A_{\inf} HAS UNCOUNTABLE KRULL DIMENSION

HENG DU

Abstract. Let $\mathcal{O}_E$ be a complete discrete valuation ring and $R$ be a perfect ring in characteristic $p$, we also assume $R$ is a complete valuation ring whose valuation group is of rank one and non-discrete, we prove the Krull dimension of the ring $W_{\mathcal{O}_E}(R)$ of $\mathcal{O}_E$-Witt vectors over $R$ is at least the cardinality of the continuum.

1. Introduction

Let $R$ be a perfect ring over $\mathbb{F}_p$, also assume $R$ complete with respect to a rank one and non-discrete valuation. In this paper, we fix $\mathcal{O}_E$ to be a complete discrete valuation ring, define $A_{\inf}$ to be the ring of $\mathcal{O}_E$-Witt vectors $W_{\mathcal{O}_E}(R)$ over $R$ as in [4, Sect. 1.2], i.e., elements in $A_{\inf}$ can be regarded as holomorphic functions in variable $\pi$, for a fixed uniformizer $\pi$ of $\mathcal{O}_E$. The main result of this paper is the following:

Theorem 1. Krull dimension of $A_{\inf}$ is at least the cardinality of the continuum.

Note that the result of the above theorem for the equal characteristic case is due to Kang-Park (cf. [6, Theorem 10]) that they prove the Krull dimension of $R[[X]]$ is at least the cardinality of the continuum for any rank one non-discrete valuation ring $R$. Their work improved a result of Arnold that shows the Krull dimension of $R[[X]]$ is infinite for such $R$, cf. [1]. In the mixed characteristic case, $A_{\inf}$ was studied in the work of Fontaine and is the core in $p$-adic Hodge theory. The ring structure of $A_{\inf}$ for general perfectoid rings is studied by Scholze[10], Kedalaya-Liu[8], Bhatt-Morrow-Scholze[3], etc, but there are still lots of unknowns, for example, mentioned in [7]. Elements in $A_{\inf}$ should be considered as formal power series in variable “$p$”, so $A_{\inf}$ should share similar properties as $R[[X]]$. There is conjecture in this direction on the Krull dimension of $A_{\inf}$ in [7] and [2, Warning 2.24], then proved to be infinite by Lang-Ludwig in [9] using a similar argument of Arnold.

The main input of this paper is the idea of using $\{v_s\}_{s \geq 0}$, the family of valuations associated with the family of Gauss norms on $A_{\inf}$, to study the ring properties of $A_{\inf}$. Those Gauss norms are crucial for the study of the (adic) geometry of $A_{\inf}$. In [4], Fargues and Fontaine carried
out the idea of viewing \( \{v_s\}_{s \geq 0} \) as a family of functions \( s \mapsto v_s(f) \) in non-negative real variable \( s \), and they could show for a fixed \( f \in \mathbb{A}_{\text{inf}} \), the function \( s \mapsto v_s(f) \) is a piecewise-linear concave increasing function with integer slopes. It can happen that for some \( f \), the slopes of \( s \mapsto v_s(f) \) could increase very rapidly when \( s \) approaches 0. In this paper, we are inspired by this observation of Fargues-Fontaine, we will see the key of our proof of Theorem 1 is an explicit construction of a chain of multiplicative subsets in \( \mathbb{A}_{\text{inf}} \) analyzing rates of convergence of \( s \mapsto v_s(f) \) at 0.

We will review some basic facts of functions \( s \mapsto v_s(f) \) in Section 2, in particular, the result of Fargues-Fontaine of the relation of \( s \mapsto v_s(f) \) with the Newton polygon of \( f \). In Section 3, we will construct a family of multiplicative subsets in \( \mathbb{A}_{\text{inf}} \) indexed by \((0,1)\). In Section 4, we prove Theorem 1 by proving \( \mathbb{A}_{\text{inf}} \) satisfying conditions in the following Theorem in [6]:

**Theorem 2 ([6, Theorem 3]).** Let \( R \) be an integral domain and assume that there is a proper chain of ideals \( \{I_\lambda\}_{\lambda \in (0,1)} \), such that

1. for \( 0 < \lambda < \mu < 1 \), we have \( I_\lambda \subseteq I_\mu \);
2. for each \( 0 < \mu < 1 \), there is \( g_\mu \in I_\mu \), such that for all \( 0 < \lambda < \mu \), and all minimal prime \( p \) over \( I_\lambda \), \( g_\mu \notin p \).

Then Krull dimension of \( R \) is at least the cardinality of the continuum.

**Remark 3.** Theorem 2 works for general \( R \), in particular, \( R \) does not need to be local. Also note that our construction of \( \{I_\lambda\}_{\lambda \in (0,1)} \) made systematic use of the theory of valuations, and the method in this paper has been generalized to prove the perfectoid Tate algebras has uncountable Krull dimension in [5].

2. Family of valuations on \( \mathbb{A}_{\text{inf}} \)

Fix a perfect complete non-discrete valuation ring \( R \) in characteristic \( p \), and let \( v \) be the valuation map to \( \mathbb{R} \cup \{\infty\} \). Fix a discrete valuation ring \( \mathcal{O}_E \), let \( \mathbb{A}_{\text{inf}} = W_{\mathcal{O}_E}(R) \) be the ring of \( \mathcal{O}_E \)-Witt vectors over \( R \). For any uniformizer \( \pi \in \mathcal{O}_E \), one can show the projection \( \mathbb{A}_{\text{inf}} \to R = \mathbb{A}_{\text{inf}}/(\pi) \) admits a unique multiplicative section \([\cdot] : R \to \mathbb{A}_{\text{inf}} \), which is independent of the choice of \( \pi \). Moreover, use the theory of strict \( \pi \)-rings, one can show that every elements \( f \in \mathbb{A}_{\text{inf}} \) has an unique \( \pi \)-expansion, i.e., any \( f \in \mathbb{A}_{\text{inf}} \) has a unique expansion \( f = \sum_{i \geq 0} [a_i] \pi^i \), with \( a_i \in R \), cf. [4, Sect. 1.2].

Fix a non-negative real number \( s \), for any element \( f = \sum_{i \geq 0} [a_i] \pi^i \in \mathbb{A}_{\text{inf}} \), define

\[
v_s(f) = \inf_{i \geq 0} \{v(a_i) + is\}.
\]
One can show that for \( f \) in \( A_{\inf} \), and \( t \geq s \geq 0 \), then \( v_t(f) \geq v_s(f) \geq 0 \), and \( v_s(f) = \infty \) if and only if \( f = 0 \).

**Proposition 4** (\[4, \text{Sect. 1.4}\]). For \( s \geq 0 \), we have

\[
v_s(fg) = v_s(f) + v_s(g), \quad v_s(f + g) \geq \min\{v_s(f), v_s(g)\}
\]

for all \( f, g \in A_{\inf} \).

2.1. **Relation with Newton polygons.** For \( f \in A_{\inf} \), let \( \mathcal{N}(f) \) be the Newton polygon of \( f \). Recall that \( \mathcal{N}(f) \) is defined to be the nonnegative convex piecewise-linear decreasing functions from \( \mathbb{R}_{\geq 0} \) to \( \mathbb{R} \cup \{\infty\} \) determined by the boundary of the decreasing convex hull of the set \( \{i, v_s(f) : s \geq 0\} \).

For a convex piecewise-linear decreasing function \( F \) from \( \mathbb{R}_{\geq 0} \) to \( \mathbb{R} \cup \{\infty\} \), we say \( x \) is a node of \( F \) if \( F(x) < \infty \) and \( F \) is not differentiable at \( x \), i.e., either \( \lim_{t \to x^-} F(t) = \infty \) or \( \partial_- F(x) \neq \partial_+ F(x) \), where \( \partial_- F(x), \partial_+ F(x) \) are the left and right differentials of \( F \) at \( x \). Note that it is easy to see that if \( n \) is a node of \( \mathcal{N}(f) \) then \( \mathcal{N}(f)(n) = v(x_n) \).

For a convex piecewise-linear function \( F \) from \( \mathbb{R}_{\geq 0} \) to \( \mathbb{R} \cup \{\infty\} \) that is not identically equal to \( \infty \), we define its Legendre transform \( \mathcal{L}(F) \) to be

\[
\mathcal{L}(F) : \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{-\infty\}
\]

\[
t \mapsto \inf\{F(x) + tx : x \in \mathbb{R}_{\geq 0}\}
\]

It is easy to see that \( \mathcal{L}(F) \) is also piecewise-linear. And when \( F \) is nonnegative and decreasing, we have the infimum in the above definition can be taken over the set of nodes of \( F \). In particular, fix a nonzero \( f \in A_{\inf} \), we have

\[
\mathcal{L}(\mathcal{N}(f))(t) = \inf\{\mathcal{N}(f)(x) + tx : x \in \mathbb{R}_{\geq 0}\}
\]

\[
= \inf\{\mathcal{N}(f)(x) + tx : x \in \mathbb{N}\}
\]

\[
= v_t(f).
\]

Moreover, by studying the nodes of \( \mathcal{N}(f) \), one can show:

**Proposition 5** (\[4, \text{Sect. 1.5}\]). Fix a nonzero \( f \) in \( A_{\inf} \), the function \( t \mapsto v_t(f) \) is equal to the Legendre transform of \( \mathcal{N}(f) \). More explicitly, let \( \{n_i\} \) be the set of nodes of \( \mathcal{N}(f) \) and let \( -s_i \) be the slope of \( \mathcal{N}(f) \) on the interval \( (n_i, n_{i+1}) \) (with the convention that \( s_m = 0 \) if there are only finitely many nodes and \( n_m \) is the maximal node). Then \( \mathcal{L}(\mathcal{N}(f)) \) is the unique piecewise-linear function from \( \mathbb{R}_{\geq 0} \) to \( \mathbb{R} \) such that

1. \( \mathcal{L}(\mathcal{N}(f))(0) = v_0(f) \),
2. \( \mathcal{L}(\mathcal{N}(f)) \) has slope \( n_{i+1} \) on the interval \( (s_{i+1}, s_i) \),
3. \( \mathcal{L}(\mathcal{N}(f)) \) has slope \( n_1 \) on the interval \( (s_1, \infty) \).
Corollary 6. Under the notations in Proposition 5 and further assume that \( \lim_{i \to \infty} s_i n_{i+1} = 0 \) or there are only finitely many nodes, then

\[
\mathcal{N}(f)(n_i) = -s_i n_i + \mathcal{L}(\mathcal{N}(f))(s_i).
\]

Proof. From Proposition 5 or Figure 1 we have

\[
\mathcal{N}(f)(n_i) = \begin{cases} 
\sum_{j \geq i}^{m-1} s_j(n_{j+1} - n_j) + v_0(f) & \text{if } n_m \text{ is the maximal node} \\
\sum_{j \geq i} s_j(n_{j+1} - n_j) + v_0(f) & \text{if there is no such } m
\end{cases}
\]

and

\[
\mathcal{L}(\mathcal{N}(f))(s_i) = \begin{cases} 
\sum_{j \geq i}^{m-1} (s_j - s_{j+1})n_{j+1} + v_0(f) & \text{if } n_m \text{ is the maximal node} \\
\sum_{j \geq i} (s_j - s_{j+1})n_{j+1} + v_0(f) & \text{if there is no such } m
\end{cases}
\]

From Abel’s lemma on summation by parts, we have

\[
\sum_{j=1}^{m} s_j(n_{j+1} - n_j) = (s_m n_{m+1} - s_i n_i) - \sum_{j=1}^{m-1} (s_{j+1} - s_j)n_{j+1}.
\]

When \( n_m \) is the maximal node, the above equation makes sense since \( s_m = 0 \) under our convention, and the left hand side equals to \( \mathcal{N}(f)(n_i) - v_0(f) \). When there are infinitely many nodes, let \( m \) go to infinity, we get the formula. \( \square \)
Remark 7.  

(1) For smooth functions, Legendre transform is related to integration by part, Corollary 6 is a discrete version of that.
(2) There are counterexamples that \( \lim_{i \to \infty} s_i n_{i+1} = 0 \) is not satisfied. But one has the limit always exists.

3. A CHAIN OF MULTIPLICATIVE SUBSET

Define \( p = \{ f \in A_{\inf} \mid v_0(f) > 0 \} \) and \( m = \{ \sum_{i \geq 0} [a_i] \pi^i \mid v(a_i) > 0 \text{ for all } i \} \). It is easy to see \( p \subset m \) and both are prime ideals using Proposition 4.

Lemma 8. We have \( m = \{ f \mid \limsup_{t \to 0^+} \frac{\mathcal{L}(\mathcal{N}(f))(t)}{t} = \infty \} \).

Proof. We are going to show their compliments are the same. If \( f \notin m \), then \( \mathcal{N}(f) \equiv 0 \) for \( t \gg 0 \), so \( \mathcal{N}(f) \) has only finitely many nodes and slopes. From Proposition 5 we have there is a neighborhood of 0 where \( \mathcal{L}(\mathcal{N}(f)) \) is linear and \( \lim_{t \to 0} \frac{\mathcal{L}(\mathcal{N}(f))(t)}{t} \) converges to the slope.

On the other hand, from Proposition 5 we have \( \frac{\mathcal{L}(\mathcal{N}(f))(t)}{t} = n_i + \frac{b_i}{t} \) on the interval \((s_i, s_{i-1})\), where \( b_i \) is the \( y \)-intercepts of the linear functions on each interval. Because \( \mathcal{L}(\mathcal{N}(f)) \) is concave, we have \( b_i > 0 \).

\( \limsup_{t \to 0^+} \frac{\mathcal{L}(\mathcal{N}(f))(t)}{t} \geq \limsup_i n_i \). Then if we assume \( \limsup_{t \to 0^+} \frac{\mathcal{L}(\mathcal{N}(f))(t)}{t} \) to be finite, then \( \limsup_i n_i \) is finite which means there can be only finite many nodes. Besides, \( \mathcal{L}(\mathcal{N}(f))(0) = v_0(f) \) has to be 0. We have in this case there is a node \( n \) such that \( v(a_n) = v_0(f) = 0 \), in particular, \( f \) is not in \( m \). \( \square \)

Definition 9. For any real number \( \lambda \in (0, 1] \), let \( \tilde{S}_\lambda = \{ f \mid \limsup_{t \to 0^+} \frac{\mathcal{L}(\mathcal{N}(f))(t)}{t^\lambda} < \infty \} \), then \( \tilde{S}_1 \) is the complement of \( m \) by the previous lemma. Define \( S_\lambda = \bigcup_{\lambda < \nu < 1} \tilde{S}_\nu \).

From the definition, one has for \( 0 < \lambda < \mu \leq 1 \), \( \tilde{S}_\mu \subseteq \tilde{S}_\lambda \). Using the fact that \( \mathcal{L}(\mathcal{N}(f))(t) = v_t(f) \) and Proposition 4 we have the following fact.

Lemma 10. For all \( \lambda \in (0, 1] \), we have

(1) \( \tilde{S}_\lambda, S_\lambda \) are closed under multiplication;
(2) \( p \cap \tilde{S}_\lambda = \emptyset \).
4. Proof of the Main Theorem

As we mentioned in the introduction, to show Theorem 1, it is enough to find a chain of ideals \( \{I_\lambda\}_{\lambda \in (0,1)} \) satisfying conditions in Theorem 2. The key result is the following lemma, which will be proved at the end of this section.

**Lemma 11.** For all \( \lambda \in (0,1) \), there is \( g_\lambda \in \tilde{S}_\lambda \), satisfying for all \( \mu \in (\lambda,1) \)
\[
\lim_{t \to 0^+} \frac{\mathcal{L}(N(g_\lambda))(t)}{t^\mu} = \infty.
\]
In particular, and \( g_\lambda \notin \tilde{S}_\mu \) for all \( \mu \in (\lambda,1) \).

**Remark 12.** The condition \( \lim_{t \to 0^+} \frac{\mathcal{L}(N(g_\lambda))(t)}{t^\mu} = \infty \) is stronger than \( g_\lambda \notin \tilde{S}_\mu \).

Fix a choice of \( \{g_\lambda\}_{\lambda \in (0,1)} \) as in Lemma 11, define \( I_\lambda = \mathfrak{p} + (g_\nu)_{\nu \in (0,\lambda)} \). We will show \( \{I_\lambda\}_{\lambda \in (0,1)} \) with the choice of \( g_\lambda \) satisfies conditions (2) in Theorem 2.

**Proposition 13.** For all \( \lambda \in (0,1) \), we have \( I_\lambda \cap S_\lambda = \emptyset \).

**Proof.** Let \( f \in I_\lambda \), then \( f = g + \sum_{\nu \leq \lambda} a_\nu g_\nu \) with \( g \in \mathfrak{p} \), and \( a_\nu \in A_{\infty} \). We have
\[
\mathcal{L}(N(f))(t) = v_t(f) \geq \min_{\nu \leq \lambda} \{v_t(g), v_t(a_\nu g_\nu)\} \geq \min_{\nu \leq \lambda} \{v_t(g), v_t(g_\nu)\}.
\]
The result follows from the facts that \( \lim_{t \to 0^+} \frac{v_t(g_\nu)}{t^\mu} = \infty \) and \( \lim_{t \to 0^+} \frac{v_t(g)}{t^\mu} = \infty \) for any \( \mu > \lambda \). \( \square \)

**Proposition 14.** For all \( 0 < \lambda < \mu < 1 \), and all minimal prime \( \mathfrak{p} \) over \( I_\lambda \), \( g_\mu \notin \mathfrak{p} \).

**Proof.** Assume not, then by the definition of minimal prime, we have \( g_\mu = hf^m \) for some \( m \in \mathbb{N}_{>0} \), \( f \in I_\lambda \) and \( h \in A_{\infty} \setminus \mathfrak{p} \). But then we will have
\[
\limsup_{t \to 0^+} \frac{\mathcal{L}(N(g_\mu))(t)}{x^\mu} = \limsup_{t \to 0^+} \frac{v_t(hf^m)}{x^\mu} \geq \limsup_{t \to 0^+} \frac{mv_t(f)}{x^\mu},
\]
which contradicts to Proposition 13. \( \square \)

The above proposition shows \( \{I_\lambda\}_{\lambda \in (0,1)} \) with the choice of \( g_\lambda \) satisfies conditions in Theorem 2 in particular, we have Theorem 1 holds. In the rest of this section, we will prove Lemma 11 by explicitly constructing \( g_\lambda \).

For any real number \( a > 1 \), let \( \mathcal{F}_a \) be the piecewise-linear function on \( \mathbb{R}_{\geq 0} \), such that \( \mathcal{F}_a(i) = \sum_{j \geq i} j^{-a} \) for all \( i \in \mathbb{N} \) and has nodes at every
positive integer. Since the valuation group of $R$ is non-discrete, we can find a $f_a \in \mathbf{A}_{\text{inf}}$, such that $\mathcal{N}(f_a)(0) = \infty$ and $|\mathcal{N}(f_a)(i) - \mathcal{F}_a(i)| < e^{-i}$ for $i \in \mathbb{N}_{>0}$. Moreover, we can choose $f_a$ so that $\mathcal{N}(f_a)$ has nodes at all positive integers, and let $-s_i$ be the slope of $\mathcal{N}(f_a)$ on the interval $(i, i+1)$.

For $t \in [s_{i+1}, s_i]$, we have

$$\mathcal{L}(\mathcal{N}(f_a))(t) \leq \mathcal{L}(\mathcal{N}(f_a))(s_i) \quad \text{and} \quad t^\lambda \geq s_{i+1}^\lambda.$$

In particular,

$$(\text{II}) \quad \limsup_{t \to 0^+} \frac{\mathcal{L}(\mathcal{N}(f_a))(t)}{t^\lambda} \leq \limsup_{i \to \infty} \frac{\mathcal{L}(\mathcal{N}(f_a))(s_i)}{s_{i+1}^\lambda}.$$

Similarly, we have

$$(\text{I}) \quad \liminf_{t \to 0^+} \frac{\mathcal{L}(\mathcal{N}(f_a))(t)}{t^\lambda} \geq \liminf_{i \to \infty} \frac{\mathcal{L}(\mathcal{N}(f_a))(s_i)}{s_i^\lambda}.$$

For $a > 1$, we have the estimation:

$$|s_i - i^{-a}| < 2e^{-i}$$

and use a standard estimation of $\sum_{j \geq i} j^{-a}$, we have

$$(a-1)^{-1}i^{1-a} - e^{-i} < \mathcal{N}(f_a)(i) < (a-1)^{-1}(i-1)^{1-a} + e^{-i}.$$ 

In particular, one can use these to check $\lim_{i \to \infty} s_i(i+1) = 0$ and $v_0(f_a) = 0$. So we can apply Corollary to $f_a$ to get

$$(\text{III}) \quad \mathcal{L}(\mathcal{N}(f_a))(s_i) = is_i + \mathcal{N}(f_a)(i).$$

Proposition 15. For all $\lambda$ between 0 and 1, choose $a > 1$ satisfying $a - 1 = \lambda$, then $g_\lambda = f_a$ satisfies the conditions in Lemma II.

Proof. From equations (I)(II)(III) and the above estimations, for any $\nu \in (0, 1)$, we have

$$\liminf_{i \to \infty} \frac{i^{\nu a+1-a}}{a-1} \leq \liminf_{t \to 0^+} \frac{\mathcal{L}(\mathcal{N}(f_a))(t)}{t^\nu}$$

and

$$\limsup_{t \to 0^+} \frac{\mathcal{L}(\mathcal{N}(f_a))(t)}{t^\nu} \leq \limsup_{i \to \infty} \frac{ai^{\nu a+1-a}}{a-1}.$$ 

Let $\nu$ be $\lambda$ and $\mu \in (\lambda, 1)$ respectively, then the result follows from a direct computation of the two limits. \qed
Acknowledgements. We thank Jaclyn Lang and Judith Ludwig for their paper on the related topic. We thank Pavel Čoupek, Kiran Kedlaya, Tong Liu, Judith Ludwig, Linquan Ma, Dongming She, and Yifu Wang for their feedback on the early versions of the paper. We also thank Kiran Kedlaya for suggesting this more accurate statement of the main theorem.

References

1. Jimmy T. Arnold, Krull dimension in power series rings, Trans. Am. Math. Soc. 177 (1973), 299–304 (English).
2. Bhargav Bhatt, Specializing varieties and their cohomology from characteristic 0 to characteristic \( p \), Algebraic geometry: Salt Lake City 2015. 2015 summer research institute in algebraic geometry, University of Utah, Salt Lake City, UT, USA, July 13–31, 2015. Proceedings. Part 2, American Mathematical Society (AMS): Cambridge, MA: Clay Mathematics Institute, Providence, RI, 2018, pp. 43–88 (English).
3. Bhargav Bhatt, Matthew Morrow, and Peter Scholze, Integral \( p \)-adic Hodge theory, Publ. Math., Inst. Hautes Étud. Sci. 128 (2018), 219–397 (English).
4. Laurent Fargues and Jean-Marc Fontaine, Courbes et fibrés vectoriels en théorie de Hodge \( p \)-adique, Astérisque, vol. 406, Société Mathématique de France (SMF), Paris, 2018 (French).
5. Jack J. Garzella, The perfectoid tate algebra has uncountable krull dimension, 2022.
6. B. G. Kang and M. H. Park, Krull-dimension of the power series ring over a nondiscrete valuation domain is uncountable, J. Algebra 378 (2013), 12–21 (English).
7. Kiran S. Kedlaya, Some ring-theoretic properties of \( \mathbb{A}_{\inf} \), p-adic Hodge theory. Proceedings of the Simons symposium, Schloss Elmau, Germany, May 7–13, 2017, Cham: Springer, 2020, pp. 129–141 (English).
8. Kiran S. Kedlaya and Ruochuan Liu, Relative \( p \)-adic Hodge theory: foundations, Astérisque, vol. 371, Société Mathématique de France (SMF), Paris, 2015 (English).
9. Jaclyn Lang and Judith Ludwig, \( k_{\inf} \) is infinite dimensional, Journal of the Institute of Mathematics of Jussieu (2020), 1–7.
10. Peter Scholze, \( p \)-adic geometry, Proceedings of the international congress of mathematicians, ICM 2018, Rio de Janeiro, Brazil, August 1–9, 2018. Volume I. Plenary lectures, World Scientific; Sociedade Brasileira de Matemática (SBM), Hackensack, NJ; Rio de Janeiro, 2018, pp. 899–933 (English).

(Heng Du) YAU MATHEMATICAL SCIENCES CENTER, TSINGHUA UNIVERSITY, BEIJING 100084, CHINA
Email address: hengdu@mail.tsinghua.edu.cn