Linear Wave Equations and Effective Lagrangians for Wigner Supermultiplets

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Abstract

The relevance of the contracted SU(4) group as a symmetry group of the pion nucleon scattering amplitudes in the large $N_c$ limit of QCD raises the problem on the construction of effective Lagrangians for SU(4) supermultiplets. In the present study we suggest effective Lagrangians for selfconjugate representations of SU(4) in exploiting isomorphism between so(6) and ist universal covering su(4). The model can be viewed as an extension of the linear $\sigma$ model with SO(6) symmetry in place of SO(4) and generalizes the concept of the linear wave equations for particles with arbitrary spin. We show that the vector representation of SU(4) reduces on the SO(4) level to a complexified quaternion. Its real part gives rise to the standard linear $\sigma$ model with a hedgehog configuration for the pion field, whereas the imaginary part describes vector meson degrees of freedom via purely transversal $\rho$ mesons for which a helical field configuration is predicted. As a minimal model, baryonic states are suggested to appear as solitons of that quaternion.

1 Introduction

The large–$N_c$ limit of QCD introduced by t’Hooft [1] in the middle of the 70ies as an approximation scheme to the gauge theory of strong interaction underlies the idea of the extension of the colour group from SU($N_c = 3$) to SU($N_c > 3$). As a direct consequence, the behaviour of the amplitudes of hadronic processes in the expansion in powers of $1/N_c$ can be investigated. Later, $n$–point functions of QCD were systematically studied by Witten [2] with the result, that the ‘large–$N_c$’ scaling of a variety of physical quantities, such as hadronic masses, the strong meson–baryon vertex and the weak hadron–lepton couplings could be predicted (so called Witten’s $N_c$ counting rules). Within this treatment, the meson masses were shown to be independent of the number of colour degrees of freedom, whereas for the baryon masses a linear dependence on $N_c$ was obtained (see [3] for a review). Further, the weak axial coupling constant $g_A$ of the nucleon was shown to scale as $O(N_c)$, and the weak decay constant $f_\pi$ of the pion was found to be of the order $O(\sqrt{N_c})$. As a consequence, the pseudovector (PV) $\pi N$ coupling $g_{\pi NN}^{PV} \sim g_A/f_\pi$ was predicted to scale as $O(\sqrt{N_c})$. 1
With these counting rules the pion–baryon scattering amplitude was evaluated on the quark level as independent of $N_c$ in accordance with the unitarity condition on the $S$–matrix. On the level of composite particles, unitarity is ensured only if the $\pi N$ scattering amplitude is associated with direct and crossed Born diagrams including besides the one–nucleon also an equal mass $P_{33}$ intermediate state and is mathematically expressed through vanishing commutator matrix elements of the type

$$\langle N | [\sigma_a \otimes \tau_b, \sigma_k \otimes \tau_l] | N \rangle \xrightarrow{N_c \to \infty} 0$$

(1.1)

in the large $N_c$ limit. The last equation describes a Wigner–Inönü contraction of the spin–flavour static group SU(4) with respect to the nine generators $\sigma_a \otimes \tau_b$ with $a, b = 1, 2, 3$. Thus the mass degenerate nucleon and $P_{33}$ states will constitute the symmetric nonstrange $\{20\}$–plet which is common both to the static and the contracted SU(4) (subsequently denoted by SU’(4)) groups. Eq. (1.1) in practice means that in the large $N_c$ limit $g_A$ and the axial coupling constant $g_A^{N\to\Delta}$ of the weak $N \to \Delta$–transition have to satisfy the constraint

$$g_A^2 - \frac{2}{9}(g_A^{N\to\Delta})^2 = 0,$$

(1.2)

an observation already reported for the case of some effective models of the nucleon in [7].

The relevance of the SU’(4) symmetry for the low energy regime of QCD raises the question on the extension of the effective models of the nucleon to such describing Wigner supermultiplets. In the present study we propose an effective Lagrangian for selfconjugate meson representations of SU(4). The model can be viewed as an extension of the linear $\sigma$ model with SO(6) symmetry in place of SO(4) and generalizes the concept of the linear wave equations for particles with arbitrary spin.

The presentation is organized as follows. Sec. 2 reviews the idea for constructing linear wave equations (LWE) on the foundation of the special orthogonal group in 5–dimensional space, SO(5). In sec. 3 we focus on the isomorphism between the Lie algebras of SO(6) and its universal covering group SU(4) and discuss realization in terms of the elements of the Dirac–Clifford algebra. We suggest SO(6) invariant LWE for the selfconjugate SU(4) irreducible vector representation. Sec. 4 starts with a brief reminiscence of the linear SO(4) symmetric $\sigma$ model. After that an SO(6) invariant effective meson Lagrangian is suggested. In sec. 5 discussion on the perspectives for SU(4) fermionic states description is given. The paper ends with a short summary.

2 Linear wave equations

The general ansatz for linear relativistic wave equations has the form

$$(\alpha_{\mu} \partial^\mu + \chi_{1_{n\times n}})\Psi_{\{\tau\}}(x) = 0$$

(2.1)
with indices \( \{ r \} = \{ 1, ..., n \} \), \( \mu = 0, 1, 2, 3 \). The field \( \Psi_{\{ r \}}(x) \) denotes a multicomponent vector state transforming as an \( n \)-dimensional irreducible representation (irrep) of the Lorentz group and \( \chi \) is a constant related to the mass. The requirement on Lorentz invariance of eq. (2.1) leads to the commutation relation

\[
[S_{\mu \nu}, \alpha_\eta] = \alpha_\mu g_{\nu \eta} - \alpha_\nu g_{\mu \eta}
\]  

(2.2)

between the \( n \times n \)-matrices \( \alpha_\mu \) and the six generators \( S_{\mu \nu} \) of the homogeneous Lorentz group. The quantities \( S_{\mu \nu} \) satisfy the Lie algebra

\[
[S_{\mu \nu}, S_{\rho \sigma}] = -g_{\mu \rho} S_{\nu \sigma} + g_{\nu \sigma} S_{\mu \rho}
\]  

(2.3)

with \( g_{\mu \nu} = (1, -1, -1, -1, -1) \). The algebra of the homogeneous Lorentz group in (3+1) space–time dimensions was extended to that in (4+1) space–time dimensions by Pauli \[1\] for the purpose of Kaluza–Klein theory. Pauli was the first who emphasized already in 1933 the relevance of \( \text{SO}(5) \) for relativistic problems. The Lie algebra of the \( \text{SO}(5) \) group is obtained by completing eq. (2.3) by the following commutation relations:

\[
[l_\mu, l_\nu] = S_{\mu \nu}, \quad [S_{\mu \nu}, l_\sigma] = g_{\nu \sigma} l_\mu - g_{\mu \sigma} l_\nu.
\]  

(2.4)

Later, in 1945, Bhabha \[12\] observed that Eq. (2.2) is satisfied if the matrices \( \alpha_\mu \) are identified with the four \( \text{SO}(5) \) generators \( l_\mu \) as

\[
\alpha_\mu = l_\mu.
\]  

(2.5)

Insertion of eq. (2.4) into eq. (2.2) leads to the algebra

\[
[[\alpha_\mu, \alpha_\nu], \alpha_\eta] = g_{\nu \eta} \alpha_\mu - g_{\mu \eta} \alpha_\nu.
\]  

(2.6)

The vector fields \( \Psi_{\{ r \}}(x) \) in eq. (2.1) therefore can be viewed as irreps of \( \text{SO}(5) \), the special orthogonal group in five dimensions \[13\]. First order wave equations for scalar and vector fields are associated with the 5– and 10–dimensional \( \text{SO}(5) \) irreps and are usually referred to as Duffin–Kemmer–Petiau equations \[14\]. Particles of arbitrary spin are related to higher dimensional \( \text{SO}(5) \) multiplets and the corresponding equations are often called ‘Bhabha equations’. For more details the interested reader is referred to the most extensive study on this subject performed in the series of papers by Krajcik and Nieto \[15\].

To construct the Lagrangian underlying eq. (2.1) it is necessary to define a conjugation operation on the fields \( \Psi_{\{ r \}}(x) \). This is done by means of a \( n \times n \) matrix \( \eta \) with the property

\[
- D (\Lambda) \eta = \eta D (\Lambda)^T.
\]  

(2.7)

We follow the discussion (\[12\], \[13\], \[15\]) given in terms of compact groups and comment later on appropriate noncompact groups.
Here, $D (\Lambda)$ stands for the $n \times n$ matrix representations of the Lorentz transformations. In the special case of the four dimensional irrep of SO(5) the Bhabha equation is identical to the Dirac equation and the matrix $\eta$ can be expressed by $\alpha_0$ as $\eta = 2\alpha_0$. It is easily proved that the quantity $\overline{\Psi}(x)\Psi(x) = \Psi^+(x)\eta\Psi(x)$ transforms as a Lorentz scalar. Thus the Lagrangian leading to eq. (2.1) can be written as

$$\mathcal{L} = \overline{\Psi}(x)\partial \cdot \alpha \Psi(x) + \chi \overline{\Psi}(x)\Psi(x).$$

The irreducible representations of the group SO(5) as introduced above can be used for the description of one-flavour states only and thus Bhabha’s equations are not directly applicable as wave equations for the spin-flavour multiplets emerging in the large $N_c$ limit of QCD. Nevertheless, the trail blazed by Bhabha’s equations can be pursued and generalized to incorporate isospin degrees of freedom. This will be the subject of the next section. There we will show that SO(6) invariant LWE can be used to describe the selfconjugate SU(4) spin–flavour $\{15\}$–plet by means of the chain SU(4) $\sim$ SO(6) $\supset$ SO(5).

3 SO(6) invariant linear wave equations and isospin degrees of freedom

To incorporate the isospin degrees of freedom into the LWE we exploit the isomorphism (denoted by $\simeq$) between the Lie algebras su(4) and so(6). As is well known from group theory, a doubly connected orthogonal group SO($n$) has a simply connected universal covering group denoted by $\text{Spin}(n)$. The four cases in which the $\text{Spin}$–groups correspond to classical groups are

$$\text{Spin}(3) \simeq \text{SU}(2),$$

$$\text{Spin}(4) \simeq \text{SU}(2) \otimes \text{SU}(2),$$

$$\text{Spin}(5) \simeq \text{Sp}(2),$$

$$\text{Spin}(6) \simeq \text{SU}(4),$$

whereas for the Lie algebras the isomorphisms

$$\text{su}(2) \simeq \text{so}(3),$$

$$\text{su}(2) \oplus \text{su}(2) \simeq \text{so}(4),$$

$$\text{sp}(2) \simeq \text{so}(5),$$

$$\text{su}(4) \simeq \text{so}(6)$$

hold.

The first three equations have relevant physical applications. For example, eq. (3.5) leads to equivalent description of rigid body rotation in three dimensional space in terms
of the Euler–angles and the Cayley–Klein parameters, respectively. Eq. (3.6) underlies the construction of effective models in Chiral Dynamics like the \( \sigma \) model \[8\], whereas eq. (3.7) builds the basis for the linear wave equations \[9\]. The aim of the present study is to associate a physical model with eq. (3.8).

Among eqs. (3.1)–(3.4) the first and fourth are the most fundamental ones because the corresponding algebras are in addition isomorphic to the ones generated by the elements of the Clifford algebras \( C_2 \) and \( C_4 \), respectively. Indeed, it was shown by Barut \[18\] that the Lie group generated by the 15 elements of the Dirac–Clifford algebra \( C_4 \) is isomorphic to the six dimensional real Lorentz group with the metric \((-1,-1,+1,-1,-1,-1)\), and is thus related to the compact group \( \text{SO}(6) \). To see this, one has first to consider the antisymmetric set of generators

\[
S_{ab} = \begin{pmatrix}
0 & \gamma_5 & -\gamma_5\gamma_0 & \gamma_5\gamma_1 & \gamma_5\gamma_2 & \gamma_5\gamma_3 \\
0 & -\gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 & 0 \\
0 & \gamma_0\gamma_1 & \gamma_0\gamma_3 & \gamma_0\gamma_3 & 0 & \gamma_2\gamma_1 \\
0 & \gamma_2\gamma_1 & \gamma_3\gamma_1 & 0 & \gamma_3\gamma_2 & 0 \\
0 & \gamma_3\gamma_2 & 0 & \gamma_3 & \gamma_2 & \gamma_1 \\
0 & \gamma_5 & \gamma_5\gamma_0 & -\gamma_5\gamma_1 & -\gamma_5\gamma_2 & \gamma_5\gamma_3 \\
0 & -\gamma_5 & -\gamma_5\gamma_0 & \gamma_5\gamma_1 & -\gamma_5\gamma_2 & \gamma_5\gamma_3 \\
0 & \gamma_5\gamma_0 & \gamma_5\gamma_1 & \gamma_5\gamma_2 & \gamma_5\gamma_3 & 0 \\
0 & \gamma_5\gamma_1 & -\gamma_5\gamma_0 & \gamma_5\gamma_1 & \gamma_5\gamma_2 & \gamma_5\gamma_3 \\
0 & \gamma_5\gamma_2 & -\gamma_5\gamma_0 & -\gamma_5\gamma_1 & \gamma_5\gamma_2 & \gamma_5\gamma_3 \\
0 & \gamma_5\gamma_3 & -\gamma_5\gamma_0 & -\gamma_5\gamma_1 & -\gamma_5\gamma_2 & \gamma_5\gamma_3 \\
0 & -\gamma_5 & \gamma_5\gamma_0 & \gamma_5\gamma_1 & \gamma_5\gamma_2 & -\gamma_5\gamma_3 \\
\end{pmatrix}
\]

with \( a, b = 1, \ldots, 6 \) and then to re-express them by the physically more convenient generators,

\[
S_{\mu\nu} = \frac{1}{2}(\gamma_\mu \gamma_\nu - g_{\mu\nu}), \quad (3.9)
\]
\[
l_\mu = \frac{1}{2} \gamma_\mu, \quad (3.10)
\]
\[
\tilde{l}_\mu = \frac{1}{2} \gamma_5 \gamma_\mu, \quad (3.11)
\]
\[
K = \frac{\gamma_5}{2}, \quad \mu = 0, 1, 2, 3. \quad (3.12)
\]

It is easy to prove the commutation relations

\[
[S_{\mu\nu}, \tilde{l}_\sigma] = g_{\nu\sigma} l_\mu - g_{\mu\sigma} \tilde{l}_\nu, \quad (3.13)
\]
\[
[\tilde{l}_\mu, \tilde{l}_\nu] = S_{\mu\nu}, \quad (3.14)
\]
\[
[l_\mu, \tilde{l}_\nu] = -g_{\mu\nu} K, \quad (3.15)
\]
\[
[K, l_\mu] = \tilde{l}_\mu, \quad (3.16)
\]
\[
[K, l_\mu] = -l_\mu, \quad (3.17)
\]
\[
[K, S_{\mu\nu}] = 0. \quad (3.18)
\]

In joining them to eqs. (2.3)–(2.4), the Lie algebra of the group \( \text{SO}(6) \) is obtained. For this reason, the matrices \( \alpha_\mu \) entering eq. (2.1) can be viewed as the generators \( l_\mu \) of the group

\(^2\)The Clifford algebra has actually 16 elements but one of them commutes with all the others and equals the identity operator.
SO(6), and the field representations $\Psi^{(r)}(x)$ will behave as irreps belonging to SO(6). It was Barut [18] who showed that the SO(6) generators $S_{12}$, $-iS_{13}$, and $-iS_{23}$ span a new rotation group disjoint from the spin group. This former group can be identified with the group of isospin. Thus the advantage of SO(6) symmetric LWE will be that their solutions can have spin–flavour content. Since only selfconjugate irreps of su(4) can be related directly to so(6) irreps, the SO(6) invariant LWE can be employed only for mesonic spin–flavour supermultiplets such as the selfconjugate $\{15\}$–plet. The linear relativistic SO(6) equation for the (SO(5) reducible) $\{15\}$–plet reads:

$$ (\partial_\mu\alpha^{\mu}_{15} + \chi_{15}\Phi_{15})\Psi^{(15)}(x) = 0. \quad (3.19) $$

The decomposition of the $\{15\}$–plet within the SO(5) basis to which the Lorentz transformation directly applies, reads:

$$ \{15\} = \{5\} \oplus \{10\}. \quad (3.20) $$

Note, that the DKP algebra $B(1)$ [13] is reducible into the two inequivalent representations of dimensions 5 and 10, respectively, which are used for the description of spin–0 and spin–1 particles [14,15]. For on shell particles the LWE for the $\{5\}$– and $\{10\}$–plets are equivalent to the Klein–Gordon and the Proca equations, respectively. Thus the so(5) decomposition of the so(6) vector $\{15\} = \{5\} \oplus \{10\}$ fits naturally into the DKP algebra.

The SU(4) particle content associated with that states will be

$$ \{5\} \rightarrow \text{col}(\omega_3, \rho_3^0, \pi^+\pi^0, \pi^-\pi^0) \quad (3.21) $$

$$ \{10\} \rightarrow \text{col}(\rho_3^0, \rho_3^+, \rho_3^-, \omega^+, \omega^-) \quad (3.22) $$

Now the idea is to interprete the spin–flavour $\{5\}$–plet in (3.21) as the solution of the standard DKP-equation for a scalar particle field $\psi^{(5)}(p) = (\chi/p_0V)^{1/2}U^{(5)}_{DKP}(p)e^{ip\cdot x}$ [10,13] which leads to the correspondence

$$ U^{(5)}_{DKP}(p) = (2\chi^2)^{-1/2} \begin{pmatrix} -\chi \phi \\ \partial_\mu \phi \\ \partial_1 \phi \\ \partial_2 \phi \\ \partial_3 \phi \end{pmatrix} = (2\chi^2)^{-1/2} \begin{pmatrix} \omega_3 \\ \rho_3^0 \\ \pi_x \\ \pi_y \\ \pi_z \end{pmatrix}. \quad (3.24) $$

The relevant degrees of freedom for so(6) are actually the real cartesian components. It is the linear character of the representations in combination with the linear wave equations which makes it possible to use equal dimensional multiplets containing complexified (i.e. charged) fields.
Here the Lorentz indices run from $\mu = 0, ..., 3$, whereas the coordinates in intrinsic isospin space are denoted by $x, y, z$. Similarly, the solution of the Duffin–Kemmer–Petiau equation for a vector particle field (denoted by $a_\mu$)

$$
\mathcal{U}^{(10)}_{DKP}(p) = (2\chi^2)^{-1/2} \begin{pmatrix}
-\partial_1 a_0 - \partial_0 a_1 \\
-\partial_2 a_0 - \partial_0 a_2 \\
-\partial_3 a_0 - \partial_0 a_3 \\
\partial_2 a_3 - \partial_3 a_2 \\
\partial_3 a_1 - \partial_1 a_3 \\
\partial_1 a_2 - \partial_2 a_1 \\
-\chi a_1 \\
-\chi a_2 \\
-\chi a_3 \\
-\chi a_0
\end{pmatrix} = (2\chi^2)^{-1/2} \begin{pmatrix}
e_1 \\
e_2 \\
e_3 \\
h_1 \\
h_2 \\
h_3 \\
-\chi a_1 \\
-\chi a_2 \\
-\chi a_3 \\
-\chi a_0
\end{pmatrix}
$$

(3.25)
can be used to predict a correlation between spin–flavour and space–time degrees of freedom for the vector mesons. The isotriplet helicity doublet $\{\rho^+, \rho^-, \rho^0\} \oplus \{\rho^+, \rho^-, \rho^0\}$ is most naturally mapped onto the $\{1, 0\} \oplus \{0, 1\}$ representation of $so(4)$, whereas the charge/spin doublets $\{\rho_1^+, \rho_5^+\} \oplus \{\omega^+, \omega^+\}$ are mapped onto $\{1/2, 1/2\}$ according to

$$
\begin{pmatrix}
\rho^+ \\
\rho^- \\
\rho^0 \\
\rho^+ \\
\rho^- \\
\rho^-
\end{pmatrix} = \begin{pmatrix}
e_1 + ih_1 \\
e_2 + ih_2 \\
e_3 + ih_3 \\
e_1 - ih_1 \\
e_2 - ih_2 \\
e_3 - ih_3
\end{pmatrix},
$$

(3.26)

and

$$
\begin{pmatrix}
\omega^+ \\
\rho_5^+
\end{pmatrix} \rightarrow -\chi \begin{pmatrix}
a_0 + a_3 & a_1 - ia_2 \\
a_1 + ia_2 & a_0 - a_3
\end{pmatrix}.
$$

(3.27)

In eq. (3.25), $\mathcal{U}^{(10)}_{DKP}(p)$ stands for a massive $DKP$ spinor field related to the solution $\Psi_{DKP}$ of the spin–1 DKP equation via $\Psi_{DKP} = \exp(\chi/p_0V)^{1/2}\mathcal{U}^{(10)}_{DKP}(p)$ [10], [15]. The mass spectrum of the irreps of the group SO(5) was studied in great detail in [15]. There, it was shown that after SO(4)–reduction of a quintuplet a quadruplet ($col(\rho_5^+, \pi^+, \pi^-, \pi^0)$ in our case) of finite mass, $\chi$, and a singlet ($\psi_{\{0\}} = \omega_3$ in our case) satisfying at rest the equation $(0 \cdot \partial_t^2 - \chi^2)\psi_{\{0\}} = 0$ and therefore of infinite mass, appear. Similar analysis shows that the 6–dimensional vector $col(\rho^+, \rho^-, \rho^0, \rho^+, \rho^-)$ will be of finite mass $\chi$, whereas an infinite mass should be attributed to the accompanying four–vector $(col(\rho_5^+, \rho_5^+, \omega^+, \omega^+)$ in our case). The latter state, the only one for which spin and isospin degrees of freedom do not decouple, is expunged from the low mass region and becomes a non–observable degree of freedom for the low energy hadron physics considered.
here. Thus the $\omega$ meson as well as the longitudinal modes of the positively and negatively charged $\rho$ mesons drop out through $\text{SO}(6) \supset \text{SO}(5) \supset \text{SO}(4)$ reduction of the defining representation. Note, that in the case of pure flavour $\text{SU}(4)$ this reduction scheme would be less natural, for the decomposition of the purely pseudoscalar flavour $\{15\}$–plet into $\text{SO}(4)$ irreps requires additional information.

In the next section we present ideas on how to construct effective mesonic field theories by means of the $\text{SO}(6)$ invariant equations developed above.

4 Effective $\text{SO}(6)$–invariant Lagrangian for the spin-flavour $\{15\}$–plet

The most famous effective theory in low energy hadron physics is the linear $\sigma$ model. It is based on a pure mesonic Lagrangian of the type $\phi^4$ which is the maximal renormalizable theory in (3+1) space–time dimensions. The underlying symmetry of the model is the chiral group $\text{SU}(2) \otimes \text{SU}(2)$ acting as the universal covering group of $\text{SO}(4)$ (compare eq. (3.2)). For this purpose a relativistic, $\text{SO}(4)$ invariant Lagrangian is constructed in terms of the four vector $\{1/2, 1/2\} \equiv \psi_{\{4\}}$ in place of the scalar $\phi$ as follows:

$$ L_{\sigma}(x) = \frac{1}{2} \partial^\mu \psi_{\{4\}}(x) \partial_\mu \psi_{\{4\}}(x) - V(x), $$

$$ V(x) = \frac{1}{2} \mu^2 ||\psi_{\{4\}}(x)||^2 - \frac{1}{4} \lambda^2 ||\psi_{\{4\}}(x)||^4 $$

with the constraint

$$ ||\psi_{\{4\}}||^2 = \sigma^2(x) + \bar{\pi}^2(x) = f^2. \quad (4.1) $$

This Lagrangian is related to unitary theories by use of a nonlinear realization

$$ ||\psi_{\{4\}}||^2 = \frac{f^2}{2} \text{tr} \left( U^+(x)U(x) \right), \quad (4.2) $$

$$ U(x) = \exp (-i\bar{\tau} \cdot \varphi(x)) = \frac{1}{f} \left( \sigma(x) 1 - \sum_{j=1}^{3} \pi_j(x) i\tau_j \right). \quad (4.3) $$

Here, $\varphi(x)$ stands for a dimensionless parameter set termed to as ‘generalized chiral angles’ which allows for the parametrization

$$ \frac{\sigma}{f} = \cos \varphi, \quad \frac{\pi_j}{f} = \frac{\varphi_j}{\varphi} \sin \varphi, \quad \varphi = ||\varphi|| \quad (4.4) $$

whereas $f$ rescales the norm of $\psi_{\{4\}}$ to unity. The interpretation of this nonlinear realization and its transformations in Chiral Dynamics becomes much more apparent when the field $U(x)$ is interpreted in terms of real quaternions.\footnote{Subsequently, we denote real, complex, quaternionic and octonionic numbers by $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$, respectively.} Embedding quaternions by Pauli...
matrices $\tau_j$ into two dimensional complex spaces,

$$q_0 = 1_{[2 \times 2]}, q_j = -i \tau_j \iff U(x) = \frac{1}{f} \left( \sigma(x)q_0 + \sum_{j=1}^{3} \pi_j(x)q_j \right), \quad (4.5)$$

$U(x) \in \text{USp}(2) \simeq U(1,q)$ denotes a normalized real quaternion. Transformations of $U(x)$ may be performed just by quaternionic multiplication, i.e. for $a, b \in U(1,q)$ we have

$$U \longrightarrow U' = aUb^+, \quad (4.6)$$

where $'+'$ denotes quaternionic conjugation $q_0 \rightarrow q_0$, $q_j \rightarrow -q_j$. The transformation (4.6) may be parametrized by six real parameters $\vec{\epsilon}_R$ and $\vec{\epsilon}_L$ according to

$$a = \exp \left( -i \vec{\tau} \cdot \vec{\epsilon}_R \right), \quad b^+ = \exp \left( i \vec{\tau} \cdot \vec{\epsilon}_L \right), \quad (4.7)$$

so that the relation to right/left transformations of the meson field $U$ used in Chiral Dynamics is obvious. The conservation of the norm of $U$ can either be calculated directly by use of eq. (4.6) or from the property of the real quaternions to form a division algebra, i.e.

$$||U'|| = ||aUb^+|| = ||a|| ||U|| ||b^+|| = ||U||. \quad (4.8)$$

Furthermore, the restriction $\epsilon_R = \epsilon_L$ leads to $a = b$ and the transformation

$$U \longrightarrow U' = aUa^+ \quad (4.9)$$

of the field $U$. Thus, SU(2) flavour as the ‘diagonal subgroup’ of Chiral Dynamics emerges automatically as automorphism group of real quaternions [19]. The Lagrangian of the linear sigma model as expressed in terms of the U–field reads

$$L_\sigma = -\frac{f^2}{4} tr (\partial \nu U(x) \partial^\nu U(x)) - \frac{1}{2} \mu^2 ||U(x)||^2 + \frac{1}{4} \lambda^2 ||U(x)||^4. \quad (4.10)$$

It is noteworthy, that the properties discussed above are special features of the $\{1/2, 1/2\}$ irrep which are in general not shared by other representations of the orthogonal group.

In the Skyrme model [20] where the Lagrangian of the linear sigma model is completed by the so called Skyrme term [20] the nucleon is described as a soliton of the $2 \times 2$ quaternion field $U(x)$ realizing at fixed times a map of the unit three space sphere $S^3$ from $\mathbb{R}^4$ onto the SU(2) group manifold $S^3$ (see [21] for a review). Once $L_\sigma$ has been written in terms of an exponentially represented SU(2) group element, its generalization to higher SU($N_F > 2$) can be considered with the drawback that the group manifolds may not longer be spheres. Such (complex) generalizations of the Skyrme model have extensively been considered in the literature (see [22] for a recent work).

Generalizing the symmetry of the linear sigma model from SO(4) to SO(6) with the 15–dimensional defining representation of so(6) in place of $\{1/2, 1/2\}$, it is necessary to
emphasize the important rôle of quaternions and their embedding in complex vector spaces. The embedding of twofold quaternions leads to USp(4) \( \simeq U(2,q) \) and the group SU(4). The vector representation of the algebra su(4) is cast into the traceless second rank tensor

\[ M^A_B(x) = \begin{pmatrix}
\left(\frac{1}{2}(\pi^0 + \omega_3 + \rho_3^0)\right) & \sqrt{\frac{1}{2}(\pi^+ + \rho_3^+)} & \sqrt{\frac{1}{2}(\omega^+ + \rho^0 \uparrow)} & \rho^+ \uparrow \\
\sqrt{\frac{1}{2}(\pi^- + \rho_3^-)} & \left(\frac{1}{2}(-\pi^0 + \omega_3 - \rho_3^0)\right) & \rho^- \uparrow & \sqrt{\frac{1}{2}(\omega^+ - \rho_3^0 \uparrow)} \\
\sqrt{\frac{1}{2}(\omega^- + \rho^0 \downarrow)} & \rho^+ \downarrow & \sqrt{\frac{1}{2}(\pi^0 - \omega_3 - \rho_3^0)} & \sqrt{\frac{1}{2}(\pi^+ - \rho_3^+)} \\
\rho^- \downarrow & \sqrt{\frac{1}{2}(\omega^- - \rho_3^0 \downarrow)} & \sqrt{\frac{1}{2}(\pi^- - \rho_3^-)} & \left(\frac{1}{2}(-\pi^0 - \omega_3 + \rho_3^0)\right)
\end{pmatrix}
\]

The exponential representation of the meson supermultiplet is given by

\[ U(x) = \exp(iM(x)) \] (4.11)

and describes finite transformations with respect to SU(4). If we take into account the isomorphism su(4) \( \leftrightarrow \) so(6), it is possible to relate the selfconjugate representations of su(4) to so(6) and treat them on the same footing. Towards the construction of a mesonic SO(6) invariant Lagrangian the following ideas can be exploited:

1. The first idea relies on the property of the unitary/orthogonal groups to preserve the norms of the corresponding irreducible representations. Because of the isomorphism between the su(4) and so(6) algebras mentioned above, the selfconjugate su(4) tensor \( M^A_B \) is associated with the vector representation of so(6) and the following effective Lagrangian can be written:

\[ L = \overline{\Psi}_{15} \alpha \cdot \partial \Psi_{15} + \mu^2 \overline{\Psi}_{15} \eta \Psi_{15} - \lambda^2 \left( \overline{\Psi}_{15} \eta \Psi_{15} \right)^2. \] (4.12)

Note, that we have omitted the space–time argument \( x \) to simplify the notation.

The constants \( \mu^2 \) and \( \lambda^2 \) have to be determined by comparison to suitably chosen experimental data. From the latter equation it follows that if we neglect the \( \{10\} \)–plet, the interaction term reduces to that of the standard sigma model with the \( \rho_3^0 \) field appearing in place of the \( \sigma \) meson. Now the correspondence between the spin–flavour and space–time degrees of freedom given in eq. (3.24) shows that for the case of a radial configuration of the scalar field \( \phi(r) \) the famous hedgehog ansatz [21] of the nonlinear sigma model \( \pi_x(r) \sim \vec{r}_1, \pi_y(r) \sim \vec{r}_2, \pi_z(r) \sim \vec{r}_3 \), is recovered.

In a similar way, eq. (3.26) will lead to a helical field configuration for the vector mesons. Detailed study of the solution of the equations proposed will be given in a forthcoming paper.

2. The second possibility is to introduce nonlinear terms along the line of conformally invariant spinor equations (see [23] for a review) via

\[ \left( \alpha \cdot \partial + F(\overline{\Psi}_{15}, \Psi_{15}) \right) \Psi_{15} = 0, \] (4.13)
with
\[ F(\Psi_{15}, \Psi_{15}) = F_1 + F_2 \eta + F_3 \alpha^\mu (\Psi_{15} \eta \alpha^\mu \Psi_{15}) + F_4 S^{\mu \nu} \left( \Psi_{15} \eta S_{\mu \nu} \Psi_{15} \right), \] (4.14)

\[ S_{\mu \nu} = \frac{i}{4} (\alpha_\mu \alpha_\nu - \alpha_\nu \alpha_\mu). \]

Here \( F_1, F_2, F_3 \) and \( F_4 \) stand for some arbitrary scalar functions of \( \overline{\Psi} \Psi \) and \( \overline{\Psi} \eta \Psi \).

The proof that such equations are solvable can be found i.e. in \([23]\).

3. A third effective model can be constructed by means of the field \( \mathcal{U}(x) \) determining the norm of \( M \) in the reduction scheme \( \text{SO}(6) \supset \text{SO}(5) \supset \text{SO}(4) \):

\[ \mathcal{U}(x) = \frac{1}{F} (u_1(x) + i u_2(x)), \] (4.15)

with the quaternions \( u_1 \) and \( u_2 \) given by:

\[ u_1(x) = \rho_3^0(x) \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix} - \vec{\pi}(x) \cdot \begin{pmatrix} -i \vec{\tau} & 0 \\ 0 & -i \vec{\tau} \end{pmatrix}, \]

\[ u_2(x) = -\sqrt{2} \begin{pmatrix} 0 & -i \vec{\tau} \cdot \vec{\rho}(x) \uparrow \\ -i \vec{\tau} \cdot \vec{\rho}(x) \downarrow & 0 \end{pmatrix}. \] (4.16)

The new constant \( F \) instead of \( f \) rescales the norm of the meson supermultiplet to unity. It is then natural to suppose that the fermionic \( SU(4) \) \{20\}–plet, occurring in the large \( N_c \) limit of QCD, could emerge as a soliton of the \( \mathcal{U}(x) \) field.

Within the schemes proposed above meson–meson scattering inclusive corrections from \{15\}–loops can be calculated.

Effective Lagrangians with linear kinetic terms have been considered e.g. in ref. \([24]\) where an \( \text{SO}(5) \) invariant Duffin–Kemmer-Petiau Lagrangian has been used for the construction of a nonlinear sigma model. There, four different \( \text{SO}(5) \) quintuplets have been in turn associated with the \( \pi^+, \pi^-, \pi^0 \) and the \( \sigma \) meson fields. Furthermore, an auxiliary \( \text{SO}(5) \) singlet state has been introduced and exploited as a Lagrange multiplier to account for the chiral circle constraint of (4.1). This field has then been organized together with the four quintuplets mentioned above into a reducible \{21\}–plet undergoing \( \text{SO}(5) \) algebra transformations. Note that no \( SU(4) \) representation could be mapped onto this \{21\}–plet. Our approach differs principally from the one presented in \([24]\) since it is based on irreducible representations common both to \( \text{SO}(6) \) and its covering group \( SU(4) \).

5 Perspectives for \( SU(4) \) fermion states description

To include explicit fermionic degrees of freedom in the Lagrangians, one has to construct spinor representations of \( SU(4) \), the universal covering group of \( \text{SO}(6) \). To benefit once
more from quaternions one can exploit isomorphism between SO(5) and USp(4), the maximal subgroup in SU(4), which allows one to consider the stereographic projection $S^4$ from $\mathbb{R}^5$ onto $\mathbb{H}$ in analogy to the stereographic projection $S^2 \rightarrow \mathbb{C}$.

Via the stereographic projection each point of the unit sphere $S^2$ of $\mathbb{R}^3$ placed on a line passing, say, through the north pole, is mapped onto the complex number $z$ determining the intersection between that line and the equatorial plane $\mathbb{R}^2$. Introducing complex homogeneous coordinates $z_1$ and $z_2$ via $z \rightarrow z_1/z_2$, it is observed that M"{o}bius transformation of the complex plane

$$z' = \frac{\alpha z + \beta}{-\beta^* z + \alpha^*} = \frac{\alpha z_1 + \beta z_2}{-\beta^* z_1 + \alpha^* z_2}$$

with $|\alpha|^2 + |\beta|^2 = 1$ are equivalent to SU(2) transformations of the two–component vectors $\psi$,

$$\psi = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \rightarrow \psi' = \begin{pmatrix} z'_1 \\ z'_2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

and thus describe rotations of $S^2$. In generalizing the M"{o}bius transformations in the complex plane to Sl(2,c), particles of arbitrary spin can be described by means of higher rank tensors (so called spin–tensors) constructed as direct products of the spinors $\psi$ and $\psi^+$ with the property to satisfy the Bargmann–Wigner equations.

In a similar way, the points of the unit sphere $S^4$ of $\mathbb{R}^5$ can be mapped via stereographic projection onto real quaternions $\mathbb{H}$, and as in the case of complex spinors one may introduce quaternionic spinors in terms of quaternionic homogeneous coordinates $q \rightarrow q_1/q_2$. Note, that the quotient $q$ is mathematically well defined since real quaternions constitute a division algebra. Depending on the metric, sesquilinear symmetric or bilinear antisymmetric $[17]$, M"{o}bius transformations of the quaternionic plane give rise to transformations of the spinor

$$\Psi = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

which can be described by the compact groups U(2,q) and Sp(2,q), respectively. The representation of such spinor transformations by means of complex vector spaces is based on the isomorphisms

$$\text{Sp}(2,q) \simeq \text{USp}(4) \simeq \text{U}(2,q).$$

A further generalization of the M"{o}bius transformations in the quaternionic plane leads to Sl(2,q) transformations of the two quaternionic homogeneous coordinates. The latter transformations can be realized on complex vector spaces by the noncompact group SU*(4) whose algebra is isomorphic to so(5,1) and thus generates transformations of real coordinates. In analogy to the stereographic projection $S^2 \rightarrow \mathbb{C}$ where the equations of motion are written in terms of the independent spinors $\psi \sim z$ and $\psi^+ \sim \bar{z}$ our approach
suggests the formulation of relativistic equations of motions in terms of the independent quaternion spinors $\Psi \sim q$ and $\Psi^+ \sim q^+$. These considerations throw some light on the reason for which SU(4) Wigner supermultiplet scheme may be favoured over the relativistic SU(6) approaches of the late 60ies solely based on the generalization of complex dimensions. Indeed, in recalling the relations [17], [27]

$$\text{su}^*(4) \simeq \text{sl}(2,q) \simeq \text{so}(5,1) \quad (5.5)$$

it becomes apparent that the advantage of SU(4) over SU(6) lies in the embedding of quaternions. Since the algebras $\text{su}^*(4)$ and $\text{su}(4)$ are related by Weyl’s unitary trick, we can use the isomorphism $\text{su}^*(4) \simeq \text{so}(5,1)$ which leads to the deSitter group via the chain $\text{SO}(5,1) \supset \text{SO}(4,1)$, and therefore to the Poincaré group as obtained from $\text{SO}(4,1)$ by contraction [28]. Thus a relativistic description of hadrons in terms of SU(4) representations and embeddings of (twofold) quaternionic homogeneous coordinates becomes possible.

Note, that real quaternions have already been successfully exploited by Finkelstein et al. [19] for formulating quantum mechanics consistently. We here expect quaternion ‘spinors’ to be the most suited mathematical tools for a quantum field theory of hadrons. For the relation of the approach suggested to Chiral Dynamics see [29]. In fact, there are different approaches possible to handle the relativistic aspects of field theories. The most elegant approach would be the construction of quaternionic polynomials and application of an appropriate quaternionic analysis which would also allow to study finite SO(5) rotations and the geometry of Kaluza–Klein theories in Pauli’s approach [11]. Unfortunately, due to the noncommutativity of the quaternions and the lack of quaternionic analysis the problem has to be treated by embedding the quaternions into complex representation spaces.

Therefore, the more realistic approach will be instead to use the SU(4) representations and their decomposition according to the chain $\text{SU}(4) \sim \text{SO}(6) \supset \text{SO}(5)$ in terms of real coordinates and the DKP–algebra. Alternatively, the reduction $\text{SU}(4) \supset \text{SU}(2) \times \text{SU}(2)$ [31] as a complex analogon to the quaternionic projective space $\mathbb{HP}(1) = \text{Sp}(2)/\text{Sp}(1) \otimes \text{Sp}(1)$ [27] can be exploited, too. In this respect we wish to quote early work by Hecht and Pang [29] where SU(4) state vectors have been constructed and the appropriate Wigner–Racah algebra has been worked out. For example, the meson fields in the vector representation $[2,1,1]$ can be attached to SU(4) irreducible tensors $T_{(S_m_s)}^{[2,1,1]}(T_m_t)$, where $(S_m_s)$ and $(T_m_t)$ denote spin and isospin quantum numbers, respectively. The nucleon and the delta resonance can be organized into the $[3,0,0]$ representation and described by means of the totally symmetric third rank tensors $T_{(S_m_s)}^{[3,0,0]}(\frac{1}{2}m_s,\frac{3}{2}m_s,\frac{3}{2}m_t)$ and $T_{(S_m_s)}^{[3,0,0]}(\frac{3}{2}m_s,\frac{1}{2}m_s,\frac{1}{2}m_t)$. It is further possible to construct a polynomial system for SU(4) in analogy to the spherical harmonics of the rotational group SO(3) [30]. These polynomials are available as a basis in which a proper SU(4) Hamiltonian can be diagonalized. More recently, SU(4) meson–fermion vertices have been presented in [31] where importance of the totally symmetric $\{20\}$–plet
as intermediate state in Born graphs for neutral pion photoproduction on the nucleon at threshold was emphasized. With respect to the relativistic aspects of our approach, it should be noted that the necessary discussion of the complexified quaternionic algebra and the associated real forms leads naturally to the last division algebra, the octonions \( \mathbb{O} \). With respect to physical ideas and recent research, we only want to point out the deep relation between \( \mathbb{O} \) and the sphere \( S^7 \). Especially the group \( \text{SO}(8) \) acting on \( S^7 \) is used in the framework of GUT theories [32]. Furthermore, for a thorough discussion on the relation of octonions to quarks and the associated symmetry groups \( G_2 \) and \( \text{SU}(3) \), we refer the interested reader to the excellent article of Günaydin and Gürsey [33].

6 Summary

In this study we propose \( \text{SO}(6) \) invariant first order wave equations describing selfconjugate spin–flavour representations of the group \( \text{SU}(4) \). In the spirit of the linear \( \sigma \) model we construct corresponding effective \( \text{SO}(6) \) invariant Lagrangians on the basis of quaternionic canonical embeddeds. We show that the finite mass \( \{4\} \)-plet emerging in the \( \text{SO}(6) \supset \text{SO}(5) \supset \text{SO}(4) \) reduction scheme corresponds to the hedgehog solution of the nonlinear \( \sigma \) model whereas for the respective \( \{6\} \)-plet (and therefore for the relevant vector mesons degrees of freedom) we predict a helical field configuration. We emphasize the important rôle of quaternionic geometry and analysis for relativistic field theory. In this context, we point out the possibility of further generalizations of effective models not by increasing the dimensions of their representation spaces but by additional complexification of the underlying division algebras in the sequence \( \mathbb{C} \to \mathbb{H} \to \mathbb{O} \) and by use of appropriate homogeneous coordinates.

We suggest that the totally symmetric fermionic \( \{20\} \)-plet of the contracted \( \text{SU}'(4) \) emerging in the large \( N_c \) limit of QCD possibly shows up as a soliton of the complexified quaternion \( \mathbb{U}(x) \).

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