Compressing the hidden variable space of a qubit

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In previously exhibited hidden variable models of quantum state preparation and measurement, the number of continuous hidden variables describing the actual state of single realizations is never smaller than the quantum state manifold dimension. We introduce a simple model for a qubit whose hidden variable space is one-dimensional, i.e., smaller than the two-dimensional Bloch sphere. The hidden variable probability distributions associated with quantum states satisfy reasonable criteria of regularity. Possible generalizations of this shrinking to an $N$-dimensional Hilbert space are discussed.

I. INTRODUCTION

A quantum state is not discernible by means of a single replica, but can be reconstructed only by performing many measurements on identically prepared systems [1]. In this sense it is analogous to a classical probability distribution that carries statistical information on the actual state of many realizations. The similarity is reinforced by the fact that both the quantum state and the probability distribution have linear dynamical equations. Furthermore, as for the probability distributions, the space of quantum states of a composite system is the tensorial product of the subsystem Hilbert spaces. These analogies suggest the idea that the quantum state does not represent the actual state of a single realization, but merely contains statistical information about an underlying hidden variable state, also named ontic state [2]. Indeed, a quantum state has the peculiar feature of containing the full statistical information concerning any measurement outcome performed on ensembles. However, unlike classical mechanics, the standard quantum framework does not provide a picture beyond this ensemble description, which unavoidably requires an exponential growth of resources in the state definition. Thus, it is reasonable to wonder if the complexity of quantum mechanics could be mitigated by filling the gap between classical and quantum representations. Indeed there is no a priori reason to believe that the ontic states must contain the full ensemble information contained in quantum states.

The relation between exponential complexity and ensemble description is well-illustrated by a classical example. On the one hand, a system of $N$ particles is described by a set of canonical variables $\{\vec{q}_i, \vec{p}_i\}$ whose number scales linearly with $N$. On the other hand, the statistical state is given by a multivariate probability distribution $\rho(\{\vec{q}_i, \vec{p}_i\})$, whose complete characterization requires generally a number of resources growing exponentially with $N$ for a given accuracy. This is not surprising, since the probability distribution contains a quantity of information that could be approached only by performing a series of measurements on a large number of replicas.

Our core question is as follows. Suppose, in accordance with Einstein’s point of view [3], that there exists a more fundamental theory that does not use quantum states, but describes each single system by means of well-defined (ontological) variables. The wave-function and the ontological variables would be analogous to the classical multivariate probability distribution $\rho$ and the coordinates $\{\vec{q}_i, \vec{p}_i\}$, respectively. Thus, the following question naturally arises: can the ontological space dimension grow polynomially with the size of the quantum system? By size here we mean the number of elements that compose the system, for example the number of spins. As will be shown in this Letter, this question is anything but trivial. Indeed, we will prove with a practical example that the ontological space dimension can be smaller than the quantum state manifold dimension. We call this dimensional reduction ontological shrinking. The ontological shrinking is a necessary condition for having an exponential compression, although not sufficient. Our result is fundamental for sweeping away any prejudice that this reduction is impossible.

The ontological shrinking is closely related to the concept of classical “weak simulation” in quantum information theory. Contrary to a “strong simulation” of quantum computers, the goal of a classical “weak simulation” is not to compute the measurement probabilities with high accuracy, but the outcomes in accordance with the probabilities. There are examples of quantum circuits that cannot be efficiently simulated in a strong way, but whose weak simulation is nevertheless tractable [4]. In a hidden variable theory with reduced sampling space, the identification of the actual state of a single realization would require less resources than the quantum state definition. Thus, the ontological shrinking could offer in a natural way an efficient method of “weak simulation” of quantum computers.

The problem of the minimal quantity of ontological resources was recently considered by different authors [6–10]. In Ref. [6], Hardy has proved that the number of ontic states is always infinite, even in the case of a finite-dimensional Hilbert space. According to this constraint, the set of ontic states is not less than countably infinite. That is a weak condition, since the Hilbert space is continuous and in general multi-dimensional. The problem of the smallest dimension of the ontological space was introduced in Ref. [7]. It was subsequently proved that the
ontological space dimension cannot be smaller than the dimension of the quantum state manifold in the case of a short memory hidden variable theory \cite{8,11}, implying an exponential growth of resources (“Short memory” means that the dynamics of the ontic state is Markovian). In Ref. \cite{10} a hidden variable model for a finite number of measurements was reported, whose number of resources saturates to the constraint of Ref. \cite{8} in the case where the whole set of possible measurements is considered. In the conclusion of Ref. \cite{8} it was noted that a possible way to overcome the exponential growth would be to reject one of the hypotheses of the theorem, e.g., the causality. Thus, the dynamics of the ontological state would not be described as a Markov succession of causes and effects. The Wheeler-Feynman absorber theory \cite{12} is an example of non-causal theory, introduced in another context with the aim of removing the self-interaction of electric charges. It is an example of retro-causality, where future events can have effects on the past. Recently, Wharton introduced a time-symmetric interpretation of the Klein-Gordon equation by imposing boundary conditions both in past and future \cite{13}.

In this Letter, we consider the problem of describing a process of state preparation and subsequent measurement completely neglecting any concern about the dynamics. We present a hidden variable model of a qubit whose ontological manifold has one dimension, that is one half the dimension of the quantum state manifold (the Bloch sphere). This is the first example of a hidden variable model for measurements with a compressed sampling space and raises the interesting question of whether an exponentially growing number of resources is in fact required to describe the hidden variable state of a quantum system. Furthermore, it illustrates that the short memory hypothesis is indeed strictly necessary for the proof of the theorem in Ref. \cite{8}. Finally, we discuss possible generalizations to an \(N\)-dimensional Hilbert space and present a model that works for a large set of trace-one projectors.

II. ONTOLOGICAL MODEL OF QUBIT

In a hidden variable theory of state preparation and measurement, the quantum state is translated into a classical language by replacing it with a probability distribution on a sampling space \(X\) of ontic states, \(|\psi\rangle \rightarrow \rho(X|\psi\rangle\), i.e., it is assumed that a quantum system is described, at a deeper and partially hidden level, by a set of classical variables that do not necessarily contain the whole information carried by \(|\psi\rangle\). Such information is contained in the ensemble distribution \(\rho\). We assume that the ontological space \(X\) is a disconnected manifold defined by a set of continuous variables \((x_1, ..., x_D) \equiv \vec{x}\) and a discrete index \(n\), labeling the components of the manifold. Its Lebesgue covering dimension is \(D\). The distribution \(\rho(\vec{x}, n|\psi\rangle\) is a regular function, in a sense that will be specified later on. This allows us to define the concept of ontological space dimension, which is the number \(D\) of continuous variables. The ontological variables are only partially hidden in the sense that they have a role in the generation of an event when a measurement is performed. Let the trace-one projector \(|\phi\rangle\langle\phi|\) be the observable that will be measured after preparing the state \(|\psi\rangle\). At the ontological level, there is a conditional probability \(P(\phi|X)\) of obtaining the event \(\phi\) given the ontic state \(X\). The ontological theory is equivalent to quantum mechanics if

\[
\sum_n \int d^D x P(\phi|\vec{x}, n) \rho(\vec{x}, n|\psi\rangle) = |\langle\phi|\psi\rangle|^2. \tag{1}
\]

It is important to note that the dimension of a manifold is a topological property and we have to introduce some conditions of regularity on \(\rho(X|\psi\rangle)\) in order that the problem of the ontological dimension is not ill-posed. Without these conditions the ontological shrinking would be a trivial task, since a multi-dimensional space has the same cardinality of a one-dimensional space. Indeed, it is always possible to construct a one-dimensional ontological model from a multi-dimensional one in a trivial way, by means of a bijective transformation. Let us consider for example the Kochen-Specker (KS) model for a qubit \cite{14}. The corresponding ontological space is a unit sphere and the states are unit three-dimensional vectors \(\vec{s}\), which can be identified by two continuous variables \(q\) and \(p\), such as the angles in the spherical coordinate system. Each quantum state \(|\psi\rangle\) is associated with a probability distribution \(\rho(q,p|\psi\rangle)\). The mapping \(|\psi\rangle \rightarrow \rho(q,p|\psi\rangle)\) in the KS model is almost regular, that is \(\rho(q,p|\psi\rangle)\) is a differentiable function for every \(|\psi\rangle\) and \((q,p)\), apart from a zero measure subset of ontological states. We can rescale \(q\) and \(p\) in such a way that they can be written in the decimal notation \(q = q_0q_1q_2q_3...\) and \(p = p_0p_1p_2p_3...\), \(q_i\) and \(p_i\) being digits. The two-dimensional ontological space can be compressed into a one-dimensional space by encoding the information carried by \(q\) and \(p\) into the single real number \(r \equiv q_0p_0q_1p_1q_2p_2q_3p_3...\) However, the distribution \(\rho(r|\psi\rangle)\), with the new topological space \(r\) as domain, will be highly irregular and physically unsuitable.

In order to rule out these trivial routes towards the ontological shrinking, we impose the condition that the function \(\rho(\vec{x}, n|\psi\rangle)\) is analytic in \(\vec{x}\) and \(|\psi\rangle\) almost everywhere, possibly containing also a collection of Dirac functions whose heights and positions are differentiable functions of \(|\psi\rangle\) almost everywhere. This gives a sufficient general condition of regularity that makes the ontological shrinking a non-trivial problem.

With these premises, let us consider a two-state quantum system and introduce a one-dimensional hidden variable model that works only for a subset of preparation states. Later on we will extend the model to the whole quantum state manifold. Our ontological space is given by a continuous variable \(x\) and a discrete index \(n\) that takes the two values 0 and 1. It is convenient to represent the quantum state \(|\psi\rangle\) and the event \(|\phi\rangle\) by means of the Bloch vectors \(\vec{v} \equiv (\langle\psi|\vec{\sigma}|\psi\rangle\) and \(\vec{w} \equiv (\langle\phi|\vec{\sigma}|\phi\rangle\), where \(\vec{\sigma} \equiv (\vec{\sigma}_x, \vec{\sigma}_y, \vec{\sigma}_z)\), the \(\vec{\sigma}_i\) being the Pauli matrices.
The probability distribution associated with the state $\vec{v}$ is
\[
\rho(x, n|\vec{v}) = \sin \theta \delta_{n,0} \delta(x-\varphi) + (1 - \sin \theta) \delta_{n,1} \delta(x - \theta),
\] (2)
where $\varphi$ and $\theta$ are respectively the azimuth and zenith angles in the spherical coordinate system
\[
v_x = \sin \theta \cos \varphi, \\
v_y = \sin \theta \sin \varphi, \\
v_z = \cos \theta.
\] (3)
Thus, when the quantum state $\vec{v}$ is prepared, the index $n$ takes the value 0 or 1 with probability $\sin \theta$ or $1 - \sin \theta$ and the continuous variable is the azimuth or zenith angle according to the value of $n$. The probability distribution $\rho(x, n|\vec{v})$ is non-negative for any $\{x, n\}$ and $\vec{v}$.

The probability distribution \(\rho(x, n|\vec{v})\) is a collection of two delta functions whose heights and positions are differentiable functions of $\theta$ and $\varphi$. Thus, they fulfill the conditions of regularity required previously. It is important to note that a single realization contains less information than the quantum state, since only one of the two angles $\theta$ and $\varphi$ is stored in the ontic state $\{x, n\}$. The whole information about $\vec{v}$ is carried by the ensemble distribution $\rho(x, n|\vec{v})$.

The conditional probability $P(\vec{w}|x, n)$ for an event $\vec{w}$ with $w_z > 0$ is defined as follows:
\[
P(\vec{w}|x, 0) = \frac{1}{2} \left( 1 + \frac{w_x \cos \varphi + w_y \sin \varphi - \sqrt{1 - w_z^2}}{2} \right),
\]
\[
P(\vec{w}|x, 1) = \frac{1}{2} \left( 1 + \frac{w_x \cos \varphi + w_y \sin \varphi + \sqrt{1 - w_z^2}}{2} \right),
\] (4)
The events $\vec{w}$ with $w_z < 0$ correspond simply to the non-occurrence of the events $-\vec{w}$ with $w_z > 0$, i.e., $P(-\vec{w}|x, n) = 1 - P(\vec{w}|x, n)$.

It is easy to prove that these probability functions fulfill the condition \(\mathbf{1}\), that is, $P(\vec{w}|\varphi, 0) \sin (\theta) + P(\vec{w}|\theta, 1)(1 - \sin \theta) = (1 + \vec{w} \cdot \vec{v})/2$. We have to check that the conditional probabilities satisfy the constraints $0 \leq P(\vec{w}|x, n) \leq 1$. It is sufficient to consider the case $w_z > 0$. The non-negativity of $P(\vec{w}|x, 0)$ for any $x$ and $\vec{w}$ is proved by the fact that its minimum is zero. The minimum with a fixed $\vec{w}$ is equal to
\[
1 - \sqrt{1 - w_z^2} \equiv m(w_z)
\] (5)
and when taken the vectors $(\cos x, \sin x)$ and $(w_x, w_y)$ are antiparallel, that is, when $\cos x = -w_x/\sqrt{w_x^2 + w_y^2}$, $\sin x = -w_y/\sqrt{w_x^2 + w_y^2}$. The overall minimum, i.e., the minimum of $m(w_z)$, is taken at $w_z = 0$ and is equal to 0. Similarly, the largest value, with $(\cos x, \sin x)$ and $(w_x, w_y)$ parallel, is equal to 1. The conditional probability $P(\vec{w}|x, 1)$ takes its maximum when the vectors $(\sqrt{1 - w_z^2}, w_z)$ and $(\cos x, \sin x)$ are parallel and is equal to 1. Its non-negativity check deserves a more detailed discussion. It is easy to realize that $P(\vec{w}|x, 1)$ is negative for some $\vec{w}$ if $\cos x < 0$, thus this model does not work for any state. Let us find the region with $\cos x > 0$ where the probability is non-negative for every $\vec{w}$. For a fixed $x$, the conditional probability is minimal at the boundary $w_z = 1$ and takes the value \(\frac{1 - 2\sin x + \cos x}{2 - 2\sin x}\). It is non-negative if
\[
\theta = x < \theta_0 \equiv \arccos(3/5) \approx 53.13\text{ degrees}.
\] (6)

Thus, the present model works only for a set of prepared states whose Bloch vector lies inside a cone with aperture $2\theta_0$, the $z$-axis being the cone symmetry axis. It is interesting to note that there is no constraint on $\vec{w}$ and the model works for any event if the prepared state fulfills the condition $\mathbf{3}$.

A simple way to extend the model to any preparation state is to increase the information contained in the ontological state. Let $\vec{n}_1, \ldots, \vec{n}_{M_b}$ be $M_b$ fixed Bloch vectors. A set of orthogonal coordinates is associated with each $\vec{n}_k$. The fixed Bloch vector $\vec{n}_k$ is the $z$-axis of the associated coordinate system. When the state $\vec{v}$ is prepared, the information on the nearest $\vec{n}_k$ is enclosed in an additional discrete index $m$ that can take $M_b$ possible values. This index does not change the ontological space dimension, which remains equal to one. Furthermore, the preparation apparatus uses the coordinate system attached to $\vec{n}_k$. By nearest vector we mean the vector $\vec{n}_k$ with the largest scalar product $\vec{n}_k \cdot \vec{v}$. The advantage of the added information is that the measurement apparatus receives the information on the closest $\vec{n}_k$ and can use the protocol previously described with the coordinate system attached to $\vec{n}_k$. If the number $M_b$ of vectors $\vec{n}_k$ is sufficiently large, the angle between the closest axis and the quantum state $\vec{v}$ is always smaller than $\theta_0$. In the case of equidistributed vertices, the smallest number $M_b$ of Bloch vectors is 12 and they correspond to the vertices of an icosahedron. For this polyhedron, the length of the edges is $L = 4/\sqrt{10 + 2\sqrt{5}}$ for a unit sphere, so the distance between the circumcenter of each face and its vertices is $d = L/\sqrt{2}$. The angle between vectors passing through a circumcenter and the closest vertex is $\theta_1 = \arcsin d \approx 37.37\text{ degrees}$. The inequality $\theta_1 < \theta_0$ guarantees that the angle between the $z$-axis and $\vec{v}$ is always smaller than $\theta_0$. Thus, the extended ontological model works for any state preparation and measurement. The patches associated with each vector are 12 congruent spherical pentagons, which give a regular tessellation of the Bloch sphere. They are the sphere projection of dodecahedron faces. We indicate the patch pointed to by $\vec{n}_k$ with $\Omega_k$.

The extended model is equivalent to the following protocol. Suppose that Bob and Alice share a common reference frame on the Bloch sphere and a set of Bloch vectors $\vec{n}_1, \ldots, \vec{n}_{M_b}$, corresponding to the icosahedron vertices. Let the first vector $\vec{n}_1$ be the $z$-axis of the reference frame. Bob prepares a quantum state $\vec{v}$, which for the moment is assumed to point at the spherical pentagon $\Omega_1$ associated with $\vec{n}_1$. He assigns to a discrete variable $n$ one of values 0 and 1, randomly generated with probabilities $\sin \theta$ and $1 - \sin \theta$, respectively. If $n = 0(1)$,
Bob sets a continuous variable \( x \) equal to the azimuth angle \( \varphi \) (zenith angle \( \theta \)) and sends both \( n \) and \( x \) to Alice. Alice generates the event \( \vec{w} \) with probability given by Eq. (4). The fact that \( \vec{v} \) points at \( \Omega_1 \) guarantees that condition (3) is always satisfied, that is, Alice always receives values of \( \{x, n\} \) such that \( 0 \leq P(\vec{w}|x, n) \leq 1 \). It is worth to stress that in a single run Alice has only a partial information about the quantum state. Indeed she knows only \( \varphi \) or \( \theta \), according to the value of \( n \). Nevertheless, she can generate with this partial information events in accordance with quantum probabilities. The model is extended to the whole Bloch sphere as follows. The task is generating the event \( \vec{w} \) with probability \( (1 + \vec{v} \cdot \vec{w})/2 \), given any state \( \vec{v} \). Let \( \vec{n}_k \) be the closest vector to \( \vec{v} \). Bob evolves the quantum state according to a unitary evolution \( \hat{U}_k \) that takes the spherical pentagon \( \Omega_k \) to \( \Omega_1 \). Let \( \hat{O}_k \) be the corresponding orthogonal transformation on the Bloch sphere. After the transformation \( \vec{v} \rightarrow \hat{O}_k \vec{v} \), Bob executes the previously described protocol and sends to Alice the pair \( \{x, n\} \) and the auxiliary index \( k \). In order to evaluate the probability of \( \vec{w} \) given \( \{x, n\} \), Alice executes the same transformation on \( \vec{w} \) evaluating the probability of \( \hat{O}_k \vec{w} \) by means of Eq. (4), that is, the conditional probability \( P(\vec{w}|x, n, k) \) of \( \vec{w} \) in the extended model is equal to \( P(\hat{O}_k \vec{w}|x, n, k) \), where \( P(\cdot|x, n) \) is given by Eq. (1). The unitary transformation performed by Bob guarantees that \( 0 \leq P(\vec{w}|x, n, k) \leq 1 \). Since \( \vec{O}_k \vec{v} \cdot \vec{O}_k \vec{w} = \vec{v} \cdot \vec{w} \), the overall protocol generates the event \( \vec{w} \), given \( \vec{v} \), with the correct quantum probability \( (1 + \vec{v} \cdot \vec{w})/2 \).

It is worth to note that, before playing the game, Bob and Alice have to agree about a common reference frame and the way the Bloch sphere is partitioned. This agreement has to be established only at the beginning and the shared information can be used for any state and measurement that Bob and Alice wish to test. It is also interesting to observe that applying this model to the concrete example of a 1/2 spin requires that we enrich the space with a preset structure that breaks the rotational symmetry. This is not surprising, since no (linear or nonlinear) representation of three-dimensional rotations exist on a one-dimensional manifold, that is, it is impossible to represent three-dimensional rotations by means of differentiable endomorphisms of a one-dimensional manifold. The symmetry breaking would occur at the ontological level in the description of the ontic state, but it would be concealed at the phenomenological level. However, the ontological shrinking program does not necessarily imply the spatial symmetry breaking, since the dimensional reduction could involve entanglement and not single particles. For example, it would be possible in principle to have an ontological theory that describes \( n \) spins by means of \( n \) vectors. The ontological space would have a reduced dimensionality, but the theory would not break the spatial symmetry. Anyway the ontological shrinking makes the representation of \( SU(N) \) on the ontological space impossible.

We have presented the first example of hidden variable model whose sampling space dimension is smaller than the quantum state manifold one. For example, as previously noted, the Kochen-Specker model \cite{14} for a two-state system uses a two-dimensional space.

III. ONTOLOGICAL SHRINKING FOR HIGHER DIMENSIONAL HILBERT SPACES

Our model raises the question whether the ontological shrinking is possible also for higher dimensions of the Hilbert space. Since in the two-dimensional model the probability distribution is the mixture of two delta distributions, a natural generalization of this distribution would be

\[
\rho(\vec{x}, n|\psi) = R(n|\psi)\delta[\vec{x} - \vec{f}_n(\psi)],
\]

where \( \vec{x} \) is a \( D \)-tuple of real variables and \( n \) a discrete index that goes from 1 to \( M \). \( D \) is the dimension of the ontological space. \( \vec{f}_n \) is a generic vectorial function and \( R(n|\psi) \) is the probability of \( n \) given \( |\psi\rangle \). The conditional probability \( P(\phi|\vec{x}, n) \) of an event \( \phi \) given \( \vec{x} \) and \( n \) is such that Eq. (1) is satisfied.

The Beltrametti-Bugajski theory has for example the above structure with \( D \) equal to twice the Hilbert space dimension and \( M = 1 \) \cite{13}. In this case there is no shrinking since the dimension \( D \) of the ontological space is not smaller than the quantum state manifold dimension. The shrinking occurs for \( D < 2N - 2 \), \( N \) being the Hilbert space dimension.

A simple model with a compressed space and working for a large class of events and states has \( M = N^2 \) and \( D = 2 \). The probability distribution \( \rho \) and the conditional probabilities for the event \( \phi \) are

\[
\rho(X, n, m|\psi) = R(n, m)\delta(X - \psi_n^*\psi_m),
\]

\[
P(\phi|X, n, m) = 1 - \frac{1}{2R(n, m)}|\phi_n^*\phi_m - X|^2,
\]

where \( n, m = 1, ..., N \), \( X \) is a complex number and \( \psi_n \equiv \langle n|\psi\rangle \). \( \{|n\} \) being a complete set of orthonormal vectors. The distribution \( R(n, m) \) is non-negative and normalized to 1. It is easy to check that the functions fulfill the condition (1). However, the conditional probabilities are positive only if

\[
|\psi_n^*\psi_m - \phi_n^*\phi_m|^2 < 2R(n, m).
\]

It is interesting to note that the selected manifold of states \( |\psi\rangle \) and events \( |\phi\rangle \) have the same dimension as the overall quantum state manifold. This is a very important property, since economical ontological models working in a zero measure region of events or states are quite trivial.

If \( R(n, m) \) is constant, it is easy to prove that the condition

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|\psi_n - \phi_n|^2 < \frac{R}{2}, \text{ for all } n
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is fulfilled only at the beginning and the shared information can be used for any state and measurement that Bob and Alice wish to test. It is also interesting to observe that applying this model to the concrete example of a 1/2 spin requires that we enrich the space with a preset structure that breaks the rotational symmetry. This is not surprising, since no (linear or nonlinear) representation of three-dimensional rotations exist on a one-dimensional manifold, that is, it is impossible to represent three-dimensional rotations by means of differentiable endomorphisms of a one-dimensional manifold. The symmetry breaking would occur at the ontological level in the description of the ontic state, but it would be concealed at the phenomenological level. However, the ontological shrinking program does not necessarily imply the spatial symmetry breaking, since the dimensional reduction could involve entanglement and not single particles. For example, it would be possible in principle to have an ontological theory that describes \( n \) spins by means of \( n \) vectors. The ontological space would have a reduced dimensionality, but the theory would not break the spatial symmetry. Anyway the ontological shrinking makes the representation of \( SU(N) \) on the ontological space impossible.

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where \( n, m = 1, ..., N \), \( X \) is a complex number and \( \psi_n \equiv \langle n|\psi\rangle \). \( \{|n\} \) being a complete set of orthonormal vectors. The distribution \( R(n, m) \) is non-negative and normalized to 1. It is easy to check that the functions fulfill the condition (1). However, the conditional probabilities are positive only if

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It is interesting to note that the selected manifold of states \( |\psi\rangle \) and events \( |\phi\rangle \) have the same dimension as the overall quantum state manifold. This is a very important property, since economical ontological models working in a zero measure region of events or states are quite trivial.

If \( R(n, m) \) is constant, it is easy to prove that the condition

\[
|\psi_n - \phi_n|^2 < \frac{R}{2}, \text{ for all } n
\]
is sufficient for positivity. Indeed, from it we have:

\[
|\psi_n^* \phi_m - \phi_n^* \phi_m|^2 = \frac{1}{4} (|\psi_n^* - \phi_n^*|^2 |\psi_m + \phi_m|^2 + (\phi_n^* + \phi_n^*)(\psi_m - \phi_m)|^2 \leq \frac{1}{4} (|\psi_n^* - \phi_n^*|^2 |\psi_m + \phi_m|^2 + |\psi_n + \phi_n|^2 |\psi_m - \phi_m|^2 \leq (|\psi_n - \phi_n| + |\psi_m - \phi_m|)^2 \leq 2R.
\]

(12)

At variance with the previous model for a qubit, the constraint is not only on the quantum states, but involves also the events. Thus, the patching method previously used is able to extend the validity of the model to the whole quantum state space, but not to the whole set of trace-one projective measurements. One could try to find the functions \(R(n, m)\) that maximize the volume of the positivity region, however it is impossible to cover the whole space of events. With the choice \(R(n, m) = 1/\sqrt{N}\), from Eq. (11) we have that a sufficient condition for positivity is \(\psi_n - \phi_n \leq \frac{1}{\sqrt{2N}}\). This inequality implies that the volume of a patch region scales at least as \((2\pi/\sqrt{N})^{N-1}\).

Since the volume of the whole quantum state manifold scales as \((\pi/\sqrt{2})^{N}\), the additional information required for the patching grows as \(N \log N\), that is, almost exponentially with respect the size of the system. This choice of \(R(n, m)\) is not optimal. For example, one could give a larger statistical weight to the events with \(n\) or \(m\) equal to zero. If \(R(0, n)\) and \(R(n, 0)\) scale as \(1/N\), it is easy to show by condition (10) that both \(\psi_0\) and \(\phi_0\) go to 1 for large \(N\). Using this property, one finds that the constraint (10) is satisfied if \(|\phi_{n \neq 0}| \lesssim 1/\sqrt{N}\) and \(|\psi_{n \neq 0}| \lesssim 1/\sqrt{N}\). This scaling of the positivity region is a necessary condition to have a non-exponential growth of the additional information required for the patching. We will not go into further details on the optimization of this model.

There is a simple argument to show that any hidden variable theory of the form in Eq. (7) with \(M\) finite and working for any measurements cannot have an ontological space whose dimension is smaller than \(2N - 3\). Thus, if there exists an ontological theory for any measurements that does not require an exponentially growing number of resources to describe a single system, then in this theory \(M\) is infinite. The proof is the following. Using Eq. (7), Eq. (11) becomes

\[
\sum_n P(\phi|\bar{x}_n, n) R(n|\psi) = |\langle \phi|\psi \rangle|^2,
\]

(13)

where

\[
\bar{x}_n = \bar{f}_n(\psi).
\]

(14)

We can assume that there exists a non-zero measure subset of the quantum state manifold where \(R(n|\psi)\) is different from zero or identically equal to zero for each \(n\). If \(R(n|\psi)\) is differentiable almost everywhere and \(M\) finite, it is always possible to find such a region. Let us consider in the following only the quantum states \(|\psi\rangle\) living in this subset. Since the terms in Eq. (13) with \(R(n|\psi) = 0\) do not contribute, we can assume \(R(n|\psi) \neq 0\) without loss of generality. The conditional probability \(P(\phi|\bar{x}_n, n)\) is zero if there exists a \(|\psi\rangle\) orthogonal to \(|\phi\rangle\) such that \(\bar{x}_n = \bar{f}_n(\psi)\). If the dimension \(D\) of the ontological space is equal to the quantum state manifold dimension \(2N - 2\) and \(\bar{x}_n\) completely identifies the quantum state \(|\psi\rangle\), then the manifold of \(|\phi\rangle\) where \(P(\phi|\bar{x}_n, n) = 0\) is \((2N - 4)\)-dimensional and contains the vectors orthogonal to the one vector \(|\psi\rangle\) satisfying Eq. (11). In the case that one ontological dimension is missed (ontological space with \(2N - 3\) dimensions), one can realize that the manifold where \(P(\phi|\bar{x}_n, n) = 0\) is \((2N - 3)\)-dimensional.

Indeed, the ontic vector \(\bar{x}_n\) identifies a one-dimensional manifold of quantum states and the conditional probability is zero if \(|\phi\rangle\) is orthogonal to one of these states. For a larger number of missed dimensions the manifold with zero probability has the same dimension of the overall manifold of \(|\phi\rangle\) vectors. This means that for an ontological space dimension smaller than \(2N - 3\) the overall probability of obtaining \(|\phi\rangle\) given \(|\psi\rangle\) is zero in a large region of the events and this region has non-zero measure if \(M\) is finite. But this is impossible since the probability of \(|\phi\rangle\) is \(|\langle \phi|\psi \rangle|^2\) and is zero only in a zero measure region of the events. Thus, \(M\) cannot be finite. It is interesting to note that this reasoning does not forbid the shrinking from \(2N - 2\) to \(2N - 3\) for \(M\) finite (the model for a qubit gives a practical example), but it forbids any shrinking in the case of an \(N\)-dimensional Hilbert space with real field. Indeed in this case the dimension of the manifold of \(|\phi\rangle\) orthogonal to \(|\psi\rangle\) is \(N - 2\), where \(N\) is the dimension of the Hilbert space. Thus, it is sufficient an \((N - 2)\)-dimensional space, with only one missed dimension, in order to have \(P(\phi|\bar{x}_n, n) = 0\) in a region of events with non-zero measure. From the above argument it is evident that the model given by Eqs. (5) does not work for any measurement, since the probability distribution contains a finite number of delta functions.

It is important to note that the reported two-dimensional model is not in contrast with the theorem proved in Ref. [8]. That theorem states that the dynamics in a theory with dimensional reduction cannot be Markovian. Indeed, for our model there does not exist a positive conditional probability \(P_U(x, n|\bar{x}, \bar{n})\) for every unitary evolution \(U\) such that

\[
\rho(x, n|\bar{U}\psi) = \sum_{\bar{n}} \int d\bar{x} P_U(x, n|\bar{x}, \bar{n}) \rho(\bar{x}, \bar{n}|\psi).
\]

(15)

Indeed, it is possible to prove that the dynamics of the probability distribution is not described by a linear equation, as required for Markov processes. For this purpose it is sufficient to assume that the Bloch vector \(\bar{v}\), defining the quantum state, is in the patch \(\Omega_t\). This allows us to neglect the additional index labeling the \(M_0\) patches. Thus, the probability distribution is given by Eq. (2) and lives on the space spanned by \(x\) and the binary discrete variable \(n\). Let us consider the Bloch vector rotation
around the $y$-axis

\[
\begin{align*}
\frac{\partial \varphi}{\partial \theta} &= v_z, \\
\frac{\partial \varphi}{\partial \alpha} &= 0, \\
\frac{\partial \varphi}{\partial \theta} &= -v_x,
\end{align*}
\]  

(16)

which correspond in spherical coordinates to

\[
\begin{align*}
\frac{\partial \varphi}{\partial \theta} &= -\cot \theta \sin \varphi, \\
\frac{\partial \varphi}{\partial \alpha} &= \cos \varphi.
\end{align*}
\]  

(17)

If the dynamics was Markovian, then the time evolution of the probability distribution $\rho(x, n, t)$ would be described by the differential equation

\[
\frac{\partial \rho(x, n, t)}{\partial t} = \sum_{\bar{n}} \int d\bar{x} K(x, \bar{n}, \bar{n}) \rho(\bar{x}, \bar{n}, t),
\]  

(18)

$K$ being a suitable kernel.

By means of Eqs. (2), this becomes

\[
\sin \theta \delta_{n,0} \frac{\partial \delta(x-\varphi)}{\partial \varphi} \frac{\partial \varphi}{\partial \theta} + \frac{\partial \delta(x-\varphi)}{\partial \varphi} \frac{\partial \varphi}{\partial \theta} = K(x, n|\varphi, 0) \sin \theta + K(x, n|\theta, 1)(1 - \sin \theta).
\]  

(19)

In particular, for $n = 0$, using Eq. (17), we have that

\[
-\cot \theta \frac{\partial \delta(x-\varphi)}{\partial \varphi} \sin \varphi + \cot \theta \delta(x-\varphi) \cos \varphi = K(x, 0|\varphi, 0) \sin \theta + K(x, 0|\theta, 1)(1 - \sin \theta).
\]  

(20)

Dividing both sides by $\sin \theta$ and differentiating with respect to $\theta$, we obtain that

\[
\frac{\partial}{\partial \theta} \left[ -\cot \theta \frac{\partial \delta(x-\varphi)}{\partial \varphi} \sin \varphi + \cot \theta \delta(x-\varphi) \cos \varphi \right] = \frac{\partial}{\partial \theta} K(x, 0|\theta, 1)(1 - \sin \theta).
\]  

(21)

There is no $K$ satisfying this equation, since the left-hand side is a function of both $\theta$ and $\varphi$, whereas the right-hand side depends only on $\theta$. Thus, the dynamical equation of the probability distribution is non-Markovian.

If we allow the dynamics to be non-causal then we should consider the possibility that in the preparation-measurement processes some information could be transferred back from the measurement apparatus. In other words, in a possible extension of the presented models, the probability distribution could contain a small quantity of information about $|\phi\rangle$. Thus, a more general question is: what is the minimal number of continuous variables that the preparation and measurement apparatuses have to exchange in order to reproduce the quantum probabilities?

IV. CONCLUSION

In conclusion, we have reported a hidden variable model of measurements where a pure quantum state is represented as a statistical mixture of ontic states living in a one-dimensional real space. This model is the first example of ontological shrinking that works for any state preparation and measurement. The Wigner and Husimi distributions are known examples of statistical representations of quantum states on a smaller space. However, none of them is everywhere non-negative or has non-negative conditional probabilities associated with measurements. It is interesting to note that, unlike the Wigner and Husimi functions, the distribution in our model is not quadratic in the quantum state. Indeed, any ontological model of measurement enjoys this property, as proved in Ref. [7]. Our practical example of shrinking may seem artificial, but is important since it raises the interesting question of whether a hidden variable description of a quantum system needs an exponentially growing number of resources. Furthermore, our model shows that the Markov hypothesis is necessary for the proof of the theorem in Ref. [8], where we stated that the ontological space dimension cannot be smaller than the Hilbert space manifold dimension. In a reversed form, this theorem states that any hidden variable theory with an ontological shrinking cannot have a short memory dynamics or, in particular, be causal. Finally, we have discussed possible extensions in $N$ dimensions and found a general property for these models. The possibility of shrinking considerably the ontological space and the introduction of the dynamics in these models are open questions, whose answer could provide a deeper explanation of the exponential complexity of quantum mechanics. Indeed, the knowledge of the resources required for a classical simulation of quantum systems is a very important step for understanding the actual computational speed-up of quantum algorithms.

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[1] W. K. Wootters, W. H. Zurek, Nature 299, 802 (1982).
[2] R. W. Spekkens, Phys. Rev. A 75, 032110 (2007).
[3] P. A. Schilpp, ed., Albert Einstein: Philosopher Scientist (Open Court, 1949); A. Einstein, Ideas and Opinions (New York: Crown Publishing Co., 1954).
[4] R. Jozsa, A. Miyake, Proc. R. Soc. A 464, 3089 (2008).
[5] M. Van den Nest, Quant. Inf. Comp. 10 0258 (2010); M. Van den Nest, arXiv:0911.1624.
[6] L. Hardy, Stud. Hist. Phil. Sci. B 35, 267 (2004).
[7] A. Montina, Phys. Rev. Lett. 97, 180401 (2006); A. Montina, J. Phys.: Conf. Ser. 67, 012050 (2007).
[8] A. Montina, Phys. Rev. A 77, 022104 (2008).
[9] N. Harrigan, T. Rudolph, S. Aaronson, arxiv:quant-ph/0709.1149.
[10] T. Paterek, B. Dakić, and C. Brukner, Phys. Rev. Lett. 101, 190402 (2008).
[11] In Ref. [8] the theorem proof needed a hypothesis of trajectory relaxation. This hypothesis is analyzed in more details in A. Montina, arXiv:1008.4415.
[12] J. A. Wheeler, R. P. Feynman, Rev. Mod. Phys. 17, 157 (1945); ibid. 21, 425 (1949).
[13] K. B. Wharton, Found. Phys. 40, 313 (2010).
[14] S. Kochen and E. P. Specker, J. Math. Mech. 17, 59 (1967).
[15] G. Beltrametti and S. Bugajski, J. Phys. A 28, 3329 (1995).
[16] E. Wigner, Phys. Rev. 40, 749 (1932).
[17] K. Husimi, Proc. Phys. Math. Soc. Jpn. 22, 264 (1940).
[18] The projective Hilbert space can be considered as a manifold, i.e., it is locally homeomorphic to Euclidean space, its topology induced by the Fubini-Study metric. Its dimension is twice the Hilbert space dimension minus two.