A Genuine Multipartite Entanglement Measure Generated by the Parametrized Entanglement Measure

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A genuine multipartite entanglement measure based on the geometric method is investigated in this paper. This measure has desirable properties for quantifying the genuine multipartite entanglement. A lower bound of the genuine multipartite entanglement measure derived with the fidelity-based method is then presented. The advantages of the measure proposed here with other measures are also presented. At last, examples are presented to show that the genuine entanglement measure has distinct entanglement ordering from other measures.

1. Introduction

Quantum entanglement is an essential feature of quantum mechanics. It plays an important role in quantum information and quantum computation theory, such as superdense coding,[2] teleportation,[3] and the speedup of quantum algorithms.[4]

Quantum entanglement is an essential feature of quantum mechanics. It plays an important role in quantum information and quantum computation theory, such as superdense coding,[2] teleportation,[3] and the speedup of quantum algorithms.[4]

One of the most important problems is to quantify the entanglement in a composite quantum system. Vedral et al. in ref. [5] presented a necessary condition for entanglement measures, that is, the amount of entanglement cannot increase under local operation and classical communication (LOCC). Then Vidal considered a stronger property, entanglement monotone, he also proposed a general mathematical framework to build entanglement monotone with functions satisfying some properties for pure states.[6] The other interesting approach with operational significance to quantify the entanglement is proposed in refs. [7, 8]. Compared with the bipartite entanglement systems, the complexity of a multipartite entanglement system grows remarkably with the increasing number of parties and the increasing dimension of the systems. The notion of multipartite entanglement measure can be refined into the so-called genuine multipartite entanglement (GME).[9,10] Substantial results have been achieved on the GME measures in the last few decades.[11–20] In ref. [12], the authors proposed a genuinely entangled measure defined as the shortest distance from a given state to the k-separable states, which was denoted as a generalized geometric measure (GGM). The other GME measure, the genuinely multipartite concurrence (GMC) was defined as the minimal bipartite concurrence among all bipartitions.[14] However, the above two measures cannot show all the conditions of entanglement among the parties, as both are defined on the minimizations of the partitions. Concurrence fill was proposed as a three-qubit GME measure,[17] it is the square of the area of the three-qubit concurrence triangle. The authors in ref. [19] generalized the above method to build a multipartite entanglement measure, however, it is hard when the parties are bigger. Hence, there is much work to do on how to understand and quantify the multipartite entanglement. Recently, another method to investigate the GME measure was proposed, it was based on the geometric mean of all bipartite entanglement measures, concurrence.[20]

In this paper, we investigate a measure $G_q(\cdot)$ for multiparticle entangled systems based on the geometric mean method in terms of a bipartite entanglement measure proposed in ref. [21]. As a one-parameter class of GME measure, it contains the measure proposed in ref. [20] as a special case up to a square root and some constant for the pure states in a given multipartite system. And it also satisfies the subadditivity and continuity for pure states. Next, we consider two important states $|\text{GHZ}_n\rangle$ and $|\text{W}_n\rangle$. The two states play key roles in numerous quantum information processing tasks, such as quantum sensing,[22,23] secret sharing,[24,25] and quantum computation.[26] Here, by comparing the values of $|\text{W}_n\rangle$ and $|\text{GHZ}_n\rangle$ in n-qubit systems, we certify that $|\text{GHZ}_n\rangle$ is more entangled than $|\text{W}_n\rangle$. Then we present the bound of this GME measure for mixed states based on the method proposed in ref. [27]. At last, by comparing with GMC and GGM, we show some advantages of the measure $G_q(\cdot)$.

This paper is organized as follows: In Section 2, we present the preliminary knowledge needed here. In Section 3, we present the main results. First, we present a GME measure based on the bipartite entanglement measure, and we show that it satisfies the subadditivity and continuity for pure states. Next, by further applying the measure proposed here to the absolutely maximally entangled states, we can certify $G_q(\cdot)$ is a proper GME measure. Besides, by comparing $G_q(|\text{GHZ}_n\rangle)$ and $G_q(|\text{W}_n\rangle)$ in terms of the measure here, we have $|\text{GHZ}_n\rangle$ is more entangled than $|\text{W}_n\rangle$. Then we present a lower bound of the measure for multipartite

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mixed states. At last, we make some comparisons between the measures proposed here and GMC and GGM. In Section 4, we end with a conclusion.

2. Preliminary Knowledge

Concurrence is one of the most important entanglement measures for bipartite quantum systems,[28] it has attracted much attention from the relevant researchers since the end of the last century.[27,29-34] For a bipartite pure state \( |\psi\rangle_{AB} = \sum_i \sqrt{\lambda_i} |i\rangle |i\rangle \), concurrence is defined as

\[
C(|\psi\rangle_{AB}) = \sqrt{2(1 - \text{Tr} \rho_A^2)}
\]

(1)

where \( \rho_{AB} \) is a mixed state, concurrence is defined as

\[
C(\rho_{AB}) = \min \sum p_i C(|\psi_i\rangle)
\]

(2)

where the minimum is taken over all the decompositions of \( \rho_{AB} = \sum p_i |\psi_i\rangle \langle \psi_i| \). For two qubit mixed states, there exists a direct link between the concurrence and the entanglement of formation.[29]

As a generalized von Neumann entropy, Tsallis-\( q \) entropy can present more properties of the entangled states.[37,38] For a pure state \( |\psi\rangle_{AB} \), then its Tsallis-\( q \) entanglement measure[39] is defined as

\[
T_q(|\psi\rangle_{AB}) = \frac{1 - \rho_A^q}{q - 1}
\]

(3)

here \( \rho_A = \text{Tr}_B \rho_{AB}, q \in (0, 1) \cup (1, \infty) \).

Motivated by concurrence and the Tsallis-\( q \) entanglement entropy, Yang et al. proposed a parametrized entanglement measure, \( q \)-concurrence, for bipartite entanglement systems.[21]

When \( |\psi\rangle_{AB} \) is a pure state, \( q \)-concurrence is defined as

\[
C_q(|\psi\rangle_{AB}) = F_q(\rho_A)
\]

(4)

here \( F_q(\rho) = 1 - \text{Tr} \rho^q, q \geq 2 \).

Due to the properties of \( F_q(\rho) \) shown in ref. [21], the maximum of \( F_q(\rho) \) is attained when its reduced density matrix of the smaller subsystem is the maximally mixed state, that is,

\[
\max_{|\psi\rangle_{AB}} F_q(\rho) = \frac{d^{q-1} - 1}{d^{q-1}}
\]

(5)

here \( d \) is the smaller dimension of the two systems.

Next we review some knowledge needed on multipartite entanglement. A multipartite pure state is biseparable if

\[
|\psi\rangle_{A_1A_2\cdots A_n} = |\phi\rangle_{S\bar{S}}\chi_{\bar{S}}
\]

(6)

for some partite \( S|\bar{S} \), here \( S \) is a subset of \( A = \{A_1, A_2, \ldots, A_n\} \), and \( \bar{S} = A - S \). Otherwise, it is genuinely entangled. Next a mixed state \( \rho \) is biseparable if it can be written as a convex combination of biseparable pure states \( \rho = \sum p_i |\psi_i\rangle \langle \psi_i| \), where the pure states in \( \{|\psi_i\rangle\} \) are biseparable with respect to some bipartition. If an \( n \)-partite state is not biseparable, it is genuinely entangled.

A genuine multipartite entanglement (GME) measure \( E \) should satisfy the following conditions:[14]

1 ) It is an entanglement monotone.
2 ) \( E(\rho) = 0 \), if \( \rho \) is biseparable.
3 ) \( E(\rho) > 0 \), if \( \rho \) is a genuinely entangled state.

Next we recall the definitions of GMC[12] and GGM.[14] Assume \( |\psi\rangle \) is an \( n \)-partite pure state in \( \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n \), where \( \dim(\mathcal{H}_i) = d_i, i = 1, 2, \ldots, n \). Its GMC is defined as

\[
E_G(|\psi\rangle) = \min_{\gamma \in \mathcal{F}} \sum_{i=1}^n \text{Tr} \rho_i^{\gamma}
\]

(7)

where \( \gamma = \gamma_i \) represents the set of all possible bipartitions \( \{A_i, \bar{A}_i\} \) of \( \{1, 2, \ldots, n\} \), and the maximum is taken over all the biseparable states in terms of \( A_i, \bar{A}_i \). Its GMC is defined as

\[
\text{GMC}(|\psi\rangle) = \min_{\gamma \in \mathcal{F}} \sum_{i=1}^n \text{Tr} \rho_i^{\gamma}
\]

(8)

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Then we present the definition of a GME measure based on the \( q \)-concurrence in terms of the geometric method.

**Definition 1.** Assume \( |\psi\rangle_{A_1A_2\cdots A_n} \) is an \( n \)-partite pure state, the geometric mean of \( q \)-concurrence (G\( q \)C) is defined as

\[
G_q(\rho) = (\prod_{i=A} C_q(\rho_{A_i}|_{\bar{A}_i}))^{1/|\alpha|}
\]

(9)

here \( \rho_{A_i} = \text{Tr}_{\bar{A}_i} |\psi\rangle \langle \psi| \), \( \alpha = \{A_i\} \) is the set that denotes all possible bipartitions \( A_i, \bar{A}_i \) of the \( n \) parties, \( c(\alpha) \) is the cardinality of \( \alpha \),

\[
c(\alpha) = \begin{cases} \sum_{i=1}^{n-1} C_{m}^{m} & \text{if } n \text{ is odd,} \\ \sum_{i=1}^{n-1} C_{m}^{m} + \frac{1}{2} C_{m}^{m} & \text{if } n \text{ is even} \end{cases}
\]

(10)

When \( \rho \) is an \( n \)-partite mixed state,

\[
G_q(\rho) = \min \sum p_i G_q(\rho_i)
\]

(11)

where the minimum is taken over all the decompositions of \( \rho = \sum p_i |\psi_i\rangle \langle \psi_i| \).

By using the \( q \)-concurrence, G\( q \)C is a class of GME measure with one parameter for multipartite states, it can reflect more properties of the multipartite entangled systems. Based on the properties of \( F_q(\cdot) \), we will present that G\( q \)C is subadditive and continuous for pure states in the next section, which are less considered for other GME measures but meaningful for entanglement measures.
3. Main Results

In this section, we present the main results of this article. In Section 3.1, we present the properties of the GqC. In Section 3.2, we present a class of examples to illustrate that GqC is a proper GME measure further. In Section 3.3, a lower bound of the GqC for an n-partite mixed state $\rho$ was presented. In Section 3.4, we make a comparison of the GqC with other GME measures.

3.1. The Properties of GqC

In this subsection, we show that the GqC is a GME measure, and it satisfies the subadditivity and continuous properties for pure states.

**Theorem 2.** For an arbitrary n-partite quantum state $|\psi\rangle_{A_1A_2...A_n}$, the GqC($q \geq 2$) is a GME measure.

**Proof.** To show GqC is a GME measure, we need to prove it satisfies the following properties: 1) it is an entanglement monotone; 2) when $\rho$ is biseparable, $G_q(\rho) = 0$; and 3) when $\rho$ is genuinely entangled, $G_q(\rho) > 0$.

For the first property, assume $\{\rho, \sigma\}$ is an ensemble after the action of an LOCC channel $\Psi$ on the state $\rho$. When $\rho$ and $\sigma$ are pure states, as $C_q(\rho)$ is biseparable, $G_q(\rho) = 0$.

For the second property, assume $\{\rho, \sigma\}$ is a mixed state.

For the third condition, as a GME pure state $|\psi\rangle$ cannot be written as product states with respect to any bipartition, then we have all the bipartite $C_q$ is bigger than 0, so $G_q(|\psi\rangle) > 0$. Due to the definition of a mixed GME state $\rho$ and the definition of GqC, we have when $\rho$ is GME, $G_q(\rho) > 0$, then we prove the condition (3).

Next we show that the GqC satisfies the subadditivity for pure states.

Assume $|\psi_1\rangle$ and $|\psi_2\rangle$ are pure states in a bipartite system $H_A \otimes H_B$, then we have

$$C_q(|\psi_1\rangle) + C_q(|\psi_2\rangle) \geq C_q(|\psi_1\rangle \otimes |\psi_2\rangle)$$

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Here we place proofs of the Theorems 3 and 4 in Appendix B.

\[\begin{align*}
\text{Theorem 3.} & \text{ Assume } |\psi_1\rangle \text{ and } |\psi_2\rangle \text{ are pure states in } H_A \otimes H_B, \text{ when } ||\psi_1|| - ||\psi_2|| \leq \epsilon, \text{ then we have} \\
|C_q(|\psi_1\rangle) - C_q(|\psi_2\rangle)| & \leq d||1 + \frac{\epsilon}{d}||^1 - 1. \\
\text{Theorem 4.} & \text{ Assume } |\psi_1\rangle \text{ and } |\psi_2\rangle \text{ are pure states in } n\text{-partite systems } H_A \otimes H_B \otimes \cdots \otimes H_d, \text{ when } ||\psi_1|| - ||\psi_2|| \leq \epsilon, \text{ then we have} \\
|G_q(|\psi_1\rangle) - G_q(|\psi_2\rangle)| & \leq \left[ \sum_{i=1}^{\infty} C_q^d[(1 + \frac{\epsilon}{d})^{1/2} - 1]\right]^{1/2} \leq \left[ \sum_{i=1}^{\infty} C_q^d[(1 + \frac{\epsilon}{d}) - 1]\right]^{1/2} \\
\end{align*}\]
3.2. Examples

In this subsection, we consider the absolutely maximally entangled states, $|GHZ_n\rangle$ and $|W_n\rangle$. These results can be seen as the other evidences that the GqC is a proper GME measure.

A pure multipartite entangled state is called absolutely maximally entangled state (AMES) if all reduced density operators obtained by tracing out at least half of the particles of the pure state are the maximally mixed.$^{[40]}$ Due to the definitions and properties of $C_q(\rho)$, when $\rho$ is an AMES, $C_q(\rho)$ gets the maximum. Then we declare that the GqC is a proper GME measure by illustrating the AMES owns the maximal entanglement to finish the quantum processing tasks better. As the reduced density matrix of smaller subsystems of the AMES is maximally mixed, they can be always used to develop the quantum secret sharing schemes$^{[41]}$ and quantum error correction codes.$^{[42,43]}$ Especially, a perfect teleportation in three qubit systems can be performed via the GHZ state, while the W state cannot.$^{[44]}$ Besides the AMES in a three-qubit system are LU equivalent to the GHZ state.$^{[45]}$ Hence, the AMES can be regarded as owning the maximal entanglement, that is, the GqC is a proper GME measure.

Next we consider two pure states in multiqubit systems that are inequivalent in terms of stochastic LOCC (SLOCC), the W states and GHZ states. Moreover, for a k-partite W and GHZ states ($k \geq 3$), the infimum asymptotic ratio from GHZ to W is 1.$^{[46]}$ However, the infimum asymptotic ratio from W to GHZ is bigger than 1.$^{[47]}$ Thus the GHZ states can be thought more entangled than the W states. Then by comparing $C_q(|W_n\rangle)$ with $C_q(|GHZ_n\rangle)$, $|GHZ_n\rangle$ attains more values than $|W_n\rangle$ in terms of GqC. This may be seen as another evidence that the GqC is a proper measure.

**Example 5.**

$|W_n\rangle = \frac{1}{\sqrt{n}}((10 \ldots 0) + |01 \ldots 0\rangle + \ldots + |00 \ldots 1\rangle)$,

$|GHZ_n\rangle = \frac{1}{\sqrt{2}}((00 \ldots 0\rangle + |11 \ldots 1\rangle)$.

Here we place the results on $C_q(|W_n\rangle)$ and $C_q(|GHZ_n\rangle)$ in Appendix A. In Figure 1, we plot the values of $C_q(|W_n\rangle)$ and $C_q(|GHZ_n\rangle)$ when $q = 3$ with the increasing number of qubits, we can see that $C_q(|W_n\rangle)$ is less than $C_q(|GHZ_n\rangle)$ for any $n$ and the values of $C_q(|W_n\rangle)$ tends to 1 with the increase of $n$, which are consistent with the results in refs. [44, 46, 47].

3.3. A Lower Bound of GqC

In this subsection, we first present a lower bound of the entanglement measure $C_q(\rho_{AB})$ for a bipartite mixed state $\rho_{AB}$, then we extend the results to the GqC for multipartite mixed systems.

**Theorem 6.** For a bipartite mixed state $\rho$ on the system $H_m \otimes H_n (m \leq n)$, $|\psi\rangle = V_{\lambda} \otimes V_{\eta} \sum_{i=1}^{n} |\sqrt{\lambda}|i\rangle$ is an arbitrary pure state in $H_m \otimes H_n$, its revised parametrized entanglement measure $C_q(\rho)$ satisfies

$$C_q(\rho) \geq \frac{m^{q-1}-1}{m^{q-2}(m-1)} (\Lambda - \frac{1}{m})$$

(20)

**Example 8.** Consider a 3-qubit W state with the white noise,

$$\rho_W = p|W\rangle\langle W| + \frac{1-p}{8} I, \ p \in (0, 1)$$

(22)

In Figure 2, we plot a lower bound of $G_3(\rho_W)$ according to Theorem 7. As $G_3(\cdot)$ is a GME measure, when the lower bound is bigger than 0, $\rho_W$ is genuinely entangled. With the increasing of $p$, the lower bound of $G_3(\rho_W)$ is bigger and bigger.

**Example 9.** Let $\rho_{GHZ}$ be a 3-qubit mixed state,

$$\rho_{GHZ} = c|GHZ\rangle\langle GHZ| + \frac{1-c}{8} I,$$

(23)

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle), \ c \in (0, 1)$$

$\Lambda = \sum_{i=1}^{n} s_i^q/m$, and $s_i$ is the largest in the set of Schmidt coefficients $\{s_i | i = 1,2, \ldots, m\}$ of $|\phi\rangle$.
Figure 2. The lower bound of $G_q(\rho_W)$ when $q = 2$. Here we use $|\psi\rangle = |W\rangle$ and $s_1 = \frac{2}{3}$.

Here we plot the lower bound of $G_qC$ for $\rho_{\text{GHZ}}$ when $q \in [2, 12]$ in Figure 3. Due to Theorem 6, when $G_q(\rho)$ is bigger than 0, $\rho$ is genuinely entangled. Then from Figure 3, when the lower bound is bigger than 0, $\rho_{\text{GHZ}}$ is genuinely entangled. And when $q$ is fixed, with the increasing of $c$, the value of $G_q(\rho_W)$ is bigger and bigger.

Example 10. In this example, we apply three experimental results of multipartite systems\cite{48-50} to present the lower bound of $G_qC$ for the produced states. All the three experiments demonstrate multipartite entanglement. In these experiments, six-ten photons are attained. It is very hard to perform quantum tomography and obtain the exact values of most entanglement measures for these states. However, Theorem 7 provides a reliable way to get the lower bound of $G_qC$ for these states. In Figure 4, we plot the lower bound of $G_q(\rho)$ with the increasing $q$. As the lower bounds of all these states are bigger than 0, we have the genuine entanglement exists in the experimentally realized states in refs. \cite{48–50}. Moreover, through the Figure 4b, for a given $|\phi\rangle$, as the fidelity of $|\phi\rangle$ and $\rho$ increases, the lower bound of $G_q(\rho)$ increases.

3.4. Comparisons with Other GME Measures

In this section, we present some examples on the $G_q(\cdot)$ and compare them with other GME measures, GMC (Equation (8)) and GGM (Equation (7)). Through comparison, we can get the difference between GqC and other GME measures, also we can get the advantages of GqC.
First we present a class of 4-qubit pure states, which shows the advantages of GqC when comparing with GGM and GMC.

**Example 11.**

\[
|\psi\rangle = \cos \theta |\phi_1\rangle + \sin \theta |0111\rangle, \quad \theta \in (0, \frac{\pi}{2})
\]  
\[
|\phi_1\rangle = \cos \frac{2\pi}{3} |0100\rangle + \sin \frac{2\pi}{3} |1000\rangle
\]  

As presented in Figure 5, when \(\theta\) increases from \([\theta_1, \theta_3]\), the values of G3C increases, while the GGM decreases from \([\theta_2, \theta_3]\) and GMC decreases from \([\theta_1, \theta_3]\). Next when \(\theta\) ranges in \([0, \theta_3]\), each G3C value corresponds to a unique state in the class (Equation (24)), while there exists pairs of states in the class (Equation (24)) with the same GGM or GMC, that is, the GME measure G3C can detects the robustness between some states while the GGM and GMC cannot. Moreover, Figure 5 show the smoothness of G3C, nevertheless, a sharp peak appears with the varying \(\theta\) when considering the GMC and GGM.

Entanglement ordering is meaningful when considering the entanglement measures.\(^{[51,52]}\) It means that if \(E_1\) and \(E_2\) are two entanglement measures, for any pair of \(\sigma_1\) and \(\sigma_2\), \(E_1(\sigma_1) \geq E_1(\sigma_2)\) derives \(E_2(\sigma_1) \geq E_2(\sigma_2)\). Next by considering the following two classes of states in three-qubit systems, we show that the entanglement ordering of G4C is different from GMC and GGM.

**Example 12.**

**Class I:**  
\[
|\psi\rangle = \frac{1}{\sqrt{2}}(\cos \theta |000\rangle + \sin \theta |001\rangle) + \frac{1}{\sqrt{2}} |111\rangle
\]  

**Class II:**  
\[
|\phi\rangle = \cos \theta |000\rangle + \sin \theta |111\rangle
\]  

From Figure 6a, we see that the G4C owns different entanglement order from the GMC. For a given state belonging to the Class I state, there are many states in Class II with larger GMC but with smaller G4C. This can be shown by drawing vertical or horizontal lines, when comparing with the intersection point, the states in the upper left area of the intersection point owns larger GMC and smaller G4C. Similarly, from the Figure 6b, the entanglement order of G4C is different from GGM. For a given state of the Class I, there are many states in Class II with larger GGM but with smaller G4C. The opposite results can be arrived at when given a Class II pure state.

4. **Conclusion**

In this paper, we have proposed and investigated a GME measure based on the geometric mean method. First, we have presented the GqC is a GME measure and satisfies the subadditivity and continuity for pure states. Next, we have pointed out that the AMES reaches the maximum in terms of \(\mathcal{G}_q(\cdot)\), which can be seen as another evidence that the GqC is a proper GME measure. Then, by comparing \(\mathcal{G}_q(\ket{\text{GHZ}_n})\) and \(\mathcal{G}_q(\ket{\text{W}_n})\), we have shown that the entanglement of the GHZ state is stronger than the W state. At last, we have presented a lower bound of the GqC for a multipartite mixed state. Due to the importance of the study of GME measures, our results can provide a reference for future work on the study of multiparty quantum entanglement.
Appendix A: Some Results On \( \mathcal{G}_q(W_n) \) and \( \mathcal{G}_q(\{GHZ_n\}) \)

An n-qubit \( W \) state can be represented as

\[
|W_n\rangle = \frac{1}{\sqrt{n}} \left( |00\cdots 1\rangle + |01\cdots 0\rangle + \cdots + |00\cdots 1\rangle \right) \tag{A1}
\]

Through computation, we have

\[
\mathcal{G}_q(W_n) = \begin{cases} 
\frac{1}{n^{q/2}} - \frac{(n-1)^q}{n^q}, & \text{if } n \text{ is odd,} \\
\frac{1}{n^{q/2}} - \frac{(n-2)^q}{n^q}, & \text{if } n \text{ is even.}
\end{cases}
\]

An n-qubit GHZ state can be represented as

\[
|GHZ_n\rangle = \frac{1}{\sqrt{2}} \left( |00\cdots 0\rangle + |11\cdots 1\rangle \right) \tag{A2}
\]

Through computation, we have

\[
\mathcal{G}_q(|GHZ_n\rangle) = 1 - \frac{1}{2^{q/2}} \tag{A3}
\]

Then we have

\[
\mathcal{G}_q(|W_n\rangle) = \begin{cases} 
\left( \prod_{i=1}^{n} \left[ 1 - \frac{k_i}{n} - \frac{(n-k_i)^q}{n^q} \right]^{1/2} \right)^{\frac{1}{2}}, & \text{if } n \text{ is odd,} \\
\left( \prod_{i=1}^{n} \left[ 1 - \frac{k_i}{n} - \frac{(n-k_i)^q}{n^q} \right]^{1/2} \right)^{\frac{1}{2}}, & \text{if } n \text{ is even.}
\end{cases}
\]

Then by using the Stolz–Cesaro theorem, we have

\[
\lim_{k \to \infty} \frac{\mathcal{G}_q(|W_{2k}\rangle)}{\mathcal{G}_q(|GHZ_{2k}\rangle)} = 1 \
\lim_{k \to \infty} \frac{\mathcal{G}_q(|W_{2k+1}\rangle)}{\mathcal{G}_q(|GHZ_{2k+1}\rangle)} = 1 
\]

Appendix B: The Proof of the Continuity of \( \mathcal{G}_q(\cdot) \) for Pure States

Here we present the proof of Lemma 3 and Theorem 4.

Lemma 3: Assume \( |\psi_1\rangle \) and \( |\psi_2\rangle \) are pure states in \( H_d \otimes H_d \), when \( ||\psi_1|| - ||\psi_2|| \leq \epsilon \), we have

\[
|\mathcal{G}_q(|\psi_1\rangle) - \mathcal{G}_q(|\psi_2\rangle)\| \leq d(1 + \frac{\epsilon}{d})^q - 1 \tag{B1}
\]

Proof. As partial trace is trace-preserving, then \( ||\rho_A - \sigma_A|| \leq \epsilon \), here

\[
\rho_A = \mathcal{N}_d(\psi_1\psi_1^\dagger), \quad \sigma_A = \mathcal{N}_d(\psi_2\psi_2^\dagger).
\]

Next as \( ||\cdot||_1 \) is unitarily invariant, then

\[
||\mathcal{E}_q(\rho_A) - \mathcal{E}_q(\sigma_A)\| \leq ||\rho_A - \sigma_A|| \leq ||\rho_A - \sigma_A||_1 \leq ||\mathcal{E}_q(\rho_A) - \mathcal{E}_q(\sigma_A)||. \tag{B2}
\]

that is, we only need to consider the classical case. Readers who are interesting in the above two inequalities please refer to ref. [53].

Assume \( \rho_A \) and \( \sigma_A \) are two diagonal density matrices with their diagonal elements \( p_i \) and \( q_i \), respectively. And \( p_i \) and \( q_i \) satisfy \( \sum_{i} p_i = \sum_{i} q_i = 1 \), and \( |p_i - q_i| \leq \epsilon \). Then

\[
|\mathcal{C}_q(p_A) - \mathcal{C}_q(q_A)| = |\text{Tr} \rho_A^q - \text{Tr} \sigma_A^q| \leq \sum_{i} |p_i - q_i|^q \tag{B3}
\]

Next let \( p_i - q_i = \epsilon_i, \sum \epsilon_i \leq \epsilon \), when \( p_i = r_i + c_i \), then

\[
|p_i - q_i|^q = |(r_i + c_i)^q - r_i^q| \leq ((1 + c_i)^q - 1) \tag{B4}
\]

the last inequality is due to that \( (1 + c_i)^q - 1 \) is increasing in terms of \( r_i \) and \( c_i \in (0, 1) \). And due to that \( (1 + c_i)^q - 1 \) is increasing in terms of \( c_i \), when \( r_i = r_i - c_i \), the above inequality is also valid. Then the inequality in Equation (B3) becomes

\[
|\mathcal{C}_q(p_A) - \mathcal{C}_q(q_A)| \leq d\left( 1 + \frac{\epsilon}{d} \right)^q - 1 \tag{B5}
\]

when all \( \epsilon_i = \frac{\epsilon}{d} \), the equality in the last inequality is valid. \( \square \)

Theorem 4: Assume \( |\psi_1\rangle \) and \( |\psi_2\rangle \) are two pure states in n-partite systems \( H_d \otimes H_d \otimes \cdots \otimes H_d \), here \( ||\psi_1|| - ||\psi_2|| \leq \epsilon \), then we have

\[
|\mathcal{G}_q(|\psi_1\rangle) - \mathcal{G}_q(|\psi_2\rangle)\| \leq \frac{\sum_{i=1}^{n} C_d^d\left( 1 + \frac{\epsilon}{d} \right)^q - 1 \right)^{1/d}}{d} \tag{B6}
\]

Proof. When \( n \) is odd, then we have

\[
|\mathcal{G}_q(|\psi_1\rangle) - \mathcal{G}_q(|\psi_2\rangle)\| = |P_d(|\psi_1\rangle) - P_d(|\psi_2\rangle)| \tag{B7}
\]

the first inequality is due to that when \( p, q, x, y \in (0, 1) \), \( |p^q - q^q| \leq |p^q - q^q| \), the second inequality is due to the following inequality, when \( x_i, y_i \in (0, 1) \), \( i = 1, 2, \ldots, n \), then we have

\[
x_1, x_2 \cdots x_n = y_1, y_2 \cdots y_n \tag{B8}
\]

\[
=x_1(1 - y_1)x_2 \cdots x_n + y_1(x_2 - y_2)x_3 \cdots x_n + y_1y_2(x_3 - y_3)x_4 \cdots x_n + \cdots + y_1y_2 \cdots y_{n-1}(x_n - y_n) \tag{B9}
\]

\[
\leq \sum_i |x_i - y_i| \tag{B10}
\]
here the last inequality is due to the triangle inequality and \( x, y \in (0, 1) \).

**Appendix C: The Proof of Theorem 6**

Here we prove Theorem 6 based on the method in ref. [33].

**Theorem 6:** For a bipartite mixed state \( \rho \) on the system \( \mathcal{H}_m \otimes \mathcal{H}_n \) (\( m \leq n \)), its revised parametrized entanglement measure \( C_q(\rho) \) satisfies

\[
C_q(\rho) \geq \frac{m^{q-1} - 1}{m^{q-1}(m - 1)} \left( \lambda - \frac{1}{m} \right)
\]

where \( \lambda = \max \left\{ \frac{\langle \phi|\phi \rangle}{s_i^m}, \frac{1}{m} \right\} \), and \( s_i \) is the largest in the set of Schmidt coefficients \( \{ s_i | i = 1, 2, \ldots, m \} \) of \( |\phi \rangle \).

**Proof:** Here we consider the following function

\[
R(\lambda) = \min_{\vec{\mu}} \{ L(\vec{\mu}) | \lambda = \frac{1}{m} \sum_i \sqrt{\mu_i}^2 \}
\]

where \( L(\vec{\mu}) = 1 - \sum_i \mu_i^2 \). Due to the results in refs. [54, 55], the minimum \( L(\vec{\mu}) \) versus \( \lambda \) is in the form

\[
\vec{\mu} = \left\{ \frac{1}{m - 1}, \frac{1 - t}{m - 1}, \ldots, \frac{1 - t}{m - 1} \right\}
\]

for \( t \in \left[ \frac{1}{m - 1}, 1 \right] \)

Therefore, we have the minimum \( L(\vec{\mu}) \) and the function \( t(\lambda) \) are

\[
L(t) = 1 - t - \frac{(1 - t)^{q-1}}{(m - 1)^{q-1}}
\]

\[
t(\lambda) = \frac{m}{1 + \sqrt{m - 1}(1 - \lambda)^{q-1}}
\]

Substituting (C5) into (C4), we have

\[
L(\lambda) = \frac{1}{m} \left( \sqrt{\lambda} + \sqrt{m - 1}(1 - \lambda) \right)^2
\]

through computation, we have

\[
L''(\lambda) = -q(1 - \lambda)^{q-2} \frac{(1 - q(1 - \lambda))^{q-2}}{(m - 1)^{q-1}} \leq 0
\]

\[
t''(\lambda) = \frac{1}{m} \left( \sqrt{\lambda} + \sqrt{m - 1}(1 - \lambda) \right)
\]

\[
\times \left( \frac{1}{\sqrt{\lambda}} + \frac{1 - m}{\sqrt{m - 1}(1 - \lambda)} \right) \leq 0
\]

as \( L'' \leq 0 \), \( L'' \) is concave. Next we prove \( L(\lambda) \) is increasing function.

**Appendix D: The Proof of Theorem 7**

Here we present the proof of Theorem 7.

**Theorem 7:** Assume \( \rho_{A_1A_2\ldots A_n} \) is a mixed state on an \( n \)-partite system. Then we have

\[
G_q(\rho) \geq \frac{m^{q-1} - 1}{m^{q-1}(m - 1)} (\lambda' - \frac{1}{m})
\]

where \( \lambda' = \max \left\{ \frac{\langle \phi|\phi \rangle}{s_i^m}, \frac{1}{m} \right\} \)

\[
t(\lambda) = \frac{m}{1 + \sqrt{m - 1}(1 - \lambda)^{q-1}}
\]

\[
\vec{\mu} = \left\{ \frac{1}{m - 1}, \frac{1 - t}{m - 1}, \ldots, \frac{1 - t}{m - 1} \right\}
\]

for \( t \in \left[ \frac{1}{m - 1}, 1 \right] \)

Therefore, we have the minimum \( L(\vec{\mu}) \) and the function \( t(\lambda) \) are

\[
L(t) = 1 - t - \frac{(1 - t)^{q-1}}{(m - 1)^{q-1}}
\]

\[
t(\lambda) = \frac{m}{1 + \sqrt{m - 1}(1 - \lambda)^{q-1}}
\]

Substituting (C5) into (C4), we have

\[
L(\lambda) = \frac{1}{m} \left( \sqrt{\lambda} + \sqrt{m - 1}(1 - \lambda) \right)^2
\]

through computation, we have

\[
L''(\lambda) = -q(1 - \lambda)^{q-2} \frac{(1 - q(1 - \lambda))^{q-2}}{(m - 1)^{q-1}} \leq 0
\]

\[
t''(\lambda) = \frac{1}{m} \left( \sqrt{\lambda} + \sqrt{m - 1}(1 - \lambda) \right)
\]

\[
\times \left( \frac{1}{\sqrt{\lambda}} + \frac{1 - m}{\sqrt{m - 1}(1 - \lambda)} \right) \leq 0
\]

as \( L'' \leq 0 \), \( L'' \) is concave. Next we prove \( L(\lambda) \) is increasing function.

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Conflict of Interest
The authors declare no conflict of interest.

Data Availability Statement
The data that support the findings of this study are available from the corresponding author upon reasonable request.

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[1] R. Horodecki, P. Horodecki, M. Horodecki, K. Horodecki, Rev. Mod. Phys. 2009, 81, 865.
[2] C. H. Bennett, S. J. Wiesner, Phys. Rev. Lett. 1992, 69, 2881.
[3] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, W. K. Wootters, Phys. Rev. Lett. 1993, 70, 1895.
[4] Y. Shimoni, D. Shapira, O. Biham, Phys. Rev. A 2005, 72, 062308.
[5] V. Vedral, M. B. Plenio, M. A. Rippin, P. L. Knight, Phys. Rev. Lett. 1997, 78, 2275.
[6] G. Vidal, Mod. Opt. 2000, 47, 355.
[7] G. Cour, M. Tomamichel, arXiv:2006.12408, 2020.
[8] X. Shi, L. Chen, Ann. Phys. 2021, 533, 2000462.
[9] O. Gühne, G. Tóth, H. J. Briegel, New J. Phys. 2005, 7, 229.
[10] M. B. Plenio, S. S. Virmani, in Quantum Information and Coherence (Eds: A. Bhar, I. Chattopadhyay, D. Sarkar), Springer, Berlin 2014, pp. 173–209.
[11] D. A. Meyer, N. R. Wallach, J. Math. Phys. 2002, 43, 4273.
[12] M. Blasone, F. Dell’Anno, S. De Siena, F. Illuminati, Phys. Rev. A 2008, 77, 062304.
[13] B. C. Hiesmayr, M. Huber, P. Krammer, Phys. Rev. A 2009, 79, 062308.
[14] Z.-H. Ma, Z.-H. Chen, J.-L. Chen, C. Spengler, A. Gabriel, M. Huber, Phys. Rev. A 2011, 83, 062325.
[15] B. Jungnitsch, T. Moroder, O. Gühne, Phys. Rev. Lett. 2011, 106, 190502.
[16] S. H. Rafsanjani, M. Huber, C. J. Broadbent, J. H. Eberly, Phys. Rev. A 2012, 86, 062304.
[17] S. Xie, J. H. Eberly, Phys. Rev. Lett. 2021, 127, 040403.
[18] J. L. Beckey, N. Gigena, P. J. Coles, M. Cerezo, Phys. Rev. Lett. 2021, 127, 140501.
[19] Y. Guo, Y. Jia, X. Li, L. Huang, J. Phys. A: Math. Theor. 2022, 55, 145303.
[20] Y. Li, J. Shang, Phys. Rev. Res. 2022, 4, 023059.
[21] X. Yang, M.-X. Luo, Y.-H. Yang, S.-M. Fei, Phys. Rev. A 2021, 103, 052343.
[22] C. L. Degen, F. Reinhard, P. Cappellaro, Rev. Mod. Phys. 2017, 89, 035002.
[23] C. D. Marciniak, T. Feldker, I. Pogorelov, R. Kaubruengger, D. V. Vasilyev, R. van Bijnen, P. Schindler, P. Zoller, R. Blatt, T. Monz, Nature 2022, 603, 604.
[24] J. Joo, J. Lee, J. Jang, Y.-J. Park, arXiv:quant-ph/0603144, 2002.
[25] M. Hillery, V. Bužek, A. Berthiaume, Phys. Rev. A 1999, 59, 1829.
[26] S. Bartolucci, P. Birchall, H. Bombin, H. Cable, C. Dawson, M. Gmene-Segovia, E. Johnston, K. Kielping, N. Nickerson, M. Pant, F. Pastawski, T. Rudolph, C. Sparrow, Nat. Commun. 2023, 14, 912.
[27] Y. Dai, Y. Dong, Z. Xu, W. You, C. Zhang, O. Gühne, Phys. Rev. Appl. 2020, 13, 054022.
[28] S. Hill, W. K. Wootters, Phys. Rev. Lett. 1997, 78, 5022.
[29] W. K. Wootters, Phys. Rev. Lett. 1998, 80, 2245.
[30] V. Coffman, J. Kundu, W. K. Wootters, Phys. Rev. A 2000, 61, 052306.
[31] F. Mintert, M. Küs, A. Buchleitner, Phys. Rev. Lett. 2004, 92, 167902.
[32] K. Chen, S. Albeverio, S.-M. Fei, Phys. Rev. Lett. 2005, 95, 040504.
[33] C. Zhang, S. Yu, Q. Chen, H. Yuan, C. Oh, Phys. Rev. A 2016, 94, 042325.
[34] M. Li, Z. Wang, J. Wang, S. Shen, S.-M. Fei, Quantum Inf. Process. 2020, 19, 130.
[35] C. Tsallis, J. Stat. Phys. 1988, 52, 479.
[36] P. T. Landsberg, V. Vedral, Phys. Rev. A 1998, 247, 211.
[37] C. Tsallis, S. Lloyd, M. Baranger, Phys. Rev. A 2001, 63, 042104.
[38] R. Rossignoli, N. Canosa, Phys. Rev. A 2002, 66, 042306.
[39] J. San Kim, Phys. Rev. A 2010, 81, 062328.
[40] W. Helwig, W. Cui, J. I. Latore, A. Riera, H.-K. Lo, Phys. Rev. A 2012, 86, 052335.
[41] W. Helwig, W. Cui, arXiv:1306.2536, 2013.
[42] M. Grässl, M. Rötteler, in 2015 IEEE Int. Symp. on Information Theory (ISIT), IEEE, Piscataway, NJ, 2015, pp. 1104–1108.
[43] D. Alsina, M. Razavi, Phys. Rev. A 2021, 103, 022402.
[44] J. Joo, Y.-J. Park, S. Oh, J. Kim, New J. Phys. 2003, 5, 136.
[45] X. Shi, L. Chen, M. Hu, Phys. Rev. A 2021, 104, 012426.
[46] N. Yu, C. Guo, R. Duan, Phys. Rev. Lett. 2014, 112, 160401.
[47] P. Vrana, M. Christandl, J. Math. Phys. 2015, 56, 022204.
[48] Y.-F. Huang, B.-H. Liu, L. Peng, Y.-H. Li, L. Li, C.-F. Li, G.-C. Guo, Nat. Commun. 2011, 2, 546.
[49] X.-C. Yao, T.-Y. Wang, P. Xu, H. Lu, G.-S. Pan, C.-Z. Peng, C.-Y. Lu, Y.-A. Chen, J.-W. Pan, Nat. Photonics 2012, 6, 225.
[50] M. Gong, M.-C. Chen, Y. Zheng, S. Wang, C. Zha, H. Deng, Z. Yan, H. Rong, Y. Wu, S. Li, F. Chen, Y. Zhao, F. Liang, J. Lin, Y. Xu, C. Guo, L. Sun, A. D. H. Wang, C. Peng, C.-Y. Lu, X. Zha, J.-W. Pan, Phys. Rev. Lett. 2019, 122, 110501.
[51] S. Virmani, M. F. Sacchi, M. B. Plenio, D. Markham, Phys. Lett. A 2001, 288, 62.
[52] K. Zyczkowski, I. Bengtsson, Ann. Phys. 2002, 295, 115.
[53] R. Bhatia, Matrix Analysis, Vol. 169, Springer Science & Business Media, Berlin 2013.
[54] B. M. Terhal, K. G. H. Vollbrecht, Phys. Rev. Lett. 2000, 85, 2625.
[55] D. W. Berry, B. C. Sanders, J. Phys. A: Math. Gen. 2003, 36, 12255.