Asymptotics for pseudo-Anosov elements in Teichmüller lattices

Joseph Maher*
April 20, 2010

Abstract

A Teichmüller lattice is the orbit of a point in Teichmüller space under the action of the mapping class group. We show that the proportion of lattice points in a ball of radius $r$ which are not pseudo-Anosov tends to zero as $r$ tends to infinity. In fact, we show that if $R$ is a subset of the mapping class group, whose elements have an upper bound on their translation length on the complex of curves, then proportion of lattice points in the ball of radius $r$ which lie in $R$ tends to zero as $r$ tends to infinity.

Subject code: 37E30, 20F65, 57M50.

Contents

1 Introduction 1
  1.1 Outline ................................................................. 2
  1.2 Acknowledgements ..................................................... 3

2 The Teichmüller geodesic flow 3

3 Limit sets 7

4 Asymptotics 11

1 Introduction

A Teichmüller lattice $\Gamma y$ is the orbit of a point $y$ in Teichmüller space under the action of the mapping class group $\Gamma$. Athreya, Bufetov, Eskin and Mirzakhani [ABEM] showed that the asymptotic growth rate of the number of lattice points in a ball of radius $r$ is

$$|\Gamma y \cap B_r(x)| \sim \Lambda(x)\Lambda(y)he^{hr}.$$ 

Here $B_r(x)$ denotes the ball of radius $r$ centered at $x$ in the Teichmüller metric, $h = 6g - 6$ is the topological entropy of the Teichmüller geodesic flow, $\Lambda$ is the Hubbard-Masur function, $|X|$ denotes

* joseph.maher@csi.cuny.edu
the number of elements in a finite set $X$, and $f(r) \sim g(r)$ means $f(r)/g(r)$ tends to one as $r$ tends to infinity. We use their work, together with some results from [Mab], to show that the number of lattice points corresponding to pseudo-Anosov elements in the ball of radius $r$ is asymptotically the same as the total number of lattice points in the ball of radius $r$. More generally, let $R \subset \Gamma$ be a set of elements for which there is an upper bound on their translation length on the complex of curves, for example, the set of non-pseudo-Anosov elements. We shall write $R_y$ for the orbit of the point $y$ under the subset $R$. We show that the proportion of lattice points $\Gamma y$ in $B_r(x)$ which lie in $R_y$ tends to zero as $r$ tends to infinity. In fact, we show a version of this result for bisectors. Let $Q$ be the space of unit area quadratic differentials on the surface $\Sigma$, and given $x \in T$, let $S(x)$ be the subset of $Q$ consisting of unit area quadratic differentials on $x$. The space $Q$ has a canonical measure, known as the Masur-Veech measure, which we shall denote $\mu$, and we will write $s_x$ for the conditional measure on $S(x)$ induced by $\mu$. We may think of $S(x)$ as the (co-)tangent space at $x$. Given $x, y \in T$, let $q_x(y)$ be the unit area quadratic differential on $x$ corresponding to the geodesic ray through $y$. Given a lattice point $\gamma y$, we will write $q(x, \gamma y)$ for the pair $(q_x(\gamma y), q_y(\gamma^{-1} x)) \in S(x) \times S(y)$. Given subsets $U \subset S(x)$ and $V \subset S(y)$, we may consider those lattice points $\gamma y$ which lie in the bisector determined by $U$ and $V$, i.e. those $\gamma y$ for which $q(x, \gamma y) \in U \times V$. If $X$ is a finite subset of $\Gamma$, we will write $|X, \text{condition}|$ to denote the number of elements $\gamma \in X$ which also satisfy condition. We say a surface of finite type is sporadic if it is a torus with at most one puncture, or a sphere with at most four punctures.

**Theorem 1.1.** Let $\Gamma$ be the mapping class group of a non-sporadic surface. Let $R \subset \Gamma$ be a set of elements of the for which there is an upper bound on their translation distance on the complex of curves. Let $x, y \in T$, and let $U \subset S(x)$ and $V \subset S(y)$ be Borel sets whose boundaries have measure zero. Then

$$|R_y \cap B_r(x), q(x, \gamma y) \in U \times V| / |\Gamma y \cap B_r(x), q(x, \gamma y) \in U \times V| \to 0, \text{ as } r \to \infty.$$

(1)

This shows that pseudo-Anosov elements are “generic” in the mapping class group, at least for one particular definition of generic, see Farb [Far06] for a discussion of similar questions. In the case in which $R$ consists of the non-pseudo-Anosov elements of the mapping class group, this result should also follow from the methods of Eskin and Mirzakhani [EM], which they use to show that the number of conjugacy classes of pseudo-Anosov elements of translation length at most $r$ on Teichmüller space is asymptotic to $e^{hr}/hr$. In the sporadic cases, the mapping class group is either finite, or already well understood, as the mapping class group is $SL(2, \mathbb{Z})$, up to finite index.

In the remainder of this section we give a brief outline of the argument. In Section 2 we describe the results we need from Athreya, Bufetov, Eskin and Mirzakhani [ABEM] and set up some notation. In Section 3 we review some useful properties of the visual boundary of Teichmüller space, and then in Section 4 we prove the main result.

### 1.1 Outline

Let $R \subset \Gamma$ be a set of elements for which there is an upper bound on their translation length on the complex of curves, for example, the set of non-pseudo-Anosov elements in the mapping class group. We wish to consider the distribution of elements of $R_y$ inside Teichmüller space $T$. In some parts of $T$ elements of $R_y$ are close together, and in other parts elements of $R_y$ are widely separated. We quantify this by by defining $R_k$ to be the $k$-dense elements of $R_y$, namely those elements of $R_y$ which are distance at most $k$ in the Teichmüller metric from some other element of $R$. If two lattice
points \(\gamma y\) and \(\gamma' y\) are a bounded Teichmüller distance apart, then \(\gamma\) and \(\gamma'\) are a bounded distance apart in the word metric on \(\Gamma\). In [Mah] we showed that the limit set of the \(k\)-dense elements in the word metric has measure zero in the Gromov boundary of the relative space, and we use this to show that the \(k\)-dense elements in Teichmüller space have a limit set in the visual boundary \(S(x)\) which has \(s_x\)-measure zero. A straightforward application of the results of [ABEM] then shows that the proportion of lattice points \(\Gamma y \cap B_r(x)\) which lie in \(R_k y\) tends to zero as \(r\) tends to infinity.

It remains to consider \(R y \setminus R_k y\), which we shall denote \(R_k' y\). We say a subset of \(T\) is \(k\)-separated, if any two elements of the set are Teichmüller distance at least \(k\) apart, so \(R_k' y\) is a \(k\)-separated subset of \(T\). Naively, one might hope that the proportion of \(k\)-separated elements of \(\Gamma y\) in \(B_r(x)\) is at most \(1/|\Gamma y \cap B_k(y)|\), as each lattice point \(\gamma y \in R_k' y\) is contained in a ball of radius \(k\) in Teichmüller space containing \(|\Gamma y \cap B_k(y)|\) other lattice points, none of which lie in \(R_k' y\). Such a bound would imply the required result, as this would give a collection of upper bounds for

\[
\lim_{r \to \infty} \frac{|R y \cap B_r(x)|}{|\Gamma y \cap B_r(x)|}
\]

which depend on \(k\), and furthermore these upper bounds would decay exponentially in \(k\), so this implies that the limit above is zero. However, this argument only works for those lattice points in the interior of \(B_r(x)\) for which \(B_k(\gamma y) \subseteq B_r(x)\). If a lattice point \(\gamma y\) is within distance \(k\) of \(\partial B_r(x)\), then many of the lattice points in \(B_k(\gamma y)\) may lie outside \(B_r(x)\), and a definite proportion of lattice points are close to the boundary, as the volume of \(B_r(x)\) grows exponentially. We use the mixing property of the geodesic flow to show that \(\partial B_r(x)\) becomes equidistributed on compact sets of the quotient space \(T/\Gamma\), and this in turn shows that the intersections of \(\partial B_r(x)\) with \(B_k(\gamma y)\) are evenly distributed. This implies that we can find an upper bound for the average number of lattice points of \(\Gamma y\) near some \(\gamma y\) close to the boundary, which do in fact lie inside \(B_r(x)\). In fact, we prove a result that works for bisectors, so we also need to show that the proportion of lattice points near the geodesics rays through \(\partial U\) tends to zero as \(r\) tends to infinity. These arguments using mixing originate in work of Margulis [Mar04], and our treatment of conditional mixing is essentially due to Eskin and McMullen [EM93], see also Gorodnik and Oh [GO07], for the higher rank case.

1.2 Acknowledgements

I would like to thank Alex Eskin, Howard Masur and Kasra Rafi for useful advice. This work was partially supported by NSF grant DMS-0706764.

2 The Teichmüller geodesic flow

In this section we review the work of Athreya, Bufetov, Eskin and Mirzakhani [ABEM] that we will use, fix notation, and use the mixing property of the geodesic flow to show a conditional mixing result, which is an analogue in Teichmüller space of a result of Eskin and McMullen [EM93] in the case of Lie groups.

Let \(\Sigma_{g,b}\) be an orientable surface of finite type, of genus \(g\) and with \(b\) punctures, which is not a torus with one or fewer punctures, or a sphere with four or fewer punctures. We will just write \(\Sigma\) for the surface if we do not need to explicitly refer to the genus or number of punctures. Let \(\Gamma\) be the mapping class group of \(\Sigma\), and we will consider \(\Gamma\) to be a metric space with the word metric coming from some fixed choice of generating set. We will write \(\mathcal{T}\) for the Teichmüller space of conformal
structures on $\Sigma$, with the Teichmüller metric, and $\mathcal{T}$ is homeomorphic to $\mathbb{R}^{6g-6+2b}$. We will write $B_r(x)$ for the ball of radius $r$, centered at $x$ in $\mathcal{T}$. A choice of basepoint $y$ for $\mathcal{T}$ determines a map from $\Gamma$ to $\mathcal{T}$, defined by $\gamma \mapsto \gamma y$. We shall write $\Gamma y$ for the image of $\Gamma$ under this map, which we shall call a Teichmüller lattice. The map $\gamma \mapsto \gamma y$ is coarsely distance decreasing, but is not a quasi-isometry.

Let $\mathcal{MF}$ be the space of measured foliations on the surface $\Sigma$, which is homeomorphic to $\mathbb{R}^{6g-6+2b}$, and let $\nu$ be the Thurston measure on $\mathcal{MF}$, which is preserved by the action of $\Gamma$. Let $Q$ be the space of unit-area quadratic differentials on $\Sigma$, let $\pi : Q \to \mathcal{T}$ be the projection from a quadratic differential to the underlying Riemann surface, and let $S(x)$ be the pre-image of $x$ in $Q$ under the projection. Hubbard and Masur [HM79] showed that for $x \in \mathcal{T}$, the map which sends a quadratic differential $q$ on $x$ to its vertical foliation $\text{Re}(q^{1/2})$ is a homeomorphism, and we shall write $\eta^+ : Q \to \mathcal{MF}$ for the restriction of this map to unit area quadratic differentials, and $\eta^- : Q \to \mathcal{MF}$ for the induced map from unit area quadratic differentials to projective equivalence classes of their vertical foliations. Similarly, the map that sends a quadratic differential to its horizontal foliation $\text{Im}(q^{1/2})$ is a homeomorphism, and we shall write $\eta^- : Q \to \mathcal{MF}$ for the restriction of this map to unit area quadratic differentials, and $\eta^- : Q \to \mathcal{MF}$ for the corresponding map to $\mathcal{MF}$. In particular, the restriction $\eta^- : S(x) \to \mathcal{MF}$ is a homeomorphism, as is the restriction of $\eta^-$. Masur [Mas82] and Veech [Vee82] showed that the space $Q$ carries a $\Gamma$-invariant smooth measure $\mu$, preserved by the Teichmüller geodesic flow, such that $\mu(Q/\Gamma)$ is finite, and this measure is unique up to rescaling, so we shall assume that $\mu(Q/\Gamma) = 1$. A quadratic differential $q$ is uniquely determined by its real and imaginary measured foliations, $\eta^+(q)$ and $\eta^-(q)$, so the map $\eta^+ \times \eta^-$ gives an embedding of $Q$ in $\mathcal{MF} \times \mathcal{MF}$. The Masur-Veech measure $\mu$ is then defined by $\mu(E) = (\nu \times \nu)(\text{Cone}(E))$, where $\text{Cone}(E)$ is the cone over $E$ based at the origin, i.e. $\{tq \mid q \in E, 0 < t \leq 1\}$. We shall write $m$ for the induced measure on Teichmüller space, i.e. $m = \pi_* \mu$.

Given a point $x \in \mathcal{T}$, the visual boundary at $x$ is the space of geodesic rays based at $x$, which may be identified with $S(x)$, the space of unit area quadratic differentials on $x$. We shall write $\mathcal{T}_x$ for $\mathcal{T} \cup S(x)$, the compactification of Teichmüller space using the visual boundary at $x$. We will write $q_\alpha(y)$ for the unit area quadratic differential on $x$ which corresponds to the Teichmüller geodesic ray starting at $x$ which passes through $y$. Given a subset $U$ of $S(x)$ we shall write $\text{Sect}_x(U)$ for the union of geodesic rays based at $x$ corresponding to quadratic differentials in $U$, so $y \in \text{Sect}_x(U)$ if and only if $q_\alpha(y) \in U$.

Let $g_\alpha$ be the Teichmüller geodesic flow on $Q$. The flow $g_\alpha$ commutes with the action of $\Gamma$, preserves the measure $\mu$, and preserves the following foliations. The strong stable foliation $\mathcal{F}^s$ has leaves of the form $\{q \in Q \mid \eta^+(q) = \text{const}\}$, and if $p$ is a point in $Q$ we will write $\alpha^s(p)$ for the leaf through $p$, i.e.

$$\alpha^s(p) = \{q \in Q \mid \eta^+(q) = \eta^+(p)\}.$$

The strong unstable foliation $\mathcal{F}^u$ has leaves of the form $\{q \in Q \mid \eta^-(q) = \text{const}\}$, and we shall write $\alpha^u(p)$ for the leaf through $p$, i.e.

$$\alpha^u(p) = \{q \in Q \mid \eta^-(q) = \eta^-(p)\}.$$

These foliations are $\Gamma$-invariant, so they descend to foliations on $Q/\Gamma$. The unstable foliation $\mathcal{F}^u$ has leaves of the form

$$\alpha^u(q) = \bigcup_{t \in \mathbb{R}} g_t \alpha^u(q),$$
while the stable foliation $\mathcal{F}^s$ has leaves of the form

$$\alpha^s(q) = \bigcup_{t \in \mathbb{R}} g_t \alpha^{ss}(q).$$

Each leaf $\alpha^+$ of the strongly unstable foliation $\mathcal{F}^uu$, as well as each leaf $\alpha^-$ of the strongly stable foliation $\mathcal{F}^{ss}$, carries a globally defined conditional measure $\mu_{\alpha^+}$, or $\mu_{\alpha^-}$, which is $\Gamma$-invariant, and has the property that

$$(g_t)_* \mu_{\alpha^+} = e^{ht} \mu_{g_t \alpha^+}$$

$$(g_t)_* \mu_{\alpha^-} = e^{-ht} \mu_{g_t \alpha^-},$$

where $h = 6g + 2b$ is the topological entropy of the geodesic flow $g_t$ on $Q/\Gamma$, with respect to $\mu$. Each leaf of $\mathcal{F}^s$ is homeomorphic to an open subset of $\mathcal{MF}$ via the map $\eta$, so the pullback of the Thurston measure $\nu$ on $\mathcal{MF}$ defines a conditional measure on $\mathcal{F}^s$. The foliations $\mathcal{F}^u$ and $\mathcal{F}^{ss}$ form a complementary pair in the sense of Margulis [Mar04], as do $\mathcal{F}^s$ and $\mathcal{F}^{uu}$.

Let $q \in Q$, and let $\alpha^u(q)$ be the leaf of the unstable foliation through $q$. By the Hubbard-Masur Theorem, the projection $\pi : Q \to \mathcal{T}$ induces a smooth bijection between $\alpha^u(q)$ and $\mathcal{T}$. The globally defined conditional measure $\mu_{\alpha^u(q)}$ on the leaf $\alpha^u(q)$ projects onto a measure on the Teichmüller space, which is absolutely continuous with respect to the smooth measure $\mathbf{m}$, so we may consider the Radon-Nikodym derivative of $\pi_*(\mu_{\alpha^u(q)})$ with respect to $\mathbf{m}$. Let $\lambda^t : Q \to \mathbb{R}$ be the function defined by

$$\frac{1}{\lambda^+(q)} = \frac{d(\pi_*(\mu_{\alpha^u(q)}))}{d\mathbf{m}},$$

and similarly define $\lambda^-$ to be the function

$$\frac{1}{\lambda^-(q)} = \frac{d(\pi_*(\mu_{\alpha^u(q)}))}{d\mathbf{m}},$$

where the Radon-Nikodym derivatives are evaluated at $\pi(q)$. Let $s_x$ be the conditional measure of $\mu$ on $S(x)$. The Hubbard-Masur function $\Lambda$ is defined to be

$$\Lambda(x) = \int_{S(x)} \lambda^+(q) \, ds_x(q) = \int_{S(x)} \lambda^-(q) \, ds_x(q).$$

The functions $\lambda^+, \lambda^-$ and $\Lambda$ are all $\Gamma$-invariant.

Athreya, Bufetov, Eskin and Mirzakhani [ABEM], showed how to count the number of images of $x$ in the ball of radius $r$ in Teichmüller space, with the Teichmüller metric. If $X$ is a finite subset of $\Gamma$, we will write $|X, \text{condition}|$ to denote the number of elements $\gamma \in X$ which also satisfy $\text{condition}$. Let $B_r(x)$ be the ball of radius $r$ in the Teichmüller metric, centered at $x$ in Teichmüller space, and let $U \subset S(x)$ and $V \subset S(y)$ be Borel sets. We shall write $\partial U$ for the boundary of $U$, which is $\overline{U} \cap S(x) \setminus U$, and we shall always require that the sets $U$ and $V$ have boundaries of measure zero, with respect to either $s_x$ or $s_y$, as appropriate. Given $\gamma \in \Gamma$, we will write $q(x, \gamma y)$ to denote the pair of quadratic differentials $(q_2(\gamma y), q_y(\gamma^{-1}x)) \in S(x) \times S(y)$, and so $\gamma y \in \text{Sect}_x(U)$ if and only if $q(x, y) \in U \times S(y)$. The following result is shown in [ABEM].
Theorem 2.1 (ABEM Theorem 7.2)]. Let \( x, y \in \mathcal{T} \), and let \( U \subset S(x) \) and \( V \subset S(y) \) be Borel sets whose boundaries have measure zero. Then as \( r \to \infty \),

\[
|\Gamma y \cap B_r(x), q(x, \gamma y) \in U \times V| \sim \frac{1}{r^2} e^{h_r} \int_U \lambda^-(q) ds_x(q) \int_V \lambda^+(q) ds_y(q).
\]

In [ABEM] the result is stated for closed surfaces, but the proof also works for non-sporadic surfaces of finite type.

Veech [Vee86] showed that the Teichmüller geodesic flow is mixing, i.e. let \( \alpha \) and \( \beta \) be in \( L^2(Q/\Gamma) \). Then

\[
\lim_{t \to \infty} \int_{Q/\Gamma} \alpha(gq) \beta(q) d\mu(q) = \int_{Q/\Gamma} \alpha(q) d\mu(q) \int_{Q/\Gamma} \beta(q) d\mu(q).
\]

Following Eskin and McMullen [EM93], we now observe that the Teichmüller geodesic flow is also mixing for conditional measures.

Proposition 2.2. Let \( x \in \mathcal{T} \), let \( \alpha \) and \( \beta \) be continuous non-negative functions on \( Q/\Gamma \), with compact support, and let \( U \subset S(x) \) be a Borel set whose boundary has measure zero. Then

\[
\lim_{t \to \infty} \int_U \alpha(gq) \beta(q) ds_x(q) = \int_{Q/\Gamma} \alpha(q) d\mu(q) \int_U \beta(q) ds_x(q).
\]

**Proof.** Suppose that \( U \) is an open set. Let \( U_\epsilon = \bigcup_{|s| < \epsilon} q(U) \), and let \( I_\epsilon \) be a continuous approximation to the characteristic function of \( U_\epsilon \), i.e. \( I_\epsilon \) has maximum value one at all points of \( U_\epsilon \) and is zero outside a small neighbourhood of \( U_\epsilon \). By the definition of conditional measure,

\[
\lim_{t \to \infty} \int_U \alpha(gq) \beta(q) ds_x(q) = \lim_{t \to \infty} \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{Q/\Gamma} \alpha(gq) I_\epsilon(q) \beta(q) d\mu(q).
\]

As \( \alpha(q) \) and \( I_\epsilon(q) \beta(q) \) are continuous functions with compact support, they are almost constant on sufficiently short segments of the geodesic flow, i.e. for all \( \delta > 0 \) there is an \( \epsilon > 0 \) such that \( |\alpha(gq) - \alpha(q)| \leq \delta \) and \( |I_\epsilon(gq) \beta(gq) - I_\epsilon(q) \beta(q)| \leq \delta \) for all \( q \in Q/\Gamma \), and for all \( |s| < \epsilon \). As the geodesic flow preserves the lengths of flow line segments, this implies that \( |\alpha(g_{t+s}q) - \alpha(gq)| \leq \delta \) for all \( q \in Q/\Gamma, |s| < \epsilon \), and for all \( t \). Therefore the inner limit convergences uniformly, independently of \( t \), and so we may swap the order of the limits. The mixing property of the Teichmüller flow implies

\[
\lim_{\epsilon \to 0} \lim_{t \to \infty} \frac{1}{2\epsilon} \int_{Q/\Gamma} \alpha(gq) I_\epsilon(q) \beta(q) d\mu(q) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{Q/\Gamma} \alpha(gq) d\mu(q) \int_{Q/\Gamma} I_\epsilon(q) \beta(q) d\mu(q),
\]

and by the definition of conditional measure, this is equal to

\[
\int_{Q/\Gamma} \alpha(q) d\mu(q) \int_U \beta(q) ds_x(q).
\]

The result now follows for Borel sets with measure zero boundaries by approximating them by open sets. \( \square \)
3 Limit sets

Let $R$ be a set of elements of $\Gamma$ for which there is an upper bound on their translation distance on the complex of curves. Let $R_k y$ be the $k$-dense subset of $R y$ in Teichmüller space, i.e. $R_k$ consists of those $\gamma \in R$ such that there is some other element $\gamma' \in R$ with $d_T(\gamma y, \gamma'y) \leq k$. We will write $R_k^c y$ for the complement of $R_k y$ in $R y$, and so $R_k y$ is a $k$-separated subset of $\mathcal{T}$, as any two elements of $R_k y$ are distance at least $k$ apart. The main aim of this section is to show that the limit set of $R_k y$ in the visual boundary $S(x)$ has $s_x$-measure zero, Lemma 3.5, where $s_x$ is the conditional measure induced by the Masur-Veech measure $\mu$ on $Q$. We will also show that the limit set of a metric $k$-neighbourhood of geodesic ray in the visual boundary is contained in the zero set of the vertical foliation of the geodesic ray, Lemma 3.6. We start by reviewing the properties of some useful spaces associated to the mapping class group.

By work of Masur and Minsky [MM99], the following three spaces associated with a surface are quasi-isometric $\delta$-hyperbolic spaces.

- The complex of curves $C(\Sigma)$, which is a simplicial complex whose vertices are isotopy classes of simple closed curves, and whose simplices are spanned by collections of disjoint simple closed curves. We shall write $d_C$ for the metric induced on the 1-skeleton of $C(\Sigma)$ by setting every edge length equal to one.

- Electrified Teichmüller space $\mathcal{T}_{el}$, which is the metric space obtained by adding a vertex $v_\alpha$ for every isotopy class of simple closed curve $\alpha$ on $\Sigma$, and then adding an edge of length one half connecting every point of $\text{Thin}_\alpha(\alpha)$ to $v_\alpha$.

- The relative space $\hat{G}$, which consists of the mapping class group $G$, with a word metric $\hat{d}$ coming from the union of a finite generating set for $G$, together with a collection of subgroups consisting of stabilizers of representatives of simple closed curves under the action of the mapping class group. We will refer to $\hat{d}(1, \gamma)$ as the relative length of $\gamma$.

Klarreich [Kla] identified the Gromov boundary of these spaces, which we now describe. Thurston constructed a boundary for $\mathcal{T}$ consisting of the space of projective measured foliations, which we shall denote $\mathcal{PMF}$. The space $\mathcal{PMF}$ is a sphere of dimension $6g - 7 + 2b$, and $\mathcal{T} \cup \mathcal{PMF}$ is homeomorphic to a ball on which $\Gamma$ acts continuously. The Thurston and visual boundaries are distinct, see for example Kerckhoff [Ker80] and Masur [Mas75], and in particular, the action of $\Gamma$ does not extend continuously to the visual boundary. There is an inclusion map from Teichmüller space $\mathcal{T}$ to electrified Teichmüller space, $\mathcal{T}_{el}$. This inclusion map does not extend continuously to the entire Thurston boundary $\mathcal{PMF}$, but Klarreich [Kla] shows that it does extend continuously to the set of minimal foliations $\mathcal{F}_{min}$ in $\mathcal{PMF}$, and that $\mathcal{F}_{min}$ is the Gromov boundary for $C(\Sigma)$, and hence for the spaces listed above which are quasi-isometric to $C(\Sigma)$. The set $\mathcal{F}_{min}$ is the set of minimal foliations in $\mathcal{PMF}$, i.e. those foliations for which no simple closed curve is contained in a (possibly singular) leaf of the foliation, and furthermore, two foliations are identified if they are topologically equivalent, even if they have different measures.

Let $Y$ be a subset of the relative space $\hat{G}$, and let $L$ be a real number. We define a relative $L$-horoball neighbourhood of $Y$, which we shall denote $\hat{O}_L(Y)$, to be the union of balls in $\hat{G}$ centered at $y \in Y$, of radius $|\hat{y}| + L$, i.e.

$$\hat{O}_L(Y) = \bigcup_{y \in Y} \hat{B}_{|\hat{y}| + L}(y).$$
We emphasize that this definition uses relative distance in $\hat{G}$. We now show that the limit set of $\hat{O}_L(Y)$ in the Gromov boundary $F_{\text{min}}$ is the same as the limit set of $Y$ in $F_{\text{min}}$.

**Lemma 3.1.** Let $Y$ be a subset of the relative space $\hat{G}$. Then the limit set of $\hat{O}_L(Y)$ in the Gromov boundary is equal to the limit set of $Y$.

**Proof.** We will choose the identity element 1 to be a basepoint in the relative space $\hat{G}$, and we will write $\delta$ for the constant of hyperbolicity of $\hat{G}$. We will write $(x|y)$ for the Gromov product $\frac{1}{\delta}(d(1,x) + d(1,y) - d(x,y))$, which is equal to the distance from 1 to a geodesic $[x,y]$, up to an additive error which only depends on $\delta$, see for example Bridson and Haefliger [BH99] Chapter III.H.

Let $x_n$ be a sequence in $\hat{O}_L(Y)$ which converges to a point in the Gromov boundary, so in particular $d(1,x_n)$ tends to infinity. By the definition of $\hat{O}_L(Y)$, each $x_n$ lies in a ball of radius $d(x_0,y_n) + L$, for some $y_n$ in $Y$. As $d(1,x_n)$ tends to infinity, this implies that $d(1,y_n)$ also tends to infinity. We now show that the sequence $y_n$ converges to the same limit point as the sequence $x_n$, by showing that the Gromov product $(x_n|y_n)$ tends to infinity. Suppose not, then possibly after passing to a subsequence, there is a number $K$ such that $(x_n|y_n) < K$, for all $n$. Let $p_n$ be the nearest point projection of $x_n$ to a geodesic $[1,y_n]$, then it is well known that in a $\delta$-hyperbolic space, $d(1,p_n)$ is equal to the distance from 1 to a geodesic $[x_n,y_n]$, up to an additive error that only depends on $\delta$, see for example [Mah10] Proposition 3.2. Therefore $d(1,p_n)$ is equal to the Gromov product $(x_n|y_n)$, and so is at most $K$, up to an additive error which depends only on $\delta$. This implies that $d(p_n,y_n)$ is roughly $d(1,y_n) - K$, and as $d(x_n,y_n) \leq d(1,y_n) + L$, and $p_n$ is a nearest point projection of $x_n$ to $[1,y_n]$, this implies that $d(p_n,y_n)$ is equal to $L + K$, up to additive error that depends only on $\delta$. Therefore $d(1,x_n)$ is at most $L + 2K$, again up to additive error that depends only on $\delta$, which contradicts the fact that $d(1,x_n)$ tends to infinity.

We now show that the limit set of the orbit of a point in Teichmüller space under a horoball neighbourhood of a centralizer has measure zero in the visual boundary.

**Lemma 3.2.** Let $\gamma$ be an element of the mapping class group $\Gamma$ which does not lie in the center of $\Gamma$. Let $H = \hat{O}_L(C(\gamma))$ be a relative horoball neighbourhood of the centralizer $C(\gamma)$ in $\hat{G}$. Then the limit set of $Hy$ in the visual boundary $\hat{T}_x$ has $s_x$-measure zero.

**Proof.** The set of quadratic differentials with uniquely ergodic initial foliations has full measure in $S(x)$ with respect to $s_x$, as shown by Masur [Mas82] and Veech [Vee82]. Therefore, it suffices to consider the subset of $\overline{Hy}$ consisting of limit points which are uniquely ergodic. It is well known that if a sequence of points $x_n$ in $T$ converges to a uniquely ergodic foliation $F \in \mathcal{PMF}$, then the corresponding sequence of initial measured foliations also converges to the same uniquely ergodic foliation, see for example Klarreich [Kla] Proposition 5.3. Therefore the uniquely ergodic limit points of $\overline{Hy}$ are precisely the uniquely ergodic foliations in the limit set of $H$ in the Gromov boundary $F_{\text{min}}$. This in turn is equal to the limit set of $\hat{O}(\gamma)$ in the Gromov boundary, which is contained in the fixed set of $\gamma$ by [Mah] Proposition 2.5. The conditional measure $s_x$ on $S(x)$ is induced from the measure $\mu$ on $Q$, which in turn is defined in terms of the Thurston measure $\nu$ on $\mathcal{MF}$, and so in order to show a subset $U$ of $S(x)$ has $s_x$-measure zero, it suffices to show that the corresponding cone over the uniquely ergodic foliations in $U$ has $\nu$-measure zero in $\mathcal{MF}$. If $\gamma$ is pseudo-Anosov, then the fixed set consists of two points, which has measure zero. If $\gamma$ is reducible, then the fixed set contains no uniquely ergodic foliations, and so has measure zero. Finally, if $\gamma$ is a non-central
Elements of the mapping class group act as simplicial isometries on the complex of curves. Recall that the translation length of an isometry $\gamma$ is

$$\tau_\gamma = \lim_{n \to \infty} \frac{1}{n} d_C(x, \gamma^n x),$$

and this is independent of the choice of point $x \in C(\Sigma)$. We now show that for the mapping class group, the translation length of an element $\gamma$ is coarsely equivalent to the shortest relative length of any conjugate of $\gamma$, which we shall denote $|\overline{\gamma}|_c$.

**Proposition 3.3.** There are constants $K_1$ and $K_2$, which depend on $\Sigma$, such that for any element $\gamma$ in the mapping class group

$$\frac{1}{K_1} \tau_\gamma - K_2 \leq |\overline{\gamma}|_c \leq K_1 \tau_\gamma + K_2,$$

where $\tau_\gamma$ is the translation length of $\gamma$ on the complex of curves, and $|\overline{\gamma}|_c$ is the shortest relative length of any conjugate of $\gamma$.

**Proof.** The quasi-isometry from $\hat{G}$ to $C(\Sigma)$ is defined to be the map which sends $\gamma$ to $\gamma x_0$, for some choice of basepoint $x_0$ in $C(\Sigma)$, so there are constants $K$ and $k$ such that

$$\frac{1}{K} \hat{d}(\gamma, \gamma') - k \leq d_C(\gamma x_0, \gamma' x_0) \leq K \hat{d}(\gamma, \gamma') + k,$$

for all $\gamma$ and $\gamma'$ in $\Gamma$.

If $\gamma$ is periodic, then the translation length of $\gamma$ on $C(\Sigma)$ is zero. There are only finitely many conjugacy classes of periodic elements in the mapping class group, so the proposition holds for periodic elements as long as $K_2$ is at least the maximum of $|\overline{\gamma}|_c$ over all periodic elements $\gamma$.

If $\gamma$ is reducible, then again its translation length on $C(\Sigma)$ is zero. Every reducible element preserves a collection of disjoint simple closed curves, so there is a simple closed curve which is moved distance at most one by $\gamma$. The mapping class group $\Gamma$ acts coarsely transitively on $C(\Sigma)$, in fact every simple closed curve may be moved to within distance one of the basepoint $x_0$, so this implies that there is a conjugate $\gamma'$ of $\gamma$ with $d_C(x_0, \gamma' x_0) \leq 3$. This implies that $\hat{d}(1, \gamma') \leq K(3 + k)$, and so $|\overline{\gamma}|_c \leq K(3 + k)$. Therefore the proposition holds for reducible elements, as long as $K_2$ is at least $K(3 + k)$.

Finally, we consider the case in which $\gamma$ is pseudo-Anosov. We may assume that we have chosen $\gamma$ such that $\hat{d}(1, \gamma) = |\overline{\gamma}|_c$. By the triangle inequality, the distance $\gamma$ moves any point in $C(\Sigma)$ is an upper bound for the translation length of $\gamma$, so $\tau_\gamma \leq d_C(x_0, \gamma x_0) \leq K \hat{d}(1, \gamma) + k$. This gives the left hand inequality in the proposition above, with $K_1 = K$, and $K_2 = k$. By work of Masur and Minsky [MM00], $\gamma$ has an axis $\alpha_\gamma$, which is a bi-infinite geodesic such that $\alpha_\gamma$ and $\gamma \alpha_\gamma$ are $2\delta$-fellow travellers, where $\delta$ is the constant of hyperbolicity for $C(\Sigma)$. The mapping class group acts coarsely transitively on $C(\Sigma)$, so with loss of generality we may assume that we have chosen $\gamma$ such that its axis $\alpha_\gamma$ passes within distance one of the basepoint $x_0$. We will let $p_n$ be a closest point on $\alpha_\gamma$ to $\gamma^n x_0$. As $\alpha_\gamma$ is an axis for any power of $\gamma$, this implies that $d_C(\gamma^n x_0, p_n) \leq 2\delta$ for all $n$, and so $d_C(q_n, q_{n+1}) \geq d_C(q_0, q_1) - 8\delta$ for each $n$. As the $q_n$ all lie on a common geodesic, this
implies that \( d_C(q_0, q_n) \geq n(d_C(q_0, q_1) - 8\delta) \), and so \( \tau_{\gamma} \geq d_C(x_0, \gamma x_0) - 10\delta - 2 \). Therefore \( \tau_{\gamma} \geq \frac{1}{K} d(1, \gamma) - k - 10\delta - 2 \). This gives the right hand inequality in the proposition, with \( K_1 = K \), and \( K_2 = K(k + 10\delta + 2) \).

We have shown that the inequalities in the proposition hold for each of the three types of elements of the mapping class group, so if we choose the constants \( K_1 \) and \( K_2 \) to be the maximum of the constants we have obtained in each of the three cases above, then the inequalities hold for all elements of the mapping class group, as required.

Proposition \([3.5]\) above shows that a set \( R \) of mapping class group elements which have an upper bound on their translation distance on \( C(\Sigma) \), is also a set of elements for which there is an upper bound on the shortest relative length of elements in the conjugacy class of each element of \( R \).

Theorem 3.4. \([3.4]\) Let \( R \) be a set of elements in the mapping class group \( \Gamma \), each of which is conjugate to an element of relative length at most \( B \). Then, given \( B \) and \( \gamma \in \Gamma \), there is a constant \( L \) such that \( R \cap R\gamma \) is contained in an \( L \)-horoball neighbourhood of the centralizer of \( \gamma \).

We now use this to show that the limit set in the visual boundary \( S(x) \) of the \( k \)-dense elements \( R_k y \) in \( \mathcal{T} \) has \( s_x \)-measure zero.

**Lemma 3.5.** For a non-sporadic surface whose mapping class group has trivial center, the closure of \( R_k y \) in \( \mathcal{T}_x \) has \( s_x \)-measure zero in \( S(x) \).

**Proof.** Let \( \rho \in R_k \), then, by the definition of \( R_k \), there is a \( \rho' \in R \setminus \rho \) such that \( d_T(\rho y, \rho' y) \leq k \), so \( \rho = \rho' \gamma \) for some non-trivial \( \gamma \) with \( d_T(y, \gamma y) \leq k \). There are only finitely many such \( \gamma \), as there are only finitely many elements of \( \Gamma y \) in \( B_k(y) \). Therefore \( R_k \) is contained in a finite union of sets of the form \( R \cap R\gamma \), for elements \( \gamma \) with \( d_T(y, \gamma y) \leq k \). By Theorem 3.3 a set of the form \( R \cap R\gamma \) is contained in a relative horoball neighbourhood of the centralizer of \( \gamma \), i.e. \( \widehat{O}_L(C(\gamma)) \), where the constant \( L \) depends on \( R \), \( d(1, \gamma) \) and \( \Gamma \). As no element \( \gamma \) is central, Lemma 3.2 implies that the limit set of \( (R \cap R\gamma) y \) has \( s_x \)-measure zero in \( S(x) \), and as the limit set of \( R_k y \) is contained in a finite union of these limit sets, the limit set of \( R_k y \) also has \( s_x \)-measure zero.

Finally, it is well known that the limit set in the visual boundary \( S(x) \) of a \( k \)-neighbourhood of a geodesic ray based at \( x \) is contained in the zero set of the vertical foliation of the geodesic ray. We will write \( Z(F) \) for the set of foliations with zero intersection number with \( F \), and given \( X \subset \mathcal{T} \) we will write \( N_k(X) \) to be a metric \( k \)-neighbourhood of \( X \), using the Teichmüller metric.

**Lemma 3.6.** Let \( q_t \) be a geodesic ray based at \( x \) with vertical foliation \( F = \mathcal{T}^+(q_t) \). Then the limit set of \( N_k(q_t) \) in the visual boundary \( S(x) \) is contained in \( Z(F) \).

**Proof.** This follows immediately from work of Ivanov \([Iva01]\), which shows that if \( p \) and \( q \) are quadratic differentials with vertical foliations with non-zero intersection number, then the corresponding Teichmüller geodesic rays \( p_t \) and \( q_t \) diverge, i.e. \( d_T(p_t, q_t) \to \infty \) as \( t \to \infty \). In fact, divergent Teichmüller rays are completely classified in terms of their vertical foliations, see Ivanov \([Iva01]\), Lenzhen and Masur \([LM10]\) and Masur \([Mas75, Mas80]\).
4 Asymptotics

Let $R$ be a subset of $\Gamma$ consisting of elements for which there is an upper bound on their translation distance on the complex of curves. In this section we will prove Theorem 1.1 i.e. we will show that the proportion of lattice points $\Gamma y$ in $B_r(x)$ which lie in $Ry$ tends to zero as $r$ tends to infinity.

As before, let $R_k$ be the subset of $R$ consisting of those elements whose images in $T$ are $k$-dense in $T$, and let $R'_k$ be the complement $Ry \setminus R_k$, and so $R'_k$ is a $k$-separated subset of $T$. In Lemma 3.5 we showed that the limit set of $R_k$ in the visual boundary $S(x)$ has $s_x$-measure zero. We now show that this implies that the proportion of lattice points in $\Gamma \cap B_r(x)$ which lie in $R_k$ tends to zero as $r$ tends to infinity.

**Lemma 4.1.** Let $x, y \in T$, and let $U \subset S(x)$ and $V \subset S(y)$ be Borel sets whose boundaries have measure zero. Let $X$ be a closed subset of $\overline{T}_x$ such that $X \cap S(x)$ has measure zero with respect to $s_x$. Then

$$\frac{|X \cap \Gamma y \cap B_r(x), \ q(x, \gamma y) \in U \times V|}{|\Gamma y \cap B_r(x), \ q(x, \gamma y) \in U \times V|} \rightarrow 0, \ \text{as} \ r \rightarrow \infty.$$  

**Proof.** Consider an open set $W \subset U$ whose closure is disjoint from $X$. As $X$ is closed, there is a number $D$, depending on $V$, such that $\text{Sect}_x(W) \setminus B_D(x)$ is disjoint from $X$. Then the proportion of lattice points inside $B_r(x) \cap \text{Sect}_x(W)$ which lie in $X$ is at most the proportion of lattice points in $B_r(x) \cap \text{Sect}_x(U)$ which lie outside $B_D(x)$, and this decays asymptotically exponentially in $r$, and in particular tends to zero. Therefore, by Theorem 2.1 the limiting proportion of lattice points which lie in $X$ inside $B_r(x) \cap \text{Sect}_x(U)$ is at most

$$\lim_{r \rightarrow \infty} \frac{|X \cap \Gamma y \cap B_r(x), \ q(x, \gamma y) \in U \times V|}{|\Gamma y \cap B_r(x), \ q(x, \gamma y) \in U \times V|} \leq \frac{\int_{U \setminus W} \lambda^-(q) ds_x(q)}{\int_U \lambda^-(q) ds_x(q)}.$$  

The function $\lambda^-$ is absolutely continuous with respect to $s_x$, and we may choose a sequence of open sets $W_n \subset U$ such that the $s_x$-measure of $W_n$ tends to the $s_x$-measure of $U$. This implies that the proportion of lattice points in $X$ tends to zero, as $r$ tends to infinity. $\square$

We now complete the proof of Theorem 1.1.

**Proof.** We will first assume that the mapping class group has trivial center, which covers all cases except for the genus two surface and the twice-punctured sphere, which we consider at the very end. Let $R_k$ be the $k$-dense subset of $Ry$ in Teichmüller space $T$, and let $R'_k$ be its complement $Ry \setminus R_k$, which is a $k$-separated set in $T$. Therefore we may rewrite the fraction in line (1) of the statement of Theorem 1.1 as

$$\frac{|R_k \cap B_r(x), \ q(x, \gamma y) \in U \times V|}{|\Gamma y \cap B_r(x), \ q(x, \gamma y) \in U \times V|} + \frac{|R'_k \cap B_r(x), \ q(x, \gamma y) \in U \times V|}{|\Gamma y \cap B_r(x), \ q(x, \gamma y) \in U \times V|}.$$  

We have shown in Lemma 3.5 that $R_k$ has $s_x$-measure zero in $S(x)$, so the first term tends to zero as $r$ tends to infinity for any $k$, by Lemma 1.1. Therefore

$$\lim_{r \rightarrow \infty} \frac{|Ry \cap B_r(x), \ q(x, \gamma y) \in U \times V|}{|\Gamma y \cap B_r(x), \ q(x, \gamma y) \in U \times V|} = \lim_{r \rightarrow \infty} \frac{|R'_k \cap B_r(x), \ q(x, \gamma y) \in U \times V|}{|\Gamma y \cap B_r(x), \ q(x, \gamma y) \in U \times V|}. \tag{2}$$

Consider the subset of Teichmüller space consisting of $B_r(x) \cap \text{Sect}_x(U)$. We divide this set into three parts, which we now describe.
• The vertical region: \( V_r = N_k(\text{Sect}_x(\partial U)) \cap B_r(x) \), where \( N_k(X) \) is a metric \( k \)-neighbourhood of \( X \subset T \).

• The interior region: \( I_r = (\text{Sect}_x(U) \setminus V_r) \cap B_{r-k}(x) \)

• The annular region: \( A_r = \text{Sect}_x(U) \setminus (I_r \cup V_r) \)

\[
\begin{array}{c}
\text{Sect}_x(U) \\
V_r \\
A_r \\
I_r \\
\gamma y \\
V_r \\
x \\
B_{r-k}(x) \\
B_r(x)
\end{array}
\]

Figure 1: Dividing a sector into three regions.

Therefore we may rewrite line (2) as a sum of three terms.

\[
\lim_{r \to \infty} \left( \frac{|R'_{k,y} \cap V_r, q_y(\gamma^{-1}) \in V|}{|\Gamma_y \cap B_r(x), q(x, \gamma y) \in U \times V|} + \frac{|R'_{k,y} \cap I_r, q_y(\gamma^{-1}) \in V|}{|\Gamma_y \cap B_r(x), q(x, \gamma y) \in U \times V|} + \frac{|R'_{k,y} \cap A_r, q_y(\gamma^{-1}) \in V|}{|\Gamma_y \cap B_r(x), q(x, \gamma y) \in U \times V|} \right)
\]

We shall consider each term in turn. In each case, we find upper bounds for the term which may depend on \( k \), or additional parameters, and we then show that there is some sequence of upper bounds which tends to zero. We start by considering the lattice points \( R'_{k,y} \) in the vertical region \( V_r \).

**Claim 4.2.** For any fixed \( k \), the proportion of lattice points in \( B_r(x) \) with \( q(x, \gamma y) \in U \times V \), which lie in both \( R'_{k,y} \) and the vertical region \( V_r \), tends to zero as \( r \to \infty \), i.e.

\[
\lim_{r \to \infty} \frac{|R'_{k,y} \cap V_r, q(x, \gamma y) \in U \times V|}{|\Gamma_y \cap B_r(x), q(x, \gamma y) \in U \times V|} = 0.
\]

**Proof.** The number of elements of \( R'_{k,y} \) in the vertical boundary \( V_r \) is at most the total number of lattice points \( \Gamma_y \) in \( V_r \), i.e.

\[
|R'_{k,y} \cap V_r, q(x, \gamma y) \in U \times V| \leq |\Gamma_y \cap V_r, q(x, \gamma y) \in U \times V|.
\]

The set \( V_r \) is contained in \( N_k(\text{Sect}_x(\partial U)) \), and by Lemma 3.6, the limit set of \( N_k(\text{Sect}_x(\partial U)) \) is contained in \( Z(\partial U) \), where \( Z(\partial U) \) is all foliations with zero intersection with some \( F \) in \( \partial U \), the frontier of \( U \). As \( \partial U \) has \( s_x \)-measure zero, the set \( Z(\partial U) \) also has \( s_x \)-measure zero. Let \( W_1 \supset W_2 \supset \cdots \) be a nested sequence of open neighbourhoods of \( Z(\partial U) \), such that \( \bigcap W_n = Z(\partial U) \). Then for each \( n \) there is a \( D_n \) such that

\[
N_k(\text{Sect}_x(\partial U)) \setminus B_{D_n}(x) \subset \text{Sect}_x(W_n),
\]

12
and \( B_{D_n}(x) \) contains only finitely many lattice points. This implies
\[
\lim_{r \to \infty} \frac{|R'_y \cap V_r, q(x, \gamma y) \in U \times V|}{|\Gamma_y \cap B_r(x), q(x, \gamma y) \in U \times V|} \leq \lim_{r \to \infty} \frac{|\Gamma_y \cap B_r(x), q(x, \gamma y) \in W_n \times V|}{|\Gamma_y \cap B_r(x), q(x, \gamma y) \in U \times V|},
\]
for any fixed \( n \). Using Theorem 2.1, we may take the limit as \( r \) tends to infinity on the right hand side, to obtain
\[
\lim_{r \to \infty} \frac{|R'_y \cap V_r, q(x, \gamma y) \in U \times V|}{|\Gamma_y \cap B_r(x), q(x, \gamma y) \in U \times V|} \leq \frac{\int_{W_n} \lambda^-(q)ds_x(q)}{\int_U \lambda^-(q)ds_x(q)},
\]
and this upper bound holds for any fixed \( n \). However, the top integral on the right hand side tends to zero as \( n \) tends to infinity, as the \( s_x \)-measure of \( W_n \) tends to zero and \( \lambda^- \) is absolutely continuous with respect to \( s_x \).

We have shown that the first term on the right hand side of (3) tends to zero as \( r \to \infty \). We now consider the lattice points \( R'_k y \) in the interior region \( I_r \). We find an upper bound for the proportion of lattice points \( \gamma y \) in \( B_r(x) \) with \( q(x, \gamma y) \in U \times V \), which lie in both \( R'_k y \) and the interior region \( I_r \). This upper bound depends on \( k \), and we show that in fact it decays exponentially in \( k \).

**Claim 4.3.** There is an upper bound for the limiting proportion of lattice points in \( I_r \), with \( q(x, \gamma y) \in U \times V \) and which are contained in \( R'_k y \), and furthermore, this upper bound decays exponentially in \( k \). In particular,
\[
\lim_{r \to \infty} \frac{|R'_y \cap I_r, q(x, \gamma y) \in U \times V|}{|\Gamma_y \cap B_r(x), q(x, \gamma y) \in U \times V|} \to 0, \quad \text{as } k \to \infty.
\]

**Proof.** Every element of \( R'_k y \) in the interior set \( I_r \) is surrounded by a ball of radius \( k \), which contains \( |\Gamma_y \cap B_k(y)| \) lattice points, at most one of which lies in \( R'_k y \). Every ball of radius \( k \) in Teichmüller space which intersects \( I_r \) is contained in \( B_r(x) \cap Sector(y) \). The terminal quadratic differentials for these lattice points need not be contained in \( V \), so we obtain an upper bound in terms of \( U \times S(y) \) instead of \( U \times V \), namely,
\[
|R'_y \cap I_r, q(x, \gamma y) \in U \times V| \leq \frac{1}{|\Gamma_y \cap B_k(y)|} |\Gamma_y \cap B_r(x), q(x, \gamma y) \in U \times S(y)|.
\]

Therefore, by Theorem 2.1 we obtained the following upper bound,
\[
\lim_{r \to \infty} \frac{|R'_y \cap I_r, q(x, \gamma y) \in U \times V|}{|\Gamma_y \cap B_r(x), q(x, \gamma y) \in U \times V|} \leq \frac{1}{|\Gamma_y \cap B_k(y)|} \frac{\int_{S(y)} \lambda^+(q)ds_y(q)}{\int_U \lambda^+(q)ds_x(q)}.
\]

The result now follows by applying Theorem 2.1 to \( |\Gamma_y \cap B_k(y)| \), which shows that the denominator of the right hand side grows exponentially in \( k \).

Finally, we consider the lattice points in the annular region \( A_r \).

**Claim 4.4.** The limiting proportion of lattice points in the annular region \( A_r \), with \( q(x, \gamma y) \in U \times V \) and which lie in \( R'_k y \), has an upper bound which depends on \( k \), and this upper bound tends to zero as \( k \to \infty \), i.e.
\[
\lim_{r \to \infty} \frac{|R'_y \cap A_r, q(x, \gamma y) \in U \times V|}{|\Gamma_y \cap B_r(x), q(x, \gamma y) \in U \times V|} \to 0, \quad \text{as } k \to \infty.
\]
Proof. We will construct a collection of upper bounds $C(k, d)$ for the term above, which depend on $k$, and an extra parameter $d$, which may be any positive number less than $k/2$. We will then show that there is a sequence of these upper bounds which tends to zero, by finding an upper bound for $\lim_{k \to \infty} C(k, d)$ which depends on $d$, and then showing that $\lim_{d \to \infty} (\lim_{k \to \infty} C(k, d)) = 0$.

We now give a brief overview of the argument, before giving the details. Each lattice point $\gamma y$ in $R_k^d \cap A_r$, is surrounded by a ball of radius $k$ disjoint from any other element of $R_k^d$, and which contains $|\Gamma y \cap B_k(y)|$ lattice points of $\Gamma y$. However, many of these points may lie outside $B_r(x)$, so we cannot use $1/|\Gamma y \cap B_k(y)|$ as an estimate for the proportion of lattice points in $\Gamma y \cap A_r$, which also lie in $R_k^d$. Recall that $q_{\gamma y}(x)$ is the terminal quadratic differential of the geodesic from $x$ to $\gamma y$, so $\pi g_{k/2 q_{\gamma y}}(x)$, is the point distance $k/2$ from $\gamma y$ along the geodesic from $x$ to $\gamma y$. The ball of radius $k/2$ centered at $\pi g_{k/2 q_{\gamma y}}(x)$ is contained in $B_k(\gamma y) \cap B_r(x)$, but $\pi g_{k/2 q_{\gamma y}}(x)$ is not necessarily an element of $\Gamma y$, and a priori, this ball need not contain any lattice points. Let $N_d(\Gamma y)$ be a metric $d$-neighbourhood of the Teichmüller lattice. If $\pi g_{k/2 q_{\gamma y}}(x) \in N_d(\Gamma y)$ then there is a lattice point $\gamma'$ distance at most $d$ from $\pi g_{k/2 q_{\gamma y}}(x)$, and so there are at least $|\Gamma y \cap B_{k/2-d}(y)|$ lattice points of $\Gamma y$ in $B_k(\gamma y) \cap B_r(x)$, as illustrated below in Figure 2.

![Figure 2: The case in which $\pi g_{k/2 q_{\gamma y}}(x)$ lies in $N_d(\Gamma y)$.](image)

Therefore, for lattice points $\gamma y \in R_k^d$ with $\pi g_{k/2 q_{\gamma y}}(x) \in N_d(\Gamma y)$, we may use $1/|\Gamma y \cap B_{k/2-d}(y)|$ as an upper bound for the proportion of lattice points $\Gamma y$ in $A_r$, which lie in $R_k^d$. The number of lattice points in $R_k^d \cap A_r$, with $\pi g_{k/2 q_{\gamma y}}(x) \notin N_d(\Gamma y)$ is bounded above by the number of lattice points in $\Gamma y \cap A_r$, with $\pi g_{k/2 q_{\gamma y}}(x) \notin N_d(\Gamma y)$, and we now describe how to estimate the proportion of lattice points with this property. If we project down to moduli space $Q/\Gamma$, then the geodesic segments from $x$ to $\gamma y$ all arrive at the same point in $Q/\Gamma$, which we shall call $y$. Theorem 2.1 implies that the limiting distribution of terminal quadratic differentials $\gamma^{-1} q_{\gamma y}(x) = q_y(\gamma^{-1} x)$ in $S(y)$ is given by $\lambda^+(q) ds_y(q)$, up to rescaling. The proportion of these quadratic differentials for which $\pi g_{k/2 q_y}(\gamma^{-1} x)$ lies in $N_d(\Gamma)$ is then described as the integral of $I_d(g_{k/2 q})$ over $S(y)$, where $I_d$ is the characteristic function for the pre-image of $N_d(\Gamma)$ in $Q$, and so we may apply Proposition 2.2 to take the limit of this integral as $k$ tends to infinity. The limit is equal to the volume of the pre-image of $N_d(\Gamma)$ in moduli space $Q/\Gamma$, and this tends to one as $d$ tends to infinity. This means that the proportion of $k$-separated lattice points $R_k^d$ for which there are at least $|\Gamma y \cap B_{k/2-d}(y)|$ nearby lattice points in $\Gamma y \cap B_r(x)$, but not in $R_k^d$, tends to one as $k$ tends to infinity, for appropriate choices of $d$ for each $k$. Therefore there is a sequence of upper bounds which tend to zero. We now give a detailed version of this argument.
Recall that \( q_x(\gamma y) \) is the unit area initial quadratic differential on \( x \) for the Teichmüller geodesic from \( x \) to \( \gamma y \), and \( q_y(x) \) is the corresponding unit area terminal quadratic differential on \( \gamma y \). Recall that \( \pi \) is the projection map \( \pi : Q \to T \), and we denote the Teichmüller geodesic flow on \( Q \) by \( g_t \), so \( \pi g_t q_{\gamma y}(x) \) is the point distance \( t \) along the geodesic ray starting at \( \gamma y \) which passes through \( x \). Let \( d < k/2 \), and let \( N_d(\Gamma y) \) be a metric \( d \)-neighbourhood of the Teichmüller lattice \( \Gamma y \). If \( \gamma y \in R'_k y \cap A_r \), and \( \pi g_{k/2} q_{\gamma y}(x) \in N_d(\Gamma y) \), then there is a lattice point \( \gamma' y \) with \( d_T(\pi g_{k/2} q_{\gamma y}(x), \gamma' y) \leq d \). In particular,
\[
B_{k/2-d}(\gamma' y) \subset B_k(\gamma y) \cap B_r(x) \cap Sect_x(U),
\]
as illustrated above in Figure 2. There are \( |\Gamma y \cap B_{k/2-d}(y)| \) lattice points of \( \Gamma y \) in \( B_{k/2-d}(\gamma' y) \), and at most one of these lattice points lies in \( R'_k y \).

Let \( Y(k, d) = \{ q \in S(y) \mid \pi g_{k/2} q \in N_d(\Gamma y) \} \). We may divide the points \( R'_k y \cap A_r \) into two sets, depending on whether or not the corresponding terminal quadratic differential \( q_y(\gamma^{-1} x) \) lies in \( Y(k, d) \). Therefore \( |R'_k y \cap A_r, q(x, \gamma y) \in U \times V \cap Y(k, d)| \) is equal to
\[
|R'_k y \cap A_r, q(x, \gamma y) \in U \times V \cap Y(k, d)| + |R'_k y \cap A_r, q(x, \gamma y) \in U \times V \setminus Y(k, d)|.
\]
We first consider the first term from line (4). For each \( \gamma y \in R'_k y \cap A_r \), with \( q_y(\gamma^{-1} x) \in Y(k, d) \), we get at least \( |\Gamma y \cap B_{k/2-d}(y)| \) lattice points of \( \Gamma y \) inside \( B_k(\gamma y) \cap B_r(x) \cap Sect_x(U) \), at most one of which lies in \( R'_k y \). The terminal quadratic differentials for these lattice points need not lie in \( V \), so we obtain an upper bound in terms of \( U \times S(y) \) instead of \( U \times V \), i.e.
\[
|\Gamma y \cap B_{k/2-d}(y)| |\Gamma y \cap B_r(x), q(x, \gamma y) \in U \times S(y)|. \quad (5)
\]
For the second term in line (4), we use the fact that \( R'_k y \subset \Gamma y \), and \( A_r \subset B_r(x) \), which gives the following upper bound,
\[
|\Gamma y \cap B_{k/2-d}(y)| \leq |\Gamma y \cap B_r(x), q(x, \gamma y) \in U \times V \setminus Y(k, d)|. \quad (6)
\]
Now applying Theorem 2.1 to lines (5) and (6) above, and adding them together as in line (4), we obtain the following upper bound,
\[
\lim_{r \to \infty} \frac{|R'_k y \cap A_r, q(x, \gamma y) \in U \times V|}{|\Gamma y \cap B_r(x), q(x, \gamma y) \in U \times V|} \leq \frac{1}{|\Gamma y \cap B_{k/2-d}(y)|} \left( \int_{S(y)} \lambda^+(q) ds_y(q) \right) + \frac{\int_{V \setminus Y(k, d)} \lambda^+(q) ds_y(q)}{\int_V \lambda^+(q) ds_y(q)}. \quad (7)
\]
and this upper bound holds for all \( k \), for all \( d < k/2 \).

By Theorem 2.1, the first term on the right hand side above tends to zero as \( k \) tends to infinity, for any fixed \( d \). We now consider the second term above, which we may rewrite as \( 1 - p(k, d) \), where
\[
p(k, d) = \frac{\int_{V \setminus Y(k, d)} \lambda^+(q) ds_y(q)}{\int_V \lambda^+(q) ds_y(q)}. \quad (8)
\]
Recall that \( Y(k, d) = \{ q \in S(y) \mid \pi g_{k/2} q \in N_d(\Gamma y) \} \), and let \( I_d \) be the characteristic function for the pre-image of \( N_d(\Gamma y) \) in \( Q \). The function \( I_d \) is \( \Gamma \)-equivariant, so we may write the expression for \( p(k, d) \) in line (8) above as
\[
p(k, d) = \frac{\int_V I_d(g_{k/2} q) \lambda^+(q) ds_y(q)}{\int_V \lambda^+(q) ds_y(q)}.
\]

15
Therefore as the Teichmüller geodesic flow is mixing for conditional measures, Proposition 2.2, this implies
\[
\lim_{k \to \infty} p(k, d) = \int_{Q/\Gamma} I_d(q) d\mu(q).
\]

Therefore, if we fix \(d\) and let \(k\) tend to infinity in line (7) above, we obtain the following upper bound,
\[
\lim_{k \to \infty} \lim_{r \to \infty} \left| R_k y \cap A_r, q(x, \gamma y) \in U \times V \right| \leq 1 - \int_{Q/\Gamma} I_d(q) d\mu(q), \tag{9}
\]
and this upper bound holds for all \(d\). However, the integral on line (9) above tends to one as \(d\) tends to infinity, as we have normalized our measures so that the volume of moduli space is one. Therefore, the term on the right hand side of (9) above tends to zero as \(d\) tends to infinity. As the right hand side of (9) is an upper bound for the left hand side, for all values of \(d\), this implies that the left hand side of (9) is zero, as required. 

Therefore, all three terms on the right hand side of (3) tend to zero as \(k\) tends to infinity. This completes the proof of Theorem 1.1 in the case that the mapping class group has trivial center.

We now deal with the two cases in which the mapping class group has non-trivial center, which are the genus two surface \(\Sigma_{2,0}\), and the twice-punctured torus \(\Sigma_{1,2}\), and we will write \(\Gamma_{g,b}\) for the mapping class group of \(\Sigma_{g,b}\). In the case of the genus two surface, the center \(Z\) of \(\Gamma_{2,0}\) is \(\mathbb{Z}/2\mathbb{Z}\), generated by the hyperelliptic involution, and the quotient of the genus two surface by the hyperelliptic involution is the six-punctured sphere. The hyperelliptic involution acts trivially on \(T(\Sigma_{2,0})\), which is isometric to \(T(\Sigma_{0,6})\) and \(\Gamma_{2,0}/Z\) is isomorphic to the mapping class group of the six-punctured sphere, so a Teichmüller lattice in \(T(\Sigma_{2,0})\) is isometric to \(T(\Sigma_{0,6})\). The hyperelliptic involution also acts trivially on the complex of curves, so \(C(\Sigma_{2,0})\) is isometric to \(C(\Sigma_{0,6})\). Therefore a set \(R\) of elements of bounded translation length length on \(C(\Sigma_{2,0})\) is also a set of elements of bounded translation length on \(C(\Sigma_{0,6})\). Theorem 1.1 holds for \(\Gamma_{0,6}\), as \(\Gamma_{0,6}\) has trivial center, and so this implies that Theorem 1.1 also holds for \(\Gamma_{2,0}\). In the case of the twice-punctured torus, the quotient surface under the hyperelliptic involution is the five-punctured sphere, and again the hyperelliptic involution acts trivially on Teichmüller space and the complex of curves. Therefore the argument above works exactly as before, except for the fact that \(\Gamma_{1,2}/Z\) is a finite index subgroup of \(\Gamma_{0,5}\). However, by Theorem 2.1 the asymptotic number of lattice points of \(\Gamma_{1,2} y \cap B_r(x)\) in a bisector is a constant multiple of the asymptotic number of lattice points of \(\Gamma_{0,5} y \cap B_r(x)\) in the same bisector, and so the proportion of points in \(R y \cap B_r(x)\) in the bisector tends to zero in either lattice. This completes the proof of Theorem 1.1. 

\[
\square
\]

References

[ABEM] J. Athreya, A. Bufetov, A. Eskin, and M. Mirzakhani, "Lattice point asymptotics and volume growth on Teichmüller space," available at arxiv:math.DS/0610715.

[BH99] Martin R. Bridson and André Haefliger, Metric spaces of non-positive curvature, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999.

[EM93] Alex Eskin and Curt McMullen, "Mixing, counting, and equidistribution in Lie groups," Duke Math. J. 71 (1993), no. 1, 181–209.

[EM] Alex Eskin and Maryam Mirzakhani, "Counting closed geodesics in Moduli space," available at arxiv:0811.2382.
[Far06] Benson Farb, *Some problems on mapping class groups and moduli space*, Problems on mapping class groups and related topics, Proc. Sympos. Pure Math., vol. 74, Amer. Math. Soc., Providence, RI, 2006, pp. 11–55.

[GO07] Alexander Gorodnik and Hee Oh, *Orbits of discrete subgroups on a symmetric space and the Furstenberg boundary*, Duke Math. J. **139** (2007), no. 3, 483–525.

[HM79] John Hubbard and Howard Masur, *Quadratic differentials and foliations*, Acta Math. **142** (1979), no. 3-4, 221–274.

[Iva01] Nikolai V. Ivanov, *Isometries of Teichmüller spaces from the point of view of Mostow rigidity*, Topology, ergodic theory, real algebraic geometry, Amer. Math. Soc. Transl. Ser. 2, vol. 202, Amer. Math. Soc., Providence, RI, 2001, pp. 131–149.

[Ker80] Steven P. Kerckhoff, *The asymptotic geometry of Teichmüller space*, Topology **19** (1980), no. 1, 23–41.

[Kla] E. Klarreich, *The boundary at infinity of the curve complex and the relative Teichmüller space*.

[LM10] Anna Lenzhen and Howard Masur, *Criteria for the divergence of pairs of Teichmüller geodesics*, Geom. Dedicata **144** (2010), 191–210.

[Mah] Joseph Maher, *Random walks on the mapping class group*, available at [arXiv:math.GT/0604433](http://arXiv:math.GT/0604433).

[Mah10] Joseph Maher, *Linear progress in the complex of curves*, Trans. Amer. Math. Soc. **362** (2010), 2963–2991.

[Mar04] Grigoriy A. Margulis, *On some aspects of the theory of Anosov systems*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2004. With a survey by Richard Sharp: Periodic orbits of hyperbolic flows; Translated from the Russian by Valentina Vladimirovna Szulikowska.

[Mas75] Howard Masur, *On a class of geodesics in Teichmüller space*, Ann. of Math. (2) **102** (1975), no. 2, 205–221.

[Mas80] Howard Masur, *Uniquely ergodic quadratic differentials*, Comment. Math. Helv. **55** (1980), no. 2, 255–266.

[Mas82] Howard Masur, *Interval exchange transformations and measured foliations*, Ann. of Math. (2) **115** (1982), no. 1, 169–200.

[MM99] Howard A. Masur and Yair N. Minsky, *Geometry of the complex of curves. I. Hyperbolicity*, Invent. Math. **138** (1999), no. 1, 103–149.

[MM00] H. A. Masur and Y. N. Minsky, *Geometry of the complex of curves. II. Hierarchical structure*, Geom. Funct. Anal. **10** (2000), no. 4, 902–974.

[Vee82] William A. Veech, *Gauss measures for transformations on the space of interval exchange maps*, Ann. of Math. (2) **115** (1982), no. 1, 201–242.

[Vee86] William A. Veech, *The Teichmüller geodesic flow*, Ann. of Math. (2) **124** (1986), no. 3, 441–530.