A K-THEORETIC CLASSIFICATION FOR CERTAIN $\mathbb{Z}/p\mathbb{Z}$ ACTIONS ON AF ALGEBRAS

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Abstract. A K-theoretic classification is given of the $C^*$-dynamical systems $\lim(A_n, \alpha_n, G)$ where $A_n$ is finite dimensional and $G$ is any cyclic group of prime order (namely, $G = \mathbb{Z}/p\mathbb{Z}$ for some prime number $p$). Such actions contain N. C. Phillips' natural examples of finite group actions on UHF algebras which don't have the tracial Rokhlin property in [8].

1. Introduction

A number of results concerning the classification of $C^*$-algebras have been obtained under the Elliott programme. However, classification of group actions on $C^*$-algebras is still a far less developed subject, partially because of K-theoretical difficulties. When the $C^*$-algebra and the group action have an inductive limit structure, then the equivariant version of Elliott's intertwining argument can be used for classifying such group actions.

Given a compact group $G$, let $A = \lim A_n$ be the inductive limit of a sequence of finite dimensional $C^*$-algebras, let $\alpha = \lim \alpha_n$ be an inductive limit action of $G$ on $A$. Then one can form the $C^*$-algebra cross product $A \rtimes \alpha G = \lim A_n \rtimes \alpha_n G$. If each $\alpha_n$ is given by inner automorphisms arisen from a unitary representation of the group $G$, then it was shown in [5] that the natural K-theory data of $A \rtimes \alpha G$ is a complete invariant for the $C^*$-dynamical system $(A, \alpha, G)$. Such actions they referred to as locally representable. In the case that $A$ is unital, the K-theory data in [5] consists of the K-group $K_0(A \rtimes \alpha G)$ together with (i) the natural order structure, (ii) the special element coming from the projection given by averaging the canonical unitaries of the cross product, (iii) the natural module structure over the representation ring $K_0(G)$. In [7], Kishimoto considered actions of finite groups on inductive limit algebras with more complicated building blocks (circles), and in [2], this study was extended to still more complicated inductive limit systems and to general compact groups.

In all of these cases above, it was assumed that the actions still satisfied a local representability condition. So it is interesting to consider the case in which the group action is not necessarily inner. Along this line, in [4], G. A. Elliott and H. Su removed this local representability hypothesis in the case where the group is $\mathbb{Z}/2\mathbb{Z}$ and the building blocks are finite dimensional. In [10], this local representability condition was also removed, where the group is still $\mathbb{Z}/2\mathbb{Z}$, but the inductive limits are certain real rank zero systems built on some subhomogeneous graph $C^*$-algebras. Then, it is an natural question that to which extent one can

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obtain a K-theoretic classification for more general group actions on C*-algebras, conceivably, finite abelian group actions will be a quite large class. To do this, from group structure theory, the p (prime) groups (groups with order being some power of p) cases will be fundamental, and among them, the cyclic groups with prime orders should be the first test case. In the present paper, a K-theoretic classification for inductive limit actions of cyclic groups with prime orders on AF (approximately finite dimensional) algebras will be obtained.

On the other hand, there is another class of group actions on C*-algebras which draw many people’s attention, namely, the group actions with (tracial) Rokhlin property. For the integer group \( \mathbb{Z} \) and \( \mathbb{Z}^d \), I. Hirshberg, W. Winter, and J. Zacharias (in [8]) and N. C. Phillips (in [9]) showed that the actions with the tracial Rokhlin property are generic for nice C*-algebras (for example, TAF algebras). For finite group action case, N. C. Phillips give natural examples in [8] that there are inductive limit actions of cyclic groups on UHF algebras which don’t have the tracial Rokhlin property. We quote his example of \( \mathbb{Z}_2 \) action here:

\[
A = \lim_{\longrightarrow} M_{2^n}, \quad \alpha = \lim_{\longrightarrow} \alpha_n, \quad \alpha_n = Ad\left(\begin{array}{cc} 12^n-1 & 0 \\ 0 & -1 \end{array}\right),
\]

where the distribution of the eigenvalues of the unitaries indicate that this action doesn’t have tracial Rokhlin property (see Example 2.9 in [8] for detail). Since such examples have inductive limit structure, they sit in our classification results.

Throughout this paper, let us denote the group \( \mathbb{Z}/p\mathbb{Z} \) by \( \mathbb{Z}_p \), where \( p \) is a prime. To state the invariant, let \( A \) be a unital C*-algebra, and let \( \alpha \) be a group action of \( \mathbb{Z}_p \) on \( A \). The invariant we need is as follows:

1. \( (K_0(A), K_0(A^+), [1_A], \alpha) \),
2. \( (K_0(A \rtimes_{\alpha} \mathbb{Z}_p), K_0(A \rtimes_{\alpha} \mathbb{Z}_p)^+, \zeta, \hat{\alpha}) \), where \( \zeta \) is the special element in \( K_0(A \rtimes_{\alpha} \mathbb{Z}_p) \) and \( \hat{\alpha} \) is the dual action of \( \hat{\alpha} \) on \( A \rtimes_{\alpha} \mathbb{Z}_p \),
3. \( \iota_* : K_0(A) \rightarrow K_0(A \rtimes_{\alpha} \mathbb{Z}_p) \), where \( \iota \) is the canonical embedding of \( A \) into \( A \rtimes_{\alpha} \mathbb{Z}_p \).

(1) and (3) are necessary, since the action may not be inner, the information in \( K_0(A) \) may not be recovered completely from \( K_0(A \rtimes_{\alpha} \mathbb{Z}_p) \), we must adjoin this, as well as the actions on the K-groups, to the invariant. We state the main theorem here.

**Theorem 1.1.** Let \( (A, \alpha, \mathbb{Z}_p) = \lim_{\longrightarrow}(A_n, \alpha_n, \mathbb{Z}_p) \) and \( (B, \beta, \mathbb{Z}_p) = \lim_{\longrightarrow}(B_n, \beta_n, \mathbb{Z}_p) \) be two approximately finite dimensional inductive limit C*-dynamical systems, let \( F \) be an scaled order preserving group isomorphism from \( (K_0(A), \alpha) \) to \( (K_0(B), \beta) \), and let \( \phi \) be an order preserving group isomorphism from \( (K_0(A \rtimes_{\alpha} \mathbb{Z}_p), \hat{\alpha}) \) to \( (K_0(B \rtimes_{\beta} \mathbb{Z}_p), \hat{\beta}) \) mapping the special element to the special element. Suppose that the following diagram commutes:

\[
\begin{array}{ccc}
K_0(A) & \longrightarrow & K_0(A \rtimes_{\alpha} \mathbb{Z}_p) \\
F & & \phi \\
K_0(B) & \longrightarrow & K_0(B \rtimes_{\beta} \mathbb{Z}_p).
\end{array}
\]

Then there is an isomorphism \( \psi \) from \( (A, \alpha, \mathbb{Z}_p) \) to \( (B, \beta, \mathbb{Z}_p) \) such that \( \psi_* = F \) and such that the extension of \( \psi \) to \( A \rtimes_{\alpha} \mathbb{Z}_p \) induces \( \phi \).
The paper is organized as follows. In Section 2, some preliminaries are given about the crossed products of finite dimensional C*-algebras with $\mathbb{Z}/p\mathbb{Z}$ actions. In Section 3, an existence result is proved, namely, morphisms between the invariant of the finite dimensional C*-dynamical systems can be lifted to morphisms between the finite dimensional C*-dynamical systems. In Section 4, a uniqueness result is obtained, namely, for any two morphisms between the finite dimensional C*-dynamical systems, if their induced maps agree on the invariant, then they are unitarily equivalent by an equivariant unitary, i.e., a unitary in the fixed point subalgebra of the codomain algebra. These two results are the main ingredients in Elliott’s intertwining argument. In Section 5, the main theorem will be proved by Elliott’s intertwining argument.

2. Preliminaries

Let $A = \bigoplus_{k=1}^{m} M_{n_k}$ be a finite dimensional C*-algebra, and let $\alpha$ be a group action of $\mathbb{Z}/p\mathbb{Z}$ on $A$. Since $\mathbb{Z}/p\mathbb{Z}$ is cyclic, then $\alpha$ is determined by the corresponding automorphism of the generator of $\mathbb{Z}/p\mathbb{Z}$. Let $\sigma$ be an order $p$ automorphism of $A$. From basic representation theory, $\sigma$ can be decomposed into a finite direct sum of irreducible actions. Each irreducible action has the form either $(M_n, \rho)$ or $(M_n \oplus \ldots \oplus M_n, \rho)$.

Let us prepare all the K-theoretic information about the irreducible actions. In the case $(M_n, \rho)$, $\rho$ is given by an order $p$ unitary $V \in M_n$: $\rho(a) = V a V^*$, $a \in M_n$.

V could be chosen to be diagonal.

Lemma 2.1. $M_n \rtimes_{\rho} \mathbb{Z}/p\mathbb{Z}$ is isomorphic to $M_n \oplus \ldots \oplus M_n$. \hspace{1cm}

Proof. The identification map is given as follows:

\[ a_0 + a_1 U_{\rho} + a_2 U_{\rho^2} + \ldots + a_{p-1} U_{\rho^{p-1}} \rightarrow (a_0 + a_1 V + a_2 V^2 + \ldots + a_{p-1} V^{p-1}, \]

\[ a_0 + e^{i2\pi \frac{a_1}{p}} a_1 V + e^{i2\pi \frac{a_2}{p}} a_2 V^2 + \ldots + e^{i2\pi \frac{2(p-1)}{p}} a_{p-1} V^{p-1}, \]

\[ a_0 + e^{i2\pi \frac{a_1}{p}} a_1 V + e^{i2\pi \frac{a_2}{p}} a_2 V^2 + \ldots + e^{i2\pi \frac{2(p-2)}{p}} a_{p-1} V^{p-1}, \]

\[ \ldots, \]

\[ a_0 + e^{i2\pi \frac{2(p-1)}{p}} a_1 V + e^{i2\pi \frac{2(p-2)}{p}} a_2 V^2 + \ldots + e^{i2\pi} a_{p-1} V^{p-1}, \]

where $U_{\rho^k}, k = 1, \ldots, p-1$ are the canonical unitaries in the cross product algebra. Then one can verify the lemma by this formula. \hfill \Box

Remark 2.2. This lemma is also true for non prime $p$.

Then $K_0(M_n) = \mathbb{Z}$ and $K_0(M_n \rtimes_{\rho} \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$, and the map from $K_0(M_n)$ to $K_0(M_n \rtimes \mathbb{Z}/p\mathbb{Z})$ sends $x$ to $(x, \ldots, x)$ and $\rho_n$ is trivial. It is well known that $\hat{\mathbb{Z}}_p = \mathbb{Z}/p\mathbb{Z}$.
and the generator of \( \hat{Z}_p \) is \( \hat{\rho} \) which takes the identity element to 1, and takes \( \rho \) to \( e^{i\frac{2\pi}{p}} \). So

\[
\hat{\rho}(\sum_{k=0}^{p-1} a_k U_\rho^k) = a_0 + e^{-i\frac{2\pi}{p}} a_1 U_\rho + e^{-i\frac{4\pi}{p}} a_2 U_\rho^2 + \ldots + e^{-i\frac{2(p-1)\pi}{p}} a_{p-1} U_\rho^{p-1}
\]

by the identification formula above, it is easily to see that \( \hat{\rho} \) and \( \hat{\rho}_* \) is the permutation given by \( (\xi_1, \xi_2, \ldots, \xi_p) \to (\xi_p, \xi_1, \ldots, \xi_{p-1}) \). Take the special element \( \zeta = (l_0, \ldots, l_{p-1}) \), where \( l_k \) is the number of the eigenvalue \( e^{i\frac{2\pi k}{p}} \) of the unitary \( V \) which implements the automorphism of \( \rho \).

In the case \( (M_n \oplus \ldots \oplus M_n, \rho) \), up to conjugacy, \( \rho \) can be chosen to have the form: \( \rho(a_1, a_2, \ldots, a_p) = (a_p, a_1, \ldots, a_{p-1}) \).

**Lemma 2.3.** \( M_n \oplus \ldots \oplus M_n \rtimes_{\alpha} Z_p \) is isomorphic to \( M_{pn} \).

**Proof.** The identification map is given as follows:

\[
(a_0^0, a_1^0, \ldots, a_{p-1}^0) + (a_0^1, a_1^1, \ldots, a_{p-1}^1) U_\rho + \ldots + (a_0^{p-1}, a_1^{p-1}, \ldots, a_{p-1}^{p-1}) U_\rho^{p-1}
\rightarrow \left( \begin{array}{cccc}
  a_0^0 & a_1^0 & \cdots & a_{p-1}^0 \\
  a_0^1 & a_1^1 & \cdots & a_{p-1}^1 \\
  \vdots & \vdots & \ddots & \vdots \\
  a_0^{p-1} & a_1^{p-1} & \cdots & a_{p-1}^{p-1}
\end{array} \right)
\]

Then \( K_0(M_n \oplus \ldots \oplus M_n \rtimes_{\alpha} Z_p) = Z \), the canonical map between the K-groups sends \( (x_1, \ldots, x_p) \) to \( \sum_{k=1}^p x_k \). Let

\[
\xi = (a_0^0, a_1^0, \ldots, a_{p-1}^0) + (a_0^1, a_1^1, \ldots, a_{p-1}^1) U_\rho + \ldots + (a_0^{p-1}, a_1^{p-1}, \ldots, a_{p-1}^{p-1}) U_\rho^{p-1},
\]

so

\[
\hat{\rho}(\xi) = (a_0^0, a_1^0, \ldots, a_{p-1}^0) + e^{-i\frac{2\pi}{p}} (a_0^1, a_1^1, \ldots, a_{p-1}^1) U_\rho + \ldots + e^{-i\frac{2(p-1)\pi}{p}} (a_0^{p-1}, a_1^{p-1}, \ldots, a_{p-1}^{p-1}) U_\rho^{p-1}.
\]

Then by the identification in Lemma 2.2, the dual action is as follows:

\[
\hat{\rho}(C) = \left( \begin{array}{cccc}
  1 & e^{i\frac{2\pi}{p}} & \cdots & e^{i\frac{2(p-1)\pi}{p}} \\
  e^{-i\frac{2\pi}{p}} & \cdots & \cdots & e^{-i\frac{2(p-1)\pi}{p}} \\
  \cdots & \cdots & \cdots & \cdots \\
  e^{i\frac{2(p-1)\pi}{p}} & e^{-i\frac{2\pi}{p}} & \cdots & \cdots
\end{array} \right) C \left( \begin{array}{cccc}
  1 & e^{-i\frac{2\pi}{p}} & \cdots & e^{-i\frac{2(p-1)\pi}{p}} \\
  e^{i\frac{2\pi}{p}} & \cdots & \cdots & e^{i\frac{2(p-1)\pi}{p}} \\
  \cdots & \cdots & \cdots & \cdots \\
  e^{-i\frac{2(p-1)\pi}{p}} & e^{i\frac{2\pi}{p}} & \cdots & \cdots
\end{array} \right),
\]

for all \( C \in M_{pn} \). Hence \( \rho_* \) is the permutation and \( \hat{\rho}_* \) is trivial. Take the special element \( \zeta \) to be \( n \).

**Remark 2.4.** Lemma 2.3 also verify the Takai Duality for \( (M_n, Z_p) \).
3. Existence

In this section, we are going to establish an existence result, which states that morphisms between the invariant of the finite dimensional C*-dynamical systems can be lifted to morphisms between the finite dimensional C*-dynamical systems. This existence theorem together with the uniqueness theorem in next section are the two main ingredients in Elliott’s intertwining argument.

**Theorem 3.1.** Let \((A_k, \alpha_k, Z_p)\) and \((B_n, \beta_n, Z_p)\) be two irreducible finite dimensional C*-dynamical systems. Let \(F_k\) be an ordered group morphism from \((K_0(A_k), [1]_{A_k}, \alpha_k)\) to \((K_0(B_n), [1]_{B_n}, \beta_n)\). Let \(\phi_k\) be an ordered group morphism from \((K_0(A_k \times_{\alpha_k} Z_p), \hat{\alpha}_k)\) to \((K_0(B_n \times_{\beta_n} Z_p), \hat{\beta}_n)\), which preserves the special element. Then there exists a homomorphism \(\psi_k\) from \((A_k, \alpha_k, Z_p)\) to \((B_n, \beta_n, Z_p)\), such that \(\psi_k = F_k\), and \(\tilde{\psi}_k\) is the natural extension of \(\psi_k\) to \(A_k \times_{\alpha_k} Z_p\).

**Proof.** We are going to prove the theorem in four different cases. Assume \(Z_p = \{\rho, \rho^2, \ldots, \rho^{p-1}, 1\}\).

(1). \(A_k = M_k, B_n = M_n\).

Suppose \(U \in M_k\) and \(V \in M_n\) are the two unitaries which implement the automorphism for \(\rho\) on \(M_k\) and \(M_n\) respectively. Since \(F_k\) preserves the scale, then \(F_k = \hat{\phi}_k\). By Lemma 2.1, \(\phi_k\) is of the form:

\[
\phi_k = \begin{pmatrix}
1_{l_1} & l_{12} & \ldots & l_{1p} \\
l_{21} & 1_{l_2} & \ldots & l_{2p} \\
\ldots & \ldots & \ldots & \ldots \\
l_{p1} & l_{p2} & \ldots & 1_{lp}
\end{pmatrix}.
\]

Moreover, \(\phi_k\) intertwines \(\hat{\alpha}_k\) and \(\hat{\beta}_n\), by calculation, one obtains that

\[
\phi_k = \begin{pmatrix}
l_{11} & l_{12} & \ldots & l_{1p} \\
l_{1p} & l_{11} & \ldots & l_{1p-1} \\
\ldots & \ldots & \ldots & \ldots \\
l_{12} & l_{13} & \ldots & l_{11}
\end{pmatrix}.
\]

By assumption, we have \(\phi_k \zeta = \zeta'\), where \(\zeta\) and \(\zeta'\) are the two special elements in \(M_k \times_{\alpha_k} Z_p\) and \(M_n \times_{\beta_n} Z_p\), then \((l_{11} + l_{12} + \ldots + l_{1p})k = n\).

Define

\[
e_1 = e_2 = \ldots = e_{l_{11}} = I_k,
\]

\[
e_{l_{11}+1} = e_{l_{11}+2} = \ldots = e_{l_{11}+l_{12}} = e^{-i\frac{2\pi}{p}} \otimes id_{\frac{p}{p}},
\]

\[
e_{l_{11}+l_{12}+1} = e_{l_{11}+l_{12}+2} = \ldots = e_{l_{11}+l_{12}+l_{13}} = e^{-i\frac{3\pi}{p}} \otimes id_{\frac{p}{p}},
\]

\[
, \ldots,
\]

set

\[
e = diag(e_1, \ldots, e_{l_{11}}, e_{l_{11}+1}, \ldots, e_{l_{11}+l_{12}}, \ldots, e_{l_{11}+\ldots+l_{1p}}),
\]

then \(e(U \otimes id_{\frac{p}{p}}) = (U \otimes id_{\frac{p}{p}})e\).

Because \(\phi_k\) preserves the special elements (and the choice of special element in this case), then the eigenvalue list of \((U \otimes id_{\frac{p}{p}})e\) is the same as \(V\), then there exists a unitary \(W\), such that \(W^*VW = (U \otimes id_{\frac{p}{p}})e\).
Define a homomorphism $\psi_k : M_k \to M_n$ by:

$$\psi_k(a) = W(a \otimes id_\mathbb{F})W^*,$$

then $(U^* \otimes id_\mathbb{F})W^*VW(a \otimes id_\mathbb{F}) = (a \otimes id_\mathbb{F})W^*VW,$ namely, $\psi_k$ intertwines $\alpha_k$ and $\beta_n$, and $\psi_{k*} = F_k$. Since the natural extension $\tilde{\psi}_k$ intertwines $\tilde{\alpha}_k$ and $\tilde{\beta}_n$, by calculation, $\tilde{\psi}_{k*} = \phi_k$.

(2). $A_k = M_k$, $B_n = \bigoplus_{p} M_n$.

Obviously, $F_k = \left(\begin{array}{c}
\frac{n}{k} \\
\vdots \\
\frac{n}{k}
\end{array}\right)$. Since $\phi_k$ intertwines $\tilde{\alpha}_{k*}$ and $\tilde{\beta}_{n*}$, by calculation, we have: $\phi_k = \left(\begin{array}{c}
l_1 \\
l_1 \\
\vdots \\
l_1
\end{array}\right)$. Moreover, because $\phi_k$ preserves the special element, then $l_1 = \frac{n}{k}$. Let $V$ be the unitary implementing the automorphism for $\rho$ on $M_k$.

Define a homomorphism $\psi_k : M_k \to M_n$ by:

$$\psi_k(a) = (W_1(a \otimes id_\mathbb{F})W_1^*, W_2(a \otimes id_\mathbb{F})W_2^*, \ldots, W_p(a \otimes id_\mathbb{F})W_p^*),$$

where $W_1 = 1 \otimes id_\mathbb{F}$, $W_2 = V^* \otimes id_\mathbb{F}$, \ldots, $W_p = (V^*)^{p-1} \otimes id_\mathbb{F}$. Then it is easily to check that $\psi_k$ intertwines $\alpha_k$ and $\beta_n$, and $\psi_{k*} = F_k$. The natural extension $\tilde{\psi}_k$ intertwines $\tilde{\alpha}_k$ and $\tilde{\beta}_n$, so $\tilde{\psi}_{k*} = \phi_k$.

(3). $A_k = M_k \oplus \ldots \oplus M_k$, $B_n = M_n$.

Since $F_k$ intertwines $\alpha_k$ and $\beta_n$, by calculation, $F_k = \left(\begin{array}{c}
\frac{n}{pk} \\
\frac{n}{pk} \\
\vdots \\
\frac{n}{pk}
\end{array}\right)$.

Since $\phi_k$ intertwines $\tilde{\alpha}_{k*}$ and $\tilde{\beta}_{n*}$, we have $\left(\begin{array}{c}
l_1 \\
l_1 \\
\vdots \\
l_1
\end{array}\right)$, moreover by the assumption on the special elements, $l_1 = \frac{n}{pk}$. Let $V$ be the unitary implementing the automorphism for $\rho$ on $M_n$. To define a homomorphism which intertwines $\alpha_k$ and $\beta_n$, we need to find a unitary $W$, such that $\text{Ad}(W^*VW)$ maps $\text{diag}(a_1 \otimes id_\mathbb{F}, a_2 \otimes id_\mathbb{F}, \ldots, a_p \otimes id_\mathbb{F})$ to $\text{diag}(a_p \otimes id_\mathbb{F}, a_1 \otimes id_\mathbb{F}, \ldots, a_{p-1} \otimes id_\mathbb{F})$ for all $(a_1, \ldots, a_p)$ in $\bigoplus_{p} M_n$.

By Lemma IV.2 in [3], this can be done.

(4). $A_k = M_k \oplus \ldots \oplus M_k$, $B_n = M_n \oplus \ldots \oplus M_n$.

Since $F_k$ intertwines $\alpha_k$ and $\beta_n$, by calculation, we have:

$$F_k = \left(\begin{array}{cccc}
l_{11} & l_{12} & \cdots & l_{1p} \\
l_{1p} & l_{11} & \cdots & l_{1p-1} \\
\vdots & \vdots & \ddots & \vdots \\
l_{12} & l_{13} & \cdots & l_{11}
\end{array}\right),$$

and $(l_{11} + l_{12} + \ldots + l_{1p})k = n$. Similarly, we also have $\phi_k = \frac{n}{k}$.
Define a homomorphism $\psi_k : M_k \oplus \ldots \oplus M_k \to M_n \oplus \ldots \oplus M_n$ by:

$$\psi_k(a_1, a_2, \ldots, a_p) = (\psi_k^1(\cdot), \psi_k^2(\cdot), \ldots, \psi_k^p(\cdot)),$$

where (\cdot) is the abbreviation of $(a_1, a_2, \ldots, a_p) \in M_n \oplus \ldots \oplus M_n$, and

$$\psi_k^1(a_1, a_2, \ldots, a_p) = \text{diag}(a_1 \otimes \text{id}_{l_1}, a_2 \otimes \text{id}_{l_2}, \ldots, a_p \otimes \text{id}_{l_p}),$$

$$\psi_k^2(a_1, a_2, \ldots, a_p) = \text{diag}(a_2 \otimes \text{id}_{l_1}, a_3 \otimes \text{id}_{l_2}, \ldots, a_1 \otimes \text{id}_{l_p}),$$

$$\psi_k^3(a_1, a_2, \ldots, a_p) = \text{diag}(a_3 \otimes \text{id}_{l_1}, a_4 \otimes \text{id}_{l_2}, \ldots, a_2 \otimes \text{id}_{l_p}),$$

$$\ldots,$$

$$\psi_k^p(a_1, a_2, \ldots, a_p) = \text{diag}(a_p \otimes \text{id}_{l_1}, a_1 \otimes \text{id}_{l_2}, \ldots, a_{p-1} \otimes \text{id}_{l_p}).$$

Then $\psi_k$ satisfies the condition. $\square$

**Corollary 3.2.** Let $(A_k, \alpha_k, Z_p)$ and $(B_n, \beta_n, Z_p)$ be two finite dimensional $C^*$-dynamical systems. Let $F_k$ be an ordered group morphism from $(K_0(A_k), [1_{A_k}], \alpha_k)$ to $(K_0(B_n), [1_{B_n}], \beta_n)$. Let $\phi_k$ be an ordered group morphism from $(K_0(A_k \rtimes \alpha_k Z_p), \alpha_k)$ to $(K_0(B_n \rtimes \beta_n Z_p), \beta_n)$, which preserves the special element, and the following diagram

$$
\begin{array}{ccc}
K_0(A_k) & \rightarrow & K_0(A_k \rtimes \alpha_k Z_p) \\
\downarrow F_k & & \downarrow \phi_k \\
K_0(B_n) & \rightarrow & K_0(B_n \rtimes \beta_n Z_p)
\end{array}
$$

commutes. Then there exists a homomorphism $\psi_k$ from $(A_k, \alpha_k, Z_p)$ to $(B_n, \beta_n, Z_p)$, such that $\psi_{k*} = F_k$, and $\psi_k = \phi_k$, where $\psi_k$ is the natural extension of $\psi_k$ to $A_k \rtimes \alpha_k Z_p$.

**Proof.**

In this section, we are going to establish the uniqueness theorem, namely, if two morphisms between the finite dimensional $C^*$-dynamical systems agree on the K-theoretic invariants, then they are unitarily equivalent by an equivariant unitary, namely, an unitary in the fixed point subalgebra of the codomain algebra.

**Theorem 4.1.** Let $\phi_k$ and $\psi_k$ be two homomorphisms from the irreducible finite dimensional $C^*$-dynamical system $(A_k, \alpha_k, Z_p)$ to $(B_n, \beta_n, Z_p)$. Denote by $\tilde{\phi}_k$ and $\tilde{\psi}_k$ the morphisms from $A_k \rtimes \alpha_k Z_p$ to $B_n \rtimes \beta_n Z_p$ induced by $\phi_k$ and $\psi_k$, respectively. If $\phi_{k*} = \psi_{k*}$ and $\tilde{\phi}_{k*} = \tilde{\psi}_{k*}$, then there exists a unitary $W$ in $B_n^{\mathbb{Z}_n}$, the fixed point subalgebra of $B_n$, such that $\phi_k = \text{Ad}W \circ \psi_k$.

**Proof.** Again we are going to prove the theorem in four cases. Assume $Z_p = \{\rho^0, \rho^1, \ldots, \rho^{p-1}, 1\}$.

1. $A_k = M_k, B_n = M_n$.

Let $U \in M_k$ and $V \in M_n$ be the two unitaries which implement the action $\alpha_k$ and $\beta_n$, respectively. Let $X$ and $Y$ be two unitaries in $B_n$ such that

$$\phi(a) = X(a \otimes \text{id}_Z)X^*, \psi(a) = Y(a \otimes \text{id}_Z)Y^*, \forall a \in M_k.$$

Since $\phi$ and $\psi$ intertwine the actions $\alpha_k$ and $\beta_n$, we have that

$$X(UaU^* \otimes \text{id}_Z)X^* = VX(a \otimes \text{id}_Z)X^*Y^*,$$

$$Y(UaU^* \otimes \text{id}_Z)Y^* = VY(a \otimes \text{id}_Z)Y^*V^*.$$
Hence,
\[
X^*V^*X(U \otimes id_{\mathbb{P}})(a \otimes id_{\mathbb{P}}) = (a \otimes id_{\mathbb{P}})X^*V^*X(U \otimes id_{\mathbb{P}}), \\
Y^*V^*Y(U \otimes id_{\mathbb{P}})(a \otimes id_{\mathbb{P}}) = (a \otimes id_{\mathbb{P}})Y^*V^*Y(U \otimes id_{\mathbb{P}}),
\]
Set \( L = X^*V^*X(U \otimes id_{\mathbb{P}}) \), \( N = Y^*V^*Y(U \otimes id_{\mathbb{P}}) \), then \( L \) and \( N \) commute with \((a \otimes id_{\mathbb{P}})\), for all \( a \in M_k \), and
\[
L^p = X^*(V^*)^pX(U^p \otimes id_{\mathbb{P}}) = I, \quad N^p = I.
\]
Note that \( L, N \) commute with all \((a \otimes id_{\mathbb{P}})\), then \( L, N \) belong to \( I_k \otimes M_{\mathbb{P}} \). Let \( S \) and \( R \) be two unitaries in \( I_k \otimes M_{\mathbb{P}} \) such that
\[
SLS^* = I_k \otimes \text{diag}(\lambda_1, ..., \lambda_{\mathbb{P}}), \\
RNR^* = I_k \otimes \text{diag}(\mu_1, ..., \mu_{\mathbb{P}}).
\]
For \( a_0 + a_1U_p + ... + a_{p-1}U_{p-1} \in M_k \otimes_{\alpha_k} Z_p \), take \( a_0 = a_2 = ... = a_{p-1} = 0 \), and \( a_1 = U^* \), by Lemma 2.1, we have that
\[
\hat{\phi}(I, e^{i\frac{2\pi}{p}}I, ..., e^{i\frac{2(p-1)\pi}{p}}I) = (X(U^* \otimes id_{\mathbb{P}})X^*V, e^{i\frac{2\pi}{p}}X(U^* \otimes id_{\mathbb{P}})X^*V, ...), \]
\[
\hat{\psi}(I, e^{i\frac{2\pi}{p}}I, ..., e^{i\frac{2(p-1)\pi}{p}}I) = (Y(U^* \otimes id_{\mathbb{P}})Y^*V, e^{i\frac{2\pi}{p}}Y(U^* \otimes id_{\mathbb{P}})Y^*V, ...)
\]
Since \( \hat{\phi}_* = \hat{\psi}_* \), then there exists a unitary \( Z \) such that \( XL^*X^* = ZYN^*Y^*Z^* \), hence, \( L = X^*ZYN^*Y^*Z^*X \), so \( \{\lambda_1, ..., \lambda_{\mathbb{P}}\} = \{\mu_1, ..., \mu_{\mathbb{P}}\} \). Then there exists a unitary \( \tilde{Z} \in I_k \otimes M_{\mathbb{P}} \) such that \( L = \tilde{Z}N\tilde{Z}^* \). Hence,
\[
X^*V^*X(U \otimes id_{\mathbb{P}}) = \tilde{Z}Y^*V^*Y(U \otimes id_{\mathbb{P}})\tilde{Z}^*,
\]
which implies that \( VX\tilde{Z}Y^* = X\tilde{Z}Y^*V \). Therefore \( X\tilde{Z}Y^* \in B_n^\beta \), put \( W = X\tilde{Z}Y^* \), then \( \phi_k = AdW \circ \psi_n \).

(2) \( A_k = M_k \otimes \underbrace{M_n \otimes ... \otimes M_n}_{p} \).

Let \( U \) be the order \( p \) unitary such that \( \rho(a) = UaU^*, \forall a \in M_k \). Let \( X_1, ..., X_p \) be the unitaries in \( M_n \) such that \( \phi(a) = (X_1 a \otimes id_{\mathbb{P}} X_1^*, ..., X_p a \otimes id_{\mathbb{P}} X_p^*) \); let \( Y_1, ..., Y_p \) be the unitaries in \( M_n \) such that \( \psi(a) = (Y_1 a \otimes id_{\mathbb{P}} Y_1^*, ..., Y_p a \otimes id_{\mathbb{P}} Y_p^*) \), \( \forall a \in M_k \). Since \( \phi \) and \( \psi \) intertwines \( \alpha_k \) and \( \beta_n \), we obtains:
\[
(X_1(UaU^* \otimes id_{\mathbb{P}})X_1^*, ... , X_p(UaU^* \otimes id_{\mathbb{P}})X_p^*) = (X_{pa} \otimes id_{\mathbb{P}} X_p^*, X_{1a} \otimes id_{\mathbb{P}} X_1^*, ..., X_{p-1a} \otimes id_{\mathbb{P}} X_{p-1}^*).
\]
Hence,
\[
X_1(UaU^* \otimes id_{\mathbb{P}})X_1^* = X_{pa} \otimes id_{\mathbb{P}} X_p^*, \\
X_2(UaU^* \otimes id_{\mathbb{P}})X_2^* = X_{1a} \otimes id_{\mathbb{P}} X_1^*, \\
\vdots \\
X_p(UaU^* \otimes id_{\mathbb{P}})X_p^* = X_{p-1a} \otimes id_{\mathbb{P}} X_{p-1}^*.
\]
This implies that
\[
X'_pX_1U \otimes \id_{\mathbb{Z}}a \otimes \id_{\mathbb{Z}} = a \otimes \id_{\mathbb{Z}}X'_pX_1U \otimes \id_{\mathbb{Z}},
\]
\[
X'_1X_2U \otimes \id_{\mathbb{Z}}a \otimes \id_{\mathbb{Z}} = a \otimes \id_{\mathbb{Z}}X'_1X_2U \otimes \id_{\mathbb{Z}},
\]
\[
\ldots
\]
\[
X'_{p-1}X_pU \otimes \id_{\mathbb{Z}}a \otimes \id_{\mathbb{Z}} = a \otimes \id_{\mathbb{Z}}X'_{p-1}X_pU \otimes \id_{\mathbb{Z}}.
\]

Similarly, we also have:
\[
Y'_pY_1U \otimes \id_{\mathbb{Z}}a \otimes \id_{\mathbb{Z}} = a \otimes \id_{\mathbb{Z}}Y'_pY_1U \otimes \id_{\mathbb{Z}},
\]
\[
Y'_1Y_2U \otimes \id_{\mathbb{Z}}a \otimes \id_{\mathbb{Z}} = a \otimes \id_{\mathbb{Z}}Y'_1Y_2U \otimes \id_{\mathbb{Z}},
\]
\[
\ldots
\]
\[
Y'_{p-1}Y_pU \otimes \id_{\mathbb{Z}}a \otimes \id_{\mathbb{Z}} = a \otimes \id_{\mathbb{Z}}Y'_{p-1}Y_pU \otimes \id_{\mathbb{Z}}.
\]

Our goal is to find a unitary \(W = (W_1, \ldots, W_p) \in B^\beta_n\), such that \(\phi = AdW \circ \psi\). Note that \(W \in B^\beta_n\) means that \((W_1, W_2, \ldots, W_p) = (W_p, W_1, \ldots, W_{p-1})\), namely, \(W_1 = W_2 = \ldots = W_p\).

Set
\[
L_1 = X'_pX_1U \otimes \id_{\mathbb{Z}}, \quad N_1 = Y'_pY_1U \otimes \id_{\mathbb{Z}},
\]
\[
\ldots
\]
\[
L_p = X'_{p-1}X_pU \otimes \id_{\mathbb{Z}}, \quad N_p = Y'_{p-1}Y_pU \otimes \id_{\mathbb{Z}},
\]
then by the calculation above, all of these \(L_i, N_i\) commute with \(a \otimes \id_{\mathbb{Z}}, \forall a \in M_n\).

Then \(N_pL_p = Y'_pY_pX'_pX_{p-1}\), which implies \(X_pY'_p = X_{p-1}L_pN_pY'_p\). Moreover, \(X_{p-2}L_{p-1}L_pN_pN_{p-1}Y_{p-2} = X_{p-2}X'_{p-2}X_{p-1}(U \otimes \id_{\mathbb{Z}})L_pN_p(U \otimes \id_{\mathbb{Z}})Y_{p-1}Y_{p-2} = X_{p-1}L_pN_pY_{p-1}\). Similarly, we have
\[
X_{p-3}L_{p-2}L_{p-1}L_pN_pN_{p-1}N_{p-2}Y_{p-3} = X_{p-2}L_{p-1}L_pN_pN_{p-1}Y_{p-2},
\]
\[
\ldots
\]
\[
X_1L_2\ldots L_pN_p^*N_{p-1}^*N_{p-2}^*Y_{1} = X_2L_3\ldots L_pN_p^*N_{p-1}^*Y_{2}.
\]

So all of these terms equal to \(X_{p-1}L_pN_pY_{p-1} = X_pY_p^*\).

Put
\[
W = (X_1L_2\ldots L_pN_p^*\ldots N_2^*Y_1^*, X_2L_3\ldots L_pN_p^*\ldots N_3^*Y_2^*, \ldots, X_pY_p^*),
\]
then \(W \in B^\beta_n\), and \(\phi_k = AdW \circ \psi_n\), since for each \(i = 1, \ldots, p - 1\), we have
\[
X_iL_{i+1}\ldots L_pN_p^*\ldots N_i^*Y_i^*(a \otimes \id_{\mathbb{Z}})Y_i^*Y_iN_{i+1}\ldots N_pL_p\ldots L_{i+1}X_i^* = X_i(a \otimes \id_{\mathbb{Z}})X_i^*.
\]

(3). \(A_k = M_k \oplus \ldots \oplus M_k, B_n = M_n\).

Let \(V\) be the unitary which implements the action of \(\beta_n\), namely, \(\rho(a) = VaV^*, \forall a \in M_n\). Let \(X, Y\) be unitaries in \(M_n\) such that \(\forall (a_1, a_2, \ldots, a_p) \in \underbrace{M_k \oplus \ldots \oplus M_k}_p\),
\[
\phi(a_1, a_2, \ldots, a_p) = X \text{diag}(a_1 \otimes \id_{\mathbb{Z}}, a_2 \otimes \id_{\mathbb{Z}}, \ldots, a_p \otimes \id_{\mathbb{Z}})X^*,
\]
\[
\psi(a_1, a_2, \ldots, a_p) = Y \text{diag}(a_1 \otimes \id_{\mathbb{Z}}, a_2 \otimes \id_{\mathbb{Z}}, \ldots, a_p \otimes \id_{\mathbb{Z}})Y^*.
\]

This is the case since both \(\phi_*\) and \(\psi_*\) intertwine the actions, and \(\phi_* = \psi_*\).
Since $\phi$ and $\psi$ intertwine the actions, we obtain that:

$$
\begin{pmatrix}
  a_p \otimes id_{\frac{n}{pk}} \\
  a_1 \otimes id_{\frac{n}{pk}} \\
  \vdots \\
  a_{p-1} \otimes id_{\frac{n}{pk}}
\end{pmatrix} X^* = V
\begin{pmatrix}
  a_1 \otimes id_{\frac{n}{pk}} \\
  a_2 \otimes id_{\frac{n}{pk}} \\
  \vdots \\
  a_p \otimes id_{\frac{n}{pk}}
\end{pmatrix} X^* V^*.
$$

and

$$
\begin{pmatrix}
  a_p \otimes id_{\frac{n}{pk}} \\
  a_1 \otimes id_{\frac{n}{pk}} \\
  \vdots \\
  a_{p-1} \otimes id_{\frac{n}{pk}}
\end{pmatrix} Y^* = V
\begin{pmatrix}
  a_1 \otimes id_{\frac{n}{pk}} \\
  a_2 \otimes id_{\frac{n}{pk}} \\
  \vdots \\
  a_p \otimes id_{\frac{n}{pk}}
\end{pmatrix} Y^* V^*.
$$

Set $P = \begin{pmatrix}
  I_{\frac{n}{pk}} \\
  \vdots \\
  I_{\frac{n}{pk}}
\end{pmatrix}$, then

$$
\begin{pmatrix}
  a_p \otimes id_{\frac{n}{pk}} \\
  a_1 \otimes id_{\frac{n}{pk}} \\
  \vdots \\
  a_{p-1} \otimes id_{\frac{n}{pk}}
\end{pmatrix} P^* =
\begin{pmatrix}
  a_1 \otimes id_{\frac{n}{pk}} \\
  a_2 \otimes id_{\frac{n}{pk}} \\
  \vdots \\
  a_p \otimes id_{\frac{n}{pk}}
\end{pmatrix}.
$$

So we have:

$$
XP^*P \begin{pmatrix}
  a_p \otimes id_{\frac{n}{pk}} \\
  a_1 \otimes id_{\frac{n}{pk}} \\
  \vdots \\
  a_{p-1} \otimes id_{\frac{n}{pk}}
\end{pmatrix} P^* PX^*.
$$
\[ VX \left( \begin{array}{ccc} a_1 \otimes \text{id}_{\frac{m}{pk}} \\ a_2 \otimes \text{id}_{\frac{m}{pk}} \\ \vdots \\ a_p \otimes \text{id}_{\frac{m}{pk}} \end{array} \right) X^*V^*, \]

namely,

\[ XP^* \left( \begin{array}{ccc} a_1 \otimes \text{id}_{\frac{m}{pk}} \\ a_2 \otimes \text{id}_{\frac{m}{pk}} \\ \vdots \\ a_p \otimes \text{id}_{\frac{m}{pk}} \end{array} \right) PX^* \]

so

\[ X^*V^*XP^* \left( \begin{array}{ccc} a_1 \otimes \text{id}_{\frac{m}{pk}} \\ a_2 \otimes \text{id}_{\frac{m}{pk}} \\ \vdots \\ a_p \otimes \text{id}_{\frac{m}{pk}} \end{array} \right) X^*V^*P^*, \]

similarly,

\[ Y^*V^*YP^* \left( \begin{array}{ccc} a_1 \otimes \text{id}_{\frac{m}{pk}} \\ a_2 \otimes \text{id}_{\frac{m}{pk}} \\ \vdots \\ a_p \otimes \text{id}_{\frac{m}{pk}} \end{array} \right) Y^*V^*P^*. \]

Put \( L = X^*V^*XP^* \), \( N = Y^*V^*YP^* \), then

\[ L = \text{diag}(L_1, \ldots, L_p), N = \text{diag}(N_1, \ldots, N_p), \]

where each \( L_i \) and \( N_i \) belongs to \( \text{id}_k \otimes M_{\frac{m}{pk}} \), which means each of them commutes with any matrix in \( M_{k \otimes \frac{m}{pk}} \). Hence,

\[ X^*V^*X = LP = \left( \begin{array}{ccc} L_1 \\ \vdots \\ L_p \end{array} \right), \]
and
\[ Y^*V^*Y = NP = \begin{pmatrix} N_2 & N_1 \\ \vdots & \vdots \\ N_p & N_p \end{pmatrix}. \]

Since \( V \) is an order \( p \) unitary, we obtain
\[ \begin{pmatrix} L_2 & L_1 \\ \vdots & \vdots \\ L_p & \end{pmatrix}^p = I, \quad \begin{pmatrix} N_2 & N_1 \\ \vdots & \vdots \\ N_p & \end{pmatrix}^p = I. \]

Form this we have that
\[ L_1L_p...L_2 = I, \]
\[ L_2L_1...L_3 = I, \]
\[ , \ldots, \]
\[ L_pL_p-1...L_1 = I, \]
similarly,
\[ N_1N_p...N_2 = I, \]
\[ N_2N_1...N_3 = I, \]
\[ , \ldots, \]
\[ N_pN_p-1...N_1 = I. \]

Set
\[ Z = \begin{pmatrix} N_1L_p...L_2 \\ N_2N_1L_p...L_3 \\ \vdots \\ N_{p-1}...N_1L_p \end{pmatrix}, \]

By using the relations above, we have that
\[ Z \begin{pmatrix} L_2 & L_1 \\ \vdots & \vdots \\ L_p & \end{pmatrix} Z^* = \begin{pmatrix} N_2 & N_1 \\ \vdots & \vdots \\ N_p & \end{pmatrix}, \]

namely, \( ZX^*V^*XZ^* = Y^*V^*Y \). Put \( W = XZ^*Y \), then \( WV = VW \), so \( W \in B_n^\beta \), and \( \phi_k = AdW \circ \psi_n \).

(4). \( A_k = M_k \oplus \ldots \oplus M_k \), \( B_n = M_n \oplus \ldots \oplus M_n \).

Since \( \phi \) intertwines the actions \( \alpha_k \) and \( \beta_n \), by calculation,
\[ \phi_k = \begin{pmatrix} l_{11} & l_{12} & \ldots & l_{1p} \\ l_{1p} & l_{11} & \ldots & l_{1p-1} \\ \vdots & \vdots & \ddots & \vdots \\ l_{12} & l_{13} & \ldots & l_{11} \end{pmatrix}. \]
where \((l_1 + l_2 + \ldots + l_p)k = n\), and \(\phi\) is of the following form:
\[
\phi(a_1, a_2, \ldots, a_p) = \left(\phi(1), \phi(2), \ldots, \phi(p)\right), \forall (a_1, a_2, \ldots, a_p) \in \bigoplus_{p} M_n,
\]
where \((.)\) is the abbreviation of \((a_1, a_2, \ldots, a_p)\), and
\[
\phi_1(a_1, a_2, \ldots, a_p) = X \text{diag} (a_1 \otimes \text{id}_{l_1}, a_2 \otimes \text{id}_{l_2}, \ldots, a_p \otimes \text{id}_{l_p}) X^*,
\]
\[
\phi_1(a_1, a_2, \ldots, a_p) = X \text{diag} (a_2 \otimes \text{id}_{l_1}, a_3 \otimes \text{id}_{l_2}, \ldots, a_1 \otimes \text{id}_{l_p}) X^*,
\]
\[
\vdots,
\]
\[
\phi_1(a_1, a_2, \ldots, a_p) = X \text{diag} (a_p \otimes \text{id}_{l_1}, a_1 \otimes \text{id}_{l_2}, \ldots, a_{p-1} \otimes \text{id}_{l_p}) X^*,
\]
here \(X\) is a unitary in \(M_n\). Since \(\phi_\ast = \psi_\ast\), similarly,
\[
\psi(a_1, a_2, \ldots, a_p) = \left(\psi(1), \psi(2), \ldots, \psi(p)\right), \forall (a_1, a_2, \ldots, a_p) \in \bigoplus_{p} M_n,
\]
where \((.)\) is the abbreviation of \((a_1, a_2, \ldots, a_p)\), and
\[
\psi_1(a_1, a_2, \ldots, a_p) = Y \text{diag} (a_1 \otimes \text{id}_{l_1}, a_2 \otimes \text{id}_{l_2}, \ldots, a_p \otimes \text{id}_{l_p}) Y^*,
\]
\[
\psi_1(a_1, a_2, \ldots, a_p) = Y \text{diag} (a_2 \otimes \text{id}_{l_1}, a_3 \otimes \text{id}_{l_2}, \ldots, a_1 \otimes \text{id}_{l_p}) Y^*,
\]
\[
\vdots,
\]
\[
\psi_1(a_1, a_2, \ldots, a_p) = Y \text{diag} (a_p \otimes \text{id}_{l_1}, a_1 \otimes \text{id}_{l_2}, \ldots, a_{p-1} \otimes \text{id}_{l_p}) Y^*,
\]
here \(Y\) is a unitary in \(M_n\).

Put \(W = (XY^*, \ldots, XY^*) \in B_n^{2n}\), then it is clear that \(\phi = \text{Ad} W \circ \psi\).

\[\square\]

**Corollary 4.2.** Let \(\phi_k\) and \(\psi_k\) be two homomorphisms from the finite dimensional \(C^\ast\)-dynamical system \((A_k, \alpha_k, Z_p)\) to \((B_n, \beta_n, Z_p)\). Denote by \(\bar{\phi}_k\) and \(\bar{\psi}_k\) the morphisms from \(A_k \times \alpha_k Z_p\) to \(B_n \times \beta_n Z_p\) induced by \(\phi_k\) and \(\psi_k\), respectively. If \(\phi_{k*} = \psi_{k*}\) and \(\bar{\phi}_{k*} = \bar{\psi}_{k*}\), then there exists a unitary \(W\) in \(B_n^{2n}\), the fixed point subalgebra of \(B_n\), such that \(\phi_k = \text{Ad} W \circ \psi_k\).

### 5. Classification

In this section, we prove the classification Theorem[1.1] by using Elliott’s intertwining arguments.

**Proof.** First of all, by standard argument, the K-theoretic invariants of the AF \(C^\ast\)-dynamical systems can be lifted to finite stages, namely, by passing to subsequences and changing notation, we could obtain the following intertwining:

\[
(K_0(A_1), \alpha_1) \longrightarrow (K_0(A_2), \alpha_2) \longrightarrow \cdots \longrightarrow (K_0(A), \alpha_\ast)
\]
\[
(K_0(B_1), \beta_1) \longrightarrow (K_0(B_2), \beta_2) \longrightarrow \cdots \longrightarrow (K_0(B), \beta_\ast)
\]
and
\[
(K_0(A_1 \times \alpha_1 Z_p), \hat{\alpha}_1) \longrightarrow (K_0(A_2 \times \alpha_2 Z_p), \hat{\alpha}_2) \longrightarrow \cdots \longrightarrow (K_0(A \times \alpha Z_p), \hat{\alpha}_\ast)
\]
\[
(K_0(B_1 \times \beta_1 Z_p), \hat{\beta}_1) \longrightarrow (K_0(B_2 \times \beta_2 Z_p), \hat{\beta}_2) \longrightarrow \cdots \longrightarrow (K_0(B \times \beta Z_p), \hat{\beta}_\ast).
\]
Second, we would like to make the two intertwinings above to be compatible. Note that we have that:

\[ K_0(A_1) \to K_0(A) \cong K_0(B) \leftarrow \cdots \leftarrow K_0(B_1) \]
\[ K_0(A_1 \rtimes_{\alpha_1} Z_p) \to K_0(A \rtimes_{\alpha} Z_p) \cong K_0(B \rtimes_{\beta} Z_p) \leftarrow \cdots \leftarrow K_0(B_1 \rtimes_{\beta} Z_p) \]

Hence, there exists \( n \), such that

\[ K_0(A_1) \to K_0(A_1 \rtimes_{\alpha_1} Z_p) \to K_0(A) \to K_0(A \rtimes_{\alpha_1} Z_p) \to K_0(B_n) \to K_0(B_n \rtimes_{\beta_1} Z_p) \to K_0(B_1 \rtimes_{\beta_1} Z_p) \]

commutes. After reindexing, the two intertwinings above could satisfy the following commutative diagrams:

\[ K_0(A_n) \to K_0(A_n \rtimes_{\alpha_n} Z_p) \to K_0(A) \to K_0(A \rtimes_{\alpha_1} Z_p) \to K_0(B_n) \to K_0(B_n \rtimes_{\beta_n} Z_p) \]

and

\[ K_0(B_n) \to K_0(B_n \rtimes_{\beta_n} Z_p) \to K_0(B_1 \rtimes_{\beta_1} Z_p) \to K_0(B) \to K_0(B \rtimes_{\beta} Z_p) \]

Also, these intertwinings can preserve the special elements and the units.

Now, we can apply the existence and uniqueness results on finite stages. By Corollary 3.2 we can lift each morphism of the invariant to a morphism between the dynamical systems. By Corollary 4.2 we can correct each morphism by an inner morphism commuting with the actions, so we obtain an intertwining of the dynamical systems:

\[ (A_1, \alpha_1) \to (A_2, \alpha_2) \to \cdots \to (A, \alpha) \]
\[ (B_1, \beta_1) \to (B_2, \beta_2) \to \cdots \to (B, \beta) \]

Hence, \((A, \alpha)\) and \((B, \beta)\) are isomorphic by an isomorphism \( \psi \) which induces \( F \) and \( \phi \).

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