NUMERICAL SOLUTION OF A NONLINEAR REACTION-DIFFUSION PROBLEM IN THE CASE OF HS-REGIME

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ABSTRACT. In this paper, the author propose a numerical method to compute the solution of the Cauchy problem: \( w_t - (w^m w_x)_x = w^p \), the initial condition is a nonnegative function with compact support, \( m > 0, 1 < p < m + 1 \). The problem is split in two parts: A hyperbolic term solved by using the Hopf and Lax formula and a parabolic term solved by a backward linearized Euler method in time and a finite element method in space. Estimates of the numerical solution are obtained and it is proved that any numerical solution is unbounded.

1. Introduction

In this paper, we study a numerical method to compute the solution of the Cauchy problem:

\[
\begin{align*}
  w_t - (w^m w_x)_x &= w^p, \quad t > 0, \quad x \in \mathbb{R} \\
  w(x, 0) &= w_0(x) \geq 0, \quad x \in \mathbb{R}
\end{align*}
\]

\( w_0 \) is a function with compact support, \( m > 0, 1 < p < m + 1 \).

Samarskii et al [10], see also [1], [2], [3], [6], [5], [4], [9] have obtained theoretical results on this problem. In the case \( 1 < p < m + 1 \), the numerical solution blows up in finite time and there is no localization (HS-regime) that is \( u(t, x) \to \infty \) in \( \mathbb{R} \) if \( t \to T_0 \).

A numerical method to solve (1.1) has been proposed in the case of S-regime \( (p = m + 1) \) in [7] and in the case of LS-regime \( (p > m + 1) \) [8]. If we denote \( \omega_L = \{ x \in \mathbb{R} / u(T_0^+, x) = \infty \} \), in the case of S-regime, \( L^* = mes(\omega_L) \) is positive while in the case of LS-regime, \( L^* = 0 \). The problem is solved by using a splitting method; for that, it is more convenient to work with the function \( u = w^m \). Problem (1.1) may be written:

\[
\begin{align*}
  u_t - \frac{1}{m} u_x^2 - uu_{xx} &= mu^{q+1}, \quad t > 0, \quad x \in \mathbb{R} \\
  u(x, 0) &= u_0(x) = w_0^m(x), \quad x \in \mathbb{R}
\end{align*}
\]

with \( q = \frac{p-1}{m}, \quad m > 0, \quad q < 1 \).

This problem is split in two parts: a hyperbolic problem which will be solved exactly at the nodes at each time step and allows the extension of the domain and a parabolic problem which will be solved by a backward linearized Euler method which allows the blow up of the solution.

In [7], [8], the convergence of the scheme has been proved in the cases \( q = 1 \) and \( q > 1 \). It has also been proved that for \( q = 1 \), the numerical solution blows up in finite time for any initial condition.
and that its support remains bounded if the initial condition is smaller than a self-similar solution. In the case, $1 \leq q < \frac{m+2}{m}$, any numerical solution is unbounded while for $q > \frac{m+2}{m}$ if $p > m + 3$ if the initial condition is sufficiently small, a global solution exists and for $q \geq \frac{m+2}{m}$ for large initial condition, the solution blows up in one point and that for $q = \frac{m+2}{m}$ also, if the initial condition is sufficiently small, a global solution exists.

Here, we generalize this method to the case $q < 1$.

An outline of the paper is as follows:

In Section 2, we present the numerical scheme. In Section 3, we obtain estimates of the approximate solution and of its derivative in space. In Section 4, we prove that if $q < 1$, any numerical solution is unbounded.

2. Definition of the numerical solution

In order to solve problem (1.2), we separate it in two parts: a hyperbolic problem

\begin{equation}
 u_t - \frac{1}{m}(u_x)^2 = 0, \ x \in \mathbb{R}, \ t > 0
\end{equation}

and a parabolic problem

\begin{equation}
 u_t - uu_{xx} = mu^{q+1}, \ x \in \mathbb{R}, \ t > 0
\end{equation}

We denote by $\Delta t_n$ the time increment between the time levels $t_n$ and $t_{n+1}$, $n \geq 0$ and by $u^n_h$ the approximate solution at the time level $t_n$. This solution will be in a finite-dimensional space which will be defined below.

Without loss of generality, we can assume that the initial condition is a continuous function with a symmetric compact support $[-s_0, s_0]$. Let $N \in \mathbb{N}$, the space step $h$ is defined by $h = \frac{s_0}{N}$; we note $x_i = ih, \ i \in \mathbb{Z}$, $I_i = (x_{i-1}, x_i)$ and we define the finite dimensional space $V^0_h$ by

\begin{equation}
 V^0_h = \{ \phi_h \in C^0(\mathbb{R})/ \phi_h(x) = 0, \ x \notin [-s_0, s_0], \ \phi_h|_{I_i} \in P_1, \ i = -N + 1, N \}
\end{equation}

For $\phi_h \in V^0_h$, we note $\phi_i = \phi_h(x_i)$, $i \in \mathbb{Z}$ and for any function $v \in C^0(\mathbb{R})$ with compact support $[-s_0, s_0]$, we define its interpolate by $\pi^0_h v \in V^0_h$ and $\pi^0_h v(x_i) = v(x_i), \ i \in \mathbb{Z}$.

The support of the solution $u^n_h$ will be denoted $[-s^n_-, s^n_+]$ and will be computed at each time level by solving (2.1).

We denote

\begin{equation}
 N^n_- = \left[ \frac{s^n_-}{h} \right], \ N^n_+ = \left[ \frac{s^n_+}{h} \right], \ h^n_- = s^n_- - (N^n_- - 1)h, \ h^n_+ = s^n_+ - (N^n_+ - 1)h
\end{equation}
We then define the finite-dimensional space $V_h^n$ by:

$$V_h^n = \left\{ \phi_h \in C^0(\mathbb{R})/\phi_h(x) = 0, \ x \not\in [-s^n_-, s^n_+] \mid \begin{array}{l}
\phi_h|_{I_i} \in P_1, i = -N^n_+, 2N^n_+ - 1 \\
\phi_h(-s^n_-, x_{-N^n_+ + 1}), \phi_h(x_{N^n_+ - 1}, s^n_+) \in P_1
\end{array} \right\}$$

and denote by $\pi_h^n v$ the Lagrange interpolate in $V_h^n$ of a function $v \in C^0(\mathbb{R})$ with a compact support in $[-s^n_-, s^n_+]$.

The solution $u_{n+1}^h$ at the time level $t_{n+1}$ is computed in two steps: knowing $u_n^h$, we compute first an approximate solution of (2.1) which will be denoted $u_{n+\frac{1}{2}}^h$; then starting with this intermediate value, we compute an approximate solution of (2.2) at time level $t_{n+1}$.

Then if $S$ is the semi-group operator associated with (2.1), we define

$$u_{n+\frac{1}{2}}^h = S(\Delta t_n)u_n^h.$$ 

The support of this function will be denoted $[-s^n_{n+1}, s^n_{n+1}]$ and the interpolate of $u_{n+\frac{1}{2}}^h$ in $V_h^{n+1}$ will be denoted $u_{n+\frac{1}{2}}^h = \pi_h^{n+1} u_{n+\frac{1}{2}}^h$.

Then starting with $u_{n+\frac{1}{2}}^h$, the function $u_{n+1}^h$ is obtained by solving (2.2) with a backward linearized Euler method in time and a $P_1$-finite element method with numerical integration in space. This function has the same support as $u_{n+\frac{1}{2}}^h$.

### 2.1. Computation of the solution of the hyperbolic problem.

The hyperbolic problem is independent of $q$; we use the same method as in [7] for the case $q = 1$. It is not necessary in this case to use $P_2$-interpolation on the last interval since there is no localization. We use the Hopf and Lax formula which gives explicitly the solution to (2.1) with the starting data $u_h^n$ at the time level $t = t_n$. Here, we simply recall the results obtained in [7].

We define the piecewise constant function $u_i^n$ by

$$v^n_i = \frac{u^n_i - u_{i-1}^n}{h} \text{ on } I_i, \quad -N^n_+ + 2 \leq i \leq N^n_+ - 1$$

$$v^n_{N^n_+} = -\frac{u^n_{N^n_+ - 1}}{h^{N^n_+}} \text{ on } (x_{N^n_+ - 1}, s^n_+)$$

$$v^n_{-N^n_-} = \frac{u^n_{-N^n_- + 1}}{h^{-N^n_-}} \text{ on } (-s^n_-, x_{-N^n_- + 1}).$$

Let us denote $r_n = \frac{\Delta t_n}{h}$, $\|v\|_s = \|v\|_{L^s(\mathbb{R})}$, $s > 0$. 

Proposition 2.1. If the following stability condition

\[(2.4) \quad r_n \|v_n^h\|_\infty \leq \frac{m}{2}\]

is satisfied, then the solution \(u_{n+1}^n\) of (2.1) is defined by

\[u_i^{n+\frac{1}{2}} = u_i^n + \frac{\Delta t_n}{m} (\max(0, -v_i^n, v_{i+1}^n))^2, \quad -N^n + 1 \leq i \leq N^n - 1\]

\[s_{+}^{n+1} = s_{+}^{n} - \frac{\Delta t_n}{m} v_{N^n}^n\]

\[s_{-}^{n+1} = s_{-}^{n} + \frac{\Delta t_n}{m} v_{-N^n+1}^n\]

If \(N_{n+1}^n = N^n + 1\), we get

\[u_{N^n_+}^{n+\frac{1}{2}} = \left(1 - \frac{h}{h^n_+}\right) u_{N^n_+ - 1}^{n} + \frac{\Delta t_n}{m} \left(v_{N^n_+}^n\right)^2\]

and we get analogous formula at the other end of the support.

2.2. Computation of the parabolic problem.

The approximate solution at \(t_{n+1}\) is now obtained by solving problem (2.2). We introduce the approximate scalar product on \(V_{n+1}^n\):

\[(\phi_h, \psi_h)_h = \frac{1}{2} (h_{n+1}^- + h) \phi_{-N^n+1}^n \psi_{-N^n+1}^n + h \sum_{i=-N^n+1+2}^{i=N^n+1-2} \phi_i \psi_i + \frac{1}{2} (h_{n+1}^- + h) \phi_{N^n+1-1} \psi_{N^n+1-1}^n.\]

We define \(u_{n+1}^n\) as the solution of the following problem:

\[(2.5) \quad \forall \phi_h \in V_{n+1}^n, \quad \begin{cases} 
(u_{n+1}^n, \phi_h)_h + \Delta t_n ((u_{n+1}^n)_x, (u_{n+1}^n_+)^q \phi_h)_x = & (u_{n+1}^{n+\frac{1}{2}}, \phi_h)_h + m q \Delta t_n \left(\left(u_{n+1}^{n+\frac{1}{2}}\right)_x, \left(u_{n+1}^{n+\frac{1}{2}}\right)_x \phi_h \right)_h \\
+ m (1 - q) \Delta t_n \left(\left(u_{n+1}^{n+\frac{1}{2}}\right)_x, \left(u_{n+1}^{n+\frac{1}{2}}\right)_x \phi_h \right)_h & 
\end{cases}\]

The second member of (2.2) is splitted in two parts in order to obtain the \(L^\infty\)-estimate of \(u_{n+1}^n\). This equation may be written:

\[\left(1 - m q \Delta t_n \left(u_{i+1}^{n+\frac{1}{2}}\right)_x\right) u_{i+1}^{n+1} + \frac{\Delta t_n}{h^2} u_{i+1}^{n+\frac{1}{2}} (2 u_{i+1}^{n+1} - u_{i-1}^{n+1} - u_{i+1}^{n+1}) =\]
\[ u_i^{n+\frac{1}{2}} + m(1 - q)\Delta t_n \left( u_i^{n+\frac{1}{2}} \right)^{q+1}, -N_-^{n+1} + 2 \leq i \leq N_+^{n+1} - 2 \]

\[
\left( 1 - mq\Delta t_n \left( u_{N_+^{n+1} - 1}^{n+\frac{1}{2}} \right)^q \right) u_{N_+^{n+1} - 1}^{n+1} + \frac{\Delta t_n}{h} u_{N_+^{n+1} - 1}^{n+\frac{1}{2}} \left( \frac{2}{h_{N_+^{n+1}}} u_{N_+^{n+1} - 1}^{n+1} - \frac{2}{h_{N_-^{n+1}}} u_{N_-^{n+1} - 2}^{n+1} \right)
\]

\[
= u_{N_+^{n+1} - 1}^{n+\frac{1}{2}} + m(1 - q)\Delta t_n \left( u_{N_+^{n+1} - 1}^{n+\frac{1}{2}} \right)^{q+1}
\]

\[
\left( 1 - mq\Delta t_n \left( u_{-N_-^{n+1} + 1}^{n+\frac{1}{2}} \right)^q \right) u_{-N_-^{n+1} + 1}^{n+1} + \frac{\Delta t_n}{h} u_{-N_-^{n+1} + 1}^{n+\frac{1}{2}} \left( \frac{2}{h_{-N_-^{n+1}}} u_{-N_-^{n+1} + 1}^{n+1} - \frac{2}{h_{-N_-^{n+1} + 2}} u_{-N_-^{n+1} + 2}^{n+1} \right)
\]

\[
= u_{-N_-^{n+1} + 1}^{n+\frac{1}{2}} + m(1 - q)\Delta t_n \left( u_{-N_-^{n+1} + 1}^{n+\frac{1}{2}} \right)^{q+1}
\]

We get immediately the result:

**Proposition 2.2.** If the hypotheses of proposition 2.2 are satisfied and if \( u_{h}^{n+\frac{1}{2}} \) satisfies

\[
(2.6) \quad mq\Delta t_n \left\| u_{h}^{n+\frac{1}{2}} \right\|^{q}_{\infty} < 1
\]

then the solution \( u_{h}^{n+1} \) of (2.2) is unique and nonnegative.

2.3. Properties of the scheme.

**Lemma 2.3.** If the hypotheses of proposition 2.2 are satisfied, then the following estimate holds:

\[
(2.7) \quad \left\| u_{h}^{n+1} \right\|_{\infty} \leq \frac{\left\| u_{h}^{n} \right\|_{\infty} (1 + m(1 - q)\Delta t_n \left\| u_{h}^{n} \right\|^{q}_{\infty})}{1 - mq\Delta t_n \left\| u_{h}^{n} \right\|^{q}_{\infty}}
\]
Proof: We get immediately from the Hopf and Lax formula that $\|u^{n+1}_h\|_\infty \leq \|u^n_h\|_\infty$. If we denote $i_0$ the index such that $u_{i_0}^{n+1} = \|u^{n+1}_h\|_\infty$, we get from (2.2), (2.2), (2.2) that

$$u_i^{n+1} \leq u_{i_0}^{n+\frac{1}{2}} \frac{1 + m(1 - q) \Delta t_n \left( u_{i_0}^{n+\frac{1}{2}} \right)^q}{1 - mq \Delta t_n \|u^n_h\|_\infty}$$

which proves the lemma.

We deduce the following theorem:

**Theorem 2.4.** Under the hypotheses of proposition 2.2, the numerical solution exists at least until the time

$$T_1 = \frac{1}{mq \|u^0_h\|^q}$$

and the following estimate holds:

$$\|u^n_h\|_\infty \leq \frac{\|u^0_h\|_\infty}{\left(1 - mq t_n \|u^0_h\|_\infty \right)^\frac{1}{q}}$$

Proof: This result is proved recurrently. It is true for $n = 0$. If we suppose that we have estimate (2.9) at the time level $t_n$, we get from (2.7), at the time $t_{n+1}$:

$$\|u_{i_0}^{n+1}\|_\infty \leq \|u^n_h\|_\infty \frac{1 + m(1 - q) \Delta t_n \|u^n_h\|_\infty}{1 - mq \Delta t_n \|u^n_h\|_\infty}$$

or

$$\|u_{i_0}^{n+1}\|_\infty \leq \|u^n_h\|_\infty \frac{1 - mq t_{n+1} \|u^n_h\|_\infty + m(1 - q) \Delta t_n \|u^n_h\|_\infty}{\left(1 - mq t_{n+1} \|u^n_h\|_\infty - \left(1 - mq t_{n+1} \|u^n_h\|_\infty \right)^\frac{1}{q} \right)^\frac{1}{q}}$$

The inequality (2.9) will be satisfied at the time $t_{n+1}$ if:

$$(1 - mq t_{n+1} \|u^n_h\|_\infty) \frac{1}{q} \leq (1 - mq t_{n+1} \|u^n_h\|_\infty) \frac{1}{q} - 1$$

By using the Taylor formula, we get:

$$(1 - mq t_n \|u^n_h\|_\infty)^\frac{1}{q} - (1 - mq t_{n+1} \|u^n_h\|_\infty)^\frac{1}{q} \geq m \Delta t_n \|u^n_h\|_\infty \left(1 - mq t_{n+1} \|u^n_h\|_\infty\right)^\frac{1}{q} - 1$$
and the inequality (2.3) is satisfied:

**Lemma 2.5.** Under the hypothesis of Proposition (2.2), we have the estimate:

\[ \| v_h^n \|_\infty \leq C \]

for \( t_n \leq T < T_1 \) where \( C \) is a constant depending on \( T \) and \( u_0 \).

**Proof:** We have the inequality ([7]): \( \| v_h^{n+\frac{1}{2}} \| \leq \| v_h^n \|_\infty \).

It remains to estimate \( \| v_h^{n+1} \|_\infty \).

From (2.2), we deduce the following equation satisfied by \( v_h^{n+1} \):

\[
\left( 1 - mq \Delta t_n \left( u_i^{n+\frac{1}{2}} \right)^q \right) v_i^{n+1} + \frac{\Delta t_n}{h^2} \left( v_{i-1}^{n+1} \left( u_i^{n+\frac{1}{2}} + u_{i-1}^{n+\frac{1}{2}} \right) - v_{i-1}^{n+1} u_i^{n+\frac{1}{2}} - v_i^{n+1} u_{i-1}^{n+\frac{1}{2}} \right) = v_i^{n+\frac{1}{2}} + mq \frac{\Delta t_n}{h} \left( u_i^{n+1} - u_{i-1}^{n+1} \right) u_i^{n+\frac{1}{2}} + m \left( 1 - q \right) \frac{\Delta t_n}{h} \left( u_i^{n+\frac{1}{2}} u_i^{n+1} - u_i^{n+\frac{1}{2}} u_{i-1}^{n+\frac{1}{2}} \right)
\]

\[-N_i^{n+1} + 3 \leq i \leq N_i^{n+1} - 2\]

and we have analogous inequalities for \( i = N_i^{n+1} - 1, N_i^{n+1}, -N_i^{n+1} + 2, -N_i^{n+1} + 1 \).

By using (2.2) for \( i - 1 \), we can replace \( u_i^{n+\frac{1}{2}} \) in the second member by its expression in function of the values of \( u_i^{n+\frac{1}{2}} \) and \( v_i^{n+1} \) and we get:

\[
\left( 1 - mq \Delta t_n \left( u_i^{n+\frac{1}{2}} \right)^q \right) v_i^{n+1} + \Delta t_n \left( v_{i-1}^{n+1} - v_i^{n+1} \right) u_i^{n+\frac{1}{2}} + \frac{1 - mq \Delta t_n \left( u_i^{n+\frac{1}{2}} - u_{i-1}^{n+\frac{1}{2}} \right)^q}{1 - mq \Delta t_n \left( u_i^{n+\frac{1}{2}} \right)^q} \]

\[
= v_i^{n+\frac{1}{2}} + mq \frac{\Delta t_n}{h} \left( u_i^{n+\frac{1}{2}} - u_{i-1}^{n+\frac{1}{2}} \right) u_i^{n+\frac{1}{2}} + m \left( 1 - q \right) \frac{\Delta t_n \left( u_i^{n+\frac{1}{2}} \right)^q}{1 - mq \Delta t_n \left( u_i^{n+\frac{1}{2}} \right)^q}
\]

\[+m(1-q)\frac{\Delta t_n}{h} \left( u_i^{n+\frac{1}{2}} \right)^{q+1} - \left( u_i^{n+\frac{1}{2}} \right)^{q+1} + m \left( 1 - q \right) \frac{\Delta t_n \left( u_i^{n+\frac{1}{2}} \right)^{q+1}}{1 - mq \Delta t_n \left( u_i^{n+\frac{1}{2}} \right)^{q+1}}
\]
Let $i_0$ the index such that $\left| v_{i_0}^{n+1} \right| = \left\| v_h^{n+1} \right\|_\infty$. From the previous equality, we get:

\[
(1 - mq\Delta t_n \left| v_h^n \right|_\infty) \left| v_{i_0}^{n+1} \right| \leq \left| v_{i_0}^{n+\frac{1}{2}} \right| + m(1 - q)\frac{\Delta t_n}{h} \left| v_{i_0}^{n+\frac{1}{2}} \right|^{q+1} - \left| v_{i_0}^{n+\frac{1}{2}} \right|^{q+1}
\]

\[+mq\frac{\Delta t_n}{h} \left| v_{i_0}^{n+\frac{1}{2}} \right|^{q} - \left( u_{i_0}^{n+\frac{1}{2}} \right)^q \frac{1 + m(1 - q)\Delta t_n}{1 - mq\Delta t_n \left| u_h^n \right|_\infty} \left| v_{i_0}^{n+\frac{1}{2}} \right|^{q+1}
\]

We easily obtain:

\[
\frac{1}{h} \left| \left( u_{i_0}^{n+\frac{1}{2}} \right)^{q+1} - \left( u_{i_0}^{n+\frac{1}{2}} \right)^{q+1} \right| \leq (q + 1) \left| u_h^0 \right|_\infty \left| v_{i_0}^{n+\frac{1}{2}} \right|
\]

and

\[
\frac{1}{h} \left| \left( u_{i_0}^{n+\frac{1}{2}} \right)^{q} - \left( u_{i_0}^{n+\frac{1}{2}} \right)^{q} \right| \left| u_{i_0}^{n+\frac{1}{2}} \right| \leq \left| v_{i_0}^{n+\frac{1}{2}} \right| \left| u_h^0 \right|_\infty
\]

and we get:

\[
(1 - mq\Delta t_n \left| u_h^0 \right|_\infty) \left\| v_h^{n+1} \right\|_\infty \leq \\
\left\| v_h^n \right\|_\infty \left( 1 + m(1 - q^2)\Delta t_n \left\| u_h^0 \right\|_\infty^q + mq\Delta t_n \left\| u_h^n \right\|_\infty^q \frac{1 + m(1 - q)\Delta t_n}{1 - mq\Delta t_n \left\| u_h^n \right\|_\infty^q} \right)
\]

By (2.9), we get easily there exist a positive constant $C$ depending on $m, q, T, \left\| u_h^0 \right\|_\infty$ such that

\[
\left\| v_h^{n+1} \right\|_\infty \leq (1 + C\Delta t_n) \left\| v_h^n \right\|_\infty
\]

Hence for $t_n \leq T < T_1$, we get $\left\| v_h^n \right\|_\infty \leq C$.

**Lemma 2.6.** Under the hypotheses of Proposition (2.2), we have the estimate:

\[
(\text{2.10}) \quad \left\| v_h^n \right\|_1 \leq C
\]

for $t_n < T < T_1$ where $C$ is a constant depending on $T$ and $u_0$.

**Proof:** From the properties of the semigroup operator $S$ [7], we get: $\left\| v_h^{n+\frac{1}{2}} \right\|_1 \leq \left\| v_h^n \right\|_1$, and by using the equations satisfied by $v_i^{n+\frac{1}{2}}, -N^{-1}_n \leq i \leq N^+_{n+1}$, we obtain:
\[
(1 - mq \Delta t_n) \left| u_h^{n+\frac{1}{2}} \right|_\infty \| v_h^{n+1} \|_1 \leq \| v_h^{n+\frac{1}{2}} \|_1 \left( 1 + m \Delta t_n \left| u_h^{n+\frac{1}{2}} \right|_\infty \left( q^2 \left| u_h^{n+1} \|_\infty + (1 - q^2) \left| u_h^{n+\frac{1}{2}} \right|_\infty \right) \right) \]
\]

and we immediately deduce the estimate (2.10).

In this case, since \( q < 1 \), the variation of \( v_h^n \) is not bounded.

### 3. Blow-up of the solution

In this part, we prove that for \( q < 1 \) the solution blows up in finite time.

#### 3.1. Construction of unbounded solutions.

Define the function

\[
\theta(x) = \begin{cases} 
1 - \frac{x^2}{a^2}, & |x| \leq a \\
0, & |x| \geq a 
\end{cases}
\]

We note \( \theta_h = \pi_h \theta \). If the initial condition is \( u_h^0 = \frac{\lambda}{(T-t_0)^\frac{q}{q}} \theta_h(\xi^0) \) with \( \xi^0 = \frac{x}{T-t_0} \), we prove that it is possible to choose \( \lambda \) and \( a \) in such a manner that

\[
\hat{u}_h^n(x) \geq \lambda \left( \frac{T - t_n}{(T - t_0)^\frac{q}{q}} \right)^\frac{1}{q} \theta_n(x), \quad \text{with } \theta_n(x) = \theta(\xi_n(x)).
\]

The support of \( \hat{u}_h^n \) is \([ -a \zeta_n, a \zeta_n ] \); its length is increasing with the time.

If \( \hat{u}_h^n \leq u_h^n \), we get \( \hat{u}_h^{n+\frac{1}{2}} \leq u_h^{n+\frac{1}{2}} \).

The support of \( \hat{u}_h^{n+\frac{1}{2}} \) is \([-\hat{s}_n, \hat{s}_n] \) with \( \hat{s}_n+1 \leq s_n+1 \leq s_n+1 \). Since \( \hat{u}_h^n \) is a symmetric function, it is sufficient to study the case \( x \geq 0, (i \geq 0) \).

We get for \( i \geq 0 \),

\[
\hat{u}_i^{n+\frac{1}{2}} = \hat{u}_i^n + \frac{\Delta t_n}{m} (v_i^n)^2
\]

for \( i \) such that \( x_i \leq a \zeta_n \) since the function \( \hat{u}_h^n \) is decreasing for \( x \geq 0 \) and we have:

\[
\hat{u}_i^n = \frac{\hat{u}_i^n - \hat{u}_{i-1}^n}{h} = \frac{\lambda}{(T-t_n)^\frac{q}{q}} \frac{\theta_i^n - \theta_{i-1}^n}{h}
\]

with \( \theta_i^n = \theta(\xi_i^n) \).

Hence, we obtain:
with $\theta^n_{i-\frac{1}{2}} = \theta(\xi^n_{i-\frac{1}{2}})$, $\xi^n_{i-\frac{1}{2}} = \frac{1}{2}(\xi^n_i + \xi^n_{i-1})$.

Then $\hat{u}^{n+1}_h$ will be a subsolution of (2.2) if

$$
\frac{\lambda}{(T-t_{n+1})^{\frac{q}{2}}} \left( 1 - mq\Delta t_n \left( \hat{u}^{n+\frac{1}{2}}_i \right)^q \right) \theta^{n+1}_i + \frac{\lambda}{(T-t_{n+1})^{\frac{q}{2}}} \frac{\Delta t_n}{h^2} \hat{u}^{n+\frac{1}{2}}_i (2\theta^{n+1}_i - \theta^{n+1}_{i-1} - \theta^{n+1}_{i+1}) 
$$

$$
\leq \hat{u}^{n+\frac{1}{2}}_i + m(1-q)\Delta t_n \left( \hat{u}^{n+\frac{1}{2}}_i \right)^{q+1}
$$

By using the equality \( \frac{1}{h^2} (2\theta^{n+1}_i - \theta^{n+1}_{i-1} - \theta^{n+1}_{i+1}) = \frac{2}{a^2\xi_{n+1}^2} \), this inequality reduces after simplifications to

$$
\frac{\lambda}{(T-t_{n+1})^{\frac{q}{2}}} \theta^{n+1}_i \leq \hat{u}^{n+\frac{1}{2}}_i \left( 1 - \frac{2\lambda\Delta t_n}{a^2 (T-t_{n+1})} \right) + \frac{\lambda}{(T-t_{n+1})^{\frac{q}{2}}} mq\Delta t_n \left( \hat{u}^{n+\frac{1}{2}}_i \right)^q \theta^{n+1}_i
$$

$$
+ m(1-q)\Delta t_n \left( \hat{u}^{n+\frac{1}{2}}_i \right)^{q+1}
$$

Noting that $\theta^n_{i-\frac{1}{2}} = \theta^n_i + \frac{h}{a^2\xi_{n+1}^{1/2}} = \theta^n_i + \eta^n_i$ with $|\eta^n_i| \leq \frac{h}{a\xi_n}$, we get:

$$
\hat{u}^{n+\frac{1}{2}}_i = \frac{\lambda}{(T-t_n)^{\frac{q}{2}}} \left( \theta^n_i \left( 1 - \frac{4\lambda}{ma^2 T-t_n} \right) + \frac{4\lambda}{ma^2 T-t_n} \Delta t_n (1 - \eta^n_i) \right)
$$

and the inequality (3.1) becomes:

$$
\frac{\theta^{n+1}_i}{(T-t_{n+1})^{\frac{q}{2}}} \leq \frac{\theta^n_i}{(T-t_n)^{\frac{q}{2}}} \left( 1 - \frac{4\lambda}{ma^2 T-t_n} - \frac{2\lambda}{ma^2 T-t_{n+1}} \left( 1 - \frac{4\lambda}{ma^2 T-t_n} \right) \right)
$$

$$
+ \frac{4\lambda}{ma^2 (T-t_{n+1})^{\frac{q+1}{2}}} \left( 1 - \frac{2\lambda}{ma^2 T-t_{n+1}} \right) (1 - \eta^n_i) + \frac{\Delta t_n}{h^2} \left( \hat{u}^{n+\frac{1}{2}}_i \right)^q \theta^{n+1}_i
$$

$$
+ m(1-q)\Delta t_n \left( \hat{u}^{n+\frac{1}{2}}_i \right)^{q+1}
$$

By using the equality $\theta^{n+1}_i = \theta^n_i \frac{\xi_{n+1}^2}{\xi_{n+1}^{1/2}} + 1 - \frac{\xi_{n+1}^2}{\xi_{n+1}^{1/2}}$, this inequality reduces to:

$$
\theta^n_i \left( 1 + \frac{4\lambda T-t_{n+1}}{ma^2 T-t_n} + \frac{2\lambda}{a^2} \left( 1 - \frac{4\lambda}{ma^2 T-t_n} \right) \right)
$$
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\[ \leq \frac{4\lambda}{ma^2} \frac{T - t_{n+1}}{T - t_n} \left( 1 - \frac{2\lambda}{a^2} \frac{\Delta t_n}{T - t_{n+1}} (1 - \eta_i^n) \right) - \left( \frac{\zeta_i^{n+1}}{\zeta_i^n} - 1 \right) \frac{T - t_n}{\Delta t_n} \]

\[ + mq \left( T - t_{n+1} \right)^{\frac{1}{n}} \left( T - t_n \right)^{\frac{1}{n}} \left( \hat{u}_{i+\frac{n}{2}} \right)^q \theta_i^{n+1} + m(1-q) \frac{T - t_{n+1}}{\lambda} \left( T - t_n \right)^{\frac{1}{n}} \left( \hat{u}_{i+\frac{n}{2}} \right)^{q+1}. \]

If we denote \( \mu_n = \frac{4\lambda}{ma^2} \Delta t_n \) since \( \hat{u}_{i+\frac{n}{2}} \geq \frac{\lambda}{(T-t_n)^{\frac{1}{n}}} \theta_i^n (1-\mu_n) \), the preceding inequality will be satisfied if:

\[ \theta_i^n \left( 1 + \frac{4\lambda}{ma^2} + \frac{2\lambda}{a^2} - \mu_n \left( 1 + \frac{2\lambda}{a^2} \right) \right) \leq \left( \frac{4\lambda}{ma^2} - \mu_n \left( 1 + \frac{2\lambda}{a^2} \right) \right) (1 - \eta_i^n) - \left( \frac{\zeta_i^{n+1}}{\zeta_i^n} - 1 \right) \frac{T - t_n}{\Delta t_n} \]

\[ + mq \lambda^q \left( \theta_i^n \right)^q (1 - \mu_n)^q \left( \frac{\zeta_i^n}{\zeta_i^{n+1}} + 1 - \frac{\zeta_i^n}{\zeta_i^{n+1}} \right) \]

(3.1)

\[ + m(1-q) \lambda^q \frac{T - t_{n+1}}{T - t_n} (\theta_i^n)^{q+1} (1 - \mu_n)^{q+1} \]

From the stability condition, we get:

\[ \mu_n \leq \frac{h}{a\xi_n} \leq \frac{h}{a\xi_0} = \delta \]

hence for \( h \) sufficiently small, we get: \( \frac{4\lambda}{ma^2} - \mu_n \left( 1 + \frac{2\lambda}{a^2} \right) > 0 \), and \( |\eta_i^n| \leq \delta \)

So the inequality (3.1) will be satisfied if :

\[ \theta_i^n \left( 1 + \frac{4\lambda}{ma^2} + \frac{2\lambda}{a^2} - \mu_n \left( 1 + \frac{2\lambda}{a^2} \right) \right) \leq \left( \frac{4\lambda}{ma^2} - \mu_n \left( 1 + \frac{2\lambda}{a^2} \right) \right) (1 - \delta) - \left( \frac{\zeta_i^{n+1}}{\zeta_i^n} - 1 \right) \frac{T - t_n}{\Delta t_n} \]

\[ + mq \lambda^q (1 - \mu_n)^q \left( \theta_i^n \right)^q \left( \frac{\zeta_i^n}{\zeta_i^{n+1}} + 1 - \frac{\zeta_i^n}{\zeta_i^{n+1}} \right) + m(1-q) \lambda^q (1 - \mu_n)^{q+1} (\theta_i^n)^{q+1} \frac{T - t_{n+1}}{T - t_n} \]

Since \( \theta_i^n \in (0, 1) \), we introduce the function \( \Phi_n(y) \) defined on \( (0, 1) \) by :

\[ \Phi_n(y) = m\lambda^q (1 - \mu_n)^q y^{q+1} \left( q + (1-q)(1 - \mu_n) \frac{T - t_{n+1}}{T - t_n} \right) \]

\[ + \left( \frac{4\lambda}{ma^2} - \mu_n \left( 1 + \frac{2\lambda}{a^2} \right) \right) (1 - \delta) - \left( \frac{\zeta_i^{n+1}}{\zeta_i^n} - 1 \right) \frac{T - t_n}{\Delta t_n} \]
or $\Phi_n(y) = A\lambda y^{q+1} + C - By$

with

$$A = m (1 - \mu_n)^q \left( q + (1 - q)(1 - \mu_n) \frac{T - t_{n+1}}{T - t_n} \right)$$

$$B = 1 + \frac{4\lambda}{ma^2} + \frac{2\lambda}{a^2} - \mu_n \left( 1 + \frac{2\lambda}{a^2} \right)$$

$$C = \left( \frac{4\lambda}{ma^2} - \mu_n \left( 1 + \frac{2\lambda}{a^2} \right) (1 - \delta) - \left( \frac{\zeta_{n+1}^2}{\zeta_n^2} - 1 \right) \frac{T - t_n}{\Delta t_n} \right)$$

A sufficient condition to satisfy (3.1) is: $\Phi_n(y) \geq 0, \ y \in (0, 1)$.

We have $\Phi_n(0) = C$, hence we get

$$\Phi_n(0) \geq \left( \frac{4\lambda}{ma^2} - \delta \left( 1 + \frac{2\lambda}{a^2} \right) \right) (1 - \delta) - \left( \frac{\zeta_{n+1}^2}{\zeta_n^2} - 1 \right) \frac{T - t_n}{\Delta t_n}$$

But since $0 < q < 1$, we get:

$$0 \leq \frac{\zeta_{n+1}^2}{\zeta_n^2} - 1 \leq \frac{1 - q}{q} \frac{\Delta t_n}{T - t_{n+1}} \frac{\zeta_n^2}{\zeta_{n+1}^2}$$

and

$$C \geq \left( \frac{4\lambda}{ma^2} - \delta \left( 1 + \frac{2\lambda}{a^2} \right) \right) (1 - \delta) - \frac{1 - q}{q}.$$

So, if the quantity $\frac{4\lambda}{ma^2}$ is such that $\frac{4\lambda}{ma^2} > \frac{1 - q}{q}$, if $h$ is sufficiently small, we get $C > 0$.

Let us define $y_0 = \frac{C}{B}$, $y_0 \in (0, 1)$ and if $y \leq y_0$, we obtain $\Phi_n(y) \geq 0$; if $y_0 \leq y \leq 1$, we obtain:

$$\Phi_n(y) \geq A\lambda y^{q+1} - B(1 - y_0)$$

This quantity will be positive if: $\lambda^q \geq \frac{B - C}{Ay_0^q}$.

Further, we have:

$$A \geq m(1 - \delta)^q \left( q + (1 - q)(1 - \delta) \left( 1 - \frac{\Delta t_n}{T - t_n} \right) \right)$$

The solution at the time level $t_{n+1}$ exists if $mq\Delta t_n \|\hat{u}_n^h\|_\infty < 1$, that is $\frac{\Delta t_n}{T - t_n} < \frac{1}{mq\lambda^q}$

and if $\lambda \geq \lambda_0 > \frac{1}{(mq)^q}$, we get: $A \geq mq(1 - \delta)^q$

and $\Phi_n(y)$ will be positive if:
\[ \lambda^q \geq \frac{1}{mq(1-\delta)^q} \left( \frac{1}{q} + \frac{2\lambda}{a^2} + \frac{4\lambda}{ma^2} \delta \right) \left( \frac{1}{ma^2} + \frac{2\lambda}{a^2} \right) \left( \frac{4\lambda}{ma^2} (1-\delta) - \delta \left( 1 + \frac{2\lambda}{ma^2} \right) - \frac{1-q}{q} \right)^{\frac{q+1}{q}} \]

The second member is a function of $\frac{\lambda}{a^2}$, hence if $\frac{\lambda}{a^2}$ is fixed such that $\frac{4\lambda}{ma^2} > \frac{1-q}{q}$, the inequality (3.2) will be satisfied if $\lambda$ is large enough and $h$ sufficiently small. Hence if the initial condition satisfies $u_0(x) \geq \frac{\lambda}{T^q} \theta_h(\xi_0)$, $\lambda$ and $a$ satisfying (3.2), the solution blows up in finite time.

3.2. Blow-up of the solution.

**Theorem 3.1.** Let $0 < q < 1$. The solution of problem (1.2) blows up in finite time.

**Proof:** Since $u_0(x) \not\equiv 0$, there exists $\rho > 0$, $\epsilon > 0$, $x_0$ such that $u_0(x) \geq \epsilon > 0$ for $x$ such that $|x-x_0| < \rho$. So we can choose $T > 0$ such that $\frac{\lambda}{T^q} < \epsilon$ and $\frac{a}{T^{q+1}} < \rho$.

We have $u_0(x) \geq \frac{\lambda}{T^q} \theta_h \left( \frac{x-x_0}{\xi_0} \right)$ and then the solution blows up at time $T_0 \leq T^*$ where

\[ T^* = \text{Max} \left( \left( \frac{\lambda}{T^q} \right)^q, \left( \frac{a}{T^{q+1}} \right)^{\frac{q+1}{q}} \right). \]

In Fig1, we present the evolution of an initial condition $u_0$ for $m = 1$, $p = 1.5$. The solution blows up in finite time.

**Figure 1.** blow-up for m=1, p=1.5
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