The Uses of Chiral Anomaly
for Determination of the Number of Colors

R. N. Rogalyov

State Research Center of Russia
"Institute for High-Energy Physics", Protvino, Russia

Abstract

The \( N_c \)-dependence of the vertices \( \bar{P} P P \gamma \), where \( P \) is a pseudoscalar meson and \( N_c \) is the number of colors, is analyzed with regard for the \( N_c \)-dependence of the quark charges. It is shown that the best processes for the determination of \( N_c \) are the reactions \( K \gamma \rightarrow K \pi \) and \( \pi^\pm \gamma \rightarrow \pi^\pm \eta \) as well as the decay \( \eta \rightarrow \pi^+ \pi^- \gamma \). The measurement of the cross section \( \sigma(\pi^\pm \gamma \rightarrow \pi^\pm \eta) \) at the VES facility at the IHEP agrees with the value \( N_c = 3 \).

The chiral anomaly \( [1] \) is a fundamental property of quantum field theories with chiral fermions such as the Standard Model (SM). The chiral anomaly offers a quantum-mechanical violation of a classical symmetry at small distances (electroweak scale) such that its manifestations at large distances (hadronic scale) are unambiguously determined. This property distinguishes the chiral anomaly from the other predictions of the SM; it makes the only effect of quark–lepton interactions at small distances which can be described in terms of hadronic fields without introducing an additional phenomenological parameter. For this reason, an experimental study of the chiral anomaly would be a test of the theoretical foundations of the elementary particle physics.

Phenomenological implications of the chiral anomaly are accounted for by the Wess–Zumino–Witten (WZW) functional \( [2, 3] \)

\[
S[U, \ell, r]^{(N_c=3)}_{\text{WZW}} = -\frac{iN_c}{48\pi^2} \int d^4x \varepsilon^{\mu
u\alpha\beta} \left( U\ell_\mu U^\dagger \ell_\alpha U^\dagger r_\beta + \frac{1}{4} U\ell_\mu U^\dagger r_\nu U\ell_\alpha U^\dagger r_\beta + \right. \\
+ i\partial_\mu \ell_\nu \ell_\alpha U^\dagger r_\beta + i\partial_\mu r_\nu \ell_\alpha U^\dagger r_\beta - i\Sigma^L_\mu \ell_\nu U\ell_\alpha U^\dagger r_\beta + \Sigma^L_\mu U^\dagger \partial_\nu r_\alpha U^\dagger r_\beta - \\
\left. -\Sigma^L_\mu \Sigma^L_\nu U^\dagger r_\alpha U\ell_\nu + \Sigma^L_\mu \ell_\nu \partial_\alpha r_\beta + \Sigma^L_\mu \partial_\nu \ell_\alpha r_\beta - i\Sigma^L_\mu \ell_\nu \ell_\alpha r_\beta + \\
+ \frac{1}{2} \Sigma^L_\mu \Sigma^L_\nu \ell_\beta - i\Sigma^L_\mu \Sigma^L_\nu \Sigma^L_\alpha \ell_\beta \right) - (L \leftrightarrow R),
\]

which should be added to the Lagrangian of the chiral perturbation theory. Here \( N_c \) is the number of colors (\( N_c=3 \)); the brackets \( \langle \ldots \rangle \) denote trace over flavor indices;

\[
\Sigma^L_\mu = U^\dagger \partial_\mu U; \quad \Sigma^R_\mu = U \partial_\mu U^\dagger; \quad U = \exp \left( i\Phi \sqrt{2}/F \right); \\
r_\mu = \ell_\mu = eA_\mu Q = A_\mu \text{ diag} \left( \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3} \right);
\]

\( ^* \)E-mail: rogalyov@mx.ihep.su

1) In formula (1) it is assumed that the charge matrix is given by the expression (3) for all \( N_c \); however, it is shown below that this assumption is untrue. For this reason, the expression (1) is valid only for \( N_c = 3 \).
$A_{\mu}$ is the electromagnetic field; $F=93$ MeV; symbol $(L \leftrightarrow R)$ denotes the substitutions $U \leftrightarrow U^\dagger$, $\ell_\mu \leftrightarrow r_\mu$ and $\Sigma^L_\mu \leftrightarrow \Sigma^R_\mu$; and

$$
\Phi = \left( \begin{array}{ccc}
\pi^0 \sqrt{2} & \pi^+ \sqrt{6} & K^+ \\
\pi^- \sqrt{2} & -\pi^0 \sqrt{6} & \bar{K}_0 \\
K^- & 2\eta^0 \sqrt{6} & \eta^0 \sqrt{3} \\
\end{array} \right).
$$

The functional (I) determines low-energy behavior of the amplitudes of the reactions $\pi^0 \to \gamma \gamma$, $\eta \to \gamma \gamma$, $\eta \to \pi^+ \pi^- \gamma$, $\pi^+ \gamma \to \pi^+ \pi^0$, $\pi^+ \gamma \to \pi^+ \eta$, $K^+ \gamma \to K^+ \pi^0$ etc. Some of these reactions ($\pi^0 \to \gamma \gamma$, $\eta \to \gamma \gamma$, $\eta \to \pi^+ \pi^- \gamma$, and $\pi^+ \gamma \to \pi^+ \pi^0$) were used for the determination of the number of colors $N_c$.

However, a recent analysis [3] of the vertices $PPP\gamma$ and $P\gamma\gamma$ ($P$ is a pseudoscalar meson) has revealed that, in a self-consistent theory, the vertices $\pi^0 \gamma \gamma$ and $\pi^0 \pi^+ \pi^- \gamma$ are independent of $N_c$, in spite of the fact that, in any textbook on elementary particle physics (see, for example, [3]), the width of the decay $\pi^0 \to \gamma \gamma$ is said to be proportional to $N_c^2$ and thus the width $\Gamma(\pi^0 \to \gamma \gamma)$ is considered to be an important source of experimental information on the value of $N_c$. The point is that the statement on the dependence of the amplitudes $A_{\pi^+ \gamma \to \pi^+ \pi^0}$ and $A_{\pi^+ \gamma \to \gamma}$ on $N_c$ stems from an implicit (and faulty) assumption that the quark charges $Q_u = 2/3, Q_d = -1/3, Q_s = -1/3$ are independent of $N_c$. If this assumption were true, the triangle anomalies in the quark sector do not cancel those in the lepton sector and thus the SM is not renormalizable. Assuming renormalizability of the SM for all $N_c$, we obtain the relations between $N_c$ and the quark charges

$$
Q_u = \frac{1}{2} \left( \frac{1}{N_c} + 1 \right), \quad Q_d = \frac{1}{2} \left( \frac{1}{N_c} - 1 \right).
$$

(4)

Using these relations as the base one can show that the amplitudes of the reactions $\pi^0 \to \gamma \gamma$, $\pi^+ \gamma \to \pi^+ \pi^0$, and $\eta \to \gamma \gamma$ are independent of $N_c$. The anomalous vertices $\gamma \pi^0 \pi^+ \pi^-$ and $\gamma \eta \pi^+ \pi^-$ have been studied theoretically (in the case $N_c = 3$) in [1] and experimentally in the processes of Coulomb production of $\pi^0$ [7] and $\eta$ [8] mesons on nuclei at the IHEP. It should be mentioned that the motivation of the experiment [1] (the measurement of the cross section $\sigma(\pi^+ \gamma \to \pi^+ \pi^0)$) was to determine the number of colors; however, according to the above, the number of colors could not be determined in this experiment. Therewith, the data obtained in the experiment [8] can well be used for a determination of $N_c$. As for now, the only vertex involving light mesons used for a determination of the number of colors $N_c$ is $\eta \pi^+ \pi^- \gamma$. It has been studied in the decay $\eta \to \pi^+ \pi^- \gamma$ [9] and in the scattering process $\pi^+ \gamma \to \pi^+ \eta$ [8]. The expression for this vertex for an arbitrary value of $N_c$ is presented below. With regard for this expression, the mentioned experiments give evidence for the value $N_c = 3$.

The present study as well as [4] does not cast doubt on the total of the experimental data used for the determination of the parameter $N_c$; we pursue rather unpretentious goals. Bär and Wiese [4] suggest that the decay $\eta \to \pi^+ \pi^- \gamma$ "should replace the textbook example $\pi^0 \to \gamma \gamma$ for lending experimental support to the fact that there are three colors in our world.” In the present study, we consider the $N_c$-dependence of the cross sections of the reactions $K\gamma \to K\pi$ and $K\gamma \to K\eta$, which can also be used for the determination of $N_c$. A detailed study of the vertices $KK\pi\gamma$ and $KK\eta\gamma$ is needed because, in spite of a sophisticated and comprehensive
analysis of the $P\gamma\gamma$ and $PPP\gamma$ vertices performed in [1], the ultimate expressions for these vertices (formula (5.11) of the cited paper) are in error. For instance, the mentioned formula in the case $N_c = 3$ disagree with that obtained from the WZW Lagrangian [1].

The effective Lagrangian for the vertices $PPP\gamma$ can be calculated by either of two ways:

1. By a straightforward computation of the group-theoretical coefficients of the quark diagrams contributing to the respective Green function with antisymmetrization over axial currents and regard for the relations (4). Strictly speaking, this procedure yields only the ratios between the coefficients of the same type; the known vertices $\pi^0\gamma\gamma$ and $\pi^+\pi^-\pi^0\gamma$ can be used as the reference values.

2. By a substitution of the explicit expression for the matrix $U$ (formula (2)) in the expression for the WZW Lagrangian (1), generalized to the case $N_c \neq 3$. Such generalization was proposed in [4]. It has the form

$$S = S^{(N_c=3)} + \left( 1 - \frac{N_c}{3} \right) S_{GW},$$

where $S_{GW}$ is the Goldstone–Wilczek current [10]:

$$S_{GW}[U, A_\mu] = \frac{e}{48\pi^2} \int d^4 x \, \epsilon^{\mu\nu\alpha\beta} A_\mu \text{Tr}[(U^\dagger \partial_\nu U)(U^\dagger \partial_\alpha U)(U^\dagger \partial_\beta U)]$$

$$- \frac{i e^2}{32\pi^2} \int d^4 x \, \epsilon^{\mu\nu\alpha\beta} A_\mu F_{\nu\alpha} \text{Tr}[Q(\partial_\beta UU^\dagger + U^\dagger \partial_\alpha U)];$$

when considering the $PPP\gamma$ vertices, the second row in formula (6) can be omitted.

The results of the calculations by these two ways coincide; the vertices sought for have the form

$$\mathcal{L}_{WZW}^{PPP\gamma} = \frac{ie}{4\pi^2 F^3} \epsilon^{\mu\nu\alpha\beta} A_\beta \left( \partial_\mu \pi^0 \partial_\nu \pi^+ \partial_\alpha \pi^- + \right.$$  

$$+ \frac{N_c}{3\sqrt{3}} \partial_\mu \eta^8 \partial_\nu \pi^+ \partial_\alpha \pi^- +$$

$$+ \frac{N_c + 3}{6} \partial_\mu \pi^0 \partial_\nu K^+ \partial_\alpha K^- +$$

$$+ \frac{N_c - 1}{2} \partial_\mu \pi^0 \partial_\nu K^0 \partial_\alpha \bar{K}^0 +$$

$$+ \frac{9 - N_c}{6\sqrt{3}} \partial_\mu \eta^8 \partial_\nu K^+ \partial_\alpha K^- +$$

$$- \frac{\sqrt{3}(N_c - 1)}{2} \partial_\mu \eta^8 \partial_\nu K^0 \partial_\alpha \bar{K}^0 -$$

$$- \frac{N_c - 3}{3\sqrt{2}} \partial_\mu \pi^- \partial_\nu K^+ \partial_\alpha \bar{K}^0 +$$

$$+ \frac{N_c - 3}{3\sqrt{2}} \partial_\mu \pi^+ \partial_\nu K^- \partial_\alpha K^0 +$$

$$+ \frac{\sqrt{6}}{9} \partial_\mu \eta^0 \partial_\nu K^+ \partial_\alpha K^- +$$

$$+ \frac{N_c \sqrt{6}}{9} \partial_\mu \eta^0 \partial_\nu \pi^+ \partial_\alpha \pi^- \right),$$
where \( \eta^0 \) and \( \eta^8 \) are the singlet and octet states:

\[
\eta = \eta^8 \cos \theta_P - \eta^0 \sin \theta_P, \quad \eta' = \eta^8 \sin \theta_P + \eta^0 \cos \theta_P, \quad \theta_P \simeq 20^\circ.
\] (8)

It should be noticed that the vertices \( K^+ K^0 \pi^- \gamma \) and \( K^- \bar{K}^0 \pi^+ \gamma \) do not appear in the anomalous action\(^2\) only in the case \( N_c = 3 \). For this reason, the near-threshold behavior of the reactions \( K^+ \gamma \rightarrow K^+ \pi^0 \) and \( K^0 \gamma \rightarrow K^+ \pi^- \) offer a good indicator of a deviation of the parameter \( N_c \) from the value \( N_c = 3 \). To put it differently, the chiral anomaly gives a contribution to the amplitudes of the reactions \( K^+ \gamma \rightarrow K^+ \pi^0 \) and \( K^0 \gamma \rightarrow K^+ \pi^0 \) and gives no contribution to the amplitudes of the reactions \( K^+ \gamma \rightarrow K^0 \pi^+ \) and \( K^0 \gamma \rightarrow K^+ \pi^- \) only at \( N_c = 3 \). As a result of this, the cross sections of the reactions \( K^+ \gamma \rightarrow K^+ \pi^0 \) and \( K^0 \gamma \rightarrow K^+ \pi^0 \) over the near-threshold region far exceed the cross sections of the reactions \( K^+ \gamma \rightarrow K^0 \pi^+ \) and \( K^0 \gamma \rightarrow K^+ \pi^- \). These cross sections were calculated (in the case \( N_c = 3 \)) in [11], where a possibility of an experimental study of these cross sections is discussed. A measurement of the cross sections of the reactions \( K^+ \gamma \rightarrow K^+ \pi^0 \), \( K^0 \gamma \rightarrow K^0 \pi^0 \), \( K^+ \gamma \rightarrow K^0 \pi^+ \), and \( K^0 \gamma \rightarrow K^+ \pi^- \) is of particular interest due to their dependence on \( N_c \), presented in formula (4).

However, the formulas (7) give an adequate description of the amplitudes only at sufficiently small momenta. To describe the reactions \( K \gamma \rightarrow K \pi, \pi^\pm \gamma \rightarrow \pi^\pm \eta \) and the decay \( \eta \rightarrow \pi^+ \pi^- \gamma \) at the physical values of momenta, one should take into account the contribution of the \( 1^- \) resonances, which can be calculated in the vector-meson dominance model. In what follows, we use the version of the vector-meson dominance model proposed in [12], because (i) in the chiral limit, it goes over into the chiral perturbation theory; (ii) the formalism proposed in [12] is well suited for taking the chiral anomaly into consideration. The Lagrangian of this model and its application to the above-mentioned processes can be found in [11, 12, 13]; here we only point to its dependence on \( N_c \). All \( N_c \)-dependence in the normal part of this Lagrangian is absorbed in the effective coupling \( g \). The anomalous terms should be multiplied by \( N_c/3 \) in order to obtain the WZW Lagrangian in the vicinity of the chiral limit; the quark charges are considered to be the functions of \( N_c \), according to (4). The analysis of the reactions \( \pi^\pm \gamma \rightarrow \pi^\pm \eta \) and the decay \( \eta \rightarrow \pi^+ \pi^- \gamma \) in the vector-meson dominance model was performed in [8]. From the expressions presented in [8] it follows that both diagrams for these processes are proportional to \( N_c(C_u - Q_d) = N_c \) and the dependence of the amplitudes on \( N_c \) is readily determined.

The reactions \( K \gamma \rightarrow K \pi \) present a challenge: a straightforward computation of the amplitudes is needed. The respective Feynman diagrams are presented in the figure.

The calculations were performed with the FORM package [14], the result has the form

\[
A_{K \gamma \rightarrow K \pi} = \frac{-ie}{16\pi^2 F^3} e^{\mu \alpha \beta} d^\mu \nu \rho_2^\alpha \epsilon_\beta \left( C_0 + \frac{C_s M_{K^*}^2}{s - M_{K^*}^2 + i \Gamma_{K^*} \sqrt{s}} + \frac{C_t M_{\rho}^2}{t - M_{\rho}^2 + \frac{C_u M_{K^*}^2}{u - M_{K^*}^2}} \right),
\] (9)

where \( \epsilon \) is the photon polarization vector; the momenta \( q, p_b, \) and \( p_2 \) are defined in the table; \( s = (q + p_b)^2, \ t = (p_b - p_2)^2, \ u = (q - p_2)^2 \); \( M_{K^*} \) and \( M_\rho \) are the masses of the \( K^* \) and \( \rho \)

\(^2\) The vertices \( K^+ K^0 \pi^- \gamma \) and \( K^- \bar{K}^0 \pi^+ \gamma \) are not presented in [4].
Figure 1: Diagrams contributing to the reactions $K\gamma \rightarrow K\pi$. 
mesons; $\Gamma_{K^*}$ is the width of the $K^*$ meson. The coefficients $C_0$, $C_s$, $C_t$, and $C_u$ for specific processes are presented in the table as the functions of $N_c$.

**Table.** Coefficients $C_s$, $C_t$, $C_u$, and $C_0$ from formula (9).

| Reaction | $C_0$ | $C_s$ | $C_t$ | $C_u$ |
|----------|-------|-------|-------|-------|
| $K^+(p_b)\gamma(q) \rightarrow K^+(p_2)\pi^0(p_1)$ | $\frac{N_c + 3}{3}$ | 1 | $N_c + 1$ | 1 |
| $K^+(p_b)\gamma(q) \rightarrow K^0(p_2)\pi^+(p_1)$ | $\sqrt{2}(N_c - 3)$ | $\sqrt{2}$ | $\sqrt{2}(N_c - 2)$ | $-2\sqrt{2}$ |
| $\pi^+(p_b)\gamma(q) \rightarrow \pi^+(p_2)\eta(p_1)$ | $\frac{2N_c\sqrt{3}}{9} P_0$ | 0 | $\frac{2N_c\sqrt{3}}{3} P_0$ | 0 |

**Note.** Taking into account the $\eta - \eta'$ mixing (8) gives rise to the factor $P_0 = \frac{F}{F_8} \cos \theta_P - F_0 \sin \theta_P$, where the difference between the decay constants $F_{\pi^\pm} = F$, $F_0$, and $F_8$ must also be taken into consideration. The values $F_0$ ($F_8$) parametrize the matrix element of the axial current between vacuum and purely singlet (octet) state. Their magnitudes were computed in the one-loop approximation of the chiral perturbation theory and are equal to $F_3$: $F_0 \approx 1.04F$, $F_8 \approx 1.30F$; the mixing angle is $\theta_P = 20^\circ$. All this makes theoretical predictions for the decays $\eta \rightarrow \pi^+\pi^-\gamma$ and $\eta' \rightarrow \pi^+\pi^-\gamma$ and the reactions $\pi^+\gamma \rightarrow \pi^+\eta(\eta')$ less accurate than those for the reactions $K\gamma \rightarrow K\pi$.

Thus a measurement of the cross sections of the reactions $K^{+}\gamma \rightarrow K^{+}\pi^0$ and $K^{+}\gamma \rightarrow K^{0}\pi^+$ would compensate for a deficiency in experimental facts giving evidence for $N_c = 3$. This deficiency is due to an exclusion of the decays $\pi^0 \rightarrow \gamma\gamma$ and $\eta \rightarrow \gamma\gamma$ and the reaction $\pi^+\gamma \rightarrow \pi^+\pi^0$ from the total of the experimental data used for a determination of $N_c$. Moreover, a measurement of the cross sections of the reactions $K\gamma \rightarrow K\pi$ would make it possible to check the phenomenological implications of the chiral anomaly in the world with three (rather than two) light quarks. It is important because the WZW Lagrangian was derived under the assumption that there are just three light quarks.

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