On a multiple Hilbert-type integral inequality involving the upper limit functions

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Abstract

By applying the weight functions, the idea of introducing parameters and the technique of real analysis, a new multiple Hilbert-type integral inequality involving the upper limit functions is given. The constant factor related to the gamma function is proved to be the best possible in a condition. A corollary about the case of the nonhomogeneous kernel and some particular inequalities are obtained.

MSC: 26D15

Keywords: Weight function; Hilbert-type integral inequality; Upper limit function; Parameter; Gamma function

1 Introduction

Assuming that \( 0 < \sum_{m=1}^{\infty} a_m^2 < \infty \) and \( 0 < \sum_{n=1}^{\infty} b_n^2 < \infty \), we have the following Hilbert’s inequality with the best possible constant factor \( \pi \) (cf. [1], Theorem 315):

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m b_n \frac{m+n}{m+n} < \pi \left( \sum_{m=1}^{\infty} a_m^2 \sum_{n=1}^{\infty} b_n^2 \right)^{1/2}.
\] (1)

If \( 0 < \int_{0}^{\infty} f^2(x) \, dx < \infty \) and \( 0 < \int_{0}^{\infty} g^2(y) \, dy < \infty \), then we still have the following integral analogue of (1), named Hilbert’s integral inequality (cf. [1], Theorem 316):

\[
\int_{0}^{\infty} \int_{0}^{\infty} \frac{f(x)g(y)}{x+y} \, dx \, dy < \pi \left( \int_{0}^{\infty} f^2(x) \, dx \int_{0}^{\infty} g^2(y) \, dy \right)^{1/2},
\] (2)

where the constant factor \( \pi \) is the best possible. Inequalities (1) and (2) play an important role in analysis and its applications (cf. [2–13]).

The following half-discrete Hilbert-type inequality was provided: If \( K(x) (x > 0) \) is a decreasing function, \( p > 1, \frac{1}{p} + \frac{1}{q} = 1, 0 < \phi(s) = \int_{0}^{s} K(x)x^{q-1} \, dx < \infty, f(x) \geq 0, 0 < \int_{0}^{\infty} f^p(x) \, dx < \infty \), then (cf. [1], Theorem 351)

\[
\sum_{n=1}^{\infty} n^{p-2} \left( \int_{0}^{\infty} K(nx)f(x) \, dx \right)^{p} < \phi\left( \frac{1}{q} \right) \int_{0}^{\infty} f^p(x) \, dx.
\] (3)
In recent years, some new extensions of (3) were provided by [14–19].

In 2006, by using Euler–Maclaurin summation formula, Krnic et al. [20] gave an extension of (1) with the kernel \( \frac{1}{(m+n)^{\lambda}} \) \( (0 < \lambda \leq 4) \). In 2019, following the result of [20], Adiyasuren et al. [21] considered an extension of (1) involving the partial sums. In 2016–2017, by applying the weight functions, Hong [22, 23] obtained some equivalent statements of the extensions of (1) and (2) with a few parameters. A few similar works were provided by [24–38].

In this paper, following the idea of [21], by using the weight functions, the way of introducing parameters and the technique of real analysis, a new multiple Hilbert-type integral inequality with the kernel \( \frac{1}{(x_1 + \ldots + x_n)^{\lambda}} \) \( (\lambda > 0) \) involving the upper limit functions is given.

The constant factor related to the gamma function is proved to be the best possible in a condition. A corollary about the case of the nonhomogeneous kernel and some particular inequalities are obtained.

2 Some lemmas

In what follows, we assume that \( n \in \mathbb{N} \setminus \{1\} := \{2, 3, \ldots, n\}, p_i, r_i > 1 \) \( (i = 1, \ldots, n) \), \( \sum_{i=1}^{n} \frac{1}{p_i} = 1, \lambda > 0, c_i := (1 - \sum_{j=1}^{n} \frac{1}{p_j}) \lambda f_i(x) \) \( (i = 1, \ldots, n) \) are nonnegative measurable functions in \( R_+ = (0, \infty) \) such that \( f_i(x) = o(e^{ax}) \) \( (t > 0; x \rightarrow \infty) \), and for any \( A = (0, a) (a > 0) \), \( f_i \in L^1(A) \), the upper limit functions are defined by \( F_i(x) := \int_{0}^{x} f_i(t) \, dt \) \( (x \geq 0) \), satisfying

\[ 0 < \int_{x_i}^{\infty} e^{-p_i^{\lambda} - c_i^{\lambda} - 1} F_i^{p_i}(x_i) \, dx_i < \infty \quad (i = 1, \ldots, n). \]

By the definition of the gamma function, for \( x_i > 0 \) \( (i = 1, \ldots, n) \), the following expression holds:

\[ \frac{1}{(\sum_{i=1}^{n} x_i)^{\lambda}} = \frac{1}{\Gamma(\lambda)} \int_{0}^{\infty} t^{\lambda-1} e^{-\sum_{i=1}^{n} x_i} \, dt. \]  (4)

**Lemma 1** For \( t > 0 \), we have the following expressions:

\[ \int_{0}^{\infty} e^{-tx} f_i(x) \, dx = t \int_{0}^{\infty} e^{-tx} F_i(x) \, dx \quad (i = 1, \ldots, n), \]  (5)

**Proof** In view of \( F_i(0) = 0 \), we find

\[ \int_{0}^{\infty} e^{-tx} f_i(x) \, dx = \int_{0}^{\infty} e^{-tx} \, dF_i(x) \]

\[ = e^{-tx} F_i(x) \bigg|_{0}^{\infty} - \int_{0}^{\infty} F_i(x) \, de^{-tx} \]

\[ = \lim_{x \rightarrow \infty} \frac{F_i(x)}{e^{tx}} + t \int_{0}^{\infty} e^{-tx} F_i(x) \, dx. \]

If \( F_i(\infty) = \text{constant} \), then \( \lim_{x \rightarrow \infty} \frac{F_i(x)}{e^{tx}} = 0 \) and (5) follows; if \( F_i(\infty) = \infty \), since \( f_i(x) = o(e^{ax}) \) \( (x \rightarrow \infty) \), we find

\[ \int_{0}^{\infty} e^{-tx} f_i(x) \, dx = \lim_{x \rightarrow \infty} \frac{F_i(x)}{e^{tx}} + t \int_{0}^{\infty} e^{-tx} F_i(x) \, dx \]
\[
\lim_{x \to \infty} \frac{f(x)}{te^{tx}} + t \int_{0}^{\infty} e^{-tx} F_i(x) \, dx = 0 + t \int_{0}^{\infty} e^{-tx} F_i(x) \, dx,
\]
and then (5) follows, too.

The lemma is proved.

\begin{lemma}
For \( x_i > 0 \) (\( i = 1, \ldots, n \)), the following expression holds:
\[
A := \prod_{i=1}^{n} \left[ x_i^{\left( \frac{1}{n} \right) - 1} \prod_{j=1}^{n} x_j^{\frac{1}{n} - 1} \right]^\frac{1}{n} = 1.
\]
\end{lemma}

\begin{proof}
We have
\[
A = \prod_{i=1}^{n} \left[ x_i^{\left( \frac{1}{n} \right) - 1} \prod_{j=1}^{n} x_j^{\frac{1}{n} - 1} \right]^\frac{1}{n} = \prod_{i=1}^{n} \left[ x_i^{\left( \frac{1}{n} \right) - 1} \prod_{j=1}^{n} x_j^{\frac{1}{n} - 1} \right]^\frac{1}{n} = 1,
\]
and then (6) follows.

The lemma is proved.

\begin{lemma}
For \( n \in \mathbb{N} \setminus \{1\} \), defining the following weight functions:
\[
\omega_{(i)}(x_i) := x_i^{\frac{\lambda}{n} - \frac{1}{n} + c_2} \prod_{j=1}^{n} x_j^{\frac{\lambda}{n} - 1} \int_{0}^{\infty} \cdots \int_{0}^{\infty} \frac{1}{(\sum_{i=1}^{n} x_i)^{\lambda}} \prod_{j=1}^{n} x_j^{\frac{1}{n} - 1} \, dx_1 \cdots dx_{i-1} \, dx_i \cdots dx_n,
\]
we have
\[
\omega_{(i)}(x_i) = k_{(i)} := \frac{\Gamma\left( \frac{\lambda}{n} - \frac{1}{n} + c_2 \right)}{\Gamma\left( \frac{\lambda}{n} - \frac{1}{n} \right)} \prod_{j=1}^{n} x_j^{\frac{\lambda}{n} - 1} \Gamma\left( \frac{\lambda}{n} \right) \in \mathbb{R}^{+}, \quad (i = 1, \ldots, n).
\]

In particular, for \( \sum_{i=1}^{n} \frac{1}{n} = 1 \), we have
\[
k_{(i)} = k := \frac{1}{\Gamma\left( \frac{\lambda}{n} \right)} \prod_{j=1}^{n} x_j^{\frac{\lambda}{n} - 1} \quad (i = 1, \ldots, n).
\]
\end{lemma}

\begin{proof}
For \( j \neq i \), setting \( u_j = \frac{x_j}{x_i} \) in (7), we have
\[
\omega_{(i)}(x_i) = \int_{0}^{\infty} \cdots \int_{0}^{\infty} \frac{1}{\left( u_1 + \cdots + u_{i-1} + 1 + u_{i+1} + \cdots + u_n \right)^{\lambda}} \prod_{j=1}^{n} u_j^{\frac{\lambda}{n} - 1} \, du_1 \cdots du_{i-1} \, du_{i+1} \cdots du_n.
\]
Then by Lemma 9.15 and (9.1.19) (cf. [2], p. 341–342), we obtain (8).

The lemma is proved.
\end{proof}
**Lemma 4** We have the following inequality:

\[
H_\lambda := \int_0^\infty \cdots \int_0^\infty \frac{1}{(\sum_{i=1}^n x_i)^\lambda} \prod_{i=1}^n F_i(x_i) \, dx_1 \cdots dx_n
\]

\[
< \prod_{i=1}^n \left( \int_0^\infty x_i^{p_i(\frac{1}{\lambda}+\epsilon)-1} F_i^p(x_i) \, dx_i \right)^{\frac{1}{p_i}}.
\]  \hspace{1cm} (10)

**Proof** By (6) and Hölder’s integral inequality (cf. [39]), we obtain

\[
H_\lambda = \int_0^\infty \cdots \int_0^\infty \frac{1}{(\sum_{i=1}^n x_i)^\lambda} \prod_{i=1}^n \left[ x_i^{(\frac{1}{p_i}-1)(1-p_i)} \prod_{j=1(j\neq i)}^n x_j^{\frac{1}{p_j}} \right] F_i(x_i) \, dx_1 \cdots dx_n
\]

\[
\leq \prod_{i=1}^n \left[ \int_0^\infty \left( \int_0^\infty \cdots \int_0^\infty \frac{1}{(\sum_{i=1}^n x_i)^\lambda} x_i^{\frac{1}{p_i}} x_j^{\frac{1}{p_j}} \cdots x_n^{\frac{1}{p_n}} \, dx_1 \cdots dx_{i-1} \, dx_{i+1} \cdots dx_n \right) \right]
\]

\[
\times x_i^{p_i(\frac{1}{\lambda}+\epsilon)-1} F_i^p(x_i) \, dx_i \right]^{\frac{1}{p_i}}
\]

\[
= \prod_{i=1}^n \left[ \int_0^\infty \omega_{i,x_i}^{(i)}(x_i) x_i^{p_i(\frac{1}{\lambda}+\epsilon)-1} F_i^p(x_i) \, dx_i \right]^{\frac{1}{p_i}}.
\]  \hspace{1cm} (11)

If (11) takes the form of an equality, then there exist constants \(C_i, C_k\) (\(i \neq k\)) such that they are not all zero and

\[
C_i x_i^{\frac{1}{\lambda}+\epsilon} \prod_{j=1(j\neq i)}^n x_j^{\frac{1}{p_j}} p_i(\frac{1}{\lambda}+\epsilon)-1 F_i^p(x_i)
\]

\[
= C_k x_k^{\frac{1}{\lambda}+\epsilon} \prod_{j=1(j\neq k)}^n x_j^{\frac{1}{p_j}} p_k(\frac{1}{\lambda}+\epsilon)-1 F_k^p(x_k) \quad \text{a.e. in } R_+.
\]

namely, \(C_i x_i^{p_i(\frac{1}{\lambda}+\epsilon)-1} F_i^p(x_i) = C_k x_k^{p_k(\frac{1}{\lambda}+\epsilon)-1} F_k^p(x_k) = C\) a.e. in \(R_+\). Assuming that \(C_i \neq 0\), we have

\[
x_i^{p_i(\frac{1}{\lambda}+\epsilon)-1} F_i^p(x_i) = \frac{C}{C_i} x_i^{p_i(\frac{1}{\lambda}+\epsilon)-1} \quad \text{a.e. in } R_+,
\]

which contradicts the fact that \(0 < \int_0^\infty x_i^{p_i(\frac{1}{\lambda}+\epsilon)-1} F_i^p(x_i) \, dx_i < \infty\), in view of \(\int_0^\infty x_i^{p_i(\frac{1}{\lambda}+\epsilon)-1} \, dx_i = \infty\). Then by (8) and (11), we have (10).

The lemma is proved. \(\square\)

**Remark 1** Replacing \(\lambda\) (resp. \(\frac{1}{\tau_i}\)) by \(\lambda + n\) (resp. \(\frac{1}{\tau_i} + 1\)) in (10), we have

\[
H_{\lambda+n} = \int_0^\infty \cdots \int_0^\infty \frac{1}{(\sum_{i=1}^n x_i)^{\lambda+n}} \prod_{i=1}^n F_i(x_i) \, dx_1 \cdots dx_n
\]

\[
< \prod_{i=1}^n \left( \int_0^\infty x_i^{p_i(\frac{1}{\lambda+n}+\epsilon)-1} F_i^p(x_i) \, dx_i \right)^{\frac{1}{p_i}}.
\]  \hspace{1cm} (12)
where we denote

\[
\tilde{k}_{i,n}(\lambda) := \frac{\Gamma(\lambda(1 - \frac{1}{ri}) + n - 1)}{\Gamma(\sum_{j=1}^{n}(\frac{1}{r_j}) + 1)} \cdot \frac{\prod_{j=1}^{n} \Gamma(\frac{1}{r_j}) + 1}{\Gamma(\lambda + n)} \in \mathbb{R}, \quad (i = 1, \ldots, n).
\]

### 3 Main results and a corollary

**Theorem 1** We have the following inequality:

\[
I := \int_0^\infty \cdots \int_0^\infty \frac{1}{(\sum_{i=1}^{n} x_i)^\lambda} \prod_{i=1}^{n} f_i(x_i) \, dx_1 \cdots dx_n
\]

\[
< \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} \prod_{i=1}^{n} \tilde{k}_{i,n}(\lambda) \int_0^\infty x_i^{-\frac{1}{ri}} e^{-x_i F_i(x_i)} \, dx_i \left( \int_0^\infty x_i^{-\frac{1}{ri}} e^{-x_i F_i(x_i)} \, dx_i \right)^{\frac{1}{ri}}.
\] (13)

In particular, for $\sum_{i=1}^{n} \frac{1}{ri} = 1$, we have

\[0 < \int_0^\infty x_i^{-\frac{1}{ri}} e^{-x_i F_i(x_i)} \, dx_i < \infty \quad (i = 1, \ldots, n),\]

and the following inequality:

\[
I = \int_0^\infty \cdots \int_0^\infty \frac{1}{(\sum_{i=1}^{n} x_i)^\lambda} \prod_{i=1}^{n} f_i(x_i) \, dx_1 \cdots dx_n
\]

\[
< \frac{1}{\Gamma(\lambda)} \prod_{i=1}^{n} \frac{\lambda}{ri} \Gamma\left(\frac{\lambda}{ri}\right) \left( \int_0^\infty x_i^{-\frac{1}{ri}} e^{-x_i F_i(x_i)} \, dx_i \right)^{\frac{1}{ri}}.
\] (14)

**Proof** By (4) and (5), we have

\[
I = \frac{1}{\Gamma(\lambda)} \int_0^\infty \cdots \int_0^\infty \int_0^\infty t^{\lambda-1} e^{-t(x_1 + \cdots + x_n)} \, dt \, dx_1 \cdots dx_n
\]

\[
= \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} \prod_{i=1}^{n} \int_0^\infty e^{-tx_i} f_i(x_i) \, dx_i \, dt
\]

\[
= \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda+n-1} \prod_{i=1}^{n} \int_0^\infty e^{-tx_i} F_i(x_i) \, dx_i \, dt
\]

\[
= \frac{1}{\Gamma(\lambda)} \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{n} F_i(x_i) \int_0^\infty t^{\lambda+n-1} e^{-t(x_1 + \cdots + x_n)} \, dt \, dx_1 \cdots dx_n
\]

\[
= \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} H_{\lambda,n}.
\]

Then by (12), we have (13).

The theorem is proved. □

**Theorem 2** The constant factor $\frac{1}{\Gamma(\lambda)} \prod_{i=1}^{n} \frac{\lambda}{ri} \Gamma\left(\frac{\lambda}{ri}\right)$ in (14) is the best possible.
Proof For any \(0 < \varepsilon < \lambda \min_{1 \leq i \leq n} \left\{ \frac{\lambda}{\eta_i} \right\} \), we set
\[
\tilde{f}_i(x_i) := \begin{cases} 
0, & 0 < x_i \leq 1, \\
\frac{\lambda}{\eta_i} - \varepsilon - 1, & x_i > 1, 
\end{cases} \quad (i = 1, \ldots, n).
\]
We obtain that \(\tilde{f}_i(x_i) = o(e^{t_{x_i}}) \) \((t > 0; x_i \to \infty)\), and \(\tilde{F}_i(x_i) \equiv 0 \ (0 < x_i \leq 1)\).
\[
\tilde{F}_i(x_i) = \int_0^{x_i} \tilde{f}_i(t) \, dt = \int_1^{x_i} t^{\frac{\lambda}{\eta_i} - 1} \, dt = \frac{x_i^{\frac{\lambda}{\eta_i} - \varepsilon}}{\frac{\lambda}{\eta_i} - \varepsilon} \left( x_i^{\frac{\lambda}{\eta_i} - \varepsilon} - 1 \right) \quad (x_i > 1; i = 1, \ldots, n).
\]
If there exists a positive constant \(M(M \leq \frac{1}{\Gamma(\lambda)} \prod_{i=1}^{n} \frac{\lambda}{\eta_i} \Gamma\left( \frac{\lambda}{\eta_i} \right))\) such that (14) is valid when replacing \(\frac{1}{\Gamma(\lambda)} \prod_{i=1}^{n} \frac{\lambda}{\eta_i} \Gamma\left( \frac{\lambda}{\eta_i} \right) \) by \(M\), then in particular, by substitution of \(f_i(x_i) = \tilde{f}_i(x_i)\) and \(F_i(x_i) = \tilde{F}_i(x_i)\), we have
\[
\tilde{I} := \int_0^\infty \cdots \int_0^\infty \frac{1}{(\sum_{i=1}^{n} x_i)^{\frac{1}{\lambda}} - \varepsilon} \prod_{i=1}^{n} f_i(x_i) \, dx_1 \cdots dx_n < M \prod_{i=1}^{n} \frac{1}{\frac{\lambda}{\eta_i} - \varepsilon}.
\]
In view of Lemma 9.1.4 (9.1.5) in [2], we find
\[
I_\varepsilon := \varepsilon \tilde{I} = \varepsilon \int_1^\infty \cdots \int_1^\infty \frac{1}{(\sum_{i=1}^{n} x_i)^{\frac{1}{\lambda}} - \varepsilon} \prod_{i=1}^{n} x_i^{\frac{1}{\lambda} - \varepsilon - 1} \, dx_1 \cdots dx_n = k_\lambda + o(1) \quad (\varepsilon \to 0^+).
\]
Hence, we have
\[
\frac{1}{\Gamma(\lambda)} \prod_{i=1}^{n} \Gamma\left( \frac{\lambda}{\eta_i} \right) + o(1) = k_\lambda + o(1) = \varepsilon \tilde{I} < M \prod_{i=1}^{n} \frac{1}{\frac{\lambda}{\eta_i} - \varepsilon}.
\]
For \(\varepsilon \to 0^+\), we find
\[
\frac{1}{\Gamma(\lambda)} \prod_{i=1}^{n} \frac{\lambda}{\eta_i} \Gamma\left( \frac{\lambda}{\eta_i} \right) \leq M,
\]
which yields that the constant factor \(M = \frac{1}{\Gamma(\lambda)} \prod_{i=1}^{n} \frac{\lambda}{\eta_i} \Gamma\left( \frac{\lambda}{\eta_i} \right)\) in (14) is the best possible.

The theorem is proved. \(\square\)

Setting \(x = \frac{1}{x_1} f(x) = x^{\lambda-2} f_1\left( \frac{1}{x} \right)\) in \(I\) of (14), we have
\[
I = \int_0^\infty \cdots \int_0^\infty \frac{f(x)}{\left( 1 + \sum_{i=2}^{n} x_i x_{i-1} \right)^{\frac{1}{\lambda}} - \varepsilon} \prod_{i=2}^{n} f_i(x_i) \, dx_2 \cdots dx_n.
\]
For $f_1(t) = t^{\lambda-2}f\left(\frac{1}{t}\right)$, we find

$$F_1(x_1) = \int_0^{x_1} f_1(t)\,dt = \int_0^{x_1} t^{\lambda-2}f\left(\frac{1}{t}\right)\,dt.$$  

Then, replacing back $x$ (resp. $f(x)$) by $x_1$ (resp. $f_1(x_1)$), we have

**Corollary 1** If $\tilde{F}_1(x_1) = \int_0^{x_1} t^{\lambda-2}f_1\left(\frac{1}{t}\right)\,dt$,

$$\tilde{F}_i(x_i) := \int_0^{x_i} f_i(t)\,dt \quad (i = 2, \ldots, n),$$

then we have the following inequality with the nonhomogeneous kernel:

$$\int_0^\infty \cdots \int_0^\infty \frac{1}{(1 + \sum_{i=2}^n x_i)^\lambda} \prod_{i=1}^n f_i(x_i)\,dx_1 \cdots dx_n < \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \frac{\lambda}{p_i} \Gamma\left(\frac{\lambda}{p_i}\right) \left(\int_0^\infty x_i^{-p_i} \tilde{F}_{p_i}^{p_i}(x_i)\,dx_i\right)^{\frac{1}{p_i}},$$

(15)

where the constant factor $\frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \frac{\lambda}{p_i} \Gamma\left(\frac{\lambda}{p_i}\right)$ in (15) is the best possible.

**Remark 2** (i) For $n = 2$, (14) reduces to (cf. [40])

$$\int_0^\infty \int_0^\infty f_1(x_1) f_2(x_2) \frac{dx_1\,dx_2}{(x_1 + x_2)^\lambda} < \frac{\lambda^2}{r_1 r_2} B\left(\frac{\lambda}{r_1}, \frac{\lambda}{r_2}\right) \left(\int_0^\infty x_1^{-p_1} \tilde{F}_{p_1}^{p_1}(x_1)\,dx_1\right)^{\frac{1}{p_1}} \left(\int_0^\infty x_2^{-p_2} \tilde{F}_{p_2}^{p_2}(x_2)\,dx_2\right)^{\frac{1}{p_2}},$$

(16)

and (15) reduces to the following new inequality:

$$\int_0^\infty \int_0^\infty f_1(x_1) f_2(x_2) \frac{dx_1\,dx_2}{(1 + x_1 x_2)^\lambda} < \frac{\lambda^2}{r_1 r_2} B\left(\frac{\lambda}{r_1}, \frac{\lambda}{r_2}\right) \left(\int_0^\infty x_1^{-p_1} \tilde{F}_{p_1}^{p_1}(x_1)\,dx_1\right)^{\frac{1}{p_1}} \left(\int_0^\infty x_2^{-p_2} \tilde{F}_{p_2}^{p_2}(x_2)\,dx_2\right)^{\frac{1}{p_2}}.$$  

(17)

(ii) For $r_i = p_i$ ($i = 1, \ldots, n$), (14) reduces to

$$\int_0^\infty \cdots \int_0^\infty \frac{1}{(\sum_{i=1}^n x_i)^\lambda} \prod_{i=1}^n f_i(x_i)\,dx_1 \cdots dx_n < \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \frac{\lambda}{p_i} \Gamma\left(\frac{\lambda}{p_i}\right) \left(\int_0^\infty x_i^{-p_i} \tilde{F}_{p_i}^{p_i}(x_i)\,dx_i\right)^{\frac{1}{p_i}},$$

(18)

and (15) reduces to

$$\int_0^\infty \cdots \int_0^\infty \frac{1}{(1 + \sum_{i=2}^n x_i)^\lambda} \prod_{i=1}^n f_i(x_i)\,dx_1 \cdots dx_n.$$
\( \frac{1}{\Gamma(\lambda)} \prod_{i=1}^{n} \frac{\lambda}{\beta_i} \Gamma\left( \frac{\lambda}{\beta_i}\right) \left( \int_{0}^{\infty} x_i^{\lambda - 1} \tilde{F}_{\pi_i}(x_i) \, dx_i \right)^{\frac{1}{\beta_i}}. \)  

(19)

The constant factors in the above inequalities are the best possible.

4 Conclusions

In this paper, following the idea of [21], by the use of the weight functions, the way of introducing parameters and the technique of real analysis, a new multiple Hilbert-type integral inequality with the kernel \( \frac{1}{(x_1 + \cdots + x_n)^\lambda} \) \((\lambda > 0)\) involving the upper limit functions is given in Theorem 1. In a condition, the best possible constant factor related to the gamma function and a few parameters is proved in Theorem 2. A corollary about the case of nonhomogeneous kernel and some particular inequalities are obtained in Corollary 1 and Remark 2. The lemmas and theorems provide an extensive account of this type of inequalities.

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The data used to support the findings of this study are included within the article.

Competing interests

The authors declare that they have no competing interests.

Authors’ contributions

BY carried out the mathematical studies, participated in the sequence alignment and drafted the manuscript. JZ participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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References

1. Hardy, G.H., Littlewood, J.E., Polya, G.: Inequalities. Cambridge University Press, Cambridge (1934)
2. Yang, B.C.: The Norm of Operator and Hilbert-Type Inequalities. Science Press, Beijing (2009)
3. Yang, B.C.: Hilbert-Type Integral Inequalities. Bentham Science, The United Arab Emirates (2009)
4. Yang, B.C.: On the norm of an integral operator and applications. J. Math. Anal. Appl. 321, 182–192 (2006)
5. Xu, J.S.: Hardy–Hilbert’s inequalities with two parameters. Adv. Math. 36(2), 63–76 (2007)
6. Yang, B.C.: On the norm of a Hilbert’s type linear operator and applications. J. Math. Anal. Appl. 325, 529–541 (2007)
7. Xie, Z.T., Zeng, Z., Sun, Y.F.: A new Hilbert-type inequality with the homogeneous kernel of degree –2. Adv. Appl. Math. Sci. 12(7), 391–401 (2013)
8. Zhen, Z., Raja Rama Gandhi, K., Xie, Z.T.: A new Hilbert-type inequality with the homogeneous kernel of degree –2 and with the integral. Bull. Math. Sci. Appl. 3(1), 11–20 (2014)
9. Xin, D.W.: A Hilbert-type integral inequality with the homogeneous kernel of zero degree. Math. Theory Appl. 30(2), 70–74 (2010)
10. Azar, L.E.: The connection between Hilbert and Hardy inequalities. J. Inequal. Appl. 2013, 452 (2013)
11. Batbold, T., Sawano, Y.: Sharp bounds for m-linear Hilbert-type operators on the weighted Morrey spaces. Math. Inequal. Appl. 20, 263–337 (2017)
12. Adiyasuren, V., Batbold, T., Krmic, M.: Multiple Hilbert-type inequalities involving some differential operators. Banach J. Math. Anal. 10, 520–537 (2016)
13. Adiyasuren, V., Batbold, T., Krmic, M.: Hilbert-type inequalities involving differential operators, the best constants and applications. Math. Inequal. Appl. 18, 111–124 (2015)
14. Rassias, M., Yang, B.C.: On half-discrete Hilbert’s inequality. Appl. Math. Comput. 220, 75–93 (2013)
15. Yang, B.C., Krmic, M.: A half-discrete Hilbert-type inequality with a general homogeneous kernel of degree 0. J. Math. Inequal. 6(3), 401–417 (2012)
16. Rassias, M., Yang, B.C.: A multidimensional half-discrete Hilbert-type inequality and the Riemann zeta function. Appl. Math. Comput. 225, 263–277 (2013)
17. Rassias, M., Yang, B.C.: On a multidimensional half-discrete Hilbert-type inequality related to the hyperbolic cotangent function. Appl. Math. Comput. 242, 800–813 (2013)
18. Huang, Z.X., Yang, B.C.: On a half-discrete Hilbert-type inequality similar to Mulholland’s inequality. J. Inequal. Appl. 2013, 290 (2013)
19. Yang, B.C., Lehnart, L.: Half-Discrete Hilbert-Type Inequalities. World Scientific, Singapore (2014)
20. Krnic, M., Pecaric, J.: Extension of Hilbert’s inequality. J. Math. Anal. Appl. 324(1), 150–160 (2006)
21. Adiyasuren, V., Batbold, T., Azar, L.E.: A new discrete Hilbert-type inequality involving partial sums. J. Inequal. Appl. 2019, 127 (2019)
22. Hong, Y., Wen, Y.: A necessary and sufficient condition of that Hilbert type series inequality with homogeneous kernel has the best constant factor. Ann. Math. 37A(3), 329–336 (2016)
23. Hong, Y.: On the structure character of Hilbert’s type integral inequality with homogeneous kernel and applications. J. Jilin Univ. Sci. Ed. 55(2), 189–194 (2017)
24. Hong, Y., Huang, Q.L., Yang, B.C., Liao, J.L.: The necessary and sufficient conditions for the existence of a kind of Hilbert-type multiple integral inequality with the non-homogeneous kernel and its applications. J. Inequal. Appl. 2017, 316 (2017)
25. Xin, D.M., Yang, B.C., Wang, A.Z.: Equivalent property of a Hilbert-type integral inequality related to the beta function in the whole plane. J. Funct. Spaces 2018, Article ID 2691816 (2018)
26. Hong, Y., He, B., Yang, B.C.: Necessary and sufficient conditions for the validity of Hilbert type integral inequalities with a class of quasi-homogeneous kernels and its application in operator theory. J. Math. Inequal. 12(3), 777–788 (2018)
27. Rassias, M., Yang, B.C.: On an equivalent property of a reverse Hilbert-type integral inequality related to the extended Hurwitz-zeta function. J. Math. Inequal. 13(2), 315–334 (2019)
28. Rassias, M., Yang, B.C.: A reverse mulholland-type inequality in the whole plane with multi-parameters. Appl. Anal. Discrete Math. 13, 290–308 (2019)
29. Luo, R.C., Yang, B.C.: Parameterized discrete Hilbert-type inequalities with intermediate variables. J. Inequal. Appl. 2019, 142 (2019)
30. He, L.P., Liu, H.Y., Yang, B.C.: Parametric Mulholland-type inequalities. J. Appl. Anal. Comput. 9(5), 1973–1986 (2019)
31. Yang, B.C., Wu, S.H., Wang, A.Z.: On a reverse half-discrete Hardy–Hilbert’s inequality with parameters. Mathematics 7, 1054 (2019)
32. Yang, B.C., Wu, S.H., Chen, Q.: On an extended Hardy–Littlewood–Polya’s inequality. AIMS Math. 5(2), 1550–1561 (2020)
33. Yang, B.C., Huang, M.F., Zhong, Y.R.: Equivalent statements of a more accurate extended Mulholland’s inequality with a best possible constant factor. Math. Inequal. Appl. 23(1), 231–244 (2020)
34. Yang, B.C., Wu, S.H., Wang, A.Z.: A new Hilbert-type inequality with positive homogeneous kernel and its equivalent form. Symmetric 12, 342 (2020). https://doi.org/10.3390/sym12030342
35. Liao, J.Q., Hong, Y., Yang, B.C.: Equivalent conditions of a Hilbert-type multiple integral inequality holding. J. Funct. Spaces 2020, Article ID 3059952 (2020)
36. Yang, B.C., Wu, S.H., Chen, Q.: A new extension of Hardy–Hilbert’s inequality containing kernel of double power functions. Mathematics 8, 339 (2020). https://doi.org/10.3390/math8060394
37. Rassias, M., Yang, B.C., Raigorodskii, A.: On the reverse Hardy-type integral inequalities in the whole plane with the extended Riemann Zeta function. J. Math. Inequal. 14(2), 525–546 (2020)
38. Yang, B.C., Zhong, Y.R.: On a reverse Hardy–Littlewood–Polya’s inequality. J. Appl. Anal. Comput. 10(5), 2220–2232 (2020)
39. Kuang, J.C.: Applied Inequalities. Shangdong Science and Technology Press, Jinan (2004)
40. Mo, H.M., Yang, B.C.: On a new Hilbert-type integral inequality involving the upper limit functions. J. Inequal. Appl. 2020, 5 (2020)