LINK CRITERION FOR LIPSCHITZ NORMAL EMBEDDING OF DEFINABLE SETS

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Abstract. We give a link criterion for normal embedding of definable sets in o-minimal structures. Namely, we prove that given a definable germ \((X, 0) \subset (\mathbb{R}^n, 0)\) connected and a continuous definable function \(\rho : (X, 0) \to \mathbb{R}_{\geq 0}\) such that \(\rho(x) \sim \|x\|\), then \((X, 0)\) is Lipschitz normally embedded (LNE) if and only if \((X, 0)\) is link Lipschitz normally embedded (LLNE) with respect to \(\rho\) (i.e., for \(r > 0\) small enough, \(X \cap \rho^{-1}(r)\) is Lipschitz normally embedded and its LNE constant is bounded by a constant \(C\) independent of \(r\)). This is a generalization of Mendes–Sampaio’s result for the subanalytic case. As an application, we give a counterexample to a question on the relation between Lipschitz normal embedding and MD Homology asked by Bobadilla et al in their paper about Moderately Discontinuous Homology.

1. Introduction

Given a connected definable set \(X \subset \mathbb{R}^n\), one can equip \(X\) with two natural metrics: the outer metric \((d_{\text{out}})\) induced by the Euclidean metric of the ambient space \(\mathbb{R}^n\) and the inner metric \((d_{\text{inn}})\) where the distance between two points in \(X\) is defined as the infimum of the lengths of rectifiable curves connecting these points. We call \(X\) Lipschitz normally embedded (LNE) if these two metrics are equivalent, i.e., there is a constant \(C > 0\) such that

\[
d_{\text{inn}}(x, y) \leq Cd_{\text{out}}(x, y), \forall x, y \in X.
\]

Any such \(C\) is referred to as a LNE constant for \(X\). We say that \(X\) is LNE at a point \(x_0 \in X\) (or the germ \((X, x_0)\) is LNE) if there is a neighbourhood \(U\) of \(x_0\) in \(\mathbb{R}^n\) such that \(X \cap U\) is LNE.

The notion of Normal Embedding first appeared in a paper of Birbrair and Mostowski [4], it has since been an active research area and many interesting results were proved in [2], [3], [4], [7], [8], [9], [14], [15]. Most of these results concern with necessary conditions for Lipschitz Normal Embeddings. Birbrair and Mendes [2] prove that Lipschitz normal embedding of a semialgebraic germ \((X, 0)\) is equivalent to the condition that for any pair of arcs parametrized by the distance to the origin, the inner and outer contacts are the same. Recently, Mendes and Sampaio [13] gave a nice criterion for a germ of subanalytic set to be LNE based on the LNE condition on the link. Namely, they prove that

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Theorem 1.1 ([13], Theorem 1.1). A subanalytic germ \((X, 0)\) in \(\mathbb{R}^n\) with \((X \setminus \{0\}, 0)\) connected is LNE if and only if \((X, 0)\) is LLNE.

Let us recall the definition of LLNE (link Lipschitz normally embedded). For \(r > 0\), the \(r\)-link of \(X\) is the set \(L_{X,r} := X \cap S^{n-1}_r\) where \(S^{n-1}_r\) denotes the \((n-1)\)-dimensional sphere the radius \(\varepsilon\) and centered at the origin. \((X, 0)\) is called LLNE if there are \(r_0 > 0\) and \(C > 0\) such that for all \(0 < r < r_0\), \(L_{X,r}\) are LNE with LNE constants bounded by \(C\). We call such constant \(C\) a LLNE constant for \((X, 0)\).

The arguments in Mendes–Sampaio's proof rely on Birbrair–Mendes's result and especially on a powerful result due to Valette [20] which states as follows:

Theorem 1.2 ([18], Theorem 4.4.8; [20], Corollary 2.2). Let \((X, 0)\) be a subanalytic germ in \(\mathbb{R}^n\). Let \(\rho : X \to \mathbb{R}\) be a Lipschitz subanalytic function such that \(\rho(x) \sim \|x\|\). Then there is a bi-Lipschitz subanalytic homeomorphism \(h : (X, 0) \to (X, 0)\) such that \(\|h(x)\| = \rho(x)\).

The above theorem implies that, up to a bi-Lipschitz subanalytic homeomorphism, the link of \(X\) can be defined as a fiber of a Lipschitz subanalytic function “preserving” the distance to the origin. By similar arguments one can check that Mendes–Sampaio's result holds true for definable sets in polynomially bounded o-minimal structures. It is because both Birbrair–Mendes' and Valette's results still work for this setting. However, there is a big obstruction with non-polynomially bounded o-minimal structures. The notion of the order between two definable arcs as in the result of Birbrair–Mendes does not make a good sense, furthermore the result of Valette is no longer valid. It is easy to find examples of a definable set in an exponential o-minimal structure whose \(r\)-Link is of different bi-Lipschitz type when \(r\) changes. It raises a question whether Theorem 1.1 is true in any o-minimal structure. Surprisingly, the answer is positive. In fact, in Section 2 we prove a more general statement:

Theorem 1.3. Let \((X, 0)\) be a definable germ in \(\mathbb{R}^n\). Let \(\rho : X \to \mathbb{R}\) be a continuous definable function such that \(\rho(x) \sim \|x\|\). Suppose that \((X \setminus \{0\}, 0)\) is connected. Then the following statements are equivalent:

(i) \((X, 0)\) is LNE.

(ii) There are \(r_0 > 0\) and \(C > 0\) such that for all \(0 < r < r_0\), \(\rho^{-1}(r) \cap X\) are LNE with LNE constants bounded by \(C\).

If a germ \((X, 0)\) satisfies the condition (ii) then we call \((X, 0)\) LLNE with respect to \(\rho\). Our idea to prove Theorem 1.3 is as follows. First by a continuous definable extension we can assume that \(\rho\) is well defined on the whole \(\mathbb{R}^n\). We show that the map \(\varphi(x) := \frac{\rho(x)}{\|x\|_\infty} x\) is a germ of a bi-Lipschitz homeomorphism. Note that \(\varphi\) takes \(X \cap \rho^{-1}(r)\) to \(\varphi(X) \cap \rho'^{-1}(r)\) where \(\rho'(x) := \|x\|_\infty\) (Lemma 2.5). This reduces our problem to proving the theorem for a germ \((X, 0)\) and \(\rho = \|x\|_\infty\). To do this, we need Lemma 2.3 which tells us the fastest way to move from a point in \(X \cap \rho^{-1}(r)\) to \(X \cap \rho^{-1}(r')\) for \(r \neq r'\). Indeed this lemma is enough to prove that (ii) implies (i). For the converse, we have to separate \((X, 0)\) into finitely many \(L\)-cells which are proved to be LLNE with respect to \(\rho\) (see Lemma 2.6). The LLNE
property of \(L\)-cells allows us to show that the length of the shortest path in \(X\) connecting two points in \(X \cap \rho^{-1}(r)\) is equivalent to the length of a path in \(X \cap \rho^{-1}(r)\) connecting these points. This is enough for us to prove that (i) implies (ii).

In [5] Bobadilla–Heinze–Pereira–Sampaio introduced a homology called Moderately Discontinuous homology (MD homology) in order to capture the singular homology of the link of given subanalytic germ \((X,0)\) after collapsing with a certain speed. The identity map \(Id: (X,0, d_{inn}) \to (X,0, d_{out})\) induces homomorphisms between groups of MD-homologies of \((X,x_0)\) for all \(x_0 \in (X,0)\). It is easy to check that if \((X,0)\) is LNE then these homomorphisms are actually isomorphisms. It is asked if the converse holds, i.e., suppose these homomorphisms are isomorphisms then is it true that \((X,0)\) is LNE. In Section 3, as an application of Theorem 1.3, we give an example showing that in general the answer is negative.

Throughout the paper, we assume that the reader is familiar with the notion of \(o\)-minimal structures on \(\mathbb{R}\). By “definable” we mean definable in the a given \(o\)-minimal structure. We will use Curve Selection ([6], Theorem 3.2), Definable Choice ([6], Theorem 3.1) without reciting the references. We refer the reader to [6], [21] for more details about the theory of \(o\)-minimal structures.

We denote by \(B^n_r, S^{n-1}_r\) respectively the \(n\)-dimensional closed ball and \((n-1)\)-dimensional sphere in \(\mathbb{R}^n\) of radius \(r\) centered at 0. Given \(X \subset \mathbb{R}^n\), we denote by \(\overline{X}\) the closure of \(X\) in \(\mathbb{R}^n\). Given non-negative functions \(f, g: X \to \mathbb{R}\), we write \(f \preceq g\) (or \(g \succeq f\)) if there is \(C > 0\) such that \(f(x) \leq Cg(x), \forall x \in X\), and write \(f \sim g\) if \(f \preceq g\) and \(g \preceq f\). For two function germs \(f, g: (X,0) \to \mathbb{R}\), we write \(f \ll g\) (or \(f = o(g)\)) if \(\lim_{x \to 0} f(x)/g(x) \to 0\) where \(g(x) \neq 0\). \(|.|\) denotes the Euclidean norm and \(||.|\|\) denotes the maximum norm. Given a definable set \(X\), sometimes instead of using the notion \(d_{inn}\) we use \(d_X\) for the inner metric on \(X\).

## 2. Link criterion of Lipschitz normally embedding

In this section, we will give a proof of Theorem 1.3. We need the following results.

### Definition 2.1

(1) We call \(A \subset \mathbb{R}^n\) a standard \(L\)-cell (with constant \(C\)) if

- \(n = 1\): \(A\) is a point or an open interval,
- \(n > 1\): \(A\) has one of the following forms
  
  (i) (Graph) \(\Gamma_\xi := \{(x,y) \in B \times \mathbb{R}: x \in B, y = \xi(x)\}\)

  (ii) (Band) \(\xi_1, \xi_2 := \{(x,y) \in B \times \mathbb{R}: x \in B, \xi_1(x) < y < \xi_2(x)\}\)

  where \(B\) is a standard \(L\)-cell (with constant \(C\)) in \(\mathbb{R}^{n-1}\) and \(\xi, \xi_1, \xi_2: B \to \mathbb{R}\) are \(C^1\) definable functions such that \(||D\xi(x)||, ||D\xi_1(x)||\) and \(||D\xi_2(x)||\) are bounded by \(C\). We call \(B\) the basis of \(A\).

(2) We call \(A \subset \mathbb{R}^n\) a \(L\)-cell if there is an orthogonal change of coordinates \(\phi: \mathbb{R}^n \to \mathbb{R}^n\) such that \(\phi(A)\) is a standard \(L\)-cell.
Lemma 2.2. Let $X \subset \mathbb{R}^n$ be a definable set. If $f, g : \mathbb{R}^n \to \mathbb{R}$ and $h : X \to \mathbb{R}$ are continuous definable functions such that $f(x) \leq g(x), \forall x \in \mathbb{R}^n$ and $f(x) \leq h(x) \leq g(x), \forall x \in X$, then there is a continuous definable extension $H : \mathbb{R}^n \to \mathbb{R}$ of $h$ such that $f(x) \leq H(x) \leq g(x), \forall x \in \mathbb{R}^n$.

Proof. Take

$$h'(x) := \inf_{a \in X} \frac{d(x, a)}{d(x, X)} x \in \mathbb{R}^n \setminus X.$$ 

It is easy to prove that $h'$ is a continuous definable function on the whole $\mathbb{R}^n$ (see [1], Lemma 6.6). Define $H : \mathbb{R}^n \to \mathbb{R}, H(x) := \max\{f(x), \min\{g(x), h'(x)\}\}$. Then $H$ satisfies the statement of the lemma. □

Lemma 2.3. Let $X = \{X_i\}_{i \in I}$ be a finite collection of definable germs at 0. Let $\rho : (\mathbb{R}^n, 0) \to (\mathbb{R}_{\geq 0}, 0)$ be a germ of a continuous definable function such that $\rho(x) \sim \|x\|$. Then, there are $t_0 > 0$, $C > 0$, a definable stratification $\Sigma$ compatible with $\{X_i \setminus \{0\}\}_{i \in I}$ and a continuous integrable stratified vector field $\xi$ on $\Sigma$ such that the flow $\Phi : (X \setminus \{0\}) \times [0, t_0) \to X \setminus \{0\}$ generated by $\xi$ has the following properties:

1. $\Phi(x, s)$ preserves the strata of the stratification (i.e., if $x \in S \in \Sigma$ then $\Phi(x, s) \in S, \forall s \in [0, t_0)$),
2. $\Phi(x, s) \in X \cap \rho^{-1}(r - s)$ for any $s < r \leq t_0$ and $x \in X \cap \rho^{-1}(r)$,
3. $\text{length}(\Phi(x \times [0, s]) \leq C s$.

Proof. Let $\Sigma$ be a Whitney ($b$)-regular stratification of $\mathbb{R}^n \setminus \{0\}$ compatible with $\{X_i \setminus \{0\}\}_{i \in I}$ such that the restriction of $\rho$ to each stratum of $\Sigma$ is of class $C^2$. The existence of Whitney stratification for definable sets is proved in [11] (see also [12], [16]). For $x \neq 0$, set $v(x) := \frac{x}{\|x\|}$. For $x \in S \in \Sigma$, set $w = P_x(v(x))$ where $P_x : \mathbb{R}^n \to T_x S$ is the orthogonal projection from $\mathbb{R}^n$ to the tangent space to $S$ at the point $x$.

We claim that for any $\varepsilon, \varepsilon' > 0$ there is $R > 0$ such that

1. $\|w(x) - v(x)\| < \varepsilon, \forall x \in B_R^\Sigma \setminus \{0\}$,
2. for $x \in S \cap B_R^\Sigma, S \in \Sigma$, we have $d_x(\rho|_S)(w(x)) > \varepsilon'$ where $d_x(\rho|_S)$ denotes the tangent map at $x$ of the restriction $\rho|_S$.

First, we give a proof for (i). Assume that (i) is not true. By Curve Selection, there are $\varepsilon > 0$, a stratum $S \in \Sigma$ and a $C^1$ definable curve $\gamma : [0, \delta) \to \mathbb{R}^n$ with $\gamma(0) = 0$, $\gamma((0, \delta)) \subset S$ such that for every $t \in (0, \delta)$:

$$\|w(\gamma(t)) - v(\gamma(t))\| \geq \varepsilon, \forall t \in (0, \delta).$$

Since $\gamma$ is a $C^1$ curve through the origin, the angle between $v(\gamma(t)) = \frac{\gamma(t)}{\|\gamma(t)\|}$ and the tangent line to $\gamma$ at $\gamma(t)$ tends to 0 as $t$ tends to 0. This implies the angle between $v(\gamma(t))$ and the tangent space $T_{\gamma(t)} S$ tends to 0 as well, which gives a contradiction.

Now we prove (ii). We will show that

$$|d_x(\rho|_S)(w(x))| > \varepsilon', \forall x \in S \cap B_R^\Sigma.$$
Assume on the contrary that (ii) fails. By Curve Selection there are $S \subseteq \Sigma$ and a $C^1$ definable curve $\alpha : [0, \delta) \to \mathbb{R}^n$ with $\alpha(0) = 0$ and $\alpha((0, \delta)) \subset S$ such that $d_x(\rho|_S)(w(\alpha(t))) \to 0$ as $t \to 0$. Reparametrizing $\alpha$ if necessary, we may assume $\|\alpha(t)\| \sim t$. Then $\alpha(t) = at + o(t)$ for some $a \in \mathbb{R}^n, a \neq 0$. Note that $\lim_{t \to 0} v(\alpha(t)) = \lim_{t \to 0} w(\alpha(t)) = \lim_{t \to 0} \frac{\alpha'(0)}{\|\alpha'(0)\|}$. Therefore,

$$0 = \lim_{t \to 0} d_{\alpha(t)}(\rho|_S)(w(\alpha(t))) = \lim_{t \to 0} d_{\alpha(t)}(\rho|_S)(\frac{\alpha'(0)}{\|\alpha'(0)\|}) = \lim_{t \to 0} \frac{1}{\|\alpha'(0)\|} \frac{d(\rho(\alpha(t)))}{dt}.$$ 

Since $\lim_{t \to 0} \|\alpha'(0)\| = \|a\| \neq 0$, $\lim_{t \to 0} \frac{d(\rho(\alpha(t)))}{dt} = 0$. This implies that $\rho(\alpha(t)) \ll t$, which contradicts the fact that $\rho(\alpha(t)) \sim \|\alpha(t)\| \sim t$.

Observe that for any $C^1$ definable curve $\beta : [0, \delta) \to \mathbb{R}^n$ with $\beta(0) = 0$ and $\beta((0, \delta)) \subset S$ for some $S \subseteq \Sigma$, $\rho \circ \beta$ is a non-negative definable function, $\rho(\beta(0)) = 0$ and $\rho(\beta(t)) \sim t > 0$ as $t \to 0$. This implies that $\rho \circ \beta$ is strictly increasing near 0. Hence $d_{\beta(t)}(\rho|_S)(w(\beta(t))) > 0$ as $t \to 0$.

Now we prove the lemma. It is showed in the proof of Lemma 3.2 in [17] that for any $\varepsilon > 0$, there are $R > 0$ and a continuous integrable stratified vector field $\mu$ on $\Sigma$ such that $\|\mu(x) - v(x)\| < \varepsilon$. By (i) in the claim, shrinking $R$ if necessary, we may assume that $\|\mu(x) - w(x)\| < 2\varepsilon$. It follows from (ii) that there is $\varepsilon' > 0$ such that $d_x(\rho|_S)(\mu(x)) > \varepsilon'$.

Set

$$\xi(x) := \frac{-\mu(x)}{d_x(\rho|_S)(\mu(x))}$$

where $x \in S \subseteq \Sigma$. When $\varepsilon$ small enough, we have $\|\mu(x)\| \sim \|w(x)\| \sim \|v(x)\|$. Since $\|v(x)\| = 1$, there is $C > 0$ such that $\|\xi\| < C$. Furthermore, since $\mu$ is integrable so is $\xi$.

Let $\Phi(x, t)$ denote the flow generated by $\xi$. Take $0 < r_0 < R$ such that $\rho^{-1}((0, r_0)) \subseteq B^0_R$ (this is possible because $\rho(x) \sim \|x\|$). We now check that $\Phi$ satisfies the conditions (1), (2) and (3). (1) is clear from the fact that $\xi$ is a stratified vector field on $\Sigma$. Since $d_x(\rho|_S)(\xi(x)) = -1$, for $x \in S \cap \rho^{-1}(r)(r \leq r_0)$ and $s < r$ one has $\Phi(x, s) \in S \cap \rho^{-1}(r-s)$. Condition (2) then follows. Finally,

$$\text{length}(\Phi(x \times [0, s])) = \int^s_0 |\xi(\Phi(x, s))| ds \leq C \int^s_0 ds = Cs,$$

and hence (3) is satisfied.

Let us denote by $\mathcal{L}$ the set of all half lines in $\mathbb{R}^n$ starting at the origin.

**Lemma 2.4.** Let $\rho : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a continuous definable function such that $\rho(x) \sim \|x\|$. There is $R > 0$ such that $\rho|_{l \cap B^0_R} : l \cap B^0_R \to \mathbb{R}_{\geq 0}$ is injective for every $l \in \mathcal{L}$.
Proof. It follows from the claim (ii) in the proof of Lemma 2.3 that there is a definable open dense subset of $\mathbb{B}^n_R$, let say $U$, such that $\rho$ is $C^2$ on $U$ and
\[
\langle \text{grad}\rho(x), \frac{x}{\|x\|} \rangle > \varepsilon, \forall x \in U,
\]
where $\text{grad}\rho(x)$ is the gradient of $\rho$ at the point $x$.

Assume on the contrary that the statement of the lemma is not true. By Curve Selection, there is a $C^1$ definable curve $\gamma : (0, \varepsilon) \to \mathbb{R}^n \times \mathbb{R}^n, t \mapsto (\gamma_1(t), \gamma_2(t))$ such that (i): $\gamma_1(0) = \gamma_2(0) = 0$, $\gamma_1(t) \neq \gamma_2(t)$ and they are both contained in some $l_t \in \mathcal{L}$; (ii): $\rho(\gamma_1(t)) = \rho(\gamma_2(t))$. Shrinking $R$ if necessary, we may assume that $\|\gamma_1(t)\| < \|\gamma_2(t)\|$.

Let $l := \lim_{t \to 0} l_t \in \mathcal{L}$. By (i), $\gamma_1$ and $\gamma_2$ are sharing the same tangent cone at 0 that coincides with $l$. By (2.1), for each $t$ we can choose $\tilde{\gamma}(t) \in U$ sufficiently close to $\gamma_2(t)$ such that $\|\rho(\tilde{\gamma}(t)) - \rho(\gamma_2(t))\| \ll |\tilde{\gamma}(t) - \gamma_1(t)|, \lim_{t \to 0} \frac{\tilde{\gamma}(t)}{\|\tilde{\gamma}(t)\|} = \lim_{t \to 0} \gamma_2(t)/\|\gamma_2(t)\|$ and the intersection $[\gamma_1(t), \tilde{\gamma}(t)] \cap U$ is dense in the segment $[\gamma_1(t), \tilde{\gamma}(t)]$. By Definable Choice, we may assume that $\tilde{\gamma}$ is a $C^1$ definable curve. It is obvious that for any $\beta(t) \in [\gamma_1(t), \tilde{\gamma}(t)] \cap U$ we have
\[
\lim_{t \to 0} \frac{\beta(t)}{\|\beta(t)\|} = \lim_{t \to 0} \frac{\tilde{\gamma}(t) - \gamma_1(t)}{\|\tilde{\gamma}(t) - \gamma_1(t)\|} = \lim_{t \to 0} \frac{\gamma_1(t)}{\|\gamma_1(t)\|}.
\]
Moreover, (2.1) shows
\[
\langle \text{grad}\rho(\beta(t)), \frac{\beta(t)}{\|\beta(t)\|} \rangle > \varepsilon.
\]
Thus, for $t$ small enough,
\[
\langle \text{grad}\rho(\beta(t)), \frac{\tilde{\gamma}(t) - \gamma_1(t)}{\|\tilde{\gamma}(t) - \gamma_1(t)\|} \rangle > \varepsilon/2.
\]
Since such $\beta(t)$ exists almost everywhere in $[\gamma_1(t), \tilde{\gamma}(t)]$,
\[
|\rho(\tilde{\gamma}(t)) - \rho(\gamma_1(t))| \geq \frac{\varepsilon}{2} |\tilde{\gamma}(t) - \gamma_1(t)|,
\]
which contradicts the fact that $\|\rho(\tilde{\gamma}(t)) - \rho(\gamma_2(t))\| = \|\rho(\tilde{\gamma}(t)) - \rho(\gamma_1(t))\| \ll |\tilde{\gamma}(t) - \gamma_1(t)|$.
This ends the proof of the lemma.

Lemma 2.5. Let $\rho : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a germ of a continuous definable function such that $\rho(x) \sim \|x\|$. Let $\varphi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ defined by $\varphi(x) := \frac{\rho(x)}{\|x\|} x$ if $x \neq 0$ and $\varphi(0) = 0$. Then $\varphi$ is a germ of a bi-Lipschitz definable homeomorphism.

Proof. First we show that $\varphi$ is a bijection. Note that $x$ and $\varphi(x)$ lie in the same half line starting at the origin. To prove the bijectivity of $\varphi$ it suffices to show that there is $R > 0$ such that for any $l \in \mathcal{L}$ the restriction of $\varphi$ to $l \cap \mathbb{B}^n_R \setminus \{0\}$ is injective.

Take $R > 0$ as in Lemma 2.4. We have the restriction of $\rho$ to $l \cap \mathbb{B}^n_R$ is injective. Let $x$ and $x'$ be in $l \cap \mathbb{B}^n_R \setminus \{0\}$, $x \neq x'$. Then, there is $t \neq 0$ such that $x = tx'$. We have
\[
\varphi(x) = \varphi(x') \iff \frac{\rho(x)}{\|x\|} x = \frac{\rho(tx)}{\|tx\|} tx.
\]
\( \Leftrightarrow \rho(x) = \rho(tx) = \rho(x') \Leftrightarrow x = x' \) (since \( \rho|_{\mathcal{B}^n_R} \) is injective).

Thus, \( \varphi|_{\mathcal{B}^n_R\setminus \{0\}} \) is injective.

To prove \( \varphi \) is bi-Lipschitz on \( \mathcal{B}^n_R \), it is enough to show that \( \frac{\|\varphi(x) - \varphi(x')\|}{\|x - x'\|} \sim 1 \) for \( x, x' \in (\mathcal{B}^n_R \setminus \{0\}) \). Let \( x, x' \in \mathcal{B}^n_R \setminus \{0\} \), we may assume \( \|x\| \geq \|x'\| \). Set \( \sigma(x) := \frac{\rho(x)}{\|x\|\infty} \). Note that if \( \sigma(x) \sim 1 \), then there are \( C_1 > C_2 > 0 \) such that \( C_2 \leq \sigma(x)/\sigma(x') \leq C_1 \) for all \( x, x' \in \mathbb{R}^n \). This implies that \( C_1 \frac{\|x''\|}{\|x''\|} \geq \|\frac{\varphi(x)}{\varphi(x')}\| \geq C_2 \frac{\|x\|}{\|x''\|} \). Put \( K := \max\{3, 3/C_2\} \). The proof is split into two cases.

Case 1: \( \frac{\|x\|}{\|x''\|} \geq K \). In this case we have \( \frac{\|\varphi(x) - \varphi(x')\|}{\|x - x''\|} \) and \( \frac{\|x\|}{\|x''\|} \) are both greater than 3.

\[
\begin{align*}
\|x - x''\|^2 &= \|x\|^2 + \|x''\|^2 - 2 \cos \alpha \|x\| \|x''\| \\
&\geq \|x\| + \|x''\|^2 - \frac{2}{3} \|x\|^2 \sim \|x\|^2,
\end{align*}
\]

where \( \alpha \) is the angle between \( \overrightarrow{0x} \) and \( \overrightarrow{0x'} \). On the other hand, \( \|x - x''\| \leq \|\|x\| - \|x''\|\| \sim \|x\| \).

Therefore, \( \|x - x''\| \sim \|x\| \).

Similarly, one can easily show that \( \frac{\|\varphi(x) - \varphi(x')\|}{\|x - x''\|} \sim \frac{\|\varphi(x)\|}{\|x\|} \sim 1 \).

Case 2: \( 1 \leq \frac{\|x\|}{\|x''\|} \leq K \). We have \( C_2 \leq \frac{\|\varphi(x)\|}{\|\varphi(x')\|} \leq KC_1 \). If \( \cos \alpha < \frac{1}{2} \), then

\[
\begin{align*}
\|x - x''\|^2 &= \|x\|^2 + \|x''\|^2 - 2 \cos \alpha \|x\| \|x''\| \\
&\geq \|x\| + \|x''\|^2 - \|x\|^2 \geq (1/K)\|x''\|^2.
\end{align*}
\]

Since \( \|x\| \sim \|x''\| \), \( \|x - x''\| \leq \|x\| \). This implies that \( \|x - x''\| \sim \|x\| \). Similarly, we also have \( \|\varphi(x) - \varphi(x')\| \sim \|\varphi(x)\| \), hence \( \frac{\|\varphi(x) - \varphi(x')\|}{\|x - x''\|} \sim 1 \).

Now we assume that \( \cos \alpha > 1/2 \), i.e., \( \alpha < \pi/3 \). Let \( l \) be the half line in \( \mathcal{L} \) containing \( x \). Let \( z \) and \( z' \) denote the images of \( x' \) and \( \varphi(x') \) under the orthogonal projection onto \( l \). Let \( \beta \) denote the angle between \( x0 \) and \( xx' \), and let \( \beta' \) denote the angle between \( \varphi(x)0 \) and \( \varphi(x)\varphi(x') \) (see Figure 1). We have \( \|x - x''\| \sin \beta = \|x' - z\| \) and \( \|\varphi(x) - \varphi(x')\| \sin \beta' = \).
\[ \| \varphi(x') - z' \|. \] It follows that
\[ \frac{\| \varphi(x) - \varphi(x') \| \sin \beta'}{\| x - x' \| \sin \beta} = \frac{\| \varphi(x') - z' \|}{\| x' - z \|} = \frac{\| \varphi(x') \|}{\| x' \|} \sim 1. \]

Hence,
\[ \frac{\| \varphi(x) - \varphi(x') \|}{\| x - x' \|} \sim \frac{\sin \beta}{\sin \beta'}. \]

Since \( \| x \| \sim \| x' \| \) and \( \| \varphi(x) \| \sim \| \varphi(x') \| \), \( \beta \) and \( \beta' \) are bounded away from 0. This implies that \( \frac{\sin \beta}{\sin \beta'} \sim 1 \), so \( \frac{\| \varphi(x) - \varphi(x') \|}{\| x - x' \|} \sim 1. \]

We define \( \mathcal{C}_n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : 0 \leq x_i \leq x_{i+1}, i = 2, \ldots, n \}. \)

**Lemma 2.6.** Let \( C \subset \mathcal{C}_n \) be a definable set such that \( \dim X \geq 1 \). Suppose \( (C, 0) \) is a germ of a L-cell. Then \( \overline{C} \) is LLNE at 0 with respect to \( \rho(x) = \| x \|_\infty \).

**Proof.** The proof is by induction on \( n \). For \( n = 1 \), the result is trivial.

Let \( B \subset \mathcal{C}_{n-1} \) denote the basis of \( C \). There are two possibilities of \( C \), either a graph or a band over \( B \).

**Case 1:** \( C = \Gamma_\xi \) (graph). Since \( \xi \) is Lipschitz, it is possible to extend it to a Lipschitz function over \( B \). We use the same notation \( \xi \) for this extension. Obviously, \( \overline{C} \) is the graph of \( \xi \) over \( B \). Note that \( \rho|_{\mathcal{C}_n} = x_1 \) and \( \overline{C} \subset \mathcal{C}_n \). It is obvious that \( \overline{C}(r) := \overline{C} \cap \rho^{-1}(r) \) is a graph of the restriction of \( \xi \) to \( \overline{B}(r) := B \cap \rho^{-1}(r) \). By inductive assumption we have \( \overline{B}(r) \) is LNE, moreover, \( \xi \) is a Lipschitz function, therefore \( \overline{C}(r) \) is LNE.

**Case 2:** \( C = (\xi_1, \xi_2) \) (band). Extend \( \xi_1, \xi_2 \) to Lipschitz functions on \( \overline{B} \), use the same notation for these extensions. It is clear that \( \overline{C} = \{(x, y) \in \overline{B} \times \mathbb{R} : \xi_1(x) \leq y \leq \xi_2(x) \} \). In this case \( \overline{C}(r) := \overline{C} \cap \rho^{-1}(r) \) is the band bounded by the graphs of \( \xi_1 \) and \( \xi_2 \) restricted to \( \overline{B}(r) := \rho^{-1}(r) \cap \overline{B} \). We may write \( \overline{C}(r) = (\xi_1|_{\overline{B}(r)}, \xi_2|_{\overline{B}(r)}) \).

Given two points \( z \) and \( z' \) in \( \overline{C}(r) \). There are finitely many points \( \{z = z_0, z_1, \ldots, z_k = z'\} \) contained in the segment \( [z, z'] \) connecting \( z \) and \( z' \) such that \( z_i \in \Gamma_{\xi_1|_{\overline{B}(r)}} \cup \Gamma_{\xi_2|_{\overline{B}(r)}} \) where \( 1 \leq i \leq k \), and for each \( i < k \), the open interval \( (z_i, z_{i+1}) \) is either in \( \overline{C}(r) \) or in \( \mathcal{C}_n \cap \{x_1 = r\} \). If \( (z_i, z_{i+1}) \subset \overline{C}(r) \) then \( \| z_i - z_{i+1} \| = \rho_{\overline{C}(r)}(z_i, z_{i+1}) \). Recall that by \( d_A \) we mean the inner metric on \( A \). If \( (z_i, z_{i+1}) \subset (\mathcal{C}_n \cap \{x_1 = r\}) \cap \overline{C}(r) \) then \( z_i \) and \( z_{i+1} \) are both in the same graph of \( \xi_j|_{\overline{B}(r)} \), \( j \in \{1, 2\} \). Since \( \overline{B}(r) \) is LNE (by inductive assumption) and \( \xi_j \) is Lipschitz, the graph of \( \xi_j|_{\overline{B}(r)} \) is LNE. This implies that \( \| z_i - z_{i+1} \| \sim \rho_{\xi_j|_{\overline{B}(r)}}(z_i, z_{i+1}) \geq \rho_{\overline{C}(r)}(z_i, z_{i+1}) \). Therefore,
\[ \| z - z' \| = \sum_{i=0}^{k-1} \| z_i - z_{i+1} \| \geq \sum_{i=0}^{k-1} \rho_{\overline{C}(r)}(z_i, z_{i+1}) \geq \rho_{\overline{C}(r)}(z, z'). \]
Note that in both cases, the LNE constant for \( C(r) \) is uniformly bounded by a constant depending only on \( \overline{B} \) and the Lipschitz constants of \( \xi, \xi_1, \xi_2 \).

\[ \Box \]

**Proof of Theorem 1.3.** By taking a continuous definable extension we may assume that \( \rho \) is defined on the whole \( \mathbb{R}^n \) (see Lemma 2.2).

Case 1: \( \rho(x) = \|x\|_{\infty} \). For \( 1 \leq k \leq n \), set \( C_k^+ := \{ x \in \mathbb{R}^n : \|x\|_{\infty} = x_k \} \) and \( C_k^- := \{ x \in \mathbb{R}^n : \|x\|_{\infty} = -x_k \} \). Then \( \mathbb{R}^n = \bigcup_{k=1}^n C_k^+ \cup C_k^- \). There is a definable partition \( \mathcal{S} = \{ S_i \}_{i \in I} \) of \( \mathbb{R}^n \) into \( L \)-cells compatible with \( \{ X, C_1^+, C_1^-, \ldots, C_n^+, C_n^- \} \) (see [10], Proposition 1.4). It is clear that for \( S_i \in \mathcal{S} \), after permuting the coordinates, we have \( S_i \subset \mathcal{C}_n \). By Lemma 2.6, the germ of \( \overline{S}_i \) at 0 is LLNE with respect to \( \rho \).

Applying Lemma 2.3 to \( \mathcal{S} \) we obtain \( r_0 > 0, C > 0 \), a stratification \( \Sigma \) compatible with \( \mathcal{S} \) and a stratified vector field \( \xi \) on \( \Sigma \) such that the flow \( \Phi : (X \setminus \{0\}) \times [0, r_0) \to X \setminus \{0\} \) generated by \( \xi \) satisfies:

(a) \( \Phi(x, s) \in X \cap \rho^{-1}(r-s) \) for every \( 0 < s < r \leq r_0 \),

(b) \( \text{length}(\Phi(x \times [0, s])) \leq C s \).

Since \( \Sigma \) is a refinement of \( \mathcal{S} \) and \( \Phi \) preserves the strata of \( \Sigma \), \( \Phi \) preserves the strata of \( \mathcal{S} \).

Let \( p, q \in X \cap \rho^{-1}([0, r_0]) \). Set \( r_1 := \|p\|_{\infty}, r_2 := \|q\|_{\infty} \) and \( s := \|r_1 - r_2\| \). We may assume that \( r_1 \leq r_2 \). We denote by \( X(r) := X \cap \rho^{-1}(r) \). Let \( \gamma \) be the integral curve of \( \Phi \) through \( p \) and let \( p' := \gamma \cap \rho^{-1}(r_2) \). It is clear that

\[ \|p - q\| \sim \|p - q\|_{\infty} \geq \|p\|_{\infty} - \|q\|_{\infty} = s. \]

By (b),

\[ \|p - p'\| \leq d_\gamma(p, p') \leq C s. \]

Thus,

\[ \|q - p'\| \lesssim \|p - q\| + \|p - p'\| \lesssim \|p - q\|. \]

First we prove \((ii) \Rightarrow (i)\). We have

\[ d_X(p, q) \leq d_\gamma(p, p') + d_X(r_2)(p', q) \lesssim \|p - q\| + K \|q - p'\| \text{ (by (2.2),(2.3) and the fact that } X \text{ is LLNE w.r.t } \rho),} \]

\[ \lesssim \|p - q\| \text{ (by (2.4)),} \]

where \( K \) is the LLNE constant for \( X \) at 0. This shows that \( X \) is LNE at 0.

Now we prove \((i) \Rightarrow (ii)\). Given two points \( p \) and \( q \) in \( X(r) \). Let \( \beta \) be a curve in \( X \) realizing the inner distance from \( p \) to \( q \). Taking \( r \ll R \) we can assume that \( \beta \) is contained in \( X \cap \rho^{-1}([0, r_0]) \). In case \( \beta \) intersects \( \overline{S}_i \in \mathcal{S} \), we denote by \( p_{i,1}, p_{i,2} \in \overline{S}_i \) respectively the starting and the ending points of \( \beta \) in \( \overline{S}_i \). Set \( \Lambda := \{ i : \beta \cap \overline{S}_i \neq \emptyset \} \). Let \( \beta_{i,j}, i \in \Lambda, j = 1, 2 \) be the integral curves of \( \Phi \) through \( p_{i,j} \) and let \( q_{i,j} := \beta_{i,j} \cap X(r) \) (see Figure 2).

Set \( l := \max_{i \in \Lambda} \{ \|p_{i,j} - q_{i,j}\| \} \). We may assume \( l = \|p_{i,1} - q_{i,1}\| \) for some \( i \in \Lambda \). By (b), we have

\[ l \leq C \|p_{i,1}\|_{\infty} - \|q_{i,1}\|_{\infty}. \]
On the other hand,
\[
\text{length}(\beta) \geq d_{\beta}(p_{i,1}, p) \geq \|p_{i,1} - p\| \sim \|p_{i,1} - p\|_{\infty} \geq \|\|p_{i,1}\|_{\infty} - \|p\|_{\infty}\|
\]
Since \( p, q_{i,1} \) lie in \( \rho^{-1}(r) \), \( \|p\|_{\infty} = \|q_{i,1}\|_{\infty} \). Hence
\[
\text{length}(\beta) \geq \|p_{i,1}\|_{\infty} - \|q_{i,1}\|_{\infty} \gtrsim l. \tag{2.5}
\]
Since \( X \) is LNE at 0, we have
\[
l \lesssim \text{length}(\beta) \sim \|p - q\|. \tag{2.6}
\]
In addition,
\[
\|q_{i,1} - q_{i,2}\| \leq \|q_{i,1} - p_{i,1}\| + \|p_{i,1} - p_{i,2}\| + \|p_{i,2} - q_{i,2}\| \leq \|p_{i,1} - p_{i,2}\| + 2l. \tag{2.7}
\]
Since \( S_i \) is LNE,
\[
\text{length}(\beta) = \sum_{i \in \Lambda} d_{S_i}(p_{i,1}, p_{i,2}) \sim \sum_{i \in \Lambda} \|p_{i,1} - p_{i,2}\|. \tag{2.8}
\]
Lemma 2.6 says that \( S_i \) is LLNE with respect to \( \rho \). Combining with (2.6), (2.7) and (2.8) we get
\[
\begin{align*}
d_{X(r)}(p, q) &\leq \sum_{i \in \Lambda} d_{S_i(r)}(q_{i,1}, q_{i,2}) \sim \sum_{i \in \Lambda} \|q_{i,1} - q_{i,2}\| \\
&\leq \sum_{i \in \Lambda} (\|p_{i,1} - p_{i,2}\| + 2l) \lesssim (2m + 1)\text{length}(\beta) \sim (2m + 1)\|p - q\|,
\end{align*}
\]
where \( m := \#\Lambda \). This implies that \( X \) is LLNE with respect to \( \rho \).

For a general \( \rho \), let \( \varphi(x) := \frac{\rho(x)}{\|x\|_{\infty}} x \) if \( x \neq 0 \) and \( \varphi(0) = 0 \). It follows from Lemma 2.5 that \( \varphi \) is a germ of a bi-Lipschitz definable homeomorphism. The proof follows from the fact that the following are equivalent:

1. \( X \) is LNE;
2. \( \varphi(X) \) is LNE;
(3) $\rho^{-1}(r) \cap \varphi(X)$ is LNE with LNE-constant uniformly bounded;
(4) $\rho^{-1}(r) \cap X$ is LNE with LNE-constant uniformly bounded.

Indeed, (1) $\iff$ (2) follows from the definition of LNE sets; (3) $\iff$ (4) follows from the fact that $\varphi$ brings $X \cap \rho^{-1}(r)$ to $X \cap \rho'^{-1}(r)$. Finally, (2) $\iff$ (3) follows from Case 1.

3. Applications

3.1. Moderately discontinuous homology. We briefly recall the definition of Moderately discontinuous homology. For more details about this theory, we refer the reader to [5].

Definition 3.1. Let $(X, x_0, d_1)$ and $(Y, y_0, d_2)$ be two metric subanalytic germs. Given a subanalytic continuous map $f : (X, x_0, d_1) \rightarrow (Y, y_0, d_1)$.

(i) $f$ is called linearly vertex approaching (l.v.a) if there is $K \geq 1$ such that
\[
\frac{1}{K} \|x - x_0\| \leq \|f(x) - y_0\| \leq K \|x - x_0\|, \forall x \in X.
\]
Such a constant $K$ is called a l.v.a constant for $f$.

(ii) $f$ is called Lipschitz linearly vertex approaching (Lipschitz l.v.a) if $f$ is l.v.a and there is $C > 1$ such that
\[
d_2(f(x), f(x')) \leq Cd_1(x, x'), \forall x, x' \in X.
\]
Such a constant $C$ which also serves as a l.v.a constant is called a Lipschitz l.v.a constant for $f$.

For $n \in \mathbb{N}$, let $\Delta_n \subset \mathbb{R}^{n+1}$ denote the standard $n$-simplex (i.e., $\Delta_n := \{(p_0, \ldots, p_n) \in \mathbb{R}^{n+1}_{\geq 0} : \sum_{i=0}^{n} p_i = 1\}$) with the orientation induced by the standard orientation in $\mathbb{R}^{n+1}$ of the convex hull of $\Delta_n \cup 0$. For $0 \leq k \leq n$, denote by $i^n_k : \Delta_{n-1} \hookrightarrow \Delta_n, (p_0, \ldots, p_{n-1}) \mapsto (p_0, \ldots, p_{k-1}, 0, p_k, \ldots, p_{n-1})$. Set
\[
\hat{\Delta}_n := \{(tx, t) \in \mathbb{R}^{n+1} \times \mathbb{R} : x \in \Delta_n, t \in [0, 1]\}
\]
and let $j^n_k : \hat{\Delta}_{n-1} \rightarrow \hat{\Delta}_n, (tx, t) \mapsto (ti^n_k(x), t)$. We identify $\hat{\Delta}_n$ with its germ at $(0, 0)$.

Definition 3.2. Given a definable germ $(X, x_0)$. A linearly vertex approaching $n$-simplex (l.v.a $n$-simplex) in $(X, x_0)$ is a subanalytic continuous map germ : $\sigma : \hat{\Delta}_n \rightarrow (X, x_0)$ such that there is $K \geq 1$ satisfying
\[
\frac{1}{K} t \leq \|\sigma(tx, t) - x_0\| \leq K t, \forall (x, t) \in \hat{\Delta}_n.
\]

Given an abelian group $A$, a linear vertex approaching $n$-chain in $(X, x_0)$ is a finite formal sum $\sum_{i \in I} a_i \sigma_i$ where $a_i \in A$ and $\sigma_i$ is a l.v.a $n$-simplex in $(X, x_0)$. Denote by $MDC^\text{pre,}\infty_n((X, x_0), d_1, A)$ the abelian group of $n$-chains. If the metric $d_1$ and the group
Definition 3.3. A homological subdivision of $\hat{\Delta}_n$ is a finite collection $\{\rho_i\}_{i \in I}$ of injective l.v.a map germs $\rho_i : \hat{\Delta}_n \to \hat{\Delta}_n$ such that there is a subanalytic triangulation $\alpha : |K| \to \hat{\Delta}_n$ with the following properties

- $\alpha$ is compatible with faces of $\hat{\Delta}_n$;
- the collection $\{T_i\}$ of maximal triangles of $\alpha$ is also indexed by $I$;
- for any $i \in I$, $\rho_i(\Delta_n) = T_i$ and the map $\alpha^{-1} \circ \rho_i$ takes faces of $\hat{\Delta}_n$ to faces of $K$.

The sign of $\rho_i$, denoted by $\text{sign}(\rho_i)$, is defined to be 1 if $\rho_i$ preserves orientation preserving, and $-1$ if it is of the opposite orientation.

Definition 3.4. Let $b \in (0, +\infty]$. Two $n$-simplices $\sigma_1$ and $\sigma_2$ in $\text{MDC}^{n,\infty}_n(X, x_0)$ are called $b$-equivalent (write $\sigma_1 \sim_b \sigma_2$) if

- for $b < \infty$: $\lim_{t \to 0} \max \{d_1(\sigma_1(tx, t), \sigma_2(tx, t)), x \in \Delta_n\} = 0$.
- for $b = \infty$: $\sigma_1(tx, t) = \sigma_2(tx, t), \forall x \in \Delta_n$.

Definition 3.5. Let $b \in (0, +\infty]$. Let $z = \sum_{i \in I} a_i \sigma_i$ and $z' = \sum_{j \in J} b_j \tau_j$ be l.v.a $n$-complex chains in $\text{MDC}^{n,\infty}_n(X, x_0)$. Write $I = \bigcup_{k \in K} I_k$ and $J = \bigcup_{k \in K} J_k$ where

(a) $i_1, i_2 \in I$ belong to the same $I_k$ iff $\sigma_{i_1} \sim_b \sigma_{i_2}$.
(b) $j_1, j_2 \in J$ belong to the same $J_k$ iff $\tau_{j_1} \sim_b \tau_{j_2}$.
(c) for $k \in K$, $i \in I_k$, $j \in J_k$ we have $\sigma_i \sim_b \tau_j$.

Then, $z$ is called $b$-equivalent to $z'$, denoted $z \sim_b z'$, if for any $k$,

$$\sum_{i \in I_k} a_i = \sum_{j \in J_k} b_j.$$

Definition 3.6. Let $b \in (0, +\infty]$. Given two l.v.a $n$-complex chains $z = \sum_{i \in I} a_i \sigma_i$ and $z' = \sum_{j \in J} b_j \tau_j$ in $\text{MDC}^{n,\infty}_n(X, x_0)$.

- We write $z \rightarrow_b z'$ (immediate relation) if for each $i \in I$ there is a homological subdivision $\{\rho_{i,k}\}_{k \in K_i}$ such that

  $$\sum_{i \in I} \sum_{k \in K_i} \text{sgn}(\rho_{i,k}) a_i \sigma \circ \rho_{i,k} \sim_b \sum_{j \in J} b_j \tau_j.$$

- $z$ and $z'$ are called homological subdivision equivalent, denoted by $z \sim_{\text{hs}} z'$, if there exist sequences of immediate sequences $z = z_0 \rightarrow_b z_1 \rightarrow_b \ldots \rightarrow_b z_l$ and

  $$z' = w_0 \rightarrow_b w_1 \rightarrow_b \ldots \rightarrow_b w_m$$

  such that $w_m \sim_b z_l$. 

$A$ are clear from the context, we shorten it as $\text{MDC}^{n,\infty}_n(X, x_0)$. The boundary of $\sigma$ is a formal sum of $(n - 1)$-simplices defined as follows:

$$\partial \sigma := \sum_{k=0}^{n} (-1)^k \sigma \circ j_k^k.$$
Definition 3.7. Let \( b \in (0, +\infty] \). The \( b \)-moderately discontinuous chain complex in \((X, x_0)\) is the quotient group \( \text{MDC}^b_\bullet(X, x_0) := \text{MDC}^\text{pre,}\infty_\bullet(X, x_0)/\sim_{S,b} \). Its homology is called \( b \)-moderately discontinuous homology, denoted by \( \text{MDH}^b_\bullet(X, x_0) \).

Given a Lipschitz l.v.a subanalytic map \( f : (X, x_0, d_1) \to (Y, y_0, d_2) \). It is easy to check that \( f_\# : \text{MDC}^b_\bullet((X, x_0, d_1), A) \to \text{MDC}^b_\bullet((Y, y_0, d_2), A), \sigma \mapsto f \circ \sigma \) satisfies \( f_\# \circ \partial = \partial \circ f_\# \). Therefore, we have the homomorphism \( f_* : \text{MDH}^b_\bullet((X, x_0, d_1), A) \to \text{MDH}^b_\bullet((Y, y_0, d_2), A) \).

In particular, if \( f \) is a subanalytic bi-Lipschitz homeomorphism then \( f_* \) is an isomorphism.

Let \((X, x_0)\) be a subanalytic germ. The identity map \( \text{Id}_{x_0} : (X, x_0, d_{inn}) \to (X, x_0, d_{out}) \) is a Lipschitz l.v.a subanalytic map. If \((X, x_0)\) is LNE, then

\[
\text{Id}_{x_0,*} : \text{MDH}^b_\bullet((X, x_0, d_{inn}), A) \to \text{MDH}^b_\bullet((X, x_0, d_{out}), A)
\]

is an isomorphism.

Question 3.8 ([5], Problem 147). Let \( X \) be a subanalytic closed subset of \( \mathbb{R}^n \). Suppose that for every \( x \in X \), the map \( \text{Id}_{x,*} : \text{MDH}^b_\bullet((X, x, d_{inn}), A) \to \text{MDH}^b_\bullet((X, x, d_{out}), A) \) is an isomorphism. Is \( X \) LNE?

We need the following result to calculate the MD homology of our example.

Definition 3.9. Let \((X, x_0, d_1)\) and \((Y, y_0, d_2)\) be metric subanalytic germs. Let \( f, g : (X, x_0, d_1) \to (Y, y_0, d_2) \) be Lipschitz l.v.a subanalytic maps. A metric homotopy between \( f \) and \( g \) is a continuous subanalytic map \( H : X \times I \to Y \) such that \( H_s := H(\cdot, s) \) is Lipschitz l.v.a with the Lipschitz l.v.a constant independent of \( s \) and \( H_0 = f \) and \( H_1 = g \). In this case, \( f \) and \( g \) are called metrically homotopic.

Theorem 3.10 ([5], Theorem 81 (2)). Let \( f, g : (X, x_0, d_1) \to (Y, y_0, d_2) \) be Lipschitz l.v.a subanalytic maps which are metrically homotopic. Then, for any \( b \in (0, \infty] \)

\[
f_\#, g_\# : \text{MDC}^b_\bullet(X, x_0) \to \text{MDC}^b_\bullet(Y, y_0)
\]

represent the same map and hence induce the same homomorphism in the MD homology.

3.2. A counterexample to Question 3.8.

Proposition 3.11. Let \( X := \{(t, x, z) \in \mathbb{R}^3, z^2 = t^2x^2, 0 \leq x \leq t\} \). Then

(1) \((X, 0)\) is not LNE.

(2) \( \text{Id}_{x,*} : \text{MDH}^b_\bullet(X, x, d_{inn}) \to \text{MDH}^b_\bullet(X, x, d_{out}) \) is isomorphic for every \( x \in (X, 0) \).

Therefore, \((X, 0)\) is an counterexample to Question 3.8.

Proof. Let \( \rho : X \to \mathbb{R}, w = (t, x, z) \mapsto t \). It is obvious that \( \rho \) is a continuous semialgebraic function with \( \rho(w) \sim \|w\| \). Set \( X(r) := X \cap \{\rho^{-1}(r)\} \).

It is easy to see that \( X(r) \) is LNE with the LNE constant \( \sim 1/r \), which tends to \( \infty \) when \( r \) tends to \( 0 \). This means \((X, 0)\) is not LLNE with respect to \( \rho \), so by Theorem 1.3, it is not LNE. Thus, (1) is proved.
We now show (2). Let $S_0 := \{0\}$, $S_1 := \{(t, 0, 0) \in \mathbb{R}^3, t > 0\}$, $S_2 := \{(t, x, z) \in X, 0 < x = t\}$ and $S_3 := X \setminus \bigcup_{i=0}^{3} S_i$ (see Figure 3).

It is clear that $\{S_i\}_{i=0}^{3}$ is a stratification of $X$. Let $x \in X$. If $x \in S_3$, then $x$ is a smooth point, hence $(X, x)$ is LNE, hence (2) is satisfied. If $x \in S_2$, the germ $(X, x)$ is a smooth manifold with boundary, so it is also LNE hence (2) is again true. If $x \in S_1$, by Valette’s Lipschitz Triviality Theorem (see [19], Theorem 2.2) in a neighbourhood of $x$, $X$ is bi-Lipschitz equivalent to $X(r) \times (0, \varepsilon)$ which is obviously LNE. This implies that (2) holds. The only case needs verifying is $x = 0$.

Consider the following map:

$$H : X \times I \to X, (t, x, z, s) \mapsto H_s(t, x, z) := (t, sx, sz).$$

We show that $H$ is a Lipschitz l.v.a metric homotopy for both the outer and the inner metrics. Fix $s$ and let $w = (t_1, x_1, z_1)$ and $w' = (t_2, x_2, z_2)$. We have

$$\|H_s(w) - H_s(w')\| = \|(t_1, sx_1, sz_1) - (t_2, sx_2, sz_2)\|$$

$$\leq |t_1 - t_2| + s(|x_1 - x_2| + |z_1 - z_2|) \lesssim \|w - w'\|.$$

This shows that $H_s$ is Lipschitz with respect to the outer metric.

Observe that $X$ consists of two branches $X_1 := \{(t, x, z) \in X, z \geq 0\}$ and $X_2 := \{(t, x, z) \in X, z \leq 0\}$ and each branch is LNE. Moreover, $H_s(\_)$ preserves these branches.

If $w$ and $w'$ are in the same branch we have

$$d_{inn}(H_s(w), H_s(w')) \sim \|H_s(w) - H_s(w')\| \lesssim \|w - w'\| \sim d_{inn}(w, w').$$

Now assume that $w \in X_1$ and $w' \in X_2$. We have

$$d_{inn}(H_s(w), H_s(w')) = d_{inn}((t_1, sx_1, sz_1), (t_2, sx_2, sz_2))$$

$$\leq d_{inn}((t_1, sx_1, sz_2), (t_2, 0, 0)) + d_{inn}((t_2, 0, 0), (t_2, sx_2, sz_2))$$

$$\lesssim |t_1 - t_2| + s(|x_1| + |z_1|) + s(|x_2| + |z_2|).$$

$$X \cap \rho^{-1}(r) = \{x = \frac{1}{r}|z|\}$$
Let $\gamma$ be a curve connecting $w$ and $w'$ which realizes the inner distance between $w$ and $w'$. Since $w$ and $w'$ lie in two different branches of $X$, $\gamma$ has to pass through the $t$-axis. Hence

$$d_{inn}(w, w') = \text{length}(\gamma) \gtrsim |t_1 - t_2| + (|x_1| + |z_1| + |x_2| + |z_2|).$$

It follows that $d_{inn}(H_s(w), H_s(w')) \lesssim d_{inn}(w, w')$. Consequently, $H_s$ is Lipschitz with respect to the inner metric. Note that Lipschitz constants for $H_s$ can be chosen to be independent of $s$.

Let $Y := \{(t, 0, 0), t \geq 0\}$ and $g : X \to Y, w \mapsto g(w) = H_0(w)$. Clearly, $H_0 = \iota \circ g$ where $\iota : Y \to X$ is the inclusion map. Since $H_1 = Id_X$ and $H_0$ are metrically homotopic for both the inner and the outer metrics, $Id_{X,*}, H_{0,*} : MDH_b^*(X, 0, d) \to MDH_b^*(X, 0, d)$ (where $d \in \{d_{inn}, d_{out}\}$) represent the same homomorphism which is actually an isomorphism (see Theorem 3.10). Since $H_{0,*} = \iota_* \circ g_*$, which is an isomorphism, and $i_*$ is injective, $g_*$ must be an isomorphism. We have the following commutative diagram:

$$
\begin{array}{ccc}
MDH^b_*(X, 0, d_{inn}) & \xrightarrow{g_*} & MDH^b_*(Y, 0, d_{inn}) \\
\downarrow Id_{X,*} & & \downarrow Id_{Y,*} \\
MDH^b_*(X, 0, d_{out}) & \xrightarrow{g_*} & MDH^b_*(Y, 0, d_{out})
\end{array}
$$

Since $(Y, 0)$ is a germ of a half line which is LNE, $Id_{Y,*}$ is an isomorphism. Thus, $Id_{X,*}$ is also an isomorphism.

**Remark 3.12.** Since our counterexample is a set with non-isolated singularity, the question is still open for the isolated case.

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