Capacity of entanglement in random pure state

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Abstract: We compute the capacity of entanglement in the bipartite random pure state model using the replica method. We find the exact expression of the capacity of entanglement which is valid for a finite dimension of the Hilbert space. We argue that in the gravitational path integral, the capacity of entanglement receives contributions only from the sub-leading saddle points corresponding to the partially connected geometries.
1 Introduction

In recent papers [1, 2], the Page curve of the Hawking radiation [3] is reproduced from the replica computation of the entanglement entropy (see also [4] for a review). As argued by Page in [5], the entropy computation of the Hawking radiation is nicely modeled by the random pure state $|\Psi\rangle$ in a bipartite Hilbert space

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B.$$  \hfill (1.1)

Here the subsystems $A$ and $B$ correspond to the Hawking radiation and the black hole, respectively. From the reduced density matrix $\rho_A$ of the subsystem $A$

$$\rho_A = Tr_B |\Psi\rangle \langle \Psi|,$$  \hfill (1.2)

we can compute the entanglement entropy $S_A$

$$S_A = -\langle Tr \rho_A \log \rho_A \rangle,$$  \hfill (1.3)

where the bracket $\langle \cdots \rangle$ denotes the ensemble average over the random pure state $|\Psi\rangle$. The exact form of $S_A$ (4.5) is obtained in [5] as a function of the dimensions $d_A, d_B$ of the Hilbert spaces (1.1)

$$d_A = \dim \mathcal{H}_A, \quad d_B = \dim \mathcal{H}_B.$$  \hfill (1.4)

As discussed in [3], $S_A$ in the random pure state model exhibits a similar behavior as the Page curve for the Hawking radiation from an evaporating black hole.

In a recent paper [6], it is argued that the capacity of entanglement $C_A$ introduced in [7] is a useful quantity to diagnose the phase transition around the Page time. $C_A$ is defined by

$$C_A = \text{Tr}(\rho_A K^2) - (\text{Tr} \rho_A K)^2,$$  \hfill (1.5)

where $K = -\log \rho_A$ is the modular Hamiltonian. In other words, $C_A$ measures the fluctuation of the modular Hamiltonian. See e.g. [8–10] for the study of $C_A$ in various models.
In this paper, we compute the capacity of entanglement in the random pure state model using the replica method. From the ensemble average of $\text{Tr} \rho^n_A$ over the random pure state, we can compute the entanglement entropy $S_A$ and the capacity of entanglement $C_A$ as the derivative with respect to the replica number $n$ at $n = 1$

$$S_A = -\partial_n \log \langle \text{Tr} \rho^n_A \rangle \bigg|_{n=1} = -\partial_n \langle \text{Tr} \rho^n_A \rangle \bigg|_{n=1},$$

$$C_A = \partial^2_n \log \langle \text{Tr} \rho^n_A \rangle \bigg|_{n=1} = \partial^2_n \langle \text{Tr} \rho^n_A \rangle \bigg|_{n=1} - (S_A)^2. \quad (1.6)$$

We find the exact form of $C_A$ as a function of the dimensions $d_A, d_B$ of the Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$. The exact expression of $C_A$ in (4.9) is the main result of this paper.

As discussed in [1, 2], fully disconnected or fully connected geometries dominate in the replica computation of the entropy $S_A$, and their contributions exchange dominance around the Page time. In the case of the capacity $C_A$, it turns out that the leading saddle point from the fully disconnected or fully connected geometries does not contribute to $C_A$, and it receives non-zero contributions only from the sub-leading saddle points corresponding to the partially connected geometries. This is consistent with the result of [10] that $C_A$ is a measure of the partial entanglement.

This paper is organized as follows. In section 2, we review the random pure state model and the known exact result of $\langle \text{Tr} \rho^n_A \rangle$. We find a new formula of $\langle \text{Tr} \rho^n_A \rangle$ in terms of the Narayana number (2.13), which is useful for the replica computation of the entanglement entropy $S_A$ and the capacity of entanglement $C_A$. In section 3, we compute $S_A$ and $C_A$ in the planar limit using the replica method. Our computation shows that the leading saddle point does not contribute to $C_A$ and it receives contributions only from sub-leading saddle points. In section 4, we compute the exact $S_A$ and $C_A$ using the replica method. Finally we conclude in section 5.

2 Random pure state model

In this section, let us briefly review the random pure state model. We consider a pure state $|\Psi\rangle$ in the bipartite Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. This models the black hole evaporation where $A$ corresponds to the Hawking radiation while $B$ corresponds to the black hole. In the model discussed in [1], $A$ and $B$ correspond to the end of the world brane and the bulk JT gravity, respectively. We can expand the state $|\Psi\rangle$ in terms of the orthonormal basis of $\mathcal{H}_A$ and $\mathcal{H}_B$

$$|\Psi\rangle = N \sum_{i=1}^{d_A} \sum_{\alpha=1}^{d_B} X_{i\alpha} |i\rangle_A \otimes |\alpha\rangle_B, \quad (2.1)$$

where $N$ is the normalization factor to ensure the unit norm of $|\Psi\rangle$

$$\langle \Psi | \Psi \rangle = 1. \quad (2.2)$$

It is useful to regard the coefficient $X_{i\alpha}$ in (2.1) as a component of the $d_A \times d_B$ complex matrix $X$

$$X = (X_{i\alpha}). \quad (2.3)$$
Then the normalization factor $\mathcal{N}$ is written as

$$\mathcal{N} = \frac{1}{\sqrt{\operatorname{Tr}(XX^\dagger)}}. \quad (2.4)$$

We are interested in the reduced density matrix $\rho_A$ defined in (1.2) obtained by tracing out $B$. In terms of the matrix $X$ in (2.3), $\rho_A$ is written as a $d_A \times d_A$ matrix

$$\rho_A = \frac{XX^\dagger}{\operatorname{Tr}(XX^\dagger)} = \frac{W}{\operatorname{Tr}W}, \quad (2.5)$$

where $W = XX^\dagger$. The ensemble average over the random pure state $|\Psi\rangle$ can be defined by the Gaussian integral over the matrix $X$

$$\langle O(W) \rangle = \frac{\int dXdX^\dagger O(W)e^{-\operatorname{Tr}(XX^\dagger)}}{\int dXdX^\dagger e^{-\operatorname{Tr}(XX^\dagger)}}. \quad (2.6)$$

As a distribution of the matrix $W = XX^\dagger$, this is known as the Wishart-Laguerre ensemble. See [11] for a nice review on this subject.

The matrix integral (2.6) can be written as the eigenvalue integral by diagonalizing the matrix $W$. In the original paper by Page [5], the entropy $S_A$ was computed by evaluating the eigenvalue integral of log $\rho_A$ directly. In this paper, we will compute the entropy $S_A$ and the capacity $C_A$ using the replica method (1.6). To do this, we need the expectation value of the moment $\operatorname{Tr} \rho_A^n$. Fortunately, the exact result of $\langle \operatorname{Tr} \rho_A^n \rangle$ is already obtained in [12].

$$\langle \operatorname{Tr} \rho_A^n \rangle = \frac{\Gamma(d_Ad_B)}{\Gamma(d_Ad_B + n + 1)} \cdot \frac{1}{n} \sum_{j=1}^{n} (-1)^{j-1} [d_A + n - j][d_B + n - j]n!}{(j-1)!(n-j)!}, \quad (2.8)$$

where $[a]_n = a(a-1)\cdots(a-n+1)$ denotes the falling factorial. For instance, the first few terms of $\langle \operatorname{Tr} \rho_A^n \rangle$ read

$$\langle \operatorname{Tr} \rho_A^2 \rangle = \frac{d_A + d_B}{d_Ad_B + 1}, \quad \langle \operatorname{Tr} \rho_A^3 \rangle = \frac{d_A^2 + 3d_Ad_B + d_B^2 + 1}{(d_Ad_B + 1)(d_Ad_B + 2)}, \quad (2.9)$$

$$\langle \operatorname{Tr} \rho_A^4 \rangle = \frac{d_A^3 + 6d_A^2d_B + 6d_Ad_B^2 + d_B^3 + 5d_A + 5d_B}{(d_Ad_B + 1)(d_Ad_B + 2)(d_Ad_B + 3)},$$

which agree with the known exact results of $\langle \operatorname{Tr} \rho_A^n \rangle$ [15, 16]. Note that $\langle \operatorname{Tr} \rho_A^n \rangle$ in (2.8) is symmetric under the exchange of $d_A$ and $d_B$, which implies that $S_A$ and $C_A$ are also symmetric functions of $d_A$ and $d_B$

$$S_A(d_A, d_B) = S_A(d_B, d_A), \quad C_A(d_A, d_B) = C_A(d_B, d_A). \quad (2.10)$$
In what follows, we will assume \( d_A \leq d_B \) without loss of generality. \( S_A \) and \( C_A \) in the opposite regime \( d_A > d_B \) can be obtained by exchanging \( d_A \) and \( d_B \) using the symmetry (2.10).

When \( n \) is a positive integer, the summation of \( j \) in (2.8) can be extended to \( j = \infty \) since the summand vanishes for \( j > n \). Then (2.8) becomes

\[
\langle \mathrm{Tr} \rho^n_A \rangle = \frac{\Gamma(d_A d_B)}{\Gamma(d_A d_B + n)} \cdot \frac{1}{n} \sum_{j=1}^{\infty} \frac{(-1)^{j-1} [d_A + n - j]_n [d_B + n - j]_n}{(j-1)! (n-j)!} 
\]

\[
= \frac{\Gamma(d_A + n) \Gamma(d_B + n) \Gamma(d_A d_B + 1)}{\Gamma(n+1) \Gamma(d_B + 1) \Gamma(d_A + 1) \Gamma(d_A d_B + n)} 
\times 3F_2(\{1 - d_A, 1 - d_B, 1 - n\}, \{1 - d_A - n, 1 - d_B - n\}; 1). \tag{2.11}
\]

The last expression makes sense for non-integer \( n \) and it defines an analytic continuation of \( \langle \mathrm{Tr} \rho^n_A \rangle \) away from the integer \( n \). One can in principle compute the derivative of the last expression in (2.11) with respect to \( n \) to find \( S_A \) and \( C_A \). However, it is not straightforward to simplify the derivative of the hypergeometric function \( 3F_2 \) in (2.11).

It turns out that it is useful to rewrite (2.11) as the following form using the identity of \( 3F_2 \) \(^2\)

\[
\langle \mathrm{Tr} \rho^n_A \rangle = \frac{\Gamma(d_B + n) \Gamma(d_A d_B + 1)}{\Gamma(d_B + 1) \Gamma(d_A d_B + n)} 
\times 3F_2(\{1 - d_A, 1 - d_B, 1 - n\}, \{2, 1 - d_B - n\}; 1). \tag{2.12}
\]

We find that this is expanded as

\[
\langle \mathrm{Tr} \rho^n_A \rangle = \sum_{k=1}^{\infty} N_{n,k} \frac{\Gamma(d_A) \Gamma(d_B + 1 + n - k) \Gamma(d_A d_B + 1)}{\Gamma(d_A + 1 - k) \Gamma(d_B + 1) \Gamma(d_A d_B + n)} , \tag{2.13}
\]

where \( N_{n,k} \) is the Narayana number

\[
N_{n,k} = \frac{1}{n} \binom{n}{k} \frac{n}{k-1} = \frac{\Gamma(n) \Gamma(n+1)}{k!(k-1)! \Gamma(1+n-k) \Gamma(2+n-k)}. \tag{2.14}
\]

Indeed, one can show that the summation in (2.13) reproduces the hypergeometric function in (2.12). This expression (2.13) makes contact with the planar limit of \( \langle \mathrm{Tr} \rho^n_A \rangle \) where the Narayana number naturally appears from the number of non-crossing permutations \([17]\).

When \( n \) is a positive integer, the summation of \( k \) in (2.13) is truncated at \( k = n \) and one can easily check that (2.13) reproduces the result (2.9) for small \( n \).

Using the analytic continuation of the Narayana number by the last expression in (2.14), we can define a natural analytic continuation of \( \langle \mathrm{Tr} \rho^n_A \rangle \) in (2.13) for non-integer \( n \). When \( d_A \leq d_B \) and \( d_A \) is a positive integer, the summation in (2.13) vanishes for \( k > d_A \) and hence (2.13) becomes

\[
\langle \mathrm{Tr} \rho^n_A \rangle = \sum_{k=1}^{d_A} N_{n,k} \frac{\Gamma(d_A) \Gamma(d_B + 1 + n - k) \Gamma(d_A d_B + 1)}{\Gamma(d_A + 1 - k) \Gamma(d_B + 1) \Gamma(d_A d_B + n)} . \tag{2.15}
\]

In section 4, we will use this expression of \( \langle \mathrm{Tr} \rho^n_A \rangle \) for the replica computation of the exact \( S_A \) and \( C_A \).

\(^2\)See the identity in the Wolfram Functions Site http://functions.wolfram.com/07.27.17.0046.01.
3 Planar limit

Before discussing the exact result of $C_A$, in this section we will compute $C_A$ in the planar limit

$$d_A, d_B \rightarrow \infty \text{ with } \alpha = \frac{d_A}{d_B} : \text{ fixed.} \quad (3.1)$$

We will assume $\alpha \leq 1$ without loss of generality. $C_A$ in the opposite regime $\alpha > 1$ can be obtained by sending $\alpha \rightarrow \alpha - 1$ using the symmetry (2.10). The computation of $C_A$ in this limit (3.1) has been already done in [6] using the planar eigenvalue density of the Wishart-Laguerre ensemble, known as the Marchenko–Pastur distribution. Here we will use the replica method to compute $C_A$, which clarifies the role of replica wormholes in $C_A$.

In the planar limit (3.1), the exact result of $\langle \text{Tr} \rho_A^n \rangle$ in (2.15) reduces to [17]

$$\langle \text{Tr} \rho_A^n \rangle_{\text{planar}} = d_A 1 - n \sum_{k=1}^{\infty} N_{n,k} \alpha^{k-1}. \quad (3.2)$$

When $n$ is a positive integer, the sum over $k$ is truncated to $1 \leq k \leq n$ since the Narayana number $N_{n,k}$ in (2.14) vanishes for $k \geq n + 1$. The first few terms of the planar expectation values of $\text{Tr} \rho_A^n$ are given by

$$\begin{align*}
\langle \text{Tr} \rho_A^2 \rangle_{\text{planar}} &= \frac{d_A + d_B}{d_A d_B}, \\
\langle \text{Tr} \rho_A^3 \rangle_{\text{planar}} &= \frac{d_A^2 + 3d_A d_B + d_B^2}{(d_A d_B)^2}, \\
\langle \text{Tr} \rho_A^4 \rangle_{\text{planar}} &= \frac{d_A^3 + 6d_A^2 d_B + 6d_A d_B^2 + d_B^3}{(d_A d_B)^3}.
\end{align*} \quad (3.3)$$

One can see that (3.3) is obtained from the planar limit of the exact result (2.9), as expected.

When $n$ is a positive integer, (3.2) is expanded as

$$\langle \text{Tr} \rho_A^n \rangle_{\text{planar}} = d_A 1 - n + \frac{1}{2} n(n - 1) d_A^{-2} - d_B^{-1} + \cdots + d_B^{-n}. \quad (3.4)$$

In the gravitational path integral, the first term $d_A 1 - n$ and the last term $d_B^{-n}$ come from fully disconnected and fully connected geometries, respectively. If we assume $d_A < d_B$, the dominant contribution is the first term $d_A 1 - n$ and $\langle \text{Tr} \rho_A^n \rangle_{\text{planar}}$ is written as

$$\langle \text{Tr} \rho_A^n \rangle_{\text{planar}} = d_A 1 - n \left[ 1 + f(n, \alpha) \right], \quad (3.5)$$

where $f(n, \alpha)$ is given by

$$f(n, \alpha) = \sum_{k=2}^{\infty} N_{n,k} \alpha^{k-1} = 2 F_1 (1 - n, -n; 2; \alpha) - 1. \quad (3.6)$$

In other words, in the gravitational picture $f(n, \alpha)$ summarizes all contributions from the sub-dominant, partially connected geometries. Note that $f(n, \alpha)$ vanishes at $n = 1$ by our definition in (3.6)

$$f(1, \alpha) = 0. \quad (3.7)$$
Plugging (3.5) into (1.6) we find
\begin{align}
S_{A, \text{planar}} &= \log d_A - f'(1, \alpha), \\
C_{A, \text{planar}} &= f''(1, \alpha) - f'(1, \alpha)^2,
\end{align}
where the prime in $f'$ and $f''$ denotes the derivative with respect to $n$. Note that the capacity is completely determined by the sub-leading contributions $f(n, \alpha)$ in (3.5). In fact, if we use the leading approximation of the trace
\[
\langle \text{Tr} \rho^n_A \rangle_{\text{planar}} \approx d_A^{1-n},
\]
the capacity vanishes
\[
C_{A, \text{planar}} = \partial_n^2 \langle \text{Tr} \rho^n_A \rangle_{\text{planar}} \bigg|_{n=1}^2 - \left( \partial_n \langle \text{Tr} \rho^n_A \rangle_{\text{planar}} \bigg|_{n=1} \right)^2 
\approx (\log d_A)^2 - (\log d_A)^2 = 0.
\]
The same conclusion holds in the opposite regime $d_A > d_B$ as well if we use the leading approximation $\langle \text{Tr} \rho^n_A \rangle_{\text{planar}} \approx d_B^{1-n}$. This implies that the dominant saddle point of gravitational path integral does not contribute to the capacity. In other words, the capacity of entanglement is sensitive to the sub-dominant saddle points corresponding to the partially connected geometries. Thus, $C_A$ is a useful probe of the contributions of replica wormholes which are not fully connected nor fully disconnected geometries, but some “intermediate” geometries. This is consistent with the result in [10] that $C_A$ takes a non-zero value for partially entangled states and $C_A$ vanishes for the pure state or a maximally entangled state. Namely, $C_A$ is a measure of partial entanglement [10].

Let us evaluate $f'(1, \alpha)$ and $f''(1, \alpha)$. From (3.6) they are written as
\begin{align}
f'(1, \alpha) &= \sum_{k=2}^{\infty} \partial_n N_{n,k} \bigg|_{n=1} \alpha^{k-1}, \\
f''(1, \alpha) &= \sum_{k=2}^{\infty} \partial_n^2 N_{n,k} \bigg|_{n=1} \alpha^{k-1}.
\end{align}
Thus, we need to compute the derivative of Narayana number $N_{n,k}$ at $n = 1$. From (2.14), one can easily show that $N_{n,k}$ is expanded around $n = 1$ as
\[
N_{n,k} = \begin{cases}
1, & (k = 1), \\
\frac{1}{2} (n - 1) + \frac{1}{2} (n - 1)^2, & (k = 2), \\
- \frac{(n - 1)^2}{k(k - 2)(k - 1)^2} + \mathcal{O}((n - 1)^3), & (k \geq 3).
\end{cases}
\]
This implies that the first derivative $\partial_n N_{n,k}$ at $n = 1$ vanishes unless $k = 2$, and $f'(1, \alpha)$ in (3.11) becomes
\[
f'(1, \alpha) = \alpha \partial_n N_{n,2} \bigg|_{n=1} = \frac{\alpha}{2}.
\]
In a similar manner, from (3.12) we find that $f''(1, \alpha)$ in (3.11) becomes

$$f''(1, \alpha) = \alpha - \sum_{k=3}^{\infty} \frac{2}{k(k-2)(k-1)^2} \alpha^{k-1}$$

$$= -1 - \frac{3\alpha}{2} + (\alpha - \alpha^{-1}) \log(1 - \alpha) + 2\text{Li}_2(\alpha).$$

Finally, plugging the result of $f'(1, \alpha)$ in (3.13) and $f''(1, \alpha)$ in (3.14) into the definition of $S_{A,\text{planar}}$ and $C_{A,\text{planar}}$ in (3.8) we find

$$S_{A,\text{planar}} = \log d_A - \frac{\alpha}{2},$$

$$C_{A,\text{planar}} = -1 - \frac{3\alpha}{2} - \frac{\alpha^2}{4} + (\alpha - \alpha^{-1}) \log(1 - \alpha) + 2\text{Li}_2(\alpha).$$

This agrees with the result in [6] obtained from the Marchenko–Pastur distribution. Our replica computation reveals the importance of the sub-leading contribution $f(n, \alpha)$ to the capacity of entanglement.

We note in passing that $C_{A,\text{planar}}$ in (3.15) takes the maximal value at $\alpha = 1$, or $d_A = d_B$ [6, 10]

$$C_{A,\text{planar}}^{(\text{max})} = C_{A,\text{planar}} \bigg|_{\alpha=1} = \frac{\pi^2}{3} - \frac{11}{4}.$$

### 4 Exact capacity of entanglement at finite $d_A, d_B$

In this section we will compute the exact $S_A$ and $C_A$ using the exact result of $\langle \text{Tr} \rho_A^n \rangle$ in (2.15). Here we assume that $d_A$ and $d_B$ are both integers and $d_A \leq d_B$.

Let us first compute the entanglement entropy $S_A$. Plugging (2.15) into the definition of $S_A$ in (1.6) we find

$$S_A = -\lim_{n \to 1} \sum_{k=1}^{d_A} \frac{\Gamma(d_A) \Gamma(d_B + 1 + n - k) \Gamma(d_A d_B + 1)}{\Gamma(d_A + 1 - k) \Gamma(d_B + 1) \Gamma(d_A d_B + n)}$$

$$\times \left[ N_{n,k} \left( \psi(d_B + 1 + n - k) - \psi(d_A d_B + n) \right) + \partial_n N_{n,k} \right],$$

where $\psi(z)$ denotes the digamma function

$$\psi(z) = \frac{d}{dz} \log \Gamma(z).$$

From the behavior (3.12) of the Narayana number $N_{n,k}$ near $n = 1$, (4.1) becomes

$$S_A = \psi(d_A d_B + 1) - \psi(d_B + 1) - \frac{1}{2} \frac{\Gamma(d_A) \Gamma(d_B)}{\Gamma(d_A + 1) \Gamma(d_B + 1)}.$$

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3The exact computation of $S_A$ by the replica method is also considered in [18] using the expression of $\langle \text{Tr} \rho_A^n \rangle$ in (2.8).
Using the property of the digamma function
\[
\psi(m + 1) = \sum_{k=1}^{m} \frac{1}{k} - \gamma, \quad (m \in \mathbb{N}), \tag{4.4}
\]
with \(\gamma\) being the Euler’s constant, we arrive at the exact entanglement entropy \(S_A\) for \(d_A \leq d_B\)
\[
S_A = \sum_{k=d_B+1}^{d_A d_B} \frac{1}{k} - \frac{d_A - 1}{2d_B}. \tag{4.5}
\]
This agrees with the famous Page’s result \([5]^{4}\). \(S_A\) in the opposite regime \(d_A > d_B\) is obtained from (4.5) by exchanging the role of \(d_A\) and \(d_B\). Note that the first term of (4.5) is written as
\[
\sum_{k=d_B+1}^{d_A d_B} \frac{1}{k} = H_{d_A d_B} - H_{d_B}, \tag{4.6}
\]
where \(H_m = \sum_{k=1}^{m} 1/k\) denotes the harmonic number.

**Figure 1**: Plot of the entanglement entropy \(S_A\) as a function of \(\log d_A\). We have set \(d_B = 20\) in this figure. The blue solid curve represents the exact result of \(S_A\) in (4.5). The orange dashed curve is the leading approximation \(S_A = \log d_A\) \((d_A \leq d_B)\) and \(S_A = \log d_B\) \((d_A > d_B)\).

In Fig. 1, we show the plot of \(S_A\) as a function of \(\log d_A\) with a fixed \(d_B\). As we can see from Fig. 1, \(S_A\) grows like \(\log d_A\) for small \(d_A\) and approaches \(\log d_B\) for large \(d_A\). At least qualitatively, this reproduces the Page curve of the Hawking radiation \([3]\) if we regard the subsystem \(A\) as the radiation and the subsystem \(B\) as the black hole and \(t = \log d_A\) as time. Around the Page time \(t = \log d_B\), the contributions from the fully disconnected and the fully connected geometries exchange dominance in the replica computation of \(S_A\) \([1, 2]\).
Next, let us compute the capacity of entanglement $C_A$ by the replica method (1.6). To do this, we need to compute the second derivative of $\langle \text{Tr} \rho_A^n \rangle$ at $n = 1$

$$\left. \partial_n^2 \langle \text{Tr} \rho_A^n \rangle \right|_{n=1} = \lim_{n \to 1} \sum_{k=1}^{d_A} \frac{\Gamma(d_A)\Gamma(d_B + 1 + n - k)\Gamma(d_A d_B + 1)}{\Gamma(d_A + 1 - k)\Gamma(d_B + 1)\Gamma(d_A d_B + n)}$$

$$\times \left[ N_{n,k} \left( \psi(d_B + 1 + n - k) - \psi(d_A d_B + n) \right)^2 
+ 2 \partial_n N_{n,k} \left( \psi(d_B + 1 + n - k) - \psi(d_A d_B + n) \right) 
+ N_{n,k} \left( \psi_1(d_A + 1 + n - k) - \psi_1(d_A d_B + n) \right) + \partial_n^2 N_{n,k} \right],$$

(4.7)

where $\psi_1(z) = \frac{d}{dz} \psi(z)$ denotes the trigamma function. Using the relation

$$\psi_1(m + 1) = \frac{\pi^2}{6} - \sum_{k=1}^{m} \frac{1}{k^2}, \quad (m \in \mathbb{N}),$$

(4.8)

and the behavior (3.12) of $N_{n,k}$ near $n = 1$, after some algebra we find the exact result of capacity $C_A$ for $d_A \leq d_B$

$$C_A = \sum_{k=d_B+1}^{d_A d_B} \frac{1}{k^2} - \frac{(d_A - 1)(d_A + 3)}{4d_B} + \frac{d_A - 1}{d_B} - \sum_{k=3}^{d_A} 2 \frac{1}{k(k-2)(k-1)^2} \prod_{i=0}^{k-2} \frac{d_A - 1 - i}{d_B - i}.$$

(4.9)

This is our main result. $C_A$ in the opposite regime $d_A > d_B$ is obtained from (4.9) by exchanging $d_A$ and $d_B$ using the symmetry (2.10). Note that the first term of (4.9) is written as

$$\sum_{k=d_B+1}^{d_A d_B} \frac{1}{k^2} = H_{d_A d_B}^{(2)} - H_{d_B}^{(2)},$$

(4.10)

where $H_{m}^{(2)} = \sum_{k=1}^{m} 1/k^2$ denotes the generalized harmonic number of the 2nd order. This is similar to the first term of $S_A$ in (4.5), but the other terms in $C_A$ are more complicated than $S_A$.

One can easily check that (4.9) reduces to $C_{A, \text{planar}}$ in (3.15) in the planar limit (3.1). Also, one can check that $C_A(d_A, d_B)$ in (4.9) for $d_A, d_B = 2, 3$ agree with the result in [10]

$$C_A(2, 2) = \frac{13}{36},$$

$$C_A(2, 3) = \frac{1169}{3600},$$

$$C_A(3, 3) = \frac{2898541}{6350400}.$$

(4.11)

From the exact result (4.9), we find the small $d_A$ and the large $d_A$ behavior of $C_A$

$$C_A \approx \begin{cases} 
\left( \frac{d_A}{d_B} - 1 \right) \frac{1}{d_B}, & (1 \leq d_A \ll d_B), \\
\left( \frac{d_B}{d_A} - 1 \right) \frac{1}{d_A}, & (d_A \gg d_B).
\end{cases}$$

(4.12)
Our exact $C_A$ in (4.9) takes the maximal value at $d_A = d_B$

$$
C_A^{(\text{max})} = \sum_{k=1}^{d_B} \frac{1}{k^2} + \sum_{k=1}^{d_B} \frac{1}{k^2} + \frac{1}{d_B} - \frac{1}{4d_B^2} - \frac{11}{4}.
$$

(4.13)

In the large $d_B$ limit this is expanded as

$$
C_A^{(\text{max})} = \frac{\pi^2}{3} - \frac{11}{4} - \frac{3}{4d_B^2} + \mathcal{O}(d_B^{-3}),
$$

(4.14)

where the first two terms agree with the maximal value of capacity in the planar limit (3.16). For finite $d_A, d_B$, one can show that the exact $C_A$ is bounded from above

$$
C_A \leq C_A^{(\text{max})} < \frac{\pi^2}{3} - \frac{11}{4}.
$$

(4.15)

---

**Figure 2**: Plot of the capacity of entanglement $C_A$ as a function of $\log d_A$. We have set $d_B = 20$ in this figure. The blue solid curve is the exact result of $C_A$ in (4.9). The orange dashed curves represent the asymptotic behavior of $C_A$ in (4.12). The dashed vertical line is at $d_A = d_B$ where $C_A$ becomes maximal.

In Fig. 2, we show the plot of the exact capacity $C_A$ in (4.9). We can see that $C_A$ vanishes for $d_A = 1$ and approaches zero at large $d_A \gg d_B$. This is qualitatively similar to the result of the planar limit found in [6]. We emphasize that our result (4.9) is exact at finite $d_A, d_B$ and (4.9) includes all the non-planar corrections. As we argued in the previous section, $C_A$ is sensitive to the sub-leading terms in $\langle \text{Tr} \rho_A^3 \rangle$ corresponding to the partially connected geometries. Indeed, $C_A$ vanishes at the early and late “time” $t = \log d_A$ where the fully connected or fully disconnected geometry is dominant. $C_A$ takes a non-zero value near the Page time $t = \log d_B$ (or $d_A = d_B$) which is interpreted that the partially connected geometries give substantial contributions to $C_A$ near the Page time.
5 Conclusions and outlook

In this paper, we have computed the exact capacity of entanglement $C_A$ (4.9) at finite $d_A, d_B$ using the replica method (1.6). At the technical level, the important ingredient in our computation is the new exact formula (2.15) of $\langle \text{Tr} \rho^n_A \rangle$ written in terms of the Narayana number $N_{n,k}$. This formula (2.15) makes the relation to the planar limit manifest. We argued that $C_A$ vanishes for the fully connected or fully disconnected geometries, and $C_A$ is sensitive to the sub-leading contributions to $\langle \text{Tr} \rho^n_A \rangle$ coming from the partial connected geometries in the gravitational path integral. This suggests that $C_A$ is a good probe of the partial entanglement, as discussed in [10].

There are several open questions. The capacity of entanglement is introduced in [7] as an analogue of the heat capacity. Indeed, if we introduce the modular Hamiltonian $K = -\log \rho_A$, the moment $\text{Tr} \rho^n_A$ looks like the partition function

$$Z_n = \text{Tr} \rho^n_A = \text{Tr} e^{-nK},$$

and $n$ plays the role of the inverse temperature $\beta$. In this picture, our definition (1.6) of $S_A$ and $C_A$ is based on the “annealed” free energy $\log \langle Z_n \rangle$

$$S_A = -\partial_n \log \langle Z_n \rangle \bigg|_{n=1}, \quad C_A = \partial^2_n \log \langle Z_n \rangle \bigg|_{n=1}. \quad (5.2)$$

One could consider the quenched version of $S_A$ and $C_A$ as well

$$S_A^{\text{qu}} = -\partial_n \langle \log Z_n \rangle \bigg|_{n=1}, \quad C_A^{\text{qu}} = \partial^2_n \langle \log Z_n \rangle \bigg|_{n=1}. \quad (5.3)$$

We leave the computation of the quenched version of $S_A$ and $C_A$ as an interesting future problem.

It would be interesting to study the gravitational picture of the capacity of entanglement. $C_A$ is related to the quantum fluctuation of the modular Hamiltonian and the prescription of the gravitational computation of $C_A$ is proposed in [8]. We have argued that $C_A$ receives contributions only from the sub-leading partially connected geometries in the replica computation. It would be interesting to related this picture to the prescription in [8].

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