Multipliers spaces and pseudo-differential operators

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7th July 2018

Abstract

Let \( \sigma(x, \xi) \) be a sufficiently regular function defined on \( \mathbb{R}^d \times \mathbb{R}^d \). The pseudo-differential operator with symbol \( \sigma \) is defined on the Schwartz class by the formula:

\[
f \rightarrow \sigma f(x) = \int_{\mathbb{R}^d} \sigma(x, \xi) \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi,
\]

where \( \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx \) is the Fourier transform of \( f \).

In this paper, we shall consider the regularity of the following type:

(a) \( |\partial_\xi^\alpha \sigma(x, \xi)| \leq A_\alpha (1 + |\xi|)^{-|\alpha|} \),

(b) \( |\partial_\xi^\alpha \sigma(x+y, \xi) - \partial_\xi^\alpha \sigma(x, \xi)| \leq A_\alpha \omega(|y|) (1 + |\xi|)^{-|\alpha|} \),

where \( \omega \) is suitable positive function and we prove boundedness results for pseudo-differential operators on multipliers spaces \( X^r = \mathcal{M}(H^r \rightarrow L^2) \) whose symbol \( \sigma(x, \xi) \) satisfies the regularity condition on \( x \).

1 Introduction

A pseudo-differential operator \( \sigma \) with symbol \( \sigma(x, \xi) \), defined initially on the Schwartz class of testing functions \( \mathcal{S}(\mathbb{R}^d) \), is given by

\[
f \rightarrow \sigma f(x) = \int_{\mathbb{R}^d} \sigma(x, \xi) \hat{f}(\xi) e^{2\pi i x \cdot \xi} \, d\xi,
\]

(1)
with
\[
\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x \cdot \xi} dx.
\]
being the Fourier transform of \(f\).

We shall consider the standard symbol class, denoted by \(S^m\), which is the most common and useful of the general symbol classes. A function \(\sigma\) belongs to \(S^m\) (and is said to be of order \(m\)) if \(\sigma(x, \xi)\) is a \(C^\infty\) function of \((x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d\) and satisfies the differential inequalities
\[
\left| \partial_\xi^\alpha \partial_\xi^\beta \sigma(x, \xi) \right| \leq A_{\alpha, \beta} (1 + |\xi|)^{m-|\alpha|},
\]
for all multi-indices \(\alpha\) and \(\beta\).

Before we state our result, we need to make precise the definition of the pseudo-differential operator (1) and the class of symbols that is used.

We call the modulus of continuity every function \(\omega: [0, +\infty[ \to [0, +\infty[\) which is continuous, increasing, concave and such that \(\omega(0) = 0\).

**Definition 1.1** Letting \(\omega\) be a modulus of continuity, we denote \(\sigma(x, \xi) \in \mathbb{S}_\omega\), if \(\sigma(x, \xi) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}\) is the continuous function and for all \(\alpha \in \mathbb{N}^d\), there exists a constant \(C_\alpha\) for which we have
\[
\left| \partial_\xi^\alpha \partial_\xi^\beta \sigma(x, \xi) \right| \leq A_{\alpha, \beta} (1 + |\xi|)^{m-|\alpha|},\quad (2)
\]
\[
\left| \partial_\xi^\alpha \sigma(x + y, \xi) - \partial_\xi^\alpha \sigma(x, \xi) \right| \leq A_{\alpha} \omega(|y|) (1 + |\xi|)^{m-|\alpha|}.
\]

**Definition 1.2** Let \(1 < p < \infty\). A measurable function \(f\) is said to belong to the weighted \(L^p\), \(L^p(\mathbb{R}^d, wdx)\), with weight function \(w\), if
\[
\int_{\mathbb{R}^d} |f(x)|^p w(x)dx < \infty.
\]

We denote the weighted \(L^p\) norm by
\[
\|f\|_{L^p_w} = \left( \int_{\mathbb{R}^d} |f(x)|^p w(x)dx \right)^{\frac{1}{p}}.
\]

For \(1 < p < \infty\), a positive weight function \(w\) is said to be in the class \(A_p\) if \(w\) is locally integrable and satisfies the condition
\[
\sup_Q \left( \frac{1}{|Q|} \int_Q w(x)dx \right)^{\frac{1}{p-1}} \left( \frac{1}{|Q|} \int_Q w^{-\frac{1}{p-1}}(x)dx \right)^{p-1} < \infty,
\]
where the supremum is taken over all cubes \(Q\) in \(\mathbb{R}^d\).

Our result is stated as follows :
Theorem 1.3 If the modulus of continuity \( \omega \) satisfies the condition \( j^2 \omega (2^{-j}) < C \), for all \( j \in \mathbb{N} \), then any pseudo-differential operator \( \sigma \) with symbol \( \sigma (x, \xi) \in \sum_{\omega} \) has a bounded extension to all of \( X^r = M (H^r \to L^2) \).

To prove this theorem, we first introduce some notations. Let \( Q \) denote any cube in \( \mathbb{R}^d \) and write \( |Q| \) for the Lebesgue measure of \( Q \). For a locally integrable function \( f \), let \( f_Q \) denote the mean value of \( f \) over \( Q \), that is

\[
f_Q = \frac{1}{|Q|} \int_Q f(x) dx.
\]

We list the several operators we use later:

(a) The Hardy-Littlewood maximal function, \( Mf \), for a locally integrable function \( f \) on \( \mathbb{R}^d \) by

\[
Mf(x) = \sup_Q \frac{1}{|Q|} \int_Q |f(y)| dy,
\]

where the supremum ranges over all cubes \( Q \) containing \( x \).

(b) Modified maximal function of \( f \) :

\[
M_\gamma f(x) = \sup_Q \left( \frac{1}{|Q|} \int_Q |f(y)|^{\gamma} dy \right)^{\frac{1}{\gamma}},
\]

where the supremum is taken over all cubes \( Q \) containing \( x \).

(c) Dyadic maximal function of \( f \) :

\[
f^*(x) = \sup_Q \frac{1}{|Q|} \int_Q |f(y)| dy
\]

where the supremum is taken over all dyadic cubes \( Q \), with sides parallel to the axes containing \( x \).

(d)

\[
f^#(x) = \sup_Q \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,
\]

where the supremum is taken over all cubes \( Q \) containing \( x \).

Lemma 1.4 (M, Lemma 2.1) Let \( w \in A_p \), then \( S (\mathbb{R}^d) \) is dense in \( L^p (R^d, wd\mu) \), \( 1 < p < \infty \).

We shall make use of this fact later.
Lemma 1.5 (CM, Theorem 9) Let $\sigma$ be a pseudo-differential operator with symbol $\sigma(x, \xi) \in \sum_\omega$. Then the following two conditions are equivalent:

$$\sum_{j=0}^{\infty} [\omega (2^{-j})]^2 < +\infty,$$

(5)

For all $p$, $1 < p < \infty$, $\sigma$ is bounded on $L^p(\mathbb{R}^d)$.

(6)

Definition 1.6 (CM, p.41) We say that $\sigma(x, \xi) \in \sum_\omega$ is a reduced symbol if there exist a constant $C_1 > 0$, a function $\phi \in C^\infty_0(\mathbb{R}^d)$ and a sequence $m_j$, $j \geq 0$ of continuous functions on $\mathbb{R}^d$ such that

$$\sigma(x, \xi) = \sum_{j=0}^{\infty} m_j(x) \phi (2^{-j} \xi),$$

(7)

where

$$\|m_j\|_{L^\infty} \leq C_1$$

(8)

$$\|m_j(x + y) - m_j(x)\|_{L^\infty} \leq C_1 \omega(|y|)$$

(9)

and

$$\phi \text{ is supported in } \frac{1}{3} \leq |\xi| \leq 3$$

(10)

and

$$|\partial_\xi^\alpha \phi(\xi)| \leq C_1 \text{ for } |\alpha| \leq d.$$  

(11)

Lemma 1.7 (CM, Proposition 5, p.46) For every symbol $\sigma(x, \xi) \in \sum_\omega$, we can find a sequence of reduced symbols $\sigma_k(x, \xi), k \in \mathbb{Z}^d$, such that

$$\sigma(x, \xi) = \tau(x, \xi) + \sum_{k \in \mathbb{Z}^d} (1 + |k|^2)^{-d} \sigma_k(x, \xi)$$

and

$$|\partial_\xi^\alpha \tau(x, \xi)| \leq C_\alpha, \quad \tau(x, \xi) = 0 \quad \text{if } |\xi| \geq 1.$$  

Lemma 1.8 (St1, p.63) Let $\phi$ be a radial, decreasing, positive and integrable function. Set $\phi_t(x) = t^{-d} \phi(t^{-1}x)$. Then

$$\sup_{t>0} |\phi_t * f(x)| \leq CMf(x) \quad \text{for } f \in \mathcal{S}(\mathbb{R}^d).$$

Lemma 1.9 (M, Lemma 2.9) Let $\phi$ be a function in definition 1.6. Then for $t \geq 0$, there is a constant $C_t$ such that the inequality

$$|y|^t \left| \int_{\mathbb{R}^d} \phi(2^{-j} \xi) e^{2\pi i x \xi} d\xi \right| \leq C_t 2^{j(d-t)}$$

holds for all $y \in \mathbb{R}^d$ and every integer $j \geq 0$. 

4
Lemma 1.10 (M, Lemma 2.7) There is a constant $C > 0$ such that

$$
\|f^*\|_{L^2_w} \leq C \|f\|_{L^2_w} \quad \text{for all } f \in L^2(\mathbb{R}^d, wdx) \cap L^1(\mathbb{R}^d).
$$

2 An $L^2(\mathbb{R}^d, wdx)$ theorem

After these preliminaries, we state the first main result which constitutes the main part of the proof of theorem 1.3.

Theorem 2.1 Suppose $1 < \gamma < \infty$ and let $\sigma$ be a pseudo-differential operator with symbol $\sigma(x, \xi) \in \sum_{\omega}$. If the modulus of continuity $\omega$ satisfies the condition

$$
j^2 \omega(2^{-j}) < C \quad \text{for all } j \geq 0,
$$

then there is a constant $C > 0$ such that the pointwise estimate

$$(\sigma f)^\#(x_0) \leq CM_f(x_0)$$

holds for all $x_0 \in \mathbb{R}^d$ and $f \in S(\mathbb{R}^d)$.

Proof. The proof is based on the idea of the proof of theorem 2.8 in [M].

Given $x_0 \in \mathbb{R}^d$, we let $Q$ be a cube containing $x_0$, with center $x'$ and diameter $D$. Fix a function $\eta \in C^\infty_0(\mathbb{R}^d)$ so that $0 \leq \eta(x) \leq 1$, with $\eta(x) = 1$ for $|x - x'| \leq 2D$, and $\eta(x) = 0$ for $|x - x'| \geq 3D$. Then, for $f \in S(\mathbb{R}^d)$

$$
\frac{1}{|Q|} \int_Q |\sigma f(x) - (\sigma f)_Q| \, dx \leq \frac{2}{|Q|} \int_Q |\sigma (\eta f)(x)| \, dx
$$

$$
+ \frac{1}{|Q|} \int_Q \left| \sigma ((1 - \eta) f)(x) - [\sigma ((1 - \eta) f)]_Q \right| \, dx.
$$

Letting $Q'$ be the cube centered at $x'$, with sides parallel to those of $Q$ and with diameter $4D$. Since the Hardy-Littlewood maximal operator is bounded on $L^\gamma(\mathbb{R}^d)$ for $1 < \gamma < \infty$, we see that the first term is dominated by

$$
\frac{2}{|Q|} \int_Q |\sigma (\eta f)(x)| \, dx \leq 2 \left( \frac{1}{|Q|} \int_Q |\sigma (\eta f)(x)|^\gamma \, dx \right)^{\frac{1}{\gamma}}
$$

$$
\leq C_\gamma \left( \frac{1}{|Q|} \int_{\mathbb{R}^d} |(\eta f)(x)|^\gamma \, dx \right)^{\frac{1}{\gamma}}
$$

$$
\leq C_\gamma \left( \frac{1}{|Q'|} \int_{Q'} |f(x)|^\gamma \, dx \right)^{\frac{1}{\gamma}} \leq C_\gamma M_f(x_0).
$$
To deal with the second term, we for simplicity write \( f \) for \((1 - \eta) f\), and we assume that \( f \) has the support in the set \( \{ x : |x - x'| \geq 2D \} \).

We begin by decomposing the symbol \( \sigma(x, \xi) \) into the sum of simpler symbols by making use of Lemma 1.7. Then we can write

\[
(\sigma f)(x) = \int_{\mathbb{R}^d} \sigma(x, \xi) \hat{f}(\xi) e^{2\pi i x \xi} d\xi
\]

\[
= \int_{\mathbb{R}^d} \tau(x, \xi) \hat{f}(\xi) e^{2\pi i x \xi} d\xi
\]

\[
+ \int_{\mathbb{R}^d} f(y) \int_{\mathbb{R}^d} \sum_{k \in \mathbb{Z}^d} \left(1 + |k|^2\right)^{-d} \sigma_k(x, \xi) e^{2\pi i (x-y) \xi} d\xi dy
\]

\[
= Tf(x) + \sum_{k \in \mathbb{Z}^d} \left(1 + |k|^2\right)^{-d} S_k f(x).
\]

\( T \) is a pseudo-differential operator whose symbol is \( \tau(x, \xi) \); the \( \xi \)-support of this symbol is contained in the set \( \{ \xi : |\xi| \leq 1 \} \), and \( \tau(x, \xi) \) has the property that

\[
|\partial_\xi^\alpha \tau(x, \xi)| \leq C_{\alpha}, \quad \alpha \in \mathbb{N}^d.
\]

Thus, we can write

\[
T f(x) = \int_{\mathbb{R}^d} f(y) K(x, x - y) dy,
\]

where we have set

\[
K(x, y) = K(x, x - y) = \int_{\mathbb{R}^d} \tau(x, \xi) e^{2\pi i x \xi} d\xi.
\]

Note that \( K(x, y) \) has the property that

\[
|K(x, y)| \leq C_m (1 + |y|)^{-m} \quad \text{for all} \ x \in \mathbb{R}^d,
\]

where \( m \) is any integer greater than \( d \), and \( C_m \) is a constant independent of \( x \).

In fact more generally we have

\[
|y^\alpha \partial_\xi^\beta K(x, y)| = A_{\alpha, \beta} \int_{\mathbb{R}^d} \tau(x, \xi) \xi^\beta \partial_\xi^\alpha e^{2\pi i y \xi} d\xi
\]

\[
\leq A_{\alpha, \beta} \int_{\mathbb{R}^d} |\partial_\xi^\beta [\tau(x, \xi) \xi^\beta]| d\xi
\]

\[
\leq A_{\alpha, \beta}.
\]
with $A_{\alpha, \beta}$ independent of $x$ and $y$. Then by lemma 1.8, we have

$$|Tf(x)| \leq \int_{\mathbb{R}^d} |f(y)| |K(x, x-y)| dy$$

$$\leq C_m \int_{\mathbb{R}^d} |f(y)| (1 + |x-y|)^{-m} dy$$

$$\leq C_m Mf(x).$$

Thus, we have

$$(Tf)^\#(x_0) \leq C_m M\gamma f(x_0),$$

and hence

$$(\sigma f)^\#(x_0) \leq C M\gamma f(x_0) + \sum_{k \in \mathbb{Z}^d} \left(1 + |k|^2\right)^{-d} (S_k f)^\#(x_0).$$

Therefore, our next task is to examine the operator $S_k$. We note that $\sigma_k(x, \xi)$ satisfies the condition (7) to (11) in definition 1.6 with $m_{j,k}, \phi_k$ in place of $m_j, \phi$ respectively, where $C'$s are independent of $k$ and also of $j$. Then for every $k$,

$$S_k f(x) = \int_{\mathbb{R}^d} f(y) \sum_{j=0}^{\infty} m_{j,k}(x) \phi_k (2^{-j} \xi) e^{2\pi i (x-y) \xi} dy$$

$$= \sum_{j=0}^{\infty} A_{j,k} f(x).$$

We now estimate $(S_k f)^\#(x_0)$.

$$\frac{1}{|Q|} \int_{Q} \left| A_{j,k} f(x) - (A_{j,k} f)_{Q} \right| dx = \frac{1}{|Q|} \int_{Q} \left| \int_{Q} A_{j,k} f(x) - (A_{j,k} f)(z) dz \right| dx$$

$$= \frac{1}{|Q|} \int_{Q} \int_{\mathbb{R}^d} \left| f(y) \int_{\mathbb{R}^d} \phi_k (2^{-j} \xi) \left[ m_{j,k}(x) e^{2\pi i (x-y) \xi} - m_{j,k}(z) e^{2\pi i (z-y) \xi} \right] d\xi dy dz \right| dx. \quad (13)$$

To estimate this quantity, we consider two cases:
Case 1. $2^j D \geq 1$. The last quantity is dominated by

$$2 \sum_{h=1}^{\infty} \frac{1}{|Q_h|} \int_{Q_{2^n}} \int_{|y-x'| \leq 2^{n+1} D} |f(y)| \left| \int_{\mathbb{R}^d} \phi_k \left( 2^{-j} \xi \right) m_{j,k}(x)e^{2\pi i(y-x')\xi} d\xi \right| dydx$$

$$\leq C \sum_{h=1}^{\infty} \frac{2^{hd}}{|Q_h|} \int_{Q_{2^n}} \int_{|y-x'| \leq 2^{n+1} D} |f(y)| |x-y|^{d+1}.$$ 

$$\cdot \int_{\mathbb{R}^d} \phi_k \left( 2^{-j} \xi \right) e^{2\pi i(y-x')\xi} d\xi \left| m_{j,k}(x) \right| dydx,$$

where $Q_h$ is the cube with center $x'$ with sides parallel to those of $Q$ and with diameter $2^{h+2} D$. The last term is bounded by

$$C \sum_{h=1}^{\infty} D^{d+2hd} (2^h D)^{-d-1} 2^{-j} \frac{1}{|Q_h|} \int_{Q_h} |f(y)| dy$$

by lemma 1.9 with $t = d+1$, and the condition (3) of $m_{j,k}$

$$C \sum_{h=1}^{\infty} D^{d+2hd} (2^h D)^{-d-1} 2^{-j} \frac{1}{|Q_h|} \int_{Q_h} |f(y)| dy \leq C (2^j D)^{-1} M f(x_0).$$

Case 2. $2^j D < 1$. In this case, (13) is dominated by

$$\frac{1}{|Q|} \int_{|Q|} \frac{1}{|Q|} \sum_{h=2^h D \leq |y-x'| \leq 2^{h+1} D} \int |f(y)| \cdot$$

$$\cdot \int_{\mathbb{R}^d} \phi_k \left( 2^{-j} \xi \right) \left[ m_{j,k}(x)e^{2\pi i(x-y)\xi} - m_{j,k}(z)e^{2\pi i(z-y)\xi} + m_{j,k}(z)e^{2\pi i(z-y)\xi} - m_{j,k}(z)e^{2\pi i(z-y)\xi} \right] d\xi \left| dydzdx \right.$$ 

$$\leq \frac{1}{|Q|} \int_{|Q|} \frac{1}{|Q|} \sum_{h=2^h D \leq |y-x'| \leq 2^{h+1} D} \int |f(y)| \cdot$$

$$\cdot \int_{\mathbb{R}^d} \phi_k \left( 2^{-j} \xi \right) \left[ e^{2\pi i(x-y)\xi} - e^{2\pi i(z-y)\xi} \right] m_{j,k}(x) d\xi \left| dydzdx \right.$$ 

$$+ \frac{1}{|Q|} \int_{|Q|} \frac{1}{|Q|} \sum_{h=2^h D \leq |y-x'| \leq 2^{h+1} D} \int |f(y)| \cdot$$

$$\cdot \int_{\mathbb{R}^d} \phi_k \left( 2^{-j} \xi \right) e^{2\pi i(z-y)\xi} \left[ m_{j,k}(x) - m_{j,k}(z) \right] d\xi \left| dydzdx \right.$$ 

$$= A + B.$$
We first estimate $A$.

$$A \leq \frac{1}{|Q|} \left[ \frac{1}{|Q|} \int_{h=2^k D \leq |y-x'| \leq 2^{h+1} D} \left( \sum_{p=1}^d (x_p - z_p) \right) \left| \int_0^1 2\pi i \xi_p e^{2\pi i (x(t) - y) \xi} d\xi \right| dydz \right] m_{j,k}(x) \, dx.$$  

where $x(t) = z + t(x - z)$

$$\leq \frac{1}{|Q|} \left[ \frac{1}{|Q|} \int_{h=2^k D \leq |y-x'| \leq 2^{h+1} D} \int_{|y-x'| \leq 2^{h+1} D} \left| \int_0^1 |f(y)| d\xi \right| \left| \sum_{p=1}^d |x_p - z_p| \right| dx \right] dtdydz \, m_{j,k}(x) \, dx.$$  

The integral with respect to $\xi$ is handled just as in the proof of Lemma 1.9 with $t = d + \frac{1}{2}$ and we see that the last member is not greater than

$$C \sum_{h=1}^{\infty} (2^h D)^d (2^h D)^{-d-1/2} D^{2\frac{d}{2}} \frac{1}{|Q_h|} \int_{Q_h} |f(y)| dy$$

$$\leq C \left( 2^d D \right)^{\frac{d}{2}} Mf(x_0).$$

Next, we estimate $B$.

$$B \leq \frac{1}{|Q|} \left[ \frac{1}{|Q|} \int_{h=2^k D \leq |y-x'| \leq 2^{h+1} D} \left( \sum_{p=1}^d (x_p - z_p) \right) \left| \int_0^1 2\pi i \xi_p e^{2\pi i (x(t) - y) \xi} d\xi \right| dydz \right] m_{j,k}(x) \, dx$$

$$\leq \frac{1}{|Q|} \left[ \frac{1}{|Q|} \int_{h=2^k D \leq |y-x'| \leq 2^{h+1} D} \int_{|y-x'| \leq 2^{h+1} D} \left| \int_0^1 |f(y)| d\xi \right| \left| \sum_{p=1}^d |x_p - z_p| \right| dx \right] dzdx \, m_{j,k}(x) \, dx$$

$$+ \frac{1}{|Q|} \left[ \frac{1}{|Q|} \int_{h=N+1 D \leq |y-x'| \leq 2^{h+1} D} \int_{|y-x'| \leq 2^{h+1} D} \left| \int_0^1 |f(y)| d\xi \right| \left| \sum_{p=1}^d |x_p - z_p| \right| dx \right] dzdx = B_1 + B_2.$$
where $N$ is the integer which satisfies $2^N D < 1 \leq 2^{N+1} D$.

\[
B_1 = \frac{1}{|Q|} \int_{Q} \frac{1}{|Q|} \int_{Q} \sum_{h=1}^{N} \int_{h^{2h} D \leq |y - x'| \leq 2^{h+1} D} \frac{|f(y)|}{|z - y|^d} |z - y|^d.
\]

\[
\cdot \int_{\mathbb{R}^d} \phi_k (2^{-j} \xi) e^{i \pi i (z - y \xi)} d\xi \left|dy \left[ m_{j,k}(x) - m_{j,k}(z) \right] \right| dz dx
\]

\[
\leq C' \sum_{h=1}^{N} (2^h D)^d (2^h D)^{-d} \omega(D) \frac{1}{|Q_h|} \int_{Q_h} |f(y)| dy
\]

by lemma 1.9 with $t = d$ and the condition (14) of $m_{j,k}$

\[
B_1 \leq C N \omega(2^{-N}) M f(x_0).
\]

\[
B_2 = \frac{1}{|Q|} \int_{Q} \frac{1}{|Q|} \int_{Q} \sum_{h=N+1}^{\infty} \int_{h^{2h} D \leq |y - x'| \leq 2^{h+1} D} \frac{|f(y)|}{|z - y|^d} |z - y|^d.
\]

\[
\cdot \int_{\mathbb{R}^d} \phi_k (2^{-j} \xi) e^{i \pi i (z - y \xi)} d\xi \left|dy \left[ m_{j,k}(x) - m_{j,k}(z) \right] \right| dz dx
\]

\[
\leq C' \sum_{h=N+1}^{\infty} (2^h D)^d (2^h D)^{-d} \omega(D) 2^{-j} \omega(1) \frac{1}{|Q_h|} \int_{Q_h} |f(y)| dy
\]

by lemma 1.9 with $t = d + 1$ and the condition (14) of $m_{j,k}$

\[
B_2 \leq C' \sum_{h=N+1}^{\infty} (2^h D)^{-1} 2^{-j} \omega(1) M f(x_0)
\]

\[
\leq C' 2^{-j} M f(x_0).
\]

Thus we have

\[
B \leq B_1 + B_2 \leq C N \omega(2^{-N}) M f(x_0) + C' 2^{-j} M f(x_0)
\]

\[
\leq \{ C N \omega(2^{-N}) + C' 2^{-j} \} M f(x_0).
\]

Putting two cases together, we have shown that if $Q$ is any cube containing $x_0$, then

\[
(S_k f)(x_0) \leq \sum_{j=0}^{\infty} (A_{j,k} f)(x)
\]

\[
\leq \left\{ C \sum_{2^j D \geq 1} (2^j D)^{-1} + C' \sum_{2^j D < 1} (2^j D)^{+} + \sum_{2^j D < 1} (C' N \omega(2^{-N}) + C' 2^{-j}) \right\} M f(x_0)
\]

\[
\leq \{ C + N^2 \omega(2^{-N}) \} M f(x_0) \leq CM f(x_0) \leq CM_2 f(x_0).
\]
We thus find that
\[(\sigma f)^\#(x_0) \leq CM_\gamma f(x_0) + \sum_{k \in \mathbb{Z}^d} \left(1 + |k|^2\right)^{-d} CM_\gamma f(x_0)\]
\[\leq CM_\gamma f(x_0).
\]

Summarizing, we have shown that if \(Q\) is any cube containing \(x_0\), then
\[\frac{1}{|Q|} \int_Q |\sigma f(x) - (\sigma f)_Q| \, dx \]
\[\leq (\sigma f)^\#(x_0) + [T((1 - \eta) f)]^\#(x_0) + \sum_{k \in \mathbb{Z}^d} \left(1 + |k|^2\right)^{-d} [S_k((1 - \eta) f)]^\#(x_0)\]
\[\leq CM_\gamma f(x_0) + CM_\gamma ((1 - \eta) f)(x_0)\]
\[\leq CM_\gamma f(x_0),\]

where the constant \(C\) is independent of \(Q, f\) and \(x_0\). When we take the supremum of the left side over all cubes \(Q\) containing \(x_0\), we finally obtain the desired inequality:
\[(\sigma f)^\#(x_0) \leq CM_\gamma f(x_0)\]
for all \(x_0 \in \mathbb{R}^d, f \in S(\mathbb{R}^d)\).

We are ready to prove a basic result about pseudo-differential operators.

**Theorem 2.2** If \(w \in A_2(\mathbb{R}^d)\) and if the modulus of continuity \(\omega\) satisfies the condition
\[j^2 \omega(2^{-j}) < C, \text{ for all } j \in \mathbb{N},\]
then any pseudo-differential operator \(\sigma\) with symbol \(\sigma(x, \xi) \in \sum\omega\), initially defined on \(S\), extends to a bounded operator from \(L^2(\mathbb{R}^d, wdx)\) to itself.

To prove the theorem, it suffices to show that
\[\|\sigma f\|_{L^2_w} \leq C \|f\|_{L^2_w}, \quad \text{whenever } f \in S,\]
with \(C\) independent of \(f\).

**Proof.** We prove this in the same way as was used by [M, theorem 2.12]. If \(f \in S(\mathbb{R}^d)\), then since \(\sigma f \in L^2(\mathbb{R}^d, wdx) \cap L^1(\mathbb{R}^d)\)
\[\|\sigma f\|_{L^2_w} \leq \|(\sigma f)^*\|_{L^2_{\omega}} \leq C \|(\sigma f)^\#\|_{L^2_{\omega}} \]
\[\leq C \|M_\gamma f\|_{L^2_w}, \quad \text{if } 1 < \gamma < \infty\]
\[\leq C \|f\|_{L^2_w}, \quad \text{if } 1 < \gamma < 2.\]

The first inequality is easy, since
\[|\sigma f(x)| \leq (\sigma f)^*(x) \quad \text{for every } x.\]
Since $\sigma f \in S$, $\sigma f \in L^2(\mathbb{R}^d, wdx) \cap L^1(\mathbb{R}^d)$; so we can apply Lemma 1.10 to prove the second inequality. The third inequality is Theorem 2.1, while the last inequality is proved like this:

$$
\| M_\gamma f \|_{L^2_w} = \left\| (1 - \Delta)^{\gamma/2} u \right\|_{L^2_w} \leq C \left( \int |f|^2 w(x) dx \right)^{1/2} \quad \text{since } \gamma < 2
$$

Because of Lemma 1.4, we can extend $\sigma$ to a bounded operator on $L^2(\mathbb{R}^d, wdx)$.

3 Pointwise multipliers $X^r$

In this section, we give a description of the multiplier space $X^r$ introduced recently by P.G. Lemarié-Rieusset in his work [Lem]. The space $X^r$ of pointwise multipliers which map $L^2$ into $H^{-r}$ is defined in the following way:

**Definition 3.1** For $0 \leq r < \frac{d}{2}$, we define the space $X^r(\mathbb{R}^d)$ as the space of functions, which are locally square integrable on $\mathbb{R}^d$ and such that pointwise multiplication with these functions maps boundedly $H^r(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$, i.e.,

$$
X^r = \{ f \in L^2_{loc} : \forall g \in H^r \ f g \in L^2 \}
$$

where we denote by $H^r(\mathbb{R}^d)$ the completion of the space $D(\mathbb{R}^d)$ with respect to the norm $\| u \|_{H^r} = \left\| (1 - \Delta)^{\gamma/2} u \right\|_{L^2}$.

The norm of $X^r$ is given by the operator norm of pointwise multiplication:

$$
\| f \|_{X^r} = \sup_{\| g \|_{H^r} \leq 1} \| f g \|_{L^2}
$$

We now turn to another way of introducing capacity.

**Definition 3.2** (Capacitary measures and capacitary potentials) The Bessel capacity $\text{cap} (e; H^r)$ of a compact set $e \subset \mathbb{R}^d$ is defined by [AH]

$$
\text{cap} (e; H^r) = \inf \left\{ \| u \|_{H^r}^2 : u \in C^\infty_0 (\mathbb{R}^d), u \geq 1 \text{ sur } e \right\}
$$

We shall show the following theorem.

**Theorem 3.3** There is a positive constant $C$ depending only on $d$ such that

$$
\int_0^\infty \text{cap} (A_t, H^r) d (t^2) \leq C \| u \|_{H^r}^2
$$

where

$$
A_t = \{ x \in \mathbb{R}^d : u(x) \geq t \} \quad \text{et } u = G_r * f \in C^\infty_0 (\mathbb{R}^d)
$$

for any nonnegative measurable function $f$. 
This theorem was established first by Hansson [Han]. Later Maz’ya ([Maz], th.8.2.3) and Adams ([Ad], th 1.6). Maz’ya and Adams used the joint measurability of \( G_r * \mu_t \) on \( \mathbb{R}^d \) where \( \mu_t \) is the capacitary measure for the set \( \{ x \in \mathbb{R}^d : u(x) \geq t \} \). However, the measurability does not seem to be obvious. We shall give an elementary proof which gets around this difficulty.

An easy corollary to theorem [3.3] we obtain the following characterization of Carleson types measures.

**Corollary 3.4** For a nonnegative measure \( \mu \), the following assertions are equivalent:

1. For any \( f \in L^2_+ (\mathbb{R}^d) \), we have
   \[
   \int (G_r * f)^2 \, d\mu \leq C_1 \int f^2 \, dx
   \]

2. For any compact set \( e \subset \mathbb{R}^d \), we have
   \[
   \mu(e) \leq C_2 \text{cap}(e)
   \]

Moreover, we have the following characterization:

\[
\int u^2 \, d\mu \leq C \| u \|_{H^r}^2 \iff \mu(e) \leq C_2 \text{cap}(e)
\]

**Remark 1** Let \( \mu \) nonnegative measure. The inequality

\[
\int_{\mathbb{R}^d} u^2 \, d\mu \leq C \| u \|_{H^r}^2
\]

for \( u \in C_0^\infty (\mathbb{R}^d) \) is called the trace inequality.

Before to prove this theorem [3.3], we prove the corollary [3.4] (and thus the theorem [3.3] gives the characterization of the multipliers spaces.

**Proof.** (1)\( \Rightarrow \) (2). This part can be proved without the capacity inequality.

Let \( e \) be a compact set. Take \( f \in L^2_+ (\mathbb{R}^d) \) such that \( G_r * f \geq 1 \) on \( e \). Then

\[
\mu(e) \leq \int (G_r * f)^2 \, d\mu \leq C_1 \int f^2 \, dx
\]

Taking the infinimum with respect to \( f \), we obtain

\[
\mu(e) \leq C_1 \text{cap}(e)
\]

(2)\( \Rightarrow \) (1). By the capacitability, we have

\[
\mu(e) \leq C_2 \text{cap}(e)
\]
for every Borel set $e$. Let $f \in L^2_+ (\mathbb{R}^d)$ and apply the above inequality to $A_t$:

$$A_t = \{ x : G_r * f(x) \geq t \}.$$ 

By theorem 3.3, we have

$$\int f^2 dx \geq \frac{1}{c} \int_{0}^{\infty} \text{cap} (A_t, H^r) d(t^2) \geq \frac{1}{c} \int_{0}^{\infty} \mu (A_t) d(t^2)$$

$$= \frac{1}{c} \int (Gr * f)^2 d\mu$$

To proof theorem 3.3, we will need several lemmas.

**Lemma 3.5** If $a_j \geq 0$ for $j \in \mathbb{Z}$, then

$$\left( \sum_{j \in \mathbb{Z}} a_j \right)^2 \leq 2 \sum_{i \in \mathbb{Z}} a_i \left( \sum_{j=i}^{\infty} a_j \right)$$

The proof is immediat.

**Lemma 3.6** Suppose $\mu_j$ are measures function such that

$$G_r * (G_r * \mu_j) \leq 1 \text{ on supp} (\mu_j) \text{ for } j \in \mathbb{Z}$$

Then

$$\int_{\mathbb{R}^d} \sum_{j \in \mathbb{Z}} (2^j G_r * \mu_j)^2 dx \leq c \sum_{j \in \mathbb{Z}} 2^{2j} \| \mu_j \|$$

Proof. Apply the equilibrium potential of $\mu_j$

$$G_r * (G_r * \mu_j) \leq K \text{ on } \mathbb{R}^d$$

to obtain

$$\int_{\mathbb{R}^d} \sum_{j \in \mathbb{Z}} (2^j G_r * \mu_j)^2 dx \leq 2 \int_{\mathbb{R}^d} \sum_{j \in \mathbb{Z}} (2^j G_r * \mu_j) \sum_{j = 1}^{+\infty} (2^j G_r * \mu_j) dx$$

$$= 2 \sum_{j \in \mathbb{Z}} 2^j \sum_{i = -\infty}^{j} 2^i \int_{\mathbb{R}^d} (G_r * G_r * \mu_i) d\mu_j$$

$$\leq 2 \sum_{j \in \mathbb{Z}} 2^j \sum_{i = -\infty}^{j} 2^i M \int_{\mathbb{R}^d} d\mu_j$$

$$= c \sum_{j \in \mathbb{Z}} 2^{2j} \| \mu_j \|. $$
Hence,
\[ \int_{\mathbb{R}^d} \left( \sum_{j \in \mathbb{Z}} (2^j G_r \ast \mu_j) \right)^2 \, dx \leq c \sum_{j \in \mathbb{Z}} 2^{2j} \| \mu_j \|. \]

The lemma follows.

**Lemma 3.7** Let \( f \) be a nonnegative continuous function of compact support and \( e \) a Borel set. Let
\[ A_j = \{ x \in e : u(x) \geq 2^j \} \]
and let \( \mu_j \) be the capacitory measure for \( A_j \), i.e.,

1. \( \text{supp}(\mu_j) \subset \overline{A_j} \),
2. \( G_r \ast (G_r \ast \mu_i) \geq 1 \), p.p. sur \( A_j \)
3. \( G_r \ast (G_r \ast \mu_i) \leq 1 \), sur \( \text{supp}(\mu_j) \)
4. \( \| \mu_j \| = \text{cap}(A_j) \).

Then
\[ \frac{3}{4} \sum_{j \in \mathbb{Z}} 2^{2j} \| \mu_j \| \leq \int_0^\infty \text{cap}(\{ x : u(x) \geq t \} ; H^r) \, d(t^2) \leq 3 \| f \|_{L^2} \left\| \sum_{j \in \mathbb{Z}} 2^j (G_r \ast \mu_j) \right\|_{L^2} \]

Proof. By definition
\[ \int_0^\infty \text{cap}(\{ x : u(x) \geq t \} ; H^r) \, d(t^2) = \sum_{j \in \mathbb{Z}} \int_{2^j}^{2^{j+1}} \text{cap}(\{ x : u(x) \geq t \} ; H^r) \, d(t^2) \]

Or
\[ \left( 2^{2(j+1)} - 2^{2j} \right) \text{cap}(A_{j+1}) \leq \int_{2^j}^{2^{j+1}} \text{cap}(\{ x : u(x) \geq t \} ; H^r) \, d(t^2) \leq \left( 2^{2(j+1)} - 2^{2j} \right) \text{cap}(A_j) \]

The left hand side is equal to
\[ 2^{2(j+1)} \left( 1 - \frac{1}{4} \right) \text{cap}(A_{j+1}) = \frac{3}{4} 2^{2(j+1)} \text{cap}(A_{j+1}) = \frac{3}{4} 2^{2(j+1)} \| \mu_{j+1} \| \]

which implies the first required inequality, i.e.,
\[ \frac{3}{4} \sum_{j \in \mathbb{Z}} 2^{2j+2} \| \mu_{j+1} \| \leq \sum_{j \in \mathbb{Z}} \int_{2^j}^{2^{j+1}} \text{cap}(\{ x : u(x) \geq t \} ; H^r) \, d(t^2) \]
The right hand side is equal to

\[
(2^{2(j+1)} - 2^{2j}) \text{cap}(A_j) = 3 \times 2^{2j} \|\mu_j\| \leq 3 \times 2^{2j} \int_{\mathbb{R}^d} \frac{G_r * f}{2^j} d\mu_j
\]

\[
= 3 \times 2^j \int_{\mathbb{R}^d} f (G_r * \mu_j) \, dx
\]

Adding this, we obtain

\[
\int_0^\infty \text{cap}(\{ x : u(x) \geq t \} ; H^r) \, d \left( t^2 \right) \leq 3 \sum_{j \in \mathbb{Z}} 2^j \int_{\mathbb{R}^d} f (G_r * \mu_j) \, dx
\]

\[
= 3 \int_{\mathbb{R}^d} f \left( \sum_{j \in \mathbb{Z}} 2^j (G_r * \mu_j) \right) \, dx
\]

Hence Hölder’s inequality yields

\[
\int_0^\infty \text{cap}(\{ x : u(x) \geq t \} ; H^r) \, d \left( t^2 \right) \leq 3 \| f \|^2_{L^2} \left\| \sum_{j \in \mathbb{Z}} 2^j (G_r * \mu_j) \right\|_{L^2}
\]

We now are in position to prove the theorem 3.3. For the preceding corollary, we are immediately deduce the main result.

Proof. In view of the monotone convergence theorem, it is sufficient to show that

\[
I = \int_0^\infty \text{cap}(\{ x \in B(0, R) : u(x) \geq t \}) \, d \left( t^2 \right) \leq C \| f \|^2_{L^2}
\]

for $R > 0$, $f \in C_0^\infty (\mathbb{R}^d)$ and $C$ is independent of $R$ and $f$. Since $G_r * f = u$ is bounded, it follows that $\{ x : u(x) \geq t \} = \emptyset$ for large $t$, say $t > T$, so that

\[
I \leq \text{cap}(B(0, R)) T^2 < \infty.
\]

Let $\mu_j$ be the capacitary measure for $A_j$

\[
A_j = \{ x \in B(0, R) : u(x) \geq 2^j \}
\]

By lemma 3.6 with $A = B(0, R)$, we have

\[
I \leq 3 \| f \|^2_{L^2} \left\| \sum_{j \in \mathbb{Z}} 2^j (G_r * \mu_j) \right\|_{L^2}
\]

Then lemma 3.6 yield

\[
\left\| \sum_{j \in \mathbb{Z}} 2^j (G_r * \mu_j) \right\|_{L^2} \leq c \left( \sum_{j \in \mathbb{Z}} 2^{2j} \| \mu_j \| \right)^{1/2} \quad \text{(lemma 3.6)}
\]
and
\[ \frac{3}{4} \sum_{j \in \mathbb{Z}} 2^{2j} \| \mu_j \|_{L^1} \leq I. \]

Then
\[ \left\| \sum_{j \in \mathbb{Z}} 2^j (G_r * \mu_j) \right\|_{L^2} \leq c I^{\frac{1}{2}} \]

Since \( I < \infty \), it follows from (14) that
\[ I \leq c \| f \|_{L^2} I^{\frac{1}{2}} \]
i.e
\[ I \leq c \| f \|_{L^2}^2 \]

Finally,
\[ \int_0^\infty \text{cap} \left( \{ x \in B(0, R) : u(x) \geq t \} \right) d \left( t^2 \right) \leq C \| f \|_{L^2}^2 \]

The theorem is proved.

Remark 2 Under this conditions and the preceding results, let \( \mu = f^2 \), it follows that
\[ \int_{\mathbb{R}^d} u^2 f^2 dx \leq C \| u \|_{H^r}^2 \]
and hence
\[ \int_e f^2 dx \leq C_2 \text{cap}(e, H^r). \]

Then, we define the norm \( \| f \|_{\mathcal{M}(H^r \to L^2)} \) by [MS]

\[ \| f \|_{\mathcal{M}(H^r \to L^2)} \sim \sup_e \left( \frac{\int_e f^2 dx}{\text{cap}(e, H^r)} \right)^{\frac{1}{2}} \]

(15)

We will need the following theorem, which shows that many operators of classical analysis are bounded in the space of multipliers.

**Theorem 3.8** Let \( 0 \leq r < \frac{d}{2} \). Suppose that a function \( h \in L^2_{\text{loc}} \) satisfies
\[ \int_e |h(x)|^2 dx \leq C \text{cap}(e) \]

(16)
for all compact set \( e \) with \( \text{cap}(e) = \text{cap}(e; H^r) \). Suppose that, for all weights \( w \in A_1 \),

\[
\int_{\mathbb{R}^d} |g(x)|^2 w(x) \, dx \leq K \int_{\mathbb{R}^d} |h(x)|^2 w(x) \, dx
\]

(17)

with a constant \( K \) depending only on \( d \) and the constant \( A \) in the Muckenhoupt condition. Then

\[
\int_{e} |g(x)|^2 \, dx \leq C\text{cap}(e)
\]

for all compact sets \( e \) with \( C = C(d, r, K) \).

To show this theorem, we need some facts from the equilibrium potential of a compact set \( e \) of positive capacity \([AH]\). The equilibrium potential of a measure \( \mu \in M^+ \) is defined by

\[
P_e(x) = P_{e, J_r(\mu)}.
\]

**Lemma 3.9 ([AH])** For any compact set \( e \subset \mathbb{R}^d \), there exists a measure \( \mu = \mu_e \) such that

(i) \( \text{supp} \mu \subset e \);

(ii) \( \mu(e) = \text{cap}(e; H^r) \);

(iii) \( ||J_r \mu ||^2_{L^2} = \text{cap}(e; H^r) \);

(iv) \( P_e(x) \geq 1 \) quasi-everywhere on \( \mathbb{R}^d \);

(v) \( P_e(x) \leq K = K(d, r) \) on \( \mathbb{R}^d \);

(vi) \( \text{cap} \{ P_e \geq t \} \leq At^{-1} \text{cap}(e; H^r) \) for all \( t > 0 \) and the constant is independent of \( e \).

The measure \( \mu_e \) associated with \( e \) is called the capacitary (equilibrium) measure of \( e \). We will also need the asymptotics (Voir [AH])

\[
G_{\alpha}(x) \simeq |x|^{\alpha-d} , \text{ if } d \geq 3, \quad |x| \to 0;
\]

\[
G_{\alpha}(x) \simeq |x|^{\frac{\alpha-d}{2}} e^{-|x|} , \text{ if } d \geq 2, \quad |x| \to +\infty;
\]

Sometimes, it will be more convenient to use a modified kernel

\[
\tilde{G}_r(x) = \max(G_r(x), 1)
\]

which does not have the exponential decay at \( \infty \). Obviously, both \( G_r \) and \( \tilde{G}_r \) are positive nonincreasing radial kernels. Moreover, \( \tilde{G}_r \) has the doubling property :

\[
\tilde{G}_r(2s) \leq \tilde{G}_r(s) \leq c(d)\tilde{G}_r(2s)
\]
The corresponding modified potential is defined by
\[ \tilde{P}(x) = \tilde{G}_r \ast \mu(x) \]
The rest of the proof of theorem 3.8 is based on the following proposition:

**Proposition 3.10** Let \( d \geq 2 \) and let \( 0 < \delta < \frac{d}{d-r} \). Then \( \tilde{P} \) lies in the Muckenhoupt class \( A_1 \) on \( \mathbb{R}^d \), i.e.,
\[ M \left( \tilde{P}^\delta (x) \right) \leq C(\delta, d) \tilde{P} (x), \text{ d}x \text{ p.p} \]
where \( M \) denotes the Hardy-Littlewood maximal operator on \( \mathbb{R}^d \), and the corresponding \( A_1 \)-bound \( C(\delta, d) \) depends only on \( d \) and \( \delta \).

**Proof.** Let \( k : \mathbb{R}_+ \to \mathbb{R}_+ \) be a nonincreasing function which satisfies the doubling condition:
\[ k(2s) \leq ck(s), \ s > 0 \]
It is easy to see that the radial weight \( k(|x|) \in A_1 \) if and only if
\[ \int_0^R k^\delta(t)t^{d-1}dt \leq cR^d k(R), \ R > 0 \quad (19) \]
Moreover, the \( A_1 \)-bound of \( k \) is bounded by a constant which depends only on \( C \) in the preceding estimate and the doubling constant \( c \) (see [St2]). It follows from that
\[ \tilde{G}_r(s) \simeq |s|^{r-d} \text{ if } d \geq 3 \text{ for } 0 < s < 1 \]
and
\[ \tilde{G}_r(s) \simeq 1 \text{ for } s \geq 1 \]
Hence, \( k(|s|) = G_r(s) \) is a radial nonincreasing kernel with the doubling property. By Jensen’ inequality, we have
\[ \tilde{G}_r \in A_1 \text{ implies } \tilde{G}_r \in A_1 \text{ if } \delta_1 \geq \delta_2 \]
Clearly \( (19) \) holds if and only if \( 0 < \delta < \frac{d}{d-r} \). Hence, without loss of generality, we assume \( 1 \leq \delta < \frac{d}{d-r} \). Then by Minkowski’s inequality and the \( A_1 \)-estimate for \( G_r \) established above, it follows
\[ M \left( \tilde{P}^\delta (x) \right) \leq M \left( \left( \tilde{G}_r^\delta \ast \mu(x) \right)^\delta \right) \]
\[ \leq C(\delta, d) \tilde{G}_r^\delta \ast \mu (x) \]
\[ = C(\delta, d) \tilde{P} (x) \]
We are now in a position to prove Theorem 3.8.

**Proof.** Suppose \( \nu_e \) is the capacitary measure of \( e \subset \mathbb{R}^d \) and let \( \varphi = P \) is its potential. Then, by Lemma 3.9, we have

(i) \( \varphi(x) \geq 1 \) quasi-everywhere on \( e \);

(ii) \( \varphi(x) \leq B = B(d, r) \) for all \( x \in \mathbb{R}^d \);

(iii) \( \text{cap} \{ \varphi \geq t \} \leq Ct^{-1} \text{cap}(e) \) for all \( t > 0 \) with the constant \( C \) is independent of \( e \).

Now, it follows from a Proposition 3.10 that \( \varphi^\delta \in A_1 \). Hence, by (17),

\[
\int_{\mathbb{R}^d} |g(x)|^2 \varphi^\delta dx \leq K \int_{\mathbb{R}^d} |h(x)|^2 \varphi^\delta dx
\]

Applying this together with (i) and (ii), we get

\[
\int_{e} |g(x)|^2 dx \leq \int_{\mathbb{R}^d} |g(x)|^2 \varphi^\delta dx \leq C \int_{\mathbb{R}^d} |h(x)|^2 \varphi^\delta dx = C \int_{0}^{B} |h(x)|^2 dxt^\delta - 1 dt
\]

By (16) and (iii),

\[
\int_{\varphi \geq t} |h(x)|^2 dx \leq C \text{cap} \{ \varphi \geq t \} \leq \frac{C}{t} \text{cap}(e)
\]

Hence,

\[
\int_{e} |g(x)|^2 dx \leq C \int_{0}^{B} t^{-1} \text{cap}(e) t^\delta - 1 dt = C \text{cap}(e) \int_{0}^{B} t^{\delta - 2} dt
\]

Clearly, for all \( 0 \leq r < \frac{d}{2} \), we can choose \( \delta > 1 \) so that \( 0 < \delta < \frac{d}{\alpha - 2r} \). Then

\[
\int_{0}^{B} t^{\delta - 2} dt = \frac{B^{\delta - 1}}{\delta - 1} < \infty
\]

which concludes

\[
\int_{e} |g(x)|^2 dx \leq C \text{cap}(e).
\]

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