Mermin inequalities for perfect correlations

Adán Cabello,1 Olaf Gühne,2,3 and David Rodríguez4

1Departamento de Física Aplicada II, Universidad de Sevilla, E-41012 Sevilla, Spain
2Institut für Quantenoptik und Quanteninformation, Österreichische Akademie der Wissenschaften, A-6020 Innsbruck, Austria
3Institut für Theoretische Physik, Universität Innsbruck, A-6020 Innsbruck, Austria
4Departamento de Física Aplicada III, Universidad de Sevilla, E-41092 Sevilla, Spain

(Dated: June 16, 2008)

Any $n$-qubit state with $n$ independent perfect correlations is equivalent to a graph state. We present the optimal Bell inequalities for perfect correlations and maximal violation for all classes of graph states with $n \leq 6$ qubits. Twelve of them were previously unknown and four give the same violation as the Greenberger-Horne-Zeilinger state, although the corresponding states are more resistant to decoherence.

PACS numbers: 03.65.Ud, 03.67.Mn, 03.67.Pp, 42.50.-p

I. INTRODUCTION

In 1989, Greenberger, Horne, and Zeilinger (GHZ) showed that no local hidden variable (LHV) theory can assign predefined local results which agree with the quantum predictions given by the quantum mechanics [3]. The amount of the violation of Mermin’s inequalities, measured by the ratio \( D \) between the quantum value of the Bell operator and its bound in LHV theories, grows exponentially with $n$. For a given $n$ (with $n$ odd), Mermin’s inequality gives the maximal possible violation of any $n$-party two-setting Bell inequality in quantum mechanics [2].

Can we extend this result to other $n$-qubit states? The essential ingredient for GHZ-type proofs and Mermin-type Bell inequalities is that they require an $n$-qubit quantum state, which is a simultaneous eigensate of $n$ commuting local observables (i.e., a stabilizer state). Any stabilizer state is, up to local rotations, equivalent to a graph state [2] (i.e., a stabilizer state whose generators can be written with the help of a graph [5]). These states are essential in quantum error correction [6] and resistant to decoherence.

Bell inequalities for graph states constitute a subject of intense study [2, 10, 11, 12, 13, 14, 15, 16]. However, the Mermin inequalities for most of them are unknown. For a given state, the Mermin inequality is the Bell inequality such that (I) the Bell operator is a sum of stabilizing operators of that state (i.e., perfect correlations), and (II) the violation is maximal. If the maximum is obtained for Bell operators with a different number of terms, then we will choose the one with the lowest number, since the other inequalities contain this inequality and require more measurements. For some graph states, the Mermin inequality is not unique due to the symmetries of the graph.

This definition is motivated by the relation between the original GHZ proof [1] and the Mermin inequality [2]. The aim of this paper is to introduce the Mermin inequalities for all graph states (or, equivalently, for all pure states with $n$ independent perfect correlations) with $n < 7$ qubits.

The graph state $|G\rangle$ is the unique $n$-qubit state that satisfies $g_i(G) = |G\rangle$, for $i = 1, \ldots, n$, where $g_i$ are the generators of the stabilizer group of the state, defined as the set \( \{s_j\}_{j=1}^{2^n} \) of all products of the generators. The perfect correlations of the graph state are

$$
\langle G | s_j | G \rangle = 1 \quad \text{for} \ j = 1, \ldots, 2^n. \quad (1)
$$

The $g_i$’s are obtained with the help of a graph $G$. For instance, the $n$-qubit GHZ state is associated to the star-shaped graph in which qubit 1 is connected to all the other qubits. This means that $g_1 = X_1 \otimes_{l \neq 1} Z_l$ and $g_i = X_1 \otimes Z_i$ for $i \neq 1$; $X_i$, $Y_i$, and $Z_i$ denote the Pauli matrices acting on the $i$th qubit (see [5] for more details).

There are many possible GHZ-like proofs for a given graph state associated to a connected graph of $n \geq 3$ qubits. All of them have the same structure. Any LHV theory assigning predefined values $-1$ or 1 to $X_i$, $Y_i$, and $Z_i$, in agreement with the quantum predictions given by (1) must satisfy

$$
s_j = 1 \quad \text{for} \ j = 1, \ldots, 2^n. \quad (2)
$$

However, if we choose a suitable subset of $q$ predictions from the set \( \{s_j\} \), and assume predefined values, either $-1$ or 1, then for some choices it happens that, at most, only $p \leq q$ of these predictions are satisfied. For the remaining $q - p$ quantum predictions, the prediction of
the LHV theory is the opposite: $s_j = -1$. This difference can be reformulated as a violation of the Bell inequality

$$\beta \leq 2p - q,$$

where the Bell operator $\beta$ is the sum of the stabilizing operators of the chosen subset. According to Eq. (1), the graph state satisfies

$$\langle G|\beta|G\rangle = q.$$  

Therefore, $|G\rangle$ violates the inequality \(\beta\) by an amount $D = q/(2p - q)$. For the GHZ proof with $n$ odd, the maximum contradiction, measured by $q/p$, and the maximum violation of the Bell inequality, measured by $D$, is obtained when $q = 2^{n-1}$ and $p = 2^{n-2} + 2^{(n-3)/2}$. This is Mermin's inequality \ref{eq:mermin}. If we take a different subset of stabilizing operators, then we can have a violation of a Bell inequality, but usually not the maximum one.

Specifically, for a given graph state associated to a connected graph of $n \geq 3$ qubits, if we consider the Bell operator consisting of the whole set of stabilizing operators, then we always have a violation of a Bell inequality \ref{eq:mermin}, but not the maximum one. A violation occurs because that Bell operator contains a simpler Bell operator giving the maximum violation.

Why are we interested in those Bell inequalities with the maximum $D^n$? $D$ is the measure of nonlocality used in Refs. \textcite{2, 3, 10}. For graph states and stabilizer Bell inequalities, it is well defined, easily computable, and the two practical measures of nonlocality, the resistance to noise and the detection efficiency for a loophole-free Bell experiment, are connected to $D$.

(i) In actual experiments, instead of a pure state $|G\rangle$, we usually have a noisy one, $\rho = V|G\rangle\langle G| + (1 - V)\mathbb{I}/2^n$, where $\mathbb{I}$ is the identity matrix in the Hilbert space of the whole system. $D$ is related to the minimum value of $\log V$ required to actually observe a violation of the Bell inequality $V_{\text{crit}}$. For graph states and stabilizer Bell inequalities, if $D$ increases, then $V_{\text{crit}}$ decreases. Specifically, a simple calculation gives that $V_{\text{crit}} = 1/D$.

(ii) An open problem in fundamental physics is achieving a loophole-free Bell experiment. A particularly important problem is the detection loophole \ref{eq:detection}. $D$ is related to the minimum detection efficiency required for a loophole-free Bell experiment $\eta_{\text{crit}}$. For graph states and stabilizer Bell inequalities, if $D$ increases, then $\eta_{\text{crit}}$ decreases. Specifically, for GHZ states and the Mermin inequality with $n$ odd, $\eta_{\text{crit}} = \frac{2 + (\log 2)/\log D}{4}$ \ref{eq:eta_crit}.

(iii) In addition, $D$ provides the relevant parameters of the underlying GHZ-type proof: $q$ and $p$. Any GHZ-type proof can be converted into an $n$-party quantum pseudotelepathy game in which a team assisted with a graph state always wins, while a team with only classical resources wins only with probability $p/q$ \ref{eq:rho}. Therefore, the higher $D$, the lower $p/q$ and the higher quantum advantage.

The knowledge of the Mermin inequalities for all graph states is important for (a) Quantum information. Graph states are essential for quantum information tasks. Mermin inequalities are useful tools for their experimental analysis. For instance, in recent experiments preparing 6-qubit graph states $V$ is around $0.5$ \textcite{20, 21, 22}, thus Bell inequalities with $D > 2$ are required to observe violation. We will show that for all 6-qubit graph states, Mermin inequalities have $D > 2$.

(b) Nonlocality vs decoherence experiments. For GHZ states, $D$ increases exponentially with $n$ \ref{eq:mermin}. However, GHZ states’ entanglement lifetime under decoherence decreases with $n$ \ref{eq:decoherence}. Therefore, a fundamental limitation seemingly exists to observe macroscopic violations of Bell inequalities with GHZ states. A natural question is: Does this limitation also hold for other types of graph states? What happens to those graph states whose lifetime under decoherence does not decrease with $n$ \ref{eq:decoherence}? To answer these questions we need to know how $D$ scales with $n$ within a family of graph states, and which graph states have higher $D$.

II. MERMIN INEQUALITIES FOR GRAPH STATES

For each graph state, our task is to obtain, from all possible Bell operators which are sums of stabilizing operators, those which provide the highest violation. The exhaustive study for $n \geq 6$ becomes computationally difficult because the number of potential Bell operators to test scales like $2^{2n}$. However, if we restrict our attention to Bell operators with the same symmetry as the underlying graph, this investigation is still computationally feasible for $n = 6$.

In Table I we present all the Mermin inequalities for all graph states with $2 < n < 6$ qubits. In Table II we present the Mermin inequalities possessing the same symmetry as the underlying graph for all graph states with $n = 6$ qubits. In both tables we have followed the classification and the labeling of the qubits of Fig. 1 (taken from Ref. \textcite{3}). $L_{n}$ ($RC_{n}$) denotes the $n$-qubit linear (ring) cluster state \ref{eq:cluster}. $Y_{6}$ denotes the 5-qubit graph state associated to the graph “Y”, $H_{6}$ the 6-qubit graph state associated to the graph “H”, etc. The quantum prediction for each Bell operator $\beta$ is $q$ (i.e., the number of terms of $\beta$); $p$ is the maximum number of the $q$ perfect correlations that a LHV theory can satisfy; $D = \frac{q}{2p - q}$ is the violation of the Bell inequality $\beta \leq 2p - q$.

Some of the inequalities in Tables I and II were previously known. For the $n$-qubit GHZ states with $n$ odd, we recover the original Mermin inequalities \ref{eq:mermin}. For the $n$-qubit GHZ states with $n$ even, the original Mermin inequalities are the sum of our two symmetric inequalities (the fist two inequalities for the GHZ$_{4}$ in Table I and the two inequalities for the GHZ$_{6}$ in Table II). Our inequalities have the same violation as Mermin’s, but only half of the terms. For $n$ even, Ardehali proposed a method giving an additional violation of $\sqrt{2}$ \textcite{23}. Ardehali’s inequalities do not use only perfect correlations. Ardehali’s
These inequalities have been recently tested in the laboratory using 3- and 4-qubit GHZ states and 6-qubit GHZ states already exist. Sources of 5- and 6-qubit GHZ states have been recently tested in the laboratory. These inequalities have been recently tested in the laboratory.

For the 4-qubit cluster state (LC₄), the Mermin inequalities in Table I contain those introduced in. These inequalities have been recently tested in the laboratory.

However, twelve of the Mermin inequalities in Table I and II are new. They include those for the RC₅, important for quantum error correction codes for the H₆, a universal resource for one-way quantum computation recently prepared in the laboratory; and for the Y₆, which allows a demonstration of anyonic statistics in the Kitaev model.

A remarkable fact is, that four 6-qubit graph states have the same violation as the GHZ₆: the graph state no. 10, the H₆, the Y₆, and the LC₅ (see Table II). This is interesting because these states are more resistant to decoherence than the GHZ₆. This proves that the nonlocality vs decoherence ratio of GHZ states is not universal: there are states with similar violations but that are more robust against decoherence.

### Acknowledgments

The authors thank H. J. Briegel, P. Moreno, and G. Vallone for useful discussions. A.C. acknowledges support from the Spanish MEC Project No. FIS2005-07689, and the Junta de Andalucía Excellence Project No. P06-FQM-02243. O.G. acknowledges support from the FWF and the EU (OLAQUI, SCALA, and QICS).

### Table I: Mermin inequalities for all graph states of n < 6 qubits.

| Graph state | gᵢ | β ≤ 2p - q | Settings | D |
|-------------|-----|------------|----------|---|
| 2 (GHZ₃)   | g₁ = X₁Z₁Z₁ | g₁(1 + g₂)(1 + g₃) ≤ 2 | 2-2-2-2 | 2 |
|             | g₂ = Z₁X₁ for i ≠ 1 | | | |
| 3 (GHZ₄)   | g₁ = X₁Z₂Z₃Z₄ | (1 + g₁)g₂(1 + g₃) ≤ 2 and g₁ → g₂g₁ | 2-2-2-2 | 2 |
|             | g₂ = Z₁X₁ for i ≠ 1 | | | |
| 4 (LC₄)    | g₁ = X₁Z₂, g₄ = Z₃X₄ | (1 + g₁)(1 + g₃) ≤ 2 and g₃ → g₃g₄ | 2-2-2-2 | 2 |
|             | g₂ = Z₁X₄ for i ≠ 1 | | | |
| 5 (GHZ₅)   | g₁ = X₁Z₂Z₃Z₄Z₅ | g₁(1 + g₂)(1 + g₃)(1 + g₄) ≤ 4 | 2-2-2-2-2 | 4 |
|             | g₂ = Z₁X₄ for i ≠ 1 | | | |
| 6 (Y₅)     | g₁ = X₁Z₂, g₄ = Z₂X₅ | g₂[(1 + g₁ + g₃)(1 + g₃ + g₄) + (1 + g₁g₃g₄)] | 3-3-3-3-2 | 5 |
|             | g₂ = Z₁X₂Z₅ | + (1 + g₁g₃)g₄(1 + g₄) ≤ 7 | | |
|             | g₃ = Z₂X₃Z₄ | g₂ → g₄g₄ | 3-3-3-3 | |
|             | g₄ = Z₃X₄ | β → g₄β and 32 nonsymmetric more | | |
| 7 (LC₅)    | g₁ = X₁Z₂, g₅ = Z₄X₅ | (1 + g₁)[(1 + g₂)g₃(1 + g₄) + g₂g₃g₄] (1 + g₅) ≤ 8 | 3-3-3-3-3 | 5 |
|             | g₂ = Z₁X₅ for i ≠ 1 | | | |
| 8 (RC₅)    | g₁ = Z₁X₁Z₁Z₁ | γ + ∑ₖ₌₁^₅ gₖgₖ₊₁ ≤ 9 | 3-3-3-3-3 | 7 |
|             | g₂ = Z₂X₂Z₂ | γ + g₂g₂g₂ + g₆g₆g₆ + g₁g₁g₁ + g₂g₂g₂ + g₃g₃g₃ + g₄g₄g₄ + g₅g₅g₅ ≤ 9 | 3-3-3-3-3 | |
|             | g₃ = Z₃X₃Z₃ | γ = ½ [(1 + g₁) - (1 + g₃)] | | |
|             | g₄ = Z₄X₄ | and 105 more | | |

[1] D. M. Greenberger, M. A. Horne, and A. Zeilinger, in *Bell’s Theorem, Quantum Theory, and Conceptions of the Universe*, edited by M. Kafatos (Kluwer Academic, Dordrecht, Holland, 1989), p. 69.
[2] N. D. Mermin, Phys. Rev. Lett. 65, 1838 (1990).
[3] R. F. Werner and M. M. Wolf, Phys. Rev. A 64, 032112 (2001).
[4] M. Van den Nest, J. Dehaene, and B. De Moor, Phys. Rev. A 69, 022316 (2004).
[5] M. Hein, J. Eisert, and H. J. Briegel, Phys. Rev. A 69, 062311 (2004).
[6] D. Gottesman, Phys. Rev. A 54, 1862 (1996).
[7] R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. 86, 5188 (2001).
[8] W. Dir and H. J. Briegel, Phys. Rev. Lett. 92, 180403 (2004).
[9] D. P. DiVincenzo and A. Peres, Phys. Rev. A 55, 4089 (1997).
[10] O. Gühne, G. Tóth, P. Hyllus, and H. J. Briegel, Phys. Rev. Lett. 95, 120405 (2005).
| Graph state | \( g_i \) | Settings |  \( D \) |
|-------------|-------------|----------|-------|
| 9 (GHZ)    | \( X_1 Z_2 Z_3 Z_4 Z_5 Z_6 \) | 1-2-2-2-2-2 | 4 |
|            | \( X_i Z_i \) for \( i \neq 1 \) | 1-2-2-2-2-2 | 4 |
| 10         | \( X_i Z_i \) for \( i = 1, 2, 3 \) | 2-2-2-1-2-2 | 4 |
|            | \( X_4 Z_5 \) | 2-2-2-1-2-2 | 4 |
|            | \( Z_4 X_5 Z_6 \) | 2-2-2-1-2-2 | 4 |
| 11 (H6)    | \( X_1 Z_6, g_2 = X_4 Z_6 \) | 1-2-3-3-3-2 | 4 |
|            | \( X_1 Z_5, g_4 = X_4 Z_5 \) | 2-1-3-3-3-2 | 4 |
|            | \( Z_2 Z_1 X_5, g_6 = Z_2 Z_1 X_6 \) | 3-3-1-2-2-3 | 4 |
| 12 (Y6)    | \( X_1 Z_2, g_6 = Z_2 X_6 \) | 2-2-1-2-2-2 | 4 |
|            | \( X_3 Z_4 Z_6, g_3 = Z_2 X_3 Z_4 \) | 2-3-3-3-3-2 | 3 |
|            | \( g_1 \) | 2-3-3-3-3-2 | 3 |
| 13 (E6)    | \( X_1 Z_2, g_2 = Z_2 Z_4 \) | 2-3-3-3-3-2 | 3 |
|            | \( g_1 \) | 2-3-3-3-3-2 | 3 |
| 14 (LC6)   | \( X_1 Z_2, g_2 = Z_2 Z_4 \) | 3-3-3-3-3-3 | 3 |
|            | \( g_1 \) | 3-3-3-3-3-3 | 3 |
| 15         | \( X_1 Z_6, g_2 = X_2 Z_4 \) | \[1 \] | \[1 \] |
|            | \( X_3 Z_4 Z_6, g_5 = Z_4 X_3 Z_6 \) | \[1 \] | \[1 \] |
|            | \( g_4 \) | \[1 \] | \[1 \] |
|            | \( g_6 = Z_3 X_6 \) | \[1 \] | \[1 \] |
| 16         | \( X_1 Z_2, g_2 = Z_4 X_5 \) | \[1 \] | \[1 \] |
|            | \( g_1 \) | \[1 \] | \[1 \] |
| 17         | \( X_1 Z_2, g_2 = Z_4 X_5 \) | \[1 \] | \[1 \] |
|            | \( g_1 \) | \[1 \] | \[1 \] |
|            | \( g_6 = Z_3 X_6 \) | \[1 \] | \[1 \] |
| 18 (RC6)   | \( X_1 Z_2, g_2 = Z_4 X_5 \) | \[1 \] | \[1 \] |
|            | \( g_1 \) | \[1 \] | \[1 \] |
|            | \( g_6 = Z_3 X_6 \) | \[1 \] | \[1 \] |

\[1\] A. Cabello, Phys. Rev. Lett. 95, 210401 (2005).

\[2\] V. Scarani, A. Acín, E. Schenck, and M. Aspelmeyer, Phys. Rev. A 71, 042325 (2005).

\[3\] G. Tóth, O. Gühne, and H. J. Briegel, Phys. Rev. A 73, 022303 (2006).

\[4\] L.-Y. Hsu, Phys. Rev. A 73, 042308 (2006).

\[5\] J. Barrett, C. M. Caves, B. Eastin, M. B. Elliott, and S. Piranion, Phys. Rev. A 75, 012103 (2007).

\[6\] C. Wu, Y. Yeo, L. C. Kwek, and C. H. Oh, Phys. Rev. A 75, 032332 (2007).

\[7\] P. M. Pearle, Phys. Rev. D 2, 1418 (1970).

\[8\] A. Cabello, D. Rodríguez, and I. Villanueva e-print [arXiv:0712.3268]

\[9\] A. Cabello, Phys. Rev. A 73, 022302 (2006).

\[10\] D. Leibfried, E. Knill, S. Seidelin, J. Britton, R. Blakestein, J. Chiaverini, D. B. Hume, W. M. Itano, J. D. Jost, C. Langer, R. Ozeri, R. Reichle, and D. J. Wineland, Nature (London) 438, 639 (2005).

\[11\] C.-Y. Lu, X.-Q. Zhou, O. Gühne, W.-B. Gao, J. Zhang, Z.-S. Yuan, A. Goebel, T. Yang, and J.-W. Pan, Nat. Phys. 3, 91 (2007).

\[12\] C.-Y. Lu, W.-B. Gao, O. Gühne, X.-Q. Zhou, Z.-B. Chen, and J.-W. Pan, e-print [arXiv:0710.2273]

\[13\] M. Ardehali, Phys. Rev. A 46, 5575 (1992).

\[14\] O. Gühne and A. Cabello, Phys. Rev. A 77, 032108 (2008).

\[15\] J.-W. Pan, D. Bouwmeester, M. Daniell, H. Weinfurter, and A. Zeilinger, Nature (London) 403, 515 (2000).

\[16\] C. A. Sackett, D. Kielpinski, B. E. King, C. Langer, V. Meyer, C. J. Myatt, M. Rowe, Q. A. Turchette, W. M. Itano, D. J. Wineland, and C. Monroe, Nature (London) 404, 256 (2000).

\[17\] Z. Zhao, T. Yang, Y.-A. Chen, A.-N. Zhang, M. Žukowski, and J.-W. Pan, Phys. Rev. Lett. 91, 180401
FIG. 1: Graphs representing all classes of $n$-qubit graph states, with $2 \leq n \leq 6$, that are inequivalent under single-qubit unitary transformations and graph isomorphism. The figure is taken from Ref. [5].

[28] Z. Zhao, Y.-A. Chen, A.-N. Zhang, T. Yang, H. J. Briegel, and J.-W. Pan, Nature (London) 430, 54 (2004).
[29] P. Walther, M. Aspelmeyer, K. J. Resch, and A. Zeilinger, Phys. Rev. Lett. 95, 020403 (2005).
[30] N. Kiesel, C. Schmid, U. Weber, G. Tóth, O. Gühne, R. Ursin, and H. Weinfurter, Phys. Rev. Lett. 95, 210502 (2005).
[31] G. Vallone, E. Pomarico, P. Mataloni, F. De Martini, and V. Berardi, Phys. Rev. Lett. 98, 180502 (2007).
[32] Y.-J. Han, R. Raussendorf, and L.-M. Duan, Phys. Rev. Lett. 98, 150404 (2007).
[33] J. K. Pachos, W. Wieczorek, C. Schmid, N. Kiesel, R. Pohlner, and H. Weinfurter, e-print arXiv:0710.0895.  

(2003).