A second note on the discrete Gaussian Free Field
with disordered pinning on $\mathbb{Z}^d$, $d \geq 2$

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February 7, 2014

Abstract

We study the discrete massless Gaussian Free Field on $\mathbb{Z}^d$, $d \geq 2$, in the presence of a disordered square-well potential supported on a finite strip around zero. The disorder is introduced by reward/penalty interaction coefficients, which are given by i.i.d. random variables.

In the previous note [4], we proved under minimal assumptions on the law of the environment, that the quenched free energy associated to this model exists in $\mathbb{R}^+$, is deterministic, and strictly smaller than the annealed free energy whenever the latter is strictly positive.

Here we consider Bernoulli reward/penalty coefficients $b \cdot e_x + h$ with $e_x \sim \text{Bernoulli}_{1/2}(-1, +1)$ for all $x \in \mathbb{Z}^d$, and $b > 0$, $h \in \mathbb{R}$. We prove that in the plane $(b, h)$, the quenched critical line (separating the phases of positive and zero free energy) lies strictly below the line $h = 0$, showing in particular that there exists a non trivial region where the field is localized though repulsed on average by the environment.

Keywords : Random interfaces, random surfaces, pinning, disordered systems, Gaussian free field.

MSC2010 : 60K35, 82B44, 82B41.

1 The model

We study the discrete Gaussian Free Field with a disordered square-well potential. For $\Lambda$ a finite subset of $\mathbb{Z}^d$, denoted by $\Lambda \subset \mathbb{Z}^d$, let $\varphi = (\varphi_x)_{x \in \Lambda}$ represent the heights over sites of $\Lambda$. The values of $\varphi_x$ can also be seen as continuous unbounded (spin) variables, we will refer to $\varphi$ as “the interface” or “the field”.

Let $\Omega = \mathbb{R}^{\mathbb{Z}^d}$ be the set of configurations. The finite volume Gibbs measure in $\Lambda$ for the discrete Gaussian Free Field with disordered square-well potential, and 0 boundary conditions, is the probability measure on $\Omega$ defined by :

$$
\mu^\varphi_{\Lambda,0}(d\varphi) = \frac{1}{Z^\varphi_{\Lambda,0}} \exp \left( -\beta \mathcal{H}_\Lambda(\varphi) + \beta \sum_{x \in \Lambda} (b \cdot e_x + h) 1_{\{\varphi_x \in [-a,a]\}} \right) \prod_{x \in \Lambda} d\varphi_x \prod_{y \in \Lambda^c} \delta_0(d\varphi_y). \tag{1}
$$

where $a, \beta, b > 0$, $h \in \mathbb{R}$ and $\mathcal{H}_\Lambda(\varphi)$ is given by

$$
\mathcal{H}_\Lambda(\varphi) = \frac{1}{4d} \sum_{\{x,y\} \cap \Lambda \neq \emptyset} (\varphi_x - \varphi_y)^2, \tag{2}
$$
where $x \sim y$ denotes an edge of the graph $\mathbb{Z}^d$ and $\mathbb{1}_{[A]}$ denotes the indicator function of $A$. An environment is denoted as $e := (e_x)_{x \in \Lambda}$. We consider here $e$ given by i.i.d. random variables $e_x \sim \text{Bernoulli}_{1/2}(-1, +1)$.

The parameter $b$ is usually called the “intensity of the disorder”, while $h$ is its average. The disordered potential attracts or repulses the field at heights belonging to $[-a,a]$. $Z^e_\Lambda$ is the partition function, i.e. it normalizes $\mu^e_\Lambda$ so it is a probability measure. The superscript $0$ reminds the boundary condition, it is added to the notation compared to $[4]$ because it will be useful below. We stress that our model contains two levels of randomness. The first one is $e$ which we refer to as “the environment”. The second one is the actual interface model whose low depends on the realization of $e$.

The questions we are addressing in this framework are the usual ones concerning statistical mechanics models in random environment: Is the quenched free energy non-random? Does it differ from the annealed one? Can we give a physical meaning to the strict positivity (resp. vanishing) of the free energy? What can be said concerning the quenched and annealed critical lines (surfaces) in the space of the relevant parameters of the system?

In the previous note $[4]$, we proved under minimal assumptions on the law of the environment, that the quenched free energy associated to this model exists in space of the relevant parameters of the system? for all $x \in \Lambda$. In other words $\mathbb{E}Z^e_\Lambda = Z^f_\Lambda$. 

In $[4]$ Fact 2.3 we proved that for any environment $e$ such that the annealed model exists, i.e. $\mathbb{E}(e^{b \cdot e_x + h}) < \infty$, both the quenched and annealed free energies are non-negative. This motivates the

\[ f^q_\Lambda(e) = |\Lambda|^{-1} \log \left( \frac{Z^e_\Lambda}{Z^{0,0}_\Lambda} \right), \quad f^a_\Lambda(e) = |\Lambda|^{-1} \log \left( \frac{\mathbb{E}Z^e_\Lambda}{Z^{0,0}_\Lambda} \right), \quad (3) \]

where $Z^{0,0}_\Lambda$ denotes the partition function of the model with no potential, $e_x \equiv 0$ (i.e. of the Gaussian free field). In the case when $\Lambda = \Lambda_n = \{0, \ldots, n-1\}^d$ we will use short forms $f^q_n(e)$ and $f^a_n(e)$. By the Jensen inequality, we have $f^q(e) \leq f^a(e)$. Moreover, it is not difficult to see that the annealed model corresponds to the model with constant (we will also say homogenous) pinning with the strength

\[ \ell(e) := \log(\mathbb{E}(e^{b \cdot e_x + h})). \quad (4) \]

for all $x \in \Lambda$. In other words $\mathbb{E}Z^e_\Lambda = Z^\ell(e)^{0,0}_\Lambda$.

In $[4]$ Fact 2.3] we proved that for any environment $e$ such that the annealed model exists, i.e. $\mathbb{E}(e^{b \cdot e_x + h}) < \infty$, both the quenched and annealed free energies are non-negative. This motivates the
following notions. We introduce the quenched (resp. annealed) critical lines, which are delimiting the
region where \( f^q(e) = 0 \) (resp. \( f^a(e) = 0 \)) from the region \( f^q(e) > 0 \) (resp. \( f^a(e) > 0 \)).

\[
h^q_c(b) := \sup\{h \in \mathbb{R} : f^q(e) = 0\} \quad \text{and} \quad h^a_c(b) := \sup\{h \in \mathbb{R} : f^a(e) = 0\}
\]

We are interested in describing the behavior of these quantities in the phase diagram described by
the plane \((b, h)\). Knowing the behavior of the homogenous model for positive pinning \([3]\), we easily
deduce that the annealed critical line is given by the equation \( f(e) = 0 \).

Note that \( f^q(e) \leq f^a(e) \) implies that \( h^q_c(b) \geq h^a_c(b) \). In Theorem 2.1, we show that the quenched
critical line lies strictly below the axis \( h = 0 \) for all \( d \geq 2 \). Our result shows in particular that there exists a non trivial region where \( h < 0, b > 0 \) and \( f^q(e) > 0 \), i.e. where the field is localized though it is repulsed on average by the environment.

Note that we don’t have any estimate on the behavior of \( h^q_c(b) - h^a_c(b) \).

**Theorem 2.1.** Let \( e \sim \otimes_{x \in \mathbb{Z}^d} \text{Bernoulli}_{1/2}(-1, +1) \). Then,

For \( d \geq 2 \), the quenched critical line is located in the quadrant \( \{(b, h) : b \geq 0, h < 0\} \).

More precisely, there exists some \( C, C' > 0 \) depending on \( d, a \) only and \( \epsilon \in (0,1) \) such that for any
environment \( e \) which fulfills \( b + h > 0, -\epsilon < -b + h < 0 \) and

\[
\begin{align*}
    h &> \frac{C'(-b+h)^2}{\log(b-h)} & \text{for } d = 2 \\
    h &> -C' \cdot (-b + h)^2 & \text{for } d \geq 3,
\end{align*}
\]

we have \( f^q(e) > 0 \).

**Remark 2.2.**

1. A sketch of these bounds in the plane \((b, h)\) can be seen on Figure 1. Moreover,
   the bound for \( d \geq 3 \) can be rewritten as \( h > -C'(d, a) \cdot b^2 \).

2. Jensen’s inequality gives us an upper bound on \( C, C' \). Indeed, as \( f^a(e) \geq f^q(e) \), if \( f^a(e) = 0 \)
   then \( f^q(e) = 0 \). In particular, we must have \( -C \leq \frac{\beta^2}{b^2} h|_{b=0} < 0 \). Our result gives thus an
   upper-bound on the behavior of the quenched critical line near \( b = 0 \).

**2.1 Related results**

The same type of result has been proven for \((1d)\) polymer models in great generality in \([\Pi]\), where
Alexander and Sidoravicius consider a polymer, with monomer locations modeled by the trajectory of
a Markov chain \( (X_i)_{i \in \mathbb{Z}} \), in the presence of a potential (usually called a “defect line”) that interacts
with the polymer when it visits \( 0 \). Formally, the model is given by weighting the realization of the
chain with the Boltzmann term

\[
\exp\left( \beta \sum_{i=1}^{n} (u + V_i) 1_{X_i = 0} \right).
\]

with \((V_i)_{i \in \mathbb{Z}} \) an i.i.d. sequence of 0-mean random variables. They studied the localization transition
in this model. We say that the polymer is pinned, if a positive fraction of monomers is at \( 0 \). In the
plane \((\beta, u)\) critical lines are defined as above: for \( \beta \) fixed, \( u^q(\beta) \) (resp. \( u^a(\beta) \)) is the value of \( u \) above
which the polymer is pinned with probability 1 (for the quenched (resp. annealed) measure). They
showed that the quenched free energy and critical point are non-random, calculated the critical point.
for a deterministic interaction (i.e. $V_i \equiv 0$) and proved that the critical point in the quenched case is strictly smaller.

When the underlying chain is a symmetric simple random walk on $\mathbb{Z}$, the deterministic critical point is $0$, so having the quenched critical point $u_c(\beta)$ strictly negative means that, even when the disorder is repulsive on average, the chain is pinned. This result was obtained by Galluccio and Graber in [7] for a periodic potential, which is frequently used in the physics literature as a “toy model” for random environments.

A much shorter proof with explicit bounds can be found in [8], in less generality, but [6] contains a revisited proof with explicit estimates and weakening the assumptions on the underlying model.

Note that for polymers, or discrete height interfaces, one need a coarse graining procedure to achieve the proof. In our case, as we will see in the next section, we can shift the continuous interface where the environment is unfavorable, and this has a small cost in dimension $d \geq 3$. The procedure is a bit more complicated in dimension 2 and we have to localize the field before by introducing a small mass.

2.2 Proof of Theorem 2.1

2.2.1 Case $d \geq 3$

We assume $-b + h < 0 < b + h$ i.e. the environment is repulsive if $e_x = -1$ while it is attractive if $e_x = +1$.

The idea is to tilt the measure such that the field $\varphi$ is shifted up of an amount $s$ on the sites $x$ for which $b \cdot e_x + h < 0$. In this way the shift of the field follows the environment. For some technical reasons, we need to work with the measure with boundary condition $a$, so perform two changes of measure (first one changing boundary condition and the second one to following the environment). Let $s > 0$ (to be fixed later).

$$f^a_{\Lambda_n}(e) = n^{-d} \log \mu^0_{\Lambda_n,e,s} \left( \frac{d\mu^0_{\Lambda_n}}{d\mu^0_{\Lambda_n,e,s}} \frac{d\mu^0_{\Lambda_n}}{d\mu^0_{\Lambda_n}} \exp \left( \sum_{x \in \Lambda_n} (b \cdot e_x + h) 1_{[\varphi_x \in [-a,a]]} \right) \right),$$

where $(\varphi_x)_{x \in \Lambda_n}$ under $\mu^0_{\Lambda_n,e,s}$ is distributed as $(\varphi_x + s 1_{[(b \cdot e_x + h) < 0]})_{x \in \Lambda_n}$ under $\mu^0_{\Lambda_n}$. More formally,
introducing $T_{e,s} : \{ (\varphi_x)_{x \in \Lambda_n} \} \mapsto (\varphi_x + s \mathbb{1}_{[b_{e_s}+h] < 0})_{x \in \Lambda_n}$, we define $\mu_{\Lambda_n,e,s}^{0,a}$ as $\mu_{\Lambda_n}^{0,a} \circ T_{e,s}^{-1}$.

Using Jensen’s inequality, we get

$$f_{\Lambda_n}^{\text{q}}(e) = n^{-d} \log \left[ \mu_{\Lambda_n,e,s}^{0,a} \exp \left( \sum_{x \in \Lambda_n} (b \cdot e_x + h) \mathbb{1}_{[\varphi_x \in [-a,a]]} + \log \frac{d\mu_{\Lambda_n}^{0,a}}{d\mu_{\Lambda_n,e,s}^{0,a}} + \log \frac{d\mu_{\Lambda_n}^{0,0}}{d\mu_{\Lambda_n}^{0,a}} \right) \right]$$

$$\geq n^{-d} \mu_{\Lambda_n,e,s}^{0,a} \left( \sum_{x \in \Lambda_n} (b \cdot e_x + h) \mathbb{1}_{[\varphi_x \in [-a,a]]} + \log \frac{d\mu_{\Lambda_n}^{0,a}}{d\mu_{\Lambda_n,e,s}^{0,a}} + \log \frac{d\mu_{\Lambda_n}^{0,0}}{d\mu_{\Lambda_n}^{0,a}} \right)$$

As $Z_{\Lambda_n,e,s}^{0,0} = Z_{\Lambda_n}^{0,0}$ (which follows by change of variables in the Gaussian integral), the first term can be written as

$$(1) = -\frac{1}{4d} \sum_{\{x,y\} \cap \Lambda_n \neq \emptyset} (\varphi_x - \varphi_y)^2 - (\hat{\varphi}_x - \hat{\varphi}_y)^2$$

where $\hat{\varphi}_x := \varphi_x + s \mathbb{1}_{[b_{e_s}+h] < 0}$. Hence, using the definition of $\mu_{\Lambda_n,e,s}^{0,a}$,

$$n^{-d} \mu_{\Lambda_n,e,s}^{0,a} ((1)) = -\frac{n^{-d}}{4d} \mu_{\Lambda_n}^{0,a} \left( \sum_{\{x,y\} \cap \Lambda_n \neq \emptyset} (\varphi_x - \varphi_y)^2 - (\varphi_x - \varphi_y)^2 \right)$$

$$= -\frac{s^2 n^{-d}}{4d} \sum_{\{x,y\} \cap \Lambda_n \neq \emptyset} (\mathbb{1}_{[b_{e_s}+h] < 0} - \mathbb{1}_{[b_{e_y}+h] < 0})^2$$

The second term contains only boundary contribution of order $n^{d-1}$. Indeed,

$$(2) = \left( 2a \sum_{x \in \partial \Lambda_n} \varphi_x - a^2 |\partial \Lambda_n| \right) + \log \left( \frac{Z_{\Lambda_n}^{0,a}}{Z_{\Lambda_n}^{0,0}} \right) \geq \left( 2a \sum_{x \in \partial \Lambda_n} \varphi_x - a^2 |\partial \Lambda_n| \right) - C n^{-1}$$

Hence,

$$n^{-d} \mu_{\Lambda_n,e,s}^{0,a} ((2)) \geq 2a \cdot n^{-d} \sum_{x \in \partial \Lambda_n} \mu_{\Lambda_n}^{0,a} (\varphi_x) - C n^{-1} \geq s \sum_{x \in \partial \Lambda_n} \mathbb{1}_{[b_{e_s}+h] < 0} - C n^{-1} \geq -C n^{-1}$$

We get

$$f_{\Lambda_n}^{\text{q}}(e) \geq n^{-d} \sum_{x \in \Lambda_n} (b \cdot e_x + h) \mu_{\Lambda_n}^{0,a} (\varphi_x \in [-a,a]) - \frac{s^2 n^{-d}}{4d} \sum_{\{x,y\} \cap \Lambda_n \neq \emptyset} (\mathbb{1}_{[b_{e_s}+h] < 0} - \mathbb{1}_{[b_{e_y}+h] < 0})^2 - C n^{-1}$$

Now we use the fact that the marginal laws of all $\varphi_x$, $x \in \Lambda_n$ under $\mu_{\Lambda_n}^{0,a}$ are Gaussian variables centered at $a$, i.e. $\varphi_x \sim \mathcal{N}(a, \sigma_n^2)$ where $\sigma_n^2 = \text{Var}_{\Lambda_n}^{0,a}(\varphi_x) \leq \text{Var}_{\Lambda_n}^{0,0}(\varphi_x) \leq c(d) < \infty$ for $d \geq 3$. Therefore,

$$\mu_{\Lambda_n}^{0,a}(\varphi_x \in [-a,a]) = \mu_{\Lambda_n}^{0,0}(\varphi_x \in [-2a,0]) + \mu_{\Lambda_n}^{0,0}(\varphi_x \in [-2a-s,-s])$$

$$= C \left( \int_{-s}^{0} - \int_{-2a-s}^{-2a} \right) e^{-y^2/2\sigma_n^2} dy \asymp s \text{ as } n \to \infty,$$
for \(c(d) \gg s\). In particular we will use that:

\[
\mu^{0,a}_{\Lambda_n} (\phi \in [-a, a]) - \mu^{0,a}_{\Lambda_n} (\phi + s \in [-a, a]) \geq C_1 (d, a) \cdot s,
\]

for some \(C_1 (d, a) > 0\).

\[
f_n^a (e) \geq n^{-d} \sum_{x \in \Lambda_n} (b \cdot e_x + h) \left( \mu^{0,a}_{\Lambda_n} (\phi \in [-a, a]) - C_1 (d, a) s \mathbf{1}_{|b \cdot e_x + h| < 0} \right)
\]

\[\quad - \frac{s^2 n^{-d}}{4d} \sum_{\{x, y\} \in \Lambda_n, \gamma \neq \emptyset} \left( \mathbf{1}_{|b \cdot e_x + h| < 0} - \mathbf{1}_{|b \cdot e_y + h| < 0} \right)^2 - Cn^{-1} \]

Observe that \(\mu^{0,a}_{\Lambda_n} (\phi \in [-a, a]) = \mu^{0,0}_{\Lambda_n} (\phi \in [-2a, 0]) \geq \mu_{\infty} (\phi \in [-2a, 0]) \geq C_2 (d, a)\) for some \(C_2 (d, a) > 0\).

By taking the expectation with respect to the environment, using the bounded convergence theorem and the fact that \(f_n^a (e) = E (f_n^a (e))\) (cf. [4, Theorem 2.1]) we get:

\[
f_n^a (e) = \lim_{n \to \infty} E f_n^a (e) \geq h C_2 (d, a) - \frac{sc_1 (d, a)}{2} (-b + h) - \frac{s^2}{16} \]

We may optimize over \(s\) as the left hand side does not depend on it. Doing this one checks that \(f_n^a (e) \gg 0\) as soon as

\[
h > - \frac{C_1 (d, a)}{C_2 (d, a)} (-b + h)^2 = -K (d, a) (-b + h)^2
\]

This gives the implicit equation in terms of the variance \(b^2\) of \(b \cdot e_x + h\):

\[
h > b - \frac{1}{2K} + \frac{1}{2} \left( \frac{1}{K^2} - \frac{8b}{K} \right) = -Kb^2 + O (b^3)
\]

The annealed critical curve as well as this bound are drawn on Figure [1]. We recall that [5] is valid under assumption that \(s\) is small. The maximum of [6] is realized at \(s_{\max} = -4C_1 \cdot (-b + h)\), thus it is enough to assume that \((-b + h)\) is small.

**2.2.2 Case \(d = 2\)**

In the case \(d = 2\), the variance of the Gaussian free field diverges with the size of the box, so we cannot use the previous estimates. To circumvent this problem we introduce the so-called massive free field. Let \(m > 0\),

\[
\mu^{0,\zeta}_{\Lambda_n, m} (d \phi) = \frac{1}{Z^{\Lambda_n, m}_{\Lambda_n, m}} \exp \left( -H \Lambda_n (\phi) - m^2 \sum_{x \in \Lambda_n} (\phi_x - \zeta)^2 \right) \prod_{x \in \Lambda_n} d \phi_x \prod_{x \in \partial \Lambda_n} \delta_0 (d \phi_x),
\]

where \(H \Lambda (\phi)\) is defined in [2]. Known facts about this model can be found in [5, Section 3.3]. In particular, the random walk representation for the massless GFF [3, (1.3)] is still true, but for a random walk \(Y_t\) that is killed with rate \(\xi (m) = \frac{m^2}{1 + m^2}\), namely at each time \(\ell\), if the walk has not already been killed, it is killed with probability \(\xi (m)\), where the killing is independent of the walk. We write its law \(P_x\) when it starts at \(x\).
Lemma 2.3. Let $d = 2$. Then,

1. There exists some $C_1 > 0$ such that for $n$ large enough, $m > 0$ small enough and all $x \in \Lambda_n$,
   \[ \mu_{\Lambda_n,m}^0(\varphi_x^2) \leq C_1 |\log(m)|. \]

2. There exists some $C_2 > 0$ such that for $n$ large enough and $m > 0$ small enough, we have
   \[ n^{-2} \log \frac{Z_{\Lambda_n,m}^0}{Z_{\Lambda_n}^0} \geq -C_2 m^2 |\log(m)|. \]

Proof. These bounds are rather standard. We give here the main steps of the proofs with some references. For the first claim, we use the random walk representation [5] to write
   \[ \mu_{\Lambda_n,m}^0(\varphi_x^2) = \sum_{\ell=0}^{\infty} P_x(Y_\ell = x, \tau_{\Lambda_n} \wedge \kappa > \ell) = \sum_{\ell=0}^{\infty} (1 - \xi(m))^\ell P_x(Y_\ell = x, \tau_{\Lambda_n} > \ell) \]
where $\tau_{\Lambda_n}$ is the first exit time of $\Lambda_n$ and $\kappa$ is the killing time of the random walk $Y_\ell$. Hence,
   \[ \mu_{\Lambda_n,m}^0(\varphi_x^2) \leq \mu_{\Lambda_n,m}^0(\varphi_0^2) = C_1 |\log(m)| (8) \]
where $X_\ell$ is a simple random walk (without killing). The projections of $X_\ell$ onto the two coordinate axis are two independent 1-dimensional random walks $X_\ell^1$ and $X_\ell^2$, then by Stirling formula,
   \[ P_0(X_{2\ell} = 0) = (P_0(X_{2\ell}^1 = 0))^2 = \left( \frac{2\ell}{\ell} \right) \left( \frac{2^{-2\ell}}{\ell} \right)^2 = \frac{1}{\ell} (1 + o(1)) \text{ as } \ell \to \infty \]
The asymptotics of (8) for small $m$ gives the desired upper-bound.
To prove the second claim we use the representation of the partition function described in [2, p.542] (it applies to the massive GFF with an obvious modification). We denote by $\tilde{P}$ the coupling of a random walk $X_n$ and a killed random walk $Y_n$ such that $Y_n = X_n$ up to its killing time $\kappa$,
   \[ |\Lambda_n|^{-1} \log \frac{Z_{\Lambda_n}^0}{Z_{\Lambda_n,m}^0} = |\Lambda_n|^{-1} \left( \frac{1}{2} \sum_{x \in \Lambda_n} \sum_{\ell=1}^{\infty} \frac{1}{2\ell} \left( \tilde{P}_x(X_{2\ell} = x, \tau_{\Lambda_n} > 2\ell) - \tilde{P}_x(Y_{2\ell} = x, \tau_{\Lambda_n} \wedge \kappa > 2\ell) \right) \right) \]
   \[ \leq \frac{1}{2} \sum_{\ell=1}^{\infty} \frac{1}{2\ell} \left( \tilde{P}_0(X_{2\ell} = 0, \tau_{\Lambda_n} > 2\ell) - \tilde{P}_0(Y_{2\ell} = 0, \tau_{\Lambda_n} \wedge \kappa > 2\ell) \right) \]
   \[ = \frac{1}{2} \sum_{\ell=1}^{\infty} \frac{1}{2\ell} \tilde{P}_0(X_{2\ell} = 0, \tau_{\Lambda_n} > 2\ell, \kappa \leq 2\ell) \]
   \[ \leq \frac{1}{2} \sum_{\ell=1}^{\infty} \frac{1}{2\ell} \tilde{P}_0(X_{2\ell} = 0) \left( 1 - (1 - \xi(m))^{2\ell} \right) \]
Using the same estimate as in [8], the asymptotics of (9) for small $m$ gives the desired upper-bound. \qed

The idea is to tilt the measure, as in the proof for $d \geq 3$, first to work with the massive measure, and second to follow the environment such that the field $\varphi$ is shifted up of an amount $s$ on the sites $x$ for which $b \cdot e_x + h < 0$. For some technical reason, we need to work with the measure with boundary
condition $a$, so we perform three changes of measure (first one for changing boundary condition, a second one for adding mass, and a third one for following the environment).

Let $s > 0$ and $m > 0$ to be fixed later.

$$f^q_{\Lambda_n}(e) = n^{-2} \log \mu^0_{\Lambda_n} \left( \exp \sum_{x \in \Lambda_n} (b \cdot e_x + h) 1_{[\phi_x \in [-a, a]]} \right)$$

$$= n^{-2} \log \mu^0_{\Lambda_n, m, e, s} \left( \exp \left( \sum_{x \in \Lambda_n} (b \cdot e_x + h) 1_{[\phi_x \in [-a, a]]} \right) + \log \frac{d\mu^0_{\Lambda_n}}{d\mu^0_{\Lambda_n, m}} + \log \frac{d\mu^0_{\Lambda_n, m}}{d\mu^0_{\Lambda_n, m, e, s}} \right)$$

where $(\phi_x)_{x \in \Lambda_n}$ under $\mu^0_{\Lambda_n, m, e, s}$ is distributed as $(\phi_x + s1_{[b-e_x+h<0]})_{x \in \Lambda_n}$ under $\mu^0_{\Lambda_n, m}$. More formally, introducing $T_{e, s}: ((\phi_x)_{x \in \Lambda_n}) \mapsto (\phi_x + s1_{[b-e_x+h<0]})_{x \in \Lambda_n}$, we define $\mu^0_{\Lambda_n, m, e, s}$ as $\mu^0_{\Lambda_n, m} \circ T_{e, s}^{-1}$. Using Jensen’s inequality we get

$$f^q_{\Lambda_n}(e) \geq n^{-2} \mu^0_{\Lambda_n, m, e, s} \left( \sum_{x \in \Lambda_n} (b \cdot e_x + h) 1_{[\phi_x \in [-a, a]]} + \log \frac{d\mu^0_{\Lambda_n}}{d\mu^0_{\Lambda_n, m}} + \log \frac{d\mu^0_{\Lambda_n, m}}{d\mu^0_{\Lambda_n, m, e, s}} \right)$$

As in the proof for $d \geq 3$, we have

$$n^{-2} \mu^0_{\Lambda_n, m, e, s}(1) \geq - Cn^{-1}.$$ 

By Lemma 2.3 we have $\frac{Z^0_{\Lambda_n, m}}{Z^0_{\Lambda_n}} = \frac{Z^0_{\Lambda_n, m}}{Z^0_{\Lambda_n}} \frac{Z^0_{\Lambda_n, m}}{Z^0_{\Lambda_n}} \geq - Cn - C_2 n^2 m^2 |\log m|$, and then

$$(2) = \log \left( \frac{Z^0_{\Lambda_n, m}}{Z^0_{\Lambda_n}} \right) + m^2 \sum_{x \in \Lambda_n} \phi_x^2 \geq - Cn - C_2 n^2 m^2 |\log m| + m^2 \sum_{x \in \Lambda_n} \phi_x^2$$

hence,

$$n^{-2} \mu^0_{\Lambda_n, m, e, s}(2) \geq - Cn^{-1} - C_2 m^2 |\log m|.$$ 

Finally, noticing that $Z^0_{\Lambda_n, m, e, s} = Z^0_{\Lambda_n, m}$ (just perform a change of variables in the Gaussian integral), we can compute the third term.

$$(3) = - \frac{1}{8} \sum_{(x,y) \cap \Lambda_n \neq \emptyset} \sum_{x \sim y} (\phi_x - \phi_y)^2 - (\phi_x - \phi_y)^2 - m^2 \sum_{x \in \Lambda_n} (\phi_x - s)^2 - (\phi_x - s)^2$$

where $\phi_x := \phi_x + s1_{[b-e_x+h<0]}$. Now we will use the fact that the marginal laws of all $\phi_x$, $x \in \Lambda_n$ under $\mu^0_{\Lambda_n,m}$ are Gaussian variables, i.e. $\phi_x \sim N(\mu_x^0, \sigma_x^2)$ where $\mu_x^0 \approx \zeta$ except for $x$ close to the boundary of the box. Indeed, by the random walk representation of the mean, there is $C > 0$ such that $|\mu_{\Lambda_n,m}(\phi_x) - \zeta| \leq C(1 + m^2)^{-1/2}$. Moreover, $\sigma_x^2 = \text{Var}_{\Lambda_n}^0(\phi_x) \leq C_1 |\log m|$. Using the definition of $\mu^0_{\Lambda_n, m, e, s}$ and computing the terms as in the proof for $d \geq 3$,

$$n^{-2} \mu^0_{\Lambda_n, m, e, s}(3) \geq - \frac{s^2}{8n^2} \sum_{(x,y) \cap \Lambda_n \neq \emptyset} \sum_{x \sim y} (1_{[b-e_x+h<0]} - 1_{[b-e_y+h<0]})^2 - \frac{m^2 s^2}{n^2} \sum_{x \in \Lambda_n} 1_{[b-e_x+h<0]} + C.$$ 

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We get, for \( n \) large enough and \( m \) small enough

\[
f^q_{\Lambda_n}(e) \geq n^{-2} \sum_{x \in \Lambda_n} (b \cdot e_x + h) \mu_{\Lambda_n,m}^{0,\xi}(\varphi_x \in [-a,a]) - \frac{s^2}{8n^2} \sum_{x,y \in \Lambda_n \neq \emptyset} (1_{|b \cdot e_x + h| < 0} - 1_{|b \cdot e_y + h| < 0})^2 \\
- \frac{m^2 s^2}{n^2} \sum_{x \in \Lambda_n} 1_{|b \cdot e_x + h| < 0} - C'm^2 \log m - Cn^{-1},
\]

for some \( C \) and \( C' > 0 \). Note that for \( O(n^2) \) sites \( x \), we have

\[
\mu_{\Lambda_n,m}^{0,\xi}(\varphi_x \in [-a,a]) - \mu_{\Lambda_n,m}^{0,\xi}(\varphi_x + s \in [-a,a]) \propto \Phi_{\xi,b}'(a) \cdot s \quad \text{as} \quad n \to \infty,
\]

for \( s \ll a \leq \xi = C_1 |\log m| \), and \( b = C_1 |\log m| \). Above \( \Phi_{\xi,b} \) stands for the p.d.f. of the above Gaussian distribution with mean \( \xi \) and variance \( b^2 \). In particular, for a positive fraction of \( x \) (close to 1) and \( m \) sufficiently small, we have the upper bound:

\[
\mu_{\Lambda_n,m}^{0,\xi}(\varphi_x \in [-a,a]) - \mu_{\Lambda_n,m}^{0,\xi}(\varphi_x + s \in [-a,a]) \geq \frac{C_1(a)}{|\log m|} \cdot s,
\]

for some \( C_1(a) > 0 \). Now we can compute:

\[
f^q_{\Lambda_n}(e) \geq n^{-2} \sum_{x \in \Lambda_n} (b \cdot e_x + h) (\mu_{\Lambda_n,m}^{0,\xi}(\varphi_x \in [-a,a]) - \frac{C_1(a)}{|\log m|} s 1_{|b \cdot e_x + h| < 0}) \\
- \frac{s^2}{8n^2} \sum_{x,y \in \Lambda_n \neq \emptyset} (1_{|b \cdot e_x + h| < 0} - 1_{|b \cdot e_y + h| < 0})^2 \\
- \frac{m^2 s^2}{n^2} \sum_{x \in \Lambda_n} 1_{|b \cdot e_x + h| < 0} - C'm^2 \log m - Cn^{-1}
\]

Observe that \( \mu_{\Lambda_n,m}^{0,\xi}(\varphi_x \in [-a,a]) \geq 2a \cdot \Phi_{\xi,b,\xi}(-a) = \frac{C_1(a)}{|\log m|} \) uniformly in \( x \in \Lambda_n \). Let us take the expectation with respect to the environment, and use the bounded convergence theorem and [4 Theorem 2.1], we get:

\[
f^q(e) = \lim_{n \to \infty} \mathbb{E} f^q_{\Lambda_n}(e) \geq h \frac{C_1(a)}{|\log m|} - s \frac{C_1(a)(-b+h)}{2 |\log m|} - \frac{s^2 m^2}{2} - \frac{s^2}{16} - C'm^2 |\log m|.
\]

Our aim now is to show that the right hand side can be positive even when \( h \) is negative. In the above expression \( s,m \) are free parameters which we may vary. However, we have to remember that both \( s \) and \( m \) need to be small enough, which makes standard optimization analysis cumbersome. We are going to show that there exists \( C > 0 \) and \( \epsilon > 0 \) such that for any \( b,h \) such that \(( -b+h ) \in ( -\epsilon,0 ) \)

\[
h := C \frac{(-b+h)^2}{\log((-b+h))},
\]

there exist small \( s \) and \( m \) such that the r.h.s. of (11) is positive. Notice that the result will imply that for any \( h \geq C \frac{(-b+h)^2}{\log((-b+h))} \) the free energy is positive. Let us choose the value of \( s \) which maximizes (11) for fixed \( m \), i.e.

\[
s = - \frac{C_1(a)(-b+h)}{(m^2 + 1/4) |\log(m)|}.
\]
and for $m$ let us take

$$m^2 = -k/(\log k)^3, \text{ where } k := -h\tilde{C}_1(a)/C'. \tag{13}$$

One can verify that with the above choice of parameters both $s$ and $m$ are as small as we want. Let us first put (12) into the r.h.s. of (11) and obtain

$$f^q(e) \geq h\frac{\tilde{C}_1(a)}{|\log m|} + \frac{C_1(a)^2(-b + h)^2}{(m^2 + 1/4)(\log m)^2} - C'm^2|\log m|.$$  

For $k$ and consequently $m$ small enough we have

$$f^q(e) \geq h\frac{\tilde{C}_1(a)}{|\log m|} + \frac{C_1(a)^2(-b + h)^2}{2(\log m)^2} - C'm^2|\log m|.$$  

Further let us multiply both sides by $(\log m)^2$ and insert (13).

$$f^q(e)(\log m)^2 \geq -h\tilde{C}_1(a)\log m + \frac{C_1(a)^2(-b + h)^2}{2} + C'm^2(\log m)^3$$

$$= -\frac{h\tilde{C}_1(a)}{2}(\log k - 3|\log k|) + \frac{C_1(a)^2(-b + h)^2}{2} - C'k(\log k - 3|\log k|)^3$$

$$= -\frac{h\tilde{C}_1(a)}{2}\log(-h\tilde{C}_1(a)/C') + \frac{C_1(a)^2(-b + h)^2}{2} + h\frac{\tilde{C}_1(a)}{8} + o(h) \quad \text{as } |h| \to 0$$

From the last claim it is straightforward to conclude existence of $C$ (sufficiently small) and $\epsilon$ with the properties described above. 

\[ \square \]

Acknowledgements. We would like to warmly acknowledge Yvan Velenik for introducing us to the topic. L.C. was partially supported by the Swiss National Foundation. P.M. was supported by a Sciex Fellowship grant no. 10.044.

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