Entropy calculation for a toy black hole

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Abstract

In this paper, we carry out the counting of states for a black hole in loop quantum gravity, assuming however an equidistant area spectrum. We find that this toy-model is exactly solvable, and we show that its behavior is very similar to that of the correct model. Thus this toy-model can be used as a nice and simplifying ‘laboratory’ for questions about the full theory.

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1. Introduction

The present paper is concerned with the description of black holes and the calculation of their entropy in the framework of loop quantum gravity (see [1, 2] for a general introduction to loop quantum gravity). The literature on this subject is large and includes (but is by no means limited to) the pioneering work [3], the introduction of a precise formalism [4, 5], the reformulation and approximate solution of the combinatorial problems involved [9, 10]. Although the basics are by now quite well understood, there are still surprises in store. One example is the structures that were found in a computer analysis of the spectrum of states [21, 22].

The calculation of the entropy of a non-rotating black hole in loop quantum gravity boils down to a rather complicated combinatorial problem. It can be treated to a very good approximation in an asymptotic regime [9, 10], and proportionality of entropy to area has been established.

In the present paper, we will develop a model that drastically simplifies the technical aspects of the entropy calculation while—as our results will show—retaining many of the qualitative features of the actual situation. In particular, the combinatorial problem for our model can be solved exactly, so any question that one may have about it can be answered with relative ease.

The simplifying assumption that we will make is a rather obvious and simple one. One of the hallmarks of loop quantum gravity is a complicated, non-equidistant area spectrum. In particular, the area eigenvalues for a non-rotating isolated horizon of a black hole are sums of numbers $A_j$,

$$A_j = 8\pi \gamma l_p^3 \sqrt{j(j+1)}, \quad j \in \mathbb{N}/2,$$

which

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where $\gamma$ is the Barbero–Immirzi parameter, and $l_p$ the Planck length. The $A_j$ are obviously not equidistant; however, they become approximately equidistant for large $j$. Our approximation in this paper consists in using

$$A_j \doteq 8\pi \gamma l_p^2 \left( j + \frac{1}{2} \right)$$

instead of (1), i.e. effectively changing the area operator of the theory. In fact, it has been argued in [6] that the equidistant area spectrum (2) is the correct area spectrum for loop quantum gravity.\(^1\) The argument given there is quite compelling in principle; however, the resulting spectrum cannot come from a cylindrically consistent operator on the Hilbert space of loop quantum gravity due to the constant nonzero contribution for $j = 0$. (This may be different for a different choice of Hilbert space.) Therefore, we prefer to interpret (2) as the first two terms in the series

$$\sqrt{j(j+1)} = j + \frac{1}{2} - \frac{1}{4(2j+1)} - \frac{1}{16(2j+1)^2} + \cdots$$

We are certainly not the first to use an approximation like this. For example, it has been used in [9] to give bounds on the Barbero–Immirzi parameter.

We should also point out that it has been argued [11] that a very similar equidistant spectrum,

$$A_j' = 8\pi \gamma l_p^2 j \quad j \in \mathbb{N}/2,$$

does arise in loop quantum gravity, upon quantizing the area of a non-rotating black hole following an alternative route. We will give results on the entropy for this modification of the area spectrum in the appendix.

The nice thing about the approximation (2) is that it simplifies the calculation of black hole entropy in the theory tremendously. We can easily calculate the generating function corresponding to the combinatorial problem of enumerating the horizon states. From the generating function, a lot of information can then be obtained, as we will demonstrate.

Due to this simplicity our model may be useful, for example to do a first quick check on some hypotheses, before attempting to check it for the actual system. We demonstrate this by studying—and ruling out—some admittedly far-fetched proposal about describing rotating black holes within this formalism (section 2.3).

Calculations of black hole entropy using (2) or (4) have been carried out before, see for example [6–8]. The present paper differs from its predecessors in a number of respects. What is new in our treatment is the use of generating functions. This makes the calculations easy and elegant. Also it enables us to treat the counting problem with a certain constraint (see (5)) coming from the Gauss constraint of the full theory implemented. This is in contrast to earlier work, where the constraint was either ignored or treated using approximations. With the present method, we can prove that the constraint has no consequences to leading order and also what its consequence is in next to leading order. A difference to other calculations is the precise combinatorial problem that we are treating. As we will explain in the next section we chose to follow [9], which in turn is based on [5]. We note that this differs in terms of degeneracy factors and statistics from [4] and papers based on that reference.

The paper is organized as follows: in the following section, we calculate the generating function for the problem and derive various results about the asymptotic growth of the number

\(^1\) The argument is in essence that there is a natural quantization map—the Duflo map—for polynomials over a Lie algebra, and when applied in the case at hand the Casimir gets ‘additively renormalized’ in such a way that the equidistant spectrum (2) results.

\(^2\) Take $\sqrt{j(j+1) + x}$, expand around $x = 1/4$ and evaluate at $x = 0$. 


of states and thus the entropy. In section 3, we will show that our results parallel those of [10] for the full spectrum, and discuss some ramifications. In the appendix, we give results for the modified area spectrum $A'_j$.

2. Counting

In the literature on the subject, slightly different things have been counted when calculating black hole entropy in loop quantum gravity [9, 12–15]. This has to do with the fact that one has to distinguish between bulk- and boundary-states and this distinction is not entirely trivial. In practice, there arise two different ways of counting the entropy, and both lead to the same results on a qualitative level.

While we do not want to commit ourselves to either of these ways of counting on physical grounds here (we may have to say more about this elsewhere), we still restrict to only one way of counting in the present paper. This is merely to keep the presentation straightforward. We do not see any problem in extending our results to the alternative way of counting [12, 13, 15].

We will follow the definitions of [5, 9]: let us call the number of surface states with an area smaller than or to equal $\text{true}_\text{true}(A)$, (the superscript ‘true’ is meant to indicate that this is with respect to the actual area spectrum (1)). $N^\text{true}_\leq(A)$ can be obtained [5] by counting ordered sequences $(b_i)$ of integers $b_i$ modulo $k$ which sum to zero,

$$\sum_i b_i = 0 \mod k$$

and satisfy certain additional requirements, namely: there exist sequences $(m_i)$, $m_i \in \mathbb{Z}/2$ and $(j_i)$, $j_i \in \mathbb{N}/2$ such that

$$b_i = -2m_i \mod k,$$

$$m_i \in \{-j_i, -j_i + 1, \ldots, j_i\}$$

as well as

$$8\pi\gamma l_p^2 \sum_i \sqrt{j_i(j_i + 1)} \leq A.$$  

(6)

According to the philosophy laid out in section 1, we will just change the area spectrum, and keep all else unchanged. We will denote by $N_\leq(A)$ the number of sequences $(b_i)$ of integers $b_i$ modulo $k$ which sum to zero and such that there are $(m_i)$, and $(j_i)$ as above, except for that we ask

$$8\pi\gamma l_p^2 \sum_i \left[j_i + \frac{1}{2}\right] \leq A$$

instead of (6). In [9], it was shown that the definition of $N^\text{true}_\leq$ is equivalent to a much simpler one: $N^\text{true}_\leq$ is the number of ordered sequences $(m_i)$, $m_i \in \mathbb{Z}/2$ such that

$$\sum_i m_i = 0 \quad \text{and} \quad 8\pi\gamma l_p^2 \sum_i \sqrt{|m_i|(|m_i| + 1)} \leq A.$$  

The same arguments can be applied to $N_\leq(A)$. It is easy to see that it is the number of ordered sequences $(m_i)$, $m_i \in \mathbb{Z}/2$ such that

$$\sum_i m_i = 0 \quad \text{and} \quad 8\pi\gamma l_p^2 \sum_i \left|m_i\right| + \frac{1}{2} \leq A.$$  

(7)

3 Barring some sort of holography (for which there is currently little evidence in LQG), there are infinitely many different bulk states for a black hole of a given area, so counting those does not even make mathematical sense.

4 $k \doteq A/(4\pi\gamma l_p^2)$ has to be an integer—it represents the level of the Chern–Simons theory on the horizon [5].
Let us also define $N(A)$, the number of such sequences that satisfy (7) with ‘$\leq$’ replaced by ‘$=$’.

It was realized in [9, 10] that the counting problem can be simplified by implementing the two conditions of (7) in separate steps. We will follow this strategy and define

$$N(a, j) = \sum_{i=1}^{a} N(i, j).$$

Similarly we define

$$N_{\leq}(a, j) = \sum_{i=1}^{a} N(i, j).$$

Note that $N_{\leq}(A) = N_{\leq}(A/ (4\pi yl^2), 0)$ etc.

A useful way to think about the counting problem for $N(a, j)$ is the following: $N(a, j)$ is the number ways to move, in an arbitrary number of steps, on the integer lattice $\mathbb{Z}$, from the point 0 to the point $j$, such that the total length of the path, plus the number of steps, is $a$.

The numbers $N(a, j)$ obey a recursion relation similar to those given in [10]. It is simple to calculate $N(a, j)$ for low $a$ using a computer. Here are the first few values:

| $a$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
|     | $0$ | $0$ | $0$ | $0$ | $0$ | $1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
|     | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
|     | $0$ | $0$ | $0$ | $0$ | $0$ | $1$ | $0$ | $1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
|     | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $1$ | $0$ | $0$ | $0$ | $1$ | $0$ | $0$ | $0$ | $0$ | $0$ |
|     | $0$ | $0$ | $0$ | $0$ | $0$ | $1$ | $1$ | $0$ | $2$ | $0$ | $1$ | $1$ | $0$ | $0$ | $0$ | $0$ |
|     | $0$ | $0$ | $0$ | $0$ | $0$ | $1$ | $2$ | $0$ | $2$ | $0$ | $2$ | $1$ | $0$ | $0$ | $0$ | $0$ |
|     | $0$ | $0$ | $0$ | $0$ | $1$ | $3$ | $1$ | $2$ | $3$ | $2$ | $3$ | $2$ | $1$ | $3$ | $1$ | $0$ |
|     | $0$ | $0$ | $0$ | $1$ | $4$ | $3$ | $2$ | $6$ | $2$ | $6$ | $2$ | $6$ | $2$ | $3$ | $4$ | $1$ |
|     | $0$ | $0$ | $1$ | $5$ | $6$ | $3$ | $9$ | $6$ | $9$ | $8$ | $9$ | $6$ | $9$ | $3$ | $6$ | $5$ |
|     | $0$ | $0$ | $1$ | $6$ | $10$ | $6$ | $12$ | $14$ | $12$ | $18$ | $12$ | $18$ | $12$ | $18$ | $14$ | $20$ |
|     | $0$ | $0$ | $1$ | $7$ | $15$ | $12$ | $16$ | $26$ | $20$ | $32$ | $25$ | $34$ | $25$ | $32$ | $26$ | $16$ |
|     | $1$ | $8$ | $21$ | $22$ | $23$ | $42$ | $38$ | $50$ | $53$ | $54$ | $58$ | $54$ | $53$ | $50$ | $38$ | $42$ |
|     | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |

where $j$ runs horizontally from -10 to 10 and $a$ vertically from 0 to 11.

We will now compute the generating function

$$G(g, z) = \sum_{a=0}^{\infty} \sum_{j=-a}^{a} N(a, j) g^a z^j.$$

To that end, we refer back to the description of $N(a, j)$ in terms of paths on $\mathbb{Z}$. Consider paths with just one step. There is just one such path from 0 to $j$ and it has total length $|j|$. Hence the one-step generating function is

$$G_1(g, z) = g \sum_{n=1}^{\infty} (gz)^n + \left(\frac{g}{z}\right)^n = g^2 \left(\frac{1}{z-g} + \frac{z}{1-gz}\right).$$

The generating function for paths with $n$ steps is just $G_1^n$, and thus we get for the generating function for our problem of interest (i.e. paths with arbitrary many steps)

$$G(g, z) = \sum_{n=1}^{\infty} (G_1(g, z))^n = \frac{g^2(z^2 - 2gz + 1)}{(g + 1)(2gz^2 - (z^2 + z + 1)g + z)}.$$
Because \( N_{\leq} (a, j) \) are partial sums of \( N(a, j) \), the generating function \( G_{\leq} (g, z) \) can be obtained as \([17]\)

\[
G_{\leq} (g, z) \doteq \sum_{a=0}^{\infty} \sum_{j=-a}^{a} N_{\leq} (a, j) g^{a} z^{j} = \frac{1}{1 - g} G(g, z). \tag{9}
\]

These generating functions contain information about the counting problem in a very compact and accessible form. In the following, we will extract some of this information.

### 2.1. The asymptotics of \( N(a, 0) \) and \( N_{\leq} (a, 0) \)

The physical states of the black hole horizon \([4, 9]\) correspond, in our simplified model, to the states with \( j = 0 \). Therefore, the numbers \( N(a, 0) \) and \( N_{\leq} (a, 0) \) are of special interest. We will calculate their generating functions \( G(j=0)(g) \), \( G_{\leq} (j=0)(g) \) and asymptotic behavior.

The generating function \( G(j=0)(g) \) is the coefficient of \( z^{0} \) in \( G(g, z) \),

\[
G(j=0)(g) = \frac{1}{2\pi i} \oint_{C} G(g, z) \frac{1}{z} dz,
\]

where \( C \) is a certain contour. Poles of \( G(g, z)/z \) are

\[
z_0 = 0, \quad z_{\pm} = \frac{-2g^2 + g \pm \sqrt{4g^4 - 4g^3 + g^2 - 2g + 1} - 1}{2g}
\]

with residues

\[
\text{Res}_{z_0} (G(g, z)/z) = -\frac{g}{g+1},
\]

\[
\text{Res}_{z_\pm} (G(g, z)/z) = \pm \frac{(1-g)g}{(g+1)\sqrt{(g-1)(2g-1)(2g^2 + g + 1)}}.
\]

Choosing the contour \( C \) around \( z_0 \) and \( z_+ \) gives

\[
G(j=0)(g) = \frac{(1-g)g}{(g+1)\sqrt{(g-1)(2g-1)(2g^2 + g + 1)}} - \frac{g}{g+1}
\]

\[
= 2g^4 + 2g^5 + 4g^6 + 10g^7 + 18g^8 + 30g^9 + 64g^{10} + 122g^{11} + \cdots.
\tag{10}
\]

According to (9), the generating function for \( N_{\leq} (a, j = 0) \) can be obtained as

\[
N_{\leq}^{(j=0)} (g) = \frac{1}{1 - g} G(j=0)(g)
\]

\[
= 2g^4 + 2g^5 + 4g^6 + 10g^7 + 18g^8 + 30g^9 + 64g^{10} + 122g^{11} + \cdots.
\]

Let us now look at the asymptotic behavior. The coefficients in the series expansion of the second term in (10) are constant, so they do not contribute at all to asymptotic growth, and we can focus on the first term. Its singularity at \( g = 1/2 \) is the one closest to the origin and thus we suspect that \( \ln(N(a, 0)) \sim \ln(2a) \) to leading order. Since the singularities are algebraic, one uses Darboux’s lemma (e.g., \([17]\)) to verify this. It states that for a function of the form \( f(z) = v(z)(1 - z)^{\beta} \) with \( \beta \not\in \mathbb{N} \) and \( v \) analytic in a region containing the unit disk, expanding \( v \) around 1 gives an asymptotic expansion of the function. In particular

\[
[z^{n}] f(z) = v(1)[z^{n}](1 - z)^{\beta} + O(n^{-\beta - 1}) = v(1) \left( \frac{n^{\beta} - 1}{n} \right) + O(n^{-\beta - 2})
\]
where we have introduced the notation $[z^n](\ldots)$ for the coefficient of $z^n$ in the series expansion around zero. Applied to our case we find

\[
[g^a]G^{(j=0)}_{\leq}(g) \sim \frac{1}{3} \sqrt{\pi a} \sqrt{\frac{2a}{\sqrt{a}}}
\]

to highest order. An almost identical calculation gives

\[
[g^a]G^{(j=0)}(g) \sim \frac{1}{3} \sqrt{\pi a} \sqrt{\frac{2a}{\sqrt{a}}}
\]

These approximations are compared with the actual $N(a, 0), N_{\leq}(a, 0)$ in figure 1.

2.2. The asymptotics of unrestricted states

Our next task is to compute the number of states without taking into account the restriction $j = 0$,

\[
T(a) = \sum_{j=-a}^{a} N(a, j), \quad T_{\leq}(a) = \sum_{j=-a}^{a} N_{\leq}(a, j).
\]

The generating function for $T(a)$ is simply $G(g, 1)$,

\[
G(g, 1) = -\frac{2g^2}{2g^2 + g - 1} = 2g^2 + 2g^3 + 6g^4 + 10g^5 + \cdots
\]

\[
= \frac{1}{3} \sum_{a=1}^{\infty} (2(-1)^a + 2^a) g^a
\]
so that we find \( T(a) = (2(-1)^a + 2^a)/3 \) and hence \( \ln(T(a)) \sim \ln(2)a \). For \( T(g) \) we find
\[
G(g, 1) = \frac{2g^2}{(1-g)(2g^2 + g - 1)} = 2g^2 + 4g^3 + 10g^4 + 20g^5 + \cdots
\]
whence \( T(g) = (-3 + (-1)^a + 2^{a+1})/3 \) and \( \ln(T(g)) \sim \ln(2)a \) as well.

2.3. Asymptotics in both variables

Finally we take a look at the joint asymptotics for \( a \) and \( j \) large and comparable. This is in part motivated by the following curious observation: in [10], the \((a, j)\) asymptotics are calculated to be
\[
\ln N(a,j) \sim c_1a + c_2j^2/a
\]
where \( c_1 \) and \( c_2 \) are certain numerical constants of order one. This is reminiscent of an expansion of the Smarr formula for the area of a Kerr black hole
\[
A(M, J) = \frac{8\pi M^2}{1 + \sqrt{1 - J^2/M^4}} = \frac{16\pi M^2}{M^2} - 4\pi \frac{J^2}{M^2} + \cdots,
\]
if one identifies \( a \) with \( M^2 \) and \( j \) with \( J \). \( j \) is bounded by \( a \) [9] so this identification would not lead out of the range of allowed spins for a Kerr black hole. It should also be noted that a similar suggestion has been made before [16]. So, could it be that the states with \( j \neq 0 \) describe rotating black holes?

There are a lot of reasons to doubt that, including that the meaning of \( a \) really should be area, not mass squared, that states with \( j \neq 0 \) are un-physical according to the framework of [4], and that there is a sophisticated treatment of rotating black holes in loop quantum gravity [18] that works quite differently. In fact we find that the asymptotics we calculate do not match this hypothesis at all, as follows.

Determining the asymptotics of multivariate sequences using generating functions is not a very well-known subject and can become quite technical. There is however a beautifully developed general theory that one can rely on (see, e.g., [19] for the generic case). We will use these results in the form presented in [20]. The upshot is that the asymptotics of the multivariate sequence in a certain direction are governed by one or more critical points, certain singular points \( z \) of the generating function. In the case of a two variable generating function \( G(g, z) = I(g, z)/J(g, z) \), critical points are given as solutions \( z = (g_*, z_*) \) to
\[
J(g, z) = 0, \quad n_2g \frac{\partial}{\partial g} J(g, z) = n_1z \frac{\partial}{\partial z} J(g, z)
\]
where \((n_1, n_2) \in \mathbb{N}^2\) gives the direction in which the asymptotics of the sequence are taken. \( J \) and \( I \) have to satisfy several properties, for which we refer the reader to [20]. Which of the critical points actually contribute to the asymptotics can be determined by a straightforward but rather tedious analysis which we will circumvent here. If a critical point \((g_*, z_*)\) does contribute to the asymptotics of the sub-sequence \( c_n \equiv c_{n, n, n, n} \) of the multivariate sequence \( c_{m, n, a} \), it does so by a factor \( g_*^{n_1}z_*^{n_2}n_3^{n_4} \) to highest order.

We are interested in the asymptotics of \( N(a, j) \) for \( \alpha := j/a \) constant. With the help of Mathematica, we compute the critical points for the problem at hand. There are eight critical
Figure 2. $\ln(N(a,j))$ (dots) and the asymptotic result (solid line) compared for two different values of $a$.

Figure 3. Comparison of $\ln(N(a,j))$ (dots) and the function $s$ from (12) for $a = 100$.

points, all depending on $\alpha$. Instead of doing a lengthy analysis as to which of these solutions contributes we simply pick the solution with the right limiting values at $\alpha = 0, 1$. It reads

$$g_* = -\frac{(\alpha - 1)((Y + 11)\alpha^2 + (Y + 4)\alpha + (8\alpha^2 - X(Y - 1))\alpha - XY)}{4\alpha(\alpha + 1)(-(Y - 1)\alpha^2 + X + Y)}$$

$$z_* = \frac{-(Y - 1)\alpha^2 + X + Y}{2(1 - \alpha^2)}$$
where we have used the abbreviation

\[ X = \sqrt{8\alpha^4 - 11\alpha^2 + 4}, \quad Y = \sqrt{\frac{\alpha^2(5\alpha^2 + 2X - 3)}{(1 - \alpha^2)^2}}. \]

The asymptotic behavior is thus given by

\[ \ln(N(a, \alpha a)) \sim -\ln(g_\ast a) - \ln(z_\ast a\alpha). \quad (11) \]

This asymptotic formula works quite well. For comparison we have plotted (11) and the actual values for two different values of \( a \).

What about the Smarr formula? We have found that, to highest order, \( \ln(N(a, j)) \) is indeed a function of the form \( af(j/a) \). However, the function \( f \) is very complicated and not what one would expect if the suggested interpretation were correct. Moreover the numerical fit is quite bad, as figure 3 demonstrates. There we have plotted the numeric result in terms of \( \alpha = j/a \) (and appropriately normalized), together with the function

\[ s(\alpha) = 1 + \sqrt{1 - \alpha^2}. \quad (12) \]

Thus we have ample reason to throw out the hypothesis that we stated in the beginning of this subsection.

3. Conclusions

Let us summarize our results and compare them to what is known [10] about the counting for black holes using the correct area spectrum (1). To start out, all the statements made in [10] for the full spectrum are also borne out in our model on a qualitative level. In particular we see the following.

- The number of states grows exponentially with area.
- The number of (un-physical) states with \( j \) arbitrary grows with the same rate as that of the physical states (\( j = 0 \)) to highest order.
- The highest order growth of \( N_\leq(A) \) and \( N(A) \) is the same.
- The next to leading order term in the logarithm of the number of \( j = 0 \)-states is \(-1/2 \ln(a)\).

This confirms the present model as a nice and simplified ‘laboratory’ for questions about the full theory. Vice versa, the present note can be read as giving confirmation of the results [10]. We should note however that all the numerical factors appearing in our model are different from (although close to) the ones for the correct spectrum. To give an example, the Barbero–Immirzi parameter obtained in [10] is \( \gamma_M \approx 0.24 \) whereas the one that would result if our model were correct is \( \ln(2)/\pi \approx 0.22 \). This is not so surprising if one remembers a result from [9] that shows that states with \( m \) higher than \( 1/2 \) certainly contribute substantially to the overall counting; however, their numbers are increasingly suppressed with increasing \( m \). Thus, among all the states to be counted, those for which our approximation (3) is relatively bad are most numerous.

It is curious that the coefficient of the \( \ln(\text{area}) \)-term in the entropy seems to be very robust, as it is \(-1/2 \) in the present framework as well as in [10], [21], and elsewhere.

There is one last point worth mentioning, one that on first sight seems trivial: since the spectrum is equidistant, the number of black hole states, and hence the entropy will grow in discrete steps. The height of the steps may vary with area, but the size of the steps is always \( 4\pi \gamma f_\mu^2 \). Surprisingly, a similar (but certainly more complex) behavior has been observed [21, 22] for the state counting in the full theory. One might at first think that this is not an accident. After all, the ‘ladder’ observed in [21, 22] may perhaps be understood as a
perturbation of the half integer steps in the present model, much as the full spectrum (1) can be viewed as a perturbation (3) around the equidistant one. We will investigate this point in much more detail in a forthcoming paper [23]. The upshot is that the explanation of the ‘ladder’ is not that simple. It is related to properties of the area spectrum, in particular also to (3), but in a rather involved way.

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Appendix A. An alternative modified area spectrum

The form of the modified area spectrum (2) in the main text was chosen so as to approximate the actual spectrum arising in loop quantum gravity as good as possible while affording drastic simplification by being equidistant. There is another equidistantly spaced modification of the area spectrum that has been considered in the literature. It differs form (2) by removing the constant \(1/2\),

\[ A_j = 8\pi \gamma l_p^2 j. \]  

(A.1)

It can be argued that this spectrum does arise in loop quantum gravity, upon quantizing the area of a non-rotating black hole following an alternative route [11].5 In this appendix, we will derive results for the asymptotic behavior of the entropy obtained with the spectrum (A.1). We will however be more terse than in the main text. We will get rid of all units and some constants by considering the quantity

\[ N(a, j) = \left\{ (m_1, m_2, \ldots), \ m_i \in \mathbb{Z}\setminus\{0\} : \sum_i m_i = j, \sum_i |m_i| = a \right\}. \]

A useful way to think about this is the following: \(N(a, j)\) is the number of ways to move, in an arbitrary number of steps, on the integer lattice \(\mathbb{Z}\), from the point 0 to the point \(j\), such that the total length of the path is \(a\).

The numbers \(N(a, j)\) obey a recursion relation similar to those given in [10]. Up to simple changes of variables, they are A035002 in [24], and they are closely related to the combinatorial problems in [25].6

It is simple to calculate \(N(a, j)\) for low \(a\) using a computer. Here are the first few values:

\[
\begin{align*}
0 & 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 \\
0 & 0 0 0 0 0 0 0 1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 2 0 2 0 0 0 0 0 0 0 0 0 4 0 5 0 5 0 4 0 0 0 0 0 0 8 0 12 0 14 0 12 0 8 0 0 0 0 0 0 16 0 28 0 37 0 37 0 28 0 16 0 0 0 32 0 64 0 94 0 106 0 94 0 64 0 32 0 64 0 144 0 232 0 289 0 289 0 232 0 144 0 64
\end{align*}
\]

where \(j\) runs horizontally and \(a\) vertically.

\footnote{This alternative quantization is only possible for the black hole horizon. For other surfaces, only (1) applies.}

\footnote{See especially section 7 in [25] where the object under study is essentially \(N(a, 0)\).}
The one-step generating function for $N$ is

$$G_1(g, z) = \sum_{n=1}^{\infty} (gz)^n + \left(\frac{g}{z}\right) = -\frac{g}{g-z} - \frac{gz}{g-z}.$$  (A.2)

The generating function for paths with $n$ steps is just $G_n^1$, and thus we get for the generating function for $N$

$$G(g, z) = \sum_{n=1}^{\infty} (G_1(g, z))^n = -\frac{g(-z^2 + 2gz - 1)}{3g^2 - 2z^2g - 2g + z}.$$

A.1. The asymptotics of $N(a, 0)$

The generating function for $N(a, 0)$ is (formally) the coefficient of $z^0$ in $G(g, z)$,

$$G(j=0)(g) = \frac{1}{2\pi i} \oint_C \frac{G(g, z) \, dz}{z},$$

where $C$ is a certain contour. Poles of $G(g, z)/z$ are

$$z_{\pm} = \frac{3g^2 \pm \sqrt{9g^4 - 10g^2 + 1} + 1}{4g}, \quad z_0 = 0$$

with residues

$$\text{Res}_{z_0}(G(g, z)/z) = -\frac{1}{2}, \quad \text{Res}_{z_{\pm}}(G(g, z)/z) = \pm \frac{1 - g^2}{2\sqrt{9g^4 - 10g^2 + 1}}.$$

Choosing the contour $C$ around $z_0$ and $z_+$ gives

$$G(j=0)(g) = \frac{1 - g^2}{2\sqrt{9g^4 - 10g^2 + 1}} - \frac{1}{2} = -\frac{\sqrt{g^2 - 1}}{2\sqrt{(g - 1/3)(g + 1/3)}} - \frac{1}{2}$$

$$= 2g^2 + 14g^4 + 106g^6 + 838g^8 + 6802g^{10} + O(g^{11}).$$

The singularities $g = \pm 1/3$ suggest that $\ln(N(a, 0)) \propto \ln(3)a$ to leading order. ‘Darboux’s lemma’ confirms this. We find indeed

$$[g^a]G(j=0)(g) = [u^{a/2}]G(j=0)(\sqrt{u}) = \frac{1}{2} 2^{a/2} [u^{a/2}](1-u)^{-1/2} \sqrt{9-u}$$

$$\propto \sqrt{23^a} \left(\frac{a/2 - 1/2}{a/2}\right) \propto \frac{2}{\sqrt{\pi}} \frac{1}{\sqrt{a}} 3^a$$

to highest order.

A.2. The asymptotics of unrestricted states

Our next task is to compute the number of states without taking into account the restriction $j = 0$,

$$T(a) = \sum_{j=-a}^{a} N(a, j).$$
The generating function for $T(a)$ is simply $G(g, 1)$, 

$$G(g, 1) = \frac{2g}{1 - 3g} = 2 \sum_{n=1}^{\infty} 3^{a-1} g^n$$

so that we find $T(a) = 2 \cdot 3^{a-1}$ and hence $\ln(T(a)) \propto \ln(3a)$.

### A.3. Extremal states

Out of curiosity, let us also calculate the number of ‘extremal configurations’, $N(a, a)$. It this is easiest by going back to (A.2) and modifying it appropriately. In this case, we only need one variable and we have

$$G_1(g) = \sum_{n=1}^{\infty} g^n = \frac{g}{1 - g}$$

whence

$$G(g) = \sum_{n=1}^{\infty} \frac{g^n}{(1 - g)^n} = \frac{g}{1 - 2g} = \sum_{a=1}^{2^{a-1}} g^a.$$  

(This is the well-known formula for the number of ordered partitions of an integer $a$.) Hence $N(a, a) = 2^{a-1}$ and $\ln(N(a, a)) \propto \ln(2a)$.

### A.4. Asymptotics in both variables

Finally we take a look at the asymptotics of $N(a, j)$ for $\alpha := j/a$ constant. With the help of Mathematica, we compute the critical points for the problem at hand. There are four critical points, all depending on $\alpha$. Instead of doing a lengthy analysis as to which of these solutions contributes we simply pick the solution with the right limiting values at $\alpha = 0, 1$. It reads

$$g_* = \frac{-L + 3\alpha}{3(\alpha + 1)} \frac{\sqrt{1 - \alpha^2}}{\sqrt{-2\alpha^2 + 2L \alpha + 4}}, \quad z_* = \frac{(L - \alpha)\alpha + 2}{2 - 2\alpha^2}.$$
where we have used the abbreviation

\[ L = \sqrt{4 - 3\alpha^2}. \]

The asymptotic behavior is thus given by

\[ \ln(N(\alpha, \alpha a)) \propto -\ln(g_*) a - \ln(z_*) \alpha a. \]

What about the Smarr formula in this case? We have found that, to highest order, \( \ln(N(a, j)) \) is indeed a function of the form \( af(j/a) \). However, the function \( f \) is very complicated and not what one would expect if the suggested interpretation were correct. However, the situation is not catastrophic either: in figure A1 we have plotted the numeric result in terms of \( \alpha = j/a \) (and appropriately normalized), together with the function

\[ s(\alpha) = 1 + \sqrt{1 - \alpha^2}. \]

(A.3)

One sees that the numerical correspondence is not bad, but certainly not convincing.

References

[1] Thiemann T 2001 Introduction to modern canonical quantum general relativity Preprint gr-qc/0110034
[2] Ashtekar A and Lewandowski J 2004 Background independent quantum gravity: a status report Class. Quantum Grav. 21 R53 (Preprint gr-qc/0404018)
[3] Rovelli C 1996 Black hole entropy from loop quantum gravity Phys. Rev. Lett. 77 3288 (Preprint gr-qc/9603063)
[4] Ashtekar A, Baez J, Corichi A and Krasnov K 1998 Quantum geometry and black hole entropy Phys. Rev. Lett. 80 904 (Preprint gr-qc/9710007)
[5] Ashtekar A, Baez J C and Krasnov K 2000 Quantum geometry of isolated horizons and black hole entropy Adv. Theor. Math. Phys. 4 1 (Preprint gr-qc/0005126)
[6] Alekseev A, Polychronakos A P and Smedback M 2003 On area and entropy of a black hole Phys. Lett. B 574 296 (Preprint hep-th/0004036)
[7] Polychronakos A P 2004 Area spectrum and quasinormal modes of black holes Phys. Rev. D 69 044010 (Preprint hep-th/0304135)
[8] Gour G and Suneeta V 2004 Comparison of area spectra in loop quantum gravity Class. Quantum Grav. 21 3405 (Preprint gr-qc/0401110)
[9] Domagala M and Lewandowski J 2004 Black hole entropy from quantum geometry Class. Quantum Grav. 21 5233 (Preprint gr-qc/0407051)
[10] Meissner K A 2004 Black hole entropy in loop quantum gravity Class. Quantum Grav. 21 5245 (Preprint gr-qc/0407052)
[11] Lewandowski J 2005 personal communication
[12] Khraplovich I B 2004 Holographic bound and spectrum of quantized black hole Preprint gr-qc/0411109
[13] Tamaki T and Nonura H 2005 Ambiguity of black hole entropy in loop quantum gravity Phys. Rev. D 72 107501 (Preprint hep-th/0508142)
[14] Ghosh A and Mitra P 2006 Counting black hole microscopic states in loop quantum gravity Phys. Rev. D 74 064026 (Preprint hep-th/0606125)
[15] Ghosh A and Mitra P 2005 An improved lower bound on black hole entropy in the quantum geometry approach Phys. Lett. B 616 144 (Preprint gr-qc/0411035)
[16] Krasnov K 1999 Quanta of geometry and rotating black holes Class. Quantum Grav. 16 L15 (Preprint gr-qc/9902015)
[17] Wilf H S 1994 Generatingfunctionology 2nd edn (Boston, MA: Academic) (also available online from the homepage of its author)
[18] Ashtekar A, Engle J and Van Den Broeck C 2005 Quantum horizons and black hole entropy: inclusion of distortion and rotation Class. Quantum Grav. 22 L27 (Preprint gr-qc/0412003)
[19] Pemantle R and Wilson M C 2002 Asymptotics of multivariate sequences: I. Smooth points of the singular variety J. Comb. Theory A 97 129
[20] Raichev A and Wilson M C 2007 A new method for computing asymptotics of diagonal coefficients of multivariate generating functions Preprint math/0702595
[21] Corichi A, Diaz-Polo J and Fernandez-Borja E 2007 Black hole entropy quantization Phys. Rev. Lett. 98 181301 (Preprint gr-qc/0609122)
[22] Diaz-Polo J and Fernandez-Borja E 2007 Note on black hole radiation spectrum in loop quantum gravity
Preprint 0706.1979

[23] Sahlmann H 2007 Toward explaining black hole entropy quantization in loop quantum gravity Phys. Rev. D 76
104050 (Preprint 0709.2433)

[24] Sloane N J A 2007 The On-Line Encyclopedia of Integer Sequences, published electronically at
www.research.att.com/~njas/sequences/

[25] Coker C 2003 Enumerating a class of lattice paths Discrete Math. 271 13