EXISTENCE OF SOLUTIONS TO THE SUPERCRITICAL HARDY-LITTLEWOOD-SOBOLEV SYSTEM WITH FRACTIONAL LAPLACIANS

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Abstract. It is known that the supercritical Hardy-Littlewood-Sobolev (HLS) systems with an integer power of Laplacian admit classic solutions. In this paper, we prove that the supercritical HLS systems with fractional Laplacians \((-\Delta)^s, s \in (0, 1)\), also admit classic solutions.

1. Introduction. For a bounded \(C^2\) function \(f : \mathbb{R}^n \to \mathbb{R}\), fractional Laplacian \((-\Delta)^s\) can be defined by,

\[
(-\Delta)^s f(x) := C_{n,s}(P.V.) \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x - y|^{n+2s}} dy,
\]

where \((P.V.)\) stands for the principal value, \(s \in (0, 1)\) and \(C_{n,s}\) is some normalization constant. Equivalently, \((P.V.)\) can be removed and \((-\Delta)^s\) can be defined as

\[
(-\Delta)^s f(x) = -\frac{1}{2} C_{n,s} \int_{\mathbb{R}^n} \frac{f(x+y) + f(x-y) - 2f(x)}{|y|^{n+2s}} dy.
\]

In general, fractional Laplacians can be defined (by duality) in a weighted \(L^1\)-space (see [22, 34]):

\[
L_s := \left\{ f : \mathbb{R}^n \to \mathbb{R} \text{ such that } \int_{\mathbb{R}^n} \frac{|f(x)|}{1 + |x|^{n+2s}} dx < \infty \right\}.
\]

The fractional Laplacian operator arises from various scientific backgrounds, e.g., phase transitions, flame propagation, stratified materials and etc. In particular, it has an equivalent definition as an infinitesimal generator of a stable Lévy-process. For a expository reference of the fractional Laplacian, we refer the reader to [35].

In this paper we shall study the following system of equations with fractional Laplacians

\[
\begin{cases}
(-\Delta)^s u = v^p & \text{in } \mathbb{R}^n, \\
(-\Delta)^s v = u^q & \text{in } \mathbb{R}^n, \\
u, v > 0 & \text{in } \mathbb{R}^n,
\end{cases}
\]

2010 Mathematics Subject Classification. Primary: 35R11, 35R09; Secondary: 45G15.

Key words and phrases. Fractional Laplacian, bifurcation theory, asymptotic analysis.

The authors are supported by NSF DMS-1601885.

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where \( n \geq 2 \) and \( p, q \) satisfy \( p, q > 1 \) and
\[
\frac{1}{p + 1} + \frac{1}{q + 1} < \frac{n - 2s}{n}.
\] (4)

For \( 0 < p, q < \infty \) and \( s \in (0, \frac{n}{2}) \), (3) is called Hardy-Littlewood-Sobolev (HLS) type system (c.f. [10]) because it has a close connection with the well-known HLS inequality. For the HLS inequality, \( p, q \) satisfy the critical condition,
\[
\frac{1}{p + 1} + \frac{1}{q + 1} = \frac{n - 2s}{n}.
\]
Hence, we call (4) the supercritical condition. The Euler-Lagrange equation of the extremal functions of the HLS inequality, after a change of variables, becomes (3). For the existence of extremal functions, see Lieb [18].

There is also a strong connection between the HLS system and theory of geometric analysis and functional analysis. For example, suppose \( u = v \) and \( p = q \), (3) reduces to a scalar equation, \((-\Delta)^s u = u^p\), and there is a large amount of literature about the classification of solutions to this equation, see [2, 4, 6, 13, 15] and the reference therein.

For a general HLS type system, \( 0 < p, q < \infty \) and \( s \in (0, \frac{n}{2}) \), the existence of solutions to the supercritical system (3)-(4) is a hard problem, and only some special cases have been settled. For instance, when \( s = 1 \), the existence of solutions for (3) can be proved by a shooting method of Serrin and Zou [30]. Later, for an integer \( s \geq 2 \), it is proved by the shooting method with a degree argument (see, c.f. [16, 17, 19]). However, for a fractional \( s \), less is known. For the single equation, i.e., (3) with \( u = v \) and \( p = q \), Chen-Li-Ou [6] have classified the critical case,
\[
p = q = \frac{n - 2s}{n + 2s}.
\]
For \( s = \frac{1}{2} \), the existence of the supercritical single equation, (3)-(4) with \( u = v \) and \( p = q \), is obtained by Chipot, et. al. [11], where the authors actually treated an elliptic problem with Neumann boundary condition in \( \mathbb{R}^{n+1}_+ \).

According to the extension method developed by Caffarelli and Silvestre [3], the problem solved by Chipot is equivalent to a Dirichlet problem with \((-\Delta)^{\frac{1}{2}} \) on \( \mathbb{R}^n \).

The main result of this paper is

**Theorem 1.1.** Problem (3) admits a classic solution \((u, v) \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\), which is radially symmetric and decreases to zero at infinity, and moreover, \( u(0) + v(0) = 1 \).

There are difficulties in dealing with a pseudo-differential operator such as the fractional Laplacian \((-\Delta)^s \) due to the nonlocal feature. As a result, many common tools in classical elliptic problems with local operators are not at our disposal. For example, when considering radial equation, the shooting method and ODE theory are very handy for (3) with an integer \( s \), but they are not applicable for a fractional \( s \). To overcome this, our main idea is to use a topological approach, namely the bifurcation theory, to obtain a solution. First, we apply the bifurcation theory to a local auxiliary problem to obtain a solution, and then prove a sequence of solutions to the auxiliary problems converges to a solution of the original problem. Again, due to the lack of locality, difficulties arise in deriving the monotonicity of the solution. We shall implement a direct method of moving plane recently developed by Chen et. al [5] to overcome this issue. For the recent development of the method, see [7, 20, 36]

The paper is organized as follows: In section 2, we prove the existence and the monotonicity of a solution to a local auxiliary problem; in section 3, we show that a sequence of solutions to the auxiliary problems converges to a classic solution to the original problem; in section 4, we use energy estimates to show that the solution decays to zero at infinity with certain rates.
2. Auxiliary problem. Heuristically, we first obtain a non-negative solution \((u, v)\) in \(\mathbb{R}^n\) to the following Dirichlet problem,

\[
\begin{aligned}
(-\Delta)^s u &= \lambda u + |v|^{p-1}v \quad \text{in } B_R, \\
(-\Delta)^s v &= \lambda v + |u|^{q-1}u \quad \text{in } B_R, \\
u = v = 0 &\quad \text{in } \mathbb{R}^n \setminus B_R, 
\end{aligned}
\]

(5)

where \(B_R\) is a ball in \(\mathbb{R}^n\) with radius \(R\) and centered at origin. Then along some sequence \(R_j \to \infty\) we obtain a sequence of solutions converging to a classic solution of (3).

Remark 1. For a Dirichlet problem involving fractional Laplacians,

\[
\begin{aligned}
(-\Delta)^s u &= g \quad \text{in } \Omega, \\
u = 0 &\quad \text{in } \mathbb{R}^n \setminus \Omega,
\end{aligned}
\]

the equation holds only inside the domain \(\Omega\). In fact \((-\Delta)^s u\) is undefined on the boundary \(\partial \Omega\) when the solution \(u\) is only \(C^{0,s}(\mathbb{R}^n)\).

To be more precise, if \(u \in C^{0,s}(\mathbb{R}^n)\) is a solution of the Dirichlet problem above and \(g \in L^\infty(\Omega)\), then for \(x \in \mathbb{R}^n \setminus \Omega\) and close to the boundary \(\partial \Omega\), by direct computations we have \(|(-\Delta)^{s/2}u(x)| \leq C|\log \delta|\), where \(\delta = \text{dist}(x, \partial \Omega)\). Moreover, \(|(-\Delta)^{s/2}u(x)| \leq C|x|^{n-s}\) for large \(|x|\). This leads to \(u \in H^s(\mathbb{R}^n)\).

Similarly, \(|(-\Delta)^s u(x)| \leq C|x|^{-n-2s}\) for large \(|x|\) and \(|(-\Delta)^s u(x)| \leq C\delta^{-s}\), \(\delta = \text{dist}(x, \partial \Omega)\) for \(x \in \mathbb{R}^n \setminus \Omega\) and close to the boundary \(\partial \Omega\). So \((-\Delta)^s u \in L^p(\mathbb{R}^n)\) with \(p \in [1, \frac{1}{s}]\).

We will look for radial solutions of (5) and define a solution space. Let

\[
\begin{aligned}
E &= \{u \in C^{0,s}(\mathbb{R}^n) : u \equiv 0 \text{ in } \mathbb{R}^n \setminus B_R \text{ and } u(x) = u(|x|)\}, \\
E_R &= E \times E.
\end{aligned}
\]

Let \(G_R(x, y)\) be the Green's function of \((-\Delta)^s\) on \(B_R\), which can be obtained by scaling \(G_1(x, y)\), the Green's function of \((-\Delta)^s\) on \(B_1\) (see [26]),

\[
G_R(x, y) = C_{n,s}|x - y|^{2s-n} \int_0^{r_0(x,y)} \frac{r^{s-1}}{(r+1)^{n/2}}dr,
\]

(8)

with

\[
r_0(x, y) = \frac{(R^2 - |x|^2)(R^2 - |y|^2)}{R^2|x - y|^2}.
\]

Definition 2.1. \(A, T_1, T_2 : E \to E\), for any \(f \in E\) and \(x \in B_R\),

\[
\begin{aligned}
Af(x) \to &\int_{B_R} G_R(x, y)f(y)dy, \\
T_1(f)(x) \to &\int_{B_R} G_R(x, y)|f(y)|^{q-1}f(y)dy, \\
T_2(f)(x) \to &\int_{B_R} G_R(x, y)|f(y)|^{p-1}f(y)dy.
\end{aligned}
\]

(9)

(10)

For \(x \in \mathbb{R}^n \setminus B_R\), simply let \(Af(x), T_1(f)(x), T_2(f)(x)\) be 0.

Obviously, the images of \(A, T_1, T_2\) are radial, and \(A, T_1, T_2\) are well defined. Indeed, we will show that they are compact operators on \(E\) (see Corollary 1). Now
For any \(\lambda\) where \(\lambda > 0\), the operator \(A\) defined in (9) is a compact operator on \(E\), which is defined in (6).

**Proof.** The key is to show the following estimate,

\[
\|Af\|_{s,\mathbb{R}^n} \leq C(n, s, R)\|f\|_{L^\infty(\mathbb{R}^n)}.
\]
Given a bounded sequence \( \{f_n\} \) in \( E_R \), by Arzelà-Ascoli Theorem, we get a uniformly convergent subsequence still denoted as \( \{f_n\} \). Then by (15), we see that \( \{Af_n\} \) must be a Cauchy sequence in \( E_R \). Hence \( A \) must be compact in \( E \).

Estimate (15) directly follows from Proposition 1.1 in [27]. For completeness and the convenience of the readers, we prove (15) by a direct computation with the Green’s function \( G_R(x, y) \) in (8).

**First step.** Since \( Af \equiv 0 \) on \( \mathbb{R}^n \setminus B_R \), we only need to exam the behavior of \( Af \) in \( B_R \). For any \( x_0 \in B_R \), since \((-\Delta)^s Af = f \) on \( B_R \), by a well-known interior estimate (see Proposition 2.9 of [34]), we have
\[
\|Af\|_{s,B_1} \leq C(n, s, \delta_0)(\|Af\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(\mathbb{R}^n)}),
\]
where \( B_1 = B_{\frac{1}{2} \delta_0}(x_0) \), and \( \delta_0 = \text{dist}(x_0, \partial B_R) \). Then (16) easily leads to
\[
\|Af\|_{s,B_1} \leq C(n, s, \delta_0, R)\|f\|_{L^\infty(\mathbb{R}^n)}.
\]

**Second step.** Let’s show that \( Af \) is Hölder continuous in the neighborhood of \( \partial B_R \). For any \( x_0 \in B_R \), let \( x_1 \in \partial B_R \) be the point such that \( \delta := \text{dist}(x_0, \partial B_R) = \text{dist}(x_0, x_1) = |R - x_0| \). Then \( Af(x_1) = 0 \) by Definition (2.1), and
\[
|Af(x_0) - Af(x_1)| = |Af(x_0)|
\leq \int_{B_R} G_R(x_0, y)f(y)dy
\leq I_1 + I_2,
\]
where
\[
I_1 = \int_{B_\delta(x_0)} G_R(x_0, y)|f(y)|dy,
I_2 = \int_{B_R \setminus B_\delta(x_0)} G_R(x_0, y)|f(y)|dy.
\]

Then
\[
I_1 \leq C(n, s) \int_{B_\delta(x_0)} |x_0 - y|^{2s-n}|f(y)|dy
\leq C(n, s)\|f\|_{L^\infty(\mathbb{R}^n)}\delta^{2s}.
\]

In \( B_R \setminus B_\delta(x_0) \), since \( \delta < |x_0 - y| \) and \( |R - y| \leq |R - x_0| + |x_0 - y| \), we can estimate \( r_0 \) in (8) by
\[
r_0(x_0, y) \leq \frac{4 \delta s |\delta + |x_0 - y||}{|x_0 - y|^2}
\leq \frac{4 \delta + 2|x_0 - y|}{|x_0 - y|^2}
\leq \frac{8\delta}{|x_0 - y|}.
\]

Hence,
\[
I_2 \leq C(n, s) \int_{B_R \setminus B_\delta(x_0)} \frac{1}{|x_0 - y|^{n-2s}} \int_0^{\frac{r_0^s}{r_0^{s-1}}} (r + 1)^{n/2} dr |f(y)|dy
\leq C(n, s) \int_{B_R \setminus B_\delta(x_0)} \frac{1}{|x_0 - y|^{n-2s}} \cdot \frac{\delta^s}{|x_0 - y|^s} |f(y)|dy
\leq C(n, s)\|f\|_{L^\infty(\mathbb{R}^n)}\delta^s(C_R - \delta^s)
\leq C(n, s, R)\|f\|_{L^\infty(\mathbb{R}^n)}\delta^s.
\]
Therefore, $Af$ is Hölder continuous in the neighborhood of $\partial B_R$, and

$$|Af(x_0)| \leq C(n, s, R)\|f\|_{L^\infty(\mathbb{R}^n)}\delta^s. \quad (18)$$

**Third step.** Given any two points $x_1, x_2 \in \overline{B_R}$, let $\delta_1 = \text{dist}(x_1, \partial B_R)$, $\delta_2 = \text{dist}(x_2, \partial B_R)$ and $\delta = \min\{\delta_1, \delta_2\}$. There are three cases: (i) $|x_1 - x_2| < \delta/2$; (ii) $|x_1 - x_2| \geq \max\{\delta_1/2, \delta_2/2\}$ and (iii) without loss of generality, $\delta_2/2 \geq |x_1 - x_2| \geq \delta_1/2$.

For case (i), the estimate (16) holds on $B_1 = B_{\delta/2}(x_1)$.

For case (ii),

$$|Af(x_1) - Af(x_2)| \leq C(n, s, R)\|f\|_{L^\infty(\mathbb{R}^n)}(\delta_1^s + \delta_2^s) \leq C(n, s, R)\|f\|_{L^\infty(\mathbb{R}^n)}|x_1 - x_2|^s.$$

For case (iii), since

$$3|x_1 - x_2| \geq \delta_1 + |x_1 - x_2| \geq \delta_2,$$

we see that $\delta_2 \leq 3|x_1 - x_2|$. Therefore,

$$|Af(x_1) - Af(x_2)| \leq C(n, s, R)\|f\|_{L^\infty(\mathbb{R}^n)}(\delta_1^s + \delta_2^s) \leq C(n, s, R)\|f\|_{L^\infty(\mathbb{R}^n)}|x_1 - x_2|^s.$$

Thus, (15) is proved.

**Corollary 1.** Operators $A$ and $T = (T_2, T_1)$ defined in (9)-(10) are compact in $E_R$, which is defined in (7).

**Proof.** By Theorem 2.4, $A$ is compact in $E$. So operator $(A,A)$, which is still denoted as $A$, is compact in $E_R = E \times E$.

Due to $q > 1$, we have $\|g\|_{s, \Omega} \leq \|g\|_{L^\infty(\Omega)}^{q-1}\|g\|_{s, \Omega}$ for $g \in E$. Since $T_1(g) = Ag$, we have

$$\|T_1(g)\|_{s, \mathbb{R}^n} = \|Ag\|_{s, \mathbb{R}^n} \leq C(n, s, R)\|g\|_{L^\infty(\mathbb{R}^n)} \leq C\|g\|_{L^\infty(\mathbb{R}^n)}^q.$$  

Similarly we get

$$\|T_2(g)\|_{s, \mathbb{R}^n} \leq C\|h\|_{L^\infty(\mathbb{R}^n)}^q.$$ 

Hence $T$ is compact in $E_R$.  

**2.2. Bifurcation analysis.** For $w = (u,v) \in E_R$, set

$$S_\lambda(w) = w - \lambda Aw - T(w), \quad (19)$$

and denote

$$\Sigma = \{(\lambda, w) \in \mathbb{R} \times E_R, w \neq 0 : S_\lambda(w) = 0\}. \quad (20)$$

Recall $\lambda_1$ is the first eigenvalue of (12), so $(I - \lambda_1 A)w = 0$ and $\lambda_1$ is also called a characteristic value of $A$. Below we quote the Rabinowitz Theorem (see [1, 23, 25]),

**Theorem 2.5.** Let $A \in L(X)$ be compact and let $T \in C^1(X, X)$ be compact and such that $T(0) = 0$ and $T'(0) = 0$. Suppose that $\lambda_1$ is a characteristic value of $A$ with odd multiplicity. Let $C$ be the connected component of $\Sigma$ containing $(\lambda_1, 0)$. Then either

(i) $C$ is unbounded in $\mathbb{R} \times E_R$, or

(ii) there exists another characteristic value of $A$, $\lambda_2$, such that $\lambda_2 > \lambda_1$ and $(\lambda_2, 0) \in C$. 

Let
\[ E^+_R = \{(u,v) \in E_R : u, v > 0 \text{ in } B_R \}, \quad E^-_R = \{(u,v) \in E_R : u, v < 0 \text{ in } B_R \}. \]

The following lemma states that \( \mathcal{C} \) only branches out from a bifurcation point to either the positive quadrant, \( E^+_R \), or the negative quadrant, \( E^-_R \). So, we only need to consider positive solutions.

**Lemma 2.6.** Let \( \mathcal{C} \) be the bifurcation branch of \( S_\lambda \) at \( (\lambda_1,0) \) defined in Theorem 2.5. Then
\[ \mathcal{C} = (\mathcal{C} \cap (\mathbb{R} \times E^+_R)) \cup (\mathcal{C} \cap (\mathbb{R} \times E^-_R)) \cup \{(\lambda_1,0)\}. \]

The proof of the Lemma is almost the same as Lemma 2.6 in [11] and Theorem 3.6 in [12], which needs a maximum principle for the fractional Laplacian (see Theorem 2.8). By this Lemma, hereinafter, we assume \( u,v > 0 \), and thus \( |u|^{q-1} u = u^q \) and \( |v|^{p-1} v = v^p \).

Now, we are going to show that the second case of Theorem 2.5 does not happen for \( S_\lambda \). Indeed, suppose \( (\lambda,w) \in \mathcal{C} \cap \Sigma \) is a solution to (11), we show that \( 0 < \lambda < \lambda_1 \). Due to the definition of \( \lambda_1 \), \( \lambda \) can not be another characteristic value of \( A \). Thus case two is ruled out.

**Lemma 2.7.** Let \( (\lambda, w) \in \mathcal{C} \cap \Sigma \) be a solution to (11), then \( \lambda > 0 \).

**Proof.** Notice that \( (\lambda, w) \) is also a solution to (5), then we can find the associated Pohožæv identity.

Since \( p,q > 1 \), \( (f_1(u,v), f_2(u,v)) = (\lambda u + v^p, \lambda v + u^q) \) is Lipschitz in the sense that its Hölder norm can be bounded by the Hölder norm of \( u,v \) (here we use the fact \( |f_i(u_2,v_2) - f_i(u_1,v_1)| \leq C(|u_2 - u_1| + |v_2 - v_1|) \)). Also, \( \Omega = B_R \) is a \( C^{1,1} \) bounded domain. Thus by a slight modification, Corollary 1.6 of [27] holds true for \( f(u,v) = (f_1(u,v), f_2(u,v)) \), hence \( \frac{u}{\delta^s}, \frac{v}{\delta^s} \) can be continuously extended to \( \overline{\Omega} \), where \( \delta = \text{dist}(x, \partial \Omega) \).

Moreover, \( (u,v) \) satisfies the condition of Proposition 1.6 in [28]. Now let \( w_1 = u + v \), \( w_2 = u - v \) and \( \nu \) be the unit outward normal vector. By Proposition 1.6 of [28], we have
\[
\int_{\Omega} (x \cdot \nabla w_i) (-\Delta)^s w_i dx = 2s - n \int_{\Omega} w_i (-\Delta)^s w_i dx - \frac{\Gamma(1+s)^s}{2} \int_{\partial \Omega} \left( \frac{w_i}{\delta^s} \right)^2(x \cdot \nu) d\sigma, \quad i = 1,2.
\]

Summing up these two identities, we get
\[
\int_{\Omega} ((x \cdot \nabla u)(-\Delta)^s v + (x \cdot \nabla v)(-\Delta)^s u) dx = 2s - n \int_{\Omega} (u(-\Delta)^s v + v(-\Delta)^s u) dx - \Gamma(1+s)^s \int_{\partial \Omega} \frac{u}{\delta^s} \frac{v}{\delta^s} (x \cdot \nu) d\sigma.
\]

(21)

Then we use (5) to substitute \( (-\Delta)^s u, (-\Delta)^s v \) and get the left side as,
\[
LS = -n \int_{\Omega} \left( \lambda uv + \frac{u^{q+1}}{q+1} + \frac{v^{p+1}}{p+1} \right) dx.
\]
For the right side, notice that $u, v \in E \cap H^s(\mathbb{R}^n)$ (see Remark 1), so
\[
\int_\Omega v(-\Delta)^s u dx = \int_{\mathbb{R}^n} (-\Delta)^{s/2} v(-\Delta)^{s/2} u dx = \int_\Omega u(-\Delta)^s v dx,
\]
and hence we have for any $\theta_1, \theta_2 \in [0, 1]$ such that $\theta_1 + \theta_2 = 1$,
\[
RS = (2s - n) \int_\Omega (\lambda uv + \theta_1 u^{q+1} + \theta_2 v^{p+1}) dx - \Gamma(1 + s) \int_\Omega \frac{u}{\delta^s} \frac{v}{\delta^s} (x \cdot \nu) d\sigma.
\]
Setting equal of both sides, we arrive at
\[
2s\lambda \int_\Omega uv dx = \left( (n - 2s)\theta_1 - \frac{n}{q+1} \right) \int_\Omega u^{q+1} dx + \left( (n - 2s)\theta_2 - \frac{n}{p+1} \right) \int_\Omega v^{p+1} dx + \Gamma(1 + s) \int_{\partial\Omega} \frac{u}{\delta^s} \frac{v}{\delta^s} (x \cdot \nu) d\sigma.
\]
Now, since $(p, q)$ satisfy (4), there exist $\theta_1, \theta_2 \in [0, 1]$ and $\theta_1 + \theta_2 = 1$ such that $(n - 2s)\theta_1 - \frac{n}{q+1} > 0$ and $(n - 2s)\theta_2 - \frac{n}{p+1} > 0$. Since $w = (u, v)$ is a solution to (5), $u, v > 0$. Hence, the equation above implies $\lambda > 0$. □

A maximum principle for the fractional Laplacian is established by Silvestre in Proposition 2.17 in [34], and a simpler proof for $u \in C^{1, 1}_{loc}(\Omega)$ can be found in Theorem 2.1 in [5] (see also [8]),

**Theorem 2.8.** Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $u$ be lower-semicontinuous function in $\bar{\Omega}$ such that $(-\Delta)^s u \geq 0$ in $\Omega$ and $u \geq 0$ in $\mathbb{R}^n \setminus \Omega$. Then $u \geq 0$ in $\mathbb{R}^n$.

Moreover, if $u(x) = 0$ for one point $x$ inside $\Omega$, then $u \equiv 0$ in $\mathbb{R}^n$.

This maximum principle (Theorem 2.8), leads us to an upper bound of $\lambda$.

**Lemma 2.9.** Let $(\lambda, w) \in \mathcal{C} \cap \Sigma$ be a solution to (11), then $\lambda < \lambda_1$.

**Proof.** Again, $(\lambda, w)$ is also a solution to (5). By Lemma 2.7, $(-\Delta)^s u, (-\Delta)^s v > 0$ in $B_R$. Meanwhile, $u, v \equiv 0$ on $\mathbb{R}^n \setminus B_R$, and hence by Hopf Lemma for fractional Laplacian (cf. [14]), $\frac{u}{\delta^s}, \frac{v}{\delta^s} \geq c > 0$ on $\partial B_R$.

Let $\phi_1$ be the first eigenfunction of $(-\Delta)^s$ on $B_R$. It follows that $t_1 \phi_1 < \min\{u, v\}$ in $B_R$ for small $t > 0$. Therefore, there exists a $t_1 > 0$ such that $t_1 \phi_1 \leq \min\{u, v\}$ and
\[
t_1 \phi_1(x_0) = \min\{u, v\}(x_0) \text{ for some } x_0 \in B_R.
\]
Now, if $\lambda \geq \lambda_1$, and without loss of the generality, suppose $t_1 \phi_1(x_0) = u(x_0)$, then we have
\[
(-\Delta)^s(u - t_1 \phi_1) = \lambda u + v^p - t_1 \lambda_1 \phi_1 \geq 0.
\]
By the maximum principle (Theorem 2.8), $u \equiv t_1 \phi_1$ on $B_R$, which is impossible since they satisfy different equations. □

Thus, we are prepared to prove the first part of Theorem 2.3,

**Proof of Part I of Theorem 2.3.** Due to $0 < \lambda < \lambda_1$ (by the previous two lemmas) and the simplicity of $\lambda_1$, $\mathcal{C}$ must be unbounded in $\Sigma$. Moreover, $\mathcal{C}$ must be unbounded in $\mathbb{R} \times C(B_R) \times C(B_R)$; otherwise, $\|w\|_{\infty} < C$, for some $C > 0$, and by the estimate (15), $\mathcal{C}$ must be bounded in $\Sigma$, a contradiction. It follows that there exists a solution $(\lambda, w) \in \mathcal{C}$ such that $\|w\|_{L^\infty(\mathbb{R}^n)} = \|u\|_{L^\infty(\mathbb{R}^n)} + \|v\|_{L^\infty(\mathbb{R}^n)} = 1$. □
2.3. **Monotonicity.** For classical elliptic problems, the monotonically decaying property usually is easy to get when the solution is radial. However, this is not the case for problems with fractional Laplacian due to the lack of locality. To show the solution monotonically decays from the origin, we use a direct method of moving plane recently developed by Chen et al. [5], and the key is a maximum principle on a narrow region for some antisymmetric function. Let

\[ \Pi_t = \{ x \in \mathbb{R}^n | x_1 < t \}, \quad x' = (2t - x_1, x') \].

**Theorem 2.10.** Let \( \Omega \) be a bounded narrow region in \( \Pi_t \), such that it is contained in

\[ \{ x | t - \delta < x_1 < t \} \]

with small \( \delta \). Suppose that \( w \in C^{1,1}_{\text{loc}}(\Omega) \) and assume it makes sense to take \((-\Delta)^s\) on \( w \) and \( w \) is lower semi-continuous on \( \overline{\Omega} \). If \( c(x) \) is bounded from below, i.e. \( c(x) \geq \Lambda \) in \( \Omega \) for some \( \Lambda \) and

\[
\begin{align*}
(\Delta)^s w + c(x)w & \geq 0 \quad \text{in } \Omega, \\
w(x) & \geq 0 \quad \text{in } \Pi_t \setminus \Omega, \\
w(x') & = -w(x) \quad \text{in } \Pi_t,
\end{align*}
\]

then for sufficient small \( \delta = \delta(n, s, \Lambda) \), we have

\[ w(x) \geq 0 \text{ in } \Omega. \]

Furthermore, if \( w(x) = 0 \) at \( x \) in \( \Omega \), then

\[ w(x) = 0 \text{ almost everywhere in } \mathbb{R}^n. \]

**Remark 2.** In [5], \( \delta \) is not explicitly specified to be independent of \( t \), but indeed it is and \( \delta = \delta(n, s, \Lambda) \). This can be seen from their proof, and \( \delta \) is determined by \( C_{n,s}/\delta^{2s} + \Lambda > 0 \).

Now we are ready prove the second part of Theorem 2.3.

**Proof of Part II of Theorem 2.3.** Let

\[ w_1^t(x) = u(x') - u(x), \]
\[ w_2^t(x) = v(x') - v(x). \]

We start to move the plane at \( t = -R \), and since \( u(x), v(x) > 0 \) in \( B_R \), \( w_1^t(x), w_2^t(x) \geq 0 \) on \( \Pi_t \). Let

\[ t_0 = \sup \{ t < 0 | w_1^t(x), w_2^t(x) \geq 0 \text{ in } \Pi_t, \mu \leq t \}. \]

We are going to show \( t_0 = 0 \) and \( w_1^{t_0}(x) \equiv w_2^{t_0}(x) \equiv 0 \) on \( \Pi_{t_0} \). Suppose \( t_0 < 0 \), we show that \( \Pi_{t_0} \) can be moved further right.

For \( t \leq t_0 \), in \( \Pi_t \),

\[
\begin{align*}
(-\Delta)^s w_1^t - \lambda w_1^t & = v^\theta(x') - v^\theta(x) \geq 0, \quad \text{in } B_R \cap \Pi_t, \\
(-\Delta)^s w_2^t - \lambda w_2^t & = u^\theta(x') - u^\theta(x) \geq 0, \quad \text{in } B_R \cap \Pi_t, \\
w_1^t, w_2^t & \geq 0, \quad \text{in } \Pi_t \setminus B_R.
\end{align*}
\]

We **claim** that on \( B_R \cap \Pi_{t_0-\varepsilon} \), \( w_1^{t_0}(x), w_2^{t_0}(x) \geq c \) for some \( c > 0 \). Then by continuous dependence of \( w_1^t(x) \) and \( w_2^t(x) \) on \( t \), for sufficiently small \( \varepsilon_1 > 0 \) we have

\[ w_1^t(x), w_2^t(x) \geq 0, \text{ in } B_R \cap \Pi_{t_0-\varepsilon_1}, \forall t \in [t_0, t_0 + \varepsilon_1]. \]
Thus, choose $\varepsilon < \min\{\varepsilon_1, \frac{\delta}{2}\}$ where $\delta$ is from Theorem 2.10, and apply Theorem 2.10 on $\Omega = (\Pi_{t_0 + \varepsilon} \setminus \Pi_{t_0 - \varepsilon}) \cap B_R$, and then we have $w_1^{t_0}(x), w_2^{t_0}(x) \geq 0$ on $\Pi_t$, for all $t \in [t_0, t_0 + \varepsilon]$. This is a contradiction to the definition of $t_0$.

Now we prove the claim. Suppose

$$w_1^{t_0}(x_*) = \min_{x \in B_R \cap \Pi_{t_0 - \varepsilon}} w_1^{t_0} = 0.$$  

Using the fact that $w_1^{t_0}(x) = -w_1^{t_0}(x^t)$ and by direct computations we get

$$(-\Delta)^s w_1^{t_0}(x_*) = C_{n,s} P.V. \int_{\Pi_{t_0}} \left( \frac{1}{|x_*-y|^{n+2s}} - \frac{1}{|x_*-y|^{n+2s}} \right) w_1^{t_0}(y)dy < 0.$$  

The last inequality is due to $w_1^{t_0}(x) \geq 0$ on $\Pi_{t_0}$ and $w_1^{t_0}(x) \neq 0$ for some $x \in \Pi_{t_0}$. To see the latter fact, notice that $u(x) > 0$ in $B_R$ and $t_0 < t$, and let’s call the mirror image of $B_R$ on the right side of $\partial \Pi_{t_0}$, $B_R^{t_0}$. Then in $B_R^{t_0} \setminus B_R$, $w_1^{t_0}(x) = u(x^s) > 0$. On the other hand,

$$(-\Delta)^s w_1^{t_0}(x_*) = \lambda w_1^{t_0}(x_*) + v^p(x_*) - v^p(x_*) \geq 0.$$  

We get a contradiction, and the claim is proved.

Since $t_0 = 0$ and $w_i(x) = w_i(|x|)$, we have obtained the monotonic decaying property of the solution due to the definition of $t_0$.

**3. Proof of Theorem 1.1.** In order to have the desired convergence in Hölder space, we need a local regularity.

**Proposition 1.** Assume that $u$ is a solution to $(-\Delta)^s u = f$ in $B_R$ with $u \in C^{0,\lambda}(\mathbb{R}^n)$, $f \in C^{0,\tau}(\mathbb{R}^n)$ and $s, \tau \in (0, 1)$. Then for any $l \in (0, R),$

$$\|u\|_{\tau+2s, B_1} \leq C(n, s, \tau, l, R) \left( \|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{\tau, B_R} \right).$$ (23)

This estimate is classic and readers may refer to the proof of Proposition 2.8 in [34].

**Proof of Theorem 1.1.** Let $\{R_j\} \to \infty$ as $j \to \infty$. By Theorem 2.2, there exists a solution to (5) on each $B_{R_j}$, $(\lambda_j, w_j) \in (0, \lambda^j_1) \times E_{R_j}$ defined in (7), such that $\|w_j\|_{L^\infty(B_{R_0})} = u_j(0) + v_j(0) = 1$.

Fixing $R_0 > 0$, then for all $j$ such that $R_j > R_0$, we have $w_j \in C^{3s}(B_l)$ for any $l < R_0$ by Proposition 1. A bootstrapping argument allows us to claim that $w_j \in C^{k,\beta}(B_{\frac{3}{2}R_0})$ for any integer $k > 0$ and $\beta \in (0, 1)$.

By Arzelà-Ascoli theorem, we obtain a subsequence of $\{w_j\}$ converging in $C^{k,\beta}(B_{R_0/2})$.

Let $R_0 \to \infty$, and by a diagonal argument we can pick a subsequence of $\{w_j\}$ converging to $w = (u, v) \in C^\infty(\mathbb{R}^n) \times C^\infty(\mathbb{R}^n)$ and $\|w\|_{L^\infty(\mathbb{R}^n)} = u(0) + v(0) = 1$. The radial symmetry and monotonic decay of $w$ can be easily seen from the radial symmetry and the monotonic decay of $u, v$. Now we need to show $w$ is a solution to (3).

First we show that $(-\Delta)^s u_j$ converges to $(-\Delta)^s u$ pointwise. Notice that $u_j's$ are uniformly bounded and indeed $|u_j(x)| \leq |u_j(0)| \leq 1$. Then for a fixed point $x,$
for any $\varepsilon > 0$ there exists a $R > 0$ such that

$$|(-\Delta)^s(u_j - u)(x)| \leq \int_{\mathbb{R}^n \setminus B_R(x)} \frac{u_j(x) - u(x) - u_j(y) + u(y)}{|x-y|^{n+2s}} dy$$

$$+ \left| \text{P.V.} \int_{B_R(x)} \frac{u_j(x) - u(x) - u_j(y) + u(y)}{|x-y|^{n+2s}} dy \right|$$

$$\leq \int_{\mathbb{R}^n \setminus B_R(x)} \frac{4dy}{|x-y|^{n+2s}} \leq C \|D^2(u_j - u)\|_{L^\infty(B_R(x))}$$

$$\leq \varepsilon + C \|D^2(u_j - u)\|_{L^\infty(B_R(x))}.$$ 

Since $\varepsilon > 0$ is arbitrary, by the convergence of $\{u_j\}$ in $C^{k,\beta}(B_R(x))$ for any $k$ and $\beta \in (0,1)$, the limit of the last term is zero. Thus, $(-\Delta)^s u_j \to (-\Delta)^s u$ pointwise. Also, since $\lambda_j \to 0$ as $R_j \to \infty$ by a scaling argument, we see $\lambda_j u_j + v_j^p \to v^p$.

A similar argument follows for the second equation of (3). Therefore, $(u,v)$ is a classic solution to (3). \hfill $\Box$

4. Asymptotic analysis. First, let us show a simple integration by parts lemma. Notice that for a smooth function $u$ with certain decay, e.g., $u \in H^{2s} (\mathbb{R}^n)$, the integration by parts holds true due to Fourier transform and Parseval’s identity. However, the solution we found may not have such decaying property, and we show the following formula holds.

**Lemma 4.1.** Let $u \in C^\infty (\mathbb{R}^n) \cap L^\infty (\mathbb{R}^n)$, $g \in W^{2s,1} (\mathbb{R}^n)$. Then

$$\int_{\mathbb{R}^n} (-\Delta)^s u(x) g(x) dx = \int_{\mathbb{R}^n} u(x)(-\Delta)^s g(x) dx. \quad (24)$$

**Proof.** First we assume $g \in C^\infty_0 (\mathbb{R}^n)$ and $\text{supp}(g) \subset B_R$. Let $\eta \in C^\infty$ be the cutoff function on $B_1$, such that $\eta \equiv 1$ on $B_{1/2}$ and $\eta \equiv 0$ on $\mathbb{R}^n \setminus B_2$. Let $\eta_j(x) = \eta(x/j)$, and pick a a sequence $\{R_j\}$ that goes to infinity. Let $u_j = \eta_j u$, hence $|u_j| \leq |u|$.

By Parseval’s formula,

$$\int_{\mathbb{R}^n} (-\Delta)^s u_j(x) g(x) dx = \int_{\mathbb{R}^n} |\xi|^{2s} \mathcal{F}(u_j) \mathcal{F}(g) d\xi$$

$$= \int_{\mathbb{R}^n} u_j(x)(-\Delta)^s g(x) dx.$$ 

Now, by the dominated convergence theorem the last term on the right converges to the right of (24). We only need to show the left term above converges to the left term of (24). For sufficiently large $j$, $\eta_j \equiv 1$ on $B_{R+1}$, then

$$\left| \int_{\mathbb{R}^n} (-\Delta)^s (u_j - u)(x) g(x) dx \right| = \int_{B_R} \int_{\mathbb{R}^n \setminus B_{R+1}} \frac{-u_j(y) + u(y)}{|x-y|^{n+2s}} dy g(x) dx.$$

Since

$$\left| \frac{-u_j(y) + u(y)}{|x-y|^{n+2s}} g(x) \right| \leq 2 \|u\|_\infty \|g\|_\infty \frac{1}{|x-y|^{n+2s}}$$

is integrable on $B_R \times \mathbb{R}^n \setminus B_{R+1}$, by the dominated convergence theorem the limit of the integral above is 0.

Since $C^\infty_0 (\mathbb{R}^n)$ is dense in $W^{2s,1} (\mathbb{R}^n)$, the lemma is proved. \hfill $\Box$
Notice that system (3) satisfy a scaling property, i.e., suppose \((u, v)\) is a solution to system (3), then
\[
u_R(x) = R^\alpha u(Rx), \quad v_R(x) = R^\beta v(Rx),
\]
is also a solution, where
\[
\alpha = \frac{2s(p+1)}{pq-1}, \quad \beta = \frac{2s(q+1)}{pq-1}.
\]
The following energy estimates are similar to estimates for the classical Laplacian for which readers can refer to [9, 29] for general \(0 < p, q < \infty\), and [21, 24] for \(p, q \geq 1\).

**Theorem 4.2.** For \(p, q > 1\), a solution \((u, v)\) to system (3) must satisfy
\[
\int_{B_R} u dx \leq CR^{n-\alpha}, \quad \int_{B_R} v dx \leq CR^{n-\beta},
\]
and
\[
\int_{B_R} u^q dx \leq CR^{n-q\alpha}, \quad \int_{B_R} v^p dx \leq CR^{n-p\beta},
\]
where \(C\) is a constant depending only on \(n, s\). As a consequence, if \((u, v)\) is a radial and monotonically decaying solution, then \(u \leq C|x|^{-\alpha}\) and \(v \leq C|x|^{-\beta}\) for some \(C > 0\).

**Proof.** We only need to show, for some \(C = C(n, s)\),
\[
\int_{B_{\frac{1}{2}}} u dx \leq C, \quad \int_{B_{\frac{1}{2}}} v dx \leq C.
\]
From (29) and by the fact \(p, q > 1\) and Hölder’s inequality, we get
\[
\int_{B_{\frac{1}{2}}} u^q dx \leq C, \quad \int_{B_{\frac{1}{2}}} v^p dx \leq C.
\]
Then (27) and (28) follows from a dilation argument.

Let \(\phi_1(x) = \phi_1(|x|)\) and \(\phi_1(x) \equiv 0\) a.e. in \(\mathbb{R}^n \setminus B_1\) be the first eigenfunction of (12) on \(B_1\). Normalize \(\phi_1(x)\) such that \(\int_{B_1} \phi_1(x) dx = 1\). Then \(\phi_1(x) \in H^s(\mathbb{R}^n)\) is non-negative and also simple (See Theorem 2 and Proposition 9 of [32]). Moreover, \(\phi_1 \in C^{0, s}(\mathbb{R}^n)\) (The Hölder continuity of eigenfunction is due to Theorem 1 in [33], which is a result of Proposition 1.1 in [27] and Proposition 4 in [31]). Hence, \(\phi_1 \in W^{2s, 1}(\mathbb{R}^n)\) (see Remark 1).

Thus, due to \((-\Delta)^s \phi_1 < 0\) on \(\mathbb{R}^n \setminus B_1\) we have,
\[
\lambda_1 \int_{B_1} u \phi_1 dx = \int_{B_1} u(-\Delta)^s \phi_1 dx \\
\geq \int_{\mathbb{R}^n} u(-\Delta)^s \phi_1 dx \\
= \int_{\mathbb{R}^n} (-\Delta)^s u \phi_1 dx \quad \text{by Lemma 4.1} \\
\geq \int_{B_1} v^p \phi_1 dx \\
\geq \left(\int_{B_1} v \phi_1 dx\right)^p,
\]
where the last inequality is due to Jensen’s inequality. Similarly,
\[ \lambda_1 \int_{B_1} v\phi_1 dx \geq \left( \int_{B_1} u\phi_1 dx \right)^q. \]

Hence,
\[ \lambda_1^{p+1} \int_{B_1} u\phi_1 dx \geq \left( \int_{B_1} u\phi_1 dx \right)^{pq}. \]

Since \( pq > 1 \) and \( \phi_1 \geq c > 0 \) on \( B_1^2 \), the above estimate implies (29). Similarly, we get the estimate for \( v \).

As a consequence, a radial and monotonically decaying solution of (3) must decay to zero at infinity and \( u \leq C|x|^{-\alpha} \) and \( v \leq C|x|^{-\beta} \) for some \( C > 0 \) with \( \alpha, \beta \) given by (26).

Acknowledgments. We would like to thank Professor Congming Li for his suggestion and comments and all the reviewers for their help and effort on improving this paper.

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Received February 2018; revised June 2018.

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