Majority-vote model on \((3,12^2), (4,6,12)\) and \((4,8^2)\) Archimedean lattices

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On \((3,12^2), (4,6,12)\) and \((4,8^2)\) Archimedean lattices, the critical properties of majority-vote model are considered and studied using the Glauber transition rate proposed by Kwak et al. [Phys. Rev. E. 75, 061110 (2007)] rather than the traditional majority-vote with noise [José Mário de Oliveira, J. Stat. Phys. 66, 273 (1992)]. The critical temperature and the critical exponents for this transition rate are obtained from extensive Monte Carlo simulations and with a finite size scaling analysis. The calculated values of the critical temperatures Binder cumulant are \(T_c = 0.363(2)\) and \(U_4^c = 0.577(4)\); \(T_c = 0.651(3)\) and \(U_4^c = 0.612(5)\); and \(T_c = 0.667(2)\) and \(U_4^c = 0.613(5)\) for \((3,12^2)\), \((4,6,12)\) and \((4,8^2)\) lattices, respectively. The critical exponents \(\beta/\nu, \gamma/\nu\) and \(1/\nu\) for this model are \(0.237(6), 0.73(10),\) and \(0.83(5); 0.105(8), 1.28(11),\) and \(1.16(5); 0.113(2), 1.60(4),\) and \(0.84(6)\) for \((3,12^2)\), \((4,6,12)\) and \((4,8^2)\) lattices, respectively. These results differ from the usual Ising model results and the majority-vote model on so-far studied regular lattices or complex networks.

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I. INTRODUCTION

The use of local majority rules to study voting systems was introduced by Galam three decades ago to study bottom-up democratic voting in hierarchical structures \cite{1}. It is one of the founding papers of sociophysics with a follow-up paper published a few years latter in the Journal of Statistical Physics \cite{2}, which 33 years later has devoted a special issue to the modelling of social systems \cite{3} including a paper by Galam extending his earlier work from two to three parties \cite{4}. Indeed, while sociophysics has been rejected by physicists in the eighties \cite{2}, it has become today an active field of research among physicists all over the world \cite{4,5,6}.

The local majority rule model has motivated a good deal of works under several names including the Majority Model and the Majority Vote Model (MVM). The nonequilibrium majority-vote model proposed by Oliveira \cite{6} defined on two-dimensional regular lattices shows second-order phase transition with critical exponents \(\beta, \gamma, \nu\) identical \cite{6,10} with those of equilibrium Ising model \cite{11,12} that agree with hypothesis of Grinstein et al. \cite{13}.

The MVM on the complex networks exhibit different behavior \cite{14,21}. Campos et al. investigated MVM on undirected small-world network \cite{14}. They found that the critical exponents \(\gamma/\nu\) and \(\beta/\nu\) differ from those of the Ising model \cite{12} and depend on the rewiring probability. Luz and Lima studied MVM on directed small-world network \cite{15} constructed using the same process described by Sánchez et al. \cite{22}. They found that the critical exponents \(\gamma/\nu\) and \(\beta/\nu\) are also different from those of the Ising model on square lattices and in this case MVM the exponents do not depend on the rewiring probability, that is contrary to results of Campos et al. \cite{14}. Pereira et al. \cite{16} and Lima et al. \cite{17} studied MVM on undirected Erdős–Rényi’s (ERU) on directed Erdős–Rényi’s (ERD) classical random graphs \cite{22} and their results obtained for critical exponents agree with the results of Pereira et al. \cite{16}, within the error bars. After Lima et al. \cite{18} also studied the MVM on random Voronoy–Delaunay lattice \cite{24} with periodic boundary conditions. Lima also \cite{19,21} studied the MVM on directed Albert–Barabási (ABD) and undirected Albert–Barabási (ABU) network \cite{23} and contrary to the Ising model on these networks \cite{26}, the order/disorder phase transition was observed in this system. However, the calculated \(\beta/\nu\) and \(\gamma/\nu\) exponents for MVM on ABD and ABU networks are different from those for the Ising model \cite{12} and depend on the mean value of connectivity \(\bar{z}\) of ABD and ABU network. Lima and Malarz \cite{27} studied the MVM on \((3, 4, 6, 4)\) and \((3^4, 6)\) Archimedean lattices (AL). They remark that the critical exponents \(\gamma/\nu\), \(\beta/\nu\) and \(1/\nu\) for MVM on \((3, 4, 6, 4)\) AL are different from the Ising model \cite{12} and differ from those for so-far studied regular two-dimensional lattices \cite{8,9}, but for \((3^4, 6)\) AL, the critical exponents are much closer to those known analytically for SL Ising model. Santos et al. \cite{28} studied the MVM on triangular \((3^6)\), honeycomb \((6^3)\) and Kagomé \((3, 6, 3, 6)\) AL. They found for \((3^6), (6^3)\) and \((3, 6, 3, 6)\) AL some critical exponents are much closer to those known analytically for square lattice Ising model, i.e. \(\beta = 1/8 = 0.125, \gamma = 7/4 = 1.75\) and \(\nu = 1\), but except for \(\nu\) they differ for more than three numerically estimated uncertainties.

The results presented in Refs. \cite{14,21} show that the MVM on various complex topologies belongs to different universality classes. Moreover, contrary for MVM on
regular lattices \([8, 9]\), the obtained critical exponents are different from those of the equilibrium Ising model \([12]\). Very recently, Yang and Kim \([29]\) showed that also for \(d\)-dimensional hypercube lattices \((3 \leq d \leq 6)\) critical exponents for MVM differ from those for SL Ising model. The same situation occurs on hyperbolic lattices \([30]\).

In this paper we study the MVM on three AL, namely \((3, 12^2)\), \((4, 6, 12)\), and \((4, 8^2)\). The AL are vertex transitive graphs that can be embedded in a plane such that every face is a regular polygon. The AL are labeled according to the sizes of faces incident to a given vertex. The face sizes are sorted, starting from the face for which the list is the smallest in lexicographical order. In this way, the lattice gets the name \((3, 12^2)\), \((4, 6, 12)\) and \((4, 8^2)\). Critical properties of these lattices were investigated in terms of site percolation \([31]\) and Ising model \([32]\).

Our main goal is to check the hypothesis of Grinstein et al. \([13]\), i.e., that non-equilibrium stochastic spin systems following the local majority \([8, 16–19, 21]\).

II. MODEL AND SIMULATION

On the original MVM \([8]\), the system dynamics is as follows. Initially, we assign a spin variable \(\sigma\) with values \(\pm 1\) at each site of the square lattice (SL). At each step we try to flip the spin of the nodes in a sequential way. The flip is accepted with probability
\[
w_i = \frac{1}{2} \left[ 1 - (1 - 2q)\sigma_i \cdot S \left( \sum_{j=1}^{k} \sigma_j \right) \right],
\]
where \(S(x)\) is the sign \(\pm 1\) of \(x\) if \(x \neq 0\) and \(S(x) = 0\) if \(x = 0\). To calculate \(w_i\) our sum runs over the number \(k\) \((k = 4\) for SL\) of nearest neighbors of \(i\)th spin. Equation (1) means that with probability \((1-q)\) the spin will adopt the same state as the majority of its neighbors. Here, the control parameter \(0 \leq q \leq 1\) plays a role similar to that of the temperature in equilibrium systems. The smaller the \(q\) the greater the probability of parallel aligning with the local majority \([8, 16–19, 21]\).

Here we study the MVM on \((3, 12^2)\), \((4, 6, 12)\) and \((4, 8^2)\) AL using an alternative probability of Eq. (1) called Glauber rate probability proposed by Kwak et al. \([33]\). The Glauber transition rates of MVM can be written as
\[
w_{GL} = \frac{1}{2} \left[ 1 - \sigma_i \cdot S \left( \sum_{j=1}^{k} \sigma_j \right) \tanh \beta_T \right],
\]
where \(\beta_T\) is the inverse of the temperature \(1/K_B T\) and \(K_B\) is the Boltzmann constant. Comparing this expression with Eq. (1), we see the correspondence between the original MVM and that with Glauber dynamics, which leads to the relation between the noise parameter \(q\) and the temperature in Glauber dynamics as \((1 - 2q) = \tanh \beta_T\).

To study the critical behavior of the model we define the variable \(m = \sum_{i=1}^{N} \sigma_i / N\). In particular, we are interested in the magnetization \(M\), susceptibility \(\chi\) and the reduced fourth-order cumulant \(U\)
\[
M(T) \equiv \langle |m| \rangle,
\]
\[
\chi(T) \equiv \langle m^2 \rangle - \langle m \rangle^2.
\]
\[ \chi(T) \equiv N \left( \langle m^2 \rangle - \langle m \rangle^2 \right), \quad \text{(3b)} \]

\[ U(T) \equiv 1 - \frac{\langle m^4 \rangle}{3\langle m^2 \rangle^2}, \quad \text{(3c)} \]

where \( \langle \cdots \rangle \) stands for a thermodynamics average. The results are averaged over the \( N_{\text{run}} \) independent simulations.

These quantities are functions of temperature \( T \) and obey the finite-size scaling relations

\[ M = L^{-\beta/\nu} f_m(x), \quad \text{(4a)} \]

\[ \chi = L^{\gamma/\nu} f_\chi(x), \quad \text{(4b)} \]

\[ \frac{dU}{dT} = L^{1/\nu} f_U(x), \quad \text{(4c)} \]

where \( \nu, \beta, \) and \( \gamma \) are the usual critical exponents, \( f_{m,\chi,U}(x) \) are the finite size scaling functions with

\[ x = (T - T_c)L^{1/\nu} \quad \text{(4d)} \]

being the scaling variable. Therefore, from the size dependence of \( M \) and \( \chi \) we obtained the exponents \( \beta/\nu \) and \( \gamma/\nu \), respectively. The maximum value of susceptibility also scales as \( L^{\gamma/\nu} \). Moreover, the value of \( T^* \) for which \( \chi \) has a maximum is expected to scale with the lattice size as

\[ T^* = T_c + bL^{-1/\nu} \quad \text{with } b \approx 1. \quad \text{(5)} \]

Therefore, the relations \( \text{(4e) and (4f)} \) may be used to get the exponent \( 1/\nu \).

We performed Monte Carlo simulation on the \((3,12^2)\), \((4,6,12)\) and \((4,8^2)\) AL with various lattice of size \((L = 21, 31, 41, 51, \text{ and } 61)\) for \((3,12^2)\) with \( N = 6L^2 \) that give \( N = 2646, 5766, 10086, 15606 \) and \( 22306; \) \((4,6,12)\) with \( N = 12L^2 \) and \( N = 5292, 11532, 20172, 31212, \) and \( 44652; \) and for \((4,8^2)\) with \( N = 4xL^2 \) and \( 3844, 6724, 10404, \) and \( 14884 \) sites. It takes \( 2 \times 10^5 \) Monte Carlo steps (MCS) to make the system reach the steady state, and then the time averages are estimated over the next \( 2 \times 10^5 \) MCS. One MCS is accomplished after all the \( N \) spins are investigated whether they flip or not. The results are averaged over \( N_{\text{run}} \) \((30 \leq N_{\text{run}} \leq 50)\).
TABLE I: Critical parameter, exponents and effective dimension for MVM model on (3, 12²), (4, 6, 12) and (4, 8²) AL. For completeness we cite data for SL Ising model as well.

|            | (3, 12²) | (4, 6, 12) | (4, 8²) | SL Ising |
|------------|----------|------------|---------|----------|
| \(T_c\)   | 0.363(2) | 0.651(3)   | 0.667(2)|          |
| \(\beta/\nu\) | 0.237(6) | 0.105(8)   | 0.113(2)| 0.125    |
| \(\gamma/\nu^a\) | 0.73(10) | 1.28(11)   | 1.60(4) | 1.75     |
| \(\gamma/\nu^b\) | 0.70(8)  | 1.44(4)    | 1.66(2) | 1.75     |
| \(1/\nu\)  | 0.83(5)  | 1.16(5)    | 0.84(6) | 1        |

\(^a\)obtained using \(\chi(N)\) at \(T = T_c\).
\(^b\)obtained using \(\chi(N)\) at \(T = T^*\)

III. RESULTS AND DISCUSSION

In Fig. 3 we show the dependence of the magnetization \(M\), Binder cumulant \(U_4\), and the susceptibility \(\chi\) on the temperature \(T\), obtained from simulations on (3, 12²), (4, 6, 12) and (4, 8²) AL with \(N\) ranging from \(N = 1764\) to 44652 sites. The shape of \(M(T)\), \(U(T)\), and \(\chi(T)\) curve, for a given value of \(N\), suggests the presence of the second-order phase transition in the system. The phase transition occurs at the value of the critical temperature \(T_c\). The critical noise parameter \(T_c\) is estimated as the point where the curves for different system sizes \(N\) intercept each other [34]. Then, we obtain \(T_c = 0.363(2)\) and \(U_4^* = 0.577(4); T_c = 0.651(3)\) and \(U_4^* = 0.612(5); T_c = 0.667(2)\) and \(U_4^* = 0.613(8)\) for (3, 12²), (4, 6, 12) and (4, 8²) AL, respectively.

In Fig. 3 we plot the dependence of the magnetization \(M^* = M(T_c)\) vs. the linear system size \(L\). The slopes of curves correspond to the exponent ratio \(\beta/\nu\) according to Eq. (4a). The obtained exponents are \(\beta/\nu = 0.237(6), 0.105(8), \) and 0.113(2), respectively for (3, 12²), (4, 6, 12) and (4, 8²) AL.

The exponents ratio \(\gamma/\nu\) at \(T_c\) are obtained from the slopes of the straight lines with \(\gamma/\nu = 0.73(10)\) for (3, 12²), \(\gamma/\nu = 1.28(11)\) for (4, 6, 12), and \(\gamma/\nu = 1.60(4)\) for (4, 8²), as presented in Fig. 4. The exponents ratio \(\gamma/\nu\) at \(T_{\chi_{max}}(N)\) are \(\gamma/\nu = 0.70(8)\) for (3, 12²), \(\gamma/\nu = 1.44(4)\) for (4, 6, 12), and \(\gamma/\nu = 1.66(2)\) for (4, 8²), as presented in Fig. 5.

To obtain the critical exponent \(1/\nu\), we used the scaling relation (5). The calculated values of the exponents \(1/\nu = 0.83(5)\) for (3, 12²) (circles), \(1/\nu = 1.16(5)\) for (4, 6, 12) (squares), and \(1/\nu = 0.84(6)\) for (4, 8²) (diamonds) (see Fig. 6).

We plot \(ML^{\beta/\nu}\) versus \((T-T_c)L^{1/\nu}\) in the Fig. 7 using the critical exponents \(1/\nu = 0.113(2)\) and \(\beta/\nu = 0.84(6)\) for size lattice \(L = 31, 41, 51,\) and 61 for (4, 8²) AL. The excellent collapse of the curves for four different system sizes corroborates the estimation for \(T_c\) and the critical exponents \(\beta/\nu\) and \(1/\nu\). In the Fig. 8 we plot \(\chi L^{-\gamma/\nu}\) versus \((T-T_c)L^{1/\nu}\) using the critical exponents \(\gamma/\nu = 1.60(4)\) and \(\beta/\nu = 0.84(6)\) for size lattice \(L = 31, 41, 51,\) and 61 for (4, 8²) AL. Again, the excellent
of the curves for four different system sizes corroborates the estimation for $T_c$ and the critical exponents $\gamma/\nu$ and $1/\nu$. The results of simulations are collected in Tab. I.

IV. CONCLUSION

Finally, we remark that the critical exponents $\gamma/\nu$, $\beta/\nu$ and $1/\nu$ for MVM on regular $(3,12^2)$, $(4, 6, 12)$ and $(4, 8^2)$ AL are similar to the MVM model on regular $(6^3)$, $(3, 6, 3, 6)$ and $(3^6)$ \cite{28} and also at $(3, 4, 6, 4)$ and $(3^2, 6)$ \cite{27} and are different from the Ising model \cite{12} and differ from those for so-far studied regular lattices \cite{8,9} and for the directed and undirected ER random graphs \cite{16,17} and for the directed and undirected AB networks \cite{19,21}. However, in the latter cases \cite{16,17,19,21} the scaling relations \cite{7} must involve the number of sites $N$ instead of linear system size $L$ as these networks in natural way do not posses such characteristic which allow for $N \propto L^d$ ($d \in \mathbb{Z}$) dependence \cite{36}. For $(3,12^2)$, $(4, 6, 12)$ and $(4, 8^2)$ AL some critical exponents are different to those known analytically for square lattice Ising model, i.e. $\beta = 1/8 = 0.125$, $\gamma = 7/4 = 1.75$ and $\nu = 1$.

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