Periodic solutions of hybrid jump diffusion processes

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Abstract We investigate periodic solutions of regime-switching jump diffusions. We first show the well-posedness of solutions to stochastic differential jump equations corresponding to the hybrid system. Then, we derive the strong Feller property and irreducibility of the associated time-inhomogeneous semigroups. Finally, we establish the existence and uniqueness of periodic solutions. Concrete examples are presented to illustrate the results.

Keywords Hybrid system, regime-switching jump diffusion, periodic solution, strong Feller property, irreducibility

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1 Introduction

A hybrid system is a dynamical system whose evolution depends on both continuous and discrete variables. The study of hybrid systems is becoming more and more important in different research areas such as biology, ecosystems, wireless communications, signal processing, engineering, and mathematical finance. In the recent years, lots of progress has been made on a class of hybrid systems called regime-switching jump diffusion processes. This model consists of two component processes \((X(t), \Lambda(t))\) with \(X(t)\) and \(\Lambda(t)\) being of continuous and discrete states, respectively. If \(\Lambda(t)\) is a Markov chain that is independent of \(X(t)\), then we have the model of Markov-switching jump diffusions; whereas if \(\Lambda(t)\) depends on \(X(t)\), then we have the model of regime-switching jump diffusions. We refer the reader to the monographs [8,17] for comprehensive studies of hybrid switching diffusions and their applications.
The model of regime-switching jump diffusions provides more flexibility in applications but also requires careful examination of the dependence between the continuous and discrete components. In the past decade, various properties of these hybrid systems have been thoroughly studied and many remarkable results have been obtained. For example, Chen et al. obtained maximum principles and Harnack inequalities in [2] and discussed the recurrence and ergodicity in [3]. Xi [13] and Xi and Yin [14] investigated asymptotic properties of the model. Nguyen and Yin [10,11], Shao [12], Xi and Zhu [16], and Xi et al. [15] considered the model whose switching component has a countably infinite state space. Note that most of the existing literature has focused on the time-homogeneous case. In this paper, we will study time-inhomogeneous regime-switching jump diffusions and investigate their periodic solutions.

Periodic solutions play an important role in the study of stochastic dynamical systems. Here we just list some previous works that are closely related to this paper. Khasminskii [6] systematically developed the theory of periodic solutions to random systems modelled by stochastic differential equations (SDEs). Zhang et al. [18] investigated periodic solutions of SDEs driven by Lévy processes. Hu and Xu [5] obtained the existence and uniqueness of periodic solutions for stochastic functional differential equations by proving the global attractivity of solutions. In our recent work [4], we established the ergodicity and uniqueness of periodic solutions for jump diffusions by proving the strong Feller property and irreducibility of the associated time-inhomogeneous semigroups. Note that none of the above mentioned works discusses periodic solutions of hybrid systems. Our present paper seems to be the first one discussing periodic solutions of regime-switching jump diffusions whose switching components can have countably infinite state spaces.

The rest of this paper is organized as follows. In Section 2, we will show the well-posedness of solutions to SDEs corresponding to the hybrid system (1) and (3) (see below). The unique strong solution is obtained by representing the switching component as a stochastic integral with respect to a Poisson random measure (see (4) below) and by using an interlacing procedure. Different from [14, Proposition 2.1], we only assume that the coefficient functions satisfy the local Lipschitz condition. Also, we remove a key assumption of [12,15,16], which requires that the transition matrix of the switching component is Hölder continuous. In Section 3, we will establish the existence and uniqueness of periodic solutions. To this end, we derive the strong Feller property and irreducibility of the associated time-inhomogeneous semigroups. Our assumptions $Q_0$ and $Q_1$ (see below) are weaker than the corresponding assumptions in [12, (A1)], [16, Assumption 5.1], and [15, Assumption 4.3]. Finally, we will give two examples in Section 4 to demonstrate our main results.

2 Existence and uniqueness of regime-switching jump diffusion processes

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space with filtration $\{\mathcal{F}_t\}_{t \geq 0}$
satisfying the usual conditions (i.e., it is increasing, right continuous and \( F_0 \) contains all \( \mathbb{P} \)-null sets). Suppose \( k, l, m \in \mathbb{N} \) with \( k \geq m \). Denote by \( \mathbb{R}_+ \) the set of all non-negative real numbers. Let \( \{B(t)\}_{t \geq 0} \) be a \( k \)-dimensional standard Brownian motion and \( N \) be an independent Poisson random measure (corresponding to a stationary point process \( p(t) \)) on \( \mathbb{R}_+ \times (\mathbb{R}^l \setminus \{0\}) \). The compensator \( \tilde{N} \) of \( N \) is given by

\[
\tilde{N}(dt, du) = N(dt, du) - \nu(du)dt,
\]

where \( \nu(\cdot) \) is a Lévy measure satisfying

\[
\int_{\mathbb{R} \setminus \{0\}} (1 \wedge |u|^2) \nu(du) < \infty.
\]

Let \( \mathcal{S} = \{1, 2, \ldots \} \).

Throughout this paper, we fix a \( \theta > 0 \). Let \( (X(t), \Lambda(t)) \) be a pair of right continuous processes with left-hand limits on \( \mathbb{R}^m \times \mathcal{S} \). The first component \( X(t) \) satisfies the following SDE:

\[
dX(t) = b(t, X(t), \Lambda(t))dt + \sigma(t, X(t), \Lambda(t))dB(t)
+ \int_{\{ |u| < 1 \}} H(t, X(t-), \Lambda(t-), u) \tilde{N}(dt, du)
+ \int_{\{ |u| \geq 1 \}} G(t, X(t-), \Lambda(t-), u)N(dt, du).
\]  

We assume that the coefficient functions

\[
b(t, x, i) : [0, \infty) \times \mathbb{R}^m \times \mathcal{S} \to \mathbb{R}^m,
\sigma(t, x, i) : [0, \infty) \times \mathbb{R}^m \times \mathcal{S} \to \mathbb{R}^{m \times k},
H(t, x, i, u), G(t, x, i, u) : [0, \infty) \times \mathbb{R}^m \times \mathcal{S} \times \mathbb{R}^l \to \mathbb{R}^m,
\]

are all Borel measurable and satisfy

\[
b(t + \theta, x, i) = b(t, x, i), \quad \sigma(t + \theta, x, i) = \sigma(t, x, i),
H(t + \theta, x, i, u) = H(t, x, i, u), \quad G(t + \theta, x, i, u) = G(t, x, i, u),
\]  

for any \( t \geq 0, x \in \mathbb{R}^m, i \in \mathcal{S}, \) and \( u \in \mathbb{R}^l \setminus \{0\} \). The second component \( \Lambda(t) \) has the state space \( \mathcal{S} \) such that

\[
\mathbb{P}\{\Lambda(t + \Delta) = j \mid \Lambda(t) = i, X(t) = x\} = \begin{cases} q_{ij}(x)\Delta + o(\Delta), & i \neq j, \\ 1 + q_{ij}(x)\Delta + o(\Delta), & i = j, \end{cases}
\]

as \( \Delta \downarrow 0 \). Hereafter, \( q_{ij}(x) \) is a Borel measurable function on \( \mathbb{R}^m \) for \( i, j \in \mathcal{S} \) such that \( q_{ij}(x) \geq 0 \) for all \( i, j \in \mathcal{S} \) with \( i \neq j \) and \( \sum_{j \in \mathcal{S}} q_{ij}(x) = 0 \) for all \( i \in \mathcal{S} \) and \( x \in \mathbb{R}^m \). We make the following assumption on the matrix \( Q = (q_{ij}(x)) \).
Q_0) \quad \sup_{x \in \mathbb{R}^m, i \in \mathcal{I}} \sum_{j \neq i} q_{ij}(x) < \infty.

We point out that \( \Lambda(t) \) can be represented as a stochastic integral with respect to a Poisson random measure. To this end, we construct a family of intervals \( \{\Delta_{ij}(x) : i, j \in \mathcal{I}, i \neq j\} \) on the positive real line as follows: for \( x \in \mathbb{R}^m \) and \( i, j \in \mathcal{I} \) with \( j \neq i \), set

\[
\Delta_{i1}(x) = [0, q_{i1}(x)), \quad \Delta_{i2}(x) = [q_{i1}(x), q_{i1}(x) + q_{i2}(x)), \quad \ldots,
\]

\[
\Delta_{ij}(x) = \left[ \sum_{s=1}^{j-1} q_{is}(x), \sum_{s=1}^{j} q_{is}(x) \right), \quad \ldots.
\]

When \( q_{ij}(x) = 0 \) for \( j \neq i \), we set \( \Delta_{ij}(x) = \emptyset \). Define

\[
L = \sup_{x \in \mathbb{R}^m, i \in \mathcal{I}} \sum_{j \neq i} q_{ij}(x).
\]

We have \( L < \infty \) by the assumption \( Q_0 \) and each value of the interval \( \Delta_{ij}(x) \) is bounded by \( L \).

Define a function

\[
h: \mathbb{R}^m \times \mathcal{I} \times [0, L] \to \mathbb{R},
\]

\[
(x, i, r) \mapsto \sum_{j \in \mathcal{I}} (j - i) 1_{\Delta_{ij}(x)}(r),
\]

that is,

\[
h(x, i, r) = \begin{cases} 
  j - i, & r \in \Delta_{ij}(x), \\
  0, & \text{otherwise}. 
\end{cases}
\]

Then \( \Lambda(t) \) can be modeled by

\[
d\Lambda(t) = \int_{[0,L]} h(X(t-), \Lambda(t-), r) N_1(dt, dr), \quad (4)
\]

where \( N_1(dt, dr) \) is a Poisson random measure (corresponding to a stationary point process \( p_1(t) \) which is adapted to \( \mathcal{F}_t \)) with characteristic measure \( \mathcal{L}(dr) \), the Lebesgue measure on \([0,L]\). The Poisson random measure \( N_1(\cdot, \cdot) \) is assumed to be independent of the Brownian motion \( B(\cdot) \) and the Poisson random measure \( N(\cdot, \cdot) \). Therefore, the processes \( (X(t), \Lambda(t)) \) modeled by (1) and (3) can be thought of as a solution to the hybrid system (1) and (4).

We use \( |x| \) to denote the Euclidean norm of a vector \( x \), use \( A^T \) to denote the transpose of a matrix \( A \), and use \( |A| := \sqrt{\text{tr}(A^T A)} \) to denote the trace norm of \( A \). Let \( C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^m \times \mathcal{I}; \mathbb{R}) \) be the space of all real-valued functions \( f(t, x, i) \) on \( \mathbb{R}_+ \times \mathbb{R}^m \times \mathcal{I} \) which are continuously differentiable with respect to \( t \) and twice continuously differentiable with respect to \( x \). For \( f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^m \times \mathcal{I}; \mathbb{R}) \), define

\[
\mathcal{A} f(t, x, i) := \mathcal{L}_f(t, x, i) + Q(x)f(t, x, \cdot)(i),
\]
Periodic solutions of hybrid jump diffusion processes

\[ L_i f(\cdot, \cdot, i)(t, x) := f_t(t, x, i) + \langle f_x(t, x, i), b(t, x, i) \rangle + \frac{1}{2} \text{tr}(\sigma^T(t, x, i)f_{xx}(t, x, i)\sigma(t, x, i)) + \int_{\{u < 1\}} [f(t, x + H(t, x, i, u), i) - f(t, x, i)] \nu(du) + \int_{\{u \geq 1\}} [f(t, x + G(t, x, u, i), i) - f(t, x, i)] \nu(du) \]

and

\[ Q(x)f(t, x, \cdot)(i) := \sum_{j \in \mathcal{S}} q_{ij}(x)[f(t, x, j) - f(t, x, i)], \]

where

\[ f_t = \frac{\partial f}{\partial t}, \quad f_x = \nabla_x f = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_m} \right), \quad f_{xx} = \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{m \times m}. \]

Define a metric \( d(\cdot, \cdot) \) on \( \mathbb{R}^m \times \mathcal{S} \) by

\[ d((x, i), (y, j)) = |x - y| + |i - j|. \]

A Borel measurable function \( f \) on \([0, \infty)\) is said to be locally integrable, denoted by \( f \in L^1_{\text{loc}}([0, \infty); \mathbb{R}) \), if

\[ \int_0^\tau |f(x)|dx < \infty, \quad \forall \tau > 0. \]

We make the following assumptions.

**A1**) For each \( i \in \mathcal{S}, \)

\[ b(\cdot, 0, i), \sigma(\cdot, 0, i) \in L^2([0, \theta); \mathbb{R}^m), \]

\[ \int_{\{u < 1\}} |H(\cdot, 0, i, u)|^2 \nu(du) \in L^1([0, \theta); \mathbb{R}^m). \tag{5} \]

For each \( n \in \mathbb{N}, \) there exists \( L_n(t) \in L^1([0, \theta); \mathbb{R}_+) \) such that for any \( t \in [0, \theta), \)

\( i \in \mathcal{S}, \) and \( x, y \in \mathbb{R}^m \) with \( |x| \vee |y| \leq n, \)

\[ |b(t, x, i) - b(t, y, i)|^2 \leq L_n(t)|x - y|^2, \]

\[ |\sigma(t, x, i) - \sigma(t, y, i)|^2 \leq L_n(t)|x - y|^2, \]

\[ \int_{\{u < 1\}} |H(t, x, i, u) - H(t, y, i, u)|^2 \nu(du) \leq L_n(t)|x - y|^2. \tag{6} \]
There exist $V_1 \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^m; \mathbb{R}_+)$ and $g_1 \in L^1_{loc}([0, \infty); \mathbb{R})$ such that
\[
\lim_{|x| \to \infty} \inf_{t \in [0, \infty)} V_1(t, x) = \infty,
\]
and for any $t \geq 0$, $i \in \mathcal{I}$, and $x \in \mathbb{R}^m$,
\[
\mathcal{L}_i V_1(t, x) \leq g_1(t).
\]

**Q1** There exists a positive increasing function $f$ on $\mathcal{I}$ satisfying
\[
\lim_{j \to \infty} f(j) = \infty, \quad \sup_{x \in \mathbb{R}^m, i \in \mathcal{I}} \sum_{j \neq i} [f(j) - f(i)] q_{ij}(x) < \infty.
\]

**Theorem 1** Suppose that assumptions $A_1$, $B_1$, $Q_0$, and $Q_1$ hold. Then the hybrid system given by (1) and (4) has a unique strong solution $(X(t), \Lambda(t))$ with initial value $(X(0), \Lambda(0)) = (x, i)$. Moreover, $\mathbb{P}\{T_\infty = \infty\} = 1$, where $T_\infty = \lim_{n \to \infty} T_n$ and $T_n = \inf\{t \geq 0: |X(t)| \vee \Lambda(t) \geq n\}$.

**Proof** Let $(x, i) \in \mathbb{R}^m \times \mathcal{I}$. By [4, Theorem 2.2], under assumptions $A_1$ and $B_1$, there exists a unique non-explosive strong solution $X^{(i)}(t)$ to the following SDE:
\[
dX^{(i)}(t) = b(t, X^{(i)}(t), i)dt + \sigma(t, X^{(i)}(t), i)dB(t)
\]
\[
+ \int_{\{|u|<1\}} H(t, X^{(i)}(t-), i, u) \tilde{N}(dt, du)
\]
\[
+ \int_{\{|u|\geq 1\}} G(t, X^{(i)}(t-), i, u) N(dt, du)
\]
with initial value $X^{(i)}(0) = x$. Let $\sigma_1 < \sigma_2 < \cdots < \sigma_n < \cdots$ be the set of all jump points of the stationary point process $p_1(t)$ corresponding to the Poisson random measure $N_1(dt, dr)$. Since $\mathcal{L}(\cdot)$ is a finite measure on $[0, L]$, $\lim_{n \to \infty} \sigma_n = \infty$ almost surely.

We now construct the processes $(X(t), \Lambda(t))$. Define
\[
(X(t), \Lambda(t)) = (X^{(i)}(t), i), \quad t \in [0, \sigma_1).
\]
Let
\[
\Lambda(\sigma_1) = i + \sum_{j \in \mathcal{I}} (j - i) 1_{\Delta_{ij}(X^{(i)}(\sigma_1-))}(p_1(\sigma_1)).
\]
Define
\[
(X(t), \Lambda(t)) = (X^{(i)}(\sigma_1), \Lambda(\sigma_1)), \quad t = \sigma_1.
\]
Note that (1) is equivalent to (9) on the time interval $[0, \sigma_1)$. Hence, the above defined processes $(X(t), \Lambda(t))$ provide the unique strong solution to the hybrid system (1) and (4) on $[0, \sigma_1)$. 

Let
\[ \tilde{B}(t) = B(t + \sigma_1) - B(t), \quad \tilde{p}(t) = p(t + \sigma_1), \quad \tilde{p}_1(t) = p_1(t + \sigma_1). \]
Set
\[ (\tilde{X}(t), \tilde{\Lambda}(t)) = (X^{(\Lambda(\sigma_1))}(t), \Lambda(\sigma_1)), \quad t \in [0, \sigma_2 - \sigma_1), \]
\[ \tilde{X}(\sigma_2 - \sigma_1) = X^{(\Lambda(\sigma_1))}(\sigma_2 - \sigma_1), \]
\[ \tilde{\Lambda}(\sigma_2 - \sigma_1) = \Lambda(\sigma_1) + \sum_{j \in \mathcal{S}} (j - \Lambda(\sigma_1))1_{\tilde{\Lambda}}(\tilde{p}_1(\sigma_2 - \sigma_1)), \]
where
\[ \tilde{\Lambda}(j) = \Delta_{\Lambda(\sigma_1), j}(X^{(\Lambda(\sigma_1))}((\sigma_2 - \sigma_1)^-)). \]
Then, we define
\[ (X(t), \Lambda(t)) = (\tilde{X}(t - \sigma_1), \tilde{\Lambda}(t - \sigma_1)), \quad t \in [\sigma_1, \sigma_2], \]
which together with (10) and (11) gives the unique strong solution on the time interval \([0, \sigma_2]\). Continuing this procedure inductively, we define \((X(t), \Lambda(t))\) on the time interval \([0, \sigma_n]\) for each \(n\). Therefore, \((X(t), \Lambda(t))\) is the unique strong solution to the hybrid system (1) and (4).

Next, we show that \((X(t), \Lambda(t))\) is non-explosive. Define
\[ \tau_0 = 0, \quad \tau_p = \inf\{t > \tau_{p-1} : \Lambda(t) \neq \Lambda(\tau_{p-1})\}, \quad \tau = \lim_{p \to \infty} \tau_p. \]
Then \(\tau\) is the first instant prior to which \(\Lambda(t)\) has infinitely many switches. Note that \(\{\tau_p\}_{p \geq 1}\) is a subsequence of \(\{\sigma_n\}_{n \geq 1}\) since \(\Lambda(t)\) can have jumps only at the time sequence \(\{\sigma_n\}_{n \geq 1}\). Define
\[ q_i(x) := -q_{ii}(x), \quad \forall \, x \in \mathbb{R}^m, \, i \in \mathcal{S}. \]
Based on the interlacing construction of \((X(t), \Lambda(t))\), we get
\[ \mathbb{P}\{\tau_{p+1} - \tau_p > t\} = \mathbb{E}\left[ \exp \left\{ -\int_0^t q_{\Lambda(\tau_p)}(X^{\Lambda(\tau_p)}(s))ds \right\} \right] \quad (12) \]
and
\[ \mathbb{P}\{\Lambda(\tau_{p+1}) = j \mid \mathcal{F}_{\tau_{p+1}-}\} = \frac{q_{\Lambda(\tau_p), j}(X(\tau_{p+1}-))}{q_{\Lambda(\tau_p)}(X(\tau_{p+1}-))} (1 - \delta_{\Lambda(\tau_p), j})1\{q_{\Lambda(\tau_p)}(X(\tau_{p+1}-)) > 0\} \]
\[ + \delta_{\Lambda(\tau_p), j}1\{q_{\Lambda(\tau_p)}(X(\tau_{p+1}-)) = 0\}. \]
By the assumption \(Q_0\), for any \(i \in \mathcal{S}\), we have
\[ q_i(x) = -q_{ii}(x) = \sum_{j \neq i} q_{ij}(x) \leq L. \]
Then, (12) implies that
\[ \mathbb{P}\{\tau_{p+1} - \tau_p > t\} \geq e^{-Lt}, \quad \forall p \in \mathbb{N}, \ t > 0. \]
Hence,
\[
\mathbb{P}\{\tau_{\infty} = \infty\} \geq \mathbb{P}\{\{\tau_{p+1} - \tau_p > t\} \text{ i.o.}\} \\
= \mathbb{P}\left\{\bigcap_{r=1}^{\infty} \bigcup_{s=r}^{\infty} \{\tau_{s+1} - \tau_s > t\}\right\} \\
= \lim_{r \to \infty} \mathbb{P}\left\{\bigcup_{s=r}^{\infty} \{\tau_{s+1} - \tau_s > t\}\right\} \\
\geq \lim_{r \to \infty} \sup_{t > 0} \mathbb{P}\{\tau_{r+1} - \tau_r > t\} \\
\geq e^{-Lt}.
\]
Letting \( t \downarrow 0 \), we get \( \mathbb{P}\{\tau_{\infty} = \infty\} = 1 \). Therefore, \((X(t), \Lambda(t))\) is the unique strong solution to the hybrid system (1) and (4) with initial value \((x, i)\) for all \( t \in [0, \infty) \).

Finally, we show that \( \mathbb{P}\{T_{\infty} = \infty\} = 1 \). If this is not true, then there exist \( \varepsilon > 0 \) and \( T' \in (0, \infty) \) such that
\[ \mathbb{P}\{T_{\infty} \leq T'\} > 2\varepsilon. \]
Hence, we can find a sufficiently large integer \( n_0 \) such that
\[ \mathbb{P}\{T_n \leq T'\} > \varepsilon, \quad \forall n \geq n_0. \quad (13) \]

Let \( f \) be a function on \( \mathcal{S} \) satisfying \( Q_1 \). Define
\[ V(t, y, k) := V_1(t, y) + f(k), \]
where \( V_1 \) is given in the assumption \( B_1 \). By Itô’s formula and (8), we get
\[
\mathbb{E}_{0,x,i}[V(T' \wedge T_n, X(T' \wedge T_n), \Lambda(T' \wedge T_n))]
\]
\[ = V(0, X(0), \Lambda(0)) + \mathbb{E}_{0,x,i}\left[ \int_0^{T' \wedge T_n} \mathcal{A} V(s, X(s), \Lambda(s))ds \right] \\
= V(0, x, i) + \mathbb{E}_{0,x,i}\left[ \int_0^{T' \wedge T_n} \mathcal{L}_\Lambda(s)V_1(s, X(s))ds \right] \\
+ \mathbb{E}_{0,x,i}\left[ \int_0^{T' \wedge T_n} \sum_{j \in \mathcal{S}} [f(j) - f(\Lambda(s))]|q\Lambda(s),j(X(s))|ds \right] \\
\leq V(0, x, i) + \int_0^{T'} |g_1(s)|ds + T' \sup_{y \in \mathbb{R}^m, i \in \mathcal{S}} \sum_{j \neq i} [f(j) - f(i)]q_{ij}(y).
\]
Set
\[ M := V(0, x, i) + \int_0^{T'} |g_1(s)|ds + T' \sup_{y \in \mathbb{R}^m, i \in \mathcal{S}} \sum_{j \neq i} [f(j) - f(i)]q_{ij}(y). \]
Then $M < \infty$ and
\[ \mathbb{E}_{0,x,i}[1\{T_n \leq T'\}V(T_n, X(T_n), \Lambda(T_n))] \leq M. \] (14)

Define
\[ \mu(n) = \inf\{V(t, y, j): (t, y, j) \in \mathbb{R}_+ \times \mathbb{R}^m \times \mathcal{I}, |y| \vee j \geq n\}. \]

Then $\lim_{n \to \infty} \mu(n) = \infty$ by (7) and our choice of $f$. However, by (13) and (14), it follows that
\[ \varepsilon \mu(n) < \mu(n)\mathbb{P}\{T_n \leq T'\} \leq M. \]

We have arrived at a contradiction. Therefore,
\[ T_\infty = \infty \quad \text{a.s.} \]

The proof is complete. \qed

3 Periodic solutions of regime-switching jump diffusion processes

In this section, we will study the existence and uniqueness of periodic solutions of the hybrid system given by (1) and (4). Denote by $\mathcal{B}(\mathbb{R}^m \times \mathcal{I})$ the Borel $\sigma$-algebra of $\mathbb{R}^m \times \mathcal{I}$ and denote by $\mathcal{B}_b(\mathbb{R}^m \times \mathcal{I})$ (resp., $C_b(\mathbb{R}^m \times \mathcal{I})$) the space of all real-valued bounded Borel functions (resp., continuous and bounded functions) on $\mathbb{R}^m \times \mathcal{I}$. Suppose that $\{Y(t), t \geq 0\}$ is a Markov process on $\mathbb{R}^m \times \mathcal{I}$. We define its transition probability function by
\[ P(s, y, t, A) = \mathbb{P}\{Y(t) \in A | Y(s) = y\}, \quad y \in \mathbb{R}^m \times \mathcal{I}, \; 0 \leq s < t, \]
and define the corresponding Markovian semigroup of linear operators $\{P_{s,t}\}$ on $\mathcal{B}_b(\mathbb{R}^m \times \mathcal{I})$ by
\[ (P_{s,t}f)(y) := \mathbb{E}_{s,y}[f(Y(t))]
\[ := \int_{\mathbb{R}^m \times \mathcal{I}} f(z)P(s, y, t, dz), \quad y \in \mathbb{R}^m \times \mathcal{I}, \; 0 \leq s < t. \]

**Definition 1**

(i) A Markov process $\{Y(t), t \geq 0\}$ on $\mathbb{R}^m \times \mathcal{I}$ is said to be $\theta$-periodic if for any $n \in \mathbb{N}$ and any $0 \leq t_1 < t_2 < \cdots < t_n$, the joint distribution of the random variables $Y(t_1 + k\theta), Y(t_2 + k\theta), \ldots, Y(t_n + k\theta)$ is independent of $k$ for $k \in \mathbb{N} \cup \{0\}$. A Markovian transition semigroup $\{P_{s,t}\}$ is said to be $\theta$-periodic if
\[ P(s, y, t, A) = P(s + \theta, y, t + \theta, A) \]
for any $0 \leq s < t$, $y \in \mathbb{R}^m \times \mathcal{I}$, and $A \in \mathcal{B}(\mathbb{R}^m \times \mathcal{I})$.

(ii) A family of probability measures $\{\mu_s, s \geq 0\}$ is said to be $\theta$-periodic with respect to the Markov semigroup $\{P_{s,t}\}$ of $Y(t)$ if
\[ \mu_s(A) = \int_{\mathbb{R}^m \times \mathcal{I}} P(s, y, s + \theta, A)\mu_s(dy), \quad \forall s \geq 0, A \in \mathcal{B}(\mathbb{R}^m \times \mathcal{I}). \]
A stochastic process \( \{Y(t), t \geq 0\} \) is said to be a \( \theta \)-periodic solution of the system given by (1) and (4) if it is a solution of the system given by (1) and (4) and is \( \theta \)-periodic.

**Definition 2** Let \( 0 \leq s_0 < t_0 < \infty \). A Markovian transition semigroup \( \{P_{s,t}\} \) is said to be regular at \( (s_0, t_0) \) if all transition probability measures \( P(s_0, y, t_0, \cdot) \), \( y \in \mathbb{R}^m \times \mathcal{I} \), are mutually equivalent. \( \{P_{s,t}\} \) is said to be strongly Feller at \( (s_0, t_0) \) if \( P(s_0, y, t_0, A) > 0 \) for any \( y \in \mathbb{R}^m \times \mathcal{I} \) and any nonempty open subset \( A \) of \( \mathbb{R}^m \times \mathcal{I} \). \( \{P_{s,t}\} \) is said to be regular, strongly Feller, irreducible at any \( (s_0, t_0) \), respectively.

### 3.1 Strong Feller and irreducible properties of time-inhomogeneous semigroups

By Theorem 1, under assumptions \( \mathbf{A}_1, \mathbf{B}_1, \mathbf{Q}_0, \) and \( \mathbf{Q}_1 \), there exists a unique strong solution \( (X(t), \Lambda(t)) \) to the hybrid system (1) and (4). By the interlacing structure, one finds that \( (X(t), \Lambda(t)) \) is a Markov process on \( \mathbb{R}^m \times \mathcal{I} \). In this subsection, we will show that the transition semigroup \( \{P_{s,t}\} \) of \( (X(t), \Lambda(t)) \) is strongly Feller and irreducible.

We make the following assumptions.

**A2** For each \( i \in \mathcal{I} \),

\[
b(\cdot, 0, i) \in L^2([0, \theta]; \mathbb{R}^m), \quad \sigma(\cdot, 0, i) \in L^\infty([0, \theta]; \mathbb{R}^m),
\]

and (5) holds. For each \( n \in \mathbb{N} \), there exists \( L_n \in L^\infty([0, \theta]; \mathbb{R}_+) \) such that for any \( t \in [0, \theta] \), \( i \in \mathcal{I} \), and \( x, y \in \mathbb{R}^m \) with \( |x| \vee |y| \leq n \), (6) holds.

**A3** For each \( i \in \mathcal{I} \), \( t \in [0, \theta] \), and \( x \in \mathbb{R}^m \), \( Q(t, x, i) := \sigma(t, x, i)\sigma^T(t, x, i) \) is invertible and

\[
\sup_{|x| \leq n, t \in [0, \theta]} |Q^{-1}(t, x, i)| < \infty, \quad \forall \ n \in \mathbb{N}, i \in \mathcal{I}. \tag{15}
\]

**B2** There exists \( V_2 \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^m; \mathbb{R}_+) \) such that

\[
\lim_{|x| \to \infty} \inf_{t \in [0, \infty)} V_2(t, x) = \infty,
\]

and

\[
\sup_{x \in \mathbb{R}^m, t \in [0, \infty)} \mathcal{L} V_2(t, x) < \infty, \quad \forall \ i \in \mathcal{I}.
\]

Obviously, **A2** implies **A1** and **B2** implies **B1**.

**Theorem 2** Suppose that assumptions **A2, A3, B2, Q0, and Q1** hold. Then the transition semigroup \( \{P_{s,t}\} \) of \( (X(t), \Lambda(t)) \) is strongly Feller.

**Proof** Denote the transition probability function of \( (X(t), \Lambda(t)) \) by

\[
\{P(s, (x, i), t, B \times \{j\}) : 0 \leq s < t, (x, i) \in \mathbb{R}^m \times \mathcal{I}, B \in \mathcal{B}(\mathbb{R}^m), j \in \mathcal{I}\}.
\]
For \((x, i) \in \mathbb{R}^m \times S\), let \(X^{(i)}(t)\) be defined by (9). We kill the process \(X^{(i)}(t)\) with rate \(q_i(\cdot)\) and obtain a subprocess \(\widetilde{X}^{(i)}(t)\) with generator \(\mathcal{L} + q_i\). Then

\[
E[f(\widetilde{X}^{(i)}(t))] = E\left[ f(X^{(i)}(t)) \exp\left\{ \int_0^t q_i(X^{(i)}(u)) du \right\} \right], \quad f \in \mathcal{B}_b(\mathbb{R}^m).
\]

Let \(\widetilde{P}^{(i)}(s, x, \cdot)\) be the transition probability function of \(\widetilde{X}^{(i)}(t)\). Then, for \(0 \leq s < t, B \in \mathcal{B}(\mathbb{R}^m)\), and \(j \in \mathcal{S}\), we obtain by the Markovian property of \((X(t), \Lambda(t))\) and conditioning on the first jump of \((\Lambda(t))\) that

\[
P(s, (x, i), t, B \times \{j\}) = \delta_{ij} \widetilde{P}^{(i)}(s, x, t, B) + \int_s^t \sum_{j_1 \in \mathcal{S} \setminus \{i\}} \int_{\mathbb{R}^m} P(t_1, (x_1, j_1), t, B \times \{j\}) q_{ij_1}(x_1) \widetilde{P}^{(i)}(s, x_1, t_1, dx_1) dt_1.
\]

Repeating this procedure, we get

\[
P(s, (x, i), t, B \times \{j\}) = \delta_{ij} \widetilde{P}^{(i)}(s, x, t, B) + \sum_{k=1}^n \Psi_k + U_n,
\]

where

\[
\Psi_k = \int_{s \leq t < \cdots < t_k < t_0 = i, j_k = j, j_i \in \mathcal{S} \setminus \{j_{i-1}\}, 1 \leq l \leq k} \int_{\mathbb{R}^m} \cdots \int_{\mathbb{R}^m} \widetilde{P}^{(i)}(s, x_1, t_1, dx_1) \cdot q_{ij_1}(x_1) \widetilde{P}^{(j_1)}(t_1, x_1, t_2, dx_2) \cdots q_{j_{k-1}j_k}(x_k) \widetilde{P}^{(j_k)}(t_k, x_k, t, B) dt_1 dt_2 \cdots dt_k
\]

and

\[
U_n = \int_{s < t_1 < \cdots < t_{n+1} < t_0 = i, j_i \in \mathcal{S} \setminus \{j_{i-1}\}, 1 \leq l \leq n+1} \int_{\mathbb{R}^m} \cdots \int_{\mathbb{R}^m} \widetilde{P}^{(i)}(s, x_1, t_1, dx_1) \cdot \cdot q_{ij_1}(x_1) \widetilde{P}^{(j_1)}(t_1, x_1, t_2, dx_2) \cdots q_{j_{n+1}j_{n+1}}(x_{n+1}) \cdot P(t_{n+1}, (x_{n+1}, j_{n+1}), t, B \times \{j\}) dt_1 dt_2 \cdots dt_{n+1}.
\]

By assumption \(Q_0\), we find that \(U_n\) does not exceed \([\left( t - s \right) L]^{n+1}/(n + 1)!\). Hence,

\[
P(s, (x, i), t, B \times \{j\}) = \delta_{ij} \widetilde{P}^{(i)}(s, x, t, B) + \sum_{n=1}^{\infty} \Psi_n.
\]  

(16)

By [4, Theorem 3.8] and assumptions \(A_2, A_3, B_2\), we know that the transition semigroup of \(X^{(i)}(t)\) is strongly Feller. Following the argument of [15, Lemma 4.5], we can show that the transition semigroup of \(\widetilde{X}^{(i)}(t)\) is also strongly Feller. Then, \(\widetilde{P}^{(i)}(s, x, t, B)\) and \(\Psi_n, n \in \mathbb{N}\), are all continuous with respect to \(x\). Note that \(\mathcal{S}\) is equipped with a discrete metric. Then the left-hand side of (16) is
lower semi-continuous with respect to \((x, i)\). Therefore, the transition semigroup \(\{P_{s,t}\}\) of \((X(t), \Lambda(t))\) is strongly Feller by \([9, Proposition \, 6.1.1]\). \(\square\)

Now, we consider the irreducibility of the transition semigroup \(\{P_{s,t}\}\) of \((X(t), \Lambda(t))\). Denote by \(B_{b,loc}(\mathbb{R}_+)\) and \(B_{b,loc}([0, \infty) \times \mathbb{R}^m; \mathbb{R}^m)\) the sets of all locally bounded Borel measurable functions on \(\mathbb{R}_+\) and maps from \([0, \infty) \times \mathbb{R}^m\) to \(\mathbb{R}^m\), respectively. Let \(g\) be a function on \([0, \infty) \times \mathbb{R}^m\). For \(\rho > 0\), we define

\[
g^*(t, x) = g(t, \rho x), \quad t \geq 0, \ x \in \mathbb{R}^m.
\]

We make the following assumptions.

\(B_3\) There exists \(V_3 \in C^{1,2}([0, \infty) \times \mathbb{R}^m; \mathbb{R}_+)\) satisfying the following conditions:

(i) \[
\lim_{|x| \to \infty} \inf_{t \in [0,\infty)} V_3(t, x) = \infty.
\]

(ii) For any \(\rho \geq 1\), there exist \(q_0 \in B_{b,loc}(\mathbb{R}_+)\) and \(W_\rho(t, x) \in B_{b,loc}([0, \infty) \times \mathbb{R}^m; \mathbb{R}^m)\) satisfying for each \(n \in \mathbb{N}\), there exists \(R_n \in L^1_{loc}([0, \infty); \mathbb{R}_+)\) such that for any \(t \in [0, \infty)\) and \(x, y \in \mathbb{R}^m\) with \(|x| \vee |y| \leq n\),

\[
|W_\rho(t, x) - W_\rho(t, y)|^2 \leq R_n(t)|x - y|^2,
\]

and for \(t \geq 0, \ x \in \mathbb{R}^m, \) and \(i \in \mathcal{I}\),

\[
\mathcal{L}_i V_3^*(t, x) \leq q_\rho(t), \quad V_3^*(t, x) \leq \langle W_\rho, \nabla_x V_3^*(t, x) \rangle.
\]

\(Q_2\) For any distinct \(i, j \in \mathcal{I}\), there exist \(j_1, j_2, \ldots, j_r \in \mathcal{I}\) with \(j_p \neq j_{p+1}\), \(j_1 = i\) and \(j_r = j\) such that the set \(\{x \in \mathbb{R}^m: q_{j_pj_{p+1}}(x) > 0\}\) has positive Lebesgue measure for \(p = 1, 2, \ldots, r - 1\).

**Theorem 3** Suppose that assumptions \(A_1, A_3, B_1, B_3, Q_0, Q_1,\) and \(Q_2\) hold. Then the transition semigroup \(\{P_{s,t}\}\) of \((X(t), \Lambda(t))\) is irreducible.

**Proof** Denote the transition probability function of \((X(t), \Lambda(t))\) by \(\{P(s, (x, i), t, B \times \{j\}): 0 \leq s < t, (x, i) \in \mathbb{R}^m \times \mathcal{I}, B \in \mathcal{B}(\mathbb{R}^m), j \in \mathcal{I}\}\). As shown in the proof of Theorem 2, the assumption \(Q_0\) implies that (16) holds. By \([4, Theorem \, 3.3.9]\), we know that for any \(i \in \mathcal{I}\), the transition semigroup of \(X^{(i)}(t)\) is irreducible under assumptions \(A_1, A_3, B_1,\) and \(B_3\). Further, we find that the subprocess \(X^{(i)}(t)\) is irreducible by the assumption \(Q_0\). This together with the assumption \(Q_2\) implies that the right-hand side of (16) is positive whenever \(B\) is a nonempty open set of \(\mathbb{R}^m\). Then, \(P(s, (x, i), t, B \times \{j\}) > 0\). Since \(B\) is arbitrary, the transition semigroup \(\{P_{s,t}\}\) of \((X(t), \Lambda(t))\) is irreducible. \(\square\)

### 3.2 Existence and uniqueness of periodic solutions

To prove the existence and uniqueness of periodic solutions to the hybrid system (1) and (4), we make the following assumptions.
P) There exists \( \tilde{V} \in C^{1,2}([0, \infty) \times \mathbb{R}^m \times \mathcal{S}; \mathbb{R}_+) \) satisfying the following conditions:

\[
\lim_{|x| \to +\infty} \inf_{t \in [0, \infty)} \tilde{V}(t, x, i) = \infty, \tag{17}
\]

\[
\lim_{|x| \to +\infty} \sup_{t \in [0, \infty)} \mathcal{A}\tilde{V}(t, x, i) = -\infty, \tag{18}
\]

and

\[
\sup_{x \in \mathbb{R}^m, i \in \mathcal{S}, t \in [0, \infty)} \mathcal{A}\tilde{V}(t, x, i) < \infty. \tag{19}
\]

B) There exists \( V \in C^{1,2}([0, \infty) \times \mathbb{R}^m; \mathbb{R}_+) \) satisfying the following conditions:

(i) \( \lim_{|x| \to \infty} \inf_{t \in [0, \infty)} V(t, x) = \infty, \)

(ii) for any \( \rho \geq 1 \), there exist \( q_0 \in B_{b, loc}(\mathbb{R}_+) \) and \( W_{\rho}(t, x) \in B_{b, loc}([0, \infty) \times \mathbb{R}^m; \mathbb{R}^m) \) satisfying for each \( n \in \mathbb{N} \), there exists \( R_n \in L_{loc}^1([0, \infty); \mathbb{R}_+) \) such that for any \( t \in [0, \infty) \) and \( x, y \in \mathbb{R}^m \) with \( |x| \lor |y| \leq n \),

\[
|W_{\rho}(t, x) - W_{\rho}(t, y)|^2 \leq R_n(t)|x - y|^2,
\]

\[
\sup_{x \in \mathbb{R}^m, i \in \mathcal{S}, t \in [0, \infty)} \mathcal{L}_i V^\rho(t, x) < \infty,
\]

and

\[
V^\rho(t, x) \leq \langle W_{\rho}, \nabla_x V^\rho(t, x) \rangle, \quad \forall t \geq 0, x \in \mathbb{R}^m.
\]

Obviously, B implies \( B_1 \sim B_3 \).

**Theorem 4** Suppose that assumptions \( A_2, A_3, B, P, Q_0, Q_1, \) and \( Q_2 \) hold. Then we have the following statements.

(i) The hybrid system given by (1) and (4) has a unique \( \theta \)-periodic solution \( (X(t), \Lambda(t)) \).

(ii) The Markovian transition semigroup \( \{P_{s,t}\} \) of \( (X(t), \Lambda(t)) \) is strongly Feller and irreducible.

(iii) Let \( \mu_s(A) = \mathbb{P}((X(t), \Lambda(t)) \in A) \) for \( A \in \mathcal{B}(\mathbb{R}^m \times \mathcal{S}) \) and \( s \geq 0 \). Then, for any \( s \geq 0 \) and \( \varphi \in L^2(\mathbb{R}^m \times \mathcal{S}; \mu_s) \), we have

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} P_{s,s+i\theta} \varphi = \int_{\mathbb{R}^m \times \mathcal{S}} \varphi d\mu_s \quad \text{in} \ L^2(\mathbb{R}^m \times \mathcal{S}; \mu_s).
\]

**Proof** Let \( (X(t), \Lambda(t)) \) be the unique strong solution to the hybrid system given by (1) and (4) with initial value \( (x, i) \in \mathbb{R}^m \times \mathcal{S} \). For \( n \in \mathbb{N} \), we define the stopping time \( T_n \) by

\[
T_n = \inf\{t \in [0, \infty): |X(t)| \lor \Lambda(t) \geq n\}.
\]
For $t \geq 0$, by Itô’s formula, we get

\[
\mathbb{E}[\tilde{V}(t \wedge T_n, X(t \wedge T_n), \Lambda(t \wedge T_n))]
= \mathbb{E}[\tilde{V}(0, X(0), \Lambda(0))] + \mathbb{E}\left[\int_{0}^{t \wedge T_n} \mathcal{A}\tilde{V}(u, X(u), \Lambda(u))du\right].
\]  

(20)

Define

\[
A_n := -\sup_{|y|+k>n, t\in[0,\infty)} \mathcal{A}\tilde{V}(t, y, k).
\]

By (18), we get

\[
\lim_{n \to \infty} A_n = \infty.
\]

(21)

We have

\[
\mathcal{A}\tilde{V}(u, X(u), \Lambda(u)) \leq -1\{|X(u)|+k\geq n\}A_n + \sup_{|y|+k<n, u\in[0,\infty)} \mathcal{A}\tilde{V}(u, y, k).
\]

Then, there exist positive constants $c_1$ and $c_2$ such that for large $n$,

\[
\mathbb{E}\left[\int_{0}^{t \wedge T_n} 1\{|X(u)|+k\geq n\}du\right] \leq \frac{c_1 t + c_2}{A_n}.
\]

(22)

Denote

\[
B_n = \{(y, k) \in \mathbb{R}^m \times \mathcal{S} : |y| < n\},
\]

\[
B^c_n = \{(y, k) \in \mathbb{R}^m \times \mathcal{S} : |y| + k \geq n\}.
\]

Letting $n \to \infty$ in (22), we obtain by (21) that

\[
\lim_{n \to \infty} \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} P(0, (x, i), u, B^c_n)du = 0.
\]

(23)

By (19), there exists $\lambda > 0$ such that

\[
\mathcal{A}\tilde{V}(t, x, i) \leq \lambda, \quad \forall t \geq 0, (x, i) \in \mathbb{R}^m \times \mathcal{S}.
\]

(24)

By (20) and (24), we get

\[
\mathbb{E}[\tilde{V}(t, X(t), \Lambda(t))] \leq \lambda t + \tilde{V}(0, x, i).
\]

Together with Chebyshev’s inequality, this implies that

\[
P(0, (x, i), t, B^c_n) \leq \frac{\lambda t + \tilde{V}(0, x, i)}{\inf_{|y|+k>n, t\in[0,\infty)} \tilde{V}(t, y, k)}.
\]

(25)

By (17) and (25), we find that there exists a sequence of positive integers $\gamma_n \uparrow \infty$ such that

\[
\lim_{n \to \infty} \sup_{(x, i) \in B_{\gamma_n}, t\in(0,\theta)} P(0, (x, i), t, B^c_n) = 0.
\]

(26)
For \( t \geq 0 \), \( C \in \mathcal{B}(\mathbb{R}^l \setminus \{0\}) \), and \( C_1 \in \mathcal{B}((0, L]) \), define
\[
\overline{B}(t) = B(t + \theta) - B(\theta)
\]
and
\[
\overline{N}(t, C) = N(t + \theta, C) - N(\theta, C), \quad \overline{N}_1(t, C_1) = N_1(t + \theta, C_1) - N_1(\theta, C_1).
\]
Then, by (1), (2), and (4), we obtain
\[
dX(t + \theta) = b(t, X(t + \theta), \Lambda(t + \theta))dt + \sigma(t, X(t + \theta), \Lambda(t + \theta))d\overline{B}(t)
+ \int_{\{|u|<1\}} H(t, X(t + \theta -), \Lambda(t + \theta -), u)\overline{N}(dt, du)
+ \int_{\{|u|\geq 1\}} G(t, X(t + \theta -), \Lambda(t + \theta -), u)\overline{N}(dt, du)
\]
and
\[
d\Lambda(t + \theta) = \int_{[0,L]} h(X(t + \theta -), \Lambda(t + \theta -), r)\overline{N}_1(dt, dr).
\]
Hence, \((X(t + \theta), \Lambda(t + \theta))\) is a weak solution of the hybrid system given by (1) and (4). From the weak uniqueness of solutions, we know that \((X(t), \Lambda(t))\) and \((X(t + \theta), \Lambda(t + \theta))\) have the same distribution. Thus,
\[
P(s, x, t, A) = P(s + \theta, x, t + \theta, A), \quad \forall s \in [0, t), \; x \in \mathbb{R}^m, \; A \in \mathcal{B}(\mathbb{R}^m \times \mathcal{F}).
\]
Combining the periodicity of the transition semigroup \(\{P_{s,t}\}\) with (23), (26), and [6, Theorem 3.2, Remark 3.1], we conclude that the hybrid system given by (1) and (4) has a \(\theta\)-periodic solution. Here, we would like to call the reader’s attention to a missing condition in [6, Theorem 3.2], which was pointed out by Hu and Xu recently. According to Hu and Xu [5, Theorem 2.1, Remark A.1], [6, Theorem 3.2, Remark 3.1] holds under the additional assumption that \(\{P_{s,t}\}\) is a Feller semigroup. By Theorem 2, \(\{P_{s,t}\}\) is a strong Feller semigroup and hence we can apply [6, Theorem 3.2, Remark 3.1] to show that the hybrid system given by (1) and (4) has a \(\theta\)-periodic solution.

By using the same argument, we can show that [4, Lemmas 3.12, 3.13] hold with the state space \(\mathbb{R}^m\) replaced by \(\mathbb{R}^m \times \mathcal{F}\). Hence, the uniqueness of the \(\theta\)-periodic solution is a direct consequence of Theorems 2 and 3. Finally, the last assertion of the theorem can be proved by following the same argument of the proof of [4, Lemma 3.13].

4 Examples

In this section, we use two examples to illustrate our main results.

Example 1  Stochastic Lorenz equation with regime switching.
We consider the following stochastic Lorenz equation ([7]) with regime switching:

\[
\begin{align*}
    dX_1(t) &= [-\alpha(t, \Lambda(t)) X_1(t) + \alpha(t, \Lambda(t)) X_2(t)]dt + \sum_{j=1}^{3} \sigma_{1j}(t, X(t), \Lambda(t)) dB_j(t) \\
    &+ \int_{\{|u|<1\}} H_1(t, X(-), \Lambda(-), u) \tilde{N}(dt, du) \\
    &+ \int_{\{|u|\geq 1\}} G_1(t, X(-), \Lambda(-), u) N(dt, du), \\
    dX_2(t) &= [\mu(t, \Lambda(t)) X_1(t) - X_2(t) - X_1(t) X_3(t)]dt \\
    &+ \sum_{j=1}^{3} \sigma_{2j}(t, X(t), \Lambda(t)) dB_j(t) \\
    &+ \int_{\{|u|<1\}} H_2(t, X(-), \Lambda(-), u) \tilde{N}(dt, du) \\
    &+ \int_{\{|u|\geq 1\}} G_2(t, X(-), \Lambda(-), u) N(dt, du), \\
    dX_3(t) &= [-\beta(t, \Lambda(t)) X_3(t) + X_1(t) X_2(t)]dt + \sum_{j=1}^{3} \sigma_{3j}(t, X(t), \Lambda(t)) dB_j(t) \\
    &+ \int_{\{|u|<1\}} H_3(t, X(-), \Lambda(-), u) \tilde{N}(dt, du) \\
    &+ \int_{\{|u|\geq 1\}} G_3(t, X(-), \Lambda(-), u) N(dt, du),
\end{align*}
\]

(27)

where \( \Lambda(t) \) takes values in \( \mathcal{S} = \{1, 2, \ldots \} \) and is generated by \( Q = (q_{ij}(x)) \) satisfying the following condition.

There exist \( k \in \mathbb{N} \) and \( M > 0 \) such that

\[
q_{ij}(x) = \begin{cases} 
0, & |i - j| > k, \\
\in (0, M], & 0 < |i - j| \leq k,
\end{cases}
\]

and

\[
\inf_{x \in \mathbb{R}^3, i > k, i - k \leq j < i} \{q_{ij}(x)\} > \sup_{x \in \mathbb{R}^3, i > k, i + k \leq j} \{q_{ij}(x)\}.
\]

(28) (29)

Suppose that

\[
\begin{align*}
\alpha(t, i), \beta(t, i), \mu(t, i) &: [0, \infty) \times \mathcal{S} \to \mathbb{R}_+, \\
\sigma(t, x, i) &: [0, \infty) \times \mathbb{R}^3 \times \mathcal{S} \to \mathbb{R}^{3 \times 3}, \\
H(t, x, i, u), G(t, x, i, u) &: [0, \infty) \times \mathbb{R}^3 \times \mathcal{S} \times \mathbb{R}^l \to \mathbb{R}^3,
\end{align*}
\]
are all Borel measurable and periodic with respect to \( t \) with period \( \theta \). We assume that \( \sigma(t, x, i) \) and \( H(t, x, i, u) \) satisfy (6) and (15), and there exist \( \gamma > 0 \) and a continuously differentiable periodic function \( a(t) \) with period \( \theta \) such that for any \( t \in [0, \theta), \, i \in \mathcal{I} \),

\[
\gamma < \alpha(t, i), \, \beta(t, i), \, \mu(t, i) \leq a(t).
\]  

(30)

Moreover, for any \( \varepsilon > 0 \), there exists \( c_\varepsilon > 0 \) such that for any \( i \in \mathcal{I} \),

\[
|\sigma(t, x, i)|^2 + \int_{\{u < 1\}} |H(t, x, i, u)|^2 \nu(du) + \int_{\{|u| \geq 1\}} |G(t, x, i, u)|^2 \nu(du) \\
\leq \varepsilon|x|^2 + c_\varepsilon.
\]

Define

\[
V(t, x) = x_1^2 + x_2^2 + (x_3 - 2a(t))^2.
\]

We will show that \( V \) satisfies the assumption \( \text{B} \). Without loss of generality, we assume that \( \rho = 1 \). The verification for the case that \( \rho > 1 \) is completely similar. Let

\[
W(t, x) = \frac{1}{2} (x_1, x_2, x_3 - 2a(t)).
\]

Then

\[
V(t, x) = \langle W, V_x(t, x) \rangle.
\]

For \( \varepsilon > 0 \) and \( i \in \mathcal{I} \), we obtain by (30) that

\[
\mathcal{L}_i V(t, x) \leq 4(2a(t) - x_3)a'(t) - 2\alpha(t, i)x_1^2 - 2x_2^2 - 2\beta(t, i)[x_3^2 - 2a(t)x_3] \\
+ |\sigma(t, x, i)|^2 + \sum_{j=1}^3 \int_{\{u < 1\}} |H_j(t, x, i, u)|^2 \nu(du) \\
+ \sum_{j=1}^2 \int_{\{|u| \geq 1\}} [(x_j + G_j(t, x, i, u))^2 - x_j^2] \nu(du) \\
+ \int_{\{|u| \geq 1\}} [(x_3 + G_3(t, x, u) - 2a(t))^2 - (x_3 - 2a(t))^2] \nu(du) \\
\leq 4(2a(t) - |x_3|)|a'(t)| - 2[\gamma x_1^2 + x_2^2 + \gamma x_3^2 - 2(a(t))^2|x_3]| + \varepsilon|x|^2 \\
+ c_\varepsilon + 2|x_1| + |x_2| + |x_3| + 2a(t)[\nu(\{|u| \geq 1\})]^{1/2}(\varepsilon|x|^2 + c_\varepsilon)^{1/2}.
\]

Therefore,

\[
\lim_{n \to \infty} \sup_{|x| > n, \, i \in \mathcal{I}, \, t \in [0, \infty)} \mathcal{L}_i V(t, x) = -\infty, \quad \sup_{x \in \mathbb{R}^m, \, i \in \mathcal{I}, \, t \in [0, \infty)} \mathcal{L}_i V(t, x) < \infty.
\]

Thus, the assumption \( \text{B} \) is satisfied.

By (28) and setting \( f(j) = j \), we know that assumptions \( \text{Q}_0, \text{Q}_1, \) and \( \text{Q}_2 \) hold. Define

\[
\tilde{V}(t, x, i) = V(t, x) + i^2.
\]
Note that
\[ \tilde{V}(t, x, i) = \mathcal{L}_i V(t, x) + \sum_{j \neq i} (j^2 - i^2) q_{ij}(x), \]
and (29) implies that for \( i > k, \)
\[
\sum_{j \neq i} (j^2 - i^2) q_{ij}(x) \leq - \sum_{j = i-k}^{i-1} (i^2 - j^2) \inf_{x \in \mathbb{R}^3, i > k, i-k \leq j < i} \{q_{ij}(x)\}
+ \sum_{j = i+1}^{i+k} (j^2 - i^2) \sup_{x \in \mathbb{R}^3, i > k, i < j \leq i+k} \{q_{ij}(x)\}
= - \sum_{p=1}^{k} (2ip - p^2) \inf_{x \in \mathbb{R}^3, i > k, i-k \leq j < i} \{q_{ij}(x)\}
+ \sum_{p=1}^{k} (2ip + p^2) \sup_{x \in \mathbb{R}^3, i > k, i < j \leq i+k} \{q_{ij}(x)\}
\rightarrow - \infty, \quad i \rightarrow \infty.
\]
Hence, the assumption \( P \) holds. Thus, all assumptions of Theorem 4 are satisfied. Therefore, the SDE (27) has a unique \( \theta \)-periodic solution \((X(t), \Lambda(t))\) and assertions (ii) and (iii) of Theorem 4 hold.

**Example 2** Stochastic equation of the lemniscate of Bernoulli with regime switching.

In this example, we consider the stochastic equation of the lemniscate of Bernoulli, which generalizes [1, Example 3.20] to the non-autonomous case with Lévy noise and regime-switching.

For \( x = (x_1, x_2) \in \mathbb{R}^2, \) define
\[
I(x) = (x_1^2 + x_2^2)^2 - 4(x_1^2 - x_2^2).
\]
Let
\[
\mathcal{V}(I) = \frac{I^2}{2(1 + I^2)^{3/4}}, \quad \mathcal{H}(I) = \frac{I}{(1 + I^2)^{3/8}}.
\]
Consider the vector field
\[
b(x) = -\left[ \mathcal{V}_x(I) + \left( \frac{\partial \mathcal{H}(I)}{\partial x_2}, -\frac{\partial \mathcal{H}(I)}{\partial x_1} \right)^T \right].
\]
We have
\[
f(I) := \frac{d\mathcal{V}(I)}{dI} = \frac{I(I^2 + 4)}{4(1 + I^2)^{7/4}}, \quad g(I) := \frac{d\mathcal{H}(I)}{dI} = \frac{I^2 + 4}{4(1 + I^2)^{11/4}},
\]
\[
\frac{\partial I}{\partial x_1} = 4x_1(x_1^2 + x_2^2) - 8x_1, \quad \frac{\partial I}{\partial x_2} = 4x_2(x_1^2 + x_2^2) + 8x_2.
\]
Then
\[ \mathcal{V}_t(I) = f(I) \left( \frac{\partial I}{\partial x_1}, \frac{\partial I}{\partial x_2} \right)^T, \]
and
\[
\begin{align*}
b_1(x) &= -f(I)(4x_1(x_1^2 + x_2^2) - 8x_1) - g(I)(4x_2(x_1^2 + x_2^2) + 8x_2), \\
b_2(x) &= -f(I)(4x_2(x_1^2 + x_2^2) + 8x_2) - g(I)(-4x_1(x_1^2 + x_2^2) + 8x_1).
\end{align*}
\]

We consider the following SDEs:
\[
\begin{align*}
dX_1(t) &= b_1(X(t))dt + \sigma_{11}(t, X(t), \Lambda(t))dB_1(t) + \sigma_{12}(t, X(t), \Lambda(t))dB_2(t) \\
&\quad + \int_{\{u < 1\}} H_1(t, X(t), \Lambda(t), u)\tilde{N}(dt, du) \\
&\quad + \int_{\{|u| \geq 1\}} G_1(t, X(t), \Lambda(t), u)N(dt, du), \\
dX_2(t) &= b_2(X(t))dt + \sigma_{21}(t, X(t), \Lambda(t))dB_1(t) + \sigma_{22}(t, X(t), \Lambda(t))dB_2(t) \\
&\quad + \int_{\{u < 1\}} H_2(t, X(t), \Lambda(t), u)\tilde{N}(dt, du) \\
&\quad + \int_{\{|u| \geq 1\}} G_2(t, X(t), \Lambda(t), u)N(dt, du),
\end{align*}
\]
where \( \Lambda(t) \) takes values in \( \mathcal{S} = \{1, 2, \ldots\} \) and is generated by \( Q(x) = (q_{ij}(x)) \) satisfying
\[
0 < \inf_{x \in \mathbb{R}^2, j \neq i} \{j^{1+\delta}q_{ij}(x)\} < \sup_{x \in \mathbb{R}^2, j \neq i} \{j^{1+\delta}q_{ij}(x)\} < \infty \quad \text{for some } \delta > 0, \quad (32)
\]
and
\[
\begin{align*}
\sigma(t, x, i) &\colon [0, \infty) \times \mathbb{R}^2 \times \mathcal{S} \to \mathbb{R}^{2 \times 2}, \\
H(t, x, i, u), G(t, x, i, u) &\colon [0, \infty) \times \mathbb{R}^2 \times \mathcal{S} \times \mathbb{R}^l \to \mathbb{R}^2,
\end{align*}
\]
are all Borel measurable and periodic with respect to \( t \) with period \( \theta \). We assume that \( \sigma(t, x, i) \) and \( H(t, x, i, u) \) satisfy (6) and (15). Moreover, for any \( \varepsilon > 0 \), there exists \( c_\varepsilon > 0 \) such that for any \( i \in \mathcal{S} \),
\[
\begin{align*}
|\sigma(t, x, i)|^2 + \int_{\{|u| < 1\}} |H(t, x, i, u)|^2 \nu(du) \\
&\quad + |x| \int_{\{|u| \geq 1\}} |G(t, x, i, u)| \nu(du) + \int_{\{|u| \geq 1\}} |G(t, x, i, u)|^2 \nu(du) \\
&\leq \varepsilon |x|^2 + c_\varepsilon.
\end{align*}
\]

By (32) and setting \( f(j) = j^{\delta/2} \), we know that assumptions \( Q_0, Q_1, \) and \( Q_2 \) hold. Define
\[
V(t, x) = \mathcal{V}(I(x)).
\]
Following the argument of [4, Example 4.3], we can check that the assumption \(B\) is satisfied. Define
\[
\tilde{V}(t, x, i) = V(t, x) + i^{\delta/2}.
\]
Note that
\[
\mathcal{A}\tilde{V}(t, x, i) = \mathcal{L}V(t, x) + \sum_{j \neq i} (j^{\delta/2} - i^{\delta/2})q_{ij}(x),
\]
and (32) implies that
\[
\sum_{j \neq i} (j^{\delta/2} - i^{\delta/2})q_{ij}(x) \leq \sup_{x \in \mathbb{R}^2, j \neq i} \left\{ j^{1+\delta} q_{ij}(x) \right\} \sum_{j=1}^{\infty} \frac{1}{j^{1+\frac{\delta}{2}}} - i^{\delta/2} \inf_{x \in \mathbb{R}^2} \sum_{j \neq i} q_{ij}(x)
\]
\[
\to -\infty, \quad i \to \infty.
\]
Hence, the assumption \(P\) holds. Thus, all assumptions of Theorem 4 are satisfied. Therefore, the SDE (31) has a unique \(\theta\)-periodic solution \((X(t), \Lambda(t))\) and assertions (ii) and (iii) of Theorem 4 hold.

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