The Promise and Limitations of Precision Gravity: Application to the Interior Structure of Uranus and Neptune

Naor Movshovitz and Jonathan J. Fortney
Department of Astronomy and Astrophysics, University of California, Santa Cruz, CA 95064 USA; nmovshov@ucsc.edu
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Abstract

We study the constraining power of a high-precision measurement of the gravity field for Uranus and Neptune, as could be delivered by a low-periapse orbiter. Our study is practical, assessing the possible deliverables and limitations of such a mission with respect to the structure of the planets. Our study is also academic, assessing in a general way the relative importance of the low-order gravity, high-order gravity, rotation rate, and moment of inertia (MOI) in constraining planetary structure. We attempt to explore all possible interior density structures of a planet that are consistent with hypothetical gravity data via MCMC sampling of parameterized density profiles. When the gravity field is poorly known, as it is today, uncertainties in the rotation rate on the order of 10 minutes are unimportant, as they are interchangeable with uncertainties in the gravity coefficients. By the same token, when the gravity field is precisely determined, the rotation rate must be known to comparable precision. When gravity and rotation are well known, the MOI becomes well constrained, limiting the usefulness of independent MOI determinations unless they are extraordinarily precise. For Uranus and Neptune, density profiles can be well constrained. However, the nonuniqueness of the relative roles of H/He, watery volatiles, and rock in the deep interior will still persist with high-precision gravity data. Nevertheless, the locations and magnitudes (in pressure space) of any large-scale composition gradient regions can likely be identified, offering a crucially better picture of the interiors of Uranus or Neptune.

Unified Astronomy Thesaurus concepts: Uranus (1751); Neptune (1096); Planetary interior (1248)

1. Introduction

The gravity field of a giant planet is perhaps our best window into its interior structure and composition. Obtaining gravity data is a simple task, though. Rough estimates can be deduced with ground-based observations of natural satellites, but precise determinations must come from tracking radio signals from probes sent to the outer solar system to orbit, or at least fly by, these worlds. Turning the hard-won gravity data into inferences about a planet’s interior is also not a straightforward task. There is no simple inversion; the usual process involves creating models of the planet’s interior and deducing features of the planet we are interested in, for example, the existence and size of a heavy-element core, based on how well the calculated gravity of model planets with such features matches the observed gravity of the real planet.

The past decade has seen the Juno mission to Jupiter and the Cassini Grand Finale orbits at Saturn revolutionize our understanding of these planets, as the precision of the gravity fields of these giant planets has improved by 2 orders of magnitude (Bolton et al. 2017; Iess et al. 2018, 2019; Durante et al. 2020). The bulk compositions of these planets are dominated by hydrogen and helium, and the typical questions involve the mass and distribution of heavy elements within the planets to better understand structure, formation, and evolution. Since these data have revolutionized our views of Jupiter and Saturn (Stevenson 2020; Mankovich & Fuller 2021), it is a natural question to wonder how similar data may be able to alter our views of Uranus and Neptune. A variety of orbiter mission concepts have been put forward with comprehensive science cases, including improving our view of the interior structure of these planets (Hofstadter et al. 2019; Fletcher et al. 2020; Rymer et al. 2021).

The advances in giant planet gravity field data have necessitated advances in modeling efforts, first in terms of new methods of ultraprecise gravity field calculations (Hubbard 2012, 2013; Nettelmann 2017; Nettelmann et al. 2021). A second effort has been in reexamining the assumptions that go into models. What may not be well appreciated are the assumptions that underlie assumptions that go into models, making virtually every inference about a giant planet’s structure strongly model-dependent.

1.1. The Role of Gravity in Interior Models

Physical interior models try to create a self-consistent description of the composition, density, pressure, and temperature at every point inside the planet. Gravity enters the problem from the requirement of hydrostatic equilibrium, connecting the pressure $p$ (really its gradient) and density $\rho$ at every point. Hydrostatic equilibrium also allows us to use a one-dimensional structure, $\rho(s)$ and $p(s)$, where $s$ is the mean (volume-equivalent) radius of a surface of constant density and pressure. Once a self-consistent $\rho(s)$ is determined, it can be integrated to yield a total planetary mass, radius, and external gravity field that can be compared with observed values to judge the model’s overall likelihood. For Uranus and Neptune, a recent reference in the physical picture is Nettelmann et al. (2013).
The gravitational potential exterior to the planet, $V_e(r)$, is described by the expansion

$$V_e(r) = -\frac{GM}{r} \left(1 - \sum_{n=1}^{\infty} \left(a_0/r\right)^{2n} J_{2n}(\cos \theta)\right),$$

and the connection with the interior mass distribution is in the coefficients

$$J_n = -\frac{1}{Ma_0^2} \int \rho(r')(r')^n P_n(\cos \theta') dr'.$$

In this expansion, $M$ is the planet’s mass; $a_0$ is a normalizing constant with dimensions of length, commonly a whole number of kilometers close to the planet’s equatorial radius; and $P_n$ are the Legendre polynomials. The integrals are carried over the volume of the planet.

Equation (1) depends on the colatitude angle $\theta$ but not azimuth and is also strictly north–south symmetric (even $n$ only). Clearly, this is an approximate, partial description of the full gravity field. Furthermore, the integrals (Equation (2)) require knowledge of the equilibrium shape of the model planet. Methods for deriving this shape (Section 2.3) rely on equilibrium between gravitational and centrifugal potential, which greatly restricts the rotation state under consideration. In fact, rigid rotation at a constant rate $\omega$ is almost always assumed. Real planets, on the other hand, exhibit more complex gravity. Nonzero odd gravity harmonics ($J_{3,5,7,9}$) have been measured for Jupiter (Iess et al. 2018) and used to infer the depth of the observed asymmetric surface flow (Kaspi et al. 2018). And in Saturn, evidence for strong differential rotation was found not in the odd $J$s but rather in the unexpected magnitude of even harmonics higher than $J_6$ (Galanti et al. 2019; Iess et al. 2019).

There is no doubt that such nonhydrostatic effects can be expected to exist in Uranus and Neptune too. Nevertheless, it is appropriate to focus on the simpler gravity field with the implied uniform rotation, as these are the most important for determining the bulk structure of the interior. We discuss nonuniform rotation again in Section 2.5, but in the rest of the text, “gravity” implies the zonal, north–south symmetric gravity of Equation (1).

1.2. The Role of the Equation of State

Pressure and density are also related via a thermodynamic equation of state (EOS). A self-consistent solution is found by iteration such that the value of the pressure satisfying hydrostatic equilibrium, which depends on gravity, matches everywhere the value given by the EOS. The EOS requires knowledge at every point of temperature and composition. Temperature can be calculated by equations of heat transfer, perhaps with input from cooling models that follow a long-term evolution of the planet. In detailed structure models, a prescribed, static temperature structure (usually adiabatic) or, equivalently, entropy structure is used. Composition cannot be calculated by an equation; it must be stipulated.

And that fact regarding composition is a limitation; the model must assume the very thing it is supposed to infer. To be sure, there are some very good assumptions that one can make. For example, if the target planet is a gas giant, we may assume that, in a large fraction of the volume of the interior, the dominant species is a mix of hydrogen and helium, although the relative proportion of hydrogen to helium should be allowed to change with depth, given helium phase separation (Stevenson & Salpeter 1977; Mankovich & Fortney 2020).

In the case of the ice giants, on the other hand, the dominant species is not so easy to guess. Relatively recent models within the physical picture (Nettelmann et al. 2013) suggest important roles for H/He, water and other volatiles, as well as rock/iron. Some models even suggest more rock than volatiles (Teanby et al. 2020). Some of the most important questions about planet formation (that are perhaps the main motivation for the model in the first place) depend most strongly on the inferred content, composition, and distribution of heavier elements, with this “metals” mass fraction referred to as $Z(s)$. Recent reviews can be found in Helld et al. (2020) and Helld & Fortney (2020). Whereas for Jupiter and Saturn, the outcome of the inferred $Z(s)$ is not overly sensitive to exactly which heavy element (i.e., volatiles versus rocks) one uses in the model, the same is not true for Uranus and Neptune.

However, such a hypothetical model is already much too complex. To resolve the interior to a meaningful degree requires discretizing the continuous variables on a fine grid in $s$ with at least hundreds and preferably thousands of grid points, resulting in thousands of model parameters and an impracticable task. Some very strong simplifying assumptions are needed. The most important one is the assumption of some kind of layering.

The most common class of models for both the gas giants and ice giants has for a long time been the three-layer model. The planet is assumed to consist of radial regions, each of homogeneous composition. This assumption is typically made for computational expediency and potentially out of physical reasoning, the latter being that thermodynamics may support, under some conditions, the existence of fully convective regions, with boundaries between them that resist mixing (Bailey & Stevenson 2021). Of course, that thermodynamics may support such configurations is no proof that these are the only possible ones. Modelers have been gradually increasing the sophistication of layered-composition models, for example, by adding layers of composition gradients, and will no doubt continue to do so, while still retaining the basic paradigm. Figure 1 shows what density

![Figure 1. Density profiles $\rho(s)$ of the three-layer models of Nettelmann et al. (2013), replotted from their data. Here $R_m$ is the mean (volume-equivalent) radius of a surface of constant density, which by hydrostatic equilibrium must also be a surface of constant pressure and potential; $s_0$ is the mean radius of the 1 bar surface.](image-url)
profiles deriving from such models of Uranus and Neptune (Nettelmann et al. 2013) may look like; the three-layer structure is clearly visible.

1.3. Composition-agnostic Models

An alternative approach that has been used is to create so-called empirical models. Rather than parameterize the planet’s composition and then solve for the pressure and density structure, these models parameterize the structure directly and make inferences about the composition from the resulting models (Helie et al. 2009, 2011a; Movshovitz et al. 2020). For example, some models may assign a synthetic (and simple) pressure–density relation to one or more regions of the planet. A popular choice is a combination of one or more polytropic regions, where \( p(s) \propto s^n \) for polytropic index \( n \), and perhaps a region of constant density approximating a core (Neuenschwander et al. 2021). A synthetic pressure–density relation can be used just like a physical EOS to derive self-consistent equilibrium shape and density profiles. A three-layer structure like those shown in Figure 1 can be approximated by using three different polytropes in three radial regions.

The density profile can also be parameterized directly, and this is our preferred approach here. A parametric mathematical representation of a curve is chosen, mapping a vector of parameter values \( s \in \mathbb{R}^n \) (hopefully \( n \) is not too large) to a density profile \( \rho(s) \) and thereby, through hydrostatic equilibrium, to a self-consistent interior structure. The advantage of being entirely divorced from an assumed EOS, real or synthetic, is that density profiles representing both layered composition and continuous composition can be represented with the same model. The disadvantage is that without an EOS, real or synthetic, to guide us, it is harder to know how to interpret the resulting density profile. Our work builds on pioneering work in this area, of “random” interior models of Uranus and Neptune (Marley et al. 1995; Podolak et al. 2000), that were motivated by an exploration to move past the three-layer model and explore the widest possible range of density distributions allowed by the gravity field.

1.4. Nonuniqueness of Models

Regardless of how a model planet is constructed, with a physical EOS, a synthetic EOS, or a synthetic \( \rho(s) \), there remains the problem of degeneracy of solutions. The problem is simply that multiple models can be constructed that all match the measured gravity field to within a specified uncertainty. In fact, there are infinitely many such models, covering an unknown region of parameter space. When modelers claim to constrain some desired property of the planet, for example, if they say that the heavy-element content of the planet is found to be between some upper and lower bounds, what they mean is that, given their chosen model design, no solutions were found with heavy-element content outside those bounds that match the gravity measurement to within some tolerance. It is typically not known how much of this constraint is due to the measured or observed properties of the planet and how much is due to the assumptions built into the model or even incomplete exploration of the model parameter space.

That is the main deficiency we aim to address in this work. Focusing on Uranus and Neptune as the target planets, we choose a parameterization designed for maximum flexibility and a sampling method designed for maximum coverage and generate representative random samples of \( \rho(s) \) from several different distributions, each constrained by a different combination of observable planetary properties. Concretely, we constrain the \( \rho(s) \) distributions by applying progressively stronger information as follows: (1) the planet’s mass and radius only; (2) mass, radius, and rotation period; (3) adding the planet’s \( J_2 \) and \( J_4 \) gravity coefficients, with uncertainty varying from that of currently available data to precision limited only by computational considerations; and (4) with nominal values of gravity coefficients of increasing order (up to \( J_{12} \)), hypothetically, as if known to great precision.

The difference between distributions of density profiles obtained with the different constraints illustrates the “constraining power” of the different observables.

Our focus on Neptune and Uranus here is motivated by a number of factors. Compared with the gas giants, Jupiter and Saturn, the vital observables of the ice giants, including and especially their gravity fields and rotation periods, are much less well known. A future mission to these worlds with one of the goals being to obtain more precise measurements of their gravity fields would be extremely valuable; this is not in doubt. What is still unknown, however, is what would be the expected relationship between the precision of such measurements and our ability to translate them into better constraints of the interior structure of Uranus and Neptune. Our work here provides motivation for such missions and also a framework to understand their limitations, such that we can better assess whether and how a precision gravity field could revolutionize our view of these planets.

We explain in detail the methods of this experiment in Section 2. Theoretical results are illustrated in Section 3. Results specific to Uranus and Neptune, demonstrating the difference between models constrained with currently available data for these planets and data improved by a plausible mission scenario, are shown in Section 4. Our conclusions are summarized in Section 5.

2. Experimental Method

2.1. Overview

In order to evaluate the potential of precise measurements of the gravity fields to improve our knowledge of the interiors of the ice giants, we generate samples of interior density profiles that are constrained by successively higher-order and/or more precisely known gravity coefficients. By comparing the range of density values and variety of density profiles attained by each sample, we get an idea of the “constraining power” of gravity as a measured observable.

We start with a sample of density profiles guided by knowledge of the mass and radius of the planet only, to use as a baseline for comparison. The oblate shape of the rotating planet is calculated because it affects the mass, but the associated gravity field is not expected to match the observation. Even so, the space of allowed \( \rho(s) \) curves is already constrained by basic physics. Trivially, \( \rho(s) \) must be monotonic strictly decreasing with \( s \) and its integral from the center to the surface should match the planet’s mass within the observational uncertainty. Additionally, the relationship between pressure and density in hydrostatic equilibrium implies \( \lim_{s \to 0^+} d\rho/ds = 0 \). By themselves, these conditions only restrict the shape of the \( \rho(s) \) curve, not its scale, but we can constrain the scale by putting
reasonable limits on the density at both the surface and the center of the planet.

On the surface, which in our models we take to mean the 1 bar surface, we get a rough estimate of density by applying the ideal gas law to a protosolar mix of hydrogen and helium (mean molecular weight 2.319 amu) at the observed temperature, $T_1$ bar. This leads to a nominal value of $\rho_1$ bar, $u_1 = 0.367$ kg m$^{-3}$ for Uranus and $\rho_1$ bar, $u_1 = 0.387$ kg m$^{-3}$ for Neptune. The 1 bar temperature itself is not a direct measurement. It is inferred from analysis of radio occultation data from the Voyager 2 mission (Lindal 1992) and includes an uncertainty of several percent. And, of course, the atmospheric composition is unknown. Combining the uncertainties from temperature and composition (mean molecular weight), we should allow some “play” in the 1 bar surface density, treating it as a sampled variable with an appropriate prior. However, in preliminary samples, we found that the resulting $\Delta \rho_1$ bar is too small to have any effect on the sample, so we keep $\rho_1$ bar fixed in all models to speed up the sampling process.

The density at the center of the planet is, of course, not directly available. We can estimate a reasonable upper bound value by considering the order of magnitude of the central pressure. We set the upper limit for both planets at 20,000 kg m$^{-3}$, comfortably above the density of pure rock at 50 Mbar.

The rotation period of the planets is also not precisely known (Jacobson 2009, 2014; Podolak and Helled 2012; Nettelmann et al. 2013). For a given density profile, changing the rotation period will change the equilibrium shape and, as the equatorial radius is held fixed, the mass and gravity. We therefore include the rotation period as one of the sampled parameters with an estimated prior, allowing the Markov Chain Monte Carlo (MCMC) procedure to sample from the space of density profiles and rotation period simultaneously. The effect of the rotation period uncertainty is discussed further in Section 3.

To summarize, the baseline sample for each planet is drawn from the space of one-dimensional profiles restricted to monotonically decreasing $\rho(s)$ with vanishing $d\rho/ds$ at the center and satisfying $\rho_{\text{sun}} = \rho_1$ bar and $\rho(s) \leq \rho_{\text{max}}$ everywhere. The prior includes information about the imperfectly known rotation period, and the likelihood function driving the sampling procedure compares the integral of the density profile with the planet’s reference mass $M$. More details about the exact sampling procedure are given in the Appendix. The values used for these baseline models are summarized in Table 1.

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### Table 1

| Uranus | Neptune |
|---|---|
| $G(10^{-11}$ m$^3$ kg$^{-1}$ s$^{-2}$) | 6.67430(15) | 6.67430(15) |
| $M(10^{24}$ kg) | 86.8127(40) | 102.4126(48) |
| $a_0$ (km) | 23.59 | 24.76 |
| $T_1$ bar (K) | 76(2) | 72(2) |
| $P$ (s) | 62.064(600) | 57.996(600) |
| $J_2$ ($\times 10^6$) | 3510.7(7) | 3536.5(47) |
| $J_3$ ($\times 10^6$) | -34.2(13) | -36.0(31) |

Note. Reference values from Tiesinga et al. (2021, 2018 CODATA) and http://ssd.jpl.nasa.gov, analysis of Voyager data from Lindal (1992), rotation period from Podolak & Helled (2012), and $J_n$ values from Jacobson (2009, 2014) renormalized to a fixed value of $a_0$.

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We then proceed to constrain the interior density profiles further by using additional information about the planet’s gravity field. Currently, an approximate determination of the gravity fields of Neptune and Uranus comes from a combination of ground-based astrometry and radio data from both Voyager missions (Jacobson 2009, 2014, and references therein). The gravity harmonics $J_2$ and $J_4$ are available, with significant uncertainty, for both planets. We use the nominal values from Table 1 to center the multidimensional Gaussian we use in the sampling likelihood function,

$$D_j^2 = \sum_{\text{even } n} \left( J_n - J_n^{\text{obs}} \right)^2 / \sigma_n^2.$$

But for the $\sigma_n$ uncertainties, we are less interested in the precision of the Voyager missions’ data; our question is about the theoretical constraining power of increasingly precise gravity. We assign, in different samples, $\sigma_n J_n = 10^{-\mu_n}$ with $\mu_n$ ranging from 2 to 6. Concretely, we obtained samples where $n_{\text{max}}$ was 2, 4, 6, and 12. For $n_{\text{max}} > 4$, we do not have measured nominal values for the Gaussian centers; we use the mean values from the sample with $n_{\text{max}} = 4$.

Comparing samples obtained with $n_{\text{max}} = 2$ but with varying values of uncertainty, $\sigma_n$, will show the expected benefit of more precise measurements of the low-order coefficients. Comparing samples obtained with increasing $n_{\text{max}}$ and fixed $\sigma_n$ will show the expected benefit of measuring higher-order coefficients, even crudely. Comparing samples obtained with the same $n_{\text{max}}$ and $\sigma_n$ but different priors set on rotation period will show the expected benefit of a more precise determination of rotation. In a real mission, of course, these are not distinct, independent measurements. Nevertheless, knowing which of the above options provides the most benefit in terms of constraining models of interior structure may help optimize the mission design.

Before we look at the resulting sample distributions, we need to define the density profile parameterization used in this work.

### 2.2. Parameterization of the Density Profiles

Consider the density profiles of the three-layer Uranus and Neptune models of Nettelmann et al. (2013), shown in Figure 1. These profiles are derived from models that solve the planetary structure equations for an assumed layered composition using physical EOSs. Our goal is to find a suitable direct parameterization of density as a function of radius, $\rho(s)$, that is capable of capturing the profiles of Figure 1 as a subset and is otherwise as flexible as possible. It is also important to keep the number of parameters small and even more important to minimize the parameter correlations. The more parameters we use and the stronger the correlations between them, the longer it will take any sampling algorithm to adequately cover the sample space.

With these requirements in mind, what are the important features of the curves in Figure 1 that we need to allow for? Each $\rho(s)$ curve is smooth and monotonic, except for two sharp discontinuities. In the physical models, these discontinuities are “built in”; the models assume layers of homogeneous composition with sharp boundaries between them. Three-layer models thus have two density discontinuities, the implicit assumption being that regions of composition gradients of length scale below the model’s resolution separate the layers. The smoothness of the $\rho(s)$ curve between these density
“jumps” suggests that the entire profile can be well represented by a piecewise polynomial function. Such a parameterization was used successfully in Movshovitz et al. (2020) to generate density profiles for Saturn, but here we utilize an alternative parameterization that is superior to the piecewise polynomial in two significant ways, both having to do with representing the density jumps.

There are two important limitations to representing $\rho(s)$ with piecewise polynomials. The first and more obvious is that this parameterization only allows for sharp density discontinuities, not gradual ones. There are two mathematical discontinuities (of the first kind) in every density profile, with parameters controlling their location, say, fractions $z_1$ and $z_2$ of the planet’s radius, and jump magnitude, say, $\delta_1$ and $\delta_2$ in density units.\(^5\) The jumps can merge ($z_1 \rightarrow z_2$), or one or both may vanish ($\delta_1 \rightarrow 0$), but they cannot approximate a more gradual transition, a gradient region detectable on the scale of the model. These sharp density jumps are features of layered-composition models, and we want our parameterization to be able to reproduce them, but we would like it to have the flexibility to capture gradient regions as well, something that composition-based models have difficulty with.

The second limitation of the piecewise polynomial parameterization is a technical one that becomes apparent when we try to use MCMC algorithms to sample from the parameters’ joint posterior. It turns out that the parameters are very strongly correlated, leading to an impractically long convergence time. One way to mitigate this problem is to fix the values of the parameters $z_1$ and $z_2$, obtain their marginal distributions by sampling the other parameter values under their conditional probability, and repeat the process for a range of reasonable values for $z_i$. A sample from the full posterior can be created by drawing from the marginal probabilities in proportion to their relative likelihoods. While this approach provides a working method, it is more cumbersome and time consuming. Worse, the additional step of combining the marginals into a single posterior involves the difficult task of calculating, at least approximately, the posterior odds ratio (also called the Bayes factor, the evidence integral, or, simply, the evidence). While this is a common and well-studied task, it still has no generally agreed upon best method (Nelson et al. 2020).

Our alternative parameterization represents $\rho(s)$ with a single continuous and continuously differentiable function:

$$
\rho(z) = \rho(s/R_m) = \sum_{n=2}^{8} a_n(z^n - 1) + \rho_0 \\
+ \sum_{n=1}^{2} \sigma_n \left( \frac{\pi}{2} + \text{arctan}(\nu_n(z - z_n)) \right). \tag{4}
$$

The first line is a degree-8 polynomial in $z = s/R_m$. It is constrained to have a vanishing derivative at $z = 0$ and pass through the point $(1, \rho_0)$. Since we reference our $J$ values to the 1 bar surface, we take $\rho_0 = \rho_{1 \text{ bar}}$ to be the 1 bar density.

The second line in Equation (4) is a sigmoid function parameterization of possible density jumps overlaid on the polynomial. In this version of the parameterization, we allow up to two such jumps. The parameter $z_d$ defines the location (in normalized radius) of the center of the inner of these. A density increase of $\sigma_1$ (in density units) is applied asymptotically and similarly around this point, the width being controlled by the nondimensional sharpness parameter $\nu_1$. The jump can be made arbitrarily sharp, to resemble a discontinuity like the ones in Figure 1, by increasing the value of $\nu_1$. Conversely, small values of $\nu_1$ result in a smooth, gradual density increase, indistinguishable from the background polynomial.

The location, scale, and sharpness of the outer density jump are similarly determined by $z_2$, $\sigma_2$, and $\nu_2$, respectively. An example is illustrated in Figure 2, using parameter values tailored to approximate the shape of the Uranus model of Nettelmann et al. (2013).

2.3. Solving for Equilibrium Shape and Gravity

There are several methods of calculating $J_n$ for a model planet. They vary in theoretical precision and, more importantly, the precision they achieve in practice when working with manageable resolution (Hubbard et al. 2014; Wisdom & Hubbard 2016; Nettelmann 2017; Debras & Chabrier 2017; Nettelmann et al. 2021). In this work, we use the theory of figures (ToF) algorithm, taken to fourth order (ToF4) where possible and seventh order (ToF7) where necessary.\(^5\) We estimate the practical precision of this method when calculating the gravity of a given density profile (as opposed to a given density–pressure relation) with a given rotation period to have a fractional error $\epsilon_n$ for $n = 10^{-6}$, $10^{-5}$, $10^{-4}$, $10^{-3}$, $10^{-2}$, $10^{-1}$ for $n = 2, 4, 6, 8, 10$, and $12$, respectively. This estimate is derived by comparison with the extremely precise benchmark $n = 1$ polytrope gravity solutions of Wisdom & Hubbard (2016); they are the residual differences that remain when the ToF7 algorithm is applied to very high resolution models. We then choose the resolution of our models (the number, $N$, of discretized density levels) such that a doubling of $N$ results in a relative difference to the resulting $J_n$ of less than a part per million. We find that $N = 4098$ is always sufficient, and for

\(^5\) For reasons of efficiency, one may choose to use some one-to-one transformation of these parameters (e.g., Movshovitz et al. 2020, Appendix B), but their physical meaning remains.

\(^5\) Simply because ToF7 is slower to run than ToF4.
consistency, we use this resolution in all models in every sample, even when high precision is not required.

2.4. Sampling

As given in Equation (4), the number of parameters required to completely define the density profile is 13 (since we take $\rho_0$ to be constant). A degree-8 polynomial with two boundary conditions takes seven parameters, and two potential density jumps take three parameters each (location, scale, and sharpness). Of course, this polynomial-plus-sigmoid parameterization can be made even more flexible by increasing the polynomial degree and/or adding more sigmoid terms. But the advantages of added flexibility must be carefully weighed against the cost of increasing the dimensionality of the sample space.

As it is, sampling from the 13-dimensional parameter space is a computationally expensive operation, although much less so than with the piecewise polynomial parameterization. The advantage comes not from the number of parameters, which is similar in both, but from the weaker parameter correlations. The polynomial-plus-sigmoid parameters are less strongly correlated, meaning a small change in one parameter while keeping the other parameters fixed results in a smaller overall change in the density profile. This significantly improves the behavior of the sampling algorithm. Even so, ensuring a large enough draw of independent samples with adequate coverage of the parameter space is not straightforward. We utilize the ensemble sampler algorithm of Goodman & Weare (2010) implemented in the emcee package (Foreman-Mackey et al. 2013), but we find it necessary to employ a “tempering” procedure, whereby sampling initially follows a modified loss function with arbitrarily lower sensitivity to the parameter values to facilitate the algorithm moving through parameter space, followed by sampling with the full loss function. Repeated application of this procedure results in a set of independent draws from what is hopefully the static but unknown posterior.

The full details of the sampling procedure are given in the Appendix.

2.5. Some Caveats

The methodology described above was designed to minimize the impact of implicit assumptions and model limitations. Nevertheless, some necessary assumptions remain. Perhaps the strongest of these is the assumption, made throughout, of rigid rotation, in which every part of the planet is assumed to be moving at a single angular rotation rate $\omega$. Many planets, including Uranus and Neptune, exhibit signs of nonrigid rotation, but this does not always invalidate the rigid rotation models. Surface phenomena that involve only a small fraction of a planet’s mass would have a negligible effect on the gravity field. But deeper-rooted latitude- and depth-dependent motion will sufficiently change the equilibrium state at the time of measurement so that the observed gravity field’s coefficients are a mix of the rigid equilibrium gravity and nonrigid wind perturbations:

$$J_n^{\text{obs}} = J_n^{\text{rigid}} + J_n^{\text{wind}}.$$  \hspace{2cm} (5)

We say perturbation, but for high-order $J_{n>6}$, the nonrigid component (also referred to as the dynamic component) may be significant or even dominant. Models that assume rigid rotation to calculate $J_n$ from a given density profile should be made to match $J_n^{\text{rigid}} = J_n^{\text{obs}} - J_n^{\text{wind}}$ instead of $J_n^{\text{obs}}$. Thus, getting a solid handle on the nonrigid rotation’s contribution to the gravity field—at least its expected magnitude, if not a nominal correction—should be considered a prerequisite to using $J_n^{\text{obs}}$ to constrain interior models.

On both Uranus and Neptune, fast east-to-west atmospheric streams are observed at cloud level, the crucial question being the depth of these winds. Analysis of the difference between $J_n^{\text{obs}}$ and the $J_n^{\text{rigid}}$ of relatively simple interior models suggests that these are relatively shallow jets involving a small fraction of the planets’ mass (Kaspi et al. 2013), but the possibility remains that a better characterization of the gravity field will require addressing the correction due to dynamics. Work on this problem continues (Kaspi et al. 2017, 2018; Galanti & Kaspi 2017; less et al. 2018), and it is clear that without a good understanding of dynamic effects, the usefulness of high-order gravity measurements would be much reduced.

A much more minor limitation comes from the use of composition-agnostic parameterization, which can theoretically generate $\rho(s)$ profiles that would be “unphysical” for one reason or another, including characteristics that are difficult to predict or detect. It is not impossible, for example, that some density profile exists in the sample that, if used with realistic EOSs to back out a temperature profile, would imply unrealistic temperature gradients somewhere in the interior. Any additional restriction of the parameterization that would guard against such features would be useful in further constraining the allowable models but would necessarily rely on imperfectly known EOSs and/or additional model assumptions.

Conversely, the specific functional form (Equation (4)), in addition to possibly being too permissive physically, may not be flexible enough mathematically. Certainly, it is not a complete basis for the space of all functions in $(0,1)$ or even for the more restricted space of functions representing a valid density profile. Just how much of that space may be “missing” is difficult to analyze. In Section 3.1, by examining a baseline sample, we offer some reassurance that the chosen functional form can indeed capture a very wide variety of density profiles.

Finally, any work that makes use of MCMC sampling must acknowledge that convergence to the correct static probability distribution is never assured. We describe our sampling procedure and our reasons for accepting the resulting samples as correct in the Appendix.

3. Constraining Power of Gravity

3.1. A Baseline Sample

We begin by looking at the baseline sample. We will look at Uranus samples first because we find it more instructive to examine different views of the same sample side by side, rather than compare the same view for both planets. Figure 3 shows three different views of the baseline Uranus sample. The left panel shows what we call the envelope view, a shaded area covering, for each radius, roughly a spread in density values obtained there by the sample. This view is helpful in providing, at a glance, a sense of the overall extent of the density values reachable under the relevant constraint. Recall that for the baseline sample, the only constraints were that each density profile integrates to the correct mass, with the boundary conditions $\rho(1) = \rho_0$ and $\rho(0) \leq \rho^\text{max}$ and the Gaussian prior set on the rotation period (Table 1).
The middle panel of Figure 3 is what we call the ensemble view. A subset of 20 density profiles from the sample is shown, where we attempted to pick ones that show the variety of possible density profiles reachable under the given constraint. This view is more helpful for seeing features like the location and scale of density jumps and identifying regions of possible layering.

Lastly, the right panel is a histogram of the moment of inertia (MOI) values from the sample. As an integrated, scalar value, it serves as a quantitative measure of the "width" of the distribution.

The distribution of density profiles evident in the baseline sample is not surprising and does not contain much information about Uranus in particular or planetary interiors in general. Figure 3 is nevertheless important because, first, every other sample will be compared against the baseline in order to gauge the effectiveness of the employed constraints, and second, it serves to validate the combination of the chosen parameterization scheme and sampling procedure. Recall that the goal was to have a parameterization flexible enough and a sampling procedure robust enough that any restriction of the resulting density profiles evident in the sample can be attributed to the constraints that were deliberately used. The baseline sample was expected to, and in fact did, cover essentially the entire range of plausible interior profiles. From the middle and, especially, the right panel it is clear that both nearly "flat" profiles and very centrally condensed ones are reachable with this sampling method. Even if we cannot guarantee the exact shape of the probability distribution (it is possible, for example, that regions of apparent low probability would fill up gradually if much larger samples were drawn; see also the Appendix), it is at least clear that the parameterization and/or sampling do not by themselves restrict the resulting samples.

3.2. Rotation Period Priors

A more precise determination of the rotation by itself is not a helpful constraint. In Figure 4, the baseline sample (rotation period prior is Gaussian with $\sigma = 10$ min; Podolak and Helled 2012) is shown again, overlaid with two more samples. One was obtained assuming a much higher uncertainty in the rotation period (Gaussian prior, $\sigma = 30$ min), and the second was obtained with a perfectly known rotation period: the relevant parameter kept constant instead of being sampled. It is perhaps surprising that these samples are essentially identical; it is surprising that the rotation period can change by as much as 1 hr with apparently no effect on the shape of the allowable density profiles. This is because density and rotation are in a sense interchangeable, when the only observable used to constrain the sample is total mass. While faster rotation increases the planet’s oblateness, which would result in a smaller mass for the same density profile (the outer mean radius is decreased, since the equatorial radius is fixed), this is easily compensated for by a small increase in density in the upper layers. It is possible that much stricter interior boundary conditions, i.e., much smaller $r_{\text{max}}$ and applying a central $r_{\text{min}}$, would reveal the limits of the allowable rotation period. We did not test this because there is no realistic prospect of independently estimating the central density.
We stress that even though the rotation period by itself is not a helpful constraint, it will become important when high-precision gravity is considered in Section 3.5.

### 3.3. Low-order Gravity

Adding information about the gravity field, even just a crudely estimated \( J_2 \) value, significantly shrinks the extent of the allowable density profiles. Looking at Figure 5, we can deduce two important, if unsurprising, lessons about the constraining power of gravity and a third that was perhaps not as obvious a priori. The first is that gravity is most sensitive to the density structure at the outer regions of the planet, gradually becoming less sensitive the deeper we look, and almost indifferent to the density near the center of the planet. This behavior is well understood and easily predicted, at least qualitatively, from the definition of the gravity coefficients as integrals over the radius. Nevertheless, we find it instructive to see a direct, quantitative illustration. This is most clearly seen in the left panel of Figure 5, while the middle and right panels better illustrate the second unsurprising fact: that gravity restricts the shape of the density better than its overall extent. For example, even the most restrictive sample, the one assuming very precisely known \( J_2 \) and \( J_4 \) (blue dashed-dotted lines in the figure), allows the central density to reach values almost as low or high as the less restrictive samples or even the baseline sample. But there are many ways in which a \( \rho(s) \) curve can reach those values, and some of these curves appear in the less restrictive samples and are notably missing from the last one. In particular, large and sharp density increases at large \( 0.3 \lesssim s / R_m \lesssim 0.6 \) radii seem to be disfavored or even disallowed by this nominal value of \( J_4 \), thereby telling us, in this case, that Uranus is unlikely to have a huge rocky core.

The last and least predictable lesson gleaned from Figure 5 is that, loosely speaking, it is better to know more \( J \) values than to know the same \( J \) value more precisely. This is probably best seen in the right panel, as well as in other samples, not shown here, with different combinations of \( J_2 \) and \( \sigma_{J_2} \). While not exactly surprising, it is nevertheless not obvious why, for example, a 4 order of magnitude improvement in the precision of the \( J_2 \) value would, by itself, amount to only a modest shrinking of the allowed range of densities, density profiles, or MOI values.

### 3.4. High-order Gravity

There are at present no usable estimates of actual \( J_n \) values for either planet for \( n > 4 \). To continue investigating the potential constraining power of high-precision, high-order gravity measurement, we need to assign some hypothetical yet reasonable values to \( J_n \) and \( \sigma_{J_n} \).

For nominal \( J_n \) values, the only sensible choice is to use the mean values from the previous samples, specifically from the most constraining one obtained so far (blue dashed-dotted lines in Figure 5). Only the \( J_2 \) and \( J_4 \) values were used in the likelihood function driving the sampling algorithm, but all values of \( J_n \) are available after the fact. Figure 6 reminds us that the distributions of \( J_n \) values for different \( n \) are highly correlated, yet there is a range of allowable values that could be further restricted.

The usable precision of gravity data, \( \sigma_{J_n} \), is limited by the worst of three factors. The first is the expected accuracy of determination by a hypothetical future orbiter mission, should it become available. For this, we can look to the recent successes of the radio science teams of the Cassini and Juno missions in reconstructing the gravity fields of Saturn and Jupiter, respectively, from the spacecraft accelerations deduced from Doppler shifts detected in the radio link between the spacecraft and tracking stations on Earth. We may take the exquisite precision obtained during those missions as a best-case bound of what might be expected from a future Uranus/Neptune mission. But recall that this truly impressive precision applies just to \( J_2^{\text{obs}} \) and not to \( J_2^{\text{true}} \) (Equation (5)). The second factor, therefore, is the expected accuracy of a correction term, \( J_2^{\text{wind}} \). It is hard to speculate what that correction would look like, although it is safe to assume it will come with an attached uncertainty greater than the formal \( \sigma_{J_2}^{\text{obs}} \).

The last limiting factor is the precision with which we are able to calculate the equilibrium shape and gravity of a model planet. It is unusual for a numeric computation to rival observational data in terms of poorer accuracy, but this is in fact the case here. For the high-order gravity samples, we solve for equilibrium shape and gravity using the ToF7 method (Section 2.3), which gives us usable values for up to \( J_{12} \) and the associated \( \sigma_{J_n} \) for the likelihood function.

With this framework, Figure 7 shows the distributions from samples constrained by progressively higher-order gravity up to \( J_{12} \). Again, we see that high-precision gravity provides an excellent constraint on the density overall but cannot, by itself, pin down the central density or even the existence of a distinct central region of high density.

The MOI factor seems to be very tightly correlated with the gravity as soon as we go beyond \( J_2 \). While correlation is expected, both gravity and MOI being essentially different integrals of the same \( \rho(s) \), it has been suggested that an
independent measurement of the MOI can provide new information not already contained in the gravity field (Helled et al. 2011b; Helled 2011). It may be true that knowledge of $J_n$ to any order and with full precision is equivalent to knowledge of $\rho(s)$ and therefore of every other quantity derived from it. Whether or not this is strictly true in a precise, mathematical sense is not all that relevant. The relevant question is how much variability is still possible in the MOI value once gravity is measured to realistic order and with realistic precision. Our answer, if this sample is representative, is about 0.1% (but see below for the importance of rotation period uncertainty).

3.5. Rotation and Gravity

A planet’s rotation period may be estimated by a variety of methods (e.g., Anderson & Schubert 2007; Gurnett et al. 2007; Read et al. 2009; Helled et al. 2015; Mankovich et al. 2019, and references therein) and with varying degrees of precision. In an example of a best-case scenario, Jupiter’s deep interior rotation is tied to the precession of the polar axis of its strong magnetic field, which can be measured to subsecond precision (Seidelmann et al. 2007). In other cases, we are not so lucky; the rotation periods of Uranus and Neptune are estimated with an uncertainty of at least 10 minutes (in each direction; Podolak & Helled 2012; Helled et al. 2010) and perhaps much higher.

In Section 3.2, we conclude that shrinking the uncertainty in the rotation period by itself does little to further constrain the space of allowable density profiles. However, the same is not true when gravity is also considered. Figure 8 shows a sample obtained assuming a precisely known rotation period. Compare this with Figure 7. The constraining power of gravity is significantly enhanced by precise knowledge of rotation. Or, said another way, ignoring uncertainty in rotation may lead to unjustifiably tight constraints being deduced from models.

Assuming a precisely known rotation period\footnote{Actually, most models, including ours, prefer to fix a dimensionless rotation parameter such as $m = \omega s_0/GM$ where $\omega = 2\pi/P$ and $s_0$ is the 1 bar level surface mean radius. This is almost equivalent to fixing the rotation period $P$ itself, but as the equilibrium shape and with it $s_0$ are allowed to change, one cannot keep the mass, equatorial radius, rotation period, and rotation parameter all fixed simultaneously. Normally benign, this subtlety can cause confusion when directly comparing models derived by different groups.} is a common,
though not universal, simplification made by modelers. We caution that such a simplifying assumption should be carefully justified or, better yet, avoided.

As is evident from the right panel of Figure 8, when both gravity and rotation period are known to high precision, the remaining variation in MOI value all but disappears (sample $\sigma/\mu \approx 10^{-6}$). An independent measurement of the MOI would have to be extremely precise if it is expected to distinguish between different models already fitting the other constraints.

### 3.6. Pressure–Density Relation

The density profile, $\rho(s)$, was the focus of our attention because it is the quantity that directly determines the gravity field and is therefore directly inferred by its measurement. But the density itself is not really what we are most interested in. What we would like to know, ultimately, is what the planet is made of, its composition, and how the various molecular species are distributed inside the planet.

Gravity by itself can never give us this information without additional information and/or assumptions. But it can get us a step closer by noting that the condition of hydrostatic equilibrium defines a one-to-one relationship between density and pressure once the gravity field is known. If, everywhere in the planet, the weight of a layer of fluid is exactly balanced by the force due to a pressure gradient, $\nabla p$, then

$$\nabla p = -\rho \nabla U$$

(6)

everywhere, where $U$ is the total potential, gravitational plus centrifugal. Note that $U$ must be known in the interior of the planet, while the measured $J_n$ values only relate to the external potential. Luckily, the process of calculating the external potential from a given $\rho(s)$ and rotation period also furnishes the potential on interior level surfaces as a useful by-product. We can numerically integrate Equation (6) starting from the 1 bar reference level and obtain $p(s)$ and therefore $\rho(p)$ on every level surface.

The pressure–density relation, also called a barotrope, is not quite enough to uniquely relate to composition (a true EOS would require the temperature profile as well), but it can already help by setting some bounds. Figure 9 shows the distribution envelope of barotropes integrated from the Uranus density profiles of Figures 7 and 8. The left panel corresponds to samples obtained with our conservative, $\sigma = 10$ min, prior on rotation period, and the right panel corresponds to samples obtained with a precisely known rotation period. Adiabats of several compositions, computed with the SCvH and ANEOS EOSs (Thompson 1990; Saumon et al. 1995), are overlaid for comparison.\(^5\) (The dotted line approximates a high-metallicity envelope by adding water to the H/He adiabat at $Z = 0.57$ mass fraction, or about 100 times the solar O:H abundance.)

We see in Figure 9 how progressively higher-order gravity is able to shrink the allowable region in $p-$$p$ space. The mass and radius of the planet (gray; baseline region) already tells us something about the possible composition, for example, that Uranus is not dense enough to have a significant iron core. The low-order $J_3$ and $J_4$, if measured with higher precision, can be used to further narrow down possible configurations, and a higher-order $J_\nu$ would help even more. But a precise determination of the underlying rotation period is necessary for maximum benefit.

### 4. Realistic Constraints on Uranus and Neptune

#### 4.1. Constraints with Presently Known Gravity

In the previous section, we investigated the ability of gravity field measurements in general to constrain the interior density distribution of a fluid planet. We used nominal values of mass, radius, and low-order $J$s for Uranus to illustrate the results, but the models presented above should not be used for making predictions about the real planet Uranus, since they use hypothetical uncertainty values. In this section, we look at samples of interior models obtained with presently available best values and uncertainties for both Uranus and Neptune. We use the same views of the resulting distributions as in Section 3 but showing side by side the same view for both planets.

Figure 10 compares the planets in envelope view. The more accurately determined gravity of Uranus compared with that of Neptune (Table 1) allows a much tighter constraint of density in the upper envelope but, as expected, does not help with the deep interior. A similar picture is evident with the ensemble view in Figure 11, while the MOI histograms in Figure 12 allow a prediction: the range of possible values of the MOI factor, should it ever be independently measured, is $0.225 \leq I/MR^2 \leq 0.229$ for Uranus and $0.234 \leq I/MR^2 \leq 0.239$ for Neptune.

The barotrope view (Figure 13) allows us to make some statements about the planets’ compositions, albeit only very generally. Both planets are too dense to not include significant amounts of heavy elements. No surprise there. But it seems that Neptune’s entire envelope must be enriched with He or heavier elements, while a large (in pressure) fraction of Uranus is

\(^5\) The plots begin at 10 bars because the SCvH EOS table does not extend down to the required temperature at 1 bar.
Figure 9. Pressure–density envelopes integrated from sampled ρ(s) profiles of Uranus. Left: samples obtained with a Gaussian prior on the rotation period. Right: samples obtained with a known rotation period. Overlaid isentropes for hypothetical homogeneous compositions extend from a common $T_{10 \text{ bars}} = 150 \text{ K}$.

Figure 10. Envelope view of sampled ρ(s) profiles of Uranus (left) and Neptune (right) matching currently available observables (Jacobson 2009, 2014). The dark shaded regions show the range of density values at every radius that lie between the 16th and 84th percentiles (roughly the sample’s 1σ spread), and the light shaded regions show the value between the 2nd and 98th percentiles, roughly the sample’s 2σ spread.

Figure 11. Ensemble view of sampled ρ(s) profiles of Uranus (left) and Neptune (right) matching currently available observables.
consistent with a solar composition H/He mixture or even a somewhat helium-poor atmosphere. This dichotomy is consistent with the one found in the models of Nettelmann et al. (2013, their Table 2), where the envelope of both planets was assumed to have a constant H/He ratio, and much higher metallicity was required in Neptune’s envelope compared with Uranus. Observational constraints on the atmospheric abundances in both planets from spectral data are inconclusive, showing similar C:H ratios in both planets, for example, but detecting signatures of CO and HCN in the atmosphere of Neptune but not Uranus (Gautier et al. 1995). Clearly, this would be one of the more important observations to improve upon should the opportunity arise.

It also appears from Figure 13 that both planets allow for significant amounts of molecules heavier than water in their central regions, suggesting the existence of rocky cores. While the above interpretations are admittedly loose and can probably be made more robust by comparisons with additional adiabatic and nonadiabatic barotropes of different compositions, they have the benefit of not being strongly model-dependent. No assumptions at all about composition or temperature profile were made in obtaining the samples and hence the blue shaded areas in Figure 13. The generality of the solutions is limited only by the flexibility of the parameterization (Section 2.2), the thoroughness of the sampling procedure (see the Appendix), and the implicit assumptions of hydrostatic equilibrium and rigid rotation at a well-known rotation rate.

4.2. A Realistic Scenario for Future Tight Gravity Field Constraints

In Section 2.5, we discuss how making use of high-order gravity, \( J_n^{\text{obs}} \), would be impossible without some robust estimate of the effect of nonrigid rotation, \( J_n^{\text{wind}} \). For this reason, we obtained one more sample for each planet with the assumption of well-known \( J_2, J_4, J_6 \), and rotation period \( P \) but unknown \( J_n \) for \( n > 6 \). These constraints correspond to a plausible scenario in which a future mission is able to obtain a high-precision gravity and rotation period, but uncertainty about the effects of deep zonal winds renders the higher-order \( J_n \) unusable. This is similar to the current situation for Saturn.
where large differential rotation strongly affects the even $J_n$ deduced from Cassini Grand Finale data, and dynamical wind models must therefore be used to predict $J_n$ wind correction (Iess et al. 2019).

With such a data set, how would our knowledge of the interior density profile for either Uranus or Neptune be improved? We find that the improvement would be quite significant. Figure 14 shows the sampled profiles of density versus radius, while Figure 15 shows density versus pressure, which is potentially the most illustrative.

There are several aspects to note. In particular, the metallicity of the outer H/He envelope ($P < 0.1$ Mbar) could potentially be reliably assessed, given any consistency or inconsistency with a solar metallicity H/He adiabatic density profile. In addition, a metallicity enhancement that would deviate from a uniform enhanced metallicity could potentially be “seen” with gravity data. This would be more readily done at higher metallicities that deviate strongly from solar.

The long-standing question of whether the “middle” layers of Uranus and Neptune are exclusively made of water and other “ices” or instead have less water but more H/He and rocks to give a similar density may not be directly answerable. However, the pressures at which changes in density structures occur and the “slope” in density-versus-pressure space of density profiles may well allow for plausible explanations for any constrained density profiles to be determined. Such profiles could be connected directly to predicted composition profiles from planet formation (Helled & Stevenson 2017; Ormel et al. 2021) and thermal evolution models (Vazan & Helled 2020; Scheibe et al. 2021; Stixrude et al. 2021). Clearly, the constraints on interior models would not only become much tighter than possible with currently available data but also tight enough in absolute terms to distinguish between different temperature/composition profiles.

5. Summary and Conclusions

In the previous sections, we describe an experiment designed to gauge the ability of a precise measurement of a planet’s gravity to constrain the possible distribution of mass in its
interior. Gravity is a long-range force, and the planet’s gravitational potential at any exterior point is determined by an integral of the mass density over the entire planet so that, in principle, knowledge of one should inform the other. In practice, the gravitational potential can be measured with varying degrees of uncertainty: crudely with the aid of natural satellites, better by tracking a flying-by spacecraft, and with potentially exquisite precision by a dedicated orbiting mission. There is no simple formula connecting the degree of precision potentially exquisite precision by a dedicated orbiting mission.

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Examining the samples presented in Section 3 led us to the conclusions summarized below. Many of these are unsurprising, being predictable at least qualitatively from the nature of the gravity integral. Nevertheless, we find useful the more direct and quantitative demonstration made possible by the sampling framework.

1. Even the most crude estimation of a planet’s gravitational potential, say, a measurement of only the $J_2$ coefficient to within 1% (Figure 5), considerably narrows down the space of allowable density profiles compared with a baseline constrained only by the planet’s mass, radius, and boundary conditions. This narrowing down is most evident in the upper 10%–20% (by radius) of the planet’s interior, very quickly disappearing with depth.

2. The degree of constraint applied to the interior $\rho(s)$ by a gravity measurement (a set of $J_n, \sigma_{J_n}, n \leq n_{\text{max}}$) can be loosely quantified by the width of the distribution of associated MOI values, a scalar quantity integrated from (and sensitive to) $\rho(s)$ over the equilibrium shape of the planet.

3. More precisely known values of the low-order gravity coefficients are not nearly as useful as adding even crudely measured values of higher-order coefficients. Compare Figures 5 and 7, especially the right panel in each. This fact presents a difficulty, however, as higher-order coefficients are increasingly sensitive to and eventually dominated by dynamic effects not captured by a simple rigid rotation rate, such as zonal winds and deep differential rotation. For high-precision high-order gravity to be useful, these dynamic effects must be accounted for.

4. If high-precision or high-order gravity is to be used to constrain interior models, then the planet’s rotation period must also be known to comparable precision. Compare Figures 7 and 8. This point is worth emphasizing, since neglecting the uncertainty in the rotation period can lead to overly confident predictions.

5. No level of precision and completeness of characterization of the gravity field and rotation state can be expected to pin down by itself the density at the center of a planet. See the yellow shaded area in Figure 8. To do better than a factor of 2 or more will require making additional assumptions.

6. A measurement of a planet’s MOI factor, in addition to and independent of the gravity field measurement, can potentially assist with constraining the interior mass distribution, but it would have to be quite a precise one.

When only the low-order $J_2$ and $J_4$ are known and only to rough precision, as is presently the case for Uranus and Neptune, the distribution of the correlated MOI values is already constrained to a large degree: about 0.6% in the case of Uranus and about 1% for Neptune. If higher-order $J_n$ become known (and recall that this also implies precise determination of the rotation period), the MOI is essentially fixed and can provide no further information.

A second goal of this work was to look at the presently available gravity field estimate of Uranus and Neptune and see what predictions can be made about their interiors that would be, as much as possible, immune to implicit model assumptions and uncertainties in the EOSs of hydrogen, helium, and heavier elements. Unsurprisingly, these predictions are general in nature and cannot replace detailed models. Nevertheless, they illustrate the potential of more complete characterization of the gravity fields, should one become available, to better direct such models.

1. As is well known, both Uranus and Neptune are much too dense to not include significant quantities of elements heavier than helium. In their central regions, both planets appear to allow significant enrichment by components denser than $\text{H}_2\text{O}$. To say anything more about the denser components will require a fuller characterization of the gravity field or making more detailed model assumptions or, very likely, both.

2. A fraction of Uranus’s envelope is consistent with an adiabatic region of $\text{H}/\text{He}$ at solar atmospheric abundances. Neptune’s envelope, however, is not but should be significantly metal-enriched or, perhaps, somehow, He-rich.

3. An orbiter mission to better characterize the gravity field and rotation of either or both planets would be of very high value. Even if high-order gravity $J_n > 0$ cannot be reliably separated into hydrostatic and dynamic parts, the interior barotrope (density versus pressure profile) could be tightly constrained. This would allow for direct comparisons of profiles of composition versus depth from formation and evolution models, opening a new era in our understanding of Uranus and Neptune.

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*Software:* *emcee* (Foreman-Mackey et al. 2013).

**Appendix**

**Sampling Procedure**

To draw a sample of $\rho(s)$ profiles for each planet, we run the ensemble sampler of the *emcee* Python package (Foreman-Mackey et al. 2013), which implements the parallel stretch-move algorithm of Goodman & Weare (2010). The goal is a draw of a large enough sample of independent realizations of the model parameters (Equation (4)) from the probability distribution dictated by the likelihood function (Equation (3)) for each combination of observables and uncertainties ($\tilde{\sigma}_n$ and $\nu_n$).
### A.1. Variable Transformations

In practice, sampling is often preceded by some isomorphic transformation of the physical model parameters into equivalent random variables whose probability distribution is predicted to be, in some sense, smoother and thus easier for the sampling algorithm to work with. The particular transformations are determined by analysis or, more often, by trial and error, advice, or general common wisdom suggestions. In the best cases, a simple transformation, for example, using a logit transformation of the physical model parameters into equivalent random variables whose probability distribution is benign (i.e., steep gradients and sharp discontinuities) if and where they are “preferred” by the data. This is, in fact, exactly what happens, very slowly. The mixing rate between walkers in our samples is very low, probably due to correlations between parameters.

In order to speed up the generation of new samples, we save and reuse density profiles (parameter values) from already-generated samples. A new ensemble of $n_w$ walkers is seeded with a random choice of $n_w$ items from a large seed bank of diverse profiles and may start out looking like the example in Figure 16. While this does not solve the problem of slow mixing, once the chains are long enough to forget their initial state (longer than their autocorrelation time), the final links from each chain are an independent sample of size $n_w$.

#### A.3. Tempering

To improve interwalker mixing, we use a common idea known as tempering. The ensemble is run for a while under an artificially widened likelihood function designed to lower the peaks and raise the valleys in the likelihood landscape. In our case, this is equivalent to assuming very poorly known values for the planet’s observables (gravity, mass, rotation period). Under this likelihood function, walkers can readily mix and do not get stuck in local regions of high likelihood. After a few autocorrelation times (much shorter now), the ensemble walkers are well mixed and spread out in parameter space following the posterior distribution but of the modified likelihood. When the real likelihood is then applied, this effectively creates an ideal seed state for the ensemble. When the walkers then continue under the real likelihood, they explore their local neighborhood more slowly and get stuck if they find a high-likelihood peak. But at this point, the last link from each walker is collected, and the process restarts, repeating the cycle as many times as necessary to produce a final sample of 1000 independent states.

We believe but did not mathematically prove that this procedure satisfies the requirement of detailed balance, which makes it an MCMC algorithm and guarantees that each $\rho(s)$ appears in the final sample in proportion to its frequency in the unknown posterior distribution. More important for the purpose of this work, however, is that the procedure is able to efficiently explore the entire parameter space (as is evident from Figure 3).
Figure 16. Density profiles of a typical ensemble seed state.

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