A FUNCTION ON THE SET OF ISOMORPHISM CLASSES IN THE STABLE CATEGORY OF MAXIMAL COHEN-MACAULAY MODULES OVER A GORENSTEIN RING: WITH APPLICATIONS TO LIASON THEORY

TONY J. PUTHENPURAKAL

Abstract. Let \((A, \mathfrak{m})\) be a Gorenstein local ring of dimension \(d \geq 1\). Let \(\text{CM}(A)\) be the stable category of maximal Cohen-Macaulay \(A\)-modules and let \(\text{ICM}(A)\) denote the set of isomorphism classes in \(\text{CM}(A)\). We define a function \(\xi : \text{ICM}(A) \to \mathbb{Z}\) which behaves well with respect to exact triangles in \(\text{CM}(A)\). We then apply this to (Gorenstein) liaison theory. We prove that if \(\dim A \geq 2\) and \(A\) is not regular then the even liaison classes of \(m^n; n \geq 1\) is an infinite set. We also prove that if \(A\) is an complete equi-characteristic simple singularity with \(A/\mathfrak{m}\) uncountable then for each \(m \geq 1\) the set \(\mathcal{C}_m = \{I | I\) is a codim 2 CM-ideal with \(e_0(A/I) \leq m\}\) is contained in finitely many even liaison classes \(L_1, \ldots, L_r\) (here \(r\) may depend on \(m\)).

1. Introduction

Let \((A, \mathfrak{m})\) be a Gorenstein local ring of dimension \(d \geq 1\) with residue field \(k\). Let \(\text{CM}(A)\) denote the full subcategory of maximal Cohen-Macaulay \(A\)-modules and let \(\text{CM}(A)\) denote the stable category of maximal Cohen-Macaulay \(A\)-modules.

Recall that objects in \(\text{CM}(A)\) are same as objects in \(\text{CM}(A)\). However the set of morphisms \(\text{Hom}_A(M, N)\) between \(M\) and \(N\) is \(= \text{Hom}_A(M, N)/P(M, N)\) where \(P(M, N)\) is the set of \(A\)-linear maps from \(M\) to \(N\) which factor through a finitely generated free module. It is well-known that \(\text{CM}(A)\) is a triangulated category with translation functor \(\Omega^{-1}\). Here \(\Omega(M)\) denotes the syzygy of \(M\) and \(\Omega^{-1}(M)\) denotes the co-syzygy of \(M\). Also recall that an object \(M\) is zero in \(\text{CM}(A)\) if and only if it is free considered as an \(A\)-module. Furthermore \(M \cong N\) in \(\text{CM}(A)\) if and only if there exists finitely generated free modules \(F, G\) with \(M \oplus F \cong N \oplus G\) as \(A\)-modules. Let \(\text{ICM}(A)\) denote the set of isomorphism classes in \(\text{CM}(A)\) and for an object \(M \in \text{CM}(A)\) we denote its isomorphism class by \([M]\).

We say a function \(\xi : \text{ICM}(A) \to \mathbb{Z}\) is a triangle function if it satisfies the following properties:

1. \(\xi([M]) \geq 0\) for all \(M \in \text{CM}(A)\).
2. \(\xi([M]) = 0\) if and only if \(M = 0\) in \(\text{CM}(A)\).
3. \(\xi([M_1 \oplus M_2]) = \xi([M_1]) + \xi([M_2])\) for all \(M_1, M_2 \in \text{CM}(A)\).
4. (sub-additivity) If \(M \to N \to L \to \Omega^{-1}(M)\) is an exact triangle in \(\text{CM}(A)\) then

\(\xi([L]) = \xi([M]) + \xi([N])\).

Date: May 7, 2014.
1991 Mathematics Subject Classification. Primary 13C40; Secondary 13C14,13D40.
Key words and phrases. stable category, Liaison theory, maximal Cohen-macaulay approximations.
(a) $\xi([N]) \leq \xi([M]) + \xi([L])$.
(b) $\xi([L]) \leq \xi([N]) + \xi([\Omega^{-1}(M)])$.
(c) $\xi([\Omega^{-1}(M)]) \leq \xi([L]) + \xi([\Omega^{-1}(N)])$.

**Remark 1.1.**  
(i) Since rotations of exact triangles are exact it follows that if $\xi$ satisfies (4)(b) for all exact triangles then it will also satisfy 4(a), (c).
(ii) Axiom (3) implies that $\xi([M]) = 0$ if $M = 0$. However note that axiom (2) also implies that if $\xi([M]) = 0$ then $M = 0$.

We have the following result on existence of triangle functions. Let $\ell(N)$ denote the length of an $A$-module $N$.

**Theorem 1.2.** Let $(A, m)$ be a Gorenstein local ring of dimension $d \geq 1$. Then the function

$$e^x_A([M]) = \lim_{n \to \infty} \frac{(d-1)!}{n^{d-1}} \ell \left( \text{Tor}_1^A(M, \frac{A}{m^{n+1}}) \right)$$

where $[M] \in \text{ICM}(A)$ is a triangle function on $\text{ICM}(A)$.

Unlike the multiplicity function which can be defined uniquely through a set of axioms, triangle functions are highly non-unique. In [29] we will construct infinitely many triangle functions. However $e^x_A$ is the simplest triangle function that we have constructed. It also behaves well with generic hyperplane sections, see Proposition 2.9 for details.

1.1. **Applications to Liaison theory.**

The existence of triangle functions has non-trivial implications in Liaison theory. In fact in Application I and II we prove our results by using any triangle function. However for application III we need some additional properties of $e^x_A$.

Let $(A, m)$ be a Gorenstein local ring. We say an ideal $q$ is a Gorenstein ideal if it is perfect and $A/q$ is a Gorenstein ring. We should remark that some authors do not require in the definition of Gorenstein ideals for $q$ to be perfect. However we will require it to be so.

We begin by recalling the definition of (Gorenstein) linkage.

**Definition 1.3.** Ideals $I$ and $J$ of $A$ are (algebraically) linked by a Gorenstein ideal $q$ if

(a) $q \subseteq I \cap J$, and
(b) $I = (q: J)$ and $J = (q: I)$.

We write it as $I \sim_q J$.

If $q$ is a complete intersection ideal then we say that $I$ is CI-linked to $J$. We say ideals $I$ and $J$ is in the same linkage class if there is a sequence of ideals $I_0, \ldots, I_n$ in $A$ and Gorenstein ideals $q_0, \ldots, q_{n-1}$ such that

(i) $I_j \sim_{q_j} I_{j+1}$, for $j = 0, \ldots, n - 1$.
(ii) $I_0 = I$ and $I_n = J$.

If $n$ is even then we say that $I$ and $J$ are evenly linked. We can analogously define CI-linkage class and even CI-linkage class.

The notion of linkage has been extended to modules, [7]. See section 4 for definition. Note that ideals $I$ and $J$ are linked as ideals if and only if the cyclic modules $A/I$ and $A/J$ are linked as modules; see [2] Proposition 1.
Application- I: Let $K$ be a field let $R = K[[X_1, \ldots, X_n]]$. Set $n = (X_1, \ldots, X_n)$. By results in [5, Theorem 3.6] it can be shown that $n^i$ is evenly linked to $n^{i-1}$ for all $i \geq 2$. Note that if $n \geq 3$ then this result does not hold for CI-liason [11, Theorem 1.1]. If $(A, m)$ is a one dimensional Gorenstein local ring then one can prove that there exists $s \geq 1$ such that $m^{sn+r}$ is evenly linked to $m^{(n-1)+r}$ for all $n \gg 0$ and $r = 0, 1, \ldots, s-1$; see Proposition [5,1]. A natural question is when is the set of ideals $\{m^n \mid n \geq 1\}$ contained in finitely many even liason classes. Our first result implies that the above two cases are essentially the only cases when the above condition holds. We prove the following more general result:

Theorem 1.4. Let $(A, m)$ be a Gorenstein local ring. Let $M$ be a finitely generated $A$-module of dimension $r \geq 2$. Let $\Lambda_M = \{M/m^n M \mid n \geq 1\}$. If there exists finitely many even liason classes of modules $L_1, L_2, \ldots, L_m$ such that

$$\Lambda_M \subseteq \bigcup_{i=1}^{m} L_i$$

then $A$ is regular.

Application-II: Assume $(A, m)$ is a complete equi-characteristic Gorenstein local ring. Let $I$ be an ideal in $A$ generated by a regular sequence. Using results in [5, Theorem 3.6] it can be proved that $I^n$ is evenly linked to $I^{n-1}$ for all $n \geq 2$, see Proposition [6,1]. Thus the modules $A/I^n$ is evenly linked to $A/I^{n-1}$ for all $n \geq 2$. It follows that if $F$ is a finitely generated free $A$-module then $F/I^n F$ is evenly linked to $F/I^{n-1} F$ for all $n \geq 2$. A natural question is whether $M/I^n M$ is evenly linked to $M/I^{n-1} M$ for all $n \gg 0$ when $M \in \text{CM}(A)$. We prove the following surprising result:

Theorem 1.5. Let $(A, m)$ be a Gorenstein local ring of dimension $d \geq 2$. Let $M \in \text{CM}(A)$. Let $x_1, \ldots, x_r$ be an $A$-regular sequence with $r \geq 2$. Let $I = (x_1, \ldots, x_r)$ and let $\Lambda^I_M = \{M/I^n M \mid n \geq 1\}$. If there exists finitely many even liason classes of modules $L_1, L_2, \ldots, L_m$ such that

$$\Lambda^I_M \subseteq \bigcup_{i=1}^{m} L_i$$

then $M$ is free.

Note that in the above result we do not assume that $A$ is complete or contains a field. We do not know whether the result holds if $r = 1$.

Application-III: Let $I$ be a perfect ideal of codimension 2. It is well-known that $I$ is licci (i.e., it is CI-linked to a complete intersection). However an arbitrary codimension two Cohen-Macaulay ideal need not be licci. For instance if $(A, m)$ is non-regular Gorenstein ring of dimension 2 then $m$ is not a licci-ideal (this is so because if $I$ is licci then projdim $A/I$ is finite.) So a natural question is whether codimension two Cohen-Macaulay ideals are contained in finitely many even liason classes. Again this is not possible. Let $(A, m)$ be a non-regular Gorenstein ring of dimension 2. Then by Theorem 1.4 the set of ideals $\{m^n \mid n \geq 1\}$ is not contained in finitely many even liason classes of ideals in $A$. Note that $\ell(A/m^n) \to \infty$ as $n \to \infty$. So we rephrase the question. Let $C_m = \{I \mid I$ is a codim 2 CM-ideal with $c_0(A/I) \leq m\}$. Here $c_0(A/I)$ is the multiplicity of the ring $A/I$ with respect to its maximal ideal. Our question is whether $C_m$ contained in finitely many even liason classes of ideals. Regular rings trivially have this property. We prove
Theorem 1.6. Let $k$ be an uncountable algebraically closed field of characteristic different from 2. Let $(P, m)$ be a regular analytic $k$-algebra, i.e., a formal or convergent (if $k$ is a complete non-trivial valuated field) power series ring $k < x_0, x_1, \ldots, x_d >$ with $d \geq 2$. Let $f \in m$ be such that $A = P/(f)$ is a simple hypersurface singularity. For $m \geq 1$ let $C_m = \{ I \mid I$ is a codim 2 CM-ideal with $e_0(A/I) \leq m \}$. Then for every $m \geq 1$ there exists finitely many even liason classes $L_1, \ldots, L_r$ (depending on $m$) such that

$$C_m \subseteq \bigcup_{i=1}^r L_i$$

In the above result we use the fact that a simple simple singularity only has finitely many indecomposable maximal Cohen-Macaulay modules, see [5]. The assumption $K$ is uncountable is a bit irritating, however it is essential in our proof. We conjecture that the converse of this theorem is also true.

We now describe in brief the contents of the paper. In section two we introduce the function $e_A^T(\cdot)$ and prove some of its basic properties. In section three we prove Theorem 1.2. In section four we discuss some results on Liaison theory of modules and discuss the notion of maximal Cohen-Macaulay approximations. In section 5,6,7 we prove Theorems 1.4, 1.5,1.6 respectively.

2. Pre-triangles in CM$(A)$

In this paper all rings are commutative Noetherian and all modules are assumed to be finitely generated. In this section $(A, m)$ is a Cohen-Macaulay local ring of dimension $d \geq 1$. Let $ICM(A)$ denote the set of isomorphism classes of maximal Cohen-Macaulay $A$-modules and for an object $M \in ICM(A)$ we denote its isomorphism class by $[M]$. In this section we study the function

$$e_A^T([M]) = \lim_{n \to \infty} \frac{(d-1)!}{n^{d-1}} \ell \left( Tor_1^A(M, \frac{A}{m^{n+1}}) \right)$$

where $[M] \in ICM(A)$.

We also abstract some of its properties and call the notion a pre-triangle function.

2.1. Let $M$ be an $A$-module. We denote its first syzygy-module by $\Omega(M)$. If we have to specify the ring then we write it as $\Omega_A(M)$. Recall $\Omega(M)$ is constructed as follows: Let $G \xrightarrow{\varphi} F \xrightarrow{\epsilon} M \to 0$ be a minimal presentation of $M$. Then $\Omega(M) = \ker \epsilon$. It is easily shown that if $G' \xrightarrow{\varphi'} F \xrightarrow{\epsilon'} M \to 0$ is another minimal presentation of $M$ then $\ker \epsilon \cong \ker \epsilon'$.

Set $\Omega^1(M) = \Omega(M)$. For $i \geq 2$ define $\Omega^i(M) = \Omega(\Omega^{i-1}(M))$. It can be easily proved that $\Omega^i(M)$ are invariant’s of $M$.

2.2. The function $e_A^T(\cdot)$ arose in the authors study of certain aspects of the theory of Hilbert functions [9], [10]. Let $N$ be an $A$-module of dimension $r$. It is well-known that there exists a polynomial $P_N(z) \in \mathbb{Q}[z]$ of degree $r$ such that $P_N(n) = \ell(N/m^{n+1}N)$ for all $n \gg 0$. We write

$$P_N(z) = \sum_{i=0}^r (-1)^i e_i(N) \binom{z + r - i}{r - i}.$$ 

Then $e_0(N), \ldots, e_r(N)$ are integers and are called the Hilbert coefficients of $N$. The number $e_0(N)$ is called the multiplicity of $N$. It is positive if $N$ is non-zero.
The number $e_1(N)$ is **non-negative** if $N$ is Cohen-Macaulay; see [9] Proposition 12]. Also note that
\[ \sum_{n \geq 0} \ell(N/m^{n+1}N)z^n = \frac{h_N(z)}{(1-z)^{r+1}}, \]
where $h_M(z) \in \mathbb{Z}[z]$ with $e_i(N) = h^{(i)}(1)/i!$ for $i = 0, \ldots, r$.

### 2.3. Let $M \in \text{CM}(A)$. In [9] Prop. 17] we proved that the function
\[ n \mapsto \ell \left( \text{Tor}_1^A(M, \frac{A}{m^{n+1}}) \right) \]
is of polynomial type, i.e., it coincides with a polynomial $t_M(z)$ for all $n \gg 0$. In [9] Theorem 18] we also proved that
\begin{enumerate}
\item $M$ is free if and only if deg $t_M(z) < d - 1$.
\item If $M$ is not free then deg $t_M(z) = d - 1$ and the normalized leading coefficient of $t_M(z)$ is $\mu(M)e_1(A) - e_1(M) - e_1(\Omega(M))$; here $\mu(M)$ denotes the minimal number of generators of $M$.
\item For any $M \in \text{CM}(A)$,
\[ e_A^T(M) = \lim_{n \to \infty} \frac{(d-1)!}{n^d-1} \ell \left( \text{Tor}_1^A(M, \frac{A}{m^{n+1}}) \right) \]
= $\mu(M)e_1(A) - e_1(M) - e_1(\Omega(M))$.
\end{enumerate}

By (1) note that $e_A^T(M) = 0$ if and only if $M$ is free. Otherwise $e_A^T(M) > 0$. In fact $e_A^T(M) \geq e_0(\Omega(M))$; [9] Lemma 19].

Our first result shows that we need not confine to minimal presentation to compute $e_A^T(M)$.

**Lemma 2.4.** Let $M \in \text{CM}(A)$ and let $0 \to N \to F \to M \to 0$ be an exact sequence in $\text{CM}(A)$ with $F$ free. Then
\[ e_A^T(M) = e_1(F) - e_1(M) - e_1(N). \]

**Proof.** By Schanuel’s Lemma [8] Lemma 3, section 19] we have $A^{\mu(M)} \oplus N \cong F \oplus \Omega(M)$. So
\[ \mu(M)e_1(A) + e_1(N) = e_1(F) + e_1(\Omega(M)). \]
The result follows. \qed

Our next result shows that $e_1(-)$ is sub-additive over short-exact sequences in $\text{CM}(A)$.

**Proposition 2.5.** Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short-exact sequence in $\text{CM}(A)$. Then
\[ e_1(M_2) \geq e_1(M_1) + e_1(M_3). \]

**Proof.** Note $e_0(M_2) = e_0(M_1) + e_0(M_3)$. For $n \geq 0$ we define modules $K_n$ by the exact sequence
\[ 0 \to K_n \to \frac{M_1}{m^{n+1}M_1} \to \frac{M_2}{m^{n+1}M_2} \to \frac{M_3}{m^{n+1}M_3} \to 0. \]
It follows that
\[ \sum_{n \geq 0} \ell(K_n)z^n = \frac{h_{M_1}(z) - h_{M_2}(z) + h_{M_3}(z)}{(1-z)^{d+1}}. \]
Since $e_0(M_2) = e_0(M_1) + e_0(M_3)$ we have that $h_{M_1}(z) - h_{M_2}(z) + h_{M_3}(z) = (1 - z)l_K(z)$ for some $l_K(z) \in \mathbb{Z}[z]$. So we have

$$
\sum_{n \geq 0} \ell(K_n)z^n = \frac{l_K(z)}{(1-z)^d}.
$$

Notice $l_K(1) = e_1(M_2) - e_1(M_1) - e_1(M_3)$. It follows that for all $n \gg 0$

$$
\ell(K_n) = (e_1(M_2) - e_1(M_1) - e_1(M_3)) \frac{n^{d-1}}{(d-1)!} + \text{lower terms in } n.
$$

So $e_1(M_2) \geq e_1(M_1) + e_1(M_3)$.

We now prove that $e^T_A(-)$ is sub-additive over short-exact sequences in CM($A$).

**Theorem 2.6.** Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short-exact sequence in CM($A$). Then

$$e^T_A(M_2) \leq e^T_A(M_1) + e^T_A(M_3).$$

**Proof.** By a standard result in homological algebra we have the following diagram with exact rows and columns; with $F_i$ free $A$-modules for $i = 1, 2, 3$:

$$
\begin{array}{cccccc}
0 & 0 & 0 & \\
0 & N_1 & N_2 & N_3 & 0 \\
0 & F_1 & F_2 & F_3 & 0 \\
0 & M_3 & M_2 & M_1 & 0 \\
0 & 0 & 0 & 0
\end{array}
$$

Note $F_2 \cong F_1 \oplus F_3$. So $e_1(F_2) = e_1(F_1) + e_1(F_3)$. However $e_1(M_2) \geq e_1(M_1) + e_1(M_3)$ and $e_1(N_2) \geq e_1(N_1) + e_1(N_3)$.

By Lemma 2.4 we have $e^T_A(M_i) = e_1(F_i) - e_1(M_i) - e_1(N_i)$ for $i = 1, 2, 3$. The result follows.

**2.7.** Let us recall the definition of superficial elements. Let $N$ be an $A$-module. An element $x \in m \setminus m^2$ is said to be $N$-superficial if there exists $c > 0$ such that $(m^{n+1}N : x) \cap m^cN = m^nN$ for all $n \gg 0$. It is well-known that superficial elements exist when the residue field $k$ of $A$ is infinite. If depth $N > 0$ then one can prove that a $N$-superficial element $x$ is $N$-regular. Furthermore $(m^{n+1}N : x) = m^nN$ for all $n \gg 0$.

**2.8. Behavior of Hilbert coefficients with respect to superficial elements:** Assume $N$ is an $A$-module with depth $N > 0$ and dimension $r \geq 1$. Let $x$ be $N$-superficial. Then by [8] Corollary 10] we have

$$e_i(N/xN) = e_i(N) \quad \text{for } i = 0, \ldots, r-1.$$

Our next result shows that $e^T_A(-)$ behaves well mod superficial elements.
Proposition 2.9. Suppose \( \dim A \geq 2 \) and let \( M \in \text{CM}(A) \). Assume the residue field \( k \) is infinite. Let \( x = A \oplus M \oplus \Omega_A(M) \)-superficial. Set \( B = A/(x) \) and \( N = M/xM \). Then
\[
e_T^B(N) = e_T^A(M).
\]

Proof. Note
\[
e_T^A(M) = e_1(A)\mu(M) - e_1(M) - e_1(\Omega_A(M)),
\]
\[
= e_1(B)\mu(N) - e_1(N) - e_1(\Omega_A(M)/x\Omega_A(M)).
\]
The result follows from observing that \( \Omega_A(M)/x\Omega_A(M) \cong \Omega_B(M/xM) \).

2.10. We now abstract some of the essential properties of \( e_T^A(\cdot) \).

We say a function \( \xi : \text{ICM}(A) \to \mathbb{Z} \) is a pre-triangle function if it satisfies the following properties:

1. \( \xi([M]) \geq 0 \) for all \( M \in \text{CM}(A) \).
2. \( \xi([M]) = 0 \) if and only if \( M \) is free.
3. \( \xi([M_1 \oplus M_2]) = \xi([M_1]) + \xi([M_2]) \) for all \( M_1, M_2 \in \text{CM}(A) \).
4. (sub-additivity) If \( 0 \to M \to N \to L \to 0 \) is an exact sequence in \( \text{CM}(A) \) then
\[
\xi([N]) \leq \xi([M]) + \xi([L]).
\]

We state our basic existence result of pre-triangle functions.

Theorem 2.11. Let \( (A, \mathfrak{m}) \) be a Cohen-Macaulay local ring of dimension \( d \geq 1 \). Then the function
\[
e_T^A([M]) = \lim_{n \to \infty} \frac{(d-1)!}{n^{d-1}l} \left( \text{Tor}_1^A(M, \frac{A}{\mathfrak{m}^{n+1}}) \right)
\]
where \( [M] \in \text{ICM}(A) \)

is a pre-triangle function on \( \text{ICM}(A) \).

Proof. Properties (1), (2) are satisfied by 2.3. Property (3) is trivially satisfied. Property (4) is satisfied by Theorem 2.6.

2.12. If \( \xi \) is a pre-triangle function then trivially \( k\xi \) is a pre-triangle function for any \( k \geq 1 \). Perhaps less-obvious is the following:

Proposition 2.13. Let \( \xi \) be a pre-triangle function. Then the function
\[
\xi^{(i)} : \text{ICM}(A) \to \mathbb{Z}
\]
defined by
\[
\xi^{(i)}([M]) = \xi([\Omega^i(M)])
\]
is a pre-triangle function for all \( i \geq 0 \).

Proof. Note \( \xi^{(0)} = \xi \). Also note that for \( i \geq 2 \) we have
\[
\xi^{(i)} = \left( \xi^{(i-1)} \right)^{(1)}.
\]
So it suffices to prove that \( \nu = \xi^{(1)} \) is a pre-triangle function.

It is very easy to prove that \( \nu \) satisfies properties (1), (2) and (3) and is left to the reader. We prove that \( \nu \) satisfies property (4). Let \( 0 \to M_1 \to M_2 \to M_3 \to 0 \) be a short exact sequence in \( \text{CM}(A) \). Note that we have a short exact sequence
\[
0 \to \Omega(M_1) \to \Omega(M_2) \oplus F \to \Omega(M_3) \to 0;
\]
Proposition 2.15. Let $\xi$ is a pre-triangle function we have
\[
\xi([\Omega(M_2)]) = \xi([\Omega(M_2) \oplus F]) \leq \xi([\Omega(M_1)]) + \xi([\Omega(M_3)])
\]
The result follows. \qed

Remark 2.14. In general $\xi^{(i)}$ will be different from $\xi$. For instance if $\xi = e_A^T(\cdot)$ and if the betti-numbers of $M$ are unbounded then note as $e_A^T(M) \geq e_0(\Omega(M)) \geq \mu(\Omega(M))$, see [9] Lemma 19], we get that for $i \gg 0$ we have $e_A^T(\Omega^i(M)) > e_A^T(M)$. So in this case $\xi^{(i)}(M) \neq \xi(M)$.

The following easy proposition (proof left to the reader) combined with 2.12 and 2.13 yields yet another abundant number of pre-triangle functions.

Proposition 2.15. Let $\xi_1, \xi_2$ be two pre-triangle functions. Then $\xi = \xi_1 + \xi_2$ is a pre-triangle function. \qed

3. Triangle functions on $\text{CM}(A)$

In this section $(A, m)$ is a Gorenstein local ring of dimension $d \geq 1$ with residue field $k$. Let $\text{CM}(A)$ denote the full subcategory of maximal Cohen-Macaulay $A$-modules and let $\text{CM}(A)$ denote the stable category of maximal Cohen-Macaulay $A$-modules. Let $\text{ICM}(A)$ denote the set of isomorphism classes in $\text{CM}(A)$ and for an object $M \in \text{CM}(A)$ we denote its isomorphism class by $[M]$. In this section we prove Theorem 1.2. We also construct a large class of triangle functions on $\text{ICM}(A)$.

3.1. Let $M \in \text{CM}(A)$. By $M^*$ we mean the dual of $M$, i.e., $M^* = \text{Hom}_A(M, A)$. Note $M \cong M^{**}$.

By $\Omega^{-1}(M)$ we mean the co-syzygy of $M$. Recall this is constructed as follows. Let $F \rightarrow G \rightarrow M^* \rightarrow 0$ be a minimal presentation of $M^*$. Dualizing we get an exact sequence $0 \rightarrow M \rightarrow G^* \rightarrow F^*$. Then $\Omega^{-1}(M) = \text{coker } e^*$. It can be easily shown that if $F^* \rightarrow G^* \rightarrow M^* \rightarrow 0$ is another minimal presentation of $M^*$ then $\text{coker } e^* \cong \text{coker } \eta^*$.

3.2. Triangulated category structure on $\text{CM}(A)$.

The reference for this topic is [2, 4.7]. We first describe a basic exact triangle. Let $f: M \rightarrow N$ be a morphism in $\text{CM}(A)$. Note we have an exact sequence $0 \rightarrow M \rightarrow Q \rightarrow \Omega^{-1}(M) \rightarrow 0$, with $Q$-free. Let $C(f)$ be the pushout of $f$ and $i$. Thus we have a commutative diagram with exact rows

```
\[
\begin{array}{ccccccccc}
0 & \rightarrow & M & \rightarrow & i & Q & \rightarrow & p & \Omega^{-1}(M) & \rightarrow & 0 \\
\downarrow \downarrow & & \downarrow f & & \downarrow i & & \downarrow p & & \downarrow j \\
0 & \rightarrow & N & \rightarrow & i' & C(f) & \rightarrow & p' & \Omega^{-1}(M) & \rightarrow & 0
\end{array}
\]
```

Here $j$ is the identity map on $\Omega^{-1}(M)$. As $N, \Omega^{-1}(M) \in \text{CM}(A)$ it follows that $C(f) \in \text{CM}(A)$. Then the projection of the sequence

```
\[
M \rightarrow N \rightarrow C(f) \rightarrow \Omega^{-1}(M)
\]
```

in $\text{CM}(A)$ is a basic exact triangle. Exact triangles in $\text{CM}(A)$ are triangles isomorphic to a basic exact triangle.
Remark 3.3. If \( 0 \rightarrow M \xrightarrow{f} N \rightarrow L \rightarrow 0 \) is an exact sequence in \( \text{CM}(A) \) then we have an exact triangle \( M \rightarrow N \rightarrow L \rightarrow \Omega^{-1}(M) \) in \( \text{CM}(A) \). To see this we do the basic construction with the map \( f \). Then note that we have an exact sequence in \( \text{CM}(A) \)

\[
0 \rightarrow Q \rightarrow C(f) \rightarrow L \rightarrow 0.
\]

As \( A \) is Gorenstein and \( Q \) is free we get \( C(f) \cong Q \oplus L \). It follows that \( C(f) \cong L \) in \( \text{CM}(A) \). The result follows.

The main result of this section is

**Theorem 3.4.** Let \( \xi \colon \text{ICM}(A) \rightarrow \mathbb{Z} \) be a pre-triangle function. Then \( \xi \) induces a triangle function \( \xi' \colon \text{ICM}(A) \rightarrow \mathbb{Z} \) defined as

\[
\xi'([M]) = \xi(<M>).
\]

(Here by \( <M> > \) we mean isomorphism class of \( M \) in \( \text{CM}(A) \)).

**Proof.** We first show that \( \xi' \) is a well-defined function. Let \([M] = [N]\). Then there exists free modules \( F,G \) such that \( M \oplus F \cong N \oplus G \). So \( <M \oplus F> = <N \oplus G> \) in \( \text{ICM}(A) \). Thus \( \xi(<M \oplus F>) = \xi(<N \oplus G>) \). But \( \xi \) is a pre-triangle function. So

\[
\xi(<M \oplus F>) = \xi(<M>) + \xi(<F>) = \xi(<M>).
\]

Similarly \( \xi(<N \oplus G>) = \xi(<N>) \). It follows that \( \xi' \) is a well-defined function.

Properties (1),(2),(3) are trivial to show and is left to the reader. We prove property (4). Let \( M \rightarrow N \rightarrow L \rightarrow \Omega^{-1}(M) \) be an exact triangle in \( \text{CM}(A) \). Then it is isomorphic to a basic triangle \( M' \xrightarrow{f} N' \xrightarrow{g} C(f) \xrightarrow{\Omega} \Omega^{-1}(M') \rightarrow 0 \). As \( \xi \) is a pre-triangle we have

\[
\xi(<C(f)>) \leq \xi(<N'>) + \xi(<\Omega^{-1}(M')>).
\]

Note \( C(f) \cong L \), \( \Omega^{-1}M \cong \Omega^{-1}(M') \) and \( N \cong N' \) in \( \text{CM}(A) \). So we have

\[
\xi'([L]) \leq \xi'([N]) + \xi'([\Omega^{-1}(M)]).
\]

Thus we have shown property 4(b) for all exact triangles. By [1.4] it follows that property 4(a),(c) are also satisfied for all exact triangles. \( \square \)

We now give

**Proof of Theorem 1.2.** This follows from Theorem 2.11 and Theorem 3.4. \( \square \)

3.5. We now give construction of infinitely many triangle functions on \( \text{CM}(A) \). Since we have one pre-triangle function on \( \text{ICM}(A) \), we constructed in 2.12, 2.13 and 2.15 infinitely many pre-triangle functions. Each of these will yield a triangle function on \( \text{CM}(A) \).

4. Some preliminaries on Liaison of Modules and Maximal Cohen-Macaulay approximation

In this section we recall the definition of linkage of modules as given in [7]. We also recall the notion of maximal Cohen-Macaulay approximations and then briefly explain its connection with Liaison theory. We also prove an easy result regarding maximal Cohen-Macaulay approximations (which we suspect is already known but we are unable to find a reference). Throughout this section \( A \) is a Gorenstein ring. Recall a Gorenstein ideal \( q \) in \( A \) is a perfect ideal \( q \) with \( A/q \) a Gorenstein ring.
4.1. Let us recall the definition of transpose of a module. Let \( F_1 \xrightarrow{\phi} F_0 \to M \to 0 \) be a minimal presentation of \( M \). Let \((-)^* = \text{Hom}(-, A)\). The transpose \( \text{Tr}(M) \) is defined by the exact sequence

\[
0 \to M^* \to F_0^* \xrightarrow{\phi^*} F_1^* \to \text{Tr}(M) \to 0.
\]

**Definition 4.2.** Two \( A \)-modules \( M \) and \( N \) are said to be **horizontally linked** if \( M \cong \Omega(\text{Tr}(N)) \) and \( N \cong \Omega(\text{Tr}(M)) \).

Next we define linkage in general.

**Definition 4.3.** Two \( A \)-modules \( M \) and \( N \) are said to be linked via a Gorenstein ideal \( q \) if

1. \( q \subseteq \text{ann} M \cap \text{ann} N \), and
2. \( M \) and \( N \) are horizontally linked as \( A/q \)-modules.

We write it as \( M \sim_q N \).

**Remark 4.4.** It can be shown that ideals \( I \) and \( J \) are linked by a Gorenstein ideal \( q \) (definition as in the introduction) if and only if the module \( A/I \) is linked to \( A/J \) by \( q \), see [7, Proposition 1].

4.5. We say \( M, N \) are in **same linkage class** of modules if there is a sequence of \( A \)-modules \( M_0, \ldots, M_n \) and Gorenstein ideals \( q_0, \ldots, q_{n-1} \) such that

1. \( M_j \sim_{q_j} M_{j+1} \), for \( j = 0, \ldots, n-1 \).
2. \( M_0 = M \) and \( M_n = N \).

If \( n \) is even then we say that \( M \) and \( N \) are **evenly linked**.

4.6. (**MCM-approximations**) An MCM approximation of an \( A \)-module \( M \) is a short exact sequence \( 0 \to Y \to X \to M \to 0 \) where \( X \) is maximal Cohen-Macaulay and \( \text{projdim} \ Y < \infty \). If \( 0 \to Y' \to X' \to M \to 0 \) is another MCM approximation of \( M \) then \( X \) and \( X' \) are stably isomorphic, i.e., there exists free modules \( F, G \) with \( X \oplus F \cong X' \oplus G \). Thus we have a well-defined object \( X_M \) in \( \text{CM}(A) \). This in fact defines a functor but we do not need it here.

The relation between Liaison theory and MCM approximation is the following result by Martsinkovsky and Strooker [7, Theorem 13]. For Cohen-Macaulay modules of codimension \( r > 0 \) this result was proved by Yoshino and Isogawa [12, Corollary 1.6].

**Theorem 4.7.** Let \((A, m)\) be a Gorenstein local ring and let \( M \) and \( N \) be two \( A \)-modules. If \( M \) is evenly linked to \( N \) then \( X_M \cong X_N \) in \( \text{CM}(A) \).

4.8. If \( M \) is Cohen-Macaulay then maximal Cohen-Macaulay approximation of \( M \) are very easy to construct. We recall this construction from [11, p. 7]. Let \( n = \text{codim} \ M = \dim A - \dim M \). Let \( M^\vee = \text{Ext}_A^n(M, A) \). It is well-known that \( M^\vee \) is Cohen-Macaulay module of codim \( n \) and \( M^{\vee \vee} \cong M \). Let \( F \) be any free resolution of \( M^\vee \) with each \( F_i \) a finitely generated free module. Note \( F \) need not be minimal free resolution of \( M \). Set \( S_n(F) = \text{image}(F_n \xrightarrow{\partial_n} F_{n-1}) \). Then note \( S_n(F) \) is a maximal Cohen-Macaulay \( A \)-module. It can be easily proved that \( X_M \cong S_n(F)^* \) in \( \text{CM}(A) \).

The following result should be well-known to the experts. We give a proof due to lack of a reference.
Proposition 4.9. Let $M, N, L$ be Cohen-Macaulay $A$-modules with $\text{codim} = n$. Suppose we have an exact sequence $0 \to M \to N \to L \to 0$. Then we have an exact triangle

$$X_M \to X_N \to X_L \to \Omega^{-1}(X_M)$$

in $\text{CM}(A)$.

Proof. Dualizing we have an exact sequence $0 \to L^\vee \to N^\vee \to M^\vee \to 0$. By a well-known theorem in homological algebra there exists a short-exact sequence of complexes $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ where $\mathcal{F}, \mathcal{G}$ and $\mathcal{H}$ are free resolutions of $L^\vee, N^\vee$ and $M^\vee$ respectively. We use notation as in 4.8. Note we have an exact sequence

$$0 \to S_n(\mathcal{F}) \to S_n(\mathcal{G}) \to S_n(\mathcal{H}) \to 0.$$

As each of the modules in the above short exact sequence is maximal Cohen-Macaulay we get an exact sequence

$$0 \to S_n(\mathcal{H})^* \to S_n(\mathcal{G})^* \to S_n(\mathcal{F})^* \to 0.$$

The result now follows from 3.3 and 4.8. □

5. Proof of Theorem 1.4

In this section $(A, \mathfrak{m})$ is a Gorenstein local ring. First we prove that for one dimensional rings the set of even liason classes of $\{m^n \mid n \geq 1\}$ is a finite set.

Proposition 5.1. Let $(A, \mathfrak{m})$ be a one-dimensional Gorenstein ring. Then there exists $s \geq 1$ such that $m^{sn+r}$ is evenly linked to $m^{(n-1)+r}$ for all $n \gg 0$ and $r = 0, 1, \ldots, s - 1$.

Proof. Let $a \in m^s \setminus m^{s+1}$ be such that image of $a$ in $m^s/m^{s+1}$ is a parameter for the associated graded ring $G = \bigoplus_{n \geq 0} m^n/m^{n+1}$. Then it can be shown that $a$ is a non-zero divisor of $A$ and $(m^{n+s}: a) = m^n$ for all $n \gg 0$. We also have that $m^{n+s} = \mathfrak{m}^n$ for all $n \gg 0$.

It is easily verified that for all $n \gg 0$ we have $(a^n: m^{sn-r}) = (a^{n-1}: m^{s(n-1)-r})$ for $r = 0, 1, \ldots, s - 1$. Therefore $m^{sn-r}$ is evenly linked to $m^{s(n-1)-r}$ for $r = 0, 1, \ldots, s - 1$ and for all $n \gg 0$. □

Remark 5.2. If the residue field of $A$ is infinite then note we can choose $s = 1$ in the above Proposition 3.1.5.12. So we get $m^n$ is evenly linked to $m^{n-1}$ for all $n \gg 0$.

5.3. By [3] Theorem 3.6] it follows that if $K$ is a field and $R = K[[X_1, \ldots, X_n]]$ then $n^i$ is evenly linked to $n^{i-1}$ for all $i \geq 2$; here $n$ is the maximal ideal of $R$. We do not know whether in general for a regular local ring $(R, \mathfrak{n})$ with $\text{dim} R \geq 3$ we have $n^i$ is evenly linked to $n^{i-1}$. We also do not know whether the set of even liason classes of $\{n^i \mid i \geq 1\}$ is a finite set.

5.4. Let $M$ be an $A$-module of dimension $r$. The function

$$H(M, n) = \ell(m^n M/m^{n+1}M) \quad n \geq 0,$$

is called the Hilbert function of $M$. It is well-known that it is of polynomial type of degree $r - 1$. In particular if $r \geq 2$ then $H(M, n) \to \infty$ as $n \to \infty$.

We now give:
Proof of Theorem 1.4. For \( n \geq 0 \) we have an exact sequence of finite length \( A \)-modules
\[
0 \to \frac{m^n M}{m^{n+1} M} \to \frac{M}{m^{n+1} M} \to \frac{M}{m^n M} \to 0.
\]
For \( n \geq 0 \), let \( X_n, Y_n \) denote the maximal Cohen-Macaulay approximations of \( \frac{m^n M}{m^{n+1} M} \) and \( \frac{M}{m^{n+1} M} \) respectively. Note \( X_n \cong X_k^{H(M,n)} \) in \( \text{CM}(A) \). By (4.9) for all \( n \geq 1 \) we have an exact triangle in \( \text{CM}(A) \)
\[(5.4.1) \quad X_n \to Y_n \to Y_{n-1} \to \Omega^{-1}(X_n).\]

Suppose if possible \( \mathcal{A} \subseteq \bigcup_{i=1}^m L_i \) for some finitely many even liaison classes \( L_1, \ldots, L_m \). Choose \( V_i \in L_i \) for \( i = 1, \ldots, m \). Then for all \( n \geq 0 \) we have \( Y_n \cong X_{V_i} \) in \( \text{CM}(A) \) for some \( n \) (depending on \( n \)). Notice we also have \( \Omega^{-1}(Y_n) \cong \Omega^{-1}(X_{V_i}) \) in \( \text{CM}(A) \).

Let \( \xi \) be any triangle function on \( \text{ICM}(A) \). Then by (5.4.1) we have
\[(5.4.2) \quad \xi(\Omega^{-1}(X_n)) \leq \xi(\Omega^{-1}(Y_{n-1})).\]

Let
\[
\alpha = \max\{\xi([X_{V_i}]) \mid i = 1, \ldots, m\},
\beta = \max\{\xi([\Omega^{-1}(X_{V_i})]) \mid i = 1, \ldots, m\}.
\]
Also note that
\[
\Omega^{-1}(X_n) = (\Omega^{-1}X_k)^{H(M,n)} \quad \text{in} \quad \text{CM}(A).
\]
By (5.4.2) we have
\[
H(M,n)\xi(\Omega^{-1}(X_k)) \leq \alpha + \beta.
\]

Since \( \dim M \geq 2 \) we have that \( H(M,n) \to \infty \) as \( n \to \infty \). It follows that \( \xi(\Omega^{-1}(X_k)) = 0 \). Therefore \( \Omega^{-1}(X_k) \) is free. It follows that \( X_k \) is free. Therefore \( \text{projdim} k < \infty \). This implies that \( A \) is regular. \( \square \)

6. Proof of Theorem 1.5

The following result follows easily from [3, Theorem 3.6]. However we give a proof as we do not have a reference. It also explains the significance of Theorem 1.5.

Proposition 6.1. Let \( (A, m) \) be a complete equi-characteristic Gorenstein local ring. Let \( I \) be an ideal generated by a regular sequence. The \( I^n \) is evenly linked to \( I^{n-1} \) for all \( n \geq 2 \).

To prove this result we need the following general result.

Lemma 6.2. Let \( \phi: (A, m) \to (B, n) \) be a faithfully flat homomorphism of Gorenstein local rings. Let \( I, J \) be ideals in \( A \) and let \( q \) be a Gorenstein ideal in \( A \) such that \( I \sim_q J \). Then
(1) \( qB \) is a Gorenstein ideal in \( B \).
(2) \( IB \sim_{qB} JB \).

Proof. (1) Tensoring a minimal free \( A \)-resolution of \( A/q \) with \( B/qB \) we get a minimal free \( B \)-resolution of \( B/qB \). Thus \( \text{projdim}_{B/qB} B/qB \) is finite.

Let \( S = B/mB \) be the fiber ring of \( \phi \). Then as \( B \) is Gorenstein we have that \( S \) is a Gorenstein ring as well, see [3, Theorem 23.4].
Note the induced map $\overline{\phi} : A/q \to B/qB$ is also flat with fiber ring $S$. As $A/q$ and $S$ are Gorenstein rings we have that $B/qB$ is also Gorenstein, see [3, Theorem 23.4]. Thus $qB$ is a Gorenstein ideal.

(2) As $I \sim q J$ we have $(q : I) = J$ and $(q : J) = I$. As $\phi$ is flat we have

$$JB = (q : I)B = (qB : IB);$$

see [8, Theorem 7.4]. Similarly $IB = (qB : JB)$. Therefore $IB \sim qB JB$. □

As an easy consequence we have

**Corollary 6.3.** Let $K$ be a field. Let $R = K[[X_1, \ldots, X_n]]$. Fix $r \geq 1$. Set $I = (X_1, \ldots, X_r)$. Then $I^n$ is evenly linked to $I^{n-1}$ for $n \geq 2$.

**Proof.** Let $T = K[[X_1, \ldots, X_r]]$ and let $m = (X_1, \ldots, X_r)$. The inclusion $T \to R$ is flat. By [5, Theorem 3.6], $m^n$ is evenly linked to $m^{n-1}$ for $n \geq 2$. By [6.2] we have that $I^n$ is evenly linked to $I^{n-1}$ for $n \geq 2$. □

We now give

**Proof of Proposition 6.1.** Let $I = (x_1, \ldots, x_r)$. Extend this regular sequence to a system of parameters $x_1, \ldots, x_d$ of $A$. Assume $A = K[[Y_1, \ldots, Y_m]]/I$. Consider the subring $B = K[[x_1, \ldots, x_d]]$ of $A$. Then note that

1. $A$ is finitely generated as a $B$-module.
2. $B \cong K[[X_1, \ldots, X_d]]$ the power series ring over $K$ in $d$-variables.
3. As $A$ is Cohen-Macaulay we have that $A$ is free as a $B$-module. Thus the inclusion $i : A \to B$ is flat.

By Corollary 6.3 we have that the $B$-ideal $J = (x_1, \ldots, x_r)$ has the property that $J^n$ is evenly linked to $J^{n-1}$ for all $n \geq 2$. By [6.2] it follows that $I^n$ is evenly linked to $I^{n-1}$ for all $n \geq 2$. □

**Remark 6.4.** (with hypotheses as in 6.1). Note that as modules, $A/I^n$ is evenly linked to $A/I^{n-1}$ for all $n \geq 2$. It follows that if $F$ is a finitely generated free $A$-module then $F/I^n F$ is evenly linked to $F/I^{n-1} F$ for all $n \geq 2$.

We now give

**Proof of Theorem 1.5.** As $M$ is a maximal Cohen-Macaulay $A$-module it follows that $x_1, \ldots, x_r$ is an $M$-regular sequence. Note that $I^n M/I^{n+1} M \cong (M/IM)^{\gamma_n}$ where $\gamma_n = \binom{n+r-1}{r-1}$, see [3, Theorem 1.1.8]. For all $n \geq 0$ we also have an exact sequence

$$(6.4.3) \quad 0 \to \frac{I^n M}{I^{n+1} M} \to \frac{M}{I^{n+1} M} \to \frac{M}{I^n M} \to 0.$$

Inductively one can prove that $M/I^n M$ is a Cohen-Macaulay $A$-module of codimension $r$. Thus [6.4.3] is an exact sequence of codimension $r$ Cohen-Macaulay $A$-modules. For $n \geq 0$ let $X_n, Y_n$ denote maximal Cohen-Macaulay approximations of $I^n M/I^{n+1} M$ and $M/I^{n+1} M$ respectively. Therefore by [4.9] for all $n \geq 1$ we have the following exact triangle in $\text{CM}(A)$

$$(6.4.4) \quad X_n \to Y_n \to Y_{n-1} \to \Omega^{-1}(X_n).$$

Suppose if possible $\Lambda^j M \subseteq \bigcup_{i=1}^m L_i$ for some finitely many even lasson classes $L_1, \ldots, L_m$. Choose $V_i \subseteq L_i$ for $i = 1, \ldots, m$. Then for all $n \geq 0$ we have $Y_n \cong X_{V_i}$
in \( \text{CM}(A) \) for some \( i \) (depending on \( n \)). Notice we also have \( \Omega^{-1}(Y_n) \cong \Omega^{-1}(X_V) \) in \( \text{CM}(A) \).

Let \( \xi \) be any triangle function on \( I\text{CM}(A) \). Then by \( 6.4.1 \) we have
\[
(6.4.5) \quad \xi([\Omega^{-1}(X_n)]) \leq \xi([\Omega^{-1}(Y_{n-1})]) + \xi([\Omega^{-1}(Y_n)]).
\]

Let
\[
\alpha = \max\{\xi([X_{V_i}]) \mid i = 1, \ldots, m\},
\]
\[
\beta = \max\{\xi([\Omega^{-1}(X_{V_i})]) \mid i = 1, \ldots, m\}.
\]

Also note that
\[
\Omega^{-1}(X_n) = (\Omega^{-1}X_{M/IM})^{\gamma_n} \quad \text{in \( \text{CM}(A) \).}
\]

By \( 6.4.5 \) we have
\[
\gamma_n \xi([\Omega^{-1}X_{M/IM}]) \leq \alpha + \beta.
\]

Since \( r \geq 2 \) we have that \( \gamma_n \to \infty \) as \( n \to \infty \). It follows that \( \xi([\Omega^{-1}X_{M/IM}]) = 0 \). Therefore \( \Omega^{-1}(X_{M/IM}) \) is free. It follows that \( X_{M/IM} \) is free. Therefore \( \text{projdim}_A M/IM < \infty \). As \( x_1, \ldots, x_r \) is an \( M \)-regular sequence it follows that \( \text{projdim}_A M \) is finite. So \( M \) is free. \( \square \)

7. Proof of Theorem 1.6

Let \( r \geq 1 \). Let \( \text{CM}'(A) \) denote the full sub-category of Cohen-Macaulay \( A \)-modules of codimension \( r \). In this section we define an invariant of modules in \( \text{CM}(A) \) and then use it to prove Theorem 1.6.

Definition 7.1. Let \( N \in \text{CM}'(A) \). Let \( X_N \) be a maximal Cohen-Macaulay approximation of \( N \). Set \( \theta_A(N) = \epsilon_A^T([X_N]) \).

As \( \epsilon_A^T(-) \) is a triangle function on \( \text{CM}(A) \) it follows that \( \theta_A(N) \) is a well-defined invariant of \( M \).

The number \( \theta_A(-) \) behaves well mod superficial sequences. Let us recall the notion of a superficial sequence. Let \( N \) be an \( A \)-module of dimension \( r \). We say \( x = x_1, \ldots, x_s \) (with \( s \leq r \)) is an \( N \)-superficial sequence if \( x_1 \) is \( N \)-superficial, \( x_i \) is \( N/(x_1, \ldots, x_{i-1})N \) superficial for \( 2 \leq i \leq s \).

Proposition 7.2. Let \( N \in \text{CM}'(A) \) with \( r \geq 1 \) and let \( \dim A = d \). Let \( 0 \to Y \to X \to N \to 0 \) be a maximal Cohen-Macaulay approximation of \( M \). Let \( x = x_1, \ldots, x_{d-r} \) be a \( A \oplus X \oplus \Omega(X) \oplus N \) superficial sequence. Set \( B = A/(x) \).

Then
\[
\theta_A(N) \leq \epsilon_0(N)\theta_{B}(k).
\]

Proof. Note \( x \) is a \( X \oplus Y \oplus N \) regular sequence. So \( 0 \to Y/xY \to X/xX \to N/xN \to 0 \) is a maximal Cohen-Macaulay approximation of the \( B \)-module \( N/xN \). Note as \( r \geq 1 \) we have that \( d - r = d - 1 \). Using \( 2.9 \) we get \( \epsilon_B^T(X/xX) = \epsilon_A^T(X) \). So we have \( \theta_A(N) = \theta_B(N/xN) \). It suffices to prove \( \theta_B(N/xN) \leq \epsilon(N)\theta_{B}(k) \). By \( 2.8 \) we get that \( N/xN \) is a \( B \)-module of length \( \epsilon_0(N) \).

Let \( L \) be a finite length \( B \)-module. We prove by induction on \( \ell(L) \) that \( \theta_B(L) \leq \ell(L)\theta_{B}(k) \). We have nothing to prove if \( \ell(L) = 1 \). So assume \( \ell(L) = m \geq 2 \) and the result is proved for all \( B \)-modules of length \( \leq m - 1 \).

We have an exact sequence \( 0 \to V \to L \to k \to 0 \) where \( \ell(V) = m - 1 \). In \( \text{CM}(B) \) we have an exact triangle
\[
X_V \to X_L \to X_k \to \Omega^{-1}(X_V)
\]
It follows that \( e_B^L(X_L) \leq e_B^T(X_V) + e_B^T(X_k) \leq m e_B^R(k) \). Thus \( \theta_B(L) \leq m \theta_B(k) \). \( \square \)

Our proof of Theorem 1.6 uses the following result by Herzog-Kühl, \( \cite{HerzogKuhl} \) Theorem 2.1.

**Theorem 7.3.** Let \( R \) be a local Gorenstein domain with infinite residue field \( k \). Let \( 0 \rightarrow F_1 \rightarrow M_1 \rightarrow I_1 \rightarrow 0 \) and \( 0 \rightarrow F_2 \rightarrow M_2 \rightarrow I_2 \rightarrow 0 \) be any two Bourbaki sequences (i.e., \( F_1, F_2 \) are free, \( M_1, M_2 \) are maximal Cohen-Macaulay modules and \( I_1, I_2 \) are Cohen-Macaulay ideals of codimension 2). Then the following two statements are equivalent:

1. \( M_1 \) and \( M_2 \) are stably isomorphic.
2. \( I_1 \) and \( I_2 \) are evenly linked by a complete intersection.

We should remark that a Bourbaki sequence is simply a maximal Cohen-Macaulay approximation of \( I \) where \( I \) is a codimension 2 Cohen-Macaulay ideal.

**7.4.** Our proof of Theorem 1.6 also uses the fact that if \( A \) is a simple hypersurface singularity then it has only finitely many indecomposable maximal Cohen-Macaulay modules; \( \cite{HerzogKuhl} \). We now give

**Proof of Theorem 1.6.** Suppose if possible for some \( m \geq 1 \) the set \( C_m \) is not contained in any collection of finitely many even liason classes. For \( j \geq 1 \) let \( I_j \) be ideals with \( e_0(A/I_j) \leq m \) such that the liason classes \( L_j \) of \( I_j \) are all distinct.

For \( j \geq 1 \) let \( 0 \rightarrow Y_j \rightarrow X_j \rightarrow A/I_j \rightarrow 0 \) be maximal Cohen-Macaulay approximation of \( A/I_j \).

**Claim:** there exists \( x = x_1, \ldots, x_{d-2} \) such that \( x \) is a \( A \oplus X_j \oplus \Omega(X_j) \oplus A/I_j \)-superficial for all \( j \geq 1 \).

To prove the claim let us recall a construction of a superficial element. Let \( E \) be an \( A \)-module. Consider the associated graded ring \( G = \bigoplus_{n \geq 0} m^n/m^{n+1} \) of \( A \) and let \( G(E) = \bigoplus_{n \geq 0} m^n E/m^{n+1} E \) be the associated graded module of \( E \). Let \( \text{Ass}(G(E)) \) be the relevant associated primes of \( G(E) \), i.e., those associated primes of \( G(E) \) which are not equal to \( G_+ = \bigoplus_{n \geq 1} m^n/m^{n+1} \). Let \( V = m/m^2 \). Note \( P \cap V \) is a proper subspace of \( V \) for every \( P \in \text{Ass}(G(E)) \). As the field \( k \) is infinite and as \( \text{Ass}(G(E)) \) is a finite set we get that

\[
\widetilde{V} = \left( V \setminus \bigcup_{P \in \text{Ass}(G(E))} P \cap V \right) \neq \emptyset.
\]

An element \( x \in m \) such that \( \overline{x} \in \widetilde{V} \) is an \( E \)-superficial element. This construction yields superficial element of a single module. Note we have only used that \( k \) is an infinite field.

To prove the claim, let \( E_j = A \oplus X_j \oplus \Omega(X_j) \oplus A/I_j \) for \( j \geq 1 \). Note \( \text{Ass}(G(E_j)) \) is a finite set. So

\[
D = \bigcup_{j \geq 1} \left( \bigcup_{P \in \text{Ass}(G(E_j))} P \cap V \right)
\]

is a union of a countable number of proper \( k \)-subspaces of \( V \). As \( k \) is an uncountable field we get that \( \widetilde{V} = V \setminus D \) is non-empty. An element \( x \in m \) such that \( \overline{x} \in \widetilde{V} \) is an \( E_j \)-superficial element for all \( j \geq 1 \). Note for construction of this superficial element we did not use that \( A \) is a simple hypersurface singularity. We only used that the
residue field of $A$ is uncountable. Iterating this procedure we get a superficial sequence $x = x_1, \ldots, x_{d-2}$ for all $E_j$.

Set $B = A/(x)$. By the claim and Proposition 7.2 we get that

$$e^T_A(X_j) = \theta_A(A/I_j) \leq m\theta_B(k).$$

Set $c = m\theta_B(k)$. Let $M_1, M_2, \ldots, M_m$ be all the indecomposable non-free maximal Cohen-Macaulay $A$-modules. Write

$$X_j = M_1^{a_{1,j}} \oplus \cdots \oplus M_m^{a_{m,j}} \oplus A^{l_j}.$$  

Here $a_{i,j}, l_j \geq 0$. Note that

$$\sum_{j=1}^m a_{i,j} \leq e^T_A(X_j) \leq c \quad \text{for all } j \geq 1.$$  

By pigeon-hole principle it follows that there exists $r, s$ with $r < s$ such that $X_r$ is stably isomorphic to $X_s$. Note $\Omega(X_r)$ (and a free summand) will give a maximal Cohen-Macaulay approximation of $I_r$. By Herzog-Kuhl’s result we have that $I_r$ is evenly linked to $I_s$. So $L_r = L_s$ a contradiction. Thus $C_m$ is contained in finitely many even liaison classes of $A$. □

REFERENCES

[1] M. Auslander and R-O. Buchweitz, The homological theory of maximal Cohen-Macaulay approximations, Colloque en l’honneur de Pierre Samuel (Orsay, 1987). Mm. Soc. Math. France (N.S.) No. 38 (1989), 5-37.

[2] R.-O. Buchweitz, Maximal Cohen-Macaulay modules and Tate cohomology over Gorenstein rings, Preprint, Univ. Hannover, (1986). available at http://hdl.handle.net/1807/16682.

[3] W. Bruns and J. Herzog, Cohen-Macaulay Rings, revised edition, Cambridge Stud. Adv. Math., vol. 39, Cambridge University Press, Cambridge, (1998).

[4] J. Herzog and M. Kühl, Maximal Cohen-Macaulay modules over Gorenstein rings and Bourbaki-sequences, Commutative algebra and combinatorics (Kyoto, 1985), 65-92, Adv. Stud. Pure Math., 11, North-Holland, Amsterdam, 1987

[5] J. O. Kleppe, J. C. Migliore, R. Miró-Roig, U. Nagel and C. Peterson, Gorenstein liaison, complete intersection liaison invariants and unobstructedness, Mem. Amer. Math. Soc. 154 (2001), no. 732, viii+116 pp.

[6] H. Knörrer, Cohen-Macaulay modules on hypersurface singularities. I, Invent. Math. 88 (1987), no. 1, 153-164.

[7] A. Martsinkovsky and J. R. Strooker, Linkage of modules, J. Algebra 271 (2004), no. 2, 587-626.

[8] H. Matsumura, Commutative ring theory, Translated from the Japanese by M. Reid. Second edition. Cambridge Studies in Advanced Mathematics, 8. Cambridge University Press, Cambridge, 1989. xiv+320 pp.

[9] T. J. Puthenpurakal, Hilbert-coefficients of a Cohen-Macaulay module, J. Algebra 264 (2003), no. 1, 82-97.

[10] The Hilbert function of a maximal Cohen-Macaulay module, Math. Z. 251 (2005), no. 3, 551-573.

[11] H. Wang, Hsin-Ju, Links of symbolic powers of prime ideals, Math. Z. 256 (2007), no. 4, 749-756.

[12] Y. Yoshino and S. Isogawa, Linkage of Cohen-Macaulay modules over a Gorenstein ring, J. Pure Appl. Algebra 149 (2000), 305-318.

DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY BOMBAY, POWAI, MUMBAI 400 076, INDIA

e-mail address: tputhen@math.iitb.ac.in