Estimating magnetar radii with an empirical meta-model

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Abstract. The presence of strong magnetic fields in neutron stars, such as in magnetars, may significantly affect their crust-core transition properties and the crust size. This knowledge is crucial in the correct interpretation of astrophysical phenomena involving magnetars, such as glitches in observed rotation frequencies, cooling, bursts and possibly tidal polarizabilities. A recently developed meta-modelling technique allows exploring the model dependence of density functional theory equation of state calculations. In this work, we extend this meta-model to investigate the effect of strong magnetic fields on spinodal instabilities of neutron star matter and the associated crust-core properties. Both Tolman-Oppenheimer-Volkov and a full self-consistent numerical calculations are performed for the neutron star structure, the results being quantitatively different for strong magnetic fields.

Keywords: magnetic fields, neutron stars

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1 Introduction

With the rapid advance of astrophysical techniques on compact objects observation, there is an increased need to improve theoretical modelling to accurately describe the associated observed phenomena. Following the recent launch of the NICER (Neutron Star Interior Composition Explorer) mission [1, 2], one expects to be able to measure neutron star radii to 5% accuracy. One may also expect improved estimates of the moment of inertia of neutron stars from gravitational wave observations [3]. However, improvements in theoretical estimates of neutron star radii or moment of inertia are limited due to the uncertainties in the properties of nuclear matter at high densities [4]. One such property is the density at which nuclear matter in neutron stars goes through a phase transition, from solid inhomogeneous matter in the crust to the homogeneous liquid phase in the core. Although the liquid-gas phase transition is well understood from the studies of homogeneous nuclear matter [5], in the case of neutron star matter the problem is more complex.

The understanding of the properties of the crust-core interface is important for accurate interpretation of several other astrophysical phenomena involving neutron stars. Explanations for glitches in the otherwise periodic rotational periods of neutron stars requires a knowledge of their elastic properties and crustal moment of inertia [6]. Thermal properties of the crust-core interface govern their cooling behavior [7]. Viscosity of the crust-core transition region also plays a major role in the emission of gravitational waves due to non-axisymmetric instabilities [8]. In 2017, the first detection of a gravitational wave, GW170817, generated by the collision of two neutron stars was confirmed by the LIGO and Virgo collaborations [9] and also observed in different regions of the electromagnetic spectrum, what was then called a multi-messenger event [10]. To reproduce GW170817 tidal polarizability [11, 12], besides the use of an appropriate equation of state, the size of the crust itself plays a non negligible
role in the calculation of the Love number, which is a necessary step to compute the tidal polarizabilities of the neutron stars belonging to a binary system [13].

In non-magnetized neutron stars, the crust is generally described by an outer and an inner crust [14–17], going through an inhomogeneous transition region commonly called “pasta phase” [18], to the homogeneous core. Different calculations and numerical techniques can be used to investigate the crust-core interface region, the most common involving binodal sections, thermodynamical and dynamical spinodals [19] and the pasta phase transition to homogeneous phase [20–22], all of them converging to comparable transition densities [23]. The crust-core transition being an equilibrium phenomenon, a quantitative estimation of the transition point in principle demands a complete pasta phase calculation. However, such modelling is complex and it relies on a proper determination of the surface tension, a matter of debate in the literature, at least for the last 40 years [24]. To circumvent the pasta calculation, another possibility is to study the transition region from the properties of the homogeneous matter and its phase transition by determining the appearance of the instability region well described by spinodals. In this case, many properties of the crust-core interface and of the crust itself can be obtained directly from the transition densities and related thermodynamical quantities. According to well established calculations [25–29], many of the crust properties, as its moment of inertia, mass and radius can be directly computed from the equation of state at the transition boundary. In particular, it was shown in ref. [30], that the approximate results for the crust extracted from the core calculation up to the transition density lead to very accurate results for both its thickness and mass.

However the different microscopic (relativistic and non-relativistic) calculations give different predictions for the crust-core transition point [19, 31, 32]. These differences may arise either due to the differences in the underlying form of the density functionals inherent to the model or due to limitations in our knowledge of the empirical nuclear parameters extracted from experiments [33, 34]. Hence, the crust thickness is generally model dependent, but very important if one wants to study different astrophysical phenomena involving isolated or binaries of neutron stars.

Magnetars are a special class of isolated neutron stars bearing surface magnetic fields three orders of magnitude larger than their non-magnetized counterparts and slow rotation. Although just about 30 of them have been clearly identified [35] so far, they display a large number of peculiar phenomena, such as antiglitches, bursts and oscillations that challenge our understanding of neutron stars. It is well known that in the presence of quantizing magnetic fields, the motion of charged particles are affected, which in turn, alters the equation of state (EoS) [36, 37] and related properties both in the core and the crust. Consequently the crust-core boundary of magnetars presents different features as compared with the usual one of neutron stars. Moreover, recent studies [38] indicate that during the cooling process of a magnetar, the crust-core transition region plays an important role in determining its final configuration. Another interesting calculation [39] points to quite a large crust (2.4 Km) for a 1.4 $M_\odot$ neutron star, with non-negligible consequences on the cooling mechanism and on the emission of gravitational waves.

The crust-core region of magnetars has recently been studied within the relativistic mean field model framework [40–42]. In these works, the effects of strong magnetic fields ($10^{15}$–$10^{17}$ G) on the instability region have been investigated with the help of the Vlasov formalism used to determine the dynamical spinodals. Although the magnetic fields at the surface of magnetars are of the order of $10^{14}$–$10^{15}$ G [35], stronger fields can be expected in their interior [43]. Assuming that the crust can be as large as it was computed in [39], fields of the order of $10^{17}$ G would be justified at the transition boundary.
In the present work, we revisit the problem of the crust-core boundary and the crust size in magnetars by studying thermodynamical and dynamical spinodal sections within different models and formalisms from the ones used in [40–42]. We use a meta-modelling technique based on the density functional theory, which consists of a general framework developed to take into account a huge variety of models usually used to describe nuclear and stellar matter [44–46]. The parameters of this meta-model are directly related to the empirical parameters that can be constrained through nuclear physics experiments [44]. This allows us to determine the most influential parameters governing the crust-core transition in the presence of strong magnetic fields, and to assess the model dependence of the results.

The magnetic field effects become noticeable at ultra-strong magnetic fields \( B \gg 10^{16} \) G (near the neutron star hydrostatic stability limit). At such high magnetic field strengths, the relation between the EoS, the transition point, and the crustal thickness is complicated by the fact that the Tolman Oppenheimer Volkov (TOV) formalism of hydrostatic equilibrium in general relativity is not correct, since they break the spherical symmetry assumed in the calculations that lead to the TOV [47]. In the present work, we employ a recently developed consistent formalism for constructing numerical models of neutron stars, considering magnetic field effects on the microscopic EoS and solving the Einstein-Maxwell and equilibrium equations to obtain the global neutron star structure.

This paper is organized as follows: in section 2, we describe the theoretical formalism for the microscopic calculations. In section 2.1 we recapitulate the details of this MetaModel (MM) approach for the equation of State (EoS), while the technique for the determination of the crust-core (CC) transition point is detailed in section 2.2. We study the effect of strong magnetic fields on the CC point in section 3, and we examine the sensitivity to the most influential empirical parameters in the same section. We show in particular (section 3.2) that the influence of the poorly constrained derivatives of the symmetry energy \( (L_{\text{sym}}, K_{\text{sym}}) \) dramatically increases with the increase of the magnetic field. The effect of magnetic fields on the neutron star structure is explored in section 4, and we show that going beyond the spherical TOV approximation is very important not only for the determination of the M(R) relation [47–49], but also for the determination of the crustal properties even for relatively low magnetic fields. Finally we infer the conclusions of the study in section 5.

2 Theoretical formalism for the microscopic model

In this section, we elaborate on the theoretical approach employed to describe the microscopic neutron star matter. In section 2.1, we highlight the fundamental aspects of the MM, that was originally proposed in [44], and applied to study neutron stars [33, 34, 45] as well as nuclei [46]. In the following section 2.2, we apply this MM to study the phase transition properties and to determine the crust-core phase boundary.

2.1 Meta-modelling approach for EoS

In previous works on the effect of the magnetic field on the crust-core phase transition [40–42], EoS functionals extracted from the relativistic mean-field approach were employed. To have a complementary view on the subject, in this work we take a wide range of reference models, some of the most widely used non-relativistic functionals, namely the Skyrme functional SLy5 and SLy4 [50], Bsk17 [51] as well as the relativistic model TM1 [52]. The choice was inspired by the fact that their values of slope of symmetry energy cover a wide range, which
is known to influence crust-core transition properties [19, 33], but one should remark that more sophisticated functionals have been proposed in the recent years [15].

In order to explore these different models, we will use the technique proposed in ref. [44], where a generic meta-modelling of the nuclear EoS was introduced. In that reference, a flexible fully analytical functional was proposed with a parameter space large enough to be able to accurately reproduce different EoS functional forms from phenomenological as well as ab-initio calculations. By changing the parameter set of the meta-functional, one is then able to switch between different models with a very limited computational effort. Moreover, since all the parameters can be independently varied, interpolations are possible between the different models and sensitivity studies can be done, in order to assess the relative influence of the parameters on a given observable. We will take advantage of this property of the meta-modelling in section 3.2.

The energy per particle of homogeneous nuclear matter at baryonic density 

\[ n = n_n + n_p \]

and asymmetry \( \delta = (n_n - n_p)/n \) is expressed as:

\[
e_{\text{HNM}}(n, \delta) = e_{\text{kin}}^p + e_{\text{kin}}^n + \sum_{\alpha=0}^{4} (a_{\alpha,\text{is}} + a_{\alpha,\text{iv}} \delta^2) \frac{1}{\alpha!} \left( \frac{n - n_{\text{sat}}}{n_{\text{sat}}} \right)^\alpha u_\alpha(n, b) . \tag{2.1}
\]

In this expression, \( n_{\text{sat}} \) is the saturation density of symmetric matter, \( e_{\text{kin}}^q, q = n, p \) is the kinetic energy functional

\[
e_{\text{kin}}^q = \frac{3\hbar^2}{10m^*_q} \left( \frac{3\pi^2 n}{2} \right)^2 (1 + \tau_3 \delta)^{2/3} \tag{2.2}
\]

and the in-medium nucleon effective masses \( m^*_q (q = n, p) \) are also expanded in terms of \((n, \delta)\) as [44]:

\[
\frac{m}{m^*_q}(n, \delta) = 1 + (\bar{m} + \tau_3 \tilde{\Delta} \delta) \frac{n}{n_{\text{sat}}}, \tag{2.3}
\]

where \( \tau_3 = 1 \) for neutrons and \(-1\) for protons. The parameters \( \bar{m} \) and \( \tilde{\Delta} \) are functions of the Landau effective mass at saturation \( m^*_{\text{sat}} \) and the isospin splitting \( m^*_{\text{sat}} \) such that [44]

\[
\bar{m} = \frac{m}{m_{\text{sat}}} - 1,
\]

\[
\tilde{\Delta} = \frac{1 - \left( 1 + \Delta m^*/m (1 + \bar{m})^2 \right)}{(\Delta m^*/m)}. \tag{2.4}
\]

The \( a_{\alpha,\text{is}} \) and \( a_{\alpha,\text{iv}} \) parameters fulfill a one-to-one mapping with the so-called empirical parameters, given by the successive derivatives around saturation \( n_{\text{sat}} \) of the isoscalar \( (E_{\text{sat}}, K_{\text{sat}}, Q_{\text{sat}}, Z_{\text{sat}}) \) and isovector \( (E_{\text{sym}}, L_{\text{sym}}, K_{\text{sym}}, Q_{\text{sym}}, Z_{\text{sym}}) \) part of the energy functional. Finally, \( u_\alpha(n, b) \) is a low-density correction that insures the correct zero density limit of the functional, and becomes negligible at a density \( n_{\text{min}} \ll n_{\text{sat}} \). The value of \( n_{\text{min}} \) is governed by the extra parameter \( b \) of the model.

This meta-functional was fitted to a large number of relativistic and non-relativistic nuclear models, and the corresponding parameter sets \( \{ \vec{X} \} = \{ n_{\text{sat}}, E_{\text{sat}}, K_{\text{sat}}, Q_{\text{sat}}, Z_{\text{sat}}, E_{\text{sym}}, L_{\text{sym}}, K_{\text{sym}}, Q_{\text{sym}}, Z_{\text{sym}}, m^*/m, \Delta m^*/m \} \) are given in table X and XI of ref. [44]. The technique was shown to give a very accurate reproduction of the equation of state of catalyzed neutron star matter [44]. In that fit, the extra parameter \( b \) of the meta-modelling was taken as a constant \( b = 10 \ln(2) \approx 6.93 \). The reason of that choice was that this parameter, governing
the behavior of the energy functional at extremely low density, plays a negligible role on the energy and pressure of symmetric and pure neutron matter. However, when looking at instability properties of β-equilibrium matter, the behavior at extremely low proton density, where the validity of the Taylor expansion around saturation breaks down, becomes crucial, and \( b \) has to be included as an extra parameter. This parameter can be fitted to the spinodal curve when one wants to reproduce a specific existing model, or can be largely varied when one wants to explore the model dependence of the predictions.

A fit of the \( b \) parameter on the spinodal curve was performed for a few reference models [34]. The best reproduction of the original models was found for \( b = 10/3 \log 2 \approx 2.31 \). We will therefore use this optimized value for the rest of this study. In addition to that, in that work, a posterior distribution of the \( b \) parameter was obtained within a complete Bayesian analysis, under the constraint that the different functionals described by the meta-modelling parameter space should all fall within the uncertainty interval obtained at low densities from the ab-initio effective field theory calculations of ref. [53].

When dealing with inhomogeneous matter as in section 2.2 below, the meta-functional is extended by adding two extra gradient terms [46]:

\[
e(n, \delta) = e_{\text{HNM}}(n, \delta) + \sum_{q=n,p} \frac{\hbar^2}{2m_q^*} \frac{\tau_{2q}}{n} + C_{\text{fin}} \frac{(\nabla n)^2}{n} + D_{\text{fin}} \frac{(\nabla(n\delta))^2}{n}.
\] (2.5)

The \( \tau_{2q} = \tau_{2q}^l + \tau_{2q}^nl \) are second order local and non-local corrections arising from an \( \hbar \) expansion of the kinetic energy density of an inhomogeneous system:

\[
\tau_{2q}^l = \frac{1}{36} \frac{(\nabla n_q)^2}{n_q} + \frac{1}{3} \Delta n_q
\] (2.6)

\[
\tau_{2q}^{nl} = \frac{1}{6} \frac{\nabla n_q \nabla f_q}{f_q} + \frac{1}{6} n_0 \frac{\Delta f_q}{f_q} - \frac{1}{12} n_q \left( \frac{\nabla f_q}{f_q} \right)^2,
\] (2.7)

with \( f_q = m/m_q^* \). The \( C_{\text{fin}} \) (\( D_{\text{fin}} \)) parameters govern the isoscalar (isovector) surface properties of the energy functional in the linear response approximation. Such terms appear explicitly in Skyrme functionals, and their values are determined in the fitting protocol of Skyrme parameters to nuclear ground state properties, together with the other parameters of the functional. In the case of the meta-modelling representation of non-Skyrme functionals, we determine the values of \( C_{\text{fin}} \) and \( D_{\text{fin}} \) through a \( \chi^2 \) fit of the masses of a set of magic and semi-magic nuclei, where the theoretical masses are calculated in the Extended-Thomas-Fermi spherical approximation with the meta-functional using a parametrized density profile [34].

The resulting bulk and surface parameters are reported in table 1 and 2 for the functionals used in this work.

| Model | \( n_{\text{sat}} \) | \( E_{\text{sat}} \) | \( K_{\text{sat}} \) | \( Q_{\text{sat}} \) | \( Z_{\text{sat}} \) | \( E_{\text{sym}} \) | \( L_{\text{sym}} \) | \( K_{\text{sym}} \) | \( Q_{\text{sym}} \) | \( Z_{\text{sym}} \) | \( m^*/m \) | \( \Delta m^*/m \) |
|-------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| SLy5  | 0.1604         | −15.98         | 230            | −364           | 1592           | 32.03          | 48.3           | −112           | 501            | −3087          | 0.70            | −0.18          |
| SLy4  | 0.1595         | −15.97         | 230            | −363           | 1587           | 32.01          | 46.0           | −120           | 521            | −3197          | 0.69            | −0.19          |
| TM1   | 0.1450         | −16.26         | 281            | −285           | 2014           | 36.94          | 111.0          | 34.0           | −67            | −1546.0        | 0.71            | −0.09          |
| Bsk17 | 0.1586         | −16.05         | 242            | −364           | 1460           | 30.00          | 36.3           | −182           | 451            | −2508          | 0.80            | 0.04           |

Table 1. Empirical parameters for reference models.
### Table 2. Surface parameters for reference models.

| Model  | $C_{\text{fin}}$(MeV fm$^5$) | $D_{\text{fin}}$(MeV fm$^5$) |
|--------|-----------------------------|-----------------------------|
| SLy5   | 153.05                      | -32.75                      |
| SLy4   | 153.99                      | -31.32                      |
| TM1    | 55.0                         | 100                         |
| Bsk17  | 35.0                         | 10                           |

#### 2.2 Crust-core phase transition (CCPT) properties

Once the meta-functional for the baryonic energy of section 2.1 is specified, the crust-core phase transition can be inferred from the instability properties of the functional. A first rough definition of the transition point can be obtained from the spinodal instability which is associated to the liquid-gas phase transition in neutral nuclear matter at sub-saturation densities [54]. The spinodal region is defined as the convex part of the energy density in the neutron-proton density plane, and can be spotted from the presence of negative eigenvalues of the following curvature matrix:

$$ C_{\text{NM}} = \begin{pmatrix} \frac{\partial \mu_n}{\partial \rho_n} & \frac{\partial \mu_n}{\partial \rho_p} \\ \frac{\partial \mu_p}{\partial \rho_n} & \frac{\partial \mu_p}{\partial \rho_p} \end{pmatrix} $$

i.e., the curvature matrix depends on the energy functional via the chemical potentials $\mu_q = \partial \epsilon/\partial n_q$, ($q = n, p$).

The thermodynamic spinodal only gives a qualitative estimation of the transition point. Indeed, in the presence of an electron background gas, the liquid-gas phase transition of nuclear matter is replaced by a transition from an homogeneous medium to a clusterized one. Such a transition is better signalled by the instability of nuclear matter against independent fluctuations of neutron, proton and electron densities with a finite spatial extension [54, 55]. The energy density curvature matrix subject to finite size density fluctuations, takes the form

$$ C_f = \begin{pmatrix} \frac{\partial \mu_n}{\partial \rho_n} & \frac{\partial \mu_n}{\partial \rho_p} & 0 \\ \frac{\partial \mu_p}{\partial \rho_n} & \frac{\partial \mu_p}{\partial \rho_p} & 0 \\ 0 & 0 & \frac{\partial \mu_e}{\partial \rho_e} \end{pmatrix} + 2k^2 \begin{pmatrix} C_{f1} & C_{f2} & 0 \\ C_{f2} & C_{f1} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{4\pi e^2}{k^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} $$

where $C_{f1} = C_{\text{fin}} + D_{\text{fin}}$ and $C_{f2} = C_{\text{fin}} - D_{\text{fin}}$. In this matrix, in addition to the bulk term that corresponds to the thermodynamic spinodal, there is a surface term proportional to the square of the wavelength $k$, while the Coulomb interaction adds a term inversely proportional to $k^2$. Then thermodynamic fluctuations are recovered at $k \to \infty$ limit of this model. The global instability region is defined as the envelope of the different $k$-spinodals corresponding to all the possible linear sizes for the density fluctuation.

For both thermodynamic eq. (2.8) and dynamic eq. (2.9) spinodals, the crust-core transition point is defined as the intersection of the instability envelope with the neutrino-less $\beta$-equilibrium curve of neutron star matter, defined by:

$$ \mu_n - \mu_p = \mu_e. $$

#### 2.3 Introduction of the magnetic field

It is well known that in the presence of strong magnetic fields, the motion of charged particles is confined to Landau levels, in a direction perpendicular to the magnetic field direction. This
leads to a modification of the energy levels of the particles, and hence of the EoS of neutron star matter [37].

In the formulation of the meta-modelling eq. (2.1), a modification of the single-particle levels directly affects the first term of the meta-functional, which physically corresponds to the kinetic energy per baryon in the non-relativistic approximation. Within this approximation, the kinetic energy per baryon of neutrons remains unaltered, i.e.

$$e_{n \text{kin}} = \frac{3h^2}{10m_n} \left( \frac{3\pi^2 n}{2} \right)^{2/3} \left( 1 + \delta \right)^{5/3}$$

(2.11)

where $m_n^*$ from eq. (2.3) is the effective neutron mass. Conversely, electrons and protons are affected by the magnetic field. For the electrons, we follow the standard relativistic treatment of refs. [36, 37]. Within the non-relativistic approximation [56, 57], the single particle energy of the $\nu$-th Landau level is given by

$$\mu_p^* = \frac{p_{z,F}^2}{2m_p^*} + \frac{\nu eB}{m_p^*},$$

(2.12)

where $p_z$ is the momentum of the proton in the $z$-direction in which the magnetic field is directed. The proton density becomes modified in presence of the magnetic field as:

$$n_p = \frac{eB}{2\pi^2} \sum_{\nu} g_{\nu} p_{z,F}(\nu)$$

(2.13)

where the maximum number of Landau levels is given by:

$$\nu_{\text{max}} = \frac{m_p^* \mu_p^*}{eB}.$$ 

(2.14)

Then the proton contribution to the kinetic energy per baryon becomes:

$$e_{p \text{kin}}^p = \frac{eB}{2n\pi^2} \sum_{\nu} m_p^* \sum_{\nu} \left[ \frac{p_{z,F}^3(\nu)}{3} + 2\nu eBp_{z,F}(\nu) \right].$$

(2.15)

When the magnetic field is high, only the lowest or a few Landau levels are occupied. The corresponding critical quantizing field $B_c$ can be calculated by equating the rest energy $mc^2$ of a particle of mass $m$ to its cyclotron energy $\hbar \omega_c$, where $\omega_c = eB/mc$ is the cyclotron frequency, i.e. $B_c = m^2c^3/e\hbar$. For the case of electrons, this gives $B_{c,e} = 4.4 \times 10^{13}$ G. For low magnetic fields, the Landau levels are numerous and closely spaced, such that one recovers the continuum limit in the absence of the field.

3 Results: magnetic field effects

3.1 Effect on the CCPT

We first analyze the effect of the magnetic field on the thermodynamical spinodal for our reference model SLy5. The results of this section are in qualitative agreement with the previous works [40–42] using relativistic mean-field functionals, and the Vlasov method to compute the dynamical spinodal.
Figure 1. Thermodynamic spinodals for relative magnetic field strengths $B_*=10^2$ (upper left), $10^3$ (upper right), $10^4$ (lower left) and $10^5$ (lower right). The continuous red line gives the spinodal envelope in the absence of magnetic field.

Taking multiples of the quantizing electron field $B_e=4.4 \times 10^{13}$ G, we express the magnetic field in units of $B_*=B/B_e$. The effect of the magnetic field on the thermodynamical spinodal is shown in figure 1. It is found to be negligible up to $10^2 B_e$. However, for fields above $10^3 B_e$, the effect of magnetic field on the thermodynamical spinodal region is quite dramatic. Instead of a single region, we now find alternate regions of stability and instability. This is due to the appearance of kinks in the EoS as a result of the confinement of the motion of the charged particles to quantized Landau levels. We study the effect of increasing magnetic field strengths in multiples of $B_e$ of $10^2$, $10^3$, $10^4$ and $10^5$, although magnetic field strengths as large as $10^5 B_e$, equivalent to $10^{18} G$, is highly unlikely to be encountered in astrophysical observations, since it is beyond the hydrostatic stability limit for most EoSs [58].
Table 3. The uncertainty in the transition point for thermodynamical spinodals of the SLy5 functional, for different magnetic field strengths (see text for more details).

| $B^*$ (fm$^{-3}$) | $\rho_1$ (fm$^{-3}$) | $\rho_2$ (fm$^{-3}$) | $\Delta \rho_{PT}$ (fm$^{-3}$) | $p_1$ (MeV/fm$^3$) | $p_2$ (MeV/fm$^3$) | $\Delta p_{PT}$ (MeV/fm$^3$) |
|------------------|----------------------|----------------------|-----------------------------|-------------------|-------------------|-----------------------------|
| $10^2$           | 0.088                | 0.088                | 0                           | 0.462             | 0.462             | 0                           |
| $10^3$           | 0.086                | 0.091                | 0.005                       | 0.442             | 0.502             | 0.06                        |
| $5 \times 10^3$ | 0.075                | 0.112                | 0.037                       | 0.332             | 0.882             | 0.550                       |
| $7 \times 10^3$ | 0.070                | 0.090                | 0.020                       | 0.300             | 0.501             | 0.201                       |
| $10^4$           | 0.048                | 0.113                | 0.065                       | 0.193             | 0.937             | 0.744                       |

Table 4. Density and pressure of the transition point for dynamical spinodals of the SLy5 functional, for different magnetic field strengths (see text for more details).

| $B^*$ (fm$^{-3}$) | $\rho_1$ (fm$^{-3}$) | $\rho_2$ (fm$^{-3}$) | $\Delta \rho_{PT}$ (fm$^{-3}$) | $p_1$ (MeV/fm$^3$) | $p_2$ (MeV/fm$^3$) | $\Delta p_{PT}$ (MeV/fm$^3$) |
|------------------|----------------------|----------------------|-----------------------------|-------------------|-------------------|-----------------------------|
| $10^2$           | 0.079                | 0.079                | 0                           | 0.356             | 0.356             | 0                           |
| $10^3$           | 0.075                | 0.077                | 0.002                       | 0.320             | 0.338             | 0.018                       |
| $5 \times 10^3$ | 0.071                | 0.085                | 0.014                       | 0.297             | 0.438             | 0.141                       |
| $7 \times 10^3$ | 0.065                | 0.085                | 0.020                       | 0.261             | 0.439             | 0.178                       |
| $10^4$           | 0.043                | 0.077                | 0.034                       | 0.165             | 0.369             | 0.204                       |

We now turn to investigate how magnetic field affects the CCPT for the thermodynamical and dynamical spinodals. As already discussed in section 2.2, the CCPT is determined from the crossing of the spinodals with the $\beta$-equilibrium line.

In figure 2 the solid blue lines determine the spinodals, the $\beta$-equilibrium lines are shown by the black points, while the crossing points are demarcated by the red dots. We can see that the qualitative behavior of the two spinodals is very similar, but the transition occurs at lower density when finite size fluctuations are taken into account (lower panels), as expected. Because of the non-monotonous behavior of the spinodals in the presence of magnetic fields, it is clear that several crossing points are possible. This is however not a general rule, as we can see from the lower right part of figure 2: a single transition point is observed even at very high magnetic fields for the dynamical spinodal of SLy5. This is at variance with the findings of refs. [40–42] and shows that the possibility of having successive layers of stable and unstable matter is model dependent.

In tables 3 and 4, we list the first and the last crossing points as obtained from the thermodynamic and the dynamic analysis, as a function of the magnetic field. This defines the density $\Delta \rho_{PT}$ and pressure $\Delta p_{PT}$ interval corresponding to the successive sequence of stable and unstable thermodynamic configurations, due to the presence of Landau levels.

As previously observed in refs. [40–42], the global effect of the magnetic field is to increase the density and pressure region where the EoS behavior is not monotonic, but the effect is non-linear. We observe that this effect is somewhat quenched in the dynamical spinodal treatment with respect with the thermodynamical one.
3.2 Sensitivity of CCPT to influential parameters

It was shown by different authors [31, 32, 59, 60] that the core-crust phase transition in ordinary non-magnetized neutron stars strongly depends on the symmetry energy, particularly on the slope parameter \( L_{\text{sym}} \). Within the MM approach, in a recent analysis [34] a systematic study was performed in order to isolate the effects of the empirical nuclear parameters and their uncertainties. It was found that the most influential parameters are the poorly known isovector parameters \( L_{\text{sym}} \) and \( K_{\text{sym}} \). In this section, we study the effect of \( L_{\text{sym}} \) and \( K_{\text{sym}} \) on the CCPT density as estimated from the thermodynamic spinodal, as a function of the magnetic field.

We take as fiducial values the ones of the SLy5 functional, \( L_{\text{sym}} = 48.3 \text{ MeV} \) and \( K_{\text{sym}} = -112 \text{ MeV} \). To study the sensitivity to those parameters, we modify the parameters...
one by one keeping all the others unchanged. For the parameter under study, we consider two extreme values corresponding to the present uncertainty from nuclear physics experiments as estimated in ref. [44], namely $L_{\text{sym}}^{\text{min}} = 47 \text{ MeV}$, $L_{\text{sym}}^{\text{max}} = 106 \text{ MeV}$, $K_{\text{sym}}^{\text{min}} = -135 \text{ MeV}$, $K_{\text{sym}}^{\text{max}} = 129 \text{ MeV}$.

We can see from the results of figures 3 and 4 that the transition point is strongly modified by a change of the slope parameter $L_{\text{sym}}$. As a general statement, a stiffer symmetry energy leads to higher values for the transition density, as already observed in the literature [31, 32, 59, 60]. We can additionally observe that the effect strongly increases for
the highest magnetic fields. For small values of $L_{\text{sym}}$ such as SLy5 or $L_{\text{sym}}^{\text{min}}$, the transition densities $\rho_1$ and $\rho_2$ coincide, but for large values of $L_{\text{sym}}$ they are quite different for different magnetic field strengths $B_\ast$. In particular, we can see that the existence of successive stable and instable regions around the transition point seems to be strongly correlated with the $L_{\text{sym}}$ parameter. Concerning $K_{\text{sym}}$, we can see that it is by far less influential than $L_{\text{sym}}$. Variations of $K_{\text{sym}}$ in the physically reasonable range $K_{\text{sym}}^{\text{min}}$ to $K_{\text{sym}}^{\text{max}}$ does not change the CCPT density considerably for different magnetic field strengths $B_\ast$. An important effect is only seen for the highest magnetic field, when two different transition points start to appear due to the highly complex behavior of the spinodal zone.

From figure 3 we can conclude that better constraints on the symmetry energy slope $L_{\text{sym}}$ are expected to strongly reduce the model dependence of the results.

4 Magnetic field effects on the mass-radius relation

In the previous section, we demonstrated how much a strong magnetic field affects the thermodynamical and dynamical spinodals and hence the crust-core phase transition in neutron stars. We can therefore expect following the previous studies [40–42] that the magnetic field will also play an important role in determining the structure of neutron stars, especially its radius and crust thickness. In order to investigate the influence of magnetic field on the static properties of strongly magnetized neutron stars, we choose as a reference model for this section the Skyrme SLy4 EoS, which is one of the most commonly used EoS in astrophysics [61].

Structure calculations of strongly magnetized neutron stars have been discussed intensively in the literature in the past few years [47, 48, 62–64]. It is well known that strong electromagnetic field affects neutron stars in two ways: firstly the EoS due to the quantization of the charged particles into Landau levels, and secondly by introducing an anisotropy in the energy momentum tensor and thereby affecting the global structure. In the presence of strong magnetic fields, the neutron star shape strongly deviates from spherical symmetry and hence the spherically symmetric Tolman-Oppenheimer-Volkov (TOV) equations are no longer applicable. The ideal method to tackle the problem is to self-consistently solve the neutron star structure equations endowed with a magnetic field, i.e. Einstein-Maxwell and equilibrium equations, with a magnetic field dependent EoS. However this is a complicated numerical problem that has only been tackled recently [47, 48]. In this section, we compute and compare the mass-radius relations of magnetized neutron stars obtained using both an isotropic TOV solution [40–42], as well as a full numerical solution.

4.1 TOV calculation

In the calculation of the core EoS, the magnetic field contribution to the energy density and pressure, $\epsilon_B = p_B = B^2 / 8\pi$, is assumed to be isotropic:

$$
\begin{align*}
\epsilon_{\text{tot}} &= n\epsilon_{\text{HNM}} + \epsilon_l + \epsilon_B; \\
p_{\text{tot}} &= p_{\text{HNM}} + p_l + p_B.
\end{align*}
$$

Here, $\epsilon_{\text{HNM}}$ ($p_{\text{HNM}}$) is the hadronic contribution to the energy (pressure) from the MM EoS eq. (2.1), modified such as to include the magnetic field contribution as in eq. (2.11). The lepton contribution $\epsilon_l$, $p_l$ is calculated considering electrons as a relativistic free Fermi gas, again modified by the magnetic field as explained in section 2.3.
Using the core EoS and the CCPT points calculated in the previous subsection for the SLy4 case, we aim to compute the mass-radius relations of magnetized neutron stars. The calculation of the structure of neutron stars using TOV requires the additional knowledge of the crust EoS. This can however be avoided by using the method proposed by Zdunik et al. [30]. In this method, one integrates from the core to the crust-core interface following the TOV prescription in order to obtain the core mass and radius. A simple algebraic calculation relates the crust mass $M_{\text{crust}}$ and the total radius $R_{NS}$ with the core mass $M_{\text{core}}$ and core radius $R_{\text{core}}$ and the CCPT density $\rho_{cc}$ and pressure $P_{cc}$ as:

$$M_{\text{crust}} = \frac{4 \pi P_{cc} R_{\text{core}}^4}{G M_{\text{core}}} \left( 1 - \frac{2 G M_{\text{core}}}{R_{\text{core}} c^2} \right)$$

$$R_{NS} = \frac{R_{\text{core}}}{1 - (\alpha - 1) \left( \frac{c^2 G M_{NS}}{R_{\text{core}}^{-2}} - 1 \right)}$$

where $\alpha = \left( \frac{\mu_{cc}}{\mu_0} \right)^2$ is the ratio of the baryon chemical potentials at the crust-core interface and that of cold catalyzed matter at zero pressure (930.4 MeV). The crust thickness is then easily obtained from the relation $l_{\text{crust}} = R_{NS} - R_{\text{core}}$.

This approximation was shown to be reasonably precise for not too low neutron star masses in ref. [30].

In figure 5 the mass-radius relation of neutron stars endowed with different magnetic field strengths computed within the spherical TOV approximation using the Zdunik method for the SLy4 EoS are shown. Notice that the maximum mass is not always reached, a limitation of the meta-modelling due to the fact that the convergence of the Taylor expansion is not perfect at the highest densities. However, this is not a problem here, since we are mainly interested in the radius, which depends mainly on the low density region. For comparison the mass-radius relation using the Douchin Haensel (DH) SLy4 EoS [61] for $B = 0$ is also plotted. This EoS is based on the same SLy4 energy functional, but the inhomogeneous crust contribution is explicitly accounted for. The close agreement between the DH result and our result at the lowest magnetic field demonstrated the quality of the Zdunik approximation. We can also see from the figure that except the case of very strong magnetic fields $B_* \sim 10^4$, the effect of magnetic field on the total mass and radius of the neutron star is almost negligible in the spherical TOV approximation, in agreement with the results of refs. [40–42]. However, the magnetic field has a non-negligible effect on the dynamical spinodals and hence the crust-core phase transition, which in turn should affect the crust-thickness. In table 5, we summarize the effect of different magnetic field strengths on the transition densities and the neutron star structure. For zero or low magnetic fields ($B_* \sim 10^2, 10^3$), there is a unique transition density. However for higher fields, as discussed in section 3.1, the $\beta$-equilibrium line crosses the dynamical spinodals at multiple points. The transition density corresponding to the first and the last crossings are denoted as $\rho_1$ and $\rho_2$ in the table. The difference in crust-core transition density leads to a difference in the corresponding total and core radii and crust thickness $l_{\text{crust}} = R_{NS} - R_{\text{core}}$. The values of $l_{\text{crust}}^1$ and $l_{\text{crust}}^2$, corresponding to $\rho_1$ and $\rho_2$, are also given in the table, along with their difference $\Delta l_{\text{crust}} = l_{\text{crust}}^2 - l_{\text{crust}}^1$. In this case, both $l_{\text{crust}}^1$ and $l_{\text{crust}}^2$ are of the order of 1 Km, much thinner than what was suggested in [39]. $\Delta l_{\text{crust}}$ measures the region in the neutron star crust where layers of homogeneous and inhomogeneous matter might coexist [40–42]. These results are in good qualitative agreement with refs. [40–42], even if some quantitative differences arise from the use of a different functional. The model dependence of the results will be discussed in section 4.3.
Figure 5. Mass-radius relation of neutron stars endowed with various magnetic field strengths computed using the isotropic TOV approximation and the Zdunik method for the SLy4 EoS. For comparison the mass-radius relation using the Douchin Haensel SLy4 EoS for $B = 0$ from ref. [61] is also plotted.

Table 5. Effect of a strong magnetic field on the transition density and crust thickness for a neutron star of gravitational mass $1.4 M_\odot$ with the SLy4 functional, in the isotropic TOV approximation. See text for details.

| $B_*$  | $\rho_1$ (fm$^{-3}$) | $R_{NS}^1$ (km) | $R_{core}^1$ (km) | $l_{crust}^1$ (km) | $\rho_2$ (fm$^{-3}$) | $R_{NS}^2$ (km) | $R_{core}^2$ (km) | $l_{crust}^2$ (km) | $\Delta l_{crust}$ (km) |
|-------|---------------------|-----------------|------------------|-------------------|---------------------|-----------------|------------------|-------------------|------------------------|
| 0     | 0.077               | 11.704          | 10.797           | 0.908             | 0.077               | 11.704          | 10.797           | 0.908             | 0                      |
| $10^2$| 0.076               | 11.771          | 10.843           | 0.928             | 0.076               | 11.771          | 10.843           | 0.928             | 0                      |
| $10^3$| 0.071               | 11.770          | 10.859           | 0.911             | 0.074               | 11.770          | 10.849           | 0.920             | 0.009                  |
| $5 \times 10^3$ | 0.070          | 11.764          | 10.865           | 0.899             | 0.084               | 11.761          | 10.8              | 0.961             | 0.062                  |
| $7 \times 10^3$ | 0.033          | 11.804          | 10.993           | 0.811             | 0.081               | 11.777          | 10.811           | 0.966             | 0.155                  |
| $10^4$| 0.041               | 11.826          | 10.97            | 0.856             | 0.074               | 11.795          | 10.855           | 0.940             | 0.084                  |

One may also calculate the fractional moment of inertia of the crust $I_{crust}/I$ using the formula [6]

$$I_{crust} = \frac{16\pi}{3} \frac{R_{core}^6 p_{PT}}{R_s^3} \left[ 1 - \frac{0.21}{(R/R_s - 1)} \right] \left[ 1 + \frac{48}{5} \left( R_{core}/R_s - 1 \right) (p_{PT}/\epsilon_{PT} + \ldots) \right], \quad (4.2)$$
\[
\Delta I_{\text{crust}} / I = \frac{\rho_1 I_{\text{crust}}^1}{I} - \frac{\rho_2 I_{\text{crust}}^2}{I},
\]

Table 6. Effect of strong magnetic field on transition density and fractional moment of inertia of the crust for a neutron star of gravitational mass 1.4 \(M_\odot\) with the SLy4 functional, in the isotropic TOV approximation. See text for details.

| \(B_*\) (\(\text{fm}^{-3}\)) | \(\rho_1 I_{\text{crust}}^1 / I\) | \(\rho_2 I_{\text{crust}}^2 / I\) | \(\Delta I_{\text{crust}} / I\) |
|----------------|----------------|----------------|----------------|
| 0              | 0.077          | 0.028          | 0              |
| \(10^2\)       | 0.076          | 0.028          | 0              |
| \(10^3\)       | 0.071          | 0.026          | 0.074          |
| \(5 \times 10^3\) | 0.070       | 0.025          | 0.084          |
| \(7 \times 10^3\) | 0.033       | 0.011          | 0.081          |
| \(10^4\)       | 0.041          | 0.015          | 0.074          |

where, the total moment of inertia of the star is

\[
\frac{I}{MR^2} = \frac{0.21}{1 - R_s / R},
\]

In the above expressions, \(R_s = 2GM\) is the Schwarzschild radius of the star. In table 6, we provide the values for the fractional moment of inertia of the crust for different magnetic fields, for a neutron star of gravitational mass 1.4 \(M_\odot\) in the isotropic TOV approximation.

4.2 Full numerical calculations

The ideal way to calculate the structure of strongly magnetized neutron stars is to solve the Einstein-Maxwell and equilibrium solutions self-consistently with a magnetic field dependent EoS. We adopt this technique as we have at our disposal the numerical library LORENE [65] previously developed for studying the structure of strongly magnetized neutron stars [47, 48, 58, 66]. To this aim, we first construct total EoSs for neutron stars for the crust and the core obtained using the same MM. This is done using the compressible liquid drop model for the crust, with surface tension parameters optimized for each functional via a fit on nuclear masses [33]. The magnetic field dependence is only included in the core EoSs while assuming a non-magnetized crust, which is a reasonable approximation.

We display the mass-radius relations corresponding to the SLy4 EoSs for increasing magnetic fields in figure 6. It is evident from the figure that while neutron star mass-radius relation for low fields \(B_* = 10^2, 10^3\) resembles that of the zero-field case, for higher magnetic fields \(B_* = 5 \times 10^3, 7 \times 10^3, 10^4\) the mass-radius relation departs strongly from the zero field case. As was discussed in [47, 48], this is primarily due to the purely magnetic field contribution and not the effect of magnetic field on the EoS.

One must also note that while for the isotropic TOV calculation one assumed a constant magnetic field, the magnetic field for the full numerical calculation is generated via a current function and hence generates a profile, from the highest central value decreasing towards the surface [49]. In figure 7 we display the magnetic field lines and enthalpy isocontours of a neutron star of gravitational mass of 1.4 \(M_\odot\) and a polar magnetic field \(B_* = 10^4\) (i.e. \(4.414 \times 10^{13} \text{T}\)). It is evident that the magnetic field is far from constant, and the neutron star surface deviates from a spherical shape, as elaborated previously. As the neutron star radius as well as the magnetic field are direction dependent, we specify for this study the radius as
Figure 6. Circumferential radius vs. gravitational mass of neutron stars endowed with strong magnetic fields computed using the complete MM (crust and core), and the full solution of the Einstein-Maxwell equations with LORENE, for the SLy4 EoS. For comparison the Douchin Haensel SLy4 for $B = 0$ is also plotted.

...the circumferential equatorial radius $R_{\text{circ}}$ (see [66]). In figure 8, we display the radial profiles of the first four even multipoles ($l = 0$, 2, 4, 6) of the magnetic field norm $b(r,\theta)$ computed for the stellar model described in figure 7. From symmetry argument odd multipoles are all zero [49]. We also plot the magnetic field norm as a function of the coordinate radius for different angular directions ($\theta = 0, \pi/4, \pi/2$). It is clear that the magnetic field structure in this case can neither be described by a constant amplitude nor a simple profile.

Similar to table 5, we summarize in table 7 the effect of different magnetic field strengths on the transition densities and the neutron star structure computed within the full numerical structure calculation via LORENE. As before, for low magnetic fields ($B_* \sim 10^2, 10^3$), there is a unique transition density. At higher fields, as discussed in section 3.1, there are multiple transition density points, of which the first and the last crossings are denoted as $\rho_1$ and $\rho_2$ in the table. The difference in crust-core transition density leads to a difference in the corresponding total circumferential radii $R_{NS}$, circumferential core radii $R_{\text{core}}$ and crust thickness $l_{\text{crust}}$. The values of $l_{\text{crust}}^1$ and $l_{\text{crust}}^2$, corresponding to $\rho_1$ and $\rho_2$, are also given in the table, along with their difference $\Delta l_{\text{crust}} = l_{\text{crust}}^2 - l_{\text{crust}}^1$.

As compared to the previous table 5, one can see here that although the total radii are comparable, the core radii and hence the crust thickness calculated numerically differ significantly from the isotropic TOV case. This is due to the difference in the treatment of the magnetic field profile as well as the pure magnetic field contribution. In particular, the...
Figure 7. Magnetic field lines (left panel) and enthalpy isocontours (right panel) in the meridional plane ($x, z$), for the static star configuration, with a gravitational mass of $1.4 \, M_\odot$ and a polar magnetic field $B_\ast = 10^4$. The stellar surface is depicted by the bold line. In the right panel, solid lines represent positive enthalpy isocontours, dashed lines negative ones.
Figure 8. Magnetic field lines (left panel) and enthalpy isocontours (right panel) in the meridional plane ($x, z$), for the static star configuration, with a gravitational mass of $1.4 \, M_\odot$ and a polar magnetic field $B_\ast = 10^4$.

Table 7. Effect of strong magnetic field on total circumferential radius, circumferential core radius and crust thickness for a neutron star of gravitational mass $1.4 \, M_\odot$ computed within a full numerical structure calculation with LORENE.
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|}
\hline
$B_*$ & $\rho_1$ & $R_{NS}^1$ & $R_{\text{core}}^1$ & $t_{\text{crust}}^1$ & $\rho_2$ & $R_{NS}^2$ & $R_{\text{core}}^2$ & $t_{\text{crust}}^2$ & $\Delta t_{\text{crust}}$ \\
\hline
$10^2$ & 0.072 & 10.095 & 8.660 & 2.335 & 0.090 & 10.995 & 8.660 & 2.335 & 0 \\
$10^3$ & 0.058 & 10.095 & 8.677 & 2.313 & 0.087 & 10.995 & 8.677 & 2.313 & 0 \\
$5 \times 10^3$ & 0.089 & 10.095 & 8.253 & 2.527 & 0.093 & 11.059 & 8.510 & 2.549 & 0.022 \\
$7 \times 10^3$ & 0.071 & 11.136 & 8.413 & 2.723 & 0.091 & 11.143 & 8.396 & 2.747 & 0.024 \\
$10^4$ & 0.048 & 11.244 & 8.179 & 3.065 & 0.115 & 11.276 & 8.090 & 3.186 & 0.121 \\
\hline
\end{tabular}
\caption{Effect of strong magnetic field on total circumferential radius, circumferential core radius and crust thickness for a neutron star of gravitational mass $1.1 M_\odot$ computed numerically, for the Bsk17 EoS.}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|}
\hline
$B_*$ & $\rho_1$ & $R_{NS}^1$ & $R_{\text{core}}^1$ & $t_{\text{crust}}^1$ & $\rho_2$ & $R_{NS}^2$ & $R_{\text{core}}^2$ & $t_{\text{crust}}^2$ & $\Delta t_{\text{crust}}$ \\
\hline
$10^2$ & 0.066 & 14.153 & 10.777 & 3.376 & 0.066 & 14.153 & 10.777 & 3.376 & 0 \\
$10^3$ & 0.057 & 14.157 & 10.781 & 3.376 & 0.063 & 14.157 & 10.781 & 3.376 & 0 \\
$5 \times 10^3$ & 0.055 & 14.252 & 10.625 & 3.627 & 0.093 & 14.251 & 10.595 & 3.656 & 0.029 \\
$7 \times 10^3$ & 0.061 & 14.357 & 10.444 & 3.913 & 0.099 & 14.357 & 10.415 & 3.942 & 0.029 \\
$10^4$ & 0.038 & 14.594 & 10.119 & 4.475 & 0.102 & 14.856 & 10.055 & 4.531 & 0.056 \\
\hline
\end{tabular}
\caption{Effect of strong magnetic field on total circumferential radius, circumferential core radius and crust thickness for a neutron star of gravitational mass $1.8 M_\odot$ computed numerically, for the TM1 EoS.}
\end{table}

difference in the crust thickness $\Delta t_{\text{crust}}$ is significantly higher. Actually, both $t_{\text{crust}}^1$ and $t_{\text{crust}}^2$ are now of the order of 3 Km, considerably thicker than what was obtained with the TOV treatment for an isotopic star and also larger than the 2.4 Km, estimated in [39].

### 4.3 Model dependence of the results

In order to generalize the results obtained for any EoS, we perform the isotropic approximation and full numerical calculation for two other reference parameter sets, TM1 and Bsk17 (see section 2.1). The corresponding mass-radius relations for various magnetic field values obtained numerically are given in the figures 9 and 10 and the main results are summarized in tables 8 and 9.

As it is already very well known, the stiffest model (TM1) produces the largest radius for a given mass. We can see from figures 9 and 10 that this model dependence is preserved with the increase of the magnetic field. The width of the crust monotonically increases with the magnetic field for all studied models, and it seems to be very well correlated with the star radius, a stiffer EoS producing a thicker crust. Conversely, the region where successive homogeneous and inhomogeneous layers are expected [40–42], measured by the variable $\Delta t_{\text{crust}}$, seems to decrease with increasing stiffness.

For the numerical calculations of the strongly magnetized neutron stars using LORENE, one may calculate the total moment of inertia of a slowly rotating neutron star from the total angular momentum, taking into account both contributions from rotation and magnetic field. However, within the numerical models (see [47] and references within), one cannot isolate
the crustal moment of inertia in a gauge-independent way. Hence we refrain from performing a calculation of the fractional crustal moment of inertia as was done in section 4.1 for the TOV calculations.

5 Final remarks

In this work, we extended a recently developed meta-modelling technique to study the crust-core phase transition properties and crustal thickness in the presence of strong magnetic fields. It was found that the magnetic field severely modifies the structure of the phase transition region, leading to the a non-negligible difference in the density and pressure of the transition from that of the zero magnetic field case. As previously observed in refs. [40–42], the most spectacular effect of the magnetic field is the fact that the unstable region with respect to density fluctuations is discontinuous if the magnetic field is sufficiently intense. As a consequence, matter in beta equilibrium can correspond to a sequence of stable and unstable regions, as a function of the density. How this feature will reflect in the composition of the neutron star is not clear. It might be that the crust is simply more extended, or that in its inner part homogeneous and inhomogeneous layers could coexist, with much higher impurity factors than usually considered for non-magnetized star.

A sensitivity analysis to the most influential isovector empirical parameters ($L_{\text{sym}}$ and $K_{\text{sym}}$) has shown that the discontinuity in the unstable region is strongly increasing with the $L_{\text{sym}}$ parameter, and the effect is amplified by the intensity of the magnetic field.

Figure 9. Circumferential radius vs. gravitational mass of neutron stars endowed with strong magnetic fields computed using the MM for Bsk17 EoS.
| Mass (M_{\odot}) | Radius (km) |
|-----------------|-------------|
| $B^* = 10^2$    | 14.5        |
| $B^* = 10^3$    | 15.5        |
| $B^* = 5 \times 10^3$ | 16        |
| $B^* = 7 \times 10^3$ | 16.5      |
| $B^* = 10^4$    | 17          |

**Figure 10.** Circumferential radius vs. gravitational mass of neutron stars endowed with strong magnetic fields computed using the MM for TM1 EoS.

In order to study the effect of the change in crust-core transition properties on the neutron star structure, we adopted two different formalisms. Firstly we calculated the mass-radius relation as well as the crust thickness using an isotropic TOV formalism, that has been commonly used in the literature. Next we performed a full self-consistent numerical computation of the neutron star structure. For this purpose, an RMF and three different Skyrme functionals were used, namely TM1, SLy4, SLy5 and BSk17, as reference models, to qualitatively explore the possible interplay with the symmetry energy. We showed that the results in the two cases (TOV vs. LORENE) vary considerably, and underlined the fact that a full numerical formalism is inevitable for the structure calculation of strongly magnetized neutron stars.

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