QUALITATIVE PROPERTIES AND BIFURCATIONS OF A LEAF-EATING HERBIVORES MODEL

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(Communicated by Sze-Bi Hsu)

Abstract. In this paper, we discuss the dynamics of a discrete-time leaf-eating herbivores model. First of all, to investigate the bifurcations of the model, we study the qualitative properties of a fixed point, including hyperbolic and non-hyperbolic. Secondly, applying the center manifold theorem, we give the conditions that the model produces a supercritical flip bifurcation and a subcritical flip bifurcation respectively, from which we find a generalized flip bifurcation point. And then, we prove rigorously that the model undergoes a generalized flip bifurcation and give three parameter regions that the model possesses two period-two cycles, one period-two cycles and none respectively. Next, computing the normal form, we prove that the model undergoes a subcritical Neimark-Sacker bifurcation and produces a unique unstable invariant circle near the fixed point. Finally, by numerical simulations, we not only verify our results but also show a saddle period-five cycle and a saddle period-six cycle on the invariant circle.

1. Introduction. The plant-herbivores interactions are the most fundamental and important processes in ecological system and play an important role in understanding community dynamics and ecosystem (see e.g., Maron, Agrawal and Schemske [20], Kariyat and Portman [10], Erb and Reymond [7] and references there). Over the past three decades, the plant-herbivores interactions have been attracting considerable attention (see e.g., Zhao et al [25], Khan, Ma and Xiao [12], Li et al [14] and references there) and are usually modeled by differential equations and difference equations (or mappings). For populations with overlapping generations, the interactions are usually modeled by differential equations because the birth processes happen continuously. Very rich and complex dynamics have been shown in continuous-time plant-herbivores systems (see e.g., Castellanos and Sánchez-Garduño [3], Li, Zhen and Jing [15], Liu et al [18], Sun et al [22], Zhao et al [25] and references there). However, for populations with non-overlapping generations, such as monocarpic plants and semelparous animals, their births occur in regular breeding seasons. So, their interactions are described by discrete-time models, difference equations or mappings (see e.g., Allen, Hannigan and Strauss [1],
Edelstein-Keshet [5] and references there). Generally, discrete-time models can exhibit more complicated dynamics than the corresponding continuous-time models (see e.g., Huang et al [9], Li and Zhang [16], Liu and Xiao [17], Lorenz [19], May [21] and references there). In [5, pp.105-106], Edelstein-Keshet introduced the following discrete-time leaf-eating herbivores model

\[
\begin{aligned}
&v_{n+1} = f v_n \exp(-ah_n), \\
&h_{n+1} = rh_n(\delta - \frac{h_n}{v_n}),
\end{aligned}
\]  

(1)

where \(v_n > 0\) and \(h_n \geq 0\) denote the leaf mass and the population size of leaf-eating herbivores on a tree at time \(n\) respectively, and the parameters \(f, a, r\) and \(\delta\) are all positive. With the replacement

\[
v_n = \frac{r \ln(f)}{(r\delta - 1)a} x_n, \quad h_n = \frac{\ln(f)}{a} y_n,
\]

Edelstein-Keshet reduce system (1) to the following system

\[
\begin{aligned}
&x_{n+1} = x_n \exp(k(1 - y_n)), \\
y_{n+1} = by_n \left(1 + \frac{1}{b} - \frac{ka}{x_n}\right),
\end{aligned}
\]  

(2)

where \(k = \ln(f)\) and \(b = r\delta - 1\) are both positive, and showed that system (2) has a unique fixed point \(E : (1, 1)\). In 2016 Khan and Qureshi investigated the local and global behavior of \(E\) and the rate of convergence of positive solutions in [11].

Up to now, it is not clear what qualitative properties the fixed point \(E\) has. It is of interest to know what bifurcations happen in the system (2). In this paper, we dealt with these questions. More specifically, in section 2, for later analysis of bifurcations, discussing the eigenvalues of fixed point \(E\), we prove the qualitative properties of the fixed point and list all non-hyperbolic cases. Section 3 deals with the flip bifurcation. Applying the center manifold theorem, we present the parameter regions where system (2) produces a period-two cycle. By the analysis in this section, we find that the system has a generalized flip bifurcation point and may produce a generalized flip bifurcation. Furthermore, computing the Schwarzian derivative, we prove the qualitative properties of \(E\) with one eigenvalue \(-1\) on the unit circle. In section 4, as the parameter varies near the generalized flip bifurcation point, computing a local center manifold of four degree and the normal form and analyzing the second iteration of the normal form, we investigate the codimension 2 bifurcation and give three regions where system (2) possesses two period-two cycles, one period-two cycle and none near the fixed point \(E\) respectively. In addition, computing the fifth order derivative of a dimension-1 mapping, which is obtained from restricting the model into a center manifold of four degree, and applying the results of [4], we prove the qualitative property of \(E\) as the parameter lies at the degenerated flip bifurcation point. Section 5 is devoted to the study of the Neimark-Sacker bifurcation. Computing the normal form and the first Lyapunov number, we prove that system (2) undergoes subcritical Neimark-Sacker bifurcation and present the parameter conditions that system (2) produces a unique unstable invariant circle near the fixed point \(E\). Furthermore, from the normal form in polar coordination, we give the qualitative properties of \(E\) with a pair of conjugate complex eigenvalues. At last, in section 6, using the software MAPLE 18, we numerically simulate a stable period-two cycle, from which we find that the model produces chaos, and a unstable invariant circle for verifying our results. Furthermore, using the software MATLAB version R2014a, we simulate a saddle period-five cycle and a saddle period-six cycle, respectively, on the invariant circle produced from the Neimark-Sacker bifurcation.
2. Qualitative properties of fixed point. In this section we investigate the qualitative properties of fixed point $E : (1, 1)$ and list all non-hyperbolic cases in order for further study of bifurcations in next sections. The dynamics of system (2) can be described by the planar mapping $F : \mathbb{R}_+^2 := \{(x, y) \in \mathbb{R}^2 | x > 0, y \geq 0 \} \rightarrow \mathbb{R}_+^2$.

\[
F(x, y) = \left( x \exp(k(1 - y)), by \left( 1 + \frac{1}{b} - \frac{y}{x} \right) \right).
\]

The qualitative properties of its fixed point $E$ are given as follows.

**Proposition 1.** The fixed point $E$ is non-hyperbolic if and only if the parameter $(k, b)$ either lies on one of curves $\mathcal{L}_1 := \{(k, b)|b = 4/(2 - k), 0 < k < 1\}$, $\mathcal{L}_2 := \{(k, b)|b = 4/(2 - k), 1 < k < 2\}$ and $\mathcal{L}_3 := \{(k, b)|k = 1, 0 < b < 4\}$ or lies at point $(1, 4)$ (see figure (1)). Otherwise, the fixed point is in one of the qualitative properties shown in Table 1.

![Figure 1. Diagram of bifurcation for system (2).](image)

| Conditions | $E_1$ | Cases |
|------------|-------|-------|
| $0 < k < 1$ | $0 < b < 4k$ | stable focus | $\mathcal{D}_5$ |
| $4k \leq b < 4/(2 - k)$ | stable node | $\mathcal{D}_1$ |
| $k = 1$ | $b > 4$ | saddle point | $\mathcal{D}_2$-I |
| $1 < k < 2$ | $0 < b < 4k$ | unstable focus | $\mathcal{D}_4$-I |
| $4k \leq b < 4/(2 - k)$ | unstable node | $\mathcal{D}_3$-I |
| $b > 4/(2 - k)$ | saddle point | $\mathcal{D}_2$-II |
| $k \geq 2$ | $0 < b < 4k$ | unstable focus | $\mathcal{D}_3$-II |
| $b \geq 2k$ | unstable node | $\mathcal{D}_3$-III |

**Table 1.** Topological types of fixed point $E$ in the hyperbolic case.
Proof. The matrix of linearization of mapping $F$ at $E$ is

$$DF(E) = \begin{bmatrix} e^{k(1-y)} & -kxe^{k(1-y)} \\ \frac{by^2}{x^2} & b + 1 - 2by \end{bmatrix}_{(k,b)=(1,1)} = \begin{bmatrix} 1 & -k \\ b & 1 - b \end{bmatrix}$$

and its eigenvalues are

$$\lambda_1 = 1 - b - \frac{1}{2}\sqrt{b^2 - 4bk}; \quad \lambda_2 = 1 - b - \frac{1}{2}\sqrt{b^2 - 4bk}. \quad (4)$$

It is known that $E$ is non-hyperbolic if and only if $\lambda_1$ or $\lambda_2$ lies on the unit circle $S^1$. In the case that $\lambda_1$ and $\lambda_2$ are both real, non-hyperbolic happens if and only if $\lambda_1 = \pm 1$ or $\lambda_2 = \pm 1$. Note that

$$\lambda_1 = 1 - \frac{1}{2}(b - \sqrt{b^2 - 4bk}) < 1, \quad \lambda_2 = 1 - \frac{1}{2}(b + \sqrt{b^2 - 4bk}) < 1 \quad (5)$$

since $k > 0$ and $b > 0$, which implies that $E$ is non-hyperbolic if and only if $\lambda_1 = -1$ or $\lambda_2 = -1$. One can check that $\lambda_1 = -1$ and $\lambda_2 = 3 - b < -1$ if $(k,b) \in \mathcal{L}_2$, that $\lambda_1 = 3 - b > -1$ and $\lambda_2 = -1$ if $(k,b) \in \mathcal{L}_1$, and that $\lambda_1 = \lambda_2 = -1$ if $(k,b) = (1,4)$.

In the case that $\lambda_1$ and $\lambda_2$ are a pair of conjugate complex, i.e., $b^2 - 4bk < 0$ and lie on $S^1$ if and only if $|\lambda_1| = |\lambda_2| = b(k-1) + 1 = 1$, implying that $k = 1$, i.e. $(k,b) \in \mathcal{L}_4$.

For hyperbolic cases, the proof is divided into four steps: $0 < k < 1$, $k = 1$, $1 < k < 2$ and $k \geq 2$.

In the case $0 < k < 1$, it is obvious that $b \neq 4/(2-k)$ because of hyperbolic. Our discussions are as follows:

(a) if $0 < b < 4k$, from (4), we see that $\lambda_1$ and $\lambda_2$ are a pair of complexes and satisfy $|\lambda_1| = |\lambda_2| = 1 + b(k-1) < 1$, means that $E$ is a stable focus (referred to the case $\mathcal{D}_5$).

(b) if $4k \leq b < 4/(2-k)$, i.e., $2 - 4/b < k \leq b/4$, then

$$-1 = 1 - \frac{1}{2}b \leq 1 - \frac{1}{2}b + \frac{1}{2}\sqrt{b^2 - 4b} < \lambda_1 < 1, \quad -1 = 1 - \frac{b}{2} - \frac{1}{2}(4-b) = 1 - \frac{b}{2} - \frac{1}{2}\sqrt{b^2 - 4b(2 - \frac{b}{2})} < \lambda_2 < 1,$$

imply that $E$ is a stable node (referred to the case $\mathcal{D}_1$).

(c) if $b > 4/(2-k)$, i.e., $k < 2 - 4/b$, then

$$\lambda_1 > 1 - \frac{b}{2} + \frac{1}{2}\sqrt{b^2 - 4b(2 - \frac{b}{2})} = 1 - \frac{b}{2} + \frac{1}{2}|b-4|, \quad \lambda_2 < 1 - \frac{b}{2} - \frac{1}{2}\sqrt{b^2 - 4b(2 - \frac{b}{2})} = 1 - \frac{b}{2} - \frac{1}{2}|b-4|,$$

from which we obtain that $\lambda_1 > 3 - b > -1$ and $\lambda_2 < -1$ for $b < 4$ and that $\lambda_1 > -1$ and $\lambda_2 < 3 - b \leq -1$ for $b \geq 4$. By (5), we have $-1 < \lambda_1 < 1$ and $\lambda_2 < -1$, implying that $E$ is a saddle point (referred to the case $\mathcal{D}_2$-I).

In the case $k = 1$, it is easy to see that $b > 4$ for hyperbolic. From

$$\frac{d}{db}\lambda_1 = \frac{2}{\sqrt{b^2 - 4b((b-2)) + \sqrt{b^2 - 4b}}} > 0$$

and

$$\frac{d}{db}\lambda_2 = \frac{1}{2}(1 + \frac{b-2}{\sqrt{b^2 - 4b}}) < 0,$$

it follow that $\lambda_1 |_{b=4} = -1$ and $\lambda_2 |_{b=4} = -1$. Hence, by (5), we obtain that $E$ is a saddle (referred to the case $\mathcal{D}_2$-II).
In the case $1 < k < 2$, it is clear that $k \neq 4/(2 - k)$ because of hyperbolic. We discuss as follows:

(i) if $0 < b < 4k$, it is easy to see that $\lambda_1$ and $\lambda_2$ are a pair of complexes and satisfy $|\lambda_1| = |\lambda_2| = 1 + b(k - 1) > 1$, implying that $E$ is an unstable focus (referred to the case $D_1$-I).

(ii) if $4k \leq b < 4/(2 - k)$, i.e., $2 - 4/b < k \leq b/4$, then

\[
\begin{align*}
\lambda_1 &= 1 - \frac{b}{2} + \frac{1}{2} \sqrt{b^2 - 4b(2 - \frac{4}{b})} = 1 - \frac{b}{2} + \frac{1}{2} (b - 4) = -1, \\
\lambda_2 &\leq 1 - \frac{b}{2} - \frac{1}{2} \sqrt{b^2 - 4b}\cdot \frac{b}{2} = 1 - \frac{1}{2}b < -1
\end{align*}
\]

since $b > 4$, implying that $E$ is a unstable node (referred to the case $D_3$-I).

(iii) if $b > 4/(2 - k)$, i.e., $k < 2 - 4/b$, then

\[
\begin{align*}
\lambda_1 &= 1 - \frac{b}{2} + \frac{1}{2} \sqrt{b^2 - 4b(2 - \frac{4}{b})} = 1 - \frac{b}{2} + \frac{1}{2} (b - 4) = -1, \\
\lambda_2 &< 1 - \frac{b}{2} - \frac{1}{2} \sqrt{b^2 - 4b(2 - \frac{4}{b})} = 1 - \frac{b}{2} - \frac{1}{2} (b - 4) = 3 - b < -1
\end{align*}
\]

because of $b > 4$. So, $E$ is a saddle point (referred to the case $D_2$-III).

In the case $k \geq 2$, if $0 < b < 4k$, $\lambda_1$ and $\lambda_2$ are a pair of conjugate complexes and $|\lambda_1| = |\lambda_2| > 1$, which means that $E$ is an unstable focus (referred to the case $D_1$-II). If $b \geq 4k$, implying that $b \geq 8$, then

\[
\begin{align*}
\lambda_1 &= 1 - \frac{b}{2} + \frac{1}{2} \sqrt{b^2 - 8b} < 1 - \frac{b}{2} + \frac{1}{2} \sqrt{b^2 - 8b + 16} = -1, \\
\lambda_2 &< 1 - \frac{b}{2} - \frac{1}{2} \sqrt{b^2 - 4b} = 1 - \frac{b}{2} \leq -3 < -1
\end{align*}
\]

implying that $E$ is an unstable node (referred to the case $D_3$-II). The proof is completed.

When the parameter $(k, b)$ crosses $L_1, L_2$ and $L_4$, mapping $F$ may produce bifurcations. In the next sections, we will discuss what bifurcations will happen. Furthermore, we will present the qualitative properties of $E$ in the case that the parameter $(k, b)$ lies in the curves $L_1, L_2$ and $L_4$. This is because the proofs of these qualitative properties rely on the center manifolds and the normal forms, which are used in computing the transversality conditions and non-degeneracy conditions of bifurcations.

3. **Flip bifurcation.** In this section, we will discuss whether system (2) undergoes flip bifurcation as $(k, b)$ crosses $L_1$ and $L_2$ respectively.

**Theorem 1.** If the parameter $(k, b)$ crosses the curve $L_1$ from the region $D_1$ to the region $D_2$, then system (2) undergoes a supercritical flip bifurcation and produces a stable period-two cycle near the fixed point $E$.

**Proof.** Let $\epsilon = b - 4/(2 - k)$. Translating $E$ to the origin $O$ with the transformation $(x, y) := (u_1 + 1, u_2 + 1)$ in mapping (3) and then expanding it in the Taylor series, we get the mapping $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$,

\[
\begin{pmatrix}
u_1 \\
u_2
\end{pmatrix} \mapsto \begin{pmatrix}
u_1 \\
u_2
\end{pmatrix} = \begin{pmatrix}
u_1 \\
u_2
\end{pmatrix} - k\nu_2 + f_1(u_1, u_2) + O(|u_1, u_2|^4) + f_2(u_1, u_2) + O(|u_1, u_2|^4),
\]

\[(6)\]

where

\[
\begin{align*}
f_1(u_1, u_2) &= -k u_1 u_2 + \frac{1}{7} k^2 u_2^2 + \frac{1}{7} k^2 u_1 u_2^2 - \frac{1}{7} k^3 u_1^2 u_2^2, \\
f_2(u_1, u_2) &= -\frac{4 + 2 \epsilon - k}{2 - k} u_1^2 + \frac{2(4 + 2 \epsilon - k)}{2 - k} u_1 u_2^2 - \frac{4 + 2 \epsilon - k}{2 - k} u_2^3 + \frac{4 + 2 \epsilon - k}{2 - k} u_1^3 - \frac{2(4 + 2 \epsilon - k)}{2 - k} u_1^2 u_2 + \frac{4 + 2 \epsilon - k}{2 - k} - u_1 u_2^2.
\end{align*}
\]
One can check that the Jacobian matrix

$$J\Phi(O) = \begin{bmatrix} 1 & -k \\ \frac{4+2\epsilon-k}{2-k} & -\frac{2+k+2\epsilon-k}{2-k} \end{bmatrix}$$

has eigenvectors $(1, V_1(\epsilon))^T$ and $(1, V_2(\epsilon))^T$, where $T$ denotes the transpose of vectors and matrices,

$$V_1(\epsilon) := \frac{4 + 2 \epsilon - \epsilon k - \sqrt{-(\epsilon k - 2 \epsilon - 4) (4 k^2 - \epsilon k - 8 k + 2 \epsilon + 4)}}{2k (2 - k)}$$

and

$$V_2(\epsilon) := \frac{4 + 2 \epsilon - \epsilon k + \sqrt{-(\epsilon k - 2 \epsilon - 4) (4 k^2 - \epsilon k - 8 k + 2 \epsilon + 4)}}{2k (2 - k)}$$

corresponding to eigenvalues

$$\lambda_1 = \frac{-2k - 2 \epsilon + \epsilon k + \sqrt{-(\epsilon k - 2 \epsilon - 4) (4 k^2 - \epsilon k - 8 k + 2 \epsilon + 4)}}{2(2 - k)}$$

and

$$\lambda_2 = \frac{-\epsilon k + 2 k + 2 \epsilon + \sqrt{-(\epsilon k - 2 \epsilon - 4) (4 k^2 - \epsilon k - 8 k + 2 \epsilon + 4)}}{2(2 - k)}$$

respectively. Applying the transformation

$$(u_1, u_2)^T = H_1(v_1, v_2)^T,$$

where

$$H_1 := \begin{bmatrix} 1 & 1 \\ V_1(\epsilon) & V_2(\epsilon) \end{bmatrix},$$

to diagonalize the linear part of mapping (6), in the case $0 < k < 1$ we get the following mapping

$$\left( \begin{array}{c} v_1 \\ v_2 \end{array} \right) \rightarrow \left( \begin{array}{c} \frac{(2-3k)}{2-k} v_1 + g_1(v_1, v_2, \epsilon) + O(|(v_1, v_2, \epsilon)|^4) \\ v_2 + g_2(v_1, v_2, \epsilon) + O(|(v_1, v_2, \epsilon)|^4) \end{array} \right),$$

where

$$g_1(v_1, v_2, \epsilon) = -\frac{k^2}{4(k-1)} v_1^2 + \frac{(k^2 - 6 k + 4) k}{(k-2)^2 (k-1)} v_1^2 - \frac{2 (2 k - 1)}{k-1} v_1 v_2 \left( \frac{(2k^3 - 15k^2 + 22k - 8) k^2}{64 (k-1)^3 v_1^2 + \frac{2}{3 (k-2)^2 (k-1)} v_1^3 \right)$$

$$+ \frac{(3k^2 - 3k + 1) (k-2)^2}{4 (k-1)^3} v_2 v_1^2 - \frac{(5k - 4) (k-2)^3}{8k (k-1)^3} v_2^2 \epsilon + \frac{2 (2k^2 - 3k + 2)}{(k-1) k} v_1 v_2 + \frac{2 (2k - 3) (k-2)}{3 (k-1) k} v_2^3, $$

and

$$g_2(v_1, v_2, \epsilon) = \frac{(3k^2 - 2k) k^2}{(k-2)^2 (k-1)} v_1^2 + \frac{(k-2)^2}{4(k-1)} v_2^2 - \frac{2 k}{(k-2) (k-1)} v_1 v_2 \left( \frac{(4 k^3 - k^2 - 8k + 4) k^2 v_1^2}{8 (k-2) (k-1)} \right) $$

$$+ \frac{(k-2)^2}{(k-1) k} v_2^2 - \frac{(4 k^3 - k^2 - 8k + 4) k^2 v_1^2}{8 (k-2) (k-1)}.$$
Choosing $\epsilon$ as the bifurcation parameter, we rewrite system (8) in the suspended form

$$\begin{pmatrix} v_1 \\ \epsilon \\ v_2 \end{pmatrix} \mapsto \begin{pmatrix} (2-3k) v_1 \\ \epsilon \\ -v_2 \end{pmatrix} + g_1(v_1, v_2, \epsilon) + O(|(v_1, v_2, \epsilon)|^4)$$

so as to involve the parameter $\epsilon$ explicitly in the discussion. Having exact two eigenvalues on the unit circle $S^1$, by the center manifold theorem [2, pp.33-35], system (9) has a two-dimensional $C^2$ center manifold, which has a second order approximate expression

$$v_1 = \phi(v_2, \epsilon) = a_{20} v_2^2 + a_{11} v_2 \epsilon + a_{02} \epsilon^2 + O(|(v_2, \epsilon)|^3)$$

near the origin, where the coefficients $a_{20}, a_{11}$ and $a_{02}$ are all indeterminate. By the invariance of center manifold, from (9) and (10), we get

$$\phi(-v_2 + g_2(\phi(v_2, \epsilon), v_2, \epsilon) + O(|(v_2, \epsilon)|^4), \epsilon)$$

$$= \frac{(2-3k)}{2-k} \phi(v_2, \epsilon) + g_1(\phi(v_2, \epsilon), v_2, \epsilon) + O(|(v_2, \epsilon)|^4),$$

which is equivalent to the following equation

$$a_{20} v_2^2 - a_{11} v_2 \epsilon + a_{02} \epsilon^2 + O(|(v_2, \epsilon)|^3)$$

$$= \frac{(3k-2) a_{11}}{k-2} v_2 + \frac{(3k-2) a_{02}}{k-2} \epsilon^2 + O(|(v_2, \epsilon)|^3).$$

Comparing the coefficients of terms $v_2^2, v_2 \epsilon$ and $\epsilon^2$ in (11), we have

$$a_{20} = \frac{(3k-5k^2+2k)}{(k-2)(k-1)k} a_{20} - \frac{k^3+6k^2-12k+8}{k-2},$$

$$-a_{11} = \frac{(3k-2) a_{11}}{k-2},$$

which imply

$$a_{20} = \frac{(k-2)^3}{2k^2(k-1)}, \quad a_{11} = 0, \quad a_{02} = 0.$$ 

Therefore, the expression (10) has the specifical form

$$v_1 = \frac{(k-2)^3}{2k^2(k-1)} v_2^2 + O(|(v_2, \epsilon)|^3).$$

Substituting (12) into the last equation of (9) yields a one-dimensional mapping

$$v_2 \mapsto G(v_2, \epsilon) := -v_2 + \frac{(k-2)^2}{4(k-1)} v_2 + \frac{(k-2)^2}{(k-1)k} v_2^2 + \frac{k^2(k-2)^3}{64(k-1)^3} v_2 \epsilon^2.$$
In the case 1

Proof. If \( k \) and \( b \) are given, then the mapping satisfies (13) and (14) respectively, implying that mapping \( G \) undergoes a flip bifurcation at \( (k,b) \). Hence, system (2) possesses a period-two cycle as \( (k,b) \) lies in region \( D_2 \) and is near the curve \( \mathcal{L}_1 \). Moreover, we see that \( 0 < k < 1 \) and \( b > 4/(2-k) \). Therefore, the period-two cycle of system (2) is stable near the fixed point \( E \). The proof is completed.

In addition, by mapping \( G \), the qualitative properties of \( E \) can be proved for \( (k,b) \in \mathcal{L}_1 \).

**Proposition 2.** If \( (k,b) \in \mathcal{L}_1 \), then the fixed point \( E \) is a stable node.

Proof. In the case \( \epsilon = 0 \), the Schwarzian derivative of \( G \) at \( v_2 = 0 \),

\[
SG(0,0) = \left\{ \frac{\partial^3 G}{\partial v_2^3} / \frac{\partial G}{\partial v_2} - \frac{1}{2} \left( \frac{\partial^2 G}{\partial v_2^2} / \frac{\partial G}{\partial v_2} \right)^2 \right\}_{(v_2,\epsilon)=(0,0)}
\]

since \( 0 < k < 1 \). By Theorem 1.16 in [6, p.32], the fixed point \( v_2 = 0 \) of mapping \( G(v_2,0) \) is stable. Moreover, from \( -1 < \lambda_1 < 1 \), it follows that the fixed point \( E \) of system (2) is a stable node in the case \( (k,b) \in \mathcal{L}_1 \). The proof is completed.

Furthermore, from system (6), we have the following results.

**Theorem 2.** If the parameter \( (k,b) \) crosses the curve \( \mathcal{L}_2 \) with \( k \neq 4 - \sqrt{16} \), then system (2) undergoes the flip bifurcation. More specifically, a subcritical (resp. supercritical) flip bifurcation occurs as \( (k,b) \) crosses the curve \( \mathcal{L}_2 \) with \( k \in (1,4-\sqrt{16}) \) (resp. \( k \in (4-\sqrt{16},2) \)) from region \( D_3 \) (resp. \( D_2 \)) to region \( D_2 \) (resp. \( D_3 \)), and an unstable period-two cycle arises near the fixed point \( E \).

Proof. In the case \( 1 < k < 2 \), by the transformation (7), system (6) is changed into the following mapping

\[
\begin{pmatrix}
v_1 \\
v_2
\end{pmatrix} \mapsto \begin{pmatrix}
-\frac{(k-2)^2}{4(k-1)}v_1 + \eta_1(v_1,v_2,\epsilon) + O(|(v_1,v_2,\epsilon)|^4) \\
\frac{3(k-2)}{2k^2}v_2 + \eta_2(v_1,v_2,\epsilon) + O(|(v_1,v_2,\epsilon)|^4)
\end{pmatrix},
\]

where

\[
\eta_1(v_1,v_2,\epsilon) = \frac{(k-2)^2}{4(k-1)}v_1 + \frac{(k-2)^2}{(k-1)k}v_2^2 - \frac{2k}{(k-2)(k-1)}v_1v_2
\]
which is equivalent to the following equation
c
where

suspended system of (15) from (16) and (17), it follows that
By the center manifold theorem, having two eigenvalues on the unit circle

\[
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}
= 3
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}
\] + O((v_1, v_2, \epsilon)^4),

\[
\eta_2(v_1, v_2, \epsilon) = \frac{(k-2)^2}{(k-1)k} v_1^2 + \frac{k^2}{4(k-1)} v_2\epsilon - \frac{2(k-2)}{k-1} v_1 v_2
\]

\[
+ \frac{(k^2-6k+4)k}{64(k-1)^3} v_2^2 - \frac{(5k-4)(k-2)^3}{8k(k-1)^3} v_1^2 \epsilon
\]

\[
+ \frac{2(2k-3)(k-2)}{3(k-1)k} v_1^3 - \frac{(k-2)^3 k^2}{64(k-1)^3} v_2^2 \epsilon
\]

\[
+ \frac{3(k^2-3k+1)(k-2)^2}{4(k-1)^3} v_1 v_2 \epsilon + \frac{2(2k^2-3k+2)}{(k-1)k} v_1^2 v_2
\]

\[
- \frac{2(k^3-15k^2+22k-8)}{8(k-2)(k-1)^3} v_2^2 \epsilon + \frac{2k^2}{(k-2)^2(k-1)} v_1 v_2^2
\]

\[
+ \frac{2(4k-3)k^2 v_2^3}{3(k-2)^2(k-1)} + O((v_1, v_2, \epsilon)^4).
\]

By the center manifold theorem, having two eigenvalues on the unit circle \(S^1\), the suspended system of (15)

\[
\begin{pmatrix}
v_1 \\
\epsilon \\
v_2
\end{pmatrix}
\rightarrow
\begin{pmatrix}
-v_1 \\
\epsilon \\
+\eta_1(v_1, v_2, \epsilon) + O((v_1, v_2, \epsilon)^4)
\end{pmatrix}
\]

has a local \(C^2\) center manifold passing through the point \((v_1, \epsilon) = (0, 0)\). Hence, the manifold has the following approximate expression

\[
v_2 = \psi(v_1, \epsilon) = c_{20}v_1^2 + c_{11}v_1 \epsilon + c_{02} \epsilon^2 + O((v_1, \epsilon)^3),
\] (17)

where \(c_{20}, c_{11}\) and \(c_{02}\) are all indetermination. By the invariance of center manifold, from (16) and (17), it follows that

\[
\psi(-v_1 + \eta_1(v_1, \psi(v_1, \epsilon), \epsilon) + O((v_1, \epsilon)^4), \epsilon)
\]

\[
= \frac{3k-2}{k-2} \psi(v_1, \epsilon) + \eta_2(v_1, \psi(v_1, \epsilon), \epsilon) + O((v_1, \epsilon)^4),
\]

which is equivalent to the following equation

\[
c_{20}v_1^2 - c_{11}v_1 \epsilon + c_{02} \epsilon^2 + O((v_1, \epsilon)^3)
\]

\[
= \frac{(3k^3 c_{20} - k^3 - 5k^2 c_{20} + 6k^2 + 2k c_{20} - 12k + 8)}{(k-2)(k-1)k} v_1^2
\]
Comparing the coefficients of terms $v_1^2, v_1\epsilon$ and $\epsilon^2$ in (18), we obtain
$$c_{20} = \frac{(k-2)^3}{2k^2(k-1)}, \quad c_{11} = 0, \quad c_{02} = 0.$$ because of $1 < k < 2$. So, the center manifold (17) has the specific form
$$v_2 = \frac{(k-2)^3}{2k^2(k-1)} v_1^2 + O(|(v_1, \epsilon)|^3).$$ Substituting (19) into the first equation of (16), we obtain a one-dimensional mapping
$$v_1 \mapsto \Psi(v_1, \epsilon) = -v_1 + \frac{(k-2)^2}{4(k-1)} v_1 + \frac{(k-2)^2}{8(k-1)^2} v_1^2 - \frac{(k-3)(k-4)}{3(k-1)^2} v_1^3 - \frac{3(k-4)(k-4)}{8(k-1)^3} v_1^3 + \frac{(k-2)^2}{64(k-1)^3} \epsilon^2 v_1 + O(|(v_1, \epsilon)|^4).$$ It is easy to compute that
$$\alpha := \left( \frac{1}{2} \left( \frac{\partial^2 \Psi}{\partial v_1^2} \right)^2 + \frac{1}{3} \left( \frac{\partial^3 \Psi}{\partial v_1^3} \right) \right) \Big|_{(v_1, \epsilon) = (0, 0)} = \frac{2(k^3 - 12k^2 + 48k - 48)}{3k^2(k-1)}$$ and
$$\left. \frac{\partial^2 \Psi}{\partial v_1 \partial \epsilon} \right|_{(v_1, \epsilon) = (0, 0)} = \frac{(k-2)^2}{4(k-1)} > 0$$ for $k \in (1, 2)$. Obviously, the sign of $\alpha$ is determined by the polynomial in the numerator in the case $1 < k < 2$. One can check that for $k \in (1, 2)$ the strictly increasing function $F(k) := k^3 - 12k^2 + 48k - 48$ has a unique zero point $k = 4 - \sqrt[3]{16} \approx 1.480158$, implying that $\alpha < 0$ in the case $1 < k < 4 - \sqrt[3]{16}$ and $\alpha > 0$ in the case $4 - \sqrt[3]{16} < k < 2$. So, if $k \neq 4 - \sqrt[3]{16}$, then the conditions (B.1) and (B.2) of Theorem 4.3 in [13, p.121] are satisfied by (20) and (21) respectively, implying that mapping $\Psi$ undergoes a flip bifurcation at $(v_1, \epsilon) = (0, 0)$ and produces a period-two cycle near the fixed point $O$. More specifically, if $1 < k < 4 - \sqrt[3]{16}$, mapping $\Psi$ undergoes a subcritical flip bifurcation and produces an unstable period-two cycle for small $\epsilon > 0$. If $4 - \sqrt[3]{16} < k < 2$, mapping $\Psi$ undergoes a supercritical flip bifurcation and produces a stable period-two cycle if $\epsilon$ is negative and near 0. Note that $\lambda_2 < -1$ in the case $1 < k < 2$ and $b > 4k$. Therefore, when $(k, b)$ is near $\mathcal{L}_2$, the period-two cycle of system (2) is unstable not only in the case $1 < k < 4 - \sqrt[3]{16}$ and $(k, b) \in \mathcal{D}_2$ but also in the case $4 - \sqrt[3]{16} < k < 2$ and $(k, b) \in \mathcal{D}_3$. The proof is completed. \[ \square \]

From mapping $\Psi$ in the above proof, we can prove the qualitative properties of $E$ in the case $(k, b) \in \mathcal{L}_2$, i.e., $\epsilon = 0$.

**Proposition 3.** If the parameter $(k, b)$ lies in $\mathcal{L}_2$ with $1 < k < 4 - \sqrt[3]{16}$, then the fixed point $E$ of mapping $F$ is an unstable node; if $(k, b)$ lies in $\mathcal{L}_2$ with $k > 4 - \sqrt[3]{16}$, then $E$ is a saddle.

**Proof.** In the case $\epsilon = 0$, the one-dimensional mapping,
$$\Psi(v_1) = -v_1 + \frac{(k-2)^2}{(k-1)} v_1^2 - \frac{(2k-3)(k-4)}{3(k-1)^2} v_1^3 + O(v_1^4),$$
has the Schwarzian derivative at \( v_1 = 0 \),
\[
S\Psi(0) = \frac{\Psi'''(0)}{\Psi'(0)} - 3 \left( \frac{\Psi''(0)}{\Psi'(0)} \right)^2 = -\frac{2(k^3 - 12k^2 + 48k - 48)}{k^2(k - 1)}.
\]
So, by Theorem 1.16 in [6], we obtain that in the case \( 1 < k < 4 - \sqrt{16} \) the fixed point \( v_1 = 0 \) of mapping \( \Psi \) is unstable since \( S\Psi(0) > 0 \) and that in the case \( 4 - \sqrt{16} < k < 2 \) it is stable because of \( S\Psi(0) < 0 \). On the other hand, note that \( \lambda_2 < -1 \) for \((k, b) \in \mathcal{L}_2 \). Therefore, if \((k, b) \in \mathcal{L}_2 \), then the fixed point \( E \) of system (2) is an unstable node if \( 1 < k < 4 - \sqrt{16} \) and a saddle if \( 4 - \sqrt{16} < k < 2 \). The proof is completed.

From theorem 2, we see that the condition (B.1) of Theorem 4.3 in [13, p.121] on flip bifurcation is not satisfied as \( k = 4 - \sqrt{16} \). In the next section, we will investigate this case.

4. Generalized flip bifurcation. From (20), we see that \( \alpha = 0 \) if \((k, b) \in \mathcal{L}_2 \) and \( k = 4 - \sqrt{16} \), i.e., \((k, b) = (4 - \sqrt{16}, 2/(\sqrt{2} - 1))\), implying that system (2) may undergoes a generalized flip bifurcation, as called in [13]. In what follows, we will investigate the codimension-2 bifurcation as \((k, b) \) varies near the point \((4 - \sqrt{16}, 2/(\sqrt{2} - 1))\).

Theorem 3. If \((k, b) \) varies in the neighborhood of the point \( GF : (4 - \sqrt{16}, 2/(\sqrt{2} - 1)) \), which lies in \( \mathcal{L}_2 \) in the figure 1, then system (2) undergoes a generalized flip bifurcation near the fixed point \( E \). More specifically, the following bifurcations occur (see figure 2):

![Figure 2. Bifurcation diagram of system (2) near the point GF.](image-url)
(1) If \((k, b)\) crosses the curve \(\mathcal{L}_{21} := \{(k, b)|b = 4/(2 - k), k < 4 - \sqrt{16}\}\) from the region 
\[ \mathcal{D}_{31} := \{(k, b)|b < \frac{4}{2 - k}, k \leq 4 - \sqrt{16}\} \cup \{(k, b)|b < T(k), k > 4 - \sqrt{16}\}, \]
where
\[ T(k) := \frac{2}{\sqrt{2} - 1} + \frac{4}{3(\sqrt{2} - 1)^2} (k - 4 + \sqrt{16}) \]
\[ - \frac{2}{8(\sqrt{2} + 16)} \cdot \frac{3 + 16k^2 + 32k}{(\sqrt{2} - 1)^2} (k - 4 + \sqrt{16})^2 \]
\[ + O((k - 4 + \sqrt{16})^3), \]

\text{to the region} \(\mathcal{D}_2\), \text{as shown in figure 1, then system (2) undergoes a subcritical flip bifurcation and produces a period-two cycle.}

(2) If \((k, b)\) crosses the curve \(\mathcal{L}_{22} := \{(k, b)|b = 4/(2 - k), k > 4 - \sqrt{16}\}\) from the region \(\mathcal{D}_2\) to the region \(\mathcal{D}_{32} := \{(k, b)|T(k) < b < 4/(2 - k), k > 4 - \sqrt{16}\}\), then system (2) possesses two period-two cycles.

(3) As \((k, b)\) crosses a curve \(\mathcal{T} := \{(k, b)|b = T(k), k > 4 - \sqrt{16}\}\) from the region \(\mathcal{D}_{22}\) to the region \(\mathcal{D}_{31}\), the two period-two cycles coincide and disappear.

\text{Proof. Using the transformation} \((x, y) = (u_1 + 1, u_2 + 1)\) to translate the fixed point \(E\) to the origin and expanding it in the Taylor series, we obtain the following mapping
\[ \left( \begin{array}{c} u_1 \\ u_2 \end{array} \right) \mapsto \left( \begin{array}{c} u_1 - ku_2 + \hat{f}_1(u_1, u_2) + O((u_1, u_2)^6) \\ bu_1 + (1 - b)u_2 + \hat{f}_2(u_1, u_2) + O((u_1, u_2)^6) \end{array} \right), \quad (22) \]

where
\[ \hat{f}_1(u_1, u_2) = -k u_2 u_1 + \frac{1}{2} k^2 u_2^2 + \frac{1}{2} k^2 u_1 u_2^2 - \frac{1}{6} k^3 u_2^3 - \frac{1}{6} k^3 u_1 u_2^3 + \frac{1}{24} k^4 u_2^4 \]
\[ + \frac{1}{24} k^4 u_1 u_2^4 - \frac{k^5 u_2^5}{120}, \]
\[ \hat{f}_1(u_1, u_2) = bu_1 + (1 - b) u_2 - bu_1^2 + 2 bu_1 u_2 - bu_2^2 + bu_1^3 - 2 bu_1^2 u_2 \]
\[ + bu_2^2 - bu_1^3 + 2 bu_1^2 u_2 - bu_2^3 + bu_1^4 - 2 bu_1^3 u_2 + bu_2^4 - 2 bu_1^4 u_2 + bu_1^3 u_2^2. \]

By the change of variables \((u_1, u_2)^T = H_2(v_1, v_2)^T\), where
\[ H_2 = \left( \begin{array}{cc} \frac{1}{b - \sqrt{b^2 - 4k}} & \frac{1}{b + \sqrt{b^2 - 4k}} \\ \frac{1}{b - \sqrt{b^2 - 4k}} & \frac{1}{b + \sqrt{b^2 - 4k}} \end{array} \right), \]

system (22) is transformed into the following mapping in the suspended form
\[ \left( \begin{array}{c} v_1 \\ v_2 \end{array} \right) \mapsto \left( \begin{array}{c} (-1 + \frac{(k - 2)^2}{4(k - 1)^2}) v_1 + \hat{g}_1(v_1, v_2) + O(|(v_1, v_2)|^6) \\ \frac{3k^2 - 2}{k - 2} - \frac{k^2}{4(k - 1)^2} v_2 + \hat{g}_2(v_1, v_2) + O(|(v_1, v_2)|^6) \end{array} \right), \quad (23) \]

where \(\epsilon = b - 4/(2 - k), \)
\[ \hat{g}_1(v_1, v_2) = \hat{c}_{20} v_1^2 + \hat{a}_{11} v_1 v_2 + \hat{a}_{02} v_2^2 + \hat{a}_{30} v_1^3 + \hat{a}_{21} v_1^2 v_2 + \hat{a}_{12} v_1 v_2^2 + \hat{a}_{03} v_2^3 \]
\[ + \hat{a}_{31} v_1^3 v_2 + \hat{a}_{22} v_1^2 v_2^2 + \hat{a}_{13} v_1 v_2^3 + \hat{a}_{04} v_2^4 \]
\[ + \hat{a}_{40} v_2^4 + \hat{a}_{41} v_1 v_2^3 + \hat{a}_{32} v_1^2 v_2^2 + \hat{a}_{23} v_1^3 v_2^2 + \hat{a}_{14} v_1^4 v_2 + \hat{a}_{05} v_2^5, \]
\[ \hat{g}_2(v_1, v_2) = \hat{c}_{20} v_1^2 + \hat{c}_{11} v_1 v_2 + \hat{c}_{02} v_2^2 + \hat{c}_{30} v_1^3 + \hat{c}_{21} v_1^2 v_2 + \hat{c}_{12} v_1 v_2^2 + \hat{c}_{03} v_2^3 + \hat{c}_{31} v_1^3 v_2 + \hat{c}_{22} v_1^2 v_2^2 + \hat{c}_{13} v_1 v_2^3 + \hat{c}_{04} v_2^4 \]
\[ + \hat{c}_{40} v_2^4 + \hat{c}_{41} v_1 v_2^3 + \hat{c}_{32} v_1^2 v_2^2 + \hat{c}_{23} v_1^3 v_2^2 + \hat{c}_{14} v_1^4 v_2 + \hat{c}_{05} v_2^5, \]
and the coefficients $\hat{a}_{ij}$s and $\hat{c}_{ij}$s are given in Appendix. Having two eigenvalues on the unit circle $S^1$, by center manifold theorem, system (23) has a two-dimensional $C^4$ center manifold, which has the following approximate expression

\begin{equation}
\begin{split}
v_2 = \mathcal{G}(v_1, \epsilon) = \sum_{j=2}^{4} \sum_{i=0}^{j} p_{ij}v_1^iv_2^{j-i} + O(v_1^5, \epsilon) = \mathcal{G}(v_1, \epsilon) + \dot{\mathcal{G}}(v_1, \epsilon) + O(v_1^5, \epsilon),
\end{split}
\end{equation}

(24)

near the point $(v_1, \epsilon) = (0, 0)$, where $\hat{p}_{ij}$s are all indetermination. By the invariance of center manifold, we have the following equation

\begin{equation}
\begin{split}
\mathcal{G} \left( -1 + \frac{(k-2)^2}{4(k-1)^2} \epsilon \right) v_1 + \dot{\mathcal{G}}(v_1, \epsilon) + O(v_1^5, \epsilon) \equiv \left( \frac{3k-2}{k-2} - \frac{k^2}{4(k-1)} \right) \mathcal{G}(v_1, \epsilon) + \dot{\mathcal{G}}(v_1, \epsilon) + O(v_1^5, \epsilon).
\end{split}
\end{equation}

(25)

Comparing the coefficients of $v_1^ie^{j-i}$ ($j = 2, 3, 4, i = 0, \ldots, j$) in (25), we obtain

\begin{align*}
\hat{p}_{1,1} &= \hat{p}_{0,2} = \hat{p}_{1,2} = \hat{p}_{0,3} = \hat{p}_{1,3} = \hat{p}_{0,4} = 0, \\
\hat{p}_{2,0} &= -\hat{c}_{20}(k-2) = \frac{(k-2)^3}{2k^2(k-1)} + \frac{(5k-4)(k-2)^4}{16k^2(k-1)^3} + O(\epsilon^2), \\
\hat{p}_{3,0} &= \frac{(2ka_{20} + kc_{11}c_{20} - 2kc_{30} - 4a_{20}c_{20} - 2c_{11}c_{20})(k-2)}{8k(k-1)} \\
&= -\frac{(k^3 - 6k^2 + 36k - 48)(k-2)^2}{12(k-1)^2k^3} - \frac{(13k^4 - 150k^3 + 576k^2 - 840k + 384)(k-2)^3}{96(k-1)^4k^3} + O(\epsilon^2), \\
\hat{p}_{2,1} &= \frac{c_{20}(k-2)^2(k^2 - 8k + 8)}{16k^2(k-1)} \\
&= -\frac{(k-2)^4(k^2 - 8k + 8)}{16(k-1)^2k^3} - \frac{(k-2)^5(5k-4)(k^2 - 8k + 8)}{128(k-1)^3k^3} + O(\epsilon^2), \\
\hat{p}_{4,0} &= -\left\{ \left( 4k^3a_{11}c_{20} - 2k^3c_{11}c_{20} - k^3a_{12}c_{12}c_{20} + 2k^3c_{12}c_{20}^2 + k^3c_{11}^2c_{20} + 6k^3a_{20}c_{30} - 8k^3a_{10}c_{20} - k^3c_{11}c_{30} - 4k^3c_{20}c_{21} - 20k^3a_{11}c_{20} + 4k^2a_{20}c_{11}c_{20} - 4k^2c_{02}c_{20} - 4k^2c_{11}c_{20}^2 + 8k^3c_{40} - 12k^2a_{20}c_{30} - 24k^2c_{30}c_{20} + 4k^2c_{11}c_{30} + 12k^2c_{20}c_{21} + 32ka_{11}c_{20} - 16ka_{20}c_{20}^2 - 4ka_{20}c_{11}c_{20} + 16ka_{02}c_{20}^2 + 4kc_{11}c_{20}^2 - 8c_{02}c_{40} - 16ka_{30}c_{20} - 8c_{20}c_{21} - 16a_{11}c_{20}^2 - 8c_{02}c_{40}^2(k-2) \right\} / \{16k^3(k-1)\}, \\
&= -\frac{(2k^6 - 29k^5 + 166k^4 - 536k^3 + 960k^2 - 912k + 384)(k-2)^2}{12(k-1)^3k^5} \\
&- \frac{\left\{ \left( (8k^8 - 3k^7 - 896k^6 + 6912k^5 - 24168k^4 + 47264k^3 - 54144k^2 + 34368k - 9216)(k-2)^3 \right) / \{192k^5(k-1)^5\} \right\} \epsilon + O(\epsilon^2), \\
\hat{p}_{3,1} &= -\left\{ \left( (k-2)^2 \left( 2k^4a_{20}c_{20} - k^4c_{11}c_{20} + 4k^4c_{30} - 22k^3a_{10}c_{20} - k^3c_{11}c_{20} - 12k^3c_{30} + 64k^2a_{20}c_{20} + 16k^2c_{11}c_{20} + 12k^2c_{30} - 72ka_{20}c_{20} - 28kc_{11}c_{20} + 32a_{20}c_{20} + 16c_{11}c_{20} \right) \right\} / \{64k^3(k-1)^3\} \\
&= \frac{(k^6 - 15k^5 + 114k^4 - 480k^3 + 984k^2 - 960k + 384)(k-2)^2}{96(k-1)^4k^4} + O(\epsilon^2),
\end{align*}
applying a near identity transformation

\[ \hat{z}_{2,2} = -\frac{c_{20} (k-2)^3 (3k^4 - 28k^3 + 104k^2 - 144k + 64)}{128k^3(k-1)^2} \]
\[ = \frac{(k-2)^5 (3k-4)(k^3 - 8k^2 + 24k - 16)}{128k^4(k-1)^3} + \frac{(5k-4)(k-2)^6 (3k-4)(k^3 - 8k^2 + 24k - 16)\epsilon}{1024k^4(k-1)^4} + O(\epsilon^2). \]

Hence the local center manifold (23) has specifically expression. Substituting (23) into the first equation of (24) yields a one-dimensional mapping

\[ v_1 \mapsto \hat{q}_1 v_1 + \hat{q}_2 v_1^2 + \hat{q}_3 v_1^3 + \hat{q}_4 v_1^4 + \hat{q}_5 v_1^5 + O(v_1^6), \]

where

\[ \hat{q}_1 = 1 + \frac{(k - 2)^2}{4(k-1)} \epsilon, \]
\[ \hat{q}_2 = \frac{(k - 2)^2}{k(k-1)} - \frac{(2k^3 - 9k^2 + 16k - 8)(k-2)^2}{8k(k-1)^2} \epsilon + O(\epsilon^2), \]
\[ \hat{q}_3 = -\frac{(2k-3)(k-4)}{3(k-1)^2} + \left\{ \frac{((3k^5 - 17k^4 + 14k^3 + 39k^2 - 60k + 16)}{4(3k^4 - 28k^3 + 104k^2 - 144k + 64)}\epsilon + O(\epsilon^2), \right. \]
\[ \hat{q}_4 = \frac{(17k^4 - 124k^3 + 216k^2 - 120k + 48)(k-2)^2}{12k^3(k-1)^3} \]
\[ -\left\{ \frac{((k - 2)^3 (78k^7 - 1027k^6 + 4612k^5 - 9532k^4 + 10392k^3)}{96k^4(k-1)^3} \epsilon + O(\epsilon^2), \right. \]
\[ \hat{q}_5 = \frac{((k - 2)^3 (83k^7 - 1084k^6 + 5874k^5 - 17388k^4 + 30480k^3)}{60k^4(k-1)^4} \epsilon + O(\epsilon^2), \]
\[ + \left\{ \frac{(340k^{10} - 6643k^9 + 54769k^8 - 256264k^7 + 754128k^6)}{4(3k^4 - 28k^3 + 104k^2 - 144k + 64)} \epsilon + O(\epsilon^2), \right. \]
\[ +1448800k^5 + 1829760k^4 - 1494560k^3 + 745280k^2 - 192000k + \frac{15360}{192000} \epsilon + O(\epsilon^2). \]

Next, we compute the normal form of (26) in the neighborhood of \((v_1, k, \epsilon) = (0, 4 - \sqrt{16}, 0)\). For this purpose, taking \(k = 4 - \sqrt{16} + \epsilon\) with small \(|\epsilon|\) and applying a near identity transformation \(v_1 = w + \delta w^2 + \theta w^4\), where

\[ \delta = \frac{\hat{q}_2}{(\hat{q}_1 - 1)\hat{q}_1} \]
\[ = \frac{8}{33} + \frac{2\sqrt{2}}{33} + \frac{\sqrt{3}}{66} - \frac{289}{726} + \frac{159\sqrt{2}}{484} + \frac{203\sqrt{3}}{726} \epsilon \]
\[ - \left( \frac{256}{3993} + \frac{398\sqrt{2}}{3993} + \frac{229\sqrt{3}}{15972} \right) \epsilon + O(|(\epsilon, \epsilon)|^2), \]
Therefore, system (2) undergoes a generalized flip bifurcation at $(\varepsilon, \epsilon)$, which means that the condition (GF.2) of Lemma 9.3 in [13, p.401] is satisfied.

To eliminate the non-resonant quadratic and quartic terms of (26), we obtain the normal form

\[ w \mapsto \hat{q}_1 w + Bw^3 + Cw^5 + O(w^6), \tag{27} \]

where

\[
\begin{align*}
B &= \frac{\hat{q_1}^2 \hat{q}_3 - \hat{q}_1 \hat{q}_3 + 2 \hat{q}_1^2 \hat{q}_3 \hat{q}_4}{\hat{q}_1 (\hat{q}_1 - 1)} \\
&= \left( \frac{67}{33} + \frac{52 \sqrt{2}}{33} + \frac{14 \sqrt{4}}{11} \right) \varepsilon + \left( \frac{29}{121} + \frac{12 \sqrt{2}}{121} + \frac{30 \sqrt{4}}{121} \right) \epsilon + O((\varepsilon, \epsilon)^2),
\end{align*}
\]

\[
\begin{align*}
C &= \frac{\hat{q}_3 + \frac{2}{\hat{q}_1 (\hat{q}_1 - 1)} (2 \hat{q}_1^2 + \hat{q}_1 + 3) \hat{q}_2 \hat{q}_4}{\hat{q}_1 (\hat{q}_1 - 1)^2 (\hat{q}_1^2 + \hat{q}_1 + 1)} + \frac{(3 \hat{q}_1^2 - \hat{q}_1 + 9) \hat{q}_2^2 \hat{q}_3}{\hat{q}_1^2 (\hat{q}_1 - 1)^2 (\hat{q}_1^2 + \hat{q}_1 + 1)} \\
&= \frac{14006 \sqrt{4}}{6655} - \frac{14006 \sqrt{4}}{59895} + \frac{6370358}{658845} + \frac{1900763 \sqrt{2}}{263538} + \frac{398572 \sqrt{4}}{658845} \varepsilon + O((\varepsilon, \epsilon)^2).
\end{align*}
\]

One can check that $B = 0$ and $D \approx -1.523079$ in the case $(\varepsilon, \epsilon) = (0, 0)$, implying that the condition (GF.1) of Lemma 9.3 in [13, p.401] holds.

Now we prove that the mapping $(\varepsilon, \epsilon) \mapsto (-\hat{q}_1 + 1, B)$ is regular at $(\varepsilon, \epsilon) = (0, 0)$. For this purpose, let

\[
(\mu_1, \mu_2) = (-\hat{q}_1 + 1, B), \tag{28}
\]

then it follows from (26) and (27) that

\[
\frac{\partial (\mu_1, \mu_2)}{\partial (\varepsilon, \epsilon)} \bigg|_{(\varepsilon, \epsilon)} = \frac{103}{363} \frac{\sqrt{2}}{363} + \frac{76 \sqrt{2}}{121} \neq 0,
\]

which means that the condition (GF.2) of Lemma 9.3 in [13, p.401] is satisfied. Therefore, system (2) undergoes a generalized flip bifurcation at $(\varepsilon, \epsilon) = (0, 0)$.

At last, we give more details about the bifurcation as $(\varepsilon, \epsilon)$ varies near the point $(0, 0)$. By the new parameter $(\mu_1, \mu_2)$, system (27) is rewritten as

\[ w \mapsto -(1 + \mu_1) w + \mu_2 w^3 + C_1 w^5 + O(w^6). \tag{29} \]
where
\[
C_1 = \frac{-4178}{6655} \cdot \frac{24914}{3985} - \frac{14006}{3985} + \left( \frac{1024000659}{198378830} - \frac{10248181}{22069670} + \frac{52052980}{66191961} \right) \mu_1 + \left( \frac{220324}{96895} + \frac{92359}{96895} \sqrt{2} \right) \nu_2 + O(|(\mu_1, \mu_2)|^2).
\]

A nonsingular scaling of the coordinate and the new parameters
\[
w = (-C_1)^{-\frac{1}{2}} \eta, \quad \mu_1 = \beta_1, \quad \mu_2 = \beta_2(-C_1)^{\frac{1}{2}}
\]
bring system (29) into the following form
\[
\eta \mapsto \eta + (1 + \beta_1) \eta + \beta_2 \eta^3 - \eta^5 + O(\eta^6).
\]

In the neighborhood of \( \beta := (\beta_1, \beta_2) = (0, 0) \), the second iteration of (31)
\[
\eta \mapsto g(\eta) := (1 + 2 \beta_1 + \beta_2^2) \eta - (2 \beta_2 + O(|\beta|) \eta^3 + (2 + 6 \beta_1 + O(|\beta|^2)) \eta^5 + O(\eta^6)
\]
has a single unstable fixed point \( \eta_0 = 0 \) as \( \beta \) is in the region \( D_1 \), where
\[
D_1 := \{(\beta_1, \beta_2) \mid \beta_1 > \frac{1}{4} \beta_2^2 + o(\beta_2^2), \beta_2 > 0 \} \cup \{(\beta_1, \beta_2) \mid \beta_1 \geq 0, \beta_2 \leq 0 \},
\]
implying system (31) has a single unstable fixed point \( \eta_0 = 0 \) in a sufficiently small neighborhood of origin (see Fig.3). The orbits of (31) leave the fixed point, “leapfrogging” around it. As \( \beta \) crosses the lower half line \( F_1 := \{(\beta_1, \beta_2) \mid \beta_1 = 0, \beta_2 < 0 \} \) from the region \( D_1 \) to a region \( D_2 := \{(\beta_1, \beta_2) \mid \beta_1 < 0 \} \), system (31) undergoes a flip bifurcation and produces an unstable period-two cycle \( \{\eta_1, \eta_2\} \), where
\[
\eta_{1,2} = \pm \sqrt{\frac{1}{2} \left( \beta_2 + \sqrt{\beta_2^2 - 4 \beta_1 - 14 \beta_1^2} \right) + O(|\beta|)}.
\]

**Figure 3.** Bifurcation diagram of system (31) for small \( |\beta| \).
while the fixed point at the origin becomes stable. As $\beta$ crosses the upper half line $F_2 := \{(\beta_1, \beta_2)|\beta_1 = 0, \beta_2 > 0\}$ from the region $D_2$ to a region $D_3$, where

$$D_3 := \{(\beta_1, \beta_2)|0 < \beta_1 < \frac{1}{4} \beta_2^2 + o(\beta_2^3), \beta_2 > 0\},$$

system (31) possesses two period-two cycles, a “big” unstable one $\{\eta_1, \eta_2\}$ and a “small” stable one $\{\eta_3, \eta_4\}$, where

$$\eta_{3, 4} = \pm \frac{1}{2} \left( \beta_2 - \sqrt{\beta_2^2 - 4 \beta_1 - 14 \beta_1^2} \right) + O(|\beta|^3),$$

Meanwhile, the fixed point $\eta_0 = 0$ becomes unstable. The two period-two cycles collide as $\beta$ lies in the curve $T := \{(\beta_1, \beta_2)|\beta_1 = \frac{1}{4} \beta_2^2 + O(\beta_2^3), \beta_2 > 0\}$ and disappear as $\beta$ enters into the region $D_1$.

Going back to the original parameters $(k, b)$ and applying the Implicit Function Theorem, one can check that the curves $F_1$, $F_2$ and $T$ in the figure 3 correspond to the the curves $\mathcal{L}_{21}, \mathcal{L}_{22}$ and $\mathcal{F}$ in the figure 2 respectively, and the regions $D_1, D_2$ and $D_3$ correspond to the regions $\mathcal{D}_{31}, \mathcal{D}_2$ and $\mathcal{D}_{32}$ respectively. Furthermore, from that system (31) possesses the period-two cycles, which lies in the invariant center manifold of system (2), near the fixed point $\eta_0 = 0$, it follows that system (2) has the same number of period-two cycle near the fixed point $E$. The proof is completed.

Furthermore, from (32), we have the following qualitative property of $E$ in the case $(h, a) = (4 - \sqrt[3]{16}, \sqrt[3]{2}/(\sqrt[3]{2} - 1))$.

**Proposition 4.** If $(k, b) \in \mathcal{L}_1$ with $k = 4 - \sqrt[3]{16}$, then $E$ is an unstable node.

**Proof.** From (32), we see that $g''(0) = g^{(4)}(0) = 0$ and $g^5(0) = 240 > 0$ in the case $(\beta_1, \beta_2) = (0, 0)$, implying that the fixed point $\eta = 0$ of mapping (31) is unstable by Theorem 5.1 in [4], from which we deduce that the fixed point $E$ of system (2) is an unstable node in the case $(k, b) \in \mathcal{L}_2$ with $k = 4 - \sqrt[3]{16}$. The proof is completed.

5. **Neimark-Sacker bifurcation.** From (4), we see that two eigenvalues

$$\lambda_1 = 1 - \frac{b}{2} + \frac{1}{2}i\sqrt{b(4 - b)} , \lambda_2 = 1 - \frac{b}{2} - \frac{1}{2}i\sqrt{b(4 - b)},$$

where $i$ denotes the imaginary unit, in the case $(k, b) \in \mathcal{L}_4$. So, a Neimark-Sacker bifurcation may happen in the system (2) as the parameter $(k, b)$ crosses the line $\mathcal{L}_4$. In this section, we investigate the bifurcation.

**Theorem 4.** If $(k, b)$ crosses the line $\mathcal{L}_4$ with $b \neq 2, 3$ from the region $\mathcal{D}_4$ to the region $\mathcal{D}_5$, then system (2) undergoes subcritical Neimark-Sacker bifurcation and produces a unique unstable invariant circle near the fixed point $E$. 

**Proof.** From (32), we see that $g''(0) = g^{(4)}(0) = 0$ and $g^5(0) = 240 > 0$ in the case $(\beta_1, \beta_2) = (0, 0)$, implying that the fixed point $\eta = 0$ of mapping (31) is unstable by Theorem 5.1 in [4], from which we deduce that the fixed point $E$ of system (2) is an unstable node in the case $(k, b) \in \mathcal{L}_2$ with $k = 4 - \sqrt[3]{16}$. The proof is completed.

Furthermore, from (32), we have the following qualitative property of $E$ in the case $(h, a) = (4 - \sqrt[3]{16}, \sqrt[3]{2}/(\sqrt[3]{2} - 1))$.

**Proposition 4.** If $(k, b) \in \mathcal{L}_1$ with $k = 4 - \sqrt[3]{16}$, then $E$ is an unstable node.
Proof. Denote \( \lambda_0 := \lambda_1 = 1 - b/2 + i\sqrt{b(4-b)}/2 \) for \( k \in \mathcal{L}_4 \), then by the condition (SH1) given in Theorem 3.5.2 of [8] we need to check \( \lambda_0^i \neq 1, i = 1, 2, 3, 4 \), the non-resonance conditions. It is clear that \( \lambda_0 \) is not a real, which means that \( \lambda_0 \neq \pm 1 \). Therefore, \( \lambda_0^i \neq 1 \) is proved for \( i = 1, 2 \). Suppose that \( \lambda_0^3 = 1 \), from \( \lambda_0^0 - 1 = (\lambda_0 - 1)(\lambda_0^3 + \lambda_0 + 1) \), it follows that \( \lambda_0^2 + \lambda_0 + 1 = 0 \), implying that the real part of \( \lambda_0 \), \( \Re(\lambda_0) \), is equal to \(-1/2\), which contradicts to the requirement \( b \neq 3 \). If \( \lambda_0^4 = 1 \), which is equivalent to \( \lambda_0^3 - 1 = (\lambda_0^0 - 1)(\lambda_0^3 + 1) = 0 \), from which we see that \( \lambda_0^2 + 1 = 0 \), implying that \( \Re(\lambda_0) = 0 \), which contradicts to the requirement \( b \neq 2 \). So, the non-resonance conditions holds true.

If \( (k, b) \) is near \( \mathcal{L}_4 \), then from (4) we see that

\[
\lambda_1 = 1 - \frac{1}{2} b + \frac{1}{2} i\sqrt{b(4k-b)} ,
\]

implying that \( |\lambda_1| = 1 - b + bk \), which yields

\[
\frac{d|\lambda_1|}{dk} \bigg|_{k=1} = b > 0
\]

since \( 0 < b < 4 \). Thus, the transversality condition (SH2) of Theorem 3.5.2 in [8] is satisfied.

At last, we check the non-degeneracy condition (SH3) given in Theorem 3.5.2 of [8]. Applying the transformation \((x, y) = (u_1 + 1, u_2 + 1)\) to translate the fixed point \( O \) and using the invertible linear transformation \((u_1, u_2)^T = H(v_1, v_2)^T\), where

\[
H = \begin{bmatrix} 1 & 0 \\ \frac{b}{2} & 1 \frac{1}{2} \sqrt{b(4-b)} \end{bmatrix}
\]

to simplify the linear part to the canonical form, we reduce system (2) with \( k = 1 \) to the following form

\[
\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto \begin{pmatrix} (1 - \frac{b}{2}) v_1 - \frac{1}{2} i\sqrt{b(4-b)} v_2 + \sum_{i=0}^{3} a_{2-i,i} v_1^{2-i} v_2^i \\ \frac{1}{2} i\sqrt{b(4-b)} v_1 + (1 - \frac{b}{2}) v_2 + \sum_{i=0}^{3} b_{2-i,i} v_1^{2-i} v_2^i \end{pmatrix} + \begin{pmatrix} \sum_{i=0}^{3} a_{3-i,i} v_1^{3-i} v_2^i + O(||(v_1, v_2)||^4) \\ \sum_{i=0}^{3} b_{3-i,i} v_1^{3-i} v_2^i + O(||(v_1, v_2)||^4) \end{pmatrix},
\]

where

\[
\begin{align*}
a_{2,0} &= \frac{1}{8} b (b - 4) , & a_{1,1} &= \frac{1}{4} (b - 2) \sqrt{b(4-b)} , & a_{3,0} &= -\frac{1}{16} b^2 (b - 6) , \\
a_{0,2} &= \frac{1}{8} b (b - 4) , & a_{1,2} &= \frac{1}{16} b (b - 4) (b - 2) , & a_{2,0} &= -\frac{b(5 b^2 - 20 b + 16)}{8 \sqrt{b(4-b)}} , \\
a_{2,1} &= -\frac{1}{16} b (b - 4) \sqrt{b(4-b)} , & a_{1,2} &= \frac{1}{16} b (b - 4) (b - 2) , & b_{0,2} &= \frac{5 b^2 (b - 4)}{8 \sqrt{b(4-b)}} , \\
a_{0,3} &= \frac{1}{35} \sqrt{b(4-b)} b (b - 4) , & b_{1,1} &= -\frac{5}{4} b + \frac{5}{2} , & b_{2,1} &= \frac{1}{16} b (b^2 + 12 b - 32) , \\
b_{1,1} &= -\frac{5}{4} b + \frac{5}{2} , & b_{3,0} &= \frac{b (b^2 - 13 b + 36)}{48 \sqrt{b(4-b)}} , & b_{0,3} &= -\frac{1}{48} b^2 (b - 4) , \\
b_{1,2} &= -\frac{1}{16} \frac{b^2 (b+6)(b-4)}{\sqrt{b(4-b)}} , & b_{2,1} &= \frac{1}{16} b (b^2 + 12 b - 32) , & b_{0,3} &= -\frac{1}{48} b^2 (b - 4) .
\end{align*}
\]
Let \( z = v_1 + iv_2 \), then system (33) is rewritten as the following complex form

\[
z \mapsto \lambda_0 z + \sum_{i=0}^{2} g_{2-i,z^i} + \sum_{i=0}^{3} g_{3-i,z^i} + O(|z|^4),
\]

where

\[
g_{2,0} = -\frac{b}{8} (2b - 3) - \frac{b^2 (2b - 7)}{8\sqrt{b(4-b)}} i, \quad g_{1,1} = \frac{-b}{\sqrt{b(4-b)}} i,
\]

\[
g_{0,2} = \frac{b}{8} (3b - 7) - \frac{b (3b^2 - 13b + 8) + (q_0,2 - \tilde{\lambda}_0^2 - \tilde{\lambda}_0) p_{0,2})}{16\sqrt{b(4-b)}} i,
\]

\[
g_{3,0} = \frac{1}{48} b (7b - 15) + \frac{b (7b^2 - 29b + 12)}{48\sqrt{b(4-b)}} i,
\]

\[
g_{2,1} = \frac{1}{16} b (2b - 3) + \frac{b (2b^2 - 9b + 12)}{16\sqrt{b(4-b)}} i,
\]

\[
g_{1,2} = -\frac{1}{16} b (2b - 5) + \frac{b (2b^2 - 9b + 12)}{16\sqrt{b(4-b)}} i.
\]

With the quadratic near-identity transformation

\[
z = w + p_{2,0} w^2 + p_{1,1} w \tilde{w} + p_{0,2} \tilde{w}^2,
\]

where \( p_{2,0}, p_{1,1} \) and \( p_{0,2} \) are all indeterminate, system (34) is changed into the following mapping

\[
w \mapsto \lambda_0 w + (g_{2,0} - (\lambda_0^2 - \lambda_0) p_{2,0}) w^2 + (g_{1,1} - (\lambda_0^2 - \lambda_0) p_{1,1}) w \tilde{w} + (g_{0,2} - (\lambda_0^2 - \lambda_0) p_{0,2}) \tilde{w}^2 + O(|w|^3).
\]

By the choice

\[
p_{2,0} := \frac{g_{2,0}}{\lambda_0^2 - \lambda_0}, \quad p_{1,1} := \frac{g_{1,1}}{\lambda_0^2 - \lambda_0}, \quad p_{0,2} := \frac{g_{0,2}}{\lambda_0^2 - \lambda_0},
\]

system (35) is reduced to the mapping

\[
w \mapsto \lambda_0 w + q_{3,0} w^3 + q_{2,1} w^2 \tilde{w} + q_{1,2} w \tilde{w}^2 + q_{0,3} \tilde{w}^3 + O(|w|^4),
\]

where

\[
q_{2,1} = \frac{1}{16} b (2b + 1)(1-b) + \frac{1}{16} \sqrt{b} (2b^2 - 5b^2 - 2b + 4) \tilde{w}^2 i.
\]

Further, by the cubic near-identity transformation

\[
w = \eta + p_{3,0} \eta^3 + p_{2,1} \eta^2 \tilde{\eta} + p_{1,2} \eta \tilde{\eta}^2 + p_{0,3} \tilde{\eta}^3
\]

with indeterminate \( p_{3,0}, p_{2,1}, p_{1,2} \) and \( p_{0,3} \), system (36) is changed into the form

\[
\eta \mapsto \lambda_0 \eta + (q_{3,0} - (\lambda_0^3 - \lambda_0) p_{3,0}) \eta^3 + (q_{2,1} - (\lambda_0^2 - \lambda_0) p_{2,1}) \eta^2 \tilde{\eta} \\
+ (q_{1,2} - (\lambda_0^2 - \lambda_0) p_{1,2}) \eta \tilde{\eta}^2 + (q_{0,3} - (\lambda_0^3 - \lambda_0) p_{0,3}) \tilde{\eta}^3 + O(|\eta|^4),
\]

which can be simplified as

\[
\eta \mapsto \lambda_0 \eta + q_{2,1} \eta^2 \tilde{\eta} + O(|\eta|^4)
\]
Proof. From (39) and (40), we see that the qualitative property of \( E \) can be reduced to the discussion of a differential equation, forms of some strong resonances, the Takens theorem, which shows that the analysis of dynamics for a map can be reduced to the discussion of a differential equation, and \( \eta = \rho e^{i\theta} \), then in the polar coordinates system (37) is written as
\[
rho e^{i\theta} \rightarrow \rho e^{i(\theta + c)} (1 + \lambda_0 q_{21} \rho^2) + O(\rho^4),
\]
which is equivalent to the mapping
\[
\begin{pmatrix} \rho \\ \theta \end{pmatrix} \rightarrow \begin{pmatrix} \rho |1 + \lambda_0 q_{21} \rho^2| + O(\rho^4) \\ \theta + \nu + \arg(1 + \lambda_0 q_{21} \rho^2) + O(\rho^4) \end{pmatrix}.
\]
(38)
Since
\[
|1 + \lambda_0 q_{21} \rho^2| = \left( (1 + \text{Re}(\lambda_0 q_{21})) \rho^2 + (\text{Im}(\lambda_0 q_{21}) \rho^2)^2 \right)^{1/2} = 1 + \text{Re}(\lambda_0 q_{21}) \rho^2 + O(\rho^4)
\]
and
\[
\arg(1 + \lambda_0 q_{21} \rho^2) = \arctan \left( \frac{\text{Im}(\lambda_0 q_{21}) \rho^2}{1 + \text{Re}(\lambda_0 q_{21}) \rho^2} \right) = \text{Im}(\lambda_0 q_{21}) \rho^2 + O(\rho^4)
\]
for small \( \rho \), mapping (38) can be presented as
\[
\begin{cases}
\hat{\rho} = \rho + c_3 \rho^4 + O(\rho^4), \\
\hat{\theta} = \theta + \nu + d_2 \rho^2 + O(\rho^4),
\end{cases}
\]
(39)
where
\[
c_3 := \text{Re}(\lambda_0 q_{21}) = \frac{1}{16} b^2 > 0
\]
(40)
because \( 0 < b < 4 \). Thus (40) gives a nonzero Lyapunov quantity \( c_3(\epsilon) \), referred to the first Lyapunov quantity of mapping (2) at \( k = 1 \) as usual.

To sum up, all conditions of Theorem 3.5.2 of \cite{8} are verified, implying that system (2) undergoes the Neimark-Sacker bifurcation. More specifically, from the results of \cite[p.520]{23}, we deduce that the fixed point \( E \) becomes stable and a unique unstable invariant cycle surrounding \( E \) arises as \((k, b)\) crosses \( \mathcal{L}_4 \) from the region \( \mathcal{D}_1 \) to the region \( \mathcal{D}_4 \) in the figure 1. This proof is completed. \( \square \)

From (39) and (40), the qualitative property of \( E \) is as follows.

**Proposition 5.** If \((k, b)\) lies in \( \mathcal{L}_4 \) with \( k \neq 2, 3 \), then the fixed point \( E \) of mapping \( F \) is an unstable focus.

**Proof.** From (39) and (40), we see that \( \hat{\rho} > \rho \) for small \( \rho \), implying that the fixed point \( E \) of system (2) is an unstable focus. The proof is completed. \( \square \)

In the cases that \((k, b)\) lies at one of the points \((1, 4), (2, 1) \) and \((3, 1)\), the qualitative properties of \( E \) are not given because the proofs involve computing the normal forms of some strong resonances, the Takens theorem, which shows that the analysis of dynamics for a map can be reduced to the discussion of a differential equation,
and the blowing-up techniques. The specific processes can be referred to the references [24, 26]. Furthermore, the system may produce 1:2 resonance, 1:4 resonance and 1:3 resonance respectively. We leave those to our next works for the more complicated computation.

6. **Numerical simulations.** In this section, two numerical simulations are shown for verifying the results in Theorems 1 and 4.

Regarding \( b \) as a bifurcation parameter, keeping \( k = 0.3 \) fixed and applying the software MAPLE 18, by Theorem 1, we plot the bifurcation diagram of system (2) with a initial value \((x_0, y_0) = (1.2, 1.2)\) in figure 4 in the \((b, x, y)\)-space. From figure 4, we see that system (2) exhibits the typical flip bifurcation route to chaos.

![Bifurcation diagram](image1.png)

(a) Bifurcation diagram in \((b, x, y)\)-space.  
(b) Lyapunov exponents corresponding to (a).

**Figure 4.** Flip bifurcation route to chaos for \( k = 0.3 \).

By Theorem 4, setting the parameter \((k, b) = (0.995, 1)\), which lies in the region \( D_5 \) and is near the Neimark-Sacker bifurcation curve \( \Sigma_4 \) in the figure 1, and taking two initial values \((x_{01}, y_{01}) = (1.1914396, 1.01)\) and \((x_{02}, y_{02}) = (1.1914397, 1.01)\), which are near the invariant circle \( \Gamma \) (see figure 5), we obtain two orbits, the green one and the gray one, in the figure 5 respectively. The green one leaves \( \Gamma \) and tends to the fixed point \( E \), and the gray one leaves \( \Gamma \) and tends to infinity.

Furthermore, we will show that system (2) has periodic orbits on the invariant circle \( \Gamma \). Taking the parameters \( k = 0.991 \) and \( b = 1.3819 \), setting four initial values

\[
(x_{01}, y_{01}) = (1.25, 0.9609), \quad (x_{02}, y_{02}) = (1.25, 0.9608), \\
(x_{03}, y_{03}) = (1.25, 0.9704205), \quad (x_{04}, y_{04}) = (1.25, 0.9704201),
\]

and using the mathematical software MATLAB version R2014a, after \( 10^5 \) steps, we obtain four orbits respectively, the blue one, the red one, the manganese purple one and the green one, all of which are at first approach a saddle period-five cycle \( \{S_1, S_2, S_3, S_4, S_5\} \) on the invariant circle \( \Gamma \) and then leave it (see figure 6). In addition, let \( k = 0.992 \) and \( b = 1.011 \) and set four initial values

\[
(x_{01}, y_{01}) = (1.141, 0.85272), \quad (x_{02}, y_{02}) = (1.141, 0.85273), \\
(x_{03}, y_{03}) = (1.141267, 0.88), \quad (x_{04}, y_{04}) = (1.141268, 0.88).
\]
Using the same software MATLAB, after $10^5$ steps, we obtain four orbits respectively, the blue one, the red one, the manganese purple one and the green one, all of which are at first approach a saddle period-six cycle $\{S_1, S_2, S_3, S_4, S_5, S_6\}$ on the invariant circle $\Gamma$ and then leave it (see figure 7). In fact, besides the period-five cycle and
the period-six cycle on the invariant circle $\Gamma$, system (2) possesses a lot of cycles with higher periods.

7. Conclusions. In this paper, the bifurcation diagram figure 1 and the qualitative properties of fixed point $E : (1, 1)$, including hyperbolic and non-hyperbolic, are presented in section 2, 3, 4 and 5 as the parameter $(k, b)$ lies in different regions and bifurcation curves. In the hyperbolic case, the qualitative properties of $E$ are proved by discussing the eigenvalues. For the non-hyperbolic cases, the proofs of qualitative properties for the fixed point are given after the proofs of bifurcations in sections 3, 4 and 5 respectively because they involve the center manifold theorem or the calculation of the normal form. From Proposition 1, we see that the leaf mass and populations of leaf-eating herbivores tend to the fixed point $E$ if the initial leaf mass and populations of leaf-eating herbivores on a tree are near the fixed point and the parameter $(k, b)$ lies in the regions $D_1$ and $D_5$. Hence the leaves and herbivores can coexist in a stable way.

In section 3, we primarily investigate the flip bifurcation and find that system (2) undergoes supercritical (resp. subcritical) flip bifurcation as $(k, b)$ crosses the curve $\mathcal{L}_1$ and $\mathcal{L}_2$ with $k > 4 - \sqrt{16}$ (resp. $\mathcal{L}_2$ with $1 < k < 4 - \sqrt{16}$) and that the system has a generalized flip bifurcation point $(k, b) = (4 - \sqrt{16}, 2/(\sqrt{2} - 1))$. Hence, in section 4, applying the center manifold theorem and computing the normal form, we investigate the generalized flip bifurcation and present three parameter regions near the flip bifurcation point such that system (2) has two period-two cycles, one period-two cycle and none respectively. From the ecological point of view, if the initial leaf mass and populations of leaf-eating herbivores on a tree are near the fixed point $E$ as the parameter $(k, b)$ lies in the regions $D_2$ and is near the bifurcation curve $\mathcal{L}_1$, the leaf mass and populations of leaf-eating herbivores tend to the stable period-two cycle. Furthermore, if the initial leaf mass and populations of leaf-eating herbivores on a tree belong to these period-two cycles produced by the supercritical
flip bifurcation, the subcritical flip bifurcation and the generalized flip bifurcation, the leaf mass and populations of leaf-eating herbivores return back the initial values automatically every two units of time. These results imply that the leaves and leaf-eating herbivores can coexist in a stable way on a tree.

In section 5, computing the normal form and the first Lyapunov quantity, we prove that system (2) undergoes the subcritical Neimark-Sacker bifurcation and produces a unique unstable invariant circle. Furthermore, in section 6, applying the numerical simulations, we not only verify our results but also show a saddle period-five cycle and a saddle period-six cycle on the invariant circle arisen from the Neimark-Sacker bifurcation. From the ecological point of view, if the initial leaf mass and populations of leaf-eating herbivores on a tree are on the invariant circle produced by the Neimark-Sacker bifurcation as the parameter \((k, b)\) lies in the region \(\mathcal{D}_5\) and is near the bifurcation curve \(\mathcal{L}_4\), the leaf mass and populations of leaf-eating herbivores oscillate near the fixed point \(E\). Furthermore, if the initial leaf mass and populations of leaf-eating herbivores belong to the 5-periodic (or 6-periodic) orbit, shown in Figure 6 (or 7), on the invariant circle, then the leaf mass and populations of leaf-eating herbivores return back the initial values automatically every five (or six) units of time. So, the leaves and the leaf-eating herbivores can coexist in the 5-periodic (or 6-periodic) or quasi-periodic orbits in the invariant circle.

**Acknowledgments.** I would like to thank the referees very much for their valuable comments and suggestions.

**Appendix: Expressions of some coefficients.** Coefficients \(\hat{a}_{ij}\)s and \(\hat{c}_{ij}\)s appearing in (23) are listed in the following:

\[
\begin{align*}
\hat{a}_{20} & = \frac{(k - 2)^2}{k (k - 1)} - \frac{(2k^3 - 9k^2 + 16k - 8) (k - 2)^2}{8k (k - 1)^3} \epsilon + O(\epsilon^2), \\
\hat{a}_{11} & = - \frac{2k}{(k - 1) (k - 2)} + \frac{(k^2 - k - 1) k^2}{4 (k - 1)^3} \epsilon + O(\epsilon^2), \\
\hat{a}_{02} & = \frac{(3k^2 - 2k^2)}{(k - 1) (k - 2)^2} - \frac{(4k^3 - k^2 - 8k + 4) k^2}{8 (k - 1)^3 (k - 2)} \epsilon + O(\epsilon^2), \\
\hat{a}_{30} & = - \frac{2k^2 - 6k + 6}{3k (k - 1)} + \frac{(3k^3 - 25k^2 + 48k - 24) (k - 2)^2}{24k (k - 1)^3} \epsilon + O(\epsilon^2), \\
\hat{a}_{21} & = - \frac{2k^2 - 2k + 2}{k (k - 1)} - \frac{(k - 2) (k^4 + 7k^3 - 30k^2 + 40k - 16)}{8k (k - 1)^3} \epsilon + O(\epsilon^2), \\
\hat{a}_{12} & = - \frac{2 (3k^2 - 6k + 4) k}{(k - 1) (k - 2)^2} + \frac{(11k^2 - 31k^2 + 34k - 16) k^2}{8 (k - 1)^3 (k - 2)} \epsilon + O(\epsilon^2), \\
\hat{a}_{03} & = \frac{2k^4}{3 (k - 1) (k - 2)^4} - \frac{(9k^3 - 19k^2 + 20k - 12) k^3}{24 (k - 1)^3 (k - 2)^2} \epsilon + O(\epsilon^2), \\
\hat{a}_{40} & = \frac{2(k^2 - 6k + 6)}{3k (k - 1)} - \frac{(k^3 - 23k^2 + 48k - 24) (k - 2)^2}{24k (k - 1)^3} \epsilon + O(\epsilon^2), \\
\hat{a}_{31} & = \frac{4(2k^3 - 11k^2 + 18k - 12)}{3k (k - 1) (k - 2)} - \frac{(2k^3 - 17k^3 + 73k^2 - 152k^2 + 144k - 48) k}{6k (k - 1)^3} \epsilon + O(\epsilon^2), \\
\hat{a}_{22} & = \frac{4(2k^3 - 7k^3 + 10k^2 - 8k + 4) k}{k (k - 1) (k - 2)^2} - \frac{(7k^6 - 33k^5 + 76k^4 - 124k^3 + 152k^2 - 112k + 32) k}{4k (k - 1)^3 (k - 2)} \epsilon + O(\epsilon^2), \\
\hat{a}_{13} & = \frac{4(k^2 - 9k^2 + 18k - 12) k}{3 (k - 1) (k - 2)^4} + \frac{(4k^4 - 7k^4 - 11k^3 + 36k - 24) k^2}{6 (k - 1)^4 (k - 2)^2} \epsilon + O(\epsilon^2), \\
\hat{a}_{04} & = \frac{2(3k^2 - 8k + 6) k^3}{3 (k - 1) (k - 2)^4} - \frac{(13k^4 - 25k^3 - 22k^2 + 80k - 48) k^3}{24 (k - 1)^3 (k - 2)^3} \epsilon + O(\epsilon^2).
\end{align*}
\]
\[
\begin{align*}
\delta_{50} &= -\frac{4(k^2 - 5k + 5)}{5k (k - 1)} + \frac{(5k^3 - 113k^2 + 240k - 120) (k - 3)^2}{120k (k - 1)^3} \epsilon + O(\epsilon^2), \\
\delta_{41} &= -\frac{4(2k - 3)(2k^2 - 5k + 6)}{3k (k - 1)(k - 2)} + \frac{(17k^5 - 125k^4 + 65k^3 - 924k^2 + 864k - 288)}{24k (k - 1)^3} \epsilon + O(\epsilon^2), \\
\delta_{32} &= -\frac{8(4k^4 - 19k^3 + 36k^2 - 36k + 18)}{3k (k - 1)(k - 2)^2} \\
&\quad + \frac{(21k^6 - 147k^5 + 490k^4 - 1024k^3 + 1368k^2 - 1008k + 288)}{12k (k - 1)^3(k - 2)} \epsilon + O(\epsilon^2), \\
\delta_{23} &= -\frac{8(k^4 - 4k^2 + 6k^2 - 4k + 2)}{k (k - 1)(k - 2)^2} + \frac{(k^8 - 43k^6 + 198k^5 - 384k^4 + 456k^3 - 336k + 96)}{12k (k - 1)^3(k - 2)} \epsilon + O(\epsilon^2), \\
\delta_{14} &= -\frac{4(5k^4 - 23k^3 + 45k^2 - 48k + 24) k}{3(k - 1)(k - 2)^4} + \frac{3k^3 - 139k^2 + 2100k^2 - 264k^2 + 336k - 192}{24(k - 1)^3(k - 2)^3} k^2 \epsilon + O(\epsilon^2), \\
\delta_{05} &= -\frac{4k^3(2k - 20k^2 + 45k - 30)}{15(k - 2)^4(k - 1)} - \frac{k^3 (15k^3 - 3k^2 + 76k^2 - 580k + 960)}{120(k - 1)^3(k - 2)^4} k + O(\epsilon^2), \\
\epsilon_{20} &= -\frac{(k^2 - 6k + 4) k}{k (k - 1)(k - 2)^2} + \frac{(5k^3 - 15k^2 + 22k - 8)}{8(k - 1)^3(k - 2)} \epsilon + O(\epsilon^2), \\
\epsilon_{11} &= -\frac{2(2k - 1)}{k - 1} - \frac{(3k^2 - 3k + 1) (k - 2)^2}{8(k - 1)^3} \epsilon + O(\epsilon^2), \\
\epsilon_{02} &= \frac{(k^2 - 6k + 4) k}{(k - 1)(k - 2)^2} + \frac{(2k^3 - 15k^2 + 22k - 8) k^2}{8(k - 1)^3(k - 2)} \epsilon + O(\epsilon^2), \\
\epsilon_{30} &= -\frac{2(2k - 3)}{(k - 1)(k - 2)} + \frac{(3k^3 - 25k^2 + 48k - 24)(k - 2)^2}{24k (k - 1)^4} \epsilon + O(\epsilon^2), \\
\epsilon_{21} &= \frac{2(2k^2 - 3k + 2)}{k (k - 1)} + \frac{(7k^3 - 17k^2 + 20k - 8)(k - 2)^2}{8(k - 1)^3} \epsilon + O(\epsilon^2), \\
\epsilon_{12} &= \frac{2k^4}{(k - 1)(k - 2)^2} + \frac{(5k^4 - 33k^3 + 62k^2 - 48k + 16) k}{8(k - 1)^3(k - 2)} \epsilon + O(\epsilon^2), \\
\epsilon_{30} &= \frac{2(2k - 3) k^2}{3(k - 1)(k - 2)^2} + \frac{(3k - 1)(5k - 6) k^3}{24(k - 1)^3(k - 2)} \epsilon + O(\epsilon^2), \\
\epsilon_{20} &= \frac{2(2k - 3)(k - 2)}{3k(k - 1)} + \frac{(5k^3 - 31k^2 + 52k - 24)(k - 2)^2}{24(k - 1)^4} \epsilon + O(\epsilon^2), \\
\epsilon_{11} &= \frac{-4(3k^2 - 7k + 6)}{3k (k - 1)} + \frac{(4k^3 - 17k^2 + 27k - 12)(k - 2)^2}{6k (k - 1)^5} \epsilon + O(\epsilon^2), \\
\epsilon_{22} &= -\frac{4(k^2 - 2k + 2)^2}{k(k - 1)(k - 2)^2} - \frac{(k^6 - 7k^5 - 4k^4 + 4k^3 - 136k^2 + 112k - 32)}{4k (k - 1)^3(k - 2)} \epsilon + O(\epsilon^2), \\
\epsilon_{13} &= -\frac{4(5k^2 - 9k + 6) k}{3(k - 1)(k - 2)^2} + \frac{(10k^3 - 29k^2 + 35k - 18) k^2}{6(k - 1)^4} \epsilon + O(\epsilon^2), \\
\epsilon_{04} &= \frac{2k^3}{3(k - 1)(k - 2)^2} - \frac{(7k^3 - 25k^2 + 40k - 24)(k - 2)^2}{24(k - 1)^3(k - 2)^2} \epsilon + O(\epsilon^2), \\
\epsilon_{50} &= -\frac{2(3k - 5) (k - 2)}{5k (k - 1)} - \frac{(25k^3 - 153k^2 + 260k - 120)(k - 2)^2}{120k (k - 1)^4} \epsilon + O(\epsilon^2), \\
\epsilon_{41} &= \frac{2(7k^2 - 19k + 18)}{3k (k - 1)} - \frac{(13k^3 - 81k^2 + 152k - 72)(k - 2)^2}{24k (k - 1)^4} \epsilon + O(\epsilon^2), \\
\epsilon_{32} &= \frac{4(7k^4 - 33k^2 + 68k^2 - 72k + 36)}{3k (k - 1)(k - 2)^2}.
\end{align*}
\]
\[ \hat{c}_{23} = \frac{4(3k^4 - 9k^3 + 12k^2 - 8k + 4)}{k(1 - k)(k - 2)^2} - \frac{12k(1 - 3k + 4k^2 - k^3)}{k(1 - k)(k - 2)^2} \epsilon + O(\epsilon^2), \]

\[ \hat{c}_{14} = \frac{2(5k^3 - 23k^2 + 36k - 24)}{3(k - 1)(k - 2)^3} \epsilon + O(\epsilon^2), \]

\[ \hat{c}_{05} = \frac{2(11k^2 - 35k + 30)}{15(k - 1)(k - 2)^3} + \frac{24(k - 1)(k - 2)^2}{120(k - 1)(k - 2)^3} \epsilon + O(\epsilon^2). \]

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Received February 2020; revised May 2020.

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