Backward errors and small sample condition estimation for ⋆-Sylvester equations

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Abstract

In this paper, we adopt a componentwise perturbation analysis for ⋆-Sylvester equations. Based on the small condition estimation (SCE), we devise the algorithms to estimate normwise, mixed and componentwise condition numbers for ⋆-Sylvester equations. We also define a componentwise backward error with a sharp and easily computable bound. Numerical examples illustrate that our algorithm under componentwise perturbations produces reliable estimates, and the new derived computable bound for the componentwise backward error is sharp and reliable for well conditioned and moderate ill-conditioned ⋆-Sylvester equations under large or small perturbations.

Keywords: ⋆-Sylvester equation, condition number, componentwise perturbation, backward error, small-sample condition estimation

2010 MSC: 15A09, 15A12, 65F35

1. Introduction

Consider the ⋆-Sylvester equation:

\[ AX \pm X^*B^* = C, \quad A, B, C \in \mathbb{C}^{n \times n}, \] (1)
where $\star$ denote the conjugate transpose of a complex matrix. The $\star$-Sylvester equation arises in the perturbation of palindromic eigenvalue problems and the solution of the $\star$-Ricati equation \([1]\):

$$AX^\star + XB + CX^\star + D = 0, \quad A, B, C, D \in \mathbb{C}^{n \times n}.$$ 

The following lemma gives the sufficient and necessary condition for the existence and uniqueness of the solution of $\star$-Sylvester equation, which appeared in \([1]\). The solvability of \([1]\) was also investigated in \([2]\). Note that the spectrum $\sigma(A, B)$ contains the ordered pairs $(a_i, b_i)$ and represents the generalized eigenvalues $\lambda_i = a_i/b_i$ of the matrix pencil $A - \lambda B$ or matrix pairs $(A, B)$.

**Lemma 1** \([1]\) For the $\star$-Sylvester equation

$$AX \pm X^\star B^\star = C,$$

where $A, B, C \in \mathbb{C}^{n \times n}$, the solution exists and is unique if and only if, for \(\{(a_{ii}, b_{ii})\} = \sigma(A, B)\), the following conditions are satisfied:

$$a_{ii}a_{jj}^\star - b_{ii}b_{jj}^\star \neq 0, \quad (\forall i \neq j);$$

and, for $\lambda_i = a_{ii}/b_{ii}$ and all $i$,

$$a_{ii} \pm b_{ii} \neq 0 \quad (\text{for } \star = T), \quad |\lambda_i| \neq 1 \quad (\text{for } \star = H).$$

In this paper, we focus on the case where $A, B, C$ are real with the plus sign in \((1)\), i.e., the $T$-Sylvester equation

$$AX + X^T B^T = C.$$  \(\text{(2)}\)

Similar results can easily be developed for the general cases and will be ignored.

For $A \in \mathbb{R}^{m \times n}$, $\text{vec}(A)$ stacks the columns of $A$ to a vector. The Kronecker Product $A \otimes B = (a_{ij}B) \in \mathbb{R}^{mp \times nq}$ for $A = (a_{ij}) \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times q}$ and \((2)\) is equivalent to

$$P \text{vec}(X) = \text{vec}(C),$$  \(\text{(3)}\)

where $P = I \otimes A + (B \otimes I)\Pi$ and $\Pi$ is the permutation matrix satisfying $\text{vec}(A^\top)\Pi = \text{vec}(A)$. Under the conditions in Lemma \([1]\), the coefficient matrix in \((3)\) is invertible.
An extension of the Bartels-Stewart algorithm \[3\] to uniquely solvable \(T\)-Sylvester equations was presented in \[1\]; see also \[2, 4\]. The algorithm first computes a generalized real Schur decomposition \[5\] of \(A - \lambda B^\top\):

\[
A = WT_AV^\top, \quad B^\top = WTV^\top, \quad (4)
\]

where \(T_A \in \mathbb{R}^{n \times n}\) is upper quasi-triangular, \(T_B\) is upper triangular and \(V, W \in \mathbb{R}^{n \times n}\) are orthogonal. Defining \(Y = V^\top XW\), the factorization in \(4\) allow us to transform \(2\) to the equivalent \(T\)-Sylvester equation

\[
T_A Y + Y^\top T_B^\top = W^\top CW.
\]

The (block) triangular structures of \(T_A\) and \(T_B\) yield \(Y\) by a simple substitution procedure and the algorithm is completed by the retrieval of \(X = V Y W^\top\). The total flop count of the algorithm is of \(O(67\frac{1}{6}n^3)\); see \[1\] for details.

In sensitivity analysis, condition numbers are important, measuring the worst-case effect of small changes in the data on the solution. A problem with a large condition number is called ill-posed \[6\], and the computed solution to the problem via any numerical algorithms cannot be reliable. For the perturbation analysis of Layapunov, (generalized) Sylvester and Ricatti equations, the readers are referred to \[7, 8, 9, 10, 11, 12\] and references therein. Componentwise perturbation analysis can give sharper error bounds than those based on normwise perturbation analysis because it can better capture the condition of the problem with respect to the scaling and sparsity of the data; see the comprehensive review \[13\]. Diao et al. \[14\] introduced componentwise perturbation analysis for Sylvester equation. The explicit expressions for normwise, mixed and componentwise condition numbers were derived. For the perturbation analysis for \(\star\)-Sylvester equation \(1\), Chiang et al. \[1\] studied the normwise perturbation analysis, both the normwise perturbation error bounds and normwise backward errors were investigated. Assume that there are perturbations \(\Delta A, \Delta B\) and \(\Delta C\) on \(A, B\) and \(C\) respectively, and when the norms of perturbation matrices are sufficiently small, the following perturbed \(T\)-Sylvester equation

\[
(A + \Delta A)(X + \Delta X) + (X + \Delta X)^\top(B + \Delta B)^\top = C + \Delta C
\]

has the unique solution \(X + \Delta X\). The normwise perturbation bound for \(X\) is given by \[1\] Sec. 2.2.3

\[
\frac{\|\Delta X\|_F}{\|X\|_F} \leq \frac{\kappa(P)}{1 - \kappa(P)\|\Delta P\|_F/\|P\|_F} \left(\frac{\|\Delta C\|_F}{\|C\|_F} + \frac{\|\Delta P\|_F}{\|P\|_F}\right),
\]
where $\|A\|_F$ is Frobenius norm of $A$, $\kappa(P) = \|P\|_F\|P^{-1}\|_F$ and $\|\Delta P\|_F = \|\Delta A\|_F + \|\Delta B\|_F$. The normwise backward error for the computed solution $Y$ of (2) is defined as

$$\eta(Y) = \min \left\{ \epsilon : (A + \Delta A)Y + Y^\top (B + \Delta B)^\top = C + \Delta C \right\}$$

where $\|A\|_F \leq \epsilon \|A\|_F$, $\|\Delta A\|_F \leq \epsilon \|\Delta A\|_F$, $\|\Delta B\|_F \leq \epsilon \|\Delta B\|_F$, $\|\Delta C\|_F \leq \epsilon \|\Delta C\|_F$. The upper bound for $\eta(Y)$ is given in [1, Sec. 2.2.2] as

$$\eta(Y) \leq \frac{(\|A\|_F + \|B\|_F)\|Y\|_F + \|C\|_F}{\left(\|A\|_F + \|B\|_F\right)\|X^{-1}\|^{-2} + \|C\|_F^2} \frac{\|R\|_F}{\|Y\|_F + \|C\|_F^2},$$

where $R = C - AY - Y^\top B^\top$. Recently, Yan [15] introduced componentwise perturbation analysis for \*-Sylvester equation, defined and obtained normwise, mixed and componentwise condition numbers for \*-Sylvester equation as follows

$$\kappa^{T-SYL} = \lim_{\epsilon_1 \to 0} \sup_{\|\Delta A\|_F \leq \epsilon_1 \|A\|_F, \|\Delta B\|_F \leq \epsilon_1 \|B\|_F, \|\Delta C\|_F \leq \epsilon_1 \|C\|_F} \frac{\|\Delta X\|_F}{\epsilon \|X\|_F} = \frac{\|P^{-1}[X^\top \otimes I, (I \otimes X^T)\Pi, -I]\|_F \vec{c}([A])\|}{\vec{c}([B])\|},$$

$$m^{T-SYL} = \lim_{\epsilon_2 \to 0} \sup_{\|\Delta A\|_F \leq \epsilon_2 \|A\|_F, \|\Delta B\|_F \leq \epsilon_2 \|B\|_F, \|\Delta C\|_F \leq \epsilon_2 \|C\|_F} \frac{\|\Delta X\|_{\max}}{\epsilon \|X\|_{\max}} = \left\| P^{-1}(X^\top \otimes I) \vec{c}([A]) + P^{-1}((I \otimes X^T)\Pi) \vec{c}([B]) + P^{-1} \vec{c}([C]) \right\|_{\infty},$$

$$c^{T-SYL} = \lim_{\epsilon_3 \to 0} \sup_{\|\Delta A\|_F \leq \epsilon_3 \|A\|_F, \|\Delta B\|_F \leq \epsilon_3 \|B\|_F, \|\Delta C\|_F \leq \epsilon_3 \|C\|_F} \frac{1}{\epsilon} \left\| \frac{\Delta X}{X} \right\|_{\max} = \left\| \frac{P^{-1}(X^\top \otimes I) \vec{c}([A]) + P^{-1}((I \otimes X^T)\Pi) \vec{c}([B]) + P^{-1} \vec{c}([C])}{\vec{c}([X])} \right\|_{\infty},$$

where $\|A\|_{\infty}$ is $\infty$ norm, $\|A\|_{\max} = \max_{i,j} |a_{ij}|$, $|\Delta A| \leq \epsilon |A|$ is interpreted componentwisely, $\Delta X/X$ is the componentwise quotient (when a denominator is zero, the corresponding numerator must be zero and the corresponding ratio is defined as zero), $I$ is identity matrix and

$$\epsilon_1 = \max \left\{ \|\Delta A\|_F, \|\Delta B\|_F, \|\Delta C\|_F \right\}.$$
The normwise, mixed and componentwise condition numbers were also studied in [16]. Explicit expressions have been derived without the corresponding reliable and efficient estimation. In this paper, we introduce the SCE-based condition estimation for the $\top$-Sylvester equation, as well as the associated componentwise backward error.

The following example from [15, Sec. 6] shows that there are big differences between $m_{\top-\text{SYL}}$, $c_{\top-\text{SYL}}$ and $\kappa_{\top-\text{SYL}}$, illustrating that the mixed and componentwise condition number better capture the condition of $\top$-Sylvester equation with respect to the scaling and sparsity of the input data.

**Example 1** Let $0 < \varepsilon < 1$, for the following $\top$-Sylvester equation

$$
\begin{bmatrix}
1 & 0 \\
0 & \varepsilon
\end{bmatrix} X + X^\top \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix} = \begin{bmatrix}
2 & 0 \\
0 & \varepsilon
\end{bmatrix},
$$

it is easy to see that $X = I_2$ is the exact solution. We have $m_{\top-\text{SYL}} = c_{\top-\text{SYL}} = 2$ and $\kappa_{\top-\text{SYL}} = \sqrt{\frac{63}{4} + \frac{15}{8}\varepsilon^2 + \frac{27}{8}\varepsilon^2} = O\left(\frac{1}{\varepsilon}\right)$ from (7).

From the above example, we see that the perturbation bounds based on normwise condition number may severely overestimate errors. Another issue is that the expressions for $\kappa_{\top-\text{SYL}}$, $m_{\top-\text{SYL}}$ and $c_{\top-\text{SYL}}$ involve the Kronecker product, which involves higher dimensions and prevents the efficient estimation of the condition numbers.

In practice, the problem of how condition numbers are estimated efficiently is critical [6, Chapter 15]. Kenny and Laub [17] developed the method of the small-sample statistics condition estimation (SCE), applicable for general matrix functions, linear equations [18], eigenvalue problems [19], linear least squares problem [20] and roots of polynomials [21]. Recently, SCE had been used to estimate the condition of Sylvester equations [22, 14]. In this paper we devise SCE algorithms to estimate the normwise, mixed and componentwise condition numbers of $\top$-Sylvester equation, which can be used to effectively estimate error bounds. Moreover, we introduce the componentwise backward error for (3) and derive a sharp and easily computable upper bound. In the following we will introduce the definition of the directional derivative, which will be used in the SCE algorithm. For a function $f : \mathbb{R}^n \mapsto \mathbb{R}^m$, the directional derivative of $f$ at $x$ along the direction $y \in \mathbb{R}^n$ is defined by

$$
Df(x; y) = \lim_{h \to 0} \frac{f(x + hy) - f(x)}{h}.
$$
We now introduce the following map

\[ \Phi : [A, B, C] \mapsto X, \]  

(8)

where \(X\) is the unique solution of (2). The following lemma gives the explicit expression of the directional derivative.

**Lemma 2** The map \(\Phi\) defined by (8) is continuous and directional differential at \(v = (\text{vec}(A)^\top, \text{vec}(B)^\top, \text{vec}(C)^\top)^\top\), and the directional derivative of \(\Phi\) at \(v\) along the direction \([E, F, G]\) is the solution of the following \(\top\)-Sylvester equation

\[ AY + Y^\top B^\top = G - EX - X^\top F^\top. \]  

(9)

**Proof.** Let \(\delta > 0\) and \(E, F\) and \(G\) be given, suppose \(X + \delta Y\) is the exact solution of the following \(\top\)-Sylvester equation

\[ (A + \delta E)(X + \delta Y) + (X + \delta Y)^\top (B + \delta F)^\top = C + \delta G. \]  

(10)

Subtracting from the unperturbed \(\top\)-Sylvester equation (2), forcing \(\delta \to 0\), and using the corresponding directional derivative, we then prove the lemma. \(\square\)

This paper is organized as follows. We conduct the componentwise backward error analysis in Section 2. In Section 3, the SCE-base condition estimation algorithms are proposed. Sections 4 and 5 contain the numerical examples and the concluding remarks.

2. Componentwise Backward Error Analysis

In this section, we introduce the componentwise backward error for \(\top\)-Sylvester equation (2), and derive the corresponding sharp and computational bounds.

**Definition 1** Suppose \(Y\) is the computed solution of the \(\top\)-Sylvester equation (2), we define the componentwise backward error as

\[ \mu(Y) = \min \{ \epsilon : (A + \Delta A)Y + Y^\top (B + \Delta B)^\top = C + \Delta C \} \]

where \(|\Delta A| \leq \epsilon |A|, |\Delta B| \leq \epsilon |B|, |\Delta C| \leq \epsilon |C|\}, \]

where \(|\Delta A| \leq \epsilon |A|\), interpreted componentwise with \(|A| = (|a_{ij}|)\).
The following transformation removes the absolute values from the constrains in Definition 1 and replaces inequalities by equalities. Let

\[ \mathbf{vec}(\Delta A) = D_1\nu_1, \mathbf{vec}(\Delta B) = D_2\nu_2, \mathbf{vec}(\Delta C) = D_3\nu_3, \]

where \( D_1 = \text{diag}(\mathbf{vec}(A)) \), \( D_2 = \text{diag}(\mathbf{vec}(B)) \), \( D_3 = \text{diag}(\mathbf{vec}(C)) \). The smallest value of \( \epsilon \) satisfying

\[ |\Delta A| \leq \epsilon |A|, \quad |\Delta B| \leq \epsilon |B| \quad \text{and} \quad |\Delta C| \leq \epsilon |C| \]

is \( \epsilon = \max\{\|\nu_1\|_\infty, \|\nu_2\|_\infty, \|\nu_3\|_\infty\} \), and so

\[ \mu(Y) = \min \left\{ \left\| \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix} \right\|_\infty : (A + \Delta A)Y + Y^T(B + \Delta B)^T = C + \Delta C, \right\}, \]

where \( \mathbf{vec}(\Delta A) = D_1\nu_1, \mathbf{vec}(\Delta B) = D_2\nu_2, \mathbf{vec}(\Delta C) = D_3\nu_3 \).

In general, this equality constrained nonlinear optimization problem has no closed form solution. In the following theorem, we give a sharp and easy-to-compute bound for \( \mu(Y) \).

**Theorem 1** Let \( Y \) and \( \mu(Y) \) be defined as in Definition 1 and

\( H = [ (Y^T \otimes I)D_1, (I \otimes Y^T)\Pi D_2, -D_3 ] \), \( \tilde{R} = C - AY - Y^T B^T \).

Assume \( H \) has full rank, let \( r = \mathbf{vec}(\tilde{R}) \) and consider the QR decomposition

\[ H^T = Q \begin{bmatrix} \tilde{R} \\ 0 \end{bmatrix}, \]

we have

\[ \mu(Y) \leq \overline{\mu}(Y) := \left\| Q \begin{bmatrix} z_1 \\ 0 \end{bmatrix} \right\|_\infty \leq \sqrt{3n} \mu(Y), \]

where \( z_1 = R^{-T}r \). When \( H \) is rank-deficient, \( \mu(Y) \) is defined being infinite.

**Proof.** Putting \( \Delta A, \Delta B \) and \( \Delta C \) to the right hand side of the perturbed equation

\[ (A + \Delta A)Y + Y^T(B + \Delta B)^T = C + \Delta C, \]

we have

\[ \Delta AY + Y^T \Delta B^T - \Delta C = \tilde{R}. \quad (11) \]

Applying the \( \mathbf{vec} \) operation, (11) has the following form:

\[ (Y^T \otimes I)\mathbf{vec}(\Delta A) + (I \otimes Y^T)\Pi \mathbf{vec}(\Delta B) - \mathbf{vec}(C) = \mathbf{vec}(\tilde{R}). \quad (12) \]
Then the above equation can be written as the following linear system

\[
\begin{bmatrix}
Y^\top \otimes I, (I \otimes Y^\top) \Pi, -I \otimes I
\end{bmatrix}
\begin{bmatrix}
\text{vec}(\Delta A) \\
\text{vec}(\Delta B) \\
\text{vec}(\Delta C)
\end{bmatrix}
= \text{vec}(\tilde{R}).
\] (13)

Recalling the diagonal matrices \(D_1 = \text{diag}(\text{vec}(A))\), \(D_2 = \text{diag}(\text{vec}(B))\) and \(D_3 = \text{diag}(\text{vec}(C))\), we have

\[
\begin{bmatrix}
Y^\top \otimes I, (I \otimes Y^\top) \Pi, -I \otimes I
\end{bmatrix}
\begin{bmatrix}
D_1 \\
D_2 \\
D_3
\end{bmatrix}
\begin{bmatrix}
\nu_1 \\
\nu_2 \\
\nu_3
\end{bmatrix}
= \text{vec}(\tilde{R}).
\] (14)

This is an underdetermined system of the form \(Hz = r\), with \(H \in \mathbb{R}^{n^2 \times 3n^2}\) and \(r = \text{vec}(\tilde{R})\). We seek the solution of minimal \(\infty\)-norm at \(\mu(Y)\).

If \(H\) is rank-deficient, then there may be no solution to \(Hz = r\), in which case the componentwise backward error \(\mu(Y)\) may be regarded as infinite. Assume, therefore, that \(H\) is full rank. Using the QR factorization of \(H^\top\), then \(Hz = r\) may be written as

\[
r = [R^\top \ 0]Q^\top z = [R^\top \ 0] \begin{bmatrix}
\tilde{z}_1 \\
\tilde{z}_2
\end{bmatrix}
= R^\top \tilde{z}_1.
\]

Thus \(\tilde{z}_1 = R^{-\top}r\) is uniquely determined and

\[
z = Q \begin{bmatrix}
\tilde{z}_1 \\
\tilde{z}_2
\end{bmatrix}.
\]

Choosing \(\tilde{z}_2\) to minimize \(\|z\|_\infty\) is equivalent to solving an overdetemined linear system in the \(\infty\)-norm sense, for which several methods are available.

We can obtain approximation to the desired \(\infty\)-norm minimum by minimizing in the 2-norm, which amounts to setting \(\tilde{z}_2 = 0\) (which yields \(z = H^\dagger r\), where \(H^\dagger\) is the pseudo-inverse of \(H\)). In view of the fact that \(s^{-1/2}\|t\|_2 \leq \|t\|_\infty \leq \|t\|_2\) for \(t \in \mathbb{R}^s\), it follows that

\[
\mu(Y) \leq \overline{\mu}(Y) \leq \sqrt{3n} \mu(Y).
\]

\(\square\)
3. Small-Sample Condition Estimations

In this section, based on a small-sample statistical condition estimation method, we present a practical method for estimating the condition numbers for the $\star$-Sylvester equations. The small-sample statistical condition estimation (SCE) is proposed by Kenny and Laub [17]. It is an efficient method for estimating the condition numbers for linear systems [20], linear least squares problems [18], eigenvalue problems [19], and roots of polynomials [21]. Based on the adjoint method and SCE, Cao and Petzold [23] proposed an efficient method for estimating the error in the solution of the Sylvester matrix equations. Diao et al. applied the SCE to the Sylvester equations [22, 14], algebraic Riccati equations [24] and the structured Tikhonov regularization problem [25].

3.1. Review on the SCE

We next briefly describe the SCE method. Given a differentiable function $f : \mathbb{R}^p \to \mathbb{R}$, we are interested in its sensitivity at some input vector $x$. From its Taylor expansion, we have

$$f(x + \delta d) - f(x) = \delta (\nabla f(x))^T d + O(\delta^2),$$

for a small scalar $\delta$, where

$$\nabla f(x) = \left[ \frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \ldots, \frac{\partial f(x)}{\partial x_p} \right]^T$$

is the gradient of $f$ at $x$. Then the local sensitivity, up to the first order in $\delta$, can be measured by $\|\nabla f(x)\|_2$. The condition number of $f$ at $x$ is asymptotically determined by the norm of the gradient $\nabla f(x)$. It is shown in [17] that if we select $d$ uniformly and randomly from the unit $p$-sphere $S_{p-1}$ (denoted $U(S_{p-1})$), then the expected value $\mathbb{E}(|(\nabla f(x))^T d|/\omega_p)$ is $\|\nabla f(x)\|_2$, where $\omega_p$ is the Wallis factor, dependent only on $p$:

$$\omega_p = \begin{cases} 1, & \text{for } p \equiv 1, \\ \frac{2}{\pi}, & \text{for } p \equiv 2, \\ \frac{1 \cdot 3 \cdot 5 \cdots (p-2)}{2 \cdot 4 \cdots (p-1)}, & \text{for } p \text{ odd and } p > 2, \\ \frac{2, 4, 6, \ldots, (p-2)}{\pi 1 \cdot 3 \cdots (p-1)}, & \text{for } p \text{ even and } p > 2, \end{cases}$$
which can be accurately approximated by

$$\omega_p \approx \sqrt{\frac{2}{\pi (p - \frac{1}{2})}}.$$

(15)

Therefore,

$$\nu = \frac{|(\nabla f(x))^T d|}{\omega_p}$$

can be used to estimate $$\|\nabla f(x)\|_2$$, an approximation of the condition number, with high probability. Specifically, for $$\gamma > 1$$, we have

$$\text{Prob}\left(\frac{\|\nabla f(x)\|_2}{\gamma} \leq \nu \leq \gamma \|\nabla f(x)\|_2\right) \geq 1 - \frac{2}{\pi \gamma} + O(\gamma^{-2}).$$

Multiple samples $$\{d_j\}$$ can be used to increase the accuracy. The $$k$$-sample condition estimation is given by

$$\nu(k) = \frac{\omega_k}{\omega_p} \sqrt{|\nabla f(x)^T d_1|^2 + |\nabla f(x)^T d_2|^2 + \cdots + |\nabla f(x)^T d_k|^2},$$

where $$d_1, d_2, \ldots, d_k$$ are orthonormalized after they are selected uniformly and randomly from $$U(S_{p-1})$$. For example,

$$\text{Prob}\left(\frac{\|\nabla f(x)\|_2}{\gamma} \leq \nu(2) \leq \gamma \|\nabla f(x)\|_2\right) \approx 1 - \frac{\pi}{4\gamma^2},$$

$$\text{Prob}\left(\frac{\|\nabla f(x)\|_2}{\gamma} \leq \nu(3) \leq \gamma \|\nabla f(x)\|_2\right) \approx 1 - \frac{32}{3\pi^2 \gamma^3},$$

$$\text{Prob}\left(\frac{\|\nabla f(x)\|_2}{\gamma} \leq \nu(4) \leq \gamma \|\nabla f(x)\|_2\right) \approx 1 - \frac{81\pi^2}{512\gamma^4}.$$

If we choose $$k = 3, \gamma = 10$$, then $$\nu(3)$$ has a probability 0.9989 to be within an order of $$\|\nabla f(x)\|_2$$ (i.e., between $$\|\nabla f(x)\|_2/10$$ and $$10 \cdot \|\nabla f(x)\|_2$$).

First we introduce unvec operator, for $$v = (v_1, v_2, \ldots, v_n)^T \in \mathbb{R}^n$$, then $$A = \text{unvec} (v)$$ with $$a_{ij} = v_{i+(j-1)n}$$. These results can be readily generalized to vector-valued or matrix-valued functions by viewing $$f$$ as a map from $$\mathbb{R}^s$$ to $$\mathbb{R}^t$$, possibly after the operations vec and unvec to transform data between matrices and vectors, where each of the $$t$$ entries of $$f$$ is a scalar-valued function. The main computational cost of the SCE is to evaluate the directional derivative of the given mapping $$f$$ at the input data $$x$$. Usually
when we solve the problem in numerical linear algebra by direct methods, we have some decompositions of the matrix which can be used to compute the directional derivative efficiently. For T-Sylvester equations, the generalized Schur algorithm in [1] had been proposed. We can utilize the generalized Schur algorithm to compute the directional derivative efficiently based on Lemma 2. In practice computation, we do not have the exact solution $X$ but we can use the computed one to approximate the directional derivative.

### 3.2. Normwise perturbation analysis

In this section, we will apply the SCE technique to the T-Sylvester equation (3) under normwise perturbations. For the solution of $X$, we are interested in the condition estimation at the point $[A, B, C]$. Let $[A B, C]$ be perturbed to $[A + \delta E \ B + \delta F, C + \delta G]$ in (10), where $\delta \in \mathbb{R}$ and $E, F, G \in \mathbb{R}^{n \times n}$ with $\|[E, F, G]\|_F = 1$. According to Section 3.1, we first need to evaluate the directional derivative $D\Phi([A, B, C]; [E, F, G])$ and for that from Lemma 2, we need to solve the T-Sylvester equation (9). When we use Algorithm TSylvesterR in [1] to solve (2), the generalized real Schur decomposition of the pencil $A - \lambda B^T$ is already available, thus (9) can be solved without minimal additional costs. We are now ready to use the SCE techniques in Section 3.1 to estimate the condition of the T-Sylvester equation (3) under normwise perturbations. Algorithm 3.2 computes a relative condition estimation matrix for the solution $X$ of T-Sylvester equation (3). Inputs to the method are the matrices $A, B \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{n \times n}$ and the computed solution $X$. The output is a relative condition estimation matrix $R_{rel}^{T-SYL,(k)}$ and a relative normwise condition number estimation $\kappa_{F,SCE}^{T-SYL,(k)}$ for $\kappa^{T-SYL}$ which is defined in (7). Again, the method requires the generalized real Schur decomposition of $A - \lambda B^T$, which is generally available after solving the T-Sylvester equation. The integer $k \geq 1$ refers to the number of samples of perturbations to the input data. When $k = 1$, there is obviously no need to orthonormalize the set of vectors in Step 1 of the algorithm.

In Table 3.2, we report the flop counts of Algorithm 3.2. We can see that the total flop counts of Algorithm 3.2 are $O(kn^3)$, which are the same order of the flop counts of the generalized Schur algorithm [1] for solving the T-Sylvester equation (2). It is generally true that solving a problem and estimating its condition involve a similar amount of work, indicating comparable levels of difficulty.
Algorithm 1 Subspace condition estimation for the solution $X$ of $\top$-Sylvester equation (2) under normwise perturbation

1. Generate pairs $(E_1, F_1, G_1), (E_2, F_2, G_2), \ldots, (E_k, F_k, G_k)$ with entry is in $\mathcal{N}(0, 1)$. Use the Modified Gram-Schmidt (MGS) orthogonalization process for

$$\begin{bmatrix}
\text{vec}(E_1) & \cdots & \text{vec}(E_k) \\
\text{vec}(F_1) & \cdots & \text{vec}(F_k) \\
\text{vec}(G_1) & \cdots & \text{vec}(G_k)
\end{bmatrix},$$



to obtain an orthonormal matrix $[q_1, q_2, \ldots, q_k]$. Convert $q_i$ to $(\tilde{E}_i, \tilde{F}_i, \tilde{G}_i)$ with the unvec operation.

2. Approximate $\omega_p$ and $\omega_k$ by (15), with $p = 3n^2$.

3. Solve the following $\top$-Sylvester equation via the generalized Schur algorithm [1]:

$$AY_i + Y_i^\top B^\top = \tilde{G}_i - \tilde{E}_i X + X^\top \tilde{F}_i^\top.$$

4. Calculate respectively the absolute condition matrix and the normwise absolute condition estimation

$$K_{\text{abs}}^\top-\text{SYL.}(k) = \frac{\omega_k}{\omega_p} \sqrt{|Y_1|^2 + |Y_2|^2 + \cdots + |Y_k|^2}, \quad n_{F,\text{SCE}} = \|K_{\text{abs}}^\top-\text{SYL.}(k)\|_F,$$

where the square root is taken for each elements of the matrix. Let the relative condition matrix $K_{\text{rel}}^\top-\text{SYL.}(k)$ be the matrix $||[A, B, C]||_F \cdot K_{\text{abs}}^\top-\text{SYL.}(k)$ divided componentwise by $X$, leaving entries of $K_{\text{abs}}^\top-\text{SYL.}(k)$ corresponding to zero entries of $X$ unchanged. Compute $\kappa_{F,\text{SCE}} = n_{F,\text{SCE}} / ||X||_F.$
### Table 1: Computational complexity of Algorithm 3.2

| Step | Flops         |
|------|--------------|
| 1    | $O(6k^2n^2)$ |
| 2    | $O(1)$       |
| 3    | $O(kn^3)$    |
| 4    | $O(3kn^2)$   |

#### 3.3. Componentwise perturbation analysis

Componentwise perturbations are relative to the magnitudes of the corresponding entries in the input arguments, where the perturbation $\Delta A$ satisfies $|\Delta A| \leq \epsilon |A|$. These perturbations may arise from input error or from rounding error, and hence are the most common perturbations encountered in practice. In fact, most of error bounds in LAPACK are considered componentwise \[26, \text{Section 4.3.2}\]. We often want to find the condition of a function with respect to componentwise perturbations. For $\Phi$ defined in \[8\], SCE is flexible enough to accurately gauge the sensitivity of matrix functions subject to componentwise perturbations. Define the linear mask function

$$h([E, F, G]) = [E, F, G] \odot [A, B, C], \quad E \in \mathbb{R}^{n \times n}, F \in \mathbb{R}^{n \times n}, G \in \mathbb{R}^{n \times n},$$

where $\odot$ denotes the Hadamard componentwise multiplication. When $\mathcal{E} \in \mathbb{R}^{n \times 3n}$ is the matrix of all ones, then $h(\mathcal{E}) = [A, B, C]$ and

$$h(\mathcal{E} + [E, F, G]) = [A, B, C] + h([E, F, G]).$$

Thus $h([E, F, G])$ is a componentwise perturbation on $[A, B, C]$, and $h$ converts a general perturbation $\mathcal{E}$ into componentwise perturbations on $[A, B, C]$. Therefore, to obtain the sensitivity of the solution with respect to relative perturbations, we simply evaluate the Fréchet derivative of

$$\Phi([A, B, C]) = \Phi(h(\mathcal{E}))$$

with respect to $\mathcal{E}$ in the direction $[\Delta A, \Delta B, \Delta C]$, which is

$$\mathbf{D}(\Phi \circ h) (\mathcal{E}; [E, F, G]) = \mathbf{D}\Phi(h(\mathcal{E}))\mathbf{D}h (\mathcal{E}; [E, F, G])$$

$$= \mathbf{D}\Phi([A, B, C])h ([E, F, G]) = \mathbf{D}\Phi ([A, B, C])h ([E, F, G]),$$

13
since $h$ is linear. Thus, to estimate the condition of $X$ for componentwise perturbations, we first generate the perturbations $E$, $F$ and $G$ and multiply them componentwise by $A$, $B$ and $C$, respectively. The remaining steps are the same as the corresponding steps in Algorithm 3.2 as shown in Algorithm 3.3.

4. Numerical Examples

In this section, we demonstrate our test results of some numerical examples to illustrate componentwise backward errors and condition estimations presented earlier. Numerical experiments were carried out on a machine with Intel i5 4590 @3.3GHz CPU, 8G RAM and 1TB hard driver running Windows 7 professional, using MATLAB 8.5 with a machine precision $\epsilon = 2.2 \times 10^{-16}$.

We generate the perturbation matrices as follows

$$
\Delta A = \epsilon \cdot \Delta A \odot A, \quad \Delta B = \epsilon \cdot \Delta B \odot B, \quad \Delta C = \epsilon \cdot \Delta C \odot C, \quad (17)
$$

where $\Delta A$, $\Delta B$ and $\Delta C$ are random matrices with each entries being uniformly distributed in $(-1, 1)$. Let $\tilde{X}$ be the solution of (5). We use Gaussian elimination with partial pivoting to solve (5) in Kroncker product form. Recall that $\Delta X = \tilde{X} - X$. Let us denote the true relative errors

$$
\gamma_k = \frac{\|\Delta X\|_F}{\|X\|_F}, \quad \gamma_m = \frac{\|\Delta X\|_{\max}}{\|X\|_{\max}}, \quad \gamma_c = \frac{\|\Delta X\|}{\|X\|_{\max}}.
$$

Clearly, from the definitions of condition numbers in (7), we have the following inequalities between the first order perturbation bounds and the corresponding exact relative errors:

$$
\gamma_k \leq \kappa^{T-SYL} \epsilon, \quad \gamma_m \leq m^{T-SYL} \epsilon, \quad \gamma_c \leq c^{T-SYL} \epsilon.
$$

Also, from Algorithms 3.2 and 3.3, we can compute the condition estimates $\kappa_{F,SCF}^{T-SYL(k)}$, $m_{SCF}^{T-SYL(k)}$ and $c_{SCF}^{T-SYL(k)}$ which can be used to approximate the posterior perturbation bounds for (2).

Example 2 This example is quoted from [1, Example 3.3]. We use MATLAB `randn(n)` to compute an $n \times n$ random matrix with entries being normal distributed. Let $n = 2, Q \in \mathbb{R}^{n \times n}$ be orthogonal, the exact solution be $X_e$,
Algorithm 2 Subspace condition estimation for the solution $X$ of $\top$-Sylvester equation (2) under componentwise perturbation

1. Generate pairs $(E_1, F_1, G_1), (E_2, F_2, G_2), \ldots, (E_k, F_k, G_k)$ with entry is in $\mathcal{N}(0, 1)$. Use the Modified Gram-Schmidt (MGS) orthogonalization process for

$$
\begin{bmatrix}
\text{vec}(E_1) & \cdots & \text{vec}(E_k) \\
\text{vec}(F_1) & \cdots & \text{vec}(F_k) \\
\text{vec}(G_1) & \cdots & \text{vec}(G_k)
\end{bmatrix},
$$

to obtain an orthonormal matrix $[q_1, q_2, \ldots, q_k]$. Convert $q_i$ to $(\widetilde{E}_i, \widetilde{F}_i, \widetilde{G}_i)$ with the unvec operation. Let $\widetilde{E}_i^c = \widetilde{E}_i \odot A$, $\widetilde{F}_i^c = \widetilde{F}_i \odot B$ and $\widetilde{G}_i^c = \widetilde{G}_i \odot C$.

2. Approximate $\omega_p$ and $\omega_k$ by (15), with $p = 3n^2$.

3. Solve the following $\top$-Sylvester equation via the generalized Schur algorithm [1]:

$$
AY_i + Y_i^\top B^\top = \widetilde{G}_i^c - \widetilde{E}_i^c X + X^\top (\widetilde{F}_i^c)^\top.
$$

4. Calculate the absolute condition matrix

$$
M_{\text{abs}}^{T-\text{SYL},(k)} = \frac{\omega_k}{\omega_p} \sqrt{|Y_1|^2 + |Y_2|^2 + \cdots + |Y_k|^2}.
$$

Compute the relative componentwise condition matrix $C_{\text{rel}}^{T-\text{SYL},(k)} = M_{\text{abs}}^{T-\text{SYL},(k)}/X$ (division carried out componentwise), leaving entries corresponding to zero entries of $X$ unchanged. Compute

$$
M_{\text{SCE}}^{T-\text{SYL},(k)} := \frac{\|M^{T-\text{SYL},(k)}\|_{\max}}{\|X\|_{\max}} \quad \text{and} \quad \epsilon_{\text{SCE}}^{T-\text{SYL},(k)} := \frac{\|M^{T-\text{SYL},(k)}\|}{\|X\|_{\max}},
$$

where $\|X\|_{\max} = \max_{ij} |X_{ij}|$. 

\[15\]
Table 2: Comparing the true normwise perturbation bounds with the first order normwise bounds

| $\epsilon$ | $m$ | $\gamma_{\kappa}$ | $\kappa_T^{-\text{SYL}} \cdot \epsilon$ | $\kappa_{F,SCE}^{-\text{SYL},(k)} \cdot \epsilon$ |
|----------|-----|-----------------|-----------------|-----------------|
| $10^{-8}$ | 2   | 9.5396e-08      | 3.4269e-06      | 2.8820e-06      |
|          | 4   | 2.4689e-04      | 3.0980e-03      | 8.7881e-04      |
|          | 6   | 9.7546e-04      | 5.5240e-02      | 3.0854e-02      |
|          | 8   | 3.5031e-01      | 4.9295e+00      | 3.5165e+00      |
|          | 10  | 1.0388e+00      | 6.3475e+02      | 4.6246e+02      |
| $10^{-16}$ | 2   | 1.8310e-17      | 1.1567e-14      | 5.7126e-15      |
|          | 4   | 4.8182e-14      | 3.4386e-12      | 1.1154e-12      |
|          | 6   | 1.8526e-12      | 4.5175e-10      | 2.8984e-10      |
|          | 8   | 6.2313e-09      | 1.5478e-07      | 1.1568e-07      |
|          | 10  | 3.7921e-07      | 4.1497e-05      | 1.6493e-05      |

where

\[
X = Q^\top \begin{bmatrix} 10^{-m} & 0 \\ 0 & 10^m \end{bmatrix} Q, \quad A = \begin{bmatrix} \text{randn}(1) & 0 \\ \text{randn}(1) & 10^{-m} \end{bmatrix} Q, \\
B = \begin{bmatrix} \text{randn}(1) & 0 \\ \text{randn}(1) & 2 \cdot 10^{-m} \end{bmatrix} Q,
\]

and $C = AX + X^\top B^\top$. For different $m$ and $\epsilon$, we compare the true relative errors with the true and estimated first order perturbation bounds in Table 2 and 3. For Algorithms 3.2 and 3.3, we choose $k = 3$. Typically the condition estimates fall reliably within the factors between a tenth and ten folds of the true condition numbers [6, Chap. 15]. From Tables 2 and 3, it is easy to see that the condition of (2) worsens as $m$ increases. The first order perturbation bounds approximate the true relative error well. On the other hands, the SCE-base condition estimates underestimate the true relative error within the factor 1/10, which is consistent with the theory of SCE.

Algorithms 3.2 and 3.3 output the condition matrix which bounds componentwise the true relative error of each entry of $X$. Let us denote the overestimation matrices

\[
O_N = \frac{R_{\text{rel}}^{-\text{SYL},(k)} \cdot \epsilon}{\Delta X/X}, \quad O_C = \frac{C_{\text{rel}}^{-\text{SYL},(k)} \cdot \epsilon}{\Delta X/X},
\]
Table 3: Comparing the true mixed, componentwise perturbation bounds with the first order mixed, componentwise bounds

| $\epsilon$ | $m$ | $\gamma_m$ | $m^{\text{T-SYL},\epsilon}$ | $m^{\text{SCF},\epsilon}$ | $\gamma_c$ | $c^{\text{T-SYL},\epsilon}$ | $c^{\text{SCF},\epsilon}$ |
|---|---|---|---|---|---|---|---|
| $10^{-8}$ | 2 | 1.0653e-07 | 6.7982e-07 | 7.0811e-08 | 1.0653e-07 | 6.9993e-07 | 7.3014e-08 |
| 4 | 2.8197e-04 | 6.1050e-04 | 3.1166e-05 | 3.5893e-04 | 7.7713e-04 | 3.9672e-05 |
| 6 | 1.1605e-03 | 3.7938e-02 | 1.7024e-03 | 1.1605e-03 | 3.7938e-02 | 1.7024e-03 |
| 8 | 3.4915e-01 | 1.3625e+00 | 9.0707e-02 | 3.5217e-01 | 1.3625e+00 | 9.0707e-02 |
| 10 | 1.0052e+00 | 1.7985e+02 | 2.0008e+01 | 1.0912e+00 | 1.7985e+02 | 2.0008e+01 |
| $10^{-16}$ | 2 | 1.7901e-17 | 2.4946e-16 | 1.9630e-17 | 5.7210e-16 | 7.3778e-16 | 8.5091e-17 |
| 4 | 4.7751e-14 | 3.0209e-13 | 2.7450e-14 | 5.4371e-13 | 3.4415e-12 | 3.1275e-13 |
| 6 | 1.5149e-12 | 2.4979e-11 | 2.3467e-12 | 1.7239e-11 | 2.8425e-10 | 2.6705e-11 |
| 8 | 5.9587e-09 | 1.0139e-08 | 1.1128e-09 | 6.7807e-09 | 1.1537e-07 | 1.2663e-08 |
| 10 | 3.3359e-07 | 2.7142e-06 | 1.3414e-07 | 2.0868e-06 | 1.5021e-05 | 7.6833e-07 |

where $\epsilon$ denotes the perturbation magnitude in (17), and $R^{\text{T-SYL},\epsilon}_{\text{rel}}$ and $C^{\text{T-SYL},\epsilon}_{\text{rel}}$ are outputs of Algorithms 3.2 and 3.3, respectively. We test 1000 samples of $(A, B, C)$ and plot the mean matrices $O^N$ and $O^C$ in Figure 1 for $k = 3$ and $\epsilon = 10^{-8}$. The X-axis of Figure 1 denotes the index of vec$(X)$. The graphs on the left and right of Figure 1 display respectively the mean values of the overestimations given by Algorithms 3.2 and 3.3. Clearly, Algorithm 3.3 gives better estimates.

**Example 3** This example came from [1]. Let $\hat{A}, \hat{B} \in \mathbb{R}^{n \times n}$ be real lower-triangular matrices with given diagonal elements (denoted by $a, b \in \mathbb{R}^n$) and random strictly lower-triangular elements. They are the reshuffled by the orthogonal matrices $Q, Z \in \mathbb{R}^{n \times n}$ to form $(\hat{A}, \hat{B}) = (Q\hat{A}Z, Q\hat{B}Z)$. In MATLAB commands, we have

$$
\hat{A} = \text{tril}(\text{randn}(n), -1) + \text{diag}(a), \quad \hat{B} = \text{tril}(\text{randn}(n), -1) + \text{diag}(b),
$$

$$
X = \text{randn}(n, n),
$$

and the right hands $C = AX + X^T B^T$. We generate 1000 samples of $A, B$ and $X$ with $n = 40$, and for each sample, the perturbations on $A, B$ and $C$ are generated as in the previous examples. We display the mean values of $O^N$ and $O^C$ over 1000 samples in Figure 2 for $k = 3$ and $\epsilon = 10^{-16}$ in (17). The X-axis of Figure 2 denotes the index of vec$(X)$. From Figure 2 the componentwise condition estimation matrix $C^{\text{T-SYL},\epsilon}_{\text{rel}}$ gives reliable perturbation bounds.
Figure 1: Example 2. Overestimation of condition over 1000 samples
The mean value of entries of $O^C$ is 0.1991 and the variance is 1.9140. On the other hand, the mean value of entries of $O^N$ is 72.2192 and the variance is $2.3450 \cdot 10^5$. So Algorithm 3.3 gives superior condition estimates.

**Example 4** In this example, we test the effectiveness of the proposed componentwise backward errors. The triples $A$, $B$ and $C$ are as in Example 2. The perturbations $\Delta A$, $\Delta B$ and $\Delta C$ are generated as in (17). Let $Y$ satisfies the perturbed $\top$-Sylvester equation

$$(A + \Delta A)Y + Y^\top (B + \Delta B)^\top = C + \Delta C,$$

which is solved in Kronecker production form by Gaussian elimination with partial pivoting. Denote

$$\epsilon^* = \min \{ \epsilon : |\Delta A| \leq \epsilon |A|, |\Delta B| \leq \epsilon |B|, |\Delta C| \leq \epsilon |C| \}.$$

We vary the perturbation magnitudes $\epsilon$ in (17) from $10^{-3}$ to $10^{-9}$ and compute $\overline{\eta}(Y)$ in Theorem 1 and $\eta(Y)$ in (6) for different values of $m$. The results are displayed in Table 4. When $m$ increases, the condition of the $\top$-Sylvester equation worsens, as indicated in Example 2. For most of cases, $\overline{\mu}(Y)$ has
the same order as or one order higher than $\epsilon^*$. For $m = 10$, when the perturbations $\epsilon$ are small, $\overline{\pi}(Y)$ seriously overestimates the true componentwise backward error. On the other hand, the normwise backward error $\eta(Y)$ does not estimate $\epsilon^*$ accurately even for well conditioned problem under small perturbations, as for $m = 6$ and $\epsilon = 10^{-6}$.

5. Concluding Remarks

We have considered the condition and errors of $\star$-Sylvester equations under componentwise perturbations. Backward errors have been defined and the small-sample condition estimation technique has been applied to estimate the condition of $\star$-Sylvester equations. Numerical experiments show our algorithm under componentwise perturbations produces accurate condition and error estimates which reflect true condition and errors accurately. Moreover, the new derived bound for the componentwise backward errors is sharp and reliable according to the numerical experiments for well-conditioned or moderate ill-conditioned problems under large or small perturbations. A possible future research topic is to apply the SCE to other type $\star$-Sylvester equation [1].

Acknowledgements

H. Diao is partially supported by the National Natural Science Foundation of China under grant 11001045.

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Table 4: Comparing $\epsilon^*$, componentwise and normwise backward errors

| $\epsilon$ | $m$    | $\epsilon^*$ | $\bar{\mu}(Y)$ | $\eta(Y)$ |
|------------|--------|---------------|-----------------|-----------|
| $10^{-4}$  |        |               |                 |           |
| 2          | 8.9398e-04 | 1.4532e-04 | 8.9601e-03      |           |
| 4          | 9.8239e-04 | 7.2812e-04 | 4.6979e-01      |           |
| 6          | 9.5328e-04 | 7.0167e-04 | 3.1161e+00      |           |
| 8          | 8.8183e-04 | 3.6163e-04 | 1.4204e+00      |           |
| 10         | 8.8183e-04 | 3.6163e-04 | 1.4204e+00      |           |
| $10^{-6}$  |        |               |                 |           |
| 2          | 9.2596e-07 | 3.2334e-07 | 4.6975e-06      |           |
| 4          | 8.6877e-07 | 5.1044e-07 | 4.2311e-04      |           |
| 6          | 9.8269e-07 | 8.6473e-07 | 3.5344e-01      |           |
| 8          | 9.9547e-07 | 7.2417e-07 | 8.0498e-01      |           |
| 10         | 9.5721e-07 | 2.6332e-07 | 4.1066e+00      |           |
| $10^{-8}$  |        |               |                 |           |
| 2          | 9.8999e-10 | 7.9092e-10 | 9.4315e-09      |           |
| 4          | 9.2491e-10 | 7.4925e-10 | 1.0222e-06      |           |
| 6          | 8.8399e-10 | 8.3860e-10 | 1.6531e-04      |           |
| 8          | 8.6229e-10 | 5.1742e-10 | 5.5369e-03      |           |
| 10         | 9.6373e-10 | 8.5358e-09 | 3.3068e-01      |           |
| $10^{-12}$ |        |               |                 |           |
| 2          | -9.8837e-13 | 8.5948e-13 | 2.1765e-11      |           |
| 4          | 8.6013e-13 | 4.6452e-13 | 3.2712e-09      |           |
| 6          | 8.1023e-13 | 4.6365e-12 | 1.8620e-07      |           |
| 8          | 7.7456e-13 | 1.6637e-09 | 3.6200e-06      |           |
| 10         | 9.2831e-13 | 2.3684e-07 | 1.7668e-05      |           |
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