RAMSEY THEORY ON INFINITE STRUCTURES AND
THE METHOD OF STRONG CODING TREES

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Abstract. This article discusses some recent trends in Ramsey theory on infinite structures. Trees and their Ramsey theory have been vital to these investigations. The main ideas behind the author’s recent method of trees with coding nodes are presented, showing how they can be useful both for coding structures with forbidden configurations as well as those with none. Using forcing as a tool for finite searches has allowed the development of Ramsey theory on such trees, leading to solutions for finite big Ramsey degrees of Henson graphs as well as infinite dimensional Ramsey theory of copies of the Rado graph. Possible future directions for applications of these methods are discussed.

1. Introduction

Logic and Ramsey theory have a long interconnected history. In 1929, Ramsey proved his celebrated theorem in order to obtain a partial solution to Hilbert’s Entscheidungsproblem. This problem, posed by Hilbert in 1928, asked for an algorithm which decides the validity of any statement of first-order logic. By Gödel’s Completeness Theorem (1929), this is equivalent to asking for an algorithm which can decide whether a given formula is provable from the logical axioms. Ramsey applied his partition theorem on the natural numbers to show that the validity of formulas with only universal quantifiers in normal form is decidable [37]. In light of the Church-Turing thesis ([2] and [44]) which precludes any complete solution to Hilbert’s Entscheidungsproblem, Ramsey’s result is all the more striking in its success.

Over the decades, this interplay between Ramsey theory and logic has continued, each subject motivating, informing, and providing techniques to solve problems in the other. An instance of this is seen in the 1966 result of Halpern and Läuchli [17]. While investigating the

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problem of whether or not the Boolean Prime Ideal Theorem (BPI) is strictly weaker than the Axiom of Choice (AC) over the Zermelo-Fraenkel Axioms (ZF), Halpern and Läuchli proved a theorem which was later interpreted to be a Ramsey theorem on products of finitely many trees. This theorem was central to the proof of Halpern and Lévy that, indeed, BPI is strictly weaker than AC over ZF \[18\]. In turn, the Halpern-Läuchli theorem provided means for proving Ramsey theorems for colorings of finite structures inside of infinite structures, such as the rationals as an ordered structure and the Rado graph.

Harrington later produced a proof of the Halpern-Läuchli Theorem using the set-theoretic method of forcing. His approach was novel in that the language and techniques of forcing are used to do unbounded searches for finite objects. This is in contrast to the more common use of forcing to obtain ZFC results by proving the existence of a \(\Sigma^1_2\) definable object in a generic extension, and then applying Shoenfield absoluteness to deduce that the object must be in the ground model. In an interesting turn of events, Harrington’s method of proof provided the backdrop for recent work of the author in \[4\], \[6\], and \[5\]. These results will be discussed in Section \(3\) and ideas of key methods developed to obtain these results will be set forth in Section \(4\).

This article discusses some recent trends and future directions in Ramsey theory on infinite structures. Section \(2\) recalls Ramsey theory on the natural numbers, and then reviews how these ideas extend to structures. Ties between Ramsey theory on structures and topological dynamics, due to Kechris, Pestov, and Todorcevic in \[21\] and a recent development due to Zucker in \[45\] provide additional motivation for these investigations. In Section \(6\) we present an overview of big Ramsey degrees and infinite dimensional Ramsey theory on infinite structures. As the focus here is Ramsey theory on infinite structures, and as the literature on finite structures is vast, we do not even attempt to do justice to Ramsey theory of finite structures in this article. Only a few of those results will be stated in order to provide the reader with some intuition.

The main tools used so far to obtain Ramsey results on infinite structures, aside from Ramsey’s Theorem itself, have been Ramsey theorems on trees in the vein of Halpern and Läuchli, and very recently, category theory (see \[27\]). In Section \(4\) we present an overview of the recent method of trees with coding nodes, first developed by the author in \[4\] to prove finite big Ramsey degrees for the universal ultrahomogeneous triangle-free graph. This method was extended in \[6\] and \[5\] to determine Ramsey theory on all the Henson graphs as well as infinite dimensional Ramsey theory on the Rado graph. Forcing seemed the
best first approach in the search for Ramsey theorems on these trees, and it turned out to work. An overview of some of the ideas involved in these proofs are provided. In Section 5 we point to future directions in Ramsey theory of infinite structures where these methods are likely to prove efficacious.

2. Finite and infinite dimensional Ramsey theory: from natural numbers to structures

Ramsey’s theorem for the natural numbers is the following:

**Theorem 2.1** (Infinite Ramsey Theorem, [37]). Given integers $k, r \geq 1$ and a coloring $c : [\omega]^k \to r$, there is an infinite set $N \subseteq \omega$ such that $c \upharpoonright [N]^k$ is constant.

Using the Hungarian arrow notation, this theorem is written as

(1) \[ \omega \to (\omega)^k_r. \]

The set $N$ is said to be **monochromatic** or **homogeneous**. Using compactness, one obtains the finite version of Ramsey’s Theorem.

**Corollary 2.2** (Finite Ramsey Theorem, [37]). Given integers $k, r, m \geq 1$ with $k \leq m$, there is an integer $n > m$ such that for any coloring $c : [n]^k \to r$, there is a subset $N \subseteq n$ of cardinality $m$ such that $c \upharpoonright [N]^k$ is constant.

Theorem 2.1 and Corollary 2.2 are referred to as **finite dimensional Ramsey theory**, since the sets being colored in these theorems all have the same fixed finite size. **Infinite dimensional Ramsey theory** is concerned with coloring infinite sets of natural numbers. Although the Axiom of Choice implies there is a coloring of all infinite subsets of natural numbers into two colors for which there is no infinite monochromatic set (see Erdős-Rado [10]), for any coloring which induces a sufficiently definable partition of the Baire space, monochromatic subsets are can be found. In the context of Ramsey theory, the simplest representation of the Baire space is $[\omega]^\omega$, the set of all infinite subsets of $\omega$, endowed with the Tychonoff topology. Nash-Williams initiated the study of infinite dimensional Ramsey theory in 1965, proving that for any clopen set $C$ in the Baire space, there is some infinite set $X$ such that $[X]^\omega$ is either contained in $C$ or else is disjoint from it [30].

A few years later, Galvin and Prikry extended this to Borel sets in a strong way. The following notation is central to studies of infinite dimensional Ramsey theory: For $s \in [\omega]^\omega$ and $A \in [\omega]^\omega$, define

(2) \[ [s, A] = \{ B \in [\omega]^\omega : s \supset B \subseteq A \}, \]

where $[s, A]$ is the set of all infinite subsets of $\omega$ that are greater than or equal to $s$ and less than or equal to $A$. This notation is useful for expressing various types of Ramsey-type results, where one seeks to find monochromatic subsets of the Baire space.
where \( s \) is finite and \( s \sqsubset B \) means that \( s \) is an initial segment of \( B \),
given their strictly increasing enumerations. We use the terminology of [42].

**Definition 2.3.** A subset \( \mathcal{X} \subseteq [\omega]^\omega \) is *Ramsey* if for each non-empty set \([s, A]\), there is a \( B \in [s, A] \) such that either \([s, B] \subseteq \mathcal{X}\) or else \([s, B] \cap \mathcal{X} = \emptyset\).

**Theorem 2.4** (Galvin and Prikry, [13]). Every Borel subset of the Baire space is Ramsey.

Soon after this, Silver showed that analytic sets are Ramsey [41]. Mathias [28] and Louveau [26] attained infinite dimensional Ramsey results for the case where the infinite set comes from a Ramsey ultrafilter, a related and rich area which is not the focus of this article. The pinnacle of infinite dimensional Ramsey theory on the Baire space was achieved by Ellentuck in 1974, who provided a topological characterization of those sets which are Ramsey. The Ellentuck topology is the topology on \([\omega]^\omega\) generated by basic open sets of the form \([s, A]\). This topology refines the Tychonoff topology on the Baire space.

**Theorem 2.5** (Ellentuck, [8]). A subset \( \mathcal{X} \subseteq [\omega]^\omega \) is Ramsey if and only if it has the property of Baire with respect to the Ellentuck topology.

Such theorems can be notated by

\[
\omega \rightarrow_* (\omega)_2^\omega,
\]

where \( \rightarrow_* \) denotes that the sets are partitioning \([\omega]^\omega\) are required to be definable in some sense with respect to some topology (see Section 11 of [21]).

Overlapping the development of infinite dimensional Ramsey theory on the Baire space, Ramsey theory on relational structures began to unfold. We review only the basics of Fraïssé theory for relational structures; more general background on Fraïssé theory can be found in Fraïssé’s original paper [12], as well as [21]. We call \( \mathcal{L} = \{ R_i \}_{i \in I} \) a relational signature if is a (countable) collection of relation symbols \( R_i \), where \( n(i) \) denotes the *arity* of \( R_i \), for each \( i \in I \). A structure for \( \mathcal{L} \) is a structure \( A = \langle |A|, \{ R_i^A \}_{i \in I} \rangle \), where \( |A| \neq \emptyset \) is the universe of \( A \) and for each \( i \in I \), \( R_i^A \subseteq A^{n(i)} \). An embedding between structures \( A, B \) for \( \mathcal{L} \) is an injection \( \iota : |A| \rightarrow |B| \) such that for all \( i \in I \), \( R_i^A(a_1, \ldots, a_{n(i)}) \leftrightarrow R_i^B (\iota(a_1), \ldots, \iota(a_{n(i)})) \). If \( \iota \) is the identity map, then we say that \( A \) is a substructure of \( B \). An isomorphism is an embedding which is onto it image. We write \( A \leq B \) exactly when there is an embedding of \( A \) into \( B \); \( A \cong B \) denotes that \( A \) and \( B \) are isomorphic.
A class $\mathcal{K}$ of finite structures for a relational signature $\mathcal{L}$ is called a *Fra"iss"e class* if it is hereditary, satisfies the joint embedding and amalgamation properties, contains (up to isomorphism) only countably many structures, and contains structures of arbitrarily large finite cardinality. These notions are recalled here for the reader’s convenience.

$\mathcal{K}$ is *hereditary* if whenever $B \in \mathcal{K}$ and $A \leq B$, then also $A \in \mathcal{K}$. $\mathcal{K}$ satisfies the *joint embedding property* if for any $A, B \in \mathcal{K}$, there is a $C \in \mathcal{K}$ such that $A \leq C$ and $B \leq C$. $\mathcal{K}$ satisfies the *amalgamation property* if for any embeddings $f : A \to B$ and $g : A \to C$, with $A, B, C \in \mathcal{K}$, there is a $D \in \mathcal{K}$ and there are embeddings $r : B \to D$ and $s : C \to D$ such that $r \circ f = s \circ g$.

Let $\mathcal{K}$ be a Fra"iss"e class. For $A, B \in \mathcal{K}$ with $A \leq B$, we use $(B_A^A)$ to denote the set of all substructures of $B$ which are isomorphic to $A$. Given structures $A \leq B \leq C$ in $\mathcal{K}$, we write

$$C \to (B)_k^A$$

to denote that for each coloring of $(C_A^A)$ into $k$ colors, there is a $B' \in (C_B^B)$ such that $(B'_A^A)$ is *monochromatic*, meaning every member of $(B'_A^A)$ has the same color.

**Definition 2.6.** A Fra"iss"e class $\mathcal{K}$ has the *Ramsey property* if for any two structures $A \leq B$ in $\mathcal{K}$ and any $k \geq 2$, there is a $C \in \mathcal{K}$ with $B \leq C$ such that $C \to (B)_k^A$.

Investigations into which Fra"iss"e classes have the Ramsey property commenced when Graham and Rothschild proved in 1971 that the class of finite Boolean algebras has the Ramsey property [16]. Soon after this, Graham, Leeb, and Rothschild showed that the class of finite vector spaces over a finite field have the Ramsey property [14], [15]. Several years later, the class of finite ordered graphs were found to have the Ramsey property; this was proved independently by Abramson and Harrington [11] and by Nešetřil and Rödl [31], [32]. The main theorem of the papers [31] and [32] furthermore proved that all set-systems of finite ordered relational structures omitting some irreducible substructure have the Ramsey property. This includes the classes of finite ordered graphs omitting $k$-cliques, denoted $\mathcal{G}_k^r$, for each $k \geq 3$. Over the past several decades, more Fra"iss"e classes were shown to have the Ramsey property. The correspondence between the Ramsey property and extreme amenability, proved by Kechris, Pestov, and Todorcevic in 2005 in [21], has propelled a recent burst of discoveries of more Fra"iss"e classes with the Ramsey property.

It is interesting that while most Fra"iss"e classes of finite unordered structures do not have the Ramsey property, often equipping the class
with an additional linear order produces the Ramsey property. In such cases, some remnant of the Ramsey property remains in the unordered reduct. Following notation in [21], given a Fraïssé class $K$, for $A \in K$, $t(A, K)$ denotes the smallest number $t$, if it exists, such that for each $B \in K$ with $A \leq B$ and for each $j \geq 2$, there is some $C \in K$ into which $B$ embeds such that for any coloring $c : (C^A) \to j$, there is a $B' \in (C^B)$ such that the restriction of $c$ to $(B'^A)$ takes no more than $t$ colors. In the arrow notation, this is written as

$$C \to (B)^A_{j,t(A,K)}.$$ 

(4)

A class $K$ has \textit{finite (small) Ramsey degrees} if for each $A \in K$ the number $t(A, K)$ exists. The number $t(A, K)$ is called the \textit{Ramsey degree} of $A$ in $K$ [11]. Notice that $K$ has the Ramsey property if and only if $t(A, K) = 1$ for each $A \in K$. For further information on the connection between Fraïssé classes with finite Ramsey degrees and ordered expansions, the reader is referred to Section 10 of [21] and [34]. In particular, the Fraïssé classes of finite unordered graphs and finite unordered graphs omitting $k$-cliques have finite small Ramsey degrees.

At this point, it is natural to ask the following: Which infinite structures have properties similar Theorem 2.1? As it is often not possible to obtain one color for all copies of a given object, the following definition extends the notion of small Ramsey degrees (rather than Ramsey property) to infinite structures.

\textbf{Definition 2.7} ([21]). Given an infinite structure $S$ and a finite substructure $A \leq S$, let $T(A, S)$ denote the least integer $T \geq 1$, if it exists, such that given any coloring of $(S^A)$ into finitely many colors, there is a substructure $S'$ of $S$, isomorphic to $S$, such that $(S'^A)$ takes no more than $T$ colors. This may be written succinctly as

$$\forall j \geq 1, \ S \to (S)^A_{j,T(A,S)}.$$ 

(5)

We say that $S$ has \textit{finite big Ramsey degrees} if for each finite substructure $A \leq S$, there is an integer $T(A,S)$ such that (5) holds.

The following question has been investigated for several decades.

\textbf{Question 2.8.} Which infinite structures have finite big Ramsey degrees?

This question has been asked most often in regard to Fraïssé limits of Fraïssé classes of finite structures, which we shall call simply \textit{Fraïssé structures}. A Fraïssé structure $\mathbb{F}$ for a Fraïssé class $\mathcal{K}$ is a countably infinite structure which is \textit{universal} for $\mathcal{K}$ (each member of $\mathcal{K}$ embeds into $\mathbb{F}$) and \textit{ultrahomogeneous} (any isomorphism between two finite
substructures of \( F \) extends to an automorphism of \( F \)). However, Question 2.8 also makes sense in the context of infinite structures which are universal for some \( K \) but not ultrahomogeneous, and we will see some recent progress in this direction as well in the next section.

While Question 2.8 has been of interest for many decades, it gained renewed traction in the early 2000’s, both because of results on the big Ramsey degree of the Rado graph in [39] and [24], and because of Kechris, Pestov, and Todorcevic’s question about finding an analogue of their correspondence for Fraïssé structures which have finite big Ramsey degrees (see Section 11 of [21]). Such an analogue was obtained by Zucker in [45]. Section 3 will give an overview of known results on big Ramsey degrees of infinite structures.

Another natural question is the following: Which infinite structures carry analogues of Theorems 2.4 or 2.5?

**Question 2.9.** For which infinite structures \( S \) does

\[
S \to^*_s (S)_2^S
\]

hold, where \( \to^*_s \) denotes that the partition is suitably definable, given some natural topology on the space \((S)_2\)?

This question was brought to light in Problem 11.2 of [21], which asked for an analogue of the KPT correspondence for Fraïssé structures with some infinite dimensional Ramsey theory, while simultaneously pointing out that very little was known in this direction. In the recent paper [5], the author showed that for the collection of subcopies of the Rado graph with a certain tree-structural property, all Borel sets are Ramsey. This will be discussed in Section 4, where trees with coding nodes will be presented.

3. **Big Ramsey degrees on infinite structures: An overview of previous results and methods**

Ramsey theory on infinite structures seeks to find out which infinite structures carry analogues of Ramsey’s Theorem. The first line of inquiry investigates which infinite structures are *indivisible*, meaning that given any partition of the universe of the structure into finitely many pieces, one of the pieces contains a copy of the structure. This is the same as saying (in the terminology of Definition 2.7) that substructures of size one have finite big Ramsey degree one.

It is straightforward to see that given a partition of the rationals into finitely many pieces, one of the pieces contains a copy of the rationals, that is, a dense linear order (without endpoints). Thus, the rationals as a linearly ordered structure is indivisible. The Rado graph, or random
graph, is also indivisible. This was proved by Henson in [19], using the extension property of the Rado graph.

In contrast, proving the indivisibility of the random $k$-clique-free graphs, denoted $\mathcal{H}_k$, required ideas beyond their extension properties. These Henson graphs were first constructed by Henson in 1971 in [19] as subgraphs of the Rado graph. In hindsight, these are seen to be Fraïssé limits of the Fraïssé classes of finite $k$-clique-free graphs. In [19], Henson proved that $\mathcal{H}_k$ is weakly indivisible: Given a partition of the vertices of $\mathcal{H}_k$ into two pieces, either the first piece of the partition contains a copy of $\mathcal{H}_k$, or else the second piece contains a copy of each finite $k$-clique-free graph. It is interesting to note that this was proved several years before Nešetřil and Rödl’s result in [31] that the Fraïssé classes of ordered $k$-clique-free graphs have the Ramsey property. The first result on the indivisibility of Henson graphs was due to Komjáth and Rödl, in [22], where they proved that $\mathcal{H}_3$ is indivisible. Soon after, El-Zahar and Sauer extended this result to all Henson graphs in [7].

A sample of other notable results regarding indivisibility are the following: Hindman showed, as a consequence of his partition theorem for finite sums of natural numbers, that the infinite dimensional vector space over the finite field $\mathbb{F}_2$ is indivisible [20]. Nguyen Van Thé and Sauer proved that for each $m \geq 1$, the countable Urysohn metric space with distances in $\{1, \ldots, m\}$ is indivisible [35]. Later, Sauer showed that for any finite set of distances, the Urysohn space with that distance set is indivisible [40]. For more on indivisible structures, the reader is referred to the excellent Habilitation of Nguyen Van Thé [34].

While the study of indivisibility of infinite structures continues to be a rich and challenging subject, it is interesting that the nascent subject of big Ramsey degrees can actually be traced back to an example of Sierpiński. Considering the rationals as a linearly ordered structure, Sierpiński showed that there is a coloring of pairs of rationals into two colors such that any infinite subset which is again a dense linear order preserves both colors on its pairsets. His coloring plays the linear ordering of the rationals against a well-ordering of the rationals as follows, and colors pairs of rationals red if the two orders agree on the pair, and blue otherwise. Both colors persist in every subset of the rationals forming a dense linear order.

This phenomenon is perhaps best seen via trees. For $s, t \in 2^{<\omega}$, define $s \triangleleft t$ iff one of the following holds: (a) $s <_{\text{lex}} t$, (b) $s \sqsubset t$ and $t(|s|) = 1$, or (c) $t \sqsupset s$ and $s(|t|) = 0$. Notice that $(2^{<\omega}, \triangleleft)$ is isomorphic
to \((\mathbb{Q}, <)\). Define the coloring

\[
    c(\{s, t\}) = \begin{cases} 
        0 & \text{if } |s| \leq |t| \text{ and } s < t \\
        1 & \text{otherwise}
    \end{cases}
\]

Then given any subset \(S \subseteq 2^{<\omega}\) for which \((S, \triangleleft) \cong (\mathbb{Q}, <)\), both colors will persist in \(S\); hence \(T(2, \mathbb{Q}) \geq 2\).

In his PhD thesis [3], Devlin found the precise big Ramsey degrees for finite sets of rationals, building on prior unpublished work of Galvin, who proved that \(T(2, \mathbb{Q}) = 2\), and of Laver, who proved existence of \(T(k, \mathbb{Q})\) for all natural numbers \(k\). Here, we will not provide much background or history, as there are already thorough expositions of Devlin’s results in [43] and [42].

A key component in finding upper bounds for the numbers \(T(k, \mathbb{Q})\) is a Ramsey theorem for trees, due to Milliken. This actually turns out to be at the heart of many big Ramsey degree results. Milliken’s theorem utilizes the following notion of strong tree and a theorem of Halpern and Läuchli which we now briefly review. A historical record of the Halpern-Läuchli Theorem and its variants can be found in [25]. For \(t \in \omega^{<\omega}\), \(|t| = \text{dom}(t)\) is the length of \(t\). In the subject of Ramsey theory on trees, we say that a subset \(T \subseteq \omega^{<\omega}\) is a tree if there is a subset \(L \subseteq \omega\) such that \(T = \{t \upharpoonright l : t \in T, l \in L\}\). Thus, in this definition, a tree is closed under initial segments at levels of the tree, but it is not necessarily closed under all initial segments in \(\omega^{<\omega}\). For \(t \in T\), the height of \(t\), \(h_T(t)\), is the order-type of the set \(\{u \in T : u \subseteq t\}\), linearly ordered by \(\subseteq\). We write \(T(n)\) to denote \(\{t \in T : h_T(t) = n\}\). For \(t \in T\), let \(\text{Succ}_T(t) = \{u \upharpoonright (|t| + 1) : u \in T \text{ and } u \supseteq t\}\), noting that \(\text{Succ}_T(t)\) will not be a set of nodes in \(T\) if \(T\) does not contain any nodes of length \(|t| + 1\).

**Definition 3.1.** Let \(T \subseteq \omega^{<\omega}\) be a finitely branching tree. A subset \(S \subseteq T\) is a **strong subtree** of \(T\) if and only if there is an increasing sequence of natural numbers \(\langle m_n : n < N\rangle\), where \(N \leq \omega\), such that \(S = \bigcup_{n<N} S(n)\), and for each \(n < N\),

1. \(S(n) \subseteq T(m_n)\), and
2. \(s \in S(n)\) and \(u \in \text{Succ}_T(s)\), there is exactly one node in \(S(n+1)\) extending \(u\).

Given \(k \geq 1\), we say that \(S\) is a **\(k\)-strong subtree** of \(T\) if \(N = k < \omega\).

For the next theorem, define the notation:

\[
    \bigotimes_{i<d} T_i := \bigcup_{n<\omega} \prod_{i<d} T_i(n). 
\]
Figure 1. The two similarity types of pairs of rationals

Figure 2. Strong tree envelopes for these pairs of rationals

The following is the strong tree version of the Halpern-Läuchli Theorem.

**Theorem 3.2** (Halpern-Läuchli, [17]). Let $T_i \subseteq \omega^{<\omega}$, $i < d$, be finitely branching trees with no terminal nodes and let $r \geq 2$. Given a coloring $c : \bigotimes_{i<d} T_i \rightarrow r$, there is an increasing sequence $\langle m_n : n < \omega \rangle$ and strong subtrees $S_i \leq T_i$ such that for all $i < d$ and $n < \omega$, $S_i(n) \subseteq T_i(m_n)$, and $c$ is constant on $\bigotimes_{i<d} S_i$.

The following theorem is proved by inductively applying Theorem 3.2 $\omega$-many times.

**Theorem 3.3** (Milliken, [29]). Let $T \subseteq \omega^{<\omega}$ be a strong tree with no maximal nodes. Let $k \geq 1$, $r \geq 2$, and $c$ be a coloring of all $k$-strong subtrees of $T$ into $r$ colors. Then there is a strong subtree $S \subseteq T$ such that all $k$-strong subtrees of $S$ have the same color.

Trees can be used in various ways to code structures. As mentioned above, the nodes in the tree $2^{<\omega}$ ordered by $\triangleleft$ produces a copy of the rationals. In this setting, Sierpiński’s coloring can be seen structurally as giving color red to pairs of nodes $s, t \in 2^{<\omega}$ with $|s| \leq |t|$ and $s \triangleleft t$, and color blue otherwise. Figure 1 gives pairs of nodes which are colored red (left) and blue (right). These two configurations are examples of **strong similarity types**, and both of them persist in any subcopy of the rationals. Figure 2 gives examples of strong trees which **envelope** such pairs of nodes.

The proof that $T(2, \mathbb{Q}) \leq 2$ goes roughly as follows: Consider pairs of nodes $s, t$ of different lengths such that $s(|t|) = 0$ if $|s| > |t|$, and $t(|s|) = 0$ if $|t| > |s|$. The two **strong similarity types** for such pairs are seen in Figure 1. Given a coloring of pairs of nodes in $2^{<\omega}$ into finitely many colors, fix the strong similarity type on the left in Figure 1. For
each such pair \( \{s, t\} \) in \( 2^{<\omega} \) with this strong similarity type, there are finitely many 3-strong trees enveloping \( s \) and \( t \). Give these 3-strong trees the coloring of the pair which they envelope. Apply Milliken’s theorem to this coloring of 3-strong trees to obtain an infinite strong subtree \( T \) where all pairs with this strong similarity type have the same color. Then do the same for the second strong similarity type, obtaining an infinite strong subtree \( T' \) of \( T \). Since the rationals are coded in any infinite strong subtree of \( 2^{<\omega} \), one can now pull out an antichain of nodes, each of which has passing number 0 at the levels of the other nodes, so that under the ordering \( < \), the nodes in this antichain represent \( \mathbb{Q} \). This produces a copy of the rationals in which all pairsets have at most two colors.

A similar but more involved strategy is behind the finite big Ramsey degrees of the Rado graph. For graphs, by interpreting the lexicographic order on ordered graphs due to Erdős, Hajnal, and Pósa in [9] in terms of trees, Sauer showed in [39] that nodes in the binary tree can code graphs. Let \( A \) be a graph. Enumerate the vertices of \( A \) as \( \langle v_n : n < N \rangle \). A set of nodes \( \{t_n : n < N\} \) in \( 2^{<\omega} \) codes \( A \) if and only if for each pair \( m < n < N \),

\[
t_n(\vert t_m \vert) = 1.
\]

The number \( t_n(\vert t_m \vert) \) is called the passing number of \( t_n \) at \( t_m \). See Figure 3, where the nodes \( t_0, t_1, t_2 \) in the tree code the graph \( v_0, v_1, v_2 \), which is a path of length two.

The Rado graph, \( \mathcal{R} = (R, E) \), is the Fraïssé limit of the class of all finite graphs; as such, it is ultrahomogeneous and universal for all finite graphs. Erdős, Hajnal, and Pósa launched the investigation of finite big Ramsey degrees of the Rado graph in 1975, when they showed that there is a 2-coloring of edges such that every subcopy of the Rado graph retains both colors [9], reminiscent of Sierpiński’s result for pairs of rationals. Two decades later, Pouzet and Sauer proved that for each coloring of edges of \( \mathcal{R} \) into finitely many colors, there is a subgraph \( \mathcal{R}' \) isomorphic to \( \mathcal{R} \) in which the edges have at most two colors [36]. In 2006, papers of Sauer [39] and of Laflamme, Sauer, and Vuksanovic [24] combined to the exact big Ramsey degrees for all finite graphs within the Rado graph. In fact, these papers find big Ramsey degrees for a collection of binary relational structures, including the random tournament. The strategy is similar to that outlined above for the rationals, but now the passing numbers at nodes code the edge/non-edge relation.

In [21], Kechris, Pestov, and Todorcevic asked, Which structures have finite big Ramsey degrees? In tandem with the results in [39]
and [24], this sparked a new wave of interest in big Ramsey degrees of Fraïssé structures. As part of his PhD work, Nguyen Van Thé proved that the countable ultrametric Urysohn space with any finite distance set has finite big Ramsey degrees [33]. While this result used Ramsey’s theorem, the next result required a new extended version of Milliken’s Theorem. Laflamme, Nguyen Van Thé, and Sauer proved in [23] that the structures $\mathbb{Q}_n$ have finite big Ramsey degrees, where $\mathbb{Q}_n$ is the rationals as a linear order with an equivalence relation with $n$ equivalence classes, each of which is dense in $\mathbb{Q}$. See [34] for an excellent exposition of these and related results.

As for the Henson graphs, Sauer proved in 1998 that the big Ramsey degree for edges in the triangle-free Henson graph is two [38]. Further results were slow in coming, mainly because of lack of analogues for $k$-clique free graphs of the following two fortunate facts related to the Rado graph. First, the graph $B$ induced by the countable binary tree $2^{<\omega}$ is universal for countable graphs. Precisely, let the vertices of $B$ be the nodes in the tree $2^{<\omega}$. Define two nodes to have an edge between them in $B$ precisely when one of the nodes is longer than the other and its passing number at the level of the shorter node is one. Second, Milliken’s theorem for strong subtrees of $2^{<\omega}$ can be applied to subcopies of $B$, and each infinite strong subtree again codes a graph which is universal for countable graphs. After finitely many applications of Milliken’s theorem to strong tree envelopes of finite antichains coding copies of a given finite graph, one can take a copy of the Rado graph where the big Ramsey degree of the finite graph under investigation is finite.

As analogues of these two facts were unknown for Henson graphs, the two main themes in the work of [4] and [6] were, first, to find means for coding the Henson graphs into trees in a way that the trees behaved like strong trees, and second, to prove Ramsey theory for such trees. In the next section, we discuss the methods developed in these two papers to
handle forbidden \( k \)-cliques. These methods seem to be robust enough to handle many types of structures, including, perhaps surprisingly, infinite dimensional Ramsey theory of the Rado graph (see [5]), which will also be discussed in the next section.

We close this section by pointing out the recent work of Mašulović in [27] which uses category theory to develop transfer principals for big Ramsey degrees. In that paper, Mašulović widened the investigation of big Ramsey degrees to universal structures, regardless of ultrahomogeneity. He has transferred several of the aforementioned results to prove finite big Ramsey degrees for some new structures; most strikingly, these include Fraïssé limits of classes of finite metric spaces with finite distance sets satisfying certain properties.

4. Trees with coding nodes, their Ramsey theory, and applications to Ramsey theory of infinite graphs

In order to investigate big Ramsey degrees of the Henson graphs, the first task was to find some way of representing \( k \)-clique-free graphs via trees. Since \( k \)-cliques are forbidden, there needs to be some method for determining which nodes should be allowed to split and which ones should not, so that \( H_k \) is represented while the splitting is 'maximal' in some sense. In particular, the trees need to be perfect in order for Ramsey theory to have any chance of development.

To achieve this, in the construction of such trees, certain nodes are distinguished to code certain vertices of \( H_k \). This way, one keeps track of the finite graph that is already coded, ensuring that one knows how to branch maximally, subject to never coding a \( k \)-clique in the future. This led to the following notions of trees with coding nodes and strong coding trees.

4.1. Strong coding trees. The following definitions and theorems are taken from [4] and [6]. A tree with coding nodes is a structure \( \langle T, N; \subseteq, <, c^T \rangle \) in the language \( L = \{\subseteq, <, c\} \), where \( \subseteq \) and \( < \) are binary relation symbols and \( c \) is a unary function symbol satisfying the following: \( T \subseteq 2^{\omega} \) and \( (T, \subseteq) \) is a tree. \( N \leq \omega \) and \( < \) is the standard linear order on \( N \). \( c^T : N \to T \) is injective, and \( m < n < N \rightarrow |c^T(m)| < |c^T(n)| \). \( c^T(n) \) is the \( n \)-th coding node in \( T \), and is usually denoted \( c_n^T \).

Notice that a collection of coding nodes \( \{c_{n_i}^T : i < k\} \) in \( T \) codes a \( k \)-clique if and only if \( c_{n_j}^T(|c_{n_i}^T|) = 1 \) for all \( i < j < k \). To ensure that a tree never codes a \( k \)-clique, the following branching criterion is introduced. We say that a tree \( T \) with coding nodes \( \langle c_n^T : n < N \rangle \) satisfies the \( K_k \)-Free Branching Criterion (\( k \)-FBC) if for each non-maximal node
Figure 4. A strong triangle-free tree $S_3$ densely coding $H_3$

Figure 5. A strong $K_4$-free tree $S_4$ densely coding $H_4$

t ∈ $T$, $t^0$ is always in $T$, and $t^1$ is in $T$ if and only if any coding
node extending $t^1$ cannot code a $k$-clique with coding nodes in $T$ of
length less than or equal to the length of $t$. It is a useful fact that
given any tree $T$ with coding nodes and no maximal nodes satisfying
the $k$-FBC, and in which the set of coding nodes are dense, the coding
nodes in $T$ code $H_k$ (Theorem 4.9, [6]).

The trees $S_3$ and $S_4$ in Figures 4. and 5. have coding nodes which
code Henson graphs $H_3$ and $H_4$, respectively. These trees feature
the main structural ideas behind strong coding trees; one can extrapolate
to envisage $S_k$, for any $k \geq 5$ (precise constructions are given in [6]).
The gray nodes $c_{-(k-2)}, \ldots, c_{-1}$ are pseudo-coding nodes, which code a
$(k-1)$-clique. They help to set up the tree structure so that subtrees of
$S_k$ coding $H_k$ can be isomorphic to $S_k$. The vertex $v_0$ is to be thought
of as forming a $(k-1)$-clique with some vertices in a larger ambient
copy of $H_k$. This has the effect that each coding node in $S_k$ does not split.
Perhaps the most illustrative way of thinking of these trees is the following: The nodes in the tree the level of $c_n$ in the tree are coding all finite partial types over the graph on vertices $\{v_{-(k-2)}, \ldots, v_{n-1}\}$. In this way, a strong coding tree is really just a means for visualizing the finite partial types over an (ordered) initial segment of the graph $H_k$.

The one small but insurmountable catch to these trees is that having coding nodes and splitting nodes at the same levels prevents the development of Ramsey theory on subtrees isomorphic to $S_k$. Ironically, the failure occurs at the most basic level: There are bad colorings of coding nodes for which no subtree isomorphic to $S_k$ has one color (see Example 3.18 in [4]). There turns out to be a simple solution: Skew the trees so that each level of the tree has at most one coding node or splitting node, but never both. Let $T_k$ denote the skewed version of $S_k$ (see Figure 6. for $T_3$).

We work with a collection $T_k$ of infinite subtrees of $T_k$, each of which is isomorphic to $T_k$ in a strong sense: Let $k \geq 3$ be given and let $S,T \subseteq T_k$ be meet-closed subsets. A bijection $f : S \rightarrow T$ is a strong similarity map if for all nodes $s, t, u, v \in S$, the following hold:

1. $f$ preserves lexicographic order.
2. $f$ preserves meets, and hence splitting nodes.
3. $f$ preserves relative lengths.
4. $f$ preserves initial segments.
5. $f$ preserves coding nodes.
(6) $f$ preserves passing numbers at coding nodes.

Given a subtree $T \subseteq \mathbb{T}_k$, let $G_T$ denote the graph represented by the coding nodes in $T$. Notice that if $T \subseteq \mathbb{T}_k$ is strongly similar to $\mathbb{T}_k$, then $G_T$ is isomorphic to $\mathcal{H}_k$ as ordered graphs.

Essentially, a strong coding tree is a subtree of $\mathbb{T}_k$ which is strongly similar to $\mathbb{T}_k$. We say essentially, because there is one more important consideration when working with forbidden $k$-cliques. Any finite subtree of $\mathbb{T}_k$ which we build needs to be extendable within $\mathbb{T}_k$ to a subtree which is strongly similar to $\mathbb{T}_k$. There are many finite subtrees of $\mathbb{T}_k$ for which this is not possible, because each remembers where it came from, coding edges with the original graph represented by the coding nodes in $\mathbb{T}_k$.

Take for example $k = 3$. If $A$ is a finite subtree of $\mathbb{T}_3$ and two nodes $s, t \in A$ have passing number 1 at some coding node $c_i$ in $\mathbb{T}_3$, then whenever $s$ is extended to some coding node $c_m$ in $\mathbb{T}_3$, then any extension of $t$ to a coding node $c_n$ in $\mathbb{T}_3$ with length greater than $|c_m|$ cannot have passing number 1 at $c_m$, as that would code a triangle; no coding of a triangle occurs in $\mathbb{T}_3$. In such a situation, there is no way to extend $A$ to a subtree of $\mathbb{T}_3$ coding all of $\mathcal{H}_3$.

To prevent this, we make a further requirement which is, roughly, as follows: Fix $a \in [3, k]$. We say that a level set $X \subseteq \mathbb{T}_k$ with nodes of length $\ell_X$ has a pre-$a$-clique if for some $I \subseteq [\omega]^{a-2}$, letting $i_* = \max(I)$ and $\ell_* = |c_{i_*}|$, we have that $\ell_* \leq \ell_X$, the set $\{c_i : i \in I\}$ codes an $(a-2)$-clique, and each node in $X^+$ has passing number 1 at $c_i$, for each $i \in I$. The idea is that pre-$a$-cliques code entanglements. Essentially, we say that a subtree $T$ of $\mathbb{T}_k$ has the Witnessing Property if for each pre-$a$-clique in $T$, $a \in [3, k]$, there is a set of $(a-2)$ many coding nodes $\{c_i : i \in I\}$ as above, all of which are coding nodes in $T$. A tree $T \subseteq \mathbb{T}_k$ is a member of $\mathcal{T}_k$ iff $T$ is strongly similar to $\mathbb{T}_k$ and has the Witnessing Property (Lemma 5.15, [6]). Some examples of unwitnessed and witnessed pre-cliques are in Figures 7 – 10.

Essentially, we define the space of strong coding trees, $\mathcal{T}_k$, to consist of those subtrees of $\mathbb{T}_k$ which are strongly similar to $\mathbb{T}_k$ and have the Witnessing Property. For the details, see [4] and [6]. This set of subtrees of $\mathcal{T}_k$ is the analogue of strong trees appropriate to the Henson graph $\mathcal{H}_k$. 

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4.2. A Ramsey theorem for strictly similar antichains. The upper bounds for big Ramsey degrees of the Henson graphs basically come from the fact that there are only finitely many ways to code a given finite graph within a strong coding tree. Analogously to the case for the Rado graph, given any strong coding tree $T \in \mathcal{T}_k$, there is an antichain of coding nodes in $T$ which code the Henson graph $\mathcal{H}_k$. Thus, one can restrict attention to colorings of antichains representing a given finite $k$-clique-free graph, say $G$. The relevant structural properties are those of strong similarity and new pre-cliques and their placement.
within the antichain. This is the idea behind strict similarity. For the precise definition, the reader is referred to [4] and [6].

**Theorem 4.1** ([4],[6]). Let $Z$ be a finite antichain of coding nodes in a strong $H_k$-coding tree $T \in T_k$, and suppose $h$ colors of all antichains $T$ which are strictly similar to $Z$ into finitely many colors. Then there is an strong $H_k$-coding tree $S \leq T$ such that all subsets of $S$ strictly similar to $Z$ have the same $h$ color.

The proof uses forcing to obtain a ZFC result, but not in the usual manner using absoluteness. Recall the Halpern-Läuchli Theorem 3.2 Harrington gave an insightful proof which uses Cohen forcing in the following way. Suppose we have $d$ many infinite strong trees $T_i$ and a coloring of the product of their level sets as in the statement of Theorem 3.2. Let $\kappa$ be large enough that $\kappa \rightarrow (\aleph_1)^2\aleph_0$ and take $\mathbb{P}$ to be $\kappa$-Cohen forcing which adds $\kappa$ new branches to each of the $d$ many trees. Harrington gave an argument guaranteeing that there are nodes $t^*_i \in T_i$, for each $i < d$, of the same length which have some color $\varepsilon \in r$; moreover, given any level sets extending these nodes, there are further extensions to level sets so that each member in their product has the same color $\varepsilon$. The forcing is used to find these finite sets successively in $\omega$ many steps; the generic filter is never actually used - one never actually passes to a generic extension. Instead, the forcing language and basic facts about forcing guarantee that certain finite level sets exist, and their finiteness guarantees that they are actually in the ground model. So, this is very much not a constructive proof, but at the same time, it is a ZFC proof where one constructs each level of the subtree separately.

The author extended this idea to the context strong coding trees. However, since strong coding trees have two types of nodes, coding and non-coding, new forcings which are not equivalent to $\kappa$-Cohen forcing had to be introduced. The general approach uses Harrington’s ideas as a starting point, but the implementation is much more involved. Another new element in this setting is that envelopes will be antichains of coding nodes which have a strong version of the Witnessing Property. This is quite different from envelopes for the rationals or the Rado graph being finite strong trees.

Theorem 4.1 is applied as follows to prove the finite big Ramsey degrees. Given a finite $k$-clique-free graph $G$, there are only finitely many strict similarity types of antichains of coding nodes representing $G$. The number of such strict similarity types is the upper bound for the big Ramsey degree of $G$ in $H_k$. 
Theorem 4.2 ([4],[6]). Suppose $k \geq 3$ and $G$ is a finite $k$-clique-free graph. Let $h$ color all copies of $G$ in $H_k$ into finitely many colors. Then there is a subgraph of $H_k$ which is isomorphic to $H_k$ in which the copies of $G$ take on no more colors than the number of strict similarity types of antichains in $T_k$ coding $G$.

4.3. Infinite dimensional Ramsey theory of the Rado graph. We now mention a recent result of the author on infinite dimensional Ramsey theory of the Rado graph. In Problem 11.2 of [21], Kechris, Pestov, and Todorcevic ask for the topological dynamics analogue of a corresponding infinite Ramsey-theoretic result for several Fraïssé structures, in particular, the rationals, the Rado graph, and the Henson graphs. By an infinite Ramsey-theoretic result, they mean a result of the form

$$F \rightarrow_\ast (\mathbb{F})^F_{l,t},$$

where equation (10) reads: “For each partition of $(\mathbb{F})$ into $l$ many definable subsets, there is an $F \in (\mathbb{F})$ such that $(\mathbb{F})$ is contained in no more than $t$ of the pieces of the partition.” Here, one assumes a natural topology on $(\mathbb{F})$ and definable refers to any reasonable class of sets definable relative to the topology, for instance, open, Borel, analytic, or property of Baire. A sub-question implicit in Problem 11.2 in [21] is the following broader version of Question 2.9:

Question 4.3. For which ultrahomogeneous structures $\mathbb{F}$ does it hold that

$$\mathbb{F} \rightarrow_\ast (\mathbb{F})^\mathbb{F}_{l,t},$$

for some positive integer $t$?

The natural topology to give such a space is the one induced by ordering the universe $F$ of $\mathbb{F}$ in order-type $\omega$, and viewing $(\mathbb{F})$ as a subspace of the product space $2^F$ with the Tychonoff topology. Kechris, Pestov, and Todorcevic pointed out that very little is known about Question 4.3.

In [5], the author set out to answer this question for the Rado graph. Since the big Ramsey degrees for copies of a finite graph inside the Rado graph grow without bound as the number of vertices in the finite graph whose copies are being colored grows implies that any positive answer to Question 4.3 for the Rado graph must restrict to a collection of Rado graphs all of whose vertices are ordered in the same order. Furthermore, it is necessary that all copies of the Rado graph being colored have the same strong similarity type. Otherwise, one may use
strong similarity types to make a coloring of the copies of the Rado graph to show that there is no bound \( t \) of the sort in (10), where \( \mathbb{F} \) is the Rado graph.

In [5], the author answered Question 4.3 for a collection of Rado graphs, each of which has the same strong similarity type. While the Rado graph can be represented by nodes in strong trees of the kind in the Milliken Theorem 3.3 that theorem by itself does not answer this question, as it is unclear in a tree without coding nodes how the strong subtrees should be thought of as coding subcopies of a given Rado graph. In order to make sure that the representations of the subgraphs were concrete, it turned out to be useful to work with trees with coding nodes, even though the Rado graph has no forbidden subgraphs.

Let \( \mathbb{R} = (R, E) \) denote the Rado graph with vertices ordered as \( \langle v_n : n < \omega \rangle \) represented by the coding nodes \( \langle c_n : n < \omega \rangle \) in the tree \( T_R \) in Figure 11. Let \( T_R \) denote the collection of all subtrees \( T \subseteq T_R \) strongly similar to \( T_R \). Given \( T \in T_R \), let \( G_T \) denote the ordered graph represented by the coding nodes of \( T \), noting that each \( G_T \) is a subcopy of the Rado graph, ordered in the same way as \( \mathbb{R} \). Let \( \mathcal{R} \) be the collection of all \( G_T \), where \( T \in T_R \). The topology on \( \mathcal{R} \) is the topology inherited from the Tychonoff topology on \( 2^\mathbb{R} \).

**Theorem 4.4 ([5]).** If \( \mathcal{X} \subseteq \mathcal{R} \) is Borel, then for each graph \( \mathbb{G} \in \mathcal{R} \), either all members of \( \mathcal{R} \) contained in \( \mathbb{G} \) are members of \( \mathcal{X} \), or else no member of \( \mathcal{R} \) contained in \( \mathbb{G} \) is a member of \( \mathcal{X} \).

A few remarks about this theorem are in order. First, we point out that Theorem 4.4 is the analogue of the Galvin-Prikry Theorem 2.4 for colorings of those copies of the Rado graph which have induced trees.
strongly similar to $\mathbb{T}_R$. Second, the methods of proof use forcing in a similar yet simpler way than that in [4]. However, these methods were not conducive to obtaining an analogue of the Ellentuck Theorem 2.5. So it remains open whether or not, with the natural Ellentuck-like topology on $\mathcal{R}$, the subsets of $\mathcal{R}$ with the property of Baire have the Ramsey property.

Third, the strong coding tree $\mathbb{T}_R$ is the simplest kind of tree coding $\mathbb{R}$. One could, however, fix any perfect tree (skew or not), say $\mathbb{T}'$, with coding nodes which are dense and code the Rado graph, and the same methods would produce the same result for those subcopies of the Rado graph induced by the coding nodes of subtrees of $\mathbb{T}'$ which are strongly similar to $\mathbb{T}'$. Lastly, the methods should produce a similar theorem for any of the binary relational structures in [39] and [24].

5. Future directions

As seen in the previous section, trees with coding nodes have provided means for solving problems on big Ramsey degrees for Henson graphs as well as infinite dimensional Ramsey theory for the Rado graph, which has no forbidden subgraphs. We envision several directions in which this idea can be developed to solve questions on Ramsey theory of infinite structures.

It is likely that trees with coding nodes will be useful in proving finite big Ramsey degrees for all Fraïssé limits of Fraïssé classes with finitely many binary relations satisfying the Ramsey property. Indeed, it may be enough to require finite small Ramsey degrees. At the conference, Unifying Themes in Ramsey Theory in Banff, 2018, Sauer suggested trying to move the forcing proofs from strong coding trees to a direct approach by forcing directly on the structures. Recall that the nodes in a strong coding tree represent a finite partial type over a finite substructure; that is, the trees are really just a means for visualizing or making a structure out of the types. One important task is to interpret the forcings in the work related in Section 4 back into the graph setting in a way that points to the natural analogues of strong coding trees for structures with relations of any finite arity. Such an approach can hopefully lead to proving the following conjecture.

**Conjecture 5.1.** Let $\mathcal{K}$ be a Fraïssé class of finite structures with finitely many finitary relations and satisfying the Ramsey property (or just having finite small Ramsey degrees). Then the Fraïssé limit of $\mathcal{K}$ has finite big Ramsey degrees.

The methods for the infinite dimensional Ramsey theory of the Rado graph are a simplified version of those developed for the big Ramsey
degrees of the Henson graphs. Indeed, it seems to be just a matter of double checking to see that the work in [6] will induce infinite dimensional Ramsey theory for Borel sets in any class of $k$-clique-free Henson graphs represented by strong coding trees with a a fixed strong similarity type. It will be interesting to see if some other methods can produce analogues of the Ellentuck Theorem for these graphs, or whether there is some essential property of these graphs which prevent this. It seems likely that whatever structures have finite big Ramsey degrees will also have infinite dimensional Ramsey theorems.

**Conjecture 5.2.** Let $\mathcal{K}$ be a Fraïssé class of finite structures with finitely many finitary relations and satisfying the Ramsey property (or just having finite small Ramsey degrees). Let $\mathbb{K}$ be the Fraïssé limit of $\mathcal{K}$ with some linear order of its universe in order-type $\omega$. Then there is some notion of strong similarity type so that the collection of all subcopies of $\mathbb{K}$ having the same strong similarity type has the property that all Borel subsets are Ramsey.

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