How Ordinary Elimination Became Gaussian Elimination

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Abstract
Newton, in an unauthorized textbook, described a process for solving simultaneous equations that later authors applied specifically to linear equations. This method — that Newton did not want to publish, that Euler did not recommend, that Legendre called “ordinary,” and that Gauss called “common” — is now named after Gauss: “Gaussian” elimination. (One suspects, he would not be amused.) Gauss’s name became associated with elimination through the adoption, by professional computers, of a specialized notation that Gauss devised for his own least squares calculations. The notation allowed elimination to be viewed as a sequence of arithmetic operations that were repeatedly optimized for hand computing and eventually were described by matrices.

Key words: algebra before 1800, Gaussian elimination, human computers, least squares method
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1. Lacking Both History and Heritage

The relocation of scientific research from academies to universities in the 19th century increased employment for mathematicians as teachers, which Grattan-Guinness [87, p. 177] notes both sped the professionalization of the subject and coincided with a preference for pure over applied mathematics. The same taste was manifest in historical scholarship, in that pure subjects became more thoroughly chronicled than applications. For example, at the beginning of the century, the notoriety of calculating where again to observe Ceres earned the youthful Gauss fame enough to realize his wish for a life free of teaching pure mathematics, see Bühler [25, p. 46] and Dunnington [52, pp. 405–410], yet by the end of that century, mathematical histories neglected Gauss’s applied work, see Cajori [27] and Matthiessen [135].

Grattan-Guinness (op. cit.) argues for the existence of two recollections of the mathematical past: history recounts the development of ideas in the context of contemporary associations, while heritage remembers interpreted work that embodies the state of mathematical knowledge. The thesis of this paper is that much of genuinely applicable mathematics lacks both: no history because applications may be deemed the purview of non-mathematical faculties, no heritage because applications might not enter or remain in the corpus. This thesis is illustrated by the neglected story of the algorithm now called Gaussian elimination. It belongs as much to the history of science and technology as to the intellectual history of mathematics.

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2. Gaussian Elimination Today

Both elementary and advanced textbooks discuss an algorithm called Gaussian elimination. A first course in algebra may solve two linear equations in two unknowns by various means, whereas precalculus algebra invariably introduces a method for solving arbitrarily large systems of linear equations. As explained by Cohen et al. [38 pp. 743+, sec. 10.2], “elementary operations” produce an “equivalent system” in “upper-triangular form” that can be solved by “back-substitution.”

\[
\begin{align*}
 x + 2y + z &= 3 & & x + 2y + z &= 3 & & x + 2y + z &= 3 & z &= 2 \\
 x + y + 2z &= 9 & \Rightarrow & -y + z &= 6 & \Rightarrow & -y + z &= 6 & \Rightarrow & y &= -4 \\
 2x + y + z &= 16 & -3y - z &= 10 & -4z &= -8 & x &= 9
\end{align*}
\]

This paper follows current usage by referring to any algorithm that is essentially equivalent to equation (1) as “Gaussian elimination” whatever its period or source. The distinguishing features are: the equations and variables may need to be rearranged so the leading equation contains the leading variable, the leading equation is used to remove the leading variable from each of those following, these steps apply recursively to the following (modified) equations until, finally, the back-substitution. The form of the algorithm employed in equation (1) is viewed as canonical: the leading equation remains unchanged while variables are removed by subtracting an appropriate multiple of it from each following equation.

Petersen and Arbenz [150 p. 107] explain this algorithm is the standard test for the speed of computers in scientific work. Its widespread use in so large a field as scientific computing results in many algorithmic variants that are collectively called, simply, Gaussian elimination. The variations are distinguished by acronyms, adjectives, and eponyms. At that level of differentiation the canonical algorithm of equation (1) is also named either “classic” Gaussian elimination or “Doolittle’s method.” Nevertheless, advanced or specialized texts always begin by stating exactly this algorithm identified as Gaussian elimination. For example see Duff et al. [50 pp. 43+], Farebrother [61 pp. 3+], Golub and Van Loan [83 pp. 92+], Higham [99 pp. 158+], Petersen and Arbenz [150 pp. 23+] and Stewart [163 pp. 148+].

Today, the technical literature as well as textbooks of all levels encourage the inference that Gauss introduced the method of equation (1) and his usage was somehow remarkable compared to prior art. The justifications offered for the Gaussian appellation thus range from simple citation to careful indirection. Cohen et al. [38 p. 743] claim “Gauss used this technique” to analyze the orbit of Pallas [72] though “the essentials” already appeared in ancient Chinese texts. The prior use in China leads Katz [114 p. 29] to add the qualification, “the method now known as Gaussian elimination,” that nevertheless allows for an independent origin in the work of Gauss. Higham [99 p. 187] attributes “the first published appearance of Gaussian elimination” to Gauss but in an earlier paper [71]. Stewart [163 p. 148] notes Gauss in reality eliminated variables from quadratic forms rather than from linear equations, but he suggests the method of equation (1) stems directly from Gauss because it resembles his “original derivation.” Only Farebrother [61 p. 3] leaves open the possibility of prior European origin, remarking that “Gauss’s formalization” of the “traditional schoolbook method” appeared in his Pallas work. His explanation begs the questions: what schoolbook did Gauss read, and what did he contribute?

3. Elimination Before Gauss

Algebra and its history are invariably of interest but more so for the polynomial equations that gave rise to incommensurate and imaginary numbers. Consideration of simultaneous linear equations is therefore comparatively rare in the secondary literature and also in the primary sources. Both are surveyed here for systems of linear equations in periods even much before Gauss.

By far the most impressive treatment known from antiquity is chapter 8 of the Nine Chapters of the Mathematical Art, a problem “book” anonymously and collectively written in China. Liu Hui wrote the first of several commentaries in the 3rd century, including comments on chapter 8, so the method described there for solving systems of linear equations is at least as old, and is believed to be much older, although no venerable text survives. Martzloff [134 pp. 128–131] explains chapters 1–5 are known from a 13th century copy, and chapters 6–9 are reconstructed from 18th century quotations of a lost 15th century encyclopedia.

Mathematicians in ancient China represented numbers by counting rods. They organized elaborate calculations by placing the rods inside squares arranged in a rectangle. For chapter 8, each column of squares
3rd Century BC: Problem 19 in book I of the Arithmetic of Diophantus [96] p. 136: find four numbers such that the sum of any three exceeds the fourth by a given amount. Solution using symbols for clarity: let \( n_i \) be the numbers, \( s \) the sum, and \( d_i \) the differences. Then \( s - n_i \) is the sum of the others, so \((s - n_i) - n_i = d_i \) or \( n_i = (s - d_i)/2 \). Summing \( s = n_1 + n_2 + n_3 + n_4 = 2s - (d_1 + d_2 + d_3 + d_4)/2 \) hence \( s = (d_1 + d_2 + d_3 + d_4)/2 \). Thus \( s \) can be evaluated from the given data, and then so can the \( n_i \).

\[
\begin{align*}
-n_1 + n_2 + n_3 + n_4 &= d_1 \\
n_1 - n_2 + n_3 + n_4 &= d_2 \\
n_1 + n_2 - n_3 + n_4 &= d_3 \\
n_1 + n_2 + n_3 - n_4 &= d_4
\end{align*}
\Rightarrow \quad n_i = \frac{d_1 + d_2 + \cdots + d_n}{4} - \frac{d_i}{2} \tag{2}
\]

Before 3rd Century AD: Problem 1 in chapter 8 of the Nine Chapters [113] pp. 391, 399–403: from 3 top-grade rice paddies, 2-medium grade, and 1 low-grade, the combined yield is 39 dou of grain, etc. What is the yield of a paddy of each grade? Solution:

\[
\begin{align*}
3x + 2y + z &= 39 \\
2x + 3y + z &= 34 \\
x + 2y + 3z &= 26
\end{align*}
\Leftrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 3 \\ 4 & 5 & 2 \\ 8 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} z = \frac{11}{3} \\ y = \frac{17}{3} \\ x = \frac{37}{6} \end{pmatrix} \tag{3}
\]

2nd Century AD: Problem 29 in chapter 2 of the Aryabhatiya of Aryabhata [35] p. 40]: to find several numbers when the results of subtracting each from their sum are known. Solution: sum the known differences and divide by the quantity of terms less one. The result is the sum of all the numbers, from which they can then be determined.

\[
\begin{align*}
0 + n_2 + n_3 + \cdots + n_4 &= d_1 \\
n_1 + 0 + n_3 + \cdots + n_4 &= d_2 \\
n_1 + n_2 + 0 + \cdots + n_4 &= d_3 \\
\vdots & \ddots \vdots \ddots \vdots \ddots \vdots \\
n_1 + n_2 + n_3 + \cdots + 0 &= d_n
\end{align*}
\Rightarrow \quad n_i = \frac{d_1 + d_2 + \cdots + d_n}{n - 1} - \frac{d_i}{2} \tag{4}
\]

Figure 1: Ancient problems with their solutions couched in modern symbolic algebra and their interpretations as simultaneous linear equations.

corresponds to a modern linear equation, so the ancient rectangle must be rotated counterclockwise by 90 degrees to obtain the coefficient tableau of modern mathematics. Problem 1 in chapter 8 is frequently displayed as representative of the solution method, see equation (3) in Figure 1. Chapter 8 solves 18 different systems of equations in this systematic way. Unlike equation (1), both columns to be combined are scaled by the leading number in the other. Subtracting the right column from the left column removes the leading number from the left while preserving integer coefficients. For more on chapter 8 see Lay-Yong and Kangshen [126]. The Nine Chapters appears to be the only general discussion of what can be interpreted as systems of linear equations before the 18th century.

Assertions that other ancient civilizations had mathematicians who solved linear systems do not bear close scrutiny. For example, O’Connor and Robertson [144], without citation, attribute to Babylonians “around 300 BC” a problem about two fields of grain. Incredibly, this problem of unknown provenance is now used to prepare teachers in California, see National Evaluation Systems [137] p. 9. Besides misdating ancient Babylon to Hellenistic or Parthian Mesopotamia, the problem of O’Connor and Robertson is inconsistent with the tablets quoted by [105] and summarized by Bashmakova and Smirnova [13] p. 3. When Babylonians did pose problems that can be interpreted as simultaneous equations, invariably at least one equation is nonlinear, reflecting an understandable Babylonian interest in areas of fields given knowledge of diagonals, perimeters, and the like.

Some claims, that the ancients solved simultaneous linear equations, ignore evidence that special solution methods were used. For example, Dedron and Itard [44] p. 303] see a linear system in a problem about 5 men and 100 loaves of bread from the Rhind papyrus. Gillings [79] pp. 170–172] discusses this problem (number 40) and the solution that is recorded in the papyrus which is based not on linear equations but rather
on arithmetic progressions. A knowledge of both arithmetic and geometric progressions, Gillings argues, was a distinguishing feature of ancient Egyptian mathematics. Several problems solved by Diophantus of Alexandria can be interpreted as simultaneous linear equations. Equation (2) in Figure 1 is the most elaborate of these. This problem is a special case that happens to be solvable by a repetitive formula and therefore is amenable to special reasoning. Hermann Hankel [92] p. 165], a prominent mathematician who also wrote on the history of mathematics, remarked that studying 100 Diophantine solutions would not suggest how to solve the 101st problem. A class of problems solved by Aryabhata in India is similarly special, see equation (3) in Figure 1. The two problems stated in this figure appear to be representative of all the simultaneous linear equations known from the periods and regions of Diophantus and Aryabhata.

The equivalent of single polynomial equations, but notably not simultaneous equations, were solved by several Arabic-speaking mathematicians in medieval times. Examples are in the work of the encyclopedist Al-Khwarizmi, from whom we have the words algebra and algorithm, and in the writings of Leonardo of Pisa, also known as Fibonacci, who travelled in Arabic-speaking lands. Hogendijk [100] regards knowledge of the medieval period “by no means complete” because many Arabic scientific manuscripts have not been studied.

Bashmakova and Smirnova [13] survey the European algebraic tradition and its sources from the ancient to the abstract. Symbolic algebra developed during the European Renaissance in the arithmetization of geometry and in the theory of equations. That undertaking was more than a stepwise development culminating in modern notation. Schmidt [159] explains the conceptual differences that separated Viète from the later, root-oriented theory of equations of Descartes and others. For these authors in translation see Descartes [46], Schmidt [158], and Viète [175]. By the end of the 16th century an audience had developed for textbooks that taught arithmetic, how to express “questions” in terms of symbolic equations, and the solution thereof.

To obtain a comprehensive picture of algebra books for education, Kloyda [116] surveyed 107 texts printed between 1550 and 1660. These number 9 Spanish, 12 English, 19 each French and Italian, and 24 each from Germany and the Netherlands. Only 4 of the 107 texts discussed simultaneous linear equations. Pelletier du Mans [149] has the earliest example, a problem said to have originated with Cardano, about three men with three sums of money. After explaining Cardano’s solution, Pelletier solved the problem by directly manipulating equations, restated in modern notation in equation (5) of Figure 2. Pelletier wrote an infix “p.” (piu) for modern +, also “m.” (meno) for −, and for equality he wrote the word. (This notation rather than being the author’s choice may of course have been imposed by limited typography.) The solution is obtained from the 10th, 5th, and 1st equations. Buteo [26] presented a more direct solution of a similar problem, again restated in modern notation in equation (6) of Figure 2. Buteo wrote “•” or “,” for +, and “=” for =. He performed the same double-multiply elimination as the Nine Chapters in equation (3), although Buteo used different equations for the back-substitution. Here the solution is from the 6th, 5th, and 3rd equations whereas the Nine Chapters and the canonical elimination of equation (1) would retain the 1st, 4th, and 6th. Gosselin [85] solved a problem with four equations by similar methods, being more or less direct, and Rahn [153] did the same for three equations.

4. “this bee omitted by all that have writ introductions”

The 18th century produced several more algebra books. A cursory inspection finds 35 printed in England alone from 1650 to 1750, including one published over the objections of Isaac Newton. Whiteside [181, 182] describes Newton’s work on algebra which extended roughly from his appointment to the Lucasian professorship in 1669 until he began composing the “Principia” in 1684. Much of this work dealt with investigations of algebraic curves, but Newton also addressed the theory of equations. No publications resulted immediately from the latter efforts. In 1669–1670 Newton wrote “observations” and amendments for a Latin version of

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1 Heath [96] pp. 54-58] translates Hankel’s comments and also cites Euler’s opinion on the generality of the methods used by Diophantus.

2 Apart from Newton’s, the books are: 1650 Moore, 1652 Oughtred, 1653 Balam, 1660 Leybourn, 1663 Brasser, Petri, and Backer, 1669 Renaldini, 1673 Kersey, 1680 Perkins, 1685 Wallis and Caswell, 1698 Ward, 1700 Moxon and Tuttell, 1702 Cocker, 1702 Harris, 1705 Parsons and Wastell, 1706 de Graaf, 1707 Berkeley, 1709 Alexander, Ditton, and Cobb, 1711 Ozanam, 1717 Kersey and Halley, 1728 Royer, 1728 Jacob, 1737 Ashby, 1738 Ronayne, 1739 Wolff, 1739 Hanna, 1740 Webster, 1741 Sauderson, 1742 Hammond, 1745 Simpson, 1746 Crosby, Wilcox, and Clark, 1748 MacLaurin, 1748 Muller, 1749 Holliday, 1750 Fenning, 1750 Loughton and Bickham.
16th Century AD: Problem of Pelletier du Mans [149]:

1. \(2R + A + B = 64\)
2. \(R + 3A + B = 84\)
3. \(R + A + 4B = 124\)

\[2^\text{nd} + 3^\text{rd} \Rightarrow 4. \quad 2R + 4A + 5B = 206\]
\[4^\text{th} - 1^\text{st} \Rightarrow 5. \quad 3A + 4B = 146\]
\[1^\text{st} + 2^\text{nd} \Rightarrow 6. \quad 3R + 4A + 2B = 148\]
\[1^\text{st} + 3^\text{rd} \Rightarrow 7. \quad 3R + 2A + 5B = 188\]
\[6^\text{th} + 7^\text{th} \Rightarrow 8. \quad 6R + 6A + 7B = 336\]
\[6 \times 3^\text{rd} \Rightarrow 9. \quad 6R + 6A + 24B = 744\]
\[9^\text{th} - 8^\text{th} \Rightarrow 10. \quad 17B = 408\]

16th Century AD: Problem of Buteo [26]:

1. \(3A + B + C = 42\)
2. \(A + 4B + C = 32\)
3. \(A + B + 5C = 40\)

\[3 \times 2^\text{nd} - 1^\text{st} \Rightarrow 4. \quad 11B + 2C = 54\]
\[3 \times 3^\text{rd} - 1^\text{st} \Rightarrow 5. \quad 2B + 14C = 78\]
\[11 \times 5^\text{th} - 2 \times 4^\text{th} \Rightarrow 6. \quad 150C = 750\]

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Figure 2: European Renaissance problems with their solutions stated in modern notation from the compilation by Kloyda [116].

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a Dutch algebra text, Kinckhuysen [115], that his acquaintance John Collins planned to publish in England. Collins abandoned the project when newer books appeared. Newton himself taught algebra at Cambridge for 11 years beginning with the 1673–1674 academic term. During that time he wrote and repeatedly revised an incomplete manuscript for his own algebra treatise that was to be named “Arithmetica Universalis.” His last algebra manuscript was prepared in 1684 when, for unknown reasons, Newton suddenly honored the requirements of the Lucasian professorship by depositing with Cambridge University his lectures for the algebra course. The bulk of those notes were transcribed by his secretary, Humphrey Newton (no relation), from Newton’s previous algebra manuscripts. After Newton left academic life, his lectures were published in their original Latin (1707, 1722) and in translation (1720, 1728) under the intended title of his aborted treatise, “Universal Arithmetic.” Newton had no claim to material that the university had paid him to prepare, nevertheless he strongly objected to its publication, as explained by Whiteside [182, v. 5, p. 11], lest the old lecture notes be misinterpreted as representing his latest research. The second English edition with Newton’s changes appeared the year after his death.

In the realm of unintended consequences it is to be anticipated that Newton’s comparatively accessible algebra textbook became, as characterized by Whiteside [182, v. 5, pp. 54–55, fn. 1], “the most widely read and influential of his writings.” Thanks to Whiteside’s impressive scholarship, a passage that is relevant to Gaussian elimination can be traced directly to Newton in the commentary on Kinckhuysen’s textbook. Newton remarked to Collins that contemporary textbooks lacked an explicit description of how to solve collections of equations.

Though this bee omitted by all that have writ introductions to this Art, yet I judge it very propper & necessary to make an introduction compleate.

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3The new chapter for Kinckhuysen’s textbook can be found in Newton’s original [182, v. 2, pp. 400–411, parallel Latin and English texts]. The material also appears in the transcribed lecture notes [182] v. 5, pp. 122–129, parallel texts]. That text was copied into the Latin and English editions of the unauthorized textbook, whose second English edition has been reproduced [181, v. 2, pp. 36–38].
Of the Transformation of two or more Equations into one, in order to exterminate the unknown Quantities[4]

When, in the Solution of any Problem, there are more Equations than one to comprehend the State of the Question, in each of which there are several unknown Quantities; those Equations (two by two, if there are more than two) are to be so connected, that one of the unknown Quantities may be made to vanish at each of the Operations, and so produce a new equation. . . . And you are to know, that by each Equation one unknown Quantity may be taken away, and consequently, when there are as many Equations and unknown Quantities, all at length may be reduc’d into one, in which there shall be only one Quantity unknown. —Newton [142 p. 60] and prior

On the pages following this rule, Newton [142 p. 61–62] offered several methods for removing a variable from two equations, including “equating” and “substituting.”

The Extermination of an unknown Quantity by an Equality of its Values.

When the quantity to be exterminated is only of one Dimension in both Equations, both its Values are to be sought by the Rules already delivered, and the one made equal to the other. Thus, putting \( a + x = b + y \) and \( 2x + y = 3b \), that \( y \) may be exterminated, the first Equation will give \( a + x = b + y \), and the second will give \( 3b - 2x = y \). Therefore \( a + x - b = 3b - 2x \), . . .

The Extermination of an unknown Quantity, by substituting its Value for it.

When, at least, in one of the Equations the Quantity to be exterminated is only of one Dimension, its Value is to be sought in that Equation, and then to be substituted in its room in the other Equation. . . . — Newton [142 pp. 61–62] and prior

The context and accompanying examples make clear Newton meant a general approach for solving simultaneous nonlinear equations. Indeed, Newton [142] considers simultaneous linear equations only in the illustration quoted above. In all his work, the only system of 3 or more linear equations appears to be a single, contrived example in the manuscript of his incomplete treatise, see Whiteside [182 v. 5, p. 567, problem 3]. Many of Newton’s exercises are motivated by Cartesian geometry as he practiced it or by classical physics as he invented it, and therefore are apparently original, and they contain no simultaneous linear equations.

In a study of algebra education in the 16th through the 18th centuries, Macomber [131, p. 132] finds Newton’s general rule for solving simultaneous equations was “the earliest appearance of this method on record.” Moreover, (op. cit., pp. 143–144) “before the death of Newton there came to be a demand for suitable text books of algebra for the public schools; and during the 18th century, a number of texts appeared, all more closely resembling the algebra of Newton than those of earlier writers.” Among the authors Macomber finds Newton influenced was the banker Nathaniel Hammond. He served as chief accountant for the Bank of England from 1760 to 1768 [154], and his successful algebra textbook went to four editions between 1742 and 1772. Hammond’s interesting introduction summarized algebraic history from ancient times to his own, as he understood it, but his lessons got down to business by emphasizing clear instructions for solving word problems, which were dreaded even then.

As the principal Difficulty in this Science, is acquiring the Knowledge of solving of Questions, I have given a great Variety of these respect to Numbers and Geometry, and their solutions I chose to give in the most particular, distinct, and plain Manner; and for which the Reader will find full and explicit Directions.

— Hammond [91 p. vii]

Thus Hammond [91] pp. 142, 219–220, 296–297] divided Newton’s rule for simultaneous equations into a progression of rules for two, three, and four equations.

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[4] Whiteside [182] v. 2, p. 401, n. 63] notes that Newton originally wrote “elimino” and then replaced it in some instances by “extermino.” Whiteside’s translation may be preferred to the text in Newton [142].
The Method of resolving Questions, which contain four Equations, and four unknown Quantities.

72. When the Question contains four Equations, and there are four unknown Quantities in each Equation; find the Value of one of the unknown Quantities in one of the given Equations, and for that unknown Quantity in the other three Equations write the Value of it, which then reduces the Question to three Equations, and three unknown Quantities.

Then find the Value of one of these three unknown Quantities in one of these three Equations, and for that unknown Quantity in the other two Equations write the Value of it, which reduces the Question to two Equations, and two unknown Quantities.

Then find the Value of one of the unknown Quantities in each of these two Equations, and make these Equations equal to one another, when we shall have an equation with only one unknown Quantity, which being reduced, will answer the Question. . . .

And in the same Manner may any other Question in the like Circumstances be answered. — Hammond [91, pp. 296–298]

Hammond [91, p. 142] echoed Newton’s terminology of “exterminating an unknown Quantity,” but unlike Newton he illustrated the progressive cases with many systems of linear equations. His method for removing variables from two equations was Newton’s “equating values” whereas from three or more equations it was Newton’s substitution. In marked contrast to both Newton and Hammond, a contemporary work by the sightless Lucasian professor, Saunderson [156] excerpted in Saunderson [157, pp. 164+], solved the three-equation problem of Cardano-Pelletier without stating a general rule for simultaneous equations.

Euler [58] also wrote an algebra textbook that was much admired for its concise style. By then totally blind himself, Euler begins with the compelling testimonial that the book was dictated for the instruction of his sight secretary, who mastered the subject from the text without additional instruction. Euler included a chapter specifically for simultaneous linear equations. To find the values for two unknowns in two equations, Euler repeated the “equating values” method:

The most natural method of proceeding . . . is, to determine, from both equations, the value of one of the unknown quantities, as for example $x$, and to consider the equality of these two values; for then we shall have an equation, in which the unknown quantity $y$ will be found by itself. . . . Then, knowing $y$, we shall only have to substitute its value in one of the quantities that express $x$.

— Euler [58, part 2, sec. 1, chap. 4, sec. 45] translated in [59, p. 206]

Euler continued with “equating values” for three equations. However, he cautioned against adopting a rote approach, and therefore did not state a general algorithm.

If there were more than three unknown quantities to determine, and as many equations to resolve, we should proceed in the same manner; but the calculation would often prove very tedious. It is proper, therefore, to remark, that, in each particular case, means may always be discovered of greatly facilitating the solution.

— Euler [58, part 2, sec. 1, chap. 4, sec. 53] translated in [59, p. 211]

Lacroix [120] wrote another in this growing series of algebra textbooks. Remembered as a minor mathematician today, he was a member of the reconstituted French Academy, and was recognized as an astute author. His masterpiece, Traité du calcul différentiel et du calcul intégral, remained in print as the standard reference for 18th century calculus even after Cauchy began to reinvent the foundations. Domingues [47, p. 3] comments that Lacroix sought to compare different approaches and to present the best in an original, uniform style that benefits the reader though it may obscure the origin of the material. A contemporary book review [171] presumed the principal sources for Lacroix’s *Elemens d’algèbre* were Bezout, Euler, and Lagrange. To this group perhaps should be added Hammond.

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5Heefer [97] traces most of Euler’s exercises to an algebra textbook by Christoff Rudolff in 1525 and reprinted by Michael Stifel in 1554. Kloyda [116, p. 132] sheds more light on the source of exercises by remarking, without citation, that Rudolff also published a problem book that went through several editions.
Lacroix began with lessons specifically for linear equations. These underwent considerable revision between his 2nd and 5th editions published in 1800 and 1804, respectively. The 2nd edition discussed one and two unknowns in the same number of equations, and then passed to a derivation of explicit formulas for the unknowns in systems of two, three, and even four equations [120, pp. 79–104, sec. 75–91]. Between these two rather different lessons, the 5th edition inserted a clear statement of how to solve simultaneous linear equations. The subscript-free notation of the time did not allow easy expression of arbitrarily many equations and unknowns, but in his lesson title Lacroix made it plain the method could be applied to any number.

Of the resolution of any given number of Equations of the First Degree, containing an equal number of Unknown Quantities.

78. . . . if these unknown quantities are only of the first degree, [then] according to the method adopted in the preceding articles, we take in one of the equations the value of one of the unknown quantities, as if all the rest were known, and substitute this value in all the other equations, which will then contain only the other unknown quantities.

This operation, by which we exterminate one of the unknown quantities, is called elimination. In this way, if we have three equations with three unknown quantities, we deduce from them two equations with two unknown quantities, which are to be treated as above; and having obtained the values of the two last unknown quantities, we substitute them in the expression for the value of the first unknown quantity.

If we have four equations with four unknown quantities, we deduce from them, in the first place, three equations with three unknown quantities, which are to be treated in the manner just described; having found the values of the three unknown quantities, we substitute them in the expression for the value of the first, and so on.

— Lacroix [121, p. 114, art. 78, original emphasis] translated in [123, p. 89, art. 78]

Lacroix followed Hammond in using what Newton called substitution to eliminate variables.

5. “Without any desire to do things which are useful”

The first-ever use for simultaneous linear equations was the method of least squares, invented just when Lacroix wrote his textbook, at the start of the 19th century. It appears to be difficult to show there was any need to solve elaborate equations of any kind, before then. None of the secondary literature argues that solving algebraic equations was needed, and several authors intimate to the contrary. Libbrecht [129, p. 416] values the information about daily life in ancient Chinese mathematics lessons, but however colorfully those word problems may have been composed, it is plain the exercises in chapter 8 of the Nine Chapters are contrived. Katz [114] finds a similar artificiality to algebra problems in Babylonian, Greek, Arabic, and European Renaissance texts. Neugebauer [139, pp. 71–72] points out that ancient economies required only arithmetic to function. Hogendijk [100] explains that Islamic civilization needed arithmetic for commerce and advanced mathematics for astronomy, but the mathematical studies that in hindsight were the most sophisticated, such as algebra, were undertaken for their own sake.

Few memorable discoveries have ever been made for their immediate utility, so there is no reason to suppose either our very distant predecessors or the inventors of algebra should have been any different. "By and large it is uniformly true in mathematics that there is a time lapse between a mathematical discovery and the moment when it is useful," John von Neumann [140] opined in his mannered prose, and in the meantime, "the whole system seems to function without any direction, without any reference to usefulness, and without any desire to do things which are useful." Thus it is unusual when a discovery is immediately useful, as was the method of least squares, which finally created a need for solving simultaneous linear equations.

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6Lacroix’s translator introduced Newton’s word “exterminate.”
6. “Méthode des moindres quarrés”

The genesis of least squares lies in a scientific question that was resolved unsatisfactorily in the 18th century. Making accurate predictions from measurements tested preconceptions about the relationship between mathematics and the sciences. Differences between astronomical observations and orbital formulas, derived from Newton’s principles, at times cast doubt even on the inverse square law of gravitation. Both Euler and Laplace speculated the law of gravity might need modification for astronomical distances. Stigler [164, pp. 17, 28] hypothesizes that fitting the orbital formulas was inconceivable for lack of conceptual grounds to justify amalgamating the errors of separate measurements. A new paradigm was adopted once Tobias Mayer successfully applied ad hoc fitting methods to predict the lunar orbit. Laplace in particular then derived fitted orbits that both vindicated Newton and established the mathematics Laplace employed, analysis.

Farebrother [62] surveys the several fitting methods available at the end of the 18th century. These successes begged the question, what was the best fitting method to use?

Two inventors are recognized for the method of least squares. Gauss claimed to have known “that the sum of the squares should be minimized,” since 1794 or 1795. Thus when the dwarf planet Ceres was sighted and lost in 1801, he quickly found its orbit by procedures that included least squares methods. Gauss was reticent about the calculations in the hasty announcement of the Ceres orbit. An explanation written in 1802 was sent to a friend, Olbers, and inexplicably returned only in 1805, yet all the while Gauss was publishing orbits for Ceres and other celestial bodies [52, pp. 53, 420–421].

Meanwhile, in an appendix to a long paper about geodesy, Legendre [127, pp. 72–74] posed the general problem of finding the most accurate parameterization furnished by a given set of observations. Stigler [164, pp. 13, 15] recommends Legendre’s short text as among the most elegant introductions of a significant mathematical concept. Legendre noted, the problem often involves many systems of equations each of the form,

\[ E = a + bx + cy + fx \text{ etc.,} \]

(his notation) where \( a, b, c, f, \ldots \) are numbers that vary among the equations, and \( x, y, z, \ldots \) are parameters common to all the equations. (As in linear models today, the numbers in each equation represent one observation; the model parameters are the unknowns.) Legendre viewed the problem as finding values for the parameters to make \( E \) small. If the number of equations exceeds that of the unknowns (so the equations cannot be solved exactly), then he suggested minimizing the sum of the squares of the \( E \)’s. He called this overall process the “méthode des moindres quarrés” (modern carrés). The solution was found by differentiating the sum of squares to derive the “equations of the minimum” (modern normal equations). These simultaneous equations were linear and equinumerous with the variables, so, Legendre said, they could be solved by “ordinary methods.”

Over a decade passed between Gauss’s own discovery and his publication. Although he intended to publish immediately after Legendre, the manuscript was delayed by the Napoleonic wars. In Theoria Motus, Gauss explained a refined process for orbital calculations, and he returned to the conceptual problem of fifty years earlier: how to justify values calculated from erroneous data. Lacking justification, Gauss [71, art. 186] intimated, the only reason to minimize squares was convenience. Rather than begin by minimizing the discrepancy in the equations as Legendre had, instead Gauss [71, art. 175] formulated “the expectation or probability that all these values will result together from observation.” Assuming the errors in the unknowns follow a Gaussian or normal distribution (later terminology), Gauss showed maximizing the expectation is what implies the sum of squares should be minimized. Gauss then echoed Legendre’s sentiment about the resulting calculation to find the parameters.

We have, therefore, as many linear equations as there are unknown quantities to be determined, from which the values of the latter will be obtained by common elimination. (\ldots per eliminationem vulgarem elicientur.)

— Gauss [71, art. 180], this translation (1857)
There followed a scientific dialogue in published work of Gauss and Laplace that explored the nature of probability and estimation. Gauss [74] gave a second and unqualified justification for the method of least squares that is now the Gauss-Markov theorem for the minimum variance linear unbiased estimator. Hald [90] pp. 98, 105–109] suggests few contemporary readers, if any, understood all that Gauss and Laplace wrote. Nevertheless, he continues, Gauss’s first proof for the method of least squares, coupled with the sufficiency in many instances of the assumption of normally distributed errors, allowed the likes of Hagen, Chauvenet, and Merriman to maintain and extend a statistically respectable methodology of estimation from the time of Gauss through the development of modern statistics. Mayer, Legendre, Laplace, and Gauss — those on whose shoulders we stand — each in their way contributed to the last mathematical prerequisite for the industrial revolution by reconciling experimental uncertainty with Newton’s deterministic physics to create the predictive models needed for engineering design.

At the end of the 19th century, Bartlett [11, p. 1] could proudly announce that “scientific investigations of all kinds” relied on a mature computational technology called “The Adjustment of Observations” or “The Method of Least Squares.” The subject divided into two cases.

case 1. The “adjustment of indirect observations” had Legendre’s original, overdetermined equations that today would be stated as \( \min \|b - Ax\|_2 \). These problems were solved by reducing them to \( A'Ax = c \) with \( c = A'b \). See Wright and Hayford [186 chapt. 4] and Bartlett [12] secs. 23–32.

case 2. Gauss [75] formulated the “adjustment of conditioned observations” to find minimum norm solutions of underdetermined equations, \( \min_{x \in \mathbb{R}^n} \|x\|_2 \). He reduced these problems to \( AA'u = b \) where \( x = A'u \). See Wright and Hayford [186 chapt. 5] and Bartlett [12] sec. 33.

Matrices were not used in these formulations, note, until the mid 20th century. Each row of the over-determined (case 1) or under-determined (case 2) equations, \( Ax = b \), was called a condition. The reduced forms of both problems were called normal equations and were solved by elimination. Stigler [165] pp. 415–420] reports that Gauss [73, p. 84] seemingly used this name first but also just once and offhand, so what Gauss meant by Normalgleichungen is unknown despite exhaustive scholarship.

7. Gauss’s Method

Dunnington [52, pp. 227-228] explains that Gauss performed meticulous calculations almost as a leisure activity throughout his life. Gauss thought calculations so important that he included them in his papers where he strove to make them brief and intuitively clear. Accordingly, his publication that mentioned “common elimination” [71] was followed with a detailed explanation [72] but of quadratic forms, not linear equations.

Quadratic forms appeared in prior work of Lagrange, Legendre, and Gauss [70, art. 222] on number theory and also in a numerical context. In his very first paper, [125] substituted new variables for linear combinations of the original variables in a quadratic form. \( x^tAx \) in modern matrix notation, to give the entries of a triangular matrix \( U \) of substitution coefficients so that \( A = U'DU \) for a diagonal matrix \( D \). This formula expressed the quadratic form as a weighted sum of squares, \( (Ux)^tD(Ux) \), which Lagrange used to ascertain local extrema. Toepitz [172, p. 102] and Wedderburn [178, p. 68] remembered Lagrange for this representation of quadratic forms almost two hundred years later. Thus Gauss likely knew of the Lagrangian provenance as well.

When Gauss first considered least squares in 1795, he borrowed from the Göttingen library the volume that begins with Lagrange’s paper on extrema. Amazingly, the books Gauss borrowed as a student are known [52, p. 398]. There is no evidence he read the paper, yet the indication is it influenced his approach to least squares. For example, Gauss did not need the paper for his major work on number theory from this period which did cite the journal [70, art. 202, fn. 9] but not the volume borrowed in 1795. Since neither [70] nor

---

10 Three short monographs with historical emphasis but on slightly different aspects of this work — statistical fitting procedures, parametric statistical inference, and classical analysis of variance — have recently been written by Farebrother [62], Hald [90], and [37], respectively. Histories with a wider scope should also be consulted by Gillispie [80, chap. 25], Goldstine [82, chaps. 4.10], and Stigler [164 chap. 4].

11 Bartlett [11] pp. 110-111] and [1915, pp. v–vi] lists English, French, German, and Italian textbooks from the end of the 19th century. See Ghilani and Wolf [78] for a treatment from the beginning of the 21st century.

12 Since Lagrange’s discovery seems not to have been incorporated in general algebra textbooks, it is tenuous to see, as some web pages do, a possible origin for Gaussian elimination in this work.
Gauss [72] wrote the overdetermined equations as

\[
\begin{align*}
\Omega &= w_1 + w_2 + \ldots
\end{align*}
\]

(\text{original notation, but "etc." replaced by "\ldots"}). The symbols \(n, a, b, c, \ldots\), with or without primes, are numbers. The purpose is to find values for the variables \(p, q, r, \ldots\) to minimize

\[
\Omega = w_1 + w_2 + \ldots
\]

Gauss introduced a bracket notation (unnamed by him, later called auxiliaries)

\[
[xy] = xy + x'y' + x''y'' + \ldots,
\]

where the letter \(x\) either is \(y\) or lexicographically precedes \(y\). This notation expressed the normal equations (name not yet introduced) as

\[
\begin{align*}
[an] + [aa]p + [ab]q + [ac]r + [ad]s + \ldots &= 0 \\
[bn] + [ab]p + [bb]q + [bc]r + [bd]s + \ldots &= 0 \\
[cn] + [ac]p + [bc]q + [cc]r + [cd]s + \ldots &= 0
\end{align*}
\]

(7)

etc.

As Legendre and he had done before, Gauss [72, p. 22] again remarked these equations could be solved by elimination, but he did not explicitly perform that calculation. Instead, he noted the brackets give the coefficients of the variables in the sum of squares, a quadratic form.

\[
\Omega = [an] + 2[an]p + 2[bn]q + 2[cn]r + 2[dn]s + \ldots \\
+ [aa]pp + 2[ab]pq + 2[ac]pr + 2[ad]ps + \ldots \\
+ [bb]qq + 2[bc]qr + 2[bd]qs + \ldots \\
+ [cc]rr + 2[cd]rs + \ldots
\]

etc.

Gauss extended his bracket notation to

\[
\begin{align*}
[xy, 1] &= [xy] - \frac{[ax][ay]}{[aa]} \\
[xy, 2] &= [xy, 1] - \frac{[bx, 1][by, 1]}{[bb, 1]} \\
[xy, 3] &= [xy, 2] - \frac{[cx, 1][cy, 1]}{[cc, 2]}
\end{align*}
\]

(8)

etc.

and so on. These values are the coefficients remaining in \(\Omega\) after successive variables have been grouped into perfect squares. The first of these combinations of variables, \(A\),

\[
\begin{align*}
A &= [an] + [aa]p + [ab]q + [ac]r + [ad]s + \ldots \\
B &= [bn, 1] + [bb, 1]q + [bc, 1]r + [bd, 1]s + \ldots \\
C &= [cn, 2] + [cc, 2]r + [cd, 2]s + \ldots
\end{align*}
\]

(9)

e tc.,

\footnote{Farebrother [62], p. 161n, attributes double subscript notation to Cauchy in 1815, first used by Gauss in 1828.}
simplifies the quadratic form by removing the variable $p$:

$$\Omega - \frac{A^2}{[aa]} + [bn, 1]q + 2[cn, 1]r + 2[dn, 1]s + \ldots + [bb, 1]qq + 2[bc, 1]qr + 2[bd, 1]qs + \ldots + [cc, 1]rr + 2[cd, 1]rs + \ldots$$

etc.

If this process is repeated with $B, C, \ldots$, then eventually,

$$\Omega - \frac{A^2}{[aa]} - \frac{B^2}{[bb, 1]} - \frac{C^2}{[cc, 2]} - \ldots = [nn, \mu],$$

where $\mu$ is the quantity of variables. Each $A, B, C, \ldots$ has one fewer unknown than the preceding. Thus $A = 0, B = 0, C = 0, \ldots$ can be solved in reverse order to obtain the values for $p, q, r, \ldots$, also in reverse order, at which $\Omega$ attains its minimum, $[nn, \mu]$. In later theoretical discussions of least squares methods, such as Gauss [75, art. 13], he always referred back to his 1811 paper for details of the calculations as transforming quadratic forms.

Gauss’s contributions to the method of least squares were known immediately. By 1819 even a gymnasium prospectus, Paucker [148], cited Legendre [127] and Gauss [71] — but not Gauss for any particular solution method. Gauss’s solution process that neatly tied together linear algebra, optimization theory, and his probabilistic justification for the method of least squares, seems to have been adopted as an algorithm slowly and by geodesists not by mathematicians.

8. Geodesy for Cartography

The original motivation for the computational developments in least squares was their use in two major scientific activities of the 19th century. One was astronomy which was pursued for navigation and timekeeping besides its intrinsic interest. Gauss [71] described the methods he had invented to derive orbital formulas from a few observations. He applied them before and after 1809 in many papers that constitute the bulk of his early work. Nievergelt [143] explains the orbital calculations used in 19th century astronomy, while Grier [89] explains the institutional history of computing groups in national observatories where the calculations were made.

Another scientific activity continually and directly sponsored by governments was geodetic research for cartography. Indeed, the first scientific agency of the United States was the Coast Survey Office founded in 1807, see Cajori [28]. Cartographers positioned major towns and landmarks, relative to one another, by using them as vertices in networks of triangles. Gauss became prominent in geodesy through the many papers he wrote during his protracted survey of Hanover. This small German kingdom, roughly coincident with modern Lower Saxony, was the ancestral possession of the British royal family, and was Gauss’s home for most of his life. Dunnington [52, chap. 10] relates that Bessel warned Gauss the survey toil detracted from his research. Nevertheless, although geodesy had inspired Legendre to invent the method of least squares, a survey officer explains it was from Gauss whom geodesists adopted the method.

If in effecting a [survey] triangulation one observed only just so many angles as were absolutely necessary to fix all the points, there would be no difficulty in calculating [the locations]; only one result would be arrived at. But it is the invariable custom to observe more angles than are absolutely needed, and it is these supernumerary angles which give rise to complex calculations. Until the time of Gauss and Bessel computers had simply used their judgement as they best could as to how to employ and utilize the supernumerary angles; the principal of least squares showed that a system of corrections ought to be applied, one to each observed bearing or angle, such that subject to the condition of harmonizing the whole work, the sum of their squares should be an absolute minimum. The first grand development of this principle is contained in this work of Bessel’s.

— Clarke [36, pp. 26–27]

Clarke refers to a Prussian triangulation that is still important in European cartography, wherein Bessel and Baeyer [16, p. 130] used the methods of Gauss [75]. This endorsement in a survey for a major government drew the attention of other geodesists such as Clarke.
As Clarke explained, a critical step in making maps was to reconcile the slightly inconsistent angle measurements that were the raw data gathered by surveyors. Gauss [75, art. 22] stipulated the true angles satisfy three kinds of conditions: (1) the sum of angles around an interior vertex equals $2\pi$, (2) the sum of angles in a triangle equals $\pi$ plus a spherical correction, and (3) side conditions that chain together linearized sine laws for circuits of triangles with common edges. The measured angles nearly satisfy these conditions, so the unknowns are perturbations intended to make the measurements true. Gauss formed a side condition from the triangles around each interior vertex, in which case (it is easily seen) even an ideal net consisting of $f$ nonoverlapping triangles and $v$ vertices has $3f$ angles (if all are measured), but only $3f - 2v + 4$ conditions. Thus, in general, the adjustment problems were under- not overdetermined. In his last major theoretical work on least squares, Gauss [75] introduced the solution method stated in case 2. He illustrated the method by readjusting a small part of the Dutch triangulation, see Figures 3 and 4. In comparison the British Isles triangulation was quite irregular [145, plate xviii]. Most surveys also had missing measurements from inaccessible stations (e.g. mountaintops) or blocked sight lines.

As the 19th century progressed, the growing use of least squares methods created a recurring need to solve dauntingly large problems. For example, the British Isles survey had 1554 angles subject to 920 conditions. Of necessity this large problem was broken into smaller subproblems.

In the principal triangulation of Great Britain and Ireland there are 218 stations, at sixteen of which there are no observations, the number of observed bearings is 1554, and the number of equations of condition 920. The reduction of so large a number of observations in the manner we have been describing (i.e. least squares case 2) would have been quite impossible, and it was necessary to have recourse to methods of approximation. . . . .

. . . the network covering the kingdom was divided into a number of blocks, each presenting a not unmanageable number of equations of condition. One of these being corrected or computed independently of the others, the corrections so obtained were substituted (as far as they entered) in the equations of condition of the next block, and the sum of the squares of the remaining equations in that figure made a minimum. The corrections thus obtained for the second block were substituted in the third and so on. Four of the blocks are independent commencements, have no corrections from adjacent figures carried into them. The number of blocks is 21: in 9 of them the number of equations of condition is not less than 50: and in one case the number is 77. These calculations — all in duplicate — were completed in two years and a half — an average of eight computers being employed. . . .

In connection with so great a work successfully accomplished, it is but right to remark how much it was facilitated by the energy and talents of the chief computer, Mr. James O’Farrell. — Clarke [36, pp. 237, 243]

Additional local color can be found in Palmer [147] and White [180]. In these situations, professional computers found it useful to emulate Gauss’s calculations of fifty years earlier.

9. Elimination After Gauss

“Numerical mathematics stayed as Gauss left it until World War II,” concluded Goldstine [81, p. 287] from his history of the subject. In those 120 years after Gauss [75] there were at best a dozen noteworthy publications about solving simultaneous linear equations. The topics were algorithmic simplifications for professional computers, and matrix interpretations. These developments overlap chronologically, but they did not influence each other until the middle of the 20th century, so they can be addressed separately in sections 10 and 11 respectively, with minimal cross-reference.

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14 Clarke [36] notes the conditions were formulated for the sphere using “Gauss’s Theorems” and “Legendre’s Theorem” from spherical trigonometry. The Ordnance Survey [145], the research paper by [138], the encyclopedia by Jordan [112], and the textbook by [186] have more explanations and examples.

15 Stigler [164, p. 158] switches the numbers, a Freudian slip no doubt, indicating the difficulty modern mathematicians have in comprehending that, originally, the important least squares problems — those governments would create bureaucracies to solve — were underdetermined not overdetermined.
Figure 3: A triangulation of Holland that Gauss [75, p. 86, art. 23] used to illustrate survey adjustments. He took the data from a French-language publication by de Krayenhof that seems to be unavailable. The same data and the map seen here are in the (likely equivalent) Dutch-language publication of Krayenhoff [118]. Courtesy of the Bancroft Library, University of California, Berkeley.

Figure 4: The portion of Figure 3 that Gauss readjusted. He had to find adjustments for 27 angles, but he had only 13 conditions for them to satisfy, consisting of 2 vertex conditions, 9 triangle conditions, and 2 side conditions. Jordan [112, p. 489] reproduces a similar figure.
The authors discussed under hand computing are Gauss, Doolittle, Cholesky, and Crout. Those under
matrix theory are Toeplitz, Banachiewicz, Frazer, Dwyer, Jensen, and von Neumann and Goldstine. Since
a picture is desired of how Gaussian elimination was practiced and developed after Gauss, the categories to
which authors have been assigned are less important than the decision to highlight a particular work. The
authors in the first group have been chosen because they demonstrably influenced professional computing
practice. Those in the second group independently interpreted Gaussian elimination in terms of matrix alge-
bra.

10. Perfecting Elimination for Professional Computers

10.1. Gauss’s Convenient Notation

The triangulation of the great Ordnance Survey [145] and the conditions of its adjustment are preserved in
detail, but how the calculations were performed is missing. Since, apparently, the details of major calculations
were not archived, whether and why Gauss’s solution method was used — and what it was considered to be —
must be inferred from sources such as Chauvenet [31]. There appear to have been three advantages to Gauss’s
own method of solving the normal equations, which has been described here in section [2]. First, the bracket
notation conceptually separated the equations from the arithmetic so the workflow could be addressed.

By whatever method of elimination is performed, we shall necessarily arrive at the same final val-
ues of the unknown quantities; but when the number of equations is considerable, the method of
substitution, with Gauss’s convenient notation, is universally followed. — Chauvenet [31, p. 514]

It may come as some surprise to learn that when Gauss performed Gaussian elimination, he simply listed
all the numbers in the order he computed them, using his brackets to identify the values: \[ \begin{bmatrix} cd \end{bmatrix} = 1.13382, \]
\[ \begin{bmatrix} cd, 1 \end{bmatrix} = 1.09773, \begin{bmatrix} cd, 2 \end{bmatrix} = 1.11063, \text{ etc.} \]
See Figure [5]. Gauss surely took an informed approach to comput-
ing because he was no academic dilettante. Dunnington [52, p. 138] echoed Bessel in regretting how much
time Gauss spent calculating for his interminable survey projects: Gauss himself estimated to have needed
a million numbers! Second, in contrast to the method of Hammond, Lacroix, and the textbook algorithm of
equation (1), Gauss realized economies by avoiding duplicate calculations for symmetric equations [16].

By means of a peculiar notation proposed by Gauss, the elimination by substitution is carried on
so as to preserve throughout the symmetry which exists in the normal equations. — Chauvenet
[31, p. 530]

Chauvenet even counted the brackets or auxiliaries: 156 for 8 normal equations, etc. Third, Gauss included
refinements such as estimates of precision, variances, and weights for the unknowns, all expressed in his
bracket notation. The difficulty of making changes to the formulas, whose complexity was compounded by
their relationship to Gauss’s statistical theories, and the demonstrable benefits of Gauss’s ideas, entailed a
reluctance to alter his computational prescriptions. His efficient method for overdetermined problems by
itself was conceptually difficult because the solution of the normal equations, \( AA' u = b \), was not the solution
of the problem, \( x = A' u \). Gauss’s bracket notation was still being taught a hundred years after his 1811 paper,
by Bartlett [12], Johnson [111], and [186], and for continued emphasis on the advantages of preserving
symmetry, see Palmer [146, pp. 84–85]. At the beginning of the 20th century, Wright and Hayford find
professional computers using either of just two methods to solve normal equations: the brackets of Gauss or
the tables of Doolittle.

10.2. Doolittle, Legendary Computer

Myrick Hascall Doolittle was a computer who solved Gauss’s normal equations to adjust triangulations at
the United States Coast and Geodetic Survey for 38 years from 1873 to 1911. See Figure [6]. A biographical
sketch has been written by Grier [89, pp. 78–79], and some primary biographical sources are quoted at length
by Farebrother [60]. Doolittle was the only human computer remembered for his professional acumen.

\[ ^{16} \text{The normal equations are symmetric in the sense that the same coefficient applies to the } j^{th} \text{ variable in the } k^{th} \text{ equation, and vice versa.} \]
Demonstrating the efficacy of his methods, Doolittle [49, p. 117] notes that he solved 41 normal equations in a week using paper and pencils. For perspective, Fox [66, p. 676] reports that four mathematicians, Alan Turing among them, needed two weeks to solve 18 equations using electric desk calculators in 1946. Nothing suggests Doolittle was a numerical savant because a colleague, Mr. J. G. Porter, duplicated the calculation in the same time to check for errors.

The speed stemmed from streamlining the work for hand computing. Among the practices Doolittle [49] described, is identifying the numbers of the calculation by their placement in tables. He presented a small numerical example that is restated here in symbolic bracket notation to clarify the calculation. Corresponding to Gauss’s equation (7), Doolittle’s normal equations were as follows.

\[
\begin{align*}
0 &= [aa]w + [ab]x + [ac]y + [ad]z + [an] \\
0 &= [ab]w + [bb]x + [bc]y + [bd]z + [bn] \\
0 &= [ac]w + [bc]x + [cc]y + [cd]z + [cn] \\
0 &= [ad]w + [bd]x + [cd]y + [dd]z + [dn]
\end{align*}
\]

Doolittle expressly arranged the calculation to derive Newton’s (back-) substitution formulas for each variable, which correspond to rearrangements of Gauss’s equations \( A = 0, B = 0, \ldots \). He called these formulas the “explicit functions” for the variables. Doolittle was able to co-locate many of the numbers by completing the formula for a given variable before undertaking any calculations for the next. He kept the coefficients of the formulas in table A, see Figure [7], while he used a second table, B, to record in columns the sums that give the values in table A. Exactly how Doolittle conducted the work may be lost. “For the sake of perspicuity,” he noted, “I have here made some slight departures from actual practice.”
The salient feature of Doolittle’s tables is a reduction in the labor of division and multiplication. All divisions reduce to multiplications through reciprocals formed just one per variable in the first column of table A. All multiplications have a single multiplier repeatedly applied to several multiplicands in a row of table A and recorded in another row of table B. For example, row 8 in table B results from a single multiplier in table A, row 2 column $y$, applied to several multiplicands in row 1, beginning at the same column, $y$, and moving rightward. The reduction of work occurs because Doolittle \[49, \text{p. 117}\] performed multiplication using the 3-digit tables of Crelle \(40\). Since Doolittle reused the multiplier for an entire row of calculations, he could open Crelle’s book just once to the table for that multiplier, where all the multiplicands could be found without turning pages. Schott \(160, \text{p. 93}\) emphasized that using multiplication tables was innovative — “logarithms are altogether dispensed with.”

Doolittle performed the back-substitution with similar economy. He distinguished between numbers used once or many times, so he copied the reciprocals and “explicit function” coefficients from table A to table C; see Figure 8. The value of $z$ was available in the final row of table A. The remaining variables were evaluated in table D where each row consists of one multiplier applied to one column, this time, of table C. The sums of the columns in table D give the other variables.

Doolittle’s method included several contributions which are not now recognized. He owed his speed in part to using just 3-digit arithmetic in the multiplication tables, and hence everywhere in the calculation, but the rounding errors so introduced would be considered severe. For example, a modern computer adhering to international standards for binary arithmetic has the decimal equivalent of roughly 7 or 14 digits \(106\). Thus, an important aspect of Doolittle’s method was the ability to correct the 3-digit approximate solution with comparatively little more work. Since the angle adjustment problem itself corrected numbers that were approximately known — the measured angles — it may have seemed natural to further correct the adjust-

\[\text{This comment confirms that computers generally did use logarithms to evaluate Gauss’s brackets. Note tables of “Gauss’s logarithms” are available for the subtraction.}\]
Table A

| step | w | x | y | z | absolute term |
|------|---|---|---|---|---------------|
| 1    |   |   |   |   |               |
| 2    | $\frac{-1}{[aa]}$ | $[ab]$ | $[ac]$ | $[ad]$ | $[an]$ |
| 5    | $[bb, 1]$ | $[bc, 1]$ | $[bd, 1]$ | $[bn, 1]$ |
| 6    | $\frac{-1}{[bb, 1]}$ | $-\frac{[bc, 1]}{[bb, 1]}$ | $-\frac{[bd, 1]}{[bb, 1]}$ | $-\frac{[bn, 1]}{[bb, 1]}$ |
| 10   | $[cc, 2]$ | $[cd, 2]$ | $[cn, 2]$ |
| 11   | $\frac{-1}{[cc, 2]}$ | $\frac{[cd, 2]}{[cc, 2]}$ | $\frac{[cn, 2]}{[cc, 2]}$ |
| 16   | $\frac{-1}{[dd, 3]}$ | $-\frac{[dn, 3]}{[dd, 3]}$ |

Table B

| step | x | y | z | absolute term |
|------|---|---|---|---------------|
| 4    | $\frac{[ab]}{[aa]} \times [ab]$ | $\frac{[ab]}{[aa]} \times [ac]$ | $\frac{[ab]}{[aa]} \times [ad]$ | $\frac{[ab]}{[aa]} \times [an]$ |
| 7    | $[cc]$ | $[cd]$ | $[cn]$ |
| 8    | $\frac{[ac]}{[aa]} \times [ac]$ | $\frac{[ac]}{[aa]} \times [ad]$ | $\frac{[ac]}{[aa]} \times [an]$ |
| 9    | $-\frac{[bc, 1]}{[bb, 1]} \times [bc, 1]$ | $-\frac{[bc, 1]}{[bb, 1]} \times [bd, 1]$ | $-\frac{[bc, 1]}{[bb, 1]} \times [bn, 1]$ |
| 12   | $[dd]$ | $[dn]$ |
| 13   | $-\frac{[ad]}{[aa]} \times [ad]$ | $-\frac{[ad]}{[aa]} \times [an]$ |
| 14   | $\frac{[bd, 1]}{[bb, 1]} \times [bd, 1]$ | $\frac{[bd, 1]}{[bb, 1]} \times [bn, 1]$ |
| 15   | $\frac{[cd, 2]}{[cc, 2]} \times [cd, 2]$ | $\frac{[cd, 2]}{[cc, 2]} \times [cn, 2]$ |

Figure 7: How Doolittle performed Gaussian elimination, transcribed from the numerical example in Doolittle [49] using Gauss’s bracket notation to identify the quantities. The step numbers indicate the order of forming the rows. Each row in table B is a multiple, by a single number, of a partial row in table A. The rows 5, 10, and 16 in table A are sums of rows 3–4, 7–9, and 12–15 in table B, respectively.

ments. Both Doolittle [49] and Schott [160, p. 93] described the correction process without giving it a name; today it is called iterative refinement.

In describing the refinement process, Doolittle wrote $w_1$, $x_1$, $y_1$, $z_1$ for the values obtained from tables A–D. When these approximations are substituted into the normal equations they give residual values.

$$\begin{align*}
r_1 &= [aa]w_1 + [ab]x_1 + [ac]y_1 + [ad]z_1 + [an]
r_2 &= [ab]w_1 + [bb]x_1 + [bc]y_1 + [bd]z_1 + [bn]
r_3 &= [ac]w_1 + [bc]x_1 + [cc]y_1 + [cd]z_1 + [cn]
r_4 &= [ad]w_1 + [bd]x_1 + [cd]y_1 + [dd]z_1 + [dn]
\end{align*}$$

Doolittle emphasized this particular calculation needed high accuracy after which the residuals $r_1$, $r_2$, $r_3$, $r_4$ (notation of this paper) could be rounded to 3 digits for the remaining steps. In table E of Figure 9 he performs the same calculations on the residuals that he performed on the constant terms of the normal equations in the final column of table B. Table F of this figure then duplicates the back-substitution of table D, but now for the corrections, which Doolittle named $w_2$, $x_2$, $y_2$, $z_2$. The fully-corrected solutions, $w = w_1 + w_2$, $x = x_1 + x_2$, etc., were accurate to 2 to 3 digits in Doolittle’s example.

Another innovation, “one of the principal advantages,” was a provision to include new equations and
variables. Wright and Hayford [186 pp. 117-118] provide a clearer text than Doolittle [49], whose description is somewhat brief. Doolittle suggests large problems could be solved by successively including conditions in the minimization problem, that is, by appending equations and variables to the normal equations. He recommends ordering the conditions so as to preserve zeroes in the elimination, and suggests an ordering based on the geometric interpretation of the conditions. Doolittle thus anticipates the work on sparse matrix factorizations that would be done a hundred years later, see George and Liu [72] and Duff et al. [50].

Dwyer [55] p. 112] remarked “from Doolittle down to the present” no formal proof was offered that Doolittle’s tables do solve the normal equations. Some justification is needed because Doolittle does not explicitly reduce to zero the coefficients of eliminated variables. For example, rows 8 and 9 of table B remove \(w\) and \(x\) from the 3rd equation, but Doolittle operates only on coefficients for the retained variables \(y\) and \(z\). This saving is possible because, thanks to the underlying symmetry, Doolittle knows the multipliers from the omitted calculations are the coefficients of his “explicit functions.” Dwyer [54] gave one proof, and other explanations could be given, such as expanding Gauss’s bracket formulas for the coefficients in his equations \(A = 0, B = 0, \ldots\). The relationship between Gauss’s brackets and symmetric elimination seems to have been generally known, as evidenced by Chauvenet [31, p. 530]. Doolittle had the training and ability to develop such methods. He taught mathematics at Antioch College after receiving a Bachelor’s degree there, he also studied under Benjamin Peirce at Harvard College, and he chaired the mathematics division of the Philosophical Society of Washington.

The reason for Doolittle’s reticence may be that publishing computing methods per se was neither appropriate (as judged by the mathematical community) nor desirable (from the standpoint of the computer). The

| \(x\) | \(y\) | \(z\) |
|------|------|------|
| \(\frac{1}{[aa]} \times [ab]\) | \(\frac{1}{[aa]} \times [ac]\) | \(\frac{1}{[aa]} \times [ad]\) |
| \(\frac{1}{[bb,1]} \times [bc,1]\) | \(\frac{1}{[bb,1]} \times [bd,1]\) | \(\frac{1}{[bb,1]} \times [bd,1]\) |
| \(\frac{1}{[cc,2]} \times [cd,2]\) | \(\frac{1}{[cc,2]} \times [cd,2]\) | \(\frac{1}{[cc,2]} \times [cd,2]\) |
| \(\frac{1}{[dd,3]} \times [dd,3]\) | \(\frac{1}{[dd,3]} \times [dd,3]\) | \(\frac{1}{[dd,3]} \times [dd,3]\) |

Figure 8: How Doolittle performed back-substitution. Table C he copied from table A. The sum of each column in table D evaluates the “explicit function” for a different variable.

| column sum = \(\sum s\) | column sum = \(\sum s\) | column sum = \(\sum s\) |
|---------------------------------|---------------------------------|---------------------------------|
| \(r_1\) \times \(\frac{1}{[aa]}\) \times [ab]\) | \(r_2\) \times \(\frac{1}{[aa]}\) \times [ac]\) | \(r_3\) \times \(\frac{1}{[aa]}\) \times [ad]\) |
| \(s_1\) \times \(\frac{1}{[bb,1]}\) \times [bc,1]\) | \(s_2\) \times \(\frac{1}{[bb,1]}\) \times [bd,1]\) | \(s_3\) \times \(\frac{1}{[bb,1]}\) \times [bd,1]\) |
| \(s_2\) \times \(\frac{1}{[cc,2]}\) \times [cd,2]\) | \(s_3\) \times \(\frac{1}{[cc,2]}\) \times [cd,2]\) | \(s_4\) \times \(\frac{1}{[cc,2]}\) \times [cd,2]\) |
| \(r_1\) \times \(\frac{1}{[aa]}\) \times [ab]\) | \(r_2\) \times \(\frac{1}{[aa]}\) \times [ac]\) | \(r_3\) \times \(\frac{1}{[aa]}\) \times [ad]\) |
| \(s_1\) \times \(\frac{1}{[bb,1]}\) \times [bc,1]\) | \(s_2\) \times \(\frac{1}{[bb,1]}\) \times [bd,1]\) | \(s_3\) \times \(\frac{1}{[bb,1]}\) \times [bd,1]\) |
| \(s_1\) \times \(\frac{1}{[cc,2]}\) \times [cd,2]\) | \(s_2\) \times \(\frac{1}{[cc,2]}\) \times [cd,2]\) | \(s_3\) \times \(\frac{1}{[cc,2]}\) \times [cd,2]\) |
| \(r_1\) \times \(\frac{1}{[aa]}\) \times [ab]\) | \(r_2\) \times \(\frac{1}{[aa]}\) \times [ac]\) | \(r_3\) \times \(\frac{1}{[aa]}\) \times [ad]\) |
| \(s_1\) \times \(\frac{1}{[bb,1]}\) \times [bc,1]\) | \(s_2\) \times \(\frac{1}{[bb,1]}\) \times [bd,1]\) | \(s_3\) \times \(\frac{1}{[bb,1]}\) \times [bd,1]\) |
| \(s_1\) \times \(\frac{1}{[cc,2]}\) \times [cd,2]\) | \(s_2\) \times \(\frac{1}{[cc,2]}\) \times [cd,2]\) | \(s_3\) \times \(\frac{1}{[cc,2]}\) \times [cd,2]\) |

Figure 9: How Doolittle performed iterative refinement. Table E corresponds to the final column of table B. The calculations in table B apply to the constant terms, while in table E they apply to the residuals. Table F is similar in construction to table D.
omission of calculating methods from the report of the Ordnance Survey [145] has been noted, as well as the neglect of Gauss’s computing methods in the histories of mathematics. Grier [89, p. 156] concluded from his study of hand computers that until the 20th century computing was a craft skill passed from masters to apprentices. A reluctance to disclose methods is consistent with Grier’s picture of journeymen computers. Indeed, Schott [160, p. 93] emphasized Doolittle’s paper was written at the express urging of the Coast Survey Director. The Washington Star [176] reported Doolittle “contributed numerous papers on his favourite subject” to the Philosophical Society of Washington, yet calculating was discussed only in this one of Doolittle’s three archived publications listed in the bibliography compiled by Gore [84, p. 365].

Doolittle’s paper made a strong impression on computers. The claims made for the “Coast Survey method” by Doolittle [49] and Schott [160] caused Werner [179] to examine it immediately and skeptically. He calculated Doolittle’s tables three different ways: with Crelle’s multiplication tables, with logarithms, and with the Thomas Arithmometer, which is discussed below. Thus iterative refinement was misunderstood and neglected by later authors. The Doolittle tables evidently passed muster because they were reprinted for many years. Jordan [112, pp. iv, 65] remarked on the “old, classic” notation of Gauss and then exhibited without attribution a calculation using logarithms and Doolittle’s table B. Wright and Hayford [186] offered the Gauss brackets and the Doolittle tables as competing approaches. They did not understand the subtleties because they touted the use of Crelle’s 3-digit tables (p. 120) yet they omitted the iterative refinement. Grier [89, pp. 159–164] relates that Howard Tolley, a computer at the Coast and Geodetic Office who became an official at the United States Department of Agriculture, politely scolded economists and statisticians when they neglected to credit the computing methods developed in geodesy. Tolley and Ezekiel [173] cited several textbooks teaching Doolittle’s method. They illustrated what were still essentially Doolittle’s tables by a calculation that, by then, was done with a Monroe four-function electric calculator, which obviated any need for iterative refinement. Tolley and Ezekiel report that Miss Helen Lee, a computer, could solve 5 equations with 6-digit numbers (two whole and four fractional) in 50 minutes in 1927 (or 40 minutes if rounded to two fractional digits).

10.3. Mechanical Calculators

By the time Doolittle wrote his paper, Gottfried Wilhelm Leibnitz’s stepped drums had been used to build several arithmetic machines. The last machine, and the first one to become commercially available, was the Arithmometer of Charles Xavier Thomas de Colmar. It was a four function calculator that operated by repeated addition to, or subtraction from, a Pascal adder that served as an accumulator. The drums were elongated gears with 9 cogs of different lengths on their surfaces. An axel parallel to each drum held a sliding gear positionable to mesh with from 0 to 9 cogs. When the drums made full rotations (powered by a hand crank) then each axel made a partial rotation (dependent on the position of its sliding gear). In this way each axel, through additional gearing, added a number from 0 through 9 to a corresponding digit of the Pascal adder. The accumulator could shift to permit scaling by powers of 10. For multiplication, the number represented by the positionable gears became the multiplicand. Repeated rotations of the drums and shifts of the accumulator gradually added the complete product to the accumulator. A sum of products could be accumulated in this way. Arithmometers were less reliable than could be hoped, yet [110] describes many uses for them in Victorian England after about 1870.

[167] reports that mass production of mechanical adders and calculators began at the end of the 19th century. The calculators were three-register machines based on Wilgot Odhner’s pinwheel design. The principle of repeated addition or subtraction remained the same, while more compact and reliable devices replaced the Leibnitz drums and the Pascal adder. The input register represented the digits of a number by coaxial wheels with variable quantities of pins protruding at their circumferences. A revolution of the pinwheels added the number to an output accumulator consisting of sprocket wheels that operated like a Pascal adder, but now rotating on a common axle, and again mounted on a moveable carriage. The multiplier was an output register that counted the revolutions at each carriage position to check how many additions the operator had performed at each power of two. For subtraction or division, the hand crank was turned backward.

[15] Jordan’s encyclopedic reference work on geodesy was constantly updated. He credited the material to scarcely anyone by name except Gauss himself. Althoen and McLaughlin [4] explain it is this Jordan, from Hanover, for whom the Gauss-Jordan algorithm for inverting matrices is misnamed.
Electrified calculators were made from about 1920. Collectively called rotary calculators, a motor replaced the hand crank, and more automated designs followed. The multiplier register became an input that automated the program of revolutions and carriage shifts. Since these fully automatic machines only became available during the Great Depression, first cost and then wartime rationing limited their use, so they were a luxury for scientific computers. For example, Grier reports that the Mathematical Tables Project gave most computers only paper and pencils because the price of an electric calculator nearly equaled their annual salary. explains that three variants survived after World War II: Friden, Monroe, and the ultra quick and quiet Marchant “silent speed” calculators from Oakland, California.

The most significant feature of all these machines for computers was the ability to accumulate a sum of products. It saved time because the individual products need not be recorded in a table before summing them, as Doolittle had done in his table B. Moreover, this feature was believed to result in more accurate calculations. The multiplicand and multiplier often had the same capacity, and then the accumulator had twice the digits of either. The decimal point locations chosen for the registers might require a number in the accumulator to be rounded before it could be reentered. Therefore, explained, summing products without reentry was “theoretically more accurate” because it gave “the approximation resulting from a number of operations rather than the combination of approximations resulting from the operations.”

10.4. Cholesky: Machine Algorithm

Mechanization cannot be why Doolittle revised elimination in 1878 because he began using a manual adding machine only in 1890. The first algorithm intended for a machine may be that of the military geodesist and World War I casualty André-Louis Cholesky. See Figure 11. A biography with a discussion of his work has been prepared by , see also .

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\[19\] This usage of program by Chase exhibits the original technical meaning: a sequence of events designed to occur in complex machinery. Grier traces the evolution to the verb to program in computer science.
Like Doolittle, Cholesky calculated Gaussian’s angle adjustments, case 2 of the least squares problem, which is stated here in section 6. Although matrices were known when Cholesky developed his algorithm, they were not used outside pure mathematics, so his invention is better understood without them. Cholesky wrote the condition equations as [14, eqn. 1],

\[
\begin{align*}
 a_1 x_1 + a_2 x_2 + a_3 x_3 + \ldots + a_n x_n + K_1 &= 0 \\
 b_1 x_1 + b_2 x_2 + b_3 x_3 + \ldots + b_n x_n + K_2 &= 0 \\
 \vdots & \quad \vdots & \quad \vdots & \quad \vdots \\
 \ell_1 x_1 + \ell_2 x_2 + \ell_3 x_3 + \ldots + \ell_n x_n + K_p &= 0
\end{align*}
\]

(10)

where \( n > p \), and he wrote the normal equations as [14, eqn. 5],

\[
\begin{align*}
 a_1^1 \lambda_1 + a_2^2 \lambda_2 + \ldots + a_n^p \lambda_p + K_1 &= 0 \\
 a_1^2 \lambda_1 + a_2^2 \lambda_2 + \ldots + a_n^p \lambda_p + K_2 &= 0 \\
 \vdots & \quad \vdots & \quad \vdots & \quad \vdots \\
 a_1^n \lambda_1 + a_2^n \lambda_2 + \ldots + a_n^n \lambda_p + K_p &= 0
\end{align*}
\]

(11)

where \( a_k^j = a_k^i \) is the sum of products of coefficients in the \( j^{th} \) and \( k^{th} \) conditions, for example, \( a_2^n = b_1 \ell_1 + \ldots + b_n \ell_n \).

Cholesky’s remarkable insight was, since many underdetermined systems share the same normal equations, for any normal equations there may be some condition equations that can be directly solved more easily [14, p. 70]. Cholesky found his alternate equations in the convenient, triangular form (introducing new unknowns, \( y \) in place of \( x \), and new coefficients, \( \beta \) in place of \( a \) )

\[
\begin{align*}
 \beta_1^1 y_1 + & + K_1 = 0 \\
 \beta_2^1 y_1 + \beta_2^2 y_2 + & + K_2 = 0 \\
 \beta_3^1 y_1 + \beta_3^2 y_2 + \beta_3^3 y_3 + & + K_3 = 0 \\
 \vdots & \quad \vdots & \quad \vdots & \quad \vdots \\
 \beta_p^1 y_1 + \beta_p^2 y_2 + \beta_p^3 y_3 + \ldots + & + \beta_p^p y_p + K_p = 0
\end{align*}
\]

(12)

Cholesky discovered the coefficients in equation (12) are given by straightforward formulas [14, p. 72].

\[
\begin{align*}
 \beta_i^1 &= \sqrt{a_i^1 - (\beta_i^1)^2 - (\beta_i^2)^2 - \ldots - (\beta_i^{i-1})^2} \\
 \beta_{i+r}^i &= \frac{a_{i+r}^i - \beta_{i+r}^1 \beta_{i+r}^1 + \beta_{i+r}^2 \beta_{i+r}^2 - \ldots - \beta_{i+r}^{i-1} \beta_{i+r}^{i-1}}{\beta_i^1}
\end{align*}
\]

(13)

The solution \( \lambda \) of the normal equations expresses the solution of the condition equations as a linear combination of the (transposed) coefficients in the condition equations. For Cholesky, these combinations become a system of equations to be solved for \( \lambda \) [14, eqn. 7].

\[
\begin{align*}
 y_1 &= \beta_1^1 \lambda_1 + \beta_2^1 \lambda_2 + \ldots + \beta_p^1 \lambda_p \\
 y_2 &= \beta_2^2 \lambda_2 + \ldots + \beta_p^2 \lambda_p \\
 \vdots & \quad \vdots & \quad \vdots \\
 y_p &= \beta_p^p \lambda_p
\end{align*}
\]

(14)

Since the original condition equations (10) and Cholesky’s equations (12) have the same normal equations (11), the quantities \( \lambda \) are the same for both problems. Once obtained, \( \lambda \) can be used to evaluate \( x \) as usual.

\[20\] Bénoit [14] unfortunately used \( a \) for the new coefficients which here has been replaced by \( \beta \) to more easily distinguish them from the original coefficients \( a \).
(from the transpose of the original coefficients), thereby solving the original condition equations. Therefore Cholesky’s method was to form his new coefficients by equation (13), then to solve (12) by forward-substitution for \( y \), next to solve (14) by back-substitution for \( \lambda \), and finally to evaluate \( x \).

\[
\begin{align*}
  x_1 &= a_1 \lambda_1 + b_1 \lambda_2 + \ldots + \ell_1 \lambda_p \\
  x_2 &= a_2 \lambda_1 + b_2 \lambda_2 + \ldots + \ell_2 \lambda_p \\
  & \vdots \\
  x_n &= a_n \lambda_1 + b_n \lambda_2 + \ldots + \ell_n \lambda_p \\
\end{align*}
\]  

(15)

In contrast, Gauss’s algorithm was to solve (11) by elimination for \( \lambda \), then to evaluate \( x \).

Bénoit [14] published his colleague’s method posthumously. A similar manuscript dated 1910 was found among Cholesky’s military papers and has recently appeared, Cholesky [32]. French geodesists evidently continued to use the method because Benoit thought to publish it in his own format with revised notation and a table showing how the calculation was conducted, see Figure 11. Benoit evaluated the whole \( p \)th column of coefficients — \( \beta_1, \beta_{2r+1}, \beta_{2r+2}, \ldots, \beta_p \) — before going on to the next column. The table is more compact than Doolittle’s tables because much of the intermediate work is not recorded: each \( \beta_{2r+1} \) is a sum of products. Benoit and Cholesky mentioned using calculating machines to accumulate these sums, and Cholesky specifically referred to the Dactyle brand of the Odhner design. Cholesky [32] reported solving 10 equations with 5-digit numbers in 4 to 5 hours.

| \( i \) | \( \beta_1^i \) | \( \beta_2^i \) | \( \beta_3^i \) | \( \beta_4^i \) | \( \beta_5^i \) | \( \beta_p^i \) | \( \lambda_1 \) | \( \lambda_2 \) | \( \lambda_3 \) | \( \lambda_4 \) | \( \lambda_5 \) | \( \lambda_p \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | \( a_1^1 \) | \( a_2^1 \) | \( a_3^1 \) | \( a_4^1 \) | \( a_5^1 \) | \( a_p^1 \) | \( K_1 \) | \( \lambda_1 \) |
| 2 | \( a_1^2 \) | \( a_2^2 \) | \( a_3^2 \) | \( a_4^2 \) | \( a_5^2 \) | \( a_p^2 \) | \( K_2 \) | \( \lambda_2 \) |
| \( i \) | \( a_1^i \) | \( a_2^i \) | \( a_3^i \) | \( a_4^i \) | \( a_5^i \) | \( a_p^i \) | \( K_i \) | \( \lambda_i \) |
| \( p \) | \( a_1^p \) | \( a_2^p \) | \( a_3^p \) | \( a_4^p \) | \( a_5^p \) | \( a_p^p \) | \( K_p \) | \( \lambda_p \) |
| \( \beta \) | \( y_1 \) | \( y_2 \) | \( y_3 \) | \( y_4 \) | \( y_5 \) | \( y_p \) |

Figure 11: How Cholesky may have organized the calculation to solve the normal equations by what is now the Cholesky factorization, from the fold-out table of Bénoit [14]. The left column and top row are labels; the other positions would be occupied by numbers. The coefficients, \( a_i \), and the constants terms, \( K \), of the normal equations are placed for reference above the diagonal. The coefficients, \( \beta_i \), of Cholesky’s manufactured condition equations are written below the diagonal. The solution, \( \lambda \), of his condition equations stands in the right column. Some additional rows and columns for arithmetic checks and for accuracy estimates have been omitted.

Cholesky’s method remained obscure, compared to Doolittle’s work, for 20 years after the publication by Bénoit [14]. Nevertheless, it was used in Nordic Europe during this time.

The normal equations have been solved by the Cholesky-Rubin method, which offers the advantage that the solution is easily effected on a calculating machine (Cholesky) and that the most probable values and their mean errors are derived simultaneously (Rubin). — Ahlmann and Rosenbaum [11] p. 30]

Rosenbaum applied Cholesky’s method to the case of normal equations, so the method was extended to the other type of least squares problem sometime between 1924 and 1933. That work has not been previously discussed in the historical literature, so evidently some primary literature remains to be identified. [21] Jensen

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[21] Thus Brezinski [20] claims incorrectly that after Bénoit [14] “a period of 20 years followed without any mentioning of the work.”
[109 p. 22], writing in a Danish geodetic publication, remarked that Cholesky’s method “ought to be more generally used.” Soon thereafter it was independently discovered in matrix form by Dwyer and was put to use in the United States.

10.5. Crout: “each element is determined by one continuous machine operation”

Prescott Crout, see Figure 12, was a professor of mathematics at the Massachusetts Institute of Technology with an interest in mathematics for electrical engineering. In a paper that is model of brevity, [42] listed the gamut of uses for linear equations that had developed after least squares. Crout invented the last algorithm specifically for hand-operated calculators.

Crout [42, p. 1239] explained “the method was originally obtained by combining the various processes which comprise Gauss’s method, and adapting them for use with a computing machine.” He wrote the coefficients in a rectangular matrix (his usage) with the constant terms in the final column. Crout’s method consisted of few terse rules for transforming the numbers. So much had changed in the application of Newton’s elimination rule that all semblance of symbolic algebra had disappeared! Crout’s instructions are here restated more expansively.

1. The first column is left unchanged. The first row to the right of the diagonal is divided by the diagonal entry.
2. An entry on or below the diagonal is reduced by the sum of products between entries to the left in its row and the corresponding entries above in its column. This calculation is permitted only after all those entries themselves have been transformed.
3. Ditto an entry above the diagonal except it is lastly divided by the diagonal entry in its row.
4. On completing the above steps, (summarizing further instructions) conduct a back-substitution using the coefficients above the diagonal and the final column of transformed constants.

These steps obviously were intended for calculators that could accumulate sums of products. Like Doolittle before him, Crout included instructions to improve the accuracy of the solution by an unnamed iterative
refinement process. He concluded the paper with a rigorous proof that the method necessarily solves the intended equations.

The method was quickly adopted by those with access to calculators. \cite{crout} reprised a version of his paper in a series of manuals for a calculator manufacturer. The Marchant Calculating Machine Company \cite{marchant} prepared even more detailed instructions in a subsequent manual. Wilson \cite[p. 352]{wilsont} particularly recommended these manuals to academic researchers. Black \cite{blacks} cited both Crout and Dwyer, who is discussed below, in describing how to use calculators for engineering work after World War II.

11. Enter Matrices

Von Neumann \cite{vonneumann} cited matrices as an example of the delays between the invention of a mathematical idea and its use outside pure mathematics. Hawkins \cite{hawkinst, hawkint, hawkintwo} found that matrix algebra was independently invented by Eisenstein (1852), Cayley (1858), Laguerre (1867), Frobenius (1878) and Sylvester (1881) to clarify subjects such as determinants, quadratic forms, and the elementary divisors that Weierstrass created to study ordinary differential equations. The first application of matrices outside mathematics was Heisenberg’s matrix (quantum) mechanics in 1925. Through the first half of the 20th century, most accounts of simultaneous linear equations lacked matrix notation, for example see Aitken \cite{aitken, aitkenthree}, Crout \cite{crout, crouttwo}, Dwyer \cite{dwyer}, Hotelling \cite{hotelling}, Bargmann et al. \cite{bargmann}, and MacDuffee \cite{macduffee}.

The relationship between Gaussian elimination and matrix algebra was not obviously useful. The few descriptions of computations in terms of matrix algebra in the first half of the 20th century produced no significantly new computational methods. Instead, matrix algebra led to a consolidation of approaches by revealing that all the several elimination algorithms were trivially related through what is now called triangular factorization. This paradigm, in the sense of Kuhn \cite{kuhn}, more so than the calculation of equation \eqref{equation}, is what “Gaussian elimination” means to computational mathematicians today.

11.1. Toeplitz: First Triangular Factors

David Hilbert’s study of integral equations inspired his students Erhard Schmidt and Otto Toeplitz to promote infinite matrices as a representation for linear operators on function spaces.\footnote{Bernkopf \cite[p. 330]{bernkopf} relates that the infinite matrix approach was shown to be fundamentally inadequate by another Hilbert protégé, John von Neumann.} As part of that research, Toeplitz \cite{toeplitz} used determinants to examine the invertibility of infinite matrices. “If one uses the symbolism of matrix calculus (see for example Frobenius),” then any finite symmetric matrix $S$ with all leading principal determinants not zero, has a matrix $U$ with

\[
U'U = S^{-1} \quad \text{equivalently} \quad U^{-1}U' = S
\]  

\hspace{1cm} (16)

(\text{original notation}) where $'$ is transposition. In this way Toeplitz first exhibited what now is called the Cholesky factor, $U^{-1}$, though he left it in the computationally useless form of determinants. It was clear from the determinant formulas that $U$ was a lower triangular matrix. Toeplitz remarked that Lagrange and Gauss had such a decomposition for quadratic forms, and he cited Jacobi \cite{jacobi} for a similar decomposition of bilinear forms. His equation \eqref{equation} appears to be the first expression of any such formula in matrix notation. Moreover, the equation may be unique to Toeplitz because his $U$ and $S$ are related by inversion. Taussky and Todd \cite{taussky} suggested basing a proof on the formulas of Gantmacher \cite[v. 1, p. 39]{gantmacher}.

11.2. Banachiewicz: Cracovian Algorithms

One path to matrix algebra is remembered in neither mathematical history nor heritage. In addition to the motivations in pure mathematics for the algebra of Cayley-Eisenstein-Frobenius-Laguerre-Sylvester, there is a motivation in computing for the Cracovian algebra of Tadeusz Banachiewicz. An astronomer and geodesist like Gauss, Banachiewicz \cite{banachiewicz} is still known for determining the orbit of Pluto. Witkowski \cite{witkowski} wrote the first of several biographical sketches of this concentration camp survivor.

Cracovians began as a notational scheme for astronomical calculations and became an arithmetic different from Cayley’s for matrices of arbitrary dimension. The distinguishing feature of the Cracovian algebra is a column-by-column product which is more natural when calculating with columns of figures by hand.
It must, however, be conceded that in practice it is easier to multiply column by column than to multiply row by column . . . . It may, in fact, be said, that the computations are made by cracovians and the theory by matrices
— Jensen [109, p. 5]

Banachiewicz posed least squares problems in terms of Cracovians for the purpose of improving computations. See Kocinski [117] for descriptions of the algorithms and additional references to the original papers. Jensen [108, pp. 3, 19] reports hearing Banachiewicz advocate this approach at meetings of the Baltic Geodetic Commission as early as 1933. Banachiewicz [8, 7] independently discovered Cholesky’s method, and although later than Cholesky, he had greater impact. Banachiewicz inspired the work of Jensen, and he was widely cited: by Jensen [109, p. 45], Dwyer [55, p. 89], Cassinis [29, p. 78], Bodewig [18, part V, p. 90], Laderman [124], and again by Dwyer [57, p. 103]. Reflecting the thesis of this paper, the influential mathematician Householder [103, 104, p. 142] would neglect the work of Banachiewicz, who traded places with Cholesky in mathematical obscurity.

11.3. Frazer, Duncan, and Collar: Elementary Matrices

The work of R. A. Frazer, W. J. Duncan, and A. R. Collar exemplifies the source for matrix algebra that Hawkins [95] finds in Weierstrass’s study of dynamical systems. In this case the immediate source was Baker [5] with whom Frazer studied. See Pugsley [152] for a biography of Frazer. Felippa [63] identifies Frazer, Duncan, and Collar as the developers of the finite element method for structural analysis. Their study of airframe vibration, flutter, thus translated into questions about matrix eigenvalues. They wrote an influential book that explained computations for dynamics in terms of matrices. Frazer’s aerodynamics section at the National Physical Laboratory also indirectly influenced mathematical research by employing Leslie Fox and James Wilkinson [187], and Olga Taussky [170], all of whom became prominent computational mathematicians after World War II.

Frazer et al. [68, pp. 96–99] viewed elimination as “building up the reciprocal matrix in stages by elementary operations” which could produce a triangular matrix “such that its reciprocal can be found easily.” They demonstrated column elimination of a $4 \times 4$ matrix $a$ (their notation), so they had $aM_1M_2M_3 = \tau$ where the $M_i$ are “post multipliers” and $\tau$ is “the final triangular matrix.” Frazer et al. remarked that $M_1M_2M_3$ was itself a triangular matrix but “opposite-handed” from $\tau$. Jensen [109, pp. 13–15] borrowed this approach to establish the connection between Gaussian elimination and triangular factoring. He restated it in a more conventional form of row operators acting on the left but using the same notation $M_i$. Modern textbooks still use this exposition to establish the relationship between Gaussian elimination and matrix factoring, although they attribute it to neither Frazer et al. nor to Jensen. For example see the text of Golub and Van Loan [83, p. 93] using, remarkably, the same notation $M_i$ but now called a “Gauss transformation.” Frazer et al. did not continue their presentation to the modern conclusion: they neither commented on matrix factoring nor did they write out a factorization such as $a = \tau (M_1M_2M_3)^{-1}$.

11.4. Dwyer: Abbreviated Doolittle Method and Square Root Method

Paul Sumner Dwyer was a professor at the University of Michigan and a president of the Institute of Mathematical Statistics. See Figure 13. His longtime colleague Cecil Craig [39] wrote a short biography. Dwyer collaborated with an official at the United States Department of Agriculture named Frederick Waugh. The USDA practiced a type of economics, known as econometrics [64], whose research methodology consists of data analysis by, essentially, the method of least squares. Thus, the partnership with Waugh exposed Dwyer to computers with government resources, so he could expect calculations to be mechanized, which placed him at the forefront of computing practice.

Dwyer [53] began from a comparison of solution methods that emphasized sources in American and English statistics. He considered 22 papers from 1927 to 1939 and suggested their bibliographies should be consulted for an even more thorough picture of the subject. Dwyer’s review painstakingly uncovered the similarities between proliferating methods distinguished by minor changes to the placement of numbers in tables. For example, he noted a method of Deming [45] was equivalent, except for superficial changes, to the “method of pivotal condensation” of Aitken [2], both of which Dwyer included under the rubric “method of single division,” see Figure 14. The differences among these methods only seem pedestrian until one must choose the most effective way to calculate and record all the numbers in the tables by hand. Nevertheless, in another paper, Waugh and Dwyer [177] summarized the field more succinctly, observing the methods are
more or less the same, except “Crout divides the elements of each row by the leading element while we divide the elements of columns.” Dwyer credited to others, including Waugh, the observation that accumulating calculators made it unnecessary to record the series terms in Doolittle’s table B. Dwyer called the streamlined procedure the “abbreviated method of single division – symmetric” or the “abbreviated Doolittle method.”

![Paul Sumner Dwyer, 1901–19??, circa 1960s, courtesy Prof. Ingram Olkin of Stanford University.](image)

|    | $x_1$ | $x_2$ | $x_3$ | $x_4$ | r. h. s. |
|----|-------|-------|-------|-------|----------|
| 1  | 1.0000| .4000 | .5000 | .6000 | .2000    |
| 2  | .4000 | 1.0000| .3000 | .4000 | .4000    |
| 3  | .5000 | .3000 | 1.0000| .2000 | .6000    |
| 4  | .6000 | .4000 | .2000 | 1.000 | .8000    |

r. h. s. original equations

|    | $x_1$ | $x_2$ | $x_3$ | $x_4$ | r. h. s. |
|----|-------|-------|-------|-------|----------|
| 1  | 1.0000| .4000 | .5000 | .6000 | .2000    |
| 2  | .8400 | .1000 | .1600 |       | .3200    |
| 3  | .1000 | .7500 | −.1000|       | .5000    |
| 4  | .1600 | −.1000| .6400 |       | .6800    |

r. h. s. elimination

|    | $x_1$ | $x_2$ | $x_3$ | $x_4$ | r. h. s. |
|----|-------|-------|-------|-------|----------|
| 1  | 1.0000| .4000 | .5000 | .6000 | .2000    |
| 2  | .7381 | −.1190|       |       | .4619    |
| 3  | −.1190| .6995 |       |       | .6190    |
| 4  | 1.0000| −.1612|       |       | .6258    |

r. h. s. back-substitution

|    | $x_1$ | $x_2$ | $x_3$ | $x_4$ | r. h. s. |
|----|-------|-------|-------|-------|----------|
| 1  | 1.0000|       |       |       | 1.1748   |
| 2  |       | 1.0000|       |       | .8152    |
| 3  |       |       | 1.0000|       | .0602    |
| 4  |       |       |       | 1.0000| −.9366   |

Figure 14: A form of Gaussian elimination that Paul Dwyer [53] called “the method of single division” is equivalent but for cosmetic changes to the “method of pivotal condensation” of Aitken [2], and to an earlier method of Deming [45]. The nonunitary numbers in rows 5, 9, and 12 are the upper diagonal entries in Crout’s table.
Dwyer [55] independently interpreted Gaussian elimination as matrix factoring. His primary interest was the case [1] least squares problem, so beginning from the coefficient matrix $A$ of the normal equations, he showed the abbreviated Doolittle method was an “efficient way of building up” some “so called triangular” matrices $S$ and $T$ with $A - ST = 0$. Dwyer remarked that this formula was the key to “a more general theory.” For the normal equations it was possible to choose $S = T$ for a “square root” method (modern Cholesky method) which Dwyer [56] developed in a later paper. He added that he found no other matrix algebra interpretation of solving equations except Banachiewicz [9] who, Dwyer reported, also had a square root method.

Dwyer especially influenced computers in the United States. [124] reported that [51] popularized Dwyer’s square root method, and that it was even used at the Mathematical Tables Project. European mathematicians such as Fox [65], however, preferred to apply Cholesky’s name. Since Dwyer’s many papers and his book on linear equations (1951) always invoked Doolittle’s name, his own name was never attached to either of the computing methods that he championed.

11.5. Jensen (and Bodewig): Literature Surveys

Two survey papers helped establish the matrix interpretation of Gaussian elimination by describing many algorithms independent of origin in a common notation. The geodesist Henry Jensen [108, pp. 3, 19] characterized his work as extending to “matrix symbolism” the results of Banachiewicz for least squares problems and for the normal equations. Jensen [109, p. 11] found three algorithms for solving the normal equations were similar in that they could be interpreted as “reducing the matrix in question to a triangular matrix:” the “Gauss’ian algorithm,” the Cracovian method, and Cholesky’s method. To emphasize the similarities among the triangular factoring algorithms, Jensen used pictograms for triangular matrices with zeroes either under $\mathbf{\nabla}$ or over $\mathbf{\Delta}$, the main diagonal (original terminology and pictures). His primary interest was the normal equations with coefficient matrix $A^*A = N$ for a rectangular matrix $A$, where $^*$ is transposition. Jensen [109, p. 15, eqn. 15; p. 22, eqn. 3] explained that the “Gauss’ian algorithm” amounted to $N = \mathbf{\nabla} \mathbf{\nabla}$, and Cholesky’s method was $N = B'B$ where $B = \mathbf{\nabla}$.

What Jensen [109, pp. 13–16] called the “Gauss’ian algorithm” was his original synthesis of three different approaches. He began with his own row-oriented version of Frazer et al.’s transformation of a matrix to triangular form, which was explained here in section 11.3. Jensen conducted the transformation symbolically by using Gauss’s brackets as the matrix entries, thereby connecting the transformations to the work of Gauss. Although Jensen did not mention Doolittle, he recommended the numbers of calculations should “be conveniently tabulated” not in matrices but rather in Doolittle’s table B.

Jensen’s survey appears to have been the conduit through which matrix interpretations were communicated to those who popularized them in computation. The use in modern textbooks of his formulation of Frazer et al.’s transformations has been noted. Similarly, Jensen [109, p. 22] wrote that Cholesky’s method “ought to me more generally used than is the case. It is due to Cholesky [14] and was later indicated by Banachiewicz.” Brezinski and Wuytak [23, p. 18] relate how researchers at the National Physical Laboratory — notably Fox et al. [67] and [174] — learned of Cholesky’s method indirectly from Jensen’s paper through John Todd.

The mathematician Ewald Bodewig took Jensen’s approach to summarize in matrix notation all methods for solving linear equations that were known through 1947. His interesting bibliography lay at the end of his five-part paper but, like Jensen’s, it was not comprehensive because he neglected authors such as Crout, Doolittle, and Dwyer. Bodewig [18, part I, pp. 444–450] took Jensen too literally because he called triangular matrices left or right to indicate the location of the zeroes instead of the nonzeros; his “right” was $\mathbf{\nabla} = \mathbf{\Sigma}$, and “left” was $\mathbf{\nabla} = \mathbf{\Delta}$, where $\mathbf{\Delta}$ stood for Dreiecksmatrix. He repeated Jensen’s row version of Frazer et al.’s presentation of Gaussian elimination, and he emphasized their summary formula using Jensen’s pictograms, $\mathbf{\nabla} \mathbf{\Delta} = \mathbf{\nabla}$. Bodewig followed Jensen [109] in describing Cholesky’s method as $\mathbf{\Sigma} = \mathbf{\Sigma}' \mathbf{\Sigma}$, for a symmetric $\mathbf{\Sigma}$, where $'$ is transpose. He remarked Cholesky essentially (?) showed $\mathbf{\Delta} = \mathbf{\Delta}' \mathbf{\Delta}$ for any matrix $\mathbf{\Delta}$, and that (in light of $\mathbf{\nabla} \mathbf{\Delta} = \mathbf{\nabla}$) the method in this form was effectively Gaussian elimination.

Footnote 23: The other, non-triangular methods that Jensen discussed were: solution by determinants, the method of equal coefficients (Gauss-Jordan elimination, which he correctly attributes to B. I. Clasen in 1888), Boltz’s method, and Kruger’s method.
11.6. Von Neumann and Goldstine: The Combination of Two Tricks

John von Neumann (1947) and his collaborator Herman Goldstine were alone, among the first authors to describe Gaussian elimination in terms of matrix algebra, to make a nontrivial use of the relationship. They established bounds on the errors of matrix inverses, calculated by a computer using a method related to Gaussian elimination, in terms of the ratio of the largest to the smallest singular values of the coefficient matrix. That result is beyond the scope of the present discussion because it marks the beginning of modern research interests in computational mathematics. Von Neumann and Goldstone’s ratio is now called the matrix condition number.

They also were the only authors to show exactly how the “traditional schoolbook method” of equation \[ A = A^{(1)}, A^{(2)}, A^{(3)}, \ldots, A^{(n)} \]

where the rows and columns of \( A \) are numbered from \( i \) to \( n \). For \( i = 1, \ldots, n \) the computation is,

\[
A_{j,k}^{(i+1)} = A_{j,k}^{(i)} - A_{j,i}^{(i)} A_{i,k}^{(i)} / A_{i,i}^{(i)} \quad \text{for } j, k > i
\]

(17)

\[
y_j^{(i+1)} = y_j^{(i)} - (A_{j,i}^{(i)} / A_{i,i}^{(i)}) y_i^{(i)} \quad \text{for } j > i.
\]

(18)

Next, the algorithm solves by substitution the equations \( B'x = z \) where the entries of \( B' \) and \( z \) are chosen from the reduced matrices and vectors (the first row of each). For a matrix \( C \) of the multipliers \( A_{j,i}^{(i)} / A_{i,i}^{(i)} \) with \( j \geq i \) (note the unit diagonal), von Neumann and Goldstine summed equation \[ (18) \]

over \( i \) and rearranged to give \( Cz = y \). From this equation and \( B'x = z \) they concluded \( CB' = A \).

We may therefore interpret the elimination method as … the combination of two tricks: First, it decomposes \( A \) into a product of two semi-diagonal matrices … [and second] it forms their inverses by a simple, explicit, inductive process.

— von Neumann and Goldstine [141] p. 1053

Note von Neumann and Goldstine wrote “semi-diagonal” for “triangular.”

Von Neumann and Goldstine found a lack of symmetry in the elimination algorithm because the first factor always had 1’s on its main diagonal. They divided the second factor by its diagonal to obtain \( B' = DB \), hence \( A = CDB \) which they said was “a new variant” of Gaussian elimination (op. cit., p. 1031) that is now written \( A = LDU \),

\[
L_{j,i} = A_{j,i}^{(i)} / A_{i,i}^{(i)} \quad D_{i,i} = A_{i,i}^{(i)} \quad U_{i,k} = A_{i,k}^{(i)} / A_{i,i}^{(i)} \quad j, k \geq i
\]

(19)

where \( A_{j,i}^{(i)} \) are the entries of the reduced matrices given by equation \[ (17) \].

What accounts in part for the multiple inventors of elimination algorithms is the triangular matrix decompositions are not unique. The choice of unnormalized factors is limited only by the requirement

\[
L_{j,i} D_{i,i} U_{i,k} = A_{j,k}^{(i)} A_{i,k}^{(i)} A_{i,i}^{(i)}.
\]

(20)

The factorizations \( A = L(DU), (D^{1/2}U)'(D^{1/2}U) \), and \( (LD)U \) that differently apportion the diagonal are known today by the names Doolittle, Cholesky, and Crout, respectively.

12. Gaussian Attribution in Summary

In summary of the primary and secondary sources, an algorithm functionally equivalent to Gaussian elimination appeared in mathematical texts from ancient China. That algorithm likely did not influence the invention of symbolic algebra in Europe, so presumably Gaussian elimination developed there independently. A few, exemplary systems of linear equations were solved in textbooks written through 1660, sometimes exhibiting and sometimes without the rote elimination of variables that distinguishes Gaussian elimination, and without discussing a general approach. Newton [142], writing circa 1670, described the successive elimination of variables as a rule for solving any simultaneous equations, and he noted this explanation was
lacking in contemporary textbooks. The 18th century sees Newton’s elimination rule repeated in algorithmic form by Hammond [91], who emphasized its application to linear equations. Euler [58] specifically addressed linear equations, and although he noted the possibility of following a fixed procedure to solve them, he recommended against it. At the very beginning of the next century, the influential textbook author Lacroix [121] described the algorithm in a manner very similar to Hammond, and further, recommends it as a general method for solving any simultaneous linear equations.

Gauss cannot be the source for Gaussian elimination in western mathematics because he was born after Newton’s and Hammond’s publications, and he did not write on the subject until after Lacroix’s publication. Moreover the canonical version, equation (1), did not appear in papers where Gauss himself discussed elimination. Thus the purported Gaussian history, or heritage, and the pedagogically compelling Gaussian appellation are simply wrong. If Gauss did not arrive at the elimination process on his own (Euler remarked it is the most natural way of proceeding), then as Farebrother [61] alludes, Gauss learnt Gaussian elimination from readily available textbooks.

Nevertheless, the use of Gauss’s special notation to solve the least squares normal equations, and his stature among astronomers and geodesists, indelibly linked his name to calculations. Whereas in the first half of the 19th century algebra textbooks referred only to elimination, from the second half of the 19th century reference books for astronomy and geodesy always cited Gauss to recommend their least squares calculations. His notation at first was regarded as particularly for the normal equations. For example, Chauvenet [31, p. 530] had the “elimination of unknown quantities from the normal equations . . . according to Gauss,” and Liagre [128, p. 557] “l’élimination des inconnues entre les équations du minimum (équations normales)” by “les coefficients auxiliaires de Gauss.” Astronomy and geodesy at one time accounted for the bulk of the linear equations solved professionally, so their terminology may have been considered authoritative. Such citations were shortened eventually to an unspecific “Gauss’s procedure.” Mathematicians misinterpreted this usage as attributing ordinary, common elimination to Gauss, but only long after World War II, see Table 1. Von Neumann (1947) apparently was the last prominent mathematician simply to write elimination as Lacroix [121] and Gauss [71] had, whereas today we mistakenly write Gaussian elimination.
Table 1: Nomenclature for solving simultaneous linear equations by elimination indicating the gradual evolution to “Gaussian elimination” over many years.

| YEAR | AUTHOR | USES GAUSS’S BRACKETS | NOMENCLATURE |
|------|--------|------------------------|--------------|
| 1728 | Newton | Transformation          | of two or more Equations into one … |
| 1752 | Hammond| The Method of resolving | the unknown Quantities |
| 1771 | Euler  | Der natürlichste Weg    | Questions, which contain four Equations, … |
| 1804 | Lacroix| elimination             | bestehebt nun darinn, … |
| 1805 | Legendre| par les méthodes        | ordinaires |
| 1809 | Gauss  | per eliminationem       | vulgarem (common elimination) |
| 1818 | Lacroix| elimination             | |
| 1822 | Euler  | the most natural method | of proceeding |
| 1831 | Ross   | continue this series of operations until a single equation … |
| 1835 | Davies | quantities may be eliminated | by the following rule |
| 1844 | Clark  | method of elimination   | when there are three or more … |
| 1868 | Chauvenet | ✓ method of substitution | according to Gauss |
| 1879 | Liagre | ✓ pour procéder à l’élaboration | … coefficients auxiliaires de Gauss |
| 1884 | Merriman | ✓ the method of substitution | due to Gauss |
| 1888 | Doolittle (not the Doolittle) | ✓ elimination | [by] the method of substitution, using |
| 1895 | Jordan | ✓ Gauss sche Elimination | a form of notation proposed by Gauss |
| 1896 | Bartlett | ✓ method of substitution | proposed by Gauss |
| 1905 | Johnson | ✓ method of substitution | as developed by Gauss |
| 1906 | Wright and Hayford | ✓ method of substitution | introduced by Gauss and the |
| 1907 | Helmert | ✓ Algorithmus von C. F. Gauss | Doolittle method |
| 1912 | Palmer | ✓ Gauss’s method for the solution of normal equations |
| 1924 | Benoit | ✓ les méthodes ordinaires y compris celle de Gauss |
| 1927 | Tolley and Enockel | ✓ the direct method of solution developed by Gauss |
| 1941 | Croft | ✓ Gauss’s method | |
| 1943 | Dwyer | ✓ mentions notation | … suggested by Gauss |
| 1944 | Jensen | ✓ Doolittle method | |
| 1947 | Bodewig | ✓ Gauss’s algorithm | |
| 1947 | von Neumann and Goldstine | ✓ Gaussian Verfahren | |
| 1948 | Fox, Huskey, and Wilkinson | ✓ Gaussian algorithm | |
| 1948 | Turing | ✓ Gauss elimination process | |
| 1953 | Householder | ✓ methods of elimination | |
| 1990 | Wilkinson | ✓ Gaussian elimination | |
| 1964 | Householder | ✓ Gaussian elimination | |
| 1977 | Goldstine | ✓ Gaussian elimination | |
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