The PowerURV algorithm for computing rank-revealing full factorizations

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Abstract: Many applications in scientific computing and data science require the computation of a rank-revealing factorization of a large matrix. In many of these instances the classical algorithms for computing the singular value decomposition are prohibitively computationally expensive. The randomized singular value decomposition can often be helpful, but is not effective unless the numerical rank of the matrix is substantially smaller than the dimensions of the matrix. We introduce a new randomized algorithm for producing rank-revealing factorizations based on existing work by Demmel, Dumitriu and Holtz [Numerische Mathematik, 108(1), 2007] that excels in this regime. The method is exceptionally easy to implement, and results in close-to optimal low-rank approximations to a given matrix. The vast majority of floating point operations are executed in level-3 BLAS, which leads to high computational speeds. The performance of the method is illustrated via several numerical experiments that directly compare it to alternative techniques such as the column pivoted QR factorization, or the QLP method by Stewart.

1 Introduction

In many applications such as in the analysis of large data sets and the numerical solution of boundary integral equations, it is necessary to compute a low-rank factorization of a matrix. One algorithm for this task that has proven effective in the past several years is the randomized singular value decomposition (RSVD) algorithm (see [5]). Given a desired truncation rank $k$, the RSVD computes a low-rank approximation by using a random projection to reduce the dimensionality of the problem, paired with a deterministic singular value decomposition (SVD) on the low-dimensional, projected problem. Often, a near-optimal low-rank approximation can be produced in only a fraction of the time required to run a deterministic SVD on the entire data matrix. However, the RSVD has limitations. One is that it requires information about the desired truncation rank prior to computation, which may not be available. Another is that it becomes uncompetitive when the target rank is not much smaller than the matrix dimensions. In these regimes, which will be the focus of this report, it is usually helpful to form a rank-revealing, full factorization of the matrix. By this we mean a factorization of the original matrix that can easily be truncated to form low-rank approximations. The quintessential example of this is the full SVD, which factors a matrix $A \in \mathbb{R}^{m \times n}$ with $m \geq n$ into the product $A = U \Sigma V^*$, where $U \in \mathbb{R}^{m \times n}$ has orthonormal columns, $V \in \mathbb{R}^{n \times n}$ is orthogonal, and $\Sigma \in \mathbb{R}^{n \times n}$ is a diagonal matrix that takes values $\Sigma(j,j) = \sigma_j$ for all $j = 1, 2, \ldots, n$ such that $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_n \geq 0$. This factorization exists for any matrix, and when it is obtained, a rank-$k$ approximation is given by $A_k = U(:,1:k) \Sigma(1:k,1:k) V^*$.

The Eckart-Young theorem guarantees optimality in the spectral and Frobenius norms of low-rank approximations obtained by truncating the SVD, but computing an SVD can be prohibitively expensive in practice. A more economical alternative is obtained through truncating a column pivoted QR factorization (CPQR). While this is a much faster algorithm from both

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the perspective of communication costs and operation count, there are no general guarantees on the quality of the resulting low-rank approximation, and indeed there are known cases where low-rank approximations obtained through truncating CPQR are arbitrarily poor [6].

A middle ground that can deliver results comparable in quality to the SVD, while maintaining comparable efficiency to CPQR, can be achieved through the use of so called UTV factorizations, which were introduced by G.W. Stewart in [8,9] and generalize the QR and SVD factorizations. In full generality, the UTV factorization factors a given matrix $A \in \mathbb{R}^{m \times n}$ with $m \geq n$ into the product

$$A = UTV^*, \quad (1)$$

where $U \in \mathbb{R}^{m \times n}$ has orthonormal columns, $V \in \mathbb{R}^{n \times n}$ is orthogonal, and $T \in \mathbb{R}^{n \times n}$ is triangular. While all of the algorithms we discuss in this report apply to both the case where $T$ in (1) is upper-triangular and lower-triangular, we will only consider the upper-triangular case for simplicity and so will only consider URV decompositions. If we are given a matrix where $m \leq n$, then we could do a ULV decomposition on $A^*$ and then transpose the decomposition to find a URV decomposition. Thus, it also suffices to only consider the case where $m \geq n$.

In order for a URV factorization to be useful for low-rank approximation, we need the decomposition to be rank-revealing. By this we mean that for all $k = 1, 2, \ldots, n$ a partition of the URV decomposition into

$$A = [U_1 \quad U_2] \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix}, \quad (2)$$

where $U_1 \in \mathbb{R}^{m \times k}$, $U_2 \in \mathbb{R}^{m \times n-k}$, $V_1 \in \mathbb{R}^{n \times k}$, $V_2 \in \mathbb{R}^{n \times n-k}$, $R_{11} \in \mathbb{R}^{k \times k}$, $R_{12} \in \mathbb{R}^{k \times n-k}$ and $R_{22} \in \mathbb{R}^{n-k \times n-k}$, has the properties $\| R_{11} \|_2 \approx \sigma_k(A)$ and $\| R_{22} \|_2 \approx \sigma_{k+1}(A)$. This is not at all guaranteed for CPQR, and although there do exist alternative pivoting strategies that guarantee some type of rank-revealing property for QR (see [4]), they are often expensive to compute and still tend to deliver suboptimal results relative to more general URV factorizations.

Given $A \in \mathbb{R}^{m \times n}$ with $m \geq n$, a URV decomposition consists of a matrix $U \in \mathbb{R}^{m \times n}$ with orthonormal columns, an orthogonal matrix $V \in \mathbb{R}^{n \times n}$, and an upper-triangular matrix $R \in \mathbb{R}^{n \times n}$ such that $A = URV^*$. A CPQR factorization of $A = QRP^*$ is a URV decomposition, where the left orthogonal factor is given by $Q$ and the right orthogonal factor is given by the permutation matrix $P$. It follows that an unpivoted QR factorization is also a URV factorization with the right orthogonal factor equal to the $n \times n$ identity matrix.

Throughout this work we will place an emphasis on communication costs. While low floating point operation counts are still crucial to efficient algorithms, they do not tell the whole story on modern computing architectures. This is illustrated in a comparison between CPQR and unpivoted QR. While both CPQR and unpivoted QR have the same leading order floating point operation count, the former is observed to be significantly slower than the latter as demonstrated in Figure [1]. This discrepancy is due to the fact that unpivoted QR can be executed almost completely with level-3 BLAS operations, whereas CPQR requires a significant number of level-2 BLAS operations to handle the pivoting. We will be especially interested in algorithms where the majority of floating point operations are in matrix-matrix multiplications, which are intrinsically low in communication costs. Moreover, matrix-matrix multiplication can easily leverage highly parallel environments, such as GPUs, and can be further accelerated with fast matrix-matrix algorithms, such as Strassen’s algorithm.

## 2 Algorithms for computing URV factorizations

As aforementioned, the QR factorizations and the SVD are specific cases of URV factorizations. We emphasize that we are primarily interested in URV factorizations in the regime where low-rank approximations produced through CPQR factorizations are too inaccurate and the SVD is too expensive to compute. Over the past several decades many algorithms have been posed to compute the URV factorization in this setting. In Section 2.1 we summarize a classical, deterministic algorithm for computing such a URV decomposition first introduced by Stewart in [10]. In Section 2.2 we present another algorithm first introduced in the context of leveraging
fast matrix multiplication to accelerate eigenvalue computations by Demmel, Dumitriu, and Holtz [2] and then studied in its own right in [1]. In Section 2.3, we outline a recent, randomized algorithm for computing the URV decomposition [7]. In Section 3, we introduce a modification of the algorithm by Demmel, Dumitriu, and Holtz which trades slightly higher computational cost for stronger rank-revealing properties.

2.1 Stewart’s QLP

One particularly effective algorithm for computing rank-revealing UTV factorizations was introduced by Stewart in [10], which produces a ULV factorization through the use of two CPQR factorizations. Since we have chosen to focus on the URV factorization in this report, we will instead consider a variation where the QLP algorithm is applied to \( A^* \) instead of \( A \). The transpose of the resulting factorization gives a URV factorization for \( A \). To be precise, given \( A \in \mathbb{R}^{m \times n} \), a CPQR factorization is performed on the transpose of \( A \),

\[
A^* = Q_1 R_1 P_1^* ,
\]

where \( Q_1 \in \mathbb{R}^{n \times n} \) is an orthogonal matrix, \( R_1 \in \mathbb{R}^{n \times m} \) is upper-trapezoidal, and \( P_1 \in \mathbb{R}^{m \times m} \) is a permutation matrix. Then another CPQR factorization is performed on the transpose of the product of the upper-trapezoidal factor and the permutation matrix

\[
(R_1 P_1^*)^* = Q_2 R_2 P_2^* ,
\]

where \( Q_2 \in \mathbb{R}^{m \times n} \), \( R_2 \in \mathbb{R}^{n \times n} \) is upper-triangular, and \( P_2 \in \mathbb{R}^{n \times n} \) is a permutation matrix. Setting \( U = Q_2 \), \( R = R_2 \) and \( V = Q_1 P_2 \) yields a URV decomposition for \( A \).

While the underlying mechanism behind this algorithm is the CPQR factorization, it is empirically observed that using the second CPQR factorization produces a significantly better rank-revealing decomposition. It was even shown that this procedure could handle Kahan’s example, for which CPQR by itself fails to produce a rank-revealing factorization [10]. Figure 2 illustrates the difference in quality of rank-revealing factorizations for two non-pathological examples.
2.2 DDH-URV

The first randomized algorithm for computing the URV decomposition that we are aware of was given in [2] as a tool for accelerating eigenvalue computations given a fast matrix-matrix multiplication algorithm. We will refer to this algorithm in this technical report as the DDH-URV algorithm, after the authors of [2]. The algorithm starts by generating a Haar distributed random matrix $V \in \mathbb{R}^{n \times n}$ which is produced by orthogonalizing a Gaussian random matrix using an unpivoted QR factorization. Then, another unpivoted QR factorization is taken of the product $AV$ to produce

$$AV = QR,$$  \hspace{1cm} (3)

where $Q \in \mathbb{R}^{m \times n}$ has orthonormal columns and $R \in \mathbb{R}^{n \times n}$ is upper-triangular. We see that equation (3) directly yields the URV factorization $A = QRV^*$.

As in Stewart’s QLP algorithm, two QR factorizations are required in this algorithm. However, both QR factorizations are done through unpivoted QR, which leads to a significant saving in communication costs. This is because unpivoted QR can operate on blocks, accumulate Householder reflectors, and then apply the reflectors to the trailing matrix using a level-3 BLAS matrix-matrix multiply. In contrast, CPQR needs a level-2 BLAS matrix-vector multiply after orthonormalizing each column to update the trailing column norms and choose the next pivot correctly. The price we pay for maintaining rank-revealing properties while removing the pivoting is a matrix-matrix multiplication with a dense Haar matrix, which can also be performed using a level-3 BLAS matrix-matrix multiplication. This matrix-matrix multiplication can be further accelerated by replacing the Haar matrix with a chain of random, structured orthogonal matrices [1].

While this algorithm is extremely computationally efficient, the quality of the rank-revealing factorization produced can be rather low. This is due to the fact that the method does not use any information of the data matrix $A$ to form the approximation of the right singular space spanned by the columns of $V$. Our numerical experiments show that low-rank approximations obtained through truncation of a URV factorization produced with this algorithm are substantially further from optimal than those produced than the other algorithms discussed in this section.

Technical, a matrix produced in this fashion will usually not be exactly Haar distributed, since this would require restricting the upper-triangular factor in QR factorization to be non-negative along the diagonal. In practice, this detail is unimportant, so we will continue to say that orthogonal matrices produced through QR factorizations on a Gaussian random matrix are Haar distributed. See [3] for more details.
2.3 randUTV

Another randomized algorithm for computing the URV decomposition in our regime of interest is the randUTV algorithm presented in [7]. This algorithm is similar in structure to a blocked algorithm for computing a QR factorization, but employs the idea of the RSVD to process the “active” block of columns. The randUTV algorithm requires the selection of a power parameter \( q > 0 \) and blocking parameter \( b > 0 \) a priori. It begins by drawing a Gaussian random matrix \( G \in \mathbb{R}^{m \times b} \), followed by applying the power iteration \( Y = (A^*A)^qA^*G \in \mathbb{R}^{n \times b} \), and then applying an unpivoted QR factorization on this result to obtain the orthogonal matrix \( W \in \mathbb{R}^{n \times n} \). Then \( A \) can be written

\[
A = \begin{bmatrix}
AW_1 & AW_2
\end{bmatrix} W^*,
\]

where \( W_1 = W(1:b,:) \in \mathbb{R}^{n \times b} \) and \( W_2 = W(b+1:n,:) \in \mathbb{R}^{n \times n-b} \). Then the SVD is taken

\[
AW_1 = \tilde{U}DV^*_s,
\]

where \( \tilde{U} \in \mathbb{R}^{m \times b} \), \( D \in \mathbb{R}^{b \times b} \) and \( V_s \in \mathbb{R}^{b \times b} \). We can then write

\[
A = \tilde{U} \begin{bmatrix}
D & \tilde{U}^* AW_2
\end{bmatrix} \tilde{V}^*, \quad \tilde{V} = W \begin{bmatrix}
V_s & 0 \\
0 & I_{n-b}
\end{bmatrix}, \tag{4}
\]

where \( I_{n-b} \) denotes the \((n-b) \times (n-b)\) identity matrix. We see from (4) that we have reduced the submatrix consisting of the first \( b \) columns of \( A \) to an upper-trapezoidal matrix through left and right multiplications by orthogonal matrices. We can then repeat the same procedure on the bottom \((m-b) \times (n-b)\) submatrix of \( \tilde{U}^* AW_2 \) to obtain a factorization of \( A \) where the submatrix consisting of the first \( 2b \) columns are upper-trapezoidal and so on, until a complete URV factorization is obtained. For more details, we refer the reader to [7].

In practice we have found that randUTV can offer close to optimal low-rank approximations at modest computational cost. With that said, it can be difficult to implement. This is in contrast to the other algorithms described here, which can all be implemented with just a few calls to standard BLAS routines. Another subtlety of randUTV is that the rank-revealing quality is affected by the choice of a blocking parameter, as opposed to just the computational time. Indeed, we have observed that a poor choice of blocking parameter can produce poor rank-revealing factorizations in addition to slowing down the computation. Since we are not aware of any rigorous heuristics for determining this parameter, we omit randUTV from our numerical experiments in Section 5.

3 PowerURV

The new PowerURV algorithm combines the power iteration in the RSVD with the DDH-URV algorithm discussed in Section 2.2. First a small integer \( q > 0 \) is fixed, which controls how many steps of power iteration will be taken. Then Gaussian random matrix \( G \in \mathbb{R}^{n \times n} \) is drawn. We then apply \( q \) steps of a power iteration to \( G \) and perform an unpivoted QR factorization to produce

\[
(A^*A)^qG = VZ, \tag{5}
\]

where \( V \in \mathbb{R}^{n \times n} \) is orthogonal and \( Z \in \mathbb{R}^{n \times n} \) is upper-triangular. An unpivoted QR factorization of the product \( AV \) is taken to obtain

\[
AV = UR, \tag{6}
\]

where \( U \in \mathbb{R}^{m \times n} \) has orthonormal columns and \( R \in \mathbb{R}^{n \times n} \) is upper-triangular. A URV factorization is then given by

\[
A = URV^*.\]

In practice during the power iteration in (5), we need to re-orthonormalize after each application of \( A \) or \( A^* \). Otherwise, we see a loss of accuracy in the quality of our low-rank approximations by \( \epsilon^{q+1} \), where \( \epsilon \) denotes machine precision. This is precisely analogous to the loss of accuracy in subspace iteration when re-orthonormalization is omitted.
The PowerURV algorithm retains the same algorithmic structure as DDH-URV and, when $q = 0$, is DDH-URV. With $q > 0$, PowerURV takes into account information of the row space of the data matrix $A$ when computing the approximation to the space spanned by the right singular vectors. This allows for a substantial improvement in quality when forming low-rank approximations. This improvement is especially significant for matrices with slow decay in the singular values. This is illustrated in Figure 3 which shows the quantities relevant for rank-revealing for two $200 \times 160$ matrices with fast and slow singular value decay, respectively (see bullets on Matrix 1 and Matrix 2 in Section 5 for the precise definitions of these matrices). In the case of the matrix with fast decay, we see that the $\sigma_k(R_{11})$ and $\sigma_k(R_{22})$ values of the factorizations produced by the PowerURV algorithm with 1 and 2 power iterations closely hug $\sigma_{k+1}(A)$ for all $k = 1, 2, \ldots, n$. On the other hand, the factorization without any of the power iterations is off by about an order of magnitude in both directions. We see similar behavior for the matrix with slow singular value decay except that there is a more discernible difference between $q = 1$ and $q = 2$.

In contrast to Stewart’s QLP algorithm, where the bulk of the operations required are in two CPQR factorizations, the majority of work in PowerURV is in level-3 BLAS matrix-matrix multiplications as in DDH-URV. This allows PowerURV to be readily parallelized and reap advantages of modern computer architectures. Another similarity to DDH-URV is that the implementation of PowerURV is relatively straightforward with access to an efficient matrix-matrix multiplication routine, an unpivoted QR factorization routine, and a random number generator. We can defer choices of blocking parameters to the BLAS routines themselves. The power iteration parameter $q$ depends only on spectral information of the data matrix, and we find that in most cases it suffices to set $q = 1$.

We also mention that the Gaussian matrix in (5) could be replaced by other random matrices such as a chain of orthogonal transforms (see [1]). However since this would come at a cost to accuracy and only speed up 1 out of the $2q + 1$ matrix-matrix multiplications, we prefer the stronger theoretical guarantees of the Gaussian over the minute performance advantages of using a single fast transform.

Figure 3: Comparison of qualities of rank-revealing factorizations obtained by PowerURV for $q = 0, 1, 2$ for Matrix 1 (left) and Matrix 2 (right), both specified in Section 5.

4 Cost analysis

Table 4 summarizes the leading order floating point operation counts necessary for each of the algorithms discussed in this report for an $m \times n$ matrix. For the SVD, we have only included the cost of bidiagonalization and the costs of forming the left and right orthogonal factors. We
note that the SVD has a significantly higher operation count than all of the other algorithms discussed. Moreover, the bidiagonalization in the SVD cannot be done in level-3 BLAS. (This is a byproduct of having to apply Householder reflectors from both the left and right one after the other). PowerURV has a significantly higher flop count than the QLP algorithm, but as discussed earlier, flop counts alone do not give the whole picture. The QLP algorithm needs two column pivoted QR factorizations, which unlike the unpivoted QR factorization, needs a level-2 BLAS column norm update for correct pivot selection. As a result, despite the higher flop count, we find in practice PowerURV to be as fast as QLP, if not faster, for large problems, as illustrated in Figure 4. This also suggests that PowerURV has a better outlook for parallelism. The comparison of the performance between randUTV and PowerURV is less clear. In our numerical experiments, we found that the performance of randUTV is highly dependent on the choice of blocking parameter, which can vary greatly based on hardware and problem size. It is clear that randUTV also has a high potential for parallelism, but we believe that developing and tuning an efficient parallel implementation would be non-trivial, whereas developing an effective parallel implementation of PowerURV could consist of just calling parallel routines for matrix-matrix multiplication and unpivoted QR factorization.

| Algorithm     | DGEMM/ QR | CPQR             | other level-2 BLAS |
|---------------|-----------|------------------|--------------------|
| Golub-Reinsch | 0         | 0                | 4m^2n + 8mn^2 + 9n^3 |
| QLP           | 0         | 2mn^2 + \frac{2}{3}n^3 |
| randUTV       | (5 + 2q)mn^2 - \frac{1}{2}(3 + 2q)n^3 | 0 |
| PowerURV      | 2(q + 1)m^2n + (4q + 2)mn^2 - \frac{2}{3}(2q + 1)n^3 | 0 |

Figure 4: Comparison of wall clock time elapsed for the DDH-URV algorithm, PowerURV, and QLP for Gaussian random matrices of size $n \times n$. Executed in MATLAB on a 16-core Intel Xeon CPU E5-2643 @ 3.30 GHz with 64 GB of memory.

5 Numerical results

In this section, we compare the accuracy of QLP, DDH-URV, and PowerURV. We follow the format in \[7\] and benchmark the algorithms against the following four matrices:

\[2\]The cost of forming the orthogonal factors has not been included.
- **Matrix 1 (Fast Decay):** This is a matrix $A = UDV^* \in \mathbb{R}^{200 \times 160}$ where $U$ and $V$ are drawn from a Haar distribution and where $D$ is a rectangular, diagonal matrix with entries $D(k,k) = (10^{-20})^{k-1}$.

- **Matrix 2 (Slow Decay):** This is a matrix $A = UDV^* \in \mathbb{R}^{200 \times 160}$ where $U$ and $V$ are drawn from a Haar distribution and where $D$ is a rectangular, diagonal matrix with entries $D(k,k) = k^{-1}$.

- **Matrix 3 (S-Shaped Decay):** This is a matrix $A = UDV^* \in \mathbb{R}^{200 \times 160}$ where $U$ and $V$ are drawn from a Haar distribution and where $D$ is a rectangular, diagonal matrix with entries $D(k,k) = 10^{-(1+\tanh(5(-1+2k/n)))}$ for $k = 1, 2, \ldots, 80$ and $D(k,k) = 10^{-2}$ for $k = 81, 82, \ldots, 150$.

- **Matrix 4 (Boundary Integral Equation):** This is a matrix $A \in \mathbb{R}^{200 \times 200}$ that is the result of discretizing the boundary integral equation for the Laplace equation on a smooth, 5-sided star.

In Figures 5, 6, 7, and 8 we have two plots for each matrix. For each URV algorithm, we plot the relative and absolute errors of the difference between the $A$ and the rank-$k$ approximation $U(:,1:k)R(1:k,:)V^*$. In all cases we see that the PowerURV algorithm with just $q = 1$ compares very favorably with the QLP algorithm. However, without any power iterations the algorithm is a factor 2-10 worse. Since the singular values of Matrix 2 and Matrix 4 do not decay below $\varepsilon^1 2q + 1$ for $q = 1, 2$ we omit the stabilizing orthogonalizations in the power iterations in these cases.

Figure 5: Rank-$k$ approximation errors to Matrix 1 in Section 5
Figure 6: Rank-\(k\) approximation errors to Matrix 2 in Section 5.

Figure 7: Rank-\(k\) approximation errors to Matrix 3 in Section 5.
6 Relationship with RSVD

The PowerURV algorithm is closely connected with the standard randomized singular value decomposition algorithm (RSVD). To describe the connection, let us briefly review the steps in the RSVD. Let $A \in \mathbb{R}^{m \times n}$ with $m \geq n$. Given an integer $\ell < n$, the RSVD builds an approximation to a truncated SVD via the following steps: A random matrix (typically Gaussian) $G_{\text{rsvd}} \in \mathbb{R}^{n \times \ell}$ is drawn and the product

$$Y_{\text{rsvd}} = A(A^*A)^{\frac{1}{2}}G_{\text{rsvd}} \in \mathbb{R}^{m \times \ell}, \quad (7)$$

is evaluated. The columns $Y_{\text{rsvd}}$ orthogonalized via an unpivoted QR factorization

$$Y_{\text{rsvd}} = Q_{\text{rsvd}}R_{\text{rsvd}}. \quad (8)$$

In other words, the columns of the $m \times \ell$ matrix $Q_{\text{rsvd}}$ form an orthogonal basis for the range of $Y_{\text{rsvd}}$. After this, a deterministic SVD of $Q_{\text{rsvd}}^*A$ is computed to obtain

$$Q_{\text{rsvd}}^*A = W_{\text{rsvd}}\Sigma_{\text{rsvd}}(V_{\text{rsvd}})^*, \quad (9)$$

where $W_{\text{rsvd}} \in \mathbb{R}^{\ell \times \ell}$ is unitary, where $V_{\text{rsvd}} \in \mathbb{R}^{n \times \ell}$ is orthogonal, and $\Sigma_{\text{rsvd}} \in \mathbb{R}^{\ell \times \ell}$ is diagonal with non-negative entries. The final step is to define the $m \times \ell$ matrix

$$U_{\text{rsvd}} = Q_{\text{rsvd}}W_{\text{rsvd}}. \quad (10)$$

The end result is an approximate singular value decomposition

$$A \approx U_{\text{rsvd}}\Sigma_{\text{rsvd}}V_{\text{rsvd}}^*,$$

The key claim in this section is that the first $\ell$ columns of the matrix $U$ resulting form PowerURV have exactly the same approximation accuracy as the columns of the matrix $U_{\text{rsvd}}$ resulting from the RSVD, provided that the same random matrix is used. To be precise, we have:
Lemma: Let $A$ be an $m \times n$ matrix, let $\ell$ be a positive integer such that $\ell < \min(m, n)$, let $q$ be a positive integer, and let $G$ be a matrix of size $n \times n$. Let $A = URV^*$ be the factorization resulting from the PowerURV algorithm, as defined by (5) and (6) and using $G$ as the starting point. Let $A \approx U_{rsvd} \Sigma_{rsvd} V_{rsvd}^*$ be the approximate factorization resulting from RSVD, as defined by (4) and (12), starting with $G_{rsvd} = G(:, 1: \ell)$. Suppose that the rank of $(A^* A)^q G_{rsvd}$ is no lower than the rank of $A$ (this holds with probability 1 when $G_{rsvd}$ is Gaussian). Then

$$U(:, 1 : \ell) U(:, 1 : \ell)^* A = U_{rsvd} U_{rsvd}^* A.$$  

Proof. We can without loss of accuracy assume that the matrix $A$ has rank at least $\ell$. (If it is rank deficient, then the proof we give will apply for a modified $\ell' = \text{rank}(A)$, and it is easy to see that adding additional columns to the basis matrices will make no difference since in this case $U(:, 1 : \ell) U(:, 1 : \ell)^* A = U_{rsvd} U_{rsvd}^* A = A$.)

We will prove that $\text{Ran}(U(:, 1 : \ell)) = \text{Ran}(U_{rsvd})$, which immediately implies that the projectors $U(:, 1 : \ell) U(:, 1 : \ell)^*$ and $U_{rsvd} U_{rsvd}^*$ are identical. Let us first observe that restricting (5) to the first $\ell$ columns, we obtain

$$(A^* A)^q G_{rsvd} = V(:, 1 : \ell) Z(1 : \ell, 1 : \ell).$$  

(11)

We can then connect $Y_{rsvd}$ and $U(:, 1 : \ell)$ via a simple computation

$$Y_{rsvd} \overset{6}{\prec} A (A^* A)^q G_{rsvd} \overset{11}{\prec} A V(:, 1 : \ell) Z(1 : \ell, 1 : \ell) \overset{\text{6}}{\prec} U(:, 1 : \ell) R(1 : \ell, 1 : \ell) Z(1 : \ell, 1 : \ell).$$  

(12)

Next we link $U_{rsvd}$ and $Y_{rsvd}$ via

$$U_{rsvd} \overset{10}{\prec} Q_{rsvd} W_{rsvd} \overset{\text{8}}{\prec} Y_{rsvd} R_{rsvd}^{-1} W_{rsvd}.$$  

(13)

Combining (12) and (13), we find that

$$U_{rsvd} = U(:, 1 : \ell) R(1 : \ell, 1 : \ell) Z(1 : \ell, 1 : \ell) R_{rsvd}^{-1} W_{rsvd}.$$  

(14)

The rank assumption implies that the $\ell \times \ell$ matrix $R(1 : \ell, 1 : \ell) Z(1 : \ell, 1 : \ell) R_{rsvd}^{-1} W_{rsvd}$ is non-singular, which establishes that the matrices $U_{rsvd}$ and $U(:, 1 : \ell)$ have the same range. ∎

The equivalency established in the Lemma between RSVD and PowerURV allows for much of the theory for analyzing the RSVD in [5] to directly apply to the PowerURV algorithm. To illustrate the theorem with a specific example, we show in Figures 9 and 10 how well the columns of $A$ rearrange the columns inside $Q_{rsvd}$ to make the leading columns much better aligned with the corresponding singular vectors.)

This effect is visible in Figures 9 and 10 by the fact that the black dots are initially much closer to the minimal errors on the red line than the blue dots. It is only at the very end that the two lines meet. An additional way that the RSVD benefits from the additional application of $A$ is that the columns of $V_{rsvd}$ end up being a far more accurate basis for the row space of $A$ than the columns of the matrix $V$ resulting from the PowerURV. This was of course expected since for $q = 0$, the matrix $V$ incorporates no information from $A$ at all. In Figures 9 and 10 we see this effect by noticing how much smaller the errors marked by the magenta lines are than the errors marked by the green lines.
Figure 9: Comparison of how well the factorizations resulting from the RSVD and PowerURV algorithms reveal numerical rank, as discussed in Section 6. The matrix $A$ is “Matrix 2” described in Section 5 and the RSVD was executed with $\ell = 60$. No power iteration was used for either method (so $q = 0$).

Figure 10: The same experiment as shown in Figure 9, but now with one step of power iteration, so that $q = 1$. 
7 Conclusion

We described the randomized algorithm PowerURV for computing a rank-revealing factorization of a general matrix. The method is computationally efficient since it relies only on matrix-matrix multiplications and unpivoted QR factorizations involving the full matrix. This makes it highly efficient on modern communication constrained hardware. It is also exceptionally simple to implement. The method builds off of existing work by Demmel, Dumtriu, and Holtz [2], and also ties in to work on the Randomized SVD [5]. We compared the speed and accuracy of PowerURV to previously proposed algorithms for computing rank-revealing factorizations, such as the full SVD, the so called QLP method by Stewart [10], and the original randomized method proposed by Demmel, Dumtriu, and Holtz. In the regime where a full factorization is sought (as opposed to cases where the numerical rank of the matrix is far smaller than the matrix dimensions), PowerURV provides an excellent compromise between computational speed on the one hand, and quality in terms of the rank-revealing properties on the other.
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