Phase transitions in the distribution of inelastically colliding inertial particles

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Abstract

It was recently suggested that the direction of particle drift in inhomogeneous temperature or turbulence depends on the particle inertia: weakly inertial particles localize near minima of temperature or turbulence intensity (effects known as thermophoresis and turbophoresis), while strongly inertial particles fly away from minima in an unbounded space. The problem of a particle near minima of turbulence intensity is related to that of two particles in a random flow, so that the localization-delocalization transition in the former corresponds to the path-coalescence transition in the latter. The transition is signaled by the sign change of the Lyapunov exponent that characterizes the mean rate of particle approach to the minimum (a wall or another particle). Here we solve analytically this problem for inelastic collisions and derive the phase diagram for the transition in the inertia-inelasticity plane. An important feature of the diagram is the region of inelastic collapse: if the restitution coefficient $\beta$ of particle velocity is smaller than the critical value $\beta_0 = \exp(-\pi/\sqrt{3})$, then the particle is localized for any inertia. We present direct numerical simulations which support the theory and in addition reveal the dependence of the transition of the flow correlation time, characterized by the Stokes number.

Keywords: inertial particles, inelastic collisions, turbophoresis

1. Introduction

Transport of inertial particles in an inhomogeneous turbulence or in a temperature gradient is of great importance for various industrial and natural processes. There is the long known
phenomenon of thermophoresis: the tendency of the particles to migrate in the direction of decreasing temperature [2, 3]. A similar phenomenon for inertial particles in non-uniform turbulence is called turbophoresis [6–8]. Conventional turbophoresis was consistently described within the gradient transport model for the particle concentration under the assumption of local equilibrium [8]. In that model, the turbophoretic velocity arises as a specific term in the expression for particle current which is directly proportional to the gradient of the local turbulence intensity. The turbophoretic velocity is directed towards decreasing turbulence level, so that, given zero concentration gradients and no net fluid flow in any direction, particles will migrate from high to low turbulence intensities. If turbophoresis is strong enough to overcome the effect of turbulent diffusion, the particles turn out to be localized near the minimum of turbulence. It was recently suggested [1] that particle migration in an unbounded space could be actually opposite—away from minima, if the particles are inertial enough. More specifically, the phenomenon of reverse turbophoresis takes place for inertial particles in the vicinity of turbulence minimum, provided that the particle mean free path is larger than the distance from this minimum. In this case, the local equilibrium assumption fails and there is no reduced description of particle transport in terms of spatial concentration. Thus, in contrast to standard turbophoresis, reverse turbophoresis is a nonlocal phenomenon, not associated with any local turbophoretic current.

The reversion of the direction of particle migration leads to localization–delocalization transition upon the change of inertia. It also means separation: at long timescales, particles with low inertia go to a minimum of turbulence and concentrate there, while sufficiently inertial particles escape to infinity. In many cases of interest, there is a wall, which corresponds to the minimum either of turbulence (in wall-bounded flows) or of temperature (in furnaces, combustion chambers and kerosene lanterns). It is not known how the inelastic particle–wall collisions affect the direction of the particle drift. Here we derive an analytical expression for the Lyapunov exponent associated with the motion of inertial particles near an inelastic wall. That allows us to describe the phase diagram of localization–delocalization transition in the inertia-inelasticity plane. A central result of our work is a phenomenon which one might call inelastic collapse: if the restitution coefficient of particle wall-normal velocity is smaller than the critical value that we determine, then the particles are localized near the wall at any inertia. The theoretical predictions are in a very good agreement with the results of numerics that we carry out.

All of our results are directly translated into the problem of the relative motion of two inertial particles in a spatially smooth and temporally random one-dimensional flow [15, 16]. That problem is of particular importance for the description of distribution and collisions of water droplets in clouds and for the description of planet formation. Here localization means that particles tend to approach each other and create clusters [15, 17]. Our findings mean that in one-dimensional flow inelastically colliding particles always create clusters when the restitution coefficient is below a threshold. That may be of importance for wide classes of phenomena in industry, geophysics and astrophysics, from clouds to planet formation. Note, however, that it will require further work to establish quantitatively how significant are the effects of inelastic inter-particle collisions in higher dimensional flows.

2. General relations

Consider the motion of a heavy particle suspended in an incompressible random fluid flow $u(r, t)$ near a flat impenetrable wall. We introduce a reference system with the $z$-axis perpendicular to the wall and assume that the fluid occupies the region $z > 0$. The particle is
assumed to be so small that the flow around it is viscous. Then, coordinate $r$ and velocity $v$ of the particle change according to

$$\frac{dr(t)}{dt} = v(t), \quad \frac{dv(t)}{dt} = \frac{u(r(t), t) - v(t)}{\tau},$$

(1)

where $\tau$ is the particle response time (Stokes time).

The boundary condition at $z = 0$ is inelastically reflecting so that at every collision the particle loses some part of its wall-normal velocity $v \rightarrow -\beta v$, where $\beta \equiv v$. In general, $\beta$ can be viewed as an effective restitution coefficient related to energy dissipation due to collision-induced plastic deformations and particle–wall hydrodynamic interaction [19]. Realistic models for these dissipation processes should involve $\beta$ which is a function of the impact velocity. Note also that the temporal or spatial irregularities at the surface of the wall may give rise to randomness of this parameter. In this study we consider an idealized model with the constant restitution coefficient $\beta$. The generalization of our analysis to the particular case of completely random $\beta$ will be discussed briefly at the end of section 4.

The fluid velocity field $u(r, t)$ is treated as chaotic. Its statistics is assumed to be homogeneous (stationary) in time, whereas there is no homogeneity in space: typical amplitude of velocity fluctuations is assumed to depend on the $z$-coordinate due to the presence of the wall. We further confine our attention to particle motion along the direction of flow inhomogeneity. The joint probability density function (PDF) of the particle’s velocity and coordinate normal to the wall is defined as

$$\rho(z, v, t) = \langle \delta(z - z(t))\delta(v - v(t)) \rangle,$$

(2)

where $z(t)$ and $v(t)$ are the solutions of equations (1), which are supplemented by the inelastic boundary condition, and the averaging is over the statistics of the flow.

There are three characteristic times for an inertial particle in spatially inhomogeneous random flow: the velocity correlation time $\tau_v$ of the fluid, the particle relaxation time $\tau$ and the time needed by the particle to feel the flow inhomogeneity $\tau_f$, specified below. We concentrate mainly on the situation when $\tau_f \ll \tau, \tau_v$, i.e. in particular, the Stokes number $St = \tau/\tau_v \gg 1$. Then the fluid velocity field can be treated as short-correlated and PDF (2) is the solution of the Fokker–Planck equation

$$\partial_t \rho = -v \partial_z \rho + \gamma \partial_v (vp) + \gamma^2 \kappa(z) \partial^2_z \rho,$$

(3)

where $\gamma = 1/\tau$ and the effective diffusivity $\kappa(z) = \int_{0}^{\infty} \langle u_z(z, t)u_z(z, 0) \rangle dt$ describes the non-uniform intensity of turbulence. The boundary condition for $\rho$ at $z = 0$ is dictated by particle number (or probability) conservation: the outcoming flux of particles at the boundary is balanced by incoming flux. Thus

$$\rho(z = 0, v, t) = \beta^{-2} \rho(z = 0, -v/\beta, t) \quad \text{for } v > 0.$$

(4)

Next we need to specify the $z$-dependence of $\kappa$. Our paper is devoted to the case of the quadratic diffusivity profile, i.e.

$$\kappa(z) = \mu z^2,$$

(5)

where $\mu$ measures the intensity of fluid velocity fluctuations. This model serves as a generic profile of the eddy diffusivity near minima of intensity of random flow. Note, however, that for large-Reynolds wall-bounded turbulence, the quadratic model can have only qualitative use, since the effective diffusivity behaves as $z^4$ in the viscous sub-layer and as $z$ in the logarithmic boundary layer [18].
Equations (3) and (5) also constitute a standard one-dimensional model to describe the PDF for relative motion of two particles at viscous scale of turbulence [14, 16]. In this case, $z$ is the inter-particle distance and $v$ is the relative velocity, so that $dz/dt = v$. Assuming $s(t)$ to be short-correlated we find equation (3) with $\kappa(z) = \mu z^2$ and $\mu = \int_0^\infty s(0)s(t)dt$. The boundary condition (4) takes into account inelastic inter-particle collisions. Thus, the results of further consideration are applicable also to the problem of relative dispersion of inelastically colliding particles in one-dimensional random flow with zero time correlation. Note that in higher dimensions the longitudinal (radial) relative motion of particles is coupled with the transversal dynamics [9–11]. For this reason, the influence of inelastic collisions on evolution of inter-particle distance in higher dimensional flows requires a separate investigation.

Our goal is to determine the long-term behavior of inertial particles near the inelastic wall. More precisely, we are interested in the direction of preferential particle migration. The time evolution of an arbitrary initial phase space distribution is determined by equations (3) and (4). However, an exact analytic solution of this non-stationary problem is known only for the trivial case of uniform diffusivity ($\kappa = \text{const}$) and ideally reflecting wall ($\beta = 1$). In general, the space-dependent $\kappa$ determines a local time scale $\tilde{\tau}(z)$ given by the time it takes the particle flying with a rms velocity to experience a difference in average turbulence intensity. The time $\tilde{\tau}(z)$ is estimated as the spatial scale $l \approx \kappa/\kappa'$ of flow inhomogeneity divided by the mean local ‘thermal’ particle velocity, which can be obtained by comparing the second and last terms in the right-hand side of (3), $\tilde{v}(z) \approx (\gamma \kappa)^{1/2}$. The validity of (3) requires that $\tau \ll \tilde{\tau}(z), \tau$. We introduce the dimensionless measure of particle inertia as

$$I(z) = \left( \frac{\tau}{\tilde{\tau}(z)} \right)^{1/3}. \tag{6}$$

Equivalently, $I(z)$ can be represented through the ratio of the particle mean free path and the scale of flow inhomogeneity. Notice that for $\kappa$ given by (5) the inertia degree is position independent, $I = (\mu \tau)^{1/3}$.

In the limit $I \ll 1$ the approximation of local equilibrium is valid: the statistics of particle velocity is described by the Maxwell distribution with $\gamma(\kappa(z))$ playing the role of local temperature. Thus, the particles adjust velocity locally and do not sense the boundary conditions at the wall in the leading-order approximation. The particle distribution in the real space $n(z, t) = \int_{-\infty}^{+\infty} dv \rho(z, v, t)$ obeys the gradient transport equation $\partial_t n = -\partial_j j$, where the current $j$ is

$$j = -\kappa(z) \partial_z n - n \frac{\partial \kappa(z)}{\partial z}. \tag{7}$$

The second term in the right-hand side is proportional to the eddy diffusivity gradient and represents usual turbophoretic current in the direction of decreasing turbulence. For quadratic $z$-dependence of effective diffusivity the fluxless steady state $n(z) \propto 1/z^2$ is normalizable at $z \to \infty$ so that particles concentrate near the minimum of turbulence.

For inertia degree $I$ of order 1 and larger the particles are far from the equilibrium with local turbulence and the gradient transport equation for the particle real space concentration is invalid. Recent theoretical analysis [1] of the quadratic model revealed a localization–delocalization transition upon the change of $I$ for inertial particles near an ideally reflecting wall: the direction of particle migration reverses as inertia degree becomes larger than some critical value $I_c \approx 0.827^{-1}$. This transition is signaled by the sign change of the particle Lyapunov
exponent as a function of $I$. In the present paper we extent the analysis to the case of inelastic particle–wall collisions calculating the exact Lyapunov exponent in its dependence on both $I$ and $\beta$.

3. Particles near the perfectly reflecting wall

We start with a description of the localization–delocalization transition in the case of a perfectly reflecting wall, i.e. $\beta = 1$. The time dependence of the particle position for given realization of turbulence can be written as

$$z(t) = z(0) \exp \left( \text{p.v.} \int_0^t \sigma(t') dt' \right),$$

where $\sigma = v/z$. The principal value (p.v.) of the integral corresponds to the moments of particle–wall collisions $t_i$, for which $z(t_i) = 0$ and, thus, the variable $\sigma$ becomes singular.

We wish to determine the direction of particle migration. The long-time evolution of $z$ is determined by the Lyapunov exponent

$$\lambda = \lim_{t \to +\infty} \frac{1}{t} \ln \frac{z(t)}{z(0)} = \lim_{t \to +\infty} \frac{1}{t} \text{p.v.} \int_0^t \sigma(t') dt' = \langle \sigma \rangle.$$

With the assumption of ergodicity, the time average converges to ensemble average, so that the mean value of $\sigma$ in (9) can be calculated as $\langle \sigma \rangle = \lim_{t \to +\infty} \int_{-\infty}^{+\infty} \sigma P(\sigma, t) d\sigma$, where $P(\sigma, t) = \langle \delta(\sigma - \sigma(t)) \rangle$. Importantly, since $z$-dependence of the effective diffusivity is given by the quadratic law (5), the dynamics of $\sigma$ decouples from that of $z$ and $v$. It follows from (3) to (5) that the probability distribution $P$ satisfies the closed equation

$$\partial_t P = \partial_\sigma \left[ U' P \right] + \mu \gamma^2 \partial^2_\sigma P$$

in which $U(\sigma) = \gamma \sigma^2 / 2 + \sigma^3 / 3$.

Next, it is easy to see that every reflection of the particle from $z = 0$ corresponds to $\sigma$ jumping from $-\infty$ to $+\infty$. Therefore, the $\sigma$-space has the topology of a circle with $\sigma = +\infty$ glued to $z = 0$. That topology allows the $\sigma$-independent flux of probability $-F$ going towards $-\infty$ and returning from $+\infty$: $U' P + \mu \gamma^2 \partial_\sigma P = F$. We then readily find the normalizable steady state

$$P(\sigma) = \frac{F}{\mu \gamma^2} e^{-U(\sigma)/\mu \gamma^2} \int_{-\infty}^{+\infty} e^{U(\sigma')/\mu \gamma^2} d\sigma',$$

which realizes the minimum of entropy production [12] and is the asymptotic state of any initial distribution [13]. The magnitude of flux $F$ is the average frequency of particle–wall collisions and behaves as [13]

$$F = \frac{1}{\sqrt{\pi} \tau} \left[ \int_0^{+\infty} \frac{dx}{\sqrt{x}} \exp \left( -\frac{x^3}{12} + \frac{x}{4 I^2} \right) \right]^{-1} = \frac{1}{\pi \tau} \left[ \text{Ai} \left( \frac{1}{4 I^2} \right)^2 + \text{Bi} \left( \frac{1}{4 I^2} \right)^2 \right]^{-1},$$

where $\text{Ai}$ and $\text{Bi}$ designate the Airy function of the first kind and second kind, respectively. This expression comes from the normalization condition $\int_{-\infty}^{+\infty} P(\sigma) d\sigma = 1$. 

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We can now calculate the Lyapunov exponent (9) by using (11)

$$\lambda(\beta = 1, I) = \int_{-\infty}^{+\infty} \sigma P(\sigma) d\sigma = \frac{\sqrt{\pi}}{2} \int_{0}^{+\infty} dx \frac{x}{x^2} \exp \left( -\frac{x^3}{12} + \frac{x}{4F^2} \right).$$

This integral is negative at $I < I_c \approx 0.827^{-1}$ and positive otherwise [15]. That sign change is a phase transition, called the localization–delocalization transition for motion of a particle near a wall (when $z$ is the distance from the wall) and the path-coalescence transition for inter-particle dynamics (when $z$ is an inter-particle distance). Note that even in the delocalized phase, despite going away exponentially, $z(t)$ returns to zero with the time-independent mean rate (12).

The model with the ideally reflecting wall also allows the analysis of the mean squares of the particle coordinate and the velocity: $\langle z^2(t) \rangle = \int_{0}^{\infty} \int_{-\infty}^{+\infty} z^2 \rho(z, v, t) dz dv$ and $\langle v^2(t) \rangle = \int_{0}^{\infty} \int_{-\infty}^{+\infty} v^2 \rho(z, v, t) dz dv$. Using equation (3), one obtains the identical closed equations

$$\frac{d^3}{dt^3} \langle z^2(t) \rangle + 3\gamma \frac{d^2}{dt^2} \langle z^2(t) \rangle + 2\gamma \frac{d}{dt} \langle z^2(t) \rangle = -4\gamma^3 I^3 \langle z^2(t) \rangle = 0,$$

$$\frac{d^3}{dt^3} \langle v^2(t) \rangle + 3\gamma \frac{d^2}{dt^2} \langle v^2(t) \rangle + 2\gamma \frac{d}{dt} \langle v^2(t) \rangle = -4\gamma^3 I^3 \langle v^2(t) \rangle = 0,$$

which have solutions in the form $Ae^{\lambda_1 t} + Be^{\lambda_2 t} + Ce^{\lambda_3 t}$. The amplitudes $A$, $B$ and $C$ can be expressed through the initial coordinate $z(0)$, initial velocity $v(0)$ and the eigenvalues $\lambda_1$, $\lambda_2$, and $\lambda_3$. The latter parameters are completely determined by the inertia degree $I$ and provide the long-term exponential growth of both moments for any initial conditions. This growth may seem surprising in the case of localization; however, realizations with particles going away exist even in this case—those realization contribute negligibly $\langle \ln z(t) \rangle$ but dominate $\langle z^2(t) \rangle$ and $\langle v^2(t) \rangle$.

4. Particles near the inelastic wall

Now let us assume that at every collision with the wall the particle instantaneously loses a definite part of its wall-normal velocity, i.e. $v(t_i + \delta t) = -\beta v(t_i - \delta t)$, where $z(t_i) = 0$ and $\delta t \to +0$. As a consequence, we obtain $z(t_i + \delta t) = \beta z(t_i - \delta t)$ and $\sigma(t_i + \delta t) = -\sigma(t_i - \delta t)$. Remarkably, the variable $\sigma$ does not feel the boundary conditions at the wall. This means that the PDF $P(\sigma, t)$ relaxes to the same steady state (11) for any value of the restitution coefficient $\beta$. This statement is also valid in the general case when $\beta$ is not constant.

It follows from the relation $z(t_i + \delta t) = \beta z(t_i - \delta t)$ that the particle coordinate for each particular realization of random flow is given by

$$z(t) = z(0) e^{N(t)} \exp \left( \text{p.v.} \int_{0}^{t} \sigma(t') dt' \right),$$

where $N(t)$ is number of bounces up to the moment $t$. The Lyapunov exponent can then be calculated as

$$\lambda(\beta, I) = \lim_{t \to +\infty} \frac{1}{t} \ln \frac{z(t)}{z(0)} = \lim_{t \to +\infty} \frac{1}{t} \text{p.v.} \int_{0}^{t} \sigma(t') dt' + \ln \beta \lim_{t \to +\infty} \frac{N(t)}{t}.$$
Since the statistics of $\sigma$ does not depend on $\beta$, the first term in the right-hand side coincides with the Lyapunov exponent (13), which corresponds to the case of an ideally reflecting wall

$$\lim_{t \to +\infty} \frac{1}{I} \text{p.v.} \int_0^t \sigma(t')dt' = \lambda(1, I),$$

while the second limit is the collision rate

$$\lim_{t \to +\infty} \frac{N(t)}{t} = F(I),$$

which is given by (12). That allows us to express the Lyapunov exponent for any inertia parameter $I$ and the restitution factor $\beta$,

$$\lambda(\beta, I) = \lambda(1, I) + F(I)\ln \beta. \quad (20)$$

Localization–delocalization phase transition corresponds to a zero Lyapunov exponent which determines the transition curve (see figure 1)

$$\ln \beta_c(I) = -\frac{\lambda(I)}{F(I)} = -\frac{\sqrt{\pi}}{2} \int_0^{+\infty} dx \frac{x - I^{-1}}{\sqrt{x}} \exp\left(-\frac{x^3}{12} + \frac{x}{4I^2}\right). \quad (21)$$

Calculating the integral for the infinite inertia parameter ($I \to \infty$) one obtains the non-zero value

$$\beta_0 = \exp\left(-\frac{\pi}{\sqrt{3}}\right). \quad (22)$$

We thus conclude that the curve of the phase transition in the plane $\beta - I$ hits $I = \infty$ at finite $\beta = \beta_0$. There is no phase transition for less elastic walls where particles with any inertia localize. By the same token, particles coalesce if their restitution coefficient is small enough. This phenomenon can be called inelastic collapse in analogy to a similar effect in driven granular matter [21]. Even though a beautiful formula (22) has been obtained within a short-correlated model, it perfectly corresponds to the direct numerical simulations presented in the next section for a synthetic velocity field with finite temporal correlations.
Interestingly enough, the above calculations can be extended to the case when \( \beta \) is a random variable whose statistics is stationary in time. In general, the particle coordinate can be written as

\[
z(t) = z(0) \exp \left( \int_0^t \sigma(t') \, dt' \right) \prod_{i=1}^{N(t)} \beta_i ,
\]

where \( \beta_i \) is the restitution coefficient associated with \( i \)th bounce. From (23) one obtains an expression for the Lyapunov exponent

\[
\lambda = \lim_{t \to +\infty} \frac{1}{t} \text{p.v.} \int_0^t \sigma(t') \, dt' + \lim_{t \to +\infty} \frac{1}{t} \sum_{i=1}^{N(t)} \ln \beta_i.
\]

As before, the mean \( \sigma \) is just the Lyapunov exponent (13) from the problem with a perfectly reflecting wall. Let us assume that the numbers \( \beta_1, \beta_2, \ldots, \beta_N \) are randomly chosen from some probability density \( p(\beta) \). Then the second term in the right-hand side of (24) becomes

\[
\lim_{t \to +\infty} \frac{1}{t} \sum_{i=1}^{N(t)} \ln \beta_i = F(I) \lim_{N \to +\infty} \frac{1}{N} \sum_{i=1}^{N} \ln \beta_i = F(I) \int_0^1 \ln \beta p(\beta) \, d\beta.
\]

That finally gives

\[
\bar{\lambda} = \lambda(1, I) + F(I) \int_0^1 \ln \beta p(\beta) \, d\beta.
\]

If \( \int_0^1 \ln \beta p(\beta) \, d\beta < -\pi/\sqrt{3} \), then the particle is localized for any inertia degree \( I \).

Finally in this section, let us consider the effect of partial boundary absorption. We assume that on arriving at the wall the particle is absorbed with probability \( \alpha \) and reflected with probability \( 1 - \alpha \). The wall-normal velocities of the particle just before and after reflection are related by \( v \to -\beta v \), where restitution coefficient \( \beta \) is a random variable or a given function of impact velocity. We are interested in the survival probability \( Q(t) \) which is the probability that a particle has not yet been absorbed by wall after a time \( t \). The decay rate of \( Q(t) \) is given by absorption probability \( \alpha \) multiplied by the collision rate \( F \): \( dQ/dt = -\alpha F \).

As stated above, the probability distribution of the variable \( \sigma \) relaxes to the steady state (11) providing the time-independent frequency of collisions \( F \) given by (12). This leads to the exponentially fast long-term decay \( Q(t) \propto \exp(-\alpha F t) \). Note that the survival probability does not depend on the restitution coefficient \( \beta \).

5. Simulations

We have conducted numerical simulations of the motion of inertial particles subjected to one-dimensional inhomogeneous random force. In numerics the fluid velocity is modeled as a telegraph noise. Namely, we set \( u = z \xi(t) \), where the random process \( \xi(t) \) is a piecewise random constant during \( \tau_c \), chosen from a Gaussian distribution with a zero mean and unit variance. Varying \( \tau_c/\tau \) we are able to explore finite-correlated flows as well. The equation of motion (1) is solved by the second-order Runge–Kutta algorithm with time steps much smaller than \( \tau_c \). Wall reflection is modeled as follows: if a particle changes the sign of \( z \) during \( \tau_c \), its coordinate is flipped to the opposite while its velocity changes sign and is multiplied by the restitution coefficient \( \beta \). To show the relevance of the described numerical scheme to our analytical model, we have used the correlation time of fluid velocity \( \tau_c = 0.002 \), which is much smaller than other time parameters of the problem, \( \tau_c \ll \tau, F^{-1}, \lambda^{-1} \). In this limit, the telegraph process reproduces the case of a white noise.
We compute Lyapunov exponents to determine the direction of particle drift

\[ \lambda = \lim_{t \to \infty} \frac{1}{t} \left( \ln \frac{z(t)}{z(0)} \right) \]  (27)

where averaging is over many (typically 1000) realization of the random process. The Lyapunov exponent is known to be a self-averaging quantity; that is, having the same value in any given long realization as after averaging over many realizations. The localization–delocalization phase transition occurs when the Lyapunov exponent changes its sign.

Figure 2 indicates that, as expected, the collision frequency \( F \) does not depend on \( \beta \) and is completely determined by the inertia degree \( I \) and relaxation time \( \tau \). Figures 3 and 4 show that the mean logarithm of particle coordinate is a linear function of time. The slope changes its sign upon the change of inertia degree or elasticity. The phase diagram of localization–delocalization transition demonstrates a very good agreement of numerics and analytic theory, see figure 1.

6. Phase transition upon the change of the Stokes number

As stated above, there are three time parameters for an inertial particle in a random flow: the velocity correlation time \( \tau_c \) of the fluid, the particle relaxation time \( \tau \) and the time needed by the particle to feel the flow inhomogeneity \( \hat{\tau} \). In the previous sections the transition upon change of inertia parameter \( I = (\tau/\tau_c)^2/3 \) was demonstrated in the limit of large Stokes number, \( \text{St} = \tau/\tau_c \gg 1 \). Here we consider \( \tau_c, \tau \ll \hat{\tau} \) and discuss the phase transition upon change of \( \text{St} \). If the time fluctuations of the fluid velocity are modeled by the Ornstein–Uhlenbeck process, the probability distribution \( n(z, t) \) of the particle position obeys the gradient transport equation \( \partial_t n = -\partial_z j \), where the turbulence induced current is [1, 22]
The first term in the right-hand side of (28) represents turbulent diffusion, while the second one is proportional to the diffusivity gradient and can be referred to as turbophoretic current. Note that the equation (7) is recovered from (28) in the limit as Stokes number goes to infinity. The opposite limit of small Stokes number corresponds to the turbulent transport of the passive tracers. For the quadratic diffusivity profile (5), we obtain

$$j = -\kappa(z)\partial_n n - \frac{\text{St}}{1 + \text{St}} n \frac{d\kappa(z)}{dz}. \quad (28)$$

The time dependence of the mean logarithm of the particle coordinate at fixed elasticity $\beta = 1$ for different degrees of inertia: (1) $I = 1.25147$, (2) 1.21545, (3) 1.20939, (4) 1.20332 and (5) 1.19144.

Figure 3.

The time dependence of the mean logarithm of the particle coordinate at fixed inertia degree $I = 7.37$ for different restitution coefficients: (1) $\beta = 0.2$, (2) 0.205, (3) 0.20, (4) 0.195 and (5) 0.19.

Figure 4.

$$\partial_t \left( \ln \frac{z(t)}{z(0)} \right) = \frac{\mu}{1 + \text{St}}. \quad (29)$$
where $\langle \ln z(t) \rangle = \int_0^\infty n(z, t) \ln z \, dz$ and the zero-flux condition $j = 0$ at the surface of the impenetrable wall is used. Therefore, at the critical Stokes number $St_c = 1$ the Lyapunov exponent $\lambda = \lim_{t \to \infty} (1/t) \langle \ln z(t)/z(0) \rangle$ vanishes and localization–delocalization transition occurs. For $St > St_c$ the standard turbophoresis is pronounced enough so that the inertial particles drift towards to the minimum of turbulence. However, at low inertia ($St < St_c$) the turbophoresis cannot compensate for the effect of turbulent diffusion, which tends to spread the particles throughout the fluid.

On the plane $x = 1$, $y = 1/St$ we expect the phase transition curve to start at $x = L_c$, $y = 0$ and monotonically go down to $x = 0$, $y = 1$. This is supported by the numerical data obtained with increasing ratio of the flow correlation time to the Stokes time. Taking $\beta = 1$, we find $L_c = 1.209$ for $St = 500$, $L_c = 1.204$ for $St = 50$ and $L_c = 1.137$ for $St = 5$. It is important to keep in mind that an account of fluid incompressibility (and multi-dimensionality) was crucial in deriving (28) by [1]. Therefore, the motion of particles with $\tau \lesssim \tau_c$ cannot be modeled numerically with a one-dimensional fluid flow.

7. Conclusion

We have studied analytically and numerically the relative motion of inertial particles in homogeneous random flow or, equivalently, the particle motion in inhomogeneous wall-bounded flow. The main part of our analysis was focused on the case of a short-correlated flow with quadratic dependence of the effective diffusivity on the distance, which corresponds to the inter-particle distance in the viscous interval of turbulence or to the distance of a single particle from a minimum of turbulence. In this model, the direction of particle migration is determined by two dimensionless parameters: the inertia degree $I$ and the restitution coefficient of particle–particle or particle–wall collisions $\beta$. We have derived analytically the phase diagram for the localization–delocalization transition in the plane $I - \beta$. Remarkably, there is a critical value of the coefficient $\beta$ below which particles localize for any value of inertia $I$. The results are confirmed by the direct numerical simulations of particle motion in a synthetic random flow.

Our results do not apply directly to the near-wall region of developed turbulence, where the turbulent diffusivity behaves as $z^4$ inside the viscous sub-layer and as $z$ in the logarithmic boundary layer. An interesting question is whether the results provided by exactly solvable quadratic model can be generalized to these cases. Recent analysis [20] of inertial particles in the viscous sub-layer of wall-bounded turbulence ($\kappa \propto z^4$) revealed the localization–delocalization transition upon change of restitution coefficient $\beta$ with a critical value given by (22). For $\kappa \propto z^m$ we thus have the same threshold for localization at $m = 2$ and $m = 4$. Moreover, the critical value (22) is the same for inelastic collapse of trajectories under the action of a spatially uniform random force, $\kappa = \text{const}$, as long as it is zero upon the contact, $\kappa(0) = 0$ [21]. To address the localization properties of inertial particles for any $m \geqslant 0$ one needs to consider the boundary value problem (3) to (4), which will be the subject of future work.

It is important to stress that our one-dimensional consideration with a short-correlated flow is a crude approximation of reality, particularly for the problem of relative motion of two particles. Further numerical and experimental work is needed to establish whether there is some analog of the collapse transition leading to clustering of particles in real flows. This seems likely as one would expect that inelastic inter-particle collisions decrease the Lyapunov exponent associated with the evolution of distance between two inertial particles below the viscous scale of turbulence.
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