Global dynamics of the integrable Armbruster-Guckenheimer-Kim galactic potential

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Abstract We study the global dynamics of the completely integrable Armbruster-Guckenheimer-Kim galactic potential. In these cases this system has two first integrals $H_1$ and $H_2$ independent and in involution. Let $I_{h_1}$ and $I_{h_2}$ be the set of points of the phase space on which $H_1$ and $H_2$ take the values $h_1$ and $h_2$, respectively. The sets $I_{h_1} \cap I_{h_2}$ are invariant by the dynamics. We characterize the global flow on these sets and we describe the foliation of the phase space by the invariant sets $I_{h_1} \cap I_{h_2}$.

Keywords Armbruster-Guckenheimer-Kim galactic potential · Invariant manifolds · Complete integrability

1 Introduction

The Armbruster-Guckenheimer-Kim potential is a galactic potential introduced in Armbruster et al. (1989) that studies the dynamics for the interchanging of nearly nondegenerate modes with square symmetry. They derived the model starting with a normal form given by a system of differential equations which represented the codimension two bifurcation problem. More precisely, the Hamiltonian function that they provided is

$$H(x, p_x, y, p_y) = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) - \frac{a}{4}(x^2 + y^2)^2 - \frac{b}{2} x^2 y^2,$$

where $a, b$ are arbitrary constants. If we add the term $-\omega (xp_y - yp_x)$ then the system describes the dynamics of rotation of a nearly axisymmetric galaxy rotating with a constant velocity $\omega$ around a fixed axis. The existence of such $\omega$ denotes that the rotation of the galaxy must be taken into account when we study the stellar orbits (see Zeeuw and Merritt 1983). Many studies concerning the integrability and non-integrability of such systems have been done (see for instance Acosta-Humánez et al. 2018; Elmandouh 2016; El-Sabaa et al. 2019) using different techniques such as the Painlevé analysis and the Morales-Ramis theory as well as the study of the existence of periodic orbits which was done in Llibre and Roberto (2012). In particular, it was proved in El-Sabaa et al. (2019) that if $b = 2a$ or $b = -a$ the system is completely integrable but the authors do not describe completely the dynamics of the integrable systems form the point of view of the Liouville-Arnold theorem (see Sect. 2). This is the main aim of this paper.

When $b = 2a$ the Hamiltonian has the form

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) - \frac{a}{4}(x^2 + y^2)^2 - ax^2 y^2.$$

Introducing the new variables

$$u = \frac{1}{\sqrt{2}}(x - y), \quad v = \frac{1}{\sqrt{2}}(x + y),
\quad p_u = \frac{1}{\sqrt{2}}(p_x - p_y), \quad p_v = \frac{1}{\sqrt{2}}(p_x + p_y),$$

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it can be written as
\[ H(x, p_x, y, p_y) = \frac{1}{2}(p_x^2 + p_y^2) + \frac{a}{4}(x^4 + y^4) - \frac{1}{2}(x^2 + y^2) = \tilde{H}_1(x, p_x) + \tilde{H}_2(y, p_y), \]

where \( a \in \mathbb{R} \), we have renamed the variables \((u, v)\) again as \((x, y)\) and
\[
\tilde{H}_1(x, p_x) = \frac{1}{2} p_x^2 + \frac{a}{4} x^4 - \frac{1}{2} x^2,
\]
\[
\tilde{H}_2(y, p_y) = \frac{1}{2} p_y^2 + \frac{a}{4} y^4 - \frac{1}{2} y^2.
\]

Note that \( \tilde{H}_1 : \mathbb{R}^2 \to \mathbb{R} \) while \( H : \mathbb{R}^4 \to \mathbb{R} \). In all the paper we will denote by \( H \) the Hamiltonian associated to a system with two degrees of freedom and so \( H = H(x, p_x, y, p_y) : \mathbb{R}^4 \to \mathbb{R} \), \( H_i = H_i(x, p_x, y, p_y) : \mathbb{R}^4 \to \mathbb{R} \) for \( i = 1, \ldots, 4 \), and we will denote by \( \tilde{H} \) the Hamiltonian associated to a system with one degree of freedom and so \( \tilde{H}_1 = \tilde{H}_1(x, p_x) : \mathbb{R}^2 \to \mathbb{R} \) and \( \tilde{H}_2 = \tilde{H}_2(y, p_y) : \mathbb{R}^2 \to \mathbb{R} \).

We observe that \( H_1 \) and \( H_2 \) are two first integrals, independent and in involution. Hence, the Hamiltonian system associated to the Hamiltonian \( H \) is
\[
\begin{align*}
\dot{x} &= p_x, & \dot{y} &= p_y, \\
\dot{p}_x &= -ax^3 + x, & \dot{p}_y &= -ay^3 + y
\end{align*}
\]
and it is completely integrable. We recall that \( H_1 \) and \( H_2 \) are independent if the matrix
\[
\begin{pmatrix}
H_{1x} & H_{1p_x} & H_{1y} & H_{1p_y} \\
H_{2x} & H_{2p_x} & H_{2y} & H_{2p_y}
\end{pmatrix}
\]
has rank 2 in any point of \( \mathbb{R}^4 \) except, perhaps in a zero Lebesgue-measure set. As usual \( H_y = \partial_x H / \partial y \). Moreover, we say that \( H_1 \) and \( H_2 \) are in involution if their Poisson bracket is zero. Finally, a Hamiltonian system with two degrees of freedom is completely integrable if it has two independent first integrals in involution.

Note that the phase space of system (1) is \( \mathbb{R}^4 \). Since \( H_1 \) and \( H_2 \) are first integrals the sets
\[
\begin{align*}
I_{h_1} &= \{(x, p_x, y, p_y) \in \mathbb{R}^4 : H_1 = h_1\} \\
&= \{(x, p_x) \in \mathbb{R}^2 : \tilde{H}_1 = \tilde{h}_1\} \times \mathbb{R}^2 \\
&= \tilde{h}_1 \times \mathbb{R}^2,
\end{align*}
\]
\[
\begin{align*}
I_{h_2} &= \{(x, p_x, y, p_y) \in \mathbb{R}^4 : H_2 = h_2\} \\
&= \{(y, p_y) \in \mathbb{R}^2 : \tilde{H}_2 = \tilde{h}_2\} \times \mathbb{R}^2 \\
&= \mathbb{R}^2 \times \tilde{h}_2,
\end{align*}
\]

are invariant by the flow of the Hamiltonian system (1). The first objective of this paper is to describe the foliations of the phase space \( \mathbb{R}^4 \) by the invariant sets \( I_{h_i} \) for \( i = 1, 2 \) as well as by \( I_{h_1 h_2} \). The foliations provide a good description of the phase portraits of the Hamiltonian flow (1) when \( a \) varies.

When \( b = -a \) the Hamiltonian has the form
\[
\begin{align*}
H(x, p_x, y, p_y) &= \frac{1}{2}(p_x^2 + p_y^2) - \frac{a}{4}(x^4 + y^4) \\
&\quad+ \frac{1}{2}(x^2 + y^2)
\end{align*}
\]
\[
\begin{align*}
\tilde{H}_3(x, p_x) &= \frac{1}{2} p_x^2 - \frac{a}{4} x^4 + \frac{1}{2} x^2,
\tilde{H}_4(y, p_y) &= \frac{1}{2} p_y^2 - \frac{a}{4} y^4 + \frac{1}{2} y^2.
\end{align*}
\]

Note that \( H_3 \) and \( H_4 \) are two first integrals, independent and in involution. Hence the Hamiltonian system
\[
\begin{align*}
\dot{x} &= p_x, & \dot{y} &= p_y, \\
\dot{p}_x &= ax^3 - x, & \dot{p}_y &= ay^3 - y
\end{align*}
\]
is completely integrable. The sets
\[
\begin{align*}
I_{h_3} &= \{(x, p_x, y, p_y) \in \mathbb{R}^4 : H_3 = h_3\} = \tilde{I}_{h_3} \times \mathbb{R}^2, \\
I_{h_4} &= \{(x, p_x, y, p_y) \in \mathbb{R}^4 : H_4 = h_4\} = \mathbb{R}^2 \times \tilde{I}_{h_3},
\end{align*}
\]
as well as
\[
\begin{align*}
I_{h_3 h_4} &= \{(x, p_x, y, p_y) \in \mathbb{R}^4 : H_3 = h_3, H_4 = h_4\} \\
&= I_{h_3} \cap I_{h_4} = \tilde{I}_{h_3} \times I_{h_4}
\end{align*}
\]
are invariant by the flow of the Hamiltonian system (2). The second main objective of the paper is to describe the foliations of \( \mathbb{R}^4 \) by the invariant sets \( I_{h_i} \) for \( i = 3, 4 \) and by the invariant sets \( I_{h_3 h_4} \). Again, these foliations provide a good description of the phase portraits of the Hamiltonian flow (2) when \( a \) varies.

The paper is organized as follows. In Sect. 2 we recall the Liouville-Arnold theory for Hamiltonians systems with
two degrees of freedom. In Sect. 3 we describe the topology of the sets \( I_{h_1} \) (since the study for \( I_{h_2} \) is analogous). For doing that and taking into account that \( I_{h_1} = I_{h_1} \times \mathbb{R}^2 \) we will only describe the topology of the sets \( I_{h_1} \) by computing the sets of singular points and critical values for \( \tilde{H}_1 \) and the Hill regions according to the different values of \( a \) and \( \tilde{h}_1 \). In Sect. 4 we study the topology of the sets \( I_{h_1 h_2} \). In Sect. 5 we describe the topology of the sets \( I_{h_1} \) (again because the study for \( I_{h_2} \) is analogous) and recalling that \( h_1 = I_{h_1} \times \mathbb{R}^2 \) we will only describe the topology of the sets \( I_{h_3} \) by computing the sets of singular points and critical values for \( \tilde{H}_3 \) and the Hill regions according to the different values of \( a \) and \( \tilde{h}_3 \). In Sect. 6 we study the topology of the sets \( I_{h_3 h_4} \).

## 2 Integrable Hamiltonian systems

In this section we recall the Liouville-Arnold theorem for the integrable Hamiltonian systems with two degrees of freedom. We recall that a flow defined on the phase space \( \mathbb{R}^4 \) is complete if its solutions are defined for all time \( t \) in \( \mathbb{R} \).

**Theorem 1** The Hamiltonian system \((1)\) (resp. system \((2)\)) defined on the phase space \( \mathbb{R}^4 \) has the Hamiltonians \( H_1 \) and \( H_2 \) (resp. \( H_3 \) and \( H_4 \)) as two independent first integrals in involution. If \( I_{h_1 h_2} \neq \emptyset \) (resp. \( I_{h_3 h_4} \neq \emptyset \)) and \( (h_1, h_2) \) (resp. \( (h_3, h_4) \)) is a regular value of the map \( (H_1, H_2) \) (resp. \( (H_3, H_4) \)) then the following statements hold.

(a) \( I_{h_1 h_2} \) (resp. \( I_{h_3 h_4} \)) is a two-dimensional submanifold of \( \mathbb{R}^4 \) invariant under the flow of system \((1)\) (resp. system \((2)\)).

(b) If the flow on a connected component \( I_{* h_1 h_2} \) (resp. \( I_{* h_3 h_4} \)) of \( I_{h_1 h_2} \) (resp. \( I_{h_3 h_4} \)) is complete, then \( I_{h_1 h_2} \) (resp. \( I_{h_3 h_4} \)) is diffeomorphic either to the torus \( S^1 \times S^1 \), to the cylinder \( S^1 \times \mathbb{R} \), or to the plane \( \mathbb{R}^2 \).

(c) Under the assumption of statement (b), the flow on \( I_{h_1 h_2} \) (resp. \( I_{h_3 h_4} \)) is conjugated to a linear flow either on \( S^1 \times S^1 \), or on \( S^1 \times \mathbb{R} \), or on \( \mathbb{R}^2 \).

Note that Theorem 1 does not provide information on the topology of the invariant sets \( I_{h_1 h_2} \) (resp. \( I_{h_3 h_4} \)) when \( (h_1, h_2) \) (resp. \( (h_3, h_4) \)) is not a regular value of the map \((H_1, H_2)\) (resp. \((H_3, H_4)\)), or how the energy levels \( I_{h_1} \) or \( I_{h_2} \) (resp. \( I_{h_3} \) or \( I_{h_4} \)) foliate \( \mathbb{R}^4 \).

In this paper we solve these problems for systems \((1)\) and \((2)\).

## 3 The topology of the invariant sets \( I_{h_1} \)

As explained in the introduction, taking into account that \( I_{h_1} = I_{h_1} \times \mathbb{R}^2 \) we will restrict all the study to \( I_{h_1} \).

A point \( (x, p_x) \in \mathbb{R}^2 \) is a singular point for the map \( \tilde{H}_1 \) if it is a solution of

\[
\frac{\partial \tilde{H}_1}{\partial p_x} = 0, \quad \frac{\partial \tilde{H}_1}{\partial x} = 0.
\]

The value \( \tilde{h}_1 \in \mathbb{R} \) is a critical value for the map \( \tilde{H}_1 \) if there is some singular point belonging to \( \tilde{H}_1^{-1}(\tilde{h}_1) = I_{h_1} \). If \( \tilde{h}_1 \) is not critical value it is said a regular value. It is well-known that if \( \tilde{h}_1 \) is a regular value of the map \( \tilde{H}_1 \) then \( I_{h_1} \) is a one-dimensional manifold (see Hirsch 1976).

Note that the singular points for the map \( \tilde{H}_1 \) are

\[
p_x = 0, \quad x(ax^2 - 1) = 0,
\]

and so the set of singular points of \( \tilde{H}_1 \) is \( (0, 0) \) if \( a \leq 0 \), and \( (0, 0) \cup (0, -1/\sqrt{a}) \cup (0, 1/\sqrt{a}) \) if \( a > 0 \).

We define the Hill region as

\[
R_{h_1} = \left\{ x \in \mathbb{R} : \frac{a}{4} x^4 - \frac{x^2}{2} \leq \tilde{h}_1 \right\}
\]

This is the region of the configuration space \( \{ x \in \mathbb{R} \} \) where the motion of all orbits of the Hamiltonian system associated to \( \tilde{H}_1 \) have energy \( \tilde{h}_1 \) takes place. By \( R_{h_1} \) \( \approx \) \( S \), we denote that \( R_{h_1} \) is diffeomorphic to \( S \). We will also denote by

\[
P_1 = \sqrt{1 - \sqrt{1 + 4ah_1}} \quad \text{and} \quad P_2 = \sqrt{1 + \sqrt{1 + 4ah_1}}
\]

have:

(i) \( R_{h_1} \approx \mathbb{R} \) if \( a = 0 \) and \( \tilde{h}_1 > 0 \),

(ii) \( R_{h_1} \approx \mathbb{R} \) but here \( \{ 0 \} \), which is a singular point for \( \tilde{H}_1 \), is in the boundary of the Hill region, if \( a = 0 \) and \( \tilde{h}_1 = 0 \),

(iii) \( R_{h_1} \approx (-\infty, -\sqrt{-2h_1}) \cup [\sqrt{-2h_1}, \infty) \) if \( a = 0 \) and \( \tilde{h}_1 < 0 \),

(iv) \( R_{h_1} \approx \mathbb{R} \) if \( a > 0 \) and \( \tilde{h}_1 > 0 \),

(v) \( R_{h_1} \approx \mathbb{R} \) but here \( \{ 0 \} \), which is a singular point for \( \tilde{H}_1 \), is in the boundary of the Hill region, if \( a > 0 \) and \( \tilde{h}_1 = 0 \),

(vi) \( R_{h_1} \approx (-\infty, -P_-) \cup [P_-, \infty) \) if \( a < 0 \) and \( \tilde{h}_1 < 0 \),

(vii) \( R_{h_1} \approx \emptyset \) if \( a > 0 \) and \( \tilde{h}_1 < -1/(4a) \),

(viii) \( R_{h_1} \approx \left[ -\frac{1}{a} \right] \cup \left[ \frac{1}{a} \right] \) which are two of the singular points for the map \( \tilde{H}_1 \), if \( a > 0 \) and \( \tilde{h}_1 = -1/(4a) \),

(ix) \( R_{h_1} \approx [P_-, P_-] \cup [P_-, P_+] \) if \( a > 0 \) and \( \tilde{h}_1 \in (-1/(4a), 0) \),

(x) \( R_{h_1} \approx \left[ -\sqrt{\frac{2}{a}}, \sqrt{\frac{2}{a}} \right] \) but here \( \{ 0 \} \), which is a singular point for \( \tilde{H}_1 \), is in the boundary of the Hill region, if \( a > 0 \) and \( \tilde{h}_1 = 0 \),

(xi) \( R_{h_1} \approx \{-P_+, P_+\} \) if \( a > 0 \) and \( \tilde{h}_1 > 0 \).
Now we compute the energy levels $I_{h_1}$. From the definition of $I_{h_1}$ we have

$$I_{h_1} = \bigcup_{x \in R_{h_1}} E_x$$

where

$$E_x = \left\{ (x, p_x) \in \mathbb{R}^2 : \frac{p_x^2}{2} + \frac{a}{4} x^4 - \frac{1}{2} x^2 = h_1 \right\}.$$ 

Clearly for each $x \in \mathbb{R}$ the set $E_x$ is either two points, or one point or the empty set, if the point $x$ is in the interior of the Hill region $R_{h_1}$, in its boundary, or it does not belong to $R_{h_1}$, respectively. Therefore, from (3) and using the Hill region, the topology of $I_{h_1}$ is:

(i) $I_{h_1} \approx \mathbb{R} \cup \mathbb{R}^2$ if $a < 0$ and $h_1 \neq 0$,
(ii) $I_{h_1} \approx X$ if $a \leq 0$ and $h_1 = 0$. Here $X$ denotes two straight lines intersecting the origin of the two straight lines,
(iii) $I_{h_1} \approx \emptyset$ if $a > 0$ and $h_1 < -1/(4a)$,
(iv) $I_{h_1} \approx \pm (\pm \sqrt{2a}, 0)$ which are the two equilibrium points of $H_1$ if $a > 0$ and $h_1 = -1/(4a)$,
(v) $I_{h_1} \approx \mathbb{S}^1 \cup \mathbb{S}^1$ if $a > 0$ and $h_1 \in (-1/(4a), 0)$,
(vi) $I_{h_1} \approx \infty$ if $a > 0$ and $h_1 = 0$. Here $\infty$ denotes two homoclinic orbits at the origin,
(vii) $I_{h_1} \approx \mathbb{S}^1$ if $a > 0$ and $h_1 < 0$.

See in Fig. 1 the phase portraits associated to the Hamiltonian system with Hamiltonian $H_1$ depending on whether $a > 0$, $a = 0$, and $a < 0$. The phase portraits in Fig. 1 are drawn in the Poincaré disc, which essentially is a unit closed disc centered at the origin of coordinates with its interior identified to $\mathbb{R}^2$ and with its boundary (the circle $\mathbb{S}^1$) identified with the infinity of $\mathbb{R}^2$, for more details on the Poincaré disc see Chap. 5 of Dumortier et al. (2006).

4 The topology of the invariant sets $I_{h_1 h_2}$

To obtain $I_{h_1 h_2}$ we recall that $I_{h_2}$ is exactly the same as $I_{h_1}$ and that $I_{h_1 h_2} = I_{h_1} \cap I_{h_2} = I_{h_1} \times I_{h_2}$. Hence, in Table 1 we have given the description of the invariant sets $I_{h_1 h_2}$ for the different values of $h_1, h_2$ and $a$.

5 The topology of the invariant sets $I_{h_3}$

As we did for the case $H_1$, we recall that $I_{h_3} = I_{h_1} \times \mathbb{R}^2$ and so we will study only $I_{h_3}$. The singular points for the map $H_3$ satisfy

$$p_x = 0, \quad x(1 - ax^2) = 0$$

and so they are $(0, 0)$ if $a \leq 0$ and $(0, 0) \cup (0, -1/\sqrt{a}) \cup (0, 1/\sqrt{a})$ if $a > 0$. The Hill region is

$$R_{h_3} = \left\{ y \in \mathbb{R} : -\frac{a}{4} y^4 + \frac{y^2}{2} \leq h_3 \right\}$$

and so taking the notation

$$Q_- = \sqrt{\frac{1 - \sqrt{1 - 4ah_3}}{a}}, \quad Q_+ = \sqrt{\frac{1 + \sqrt{1 - 4ah_3}}{a}}$$

we have

(i) $R_{h_3} \approx \emptyset$ if $a = 0$ and $h_3 < 0$,
(ii) $R_{h_3} \approx \mathbb{R}$ if $a = 0$ and $h_3 = 0$, then
(iii) $R_{h_3} \approx [-\sqrt{2h_3}, \sqrt{2h_3}]$ if $a = 0$ and $h_3 > 0$ then
(iv) $R_{h_3} \approx \emptyset$ if $a < 0$ and $h_3 < 0$,
(v) $R_{h_3} \approx \{0\}$ if $a < 0$ and $h_3 = 0$,
(vi) $R_{h_3} \approx [-Q_+, Q_-]$ if $a < 0$ and $h_3 > 0$,
(vii) $R_{h_3} \approx \mathbb{R}$ if $a > 0$ and $h_3 > 1/(4a)$,
(viii) $R_{h_3} \approx \mathbb{R}$, but here $\pm \sqrt{\frac{1}{4a}}$, which are singular points for $H_1$, are in the boundary of the Hill region, if $a > 0$ and $h_3 = 1/(4a)$,
(ix) $R_{h_3} \approx (-\infty, -Q_+ \cup [-Q_-, Q_-] \cup [Q_+, +\infty)$ if $a > 0$ and $h_3 \in (0, 1/(4a))$,
(x) $R_{h_3} \approx \mathbb{R}$, but here $\{0\}$, which is a singular point for $H_1$, is in the boundary of the Hill region, if $a > 0$ and $h_3 = 0$,
(xi) $R_{h_3} \approx (-\infty, -Q_+ \cup [Q_+, +\infty)$ if $a > 0$ and $h_3 < 0$.

Now we compute the energy levels $I_{h_3}$, From the definition of $I_{h_3}$ we have

$$I_{h_3} = \bigcup_{y \in R_{h_3}} E_y$$

where

$$E_y = \left\{ (y, p_y) \in \mathbb{R}^2 : \frac{p_y^2}{2} - \frac{a}{4} y^4 + \frac{1}{2} y^2 = h_3 \right\}.$$
The invariant sets $I_{h_1 h_2}$ for the different values of $h_1$, $h_2$, and $a$

| $a$ | $h_1$ | $h_2$ | $I_{h_1 h_2}$ |
|-----|-------|-------|--------------|
| $\leq 0$ | $\neq 0$ | $\neq 0$ | $(\mathbb{R} \cup \mathbb{R}) \times (\mathbb{R} \cup \mathbb{R})$ |
| $\leq 0$ | $\neq 0$ | $= 0$ | $(\mathbb{R} \cup \mathbb{R}) \times X$ |
| $\leq 0$ | $= 0$ | $\neq 0$ | $X \times (\mathbb{R} \cup \mathbb{R})$ |
| $\leq 0$ | $= 0$ | $= 0$ | $X \times X$ |
| $> 0$ | $<-1/(4a)$ | $\in \mathbb{R}$ | $\emptyset$ |
| $> 0$ | $= -1/(4a)$ | $<-1/(4a)$ | $\emptyset$ |
| $> 0$ | $= -1/(4a)$ | $= -1/(4a)$ | $(\pm \sqrt{1/4a}, 0) \times (\pm \sqrt{1/4a}, 0)$ |
| $> 0$ | $= -1/(4a)$ | $\in (-1/(4a), 0)$ | $(\pm \sqrt{1/4a}, 0) \times (S^1 \cup S^1)$ |
| $> 0$ | $= -1/(4a)$ | $= 0$ | $(\pm \sqrt{1/4a}, 0) \times \infty$ |
| $> 0$ | $= -1/(4a)$ | $< 0$ | $(-\sqrt{1/4a}, 0) \times S^1$ |
| $> 0$ | $\in (-1/(4a), 0)$ | $<-1/(4a)$ | $\emptyset$ |
| $> 0$ | $\in (-1/(4a), 0)$ | $= -1/(4a)$ | $(S^1 \cup S^1) \times (\pm \sqrt{1/4a}, 0)$ |
| $> 0$ | $\in (-1/(4a), 0)$ | $\in (-1/(4a), 0)$ | $(S^1 \cup S^1) \times (S^1 \cup S^1)$ |
| $> 0$ | $\in (-1/(4a), 0)$ | $= 0$ | $(S^1 \cup S^1) \times \infty$ |
| $> 0$ | $\in (-1/(4a), 0)$ | $< 0$ | $(S^1 \cup S^1) \times S^1$ |
| $> 0$ | $= 0$ | $<-1/(4a)$ | $\emptyset$ |
| $> 0$ | $= 0$ | $= -1/(4a)$ | $\infty \times (\pm \sqrt{1/4a}, 0)$ |
| $> 0$ | $= 0$ | $\in (-1/(4a), 0)$ | $\infty \times (S^1 \cup S^1)$ |
| $> 0$ | $= 0$ | $= 0$ | $\infty \times S^1$ |
| $> 0$ | $< 0$ | $<-1/(4a)$ | $\emptyset$ |
| $> 0$ | $< 0$ | $= -1/(4a)$ | $S^1 \times (\pm \sqrt{1/4a}, 0)$ |
| $> 0$ | $< 0$ | $\in (-1/(4a), 0)$ | $S^1 \times (S^1 \cup S^1)$ |
| $> 0$ | $< 0$ | $= 0$ | $S^1 \times \infty$ |
| $> 0$ | $< 0$ | $< 0$ | $S^1 \times S^1$ |

Clearly for each $y \in \mathbb{R}$ the set $E_y$ is either two points, or one point or the emptyset, if the point $y$ is in the interior of the Hill region $R_{h_3}$, in its boundary, or it does not belong to $R_{h_3}$, respectively. Therefore, from (4) and using the Hill region, the topology of $I_{h_3}$ is:

(i) $I_{h_3} \approx \emptyset$ if $a \leq 0$ and $\tilde{h}_3 < 0$,
(ii) $I_{h_3} \approx \{(0, 0)\}$ if $a \leq 0$ and $\tilde{h}_3 = 0$,
(iii) $I_{h_3} \approx S^1$ if $a \leq 0$ and $\tilde{h}_3 > 0$,
(iv) $I_{h_3} \approx \mathbb{R} \cup \mathbb{R}$ if $a > 0$ and $\tilde{h}_3 > 1/(4a)$,
(v) $I_{h_3} \approx P$ if $a > 0$ and $\tilde{h}_3 = 1/(4a)$. Here $P$ denotes two curves with the shape of a parabola intersecting in two different points (the points are the two singular points),
(vi) $I_{h_3} \approx \mathbb{R} \cup S^1 \cup \mathbb{R}$ if $a > 0$ and $\tilde{h}_3 \in (0, 1/(4a))$,
(vii) $I_{h_3} \approx \mathbb{R} \cup \{(0, 0)\} \cup \mathbb{R}$ if $a > 0$ and $\tilde{h}_3 = 0$,
(viii) $I_{h_3} \approx \mathbb{R} \cup \mathbb{R}$ if $a > 0$ and $\tilde{h}_3 < 0$.

See the phase portrait associated to $\tilde{H}_3$ depending on whether $a > 0$, $a = 0$, or $a < 0$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{phase_portraits}
\caption{Phase portraits associated to the Hamiltonian system with Hamiltonian $H_3$ depending on whether $a > 0$ or $a \leq 0$}
\end{figure}

See in Fig. 2 the phase portraits associated to the Hamiltonian system with Hamiltonian $\tilde{H}_3$ depending on whether $a > 0$ and $a \leq 0$.

6 The topology of the invariant sets $I_{h_3 h_4}$

To obtain $I_{h_3 h_4}$ we recall that $I_{h_3}$ is exactly the same as $I_{h_3}$ and that $I_{h_3 h_4} = I_{h_3} \cap I_{h_4} = I_{h_3} \times I_{h_4}$. Hence, in Table 2 we have given the description of the invariant sets $I_{h_3 h_4}$ for the different values of $h_3$, $h_4$ and $a$. 

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