Universal construction of $\mathcal{W}_{q,p}$ algebras

J. Avan

LPTHE, CNRS-URA 280, Universités Paris VI/VII, France

L. Frappat, M. Rossi, P. Sorba

Laboratoire d’Annecy-le-Vieux de Physique Théorique LAPTH, CNRS-URA 1436
LAPP, BP 110, F-74941 Annecy-le-Vieux Cedex, France

Abstract

We present a direct construction of abstract generators for $q$-deformed $\mathcal{W}_N$ algebras. This procedure hinges upon a twisted trace formula for the elliptic algebra $A_{q,p}(\hat{sl}(N)_c)$ generalizing the previously known formulae for quantum groups.

MSC number: 81R50, 17B37
1 Introduction

The connection between $q$-deformed Virasoro, and more generally $\mathcal{W}$ algebras, and elliptic quantum $A_{q,p}(\hat{sl}(N)_c)$ algebras, was investigated in our recent papers [1–4]. It was shown that $q$-deformed Virasoro and $\mathcal{W}$ structures [5–7] were present inside the $A_{q,p}(\hat{sl}(N)_c)$ elliptic algebra [8,9]; at the quantum level, when a particular relation existed between the central charge $c$, the elliptic nome $p$ and the deformation parameter $q$: $(-p^{\frac{1}{N}})^M = q^{-c-N}$ for some integer $M$, and at the classical limit, obtained when setting an additional relation $p = q^{Nh}$ for some integer $h$. In this way, one obtained directly a set of quantizations of the classical $q$-deformed Poisson algebras, interestingly distinct from the original quantization [7] obtained from explicit derivation.

The construction was achieved at the abstract level in that only the abstract algebraic relations for $A_{q,p}(\hat{sl}(N)_c)$, defined by the eight vertex model $R$-matrix [10], were used to derive the $q$-deformed structures. It was assumed throughout the derivations that the initial formal series relations [8] were in fact extended to the level of analytic relations, thereby leading from one single exchange relation for this generating operator functional of the algebras to an infinite $\mathbb{Z}$-labeled set of exchange relations for the modes, depending upon the choice of a relevant series expansion in a crown-shaped sector for the ratio of spectral parameters in the elliptic structure function.

In our original approach [3], the extension of the construction to $sl(N)$ was achieved by defining the abstract generators of higher spin simply as shifted ordered products of the spin one generators $t(z) = \text{Tr} \left[ L^+(zq^{c/2})(L^-(z))^{-1} \right]$. In this respect, the first derivation cannot be considered as the $q$-deformed version of the $\mathcal{W}_N$ algebra construction [11] which takes as generators combinations of the current algebra generators from which one then extracts $sl(N)$ scalar objects; the detailed study developed in [11] allows to appreciate the successes and the difficulties of this approach. The question of an universal construction of $\mathcal{W}_{q,p}$ algebras from elliptic algebras thus remained open, although the construction [3] gave rise to perfectly consistent non-trivial algebraic structures, due to the shift in the spectral parameters which did no allow to interprete such $\mathcal{W}_{q,p}$ algebras as simply enveloping algebras of $Vir_q(sl(2))$. In particular, the classical limit of our algebra $\mathcal{W}_{q,p}$ did lead to the original classical $q$-$\mathcal{W}$ Poisson algebra, characterizing the quantum structure as a genuine $q$-deformed $\mathcal{W}_N$ algebra.

We address here this question. We shall rely on the algebraic structures derived from the properties of the operator $\mathfrak{T}(z) \equiv L^+(q^{\frac{c}{2}}z)(L^-(z))^{-1}$ which was the fundamental object in our previous derivation.

In a first part, we prove that $\mathfrak{T}(z)$ obeys an exchange relation of the type $R' \mathfrak{T} R \mathfrak{T} = R \mathfrak{T} R' \mathfrak{T}$. Originally derived and discussed in [13,14], these exchange relations then lead us to define new surfaces in the $(p,q,c)$ space on which quantum, then classical, $q$-Virasoro algebras of the same type as in [11] arise.

The classical structures are the same as in [1]. The quantum structures by contrast are more general, for $N \geq 3$, than the original algebras derived in [3], which one recovers as particular cases. One cannot however directly derive higher order generators from such an exchange algebra, contrary to the simpler case $RLL = LLR$ where a famous twisted trace formula exists [13,14] to generate quantum commuting Hamiltonians. But since the definition of the basic elliptic algebra involves two distinct $R$-matrices as $RLL = LLR^*$, one cannot apply [13,16] to it either.

In a second part, we show how to define a suitable twisted trace formula, involving $\mathcal{L}(z) \equiv L^+(q^{\frac{c}{2}}z)^t(L^-(z)^{-1})^t$ and the $R$-matrix of the algebra $A_{q,p}(\hat{sl}(N)_c)$, and leading to closed exchange algebras of the quantum $\mathcal{W}_{q,p}$ type, for the generalized relation $(-p^{\frac{1}{N}})^n = q^{-c-N}$ where $n$ is any integer, not necessarily multiple of $N$; then to classical $q$-$\mathcal{W}$ Poisson algebras when $p = q^{Nh}$.

The quantum and classical algebras $\mathcal{W}_{q,p}[sl(N)]$ thus constructed contain in particular the struc-
The elliptic algebra $A_{q,p}(\widehat{sl}(N)_c)$ and the algebras $Vir_{q,p}(sl(N))$

We start by defining the $R$-matrix of the $\mathbb{Z}_N$-vertex model ($\mathbb{Z}_N$ is the congruence ring modulo $N$):\[ \tilde{R}(z,q,p) = z^{2/N-2} \frac{1}{\kappa(z^2)} \theta \left[ \frac{1}{2} \right] (\zeta, \tau) \sum_{(\alpha_1, \alpha_2) \in \mathbb{Z}_N \times \mathbb{Z}_N} W_{(\alpha_1, \alpha_2)}(\xi, \zeta, \tau) \cdot I_{(\alpha_1, \alpha_2)} \otimes I_{(\alpha_1, \alpha_2)}^{-1}, \] (2.1)

where the variables $z, q, p$ are related to the $\xi, \zeta, \tau$ variables by\[ z = e^{i\pi \xi}, \quad q = e^{i\pi \zeta}, \quad p = e^{2i\pi \tau}. \] (2.2)

\( \theta \) are the Jacobi theta functions with rational characteristics $(\gamma_1, \gamma_2) \in \mathbb{Z}/N \times \mathbb{Z}/N$:
\[ \theta \left[ \begin{array}{c} \gamma_1 \\ \gamma_2 \end{array} \right] (\xi, \tau) = \sum_{m \in \mathbb{Z}} \exp \left( i\pi (m + \gamma_1)^2 \tau + 2i\pi (m + \gamma_1)(\xi + \gamma_2) \right). \] (2.3)

The normalization factor is given by:
\[ \frac{1}{\kappa(z^2)} = \frac{(q^{2N}z^{-2}; p, q^{2N})_\infty (p^{2N}z^{-2}; p, q^{2N})_\infty (pq^{2N-2}z^{-2}; p, q^{2N})_\infty}{(q^{2N}z^{-2}; p, q^{2N})_\infty (q^{2N}; p, q^{2N})_\infty (pq^{2N-2}z^{-2}; p, q^{2N})_\infty}. \] (2.4)

The functions $W_{(\alpha_1, \alpha_2)}$ are defined as follows:
\[ W_{(\alpha_1, \alpha_2)}(\xi, \zeta, \tau) = \frac{1}{N} \theta \left[ \begin{array}{c} 1 + \alpha_1/N \\ 1 + \alpha_2/N \end{array} \right] (\xi + \zeta/N, \tau) \theta \left[ \begin{array}{c} 1 + \alpha_1/N \\ 1 + \alpha_2/N \end{array} \right] (\zeta/N, \tau), \] (2.5)

and the matrices $I_{(\alpha_1, \alpha_2)}$ by:
\[ I_{(\alpha_1, \alpha_2)} = g^{\alpha_2 h_{\alpha_1}}, \] (2.6)

where $g_{ij} = \omega^i \delta_{ij}, \ h_{ij} = \delta_{i+1,j}$ are $N \times N$ matrices (the addition of indices being understood modulo $N$) and $\omega = e^{2i\pi/N}$.

The $R$-matrix $\tilde{R}$ is $\mathbb{Z}_N$-symmetric:
\[ \tilde{R}_{a+s, b+s}^{c+s, d+s} = \tilde{R}_{c, d}^{a, b} \quad a, b, c, d, s \in \mathbb{Z}_N. \] (2.7)
We introduce the “gauge-transformed” matrix:

\[ R(z, q, p) = (g^\frac{z}{2} \otimes g^\frac{z}{2}) \tilde{R}(z, q, p)(g^{-\frac{z}{2}} \otimes g^{-\frac{z}{2}}) \]  

(2.8)

It satisfies the following properties:
- Yang–Baxter equation:

\[ R_{12}(z) R_{13}(w) R_{23}(w/z) = R_{23}(w/z) R_{13}(w) R_{12}(z), \]  

(2.9)

- unitarity:

\[ R_{12}(z) R_{21}(z^{-1}) = 1, \]  

(2.10)

- crossing symmetry:

\[ R_{12}(z) t^z R_{21}(q^{-N} z^{-1}) t^z = 1, \]  

(2.11)

- antisymmetry:

\[ R_{12}(-z) = \omega (g^{-1} \otimes I) R_{12}(z) (g \otimes I), \]  

(2.12)

- quasi-periodicity:

\[ \tilde{R}_{12}(-p^z z) = (g^\frac{z}{2} h g^\frac{z}{2} \otimes I)^{-1} \left( \tilde{R}_{21}(z^{-1}) \right)^{-1} (g^\frac{z}{2} h g^\frac{z}{2} \otimes I), \]  

(2.13)

where

\[ \tilde{R}_{12}(x) = \tau_N(q^{1/2} x^{-1}) R_{12}(x), \]  

(2.14)

and

\[ \tau_N(z) = z^{\frac{N}{2}-2} \frac{\Theta_{q^N}(qz^z)}{\Theta_{q^N}(qz^{-2})}. \]  

(2.15)

The function \( \tau_N(z) \) is periodic with period \( q^N \): \( \tau_N(q^N z) = \tau_N(z) \) and satisfies \( \tau_N(z^{-1}) = \tau_N(z)^{-1} \).

### 2.2 Definition of \( A_{q,p}(\widehat{sl}(N)c) \)

We now recall the definition of the elliptic quantum algebra \( A_{q,p}(\widehat{sl}(N)c) \) \[^{8,9}\]. It is an algebra of operators \( L_{ij}(z) \equiv \sum_{n \in \mathbb{Z}} L_{ij}(n) z^n \) where \( i, j \in \mathbb{Z}_N \):

\[ L(z) = \begin{pmatrix} L_{11}(z) & \cdots & L_{1N}(z) \\
\vdots & \ddots & \vdots \\
L_{N1}(z) & \cdots & L_{NN}(z) \end{pmatrix}. \]  

(2.16)

The \( q \)-determinant is given by \( (\varepsilon(\sigma) \) being the signature of the permutation \( \sigma) \):

\[ q \text{-det} L(z) \equiv \sum_{\sigma \in \mathbf{S}_N} \varepsilon(\sigma) \prod_{i=1}^{N} L_{i, \sigma(i)}(z q^{i-N-1}). \]  

(2.17)

\( A_{q,p}(\widehat{gl}(N)c) \) is defined by imposing the following constraints on the \( L(z) \) generators:

\[ \tilde{R}_{12}(z/w) L_{1}(z) L_{2}(w) = L_{2}(w) L_{1}(z) \tilde{R}_{12}(z/w), \]  

(2.18)

where \( L_{1}(z) \equiv L(z) \otimes I, L_{2}(z) \equiv I \otimes L(z) \) and \( \tilde{R}_{12}^{*}(z, q, p) \equiv \tilde{R}_{12}(z, q, p^*) \equiv \tilde{R}_{12}(z, q, p^* = pq^{-2c}). \)

The \( q \)-determinant is in the center of \( A_{q,p}(\widehat{gl}(N)c) \) and one sets

\[ A_{q,p}(\widehat{sl}(N)c) = A_{q,p}(\widehat{gl}(N)c)/\langle q \text{-det} L - q^{\frac{N}{2}} \rangle. \]  

(2.19)
2.3 \textit{Vir}_{q,p}(sl(N)) algebras from elliptic algebras

It is here useful to introduce the following two matrices:

\begin{align}
L^+(z) & \equiv L(q^z), & L^-(z) & \equiv (g^zh^z) L(-p^z) (g^zh^z)^{-1},
\end{align}

which obey coupled exchange relations following from (2.18) and periodicity/unitarity properties of the matrices $\hat{R}_{12}$ and $\hat{R}^*_{12}$:

\begin{align}
\hat{R}_{12}(z/w) L^+_1(z) L^+_2(w) = L^+_2(w) L^+_1(z) \hat{R}_{12}(z/w), \\
\hat{R}_{12}(q^z/w) L^-_1(z) L^-_2(w) = L^-_2(w) L^-_1(z) \hat{R}_{12}(q^{-z}/w). 
\end{align}

We now recall some of the main results of refs. [1–3]:

\textbf{Theorem 1} In the three-dimensional parameter space generated by $p, q, c$, one defines a two-dimensional surface $\Sigma_{N, NM}$ for any integer $M \in \mathbb{Z}$ by the set of triplets $(p, q, c)$ connected by the relation $(-p^z)^{NM} = q^{-c-N}$. One defines the following operators in $A_{q,p}(\widehat{sl}(N)_c)$:

\begin{align}
t(z) & \equiv \text{Tr}(L^+(q^z) L^-(z)^{-1}) = \text{Tr}(L^+(q^z)^t \tilde{L}^-(z)) \equiv \text{Tr}(L(z)),
\end{align}

where $\tilde{L}^-(z) \equiv (L^-(z)^{-1})^t$.

1) On the surface $\Sigma_{N, NM}$, the operators $t(z)$ realize an exchange algebra with the generators $L(w)$ of $A_{q,p}(\widehat{sl}(N)_c)$:

\begin{align}
t(z) L(w) = F_N(NM, w) L(w) t(z)
\end{align}

where

\begin{align}
F_N(r, x) = \begin{cases}
q^{2r(1-\frac{1}{N})} \prod_{k=0}^{r-1} \frac{\Theta_{q^2N}(x^{-2}p^{-k}) \Theta_{q^2N}(x^2p^k)}{\Theta_{q^2N}(x^{-2}q^2p^{-k}) \Theta_{q^2N}(x^2q^2p^k)} & \text{for } r > 0, \\
q^{-2|r(1-\frac{1}{N})|} \prod_{k=1}^{|r|} \frac{\Theta_{q^2N}(x^{-2}q^2p^{-k}) \Theta_{q^2N}(x^2q^2p^k)}{\Theta_{q^2N}(x^{-2}p^{-k}) \Theta_{q^2N}(x^2p^k)} & \text{for } r < 0.
\end{cases}
\end{align}

2) On the surface $\Sigma_{N, NM}$, $t(z)$ closes a quadratic subalgebra:

\begin{align}
t(z)t(w) = \mathcal{Y}_N(NM, w) t(w)t(z)
\end{align}

where

\begin{align}
\mathcal{Y}_N(r, x) = \begin{cases}
\prod_{k=1}^{r} \frac{\Theta^2_{q^2N}(x^2p^{-k}) \Theta_{q^2N}(x^2q^2p^k) \Theta_{q^2N}(x^2q^{-2}p^k)}{\Theta^2_{q^2N}(x^2p^k) \Theta_{q^2N}(x^2q^2p^{-k}) \Theta_{q^2N}(x^2q^{-2}p^{-k})} & \text{for } r > 0, \\
\prod_{k=0}^{|r|-1} \frac{\Theta^2_{q^2N}(x^2p^{-k}) \Theta_{q^2N}(x^2q^2p^k) \Theta_{q^2N}(x^2q^{-2}p^k)}{\Theta^2_{q^2N}(x^2p^k) \Theta_{q^2N}(x^2q^2p^{-k}) \Theta_{q^2N}(x^2q^{-2}p^{-k})} & \text{for } r < 0.
\end{cases}
\end{align}

3) In particular, at the critical level $c = -N$, the operators $t(z)$ lie in the center of $A_{q,p}(\widehat{sl}(N)_c)$ and commute with each other.
3 The new \( \text{Vir}_{q,p}(sl(N)) \) algebras

We first prove the main basic result of this section.

3.1 A generalized quadratic exchange algebra

**Theorem 2** The operators \( \Xi(z) \) defined by

\[
\Xi(z) \equiv L^+(q^z z) L^-(z)^{-1} \tag{3.1}
\]

satisfy the following exchange relation:

\[
\hat{R}_{12}(z/w) \Xi_1(z) \hat{R}_{21}(q^c w/z) \Xi_2(w) = \Xi_2(w) \hat{R}_{12}(q^c z/w) \Xi_1(z) \hat{R}_{21}(w/z) \tag{3.2}
\]

**Proof:** One can derive from eqs. (2.21) further exchange relations between the operators \( L^+ \) and \( L^- \). One has therefore:

\[
\Xi_1(z) \hat{R}_{21}(q^c w/z) \Xi_2(w) = L_1^+(q^z z) \tilde{L}_1^-(z)^{t_1} \hat{R}_{21}(q^c w/z) L_2^+(q^z w) \tilde{L}_2^-(w)^{t_2}
\]

\[
= L_1^+(q^z z) L_2^+(q^z w) \hat{R}_{21}(w/z) \tilde{L}_1^-(z)^{t_1} \tilde{L}_2^-(w)^{t_2}
\]

\[
= \hat{R}_{12}^1(z/w) L_2^+(q^z w) L_1^+(q^z z) \hat{R}_{12}^1(z/w) \tilde{L}_1^-(z)^{t_1} \tilde{L}_2^-(w)^{t_2}
\]

\[
= T \hat{R}_{12}^1(z/w) L_2^+(q^z w) L_1^+(q^z z) \hat{R}_{12}^1(z/w) \tilde{L}_1^-(z)^{t_1} \tilde{L}_2^-(w)^{t_2}
\]

\[
\Xi_1(z) \hat{R}_{21}(q^c z/w) \Xi_2(w)
\]

where \( T \) stands for \( \tau_N(q^z z) \tau_N(q^z w) \) and we used the relations \( \hat{R}_{12}^1(z/w) \hat{R}_{21}^1(w/z) = T \) and \( T \hat{R}_{12}^1(z/w) = \hat{R}_{21}(w/z) \).

Multiplying the last equality of (3.3) by \( \hat{R}_{12}(z/w) \) on the left, one obtains the desired equation (3.2). \( \blacksquare \)

3.2 An alternative construction: new surfaces in \( \mathcal{A}_{q,p}(\hat{sl}(N)) \)

Mixed exchange relations of the type described in Theorem 2 were considered in [13, 14]. It was then shown in [14] that, provided that a \( c \)-number matrix exist such that \( \hat{R}_{12}(z/w) \gamma_1 \hat{R}_{21}(q^c w/z) \gamma_2 = \gamma_2 \hat{R}_{12}(q^c z/w) \gamma_1 \hat{R}_{21}(w/z) \), one may construct commuting generators defined as \( Q \equiv \text{tr}(\hat{\gamma}^t \Xi) \) where \( \hat{\gamma} \) is matrix dual to \( \gamma \). This, together with the properties of quasi-periodicity and unitarity of the \( R \)-matrix, leads us to consider the following operator:

\[
t(z) \equiv \text{Tr}

\left[

a^{-n} \Xi(z)
\right] = \text{Tr}

\left[

a^{-n} L^+(q^{c/2} z) \tilde{L}^-(z)^t
\right], \tag{3.4}
\]

where \( n \in \mathbb{Z} \) and the matrix \( a \) is given by:

\[
a = g^{1/2} h g^{1/2}. \tag{3.5}
\]

By analogy with the construction in Theorem 1, we expect that the mechanism of construction of directly commuting Hamiltonians in [14] will here turn into a two-step procedure, with a first
constraint on \(p, q, c\) leading to a closed exchange algebra and a second constraint leading to commuting operators and a subsequent Poisson structure.

Indeed we first establish the exchange properties of (3.4) on the surfaces \(\Sigma_{N,n}\) of the three-dimensional space of parameters \(p, q, c\), given by the equation \((n \in \mathbb{Z}, n \neq 0)\):

\[
(-p^{1/2})^n = q^{-c-N}.
\]  

(3.6)

It is relevant for later purposes to rewrite the operator \(t(z)\) as \(t(z) = \text{Tr} \left[ \mathcal{L}(n)(z) \right] \) where \(\mathcal{L}(n)(z)\) is defined by

\[
\mathcal{L}(n)(z) = (a^{-n}L^+(q^{c/2}z))^t \tilde{L}^{-}(z).
\]  

(3.7)

We prove the following lemma.

**Lemma 1** On the surfaces \(\Sigma_{N,n}\), the operators \(\mathcal{L}(n)(z)\) defined by (3.7) have the following exchange properties with the generators \(L(w)\) of \(A_{q,p}(\hat{sl}(N))\):

\[
\mathcal{L}^-(1)(z) L_2(w) = F_N \left(n, \frac{w}{z}\right) L_2(w) \left(\hat{R}^*_{21}(q^{-c}w/z)^{-1}\right)^{t_1} \mathcal{L}^-(1)(z) \hat{R}^*_{21}(q^{-c}w/z)^{t_1}. \]  

(3.8)

**Proof:** It is easier to formulate the proof in terms of \(L^+(w)\). One has:

\[
\mathcal{L}^+(1)(z) L^+_2(w) = L^+_1(zq^{c/2})^{t_1} (a_1^{-n})^{t_1} \tilde{L}^{-}(z) L^+_2(w). \]  

(3.9)

To exchange \(t(z)\) with \(L^+(w)\), we need the following exchange relations, coming directly from (2.21):

\[
\left(\hat{R}^*_{21}(q^{c}w/z)^{t_1}\right)^{-1} L^+_2(w) \tilde{L}^{-}(z) = \tilde{L}^{-}(z) L^+_2(w) \left(\hat{R}^*_{21}(q^{-c}w/z)^{t_1}\right)^{-1}, \]  

\[
L^+_1(z)^{t_1} \hat{R}^*_{12}(z/w)^{t_1} L^+_2(w) = L^+_2(w) \hat{R}^*_{12}(z/w)^{t_1} L^+_1(z)^{t_1}. \]  

(3.10)

(3.11)

Using (3.10) we have:

\[
\mathcal{L}^+(1)(z) L^+_2(w) = L^+_1(q^{c/2})^{t_1} (a_1^{-n})^{t_1} \left(\hat{R}^*_{21}(q^{c}w/z)^{t_1}\right)^{-1} L^+_2(w) \tilde{L}^{-}(z) \hat{R}^*_{21}(q^{-c}w/z)^{t_1}
\]

\[
= L^+_1(q^{c/2})^{t_1} (a_1^{-n})^{t_1} \left(\hat{R}^*_{21}(q^{c}w/z)^{t_1}\right)^{-1} (a_1^{n})^{t_1} L^+_2(w) (a_1^{-n})^{t_1} \tilde{L}^{-}(z) \hat{R}^*_{21}(q^{-c}w/z)^{t_1}. \]  

(3.12)

On the other hand, using the crossing-symmetry property we have:

\[
(a_1^{-n})^{t_1} \left(\hat{R}^*_{21}(q^{c}w/z)^{t_1}\right)^{-1} (a_1^{n})^{t_1} = (a_1^{-n})^{t_1} \left(\hat{R}^*_{21}(q^{c+N}w/z)^{-1}\right)^{t_1} (a_1^{n})^{t_1}
\]

\[
= (a_1^{n} \hat{R}^*_{21}(q^{c+N}w/z)^{-1} a_1^{-n})^{t_1}. \]  

(3.13)

We now apply \(n\) times the following relation coming from unitarity and quasi-periodicity:

\[
\hat{R}^*_{21} \left(z^{-1}(-p^{1/2})\right)^{-1} = \tau_N(q^{1/2}z)\tau_N(q^{1/2}z^{-1}) a_1 \hat{R}^*_{21} (z^{-1})^{-1} a_1^{-1}. \]  

(3.14)

We see here the role of the quasi-periodicity operator in implementing the general power of \(-p^{1/2}\) in the \(R\) matrix, leading to:

\[
\hat{R}^*_{21} \left(z^{-1}(-p^{1/2})^n\right)^{-1} = G_N(n, z) a_1^n \hat{R}^*_{21} (z^{-1})^{-1} a_1^{-n}, \]  

(3.15)
where:

\[ G_N(n, z) = \prod_{k=0}^{n-1} \tau_N \left[ zq^{\frac{1}{2}}(-p^\frac{1}{2})^{-k} \right] \tau_N \left[ z^{-1}q^{\frac{1}{2}}(-p^\frac{1}{2})^{k} \right] \quad \text{for } n > 0, \]

\[ G_N(n, z) = \prod_{k=1}^{\lfloor n/2 \rfloor} \tau_N^{-1} \left[ zq^{\frac{1}{2}}(-p^\frac{1}{2})^{k} \right] \tau_N^{-1} \left[ z^{-1}q^{\frac{1}{2}}(-p^\frac{1}{2})^{-k} \right] \quad \text{for } n < 0. \]

Applying (3.15) to (3.13) and using the equation (3.6), we have:

\[ (a_1^{-n})^{t_1} \left( \hat{R}_{21}(q^{\frac{z}{w}}w/z)^{t_1} \right)^{-1} (a_1^{n})^{t_1} = G_N^{-1} \left( n, q^{\frac{z}{w}}(-p^\frac{1}{2})^{n}z/w \right) \left( \hat{R}_{21}(q^{-\frac{z}{w}}w/z)^{-1} \right)^{t_1}. \]  

(3.17)

Remark that from the definition of \( \tau_N(x) \) (2.15) and the relation (3.3), it follows that:

\[ G_N^{-1} \left( n, q^{\frac{z}{w}}(-p^\frac{1}{2})^{n}x^{-1} \right) = F_N \left( n, q^{\frac{z}{w}}x \right), \]  

(3.18)

where \( F_N \) is given by (2.24).

Inserting (3.17), (3.18) into (3.12), we have:

\[ L_1^{(n)}(z) L_2^+(w) = F_N \left( n, q^{\frac{z}{w}} \frac{w}{z} \right) L_1^+(q^{c/2}z)^{t_1} \left( \hat{R}_{21}(q^{-\frac{z}{w}}w/z)^{-1} \right)^{t_1} L_2^+(w)(a_1^{-n})^{t_1} \tilde{L}_1^{-1}(z) \hat{R}_{21}(q^{-\frac{z}{w}}w/z)^{t_1}. \]  

(3.19)

Now we use equation (3.11) to obtain:

\[ L_1^{(n)}(z) L_2^+(w) = F_N \left( n, q^{\frac{z}{w}} \frac{w}{z} \right) L_2^+(w) \left( \hat{R}_{21}^{*}(q^{-\frac{z}{w}}w/z)^{-1} \right)^{t_1} L_1^+(q^{c/2}z)^{t_1}(a_1^{-n})^{t_1} \tilde{L}_1^{-1}(z) \hat{R}_{21}^{*}(q^{-\frac{z}{w}}w/z)^{t_1}. \]  

(3.20)

We are now able to state the following theorem:

**Theorem 3** On the surfaces \( \Sigma_{N,n} \), the operators \( t(z) \) defined by (3.4) satisfy the following exchange relations with the generators \( L(w) \) of \( \mathfrak{A}_{q,p}(\hat{sl}(N)_c) \):

\[ t(z) L(w) = F_N \left( n, \frac{w}{z} \right) L(w) t(z). \]  

(3.21)

**Proof:** We formulate the proof in terms of \( L^+(w) \). One has

\[ t(z) L_2^+(w) = \text{Tr}_1 \left[ L_1^+(zq^{c/2})^{t_1} (a_1^{-n})^{t_1} \tilde{L}_1^{-1}(z) L_2^+(w) \right]. \]  

(3.22)

From eq. (3.20), one obtains immediately:

\[ t(z) L_2^+(w) = F_N \left( n, q^{\frac{z}{w}} \frac{w}{z} \right) L_2^+(w) \text{Tr}_1 \left[ \left( \hat{R}_{21}^{*}(q^{-\frac{z}{w}}w/z)^{-1} \right)^{t_1} L_1^+(q^{c/2}z)^{t_1}(a_1^{-n})^{t_1} \tilde{L}_1^{-1}(z) \hat{R}_{21}^{*}(q^{-\frac{z}{w}}w/z)^{t_1} \right]. \]  

(3.23)

Using the very useful property:

\[ \text{Tr}_1 \left( R_{21}Q_1 R_{21}^{*} \right) = \text{Tr}_1 \left( Q_1 R_{21}^{t_2} R_{21}^{t_2} \right)^{t_1}, \]  

(3.24)
we get:
\[ t(z) \mathcal{L}^+(w) = F_N \left( n, q^{\frac{w}{z}} \right) L^+(w) \text{Tr} \left[ \mathcal{L}^+(q^{\frac{w}{z}} z) (a^{-n})^{\frac{w}{z}} \tilde{L}^{-}(z) \hat{R}_{21} (q^{-\frac{w}{z}} w / z)^t \left( \hat{R}_{21} (q^{-\frac{w}{z}} w / z)^{-1} \right)^t \right] \]
\[ (3.25) \]

Now the product of the two \( R \)-matrices in (3.25) vanishes altogether, leaving:
\[ t(z) \mathcal{L}^+(w) = F_N \left( n, q^{\frac{w}{z}} \right) L^+(w) t(z) . \]
\[ (3.26) \]

From definitions (2.20) Theorem 3 follows then immediately.

In particular since \( a^N \) is proportional to the identity one recovers the exchange algebras in (8) when \( n = NM, \ M \in \mathbb{Z} \).

A simple corollary of Theorem 3 is the exchange relation between \( t(z) \) and \( t(w) \). Indeed using the relations:
\[ \tilde{L}^{-}(z) = \left( \mathcal{L}^{-}(z)^t \right)^{-1} , \quad \mathcal{L}^{-}(w) = a \mathcal{L}^+(-p^{\frac{1}{2}} q^{-\frac{w}{z}} w) a^{-1} , \]
\[ (3.27) \]
we derive from (3.21) the following result:

**Corollary 1** On the surface \( \Sigma_{N,n} \) the operators \( t(z) \) defined by (3.4) satisfy the following algebra:
\[ t(z) t(w) = \mathcal{Y}_N(n, w/z) t(w) t(z) , \]
\[ (3.28) \]
where \( \mathcal{Y}_N \) is the function defined by (2.27).

**Remark:** For \( n = -1 \) equation (3.28) reads as:
\[ t(z) t(w) = t(w) t(z) , \]
\[ (3.29) \]
that is, on the surface \( q^{-c-N} = (-p^{\frac{1}{2}})^{-1} \), the operators \( t(z) \) commute. However they do not belong to the center of \( \mathcal{A}_{q,p}(\hat{sl}(N)_c) \), because the exchange factor of (3.21) is different from unity.

As when \( c = -N \) this is an occurrence of a one-step mechanism where one obtains directly a commuting algebra of operators with one single constraint on \( p, q, c \). However, contrary to that previous case, where the elliptic algebra and the quantum group shared this feature, this one is characteristic of the elliptic algebra structure, involving as it does the elliptic nome \( p \).

Exchange relations (3.28) are to be understood as realizations of \( \text{Vir}_{q,p}(sl(N)) \) algebras in the framework of \( \mathcal{A}_{q,p}(\hat{sl}(N)_c) \). This conclusion derives from the following results.

**Theorem 4** On the surface \( \Sigma_{N,n} \), when \( p = q^{Nh} \) with \( h \in \mathbb{Z} \setminus \{0\} \), the function \( \mathcal{Y}(n, x) \) is equal to 1. Hence \( t(z) \) realizes an Abelian subalgebra in \( \mathcal{A}_{q,p}(\hat{sl}(N)_c) \).

**Proof:** The theorem 4 is easily proved using the periodicity properties of the \( \Theta_{q^N} \) functions. \[ \blacksquare \]

**Theorem 5** Setting \( q^{Nh} = p^{1-\beta} \) for any integer \( h \neq 0 \), the \( h \)-labeled Poisson structure defined by:
\[ \{ t(z), t(w) \}^{(h)} = \lim_{\beta \to 0} \frac{1}{\beta} \left( t(z) t(w) - t(w) t(z) \right) \]
\[ (3.30) \]
has the following expression:

\[ \{t(z), t(w)\}^{(h)} = f_h(w/z) \cdot t(z) \cdot t(w) \]  

(3.31)

where

\[
f_h(x) = 2N h \ln q \left[ \sum_{\ell \geq 0} E\left(\frac{n}{2}\right) E\left(\frac{n}{2} + 1\right) \left( \frac{2x^2 q^{2N\ell}}{1 - x^2 q^{2N\ell}} - \frac{x^2 q^{2N\ell+2}}{1 - x^2 q^{2N\ell+2}} - \frac{x^2 q^{2N\ell-2}}{1 - x^2 q^{2N\ell-2}} \right) + E\left(\frac{n+1}{2}\right)^2 \left( \frac{2x^2 q^{2N\ell+N}}{1 - x^2 q^{2N\ell+N}} - \frac{x^2 q^{2N\ell+N+2}}{1 - x^2 q^{2N\ell+N+2}} - \frac{x^2 q^{2N\ell+N-2}}{1 - x^2 q^{2N\ell+N-2}} \right) \right] 
\]

for \( h \) odd,

\[
N/N (n + 1) \cdot \ln q \left[ \sum_{\ell \geq 0} \left( \frac{2x^2 q^{2N\ell}}{1 - x^2 q^{2N\ell}} - \frac{x^2 q^{2N\ell+2}}{1 - x^2 q^{2N\ell+2}} - \frac{x^2 q^{2N\ell-2}}{1 - x^2 q^{2N\ell-2}} \right) - \frac{1}{2} \left( \frac{2x^2}{1 - x^2} - \frac{x^2 q^2}{1 - x^2 q^2} - \frac{x^2 q^{-2}}{1 - x^2 q^{-2}} \right) \right] \quad \text{for } h \text{ even.}
\]

Here the notation \( E(m) \) means the integer part of the number \( m \).

**Proof:** By direct calculation.

Formula (3.32) is a trivial generalization of formulas (5.3) of [3] provided that the formal substitution \( NM \to n \) be done. Therefore the discussion between Theorem 7 and Proposition 5 of [3] is still valid and the Poisson bracket in the sector \( k = 0 \) corresponding to (3.32) is given, modulo the substitution indicated, by the formula in Proposition 5 of [3], which for even \( h \) yields the algebra \( Vir_q(sl(N)) \) [3]. Therefore we may conclude that (3.28) realizes the exchange relations of \( bona fide \) quantum \( Vir_{q,p}(sl(N)) \) algebras.

When \( N = 2 \) the construction does not lead to new \( q \)-deformed Virasoro algebras. However when \( N \geq 3 \), one obtain new exchange algebraic structures corresponding to the surfaces \( \Sigma_{N,n} \) when \( n \neq NM, \; M \in \mathbb{Z} \), and these structures will now be extended to complete \( q \)-deformed \( \mathcal{W}_N \) algebras.

### 4 Universal construction of \( \mathcal{W}_{q,p} \) algebras

Extending the construction to higher spin currents for \( N \geq 3 \) compels to use the direct exchange relation in Lemma 1 instead of the quadratic intertwined exchange relation in Theorem 2, for which no generalizations to higher powers of \( \mathbb{Z} \) exist.

#### 4.1 Quantum \( \mathcal{W}_{q,p} \) algebras

**Theorem 6** We define the operators \( w_s(z) \; (s = 1, \ldots, N - 1) \) by:

\[
w_s(z) \equiv Tr \left[ \left( \prod_{1 \leq i \leq s} \prod_{j > i} P_{ij} \right) \prod_{1 \leq i \leq s} \left( \mathcal{L}^{(n)}_i(z_i) \prod_{j > i} \hat{R}^*_i(j - N z_i \cdot z_j)^k t_i j \right) \right].
\]

(4.1)
where
\[
\mathcal{L}_i^{(n)}(z) \equiv (a^{-n} L_i^+(q^{i \frac{z}{2}} z))^{t_i} \tilde{L}_i(z) \equiv \bigotimes_{i=1}^{s-i} \otimes (a^{-n} L_i^+(q^{i \frac{z}{2}} z))^{t_i} \tilde{L}_i(z) \otimes \bigotimes_{i=1}^{s-i} \oplus (a^{-n} L_i^+(q^{i \frac{z}{2}} z))^{t_i} \tilde{L}_i(z) \otimes \bigotimes_{i=1}^{s-i} \tag{4.2}
\]
with \( n \in \mathbb{Z}, \ z_i = z q^{-\frac{s+i+1}{2}}, \) and \( P_{ij} \) is the permutation operator between the spaces \( i \) and \( j \) including the spectral parameters.

On the surface \( \Sigma_{N,n} \) defined by \((-p^{\frac{1}{2}})^n = q^{-c-N} \), the operators \( w_s(z) \) realize an exchange algebra with the generators \( L(w) \) of \( \mathcal{A}_{q,p}(\widehat{sl}(N)_c) \):
\[
w_s(z) L(w) = F^{(s)}_N \left( n, \frac{w}{z} \right) L(w) w_s(z), \tag{4.3}
\]
where
\[
F^{(s)}_N \left( n, \frac{w}{z} \right) = \prod_{i=1}^{s} F_N \left( n, \frac{w}{z_i} \right). \tag{4.4}
\]

**Proof:** For simplicity, we will only prove the theorem for \( w_2(z) \) and \( w_3(z) \) (the proof for \( w_1(z) \equiv t(z) \) has been done in [2, 3], see theorem 1 above).

The proof is based on the exchange relation (3.20) between \( \mathcal{L}^{(n)}(z) \) and \( L^+(w) \) on the surface \( \Sigma_{N,n} \) defined by \((-p^{\frac{1}{2}})^n = q^{-c-N} \):
\[
\mathcal{L}_i^{(n)}(z) L^+_\alpha(w) = F_N \left( n, q^{i \frac{z}{2}} \right) L^+_\alpha(w) \left( \tilde{R}_{\alpha}^s(q^{-\frac{z}{2}} w/z)^{-1} \right)^{t_i} \mathcal{L}_i^{(n)}(z) \tilde{R}_{\alpha}^s(q^{-\frac{z}{2}} w/z)^{t_i}. \tag{4.5}
\]

Consider the operator \( w_2(z) \). By definition (with \( z_1 = z q^{-\frac{1}{2}} \) and \( z_2 = z q^{\frac{1}{2}} \)):
\[
w_2(z) = \text{Tr} \left[ P_{12} \mathcal{L}_1^{(n)}(z_1) \tilde{R}_{12}^s(q^{-N} z_1/z_2)^{t_1 t_2} \mathcal{L}_2^{(n)}(z_2) \right]. \tag{4.6}
\]

From the exchange relation (4.5) between \( \mathcal{L}^{(n)}(z) \) and \( L^+(w) \), one gets immediately:
\[
w_2(z) L^+_\alpha(w) = \text{Tr}_{12} \left[ P_{12} \mathcal{L}_1^{(n)}(z_1) \tilde{R}_{12}^s(q^{-N} z_1/z_2)^{t_1 t_2} \mathcal{L}_2^{(n)}(z_2) \right] L^+_\alpha(w) = F^{(2)}_N \left( n, q^{\frac{z}{2}} \right) L^+_\alpha(w) \text{Tr}_{12} \left[ P_{12} \left( \tilde{R}_{\alpha}^s(q^{-\frac{z}{2}} w/z_1)^{-1} \right)^{t_1} \mathcal{L}_1^{(n)}(z_1) \tilde{R}_{\alpha}^s(q^{-\frac{z}{2}} w/z_1)^{t_i} \right] \tilde{R}_{12}^s(q^{-N} z_1/z_2)^{t_1 t_2} \left( \tilde{R}_{\alpha}^s(q^{-\frac{z}{2}} w/z_2)^{-1} \right)^{t_2} \mathcal{L}_2^{(n)}(z_2) \tilde{R}_{\alpha}^s(q^{-\frac{z}{2}} w/z_2)^{t_1 t_2}, \tag{4.7}
\]
where \( F^{(2)}_N \left( n, q^{\frac{z}{2}} \right) \) is given by:
\[
F^{(2)}_N \left( n, q^{\frac{z}{2}} \right) = F_N \left( n, q^{\frac{w}{z}} \right) F_N \left( n, q^{\frac{w}{z_1}} \right). \tag{4.8}
\]

In order to reorganize the \( R \) matrices in (4.7), one uses the Yang–Baxter equation for the matrix \( \tilde{R}^s \) (equation (4.3) is a consequence of (2.9)), the normalization factor entering in the definition (2.14) of the \( \tilde{R} \) matrices being the same in the l.h.s. and in the r.h.s. of (4.3):
\[
\tilde{R}_{\alpha_1}^s(x_1) \tilde{R}_{\alpha_2}^s(x_2) \tilde{R}_{12}^s(x_2/x_1) = \tilde{R}_{12}^s(x_2/x_1) \tilde{R}_{\alpha_2}^s(x_2) \tilde{R}_{\alpha_1}^s(x_1), \tag{4.9}
\]
from which it follows (with a shift \( x_2 \rightarrow q^{-N} x_2 \))
\[
\tilde{R}_{\alpha_1}^s(x_1)^{t_1} \tilde{R}_{12}^s(q^{-N} x_2/x_1)^{t_1 t_2} \left( \tilde{R}_{\alpha_2}^s(x_2)^{-1} \right)^{t_2} = \left( \tilde{R}_{\alpha_2}^s(x_2)^{-1} \right)^{t_2} \tilde{R}_{12}^s(q^{-N} x_2/x_1)^{t_1 t_2} \tilde{R}_{\alpha_1}^s(x_1)^{t_1}. \tag{4.10}
\]
Therefore, one obtains
\[
w_2(z) L_\alpha^+(w) = F_N^{(2)} \left( n, q^2 \frac{w}{z} \right) L_\alpha^+(w) \text{Tr}_{12} \left[ P_{12} \left( \hat{R}_{\alpha}^*(q^{-\frac{2}{z}} w/z_1)^{-1} \right) ^{t_1} L_1^{(n)}(z_1) \left( \hat{R}_{\alpha}^*(q^{-\frac{2}{z}} w/z_2)^{-1} \right) ^{t_2} \right. \\
\left. \hat{R}_{12}^*(q^{-N} z_1/z_2)^{t_1t_2} \hat{R}_{\alpha}^*(q^{-\frac{2}{z}} w/z_1)^{t_1} L_2^{(n)}(z_2) \hat{R}_{\alpha}^*(q^{-\frac{2}{z}} w/z_2)^{t_2} \right] \\
= F_N^{(2)} \left( n, q^2 \frac{w}{z} \right) L_\alpha^+(w) \text{Tr}_{12} \left[ P_{12} \left( \hat{R}_{\alpha}^*(q^{-\frac{2}{z}} w/z_1)^{-1} \right) ^{t_1} \left( \hat{R}_{\alpha}^*(q^{-\frac{2}{z}} w/z_2)^{-1} \right) ^{t_2} \right. \\
\left. \hat{R}_{12}^*(q^{-N} z_1/z_2)^{t_1t_2} \hat{R}_{\alpha}^*(q^{-\frac{2}{z}} w/z_1)^{t_1} \hat{R}_{\alpha}^*(q^{-\frac{2}{z}} w/z_2)^{t_2} \right] \\
= F_N^{(2)} \left( n, q^2 \frac{w}{z} \right) L_\alpha^+(w) \text{Tr}_{12} \left[ \left( \hat{R}_{\alpha}^*(q^{-\frac{2}{z}} w/z_1)^{-1} \right) ^{t_2} \left( \hat{R}_{\alpha}^*(q^{-\frac{2}{z}} w/z_1)^{-1} \right) ^{t_1} P_{12} \right. \\
\left. \hat{R}_{12}^*(q^{-N} z_1/z_2)^{t_1t_2} \hat{R}_{\alpha}^*(q^{-\frac{2}{z}} w/z_1)^{t_1} \hat{R}_{\alpha}^*(q^{-\frac{2}{z}} w/z_2)^{t_2} \right]. \tag{4.11}
\]
the last equality being obtained by action of the permutation operator \( P_{12} \).

One then uses the fact that under a trace over the space \( \beta \), for any \( c \)-number matrices \( A_{\alpha\beta} \) and \( B_{\alpha\beta} \) and operator matrix \( Q_\alpha \), one has
\[
\text{Tr}_\beta \left[ A_{\alpha\beta} Q_\beta B_{\alpha\beta} \right] = \text{Tr}_\beta \left[ Q_\beta \left( (B_{\alpha\beta})^{t_\alpha} (A_{\alpha\beta})^{t_\alpha} \right) \right], \tag{4.12}
\]
Applying (4.12) to \( \beta = 1 \otimes 2, A_{\alpha\beta} \equiv R_{\alpha 2} R_{\alpha 1} \) and \( B_{\alpha\beta} \equiv R_{\alpha 1}^r R_{\alpha 2}^r \), one gets
\[
w_2(z) L_\alpha^+(w) = F_N^{(2)} \left( n, q^2 \frac{w}{z} \right) L_\alpha^+(w) \text{Tr}_{12} \left[ P_{12} \left( L_1^{(n)}(z_1) \hat{R}_{12}^*(q^{-N} z_1/z_2)^{t_1t_2} L_2^{(n)}(z_2) \left( \hat{R}_{\alpha}^*(q^{-\frac{2}{z}} w/z_2)^{t_2} \right) \hat{R}_{\alpha}^*(q^{-\frac{2}{z}} w/z_1)^{t_1} \right. \right. \\
\left. \left. \hat{R}_{12}^*(q^{-N} z_1/z_2)^{t_1t_2} \hat{R}_{\alpha}^*(q^{-\frac{2}{z}} w/z_1)^{t_1} \hat{R}_{\alpha}^*(q^{-\frac{2}{z}} w/z_2)^{t_2} \right] \right]. \tag{4.13}
\]
It follows that
\[
w_2(z) L_\alpha^+(w) = F_N^{(2)} \left( n, q^2 \frac{w}{z} \right) L_\alpha^+(w) w_2(z). \tag{4.14}
\]
Recalling the fact that \( L_\alpha^+(w) = L_\alpha(q^2 w) \), one gets the desired result.

Consider now the case of \( w_3(z) \). By definition (with \( z_1 = zq^{-1}, z_2 = z \) and \( z_3 = zq \)):
\[
w_3(z) = \text{Tr} \left[ P_{12} P_{13} P_{23} L_1^{(n)}(z_1) \hat{R}_{12}^*(q^{-N} z_1/z_2)^{t_1t_2} \hat{R}_{13}^*(q^{-N} z_1/z_3)^{t_1t_3} L_2^{(n)}(z_2) \hat{R}_{23}^*(q^{-N} z_2/z_3)^{t_2t_3} L_3^{(n)}(z_3) \right]. \tag{4.15}
\]
From the exchange relation (4.5) between \( L^{(n)}(z) \) and \( L^+(w) \), one gets:
\[
w_3(z) L_\alpha^+(w) = \text{Tr}_{123} \left[ P_{12} P_{13} P_{23} L_1^{(n)}(z_1) \hat{R}_{12}^*(q^{-N} z_1/z_2)^{t_1t_2} \hat{R}_{13}^*(q^{-N} z_1/z_3)^{t_1t_3} \right. \\
\left. \hat{L}_2^{(n)}(z_2) \hat{R}_{23}^*(q^{-N} z_2/z_3)^{t_2t_3} L_3^{(n)}(z_3) \right] L_\alpha^+(w) \\
= F_N^{(3)} \left( n, q^2 \frac{w}{z} \right) L_\alpha^+(w) \text{Tr}_{123} \left[ P_{12} P_{13} P_{23} \left( \hat{R}_{\alpha}^*(q^{-\frac{2}{z}} w/z_1)^{-1} \right) ^{t_1} L_1^{(n)}(z_1) \hat{R}_{\alpha}^*(q^{-\frac{2}{z}} w/z_1)^{t_1} \right. \\
\left. \hat{R}_{12}^*(q^{-N} z_1/z_2)^{t_1t_2} \hat{R}_{13}^*(q^{-N} z_1/z_3)^{t_1t_3} \left( \hat{R}_{\alpha}^*(q^{-\frac{2}{z}} w/z_2)^{-1} \right) ^{t_2} L_2^{(n)}(z_2) \hat{R}_{\alpha}^*(q^{-\frac{2}{z}} w/z_2)^{t_2} \right. \\
\left. \hat{R}_{23}^*(q^{-N} z_2/z_3)^{t_2t_3} \left( \hat{R}_{\alpha}^*(q^{-\frac{2}{z}} w/z_3)^{-1} \right) ^{t_3} L_3^{(n)}(z_3) \hat{R}_{\alpha}^*(q^{-\frac{2}{z}} w/z_3)^{t_3} \right]. \tag{4.16}
\]
where \( F^{(3)}_{N}(n, q^{\frac{w}{z}}) \) is given by:

\[
F^{(3)}_{N}(n, q^{\frac{w}{z}}) = F_{N}(n, q^{\frac{w}{z_1}}) F_{N}(n, q^{\frac{w}{z_2}}) F_{N}(n, q^{\frac{w}{z_3}}).
\]  

(4.17)

Applying three times the Yang–Baxter equation \((4.10)\), one obtains:

\[
w_3(z) L^+_\alpha(w) = F^{(3)}_{N}(n, q^{\frac{w}{z}}) L^+_\alpha(w) \text{Tr}_{123} \left[ P_{12} P_{13} P_{23} \left( R_{12}^{*}(q^{-\frac{z}{w}}/z_1)^{-1} \right)^{t_1} \left( R_{12}^{*}(q^{-\frac{z}{w}}/z_2)^{-1} \right)^{t_2} \left( R_{12}^{*}(q^{-\frac{z}{w}}/z_3)^{-1} \right)^{t_3} \alpha_i \right] \nonumber
\]

\[
\left( R_{31}(q^{-\frac{w}{z}}/z_1)^{t_1} \right)^{n_i} \left( R_{31}(q^{-\frac{w}{z}}/z_2)^{t_1} \right)^{n_1} \left( R_{31}(q^{-\frac{w}{z}}/z_3)^{t_1} \right)^{n_2} L^{(n)}(z) L^{(n)}(z) L^{(n)}(z)
\]

\[
= R_{23}(q^{-\frac{w}{z}}/z_1)^{t_1} \left( R_{23}(q^{-\frac{w}{z}}/z_2)^{t_1} \right)^{n_1} \left( R_{23}(q^{-\frac{w}{z}}/z_3)^{t_1} \right)^{n_2} L^{(n)}(z) L^{(n)}(z) L^{(n)}(z).
\]  

(4.18)

Finally, after action of the permutation operators and using \((4.12)\) applied to \(\beta \equiv 1 \otimes 2 \otimes 3\), the \(R\) matrices \(R_{\alpha i}^{*}\) in \((4.18)\) simplify. It follows that:

\[
w_3(z) L^+_\alpha(w) = F^{(3)}_{N}(n, q^{\frac{w}{z}}) L^+_\alpha(w) w_3(z).
\]  

(4.19)

Again, one gets the desired result since \(L^+_\alpha(w) = L_\alpha(q^{\frac{w}{z}})\).

The proof for a generic operator \(w_s(z)\) is obtained by using the basic exchange relation \((4.5)\) between \(L^{(n)}(z)\) and \(L^+(w)\) and applying \(\frac{1}{2}s(s-1)\) times the Yang–Baxter equation in the form \((4.10)\). Successive uses of this relation yields an expression involving: the group of permutation operators; the product of all \(R\) matrices appearing at the left of the \(L^{(n)}\) operator in Lemma 1; the terms of the monomial \(w_s(z)\); finally the product of all \(R\) matrices appearing at the right of the \(L^{(n)}\) operator in Lemma 1. Commutation of the permutation operators with the “left” \(R\) matrices then brings this group of \(R\) matrices in position to use the transposition procedure \((4.12)\), and precisely rearranges the indices of these \(R\) matrices in the exact way required to cancel the two “left” and “right” groups of exchange-generated \(R\) matrices once the generalization of the procedure \((4.12)\) is used. Therefore the exchange relation becomes an exchange algebra with purely scalar functional structure coefficients.

An immediate consequence is therefore the exchange relation between the operators \(w_s(z)\) and the generators \(L^{(n)}(w)\):

\[
w_s(z) L^{(n)}(w) = \frac{F^{(s)}_{N}(n, q^{\frac{w}{z}})}{F^{(s)}_{N}(n, -p^{\frac{w}{z}})} L^{(n)}(w) w_s(z) = \prod_{i=1}^{s} \gamma_N \left( n, \frac{w}{z_i} \right) L^{(n)}(w) w_s(z).
\]  

(4.20)

where \(\gamma_N\) is the function defined by \((2.20)\) and \(z_i = z q^{i-\frac{i(i+1)}{2}}\).

It is now immediate to derive the exchange algebra among the operators \(w_s(z)\). One gets the following theorem:

**Theorem 7** On the surface \(\Sigma_{N,n}\), the operators \(w_s(z)\) realize an exchange algebra

\[
w_i(z) w_j(w) = \prod_{u=-\frac{N-1}{2}}^{\frac{N-1}{2}} \prod_{v=-\frac{N-1}{2}}^{\frac{N-1}{2}} \gamma_N \left( n, q^{u-v} \frac{w}{z} \right) w_j(w) w_i(z).
\]  

(4.21)
The proof is obvious and follows immediately from definition 4.1 and equation (4.20).

Remarks:
1) The critical level \( c = -N \) can be seen as a limiting case of the relation \((-p^\frac{1}{2})^n = q^{-c-N}\) by taking \( n = 0, \ p \) and \( q \) having arbitrary generic values (\( |q| < 1, |p| < 1 \)). In this limiting case, it is easy to note that the factor \( G_N(0, x) \) is equal to 1 (see e.g. (4.13)) and hence \( F(0, x) = 1 \). Therefore, at the critical level, the operators \( w_s(z) \) provide a set of commuting quantities belonging to the center of the elliptic quantum \( \mathcal{A}_{q,p}(\widehat{sl}(N)_c) \) algebra.

2) When one chooses \( n = NM \) for \( M \in \mathbb{Z} \), one recovers the structure functions of the \( q \)-\( \mathcal{W}_N \) algebras constructed in [3]. General values of \( n \) lead to original structure functions. This time however the construction does not go through the two steps of first constructing the trace \( s(z) \) and then shift-multiply the same derived abstract generator to get the full \( q \)-\( \mathcal{W}_N \) algebra: the trace procedure and shifted multiplication are applied in one single stroke to the original elliptic algebra generators, hence our denomination of this as a “universal” procedure.

4.2 Classical limit

**Theorem 8** On the surface \( \Sigma_{N,n} \), when an additional relation \( p = q^{Nh} \) with \( h \in \mathbb{Z} \setminus \{0\} \) is imposed, the function \( Y \) is equal to 1. Hence, the operators \( w_s(z) \) realize an Abelian subalgebra in \( \mathcal{A}_{q,p}(\widehat{sl}(N)_c) \).

**Proof:** The proof is straightforward by using the explicit expression of the function \( F_s(M, x) \) and the periodicity properties of the \( \Theta_{q^2N} \) functions.

**Remark:** Although the \( w_s(z) \) realize an Abelian algebra, they do not belong to the center of \( \mathcal{A}_{q,p}(\widehat{sl}(N)_c) \), in contrast to the case of the critical level.

The theorem 8 allows us to define Poisson structures on the corresponding Abelian subalgebras. As usual, they are obtained as limits of the exchange algebras when \( p = q^{Nh} \) with \( h \in \mathbb{Z} \setminus \{0\} \).

**Theorem 9** Setting \( q^{Nh} = p^{1-\beta} \) for any integer \( h \neq 0 \), the \( h \)-labeled Poisson structure defined by:

\[
\{w_i(z), w_j(w)\}_{(h)} = \lim_{\beta \to 0} \frac{1}{\beta} \left( w_i(z) w_j(w) - w_j(w) w_i(z) \right)
\]

has the following expression:

\[
\{w_i(z), w_j(w)\} = \sum_{u=-(i-1)/2}^{(i-1)/2} \sum_{v=-(j-1)/2}^{(j-1)/2} f_h\left(q^{v-u} \frac{w}{z} \right) w_i(z) w_j(w),
\]

where \( f_h(x) \) is given by (3.32).

One recovers here the same Poisson algebra as in [3], identifying therefore the exchange algebras in Theorem 7 as new \( \mathcal{W}_{p,q}[sl(N)] \) algebras. The main issue now is to obtain explicit realizations of these algebras. Curiously enough the only known (\( q \)-boson type) explicit realization achieved in [5] corresponds to a \( \mathcal{W}_N \) algebra which does not belong to the set constructed here. We hope to address this issue in the next future.
References

[1] J. Avan, L. Frappat, M. Rossi, P. Sorba, Poisson structures on the center of the elliptic algebra \( \mathcal{A}_{q,p}(\hat{sl}(2)_c) \), Phys. Lett. A 235 (1997) 323.

[2] J. Avan, L. Frappat, M. Rossi, P. Sorba, New \( W_{q,p}(sl(2)) \) algebras from the elliptic algebra \( \mathcal{A}_{q,p}(\hat{sl}(2)_c) \), Phys. Lett. A 239 (1998) 27.

[3] J. Avan, L. Frappat, M. Rossi, P. Sorba, Deformed \( W_N \) algebras from elliptic \( sl(N) \) algebras, to appear in Commun. Math. Phys., math.QA/9801105.

[4] J. Avan, L. Frappat, M. Rossi, P. Sorba, Central extensions of classical and quantum \( q \)-Virasoro algebras, submitted to Phys. Lett. A, math.QA/9806065.

[5] E. Frenkel, N. Reshetikhin, Quantum affine algebras and deformations of the Virasoro and \( W \)-algebras, Commun. Math. Phys. 178 (1996) 237.

[6] H. Awata, H. Kubo, S. Odake, J. Shiraishi, Quantum \( W_N \) algebras and Macdonald polynomials, Commun. Math. Phys. 179 (1996) 401.

[7] B. Feigin, E. Frenkel, Quantum \( W \) algebras and elliptic algebras, Commun. Math. Phys. 178 (1996) 653.

[8] O. Foda, K. Iohara, M. Jimbo, R. Kedem, T. Miwa, H. Yan, An elliptic quantum algebra for \( \hat{sl}_2 \), Lett. Math. Phys. 32 (1994) 259.

[9] M. Jimbo, H. Konno, S. Odake, J. Shiraishi, Quasi-Hopf twistors for elliptic quantum groups, q-alg/9712029.

[10] R.J. Baxter, Exactly solved models in statistical mechanics, Academic Press, London, 1982.

[11] F. Bais, P. Bouwknegt, K. Schoutens, M. Surridge, Extensions of the Virasoro algebra constructed from Kac–Moody algebras using higher order Casimir invariants, Nucl. Phys. B 304 (1988) 348, and Coset construction for extended Virasoro algebras, Nucl. Phys. B 304 (1988) 371.

[12] A.A. Belavin, Dynamical symmetry of integrable quantum systems, Nucl. Phys. B 180 (1981) 189.

[13] N.Yu. Reshetikhin, M.A. Semenov-Tian-Shansky, Central extensions of quantum current groups, Lett. Math. Phys. 19 (1990) 133.

[14] L. Freidel, J.M. Maillet, Quadratic algebras and integrable systems, Phys. Lett. B 262 (1991) 278.

[15] J.M. Maillet, Lax equations and quantum groups, Phys. Lett. B 245 (1990) 480.

[16] J. Avan, O. Babelon, E. Billey, The Gervais–Neveu–Felder equation and the quantum Calogero–Moser systems, Commun. Math. Phys. 178 (1996) 281.