On repetitiveness measures of Thue-Morse words

Kanaru Kutsukake\textsuperscript{1}, Takuya Matsumoto\textsuperscript{1},
Yuto Nakashima\textsuperscript{1,2},
Shunsuke Inenaga\textsuperscript{1,2},
Hideo Bannai\textsuperscript{3},
and Masayuki Takeda\textsuperscript{1}

\textsuperscript{1} Department of Informatics, Kyushu University, Fukuoka, Japan
\textsuperscript{2} PRESTO, Japan Science and Technology Agency, Kawaguchi, Japan
\textsuperscript{3} M&D Data Science Center, Tokyo Medical and Dental University, Tokyo, Japan
\{kutsukake.kanaru,matsumoto.takuya,yuto.nakashima,
inenaga,takeda\}@inf.kyushu-u.ac.jp
hdbn.dsc@tmd.ac.jp

Abstract. We show that the size $\gamma(t_n)$ of the smallest string attractor of the $n$th Thue-Morse word $t_n$ is 4 for any $n \geq 4$, disproving the conjecture by Mantaci et al. [ICTCS 2019] that it is $n$. We also show that $\delta(t_n) = \frac{10^n}{3+2^n}$ for $n \geq 3$, where $\delta(w)$ is the maximum over all $k = 1, \ldots, |w|$, the number of distinct substrings of length $k$ in $w$ divided by $k$, which is a measure of repetitiveness recently studied by Kociumaka et al. [LATIN 2020]. Furthermore, we show that the number $z(t_n)$ of factors in the self-referencing Lempel-Ziv factorization of $t_n$ is exactly $2n$.

Keywords: String attractors · Thue-Morse words

1 Introduction

Measures which indicate the repetitiveness in a string is a hot and important topic in the field of string compression. For example, given string $w$, the size $g(w)$ of the smallest grammar that derives solely $w$ [5], the number $z(w)$ of factors in the Lempel-Ziv factorization [12], the number $r(w)$ of runs in the Burrows-Wheeler transform [4] (RLBWT), and the size $b(w)$ of the smallest bidirectional scheme (or macro schemes) [18]. Recently, Kempa and Prezza proposed the notion of string attractors [10], and showed that the size $\gamma(w)$ of the smallest string attractor of $w$ is a lower bound on the size of the compressed representation for these dictionary compression schemes. While $z(w)$ and $r(w)$ are known to be computable in linear time, it is NP-hard to compute $g(w), b(w), \gamma(w)$ [11][13][10].

To further understand these measures, Mantaci et al. [13] studied the size of the smallest string attractor in several well known family of strings. In particular, they showed a size-2 string attractor for standard Sturmian words which is the smallest possible. They further showed a string attractor of size $n$ for the $n$th Thue-Morse word $t_n$, and conjectured it to be the smallest.

In this paper, we continue this line of work, and investigate the exact values of various repetitive measures of the $n$th Thue-Morse word $t_n$. More specifically,
we show that the size $\gamma(t_n)$ of the smallest string attractor of $t_n$ is 4 for $n \geq 4$, disproving Mantaci et al.’s conjecture. Furthermore, we give the exact value $\delta(t_n) = \frac{10}{3 + 2^{4-n}}$ for $n \geq 3$, of the repetitiveness measure recently studied by Kociumaka et al. [11], and the size $z(t_n) = 2n$ of the self-referencing LZ77 factorization.

We note that for any standard Sturmian word $s$, $z(s) = \Theta(\log |s|)$ [1], while the size $r(s)$ of the RLBWT is always constant [14]. On the other hand, $z(t_n)$ and $r(t_n)$ are both $\Theta(n)$, i.e., logarithmic in the length $|t_n|$ (the former due to [1] as well as this work, and the latter due to [3]). This shows that Thue-Morse words are an example where the size of smallest string attractor is not a tight lower bound on the size of the smallest of the known efficiently computable dictionary compressed representation, namely, $\min\{z(w), r(w)\}$. We also conjecture that $b(t_n) = \Theta(n)$, which would seem to imply that the size of the smallest string attractor is not a tight lower bound for all currently known dictionary compression schemes.

Let $\ell(w)$ denote the size of the Lyndon factorization [9] of $w$. It is known that for any $w$, $\ell(w) = O(g(w))$ [8] and $\ell(w) = O(z(w))$ [20], although it can be much smaller. Interestingly, it is also known that $\ell(t_n) = \Theta(n)$ (Theorem 3.1, Remark 3.8 of [9]). Thus, if $b(t_n) = \Theta(n)$, then $\ell(t_n)$ would be an asymptotically tight lower bound for the smallest size of known dictionary compression schemes for $t_n$, while $\gamma(t_n)$ is not.

Table 1 summarizes what we know so far.

| measure  | description                                      | value                  | reference |
|----------|--------------------------------------------------|------------------------|-----------|
| $z(t_n)$ | Size of Lempel-Ziv factorization with self-reference | $2n$                   | [1], this work |
| $r(t_n)$ | Number of same-character runs in BWT              | $2n - 2$               | [3]       |
| $\ell(t_n)$ | Size of Lyndon factorization                   | $\frac{3n - 2}{2}$    | [9]       |
| $b(t_n)$ | Size of smallest bidirectional scheme            | open                   | N/A       |
| $\gamma(t_n)$ | Size of smallest string attractor                | 4 ($n \geq 4$)         |           |
| $\delta(t_n)$ | maximum of subword complexity divided by subword length | $\frac{10}{3 + 2^{4-n}}$ ($n \geq 3$) | this work |

2 Preliminaries

Let $\Sigma$ denote a set of symbols called the alphabet. An element of $\Sigma^*$ is called a string. For any $k \geq 0$, let $\Sigma^k$ denote the set of strings of length exactly $k$. For any string $w$, the length of $w$ is denoted by $|w|$. For any $1 \leq i \leq |w|$, let $w[i]$ denote the $i$th symbol of $w$, and for any $1 \leq i \leq j \leq |w|$, let $w[i..j] = w[i]w[i+1]\ldots w[j]$. 


If $w = xyz$ for strings $x, y, z \in \Sigma^*$, then $x, y, z$ are respectively called a prefix, substring, suffix of $w$. We denote by $\text{Substr}(w)$, the set of substrings of $w$.

In this paper, we will only consider the binary alphabet $\Sigma = \{a, b\}$. For any string $w \in \Sigma^*$, let $\overline{w}$ denote the string obtained from $w$ by changing all occurrences of $a$ (resp. $b$) to $b$ (resp. $a$).

**Definition 1 (Thue-Morse Words [16,19,15]).** The $n$-th Thue-Morse word $t_n$ is a string over a binary alphabet $\{a, b\}$ defined recursively as follows: $t_0 = a$, and for any $n > 0$, $t_n = t_{n-1}t_{n-1}$.

It is a simple observation that $|t_n| = 2^n$ for any $n \geq 0$.

Below, we define the repetitiveness measures used in this paper:

**String attractors [10]** For any string $w$, a set $\Gamma$ of positions in $w$ is a string attractor of $w$, if, for any substring $x$ of $w$, there is an occurrence of $x$ in $w$ that contains a position in $\Gamma$. For any string $w$, we will denote the size of a smallest string attractor of $w$ as $\gamma(w)$.

**$\delta$ [17,11]**

For any string $w$, $\delta(w) = \max_{k=1,\ldots,|w|} (|\Sigma^k \cap \text{Substr}(w)|/k)$.

**LZ factorization [12]** For any string $w$, the LZ factorization of $w$ is the sequence $f_1, \ldots, f_z$ of non-empty strings such that $w = f_1 \cdots f_z$, and for any $1 \leq i \leq z$, $f_i$ is the longest prefix of $f_1 \cdots f_z$ which has at least two occurrences in $f_1 \cdots f_i$, or, $|f_i| = 1$ otherwise. We denote the size of the LZ factorization of string $w$ as $z(w)$.

It is known that $\delta(w) \leq \gamma(w) \leq z(w)$, $r(w)$ for any $w$ [7,10].

3 Repetitive Measures of Thue-Morse Words

3.1 $\gamma(t_n)$

Mantaci et al. [13] showed the following explicit string attractor of size $n$ for the $n$-th Thue-Morse word.

**Theorem 1 (Theorem 8 of [13]).** A string attractor of the $n$-th Thue Morse word, with $n \geq 3$ is

$$\{2^{n-1} + 1\} \cup \{3 \cdot 2^{i-2} \mid i = 2, \ldots, n\}.$$

To prove our new upperbound of 4 for the smallest string attractor of $t_n$ for $n \geq 4$, we first show the following lemma.

**Lemma 1.** Let

$$N_n = \{t_{n-1}t_{n-1}\} \cup \left(\bigcup_{k=0}^{n-2} \{t_k\overline{t_k}, \overline{t_k}t_k\}\right).$$

Then, for any substring $w$ and $n \geq 2$, there exists $s \in N(n)$ such that the occurrence of $w$ in $s$ contains the center of $s$ (i.e., position $|s|/2$).
**Proof.** Consider the recursively defined perfect binary tree with $t_n$ as the root, with $t_{n-1}$ and $t_{n-1}$ respectively as its left and right children (See Fig. 1). The leaves consist of either $t_0$ or $t_{2n}$, each corresponding to a position of $t_n$. If $|w| = 1$, then, we can choose $t_1 = t_0t_0 = ab$ for $a$ and $t_2 = t_1t_1 = abba$ for $b$. For any substring $w = t_n[i..j]$ of length at least 2, consider the lowest common ancestor of leaves corresponding to $t_n[i]$ and $t_n[j]$. Each node of the tree is $t_n = t_{n-1}t_{n-1}$ if it is the root, or otherwise, either $t_k+1 = t_kt_k$ or $t_k+1 = t_kt_k$ for some $0 \leq k \leq n-2$. Since $w$ is a substring that starts in the left child and ends in the right child of the lowest common ancestor, the occurrence of $w$ must contain the center, and the lemma holds.

![Fig. 1. A representation of $t_n$ as a perfect binary tree (shown to depth 4) introduced in the proof of Lemma 1](image)

For each level where segments are labeled with $t_k$, non-labeled segments represent $t_k$. The black circles depict the four positions in $K_n$ defined in Theorem 2 at the node at which the center of the parent coincides with the position.

**Theorem 2.** For any $n \geq 4$, the set

$$K_n = \{2^{n-2}, 3 \cdot 2^{n-3}, 2^{n-1}, 3 \cdot 2^{n-2}\}$$

is a string attractor of $t_n$.

**Proof.** Let $w$ be an arbitrary substring of $t_n$. From Lemma 1, it suffices to show that any element in $N_n$ has an occurrence in $t_n$ whose center coincides with a position in $K_n$. For $t_n - t_{n-1}$, $t_n - 2t_{n-2}$, $t_n - 3t_{n-3}$, and $t_n - 3t_{n-3}$, it is clear from Fig. 1 that their centers respectively coincide with the four elements of $K_n$. Furthermore, there is an occurrence of $t_n - 3t_{n-3}$ whose center coincides with that of $t_n - 1t_{n-1}$, and thus with an element of $K_n$. More generally, for any $2 \leq k \leq n - 2$, each occurrence of $t_kt_k$ implies an occurrence of $t_k - 2t_k - 2$ whose centers coincide. This is because

$$t_kt_k = t_k - 1t_k - 1t_k - 1t_k - 1$$

$$= t_k - 1t_k - 2t_k - 2t_k - 2t_k - 2t_k - 2.$$
The same argument holds for $t_{k-2}t_{k-2}$ by considering $t_k t_k$. The theorem follows from a simple induction.

**Theorem 3.** $\gamma(t_n) = 4$ for any $n \geq 4$.

**Proof.** Theorem 2 implies $\gamma(t_n) \leq 4$. From Theorem 1 shown in the next subsection, we have $\delta(t_n) > 3$ for $n \geq 6$. Since $\gamma(t_n)$ is an integer which cannot be smaller than $\delta(t_n)$, it follows that $\gamma(t_n) \geq 4$ for $n \geq 6$. For $n = 4, 5$, it can be shown by exhaustive search that there is no string attractor of size 3.

### 3.2 $\delta(t_n)$

Brlek [2] investigated the number of distinct substrings of length $m$ in $t_n$, and gave an exact formula. Below is a summary of his result which will be a key to computing $\delta(t_n)$.

**Lemma 2 (Proposition 4.2, Corollary 4.2.1, Proposition 4.4 of [2]).** The number $P_n(m)$ of distinct substrings of length $m \geq 3$ in $t_n$ ($n \geq 3$) is:

$$P_n(m) = \begin{cases} 2^n - m + 1 & 2^n - 2 + 1 \leq m \leq 2^n \\ 6 \cdot 2^{q-1} + 4p & 3 \leq m \leq 2^n - 2, 0 < p \leq 2^{q-1} \\ 8 \cdot 2^{q-1} + 2p & 3 \leq m \leq 2^n - 2, 2^{q-1} < p \leq 2^q \end{cases}$$

where $p, q$ are values uniquely determined by $m = 2^q + p + 1$ and $0 < p \leq 2^q$.

**Theorem 4.**

$$\delta(t_n) = \begin{cases} 1 & n = 0 \\ 2 & n = 1, 2 \\ \frac{10}{3+2^{-n}} & n \geq 3 \end{cases}$$

**Proof.** We only consider $n \geq 3$ below. The number of distinct substrings of length 1 and 2 in $t_n$, are respectively 2 and 4. For $2^n - 2 + 1 \leq m \leq 2^n$,

$$\max_{2^n - 2 + 1 \leq m \leq 2^n} P_n(m) = \max_{2^n - 2 + 1 \leq m \leq 2^n} \left\{ \frac{2^n + 1}{m} \right\} - 1 = \frac{2^n + 1}{2^{n-2} - 1} - 1 = \frac{3}{1 + 2^{2-n}}.$$  

For $3 \leq m \leq 2^n - 2$ and fixed $q$, it is easy to verify that $P_n(m)/m$ is increasing when $0 < p \leq 2^{q-1}$, and non-increasing when $2^{q-1} < p \leq 2^q$, because

$$\left( \frac{6 \cdot 2^{q-1} + 4p}{2^q + p + 1} \right)' = \frac{4(2^q + p + 1) - (6 \cdot 2^{q-1} + 4p)}{(2^q + p + 1)^2} = \frac{2^q + 4}{(2^q + p + 1)^2} > 0$$

and

$$\left( \frac{8 \cdot 2^{q-1} + 2p}{2^q + p + 1} \right)' = \frac{2(2^q + p + 1) - (8 \cdot 2^{q-1} + 2p)}{(2^q + p + 1)^2} = \frac{(2 - 4 \cdot 2^{q-1})}{(2^q + p + 1)^2} \leq 0.$$  

Therefore, for a fixed $q$, the maximum value of $P_n(m)/m$ is obtained when $p = 2^{q-1}$, i.e., \( \frac{6 \cdot 2^{q-1} + 4 \cdot 2^{q-1}}{2^q + 2^{q-1} + 1} = \frac{10 \cdot 2^{q-1}}{3 \cdot 2^{q-1} + 1} = \frac{10}{3 + 2^{-n}} \). Since this is increasing in $q$, we have that $\max_{3 \leq m \leq 2^n - 2} P_n(m)/m$ is obtained by choosing the largest possible $q = n - 3$ (where $p = 2^{q-1} = 2^{n-4}$, and thus $m = 2^n - 3 + 2^{n-4} + 1 = 3 \cdot 2^{n-4} + 1 \leq 2^{n-2}$), which gives us the final result $\delta(t_n) = \max \{ \frac{2}{3}, \frac{4}{3\cdot 2^{-n}}, \frac{10}{3 + 2^{-n}} \} = \frac{10}{3 + 2^{-n}}$. 

\( \square \)
3.3 LZ77

We consider the size \( z(t_n) \) of the LZ factorization. Although Berstel and Savelli [1] have given a complete characterization of the LZ factorization for the infinite Thue-Morse word, we show an alternate proof in terms of the \( n \)-th Thue-Morse word. Below is an important lemma, again by Brlek, we will use.

**Lemma 3 (Corollary 4.1.1 of [2]).** The word \( t_n \) has one and only one occurrence of every factor \( w \) such that \( |w| \geq 2^n - 2 + 1 \).

**Theorem 5.** For any \( n \geq 1 \), \( z(t_n) = 2n \).

**Proof.** Clearly, \( z(t_1) = 2 \). Since \( t_k = t_{k-1}t_{k-1}t_{k-2}t_{k-2}t_{k-2}t_{k-2} \), it is easy to see that \( z(t_k) \leq z(t_{k-1}) + 2 \), because \( t_{k-2} \) and \( t_{k-2} \) respectively have earlier occurrences in \( t_k \). Thus, \( z(t_n) \leq 2n \). On the other hand, Lemma 3 implies that the substring \( t_k[2^{k-1} \ldots 3 \cdot 2^{k-2}] \) of length \( 2^{k-2} + 1 \) cannot be a single LZ factor, implying that position \( 2^{k-1}(=|t_{k-1}|) \) and position \( 3 \cdot 2^{k-2}(>|t_{k-1}|) \) belong to different factors. Similarly, the substring \( t[3 \cdot 2^{k-2} \ldots 2^k] \) of length \( 2^{k-2} + 1 \) cannot be a single LZ factor, implying that position \( 3 \cdot 2^{k-2} \) and position \( 2^k \) belong to different factors. Thus, \( z(t_{k+1}) \geq z(t_k) + 2 \), implying \( z(t_n) \geq 2n \). \( \square \)

**Acknowledgments**

This work was supported by JSPS KAKENHI Grant Numbers JP18K18002 (YN), JP17H01697 (SI), JP16H02783, JP20H04141 (HB), JP18H04098 (MT), and JST PRESTO Grant Number JPMJPR1922 (SI).
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