-(s; T)$) is just the number of the eigenvalues of $T$ larger than $s$ (resp., smaller than $-s$), and counted with the multiplicities. If $T_j = T_j^* \in S_\infty(X)$, $j = 1, 2$, then the Weyl inequalities
\[ n_\pm(s_1 + s_2; T_1 + T_2) \leq n_\pm(s_1; T_1) + n_\pm(s_2; T_2) \]
hold for $s_j > 0$, $j = 1, 2$, (see e.g. [2, Theorem 9, Section 2, Chapter 9]). For $T \in S_\infty(X)$ and $s > 0$ put
\[ n_+(s; T) := n_+(s^2; T^* T). \]
Thus, $n_+(s; T)$ is the number of the singular values of $T$ larger than $s$, and counted with the multiplicities. Evidently, if $T = T^*$, then
\[ n_\pm(s; T) \leq n_\pm(s; T), \quad s > 0. \]
Moreover, if $T_j \in S_\infty(X)$, $j = 1, 2$, then the Ky Fan inequalities
\[ n_\pm(s_1 + s_2; T_1 + T_2) \leq n_\pm(s_1; T_1) + n_\pm(s_2; T_2) \]
hold for $s_j > 0$, $j = 1, 2$, (see e.g. [2, Eq. (17), Section 1, Chapter 11]).

Further, we recall an abstract version of the Birman-Schwinger principle, suitable for our purposes.

**Lemma 3.1.** ([1, Lemma 1.1]) Let $T = T^* \geq 0$, and let $S = S^*$ be $T$-relatively compact in the sense of the quadratic forms. Then we have
\[ \text{Tr} 1_{(-\infty, -\lambda)}(T - rS) = n_+(r^{-1}; (T + \lambda)^{-1/2} S)(T + \lambda)^{-1/2} \]
for any $r > 0$ and $\lambda > 0$.

**Lemma 3.2.** Let $f, g \in L^\infty(\mathbb{R}^d)$, $d \geq 1$, and $\lim_{|x| \to \infty} f(x) = \lim_{|x| \to \infty} g(x) = 0$. Then the operator $f F^* g$ is compact in $L^2(\mathbb{R}^d)$.

**Proof.** Let $R \in (0, \infty)$, and let $\chi_{B_R}$ be the characteristic function of the ball $B_R$. Then for any fixed $R \in (0, \infty)$, the operator $\chi_{B_R} f F^* g \chi_{B_R}$ is Hilbert-Schmidt, and hence compact. On the other hand, $\lim_{R \to \infty} \| F^* g - \chi_{B_R} f F^* g \chi_{B_R} \| = 0$. Therefore, $f F^* g$ is compact.

4. **Proof of Theorem 2.1**

By the Birman-Schwinger principle (see Lemma 3.1),
\[ N(E; \eta W) = n_-(1; (-\Delta - E)^{-1/2} \eta W(-\Delta - E)^{-1/2}), \quad E < 0. \]

By the unitarity of the Fourier transform,
\[ n_-(1; (-\Delta - E)^{-1/2} \eta W(-\Delta - E)^{-1/2}) = n_-(1; a(E) F \eta W F^* a(E)), \quad E < 0. \]
where \(a(\xi; E) := (|\xi|^2 - E)^{-1/2}, \xi \in \mathbb{R}^d, E \leq 0\). Denote by \(\chi_1\) the characteristic function of the ball \(B_\delta\) with \(\delta \in (0, r/2)\) and \(r := \inf_{\xi \in B_\delta} |\xi| > 0\) (see (2.3)). Set \(\chi_2 := 1 - \chi_1\). Then

\[
a(E)\mathcal{F}\eta W F^* a(E) = \eta_0 a(E)\mathcal{F} W F^* a(E) + \\
\sum_{j=1,2} a(E)\chi_j \mathcal{F}(\eta - \eta_0) W F^* \chi_j a(E) + 2 \text{Re} a(E)\chi_1 \mathcal{F}(\eta - \eta_0) W F^* \chi_2 a(E).
\]

(4.3)

Further, for any \(u \in L^2(\mathbb{R}^d),\)

\[
((2 \text{Re} a(E)\chi_1 \mathcal{F}(\eta - \eta_0) W F^* \chi_2 a(E)) u, u)_{L^2(\mathbb{R}^d)} = 2 \text{Re} (f, g)_{L^2(\mathbb{R}^d)}
\]

where \((\cdot, \cdot)_{L^2(\mathbb{R}^d)}\) is the scalar product in \(L^2(\mathbb{R}^d)\), and

\[
f := \omega^{1/2}_\mu F^* \chi_1 a(E) u, \quad g := \omega^{1/2}_{-\mu} (\eta - \eta_0) W \omega^{-1} F^* \chi_2 a(E) u, \quad \mu \in (0, 1),
\]

with

\[
\omega_\mu(x) := (1 + |x|^2)^{-\rho(1+\mu)/2}, \quad x \in \mathbb{R}^d, \quad \mu \in \mathbb{R}.
\]

Evidently, since \(\omega_\mu(x) \leq \omega_{-\mu}(x)\) for \(x \in \mathbb{R}^d\) and \(\mu \in (0, 1),\) we have

\[
-\|\omega^{1/2}_{\mu} F^* a(E) u\|^2 - (1 + 2C^2)\|\omega^{1/2}_{-\mu} F^* \chi_2 a(E) u\|^2 \leq -\frac{1}{2}\|f\|^2 - 2\|g\|^2 \leq \\
2 \text{Re} (f, g)_{L^2(\mathbb{R}^d)} \leq \\
\frac{1}{2}\|f\|^2 + 2\|g\|^2 \leq \|\omega^{1/2}_{\mu} F^* a(E) u\|^2 + (1 + 2C^2)\|\omega^{1/2}_{-\mu} F^* \chi_2 a(E) u\|^2, \quad \mu \in (0, 1),
\]

where

\[
C := \sup_{x \in \mathbb{R}} |\eta(x) - \eta_0| \omega_0(x)^{-1}|W(x)|.
\]

Therefore,

\[
-a(E)\mathcal{F}\omega_\mu F^* a(E) - (1 + 2C^2) a(E)\chi_2 \mathcal{F}\omega_{-\mu} F^* \chi_2 a(E) \leq \\
2 \text{Re} a(E)\chi_1 \mathcal{F}(\eta - \eta_0) W F^* \chi_2 a(E) \leq \\
(4.4)
\]

\[
a(E)\mathcal{F}\omega_\mu F^* a(E) + (1 + 2C^2) a(E)\chi_2 \mathcal{F}\omega_{-\mu} F^* \chi_2 a(E), \quad \mu \in (0, 1).
\]

Similarly, since \(\omega_0(x) \leq \omega_{-\mu}(x)\) for \(x \in \mathbb{R}^d\) and \(\mu \in [0, 1),\) we have

\[
-Ca(E)\chi_2 \mathcal{F}\omega_{-\mu} F^* \chi_2 a(E) \leq -Ca(E)\chi_2 \mathcal{F}\omega_0 F^* \chi_2 a(E) \leq \\
a(E)\chi_2 F(\eta - \eta_0)\epsilon F^* \chi_2 a(E) \leq \\
(4.5)
\]

\[
Ca(E)\chi_2 \mathcal{F}\omega_0 F^* \chi_2 a(E) \leq Ca(E)\chi_2 \mathcal{F}\omega_{-\mu} F^* \chi_2 a(E), \quad \mu \in [0, 1).
\]

Now it follows from (4.3) - (4.5) that

\[
a(E)\mathcal{F}(\eta_0 W - \omega_\mu) F^* a(E) + a(E)\chi_1 \mathcal{F}(\eta - \eta_0) W F^* \chi_2 a(E) - \\
(1 + C + 2C^2) a(E)\chi_2 \mathcal{F}\omega_{-\mu} F^* \chi_2 a(E) \leq \\
a(E)\mathcal{F}\eta_0 W F^* a(E) \leq \\
a(E)\mathcal{F}(\eta_0 W + \omega_\mu) F^* a(E) + a(E)\chi_1 \mathcal{F}(\eta - \eta_0)\epsilon F^* \chi_2 a(E) + \\
(1 + C + 2C^2) a(E)\chi_2 \mathcal{F}\omega_{-\mu} F^* \chi_2 a(E).
\]

(4.6)
Applying the mini-max principle, the Weyl inequalities, and the Birman–Schwinger principle, we find, taking into account that
\[ a(\xi; E) \leq a(0; E), \quad \xi \in \text{supp } \chi_2, \quad E \leq 0, \]
that (4.6) implies
\[ N(E; (1 + s)^{-1}(\eta_0 W + \omega_{\mu}))- \]
\[ n_+(s/2; a(E)\chi_1 F(\eta - \eta_0)WF*\chi_1 a(E)) - n_+(\sqrt{s/(2(1 + C + 2C^2)); \omega_+^{1/2}F*\chi_2 a(0)}) \leq \]
\[ n_-(1; a(E)F\eta WF* a(E)) \leq \]
\[ N(E; (1 - s)^{-1}(\eta_0 W - \omega_{\mu}))+ \]
\[ n_-(s/2; a(E)\chi_1 F(\eta - \eta_0)WF*\chi_1 a(E)) + n_+(\sqrt{s/(2(1 + C + 2C^2)); \omega_+^{1/2}F*\chi_2 a(0)}) \]
for \( s \in (0, 1) \). In the case \( \eta_0 = 0 \), we will need also a slightly different estimate of \( n_-(1; a(E)F\eta WF* a(E)) \), inspired by [11]. Namely, if \( \eta_0 = 0 \), then (4.3) and the Weyl inequalities imply
\[ n_-(1; a(E)F\eta WF* a(E)) \leq \]
\[ \sum_{j=1,2} n_-(1/4; a(E)\chi_j F\eta WF*\chi_2 a(E)) + n_-(1/2; 2Re a(E)\chi_1 F\eta WF*\chi_2 a(E)). \]
By (3.3) and the Ky Fan inequalities,
\[ n_-(1/2; 2Re a(E)\chi_1 F\eta WF*\chi_2 a(E)) \leq 2n_+(1/4; a(E)\chi_1 F\eta WF*\chi_2 a(E)) \]
Using elementary operator norms and the Birman-Schwinger principle, we get
\[ n_+(1/4; a(E)\chi_1 F\eta WF*\chi_2 a(E)) \leq n_+(1/4; C\delta^{-1} a(E)F\omega_0) = \]
\[ n_+(1/16; C^2\delta^{-2} a(E)F\omega_1 F* a(E)) = N(E, -C_1 \omega_1), \quad E < 0, \]
where \( C_1 = C_1(\delta) := 16C^2\delta^{-2} \). Putting together (4.5), and (4.8)–(4.10), we get
\[ n_-(1; a(E)F\eta WF* a(E)) \leq N(E; -C_1 \omega_1)+ \]
\[ n_-(1/4; a(E)\chi_1 F\eta WF*\chi_1 a(E)) + n_+(1/2; C\omega_0^{1/2}F*\chi_2 a(0)). \]
Let us now show that under the hypotheses of Theorem 2.1, the quantity
\[ n_+(s; a(E)\chi_1 F\eta WF*\chi_1 a(E)) \] with \( s > 0 \) remains bounded as \( E \uparrow 0 \).

**Proposition 4.1.** Let \( d \geq 1, \rho \in (0, \infty) \). Suppose that \( \eta \in A(\mathbb{R}^d; \mathbb{R}), \) and \( W \in S_{2n, \rho}(\mathbb{R}^d) \) with \( n \in \mathbb{N}, n > \nu(d, \rho), \) the function \( \nu(d, \rho) \) being defined in (2.4). Then for any \( s > 0 \) we have
\[ n_+(s; a(E)\chi_1 F(\eta - \eta_0)WF*\chi_1 a(E)) = O_s(1), \quad E \uparrow 0. \]
In order to prove Proposition 4.1 we need the following
Lemma 4.2. Let $d \in \mathbb{N}$, $m \in \mathbb{Z}_+$, $\rho \in (0, \infty)$, and $n \in \mathbb{Z}_+$, $n > \frac{m+d-\rho}{2}$. Assume that $W \in S_{2n,\rho}(\mathbb{R}^d)$. Then for each $\varepsilon > 0$ we have

\begin{equation}
\hat{W} \in C^m(\mathbb{R}^d \setminus B_{\varepsilon}). \tag{4.13}
\end{equation}

Moreover,

\begin{equation}
\sup_{\xi \in \mathbb{R}^d, |\xi| \geq \varepsilon} |(D^\alpha \hat{W})(\xi)| < \infty, \tag{4.14}
\end{equation}

for each $\alpha \in \mathbb{Z}_+^d$ with $0 \leq |\alpha| \leq m$, and $\varepsilon > 0$.

Proof. For $\gamma \in \mathbb{Z}_+^d$ with $0 \leq |\gamma| \leq m$ set

\[ \Phi_{\gamma,n}(x) := x^\gamma \Delta^n W(x), \quad x \in \mathbb{R}^d. \]

The condition $W \in S_{2n,\rho}(\mathbb{R}^d)$ with $n > \frac{m+d-\rho}{2}$ implies that $\Phi_{\gamma,n} \in L^1(\mathbb{R}^d)$. Therefore,

\begin{equation}
\hat{\Phi}_{\gamma,n} \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d). \tag{4.15}
\end{equation}

On the other hand, for each $\xi \in \mathbb{R}^d \setminus \{0\}$, and $\alpha \in \mathbb{Z}_+^d$, $0 \leq |\alpha| \leq m$, we have

\begin{equation}
D^\alpha \hat{W}(\xi) = D^\alpha \left(|\xi|^{-2n} |\xi|^{2n} \hat{W}(\xi)\right) = \sum_{\beta + \gamma = \alpha} \frac{\alpha!}{\beta! \gamma!} D^\beta (|\xi|^{-2n}) D^\gamma \left(|\xi|^{2n} \hat{W}(\xi)\right) = \sum_{\beta + \gamma = \alpha} (-i)^{|\gamma|} (-1)^n \frac{\alpha!}{\beta! \gamma!} D^\beta (|\xi|^{-2n}) \hat{\Phi}_{\gamma,n}(\xi). \tag{4.16}
\end{equation}

Evidently, for any $\varepsilon > 0$, and $\beta \in \mathbb{Z}_+^d$, we have

\begin{equation}
\sup_{\xi \in \mathbb{R}^d, |\xi| \geq \varepsilon} |D^\beta (|\xi|^{-2n})| \leq c_{\beta,n} \varepsilon^{-2n-|\beta|}, \tag{4.17}
\end{equation}

where, for $d \geq 1$ fixed, the constants $c_{\beta,n}$ depend only on $\beta$ and $n$. Now (4.13) - (4.14) follow from (4.16), (4.17), and (4.15). \hfill \Box

Proof of Proposition 4.1. The operator $a(E) \chi_1 \mathcal{F}(\eta - \eta_0) W \mathcal{F}^* \chi_1 a(E)$ admits an integral kernel

\begin{equation}
(2\pi)^{-d} a(\xi; E) \chi_1(\xi) \sum_{\ell \in J_0} \eta \hat{W}(\xi - \xi') \chi_1(\xi') a(\xi'; E), \quad \xi, \xi' \in \mathbb{R}^d. \tag{4.18}
\end{equation}

Let $\varkappa \in (\delta, r/2)$, and let $\Theta \in C_0^\infty(\mathbb{R}^d; [0, 1])$ with $\text{supp} \Theta = \overline{B}_{2\varkappa}$ and $\text{supp} (1 - \Theta) \subset \mathbb{R}^d \setminus B_{2\delta}$, be a real even function; in particular, $\mathcal{F} \Theta = \mathcal{F}^* \Theta$. We can multiply the integral kernel in (4.18) by $\Theta(\xi - \xi')$ without modifying it. Hence, by the mini-max principle and the Birman–Schwinger principle,

\begin{equation}
n_{\pm}(s; a(E) \chi_1 \mathcal{F}(\eta - \eta_0) W \mathcal{F}^* \chi_1 a(E)) \leq \n_{\pm}(s; (2\pi)^{-d/2} a(E) \mathcal{F}(\eta - \eta_0) W \mathcal{F}^* a(E)) = N(E; \mp s^{-1} \mathcal{V}), \quad s > 0, \quad E < 0, \tag{4.19}
\end{equation}

where $N(E; \mp s^{-1} \mathcal{V})$ is the number of negative eigenvalues of $a(E) \chi_1 \mathcal{F}(\eta - \eta_0) W \mathcal{F}^* \chi_1 a(E)$ with respect to $\mathcal{V}$.
where $\mathcal{V} := (2\pi)^{-d/2}((\eta - \eta_0)W \ast \Theta)$. For any $d \geq 1$ we have
\[ \hat{\mathcal{V}}(\xi) = \sum_{\ell \in J_0} \eta_\ell \hat{W}(\xi - \xi_\ell)\Theta(\xi), \quad \xi \in \mathbb{R}^d. \]

Since $\{\eta_\ell\}_{\ell \in J_0} \in \ell^1(J_0)$, supp $\Theta \subset B_{2r}$, and $\inf_{\ell \in J_0} |\xi_\ell| = r > 2 \kappa$, we find, using Lemma 4.2, that $\hat{\mathcal{V}} \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Moreover, evidently, supp $\hat{\mathcal{V}}$ is compact. Hence, $\hat{\mathcal{V}} \in L^p(\mathbb{R}^d)$ for any $p \in [1, \infty)$, and
\[ \|\hat{\mathcal{V}}\|_{L^p(\mathbb{R}^d)} \leq \sum_{\ell \in J_0} |\eta_\ell| \sup_{\xi \in \mathbb{R}^d, |\xi| \geq r} |\hat{W}(\xi)| \|\Theta\|_{L^p(\mathbb{R}^d)}. \]

Let $d \geq 4$. By the Hausdorff–Young inequality
\[ \|\mathcal{V}\|_{L^{d/2}(\mathbb{R}^d)} \leq c'_d \|\hat{\mathcal{V}}\|_{L^{d/(d-2)}(\mathbb{R}^d)} \]
with a constant $c'_d$ which depends only on $d$. Now, (4.12) in the case $d \geq 4$ follows from (4.19), (1.3), and (4.20).

Assume $d = 1$ or $d = 3$. In this case Lemma 4.2 easily implies that $\hat{\mathcal{V}} \in C^2(\mathbb{R}^2)$, $\Delta \hat{\mathcal{V}} \in L^1(\mathbb{R}^d)$, and
\[ |x|^2 \mathcal{V}(x) = -(2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix \cdot \xi} \Delta \hat{\mathcal{V}}(\xi) d\xi. \]

In particular,
\[ \lim_{|x| \to \infty} |x|^2 \mathcal{V}(x) = 0. \]

Thus, (4.12) for the cases $d = 1, 3$ follows from (4.19), (1.3), and Proposition 1.3.

Assume finally $d = 2$. By Lemma 4.2 we have $\mathcal{V} \in C^4(\mathbb{R}^2)$, $\Delta^2 \hat{\mathcal{V}} \in L^1(\mathbb{R}^d)$, and
\[ |x|^4 \mathcal{V}(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix \cdot \xi} \Delta^2 \hat{\mathcal{V}}(\xi) d\xi, \]
and
\[ \sup_{x \in \mathbb{R}^d} |x|^4 \mathcal{V}(x) < \infty. \]

Hence, (4.12) for the case $d = 2$ follows from (4.19), (4.22), and Proposition 1.1.

□

Next, the operator $\omega_{-\mu}^{1/2} F^* \chi_2 a(0)$ with $\mu < 1$ is compact by Lemma 3.2. Therefore,
\[ n_s(s; \omega_{-\mu}^{1/2} F^* \chi_2 a(0)) < \infty, \quad s > 0. \]

Putting together (4.1), (4.2), (4.12), and (4.23), we find that (4.7) implies
\[ N(E; (1+s)^{-1}(\eta_0 W + \omega_{-\mu})) + O_s(1) \leq N(E; \eta W) \leq N(E; (1-s)^{-1}(\eta_0 W - \omega_{-\mu})) + O_s(1), \quad E \uparrow 0, \]
for any $s \in (0, 1)$ and $\mu \in (0, 1)$, while (4.11) implies
\[ N(E; \eta W) \leq N(E; -C_1(\delta) \omega_1) + O_s(1), \quad E \uparrow 0, \]
for \( \eta_0 = 0 \) and \( \delta > 0 \).

Now Theorem 2.1 follows easily from estimates (4.24), (4.25), and Propositions 1.1 - 1.3. Assume first the hypotheses of part (i) of the theorem. If \( \eta_0 > 0 \), then (4.24) and (1.6) imply

\[
\frac{\tau_d}{(2\pi)^d} \int_{\mathbb{R}^d} ((1 + s)^{-1}(\eta_0 W + \omega_\mu) - E)^{d/2} dx (1 + o(1)) \leq N(E; \eta W) \leq \frac{\tau_d}{(2\pi)^d} \int_{\mathbb{R}^d} ((1 - s)^{-1}(\eta_0 W - \omega_\mu) - E)^{d/2} dx (1 + o(1)), \quad E \uparrow 0.
\]

(4.26)

It is not difficult to show that

\[
\lim_{s \downarrow 0} \limsup_{E \uparrow 0} \left| \frac{\int_{\mathbb{R}^d}((1 \pm s)^{-1}(\eta_0 W \pm \omega_\mu) - E)^{d/2} dx}{\int_{\mathbb{R}^d}((\eta_0 W - E)^{d/2} dx} - 1 \right| = 0
\]

(see e.g. [6, Subsection 3.7] for a similar argument). Now (2.6) follows from (4.26) - (4.27). If \( \eta_0 < 0 \), then the support of \((\eta_0 W - \omega_\mu)_-\) with \( \mu > 0 \) is compact, so that the upper bound in (4.24) combined with (1.2), implies (2.7).

Assume now the hypotheses of Theorem 2.1 (ii). Then (2.8) follows from (4.25) and Propositions 1.1 - 1.3.

Finally, assume the hypotheses of Theorem 2.1 (iii). Bearing in mind (4.24) and (1.7), we find that for every \( s \in (0, 1) \) we have

\[
C_d((1 + s)^{-1}\eta_0 L) \leq \liminf_{E \uparrow 0}(\ln |E|)^{-1}N(E; \eta W) \leq \limsup_{E \uparrow 0}(\ln |E|)^{-1}N(E; \eta W) \leq C_d((1 - s)^{-1}\eta_0 L).
\]

(4.28)

Due to the continuity of the function \( \mathbb{R} \ni L \mapsto C_d(L) \), we conclude that (2.9) follows from (4.28). Moreover, if \( \eta_0 L > -\frac{(d-2)^2}{4} \), then there exists \( s \in (0, 1) \) such that \((1 - s)^{-1}\eta_0 L > -\frac{(d-2)^2}{4}\). Hence, the upper bound in (4.24), and Proposition 1.3 imply again (2.7) in this case.

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