Moment and tail estimation for U-statistics

with positive kernels

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Abstract

We deduce the non-asymptotical (bilateral) estimates for moment inequalities for multiple sums of non-negative (more precisely, non-negative) independent random variables, on the other words, the well known U or V-statistics. Our consideration based on the correspondent estimates for the one-dimensional case by means of the so-called degenerate approximation.

We apply also the theory of Bell functions as well as the properties of the Poisson distribution and the theory of the so-called Grand Lebesgue Spaces (GLS).

Key words and phrases: Arbitrary and independent random variables (r.v.), Bell’s numbers and function, triangle inequality, degenerate function and approximation, Grand Lebesgue Spaces (GLS), Rosenthal estimate, Poisson distribution, asymptotic estimate and expansion, Stirling’s formula, bilateral non-asymptotic estimates, moment generating function (MGF), optimization, slowly varying functions, upper and lower evaluate.

Mathematics Subject Classification 2000. Primary 42Bxx, 4202, 68-01, 62-G05, 90-B99, 68Q01, 68R01; Secondary 28A78, 42B08, 68Q15.

1 Definitions. Notations. Previous results. Statement of problem.

Let $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{C}, \nu)$ be two non-trivial probability spaces: $\mu(X) = \nu(Y) = 1$. We will denote by $|g|_p = |g|L(p)$ the ordinary Lebesgue-Riesz $L(p)$ norm of arbitrary measurable numerical valued function $g : X \to R$:

$$|g|_p = |g|L(p) = |g|L_p(X, \mu) := \left[ \int_X |g(x)|^p \mu(dx) \right]^{1/p}, \ p \in [1, \infty)$$

analogously for the (also measurable) function $h : Y \to R$.
\[ |h|_p = |h|L_p = |h|L_p(Y, \nu) := \left[ \int_Y |h(y)|^p \nu(dy) \right]^{1/p}; \]

and for arbitrary integrable function of two variables \( f : X \otimes Y \to R \)

\[ |f|_p = |f|L_p = |f|L_p(X, Y) := \left[ \int_X \int_Y |f(x, y)|^p \mu(dx) \nu(dy) \right]^{1/p}, \ p \in [1, \infty). \]

Let \( Z_+ = \{1, 2, 3, \ldots\} \) and denote \( Z_+^2 = Z_+ \otimes Z_+, \ Z_+^d = \otimes_{k=1}^d Z_+ \). Let also \( \{\xi(i)\} \) and \( \{\eta(j)\}, \ i, j = 1, 2, \ldots, \xi := \xi(1), \eta := \eta(1) \) be common independent random variables defined on certain probability space \((\Omega, \mathcal{M}, P)\) with distributions correspondingly \( \mu, \nu \):

\[
P(\xi(i) \in A) = \mu(A), \ A \in \mathcal{B};
\]

\[
P(\eta(j) \in F) = \nu(F), \ F \in \mathcal{C},
\]

so that

\[
E|g(\xi)|^p = |g|^p_p = \int_X |g(x)|^p \mu(dx), \ E|h(\eta)|^p_p = |h|^p_p = \int_Y |h(y)|^p \nu(dy),
\]

and

\[
E|f(\xi, \eta)|^p = |f|^p_p = \int \int_{X \otimes Y} |f(x, y)|^p \mu(dx) \nu(dy).
\]

Let also \( L \) be arbitrary non-empty finite subset of the set \( Z_+^2 \); denote by \( |L| \) a numbers of its elements (cardinal number): \( |L| := \text{card}(L) \). It is reasonable to suppose in what follows \( |L| \geq 1 \).

Define for any integrable function \( f : X \otimes Y \to R \), i.e. for which

\[
|f|_1 = E|f(\xi, \eta)| = \int_X \int_Y |f(x, y)| \mu(dx) \nu(dy) < \infty,
\]

the following normalized sum

\[
S_L[f] \overset{\text{def}}{=} |L|^{-1} \sum_{(k(1), k(2)) \in L} f(\xi(k(1), \eta(k(2))), \quad (1.1)
\]

which is a slight generalization of the classical \( U \) and \( V \) statistics, see the classical monograph of Korolyuk V.S and Borovskikh Yu.V. [18]. Offered here report is the direct generalization of a recent article [30], but we apply here other methods.

The reasonableness of this norming function \( |L|^{-1} \) implies that in general, i.e. in the non-degenerate case \( \text{Var}(S_L) \approx 1, \ |L| \geq 1 \). This propositions holds true still in the multidimensional case.

Our notations and some previous results are borrowed from the works of S.Klesov [14] - [17], see also [18].
We will suppose in what follows that the function \( f = f(x, y), x \in X, y \in Y \), as well as both the functions \( g, h \) are non-negative (a.e.).

The so-called centered case, i.e. when

\[
\mathbf{E}f(\xi, \eta) = \int_X \int_Y f(x, y) \mu(dx) \nu(dy) = 0,
\]

was investigated in many works, see e.g. the classical monograph of Korolyuk V.S and Borovskikh Yu.V. [18]; see also [14]-[17], [26]-[27], [28] etc. The one-dimensional case was studied in the recent preprint [30].

Our claim in this report is to derive the moment and exponential bounds for tail of distribution for the normalized sums of multi-indexed independent non-negative random variables from (1.1).

Offered here results are generalizations of many ones obtained by S.Klesov in an articles [14]-[17], see also [18], [26]-[27], where was obtained in particular the CLT for the sums of centered multiple variables.

The multidimensional case, i.e. when \( \vec{k} \in \mathbb{Z}_+^d \), will be considered further.

The paper is organized as follows. In the second section we describe and investigate the notion of the degenerate functions and approximation. In the next sections we obtain one of the main results: the moment estimates for the multi-index sums with non-negative kernel in the two-dimensional case.

The so-called non-rectangular case is considered in the 4th section. The fifth section is devoted to the multivariate case. In the following section we obtain the exponential bounds for distribution of positive multiple sums.

We show in the seventh section the upper bounds for these statistics. The last section contains as ordinary the concluding remarks.

We reproduce here for readers convenience some results concerning the one-dimensional case, see an article [30].

Denote for any r.v. - s. \( \eta_j, j = 1, 2, \ldots \) its \( k^{th} \) absolute moment by \( m_j(k) : \)

\[
m_j(k) := \mathbf{E}|\eta_j|^k, \ k \geq 1;
\]

so that

\[
|\eta_j|_k = [ \ m_j(k) ]^{1/k}.
\]

We deduce applying the triangle inequality for the \( L(p, \Omega) \) norm a very simple estimation

\[
|\sum_{j=1}^n \eta_j|_p \leq \sum_{j=1}^n [ \ m_j(p) ]^{1/p},
\]

and we do not suppose wherein that the r.v. \( \eta_j \) are non-negative and independent.
In order to describe a more fine estimations, we need to introduce some notations.

Let us define the so-called Bell’s function of two variables as follows.

\[ B(p, \beta) \overset{\text{def}}{=} e^{-\beta} \sum_{k=0}^{\infty} \frac{k^p \beta^k}{k!}, \quad p \geq 2, \beta > 0, \]

and put \( B(p) = B(p,1) \), so that

\[ B(p) \overset{\text{def}}{=} e^{-1} \sum_{k=0}^{\infty} \frac{k^p}{k!}, \quad p \geq 0. \]

The sequence of the numbers \( B(0) = 1, B(1), B(2), B(3), B(4), \ldots \) are named as Bell numbers; they appears in combinatorics, theory of probability etc., see [30], [25].

Let the random variable (r.v.) \( \tau = \tau[\beta] \), defined on certain probability space \( (\Omega, F, P) \) with expectation \( E \), has a Poisson distribution with parameter \( \beta, \beta > 0 \); write \( \text{Law}(\tau) = \text{Law}_{\tau[\beta]} = \text{Poisson}(\beta) : \)

\[ P(\tau = k) = e^{-\beta} \frac{\beta^k}{k!}, \quad k = 0, 1, 2, \ldots, \]

It is worth to note that \( B(p, \beta) = E(\tau[\beta])^p, \quad p \geq 0. \) (1.2)

In detail, let \( \eta_j, j = 1, 2, \ldots \) be a sequence of non-negative independent random (r.v.); the case of centered or moreover symmetrical distributed r.v. was considered in many works, see e.g. [9], [14]-[17], [26]-[27], [30] and so one.

The following inequality holds true

\[ E\left(\sum_{j=1}^{n} \eta_j\right)^p \leq B(p) \max \left\{ \sum_{j=1}^{n} E\eta_j^p, \left( \sum_{j=1}^{n} E\eta_j \right)^p \right\}, \quad p \geq 2, \] (1.3)

where the ”constant” \( B(p) \) in (1.2) is the best possible, see [30].

One of the interest applications of these estimates in statistics, more precisely, in the theory of \( U \) statistics may be found in the articles [9], [18].

Another application. Let \( n = 1, 2, 3, \ldots; \quad a, b = \text{const} > 0; \quad p \geq 2, \mu = \mu(a,b;p) := a^{p/(p-1)} b^{1/(1-p)}. \) Define the following class of the sequences of an independent non-negative random variables

\[ Z(a, b) \overset{\text{def}}{=} \left\{ \eta_j, \eta_j \geq 0, \sum_{j=1}^{n} E\eta_j = a; \sum_{j=1}^{n} E\eta_j^p = b \right\}. \] (1.4)

G.Schechtman in proved that
\[
\sup_{n=1,2,\ldots} \mathbb{E} \left( \sum_{j=1}^{n} \eta_j \right)^p = \left( \frac{b}{a} \right)^{p/(p-1)} B(p, \mu(a, b; p)). \tag{1.5}
\]

The introduced above Bell’s function allows in turn a simple estimation, see [30], which may be used by practical using. Indeed, let us introduce the following auxiliary function

\[
g_\beta(p) \overset{def}{=} \frac{p}{e} \inf_{\lambda > 0} \left[ \lambda^{-1} \exp \left( \beta \left( e^\lambda - 1 \right) \right) \right]^{1/p}, \beta, p > 0. \tag{1.6}
\]

It is proved in particular in [30] that

\[
B^{1/p}(p, \beta) \leq g_\beta(p), \quad p, \beta > 0. \tag{1.7}
\]

Let us introduce also the following function

\[
h_0(p, \beta) \overset{def}{=} \sup_{k=1,2,\ldots} e^{-\beta} \left\{ \frac{k^p \beta^k}{k!} \right\}; \tag{1.8}
\]

therefore

\[
B(p, \beta) \geq h_0(p, \beta), \quad p, \beta > 0. \tag{1.9}
\]

The last estimate may be simplified in turn as follows. We will apply the following version of the famous Stirling’s formula

\[
k! \leq \zeta(k), \quad k = 1, 2, \ldots,
\]

where

\[
\zeta(k) \overset{def}{=} \sqrt{2\pi k} \left( \frac{k}{e} \right)^k e^{1/(12k)}, \quad k = 1, 2, \ldots \tag{1.10}
\]

Define a new function

\[
h(p, \beta) \overset{def}{=} \sup_{x \in (1, \infty)} \left\{ e^{1/(6p^2)} \cdot \left[ \frac{e^{x-\beta} x^{p-x-1/2}}{\sqrt{2 \pi x^x}} \right]^{1/p} \right\}. \tag{1.11}
\]

We obtained in [30] really the following lower simple estimate for the Bell’s function

\[
B^{1/p}(p, \beta) \geq h_0(p, \beta), \quad B^{1/p}(p, \beta) \geq h(p, \beta), \quad p, \beta > 0. \tag{1.12}
\]

These estimates may be in turn simplified as follows. Assume that \( p \geq 2\beta, \beta > 0, \ p \geq 1; \) then
\[ B^{1/p}(p, \beta) \leq \frac{p/e}{\ln(p/\beta) - \ln \ln(p/\beta)} \cdot \exp \left\{ \frac{1}{\ln(p/\beta)} - \frac{1}{p/\beta} \right\}. \quad (1.13) \]

For example,
\[ B^{1/p}(p) \leq \frac{p/e}{\ln p - \ln \ln p} \cdot \exp \left\{ 1/\ln p - 1/p \right\}, \ p \geq 2. \quad (1.14) \]

The estimate (1.13) may be simplified as follows
\[ B^{1/p}(p) \leq \frac{p}{e \ln(p/\beta)} \cdot \left[ 1 + C_1(\beta) \cdot \frac{\ln \ln(p/\beta)}{\ln(p/\beta)} \right], \quad (1.15) \]
where \( C_1(\beta) = \text{const} \in (0, \infty) \), and we recall that \( p \geq 1, \ p \geq 2\beta. \)

For example,
\[ B^{1/p}(p) \leq \frac{p/e}{\ln p - \ln \ln p} \cdot \exp \left\{ 1/\ln p - 1/p \right\}, \ p \geq 2. \quad (1.16) \]

The lower estimate for Bell’s function has a form
\[ B^{1/p}(p, \beta) \geq \]
\[ \beta^{1/\ln(pe/\beta)} \cdot \frac{p}{\ln(pe/\beta)} \cdot \left\{ \exp \left[ \frac{\ln p - \ln(pe) - \beta}{\ln(pe/\beta)} \right] \right\}^{-1} \]
\[ p, \beta > 0, \ p/\beta \geq 2. \]

It may be simplified as follows
\[ B^{1/p}(p) \geq \frac{p}{e \ln(p/\beta)} \cdot \left[ 1 - C_2(\beta) \cdot \frac{\ln \ln(p/\beta)}{\ln(p/\beta)} \right], \quad (1.17) \]
where \( C_2(\beta) = \text{const} \in (0, \infty) \), and we recall that \( p \geq 2\beta. \)

We suppose hereafter that both the variables \( p \) and \( \beta \) are independent but such that
\[ p \geq 1, \ \beta > 0, \ p/\beta \leq 2. \quad (1.18) \]

It is known [30] that in this case there exist two absolute positive constructive finite constants \( C_3, C_4, C_3 \leq C_4 \) such that
\[ C_3 \beta \leq B^{1/p}(p, \beta) \leq C_4 \beta. \quad (1.19) \]
To summarize.

Define the following infinite dimensional random vector

\[ \eta = \vec{\eta} = \{ \eta_1, \eta_2, \eta_3, \ldots \} \]  \hspace{1cm} \text{(1.20)}

Recall that here the r.v. \{\eta_j\} are non-negative and independent. One can suppose also that \( m_j(p) = |\eta_j|_p < \infty \) for some value \( p \in [2, \infty) \).

Put also

\[ Z_{p,n} = Z_{p,n}[\eta] := n^{-1} \sum_{j=1}^{n} [m_j(p)]^{1/p}, \]
\[ V_{p,n} = V_{p,n}[\eta] := n^{-1} \cdot B^{1/p}(p) \cdot \max \left\{ \left( \sum_{j=1}^{n} m_j(p) \right)^{1/p}, \left( \sum_{j=1}^{n} m_j(1) \right) \right\}, \]

and ultimately

\[ \Theta_{p,n} = \Theta_{p,n}[\eta] := \min \{ Z_{p,n}[\eta], V_{p,n}[\eta] \}, \]  \hspace{1cm} \text{(1.21)}

\[ \Theta_p = \Theta_p[\eta] := \sup_{n=1,2,3,...} \Theta_{p,n}, \] \hspace{1cm} \text{(1.22)}

with described above correspondent upper estimations for the function \( B^{1/p}(p) \).

**Proposition 1.1.** We deduce under formulated conditions

\[ \left| n^{-1} \sum_{j=1}^{n} \eta_j \right|_p \leq \Theta_{p,n}[\eta], \] \hspace{1cm} \text{(1.23)}

As a slight consequence

\[ \sup_{n=1,2,3,...} \left| n^{-1} \sum_{j=1}^{n} \eta_j \right|_p \leq \Theta_p[\eta]. \] \hspace{1cm} \text{(1.23a)}

**Remark 1.1.** Of course, one can use in the practice in the estimate (1.21) and in the next relations instead the variable \( B^{1/p}(p) \) its upper bounds from the inequality (1.16).

**2 Degenerate functions and approximation.**

**Definition 2.1,** see [26], [27], [30]. The measurable function \( f : X \otimes Y \rightarrow R \) is said to be *degenerate*, if it has a form
\[ f(x, y) = \sum_{k(1)=1}^{M} \sum_{k(2)=1}^{M} \lambda_{k(1), k(2)} g_{k(1)}(x) h_{k(2)}(y), \quad (2.1) \]

where \( \lambda_{i,j} = \text{const} \in \mathbb{R}, \) \( M = \text{const} = 1, 2, \ldots, \infty. \)

One can distinguish two cases in the relation (2.1): ordinary, or equally finite degenerate function, if in (2.1) \( M < \infty, \) and infinite degenerate function otherwise.

The degenerate functions (and kernels) of the form (2.1) are used, e.g., in the approximation theory, in the theory of random processes and fields, in the theory of integral equations, in the game theory etc.

A particular application of this notion may be found in the authors articles [26], [27], [30].

Denotation: \( M = M[f] \overset{\text{def}}{=} \deg(f); \) of course, as a capacity of the value \( M \) one can understood its constant minimal value.

Two examples. The equality (2.1) holds true if the function \( f(\cdot, \cdot) \) is trigonometrical or algebraical polynomial.

More complicated example: let \( X \) be compact metrisable space equipped with the non-trivial probability Borelian measure \( \mu. \) This imply that an arbitrary non-empty open set has a positive measure.

Let also \( f(x, y), x, y \in X \) be continuous numerical valued non-negative definite function. One can write the famous Karunen-Loev’s decomposition

\[ f(x, y) = \sum_{k=1}^{M} \lambda_k \phi_k(x) \phi_k(y), \quad (2.2) \]

where \( \lambda_k, \phi_k(x) \) are correspondingly eigen values and eigen orthonormal function for the function (kernel) \( f(\cdot, \cdot); \)

\[ \lambda_k \phi_k(x) = \int_X f(x, y) \phi_k(y) \mu(dy). \]

We assume without loss of generality

\[ \lambda_1 \geq \lambda_2 \geq \ldots \lambda_k \geq \ldots \geq 0. \]

It will be presumed in this report, i.e. when the function \( f = f(x, y) \) is non negative, in addition to the expression (2.1) that all the functions \( \{g_i\}, \{h_j\} \) are also non-negative:

\[ \forall x \in X \; g_i(x) \geq 0, \; \forall y \in Y \; h_j(y) \geq 0. \quad (2.3) \]

Further, let \( B_1, B_2, B_3, \ldots, B_M \) be some rearrangement invariant (r.i.) spaces builded correspondingly over the spaces \( X, Y; \; Z, W, \ldots, \) for instance, \( B_1 = L_p(X), \) \( B_2 = L_q(Y), 1 \leq p, q \leq \infty. \) If \( f(\cdot) \in B_1 \otimes B_2, \) we suppose also in (2.1) \( g_i \in B_1, \; h_j \in B_2; \) and if in addition in (2.1) \( M = \infty, \) we suppose that the series in (2.1) converges in the norm \( B_1 \otimes B_2 \)
\[ \lim_{m \to \infty} \| f(\cdot) - \sum_{i=1}^{m} \sum_{j=1}^{m} \lambda_{i,j} g_i(\cdot) h_j(\cdot) \|_{B_1 \otimes B_2} = 0. \]  

(2.3b)

The condition (2.3b) is satisfied if for example \( \|g_i\|_{B_1} = \|h_j\|_{B_2} = 1 \) and

\[ \sum_{i,j=1}^{M} |\lambda_{i,j}| < \infty, \]  

(2.4)

or more generally when

\[ \sum_{i,j=1}^{M} |\lambda_{i,j}| \cdot \|g_i\|_{B_1} \cdot \|h_j\|_{B_2} < \infty. \]  

(2.4a)

The function of the form (2.1) with \( M = M[f] = \deg(f) < \infty \) is named \textit{degenerate}, notation \( f \in D[M] \); we put also \( D := \bigcup_{M<\infty} D[M] \). Obviously,

\[ B_1 \otimes B_2 = D[\infty]. \]

Define also for each \textit{non-negative} such a function \( f \in D \) the following quasi-norm, also non-negative:

\[ \|f\|_{D^+(B_1, B_2)} \overset{def}{=} \inf \left\{ \sum_{i,j=1}^{M} \sum_{i,j=2,\ldots,M[f]} |\lambda_{i,j}| \cdot \|g_i\|_{B_1} \cdot \|h_j\|_{B_2} \right\}, \]  

(2.5)

where all the arrays \( \{\lambda_{i,j}\} \), \( \{g_i\} \), \( \{h_j\} \) are taking from the representation 2.1; and in addition,

\[ g_i(x) \geq 0, \ h_j(y) \geq 0; \ x \in X, \ y \in Y. \]

We will write for brevity \( \|f\|_{D_p} := \)

\[ \|f\|_{D^+(L_p(X), L_p(Y))} = \inf \left\{ \sum_{i,j=1,2,\ldots,M[f]} |\lambda_{i,j}| \cdot \|g_i\|_{L_p} \cdot \|h_j\|_{L_p} \right\}, \]  

(2.5a)

where all the arrays \( \{\lambda_{i,j}\} \), \( \{g_i\} \), \( \{h_j\} \) are taking from the representation 2.1, of course with non-negative functions \( g_i, h_j \).

Further, let the function \( f \in B_1 \otimes B_2 \) be given. The error of a degenerate approximation of the non-negative function \( f : X \otimes Y \to R \) by the degenerate ones of the degree \( M \), also with non-negative summands, will be introduced as follows

\[ Q_M^+[f](B_1 \otimes B_2) \overset{def}{=} \inf_{\tilde{f} \in D[M]} \|f - \tilde{f}\|_{B_1 \otimes B_2} = \min_{\tilde{f} \in D^+[M]} \|f - \tilde{f}\|_{B_1 \otimes B_2}. \]  

(2.6)

Obviously, \( \lim Q_M^+[f](B_1 \otimes B_2) = 0, \ M \to \infty. \)

For brevity:
\[ Q_M^+[f]_p \overset{\text{def}}{=} Q_M^+[f](L_p(X) \otimes L_p(Y)). \] (2.6a)

The function \( \tilde{f} \) which realized the minimum in (2.6), obviously, non-negative, not necessary to be unique, will be denoted by \( Z^+_M[f](B_1 \otimes B_2) : \)

\[ Z^+_M[f](B_1 \otimes B_2) := \arg\min_{\tilde{f} \in D^+[M]} \| f - \tilde{f} \| B_1 \otimes B_2, \] (2.7)

so that

\[ Q^+_M[f](B_1 \otimes B_2) = \| f - Z^+_M[f] \| (B_1 \otimes B_2). \] (2.8)

For brevity:

\[ Z^+_M[f]_p := Z^+_M[f](L_p(X) \otimes L_p(Y)). \] (2.9)

Let for instance again \( f(x,y), x,y \in X \) be continuous numerical valued non-negative definite function, non necessary to be non-negative, see (2.3) and (2.3a).

It is easily to calculate

\[ Q_M[f](L_2(X) \otimes L_2(X)) = \sum_{k=M+1}^{\infty} \lambda_k. \]

3 Moment estimates for multi-index sums.

Two-dimensional case.

A trivial estimate.

The following simple estimate based only on the triangle inequality, may be interpreted as trivial:

\[ |S_L|(L_p(X) \otimes L_p(Y)) \leq |f|(L_p(X) \otimes L_p(Y)) = |f|_p, \] (3.1)

even without an assumption of the non-negativity of the function \( f \) and the independents of the r.v \( g_k(\xi(i)), h_l(\eta(j)). \)

Hereafter \( p \geq 2. \)

The two-dimensional degenerate case.

In this subsection the non-negative kernel-function \( f = f(x,y) \) will be presumed to be degenerate with minimal constant possible degree \( M = M[f] = 1, \) on the other words, factorizable:
\[ f(x, y) = g(x) \cdot h(y), \quad x \in X, \ y \in Y, \] (3.2)
of course, with non-negative factors \( g, h. \)

Further, we suppose that the set \( L \) is integer constant rectangle:
\[
L = [1, 2, \ldots, n(1)] \otimes [1, 2, \ldots, n(2)], \ n(1), n(2) \geq 1.
\]

Let us consider the correspondent double sum \( S_L[f] = S_L^{(2)}[f] := \)
\[
|L|^{-1} \sum_{i,j \in L} g(\xi_i) \cdot h(\eta_j),
\] (3.3),
where
\[
n = \vec{n} = (n(1), n(2)) \in L, \ n_1, n_2 \geq 1.
\] (3.3)

We have denoting
\[
\vec{g} = \{g(\xi(i))\}, \ i = 1, 2, \ldots, n(1); \ \vec{h} = \{h(\eta(j))\}, \ j = 1, 2, \ldots, n(2):\]
\[
S_L[f] = \left[n(1)^{-1} \sum_{i=1}^{n(1)} g(\xi(i))\right] \cdot \left[n(2)^{-1} \sum_{j=1}^{n(2)} h(\eta(j))\right].
\]

Since both the factors in the right-hand size of the last inequality are independent, we deduce applying the one-dimensional estimates (1.23), (1.23a):
\[
|S_L|_p \leq \Theta_{p,n(1)}[\vec{g}] \cdot \Theta_{p,n(2)}[\vec{h}],
\] (3.4)
and hence
\[
\sup_{L: |L| \geq 1} |S_L|_p \leq \Theta_p[\vec{g}] \cdot \Theta_p[\vec{h}], \quad (3.4a)
\]

**Estimation for an arbitrary degenerate kernel.**

In this subsection the function \( f(\cdot, \cdot) \) is non-negative and degenerate, as well as all the functions \( g_{k(1)}(\cdot), \ h_{k(2)}(\cdot) : \)
\[
f(x, y) = \sum_{k(1),k(2)=1}^{M} \lambda_{k(1),k(2)} g_{k(1)}(x) \cdot h_{k(2)}(y), \quad (3.5)
\]
where \( 1 \leq M \leq \infty, \)
\[
g_{k(1)}(\cdot) \in L_p(X), \ h_{k(2)}(\cdot) \in L_p(Y),
\]
and as before \( g_{k(1)}(x) \geq 0, \ h_{k(2)}(y) \geq 0. \)
Denote also by $R = R(f)$ the set of all such the functions \{g\} = \vec{g}$ and \{h\} = \vec{h}$ as well as the sequences of coefficients \{\lambda\} = \{\lambda_{k(1), k(2)}\} from the representation (3.5):

$$R[f] := \{\lambda\}, \{g\} = \vec{g}, \{h\} = \vec{h} :$$

$$f(x, y) = \sum_{k(1), k(2)=1}^{M} \lambda_{k(1), k(2)} g_{k(1)}(x) h_{k(2)}(y). \quad (3.5a)$$

We must impose on the series (3.5) in the case when $M = \infty$ the condition of its convergence in the norm of the space $L_p(X) \otimes L_p(Y)$.

Let us investigate the introduced before statistics

$$S_L^{(\lambda)} = S_L^{(\lambda)}[f] := |L|^{-1} \sum_{i,j \in L} f(\xi(i), \eta(j)) =$$

$$|L|^{-1} \sum_{i,j \in L} \left[ \sum_{k(1), k(2)=1}^{M} \lambda_{k(1), k(2)} g_{k(1)}(\xi(i)) h_{k(2)}(\eta(j)) \right] =$$

$$|L|^{-1} \sum_{k(1), k(2)=1}^{M} \lambda_{k(1), k(2)} \left[ \sum_{i,j \in L} g_{k(1)}(\xi(i)) h_{k(2)}(\eta(j)) \right] =$$

$$\sum_{k(1), k(2)=1}^{M} \lambda_{k(1), k(2)} \cdot \left[ (n(1))^{-1} \sum_{i=1}^{n(1)} g_{k(1)}(\xi(i)) \right] \times$$

$$\left[ (n(2))^{-1} \sum_{j=1}^{n(2)} h_{k(2)}(\eta(j)) \right]. \quad (3.6)$$

We have using the triangle inequality and the estimate (3.4)

$$\left| S_L^{(\lambda)}[f] \right| _p \leq \sum_{k(1), k(2)=1}^{M} \left| \lambda_{k(1), k(2)} \cdot \Theta_{p,n(1)}(\vec{g}_{k(1)}) \cdot \Theta_{p,n(2)}(\vec{h}_{k(2)}) \right|. \quad (3.7)$$

This estimate remains true in the case when $M = \infty$, if of course the right-hand side of (3.7) is finite; in the opposite case it is nothing to make.

To summarize, we introduce a new weight norm on the (numerical) array $\vec{\lambda} = \{\lambda_{k(1), k(2)}\}$, more exactly, the sequence of the norms

$$f \in D(M) \Rightarrow |||f|||_{\Theta_p} = |||\vec{\lambda}|||_{\Theta_p} =$$

$$|||\vec{\lambda}|||_{\Theta_p^{(2)}} = |||\vec{\lambda}|||_{\Theta(p; n(1), n(2), \{g\}, \{h\})} \overset{df}{=}$$
\[ \sum_{k(1), k(2) = 1}^{M} | \lambda_{k(1), k(2)} | \cdot \Theta_{p, n(1)} \left[ \tilde{g}_{k(1)} \right] \cdot \Theta_{p, n(2)} \left[ \tilde{h}_{k(2)} \right]. \] 

(3.8)

**Proposition 3.1.** If \( f \in D(M) \), then

\[ \left| S_{L}^{(\lambda)}[f] \right|_p \leq \left| ||f|| \Theta_{p} = \left| ||\lambda|| \Theta_{p} = \right. \right. \]

and as a consequence

\[ \sup_{L: |L| \geq 1} \left| S_{L}^{(\lambda)}[f] \right|_p \leq \sup_{n(1), n(2)} ||f|| \Theta_{p} = \sup_{n(1), n(2)} \left| ||\lambda|| \Theta_{p} \{f\} = \right. \]

\[ \sup_{n(1), n(2)} \left| ||\lambda|| \Theta(p; n(1), n(2), \{g\}, \{h\}) \right|. \] 

(3.9)

**Main result.** Degenerate approximation approach.

**Theorem 3.1.** Let \( f = f(x, y) \) be arbitrary function from the space \( L_{p}(X) \otimes L_{p}(Y), \ p \geq 2 \). Then \( |S_{L}[f]|_p \leq W_{L}[f](p) \), where

\[ W_{L}[f](p) = W_{L}[f; \{\lambda\}, \{g\}, \{h\}](p) \overset{def}{=} \left. \inf_{M \geq 1} \left[ \left| \left| Z_{M}[f] \right| \right| \Theta_{p} + Q_{M}^{+}[f] \right] \right. \]

where in turn the vector triple \( \{\lambda\}, \{g\}, \{h\} \) is taken from the representation (3.5): \( \{\lambda\}, \{g\}, \{h\} \in R[f] \).

As a slight consequence: \( \sup_{L: |L| \geq 1} |S_{L}[f]|_p \leq W[f](p) \), where

\[ W[f](p) \overset{def}{=} \sup_{L: |L| \geq 1} W_{L}[f](p) = \]

\[ \sup_{L: |L| \geq 1} \inf_{M \geq 1} \left[ \left| \left| Z_{M}[f] \right| \right| \Theta_{p} + Q_{M}^{+}[f] \right] \] 

(3.10a)

and the next consequence

\[ \sup_{L: |L| \geq 1} |S_{L}[f]|_p \leq \inf_{\{\lambda\}, \{g\}, \{h\} \in R[f]} W_{L}[f; \{\lambda\}, \{g\}, \{h\}](p). \] 

(3.10b)

**Proof** is very simple, on the basis of previous results of this section. Namely, let \( L \) be an arbitrary non-empty set. Consider a splitting

\[ f = Z_{M}^{+}[f] + (f - Z_{M}^{+}[f]) =: \Sigma_{1} + \Sigma_{2}. \]
We have

\[ |\Sigma_1|_p \leq |||Z_M[f]|||_p. \]

The member \(|\Sigma_2|_p\) may be estimated by virtue of inequality (3.1):

\[ |\Sigma_2|_p \leq |L| \ |f - Z^+_M[f]|_p = Q^+_M[f]_p. \]

It remains to apply the triangle inequality and minimization over \(M\).

**Example 3.1.** We deduce from (3.10) as a particular case

\[ \sup_{L:|L|\geq 1} \left| S^{(\lambda)}_L[f] \right|_p \leq \|f\|D^+_p, \tag{3.11} \]

if of course the right-hand side of (3.11) is finite for some value \(p, p \geq 2\).

Recall that in this section \(d = 2\).

## 4 Non-rectangular case.

We denote by \(\pi^+(L)\) the set of all rectangular’s which are circumscribed about the set \(L: \pi^+(L) = \{L^+\}\), where

\[ L^+ = \{[n(1)^+, n(1)^{++}] \otimes [n(2)^+, n(2)^{++}] \mid L^+ \supset L, \tag{4.1} \]

and

\[ 1 \leq n(1)^+ \leq n(1)^{++} < \infty, \ n(1)^+, n(1)^{++} \in \mathbb{Z}_+, \]

\[ 1 \leq n(2)^+ \leq n(2)^{++} < \infty, \ n(2)^+, n(2)^{++} \in \mathbb{Z}_+. \]

**Proposition 4.1.**

\[ |S_L|_p \leq \inf_{L^+: L \subset L^+} \left\{ \frac{|L^+|}{|L|} \cdot W_{L^+}[f; \{\lambda\}, \{g\}, \{h\}]_p \right\}. \tag{4.2} \]

**Proof** is very simple. We have

\[ |L| \ S_L = \sum_{i,j \in L} f(\xi(i), \eta(j)), \]

therefore

\[ \left| |L| \ S_L \right|_p = \left| \sum_{i,j \in L} f(\xi(i), \eta(j)) \right|_p \leq \]
Moment estimates for multi-index sums.

Multidimensional generalization.

Let now \((X_m, B_m, \mu_m), \ m = 1, 2, \ldots, d, \ d \geq 3\) be a family of probability spaces: \(\mu_m(X_m) = 1\); \(X := \otimes^d_{m=1}X_m\); \(\xi(m)\) be independent random variables having the distribution correspondingly \(\mu_m: P(\xi(m) \in A_m) = \mu_m(A_m), A_m \in B_m; \ 
\xi_i(m), \ i = 1, 2, \ldots, n(m); n(m) = 1, 2, \ldots, n(m) < \infty\) be independent copies of \(\xi(m)\) and also independent on the other vectors \(\xi_i(s), s \neq m\), so that all the random variables \(\{\xi_i(m)\}\) are common independent.

Another notations, conditions, restrictions and definitions. \(L \subset \mathbb{Z}^d_+, |L| = \text{card}(L) > 1; \ j = \overline{j} \in L; \ k = \overline{k} = (k(1), k(2), \ldots, k(d)) \in \mathbb{Z}^d_+; N(\overline{k}) := \max_{j=1,2,\ldots,d} k(j); \) (5.0)

\[ \overline{\xi} := \{\xi(1), \xi(2), \ldots, \xi(n(m))\}; \ \overline{\xi}_i := \{\xi_i(1), \xi_i(2), \ldots, \xi_i(n(m))\}; \ X := \otimes^d_{i=1}X_i, \ \ f: X \rightarrow R \text{ be measurable non-negative function, i.e. such that } f(\overline{\xi}) \geq 0; \]

\[ S_L[f] := |L|^{-1} \sum_{\overline{k} \in L} f(\overline{\xi}_k). \]

The following simple estimate is named as before trivial:

\[ |S_L[f]|_p \leq |f|_L^p. \] (3.0a)

Recall that by-still hereafter \(p \geq 2\).

By definition, as above, the function \(f: X \rightarrow R\) is said to be degenerate, iff it has the form

\[ f(\overline{x}) = \sum_{\overline{k} \in \mathbb{Z}^d_+, N(\overline{k}) \leq M} \lambda(\overline{k}) \prod_{s=1}^d g^{(s)}_k(x(s)), \]

for some integer constant value \(M,\) finite or not, where all the functions \(g^{(s)}_k(\cdot)\) are in turn non-negative: \(g^{(s)}_k(\xi(k)) \geq 0\). Denotation: \(M = \text{deg}[f]\).

Define also as in the two-dimensional case for each such a function \(f \in D^+\) the following non-negative quasi-norm

\[ |f|_L \leq |f|_L^p. \]
\[ ||f||D_p^+ \overset{df}{=} \inf \left\{ \sum_{\vec{k} \in \mathbb{Z}_d^+, \, N(\vec{k}) \leq M[f]} |\lambda(\vec{k})| \cdot \prod_{s=1}^{d} |g_{k_s}(\xi(s))|_p \right\} \quad (5.3) \]

where all the arrays \( \{\lambda(\vec{k})\} \), \( \{g_j\} \), are taking from the representation 5.2, in particular, all the summands are non-negative.

The last assertion allows a simple estimate: \[ ||f||D_p^+ \leq ||f||D_p^{+o}, \]

and if we denote

\[ G(p) := \prod_{j=1}^{d} |g_{k_j}(\xi_j)|_p, \quad p \geq 1; \quad ||\lambda||_1 := \sum_{\vec{k} \in \mathbb{Z}_d^+} |\lambda(\vec{k})|, \]

then

\[ ||f||D_p^+ \leq ||f||D_p^{+o} \leq G(p) \cdot ||\lambda||_1. \quad (5.3b) \]

Further, let the non-negative function \( f \in B_1 \otimes B_2 \otimes \ldots \otimes B_d \) be given. Here \( B_r, \, r = 1, 2, \ldots , d \) are some Banach functional rearrangement invariant spaces builded correspondingly over the sets \( X_m \).

The error of a degenerate approximation of the function \( f \) by the degenerate and non-negative ones of the degree \( M \) will be introduced as before

\[ Q_M[f](B_1 \otimes B_2 \otimes \ldots \otimes B_d) \overset{df}{=} \inf_{f \in D^+[M]} ||f - \tilde{f}||B_1 \otimes B_2 \otimes \ldots \otimes B_d = \min_{f \in D^+[M]} ||f - \tilde{f}||B_1 \otimes B_2 \otimes \ldots \otimes B_d. \quad (5.4) \]

Obviously, \( \lim_{M \to \infty} Q_M[f](B_1 \otimes B_2 \otimes \ldots \otimes B_d) = 0 \), \( M \to \infty \).

For brevity:

\[ Q_M[f]_p \overset{df}{=} Q_M[f](L_p(X_1) \otimes L_p(X_2) \otimes \ldots \otimes L_p(X_d)). \quad (5.5) \]

The function \( \tilde{f} \) which realized the minimum in (5.4), not necessary to be unique, will be denoted by \( Z_M[f](B_1 \otimes B_2 \otimes \ldots \otimes B_d) = Z_M^+[f](B_1 \otimes B_2 \otimes \ldots \otimes B_d) : \)

\[ Z_M^+[f](B_1 \otimes B_2 \otimes \ldots \otimes B_d) := \text{argmin}_{f \in D^+[M]} ||f - \tilde{f}||B_1 \otimes B_2 \otimes \ldots \otimes B_d, \quad (5.6) \]

so that

\[ Q_M^+[f](B_1 \otimes B_2 \otimes \ldots \otimes B_d) = ||f - Z_M^+[f]||B_1 \otimes B_2 \otimes \ldots \otimes B_d. \quad (5.7) \]
For brevity:

\[ Z^{+}_M[f] = Z_M[f]_p := Z_M[f](L_p(X_1) \otimes L_p(X_2) \otimes \ldots \otimes L_p(X_d)). \]  

(5.8)

Denote as in the third section for \( f \in D(M) \) in the multivariate d-dimensional case

\[ |||f|||_{\Theta_p} = |||f|||_{\Theta_p,L} \overset{def}{=} \sum_{N(k) \leq M} \lambda_k \prod_{s=1}^{d} \Theta_{p,n(s)}^{(s)} \left[ q_{k(s)}^{(s)} \right], \]  

(5.9a)

\[ W_L[f](p) = W^{(d)}_L[f](p) \overset{def}{=} \inf_M \left[ |||Z^{+}_M[f] + Q^{+}_M[f]_p||| \right], \]  

(5.9b)

\[ W[f]^{(d)}(p) \overset{def}{=} \sup_{L:|L| \geq 1} W^{(d)}_L[f](p). \]  

(5.9c)

We deduce analogously to the third section

**Proposition 5.1.** If \( f \in D(M) \), then

\[ |S^{(\lambda)}_L[f]|_p \leq |||f|||_{\Theta_p,L}, \]  

and of course

\[ \sup_{L:|L| \geq 1} |S^{(\lambda)}_L[f]|_p \leq \sup_{L:|L| \geq 1} |||f|||_{\Theta_p,L}, \]  

(5.10a)

**Theorem 5.1.** Let \( f = f(x) = f(\vec{x}), \ x \in X \) be arbitrary non-negative function from the space \( L_p(X_1) \otimes L_p(X_2) \otimes \ldots \otimes L_p(X_d) \), \( p \geq 2 \). Then

\[ |S_L[f]|_p \leq W^{(d)}_L[f](p), \]  

(5.11)

\[ \sup_{L:|L| \geq 1} |S_L[f]|_p \leq \sup_{L:|L| \geq 1} W^{(d)}_L[f](p) = W^{(d)}[f](p). \]  

(5.11a)

**Example 5.1.** We deduce alike the example 3.1 as a particular case

\[ \sup_{L:|L| \geq 1} |S_L[f]|_p \leq (2/3)^d \cdot \left[ \frac{p}{e \cdot \ln p} \right]^d \cdot |||f|||_{D_p}, \]  

(5.12)

if of course the right-hand side of (5.9a) is finite for some value \( p, p \geq 2 \).

**Remark 5.1.** Notice that the last estimates (5.10), (5.11), and (5.12) are essentially non-improvable. Indeed, it is known still in the one-dimensional case
\[ d = 1; \text{ for the multidimensional one it is sufficient to take as a trivial factorizable function; say, when } \, d = 2, \text{ one can choose} \]

\[ f_0(x, y) := g_0(x) \, h_0(y), \, x \in X, \, y \in Y. \]

6  **Exponential bounds for distribution of positive multiple sums.**

We intend to derive in this section the uniform relative the amount of summand \(|L|\) exponential bounds for tail of distribution of the r.v. \(S_L\), based in turn on the moments bound obtained above as well as on the theory of the so-called Grand Lebesgue Spaces (GLS). We recall now for readers convenience some facts about these spaces and supplement more.

These spaces are Banach functional space, are complete, and rearrangement invariant in the classical sense, see [4], chapters 1, 2; and were investigated in particular in many works, see e.g. [5], [6]-[7], [13], [19]-[20], [21], [22]-[25], and so one.

They are closely related with the so-called exponential Orlicz spaces, see [5], [6], [7], [22], [23]-[25] etc.

Denote for simplicity

\[ \nu_L(p) := W_L^{(d)}[f](p), \, \nu(p) := \sup_{L:|L| \geq 1} \psi_L(p), \]

and suppose

\[ \exists b = \text{const} \in (1, \infty); \quad \forall p \in (1, b) \quad \Rightarrow \psi(p) < \infty. \] (6.2)

Recall that the norm of the random variable \(\xi\) in the so-called Grand Lebesgue Space \(G\psi\) is defined as follows

\[ \|\|\xi\|\|_{G\psi} \overset{\text{def}}{=} \sup_{p \in (1, b)} \left\{ \frac{|\xi|_p}{\psi(p)} \right\}. \] (6.3)

Here the generating function for these spaces \(\psi = \psi(p)\) will be presumed to be continuous inside the open interval \(p \in (1, b)\) and such that

\[ \inf_{p \in (1, b)} \psi(p) > 0. \]

The inequalities (5.11) and (5.11a) may be rewritten as follows

\[ \|S_L[f]\|_{G\nu_L} \leq 1; \quad \sup_L \|S_L[f]\|_{G\nu} \leq 1. \] (6.4)
The so-called tail function $T_f(y)$, $y \geq 0$ for arbitrary (measurable) numerical valued function (random variable, r.v.) $f$ is defined as usually

$$T_f(y) \overset{def}{=} \max(P(f \geq y), \ P(f \leq -y)), \ y \geq 0.$$ 

Obviously, if the r.v. $f$ is non-negative, then

$$T_f(y) = P(f \geq y), \ y \geq 0.$$ 

It is known that and if $f \in G_\psi$, $||f||G_\psi = 1$, then

$$T_f(y) \leq \exp \left( -\zeta_\psi'(\ln(y)) \right), \ y \geq e$$ (6.5)

where

$$\zeta(p) = \zeta_\psi(p) := p \ln \psi(p).$$

Here the operator (non-linear) $f \rightarrow f^*$ will denote the famous Young-Fenchel, or Legendre transform

$$f^*(u) \overset{def}{=} \sup_{x \in \text{Dom}(f)} (x u - f(x)).$$

We deduce by means of theorem 5.1 and property (6.5)

**Proposition 6.1.**

$$T_{SL[f]}(y) \leq \exp \left( -\nu_L \left( \ln(y) \right) \right), \ y \geq e; \quad (6.6)$$

$$\sup_L T_{SL[f]}(y) \leq \exp \left( -\nu \left( \ln(y) \right) \right), \ y \geq e. \quad (6.6a)$$

**Example 6.1.**

Let us bring an example, see [30] for the centered r.v. Let $m = \text{const} > 1$ and define $q = m' = m/(m - 1)$. Let also $R = R(y)$, $y > 0$ be positive continuous differentiable slowly varying at infinity function such that

$$\lim_{\lambda \to \infty} \frac{R(y/R(y))}{R(y)} = 1.$$ (6.7)

Introduce a following $\psi$ - function

$$\psi_{m,R}(p) \overset{def}{=} p^{1/m} R^{-1/(m-1)} \left( p^{(m-1)^2/m} \right), \ p \geq 1, m = \text{const} > 1,$$ (6.7a)

Suppose

$$\nu(p) \leq \psi_{m,R}(p), \ p \in [1, \infty);$$
then [19]-[20], [30] the correspondent exponential tail function has a form

\[ T^{(m,R)}(y) \overset{\text{def}}{=} \exp \left\{ -C(m,R) \ y^m \ R^{m-1} \left( y^{m-1} \right) \right\}, \ C(m,R) > 0, \ y \geq 1; \quad (6.7b) \]

so that

\[ \sup_L T_{S_L}(y) \leq T^{(m,R)}(y), \ y \geq 1. \quad (6.8) \]

A particular cases: \( R(y) = \ln(y+e), \ r = \text{const}, \ y \geq 0; \) then the correspondent generating functions has a form (up to multiplicative constant)

\[ \psi_{m,r}(p) = p^{1/m} \ln^{-r}(p), \ p \in [2, \infty), \quad (6.9a) \]

and the correspondent tail function has a form

\[ T^{m,r}(y) = \exp \left\{ -K(m,r) \ y^m \ (\ln y)^r \right\}, \ K(m,r) > 0, \ y \geq e. \quad (6.9b) \]

Many other examples may be found in [19], [20], [22], [30] etc.

**Example 6.2.** Let the function \( f : X = \bigotimes_{s=1}^d X_s \to R \) be from the degenerate representation

\[ f(\vec{x}) = \sum_{\vec{k} \in \mathbb{Z}_+^d, \ N(\vec{k}) \leq M} \lambda(\vec{k}) \prod_{j=1}^d g_{k_j}(x_j), \quad (5.2a) \]

for some constant integer value \( M, \) finite or not, where all the functions \( g_{k_j}(\cdot) \) are in turn non-negative: \( g_{k_j}(\xi(\vec{k})) \geq 0. \) Recall the denotation: \( M = \deg[f]. \)

*Suppose here and in what follows in this section that

\[ \sum_{\vec{k} \in \mathbb{Z}_+^d, N(\vec{k}) \leq M} |\lambda(\vec{k})| \leq 1 \]

and that each the non-negative r.v. \( g_{k_j}(\xi(\vec{k})) \) belongs to some \( G\psi_k \) - space uniformly relative the index \( j : \)

\[ \sup_j |g_{k_j}(\xi(\vec{k}))|_p \leq \psi_k(p). \quad (6.11) \]

Of course, as a capacity of these functions may be picked the natural functions for the r.v. \( g_k(\xi(\vec{k})): \)

\[ \psi_k(p) \overset{\text{def}}{=} \sup_j |g_{k_j}(\xi(\vec{k}))|_p, \]

if the last function is finite for some non-trivial interval \([2,a(\vec{k}))\), where \( a(\vec{k}) \in (2, \infty]. \)

Obviously,
\[ |f(\xi)|_p \leq \prod_{k=1}^d \psi_k(p), \]

and the last inequality is exact if for instance \( M = 1 \) and all the functions \( \psi_k(p) \) are natural for the family of the r.v. \( g_k^{(j)}(\xi(k)) \).

Define the following \( \Psi - \) function

\[ \beta(p) = \kappa_d\lambda(\xi)(p) = \prod_{k=1}^d \psi_k(p). \]

The assertion of proposition (5.1) gives us the estimations

\[ \sup_{L:|L| \geq 1} ||S_L[f]||G\kappa \leq 1 \quad (6.12) \]

and hence

\[ \sup_{L:|L| \geq 1} T_{S_L[f]}(u) \leq \exp \left( -v^*_k(\ln u) \right), \quad u \geq e, \quad (6.12b) \]

with correspondent Orlicz norm estimate.

**Example 6.3.**

Suppose again that

\[ \sum_{\tilde{k} \in \mathbb{Z}_+^d, N(\tilde{k}) \leq M} |\lambda(\tilde{k})| \leq 1 \]

and that the arbitrary r.v. \( g_k^{(j)}(\xi(k)) \) belongs uniformly relative the index \( j \) to the correspondent \( G\psi_{m(k),\gamma(k)} \) space:

\[ \sup_j \ |g_k^{(j)}(\xi(k))|_p \leq p^{1/m(k)} [\ln p]^{\gamma(k)}, \quad p \geq 2, \ m(k) > 0, \ \gamma(k) \in \mathbb{R}, \quad (6.13) \]

or equally

\[ \sup_j T_{g_k^{(j)}(\xi(k))}(u) \leq \exp \left( -C(k) \ u^{m(k)} [\ln u]^{-\gamma(k)} \right), \quad u \geq e. \quad (6.13a) \]

Define the following variables:

\[ m_0 := \left[ \sum_{k=1}^d 1/m(k) \right]^{-1}, \quad \gamma_0 := \sum_{k=1}^d \gamma(k), \]

\[ \hat{S}_L = \hat{S}_L[f] := e^d C_{R}^{-d} S_L. \quad (6.15) \]

We conclude by means of the proposition 5.1
\[
\sup_{L:|L|\geq 1} \left\| \hat{S}_L \right\| G_{\psi_{m_0,\gamma_0}} \leq 1 \tag{6.16}
\]

and therefore
\[
\sup_{L:|L|\geq 1} T_{S_L}(u) \leq \exp \left\{ -C(d, m_0, \gamma_0) \ u^{m_0} \ (\ln u)^{-\gamma_0} \right\}, \ u > e. \tag{6.17}
\]

**Example 6.4.**

Let us consider as above the following \( \psi_\beta(p) \) function
\[
\psi_{\beta,C}(p) := \exp \left( C p^\beta \right), \ C, \ \beta = \text{const} > 0, \ p \in [1, \infty). \tag{6.18}
\]
see example 6.3, (6.15)-(6.17).

Let \( g_k^{(j)}(\xi(k)) \) be non-negative independent random variables belonging to the certain \( G_{\psi_{\beta,C}(\cdot)} \) space uniformly relative the indexes \( k, j \):
\[
\sup_j \sup_k \left\| g_k^{(j)}(\xi(k)) \right\| G_{\psi_{\beta,C}} = 1, \tag{6.19}
\]
or equally
\[
\sup_j \sup_k T_{g_k^{(j)}(\xi(k))}(y) \leq \exp \left( -C_1(C, \beta) \ [\ln(1 + y)]^{1+1/\beta} \right), \ y > 0. \tag{6.20}
\]

Then
\[
\sup_{L: |L|\geq 1} T_{S_L}(y) \leq \exp \left( -C_2(C, \beta) \ [\ln(1 + y)]^{1+1/\beta} \right), \ y > 0, \tag{6.20a}
\]
or equally
\[
\sup_{L: |L|\geq 1} \left\| S_L[f] \right\| G_{\psi_{\beta,C_4}(C, \beta)} = C_4(C, \beta) < \infty. \tag{6.20a}
\]

**Example 6.5.** Suppose now that the each non-negative random variable \( g_k^{(j)}(\xi(k)) \) belongs uniformly relative the index \( j \) to certain \( G_{\psi^{<b(k),\theta(k)>}} \) space, where \( b(k) \in (2, \infty), \ \theta(k) \in R \):
\[
\sup_j \left\| g_k^{(j)}(\xi(k)) \right\| G_{\psi^{<b(k),\theta(k)>}} < \infty,
\]
where by definition
\[
\psi^{<b(k),\theta(k)>}(p) \overset{\text{def}}{=} C_1(b(k), \theta) \ (b(k) - p)^{-(\theta(k)+1)/b(k)}, \ 1 \leq p < b(k).
\]

This case is more complicates than considered before.

Note that if the r.v. \( \eta \) satisfies the inequality
\[
T_{\eta}(y) \leq C \ y^{-b(k)} \ [\ln y]^\theta(k), \ y \geq e,
\]
then $\eta \in G^{<b(k),\theta(k)>}$, see the example 6.2.

One can assume without loss of generality

$$b(1) \leq b(2) \leq b(3) \leq \ldots b(d).$$

Denote

$$\nu_k(p) := \psi^{<b(k),\theta(k)>}(p), \ b(0) := \min_k b(k),$$

so that $b(0) = b(1) =$

$$b(2) = \ldots = b(k(0)) < b(k(0) + 1) \leq \ldots \leq b(d), \ 1 \leq k(0) \leq d;$$

$$\Theta := \sum_{k=1}^{k(0)} (\theta(k) + 1)/b(0),$$

$$v(p) = v_\xi[f](p) \triangleq \prod_{l=1}^{k(0)} \nu_l(p) = C \cdot \left[ b(0) - p \right]^{-\Theta}, \ 2 \leq p < b(0).$$

Obviously,

$$\prod_{k=1}^{d} \nu_k(p) \leq C(d) \ v(p) = C \left[ b(0) - p \right]^{-\Theta}, \ C = C_d(\xi, \vec{b}, \vec{\theta}, k(0)).$$

Thus, we obtained under formulated above conditions

$$\sup_{L:|L| \geq 1} |S_L|_p \leq C_2 \left( b(0) - p \right)^{-\Theta}, \ p \in [2, b(0))$$

with the correspondent tail estimate

$$\sup_{L:|L| \geq 1} T_{S_L}(y) \leq C_3 \ y^{-b(0)} \left[ \ln y \right]^{b(0)} \Theta, \ y \geq e.$$

7 Upper bounds for these statistics.

A. A simple lower estimate in the Klesov’s (3.4) inequality may has a form

$$\sup_{L:|L| \geq 1} \left| S^{(2)}_L \right|_p \geq \left| S^{(2)}_1 \right|_p = \left| g(\xi) \right|_p \left| h(\eta) \right|_p, \ p \geq 2,$$

as long as the r.v. $g(\xi), h(\eta)$ are independent.

Suppose now that $g(\xi) \in G^{\psi_1}$ and $h(\eta) \in G^{\psi_2}$, where $\psi_j \in \Psi(b), \ b = \text{const} \in (2, \infty)$; for instance $\psi_j, \ j = 1, 2$ must be the natural functions for these r.v. Put $v(p) = \psi_1(p) \ \psi_2(p)$; then
\[ \nu(p) \leq \sup_{|L| \geq 1} \left| S^{(2)}_L \right|_p \leq K_1^d \cdot \nu(p), \quad K_1 < \infty. \]  

(7.2)

Assume in addition that \( b < \infty; \) then \( K_1 \leq C(b) < \infty. \) We get to the following assertion.

**Proposition 7.1.** We deduce under formulated above in this section conditions

\[ 1 \leq \frac{\sup_{|L| \geq 1} |S_L|_p}{\nu(p)} \leq C^d(b) < \infty, \quad p \in [2, b). \]  

(7.3)

**B. Tail approach.** We will use the example 6.2 (and notations therein.) Suppose in addition that all the (independent) r.v. \( \xi(k) \) have the following tail of distribution

\[ T_{g(\xi(k))}(y) = \exp \left( - \frac{\ln(1 + y)}{1+1/\beta} \right), \quad y \geq 0, \quad \beta = \text{const} > 0, \]

i.e. an unbounded support. As we knew,

\[ \sup_{L: |L| \geq 1} T_{S_L}(y) \leq \exp \left( -C_5(\beta, d) \frac{\ln(1 + y)}{1+1/\beta} \right), \quad y > 0, \]

On the other hand,

\[ \sup_{L: |L| \geq 1} T_{S_L}(y) \geq T_{S_1}(y) \geq \exp \left( -C_6(\beta, d) \frac{\ln(1 + y)}{1+1/\beta} \right), \quad y > 0. \]  

(7.4)

**C. An example.** Suppose as in the example 6.1 that the independent centered r.v. \( g_k^{(j)}(\xi(k)) \) have the standard Poisson distribution: \( \text{Law}(\xi(k)) = \text{Poisson}(1), \) \( k = 1, 2, \ldots, d. \) Assume also that in the representation (5.2a) \( M = 1 \) (a limiting degenerate case). As long as

\[ |g_k^{(j)}(\xi(k))|_p \leq C \frac{p}{\ln p}, \quad p \geq 2, \]

we conclude by virtue of theorem 5.1

\[ \sup_{L: |L| \geq 1} |S_L|_p \leq C_2^d \frac{p^{2d}}{[\ln p]^{2d}}, \quad p \geq 2, \]  

(7.5)

therefore

\[ \sup_{L: |L| \geq 1} T_{S_L}(y) \leq \exp \left( -C_1(d) \frac{y^{1/(2d)}}{[\ln y]^{2d}} \right), \quad y \geq e. \]  

(7.6)

On the other hand,

\[ \sup_{L: |L| \geq 1} |S_L|_p \geq |S_1|_p \geq C_3(d) \frac{p^d}{[\ln p]^d}, \]
and following

$$\sup_{L:|L| \geq 1} T_{S_L}(y) \geq \exp \left( -C_4(d) \ y^{1/d} \ [\ln y]^d \right), \ y \geq e. \quad (7.7)$$

8 Concluding remarks.

A. It is interest by our opinion to generalize obtained in this report results onto the mixing sequences or onto martingales, as well as onto the multiple integrals instead sums.

B. Perhaps, a more general results may be obtained by means of the so-called method of majorizing measures, see [1]-[3], [11], [29], [31]-[35].

C. Possible applications: statistics and Monte-Carlo method, alike [8], [10] etc.

D. It is interest perhaps to generalize the assertions of our theorems onto the sequences of domains \{L\} tending to “infinity” in the van Hove sense, in the spirit of an articles [26]-[27], [30].

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