Brans-Dicke Theory on $M_4 \times Z_2$ Geometry

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Abstract

The gauge theory on $M_4 \times Z_2$ geometry is applied to the Brans-Dicke(BD) theory, where $M_4$ is the four dimensional space-time and $Z_2$ is a discrete space with two points. This approach had been previously proposed by Konisi and Saito without recourse to noncommutative geometry(NCG). Since our approach is geometrically simpler and clearer than NCG, one can see more directly the effect of the $Z_2$ space in obtaining the BD theory.

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§1. Introduction

The noncommutative geometry (NCG) of Connes\cite{1, 2} has been successful in giving a geometrical interpretation of the standard model as well as some grand unification models. In this interpretation the Higgs fields are regarded as gauge fields along directions in the discrete space. The bosonic parts of actions are just the pure Yang-Mills actions containing gauge fields on both continuous and discrete spaces. They contain Higgs potentials which automatically generate the spontaneous symmetry breaking. The Yukawa coupling is regarded as a kind of gauge interactions of fermions.

There are now various alternative versions of NCG\cite{3}. Any NCG, however, has so far been algebraic rather than geometric. Nobody has considered the original geometric meaning such as covariant differences, parallel transportations of vector fields, curvatures and so on in the discrete space. In a previous paper\cite{4} two of the authors (G.K. and T.S.) have considered such a geometric meaning of NCG and proposed a gauge theory on $M_4 \times Z_N$ geometry without recourse to any knowledge of such NCG. This approach appears to be geometrically much simpler and clearer than NCG. Here the Higgs fields are introduced as mapping functions between any pair of vector fields belonging independently to the $N$-sheeted space-time, just as the Yang-Mills fields are so between both vectors on $x$ and $x + \Delta x$. This approach has been applied to the Weinberg-Salam model for electroweak interactions\cite{4} and to the $N = 2$ and 4 super Yang-Mills theories\cite{4}.

Besides model buildings in particle physics there are some works on gravity in terms of NCG on $M_4 \times Z_2$\cite{5}–\cite{10}, where a scalar field is coupled to gravity. However, the scalar fields in those papers are different from each other. The reason of this owes partly to different torsion-free conditions used in their papers. The so-called “partial” torsion-free condition used in Refs.\cite{9, 10} exactly leads to the Brans-Dicke (BD) theory\cite{11}. However, the use of “total” torsion-free condition makes NCG into the usual commutative geometry and leads to the other kinds of scalar-tensor theory (including electromagnetism of the Kaluza-Klein type).

Another reason owes mostly to the lack of a proper formalism of gravity in terms of NCG. In this paper we consider the derivation of the BD theory from the gauge theory on $M_4 \times Z_2$ without recourse to NCG. For this purpose we need the equivalence assumption proposed in the previous work\cite{10}. This is stated as follows: The manifold $M_4 \times Z_2$ may be regarded as a pair of the $M_4$’s, each at the point $p = +$ or $p = -$ on $Z_2$. The physics requires that these two $M_4$-pieces should be equivalent to each other. We assume that the equivalence is attained by a limiting process with some parameter $\varepsilon$ which tends to zero. As a technical tool to express the limiting process we introduce a coordinate $z(p)$ on $Z_2(p = \pm)$ such that
a difference
\[ \Delta z(p) = z(p) - z(p') \sim \varepsilon \quad (p \neq p') \] (1.1)
is proportional to the limiting process parameter \( \varepsilon \). Let \( \psi(z(p)) \) be any function on \( Z_2 \). Since \( \psi(z(p)) \) is a linear function of \( z(p) \), one may put
\[ \psi(z(p)) = A + Bz(p), \]
(1.2)
where \( A \) and \( B \) are some constants. From this we find that its Taylor expansion is cut off only up to the first order \( \Delta z(p) \)
\[ \psi(z(p)) = \psi(z(p')) + B\Delta z(p). \] (1.3)
Now, by the equivalence we mean that
\[ \psi(x, z(p)) \rightarrow \psi(x, z(p')) \quad \text{as} \quad \varepsilon \rightarrow 0, \] (1.4)
where the coordinate \( x \) on \( M_4 \) is inserted.

In the next section, by using this equivalence assumption we derive the BD theory. Since our approach is geometrically simple, one can see more directly the effect of the \( Z_2 \) space. The final section is devoted to concluding remarks.

§2. Brans-Dicke Theory on \( M_4 \times Z_2 \)

We regard the manifold \( M_4 \times Z_2 \) to be the Kaluza-Klein like space where the fifth continuous dimension is replaced by two discrete points \( z(+) \) and \( z(-) \). The line element \( \Delta s \) of this space is assumed to be
\[ \Delta s^2 = g_{\mu\nu}(x)\Delta x^\mu \Delta x^\nu + \lambda^2(x)\Delta z^2 \]
\[ = G_{MN}(x)\Delta x^M \Delta x^N, \] (2.1)
where
\[ \Delta x^N = (\Delta x^\mu, \Delta x^5 \equiv \Delta z), \quad \mu = 0, 1, 2, 3 \] (2.2)
and \( \Delta z = z(+) - z(-) \). Here we have considered a simple case that the four-dimensional metric \( g_{\mu\nu}(x) \) and the scalar field \( \lambda(x) \) are independent of \( z(\pm) \) on \( Z_2 \) and are functions only of \( x \in M_4 \). The \( G_{MN}(x) \) is regarded as the five-dimensional metric of \( M_4 \times Z_2 \).
Let $\psi^M(x, p)$ be a vector field on $M_4 \times Z_2$ with $p = (+, -)$. We first consider the parallel transportation of $\psi^M(x, p)$ from $(x, p)$ to $(x + \Delta x, p)$. As usual this is achieved by the mapping function $H^M_N(x + \Delta x, x, p)$

$$\psi^M(x + \Delta x, p) = H^M_N(x + \Delta x, x, p)\psi^N(x, p).$$

(2.3)

In the familiar notation with the affine connection $\Gamma^M_{N\mu}(x)$ we set

$$H^M_N(x + \Delta x, x, p) = \delta^M_N - \Gamma^M_{N\mu}(x)\Delta x^\mu + \frac{1}{2}C^M_{N\mu\nu}(x)\Delta x^\mu \Delta x^\nu + \cdots,$$

(2.4)

where $\Gamma^M_{N\mu}(x)$ and $C^M_{N\mu\nu}(x)$ are assumed to be independent of $p = \pm \in Z_2$. We then consider the Riemann curvature of the usual type corresponding to Fig.1. The parallel transportations of $\psi^M(x, p)$ from $x$ to $x + \Delta_1 x + \Delta_2 x$ through $x + \Delta_1 x$ is given by

$$C_1 = H^M_L(x + \Delta_1 x + \Delta_2 x, x + \Delta_1 x)H^L_N(x + \Delta_1 x, x)\psi^N(x, p).$$

(2.5)

In the same way the other parallel transportations along the path $C_2$ is

$$C_2 = H^M_L(x + \Delta_1 x + \Delta_2 x, x + \Delta_2 x)H^L_N(x + \Delta_2 x, x)\psi^N(x, p).$$

(2.6)

The difference between two parallel transportations $C_1$ and $C_2$ gives the Riemann curvature $R^M_{N\mu\nu}$. Substituting (2.4) into $C_1$ and $C_2$, we have

$$C_1 - C_2 = -R^M_{N\mu\nu}(x)\psi^N(x, p)\Delta_1 x^\mu \Delta_2 x^\nu,$$

(2.7)

where

$$R^M_{N\mu\nu}(x) = \partial_\mu \Gamma^M_{N\nu} - \partial_\nu \Gamma^M_{N\mu} + \Gamma^M_{L\mu} \Gamma^L_{N\nu} - \Gamma^M_{L\nu} \Gamma^L_{N\mu}.$$

(2.8)

The Riemann curvature of the second type corresponds to Fig.2. The parallel transportations of $\psi^M(x, +)$ from $(x, +)$ to $(x + \Delta x, -)$ through $(x, -)$ and $(x + \Delta x, +)$ are given by

$$C_3 = H^M_L(x + \Delta x, x, -)H^L_N(x, -, +)\psi^N(x, +)$$

(2.9)
and

\[ C_4 = H^M_L(x + \Delta x, -)H^L_N(x + \Delta x, x, +)\psi^N(x, +), \]  

(2.10)

respectively, where \( H(x, -, +) \) and \( H(x + \Delta x, -, +) \) are mapping functions and

\[ H^L_N(x, -, +)\psi^N(x, +) \]  

(2.11)

shows the parallel transportation of \( \psi^N(x, +) \) from \((x, +)\) to \((x, -)\). The difference between two parallel transportations \( C_3 \) and \( C_4 \) gives the Riemann curvature \( R^M_{N\mu5} \). Introducing the affine connection \( \Gamma^L_{N5}(x) \), which is also assumed to be independent of \( p \), and from the equivalence assumption, one may put

\[ H^L_N(x, x, -) = \delta^L_N - \Gamma^L_{N5}(x)\Delta z, \]  

(2.12)

\[ H^L_N(x, +, -) = \delta^L_N + \Gamma^L_{N5}(x)\Delta z, \]  

(2.13)

so that

\[ C_3 - C_4 = R^M_{N\mu5}(x)\psi^N(x, +)\Delta x^\mu\Delta z, \]  

(2.14)

where

\[ R^M_{N\mu5}(x) = \partial_\mu \Gamma^M_{N5} + \Gamma^M_{L\mu} \Gamma^L_{N5} - \Gamma^M_{L5} \Gamma^L_{N\mu}. \]  

(2.15)

Note that in Eqs. (2.12) and (2.13) there are no term of \( O(\Delta z^2) \) as shown in Appendix A.

The Riemann curvature of the third type corresponds to Fig.3. Namely, \( \psi^M(x, +) \) is compared with \( \psi^M_\parallel(x, +) \) which is the parallel-transported vector of \( \psi^M(x, +) \) from \((x, +)\) to \((x, -)\) and then returning to \((x, +)\), i.e.,

\[ \psi^M(x, +) - \psi^M_\parallel(x, +) = [\delta^M_N - H^M_L(x, +, -)H^L_N(x, -, +)]\psi^N(x, +) \]
\[ = [\delta^M_N - (\delta^M_L + \Gamma^M_{L5}\Delta z)(\delta^L_N - \Gamma^L_{N5}\Delta z)]\psi^N(x, +) \]
\[ = \Gamma^M_{L5}(x)\Gamma^L_{N5}(x)\psi^N(x, +)\Delta z\Delta z. \]  

(2.16)

This gives the Riemann curvature of the third type

\[ R^M_{N55}(x) = \Gamma^M_{L5}(x)\Gamma^L_{N5}(x). \]  

(2.17)
On $M_4$ we know that there is no curvature of the similar type corresponding to two paths: $x \rightarrow x + \Delta x \rightarrow x$ and $x \rightarrow x$. On $Z_2$, however, we find non-vanishing curvature of the third type (see Appendix B).

Now we would like to express the affine connection $\Gamma^L_{MN}(x)$ in terms of the metric $G_{MN}(x)$. This is achieved in the usual way if we use the covariant-free equation

$$\nabla_K G_{MN} \equiv \partial_K G_{MN} - \Gamma_{MKN} - \Gamma_{NKM} = 0, \quad (2.18)$$

where $\Gamma_{MKN} \equiv G_{ML} \Gamma^L_{KN}$, and if $\Gamma_{MKN}$ is symmetric with respect to $K$ and $N$, i.e., $\Gamma_{MKN} = \Gamma_{MKN}$. The equation (2.18) will follow from the requirement that the length of any vector field is invariant under the parallel transportation, i.e.,

$$G_{MN}(X + \Delta X) \psi^M(X + \Delta X) = G_{MN}(X) \psi^M(X) \psi^N(X), \quad (2.19)$$

where $X = (x^\mu, z(p))$. The result is known to be

$$\Gamma^L_{MN} = \Gamma^L_{NM} = \frac{1}{2} G^{LK}(\partial_M G_{KN} + \partial_N G_{KM} - \partial_K G_{MN}). \quad (2.20)$$

In the present case one may set $\partial_5 G_{MN} = 0$, because $G_{MN}$ is independent of $z(p)$.

Since the metric $G_{MN}$ and its inverse $G^{MN}$ are given by

$$G_{MN} = \begin{pmatrix} g_{\mu\nu}(x) & 0 \\ 0 & \lambda^2(x) \end{pmatrix}, \quad (2.21)$$

$$G^{MN} = \begin{pmatrix} g^{\mu\nu}(x) & 0 \\ 0 & \lambda^{-2}(x) \end{pmatrix}, \quad (2.22)$$

we get all components of $\Gamma^L_{MN}$

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}), \quad (2.23a)$$

$$\Gamma^5_{\mu\nu} = 0, \quad (2.23b)$$

$$\Gamma^5_{5\nu} = -\lambda \partial^\rho \lambda, \quad (2.23c)$$

$$\Gamma^5_{5\mu} = 0, \quad (2.23d)$$

$$\Gamma^5_{55} = \frac{1}{\lambda} \partial_\nu \lambda, \quad (2.23e)$$

$$\Gamma^5_{55} = 0. \quad (2.23f)$$
Substituting (2.23) into (2.8), (2.15) and (2.17), one obtains the Riemann curvature components

\[
R^\rho{}_{\sigma \mu \nu} = \partial_\mu \Gamma^\rho{}_{\sigma \nu} - \partial_\nu \Gamma^\rho{}_{\sigma \mu} + \Gamma^\rho{}_{\lambda \mu} \Gamma^\lambda{}_{\sigma \nu} - \Gamma^\rho{}_{\lambda \nu} \Gamma^\lambda{}_{\sigma \mu} = -R^\rho{}_{\sigma \nu \mu},
\]

(2.24a)

\[
R^5{}_{\mu \nu} = -\overset{\circ}{\nabla}_\nu (\frac{1}{\lambda} \partial_\mu \lambda) - \frac{1}{\lambda^2} \partial_\nu \partial_\mu \lambda = -\frac{1}{\lambda} \overset{\circ}{\nabla}_\nu \partial_\mu \lambda = -R^5{}_{\mu \nu 5},
\]

(2.24b)

\[
R^\mu{}_{5 \nu 5} = \overset{\circ}{\nabla}_\nu (\lambda \partial^\mu \lambda) - \partial^\mu \lambda \partial_\nu \lambda = \lambda \overset{\circ}{\nabla}_\nu \partial^\mu \lambda = -R^\mu{}_{5 \nu 5},
\]

(2.24c)

\[
R^\mu{}_{5 55} = -\partial^\mu \lambda \partial_\nu \lambda,
\]

(2.24d)

\[
R^5{}_{5 55} = -\partial_\mu \lambda \partial^\mu \lambda,
\]

(2.24e)

and all other components are zero, where \(\overset{\circ}{\nabla}_\nu\) is the covariant derivative on \(M_4\), i.e.,

\[
\overset{\circ}{\nabla}_\nu V^\mu \equiv \partial_\nu V^\mu + \Gamma^\rho{}_{\mu \nu} V^\rho,
\]

(2.25)

\[
\overset{\circ}{\nabla}_\nu V_\mu \equiv \partial_\nu V_\mu - \Gamma^\rho{}_{\mu \nu} V_\rho.
\]

(2.26)

The Ricci curvature on \(M_4 \times Z_2\) is, therefore, given by

\[
R_{MN} = \sum_L R^L_{MLN}
\]

= \(R^\rho{}_{M \rho \nu} + R^5{}_{M 5 \nu}\),

(2.27a)

hence

\[
R_{\mu \nu} = \overset{\circ}{R}_{\mu \nu} + R^5{}_{\mu 5 \nu}
\]

= \(\overset{\circ}{R}_{\mu \nu} - \frac{1}{\lambda} \overset{\circ}{\nabla}_\nu \partial_\mu \lambda\),

(2.27b)

\[
R_{5 \nu} = R_{5 \nu 5} = 0,
\]

(2.27c)

\[
R_{5 5} = R^\rho{}_{5 \rho 5} + R^5{}_{5 5 5}
\]

= \(-\lambda \overset{\circ}{\nabla}_\rho \partial^\rho \lambda - \partial_\mu \lambda \partial^\mu \lambda\),

(2.27d)

where

\[
\overset{\circ}{R}_{\mu \nu} \equiv R^\rho{}_{\mu \rho \nu}
\]

(2.27e)

is the usual four-dimensional Ricci curvature on \(M_4\). Finally, therefore, we get the scalar curvature on \(M_4 \times Z_2\)

\[
R = G^{MN} R_{MN} = g^{\mu \nu} R_{\mu \nu} + G^{55} R_{5 5}
\]

= \(\overset{\circ}{R} - 2 \frac{1}{\lambda} \overset{\circ}{\nabla}_\nu \partial^\nu \lambda - \frac{1}{\lambda^2} \partial_\mu \lambda \partial^\mu \lambda\).

(2.28)
By using \( \det(G_{MN}) = \det(g_{\mu\nu})^2 \equiv g^2 \lambda^2 \), we obtain the gravity action

\[
I = \int_{M_4} \int_{Z_2} \sqrt{-\det(G_{MN})} \, R
\]

\[
= \int_{M_4} \sqrt{-g} \lambda \left[ \hat{R} - 2 \frac{\partial}{\partial x^\mu} \partial^\mu \lambda - \frac{1}{\lambda^2} \partial_\mu \lambda \partial^\mu \lambda \right]
\]

\[
= \int_{M_4} \sqrt{-g} \left[ \lambda \hat{R} - 2 \partial_\mu \lambda \partial^\mu \lambda - \frac{1}{\lambda} \partial_\mu \lambda \partial^\mu \lambda \right]
\]

\[
= \int_{M_4} \sqrt{-g} \left[ \lambda \hat{R} - \frac{1}{\lambda} \partial_\mu \lambda \partial^\mu \lambda \right],
\]

which is just the BD theory obtained in the previous works\[9, 10\].

§3. Concluding remarks

On the basis of the equivalence assumption stated in the introduction we have derived the Brans-Dicke theory on the manifold \( M_4 \times Z_2 \). We have used the gauge theory on \( M_4 \times Z_2 \) geometry without recourse to NCG. The BD kinetic term comes from the Riemann curvature of the third type corresponding to Fig.3. This sharply differs from the Kaluza-Klein theory with the scalar field, in which there is no BD kinetic term.

On the continuous manifold one can see that there is no curvature coming from the difference between two paths: \( x \rightarrow x + \Delta x \rightarrow x \) and \( x \rightarrow x \). On the discrete manifold, however, we find non-vanishing curvatures of the third type, which corresponds to two paths: \( A \rightarrow B \rightarrow A \) and \( A \rightarrow A \). These have been discussed in Appendix B. The result shows that the BD scalar field is related to the \( Z_2 \) discrete space and is interpreted to describe the distance between the two points in this space. The affine connection \( \Gamma^M_{N\delta}(x) \) has been introduced as a gauge field along the \( Z_2 \) direction. Its non-zero components are given in (2.23c,e) in terms of the BD scalar field.

Finally we consider the torsion of \( M_4 \times Z_2 \). There are three kinds of torsions: \( \Gamma^M_{\mu\nu} - \Gamma^M_{\nu\mu} \), \( \Gamma^M_{\mu\delta} - \Gamma^M_{\delta\mu} \) and \( \Gamma^M_{55} \). Since the affine connection \( \Gamma^M_{LN} \) is symmetric with respect to \( L \) and \( N \), the first and second torsions vanish. However, the last torsion \( \Gamma^M_{55} = -\delta^M_5 \lambda \partial^\mu \lambda \) is not zero except for \( \lambda = \text{constant} \). This is undoubtedly related to the \( Z_2 \) space and \( \lambda \).
Appendix A

Since any function on $\mathbb{Z}_2$ is a linear function of coordinates $z(+) = z$ and $z(-) = z'$, one may put

$$H(z, z') = A + Bz + B'z' + Czz',$$

(A.1)

where $A, B, B'$ and $C$ are independent of $z$ and $z'$. From $H(z, z) = 1$ one gets

$$A + (B + B')z + Cz^2 = 1.$$  

(A.2)

Without loss of generality we can put $z = \pm \varepsilon$, where $\varepsilon$ is the limiting process parameter $|1.1|$. Then it follows that

$$A = 1 - C\varepsilon^2, \quad B' = -B,$$

(A.3)

hence

$$H(z, z') = 1 + B(z - z') + C(z^2 - z'^2)$$

$$= 1 + (B - Cz)\Delta z$$

$$= 1 + \Gamma\Delta z.$$  

(A.4)

In (2.12) and (2.13) the affine connection $\Gamma$ is assumed to be independent of $z$, i.e., $\Gamma = B$ and $C = 0$. In this Appendix we have omitted suffices of $H^M_N$ and $\Gamma^M_{N5}$.

Appendix B

Let us consider two paths on $M_4$

$$x \rightarrow x + \Delta_1 x \rightarrow (x + \Delta_1 x) + \Delta_2 x,$$  

(B.1)

$$x \rightarrow x + (\Delta_1 x + \Delta_2 x).$$  

(B.2)

The difference $\Delta$ between two mapping functions is given by

$$\Delta = H((x + \Delta_1 x) + \Delta_2 x, x + \Delta_1 x)H(x + \Delta_1 x, x) - H(x + (\Delta_1 x + \Delta_2 x), x).$$

(B.3)
Substituting Eq. (2.4), i.e.,
\[ H(x + \Delta x, x) = 1 - \Gamma_{\mu}(x)\Delta x^{\mu} + \frac{1}{2} C_{\mu\nu}(x)\Delta x^{\mu}\Delta x^{\nu} + \cdots \] (B.4)
into (B.3), we have
\[ \Delta = (\Gamma_{\mu\nu} - \Gamma_{\nu\mu} - C_{\mu\nu}) \Delta_{1} x^{\mu} \Delta_{2} x^{\nu}. \] (B.5)

If we choose \( \Delta_{2} x^{\mu} = \alpha \Delta_{1} x^{\mu} (\alpha > 0) \), two paths (B.1) and (B.2) become the same. In this case the difference \( \Delta \) vanishes so that
\[ C_{\mu\nu} = \frac{1}{2} (\Gamma_{\mu\nu} - \Gamma_{\nu\mu}). \] (B.6)

Substituting this into (B.5) we find
\[ \Delta = \frac{1}{2} (\Gamma_{\mu\nu} - \Gamma_{\nu\mu} + \Gamma_{\nu\mu}) \Delta_{1} x^{\mu} \Delta_{2} x^{\nu} = \frac{1}{2} R_{\mu\nu} \Delta x^{\mu} \Delta x^{\nu}, \] (B.7)
where \( R_{\mu\nu} \) is the Riemann curvature.

Let us now choose \( \alpha = -1 \), i.e., \( \Delta_{2} x^{\mu} = -\Delta_{1} x^{\mu} = -\Delta x^{\mu} \). Then Eq. (B.3) is reduced to
\[ \Delta = H(x, x + \Delta x) H(x + \Delta x, x) - 1 = -\frac{1}{2} R_{\mu\nu} \Delta x^{\mu} \Delta x^{\nu} = 0, \] (B.8)
because \( R_{\mu\nu} \) is antisymmetric with respect to \( \mu \) and \( \nu \). This shows that on \( M_4 \) there is no curvature corresponding to two paths: \( x \rightarrow x + \Delta x \rightarrow x \) and \( x \rightarrow x \).

In the discrete space, say \( Z_N \), there is no case such that two paths \( A \rightarrow B \rightarrow C \) and \( A \rightarrow C \) become the same. So we calculate in \( Z_2 \) directly \( \Delta \) defined by
\[ \Delta = 1 - H(+, -) H(-, +). \] (B.9)
Substituting (A.4) into (B.9) we find (2.17), the Riemann curvature of the third type.
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