Homotopy invariance of higher \( K \)-theory for abelian categories

Satoshi Mochizuki  Akiyoshi Sannai

Abstract

The main theorem in this paper is that the base change functor from a noetherian abelian category \( \mathcal{A} \) to \( \mathcal{A}[[t]] \) the noetherian polynomial category of \( \mathcal{A} \), – \( \otimes_{\mathcal{A}} \mathbb{Z}[t] : \mathcal{A} \to \mathcal{A}[[t]] \) induces an isomorphism on \( K \)-theory. The main theorem implies the well-known fact that \( A^1 \)-homotopy invariance of \( K' \)-theory for noetherian schemes.

Classification 18E10, 19D35.
Keywords higher \( K \)-theory Abelian categories.

1 Introduction

Contrary to the importance of \( A^1 \)-homotopy invariance in the motivic homotopy theory \[ \text{[Voe98], [MV99] and [Voe00]} \], the homotopy invariance of \( K' \)-theory for noetherian schemes still has been mysterious in the following sense. Recall the footstep of \( K \)-theory in the viewpoint of axiomatic characterization. The observation that the additivity theorem is the fundamental theorem of connective algebraic \( K \)-theory is implicit in Waldhausen \[ \text{[Wal85]} \] (See also \[ \text{[GSVW92]} \]) and was known to Grayson \[ \text{[Gra87]} \], Staffeldt \[ \text{[Ste87]} \] and McCarthy \[ \text{[McC93]} \]. Recently the connective \( K \)-theory is regarded as the universal additive invariant by Tabuada \[ \text{[Tab08]} \] and Barwick \[ \text{[Bar14]} \]. After Thomason \[ \text{[TT90]} \], derived invariance and localizing properties of non-connective \( K \)-theory are emphasized by many authors \[ \text{[Nee92], [Kel99], [TV04], [Cis10], [BM11], [Sch06], [Sch11] and [Moc13]} \]. In the landscape of non-commutative motive theory \[ \text{[CT11]} \] or motive theory for \( \infty \)-categories \[ \text{[GT13]} \], non-connective \( K \)-theory is the universal localizing invariant. To relate motivic homotopy theory with motive theory for DG or \( \infty \)-categories, it is important to make clear that what additional axiom implies the homotopy invariance property. Many authors have already defined and studied affine lines over certain categories as in \[ \text{[Alm74], [Alm78], [Gra77], [Haz83], [GM96], [Yao96] and [Sch06]} \]. (See also \[ \text{[Tab14]} \]). The main objective in this paper is to examine the homotopy invariance of \( K \)-theory for abelian categories by taking Schlichting polynomial categories. We recall the definition of the polynomial categories. For a category \( \mathcal{C} \), we let \( \text{End} \mathcal{C} \) denote the category of endomorphisms in \( \mathcal{C} \). Namely, an object in \( \text{End} \mathcal{C} \) is a pair \((x,\phi)\) consisting of an object \( x \in \mathcal{C} \) and a morphism \( \phi : x \to x \) in \( \mathcal{C} \) and a morphism between \((x,\phi)\) and \((y,\psi)\) is a morphism \( f : x \to y \) in \( \mathcal{C} \) such that \( \psi f = f \phi \). (See Notation 2.1). From now on, let \( \mathcal{A} \) be an abelian category. We write \( \text{Lex} \mathcal{A} \) for the category of left exact functors from \( \mathcal{A}^{op} \) to \( \text{Ab} \) the category of abelian groups. The category \( \text{Lex} \mathcal{A} \) is a Grothendieck abelian category and the Yoneda embedding \( y : \mathcal{A} \to \text{Lex} \mathcal{A} \) is exact and reflects exactness. We say an object \( x \in \mathcal{A} \) is noetherian if every ascending filtration of subobjects of \( x \) is stationary. We say \( \mathcal{A} \) is noetherian if every object in \( \mathcal{A} \) is noetherian. (See Definition 2.4).

We assume that \( \mathcal{A} \) is a noetherian abelian category and we write \( \mathcal{A}[t] \) for the full subcategory of noetherian objects in \( \text{End} \text{Lex} \mathcal{A} \) and call it the noetherian polynomial category over \( \mathcal{A} \). (See Definition 2.13). We can prove that \( \mathcal{A}[t] \) is an abelian category. (See Lemma 2.5). For an object \( a \) in \( \mathcal{A} \), let us define an object \( a[t] = (a[t], t) \) in \( \text{End} \text{Lex} \mathcal{A} \) as follows. The underlying object \( a[t] \) is \( \bigoplus_{n=0}^{\infty} a[t] \) where \( a[t] \) is a copy of \( a \). The endomorphism \( t : a[t] \to a[t] \) is defined by the identity morphisms \( at^i \to at^{i+1} \) in each components. We can prove that if \( a \) is noetherian in \( \mathcal{A} \), then \( a[t] \) is noetherian in \( \mathcal{A}[t] \). (See Theorem 2.17). We call the association \( \otimes_{\mathcal{A}} \mathbb{Z}[t] : \mathcal{A} \to \mathcal{A}[t], a \mapsto a[t] \) the base change functor which is an exact functor. One of the consequence of the main theorem is the following:
Theorem 1.1 (A part of Theorem 5.1). Let $A$ be a noetherian abelian category. The functor $- \otimes_A \mathbb{Z}[t] : A \to A[t]$ induces a homotopy invariance of spectra on $K$-theory

$$K(A) \xrightarrow{\sim} K(A[t]).$$

The key idea of how to prove the main theorem is, roughly speaking, that we recognize an affine space as to be a rudimental projective space

$$A^n = \mathbb{P}^n \setminus \{[x_0 : \cdots : x_n] \in \mathbb{P}^n; x_n = 0\}(= \mathbb{P}^n \setminus \mathbb{P}^{n-1}).$$

(Compare the equation above with the formula (2) below). To give more precise explanation, for a scheme $X$ which has an ample family of line bundles and a closed subset $Y$ of $X$, we write $[X^Y]$ and $K(\text{Coh}_Y X)$ for the derived category and the non-connective $K$-theory of bounded complexes of coherent sheaves $E^*$ on $X$ such that $\bigcup \text{Supp} H^i(E^*) \subset Y$ respectively and denote $[X^X]$ and $K(\text{Coh}_X X)$ by $[X]$ and $K(\text{Coh}_X X)$ respectively. Then the following three formulas imply $A^1$-homotopy invariance of $K$-theory of coherent sheaves over noetherian schemes:

1. (Derived projective bundle formula). $[\mathbb{P}^{n-1}_X]/[\mathbb{P}^{n-2}_X] \xrightarrow{\sim} [X]$.

2. (Localization formula). $[\mathbb{P}^{n-1}_X]/[\mathbb{P}^{n-2}_X(\mathbb{P}^{n-1}_X)] \xrightarrow{\sim} [A^{n-1}_X]$.

3. (Purity). We have the isomorphism

$$K(\text{Coh}_{\mathbb{P}^{n-1}_X}, \mathbb{P}^n_X) \xrightarrow{\sim} K(\text{Coh}_{X} \mathbb{P}^{n-1}_X).$$

In this paper, we trace parallel arguments above in categorical setting. Projective spaces are replaced with graded categories over categories which is introduced in §3. The formulas (1) and (2) above correspond to Theorem 4.24 and Theorem 5.6 respectively. Finally the formula (3) above is replaced with Proposition 5.14 which is a consequence of the d´evissage theorem. A geometric meaning of the d´evissage theorem in the view of categorical algebraic geometry based on the support varieties theory as in [Bal07], [BKS07] and [Gar09] will be studied in the first author’s subsequent papers. See [3.4] for an ad hoc axiomization of d´evissage property. In the final subsection, we propose a generalized Vorst problem. Recall that Vorst conjecture in [Vor79] which says that for any affine scheme $X$, $A^1$-homotopy invariance of $K$-theory for $X$, characterizes the regularity of $X$, has been recently proved in [CHW08] and [GH12]. To attack generalized Vorst conjecture, the first author hope to extends the arguments in ibid to categorical algebraic geometry setting.

Convention. In this note, basically we follow the notation of exact categories for [Kel90] and algebraic $K$-theory for [Qui73] and [Wal85]. For example, we call admissible monomorphisms (resp. admissible epimorphisms and admissible short exact sequences) inflations (resp. deflations, con- flations). We also call a category with cofibrations and weak equivalences a Waldhausen category. Let us denote the set of all natural numbers by $\mathbb{N}$. We regard it as a totally ordered set with the usual order. For a Waldhausen category, we denote the specific zero object by the same letter $*$. We denote the 2-category of essentially small categories by $\text{Cat}$, the category of sets by $\text{Set}$ and the category of essentially small abelian (resp. exact) categories by $\text{AbCat}$ (resp. $\text{ExCat}$). For any non-negative integer $n$, we denote the set of all integers $k$ such that $0 \leq k \leq n$ by $[n]$. For categories $\mathcal{X}, \mathcal{Y}$, we denote the (large) category of functors from $\mathcal{X}$ to $\mathcal{Y}$ by $\mathcal{HOM}(\mathcal{X}, \mathcal{Y})$. For any ring with unit $A$, we denote the category of right $A$-modules (resp. finitely generated right $A$-modules) by $\text{Mod}_*(A)$ (resp. $\mathcal{M}_*(A)$). Throughout the paper, we use the letter $A$ to denote an essentially small abelian category. For an object $x$ in $A$ and a finite family $\{x_i\}_{1 \leq i \leq m}$ of subobjects of $x$, $\sum_{i=1}^m x_i$ means the minimum subobject of $x$ which contains all $x_i$. For an additive category $B$, we write $\text{Ch}(B)$ for the category of chain complexes on $B$.

2 Polynomial categories

In this section, we recall the notation of polynomial abelian categories from [Sch00] or [Sch06].
2.1 End categories

Definition 2.1. For a category $C$, we denote the category of endomorphisms in $C$ by $\text{End} C$. Namely, an object in $\text{End} C$ is a pair $(x, \phi)$ consisting of an object $x$ in $C$ and a morphism $\phi : x \to x$ in $C$ and a morphism between $(x, \phi) \to (y, \psi)$ is a morphism $f : x \to y$ in $C$ such that $\psi f = f \phi$. For any functor $F : C \to C'$, we have a functor $\text{End} F : \text{End} C \to \text{End} C'$ which sends $(x, \phi)$ to $(Fx, F\phi)$. Moreover for any natural transformation $\theta : F \to F'$ between functors $F, F : C \to C'$, we have a natural transformation $\text{End} \theta : \text{End} F \to \text{End} F'$ defined by the formula $\text{End} \theta(x, \phi) := \theta(x)$ for any object $(x, \phi)$ in $\text{End} C$. This association gives a 2-functor

$$\text{End} : \text{Cat} \to \text{Cat}.$$ 

We have natural transformations $i : id_{\text{Cat}} \to \text{End}$ and $U : \text{End} \to id_{\text{Cat}}$ defined by $i(C) : C \to \text{End} C, x \mapsto (x, \text{id}_x)$ and $U(C) : \text{End} C \to C, (x, \phi) \mapsto x$ for each category $C$.

Remark 2.2. Let $C$ be a category and $F : I \to \text{End} C$, $i \mapsto (x_i, \phi_i)$ be a functor. Let us assume that there is a limit $\lim_i x_i$ (resp. colimit $\text{colim}_i x_i$) in $C$. Then we have $\lim F_i = (\lim_i x_i, \lim_i \phi_i)$ (resp. $\text{colim} F_i = (\text{colim}_i x_i, \text{colim}_i \phi_i)$). In particular, if $C$ is additive (resp. abelian), then $\text{End} C$ is also additive (resp. abelian). Moreover if $C$ is an exact category (resp. a category with cofibration), then $\text{End} C$ naturally becomes an exact category (resp. a category with cofibration). Here a sequence $(x, \phi) \to (y, \psi) \to (z, \xi)$ is a conflation if and only if $x \to y \to z$ is a conflation in $C$. (resp. a morphism $(x, \phi) \to (y, \psi)$ is a cofibration if and only if $u : x \to y$ is a cofibration in $C$.) Moreover if $w$ is a class of morphisms in $C$ which satisfies the axioms of Waldhausen categories (and its dual), then the class of all morphisms in $\text{End} C$ which is in $w$ also satisfies the axioms of Waldhausen categories (and its dual).

Remark 2.3. In [GM96, III. 5.15], for a category $C$, the category $\text{End} C$ is called the polynomial category over $C$ and denoted by $C[T]$. For any ring with unit $A$, we have the canonical category isomorphism

$$\text{Mod}(A[T]) \xrightarrow{\sim} (\text{Mod}(A))[T], \ M \mapsto (M, T)$$

where $A[T]$ is the polynomial ring over $A$ and $T$ means an endomorphism $T : M \to M$ which sends an element $x$ in $M$ to an element $xT$ in $M$. Moreover in general for any abelian category $A$, we have the equality

$$\text{hdim} A[T] = \text{hdim} A + 1$$

where $\text{hdim} A$ is the homological dimension of $A$ which is defined by

$$\text{hdim} A := \max \{n; \text{Ext}^n(x, y) \neq 0 \text{ for any objects } x, y\}.$$ 

But obviously for any right noetherian ring $A$, $(\mathcal{M}_A)[T]$ and $\mathcal{M}_A[T]$ are different categories. The main reasons is that $A[T]$ is not finitely generated as an $A$-module. In particular, the object $(A[T], T)$ is in $(\text{Mod}(A))[T]$ but not in $(\mathcal{M}_A)[T]$. In the subsection 2.4 we define the noetherian polynomial categories over noetherian abelian category which is introduced by Schlichting in [Sch06]. In this notion, we have the canonical category equivalence between $\mathcal{M}_A[t]$ and $(\mathcal{M}_A)[t]$. See Example 2.21.

2.2 Noetherian objects

In this subsection, we develop the theory of noetherian objects in exact categories which is slightly different from the usual notation in the category theory.

Definition 2.4. Let $E$ be an exact category and $x$ an object in $E$. We say $x$ is a noetherian object if any ascending filtration of admissible subobjects of $x$

$$x_0 \hookrightarrow x_1 \hookrightarrow x_2 \hookrightarrow \cdots$$

is stational. We say $E$ is a noetherian category if all objects in $E$ are noetherian.

We can easily prove the following lemmata.
Lemma 2.5. Let $E$ be an exact category. Then

(1) Let $x \rightarrow y \rightarrow z$ be a conflation in $E$. If $y$ is noetherian, then $x$ and $z$ are also noetherian.

(2) For noetherian objects $x, y$ in $E$, $x \oplus y$ is also noetherian.

(3) Moreover, assume that $E$ is abelian, then the converse of (1) is true. Namely, in the notation (1), if $x$ and $z$ are noetherian, then $y$ is also noetherian.

Lemma 2.6. For any exact faithful functor $F : A \rightarrow B$ between abelian categories and an object $x$ in $A$, if $Fx$ is noetherian, then $x$ is also noetherian.

2.3 Grothendieck category

In this subsection, we briefly review the notion of Grothendieck categories.

Definition 2.7 (Generator). An object $u$ in a category $C$ is said to be a generator if the corepresentable functor $\text{Hom}(u, -) : C \rightarrow \text{Set}$ associated with $u$ is faithful.

Definition 2.8 (finite type). Let $B$ be an additive category and $x, y$ objects in $B$. We say that $y$ is of $x$-finite type (in $B$) if there exists a positive integer $n$ and an epimorphism $x^{\oplus n} \twoheadrightarrow y$ in $B$.

Example 2.9. Let $R$ be a ring with unit. An object $M$ in $\text{Mod}(R)$ is a finitely generated $R$-module if and only if $M$ is of $R$-finite type.

Lemma 2.10. (1) Let $f : B \rightarrow C$ be an exact functor from an abelian category $B$ to an exact category $C$ and $x, y$ objects in $B$. If $y$ is of $x$-finite type, then $f(y)$ is of $f(x)$-finite type.

(2) Let $B$ be an abelian category which has a generator $u$. Then any noetherian objects in $B$ are of $u$-finite type.

Proof. (1) There exists a positive integer $n$ and an epimorphism $p : x^{\oplus n} \twoheadrightarrow y$. Then we have an epimorphism $f(p) : f(x)^{\oplus n} \twoheadrightarrow f(y)$. Hence $f(y)$ is of $f(x)$-finite type.

(2) Let $x$ be a noetherian object in $B$ and we put $\Lambda := \text{Hom}(u, x)$. For any $\lambda \in \Lambda$, we write $u_\lambda$ for a copy of $u$. Then $\{\lambda : u_\lambda \rightarrow x\}_{\lambda \in \Lambda}$ induces a morphism $p : \bigoplus_{\lambda \in \Lambda} u_\lambda \rightarrow x$.

Claim. $p$ is an epimorphism.

Proof of claim. Let $\alpha : x \rightarrow y$ be a non-zero morphism in $B$. Since $u$ is a generator, $\text{Hom}(u, \alpha)$ is a non-zero map. Therefore there exists a morphism $\lambda_0 : u_{\lambda_0} \rightarrow x$ such that $\alpha \lambda_0 \neq 0$. In particular $\alpha p \neq 0$ and $p$ is an epimorphism.

If $\Lambda$ is a finite set, then we get the desired result. If $\Lambda$ is an infinite set, then there exists an injection $\omega : \mathbb{N} \rightarrow \Lambda$. We put $x_n = p( \bigoplus_{\alpha \in \omega([n])} u_\alpha)$ where $[n]$ is the set $\{0, 1, \ldots, n\}$. Then the family $\{x_n\}_{n \in \mathbb{N}}$ is an ascending chain of subobjects of a noetherian object $x$ and therefore it is stational. Say $x_k = x_{k+1} = \cdots$. Then the restriction of $p$ to $\bigoplus_{\alpha \in \omega([k])} u_\alpha$, $\bigoplus_{\alpha \in \omega([k])} u_\alpha \rightarrow x$ is an epimorphism.

Definition 2.11 (Grothendieck category). We say that an abelian category $B$ is Grothendieck if the following conditions hold.

(1) $B$ has a generator.

(2) $B$ is cocomplete. Namely for any small category $I$, we define the diagonal functor $\Delta_I : B \rightarrow \text{HOM}(I, B)$ by sending an object $x$ in $B$ to a constant functor $I \rightarrow B$ which sends all objects in $I$ to $x$ and all morphisms in $I$ to $x$. Then $\Delta_I$ admits a left adjoint functor $\text{colim}_I : \text{HOM}(I, B) \rightarrow B$.

(3) All small direct limits in $B$ is exact. Namely for any filtered small category $I$, the colimit functor $\text{colim}_I : \text{HOM}(I, B) \rightarrow B$ is exact.

2.12. For an essentially small exact category $E$, we denote the category of left exact functors from $E^{op}$ to the category of abelian groups $\text{Ab}$ by $\text{Lex} E$. It is well-known that the category $\text{Lex} E$ is a Grothendieck category and the Yoneda embedding $y : E \rightarrow \text{Lex} E$ which sends $x$ to the representable functor associated with $x$, $\text{Hom}(-, x) : E^{op} \rightarrow \text{Ab}$ is exact and reflects exactness. (cf.
For example, let $A$ be a ring with unit, then the composition of the Yoneda embedding $\text{Mod}(A) \to \text{Lex} \text{Mod}(A)$ and the restriction $\text{Lex} \text{Mod}(A) \to \text{Lex} M_A$ induced from the inclusion functor $M_A \hookrightarrow \text{Mod}(A)$ is an equivalence

$$\text{Mod}(A) \xrightarrow{\sim} \text{Lex} M_A$$

where the inverse functor is given by sending an object $F$ in $\text{Lex} M_A$ to an object $F(A)$ in $\text{Mod}(A)$.

**Theorem 2.13 (Embedding theorem).** (cf. [GP64].) Let $B$ be a Grothendieck category with a generator $u$. We put $R := \text{Hom}_B(u,u)$. $R$ is a ring with unit by taking multiplication as composition of morphisms. Then the corepresentable functor $\text{Hom}(u,-) : B \to \text{Mod} - R$ associated with $u$ is fully faithful.

**Corollary 2.14.** Let $A$ be an essentially small noetherian abelian category. Then there exists a ring with unit $R_A$ and an exact fully faithful functor $i_A : A \to M_{R_A}$.

**Proof.** Let $u$ be a generator of $\text{Lex} A$ and put $R_A := \text{Hom}(u,u)$. Then we have an exact fully faithful functor $i_A : A \hookrightarrow \text{Mod}(R_A)$ defined by composing a corepresentable functor associated with $u$, $\text{Hom}(u,-) : \text{Lex} A \to \text{Mod}(R_A)$ and the Yoneda embedding $y_A : A \hookrightarrow \text{Lex} A$. We claim that $i_A$ factors through $A \hookrightarrow M_{R_A}$. For any object $x$ in $A$, $y_A(x)$ is a noetherian object by [Pop73, 5.8.8, 5.8.9]. Therefore by Lemma 2.10(2), $y_A(x)$ is of $\mu$-finite type and hence $i_A(x)$ is a finitely generated $R_A$-module by Example 2.9 and Lemma 2.10(1). We obtain the desired result.

### 2.4 Schlichting polynomial category

In this subsection, we introduce noetherian polynomial categories for noetherian abelian categories.

**2.15.** For an object $a$ in an additive category with countable coproducts $\mathcal{B}$, we define an object $a[t](= (a[t], t))$ in $\text{End} \mathcal{B}$ as follows. The underlying object $a[t]$ is $\bigoplus_{n=0}^{\infty} at^n$ where $at^n$ is a copy of $a$. The endomorphism $t : a[t] \to a[t]$ is defined by the identity morphisms $at^n \to at^{n+1}$ in each components. We call the object $a[t]$ in $\text{End} \mathcal{B}$ the polynomial object of $a$. For an object $a$ in an essentially small exact category $\mathcal{E}$, we similarly define an object $a[t]$ in $\text{End} \text{Lex} \mathcal{E}$.

**Lemma 2.16.** Let $\mathcal{B}$ be an additive category with countable coproducts and $a$ an object in $\mathcal{B}$. We denote the induced morphism from the identity morphisms $at^n \to a$ for non-negative integers $i$ by $\nabla_a : a[t] \to a$. Then the sequence

$$a[t] \xrightarrow{id_a[t]} a[t] - t \xrightarrow{\sum x_i} a$$

is a split exact sequence in $\mathcal{B}$.

**Proof.** We write $i_a : a \to a[t]$ for an inclusion functor $a = at^0 \to \bigoplus_{i \geq 0} at^i$ and we define $q_a : a[t] \to a[t]$ to be a morphism in $\mathcal{B}$ by sending $(x_k)_k$ to $\left( - \sum_{i \geq k+1} x_i \right)_k$. Then we can easily check the equalities $q_a(id_a[t]t) = id_a[t]$, $\nabla_a q_a = id_a$, $\nabla_a (id_a[t]t) = 0$, $q_a i_a = 0$ and $i_a \nabla_a + (id_a[t]t)q_a = id_a[t]$. Hence the sequence (1) is a split exact sequence.

The following theorem is proved in [Sch00, 9.10 b].

**Theorem 2.17 (Abstract Hilbert basis theorem).** For any noetherian object $a$ in an essentially small abelian category $A$, $a[t]$ is also a noetherian object in $\text{End} \text{Lex} A$.

**Definition 2.18 (Schlichting polynomial category).** Let us assume that $\mathcal{A}$ is an essentially small noetherian abelian category and we denote the full subcategory of noetherian objects in $\text{End} \text{Lex} \mathcal{A}$ by $\mathcal{A}[t]$ and call $\mathcal{A}[t]$ the noetherian polynomial category over $\mathcal{A}$. By virtue of Lemma 2.15 and Theorem 2.17 we acquire the assertion that $\mathcal{A}[t]$ is a noetherian abelian category.

**Remark 2.19.** We can prove that an object $x$ in $\text{End} \text{Lex} A$ is in $\mathcal{A}[t]$ if and only if there exists a deflation $a[t] \twoheadrightarrow x$ for some object $a$ in $\mathcal{A}$. 

[TT90, A.7.1, A.7.5].
In this subsection, we prove an abstract version of Artin-Rees lemma.

**2.5 Abstract Artin-Rees lemma**

**Definition 2.22**

More precisely, by Remark 2.3 and 2.12, we have the equivalences of categories

\[(1) \quad \text{A decreasing filtration} \quad \xrightarrow{\sim} \quad \text{restriction} \quad \text{Bl} \]

We define an object \( \sum \) exists a positive integer \( m \) such that \( f(a) \) is in \( \bigoplus_{i=1}^m b^i \). Since the morphism \( f \) is determined by morphisms \( c_i : a \to b \) \((0 \leq i \leq m)\) in \( A \). We write \( f \) by \( \sum_{i=1}^m c_i t^i \).

**Example 2.20.** For any noetherian objects \( a, b \) in \( A \) and a morphism \( f : a[t] \to b[t] \) in \( A[t] \), there exists a positive integer \( m \) such that \( f(a) \) is in \( \bigoplus_{i=1}^m b^i \). Since the morphism \( f \) is recovered by the restriction \( a \to a[t] \xrightarrow{f} b[t] \), \( f \) is determined by morphisms \( c_i : a \to b \) \((0 \leq i \leq m)\) in \( A \). We write \( f \) by \( \sum_{i=1}^m c_i t^i \).

**Example 2.21.** Let \( A \) be a ring with unit. Then we have the category equivalence

\[ \mathcal{M}_A[t] \xrightarrow{\sim} (\mathcal{M}_A)[t], \quad M \mapsto (M, t). \]

More precisely, by Remark 2.3 and 2.12 we have the equivalences of categories

\[ \text{Mod}(A[t]) \xrightarrow{\sim} \text{End Mod}(A) \xrightarrow{\sim} \text{End Lex} \mathcal{M}_A. \]

By considering the full subcategories of consisting of those noetherian objects, we get the desired result.

### 2.5 Abstract Artin-Rees lemma

In this subsection, we prove an abstract version of Artin-Rees lemma.

**Definition 2.22** \((t\text{-filtration})\). Let \( E \) an exact category and \( X = (x, t) \) an object in \( \text{End} E \).

1. A decreasing filtration \( \mathfrak{t} = \{X_n = (x_n, t_n)\}_{n \geq 0} \) of \( X \) in \( \text{End} E \),

\[ X = X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_n \leftarrow \cdots \]

is a \( t \)-filtration if \( \text{Im}(t_n : x_n \to x_n) \subset x_{n+1} \) for any \( n \geq 0 \).

2. A \( t \)-filtration \( \mathfrak{t} = \{X_n = (x_n, t_n)\}_{n \geq 0} \) is stable if there exists an integer \( n_0 \geq 0 \) such that \( \text{Im}(t_n : x_n \to x_n) \subset x_{n+1} \) for any \( n \geq 0 \).

**Definition 2.23** (Blow up). Let \( B \) an abelian category, \( X = (x, t) \) an object in \( \text{End} B \) and \( \mathfrak{t} = \{X_n = (x_n, t_n)\}_{n \geq 0} \) a \( t \)-filtration of \( X \).

1. We define an object \( B \mathfrak{t} X \) in \( \text{End Lex} B \) as follows. For any \( n, t_n : x_n \to x_n \) induces a morphisms \( t_n : x_n \to x_{n+1} \) and \( \bigoplus_{n \geq 0} t_n : \bigoplus_{n \geq 0} x_n \to \bigoplus_{n \geq 0} x_n \). We put \( B \mathfrak{t} X := \left( \bigoplus_{n \geq 0} x_n, \bigoplus_{n \geq 0} t_n \right) \). We call \( B \mathfrak{t} X \) a blow up object of \( X \) along \( \mathfrak{t} \).

2. For each \( n, id_{x_n} : x_n \to x_n \) and the morphisms \( t_{n+p-1} \cdots t_{n+1} t_n : x_n \to x_{n+p} \) for \( p > 1 \) induce a morphism

\[ \eta^n : X_n \to B \mathfrak{t} X \]

in \( \text{End Lex} B \).

**Lemma 2.24.** Let \( A \) be a noetherian abelian category, \( X = (x, t) \) an object in \( A[t] \) and \( \mathfrak{t} = \{X_n = (x_n, t_n)\}_{n \geq 0} \) a \( t \)-filtration in \( A[t] \). Then the following conditions are equivalent:

1. \( X \) is stable.
2. There exists an integer \( m \geq 0 \) such that the canonical morphism induced by \( \eta^k \) \((0 \leq k \leq m)\),

\[ \bigoplus_{k=0}^m X_k \to B \mathfrak{t} X \]

is an epimorphism.
3. \( B \mathfrak{t} X \) is an object in \( A[t] \), namely a noetherian object in \( \text{End Lex} \mathcal{A} \).

**Proof.** We assume that there exists an integer \( m \geq 0 \) such that \( \text{Im}(t_n : x_n \to x_n) = x_{n+1} \) for any \( n \geq m \). Then obviously the canonical morphism \( \bigoplus_{k=0}^m X_k \to B \mathfrak{t} X \) is an epimorphism.

Next assume the condition (2). Since \( B \mathfrak{t} X \) is a quotient of finite direct sum of noetherian objects in \( \text{End Lex} \mathcal{A} \), \( B \mathfrak{t} X \) is noetherian by Lemma 2.5.
Finally we assume that Bl₁ X is noetherian. We put \( z_m := \text{Im} \left( \bigoplus_{k=0}^{m} X_k \to \text{Bl}_1 X \right) \). Then the sequence \( z_0 \to z_1 \to z_2 \to \cdots \) is stationary. Say \( z_{n_0} = z_{n_0+1} = \cdots \). Then for any \( n \geq n_0 \), we have

\[
x_{n+1} \subset z_{n+1} \cap x_{n+1} = z_{n_0} \cap x_{n+1} \subset \sum_{i=0}^{n_0} \text{Im}(t_n t_{n-1} \cdots t_i : x_i \to x_{n+1}) \subset \text{Im}(t_n : x_n \to x_{n+1}).
\]

Hence \( x \) is stable.

\[\blacksquare\]

**Corollary 2.25 (Abstract Artin-Rees lemma).** Let \((x, t_x)\) be an object in \( A[t] \) and \((y, t_y)\) a subobject of \((x, t_x)\). Then there exist an integer \( n_0 \geq 0 \) such that

\[
\text{Im}(t^n_x : x \to x) \cap y = \text{Im}(t^{n-n_0}_x : (\text{Im}(t^{n_0}_x : x \to x) \cap y) \to y)
\]

for any \( n \geq n_0 \) in \( \text{Lex} \, A \).

**Proof.** Consider the \( t \)-stable filtration \( x = \{ \text{Im}(t^n_x : x \to x), t_x \}_{n \geq 0} \) of \((x, t_x)\) and the induced \( t \)-filtration \( y = \{ \text{Im}(t^n_x : x \to x) \cap y, t_y \}_{n \geq 0} \) of \((y, t_y)\). Then \( \text{Bl}_n(y, t_y) \) is a subobject of \( \text{Bl}_n(x, t_x) \). Since \( \text{Bl}_1(x, t_x) \) is noetherian by Lemma 2.24, \( \text{Bl}_n(y, t_y) \) is also noetherian and by Lemma 2.24 again, we learn that \( y \) is stable. Hence we obtain the result. \[\blacksquare\]

## 3 Non-commutative motive theory over relative exact categories

In this section, we will review the notions of additive and localizing theories over relative exact categories. Moreover we introduce a notion of nilpotent invariance.

### 3.1 Relative exact categories

In this subsection, we recall jargons of relative exact categories from [Moc13] and [HM13].

**3.1 (Relative exact categories).** (1) A relative exact category \( E = (\mathcal{E}, w) \) is a pair of an exact category \( \mathcal{E} \) with a specific zero object \( 0 \) and a class of morphisms \( w \) in \( \mathcal{E} \) which satisfies the following two axioms.

- **(Identity axiom).** For any object \( x \) in \( \mathcal{E} \), the identity morphism \( id_x \) is in \( w \).

- **(Composition closed axiom).** For any composable morphisms \( a \circ b \in \mathcal{E} \), if \( a \) and \( b \) are in \( w \), then \( ab \) is also in \( w \).

(2) A relative exact functor between relative exact categories \( f : E = (\mathcal{E}, w) \to (\mathcal{F}, v) \) is an exact functor \( f : \mathcal{E} \to \mathcal{F} \) such that \( f(w) \subset v \) and \( f(0) = 0 \). We denote the category of relative exact categories and relative exact functors by \( \text{RelEx} \).

(3) We write \( \mathcal{E}^w \) for the full subcategory of \( \mathcal{E} \) consisting of those object \( x \) such that the canonical morphism \( 0 \to x \) is in \( w \). We consider the following axioms.

- **(Strict axiom).** \( \mathcal{E}^w \) is an exact category such that the inclusion functor \( \mathcal{E}^w \hookrightarrow \mathcal{E} \) is exact and reflects exactness.

- **(Very strict axiom).** \( \mathcal{E} \) satisfies the strict axiom and the inclusion functor \( \mathcal{E}^w \hookrightarrow \mathcal{E} \) induces a fully faithful functor \( D'(\mathcal{E}^w) \to D(\mathcal{E}) \) on the bounded derived categories. We denote the category of strict (resp. very strict) relative exact categories by \( \text{RelEx}_{\text{strict}} \) (resp. \( \text{RelEx}_{\text{very}} \)).

(4) A relative natural equivalence \( \theta : f \to f' \) between relative exact functors \( f, f' : E = (\mathcal{E}, w) \to E' = (\mathcal{E}', w') \) is a natural transformation \( \theta : f \to f' \) such that \( \theta(x) \) is in \( w' \) for any object \( x \) in \( \mathcal{E} \). Relative exact functors \( f, f' : E \to E' \) are weakly homotopic if there is a zig-zag sequence of relative natural equivalences connecting \( f \) to \( f' \). A relative exact functor \( f : E \to E' \) is a homotopy equivalence if there is a relative exact functor \( g : E' \to E \) such that \( gf \) and \( fg \) are weakly homotopic to identity functors respectively.

(5) A functor \( F \) from a full subcategory \( R \) of \( \text{RelEx} \) to a category \( C \) is categorical homotopy invariant if for any relative exact functors \( f, f' : E \to E' \) in \( R \) such that \( f \) and \( f' \) are weakly homotopic, we have the equality \( F(f) = F(f') \).
3.2 (Derived category). We define the derived categories of a strict relative exact category \( E = (\mathcal{E}, w) \) by the following formula
\[
\mathcal{D}_b(E) := \text{Coker}(\mathcal{D}_b(\mathcal{E}^w) \rightarrow \mathcal{D}_b(\mathcal{E}))
\]
where \( # = b, \pm \) or nothing. Namely \( \mathcal{D}_b(E) \) is a Verdier quotient of \( \mathcal{D}_b(\mathcal{E}) \) by the thick subcategory of \( \mathcal{D}_b(\mathcal{E}) \) spanned by the complexes in \( \mathcal{D}_b(\mathcal{E}^w) \).

3.3 (Quasi-weak equivalences). Let \( P_\# : \text{Ch}_\#(\mathcal{E}) \rightarrow \mathcal{D}_\#(E) \) be the canonical quotient functor. We denote the pull-back of the class of all isomorphisms in \( \mathcal{D}_\#(E) \) by \( \text{qw}_\# \) or simply \( \text{qw} \). We call a morphism in \( \text{qw} \) a quasi-weak equivalence. We write \( \text{Ch}_\#(E) \) for a pair \( (\text{Ch}_\#(\mathcal{E}), \text{qw}) \). We can prove that \( \text{Ch}_\#(E) \) is a complicial Waldhausen category in the sense of [TT90, 1.2.11]. In particular, it is a relative exact category. The functor \( P_\# \) induces an equivalence of triangulated categories \( T(\text{Ch}_\#(\mathcal{E}), \text{qw}) \sim \mathcal{D}_\#(E) \) where the category \( T(\text{Ch}_\#(\mathcal{E}), \text{qw}) \) is the triangulated category associated with the category \( (\text{Ch}_\#(\mathcal{E}), \text{qw}) \) (See [Sch11, 3.2.17]). If \( w \) is the class of all isomorphisms in \( \mathcal{E} \), then \( \text{qw} \) is just the class of all quasi-isomorphisms in \( \text{Ch}_\#(\mathcal{E}) \) and we denote it by \( \text{qis} \).

3.4 (Consistent axiom). Let \( E = (\mathcal{E}, w) \) be a strict relative exact category. There exists the canonical functor \( \iota^\#: \mathcal{E} \rightarrow \text{Ch}_\#(\mathcal{E}) \) where \( \iota^\#: x^k \) is \( x \) if \( k = 0 \) and \( 0 \) if \( k \neq 0 \). We say that \( w \) (or \( E \)) satisfies the consistent axiom if \( \iota^\#: w \) is a consistent relative exact category. We denote the full subcategory of consistent relative exact categories (resp. very strict consistent relative exact categories) in RelEx by RelEx_{consistent} (resp. RelEx_{vs, consistent}).

Example 3.5. (cf. [Moc13]). (1) A pair \( (\mathcal{E}, i_\mathcal{E}) \) of an exact category \( \mathcal{E} \) with the class of all isomorphisms \( i_\mathcal{E} \) is a very strict consistent relative exact category. We regard the category of essentially small exact categories \( \text{ExCat} \) as the full subcategory of \( \text{RelEx}_{\text{vs}, \text{consistent}} \) by the fully faithful functor \( \text{ExCat} \rightarrow \text{RelEx}_{\text{vs}, \text{consistent}} \) which sends an exact category \( \mathcal{E} \) to a relative exact category \( (\mathcal{E}, i_\mathcal{E}) \). For simplicity, we sometimes write \( \mathcal{E} \) for \( (\mathcal{E}, i_\mathcal{E}) \).

(2) In particular we denote the trivial exact category by \( 0 \) and we also write \( (0, i_0) \) for \( 0 \). \( 0 \) is the zero objects in the category of consistent relative exact categories.

(3) A complicial exact category with weak equivalences in the sense of [Sch11, 3.2.9] is a consistent relative exact category. In particular for any relative exact category \( E \), \( \text{Ch}_\#(\mathcal{E}, \text{qw}) \) is a very strict consistent relative exact category.

3.6 (Derived equivalence). An exact functor \( f : E \rightarrow F \) is a derived equivalence if \( f \) induces an equivalence of triangulated categories on the bounded derived categories \( \mathcal{D}_b(E) \sim \mathcal{D}_b(F) \).

We give an example of derived equivalence exact functor by the proof of Corollary 3 of resolution theorem in [Qui73] and [Sch11, 3.2.8]:

3.7 (Homology theory and acyclic objects). (1) A homology theory on an exact category \( \mathcal{E} \) to an abelian category \( B \) is an exact connected sequence of functors \( T = \{ T_n \}_{n \geq 1} \) from \( \mathcal{E} \) to \( B \). Namely for any conflations \( x \rightarrow y \rightarrow z \) in \( \mathcal{E} \), we have a long exact sequence
\[
\cdots \rightarrow T_2z \rightarrow T_1x \rightarrow T_1y \rightarrow T_1z.
\]

(2) Let \( T = \{ T_n \}_{n \geq 1} \) be a homology theory on an exact category \( \mathcal{E} \). An object \( x \) is \( T \)-acyclic if \( T_nx = 0 \) for all \( n \geq 1 \).

Lemma 3.8. Let \( \mathcal{E} \) be an exact category and \( T \) a homology theory on \( \mathcal{E} \) and \( \mathcal{E}_{T-\text{acy}} \) the full subcategory of \( T \)-acyclic objects in \( \mathcal{E} \). Assume for each \( x \) in \( \mathcal{E} \) that there exists a deflation \( y \rightarrow x \) with \( y \) in \( \mathcal{E}_{T-\text{acy}} \), and that \( T_nx \) is trivial for \( n \) sufficiently large. Then the inclusion functor \( \mathcal{E}_{T-\text{acy}} \hookrightarrow \mathcal{E} \) is a derived equivalence.

3.9 (Non-connective \( K \)-theory for (consistent) relative exact categories). (cf. [Moc13]). For a consistent relative exact category \( E = (\mathcal{E}, w) \), we define the non-connective \( K \)-theory \( K(E) \) by the formula \( K(E) = K^S(\text{Ch}_b(E)) \) where \( K^S \) means the non-connective \( K \)-theory defined and studied by Schlichting in [Sch05] or [Sch11]. If either \( w \) is the class of all isomorphisms or \( E \) is a complicial exact category with weak equivalences in the sense of [Sch11], then the canonical morphism \( E \rightarrow \text{Ch}_b(E) \) induces an equivalence of spectra \( K^S(E) \sim K(E) \). The operation \( K \) becomes a functor from the category of essentially small consistent relative exact categories to the stable category of spectra.
3.2 Additive theory

3.10. Let $E = (\mathcal{E}, w)$ be a relative exact category. We denote the exact category of admissible short exact sequences in $\mathcal{E}$ by $E(\mathcal{E})$. There exist three exact functors $s$, $t$, and $q$ from $E(\mathcal{E}) \to \mathcal{E}$ which send an admissible exact sequence $x \to y \to z$ to $x$, $y$ and $z$ respectively. We write $w_{E(\mathcal{E})}$ for the class of morphisms $s^{-1}(w) \cap t^{-1}(w) \cap q^{-1}(w)$ and put $E(\mathcal{E}) := (E(\mathcal{E}), w_{E(\mathcal{E})})$. We can easily prove that $E(\mathcal{E})$ is a relative exact category and the functors $s$, $t$ and $q$ are relative exact functors from $E(\mathcal{E})$ to $E$. Moreover we can easily prove that if $E$ is consistent, then $E(\mathcal{E})$ is also consistent.

Definition 3.11 (Additive theory). (1) A full subcategory $\mathcal{R}$ of $\operatorname{RelEx}$ is closed under extensions if $\mathcal{R}$ contains the trivial relative exact category 0 and if for any $E$ in $\mathcal{R}$, $E(\mathcal{E})$ is also in $\mathcal{R}$.
(2) Let $\mathcal{A}$ be a functor from a full subcategory $\mathcal{R}$ of $\operatorname{RelEx}$ closed under extensions to an additive category $\mathcal{B}$. We say that $\mathcal{A}$ is an additive theory if for any relative exact category $E$ in $\mathcal{R}$, the following projection is an isomorphism

$$
\left( \begin{array}{c}
\mathcal{A}(s) \\
\mathcal{A}(t)
\end{array} \right) : \mathcal{A}(E(\mathcal{E})) \to \mathcal{A}(E) \oplus \mathcal{A}(E).
$$

By the proof of Corollary 2 of the additivity theorem in [Qui73], we get the additivity for characteristic filtration:

3.12 (Characteristic filtration). A characteristic filtration of a functor $f$ between exact categories $\mathcal{E}' \to \mathcal{E}$ is a finite sequence $0 = f_0 \to f_1 \to \cdots f_n = f$ of natural transformations between exact functors from $\mathcal{E}'$ to $\mathcal{E}$ such that $f_{p-1}(x) \to f_p(x)$ is an inflation in $\mathcal{E}$ for every $x$ in $\mathcal{E}'$ and $1 \leq p \leq n$, and induced quotient functors $f_p/f_{p-1}$ are exact for $1 \leq p \leq n$.

Lemma 3.13 (Additivity for characteristic filtration). Let $\mathcal{A} : \operatorname{ExCat} \to \mathcal{B}$ be a categorical homotopy invariant additive theory and $f : \mathcal{E} \to \mathcal{E}'$ be an exact functor between exact categories equiped with a characteristic filtration $0 = f_0 \subset \cdots \subset f_n = f$. Then

$$
\mathcal{A}(f) = \sum_{p=1}^{n} \mathcal{A}(f_p/f_{p-1}).
$$

3.3 Localizing theory

Definition 3.14 (Exact sequence). (1) We say that a sequence of triangulated categories $\mathcal{T} \xrightarrow{i} \mathcal{T}' \xrightarrow{j} \mathcal{T}''$ is exact if $i$ is fully faithful, the composition $ji$ is zero and the induced functor from $j$, $\mathcal{T}'/\mathcal{T} \to \mathcal{T}''$ is cofinal. The last condition means that it is fully faithful and every object of $\mathcal{T}''$ is a direct summand of an object of $\mathcal{T}'/\mathcal{T}$.
(2) A sequence $E \xrightarrow{i} F \xrightarrow{j} G$ of strict relative exact categories is derived exact if the induced sequence of triangulated categories $D_b(E) \xrightarrow{D_b(i)} D_b(F) \xrightarrow{D_b(j)} D_b(G)$ is exact. We sometimes denote the sequence above by $(u, v)$. For a full subcategory $\mathcal{R}$ of $\operatorname{RelEx}_{\text{strict}}$, we let $E(\mathcal{R})$ denote the category of exact sequences in $\mathcal{R}$. We define three functors $s^\mathcal{R}$, $m^\mathcal{R}$ and $q^\mathcal{R}$ from $E(\mathcal{R})$ to $\mathcal{R}$ which send an exact sequence $E \to F \to G$ to $E$, $F$ and $G$ respectively.

Example 3.15 (Exact sequence of abelian categories). Let $S$ be a Serre subcategory of an abelian category $B$. Then the canonical sequence

$$
S \to B \to B/S
$$

is derived exact if $S$ and $B$ satisfy the following condition $(\ast)$:

$(\ast)$ For any monomorphism $x \to y$ in $B$ with $x$ in $S$, there exists a morphism $y \to z$ with $z$ in $S$ such that the composition $x \to y \to z$ is a monomorphism. (See [Gro77 4.1] and [Kel99 1.15]).
Definition 3.16 (Localizing theory). (1) A localizing theory $(\mathcal{L}, \partial)$ from a full subcategory $\mathcal{R}$ of $\text{RelEx}_{\text{strict}}$ to a triangulated category $(\mathcal{T}, \Sigma)$ is a pair of functors $\mathcal{L} : \mathcal{R} \to \mathcal{T}$ and a natural transformation $\partial : \mathcal{L}_q \to \Sigma \mathcal{L}_s$ between functors $E(\mathcal{R}) \overset{\cdot}{\to} \mathcal{R} \overset{\mathcal{L}}{\to} \mathcal{T}$ which sends a derived exact sequence $\mathcal{E} \overset{\cdot}{\to} \mathcal{F} \overset{\cdot}{\to} \mathcal{G}$ in $\mathcal{R}$ to a distinguished triangle $\mathcal{L}(\mathcal{E}) \overset{\mathcal{L}(\partial)}{\to} \mathcal{L}(\mathcal{F}) \overset{\mathcal{L}(\partial)}{\to} \Sigma \mathcal{L}(\mathcal{E})$ in $\mathcal{T}$.

(2) A localizing theory $(\mathcal{L}, \partial)$ is fine if $\mathcal{L}$ is a categorical homotopy invariant functor and $\mathcal{L}$ commutes with filtered colimits.

Remark 3.17. (1) The non-connective $K$-theory on $\text{RelEx}_{\text{consist}}$ studied in [Sch11], [Moc13] is a fine localization theory.

(2) (cf. [Moc13, 7.9]). Let $\mathcal{L}$ be a localization theory on a full subcategory $\mathcal{R}$ of $\text{RelEx}_{\text{strict}}$. Then

(i) $\mathcal{L}$ is a derived invariant functor. Namely if a morphism $\mathcal{E} \to \mathcal{F}$ in $\mathcal{R}$ is a derived equivalence, then the induced morphism $\mathcal{L}(\mathcal{E}) \to \mathcal{L}(\mathcal{F})$ is an isomorphism.

(ii) If further we assume that $\mathcal{R}$ is closed under extensions and if $\mathcal{L}$ is categorical homotopy invariant, then we can easily prove that $\mathcal{L}$ is an additive theory.

3.4 Nilpotent invariance

In this subsection, we define the notion about nilpotent invariant functors.

Definition 3.18 (Serre radical). Let $\mathcal{B}$ be an abelian category and $\mathcal{F}$ a full subcategory of $\mathcal{B}$. We write $\sqrt{\mathcal{F}}$ for intersection of all Serre subcategories which contain $\mathcal{F}$ and call it the Serre radical of $\mathcal{F}$.

For noetherian abelian categories, we give a characterization of Serre radicals of full subcategories.

Definition 3.19 (Admissible subquotient). Let $\mathcal{E}$ be an exact category and $a$ and $b$ objects in $\mathcal{E}$. We say that $a$ is an admissible subquotient of $b$ if there exists a filtration of inflations $b = b_0 \leftarrow b_1 \leftarrow b_2$ such that $b_1/b_2 \sim a$.

Proposition 3.20. (cf. [Her97, 3.1], [Gar09, 2.2]). Let $\mathcal{B}$ be a noetherian abelian category, $\mathcal{F}$ a full subcategory of $\mathcal{B}$ and $x$ an object in $\mathcal{B}$. Then $x$ is in $\sqrt{\mathcal{F}}$ if and only if there exists a finite filtration of admissible subobjects $x = x_0 \leftarrow x_1 \leftarrow x_2 \leftarrow x_3 \leftarrow \cdots \leftarrow x_n = 0$ such that for every $i < n$, $x_i/x_{i+1}$ is an admissible subquotient of an object $\mathcal{F}$.

Definition 3.21 (Nilpotent invariance). Let $\mathcal{R}$ be a full subcategory of $\text{RelEx}$ which contains $\text{AbCat}$ the category of essentially small abelian categories. A functor $\mathcal{N} : \mathcal{R} \to \mathcal{C}$ is nilpotent invariant if for any noetherian abelian category $\mathcal{B}$ and any full subcategory $\mathcal{F}$ such that $\sqrt{\mathcal{F}} = \mathcal{B}$ and $\mathcal{F}$ is closed under finite direct sums, sub- and quotient objects, the inclusion functor $\mathcal{F} \to \mathcal{B}$ induces an isomorphism $\mathcal{N}(\mathcal{F}) \sim \mathcal{N}(\mathcal{B})$ in $\mathcal{C}$.

Example 3.22. The connective and the non-connective $K$-theory are nilpotent invariant by dévissage theorem in [Qui73] and Theorem 7 in [Sch06].

4 Graded categories

In this section, we will introduce the notion of (noetherian) graded categories over categories and calculate a fine localizing theory of noetherian graded categories over noetherian abelian categories.
4.1 Fundamental properties of graded categories

As in the results [Ser55], [AZ94], [Pol05] and [GP08], the category of finitely generated graded objects underestudies the category of coherent sheaves over projective spaces. We define the notions of graded categories over categories and study the fundamental properties. It is an abstract version of graded modules. See for the motivational Example 4.17.

4.1. For a positive integer \( n \), we define the category \( \langle n \rangle \) as follows. The class of objects of \( \langle n \rangle \) is just the set of all natural numbers \( \mathbb{N} \). The class of morphisms of \( \langle n \rangle \) is generated by morphisms \( \psi^i_m : m \to m + 1 \) for any \( m \in \mathbb{N} \) and \( 1 \leq i \leq n \) which subject to the equalities \( \psi^{i+1}_m \psi^i_m = \psi^{i+1}_{m+1} \psi^i_m \) for each \( m \in \mathbb{N} \) and \( 1 \leq i, j \leq n \).

**Definition 4.2 (Graded categories).** For any positive integer \( n \) and any category \( C \), we put \( C_{gr}[n] := \mathcal{HOM}(\langle n \rangle, C) \) and call it the category of \( (n-) \)-graded category over \( C \). For any object \( x \) and any morphism \( f : x \to y \) in \( C_{gr}[n] \), we denote \( x(m) \), \( x(\psi^i_m) \) and \( f(m) \) by \( x_m \), \( \psi^i_m \) or \( f_m \) respectively.

**Remark 4.3.** We can calculate a (co)limit in \( C_{gr}[n] \) by term-wise (co)limit in \( C \). In particular, if \( C \) is additive (resp. abelian) then \( C_{gr}[n] \) is also additive (resp. abelian). Moreover if \( C \) is a category with cofibration (resp. an exact category), then \( C_{gr}[n] \) naturally becomes a category with cofibration (resp. an exact category). Here a sequence \( x \to y \to z \) is a conflation (resp. a morphism \( x \to y \) is a cofibration) if it is term-wisely in \( C \). Moreover if \( w \) is a class of morphisms in \( C \) which satisfies the axioms of Waldhausen categories (and its dual), then the class of all morphisms \( lw \) in \( C_{gr}[n] \) consisting of those morphisms \( f \) such that \( f_m \) is in \( w \) for all natural number \( m \) also satisfies the axioms of Waldhausen categories (and its dual).

We can prove the following lemma and corollary:

**Lemma 4.4.** Let \( C, D \) and \( I \) be categories and \( f : C \to D \) a functor. If \( f \) is faithful (resp. fully faithful), then \( \mathcal{HOM}(I, f) : \mathcal{HOM}(I, C) \to \mathcal{HOM}(I, D) \) is faithful (resp. fully faithful).

**Corollary 4.5.** Let \( f : C \to D \) be a functor between categories and \( n \) a positive integer. If \( f \) is faithful (resp. fully faithful), then the induced functor \( f_{gr}[n] : C_{gr}[n] \to D_{gr}[n] \) is faithful (resp. fully faithful).

4.6. For an exact category \( E \) and a positive integer \( n \), we denote the full subcategory of all noetherian objects in \( E_{gr}[n] \) by \( E'_{gr}[n] \). In particular if \( E \) is an abelian category then \( E'_{gr}[n] \) is a noetherian abelian category by Lemma 2.5. In this case, we call \( E'_{gr}[n] \) the noetherian \((n-) \)-graded category over \( E \).

**Definition 4.7 (Degree shift).** Let \( C \) be a category with a specific zero object \( 0 \) and \( k \) an integer. We define the functor \( (k) : C_{gr}[n] \to C_{gr}[n], x \mapsto x(k) \). For any object \( x \) and any morphism \( f : x \to y \) in \( C_{gr}[n] \), we define an object \( x(k) \) and a morphism \( f(k) : x(k) \to y(k) \) in \( C_{gr}[n] \) as follows. We put

\[
x(k)_m = \begin{cases} x_{m+k} & \text{if } m \geq -k \\ 0 & \text{if } m < -k \
\end{cases}, \quad \psi^{i,x}_m := \begin{cases} \psi^{i,x}_{m+k} & \text{if } m \geq -k \\ 0 & \text{if } m < -k \
\end{cases} \quad \text{and} \quad f(k)_m := \begin{cases} f_{m+k} & \text{if } m \geq -k \\ 0 & \text{if } m < -k \
\end{cases}
\]

For any object \( x \) in \( C_{gr}[n] \) and any positive integer \( k \), we have the canonical morphism \( \psi^k_{m+k}(= \psi^k) : x(-k) \to x(-k+1) \) defined by \( \psi^k_{m-k} : x(-k)_m = x_{m-k} \to x(-k+1)_m = x_{m-k+1} \) for each \( m \in \mathbb{N} \). We consider a pair \( (C_{gr}[n], (-1)) \) as an object in \( \text{EndCat} \).

**Remark 4.8.** If \( E \) is an exact category, then for any integer \( k \), the functor \( (k) : E_{gr}[n] \to E_{gr}[n] \) is exact. Moreover this functor induce the exact functor \( (k) : E'_{gr}[n] \to E'_{gr}[n] \).

**Definition 4.9.** For any natural numbers \( m \) and \( k \), any object \( x \) in \( C_{gr}[n] \) and any multi index \( i = (i_1, \cdots, i_n) \in \mathbb{N}^n \), we define the morphism \( \psi^k_i \) by

\[
\psi^i = (\psi^n)^{i_n}(\psi^{n-1})^{i_{n-1}} \cdots (\psi^2)^{i_2}(\psi^1)^{i_1}.
\]
Definition 4.10 (Free graded object). Let $\mathcal{C}$ be an additive category and $n$ a positive integer. We define the functor $F_{\mathcal{C}}[n] = F[n] : \mathcal{C} \to \mathcal{C}_{\text{gr}}[n]$ in the following way. For any object $x$ in $\mathcal{C}$, we define the object $F[n](x) = x[\{\psi^1\}_{1 \leq i \leq m}]$ in $\mathcal{C}_{\text{gr}}[n]$ as follows. We put

$$F[n](x)_m := \bigoplus_{i=(i_1, \cdots, i_n) \in \mathbb{N}^n} x_i$$

where $x_i$ is a copy of $x$. $x_i \left( \sum_{j=1}^n j_i = m \right)$ components of the morphisms $\psi^m_{x_i} F[n](x) : F[n](x)_m \to F[n](x)_{m+1}$ defined by $i : x_i \to x_{i+e}$ where $e_k$ is the $k$-th unit vector.

4.11. Let $\mathcal{C}$ be an additive category and $k$ a natural number. For any object $x$ in $\mathcal{C}_{\text{gr}}[n]$, we have the canonical morphism $F[n](x)(-k) \to x$ which is defined as follows. For any $m \geq k$ and any $i = (i_1, \cdots, i_n) \in \mathbb{N}^n$ such that $\sum_{j=1}^n j_i = m - k$, on the $x_i$ component of $F[n](x)(-k)_m$, the morphism is defined by $\psi^m_{x_i} : x_i \to x_m$.

Remark 4.12. Let $\mathcal{C}$ be an additive category. Then the functor $F[n] : \mathcal{C} \to \mathcal{C}_{\text{gr}}[n]$ is the left adjoint functor of the functor $\mathcal{C}_{\text{gr}}[n] \to \mathcal{C}$, $y \mapsto y_n$. Namely for any object $x$ in $\mathcal{C}$ and any object $y$ in $\mathcal{C}_{\text{gr}}[n]$, we have a functorial isomorphism $\text{Hom}_C(x, y_0) \cong \text{Hom}_{\mathcal{C}_{\text{gr}}}[n](F[n](x), y)$, which sends $f$ to $(F[n](x) F[n](f)) F[n](y_0) \to y$.

Example 4.13. For any objects $x$ and $y$ in an additive category $\mathcal{C}$, any positive integer $k$, and any family of morphisms $\{c_i = (i_1, \cdots, i_n) \in \mathbb{N}^n \text{ s.t. } \sum_{j=1}^n j_i = k \}$ from $x$ to $y$, we define the morphism $\sum_{j=1}^n c_j \psi^j : F[n](x)(-k) \to F[n](y)$ by $c_j : x_j \to x_{j+k}$ on its $x_i$ component to $x_{i+k}$ component.

Lemma 4.14. Let $\mathcal{A}$ be a noetherian abelian category and $n$ a positive integer. Then

1. For any object $x$ in $\mathcal{A}$, $F[n](x)$ is a noetherian object in $\mathcal{A}_{\text{gr}}[n]$. In particular, we have the exact functor $F_{\mathcal{A}}[n] : \mathcal{A} \to \mathcal{A}_{\text{gr}}[n]$.
2. For any object $x$ in $\mathcal{A}_{\text{gr}}[n]$, there exists a natural number $m$ such that the canonical morphism as in 4.11

$$\bigoplus_{k=0}^m F[n](x)(-k) \to x$$

is an epimorphism.

Proof. (1) We define the functor

$$F : \mathcal{A}_{\text{gr}}[n] \to \text{End}^n \text{Lex} \mathcal{A}, \ x \mapsto \left( \bigoplus x_i, \bigoplus \psi_m^1, \cdots, \bigoplus \psi_m^n \right)$$

where $\text{End}^n$ means the $n$-times iteration of the functor $\text{End}$. Since $\text{Lex} \mathcal{A}$ is Grothendieck abelian, the functor $\bigoplus$ is exact and therefore $\Gamma$ is an exact functor. Moreover for any morphism $f : x \to y$ in $\mathcal{A}_{\text{gr}}[n]$, the condition $\Gamma(f) = 0$ obviously implies the condition $f = 0$. Hence $\Gamma$ is faithful. We can easily check that for any object $x$ in $\mathcal{A}$, we have the canonical isomorphism $\Gamma(F[n](x)) \cong x[t_1, \cdots, t_n]$ and $x[t_1, \cdots, t_n]$ is a noetherian object in $\text{End}^n \text{Lex} \mathcal{A}$ by Theorem 2.17. Therefore $F[n](x)$ is noetherian in $\mathcal{A}_{\text{gr}}[n]$ by Lemma 2.6.

(2) We put $z_l = \text{Im} \left( \bigoplus_{k=0}^l F[n](x)(-k) \to x \right)$. Let us consider the ascending chain of subobjects in $x$

$$z_1 \hookrightarrow z_2 \hookrightarrow \cdots \hookrightarrow x.$$

Since $x$ is a noetherian object, there exists a natural number $m$ such that $z_m = z_{m+1} = \cdots$. We claim that the canonical morphism

$$y := \bigoplus_{k=0}^m F[n](x)(-k) \to x$$
is an epimorphism. If \( k \leq m \), \( y_k \to x_k \) is obviously an epimorphism. If \( k > m \), then we have the equalities
\[
\text{Im}(y_k \to x_k) = (z_m)_k = (z_k)_k = x_k.
\]
Therefore we get the desired result. \( \square \)

**Definition 4.15 (Finitely generated objects).** Let \( \mathcal{E} \) be an exact category.

1. An object \((x, u)\) in \( \text{End} \mathcal{E} \) is **finitely generated** if there exists an object \( y \in \mathcal{E} \) and an epimorphism \((y[t], t) \to (x, u)\) in \( \text{End}(\text{Lex} \mathcal{E}) \). Let us write \( \text{End}(\text{Lex} \mathcal{E})_f \) for the full subcategory of \( \text{End}(\text{Lex} \mathcal{E}) \) consisting of those finitely generated objects in \( \text{End}(\text{Lex} \mathcal{E}) \).
2. An object \( x \in \mathcal{E}_{gr}[n] \) is **finitely generated** if there exists a non-negative integer \( n \) such that the canonical morphism \( \bigoplus_{k=0}^{n} F[n](x_{-k}) \to x \) as in Remark 4.12 is an epimorphism. We denote the full subcategory of \( \mathcal{E}_{gr}[n] \) consisting of those finitely generated objects in \( \mathcal{E}_{gr}[n] \) by \( \mathcal{E}_{gr}[n]_f \).

**Remark 4.16.** Let \( f : B \to C \) be an exact functor from an exact category \( B \) to an exact category \( C \). Then
\[
\text{(1) For any object } x \in B, \text{ we have the equality } f_{gr}[n](x) = f(x).
\]
\[
\text{(2) Therefore if } B \text{ is an abelian category, then } f \text{ induces an exact functor } f_{gr}[n] : B_{gr}[n]_f \to C_{gr}[n]_f.
\]
\[
\text{(3) Moreover if } B \text{ is an essentially small noetherian abelian category, then we have } B_{gr}[n]_f = B'_{gr}[n]
\]
and \( \text{End}(\text{Lex} B)_{fg} = B[f] \) by Lemma 2.19 and Remark 2.19.

**Example 4.17.** For a ring with unit \( A \) and \( \mathcal{E} = \mathcal{M}_A, \mathcal{E}_{gr}[n]_f \) is just the category of finitely generated graded right \( A[t_1, \cdots, t_n] \)-modules \( \mathcal{M}_A[t_1, \cdots, t_n]_{gr} \).

**Proof.** Any object \( x \in \mathcal{M}_A[t_1, \cdots, t_n]_{gr} \) is considered to be an object in \( \mathcal{E}_{gr}[n]_f \) in the following way. Let us define the functor \( x' : (n) \to \mathcal{E} \) by \( k \mapsto x_k \) and \((\psi : k \to k+1) \mapsto (t_i : x_k \to x_{k+1})\). The association \( x \mapsto x' \) induces a category equivalence \( \mathcal{M}_A[t_1, \cdots, t_n]_{gr} \sim \mathcal{E}_{gr}[n]_f \).

**Definition 4.18 (Canonical filtration).** For any object \( x \in \mathcal{A}_{gr}[n] \), we define the canonical filtration \( F^*_x \) as follows. \( F_{-x} = 0 \) and for any \( m \geq 0 \),
\[
(F_m x)_k = \begin{cases} 
   x_k & \text{if } k \leq m \\
   \sum_{\sum_{j} i_j = k-m} \text{Im} \psi_i & \text{if } k > m
\end{cases}
\]

**Remark 4.19.** Since every object \( x \in \mathcal{A}'_{gr}[n] \) is noetherian, there is the minimal integer \( m \) such that \( F_m x = F_{m+1} x = \cdots \). In this case, we can easily prove that \( F_m x = x \). We call \( m \) **degree** of \( x \) and denote it by \( \deg x \).

### 4.2 Koszul homologies

In this subsection, we define the Koszul homologies of objects in \( \mathcal{A}'_{gr}[n] \) and as an application of the notion about Koszul homologies, we study a fine localizing theory of \( \mathcal{A}_{gr}[n] \). There is the ideology about Koszul duality for example \( \text{[Bei78]} \) in this subsection behind the use of the Koszul homologies.

**Definition 4.20 (Koszul complex).** Let \( \mathcal{C} \) be an additive category and \( n \) a positive integer. For any object \( x \in \mathcal{C}_{gr}[n] \), we define the **Koszul complex** \( \text{Kos}(x) \) associated with \( x \) as follows. \( \text{Kos}(x) \) is a chain complex in \( \mathcal{C}_{gr}[n] \) concentrated in degrees \( 0, \cdots, n \) whose component at degree \( k \) is given by \( \text{Kos}(x)_k : = \bigoplus_{i=(i_1, \cdots, i_n) \in [1]^n} x_i \) where \([1]^n\) is the totally ordered set \([0, 1]^n\) with the natural order and \( x_i \)
is a copy of \( x(-\sum_{j=1}^{n} i_j) \) and whose boundary morphism \( d_k^{\text{Kos}(x)} : \text{Kos}(x)_k \to \text{Kos}(x)_{k-1} \) is defined by \((-1)^{\sum_{j=1}^{n} i_j} x_j : x_i \to x_{i_1 + \cdots + i_n} \) on its \( x_i \) to \( x_{i_1 + \cdots + i_n} \) component where \( i_j \) is the \( j \)-th unit vector. The association \( x \mapsto \text{Kos}(x) \) defines the exact functor
\[
\text{Kos} : \mathcal{C}_{gr}[n] \to \text{Ch}(\mathcal{C}_{gr}[n]).
\]
Definition 4.21 (Koszul homologies). Let $\mathcal{E}$ be an idempotent complete exact category and $n$ a positive integer. We put $B := \text{Lex} \mathcal{E}$. We define the family of functors $\{ T_i : \mathcal{E}_{gr}[n] \to B_{gr}[n] \}$ by $T_i(x) := H_i(\text{Kos}(x))$ for each $x$. $T_i(x)$ is said to be the $i$-th Koszul homology of $x$. Let us notice that for any conflation $x \rightarrow y \rightarrow z$ in $\mathcal{E}_{gr}[n]$, we have a long exact sequence

$$\cdots \rightarrow T_i+1(z) \rightarrow T_i(x) \rightarrow T_i(y) \rightarrow T_i(z) \rightarrow T_{i-1}(x) \rightarrow \cdots.$$ 

Definition 4.22 (Torsion free objects). An object $x$ in $\mathcal{A}_{gr}'[n]$ is torsion free if $T_i(x) = 0$ for any $i > 0$. For each non-negative integer $m$, we denote the category of torsion free objects (of degree less than $m$) in $\mathcal{A}_{gr}'[n]$ by $\mathcal{A}_{gr,tf}'[n]$ (resp. $\mathcal{A}_{gr,tf,m}'[n]$). Since $\mathcal{A}_{gr,tf}'[n]$, $\mathcal{A}_{gr,tf,m}'[n]$ are closed under extensions in $\mathcal{A}_{gr}'[n]$, they become exact categories in the natural way.

Proposition 4.23. For any objects $x$ in $\mathcal{A}_{gr}'[n]$ and $y$ in $\mathcal{A}$, we have the following assertions.
(1) For any natural number $k$, $\mathcal{F}[n](y)(-k)$ is torsion free.
(2) For any positive integer $s$, the assertion $T_0(x)_k = 0$ for any $k \leq s$ implies $x_k = 0$ for any $k \leq s$.
(3) We have the equality

$$T_0(F_p x)_k = \begin{cases} 0 & \text{if } k > p \\ T_0(x)_k & \text{if } k \leq p. \end{cases}$$

(4) For any natural number $p$, there exists a canonical epimorphism

$$\alpha^p : \mathcal{F}[n](T_0(x)_p)(-p) \rightarrow F_p x/F_{p-1} x.$$ 

(5) For any natural number $p$, $T_0(\alpha^p)$ is an isomorphism.
(6) If $T_1(x)$ is trivial, then $\alpha^p$ is an isomorphism.

Proof. (1) Since the degree shift functor is exact, we have the equality $T_i(x(-k)) = T_i(x)(-k)$ for any natural numbers $i$ and $k$. Therefore we shall just check that $\mathcal{F}[n](y)$ is torsion free. If $\mathcal{A}$ is the category of finitely generated free $\mathbb{Z}$-modules $\mathcal{P}_{\mathbb{Z}}$ and $y = \mathbb{Z}$, then $\mathcal{F}[n](y)$ is just the $n$-th polynomial ring over $\mathbb{Z}$, $\mathbb{Z}[t_1, \cdots, t_n]$ and $T_i(\mathcal{F}[n](y))$ is the $i$-th homology group of the Koszul complex associated with the regular sequence $t_1, \cdots, t_n$. In this case, it is well-known that $T_i(\mathcal{F}[n](y)) = 0$ for $i > 0$. For general $\mathcal{A}$ and $y$, there exists an exact functor $\mathcal{P}_{\mathbb{Z}} \rightarrow \mathcal{A}$ which sends $\mathbb{Z}$ to $y$ and which induces $\text{Ch}(\mathcal{F}_{gr}[n]) \rightarrow \text{Ch}(\mathcal{A}_{gr}[n])$ and $\text{Kos}(\mathcal{F}[n](\mathbb{Z}))$ goes to $\text{Kos}(\mathcal{F}[n](y))$ by this exact functor.

Hence we obtain the equality $T_i(\mathcal{F}[n](y)) = 0$ for any positive integer $i$.

(2) First notice that we have the equalities

$$T_0(x)_k = \begin{cases} x_0 & \text{if } k = 0 \\ x_k/\text{Im}(\psi^1, \cdots, \psi^n) & \text{if } k > 0. \end{cases}$$

Therefore if $T_0(x)_k = 0$ for $k \leq s$, then we have $x_0 = 0$ and $x_k = \text{Im}(\psi^1, \cdots, \psi^n)$ for $k \leq s$. Hence inductively we notice that $x_k = 0$ for $k \leq s$.

Assertion (3) follows from direct calculation.

(4) We have the equality

$$(F_p x/F_{p-1} x)_k \overset{\sim}{\rightarrow} \begin{cases} 0 & \text{if } k < p \\ x_p/\text{Im}(\psi^1, \cdots, \psi^n) = T_0(x)_p & \text{if } k = p. \end{cases}$$

Therefore by Remark 4.12, we have the canonical morphism

$$\alpha^p : \mathcal{F}[n](T_0(x)_p)(-p) \rightarrow ((F_p x/F_{p-1} x)(p))(-p) = F_p x/F_{p-1} x.$$ 

One can check that the morphism is an epimorphism.

(5) By (1), we have the equalities

$$F_p x/F_{p-1} x \overset{\sim}{\rightarrow} T_0(\mathcal{F}[n](T_0(x)_p)(-p))_k \overset{\sim}{\rightarrow} \begin{cases} 0 & \text{if } k \neq p \\ x_p/\text{Im}(\psi^1, \cdots, \psi^n) & \text{if } k = p. \end{cases}$$
and $T_0(\alpha^p)_p = \text{id}$. Hence we get the assertion.

Let $K^p$ be the kernel of $\alpha^p$, we have short exact sequences

$$K^p \to \mathcal{F}[n](T_0(x))_p(-p) \to F_p x / F_{p-1} x,$$

$$F_{p-1} x \to F_p x \to F_{p-1} x.$$  

We call the long exact sequences of Koszul homologies associated with short sequences above (1), (II) respectively. By (1) and assertions (1) and (5), we have the isomorphism

$$T_1(F_p x / F_{p-1} x) \sim T_0(K^p).$$

We claim that the following assertion.

Claim. $T_1(F_p x / F_{p-1} x) = 0$ and $T_1(F_p x) = 0$.

We prove the claim by descending induction of $p$. For sufficiently large $p$, we have $T_1(F_p x) = T_1(x)$ and therefore it is trivial by the assumption. Then by (II) and (3), we have

$$T_0(K^p) = T_1(F_p x / F_{p-1} x) = 0.$$

Therefore by (2), we have $K^p = 0$. By (I) and (1), we have isomorphisms

$$0 = T_2(\mathcal{F}[n](T_0(x))_p(-p)) \sim T_2(F_p x / F_{p-1} x).$$

By (II), we get $T_1(F_{p-1} x) = 0$. Hence we prove the claim and by (2), we get the desired result. 

For an object $x$ in an additive category $B$, recall the definition of the polynomial object $x[t]$ in $\text{End} \ B$ from 2.15. We regard $\text{ExCat}$ the category of essentially small exact categories as the full subcategory of $\text{RelEx}_{\text{consist}}$. (See Example 3.5(1)).

**Theorem 4.24.** (1) The inclusion functor $A_{\text{gr}, tf}^c[n] \hookrightarrow A_{\text{gr}}^c[n]$ is a derived equivalence.  
(2) Let $\mathfrak{A} : \text{ExCat} \to \mathcal{C}$ be a categorical homotopy invariant additive theory. Then for any natural number $m$, the exact functor $a : A_{\text{gr}, tf, m}^c[n] \to A_{\text{gr}}^c[n] \times m + 1$ which is defined by sending an object $x$ in $A_{\text{gr}, tf, m}^c[n]$ to $(T_0(x)_k)_{0 \leq k \leq m}$ in $A_{\text{gr}}^c[n] \times m + 1$ induces an isomorphism

$$\mathfrak{A}(A_{\text{gr}, tf, m}^c[n]) \sim \bigoplus_{k=0}^m \mathfrak{A}(A)$$

in $\mathcal{C}$.

(3) Let $\mathcal{T}$ be a triangulated category closed under countable coproducts and $\mathfrak{L} : \text{RelEx}_{\text{consist}} \to \mathcal{T}$ a fine localizing theory. Then we have the canonical isomorphism between the polynomial object $\mathfrak{L}(A)$ and $(\mathfrak{L}(A_{\text{gr}}^c[n]), \mathfrak{L}((-1)))$:

$$\lambda_{A, n} : \mathfrak{L}(A)[t] \sim (\mathfrak{L}(A_{\text{gr}}^c[n]), \mathfrak{L}((-1)))$$

in $\text{End} \ T$ which makes the diagram in $\mathcal{T}$ below commutative for any natural number $k$:

$$\begin{array}{ccc}
\mathfrak{L}(A) & \sim & \mathfrak{L}(\mathcal{F}[n](-k)) \\
\mathfrak{L}(A)t^m & \sim & \mathfrak{L}(A_{\text{gr}}^c[n]).
\end{array}$$

**Proof.** (1) We apply Lemma 3.8 to $A_{\text{gr}}^c[n]$ and Koszul homologies. The assumption of Lemma 3.8 follows from Lemma 3.14 (2) and Proposition 4.23 (1).

(2) We define the exact functor $b : A_{\text{gr}}^c[n] \times m + 1 \to A'_{\text{gr}, tf, m}^c[n]$ by sending an object $(x_k)_{0 \leq k \leq m}$ in $A_{\text{gr}}^c[n] \times m + 1$ to $\bigoplus_{k=0}^m \mathcal{F}[n](x_k) \sim (-k)$ in $A'_{\text{gr}, tf, m}^c[n]$. By virtue of Proposition 4.23 (1), the functor $ab$ is canonically
isomorphic to the identity functor on $\mathcal{A} \times m + 1$. On the other hand, the identity functor on $\mathcal{A}_{\text{gr}, tf, m}^\prime[n]$ has an exact characteristic filtration $F_\bullet$ with $F_p x / F_{p-1} x \sim \mathcal{F}[n](T_0(x)_p)(-p)$ for any object $x$ in $\mathcal{A}_{\text{gr}, tf, m}^\prime[n]$ by Proposition 4.23 (6), so applying Lemma 3.13 we have the equalities

$$\text{id}_{\mathcal{A}_{\text{gr}, tf, m}^\prime[n]} = \sum_{p=1}^{m} \mathcal{A}(F_p / F_{p-1}) = \sum_{p=0}^{m} \mathcal{A}(\mathcal{F}[n](T_0(-)_p)(-p)) = \mathcal{A}(ba).$$

Therefore we have an isomorphism

$$\mathcal{A}(\mathcal{A}_{\text{gr}, tf, m}^\prime[n]) \xrightarrow{\sim} \bigoplus_{i=0}^{m} \mathcal{A}(\mathcal{A}).$$

(3) By Remark 3.17 (2) (i), for any integer $m$, we have an isomorphism

$$\mathcal{L}(\mathcal{A}_{\text{gr}, tf, m}^\prime[n]) \xrightarrow{\sim} \bigoplus_{i=0}^{m} \mathcal{L}(\mathcal{A}).$$

Finally by taking the filtered inductive limit and utilizing assertion (1) and Remark 3.17 (2) (i), we get the desired isomorphism.

$$\bigoplus_{i=0}^{\infty} \mathcal{L}(\mathcal{A}) \xrightarrow{\sim} \colim_{m \to \infty} \bigoplus_{i=0}^{m} \mathcal{L}(\mathcal{A}) \xrightarrow{\sim} \colim_{m \to \infty} \mathcal{L}(\mathcal{A}_{\text{gr}, tf, m}^\prime[n]) \xrightarrow{\sim} \mathcal{L}(\text{colim}_{m \to \infty} \mathcal{A}_{\text{gr}, tf, m}^\prime[n]) \xrightarrow{\sim} \mathcal{L}(\mathcal{A}_{\text{gr}, tf}^\prime[n]) \xrightarrow{\sim} \mathcal{L}(\mathcal{A}_{\text{gr}}^\prime[n])$$

\[\square\]

5 The main theorem

In this section, let us fix an essentially small noetherian abelian category $\mathcal{A}$. We consider the functor $- \otimes_{\mathcal{A}} \mathbb{Z}[t]$ from $\mathcal{A}$ to $\mathcal{A}[t]$ defined by sending an object $a$ in $\mathcal{A}$ to an object $a[t]$ in $\mathcal{A}[t]$. Since Lex $\mathcal{A}$ is Gorthendieck, we can easily check that the functor $- \otimes_{\mathcal{A}} \mathbb{Z}[t]$ is exact. Recall that we regard $\text{ExCat}$ the category of essentially small exact categories as the full subcategory or $\text{RelEx}_{\text{consist}}$. (See Example 3.5 (1)). The purpose of this section is to study the induced map from $- \otimes_{\mathcal{A}} \mathbb{Z}[t]$ on $K$-theory. More generally, we will prove the following theorem:

**Theorem 5.1.** Let $(\mathcal{T}, \Sigma)$ be a triangulated category closed under countable coproducts, $\mathcal{R}$ a full subcategory of $\text{RelEx}_{\text{consist}}$ which contains $\text{AbCat}$ the category of essentially small abelian categories and $\mathcal{L}: \mathcal{R} \to \mathcal{T}$ a nilpotent invariant fine localizing theory. Then the base change functor $- \otimes_{\mathcal{A}} \mathbb{Z}[t]: \mathcal{A} \to \mathcal{A}[t]$ induces an isomorphism

$$\mathcal{L}(\mathcal{A}) \xrightarrow{\sim} \mathcal{L}(\mathcal{A}[t]).$$

By taking $\mathcal{T}$, $\mathcal{R}$ and $\mathcal{L}$ to the stable category of spectra, $\text{ExCat}$ the category of essentially small exact categories and the non-connective $K$-theory, we get Theorem 1.1 from Theorem 5.1. From now on, let $(\mathcal{T}, \Sigma)$ be a triangulated category closed under countable coproducts and $(\mathcal{L}, \partial)$ a fine localizing theory $\mathcal{L}: \text{ExCat} \to \mathcal{T}$.

5.1 Nilpotent objects in $\mathcal{A}_{\text{gr}}^\prime[2]$

In this subsection, we will define the category $\mathcal{A}_{\text{gr}, nil}^\prime[2]$ of nilpotent objects in $\mathcal{A}_{\text{gr}}^\prime[2]$. We also study the relationship $\mathcal{A}_{\text{gr}}^\prime[2]$ with $\mathcal{A}[t]$ and calculate the $K$-theory of $\mathcal{A}_{\text{gr}, nil}^\prime[2]$. Recall from the introduction that Theorem 5.6 and Proposition 5.14 correspond to geometric motivational formulas, namely the localization and the purity formulas in the introduction respectively. For simplicity in this subsection, we write $\psi$ and $\phi$ for $\psi^1$ and $\psi^2$ respectively and for any object $x$ in $\mathcal{A}$ and we write $x[\psi, \phi]$ for a $\mathcal{F}[2](x)$. 

Definition 5.2. Let \( \mathcal{E} \) be an exact category. An object \( x \) in \( \mathcal{E}_{\text{gr}}[2] \) is \((\psi^*)\) nilpotent if there exists an integer \( n \) such that
\[
\psi^k_n = 0
\]
for any non-negative integer \( k \). We write \( \mathcal{E}_{\text{gr,nil}}[2] \) (resp. \( \mathcal{E}'_{\text{gr,nil}}[2], \mathcal{E}_{\text{gr,nil}}[2]_{\text{fg}} \)) for the full subcategory of \( \mathcal{E}_{\text{gr}}[2] \) (resp. \( \mathcal{E}'_{\text{gr}}[2], \mathcal{E}_{\text{gr}}[2]_{\text{fg}} \)) consisting of all nilpotent objects.

Lemma 5.3. The category \( \mathcal{A}'_{\text{gr,nil}}[2] \) is a Serre subcategory of \( \mathcal{A}'_{\text{gr}}[2] \). In particular \( \mathcal{A}'_{\text{gr,nil}}[2] \) is an abelian category.

Proof. The assertion that \( \mathcal{A}'_{\text{gr,nil}}[2] \) is closed under sub- and quotient objects and finite direct sum is easily proved. We can also easily prove the following assertion. For a short exact sequence \( x \rightarrow y \rightarrow z \) in \( \mathcal{A}'_{\text{gr}} \), let \( i \) and \( j \) be integers such that \( \psi^i_x = 0 \) and \( \psi^j_y = 0 \). Then we can easily prove that \( \psi^{i+j}_z = 0 \). Therefore \( \mathcal{A}'_{\text{gr,nil}}[2] \) is closed under extensions in \( \mathcal{A}'_{\text{gr}}[2] \).

Definition 5.4. Let \( \mathcal{E} \) be an essentially small exact category. We define the functor
\[
\bar{\Theta}_\mathcal{E}(= \bar{\Theta}) : \mathcal{E}_{\text{gr}}[2] \rightarrow \text{End Lex} \mathcal{E}
\]
which sends an object \( x \) in \( \mathcal{E}_{\text{gr}}[2] \) to an object \((\text{colim}_x, \text{colim} \phi_n)\) in \( \text{End Lex} \mathcal{E} \) where \( \text{colim}_x \) is an inductive limit of an ind system \((x_0 \xrightarrow{\psi_0} x_1 \xrightarrow{\psi_1} x_2 \xrightarrow{\psi_2} \cdots)\), namely \( \text{Coker}(\bigoplus_{n=0}^{\infty} x_n \xrightarrow{\text{id} - \psi_n} \bigoplus_{n=0}^{\infty} x_n) \) and \( \text{colim} \phi_n \) is an inductive limit of \( \{\phi_n\} \), namely, a morphism which is induced from \( \bigoplus_{n=0}^{\infty} \phi_n \).

Lemma 5.5. Let \( \mathcal{E} \) be an essentially small exact category. Then
(1) The functor \( \Theta : \mathcal{E}_{\text{gr}}[2] \rightarrow \text{End Lex} \mathcal{E} \) is an exact functor. Moreover if \( u : x \rightarrow y \) is an epimorphism in \( \mathcal{E}_{\text{gr}}[2] \), then \( \Theta(u) : \Theta(x) \rightarrow \Theta(y) \) is also an epimorphism in \( \text{End Lex} \mathcal{E} \).
(2) For any object \( x \) in \( \mathcal{E}_{\text{gr,nil}}[2] \), \( \Theta(x) \) is a zero object.
(3) For any object \( x \) in \( \mathcal{E} \) and any positive integer \( k \), \( \psi^k : x(-k) \rightarrow x \) induces an isomorphism \( \bar{\Theta}(\psi^k) = \Theta(x(-k)) \rightarrow \Theta(x) \) in \( \text{End Lex} \mathcal{E} \).
(4) For any object \( x \) in \( \mathcal{E}_{\text{gr}}[2] \) and any positive integer \( k \), \( \bar{\Theta}(x[\psi, \phi](-k)) \) is canonically isomorphic to \( x[k] \).
(5) For any object \( x \) in \( \mathcal{E}_{\text{gr}}[2]_{\text{fg}} \), \( \bar{\Theta}(x) \) is in \( \text{End Lex} \mathcal{E} \)_{\text{fg}}. We denote the induced functor \( \mathcal{E}_{\text{gr}}[2]_{\text{fg}} \rightarrow \text{End Lex} \mathcal{E} \)_{\text{fg}} by \( \bar{\Theta}_{\text{fg}} \).

In particular \( \bar{\Theta}_{\mathcal{A}'}_{\text{fg}} \) induces the exact functor \( \Theta : \mathcal{A}'_{\text{gr}}[2] / \mathcal{A}'_{\text{gr,nil}}[2] \rightarrow \mathcal{A}[\text{t}] \).

Proof. (1) The functor \( \Theta \) factors through the functor \( y_{\text{gr}}[2] : \mathcal{E}_{\text{gr}}[2] \rightarrow \text{Lex} \mathcal{E} \) which is induced from the yoneda embedding \( y : \mathcal{E} \rightarrow \text{Lex} \mathcal{E} \) and the colimit functor \( \text{colim}_0 : (\text{Lex} \mathcal{E})_{\text{gr}}[2] \rightarrow \text{End Lex} \mathcal{E} \). Obviously \( y_{\text{gr}}[2] \) is exact and preserves epimorphisms. Since \( \text{Lex} \mathcal{E} \) is a Grothendieck category, the functor \( \text{colim}_0 : \mathcal{HOM}(\text{N}, \text{Lex} \mathcal{E}) \rightarrow \text{Lex} \mathcal{E} \) is exact. In particular, we acquire the assertion that the functor \( \Theta \) is an exact functor and preserves epimorphisms.

(2) For any object \( x \) in \( \mathcal{E}_{\text{gr,nil}}[2] \), assume that \( \psi_x^{m,k} = 0 \) for any non-negative integer \( k \). Then \( \sum_{i=0}^{m-1} \psi_x^i \) is the inverse morphism of \( \text{id} - \bigoplus_{n=0}^{\infty} \psi_n \). Therefore \( \bar{\Theta}(x) = \text{Coker}(\bigoplus_{n=0}^{\infty} x_n \xrightarrow{\text{id} - \psi_n} \bigoplus_{n=0}^{\infty} x_n) \) is trivial.

(3) Obviously \( \ker(\psi^k : x(-k) \rightarrow x) \) and \( \text{Coker}(\psi^k : x(-k) \rightarrow x) \) are \( \psi \)-nilpotent in \( \text{Lex} \mathcal{E}_{\text{gr}}[2] \). Therefore \( \bar{\Theta} \) induces an isomorphism \( \bar{\Theta}(\psi^k) \) by the observation in the proof of (2).

(4) By assertion (3), we shall assume that \( k = 0 \). In this case we have the canonical isomorphisms
\[
\bar{\Theta}(x[\psi, \phi]) \sim \text{Coker}(\bigoplus_{n=0}^{\infty} x^n \xrightarrow{\psi^i \phi^j} \bigoplus_{n=0}^{\infty} x^n \xrightarrow{\psi^i \phi^j} \bigoplus_{n=0}^{\infty} x^n \xrightarrow{\psi^i \phi^j}) \sim \bigoplus_{n=0}^{\infty} x\phi^n
\]
where \( x\psi^i \phi^j \) and \( x\phi^n \) are copies of \( x \).
(5) For any object $x$ in $\mathcal{E}_{gr}[2]_{\mathfrak{fg}}$, there exists a non-negative integer $n$ such that the canonical morphism $\bigoplus_{k=0}^{n} x_k[\psi, \phi](-k) \to x$ is an epimorphism in $\mathcal{E}_{gr}[2]$. Then by (1) and (4), we have an epimorphism $\bigoplus_{k=0}^{n} t_k \to \Theta(x)$ in $\text{End } \mathcal{E}$. Therefore by Remark 2.19, $\Theta(x)$ is in $(\text{End } \mathcal{E})_{\mathfrak{fg}}$. 

**Theorem 5.6.** The functor $\Theta: \mathcal{A}'_{gr}[2]/\mathcal{A}'_{gr, nil}[2] \to \mathcal{A}[t]$ is an equivalence of categories.

To prove Theorem 5.6 we need to the following lemmata:

**Lemma 5.7.** Let $R$ be a ring with unit and let us consider the polynomial ring $R[t]$ over $R$ and let $M = \bigoplus_{n=0}^{\infty} M_n$ be a finitely generated graded right $R[t]$-module. If the map $1-t : M \to M$ is surjective, then $M$ is $t$-nilpotent.

**Proof.** Since $M$ is finitely generated by homogenous elements, we shall just check that for any homogenous element $y$ in $M_k$, there exists a positive integer $l$ such that $yt^l = 0$. By assumption, there exists an element $x = \sum_{j=0}^{m} x_j$ in $M$ such that we have the equality

$$x(1-t) = y$$

(2)

where $x_j$ is the $j$th homogenous component of $x$. By comparing the homogenous components of the equality (2), we notice that $x_j$ is equal to 0 if $0 \leq j \leq k - 1$ or $j = m$, if $j = k$ and $x_{j-1}t$ if $k+1 \leq j \leq m-1$. Therefore if $m \leq k$, we have $y_k = 0$ and if $m > k$, we have $yt^{m-k} = x_{k+1}t^{m-k-1} = \cdots = x_m = 0$. Hence we get the desired result. 

**5.8.** We prove that $\Theta$ is faithful. By Corollary 2.14 there exists a ring with unit $R$ and an exact fully faithful embeddings $j: \mathcal{A} \hookrightarrow \mathcal{M}_R$ and $k: \text{Lex } \mathcal{A} \hookrightarrow \text{Mod}(R)$ which makes the diagram below commutative:

$$\begin{array}{ccc}
\mathcal{A} & \xrightarrow{y_A} & \text{Lex } \mathcal{A} \\
\downarrow{j} & & \downarrow{k} \\
\mathcal{M}_R & \xrightarrow{\iota} & \text{Mod}(R)
\end{array}$$

where the functor $y_A$ is the yoneda embedding functor and $\iota$ is the canonical inclusion functor. Then the functor $j$ induces the fully faithful embedding

$$j': = j_{gr}[2]_{\mathfrak{fg}}: \mathcal{A}'_{gr}[2]_{\mathfrak{fg}} = \mathcal{A}'_{gr}[2] \hookrightarrow (\mathcal{M}_R)_{gr}[2]_{\mathfrak{fg}} = \mathcal{M}_{R[t_1, t_2]_{gr}}$$

which makes the diagram below commutative by virtue of Remark 4.16 and Example 4.17.

$$\begin{array}{ccc}
\mathcal{A}_{gr}[2] & \xrightarrow{i_{gr}[2]_{\mathfrak{fg}}} & \mathcal{M}_{R[t_1, t_2]_{gr}} \\
\downarrow{\Theta_{A, \mathfrak{fg}}} & & \downarrow{\Theta_{M_R, \mathfrak{fg}}} \\
\mathcal{A}[t] = (\text{End } \mathcal{A})_{\mathfrak{fg}} & \xrightarrow{\text{End}(k)} & \text{End } \text{Mod}(R).
\end{array}$$

For an object $x$ in $\mathcal{A}'_{gr}[2]$, assume that $\Theta_{A, \mathfrak{fg}}(x)$ is a zero object. Then by Lemma 5.7, $j'(x)$ is a $t_1$-nilpotent $R$-module and therefore $x$ is $\psi$-nilpotent. Hence $\Theta_A$ is faithful.

**Definition 5.9.** ($\psi$-free object). An object $z$ in $A'_{gr}[2]$ is $\psi$-free if a morphism $\psi_{n}: z_n \to z_{n+1}$ is a monomorphism for any non-negative integer $n$.

**Lemma 5.10.** For any object $y$ in $A'_{gr}[2]$, there exists a $\psi$-free object $z$ in $A'_{gr}[2]$ and an epimorphism $u: y \to z$ in $A'_{gr}[2]$ such that the object $\ker u$ is in $A'_{gr, nil}[2]$. 

18
Proof. For any non-negative integer \( n \), we denote the canonical morphism from \( y_n \) to \( \text{colim} y_j = \theta(y) \) by \( \iota_n : y_n \to \theta(y) \) and we put \( z_n := \text{Im} \iota_n \). Then we have the commutative diagrams below

\[
\begin{array}{ccc}
y_n & \xrightarrow{\psi_n} & y_{n+1} \\
\downarrow \iota_n & & \downarrow \iota_{n+1} \\
\text{colim} \psi_j & \xrightarrow{\Theta(y)} & \text{colim} \psi_{j+1} \\
\end{array}
\quad
\begin{array}{ccc}
y_n & \xrightarrow{\phi_n} & y_{n+1} \\
\downarrow \iota_n & & \downarrow \iota_{n+1} \\
\Theta(y) & \xrightarrow{\Theta(y)} & \Theta(y) \\
\end{array}
\]

Therefore \( \psi_n \) and \( \phi_n \) induce a morphism \( z_n \xrightarrow{\phi_n} z_{n+1} \) and a monomorphism \( z_n \xrightarrow{\iota_n} z_{n+1} \) for any non-negative integer \( n \). Then \( z = \{ z_n, \psi_n, \phi_n \} \) is a \( \psi \)-free object in \( A'_{gr}[2] \) and there exists a canonical short exact sequence

\[
\ker \mu \to y \xrightarrow{\mu} z.
\]

Notice that \( y \) is in \( A'_{gr}[2] \) and therefore \( z \) is also in \( A'_{gr}[2] \). Obviously \( \Theta(y) \xrightarrow{\Theta(\mu)} \Theta(z) = \Theta(y) \) is an isomorphism in \( A[t] \). Hence by \ref{5.8}, the object \( \ker \mu \) is in \( A'_{gr, nil}[2] \). \( \square \)

Lemma 5.11. (1) For any object \( x \) in \( A \), any \( \psi \)-free object \( y \) in \( A'_{gr}[2] \) and any morphism \( \Theta(x[\psi, \phi]) = x[t] \xrightarrow{\alpha} \Theta(y) \), there exists a non-negative integer \( k \) and a morphism \( u : x[\psi, \phi](-k) \to y \) in \( A'_{gr}[2] \) such that \( a = \Theta(x[\psi, \phi])^{\psi,\phi}_{\psi,\phi} x[\psi, \phi](-k) \xrightarrow{\alpha} \Theta(y) \).

(2) For any \( \psi \)-free object \( x \) and any object \( z \) in \( A'_{gr}[2] \) and any morphism \( \Theta(z) \xrightarrow{\alpha} \Theta(y) \), there exists a non-negative integer \( k \) and a morphism \( u : z[\psi, \phi](-k) \to y \) in \( A'_{gr}[2] \) such that \( \Theta(u) \) makes the diagram below commutative

\[
\begin{array}{ccc}
\Theta(z[\psi, \phi](-n-k)) & \xrightarrow{\Theta(\alpha(-k))} & \Theta(z(-k)) \\
\downarrow & & \downarrow \\
\Theta(z[\psi, \phi](-k)) & \xrightarrow{\Theta(\psi k)} & \Theta(z) \\
\downarrow \quad \downarrow \quad \downarrow \\
\Theta(y) & \xrightarrow{\Theta(\psi)} & \Theta(z) \\
\end{array}
\]

where the morphism \( \alpha : z[\psi, \phi](-n) \to z \) is the canonical morphism as in Remark \ref{4.12}.

Proof. (1) We denote the composition of morphisms \( x \to x[t] \xrightarrow{\psi} \Theta(y) \) in \( \text{Lex} \ A \) by \( a \) and the canonical morphism from \( y_n \) to \( \Theta(y) \) by \( \iota_n : y_n \to \Theta(y) \) for any non-negative integer \( n \). Since \( \text{Im} a \) is a quotient of \( x \) in \( \text{Lex} \ A \), it is noetherian by Lemma \ref{2.5} (1) and therefore an ascending chain of subobjects of \( \text{Im} a \), \( \{ \text{Im} a \cap \text{Im} \iota_n \}_{n \in \mathbb{N}} \) is stational, say \( \text{Im} a \cap \text{Im} \iota_k = \text{Im} a \cap \text{Im} \iota_{k+1} = \cdots \). Then since \( \text{Lex} \ A \) is Grothendieck, we have \( \text{Im} a = \text{colim} \text{Im} a \cap \text{Im} \iota_k = \text{Im} a \cap \text{Im} \iota_k \). Therefore the morphism \( a \) factors through morphisms \( x \xrightarrow{a'} y_k \) and \( y_k \xrightarrow{\iota_k} \Theta(y) \). By Remark \ref{4.12}, \( a' \) induces the desired morphism \( u : x[\psi, \phi](-k) \to y \).

(2) By applying assertion (1) to the morphism \( a \Theta(\alpha) : z[\psi, \phi](-n) \to \Theta(y) \), we get the assertion. \( \square \)

5.12. We prove that \( \Theta \) is full. Namely, for any objects \( x, y \) in \( A'_{gr}[2] \), we prove that the map

\[
\Theta : \text{Hom}_{A'_{gr}[2]}(x, y) \to \text{Hom}_{A[t]}(\Theta(x), \Theta(y))
\]

is surjective. By Lemma \ref{5.10}, we may assume that \( y \) is \( \psi \)-free. By Lemma \ref{4.14} (2), there exists a non-negative integer \( m \) such that the canonical morphism

\[
z := \bigoplus_{k=0}^{m} x_k[\psi, \phi](-k) \to x
\]

is an
epimorphism. Let \( u : \Theta(x) \to \Theta(y) \) be a morphism in \( \mathcal{A}[t] \). Then by Lemma 5.11 (2), there exists a non-negative integer \( l \) and a morphism \( u : z(-l) \to y \) which makes the right diagram below commutative

\[
\begin{array}{c}
\ker P(-l) & \xrightarrow{j} & z(-l) & \xrightarrow{\Theta(j)} & \Theta(\ker P(-l)) & \xrightarrow{\Theta(y)} & \Theta(z(-l)) & \xrightarrow{\Theta(x(-l))} & \Theta(x(-l)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & & & \Theta(u) & & \Theta(y) & & \Theta(y) \\
& & & & \Theta(\psi)(u) & & \Theta(x) & & \Theta(x) \\
\end{array}
\]

Since \( \Theta \) is faithful, \( uj \) is the zero morphism, \( u \) induces a morphism \( \bar{u} : x(-l) \to y \) in the left commutative diagram above and we have the equality \( a = \Theta(x \xleftarrow{\psi} x(-l) \xrightarrow{\bar{u}} y) \). Hence we get the desired result.

**Corollary 5.13.** The sequence

\[ \mathcal{A}_{\text{gr,nil}}'[2] \to \mathcal{A}_{\text{gr}}'[2] \xrightarrow{\Theta} \mathcal{A}[t]. \]

is derived exact. In particular, there exists a distinguished triangle

\[ \mathcal{L}(\mathcal{A}_{\text{gr,nil}}'[2]) \to \mathcal{L}(\mathcal{A}_{\text{gr}}'[2]) \xrightarrow{\Theta} \mathcal{L}(\mathcal{A}[t]) \to \mathcal{S}\mathcal{L}(\mathcal{A}_{\text{gr,nil}}'[2]) \]

in \( \mathcal{T} \).

**Proof.** We check the condition (*) in Example 3.15. Let \( y \to x \) be a monomorphism in \( \mathcal{A}_{\text{gr}}'[2] \) with \( y \) in \( \mathcal{A}_{\text{gr,nil}}'[2] \).

**Claim.** There exists an integer \( n \geq 0 \) such that \( y \cap \text{Im}(\psi^n : x(-n) \to x) \) is trivial.

If we prove the claim, then the composition \( y \to x \to \text{Coker}(\psi^n : x(-n) \to x) \) is the desired monomorphism.

**Proof of Claim.** Consider the faithful exact functor \( \Gamma : \mathcal{A}_{\text{gr}}'[2] \to \mathcal{A}[t_1, t_2], x \mapsto (\bigoplus x_n, \bigoplus \phi_n, \bigoplus \psi_n) \) in the proof of Lemma 4.14 (1). Then by the abstract Artin-Rees lemma 2.25, there exists an integer \( n_0 \geq 0 \) such that

\[ \text{Im}(t^n_{2, \Gamma(x)} : \Gamma(x) \to \Gamma(x)) \cap \Gamma(y) \supseteq \text{Im}(t^{n-n_0}_{2, \Gamma(x)} : \Gamma(x) \to \Gamma(x)) \cap \Gamma(y) \to \Gamma(y) \]

for any \( n \geq n_0 \). Moreover since \( y \) is a \( \psi \)-nilpotent object, the right hand side of the equality (3) is trivial for sufficiently large \( n \). By the faithfulness of \( \Gamma \), we obtain the result.

By Example 3.15 and Theorem 5.6 the sequence \( \mathcal{A}_{\text{gr,nil}}'[2] \to \mathcal{A}_{\text{gr}}'[2] \to \mathcal{A}[t] \) is derived exact.

Recall the definition of Serre radical from Definition 3.18.

**Proposition 5.14.** We regard \( \mathcal{A}_{\text{gr}}'[1] \) as a full subcategory of \( \mathcal{A}_{\text{gr,nil}}'[2] \) by the exact functor defined by sending an object \( (x, \psi^1) \) in \( \mathcal{A}_{\text{gr}}'[1] \) to an object \( (x, 0, \psi^1) \) in \( \mathcal{A}_{\text{gr,nil}}'[2] \). Then \( \mathcal{A}_{\text{gr}}'[1] \) is closed under taking finite direct sums, admissible sub- and quotient objects in \( \mathcal{A}_{\text{gr,nil}}'[2] \) and the inclusion functor \( \mathcal{A}_{\text{gr}}'[1] \hookrightarrow \mathcal{A}_{\text{gr,nil}}'[2] \) induces an isomorphism \( \mathcal{L}(\mathcal{A}_{\text{gr}}'[1]) \cong \mathcal{L}(\mathcal{A}_{\text{gr,nil}}'[2]) \).

**Proof.** Obviously \( \mathcal{A}_{\text{gr}}'[1] \) is closed under taking finite direct sums, admissible sub- and quotient objects in \( \mathcal{A}_{\text{gr,nil}}'[2] \). Moreover for any \( x \) in \( \mathcal{A}_{\text{gr,nil}}'[2] \), let us consider the filtration \( \{ \text{Im} \psi^k \}_{k \in \mathbb{N}} \) of \( x \). Then for each \( k \), \( \text{Im} \psi^k / \text{Im} \psi^{k+1} \) is isomorphic to an object in \( \mathcal{A}_{\text{gr}}'[1] \). The last assertion follows from nilpotent invariance of \( \mathcal{L} \).

For an object \( x \) in an additive category \( \mathcal{B} \), recall the definition of the polynomial object \( x[t] \) in \( \text{End} \mathcal{B} \) from 2.15.
Corollary 5.15. We have the canonical isomorphism between the polynomial object \( \mathcal{L}(A)[t] \) of \( \mathcal{L}(A) \) and \( \mathcal{L}(A'_{\text{gr,nil}}[2]) \):

\[ \lambda': \mathcal{L}(A)[t] \xrightarrow{\sim} \mathcal{L}(A'_{\text{gr,nil}}[2]). \]

Here for any non-negative integer \( m \), \( \mathcal{L}(A)t^m \rightarrow \mathcal{L}(A'_{\text{gr,nil}}[2]) \) is induced from an exact functor \( A \rightarrow A'_{\text{gr,nil}}[2] \) which sends an object \( a \) in \( A \) to \( (a[\phi](m), 0, \phi) \) in \( A'_{\text{gr,nil}}[2] \).

5.2 The proof of the main theorem

In this subsection, we will finish the proof of Theorem 5.1. For an object \( x \) in an additive category \( B \), recall the definition of the polynomial object \( x[t] \) in \( \text{End} \, B \) from 2.15. The key lemma is the following:

Lemma 5.16. There exists the commutative diagram below

\[
\begin{array}{ccc}
\mathcal{L}(A)[t] & \xrightarrow{\lambda'} & \mathcal{L}(A'_{\text{gr,nil}}[2]) \\
\text{id} - t & \downarrow & \\
\mathcal{L}(A)[t] & \xrightarrow{\lambda_{A,2}} & \mathcal{L}(A'_{\text{gr}}[2])
\end{array}
\]

where \( \mathcal{L}(A)[t] \) is the polynomial object of \( \mathcal{L}(A) \) and the right vertical morphism are induced from the inclusion functor \( A'_{\text{gr,nil}}[2] \rightarrow A'_{\text{gr}}[2] \).

Proof. For each \( k \geq 0 \), we consider the commutativity of diagram below

\[
\begin{array}{ccc}
\mathcal{L}(A)t^k & \rightarrow & \mathcal{L}(A'_{\text{gr,nil}}[2]) \\
\text{id} - t & \downarrow & \\
\mathcal{L}(A)[t] & \xrightarrow{\lambda_{A,2}} & \mathcal{L}(A'_{\text{gr}}[2])
\end{array}
\]

An object \( a \) in \( A \) goes to \( (a[\phi](-k), 0, \phi) \) by the compositions of the functors \( A \rightarrow A'_{\text{gr,nil}}[2] \rightarrow A'_{\text{gr}}[2] \) and notice that the functor \( F[2](-k) : A \rightarrow A'_{\text{gr}}[2] \) induces \( \mathcal{L}(A) \xrightarrow{t^k} \mathcal{L}(A)[t] \xrightarrow{\sim} \mathcal{L}(A'_{\text{gr}}[2]) \) by Theorem 4.24. On the other hand, for any object \( a \) in \( A \), there exists an exact sequence in \( A'_{\text{gr}}[2] \)

\[ a[\phi, \psi](-k - 1) \xrightarrow{\psi} a[\phi, \psi](-k) \rightarrow (a[\phi](-k), 0, \phi). \]

By Lemma 3.13, this implies that the diagram (1) is commutative.

Proof of Theorem 5.1. Consider the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{L}(A)[t] & \xrightarrow{\text{id} - t} & \mathcal{L}(A)[t] \\
\lambda' & \downarrow & \lambda_{A,2} & \downarrow & \lambda \downarrow & \downarrow \\
\mathcal{L}(A'_{\text{gr,nil}}[2]) & \xrightarrow{\mathcal{L}(\Delta)} & \mathcal{L}(A'_{\text{gr}}[2]) & \xrightarrow{\mathcal{L}(\Delta)} & \mathcal{L}(A)[t] & \xrightarrow{\partial} & \mathcal{L}(A'_{\text{gr,nil}}[2]).
\end{array}
\]

Since top horizontal line is a split exact sequence by Lemma 2.16 and \( \lambda_{A,2} \) and \( \lambda' \) are isomorphisms by Theorem 4.24 and Corollary 5.15, the bottom distinguished triangle is also split, namely \( \partial = 0 \). Hence \( \mathcal{L}(\Delta) \) is an isomorphism by the five lemma.
5.3 Generalized Vorst problem

In this subsection, we propose the generalized Vorst problem. We start by defining the notion of regularity for abelian categories. Recall the definition of homological dimensions of abelian categories from §2.3.

**Definition 5.17 (Regular abelian category).** Let \( B \) be an abelian category.

1. We denote the full subcategory of projective objects in \( B \) by \( P_B \).
2. \( B \) is regular if it is noetherian and the inclusion functor \( P_B \hookrightarrow B \) is a derived equivalence. The last condition is equivalent to the condition that \( \text{hdim } B \) is finite.

**Proposition 5.18.** (cf. [GM96, III 5.16, 5.19]). Let \( A \) be a noetherian abelian category.

1. For any projective object \( p \) in \( A \), \( p[t] \) is also a projective object in \( A[t] \). In particular, there exists the base change functor \( - \otimes_A Z[t] : P_A \to P_A[t] \) which sending an object \( p \) to \( p[t] \).
2. If \( A \) is regular, then \( A[t] \) is also regular.

**Proof.** (1) is proven in [GM96, III 5.19]. (2) follows from the proof of [GM96, III 5.20] and Theorem 2.17.

**Corollary 5.19.** Let \( T \) be a triangulated category, \( \mathcal{L} : \text{ExCat} \to T \) a nilpotent invariant localizing theory and \( A \) a regular noetherian essentially small abelian category. Then the base change functor \( - \otimes_A Z[t] : P_A \to P_A[t] \) induces an isomorphism

\[
\mathcal{L}(P_A) \cong \mathcal{L}(P_A[t]).
\]

**Proof.** The inclusion functors \( P_A \hookrightarrow A \) and \( P_A[t] \hookrightarrow A[t] \) induce the commutative diagram below:

\[
\begin{array}{ccc}
\mathcal{L}(P_A) & \xrightarrow{\mathcal{L}( - \otimes_A Z[t] )} & \mathcal{L}(P_A[t]) \\
\downarrow & & \downarrow \\
\mathcal{L}(A) & \xrightarrow{\mathcal{L}( - \otimes_A Z[t] )} & \mathcal{L}(A[t]).
\end{array}
\]

Here the vertical morphisms and the bottom horizontal morphism are isomorphisms by regularity of \( A \), Proposition 5.18 (2) and Theorem 5.1 respectively. Hence we obtain the result.

**Problem 5.20 (Generalized Vorst problem).** Is the converse of Corollary 5.19 true or not? More precisely, let \( A \) be a noetherian abelian category which has enough projective objects. Assume that for any positive integer \( r \), the base change functor \( P_A \to P_A[t_1, \ldots, t_r] \) induces an isomorphism on (connective) \( K \)-theory:

\[
K(P_A) \cong K(P_A[t_1, \ldots, t_r]).
\]

Then is \( A \) regular or not?

The problem has an affirmative answer if \( A \) is a category of finitely generated modules over a noetherian commutative ring essentially of finite type over a base field and if we assume the resolution of singularities. (See [CHW08] and [GH12].

**Acknowledgements.** The authors wish to express their deep gratitude to Marco Schlichting for instructing them in the proof of the abstract Hilbert basis theorem [2.17]. They also very thank to the referee for giving valuable comments.

**References**

[Alm74] G. Almkvist, *The Grothendieck ring of the category of endomorphisms*, J. Algebra 28 (1974), p.375-388.

[Alm78] G. Almkvist, *K-theory of endomorphisms*, J. Algebra 55 (1978), p.308-340.
[AZ94] M. Artin and J. J. Zhang, Noncommutative projective schemes, Advanced in Mathematics 109 (1994), p.228-287.

[Bar14] C. Barwick, On the algebraic \( K \)-theory of higher categories, preprint [arXiv:1204.3607v5] (2014).

[Bei78] A. A. Beilinson, Coherent sheaves on \( \mathbb{P}^n \) and problems linear algebras, Funct. Anal. Appl. 12 (1978), p.214-216.

[BM11] A. J. Blumberg and M. A. Mandell, Algebraic \( K \)-theory and abstract homotopy theory, Adv. Math. 226 (2011), p.3760-3812.

[BGT13] A. J. Blumberg, D. Gepner and G. Tabuada, A universal characterization of higher algebraic \( K \)-theory, Geom. Topol. 17 (2013), p.733-838.

[BKS07] A. B. Buan, H. Krause and O. Solberg, Support varieties: An ideal approach, Homology, Homotopy and Applications. vol. 9 (2007), p.45-74.

[Cis10] D. C. Cisinski, Invariance de la \( K \)-théorie par équivalences dérivées, J. \( K \)-theory 6 (2010), p.505-546.

[CT11] D. C. Cisinski and G. Tabuada, Non connective \( K \)-theory via universal invariants, Compos. Math. 147 (2011), p.1281-1320.

[CHW08] G. Cortiñas, C. Hasemeyer and C. A. Weibel, \( K \)-regularity, cdh-fibrant Hochschild homology and a conjecture of Vorst, J. Amer. Math. Soc. 21 (2008), p.547-561.

[GP08] G. A. Garkusha and M. Prest, Reconstructing projective schemes from Serre subcategories, J. of Algebra 319 (2008), p.1132-1153.

[GH12] T. Geisser and L. Hesselholt, On a conjecture of Vorst, Math. Z. 270 (2012), p.445-452.

[GM96] S. I. Gelfand and Yu. I. Manin, Methods of homological algebra, Springer-Verlag Berlin Heidelberg New York (1996).

[Gra77] D. Grayson, The \( K \)-theory of endomorphisms, J. Algebra 48 (1977), p.439-446.

[Gra87] D. Grayson, Exact sequences in algebraic \( K \)-theory, Illinois Journal of Mathematics, 31 (1987), p.598-617.

[Gro77] A. Grothendieck, Groupes de classes des catégories abéliennes et, triangulées, Complexes parfaits, SGA 5, Exposé VIII, Springer LNM 589 (1977), p.351-371.

[GSVW92] T. Gunnarsson, R. Schwänzl, R. M. Vogt and F. Waldhausen, An un-delooped version of algebraic \( K \)-theory, Journal of Pure and Applied Algebra 79 (1992), p.255-270.

[Her97] I. Herzog, The Ziegler spectrum of a locally coherent Grothendieck category, Proc. London Math. Soc. 74 (1997), p.503-558.

[HM13] T. Hiranouchi and S. Mochizuki, Delooping of relative exact categories, available at [arXiv:1304.0557] (2013).
[Wald85] F. Waldhausen, *Algebraic K-theory of spaces*, In Algebraic and geometric topology, Springer Lect. Notes Math. 1126 (1985), p.318-419.

[Yao96] D. Yao, *The K-theory of vector bundles with endomorphisms over a scheme*, J. Algebra 184 (1996), p.407-423.

SATOSHI MOCHIZUKI
*DEPARTMENT OF MATHEMATICS, CHUO UNIVERSITY, BUNKYO-KU, TOKYO, JAPAN.*
e-mail: mochi@gug.math.chuo-u.ac.jp

AKIYOSHI SANNAI
*GRADUATE SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF TOKYO, 3-8-1 KOMABA, MEGURO-KU, TOKYO, JAPAN.*
e-mail: sannai@ms.u-tokyo.ac.jp