Positive solutions of an initial value problem of a delay-self-reference nonlinear differential equation

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Abstract
In this paper we study the existence of positive solutions for an initial value problem of a delay-state-dependent nonlinear differential equation. The continuous dependence of the unique solution on the initial data and the delay-state-dependent function will be proved. Some especial cases and examples will be given.

Keywords
Delay-state-dependent, nonlinear differential equation, existence of solutions, continuous dependence, Arzela-Ascoli Theorem, Schauder fixed point Theorem.

AMS Subject Classification
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1. Introduction

Many authors studied the differential and integral equations with deviating arguments only the in time itself, however, the case of the deviating arguments depend on both the state variable $x$ and the time $t$ is important in theory and practice, see for example [1]-[4], [7], [10], [11], [13]-[19].

In [4], the author studied the existence of a unique solution $x \in C[0,T]$ and its continuous dependence on the initial data of the initial value problem of the self-refered differential equation

$$\frac{d}{dt}x(t) = f(t,x(x(t))), \quad t \in [0,T] \quad \text{and} \quad x(0) = x_0$$

where $f \in (C[a,b],C[a,b])$.

In [8], the authors studied the existence of solutions $x \in C[0,T]$ and the continuous dependence of the unique solution on the initial data and the function $g$ of the initial value problem of functional integro-differential equation of self-reference ($\phi(t) = t$) and state-dependence ($\phi(t) \leq t$)

$$\frac{d}{dt}x(t) = f(t,x(t)\int_0^{\phi(t)} g(t,x(t))ds), \quad t \in (0,T]$$

and $x(0) = x_0$.

where $g : [0,T] \times R^+ \rightarrow [0,T]$ is continuous and $g(t,x(t)) \leq 1$. The authors in [9] proved, when $g : [0,T] \times R^+ \rightarrow [0,T]$ is continuous and $g(t,x(t)) \leq t$, the existence of positive solutions $x \in C[0,T]$ and the continuous dependence of the unique solution on the initial data and the function $g$ of the initial value problem

$$\frac{d}{dt}x(t) = f(t,x(g(t,x(t)))), \quad a.e., \quad t \in (0,T], \quad (1.1)$$

$$x(0) = x_0 \in [0,T]. \quad (1.2)$$

Let $C[0,T]$ be the class of continuous functions defined on $[0,T]$ with norm

$$||x|| = \sup_{t \in [0,T]} |x(t)|, \quad x \in C[0,T].$$
Let \( g : [0, T] \times \mathbb{R}^+ \rightarrow [0, T] \) be continuous and \( g(t, x(t)) \leq x(t) \). Consider the initial value problem of the delay-state-dependent nonlinear differential equation
\[
\frac{d}{dt}x(t) = f(t, x(g(t, x(t)))), \quad a.e., \quad t \in (0, T) \tag{1.3}
\]
\[
x(0) = x_0 \in [0, T]. \tag{1.4}
\]
Our aim in this work is to prove the existence of positive solutions \( x \in C[0, T] \) of the initial value problem (1.3)-(1.4). The continuous dependence of the unique solution on the initial data \( x_0 \) and the delay-state-dependent function \( g \) will be studied.

### 2. Main Results

In this section, we deal with the existence and uniqueness of solution for the initial value problem (1.3)-(1.4). Also we prove that the solution depends continuously on the initial data and the function \( g \). Now, we consider the following assumptions to establish the existence results:

1. \( f : [0, T] \times [0, T] \rightarrow \mathbb{R} \) satisfies Carathéodory condition i.e. \( f \) is measurable in \( t \) for all \( x \in C[0, T] \) and continuous in \( x \) for almost all \( t \in [0, T] \).
2. There exists a measurable bounded function \( m(t) \) and a constant \( b \geq 0 \) such that \( |f(t, x)| \leq m(t) + b|x| \).
3. \( g : [0, T] \times \mathbb{R}^+ \rightarrow [0, T] \) is continuous such that \( g(t, x(t))) \leq x(t) \).
4. \( L = M + b T < 1 \).
5. \( LT + |x(0)| \leq T \).

**Some examples for the function \( g \)**

1. \( g(t, x(t)) = \frac{x(t)}{1 + e^{ax(t)}} \), \( a \geq 0 \)
2. \( g(t, x(t)) = \frac{x(t)}{1 + e^{bx^2(t)}} \), \( a, b \geq 0 \)
3. \( g(t, x(t)) = \frac{x(t)}{1 + e^{-x(t)}} \), \( a \geq 0 \).

### 2.1 Existence theorem

**Theorem 2.1.** Let the assumptions (1) – (5) be satisfied, then the initial value problem (1.3), (1.4) has at least one solution \( x \in S_L \subset C[0, T] \).

**Proof.** Let \( x \) be a solution of the problem (1)-(2). Integrating the differential equation (1) we obtain the corresponding integral equation
\[
x(t) = x_0 + \int_0^t f(s, x(g(s, x(s)))) \, ds > 0, \quad t \in [0, T]. \tag{2.1}
\]
Define the set \( S_L \) by
\[
S_L = \{ x \in C[0, T] : |x(t_2) - x(t_1)| \leq L|t_2 - t_1| \} \subset C[0, T].
\]
where \( L = M + b T \).

It can be established that \( S_L \) is nonempty, closed, bounded and convex subset of \( C[0, T] \).

Define the operator \( F \) associated with equation (1.3) by
\[
F(x) = x_0 + \int_0^t f(s, x(g(s, x(s)))) \, ds \quad t \in [0, T]
\]
Firstly, we prove that \( F \) is uniformly bounded. Let \( x \in C[0, T] \), then we get
\[
|F(x(t))| \leq |x_0| + \int_0^t |f(s, x(g(s, x(s))))| \, ds \quad t \in [0, T]
\]
\[
\leq |x_0| + \int_0^t \{m(s)+b|x(g(s, x(s)))|\} \, ds
\]
\[
\leq |x_0| + \int_0^t \{M+b|x(g(s, x(s)))|\} \, ds.
\]
But
\[
x(\phi(x(t)))) - |x_0| \leq L|x(g(t,x(t)))-x(0)| \leq L|g(t,x(t))|
\]
then
\[
x(\phi(x(t))) \leq L|g(t,x(t))| + |x_0|.
\] (2.2)

Using (2.2) we obtain
\[
|F(x(t))| \leq |x_0| + \int_0^t \{M+bL|x(s)|+|x_0)|\} \, ds
\]
\[
\leq |x_0| + \int_0^t \{M+b(2L+|x_0)|}\, ds
\]
\[
\leq |x_0| + (M+2bL)|x_0|\, t
\]
\[
\leq |x_0| + (M+2b)|x_0| \leq L|t_2-t_1|.
\]
This proves the class functions \( \{Fx\} \) is equi-continuous. Secondly, we will show that \( F : S_L \rightarrow S_L \) and the class of functions \( \{Fx\} \) is equi-continuous.

Let \( x \in S_L \) and \( t_1, t_2 \in [0, T] \) with \( t_1 < t_2 \) such that \( |t_2-t_1| < \delta \), then
\[
|F(x(t_2)) - F(x(t_1))| = \int_{t_1}^{t_2} f(s, x(g(s, x(s)))) \, ds |
\]
\[
\leq \int_{t_1}^{t_2} |f(s, x(g(s, x(s))))| \, ds
\]
\[
\leq \int_{t_1}^{t_2} \{M+b|x(g(s, x(s)))|\} \, ds
\]
\[
\leq \int_{t_1}^{t_2} \{M+bL|x(s)|+|x_0)|\} \, ds
\]
\[
\leq \int_{t_1}^{t_2} \{M+bL|x(s)|+|x_0)|\} \, ds
\]
\[
\leq \int_{t_1}^{t_2} (M+bT) \, ds \leq L|t_2-t_1|.
\]
This proves that $F : S_L \rightarrow S_L$ and the class of functions $\{Fx\}$ is equi-continuous.

Now by Arzela-Ascoli Theorem [5] $F$ is compact.

Finally, let $\{x_n\} \subset S_L$ such that $x_n \rightarrow x$ on $[0,T]$, then we have

$$
\lim_{n \rightarrow \infty} x_n (g(t, x_n(t))) = x (g(t, x(t)))
$$

which implies that

$$
x_n (g(t, x_n(t))) \rightarrow x (g(t, x(t))) \text{ in } S_L.
$$

Also, from the continuity the function $f$ we obtain

$$
f \left( t, x_n (g(t, x_n(t))) \right) \rightarrow f \left( t, x (g(t, x(t))) \right).
$$

Using assumption (2) and Lebesgues dominated convergence theorem [6] we deduce that

$$
\lim_{n \rightarrow \infty} x_n(t) = x_0 + \lim_{n \rightarrow \infty} \int_0^t f \left( s, x_n (g(s, x_n(s))) \right) ds
$$

$$
= x_0 + \int_0^t f \left( s, x (g(s, x(s))) \right) ds.
$$

Then $F$ is continuous.

Now all conditions of Schauder fixed point Theorem [5], are satisfied, then the operator $F$ has at least one fixed point $x \in S_L$. Consequently there exist at least one solution $x \in C[0, T]$ of the integral equation (2.1).

Now, to complete the proof, differentiating the integral equation (3) we obtain the differential equation (1).

Also letting $t = 0$ in (3) we obtain the initial data (2).

This completes the proof of the equivalence between the initial value problem (1)-(2) and the integral equation (3).

Hence the initial value problem (1)-(2) has at least one positive solution $x \in C[0, T]$ which completes the proof.

2.2 Uniqueness of the solution

In this section we prove the uniqueness of the solution for the integral equation (2.1). For this aim we assume that:

(1') $|f(t,x) - f(t,y)| \leq b \|x-y\|

(2') $|f(t,0)| \leq M,$

(3') $|g(t,x) - g(t,y)| \leq k|x-y|$

Theorem 2.3. Let the assumptions (1),(3),(4) of Theorem 2.1 and (1'), (2') and (3') be satisfied, if $bT^2 L k < 1$, then the solution of equation (2.1) is unique.

Proof. Assumption (2) of Theorem 2.1 can be deduced from assumptions (1') and (2'). By putting $y = 0$ in (1') we get

$$
|f(t,x)| \leq b \|x| + |f(t,0)|
$$

hence we deduce that all assumptions of Theorem 2.1 are satisfied. Then the solution of equation (2.1) exists. Now let $x, y$ be two solutions of (2.1), then

$$
|x(t) - y(t)| = |\int_0^t f(s, x(s), x(s)) ds - \int_0^t f(s, y(s), y(s)) ds|$

$$
\leq \int_0^t |f(s, x(s), x(s)) - f(s, y(s), y(s)))| ds$

$$
\leq b \int_0^t |x(s, x(s)) - y(s, y(s)))| ds$

$$
\leq b \int_0^t |x(s, x(s))) - x(s, y(s)))| ds$

$$
+ b \int_0^t |x(s, y(s))) - y(s, y(s)))| ds$

$$
\leq b L \int_0^t |x(s, y(s))) - y(s, y(s)))| ds$

$$
+ b \int_0^t |x(s, x(s))) - x(s, y(s)))| ds$

$$
\leq b L k \int_0^t |x(s, x(s))) - x(s, y(s)))| ds$

$$
\leq b L k T \|x-y\| + b T \|x-y\|$

$$
= b T (L k + 1) \|x-y\|,
$$

then we obtain

$$
\|x-y\| \leq b T (L k + 1) \|x-y\|.
$$

Since $b T (L k + 1) < 1$, then we deduce that $x(t) = y(t)$ and hence the solution of (2.1) is unique.

Corollary 2.2. Let the assumptions of Theorem 1.3 be satisfied, if $g(t, x(t)) = x(t)$, then the integral equation

$$x(t) = x_0 + \int_0^t f(s, x(s)) ds,
$$

$t \in [0, T],$

has at least one solution $x \in C[0, T]$. Consequently the initial value problem

$$
\frac{dx(t)}{dt} = f(t, x(t)) \text{ a.e. } t \in (0, T]
$$

$$
x(0) = x_0
$$

has at least one solution $x \in AC[0, T]$. 

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2.3 Continuous dependence

Here we prove that the solution of equation (2.1) depends continuously on the initial data $x_0$.

Definition 2.4.

The solution of the integral equation (2.1) depends continuously on the initial data $x_0$ if $\forall \varepsilon > 0 \exists \delta(\varepsilon) > 0$ such that,

$$|x_0 - x^*_0| \leq \delta \Rightarrow ||x - x^*|| \leq \varepsilon$$

(2.3)

where $x^*$ is the unique solution of the equation where $x^*$ is the unique solution of the integral equation

$$x^*(t) = x_0^* + \int_0^t f(s,x^*(g(s,x^*(s))))ds, \quad t \in [0,T].$$

(2.4)

Theorem 2.5. Let the assumptions of Theorem 2.3 be satisfied, then the solution of (2.1) depends continuously on the initial data $x_0$.

Proof. Let $x, x^*$ be the solution of the integral equation (3) and (2.4). Then, we have

$$|x(t) - x^*(t)|$$

$$= |x_0 + \int_0^t f(s,x(g(s,x(s))))ds - x_0^*|$$

$$+ \int_0^t |f(s,x(g(s,x(s))) - f(s,x^*(g(s,x^*(s)))))|ds$$

$$\leq |x_0 - x_0^*| + b \int_0^t |x(g(s,x(s))) - x^*(g(s,x^*(s)))|ds$$

$$\leq |x_0 - x_0^*| + b L \int_0^t |g(s,x(s)) - g(s,x^*(s))|ds$$

$$+ b \int_0^t |x(g(s,x^*(s))) - x^*(g(s,x^*(s)))|ds$$

$$\leq |x_0 - x_0^*| + b L \left[ T + b L \right] ||x-x^*|| T + b \|x-x^*\| T$$

and

$$||x-x^*|| \leq \delta + b T (L k + 1)||x-x^*||,$$

then

$$||x-x^*|| \leq \frac{\delta}{1 - b T (L k + 1)}.$$ 

Since $b T (L k + 1) < 1$ it follows that the solution of (2.1) depends continuously on the initial data $x_0$.

Definition 2.6.

The solution of the integral equation (2.1) depends continuously on the function $g$ if, $\forall \varepsilon > 0 \exists \delta(\varepsilon) > 0$ such that

$$|g(t,x(t)) - g^*(t,x(t))| \leq \delta \Rightarrow ||x - x^*|| \leq \varepsilon$$

(2.5)

where $x^*$ is the unique solution of the integral equation

$$x^*(t) = x_0^* + \int_0^t f(s,x^*(g(s,x^*(s))))ds, \quad t \in [0,T].$$

(2.6)

Theorem 2.7. Let the assumptions of Theorem 2.3 be satisfied, then the solution of (2.1) depends continuously on the function $g$.

Proof. Let $x, x^*$ be the solution of the integral equation (3) and (2.6). Then, we have Let $x, x^*$ be the solution of the integral equation (3) and (2.4). Then, we have

$$|x(t) - x^*(t)|$$

$$= |x_0 + \int_0^t f(s,x(g(s,x(s))))ds - x_0^*|$$

$$\leq |x_0 - x_0^*| + b \int_0^t |x(g(s,x(s))) - x^*(g(s,x^*(s)))|ds$$

$$\leq |x_0 - x_0^*| + b L \int_0^t |g(s,x(s)) - g(s,x^*(s))|ds$$

$$+ b \int_0^t |x^*(g(s,x^*(s))) - x^*(g(s,x^*(s)))|ds$$

$$\leq |x_0 - x_0^*| + b L \left[ T + b L \right] ||x-x^*|| T + b \|x-x^*\| T$$

and

$$||x-x^*|| \leq \delta + b T (L k + 1)||x-x^*||,$$

then

$$||x-x^*|| \leq \frac{\delta}{1 - b T (L k + 1)}.$$ 

Since $b T (L k + 1) < 1$ it follows that the solution of (2.1) depends continuously on the function $g$. 

$$\Box$$
Therefore, by applying to Theorem 2.1, the initial value problem (3.1)-(3.2) has a continuous solution.

3. Example

Example 3.1. Consider the nonlinear differential equation
\[
\frac{dx}{dt} = \frac{1}{5}(1+t) + \frac{1}{7} x \left( \frac{x(t) e^{-x^2(t)}}{1 + \sin^2 x(t)} \right), \quad t \in (0,2],
\]
with the initial condition
\[
x(0) = \frac{1}{5},
\]
Set
\[
f(t,x(\phi(x(t)))) = \frac{1}{5}(1+t) + \frac{1}{7} x \left( \frac{x(t) e^{-x^2(t)}}{1 + \sin^2 x(t)} \right)
\]
thus
\[
|f(t,x)| \leq \frac{1}{5}(1+t) + \frac{1}{7} |x|
\]
here we have \(m(t) = \frac{1}{5}(1+t)\) which is measurable and bounded function with bound \(M = 3/5\) and \(b = 1/7\), \(x(0) = \frac{1}{5}\), then \(L_2 = M + b T = 31/35 < 1\) and \(LT + |x(0)| \approx 1.97 < T = 2\).

Therefore, by applying to Theorem 2.1, the initial value problem (3.1)-(3.2) has a continuous solution.

Example 3.2. Consider the nonlinear differential equation
\[
\frac{dx}{dt} = \frac{1}{9} t^3 \sin(r^2) + \frac{1}{4} x \left( \frac{x(t) e^{-x^2(t)}}{1 + x^2(t)} \right), \quad t \in (0,1],
\]
with the initial condition
\[
x(0) = \frac{1}{2},
\]
Set
\[
f(t,x(\phi(x(t)))) = \frac{1}{9} t^3 \sin(r^2) + \frac{1}{4} x \left( \frac{x(t) e^{-x^2(t)}}{1 + x^2(t)} \right)
\]
thus
\[
|f(t,x)| \leq \frac{1}{9} t^3 \sin(r^2) + \frac{1}{4} |x|
\]
here we have \(m(t) = \frac{1}{9} t^3 \sin(r^2)\) which is measurable and bounded function with bound \(M = 1/9\) and \(b = 1/4\), \(x(0) = \frac{1}{2}\), then \(L_2 = M + b T = 13/36 < 1\) and \(LT + |x(0)| = 31/36 < T = 1\).

Therefore, by applying to Theorem 2.1, the initial value problem (3.3)-(3.4) has a continuous solution.

4. Conclusion

In this paper, we prove the existence, the uniqueness and the continuous dependence of positive continuous solution \(x \in C[0,T]\) of an initial value problem of a delay-self-reference nonlinear differential equation under a suitable assumptions. Here we relax the assumptions and generalize the results in [4,8], also we introduced some examples and applications to indicate our results.

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