Noncommutative Gröbner Bases for
Almost Commutative Algebras*

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Abstract. Let $K$ be an infinite field and $K\langle X \rangle = K\langle X_1, \ldots, X_n \rangle$ the free
associative algebra generated by $X = \{X_1, \ldots, X_n\}$ over $K$. It is proved that
if $I$ is a two-sided ideal of $K\langle X \rangle$ such that the $K$-algebra $A = K\langle X \rangle/I$ is
almost commutative in the sense of [3], namely, with respect to its standard
$\mathbb{N}$-filtration $FA$, the associated $\mathbb{N}$-graded algebra $G(A)$ is commutative, then
$I$ is generated by a finite Gröbner basis. Therefore, every quotient algebra of
the enveloping algebra $U(g)$ of a finite dimensional $K$-Lie algebra $g$ is, as a
noncommutative algebra of the form $A = K\langle X \rangle/I$, defined by a finite Gröbner
basis in $K\langle X \rangle$.

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1. Introduction

Let $K$ be an infinite field and let $K\langle X \rangle = K\langle X_1, X_2, \ldots, X_n \rangle$ be the free associative $K$-algebra
generated by $X = \{X_1, X_2, \ldots, X_n\}$ over $K$. It is a well-known fact that even if a two-sided ideal
$J$ of $K\langle X \rangle$ is finitely generated, $J$ does not necessarily have a finite Gröbner basis in the sense
of [9]. However, it was proved in [4] that if the quotient algebra $A = K\langle X \rangle/J$ is commutative,
then, after a general linear change of variables (if necessary), $J$ has a finite Gröbner basis in
$K\langle X \rangle$ (the construction of a Gröbner basis for $J$ is mentioned in the beginning of section 3
below). This, indeed, makes another algorithmic way to study a commutative algebra via its
noncommutative Gröbner presentation, and its effectiveness may be illustrated, for example, by

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the work of [1], [2], [5] and [11]. In this note, we use the result of [4] and the filtered-graded transfer of Gröbner bases [7] to show that for a two-sided ideal \( I \) of \( K\langle X \rangle \), if the \( K \)-algebra \( A = K\langle X \rangle/I \) is almost commutative in the sense of [3], namely, with respect to its standard \( \mathbb{N} \)-filtration \( FA \), the associated \( \mathbb{N} \)-graded algebra \( G(A) \) is commutative, then \( I \) is generated by a finite Gröbner basis. After reaching the main result in section 2, we discuss in section 3 how this result may be realized computationally.

Throughout the note, we fix the infinite field \( K \), the finite set \( X = \{X_1, X_2, ..., X_n\} \) and the corresponding free \( K \)-algebra \( K\langle X \rangle \).

2. The Main Result

Since the free \( K \)-algebra \( K\langle X \rangle = K\langle X_1, X_2, ..., X_n \rangle \) has its standard \( K \)-basis \( B \) consisting of all monomials (words) \( X_{j_1}X_{j_2} \cdots X_{j_s} \), \( s \in \mathbb{N} \), to be convenient, we denote monomials in \( B \) by \( u, v, w, \ldots \). Consider the \( \mathbb{N} \)-gradation \( K\langle X \rangle = \bigoplus_{p \in \mathbb{N}} K\langle X \rangle_p \) of \( K\langle X \rangle \) with \( K\langle X \rangle_p = K\text{-Span}\{u \in B \mid d(u) = p\} \), where \( d(u) \) stands for the degree (length) of \( u \). If \( I \) is a two-sided ideal of \( K\langle X \rangle \), then the \( K \)-algebra \( A = K\langle X \rangle/I \) has the standard \( \mathbb{N} \)-filtration

\[
F_0A = K \subset F_1A \subset \cdots \subset F_pA \subset \cdots
\]

where for each \( p \in \mathbb{N} \),

\[
F_pA = (\bigoplus_{k \leq p} K\langle X \rangle_k + I)/I,
\]

which defines the associated \( \mathbb{N} \)-graded \( K \)-algebra \( G(A) = \bigoplus_{p \in \mathbb{N}} G(A)_p \) of \( A \) with \( G(A)_p = F_pA/F_{p-1}A \). Recall from the literature [3] that \( A \) is called an almost commutative algebra if the associated graded \( K \)-algebra \( G(A) \) of \( A \) is commutative. In [3] it was proved that \( A \) is almost commutative if and only if \( A \) is a homomorphic image of the universal enveloping algebra \( U(\mathfrak{g}) \) of some finite dimensional \( K \)-Lie algebra \( \mathfrak{g} \). For instance, the \( n \)-th Weyl algebra \( A_n(K) \) is an almost commutative algebra and it is a homomorphic image of the \( 2n+1 \)-dimensional Heisenberg Lie algebra. So the class of almost commutative \( K \)-algebras consists of quotient algebras of enveloping algebras of finite dimensional \( K \)-Lie algebras. It is well-known that the study of quotient algebras of enveloping algebras (for example, the primeness, primitivity, simplicity, etc) has been very important in the finite dimensional Lie-theory. The reader is referred to ([10] Chapter 8 and Chapter 14) for details on this topic. The result and its proof given below provide a way to study the structural properties of almost commutative \( K \)-algebras algorithmically, or more precisely, to study such algebras by using both commutative and noncommutative Gröbner bases.

2.1. Theorem If the \( K \)-algebra \( A = K\langle X \rangle/I \) is almost commutative, then \( I \) has a finite Gröbner basis. Equivalently, for a finite dimensional \( K \)-Lie algebra \( \mathfrak{g} \), every quotient algebra of the enveloping algebra \( U(\mathfrak{g}) \) of \( \mathfrak{g} \), viewed as a quotient of some free \( K \)-algebra, is defined by a finite Gröbner basis.
Proof For an element \( f \in K\langle X \rangle \), if \( f = F_p + F_{p-1} + \cdots + F_0 \) with \( F_i \in K\langle X \rangle_i \) and \( F_p \neq 0 \), then we write \( d(f) = p \) for the degree of \( f \) and \( \text{LH}(f) = F_p \) for the leading homogeneous part of \( f \). Consider the standard \( \mathbb{N} \)-filtration \( FA \) of \( A \) and its associated \( \mathbb{N} \)-graded algebra \( G(A) \), as mentioned before. By ([7] Chapter III Proposition 3.1, or [8]), \( G(A) \cong K\langle X \rangle/\langle \text{LH}(I) \rangle \), where \( \langle \text{LH}(I) \rangle \) is the graded two-sided ideal of \( K\langle X \rangle \) generated by \( \text{LH}(I) = \{ \text{LH}(f) \mid f \in I \} \). Since \( A \) is almost commutative, \( K\langle X \rangle/\langle \text{LH}(I) \rangle \) is a commutative \( K \)-algebra.

Now, let \( K[x] = K[x_1, x_2, \ldots, x_n] \) be the commutative polynomial \( K \)-algebra in variables \( x_1, x_2, \ldots, x_n \) and consider the natural \( K \)-algebra homomorphism \( \gamma : K\langle X \rangle \to K[x] \) with \( \gamma(x_i) = x_i \). Then \( G(A) \cong K\langle X \rangle/\langle \text{LH}(I) \rangle \cong K[x]/\gamma(\text{LH}(I)) \) and by ([4] Theorem 2.1, Corollary 1.1), a finite Gröbner basis of \( \langle \text{LH}(I) \rangle \) may be obtained by using a finite Gröbner basis of \( \gamma(\langle \text{LH}(I) \rangle) \). Our aim is to lift the obtained Gröbner basis of \( \langle \text{LH}(I) \rangle \) to a finite Gröbner basis of \( I \) as described in ([7] Chapter III section 3). To this end, we need to use a graded monomial ordering. Note that \( \gamma \) does not change the degree (length) of monomials, that is, for any two monomials \( u, v \) in the standard \( K \)-basis \( B \) of \( K\langle X \rangle \),

\[
d(u) < d(v) \text{ if and only if } d(\gamma(u)) < d(\gamma(v)).
\]

It turns out that if we fix a certain graded monomial ordering \( \prec \) (for example, the graded lexicographic ordering) on \( K[x] \), then the lexicographic extension \( \prec_{et} \) of \( \prec \) to \( K\langle X \rangle \) (in the sense of [4]), which is defined for \( u, v \in B \) by

\[
u \prec_{et} v \text{ if } \begin{cases} \gamma(u) < \gamma(v) \text{ or} \\ \gamma(u) = \gamma(v) \text{ and } u \text{ is lexicographically smaller than } v, \end{cases}
\]

is a graded monomial ordering. Thus, it follows from ([4] Theorem 2.1, Corollary 1.1) that \( \langle \text{LH}(I) \rangle \) has a finite Gröbner basis \( G \) with respect to \( \prec_{et} \). Since \( \gamma(\langle \text{LH}(I) \rangle) \) is a graded ideal of \( K[x] \), by the construction of \( G \) given in ([4] Theorem 2.1, or see the description given in the beginning of section 3 below), we may assume that \( G = \{ G_1, G_2, \ldots, G_s \} \) consists of homogeneous elements in \( \langle \text{LH}(I) \rangle \). Suppose for \( i = 1, 2, \ldots, s \),

\[
G_i = \sum H_i \text{LH}(f_i)T_i, \text{ where } H_i, T_i \text{ are homogeneous elements and } f_i \in I.
\]

If we put \( f_i = \text{LH}(f_i) + f_i' \), where \( d(f_i') < d(f_i) \), then \( \sum H_i f_i T_i = g_i \in I \) and

\[
g_i = \sum H_i f_i T_i = \sum H_i \text{LH}(f_i)T_i + \sum H_i f_i' T_i = G_i + \sum H_i f_i' T_i
\]

has its leading homogeneous part \( \text{LH}(g_i) = G_i, i = 1, 2, \ldots, s \). As \( \prec_{et} \) is a graded monomial ordering on \( K\langle X \rangle \), it follows from ([7] Chapter III Theorem 3.7) that \( G = \{ g_1, g_2, \ldots, g_s \} \) is a finite Gröbner basis of \( I \) with respect to \( \prec_{et} \). This completes the proof.

3. Further Discussion

Concerning the computational realization of Theorem 2.1, we first recall from [4] how a noncommutative Gröbner basis may be constructed by using a commutative version. Let
Theorem 2.1 is only theoretical.

Given by the Lie-brackets of generators of the finite dimensional Lie algebra $\mathfrak{g}$.

$S = \gamma U$

Suppose (see [7] Chapter III). In other words, if we cannot find a generating set of $\langle \gamma \rangle$.

Again by the proof of Theorem 2.1, in order to obtain a finite Gröbner basis of $\langle \gamma \rangle$ together with the elements $\delta(u \cdot g_i)$ for each $g_i \in \Gamma$ and each $u \in U(\text{LH}(I))(\text{LM}(g_i))$, where $\text{LM}(g_i)$ stands for the $<-$leading monomial of $g_i$ and $\langle \text{LM}(I) \rangle$ is the monomial ideal generated by $\text{LM}(I) = \{\text{LM}(f) \mid f \in I\}$.

In view of the characterization of an almost commutative algebra [3], to realize Theorem 2.1 computationally, we have to consider two coherent cases separately. In what follows, notations used in section 2 are maintained.

I. $A = K\langle X \rangle/I$, $I = \langle S \rangle$ and $G(A)$ is commutative.

Since $G(A)$ is commutative, it follows from the proof of Theorem 2.1 that we have

$$G(A) \cong K\langle X \rangle/\langle \text{LH}(I) \rangle \cong K[x]/\gamma(\langle \text{LH}(I) \rangle).$$

Again by the proof of Theorem 2.1, in order to obtain a finite Gröbner basis of $I$ algorithmically, we need to have a finite Gröbner basis of $\gamma(\langle \text{LH}(I) \rangle)$ so that we may use it to construct a Gröbner basis for $\langle \text{LH}(I) \rangle$ as remarked above. While this implies that we need first to know a generating set of $\gamma(\langle \text{LH}(I) \rangle)$. But the fact is that even if the generating set of $I$ is finite, say $S = \{f_1, f_2, \ldots, f_s\}$, the equality $\langle \text{LH}(I) \rangle = \langle \text{LH}(f_1), \text{LH}(f_2), \ldots, \text{LH}(f_s) \rangle$ is not necessarily true (see [7] Chapter III). In other words, if we cannot find a generating set of $\gamma(\langle \text{LH}(I) \rangle)$ effectively, Theorem 2.1 is only theoretical.

As an example, let us point out that if we know some (finite or infinite) Gröbner basis $\mathcal{G} = \{g_i \mid i \in \Omega\}$ of $I$ and $\mathcal{G}$ contains all commutators $X_iX_j - X_jX_i$, $1 \leq i < j \leq n$, then $\langle \text{LH}(I) \rangle = \langle \text{LH}(g_i) \mid i \in \Omega \rangle$ by ([7] Chapter III).

II. $A = U(g)/I$ and $I = \langle S \rangle$.

Suppose $U(g) = K\langle X \rangle/J$ with $J = \langle [X_i, X_j] - X_iX_j - X_jX_i \mid 1 \leq i < j \leq n \rangle$, where $[X_i, X_j]$ is given by the Lie-brackets of generators of the finite dimensional $K$-Lie algebra $g$. Let $\mathcal{T}$ be the two-sided ideal of $K\langle X \rangle$ such that $I = \mathcal{T}/J$. Then $A = U(g)/I \cong K\langle X \rangle/\mathcal{T}$. By the foregoing discussion, if we can find a finite Gröbner basis of $\gamma(\langle \text{LH}(\mathcal{T}) \rangle)$, then a finite Gröbner basis for $\mathcal{T}$ may be constructed.

Note that the standard filtration $FU(g)$ induces the standard filtration $FA$. Consider the filtration $FI$ on $I$ induced by $FU(g)$ and its associated graded ideal $G(I)$ in $G(A)$. Then it is
well known that
\[ G(A) = G(U(g)/I) \cong G(U(g))/G(I) \cong K[x]/\overline{G(I)}, \]
where \( \overline{G(I)} \) stands for the counterpart of \( G(I) \) in \( K[x] \). Thus, since both graded algebra epimorphisms
\[ \varphi : k[x] \twoheadrightarrow K[x]/\overline{G(I)} \cong G(A) \]
\[ \psi : k[x] \twoheadrightarrow K[x]/\gamma((LI(\overline{I}))) \cong G(A) \]
agree on the generators \( x_1, x_2, ..., x_n \), we have \( \overline{G(I)} = \gamma((LI(\overline{I}))) \). This makes the chance for us to have a finite Gröbner basis of \( \gamma((LI(\overline{I}))) \) by using the generating set \( S \) of \( I \). To see this, first recall that \( U(g) \) is a solvable polynomial algebra in the sense of [6]. Hence, starting with \( S \), a noncommutative version of Buchberger Algorithm produces a finite Gröbner basis \( G = \{g_1, g_2, ..., g_m\} \) for \( I \). Furthermore, if the monomial ordering used in producing \( G \) is a graded monomial ordering, then it follows from ([7] Chapter IV Theorem 2.1) that \( \sigma(G) = \{\sigma(g_1), \sigma(g_2), ..., \sigma(g_m)\} \) is a Gröbner basis for \( G(I) \) with respect to the same type of graded monomial ordering, where for each \( g_i \), if \( g_i \in F_p I - F_{p-1} I \), then \( \sigma(g_i) \) is the homogeneous element in \( G(I)_p \) represented by \( g_i \). Finally, by passing to \( \overline{G(I)} \) in \( K[x] \), we obtain a finite Gröbner basis \( \{g_1, g_2, ..., g_m\} \) of \( \gamma((LI(\overline{I}))) \). Note that \( U(g) = K\langle X \rangle/J, I = T/J \). Hence the preimage of \( G \) is contained in \( T \). Thus, by the definition of \( \delta(u \cdot \overline{I}) \) described in the beginning of this section, the last step of the proof of Theorem 2.1 can be realized to give a Gröbner basis for \( T \).

In this case, the good news is that nowadays there has been the well-developed computer algebra system SINGULAR: PLURAL (http://www.singular.uni-kl.de/index.html) which provides a programme named twostd for computing a two-sided Gröbner basis of a two-sided ideal in a solvable polynomial algebra.

We end this note by employing an easy example to illustrate the procedure demonstrated in part I and II above. Let \( U(s\ell_2) \) be the enveloping algebra of the 3-dimensional \( K \)-Lie algebra \( s\ell_2 = Ke \oplus Kf \oplus Kh \) subject to the relations \( [e, f] = h, [h, e] = 2e, [h, f] = -2f \), that is, \( U(s\ell_2) = K[e, f, h] \cong K\langle X, Y, Z \rangle/P \) with \( X \mapsto e, Y \mapsto f \) and \( Z \mapsto h \), where
\[ P = \langle YX - XY + Z, ZX - XZ - 2X, ZY - YZ + 2Y \rangle. \]
Consider the two-sided ideal \( I = \langle e^3, f^3, h^3 - 4h \rangle \) of \( U(g) \), then \( U(g)/I \cong K\langle X, Y, Z \rangle/\overline{T} \), where
\[ T = \langle YX - XY + Z, ZX - XZ - 2X, ZY - YZ + 2Y, X^3, Y^3, Z^3 - 4Z \rangle. \]
Let \( \prec \) be the graded reverse lexicographic ordering on \( U(g) \) defined by \( e \prec f \prec h \). Then twostd produces a two-sided Gröbner basis
\[ G = \left\{ e^3, f^3, h^3 - 4h, eh^2 + 2eh, f^2h - 2fh, 2efh - h^2 - 2h, e^2f - eh - 2e, ef^2 - fh, e^2h + 2e^2, f^2h - 2f^2 \right\}. \]
for $I$ (Singular Manual A.6.1). Hence

$$\sigma(G) = \left\{ \sigma(e)^3, \sigma(f)^3, \sigma(h)^3, \sigma(e)\sigma(h)^2, \sigma(f)\sigma(h)^2, \
2\sigma(e)\sigma(f)\sigma(h), \sigma(e)\sigma(f)^2, \sigma(e)^2\sigma(h), \sigma(f)^2\sigma(h) \right\}$$

is a Gröbner basis of $G(I)$. If we use the graded reverse lexicographic ordering $x \prec y \prec z$ on the polynomial $K$-algebra $K[x, y, z]$, then previous part II yields a Gröbner basis

$$\{ x^3, y^3, z^3, xz^2, yz^2, 2xyz, x^2y, xy^2, x^2z, y^2z \}$$

for $\gamma(\langle LH(I) \rangle)$. By the construction of $\delta(u \cdot g)$ described in the beginning of this section, it may be checked directly that $\langle LH(\overline{I}) \rangle$ has a Gröbner basis

$$\left\{ YX - XY, ZY - YZ, ZX - XZ, X^3, Y^3, \
Z^3, XZ^2, YZ^2, 2XYZ, X^2Y, XY^2, X^2Z, Y^2Z \right\}.$$  

Note that $\overline{I}$ contains the preimage of $G$. It follows from the last step of the proof of Theorem 2.1 that $\overline{I}$ has the Gröbner basis

$$\left\{ YX - XY, ZY - YZ, ZX - XZ, \
X^3, Y^3, Z^3 - 4Z, XZ^2 + 2XZ, \
YZ^2 - 2YZ, 2XYZ - Z^2 - 2Z, \
X^2Y - XZ - 2X, XY^2 - YZ, X^2Z + 2X^2, Y^2Z - 2Y^2 \right\}$$

with respect to $x \prec_{et} y \prec_{et} z$.

References

[1] D. J. Anick, On the homology of associative algebras, *Trans. Amer. Math. Soc.*, Vol. 296, 2(1986), 641–659.

[2] D. Anick and G.-C. Rota, Higher-order syzygies for the bracket algebra and for the ring of coordinates of the Grassmannian, *Proc. Nat. Acad. Sci. U.S.A.*, 88(1991), 8087–8090.

[3] M. Duflo, Certaines algèbres de type finisont des algèbres de Jacobson, *J. Algebra*, 27(1973), 358–365.

[4] D. Eisenbud, I. Peeva and B. Sturmfels, Non-commutative Gröbner bases for commutative algebras, *Proc. Amer. Math. Soc.*, 126(1998), 687-691.

[5] E. Green and R.Q. Huang, Projective resolutions of straightening closed algebras generated by minors, *Adv. in Math.*, 110(1995), 314C333.

[6] A. Kandri-Rody and V. Weispfenning, Non-commutative Gröbner bases in algebras of solvable type, *J. Symbolic Comput.*, 9(1990), 1–26.

[7] Huishi Li, *Noncommutative Gröbner Bases and Filtered-Graded Transfer*, LNM, 1795, Springer-Verlag, 2002.

[8] Huishi Li, The general PBW property, *Algebra Colloquium*, in press.

[9] T. Mora, An introduction to commutative and noncommutative Gröbner bases, *Theoretical Computer Science*, 134(1994), 131–173.
[10] J.C. McConnell and J.C. Robson, *Noncommutative Noetherian Rings*, John Wiley & Sons, 1987.

[11] I. Peeva, V. Reiner and B. Sturmfels, How to shell a monoid, *Math. Ann.*, 310(1998), 379C393.