ASYMPTOTIC BEHAVIOR OF NON-AUTONOMOUS FRACTIONAL STOCHASTIC LATTICE SYSTEMS WITH MULTIPLICATIVE NOISE

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Abstract. In this paper, we study the asymptotic behavior of non-autonomous fractional stochastic lattice systems with multiplicative noise. The considered systems are driven by the fractional discrete Laplacian, which features the infinite-range interactions. We first prove the existence of pullback random attractor in $\ell^2$ for stochastic lattice systems. The upper semicontinuity of random attractors is also established when the intensity of noise approaches zero.

1. Introduction. It is well known that lattice dynamical systems can be found in many theories such as physics, biology, chemical reaction and so on, see, e.g., [8, 9, 14]. Recently, there are many works on lattice systems, like traveling waves [3, 16, 36], the chaotic properties of solutions [15], and global attractors [5]. Stochastic lattice dynamical systems (SLDSs) arise naturally when uncertainties or random influences are taken into account. In the last few years, there are many publications concerning the dynamics of SLDSs. In particular, random attractors for SLDSs have been extensively studied, see [4, 17, 18, 19, 25, 42] for the autonomous case, and [6, 40, 41] for the non-autonomous case. We remark that the diffusion of SLDSs considered in [4, 6, 17, 18, 19, 25, 40, 41, 42, 43] is represented by discrete Laplacian which implies diffusive nearest neighbor interaction (or called local interaction) of the nodes.

Long-range interaction (or called nonlocal interaction) means interaction with nodes or other objects distanced arbitrary far from each others. For example, in neural networks, neurons interact with each other over large distances through their

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interconnecting nerve axons [2]. Generally, according to the coupling range, long-range interaction includes finite-range interactions and infinite-range interactions. Up to now, long-range interaction has been studied for different physical systems. For examples, traveling wave solution for lattice model with infinite-range interaction can be found in [3, 36], the continuum limit for discrete nonlinear Schrödinger equations with long-range lattice interactions are considered in [26, 27], attractors of neural lattice model are studied in [23, 24, 34] for finite-range interaction and [22, 39] for infinite-range interaction.

We emphasize that infinite-range interactions also arise naturally when considering the fractional discrete Laplacian. Fractional discrete Laplacian which is the fractional powers of the discrete Laplacian has been considered in [10, 11, 12, 13, 29]. The analysis of nonlocal discrete equations driven by fractional discrete Laplacian is performed in [13], where the pointwise formula and some properties concerning this operator are obtained, as well as the error estimates for the approximation problem and Schauder estimates in discrete Hölder spaces. By theories of analytic semigroups and cosine operators, the existence and uniqueness of almost periodic solution to the heat, wave and Schrödinger equations involving the fractional discrete Laplacian were established in [29]. The connection between fractional powers of the discrete Laplacian and the fractional derivative in the sense of Liouville was investigated in [11].

On the other hand, partial differential equation involving fractional Laplacian have attracted considerable attention recently due to their wide range of applications in turbulence, anomalous diffusion, finance, porous media flow, etc. For example, the extension problems for fractional Laplacian were studied in [7], and the asymptotic behavior of partial differential equation involving fractional Laplacian can be found in [32] for the deterministic case and [20, 21, 30, 31, 37, 38] for the stochastic case.

In this paper, we are interested in the long term behavior of solutions for the stochastic lattice equation with fractional discrete Laplacian. Given $s \in (0, 1)$ and $\tau \in \mathbb{R}$, we consider the following non-autonomous stochastic lattice system:

$$
\begin{align*}
\frac{du_i}{dt} + (-\Delta_d)^s u_i + \lambda u_i &= f_i(u_i, t) + g_i(t) + \alpha u_i \circ dW, \\
u_i(\tau) &= u_{\tau,i}, & i \in \mathbb{Z},
\end{align*}
$$

(1)

where $u_i \in \mathbb{R}$, $\lambda$ and $\alpha$ are positive numbers, $(-\Delta_d)^s$ is the fractional discrete Laplacian, $f(u, t) = (f_i(u_i, t))_{i \in \mathbb{Z}}$ is a nonlinearity, $g(t) = (g_i(t))_{i \in \mathbb{Z}} \in L^2_{\text{loc}}(\mathbb{R}, \ell^2)$, $W$ is a two-sided real-valued Wiener process on a probability space and $\circ$ means that the stochastic equation is understood in the sense of Stratonovich’s integration.

The fractional discrete Laplacian $(-\Delta_d)^s$ reduces to the discrete Laplacian $-\Delta_d$ if $s = 1$. As far as the authors are aware, there is no result available in the literature on the asymptotic behavior for fractional SLDSs (1). In this paper, we will investigate this problem and prove that equation (1) has a unique tempered random attractor for $s \in (0, 1)$. Finally, we establish the upper semicontinuity of random attractors when the intensity of noise approaches zero. It is worth mentioning that the fractional discrete Laplacian $(-\Delta_d)^s$ is nonlocal and hence deriving uniform estimates on the tails of solutions are much more involved than the discrete Laplacian $-\Delta_d$. In addition, lattice with fractional discrete Laplacian is different from the neural networks lattice with long-range interaction that are considered in [22, 23, 24, 34, 39].
The paper is organized as follows. In the next section, we present some definitions and fundamental results on fractional discrete Laplacian and introduce some assumptions. In Section 3, we derive uniform estimates on the solutions of system (1) and establish the existence of random attractor. The upper semicontinuity of random attractors is discussed in the last section.

2. Preliminaries. In this section, we first review some concepts and results for fractional discrete Laplace operator $(-\Delta_d)^s$. And then we give some assumptions for $f$ and $g$, which are important to prove the existence of pullback random attractor.

Denote by $\mathbb{Z}$ the set of integers. And we denote by $\ell^p (1 \leq p \leq \infty)$ the space of sequences $\{a_i\}_{i \in \mathbb{Z}}$ with the norm

$$
\|a\|_p := \left( \sum_{i \in \mathbb{Z}} |a_i|^p \right)^{1/p} < \infty, \quad 1 \leq p < \infty, \quad \|a\|_\infty := \sup_{i \in \mathbb{Z}} |a_i| < \infty, \quad p = \infty.
$$

In particular, $\ell^2$ is a Hilbert space with the inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$ given by

$$
(u, v) = \sum_{i \in \mathbb{Z}} u_i v_i, \quad \|u\| = \left( \sum_{i \in \mathbb{Z}} |u_i|^2 \right)^{1/2},
$$

for any $u = (u_i)_{i \in \mathbb{Z}}, v = (v_i)_{i \in \mathbb{Z}} \in \ell^2$.

For $0 \leq s \leq 1$, define $\ell_s$ by

$$
\ell_s = \{ u : \mathbb{Z} \to \mathbb{R} | \|u\|_{\ell_s} := \sum_{m \in \mathbb{Z}} \frac{|u_m|}{(1 + |m|)^{1+2s}} < \infty \}.
$$

Obviously, $\ell^p \subset \ell^q \subset \ell_s$ if $1 \leq p \leq q \leq \infty$ and $0 \leq s \leq 1$.

Now, we introduce the definitions of fractional discrete Laplace operator $(-\Delta_d)^s$ with $s \in (0, 1)$. As the fractional Laplace operator given in [28], there are numerous approaches to define the fractional discrete Laplace operator. The approaches mainly include: semigroup method (or Bochner subordination) [35], semidiscrete Fourier transform [11], extension problem [13] and Balakrishnan formula [33, Chapter 3]. For simplicity, we consider the case that the size of mesh equals 1 in this paper. For a function $u : \mathbb{Z} \to \mathbb{R}$, we use the notation $u_i = u(i)$ to denote the value of $u$ at the mesh point $i \in \mathbb{Z}$. The discrete Laplacian $-\Delta_d$ is given by,

$$
-\Delta_d u_i = 2u_i - u_{i-1} - u_{i+1}.
$$

For $0 < s < 1$ and $u_j \in \mathbb{R}$, the fractional discrete Laplacian $(-\Delta_d)^s$ is defined with the semigroup method (or Bochner subordination) as

$$
(-\Delta_d)^s u_j = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{t\Delta_d} u_j - u_j) \frac{dt}{t^{1+s}}, \quad (2)
$$

where $\Gamma$ denotes the Gamma function with $\Gamma(-s) = \int_0^\infty (e^{-r} - 1) \frac{dr}{r^{1+s}} < 0$ and $w_j(t) = e^{t\Delta_d} u_j$ is the solution to the semidiscrete heat equation

$$
\begin{align*}
\partial_t w_j &= \Delta_d w_j, & \text{in } \mathbb{Z} \times (0, \infty), \\
w_j(0) &= u_j, & \text{on } \mathbb{Z}.
\end{align*} \quad (3)
$$

It follows from the semidiscrete Fourier transform that the solution of (3) can be written as

$$
e^{t\Delta_d} u_j = \sum_{m \in \mathbb{Z}} G(j - m, t) u_m = \sum_{m \in \mathbb{Z}} G(m, t) u_{j-m}, \quad t \geq 0, \quad (4)
$$

where $G$ is the Green’s function associated with the discrete heat equation.

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where the semidiscrete heat kernel $G(m,t) = e^{-2t}I_{m}(2t)$ and $I_{\nu}$ is the modified Bessel function of order $\nu$.

By (2) and (4), we have the pointwise formula for $(-\Delta_d)^s$ presented in the following statement.

**Theorem 2.1.** [13, Theorem 1.1] Let $0 < s < 1$ and $u = (u_j)_{j \in \mathbb{Z}} \in \ell_s$. Then we have

$$(-\Delta_d)^s u_j = \sum_{m \in \mathbb{Z}, m \neq j} (u_j - u_m)K_s(j - m),$$

(5)

where the discrete kernel $K_s$ is given by

$$K_s(m) = \begin{cases} 4^s \Gamma(\frac{1}{2} + s) \cdot \frac{\Gamma(|m| - s)}{\Gamma(|m| + 1 + s)}, & m \in \mathbb{Z} \setminus \{0\}, \\ 0, & m = 0. \end{cases}$$

In addition, there exist positive constants $\hat{c}_s \leq \tilde{c}_s$ such that for any $m \in \mathbb{Z} \setminus \{0\}$,

$$\frac{\hat{c}_s}{|m|^{1+2s}} \leq K_s(m) \leq \frac{\tilde{c}_s}{|m|^{1+2s}}.$$

Theorem 2.1 shows that the fractional discrete Laplacian is a nonlocal operator on $\mathbb{Z}$ of order $2s$. Furthermore, it follows from Theorem 2.1 that $(-\Delta_d)^s u$ is a well defined bounded function whenever $u \in \ell^p$ ($1 \leq p \leq \infty$). In particular, we also find that, for $0 < s < 1$, if $u \in \ell^2$ then

$$(-\Delta_d)^s u \in \ell^2, \text{ satisfying } \|(-\Delta_d)^s u\| \leq 4^s\|u\|. \tag{6}$$

On the other hand, fractional discrete Laplacian can be given by means of the discrete Fourier transform. To introduce this method, we denote the following function of order $\sigma > 0$:

$$R_\sigma(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(4\sin^2\left(\frac{\theta}{2}\right)\right)^\sigma e^{-in\theta} d\theta, \quad n \in \mathbb{Z}. \tag{7}$$

A computation shows that the function $R_\sigma(n)$ can be explicitly written as

$$R_\sigma(n) = \frac{(-1)^n \Gamma(2\sigma + 1)}{\Gamma(1 + \sigma + n) \Gamma(1 + \sigma - n)}, \quad n \in \mathbb{Z}, \quad \sigma \in (0, \infty) \setminus \mathbb{N}.$$

Therefore, by using the semidiscrete Fourier transform and (7), we have the following expression for the fractional discrete Laplacian (please see [29] for more details).

**Theorem 2.2.** Let $0 < s < 1$. If $u \in \ell^2$ then

$$(-\Delta_d)^s u_j = (F_{\mathbb{Z}}^{-1}(4\sin^2\left(\frac{\theta}{2}\right))^s * u)_j$$

$$= \sum_{m \in \mathbb{Z}} R_s(j - m)u_m, \tag{8}$$

where $F_{\mathbb{Z}}^{-1}$ denotes the inverse semidiscrete Fourier transform and $*$ means convolution. Furthermore, $K_s(j) = -R_s(j)$ for all $j \in \mathbb{Z} \setminus \{0\}$ and $K_s(0) = \sum_{m \in \mathbb{Z}} R_s(m) = 0.$

In view of (6) and (8), by using Fubini’s Theorem, we have the following result of $(-\Delta_d)^s$. More detailed information can be found in [13, Lemma 6.2].
Lemma 2.3. Let \( u, v \in \ell^2 \). Then for every \( s \in (0, 1) \),
\[
((-\Delta_d)^s u, v) = ((-\Delta_d)^{\frac{s}{2}} u, (-\Delta_d)^{\frac{s}{2}} v) = \frac{1}{2} \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{Z}, m \neq j} (u_j - u_m)(v_j - v_m) \tilde{K}_s(j - m).
\]

Obviously, by Lemma 2.3 we have
\[
\|(-\Delta_d)^{\frac{s}{2}} u\|^2 = \frac{1}{2} \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{Z}, m \neq j} |u_j - u_m|^2 \tilde{K}_s(j - m) \quad \text{for} \quad u \in \ell^2. \tag{9}
\]

We now specify the probability space. Let \((\Omega, \mathcal{F}, P)\) be a probability space, where \( \Omega := \{ \omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0 \} \), the Borel \( \sigma \)-algebra \( \mathcal{F} \) is generated by the compact open topology of \( \Omega \), and \( P \) is the corresponding Wiener measure on \((\Omega, \mathcal{F})\). The Brownian motion has the form \( W(t, \omega) = \omega(t) \) for \( \omega \in \Omega \). Consider the Wiener shift \( \theta_t \) defined on the probability space \((\Omega, \mathcal{F}, P)\) by
\[
\theta_t \omega(\cdot) = \omega(t + \cdot) - \omega(t), \quad \omega \in \Omega, \quad t \in \mathbb{R}.
\]
It is known that the probability measure \( P \) is an ergodic invariant measure for \( \theta_t \). Then \((\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})\) forms a metric dynamical system (see [1]). It follows from [1] that there exists a \( \{\theta_t\}_{t \in \mathbb{R}} \)-invariant subset \( \tilde{\Omega} \subseteq \Omega \) of full measure such that for each \( \omega \in \tilde{\Omega} \),
\[
\frac{\omega(t)}{t} \to 0 \quad \text{as} \quad t \to \pm \infty.
\]

For the sake of convenience, from now on, we will abuse the notation slightly and write the space \( \tilde{\Omega} \) as \( \Omega \).

From now on, we assume \( f \) and \( g \) satisfy the following conditions. For each \( i \in \mathbb{Z} \), we assume the nonlinear function \( f_i(x, t) : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is continuous, differentiable in \( x \) and \( f_i(0, t) = 0 \) for all \( t \in \mathbb{R} \). We further assume that \( f_i(x, t) \) is locally Lipschitz continuous with respect to \( x \) and satisfies, for all \( x, t \in \mathbb{R}, i \in \mathbb{Z} \),
\[
f_i(x, t) x \leq -\alpha_1 |x|^p + \eta_i(t), \tag{10}
\]
\[
|f_i(x, t)| \leq \alpha_2 |x|^{p-1} + \xi_i(t), \tag{11}
\]
\[
\frac{\partial}{\partial x} f_i(x, t) \leq \gamma_i(t), \tag{12}
\]
where \( p \geq 2 \), \( \alpha_1 \) and \( \alpha_2 \) are positive constants, \( \eta_i(\cdot) = (\eta_i(\cdot))_{i \in \mathbb{Z}} \in L_{\text{loc}}^{p}(\mathbb{R}, \ell^2) \), \( \xi_i(\cdot) = (\xi_i(\cdot))_{i \in \mathbb{Z}} \in L_{\text{loc}}^{q}(\mathbb{R}, \ell^2) \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), and \( \gamma_i(\cdot) = (\gamma_i(\cdot))_{i \in \mathbb{Z}} \in L_{\text{loc}}^{r}(\mathbb{R}, \ell^\infty) \).

When deriving uniform estimates on solutions, we need the following conditions:
\[
\int_{-\infty}^{0} e^{\lambda t}(\|g(l + \tau)\|^2 + \|\eta(l + \tau)\|_1) d\tau < \infty, \quad \forall \tau \in \mathbb{R}, \tag{13}
\]
and
\[
\lim_{r \to +\infty} e^{-c r} \int_{-\infty}^{0} e^{\lambda t}(\|g(l - r)\|^2 + \|\eta(l - r)\|_1) d\tau = 0, \quad \forall c > 0. \tag{14}
\]

In this paper, we let \( D = \{ D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \) be a family of bounded nonempty subsets of \( \ell^2 \) and use \( D \) to denote the collection of all families of tempered nonempty subsets of \( \ell^2 \), i.e.,
\[
D = \{ D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} : \lim_{t \to +\infty} e^{-ct}\|D(\tau - t - \tau_t, \omega)\| = 0, \quad \forall c > 0 \}.
\]
3. Random attractors of fractional stochastic lattice system. In this section, we first show uniform estimates on the solutions of fractional stochastic lattice equation. And then we establish the existence of pullback random attractor.

By assumptions on \( f \), we find that \((f_i(u_i, t))_{i \in \mathbb{Z}} \in \ell^2\) for all \( t \in \mathbb{R} \) and \( u = (u_i)_{i \in \mathbb{Z}} \in \ell^2\). Therefore, for \( s \in (0, 1), \tau \in \mathbb{R} \) and \( u = (u_i)_{i \in \mathbb{Z}} \in \ell^2\), the problem (1) can be written as the following one in \( \ell^2\):

\[
\begin{dcases}
\frac{du}{dt} + (-\Delta_d)^s u + \lambda u = f(u, t) + g(t) + \alpha u \circ dW, & t > \tau, \\
u(\tau) = u_\tau = (u_{\tau, i})_{i \in \mathbb{Z}} \in \ell^2.
\end{dcases}
\] (15)

In order to convert (15) into a pathwise deterministic equation, we use a standard transformation

\[v(t, \tau, \omega, v_\tau) = e^{-\alpha z(\theta_\tau \omega)} u(t, \tau, \omega, u_\tau)\] (16)

with \( v_\tau = e^{-\alpha z(\theta_\tau \omega)} u_\tau \), where the process \( z(\theta_\tau \omega) \) satisfies

\[dz(\theta_\tau \omega) + z(\theta_\tau \omega)dt = dW\]

and \( z(\omega) = -\int_{-\infty}^{0} e^{\tau} \omega(\sigma)d\sigma \) for \( \omega \in \Omega \). Furthermore, by [1], there exists a \( \theta_\tau \)-invariant set of full measure (still denoted by \( \Omega \)) such that \( z(\theta_\tau \omega) \) is continuous for every \( \omega \in \Omega \) and

\[
\lim_{t \to \pm \infty} \frac{|z(\theta_\tau \omega)|}{|t|} = 0, \quad \lim_{t \to \pm \infty} \frac{1}{t} \int_{0}^{t} z(\theta_\tau \omega)ds = 0.
\] (17)

Then \( v = (v_i)_{i \in \mathbb{Z}} \) satisfies

\[
\begin{dcases}
\frac{dv}{dt} + (-\Delta_d)^s v + \lambda v = \alpha z(\theta_\tau \omega)v + e^{-\alpha z(\theta_\tau \omega)} f(e^{\alpha z(\theta_\tau \omega)}v, t) + e^{-\alpha z(\theta_\tau \omega)} g(t), & t > \tau, \\
v(\tau) = v_\tau = (v_{\tau, i})_{i \in \mathbb{Z}} \in \ell^2.
\end{dcases}
\] (18)

For \( s \in (0, 1) \), it follows from (6) that the fractional discrete Laplace operator \((-\Delta_d)^s\) is bounded from \( \ell^2 \) to \( \ell^2 \). Moreover, from the assumption on \( f \), we know that the nonlinearity in (18) is locally Lipschitz continuous from \( \ell^2 \) to \( \ell^2 \). Therefore, by the standard theory of ordinary differential equations, under conditions (10)-(12) we find that problem (18) is well-posed in \( \ell^2 \) for every \( \omega \in \Omega \). Moreover, one can define a continuous cocycle for problem (15) in \( \ell^2 \) which is given by \( \Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times \ell^2 \to \ell^2 \) and for every \( t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega \) and \( u_\tau \in \ell^2 \),

\[\Phi(t, \tau, \omega, u_\tau) = u(t + \tau, \tau, \theta_{-\tau} \omega, u_\tau) = e^{\alpha z(\theta_\tau \omega)} v(t + \tau, \tau, \theta_{-\tau} \omega, v_\tau),\] (19)

with \( v_\tau = e^{-\alpha z(\omega)} u_\tau \).

Next, we derive uniform estimates of solutions for equation (18).

**Lemma 3.1.** Suppose (10)-(13) hold. Let \( \alpha_0 > 0, \tau \in \mathbb{R}, \omega \in \Omega \) and \( D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D} \). Then there exists \( T = T(\tau, \omega, D, \alpha_0) > 0 \) such that for all \( t \geq T \)
and $0 < \alpha \leq \alpha_0$, the solution $v$ of problem (18) satisfies
\[
\|v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 + \int_{-t}^{0} e^{\frac{5}{4}\lambda t - 2\alpha \int_{0}^{t} z(\theta_s \omega) \, ds} \|v(l + \tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 \, dl \\
+ \int_{-t}^{0} e^{(p-2)\alpha \int_{0}^{t} z(\theta_s \omega) \, ds} \|v(l + \tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|^p \, dl \\
+ \int_{-t}^{0} e^{\frac{5}{2}\lambda t - 2\alpha \int_{0}^{t} z(\theta_s \omega) \, ds} \|(-\Delta_d)^{\frac{1}{2}} v(l + \tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 \, dl \\
\leq M \int_{-\infty}^{0} e^{-2\alpha \int_{0}^{t} z(\theta_s \omega) \, ds} \|v\|^{2} + \|\eta(l + \tau)\|_1) \, dl,
\]
where $e^{\alpha \int_{0}^{t} z(\theta_s \omega) \, ds} \in D(\tau - t, \theta_{-\tau}\omega)$ and $M$ is a positive constant independent of $\tau, \omega, D$ and $\alpha$.

Proof. Taking the inner product of equation (18) with $v$, we obtain
\[
\frac{d}{dt}\|v\|^2 + 2\|(-\Delta_d)^{\frac{1}{2}} v, v\|^2 + 2\lambda \|v\|^2 = 2\alpha \int_{0}^{t} e^{\alpha \int_{0}^{t} z(\theta_s \omega) \, ds} v(t, v) \\
+ 2e^{-\alpha \int_{0}^{t} z(\theta_s \omega) \, ds} \|f(e^{\alpha \int_{0}^{t} z(\theta_s \omega) \, ds} v(t), v) + 2e^{-\alpha \int_{0}^{t} z(\theta_s \omega) \, ds} (g(t), v). \tag{20}
\]
For the nonlinear term in the above, by (10) one has
\[
e^{-\alpha \int_{0}^{t} z(\theta_s \omega) \, ds} \|f(e^{\alpha \int_{0}^{t} z(\theta_s \omega) \, ds} v(t), v) \leq \sum_{i \in \mathbb{Z}} e^{-\alpha \int_{0}^{t} z(\theta_s \omega) \, ds} \|f_i(e^{\alpha \int_{0}^{t} z(\theta_s \omega) \, ds} v_i, t) v_i \\
\leq \sum_{i \in \mathbb{Z}} e^{-\alpha \int_{0}^{t} z(\theta_s \omega) \, ds} \|f \|_{L^p} \|v\|^p + \sum_{i \in \mathbb{Z}} e^{-\alpha \int_{0}^{t} z(\theta_s \omega) \, ds} \|g(t)\| \leq \frac{\lambda}{8} \|v\|^2 + \frac{2}{\lambda} e^{-\alpha \int_{0}^{t} z(\theta_s \omega) \, ds} \|g(t)\|^2.
\]
From the above and (9), we obtain
\[
\frac{d}{dt}\|v\|^2 + \left(\frac{5}{4} \lambda - 2\alpha \int_{0}^{t} z(\theta_s \omega) \, ds\right) \|v\|^2 + \frac{\lambda}{2} \|v\|^2 + 2\|(-\Delta_d)^{\frac{1}{2}} v\|^2 + 2\alpha \int_{0}^{t} e^{(p-2)\alpha \int_{0}^{t} z(\theta_s \omega) \, ds} \|v\|^p \, dl \\
\leq e^{-2\alpha \int_{0}^{t} z(\theta_s \omega) \, ds} \|\eta(t)\|_1 + \frac{4}{\lambda} e^{-2\alpha \int_{0}^{t} z(\theta_s \omega) \, ds} \|g(t)\|^2. \tag{21}
\]
Multiplying (21) by $e^{\frac{5}{4}\lambda t - 2\alpha \int_{0}^{t} z(\theta_s \omega) \, ds} \, dl$ and then integrating the inequality on $(\tau - t, \tau)$, we get
\[
\|v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 + \frac{\lambda}{2} \int_{-t}^{0} e^{\frac{5}{4}\lambda t - 2\alpha \int_{0}^{t} z(\theta_s \omega) \, ds} \|v(l + \tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 \, dl \\
+ 2\alpha \int_{-t}^{0} e^{(p-2)\alpha \int_{0}^{t} z(\theta_s \omega) \, ds} \|v(l + \tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|^p \, dl \\
+ 2\int_{-t}^{0} e^{\frac{5}{2}\lambda t - 2\alpha \int_{0}^{t} z(\theta_s \omega) \, ds} \|(-\Delta_d)^{\frac{1}{2}} v(l + \tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 \, dl \\
\leq e^{\frac{5}{4}\lambda t - 2\alpha \int_{0}^{t} z(\theta_s \omega) \, ds} \|v_{\tau-t}\|^2 + 2\int_{-t}^{0} e^{-\alpha \int_{0}^{t} z(\theta_s \omega) \, ds} e^{\frac{5}{2}\lambda t - 2\alpha \int_{0}^{t} z(\theta_s \omega) \, ds} \|\eta(l + \tau)\|_1 \, dl \\
+ \frac{4}{\lambda} \int_{-t}^{0} e^{-2\alpha \int_{0}^{t} z(\theta_s \omega) \, ds} e^{\frac{5}{4}\lambda t - 2\alpha \int_{0}^{t} z(\theta_s \omega) \, ds} \|g(l + \tau)\|^2 \, dl. \tag{22}
\]
By (17), we find that there exists $T_1 = T_1(\omega, \alpha_0) > 0$ such that for all $t \geq T_1$,
\[
|z(\theta_{-\omega})| \leq \frac{1}{16} \frac{\lambda}{\alpha_0} t, \quad |\int_{-t}^{-l} z(\theta_{\omega}) dr| \leq \frac{1}{16} \frac{\lambda}{\alpha_0} l.
\] (23)

Therefore, for all $t \geq T_1$,
\[
\int_{-t}^{-T_1} e^{-2\alpha z(\theta_{\omega})} e^{\frac{t}{2} \lambda l - 2\alpha \int_0^t z(\theta_{\omega}) dr} \|\eta(l + \tau)\|_1 dl \\
\leq \int_{-t}^{-T_1} e^{\lambda l} \|\eta(l + \tau)\|_1 dl \leq \int_{-\infty}^{-T_1} e^{\lambda l} \|\eta(l + \tau)\|_1 dl,
\]
which along with (13) implies that for all $t \geq T_1$,
\[
2 \int_{-t}^{0} e^{-2\alpha z(\theta_{\omega})} e^{\frac{t}{2} \lambda l - 2\alpha \int_0^t z(\theta_{\omega}) dr} \|\eta(l + \tau)\|_1 dl \\
\leq 2 \int_{-t}^{0} e^{-2\alpha z(\theta_{\omega})} e^{\frac{t}{2} \lambda l - 2\alpha \int_0^t z(\theta_{\omega}) dr} \|\eta(l + \tau)\|_1 dl.
\] (24)

For the last term in (22), by the same argument as for (24), we obtain that for all $t \geq T_1$,
\[
\frac{4}{\lambda} \int_{-t}^{0} e^{-2\alpha z(\theta_{\omega})} e^{\frac{t}{2} \lambda l - 2\alpha \int_0^t z(\theta_{\omega}) dr} ||g(l + \tau)||^2 dl \\
\leq \frac{4}{\lambda} \int_{-\infty}^{0} e^{-2\alpha z(\theta_{\omega})} e^{\frac{t}{2} \lambda l - 2\alpha \int_0^t z(\theta_{\omega}) dr} ||g(l + \tau)||^2 dl,
\] (25)

where we have used (13).

Since $e^{\alpha z(\theta_{-\omega})} v_{t-\tau} \in D(\tau - t, \theta_{-\omega})$ and $0 < \alpha \leq \alpha_0$, by (23), we obtain that for all $t \geq T_1$,
\[
e^{-\frac{t}{2} \lambda l - 2\alpha \int_0^t z(\theta_{\omega}) dr} ||v_{t-\tau}||^2 = e^{-\frac{t}{2} \lambda l} e^{-2\alpha \int_0^t z(\theta_{\omega}) dr} \|z(\theta_{\omega})\|^2 \\
\leq e^{-\frac{t}{2} \lambda l} e^{2\alpha_0 \int_0^t z(\theta_{\omega}) dr} \|z(\theta_{\omega})\|^2 ||D(\tau - t, \theta_{-\omega})||^2 \\
\leq e^{-\frac{t}{2} \lambda l} ||D(\tau - t, \theta_{-\omega})||^2.
\]

By the fact that $D$ is tempered, thus there exists $T_2 = T_2(\tau, \omega, D, \alpha_0) > T_1$ such that for all $t \geq T_2$,
\[
e^{-\frac{t}{2} \lambda l - 2\alpha \int_0^t z(\theta_{\omega}) dr} ||v_{t-\tau}||^2 \\
\leq \int_{-\infty}^{0} e^{-2\alpha z(\theta_{\omega})} e^{\frac{t}{2} \lambda l - 2\alpha \int_0^t z(\theta_{\omega}) dr} \left( \frac{4}{\lambda} ||g(l + \tau)||^2 + 2 ||\eta(l + \tau)||_1 \right) dl,
\] (26)

which together with (22), (24) and (25) completes the proof.

Based on Lemma 3.1, we now prove the existence of tempered pullback absorbing set for $\Phi$ in $\ell^2$.

**Lemma 3.2.** Suppose (10)-(14) hold. Then for each $0 < \alpha \leq \alpha_0$, the continuous cocycle $\Phi$ of problem (15) has a closed measurable $D$-pullback absorbing set $K_\alpha = \{K_\alpha(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D$, which is given by, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,
\[
K_\alpha(\tau, \omega) = \{u \in \ell^2 : \|u\|^2 \leq R_\alpha(\tau, \omega)\}
\] (27)

where
\[
R_\alpha(\tau, \omega) = Me^{2\alpha z(\omega)} \int_{-\infty}^{0} e^{-2\alpha z(\theta_{\omega})} e^{\frac{t}{2} \lambda l - 2\alpha \int_0^t z(\theta_{\omega}) dr} (||g(l + \tau)||^2 + ||\eta(l + \tau)||_1) dl,
\] (28)

with $M$ being the same number as in Lemma 3.1.
Proof. By Lemma 3.1 we get
\[ \|v(\tau, \tau - t, \theta, \omega, v_{\tau - t})\|^2 \leq M \int_{-\infty}^{0} e^{-2\alpha z(\theta, \omega)} z^{\lambda - 2\alpha} \int_{0}^{\infty} e^{z(l + \tau)} (\|g(l + \tau)\|^2 + \|\eta(l + \tau)\|_1) \, dl. \] (29)

By (16) we have
\[ u(\tau, \tau - t, \theta, \omega, u_{\tau - t}) = e^{\alpha z(\omega)} v(\tau, \tau - t, \theta, \omega, v_{\tau - t}) \] (30)
with \( u_{\tau - t} = e^{\alpha z(\theta, \omega)} v_{\tau - t} \).

If \( u_{\tau - t} \in D(\tau - t, \theta, \omega) \), then \( e^{\alpha z(\theta, \omega)} v_{\tau - t} \in D(\tau - t, \theta, \omega) \), which together with (29) and (30) implies that there exists \( T = T(\tau, \omega, D, \alpha) > 0 \) such that for all \( t \geq T \),
\[ \|u(\tau, \tau - t, \theta, \omega, u_{\tau - t})\|^2 \leq Me^{2\alpha z(\omega)} \int_{-\infty}^{0} e^{-2\alpha z(\theta, \omega)} z^{\lambda - 2\alpha} \int_{0}^{\infty} e^{z(l + \tau)} (\|g(l + \tau)\|^2 + \|\eta(l + \tau)\|_1) \, dl. \]

From (19), we have
\[ \Phi(t, \tau - t, \theta, \omega, u_{\tau - t}) = u(\tau, \tau - t, \theta, \omega, u_{\tau - t}). \]
Thus, \( \Phi(t, \tau - t, \theta, \omega, u_{\tau - t}) \in K_\alpha(\tau, \omega) \) for all \( t \geq T \), and hence \( K_\alpha \) absorbs all elements of \( D \). Similar to the Lemma 3.2 of [37], one can easily check that \( K_\alpha \) is tempered. Moreover, for each \( \tau \in \mathbb{R} \), \( R_\alpha(\tau, \cdot) \) is \( (\mathcal{F}, \mathcal{B}(\mathbb{R})) \) measurable, so \( K_\alpha(\tau, \omega) \) is also measurable. This completes the proof. \( \square \)

To derive uniform estimates on the tails of solutions, we introduce a smooth function \( \rho(\zeta) \) defined for \( 0 \leq \zeta < \infty \) such that \( 0 \leq \rho(\zeta) \leq 1 \) for \( \zeta \geq 0 \) and
\[ \rho(\zeta) = \begin{cases} 0, & 0 \leq \zeta \leq 1, \\ 1, & \zeta \geq 2. \end{cases} \] (31)

Then there exists a positive constant \( \rho_0 \) such that \( |\rho'(\zeta)| \leq \rho_0 \) for all \( \zeta \geq 0 \). In addition, the cut-off function \( \rho \) has the following property.

Lemma 3.3. Let \( \rho \) be the smooth function defined in (31) and \( s \in (0, 1) \). Then for every \( j \in \mathbb{Z} \) and \( k \in \mathbb{N} \), we have
\[ \sum_{m \in \mathbb{Z} \atop m \neq j} |\rho\left(\frac{|j|}{k}\right) - \rho\left(\frac{|m|}{k}\right)|^2 K_s(j - m) \leq L_s^2 \frac{k^s}{n^s}, \] (32)
where \( L_s \) is a positive constant independent of \( j \) and \( k \).
Proof. Given \( j \in \mathbb{Z} \) and let \( n = j - m \), by Theorem 2.1, we have
\[
\sum_{m \in \mathbb{Z}} |\rho\left(\frac{|j|}{k}\right) - \rho\left(\frac{|m|}{k}\right)|^2 K_s(j-m) \leq \hat{c}_s \sum_{m \in \mathbb{Z}, m \neq j} \frac{|\rho\left(\frac{|j|}{k}\right) - \rho\left(\frac{|m|}{k}\right)|^2}{|j-m|^{1+2s}} + \hat{c}_s \sum_{|n| > k} \frac{|\rho\left(\frac{|j|}{k}\right) - \rho\left(\frac{|j-n|}{k}\right)|^2}{|n|^{1+2s}}.
\]
\[
\leq \hat{c}_s \sum_{m \in \mathbb{Z}, m \neq j} \frac{|\rho\left(\frac{|j|}{k}\right) - \rho\left(\frac{|m|}{k}\right)|^2}{|j-m|^{1+2s}} + \hat{c}_s \sum_{|n| > k} \frac{|\rho\left(\frac{|j|}{k}\right) - \rho\left(\frac{|j-n|}{k}\right)|^2}{|n|^{1+2s}}.
\]
\[
\leq \hat{c}_s \sum_{0 < |n| \leq k} \frac{1}{|n|^{2s-1}} + 4\hat{c}_s \sum_{|n| > k} \frac{1}{|n|^{1+2s}}
\]
\[
= \frac{2\hat{c}_s \rho_0^2}{k^2} \sum_{n \leq k, n \in \mathbb{Z}^+} \frac{1}{n^{2s-1}} + 8\hat{c}_s \sum_{n > k, n \in \mathbb{Z}^+} \frac{1}{n^{1+2s}}.
\]

For \( s \in (0, \frac{1}{2}) \), we have
\[
\sum_{n \leq k, n \in \mathbb{Z}^+} \frac{1}{n^{2s-1}} \leq \sum_{n=1}^{k-1} \frac{1}{n^{2s-1}} \leq \frac{1}{k^{2s-1}} \int_1^k \frac{1}{x^{2s-1}} dx + \frac{1}{k^{2s-1}} \leq \int_1^k \frac{1}{x^{2s-1}} dx + \frac{1}{k^{2s-1}} \leq \int_1^k \frac{1}{x^{2s-1}} dx + \frac{1}{k^{2s-1}} \leq \int_1^k \frac{1}{x^{2s-1}} dx + \frac{1}{k^{2s-1}} \leq (1 + \frac{1}{2} - 2s) \frac{1}{k^{2s-2}}.
\]
(34)

For \( s \in \left[\frac{1}{2}, 1\right) \), one has
\[
\sum_{n \leq k, n \in \mathbb{Z}^+} \frac{1}{n^{2s-1}} \leq 1 + \sum_{n=2}^{k} \int_{n-1}^{n} \frac{1}{x^{2s-1}} dx \leq \frac{1}{2} - 2s \frac{1}{k^{2s-2}}.
\]
(35)

It follows from (34) and (35) that for all \( s \in (0,1) \),
\[
\frac{2\hat{c}_s \rho_0^2}{k^2} \sum_{n \leq k, n \in \mathbb{Z}^+} \frac{1}{n^{2s-1}} \leq \hat{c}_s \rho_0^2 \left(2 + \frac{1}{2 - 2s} \right) \frac{1}{k^{2s}}.
\]
(36)

In addition, for \( s \in (0,1) \), we have
\[
\sum_{n > k, n \in \mathbb{Z}^+} \frac{1}{n^{1+2s}} \leq \sum_{n > k, n \in \mathbb{Z}^+} \int_{n-1}^{n} \frac{1}{x^{1+2s}} dx = \int_{k}^{k+m} \frac{1}{x^{1+2s}} dx,
\]
which implies that
\[
\sum_{n > k, n \in \mathbb{Z}^+} \frac{1}{n^{1+2s}} \leq \lim_{m \to \infty} \int_{k}^{k+m} \frac{1}{x^{1+2s}} dx \leq \frac{1}{2s} \frac{1}{k^{2s}}.
\]
(37)
It follows from (33), (36) and (37) that for all \( s \in (0, 1) \),

\[
\sum_{m \in \mathbb{Z} \atop m \neq j} |\rho\left(\frac{|j|}{k}\right) - \rho\left(\frac{|m|}{k}\right)|^2 \mathcal{K}_s(j - m) \leq \hat{c}_s \left( \rho_0^2 \left( 2 + \frac{1}{1 - s} \right) + \frac{4}{s} \right) \frac{1}{k^{2s}},
\]

which completes the proof. \( \square \)

Next, we establish uniform estimates on the tails of solutions in \( \ell^2 \), which play an important role in proving the asymptotic compactness of solutions.

**Lemma 3.4.** Suppose (10)-(13) hold. Let \( \alpha_0 > 0 \), \( \varepsilon > 0 \), \( \tau \in \mathbb{R} \), \( \omega \in \Omega \) and \( D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D} \). Then there exist \( T = T(\tau, \omega, D, \varepsilon) \geq 1 \) and \( K = K(\tau, \omega, \varepsilon) \geq 1 \) such that for all \( 0 < \alpha \leq \alpha_0 \), \( t \geq T \) and \( k \geq K \), the solution \( v \) of problem (18) satisfies

\[
\sum_{|i| \geq k} |v_i(\tau - t, \theta_{-\tau}\omega, v_{\tau-t,i})|^2 \leq \varepsilon,
\]

where \( e^{\alpha z(\theta_{-\tau}\omega)} v_{\tau-t} \in D(\tau - t, \theta_{-\tau}\omega) \).

**Proof.** Let \( \rho \) be the function defined by (31) and \( y = (y_i)_{i \in \mathbb{Z}} \) with \( y_i = \rho\left(\frac{|i|}{k}\right)v_i \).

Taking the inner product of equation (18) with \( y \), we get

\[
\frac{d}{dt} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right)|v_i|^2 + 2((-\Delta_d)^s v, y) + 2(\lambda - \alpha z(\theta_t\omega)) \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right)|v_i|^2
\]
\[
= 2e^{-\alpha z(\theta_t\omega)}(f(u, t), y) + 2e^{-\alpha z(\theta_t\omega)}(g(t), y). \tag{38}
\]

For the second term in (38), by Lemma 2.3 we have

\[
-((-\Delta_d)^s v, y) = -((-\Delta_d)^{\tilde{z}} v, -\Delta_d)^{\tilde{z}} y)
\]
\[
= -\frac{1}{2} \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{Z} \atop m \neq j} (v_j - v_m)(y_j - y_m) \mathcal{K}_s(j - m)
\]
\[
= -\frac{1}{2} \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{Z} \atop m \neq j} (v_j - v_m)\rho\left(\frac{|j|}{k}\right)v_j - \rho\left(\frac{|m|}{k}\right)v_m) \mathcal{K}_s(j - m)
\]
\[
= -\frac{1}{2} \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{Z} \atop m \neq j} (\rho\left(\frac{|j|}{k}\right) - \rho\left(\frac{|m|}{k}\right))(v_j - v_m)v_j \mathcal{K}_s(j - m)
\]
\[
-\frac{1}{2} \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{Z} \atop m \neq j} \rho\left(\frac{|m|}{k}\right)(v_j - v_m)^2 \mathcal{K}_s(j - m). \tag{39}
\]
By (9) and (32), one has

\[
\frac{1}{2} \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{Z}, m \neq j} \left| (\rho \left( \frac{|j|}{k} \right) - \rho \left( \frac{|m|}{k} \right) ) (v_j - v_m) v_j \tilde{K}_s (j - m) \right|
\leq \frac{1}{2} \|v\| \left[ \sum_{j \in \mathbb{Z}} \left( \sum_{m \in \mathbb{Z}, m \neq j} \left| \rho \left( \frac{|j|}{k} \right) - \rho \left( \frac{|m|}{k} \right) \right|^2 \tilde{K}_s (j - m) \right]
\times \left( \sum_{m \in \mathbb{Z}, m \neq j} \left| v_j - v_m \right|^2 \tilde{K}_s (j - m) \right)^{1/2}
\leq \frac{1}{2} \|v\| \frac{L_s}{k^s} \left( \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{Z}, m \neq j} \left| v_j - v_m \right|^2 \tilde{K}_s (j - m) \right)^{1/2}
= \frac{\sqrt{2}}{2} \frac{L_s}{k^s} \|v\| \left\| (-\Delta_d)^{\frac{3}{2}} v \right\|
\leq \frac{\sqrt{2}}{4} \frac{L_s}{k^s} (\|v\|^2 + \left\| (-\Delta_d)^{\frac{3}{2}} v \right\|^2).
\] (40)

Then it follows from (39) and (40) that

\[-((-\Delta_d)^s v, y) \leq \frac{\sqrt{2}}{4} \frac{L_s}{k^s} (\|v\|^2 + \left\| (-\Delta_d)^{\frac{3}{2}} v \right\|^2).
\] (41)

We now estimate the nonlinear term in (38), by (10) one has

\[
e^{-\alpha z (\theta_t \omega)} (f(u, t), y)
= e^{-\alpha z (\theta_t \omega)} \sum_{i \in \mathbb{Z}} \rho \left( \frac{|i|}{k} \right) f_i (e^{\alpha z (\theta_t \omega)} v_i, t) v_i
\leq -\alpha_1 e^{-2\alpha z (\theta_t \omega)} \sum_{i \in \mathbb{Z}} \rho \left( \frac{|i|}{k} \right) |e^{\alpha z (\theta_t \omega)} v_i|^p + e^{-2\alpha z (\theta_t \omega)} \sum_{i \in \mathbb{Z}} \rho \left( \frac{|i|}{k} \right) \eta_i (t)
\leq e^{-2\alpha z (\theta_t \omega)} \sum_{i \in \mathbb{Z}} \rho \left( \frac{|i|}{k} \right) |\eta_i (t)|.
\] (42)

For the last term in (38), by Young’s inequality we have

\[
e^{-\alpha z (\theta_t \omega)} (g(t), y) = e^{-\alpha z (\theta_t \omega)} \sum_{i \in \mathbb{Z}} \rho \left( \frac{|i|}{k} \right) g_i (t) v_i
\leq \frac{2}{3\alpha} e^{-2\alpha z (\theta_t \omega)} \sum_{i \in \mathbb{Z}} \rho \left( \frac{|i|}{k} \right) |g_i (t)|^2 + \frac{3}{8} \sum_{i \in \mathbb{Z}} \rho \left( \frac{|i|}{k} \right) |v_i|^2.
\] (43)
Substituting (41)-(43) into (38), we deduce
\[
\frac{d}{dt} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) |v_i|^2 + \left(\frac{5}{4} \lambda - 2 \alpha z(\theta, \omega)\right) \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) |v_i|^2 \\
\leq \frac{\sqrt{2} L_\sigma}{k^\alpha} (\|v\|^2 + \|(-\Delta_d)^{\frac{\alpha}{2}} v\|^2) + 2 e^{-2 \alpha z(\theta, \omega)} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) |\eta_i(t)| \\
+ \frac{4}{3\lambda} e^{-2 \alpha z(\theta, \omega)} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) |g_i(t)|^2.
\] (44)

Since \( s \in (0, 1) \) and \( L_\sigma \) is independent of \( k \), given \( \varepsilon > 0 \), there exists \( K_1 = K_1(\varepsilon) \geq 1 \) such that for all \( k \geq K_1 \),
\[
\frac{\sqrt{2} L_\sigma}{k^\alpha} (\|v\|^2 + \|(-\Delta_d)^{\frac{\alpha}{2}} v\|^2) \leq \varepsilon (\|v\|^2 + \|(-\Delta_d)^{\frac{\alpha}{2}} v\|^2),
\]
which along with (44) implies that for all \( k \geq K_1 \),
\[
\frac{d}{dt} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) |v_i|^2 + \left(\frac{5}{4} \lambda - 2 \alpha z(\theta, \omega)\right) \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) |v_i|^2 \\
\leq \varepsilon (\|v\|^2 + \|(-\Delta_d)^{\frac{\alpha}{2}} v\|^2) + 2 e^{-2 \alpha z(\theta, \omega)} \sum_{|i| \geq k} \left(\frac{2}{3\lambda} |g_i(t)|^2 + |\eta_i(t)|\right). 
\] (45)

Given \( t \in \mathbb{R}^+ \), \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), multiplying (45) by \( e^{\frac{\xi}{4} \lambda t - 2 \alpha \int_0^t z(\theta, \omega) dr} \) and integrating over \((\tau - t, \tau)\), we get, for all \( k \geq K_1 \),
\[
\sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) |v_i(\tau, \tau - t, \omega, v_{\tau-t,i})|^2 \leq e^{-\frac{\xi}{4} \lambda t - 2 \alpha \int_0^t z(\theta, \omega) dr} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) |v_{\tau-t,i}|^2 \\
+ \varepsilon \int_{\tau-t}^{\tau} e^{\frac{\xi}{4} \lambda (\tau - \tau - t, \omega) \int_0^{\tau-t} z(\theta, \omega) dr} (\|v(l, \tau - t, \omega, v_{\tau-t-i})\|^2 + \|(-\Delta_d)^{\frac{\alpha}{2}} v(l, \tau - t, \omega, v_{\tau-t-i})\|^2) dl \\
+ 2 \int_{\tau-t}^{\tau} e^{-2 \alpha z(\theta, \omega)} e^{\frac{\xi}{4} \lambda (\tau - \tau - t, \omega) \int_0^{\tau-t} z(\theta, \omega) dr} \sum_{|i| \geq k} \left(\frac{2}{3\lambda} |g_i(l)|^2 + |\eta_i(l)|\right) dl.
\]

Replacing \( \omega \) by \( \theta_{\tau-t,\omega} \) in above, we get, for all \( k \geq K_1 \),
\[
\sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) |v_i(\tau, \tau - t, \theta_{\tau-t,\omega}, v_{\tau-t,i})|^2 \leq e^{-\frac{\xi}{4} \lambda t - 2 \alpha \int_0^t z(\theta, \omega) dr} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) |v_{\tau-t,i}|^2 \\
+ \varepsilon \int_{\tau-t}^{\tau} e^{\frac{\xi}{4} \lambda (\tau - \tau - t, \theta_{\tau-t,\omega}) \int_0^{\tau-t} z(\theta, \omega) dr} (\|v(l, \tau - t, \theta_{\tau-t,\omega}, v_{\tau-t-i})\|^2 + \|(-\Delta_d)^{\frac{\alpha}{2}} v(l, \tau - t, \theta_{\tau-t,\omega}, v_{\tau-t-i})\|^2) dl \\
+ 2 \int_{\tau-t}^{\tau} e^{-2 \alpha z(\theta, \omega)} e^{\frac{\xi}{4} \lambda (\tau - \tau - t, \theta_{\tau-t,\omega}) \int_0^{\tau-t} z(\theta, \omega) dr} \sum_{|i| \geq k} \left(\frac{2}{3\lambda} |g_i(l + \tau)|^2 + |\eta_i(l + \tau)|\right) dl. 
\] (46)

We now estimate the right-hand side of (46). For the first term, since \( e^{\alpha z(\theta, \omega) v_{\tau-t}} \in D(\tau - t, \theta_{\tau-t,\omega}) \) and \( D \) is tempered, which together with (23) implies that for all
0 < \alpha \leq \alpha_0,
\begin{align*}
e^{-\frac{\lambda}{2} t - 2\alpha f_0^{-1} z(\theta, \omega) dr} \sum_{i \in Z} \rho\left(\frac{|i|}{k}\right) |v_{\tau-t, i}|^2 \\
\leq e^{-\frac{\lambda}{2} t - 2\alpha f_0^{-1} z(\theta, \omega) dr} \epsilon^2 D(\tau - t, \theta - \omega) \|D(\tau-t, \theta-\omega)\|^2 \\
\leq e^{-\lambda t} \|D(\tau-t, \theta-\omega)\|^2 \to 0 \quad \text{as} \quad t \to +\infty.
\end{align*}

Thus, there exists \( T_3 = T_3(\tau, \omega, D, \epsilon) > 0 \) such that for all \( t \geq T_3 \) and \( 0 < \alpha \leq \alpha_0 \),
\begin{equation}
\begin{align*}
e^{-\frac{\lambda}{2} t - 2\alpha f_0^{-1} z(\theta, \omega) dr} \sum_{i \in Z} \rho\left(\frac{|i|}{k}\right) |v_{\tau-t, i}|^2 \leq \epsilon.
\end{align*}
\end{equation}

By (23) we get, for all \( 0 < \alpha \leq \alpha_0 \),
\begin{align*}
&\int_{-\infty}^{\infty} e^{-2\alpha z(\theta, \omega) - \frac{\lambda}{2} t - 2\alpha f_0^{-1} z(\theta, \omega) dr} \|g(l + \tau)^2 + \|\eta(t + \tau)\|_1\| \, dl \\
&= \int_{-\infty}^{-T} e^{-2\alpha z(\theta, \omega) - \frac{\lambda}{2} t - 2\alpha f_0^{-1} z(\theta, \omega) dr} \|g(l + \tau)^2 + \|\eta(t + \tau)\|_1\| \, dl \\
&\quad + \int_{-T}^{0} e^{-2\alpha z(\theta, \omega) - \frac{\lambda}{2} t - 2\alpha f_0^{-1} z(\theta, \omega) dr} \|g(l + \tau)^2 + \|\eta(t + \tau)\|_1\| \, dl \\
&\quad \leq \int_{-\infty}^{-T} e^{\lambda t} \|g(l + \tau)^2 + \|\eta(t + \tau)\|_1\| \, dl \\
&\quad + e^{C_1} \int_{-T}^{0} e^{\lambda t} \|g(l + \tau)^2 + \|\eta(t + \tau)\|_1\| \, dl \\
&\quad \leq e^{C_1} \int_{-\infty}^{0} e^{\lambda t} \|g(l + \tau)^2 + \|\eta(t + \tau)\|_1\| \, dl,
\end{align*}

where \( C_1 = \frac{1}{2} T_1 + 2\alpha_0 \max_{-T_1 \leq t \leq 0} |z(\theta, \omega)|(1 + T_1). \)

By (13) and (48), there exists \( K_2 = K_2(\tau, \omega) \geq K_1 \) such that for all \( k \geq K_2 \) and \( 0 < \alpha \leq \alpha_0 \),
\begin{align*}
2\int_{-\infty}^{0} e^{-2\alpha z(\theta, \omega) - \frac{\lambda}{2} t - 2\alpha f_0^{-1} z(\theta, \omega) dr} \sum_{|i| \geq k} \frac{1}{2} |g_i(t + \tau)|^2 + |\eta_i(t + \tau)| \, dl \leq \epsilon.
\end{align*}

By Lemma 3.1 and (48), there exists \( T_4 = T_4(\tau, \omega) > 0 \) such that for all \( t \geq T_4 \) and \( 0 < \alpha \leq \alpha_0 \),
\begin{align*}
&\int_{-t}^{0} e^{\frac{\lambda}{2} t - 2\alpha f_0^{-1} z(\theta, \omega) dr} \|v(t + \tau, \tau - t, \theta - \omega, v_{\tau-t})\|^2 \\
&\quad + \|(-\Delta) v(t + \tau, \tau - t, \theta - \omega, v_{\tau-t})\|^2 \) \, dl \\
&\leq M \int_{-\infty}^{0} e^{-2\alpha z(\theta, \omega) - \frac{\lambda}{2} t - 2\alpha f_0^{-1} z(\theta, \omega) dr} \|g(l + \tau)^2 + \|\eta(t + \tau)\|_1\| \, dl \\
&\leq M e^{C_1} \int_{-\infty}^{0} e^{\lambda t} \|g(l + \tau)^2 + \|\eta(t + \tau)\|_1\| \, dl,
\end{align*}
which along with (13) implies that for all \( t \geq T_4 \) and \( 0 < \alpha \leq \alpha_0 \),
\[
\varepsilon \int_{-\tau}^{0} e^{\frac{\tau}{2}L-2\alpha} \int_{0}^{\tau} z(\theta, \omega) d\theta \left( \|v(l + \tau, \tau - t, \theta, v_{\tau-t})\|^2 \right)
\]
\[
+ \|(-\Delta_d)^2 v(l + \tau, \tau - t, \theta, v_{\tau-t})\|^2 \right) dl \leq \varepsilon C_2(\tau, \omega),
\]
where \( C_2(\tau, \omega) \) is a positive number depending on \( \tau \) and \( \omega \).

Let \( T = T(\tau, \omega, D, \varepsilon) = \max\{T_3, T_4\} \), it follows from (46), (47), (49) and (50) that for all \( t \geq T \), \( k \geq K_2 \) and \( 0 < \alpha \leq \alpha_0 \),
\[
\sum_{|i| \geq 2k} |v_i(\tau, \tau - t, \theta, v_{\tau-t})|^2 \leq \varepsilon (2 + C_2(\tau, \omega)),
\]
which concludes the proof. \( \square \)

Finally, we establish the existence of \( D \)-pullback random attractor for \( \Phi \) in \( \ell^2 \).

**Theorem 3.5.** Suppose (10)-(14) hold. Then for each \( 0 < \alpha \leq \alpha_0 \), the continuous cocycle \( \Phi \) associated with problem (15) has a unique \( D \)-pullback attractor \( \mathcal{A}_\alpha = \{\mathcal{A}_\alpha(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D \) in \( \ell^2 \).

**Proof.** Note that Lemma 3.2 shows that for each \( 0 < \alpha \leq \alpha_0 \) the cocycle \( \Phi \) has a closed measurable \( D \)-pullback absorbing set \( K_\alpha(\tau, \omega) \) as given by (27), and we can prove that \( \Phi \) is asymptotically null in \( \ell^2 \) by Lemma 3.4. Therefore, the existence of \( D \)-pullback attractor \( \mathcal{A}_\alpha(\tau, \omega) \) follows immediately from Theorem 3.6 in [6]. \( \square \)

4. **Upper semicontinuity of random attractors.** In this section, we consider the limiting behavior of the \( D \)-pullback random attractor \( \mathcal{A}_\alpha \) as \( \alpha \to 0 \). In the following, we write the continuous cocycle of problem (15) as \( \Phi_\alpha \) to indicate its dependence on \( \alpha \) and assume \( \alpha \in (0, 1] \). For the rest of this section, we denote by \( C_i \) \( (i = 1, 2, \ldots) \) the generic positive constants.

The lattice equation (1) with \( \alpha = 0 \) is given by
\[
\begin{align*}
\frac{du_i}{dt} + (-\Delta_d) u_i + \lambda u_i &= f_i(u_i, t) + g_i(t), \quad t > \tau, \\
u_i(\tau) &= u_{\tau-i}, \quad i \in \mathbb{Z},
\end{align*}
\]
which is the deterministic case.

For \( u = (u_i)_{i \in \mathbb{Z}} \in \ell^2 \), \( f(u, t) = (f_i(u_i, t))_{i \in \mathbb{Z}} \) and \( g(t) = (g_i(t))_{i \in \mathbb{Z}} \), the problem (51) can be written as the following one in \( \ell^2 \):
\[
\begin{align*}
\frac{du}{dt} + (-\Delta_d) u + \lambda u &= f(u, t) + g(t), \quad t > \tau, \\
u(\tau) &= u_{\tau}, \quad (u_{\tau-i})_{i \in \mathbb{Z}} \in \ell^2.
\end{align*}
\]

Similar to the discussion on system (15), one can prove that problem (52) generates a continuous cocycle \( \Phi_0 \) in \( \ell^2 \). Let \( \mathcal{D}_0 \) be a collection of families of nonempty subsets of \( \ell^2 \):
\[
\mathcal{D}_0 = \{\mathcal{D} = \{D(\tau) \subseteq \ell^2 : \tau \in \mathbb{R}\} : \lim_{t \to +\infty} e^{-ct}\|D(\tau - t)\| = 0, \forall c > 0\}.
\]
Note that all results in Section 3 are also valid for \( \alpha = 0 \). Therefore, \( \Phi_0 \) has a unique \( \mathcal{D}_0 \)-pullback attractor \( \mathcal{A}_0 = \{\mathcal{A}_0(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}_0 \) in \( \ell^2 \) and has a \( \mathcal{D}_0 \)-pullback absorbing set
\[
K_0(\tau) = \{u \in \ell^2 : \|u\|^2 \leq R_0(\tau)\},
\]
where

$$R_0(\tau) = M \int_{-\infty}^{\tau} e^{\frac{2}{\lambda}(\|g(l + \tau)\|^2 + \|\eta(l + \tau)\|_1)} dl,$$  \hspace{1cm} (54)$$

with $M$ is a positive constant.

By Lemma 3.2, (53) and (54), one has for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\limsup_{\alpha \to 0} \|K_\alpha(\tau, \omega)\| \leq \|K_0(\tau)\|.$$  \hspace{1cm} (55)$$

Next, we establish the convergence of solutions of (15) as $\alpha \to 0$. To this end, we need an additional condition on $f$: there exist $\alpha_3 > 0$ and $\kappa(t) = (\kappa_i(t))_{i \in \mathbb{Z}} \in L^{q_i}_1(\mathbb{R})$ with $q_1 = \frac{p}{p-2}$ for $p > 2$ and $q_1 = \infty$ for $p = 2$ such that for all $i \in \mathbb{Z}$ and $t, x \in \mathbb{R}$,

$$|\frac{\partial f_i}{\partial x}(x, t)| \leq \alpha_3|x|^{p-2} + \kappa_i(t).$$  \hspace{1cm} (56)$$

Let $\bar{v} = v_\alpha - u$, where $u_\alpha$ and $u$ are the solutions of problem (18) and (52), respectively. Thus we have

$$\left\{ \begin{array}{l}
\frac{d}{dt} \bar{v} + (-\Delta_d)^s \bar{v} + \lambda \bar{v} = \alpha z(\theta_t \omega) \bar{v} + \alpha z(\theta_t \omega) u \\
+ e^{-\alpha z(\theta_t \omega)} f(e^{\alpha z(\theta_t \omega)} v_\alpha, t) - f(u, t) + (e^{-\alpha z(\theta_t \omega)} - 1) g(t), \quad t > \tau, \\
\bar{v}(\tau) = \bar{v}_\tau = (\bar{v}_{\tau, i})_{i \in \mathbb{Z}} \in \ell^2, \quad i \in \mathbb{Z}.
\end{array} \right.$$  \hspace{1cm} (57)$$

**Lemma 4.1.** Suppose (10)-(12) and (56) hold. Let $u_\alpha(t, \tau, \omega, u_{\alpha, \tau})$ and $u(t, \tau, u_\tau)$ be the solutions of (15) and (52), respectively. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, $T > 0$ and $\varepsilon \in (0, 1)$, there exists $\hat{\alpha}_0 = \hat{\alpha}_0(\tau, \omega, T, \varepsilon) > 0$ such that for all $\alpha \leq \hat{\alpha}_0$ and $t \in [\tau, \tau + T]$,

$$\|u_\alpha(t, \tau, \omega, v_{\alpha, \tau}) - u(t, \tau, u_\tau)\|^2 \leq C\|u_{\alpha, \tau} - u_\tau\|^2 + C\varepsilon(1 + \|u_{\alpha, \tau}\|^2 + \|u_\tau\|^2),$$  \hspace{1cm} (58)$$

where $C$ is a positive constant independent of $\varepsilon$ and $\alpha$.

**Proof.** Taking the inner product of (57) with $\bar{v}$ in $\ell^2$, we obtain

$$\frac{d}{dt} \|\bar{v}\|^2 + 2((-\Delta_d)^s \bar{v}, \bar{v}) + 2\lambda \|\bar{v}\|^2 = 2\alpha z(\theta_t \omega)\|\bar{v}\|^2 + 2\alpha z(\theta_t \omega)(u, \bar{v})$$

$$+ 2\left( e^{-\alpha z(\theta_t \omega)} f(e^{\alpha z(\theta_t \omega)} v_\alpha, t) - f(u, t), \bar{v} \right) + 2(e^{-\alpha z(\theta_t \omega)} - 1)(g(t), \bar{v}).$$  \hspace{1cm} (59)$$

For each $\tau \in \mathbb{R}$, $\omega \in \Omega$, $T > 0$ and $\varepsilon \in (0, 1)$, by the continuity of $z(\theta_t \omega)$, there exists $\hat{\alpha}_1 = \hat{\alpha}_1(\tau, \omega, T, \varepsilon) > 0$ such that for all $\alpha \in (0, \hat{\alpha}_1)$ and $t \in [\tau, \tau + T]$,

$$|e^{\alpha z(\theta_t \omega)} - 1| \leq \varepsilon \quad \text{and} \quad |e^{-\alpha z(\theta_t \omega)} - 1| \leq \varepsilon,$$  \hspace{1cm} (60)$$
which along with (11), (12) and (56) imply that for all \( \alpha \in (0, \bar{\alpha}_1) \), \( p > 2 \) and \( t \in [\tau, \tau + T] \),
\[
\left( e^{-\alpha z(\theta, \omega)} f(e^{\alpha z(\theta, \omega)} v_\alpha, t) - f(u, t), \bar{v} \right) \\
= \sum_{i \in \mathbb{Z}} e^{-\alpha z(\theta, \omega)} \left( f_i(e^{\alpha z(\theta, \omega)} v_{\alpha, i}, t) - f_i(e^{\alpha z(\theta, \omega)} u_i, t) \right) \bar{v}_i \\
+ \sum_{i \in \mathbb{Z}} (e^{-\alpha z(\theta, \omega)} - 1) f_i(e^{\alpha z(\theta, \omega)} u_i, t) \bar{v}_i + \sum_{i \in \mathbb{Z}} (f_i(e^{\alpha z(\theta, \omega)} u_i, t) - f_i(u_i, t)) \bar{v}_i \\
\leq \|\gamma(t)\|_\infty \|\bar{v}\|^2 + \varepsilon \sum_{i \in \mathbb{Z}} (\alpha_2 e^{\alpha(p-1)z(\theta, \omega)} |u_i|^{p-1} |\bar{v}_i| + \xi_i(t) |\bar{v}_i|) \\
+ \varepsilon \sum_{i \in \mathbb{Z}} (\alpha_3 e^{\alpha z(\theta, \omega)} + |p-2| |u_i|^{p-1} |\bar{v}_i| + \kappa_i(t) |u_i| |\bar{v}_i|) \\
\leq C_1 \|\bar{v}\|^2 + C_1 \varepsilon (\|u\|_p^p + \|v_{\alpha}\|_p^p + \|\xi(t)\|_p^p + \|\kappa(t)\|_q^q + \|g(t)\|_q^q),
\]
where \( C_1 \) is a positive constant independent of \( \varepsilon \) and \( \alpha \). For the case of \( p = 2 \), it is easy to prove that (61) also holds.

It follows from (59) and (61) that there exists \( \bar{\alpha}_2 > 0 \) such that for all \( \alpha \in (0, \bar{\alpha}_2) \) and \( t \in [\tau, \tau + T] \),
\[
\frac{d}{dt} \|\bar{v}\|^2 \leq C_0 \|\bar{v}\|^2 + C_2 \varepsilon (\|u\|_p^p + \|v_{\alpha}\|_p^p + \|\xi(t)\|_p^p + \|\kappa(t)\|_q^q + \|g(t)\|_q^q).
\]
Applying Gronwall’s inequality on \( (\tau, t) \) we get, for all \( \alpha \in (0, \bar{\alpha}_2) \) and \( t \in [\tau, \tau + T] \),
\[
\|\bar{v}(t, \tau, \omega, \bar{v})\|^2 \leq C_3 \|\bar{v}_\tau\|^2 + C_3 \varepsilon + C_3 \varepsilon \int_\tau^t \|u(l, \tau, u_\tau)\|_p^p + \|v_{\alpha}(l, \tau, \omega, v_{\alpha, \tau})\|_p^p dl.
\]
It follows from the proof of Lemma 3.1 that for all \( \alpha \in (0, 1) \) and \( t \in [\tau, \tau + T] \),
\[
\|v_{\alpha}(t, \tau, \omega, v_{\alpha, \tau})\|^2 + \int_\tau^t \|v_{\alpha}(l, \tau, \omega, v_{\alpha, \tau})\|_p^p dl \leq C_4 \|v_{\alpha, \tau}\|^2 + C_4.
\]
Similarly, we also see that for all \( t \in [\tau, \tau + T] \),
\[
\|u(t, \tau, u_\tau)\|^2 + \int_\tau^t \|u(l, \tau, u_\tau)\|_p^p dl \leq C_5 \|u_\tau\|^2 + C_5.
\]
Let \( \bar{\alpha}_3 = \min\{1, \bar{\alpha}_2\} \). By the above estimates we know, for all \( \alpha \in (0, \bar{\alpha}_3) \) and \( t \in [\tau, \tau + T] \),
\[
\|v_{\alpha}(t, \tau, \omega, v_{\alpha, \tau}) - u(t, \tau, u_\tau)\|^2 \leq C_3 \|v_{\alpha, \tau} - u_\tau\|^2 + C_6 \varepsilon (1 + \|v_{\alpha, \tau}\|^2 + \|u_\tau\|^2),
\]
which along with (16) and (60) implies (58).

In what follows, we establish the precompactness of the union of pullback attractors in \( \ell^2 \).

**Lemma 4.2.** Suppose (10)-(14) hold. Then for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), there exists \( \alpha_0 \in (0, 1) \) such that \( \bigcup_{0 < \alpha \leq \alpha_0} A_\alpha(\tau, \omega) \) is precompact in \( \ell^2 \).

**Proof.** Let \( \alpha_0 \in (0, 1) \) be the number such that all estimates in Section 3 hold true. Given \( \varepsilon > 0 \), we will prove that \( \bigcup_{0 < \alpha \leq \alpha_0} A_\alpha(\tau, \omega) \) has a finite covering of balls of radius less than \( \varepsilon \).
Given \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), define
\[
R(\tau, \omega) = M e^{2\alpha_0|z(\omega)|} \int_{-\infty}^{0} e^{2\alpha_0|z(\theta \omega)|} e^{\alpha_0^2 \lambda l + 2\lambda l + \int_0^l z(\theta \omega) d\theta} \times (\|g(l + \tau)\|^2 + \|\eta(l + \tau)\|_1) dl
\]
and
\[
K(\tau, \omega) = \{ u \in \ell^2 : \|u\|^2 \leq R(\tau, \omega) \},
\]
where \( M \) is the same constant as in (28). By (27) we have, for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),
\[
\begin{align*}
0 < \alpha_0 &\leq \alpha_0 \subseteq \mathcal{A}_{\alpha}(\tau, \omega) \subseteq \mathbb{K}_{\alpha}(\tau, \omega) \subseteq \mathbb{K}(\tau, \omega), \\
\sum_{|i| \geq K} |u_i|^2 &\leq \frac{\varepsilon}{2}, \text{ for all } u = (u_i)_{i \in \mathbb{Z}} \in \bigcup_{0 < \alpha_0 \leq \alpha_0} \mathcal{A}_{\alpha}(\tau, \omega). \tag{64}
\end{align*}
\]
which along with Lemma 3.4 and the invariance of \( \mathcal{A}_{\alpha} \), we find that for every \( \varepsilon > 0 \) there exists \( K = K(\tau, \omega, \varepsilon) \geq 1 \) such that
\[
\sum_{|i| \geq K} |u_i|^2 \leq \frac{\varepsilon}{2}, \text{ for all } u = (u_i)_{i \in \mathbb{Z}} \in \bigcup_{0 < \alpha_0 \leq \alpha_0} \mathcal{A}_{\alpha}(\tau, \omega).
\]
On the other hand, by (63) we find that the set \( \{(u_i)_{|i| < K} : u \in \bigcup_{0 < \alpha_0 \leq \alpha_0} \mathcal{A}_{\alpha}(\tau, \omega)\} \)

is bounded in a finite dimensional space and hence is precompact, which together with (64) completes the proof.

Finally, we present the upper semicontinuity of \( \mathcal{A}_{\alpha} \) as \( \alpha \to 0 \).

**Theorem 4.3.** Suppose (10)-(14) and (56) hold. Then for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),
\[
\lim_{\alpha \to 0} d_{\ell^2}(\mathcal{A}_{\alpha}(\tau, \omega), \mathcal{A}_0(\tau)) = 0.
\tag{65}
\]

**Proof.** Let \( \alpha_n \to 0 \) and \( u_{\alpha_n, \tau} \to u_\tau \) in \( \ell^2 \), by Lemma 4.1 we find that for every \( t \in \mathbb{R}^+, \tau \in \mathbb{R} \) and \( \omega \in \Omega \),
\[
\Phi_{\alpha_n}(t, \tau, \omega, u_{\alpha_n, \tau}) \to \Phi(t, \tau, \omega) \quad \text{in } \ell^2,
\]
which along with (55) and Lemma 4.2, we see that all conditions of Proposition 2.3 in [6] are fulfilled and thus (65) follows immediately.

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