Visualization of chaotic attractors in 3D as motivating tool for introductory physics course

P Nagy¹ and P Tasnádi²

¹ Faculty of Mechanical Engineering and Automation of Kecskemét College, Izsáki u. 10. Kecskemét, 6000 Hungary
² Faculty of Science of Eötvös Loránd University, Pázmány Péter Sétány 1/A, Budapest, 1117, Hungary

ttasipeter@gmail.com

Abstract. It is well known that simple nonlinear systems can produce inherently unpredictable behaviour, which is called chaotic motion. Interest in the theory behind it has risen rapidly and much effort has been invested in integrating it into graduate and undergraduate curricula. However, the concepts and ideas that are appropriate for the description of these motions are very abstract and intricate. Human beings are visual creatures in nature; therefore all aids and procedures which give a visible demonstration of an abstract idea are of great value from a didactic point of view. Our paper offers the teachers a method for the demonstration of strange attractors which play a central role in the theory of chaotic systems. The images of the 2D projection of the strange attractors are well known and common, however their 3D visualisation gives a real sensation for the human brain so they are much more suitable for capturing the students’ interest. With the help of our downloadable MAPLE program the DXF files of strange attractors can be easily generated and their images can be engraved in glass. E-materials belonging to the method can be downloaded from our webpage.

1. Introduction

Ever since Edward Lorenz discovered that simple nonlinear systems can produce inherently unpredictable behaviour, which is called chaotic motion, the interest in the theory behind it has risen rapidly. Much effort has been invested in integrating it into graduate and undergraduate curricula. Excellent introductory monographs are available that explain the basic ideas and concepts [1] and deploy a wide variety of simple mechanical systems producing chaotic behaviour [2–4]. The chaotic properties of simple mechanical systems have been investigated from an educational point of view by the authors previously [5–6].

In this paper an attractive visualization of the chaotic motion is presented. An account is also given of the use of this method in teaching the basics of chaotic systems to engineering students. Visualization is focused on the demonstration of the marvellous strange attractors which play a central role in the theory of chaotic systems. E-materials belonging to the method can be downloaded from our web page [7]. The education of engineering students is based on classical physics (mechanics, electricity and thermodynamics); therefore, topics of modern physics can be discussed in a relatively short time.
Computers have opened up a new dimension for physics experimentation. A completely new method, computerized experimental physics (numerical simulations) has been developed for the quantitative investigation of those systems that previously could be studied only qualitatively. One of the most important and best-known fields of the numerical simulations is the study of chaotic systems. The investigation of chaotic systems itself is very interesting but the subject could be an excellent didactic tool in raising the interest of the students and motivate them to learn modern physics. The study of relatively simple chaotic systems can provide a deep insight into the deterministic and probabilistic behaviour of the natural processes already at introductory physics courses. However, the concepts and ideas which are proper for the description of these motions are very abstract and intricate. Therefore their demonstration or, if possible, their visualization, can be a very nice didactic tool for the capture of the students’ interest. The teaching of chaotic systems even at introductory level needs the knowledge of some basic ideas, such as phase space, chaotic motion, trajectories, Poincaré map, attractors and so on; moreover students should be familiar at introductory level with systems of differential equations. Taking into account that this paper does not serve as a systematic introduction of the chaos theory we summarise these concepts in Appendix A. Our main goal is to offer a smooth, eye opening introduction to chaos theory. Therefore we use programs which do not need advanced programming skills and we also strive to limit the use of calculus.

2. Visualizations of 3D attractors

Deterministic systems can exhibit chaotic behaviour if they are described with at least three nonlinear differential equations [1]; consequently, their phase space should be at least also three dimensional. Since it is not easy to follow their trajectories in multidimensional phase space the trajectories are usually projected to a two dimensional phase plane (typically on the display of a computer). The most important possibility for displaying the motion on the trajectory is the Poincaré map (or Poincaré section), where the motion in phase space is represented only by the points of intersections of the trajectory with a carefully chosen two dimensional surface (almost every trajectory should intersect this surface). The intersections do not form a continuous curve but a series of points (with fractal geometry).

Why is it important to study the geometry of attractors? One of the fundamental statements of chaos theory is the following: there is a relationship between the properties and behaviour of a dynamical system and the topology of the attractor associated with that dynamical system (Figure 1) [1]. Visualization of the attractors helps us to recognise these connections.

![Figure 1. Connection between the type of dynamics, and type of chaos and fractal geometry [1.](image)](image)
Human beings are visual creatures in nature therefore every aid and procedure that is a visible demonstration of an abstract idea is of great value from a didactic point of view. The images of the 2D projection of the strange attractors are very well known and common; however, their 3D visualizations give a real sensation for the human brain, so they are much more suitable for capturing the students’ interest.

Glass engraving, also called laser engraving, is a method for the preparation and presentation of 3D images materialized inside of a transparent solid such as glass or crystals which were made about real objects [8]. To generate a point in the glass, the laser beam is controlled by a scanner with two mirrors which are moved by galvanometers along the x- and y-axes. The beam is focused by a lens onto the chosen points of the glass body. At the focus the spatial and temporal energy density of the laser beam so high that the glass in this point thermally destroyed by ionization and formation of plasma (cracking, melting and evaporation). The resulting small dot (a few microns in diameter) is visible in daylight as a white dot due to light refraction and scattering. In the past 15-20 years, use of subsurface laser engraving (SSLE) has become more cost effective to produce 3D images in souvenir ‘crystals’. A number of companies offer custom made souvenirs by taking 3D pictures or photos and engraving them into the crystal. For example since the early 2000s, artist Bathsheba Grossman has been using 3D printing and subsurface laser engraving to produce 3D physical visualizations of data from astronomy, biology, math and physics [9].

![Figure 2. A laser engraving machine and DNA model in glass](image)

The laser engraving machines accept .stl, .dxf, .obj, .3ds, and 3dmax 3dfile formats. We have used the DXF (Drawing Exchange Format) file, because it has a basic standard organization of graphic data in simply ASCII format [10]. The dataset should be given in a DXF file containing a point cloud the points of which are laying on a line in order of the steps of the solution of the equations which describe the dynamics of the system. We have written a MAPLE program to generate DXF files of 3D attractors. Using these DXF files anybody who has laser engraving equipment is able to produce the 3D visualisation of the attractor we have selected.

Attractors stored in DXF files can be displayed on the monitor of a computer (the images can be rotated, resized and changed with many other way) by running the DWG TrueView (freely downloadable from [11]) program. (All sample programs necessary for the visualization and investigation of the attractors can be downloaded in the form of a ZIP file from our webpage [7].) A brief description of the MAPLE program can be found in the Appendix B.

For example on the webpage [12] the detailed description of many spectacular 3D strange attractors with fractal geometry can be found. With the help of our downloadable MAPLE program the DXF files of these attractors can be easily generated and their images can be displayed. As an example Fig. 3 shows the DXF vector-graphic image of the famous Lorenz attractor. Its DXF file is generated by our
program (attractor_3d_lorenz_100-150_5000.dxf) and it is displayed by the DWG TrueView program. (Remark: on our webpage [7] can be seen two 3D animation made on the basis the generated vector-graphic files.)

Figure 3. 3D vector-graphic image of Lorenz attractor (fractal dimension 2.063) in DWG-TrueView.

3. Examples
In the following four examples of the use of the proposed visualization method is shown. Namely, the well-known Lorentz model, the so-called Duffing oscillator, the forced Zeeman oscillator elaborated earlier by us, and the Halvorsen model are presented. On our web page a downloadable e-material.zip file can be found for every model which contains the Maple file which can be parametrized and run. Moreover in this e-material.zip file one can find an example for the visualization of an attractor with the necessary DXF datafile and also a video about its demonstration with DWG TrueView.

3.1. The Lorenz-model
The Lorenz equations (chapter 5.7 in [1]) describe a very simple model of atmospheric convection which was derived by Edward Lorenz [13]. It was the starting point of the research of the chaotic systems. The dynamical equations of the Lorentz model in the standard form are:

\[
\begin{align*}
\dot{x} &= \sigma \left( y - x \right) \\
\dot{y} &= x \cdot (r - z) - y \\
\dot{z} &= x \cdot y - b \cdot z
\end{align*}
\]

where \( \sigma, r, b \) are positive parameters (for example in case of Figs. 3 and 5; \( \sigma=10, r=27 \) and \( b=2.6667 \)).

3.2. The Duffing oscillator
The Duffing oscillator is a periodically forced and damped anharmonic oscillator [14].
Figure 4. Sketch of a realization of Duffing oscillator.

The dynamic equations of Duffing oscillator are (in standard form, see in Appendix A (A.1)) the following:

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -a \cdot x - b \cdot x^3 - c \cdot v + \cos(z) \\
\dot{z} &= \frac{2\pi}{T}
\end{align*}
\]

where \(x\), \(y\) and \(z\) are the displacement, velocity and phase of the force with period \(T\), respectively.

3.3. The Zeeman Catastrophe Machine

The machine was created by C. Zeeman to illustrate and study catastrophic phenomena. The device is very simple (anyone can build it). A flat disc is fixed with an axle at a point \(O\) (0;0) of a rigid sheet and after this the midpoint of a rubber string is fastened to any point \(P\) of the circumference of the disk. One end of the slightly stretched string is fastened in point \(A\) (-\(a\);0) of the sheet, the other end \(B\) can be moved freely in the plane of the sheet (Fig.5).

The catastrophe theory studies the quasi-static properties of the Zeeman Machine and the sudden changes of its equilibrium state. The authors of this article have studied the dynamics of the motion can be created if a periodic driving force is applied at the \(B\) end of the string (forced Zeeman Machine) and found very typical chaotic behaviour.

Figure 5. The sketch of Zeeman machine and its realization.
The system of differential equations of the forced Zeeman Machine:

\[
\begin{align*}
\frac{d\Phi}{dt} &= f_1(\Phi, \omega, \Theta) = \omega \\
\frac{d\omega}{dt} &= f_2(\Phi, \omega, \Theta) = \\
&= c \left[ \frac{l_1 - l_0}{l_1} \cdot a \cdot \sin \Phi + \frac{l_2(\Theta) - l_0}{l_2(\Theta)} \cdot (y(\Theta) \cdot \cos \Phi - x(\Theta) \cdot \sin \Phi) \right] - \omega \\
\frac{d\Theta}{dt} &= f_3(\Phi, \omega, \Theta) = \frac{2\pi}{T_p}
\end{align*}
\]

where:

\[
\begin{align*}
l_1 &= \sqrt{(\cos \Phi + a)^2 + (\sin \Phi)^2} \\
l_2(\Theta) &= \sqrt{(x(\Theta) - \cos \Phi)^2 + (y(\Theta) - \sin \Phi)^2}
\end{align*}
\]

and \( c = \frac{I \cdot R^2 \cdot k}{\gamma^2} \),

where \( l_0 \) is the unstretched length of the rubber strings and \( k \) is their stiffness constant, \( I \) is the moment of inertia, \( R \) is the radius of the disc and \( \gamma \) is the damping coefficient.

3.4. The Halvorsen model

Our last example is the chaotic attractor of the Halvorsen model which is very spectacular and therefore it is a favourite of our students. Halvorsen’s model is a simple example of circulant systems [15]. The dynamic equations of the model are:

\[
\begin{align*}
\dot{x} &= -a \cdot x - 4y - 4z - y^2 \\
\dot{y} &= -a \cdot y - 4z - 4x - z^2 \\
\dot{z} &= -a \cdot z - 4x - 4y - x^2
\end{align*}
\]

where \( a \) is a positive parameter (for example in the version shown in Fig. 5, \( a=1.4 \)).

Figure 6 shows the laser engraved 3D image of the famous “butterfly attractor” of Lorenz, and the attractor taken from forced Zeeman machines, as well as the Halvorsen attractor (all were made by the authors and their students). In [16] and in our web page [7] the 3D picture of the Zeeman attractor which was engraved into a glass cube can be watched in a video film.

**Figure 6.** The Lorenz attractor, our Zeeman attractor and the Halvorsen attractor in glass.
4. Students’ project
The visualization method suggested above was applied in the special course “The view of modern physics” which was held for undergraduate students by one of the authors of this paper. This course deals partly with introductory chaos theory and suggests the following project for students on the basis of the procedure described above for studying chaotic systems of 3D phase space.

The short guidance of the project:
1. Choose a chaotic system of 3D phase space (a real system or from the webpage [12]).
2. Adapt the enclosed MAPLE program for the chosen model (very easy, rewrite few rows in the first execution group only, please see in the Appendix B).
3. Run the first execution group: make experiments with changing the parameters and ranges of variables, as well as resolution, observe 3D image of the attractor in MAPLE.
4. Generate the vector-graphic DXF file with the enclosed MAPLE program (only one row in the second execution group of the MAPLE program should be modified).
5. The 3D image of the attractor of the chosen system can be studied with the use of the DWG TrueView program.
6. If you are satisfied with the beauty of the attractor’s image the generated DXF file could be sent to engraving into glass.

The students were highly favorable towards the project suggested and enjoyed spending their time to experiment with attractors even if they did not want to make engraved glass cube. However, as a result of the course a lot of beautiful engraved glass cubes were made.

Finally we cite some students’ opinion about the visualization method of the attractors:
“I wish all physics courses would be like that!”
“I have enjoyed highly this work. It has given me real experience of the research and creation.”
“I have always liked physics, but I did not imagine that I would ever be able to prepare a very special, unique personal gift for my girlfriend in a physics lesson. I had a great success with it, thank you.”

At the end the simplest comment: “☺”.

APPENDIX A. Basic concepts of chaos theory
Chaos is a particular motion (more generally an evolution in time) of deterministic systems which is:

- irregular (it is not periodic and does not repeat itself),
- extremely sensitive for the initial conditions so it is unpredictable and only probabilistic statements can be done for its long time behaviour, and
- in the phase space its geometry exhibits fractal structure.

Phase space: A multidimensional abstract space of the state variables of a system \( \mathbf{x} = \{x_1, x_2, \ldots, x_n \} \). The number of the variables should be the minimum which is necessary to the unique description of the state of the system at a given instant.

Trajectory: the dynamical state of a system is represented by a point in the phase space. During the time evolution of the system this point is moving on a trajectory in the phase space. It can be easily shown that the trajectory of a deterministic system cannot be a self-intersecting one.

Dynamics gives the time evolution of a system. In continuous time the variation of the state of a system is given with a system of first order differential equations. It needs generally the introduction of velocity variables to make the equations be first order.
It can be shown that the necessary condition of the existence of chaotic motion is the following: if time is continuously changing, the equations of the motion of the investigated system should be nonlinear and should have at least a three dimensional phase space [1].

**Attractor:** the attractor of a dynamical system is a subset of the state space to which orbits originating from typical initial conditions tend as time increases. Attractors can be classified in three groups:

- **point attractor** means a stable state of the system,
- **periodic attractor** is a stable limit cycle, when it is reached, the system begins to oscillate and it behaves periodically,
- **strange attractor** is a special attractor characterizing the chaotic behaviour of a system. Trajectories coming from outside do not enter into the strange attractor they are only touching it.

**Fractals** are infinitely complex patterns which exhibit self-similarity across different scales. Plotting a characteristic variable of a fractal as a function of the scale a straight line can be obtained the slope of which generally a fraction. This number is called the fractal dimension.

**Visualization:** Attractors are subsets in an abstract multidimensional phase space and for their two dimensional visualization (e.g. on a display of a computer) we have basically two possibility:

(a) Their projection to a phase plane, or subspace (during this we lose information for example the projection of a trajectory can intersect itself).

(b) Making a Poincare map: Poincare maps shows only the intersections of the trajectory with a carefully chosen plane of the phase space. These points form a series of discrete points. The stroboscopic map used in the case of excited systems is also a special Poincare one. The points of the stroboscopic map are generated by periodic sampling from the trajectory. The period of the sampling agrees with the period of the excitation.

In case of permanent chaos the phase points of the system never leave the attractor. **Attractors are called chaotic or strange if they have a fractal structure in the Poincaré-map.**

**Appendix B. Using of our Maple program**

In Fig. 7 the core of the Maple program which is a part of the downloadable e-material.zip file can be seen. The program contains two execution groups, the first group is the experimenting part, and the second one is for producing the standard DXF files which are necessary to fabricate the 3D pictures. With the help of the first group the equations of motion of the chosen system can be solved and the solution can be studied through the change of the parameters. It makes also possible to display the 3D picture of the attractors on the screen of the monitor of a computer (clicking to the screen the picture can be rotated). If you get into favour with the display of an attractor then with running the second execution group the attractor can be saved into a standard DXF vector graphic file. On the basis of this file every 3D engraving or rapid prototyping device can fabricate the materialized version of the attractor. In the list of the program the rows the user should rewrite to obtain his (or her) own attractor

\[
\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})
\]

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2, ..., x_n) \\
\dot{x}_2 &= f_2(x_1, x_2, ..., x_n) \\
\vdots & \quad \vdots \\
\dot{x}_n &= f_n(x_1, x_2, ..., x_n)
\end{align*}
\]
are marked by red numbers. (The comments of blue typing give guidance for the necessary changes of
the rows.)

Figure 7. The key part of our MAPLE program.
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