An integrated production-inventory model for the single-vendor two-buyer problem with partial backorder, stochastic demand, and service level constraints

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Abstract. This paper presents an integrated single-vendor two-buyer production-inventory model with stochastic demand and service level constraints. Shortage is permitted in the model, and partial backordered partial lost sale. The lead time demand is assumed follows a normal distribution and the lead time can be reduced by adding crashing cost. The lead time and ordering cost reductions are interdependent with logarithmic function relationship. A service level constraint policy corresponding to each buyer is considered in the model in order to limit the level of inventory shortages. The purpose of this research is to minimize joint total cost inventory model by finding the optimal order quantity, safety stock, lead time, and the number of lots delivered in one production run. The optimal production-inventory policy gained by the Lagrange method is shaped to account for the service level restrictions. Finally, a numerical example and effects of the key parameters are performed to illustrate the results of the proposed model.

Keywords: Integrated model, partial backorder, service level, lead time

1. Introduction

In the past few decades, firms have realized that an efficient management across the different parties in a supply chain is critical to reducing inventory costs. This efficient management can be achieved through greater cooperation and better coordination among the different parties. This means that the vendor and the buyers should work cooperatively towards minimizing their costs. Integrated inventory management has received a great deal of attention. Probably, Goyal [5] has been one of the first pioneers in studying the joint optimization problem consisting of a single vendor and a single buyer where the rate of production is infinite for the vendor. The Goyal’s [5] model has been generalized by Banerjee [3] by considering the finite production rate. Later, Goyal and Nebebe [6] proposed new solution approaches for the single-vendor single-buyer model under different shipment strategies. Pan and Yang [11] also investigated the single-vendor single-buyer integrated model under assumption that the lead time demand follows normal distribution function. Further, Ouyang et al. [10] improve the Pan and Yang’s [11] model by assuming that the shortage is permitted during the lead time. Recently, the single-
vendor single-buyer integrated models have been developed by many scholars in different assumptions, for example in Rad et al. [12], An and Lee [2], and Giri and Bardhan [4]. Giri and Bardhan [4] proposed a single-vendor single-buyer model by considering the space limitation at the buyer’s end.

Further, only few researchers have addressed the integrated single-vendor multi-buyer system. In general, in the system the vendor has a problem with determining optimum production quantity and shipping schedule, and the buyer has a problem with determining the order quantity, which minimize the join total operating cost. During the last three decades, researchers have been developing the model and the solution to these problems. Lu [9] proposed the single-vendor multi-buyer inventory model with the vendor manufacturing at a finite rate. Yao and Chiou [15] then introduced a heuristic for the single-vendor multi-buyer problem. Other related papers have been developed by Abdul-Jalbar et al. [1], Rad et al. [13], and Jha and Shanker [7]. Abdul-Jalbar et al. [1] proposed an integrated inventory model for the single-vendor two-buyer problem. In the model, it is assumed that the vendor manufactures the item at a finite rate, each buyer faces a constant deterministic demand, and the shortages are not allowed. Rad et al. [13] considered a two-echelon supply chain model with a single vendor and two buyers. The mathematical model is developed for the integrated vendor-managed inventory (VMI) policy. Later, in particular, Jha and Shanker [7] analyzed an integrated production-inventory model with controllable lead time and stochastic demand during the lead time. They assumed that the unsatisfied demand at the buyers is completely backordered and a service level constraint corresponding to each buyer is included in the model, which limits that the stock-out level per cycle of each buyer is bounded.

In this paper, we address an integrated production-inventory model for the single-vendor two-buyer problem with controllable lead time and service level constraints. Unlike Jha and Shanker [7], we assume that the partial backorder situation is considered. This means that the shortages are partial backordered and partial lost sale with a certain backorder rate. Further, the lead time and ordering cost reductions are interdependent. According to Kurdhi et al. [8], the implementation of electronic data interchange (EDI) may reduce the lead time and ordering cost simultaneously. Hence, it is more close to the real situation that ordering cost reductions vary according to different lead times. It is also assumed that both buyers order the same item to the vendor and the demand from customers to each buyer is stochastic. When the stochastic demand is considered, lead time becomes an important issue. The controllable lead time leads to many benefits such as improves customer service level and increases the competitive advantage of business. In many practical situations, lead time can be controlled at the expense of extra cost which is known as lead time crashing cost. In the present study, we also assume that the lead time of each buyer has several components in which all components can be shortened at an added crashing cost. We have focused on the single-vendor two-buyer system because it is the simplest case within the single-vendor multi-buyer problems. In this system, we can show a guarantee that the optimal solution obtained satisfies the second order sufficient condition for the minimizing problem with two service level constraints. We also believe that the results of this study will offer the possible strategies which can be analyzed for the multi-buyer case.

The rest of this paper is organized as follows: The notations and assumptions used in this paper are introduced in Section 2. In Section 3, we formulate the integrated production-inventory model containing single vendor and two buyers with controllable lead time, stochastic demand, and a service level constraint on each buyer in the partial backorder case. The Lagrangian multiplier technique and a detailed solution procedure to solve the proposed model are presented in Section 4. In Section 5, a numerical example and discussion of the results are provided. Finally, some conclusions and suggestions for some future research are given in Section 6.

2. Notations and Assumptions
The following notations and assumptions are used in developing mathematical model.

2.1. Notations

ith buyer parameter (i = 1, 2)

\[ D_i \] Average demand per unit time
\( C_{bi} \) Unit purchase cost
\( A_{bi} \) Ordering cost per order
\( S_{i} \) Safety stock
\( k_{i} \) Safety factor (decision variable)
\( h_{bi} \) Holding cost rate (per monetary unit invested in inventory) per unit time
\( L_{i} \) Length of lead time (decision variable)
\( Q_{i} \) Order quantity (decision variable)
\( \tau_{i} \) Reorder point
\( X_{i} \) Lead time demand, which is normally distributed with finite mean \( D_{i} \mathcal{L}_{i} \) and standard deviation \( \sigma_{i} \sqrt{\mathcal{L}_{i}} \), where \( \sigma_{i} \) denotes the standard deviation of demand per unit time,
\( X_{i} \sim N(D_{i} \mathcal{L}_{i}, \sigma_{i} \sqrt{\mathcal{L}_{i}}) \)
\( \beta_{i} \) Fraction of the shortage that will be backordered
\( \varepsilon_{i} \) Proportion of demands that cannot be met by stock, \((1 - \varepsilon_{i})\) is the service level

Vendor parameter
\( P \) Production rate, \( P > D \) (\( D = \sum_{i=1}^{2} D_{i} \))
\( C_{v} \) Unit production cost (\( C_{v} < C_{bi}, \forall i \))
\( h_{v} \) Holding cost rate (per monetary unit invested in inventory) per unit time
\( m \) Number of lots delivered from the vendor to each buyer in a production cycle (same for all the buyers), a positive integer (decision variable)
\( A_{p} \) Setup cost per setup
\( Q \) Shipment lot size in each delivery to meet the demand of all the buyers, \( Q = \sum_{i=1}^{2} Q_{i} \).

2.2. Assumptions
1. The system consists of two buyers who are supplied with a single-item by a single-vendor.
2. Buyer \( i \) orders a lot of size \( Q_{i} \) (\( Q = \sum_{i=1}^{2} Q_{i} \)) and the vendor manufactures \( mQ \) units with a finite production rate \( P \) \((P > D)\) in one setup but ships in quantity \( Q \) over \( m \) times to meet the demands of all the buyers such that \( Q_{i} = D_{i}Q/D \).
3. The items are delivered by the vendor at the same time to buyer 1 and buyer 2.
4. The lead time demand \( X_{i} \) has finite mean \((D_{i} \mathcal{L}_{i})\) and it follows a normal distribution with standard deviation \( \sigma_{i} \sqrt{\mathcal{L}_{i}} \).
5. Each buyer reviews inventory using continuous review policy and places an order whenever inventory level falls to the reorder point.
6. The reorder point \( \tau_{i} = \) expected demand during lead time \((D_{i} \mathcal{L}_{i})\) + safety stock \( (S_{i}) \), and \( S_{i} = k_{i} \sigma_{i} \sqrt{\mathcal{L}_{i}} \), \( \mathcal{L}_{i} = D_{i} \mathcal{L}_{i} + k_{i} \sigma_{i} \sqrt{\mathcal{L}_{i}} \), where \( k_{i} \) is the safety factor.
7. The lead time \( L_{i} \) of buyer \( i \) has \( n_{i} \) mutually independent components. The \( r \)th component of lead time of buyer \( i \) has a minimum \( a_{i,r} \), normal duration \( b_{i,r} \) and a crash cost per unit time \( c_{i,r} \), where \( c_{i,1} \leq c_{i,2} \leq \cdots \leq c_{i,n_{i}} \). Let \( L_{i,r} \) be the length of the lead time with components \( 1,2,...,r \) crashed to their minimum duration, then \( L_{i,r} \) can be expressed as
\[ L_{i,r} = \sum_{j=r+1}^{n_{i}} b_{i,j} - \sum_{j=1}^{r} a_{i,j}, \quad r = 1,2,...,n_{i} \]
The lead time crashing cost \( C_{i}(L_{i}) \) per cycle for a given \( L_{i} \in [L_{i,r},L_{i,r-1}] \) is
\[ C_{i}(L_{i}) = c_{i,r}(L_{i,r-1} - L_{i}) + \sum_{j=1}^{r-1}(b_{i,j} - a_{i,j}) \]
8. The lead time and ordering cost reductions have the following logarithmic functional relationship as:
\[ \frac{A_{i,\Delta} - A_{i}}{A_{i}} = \tau \ln \left( \frac{L_{i}}{L_{i,\Delta}} \right) \]
where \( \tau < 0 \) is constant scaling parameter for the logarithmic relationship between percentages of reductions in lead time and ordering cost.
3. Model Formulation

In this research, there are two buyers who ordered single-item to the same vendor. Buyer $i$ have a number of orders ($Q_i$) and lead time demand ($X_i$). However, both the buyers and vendor work together to determine the optimal $Q$, where $Q = \sum_{i=1}^{2} Q_i$. Since the setup cost for production is considered quite expensive, the vendor producing $mQ$ units, but sending a number of $Q$ units any buyer to order. Single-item that have been produced will be sent to each buyer with lead time demand $L_i$ weeks, where the lead time $L_i$ can be reduced by crashing cost. A mathematical model of the single-vendor two-buyer system will be formulated to minimize the joint total annual expected cost while satisfying the service level constraint on each buyer by determining the optimal order quantity, lead time, safety factor of the buyers and the number of shipments in a production cycle between the vendor and the buyer. The inventory pattern of the system can be seen in Figure 1.

![Figure 1. The inventory pattern for the single-vendor two-buyer system](image)

3.1. Vendor’s expected total annual cost
When the buyer \( i \) ordering single-item to vendor amounted \( Q_i \) units, the vendor manufactures \( mQ \) at one setup with a finite production rate \( P \). The length of system cycle is \( \frac{mQ}{D} \), \( m \) lots of \( Q \) size are delivered from the vendor to all the buyers. To obtain a minimum vendor total cost, vendor will determine the optimal values of the frequency of delivery (\( m \)). The resulting expected total annual cost for the vendor, which is composed of setup cost and holding cost, is expressed as

\[
TEC_v(Q, m) = \frac{A_vD}{mQ} + \frac{Q}{2} h_vC_v \left[ m \left( 1 - \frac{D}{P} \right) - 1 + \frac{2D}{P} \right].
\]  

(1)

3.2. Buyer’s expected total annual cost

The lead time demand from customers to each buyer follows a normal distribution with mean \( D_i L_i \) and standard deviation \( \sigma_i \sqrt{L_i} \). The expected demand because of the occurrence of stockout is given by

\[
E(X_i - \eta_i) = \sigma_i \sqrt{L_i} \phi(k_i), \quad \text{where } \phi(k_i) = \phi(k_i) - k_i [1 - \Phi(k_i)], \quad \text{and } \phi, \Phi \text{ are the standard normal probability density function and cumulative distribution function}.
\]

Thus, the expected number of backorders per cycle is \( \beta \sigma_i \sqrt{L_i} \phi(k_i) \) and the expected loss in sales per ordering cycle is \( (1 - \beta) \sigma_i \sqrt{L_i} \phi(k_i) \), where \( \beta \) is the fraction of the shortage that will be backordered. The resulting expected total annual cost for each buyer, which is composed of ordering cost, holding cost, and backorders per cycle is

\[
\frac{A_i}{D_i} + k_i \begin{cases}
\sigma_i \sqrt{L_i} \phi(k_i) & \text{if } k_i < 1 - \frac{D_i}{P}, \\
\frac{D_i}{P} & \text{if } k_i \geq 1 - \frac{D_i}{P}
\end{cases}
\]

(2)

Further, each buyer use service level constraint to limit the proportion of demands not met from stock, which should not exceed a certain value. The service level constraint for buyer \( i \) can be expressed as

\[
\frac{\frac{D_i \sigma_i \sqrt{L_i} \phi(k_i)}{Q_i}}{\sigma_i \sqrt{L_i} \phi(k_i)} \leq \epsilon_i, 
\]

i.e.,

\[
\frac{\sigma_i \sqrt{L_i} \phi(k_i)}{Q_i} \leq \epsilon_i, \text{ so } 1 - \epsilon \text{ is the service level. Substitution of } \frac{Q_i}{\sigma_i \sqrt{L_i} \phi(k_i)} \text{ in the inequality gives}
\]

\[
\frac{D_i}{Q_i} \leq \epsilon_i.
\]  

(4)

3.3. Joint expected total annual cost

The joint expected total annual cost is the sum of the expected total annual cost for the vendor in (1) and the expected total annual cost for the buyer 1 and 2 in (3). The service level constraint in (4) indicates that the shortage per cycle for buyer \( i \) is limited. Hence, the problem that must be resolved is

Minimize \( JTEC(Q, k_1, k_2, L_1, L_2, m) \)

\[
\begin{align*}
\sum_{i=1}^{2} TEC_v(Q, k_i, L_i) + & TEC_v(Q, m) \\
= & \frac{D}{Q} \left[ \frac{A_v}{m} + \sum_{i=1}^{2} A_i(L_i) + C_i(L_i) \right] + \sum_{i=1}^{2} h_v C_v \left[ m \left( 1 - \frac{D}{P} \right) - 1 + \frac{2D}{P} \right] \\
& + \frac{Q}{2} h_v C_v \left[ m \left( 1 - \frac{D}{P} \right) - 1 + \frac{2D}{P} \right] \times \sigma_i \sqrt{L_i} \phi(k_i)
\end{align*}
\]

subject to \( \frac{D_i \sigma_i \sqrt{L_i} \phi(k_i)}{Q_i} \leq \epsilon_i, \forall i. \)

4. Solution Technique

The problem (5) can be solved first by adding slack variables \( H^2 \geq 0 \) and \( H^2 \geq 0 \) to convert the

\[
\begin{align*}
\sum_{i=1}^{2} \frac{\sigma_i \sqrt{L_i} \phi(k_i)}{Q_i} \leq \epsilon_1 \text{ and } & \sum_{i=1}^{2} \frac{\sigma_i \sqrt{L_i} \phi(k_i)}{Q_i} \leq \epsilon_2, \text{ respectively, to equality. Then, the Lagrange function of (5) is}
\end{align*}
\]

\[
JTEC(Q, k_1, k_2, L_1, L_2, \lambda_1, \lambda_2, H_1, H_2, m)
\]
Thus, subtituting Equations (11) and (12) to Equation (7), we get

\[
\frac{\partial}{\partial \lambda_1} \left[ \mathcal{J}_{TTEC}(Q, k_1, k_2, L_1, L_2, \lambda_1, \lambda_2, H_1, H_2) \right] = -h_b C_b k_i \sigma_i \frac{1}{\sqrt{L_i}} - h_b C_b (1 - \beta_i) \sigma_i \frac{1}{\sqrt{L_i}} \phi(k_i)
\]

Consequently, if \( Q, k_1, k_2, \lambda_1, \lambda_2, H_1, H_2, \) and \( m \) fixed, \( \mathcal{J}_{TTEC}(Q, k_1, k_2, L_1, L_2, \lambda_1, \lambda_2, H_1, H_2, m) \) is concave in \( L_i \in [L_{i,r}, L_{i,r-1}], \forall i \). Hence, for fixed \( Q, k_1, k_2, \lambda_1, \lambda_2, H_1, H_2, m \), the minimum joint total expected cost will occur at the end points of the interval \([L_{i,r}, L_{i,r-1}], \forall i\).

Taking partial derivatives of \( \mathcal{J}_{TTEC}(Q, k_1, k_2, L_1, L_2, \lambda_1, \lambda_2, H_1, H_2, m) \) in (6) with respect to \( Q, k_1, k_2, \lambda_1, \lambda_2, H_1, H_2, \) respectively, and equalizing the results to zero, we have

\[
\frac{\partial}{\partial Q} \left[ \mathcal{J}_{TTEC}(Q, k_1, k_2, L_1, L_2, \lambda_1, \lambda_2, H_1, H_2, m) \right] = -\frac{D}{Q^2} \left[ \frac{A_v}{m} + \sum_{i=1}^2 A_i(L_i) + C_i(L_i) \right] + \sum_{i=1}^2 h_b C_b \left( \frac{D_i}{2D} \right) - h_b C_b \left( 1 - \beta_i \right) \sigma_i \frac{1}{\sqrt{L_i}} \phi(k_i)
\]

From Equation (10), we obtain \( \lambda_i = 0 \) or \( H_i = 0, \forall i \). If \( H_i = 0 \) and \( \lambda_i = 0 \), one has \( \Phi(k_i) = -\frac{\beta_i}{1 - \beta_i}, \forall i \). Since \( \Phi(k_i) \) cannot be negative, then \( H_i = 0, \lambda_i \neq 0, \forall i \). Hence, the service level constraints are active when the optimal solution is obtained. Furthermore, solving Equations (8) and (9), we have the following results:

\[
\lambda_i = \frac{h_b C_b}{D(1 - \Phi(k_i))} - \frac{h_b C_b (1 - \beta_i)}{D}, \forall i
\]

\[
\phi(k_i) = \frac{\epsilon_i D Q_i}{D \sigma_i \sqrt{L_i}}, \forall i.
\]

Thus, subtituting Equations (11) and (12) to Equation (7), we get

\[
Q^* = \left( \frac{D \left[ \frac{A_v}{m} + \sum_{i=1}^2 A_i(L_i) + C_i(L_i) \right] + \sum_{i=1}^2 h_b C_b \left( \frac{D_i}{2D} \right) }{h_b C_b \left( \frac{D_i}{2D} \right) - h_b C_b \left( 1 - \beta_i \right) \sigma_i \frac{1}{\sqrt{L_i}} \phi(k_i) } \right)^{\frac{1}{2}}
\]

The following proposition shows that point \((Q^*, k_1^*, k_2^*)\) is the local optimal solution, which minimizes the joint expected total annual cost \( \mathcal{J}_{TTEC}(Q, k_1, k_2, L_1, L_2, m) \) and satisfies the service level constraints \( \frac{D \sigma_i \sqrt{L_i} \phi(k_i)}{D Q} \leq \epsilon_i, \forall i \).

**Proposition.** For given \( m \) and \( L_i \in [L_{i,r}, L_{i,r-1}], \forall i \), the point \((Q^*, k_1^*, k_2^*)\) satisfies the second order sufficient condition (SOCS) for the minimizing problem with two constraints.

**Proof.** For given \( m \) and \( L_i \in [L_{i,r}, L_{i,r-1}], \forall i \), we first obtain the Bordered Hessian matrix \( H^B \) as is follows:
Step 0: Set $m = 1$.

Step 1: For each buyer $i = 1, 2$, perform Step 2.

Step 2: For each $L_{i, r}$, $r = 0, 1, ..., n_i$, perform (i) to (v).

(i) Start with $k_{i,r}^0 = 0$, and obtain $\phi(k_{i,r}^0) = 0.39894$ and $\phi(k_{i,r}^0) = 0.5$.

(ii) Substituting $\phi(k_{i,r}^0)$ into Equation (12) to obtain $Q_0^0$. 

The following algorithm is developed to describe the solution procedure for the integrated single-vendor two-buyer model to obtain the optimal solution of the buyer’s order quantity, safety factor and lead time and number of shipments to each buyer in each production cycle of the vendor.
(iii) Substituting $Q_r^2$ into Equation (11) to evaluate $\varphi(k_{i,r}^1)$. Check the value of $\varphi(k_{i,r}^1)$ in the standard normal table in Silver and Peterson [14] to find $k_{i,r}^1$, $\Phi(k_{i,r}^1)$, and $\Phi(k_{i,r}^1)$.

(iv) Repeat (ii) and (iii) until no change occurs in the values of $Q_r$ and $k_{i,r}$. Then denote the value of $Q_r$ and $k_{i,r}$ by $Q_r^*$ and $k_{i,r}^*$.

(v) Compute $JTEC(Q_r^*, k_{1,r}^*, k_{2,r}^*, L_{1,r}, L_{2,r}, m)$ using Equation (5).

Step 3: Set $C(Q_m^*, k_{1,m}^*, k_{2,m}^*, L_{1,m}^*, L_{2,m}^*, m) = \min_{r=0,1,...,n_t} \{JTEC(Q_r^*, k_{1,r}^*, k_{2,r}^*, L_{1,r}, L_{2,r}, m)\}$, then $(Q_r^*, k_{1,r}^*, k_{2,r}^*, L_{1,r}, L_{2,r}, m)$ is the optimal solution for $m$ value.

Step 4: Set $m = m + 1$ and repeat Step (1) to (3) to get $JTEC(Q_m^*, k_{1,m}^*, k_{2,m}^*, L_{1,m}^*, L_{2,m}^*, m)$.

Step 5: If $JTEC(Q_m^*, k_{1,m}^*, k_{2,m}^*, L_{1,m}^*, L_{2,m}^*, m) \leq JTEC(Q_{m-1}^*, k_{1,m-1}^*, k_{2,m-1}^*, L_{1,m-1}^*, L_{2,m-1}^*, m-1, m-1)$ then go to step 4, otherwise go to step 6.

Step 6: Set $JTEC(Q_r^*, k_{1,r}^*, k_{2,r}^*, L_{1,r}^*, L_{2,r}^*, m^*) = JTEC(Q_{m-1}^*, k_{1,m-1}^*, k_{2,m-1}^*, L_{1,m-1}^*, L_{2,m-1}^*, m-1, m-1)$, then $(Q_r^*, k_{1,r}^*, k_{2,r}^*, L_{1,r}^*, L_{2,r}^*, m^*)$ is the optimal solution.

Step 7: Determine the optimal order quantity of each buyer using the relationship $Q_r^* = \frac{D_rQ^*}{\delta}$.

5. A numerical example
In this section, we present an instance of a single-vendor two-buyer system to illustrate the solution procedure developed in Section 4. Consider a system consisting of two buyers and a vendor with the following parameter values of the vendor: $P = 25000$ units per year, $A_v = 400$ per setup, $C_v = 15$ per unit, and $h_v = 0.2$. The parameter values for the buyers are given in Table 1, whereas the lead time of each buyer has three components with the data shown in Table 2.

| Buyer $i$ | $D_i$ (units per year) | $A_{b_i} ($) | C_{b_i} ($) | h_{b_i} | q_i (units per weak) | $1 - \epsilon_i (%)$
|-----------|------------------------|--------------|------------|--------|----------------------|------------------|
| 1         | 1000                   | 20           | 25         | 0.2    | 20                   | 96               |
| 2         | 5000                   | 30           | 20         | 0.2    | 50                   | 97               |

| Buyer $i$ | Lead time component $r$ | Normal duration $b_{i,r}$ (days) | Minimum duration $a_{i,r}$ (days) | Unit crashing cost $c_{i,r}$ ($) (days) |
|-----------|-------------------------|---------------------------------|---------------------------------|------------------------------------------|
| 1         | 1                       | 20                              | 6                               | 0.4                                      |
|           | 2                       | 20                              | 6                               | 1.2                                      |
|           | 3                       | 16                              | 9                               | 5.0                                      |
| 2         | 1                       | 20                              | 6                               | 0.5                                      |
|           | 2                       | 16                              | 9                               | 1.3                                      |
|           | 3                       | 13                              | 6                               | 5.1                                      |

We solve the case when $\tau = -0.5$. By applying the algorithm procedure, the results can be seen in Table 3. From the table, it can be observed that $JTEC$ has the minimum value of $5237.63$ for the optimal number of shipments to each buyer as $m = 2$, the optimal shipment lot size as $Q = 699.07$ units, the lead time of the buyers as $L_1 = 6, L_2 = 4$ weeks, and the safety factor of the buyers as $k_1 = 0.929, k_2 = 0.578$. Using the relationship mentioned in Step 7 of the algorithm, the optimal order quantity for the buyers can be calculated as $Q_1 = 116.512, Q_2 = 582.558$ units.

| $r$ | $L_{i,r}$ | $Q$ | $k_1$ | $k_2$ | $m$ | $JTEC(\cdot)$ |
|-----|----------|-----|-------|-------|-----|----------------|


For example, for $\tau = -0.5$, we obtain $\bar{a}_{b_1} = 0.132$ and $\bar{a}_{b_2} = 0.305$. Then the allocation of the total annual cost to the buyer 1, buyer 2, and the vendor are $691.96$, $1598.525$, and $2947.15$, respectively.

Table 4. Cost comparison for non-integrated model, partial-integrated model, and integrated model for $\tau = -0.5$

|                         | Non-integrated model | Partial-integrated model | Integrated model |
|-------------------------|----------------------|--------------------------|-----------------|
| Order quantity of buyer 1 | 111.626              | 111.626                  | 116.512         |
| Order quantity of buyer 2 | 338.94               | 442.33                   | 582.558         |
| Lead time of buyer 1    | 6                    | 6                        | 6               |
| Lead time of buyer 2    | 5                    | 5                        | 4               |
| Reorder point of buyer 1| 162                  | 162                      | 162             |
| Reorder point of buyer 2| 587                  | 571                      | 442             |
| Vendor’s production quantity | 1351.698             | 1661.87                  | 1398.138        |
| Total annual cost of buyer 1 | 696.16              | 696.16                   | 696.561         |
| Allocated annual cost of buyer 1 | -                  | -                        | 691.96          |
|                              | Total annual cost of buyer 2 | Allocated annual cost of buyer 2 | Total annual cost of vendor | Allocated annual cost of vendor | Join cost |
|------------------------------|------------------------------|---------------------------------|----------------------------|---------------------------------|-----------|
|                              | 1608.23                      | -                               | 2965.04                    | -                               | 5269.43   |
|                               | 1649.3                       | -                               | 2906.6                     | -                               | 5252.06   |
|                               | 1775.89                      | 1598.525                        | 2765.17                    | 2947.15                         | 5237.63   |

6. Conclusions

In this paper, an integrated production-inventory model for single-vendor two-buyer problem has been studied with partial backorder and controllable lead time under independent normally distributed demand on the buyers. Additionally, a service level constraint corresponding to each buyer is included in the model to limit the shortages. Minimizing the joint expected total annual cost function and satisfying the service level constraint on each buyer, a Lagrangian multiplier technique and an algorithm procedure are proposed to determine the optimal order quantity, safety factor, lead time of the buyers and number of lots delivered from the vendor to the buyers in a production cycle. Numerical results show that the optimal solution obtained satisfies the service level constraint on all the buyers. Moreover, we show that by viewing the vendor and the two buyers as a system rather than as separate individuals, total system cost can be reduced significantly.

Regarding some future research, we can extend the present model by considering the imperfect production process on the vendor and the inspection errors on the buyers. In order to show the uncertainty and vagueness, the lead time demand can be considered as a fuzzy random variable.

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