ON THE STEINBERG CHARACTER OF A REDUCTIVE P-ADIC GROUP

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Abstract. The aim of this paper is to give a generalization of the construction of the Steinberg tempered character on a connected reductive p-adic group. We prove that this character is invariant by the weak restriction of the Jacquet module by analogy to finite reductive groups.

Keywords. Parabolic induction, duality, weak constant term, character of Steinberg.

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1. Introduction

Let $G$ be a connected, reductive p-adic group, and $\mathcal{V}(G)$ the set of tempered virtual characters of $G$ [1], [14], that is the set of finite linear combinations of characters of irreducibles tempered representations of $G$. In [6], K. Bettaïeb has defined an involution on $\mathcal{V}(G)$, denoted by $D^G$ similar to the Curtis-Alvis duality for the characters of finite reductive groups [10], and to A.M. Aubert [3] in the Grothendieck group of the category of smooth finite length representations of reductive p-adic group. If $M$ is a Levi subgroup of $G$, [6] this involution commutes on the one hand with the induction functor, $i_{G,M} : \mathcal{V}(M) \to \mathcal{V}(G)$, and on the other hand with the Jacquet’s weak restriction functor, $r^t_{M,G} : \mathcal{V}(G) \to \mathcal{V}(M)$ [14].

Let $P = MN$ be a standard parabolic subgroup of $G$, and $\sigma \in \Pi_2(M)$, the set of equivalence class of discrete series representation of $M$. We denote by $i_{G,M}(\sigma)$ the set of equivalence class of the tempered representation of $G$ parabolically induced from $\sigma$, $\Pi_\sigma(G)$ the set of all its irreducible constituents, and $R_\sigma := R^G_\sigma$ its corresponding Knapp-Stein R-group [15]. It is a finite groupe defined in terms of Plancherel measures, and is the key of the determination of the intertwining algebra $C(\sigma)$ which is isomorphic to $\mathbb{C}[R_\sigma][\eta]$ the complex algebra of $R_\sigma$, with multiplication twisted by a cocycle $\eta$. Let $1 \to Z_\sigma \to \widetilde{R}_\sigma \to R_\sigma \to 1$, be a fixed central extension of $R_\sigma$, on which the cocycle associated to the intertwining operators splits. J. Arthur [1], showed that there is a bijection: $\rho \leftrightarrow \pi_\rho$ between $\Pi_\sigma(G)$ and $\Pi(\widetilde{R}_\sigma, \chi_\sigma)$, the set of equivalence classes of irreducible representations of $\widetilde{R}_\sigma$ with $\chi_\sigma$ as $Z_\sigma$-central character. Given any $\rho \in \Pi(\widetilde{R}_\sigma, \chi_\sigma)$, we denote by $\theta_\rho$ its character and by $\Theta_{\pi_\rho}$ the character of $\pi_\rho$, then this bijection induces an isomorphism $I^G$ between the respective characters of these representations, i.e, $I^G(\theta_\rho) = \Theta_{\pi_\rho}$, we say that $\Theta_{\pi_\rho}$ corresponds to $\theta_\rho$, and

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that $D_G^\ell(\Theta_{\pi_r})$, is the dual of $\Theta_{\pi_r}$.

Similar to the duality $D_G^\ell$ on $\mathcal{V}(G)$, we establish another duality on the space $C(\Pi(\overline{R}_\sigma, \chi_\sigma))$ of characters of irreducible representations in $\Pi(\overline{R}_\sigma, \chi_\sigma)$. We prove that the sign application $\xi_{\overline{R}_\sigma}$ of $\overline{R}_\sigma$, defined by $\xi_{\overline{R}_\sigma}(\tilde{r}) := (-1)^{\dim(A_{L_\sigma})}$, where $\tilde{r} \in \overline{R}_\sigma$ is the inverse image of $r \in R_\sigma$, and $L_\sigma$ is a Levi subgroup of $G$ which verifies certain condition, is the character of an irreducible representation in $\Pi(\overline{R}_\sigma, \chi_\sigma)$.

The second result of this paper is to define a special character in $\mathcal{V}(G)$, similar to that of Steinberg,(c.f. [12], [9, §4.7]), which will be called the Steinberg character associated to a pair $(M, \sigma)$, with $M$ is a Levi subgroup of $G$ and $\sigma \in \Pi_2(M)$ (such pairs are called discrete pairs), and denoted by $St^G_{\sigma}(M)$, where $\sigma$ is $G$-rational points of $G$ and $\sigma$ is $\overline{G}$-rational points of $G$, similar to $\overline{G}$-rational points of $G$. We prove that $\sigma$ is elliptic, and that its dual is $\Theta_{\pi_1}$, where $\pi_1 \in \Pi_2(G)$ corresponds to the trivial representation. In addition if $L$ is a Levi subgroup of $G$, containing $M$ and satisfies certain condition then:

$$r^\ell_{L,G}(\sigma)(St^G_{\sigma}(M)) = St^L_{\sigma}(M)$$

where $r^\ell_{L,G}(\sigma)$ is the projection of $r^\ell_{L,G}$ on $\Pi_2(L)$.

2. Preliminaires

Let $F$ be a locally compact, non-discrete, nonarchimedean field of characteristic zero, and $G$ is a connected reductive algebraic group over $F$. Let $G := G(F)$ be the group of $F$-rational points of $G$, and $A_0$ the split component of $G$, i.e, the maximal $F$-split torus lying in the center of $G$. We fix a Levi subgroup $M_0$ of a certain minimum parabolic subgroup $P_0$ of $G$ defined over $F$, and $A_0$ its split component. Any parabolic subgroup $P$ of $G$ which is defined over $F$, has a unique Levi decomposition, $P = M_P \rtimes N_P = M_P.N_P$, where $M_P$ is connected, reductive subgroup of $G$, and $N_P$ is the unipotent radical of $P$, if $P$ contains $P_0$, we say that it is standard. The Levi subgroups of $G$, are the centralizers in $G$ of their splits components, therefore they are uniquely determined by these components and vice versa.

Let $\Phi(A_0)$ be the set of reduced roots of $A_0$ in the Lie algebra of $G$, and $\Delta \subset \Phi^+(G, A_0)$ the collection of simple roots, then the conjugacy classes of standards parabolics subgroups of $G$ are in one to one correspondence with subsets of $\Delta$, [2, Lemma 1.2], written as follows:

$$I \subset \Delta \longleftrightarrow P_I = M_I N_I$$

such that:

$$A_{M_I} = A_I = (\bigcap_{\alpha \in I} \ker \alpha \cap A_0)^0, \text{ and } M_I = Z_G(A_I)$$

where $Z_G(A_I)$ is the centralizer of $A_I$ in $G$. We refer the reader to [8] for more details. We write $\mathcal{L}$ for the finite set of Levi subgroups of $G$. Given any $M \in \mathcal{L}$, with split component $A_M$, we write $\mathcal{L}(M) := \mathcal{L}^G(M)$ for the set of Levi subgroups of $G$ which contain $M$, and $\mathcal{P}(M)$ for the set of parabolic subgroups $P$, with Levi component $M_P = M$, any $P \in \mathcal{P}(M)$ admits a unique opposite parabolic subgroup $\overline{P} = M \overline{N}_P$, \linebreak
such that $P \cap \mathcal{P} = M$

For a fixed $M \in \mathcal{L}$, let $X(M)_F$ be the group of $F$-rational characters of $M$, and

$$a_M = \text{Hom}(X(M)_F, \mathbb{R})$$

$$a_M^* = X(M)_F \otimes_{\mathbb{Z}} \mathbb{R}$$

then $a_M$ is the real Lie algebra of $A_M$, and $a_M^*$ is its dual. We denote by $W_0 := W_0^G := N_G(A_0)/M_0$ the Weyl group of $G$ with respect to $A_0$, and $W := W^G$ for the Weyl group of $G$ with respect to $A_M$, where $N_G(A_0)$ is the normalizer of $A_0$ in $G$.

An element $x \in G$ is called regular if $D_G(x) \neq 0$, where $D_G$ is the standard discriminant factor defined in [12], we denote by $G_{reg}$ the set of regular elements of $G$. An element of $G_{reg}$ is elliptic if its centralizer is compact modulo $A_G$, we write $G_{ell}$ for the set of regular elliptic elements of $G$. Let $\Pi(G)$ be the set of equivalence classes of irreducible tempered representation of $G$, and $\Pi_2(G)$ the subset of $\Pi(G)$ consisting of discrete series representations of $G$. If there is no confusion, we do not make a distinction between each equivalence class and its representative. Given any $\pi \in \Pi(G)$, we write $\Theta_\pi$ for the character of $\pi$ which is locally integrable function on $G$ [12], and $\Theta^e_\pi$ for its restriction to $G_{ell}$. The representations $\pi$ is said to be elliptic, if $\Theta^e_\pi \neq 0$

2.1. The weak constant term. Let $(\pi, V)$ be an admissible representation of $G$ and $P \in \mathcal{P}(M)$. For any quasi-character $\chi$ of $A_G$, we denote by:

$$V_\chi = \{v \in V; \exists d \in \mathbb{N}, \forall a \in A_G, (\pi(a) - \chi(a).1d)^d v = 0\}$$

and

$$\text{Exp}(\pi) = \{\chi; V_\chi \neq 0\}$$

$\text{Exp}(\pi)$ is called the set of exponent of $\pi$, we get then:

$$V = \bigoplus_{\chi \in \text{Exp}(\pi)} V_\chi$$

let $\mathcal{P} = M\mathcal{N}$ the opposite parabolic subgroup of $P$, and $V_N := V/V(\mathcal{N})$ where $V(\mathcal{N})$ is the $M$–invariant subspace of $V$ generated by $\{\pi(\mathcal{N})v - v; \pi \in \mathcal{N}, v \in V\}$. For all $m \in M$ and $v \in V$, we define:

$$\pi_N(m)p(v) = \delta_{\mathcal{P}}(m)^{-1/2}p(\pi(m)v)$$

with $p : V \longrightarrow V_N$ is the canonical projection and $\delta_{\mathcal{P}}$ is the modular function of $\mathcal{P}$. The representation $(\pi_N, V_N)$ is called the normalized Jacquet module of $(\pi, V)$ corresponding to $\mathcal{P}$, which is admissible and of finite length [17, §2.3]. We write $(\Theta_\pi)_M := \Theta_{\pi_N}$ for the character of $\pi_N$, and we call it the constant term of $\Theta_\pi$ along $\mathcal{P}$, if $\pi$ is tempered, we write $(\Theta_\pi)_M^w := \Theta_{\pi_N}^w$ for the character of $\pi_N^w$ which is the maximum tempered quotient of the normalized Jacquet module $\pi_N$ corresponding to $\mathcal{P}$, defined by:

$$V_N^w = \bigoplus_{\chi \in \text{Exp}(\mathcal{P}) \cap \mathcal{M}} V_N,\chi$$

we call $(\Theta_\pi)_M^w$, the weak constant term of $\Theta_\pi$ along $\mathcal{P}$. 


2.2. The set $\mathcal{V}(G)$. A character $\Theta$ of $G$, is called virtual, if there are a finite number of $\pi_1, ..., \pi_k \in \Pi(G)$ and $c_i \in \mathbb{C}$, for all $1 \leq i \leq k$, such that:

$$\Theta = \sum_{1 \leq i \leq k} c_i \Theta_{\pi_i}.$$  

We denote by $\mathcal{V}(G)$ the set of such characters, which is the set of finite linear combinations of characters of irreducible tempered representations of $G$, we say that a character in $\mathcal{V}(G)$ is irreducible if it is the character of an irreducible representation in $\Pi(G)$. Let $\mathcal{V}_{st}(G)$ be the subset of $\mathcal{V}(G)$ formed by so-called supertempered characters [13], and defined according to [14] by:

$$\mathcal{V}_{st}(G) = \{ \Theta \in \mathcal{V}(G); \Theta_w^L = 0, \forall L \in \mathcal{L}, L \neq G \}.$$  

$E_G$ is the endomorphism of $\mathcal{V}(G)$, which extends the character of $\pi$ to the character of its contragredient, defined by:

$$E_G(\Theta) = \sum_{1 \leq i \leq k} c_i \Theta_{\pi_i}^\vee$$

where $\pi_i^\vee \in \Pi(G)$ is the contragredient of $\pi_i$. Let $\Theta' \in \mathcal{V}(M)$ then by linearity we define the induction functor:

$$i_{G,M} : \mathcal{V}(M) \rightarrow \mathcal{V}(G).$$

We define on the other hand using the constant and weak constant term, the functors of restriction and weak restriction of Jacquet:

$$r_{M,G} : \mathcal{V}(G) \rightarrow \mathcal{V}(M), \Theta \mapsto r_{M,G}(\Theta) = \Theta_M = \sum_{1 \leq i \leq k} c_i (\Theta_{\pi_i})_M.$$  

$$r_{M,G}^w : \mathcal{V}(G) \rightarrow \mathcal{V}(M), \Theta \mapsto r_{M,G}^w(\Theta) = \Theta_M^w = \sum_{1 \leq i \leq k} c_i (\Theta_{\pi_i})_M^w.$$  

Let $M, L \in \mathcal{L}$, we denote by:

$$W^{M,L} = \{ w \in W_0 : w(M \cap P_0) \subset P_0, w^{-1}(L \cap P_0) \subset P_0 \}$$

the subgroup of $W_0$, such that in each double class $W_0^L w W_0^M$, there exist a unique element of $W^{M,L}$. The following theorem due to I.N. Bernstein and A.V. Zelevinsky [4], gives a good description of the composition of functors $r_{L,G}$ and $i_{G,M}$.

**Theorem 2.1.** [4] If $M, L \in \mathcal{L}$, then $r_{L,G} \circ i_{G,M}$ has a filtration consisting of subfunctors:

$$i_{L,L_w} \circ w \circ r_{w^{-1}L_w,M}$$

where $L_w = wM \cap L \subset L$, and $w \in W^{M,L}$.

In particular, in the appropriate space of virtuals characters, if $\Theta' \in \mathcal{V}(M)$ then:

$$r_{L,G} \circ (i_{G,M}(\Theta')) = \sum_{w \in W^{M,L}} i_{L,L_w} (r_{L_w,wM}(w\Theta')).$$
2.3. **The duality on** $\mathcal{V}(G)$. In [6], K. Bettaïeb has defined an operator $D^t_G$ on the space $\mathcal{V}(G)$ as follows:

\[(2.3)\quad D^t_G = \sum_{M \in \mathcal{L}} (-1)^{\dim(A_M)} i_{G,M} \circ r^t_{M,G}.\]

**Theorem 2.2.** [6] The operator $D^t_G$ has the following properties:

1- If $L \in \mathcal{L}$, then :
\[D^t_G \circ i_{G,L} = i_{G,L} \circ D^t_L\text{ and } r^t_{L,G} \circ D^t_G = D^t_L \circ r^t_{L,G}.\]

2- $D^t_G$ is an involution, i.e $D^t_G 2 = \text{id}$. 

3- If $\pi \in \Pi_2(G)$, then $D^t_G(\Theta_\pi) = \pm \Theta_\pi$. 

4- $D^t_G$ takes irreducible representations to irreducible representations.

5- If $M \in \mathcal{L}$, and $\sigma \in \Pi_2(M)$, then $D^t_G$ preserves the irreducible characters of $i_{G,M}(\Theta_\sigma)$ up to sign.

We can also add some additional properties of $D^t_G$ illustrate in the next lemma.

**Lemma 2.3.** Let $\pi$ be an irreducible tempered representation of $G$.

1- If $\chi$ is a character of $G$. Then :
\[D^t_G(\chi \otimes \pi) = \chi \otimes D^t_G(\Theta_\pi)\]

where $\chi \otimes \pi$ denotes the twist of $\pi$ by $\chi$.

2- $D^t_G \circ E_G = E_G \circ D^t_G$.

3- Let $\Theta \in \mathcal{V}(G)$, then $D^t_G(\Theta)$ is a finite linear combination of induced of super tempered characters up to sign.

**Proof.** Let $P = MN$ a standard parabolic subgroup of $G$ and $\sigma$ is a tempered representation of $M$, then proposition 1.9 of [4] implies:

\[i_{G,M}(\Theta_{\chi \otimes \sigma}) = \chi \otimes i_{G,M}(\Theta_\sigma), \text{ and } r^t_{M,G}(\Theta_{\chi \otimes \sigma}) = \chi \otimes r^t_{G,M}(\Theta_\pi)\]

Consequently,

\[r^t_{M,G}(\Theta_{\chi \otimes \sigma}) = \chi \otimes r^t_{G,M}(\Theta_\pi).\]

The first assertion now follows from the definition of $D^t_G$.

For the second assertion, given $\pi \in \Pi_2(G)$ then according to proposition 3.1.2 and corollary 4.2.5 of [9], under the assumption that $\Theta_{\pi^w}^{\pi} := \Theta_{\pi^w}^{\pi}$ is independent of all choices of parabolic subgroups with Levi component $M$ [14], we get:

\[E_G \circ D^t_G(\Theta_\pi) = E_G \circ \sum_{M \in \mathcal{L}} (-1)^{\dim(A_M)} i_{G,M} \circ r^t_{M,G}(\Theta_\pi)\]

\[= \sum_{M \in \mathcal{L}} (-1)^{\dim(A_M)} E_G \circ i_{G,M} \circ r^t_{M,G}(\Theta_\pi)\]

\[= \sum_{M \in \mathcal{L}} (-1)^{\dim(A_M)} i_{G,M} \circ E_M \circ r^t_{M,G}(\Theta_\pi)\]

\[= \sum_{M \in \mathcal{L}} (-1)^{\dim(A_M)} i_{G,M} \circ r^t_{M,G} \circ E_G(\Theta_\pi)\]

\[= D^t_G \circ E_G(\Theta_\pi).\]

In particular:

\[D^t_G \circ E_G(\Theta_{\chi \otimes \pi}) = \chi^{-1} \otimes D^t_G(E_G(\Theta_\pi)).\]
Let \( \Theta \in \mathcal{V}(G) \), then there is an unique finite family \( \{ L_i, \Theta_i \}_{1 \leq i \leq k} \), where \( L_i \in \mathcal{L} \) and \( \Theta_i \in \mathcal{V}_{st}(G) \), such that [5, corollaire 7]:

\[
\Theta = \sum_i i_{G,L_i}(\Theta_i)
\]

so from (2.1) and that \( D^l_G \) commutes with the induction functor, we find:

\[
D^l_G(\Theta) = \sum_i i_{G,L_i} \circ D^l_{L_i}(\Theta_i) = \pm \sum_i i_{G,L_i}(\Theta_i)
\]

\( \square \)

In the next section, we will define by analogy to [6], a duality on the space of all characters of irreducible representations in \( \Pi(\tilde{R}_\sigma, \chi_\sigma) \).

3. Duality in \( \mathbb{C}(\Pi(\tilde{R}_\sigma, \chi_\sigma)) \)

We say that \((M, \sigma)\) is a discrete pair of \( G \) if \( M \in \mathcal{L} \) and \( \sigma \in \Pi_2(M) \). Let \((M, \sigma)\) be a fixed discrete pair of \( G \), \( i_{G,M}(\sigma) \) the tempered (normalized) representation parabolically induced from \( \sigma \) and \( \Pi_\sigma(G) \) the set of all its irreducible constituents. We write:

\[
W_\sigma = \{ w \in W : w\sigma \simeq \sigma \}
\]

for the isotropy group of \( \sigma \), and:

\[
\Delta_\sigma = \{ \alpha \in \Phi^+(P, A_M) : \mu_\alpha(\sigma) = 0 \}
\]

where \( \mu_\alpha(\sigma) \) is the rank one Plancherel measure for \( \sigma \) attached to \( \alpha \) [11; p 1108]. To each \( w \in W_\sigma \), there exist an intertwining operator of the representation \( i_{G,M}(\sigma) \) in itself. Let \( \mathcal{C}(\sigma) \) be the commuting algebra of \( i_{G,M}(\sigma) \), and \( W_\sigma^0 \) the subgroup of \( W_\sigma \), generated by the reflections in the roots of \( \Delta_\sigma \), then the \( R \)-group of \( i_{G,M}(\sigma) \), [11],[15]:

\[
R_\sigma = \{ r. \in W_\sigma ; r\Delta_\sigma = \Delta_\sigma \} \simeq W_\sigma / W_\sigma^0
\]

has the property that \( \mathcal{C}(\sigma) \) is isomorphic to the complex group algebra \( \mathbb{C}[R_\sigma] \) twisted by a cocycle \( \eta \). As in[1, §2], let:

\[
1 \rightarrow Z_\sigma \rightarrow \tilde{R}_\sigma \rightarrow R_\sigma \rightarrow 1
\]

the central extension of \( R_\sigma \), over which the cocycle associated to the intertwining operator split. There is a character \( \chi_\sigma \) of \( Z_\sigma \) such that \( \Pi_\sigma(G) \) is parametrize by \( \Pi(\tilde{R}_\sigma, \chi_\sigma) \), the set of irreducible representations \( \rho \) of \( \tilde{R}_\sigma \) having \( \chi_\sigma \) as \( Z_\sigma \)-central character. Let \( \mathbb{C}(\Pi(G)) \) (resp \( \mathbb{C}(\Pi(\tilde{R}_\sigma, \chi_\sigma)) \)) the complex vector space generated by the characters of representations in \( \Pi_\sigma(G) \) (resp. \( \Pi(\tilde{R}_\sigma, \chi_\sigma) \)). Hence the bijection \( \rho \mapsto \pi_\rho \) between \( \Pi(\tilde{R}_\sigma, \chi_\sigma) \) and \( \Pi_\sigma(G) \) induces an isomorphism:

\[
I^G : \mathbb{C}(\Pi(\tilde{R}_\sigma, \chi_\sigma)) \rightarrow \mathbb{C}(\Pi_\sigma(G)), \ \theta_\rho \mapsto I^G(\theta_\rho) = \Theta_{\pi_\rho}
\]

described in term of intertwining algebra as in [1, p 88].

Let \( L \in \mathcal{L}(M) \), we say that \( L \) satisfies the Compatibility condition of Arthur if \( a_L \cap a_\Delta^+ \) contains an open subset of \( a_L \), where \( a_\Delta^+ := \{ X \in a_M ; \alpha(X) > 0, \forall \alpha \in \Delta_\sigma \} \), denotes the positive chamber corresponding to \( \Delta_\sigma \). We denote by \( \mathcal{L}_A(M) \) the set of Levi subgroups \( L \in \mathcal{L}(M) \) which satisfies this condition, so if \( L \in \mathcal{L}_A(M) \), then
\( R^L_{\sigma} := R_\sigma \cap W^L \) is the \( R \)-group of \( i_{L,M}(\sigma) \). Therefore as at \( G \), we obtain a bijection \( \rho_L \mapsto \tau_{\rho_L} \) between \( \Pi(\tilde{R}^L_{\sigma}, \chi_\sigma) \) and \( \Pi_\sigma(L) \) which induces again the isomorphism:

\[
I^L : \mathbb{C}(\Pi(\tilde{R}^L_{\sigma}, \chi_\sigma)) \to \mathbb{C}(\Pi_\sigma(L)).
\]

Notice that the Jacquet-weak restriction \( r^L_{L,G} \) does not send \( \mathbb{C}(\Pi_\sigma(G)) \) into \( \mathbb{C}(\Pi_\sigma(L)) \), [14, Lemma 3.5], for that we denote by \( r^L_{L,G}(\sigma) \) the projection of \( r^L_{L,G} \) on \( \mathbb{C}(\Pi_\sigma(L)) \). Let \( L \in \mathcal{L}_A(M) \), we consider the following two functors of induction and restriction:

\[
\text{Ind}^L_{\tilde{R}^L_{\sigma}} : \mathbb{C}(\Pi(\tilde{R}^L_{\sigma}, \chi_\sigma)) \to \mathbb{C}(\Pi(\tilde{R}^L_{\sigma}, \chi_\sigma))
\]

and

\[
\text{Res}^L_{\tilde{R}^L_{\sigma}} : \mathbb{C}(\Pi(\tilde{R}^L_{\sigma}, \chi_\sigma)) \to \mathbb{C}(\Pi(\tilde{R}^L_{\sigma}, \chi_\sigma))
\]

then using (2.2) and the transitivity of induction and restriction functors we get the following commutative diagram [6, Théorème 6]:

\[
\begin{array}{ccc}
\mathbb{C}(\Pi(\tilde{R}^L_{\sigma}, \chi_\sigma)) & \xrightarrow{\text{Ind}^L_{\tilde{R}^L_{\sigma}}} & \mathbb{C}(\Pi(\tilde{R}^L_{\sigma}, \chi_\sigma)) \\
\downarrow{I^L} & & \downarrow{I^G} \\
\mathbb{C}(\Pi_\sigma(L)) & \xrightarrow{\text{Res}^L_{\tilde{R}^L_{\sigma}}} & \mathbb{C}(\Pi_\sigma(G)) \xrightarrow{r^L_{L,G}(\sigma)} \mathbb{C}(\Pi_\sigma(L)) \\
\end{array}
\]

Let \( r \in R_\sigma \), and \( \tilde{r} \) its inverse image in \( \tilde{R}_\sigma \), we say that the triplet \((M, \sigma, \tilde{r})\) is a \( \tilde{R}_\sigma \)-virtual triplet of \( G \). To each triplet \((M, \sigma, \tilde{r})\), J. Arthur [1, §3] corresponds a distribution-character \( \Theta(M, \sigma, \tilde{r}) := \Theta^G(M, \sigma, \tilde{r}) \), called the virtual character of \( G \) and is written in the following form:

\[
\Theta(M, \sigma, \tilde{r}) = \sum_{\rho \in \Pi(\tilde{R}_\sigma, \chi_\sigma)} \theta_{\rho^\vee}(\tilde{r}) \Theta_{\rho^\vee}.
\]

By inverting (3.2) we will have:

\[
\Theta_{\rho^\vee} = |\tilde{R}_\sigma|^{-1} \sum_{\tilde{r} \in \tilde{R}_\sigma} \theta_\rho(\tilde{r}) \Theta(M, \sigma, \tilde{r}).
\]

Those triplets are \( W_0 \)-invariant, and if \( z \in Z_\sigma \), then:

\[
\Theta(z(M, \sigma, \tilde{r})) = \chi_\sigma^{-1}(z) \Theta(M, \sigma, \tilde{r})
\]

So \( \Theta(M, \sigma, \tilde{r}) \) can be vanished. Following Arthur [1, p 92], the \( \tilde{R}_\sigma \)-triplet \((M, \sigma, \tilde{r})\) of \( G \) is said to be essential if, \( \chi_\sigma \equiv 1 \) on \( \{ z \in Z_\sigma, z.\text{cl}(\tilde{r}) \subset \text{cl}(\tilde{r}) \} \), where \( \text{cl}(\tilde{r}) \) is the \( \tilde{R}_\sigma \)-conjugacy class of \( \tilde{r} \). Therefore, \( \Theta(M, \sigma, \tilde{r}) \) is non-zero if and only if the \( \tilde{R}_\sigma \)-virtual triplet is essential. The collection of distributions relative to this type of triplets forms a basis of \( \mathcal{V}(G) \) [1, §3].

For each \( r \in R_\sigma \), we define:

\[
a^r_M = \{ H \in a_M : r.H = H \}
\]
Let
\[ R_{\sigma,\text{reg}} = \{ r \in R_\sigma; \ a_M^r = a_G \} \]
the set of regulars elements in \( R_\sigma \), and:
\[ \mathcal{L}(R_\sigma) = \{ S \in \mathcal{L}(M); \ L_r = S, \ \text{for some } r \in R_\sigma \} \]
so it is an immediate consequence that
\[ R_\sigma = \bigcup_{S \in \mathcal{L}(R_\sigma)} R_{\sigma,\text{reg}}^S \]
the union is disjoint. If \( R_{\sigma,\text{reg}} \) is not empty, we say that \( R_\sigma \) and the triplet \((M, \sigma, \tilde{r})\), \( \tilde{r} \in \tilde{R}_{\sigma,\text{reg}} \) the inverse image of \( R_{\sigma,\text{reg}} \) in \( \tilde{R}_\sigma \) are both elliptic, and so \( \Theta(M, \sigma, \tilde{r}) \) is a supertempered virtual character of \( G \) [14].

**Lemma 3.1.** Let \( \rho \in \Pi(\tilde{R}_\sigma, \chi_\rho) \), then for every \( L \notin \mathcal{L}_A(M) \): \( (\Theta_{\pi_\rho})_L^w = 0 \).

**Proof.** Let \( \rho \in \Pi(\tilde{R}_\sigma, \chi_\rho) \), and \( L \in \mathcal{L} \), then under the assumption that \( \pi_\rho \) is tempered, we have:
\[ (\Theta_{\pi_\rho})_L^w = [\tilde{R}_\sigma]^{-1} \sum_{\tilde{r} \in \tilde{R}_\sigma} \theta_\rho(\tilde{r}) \Theta^G(M, \sigma, \tilde{r})_L^w \]
\[ = \sum_{S \in \mathcal{L}(R_\sigma)} [\tilde{R}_{\sigma,\text{reg}}^S]^{-1} \sum_{\tilde{r} \in \tilde{R}_{\sigma,\text{reg}}^S} \theta_\rho(\tilde{r}) \Theta^G(M, \sigma, \tilde{r})_L^w. \]

For \( S \in \mathcal{L}(R_\sigma) \), if \( \tilde{r} \in \tilde{R}_{\sigma,\text{reg}}^S \), then \((M, \sigma, \tilde{r})\) is an elliptic virtual triplet of \( S \) therefore:
\[ \Theta^G(M, \sigma, \tilde{r}) = \sum_{\tau \in \Pi(S)} \theta_{\rho_\tau}(\tilde{r}) i_{G,S}(\Theta_\tau) \]
\[ = i_{G,S}(\Theta^S(M, \sigma, \tilde{r})) \]
hence:
\[ (\Theta_{\pi_\rho})_L^w = \sum_{S \in \mathcal{L}(R_\sigma)} [\tilde{R}_{\sigma,\text{reg}}^S]^{-1} \sum_{\tilde{r} \in \tilde{R}_{\sigma,\text{reg}}^S} \theta_\rho(\tilde{r}) i_{G,S}(\Theta^S(M, \sigma, \tilde{r}))_L^w \]
\[ = \sum_{S \in \mathcal{L}(R_\sigma)} [\tilde{R}_{\sigma,\text{reg}}^S]^{-1} \sum_{\tilde{r} \in \tilde{R}_{\sigma,\text{reg}}^S} \theta_\rho(\tilde{r}) r_{L,G}^t \circ i_{G,S}(\Theta^S(M, \sigma, \tilde{r})). \]

By using (2.2) we get:
\[ (\Theta_{\pi_\rho})_L^w = \sum_{S \in \mathcal{L}(R_\sigma)} [\tilde{R}_{\sigma,\text{reg}}^S]^{-1} \sum_{\tilde{r} \in \tilde{R}_{\sigma,\text{reg}}^S} \theta_\rho(\tilde{r}) \sum_{w \in W_{S,L}} i_{L,w}(r_{L,w}^t w \Theta^S(M, \sigma, \tilde{r}))) \]
where, \( L_w = wS \cap L \in \mathcal{L}(L) \). Since \( \Theta^S(M, \sigma, \tilde{r}) \in \mathcal{V}_{st}(S) \), we have \( w \Theta^S(M, \sigma, \tilde{r}) \in \mathcal{V}_{st}(wS) \), hence \( r_{L,w}^t w \Theta^S(M, \sigma, \tilde{r}) = 0 \), unless there is \( S \in \mathcal{L}(R_\sigma) \), and \( w \in W_{S,L} \) such that, \( L_w = wS \).

So, \( r_{L,w}^t w \Theta^S(M, \sigma, \tilde{r}) = 0 \), unless there is \( w \in W_{S,L} \) such that, \( S \subset w^{-1}L \), if this is the case, we get:
\[ w^{-1}a_L = a_{w^{-1}L} \subset a_S = a_M^r = a_{L_r} \subset a_M \]
for some \( r \in R_\sigma \). Therefore without loss of generality, and since \( L_r \in \mathcal{L}_A(M) \), we assume that \( L \in \mathcal{L}_A(M) \), [1, p 90].

We define formally the sign application by:
\[
\xi_{\tilde{R}_\sigma} : \tilde{R}_\sigma \to \{ \pm 1 \} : \xi_{\tilde{R}_\sigma}(\tilde{r}) = (-1)^{\dim(A_L)}.
\]

By analogy, to \( D^r_G \) we define an operator \( D_{\tilde{R}_\sigma, \chi} \) on \( \mathbb{C}(\Pi(\tilde{R}_\sigma, \chi)) \) by:
\[
D_{\tilde{R}_\sigma, \chi} = \sum_{L \in \mathcal{L}_A(M)} (-1)^{\dim(A_L)} \text{Ind}_{\tilde{R}_\sigma}^{\tilde{R}_L} \circ \text{Res}_{\tilde{R}_\sigma}^{\tilde{R}_L}
\]

If the cocycle splits, then the operator is simply \( D_{R_\sigma} \). The next theorem presents some properties of this operator.

**Theorem 3.2.** The operator \( D_{\tilde{R}_\sigma, \chi} \) has the following properties:

1- Let \( L \in \mathcal{L}_A(M) \), then the following diagram is commutative:

\[
\begin{array}{cccc}
\mathbb{C}(\Pi(\tilde{R}_\sigma, \chi)) & \xrightarrow{\text{Res}_{\tilde{R}_\sigma}^{\tilde{R}_L}} & \mathbb{C}(\Pi(\tilde{R}_L, \chi)) & \xrightarrow{\text{Ind}_{\tilde{R}_\sigma}^{\tilde{R}_L}} & \mathbb{C}(\Pi(\tilde{R}_\sigma, \chi)) \\
\downarrow{D_{\tilde{R}_\sigma, \chi}} & & \downarrow{\text{Res}_{\tilde{R}_\sigma}^{\tilde{R}_L}} & & \downarrow{\text{Ind}_{\tilde{R}_\sigma}^{\tilde{R}_L}} \\
\mathbb{C}(\Pi(\tilde{R}_\sigma, \chi)) & & \mathbb{C}(\Pi(\tilde{R}_L, \chi)) & & \mathbb{C}(\Pi(\tilde{R}_\sigma, \chi)) \xrightarrow{I^G} \mathbb{C}(\Pi_\sigma(G))
\end{array}
\]

2- \( D_{\tilde{R}_\sigma, \chi} \) is an involution.

*Proof.*

1- Let \( L \in \mathcal{L}_A(M) \), according to the last diagram:
\[
r^l_{L,G}(\sigma) \circ I^G = I^L \circ \text{Res}_{\tilde{R}_\sigma}^{\tilde{R}_L}.
\]

The composition on both sides of this last expression with the induction functor \( i_{G,L} \) gives us:
\[
i_{G,L} \circ r^l_{L,G}(\sigma) \circ I^G = i_{G,L} \circ I^L \circ \text{Res}_{\tilde{R}_\sigma}^{\tilde{R}_L}
\]

Since
\[
i_{G,L} \circ I^L = I^G \circ \text{Ind}_{\tilde{R}_\sigma}^{\tilde{R}_L}
\]

then
\[
i_{G,L} \circ r^l_{L,G}(\sigma) \circ I^G = I^G \circ \text{Ind}_{\tilde{R}_\sigma}^{\tilde{R}_L} \circ \text{Res}_{\tilde{R}_\sigma}^{\tilde{R}_L}
\]

If we apply the sum on \( \mathcal{L} \), we obtain:
\[
\sum_{L \in \mathcal{L}} (-1)^{\dim(A_L)} i_{G,L} \circ r^l_{L,G}(\sigma) \circ I^G =
\]
\[
I^G \circ \sum_{L \in \mathcal{L}} (-1)^{\dim(A_L)} \text{Ind}_{\tilde{R}_\sigma}^{\tilde{R}_L} \circ \text{Res}_{\tilde{R}_\sigma}^{\tilde{R}_L}
\]

and since \( \mathcal{L}_A(M) \subset \mathcal{L} \), then by decomposing the sum we get:
\[
D^r_G \circ I^G = I^G \circ D_{\tilde{R}_\sigma, \chi} + \left[ \sum_{L \not\in \mathcal{L}_A(M)} (-1)^{\dim(A_L)+1} i_{G,L} \circ r^l_{L,G}(\sigma) \right] \circ I^G
\]
by lemma 3.1: \[ \sum_{L \in \mathcal{L}_A(M)} (-1)^{\dim(A_L)+1} \iota_{G,L} \circ r^i_{L,G}(\sigma) \] vanishes, hence:

\[ D^i_G \circ I^G = I^G \circ D_{\mathbb{R}^e, \chi_e} \]

Let now \( L \in \mathcal{L}_A(M) \), then following the last result, we have on the one hand:

\[ I^L \circ \text{Res}_{\mathbb{R}^e_{R^L}} \circ D_{\mathbb{R}^e, \chi_e} \circ r^i_{L,G}(\sigma) \circ I^G \circ D_{\mathbb{R}^e, \chi_e} = r^i_{L,G}(\sigma) \circ D^i_G \circ I^G \]

and on the other hand, following theorem 2.2.1, and again the last result but applied to \( L \), we get:

\[ I^L \circ D_{\mathbb{R}^e_{R^L}, \chi_e} \circ \text{Res}_{\mathbb{R}^e_{R^L}} = D^L_G \circ I^L \circ \text{Res}_{\mathbb{R}^e_{R^L}} = D^t_G \circ r^i_{L,G}(\sigma) \circ I^G \]

which implies:

\[ I^L \circ \text{Res}_{\mathbb{R}^e_{R^L}} \circ D_{\mathbb{R}^e, \chi_e} = I^L \circ D_{\mathbb{R}^e_{R^L}, \chi_e} \circ \text{Res}_{\mathbb{R}^e_{R^L}} \]

Since the operator \( I^L \) is an isomorphism for all \( L \in \mathcal{L}_A(M) \), we will have:

\[ \text{Res}_{\mathbb{R}^e_{R^L}} \circ D_{\mathbb{R}^e, \chi_e} = D_{\mathbb{R}^e_{R^L}, \chi_e} \circ \text{Res}_{\mathbb{R}^e_{R^L}} \]

In the same way we have:

\[ I^G \circ \text{Ind}_{\mathbb{R}^e_{R^L}} \circ D_{\mathbb{R}^e_{R^L}, \chi_e} = \iota_{G,L} \circ I^L \circ D_{\mathbb{R}^e_{R^L}, \chi_e} \]

\[ = \iota_{G,L} \circ D^t_G \circ I^L \]

\[ = D^t_G \circ \iota_{G,L} \circ I^L \]

\[ = D^t_G \circ I^G \circ \text{Ind}_{\mathbb{R}^e_{R^L}} \]

\[ = I^G \circ D_{\mathbb{R}^e, \chi_e} \circ \text{Ind}_{\mathbb{R}^e_{R^L}} \]

2- By definition of the operator \( D_{\mathbb{R}^e, \chi_e} \), and the transitivity of the induction and restriction functors, we have:

\[ D^2_{\mathbb{R}^e} = D_{\mathbb{R}^e, \chi_e} \circ \left[ \sum_{L \in \mathcal{L}_A(M)} (-1)^{\dim(A_L)} \iota_{G,L} \circ r^i_{L,G}(\sigma) \right] \]

\[ = \sum_{L \in \mathcal{L}_A(M)} (-1)^{\dim(A_L)} D_{\mathbb{R}^e, \chi_e} \circ \text{Ind}_{\mathbb{R}^e_{R^L}} \circ \text{Res}_{\mathbb{R}^e_{R^L}} \]

\[ = \sum_{L \in \mathcal{L}_A(M)} (-1)^{\dim(A_L)} \iota_{G,L} \circ D^t_G \circ \text{Ind}_{\mathbb{R}^e_{R^L}} \circ \text{Res}_{\mathbb{R}^e_{R^L}} \]

\[ = \sum_{L \in \mathcal{L}_A(M)} (-1)^{\dim(A_L)} \iota_{G,L} \circ \sum_{K \in \mathcal{L}_A(M)} (-1)^{\dim(A_K)} \iota_{G,K} \circ \text{Ind}_{\mathbb{R}^e_{R^K}} \circ \text{Res}_{\mathbb{R}^e_{R^K}} \]

\[ = \sum_{L \in \mathcal{L}_A(M)} (-1)^{\dim(A_L)} \sum_{K \in \mathcal{L}_A(M)} (-1)^{\dim(A_K)} \iota_{G,L} \circ \text{Ind}_{\mathbb{R}^e_{R^L}} \circ \text{Res}_{\mathbb{R}^e_{R^K}} \]

\[ = \sum_{K \in \mathcal{L}_A(M)} \sum_{L \in \mathcal{L}_A(K)} (-1)^{\dim(A_K)-\dim(A_L)} \iota_{G,L} \circ \text{Ind}_{\mathbb{R}^e_{R^L}} \circ \text{Res}_{\mathbb{R}^e_{R^K}} \]
Proposition 3.3. Let $(M, \sigma)$ be a discrete pair of $G$

1- For all $\rho \in \Pi(\widetilde{R}_\sigma, \chi_\sigma)$, $D_{\widetilde{R}_\sigma, \chi_\sigma}(\theta_\rho)$ is irreducible.

2- The dual in $\widetilde{R}_\sigma$ of the trivial irreducible character, $1_{\widetilde{R}_\sigma} \in \mathbb{C}(\Pi(\widetilde{R}_\sigma, \chi_\sigma))$, is equal to the sign application $\xi_{\widetilde{R}_\sigma}$.

Proof. Let $\theta_\rho \in \mathbb{C}(\Pi(\widetilde{R}_\sigma, \chi_\sigma))$, according to the last theorem we have:

$$I^G(D_{\widetilde{R}_\sigma, \chi_\sigma}(\theta_\rho)) = D_G^t(I^G(\theta_\rho))$$

since the operator $D_G^t$ preserves up to isomorphism, the irreducibility of the characters irreducible, and $I^G$ is an isomorphism, it follow that $D_{\widetilde{R}_\sigma, \chi_\sigma}(\theta_\rho)$ is also irreducible. For the second assertion, we have by definition:

$$D_{\widetilde{R}_\sigma, \chi_\sigma}(1_{\widetilde{R}_\sigma}(\tilde{r})) = \sum_{L \in \mathcal{L}_A(M)} (-1)^{\dim(A_L)} \text{Ind}_{R_\sigma^L}^{\widetilde{R}_\sigma}(1_{\widetilde{R}_\sigma}(\tilde{r})), \quad \tilde{r} \in \widetilde{R}_\sigma$$

notice that if: $\alpha_M^r = \alpha_G$ for a certain $r \in R_\sigma$ then $\tilde{r} \in \widetilde{R}_\sigma$ is not conjugated to any element of $\widetilde{R}_\sigma^L$, $L \in \mathcal{L}_A(M)$ therefore $\text{Ind}_{R_\sigma^L}^{\widetilde{R}_\sigma}(1_{\widetilde{R}_\sigma}(\tilde{r})) = 0$, hence:

$$(*) \quad D_{\widetilde{R}_\sigma, \chi_\sigma}(1_{\widetilde{R}_\sigma}(\tilde{r})) = (-1)^{\dim(A_G)} \quad \text{si} \quad \alpha_M^r = \alpha_G.$$

Now if $r \in R_\sigma$ such that $\alpha_M^r = \alpha_L$, then its inverse image $\tilde{r} \in \widetilde{R}_\sigma^L$, so:

$$D_{\widetilde{R}_\sigma, \chi_\sigma}(1_{\widetilde{R}_\sigma}(\tilde{r})) = \text{Res}_{R_\sigma^L}^{R_\sigma}(D_{\widetilde{R}_\sigma, \chi_\sigma}(1_{\widetilde{R}_\sigma}(\tilde{r}))) = D_{R_\sigma^L}(\text{Res}_{R_\sigma^L}^{R_\sigma}(1_{\widetilde{R}_\sigma}(\tilde{r}))) = D_{R_\sigma^L}(1_{R_\sigma^L}(\tilde{r})) = (-1)^{\dim(A_L)} = \xi_{R_\sigma^L}(\tilde{r})$$

the last equality is due to the expression $(*)$ applied to $L$, instead of $G$. \hfill \Box

4. THE CHARACTER OF STEINBERG

The purpose of this section is to define a special character, belonging to $V(G)$. After that, we will see some of its remarkable properties. Let $(M, \sigma)$ be a discrete pair of $G$, and $L \in \mathcal{L}_A(M)$, for a representation $\tau_1 \in \Pi_\sigma(L)$ corresponding to the trivial representation $1_{\widetilde{R}_\sigma^L} \in \Pi(\widetilde{R}_\sigma^L, \chi_\sigma)$, we call character of Steinberg associated to $(M, \sigma)$ the character:

$$S_{(M, \sigma)}^G = \sum_{L \in \mathcal{L}_A(M)} (-1)^{\dim(A_L)} i_{G,L}(\Theta_{\tau_1}).$$

Theorem 4.1. Let $(M, \sigma)$ be a discrete pair of $G$, and $\pi_1 \in \Pi_\sigma(G)$ corresponding to the trivial representation, then the character of $\pi_1$ is equal to the dual of $S_{(M, \sigma)}^G$. In particular $S_{(M, \sigma)}^G$ is an irreducible character of $i_{G,M}(\Theta_\sigma)$.\hfill \Box
Proof. Let $\Theta_{\pi_1}$ be the character of $\pi_1 \in \Pi(G)$ corresponding to the trivial character $1_{\widetilde{R}_s}$, then: $\Theta_{\pi_1} = I^G(1_{\widetilde{R}_s})$.

According to the previous proposition and theorem 3.2.1, we have:

$$D^t_G(\Theta_{\pi_1}) = D^t_G \circ I^G(1_{\widetilde{R}_s}) = I^G(D_{\widetilde{R}_s, \chi^s}(1_{\widetilde{R}_s})) = I^G(\xi_{\widetilde{R}_s})$$

We have also:

$$D^t_G(\Theta_{\pi_1}) = I^G \circ D_{\widetilde{R}_s, \chi^s}(1_{\widetilde{R}_s}) = I^G \circ \sum_{L \in \mathcal{L}(M)} (-1)^{\dim(AL)} \text{Ind}_{\widetilde{R}_s}^{\mathbb{R}} \circ \text{Res}_{\widetilde{R}_s}^\mathbb{R}(1_{\widetilde{R}_s})$$

$$= \sum_{L \in \mathcal{L}(M)} (-1)^{\dim(AL)} I^L(1_{\widetilde{R}_s})$$

$$= \sum_{L \in \mathcal{L}(M)} (-1)^{\dim(AL)} i_{G,L}(\Theta_{\pi_1})$$

$$= St^G_{(M,\sigma)}.$$ 

It follows that $St^G_{(M,\sigma)}$ is an irreducible character of $i_{G,M}(\Theta_{\sigma})$. 

\[\square\]

**Remark 4.2.** We have just found, that we have:

$$St^G_{(M,\sigma)} = I^G(\xi_{\widetilde{R}_s})$$

which implies that $St^G_{(M,\sigma)}$ corresponding to $\xi_{\widetilde{R}_s}$. Therefore $\xi_{\widetilde{R}_s}$ is an irreducible character, it is defined as a sign character of the group $\widetilde{R}_s$.

**Corollary 4.3.** Let $(M, \sigma)$ be a discrete pair of $G$ and $L \in \mathcal{L}(M)$ then:

$$r^t_{L,G}(\sigma)(St^G_{(M,\sigma)}) = St^G_{(M,\sigma)}.$$ 

**Proof.** It is clear that:

$$St^G_{(M,\sigma)} = I^G(D_{\widetilde{R}_s, \chi^s}(1_{\widetilde{R}_s})).$$

The composition with the projection $r^t_{L,G}(\sigma)$, gives us:

$$r^t_{L,G}(\sigma)(St^G_{(M,\sigma)}) = r^t_{L,G}(\sigma) \circ I^G \circ D_{\widetilde{R}_s, \chi^s}(1_{\widetilde{R}_s}) = I^L \circ \text{Res}_{\widetilde{R}_s}^\mathbb{R} \circ D_{\widetilde{R}_s, \chi^s}(1_{\widetilde{R}_s})$$

$$= I^L \circ D_{\widetilde{R}_s, \chi^s}(\text{Res}_{\widetilde{R}_s}^\mathbb{R}(1_{\widetilde{R}_s})) = I^L(D_{\widetilde{R}_s, \chi^s}(1_{\widetilde{R}_s})) = St^G_{(M,\sigma)}$$

where the last equality is due to the expression $\star$ applied to $L$. 

\[\square\]

**Corollary 4.4.** Under the hypotheses of the theorem 4.1, we have :

$$(St^G_{(M,\sigma)})^e = \pm(\Theta_{\pi_1})^e.$$ 

**Proof.** We have $D^t_G(\Theta_{\pi_1}) = St^G_{(M,\sigma)}$, then by restriction to $G_{\ell l}$, we get:

$$(St^G_{(M,\sigma)})^e = (D^t_G(\Theta_{\pi_1}))^e = \left( \sum_{L \in \mathcal{L}(M)} (-1)^{\dim(AL)} i_{G,L} \circ r^t_{L,G}(\Theta_{\pi_1}) \right)^e.$$ 

Since, the restriction of a character properly induced to $G_{\ell l}$ is zero, then

$$(D^t_G(\Theta_{\pi_1}))^e = \pm(\Theta_{\pi_1})^e$$

\[\square\]
Proposition 4.5. Let $(M, \sigma)$ be a discrete pair of $G$. Then $St_{(M, \sigma)}^G$ is elliptic if and only if $R_\sigma$ is elliptic.

Proof. Since $St_{(M, \sigma)}^G$ is an irreducible character of $i_{G,M}(\Theta_\sigma)$ and corresponding to $\xi_{\overline{R}_\sigma}$, then from [1, Proposition 2.1], $St_{(M, \sigma)}^G$ is elliptic if and only if the restriction of $\xi_{\overline{R}_\sigma}$ to $\overline{R}_{\sigma,\text{reg}}$ is non-zero, or for all $\bar{r} \in \overline{R}_{\sigma,\text{reg}}$: 

$$\xi_{\overline{R}_\sigma}(\bar{r}) = (-1)^{\text{dim}(\Delta)}$$

hence the result. □

We end this note with a direct application. We know that the choice of a minimum parabolic subgroup $P_0$ of $G$ determines a base $\Delta$ of the set of reduced roots $\Phi(G, A_0)$. If $I \subset \Delta$, let $M_I$ the Levi subgroup of $G$ defined by $I$. We have the following proposition.

Proposition 4.6. Let $(M_I, \sigma)$ be a discrete pair of $G$, where $I \subset \Delta$. If $I = \Delta - \{\alpha\}$, for a simple root $\alpha$ and $i_{G,M_I}(\Theta_\sigma)$ is reducible, then $\Theta_{\pi_I}$ and $\pm St_{(M_I, \sigma)}^G$ are the only irreducible characters of $i_{G,M_I}(\Theta_\sigma)$. Moreover these two irreducible characters are elliptic.

Proof. If $i_{G,M_I}(\Theta_\sigma)$ is reducible, then the last theorem and the decomposition of $St_{(M_I, \sigma)}^G$ show that $\Theta_{\pi_I}$ and $\pm St_{(M_I, \sigma)}^G$ are the only irreducibles characters of $i_{G,M_I}(\Theta_\sigma)$. On the other hand, if $i_{G,M_I}(\Theta_\sigma)$ is reducible then the reflexion $s_\alpha \in R_\sigma$, it follow from corolaire 4.4 and proposition 4.5 that the irreducibles characters $\Theta_{\pi_I}$ and $\pm St_{(M_I, \sigma)}^G$ are elliptics because $R_{\sigma,\text{reg}} \neq \emptyset$. □

Remark 4.7. We fix a discrete pair $(M, \sigma)$ of $G$. We saw that $St_{(M, \sigma)}^G$ is an irreducible character of $i_{G,M}(\Theta_\sigma)$ and it is elliptic if and only if $R_\sigma$ is also elliptic. With the proposition 4.6, we want to extend this result to all the characters of $i_{G,M}(\Theta_\sigma)$ and thus get out of the choices imposed by J. Arthur (fixed central extension over which the cocycle $\chi_\sigma$ splits, essential virtual triplet, positive chamber, ...), as in [5] for the classification of irreducible, tempered representations (of reductive $p$-adic groups).

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