The isotropy constant and boundary properties of convex bodies

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Abstract

Let $\mathcal{K}^n$ be the set of all convex bodies in $\mathbb{R}^n$ endowed with the Hausdorff distance. We prove that if $K \in \mathcal{K}^n$ has positive generalized Gauss curvature at some point of its boundary, then $K$ is not a local maximizer for the isotropy constant $L_K$.

1 Introduction and statement of the main result.

Let $K$ be a convex body in $\mathbb{R}^n$ endowed with its canonical scalar product and Euclidean norm denoted by $| \cdot |$. It is well known (as a standard reference to the subject we refer to [BGVV]; another, earlier, comprehensive reference is [MP]) that there exists a unique (up to orthogonal transformations) affine, volume preserving, mapping $A : \mathbb{R}^n \to \mathbb{R}^n$ such that for some constant $M_K > 0$, depending on $K$, one has for every $y \in \mathbb{R}^n$

$$\int_{AK} \langle x,y \rangle dx = 0 \quad \text{and} \quad \int_{AK} \langle x,y \rangle^2 dx = M_K^2 |y|^2.$$ 

We say that $K$ is in isotropic position (or that $K$ is isotropic) if $A$ is the identity on $\mathbb{R}^n$. The isotropy constant $L_K$ of $K$ is defined by

$$L_K = \frac{M_K}{|K|^{\frac{1}{2n}}}.$$ 

where $|B|$ denotes the volume of a Borel subset $B$ of $\mathbb{R}^n$. Note that it is customary to assume, as part of the definition of isotropic position, that $|AK| = 1$; for the sake of convenience in our proofs, we prefer not to include this assumption in the definition.

The famous Slicing Problem asks whether there exists a universal constant $C > 0$ such that, for any $n$, any convex body $K$ in $\mathbb{R}^n$ has a hyperplane section $K \cap H$ such that

$$\text{vol}_{n-1}(K \cap H) \geq C \text{vol}_n(K)^{\frac{n-1}{n}}.$$ 

This problem is equivalent to the existence of an upper bound $D > 0$ for $L_K$, independent of the dimension. J. Bourgain proved in [B] that $L_K \leq Cn^{1/4}\log(n)$, this bound was improved by B. Klartag in [K] to $L_K \leq Cn^{1/4}$, where $C$ is an absolute constant. Note that

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the minimum of $L_K$ is obtained only for ellipsoids (for an interesting discussion of stability in that inequality, see [AB]).

Since the exact upper bound for $L_K$ is still an open problem, it is interesting to investigate what are the properties of the maximizers for this quantity (a compactness argument shows that, for a fixed $n$, maximizers for $L_K$ exist among convex bodies in $\mathbb{R}^n$). We say that a convex body $K$ in $\mathbb{R}^n$ is a local maximizer (resp. local minimizer) for $L_K$ if for some $\varepsilon > 0$ one has $L_{K'} \leq L_K$ (resp. $L_{K'} \geq L_K$) for all convex bodies $K'$ in $\mathbb{R}^n$ such that $d(K', K) < \varepsilon$ ($d$ may denote here the Hausdorff or the Banach–Mazur distance). L. Rademacher proved in [R] that if a simplicial polytope is a maximizer for $L_K$, then it must be a simplex. Campi, Colesanti and Gronchi showed in [CCG], using shadow movements, that if $K$ has an open subset of its boundary which is $C^2$ with positive Gauss curvature, then $K$ can not be a (local) maximizer of $L_K$ in $\mathbb{R}^n$.

The main result of this paper is the following strong version of the result of [CCG]:

**Theorem 1.** If a convex body $K$ in $\mathbb{R}^n$ is a local maximizer for $L_K$, then it has no positive generalized Gauss curvature at any point of its boundary. The same is true for a centrally symmetric $K$ which is a local maximizer for $L_K$ among centrally symmetric convex bodies.

An open problem is whether a maximizer for $L_K$ is necessarily a polytope. Our result is a step in this direction, because it shows that a maximizer has generalized Gauss curvature equal to 0 almost everywhere and never positive on its boundary. To prove theorem 1 we shall suppose that a convex body $K$ has a positive generalized curvature at some point $X_0$ of its boundary (see Definition 2 below), modify slightly $K$ in a neighborhood of $X_0$, from inside and from outside to get a body $K'$ for which we shall estimate $L_{K'}$. The paper is organized as follows. In section 2, after presenting some notations, we study the effect of such modifications, that are described in the general case in Lemmas 1 and 3 and in the neighborhood of some special points of the boundary of $K$ in Proposition 4 and Lemma 5. Corollary 6 is a generalization of [CCG]'s result, replacing positive curvature by strict convexity on an open subset of the boundary. To estimate carefully the asymptotic behavior of $L_{K'}$, we prove the geometric Lemma 7 and we get in Lemma 8 a special property of potential maximizers of $L_K$. Finally section 3 is devoted to the proof of theorem 1 which needs some technical and very precise computations of volumes.

In connection to Theorem 1 one should mention the paper [RSW], by Reisner, Schütt and Werner, where an analogous result is proved related to Mahler’s conjecture. Namely: a minimizer $K$ of the volume-product can not have a point of positive generalized Gauss curvature on its boundary (see also [GM]).

## 2 Notations and preliminary results.

Let $K$ be a convex body in $\mathbb{R}^n$. It is not hard to show, and is well known, that for any convex body $K$, denoting by $g(K)$ the centroid of $K$, one has

$$ M_{K}^{2n} = \frac{1}{n!} \int_{K-g(K)} \cdots \int_{K-g(K)} \left( \det(X_1, \ldots, X_n) \right)^2 dX_1 \cdots dX_n $$

$$ = \frac{1}{n!} \int_{K} \cdots \int_{K} \left( \det(X_1 - g(K), \ldots, X_n - g(K)) \right)^2 dX_1 \cdots dX_n. $$
Let $X_0 \in \partial K$. For $r > 0$, denote $B(X_0, r)$ the Euclidean ball of center $X_0$ and radius $r$.

**Definition 1.** We say that $K$ has positive generalized (Gauss) curvature at $X_0$, if there exists an inner normal $N$ of $K$ at $X_0$ and a positive definite quadratic form $q$ on $N^\perp = \{ x \in \mathbb{R}^n; \langle x, N \rangle = 0 \}$ such that for every $\varepsilon > 0$, there exists $a > 0$, such that whenever $Y \in N^\perp$ and $y \in \mathbb{R}$ satisfy

$$ X_0 + Y + yN \in \partial K \cap B(X_0, a), $$

then

$$ (1 - \varepsilon)q(Y) \leq y \leq (1 + \varepsilon)q(Y). $$

Of course, this normal $N$ and the quadratic form $q$ are then unique. Observe that if $K$ is $C^2$ with positive curvature, then $K$ has positive generalized curvature at any point $X$ of its boundary, but that positive generalized curvature at some point $X_0$ does not imply any regularity at any point of $\partial K$ other than $X_0$. We refer to [SW] for more details on positive generalized curvature.

The following two lemmas show the effect of local slight modifications of an isotropic body $K$ on $\int_K \ldots \int_K (\det(X_1, \ldots, X_n))^2 dX_1 \ldots dX_n$.

**Lemma 1.** Let $K$ be an isotropic convex body. Suppose that $C_m, m \geq 1$ is a sequence of Borel subsets of $\mathbb{R}^n$ such that $C_m \cap \text{int}(K) = \emptyset$, $|C_m| > 0$, $|C_m| \to 0$ and $K_m := K \cup C_m$ is a convex body. Then, when $m \to +\infty$,

$$ \frac{1}{n!} \int_{K_m} \ldots \int_{K_m} (\det(X_1, \ldots, X_n))^2 dX_1 \ldots dX_n = M_{2n}^2 + M_{2(n-1)}^2 \int_{C_m} |X|^2 dX + O(|K_m \setminus K|^2). $$

**Lemma 2.** Let $K$ be an isotropic convex body. Suppose that $D_m, m \geq 1$ is a sequence of Borel subsets of $\mathbb{R}^n$ such that $D_m \subset K$, $|D_m| > 0$, $|D_m| \to 0$ and $K'_m := K \setminus D_m$ is a convex body. Then, when $m \to +\infty$,

$$ \frac{1}{n!} \int_{K'_m} \ldots \int_{K'_m} (\det(X_1, \ldots, X_n))^2 dX_1 \ldots dX_n = M_{2n}^2 - M_{2(n-1)}^2 \int_{D_m} |X|^2 dX + O(|K \setminus K'_m|^2). $$

**Proof of Lemma 1 and Lemma 2:**

One has

$$ \frac{1}{n!} \int_{K_m} \ldots \int_{K_m} (\det(X_1, \ldots, X_n))^2 dX_1 \ldots dX_n $$

$$ = \frac{1}{n!} \left( \int_K \ldots \int_K (\det(X_1, \ldots, X_n))^2 dX_1 \ldots dX_n \right) $$

$$ + n \int_{C_m} \int_K \ldots \int_K (\det(X_1, \ldots, X_n))^2 dX_1 \ldots dX_n + O(|K_m \setminus K|^2) \right). $$

Now

$$ \int_{C_m} \int_K \ldots \int_K (\det(X_1, \ldots, X_n))^2 dX_1 \ldots dX_n $$
\[
= \int_{C_m} \int_{K} \cdots \int_{K} \left( \sum_{\sigma \in S_n} \sum_{\tau \in S_n} (-1)^{\varepsilon(\sigma)\varepsilon(\tau)} \prod_{i=1}^{n} X_{\sigma(i)}X_{\tau(i)} \right) dX_1 \ldots dX_n.
\]

Since \( K \) is isotropic one has
\[
\int_{K} X_{\sigma(i)}X_{\tau(i)} dX_i = 0 \text{ if } \sigma(i) \neq \tau(i).
\]

It follows that
\[
\int_{C_m} \int_{K} \cdots \int_{K} \left( \det(X_1, \ldots, X_n) \right)^2 dX_1 \ldots dX_n = (n-1)! \ M_{K}^{2(n-1)} \sum_{m=1}^{n} X_{1m}^2 dX_1
\]
\[
= (n-1)! \ M_{K}^{2(n-1)} \int_{C_m} \sum_{m=1}^{n} \left| X \right|^2 dX.
\]

We can thus conclude. The proof of lemma 2 is analogous. \( \square \)

In the next lemma, we investigate, under the hypotheses of lemmas 1 and 2 how \( M_{K_m} \) differ from \( M_K \).

**Lemma 3.** Under the hypotheses of Lemma 1 or respectively of Lemma 2, one has
\[
M_{K_m}^{2n} = \frac{1}{n!} \int_{K_m} \cdots \int_{K_m} \left( \det(X_1, \ldots, X_n) \right)^2 dX_1 \ldots dX_n + O(\left| K_m \setminus K \right|^2)
\]

or respectively,
\[
M_{K_m}^{2n} = \frac{1}{n!} \int_{K_m'} \cdots \int_{K_m'} \left( \det(X_1, \ldots, X_n) \right)^2 dX_1 \ldots dX_n + O(\left| K \setminus K_m' \right|^2).
\]

**Proof:** We assume throughout the proof that \( K \) is isotropic but, a posteriori, the equalities stated in the lemma remain true under invertible linear transformations.

Let \( g_m \) be the centroid of \( K_m \). One has:
\[
M_{K_m}^{2n} = \frac{1}{n!} \int_{K_m - g_m} \cdots \int_{K_m - g_m} \left( \det(X_1, \ldots, X_n) \right)^2 dX_1 \ldots dX_n.
\]

Since the centroid of \( K \) is at 0. One has for every \( u \in S^{n-1}, \)
\[
\langle g_m, u \rangle = \frac{1}{|K| + |C_m|} \int_{K} \langle X, u \rangle dX + \int_{C_m} \langle X, u \rangle dX = \frac{1}{|K| + |C_m|} \int_{C_m} \langle X, u \rangle dX,
\]
and thus \( |g_m| = O(|C_m|) \) (observe that the hypotheses imply that the \( C_m, m \geq 1, \) are uniformly bounded).
We have
\[ n! M_{K_m}^{2n} = \int_{K_m} \cdots \int_{K_m} (\det(Y_1 - g_m, \ldots, Y_n - g_m))^2 dY_1 \cdots dY_n \]
\[ = \int_{K_m} \cdots \int_{K_m} (\det(Y_1, \ldots, Y_m) - \sum_{k=1}^{n} \det(Y_1, \ldots, Y_{k-1}, g_m, Y_{k+1}, \ldots, Y_n))^2 dY_1 \cdots dY_n \]
\[ = A - B + C. \]

where
\[ A := \int_{K_m} \cdots \int_{K_m} (\det(Y_1, \ldots, Y_n))^2 dY_1 \cdots dY_n \]
\[ B := 2 \sum_{k=1}^{n} \int_{K_m} \cdots \int_{K_m} \det(Y_1, \ldots, Y_n) \det(Y_1, \ldots, Y_{k-1}, g_m, Y_{k+1}, \ldots, Y_n) dY_1 \cdots dY_n \]
\[ C := \int_{K_m} \cdots \int_{K_m} (\sum_{k=1}^{n} \det(Y_1, \ldots, Y_{k-1}, g_m, Y_{k+1}, \ldots, Y_n))^2 dY_1 \cdots dY_n \]

The term \( A \) has been treated already:
\[ \frac{A}{n!} = M_{K}^{2n} + M_{K}^{2(n-1)} \int_{C_{m}} |X|^2 dX + O(|C_m|^2). \]

Since \( |g_m| = O(|C_m|) \), it is clear that
\[ C = O(|C_m|^2). \]

For \( B \) we write
\[ \frac{B}{2} = D + E + O(|C_m|^2) \]

where
\[ D := \sum_{k=1}^{n} \int_{K} \cdots \int_{K} \det(Y_1, \ldots, Y_n) \det(Y_1, \ldots, Y_{k-1}, g_m, Y_{k+1}, \ldots, Y_n) dY_1 \cdots dY_n \]

and
\[ E := \sum_{k=1}^{n} \int_{K} \cdots \int_{C_{m}} \int_{K} \cdots \int_{K} \det(Y_1, \ldots, Y_n) \det(Y_1, \ldots, Y_{k-1}, g_m, Y_{k+1}, \ldots, Y_n) dY_1 \cdots dY_n. \]

It is easily seen that \( D = 0 \), because of the isotropicity of \( K \). Now, once again since \( g_m = O(|C_m|) \), one has \( E = O(|C_m|^2) \).

The corresponding result for \( K_m' \) is proved in the same way. □

**Proposition 4.** Under the assumptions of Lemma 1 or, respectively, Lemma 2 on \( K \) one has
\[ L_{K_m}^{2n} = L_{K}^{2n} \left[ 1 + \frac{\int_{K} |X|^2 dX}{M_{K}^2} - (n + 2) \frac{|K_{m} \setminus K|}{|K|} + O(|K_{m} \setminus K|^2) \right] \tag{2} \]

or, respectively,
\[ L_{K_m}^{2n} = L_{K}^{2n} \left[ 1 - \frac{\int_{K \setminus K_{m}'} |X|^2 dX}{M_{K}^2} + (n + 2) \frac{|K \setminus K_{m}'|}{|K|} + O(|K \setminus K_{m}'|^2) \right]. \tag{3} \]
Proof: By Lemma 1 and Lemma 3 we have
\[ L_{K_m}^{2n} = M_{K_m}^{2n} + M_{K_m}^{2(n-1)} \int_{K_m \setminus K} |X|^2 dX + O(|K_m \setminus K|^2). \]
From this (2) follows. The equality (3) is proved in a similar way.

Lemma 5. Suppose that \( K \) is an isotropic convex body and that, in addition to the conditions of Proposition 4, there exists \( X_0 \in \partial K \) such that \( X_0 \) is in the closure of \( C_m \) for all \( m \) and \( \text{diam}(C_m) \rightarrow 0 \) and also, \( X_0 \) is in the closure of \( D_m \) for all \( m \) and \( \text{diam}(D_m) \rightarrow 0 \).

Then, if \( K \) is a local maximizer or a local minimizer for \( L_K \), we have
\[ |X_0|^2 |K| = (n + 2) M_K^2. \]  
(4)

Proof: The conditions of the lemma imply that, when \( m \rightarrow +\infty \), one has:
\[ \int_{K_m \setminus K} |X|^2 dX \sim |X_0|^2 |K_m \setminus K| \]  
and
\[ \int_{K \setminus K_m} |X|^2 dX \sim |X_0|^2 |K \setminus K_m|. \]  
(5)
thus the result follows from Proposition 4.

Remarks
1) A common example of a point \( X_0 \) that satisfies the assumptions of Lemma 5 is the following: Let \( X_0 \in \partial K \). We say that \( \partial K \) is locally strictly convex at \( X_0 \) or that \( X_0 \) is a point of local strict convexity of \( \partial K \), if there exists no non-degenerate line segment \( I \subset \partial K \) such that \( X_0 \in I \) (even as an end-point). The following claim is easy to prove:
Claim. Let \( X_0 \) be a point of local strict convexity of \( \partial K \) and let \( N \in S^{n-1} \) be an outer normal of \( K \) at \( X_0 \). Then the sets
\[ C_m = \text{conv}(K \cup (X_0 + \frac{1}{m} N)) \setminus K \]
and
\[ D_m = \{ X \in K; \langle X, N \rangle \geq \langle X_0, N \rangle - \frac{1}{m} \} \]
satisfy the conditions of Lemma 5.

2) If \( X_0 \in \partial K \) is a point of positive generalized curvature of \( \partial K \) then it is a point of local strict convexity and thus satisfies the conditions of Lemma 5.

As a corollary of Lemma 5 and of [CCC] (or of our Theorem 1) we get the following strengthening of a result of [CCC]:

Corollary 6. Suppose that there exists an open neighborhood \( U \) in \( \partial K \) which is strictly convex (that is, every point in \( U \) is a point of local strict convexity). Then \( K \) is not a local maximizer for \( L_K \).

Proof: We may assume that \( K \) is isotropic. By Lemma 5 and the Claim following it, all the points in \( U \) have the same Euclidean norm. Thus \( U \) is an open neighborhood on a Euclidean sphere. The result of [CCC] or Theorem 1 now complete the proof. □

We shall later need the following geometric lemma.
Lemma 7. Suppose that $K$ is a convex body containing 0 in its interior and that $\partial K$ has positive generalized curvature at some point $X_0$. Assume that the normal vector of $K$ at $X_0$ is not parallel to the vector $X_0$. Then there exists $u \in S^{n-1}$ and $\alpha > 0$ such that if 

$$K(\alpha, u) = \{X \in K; \langle X, u \rangle \geq \alpha \},$$

then $K(\alpha, u)$ is a cap of $K$ with non-empty interior and 

$$\max_{X \in K(\alpha, u)} |X| < |X_0|.$$ 

Proof: After an affine change of variables in $\mathbb{R}^n$, transforming 0 into $X_0$, we may suppose that for $|Z| \leq a$, the boundary of $K$ is described by $z = g(Z)$ with $(Z, z) \in \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$, and 

$$(1 - \varepsilon)|Z|^2 \leq g(Z) \leq (1 + \varepsilon)|Z|^2.$$ 

This affine change of variables transforms $B(0, |X_0|)$ into an ellipsoid $E$ with $0 \in \partial E$, whose inner normal $N$ at 0 is not $e_n$. We may suppose that $N = \cos(\theta)e_1 + \sin(\theta)e_n$ for some angle $\theta \in [0, \frac{\pi}{2}]$. Also, since $E$ has positive curvature at 0, one can find some positive constants $b$ and $C$ such that 

$$B(0, b) \cap E \subset B(0, b) \cap E \quad (6)$$

where $P$ is the paraboloid defined by 

$$P = \{M := xe_1 + Y + ze_n; \langle OM, N \rangle \geq C(|OM|^2 - \langle OM, N \rangle^2)\}.$$ 

Let $0 < x_0 < a$. The hyperplane $H$ tangent to the upper paraboloid ($z = (1 + \varepsilon)|Z|^2$) at $M_0 = x_0e_1 + (1 + \varepsilon)|x_0|^2e_n$ has the equation 

$$z = (1 + \varepsilon)(2xx_0 - x_0^2),$$

where $M = xe_1 + Y + ze_n$ is a point in $\mathbb{R}^n$, with $Y \in \{e_1, e_n\}^\perp$. The zone $A$ between the hyperplane $H$ and the lower paraboloid ($z = (1 - \varepsilon)|Z|^2$) is described by 

$$A = \{M : xe_1 + Y + ze_n; (1 - \varepsilon)(x^2 + |Y|^2) \leq z \leq (1 + \varepsilon)(2xx_0 - x_0^2)\}.$$ 

Thus for $M \in A$, one has 

$$x^2 - 2\frac{1 + \varepsilon}{1 - \varepsilon}xx_0 + \frac{1 + \varepsilon}{1 - \varepsilon}x_0^2 \leq 0$$

which says that 

$$\left(x - \frac{1 + \varepsilon}{1 - \varepsilon}x_0\right)^2 \leq \frac{1 + \varepsilon}{1 - \varepsilon}\left(\frac{1 + \varepsilon}{1 - \varepsilon} - 1\right)x_0^2$$

or 

$$\left(\frac{1 + \varepsilon}{1 - \varepsilon} - \frac{\sqrt{2\varepsilon(1 + \varepsilon)}}{1 - \varepsilon}\right)x_0 \leq x \leq \left(\frac{1 + \varepsilon}{1 - \varepsilon} + \frac{\sqrt{2\varepsilon(1 + \varepsilon)}}{1 - \varepsilon}\right)x_0.$$ 

It follows that for $\varepsilon$ small enough one has for $M = xe_1 + Y + ze_n \in A$: $x < 2x_0$ and $x^2 + |Y|^2 \leq 3x_0^2$. Thus, for $x_0$ small enough, $A \cap \{xe_1 + Y + ze_n; z \geq g(x, Y)\}$ is a cap of $K$, passing through 0, with normal $N = \cos(\theta)e_1 + \sin(\theta)e_n$.

By (6), it is sufficient to show that for $x_0$ small enough, one has 

$$A \subset P \cap B(0, b).$$
First it is easy to choose \( x_0 \) small enough such that \( A \subset B(0, b) \). Observe then that
\[
P = \{ xe_1 + Y + ze_n; x \cos(\theta) + z \sin(\theta) \geq C(x^2 + |Y|^2 - (x \cos(\theta) + z \sin(\theta))^2) \}
\]
and that setting \( x = x_0 u, Y = x_0 V \) and \( z = x_0^2 w \), one gets
\[
A = \{ x_0(u + V + x_0 w); (1 - \varepsilon)(u^2 + |V|^2) \leq w \leq (1 + \varepsilon)(2u - 1) \}
\]

Thus we need only to prove that if \( (1 - \varepsilon)(u^2 + |V|^2) \leq w \leq (1 + \varepsilon)(2u - 1) \) then
\[
\frac{u \cos(\theta) + x_0 w \sin(\theta)}{|V|^2 + (u \sin(\theta) + x_0 w \cos(\theta))^2} \geq Cx_0.
\]
which is clear when \( x_0 \to 0 \) because \( u \sim 1 \) and \( w \) is uniformly bounded.

Observe finally that if we have the singular case that the point of tangency \( M = x_0 e_1 + (1 + \varepsilon)|x_0|^2 \) of the upper paraboloid with the tangent hyperplane \( H \) is on \( \partial K \), then we get a cap of \( K \) by pushing \( H \) a small distance into the upper paraboloid in the direction of its inner normal. \( \square \)

**Lemma 8.** Under the assumptions of Lemma 5, if \( K \) is a local maximizer for \( L_K \) and \( \partial K \) has positive generalized curvature at \( X_0 \) then the outer normal \( N(K, X_0) \) of \( K \) at \( X_0 \) is parallel to the vector \( X_0 \).

**Proof:** We assume that \( L_K \) is maximal, \( \partial K \) has positive generalized curvature at \( X_0 \) and the normal vector of \( K \) at \( X_0 \) is not parallel to \( X_0 \).

Using Lemma 7 we continue as follows: Let \( u \in S^{n-1} \) and \( \alpha > 0 \) be taken from Lemma 7. Let \( H = \{ X; \langle X, u \rangle = \alpha \} \) and \( H^+ = \{ X; \langle X, u \rangle \geq \alpha \} \). Let \( M = \max \{|X|; X \in H^+ \cap K \} \). Then \( M < |X_0| \). Let \( d \) be the distance from 0 to \( H \), \( h = h_K(u) - d \) and, for \( m \geq 1 \), let
\[
D'_m = \{ X \in K; h_K(u) - \frac{h}{m} \leq \langle X, u \rangle \leq h_K(u) \}.
\]
Then the sequence \( D'_m \) satisfies the conditions of Lemma 2. We have
\[
\int_{D'_m} |X|^2 \leq M^2 |D'_m|.
\]
Now, since \( L_K \) is maximal, we have, combining the above with 3, for \( m \) big enough,
\[
-M^2 |D'_m| + (n + 2) \frac{|D'_m|}{|K|} \leq O(|D'_m|^2).
\]
Combining the last inequality with 4 we get, passing to the limit as \( m \to \infty \),
\[
|X_0|^2 = \frac{(n + 2)M^2}{|K|} \leq 2 M^2 < |X_0|^2,
\]
which is a contradiction. \( \square \)
3 Proof of Theorem 1

Assume that $K$ is a local maximizer of $L_K$ and $X_0 \in \partial K$ is a point of positive generalized curvature of $\partial K$. We may assume that $K$ is in isotropic position.

By Lemma 8 we know that $u = \frac{X_0}{|X_0|}$ is the external normal of $K$ at $X_0$. We choose for $K_m$ and $K'_m$, $m \geq 1$, the following sets:

$$K_m = \text{conv}(X_0 + \frac{u}{m}, K)$$

and

$$K'_m = \{X \in K; \langle X, u \rangle \leq \langle X_0, u \rangle - \frac{1}{m}\}.$$

By Remark 2) following Lemma 5 the sets $K_m \setminus K$ and $K \setminus K'_m$ satisfy the conditions of Lemma 5 and, of course, of Proposition 4. In view of Lemma 5, it is essential to have an accurate estimation of

$$\int_{K_m \setminus K} |X|^2 dX - |X_0|^2 |K_m \setminus K| = \int_{K_m \setminus K} (|X|^2 - |X_0|^2) dX$$

and

$$\int_{K \setminus K'_m} |X|^2 dX - |X_0|^2 |K \setminus K'_m| = \int_{K \setminus K'_m} (|X|^2 - |X_0|^2) dX.$$

For having such estimation it would be convenient to assume that the standard approximating ellipsoid of $K$ at $X_0$ is a Euclidean ball rather than just an ellipsoid.

Let $u_1, \ldots, u_n$ be an orthonormal system in $\mathbb{R}^n$, with $u_n = \frac{X_0}{|X_0|}$ and such that $u_1, \ldots, u_{n-1}$ are the directions of the principal radii of the quadratic form $q$ associated with $X_0$ (see Definition 1). Let $T \in SL(n)$ be a volume preserving linear transformation of the form

$$T(\sum_{j=1}^n x_j u_j) = \sum_{j=1}^n \lambda_j x_j u_j; \quad \prod_{j=1}^n \lambda_j = 1$$

(we write in short $T(X) = \Lambda X$ and $T^{-1}(X) = \Lambda^{-1}X$ assuming $X$ is written using the basis $u_1, \ldots, u_n$). Choose $T$ so that the standard approximating ellipsoid of $\tilde{K} = T(K)$ at $T(X_0)$ is a Euclidean ball of radius $R$.

Denoting $\tilde{K}_m = T(K_m)$ and $\tilde{K}'_m = T(K'_m)$ we get

$$\int_{\tilde{K}_m \setminus \tilde{K}} (|X|^2 - |X_0|^2) dX = \int_{\tilde{K}_m \setminus \tilde{K}} (|\Lambda^{-1}Y|^2 - |\Lambda^{-1}Y_0|^2) dY$$

and

$$\int_{\tilde{K} \setminus \tilde{K}'_m} (|X|^2 - |X_0|^2) dX = \int_{\tilde{K} \setminus \tilde{K}'_m} (|\Lambda^{-1}Y|^2 - |\Lambda^{-1}Y_0|^2) dY.$$

We shall use a temporary coordinate system that satisfies:

1) $T(X_0) = 0$
2) The outer normal vector of $\tilde{K}$ at 0 is $-e_n$ ($e_n$ is the $n$-th coordinate vector), thus $\tilde{K} \subset \{ X \in \mathbb{R}^n ; \langle X, e_n \rangle \geq 0 \}$

We write $X = (Y, y) \in \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$. Let $G = g(\tilde{K})$ be the centroid of $\tilde{K}$. In our temporary coordinates $G = (0, b)$ with $b > 0$ (in view of Lemma 8). For $a > 0$, small enough, define

$$C_a = \text{conv}(\tilde{K}, (-a, 0)) \setminus \tilde{K}$$

$$D_a = \{(Y, y) \in \tilde{K}; y \leq a\}.$$  

By the above discussion, we have to estimate for $\tilde{K}_m \setminus \tilde{K} = C_a$ and $\tilde{K} \setminus \tilde{K}_m' = D_a$ ($a = \frac{1}{m}$), the following quantities in terms of $a > 0, a \to 0$:

$$\phi(a) = \int_{C_a} (|\Lambda^{-1}(X-G)|^2 - |\Lambda^{-1}G|^2) dX$$

$$\psi(a) = \int_{D_a} (|\Lambda^{-1}(X-G)|^2 - |\Lambda^{-1}G|^2) dX.$$  

The equation of the boundary of the body, in a neighborhood of 0 can be written as

$$y = \frac{|Y|^2}{2R} + o(|Y|^2),$$

With these notations

$$\phi(a) = \int_{(Y,y) \in C_a} \left( \sum_{j=1}^{n-1} \left( \frac{Y_j}{\lambda_j} \right)^2 + \left( \frac{y}{\lambda_n} \right)^2 - 2 \frac{yb}{\lambda_n^2} \right) dYdy,$$

$$\psi(a) = \int_{(Y,y) \in D_a} \left( \sum_{j=1}^{n-1} \left( \frac{Y_j}{\lambda_j} \right)^2 + \left( \frac{y}{\lambda_n} \right)^2 - 2 \frac{yb}{\lambda_n^2} \right) dYdy.$$  

We first estimate $\phi(a)$ and $\psi(a)$ under the hypothesis that in some neighborhood of 0 the equation of the boundary of $K$ is actually

$$y = \frac{|Y|^2}{2R}.$$  

Then we shall see that this approximation is actually good.

1) We suppose that $y = \frac{|Y|^2}{2R}$. One has

$$D_a = \{(Y,y) \in \mathbb{R}^n; |Y| \leq \sqrt{2Ra}, \frac{|Y|^2}{2R} \leq y \leq a\}.$$  

Since $D_a$ is circular with respect to $Y$, we have

$$\int_{(Y,y) \in D_a} Y_j^2 dY dy = \frac{1}{n-1} \int_{(Y,y) \in D_a} |Y|^2 dY dy.$$
Substituting \( \alpha_n = \frac{1}{n+1} \sum_{j=1}^{n-1} \lambda_j^{-1} \) we get with a change of variable to polar coordinates in \( \mathbb{R}^{n-1} \) and denoting by \( v_k \) the volume of the Euclidean ball in \( \mathbb{R}^k \),

\[
\psi(a) = (n-1)v_{n-1} \int_{S_{n-2}} \int_0^{2\pi a} \left( \int_0^a (\alpha_n r^2 + \lambda_n^{-1}(y^2 - 2y)) dy \right) r^{n-2} dr d\theta.
\]

Setting \( r = \sqrt{2Ra} \) and \( y = az \) we get

\[
\psi(a) = (n-1)v_{n-1}(2Ra)^{\frac{n-1}{2}} a^{\frac{n+1}{2}} \int_{S_{n-2}} \int_0^1 \left( \int_0^1 (2\alpha_n Ras^2 + \lambda_n^{-1}(a^2 z^2 - 2abz)) ds \right) s^{n-2} ds d\theta
\]

\[
= (n-1)v_{n-1}(2R)^{\frac{n-1}{2}} a^{\frac{n+1}{2}} \int_0^1 \left( 2\alpha_n Ras^2 + \lambda_n^{-1}(a^2 s^2 - 2abz) \right) ds
\]

\[
= (n-1)v_{n-1}(2R)^{\frac{n-1}{2}} a^{\frac{n+1}{2}} \int_0^1 \left( 2\alpha_n Ras^2 + \lambda_n^{-1}(a^2 s^2 - 2abz) \right) ds
\]

\[
= 4(n-1)v_{n-1}(2R)^{\frac{n-1}{2}} a^{\frac{n+1}{2}} \left( \frac{\alpha_n R}{n+1} - \frac{\lambda_n^{-1} b}{n+1} \right) + O(a)
\]

\[
= 4(n-1)v_{n-1}(2R)^{\frac{n-1}{2}} a^{\frac{n+1}{2}} \left( \frac{\alpha_n R}{n+1} - \frac{\lambda_n^{-1} b}{n+1} \right) + O(a)
\]

\[
\frac{\alpha_n R}{n+1} - \frac{\lambda_n^{-1} b}{n+1} + O(a).
\]

We shall need also to compute \( |D_a| \). One has

\[
|D_a| = (n-1)v_{n-1}a(2Ra)^{\frac{n-1}{2}} \int_{S_{n-2}} \int_0^1 (1 - s^2) s^{n-2} ds d\theta
\]

\[
= (n-1)v_{n-1}(2R)^{\frac{n-1}{2}} a^{\frac{n+1}{2}} \left( \frac{1}{n-1} - \frac{1}{n+1} \right) = \frac{2v_{n-1}}{n+1} (2R)^{\frac{n-1}{2}} a^{\frac{n+1}{2}}.
\]

2) We still suppose that the boundary of \( \tilde{K} \) in a neighborhood of 0 is given by \( y = \frac{|Y|^2}{2R} \). Then the tangent hyperplanes to \( \tilde{K} \) through \((0, -a)\), indexed by \( \theta \in S^{n-2} \), the direction of the projection of their point of tangency with \( \tilde{K} \), are given by the equations

\[
y = -a + \sqrt{\frac{2a}{R}} \langle \theta, Y \rangle.
\]

It follows that

\[
C_a = \left\{(Y, y) \in \mathbb{R}^n; \ |Y| \leq \sqrt{2Ra}, \ -a + \sqrt{\frac{2a}{R}} |Y| \leq y \leq \frac{|Y|^2}{2R} \right\}
\]

\[
= \{ (\sqrt{2Ra} Z, az) \in \mathbb{R}^n; \ |Z| \leq 1, \ 2|Z| - 1 \leq z \leq |Z|^2 \}.
\]

Thus, using the same rotation invariance as in (1),

\[
\phi(a) = (n-1)v_{n-1}a(2Ra)^{\frac{n-1}{2}} \int_{S_{n-2}} \int_0^1 \left( \int_{2s-1}^{s^2} (2\alpha_n Ras^2 + \lambda_n^{-1}(a^2 z^2 - 2abz)) s^{n-2} ds \right)
\]
Remark. The importance of Lemma \([8]\) comes in step \(3\) above. Here, if the normal vector of \(K\) at 0 were not parallel to the \(y\)-axis, we would get an extra error term of order that could be estimated only by \(a^{\frac{n+1}{2}}o(a)\). For our proof of Theorem \([11]\) to work we would need an estimate of order \(a^{\frac{n+1}{2}}o(a)\) for this term.
To conclude, using Proposition 4 and Lemma 5 (including (5) in its proof) and replacing $K_m \setminus K$ by $T^{-1}(C_a)$ and $K \setminus K'_m$ by $T^{-1}(D_a)$, the above computations show that for some functions $c(n, R)$ and $d(n, R)$ depending only of $n$ and $R$, 

$$L_{K_m}^{2n} = L_{C_m}^{2n} \left( 1 + c(n, R) a^\frac{n+1}{2} \left( \alpha_n R - \frac{(n+2)(n-3)}{n(n-1)} \lambda_n^{-1} b + O(a) \right) \right)$$

and 

$$L_{K'_m}^{2n} = L_{D_m}^{2n} \left( 1 - d(n, R) a^\frac{n+1}{2} \left( \alpha_n R - \frac{n+1}{n-1} \lambda_n^{-1} b + O(a) \right) \right).$$

Thus one has both 

$$\alpha_n \lambda_n R \leq \frac{(n+2)(n-3)}{n(n-1)} b \quad \text{and} \quad \alpha_n \lambda_n R \geq \frac{n+1}{n-1} b,$$

So that 

$$\frac{(n+2)(n-3)}{n(n-1)} \geq \frac{n+1}{n-1}$$

which gives a contradiction.

Note that in the case that $K$ is centrally symmetric, a similar argument, using $C_m$ and $-C_m$ together and $D_m$ and $-D_m$ together will work in the same way, keeping $K_m$ and $K'_m$ centrally symmetric. This observation takes care of the centrally symmetric part of Theorem 1. There the use of lemma 3 is not needed, due to symmetry. □

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