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The Spectral Action for Dirac Operators
with skew-symmetric Torsion

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Abstract

We derive a formula for the gravitational part of the spectral action for Dirac operators on 4-dimensional manifolds with totally anti-symmetric torsion. We find that the torsion becomes dynamical and couples to the traceless part of the Riemann curvature tensor. Finally we deduce the Lagrangian for the Standard Model of particle physics in presence of torsion from the Chamseddine-Connes Dirac operator.

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Introduction

In classical Relativity one presumes that the gravitational degrees of freedom are encoded only in the choice of the metric and one can therefore restrict to Levi-Civita connections. This is usually justified by Einstein-Cartan theory because the critical points of the corresponding action are space-times with torsion free connections.

If we want to incorporate the other forces of Nature (i.e. the electro-weak and the strong force) and the known fermions (Leptons and Quarks) we have to consider unified theories. A geometrical approach to such a unification is offered by Connes’ noncommutative geometry [1]. The natural action to consider in this framework is given by the spectral action principle [2].

In the language of noncommutative geometry any geometry is encoded by a spectral triple which consists of an algebra, a module over this algebra and a first order differential operator. Here one should picture one of the simplest examples given by compact Riemannian spin manifolds: the algebra is formed by the smooth functions, the module is given by the square-integrable spinor fields and the first order differential operator is the classical Dirac operator. To reconstruct the manifold one needs further ingredients (a real structure, a chirality operator) [1]. We will omit any details here, since the only relevant objects for this article turn out to be classical twisted Dirac operators.

In combination with product geometries based on spectral triples for Riemannian manifolds and finite geometries one even finds a conceptual explanation for the Standard Model of particle physics [3]. Still, the relevant Dirac operators are classical twisted Dirac operators and the module consists of twisted spinor fields. For these so called almost-commutative geometries, the spectral action precisely predicts the Lagrangian of corresponding Einstein-Yang-Mills-Higgs model. It has also successfully predicted scale invariant Lagrangians with dilaton fields [4], and quantum gravity boundary terms [5] in the case of manifolds with boundary.

In this paper we take the spectral geometric point of view, in the sense that we are only dealing with classical objects where the noncommutative input resides in the exact structure of the twist bundle. We expand the known calculations of the spectral action (with and without the standard model) by implementing skew-symmetric torsion into the relevant twisted Dirac operators. As a result we find that in this context the Einstein-Cartan theory does not apply. One finds an action that suggests critical points with nonzero torsion which might even be dynamical.

The spectral action for pure gravity with torsion

In this first section we concern ourselves only with the gravitational part of the spectral action for a 4-dimensional closed Riemannian manifolds $M$ with spin structures. Let us now briefly recall the basic notions of connections and Dirac operators with torsion. Each connection on the tangent bundle of a manifold can be written as a sum of the Levi-Civita connection $\nabla^{LC}$ and a $(2,1)$-tensor field $A$, i.e. $\nabla X Y = \nabla^{LC} X Y + A(X,Y)$. For such a general connection $\nabla$ the torsion 3-form is by definition $T(X,Y,Z) = \langle \nabla_X Y - \nabla_Y X - [X,Y], Z \rangle$.

The connection $\nabla$ is compatible with the Riemannian metric $\langle \cdot, \cdot \rangle$ and has the same geodesics as $\nabla^{LC}$ if and only if $A(X,Y,Z) = \langle A(X,Y), Z \rangle$ is totally anti-symmetric. We will only consider this case. Then the torsion of $\nabla$ is given by $T = 2A$, and hence

$$\nabla_X Y = \nabla^{LC}_X Y + \frac{1}{2} T(X,Y,\cdot)\#.$$  \hspace{1cm} (1)

Note the $T(X,Y,\cdot)\#$ equals the vector valued torsion 2-form of E. Cartan, which is defined as the exterior covariant derivative of the soldering form.

Since we assume that the manifold carries a spin structure, we can consider spinor fields $\psi$ and the spin connection induced by $\nabla$ can be expressed as

$$\nabla_X \psi = \nabla^{LC}_X \psi + \frac{1}{4} (X \cdot T) \cdot \psi.$$  \hspace{1cm} (2)
Here \((X,J)\) denotes Clifford multiplication\(^1\) by the 2-form \(T(X,\cdot,\cdot)\). This spin connection yields a Dirac operator \(D\) which one can write as \(D\psi = \sum_i e_i \cdot \nabla_{e_i} \psi\), for any orthonormal frame \(e_i\).

From [6 Thm. 6.2] we deduce the Bochner form of the square of this Dirac operator:

\[
D^2 = \Delta + \frac{3}{4} dT + \frac{1}{4} R - \frac{9}{8} T^2_0,
\]

where \(\Delta\) is the Laplacian associated to the spin connection \(\nabla_X\psi = \nabla_X^LC\psi + \frac{3}{4} (X,J)\cdot \psi\),

\[
dT \text{ is the exterior differential of the 3-form } T, \quad R \text{ is the scalar curvature of the Riemannian manifold (in our convention spheres have positive curvature, i.e. } R = 12 \text{ for the 4-dimensional sphere) and } T^2_0 = \frac{1}{6} \sum_{i,j=1}^n \|T(e_i,e_j,\cdot)\|^2.
\]

For the Dirac operator \(D\) we will calculate the bosonic part of the spectral action. It is defined to be the number of Eigenvalues of \(D\) in the interval \([-\Lambda, \Lambda]\) with \(\Lambda \in \mathbb{R}^+\). In [2] it is expressed as

\[
I = \text{tr } F \left( \frac{D^2}{\Lambda^2} \right)
\]

Here \(\text{tr}\) denotes the operator trace in the Hilbert space of \(L^2\)-spinor fields, and \(F: \mathbb{R}^+ \to \mathbb{R}^+\) is a cut-off function with support in the interval \([0, +1]\) which is constant near the origin. Here we follow the notation of [7].

For \(t \to 0\) one has the heat trace asymptotics [8]

\[
\text{tr } \left( e^{-tD^2} \right) \sim \sum_{n \geq 0} t^{n-2} a_{2n}(D^2)
\]

One uses the Seeley-deWitt coefficients \(a_{2n}(D^2)\) and \(t = \Lambda^{-2}\) to obtain an asymptotics for the spectral action [2] [9]

\[
I = \text{tr } F \left( \frac{D^2}{\Lambda^2} \right) \sim \Lambda^4 F_4 a_0(D^2) + \Lambda^2 F_2 a_2(D^2) + \Lambda^0 F_0 a_4(D^2) \quad \text{as } \Lambda \to \infty
\]

with the first three moments of the cut-off function which are given by \(F_4 = \int_0^\infty s \cdot F(s) ds\), \(F_2 = \int_0^\infty F(s) ds\) and \(F_0 = F'(0)\). Note that these moments are independent of the geometry of the manifold.

Setting \(E = -\frac{1}{4} dT - \frac{1}{2} R + \frac{9}{2} T^2_0\), we get \(D^2 = \Delta - E\) from [3]. We use [8 Thm. 4.1.6] to obtain the first three coefficients of the heat trace asymptotics:

\[
a_0(D^2) = \frac{1}{4\pi^2} \int_M dvol
\]

\[
a_2(D^2) = \frac{1}{96\pi^2} \int_M (6\text{tr}(E) + 4R) dvol
\]

\[
a_4(D^2) = \frac{1}{5760\pi^2} \int_M \left( \text{tr} \left( 60\Delta E + 60RE + 180E^2 + 30\Omega_{ij}\Omega_{ij} \right) + 48\Delta^LC R + 20R^2 - 8\|Ric\|^2 + 8\|Riem\|^2 \right) dvol
\]

Here \(Ric\) and \(Riem\) denote the Ricci curvature and the Riemannian curvature tensors of the metric, and \(\Omega_{ij} = \nabla_{e_i} \nabla_{e_j} - \nabla_{e_j} \nabla_{e_i} - \nabla_{[e_i,e_j]}\) is the curvature of \(\nabla\).

We evaluate the \(a_1(D^2)\) above and take into account that \(\text{tr}(dT) = 0\) due to Clifford relations and cyclicity of the trace. For the first two coefficients we get

\[
a_0(D^2) = \frac{1}{4\pi^2} \int_M dvol, \quad a_2(D^2) = \frac{1}{16\pi^2} \int_M \left( \frac{9}{2} T^2_0 - \frac{1}{3} R \right) dvol.
\]

\(^1\)Here we use the Clifford relations \(X \cdot Y + Y \cdot X = -2(X,Y)\) for tangent vectors \(X, Y\). Any 2-form \(X^\flat \wedge Y^\flat\) acts as \(\frac{1}{2} X \cdot Y^\flat\) on the spinor module.
For $a_4(D^2)$ we use that $\text{tr}(\Delta E) = \Delta^{LC} \text{tr}(E)$ and $\Delta^{LC} R$ vanish after integration over the closed manifold $M$. 

$$a_4(D^2) = \frac{1}{16\pi^2} \int_M \left( \frac{1}{72} R^2 - \frac{1}{45} \|\text{Ric}\|^2 + \frac{1}{45} \|\text{Riem}\|^2 - \frac{3}{8} R T_0^2 + \frac{9}{8} \|dT\|^2 + \frac{81}{32} (T_0^2)^2 + \frac{12}{12} \sum_{i,j} \text{tr}(\Omega_{ij} \Omega_{ij}) \right) dvol$$

Similar calculations have been done in [10]. For the curvature $\Omega_{ij}$ of the connection $\tilde{\nabla}$ we proceed by computing 

$$\Omega_{ij} = \sum_{a,b} \left( \frac{1}{4} \langle R(e_i, e_j)e_a, e_b \rangle + \frac{3}{8} a(\nabla T)(e_i, e_j, e_a, e_b) + \frac{9}{16} \sum_c (T(e_i, e_c, e_a)T(e_j, e_c, e_b) - T(e_j, e_c, e_a)T(e_i, e_c, e_b)) \right) e_a e_b,$$

where $a(\nabla T)$ denotes the anti-symmetrisation in the first two entries of $\nabla T$, i.e. $a(\nabla T)(e_i, e_j, e_a, e_b) = \nabla_{e_i} T(e_j, e_a, e_b) - \nabla_{e_j} T(e_i, e_a, e_b)$. Using the identity $\text{tr}(e_k e_l e_i) = 4(\delta_{ik} \delta_{kl} - \delta_{il} \delta_{kj})$ we obtain 

$$\sum_{i,j} \text{tr}(\Omega_{ij} \Omega_{ij}) = -8 \sum_{a,b,c} \left( \frac{1}{4} \langle R(e_i, e_j)e_a, e_b \rangle + \frac{3}{8} a(\nabla T)(e_i, e_j, e_a, e_b) + \frac{9}{8} \langle T(e_i, e_j, e_a, e_b) \rangle \right)^2,$$

where $c(T)(e_i, e_j, e_a, e_b) = \sum_c T(e_i, e_c, e_a)T(e_j, e_c, e_b)$. This term equals the square of a norm in the space of $(4,0)$-tensors. We use representation theory [11] Chap. 4 of $O(4)$ to decompose these tensors into irreducible components. We note that $c(T)$ is a formal curvature tensor after interchanging the second and the third entry and hence we may write $Riem, c(T) \in \mathbb{R} \oplus \text{Sym}^2 \oplus \text{Weyl}$. Here we consider $\mathbb{R} \oplus \text{Sym}^2 \oplus \text{Weyl} \subset \text{Sym}^2(\Lambda^2)$. Furthermore $a: \Lambda^1 \otimes \Lambda^3 \rightarrow a(\Lambda^1 \otimes \Lambda^3) \subset \Lambda^1 \otimes \Lambda^2$ is an isomorphism of $O(4)$-representations. The image splits into $a(\Lambda^1 \otimes \Lambda^3) = \Lambda^4 \otimes \text{Sym}^2_0$, where $\Lambda^4 \subset \text{Sym}^2(\Lambda^2)$ and $\Lambda^2 \otimes \text{Sym}^2_0 \subset \Lambda^2(\Lambda^2)$. As the above decompositions are orthogonal, we conclude that $c(T) \perp a(\nabla T)$ and $Riem \perp a(\nabla T)$ in the space of $(4,0)$-tensors.

After identification by the above isomorphisms, this yields that the norm of the $\Lambda^4$-component of $a(\nabla T)$ equals $\sqrt{\|dT\|^2}$ and norm of the $\Lambda^2$-component is $2\|dT\|$. The norm of the remaining component in $\text{Sym}^2_0$ is denoted by $2\|\text{sym}^2_0(\nabla T)\|$. We compute:

$$\frac{1}{12} \sum_{i,j} \text{tr}(\Omega_{ij} \Omega_{ij}) = -\frac{1}{24} \|\text{Riem}\|^2 - \frac{3}{8} \|dT\|^2 - \frac{9}{16} \|dT\|^2 - \frac{27}{32} \|c(T)\|^2 - \frac{3}{8} P(T),$$

where we abbreviate

$$P(T) := \sum_{i,j} \langle R(e_i, e_j)e_a, e_b \rangle T(e_i, e_c, e_a)T(e_j, e_c, e_b).$$

Once more we insert the Ricci decomposition of the curvature tensor into the scalar curvature, the traceless Ricci tensor and the Weyl tensor $W$ and obtain

$$P(T) = -RT_0^2 - \sum_{i,j} \text{Ric}(e_j, e_a)T(e_j, e_c, e_a)T(e_j, e_c, e_i) + \sum_{i,j} \langle W(e_i, e_j)e_a, e_b \rangle T(e_i, e_c, e_a)T(e_j, e_c, e_b).$$

Finally we obtain for the fourth heat coefficient with $R(T) = P(T) + RT_0^2$

$$a_4(D^2) = \frac{1}{16\pi^2} \int_M \left( \frac{1}{72} R^2 - \frac{1}{45} \|\text{Ric}\|^2 - \frac{7}{360} \|\text{Riem}\|^2 + \frac{81}{32} (T_0^2)^2 - \frac{27}{32} \|c(T)\|^2 + \frac{9}{16} \|dT\|^2 - \frac{3}{8} \|dT\|^2 - \frac{3}{8} \|\text{sym}^2_0(\nabla T)\|^2 - \frac{3}{8} R(T) \right) dvol$$

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Neglecting the term $a_4$ in the spectral action, we would obtain the classical Einstein-Cartan-action which has only torsion free critical points upon variation of metric and torsion 3-forms. A similar action functional for Dirac operators with totally anti-symmetric torsion has already been considered in [12, 13], where the authors used the Wodzicki residue as the bosonic action. This gives an action involving only the second Seeley-deWitt coefficient.

Considering the full spectral action (5), which also includes $a_4(D^2)$, we observe that the term $R(T)$ generically couples torsion and the trace free component of the curvature tensor. Therefore we expect critical points of the spectral action with non-zero torsion. Furthermore, due to the derivative terms of $T$, the torsion becomes dynamical.

The spectral action for the Standard Model with torsion

In the noncommutative approach to the Standard Model of particle physics, the fermionic Hilbert space is the product space of the Hilbert space of $L^2$-sections in the spinor bundle $S$ and a finite dimensional Hilbert space $\mathcal{H}_f$ (called the finite or internal Hilbert space). The specific particle model is encoded in $H = L^2(M,S) \otimes \mathcal{H}_f$, where we follow the notation in [14].

On the twisted spinor bundle $S \otimes \mathcal{H}_f$ one considers a connection $\nabla^{SM} = \nabla \otimes \text{id}_{\mathcal{H}_f} + \text{id}_S \otimes \nabla^{H_f}$, where $\nabla$ is a connection with skew-symmetric torsion as in [2] and $\nabla^{H_f}$ is a covariant derivative in the trivial bundle $\mathcal{H}_f$ induced by gauge fields. The associated Dirac operator to $\nabla^{SM}$ is called $D^{\nabla^{SM}}$. The generalised Dirac operator of the Standard Model $D_\Phi$ contains the Higgs boson, Yukawa couplings, neutrino masses and the CKM-matrix encoded in a field $\Phi$ of endomorphisms of $\mathcal{H}_f$. We follow the conventions of Chamseddine and Connes [2, 3] and define $D_\Phi$ for sections $\psi \otimes \chi \in \mathcal{H}$ as

$$D_\Phi(\psi \otimes \chi) = \hat{D}_\Phi^{SM}(\psi \otimes \chi) + \gamma_5 \psi \otimes \Phi \chi$$

where $\gamma_5 = e_0 e_1 e_2 e_3$ is the volume element and $\hat{D}_\Phi^{SM}$ is the twisted Dirac operator (compare (18) in the appendix). We note that $D_\Phi$ is required to be a self-adjoint operator, consistent with the axioms of noncommutative geometry [4]. From this one gets restrictions on $\Phi$, in particular it has to be self-adjoint and compatible with the real structure $J$ and the chirality operator. We choose the same $\Phi$ as Chamseddine and Connes [2, 3] since the torsion does not effect these relations.

The bosonic part of the Lagrangian of the Standard Model is obtained by replacing $D$ by $D_\Phi$ in (6). First we need to calculate the Bochner formula for the square of $D_\Phi$. We use the results from the appendix (e.g. the definition of $\nabla$ (16) and the Bochner formula for $D^{\nabla^{SM}}$ (17)) and get the following Bochner formula

$$D_\Phi^2(\psi \otimes \chi) = \left(\hat{D}_\Phi^{SM}\right)^2(\psi \otimes \chi) + \sum_{i=1}^{n} \left\{ \gamma_5 e_i \cdot \nabla_{e_i} \psi \otimes (\Phi \chi) + (e_i \cdot \gamma_5 \nabla_{e_i} \psi) \otimes \Phi \chi \right\}$$

$$+ \sum_{i=1}^{n} \left\{ \gamma_5 e_i \cdot \Phi \nabla_{e_i}^{H_f} \chi + e_i \cdot \gamma_5 \psi \otimes \nabla_{e_i}^{H_f}(\Phi \chi) \right\} + (\gamma_5)^2 \psi \otimes (\Phi^2) \chi$$

$$= \left(\hat{D}_\Phi^{SM}\right)^2(\psi \otimes \chi) - \sum_{i=1}^{n} \gamma_5 e_i \cdot \psi \otimes \left[ \nabla_{e_i}^{H_f}, \Phi \right] \chi + \psi \otimes (\Phi^2) \chi$$

$$= \Delta \psi \otimes \chi - E_\Phi(\psi \otimes \chi),$$

where the potential is defined as

$$E_\Phi(\psi \otimes \chi) = \hat{E}(\psi \otimes \chi) + \sum_{i=1}^{n} \gamma_5 e_i \cdot \psi \cdot \left[ \nabla_{e_i}^{H_f}, \Phi \right] \chi - \psi \cdot (\Phi^2) \chi$$

with $\hat{E}$ as in (19).

\footnote{The Clifford multiplication by a tangent vector $X$ acts as $X \cdot (\psi \otimes \chi) = (X \cdot \psi) \otimes \chi$. Note that the twisted connection $\nabla^{SM}$ is compatible with the Clifford multiplication.}
We denote the trace on \( H \) and on \( H_f \) as \( \text{Tr} \) and \( \text{tr}_f \), respectively (both pointwise and the \( L^2 \)-sense). As above \( \text{tr} \) is the trace for spinorial part \( S \). From (10) one obtains the trace \( \text{Tr}(E) = \text{rank} \ H_f \cdot \text{tr}(E) - 4 \text{tr}_f(\Phi^2) \), since the endomorphism \( \sum_{i \neq j} (e_i \cdot e_j \cdot \psi) \otimes (\Omega^{H_f}_{ij} \chi) \) in (10) is skew-symmetric and hence traceless.

For the Seeley-deWitt coefficient \( a_4(D^2_\Phi) \) we also need to calculate
\[
(E_\Phi)^2(\psi \otimes \chi) = (E^2 \psi) \otimes \chi + \frac{1}{4} \sum_{i \neq j} (e_i \cdot e_j \cdot e_k \cdot e_l \cdot \psi) \otimes \left( \Omega^{H_f}_{ij} \Omega^{H_f}_{kl} \chi \right)
\]
\[
+ \sum_{i,j=1}^n \gamma_5 e_i \cdot \gamma_5 e_j \cdot \psi \otimes [\nabla^{H_f}_{e_i}, \Phi] [\nabla^{H_f}_{e_j}, \Phi] \chi + \psi \otimes (\Phi^4) \chi - 2 E \psi \otimes (\Phi^2) \chi
\]
\[
+ \sum_{i \neq j} \nabla^{H_f}_{e_i} \psi \otimes \left( \Omega^{H_f}_{ij} \chi \right) + \sum_{i=1}^n (E \gamma_5 e_i \cdot \psi + \gamma_5 e_i \cdot E \psi) \otimes [\nabla^{H_f}_{e_i}, \Phi] \chi
\]
\[
+ \frac{1}{2} \sum_{i \neq j} (e_i \cdot e_j \cdot \gamma_5 e_k \cdot \psi) \otimes \left( \Omega^{H_f}_{ij} [\nabla^{H_f}_{e_k}, \Phi] \chi \right)
\]
\[
+ \frac{1}{2} \sum_{i \neq j} (\gamma_5 e_k \cdot e_i \cdot e_j \cdot \psi) \otimes \left( [\nabla^{H_f}_{e_k}, \Phi] \Omega^{H_f}_{ij} \chi \right)
\]
\[
- \frac{1}{2} \sum_{i \neq j} (e_i \cdot e_j \cdot \psi) \otimes \left( (\Phi^2) [\nabla^{H_f}_{e_i}, \Phi] + [\nabla^{H_f}_{e_i}, \Phi] (\Phi^2) \right) \chi.
\]

Only the first five summands on the right-hand side contribute to the trace of \( (E_\Phi)^2 \). In four dimensions \( E \) consists of two summands proportional to the identity and one summand proportional to \( \gamma_5 \). Therefore the endomorphism defined by the sixth and seventh summand of (11)
\[
\psi \otimes \chi \mapsto \sum_{i \neq j} (E e_i \cdot e_j \cdot \psi) \otimes (\Omega^{H_f}_{ij} \chi)
\]
\[
+ \sum_{i=1}^n (E \gamma_5 e_i \cdot \psi + \gamma_5 e_i \cdot E \psi) \otimes [\nabla^{H_f}_{e_i}, \Phi] \chi
\]
is traceless due to Clifford relations and cyclicity of the trace. The trace of the endomorphism given by the remaining summands of (11) vanishes due to Clifford relations without employing that the dimension of the manifold is four (in other dimensions \( \gamma_5 \) is then the volume element). Thus we find for the trace
\[
\text{Tr}(E^2_\Phi) = \text{rank} \ H_f \cdot \text{tr}(E^2) + \text{tr}_f(\Omega^{H_f}_{ij} \Omega^{H_f}_{ij}) + 4 \cdot \text{tr}_f([\nabla^{H_f}, \Phi]^2) + 4 \cdot \text{tr}_f(\Phi^4) - 2 \cdot \text{tr}(E) \cdot \text{tr}_f(\Phi^2).
\]

The last ingredient we need to calculate \( a_4(D^2_\Phi) \) is the trace the squared curvature tensor \( \Omega^{\nabla}_{ij} \), associated to the connection in the Bochner formula (17). We note that \( \text{Tr}(\Omega_{ij} \otimes \Omega^{H_f}_{ij}) = 0 \) and therefore we find
\[
\sum_{i,j} \text{Tr} \left( \Omega^{\nabla}_{ij} \Omega^{\nabla}_{ij} \right) = \sum_{i,j} \text{Tr} \left( (\Omega_{ij} \Omega_{ij}) \otimes 1_{H_f} + 14 \otimes (\Omega^{H_f}_{ij} \Omega^{H_f}_{ij}) + 2 \Omega_{ij} \otimes \Omega^{H_f}_{ij} \right)
\]
\[
= \text{rank} \ H_f \cdot \sum_{i,j} \text{tr} (\Omega_{ij} \Omega_{ij}) + 4 \cdot \sum_{i,j} \text{tr}_f \left( \Omega^{H_f}_{ij} \Omega^{H_f}_{ij} \right).
\]

We choose the finite space \( H_f \) according to the construction of the noncommutative Standard Model [2 13 14], i.e. \( \text{rank} \ H_f = 96 \) and \( \nabla^{H_f} \) is the appropriate covariant derivative associated to the Standard Model gauge group \( U(1)_Y \times SU(2)_L \times SU(3)_c \).

Inserting the above results into the spectral action (5) we obtain for the bosonic Lagrangian of the Standard Model coupled to gravity and torsion:
\[
I_{\text{bos.}} = \text{Tr} \left( \frac{D^2_\Phi}{\Lambda^2} \right) = \Lambda^4 F_4 a_0(D^2_\Phi) + \Lambda^2 F_2 a_2(D^2_\Phi) + \Lambda^0 F_0 a_4(D^2_\Phi) + O(\Lambda^{-2})
\]
\[
I_{\text{bos.}} = \frac{24\Lambda^4 F_4}{\pi^2} \int_M \text{dvol} \\
+ \frac{\Lambda^2 F_2}{96\pi^2} \int_M \{ 27 T_0^2 - 2 R - a|\varphi|^2 - \frac{3}{2}c \} \text{dvol} \\
+ \frac{F_0}{576\pi^2} \int_M \left\{ \frac{1}{6} \left[ R^2 - \frac{4}{15} \|\text{Ric}\|^2 - \frac{7}{30} \|\text{Riem}\|^2 + \frac{243}{8} (T_0^2)^2 - \frac{27}{4} \|c(T)\|^2 \right] \\
+ \frac{27}{4} \|dt\|^2 - \frac{9}{2} \|d^*T\|^2 - \frac{9}{2} \|\text{sym}_0(\nabla T)\|^2 - \frac{9}{2} R(T) \\
+ g_3^2 \|G\|^2 + g_2^2 \|F\|^2 + \frac{5}{3} g_1^2 \|B\|^2 \\
+a|D_\nu \varphi|^2 + b|\varphi|^4 + 2c|\varphi|^2 + 1 \frac{1}{2} d + \frac{1}{6} R \left( a|\varphi|^2 + \frac{1}{2} c \right) \right\} \text{dvol} + O(\Lambda^{-2})
\]

To cast this action into a more familiar form we use the standard formulas to express \( \text{tr}_f(\Omega^H_{i,j}) \) in terms of the norms of the gauge field strengths \( \|G\|^2 = \sum_{\mu,\nu,i} G^i_{\mu,\nu} G^i_{\mu,\nu}, \|F\|^2 = \sum_{\mu,\nu,\alpha} F^i_{\mu,\nu,\alpha} F^{i,\mu,\nu,\alpha}, \|B\|^2 = \sum_{\mu,\nu} B_{\mu,\nu} B^{\mu,\nu} \). Here \( G^i_{\mu,\nu} \) is the curvature of the \( U(3)_c \)-connection with coupling \( g_3 \), \( F^i_{\mu,\nu} \) is the curvature of the \( U(1)_Y \)-connection with coupling \( g_1 \), and \( B_{\mu,\nu} \) is the curvature of the \( \text{SU}(2)_W \)-connection with coupling \( g_2 \).

We also calculate the traces of of the Higgs endomorphisms \( \Phi^2 \) and \( \Phi^4 \) explicitly in terms of the Higgs doublet \( \varphi \) and obtain \( \text{tr}_f(\Phi^2) = 4a|\varphi|^2 + 2c \) and \( \text{tr}_f(\Phi^4) = 4b|\varphi|^4 + 8e|\varphi|^2 + 2d \). The coefficients \( a, b, c, d, e \) are traces of the \( 3 \times 3 \) Yukawa matrices for the quarks \( (k_u \text{ and } k_d) \), the leptons \( (k_e \text{ and } k_\nu) \) and the Majorana mass matrix for the right-handed neutrinos \( (k_{\nu_R}) \) given by

\[
\begin{align*}
\lambda &= \text{tr}_3(3|k_u|^2 + 3|k_d|^2 + |k_e|^2 + |k_\nu|^2), \\
b &= \text{tr}_3(3|k_u|^4 + 3|k_d|^4 + |k_e|^4 + |k_\nu|^4), \\
c &= \text{tr}_3(|k_{\nu_R}|^2), \\
d &= \text{tr}_3(|k_{\nu_R}|^4), \\
e &= \text{tr}_3(|k_{\nu_R}|^2)|k_{\nu_R}|^2.
\end{align*}
\]

We conclude that the spectral action principle predicts the following form of the bosonic Lagrangian for the Standard model in the presence of skew-symmetric torsion:
Here we were able to use the standard results from the torsion free case, see [7, p.22] or [3, 14]. As in the pure gravity+torsion case the torsion becomes dynamical and couples only with the trace free part of the Riemann curvature tensor.

In presence of the Standard Model fields we obtained essentially one new term (apart from the usual suspects) coupling the torsion to the Higgs field

\[ I_{\text{new}} = -\frac{9aF_0}{8\pi^2} \int_M T_0^2 |\varphi|^2 \, d\text{vol}. \]  

(14)

This is another amazing feature of the spectral action principle: it supports the interpretation of the Higgs field as the gravitational field of the internal space in the noncommutative product geometry [1].

The full Standard Model action is given by

\[ I_{\text{SM}} = \text{Tr} F \left( \frac{D^2 \Phi}{\Lambda^2} \right) + \frac{1}{2} \langle J\Psi, D\Phi \Psi \rangle \text{ with } \Psi \in \mathcal{H} \]  

(15)

where the fermionic action \( \frac{1}{2} \langle J\Psi, D\Phi \Psi \rangle \) contains a coupling between torsion and the fermions, and \( J \) is the real structure of the spectral triple. This action takes care of the fermion doubling problem, compare [7, (5.9)].

Conclusions

We have calculated the spectral action for Dirac operators arising from geometries with skew-symmetric torsion and their twisted version originating in the noncommutative approach to the standard model. In both cases we find that torsion couples to the trace free part of the Riemann curvature tensor and in the latter case to the Higgs boson of the standard model. Furthermore the torsion becomes dynamical due to derivative terms in the action.

Now one certainly has to wonder about possible experimental signatures of these new phenomena, both on local scales (Earth and the solar system) and cosmological scales.

We may assume for the moment that the Schwarzschild metric is a good approximation to the gravitational field of the Earth (even for the “new” spectral action with torsion). The Weyl curvature of the Schwarzschild metric is non-zero and hence torsion for the corresponding critical point of the action seems probable. Then we would expect effects on freely falling particles or atoms with different spins. This might lead to measurable effects in atom interferometry experiments.

The cosmological consequences are much more speculative. It has been noted [15, 16] that torsion induces four-fermion interactions which in turn may provide a possible solution to the problem of the enormously large cosmological constant \( \Lambda_c \sim \Lambda \sim 10^{17} \text{GeV} \) predicted by the spectral action (without torsion) [17]. In this framework also a natural mechanism for inflation appears naturally.

To obtain more rigorous results it will be necessary to investigate the Euler-Lagrange equations of the spectral action with torsion. It would be interesting to find exact solutions with non-vanishing torsion and compare them with the known solutions of Einstein’s equations.

Appendix: Bochner formula for twisted Dirac operators in presence of torsion

We consider a closed Riemannian spin manifold \( M \) of dimension \( n \) and a connection \( \nabla \) on the tangent bundle which is compatible with the metric and has totally anti-symmetric torsion \( T \). By \( \nabla \) we also denote the induced connection on the spinor bundle \( S \) of \( M \) which can be expressed as in [2]. Given a vector bundle \( \mathcal{H} \) over \( M \) with connection \( \nabla^\mathcal{H} \) we consider the tensor connection \( \nabla = \nabla \otimes \text{id}_\mathcal{H} + \text{id}_S \otimes \nabla^\mathcal{H} \) and the associated twisted Dirac operator \( D^\mathcal{V} \) acting on sections of \( S \otimes \mathcal{H} \). We define another tensor connection

\[ \nabla = \nabla \otimes \text{id}_\mathcal{H} + \text{id}_S \otimes \nabla^\mathcal{H}, \]  

(16)
where \( \nabla \) is the spin connection from (4), and we claim the Bochner formula

\[
(D^{\psi})^2 = \Delta^{\nabla} - \tilde{E}
\]

(17)

where \( \Delta^{\nabla} \) denotes the horizontal Laplacian associated to \( \nabla \) and \( \tilde{E} \) denotes an endomorphism field of \( S \otimes \mathcal{H} \), which still has to be determined.

To that end we fix an arbitrary point \( p \in M \), and we choose a local orthonormal basis of vector fields \( e_1, \ldots, e_n \), with \( \nabla e_i = 0 \) in \( p \) (for any \( i \)). From (1) we get that \( \nabla^L e_i = \nabla e_i \) in \( p \) (for any \( i \)) and the Lie bracket \( [e_i, e_j] = \nabla^L e_j - \nabla^L e_i = -T(e_i, e_j) \) in \( p \) (for any \( i, j \)). For any section \( \psi \) of \( S \) and any section \( \chi \) of \( \mathcal{H} \) we get in \( p \):

\[
D^{\psi} (\psi \otimes \chi) = \sum_{j=1}^{n} \left( (e_i \cdot \nabla e_i \psi) \otimes (e_i \cdot \psi) \otimes \nabla^H e_i \chi \right)
\]

(18)

\[
\left( D^{\psi} \right)^2 (\psi \otimes \chi) = (D^2 \psi) \otimes \chi + \sum_{i,j=1}^{n} (e_j \cdot e_i \cdot \psi) \otimes \nabla^H e_j \nabla^H e_i \chi
\]

+ \sum_{i,j=1}^{n} \left( (e_j \cdot e_i \cdot \psi) \otimes \nabla^H e_j \nabla^H e_i \chi + (e_j \cdot e_i \cdot \psi) \otimes \nabla^H e_j \nabla^H e_i \chi \right)

\[
= (D^2 \psi) \otimes \chi + \sum_{i,j=1}^{n} (e_j \cdot e_i \cdot \psi) \otimes \nabla^H e_j \nabla^H e_i \chi - 2 \cdot \sum_{i=1}^{n} \nabla e_i \psi \otimes \nabla^H e_i \chi,
\]

where the last equation holds due to Clifford relations.

As before, let \( \Delta \) denote the Laplacian associated to the spin connection \( \nabla \). In \( p \) we obtain for the Laplacian associated to \( \nabla \):

\[
\Delta^{\nabla} (\psi \otimes \chi) = (\Delta \psi) \otimes \chi - \sum_{i=1}^{n} \psi \otimes \nabla^H e_i \nabla^H e_i \chi - 2 \cdot \sum_{i=1}^{n} \nabla e_i \psi \otimes \nabla^H e_i \chi
\]

\[
= (\Delta \psi) \otimes \chi + \sum_{i=1}^{n} e_i \cdot e_i \cdot \psi \otimes \nabla^H e_i \nabla^H e_i \chi - 2 \cdot \sum_{i=1}^{n} \nabla e_i \psi \otimes \nabla^H e_i \chi - \sum_{i=1}^{n} ((e_i \cdot T) \cdot \psi) \otimes \nabla^H e_i \chi,
\]

where we have used (2) and (4). In \( p \) we observe

\[
\sum_{i,j=1}^{n} (e_i \cdot e_j \cdot \psi) \otimes \nabla^H e_i e_j \chi = - \sum_{i,j=1}^{n} (e_i \cdot e_j \cdot \psi) \otimes \nabla^H T(e_i, e_j, \cdot) \chi = - \sum_{i,j,k=1}^{n} (e_i \cdot e_j \cdot \psi) \otimes \nabla^H T(e_i, e_j, e_k) e_k \chi
\]

\[
= - \sum_{i,j,k=1}^{n} (T(e_i, e_j, e_k) e_i \cdot e_j \cdot \psi) \otimes \nabla^H e_k \chi = - 2 \cdot \sum_{k=1}^{n} ((e_k \cdot T) \cdot \psi) \otimes \nabla^H e_k \chi.
\]

Putting all this together we get in \( p \) that

\[
\left( D^{\psi} \right)^2 (\psi \otimes \chi) - \Delta^{\nabla} (\psi \otimes \chi) = (D^2 \psi - \Delta \psi) \otimes \chi
\]

+ \sum_{i \neq j}^{n} (e_i \cdot e_j \cdot \psi) \otimes \nabla^H e_i \nabla^H e_j \chi - \frac{1}{2} \cdot \sum_{i \neq j}^{n} (e_i \cdot e_j \cdot \psi) \otimes \nabla^H \Omega^H e_i e_j \chi
\]

\[
= (D^2 \psi - \Delta \psi) \otimes \chi - \frac{1}{2} \cdot \sum_{i \neq j}^{n} (e_i \cdot e_j \cdot \psi) \otimes (\Omega^H e_i e_j \chi),
\]

where \( \Omega^H e_i e_j = \nabla^H e_i \nabla^H e_j - \nabla^H e_j \nabla^H e_i - \nabla^H [e_i, e_j] \) is the curvature endomorphism of the twist bundle \( \mathcal{H} \). Taking (3) into account we can identify the endomorphism field \( \tilde{E} \) in the following Bochner formula (17) as

\[
\tilde{E} (\psi \otimes \chi) = \left( -\frac{3}{4} dT - \frac{1}{4} R + \frac{9}{8} T^2 \right) \psi \otimes \chi + \frac{1}{2} \cdot \sum_{i \neq j}^{n} (e_i \cdot e_j \cdot \psi) \otimes (\Omega^H e_i e_j \chi)
\]

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\[\eta(E \psi) \otimes \chi + \frac{1}{2} \sum_{i \neq j} (e_i \cdot e_j \cdot \psi) \otimes (\Omega_{ij}^H \chi).\] (19)

Finally we remark that the curvature of the connection \(\nabla = \tilde{\nabla} \otimes \text{id}_H + \text{id}_S \otimes \nabla^H\) is given by

\[\Omega_{ij}^\nabla (\psi \otimes \chi) = (\Omega_{ij}^H \psi) \otimes \chi + \psi \otimes (\Omega_{ij}^H \chi),\] (20)

where \(\Omega_{ij}\) is the curvature of the spin connection \(\tilde{\nabla}\).

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