HARMONIC MAPPINGS AND CONFORMAL MINIMAL IMMERSIONS OF RIEMANN SURFACES INTO $\mathbb{R}^N$

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ABSTRACT. We prove that for any open Riemann surface $N$, natural number $N \geq 3$, non-constant harmonic map $h : N \to \mathbb{R}^{N-2}$ and holomorphic 2-form $\mathcal{H}$ on $N$, there exists a weakly complete harmonic map $X = (X_j)_{j=1,\ldots,N} : N \to \mathbb{R}^N$ with Hopf differential $\mathcal{H}$ and $(X_j)_{j=3,\ldots,N} = h$. In particular, there exists a complete conformal minimal immersion $Y = (Y_j)_{j=1,\ldots,N} : N \to \mathbb{R}^N$ such that $(Y_j)_{j=3,\ldots,N} = h$.

As some consequences of these results:

- There exists complete full non-decomposable minimal surfaces with arbitrary conformal structure and whose generalized Gauss map is non-degenerate and fails to intersect $N$ hyperplanes of $\mathbb{CP}^{N-1}$ in general position.
- There exists complete non-proper embedded minimal surfaces in $\mathbb{R}^N$, $\forall N > 3$.

1. INTRODUCTION

In this paper we use methods coming from the study of minimal surfaces to construct harmonic mappings of Riemann surfaces into $\mathbb{R}^N$ with prescribed geometry. A basic reference for this topic is, for instance, Klotz’s work [K].

Our main result states (see Corollary 4.5):

**Theorem A.** For any open Riemann surface $N$, natural number $N \geq 3$, non-constant harmonic map $h : N \to \mathbb{R}^{N-2}$ and holomorphic 2-form $\mathcal{H}$ on $N$, there exists a weakly complete harmonic map $X = (X_j)_{j=1,\ldots,N} : N \to \mathbb{R}^N$ with Hopf differential $\mathcal{H}$ and $(X_j)_{j=3,\ldots,N} = h$.

Recall that the Hopf differential $Q_X$ of a harmonic map $X : N \to \mathbb{R}^N$ is the holomorphic 2-form given by $Q_X := (\partial_{\mathbb{C}}X, \partial_{\mathbb{C}}X)$, where $\partial_{\mathbb{C}}$ means complex differential. By definition, $X$ is said to be weakly complete if $\Gamma_X := |\partial_{\mathbb{C}}X|^2$ is a complete conformal Riemannian metric in $N$ (see [K]).

The fact that conformal minimal immersions are harmonic maps strongly influences the global theory of this kind of surfaces. It is well known that a harmonic immersion $X : N \to \mathbb{R}^N$ is minimal if and only if it is conformal, or equivalently, $Q_X = 0$. Weakly completeness is equivalent to Riemannian completeness under minimality assumptions. The geometry of complete minimal surfaces in $\mathbb{R}^N$, specially those properties regarding the Gauss map, has been the object of extensive study over the last past decades (see for instance [O1, CO, C, F3, R]).

In the recent paper [AFL], the authors constructed complete minimal surfaces in $\mathbb{R}^3$ with arbitrarily prescribed conformal structure and non-constant third coordinate function (see also [AF]). As a consequence, any open Riemann surface admits a complete conformal minimal immersion in

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\[ \mathbb{R}^3 \text{ whose Gauss map omits two antipodal points of the unit sphere.} \]

**Theorem B.** For any open Riemann surface \( \mathcal{N} \), natural number \( N \geq 3 \) and non-constant harmonic map \( h : \mathcal{N} \to \mathbb{R}^{N-2} \), there exists a complete conformal minimal immersion \( X = (X_j)_{j=1,\ldots,N} : \mathcal{N} \to \mathbb{R}^N \) with \( (X_j)_{j=3,\ldots,N} = h \).

Under some compatibility conditions depending on the map \( h \), the flux map of the immersion \( X \) can be also prescribed. Recall that the flux map of a conformal minimal immersion \( X : \mathcal{N} \to \mathbb{R}^N \) is given by \( p_X(\gamma) = \text{Im} \int_{\gamma} \partial_z X \) for all \( \gamma \in \mathcal{H}_1(\mathcal{N}, \mathbb{Z}) \). In particular, if \( h \) is the real part of a holomorphic map \( H : \mathcal{N} \to \mathbb{C}^{N-2} \), Theorem B provides a complete null holomorphic curve \( F = (F_j)_{j=1,\ldots,N} : \mathcal{N} \to \mathbb{C}^N \) such that \( (F_j)_{j=3,\ldots,N} = H \). Likewise, \( Y = (F_j)_{j=2,\ldots,N} : \mathcal{N} \to \mathbb{C}^{N-1} \) is a complete holomorphic immersion whose last \( N - 2 \) coordinates coincide with \( H \).

Theorem B also includes some information about the Gauss map of minimal surfaces in \( \mathbb{R}^N \). Given a conformal minimal immersion \( X : \mathcal{N} \to \mathbb{R}^N \), its generalized Gauss map \( G_X : \mathcal{N} \to \mathbb{C}P^{N-1} \), \( G_X(P) = \partial_z X(P) \), is holomorphic and takes values on the complex hyperquadric \( \{ \sum_{j=1}^N w_j^2 = 0 \} \). Chern and Osserman [C, CO] showed that if \( X \) is complete then either \( X(\mathcal{N}) \) is a plane or \( G_X(\mathcal{N}) \) intersects a dense set of complex hyperplanes. Even more, Ru [R] proved that if \( X \) is complete and non-flat then \( G_X \) cannot omit more than \( N(N + 1)/2 \) hyperplanes in \( \mathbb{C}P^{N-1} \) located in general position (see also the works of Fujimoto [F2, F3] for a good setting). Under the non-degeneracy assumption on \( G_X \), this upper bound is sharp for some values of \( N \), see [F4]. However, the number of exceptional hyperplanes strongly depends on the underlying conformal structure of the surface. Indeed, Ahlfors [A] proved that any holomorphic map \( G : C \to \mathbb{C}P^{N-1}, N \geq 3 \), avoiding \( N + 1 \) hyperplanes of \( \mathbb{C}P^{N-1} \) in general position is degenerate, that is to say, \( G(C) \) lies in a proper projective subspace of \( \mathbb{C}P^{N-1} \) (see [W, Chapter 5, §5] and [F1] for further generalizations). So, it is natural to wonder whether any open Riemann surface admits a complete conformal minimal immersion in \( \mathbb{R}^N \) whose generalized Gauss map is non-degenerate and omits \( N \) hyperplanes in general position. An affirmative answer to this question can be found in the following (see Corollary 4.8)

**Theorem C.** Let \( \mathcal{N} \) be an open Riemann surface, and let \( p : \mathcal{H}_1(\mathcal{N}, \mathbb{Z}) \to \mathbb{R}^N \) be a group morphism, \( N \geq 3 \). Then there exists a complete conformal full non-decomposable minimal immersion \( X : \mathcal{N} \to \mathbb{R}^N \) with \( p_X = p \) and whose generalized Gauss map is non-degenerate and omits \( N \) hyperplanes in general position.

On the other hand, Theorem B leads to some interesting consequences regarding Calabi-Yau conjectures. The embedded Calabi-Yau problem for minimal surfaces asks for the existence of complete bounded embedded minimal surfaces in \( \mathbb{R}^3 \). Complete embedded minimal surfaces in \( \mathbb{R}^3 \) with finite genus and countably many ends are proper in space [MPR, CM]. However, this result fails to be true for arbitrary higher dimensions. For instance, taking \( \mathcal{N} \) the unit disc \( \mathbb{D} \) in \( C \) and \( h : \mathbb{D} \to \mathbb{R}^2 \) the map \( h(z) = (\text{Re}(z), \text{Im}(z)) \), Theorem B generates complete non-proper embedded minimal discs in \( \mathbb{R}^4 \) (so in \( \mathbb{R}^N \) for all \( N \geq 4 \)), see Corollary 4.7 for more details.

The paper is laid out as follows. In Section 2 we introduce the necessary background and notations. In Section 3 we prove a basic approximation result by holomorphic 1-forms in open Riemann surfaces (Lemma 3.3), which is the key tool for proving our main results. Finally, in Section 4 we state and prove Theorems A, B and C. It is worth mentioning that all these theorems actually follows from the more general result Theorem 4.4 in Section 4.
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2. PRELIMINARIES

Given a topological manifold $M$, $\partial M$ will denote the one dimensional topological manifold determined by the boundary points of $M$. Given $S \subset M$, call by $S^0$ and $\bar{S}$ the interior and the closure of $S$ in $M$, respectively. Open connected subsets of $M - \partial M$ will be called domains, and those proper topological subspaces of $M$ being surfaces with boundary are said to be regions. The surface $M$ is said to be open if it is non-compact and $\partial M = \emptyset$.

If $M$ is a Riemann surface, $\partial \nu$ will denote the global complex operator given by $\partial \nu|_U = \frac{1}{\partial z} dw$ for any conformal chart $(U, w)$ on $M$.

**Remark 2.1.** Throughout this paper $\mathcal{N}$ and $\mathbb{N}$ will denote a fixed but arbitrary open Riemann surface and natural number greater than or equal to three, respectively.

Let $S$ denote a subset of $\mathcal{N}$, $S \neq \emptyset$. We denote by $\mathcal{F}_0(S)$ as the space of continuous functions $f : S \rightarrow \mathbb{C}$ which are holomorphic on an open neighborhood of $S$ in $\mathcal{N}$. Likewise, $\mathcal{F}_0(S)$ will denote the space of continuous functions $f : S \rightarrow \mathbb{C}$ being holomorphic on $S^0$.

As usual, a 1-form $\theta$ on $S$ is said to be of type $(1, 0)$ if for any conformal chart $(U, z)$ in $\mathcal{N}$, $\theta|_{U \cap S} = h(z) dz$ for some function $h : U \cap S \rightarrow \mathbb{C}$. We denote by $\Omega_0(S)$ as the space of holomorphic 1-forms on an open neighborhood of $S$ in $\mathcal{N}$. We call $\Omega_0(S)$ as the space of 1-forms $\theta$ of type $(1, 0)$ on $S$ such that $(\theta|_U)/dz \in F_0(S \cap U)$ for any conformal chart $(U, z)$ on $\mathcal{N}$.

We denote by $\tilde{\Omega}_0(S)$ as the space of holomorphic 2-forms on an open neighborhood of $S$ in $\mathcal{N}$.

Let $\mathfrak{D}(S)$ denote the free commutative group of divisors of $S$ with multiplicative notation. A divisor $D \in \mathfrak{D}(S)$ is said to be integral if $D = \prod_{i=1}^{n} Q_i^{n_i}$ and $n_i \geq 0$ for all $i$. Given $D_1$, $D_2 \in \mathfrak{D}(S)$, we write $D_1 \geq D_2$ if and only if $D_1 D_2^{-1}$ is integral. For any $f \in F_0(S)$ we denote by $(f)$ its associated integral divisor of zeros in $S$. Likewise we define $(\theta)$ for any $\theta \in \Omega_0(S)$.

In the sequel we will assume that $S$ is a compact subset of $\mathcal{N}$.

A compact Jordan arc in $\mathcal{N}$ is said to be analytical if it is contained in an open analytical Jordan arc in $\mathcal{N}$. By definition, a connected component $V$ of $\mathcal{N} - S$ is said to be bounded if $\overline{V}$ is compact, where $\overline{V}$ is the closure of $V$ in $\mathcal{N}$. Moreover, a subset $K \subset \mathcal{N}$ is said to be Runge (in $\mathcal{N}$) if $\mathcal{N} - K$ has no bounded components.

**Definition 2.2.** A compact subset $S \subset \mathcal{N}$ is said to be admissible if and only if (see Figure 2.1):

- $S$ is Runge,
- $M_S := S^0$ consists of a finite collection of pairwise disjoint compact regions in $\mathcal{N}$ with $C^0$ boundary,
- $C_S := S - M_S$ consists of a finite collection of pairwise disjoint analytical Jordan arcs, and any component $\alpha$ of $C_S$ with an endpoint $P \in M_S$ admits an analytical extension $\beta$ in $\mathcal{N}$ such that the unique component of $\beta - \alpha$ with endpoint $P$ lies in $M_S$.

Let $W$ be a domain in $\mathcal{N}$, and let $S$ be either a compact region or an admissible subset in $\mathcal{N}$. $W$ is said to be a tubular neighborhood of $S$ if $S \subset W$ and $W - S$ consists of a finite collection of pairwise disjoint open annuli. In addition, if $\overline{W}$ is a compact region isotopic to $W$ then $\overline{W}$ is said to be a compact tubular neighborhood of $S$. Here isotopic means that $j_* : \mathcal{H}_1(W, \mathbb{Z}) \rightarrow \mathcal{H}_1(\overline{W}, \mathbb{Z})$ is an isomorphism, where $j : W \rightarrow \overline{W}$ is the inclusion map.
Let $W \subset \mathcal{N}$ be a domain with $S \subset W$. We shall say that a function $f \in \mathcal{F}_0^\theta(S)$ can be uniformly approximated on $S$ by functions in $\mathcal{F}_0(W)$ if there exists $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{F}_0(W)$ such that $\{|f_n - f|\}_{n \in \mathbb{N}} \rightarrow 0$ uniformly on $S$. A 1-form $\theta \in \Omega_0^\theta(S)$ can be uniformly approximated on $S$ by 1-forms in $\Omega_0(W)$ if there exists $\{\theta_n\}_{n \in \mathbb{N}} \subset \Omega_0(W)$ such that $\{|\theta_n - \theta|\}_{n \in \mathbb{N}} \rightarrow 0$ uniformly on $S \cap U$, for any conformal closed disc $(U, dz)$ on $W$.

Given an admissible compact set $S \subset W$, a function $f : S \to C^n$, $n \in \mathbb{N}$, is said to be smooth if $f|_{M_5}$ admits a smooth extension $f_0$ to an open domain $V$ in $W$ containing $M_5$, and for any component $\alpha$ of $C_5$ and any open analytical Jordan arc $\beta$ in $W$ containing $\alpha$, $f$ admits an smooth extension $f_\beta$ to $\beta$ satisfying that $f_\beta|_{V \cap \beta} = f_0|_{V \cap \beta}$. Likewise, an 1-form $\theta$ of type $(1,0)$ on $S$ is said to be smooth if for any closed conformal disc $(U, z)$ on $\mathcal{N}$ such that $S \cap U$ is admissible, the function $\frac{\theta}{dz}$ is smooth on $S \cap U$. Given a smooth $f \in \mathcal{F}_0^\theta(S)$, we set $df \in \Omega_0^\theta(S)$ as the smooth 1-form given by $df|_{M_5} = d(f|_{M_5})$ and $df|_{\alpha \cap U} = (f \circ \alpha)'|dz|_{\alpha \cap U}$, where $(U, z = x + iy)$ is a conformal chart on $W$ such that $\alpha \cap U = z^{-1}(\mathbb{R} \cap U)$. Obviously, $df|_{\alpha}(t) = (f \circ \alpha)'|dt$ for any component $\alpha$ of $C_5$, where $t$ is any smooth parameter along $\alpha$. This definition makes sense also for smooth functions with poles in $\mathbb{S}^n$.

A smooth 1-form $\theta \in \Omega_0^\theta(S)$ is said to be exact if $\theta = df$ for some smooth $f \in \mathcal{F}_0^\theta(S)$, or equivalently if $\int_\gamma \theta = 0$ for all $\gamma \in \mathcal{H}_1(S, \mathbb{Z})$.

### 2.1. Harmonic maps and minimal surfaces in $\mathbb{R}^N$.

Given a non-constant harmonic map $X = (X_j)_{j=1,...,N} : \mathcal{N} \to \mathbb{R}^N$, the holomorphic quadratic differential

$$Q_X := \langle \partial_\gamma X, \partial_\gamma X \rangle = \sum_{j=1}^N (\partial_\gamma X_j)^2$$

is said to be the Hopf differential of $X$. We also consider the conformal metric, possibly with isolated singularities,

$$\Gamma_X := \frac{1}{2} \sum_{j=1}^N |\partial_\gamma X_j|^2.$$ 

It is clear that $2\Gamma_X$ is greater than or equal to the Riemannian metric on $\mathcal{N}$ (possibly with singularities) induced by $X$. When $X$ is an immersion then $\Gamma_X$ is a Riemannian metric, and if in addition $X$ is complete then $\Gamma_X$ is complete as well [K]. However, the reciprocal does not hold in general.

**Definition 2.3.** We say that a harmonic map $X : \mathcal{N} \to \mathbb{R}^N$ is weakly complete (or complete in the sense of [K]) if $\Gamma_X$ is a complete metric on $\mathcal{N}$.

We also associate to $X$ the group morphism

$$p_X : \mathcal{H}_1(\mathcal{N}, \mathbb{Z}) \to \mathbb{R}^N, \quad p_X(\gamma) = \text{Im} \int_\gamma \partial_\gamma X.$$
Remark 2.4. If $Q_X = 0$ and $\Gamma_X$ never vanishes, then $X$ is a conformal minimal immersion, $\Gamma_X$ is the metric induced on $\mathcal{N}$ by $X$, and $p_X$ is the flux map of $X$.

If in addition $X$ is a conformal minimal immersion and we write $\partial_\zeta X_j = f_j d\zeta$ in terms of a local parameter $\zeta$ on $\mathcal{N}$, $j = 1, \ldots, N$, then the (generalized) Gauss map of $X$ is given by

$$G_X : \mathcal{N} \to \mathbb{CP}^{N-1}, \quad G_X(\zeta) = [(f_j(\zeta))]_{j=1, \ldots, N},$$

where $[w]$ is the class of $w$ in $\mathbb{CP}^{N-1}$, $\forall w \in \mathbb{C}^N$. It is well known that $G_X$ is a holomorphic map taking values in the complex quadric $\{[(w)]_{j=1, \ldots, N} \in \mathbb{CP}^{N-1} | \sum_{j=1}^{N} w_j^2 = 0\}$.

A set of hyperplanes in $\mathbb{CP}^{N-1}$ is said to be in general position if each subset of $k$ hyperplanes, with $k \leq N - 1$, has an $(N - 1 - k)$-dimensional intersection.

Definition 2.5 ([O2]). Let $X : \mathcal{N} \to \mathbb{R}^N$ be a conformal minimal immersion.

- $X$ is said to be decomposable if, with respect to suitable rectangular coordinates in $\mathbb{R}^N$, one has $\sum_{k=1}^{n} (\partial_k X_k)^2 = 0$ for some $m < n$.
- $X$ is said to be full if $X(\mathcal{N})$ is contained in no hyperplane of $\mathbb{R}^N$.
- The Gauss map $G_X$ is said to be degenerate if $G_X(\mathcal{N})$ lies in a hyperplane of $\mathbb{CP}^{N-1}$.

When $N = 3$, decomposable, non-full and degenerate are equivalent. However, if one passes to higher dimensions then no two of these conditions are equivalent (see [O2]).

3. The Approximation Lemma

The next two lemmas are the key tools in the proof of the main result of this section (Lemma 3.3). They represent a slight generalization of Lemmas 2.4 and 2.5 in [AL].

From now on, $i$ denotes the imaginary unit and the symbol $\neq 0$ means non-identically zero.

Lemma 3.1. Let $W \subset \mathcal{N}$ be a domain with finite topology and $S \subset \mathcal{N}$ an admissible subset with $S \subset W$. Consider $f \in \mathcal{F}_0^+(S) \cap \mathcal{F}_0(M_S)$ with $f|_{M_S} \neq 0$.

Then $f$ can be uniformly approximated on $S$ by functions $\{f_n\}_{n \in \mathbb{N}}$ in $\mathcal{F}_0^0(W)$ satisfying that $(f_n) = (f|_{M_S})$ on $W$. In particular, $f_n$ never vanishes on $W - M_S$ for all $n$.

Proof. Let $\{M_k\}_{k \in \mathbb{N}}$ be a sequence of compact tubular neighborhoods of $M_S$ in $W$ such that $M_k \subset M_{k-1}$ for any $k$, $\cap_{k \in \mathbb{N}} M_k = M_S$ and $f$ holomorphically extends (with the same name) to $M_k$ and has no zeros on $M_1 - M_S$ (take into account that $f|_{M_S} \neq 0$). Choose $M_k$ so that, in addition, the compact set $S_k := M_k \cup C_S \subset W$ is admissible and $S_k \subset M_{k+1}$ is a (non-empty) Jordan arc for any component $\alpha$ of $C_S$. In particular, $M_{S_k} = M_k$ and $C_{S_k} = C_S - M_k$, $k \in \mathbb{N}$.

For any $k \in \mathbb{N}$ take $g_k \in \mathcal{F}_0^0(S_k) \cap \mathcal{F}_0(M_{S_k})$ satisfying

- $g_k|_{M_{S_k}} = f|_{M_{S_k}}$,
- $g_k$ never vanishes on $S_k - S_k^*$ (recall that $f$ has no zeros on $M_1 - M_S$), and
- the sequence $\{g_k|_{S_k}\}_{k \in \mathbb{N}}$ uniformly converges to $f$ on $S$.

The construction of such functions is standard, we omit the details. Since $g_k$ satisfies the hypotheses of Lemma 2.4 in [AL], it can be uniformly approximated on $S_k$ by a sequence $\{g_{k,n}\}_{n \in \mathbb{N}} \subset \mathcal{F}_0^0(W)$ with $(g_{k,n}) = (g_{k}|_{M_{S_k}}) = (f|_{M_S})$ on $W$, for any $k$. A standard diagonal argument concludes the proof.

Lemma 3.2. Let $W \subset \mathcal{N}$ be a domain with finite topology and $S \subset \mathcal{N}$ an admissible subset with $S \subset W$. Consider $\theta \in \Omega_0^+(S) \cap \Omega_0(M_S)$ with $\theta|_{M_S} \neq 0$.
Then $\theta$ can be uniformly approximated on $S$ by 1-forms $\{\theta_n\}_{n \in \mathbb{N}}$ in $\Omega_0(W)$ satisfying that $(\theta_n) = (\theta|_{M_S})$ on $W$. In particular, $\theta_n$ never vanishes on $W - M_S$ for all $n$.

**Proof.** Let $\theta$ be a never vanishing 1-form in $\Omega_0(W)$. Define $f := \theta/\theta \in F_0^*(S) \cap F_0(M_S)$. By Lemma 3.1, $f$ can be uniformly approximated on $S$ by a sequence $\{f_n\}_{n \in \mathbb{N}}$ in $F_0(W)$ satisfying that $(f_n) = (f|_{M_S})$ on $W$ for all $n$. It suffices to set $\theta_n := f_n\theta$, $n \in \mathbb{N}$.

**Lemma 3.3.** Let $W \subseteq N$ be a domain with finite topology and $S \subseteq N$ an admissible subset with $S \subseteq W$. Let $\Theta \in \Omega_0(W)$ and $\Phi = (\phi_1, \phi_2)$ be a smooth pair in $\Omega_0(S)^2 \cap \Omega_0(M_S)^2$ satisfying $\phi_1^2 + \phi_2^2 = \Theta|_S$ and either of the following conditions:

(A) $\phi_1|_{M_S}$ and $\phi_2|_{M_S}$ are linearly independent in $\Omega_0(M_S)$ and $\Theta$ has no zeros on $C_S$.

(B) $\Theta = 0$ and $\phi_1|_{M_S} \neq 0$.

Then $\Phi$ can be uniformly approximated on $S$ by a sequence $\{\Phi_n = (\phi_{1,n}, \phi_{2,n})\}_{n \in \mathbb{N}} \subseteq \Omega_0(W)^2$ satisfying

(a) $\phi_{1,n}^2 + \phi_{2,n}^2 = \Theta$,

(b) $\Phi_n - \Phi$ is exact on $S$, and

(c) the zeros of $\Phi_n$ on $W$ are those of $\Phi$ on $M_S$ (in particular, $\Phi_n$ never vanishes on $W - M_S$).

**Proof.** Assume (A) holds.

**Claim 3.4.** Without loss of generality it can be assumed that $\phi_1$, $\phi_2$ and $d\xi$ never vanish on $C_S$, where $\xi := \phi_1/\phi_2$.

**Proof.** Assume for a moment that the conclusion of the lemma holds when $\phi_1$, $\phi_2$ and $d\xi$ never vanish on $C_S$.

Take a sequence $\{M_k\}_{k \in \mathbb{N}}$ as in the proof of Lemma 3.1 such that $\Phi$ holomorphically extends (with the same name) to $M_k$, and $\phi_1$, $\phi_2$ and $d\xi$ never vanish on $M_k - M_S$, for all $n$ (take into account (A)). Recall that $S_k := S \cup C_S \subset W^o$ is an admissible set and $C_{S_k} = C_S - M_k^c$, $k \in \mathbb{N}$.

Since $\Theta$ never vanishes on $C_S$, which consists of a finite collection of pairwise disjoint analytic Jordan arcs, then we can find $\theta \in \Omega_0(C_S)$ with $\theta^2 = \Theta|_{C_S}$. Consider $f_j : C_S \to C$, $f_j = \phi_1/\theta$, $j = 1, 2$, and notice that $f_1^2 + f_2^2 = 1$ and $f_1|_{C_S} = f_1/f_2$. Consider a sequence $\{(f_{1,k}, f_{2,k})\}_{k \in \mathbb{N}}$ of pairs of smooth functions on $C_{S_k}$ satisfying:

i) $f_{1,k}$, $f_{2,k}$ and $df_{1,k}$ never vanish on $C_{S_k}$,

ii) $f_{1,k}^2 + f_{2,k}^2 = 1$,

iii) the function $g_{j,k}$ given by $g_{j,k}|_{M_S} = f_j$, $g_{j,k}|_{C_S} = f_{j,k}$, lies in $F_0^*(S_k)$ and is smooth, $j = 1, 2$,

iv) $\{f_{j,k}\}_{k \in \mathbb{N}}$ uniformly converges to $f_j$ on $C_S$, $j = 1, 2$, and

v) $\Psi_k|_S - \Phi$ is exact on $S$, where $\Psi_k := (g_{j,k})_{j=1,2} \in \Omega_0^*(S_k)^2 \cap \Omega_0(M_S)^2$.

The construction of this data is standard, we omit the details. Write $\Psi_k = (\psi_{j,k})_{j=1,2}$ and $\xi_k = \psi_{1,k}/\psi_{2,k}$. From i), ii) and the definition of $\theta$ follow that $\phi_{1,k}^2 + \phi_{2,k}^2 = \Theta$ and $d\xi_k$ never vanishes on $C_{S_k}$. Moreover, given $\{\Psi_k|_S\}_{k \in \mathbb{N}}$ uniformly converges to $\Phi$ on $S$.

By hypothesis, Lemma 3.3 holds for any $\Psi_k$, then there exists a sequence $\{\Psi_{k,n}\}_{n \in \mathbb{N}}$ uniformly converging to $\Psi_k$ on $S_k$ and satisfying (a), (b) and (c) of Lemma 3.3 for $\Phi = \Psi_k$ and $S = S_k$. Using that $\{\Psi_k|_S\}_{k \in \mathbb{N}}$ converges to $\Phi$, the zeros of $\Phi_k$ in $M_{S_k}$ are those of $\Phi$ in $M_{S_k}$, and a standard diagonal argument, we can obtain a sequence satisfying the conclusion of the lemma, proving the claim.

In the sequel we will assume that $\phi_1$, $\phi_2$ and $d\xi$ never vanish on $C_S$. 


Label $\eta = \phi_1 - i\phi_2 \in \Omega^2(S) \cap \Omega^0(M_S)$ and observe that $\Theta / \eta = \phi_1 + i\phi_2 \in \Omega^2(S) \cap \Omega^0(M_S)$.

Notice that $(\Theta / \eta)|_{M_S} \neq 0$,

$$\phi_1 = \frac{1}{2} \left( \eta + \Theta \right) \quad \text{and} \quad \phi_2 = \frac{i}{2} \left( \eta - \Theta \right).$$

Let $\mathcal{B}_S$ be a homology basis of $\mathcal{H}(S, \mathbb{Z})$ and label $\nu$ as its cardinal number. Consider in $\mathcal{F}_0^*(S)$ the maximum norm and the Fréchet differentiable map

$$\mathcal{P} : \mathcal{F}_0^*(S) \to \mathbb{C}^{2\nu}, \quad \mathcal{P}(f) = \left( \int_{\Sigma} \left( e^f \eta + e^{-f} \Theta \right) \phi_1, e^f \eta - e^{-f} \Theta \phi_2 \right)_{c \in \mathcal{B}_S}.$$

Label $\mathcal{A} : \mathcal{F}_0^*(S) \to \mathbb{C}^{2\nu}$ as the Fréchet derivative of $\mathcal{P}$ at 0.

**Claim 3.5.** $\mathcal{A}|_{\mathcal{F}_0(W)}$ is surjective.

**Proof.** Reason by contradiction and assume that $\mathcal{A}(\mathcal{F}_0(W))$ lies in a complex subspace $\mathcal{U} = \{(x, y, z) \in \mathbb{C}^{2\nu} | \sum_{c \in \mathcal{B}_S} (A_c x + B_c y) = 0\}$, where $A_c, B_c \in \mathbb{C}$, $\forall c \in \mathcal{B}_S$, and

$$\sum_{c \in \mathcal{B}_S} (|A_c| + |B_c|) \neq 0. \quad \text{(3.1)}$$

Then, writing $\Gamma_1 = \sum_{c \in \mathcal{B}_S} A_c c$ and $\Gamma_2 = \sum_{c \in \mathcal{B}_S} B_c c$, we have

$$\int_{\Gamma_1} f \phi_2 + i \int_{\Gamma_2} f \phi_1 = 0, \quad \forall f \in \mathcal{F}_0(W). \quad \text{(3.2)}$$

Denote by $\Sigma = \{f \in \mathcal{F}_0(W) | (f) \geq (\phi_2)|_{M_S}\}^2$ (recall that $\phi_2$ never vanishes on $C_S$). Then for any $f \in \Sigma$ the function $df \phi_2 \in \mathcal{F}_0^*(S) \cap \mathcal{F}_0(M_S)$, so it can be uniformly approximated on $S$ by functions in $\mathcal{F}_0(W)$. This fact is trivial when $f$ is constant, otherwise use Lemma 3.1. Hence equation (3.2) applies and gives

$$0 = \int_{\Gamma_2} \xi df = \int_{\Gamma_2} f \phi_1, \quad \forall f \in \Sigma, \quad \text{(3.3)}$$

where we have used integration by parts (notice that $f\xi, \xi df$ and $f d\xi$ are smooth).

Suppose $\Gamma_2 \neq 0$ and take $[\tau] \in \mathcal{H}^1_{\text{hol}}(W)$ (the first holomorphic De Rham cohomology group of $W$) and $g \in \mathcal{F}_0(W)$ so that $\int_{\Gamma_2} \tau \neq 0$, the function $f := (\tau + dg) / d\xi \in \mathcal{F}_0^*(S) \cap \mathcal{F}_0(M_S)$ and $(f)|_{M_S} \geq (\phi_2)|_{M_S}$ by substitution. The existence of such 1-form and function follows from well known arguments on Riemann surfaces theory (take into account that $A$ implies $d\xi|_{M_S} \neq 0$). By Lemma 3.1, $f$ can be uniformly approximated on $S$ by functions in $\Sigma$, so equation (3.3) applies and shows that $0 = \int_{\Gamma_2} f \phi_2 = \int_{\Gamma_2} (\tau + dg) = \int_{\Gamma_2} \tau \neq 0$, a contradiction. Therefore $\Gamma_2 = 0$.

Replacing $(\xi, \phi_1, \phi_2, \Gamma_1, \Gamma_2)$ by $(1/\xi, \phi_2, \phi_1, \Gamma_2, \Gamma_1)$ and using a symmetric argument, we can prove that $\Gamma_1 = 0$. This contradicts (3.1) and concludes the proof. \(\Box\)

Let $\{e_1, \ldots, e_{2\nu}\}$ be a basis of $\mathbb{C}^{2\nu}$, fix $f_i \in \mathcal{A}^{-1}(e_i) \cap \mathcal{F}_0(W)$ for all $i$, and set $\mathcal{Q} : \mathbb{C}^{2\nu} \to \mathbb{C}^{2\nu}$ as the analytical map given by

$$\mathcal{Q}(z_i)_{i=1,\ldots,2\nu} = \mathcal{P}(\sum_{i=1,\ldots,2\nu} z_i f_i).$$

By Claim 3.5 the differential $d\mathcal{Q}_0$ of $\mathcal{Q}$ at 0 is an isomorphism, then there exists a closed Euclidean ball $U \subset \mathbb{C}^{2\nu}$ centered at the origin such that $\mathcal{Q} : U \to \mathcal{Q}(U)$ is an analytical diffeomorphism. Furthermore, notice that $0 = \mathcal{Q}(0) \in \mathcal{Q}(U)$ is an interior point of $\mathcal{Q}(U)$.

Consider a sequence $\{\eta_n\}_{n \in \mathbb{N}} \subset \Omega_0(W)$ uniformly approximating $\eta$ on $S$ and with $(\eta_n) = (\eta)|_{M_S}$ for all $n$ (recall that $\eta|_{M_S} \neq 0$ and see Lemma 3.2).
Label $\mathcal{P}_n : \mathcal{F}_0^0(S) \to C^{2v}$ as the Fréchet differentiable map given by

$$\mathcal{P}_n(f) = \left( \int \left( e^f \theta_n + e^{-f} \frac{\partial}{\partial \theta_n} - 2 \phi_1, e^f \theta_n - e^{-f} \frac{\partial}{\partial \theta_n} + 2i \phi_2 \right) \right)_{c \in B_S}, \quad \forall n \in \mathbb{N}.$$ 

Call $Q_n : C^{2v} \to C^{2v}$ as the analytical map $Q_n((z_i)_{i=1,\ldots,2v}) = Q_n(\sum_i z_i f_i)$ for all $n \in \mathbb{N}$. Since $\{Q_n\}_{n \in \mathbb{N}} \to Q$ uniformly on compacts subsets of $C^{2v}$, without loss of generality we can suppose that $Q_n : U \to Q_n(U)$ is an analytical diffeomorphism and $0 \in Q_n(U)$ for all $n$. Label $a_n = (a_1, \ldots, a_{2v})$ as the unique point in $U$ such that $Q_n(a_n) = 0$ and note that $\{a_n\}_{n \in \mathbb{N}} \to 0.$

Set

$$\eta_n := e^{\sum_{i=1}^{2v} a_i_n f_i} \theta_n, \quad \phi_{1,n} := \frac{1}{2} \left( \eta_n + \frac{\partial}{\partial \eta_n} \right) \quad \text{and} \quad \phi_{2,n} := \frac{i}{2} \left( \eta_n - \frac{\partial}{\partial \eta_n} \right), \quad \forall n \in \mathbb{N}$$

and let us check that the sequence $\{\Phi_n = (\phi_{1,n}, \phi_{2,n})\}_{n \in \mathbb{N}}$ satisfies the conclusion of the lemma.

Indeed, since $(\eta_n) = (\theta_n) = (\eta|_{M_0})$ one has $\Theta/\eta_n \in \Omega_0(W)$ and so $\Phi_n \in \Omega_0(W)^2$. The convergence of $\{\Phi_n\}_{n \in \mathbb{N}}$ to $\Phi$ on $S$ follows from the ones of $\{\theta_n\}_{n \in \mathbb{N}}$ to $\eta$ and of $\{a_n\}_{n \in \mathbb{N}}$ to $0$. A straightforward computation gives (a). The fact that $Q_n(a_n) = 0, n \in \mathbb{N}$, implies (b). Finally, $(\eta_n) = (\eta|_{M_0})$ for all $n$ implies (c).

The proof of the lemma in case (B) goes as follows.

Notice that $\Theta = 0$ is nothing but $\beta \phi_1$, where $\beta \in \{i, -i\}$.

As above, we can assume without loss of generality that $\phi_1$ never vanishes on $C_S$ (we omit the details). Reasoning as in case (A), we can prove that $\hat{A}|_{\mathcal{F}_0^0(W)} : \mathcal{F}_0^0(W) \to C^v$ is surjective, where $\hat{A}$ is the Fréchet derivative of $\hat{\mathcal{P}} : \mathcal{F}_0^0(S) \to C^v, \hat{\mathcal{P}}(f) = \left( \int \left( e^f - 1 \right) \phi_1 \right)_{c \in B_S}, \text{ at } 0$. Take $\hat{f}_i \in \hat{A}^{-1}(\hat{\mathcal{P}}) \cap \mathcal{F}_0^0(W)$ for all $i$, where $B_S = \{\hat{\xi}_1, \ldots, \hat{\xi}_v\}$ is a basis of $C^v$, and define $\hat{Q} : C^v \to C^v$ by $\hat{Q}((z_i)_{i=1,\ldots,v}) = \hat{\mathcal{P}}(\sum_i z_i \hat{f}_i)$. Now, consider a sequence $\{\hat{\theta}_n\}_{n \in \mathbb{N}} \subset \Omega_0(W)$ that uniformly approximates $\phi_1$ on $S$ and $\hat{\theta}_n = (\phi_1|_{M_0})$ for all $n$ (as above, recall that $\phi_1|_{M_0} \neq 0$ and see Lemma 3.2). Set $\hat{\mathcal{P}}_n : \mathcal{F}_0^0(S) \to C^v$ by $\hat{\mathcal{P}}_n(f) = \left( \int \left( e^f \hat{\theta}_n - \phi_1 \right) \right)_{c \in B_S}, \text{ and call } Q_n : C^v \to C^v$ as the analytical map $Q_n((z_i)_{i=1,\ldots,v}) = Q_n(\sum_i z_i f_i)$ for all $n \in \mathbb{N}$. To finish, reason as in case (A) but setting $\phi_{1,n} := e^{\sum_{i=1}^{2v} a_i_n \hat{f}_i} \hat{\theta}_n$ and $\phi_{2,n} := \beta \phi_1$, where $\hat{a}_n = (\hat{a}_1, \ldots, \hat{a}_v)$ is chosen so that $Q_n(\hat{a}_n) = 0$ and $\{\hat{a}_n\}_{n \in \mathbb{N}} \to 0$. \hfill \QED 

4. MAIN RESULTS

The main results of this paper follow as consequence of Lemma 4.1 below. Although the proof of this lemma is inspired by the technique developed in [AFL, Lemma 3.1], it represents a wide generalization of that result.

We need the following notations and definitions.

Fix a nowhere zero $\tau_0 \in \Omega_0(N)$ (the existence of such a $\tau_0$ is well known, anyway see [AFL] for a proof). Then for any compact subset $K \subset N$ and any $\theta \in \Omega_0^0(K)$ we set $|\theta|| := \max_K \{|\theta/\tau_0|\}$. This norm induces the topology of the uniform convergence on $\Omega_0^0(K)$.

Let $K \subset N$ be a connected compact region and $\sigma^2$ a Riemannian metric on $K$ possibly with singularities. Given $P, Q \in K$ we denote by $\text{dist}_{\{K, \sigma^2\}}(P, Q) = \min \{|\text{length}_K(a)| \in K \text{ joining } P \text{ and } Q\}$. If $K_1$ and $K_2$ are two compact sets in $K$ we set $\text{dist}_{\{K, \sigma^2\}}(K_1, K_2) = \min \{|\text{dist}_{\{K, \sigma^2\}}(P, Q) | P \in K_1, Q \in K_2\}$.

**Lemma 4.1.** Let $M_1, K$ be two compact regions in $N$ with $M \subset K^0$. Assume that $M$ is Runge, $K$ is connected and consider $\theta_0 \in \Omega_0^0(M)$. Let $T$ be a conformal Riemannian metric on $K$ possibly with isolated singularities. Let $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2) : \mathcal{H}_1(K, \mathcal{Z}) \to C^2$ be a group homomorphism, $\Theta \in \Omega_0^0(K)$ and
Consider a large $\Phi = (\phi_1, \phi_2) \in \Omega_0(M)^2$ satisfying
\[ \phi_1^2 + \phi_2^2 = \Theta|_M, \quad f(\gamma) = \int_\gamma \Phi, \forall \gamma \in \mathcal{H}_1(M, \mathbb{Z}), \]
and either of the following conditions:

(A) $\phi_1$ and $\phi_2$ are linearly independent in $\Omega_0(M)$.

(B) $\Theta = 0$, $\phi_1 \neq 0$ and there is $\beta \in \{1, -1\}$ such that $\epsilon_2 = \beta \epsilon_1$ and $\phi_2 = \beta \phi_1$.

Then, for any $\epsilon > 0$ there exists $\Psi = (\psi_1, \psi_2) \in \Omega_0(K)^2$ so that

(L1) $\|\Psi - \Phi\| < \epsilon$ on $M$,

(L2) $\phi_1^2 + \phi_2^2 = \Theta$,

(L3) $f(\gamma) = \int_\gamma \Psi, \forall \gamma \in \mathcal{H}_1(K, \mathbb{Z}),$

(L4) $\text{dist}_{(K,\mathbb{C},\mathbb{Z})}(P_0, \partial K) > 1/\epsilon$, where $\sigma^2_{K,\mathbb{Z}} := |\psi_1|^2 + |\psi_2|^2 + I$, and

(L5) the zeros of $\Psi$ on $K$ are those of $\Phi$ on $M$ (in particular, $\Psi$ never vanishes on $K - M$).

**Proof.** The proof goes by induction on minus the Euler characteristic of $W - M^\circ$. Since $M$ is Runge then no component of $K - M^\circ$ is a closed disc, and so $-\chi(K - M^\circ) \geq 0$. The basis of the induction is proved in the following

**Claim 4.2.** Lemma 4.1 holds if $\chi(K - M^\circ) = 0$.

**Proof.** In this case $K^\circ - M = \bigcup_{j=1}^k A_j$, where $A_j$ are pairwise disjoint open annuli, $k \in \mathbb{N}$. On each $A_j$ we construct a Jorge-Xavier’s type labyrinth of compact sets as follows (see [JX]). Let $z_j : A_j \rightarrow \mathbb{C}$ be a conormal parametrization, and let $C_j \subset A_j$ be a compact region such that $C_j$ contains no singularities of $\mathcal{I}$, $z_j(C_j)$ is a compact annulus of radii $r_j$ and $R_j$, where $r_j < R_j$, and $z_j(C_j)$ contains the homology of $z_j(A_j)$. This choice is possible since the singularities of $\mathcal{I}$ are isolated. Since $\mathcal{I}|C_j$ has no singularities, we can find a positive constant $\mu$ with

\[
(4.1) \quad \mathcal{I} > \mu^2 |dz_j|^2 \quad \text{on} \quad C_j, \quad j = 1, \ldots, k.
\]

Consider a large $m \in \mathbb{N}$ (to be specified later) such that $2/m < \min\{R_j - r_j \mid j = 1, \ldots, k\}$. For any $j \in \{1, \ldots, k\}$ label $s_{j,0} := R_j$ and for any $n \in \{1, \ldots, 2m^2\}$ set $s_{j,n} := R_j - n/m^2$ and consider the compact set in $C_j$ (see Figure 4.1):

\[
K_{j,n} = \left\{ P \in A_j \mid s_{j,n} + \frac{1}{4m^3} \leq |z_j(P)| \leq s_{j,n-1} - \frac{1}{4m^3}, \quad \frac{1}{m^2} \leq \arg((-1)^n z_j(P)) \leq 2\pi - \frac{1}{m^2} \right\}.
\]

Then, define
\[
\mathcal{K}_j = \bigcup_{n=1}^{2m^2} K_{j,n} \quad \text{and} \quad \mathcal{K} = \bigcup_{j=1}^k \mathcal{K}_j.
\]

Consider the pair $\Xi = (\phi_1, \phi_2) \in \Omega_0(M \cup \mathcal{K})^2$ given by
\[
\Xi|_M = \Phi, \quad \Xi|_{\mathcal{K}_j} = \begin{cases}
(\frac{1}{2}(\lambda dz_j + \Theta), \frac{1}{2}(\lambda dz_j - \Theta)) & \text{if (A) holds} \\
(\lambda dz_j, \beta \lambda dz_j) & \text{if (B) holds},
\end{cases} \quad j = 1, \ldots, k,
\]

where $\lambda > \sqrt{2} \mu m^4$ is a constant. Notice that $\phi_1^2 + \phi_2^2 = \Theta|_{M \cup \mathcal{K}}$.

Let $W \subset \mathcal{N}$ be a domain with finite topology containing $K$. Applying Lemma 3.3 to the data
\[
\hat{W} = W, \quad \hat{S} = M \cup \mathcal{K}, \quad \hat{\Theta} = \Theta, \quad \text{and} \quad \hat{\Phi} = \Xi,
\]
we obtain a pair $\Psi = (\psi_1, \psi_2) \in \Omega_0(K)^2$ satisfying (L1), (L2), (L3), (L5) and

\[(4.2) \quad \psi_1^2 + |\psi_2|^2 > \mu^2 m^8 |dz_j|^2 \quad \text{on } K_j, \ j = 1, \ldots, k. \]

Then, taking into account (4.1), (4.2) and the definition of $K_j$, it is straightforward to check the existence of a positive constant $\rho_j$ depending neither on $\mu$ nor $m$ such that

$$\text{length}_{\nu(\mathcal{F})}(\alpha) > \rho_j \cdot \mu \cdot m$$

for any $\alpha$ curve in $C_j$ joining the two components of $\partial C_j$. Thus, we can choose $m$ large enough so that $\rho_j \cdot \mu \cdot m > 1/\epsilon$ for any $j = 1, \ldots, k$. This choice gives (L4) and we are done. \qed

The inductive step and so Lemma 4.1 are proved in the following

Claim 4.3. Consider $n > 0$ and assume that Lemma 4.1 holds if $-\chi(K - M^\circ) < n$. Then it also holds if $-\chi(K - M^\circ) = n$.

Proof. Since $M$ is Runge, $j_* : \mathcal{H}_1(M, \mathbb{Z}) \to \mathcal{H}_1(K, \mathbb{Z})$ is a monomorphism, where $j : M \to K$ is the inclusion map. Up to this natural identification we will consider $\mathcal{H}_1(M, \mathbb{Z}) \subset \mathcal{H}_1(K, \mathbb{Z})$. Since $-\chi(K - M^\circ) = n > 0$, there exists $\hat{\gamma} \in \mathcal{H}_1(K, \mathbb{Z}) - \mathcal{H}_1(M, \mathbb{Z})$ intersecting $K - M^\circ$ in a compact Jordan arc $\gamma$ with endpoints $P_1, P_2 \in \partial M$ and otherwise disjoint from $\partial M \cup \partial K$, and such that $S := M \cup \gamma$ is admissible. Notice that in this case $\gamma = C_S$ and $M = M_S$.

Assume (A) holds, and in addition choose $\hat{\gamma}$ so that $\Theta$ never vanishes on $\gamma$. Consider a pair $\Phi = (\phi_1, \phi_2) \in \Omega_0^1(S)^2 \cap \Omega_0(M_S)^2$ satisfying $\Phi|_M = \Phi, \phi_1^2 + \phi_2^2 = \Theta|_S$ and $\int_\gamma \Phi = f(\hat{\gamma})$ (we leave the details to the reader). By Lemma 3.3, case (A), applied to $\Phi, S, \Theta$ and $K^\circ$, we can find a compact tubular neighborhood $U$ of $S$ in $K^\circ$ and $\Xi = (\phi_1, \phi_2) \in \Omega_0(U)^2$ such that $\phi_1$ and $\phi_2$ are linearly independent in $\Omega_0(U)^2$, $\|\Xi - \Phi\| < \epsilon/2$ on $M$, $\phi_1^2 + \phi_2^2 = \Theta|_U$, the zeros of $\Xi$ on $U$ are those of $\Phi$ on $M$, and $\Xi + \Phi$ is exact on $S$. Since $-\chi(K - U^\circ) < n$, the induction hypothesis applied to $\Xi$ and $\epsilon/2$ gives the existence of a pair $\Psi \in \Omega_0(K)^2$ satisfying the conclusion of the lemma.

Assume now that (B) holds, and take a function $\hat{\phi}_1 \in \Omega_0^1(S) \cap \Omega_0(M_S)$ such that $\hat{\phi}_1|_M = \phi_1$ and $\int_\gamma \hat{\phi}_1 = f_1(\hat{\gamma})$. Apply Lemma 3.3, case (B), to the data $K^\circ$, $S$ and $(\phi_1, \hat{\rho}_1\hat{\phi}_1)$, and obtain a compact tubular neighborhood $U$ of $S$ in $K^\circ$ and a 1-form $\hat{\phi}_1 \in \Omega_0(U)$ such that $\|\hat{\phi}_1 - \phi_1\| < \epsilon/4$ on

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{image}
\caption{The labyrinth of compact sets on the annulus $z_j(C_j)$.}
\end{figure}
$M$, the zeros of $\varphi_1$ on $U$ are those of $\varphi_1$ on $M$, and $\varphi_1 - \varphi_1$ is exact on $S$. As above, the induction hypothesis applied to $(\varphi_1, \beta \varphi_1)$ and $\varepsilon/2$ gives a pair $\Psi \in \Omega_0(K)^2$ proving the claim.

This finishes the proof of the lemma.

Now we can state and prove the main theorem of this paper.

**Theorem 4.4.** Let $M \subseteq N$ be a Runge compact region. Let $I$ be a conformal Riemannian metric on $N$ possibly with isolated singularities. Consider $\mathfrak{f} = (f_1, f_2) : H_1(N, Z) \rightarrow C^2$ be a group homomorphism, $\Theta \in \Omega_0(N)$ and $\Phi = (\varphi_1, \varphi_2) \in \Omega_0(M)^2$ satisfying

$$\varphi_1^2 + \varphi_2^2 = \Theta|_M, \quad \mathfrak{f}(\gamma) = \int_\gamma \Phi, \forall \gamma \in H_1(M, Z),$$

and either of the following conditions:

(A) $\varphi_1$ and $\varphi_2$ are linearly independent in $\Omega_0(M)$.

(B) $\Theta = 0, \varphi_1 \neq 0$ and there is $\beta \in \{i, -i\}$ such that $f_2 = \beta f_1$ and $\varphi_2 = \beta \varphi_1$.

Then, for any $\varepsilon > 0$ there exists $\Psi = (\psi_1, \psi_2) \in \Omega_0(N)^2$ so that

(T1) $\|\Psi - \Phi\| < \varepsilon$ on $M$,

(T2) $\varphi_1^2 + \varphi_2^2 = \Theta$,

(T3) $\mathfrak{f}(\gamma) = \int_\gamma \Psi, \forall \gamma \in H_1(N, Z)$,

(T4) $|\varphi_1|^2 + |\varphi_2|^2 + I$ is a complete conformal Riemannian metric on $N$ with singularities at the zeros of $|\varphi_1|^2 + |\varphi_2|^2 + I$ on $M$, and

(T5) the zeros of $\Psi$ on $N$ are those of $\Phi$ on $M$ (in particular, $\Psi$ never vanishes on $N - M$).

**Proof.** Label $M_1 = M$ and let $\{M_n | n \geq 2\}$ be an exhaustion of $N$ by Runge connected compact regions with $M_n \subseteq M_{n+1}$ for all $n \in \mathbb{N}$. Fix a base point $p_0 \in M^0$ and a positive $\varepsilon < \min\{\varepsilon, 1\}$ which will be specified later.

Label $\Phi_1 = \Phi$, and by Lemma 4.1 and an inductive process, construct a sequence of pairs $\{\Phi_n = (\varphi_{jn,n})_{j=1,2}\}_{n \in \mathbb{N}}$ satisfying that

(a) $\Phi_n \in \Omega_0(M_n^2), \forall n \in \mathbb{N}$,

(b) $\|\Phi_n - \Phi_{n-1}\| < \varepsilon/2^n$ on $M_{n-1}, \forall n \geq 2$,

(c) $\varphi_{1n,n}^2 + \varphi_{2n,n}^2 = \Theta|_{M_n}, \forall n \in \mathbb{N}$,

(d) $\mathfrak{f}(\gamma) = \int_\gamma \Phi_n, \forall \gamma \in H_1(M_n, Z), \forall n \in \mathbb{N}$,

(e) $\text{dist}_{(M_n, \partial(M_n, Z))}(p_0, \partial M_n) > 2^n$, where $\sigma_{(\varphi_{jn,n})}^2 = |\varphi_{1n,n}|^2 + |\varphi_{2n,n}|^2 + I, \forall n \geq 2$, and

(f) the zeros of $\varphi_n$ on $M_n$ are those of $\Phi$ on $M, \forall n \in \mathbb{N}$.

Since $\cup_{n \in \mathbb{N}} M_n = N$, items (a) and (b) and Harnack's theorem, then the sequence $\{\Phi_n\}_{n \in \mathbb{N}}$ uniformly converges on compact subsets of $N$ to a pair $\Psi = (\psi_{j=1,2}) \in \Omega_0(N)$ satisfying (T1). Items (c) and (d) directly give (T2) and (T3), respectively. Since $\{\Phi_n\}_{n \in \mathbb{N}}$ uniformly converges to $\Psi$ and (f), Hurwitz's theorem gives that either the zeros of $\Psi$ on $N$ are those of $\Phi$ on $M$ or $\psi_1 = 0$ or $\psi_2 = 0$. However, (b) gives $\|\Psi - \Phi\| \leq \varepsilon$ on $M$ and so $\psi_j|_M \neq 0, j = 1, 2$, provided that $\varepsilon$ is small enough. This proves (T5). Finally (T5) and (e) imply (T4) and we are done.

**Corollary 4.5.** Let $\varphi_i, X = (X_i)_{i=3,\ldots,N} : N \rightarrow \mathbb{R}^{N-2}$ and $\mathfrak{p} = (\mathfrak{p}_i)_{i=1,\ldots,N} : H_1(N, Z) \rightarrow \mathbb{R}^N$ be a 2-form in $\Omega_0(N)^2$, a non-constant harmonic map and a group homomorphism, respectively, satisfying that

- $\mathfrak{p}_1(\gamma) = \text{Im} \int_\gamma \partial_2 X_i, \forall \gamma \in H_1(N, Z), \forall i = 3, \ldots, N,$ and
- $\mathfrak{p}_1 = \mathfrak{p}_2 = 0$ when $\varphi_i = \sum_{i=3}^N (\partial_2 X_i)^2$.

Then there exists a weakly complete harmonic map $Y = (Y_j)_{j=1,\ldots,N} : N \rightarrow \mathbb{R}^N$ with
(I) \((Y_i)_{i=3,...,N} = X\),

(II) \(p_Y = p\), and

(III) \(Q_Y = \delta_t\).

Furthermore, if \(X\) is full then \(Y\) can be chosen to be full, and if \(X\) is an immersion then \(Y\) is.

**Proof.** Label \(\Theta := \delta_t - \sum_{i=3}^{N} (\partial_i X_i)^2\), and assume for a moment that \(\Theta \neq 0\). Consider a compact disc \(K \subset \mathcal{N}\) and \(\eta \in \Omega_0(K)\) such that both \(\eta\) and \(\phi_1\) never vanish on \(K\), and \(\phi_1\) and \(\phi_2\) are linearly independent in \(\Omega_0(K)\), where \(\phi_1 := \frac{\Theta}{\eta + \Theta/\eta}\) and \(\phi_2 := \frac{\Theta}{\eta - \Theta/\eta}\). Consider a pair \(\Psi = (\phi_1, \phi_2)\) obtained from Theorem 4.4, case (A), applied to the data

\[
N, \quad M = K, \quad I = \sum_{i=3}^{N} |\partial_i X_i|^2, \quad \Theta, \quad \Phi = (\phi_1, \phi_2), \quad f = t(p_1, p_2)
\]

and \(\epsilon > 0\) to be specified later. Fix a point \(p_0 \in \mathcal{N}\) and define \(Y_k(P) = \text{Re} \int_{p_0}^{P} \psi_i, \forall P \in \mathcal{N}, k = 1, 2, \) and \(Y_k = X_k, \forall k = 3, \ldots, N\).

Statements (I), (II) and (III) trivially follow from the definition of \(\Theta\) and \(f\), and properties (T2) and (T3). Moreover, (T4) and the fact that \(\phi_1\) never vanishes on \(K\) give that \(\sum_{i=3}^{N} |\partial_i Y_i|^2\) is a complete conformal metric on \(\mathcal{N}\), and so \(Y\) is weakly complete. Finally, if \(X\) is full then we can choose \(\eta\) so that the map

\[
K \to \mathbb{R}^N, \quad P \mapsto \left(\int_{p_0}^{P} \phi_1, \int_{p_0}^{P} \phi_2, X(P)\right)
\]

is full as well. Then (T1) gives the fullness of \(Y\) provided that \(\epsilon\) is chosen small enough.

Assume now that \(\Theta = 0\). Take an exact \(\phi_1 \in \Omega_0(M), \phi_1 \neq 0\), and consider a pair \(\Psi\) obtained by applying Theorem 4.4, case (B), to the data

\[
N, \quad M = K, \quad I = \sum_{i=3}^{N} |\partial_i X_i|^2, \quad \Theta = 0, \quad \Phi = (\phi_1, \phi_1), \quad f = 0
\]

and \(\epsilon > 0\). To finish argue as above.\(\square\)

**Corollary 4.6.** Let \(X = (X_i)_{i=3,...,N} : \mathcal{N} \to \mathbb{R}^{N-2}\) and \(p = (p_i)_{i=3,...,N} : \mathcal{H}_1(\mathcal{N}, \mathbb{Z}) \to \mathbb{R}^N\) be a non-constant harmonic map and a group homomorphism, respectively, satisfying that

- \(p_i(\gamma) = \text{Im} \int_{\gamma} \partial_i X_i, \forall \gamma \in \mathcal{H}_1(\mathcal{N}, \mathbb{Z}), \forall i = 3, \ldots, N,\)
- \(p_1 - p_1 = 0\) when \(\sum_{i=3}^{N} (\partial_i X_i)^2 = 0\).

Then there exists a complete conformal minimal immersion \(Y = (Y_i)_{i=3,...,N} : \mathcal{N} \to \mathbb{R}^N\) with \((Y_i)_{i=3,...,N} = X\) and \(p_Y = p\). Furthermore, \(Y\) can be chosen full provided that \(X\) is.

**Proof.** Apply Corollary 4.5 for \(\delta_t = 0\) and see Remark 2.4.\(\square\)

**Corollary 4.7.** Let \(\mathcal{N}\) be a bounded planar domain. Then there exists a complete non-proper holomorphic embedding of \(\mathcal{N}\) in \(\mathbb{C}^2\).

**Proof.** Consider \(X = (X_3, X_4) : \mathcal{N} \to \mathbb{R}^2 \equiv \mathbb{C}\) given by \(X(z) = z\). Let \(Y = (Y_i)_{i=3,...,4} : \mathcal{N} \to \mathbb{R}^4\) be an immersion obtained from Corollary 4.6 applied to the data \(\mathcal{N}, X\) and \(p = 0\). Since \(X\) is injective, \(Y\) is an embedding. Finally, observe that \(Y\) is non-proper. Indeed, otherwise the holomorphic function \(Y_1 + i Y_2\) would be proper on \(\mathcal{N}\), contradicting that \(\mathcal{N}\) is hyperbolic.\(\square\)

**Corollary 4.8.** Let \(p : \mathcal{H}_1(\mathcal{N}, \mathbb{Z}) \to \mathbb{R}^N\) be a group homomorphism.

Then there exists a conformal complete minimal immersion \(Y : \mathcal{N} \to \mathbb{R}^N\) satisfying

- \(p_Y = p\).
• $Y$ is non-decomposable and full,
• $G_Y$ is non-degenerate, and
• $G_Y$ fails to intersect $\mathcal{N}$ hyperplanes of $\mathbb{CP}^{N-1}$ in general position.

Proof. We need the following

Claim 4.9 ([AFL, Theorem 4.2]). For any group homomorphism $\tilde{\phi} : \mathcal{H}(\mathcal{N}, \mathbb{Z}) \to \mathbb{R}$ there exists a never vanishing $\phi \in \Omega_0(\mathcal{N})$ with $\int_{\gamma} \phi = \int_{\gamma} \tilde{\phi}(\gamma), \forall \gamma \in \mathcal{H}(\mathcal{N}, \mathbb{Z})$.

Assume first that $N$ is even.

Consider a nowhere zero $\phi \in \Omega_0(\mathcal{N})$ (see Claim 4.9) and a compact disc $M \subset \mathcal{N}$. Fix $p_0 \in M^o$ and take $\lambda_j \in \mathbb{C} \setminus \{0\}$ and $\Phi_j = (\phi_{j,1}, \phi_{j,2}) \in \Omega_0(\mathcal{M})^2, j = 1, \ldots, N/2$, so that

• $\sum_{j=1}^{N/2} \lambda_j^2 = 0$,
• $\phi_{j,1}$ and $\phi_{j,2}$ are linearly independent in $\Omega_0(M)$ and $\phi_{j,1}^2 + \phi_{j,2}^2 = \lambda_j^2 \phi_j^2 |_{M}, \forall j = 1, \ldots, N/2$,
• the minimal immersion $X: M \to \mathbb{R}^N, X(P) = \text{Re}(\int_{p_0}^P (\Phi_j)_{j=1, \ldots, N/2})$ is non-decomposable and full, and
• $G_X$ is non-degenerate.

Write $p = (p_k)_{k=1, \ldots, N}$, and for any $j = 1, \ldots, N/2$ consider $\Psi_j = (\psi_{j,1}, \psi_{j,2}) \in \Omega_0(\mathcal{N})^2$ given by Theorem 4.4, case (A), applied to the data

$\mathcal{N}, M, I = |\phi|^2, f = i(p_{2j-1}, p_{2j}), \Theta = \lambda_j^2 \phi_j^2, \Phi = \Phi_j$,

and $\epsilon > 0$ which will be specified later. Define $Y : \mathcal{N} \to \mathbb{R}^N, Y(P) = \text{Re} \left( \int_{p_0}^P (\Psi_j)_{j=1, \ldots, N/2} \right)$.

Statement (T3) in Theorem 4.4 gives that $Y$ is well defined. From (T2) follows that $\sum_{j=1}^{N/2} (\psi_{j,1}^2 + \psi_{j,2}^2) = 0$, and so $Y$ is conformal. Moreover, $\sum_{j=1}^{N/2} (|\psi_{j,1}|^2 + |\psi_{j,2}|^2) \geq |\psi_{1,1}|^2 + |\psi_{1,2}|^2 \geq \frac{1}{|\lambda_1|^2} (|\psi_{1,1}|^2 + |\psi_{1,2}|^2)$ that is a complete Riemannian metric on $\mathcal{N}$ (take into account (T4)). Therefore, $Y$ is a complete conformal minimal immersion. Item (T3) implies that $p_Y = p$. Since $X$ is non-decomposable and full and $G_X$ is non-degenerate, then $Y$ and $G_Y$ are, provided that $\epsilon$ is chosen small enough (see (T1)). Finally, observe that $\psi_{j,1}^2 + \psi_{j,2}^2$ never vanishes on $\mathcal{N}$ for all $j = 1, \ldots, N/2$, hence $G_Y$ fails to intersect the hyperplanes

$$\Pi_{j,\delta} := \left\{ \left( (w_k)_{k=1, \ldots, N} \in \mathbb{CP}^{N-1} \mid w_{2j-1} + (-1)^j i w_{2j} = 0 \right) \right\}, \forall (j, \delta) \in \{1, \ldots, N/2\} \times \{0, 1\},$$

which are located in general position.

Assume now that $N$ is odd.

Write $p = (p_k)_{k=1, \ldots, N}$ and consider a nowhere zero $\phi \in \Omega_0(\mathcal{N})$ with $\int_{\gamma} \phi = \int_{\gamma} \tilde{\phi}(\gamma), \forall \gamma \in \mathcal{H}(\mathcal{N}, \mathbb{Z})$ (see Claim 4.9). Fix a compact disc $M \subset \mathcal{N}$ and a point $p_0 \in M^o$. Take $\lambda_j \in \mathbb{C} \setminus \{0\}$ and $\Phi_j = (\phi_{j,1}, \phi_{j,2}) \in \Omega_0(\mathcal{M})^2, j = 1, \ldots, (N - 1)/2$ so that:

• $\sum_{j=1}^{(N-1)/2} \lambda_j^2 = -1$,
• $\phi_{j,1}$ and $\phi_{j,2}$ are linearly independent in $\Omega_0(M)$ and $\phi_{j,1}^2 + \phi_{j,2}^2 = \lambda_j^2 \phi_j^2 |_{M}, \forall j = 1, \ldots, (N - 1)/2$,
• the minimal immersion $X: M \to \mathbb{R}^N, X(P) = \text{Re}(\int_{p_0}^P (\Phi_j)_{j=1, \ldots, (N-1)/2}, \phi)$ is non-decomposable and full, and
• $G_X$ is non-degenerate.
For any \( j = 1, \ldots, (N-1)/2 \) consider \( \Psi_j = (\psi_{j,1}, \psi_{j,2}) \in \Omega_0(N)^2 \) given by Theorem 4.4, case (A), applied to the data

\[
\mathcal{N}, \ M, \ I = |\psi|^2, \ f = i(p_{j,-1}, p_{j}) \quad \Theta = \lambda^2|\psi|^2, \ \Phi = \Phi_j,
\]
and \( \varepsilon > 0 \) which will be specified later.

As above

\[
Y : \mathcal{N} \to \mathbb{R}^N, \quad Y(P) = \text{Re} \left( \int_{p_0}^P (\Psi_j)_{j=1,\ldots,(N-1)/2} \frac{\Theta}{\Phi} \right)
\]
is the immersion we are looking for, provided that \( \varepsilon \) is small enough. In this case \( G_Y \) fails to intersect the following hyperplanes of \( \mathbb{CP}^{N-1} \) located in general position:

\[
\Pi_{j,\delta} := \left\{ \left( [w_k]_{k=1,\ldots,N} \right) \in \mathbb{CP}^{N-1} \; | \; w_{2j-1} + (-1)^\delta i w_{2j} = 0 \right\},
\]
\( \forall (j, \delta) \in \{1, \ldots, (N-1)/2\} \times \{0, 1\} \), and

\[
\Pi := \left\{ \left( [w_k]_{k=1,\ldots,N} \right) \in \mathbb{CP}^{N-1} \; | \; w_N = 0 \right\}.
\]

The proof is done. \( \square \)

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