ON BEHAVIOR OF HOMEOMORPHISMS WITH INVERSE MODULUS CONDITIONS

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Abstract

We consider some class of homeomorphisms of domains of Euclidean space, which are more general than quasiconformal mappings. For these homeomorphisms, we have obtained theorems on local behavior of its inverse mappings in a given domain. Under some additional conditions, we proved results about behavior of mappings mentioned above in the closure of the domain.

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1 Introduction

In Euclidean space, questions connected with the equicontinuity of quasiconformal mappings and some of their generalizations are relatively well studied (see., e.g., [1, Theorem 19.2], [2, Theorem 3.17] and [3, Lemma 3.12, Corollary 3.22]). The behavior of such classes is also investigated when this domain is closed (see., e.g., [4, Theorem 3.1] and [5, Theorem 3.1]). The passage to inverse mappings in the latter case does not present difficulties, since, as it is known, the quasiconformality of a direct mapping \( f \) implies the quasiconformality of the mapping \( f^{-1} \) (moreover, the quasiconformality constant of the mappings is one and the same, see. e.g., [1, Corollary 13.3]; see. also [1, Theorem 34.3]). In other words, the study of mappings, inverse to quasiconformal, does not bring anything new in comparison with investigation of quasiconformal mappings.

The situation essentially changes if instead of quasiconformal mappings we consider some more general class of homeomorphisms. Let \( M \) means modulus of curve family (see [2]) and \( dm(x) \) corresponds to Lebesque measure in \( \mathbb{R}^n \). Suppose that mapping \( f : D \rightarrow \mathbb{R}^n \), is defined in domain \( D \subset \mathbb{R}^n \), \( n \geq 2 \), and it is satisfying

\[
M(f(\Gamma)) \leq \int_D Q(x) \cdot \rho^n(x) \, dm(x) \quad \forall \rho \in \text{adm} \Gamma \tag{1.1}
\]
where \( Q : D \to [1, \infty] \) is a certain (given) fixed function (see, e.g., [6]). Recall that \( \rho \in \text{adm} \Gamma \) if and only if
\[
\int_\gamma \rho(x)|dx| \geq 1 \quad \forall \gamma \in \Gamma.
\]
In particular, all conformal and quasiconformal mappings satisfy (1.1), where function \( Q \) equals 1 or some constant, respectively (see, e.g., [7] Theorems 4.6 and 6.10). Note that in case of particular (unbounded) function \( Q \) we, generally speaking, can not replace \( f \) by \( f^{-1} \) in (1.1). (For this occasion, see the example 2, cited at the end of this work). The study of mappings \( g \), the inverses of which satisfy the relation (1.1) is a separate topic for research. In this note we are interested in the local behavior of such mappings \( g \) in the domain \( D' = f(D) \), \( f = g^{-1} \), and also in \( \overline{D'} \).

It is necessary to take into the early results of the first author [8], where mappings \( g \) with similar conditions were also studied. The main result is contained in [8, Theorem 6.1] and it is proved under the condition that two points of the domain are fixed by mappings, that it is difficult to call an optimal constraint. In particular, among linear fractional automorphisms of the unit circle onto itself is at most one such mapping, in view of which the indicated condition turns out to be meaningless. Our main goal is to study analogous families of mappings with a rejection of any conditions normalization. As example 1 shows at the end of the paper, it essentially enriches the results obtained in the article from the point of view of applications.

Main definition and denotes used below can be found in monographs [1] and [9] and therefore omitted. Let \( E, F \subset \mathbb{R}^n \) are arbitrary sets. Further \( \Gamma(E, F, D) \) we denote the family of all path \( \gamma : [a, b] \to \mathbb{R}^n \), that connect \( E \) and \( F \) in \( D \), i.e \( \gamma(a) \in E, \gamma(b) \in F \) and \( \gamma(t) \in D \) for \( t \in (a, b) \). Recall that the domain \( D \subset \mathbb{R}^n \) is called \textit{locally connected at the point} \( x_0 \in \partial D \), if for every neighborhood \( U \) of a point \( x_0 \) there is a neighborhood \( V \subset U \) of a point \( x_0 \) such that \( V \cap D \) is connected. The domain \( D \) is locally connected in the \( \partial D \), if \( D \) is locally connected at every point \( x_0 \in \partial D \). The boundary of \( D \) is called \textit{weakly flat} at a point \( x_0 \in \partial D \), if for every \( P > 0 \) and every neighborhood \( U \) of the point \( x_0 \), there is a neighborhood \( V \subset U \) of \( x_0 \) such that \( M(\Gamma(E, F, D)) > P \) for all continua \( E, F \subset D \), intersecting \( \partial U \) and \( \partial V \). The boundary of the domain \( D \) is weakly flat, if it is weakly flat at every point of boundary of \( D \).

For domains \( D, D' \subset \mathbb{R}^n, n \geq 2 \), and arbitrary Lebesgue measurable function \( Q : \mathbb{R}^n \to [1, \infty], Q(x) \equiv 0 \) for \( x \notin D \), denote \( \mathfrak{R}_Q(D, D') \) the family of all mappings \( g : D' \to D \) such that \( f = g^{-1} \) is homeomorphism of the domain \( D \) onto \( D' \) satisfying (1.1). The following assertion is valid.

**Theorem 1.1.** Suppose that \( \overline{D} \) and \( \overline{D'} \) are a compacts in \( \mathbb{R}^n \). If \( Q \in L^1(D) \), then the family \( \mathfrak{R}_Q(D, D') \) is equicontinuous in \( D' \).

For the number \( \delta > 0 \), domains \( D \) and \( D' \subset \mathbb{R}^n, n \geq 2 \), continuum \( A \subset D \) and arbitrary Lebesgue measurable function \( Q : \mathbb{R}^n \to [1, \infty], Q(x) \equiv 0 \) for \( x \notin D \), denote by \( \mathfrak{G}_{\delta, A, Q}(D, D') \)
the family of all mappings \( g : D' \to D \) such that \( f = g^{-1} \) is a homeomorphism of the domain \( D \) onto \( D' \) satisfying (1.1), wherein \( \text{diam } f(A) \geq \delta \). The following assertion is valid.

**Theorem 1.2.** Suppose that the domain \( D \) is locally connected at all boundary points, \( \overline{D} \) and \( \overline{D'} \) are compacts in \( \mathbb{R}^n \), and the domain \( D' \) has a weakly flat boundary. We also suppose that any path-connected component \( \partial D' \) has a non-degenerate continuum. If \( Q \in L^1(D) \), then each mapping \( g \in \mathfrak{G}_{\delta,A,Q}(D,D') \) extends by continuity to the mapping \( \overline{g} : \overline{D'} \to \overline{D} \), \( \overline{g}|_{D'} = g \), in addition, \( \overline{g}(\overline{D'}) = \overline{D} \) and family \( \mathfrak{G}_{\delta,A,Q}(\overline{D},\overline{D'}) \), consisting of all extended mappings \( \overline{g} : \overline{D'} \to \overline{D} \), is equicontinuous in \( \overline{D'} \).

## 2 Auxiliary information

First of all, we establish two elementary statements that play an important role in the proof of the main results. Let \( I \) be an open, closed or half-open interval in \( \mathbb{R} \). As usual, for a curve \( \gamma : I \to \mathbb{R}^n \) suppose:

\[
|\gamma| = \{ x \in \mathbb{R}^n : \exists t \in [a,b] : \gamma(t) = x \},
\]

wherein, \( |\gamma| \) is called *carrier (image) of the curve \( \gamma \). We say that the curve \( \gamma \) lies in the domain \( D \), if \( |\gamma| \subset D \), in addition, we will say that the curves \( \gamma_1 \) and \( \gamma_2 \) do not intersect if their carriers do not intersect. The curve \( \gamma : I \to \mathbb{R}^n \) is called *Jordan arc*, if \( \gamma \) is a homeomorphism on \( I \). The following (almost obvious) assertion is valid.

**Lemma 2.1.** Let \( D \) be a domain in \( \mathbb{R}^n \), \( n \geq 2 \), locally connected on its boundary. Then any two pairs of different points \( a \in D, b \in \overline{D} \) or \( c \in D, d \in \overline{D} \) can be joined by disjoint curves \( \gamma_1 : [0,1] \to \overline{D} \) and \( \gamma_2 : [0,1] \to \overline{D} \), so, that \( \gamma_i(t) \in D \) for all \( t \in (0,1) \), \( i = 1, 2 \), \( \gamma_1(0) = a, \gamma_1(1) = b, \gamma_2(0) = c, \gamma_2(1) = d \).

**Proof.** Notice, that the points of the domain are locally connected on the boundary and are accessible from within the domain by means of curves (see, e.g., [2] Proposition 13.2]). In this case, if \( n \geq 3 \), we connect the points \( a \) and \( b \) by an arbitrary Jordan arc \( \gamma_1 \) in the domain \( D \), not passing through the points \( c \) and \( d \) (which is possible in view of the local connection of \( D \) on the boundary and the transition from the curve to the broken line if it is necessary). Then \( \gamma_1 \) does not divide the domain \( D \) as a set of topological dimension 1 (see [10], Corollary 1.5.IV]), which ensures the existence of the desired curve \( \gamma_2 \). Thus, in the case of \( n \geq 3 \) the assertion of Lemma 2.1 is established.

Now let \( n = 2 \), then again the points \( c \) and \( d \) does not divide the domain \( D \) (see [10], Corollary 1.5.IV]). In this case, you can also connect points \( a \) and \( b \) by a Jordan arc \( \gamma_1 \) in \( D \), that does not pass through the points \( c \) and \( d \). In view of the Antoine theorem (see [11] Theorem 4.3, §4]) the domain \( D \) can be mapped onto some domain \( D^* \) by means of a flat homeomorphism \( \varphi : \mathbb{R}^2 \to \mathbb{R}^2 \), so, that \( \varphi(\gamma_1) = J \) and \( J \) is a segment in \( D^* \). We also note that the boundary points of the domain \( D^* \) are reachable from within \( D^* \) by means of curves. In this way, we can connect points \( \varphi(c) \) and \( \varphi(d) \) in \( D^* \) by a Jordan arc \( \alpha_2 : [0,1] \to \overline{D^*} \), which lies entirely in \( D^* \), except perhaps its end point \( \alpha_2(1) = \varphi(d) \).
It remains to show that the curve $\alpha_2$ can be chosen so that it does not intersect the segment $J$. In fact, let $\alpha_2$ crosses $J$, and let $t_1$ and $t_2$ are, respectively, the largest and the smallest values $t \in [0, 1]$, for which $\alpha_2(t) \in |J|$. Suppose also that

$$J = J(s) = \varphi(a) + (\varphi(b) - \varphi(a))s, \quad s \in [0, 1]$$

is a parametrization of the interval $J$. Let $\tilde{s}_1$ and $\tilde{s}_2 \in (0, 1)$ be such that $J(\tilde{s}_1) = \alpha_2(t_1)$ and $J(\tilde{s}_2) = \alpha_2(t_2)$. Suppose $s_2 = \max\{\tilde{s}_1, \tilde{s}_2\}$. Let $e_1 = \varphi(b) - \varphi(a)$ and $e_2$ is a unit vector, orthogonal to $e_1$, then the set

$$P_\varepsilon = \{ x = \varphi(a) + x_1 e_1 + x_2 e_2, \quad x_1 \in (-\varepsilon, s_2 + \varepsilon), \quad x_2 \in (-\varepsilon, \varepsilon) \}, \quad \varepsilon > 0,$$

is a rectangle containing $|J_1|$, where $J_1$ is a restriction of $J$ to a segment $[0, s_2]$ (see picture 1). We choose that $\varepsilon > 0$ so that $\varphi(c) \notin P_\varepsilon$, $\text{dist}(P_\varepsilon, \partial D^*) > \varepsilon$. In view of [12, Theorem 1.1,

Figure 1: The possibility of connecting two pairs of points by curves in the domain

ch. 5, § 46]) the curve $\alpha_2$ crosses $\partial P_\varepsilon$ for some $T_1 < t_1$ and $T_2 > t_2$. Let $\alpha_2(T_1) = y_1$ and $\alpha_2(T_2) = y_2$. Since $\partial P_\varepsilon$ is a connected set, it is possible to connect points $y_1$ and $y_2$ of the curve $\alpha^*(t) : [T_1, T_2] \to \partial P_\varepsilon$. Finally, we put

$$\alpha_2^*(t) = \begin{cases} \alpha_2(t), & t \in [0, 1] \setminus [T_1, T_2], \\ \alpha^*(t), & t \in [T_1, T_2] \end{cases}$$

and $\gamma_2^* := \varphi^{-1} \circ \alpha_2^*$. Then $\gamma_1$ connects $a$ and $b$ in $D$, and $\gamma_2^*$ connects $c$ and $d$ in $D$, while $\gamma_1$ and $\gamma_2^*$ do not intersect, which should be established. □

Above we introduced the concept of a weak plane of the boundary of the region, without mentioning, at the same time, internal points. The following lemma contains the assertion that at the indicated points the property of the «weak plane» always takes place.

**Lemma 2.2.** Let $D$ be a domain in $\mathbb{R}^n$, $n \geq 2$, and $x_0 \in D$. Then for every $P > 0$ and for any neighborhood $U$ of the point $x_0$ there is a neighborhood $V \subset U$ of the same point such that $M(\Gamma(E, F, D)) > P$ for arbitrary continua $E, F \subset D$, intersecting $\partial U$ and $\partial V$. 

Proof. Let $U$ be an arbitrary neighborhood of $x_0$. Let’s choose $\varepsilon_0 > 0$ so that $B(x_0, \varepsilon_0) \subset D \cap U$. Let $c_n$ be a positive constant, defined in the relation (10.11) in [1], and the number $\varepsilon \in (0, \varepsilon_0)$ is so small that $c_n \cdot \log \frac{\varepsilon_0}{\varepsilon} > \delta$. Suppose $V := B(x_0, \varepsilon)$. Let $E, F$ be arbitrary continua intersecting $\partial U$ and $\partial V$, then also $E$ and $F$ intersecting $S(x_0, \varepsilon_0)$ and $\partial V$ (see [12 Theorem 1.I, ch. 5, § 46]). The necessary conclusion follows on the basis of [1] par. 10.12], because the

$$M(\Gamma(E, F, D)) \geq c_n \cdot \log \frac{\varepsilon_0}{\varepsilon} > \delta. \quad \square$$

3 Proof of Theorem 1.1

We prove the theorem [1.1] by contradiction. Suppose, the family $\mathcal{R}_Q(D, D')$ is not equicontinuous at some point $y_0 \in D'$, in other words, there are $y_0 \in D'$ and $\varepsilon_0 > 0$, such that for any $m \in \mathbb{N}$ there exists an element $y_m \in D'$, $|y_m - y_0| < 1/m$, and a homeomorphism $g_m \in \mathcal{R}_Q(D, D')$, for which

$$|g_m(y_m) - g_m(y_0)| \geq \varepsilon_0. \quad (3.1)$$

We draw a line $r = r_m(t) = g_m(y_0) + (g_m(y_m) - g_m(y_0))t$, $-\infty < t < \infty$ through $g_m(y_m)$ and $g_m(y_0)$ (see picture 2). Note that this line $r = r_m(t)$ for $t \geq 1$ must intersect the domain $D$ in view of [12 Theorem 1.I, ch. 5, § 46]), since the domain $D$ is bounded; thus, there exists $t_1^m \geq 1$ such that $r_m(t_1^m) = x_1^m \in \partial D$. Without loss of generality we can assume that $r_m(t) \in D$ for all $t \in [1, t_1^m)$, then the segment $\gamma_1^m(t) = g_m(y_0) + (g_m(y_m) - g_m(y_0))t$, $t \in [1, t_1^m]$, belongs to $D$ for all $t \in [1, t_1^m]$, $\gamma_1^m(t_1^m) = x_1^m \in \partial D$ and $\gamma_1^m(1) = g_m(y_m)$. In view of analogous considerations, there are $t_2^m < 0$ and a segment $\gamma_2^m(t) = g_m(y_0) + (g_m(y_m) - g_m(y_0))t$, $t \in [t_2^m, 0]$, such that $\gamma_2^m(t_2^m) = x_2^m \in \partial D$, $\gamma_2^m(0) = g_m(y_0)$ and $\gamma_2^m(t)$ belongs to $D$ for all $t \in (t_2^m, 0]$. Put $f_m := g_m^{-1}$. Since $f_m$ is a homeomorphism, for each fixed $m \in \mathbb{N}$ the limit sets $C(f_m, x_1^m)$ and $C(f_m, x_2^m)$ of mappings $f_m$ at the corresponding boundary points $x_1^m, x_2^m \in \partial D$
lie on \( \partial D' \) (see \[9\], Proposition 13.5)). Consequently, there is a point \( z^m_1 \in D \cap |\gamma^m_1| \) such that \( \text{dist} (f_m(z^m_1), \partial D') < 1/m \). As \( \overline{D'} \) is compact, it can be assumed that the sequence \( f_m(z^m_1) \to p_1 \in \partial D' \) for \( m \to \infty \). Similarly, there is a sequence \( z^m_2 \in D \cap |\gamma^m_2| \) such that \( \text{dist} (f_m(z^m_2), \partial D') < 1/m \) and \( f_m(z^m_2) \to p_2 \in \partial D' \) for \( m \to \infty \).

Let \( P_m \) be the part of the interval \( \gamma^m_1 \), enclosed between the points \( g_m(y_m) \) and \( z^m_1 \), and \( Q_m \) be the part of the interval \( \gamma^m_2 \), enclosed between the points \( g_m(y_0) \) and \( z^m_2 \). By construction and by (3.1), \( \text{dist} (P_m, Q_m) \geq \varepsilon_0 > 0 \). Let \( \Gamma_m = \Gamma(P_m, Q_m, D) \), then the function

\[
\rho(x) = \begin{cases} \frac{1}{\varepsilon_0}, & x \in D, \\ 0, & x \notin D \end{cases}
\]

is admissible for the family \( \Gamma_m \), since for an arbitrary (locally rectifiable) curve \( \gamma \in \Gamma_m \) it is completed \( \int \rho(x)dx \geq \frac{l(\gamma)}{\varepsilon_0} \geq 1 \) (where \( l(\gamma) \) denotes the length of the curve \( \gamma \)). Since by the hypothesis the mappings \( f_m \) satisfy (1.1) we obtain:

\[
M(f_m(\Gamma_m)) \leq \frac{1}{\varepsilon_0} \int_D Q(x)dm(x) := c < \infty,
\]

as \( Q \in L^1(D) \). On the other hand, \( \text{diam } f_m(P_m) \geq |y_m - f_m(z^m_1)| \geq (1/2) \cdot |y_0 - p_1| > 0 \) and \( \text{diam } f_m(Q_m) \geq |y_0 - f_m(z^m_2)| \geq (1/2) \cdot |y_0 - p_2| > 0 \) for large \( m \in \mathbb{N} \), in addition

\[
\text{dist} (f_m(P_m), f_m(Q_m)) \leq |y_m - y_0| \to 0, \quad m \to \infty.
\]

Then, in view of Lemma 2.2

\[
M(f_m(\Gamma_m)) = M(f_m(P_m), f_m(Q_m), D') \to \infty, \quad m \to \infty,
\]

which contradicts relation (3.2). This contradiction indicates that the assumption in (3.1) is erroneous, which completes the proof of the theorem. \( \square \)

4 On the behavior of mappings in the closure of a domain

Let us pass to the question of the global behavior of mappings. The following assertion indicates that for sufficiently good domains and mappings with condition (1.1) the image of a fixed continuum under these mappings can not approach the boundary of the corresponding domain as soon as the Euclidean of the diameter of this continuum is bounded from below (see also \[1\], Theorems 21.13 and 21.14).

**Lemma 4.1.** Suppose that the domain \( D \) is locally path-connected on \( \overline{D}, \overline{D} \) and \( \overline{D'} \) are compact sets in \( \mathbb{R}^n \), \( n \geq 2 \), \( D' \) has a weakly flat boundary, \( Q \in L^1(D) \) and there is no connected component of the boundary \( \partial D' \) degenerating to a point. Let \( f_m : D \to D' \) be a sequence of homeomorphisms of the domain \( D \) onto the domain \( D' \) with the condition (1.1). Let there also be a continuum \( A \subset D \) and a number \( \delta > 0 \) such that \( \text{diam } f_m(A) \geq \delta > 0 \) for all \( m = 1, 2, \ldots \). Then there exists \( \delta_1 > 0 \) such that

\[
\text{dist} (f_m(A), \partial D') > \delta_1 > 0 \quad \forall \ m \in \mathbb{N}.
\]
Proof. Suppose, the contrary situation, that for each $k \in \mathbb{N}$ there exists $m = m_k : \text{dist} (f_{m_k} (A), \partial D') < 1/k$. Without loss of generality we can assume that the sequence $m_k$ is monotonically increasing. By condition $\overline{D'}$ is compact, therefore $\partial D'$ is also compact as a closed subset of the compactum $\overline{D'}$. In addition, $f_{m_k} (A)$ is compact as a continuous image of the compactum $A$ under the mapping $f_{m_k}$. Then there are $x_k \in f_{m_k} (A)$ and $y_k \in \partial D'$ such that $\text{dist} (f_{m_k} (A), \partial D') = |x_k - y_k| < 1/k$ (see picture 3). As $\partial D'$ since $A$ is compact, we can assume that $y_k \rightarrow y_0 \in \partial D'$, $k \rightarrow \infty$; then also

$$x_k \rightarrow y_0 \in \partial D', \quad k \rightarrow \infty.$$  

Let $K_0$ be a connected component $\partial D'$, containing the point $y_0$, then, obviously $K_0$ is a continuum in $\mathbb{R}^n$. Since $D'$ has a weakly flat boundary, for each $k \in \mathbb{N}$ the mapping $g_{m_k} := f_{m_k}^{-1}$ extends to a continuous mapping $g_{m_k} : \overline{D'} \rightarrow \overline{D}$ (see [9, Theorem 4.6]), furthermore, $g_{m_k}$ uniformly continuous on $\overline{D'}$ as a mapping that is continuous on a compactum. Then for every $\varepsilon > 0$ there is $\delta_k = \delta_k (\varepsilon) < 1/k$ such that

$$|g_{m_k} (x) - g_{m_k} (x_0)| < \varepsilon \quad \forall \ x, x_0 \in \overline{D'}, \quad |x - x_0| < \delta_k, \quad \delta_k < 1/k. \quad (4.1)$$  

Let $\varepsilon > 0$ be an arbitrary number with the condition

$$\varepsilon < (1/2) \cdot \text{dist} (\partial D, A), \quad (4.2)$$  

where $A$ is a continuum from the condition of the lemma. For each fixed $k \in \mathbb{N}$ we consider the set

$$B_k := \bigcup_{x_0 \in K_0} B(x_0, \delta_k), \quad k \in \mathbb{N}.$$  

Note that $B_k$ is an open set containing $K_0$, in other words, $B_k$ is a neighborhood of the continuum $K_0$. In view of [13, Lemma 2.2] there exists a neighborhood $U_k \subset B_k$ of the
continuum $K_0$, such that $U_k \cap D'$ is connected. Without loss of generality, we can assume that $U_k$ is an open set, then $U_k \cap D'$ is also linearly connected (see [10, Proposition 13.1]).

Let $\text{diam} K_0 = m_0$, then there exist $z_0, w_0 \in K_0$ such that $\text{diam} K_0 = |z_0 - w_0| = m_0$. Hence, we can choose sequences $\overline{y}_k \in U_k \cap D'$, $z_k \in U_k \cap D'$ and $w_k \in U_k \cap D'$ such that $z_k \to z_0$, $\overline{y}_k \to y_0$ and $w_k \to w_0$ for $k \to \infty$. We can assume that

$$|z_k - w_k| > m_0/2, \quad \forall \ k \in \mathbb{N}.$$  

(4.3)

We connect consecutively the points $z_k$, $\overline{y}_k$ and $w_k$ of the curve $\gamma_k$ in $U_k \cap D'$ (this is possible, since $U_k \cap D'$ is path-connected). Let $|\gamma_k|$ be, as usual, the carrier (image) of the curve $\gamma_k$ in $D'$. Then $g_{mk}(|\gamma_k|)$ is a compact set in $D$. Let $x \in |\gamma_k|$, then there is $x_0 \in K_0 : x \in B(x_0, \delta_k)$. We will fix $\omega \in A \subset D$. Because the $x \in |\gamma_k|$, then $x$ is an interior point of the domain $D'$, so we have the right to write $g_{mk}(x)$ instead of $\overline{g}_{mk}(x)$ for the indicated $x$. In this case, from (4.1) and (4.2), in view of the triangle inequality, for large $k \in \mathbb{N}$ we obtain:

$$|g_{mk}(x) - \omega| \geq |\omega - \overline{g}_{mk}(x_0)| - |\overline{g}_{mk}(x_0) - g_{mk}(x)| \geq$$

$$\geq \text{dist} (\partial D, A) - (1/2) \cdot \text{dist} (\partial D, A) = (1/2) \cdot \text{dist} (\partial D, A) > \varepsilon.$$  

(4.4)

Passing to (4.4) to $\inf$, over all $x \in |\gamma_k|$ and all $\omega \in A$, we obtain:

$$\text{dist} (g_{mk}(|\gamma_k|), A) > \varepsilon, \quad \forall \ k = 1, 2, \ldots.$$  

(4.5)

In view of (4.5) the length of an arbitrary curve joining compacta $g_{mk}(|\gamma_k|)$ and $A$ in $D$, not less than $\varepsilon$. Put $\Gamma_k := \Gamma(g_{mk}(|\gamma_k|), A, D)$, then the function $\rho(x) = 1/\varepsilon$ for $x \in D$ and $\rho(x) = 0$ for $x \notin D$ is admissible for $\Gamma_k$, since $\int_{\gamma} \rho(x) |dx| \geq \frac{l(\gamma)}{\varepsilon} \geq 1$ for $\gamma \in \Gamma_k$ (where $l(\gamma)$ denotes the length of the curve $\gamma$). By the definition of mappings $f_{mk}$ in (1.1), we have:

$$M(f_{mk}(|\Gamma_k|)) \leq \frac{1}{\varepsilon^n} \int_D Q(x) \ dm(x) = c = c(\varepsilon, Q) < \infty.$$  

(4.6)

Since by hypothesis $Q \in L^1(D)$.

We now show that we arrive at a contradiction with (4.6) in view of the weak boundary plane $\partial D'$. We choose at the point $y_0 \in \partial D'$ the ball $U := B(y_0, r_0)$, where $r_0 > 0$ and $r_0 < \min \{\delta/4, m_0/4\}$, $\delta$ is a number from the condition of the lemma and $\text{diam} K_0 = m_0$. Notice, that $|\gamma_k| \cap U \neq \emptyset \neq |\gamma_k| \cap (D' \setminus U)$ for sufficiently large $k \in \mathbb{N}$, because the $\text{diam} |\gamma_k| \geq m_0/2 > m_0/4$ and $\overline{y}_k \in |\gamma_k|$, $\overline{y}_k \to y_0$ for $k \to \infty$. In view of the same considerations $f_{mk}(A) \cap U \neq \emptyset \neq f_{mk}(A) \cap (D' \setminus U)$. As $|\gamma_k|$ and $f_{mk}(A)$ are continua, then

$$f_{mk}(A) \cap \partial U \neq \emptyset, \quad |\gamma_k| \cap \partial U \neq \emptyset,$$  

(4.7)

see [12, Theorem 1.1, r.l. 5, § 46]. For a fixed $P > 0$, let $V \subset U$ is a neighborhood of the point $y_0$, corresponding to the definition of a weakly flat boundary, that is, such that for any
continua $E, F \subset D'$ with condition $E \cap \partial U \neq \emptyset \neq E \cap \partial V$ and $F \cap \partial U \neq \emptyset \neq F \cap \partial V$ is satisfied the inequality

$$M(\Gamma(E,F,D')) > P.$$  
(4.8)

We note that for sufficiently large $k \in \mathbb{N}$

$$f_{m_k}(A) \cap \partial V \neq \emptyset, \quad |\gamma_k| \cap \partial V \neq \emptyset.$$  
(4.9)

Indeed $\overline{y_k} \in |\gamma_k|, x_k \in f_{m_k}(A)$, where $x_k, \overline{y_k} \to y_0 \in V$ for $k \to \infty$, therefore $|\gamma_k| \cap V \neq \emptyset \neq f_{m_k}(A) \cap V$ for large $k \in \mathbb{N}$. Besides $\text{diam} V \leq \text{diam} U = 2r_0 < m_0/2$ and, since $\text{diam} |\gamma_k| > m_0/2$ in view of (4.3), then $|\gamma_k| \cap (D' \setminus V) \neq \emptyset$. Then $|\gamma_k| \cap \partial V \neq \emptyset$ (see [12 Theorem 1.1, ch. 5, § 46]). Similarly, $\text{diam} V \leq \text{diam} U = 2r_0 < \delta/2$ and, since $\text{diam} f_{m_k}(A) > \delta$ by hypothesis, then $f_{m_k}(A) \cap (D' \setminus V) \neq \emptyset$. In view of [12 Theorem 1.1, ch. 5, § 46] we have: $f_{m_k}(A) \cap \partial V \neq \emptyset$. The relations in (4.9) are established.

Thus, according to (4.7), (4.8) and (4.9), we get that

$$M(\Gamma(f_{m_k}(A), |\gamma_k|, D')) > P.$$  
(4.10)

Notice, that $\Gamma(f_{m_k}(A), |\gamma_k|, D') = f_{m_k}(\Gamma(A,g_{m_k}(|\gamma_k|), D)) = f_{m_k}(\Gamma_k)$, so that inequality (4.10) can be rewritten in the form

$$M(\Gamma(f_{m_k}(A), g_{m_k}(|\gamma_k|), D)) = M(f_{m_k}(\Gamma_k)) > P,$$

which contradicts inequality (4.6). The resulting contradiction indicates the incorrectness of the original assumption $\text{dist}(f_{m_k}(A), \partial D') < 1/k$. The lemma is proved. \(\Box\)

**Proof of Theorem 1.2** Since $D'$ has a weakly flat boundary, each $g \in \mathcal{S}_{\delta,A,Q}(D,D')$ extends to a continuous mapping $\overline{g} : \overline{D'} \to \overline{D}$ (see [9, Theorem 4.6]).

We verify equality $\overline{\gamma(D')} = \overline{D}$. In fact, by definition $\overline{\gamma(D')} \subset \overline{D}$. It remains to show the converse inclusion $\overline{D} \subset \overline{\gamma(D')}$. Let $x_0 \in \overline{D}$, then we show that $x_0 \in \overline{\gamma(D')}$. If $x_0 \in \overline{D}$, then either $x_0 \in D$, or $x_0 \in \partial D$. If $x_0 \in D$, then there is nothing to prove, since by hypothesis $\overline{\gamma(D')} = D$. Now let $x_0 \in \partial D$, then there be $x_k \in D$ and $y_k \in D'$ such that $x_k = \overline{g}(y_k)$ and $x_k \to x_0$ for $k \to \infty$. Since $\overline{D'}$ is compact, we can assume that $y_k \to y_0 \in \overline{D'}$ for $k \to \infty$. Since $f = g^{-1}$ is a homeomorphism, then $y_0 \in \partial D'$. Since $\overline{g}^{-1}$ is continuous in $\overline{D'}$, $\overline{g}(y_k) \to \overline{g}(y_0)$. However, in this case, $\overline{g}(y_0) = x_0$, since $\overline{g}(y_k) = x_k$ and $x_k \to x_0$, $k \to \infty$. Hence, $x_0 \in \overline{\gamma(D')}$. The inclusion $\overline{D} \subset \overline{\gamma(D')}$ is proved and, hence, $\overline{D} = \overline{\gamma(D')}$, as required.

Equivicontinuity of curve family $\mathcal{S}_{\delta,A,Q}(\overline{D}, \overline{D'})$ at interior points $D'$ is the result of the theorem 1.1. It remains to show that this family is equicontinuous at the boundary points.

We carry out the proof by contradiction. Suppose we find a point $z_0 \in \partial D'$, a number $\varepsilon_0 > 0$ and sequences $z_m \in \overline{D'}, z_m \to z_0$ for $m \to \infty$ and $\overline{g}_m \in \mathcal{S}_{\delta,A,Q}(\overline{D}, \overline{D'})$ such that

$$|\overline{g}_m(z_m) - \overline{g}_m(z_0)| \geq \varepsilon_0, \quad m = 1, 2, \ldots$$  
(4.11)

Put $g_m := \overline{g}_m|_{D'}$. Since $g_m$ extends by continuity to the boundary of $D'$, we can assume that $z_m \in D'$ and, hence, $\overline{g}_m(z_m) = g_m(z_m)$. In addition, there is one more sequence $z'_m \in D'$,
$z'_m \to z_0$ for $m \to \infty$, such that $|g_m(z'_m) - \bar{g}_m(z_0)| \to 0$ for $m \to \infty$. Since $\bar{D}$ is compact, we can assume that the sequences $g_m(z_m)$ and $\bar{g}_m(z_0)$ are convergent for $m \to \infty$. Let $g_m(z_m) \to \bar{x}_1$ and $\bar{g}_m(z_0) \to \bar{x}_2$ for $m \to \infty$. By continuity of the modulus from (4.11) it follows that $\bar{x}_1 \neq \bar{x}_2$, moreover, since the homeomorphisms preserve the boundary, $\bar{x}_2 \in \partial D$.

Let $x_1$ and $x_2$ be arbitrary distinct points of the continuum $A$, none of which coincide with $c \bar{x}_1$. By Lemma 2.1 we can join points $x_1$ and $\bar{x}_1$ by the path $\gamma_1 : [0, 1] \to \bar{D}$, and points $x_2$ and $\bar{x}_2$ by the curve $\gamma_2 : [0, 1] \to \bar{D}$ such that $|\gamma_1| \cap |\gamma_2| = \emptyset$, $\gamma_i(t) \in D$ for all $t \in (0, 1)$, $i = 1, 2$.

By continuity of the modulus from (4.11) it follows that $\gamma_i(0) = x_1$, $\gamma_i(1) = \bar{x}_i$, $\gamma_2(0) = x_2$ and $\gamma_2(1) = \bar{x}_2$. Since $D$ is locally connected on its boundary there are neighborhoods $U_1$ and $U_2$ of points $\bar{x}_1$ and $\bar{x}_2$, whose closures do not intersect, such that $W_i := D \cap U_i$ is a path-connected set. By reducing the neighborhood $U_i$, if necessary, we can assume that $\bar{U}_1 \cap |\gamma_2| = \emptyset = \bar{U}_2 \cap |\gamma_1|$. Without loss of generality, we can assume that $g_m(z_m) \in W_1$ and $g_m(z'_m) \in W_2$ for all $m \in \mathbb{N}$. Let $a_1$ and $a_2$ be arbitrary points belonging to $|\gamma_1| \cap W_1$ and $|\gamma_2| \cap W_2$. Let $t_1, t_2$ be such that $\gamma_1(t_1) = a_1$ and $\gamma_2(t_2) = a_2$.

We connect the point $a_1$ with the point $g_m(z_m)$ by the curve $\alpha_m : [t_1, 1] \to W_1$ such that $\alpha_m(t_1) = a_1$ and $\alpha_m(1) = g_m(z_m)$. Similarly, we connect a point $a_2$ with the point $g_m(z'_m)$ by the curve $\beta_m : [t_2, 1] \to W_2$ such that $\beta_m(t_2) = a_2$ and $\beta_m(1) = g_m(z'_m)$ (see picture 4).

We now put

$$C^1_m(t) = \begin{cases} \gamma_1(t), & t \in [0, t_1], \\ \alpha_m(t), & t \in [t_1, 1], \end{cases} \quad C^2_m(t) = \begin{cases} \gamma_2(t), & t \in [0, t_2], \\ \beta_m(t), & t \in [t_2, 1]. \end{cases}$$

Let, as usual, $|C^1_m|$ and $|C^2_m|$ are the carriers of the curves $C^1_m$ and $C^2_m$, respectively. We note that, by construction, $|C^1_m|$ and $|C^2_m|$ are two disjoint continua in $D$, and $\text{dist}(|C^1_m|, |C^2_m|) > l_0 > 0$ for all $m = 1, 2, \ldots$. You can take, for example,

$$l_0 = \min\{\text{dist}(|\gamma_1|, |\gamma_2|), \text{dist}(|\gamma_1|, U_2), \text{dist}(|\gamma_2|, U_1), \text{dist}(U_1, U_2)\}.$$
Now let $\Gamma_m$ be a family of curves connecting $|C^1_m|$ and $|C^2_m|$ in $D$. Then the function

$$\rho(x) = \begin{cases} \frac{1}{l_0}, & x \in D \\ 0, & x \notin D \end{cases}$$

is admissible for the family $\Gamma_m$, since $\int_\gamma \rho(x)|dx| \geq \frac{t(\gamma)}{l_0} \geq 1$ for $\gamma \in \Gamma_m$ (where $l(\gamma)$ denotes the length of the curve $\gamma$). By hypothesis, the mappings $f_m, f_m = g_m^{-1}$, satisfy (1) for $Q \in L^1(D)$, so that we obtain:

$$M(f_m(\Gamma_m)) \leq \frac{1}{n_0} \int_Q Q(x) \, dm(x) := c = c(l_0, Q) < \infty.$$  \hspace{1cm} (4.12)

On the other hand, by Lemma 4.1 there is a number $\delta_1 > 0$ such that $\text{dist} (f_m(A), \partial D') > \delta_1 > 0$, $m = 1, 2, \ldots$. From this we get that

$$\text{diam } f_m(|C^1_m|) \geq |z_m - f_m(x_1)| \geq (1/2) \cdot \text{dist} (f_m(A), \partial D') > \delta_1/2,$$

$$\text{diam } f_m(|C^2_m|) \geq |z_m' - f_m(x_2)| \geq (1/2) \cdot \text{dist} (f_m(A), \partial D') > \delta_1/2 \tag{4.13}$$

for some $M_0 \in \mathbb{N}$ and for all $m \geq M_0$. We choose at the point $z_0 \in \partial D'$ the ball $U := B(z_0, r_0)$, where $r_0 > 0$ and $r_0 < \delta_1/4$, where $\delta_1$ is the number of the relations in (4.13). Notice, that $f_m(|C^1_m|) \cap U \neq \emptyset \neq f_m(|C^1_m|) \cap (D' \setminus U)$ for sufficiently large $m \in \mathbb{N}$, since $\text{diam } f_m(|C^1_m|) \geq \delta_1/2$ and $z_m \in f_m(|C^1_m|)$, $z_m \to z_0$ for $m \to \infty$. In view of the same considerations $f_m(|C^2_m|) \cap U \neq \emptyset \neq f_m(|C^2_m|) \cap (D' \setminus U)$. Since $f_m(|C^1_m|)$ and $f_m(|C^2_m|)$ are continua

$$f_m(|C^1_m|) \cap \partial U \neq \emptyset, \quad f_m(|C^2_m|) \cap \partial U \neq \emptyset, \tag{4.14}$$

see [12] Theorem 1.I, ch. 5, § 46]. For a fixed $P > 0$, let further $V \subset U$ be a neighborhood of the point $z_0$, corresponding to the definition of a weakly flat boundary, that is, such that for any continua $E, F \subset D'$ with the condition $E \cap \partial U \neq \emptyset \neq E \cap \partial V \cup F \cap \partial U \neq \emptyset \neq F \cap \partial V$ the inequality holds

$$M(\Gamma(E, F, D')) > P. \hspace{1cm} (4.15)$$

We note that for sufficiently large $m \in \mathbb{N}$

$$f_m(|C^1_m|) \cap \partial V \neq \emptyset, \quad f_m(|C^2_m|) \cap \partial V \neq \emptyset \tag{4.16}$$

Indeed, $z_m \in f_m(|C^1_m|), z'_m \in f_m(|C^2_m|)$, where $z_m, z'_m \to z_0 \in V$ for $m \to \infty$, therefore $f_m(|C^1_m|) \cap V \neq \emptyset \neq f_m(|C^2_m|) \cap V$ for large $m \in \mathbb{N}$. In addition, $\text{diam } V \leq \text{diam } U = 2r_0 < \delta_1/2$ and, since $\text{diam } f_m(|C^1_m|) > \delta_1/2$ in view of (4.13), then $f_m(|C^1_m|) \cap (D' \setminus U) \neq \emptyset$. Then $f_m(|C^1_m|) \cap \partial V \neq \emptyset$ (see [12] Theorem 1.I, ch. 5, § 46]). Similarly, $\text{diam } V \leq \text{diam } U = 2r_0 < \delta_1/2$ and since $\text{diam } f_m(|C^2_m|) > \delta$ in view of (4.13), then $f_m(|C^2_m|) \cap (D' \setminus U) \neq \emptyset$. Then by [12] Theorem 1.I, ch. 5, § 46] we have: $f_m(|C^1_m|) \cap \partial V \neq \emptyset$. Thus, (4.16) is proved.

According to (4.15) and taking into account (4.14) and (4.16), we get that

$$M(f_m(\Gamma_m)) = M(\Gamma(f_m(|C^1_m|), f_m(|C^2_m|), D')) > P,$$

which contradicts inequality (4.12). This contradiction indicates the incorrectness of the original assumption made in (4.11). The theorem is proved. \(\square\)
5 Some examples

We begin with a simple example of mappings on the complex plane.

Example 1. As it is known, the linear-fractional automorphisms of the unit disk \( \mathbb{D} \subset \mathbb{C} \) onto itself are given by the formula \( f(z) = e^{i\theta} \frac{z-a}{1-\overline{a}z}, \ z \in \mathbb{D}, \ a \in \mathbb{C}, \ |a| < 1, \ \theta \in [0, 2\pi) \). The indicated mappings are 1-homeomorphisms; all the conditions of Theorem \[1.2\] are satisfied, except for the condition \( \text{diam} f(A) \geq \delta \), which, in general, can be violated.

If, for example, \( \theta = 0 \) and \( a = 1/n, n = 1, 2, \ldots \), then \( f_n(z) = \frac{z-1/n}{1-z/n} = \frac{n^2-1}{n^2-z} \). Let \( A = [0, 1/2] \), then \( f_n(0) = -1/n \to 0 \) and \( f_n(1/2) = \frac{n-2}{2n-1} \to 1/2, \ n \to \infty \). Hence we see that the sequence \( f_n \) satisfies condition \( \text{diam} f_n(A) \geq \delta \); for example, for \( \delta = 1/4 \). By direct calculations we are convinced that \( f_n^{-1}(z) = \frac{z^{n+1}}{1+z/n} \) and, hence, \( f_n^{-1} \) converge uniformly to \( f^{-1}(z) = z \). Thus, the sequence \( f_n^{-1}(z) \) is equicontinuous in \( \mathbb{D} \).

If we put \( f_n^{-1}(z) = \frac{z^{-(n-1)/n}}{1-z^{1/(n-1)/n}} = \frac{n^2-z+1}{n^2-nz+1} \), then, as it is easy to see, such a sequence converges locally uniformly to \( -1 \) inside \( \mathbb{D} \); in the same time, \( f_n^{-1}(1) = 1 \). Taking this into account, by direct computations, we conclude that the sequence \( f_n^{-1} \) is not equicontinuous at the point 1; in this case \( f_n(z) = \frac{z^{n+1}}{1+z(n-1)/n} \) and condition \( \text{diam} f_n(A) \geq \delta \) for any \( \delta > 0 \), not depending on \( n \), can be satisfied in view of Theorem \[1.2\] if we put \( f_n^{-1}(z) \).

From what has been said, it follows that in the conditions of Theorem \[1.2\] in general, one cannot refuse the additional requirement that \( \text{diam} f(A) \geq \delta \).

Example 2. Let \( \rho \geq 1 \) be so large that the number \( n/p(n-1) \) is less than 1, and let, in addition, \( \alpha \in (0, n/p(n-1)) \) be an arbitrary number. We define the sequence of mappings \( f_m : \mathbb{B}^n \rightarrow B(0, 2) \) of the ball \( \mathbb{B}^n \) onto the ball \( B(0, 2) \) in the following way:

\[
f_m(x) = \begin{cases} \frac{1+|x|^\alpha}{|x|} \cdot x, & 1/m \leq |x| \leq 1, \\ \frac{1+(1/m)^\alpha}{1/m} \cdot x, & 0 < |x| < 1/m. \end{cases}
\]

Notice, that \( f_m \) satisfies (5.1) for \( Q = \left( \frac{1+|x|^\alpha}{|x|^\alpha} \right)^{n-1} \in L^1(\mathbb{B}^n) \) (see [8] proof of Theorem 7.1) and that \( B(0, 2) \) has a weakly flat boundary (see [14] Lemma 4.3). By construction of the mappings \( f_m \) fixes an infinite number of points of the unit ball for all \( m \geq 2 \).

We establish the equicontinuous of mappings \( g_m := f_m^{-1} \in B(0, 2) \) (for convenience we use the notation \( g_m \) also for continuous extension of the mapping \( g_m \) in \( B(0, 2) \)). It is not hard to see that

\[
g_m(y) := f_m^{-1}(y) = \begin{cases} \frac{y}{|y|}(|y|-1)^{1/\alpha}, & 1 + 1/m^\alpha \leq |y| < 2, \\ \frac{1+(1/m)^\alpha}{1/m} \cdot y, & 0 < |y| < 1 + 1/m^\alpha. \end{cases}
\]

The mappings \( g_m \) map \( B(0, 2) \) onto \( \mathbb{B}^n \). We fix \( y_0 \in B(0, 2) \). The following three situations are possible:

1) \( |y_0| < 1 \). We choose \( \delta_0 = \delta_0(y_0) \) such that \( B(y_0, \delta_0) \subset B(0, 1) \). For the number \( \varepsilon > 0 \) we put \( \delta_1 = \delta_1(\varepsilon, y_0) := \min\{\delta_0, \varepsilon\} \). In this case, with \( y \in B(y_0, \delta_1) \) and all \( m = 1, 2, \ldots \) we
have that \( |g_m(y) - g_m(y_0)| = \frac{(1/m)}{1 + (1/m)^\alpha} |y - y_0| < |y - y_0| < \varepsilon \), which proves the equicontinuity of the family \( g_m \) at the point \( y_0 \).

2) \( |y_0| > 1 \). By the definition of mappings \( g_m \) one can find \( m_0 = m_0(y_0) \in \mathbb{N} \) and \( \delta_0 = \delta_0(y_0) > 0 \) such that \( g_m(y) = \frac{y}{|y|} (|y| - 1)^{1/\alpha} \) for all \( B(y, \delta_0) \cap B(0, 2) \) and all \( m \geq m_0 \). We take \( \varepsilon > 0 \). Putting \( g(y) = \frac{y}{|y|} (|y| - 1)^{1/\alpha} \), we note that \( |g_m(y) - g_m(y_0)| = |g(y) - g(y_0)| < \varepsilon \) for \( m \geq m_0 \) and some \( \delta = \delta(\varepsilon, y_0) \), \( \delta < \delta_0 \), since the mapping \( g(y) = \frac{y}{|y|} (|y| - 1)^{1/\alpha} \) since a is continuous in \( B(0, 2) \).

3) Finally, consider the «borderline» case \( y_0 \in \mathbb{S}^{n-1} = \partial \mathbb{B}^n \). Let \( \delta_0 = \delta_0(y_0) \) be such that \( B(y, \delta_0) \subset B(0, 2) \). By definition, we have \( g_m(y_0) = \frac{(1/m)}{1 + (1/m)^\alpha} \cdot y_0, m = 1, 2, \ldots \). Notice, that

\[
|g_m(y) - g_m(y_0)| \leq \max \left\{ \left| \frac{(1/m)}{1 + (1/m)^\alpha} \cdot y_0 - \frac{y}{|y|} (|y| - 1)^{1/\alpha} \right|, \frac{(1/m)}{1 + (1/m)^\alpha} |y - y_0| \right\}.
\]

For the number \( \varepsilon > 0 \) we find the number \( m_1 = m_1(\varepsilon) > 0 \), such that \( 1/m < \varepsilon/2 \). Put \( \delta_0 = \delta_0(\varepsilon, y_0) = \min\{1, \varepsilon/2, \delta_0\} \). Using the triangle inequality, and the fact that \( 1/\alpha > 1 \), we get:

\[
\left| \frac{y}{|y|} (|y| - 1)^{1/\alpha} - \frac{(1/m)}{1 + (1/m)^\alpha} \cdot y_0 \right| \leq (|y| - 1)^{1/\alpha} + 1/m < \varepsilon/2 + \varepsilon/2 = \varepsilon \text{ for } m > m_1
\]

and \( |y - y_0| < \delta_0 \). The last relation for \( 1 \leq m \leq m_1 \) is also satisfied for \( |y - y_0| < \delta_m \) and some \( \delta_m = \delta_m(\varepsilon, y_0) > 0 \) in view of the continuity of the mappings \( g_m \). Obviously, the same way \( \frac{(1/m)}{1 + (1/m)^\alpha} |y - y_0| < \varepsilon \) for \( |y - y_0| < \delta_0 \) and all \( m = 1, 2, \ldots \). Finally, we have: \( |g_m(y) - g_m(y_0)| < \varepsilon \) for all \( m \in \mathbb{N} \) and \( y \in B(y_0, \delta) \), where \( \delta := \{\delta_0, \delta_1, \ldots, \delta_m\} \). Equicontinuity of \( g_m \) in \( B(0, 2) \) is established.

It should be noted that the family \( \mathcal{G} = \{g_m\}_{m=1}^\infty \) is equicontinuous in \( B(0, 2) \), and the family «inverse» to it is not \( \mathcal{F} = \{f_m\}_{m=1}^\infty \) (indeed, \( |f_m(x_m) - f(0)| = 1 + 1/m \not\to 0 \) for \( m \to \infty \), where \( |x_m| = 1/m \)).

The family \( \mathcal{G} \) contains an infinite number of mappings \( g_{m_k} := f^{-1}_{m_k} \), \( f_{m_k} \in \mathcal{F} \), that do not satisfy the relation (1.3). In fact, otherwise, according to Theorem 1.1 «the inverse» to \( \mathcal{G} \) family \( \mathcal{F} \) would be equicontinuous in \( \mathbb{B}^n \).

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