BETHE’S STATES FOR GENERALIZED XXX AND XXZ MODELS

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Abstract

For any rational number $p_0 \geq 1$ we prove an identity of Rogers–Ramanujan–Gordon–Andrews’ type. Bijection between the space of states for XXZ model and that of XXX model is constructed.

1 Introduction

The main goal of our paper is to study a combinatorial relationship between the space of states for generalized XXZ model and that for XXX one. In our previous paper [4] we gave a combinatorial description of states for generalized XXZ model in terms of the so–called $sl(2)$–XXZ rigged configurations. On the other hand it is well–known that when the anisotropy parameter $p_0$ of XXZ model goes to infinity then the XXZ model under consideration transforms to the XXX one. We are going to describe this transformation from combinatorial point of view in the case when $p_0$ is an integer.

A combinatorial completeness of Bethe’s states for generalized XXX–model had been proved in [1] and appeared to be a starting point for numerous applications to combinatorics of Young tableaux and representation theory of symmetric and general linear groups, see e.g. [2]. Here we mention only a ”fermionic” formula for the Kostka–Foulkes polynomials, see e.g. [2], and the relationship of the latter with $\hat{sl}(2)$–branching functions $b_k^{\Lambda_0}(q)$, see e.g. [3]. We will show in Section 1, Theorem 2.3 and Remark 2.5, that $q$–counting of the number of XXZ states using Bethe’s ansatz approach [5, 6], gives rise to the Rogers–Ramanujan–Gordon–Andrews’ type identity for any rational number $p_0 \geq 1$.

It seems an interesting problem to find a polynomial version of the Rogers–Ramanujan type identity (2.12) from our Theorem 2.3.

Another question which we are interested in is to understand a combinatorial nature of the limit

$$XXZ \underset{p_0 \to +\infty}{\longrightarrow} XXX.$$  

In Section 3 we describe a combinatorial rule which shows how the XXZ–configurations fall to the XXX pieces. For simplicity we consider in our paper only the case $p_0 > \sum_m s_m$. General case will be considered elsewhere.
2 Rogers–Ramanujan’s type identity

This paper is a continuation of our previous work [4]. Let us remind the main definitions, notation and results from [4].

For fixed $p_0 \in \mathbb{R}, p_0 \geq 1$ let us define (cf. [5]) a sequence of real numbers $p_i$ and sequences of integer numbers $\nu_i, m_i, y_i, z_i$:

\begin{align*}
p_0 &:= p_0, \quad p_1 = 1, \quad \nu_i = \left[ \frac{p_i}{p_{i+1}} \right], \quad p_{i+1} = p_{i-1} - \nu_{i-1}p_i, \quad i = 1, 2, \ldots \quad (2.1) \\
y_{i-1} = 0, \quad y_0 = 1, \quad y_1 = \nu_0, \quad y_{i+1} = y_{i-1} + \nu_i y_i, \quad i = 0, 1, 2, \ldots \quad (2.2) \\
z_{i-1} = 0, \quad z_0 = 1, \quad z_1 = \nu_1, \quad z_{i+1} = z_{i-1} + \nu_{i+1} z_i, \quad i = 0, 1, 2, \ldots \quad (2.3) \\
m_0 = 0, \quad m_1 = \nu_0, \quad m_{i+1} = m_i + \nu_i, \quad i = 0, 1, 2, \ldots \quad (2.4) \\
r(j) = i, \text{ if } m_i \leq j < m_{i+1}, \quad j = 0, 1, 2, \ldots \quad (2.5)
\end{align*}

It is clear that integer numbers $\nu_i$ define the decomposition of $p_0$ into continuous fraction

\[ p_0 = [\nu_0, \nu_1, \nu_2, \ldots] = \nu_0 + \frac{1}{\nu_1 + \frac{1}{\nu_2 + \ldots}}. \]

Let us define (see Fig. 1) a piecewise linear function $n_j, j \geq 0$,

\[ n_j := y_{i-1} + (j - m_i)y_i, \text{ if } m_i \leq j < m_{i+1}. \quad (2.6) \]

It is clear that for any integer $n > 1$ there exists the unique rational number $t$ such that $n = nt$.

Let us introduce additionally the following functions (see [4])

\[ q_j = (-1)^j (p_i - (j - m_i)p_{i+1}), \text{ if } m_i \leq j < m_{i+1}, \quad (2.7) \]

\[ \Phi_{k,2s} = \begin{cases} 
\frac{1}{2p_0}(q_k - q_k n_\chi), & \text{if } n_k > 2s, \\
\frac{1}{2p_0}(q_k - q_\chi n_k) + \frac{(-1)^{(k-1)}}{2}, & \text{if } n_k \leq 2s,
\end{cases} \]

where $2s = n_\chi - 1$.

In what follows we assume that the anisotropy parameter $p_0 \geq 1$ is a rational number, and $p_0 = [\nu_0, \nu_1, \ldots, \nu_\alpha]$ denotes its decomposition into continuous fraction. It is not difficult to see that

\[ p_0 = [\nu_0, \nu_1, \ldots, \nu_\alpha] = \frac{y_{\alpha+1}}{z_\alpha}, \]

\[ \bar{p}_0 := [\nu_0, \nu_1, \ldots, \nu_{\alpha-1}] = \frac{y_\alpha}{z_{\alpha-1}}, \]

where the numbers $\{y_j\}_{j=0}^{\alpha+1}$ and $\{z_j\}_{j=0}^{\alpha}$ are defined by (2.2) and (2.3) correspondingly. We assume that if $\alpha > 0$, then $\nu_\alpha \geq 2$. 
In order to formulate our main result of the paper [4] about the number of Bethe’s states for generalized XXZ model, let us consider the following symmetric matrix $\Theta^{-1} = (c_{ij})_{1 \leq i,j \leq m_\alpha+1}$:

\begin{enumerate}
  \item $c_{ij} = c_{ji}$ and $c_{ij} = 0$, if $|i - j| \geq 2$.
  \item $c_{j-1,j} = (-1)^{i-1}$, if $m_i \leq j < m_{i+1}$.
  \item $c_{jj} = \begin{cases} 2(-1)^i, & \text{if } m_i \leq j < m_{i+1} - 1, \quad i \leq \alpha, \\
                   (-1)^i, & \text{if } j = m_{i+1} - 1, \quad i \leq \alpha, \\
                   (-1)^{\alpha+1}, & \text{if } j = m_\alpha+1. \end{cases}$
\end{enumerate}

**Example 1** Let us take $p_0 = \frac{16}{7}$, then $p_0 = [2, 3, 2]$, $\alpha = 2$, $\tilde{p} = [2, 3] = \frac{7}{3}$,

\begin{align*}
  \nu_0 &= 2, \quad \nu_1 = 3, \quad \nu_2 = 2; \\
  m_0 &= 0, \quad m_1 = 2, \quad m_2 = 5, \quad m_3 = 7; \\
  y_0 &= 1, \quad y_1 = 2, \quad y_2 = 7, \quad y_3 = 16; \\
  z_0 &= 1, \quad z_1 = 3, \quad z_2 = 7.
\end{align*}
Therefore, $p_0 = \frac{y_{\alpha+1}}{z_\alpha}$, $\overline{p}_0 = \frac{y_\alpha}{z_{\alpha-1}}$, and

$$n_j = \begin{cases} 
  j, & \text{if } 0 \leq j < 2, \\
  1 + 2(j - 2), & \text{if } 2 \leq j < 5, \\
  2 + 7(j - 5), & \text{if } 5 \leq j < 7, \\
  7 + 16(j - 7), & \text{if } 7 \leq j.
\end{cases}$$

Finally,

$$\Theta = \frac{1}{16} \begin{pmatrix}
9 & 7 & 5 & 3 & 2 & 1 & 1 \\
7 & -7 & -5 & -3 & -2 & -1 & -1 \\
5 & -5 & -15 & -9 & -6 & -3 & -3 \\
3 & -3 & -9 & -15 & -10 & -5 & -5 \\
2 & -2 & -6 & -10 & 4 & 2 & 2 \\
1 & -1 & -3 & -5 & 2 & 9 & 9 \\
1 & -1 & -3 & -5 & 2 & 9 & -7
\end{pmatrix},$$

$$\Theta^{-1} = \begin{pmatrix}
1 & 1 \\
1 & -2 & 1 \\
1 & -2 & 1 \\
1 & -1 & -1 \\
-1 & 2 & -1 \\
-1 & 1 & 1 \\
1 & -1
\end{pmatrix},$$

and $\det |\Theta^{-1}| = 16$.

Note, see [4], Theorem 4.7, that the absolute value of the determinant $\det(\Theta^{-1})$ is equal to $y_{\alpha+1}$, i.e. the numerator of $p_0$.

To continue, let us consider the matrix $E = (e_{jk})_{1 \leq j,k \leq m_{\alpha+1}}$, where

$$e_{jk} = (-1)^{r(k)} (\delta_{j,k} - \delta_{j,m_{\alpha+1}} \cdot \delta_k - \delta_{j,m_{\alpha+1}} \cdot \delta_k - \delta_{j,m_{\alpha+1}} \cdot \delta_k - \delta_{j,m_{\alpha+1}} \cdot \delta_k).$$

One can check that the vacancy numbers $P_j(\lambda)$, see [4], (3.9), can be computed as follows

$$P_j(\lambda) + \lambda_j = ((E - 2\Theta)\tilde{x}^t + \tilde{b}^t)_j, \quad 1 \leq j \leq m_{\alpha+1},$$

where the $j$-th component of vector $b = (b_1, \ldots, b_{m_{\alpha+1}})$ is defined by the following formula

$$b_j = (-1)^{r(j)} \left( n_j \left\{ \frac{\sum 2s_m N_m - 2l}{p_0} \right\} - \sum m 2\Phi_{j,2s_m} \cdot N_m \right),$$

and for any sequence of integer numbers $\lambda = (\lambda_1, \ldots, \lambda_{m_{\alpha+1}})$ we denote by $\tilde{\lambda}$ the sequence $(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{m_{\alpha+1}})$, where $\tilde{\lambda}_j = (-1)^{r(j)} \lambda_j, \quad 1 \leq j \leq m_{\alpha+1}$. 4
Theorem 2.1 \([4]\) The number of Bethe’s states \(Z^{XXZ}(N, s \parallel l)\) of the generalized XXZ model is equal to

\[
\sum_{\lambda} \prod_{j} \left( \frac{(E - B) \tilde{\lambda}^t + b_j^t}{\lambda_j} \right),
\]

(2.8)

where summation is taken over all sequences of non-negative integer numbers \(\lambda = \{\lambda_k\}_{k=1}^{m_{a+1}}\) such that

\[
\sum_{k=1}^{m_{a+1}} n_k \lambda_k = l, \quad \lambda_k \geq 0;
\]

\[
\tilde{\lambda} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{m_{a+1}}), \quad \tilde{\lambda}_j = (-1)^{r(j)} \lambda_j, \quad B = 2\Theta.
\]

Recall, see \([4]\), or Section 3, that \(N\) and \(s\) in the above formula for the number of states \(Z^{XXZ}(N, s \parallel l)\) denote vectors \(N = (N_1, \ldots, N_k)\) and \(s = (s_1, \ldots, s_1, \ldots, s_k, \ldots, s_k)\), i.e. \(N_m\) is equal to the number of spins in the XXZ–chain which are equal to \(s_m\).

One of the main goal of the present paper is to consider a natural \(q\)–analog for (2.8).

Namely, let us define the following \(q\)–analog of the sum (2.8)

\[
\sum_{\lambda} q^{\frac{1}{2} \tilde{\lambda} B \tilde{\lambda}^t} \prod_{j} \left[ \frac{(E - B) \tilde{\lambda}^t + b_j^t}{\lambda_j} \right]_{q_{\epsilon_j}},
\]

(2.9)

where \(\epsilon_j = (-1)^{r(j)}\).

Let us recall that \(\begin{bmatrix} M \\ N \end{bmatrix}_q\) is the Gaussian \(q\)–binomial coefficient:

\[
\begin{bmatrix} M \\ N \end{bmatrix}_q = \begin{cases} \frac{(q; q)_M}{(q; q)_N(q; q)_{M-N}}, & \text{if } 0 \leq N \leq M, \\ 0, & \text{otherwise.} \end{cases}
\]

Remark 2.2 In our previous paper \([4]\), see (5.1) and (5.2), we had considered another \(q\)–analog of (2.8). It turned out however that the \(q\)–analog (5.1) from \([4]\), probably, does not possess good combinatorial properties.

One of the main results of the present paper is the following:

Theorem 2.3 Assume that \(p_0 \geq 1\) be a rational number, and consider a rational function

\[
V_{l}(q) := V_{l}(^{(p_0)})(q) = \sum_{\lambda} q^{\frac{1}{2} \tilde{\lambda} B \tilde{\lambda}^t} \prod_{j} \frac{(q^{(j)}; q^{(j)})_{\lambda_j}}{(q^{(j)}; q^{(j)})_{\lambda_j}},
\]

(2.10)
summation in (2.10) is taken over all sequences of non-negative integer numbers \( \lambda = \{ \lambda_k \}_{k=1}^{m_{\alpha+1}} \) such that
\[
l = \sum_{k \geq 1} n_k \lambda_k, \quad \lambda_k \geq 0.
\]

Then we have
\[
\sum_{l \geq 0} q^{p_0 l} V^{(p_0)}_l(q) = 1 + \sum_{k \geq 0} \sum_{m \geq 0} \frac{(-1)^{1+(-1)^{\alpha}}}{2} k+m q^{k \alpha+1} y \alpha+m y \alpha+1+m z \alpha+1+\Delta \alpha(k,m) \frac{Q^{(-1)^{\alpha}}_{k,m}(q)}{(q; q)_k},
\]

where
\[
Q^{(\pm)}_{k,m}(q) := \frac{Q^{(\pm)}_{k,m}(q)}{(-1)^{1+(-1)^{\alpha}}(k+m)},
\]
\[
\Delta \alpha(k,m) = \frac{1 + (-1)^{\alpha}}{2} \left( k \alpha m \right) + \frac{1 - (-1)^{\alpha}}{2} \left( m \alpha k \right).
\]

Let us emphasize that polynomials \( Q^{(\pm)}_{k,m}(q) \) are the same for all rational numbers \( p_0 \geq 1 \).

Taking the sum with respect to the index \( m \) in the RHS (2.11), we obtain

**Corollary 2.4** (Rogers–Ramanujan–Gordon–Andrews’ type identity)
\[
\sum_{l \geq 0} q^{p_0 l} V^{(p_0)}_l(q) = 1 + \sum_{k \geq 0} (-1)^{1+(-1)^{\alpha}} k q^{k \alpha+1} y \alpha+m y \alpha+1+m z \alpha+1+\Delta \alpha(k,k-m) Q^{(-1)^{\alpha}}_{k,k-m}(q).
\]

A proof of identity (2.11) is a “\( q \)-version” of that given in [4], Theorem 4.1.

**Remark 2.5** (Gordon–Andrews’ type identity) Let \( p_0 \geq 1 \) be an integer, then \( \alpha = 0 \), \( y_1 = p_0 \), \( y_0 = 1 \), \( z_0 = 1 \), \( z_{-1} = 0 \), and the RHS of (2.11) takes the following form
\[
1 + \sum_{k \geq 0} \sum_{m \geq 0} \frac{(-1)^{k+m}}{(k+m)} q^{k \alpha+1} (k \alpha m) + \Delta \alpha(k,k-m) Q^{(-1)^{\alpha}}_{k,k-m}(q).
\]

(2.13)
It is not difficult to see that if \( k > 0 \), the sum in the brackets (2.13) is equal to \((1 + q^k)(q; q)_k\). Hence, if \( p_0 \geq 1 \) is an integer, then we come to the following identity:

\[
\sum_{l \geq 0} q^{l^2} V^{(p_0)}_l(q) = 1 + \sum_{k>0} (-1)^k q^{k^2p_0 + \frac{k(k-1)}{2}} (1 + q^k).
\] (2.14)

Using the Jacobi triple identity, one can rewrite (2.14) in the following forms

\[
\sum_{l \geq 0} q^{l^2} V^{(p_0)}_l(q) = \prod_{n \geq 1} (1 - q^{(2p_0+1)n})(1 - q^{(2p_0+1)n-p_0-1})(1 - q^{(2p_0+1)n-p_0}),
\] (2.15)

\[
\frac{1}{(q; q)_{\infty}} \sum_{l \geq 0} q^{l^2} V^{(p_0)}_l(q) = \prod_{n \neq 0, p_0, p_0+1 \text{ (mod 2p_0+1)}} (1 - q^n)^{-1}.
\] (2.16)

It looks very challenging task to find for any rational number \( p_0 \geq 1 \) an explicit product formula (Weyl’s denominator identity) for the right hand side of identity (2.11).

We consider the identity (2.12) as an identity between bosonic and fermionic formulae for the character of ”vacuum representation” of the generalized Kac–Moody algebra corresponding to the matrix \( \Theta^{-1} \).

3 \text{ XXX → X XX} \text{ bijection}

In this section we are going to describe a bijection between the space of states for \( \text{XXZ} \)-model and that of \( \text{XXX} \)-model. Let us formulate the corresponding combinatorial problem more explicitly. First of all as it follows from the results of our previous paper, the combinatorial completeness of Bethe’s states for the \( \text{XXZ} \) model is equivalent to the following identity

\[
\prod_m (2s_m + 1)^{N_m} = \sum_{l=0}^N Z_{\text{XXZ}}(N, s \mid l),
\] (3.1)

where \( N = \sum_m 2s_m N_m \) and the numbers \( Z_{\text{XXZ}}(N, s \mid l) \) are given by (2.8). On the other hand it follows from the combinatorial completeness of Bethe’s states for \( \text{XXX} \) model (see [1]) that

\[
\prod_m (2s_m + 1)^{N_m} = \sum_{l=0}^N (N - 2l + 1) Z_{\text{XXX}}(N, s \mid l),
\] (3.2)

where the number \( Z_{\text{XXX}}(N, s \mid l) \) stands for the multiplicity of \( \left( N - \frac{l}{2} \right) \)-spin irreducible representation of \( \text{sl}(2) \) in the tensor product

\[
V_{s_1}^\otimes N_1 \otimes \cdots \otimes V_{s_m}^\otimes N_m.
\]

Let us remark that both numbers \( Z_{\text{XXZ}}(N, s \mid l) \) and \( Z_{\text{XXX}}(N, s \mid l) \) admit a combinatorial interpretation in terms of rigged configurations. The difference between the
space of states of $XXX$ model and that of $XXZ$ model is the availability of the so-called $1^-$-configurations (or $1^-$ string) in the space of states for the latter model. The presence of $1^-$-strings in the space of states for $XXZ$-model is a consequence of broken $sl(2)$-symmetry of the $XXZ$-model. Our goal in this section is to understand from a combinatorial point of view how the anisotropy of $XXZ$ model breaks the symmetry of the $XXX$ chain. More exactly, we suppose to describe a bijection between $XXZ$-rigged configurations and $XXX$-rigged configurations. Let us start with recalling a definition of rigged configurations.

We consider at first the case of $sl(2)$ $XXX$-magnet. Given a composition $\mu = (\mu_1, \mu_2, \ldots)$ and a natural integer $l$, by definition a $sl(2)$-configuration of type $(l, \mu)$ is a partition $\nu \vdash l$ such that all vacancy numbers

$$P_n(\nu; \mu) := \sum_k \min(n, \mu_k) - 2 \sum_{k \leq n} \nu'_k$$

are nonnegative. Here $\nu'$ denotes the conjugate partition to that $\nu$. A rigged configuration of type $(l, \mu)$ is a configuration $\nu$ of type $(l, \mu)$ together with the collection of integer numbers $\{J_\alpha\}_{\alpha=1}^{m_n(\nu)}$ which satisfy the following inequalities

$$0 \leq J_1 \leq J_2 \leq \cdots \leq J_{m_n(\nu)} \leq P_n(\nu; \mu).$$

Here $m_n(\nu)$ is equal to the number of parts of the partition $\nu$ which are equal to $n$. It is clear that the total number of rigged configurations of type $(l, \mu)$ is equal to the following number

$$Z(l \mid \mu) := \sum_{\nu \vdash l} \prod_{n \geq 1} \left( \frac{P_n(\nu; \mu) + m_n(\nu)}{m_n(\nu)} \right).$$

The following result had been proved in \cite{[1]}.

**Theorem 3.1** Multiplicity of $(N-2l+1)$-dimensional irreducible representation of $sl(2)$ in the tensor product $V_{s_1}^{\otimes N_1} \otimes \cdots \otimes V_{s_m}^{\otimes N_m}$ is equal to the number $Z \left( l \mid \underbrace{2s_1, \ldots, 2s_1}_{N_1}, \ldots, \underbrace{2s_m, \ldots, 2s_m}_{N_m} \right)$.

**Example 2** One can check that

$$V_1^{\otimes 5} = 6V_0 + 15V_1 + 15V_2 + 10V_3 + 4V_4 + V_5.$$

In our case we have $\mu = (2^5)$. Let us consider $l = 5$. It turns out that there exist three configurations of type $(3, (2^5))$, namely

\[ \begin{array}{cccccc} \otimes & \otimes & \otimes & 0 \\ 1 & \otimes & \otimes & \otimes & 0 \\ \otimes & \otimes & \otimes & \otimes & \otimes & 0 \end{array} \]
Hence, $Z(3 \mid (2^5)) = 1 + 2 + 3 = 6 = \text{Mult}_{v_0} \left( V_1^{\otimes 5} \right)$.

Now let us give a definition of $sl(2)$–XXZ configuration. We consider in this Section only the case when the anisotropy parameter $p_0$ is an integer, $p_0 \in \mathbb{Z}_{\geq 2}$. Under this assumption the formulae (2.6) and (2.7) take the following form:

- $n_j = j$, if $1 \leq j < p_0$, $v_j = +1$;
- $n_{p_0} = 1$, $v_{p_0} = -1$;
- $2\Phi_{k,2s} = \frac{2sk}{p_0} - \min(k, 2s)$, if $1 \leq k < p_0$, $2s + 1 < p_0$;
- $2\Phi_{p_0,2s} = \frac{2s}{p_0}$, if $2s + 1 < p_0$;
- $b_{kj} = k - j$, if $1 \leq j \leq k < p_0$;
- $b_{kp_0} = 1$, if $1 \leq k < p_0$;
- $a_j := a_j(l \mid \mu) = \sum_m \min(j, \mu_m) - 2l - j \left[ \frac{\sum_m \mu_m - 2l}{p_0} \right]$, if $1 \leq j < p_0$;
- $a_{p_0}(l \mid \mu) = \left[ \frac{\sum_m \mu_m - 2l}{p_0} \right]$.

**Definition 3.2** A $sl(2)$–XXZ configuration of type $(l, \mu)$ is a pair $(\lambda, \lambda_{p_0})$, where $\lambda$ is a composition with all parts strictly less than $p_0$, $\sum_{j<p_0} j\lambda_j + \lambda_{p_0} = l$, and such that all vacancy numbers $P_j(\lambda \mid \mu)$ are nonnegative.

Let us recall [4] that if the anisotropy parameter $p_0 \geq 2$ is an integer, then

$$P_j(\lambda \mid \mu) := a_j(l \mid \mu) + 2 \sum_{j<k<p_0} (k - j)\lambda_k + \lambda_{p_0}, \text{ if } j < p_0 - 1; \quad (3.4)$$

$$P_{p_0-1}(\lambda \mid \mu) := a_{p_0-1}(l \mid \mu) + \lambda_{p_0};$$

$$P_{p_0}(\lambda \mid \mu) := a_{p_0}(l \mid \mu) + \lambda_{p_0-1}.$$ 

In the sequel we are displaying a configuration

$$(\lambda, \lambda_{p_0}) = (\lambda_1, \lambda_2, \ldots, \lambda_{p_0-1}, \lambda_{p_0})$$

as the diagram of the following partition $(1^{\lambda_1 + \lambda_{p_0}}, 2^{\lambda_2}, \ldots, (p_0 - 1)^{\lambda_{p_0-1}})$.

**Example 3** Let us consider $p_0 = 6$, $s = \frac{3}{2}$, $N = 5$, $l = 5$. The total number of type $(5, (3^5))$ $sl(2)$–XXZ configurations is equal to 12.
The total number of type $(5, (3^5))$ rigged configurations is equal to

$$Z^{XXZ} (5 \mid (3^5)) = 101 = 1 + 4 + 3 + 7 + 10 + 10 + 6 + 16 + 8 + 12 + 18 + 6.$$  

Here we have used the symbol ♣ to mark the $1^{-}$-strings.

Now we are ready to describe a map $\Pi$ from the space of states for $XXZ$ model to that of $XXX$ one. More exactly we are going to describe a rule how a $XXZ$–configuration falls to the $XXX$–pieces. At first we describe this rule schematically:

\[
\begin{align*}
&\begin{array}{c}
\lambda \\
\downarrow \\
\lambda \\
\downarrow \\
\vdots \\
\lambda
\end{array}
\begin{array}{c}
m \\
\downarrow \\
m-1 \\
\downarrow \\
\vdots \\
m-1
\end{array} \\
\rightarrow \\
\begin{array}{c}
\lambda \\
\downarrow \\
\lambda \\
\downarrow \\
\vdots \\
\lambda
\end{array}
\begin{array}{c}
m-k \\
\downarrow \\
m-k \\
\downarrow \\
\vdots \\
m-k
\end{array}
+ \\
\begin{array}{c}
\lambda \\
\downarrow \\
\lambda \\
\downarrow \\
\vdots \\
\lambda
\end{array}
\begin{array}{c}
m-1 \\
\downarrow \\
m-1 \\
\downarrow \\
\vdots \\
m-1
\end{array} + \\
\ldots \\
+ \\
\begin{array}{c}
\lambda \\
\downarrow \\
\lambda \\
\downarrow \\
\vdots \\
\lambda
\end{array}
\begin{array}{c}
m-1 \\
\downarrow \\
m-1 \\
\downarrow \\
\vdots \\
m-1
\end{array}
\end{align*}
\]

This decomposition corresponds to the well–known identity

\[
\left[ \begin{array}{c}
m+k \\
\downarrow \\
k
\end{array} \right]_q = \sum_{j=0}^{k} q^j \left[ \begin{array}{c}
m+j-1 \\
\downarrow \\
j
\end{array} \right]_q.
\]

In what follows we will assume that $p_0 > \sum_m s_m$.

**Theorem 3.3** The map $\Pi$ is well–defined and gives rise to a bijection between the space of states of $XXZ$–model and that of $XXX$ one.

**Proof.** Let us start with rewriting the formulae (3.4) for the $XXZ$–vacancy numbers in more convenient form, namely,

\[
P^{XXZ}_j (\tilde{\nu} \mid \mu) = \sum_m \min(j, 2s_m) - 2 \sum_{k \leq j} \nu'_k - j \left[ \sum_m \frac{2s_m - 2l}{p_0} \right], \text{ if } 1 \leq j < p_0 - 1;
\]
\[ P_{p_0-1}^{XXZ}(\bar{\nu} \mid \mu) = p_0 \left\{ \frac{\sum_m 2s_m - 2l}{p_0} \right\} + \left[ \frac{\sum_m 2s_m - 2l}{p_0} \right] + \lambda_{p_0}; \quad (3.5) \]

\[ P_{p_0}^{XXZ}(\bar{\nu} \mid \mu) = \left[ \frac{\sum_m 2s_m - 2l}{p_0} \right] + m_{p_0-1}(\nu). \]

Here \( \mu = (2s_1, \ldots, 2s_m) \) and \( \bar{\nu} \) is a pair \( \bar{\nu} = (\nu, \lambda_{p_0}) \), where \( \nu \) is a partition such that \( l(\nu) \leq p_0 - 1 \), \( |\nu| + \lambda_{p_0} = l \). Relationship between \( \lambda \) from Definition 1 and \( \nu \) is the following

\[ m_j(\nu) = \lambda_j, \quad \text{i.e. } \nu = (1^{\lambda_1}2^{\lambda_2}\ldots(p_0-1)^{\lambda_{p_0-1}}). \]

Next, let us consider an integer \( l \leq \sum m_s \) and let \( \nu \vdash l \) be a \( XXX \)-configuration. Let \( \lambda_{p_0} \) be an integer such that \( 2\sum s_m - 2l - p_0 < \lambda_{p_0} \leq \sum s_m - l \) and consider the pair \( \bar{\nu} = (\nu, \lambda_{p_0}) \). It is easy to check that

\[ P_{p_0}^{XXZ}(\bar{\nu} \mid \mu) = \sum_m \min(j, 2s_m) - 2 \sum_{k \leq j} \nu'_k = P_j^{XXX}(\nu \mid \mu) \geq 0, \quad \text{if } 1 \leq j < p_0 - 1; \]

\[ P_{p_0-1}^{XXZ}(\bar{\nu} \mid \mu) = \sum_m 2s_m - 2l + \lambda_{p_0} \geq 0; \]

\[ P_{p_0}^{XXZ}(\bar{\nu} \mid \mu) = \lambda_{p_0-1} \geq 0. \]

Thus the pair \( \bar{\nu} = (\nu, \lambda_{p_0}) \) is a \( XXZ \)-configuration.

Furthermore it follows from our assumptions (namely, \( \sum_m s_m < p_0 \), \( \lambda_{p_0} > 0 \)) that \( \lambda_{p_0-1} = 0 \) and both \( 1^- \)-strings and \( (p_0 - 1)^- \)-strings do not give a contribution to the space of \( XXZ \)-states. Thus we see that both \( XXX \)-configuration \( \nu \) and \( XXZ \)-configuration \( \bar{\nu} = (\nu, \lambda_{p_0}) \) define the same number of states. Now, if \( \bar{\nu} = (\nu, \mu) \) is a \( XXZ \)-configuration then \( \nu \) is a \( XXX \) configuration as well. This is clear because (see (3.5))

\[ P_j^{XXX}(\nu \mid \mu) \geq P_j^{XXX}(\bar{\nu} \mid \mu), \quad 1 \leq j \leq p_0 - 1. \]

By the similar reasons if \( (\bar{\nu}, \lambda_{p_0}) \) is a \( XXZ \)-configuration, then for any integer \( k, 0 \leq k \leq \lambda_{p_0} \), the pair \( (\bar{\nu}, \lambda_{p_0} - k) \) is also a \( XXZ \)-configuration. It follows from the above considerations that \( \Pi \) is a well-defined map. Furthermore there exists one to one correspondence between the space of \( XXX \)-configurations and that of \( XXZ \)-configurations, namely,

\[ \nu \leftrightarrow \bar{\nu} = (\nu, \lambda_{p_0}), \]

where \( \lambda_{p_0} = [\sum m s_m - |\nu|] \).

All others \( XXZ \)-configurations \( (\nu, k) \) with \( 0 \leq k < \sum m s_m - |\nu| - 1 \) give a contribution to the space of descendants for \( \nu \leftrightarrow \bar{\nu} \).

\[ \blacksquare \]
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