Given a – possibly – irregular sequence \((a_k)_k\), its summatory function \(F(x) := \sum_{0 \leq k \leq n} a_n\) can behave in a “better” way. The authors of this nice paper study the case where the sequence \((a_k)_k\) is \(q\)-regular (see [J.-P. Allouche and J. Shallit, Theor. Comput. Sci. 98, No. 2, 163–197 (1992; Zbl 0774.68072); Theor. Comput. Sci. 307, No. 1, 3–29 (2003; Zbl 1058.68066)]. One of the most cited paper on a sequence of this kind is probably a 1975 paper of H. Delange [Enseign. Math. (2) 21, 31–47 (1975; Zbl 0306.10005)] where \(a_k\) is the sum of digits of \(k\) in a fixed integer base \(q \geq 2\). Since then, many papers have been devoted to the subject, as can be seen in the large bibliography given in the paper under review.

Here, the authors propose a general approach of the question, proving that the summatory function of a \(q\)-regular sequence can asymptotically be decomposed as a finite sum of periodic fluctuations multiplied by a scaling factor. Methods and results involve eigenvalues of sums of matrices, joint spectral radius, Fourier coefficients, Dirichlet series, Mellin-Perron summations, etc. While classical papers on the subject (such as Delange’s) look at the (non-)differentiability (almost-)everywhere, of the oscillatory terms, the authors here are rather interested in the Hölder regularity of these terms. In the last part of the paper two explicit examples are addressed in detail, namely the number of esthetic numbers less than a given integer and the number of odd entries in the rows of Pascal’s rhombus.

It is of course not possible to give all references related to the subject: one might add to the list given by the authors, e.g., the doctoral thesis of E. Cateland (available at https://tel.archives-ouvertes.fr/tel-00845511] where particular regular sequences – called “digital” – are studied, and the paper of G. Tenenbaum [Algorithmes Comb. 13, 117–128 (1997; Zbl 0869.11019)] where a general method for studying derivability of the oscillatory terms is developed on particular examples, as well as references in Cateland’s work and in Tenenbaum’s work. One could also cite a paper by M. Coons [J. Théor. Nombres Bordx. 22, No. 2, 339–352 (2010; Zbl 1223.11115)], where Theorem 3.1 can be compared with Theorem D in the paper under review, and a paper by J. P. Bell et al. [Bull. Aust. Math. Soc. 90, No. 2, 195–203 (2014; Zbl 1348.11023)], where a lower gap property for the growth of an unbounded \(k\)-regular sequence is given, which could be compared with the results in the present paper (since the summatory function of a regular sequence is also regular).

To conclude, it is quite clear that this paper is definitely a must-read.

Reviewer: Jean-Paul Allouche (Paris)
J. Combin., 22, 2, 1-53 (2015) · Zbl 1338.60068

[32] Hwang, H-K; Janson, S.; Tsai, T-H, Exact and asymptotic solutions of a divide-and-conquer recurrence dividing at half: theory and applications, ACM Trans. Algorithms, 13, 4, 47 (2017) · Zbl 1451.68359 · doi:10.1145/3127585

[33] Johansson, F., Arb: efficient arbitrary-precision midpoint-radius interval arithmetic, IEEE Trans. Comput., 66, 8, 1281-1292 (2017) · Zbl 1388.65037 · doi:10.1109/TC.2017.2690633

[34] Jungers, R., The Joint Spectral Radius. Theory and Applications. Lecture Notes in Control and Information Sciences (2009), Berlin: Springer, Berlin

[35] Lagarias, Jc; Wang, Y., The finiteness conjecture for the generalized spectral radius of a set of matrices, Linear Algebra Appl., 214, 17-42 (1995) · Zbl 0818.15007 · doi:10.1016/0024-3795(93)00052-2

[36] NIST Digital Library of Mathematical Functions. http://dlmf.nist.gov/. Release 1.0.16 of 2017-09-18 (2017), Olver, F.W.J., Daalhuis, A.B.O., Lozier, D.W., Schneider, B.I., Boisvert, R.F., Clark, C.W., Miller, B.R., Saunders, B.V. (eds.)

[37] Rota, G-C; Strang, G., A note on the joint spectral radius, Indag. Math., 22, 379-381 (1960) · Zbl 0095.09701 · doi:10.1016/S1385-7258(60)50046-1

[38] SageMath Developers: SageMath Mathematics Software (Version 8.3) (2018). http://www.sagemath.org

[39] The On-Line Encyclopedia of Integer Sequences (2018). http://oeis.org · Zbl 1439.11001

[40] Zygmund, A., Trigonometric Series (2002), Cambridge: Cambridge University Press, Cambridge · Zbl 1084.42003

This reference list is based on information provided by the publisher or from digital mathematics libraries. Its items are heuristically matched to zbMATH identifiers and may contain data conversion errors. It attempts to reflect the references listed in the original paper as accurately as possible without claiming the completeness or perfect precision of the matching.