Reconstruction and Stability Analysis of Some Cosmological Bouncing Solutions in $F(R, T)$ Theory

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The present article investigates the possibility of reconstruction of the generic function in $F(R, T)$ gravitational theory by considering some well-known cosmological bouncing models namely exponential evaluation, oscillatory, power law and matter bounce model, where $R$ and $T$ are Ricci scalar and trace of energy-momentum tensor, respectively. Due to the complexity of dynamical field equations, we propose some ansatz forms of function $F(R, T)$ in perspective models and examine that which type of Lagrangian is capable to reproduce bouncing solution via analytical expression. It is seen that for some cases of exponential, oscillatory and matter bounce models, it is possible to get analytical solution while in other cases, it is not possible to achieve exact solutions so only complementary solutions can be discussed. However, for power law model, all forms of generic function can be reconstructed analytically. Further, we analyze the energy conditions and stability of these reconstructed cosmological bouncing models which have analytical forms. It is found that these models are stable for linear forms of Lagrangian only but the reconstructed solutions for power law are unstable for some non-linear forms of Lagrangian.

Keywords: Bouncing Cosmology, Modified theories, Reconstruction scheme, Stability.

I. INTRODUCTION

In cosmological theories, the bouncing cosmology is one of the most interesting scenario which deals with those solutions having initial singularity problems [1–4]. The primary objective of such solutions is to resolve the initial big bang singularity which is regarded as one of the biggest issues in cosmology [2]. According to bouncing cosmology, initially our universe contracts up to a minimal radius, it bounces off at that point and than it starts to expand. Thus our universe can never collapse to a singular point and therefore initial singularity can be avoided. Additionally, cosmological bouncing is considered as a fascinating alternative to standard inflationary cosmology and have been appeared in loop quantum cosmology [6], matter bounce [7] and scalar field theories [1–4, 7, 8]. Actually, inflationary scenario solve some early universe issues like flatness, horizon, initial singularity and baryon asymmetry issues and provides a complete picture of structure formation [7, 9–11]. Although, inflationary model faces both the trans-Plankian as well as singularity issues during fluctuations, however, before inflation when our universe undergoes the exponential expansion, singularity produces and than inflationary scenario cannot describe the complete picture of universe. At this stage, matter bouncing scenario challenges these issues which are faced by the inflationary models [12].

In Einstein’s theory and its modifications, bouncing scenarios have grasped attentions of researchers and in literature, much work have been done on this subject. In [13], Myrzakulov and Sebastiani presented the bouncing solutions using viscous fluid in FRW flat space time by taking different scale factors (exponential and power law) along with some kinds of fluid into account. They discussed a relationship between finite singularity and bounce and extended their work to $f(R)$ gravity. Cheung et al. [14] proposed a new scenario for the production of dark matter regarding bouncing cosmology where dark matter has been produced from plasma during contracting and expanding cosmic phases. In another study [15], authors discussed the ΛCDM bouncing scenario in which they studied that contracting universe contained radiation as well as cold dark matter with positive cosmological constant. They assumed that loop quantum cosmology captured the spacetime in the context of high curvature dynamics which guaranteed the removal and replacement of big bang singularity by bounce. Further they used bounce to calculate the perturbations and examined that these are nearly scale invariant. Bandyopadhyay and Debnath [16] discussed the cosmological bouncing in entropy corrected models namely logarithmic and power law corresponding to Horava-Lifshitz gravity as well as fractal universe and also analyzed the null energy condition near the bounce.

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During the last few decades, substantial attempts have been made to develop the non-standard theories of gravity by introducing some modifications in the action of Einstein gravity. Some noteworthy examples of such modified formulations include $F(R)$, $F(R, T)$ (where $T$ represents the trace of $T_{ij}$), $F(T)$, where $T$ represents torsion, Gauss-Bonnet gravity and Brans-Dicke theory. The bouncing cosmology has been explained in such modified gravitational frameworks by numerous researchers, for reference one can see. In this respect, Oikonomou and Odintsov explained the bouncing scenario along with type IV singularity at bouncing point in Gauss Bonnet $F(G)$ and $F(R)$ gravity. They also analyzed the stability of solutions corresponding to $F(G)$ and $F(R)$ theories. In another paper, same authors discussed $\Lambda$CDM bouncing model in $F(R)$ theory and checked stability properties as well as gravitational particle production which are essential features for checking viability of the $\Lambda$CDM bounce. Further, some bouncing scenarios namely matter bounce, super bounce, singular bounce and symmetric bounce in unimodular $F(R)$ gravity have been explored by researchers and they presented the behavior of Hubble radius for all bouncing scenarios. Elizalde et al. discussed the matter bounce in extended form by considering the framework of ghost free $F(R, G)$ gravity, while Boisseau et al. produced the bouncing universe in scalar-tensor gravity having negative scalar field potential.

Modified gravitational frameworks (geometrical modification of GR action) are considered as viable dark energy candidates and predict the late time cosmic acceleration successfully. In this context, Harko et al. have proposed the $F(R, T)$ gravity (with $R$ and $T$ as Ricci scalar and energy-momentum tensor trace) in which they have suggested different forms of $F(R, T)$ such as $F(R, T) = R + 2f(T), F(R, T) = F(R) + G(T)$ and $F(R, T) = F_1(R) + F_2(R)F_3(T)$, where $F(R)$, $G(T)$, $F_1(R)$, $F_2(R)$ and $F_3(T)$ are some arbitrary functions of $R$ and $T$. Many researchers have utilized these forms of $F(R, T)$ function and addressed the phenomenon of late time cosmic acceleration. In this regard, Singh et al. explained the bouncing cosmology to obtain a specific form of Hubble parameter in $F(R, T)$ theory. Shabani and Ziaie studied the cosmological bouncing solutions using perfect fluid in $F(R, T)$ gravity. They explored the properties of such bouncing solutions, checked the stability and examined the validity of energy constraints near bouncing point. They obtained those cosmological scenarios which can exhibit the non-singular bounce before and after de-Sitter cosmic phase and also proposed the general solution for matter bounce. In the present work, we shall also adopt the same form of Lagrangian function $F(R, T)$ to reconstruct bouncing models.

In modified theories, reconstruction of cosmological models is an open problem. Many reconstruction schemes have been proposed in literature which are substantially used to understand the inter-conversion of matter dominated and DE cosmic phases in $F(R)$ gravity. By assuming some known cosmic evolution, the corresponding form of Lagrangian can be calculated which can reproduce the same background evolution. In literature, researchers have discussed various cosmological techniques of reconstruction in $F(R, T)$ gravity. Bamba et al. explained the reconstruction scenario for two bouncing (exponential and power law) solutions in $F(R)$ and $F(G)$ gravity theories and examined the stability of reconstructed bouncing models in both theories. Further, these works are extended by Bamba et al. where they discussed bouncing cosmology in a gravitational framework involving interaction of Gauss-Bonnet invariant and dynamical scalar field and reconstructed the forms of potential and Gauss-Bonnet coupling function of scalar field using hyperbolic and exponential forms of scale factor. By using conformal transformation, they also explained the link between the bouncing behaviors in Jordan and Einstein frames. Odintsov et al. studied the bouncing cosmology in the framework of $F(R)$ gravity theory via reconstruction scheme, and confronted their model with the recent observations. They solved dynamical equation numerically and discussed different qualitative features and observable measures of the proposed model. In another work, Nojiri et al. executed such reconstruction technique to obtain the $F(R)$ gravity model, which were adopted in $F(R, G)$ and Gauss-Bonnet theories. Cruz-Dombriz et al. investigated the existence of analytical bouncing solutions in extended teleparallel gravity namely $f(T, T_G)$ theory using FRW model for symmetric, oscillatory, superbounce, matter bounce, and singular bounce. Caruana et al. explored possibility of reconstruction of analytical form of Lagrangian function using some well-known bouncing models like symmetric, oscillatory, superbounce, matter bounce, and singular bounce for flat FRW geometry in $f(T, B)$ theory and obtained significant solutions.

Zubair and Kousar explained the reconstruction scenario in $F(R, R_{a\beta}R^{a\beta}, \phi)$ theory and discussed the validity of all energy bounds graphically. Mishra et al. also investigated different scenarios of anisotropic cosmological reconstruction in $F(R, T)$ gravity and explored the effect of coupling constant and anisotropy on the cosmic dynamics using more general approach. In a recent paper, Shamir studied some bouncing models in $F(G, T)$ gravity (with $G$ as the Gauss-Bonnet term and $T$ as the trace of energy-momentum tensor) by taking well-famed EoS parameter along with two forms of generic functions $F(G, T)$ involving logarithmic and linear trace terms and concluded that their discussed bouncing solutions are significant. In the context of modified theories, the stability of cosmological solutions is regarded as a captivating issue and numerous studies are already available on this topic. In stability analysis, both Hubble parameter $H$ and energy density $\rho$ are perturbed by introducing the isotropic and homogeneous perturbations. In this respect, stability of de-Sitter, power law, phantom and non-phantom matter fluid solutions have been analyzed in $F(G, T)$ gravity. Salako et al. investigated the reconstruction, thermodynamics and stability of $\Lambda$CDM model in the framework of teleparallel theory. In another paper, authors performed the stability analysis for
cosmological models using dynamical system analysis (fixed point theory) in $F(R)$ gravity, while Shabani et al. \[56\] examined the stability of Einstein static universe in Einstein-Carton-Brans Dicke gravity. In another work \[57\], authors presented bouncing scenario in $F(R, T)$ gravity and explored the energy conditions near the bouncing point and discussed the stability of obtained solutions. Godani and Samanta \[58\] estimated the cosmological parameters (Hubble and deceleration) to analyze the stability and energy conditions in a modified gravity.

In our work, we shall employ reconstruction scenario in $F(R, T)$ theory to analyze some bouncing cosmological models namely exponential, oscillatory, power law and matter bounce models. In all models, we shall choose minimal and non-minimal coupling of $R$ and $T$ in Lagrangian and reconstruct their solutions. In three models, only complementary solutions exist for non-minimal coupling and for power law model, all forms of Lagrangian have analytical solution. Further, we shall analyze the stability for these reconstructed bouncing solutions. This paper is organized in this pattern. Section II provides the set of field equations for FRW metric along with some basic assumptions used for this work. Section III comprises the reconstruction of generic function $F(R, T)$ by using exponential, oscillatory, power law and matter bounce models with different ansatz of this function. Section IV deals with the validity of energy conditions for these reconstructed models. Further, we analyze the stability of these reconstructed solutions against perturbations of FRW universe model in Section V. Final section will conclude and present the obtained results briefly.

II. BASIC MATHEMATICAL STRUCTURE OF $F(R, T)$ GRAVITY AND ITS FIELD EQUATIONS

Here, we shall present a brief review of the $F(R, T)$ gravity, its field equations along with some necessary assumptions used for this work. Let us start from the action \[22\] of this theory defined as follows

$$S = \frac{1}{2k^2} \int F(R, T) \sqrt{-g} d^4x + \int L_m \sqrt{-g} d^4x,$$  \hspace{1cm} (1)

where $L_m$ stands for the ordinary matter Lagrangian density and $g$ represents the determinant of $g_{ij}$. By varying the above action with respect to metric tensor, the following set of field equations can be obtained:

$$8\pi T_{ij} - F_T(R, T) T_{ij} - F_T(R, T) \Theta_{ij} = F_R(R, T) R_{ij} - \frac{1}{2} F(R, T) g_{ij} + (g_{ij} \Box - \nabla_i \nabla_j) F_R(R, T),$$  \hspace{1cm} (2)

where $F_R(R, T) = \frac{\partial F(R, T)}{\partial R}$, $F_T(R, T) = \frac{\partial F(R, T)}{\partial T}$, while $\Box = g^{ij} \nabla_i \nabla_j$. Also, $\nabla_i$ symbolizes the covariant derivative and the mathematical expression of $\Theta_{ij}$ is defined by

$$\Theta_{ij} = \frac{\delta^{\alpha \beta}}{\delta g^{ij}} g^{\alpha \beta} = -2T_{ij} + g_{ij} L_m - 2g^{\alpha \beta} \frac{\partial^2 L_m}{\partial g^{ij} \partial g^{\alpha \beta}}.$$

It is worthy to mention here that the energy-momentum tensor in $F(R, T)$ gravity is not conserved and yields the following relation:

$$\nabla^i T_{ij} = \frac{F_T}{K^2 - F_T} \left( T_{ij} + \Theta_{ij} \right) \nabla^i \ln F_T + \nabla^i \Theta_{ij} - \frac{1}{2} g_{ij} \nabla^i T.$$  \hspace{1cm} (3)

In this work, we assume the matter Lagrangian as $L_m = -p$, and consequently, $\Theta_{ij}$ takes the form as

$$\Theta_{ij} = -2T_{ij} - pg_{ij}$$  \hspace{1cm} (4)

and then field equations and energy-momentum conservation take the following form:

$$k^2 T_{ij} + F_T(R, T) T_{ij} + p F_T(R, T) g_{ij} = R_{ij} F_R(R, T) - \frac{1}{2} g_{ij} F(R, T) - (\nabla_i \nabla_j - g_{ij} \Box) F_R(R, T),$$  \hspace{1cm} (5)

$$\nabla^i T_{ij} = \frac{-1}{k^2 + f_T} \left[ T_{ij} \nabla^i f_T + g_{ij} \nabla^i (\frac{f}{2} + pf_T) \right].$$  \hspace{1cm} (6)

The flat FRW metric in spherical coordinates with $a(t)$ as expansion radius can be defined as

$$ds^2 = dt^2 - a^2(t) \left[ dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right].$$  \hspace{1cm} (7)
The energy-momentum tensor is considered as perfect fluid and is defined as

\[ T_{ij} = (\rho + p)u_iu_j - pg_{ij}. \]

For this geometry, the corresponding 00 component of field equation is given by

\[ k^2 \rho + (\rho + p)F_T + \frac{F}{2} + 3(\dot{H} + H^2)F_R - 3H(\dot{\mathcal{R}}f_{RR} + \ddot{F}_{RT}) = 0, \tag{8} \]

Here, the dot indicates time derivative while the Hubble parameter, Ricci scalar and trace of energy-momentum tensor can be, respectively, defined as \( H = \frac{\dot{a}}{a}, \) \( \mathcal{R} = -6(\dot{H} + 2H^2) \) and \( T = \rho - 3p. \) The continuity equation can be re-written as

\[ \dot{\rho} + 3H(\rho + p) = -\frac{1}{k^2 + F_T} \left[ (\rho + p)\ddot{F}_{TT} + \dot{p}F_T + \frac{1}{2}\ddot{F}_T \right]. \tag{9} \]

which represents that \( \nabla^i T_{ij} \neq 0 \) in \( F(\mathcal{R}, T) \) gravity. To obtain the continuity equation in standard form, one needs to impose \( \nabla^i T_{ij} = 0 \) which implies the constraint

\[ \frac{1}{k^2 + F_T} \left[ (\rho + p)\ddot{F}_{TT} + \dot{p}F_T + \frac{1}{2}\ddot{F}_T \right] = 0 \]

to be satisfied. Applying \( \text{EoS } p = \omega \rho \) in the above equation, we obtain:

\[ \dot{\rho} + 3H\rho(1 + \omega) = 0, \tag{10} \]

\[ (1 + \omega)\ddot{F}_{TT} + \frac{1}{2}(1 - \omega)F_T = 0. \tag{11} \]

From Eq. (9), the two Friedmann equations can be written as

\[ 3H^2F_R = k^2 \rho + \frac{1}{2}(\mathcal{R}F_R - F) - 3H(\dot{\mathcal{R}}f_{RR} + \ddot{F}_{RT}), \tag{12} \]

\[ (3H^2 + 2\dot{H})F_R = k^2\rho - \frac{1}{2}(\mathcal{R}F_R - F) + 2H(\dot{\mathcal{R}}f_{RR} + \ddot{F}_{RT}) + \dot{F}_{RR} + (\dot{\mathcal{R}})^2F_{RR} + 2\dot{\mathcal{R}}\ddot{F}_{RT} + \ddot{F}_{RT} \]

\[ + (\ddot{T})^2F_{RTT}. \tag{13} \]

The above equations can be re-written in terms of effective energy \( (\rho^{eff}) \) and effective pressure \( (p^{eff}) \) by comparing them with the standard Friedmann equations, i.e., \( \rho^{eff} = 3\left(\frac{H}{k}\right)^2 \) and \( p^{eff} = -\frac{3H^2 + 2\dot{H}}{k^2} \), where the effective terms are given by

\[ \rho^{eff} = \frac{1}{k^2 + F_T} \left[ (k^2 + F_T)\rho + \frac{1}{2}(\mathcal{R}F_R - F) - 3H(\dot{\mathcal{R}}f_{RR} + \ddot{F}_{RT}) + 3(1 - F_R)H^2 \right], \tag{14} \]

\[ p^{eff} = \frac{1}{k^2 + F_T} \left[ (k^2 + F_T)p - \frac{1}{2}(\mathcal{R}F_R - F) + 2H(\dot{\mathcal{R}}f_{RR} + \ddot{F}_{RT}) + \dot{F}_{RR} - (\dot{T})^2F_{RTT} - (1 - F_R)(3H^2 + 2\dot{H}) \right]. \tag{15} \]

The corresponding effective EoS parameter can be obtained by dividing \( p^{eff} \) with \( \rho^{eff} \), i.e., \( \omega^{eff} = \frac{p^{eff}}{\rho^{eff}} \). For the sake of simplicity in further calculations, we interchange the cosmic time with a new variable namely e-folding parameter given by \( N = \ln\left(\frac{a}{a_0}\right) = -\ln(1 + Z) \) and further, the time derivative operator can be converted in terms of new parameter by the relation \( \frac{d}{dN} = \frac{H}{\dot{a}} \frac{d}{d\tau} \). Consequently, the effective terms can be shifted to new variable as follows

\[ \rho^{eff} = \frac{1}{k^2 + F_T} \left[ (k^2 + F_T)\rho + \frac{F}{2} + 3HH' + H^2F_R + 18H(H^2H'' + H(H')^2 + 4H^2H')F_{RR} + T'F_{RT} \right. \]

\[ + 3(1 - F_R)H^2 \], \tag{16} \]

\[ p^{eff} = \frac{1}{k^2 + F_T} \left[ (k^2 + F_T)p - \frac{F}{2} + 3HH' + H^2F_R - 12H(H^2H'' + H(H')^2 + 4H^2H')F_{RR} + T'F_{RT} - 6(H'')^2 \right. \]

\[ + 3HH'H'' + H(H')^3 + 4H^3H'' + 8H^2(H')^2F_{RR} + 36(H^2H'' + H(H')^2 + 4H^2H')F_{RR} - 12(H^2H'') \]

\[ + H(H')^2 + 4H^2H')T'F_{RR} + T''F_{RT} + (T')^2F_{RTT} - (1 - F_R)(3H^2 + 2HH')) \right], \tag{17} \]

where prime indicates derivative with respect to \( N \).
In this section, we shall adopt the well-known reconstruction scheme to reconstruct the forms of generic function \( F(R, T) \) by taking some interesting bouncing cosmological models defined by

1. Exponential evaluation model: \( a(t) = A \exp(\sigma \frac{t^2}{t_c^2}) \), where \( A \) and \( \sigma \) are free constants;

2. Oscillatory model: \( a(t) = A \sin^2(B \frac{t}{t_c}) \), where \( A \) and \( B \) are arbitrary constants;

3. Power law model: \( a(t) = \left( \frac{t - t_0}{t_s - t_0} \right) c \), where \( t_s, t_0 \) and \( c \) are free constants;

4. Matter bounce model: \( a(t) = A \left( \frac{3}{4} \rho_{cr} t^2 + 1 \right)^{\frac{1}{3}} \), where \( A \) and \( \rho_{cr} \) are constants.

The graphical behavior of these scale factors along with Hubble parameters are presented in the graphs of Figure 1

III. RECONSTRUCTION OF BOUNCING MODELS

In this subsection, we shall reconstruct the form of generic function \( F(R, T) \) by considering the above given four cosmological bouncing solutions.

A. Exponential Evaluation Bouncing Model

Here, we consider the bouncing solution in exponential form which is defined by the following scale factor

\[
a(t) = A \exp(\sigma \frac{t^2}{t_c^2}),
\]

where \( t_s \) being the arbitrary time and \( A, \sigma \) are positive constants. The corresponding Hubble parameter takes the form

\[
H = \frac{2\sigma t}{t_c^2}.
\]
At \( t = 0 \), it can be seen that \( a(0) = A \), while the Hubble parameter’s value at \( t = 0 \) indicates that it represents a bouncing point as \( H = 0 \) there, and \( H \) is negative when \( t < 0 \) whereas it becomes positive for \( t > 0 \) as shown in the first column of Figure 1. Thus the 00 component of FRW metric in Eq. (5) is given by

\[
k^2 \rho + (\rho + p) F_T + \frac{F}{2} + \frac{1}{4}(a_1 - R) F_R - a_1 R (1 + a_1) F_{RR} - \frac{3}{4}(1 + \omega)(R + a_1) T F_{RT} = 0, \tag{20}
\]

where \( a_1 = 6 \dot{H}, \ \dot{H} = \frac{2a}{a^2} \), and \( \dot{T} = -3H(1 + \omega)T \). This is a partial differential equation whose solution is difficult to find, therefore one can consider the above ansatz form of functions. By assuming (i) form of function \( F(R, T) \), it can be easily checked that the above equation becomes separable and the resulting two differential equations (DEs) can be written as

\[
\frac{\beta_2 g(R)}{2} + \frac{\beta_1}{4}(a_1 - R)g_R - a_1 R (1 + a_1)\beta_1 g_R R = 0, \tag{21}
\]

\[
k^2 T + (1 + \omega)\beta_2 Th_T + \frac{\beta_2 (1 - 3\omega) h(T)}{2} = 0. \tag{22}
\]

On solving these DEs, the solution for \( g \) is given by

\[
g(R) = \frac{(a_1^2 + 6a_1 R + R^2)}{a_1^2} \left[ c_1 - \frac{\exp(-(R/(4a_1)))\sqrt{(a_1 + R)(3a_1 + R)c_2}}{16a_1^2(a_1^2 + 6a_1 R + R^2)} - \frac{\exp(1/4)\sqrt{\pi}c_2 Erf\left(\frac{\sqrt{(a_1 + R)}}{2\sqrt{a_1}}\right)}{32a_1^{5/2}} \right], \tag{23}
\]

and the solution for \( h(T) \) can be found by using constraint (11) as follows

\[
h(T) = -\frac{2k^2 T}{\beta_2 - 3\beta_2 \omega} + T^{1/2} - \frac{\sqrt{-1 + (\omega)(3\omega - 3\omega^2)}}{(1 + \omega)^{1/2}} \frac{c_3 + T^{1/2} + \sqrt{-1 + (\omega)(3\omega - 3\omega^2)}}{(1 + \omega)^{1/2} \beta_2 c_4} \frac{c_4}{c_1}, \tag{24}
\]

Thus \( F(R, T) \) function can be written as:

\[
F(R, T) = \frac{\beta_1(a_1^2 + 6a_1 R + R^2)}{a_1^2} \left[ c_1 - \frac{\exp(-(R/(4a_1)))\sqrt{(a_1 + R)(3a_1 + R)c_2}}{16a_1^2(a_1^2 + 6a_1 R + R^2)} - \frac{\exp(1/4)\sqrt{\pi}c_2 Erf\left(\frac{\sqrt{(a_1 + R)}}{2\sqrt{a_1}}\right)}{32a_1^{5/2}} \right] - \frac{2k^2T}{1 - 3\omega} \beta_2 c_3 + T^{1/2} \frac{\sqrt{-1 + (\omega)(3\omega - 3\omega^2)}}{(1 + \omega)^{1/2} \beta_2 c_4}, \tag{25}
\]

where all \( c_1 \)’s are constants of integration. Now, we will check the possibility of vacuum solutions, i.e., \( F(0, 0) = g(0) + h(0) = 0 \). Here, it is easy to verify that \( h(T = 0) = 0 \), thus we must impose \( g(R = 0) = 0 \) which further leads to the relation: \( 32(a_1)^2 c_1 = (6 + e^{1/4} \sqrt{\pi} Erf(1/2))c_2 = 0 \). Thus, the compatibility with the vacuum condition is possible only when the condition: \( c_1 = \frac{(6 + e^{1/4} \sqrt{\pi} Erf(1/2))c_2}{32(a_1)^2} \) holds.

For (ii) form of \( F(R, T) \) function, the 00 component results in a similar equation as given in (20) with an additional term \( R \). This equation is a second-order partial differential equation which can also be separated by introducing this \( F(R, T) \) function. Consequently, the solution for \( F(R, T) \) can be obtained as

\[
F(R, T) = \frac{27a_1^2 + 34a_1 R + 27R^2}{128a_1} + d_1 \beta_1 \frac{(a_1^2 + 6a_1 R + R^2)}{a_1^2} - \frac{\beta_1}{32a_1^{9/2}} \left[ \exp(-(R/(4a_1)))d_2(2\sqrt{a_1 + R}) \right] - \frac{2k^2 T}{1 - 3\omega} \beta_2 d_3 + T^{1/2} \frac{\sqrt{-1 + (\omega)(3\omega - 3\omega^2)}}{(1 + \omega)^{1/2} \beta_2 d_4}, \tag{26}
\]

where \( d_1 \)’s are integration constants. Since \( h(0) = 0 \) and hence to verify the vacuum condition, the condition: \( g(R = 0) = 0 \) must be satisfied. It is easy to check that \( g(R = 0) = 0 \) will hold only if \( \frac{d_1}{32(a_1)^2} (6 + e^{1/4} \sqrt{\pi} Erf(1/2)) - \frac{27a_1^2}{128a_1} = d_1 \).

For (iii) form of \( F(R, T) \) function (re-scaling type model), Eq. (11) leads to

\[
B_1 T^2 (A_1 \ln(T) + A_2) g_{TT} - B_2 (\ln(T) - A_3) g(T) = -k^2 T. \tag{27}
\]
The general solution of this DE is not possible to find, however one can find its complementary solution as follows

\[ g(T) = \frac{1}{A_1 B_2} \exp(-\frac{B_2 \sqrt{(1+4B_2/A_1 B_1)/B_2^2}(A_2 + A_1 \ln(T)))}{2A_1} \sqrt{T} \left[ c_1 \text{Hypergeometric1F1} \left[ \left( -A_2 + A_1 (-A_3 \right) \right] + A_1 B_1 \sqrt{(1+4B_2/A_1 B_1)/B_2^2})/ (A_1^2 B_1 \sqrt{(1+4B_2/A_1 B_1)/B_2^2}) \right], 2, \]

\[ + C_1 \text{HypergeometricU} \left[ \left( -A_2 + A_1 (-A_3 + A_1 B_1 \sqrt{(1+4B_2/A_1 B_1)/B_2^2}) \right)/ (A_1^2 B_1 \sqrt{(1+4B_2/A_1 B_1)/B_2^2}) \right], 2, \]

\[ - \frac{B_2 \sqrt{(1+4B_2/A_1 B_1)/B_2^2}(A_2 + A_1 \ln(T)))}{A_1} \sqrt{T} \right] (A_2 + A_1 \ln(T)), \]

(28)

where the involved constants can be defined as \( B_1 = \frac{8 \sigma (1 + \omega)}{(2 \pi - 1)^2}, A_1 = 1 + 9 \sigma, A_2 = 2B - 4 \ln(1 - 3 \omega) + 3(1 - 3 \omega) \ln(1 - 3 \omega) \rho_0, B_2 = -\frac{3}{2}(1 + \omega) - 2 \ln \rho_0, A_3 = \ln(1 - 3 \omega) \rho_0 \) while \( c_1 \) and \( c_2 \) are integrating constants. Since the general solution does not exist, therefore we can not check the vacuum condition in this case.

Using form (iv) of \( F(R, T) \) function (product form), the Friedman equation \[20\] can be re-written as

\[ \frac{k^2 T}{1 - 3 \omega} + \frac{(1 + \omega) T}{1 - 3 \omega} F(R) h_T + \frac{F(R) h(T)}{2} + \frac{1}{4} (a_1 - R) F_R h(T) - a_1 (R + a_1) F_R h(T) - \frac{3}{4} (1 + \omega) (R + a_1) T F_R h_T = 0, \]

which is a second-order partial differential equation that cannot be separated and hence its analytic solution is not possible to find. For function form (v), the Friedman equation takes the form as

\[ \dot{a}_1 g(R) + \dot{a}_3 b_1 - R b_2 g(R) + \dot{a}_4 (R + a_1) g_R R = -k^2 - \frac{1}{4 \rho_0} \exp(\dot{a}_2 (R + a_1))(R + a_1), \]

(29)

where \( \dot{a}_1 = \frac{(3 - \omega)}{2}, \dot{a}_3 = \frac{1 - 3 \omega}{4}, b_1 = 6 \dot{H}(9 \omega - 2), b_2 = (4 - 9 \omega), \dot{a}_4 = 6 \dot{H}(1 - 3 \omega) \) and \( \dot{a}_2 = -\frac{9 \sigma^2 (1 + \omega)}{(1 - 3 \omega) \sigma} \). This equation is second-order DE whose analytical solution (general solution) does not exist but the complementary solution can be obtained as follows

\[ g(R) = (a_1 + R)^{\left( a_4 - a_3 (b_1 + b_2 a_1) \right)/a_4} \] \[ \text{HypergeometricU} \left[ \left( -\frac{\dot{a}_1 a_4 - \dot{a}_3 a_4 b_1 + \dot{a}_3^2 b_1 b_2 + \dot{a}_3^2 b_2 a_1}{\dot{a}_3 a_4 b_2}, 1 + \frac{\dot{a}_4 - \dot{a}_3 a_4 b_1 - \dot{a}_3 b_2 a_1 + \dot{a}_3 b_2 R}{\dot{a}_4} \right] + (a_1 + R)^{\left( a_4 - a_3 (b_1 + b_2 a_1) \right)/a_4} \text{LaguerreL} \left[ \right. \frac{\dot{a}_1 a_4 - \dot{a}_3 a_4 b_1 + \dot{a}_3^2 b_1 b_2 + \dot{a}_3^2 b_2 a_1}{\dot{a}_4}, \frac{\dot{a}_4 - \dot{a}_3 a_4 b_1 - \dot{a}_3 b_2 a_1 + \dot{a}_3 b_2 R}{\dot{a}_4} \right], \]

(30)

with \( a_1, a_2 \) being integrating constants. It is worthy to mention here that nothing can be inferred about the vacuum condition due to unavailability of general solution. In case of form (vi), the Friedman equation can be re-defined as

\[ s_2 T^2 (a_1 + s_3 (\ln(T) - s_1)) f_T T + \frac{3 A}{A} (\ln(T) - s_1) f_T = \frac{k^2}{1 - 3 \omega} T - \frac{3 A}{A} (\ln(T) - s_1), \]

(31)

where \( a_1 = 6 \dot{H}, s_2 = 2\frac{(1 + \omega)^2}{(1 - 3 \omega)^2}, s_1 = \ln(\rho_0 (1 - 3 \omega)), s_3 = \frac{3 (1 + 9 \omega)}{A} \) and \( A = \frac{3 T^2 (1 + \omega)}{4 s_3} \). Again, general solution to this DE is not possible to determine, therefore one can only obtain only its complementary solution which is given by

\[ F(T) = \frac{1}{s_3} \exp\left( \frac{\sqrt{1 - 12/(A s_2 s_3)(-a_1 + s_1 s_3 - s_3 \ln(T))}}{2 s_3} \right) \sqrt{T} \left[ c_2 \text{Hypergeometric1F1} \left[ 1 + \frac{3 a_1 \sqrt{1 - 12}{s_3(-12 + A s_2 s_3)}{s_3}}{s_3}, \right] + c_1 \text{HypergeometricU} \left[ 1 + \frac{3 a_1 \sqrt{1 - 12}{s_3(-12 + A s_2 s_3)}{s_3}}{s_3}, 2, \right] \right], \]

\[ \frac{\sqrt{1 - 12}{s_3(-12 + A s_2 s_3)}{s_3}}{s_3} \right] (a_1 - s_1 s_3 + s_3 \ln(T)), \]

where \( c_1 \) and \( c_2 \) are integrating constants.
Next we utilize (vii) form of \(F(\mathcal{R}, T)\) function in the Friedman equation which yields the following DE:

\[
\frac{k^2 T}{1 - 3\omega} + \frac{(1 + \omega) T}{1 - 3\omega} F_2(\mathcal{R}) F_{3T} + \frac{F_1(\mathcal{R}) + F_2(\mathcal{R}) F_3(T)}{2} + \frac{1}{4} (a_1 - \mathcal{R})(F_{1\mathcal{R}} + F_{2\mathcal{R}} F_3(T)) \\
- a_1(\mathcal{R} + a_1)(F_{1\mathcal{R}\mathcal{R}} + F_{2\mathcal{R}\mathcal{R}} F_3(T)) - \frac{3}{4} (1 + \omega)(\mathcal{R} + a_1) TF_{2\mathcal{R}} F_{3T} = 0,
\]

which can be separated into two DEs given by

\[
\frac{k^2 T}{1 - 3\omega} + \frac{(1 + \omega) T}{1 - 3\omega} F_2(\mathcal{R}) F_{3T} + \frac{F_1(\mathcal{R}) + F_2(\mathcal{R}) F_3(T)}{2} + \frac{1}{4} (a_1 - \mathcal{R}) F_{2\mathcal{R}} F_3(T) - a_1(\mathcal{R} + a_1) F_{2\mathcal{R}\mathcal{R}} F_3(T) \\
- \frac{3}{4} (1 + \omega)(\mathcal{R} + a_1) TF_{2\mathcal{R}} F_{3T} = 0,
\]

\[
\frac{F_1(\mathcal{R})}{2} + \frac{(a_1 - \mathcal{R})}{4} F_{1\mathcal{R}} - a_1(a_1 + \mathcal{R}) F_{1\mathcal{R}\mathcal{R}} = 0.
\]

In the first DE, the unknowns depending on \(\mathcal{R}\) and \(T\) can not be separated, therefore we can not find its solution. The second ODE can be solved analytically and its solution for \(F_1(\mathcal{R})\) is quite similar to the solution of \(g(\mathcal{R})\) as given in Eq. (23). For (viii) form of generic function, the Friedman’s equation reduces to

\[
\mu \left( \frac{R}{R_0} \right)^\beta \left( \frac{T}{T_0} \right)^\gamma \left[ \frac{1 + \omega}{1 - 3\omega} \right] - \frac{1}{2} + \frac{\beta(12\sigma - R t^2)}{4R t^2} - \frac{12\sigma \beta(\beta - 1)(12\sigma + R t^2)}{R^2 t^4} - \frac{3(1 + \omega) \beta \gamma(12\sigma + R t^2)}{4R t^2}
\]

\[
= -T_0 \sum_i \Omega_{\omega_i, 0} a^{-3(1 + \omega)},
\]

where \(\mu, \beta\) and \(\gamma\) are arbitrary constants. In this case, the vacuum condition is satisfied if \(\gamma > 0\). At times, when \(R = T = 0\), the Friedman’s equation yields the condition, i.e., \(\sum_i \Omega_{\omega_i, 0} A^{-3(1 + \omega)} = 0\), only exists in vacuum. Which implies that \(\mu\) is equal to zero, a contradiction. Thus, there must exist a fluid with \(\omega_i = -1\). Then, by calculating the value of \(\mu\) at current times, we obtain:

\[
\mu = \frac{1}{(1 + \omega)} \left[ \frac{1}{1 - 3\omega} \right] - \frac{1}{2} + \frac{\beta(12\sigma - R t^2)}{4R_0 t^2} - \frac{12\sigma \beta(\beta - 1)(12\sigma + R t^2)}{R_0 t^4} - \frac{3(1 + \omega) \beta \gamma(12\sigma + R t^2)}{4R_0 t^2}
\]

\[
= -T_0 \sum_i \Omega_{\omega_i, 0} \frac{1}{\nu},
\]

where \(\nu \neq 0\). By inserting the value of \(\mu\) in Eq. (33), we get

\[
\nu = \left( \frac{R}{R_0} \right)^\beta \left( \frac{T}{T_0} \right)^\gamma \left[ \frac{1}{1 - 3\omega} \right] - \frac{1}{2} + \frac{\beta(12\sigma - R t^2)}{4R t^2} - \frac{12\sigma \beta(\beta - 1)(12\sigma + R t^2)}{R^2 t^4} - \frac{3(1 + \omega) \beta \gamma(12\sigma + R t^2)}{4R t^2}
\]

Since \(\nu\) is constant, therefore all time dependent terms must vanish. It is significant to mention here that there is no possible choice of parameters which make it constant and hence satisfies the Friedman’s equation. Hence, this power law model does not describe the symmetric bouncing cosmology.

**B. Oscillatory Bouncing Model**

Here we shall reconstruct the form of \(F(\mathcal{R}, T)\) function by using oscillatory bouncing model which is defined by the following scale factor [48, 49]

\[
a(t) = A \sin^2(B \frac{t}{t_*}),
\]

where \(t_* > 0\) represents the reference time while \(A\) and \(B\) are positive constants. For such model, the Hubble parameter is defined as

\[
H = \frac{2B}{t_*} \cot(B \frac{t}{t_*}),
\]

This model results into two types of bouncing behavior: firstly \(t = \frac{\pi t}{B}\), \(n \in \mathbb{Z}\) implies that \(a(t) = 0, H = -\infty\) or \(\infty\) which corresponds to super bounce and secondly, for \(t = \frac{2(\pi + 1)t}{2B}\), \(n \in \mathbb{Z}\), we obtain: \(a(t) = A, H = 0\) which
indicates the simple bounce where universe comes to its maximum size. Using this form of scale factor along with Hubble parameter in Eq. (8), we can re-write the Friedman equation as follows

\[ k^2 \rho + (1 + \omega) \rho F_T + \frac{F}{2} - \left(\frac{R}{6} + c_2\right) F_R + 6\left(\frac{R}{3} - 2a_2\right)(-\frac{R}{3} + a_2)FRR + 9T(1 + \omega)(-\frac{R}{9} + a_2)FR_T = 0, \]  

(35)

where \(a_2 = 4 \left(\frac{R}{6}\right)^2\). Again it is checked that the solution of above equation is not possible to find, so for simplicity reasons, we use specific 7 forms of \(F(R, T)\) function as listed before.

Applying (i) form of Lagrangian function, the above equation can be easily separated into two ODEs for unknowns \(g\) and \(h\) whose solutions are given by

\[
g(R) = \frac{1}{243a_2^{5/2}\sqrt{-3a_2+R}} \left[ -2(-3a_2 + R)^2 \text{ArcTanh} \left( \frac{\sqrt{-3a_2 + R}}{3\sqrt{a_2}} \right) c_2 + 3\sqrt{a_2} (729a_4^2c_1 - 486a_2^{-1/2}R c_1 + 81a_2^{-1} \sqrt{-3a_2 + R} c_2) \right],
\]

(36)

while the solution for \(h(T)\) is same as given in Eq. (24). Thus, the form of \(F(R, T)\) function is given by

\[
F(R, T) = \frac{1}{243a_2^{5/2}\sqrt{-3a_2+R}} \left[ -2(-3a_2 + R)^2 \text{ArcTanh} \left( \frac{\sqrt{-3a_2 + R}}{3\sqrt{a_2}} \right) c_2 + 3\sqrt{a_2} (729a_4^2c_1 - 486a_2^{-1/2}R c_1 + 81a_2^{-1} \sqrt{-3a_2 + R} c_2) \right] - \frac{2k^2T}{1 - 3\omega} + T^{1/2 - \sqrt{\frac{1}{1+\beta c_1} - \frac{3a_2}{R}}} \beta_2 c_3 + T^{1/2 + \sqrt{\frac{1}{1+\beta c_1} - \frac{3a_2}{R}}} \beta_2 c_4.
\]

(37)

The vacuum limit \(T \to 0\) implies that \(h(0) = 0\), whereas in the limit \(R \to 0\), the function \(g(0) = 0\) is only possible when \(c_1 = \frac{18a_2^{-1}}{2187a_2^{-1/2}}\). Hence vacuum condition can be satisfied by imposing this constraint on constants \(c_1\) and \(c_2\).

Next we use (ii) of Lagrangian function in the Friedman equation and obtain second-order separable partial differential equation whose solution is given by

\[
F(R, T) = R + \frac{1}{486a_2^{3/2}\sqrt{-3a_2+R}} \left[ 2(-3a_2 + R)^2 \text{ArcTanh} \left( \frac{\sqrt{-3a_2 + R}}{3\sqrt{a_2}} \right) (9a_2^2 - 2\beta_1 c_2) + 81a_2^{-1} \right]
\]

\[
\times (-3a_2 + R)^2 \ln(-3\sqrt{a_2} - \sqrt{-3a_2 + R}) + 3\sqrt{a_2} (2(729a_4^2\beta_1 c_1 - 486a_2^{-1/2}R \beta_1 c_1 - 9(a_2)^2) c_2)
\]

\[
\times (\sqrt{-3a_2 + R} - 9\beta_1 c_1) + 2R \sqrt{-3a_2 + R} \beta_1 c_2) - 27a_2^{3/2}(-3a_2 + R)^2 \ln(3\sqrt{a_2} + \sqrt{-3a_2 + R})
\]

\[
- \frac{2k^2T}{1 - 3\omega} + T^{1/2 - \sqrt{\frac{1}{1+\beta c_1} - \frac{3a_2}{R}}} \beta_2 c_3 + T^{1/2 + \sqrt{\frac{1}{1+\beta c_1} - \frac{3a_2}{R}}} \beta_2 c_4 \right].
\]

(38)

In the vacuum limit: \(R \to 0\), we get \(g(0) = \frac{1}{486\sqrt{3a_2}}[18a_2^{-1}(u/\sqrt{3})(9a_2^2 - 2\beta_1 c_2) + 729a_4^2(\ln(-3\sqrt{a_2} + \sqrt{3a_2})) + 4374a_2^{3/2}c_1\beta_1] \neq 0\). Here, the vacuum condition is only satisfied when \(c_1 = \frac{1}{4374a_2^{3/2}}[18a_2^{-1}(u/\sqrt{3})(9a_2^2 - 2\beta_1 c_2) + 729a_4^2(\ln(-3\sqrt{a_2} + \sqrt{3a_2}))\). If we consider (iii) form of \(F(R, T)\) function, the Friedman equation along with constraint (11) yields

\[
s_2 T^2 (s_3 T^6 + s_4) F_{TT} + (s_5 T^6 + 3a_2) F(T) = -\ddot{a} T,
\]

(39)

where \(s_2 = \frac{2(1 + \omega)^2}{\omega - 1}\), \(s_3 = -\frac{10k^2}{1 - 3\omega} \frac{B^2}{(1 - 3\omega)^4} \), \(s_4 = \frac{144k^2}{(1 - 3\omega)} \frac{B^2}{(1 - 3\omega)^4} + a_2\), \(s_5 = \frac{3a_2}{(1 - 3\omega)^4} \), \(\ddot{a} = \frac{k^2}{1 - 3\omega}\) and \(c = \frac{1}{3 + 3\omega}\). Again, it is non-homogeneous ODE of second-order whose analytical solution is not possible, however its complementary solution
is given by

\[
F(T) = t \left( \frac{c_2 s_4 - c_2 s_4 (1 - 2a c)}{s_4} \right)^{3} \times \left( \frac{c_2 s_4 - c_2 s_4 (1 - 2a c)}{s_4} \right)^{4} \times c_2 s_4 - c_2 s_4 (1 - 2a c) s_4 \left( T^c \right)
\]

\[
C_1 \text{Hypergeometric}_2 F_1 \left[ -\sqrt{\frac{-12a c^2 s_2 s_4 + c^2 s_2^2 s_4}{2c^2 s_2 s_4}}, \sqrt{\frac{-12a c^2 s_2 s_4 + c^2 s_2^2 s_4}{2c^2 s_2 s_4}} \right]
\]

\[
1 - \frac{\sqrt{-12a c^2 s_2 s_4 + c^2 s_2^2 s_4}}{2c^2 s_2 s_4}, \frac{s_3 T^c}{s_4} + t \left( \frac{c_2 s_4 + c_2 s_4 (1 - 2a c)}{s_4} \right)^{3} \times \frac{c_2 s_4 + c_2 s_4 (1 - 2a c)}{s_4} \left( T^c \right)
\]

\[
C_1 \text{Hypergeometric}_2 F_1 \left[ -\sqrt{\frac{-12a c^2 s_2 s_4 + c^2 s_2^2 s_4}{2c^2 s_2 s_4}}, \sqrt{\frac{-12a c^2 s_2 s_4 + c^2 s_2^2 s_4}{2c^2 s_2 s_4}} \right]
\]

\[
\left( 1 - \frac{\sqrt{-12a c^2 s_2 s_4 + c^2 s_2^2 s_4}}{2c^2 s_2 s_4}, \frac{s_3 T^c}{s_4} \right)
\].

(40)

It is easy to check that in the limit \( T \to 0 \), the above solution results in \( F(0) = 0 \) but nothing can be inferred about general solution of \( F(T) \). Therefore, vacuum condition is not valid in this type of Lagrangian. Similarly, imposing \((iv)\) form of \( F(R, T) \) in Friedman equation, we obtain a new partial differential equation whose analytical solution is not possible.

In case of \((v)\) form of function \( F(R, T) \), the Friedman equation becomes

\[
\frac{k^2 T}{1 - 3\omega} + T(1 + \omega)\left( \frac{3\omega R}{1 - 3\omega} + \frac{12B^2}{t_*^2} \right) + \frac{\left( R - \frac{4B^2}{t_*^2} \right)}{3 - 3\omega} = 0.
\]

Using equation of continuity, we obtain a relation between \( R \) and \( T \) given by \( R = \frac{-36B^2}{t_*^2} \left( \frac{T}{(1 - 3\omega) t_*^2} \right)^{\frac{1}{1 - 3\omega}} + \frac{48B^2}{t_*^2} \). Inserting value of \( R \) and applying the constraint \((11)\), the above DE can also be expressed in terms of \( T \) and can be written as

\[
\frac{r_1 T^2 (r_2 T^c + r_3)}{r_4} - r_4 (r_5 T^c - 1) F(T) = r_4 (r_5 T^c - 1) - \frac{k^2 T}{1 - 3\omega},
\]

(42)

which is non-homogeneous DE and its general solution does not exist, but one can obtain the following complementary solution:

\[
F(T) = r_2 \left( \frac{r_1 T^2 - r_2 + r_3}{r_3} \right)^{-\frac{r_1 T^2 - r_2 + r_3}{r_3}} \cdot \left( T^c \right) \cdot \text{Hypergeometric}_2 F_1 \left[ -\sqrt{\frac{r_1 T^2 - r_2}{r_3}}, \sqrt{\frac{r_1 T^2 - r_2}{r_3}} \right]
\]

\[
1 - \frac{\sqrt{r_1 T^2 - r_2}}{r_3}, \frac{s_3 T^c}{s_4} + r_2 \left( \frac{r_1 T^2 - r_2 + r_3}{r_3} \right)^{-\frac{r_1 T^2 - r_2 + r_3}{r_3}} \cdot \text{Hypergeometric}_2 F_1 \left[ -\sqrt{\frac{r_1 T^2 - r_2}{r_3}}, \sqrt{\frac{r_1 T^2 - r_2}{r_3}} \right]
\]

\[
1 - \frac{\sqrt{r_1 T^2 - r_2}}{r_3}, \frac{s_3 T^c}{s_4} \right),
\]

(43)

where \( r_1 = \frac{2(1 + \omega)^2}{2 - 1} \), \( r_2 = \frac{-108c^2 B^2}{(1 - 3\omega) t_*^2} \), \( r_3 = \frac{6B^2}{(1 - 3\omega) t_*^2} \), \( r_4 = \frac{12B^2}{t_*^2} \), \( r_5 = \frac{2}{1 - 3\omega} \), \( c = \frac{1}{3 + 3\omega} \), while \( c_1 \) and \( c_2 \) are integrating constants. Since the analytical solution for \( F_{\text{part}}(T) \) is not obtained so vacuum condition is not satisfied. Similarly, inserting the form \((vi)\) of generic function in Friedman equation, we obtain a new partial differential equation whose analytic solution is impossible to find.

Taking \((viii)\) power law type of Lagrangian function, the Friedman’s equation becomes

\[
\mu \left( \frac{R}{R_0} \right)^{\frac{\beta}{(T/T_0)}} \left( T/T_0 \right)^{\gamma} \left( 1 + \omega \right) \left( 1 - 3\omega \right)^{\gamma} + \frac{\beta(24B^2 + R_0^2 t_0^2)}{6R_0^2 t_0^2} + \frac{6\beta(\beta - 1)(12B^2 - R_0^2 t_0^2)}{18R_0^2 t_0^2} + \frac{9(1 + \omega) c^2 (24B^2 - R_0^2 t_0^2)}{6R_0^2 t_0^2}
\]

\[
\times (24B^2 - R_0^2 t_0^2) \right] = -T_0 \sum_i \Omega_{\omega_i,0} a^{-3(1 + \omega_i)},
\]

In this case, the vacuum condition is satisfied if \( \gamma > 0 \). At times when \( R = T = 0 \), the Friedman’s equation yields the condition: \( \sum_i \Omega_{\omega_i,0} A^{-3(1 + \omega_i)} = 0 \), which is possible in vacuum. Then, the value of \( \mu \) at the current time is given by

\[
\mu = \frac{1 - \omega}{(1 - 3\omega)^{\gamma} + 1} \left( \frac{1}{2} \beta(24B^2 + R_0^2 t_0^2) \right) \frac{6R_0^2 t_0^2}{18R_0^2 t_0^2} + \frac{6\beta(\beta - 1)(12B^2 - R_0^2 t_0^2)}{18R_0^2 t_0^2} + \frac{9(1 + \omega) c^2 (24B^2 - R_0^2 t_0^2)}{6R_0^2 t_0^2}
\]

\[
\times (24B^2 - R_0^2 t_0^2) \right] = -T_0 \sum_i \Omega_{\omega_i,0} a^{-3(1 + \omega_i)},
\]
where \( \nu \neq 0 \) and \( \omega_i = -1, \forall i \). By inserting the value of \( \mu \) in the above Friedman's equation, we obtain:

\[
\nu = \left( \frac{R}{R_0} \right)^\beta \left( \frac{T}{T_0} \right)^\gamma \left[ \frac{1 + \omega}{1 - 3\omega} \right]^{-1/2} + \frac{\beta(24B^2 + Rt_s^2)}{6Rt_s^2} + \frac{6\beta(\beta - 1)(12B^2 - R_t^2)(R_t^2 - 4B^2)}{18R^2t_s^4} + \frac{9(1 + \omega)\beta\gamma(24B^2 - R_t^2)}{6Rt_s^2}.
\]

Since \( \nu \) is a constant, therefore all time dependent terms must vanish. Clearly there is no possible choice of parameters which can make it constant and satisfy the Friedman's equation. Hence, it can be concluded that this power law model does not describe the oscillatory bouncing cosmology.

### C. Power Law Model

Here we shall reconstruct the form of generic function by taking power law bouncing model into account. It is defined by the following scale factor:

\[
a(t) = \left( \frac{t_s - t}{t_0} \right)^{\frac{2}{\alpha}},
\]

where \( t_s \) represents the bouncing point, \( t_0 \) is the arbitrary time and \( c \) is an arbitrary constant. Here, by re-scaling \( t_1 = t_s - t \) and \( \alpha = \frac{2}{a} \), we can re-write above scale factor as follows

\[
a(t_1) = \left( \frac{t_1}{t_0} \right)^\alpha \rightarrow H = -\frac{\alpha}{t_1} \rightarrow R = 6\alpha(1 - 2\alpha)t_1^2.
\]

Here \( \alpha > 0 \). Using this scale factor along with Hubble parameter in 00 component of field equation, we obtain

\[
k^2\rho + (1 + \omega)\rho F_T + \frac{F}{2} + \frac{1 - \alpha}{2(2\alpha - 1)} R F_R + \frac{1}{1 - 2\alpha} R^2 F_{RR} + \frac{3\alpha(1 + \omega)}{2(1 - 2\alpha)} R T F_{RT} = 0.
\]

Since this equation is difficult to solve for unknown, therefore we impose \( F(R, T) \) model (i) to find its solution. The corresponding DEs can be written as

\[
\beta_1 R^2 F_{RR} + \beta_1 \left( \frac{\alpha - 1}{2} \right) R F_R + \beta_1 \frac{1 - 2\alpha}{2} F(R) = 0,
\]

which is Euler's equation whose solution is given by

\[
F(R) = w_1 R^{\mu_+} + w_2 R^{\mu_-},
\]

where \( \mu_\pm = \frac{3 - 7\alpha + 2\alpha^2}{4(1 - 2\alpha)} \pm \sqrt{1 + \alpha(10 + \alpha)} \) and further, by using perfect fluid with \( \omega = \frac{\rho}{\rho} \) in the DE for unknown \( h(T) \) which implies

\[
F(R, T) = \beta_1 w_1 R^{\mu_+} + \beta_1 w_2 R^{\mu_-} + \frac{2k^2}{-1 + 3\omega} T + T^{1/2 - \sqrt{1 + \alpha(10 + \alpha)}} \beta_2 w_3 + T^{1/2 + \sqrt{1 + \alpha(10 + \alpha)}} \beta_2 w_4.
\]

In the vacuum limit: \( R \rightarrow 0, T \rightarrow 0 \), the condition \( F(0, 0) = 0 \) trivially holds.

Next let us assume the Lagrangian of type (ii), and consequently, the Friedman equation becomes separable for unknowns \( g(R) \) and \( h(T) \) which yield following solution:

\[
F(R, T) = R^{1/4(3 - \alpha - \sqrt{1 + \alpha(10 + \alpha)}(1 - 2\alpha))} \beta_1 c_1 + R^{1/4(3 - \alpha + \sqrt{1 + \alpha(10 + \alpha)}(1 - 2\alpha))} \beta_1 c_2 + \beta_2 \left[ -\frac{2k^2 T}{\beta_2 - 3\beta_2 \omega} + T^{1/2 - \sqrt{1 + \omega(10 + \alpha)} - \sqrt{1 + \alpha(10 + \alpha)}} \beta_2 c_3 + T^{1/2 + \sqrt{1 + \omega(10 + \alpha)} + \sqrt{1 + \alpha(10 + \alpha)}} \beta_2 c_4 \right].
\]

It is interesting to mention here that the vacuum condition also holds trivially in this case. Applying (iii) form of generic function in first field equation and using constraint \( \frac{d^2 A_1}{dT^2} \), we obtain the following DE:

\[
A_1 T^2 h_{TT} + A_2 h(T) = -A_3 T^D,
\]
where $A_1 = 2\left(\frac{1}{1+\omega} + \frac{3\omega}{2(1-2\omega)}\right)^{1+\omega}$, $A_2 = \frac{3\omega}{2(1-2\omega)}$, $A_3 = \frac{k^2\rho_0(1-3\omega)\omega}{6\alpha(1-2\omega)}$ and $D = 1 - d$, where $d = \frac{2}{3\alpha(1+\omega)}$. This is non-homogeneous ODE whose solution can be written as

$$h(T) = A_1\frac{T^{2(2-D)}(1+\frac{1}{2(2-D)})}{\Gamma(2-D)} A_2 \frac{T^{2(2-D)}(1+\frac{1}{2(2-D)})}{\Gamma(2-D)} A_3 \frac{T^{2(2-D)}(1+\frac{1}{2(2-D)})}{\Gamma(2-D)} (2+D)^{(2+D)-1+\frac{1}{2(2-D)}}$$

$$\times \left(4A_1A_3 + 4A_1A_3D + A_1A_3D^2\right) T^{2-D} (2+D)^{-1+\frac{1}{2(2-D)}} BesselJ\left[-\frac{1}{D} \frac{2\sqrt{A_2(T(2+D))^}\frac{T}{2(2-D)}}{\sqrt{4A_1A_3 + 4A_1A_3D + A_1A_3D^2(1+\frac{2}{2-D})}}\right] c_2\Gamma_1\left[-\frac{1}{D} \frac{2\sqrt{A_2(T(2+D))^}\frac{T}{2(2-D)}}{\sqrt{4A_1A_3 + 4A_1A_3D + A_1A_3D^2(1+\frac{2}{2-D})}}\right]$$

$$\times c_1\Gamma_1\left[1 + \frac{1}{D}\right], \quad (48)$$

with $c_1$ and $c_2$ as integrating constants. Applying vacuum limit in $h(T)$ yields zero which shows that vacuum condition is satisfied in this case. Applying (vi) form of generic function, the first Friedman equation yields

$$\frac{k^2T}{1-3\omega} + \frac{3-\omega}{2(1-3\omega)} TF(R) + \frac{R\alpha}{2(-1+2\alpha)} + \frac{(-1+4\alpha + 3\alpha\omega)\beta T}{(2-1+2\alpha)} - \frac{R^2T}{(1-2\alpha)} F_{RR} = 0. \quad (49)$$

From the equation of continuity, we get $T = (\rho_0 - 3\omega) [\frac{R}{6\alpha(1-2\alpha)}]^{\frac{3\omega(1+\omega)}{2(1-2\omega)}}$, and further by using this relation, we can re-write the first Friedman equation as follows

$$k_1 + k_2 f(R) - k_3 R^k_4 + k_5 R F_R + k_6 R^2 F_{RR} = 0, \quad (50)$$

whose solution is given by

$$F(R) = \frac{k_1}{k_2} + \frac{k_3 R^k_4}{k_2 + k_4(k_5 + (-1+k_4)k_6)} + \frac{R}{\sqrt{((k_5-k_4)k_6+k_2)^{(k_2)}}} c_1 + R \frac{\sqrt{((k_5-k_4)k_6+k_2)^{(k_2)}}}{k_2} c_2,$$

where $k_1 = \frac{k^2}{2}$, $k_2 = \frac{3-\omega}{2(1-3\omega)}$, $k_3 = \sqrt{\frac{3\omega(1+\omega)}{2\rho_0(1-1-2\omega)}}$, $k_4 = 1 - 3/2\alpha(1+\omega)$, $k_5 = \frac{3\omega(1-3\omega)}{2\alpha(1-2\omega)}$, $k_6 = 1/(1-2\alpha)$ while $c_1$ and $c_2$ are integrating constants. Thus we get

$$F(R, T) = R + T \left[\frac{k_1}{k_2} + \frac{k_3 R^k_4}{k_2 + k_4(k_5 + (-1+k_4)k_6)} \frac{R}{\sqrt{((k_5-k_4)k_6+k_2)^{(k_2)}}} c_1 + R \frac{\sqrt{((k_5-k_4)k_6+k_2)^{(k_2)}}}{k_2} c_2\right].$$

The vacuum limit $R \rightarrow 0$ implies that $g(0) = \frac{2k^2(3\omega-1)}{1+\omega(3-\omega)}$. Consequently, it is concluded that the vacuum condition is only possible when $\omega = 1/3$. Next we impose (vi) form of Lagrangian function in the Friedman equation which turns out as

$$\frac{k^2\rho}{R} + Tconsth_T + const_2(1 + h(T)) = 0. \quad (52)$$

Here by using equation of continuity, the relation: $R = 6\alpha(1-2\alpha) [\frac{T}{\rho_0(1-3\omega)}]^{2/3\omega(1+\omega)}$ and constraint (11), the above equation results into a DE whose solution is given by

$$F(R, T) = R + T \left[\frac{1}{-2A_2 A_3 + A_2^2 A_3 + A_1 T^{2/3\omega} \left(\frac{\sqrt{A_1 T^{2/3\omega}}}{\sqrt{A_1 T^{2/3\omega}}} \left(-A_1 T^{2/3\omega} + \sqrt{\frac{\sqrt{A_1 T^{2/3\omega}} A_3 - 4A_3}{\sqrt{A_1 T^{2/3\omega}} C_2}}\right) \right] + A_4 \left(\frac{T}{\sqrt{A_1 T^{2/3\omega}} + \sqrt{TC_1 + T^{1/2+\sqrt{A_1 T^{2/3\omega}}}} C_2})\right)\right].$$

\( (53) \)
Here $A_1 = \frac{K^2 \beta^{3/2(1+\omega)}}{(1-3\omega)(1-3\omega)}$, $A_2 = 1 - 2/3\alpha (1 + \omega)$, $A_3 = \frac{2(1+\omega)}{(w-1)}$ and $A_4 = const_2 = \frac{-\alpha}{2(1-2\alpha)}$. Also, $const_1 = \frac{13}{22} + \frac{3\alpha(1+\omega)}{2(1-2\alpha)}$. Here, the vacuum condition is trivially satisfied in the limit of $R \to 0$, $T \to 0$.

Now using (vi) form of function in the Friedman’s equation, we obtain the following DE:

$$
\begin{align*}
 k^2 \rho + (1 + \omega) \rho F_2(R) F_{2T} + & \frac{F_1(R) + F_2(R) F_3(T)}{2} + \frac{1 - \alpha}{2(2\alpha - 1)} R (F_1R + F_2R F_3(T)) \\
 + & \frac{1}{1 - 2\alpha} R^2 (F_1R + F_2R R F_3(T)) + \frac{3\alpha(1 + \omega) \beta \gamma}{2(1 - 2\alpha)} R T F_2 R F_{3T} = 0.
\end{align*}
$$

(54)

It can be transformed into set of two DEs whose solution is not possible to find. Applying (viii) form of Lagrangian function in the Friedman’s equation, we obtain

$$
\mu \left( \frac{R}{R_0} \right)^{\beta} \left( \frac{T}{T_0} \right)^{\gamma} \sum_i \Omega_{\omega_i,0} = \sum_i \Omega_{\omega_i,0} \left( \frac{R}{R_0} \right)^{\frac{3}{2}(1+\omega_i)}.
$$

The above expression is satisfied when the time dependent terms and their powers will be canceled. It further leads to the conditions, i.e., $\gamma = 0$ and $3\alpha(1 + \omega) = 2\beta$, $\forall i$, where the second condition will be satisfied when a fluid is present. Also, since $\alpha > 0$, the EoS parameter must be greater than $-1$, i.e., $\omega > -1$. Hence this Lagrangian can describe the super-bounce cosmology.

D. Matter Bounce Model

The matter bounce model is defined by the following scale factor $^{32, 36, 48, 49, 57}$

$$
a(t) = A \left( \frac{3}{\rho_{cr}} t^2 + 1 \right)^{\frac{1}{2}},
$$

where $\rho_{cr}$ stands for critical density and $A$ is an arbitrary positive constant. Also, the corresponding Hubble parameter and Ricci scalar are given by

$$
H = \frac{2t \rho_{cr}}{2 + 3\rho_{cr} t^2}, \quad R = \frac{-12 \rho_{cr} (2 + \rho_{cr} t^2)}{(2 + 3\rho_{cr} t^2)^2}.
$$

(55)

Here, bouncing point is $t = 0 \Rightarrow H = 0$. Since Eq. (53) is partial differential equation whose solution is difficult to find, therefore, one can consider the forms of generic function as considered in the last sections. By assuming (i) form of Lagrangian function, we obtain two DEs which are given by

$$
k^2 T + (1 + \omega) \beta_2 T h_T + \frac{\beta_2 (1 - 3\omega) g(T)}{2} = 0,
$$

$$
g(R) + \frac{3(H + H^2) g_R - 3H (R g_{RR})}{2} = 0.
$$

Here the first DE is similar to Eq. (22) whose solution is given in Eq. (24). By inserting values of $H$, $\dot{H}$ and $R$ in the second DE, we obtain

$$
g(t) - \frac{(2 + 3t^2 \rho_{cr})(-20 + 68t^2 \rho_{cr} + 15t^4 (\rho_{cr})^2)}{t \rho_{cr} (10 + 3t^2 \rho_{cr})^2} \ddot{g}(t) - \frac{(2 + 3t^2 \rho_{cr})^2}{2 \rho_{cr} (10 + 3t^2 \rho_{cr})} \dddot{g}(t) = 0,
$$

(56)
which can be solved by Frobenius method as follows

$$g(t) = \sum_{n=0}^{\infty} a_n t^{n+r}.$$ 

For $r = 3$ and $r = 0$, the following recurrence relation can be obtained

$$40(n + r + 1)(n + r - 2) = -4\rho_cr_{n-1}(33(n + r - 1)^2 - 50) - 6\rho^2_cr_{n-3}(n + r - 3)(21(n + r - 3) + 15\rho_cr + 57) + 20) - 9a_{n-5}\rho^3_c(3(n + r - 5)(n + r - 6) + 2).$$

with $a_{-1} = a_{-3} = a_{-5} = 0$ and consequently, we get the following solution:

$$g(t) = c_1 \left[ t^3 - \frac{131}{50}\rho_cr_t^5 + \frac{\rho^2_c(2188 + 135\rho_c r)}{560}t^7 + \ldots + \infty \right] + c_2 \left[ 160 \ln(t) - 80\rho_cr(2 + 5\ln(t))t^2 + ((339 + 350\ln(t))\rho^2_c + 90\rho^3_c)t^4 + \ldots + \infty \right],$$

where $t = \pm \sqrt{\frac{\sqrt{1 - \frac{1}{\rho_c r}} + \sqrt{(4\rho^2_c \rho - \rho_c)^2 - 4\rho^4_c r^2}}{\rho^2_c r^2}}$, which yields the $g(R)$. Now to check vacuum condition, we put vacuum limit $T \to 0$, $R \to 0$ which implies that $h(0) = 0$ while $g(0)$ diverges. Therefore, vacuum condition is not satisfied for this solution.

Similarly, by applying $(ii)$ form of Lagrangian function in the first Friedman’s equation, we obtain same DEs with an extra term, i.e., $g(R) = -\frac{4\rho^2_c}{3}$, whose complementary solution is identical to the previous one but general solutions is not possible. Therefore, nothing can be concluded about $g_{part}(0) = 0$ and thus vacuum condition does not hold and also $g_{horn}(0)$ diverges.

Further we insert $(iii)$ form of generic function in the first Friedman’s equation and by using values of $a(t)$, $H$ and $R$, we obtain

$$\frac{24\rho^2_c t^2}{(2 + 3t^2\rho_c)}g(t) + \frac{2(2 + \rho_c t^2(-2 + 9\omega))}{(3\omega - 1)(2 + 3t^2\rho_c)}g(t) = k^2\rho_0 \left( A \left( 1 + 3t^2\rho_c \right)^{1/3} \right)^{-3(1+\omega)}r^2,$$

whose solution is given by

$$g(t) = C_1 \exp \left( \frac{2(1 - 3\omega)(9\omega - 2) \ln(2 + 3t^2\rho_c) - 3 \ln(2 + (9\omega - 2)t^2\rho_c))}{(9\omega - 5)(9\omega - 2)} \right) + \frac{1}{A^3(7 - 15\omega + \omega(-5 + 9\omega))\rho_c}$$

$$2^{\omega - \frac{2(7 + 3\omega + 8\omega^2)}{(-5 + 9\omega) - 2 + 9\omega}} \exp \left( \frac{2(1 - 3\omega)(9\omega - 2) \ln(2 + 3t^2\rho_c) - 3 \ln(2 + (9\omega - 2)t^2\rho_c))}{(9\omega - 5)(9\omega - 2)} \right) k^2(3\omega - 1)$$

$$Hypergeometric2F1 \left[ \frac{7 - 20\omega + 9\omega^2}{5 - 9\omega}, 1 + \frac{6(-1 + 3\omega)}{(-5 + 9\omega)(-2 + 9\omega)}, \frac{12 - 29\omega + 9\omega^2}{5 - 9\omega}, \frac{(-2 + 9\omega)(2 + 3t^2\rho_c)}{2(9\omega - 5)} \right] \rho_0,$$

$$(A^3(2 + 3t^2\rho_c))^{\omega - (2 + 3t^2\rho_c)} \frac{\tilde{\omega}^{\frac{7 - 15\omega}{63\omega}}}{\omega - 5} \frac{-6 + 3t^2(2 - 9\omega)\rho_c}{9\omega - 5} \left( 2 + t^2(-2 + 9\omega)\rho_c \right) \frac{6(1 + 3\omega)}{10 - 63\omega + 81\omega^2}$$

where $2 + 3t^2\rho_c = \frac{2}{\rho_0} \left( \frac{T}{\rho_0(1 - 3\omega)} \right)^{\frac{-1}{1+\omega}}$, which yields the following solution for $g(T)$

$$g(T) = \exp \left( - \frac{2(3\omega - 1)(\ln(27) + (9\omega - 2) \ln(\frac{2(1 - 3\omega)}{A^3})^{\frac{1}{1+\omega}}) - 3 \ln(6 + (9\omega - 2)(\frac{2(1 - 3\omega)}{A^3})^{\frac{1}{1+\omega}} - 2)}{(9\omega - 5)(-2 + 9\omega)} \right)$$

$$\times \left[ C_1 + \frac{1}{A^3(\omega(9\omega - 20) + 7)r_\omega} \frac{8\pi^2}{9\omega} k^2(3\omega - 1) Hypergeometric2F1 \left[ -\omega + \frac{4}{3} \frac{6(3\omega - 1)}{81\omega^2 - 63\omega + 10} \right] \rho_0 \left( A^3(9\omega - 5) \right) \right]$$

$$\times \left[ 1 + \frac{4}{3} \frac{1}{9\omega - 5} + 2 - \omega, \frac{(9\omega - 2)(\frac{T}{\rho_0 - 3\rho_0\omega})^{\frac{-1}{1+\omega}}}{A^3(9\omega - 5)} \rho_0(A^3(9\omega - 5)) \left( \frac{T}{\rho_0 - 3\rho_0\omega} \right)^{\frac{-1}{1+\omega}} \right]$$

$$\times \left[ \frac{6(9\omega - 2)(\frac{T}{\rho_0 - 3\rho_0\omega})^{\frac{-1}{1+\omega}}}{A^3(9\omega - 5)} \left( \frac{6(9\omega - 1)}{81\omega^2 - 63\omega + 10} \right) \right] \left( \frac{9\omega - 2}{(2(\frac{T}{\rho_0 - 3\rho_0\omega})^{\frac{-1}{1+\omega}} - 2) + 6} \right).$$
The vacuum limit $T \to 0$ implies that $g(o)$ diverges, so the vacuum condition does not hold in this case. By applying $(vi)$ of Lagrangian function in the first Friedman’s equation, we obtain

$$k p_0 \left( A^{3/2 \rho_{ct}} \right) - 3(\omega - 1) - g(t) \left( \frac{6t \rho_{ct}}{3t^2 \rho_{ct} + 2} - \frac{2(t^2 \rho_{ct} + 2)}{t(1 - 3\omega)(3t^2 \rho_{ct} + 2)} \right) - \frac{12t^2 \rho_{ct}^2 g(t)}{(3t^2 \rho_{ct} + 2)^2} = 0,$$

whose analytical solution is given by

$$g(t) = u \left[ c_1 + \frac{(3\omega - 1)(A^{3/2 \rho_{ct}} + 2)}{4A^3(5 - 9\omega)^2} \left( 1 + \frac{2}{81 \omega^2 - 63 \omega + 10} \right) \rho_{ct} - \frac{2(9\omega^2 - 17\omega + 6)}{3(5 - 9\omega)(5 - 9\omega)} \right] \left( 9\omega^2 - 17\omega + 6 \right) \rho_{ct} - \frac{2}{81 \omega^2 - 63 \omega + 10} \left( 1 + \frac{2}{81 \omega^2 - 63 \omega + 10} \right) \right] \left( 9\omega - 5 \right) \rho_{ct} - \frac{2}{81 \omega^2 - 63 \omega + 10} \left( 1 + \frac{2}{81 \omega^2 - 63 \omega + 10} \right) \right],$$

where

$$u = \exp \left( \frac{1}{9\omega^2 - 17\omega + 6} \left( 9\omega^2 - 3 \left( \frac{9\omega^2 - 26\omega + 11}{5 - 9\omega} \right) + 1 \right)(9\omega - 2) \right),$$

$$u_1 = 2F_1 \left( \frac{9\omega^2 - 3 \left( \frac{9\omega^2 - 26\omega + 11}{5 - 9\omega} \right) + 1}{5 - 9\omega} ; 9\omega - 5 \right) \left( 9\omega - 2 \right)(5 - 9\omega) \left( 9\omega - 2 \right)(3p_{ct}t^2 + 2) \right),$$

$$u_2 = 2F_1 \left( \frac{6 - 12\omega}{5 - 9\omega} ; 9\omega - 5 \right) \left( 9\omega - 2 \right)(5 - 9\omega) \left( 9\omega - 2 \right)(3p_{ct}t^2 + 2) \right),$$

$$u_3 = 2F_1 \left( \frac{4 - 6\omega}{5 - 9\omega} ; 5 - 9\omega \right) \left( 81 \omega^2 - 63 \omega + 10 \right) \left( 9\omega^2 - 17\omega + 6 \right) \left( 1 + \frac{2}{81 \omega^2 - 63 \omega + 10} \right) \right].$$

By inserting $2 + 3\rho_{ct}$ in terms of $T$ yields $g(T)$. In the vacuum limit $T \to 0$, $g(0) \to \infty$. Thus, vacuum condition is not satisfied. Similarly, by inserting $(iv)$, $(v)$, $(vii)$ and $(viii)$ forms of Lagrangian function in the first Friedman’s equation, we obtain very complicated DEs whose analytical solutions are not possible to find therefore, nothing can be inferred about vacuum condition in these cases.

### IV. ENERGY CONDITIONS

The energy conditions are some well-known constraints which are based on energy-momentum tensor and possess some physical features used to examine the physical consistency of cosmic models. These can be categorized into four types namely: the null, weak, strong and dominant energy conditions. In $F(R, T)$ gravity, the energy conditions for effective fluid can be defined as

- NEC: $\rho^{\text{eff}} + p^{\text{eff}} \geq 0$,
- WEC: $\rho^{\text{eff}} \geq 0$, $\rho^{\text{eff}} + p^{\text{eff}} \geq 0$,
- SEC: $\rho^{\text{eff}} \geq 0$, $\rho^{\text{eff}} + 3p^{\text{eff}} \geq 0$,
- DEC: $p^{\text{eff}} \geq 0$, $\rho^{\text{eff}} - p^{\text{eff}} \geq 0$.

Here, we will discuss the graphical behavior of these energy conditions for all reconstructed bouncing models of previous section. The plots of energy constraints for all bouncing models with $(i)$ form of Lagrangian are shown in Figure 2. From the graphs, it is clear that NEC and SEC violate for exponential and power law models when
consistent with solution (60), then following equation of motion and bouncing solutions. For this purpose, let us suppose a solution field equations and continuity equation using FRW universe model for exponential, oscillatory, power law and matter compatible with the background field equations which further leads to form of Ricci scalar given by

\[ R = \frac{1}{2} (1 + 3\alpha) \frac{\dot{H}}{H^2} + \frac{1}{2} \frac{\dot{H}^2}{H^3} + \frac{1}{3} \frac{\dot{H}^2}{H^4} \]

It is seen that all energy constraints are valid in this case. In case of oscillatory and power law models, NEC, WEC and SEC are valid for \( \alpha = 2/3 \) while DEC violates. Similarly, we analyze the energy conditions for these bouncing models for (ii) form of generic function and similar behavior can be obtained. In case of exponential and oscillatory models with other ansatz forms of function \( F(R, T) \), we cannot analyze their energy constraints as analytical solution was not possible in these cases. Since we have obtained analytical solutions for other forms of function in case of power law, therefore, we have analyzed all energy constraints for this model with (v) and (vi) forms which are given in by the graphs of Figure 3. It is observed that for accelerating universe (\( \alpha = 2 \)) all energy conditions are valid. In case of dust dominated era (\( \alpha = 2/3 \)) for (v) form of function, it is seen that NEC and SEC are violated while DEC and WEC are valid. For the reconstructed function using (vi) form, graphs indicate that WEC, NEC and DEC may violate near bouncing point. It is worthy to mention here that for (vi) and (iii) forms of reconstructed function \( F(R, T) \), imaginary solutions are obtained and hence graphical analysis of energy constraints is not possible. For matter bounce model, we have obtained analytical solutions for (i), (iii) and (vi) ansatz forms but it is found that for (i) and (vi) solutions, energy constraints involve imaginary terms and hence cannot be discussed graphically. While for (iii) Lagrangian, energy constraints are shown in Figure 3 where it is seen that all energy constraints are valid in this case.

V. PERTURBATIONS AND STABILITY

In this section, we shall examine the stability of reconstructed bouncing solutions by applying linear and homogeneous perturbations in \( F(R, T) \) gravitational framework. In this respect, we find the corresponding perturbed field equations and continuity equation using FRW universe model for exponential, oscillatory, power law and matter bouncing solutions. For this purpose, let us suppose a solution

\[ H(t) = H_*(t) \]

compatible with the background field equations which further leads to form of Ricci scalar given by \( R_* = -6H_*(H'_* + 2H_*^2) \), where prime indicates derivative with respect to e-folding parameter. For a particular \( F(R, T) \) model which is consistent with solution (60), then following equation of motion and \( \nabla^i T_{ij} \neq 0 \) must hold:

\[
k^2 \rho_* + (1 + \omega)\rho_* F^*_{\tau} + \frac{F^*}{2} + 3(H_*, H'_* + H_*^2)F_{R_\tau} + 3H_*[(6H_*^2H''_* + H_*H''_* + 4H_*^2H'_*)F_{R_{RR}} - H_*T'_*F_{TT}] = 0, \]

\[
\rho'_* + 3(1 + \omega)\rho_* = -\frac{1}{k^2 + F^*_{\tau}} \left[ (1 + \omega)\rho_* F^*_{T\tau} + \omega \rho F^*_{\tau} + \frac{1}{2} T'_* F^*_{TT} \right].
\]
where $F^*$ indicates the function corresponding to solution (60). In case the usual conservation law holds (generally it does not remain valid $F(R, T)$ gravity), we find $\rho$ in terms of $H_*(t)$ as follows

$$\rho_*(t) = \rho_0 e^{(-3(1+\omega) \int H_*(t)dt)} = \rho_0 e^{(-3(1+\omega)N)}.$$ 

Further, the perturbed $H(t)$ and $\rho(t)$ can be defined as

$$H(t) = H_*(t)(1 + \delta(t)), \quad \rho(t) = \rho_*(t)(1 + \delta_m(t)),$$

where $\delta_m(t)$ and $\delta(t)$ stand for perturbation functions. In order to analyze introduced perturbations about the solution (60), let us expand $F(R, T)$ function as a series in terms of Ricci scalar and energy-momentum tensor trace given by

$$F(R, T) = F^* + F_0^*(R - R_*) + F_0^*(T - T_*) + \sigma^2.$$ 

(64)

Here only the linear terms are to be considered for further calculations while $\sigma^2$, denoting the quadratic or higher power terms of $R$ and $T$, will be ignored. Using Eqs. (63) and (64) in FRW equation, we obtain the following perturbed field equation:

$$b_2 \delta'' + b_1 \delta' + b_0 \delta = c_{m1}\delta_m + c_{m2}\delta'_m,$$

(65)

where $b_i$’s and $c_{mi}$’s are listed in Appendix A. Applying above perturbations in Eq. (6), we obtain the perturbed form of continuity equation as

$$d_1 \delta_m' + d_2 \delta_m + d_3 \delta' + d_4 \delta = 0,$$

(66)

where $d_i$’s are also defined in Appendix A. If the usual conservation law holds, then Eq. (65) reduces to

$$\delta''_m + 3\delta = 0.$$ 

(67)

These perturbed Eqs. (65) and (66) will play the role of key equations in analyzing the stability of FRW model. For $F(R, T) = F_1(R) + F_2(T)$ type models, these perturbed equations reduce to

$$b'_2 \delta'' + b'_1 \delta' + b'_0 \delta = c_{m1}\delta_m,$$

(68)

where $b'_i$’s and $d'_i$’s, $i = 0, 1, 2, 3$ are given in Appendix A. In following sections, we present the stability of exponential, oscillatory, matter bounce and power law models for this form of generic function.

### A. Stability of Exponential Solutions

Consider the exponential evaluation solution with dust matter, the perturbed equation (64) takes the form

$$18H^4F_{RR} \delta'' + \{ 6\rho_0H^2F_{RRT} + (36H^2H'_* + 54H^4)F_{RR} - 108H^3(H^2H''_* + H_1H'_* + 4H^2H'_*)F_{RRR} + 18H^4\rho_0F_{HTT} \} \delta'$$

$$+ \{ -6H^2f'_R - (6\rho_0H_1H'_* + 4H^2)F_{RT} + 18(2H^3H'_* + H^2H'_* + 7H^2H'_* - 4H_1^2)F_{RRT} - 108(H^3H''_* + H^2H'_*)F_{RR}^{-1} + 4H^3H'_*(H_1H'_* + 4H^2)^2F_{RRR}$$

$$+ 18(H^3H'_*)\rho_0F_{RRR} + 18(H^3H'_*)\rho_0F_{RRT} \delta + \{ k^2\rho_0 + \frac{3\rho_0f'_T^2}{2} + \rho_0F_{TT}^2 + (\rho_0(3H_1H'_* + H^2)^2)$$

$$- 3H^2\rho_0F_{TT}^2 + 18\rho_0(H^3H''_* + H^2H'_* + 4H^3H'_*)F_{RRR} - 3H^2\rho_0\rho_0F_{RRT} \} \delta_m - 3H^2F_{RRT} \delta'_m = 0.$$ 

We consider Eq. (67) to examine stability and $F(R, T)$ function which are reconstructed by utilizing the constraint (11). To solve this equation and Eq. (67), we use numerical technique for model (23). For this model, above equation yields

$$18H^4F_{RR} \delta'' + \{ (36H^3H'_* + 54H^4)F_{RR} - 108H^3(H^2H''_* + H_1H'_* + 4H^2H'_*)F_{RRR} \} \delta'$$

$$\times \{ -6H^2f'_R + 18(2H^3H''_* + H^2H'_* + 7H^2H'_* - 4H_1^2)F_{RR} - 108(H^3H''_* + H^2H'_*)F_{RR}^{-1} + 4H^3H'_*(H_1H'_* + 4H^2)^2F_{RRR}$$

$$+ 18(H^3H'_*)\rho_0F_{RRR} + 18(H^3H'_*)\rho_0F_{RRT} \delta + \{ k^2\rho_0 + \frac{3\rho_0f'_T^2}{2} + \rho_0F_{TT}^2 \} \delta_m = 0.$$ 

We set the constants $\beta_1 = \beta_2 = 2$ and numerically solve above equation along with (67) for reconstructed function (25) and analyze its stability by plotting graphs as shown in Figure 3. The development of perturbation $\delta$ and $\delta_m$
FIG. 4: Stability of $F(R, T) = \beta_1 g(R) + \beta_2 h(T)$ model for exponential case with $c_i = 2$. Distribution of $\delta(N)$ and $\delta_m(N)$ verses $N$ with $\omega = 0$ is presented. We set (a) $\delta_m = \delta_0 = 0.1$ and vary $\delta_0$, (b) $\delta_0 = 0.1 = \delta_m$ and vary $\delta_0$, (c) $\delta_m$ is varied with same initial values for $\delta_0$ and $\delta_0'$. These curves show smooth behavior having small value near $z = 0$.

regarding to all different initial conditions is presented in Figure 4(a-c), which indicates that these perturbations incline at small value, i.e., $z = 0.010052$ independent of choice of initial conditions. Hence it can be concluded that reconstructed function exhibit smooth behavior against the introduced perturbations. The stability of $F(R, T)$ model (26) with $\beta_1 = \beta_2 = 1$ is shown in Figure 5 which leads to similar results and hence corresponds to stable behavior. It is worthy to mention here that we cannot analyze the stability of other reconstructed models like $F(R, T) = R g(T), F(R, T) = R + T g(R)$ and $F(R, T) = R + R F(T)$ for exponential bouncing as we are unable to find analytical solutions in these cases.

B. Stability of Oscillatory Solutions

Here we will explore the stability of reconstructed solutions for oscillatory model given in Eqs.(37) and (38) as these are the only exact solutions we have found in Section III. For these models, the system of perturbed equations can be solve numerically by assuming $\beta_1 = \beta_2 = 2$ and applying varied initial conditions. We investigate that results of these perturbations for all initial conditions are same as provided in the graphs of Figures 6 and 7. These perturbations
FIG. 5: This plot shows the stability of exponential model for $F(R, T) = R + \beta_1 g(R) + \beta_2 h(T)$ with $c_i = .2$. Plots for $\delta(N)$ and $\delta_m(N)$ versus $N$ with $\omega = 0$ are presented. Thus behavior of such perturbations with defined initial conditions is shown in Fig.4 which smooth behavior indicates that model is stable near $Z = 0$.

approach to small value and shows the stable behavior near $z = 0$ irrespective of initial conditions.

C. Stability of Power Law Solutions

In this subsection, we shall present the stability of matter dominated as well as late time epochs of power law solution. The stability of reconstructed power law model has been discussed in Ref. 42. Now, we first consider the model and Eqs. (65) and (67). At the first place, we consider the dust dominated era for $\alpha = 2/3$ and $\alpha = 0$. Figure 5 indicates the stability of perturbations $\delta(N), \delta_m(N)$) for dust dominated era with $\beta_1 = \beta_2 = 2$. Similarly, the evolution of perturbation with $\alpha = 2$ and $\alpha = 0$ for accelerating universe is presented in Figure 6. Secondly, we analyze the stability of reconstructed solution about linear perturbations. Figure 9 presents the behavior of such perturbations in cosmic decelerated phase with $\omega = 0$ and $\alpha = 2/3$. Here, we choose $c_1 = c_2 = .2$. It is to be noted that small oscillations are produced initially and converges to zero so it shows stable behavior. Figure 10 explains the graph of perturbations for model in accelerating era with $\alpha = 2$. Initially, it shows the oscillation and then decays in future, so the solution becomes stable. Thirdly, we explored the stability of reconstructed solution for model $F(R, T) = R h(T)$ against introduced perturbations. Here, it is seen that for radiation ($\alpha = 2/3$ and $\omega = 0$) and matter ($\alpha = 1/2$ and $\omega = 0$) dominated eras, stability cannot be explained for this model as singularity appears which is a non-physical case. Similarly, for cosmic era with $\alpha = 2$ and $\omega = 0$, this model can not explain the cosmological
evolution because complex terms appear. Lastly, we investigate the stability of Eq. (53) for $\alpha = 2/3, 5/2$ with $\omega = 0$. The evolution of perturbations is presented in graphs of Figure 9 by applying different initial conditions. It is easy to check that the perturbations ($\delta(N), \delta_m(N)$) show fluctuating behavior for accelerating universe with $\alpha = 5/2$ and even present in late cosmic epochs, therefore solutions are unstable in this case.

D. Stability of Matter Bounce Solutions

In this section, we shall analyze the stability of reconstructed matter bounce solution (58) as shown in Figure 10. One can easily see that it converges to zero so this solution is stable. In case of solutions (57) and (59), it is seen that the stability of reconstructed solutions through perturbation parameters cannot be achieved as singularity or complex terms appear in these cases. Similarly, stability of perturbations for Lagrangians (ii), (iv), (v), (vii) and (viii) can not be investigated as the analytical solutions were not possible to obtain in these cases (as shown in Section III).

VI. CONCLUDING REMARKS

The search for a form of Lagrangian which can propose the cosmic evolution in an appropriate way is still under consideration in the present day cosmology. In this regard, the use of reconstruction schemes has fascinated many
researchers due to its successful applications in explaining different features of cosmology. The $F(R, T)$ gravity plays a vital role to explain the impact of dark energy on accelerated cosmic expansion. Such modified theory, involving contribution of both matter Lagrangian and curvature part, has a great importance as a source of dark energy component. Many authors have explored the cosmological reconstruction in $F(R, T)$ gravity by taking different scenarios into account. This paper is another valuable addition in this respect by presenting the reconstruction of different isotropic cosmological bouncing models in $F(R, T)$ gravitational framework. The basic idea of such theory is the inclusion of coupling between matter and curvature which produces $\Delta^i T_{ij} \neq 0$ and hence in order to get the usual continuity equation ($\Delta^i T_{ij} = 0$), one needs to impose an additional condition. In the present paper, we have adopted reconstruction approach to obtain the form of generic function $F(R, T)$ by using four well-known bouncing models namely exponential, oscillatory, power law and matter bounce models. Due to the complex form of dynamical equations, we have considered seven simple forms of generic function and explored the corresponding unknown functions. We can summarize our discussion as follows:

- In exponential and oscillatory models, only first two ansatz forms of function have analytical solutions while for other forms, we were unable to find analytical solution. In case of power law model, all assumed forms of function have analytical solution. For matter bounce, we have obtained analytical solutions for Lagrangian function using three forms of $F(R, T)$. Thus it can be concluded that for these bouncing models, more general forms of generic function, e.g., $F(R, T) = F(R)g(T)$ and $F(R, T) = F_1(R) + F_2(R)F_3(T)$ cannot be reconstructed as explained in Table I.

FIG. 7: Stability of $F(R, T) = R + \beta_1 g(R) + \beta_2 h(T)$ form for oscillatory model with $c_i = .2$. This model is stable due to its smooth behavior near origin.
FIG. 8: Stability of $F(R, T) = R + \beta_1 g(R) + \beta_2 h(T)$ model in power law case with $c_i = .2$.  

FIG. 9: Evolution of $\delta(N)$ and $\delta_m(N)$ for power law model with function (51) and (53). First two plots represent the perturbation parameters for function (51) with $\alpha = 2/3$ and $\alpha = 2$ and last two plots indicate the perturbation parameter for function (53) with $\alpha = 2/3$ and $\alpha = 5/2$. Here, $\omega = 0$.

- Next we have analyzed the behavior of energy conditions for all reconstructed models graphically which have analytical solution. Here, we have found that NEC and SEC violate while other energy constraints remain valid. Similarly, in non-phantom region, the NEC, WEC and SEC are valid but DEC is invalid.
- We have investigated the stability or instability of different forms of generic function having analytical solutions for classifying them on physical grounds. Here we have explained the stability of $F(R, T)$ solutions which reproduce the exponential, oscillatory and power law expansion history. For this purpose, in modified theory, we have explored a set of perturbation equations and explicit form of coefficients which are presented in Appendix.
FIG. 10: Evolution of $\delta(N)$ and $\delta_m(N)$ for matter bounce model with function (55). Here, $\omega = 0$.

| Models                  | $F(R, T)$                          | Solutions                                      | Vacuum solutions constraints                              |
|-------------------------|------------------------------------|------------------------------------------------|--------------------------------------------------------|
| Exponential evaluation  | $\beta_1 g(R) + \beta_2 h(T)$     | Analytical solution exist                       | $32(a_1)^2 c_1 = (6 + e^{1/4} \sqrt{\pi} \text{Erf}(\frac{1}{4})) c_2$ |
|                         | $R + \beta_1 g(R) + \beta_2 h(T)$ | Analytical solution exist                       | $\frac{d_2}{32(a_1)^2} (6 + \sqrt{\pi} e^{1/4} \text{Erf}(\frac{1}{2})) - \frac{27\pi}{12961} = d_1$ |
|                         | $R g(T)$                           | No analytical solution exist                    |                                                        |
|                         | $R + T g(R)$                       | No analytical solution exist                    |                                                        |
|                         | $R + R F(T)$                       | No analytical solution exist                    |                                                        |
|                         | $F_1(R) + F_2(R) F_3(T)$           | No solution exist                              |                                                        |
|                         | $\mu \left( \frac{R}{\pi a_0} \right)^\beta \left( \frac{T}{\pi} \right)^\gamma$ | No analytical solution                         |                                                        |
| Oscillatory             | $\beta_1 g(R) + \beta_2 h(T)$     | Analytical solution exist                       | $2187 a_2^{-1/2} c_1 = 18c_2 \text{tanh}^{-1}(\frac{\sqrt{3}}{3})$ |
|                         | $R + \beta_1 g(R) + \beta_2 h(T)$ | Analytical solution exist                       | $4374 a_2^{-1/2} \beta_1 c_1 = -[18a_2 \text{tanh}^{-1}(1/\sqrt{3})] (9a_2^3 - 2\beta_1 c_1) + 729 a_2^3 (\ln(\frac{3\sqrt{2} + \sqrt{3a_2^2 + 3a_1^2}}{\sqrt{3a_2^2 + 3a_1^2}}))$ |
|                         | $R g(T)$                           | No analytical solution exist                    |                                                        |
|                         | $R + T g(R)$                       | No analytical solution exist                    |                                                        |
|                         | $R + R F(T)$                       | No analytical solution exist                    |                                                        |
|                         | $F_1(R) + F_2(R) F_3(T)$           | No solution exist                              |                                                        |
|                         | $\mu \left( \frac{R}{\pi a_0} \right)^\beta \left( \frac{T}{\pi} \right)^\gamma$ | No analytical solution                         |                                                        |
| Power law               | $\beta_1 g(R) + \beta_2 h(T)$     | Analytical solution exist                       | Always satisfied                                        |
|                         | $R + \beta_1 g(R) + \beta_2 h(T)$ | Analytical solution exist                       | Always satisfied                                        |
|                         | $R g(T)$                           | Analytical solution exist                       | Always satisfied                                        |
|                         | $R + T g(R)$                       | Analytical solution exist                       | Holds for $\omega = 1/3$                                |
|                         | $R + R F(T)$                       | Analytical solution exist                       |                                                        |
|                         | $F_1(R) + F_2(R) F_3(T)$           | No solution exist                              |                                                        |
|                         | $\mu \left( \frac{R}{\pi a_0} \right)^\beta \left( \frac{T}{\pi} \right)^\gamma$ | Analytical solution exist                       |                                                        |
| Matter bounce           | $\beta_1 g(R) + \beta_2 h(T)$     | Analytical solution exist                       | Holds for $\gamma = 0$, $3a(1 + \omega) = 2\beta$, $\forall i$ |
|                         | $R + \beta_1 g(R) + \beta_2 h(T)$ | No Analytical solution exist                    | Not satisfied                                           |
|                         | $R g(T)$                           | No Analytical solution exist                    | Not satisfied                                           |
|                         | $R + T g(R)$                       | No solution exist                              | Not possible                                            |
|                         | $R + R F(T)$                       | Analytical solution exist                       | Not satisfied                                           |
|                         | $F_1(R) + F_2(R) F_3(T)$           | No solution exist                              | Not possible                                            |
|                         | $\mu \left( \frac{R}{\pi a_0} \right)^\beta \left( \frac{T}{\pi} \right)^\gamma$ | No analytical solution exist                    | Not possible                                            |

On stability perturbations, we impose the initial conditions at $z = 0.0100502$ and vary the range as $\{-0.1, 0.1\}$. The results for stability of exponential solutions have been presented in Figures 4 and 5 which showed the smooth behavior near $z = 0$. Similarly, the stability of oscillatory solutions are provided in Figures 6 and 7.

- For power law solutions, we discussed the perturbations for both cases dust as well as accelerated expansion. We have chosen integration constants as $c_i$’s $s=-2$. The graphs of Figures 8, 9 and 11 showed the stability of power law models in $F(R, T)$ gravity. In power law case, we fixed initial conditions at $z = 0$ and ranges vary between $\{-1.1, 1\}$. We concluded that reconstructed solutions $F(R, T) = Rh(T)$ showed unstable behavior for
linear perturbations regarding both accelerating and decelerating eras. Similarly $F(R, T) = \mathcal{R} + \mathcal{R}h(T)$ function presented the unstable behavior for accelerating universe.

- In case of matter bounce, we have analyzed the stability of solution \((58)\) and found that it indicates stable behavior. For solutions \((57)\) and \((59)\), singularity or complex terms may appear, so we can not analyze them for stability.

- Since numerous models can be reconstructed mathematically in different gravity theories, so in order to distinguish valid reconstructed solutions, an additional constraint can be imposed. For example, one interesting and viable constraint on such solutions is the requirement that the reconstructed Lagrangian must recover vacuum solutions such as Minkowski spacetime. In this paper, for all presented solutions, we have checked this constraint and the corresponding conditions have been obtained and presented in the last column of Table I.

It would be interesting to reconstruct these bouncing models in other modified gravity theories especially in torsion based formulations.

**VII. APPENDIX**

\[
\begin{align*}
b_0 &= -6H^2f_{RR} - (6\rho_+(1 + \omega)(\dot{H}_+ H_+^2 + 4H_+^2) + 3(1 - 3\omega)H^2\rho_+)f_{RT} + 18(2H_+^2 H_+^2 + H_+^2 H_+^2 + 7H_+^2 H_+^2 - 4H_+^2)f_{RR} \\
&\quad - 108(H_+^2 H_+^2 + H_+^2 H_+^2 + 4H_+^2 H_+^2)(H_+^2 H_+^2 + 4H_+^2)f_{RR} + 18(1 - 3\omega)(H_+^2 H_+^2 + 4H_+^2)\rho_+ f_{RT}, \\
b_1 &= -6(1 + \omega)\rho_+ H^2 f_{RT} + (36H_+^2 H_+^2 + 54H_+^2) f_{RR} - 108H_+^3 (H_+^2 H_+^2 + H_+^2 H_+^2 + 4H_+^2 H_+^2)f_{RR} + 18(1 - 3\omega)H_+^4 \rho_+ f_{RT}, \\
b_2 &= 18H_+^4 f_{RR}, \\
c_{m1} &= k^2\rho_+ + \frac{(3 - \omega)\rho_+ f_T^2}{2} + (1 + \omega)(1 - 3\omega)\rho_+^2 f_{TT} + (1 - 3\omega)(\rho_+ (3H_+ H_+^2 + H_+^2) - 3H_+^2 \rho_+ f_{RT} + 18(1 - 3\omega)\rho_+ f_{RT} \\
c_{m2} &= 3(1 - 3\omega)H_+^2 f_{RT}, \\
d_1 &= \rho_+ H_+ \{ k^2 + \frac{(3 - \omega)f_T^2}{2} + 3(1 + \omega)(1 - 3\omega)\rho_+ f_{TT} \}, \\
d_2 &= (1 - 3\omega)\rho_+ \{ (1/2)(5 + \omega)\rho_+ H_+ + 3(1 + \omega)\rho_+ H_+ f_{TT} + (1 + \omega)(1 - 3\omega)\rho_+ H_+ f_{TT} \}, \\
d_3 &= (3(\omega - 3)H_+ - 18(1 + \omega)\rho_+ H_+^2) f_{RT} - 6(1 + \omega)(1 - 3\omega)\rho_+^{\prime} (H_+^2 H_+^2 + 4H_+^2) f_{RT}, \\
d_4 &= 3(1 + \omega)\rho_+ H_+ (k^2 + f_T^2) - (33 - \omega)\rho_+^\prime + 18(1 + \omega)\rho_+ (H_+^2 H_+^2 + 4H_+^2) f_{RT} - 6(1 + \omega)(1 - 3\omega) \\
&\quad \times \rho_+^{\prime} (H_+^2 H_+^2 + 4H_+^2) f_{RT}, \\
b_0 &= -6H^2 f_{RR} + 18(2H_+^2 H_+^2 + H_+^2 H_+^2 + 7H_+^2 H_+^2 - 4H_+^2)f_{RR} - 108(H_+^2 H_+^2 + H_+^2 H_+^2 + 4H_+^2 H_+^2)(H_+^2 H_+^2 + 4H_+^2)f_{RR}, \\
b_1 &= (36H_+^2 H_+^2 + 54H_+^2) f_{RR} - 108H_+^3 (H_+^2 H_+^2 + H_+^2 H_+^2 + 4H_+^2 H_+^2)f_{RR}, \\
b_2 &= 18H_+^4 f_{RR}, \\
c_{m1} &= \left[ k^2\rho_+ + \frac{(3 - \omega)\rho_+ f_T^2}{2} + (1 + \omega)(1 - 3\omega)\rho_+^2 f_{TT} \right], \\
d_1 &= \rho_+ H_+ \{ k^2 + \frac{(3 - \omega)f_T^2}{2} + 3(1 + \omega)(1 - 3\omega)\rho_+ f_{TT} \}, \\
d_2 &= (1 - 3\omega)\rho_+ \{ (1/2)(5 + \omega)\rho_+ H_+ + 3(1 + \omega)\rho_+ H_+ f_{TT} + (1 + \omega)(1 - 3\omega)\rho_+ H_+ f_{TT} \}, \\
d_4 &= 3(1 + \omega)\rho_+ H_+ (k^2 + f_T^2), \\
\end{align*}
\]

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FIG. 11: Stability of \( F(R, T) = R + \beta_1 g(R) + \beta_2 h(T) \) function in power law model. The distribution of perturbations for accelerated phase is same as for dust dominated era in Fig. S

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