CONFORMAL NETS III: FUSION OF DEFECTS

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Abstract. Conformal nets provides a mathematical model for conformal field theory. We define a notion of defect between conformal nets, formalizing the idea of an interaction between two conformal field theories. We introduce an operation of fusion of defects, and prove that the fusion of two defects is again a defect, provided the fusion occurs over a conformal net of finite index. There is a notion of sector (or bimodule) between two defects, and operations of horizontal and vertical fusion of such sectors. Our most difficult technical result is that the horizontal fusion of the vacuum sectors of two defects is isomorphic to the vacuum sector of the fused defect. Equipped with this isomorphism, we construct the basic interchange isomorphism between the horizontal fusion of two vertical fusions and the vertical fusion of two horizontal fusions of sectors.

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There are various different mathematical notions of field theories. For many of these there is also a notion of defects that formalizes interactions between different field theories. See for example [9, 13, 21, 25] and references therein. Depending on the context, sometimes the terminology ‘surface operator’ or ‘domain wall’ is used in place of ‘defect’. Often field theories are described as functors from a bordism category whose objects are \((d-1)\)-manifolds and morphisms are \(d\)-dimensional bordisms (usually with additional geometric structure) to a category of vector spaces. Defects allow the extension of such functors to a larger bordism category, where the manifolds may be equipped with codimension-1 submanifolds that split the manifolds into regions labeled by field theories. The codimension-1 submanifold itself is labeled by a defect between the field theories labeling the neighboring regions.

In this paper we give a definition of defects for conformal nets. Conformal nets are often viewed as a particular model for conformal field theory. Our main result is that under suitable finiteness assumptions there is a composition for defects that we call fusion. We also extend the notion of representations of conformal nets, also known as sectors, to the context of defects. Sectors between defects are a simultaneous generalization of the notion of representations of conformal nets, and of bimodules between von Neumann algebras. Ultimately, this will lead to a 3-category whose objects are conformal nets, whose 1-morphisms are defects, whose 2-morphisms are sectors, and whose 3-morphisms are intertwiners between sectors. The lengthy construction of this 3-category in all formal details is postponed to [4], but the key ingredients of this 3-category will all be presented here. In [4] we will use the language of internal bicategories developed in [8], but we expect that the results of the present paper also provide all the essential ingredients to construct a 3-category of conformal nets, defects, sectors, and intertwiners in any other sufficiently weak model of 3-categories.

Conformal nets. Conformal nets grew out of algebraic quantum field theory and have been intensively studied; see for example [6, 10, 15, 29, 30]. In this paper we will use our (non-standard) coordinate-free definition of conformal nets [2]. A conformal net in this sense is a functor
\[ \mathcal{A} : \text{INT} \to \text{VN} \]
from the category of compact oriented intervals to the category of von Neumann algebras, subject to a number of axioms. The precise definition and properties of conformal nets are recalled in Appendix B. In contrast to the standard definition, in our coordinate-free definition there is no need to fix a vacuum Hilbert space at the outset—this feature will be useful in developing our definition of defects. Nevertheless, the vacuum Hilbert space can be reconstructed from the functor \(\mathcal{A}\). The main ingredient for this reconstruction is Haagerup’s standard form \(L^2(A)\), a bimodule that is canonically associated to any von Neumann algebra \(A\). This standard form and various facts about von Neumann algebras that are used throughout this paper are reviewed in Appendix A.

Defects. To define defects we introduce the category \(\text{INT}_{\bullet\circ}\) of bicolored intervals. Its objects are intervals \(I\) that are equipped with a covering by two subintervals \(I_\circ\) and \(I_\bullet\). If \(I\) is not completely white (\(I_\circ = I, I_\bullet = \emptyset\)) or black (\(I_\bullet = I, I_\circ = \emptyset\)) then we require that the white and the black subintervals meet in exactly one point and

1Note, however, that the precise relation of conformal nets to conformal field theory in the sense of Segal [26] is not at present clear.
we also require the choice of a local coordinate around this point. For conformal nets $\mathcal{A}$ and $\mathcal{B}$, a defect between them is a functor

$$D : \text{INT}_{\bullet} \to \text{VN}$$

such that $D$ coincides with $\mathcal{A}$ on white intervals and with $\mathcal{B}$ on black intervals and satisfies various axioms similar to those of conformal nets. Often we write $\mathcal{A} D \mathcal{B}$ to indicate that $D$ is a defect from $\mathcal{A}$ to $\mathcal{B}$, also called an $\mathcal{A}$-$\mathcal{B}$-defect. A defect from the trivial net to itself is simply a von Neumann algebra (Proposition 1.23), so our notion of defect is a generalization of the notion of von Neumann algebra. The precise definition and some basic properties of defects are given in Section 1.

Certain defects have already appeared in disguise in the conformal nets literature, through the notion of ‘solitons’ [5, 14, 18, 19]. For a conformal net $\mathcal{A}$ defined on subintervals $I$ of the real line (half-infinite intervals allowed), an endomorphism of the C$^*$-algebra $A_R := \text{colim}_{[a,b]} A([a,b])$ is called a soliton if it is localized in a half-line, that is, if it acts as the identity on elements in the image of the complementary half-line. We bicolor any subinterval $I = [a,\infty) \cup (-\infty,b]$ of the projective line by $I_\circ = [a,\infty)$ and $I_\bullet = (-\infty,b]$. Give a soliton $\sigma$, we can consider the von Neumann algebra $\mathcal{D}(I) := \sigma A([a,\infty]) \vee \sigma A([\infty,b])$ generated by $\sigma A([a,\infty])$ and $\sigma A([\infty,b])$ acting on the vacuum sector. We believe that, under certain conditions, this construction associates an $\mathcal{A}$-$\mathcal{A}$-defect to a soliton. The exact relationship between solitons and defects is, however, not yet clear.

**Sectors.** We will use the boundary of the square $S^1 := \partial [0,1]^2$ as our standard model for the circle. (The unit speed parametrization gives this circle a canonical smooth structure.) We equip the circle with the bicoloring defined by $S^1_\circ = S^1 \cap [0,\frac{1}{2}] \times [0,1]$ and $S^1_\bullet = S^1 \cap [\frac{1}{2},1] \times [0,1]$. Let $D$ and $E$ be $\mathcal{A}$-$\mathcal{B}$-defects. A $D$-$E$-sector is a Hilbert space equipped with compatible actions of the algebras $\mathcal{D}(I)$, for subintervals $I \subset S^1$ with $\left(\frac{1}{2},0\right) \notin I$, and $E(I)$, for subintervals $I \subset S^1$ with $\left(\frac{1}{2},1\right) \notin I$. Pictorially we draw a $D$-$E$-sector as follows:

$$\begin{array}{c}
\mathcal{A} & \xrightarrow{H} & \mathcal{B} \\
\mathcal{E} & \xleftarrow{D} & \end{array}$$

The thin line — should be thought of as white and stands for the conformal net $\mathcal{A}$ and the thick line — should be thought of as black and stands for $\mathcal{B}$. Usually we simplify the picture further by dropping the letters. The precise definition and some basic properties of sectors are given in Section 2.

**The vacuum sector of a defect.** For any defect $D$ we can evaluate $D$ on the top half $S^1_\circ := \uparrow$ of the circle. Applying the $L^2$ functor we obtain the Hilbert space $H_0(S^1_\circ,D) := L^2(D(S^1_\circ))$ and, as a consequence of the vacuum axiom in the definition of defects, this Hilbert space is a sector for $D$, called the vacuum sector for $D$. In our 3-category, the vacuum sector is the identity 2-morphism for the 1-morphism $D$. We often draw it as

$$\begin{array}{c}
\mathcal{A} & \xrightarrow{H_0} & \mathcal{B} \\
\mathcal{D} & \xleftarrow{D} & \end{array}$$

This darker shading is reserved for vacuum sectors.
Composition of defects. Let \( D = \mathcal{A} \mathcal{B} \) and \( E = \mathcal{B} \mathcal{C} \) be defects. Their composition or fusion \( D \oplus_B E \) is defined in Section 1.6. The definition is quite natural, but, surprisingly, it is not easy to see that \( D \oplus_B E \) satisfies all the axioms of defects.

We outline the definition of the fusion \( D \oplus_B E \). In our graphical notation, double lines \( \equiv \) will now correspond to \( \mathcal{A} \), thin lines \( \equiv \) to \( \mathcal{B} \), and thick lines \( \equiv \) to \( \mathcal{C} \). Let us concentrate on the evaluation of \( D \oplus_B E \) on \( S^1_+ \). Denote by \( S^1_+ := \partial([1, 2] \times [0, 1]) \) the translate of the standard circle and by \( S^1_{+\uparrow} \) its top half. As with the vacuum sector \( H_0(S^1_+, D) = L^2(D(S^1_+)) \) for \( D \) on \( S^1 \), we can form the vacuum sector \( H_0(S^1_+, E) := L^2(E(S^1_{+\uparrow})) \) for \( E \) on \( S^1_{+\uparrow} \). Let \( I = \{1\} \times [0, 1] \) be the intersection of the two circles \( S^1 \) and \( S^1_{+\uparrow} \), equipped with the orientation inherited from \( S^1_+ \). The Hilbert space \( H_0(S^1_+, D) \) is a right \( \mathcal{B}(I) \)-module, while \( H_0(S^1_{+\uparrow}, E) \) is a left \( \mathcal{B}(I) \)-module. Consequently, we can form their Connes fusion \( H_0(S^1_+, D) \otimes_{\mathcal{B}(I)} H_0(S^1_{+\uparrow}, E) \), drawn as

\[
\begin{array}{c c c}
\mathcal{A} & \mathcal{B} & \mathcal{E} \\
H_0 & H_0 & \mathcal{C} \\
\mathcal{D} & \mathcal{B} & \mathcal{E}
\end{array}
\]  
(0.1)

In this picture the middle vertical line \( \mid \) corresponds to \( I \). By the axioms for defects the actions of \( D(\mathcal{F}) \) and of \( \mathcal{B}(I) \) on \( H_0(S^1_+, D) \) commute. Consequently, we obtain an action of \( D(\mathcal{F}) \) on the Connes fusion (0.1). Similarly, there is an action of \( E(\mathcal{T}) \) on (0.1). Now \( (D \oplus_B E)(S^1_{+\uparrow}) \) is defined to be the von Neumann algebra generated by \( D(\mathcal{F}) \) and \( E(\mathcal{T}) \) acting on the Hilbert space (0.1). A similar construction, using the local coordinate, is used to define the evaluation of \( D \oplus_B E \) on arbitrary bicolored intervals.

A main result of this paper, Theorem 1.42, is a proof that if the net \( \mathcal{B} \) has finite index (see Appendix B.11), then \( D \oplus_B E \) is in fact a defect.

Towards a 3-category of conformal nets. The purpose of our next paper [4] is to construct the symmetric monoidal 3-category of conformal nets. More precisely, we will construct an internal dicategory \( \mathcal{C} \) object \( (\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2) \) in the 2-category of symmetric monoidal categories [6, Definition 3.3]. In this paper, we develop the essential ingredients of that 3-category, but we do not check all the axioms. These ingredients are:

- A symmetric monoidal category \( \mathcal{C}_0 \) whose objects are the conformal nets with finite index, and whose morphisms are the isomorphisms between them.
- A symmetric monoidal category \( \mathcal{C}_1 \) whose objects are the defects between conformal nets of finite index, and whose morphisms are the isomorphisms.
- A symmetric monoidal category \( \mathcal{C}_2 \) whose objects are sectors (between defects between conformal nets of finite index), and whose morphisms are those homomorphisms of sectors that cover isomorphisms of defects and of conformal nets.
- These come with source and target functors \( s, t: \mathcal{C}_1 \to \mathcal{C}_0 \) and \( s, t: \mathcal{C}_2 \to \mathcal{C}_1 \) subject to the identities \( s \circ s = s \circ t \) and \( t \circ s = t \circ t \).
- A symmetric monoidal functor composition \( \mathcal{C}_1 \times \mathcal{C}_0 \to \mathcal{C}_1 \) that describes the composition (or fusion) of defects (1.40). That the composition of defects exists is the content of Theorem 1.42.

\[\text{2Following [8], a dicategory is a bicategory where the associators } (fg)h = f(gh) \text{ are strict (i.e., they are identity 2-morphisms) but the unitors } 1f = f \text{ and } f1 = f \text{ are not necessarily strict.}\]


- A symmetric monoidal functor $\text{fusion}_h : C_2 \times_{C_0} C_2 \to C_2$ providing the horizontal composition of sectors (2.12).
- A symmetric monoidal functor $\text{fusion}_v : C_2 \times_{C_1} C_2 \to C_2$ providing the vertical composition of sectors (2.15).
- Symmetric monoidal functors $\text{identity} : C_0 \to C_1$ and $\text{identity}_v : C_1 \to C_2$ providing the identity defects (1.37) and identity sectors (2.5).
- A monoidal natural transformation $\text{associator} : C_1 \times_{C_0} C_1 \times_{C_0} C_1 \otimes C_1$ that is an associator for composition (1.56).
- A monoidal natural transformation $\text{associator}_h : C_2 \times_{C_0} C_2 \times_{C_0} C_2 \otimes C_2$ that is an associator for $\text{fusion}_h$ (2.13).
- A monoidal natural transformation $\text{associator}_v : C_2 \times_{C_1} C_2 \times_{C_1} C_2 \otimes C_2$ that is an associator for $\text{fusion}_v$ (2.17).
- Two monoidal natural transformations $\text{unitor}_t, \text{unitor}_b : C_2 \otimes C_2$ that relate $\text{fusion}_v$ and $\text{identity}_v$ (2.18).
- The coherences for composition and identity are “weak”: instead of natural transformations $C_1 \otimes C_1$, we have four functors $\text{unitor}_t, \text{unitor}_b, \text{unitor}_r, \text{unitor}_r : C_1 \to C_2$ (3.2, 3.3). This weakness, which is an intrinsic feature of conformal nets and defects, is what forces us to use the notion of internal bicategory [8, Def. 3.3] instead of the simpler notion of internal 2-category [8, Def. 3.1].
- The coherence between composition and identity $v$ is a monoidal natural transformation $C_1 \times_{C_0} C_1 \otimes C_2$ (6.1, 6.3). This is the most difficult construction of this paper and it is also the one that forces us to restrict the morphisms in the category $C_1$ of defects to be isomorphisms. We call this natural transformation the “$1 \otimes 1$-isomorphism” because its domain is a Connes fusion of two identity sectors.
- Finally, the crucial interchange isomorphism, a coherence between $\text{fusion}_h$ and $\text{fusion}_v$, is a monoidal natural transformation

$$(C_2 \times_{C_1} C_2) \times_{C_0} (C_2 \times_{C_1} C_2) \otimes C_2 \quad (0.25).$$

Its definition relies crucially on the $1 \otimes 1$-isomorphism.

**The $1 \otimes 1$ isomorphism.** This isomorphism provides a canonical identification of the Hilbert space (0.1), used to define the defect $D \otimes_B E$, with the vacuum sector for $D \otimes_B E$. By definition the vacuum sector is $H_0(S^1, D \otimes_B E) := L^2(D \otimes_B E(S^1_+))$. By construction the algebra $D \otimes_B E(S^1_+)$ contains $D(\rightarrow)$ and $E(\rightarrow)$ as two commuting subalgebras and is generated by those subalgebras. We can think of the algebra $D \otimes_B E(S^1_+)$ as associated to the tricolored interval $\overset{\rightarrow}{\overset{\rightarrow}{\text{tricolored interval}}}$. which is the upper half of the circle $\partial([0, 2] \times [0, 1])$; it is therefore natural to draw the vacuum sector for $D \otimes_B E$ as

$$(0.2)$$

\[
\begin{array}{c}
\text{A} \\
\text{H}_0 \\
\text{C} \\
\text{D} \otimes_B E
\end{array}
\]

In the language of the 3-category the vacuum sector for a defect $D$ is the identity 2-morphism, and a fusion along the middle vertical line as in (0.1) is the horizontal composition of 2-morphisms. Thus (0.1) is the composition of the identities for the defects $D$ and $E$, while (0.2) is the identity for the composition $D \otimes_B E$. For this
reason, we refer to the desired isomorphism between (0.1) and (0.2) as the “one times one isomorphism”.

The construction of the $1 \boxtimes 1$-isomorphism is quite involved and is carried out in Sections 4, 5, and 6. Section 6 also contains a short summary, on page 52, collecting all the necessary ingredients in one place. The existence and construction of the $1 \boxtimes 1$-isomorphism, completed in Theorem 6.2, is one of the main results of this paper.

**Construction of the $1 \boxtimes 1$-isomorphism.** For any von Neumann algebra $A$ the standard form $L^2(A)$ carries commuting left and right actions of $A$, i.e., $L^2(A)$ is an $A$-$A$-bimodule. In the case of the vacuum sector $H_0(S^1, D) = L^2(D([0,1]))$ these two actions correspond to the left actions of $D([0,1])$ and of $D([1,\infty])$. One difficulty in understanding the Connes fusion (0.1) comes from the fact that the algebra $B(I)$, over which the Connes fusion is taken, intersects both $D([0,1])$ and $D([1,\infty])$. To simplify the situation we can consider a variation of (0.1) with a hole in the middle:

(0.3)

This Hilbert space is built from vacuum sectors for $D$ and $E$ together with two (small) copies of the vacuum sector for $B$. Its formal definition is given in Section 4; see in particular (4.8). The Connes fusion of $B(I)$ is now replaced by four Connes fusion operations along smaller algebras. This allows us to identify, in Theorem 4.11, the Hilbert space (0.3) with the $L^2$-space of a certain von Neumann algebra that we represent by the graphical notation \( \mathcal{B}(\mathbb{R}) \). It is generated by algebras $D(\mathbb{R})$, $\mathcal{B}(\mathbb{R})$, and $E(\mathbb{R})$ acting on the Hilbert space $\mathbb{R}^2$, here $\mathcal{B}(\mathbb{R})$ is a certain enlargement of the algebra $B(\mathbb{R})$ that we abbreviate graphically by $\mathcal{B}$. We refer to (4.9, 4.10) for the details of the definitions, and to 3.10 for an explanation of the notation $\mathcal{B}$. Theorem 4.11 then reads

(0.4)

At this point we blur the distinction between intervals and algebras in our graphical notation and often draw only an interval to denote an algebra. For example we abbreviate $\mathcal{B}(\mathbb{R})$ as simply $\mathcal{B}$, and $D \boxtimes_E E(S^1)$ as $\mathcal{B}(\mathbb{R})$. We therefore write, for instance $D \boxtimes_E E(S^1) \otimes \mathcal{B}(\mathbb{R})$ as $\mathcal{B}(\mathbb{R})$. As the notation indicates, this tensor product is a subalgebra of $\mathcal{B}(\mathbb{R})$. (Note the additional dotted line in the middle.) If $\mathcal{B}$ has finite index, then we show in Corollary 4.16 that this inclusion $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{B}(\mathbb{R})$ is a finite homomorphism of von Neumann algebras. As the $L^2$-construction is functorial for such homomorphisms $\mathcal{B}$, we can apply $L^2$ to it. Combining this with (0.4) we obtain a map

(0.5)

In the next step, we need to fill the hole in (0.3). Formally, this is done by applying Connes fusion with a further (small) vacuum sector for $B$. On the domain of (0.3) this cancels the algebra $\mathcal{B}$. On the target, we simply denote the result

\[ L^2 \left( \begin{array}{c} \hline \hline \hline \hline \hline \end{array} \right) \approx \left( \begin{array}{c|c|c} \hline \hline \hline \hline \hline \end{array} \right) \]

3 The reflection along the horizontal axis $\mathbb{R} \times \{ \frac{1}{2} \}$ provides an orientation reversing identification $\mathcal{B}(\mathbb{R}) \cong \mathcal{B}(\mathbb{R})$ and this accounts for the fact that the right action of $D([0,1])$ on $L^2(D([0,1]))$ corresponds to a left action of $D([0,1])$ on $H_0(S^1, D)$.
by filling the hole with a (small) vacuum sector for $B$. In this way we obtain, in Proposition 4.18, an isometric embedding

$$L^2 \left( \bigcap \right) \rightarrow L^2 \left( \bigcap \right).$$

The existence of this isometric embedding enables us to prove that $D \otimes_B E$ is a defect.

To produce the $1 \boxtimes 1$-isomorphism from (0.6) requires two further steps. We first, in Proposition 4.29, construct an isomorphism

$$\sim \Rightarrow$$

and then define the “$1 \boxtimes 1$-isomorphism” $\Omega$ as the composite of the two maps (0.6) and (0.7).

In Theorem 6.2, we prove that the composite of (0.6) and (0.7) is indeed an isomorphism. The proof of that theorem proceeds as follows: both the domain and the target of $\Omega$ carry commuting actions of the algebras $\bigcap$ and $\bigcap$. On $\bigcap = L^2 \left( \bigcap \right)$ these two actions are clearly each other’s commutants and so to prove that $\Omega$ is an isomorphism it suffices to show that the same holds for $\bigcap$. This is a kind of Haag duality for fusion of defects. It appears as Theorem 5.2 and is one of the main technical results of this paper. All of Section 5 is devoted to its proof.

Remark. In constructing the 3-category of conformal nets, it is essential to know that the $1 \boxtimes 1$-isomorphism $\Omega$ satisfies certain axioms, such as associativity. In Lemma 4.32 we prove that the isomorphism is appropriately associative, but unfortunately this is done directly by tracing through the entire construction of $\Omega$. Better would be to use a characterization of $\Omega$ (and thus of composites of multiple $\Omega$ maps) as the unique map satisfying certain properties. Haagerup’s standard form (that is, the $L^2$-space of a von Neumann algebra) does admit such a characterization: it is determined up to unique unitary isomorphism by the module structure, the modular conjugation, and a self-dual cone. There is a natural choice of modular conjugation on $\bigcap$. Thus, to characterize the isomorphism $\Omega$, it suffices to specify a self-dual cone in that fusion of vacuum sectors. Unfortunately, we do not know how to construct such a self-dual cone from the self-dual cones of $\bigcap$ and of $\bigcap$.

Further structure maps. As an application of the $1 \boxtimes 1$-isomorphism, we construct in Section 6.2 the interchange isomorphism between horizontal and vertical composition of sectors. We also prove in Lemma 6.15 a compatibility between the $1 \boxtimes 1$-isomorphism and the unit map for identity defects. Section 6.3 contains the construction of two further structure maps concerning units that will be needed for the detailed construction of the 3-category in the sequel [4].

Summary of results. Let $A \otimes_B$ be defects between irreducible conformal nets, and assume $B$ has finite index. Let $S^+_1$ and $S^-_1$ denote respectively the top and bottom halves of the standard circle, and let $I$ denote an interval identified as necessary with the left or with the right quarter of the standard circle. The three main theorems in this paper are the following.

Theorem A (Existence of fusion of defects). The fusion $D \otimes_B E$ of two defects is again a defect.

Theorem B (Haag duality for fusion of defects). The algebras $(D \otimes_B E)(S^+_1)$ and $(D \otimes_B E)(S^-_1)$, associated by the defect $D \otimes_B E$ to the two halves of the circle, are
each other’s commutants in their action on the fusion \(H_0(D) \otimes_{\mathcal{B}(I)} H_0(E)\) of the vacuum sectors of the defects.

**Theorem C** (The 1 \(\otimes\) 1-isomorphism). There is a canonical isomorphism between the vacuum sector \(H_0(D \otimes_B E)\) of the fused defect \(D \otimes_B E\) and the Connes fusion \(H_0(D) \otimes_{\mathcal{B}(I)} H_0(E)\) of the two vacuum sectors of the defects.

These are established in the text as, respectively, Theorem 1.42, Theorem 5.2 (see also Corollary 5.9), and Theorem 6.2.

1. Defects

**1.A. Bicolored intervals and circles.** An interval is a smooth oriented 1-manifold diffeomorphic to \([0, 1]\). We write \(\text{Diff}(I)\) for the group of diffeomorphisms of \(I\) and \(\text{Diff}_0(I)\) for the subgroup that fixes a neighborhood of \(\partial I\). A **bicolored interval** is an interval \(I\) (always oriented) equipped with a cover by two closed, connected, possibly empty subsets \(I_\circ, I_\bullet \subset I\) with disjoint interiors, along with a local coordinate (that is, an embedding \((-\varepsilon, \varepsilon) \hookrightarrow I\) at \(I_\circ \cap I_\bullet\)). We disallow the cases when \(I_\circ\) or \(I_\bullet\) consist of a single point. The local coordinate does not need to preserve the orientation, but is required to send \((-\varepsilon, 0]\) into \(I_\circ\) and \([0, \varepsilon)\) into \(I_\bullet\). A bicolored interval necessarily falls in one of the following three classes:

(1.1) \(I_\circ, I_\bullet\) are intervals and \(I_\circ \cap I_\bullet\) is a point; the local coordinate is a smooth embedding \((-\varepsilon, \varepsilon) \hookrightarrow I\) that sends \((-\varepsilon, 0]\) to \(I_\circ\) and \([0, \varepsilon)\) to \(I_\bullet\);

(1.2) \(I_\circ = I, I_\bullet = \emptyset\), and there is no data of local coordinate;

(1.3) \(I_\bullet = I, I_\circ = \emptyset\), and there is no data of local coordinate.

An embedding \(f: J \hookrightarrow I\) from one bicolored intervals to another is called color preserving if \(f^{-1}(I_\circ) = J_\circ\) and \(f^{-1}(I_\bullet) = J_\bullet\). The bicolored intervals form a category \(\text{INT}_\circ\bullet\), whose morphisms are the color preserving embeddings that respect the local coordinates (that is, such that the embedding intertwines the local coordinates on a sufficiently small neighborhood of 0). We let \(\text{INT}_\circ\) and \(\text{INT}_\bullet\) be the full subcategories on the objects of the form (1.2) and (1.3), respectively. Both of them are canonically isomorphic to \(\text{INT}\), the category of unclored intervals. Elements of \(\text{INT}_\circ\) and \(\text{INT}_\bullet\) are called white intervals and black intervals, respectively. Those of type (1.1) are called genuinely bicolored intervals. The full subcategory of genuinely bicolored intervals is denoted \(\text{INT}_\bullet\).

Similarly, a **bicolored circle** \(S\) is a circle (always oriented) equipped with a cover by two closed, connected, possibly empty subsets with disjoint interiors \(S_\circ, S_\bullet \subset S\), along with local coordinates in the neighborhood of \(S_\circ \cap S_\bullet\). We disallow the cases when \(S_\circ\) or \(S_\bullet\) consists of a single point. A bicolored circle necessarily falls in one of the following three categories:

(1.4) \(S_\circ, S_\bullet \subset S\) are intervals. The intersection \(S_\circ \cap S_\bullet\) consist of two points and the two local coordinates are embeddings \((-\varepsilon, \varepsilon) \hookrightarrow I\) sending \((-\varepsilon, 0]\) to \(I_\circ\) and \([0, \varepsilon)\) to \(I_\bullet\);

(1.5) \(S_\circ = S, S_\bullet = \emptyset\), and there are no local coordinates;

(1.6) \(S_\bullet = S, S_\circ = \emptyset\), and there are no local coordinates.

Bicolored circles of type (1.4) are called genuinely bicolored.

**1.B. Definition of defects.** Let \(\text{VN}\) be the category whose objects are von Neumann algebras with separable preduals, and whose morphisms are C-linear homomorphisms, and C-linear antihomomorphisms\(^4\).

Recall our definition of conformal nets (see Appendix B). For the following definition of defect, we do not require that the conformal nets \(\mathcal{A}\) and \(\mathcal{B}\) are irreducible:

\(^4\) An antihomomorphism is a map satisfying \(f(1) = 1\) and \(f(ab) = f(b)f(a)\).
Definition 1.7. Let \( \mathcal{A} \) and \( \mathcal{B} \) be two conformal nets. A defect from \( \mathcal{A} \) to \( \mathcal{B} \) is a functor
\[
D : \text{INT}_{\bullet} \to \text{VN}
\]
that assigns to each bicolored interval \( I \) a von Neumann algebra \( \mathcal{A}(I) \), and whose restrictions to \( \text{INT}_a \) and \( \text{INT}_s \), are given by \( \mathcal{A} \) and \( \mathcal{B} \), respectively. It sends orientation-preserving embeddings to \( \mathbb{C} \)-linear homomorphisms, and orientation-reversing embeddings to \( \mathbb{C} \)-linear antihomomorphisms. The functor \( D \) is subject to the following axioms:

(i) **Isotony:** If \( I \) and \( J \) are genuinely bicolored intervals and \( f : J \mapsto I \) is an embedding, then \( D(f) : D(J) \to D(I) \) is injective.

(ii) **Locality:** If \( J \subset I \) and \( K \subset I \) have disjoint interiors, then the images of \( D(J) \) and \( D(K) \) are commuting subalgebras of \( D(I) \).

(iii) **Strong additivity:** If \( I = J \cup K \), then the images of \( D(J) \) and \( D(K) \) topologically generate \( D(I) \).

(iv) **Vacuum sector:** Let \( S \) be a genuinely bicolored circle, \( I \subset S \) a genuinely bicolored interval, and \( j : S \to S \) a color preserving orientation reversing involution that fixes \( \partial I \). Equip \( I' := j(I) \) with the orientation induced from \( S \), and consider the following two maps of algebras:
\[
\alpha : D(I) \to B(L^2 D(I))
\]
\[
\beta : D(I') \xrightarrow{D(j)} D(I)^{\text{op}} \to B(L^2 D(I))
\]
(Here \( \alpha \) is the left action of \( D(I) \) on \( L^2 D(I) \), and \( \beta \) the map \( D(I)^{\text{op}} \to B(L^2 D(I)) \) is the right action of \( D(I) \) on \( L^2 D(I) \).) Let \( J \in \text{INT}_a \cup \text{INT}_s \) be a subinterval of \( I \) such that \( J \cap \partial I \) consists of a single point, and equip \( J' := j(J) \) with the orientation induced from \( S \). We then require that the action
\[
\alpha \otimes \beta : D(J) \otimes_{\text{alg}} D(J') \to B(L^2 D(I))
\]
of the algebraic tensor product extends to an action of \( D(J \cup J') \).

We will write \( \mathcal{A} D \mathcal{B} \) to indicate that \( D \) is a defect from \( \mathcal{A} \) to \( \mathcal{B} \).

The following properties might naturally have been added as axioms in the definition of a defect, but are in fact consequences of the listed axioms and the corresponding properties of conformal nets: inner covariance (Proposition 1.10), the split property (Proposition 1.11), Haag duality (Proposition 1.17), and continuity (Proposition 1.22).

**Inner covariance and the split property.** Recall that \( \text{Diff}_0(I) \) is the subgroup of diffeomorphisms of \( I \) that fix some neighborhood of \( \partial I \).

**Proposition 1.10** (Inner covariance for defects). Let \( I \) be a genuinely bicolored interval, and let \( \varphi \in \text{Diff}_0(I) \) be a diffeomorphism that preserves the bicoloring and the local coordinate. Then \( D(\varphi) \) is an inner automorphism of \( D(I) \).

**Proof.** Write \( \varphi = \varphi_\circ \varphi_\circ \) with \( \text{supp}(\varphi_\circ) \subset I_a \) and \( \text{supp}(\varphi_\circ) \subset I_s \). Let \( \{J, K, L\} \) be a cover of \( I \) such that \( J \) is a white interval, \( K \) a genuinely bicolored interval, \( L \) is a black interval, \( \text{supp}(\varphi_\circ) \) contained in the interior of \( J \), \( \text{supp}(\varphi_\circ) \) contained in the interior of \( L \), and \( \varphi \) acts as the identity on \( K \). By inner covariance for the nets \( \mathcal{A} \) and \( \mathcal{B} \) (see Appendix B.4), there are unitaries \( u \in \mathcal{A}(J) \) and \( v \in \mathcal{B}(L) \) that implement \( \varphi_\circ \) and \( \varphi_\circ \). Let \( w \) be their product in \( D(I) \). Then \( waw^* = D(\varphi)w \) holds for every \( a \in D(I) \) that is in the image of \( \mathcal{A}(J) \), of \( D(K) \), or of \( \mathcal{B}(L) \). By strong additivity, it therefore holds for every element of \( D(I) \).
Proposition 1.11 (Split property for defects). If \( J \subset I \) and \( K \subset I \) are disjoint, then the map \( D(J) \otimes_{\text{alg}} D(K) \to D(I) \) extends to the spatial tensor product \( D(J) \otimes D(K) \).

Proof. We assume without loss of generality that the interval \( J \) is entirely white and that it does not meet the boundary of \( I \) (otherwise, replace \( I \) by a slightly larger interval). Let \( J^+ \subset I \) be a white interval that contains \( J \) in its interior and that does not intersect \( K \). Finally, let \( \iota : A(J^+) \to D(I) \) be the map induced by the inclusion \( J^+ \hookrightarrow I \). By the split property and Haag duality for conformal nets, the inclusion \( \iota A(J) \subseteq \iota A(J^+) \) is split in the sense of Definition A.29. As \( A(J^+) \) commutes with \( D(K) \), the inclusion \( \iota A(J) \to D(K) \) is then also split, where the commutant is taken in any faithful representation of \( D(I) \). Thus,

\[
D(J) \otimes_{\text{alg}} D(K) = \left( \iota A(J) \oplus \ker \iota \right) \otimes_{\text{alg}} D(K) \to D(I)
\]

extends to the spatial tensor product \( D(J) \otimes D(K) \).

Vacuum properties. Let \( S \) be a genuinely bicolored circle, along with an orientation reversing diffeomorphism \( j : S \to S \), compatible with the bicoloring and with the local coordinates. Let \( I \subset S \) be an interval whose boundary is fixed by \( j \) and let \( I' := j(I) \). The Hilbert space \( H_0 := L^2(D(I)) \) is called the vacuum sector of \( D \) associated to \( S, I \), and \( j \). It is endowed with actions of \( D(J) \) for every bicolored intervals \( J \subset S \), as follows. (Recall that bicolored intervals contain at most one color-change point.) The maps \( \rho_j \) provide natural actions of \( D(J) \) on \( H_0 \) for all subintervals \( J \subset I \) and \( J \subset I' \). By the vacuum axiom for defects, these extend to the algebras \( D(J) \) associated to white and to black subintervals of \( S \). To define the action \( \rho_j : D(J) \to B(H_0) \) of an arbitrary genuinely bicolored interval \( J \subset S \), pick a white interval \( K_1 \subset S \), a black interval \( K_2 \subset S \), and diffeomorphisms \( \varphi_1, \varphi_2 \in \text{Diff}_0(K_i) \) such that \( \varphi_1 \varphi_2(J) \) does not cross \( \partial I \). If \( u_1 \in \mathcal{A}(K_1) \) and \( u_2 \in \mathcal{B}(K_2) \) are unitaries implementing \( \varphi_1 \) and \( \varphi_2 \), then the action on \( H_0 \) of an element \( a \in D(J) \) is defined by

\[
\rho_j(a) := u_1^* u_2^* \rho_{\varphi_1 \varphi_2(j)}(D(\varphi_1 \varphi_2)(a)) u_1 u_2.
\]

This action is compatible with the actions associated to other intervals, and is independent of the choices of \( \varphi_1 \), \( \varphi_2 \) and \( u_1 \), \( u_2 \) (see Lemma 2.5 for a similar construction in a more general context).

The following result, constructing isomorphisms between different vacuum sectors, is a straightforward generalization of [2, Cor. 1.15]:

Lemma 1.13. Let \( S \) be a genuinely bicolored circle. Let \( I_1 \) and \( I_2 \) be genuinely bicolored subintervals and let \( j_1 \) and \( j_2 \) be involutions fixing \( \partial I_1 \) and \( \partial I_2 \). Then the corresponding vacuum sectors \( L^2(D(I_1)) \) and \( L^2(D(I_2)) \) are non-canonically isomorphic as representations of the algebras \( D(J) \) for \( J \subset S \).

Proof. If \( I_1 \) and \( I_2 \) contain the same color-change point, then let \( \varphi \in \text{Diff}(S) \) be a diffeomorphism that sends \( I_1 \) to \( I_2 \), that intertwines \( j_1 \) and \( j_2 \), and that can be written as \( \varphi = \varphi_\circ \varphi_\bullet \) where \( \varphi_\circ \) acts on the white part only and \( \varphi_\bullet \) acts on the black part only. Let \( K \) be a white interval that contains \( \text{supp}(\varphi_\circ) \) in its interior and let \( L \) be a black interval that contains \( \text{supp}(\varphi_\bullet) \) in its interior. Finally, let \( u \in \mathcal{A}(K) \) and \( v \in \mathcal{B}(L) \) be unitaries implementing \( \varphi_\circ \) and \( \varphi_\bullet \). Then

\[
L^2(D(I_1)) \xrightarrow{L^2(D(\varphi))} L^2(D(I_2)) \xrightarrow{u^* u^*} L^2(D(I_3))
\]

is the desired isomorphism.

If \( I_1 \) and \( I_2 \) contain opposite color-change points, then we may assume without loss of generality that \( j_1 = j_2 \) and \( I_2 = j_1(I_1) \). The isomorphism from \( L^2(D(I_1)) \) to \( L^2(D(I_2)) \) is then given by \( L^2(D(j_1)) \).

□
Notation 1.14. Given a genuinely bicolored circle $S$ and a defect $\mathcal{A}D_B$, we denote by $H_0(S, D)$ the vacuum sector associated to some interval $I \subset S$ and some involution $j$ fixing $\partial I$. By the previous lemma, that Hilbert space is well defined up to non-canonical unitary isomorphism.

Remark 1.15. If $S$ is a circle that is either entirely white (or entirely black), then the above description of $H_0(S, D)$ still makes sense and recovers the notion of vacuum sector of a conformal net $H_0(S, A)$ (or $H_0(S, B)$) [2 Definition 1.16].

Our next result, concerning the gluing of vacuum sectors, is a straightforward generalization of [2 Cor. 1.33] in the presence of defects (compare Appendix B.III). Let $S_1$ and $S_2$ be bicolored circles, let $I_i \subset S_i$ be bicolored intervals (whose boundaries do not touch the color change points), and let $I'_i$ be the closure of $S_i \setminus I_i$. Assume that there exists an orientation reversing diffeomorphism $\varphi : I_2 \to I_1$ compatible with the bicolorings, and let $S_3 := I'_1 \cup_{\partial I_2} I'_2$. Assume that $(S_3)_c$ and $(S_3)_b$ are connected and non-empty. Then, up to exchanging $S_1$ and $S_2$, we are in one of the following three situations:

- $S_1 \cup_{I_2} S_2 \sim S_3$,
- $S_1 \cup_{I_2} S_2 \rightarrow S_3$,
- $S_1 \cup_{I_2} S_2 \rightarrow S_3$.

Equip $S_1 \cup_{I_2} S_2$ with a smooth structure compatible with the given smooth structures on $S_1$ and $S_2$ [3 Definition 1.4]. That is, provide smooth structures on $S_1$, $S_2$, and $S_3$ such that there exists an action of the symmetric group $\mathfrak{S}_3$ on $S_1 \cup_{I_2} S_2$ (with no compatibility with the bicoloring) that permutes the three circles and has $\pi|_{S_a}$ smooth for every $\pi \in \mathfrak{S}_3$ and $a \in \{1, 2, 3\}$.

When $\mathcal{A}D_B$ is a defect, it will be convenient to write $H_0(S, D) := H_0(S, A)$ if $S$ is entirely white and $H_0(S, D) := H_0(S, B)$ if $S$ is entirely black.

Lemma 1.16. Let $S_1$, $S_2$, $S_3$, and $\varphi$ be as above, and let $D$ be a defect. Use the map $D(\varphi)$ to equip $H_0(S_1, D)$ with the structure of a right $D(I_2)$-module. Then there exists a non-canonical isomorphism

$$H_0(S_1, D) \boxtimes_{D(I_2)} H_0(S_2, D) \cong H_0(S_3, D),$$

compatible with the actions of $D(J)$ for $J \subset S_3$.

Proof. Depending on the topology of the bicoloring, we can either identify $H_0(S_1, D)$ with $L^2D(I_1)$ or identify $H_0(S_2, D)$ with $L^2D(I_2)$. We assume without loss of generality that we are in the first case.

Let $j \in \text{Diff}_-(S_1)$ be an involution that is compatible with the bicoloring and that fixes $\partial I_1$, and let $H_0(S_1, D) = L^2D(I_1)$ be the vacuum sector associated to $S_1$, $I_1$, and $j$. We then have

$$L^2D(I_1) \otimes_{D(I_1)} H_0(S_2, D) \cong L^2D(I_2) \otimes_{D(I_2)} H_0(S_2, D) \cong H_0(S_2, D) \cong H_0(S_3, D),$$

where the first isomorphism uses $L^2(D(\varphi)) : L^2(D(I_2)) \to L^2(D(I_1))$ and the third one is induced by the map $(j \circ \varphi) \cup_{I_2} : S_2 \to S_3$. □

Haag duality. In certain cases, the geometric operation of complementation corresponds to the algebraic operation of relative commutant:

Proposition 1.17 (Haag duality). (1) Let $S$ be a genuinely bicolored circle, let $I \subset S$ be a genuinely bicolored interval, and let $I'$ be the closure of the complement of $I$ in $S$. Then the algebras $D(I)$ and $D(I')$ are each other’s commutants on $H_0(S, D)$.

(2) Let $\mathcal{A}D_B$ be a defect and let $J \in \text{INT}_b$ and $K \in \text{INT}_c \cup \text{INT}_b$ be subintervals of $I \in \text{INT}_c$. Assume that $J \cup K = I$ and that $J \cap K$ is a point. Then $D(J)$ is the relative commutant of (the image of) $D(K)$ in $D(I)$. 
Proof. (1) Let $j \in \text{Diff}_-(S)$ be an involution that exchanges $I$ and $I'$ and that is compatible with the bicoloring and the local coordinates. By definition, we may take $H_0(S, D) = L^2(D(I))$ with the actions of $D(I)$ and $D(I')$ provided by (1.3). The result follows, as the left and right actions of $D(I)$ on $L^2(D(I))$ are each other’s commutants.

(2) We assume without loss of generality that $K \in \text{INT}_\sigma$. Let $S := I \cup_{\partial I} (\bar{I})$ be a circle formed by gluing two copies of $I$ along their boundary, such that there is a smooth involution $j$ that exchanges them:

\[
\begin{array}{c}
\begin{array}{c}
\circlearrowleft \\
K \quad I \\
\end{array} \\
\begin{array}{c}
J \\
\end{array}
\end{array}
\]

By strong additivity and the first part of the proposition, and considering actions on $H_0(S, D)$ we then have

\[
D(K)' \cap D(I) = D(K)' \cap D(\bar{I})' = (D(K) \vee D(\bar{I}))' = D(K \cup \bar{I})' = D(J).
\]

Canonical quantization. Let $S$ be a bicolored circle and $I \subset S$ a genuinely bicolored interval. Let $j \in \text{Diff}_-(S)$ be an involution that fixes $\partial I$ and that is compatible with the bicoloring and the local coordinates. Also let $K \subset S$ be a white interval such that $j(K) = K$. We call a diffeomorphism $\varphi \in \text{Diff}_0(K) \subset \text{Diff}(S)$ symmetric if it commutes with $j$, and set

\[
\text{Diff}^{\text{sym}}_0(K) := \{ \varphi \in \text{Diff}_0(K) \mid \varphi j = j \varphi \}.
\]

Given a symmetric diffeomorphism $\varphi$, we also write $\varphi_0 \in \text{Diff}(I)$ for $\varphi|_I$; to be precise, $\varphi_0 := \varphi|_{I \cap K} \cup \text{id}_{I \setminus K}$.

For an irreducible defect $A_{DB}$, we want to understand the automorphism $L^2D(\varphi_0)$ of $H_0(S, D) := L^2D(I)$, and its relation to the automorphism $L^2A(\varphi_0)$ of $H_0(S, A) := L^2A(I)$, where in these expressions involving $A$ the circle $S$ has now been painted all white.

By [2] Lem. 2.7| the unitary $u_\varphi := L^2A(\varphi_0)$ on $L^2A(I)$ implements $\varphi$, that is,

\[
A(\varphi)(a) = u_{\varphi} a u_{\varphi}^* \quad \text{for all intervals } J \subseteq S \text{ and all } a \in A(J).
\]

Let $K'$ be the closure of the complement of $K$ in $S$. Since $u_{\varphi}$ commutes with $A(K')$, we have $u_\varphi \in A(K)$ by Haag duality (Proposition [11], 1.4). We call $u_\varphi$ the canonical quantization of the symmetric diffeomorphism $\varphi$.

The map $\text{Diff}^{\text{sym}}_0(K) \to \text{Diff}_+(I)$ given by $\varphi \mapsto \varphi_0$ is continuous for the $C^\infty$-topology. The map $A : \text{Diff}_+(I) \to \text{Aut}(A(I))$ is continuous because $A$ is a continuous functor. The map $\text{Aut}(A(I)) \to U(L^2(A(I)))$ given by $\psi \mapsto L^2(\psi)$ is continuous by [11] Prop. 3.5. Therefore, altogether, $\varphi \mapsto u_{\varphi}$ defines a continuous map from the group of symmetric diffeomorphisms of $K$ to $U(A(K))$.

Lemma 1.19. Let $S$, $I$, $K$, $\varphi$, $\varphi_0$, and $u_{\varphi}$ be as above, let $A_{DB}$ be an irreducible defect, and let $H_0 := L^2D(I)$ be the vacuum sector of $D$ associated to $S$, $I$, and $j$. Then, letting $\rho_K$ be the action of $A(K)$ on $H_0$ (given by the vacuum axiom), we have $L^2(D(\varphi_0)) = \rho_K(u_{\varphi})$.

\[
\begin{array}{c}
\begin{array}{c}
\circlearrowleft \\
\end{array} \\
\begin{array}{c}
\circlearrowleft \\
K \quad S \\
\end{array}
\end{array}
\]

5 This refers to Haagerup’s $u$-topology on $\text{Aut}(A(I))$, see [11] Def. 3.4| or [2] Appendix.
Proof. We first show that the map
\[
\text{Diff}^\text{sym}_0(K) \to \text{Aut}(D(I))
\]
\[
\varphi \mapsto D(\varphi_0)
\]
(1.20)
is continuous for the \(C^\infty\) topology on \(\text{Diff}^\text{sym}_0(K)\) and the \(u\)-topology on \(\text{Aut}(D(I))\). Since \(\text{Ad}(u_\varphi) = A(\varphi)\), the operator \(\rho_K(u_\varphi)\) implements \(\varphi\) on \(H_0\). In particular, \(D(\varphi_0)\) is the restriction of \(\text{Ad}(\rho_K(u_\varphi))\) under the embedding \(D(I) \hookrightarrow B(H_0)\). The map
\[
\text{Diff}^\text{sym}_0(K) \to U(A(K)) \to U(H_0), \quad \varphi \mapsto u_\varphi \mapsto \rho_K(u_\varphi)
\]
is continuous and lands in the subgroup \(N := \{u \in U(H_0) \mid uD(I)u^* = D(I)\}\). Since \(D(\varphi_0) = \text{Ad}(\rho_K(u_\varphi))\) and \(\text{Ad} : N \to \text{Aut}(D(I))\) is continuous [2, A.18], the map (1.20) is therefore also continuous. Recalling [11] Prop. 3.5 that \(L^2 : \text{Aut}(D(I)) \to U(L^2D(I))\) is continuous, we have therefore shown that
\[
\text{Diff}^\text{sym}_0(K) \to B(H_0), \quad \varphi \mapsto L^2(D(\varphi_0))
\]
is a continuous homomorphism.

Recall that \(\rho_K(u_\varphi)\) implements \(\varphi\). By the same argument as in [2, Lem. 2.7], \(L^2(D(\varphi_0))\) also implements \(\varphi\). It follows that
\[
L^2(D(\varphi_0)) = \lambda_\varphi\rho_K(u_\varphi)
\]
for some scalar \(\lambda_\varphi \in S^1\). Thus, we get a continuous map \(\varphi \mapsto \lambda_\varphi\) from the group of symmetric diffeomorphisms of \(K\) into \(U(1)\). Our goal is to show that \(\lambda_\varphi = 1\).

Let \(J_A\) and \(J_B\) be the modular conjugations on \(L^2(A(I))\) and \(L^2(D(I))\), and let \(\pi_K\) be the natural action of \(A(K)\) on \(L^2(A(I))\). Since \(\pi_K(u_\varphi) = L^2(A(\varphi_0))\) commutes with \(J_A\), we have \(J_A \pi_K(u_\varphi) J_A = \pi_K(u_\varphi)\). Combined with [2, Lem. 2.5], this implies the equation
\[
(1.21)
\]
\[
\text{A}(j)(u_\varphi^*) = u_\varphi.
\]
Applying \(\rho_K\) to (1.21), then, by a straightforward analog of [2, Lem. 2.5], we learn that \(J_B \rho_K(u_\varphi) J_D = \rho_K(u_\varphi)\). Since both \(L^2(D(\varphi_0))\) and \(\rho_K(u_\varphi)\) commute with \(J_D\) and since the latter is antilinear, the phase factor \(\lambda_\varphi\) must be real. It follows that \(\lambda_\varphi \in \{\pm 1\}\).

To finish the argument, note that \(\text{Diff}^\text{sym}_0(K)\) is connected and that \(\{\pm 1\}\) is discrete. The map \(\varphi \mapsto \lambda_\varphi\) being continuous, it must therefore be constant. \(\square\)

**Continuity.** Given genuinely bicolored intervals \(I\) and \(J\), and a neighborhood \(N\) of \(I_\circ \cap \overset{\cdot}{J}\), let \(\text{Hom}^{(N)}(I, J)\) denote the set of embeddings \(I \to J\) that preserve the local coordinate on the whole of \(N\) (this only makes sense if \(N\) is contained in the domain of definition of the local coordinate). We equip \(\text{Hom}_{\text{INT}_\circ}(I, J)\) with the colimit of the \(C^\infty\) topologies on \(\text{Hom}^{(N)}(I, J)\).

Given two von Neumann algebras \(A\) and \(B\), the Haagerup \(u\)-topology on \(\text{Hom}_\text{VN}(A, B)\) is the topology of pointwise convergence for the induced map on preduals [2, Appendix].

**Proposition 1.22** (Continuity for defects). Let \(D : \text{INT}_\circ \to \text{VN}\) be a defect. Then \(D\) is a continuous functor: for bicolored intervals \(I\) and \(J\) the map
\[
\text{Hom}_{\text{INT}_\circ}(I, J) \to \text{Hom}_\text{VN}(D(I), D(J))
\]
is continuous for the above topology on \(\text{Hom}_{\text{INT}_\circ}(I, J)\) and Haagerup’s \(u\)-topology on \(\text{Hom}_\text{VN}(D(I), D(J))\).

**Proof.** For every \(N\) as above, we need to show that the map \(D : \text{Hom}^{(N)}(I, J) \to \text{Hom}_\text{VN}(D(I), D(J))\) is continuous. We argue as in [2, Lem 4.4]. Pick a bicolor interval \(K\), and identify \(I\) and \(J\) with subintervals of \(K\) via some fixed embeddings into its interior. Given a generalized sequence \(\varphi_i \in \text{Hom}^{(N)}(I, J)\) with limit \(\varphi\), and
given a vector $\xi$ in the predual of $D(J)$, we need to show that $D(\varphi_i)_* (\xi)$ converges to $D(\varphi)_* (\xi)$ in $D(I)_*$.

Let $\text{Diff}^{(N)}_0(K)$ be the subgroup of diffeomorphisms of $K$ that fix $N$ and also fix a neighborhood of $\partial K$. Pick an extension $\hat{\varphi} \in \text{Diff}^{(N)}_0(K)$ of $\varphi$, and let $\hat{\varphi}_{n,i} \in \text{Diff}^{(N)}_0(K)$, $n \in \mathbb{N}$, be extensions of $\varphi_i$ such that $\|\hat{\varphi}_{n,i} - \hat{\varphi}\|_{C^n} < \|\varphi_i - \varphi\|_{C^n}$, where $\|\cdot\|_{C^n}$ is any norm that induces the $C^n$ topology. Letting $F$ be the filter on $\mathbb{N} \times \mathcal{I}$ generated by the sets $\{(n, i) \in \mathbb{N} \times \mathcal{I} \mid n \geq n_0, i \geq i_0(n)\}$ (see [2 Lem 4.4]), then $F$-lim $\hat{\varphi}_{n,i} = \hat{\varphi}$ in the $C^\infty$-topology.

Write $\text{Diff}^{(N)}_0(K)$ as $\text{Diff}_\circ \times \text{Diff}_\bullet$, were $\text{Diff}_\circ$ is the subgroup of $\text{Diff}^{(N)}_0(K)$ consisting of diffeomorphisms whose support is contained in the white part, and $\text{Diff}_\bullet$ is the subgroup of diffeomorphisms whose support is contained in the black part. The continuity of [1,20] shows that the map $\text{Diff}^{(N)}_0(K) \to \text{Aut}(D(K)) : \psi \mapsto D(\psi)$ is continuous when restricted to either $\text{Diff}_\circ$ or $\text{Diff}_\bullet$. The composite

$$
\text{Diff}^{(N)}_0(K) = \text{Diff}_\circ \times \text{Diff}_\bullet \to \text{Aut}(D(K)) \times \text{Aut}(D(K)) \xrightarrow{\text{mult}} \text{Aut}(D(K))
$$

is therefore also continuous. It follows that $F$-lim $D(\hat{\varphi}_{n,i}) = D(\hat{\varphi})$ in the $u$-topology on $\text{Aut}(D(K))$. Given a lift $\hat{\xi} \in D(K)_*$ of $\xi$, the vectors $D(\hat{\varphi}_{n,i})_*(\hat{\xi})$ therefore converge to $D(\hat{\varphi})_*(\hat{\xi})$. Composing with the projection $\pi : D(K)_* \to D(I)_*$, it follows that $D(\varphi_i)_*(\xi) = \pi(D(\hat{\varphi}_{n,i})_*(\hat{\xi}))$ converges to $\pi(D(\hat{\varphi})_*(\hat{\xi})) = D(\varphi)_*(\xi)$. \qed

1.c. **Examples of defects.**

**Algebras as defects, forgetful defects, and embedding defects.** The trivial conformal net $\mathbb{C}_\circ$ evaluates to $\mathbb{C}$ on every interval [2 Eg. 1.3].

**Proposition 1.23.** There is a one-to-one correspondence (really an equivalence of categories) between $\mathbb{C}_\circ\mathbb{C}$-defects and von Neumann algebras.

**Proof.** Given a von Neumann algebra $A$, the associated defect is

$$
\mathfrak{A}(I) := \begin{cases} 
\mathbb{C} & \text{if } I \in \text{INT}_\circ \text{ or } I \in \text{INT}_\bullet \\
A & \text{if } I \in \text{INT}_\circ \text{ and the local coordinate is orientation preserving} \\
\overline{A} & \text{if } I \in \text{INT}_\bullet \text{ and the local coordinate is orientation reversing}
\end{cases}
$$

where $\overline{A}$ denotes the complex conjugate of $A$.

Conversely, let $D$ be a $\mathbb{C}_\circ\mathbb{C}$-defect. Given a bicolored interval $I$, the orientation reversing map $\text{Id}_I : I \to -I$ identifies $D(-I)$ with $D(I)$, where $-I$ denotes $I$ with opposite orientation. So we just need to show is that the restriction of $D$ to the subcategory of genuinely bicolored intervals with orientation preserving maps (compatible with the local coordinates) is equivalent to a constant functor. By applying Proposition 1.17, we see that every embedding $J \to I$ between two such intervals induces an isomorphism $D(J) \to D(I)$.

To finish the proof, we need to check that $D(\phi) = \text{Id}_{D(I)}$ for any $\phi : I \to I$. Pick a neighborhood $J \subset I$ of $I_\circ \cap I_\bullet$ on which $\phi$ is the identity. Then the two arrows $D(J) \to D(I)$ in the commutative diagram

$$
\begin{array}{ccc}
D(I) & \xrightarrow{D(\phi)} & D(I) \\
\cong & & \cong \\
D(J) & \xrightarrow{\cong} & D(J)
\end{array}
$$

are equal to each other, showing that $D(\phi) = \text{Id}$. \qed
Proposition 1.24. Let $\mathcal{A}$ be a conformal net. Then the functor $\tilde{\mathcal{A}} : \text{INT}_\circ \to \text{VN}$ given by
\[
\tilde{\mathcal{A}}(I) := \begin{cases} A(I) & \text{if } I \neq \emptyset, \\ \mathbb{C} & \text{otherwise} \end{cases}
\]
is an $\mathcal{A}$-defect.

Proof. The axioms for defects follow immediately from the corresponding axioms for $A$ (Appendix B.1). □

Conformal embeddings provide examples of defects. Recall that a morphism of conformal nets $\tau : \mathcal{A} \to \mathcal{B}$ is called a conformal embedding [3 Section 1.6] if
\[
\text{Ad}(u) = A(\varphi) \Rightarrow \text{Ad}(\tau_I(u)) = B(\varphi)
\]
for every diffeomorphism $\varphi \in \text{Diff}_0(I)$ and unitary $u \in A(I)$.

Proposition 1.25. Let $\mathcal{A}$ and $\mathcal{B}$ be conformal nets and let $\tau : \mathcal{A} \to \mathcal{B}$ be a conformal embedding. Then
\[
D_\tau(I) := \begin{cases} A(I) & \text{for } I \in \text{INT}_\circ \\ B(I) & \text{for } I \in \text{INT}_\bullet \cup \text{INT}_\circ \end{cases}
\]
is an $\mathcal{A}$-$\mathcal{B}$-defect.

Proof. The axioms of isotony, locality, and vacuum sector follow directly from the corresponding axioms for $B$. It remains to prove strong additivity. We need to show that
\[
\tau_{[0,1]}(A([0,1])) \vee B([1,2]) = B([0,2]).
\]
For every point $x \in (0,1)$, pick a diffeomorphism $\varphi_x \in \text{Diff}_0([0,2])$ sending 1 to $x$, and let $u_x \in A([0,2])$ be a unitary implementing $A(\varphi_x)$. Since $\tau$ is a conformal embedding, we then have $u_x b u_x^* = B(\varphi)(b)$ for all $b \in B([0,2])$. Moreover, since $u_x \in A([0,2]) = A([0,1]) \vee A([1,2]) \subset A([0,1]) \vee B([1,2])$ and since $u_x$ conjugates $B([1,2])$ to $B([x,2])$, we have $B([x,2]) \subset A([0,1]) \vee B([1,2])$. The argument being applicable to any $x \in (0,1)$, it follows from [3 Lem. 1.4] that
\[
B([0,2]) = \bigvee_{x \in (0,1)} B([x,2]) \subset A([0,1]) \vee B([1,2]).
\]

Sums and integrals of defects.

Lemma 1.27. Let $D_1$ and $D_2$ be $\mathcal{A}$-$\mathcal{B}$-defects. Then their direct sum $E := D_1 \oplus D_2$ is also an $\mathcal{A}$-$\mathcal{B}$-defect. Here, $E$ is defined by
\[
E(I) = D_1(I) \oplus D_2(I) \text{ for } I \in \text{INT}_\circ,
\]
\[
E(I) = A(I) \text{ for } I \in \text{INT}_\circ \cup \text{INT}_\bullet = B(I) \text{ for } I \in \text{INT}_\bullet.
\]

Proof. The only non-trivial axiom is strong additivity. Consider the situation where $I = K \cup J$, with $J$ genuinely bicolored and $K$ white. Letting $\Delta : \mathcal{A} \to \mathcal{A}^{\oplus 2}$ denote the diagonal map, we need to show that $E(I)$ is equal to the subalgebra
\[
\Delta A(K) \vee E(J) \subset E(I)
\]
generated by the images of $\Delta A(K)$ and $E(J)$. (Note that our notation is a little bit misleading, as the map $\Delta A(K) \to E(I)$ might fail to be injective). Pick a white interval $L \subset J$ that touches $K$ in a point. Since $\Delta$ is a conformal embedding, it follows from the previous proof that $\Delta A(K) \vee \mathcal{A}^{\oplus 2}(L) = \mathcal{A}^{\oplus 2}(K \cup L)$.
Thus, we have the following equalities between subalgebras of $E(I)$:

$$\Delta A(K) \lor E(J) = \Delta A(K) \lor A^{\oplus 2}(L) \lor E(J) = A^{\oplus 2}(K \cup L) \lor E(J) = E(I).$$

\[\square\]

**Remark 1.28.** By the same argument as above, one can also show that a direct integral of $A$-$B$-defects is an $A$-$B$-defect.

**Disintegrating defects.**

**Definition 1.29.** An $A$-$B$-defect $D$ is said to be irreducible if $D(I)$ is a factor for every genuinely bicolored interval $I$. A finite direct sum of irreducible defects is called semisimple.

Let $D$ be a defect and let $f : J \to I$ be an embedding of genuinely bicolored intervals. Then one can show as follows that $D(f)$ induces an isomorphism between $Z(D(J))$ and $Z(D(I))$; compare the proof of [2, Proposition 1.39]. We may as well assume that $I$ and $J$ share a boundary point. Let $K$ be the closure of $I \setminus J$. The image of $Z(D(J))$ in $D(I)$ commutes with both $D(J)$ and $D(K)$, and so is in $Z(D(I))$ by strong additivity. Conversely, $Z(D(I))$ commutes with $D(K)$, and is therefore contained in the image of $Z(D(J))$ by Haag duality (Proposition 1.17).

As in the case for conformal nets [2, Sec. 1.D], we can then introduce an algebra $Z(D)$ that only depends on $D$, and that is canonically isomorphic to $Z(D(I))$ for every genuinely bicolored interval $I$. Disintegrating each $D(I)$ over that algebra, we can then write

$$D(I) = \int_x^{\oplus} D_x(I) \quad \text{for every } I \in \text{INT},$$

where $X$ is any measure space with an isomorphism $L^\infty X \cong Z(D)$.

Recall that a conformal net is called semisimple if it is a finite direct sums of irreducible conformal nets (Appendix B.I).

**Lemma 1.30.** Any $A$-$B$-defect between semisimple conformal net $R$ is isomorphic to a direct integral of irreducible $A$-$B$-defects.

**Proof.** The algebra $D(I)$ disintegrates as above. We need to show that for $K \subset I$ a white subinterval (respectively a black subinterval), the map $A(K) \to D(I)$ (respectively $B(K) \to D(I)$) similarly disintegrates. It suffices to see that $A(K) \to D(I)$ induces maps $A(K) \to D_x(I)$ for almost every $x$.

Note that it is in general not true that a map $N \to \int_x^{\oplus} M_x$ from a von Neumann algebra $N$ into a direct integral induces maps $N \to M_x$ for almost every $x$. This is however true when $N$ is a direct sum of type $I$ factors. Indeed, letting $K \subset N$ be the ideal of compact operators, we obtain maps $K \to M_x$ by standard separability arguments. One then uses the fact that a $C^*$-algebra homomorphism from $K$ into a von Neumann algebra extends uniquely to a von Neumann algebra homomorphism from $N$.

We can leverage this observation about direct sums of type $I$ factors to construct the desired maps $A(K) \to D_x(I)$. Consider a slightly larger interval $I^+$ that contains $I$, and let $K^+ \subset I^+$ be a white interval that contains $K$ in its interior.

\[
\begin{array}{c}
I^+ \\
K \\
I
\end{array}
\]

By the split property and the semisimplicity of $A$, we can find an intermediate algebra $A(K) \subset N \subset A(K^+)$ that is a direct sum of type $I$ factors. The map $N \to D(I^+) = \int_x^{\oplus} D_x(I^+)$ induces maps $\tilde{\iota}_x : N \to D_x(I^+)$ for almost every $x$; let

---

6Arbitrary direct sums of irreducible conformal nets would also work.
\( \iota_x \) denote the restriction of \( \tilde{\iota}_x \) to \( \mathcal{A}(K) \). The map \( f^\oplus \iota_x : \mathcal{A}(K) \to \int^\oplus D_x(I^+) \) is the composite of our original map \( \mathcal{A}(K) \to D(I) \) with the inclusion \( D(I) \hookrightarrow D(I^+) \). The image of \( f^\oplus \iota_x \) is contained in \( \int^\oplus D_x(I) \). For almost every \( x \) the image of \( \iota_x \) is therefore contained in \( D_x(I) \), and we have our desired maps \( \mathcal{A}(K) \to D_x(I) \).

The isotony, locality, and strong additivity axioms for \( D_x \) are immediate, and we omit their proofs.

The vacuum sector axiom requires a little bit more work. Let \( S, I, \text{ and } J \) be as in the formulation of the axiom, and let us assume without loss of generality that \( J \) is white. We need to show that, for almost every \( x \), a certain representation of \( \mathcal{A}(J) \otimes_{\text{alg}} \mathcal{A}(\bar{J}) \) on \( H_0(S, D_x) \) extends to \( \mathcal{A}(J \cup \bar{J}) \). We know that the corresponding representation of \( \mathcal{A}(J) \otimes_{\text{alg}} \mathcal{A}(\bar{J}) \) on \( H_0(S, D) = \int^\oplus H_0(S, D_x) \) does extend to \( \mathcal{A}(J \cup \bar{J}) \), and we want to show that the latter disintegrates into actions of \( \mathcal{A}(J \cup \bar{J}) \) on \( H_0(S, D_x) \). Clearly, the action of \( \mathcal{A}(J \cup \bar{J}) \) on \( H_0(S, D) \) commutes with that of \( Z(D) \), but that is not enough to guarantee actions of the individual fibers \( H_0(S, D_x) \).

Pick a white interval \( K \subset S \) that contains \( J \cup \bar{J} \) in its interior, and an intermediate algebra \( \mathcal{A}(J \cup \bar{J}) \subset N \subset \mathcal{A}(K) \) that is a direct sum of type \( I \) factors. By the same argument as before, the action of \( N \) on \( \int^\oplus H_0(S, D_x) \) disintegrates into actions on \( H_0(S, D_x) \), and therefore so does the action of \( \mathcal{A}(J \cup \bar{J}) \). \( \square \)

**Irreducible defects over semisimple nets.** In Section 1.3.3, we will define the operation of fusion of defects, which is the composition of 1-morphisms in the 3-category of conformal nets. That operation does not preserve irreducibility (even if the conformal nets are irreducible) and so, unlike for conformal nets, it is not advisable to restrict attention to irreducible defects.

We call a defect \( D \) faithful if the homomorphisms \( D(f) \) are injective for every embedding \( f : I \to J \) of bicolored intervals.

**Lemma 1.31.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be conformal nets, and let \( D \) be an \( \mathcal{A} \)-\( \mathcal{B} \)-defect that is irreducible and faithful. Then \( \mathcal{A} \) and \( \mathcal{B} \) are irreducible.

**Proof.** Let \( S \) be a genuinely bicolored circle, \( I \subset S \) a white interval and \( I' \) the closure of its complement. Since \( D \) is irreducible, the vacuum sector \( H_0(S, D) \) is acted on jointly irreducibly by the algebras \( \mathcal{A}(J), J \subset S \).

Since \( D \) is faithful, \( \mathcal{A}(I) \) acts faithfully on \( H_0(S, D) \). A non-trivial central projection \( p \in \mathcal{A}(I) \) would thus induce a non-trivial direct sum decomposition of \( H_0(S, D) \), contradicting the fact that it is irreducible. Indeed, for a bicolored interval \( J \subset S \), the projection \( p \) commutes with both \( D(J \cap I) \) and \( D(J \cap I') \). By strong additivity, \( p \) therefore commutes with \( D(J) \). \( \square \)

Here, as for conformal nets [2 Sec. 3.A], we have used the split property to extend the functor \( D \) to disjoint unions of bicolored intervals by setting

\[
D(I_1 \cup \ldots \cup I_n) := D(I_1) \otimes \ldots \otimes D(I_n).
\]

**Corollary 1.33.** Let \( \mathcal{A} = \bigoplus \mathcal{A}_i \) and \( \mathcal{B} = \bigoplus \mathcal{B}_j \) be semisimple conformal nets, where \( \mathcal{A}_i \) and \( \mathcal{B}_j \) are irreducible. Let \( D \) be an irreducible \( \mathcal{A} \)-\( \mathcal{B} \)-defect. Then there exist indices \( i \) and \( j \) such that \( D \) is induced from a faithful irreducible \( \mathcal{A}_i \)-\( \mathcal{B}_j \)-defect under the projections maps \( \mathcal{A} \to \mathcal{A}_i \) and \( \mathcal{B} \to \mathcal{B}_j \), respectively. \( \square \)

The above discussion shows that defects between semisimple conformal nets can be entirely understood in terms of defects between irreducible conformal nets. In the rest of this paper, we will therefore mostly restrict attention to irreducible conformal nets.

1.D. The category \( \text{CN}_1 \) of defects.
Definition 1.34. Defects form a symmetric monoidal category $\text{CN}_1$. An object in that category is a triple $(\mathcal{A}, \mathcal{B}, D)$, where $\mathcal{A}$ and $\mathcal{B}$ are semisimple conformal nets, and $D$ is a defect from $\mathcal{A}$ to $\mathcal{B}$. A morphism between the objects $(\mathcal{A}, \mathcal{B}, D)$ and $(\mathcal{A}', \mathcal{B}', D')$ is triple of natural transformations $\alpha: \mathcal{A} \to \mathcal{A}'$, $\beta: \mathcal{B} \to \mathcal{B}'$, $\delta: D \to D'$, with the property that $\delta|_{\text{INT}_\mathcal{A}} = \alpha$ and $\delta|_{\text{INT}_\mathcal{B}} = \beta$. The symmetric monoidal structure on this category is given by objectwise spatial tensor product.

Recall that a map between von Neumann algebras with finite-dimensional centers is said to be finite if the associated bimodule $A \otimes B$ is dualizable (Appendix A.1).

Definition 1.35. A natural transformation $\tau: D \to E$ between semisimple defects $\mathcal{A}D_B$ and $cE_D$ is called finite if $\tau_I : D(I) \to E(I)$ is a finite homomorphism for every $I \in \text{INT}_\mathcal{A}$.

Remark 1.36. We believe that the condition of having finite-dimensional centers is not really needed to define the notion of finite homomorphism between von Neumann algebras [1, Conjecture 6.17]. If that is indeed the case, then we can extend the notion of finite natural transformations to non-semisimple defects.

Recall from Appendix B.1 that $\text{CN}_0$ denotes the symmetric monoidal category of semisimple conformal nets and their natural transformations, and $\text{CN}_0'$ denotes the symmetric monoidal category of semisimple conformal nets all of whose irreducible summands have finite index, together with finite natural transformations. Later on, we will denote by $\text{CN}_1'$ the symmetric monoidal category of semisimple defects (between semisimple conformal nets all of whose irreducible summands have finite index), together with finite natural transformations. The category $\text{CN}_1$ is equipped with two forgetful functors

$$\text{source}: \text{CN}_1 \to \text{CN}_0 \quad \text{target}: \text{CN}_1 \to \text{CN}_0$$

given by $\text{source}(\mathcal{A}, \mathcal{B}, D) := \mathcal{A}$ and $\text{target}(\mathcal{A}, \mathcal{B}, D) := \mathcal{B}$, respectively, and a functor (1.37)

$$\text{identity}: \text{CN}_0 \to \text{CN}_1$$

given by $\text{identity}(\mathcal{A})(I) := \mathcal{A}(I)$, where we forget the bicoloring of $I$ in order to evaluate $\mathcal{A}$. We sometimes abbreviate $\text{identity}(\mathcal{A})$ by $L_\mathcal{A}$. Note that the above functors are all compatible with the symmetric monoidal structure.

Remark 1.38. A conformal net $\mathcal{A}$ also has a weak identity given on genuinely bi-colored intervals $I$ by $I \mapsto \mathcal{A}(I_\circ \cup [0,1] \cup I_\bullet)$. That defect is not isomorphic to $L_\mathcal{A}$ in the category $\text{CN}_1$. It is nevertheless equivalent to $L_\mathcal{A}$ in the sense that there is an invertible sector between them; see Example 3.5.

1.e. Composition of defects. Given conformal nets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and defects $\mathcal{A}D_B$ and $B_E \mathcal{C}$, we will now define their fusion $D \otimes_B E$, which is an $\mathcal{A}$-$\mathcal{C}$-defect if the conformal net $\mathcal{B}$ has finite index. If $\mathcal{B}$ does not have finite index, then $D \otimes_B E$ might still be a defect, but we do not know how to prove this.

If $I$ is in $\text{INT}_\mathcal{A}$ or $\text{INT}_\mathcal{C}$, then $(D \otimes_B E)(I)$ is given by $\mathcal{A}(I) \cup \mathcal{C}(I)$, respectively. If $I$ is genuinely bicolored, then we use the local coordinate to construct intervals

$I^+_\circ := I_\circ \cup [0, \frac{1}{2}], \quad I^{++}_\circ := I_\circ \cup [0, \frac{3}{2}], \quad I^- := [-\frac{1}{2}, 0] \cup I_\bullet, \quad I^{++} := [-\frac{3}{2}, 0] \cup I_\bullet$,

bicolored by

$I^+_\circ = I^{++}_\circ = I_\circ, \quad I^+_\circ = [0, \frac{1}{2}], \quad I^{++}_\circ = [0, \frac{3}{2}]$,

$I^- = I^{++} = I_\bullet, \quad I^- = [-\frac{1}{2}, 0], \quad I^{++} = [-\frac{3}{2}, 0]$.

Let $J := [0,1]$ and consider the maps $J \to I^{++}_\circ \hookrightarrow I^{++}$ and $J \to I^{++}_\circ \hookrightarrow I^{++}$ given by $x \mapsto \frac{1}{2} - x$ and $x \mapsto x - \frac{3}{2}$, respectively. These embeddings induce
homomorphisms $D(I^{++}) \leftarrow B(J)^{op}$ and $B(J) \to E^{(++)I}$ that we use to form the fusion of the von Neumann algebras $D(I^{++})$ and $E^{(++)I}$ (Definition A.3). We define

$$
(D \otimes_B E)(I) := \begin{cases} 
A(I) & \text{for } I \in \text{INT} \circ \,
D(I^{++}) \otimes_{B(J)} E^{(++)I} & \text{for } I \in \text{INT}.
\end{cases}
$$

Pictorially, this is

$$
(D \otimes_B E) \begin{array}{c}
\begin{array}{c}
\hline
\hline
\end{array}
\end{array} := D \begin{array}{c}
\begin{array}{c}
\hline
\hline
\end{array}
\end{array} \otimes_{B(J)} E \begin{array}{c}
\begin{array}{c}
\hline
\hline
\end{array}
\end{array}
$$

If $I$ is genuinely bicolored then, by Proposition 1.17, we have

$$(D \otimes_B E)(I) = (D(I^{++}) \cap B(J)^{op}) \lor (E^{(++)I} \cap B(J)') = D(I^{++}) \lor E^{(++)I},$$

where the algebras act on $H \boxtimes_{B(J)} K$ for some faithful $D(I^{++})$-module $H$ and some faithful $E^{(++)I}$-module $K$. Therefore, we obtain the following equivalent definition of composition of defects:

**Definition 1.41.** The algebra $(D \otimes_B E)(I)$ is the completion of the algebraic tensor product $D(I^{++}) \otimes_{alg} E^{(++)I}$ inside $B(H \boxtimes_{B(J)} K)$, where $H$ is a faithful $D(I^{++})$-module, and $K$ is a faithful $E^{(++)I}$-module.

We conjecture that that $D \otimes_B E$ is always an $A$-$C$-defect. Our first main theorem says that this holds when $B$ has finite index.

**Main Theorem 1.42.** Let $A$, $B$, and $C$ be irreducible conformal nets, and let us assume that $B$ has finite index. If $D$ is a defect from $A$ to $B$, and $E$ a defect from $B$ to $C$, then $D \otimes_B E$ is a defect from $A$ to $C$.

**Proof.** We first prove isotony. Let $I_1 \subset I_2$ be genuinely bicolored intervals, let $H$ be a faithful $D(I_2^{++})$-module and let $K$ be a faithful $E^{(++)I_2}$-module. By the isotony property of $D$ and $E$, the actions of $D(I_1^{++})$ on $H$ and of $E^{(++)I_1}$ on $K$ are faithful. Therefore, both $(D \otimes_B E)(I_1)$ and $(D \otimes_B E)(I_2)$ can be defined as subalgebras of $B(H \boxtimes_{B(J)} K)$. It is then clear that $(D \otimes_B E)(I_1)$ is a subalgebra of $(D \otimes_B E)(I_2)$.

We next show locality and strong additivity. Let $J \subset I$ and $K \subset I$ be bicolored intervals whose union is $I$ and that intersect in a single point. We assume without loss of generality that $K$ is white and that $J$ and $I$ are genuinely bicolored. In particular, we then have $I^{++} = J^{++}$. By the strong additivity of $D$, we have

$$A(K) \lor (D \otimes_B E)(J) = A(K) \lor D(J^{++}) \lor E^{(++)J} = D(I^{++}) \lor E^{(++)I} = (D \otimes_B E)(I),$$

which proves that $D \otimes_B E$ is also strongly additive. Since $D$ satisfies locality, the images of $A(K)$ and $D(J^{++})$ commute in $D(I^{++})$. The algebra $D(I^{++})$ commutes with $E^{(++)I} = E^{(++)J}$ by the definition of $\otimes$. It follows that all three algebras $A(K)$, $D(J^{++})$, and $E^{(++)J}$ commute with one another. The algebras $A(K)$ and $(D \otimes_B E)(J)$ therefore also commute, as required.

The vacuum axiom is much harder. Let us first assume that $D$ and $E$ are irreducible. Let $J \subset I$ be as in the formulation of the vacuum axiom (Definition 1.7), and let us assume without loss of generality that $J$ is white. We need to show that the $A(J) \otimes_{alg} A(J)$-module structure on $L^2((D \otimes_B E)(I))$ given by (1.8, 1.9) extends to an action of $A(J \cup J)$. This will follow from the existence of an injective homomorphism from $L^2((D \otimes_B E)(I))$ into some other $A(J) \otimes_{alg} A(J)$-module that is visibly an $A(J \cup J)$-module. The desired homomorphism is (1.19) and will be constructed in Proposition 1.18. The fact that $A(J \cup J)$ acts on the codomain of (1.19) is an immediate consequence of the vacuum axiom for $D$. 


For general defects $D$ and $E$, write them as direct integrals $D = \int D_x$ and $E = \int E_y$ of irreducible defects, and note that $D \otimes_B E = \int \int D_x \otimes_B E_y$ is a defect by Remark 1.28.

In view of Corollary 1.33 and the fact that any defect between semisimple conformal nets can be disintegrated into irreducible defects (Lemma 1.30), the above theorem generalizes in a straightforward way to the situation where $A$, $B$, and $C$ are not necessarily irreducible but merely semisimple: in this case, if all the irreducible summands of $B$ have finite index, then the composition of an $A$-$B$-defect with a $B$-$C$-defect is an $A$-$C$-defect.

One might hope that composition of defects induces a functor

$$\text{composition}: \mathsf{CN}_1 \times_{\mathsf{CN}_0} \mathsf{CN}_1 \to \mathsf{CN}_1.$$  

However, some caution is needed. First, we used the finite index condition on $B$ for our proof that $D \otimes_B E$ is a defect. Second and more important, the operation of fusion of von Neumann algebras is only functorial with respect to isomorphisms of von Neumann algebras: given homomorphisms $A_1 \leftarrow C_{1}^{\text{opp}}, C_1 \to B_1$ and $A_2 \leftarrow C_{2}^{\text{opp}}, C_2 \to B_2$ it is not true that a triple of maps $a : A_1 \to A_2, b : B_1 \to B_2, c : C_1 \to C_2$ (subject to the obvious compatibility conditions) induces a map

$$a \circ_c b : A_1 \otimes_{C_1} B_1 \to A_2 \otimes_{C_2} B_2. \tag{1.44}$$

Moreover, requiring that the maps $a, b,$ and $c$ be finite homomorphisms does not help to construct the map (1.44). However, unlike the fusion of von Neumann algebras, the composition of defects is functorial for more than just isomorphisms.

**Proposition 1.45.** Let $a : A_1 \to A_2$, $b : B_1 \to B_2$, and $c : C_1 \to C_2$ be natural transformations between irreducible conformal nets. Let $\mathcal{A}_1, D_{1, B_1}, A_2, D_{2, B_2}, E_1, C_1$, and $E_2, C_2$ be defects, and let $d : D_1 \to D_2$, $e : E_1 \to E_2$ be natural transformations such that $d|_{\text{INT}_{\mathcal{A}_1}} = a, d|_{\text{INT}_{\mathcal{A}_2}} = c|_{\text{INT}_{\mathcal{A}_1}} = b$, and $e|_{\text{INT}_{\mathcal{B}_1}} = c$. Assume moreover that $b$ is finite (Appendix B.IV and Appendix A.IV). Then the above maps induce a natural transformation $D_1 \otimes_{B_1} E_1 \to D_2 \otimes_{B_2} E_2$.

Moreover, if $D$ and $E$ are semisimple and if $d$ and $e$ are finite, then the defects $D_1 \otimes_{B_1} E_1$ are semisimple and the above natural transformation is finite.

**Proof.** The semisimplicity of $D_1 \otimes_{B_1} E_1$ is the content of Theorem 3.6. Given a genuinely bicolored interval $I$, we need to construct a homomorphism $(D_1 \otimes_{B_1} E_1)(I) \to (D_2 \otimes_{B_2} E_2)(I)$. We assume without loss of generality that $d$ and $e$ are faithful (otherwise, their kernels are direct summands). Let $H$ be a faithful $D_2(I^{+})$-module, and let $K$ be a faithful $E_2(\overleftarrow{I})$-module. By [1, Thm 6.23], the natural transformation $b$ induces a bounded linear map $H \boxtimes_{B_2(J)} K \to H \boxtimes_{B_2(J)} K$ which is surjective by construction. This map is equivariant with respect to the homomorphism $D_1(I^+) \otimes_{\text{alg}} E_1(\overrightarrow{I}) \to D_2(I^+) \otimes_{\text{alg}} E_2(\overrightarrow{I})$, and therefore induces a map from the completion of $D_1(I^+) \otimes_{\text{alg}} E_1(\overrightarrow{I})$ in $\mathcal{B}(H \boxtimes_{B_2(J)} K)$ to the completion of $D_2(I^+) \otimes_{\text{alg}} E_2(\overrightarrow{I})$ in $\mathcal{B}(H \boxtimes_{B_2(J)} K)$. The resulting homomorphism $(D_1 \otimes_{B_1} E_1)(I) \to (D_2 \otimes_{B_2} E_2)(I)$ is finite by [1, Lem. 7.18].

We suspect that the functor (1.43) does not exist as stated. However, instead of trying to compose over the full category $\mathsf{CN}_0$ of semisimple conformal nets, we can restrict attention to the subcategory $\mathsf{CN}_0^{\text{fin}} \subset \mathsf{CN}_0$ of semisimple conformal nets all of whose irreducible summands have finite index, together with their finite index.

\[\text{Note that there is a missing assumption in [1, Lem. 7.18]: the map } h : H_1 \to H_2 \text{ should be surjective.}\]
natural transformations. If we let $\text{CN}_1 \times \text{CN}_3 \text{CN}_4$ be a shorthand notation for $\text{CN}_1 \times \text{CN}_3 \text{CN}_4 \times \text{CN}_4 \text{CN}_4$, then the composition functor

\begin{equation}
\text{composition} : \text{CN}_1 \times \text{CN}_3 \text{CN}_4 \rightarrow \text{CN}_4
\end{equation}

exists by Theorem 1.4.3 and Proposition 1.4.3.

1.6. Associativity of composition. It will be convenient to work with the square model $S^1 := \partial[0,1]^2$ of the “standard circle” (see the beginning of Section 2.1) and to use the following notation.

**Notation 1.4.7.** Given real numbers $a < b$ and $c < d$ and $M = [a, b] \times [c, d]$, we let

\begin{align*}
\partial^C M & := \{(a) \times [c, d] \cup ([a, b] \times \{c, d\}) \}, \\
\partial^F M & := \{(a) \times [c, d] \cup ([a, b] \times \{d\}) \}, \\
\partial^L M & := \{(b) \times [c, d] \cup ([a, b] \times \{c\}) \}, \\
\partial^R M & := \{(b) \times [c, d] \cup ([a, b] \times \{d\}) \}, \\
\partial^I M & := \{(a, b) \times [c, d] \cup ([a, b] \times \{c\}) \}, \\
\partial^J M & := \{(b, c) \times [c, d] \cup ([a, b] \times \{d\}) \}
\end{align*}

be the subsets of $\partial M$ hinted by the pictorial superscript.

**Definition 1.4.8.** The standard bicolored circle is $S^1 = \partial[0,1]^2$ with bicoloring $S^1_{\uparrow} = \partial^C([0, 1/2] \times [0, 1])$ and $S^1_{\downarrow} = \partial^R([1/2, 1] \times [0, 1])$. The upper half of this circle is $S^1_{\uparrow} = \partial^F([0, 1] \times [1/2, 1])$, and the standard involution is $(x, y) \mapsto (x, 1 - y)$. The vacuum sector associated to the standard bicolored circle, its upper half, and its standard involution, is

$$H_0(D) := L^2(D(S^1_{\uparrow})).$$

It has left actions of $D(I)$ for every interval $I \subset S^1$ such that $(\frac{1}{2}, 0)$ and $(\frac{1}{2}, 1)$ are not both in $I$ and $\{(\frac{1}{2}, 0), (\frac{3}{2}, 1)\} \cap \partial I = \emptyset$.

The fiber product $\otimes$ of von Neumann algebras was studied in [28]. It is an alternative to the fusion $\oplus$ of von Neumann algebras which remedies the formal shortcomings of the fusion operation (see Appendix A.14). In view of this, one might rather have defined the composition of defects as

\begin{equation}
(D \ast_B E)(I) := \begin{cases} A(I) & \text{for } I \in \text{INT}_o \\
D(I^{++}) \ast_{B,J} E(I^{++}) & \text{for } I \in \text{INT}_s \\
C(I) & \text{for } I \in \text{INT}_s \end{cases}
\end{equation}

where $I^{++}$, $++I$, and $J$ are as in (1.3.9). This is related to the previous definition (1.3.9) as follows. Let $S^A$ be the standard bicolored circle (see Definition 1.48), with upper and lower halves $S^A_{\uparrow}$ and $S^A_{\downarrow}$.

**Lemma 1.5.0.** Let $D_B$ and $E_C$ be defects, with corresponding vacuum sectors $H := H_0(D)$ and $K := H_0(E)$. Viewed as algebras acting on $H \boxtimes_B K$, we then have

$$\begin{align*}
(D \ast E)(S^A_{\uparrow}) & = ((D \oplus E)(S^A_{\uparrow}))' \\
(D \ast E)(S^A_{\downarrow}) & = ((D \oplus E)(S^A_{\downarrow}))'.
\end{align*}$$

**Proof.** Using a graphical representation as in (1.3.9), we have:

\begin{align*}
(D \ast E)(S^A_{\uparrow}) & = D(S^A_{\uparrow}) \ast_{B,C} E(S^A_{\uparrow}) \\
& = \left(D(S^A_{\uparrow}) \otimes \text{alg} E(S^A_{\uparrow})\right)' \quad \text{(commutants taken on } H, K, \text{ and } H \boxtimes_B K, \text{ respectively}) \\
& = \left(D(S^A_{\uparrow}) \vee E(S^A_{\downarrow})\right)' = ((D \oplus E)(S^A_{\downarrow}))'.
\end{align*}
where the third equality follows by Haag duality (Proposition 1.17). The second equation is similar. □

When the conformal net $\mathcal{B}$ has finite index (and conjecturally even without that restriction), the two definitions of fusion (1.53) and (1.49) actually agree:

**Theorem 1.51.** Let $\mathcal{A}_D\mathcal{B}$ and $\mathcal{B}_E\mathcal{C}$ be defects. If $\mathcal{B}$ has finite index, then for every bicolored interval $I$ the inclusion

\begin{equation}
(D \circ\mathcal{B} E)(I) \hookrightarrow (D \ast\mathcal{B} E)(I)
\end{equation}

is an isomorphism.

**Proof.** If $I \in \text{INT}_\circ$ or $I \in \text{INT}_\bullet$, then there is nothing to show. Recall the Notation (1.47): for $I := \partial^\circ((0,1],[\frac{1}{2},1])$, with bicoloring $I_\circ := I_{x\leq \frac{1}{2}}$ and $I_\bullet := I_{x\geq \frac{1}{2}}$, the equality $(D \circ\mathcal{B} E)(I) = (D \ast\mathcal{B} E)(I)$ is the content of Corollary 5.9. The result follows as every genuinely bicolored interval is isomorphic to this interval. □

Using the above theorem, the associator

\begin{equation}
(D \circ\mathcal{B} E) \circ\mathcal{C} F \equiv D \circ\mathcal{B} (E \circ\mathcal{C} F)
\end{equation}

is then induced from the associator for the operation $\ast$ of fiber product of von Neumann algebras. If $I$ is a genuinely bicolored interval, then evaluating the two sides of (1.53) on $I$ yields

\begin{equation}
(D(I^{++}) \ast_{\mathcal{B}(I)} E(K)) \ast_{\mathcal{C}(J')} F^{(++)}(I) \quad \text{and} \quad D(I^{++}) \ast_{\mathcal{B}(I)} (E(K) \ast_{\mathcal{C}(J')} F^{(++)}(I)),
\end{equation}

where

\[
I^{++} = I_\circ \cup [0,\frac{3}{4}], \quad K = [-\frac{3}{4},\frac{3}{4}], \quad ++I = [-\frac{3}{4},0] \cup I_\bullet, \quad J = J' = [0,1],
\]

and the embeddings $I^{++} \hookrightarrow J \hookrightarrow K \hookrightarrow J' \hookrightarrow ++I$ are as in (1.39). The associator relating the two sides of (1.54) (see [28] Prop. 9.2.8) for a construction) is the desired natural isomorphisms (1.53). The properties of (1.53) can then be summarized by saying that it provides a natural transformation

\begin{equation}
\text{associator: } \text{CN}_1 \times_{\text{CN}_0} \text{CN}_1 \times_{\text{CN}_0} \text{CN}_1 \rightarrow \text{CN}_1
\end{equation}

that is an associator for the composition (1.46). This associator satisfies the pentagon identity by the corresponding pentagon identity for the operation $\ast$.

2. **Sectors**

We will use the constant speed parametrization to identify the standard circle $\{z \in \mathbb{C} : |z| = 1\}$ with the boundary of the unit square $\partial [0,1]^2$. Under our identification, the points 1, $i$, $-1$, and $-i$ get mapped to (1, $\frac{1}{2}$), ($\frac{1}{2}$, 1), (0, $\frac{1}{2}$), and ($\frac{1}{2}$, 0), respectively.

Recall the Notation (1.47): Our standard circle $S^1 = \partial [0,1]^2$ has a standard bicoloring given by $S^1_\circ := \partial^\circ((0,\frac{1}{2}) \times [0,1])$ and $S^1_\bullet := \partial^\bullet((\frac{1}{2},1] \times [0,1])$. Let $\text{INT}_{S^1}$ be the poset of subintervals of $S^1$, and let $\text{INT}_{S^1,\bullet}$ be the sub-poset of intervals $I \subset S^1$ such that $(I \cap S^1_\circ, I \cap S^1_\bullet)$ is a bicoloring. Thus, an interval $I$ is in $\text{INT}_{S^1,\bullet}$ if neither of the color-change points ($\frac{1}{2},0$) and ($\frac{1}{2},1$) are in its boundary, and if both $I \cap S^1_\circ$ and $I \cap S^1_\bullet$ are connected (possibly empty). We view $\text{INT}_{S^1,\bullet}$ as a (non-full) subcategory of $\text{INT}_\bullet$. 
2. A. The category $\text{CN}_2$ of sectors. The elements of $\text{INT}_{S^1,\circ}$ naturally fall into four classes:

$$\text{INT}_{S^1,\circ} := \{ I \subseteq S^1 | I \cap S^1_\ast = \emptyset \}$$
$$\text{INT}_{S^1,\ast} := \{ I \subseteq S^1 | I \cap S^1_\circ = \emptyset \}$$
$$\text{INT}_{S^1,\top} := \{ I \subseteq S^1 | (\frac{1}{2}, 1) \in I \text{ and } (\frac{1}{2}, 0) \notin I \}$$
$$\text{INT}_{S^1,\bot} := \{ I \subseteq S^1 | (\frac{1}{2}, 1) \notin I \text{ and } (\frac{1}{2}, 0) \in I \}.$$  

(2.1)

**Definition 2.2.** Let $\mathcal{A}$ and $\mathcal{B}$ be conformal nets, and let $\mathcal{A}D_B$ and $\mathcal{A}E_B$ be $A$-$B$-defects. A $D$-$E$-sector is a Hilbert space $H$, equipped with homomorphisms

$$\rho_1: A(I) \to B(H), \quad \text{for } I \in \text{INT}_{S^1,\circ}$$
$$\rho_2: D(I) \to B(H), \quad \text{for } I \in \text{INT}_{S^1,\top}$$
$$\rho_3: B(I) \to B(H), \quad \text{for } I \in \text{INT}_{S^1,\ast}$$
$$\rho_4: E(I) \to B(H), \quad \text{for } I \in \text{INT}_{S^1,\bot}$$

(2.3)

subject to the condition $\rho_2 | J = \rho_3$ whenever $J \subset I$. Moreover, if $I \in \text{INT}_{S^1,\top}$ and $J \in \text{INT}_{S^1,\bot}$ are intervals with disjoint interiors, then $\rho_2(D(I))$ and $\rho_3(E(J))$ are required to commute with each other. We write $D_H E$ to indicate that $H = (H, \rho)$ is a $D$-$E$-sector. If $D = E$, then we say that $H$ is a $D$-sector.

Pictorially we will draw a $D$-$E$-sector as follows:

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\end{array} \quad \begin{array}{c}
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\top
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\bot
\end{array}
\end{array}
\end{array}$$

The thin line stands for the conformal net $\mathcal{A}$ and the thick line stands for $\mathcal{B}$.

**Remark 2.4.** If $I \in \text{INT}_{S^1,\top}$ and $J \in \text{INT}_{S^1,\bot}$ are disjoint intervals, we do not require that the action of $D(I) \otimes \text{alg } E(J)$ extends to an action of $D(I) \otimes E(J)$.

Recall from Proposition 1.23 that if $\mathcal{A}$ and $\mathcal{B}$ are both equal to the trivial conformal net $C_\infty$, then a $C_\infty$-$C_\infty$-defect may be viewed simply as a von Neumann algebra. A $D$-$E$-sector between two such defects is given by a bimodule between the corresponding von Neumann algebras.

The following lemma is a straightforward analog of [24 Lem. 1.9].

**Lemma 2.5.** Let $S^1$ be the standard bicolored circle, and let $\{ I_i \subseteq S^1 \}$ be bicolored intervals whose interiors cover $S^1$. Suppose that we have actions

$$\rho_1: A(I_i) \to B(H), \quad \text{for } I_i \in \text{INT}_{S^1,\circ}$$
$$\rho_2: D(I_i) \to B(H), \quad \text{for } I_i \in \text{INT}_{S^1,\top}$$
$$\rho_3: B(I_i) \to B(H), \quad \text{for } I_i \in \text{INT}_{S^1,\ast}$$
$$\rho_4: E(I_i) \to B(H), \quad \text{for } I_i \in \text{INT}_{S^1,\bot}$$

subject to the following two conditions: 1. $\rho_2 |_{I_i \cap I_j} = \rho_3 |_{I_i \cap I_j}$, 2. if $I_i \cap I_j = \emptyset$, then the images of $\rho_1$ and $\rho_2$ commute. Then these actions endow $H$ with the structure of a $D$-$E$-sector.

**Proof.** Given an interval $J \subset S^1$, pick a diffeomorphism $\varphi \in \text{Diff}_+(S^1)$ that is trivial in a neighborhood $N$ of the two color changing points, and such that $\varphi(J) \subset I_{u_0}$ for some $I_{u_0}$ in our cover. Write $\varphi$ as $\varphi_1 \circ \cdots \circ \varphi_n$ for diffeomorphisms $\varphi_k$ that are trivial on $N$ and whose supports lie in elements of the cover. Let $u_k$ be unitaries implementing $\varphi_k$ (Proposition 1.10). Upon identifying $u_k$ with its image (under the relevant $\rho_k$) in $B(H)$, we set

$$\rho_f(a) := u_n^* \cdots u_1^* \rho_{u_0}(\varphi(a)) u_1 \cdots u_n.$$  

(2.6)
Here we have used \( \varphi(a) \) as an abbreviation for \( A(\varphi)(a), D(\varphi)(a), B(\varphi)(a), \) or \( E(\varphi)(a) \), depending on whether \( J \) is a white, top, black, or bottom interval. Finally, as in the proof of \([2, \text{Lem. 1.9}]\), one checks that \( \rho_{J|K} = \rho_{J|K}^I \) for any sufficiently small interval \( K \subset J \cap I \), and then uses strong additivity to conclude that \( \rho_{J|J \cap I} = \rho_{J|J \cap I}^I \).

As before let \( S^1_+ = \partial^+(\{0,1\} \times [0,1]) \) be the upper and lower halves of the standard bicolored circle.

**Definition 2.7.** Sectors form a category that we call \( \text{CN}_2 \). Its objects are quintuples \((A, B, D, E, H)\), where \( A, B \) are semisimple conformal nets, \( D, E \) are \( A\)-\( B\)-defects, and \( H \) is a \( D\)-\( E\)-sector. A morphism from \((A, B, D, E, H)\) to \((A’, B’, D’, E’, H’)\) consists of four compatible invertible natural transformations \[ \alpha : A \to A’, \quad \beta : B \to B’, \quad \delta : D \to D’, \quad \varepsilon : E \to E’, \] along with a bounded linear map \( h : H \to H’ \) that is equivariant in the sense that \[ \rho_{J}(\alpha(a)) \circ h = h \circ \rho_{J}(a), \quad \rho_{J}(\beta(b)) \circ h = h \circ \rho_{J}(b), \quad \rho_{J}(\delta(d)) \circ h = h \circ \rho_{J}(d), \quad \rho_{J}(\varepsilon(e)) \circ h = h \circ \rho_{J}(e), \] for \( a \in \mathcal{A}(I), b \in \mathcal{B}(I), d \in \mathcal{D}(I), e \in \mathcal{E}(I), \) and \( I \in 1 \text{NT}_{S_{1,\infty}}, 1 \text{NT}_{S_{1,\bullet}}, 1 \text{NT}_{S_{1,\text{top}}}, 1 \text{NT}_{S_{1,\text{bot}}} \), respectively.

There is also a symmetric monoidal structure on \( \text{CN}_2 \) given by objectwise spatial tensor product for the functors \( A, B, D, E, \) and by tensor product of Hilbert spaces.

The category \( \text{CN}_2 \) is equipped with two forgetful functors

\[
\text{source}_*: \text{CN}_2 \to \text{CN}_1 \quad \text{target}_*: \text{CN}_2 \to \text{CN}_1
\]

called ‘vertical source’ and ‘vertical target’, given by \( \text{source}_*(A, B, D, E, H) = (A, B, D) \) and \( \text{target}_*(A, B, D, E, H) = (A, B, E) \). They satisfy

\[
\text{source} \circ \text{source}_* = \text{source} \circ \text{target}_* \quad \text{and} \quad \text{target} \circ \text{source}_* = \text{target} \circ \text{target}_*.
\]

Provided we restrict to the subcategory \( \text{CN}_1^I \subset \text{CN}_1 \) whose objects are semisimple defects between semisimple conformal nets and whose morphisms are finite natural transformations (another option is to allow all defects between semisimple conformal nets but restrict the morphisms to be only the isomorphisms), there is also a ‘vertical identity’ functor

\[
\text{id}_{\text{source}} : \text{CN}_1^I \to \text{CN}_2
\]

that sends an \( A\)-\( B\)-defect \( D \) to the object \((A, B, D, D, H_0(D))\) of \( \text{CN}_2 \). Here, the vacuum sector \( H_0(D) := L^2(D(S^1_+)) = L^2(D(\{0,1\})) \) is as described in Definition \(1.48\). We represent it pictorially as follows:

\[
\text{id}_{\text{source}}(A D_B) = \begin{array}{c}
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Remark 2.10. We will see later, in Warning 6.8, that we will have to further restrict our morphisms, and only allow natural isomorphisms between defects (even if the defects are semisimple). This will render otiose the subtleties related to Conjecture 6.17; in particular, there is no need to restrict to semisimple defects.

2.B. Horizontal fusion. Consider the translate $S^1_+:=[0,1] \subset \mathbb{R}^2$ of the standard circle $S^1 = \partial [0,1]^2$, and let $\text{INT}_{S^1_+}$, $\text{INT}_{S^1_+}$, $\text{INT}_{S^1_+}$ be the obvious analogs of (2.1). Given conformal nets $A, B, C$, defects $\mathcal{A}D_B, \mathcal{A}E_B, B \mathcal{F}_C, B \mathcal{G}_C$, and sectors $D_{H_E}, F_{K_G}$, let us replace the structure maps (2.3) of $K$ by actions

$$\rho_I : B(I) \to B(K) \text{ for } I \in \text{INT}_{S^1_+},$$

$$\rho_I : C(I) \to B(K) \text{ for } I \in \text{INT}_{S^1_+},$$

$$\rho_I : G(I) \to B(K) \text{ for } I \in \text{INT}_{S^1_+},$$

by precomposing with the translation. Let $J$ be $\{1\} \times [0,1] = S^1 \cap S^1_+$, with the orientation inherited from $S^1_+$. The algebra $B(J)$ has actions of opposite variance on $H$ and on $K$, so it makes sense to take the Connes fusion

$$H \boxtimes_B K := H \boxtimes_{B(J)} K.$$

We now show that $H \boxtimes_B K$ is a $(D \boxplus_B F)-(E \boxplus_B G)$-sector. Given $I \in \text{INT}_{S^1_+}$, let $I_+$ be the subinterval of $\partial ([0,2] \times [0,1])$ given by

$$I_+ := I \text{ if } I \in \text{INT}_{S^1_+},$$

$$I_+ := I + (1,0) \text{ if } I \in \text{INT}_{S^1_+},$$

$$I_+ := I_0 \cup \left( (\frac{1}{2}, \frac{3}{2}) \times \{1\} \right) \cup (I_+ + (1,0)) \text{ if } I \in \text{INT}_{S^1_+},$$

$$I_+ := I_0 \cup \left( (\frac{1}{2}, \frac{3}{2}) \times \{0\} \right) \cup (I_+ + (1,0)) \text{ if } I \in \text{INT}_{S^1_+}.$$

For $I \in \text{INT}_{S^1_+}$ and $I \in \text{INT}_{S^1_+}$, the structure maps (2.3) are given by the obvious actions of $\mathcal{A}(I_+)$ and $\mathcal{C}(I_+)$ on the Hilbert space $H \boxtimes_B K$. For an interval $I \in \text{INT}_{S^1_+}$ or $I \in \text{INT}_{S^1_+}$, the algebras that act on $H \boxtimes_B K$ are given by

$$D\left((I_+ \cap S^1) \cup J\right) \boxplus_{B(J)} F\left((I_+ \cap S^1) \cup J\right)$$

and

$$E\left((I_+ \cap S^1) \cup J\right) \boxplus_{B(J)} G\left((I_+ \cap S^1) \cup J\right)$$

respectively—see Appendix A.4. Upon identifying the intervals $(I_+ \cap S^1) \cup J$ and $J \cup (I_+ \cap S^1)$ of (2.11) with the intervals $I^{++}$ and $++I$ of (1.39), we see that the algebras (2.11) are equal to $(D \boxplus_B F)(I)$ and $(E \boxplus_B G)(I)$, respectively. We can now define the functor

$$(2.12) \quad \text{fusion}_h : \text{CN}_2 \times \text{CN}_0 \to \text{CN}_2$$

by $\text{fusion}_h \left( (A, B, D, E, H), (B, C, F, G, K) \right) := (A, C, D \boxplus_B F, E \boxplus_B G, H \boxtimes_B K)$. Here, as in (1.40), $\text{CN}_2 \times \text{CN}_0 \text{CN}_2$ is a shorthand notation for $\text{CN}_2 \times \text{CN}_0 \text{CN}_0 \times \text{CN}_0 \text{CN}_2$, and the relevant maps $\text{CN}_2 \to \text{CN}_0$ are target $\circ \text{source}_v$ and $\text{source} \circ \text{source}_v$, respectively.

Pictorially, we understand the functor $\text{fusion}_h$ as the operation of gluing two squares along a common edge as follows:

$$(2.12) \quad \text{fusion}_h \left( \begin{array}{c} D \\ E \\ H \\ B \\ F \\ K \\ C \end{array} \right) = \begin{array}{c} D \boxtimes_B F \\ E \boxtimes_B G \\ H \boxtimes_B K \end{array}$$
The associator for fusion is denoted by the usual associator for Connes fusion. It consists of a natural transformation 
\begin{equation}
\text{assoc}_{\text{fusion}} : CN_2 \times_{CN_0'} CN_2 \times_{CN_0'} CN_2 \rightarrow CN_2,
\end{equation}
and satisfies the pentagon identity.

2.c. **Vertical fusion.** We now describe the functor fusion of vertical fusion. Given conformal nets $A$, $B$, defects $AD_A$, $AE_B$, $AF_B$, and sectors $D_H$, $E_{KF}$, we want to construct a $D$-$F$-sector $H \boxtimes_E K$. Let $S_1^T = \partial^T([0,1] \times \frac{1}{2}, 1)$ and $S_1^B = \partial^B([0,1] \times \frac{1}{2}, \frac{1}{2})$ be the top and bottom halves of our standard circle $\partial[0,1]^2$, and let $j : S_1^T \rightarrow S_1^B$ be the reflection map along the horizontal symmetry axis. The algebra $E(S_1^T)$ has two actions
\begin{align}
E(S_1^T) & \rightarrow B(H) \\
E(S_1^B) & \rightarrow B(K)
\end{align}
of opposite variance on $H$ and $K$, and so it makes sense to take the Connes fusion
\[ H \boxtimes_E K := H \boxtimes_{E(S_1^T)} K. \]

To see that $H \boxtimes_E K$ is a $D$-$F$-sector, we have to show that the algebras $A(I)$, $B(I)$, $D(I)$, and $F(I)$ act on it for $I \in \text{INT}_{S_1^T, o}$, $\text{INT}_{S_1^B, o}$, $\text{INT}_{S_1^B, \bullet}$, and $\text{INT}_{S_1^B, \nu}$, respectively.

We first treat the case $I \in \text{INT}_{S_1^T, o}$. If $I$ is contained in $S_1^T$ (or $S_1^B$), then the action of $A(I)$ on $H \boxtimes_E K$ is induced by its action on $H$ (or $K$). If $I$ contains the point $(0, \frac{1}{2})$ in its interior, then the algebra
\begin{equation}
E(I \cup S_1^T) \otimes_{E(S_1^T)} E(I \cup S_1^B)
\end{equation}
acts on $H \boxtimes_{E(S_1^T)} K$, where the homomorphism $E(S_1^T) \rightarrow (E(I \cup S_1^T))^{op}$ implicit in (2.14) is given by $E(j)$. We observe, as follows, that there is a canonical homomorphism (typically not an isomorphism) from $A(I)$ to the algebra (2.14). In the definition of that fusion product, we are free to chose any faithful $E(I \cup S_1^T)$-module and any faithful $E(I \cup S_1^B)$-module (see Appendix A.IV): let us take both of them to be the vacuum $H_0(E)$. Then, by definition, the algebra (2.14) is generated on
\[ H_0(E) \boxtimes_{E(S_1^T)} H_0(E) \cong H_0(E) \]
by $E(I \cup S_1^T) \cap E(S_1^T)'$ and $E(I \cup S_1^B) \cap E(S_1^B)'$. By the vacuum and locality axioms, we have natural homomorphisms
\[ A(I) \rightarrow E(I \cap S_1^T) \vee E(I \cap S_1^B) \]
\[ \cong (E(I \cap S_1^T) \cap E(S_1^T)') \vee (E(I \cap S_1^B) \cap E(S_1^B)') \]
\[ = E(I \cap S_1^T) \otimes_{E(S_1^T)} E(I \cap S_1^B). \]

Composing this composite with the action of (2.14) on $H \boxtimes_E K$ gives our desired action of $A(I)$.

By the same argument, we also have actions of $B(I)$ on $H \boxtimes_E K$ for $I \in \text{INT}_{S_1^B, \bullet}$. Furthermore, there are actions of $D(S_1^T)$ and $F(S_1^B)$ on $H \boxtimes_E K$ coming from their respective actions on $H$ and on $K$. We can therefore apply Lemma 2.5 to all the actions constructed so far, and conclude that $H \boxtimes_E K$ is a $D$-$F$-sector.

One might expect vertical fusion to be a functor $CN_2 \times_{CN_1} CN_2 \rightarrow CN_2$. However, as the vertical identity (2.13) is only a functor on the smaller category $CN_1'$, and the horizontal fusion is only a functor on the restricted product $CN_2 \times_{CN_1'} CN_2$, so too vertical fusion only gives a functor on the restricted product:
\begin{equation}
fusion : CN_2 \times_{CN_1'} CN_2 \rightarrow CN_2
\end{equation}
\[ \text{fusion}_v((A, B, D, E, H), (A, B, E, F, K)) := (A, B, D, F, H \boxtimes_E K). \]
The restriction is necessary to ensure the Connes fusion $H \boxtimes_E K$ is functorial with respect to the relevant natural transformations of the defect $E$ \[.\]

Unlike horizontal fusion, vertical fusion is not the operation of gluing two squares along a common edge. Rather, it consists of gluing those two squares along half of their boundary:

$$\text{fusion}_v \left( \begin{array}{c} D \\ H \\ E \\ B \end{array} , \begin{array}{c} A \\ E \\ F \\ K \end{array} \right) = \begin{array}{c} D \\ H \\ F \\ K \end{array}.$$  

Definition 2.16. A $D$-$E$-sector $H$ between semisimple defects is called dualizable if it is dualizable \[ Def. 4.4\] as an $S^1_\cdot (D)$-$S^1_\cdot (E)$-bimodule.

The associator for vertical fusion
\[(2.17)\]  
$$\text{associator}_v : \text{CN}_2 \times_{\text{CN}_1} \text{CN}_2 \times_{\text{CN}_1} \text{CN}_2 \boxtimes \text{CN}_2$$
comes from the associator of Connes fusion and satisfies the pentagon identity.

There are also ‘top’ and ‘bottom’ identity natural transformations,
\[(2.18)\]  
$$\text{unitor}_t : \text{CN}_2 \boxtimes \text{CN}_2, \quad \text{unitor}_b : \text{CN}_2 \boxtimes \text{CN}_2$$
that describe the way fusion$_v$ and identity$_v$ interact. Given a sector $DHE$, they provide natural isomorphisms
\[(2.19)\]  
$$D H_0(D) \boxtimes_D H_E \cong D H_E \quad \text{and} \quad D H \boxtimes_E H_0(E) \cong D H_E$$
subject to the usual triangle axioms. Strictly speaking, the source functor of unitor$_t$ is only defined on the subcategory $\text{CN}_1 \times_{\text{CN}_1} \text{CN}_2$ of $\text{CN}_2$, and so the transformation unitor$_t$ itself is only defined on that subcategory. Similarly, unitor$_b$ is only defined on the subcategory $\text{CN}_2 \times_{\text{CN}_1} \text{CN}_1$.

3. Properties of the composition of defects

3.1. Left and right units. Units are a subtle business. One might guess that the left unit is a natural isomorphism $\text{CN}_1 \boxtimes \text{CN}_1$ whose source is the functor composition $\circ (\text{id} \circ \text{source}) \times \text{id}_{\text{CN}_1}$ and whose target is the identity functor. (Here $\text{id}_{\text{CN}_1} : \text{CN}_1 \to \text{CN}_1$ is the identity functor and $\text{id} : \text{CN}_0 \to \text{CN}_1$ takes a net to the identity defect, as in (1.37).) But, unfortunately, in general there is no such natural isomorphism. Instead, we have the following ‘weaker’ piece of data: a functor

$$\text{unitor}_t : \text{CN}_1 \to \text{CN}_2$$
(‘tl’ stands for top left) with the property that

$$\text{source}_v \circ \text{unitor}_t = \text{composition} \circ (\text{id} \circ \text{source}) \times \text{id}_{\text{CN}_1},$$  
and  
$$\text{target}_v \circ \text{unitor}_t = \text{id}_{\text{CN}_1}.$$  

The construction of this functor is based on the following lemma.

Lemma 3.1. Let $A D B$ be a defect. Then $1 \circ D := \text{id}(A) \circ_A D$ is given on genuinely bicolored intervals $I$ by

$$(1 \circ D)(I) = D(<I)$$
where $<I := I_\circ \cup [0, 1] \cup I_\bullet$ with bicoloring $<I_\circ := I_\circ \cup [0, 1]$ and $<I_\bullet := I_\bullet.$
Similarly, on genuinely bicolored intervals we have

\[(D \otimes 1)(I) = D(I^+)\]

where \(I^+ := I_0 \cup [0, 1] \cup I_*\) with bicoloring \(I^+_0 := I_0\) and \(I^*_0 := [0, 1] \cup I_*\).

Proof. We prove the first statement; the second one is entirely similar. Consider the intervals \(K := \{\frac{1}{2}\} \times [0, 1], J := I_0 \cup [0, \frac{1}{2}] \times \{1\}, J^+ := J \cup K, I^+ := [\frac{1}{2}, 1] \times \{1\} \cup I_*\), and \(I^+ := I \cup K\). These intervals are bicolored by \(I^+_1 = \frac{1}{2} \oplus I_* \equiv I_*\) and \(I^*_0 = J_* \equiv 0\).

Extend the map \([0, \frac{1}{2}] \times \{1\} \to K\) to \((t, 1) \mapsto \left(\frac{1}{2}, t + \frac{1}{2}\right)\) to an embedding \(f : J \to K\) so that \(K \setminus f(J)\) is non-empty. Using \(A(f)\), we can then equip \(L^2(A(K))\) with a left action of \(A(J)\). Combining this left action with the natural right action of \(A(K)\), we get a faithful action of \(A(J) \otimes_{op} A(K)^{op}\) on \(L^2(A(K))\), which extends to \(A(J^+)\) by the vacuum axiom for conformal nets (see Appendix 13). Pick a faithful \(D(I^+)\)-module \(H\). By definition,

\[(1 \otimes D)(I) = A(J^+) \otimes_{op} A(K) D(I^+)\]

is the von Neumann algebra generated by \(A(J)\) and \(D(I)\) on the Hilbert space \(L^2(A(K)) \otimes A(K) H \cong H\). This algebra is equal to \(D(J \cup I^+) = D(\text{int})\) by strong additivity.

Recall that our standard circle \(S^1\) is the square \(\partial[0, 1]^2\). Let

\[S^1_1 := \partial^+(\{0, 1/2\} \times \{1/2, 1\}), \quad S^1_2 := \partial^+(\{1/2, 1\} \times \{1/2, 1\}), \quad S^1_0 := \partial^-(\{0, 1/2\} \times \{0, 1/2\}), \quad S^1_1 := \partial^-(\{1/2, 1\} \times \{0, 1/2\})\]

be the four “quarter circles”. Let us also pick, once and for all, a diffeomorphism \(\phi_* : S^1 \cup [0, 1] \to S^1\) (here \((1/2, 1) \in S^1\) is glued to \(0 \in [0, 1]\) whose derivative is equal to one in a neighborhood of the boundary. The three mirror images of \(\phi_*\) are called \(\phi_- : [0, 1] \cup S^1 \to S^1, \phi^- : S^1 \cup [0, 1] \to S^1, \text{ and } \phi_+ : [0, 1] \cup S^1 \to S^1\):

We are now ready to define the functor

\[(3.2) \quad \text{unitor} \_D : \CN_1 \to \CN_2.\]

It assigns to every \(\mathcal{A}\)-\(\mathcal{B}\)-defect \(D\) an invertible \((1 \otimes D)\)-\(\text{sector}\). As a Hilbert space, \(\text{unitor}_D(D)\) is simply the vacuum sector \(H_0(D)\). Let \(\mathcal{S}\) be the bicolored circle with white half \(\mathcal{S}_0 := S^1_0 \cup \{\frac{1}{2}\} [0, 1]\) and black half \(\mathcal{S}_1 := S^1_1\). One should imagine \(\mathcal{S}\) as being the standard bicolored circle \(S^1\), to which an extra white interval \([0, 1]\) has been inserted at the top—see 6.4. In view of Lemma 5.1 a \((1 \otimes D)\)-\(\text{sector}\) is the same thing as a Hilbert space \(H\) equipped with compatible actions of \(D(I)\) for every bicolored interval \(I \subset \mathcal{S}\). Let \(\hat{\phi}_* : \mathcal{S} \to S^1\) be the diffeomorphism given by \(\phi_*\) on \([0, 1]\), and by the identity on its complement. The \((1 \otimes D)\)-\(\text{sector}\) structure on \(H_0(D) = \text{unitor}_D(D)\) is given by letting \(D(I)\) act by the composition of \(D(\hat{\phi}_*): D(I) \to D(\hat{\phi}_*(I))\) with the natural action of \(D(\hat{\phi}_*(I))\) on \(H_0(D)\).
We also have functors

\[
\text{unitor}_{tr} : \text{CN}_1^f \to \text{CN}_2
\]
(3.3)
\[
\text{unitor}_{bd} : \text{CN}_1^f \to \text{CN}_2
\]
\[
\text{unitor}_{br} : \text{CN}_1^f \to \text{CN}_2
\]

that are defined in a similar fashion. The underlying Hilbert spaces of \(\text{unitor}_{tr}(D)\), \(\text{unitor}_{bd}(D)\), and \(\text{unitor}_{br}(D)\) are all \(H_0(D)\), and they are equipped with the structures of \((D \oplus 1)-D\)-sector, \(D-(1 \oplus D)\)-sector, and \(D-(D \oplus 1)\)-sector, respectively. Let \(S \), \(S_0\), and \(S^4\) be the bicolored circles given by \(S_0 := S_0^1 \cup S_0^\ast = [0, 1] \cup (\frac{1}{2}, 1)\), \(S_1 := S_1^0 \cup S_1^\ast = [0, 1] \cup (\frac{1}{2}, 0)\), and \(S^4 := S^4_0, S^4_1 := [0, 1] \cup (\frac{1}{2}, 0)\) respectively:

\[
S = \bigcup_{\text{genuinely bicolored}} (1_A \oplus_A 1_A)(J) = A([0, 1])
\]

is non-trivial, which is not the case if \(1_A \oplus_A 1_A\) is replaced by \(1_A\) in the intersection expression.

The invertible sector between \(1_A\) and \(1_A \oplus_A 1_A\) is the vacuum module of \(A\) associated to the “circle” \(\bigcup\) constructed by inserting a copy of \([0, 1]\) at the point \((\frac{1}{2}, 1) \in \partial[0, 1]^2\).

3.8. **Semisimplicity of the composite defect.** Given two semisimple defects, we can ask whether their fusion is again a semisimple defect. From now on, we always assume that our conformal nets are irreducible. The purpose of this section is to prove the following theorem:

**Theorem 3.6.** Let \(A_D B\) and \(B_E C\) be semisimple defects. If the conformal net \(B\) has finite index, then \(A_D \oplus_B E_C\) is a semisimple \(A-C\)-defect.

**Detecting semisimplicity.** We begin with a few lemmas.

**Lemma 3.7.** Let \(A, B\) be von Neumann algebras and let \(H\) be a faithful \(A-B\)-bimodule. If the algebra of \(A-B\)-bimodule endomorphisms of \(H\) is finite-dimensional, then \(H\) is a finite direct sum of factors.

**Proof.** The center of \(A\) acts faithfully by \(A-B\)-bimodule endomorphisms. It is therefore finite-dimensional. \(\square\)

From now on, we fix a faithful defect \(A_D B\), and denote its vacuum sector \(H_0 = H_0(D)\). Recall that our standard circle is \(S^1 := \partial[0, 1]^2\), and that its top and bottom halves are denoted \(S^1_+\) and \(S^1_-\).
Notation 3.8. Given an interval $I \subset S^1$ that contains the two color-change points $(\frac{1}{2}, 0)$ and $(\frac{1}{2}, 1)$ in its interior, we define an algebra $\hat{D}(I) \subset \mathcal{B}(H_0)$ as follows. It is the algebra generated by $D(I_1)$ and $D(I_2)$, where $I_1$ and $I_2$ are any two intervals covering $I$ with the property that $(\frac{1}{2}, 1) \notin I_1$ and $(\frac{1}{2}, 0) \notin I_2$. By strong additivity, the algebra $\hat{D}(I)$ does not depend on the choice of covering.

Lemma 3.9. Let $I \subset S^1$ be an interval containing both color-change points in its interior. If $D$ is an irreducible defect, then $\hat{D}(I)$ is a factor.

Proof. Let $I'$ be the closure of $S^1 \setminus I$. The center of $\hat{D}(I)$ commutes with both $D(I)$ and $D(I')$. Since $D(S^1 \setminus I)$ and $D(S^1 \setminus I')$ generate $D(S^1)$, $Z(\hat{D}(I))$ commutes with $D(S^1)$. Similarly, $Z(\hat{D}(I))$ commutes with $D(S^1)$. Therefore, $Z(\hat{D}(I))$ acts on $H_0$ by $D(S^1)\triangleright D(S^1)^{op}$-bimodule automorphisms. If $D(I)$ was not a factor, that action could be used to construct a non-trivial direct sum decomposition of $H_0 = L^2(D)$, contradicting the irreducibility of $\hat{D}(I)$.

Finiteness of the defect vacuum as a 4-interval bimodule. (splitting)

Notation 3.10. Let $S^1 = I_1 \cup I_2 \cup I_3 \cup I_4$ be a partitioning of the standard bicolored circle into four intervals so that $I_1$ and $I_4$ are genuinely bicoloried, and so that each intersection $I_i \cap I_{i+1}$ (cyclic numbering) is a single point. For such a partition, we let $\hat{D}(I_1 \cup I_4)$ denote the commutant of $D(I_2 \cup I_4) = D(I_2) \otimes D(I_4)$ acting on the vacuum sector $H_0(D)$.

Similarly, if $\mathcal{A}$ is a conformal net and $S^1 = I_1 \cup I_2 \cup I_3 \cup I_4$ is a partitioning of the standard (not bicolored) circle, we let $\hat{\mathcal{A}}(I_1 \cup I_4)$ denote the commutant of $\mathcal{A}(I_2 \cup I_4) = \mathcal{A}(I_2) \otimes \mathcal{A}(I_4)$ on the vacuum sector $H_0(\mathcal{A})$.

It should be noted that the choice of ambient circle does not affect the definition of the algebras $\hat{D}(I_1 \cup I_3)$ and $\hat{\mathcal{A}}(I_1 \cup I_3)$: they only depend (up to canonical isomorphism) on the intervals $I_1$ and $I_3$, and on their bicoloring.

Lemma 3.11. Let $I_1, I_2, I_3, I_4$ be as in Notation 3.10. Assume furthermore that $I_2$ and $I_3$ are white. Write $I_1 = J_1 \cup J_2$, with $J_1$ genuinely bicoloried, $J_2$ white, and $J_1 \cap J_2$ a single point:

$$
\begin{tikzpicture}
    \draw (-1,0) to (0,0);
    \draw (0,0) to (1,0);
    \draw (1,0) to (0,0);
    \draw (0,0) to (-1,0);
    \draw (-0.5,0) to (-0.5,-1);
    \draw (0.5,0) to (0.5,-1);
    \draw (-0.5,-1) to (0.5,-1);
    \draw (-0.5,-1) to (-0.5,1);
    \draw (0.5,-1) to (0.5,1);
    \draw (-0.5,1) to (-0.5,-1);
    \draw (0.5,1) to (0.5,-1);
    \draw (-0.5,-1) to (0.5,-1);
    \node at (0,0) {$\mathcal{I}$};
    \node at (-1,0) {$I_1$};
    \node at (1,0) {$I_4$};
    \node at (0,-1) {$I_3$};
    \node at (0,1) {$I_2$};
    \end{tikzpicture}
$$

Let $\hat{\mathcal{A}}(J_2 \cup I_3)$ be the commutant of $\mathcal{A}(J_2) \otimes \mathcal{A}(S^1 \setminus (J_2 \cup I_2 \cup I_3))$ acting on $H_0(\mathcal{A})$. Then there is a natural action of the algebra $\hat{\mathcal{A}}(J_2 \cup I_3)$ on the vacuum sector $H_0(D)$, and we have $\hat{D}(I_1 \cup I_3) = D(J_1) \vee \hat{\mathcal{A}}(J_2 \cup I_3)$.

Proof. We assume that $D$ is faithful (otherwise $D = 0$, and there is nothing to show). By Haag duality for $\mathcal{A}$, the algebra $\hat{\mathcal{A}}(J_2 \cup I_3)$ is the relative commutant of $\mathcal{A}(I_2)$ inside $\mathcal{A}(J_2 \cup I_2 \cup I_3)$. The latter acts naturally on $H_0(D)$, and therefore so does $\hat{\mathcal{A}}(J_2 \cup I_3)$. As an algebra on $H_0(D)$, $\hat{\mathcal{A}}(J_2 \cup I_3)$ is given by the same expression $\mathcal{A}(J_2 \cup I_2 \cup I_3) \cap \mathcal{A}(I_2)'$, where the commutant is now interpreted on $H_0(D)$.

By Lemma 3.10,

$$D(J_1) \vee \hat{\mathcal{A}}(J_2 \cup I_3) = D(J_1) \vee (\mathcal{A}(J_2 \cup I_2 \cup I_3) \cap \mathcal{A}(I_2)')$$
is equal to
\[ (D(J_1) \vee \mathcal{A}(J_2 \cup I_2 \cup I_3)) \cap \mathcal{A}(I_2) = D(I_1 \cup I_2 \cup I_3) \cap \mathcal{A}(I_2) = (D(I_1 \cup I_2 \cup I_3) \vee \mathcal{A}(I_2))'. \]
The latter is equal to \( (D(I_4) \vee \mathcal{A}(I_2))' = D(I_2 \cup I_3)' \) by Haag duality for defects (Proposition 1.17).

**Lemma 3.13.** Let \( I_1, I_2, I_3, I_4 \) be arranged as in (3.12). Assuming \( D \) is irreducible, then \( \mathcal{A}(I_2) \) is the relative commutant of \( D(I_1 \cup I_3) \) inside \( D(I_1 \cup I_2 \cup I_3) \).

**Proof.** By Lemma A.32 we have \( \mathcal{A}(I_2) = (\mathcal{A}(I_2) \vee D(I_1 \cup I_2 \cup I_3)) \cap D(I_1 \cup I_2 \cup I_3) \).

The latter is equal to \( (\mathcal{A}(I_2) \vee D(I_1)) \cap D(I_1 \cup I_2 \cup I_3) = D(I_1 \cup I_3)' \cap D(I_1 \cup I_2 \cup I_3) \). \( \square \)

In the next Lemma we will use the notion of minimal index \([A : B]\) of a subfactor \( B \subseteq A \); see Appendix A.VIII for a definition.

**Lemma 3.14.** Let \( \mathcal{A} \) be a conformal net with finite index \( \mu(\mathcal{A}) \), and let \( \mathcal{A} \mathcal{D}_B \) be an irreducible defect. Let \( I_1, I_2, I_3, I_4 \) be arranged as in (3.12). Then \( [\hat{D}(I_1 \cup I_3) : D(I_1 \cup I_3)] \leq \mu(\mathcal{A}) \).

**Proof.** Note that \( \hat{D}(I_1 \cup I_3) \) and \( D(I_1 \cup I_3) = \mathcal{A}(I_3) \otimes D(I_1) \) are both factors.

Let us decompose \( I_1 \) into intervals \( J_1, J_2 \) as in (3.12). By Lemma 3.11 we have
\[ \hat{A}(J_2 \cup I_3) \vee D(J_1) = \hat{D}(I_1 \cup I_3). \]
We also have
\[ \mathcal{A}(J_2 \cup I_3) \vee D(J_1) = D(I_1 \cup I_3). \]
By definition \( \mu(\mathcal{A}) = [\hat{A}(J_2 \cup I_3) : \mathcal{A}(J_2 \cup I_3)] \). The result follows because the minimal index cannot increase under the operation \( - \vee D(J_1) \), see (A.24) in Appendix A.VIII. \( \square \)

**Remark 3.15.** We will see later, in Corollary 5.20, that in fact we have an equality \([\hat{D}(I_1 \cup I_3) : D(I_1 \cup I_3)] = \mu(\mathcal{A})\).

**Finiteness implies semisimplicity.** We can now prove the semisimplicity of the fusion of semisimple defects.

**Proof of Theorem 3.6.** Because the defects \( D \) and \( E \) are semisimple, we may write them as finite direct sums of irreducible defects: \( D = \bigoplus D_i \) and \( E = \bigoplus E_j \). Fusion of defects is compatible with direct sums
\[ (\bigoplus D_i) \otimes_B (\bigoplus E_j) = \bigoplus_{ij} D_i \otimes_B E_j. \]
It therefore suffices to assume \( D \) and \( E \) are irreducible, and to show that for \( I \) genuinely bicolored, the von Neumann algebra \((D \oplus E)(I)\) has finite-dimensional center.

Without loss of generality, we assume that \( I = S_1^+ \). Let \( H := H_0(D) \) and \( K := H_0(E) \) be the vacuum sectors of \( D \) and \( E \). The algebra \( \hat{D}(\partial^-[0,1]) \) acts on \( H \) (see notation 3.8). Similarly, the algebra \( \hat{E}(\partial^-[0,1]) \) acts on \( K \). Let us denote those two algebras graphically by \( \hat{D}(\square^-) \) and \( \hat{E}(\square^-) \).

The Hilbert space \( H \otimes_B K \) is a faithful \((D \oplus E)(S_1^\perp) \cdot (D \oplus E)(S_1^\perp)\)-bimodule. So by Lemma 3.11 it is enough to show that the algebra of bimodule endomorphisms of \( H \otimes K \) is finite-dimensional. This algebra of endomorphisms is equal to the algebra of \( \hat{D}(\square^-) \cdot \hat{E}(\square^-) \)-endomorphisms of \( H \otimes K \). Note that the algebras \( \hat{D}(\square^-) \) and \( \hat{E}(\square^-) \) are factors by Lemma 3.9. If a bimodule has finite statistical dimension (see Appendix A.VIII), then its algebra of bimodule endomorphisms is
finite dimensional [1] Lem. 4.10. It is therefore enough to show that the statistical
dimension of \( H \mathbb{E}_B K \) as a \( \tilde{D}(\mathbb{E})-\mathbb{E}(\mathbb{E}) \)-bimodule is finite.

Using the compatibility of statistical dimension with Connes fusion \( \text{A.13} \) the
dimension in question can be computed as

\[
\dim \left( \tilde{D}(\mathbb{E}) H \mathbb{E}_B K_{\mathbb{E}(\mathbb{E})} \right) = \dim \left( \tilde{D}(\mathbb{E}) H_B(1) \right) \cdot \dim \left( \mathbb{E}(\mathbb{E}) K_{\mathbb{E}(\mathbb{E})} \right).
\]

So it suffices to argue that the dimension of \( H \) as a \( \tilde{D}(\mathbb{E})-\mathbb{E}(\mathbb{E}) \)-bimodule and the
dimension of \( K \) as a \( \mathbb{E}(\mathbb{E})-\mathbb{E}(\mathbb{E}) \)-bimodule are finite. This is the content of Lemma 3.16 below.

Before proceeding, let us fix some new names for certain subintervals of our
standard circle:

\[
\begin{align*}
I_1 &:= \partial L([0, \frac{1}{2}] \times [0, \frac{1}{2}]), & I_2 &:= [\frac{1}{2}, 1] \times \{0\}, \\
I_3 &:= \{1\} \times [0, 1], & I_4 &:= \partial L([0, 1] \times [\frac{1}{2}, 1]).
\end{align*}
\]

Given a defect \( \mathcal{A} D_B \), let us also introduce the following shorthand notations:

\[
\begin{align*}
D_{234} &:= D(I_2 \cup I_3 \cup I_4), & D_{24} &:= B(I_2) \otimes D(I_4), & B_3 &:= B(I_3), \\
\tilde{D}_{24} &:= (D(I_1) \otimes B(I_3))^\prime, & \tilde{D}_{412} &:= D(I_4) \vee D(I_1 \cup I_2).
\end{align*}
\]

**Lemma 3.16.** Let \( B \) be a conformal net with finite index and let \( \mathcal{A} D_B \) be an irreducible defect. Then \( H_0(D) \) has finite statistical dimension as a \( \tilde{D}_{412}-B_3 \)-bimodule.

**Proof.** The statistical dimension of \( H_0(D) \) as a \( \tilde{D}_{412}-B_3 \)-bimodule is the square root of the minimal index of the subfactor \( B_3 \subseteq \tilde{D}_{412} \), see \( \text{A.19} \) and the definition of minimal index in Appendix \( \text{A.1VIII} \). Therefore, we need to show that \( [\tilde{D}_{412} : B_3] < \infty \). We know by Lemma 3.14 (with \( \mathcal{A} \) and \( B \) interchanged) that the algebra \( B_3 \) is the relative commutant of \( D_{24} \) inside \( D_{234} \). The algebra \( \tilde{D}_{412} \) is the relative commutant of \( D_{24} \) inside \( D_{234} \), as can be seen by taking the commutant of the equation \( (D_{24} \cap D_{34})^\prime = D_{24} \vee D_{234} = \tilde{D}_{412} \). The index is unchanged by taking commutants, and can only decrease under the operation \( \cap \) \( D_{234} \) by \( \text{A.23} \). Thus

\[
[\tilde{D}_{412} : B_3] = [D_{24} \cap D_{34} : \tilde{D}_{24} \cap D_{34}] \leq [\tilde{D}_{24} : D_{24}],
\]

and we have already seen in Lemma 3.14 that \( [\tilde{D}_{24} : D_{24}] \leq \mu(B) < \infty \). \( \square \)

**Finiteness of the defect vacuum as a 4-interval bimodule. (splitting \( \bigcirc \) )** We record the following finiteness result, somewhat similar to Lemma 3.14 for future reference.

Let \( I_1, I_2, I_3, I_4 \) now be the four sides of our standard bicolored circle:

\[
\begin{array}{c}
I_1 \\
II \\
I_3 \\
I_4
\end{array}
\]

The intervals \( I_1 \) and \( I_3 \) are genuinely bicolored, \( I_2 \) is white, and \( I_4 \) is black.
Lemma 3.17. Let $\mathcal{A}$ and $\mathcal{B}$ be conformal nets with finite index, and let $\mathcal{A}D\mathcal{B}$ be an irreducible defect. Then the vacuum sector $H_0(D)$ has finite statistical dimension as a $D(I_1) \vee D(I_3) - (\mathcal{A}(I_2) \vee \mathcal{B}(I_4))' \text{-bi-module}$.

Proof. Consider the following intervals:

- $J_1 := [\frac{3}{4}, 1] \times \{0\}$
- $J_2 := I_4$
- $J_3 := [\frac{3}{4}, 1] \times \{1\}$
- $J_4 := [\frac{1}{4}, \frac{3}{4}] \times \{1\}$
- $J_5 := [0, \frac{1}{4}] \times \{1\}$
- $J_6 := I_2$
- $J_7 := [0, \frac{1}{4}] \times \{0\}$
- $J_8 := [\frac{1}{4}, \frac{3}{4}] \times \{0\}$

which we draw here:

![Diagram](image)

It will be convenient to introduce a graphical notation for the subalgebras of $\mathcal{B}(H_0(D))$ used in this proof:

- $\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array} = D(J_1 \cup J_2 \cup J_3 \cup J_4 \cup J_5 \cup J_6 \cup J_7)$
- $\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array} = D(J_4)$
- $\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array} = B(J_1 \cup J_2 \cup J_3) \vee A(J_5 \cup J_6 \cup J_7)$
- $\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array} = D(J_6)$
- $\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array} = B(J_1) \vee D(J_3 \cup J_4 \cup J_5) \vee A(J_7)$
- $\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array} = D(J_8)$
- $\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array} = B(J_1 \cup J_3) \vee A(J_5 \cup J_7)$
- $\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array} = B(J_2) \vee A(J_6) \vee D(J_8)$
- $\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array} = B(J_1 \cup J_3) \vee \hat{A}(J_5 \cup J_7)$
- $\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array} = B(J_2) \vee A(J_6) \vee D(J_8)$

where, as in 3.10, $\hat{A}(J_5 \cup J_7)$ is the relative commutant of $A(J_6)$ in $A(J_5 \cup J_6 \cup J_7)$ and $\hat{B}(J_1 \cup J_3)$ is the relative commutant of $B(J_2)$ in $B(J_1 \cup J_2 \cup J_3)$. Note that $\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array}$ is the commutant of $\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array}$ since, by Lemma A.30, we have

\[
\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array} = \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array} \cap (\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array} \cap (\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array}))' = (\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array} \cap (\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array}))' = (\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array} \cap (\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array}))' = (\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array} \cap (\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array}')).
\]

In particular, the algebra $\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array}$ is a factor.

We have to show that

\[
[D(I_1) \vee D(I_3)]' : A(I_2) \vee B(I_4) \subset \infty.
\]

Using Haag duality and strong additivity, note that the algebra $(D(I_1) \vee D(I_3))'$ is the relative commutant of $\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array}$ inside $\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array}$. Similarly, it follows from Lemma A.22 that the algebra $\mathcal{A}(I_2) \vee \mathcal{B}(I_4)$ is the relative commutant of $\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array}$ in $\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array}$:

\[
\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array} = (\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array} \cap (\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array}))' \cap \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array} = \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array} \cap \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array} = (\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array} \cap \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{array}').\]
By (A.23) and (A.24), we then have
\[
\left( (D(I_1) \cup D(I_3))' : A(I_2) \cup B(I_4) \right) = \left[ \begin{array}{c} \square \cap \left( \square' \right) : \square \cap \left( \square' \right) \end{array} \right] \\
\leq \left[ \begin{array}{c} \square \cup \square : \square \cup \square \end{array} \right] = \left[ \begin{array}{c} \square \cup \square : \square \cup \square \end{array} \right] = \mu(A)\mu(B). \quad \Box
\]

4. A VARIANT OF HORIZONTAL FUSION

In Section 2.3 we saw how to define the horizontal fusion of two sectors. We will now define a variant of the horizontal fusion, called keystone fusion, which itself depends on an intermediate construction we refer to as keyhole fusion. In Section 4.3 we will show that horizontal fusion and keystone fusion are in fact naturally isomorphic, and we will construct a canonical isomorphism Φ between them. That isomorphism will be essential in our construction of the $1\boxtimes 1$-isomorphism $\Omega$ (4.31).

Recall that we implicitly assume that all our conformal nets are irreducible.

4.A. The keyhole and keystone fusion. Recall the Notation 1.17. Consider the intervals $I_t := \partial^2([2/3, 1] \times [0, 1])$, $I_r := \partial^2([1, 4/3] \times [0, 1])$, and $I := I_t \cap I_r$. Orient $I_t$ and $I_r$, counterclockwise, and orient $I$ so that the inclusion $I \hookrightarrow I_t$ is orientation preserving—see (1.1). The inclusion $I \hookrightarrow I_t$ is then orientation reversing. Let $J$ be the closure of $(I_t \cup I_r) \setminus I$. We orient $J$ so that it agrees with the orientation of $I_t$ on $J \cap I_t$. We draw these intervals as follows:

\begin{align}
(4.1) \quad I_t &= \begin{array}{c} \square \end{array}, \quad I_r = \begin{array}{c} \square \end{array}, \quad I = \begin{array}{c} \square \end{array} \quad \text{and} \quad J = \begin{array}{c} \square \end{array}.
\end{align}

Given a conformal net $A$ with finite index, we will define three functors

\[ F, G, G_0 : A(I_t)-\text{modules} \times A(I_r)-\text{modules} \to A(J)-\text{modules}. \]

These operations will be called respectively the fusion, the keyhole fusion, and the keystone fusion, and will be denoted graphically as follows:

\[ F(H_t, H_r) = \begin{array}{c} H_t \square H_r \end{array}, \quad G_0(H_t, H_r) = \begin{array}{c} H_t \square H_r \end{array}, \quad G(H_t, H_r) = \begin{array}{c} H_t \square H_r \end{array}. \]

When we want to stress the dependence on the conformal net $A$, we will denote these functors $F_A, G_{0,A}, G_A$.

The ordinary horizontal fusion. The functor $F$ is defined by fusion over $A(I)$: using the orientation preserving inclusion $I \hookrightarrow I_r$, any left $A(I_r)$-module is also a left $A(I)$-module, and using the orientation reversing inclusion $I_0 \hookrightarrow I_t$, any left $A(I_t)$-module is also a right $A(I)$-module. We can therefore define the horizontal fusion functor as follows:

\[ F(H_t, H_r) := H_t \boxtimes_{A(I)} H_r. \]

Write $J$ as $J_1 \sqcup J_2$: we obtain actions of $A(J_1)$ and $A(J_2)$ on $H_t \boxtimes_{A(I)} H_r$, by [2 Cor. 1.28]. Note that in the case $H_t = L^2(A(I_t)$ and $H_r = L^2(A(I_r)$, the actions of $A(J_1)$ and $A(J_2)$ extend to an action of $A(J) = A(J_1) \boxtimes A(J_2)$; the same therefore holds for arbitrary $H_t$ and $H_r$. The only difference between the functor $F$ and the functor fusion, from (2.12) is that they have somewhat different source and target categories—the main construction is identical in both functors.
The keyhole fusion. We will need to name a few more manifolds. Let
\[ I_l := \partial \left( [2/3, 5/6] \times [0, 1] \right) \quad \text{and} \quad I_r := \partial \left( [7/6, 4/3] \times [0, 1] \right) \]
\[ S_u := \partial \left( [5/6, 7/6] \times [2/3, 1] \right) \quad \text{and} \quad S_m := \partial \left( [5/6, 7/6] \times [1/3, 2/3] \right) \]
\[ S_d := \partial \left( [5/6, 7/6] \times [0, 1/3] \right) \quad \text{and} \quad K := (S_u \cup S_d) \cap (I_l \cup I_r). \]
We draw these as
\[ (4.2) \quad I_l = , \quad I_r = , \quad S_u = \square, \quad S_m = \square, \quad S_d = \square \quad \text{and} \quad K = 4^4 4^4. \]
The intervals \( I_l \) and \( I_r \) are oriented counterclockwise, as were \( I_l \) and \( I_r \). The manifolds \( S_u, S_m \) and \( S_d \) are conformal circles via their constant speed parametrizations and are also oriented counterclockwise. (A conformal circle is a circle together with a homeomorphism with \( S^1 \) that is only determined up to orientation preserving conformal diffeomorphisms of \( S^1 \).) Finally, the manifold \( K \) inherits its orientation from \( S_u \cup S_d \). Note that the inclusion \( K \hookrightarrow I_l \cup I_r \) is orientation reversing. We will also need the reflection \( j \) along the horizontal axis \( y = 1/2 \).

Let us fix orientation preserving identifications \( \phi_l : I_l \rightarrow I_l \) and \( \phi_r : I_r \rightarrow I_r \) that are symmetric with respect to the reflection \( j \), restrict to the identity in a neighborhood of \( \partial I_l = \partial I_l \) and \( \partial I_r = \partial I_r \), and satisfy \( \phi_l(5/6, t) = (1, t) \) and \( \phi_r(7/6, t) = (1, t) \) for all \( t \in [0, 1] \). Using these identifications, any \( \mathcal{A}(I_l) \)-module becomes an \( \mathcal{A}(I_l) \)-module and any \( \mathcal{A}(I_r) \)-module becomes an \( \mathcal{A}(I_r) \)-module. We can now define the keyhole fusion functor as follows:
\[ G_0(H_l, H_r) := (H_l \otimes H_r) \boxtimes_{\mathcal{A}(K)} (H_0(S_u) \otimes H_0(S_d)), \]
where \( H_0(S_u) \) and \( H_0(S_d) \) are the canonical vacuum sectors provided by [2, Thm. 2.13]. The right hand side is an instance of what we call cyclic fusion—see Appendix A.III.

In the notation of cyclic fusion, we have
\[ G_0(H_l, H_r) = \begin{array}{c}
\mathcal{A}(K_1) \boxtimes H_l \boxtimes H_0(S_u) \boxtimes H_r \boxtimes H_0(S_d) \boxtimes \mathcal{A}(K_2) \boxtimes \mathcal{A}(K_3) \boxtimes \mathcal{A}(K_4)
\end{array} \]
where \( K_1 = I_l \cap S_u, K_2 = I_r \cap S_u, K_3 = I_r \cap S_d, \) and \( K_4 = I_l \cap S_d \), appropriately oriented. It follows from [2, Cor. 1.28] that the algebras \( \mathcal{A}(J \cap (I_l \cup I_r)) \) and \( \mathcal{A}(J \cap (S_u \cup S_d)) \) generate an action of \( \mathcal{A}(J) \) on \( G_0(H_l, H_r) \).

The keystone fusion. Note that the algebra
\[ \mathcal{A} \left( \begin{array}{c} I_l \end{array} \right) \boxtimes_{\mathcal{A}(K)} \mathcal{A} \left( \begin{array}{c} I_r \end{array} \right) \]
has a natural left action on \( G_0(H_l, H_r) \) commuting with the action of \( \mathcal{A}(J) \). This algebra can be identified with \( \mathcal{A}(S_m)^{op} \cong \mathcal{A}(-S_m) \), where \(-S_m\) denotes the circle \( S_m \) equipped with the opposite (i.e., clockwise) orientation. Here, we use the extension of \( \mathcal{A} \) from intervals to 1-manifolds constructed in [3, Sec. 1.B]; see also Appendix B.VI. By definition, the algebra \( \mathcal{A}(S_m) \) is generated by all the \( \mathcal{A}(I) \) for \( I \subset S_m \) on the Hilbert space \( \mathcal{H} \).

By Theorem B.9, the algebra \( \mathcal{A}(S_m) \) contains a direct summand that is canonically isomorphic to \( \mathcal{B}(H_0(S_m, \mathcal{A})) \). We can therefore define the keystone fusion functor as follows:
\[ G(H_l, H_r) := G_0(H_l, H_r) \boxtimes_{\mathcal{A}(S_m)} H_0(S_m). \]
Moreover, since \( \mathcal{B}(H_0(S_m, \mathcal{A})) \) and \( \mathcal{A}(J) \) commute on \( G_0(H_l, H_r) \), there is a residual action of \( \mathcal{A}(J) \) on the Hilbert space \( G(H_l, H_r) \).

---

\(^8\)For this, we implicitly identify the surfaces \( \square \) and \( S_m \times [0, 1] \).
Fusion and keystone fusion are isomorphic. We will show presently that the functors
\[ F, G : \mathcal{A}(I_1) \text{-} \text{modules} \times \mathcal{A}(I_2) \text{-} \text{modules} \rightarrow \mathcal{A}(J) \text{-} \text{modules} \]
are naturally isomorphic to one another, and then later (in Proposition 4.3) construct a specific such natural isomorphism.

We use the following straightforward generalization of Lemma A.28.

**Lemma 4.3.** Let \( F, G : A_1 \text{-} \text{modules} \times A_2 \text{-} \text{modules} \rightarrow B \text{-} \text{modules} \) be continuous additive functors, and let \( M_i \) be a faithful \( A_i \text{-} \text{module} \), for \( i = 1, 2 \). Then, in order to uniquely define a natural transformation \( a : F \rightarrow G \), it is enough to specify its value on \((M_1, M_2)\) and to check that for each pair \((r_1, r_2)\) with \( r_i \in \text{End}_A(M_i) \), the diagram
\[ \begin{array}{ccc}
F(M_1, M_2) & \xrightarrow{F(r_1, r_2)} & F(M_1, M_2) \\
\downarrow{\alpha_{M_1, M_2}} & & \downarrow{\alpha_{M_1, M_2}} \\
G(M_1, M_2) & \xrightarrow{G(r_1, r_2)} & G(M_1, M_2)
\end{array} \]
commutes. \( \square \)

Using [2, Thm. 3.23] and the above lemma, we prove that the two different version of horizontal fusion are naturally isomorphic to each other:

**Proposition 4.4.** There exists a natural isomorphism between the fusion functor \( F \) and the keystone fusion functor \( G \).

**Proof.** Consider the circles
\[
S_l := \partial([0, 1] \times [0, 1]), \quad S_r := \partial([1, 2] \times [0, 1]), \quad S_b := \partial([0, 2] \times [1, 2])
\]
\[
\bar{S}_l := \partial([0, 5/6] \times [0, 1]), \quad \bar{S}_r := \partial([7/6, 2] \times [0, 1])
\]
\[
\hat{S}_l := \partial([0, 5/6] \times [0, 1] \cup [5/6, 7/6] \times [0, 1/3] \cup [5/6, 7/6] \times [2/3, 1])
\]
which we draw as follows:
\[
\text{(4.5)}
\]
\[
S_l = \begin{array}{c}
\text{□}
\end{array}, \quad S_r = \begin{array}{c}
\text{□}
\end{array}, \quad S_b = \begin{array}{c}
\text{□}
\end{array}
\]
\[
\bar{S}_l = \begin{array}{c}
\text{□}
\end{array}, \quad \bar{S}_r = \begin{array}{c}
\text{□}
\end{array}, \quad \hat{S}_l = \begin{array}{c}
\text{□}
\end{array}
\]

\(\text{‘b’ stands for big.}\) The identifications \( \phi_l : \hat{I}_l \cong I_l \) and \( \phi_r : \hat{I}_r \cong I_r \) induce isomorphisms \( H_0(\hat{S}_l) \cong H_0(S_l) \) and \( H_0(\hat{S}_r) \cong H_0(S_r) \) that are equivariant with respect to \( \mathcal{A}(I) \) and \( \mathcal{A}(I') \) (here, \( I' \) and \( I'' \) are the closures of \( S_l \setminus I_l \) and \( S_r \setminus I_r \), respectively). From the isomorphism \( H_0(\hat{S}_l) \cong H_0(S_l) \otimes A_{(K_1)} H_0(\hat{S}_r) \otimes A_{(K_2)} H_0(S_r) \) it follows that \( G_0(H_0(S_l), H_0(S_r)) \) represents the Hilbert space of an annulus; see Appendix B. Using Theorem 11.3 we therefore have
\[
G_0(H_0(S_l), H_0(S_r)) \cong \bigoplus_{\lambda \in \Delta} H_\lambda (-S_m) \otimes H_\lambda (S_b).
\]

We draw the above isomorphisms as follows:
\[
G_0(H_0(S_l), H_0(S_r)) \cong \bigoplus_{\lambda \in \Delta} \lambda \otimes \lambda.
\]
Note that the two isomorphisms intertwine the natural \( \{\mathcal{A}(I)\}_{I \in (S_l \cup -S_m)} \)-actions. We can now compute
\[
G(H_0(S_l), H_0(S_r)) := G_0(H_0(S_l), H_0(S_r)) \boxtimes_{\mathcal{A}(S_m)} H_0(S_m)
\]
\[
\cong \left( \bigoplus_{\lambda \in \Delta} H_\lambda(-S_m) \otimes H_\lambda(S_m) \right) \boxtimes_{\mathcal{A}(S_m)} H_0(S_m)
\]
(4.6)
\[
\cong H_0(S_m) \otimes \left( H_0(-S_m) \boxtimes_{\mathcal{B}(H_0(S_m))} H_0(S_m) \right)
\]
\[
\cong H_0(S_m) \otimes C \cong H_0(S_b)
\].

Combining (4.6) with the non-canonical isomorphism \( F(H_0(S_l), H_0(S_r)) \cong H_0(S_b) \) of \([2]\) Cor. 1.33, we get an isomorphism
\[
\varphi : G(H_0(S_l), H_0(S_r)) \cong F(H_0(S_l), H_0(S_r))
\]
of \( \mathcal{A}(I') \otimes_{alg} \mathcal{A}(I'_r) \)-sectors.

Since \( H_0(S_l) \) and \( H_0(S_r) \) are faithful \( \mathcal{A}(I') \)- and \( \mathcal{A}(I'_r) \)-modules, we can use Lemma \([13]\) to finish the argument: it remains only to check that \( \varphi \) is equivariant with respect to all \( r_1 \in \text{End}_{\mathcal{A}(I)}(H_0(S_l)) \) and \( r_2 \in \text{End}_{\mathcal{A}(I'_r)}(H_0(S_r)) \). That equivariance follows immediately from Haag duality for nets (Proposition \([3, 4]\)) and the fact that \( \varphi \) is equivariant with respect to \( \mathcal{A}(I'_r) \) and \( \mathcal{A}(I'_l) \).

Unfortunately, the above proposition is not sufficient for our purposes: it does not construct a natural isomorphism \( \Phi_A : F_A \to G_A \), but only proves that one exists. This leaves unsettled, for instance, the question of whether these natural isomorphisms can be chosen so that \( \Phi_A \otimes B = \Phi_A \otimes \Phi_B \). In the following sections, we will construct a canonical choice of such natural isomorphisms for which the desired symmetric monoidal property is clear.

### 4.8. The keyhole fusion of vacuum sectors of defects

Let \( S_l, S_r, S_b, \tilde{S}_l, \tilde{S}_r, S_u, S_m, S_d, I_l, I_r, I, \tilde{I}, I, K \) be as in \([1, 1]\), \([12]\), and \([15]\). We bicolor \( S_l, S_r, \tilde{S}_l, \tilde{S}_r \) by setting
\[
(S_l)_o := (S_l)_{x \leq \frac{1}{4}} \quad (S_l)_\bullet := (S_l)_{x \geq \frac{1}{4}} \quad (S_r)_o := (S_r)_{x \leq \frac{1}{4}} \quad (S_r)_\bullet := (S_r)_{x \geq \frac{1}{4}}
\]
\[
(\tilde{S}_l)_o := (\tilde{S}_l)_{x \leq \frac{1}{4}} \quad (\tilde{S}_l)_\bullet := (\tilde{S}_l)_{x \geq \frac{1}{4}} \quad (\tilde{S}_r)_o := (\tilde{S}_r)_{x \leq \frac{1}{4}} \quad (\tilde{S}_r)_\bullet := (\tilde{S}_r)_{x \geq \frac{1}{4}}.
\]
Denote by \( j \) the reflection across the horizontal axis \( y = 1/2 \), and let
\[
S_{l, \top} := (S_l)_{y \geq \frac{1}{2}} \quad S_{r, \top} := (S_r)_{y \geq \frac{1}{2}} \quad S_{b, \top} := (S_b)_{y \geq \frac{1}{2}}
\]
\[
\tilde{S}_{l, \top} := (\tilde{S}_l)_{y \geq \frac{1}{2}} \quad \tilde{S}_{r, \top} := (\tilde{S}_r)_{y \geq \frac{1}{2}} \quad \tilde{I}_{l, \top} := (\tilde{I}_l)_{y \geq \frac{1}{2}} \quad \tilde{I}_{r, \top} := (\tilde{I}_r)_{y \geq \frac{1}{2}}.
\]
Let \( A, B, C \) be conformal nets, and let \( A D_B \) and \( B C \) be defects. We are interested in evaluating \( H_0 := G_{0, B} \) on the vacuum sectors
\[
H_0(S_l, D) := L^2(D(S_{l, \top})) \quad \text{and} \quad H_0(S_r, E) := L^2(E(S_{r, \top}))
\]
from Definition \([1, 48]\). These have compatible actions of the algebras \( \{ D(I) \}_{I \in S_l} \) and \( \{ E(I) \}_{I \in S_r} \), respectively. In particular, they are respectively \( \mathcal{B}(I_l) \)- and \( \mathcal{B}(I_r) \)-modules, and so we can apply the functor \( G_0 \).

Let us also define
\[
H_0(\tilde{S}_l, D) := L^2(D(\tilde{S}_{l, \top})) \quad \text{and} \quad H_0(\tilde{S}_r, E) := L^2(E(\tilde{S}_{r, \top}))
\].

Recall that the definition of the functor \( G_0 \) uses identifications \( \phi_l : \tilde{I}_l \cong I_l \) and \( \phi_r : \tilde{I}_r \cong I_r \), to endow \( H_0(S_l, D) \) with a \( \mathcal{B}(I_l) \)-action, and \( H_0(S_r, D) \) with a \( \mathcal{B}(I_r) \)-action. We write \( \phi_l^* H_0(S_l, D) \) and \( \phi_r^* H_0(S_l, D) \) for the resulting \( \tilde{S}_l \)-sector of \( D \) and
\( \tilde{S}_r \)-sector of \( E \). Recall that the maps \( \phi_l \) and \( \phi_r \) were chosen to commute with \( j \), and to be the identity near to the boundary. Let us call

\[
\phi_{l,T} : \tilde{S}_{l,T} \cong S_{l,T}, \quad \phi_{r,T} : \tilde{S}_{r,T} \cong S_{r,T}
\]

the extension by the identity of the maps \( \phi_l|_{\tilde{I}_{l,T}} \) and \( \phi_r|_{\tilde{I}_{r,T}} \). We then have canonical identifications

\[
L^2(D(\phi_{l,T})) : H_0(\tilde{S}_l, D) \to \phi_l^* H_0(S_l, D)
\]

\[
L^2(D(\phi_{r,T})) : H_0(\tilde{S}_r, D) \to \phi_r^* H_0(S_r, D)
\]

of \( \tilde{S}_l \)-sectors of \( D \) and \( \tilde{S}_r \)-sectors of \( E \).

We now have an isomorphism

\[
G_{0,B}(H_0(S_l, D), H_0(S_r, E))
\]

\[
\cong (H_0(\tilde{S}_l, D) \otimes H_0(\tilde{S}_r, E)) \otimes_{B(K)} (H_0(S_u, B) \otimes H_0(S_d, B))
\]

\[
\cong \{ H_0(\tilde{S}_l, D) \otimes_{B(K_1)} H_0(S_u, B) \otimes_{B(K_2)} H_0(\tilde{S}_r, E) \otimes_{B(K_3)} H_0(S_d, B) \otimes_{B(K_4)} - \}
\]

which we draw as follows:

(4.8) \[ G_{0,B}(H_0(S_l, D), H_0(S_r, E)) \cong \begin{array}{c|c|c|c}
\end{array} \]

Here, the lines \( \equiv \), \( \equiv \), \( \equiv \), and \( \equiv \) correspond to the conformal nets \( A, B, \) and \( C \), and the transition points \( \equiv \), \( \equiv \) indicate the defects \( D \) and \( E \).

**Keyhole fusion as an \( L^2 \)-space.** We need to introduce yet more manifolds. We have already encountered \( K_1 = \tilde{S}_l \cap S_u \) and \( K_2 = \tilde{S}_r \cap S_u \). We define \( K_u := K_1 \cup K_2 \) and \( J_u := J_1 \cup J_2 \), where \( J_1 := S_b \cap S_u \) and \( J_2 := S_u \cap S_m \). We orient \( K_u \) and \( J_u \) compatibly with \( S_u \). Let \( J_1 \) be the closure of \( \tilde{S}_{l,T} \setminus K_1 \) and, similarly, let \( J_r \) be the closure of \( \tilde{S}_{r,T} \setminus K_2 \). The orientations and the bicolorings of \( J_l \) and \( J_r \) are inherited from \( \tilde{S}_l \) and \( \tilde{S}_r \). We include pictures of these manifolds:

(4.9) \[ J_l = \begin{array}{c|c|c|c}
\end{array}, \quad J_r = \begin{array}{c|c|c|c}
\end{array}, \quad K_u = \begin{array}{c|c|c|c}
\end{array}, \quad J_u = \begin{array}{c|c|c|c}
\end{array}. \]

Following Notation 3.10, we let \( \hat{B}(J_u) \) denote the commutant of \( B(K_u) \) on \( H_0(S_u, B) \). Our computation of the keyhole fusion will be in terms of the algebra

\[
D(J_l) \vee \hat{B}(J_u) \vee E(J_r) \subset B(H_0(\tilde{S}_l, D) \otimes_{B(K_1)} H_0(S_u, B) \otimes_{B(K_2)} H_0(\tilde{S}_r, E))
\]

which we denote pictorially by

(4.10) \[ D(J_l) \vee \hat{B}(J_u) \vee E(J_r) = \begin{array}{c|c|c|c}
\end{array}. \]

The dotted line in this picture picture serves to remind us that \( \hat{B}(J_u) \) was used instead of \( B(J_u) \). Note that the algebra \( \{1,1\} \) also acts on \( G_{0,B}(H_0(S_l, D), H_0(S_r, E)) \) because the latter is obtained from

\[
\begin{array}{c|c|c|c}
\end{array} = H_0(\tilde{S}_l, D) \otimes_{B(K_1)} H_0(S_u, B) \otimes_{B(K_2)} H_0(\tilde{S}_r, E)
\]

by fusing it over \( B(K_3 \cup K_4) \) with \( H_0(S_d, B) \).

Let

\[
S_{b,T} := \{(x, y) \in S_b | x - y < \frac{1}{2}\}
\]

\[
S_{b,L} := \{(x, y) \in S_b | x + y < \frac{1}{2}\}
\]

\[
S_{b,T} := \{(x, y) \in S_b | x + y > \frac{1}{2}\}
\]

\[
S_{b,L} := \{(x, y) \in S_b | x - y > \frac{1}{2}\},
\]
with orientations and bicolorings as in the following pictures

\[ S_{b,r} = \quad , \quad S_{b,L} = \quad , \quad S_{b,\gamma} = \quad , \quad S_{b,J} = \quad . \]

Note that these manifolds do not include their boundary points.

**Theorem 4.11.** Let \( \mathcal{A}, \mathcal{B}, \mathcal{C} \) be conformal nets, and let \( \mathcal{A}D_{\mathcal{B}} \) and \( \mathcal{B}E_{\mathcal{C}} \) be defects. Then there is a canonical unitary isomorphism

\[
\Psi_0 : \quad L^2 \left( \begin{array}{c} \mathcal{A} \\ \mathcal{B} \end{array} \right) \cong \quad .
\]

In formulas, this is a map

\[
\Psi_0 = (\Psi_0)_{D,E} : \quad L^2(D(J_l) \vee \mathcal{B}(J_u) \vee E(J_r)) \cong G_{0,B}(H_0(S_t, D), H_0(S_r, E)),
\]

where \( S_t, S_r, J_l, J_u, J_r \) are as in (4.7) and (4.9). The map \( \Psi_0 \) is equivariant for the natural left actions of \( \{D(I)\}_{I \subset S_{b,r}}, \{D(I)\}_{I \subset S_{b,L}}, \{E(I)\}_{I \subset S_{b,\gamma}} \) and \( \{E(I)\}_{I \subset S_{b,J}} \), and for the natural right actions of \( \{B(I)\}_{I \subset S_{b,m}} \).

Proof. Since the Haagerup \( L^2 \)-space is uniquely determined up to unique unitary isomorphism (see Appendix A.1), this follows from a special case of Proposition A.13 with \( M := D(\tilde{S}_l) \otimes E(\tilde{S}_r) \), \( M_0 := D(J_l) \otimes E(J_r) \), \( A = B(K_u) \), and \( H = H_0(S_t, B) \). In pictures, these are

\[
(4.13) \quad M := \quad , \quad M_0 := \quad , \quad A := \quad , \quad H := \quad .
\]

The equivariance of \( \Psi_0 \) is clear for intervals \( I \) that are contained in the upper half \( \{(x,y) | y \geq \frac{1}{2} \} \) or in the lower half \( \{(x,y) | y \leq \frac{1}{2} \} \), and follows by strong additivity for more general intervals.

**Associativity of the standard form identification.** The isomorphism \( \Psi_0 \) is in an appropriate sense associative, as follows. Suppose that we have three defects \( \mathcal{A}D_{\mathcal{B}}, \mathcal{B}E_{\mathcal{C}}, \) and \( \mathcal{C}F_{\mathcal{D}} \). We then have various applications of \( \Psi_0 \) forming the square

\[
\begin{array}{c}
\begin{array}{c}
L^2(\begin{array}{c} \mathcal{F} \\ \mathcal{A} \end{array}) \\
L^2(\begin{array}{c} \mathcal{F} \\ \mathcal{B} \end{array}) \\
L^2(\begin{array}{c} \mathcal{B} \\ \mathcal{C} \end{array}) \\
L^2(\begin{array}{c} \mathcal{A} \\ \mathcal{B} \end{array})
\end{array}
\end{array}
\cong
\begin{array}{c}
\begin{array}{c}
L^2(\begin{array}{c} \mathcal{B} \\ \mathcal{C} \end{array}) \\
L^2(\begin{array}{c} \mathcal{C} \\ \mathcal{D} \end{array}) \\
L^2(\begin{array}{c} \mathcal{D} \\ \mathcal{E} \end{array}) \\
L^2(\begin{array}{c} \mathcal{A} \\ \mathcal{B} \end{array})
\end{array}
\end{array}
\end{array}
\cong
\begin{array}{c}
\begin{array}{c}
L^2(\begin{array}{c} \mathcal{A} \\ \mathcal{B} \end{array}) \\
L^2(\begin{array}{c} \mathcal{B} \\ \mathcal{C} \end{array}) \\
L^2(\begin{array}{c} \mathcal{C} \\ \mathcal{D} \end{array}) \\
L^2(\begin{array}{c} \mathcal{A} \\ \mathcal{B} \end{array})
\end{array}
\end{array}
\end{array}
\]

This diagram commutes by Proposition A.16.

We explain the meaning of the pictures in this square. The conformal nets \( \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \) are indicated by lines of various thickness \( -\), \( -\), \( -\), \( -\), and the defects \( \mathcal{A}D_{\mathcal{B}}, \mathcal{B}E_{\mathcal{C}}, \mathcal{C}F_{\mathcal{D}} \) are indicated by the transitions \( -\), \( -\), \( -\), and \( -\). The notations \( L^2(\mathcal{F}), L^2(\mathcal{B}), L^2(\mathcal{C}), \) stand for \( L^2(D(S^1_{l})), L^2(E(S^1_{l})), L^2(F(S^1_{l})) \). The lower right corner of (4.14) is

\[
L^2(\mathcal{A})L^2(\mathcal{B})L^2(\mathcal{C})L^2(\mathcal{D}) = G_{0,B}(L^2(D(S^1_{l})), G_{0,C}(L^2(E(S^1_{l})), L^2(F(S^1_{l})))).
\]

Note that, following (4.8), this Hilbert space is also denoted \( \begin{array}{c} \mathcal{A} \mathcal{B} \mathcal{C} \mathcal{D} \end{array} \).

As in (4.10), \( L^2(\begin{array}{c} \mathcal{F} \\ \mathcal{A} \end{array}) \) and \( L^2(\begin{array}{c} \mathcal{F} \\ \mathcal{B} \end{array}) \) denote the Hilbert spaces \( L^2(D(J_l) \vee \mathcal{B}(J_u) \vee E(J_r)) \) and \( L^2(E(J_l) \vee \mathcal{C}(J_u) \vee F(J_r)) \), respectively. The upper right and lower left corners of (4.14) are therefore given by

\[
L^2(\begin{array}{c} \mathcal{A} \\ \mathcal{B} \end{array})L^2(\begin{array}{c} \mathcal{B} \\ \mathcal{C} \end{array}) = G_{0,C}(L^2(D(J_l) \vee \mathcal{B}(J_u) \vee E(J_r)), L^2(F(S^1_{l}))).
\]

\[
L^2(\begin{array}{c} \mathcal{F} \\ \mathcal{A} \end{array})L^2(\begin{array}{c} \mathcal{F} \\ \mathcal{B} \end{array}) = G_{0,B}(L^2(D(J_l) \vee \mathcal{B}(J_u) \vee E(J_r), L^2(S^1_{l})).
\]
and
\[ L^2([\mathbb{R}^\infty]) \to L^2([\mathbb{R}^\infty]) = G_{0,B}(L^2(D(S^1)), L^2(E(J_1)) \vee \hat{\mathcal{C}}(J_a) \vee F(J_*))). \]
Finally, the vector space \( L^2([\mathbb{R}^\infty]) \) that appears in the upper left corner of (4.14) is the \( L^2 \) space of the von Neumann algebra
\[
\begin{align*}
\begin{array}{c}
\hspace{1cm}
\end{array}
\end{align*}
\]
where the completion is taken on the Hilbert space \( \begin{array}{c}
\end{array} \) or, equivalently, on the Hilbert space \( \begin{array}{c}
\end{array} \).

4.c. \textbf{The keystone fusion of vacuum sectors of defects.} In this section, the defects \( D_B \) and \( E_C \) are assumed to be irreducible. As before, the conformal net \( \mathcal{B} \) is taken to be of finite index.

Recall the algebra \( \begin{array}{c}
\end{array} \) from (4.10). Let us also introduce
\[
\begin{array}{c}
\end{array} \] is a factor, as can be seen by applying Lemma A.15 from the situation of (4.13), but its subalgebra \( \begin{array}{c}
\end{array} \) will typically not be a factor. However, since \( \mathcal{B} \) has finite index, we know by Theorem 3.6 that the subalgebra is at least semisimple.

\textbf{Lemma 4.15.} Let \( \mathcal{B} \) be a conformal net with finite index \( \mu(\mathcal{B}) \), and let \( D_B \) and \( E_C \) be irreducible defects. Let \( p_1, \ldots, p_n \) be the minimal central projections of the algebra \( \begin{array}{c}
\end{array} \). Then we have
\[
\sum_i \left[ p_i \begin{array}{c}
\end{array} : p_i \begin{array}{c}
\end{array} \right] \leq \mu(\mathcal{B}).
\]

\textbf{Proof.} To simplify the notation we abbreviate \( N := \mathcal{B} \begin{array}{c}
\end{array} \), \( M := \mathcal{B} \begin{array}{c}
\end{array} \), and \( A := D \begin{array}{c}
\end{array} = M \vee A \) and \( \begin{array}{c}
\end{array} = N \vee B \).

By definition, \( \mu(\mathcal{B}) = [M : N] \) is the square of \( [M : N] \), the latter being our notation for the statistical dimension of \( N L^2(M)_M \).

Also, \( [p_i(M \vee A)p_i : p_i(N \vee A)] \) is the square of the statistical dimension of \( p_i(M \vee A)p_i L^2(M \vee A)_{M \vee A} \), as can be seen using (A.19); cf. the proof of [1] Prop. 5.18]. The vector whose entries are the statistical dimensions of \( p_i(M \vee A)p_i L^2(M \vee A)_{M \vee A} \) is denoted by \( [M \vee A : N \vee A] \).

By (A.21)
\[
\| [M \vee A : N \vee A] \|_2 \leq [M : N]
\]
and the result follows.

\textbf{Corollary 4.16.} The inclusion \( \begin{array}{c}
\end{array} \to \begin{array}{c}
\end{array} \) is a finite homomorphism of von Neumann algebras with finite-dimensional center (see Appendix A.7).

\textbf{Proof.} Let \( X := \begin{array}{c}
\end{array} \), with minimal central projections \( p_1, \ldots, p_n \), and let \( Y := \begin{array}{c}
\end{array} \). Recall that \( Y \) is a factor. By definition, the inclusion \( X \to Y \) is finite iff the bimodule \( \chi L^2_Y Y \) is dualizable if its summands \( p_i X p_i L^2_Y Y \) are dualizable.

Indeed, the commutant of \( Y \) on \( p_i L^2_Y \) is \( p_i Y p_i \), and the inclusion \( p_i X \to p_i Y p_i \) is finite by the previous lemma.

\hfill \Box
Keystone fusion contains vacuum sector of fused defect. Let $\mathcal{B}$ be a conformal net with finite index. Recall from Section 4.10 that, given a $\mathcal{B}(R)$-module $H_r$ and a $\mathcal{B}(I_r)$-module $H_I$, the keystone fusion $G_\mathcal{B}(H_r, H_I)$ is defined by

$$G_\mathcal{B}(H_r, H_I) := G_0 \mathcal{B}(H_r, H_I) \boxtimes_{\mathcal{B}(S_m)} H_0(S_m, \mathcal{B}).$$

This construction uses the isomorphism $\mathcal{B}(S_m) \cong \bigoplus_{\lambda \in \Delta} \mathcal{B}(H_\lambda(S_m, \mathcal{B}))$ from Theorem 4.9.

**Lemma 4.17.** There is a canonical isomorphism

$$H_0(S_m, \mathcal{B}) \boxtimes_{\mathcal{B}(S_m)} H_0(S_m, \mathcal{B}) \cong \mathbb{C}.$$

**Proof.** The two actions of $\mathcal{B}(S_m)$ factor through its summand $\mathcal{B}(H_0(S_m, \mathcal{B}))$. The result is therefore a special case of the general isomorphism $\mathcal{H} \boxtimes_{\mathcal{B}(H)} \mathcal{H} \cong \mathbb{C}$. □

**Proposition 4.18.** Let $\mathcal{B}$ be a conformal net with finite index, and let $\mathcal{A}D_\mathcal{B}$ and $\mathcal{B}E_\mathcal{C}$ be irreducible defects. Then there is an isometric embedding

$$(4.19) \quad \Psi : L^2\left(\begin{array}{c}
\mathcal{A}D_\mathcal{B} \\
\mathcal{B}E_\mathcal{C}
\end{array}\right) \to \begin{array}{c}
\mathcal{A}D_\mathcal{B} \\
\mathcal{B}E_\mathcal{C}
\end{array}.$$

In formulas, this is a map

$$\Psi = \Psi_{D,E} : L^2\left((D \boxtimes \mathcal{B} E)(S^1_+)\right) \to G_\mathcal{B}(H_0(S_1, D), H_0(S_1, E)).$$

As in Theorem 4.11, the map $\Psi$ is equivariant with respect to the left actions of $\{D(I)\}_{I \subseteq S_{\mathcal{B}, r}}$, $\{E(I)\}_{I \subseteq S_{\mathcal{B}, \lambda}}$, and $\{E(I)\}_{I \subseteq S_{\mathcal{B}, \lambda}^r}$.

**Proof.** By the split property, we can identify $\begin{array}{c}
\mathcal{A}D_\mathcal{B} \\
\mathcal{B}E_\mathcal{C}
\end{array}$ with $\begin{array}{c}
\mathcal{A}D_\mathcal{B} \\
\mathcal{B}E_\mathcal{C}
\end{array}$ and $\begin{array}{c}
\mathcal{A}D_\mathcal{B} \\
\mathcal{B}E_\mathcal{C}
\end{array}$, and thus $L^2\left(\begin{array}{c}
\mathcal{A}D_\mathcal{B} \\
\mathcal{B}E_\mathcal{C}
\end{array}\right)$ with

$$L^2\left(\begin{array}{c}
\mathcal{A}D_\mathcal{B} \\
\mathcal{B}E_\mathcal{C}
\end{array}\right) \oplus L^2\left(\begin{array}{c}
\mathcal{A}D_\mathcal{B} \\
\mathcal{B}E_\mathcal{C}
\end{array}\right) = L^2\left((D \boxtimes \mathcal{B} E)(S^1_+)\right) \otimes H_0(-S_m, \mathcal{B}) = L^2\left((D \boxtimes \mathcal{B} E)(S^1_+)\right) \otimes H_0(S_m, \mathcal{B}).$$

Fusing with $H_0(S_m, \mathcal{B})$ and applying Lemma 4.17, we get a canonical isomorphism

$$U : L^2\left(\begin{array}{c}
\mathcal{A}D_\mathcal{B} \\
\mathcal{B}E_\mathcal{C}
\end{array}\right) = L^2\left((D \boxtimes \mathcal{B} E)(S^1_+)\right) \cong L^2\left((D \boxtimes \mathcal{B} E)(S^1_+)\right) \otimes H_0(-S_m, \mathcal{B}) \boxtimes_{\mathcal{B}(S_m)} H_0(S_m, \mathcal{B}) \cong L^2\left(\begin{array}{c}
\mathcal{A}D_\mathcal{B} \\
\mathcal{B}E_\mathcal{C}
\end{array}\right) \boxtimes_{\mathcal{B}(S_m)} H_0(S_m, \mathcal{B}).$$

Recall from Appendix A.11 that the $L^2$-space construction is functorial for finite homomorphisms between von Neumann algebras with finite-dimensional center. By Corollary 4.10, the inclusion $\iota : \begin{array}{c}
\mathcal{A}D_\mathcal{B} \\
\mathcal{B}E_\mathcal{C}
\end{array} \to \begin{array}{c}
\mathcal{A}D_\mathcal{B} \\
\mathcal{B}E_\mathcal{C}
\end{array}$ therefore induces a map $L^2(\iota) : L^2\left(\begin{array}{c}
\mathcal{A}D_\mathcal{B} \\
\mathcal{B}E_\mathcal{C}
\end{array}\right) \to L^2\left(\begin{array}{c}
\mathcal{A}D_\mathcal{B} \\
\mathcal{B}E_\mathcal{C}
\end{array}\right)$. Let $L^2(\iota)_{iso}$ be the isometry in the polar decomposition of $L^2(\iota)$. We set $\Psi$ to be the composite

$$\Psi : L^2\left(\begin{array}{c}
\mathcal{A}D_\mathcal{B} \\
\mathcal{B}E_\mathcal{C}
\end{array}\right) \xrightarrow{U} L^2\left(\begin{array}{c}
\mathcal{A}D_\mathcal{B} \\
\mathcal{B}E_\mathcal{C}
\end{array}\right) \boxtimes_{\mathcal{B}(S_m)} H_0(S_m, \mathcal{B}) \quad (4.20)$$

$$\xrightarrow{L^2(\iota)_{iso} \otimes \text{id}} L^2\left(\begin{array}{c}
\mathcal{A}D_\mathcal{B} \\
\mathcal{B}E_\mathcal{C}
\end{array}\right) \boxtimes_{\mathcal{B}(S_m)} H_0(S_m, \mathcal{B}) \xrightarrow{\Psi_0 \otimes \text{id}} H_0(S_m, \mathcal{B}) = G(H_0(D), H_0(E)),$$

where $\Psi_0$ is the unitary isomorphism from Theorem 4.11. □

We will prove later, in Theorem 5.2 concerning the composite map (4.31), that the map $\Psi$ is actually an isomorphism. We can already observe the following special case of that result:
Lemma 4.21. If $D$ and $E$ are the identity defects of some finite-index conformal net $A$, then $\Psi_{D,E}$ is a unitary isomorphism.

Proof. We need to show that the map

$$((\Psi_0 \circ L^2(t_{iso}) \otimes \text{id}) \circ U : H_0(S_b) = L^2(\bigotimes_{A(S_m)} H_0(S_m))$$

is an isomorphism. By the computation (4.6), we know that the right hand side is isomorphic to $H_0(S_b)$, and thus is irreducible as an $S_b$-sector of $A$. The above map is a homomorphism of $S_b$-sectors and is injective by the previous proposition. It is therefore an isomorphism.

Associativity for the inclusion of the vacuum sector. Using the isometric embedding $\Psi$ from (4.20) in place of the unitary isomorphism $\Psi_0$ from (4.12), we can form the following diagram analogous to (4.14):

$$\begin{array}{c}
L^2(\bigotimes_{A(S_m)} H_0(S_m)) \\
\downarrow
\end{array} \begin{array}{c}
L^2(\bigotimes_{B(S_m)} H_0(S_m)) \\
\downarrow
\end{array} \begin{array}{c}
L^2(\bigotimes_{C(S_m)} H_0(S_m))
\end{array}$$

(4.22)

Lemma 4.23. Let $B$ and $C$ be finite-index conformal nets, and let $\Delta D_B$, $\Delta E_C$, and $\Delta F_D$ be irreducible defects. Then the diagram (4.22) is commutative.

Proof. By definition, each side of (4.22) is a composite of three maps. Replacing each side by its definition, that diagram can be expanded to a $4 \times 4$ grid

that contains 9 squares. The four upper left squares of that grid are given by

$$\begin{array}{c}
L^2(\bigotimes_{A(S_m)} H_0(S_m)) \\
\downarrow
\end{array} \begin{array}{c}
L^2(\bigotimes_{B(S_m)} H_0(S_m)) \\
\downarrow
\end{array} \begin{array}{c}
L^2(\bigotimes_{C(S_m)} H_0(S_m))
\end{array}$$

where $\bigotimes_{A(S_m)}$ and $\bigotimes_{B(S_m)}$ stand for $\bigotimes_{A(S_m)} H_0(S_m, B)$ and $\bigotimes_{B(S_m)} H_0(S_m, C)$, respectively, and $\bigotimes_{C(S_m)}(a \otimes b)$ stands for $\bigotimes_{B(S_m)}(a \otimes c) H_0(S_m, B) \otimes H_0(S_m, C)$. Here, $S_m$ denotes a translated copy of the circle $S_m$.

The squares $\Box$, $\Box$, and $\Box$ clearly commute. To see that $\Box$ commutes, note first that $\bigotimes_{A(S_m)}$ is a factor, as can be seen by applying Lemma A.15 twice. That square then commutes by the functoriality of $L^2(-)_{iso}$—see [1, Prop. 6.22] and note that the necessary conditions for that functoriality are satisfied by Corollary 4.16 and
by Lemma 4.24 below. The upper right squares of our \(4 \times 4\) grid are given by

\[
\begin{array}{c}
L^2(\mathcal{A} \otimes \mathcal{B}) \\
L^2(\mathcal{C} \otimes \mathcal{D}) \\
L^2(\mathcal{E} \otimes \mathcal{F})
\end{array}
\]

and their commutativity is unproblematic. We refrain from drawing the last row of the grid. The squares \(7\) and \(8\) are similar to \(3\) and \(6\). The commutativity of \(9\) follows from that of (4.14).

\[\square\]

**Lemma 4.24.** Let \(\mathcal{A} \to \mathcal{B} \to \mathcal{C} \to \mathcal{D}\) be as in Lemma 4.23. Let \(\mathcal{A} := \mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C} \otimes \mathcal{D}\), and let \(\iota : \mathcal{A} \to \mathcal{B}\) denote the inclusion. Then \(Z(\mathcal{B}) \subset \iota(\mathcal{A})\).

**Proof.** By the split property, we have isomorphisms \(\ell^2 \otimes \mathcal{H} \cong \ell^2 \otimes \mathcal{M}\) and \(\ell^2 \otimes \mathcal{H} \cong \ell^2 \otimes \mathcal{M}'\). It is therefore sufficient to show that \(Z(\mathcal{H}) \subset \iota(\mathcal{M})\).

Let \(S_u, K_u, J_u\) be as in (4.2) and (4.9), and let \(H := H_0(S_u, B)\), and \(M := B(K_u)\), with commutant \(M' = B(J_u)\). Since \(H\) is a faithful \(M\)-module, we can pick an \(M\)-linear isomorphism \(\ell^2 \otimes H \cong \ell^2 \otimes \mathcal{L}(M)\)\(^9\). Under the corresponding isomorphism of Hilbert spaces \(\ell^2 \otimes H \cong \ell^2 \otimes \mathcal{M} \otimes \mathcal{M}'\), the algebra

\[
\mathcal{B}(\ell^2 \otimes H) \cong \mathcal{B}(\ell^2 \otimes M') \vee \mathcal{A}
\]

corresponds to

\[
(\mathcal{B}(\ell^2 \otimes M') \vee \mathcal{A}) = \mathcal{B}(\ell^2 \otimes M) \vee \mathcal{A}.
\]

It follows that

\[
Z(\mathcal{A}) \cong Z(\mathcal{B}) \cong Z(\mathcal{C}) \cong Z(\mathcal{D}) \subset \mathcal{A}.
\]

where the last equality is because \(\mathcal{A} \mathcal{D}\) is irreducible.

We now argue that the natural inclusion \(\mathcal{A} \to \mathcal{A}\) induces an isomorphism of centers. By Theorem 3.6, the center of these algebras is finite-dimensional. The center \(Z(\mathcal{A})\) certainly maps to the center \(Z(\mathcal{A})\) and that the map is injective. It is therefore an isomorphism. The claim now follows, as

\[
Z(\mathcal{A}) \cong Z(\mathcal{B}) \cong Z(\mathcal{C}) \cong Z(\mathcal{D}) \subset \mathcal{A}.
\]

\[\square\]

**Remark 4.25.** All the defects in this section were assumed to be irreducible. However, using the compatibility of direct integrals with various operations, it is straightforward to extend Proposition 4.18 and Lemma 4.23 to arbitrary defects.

\[\text{Here, } \ell^2 := \ell^2(\mathbb{N}) \text{ could be removed from this isomorphism if we knew that } M \text{ was a type III factor, a fact which is likely to be true (unless } B \text{ is trivial) but which we haven't proven in our setup.} \]
4.D. Comparison between fusion and keystone fusion. Let \( \mathcal{A} \) be a conformal net with finite index (implicitly irreducible as before). In this section, we will define a unitary natural transformation \( \Phi_\mathcal{A} : F_\mathcal{A} \to G_\mathcal{A} \) between the functors introduced in Section 4.3. Graphically, this natural transformation is denoted

\[
\Phi : \begin{array}{c}
\mathcal{H}_I \\
\mathcal{H}_I'
\end{array} \to \begin{array}{c}
\mathcal{H}_I \\
\mathcal{H}_I'
\end{array}.
\]

Recall the circles \( S_I, S_r, \) and \( S_b \) introduced in (4.15):

\[
(4.26) \quad S_I = \begin{array}{c}
\mathcal{H}_I \\
\mathcal{H}_I'
\end{array}, \quad S_r = \begin{array}{c}
\mathcal{H}_I \\
\mathcal{H}_I'
\end{array}, \quad \text{and} \quad S_b = \begin{array}{c}
\mathcal{H}_I \\
\mathcal{H}_I'
\end{array}.
\]

As before, we let \( I := S_I \cap S_r, \) with orientation inherited from \( S_r. \) The circles \( S_I \) and \( S_r \) are given conformal structures by their unit speed parametrizations. The circle \( S_b \) is also given a conformal structure, as follows. Let \( j_l \in \text{Conf}(S_I) \) and \( j_r \in \text{Conf}(S_r) \) be the involutions fixing \( \partial I. \) The conformal structure on \( S_b \) is the one making \( \epsilon_l := j_l \bigcup \text{Id}_{j_r(S_r \setminus I)} : S_r \to S_b \) into a conformal map. Equivalently, it is the one for which \( \epsilon_r := j_r \bigcup \text{Id}_{j_l(S_I \setminus I)} : S_I \to S_b \) is a conformal map.

**Warning 4.27.** The conformal structure on \( S_b \) is not the one induced by its constant speed parametrization. Nevertheless, the reflection along the horizontal and vertical symmetry axes of \( S_b \) are conformal involutions.

Consider the vacuum sectors \( H_0(S_I), H_0(S_r), \) and \( H_0(S_b) \) for the net \( \mathcal{A} \) on the circles (4.26). By the construction (4.6), there is a canonical identification \( \Upsilon : H_0(S_I) \boxtimes_{\mathcal{A}(I)} H_0(S_r) \cong H_0(S_b) \) that is equivariant with respect to the actions of \( \mathcal{A}(J) \) for every \( J \subset S_b. \) We denote it graphically as

\[
(4.28) \quad \Upsilon : \begin{array}{c}
\mathcal{H}_I \\
\mathcal{H}_I'
\end{array} \cong \begin{array}{c}
\mathcal{H}_I \\
\mathcal{H}_I'
\end{array}.
\]

The following proposition improves on Proposition 4.14 by providing an explicit construction of the natural isomorphism \( \Phi : F \to G. \) In particular, with this new construction, it is obvious that \( \Phi_{\mathcal{A} \otimes \mathcal{B}} = \Phi_\mathcal{A} \otimes \Phi_\mathcal{B}. \)

**Proposition 4.29.** There is a unitary natural isomorphism \( \Phi = \Phi_\mathcal{A} \) between the fusion functor \( F_\mathcal{A} \) and the keystone fusion functor \( G_\mathcal{A}. \)

\[
\Phi : \begin{array}{c}
\mathcal{H}_I \\
\mathcal{H}_I'
\end{array} \to \begin{array}{c}
\mathcal{H}_I \\
\mathcal{H}_I'
\end{array}.
\]

**Proof.** Let \( I_l \) and \( I_r \) be as in (4.11), and let \( I'_l \) and \( I'_r \) be the closures of their complements in \( S_I \) and \( S_r, \) respectively. Since the actions of \( \mathcal{A}(I_l) \) and \( \mathcal{A}(I_r) \) on \( H_0(S_I) \) and \( H_0(S_r) \) are faithful, by Lemma 4.13 it is enough to define the isomorphism \( \Phi_{H_0(S_I), H_0(S_r)} : \begin{array}{c}
\mathcal{H}_I \\
\mathcal{H}_I'
\end{array} \to \begin{array}{c}
\mathcal{H}_I \\
\mathcal{H}_I'
\end{array}, \) and to check that it commutes with the natural actions of \( \mathcal{A}(I'_l) = \mathcal{A}(I'_l)' \) and \( \mathcal{A}(I_r) = \mathcal{A}(I'_r). \) We define this isomorphism as the composite

\[
(4.30) \quad \begin{array}{c}
\mathcal{H}_I \\
\mathcal{H}_I'
\end{array} \cong \begin{array}{c}
\mathcal{H}_I \\
\mathcal{H}_I'
\end{array} \to L^2\left( \begin{array}{c}
(4.12)
\end{array} \right) \cong \begin{array}{c}
\mathcal{H}_I \\
\mathcal{H}_I'
\end{array},
\]

where \( \varphi_{S_b, \Upsilon} : H_0(S_b, \mathcal{A}) \to L^2(\mathcal{A}(S_b, \Upsilon)) \) is the canonical unitary isomorphism (4.12) associated to the upper half \( S_b, \Upsilon \) of the conformal circle \( S_b, \) and \( \Psi \) is the unitary isomorphism from Lemma 4.21. \( \square \)

Let \( \mathcal{A} \) and \( \mathcal{C} \) be conformal nets, let \( \mathcal{B} \) be a conformal net with finite index, and let \( \mathcal{A}D_\mathcal{B} \) and \( \mathcal{C}E_\mathcal{B} \) be defects. Let us introduce the notation \( \begin{array}{c}
\mathcal{H}_I \\
\mathcal{H}_I'
\end{array} \) for the Hilbert space \( L^2(\mathcal{B}) = L^2((D \otimes \mathcal{B})(S_I^1)) \) that appears in the left hand side of (4.19).

Combining Proposition 4.18 (see also Remark 4.22 and Proposition 4.29) we can construct an isometric map

\[
(4.31) \quad \Omega : \begin{array}{c}
\mathcal{H}_I \\
\mathcal{H}_I'
\end{array} \to \begin{array}{c}
\mathcal{H}_I \\
\mathcal{H}_I'
\end{array}.
\]
CONFORMAL NETS III: FUSION OF DEFECTS

where Φ stands for Φ_{H_0(S_l,D),H_0(S_r,E)}. We will show later, in Theorem 6.2, that the
map Ω = Ω_{D,E} is in fact an isomorphism. This map is the fundamental “1 ⊠ 1 = 1
isomorphism” comparing 1_{D ⊗ E} with 1_D ⊠ 1_E.

Lemma 4.32. Let A, B, C, D be conformal nets, of which the second and third
are assumed to be finite, and let _A^D_B, B^E_C, C^F_D be defects. Then the maps

Ω_{D,E,F} ⊠ id

form a commutative diagram.

Proof. By the definition of Ω, the above diagram can be expanded to

The upper left square commutes by Lemma 4.23 (see also Remark 4.25). The
remaining three squares commute by the naturality of Φ^{-1}. □

5. HAAG DUALITY FOR COMPOSITION OF DEFECTS

Throughout this section we fix conformal nets A, B, and C, always assumed
irreducible, and irreducible defects _A^D_B and B^E_C. In our pictures, we will use the
notation _ for intervals on which we evaluate A, we will use _ for intervals on
which we evaluate B, and _ for intervals on which we evaluate C. We will also use
_ and _ for bicolored intervals on which we evaluate D, and _ and _ for bicolored intervals
on which we evaluate E.

Let S_I and S_j be as in (1.4) and (1.7), with intersection I oriented like S_r. As
before, we use the notation _ := H_0(S_I, D) = L^2([0,1]), similarly _ := H_0(S_j, E) =
L^2([0,1]), and [ ] := H_0(S_I, D) ⊗_{B(I)} H_0(S_j, E). We will again be using the Notation 1.4.

Letting

\begin{align}
I_1 &:= \partial \overline{[0,1] \times [1/2,1]} & (I_1)_\circ &:= (I_1)_{x \leq \frac{1}{2}} & (I_1)_\bullet &:= (I_1)_{x \geq \frac{1}{2}} \\
I_2 &:= \partial \overline{[1,2] \times [1/2,1]} & (I_2)_\circ &:= (I_2)_{x \leq \frac{1}{2}} & (I_2)_\bullet &:= (I_2)_{x \geq \frac{1}{2}} \\
I_3 &:= \partial \overline{[0,1] \times [0,1/2]} & (I_3)_\circ &:= (I_3)_{x \leq \frac{1}{2}} & (I_3)_\bullet &:= (I_3)_{x \geq \frac{1}{2}} \\
I_4 &:= \partial \overline{[1,2] \times [1/2,1]} & (I_4)_\circ &:= (I_4)_{x \leq \frac{1}{2}} & (I_4)_\bullet &:= (I_4)_{x \geq \frac{1}{2}},
\end{align}

we will write D( ), E( ), D(I_\circ), E(I_\circ) for D(I_1), E(I_2), D(I_3), E(I_4),
respectively.

Main Theorem 5.2. Assuming B has finite index, then on the Hilbert space _ [ ] , we have

\begin{align}
D \left( \boxed{\smiley} \right) \lor E \left( \boxed{\frown} \right) &= \left( D \left( \boxed{\frown} \right) \lor E \left( \boxed{\smiley} \right) \right)',
\end{align}
Proof. Let us introduce some notation for various algebras that act on the Hilbert space $\mathcal{H}$. The main algebras of interest are $\mathcal{A} = (D \otimes B)(S^\perp_1)$ and $\mathcal{B} = (D \otimes B)(S^\perp_1)$, and our goal is to show that the inclusion

(5.4) $\mathcal{A} \subseteq (\mathcal{B})'$

is an isomorphism. Let us fix once and for all a small number $\epsilon$. Consider the 1-manifolds

$J_0 := (\{0\} \times [1/2, 1]) \cup ([0, 1/2 + \epsilon] \times \{1\}),$

$J_1 := ([1/2 + \epsilon, 2/3] \cup [5/6, 1]) \times \{1\},$

$J_2 := ([1, 3/2 - \epsilon] \times \{1\}) \cup ([7/6, 4/3] \times \{0\}),$

$J_3 := ([3/2 - \epsilon, 2] \times \{1\}) \cup ([2] \times [1/2, 1]).$

We will use the following algebras:

(5.5) $\mathcal{A} := D(J_0) \vee B(J_1) \vee B(J_2) \vee E(J_3) = D(J_0 \cup J_1) \vee E(J_2 \cup J_3)$

(5.6) $\mathcal{B} := D(J_0) \vee \tilde{B}(J_1) \vee B(J_2) \vee E(J_3) = \tilde{D}(J_0 \cup J_1) \vee E(J_2 \cup J_3)$

(5.7) $\mathcal{C} := D(J_0) \vee B(J_1) \vee \tilde{B}(J_2) \vee E(J_3) = D(J_0 \cup J_1) \vee \tilde{E}(J_2 \cup J_3)$

(5.8) $\mathcal{D} := D(J_0) \vee \tilde{B}(J_1) \vee \tilde{B}(J_2) \vee E(J_3) = \tilde{D}(J_0 \cup J_1) \vee \tilde{E}(J_2 \cup J_3)$

Here $\tilde{B}$, $\tilde{D}$, and $\tilde{E}$ are as in 3.10, and $J_0$ and $J_3$ are bicolored as in (4.7). By Lemma 3.11, the algebras $\tilde{B}(J_1)$ and $\tilde{B}(J_2)$ act on $\mathcal{A}$ and $\mathcal{B}$ respectively, and satisfy $D(J_0) \vee \tilde{B}(J_1) = \tilde{D}(J_0 \cup J_1)$ and $\tilde{B}(J_2) \vee E(J_3) = \tilde{E}(J_2 \cup J_3)$. The equalities in (5.6)–(5.8) for actions on $\mathcal{A}$ follow.

In Section 5.7 below we will obtain some purchase on the Haag inclusion $\mathcal{A} \subseteq (\mathcal{B})'$ by showing that its statistical dimension is the same as that of the inclusion $\mathcal{B} \subseteq (\mathcal{D})'$. (Here the algebra $\mathcal{D}$ is defined similarly to $\mathcal{A}$.) We can compute the statistical dimension of that latter inclusion by squeezing it into a sequence of simpler inclusions of von Neumann algebras, as follows:

Because $D$ and $E$ are irreducible, the algebra (5.5) is a factor. Using Lemma 3.11 note that the algebra (the right connected component of the picture (5.7)) is the commutant of a factor acting on a vacuum sector for $E$; it follows that (5.7) is a factor. More difficult is the fact that (5.8) is a factor—that is the content of Corollary 5.16 following from Lemma 5.13 below. The algebra (5.6) is not a factor, but combining Lemma 5.10 below and Theorem 3.6 we will learn that it does have finite-dimensional center; let $n$ be the dimension of this center.
Let $\nu_1, \ldots, \nu_6$ be the matrices of statistical dimensions (Appendix A.VIII) of the various inclusions in the above diagram:

\[
\nu_1 := \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} : \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \\
\nu_2 := \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} : \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \\
\nu_3 := \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} : \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \\
\nu_4 := \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} : \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \\
\nu_5 := \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \end{bmatrix} \end{bmatrix} : \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \end{bmatrix} \end{bmatrix} \\
\nu_6 := \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \end{bmatrix} \end{bmatrix} : \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \end{bmatrix} \end{bmatrix}.
\]

Note that $\nu_3$ and $\nu_4$ are scalars, $\nu_1$ is a row vector, $\nu_2$ and $\nu_6$ are column vectors, and $\nu_5$ is an $n \times n$ matrix. As the statistical dimension is multiplicative (A.21), it follows that $\nu_1 \nu_2 = \nu_3 \nu_4$, and that $\nu_2 = \nu_5 \nu_6$.

We will need the following facts about the statistical dimensions $\nu_1, \ldots, \nu_6$:

(i) There is an equality of $n \times n$ matrices

\[
\nu_5 \equiv \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \end{bmatrix} \end{bmatrix} : \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \end{bmatrix} \end{bmatrix} : \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \end{bmatrix} \end{bmatrix}.
\]

This is proven in Lemma 5.10 below.

(ii) The map (4.31) exhibits $L^2(\mathcal{E}K)$ as a sub-bimodule of $\hat{\mathcal{E}}$. Using the additivity of statistical dimension (A.17), we get the entrywise matrix inequality

\[
\nu_5 = \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \end{bmatrix} \end{bmatrix} : \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \end{bmatrix} \end{bmatrix} \geq 1_n,
\]

where $1_n$ denotes the identity matrix.

(iii) In Corollary 5.17 we will show that $\nu_1$ is the transpose of $\nu_6$.

(iv) In Corollary 5.21 we will show that $\nu_3 = \sqrt{\mu}$ and $\nu_4 = \sqrt{\mu}$, where $\mu = \mu(B)$ is the index of the conformal net $B$.

Using these results and the fact that the matrices of statistical dimensions have only non-negative entries, we can now compute that

\[
\|\nu_2\|^2 = \nu_2^* \nu_2 = \nu_3^* \nu_3 \nu_5 \nu_6 = \nu_1 \nu_2^* \nu_3 \nu_4 \geq \nu_1 \nu_5 \nu_6 = \nu_3 \nu_4 = \mu,
\]

with equality if and only if $\nu_5 = 1_n$. However, by applying (A.21) to the algebras $M := \hat{B}(J_2)$, $N := B(J_2)$, and $A := D(J_0) \vee \hat{B}(J_1) \vee E(J_3)$, we obtain the reverse inequality

\[
\|\nu_2\| \leq \sqrt{\mu}.
\]

It follows that $\nu_5 = 1_n$, and by [1, Prop. 5.2] the inclusion (5.3) is therefore an isomorphism.

In Theorem 5.2, the defects $D$ and $E$ were assumed to be irreducible, but the statement holds in general:

**Corollary 5.9.** Let $\mathcal{A}D_B$ and $\mathcal{B}E_C$ be defects. If the conformal net $B$ has finite index, then the algebra $D(\mathcal{E}K) \vee E(\mathcal{E}K)$ is the commutant of $D(\mathcal{E}K) \vee E(\mathcal{E}K)$ on $\mathcal{E}K$.

**Proof.** We need to show that $D(I_1) \vee E(I_2) = (D(I_3) \vee E(I_4))'$, where the intervals $I_1, I_2, I_3, I_4$ are as in (5.1). Disintegrating

\[
D = \int^{\oplus} D_x \quad \text{and} \quad E = \int^{\oplus} E_y
\]

into irreducible defects, the Hilbert space $\mathcal{E}K = H_0(S_1, D) \boxtimes_{\mathcal{B}(I)} H_0(S_1, E)$ decomposes correspondingly as $\int\int^{\oplus} H_0(S_1, D_x) \boxtimes_{\mathcal{B}(I)} H_0(S_1, E_y)$. This induces direct integral decompositions

\[
D(I_1) \vee E(I_2) = \int^{\oplus} D_x(I_1) \vee E_y(I_2),
\]
\[
D(I_3) \vee E(I_4) = \int^{\oplus} D_x(I_3) \vee E_y(I_4),
\]

\[
D(I_1) \vee E(I_2) = \int^{\oplus} D_x(I_1) \vee E_y(I_2),
\]
\[
D(I_3) \vee E(I_4) = \int^{\oplus} D_x(I_3) \vee E_y(I_4),
\]

Thus, the statement holds. \qed
and therefore also $(D(I_3) \lor E(I_4))' = \int_0^\infty (D_x(I_3) \lor E_y(I_4))'$, where the commutant of $D_x(I_3) \lor E_y(I_4)$ is taken on $H_0(S_t, D_x) \boxtimes H_0(S_t, E_y)$. By Theorem 5.2 we have $D_x(I_3) \lor E_y(I_4) = (D_x(I_3) \lor E_y(I_4))'$, and the result follows.

5. A. The dimension of the Haag inclusion. Recall the notion of center of a defect from Section 1. For a genuinely bicolored interval $I$, the algebra $Z(D(I))$ is independent of $I$ (up to canonical isomorphism), and denoted $Z(D)$.

From now on, the defects $D$ and $E$ are again assumed irreducible:

**Lemma 5.10.** Let $X := \text{Spec}(Z(D \boxtimes E))$ be the set of irreducible summands of $D \boxtimes E$ [4]. We have a canonical identification of centers $Z(D \boxtimes E) = Z_D = Z(E) = Z(D \boxtimes E)$, and we have the following equality of $X \times X$ matrices:

$$
\begin{pmatrix}
\text{Spec}(Z(D \boxtimes E))' : & \text{Spec}(Z(D \boxtimes E)) \\
\end{pmatrix}
= \begin{pmatrix}
\text{Spec}(Z(D))' : & \text{Spec}(Z(D)) \\
\end{pmatrix}
= \begin{pmatrix}
\text{Spec}(Z(E))' : & \text{Spec}(Z(E)) \\
\end{pmatrix}
= \begin{pmatrix}
\text{Spec}(Z(D \boxtimes E))' : & \text{Spec}(Z(D \boxtimes E)) \\
\end{pmatrix}
$$

**Proof.** Note that $[A : B] = [A \otimes C : B \otimes C]$ whenever $C$ is a factor [1A22]. The algebras $B([7/6, 4/3])$ and $B([7/6 - \epsilon, 4/3 + \epsilon])'$ are split on $H_0(S_t, B)$, and hence on any $B([7/6 - \epsilon, 4/3 + \epsilon])$-module. Since $(\quad \quad)$' $\subset B([7/6 - \epsilon, 4/3 + \epsilon])'$ on $\mathbb{H}$, it follows that $(\quad \quad) \lor _\quad = (\quad \quad) \otimes _\quad$, where the line _ stands for $B([7/6, 4/3] \times \{0\})$. We conclude that

$$
\begin{pmatrix}
\text{Spec}(Z(D \boxtimes E))' : & \text{Spec}(Z(D \boxtimes E)) \\
\end{pmatrix}
= \begin{pmatrix}
\text{Spec}(Z(D))' : & \text{Spec}(Z(D)) \\
\end{pmatrix}
= \begin{pmatrix}
\text{Spec}(Z(E))' : & \text{Spec}(Z(E)) \\
\end{pmatrix}
= \begin{pmatrix}
\text{Spec}(Z(D \boxtimes E))' : & \text{Spec}(Z(D \boxtimes E)) \\
\end{pmatrix}
$$

the last equality follows by applying Lemma 1A30 with $B := \quad \quad$, $A := \quad$, $A_0 := \quad$, and $A \cap A_0' = \quad$. Similarly, we have

$$
\begin{pmatrix}
\text{Spec}(Z(D \boxtimes E))' : & \text{Spec}(Z(D \boxtimes E)) \\
\end{pmatrix}
= \begin{pmatrix}
\text{Spec}(Z(D))' : & \text{Spec}(Z(D)) \\
\end{pmatrix}
= \begin{pmatrix}
\text{Spec}(Z(E))' : & \text{Spec}(Z(E)) \\
\end{pmatrix}
= \begin{pmatrix}
\text{Spec}(Z(D \boxtimes E))' : & \text{Spec}(Z(D \boxtimes E)) \\
\end{pmatrix}
$$

Taking commutants transposes the matrix of statistical dimensions [1, Cor. 5.17]. Thus 5.12 implies

$$
\begin{pmatrix}
\text{Spec}(Z(D \boxtimes E))' : & \text{Spec}(Z(D \boxtimes E)) \\
\end{pmatrix}
= \begin{pmatrix}
\text{Spec}(Z(D))' : & \text{Spec}(Z(D)) \\
\end{pmatrix}
$$

which, combined with 5.11, proves the Lemma. □

---

10Here, we use “Spec” in the sense of algebraic geometry.
5B. The double bridge algebra is a factor. Let \( S_l \) and \( S_r \) be as in (4.6) and now let \( \tilde{S}_l := \partial([5/4, 2] \times [0, 1]) \) and \( \tilde{S}_r := \partial([5/4, 2] \times [0, 1]) \); give these circles the bicoloring
\[
(\tilde{S}_l)_r := (\tilde{S}_l)_{x \leq \frac{3}{2}} \quad (\tilde{S}_l)_o := (\tilde{S}_l)_{x > \frac{3}{2}} \quad (\tilde{S}_r)_l := (\tilde{S}_r)_{x \leq \frac{3}{2}} \quad (\tilde{S}_r)_o := (\tilde{S}_r)_{x > \frac{3}{2}}
\]
Let \( I := S_l \cap S_r \) and \( \tilde{I} := \tilde{S}_l \cap \tilde{S}_r \). The vacuum sector \( H_0(S, \mathcal{B}) \) of \( \mathcal{B} \) on any circle conformal \( S \) is a unit for Connes fusion over \( \mathcal{B}(I) \) for any interval \( I \subset S \); see (3.2). By applying this fact twice, we can construct a non-canonical isomorphism
\[
H_0(S_l, D) \ast_{\mathcal{B}(I)} H_0(S_r, E) \cong H_0(\tilde{S}_l, D) \ast_{\mathcal{B}(\tilde{I})} H_0(\tilde{S}_r, E),
\]
equivalent with respect to the actions of \( D(J) \) for every \( J \subset \bigcup \) and \( J \subset \bigcup \), and with respect to the actions of \( E(J) \) for every \( J \subset \bigcup \) and \( J \subset \bigcup \).
Recall the fiber product operation \( \ast \) from Appendix [A.IV]. Let \( J := [7/6, 4/3] \times [0, 1], \quad K_l := \partial_-[7/6, 5/4] \times [0, 1], \quad K_r := \partial_-[5/4, 4/3] \times [0, 1], \quad K_l' := \partial_+[4/3, 2] \times [0, 1], \quad K_r' := \partial_+[7/6, 0] \times [0, 1], \quad K_r' := K_r \cup K_r'. \)
which we draw for convenience:
\[
J = \bigcup, \quad K_l = \bigcup, \quad K_r = \bigcup, \quad K_l' = \bigcup, \quad K_r' = \bigcup.
\]
We then have \( \tilde{S}_l = K_l \cup K_l' \) and \( \tilde{S}_r = K_r \cup K_r' \). We use \( H = H_0(\mathcal{B}, \tilde{S}_l) \) and \( K = H_0(\mathcal{B}, \tilde{S}_r) \) in the definition [A.10] of the fiber product \( B(K_l) \ast_{\mathcal{B}(I)} B(K_r) = (B(K_l)' \vee B(K_r)')' \). By Haag duality, we have \( B(K_l)' = B(K_l) \) and \( B(K_r)' = B(K_r') \). These algebras all act on \( H_0(\mathcal{B}, \tilde{S}_l) \otimes_{\mathcal{B}(\tilde{I})} H_0(\mathcal{B}, \tilde{S}_r) \), which can be identified with \( H_0(\mathcal{B}, S_0) \) by [2] Cor. 1.33. Altogether we obtain
\[
B(J) = (B(K_l)' \vee B(K_r)')' = B(K_l)' \bullet B(K_r)' = B(K_l)' \ast_{\mathcal{B}(\tilde{I})} B(K_r').
\]
We denote the above equation graphically by \( \bigcup = \bigcup' \).

Lemma 5.13. We have the following equality of subalgebras of \( \mathcal{B}(\bigcup \bigcup) \):
\[
(\bigcup \bigcup)' = (\bigcup \bigcup),
\]
\[
(\bigcup \bigcup)' = (\bigcup \bigcup).
\]

Proof. By Lemma 3.11 and Proposition 4.17 respectively, we have
\[
\bigcup = (\bigcup)' \quad \text{on} \quad \bigcup, \quad \text{and} \quad \bigcup = (\bigcup)' \quad \text{on} \quad \bigcup,
\]
where \( \bigcup \) stands for \( H_0(\tilde{S}_l, D) \), and \( \bigcup \) stands for \( H_0(\tilde{S}_r, E) \).
We have the following sequence of equalities
\[
\bigcup \bigcup = (\bigcup \bigcup) \vee (\bigcup \bigcup) \quad \text{on} \quad \bigcup, \quad \text{and} \quad \bigcup = (\bigcup \bigcup) \ast (\bigcup \bigcup) \quad \text{on} \quad \bigcup.
\]
Here the third equality uses Lemma 4.34. By Lemma 3.11 we also have
\[
\bigcup \bigcup = (\bigcup \bigcup) \quad \text{on} \quad \bigcup.
\]
We therefore similarly have
\[
\begin{align*}
\mathcal{F}\mathcal{E}\mathcal{F}\mathcal{E} \mathcal{F} = (\mathcal{F}\mathcal{E}\mathcal{F}) \vee (\mathcal{F}) \vee (\mathcal{E}) \\
= (\mathcal{F}\mathcal{E}\mathcal{F}) \vee (\mathcal{E}) \vee (\mathcal{F}) = \mathcal{F}\mathcal{E}\mathcal{F} \mathcal{F} \mathcal{E} \mathcal{F} \mathcal{F} \\
= (((\mathcal{E}) \vee (\mathcal{F}))')' = (\mathcal{E} \mathcal{F})'.
\end{align*}
\]

\[\square\]

Corollary 5.16. The algebra \(\mathcal{F}\mathcal{E}\mathcal{F}\mathcal{E}\mathcal{F}\mathcal{E}\) is a factor. \[\square\]

Corollary 5.17. We have
\[
\text{(5.18)} \quad \llbracket \rho_{\mathcal{F}\mathcal{E}\mathcal{F}\mathcal{E}\mathcal{F}} : \rho_{\mathcal{E}\mathcal{F}\mathcal{F}\mathcal{E}} \rrbracket = \llbracket \rho_{\mathcal{F}\mathcal{E}\mathcal{F}} : (\mathcal{E} \mathcal{F})' \rrbracket'
\]

Proof. By (A.20) and by Lemma 5.13, the right hand side of (5.18) is equal to
\[
\llbracket \rho_{\mathcal{E} \mathcal{F}} : (\mathcal{F}\mathcal{E}\mathcal{F}\mathcal{E}\mathcal{F})' \rrbracket = \llbracket \rho_{\mathcal{E} \mathcal{F}} : \rho_{\mathcal{E} \mathcal{F}} \rrbracket.
\]

The algebras \(\rho_{\mathcal{E} \mathcal{F}}\) and \(\rho_{\mathcal{E} \mathcal{F}}\) are related to those on the left hand side of (5.18) by the action of orientation reversing diffeomorphisms of the underlying 1-manifolds: these diffeomorphisms induce algebra isomorphisms \(\rho_{\mathcal{E} \mathcal{F}} \cong (\rho_{\mathcal{F}\mathcal{E}\mathcal{F}\mathcal{E}\mathcal{F}})^{op}\) and \(\rho_{\mathcal{E} \mathcal{F}} \cong (\rho_{\mathcal{F}\mathcal{E}\mathcal{F}\mathcal{E}\mathcal{F}})^{op}\). The result now follows since \([A : B] = [A^{op} : B^{op}]\). \[\square\]

5.c. The dimension of the bridge inclusions.

Lemma 5.19. We have the following equalities of statistical dimensions:
\[
\llbracket \rho_{\mathcal{F}\mathcal{E}\mathcal{F}} : \rho_{\mathcal{E}\mathcal{F}\mathcal{F}} \rrbracket = \llbracket \rho_{\mathcal{F}\mathcal{E}\mathcal{F}} : (\mathcal{E} \mathcal{F})' \rrbracket' = \sqrt{\mu(B)}.
\]

Proof. We have
\[
\llbracket \rho_{\mathcal{F}\mathcal{E}\mathcal{F}} : \rho_{\mathcal{E}\mathcal{F}\mathcal{F}} \rrbracket = \llbracket \rho_{\mathcal{F}\mathcal{E}\mathcal{F}} : \rho_{\mathcal{F}\mathcal{E}\mathcal{F}} \rrbracket = \llbracket \rho_{\mathcal{E} \mathcal{F}} : \rho_{\mathcal{E} \mathcal{F}} \rrbracket = \sqrt{\mu(B)},
\]

where the first equality is obtained by using an appropriate diffeomorphism, the second one follows from (A.20) and the special case of Lemma 5.13 when \(D\) is an identity defect, and the third one uses (A.22).

Let us introduce the auxiliary quantity
\[
\nu := \llbracket \rho_{\mathcal{F}\mathcal{E}\mathcal{F}} : \rho_{\mathcal{E}\mathcal{F}\mathcal{F}} \rrbracket.
\]

By Lemma 3.14 we know that \(\nu \leq \sqrt{\mu(B)}\); in particular \(\nu < \infty\). By (A.22), we have
\[
\llbracket \rho_{\mathcal{F}\mathcal{E}\mathcal{F}} : \rho_{\mathcal{E}\mathcal{F}\mathcal{F}} \rrbracket = \llbracket \rho_{\mathcal{F}\mathcal{E}\mathcal{F}} : \rho_{\mathcal{E}\mathcal{F}\mathcal{F}} \rrbracket = \nu
\]
and
\[
\llbracket \rho_{\mathcal{F}\mathcal{E}\mathcal{F}} : \rho_{\mathcal{E}\mathcal{F}\mathcal{F}} \rrbracket = \llbracket \rho_{\mathcal{F}\mathcal{E}\mathcal{F}} : \rho_{\mathcal{E}\mathcal{F}\mathcal{F}} \rrbracket = \nu.
\]

Next observe that
\[
\begin{align*}
\left[ \begin{array}{c|c}
\ast & \ast \\
\ast & \ast \\
\end{array} \right] & = \left[ \begin{array}{c|c}
\ast & \ast \\
\ast & \ast \\
\end{array} \right] \\
& = \left[ \begin{array}{c|c}
\ast & \ast \\
\ast & \ast \\
\end{array} \right] \\
& = \left[ \begin{array}{c|c}
\ast & \ast \\
\ast & \ast \\
\end{array} \right] \\
& = \left[ \begin{array}{c|c}
\ast & \ast \\
\ast & \ast \\
\end{array} \right] \\
& = \left[ \begin{array}{c|c}
\ast & \ast \\
\ast & \ast \\
\end{array} \right] = \nu,
\end{align*}
\]
where the fourth and fifth equalities follow from Lemmas A.34 and A.36, respectively.

We conclude the argument by noting that, by (A.21),
\[
\left[ \begin{array}{c|c}
\ast & \ast \\
\ast & \ast \\
\end{array} \right] \left[ \begin{array}{c|c}
\ast & \ast \\
\ast & \ast \\
\end{array} \right] = \left[ \begin{array}{c|c}
\ast & \ast \\
\ast & \ast \\
\end{array} \right] \left[ \begin{array}{c|c}
\ast & \ast \\
\ast & \ast \\
\end{array} \right].
\]

In light of the above computations, that equation gives \( \nu \sqrt{\mu(B)} = \nu^2 \); since \( \nu \) is finite, we must have \( \nu = \sqrt{\mu(B)} \), as required. \( \square \)

As a corollary, we obtain the following improvement on Lemma 3.14:

**Corollary 5.20.** We have
\[
\left[ \begin{array}{c|c}
\ast & \ast \\
\ast & \ast \\
\end{array} \right] = \sqrt{\mu(B)}.
\]

**Corollary 5.21.** We have the following two equalities:
\[
\left[ \begin{array}{c|c}
\ast & \ast \\
\ast & \ast \\
\end{array} \right] = \sqrt{\mu(B)},
\]
\[
\left[ \begin{array}{c|c}
\ast & \ast \\
\ast & \ast \\
\end{array} \right] = \sqrt{\mu(B)}.
\]

**Proof.** The first equality follows immediately from Corollary 5.20. For the second equality, note that \( \left[ \begin{array}{c|c}
\ast & \ast \\
\ast & \ast \\
\end{array} \right] = \left[ \begin{array}{c|c}
\ast & \ast \\
\ast & \ast \\
\end{array} \right] \) by Lemma 5.13, the result follows by a version of Corollary 5.20 in which the roles of the nets \( A \) and \( C \) have been interchanged. \( \square \)

### 6. The \( 1 \boxtimes 1 \) isomorphism

We are now in a state to prove that the map \( \Omega \) (3.13), from the vacuum sector of the composition of two defects to the fusion of the vacuum sectors of the individual defects, is an isomorphism. This isomorphism provides the modification

\[
(6.1)
\]

that one expects in any 3-category. More importantly, it also provides the basis for our construction of the fundamental interchange modification

\[
\begin{align*}
\left( \begin{array}{c|c}
D & E \\
\hline
\ast & \ast \\
\ast & \ast \\
\end{array} \right) & \quad \Rightarrow \quad \left( \begin{array}{c|c}
D & E \\
\hline
\ast & \ast \\
\ast & \ast \\
\end{array} \right)
\end{align*}
\]
6. A. The $1 \boxtimes 1$ map is an isomorphism. Let $\mathcal{A}, \mathcal{B},$ and $\mathcal{C}$ be conformal nets, always assumed irreducible, and let $\mathcal{A}D_B$ and $\mathcal{B}E_C$ be defects. Assume furthermore that $\mathcal{B}$ has finite index. As before, we let $\square$ represent the Hilbert space $L^2((D \oplus E)(S_1^1))$, and let $\mathcal{B}$ stand for the fusion $L^2(D(S_1^1)) \boxtimes_B L^2(E(S_1^1))$, where $I$ is the vertical interval as in (4.1).

**Main Theorem 6.2.** Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D},$ and $\mathcal{E}$ be as above. Then the map

$$
\Omega_{D,E} : \square \rightarrow \mathcal{B}
$$

constructed in (1.31) is a unitary isomorphism.

**Proof.** As both sides of (6.3) are compatible with direct integrals, we may assume without loss of generality that $D$ and $E$ are irreducible. By construction $\Omega_{D,E}$ is an isometry. The algebras $\mathcal{B} = (D \oplus E)(S_1^1)$ and $\mathcal{B} = (D \oplus E)(S_1^1)$ act faithfully on both sides of (6.3). By definition, they are each other’s commutants on $\mathcal{B}$. Therefore, when viewed as $(D \oplus E)(S_1^1)$, both sides of (6.3) have a matrix of statistical dimensions that is the identity matrix, and $\Omega_{D,E}$ is an isomorphism.

Given the crucial importance of the “1 times 1 isomorphism” $\Omega$, we collect in one place the main ingredients used in its definition. These are the unitary isomorphisms (1.20) and (1.30), and the composite (1.31):

*Definition of the unitary $\Omega_*$:*

| $\Psi$ | $\Omega_{D,E}$ | $\Phi$ |
|---|---|---|
| $\square \rightarrow L^2(\square) : \square \rightarrow L^2(\square) : \square \rightarrow \mathcal{B}$ | $\Omega_{D,E} : \square \rightarrow \mathcal{B}$ | $\square \rightarrow \mathcal{B}$ |
| $\Phi : \mathcal{H}_1 \boxtimes \mathcal{H}_1$ | $\Psi_{D,E}$ | $\Phi^{-1}$ |

Here, the symbol “$: \square$” is an abbreviation for $- \boxtimes_{B(S_m)} H_0(S_m, B)$. 

*The 1 $\boxtimes$ 1 isomorphism is monoidal.* Next, we discuss the compatibility of $\Omega$ with the symmetric monoidal structure on conformal nets and defects. Let $\mathcal{A}_1(D_1)B_1$, $\mathcal{B}_1(E_1)C_1$, and $\mathcal{A}_2(D_2)B_2$ and $\mathcal{B}_2(E_2)C_2$ be defects. We abbreviate $D_1 := D_1(S_2^2)$, $D_2 := D_2(S_2^2)$, $D_1 \otimes D_2 := (D_1 \otimes D_2)(S_2^2)$, $E_1 := E_1(S_2^2)$, $E_2 := E_2(S_2^2)$, $E_1 \otimes E_2 := (E_1 \otimes E_2)(S_2^2)$, and $B_1 := B_1(I)$, $B_2 := B_2(I)$, $B_1 \otimes B_2 := (B_1 \otimes B_2)(I)$. Because of the canonical isomorphisms from Appendix A.V there are canonical isomorphisms

$$
L^2(D_1 \otimes B_1, E_1) \otimes L^2(D_2 \otimes B_2, E_2) \cong L^2((D_1 \otimes D_2) \otimes B_1 \otimes B_2, (E_1 \otimes E_2)),
$$

(6.4)

$$
L^2(D_1) \boxtimes_{B_1} L^2(E_1) \otimes L^2(D_2) \boxtimes_{B_2} L^2(E_2) \cong L^2(D_1 \otimes D_2) \boxtimes_{B_1 \otimes B_2} L^2(E_1 \otimes E_2),
$$

(6.5)
6.8. It appears that \( \Omega \) is not a natural transformation! More precisely, there seem to exist irreducible defects \( \mathcal{A}D_B, \mathcal{B}E_C, \mathcal{A}E_B, \mathcal{B}G_C \) and finite natural transformation \( \tau : D \to F, \sigma : E \to G \) (Definition 1.35) for which the diagram

\[
L^2(D \oplus_B E) \xrightarrow{\Omega_{D,E}} L^2(D) \boxtimes_B L^2(E)
\]

fails to be commutative. This problem can be blamed on the bad functorial properties of \( L^2_{\text{iso}} \) (used in the definition of \( \Psi \)). However, \( \Omega \) is still natural with respect to natural isomorphisms of defects. There are two ways of dealing with the above situation: 1. Restrict to the groupoid parts of \( \mathcal{CN}_0 \) and \( \mathcal{CN}_1 \); 2. Replace the \( L^2_{\text{iso}} \) by \( L^2 \) in the definition of \( \Psi \); the price to pay is that \( \Omega \) is then no longer unitary.

6.B. Compatibility with left and right units. We now show that, if one of the two defects in (6.3) is an identity defect, then the \( 1 \boxtimes 1 \) isomorphism admits a much simpler description, in terms of certain natural transformations that we describe below.

Let \( S_l, S_r, S_b, I, j_l, j_r \) be as in Section 4.13 and let \( I_l := j_l(I), I_r := j_r(I) \). We draw them once more:

\[
\begin{align*}
S_l &= \quad, & S_r &= \quad, & S_b &= \quad, & I &= \quad, \\
j_l : I_l &= \quad, & j_r : I_r &= \quad
\end{align*}
\]

Recall that we equipped \( S_b \) with a conformal structure that makes \( j_l|_{I_l} \cup \text{Id}_{I_l} : S_b \to S_c \) and \( \text{Id}_{I_l} \cup j_r|_{I_r} : S_b \to S_l \) conformal (and therefore smooth).

Fix a small number \( \varepsilon \). Then \( \Upsilon^l \) and \( \Upsilon^r \) are invertible natural transformations \( \mathcal{A}(\partial^C_3 [(1, 3/2 - \varepsilon) \times [0, 1]]) \)-modules \( \cong \mathcal{A}(\partial^C_3 [(0, 3/2 - \varepsilon) \times [0, 1]]) \)-modules and \( \mathcal{A}(\partial^C_3 ([1/2 + \varepsilon, 1] \times [0, 1])) \)-modules \( \cong \mathcal{A}(\partial^C_3 ([1/2 + \varepsilon, 2] \times [0, 1])) \)-modules, equivalently:

\[
\begin{align*}
\Upsilon^l : \mathcal{A}(\quad)-\text{modules} & \cong \mathcal{A}(\quad)-\text{modules} \\
\Upsilon^r : \mathcal{A}(\quad)-\text{modules} & \cong \mathcal{A}(\quad)-\text{modules}
\end{align*}
\]

11See equations (4.2) and (4.3) in [1].
The natural transformation $\Upsilon$ goes from the functor $H_0(S_r, A) \boxtimes A(I)$ to the functor of restriction along $\mathcal{A}(j_{!}|I \cup \text{Id}) : \mathcal{A}(\square) \xrightarrow{\sim} \mathcal{A}(\bigtriangledown)$. Its value on an $A(I)$-module $K$ is given by

$$\Upsilon_K : H_0(S_r, A) \boxtimes A(I) K \xrightarrow{w \otimes 1} L^2(A(I)) \boxtimes A(I) K \cong K,$$

where $w$ is the isomorphism gotten by composing $v_i : H_0(S_l, A) \rightarrow L^2(A(I))$ from Appendix B.11 and the map $L^2(A(I)) \rightarrow L^2(A(I))$ induced by $j_i : I_l \rightarrow I$. Similarly, the natural transformation $\Upsilon$ goes from the functor $- \boxtimes A(I) H_0(S_r, A)$ to the functor of restriction along $\mathcal{A}(j_{!}|I \cup \text{Id}) : \mathcal{A}(\square) \xrightarrow{\sim} \mathcal{A}(\bigtriangledown)$. It is given by

$$\Upsilon_K : K \boxtimes A(I) H_0(S_l, A) \xrightarrow{1 \otimes v} K \boxtimes A(I) L^2(A(I)) \cong K,$$

where $v = v_I$ is the identification $H_0(S_r, A) \cong L^2(A(I))$ from Appendix B.11.

The transformations $\Upsilon'$ and $\Upsilon''$ generalize the map $\Upsilon : \square \rightarrow \bigtriangledown$ from B.28 and B.6:

**Lemma 6.12.** Let $\epsilon_l := j_{!}|I \cup \text{Id} \in \text{Conf}(S_r, S_b)$ and $\epsilon_r := j_{!}|I \cup \text{Id} \in \text{Conf}(S_l, S_b)$ be as in Section 1.5. Then the two maps

$$H_0(\epsilon_l, A) \circ \Upsilon''_{H_0(S_r, A)} , \quad H_0(\epsilon_r, A) \circ \Upsilon''_{H_0(S_l, A)} : \square \xrightarrow{\sim} \bigtriangledown$$

are equal to each other, and are equal to $\Upsilon$.

**Proof.** The equality $\Upsilon = H_0(\epsilon_r, A) \circ \Upsilon''_{H_0(S_l, A)}$ follows from the commutativity of the diagram

$$
\begin{array}{c}
H_0(S_l) \boxtimes A(I) H_0(S_r) \xrightarrow{v_i \otimes v_j} L^2(A(I)) \boxtimes A(I) L^2(A(I)) \cong L^2(A(I)) \xrightarrow{v_i} H_0(S_l) \\
H_0(S_l) \boxtimes A(I) H_0(S_r) \xrightarrow{1 \otimes v_j} H_0(S_l) \boxtimes A(I) L^2(A(I)) \cong H_0(S_l) \xrightarrow{H_0(\epsilon_r)} H_0(S_b)
\end{array}
$$

where the top row is $\Upsilon$ and the bottom row is $H_0(\epsilon_r, A) \circ \Upsilon''_{H_0(S_l, A)}$. The rightmost square commutes by the naturality of the maps $v_I$, see B.3.

To see that $H_0(\epsilon_l) \circ \Upsilon''_{H_0(S_r, A)} = H_0(\epsilon_r) \circ \Upsilon''_{H_0(S_l, A)}$, one contemplates the diagram

$$
\begin{array}{c}
L^2(A(I)) \boxtimes A(I) H_0(S_r) \xrightarrow{L^2(A(j_{!})) \otimes 1} L^2(A(I)) \boxtimes A(I) H_0(S_r) \cong H_0(S_r) \xrightarrow{H_0(\epsilon_l)} H_0(S_b) \\
L^2(A(I)) \boxtimes A(I) H_0(S_l) \xrightarrow{1 \otimes v_j} L^2(A(I)) \boxtimes A(I) H_0(S_l) \cong H_0(S_l) \xrightarrow{H_0(\epsilon_r)} H_0(S_b)
\end{array}
$$

where $\tau := j_{!}|I \cup j_{!}|I = \epsilon_l^{-1} \circ \epsilon_r \in \text{Conf}_+(S_l, S_r)$. \hfill $\square$
Lemma 6.13. The map Υ satisfies the following version of associativity:

\[
\begin{array}{ccc}
1 & \otimes & \Upsilon \\
\Downarrow & & \Downarrow \\
H_0(\gamma_r, A) & \circ & \Upsilon \\
\end{array}
\]

where \( \gamma_l : \partial([0, 3] \times [0, 1]) \to \partial([0, 2] \times [0, 1]) \) and \( \gamma_r : \partial([1, 3] \times [0, 1]) \to \partial([0, 3] \times [0, 1]) \) are the maps

\[
\gamma_l = j_l \cup \text{Id} \quad \text{and} \quad \gamma_r = \text{Id} \cup j_r^+, 
\]

given by \( \gamma_l = j_l \cup \text{Id} \Rightarrow \) and \( \gamma_r = \text{Id} \cup j_r^+ \), and \( j_r^+ \) is obtained by conjugating \( j_r \) by \((x, y) \mapsto (x + 1, y)\).

Proof. Using Lemma 6.12 twice, we can expand (6.14) into the following diagram:

The lower right square commutes by the functoriality of \( H_0 \), see (B.1). The remaining three squares commute by the fact that \( \Upsilon_l \) and \( \Upsilon_r \) are natural transformations. \( \Box \)

Let \( \varepsilon_l \) and \( \varepsilon_r \) be as above, and let \( \varepsilon_l, \Upsilon : S_{\varepsilon_l, \Upsilon} \to S_{\varepsilon_l, \Upsilon} \) and \( \varepsilon_r, \Upsilon : S_{\varepsilon_r, \Upsilon} \to S_{\varepsilon_r, \Upsilon} \) be their restrictions to the upper halves of \( S_{\varepsilon_l} \) and \( S_{\varepsilon_r} \), respectively.

Lemma 6.15. Let \( A \) be a conformal net with finite index, and let \( \mathcal{D} \) be an irreducible defect. Let \( H_r := H_0(S_r, D) \), where the circle \( S_r \) is bicolored as in (4.7). Then the map \( \Omega_{\mathcal{D}, \text{id}_A} : \mathbb{1} \to \mathbb{1} \) is the inverse of \( L^2(D(\varepsilon_l, \Upsilon)) \circ \Upsilon_l^l \).

Similarly, assuming \( B \) has finite index, the map \( \Omega_{\mathcal{D}, \text{id}_B} : \mathbb{1} \to \mathbb{1} \) is the inverse of \( L^2(D(\varepsilon_r, \Upsilon)) \circ \Upsilon_r^r \).

Proof. We only treat the first equation \( \Omega_{\mathcal{D}, \text{id}_A}^{-1} = L^2(D(\varepsilon_l, \Upsilon)) \circ \Upsilon_l^l \). We first prove it in the case when \( D = \text{id}_A \). By definition, \( \Omega_{\mathcal{D}, \text{id}_A} \) is the composite

\[
\begin{array}{ccc}
\mathbb{1} & \xrightarrow{\Phi_{\text{id}_A, \text{id}_A}} & \mathbb{1} \\
\Downarrow & & \Downarrow \\
\Phi_{H_0(A), H_0(A)}^{-1} \circ H_0(A) & \rightarrow & \mathbb{1} \\
\end{array}
\]

and \( \Phi_{H_0(A), H_0(A)} \) is the composite

\[
\begin{array}{ccc}
\mathbb{1} & \xrightarrow{\Upsilon} & \mathbb{1} \\
\Downarrow & & \Downarrow \\
\Phi_{\text{id}_A, \text{id}_A} & \rightarrow & \mathbb{1} \\
\end{array}
\]

It follows that \( \Omega_{\text{id}_A, \text{id}_A} = \Upsilon^{-1} \) and we are done by Lemma 6.12.
We now treat the general case. As a special case of Lemma 4.32 (with the defects \( \text{id}_A, \text{id}_A, \) and \( D \)), we get the commutativity of the following diagram:

\[
\begin{array}{c}
\begin{array}{c}
\text{\( \Omega_{\text{id}_A \#_A D} \)} \\
\text{\( 1 \otimes \Omega_{\text{id}_A \#_A D} \)}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\Omega_{\text{id}_A \#_A D} \otimes 1 = \Upsilon^{-1} \otimes 1
\end{array}
\end{array}
\]

\begin{equation}
(6.16)
\end{equation}

Consider the circles \( S_l := \partial([0,1] \times [0,1]), S_m := \partial([1,2] \times [0,1]), S_r := \partial([2,3] \times [0,1]), S_{lm} := \partial([0,2] \times [0,1]), S_{mr} := \partial([1,3] \times [0,1]), S_{lmr} := \partial([0,3] \times [0,1]) \), and the corresponding half-circles \( \alpha_{l,T} := (\alpha_x)_{y \geq \frac{4}{3}} \) for \( \alpha \in \{ l, m, r, lm, mr, lmr \} \). Let also \( \varphi : \left[ \frac{1}{5} - \epsilon, \frac{4}{5} + \epsilon \right] \to \left[ \frac{1}{5} - \epsilon, \frac{4}{5} + \epsilon \right] \) and \( \psi : \left[ \frac{4}{5} - 2 \epsilon, \frac{2}{5} - \epsilon \right] \to \left[ \frac{4}{5} - 2 \epsilon, \frac{2}{5} - \epsilon \right] \) be diffeomorphisms whose derivative is 1 in a neighborhood of the boundary, where \( \epsilon \) is a fixed small number. These extend to diffeomorphisms
\[
\varphi_{lm} : S_{lm,T} \to S_{lm,T}, \quad \varphi_{l} : S_{l,T} \to S_{lm,T},
\]
\[
\psi_{lm} : S_{lm,T} \to S_{lm,T}, \quad \psi_{m} : S_{m,T} \to S_{lm,T},
\]
whose derivative is 1 outside the domains of \( \varphi \) and \( \psi \), respectively. Let also \( \chi := \psi_{lm}^{-1} \circ \varphi_{lm} \). We will use later on that \( \chi(x,y) = (x,y) \) for \( y \geq \frac{1}{3} \). Since the construction of \( \Omega \) is natural with respect to isomorphisms, the squares \( \square \) are commutative:

\[
\begin{array}{c}
\begin{array}{c}
\text{\( L^2(D(\chi)) \)} \\
\text{\( \Omega_{\text{id}_A \#_A D} \)}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{\( L^2(D(\varphi_{lm})) \)} \\
\text{\( \Omega_{\text{id}_A \#_A D} \)}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{\( L^2(D(\psi_{lm})) \)} \\
\text{\( 1 \otimes L^2(D(\psi_{lm})) \)}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{\( L^2(D(\psi_{m})) \)} \\
\text{\( 1 \otimes L^2(D(\psi_{m})) \)}
\end{array}
\end{array}
\]

\begin{equation}
(6.17)
\end{equation}

The remaining two squares of \( 6.17 \) are commutative by \( 6.16 \), and by the definition of \( \chi \). We now consider the following diagram

\[
\begin{array}{c}
\begin{array}{c}
\text{\( L^2(A(\chi)) \)} \\
\text{\( Y^{-1} \)}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{\( L^2(A(\psi_{m})) \)} \\
\text{\( 1 \otimes L^2(A(\psi_{m})) \)}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{\( L^2(A(\psi_{lm})) \)} \\
\text{\( 1 \otimes L^2(A(\psi_{lm})) \)}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{\( L^2(A(\psi_{lm})) \)} \\
\text{\( 1 \otimes L^2(A(\psi_{lm})) \)}
\end{array}
\end{array}
\]

\begin{equation}
(6.18)
\end{equation}

where \( \tau = \epsilon_{l,T}^{-1} \circ \chi \circ \epsilon_{l,T}, \sigma = \epsilon_{l,T}^{-1} \circ \psi_{m} \), the lower right corner stands for the fusion of \( H_0(S_l) \) with \( H_0(S_{r}) \) along \( A(j_1^+|_{\{1\} \times [0,1]} \), and \( j_1^+ \) is obtained by conjugating \( j_1 \) by \( (x,y) \mapsto (x+1,y) \):

\[
\begin{array}{c}
\begin{array}{c}
\text{\( j_1^+ : \square \)} \\
\text{\( j_1^+|_{\{1\} \times [0,1]} : \square \)}
\end{array}
\end{array}
\]
Using the identity $\Omega_{\text{d}_A, \text{id}_A} = \Upsilon^{-1}$ proved earlier, the case $D = \text{id}_A$ of (6.17) implies the commutativity of $\square$. The triangles $\square$ commute by virtue of Lemma 6.12 and so the whole diagram (6.18) is commutative.

Let $\hat{\tau}, \hat{\sigma} \in \text{Diff}(\partial[0, 1]^2)$ be the symmetric extensions of $\tau$ and $\sigma$, so that $|\hat{\tau}|_{\partial[0, 1]} = \tau$ and $|\hat{\sigma}|_{\partial[0, 1]} = \sigma$, and they both commute with $(x, y) \mapsto (x, 1 - y)$. From the fact that $\chi(x, y) = (x, y)$ for $y \geq \frac{1}{2}$, it follows that $\hat{\sigma}(x, y) = \hat{\tau}(x, y) = (x, y)$ for $y \geq \frac{1}{2}$. Let $u, v \in A(\square)$ be the canonical quantizations as in (1.18) of the symmetric diffeomorphism $\hat{\tau}$ and $\hat{\sigma}$. By definition, we then have $L^2(A(\tau)) = \pi(u)$ and $L^2(A(\sigma)) = \pi(v)$, where $\pi$ is the action of $A(\square)$ on $\square = H_0(A)$.

We now consider the following diagram of natural transformations between functors from $A(\square)$-modules to Hilbert spaces:

(6.19)

When evaluated on $H_0(A)$, the above diagram commutes by (6.18). Therefore, by Lemma A.28 since $H_0(A)$ is a faithful $A(\square)$-module, the diagram (6.19) commutes regardless of the module one evaluates it on. We now consider the following variant of diagram (6.18):

Our goal is to show is that the triangles $\square$ are commutative. Since $D$ is irreducible, there exists an invertible complex number $\lambda$ such that

$$\Omega_{\text{d}_A, D} \circ L^2(D(\epsilon_1, \tau)) = \lambda (\Upsilon^{\square})^{-1}.$$ 

The 7-gon $\square$ is simply (6.17), and it is therefore commutative. The triangle $\square$ occurs two times with a given orientation, and once with the opposite orientation: the outer 7-gon therefore commutes up to a factor of $\lambda$. But that outer 7-gon is an instance of (6.19) by Lemma 1.19 and is therefore commutative. It follows that $\lambda = 1$. $\square$

6.c. Unitors for horizontal fusion of sectors. In this section, we will introduce certain variants of the transformations $\Upsilon'$ and $\Upsilon''$ that will be more convenient for the full verification $[4]$ that conformal nets form a 3-category (more precisely, an internal dicategory in the 2-category of symmetric monoidal categories $[8, \text{Definition 3.3}]$).

We will again be using the circles $S_1$, $S_r$, $S_b$, the intervals $I$, $I_1$, $I_r$, and the involutions $j_1 \in \text{Conf}^-(S_1)$, $j_r \in \text{Conf}^-(S_r)$ from (6.10). Let $\alpha_l := j_1|_{I_l} \cup \text{id}_{I_l} : S_b \to S_l$ and $\alpha_r := \text{id}_{I_r} \cup j_r|_{I_r} : S_b \to S_l$ be the diffeomorphisms used in the definition of $\Upsilon'$ and $\Upsilon''$—their inverses appeared in Lemma 6.12 under the names $\epsilon_1$ and $\epsilon_r$. 

The restriction $\alpha_I|_{\{0\}\times[0,1]}$ is not the map $(0,y)\mapsto (1,y)$ and, as a consequence, the way $\Upsilon^l$ interacts with horizontal fusion is somewhat complicated to describe. (see the lower right corner of $\Upsilon$ and the vertical arrow from it). A similarly story holds for the restriction of $\alpha_r$ to $\{2\}\times[0,1]$. Our next goal is to introduce diffeomorphisms $\beta_l: S_l \to S_r$ and $\beta_r: S_l \to S_l$ to replace of $\alpha_l$ and $\alpha_r$, so that the corresponding variants of $\Upsilon^l$ and $\Upsilon^r$ avoid this complication with horizontal fusion.

Pick intervals $I^+_l, I^+_r \subset S_l$ that are slightly larger than $I_l$ and $I_r$, and diffeomorphisms $\beta_l$ and $\beta_r$ that satisfy
\begin{equation}
\beta_l(0, y) = (1, y), \quad \beta_l \circ j = j \circ \beta_l, \quad \beta_l|_{S_l \setminus I^+_l} = \text{id}.
\end{equation}
\begin{equation}
\beta_r(2, y) = (1, y), \quad \beta_r \circ j = j \circ \beta_r, \quad \beta_r|_{S_l \setminus I^+_r} = \text{id},
\end{equation}
where $j(x, y) = (x, 1 - y)$:
\[
\begin{align*}
\alpha_l : & \quad \begin{array}{c}
\text{Upsilon} \\
\end{array}, & \quad \beta_l : & \quad \begin{array}{c}
\text{Diffeo} \\
\end{array}, & \quad I_l = & \quad \begin{array}{c}
\text{Interval} \\
\end{array} & \quad \subset I^+_l = & \quad \begin{array}{c}
\text{Interval} \\
\end{array} \\
\alpha_r : & \quad \begin{array}{c}
\text{Upsilon} \\
\end{array}, & \quad \beta_r : & \quad \begin{array}{c}
\text{Diffeo} \\
\end{array}, & \quad I_r = & \quad \begin{array}{c}
\text{Interval} \\
\end{array} & \quad \subset I^+_r = & \quad \begin{array}{c}
\text{Interval} \\
\end{array}.
\end{align*}
\]
The illustrations of $\beta_l$ and $\beta_r$ are somewhat crude, and do not reflect that they are contracting on the horizontal parts of $I^+_l$ and $I^+_r$.

The composition $\beta_l \circ \alpha_l^{-1}: S_l \to S_r$ is symmetric with respect to $j$, and restricts to the identity on the complement of $\alpha_l(I^+_l)$. As explained in the discussion preceding Lemma 6.21, there is then a canonical implementation $u_l$ of the diffeomorphism $\beta_l \circ \alpha_l^{-1}$ on $H_0(S_r)$. Similarly, there is a canonical implementation $u_r$ of $\beta_r \circ \alpha_r^{-1}$ on $H_0(S_l)$.

Given a $D\dash E$-sector $H$ between $A\dash B$-defects $D$ and $E$, then pulling back $H$ along $A(\alpha_l)$ produces an $(\text{id}_A \oplus D)\dash (\text{id}_A \oplus E)$-sector. This operation is a functor $\alpha_l^* : \mathcal{CN}_2 \to \mathcal{CN}_2$. Similarly, we have functors $\beta_l^*, \alpha_r^*, \beta_r^* : \mathcal{CN}_2 \to \mathcal{CN}_2$. Multiplication by $u_l$ and $u_r$ then provide natural isomorphisms
\[
U^l : \alpha_l^* \cong \beta_l^* : \mathcal{CN}_2 \boxtimes \mathcal{CN}_2, \quad U^r : \alpha_r^* \cong \beta_r^* : \mathcal{CN}_2 \boxtimes \mathcal{CN}_2
\]
between the functors $\alpha_l^*$ and $\beta_l^*$, and between the functors $\alpha_r^*$ and $\beta_r^*$. Recall that $\Upsilon^l$ and $\Upsilon^r$ yield natural isomorphisms $H_0(A) \boxtimes A - \cong \alpha_l^*$ and $- \boxtimes B H_0(B) \cong \alpha_r^*$ of functors from $\mathcal{CN}_2$ to $\mathcal{CN}_2$.

**Definition 6.20.** The left and right *unitors* are the natural transformations $\bar{T}^l := U^l \circ \Upsilon^l : H_0(A) \boxtimes A - \to \beta_l^*$ and $\bar{T}^r := U^r \circ \Upsilon^r : - \boxtimes B H_0(B) \to \beta_r^*$. We draw them as
\[
\begin{align*}
\bar{T}^l_H : \begin{array}{c}
\text{H} \\
\end{array} & \quad \to \quad \begin{array}{c}
\text{H} \\
\end{array} \\
\bar{T}^r_H : \begin{array}{c}
\text{H} \\
\end{array} & \quad \to \quad \begin{array}{c}
\text{H} \\
\end{array}.
\end{align*}
\]
(Of course, $\bar{T}^l$ and $\bar{T}^r$ depend on our earlier unspecified choices of $\beta_l$ and $\beta_r$.)

We record some properties of the unitors that will be used in our later paper [4]. First of all, they are compatible with the $1 \boxtimes 1$-isomorphism $\Omega$:

**Lemma 6.21.** Let $B$ be a conformal net with finite index, and let $A\dash D\oplus B$ and $B\oplus E$ be defects. Then
\[
\begin{align*}
\beta_l^* \Omega_{D\oplus E} \circ \bar{T}^l_{H_0(D\oplus B)E} & = \bar{T}^l_{H_0(D)\boxtimes B H_0(E)} \circ (1_{H_0(A)} \boxtimes A \Omega_{D,E}), \\
\beta_r^* \Omega_{D\oplus E} \circ \bar{T}^r_{H_0(D\oplus B)E} & = \bar{T}^r_{H_0(D)\boxtimes B H_0(E)} \circ (\Omega_{D,E} \boxtimes A 1_{H_0(B)}).
\end{align*}
\]
Equivalently, the two diagrams commute.

Proof. We can view $\hat{\Upsilon}^l$ and $\hat{\Upsilon}^r$ as a natural transformations $\hat{\Upsilon}^l : \mathcal{A}(\_\_)$-modules $\xrightarrow{\cong} \mathcal{A}(\_\_)$-modules $\hat{\Upsilon}^r : \mathcal{C}(\_\_)$-modules $\xrightarrow{\cong} \mathcal{C}(\_\_)$-modules. The asserted equalities are then instances of this naturality.

Let $\alpha_l, \beta_l : S_b \rightarrow S_l$ and $\alpha_r, \beta_r : S_b \rightarrow S_r$ be the restrictions of $\alpha_l, \beta_l, \alpha_r, \beta_r$ to the upper half of $S_b$.

Lemma 6.22. Let $A_D$ be a defect, color the circles $S_l$ and $S_r$ as in (4.7), and let $H_r := H_0(S_l, D)$ and $H_l := H_0(S_r, D)$. Then $\hat{\Upsilon}^l_H \circ \Omega_{\text{id}, D} : \mathcal{C} \rightarrow \mathcal{C}$ coincides with $L^2(D(\beta_l, \_\_))$, and $\hat{\Upsilon}^r_H \circ \Omega_{D, \text{id}B} : \mathcal{C} \rightarrow \mathcal{C}$ coincides with $L^2(D(\beta_r, \_\_))$.

Proof. From Lemma 6.15 (recall that $\epsilon_l = \alpha_l^{-1}$ and $\epsilon_r = \alpha_r^{-1}$), we know that $\Upsilon^l_{H_r} \circ \Omega_{\text{id}, A, D} = L^2(D(\alpha_l, \_\_))$ and $\Upsilon^r_{H_l} \circ \Omega_{D, \text{id}B} = L^2(D(\alpha_r, \_\_))$.

We also have $U^l_{H_r} = L^2(D(\beta_l, \alpha_l^{-1}))$ and $U^r_{H_l} = L^2(D(\beta_r, \alpha_r^{-1}))$ by Lemma 1.19.

Finally, recall that $\hat{\Upsilon}^l = U^l \circ \Upsilon^l$ and $\hat{\Upsilon}^r = U^r \circ \Upsilon^r$. We therefore get

$\hat{\Upsilon}^l_{H_r} \circ \Omega_{\text{id}, A, D} = U^l_{H_r} \circ \Upsilon^l_{H_r} \circ \Omega_{\text{id}, A, D}$

$= L^2(D(\beta_l, \alpha_l^{-1})) \circ L^2(D(\alpha_l, \_\_)) = L^2(D(\beta_l, \_\_))$

and

$\hat{\Upsilon}^r_{H_l} \circ \Omega_{D, \text{id}B} = U^r_{H_l} \circ \Upsilon^r_{H_l} \circ \Omega_{D, \text{id}B}$

$= L^2(D(\beta_r, \alpha_r^{-1})) \circ L^2(D(\alpha_r, \_\_)) = L^2(D(\beta_r, \_\_))$

Combining the above lemma with the factorization of $L^2(f)$ from (A.7) for the maps $f = D(\beta_l, \_\_)$ and $D(\beta_r, \_\_)$, we obtain the commutativity of the following two diagrams:

Here, the stacked pictures in the upper right hand corners signify the Connes fusion along the algebras $D(\_\_)$ and $D(\_\_)$ respectively, and we have suppressed the isomorphism (A.6).
6. The interchange isomorphism. In a 2-category, the interchange law says that the two ways of evaluating the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{F} & B \\
\| & H & \downarrow \\
P & \xleftarrow{L} & Q \\
\end{array}
\quad \begin{array}{ccc}
E & \xleftarrow{G} & C \\
\| & K & \downarrow \\
\end{array}
\]

are equal to each other: if one first performs the two vertical compositions and then composes horizontally, or one first composes horizontally and then vertically, one should obtain the same answer. In our case, the two ways of fusing four sectors

\[
\begin{array}{ccc}
D & \xrightarrow{H} & B \\
\| & \downarrow & \| \\
\| & L & \downarrow \\
\end{array}
\quad \begin{array}{ccc}
A & \xleftarrow{F} & C \\
\| & K & \downarrow \\
\| & M & \downarrow \\
\| & G & \downarrow \\
\end{array}
\]

are equal to each other up to isomorphism. In other words, we need a unitary natural isomorphism

\[
\begin{array}{ccc}
D & \xrightarrow{H} & B \\
\| & \downarrow & \| \\
\| & L & \downarrow \\
\end{array}
\quad \begin{array}{ccc}
A & \xleftarrow{F} & C \\
\| & K & \downarrow \\
\| & M & \downarrow \\
\| & G & \downarrow \\
\end{array}
\]

\[
\Rightarrow \quad \begin{array}{ccc}
D & \xrightarrow{H} & B \\
\| & \downarrow & \| \\
\| & L & \downarrow \\
\end{array}
\quad \begin{array}{ccc}
A & \xleftarrow{F} & C \\
\| & K & \downarrow \\
\| & M & \downarrow \\
\| & G & \downarrow \\
\end{array}
\]

\]

\[
\Rightarrow \quad \begin{array}{ccc}
D & \xrightarrow{H} & B \\
\| & \downarrow & \| \\
\| & L & \downarrow \\
\end{array}
\quad \begin{array}{ccc}
A & \xleftarrow{F} & C \\
\| & K & \downarrow \\
\| & M & \downarrow \\
\| & G & \downarrow \\
\end{array}
\]

\[
\Rightarrow \quad \begin{array}{ccc}
D & \xrightarrow{H} & B \\
\| & \downarrow & \| \\
\| & L & \downarrow \\
\end{array}
\quad \begin{array}{ccc}
A & \xleftarrow{F} & C \\
\| & K & \downarrow \\
\| & M & \downarrow \\
\| & G & \downarrow \\
\end{array}
\]

\]

\[
\Rightarrow \quad \begin{array}{ccc}
D & \xrightarrow{H} & B \\
\| & \downarrow & \| \\
\| & L & \downarrow \\
\end{array}
\quad \begin{array}{ccc}
A & \xleftarrow{F} & C \\
\| & K & \downarrow \\
\| & M & \downarrow \\
\| & G & \downarrow \\
\end{array}
\]

\]

\[
\Rightarrow \quad \begin{array}{ccc}
D & \xrightarrow{H} & B \\
\| & \downarrow & \| \\
\| & L & \downarrow \\
\end{array}
\quad \begin{array}{ccc}
A & \xleftarrow{F} & C \\
\| & K & \downarrow \\
\| & M & \downarrow \\
\| & G & \downarrow \\
\end{array}
\]

should yield the same answer up to isomorphism. In other words, we need a unitary natural transformation

\[
\tau : (CN_2 \times_{CN_1} CN_2) \times_{CN_0} (CN_2 \times_{CN_1} CN_2) \xrightarrow{\cong} CN_2
\]

between the functors \(fusion_h \circ (fusion_v \times fusion_v)\) and \(fusion_v \circ (fusion_h \times fusion_h)\) \(\circ \tau\), where

\[
\tau : (CN_2 \times_{CN_1} CN_2) \times_{CN_0} (CN_2 \times_{CN_1} CN_2) \Rightarrow (CN_2 \times_{CN_1} CN_2) \times_{CN_0} (CN_2 \times_{CN_1} CN_2)
\]

is the isomorphism that exchanges the two middle factors.

More concretely, given sectors \(D H F, E K G, F L P, G M Q\) as in (6.24), we are looking for a unitary isomorphism

\[
(H \boxtimes F L) \boxtimes B (K \boxtimes G M) \xrightarrow{\cong} (H \boxtimes_B K) \boxtimes_{F \boxtimes_B G} (L \boxtimes_B M)
\]

of \(D \boxtimes_B E - P \boxtimes_B Q\)-sectors.

We can view \((H, K, L, M)\) as an object of the category

\[
\text{C} := F\left(\begin{array}{c}
\end{array}\right)-\text{modules} \times G\left(\begin{array}{c}
\end{array}\right)-\text{modules}
\]

\[
\times F\left(\begin{array}{c}
\end{array}\right)-\text{modules} \times G\left(\begin{array}{c}
\end{array}\right)-\text{modules}
\]

\footnote{Here, as in the 1 \(\boxtimes 1\)-isomorphism \(\Omega\), we restrict to the groupoid parts of \(CN_1\) and \(CN_0\).}
The forgetful functor \((\text{CN}_2 \times_F \text{CN}_2) \times (\text{CN}_2 \times_G \text{CN}_2) \to \mathbb{C}\) is faithful. In order to construct the natural transformation (6.25), it is therefore enough to produce corresponding natural transformations
\[(6.27) \quad \mathbb{C} \xrightarrow{\rho} \text{Hilbert Spaces}\]
for every \(F\) and \(G\). The fact that (6.26) intertwines the actions of \(F(\varpi)\), \(G(\varpi)\), \(F(\underline{\varpi})\), and \(G(\underline{\varpi})\), i.e., that it is a morphism of \(D \oplus_B E - P \oplus_B Q\)-sectors, will then follow from the naturality of (6.27).

Since the object \((H_0(F), H_0(G), H_0(F), H_0(G)) \in \mathbb{C}\) consists of faithful modules, the obvious analog of Lemma 1.3 (itself a generalization of Lemma A.28) applies, and so it is enough to construct the natural transformation (6.27) on the above object. Using the isomorphisms (6.25) and (6.19), the latter is given by
\[\begin{align*}
\Omega^{-1} &\quad \longrightarrow H_0(F) \otimes_B G \xrightarrow{\sim} H_0(F) \otimes_B (H_0(G) \boxtimes H_0(G)) \\
\Omega^{-1} &\quad \longrightarrow H_0(F) \otimes_B G \xrightarrow{\sim} H_0(F) \otimes_B G \otimes_B (H_0(G) \boxtimes H_0(G)).
\end{align*}\]
Using the compatibility of \(\Omega\) with the monoidal structure (Proposition 6.10), the same can be deduced for the exchange isomorphism: (6.25) is a monoidal natural transformation.

**Appendix A. Von Neumann Algebras**

Given a Hilbert space \(H\), we let \(\mathcal{B}(H)\) denote its algebra of bounded operators. The ultraweak topology on \(\mathcal{B}(H)\) is the topology of pointwise convergence with respect to the pairing with its predual, the trace class operators.

**Definition A.1.** A von Neumann algebra, is a topological \(*\)-algebra (no compatibility between the topology and the algebra structure!) that is embeddable as closed subalgebra of \(\mathcal{B}(H)\) with respect to the ultraweak topology.

The spatial tensor product \(A_1 \boxtimes A_2\) of von Neumann algebras \(A_i \subset \mathcal{B}(H_i)\) is the closure in \(\mathcal{B}(H_1 \otimes H_2)\) of their algebraic tensor product \(A_1 \hat{\otimes}_{\text{alg}} A_2\).

**Definition A.2.** Let \(A\) be a von Neumann algebra. A left (right) \(A\)-module is a Hilbert space \(H\) equipped with a continuous homomorphism from \(A\) (respectively \(A^{op}\)) to \(\mathcal{B}(H)\). We will use the notation \(\_A H\) (respectively \(H_A\)) to denote the fact that \(H\) is a left (right) \(A\)-module.

We now review the parts of our earlier publications [1], [2], [3] that are used in the present paper. For further details, we refer the reader to [1] §2 and §6 for Section A.1 to [1] §3 for Section A.11 to [3] Appendix A, for Section A.11 to [2] §1C for Section A.12 to [1] §4 for Section A.14, and to [1] §5 for Section A.18.

**A.1. The Haagerup \(L^2\)-space.** A faithful left module \(H\) for a von Neumann algebra \(A\) is called a *standard form* if it comes equipped with an antilinear isometric involution \(J\) and a selfdual cone \(P \subset H\) subject to the properties
\[(i) \quad JAJ = A' \quad \text{on} \ H,\]
\[(ii) \quad JcJ = c^* \quad \text{for all} \ c \in Z(A),\]
\[(iii) \quad J\xi = \xi \quad \text{for all} \ \xi \in P,\]
\[(iv) \quad aJaJ(P) \subseteq P \quad \text{for all} \ a \in A\]
where \(A'\) denotes the commutant of \(A\). The operator \(J\) is called the *modular conjugation*. The standard form is an \(A\)-\(A\)-bimodule, with right action \(\xi a := Ja^*J\xi\). It is unique up to unique unitary isomorphism [1].
The space of continuous linear functionals \( A \to \mathbb{C} \) forms a Banach space \( A_\ast = L^1(A) \) called the predual of \( A \). It comes with a positive cone \( L^2_+(A) := \{ \phi \in A_\ast \mid \phi(x) \geq 0 \ \forall x \in A_+ \} \) and two commuting \( A \)-actions given by \((ab)(x) := \phi(bxa)\). Given a von Neumann algebra \( A \) there is a canonical construction of a standard form for \( A \) \[17\]. It is the completion of

\[
\bigoplus_{\phi \in L^1_+(A)} \mathbb{C} \sqrt{\phi}
\]

with respect to some pre-inner product, and is denoted \( L^2(A) \). The positive cone in \( L^2(A) \) is given by \( L^2_+(A) := \{\sqrt{\phi} \mid \phi \in L^1_+(A)\} \). The modular conjugation \( J_A \) sends \( \lambda \sqrt{\phi} \) to \( \overline{\lambda} \sqrt{\overline{\phi}} \) for \( \lambda \in \mathbb{C} \).

If \( f : A \to B \) is an isomorphism, then we write \( L^2(f) : L^2(A) \xrightarrow{\cong} L^2(B) \) for the induced unitary isomorphism. The standard form is in fact functorial with respect to a bigger class of maps, see Subsection A.VI.

### A.II. Connes fusion.

**Definition A.3.** Given two modules \( H_A \) and \( A K \) over a von Neumann algebra \( A \), their Connes fusion \( H \otimes_A K \) is the completion \[7\] \[22\] \[29\] of

\[
\text{Hom}(L^2(A)_A, H_A) \otimes_A L^2(A) \otimes_A \text{Hom}(A L^2(A), A K)
\]

with respect to the inner product \( \langle \phi_1 \otimes \xi_1 \otimes \psi_1, \phi_2 \otimes \xi_2 \otimes \psi_2 \rangle := \langle (\phi^*_1 \phi_1) \xi_1 (\psi^*_1 \psi_1), \xi_2 \rangle \). Here, we have written the action of \( \psi_1 \) on the right, which means that \( \psi_1 \psi^*_2 \) stands for the composite \( L^2(A) \xrightarrow{\psi_1} K \xrightarrow{\psi^*_2} L^2(A) \).

The \( L^2 \) space is a unit for Connes fusion in the sense that there are canonical unitary isomorphisms

\[
A L^2(A) \otimes_A H \cong_A H \quad \text{and} \quad H \otimes_A L^2(A) \cong H_A
\]

defined by \( \phi \otimes \xi \otimes \psi \mapsto (\phi \xi) \psi \) and \( \phi \otimes \xi \otimes \psi \mapsto \phi (\xi \psi) \). If \( f : A \to B \) is an isomorphism of von Neumann algebras, \( H_A \) and \( \hat{K} \) are modules, then

\[
(H_A)_{f^{-1}} \otimes_B \hat{K} \cong H_A \otimes_B f(B K)
\]

via \( \phi \otimes \xi \otimes \psi \mapsto (\phi \circ L^2(f)) \otimes (\xi \circ (\psi \circ L^2(f))) \). Here the indices \( f^{-1} \) and \( f \) indicate restrictions of actions along the isomorphisms \( f \) and \( f^{-1} \). Using (A.5), \( L^2(f) \) is

\[
L^2(A) \cong L^2(A)_{f^{-1}} \otimes_B L^2(B) \cong L^2(A) \otimes_A f L^2(B) \cong L^2(B).
\]

### A.III. Cyclic fusion.

Let \( n \geq 2 \) be some number. For each \( i \in \{1, \ldots, n\} \), let \( A_i \) be a von Neumann algebra, and let \( H_i \) be an \( A_i \otimes A_i^\text{op} \)-module (cyclic numbering). Then for each \( i, j \in \{1, \ldots, n\} \), we can form the fusion of \( H_i \otimes_{A_{i+j}} \cdots \otimes_{A_{j-1}} H_{j-1} \) (cyclic numbering) with \( H_j \otimes_{A_{j+i}} \cdots \otimes_{A_{i-1}} H_{i-1} \) over the algebra \( A_i^\text{op} \otimes A_j \). Under the above conditions, the Hilbert space

\[
(H_i \otimes_{A_{i+j}} \cdots \otimes_{A_{j-1}} H_{j-1}) \otimes_{A_i^\text{op} \otimes A_j} (H_j \otimes_{A_{j+i}} \cdots \otimes_{A_{i-1}} H_{i-1})
\]

is independent, up to canonical unitary isomorphism, of the choices of \( i \) and \( j \) Appendix A]. We call the above Hilbert space the cyclic fusion of the \( H_i \)’s, and denote it by

\[
\cdots H_1 \otimes_{A_2} \cdots \otimes_{A_n} H_n \otimes_{A_2} \cdots.
\]
A.IV. Fusion and fiber product of von Neumann algebras.

Definition A.8. Let \( A \leftarrow C^{op} \), \( C \rightarrow B \) be two homomorphisms between von Neumann algebras, and let \( \_A H \) and \( B K \) be faithful modules. Viewing \( H \) as a right \( C \)-module, we may form the Connes fusion \( H \boxtimes_C K \). One then defines the fusion of \( A \) and \( B \) over \( C \) as
\[
A \boxtimes_C B := (A \cap C^{op'}) \vee (C' \cap B) \subset B(H \boxtimes_C K),
\]
where the commutants of \( C^{op} \) and \( C \) are taken in \( H \) and \( K \), respectively.

The fusion is independent, up to canonical isomorphism, of the choice of modules \( H \) and \( K \) \cite[Prop. 1.21]{2}. If those modules are not faithful, then there is still an action, albeit non-faithful, of \( A \otimes C B \) on \( H \boxtimes_C K \) \cite[Lem. 1.23]{2}. Note that the operation \( \otimes \) is not associative \cite[Warn. 1.21]{2}.

The fiber product of von Neumann algebras (introduced in \cite{23} when \( C \) is abelian, and in \cite[Def. 10.2.4]{28} in general) is better behaved:

Definition A.10. In the situation of Definition A.8, the fiber product of \( A \) and \( B \) over \( C \) is given by
\[
A \star_C B := (A' \otimes_{alg} B')',
\]
where the commutants \( A' \) and \( B' \) are taken in \( \mathcal{B}(H) \) and \( \mathcal{B}(K) \) respectively, while the last one is taken in \( \mathcal{B}(H \boxtimes_C K) \).

The fiber product is independent of the choice of modules \( H \) and \( K \) \cite[Prop. 1.21]{24} and there is an associator \( A \star_C (B \star_D E) \rightarrow (A \star_C B) \star_D E \) that satisfies the pentagon identity \cite[Prop. 10.2.8]{28}.

If \( C = \mathbb{C} \), then \( A \star_C B = A \otimes_C B \) is the spatial tensor product \( A \otimes B \) of von Neumann algebras.

A.V. Compatibility with tensor products. There is a canonical isomorphism \cite[24]{24}
\[
L^2(A) \otimes L^2(B) \cong L^2(A \otimes B)
\]
that sends \( \sqrt{\phi} \otimes \sqrt{\psi} \) to \( \sqrt{\phi} \otimes \sqrt{\psi} \). This isomorphism provides a natural compatibility between Connes fusion and tensor products
\[
(H_1 \boxtimes_A H_2) \otimes (K_1 \boxtimes_B K_2) \cong (H_1 \otimes K_1) \boxtimes_{A \otimes B} (H_2 \otimes K_2).
\]
This later isomorphism can then be used to construct natural compatibilities isomorphisms for the spatial tensor product and the fusion respectively the fiber product of von Neumann algebras:
\[
(A_1 \circ_C B_1) \otimes (A_2 \circ_C B_2) \cong (A_1 \circ A_2) \circ_{C_1 \circ C_2} (B_1 \circ B_2),
\]
where the latter also relies on the equation \( (A \otimes B)' = A' \otimes B' \) \cite[Thm. 12.3]{27}.

A.VI. Dualizability. A von Neumann algebra whose center is \( \mathbb{C} \) is called a factor. Von Neumann algebras with finite-dimensional center are direct sums of factors.

Definition A.11. Let \( A \) and \( B \) be von Neumann algebras with finite-dimensional center. Given an \( A \)-\( B \)-bimodule \( H \), we say that a \( B \)-\( A \)-bimodule \( \tilde{H} \) is dual to \( H \) if it comes equipped with maps
\[
R: A L^2(A)_A \rightarrow A H \boxtimes_B \tilde{H}_A \quad \text{and} \quad S: B L^2(B)_B \rightarrow B \tilde{H} \boxtimes_A H_B
\]
satisfying the duality equations \((R^* \otimes 1)(1 \otimes S) = 1, (S^* \otimes 1)(1 \otimes R) = 1, \) and to the normalization \( R^*(x \otimes 1)R = S^*(1 \otimes x)S \) for all \( x \in \text{End}(A) \). A bimodule whose dual module exists is called dualizable.
If \( AH_B \) is a dualizable bimodule, then its dual bimodule is well defined up to canonical unitary isomorphism \([1]\) Thm 4.22. Moreover, the dual bimodule is canonically isomorphic to the complex conjugate Hilbert space \( \overline{H} \), with the actions \( b^*a := a^*b^* \) \([1]\) Cor 6.12.

A homomorphism \( f: A \to B \) between von Neumann algebras with finite-dimensional center is said to be finite if the associated bimodule \( A L^2(B) A \) is dualizable. If \( f: A \to B \) is a finite homomorphism, then there is an induced map \( L^2(f): L^2(A) \to L^2(B) \), and we have \( L^2(f \circ g) = L^2(f) \circ L^2(g) \). In other words, Haagerup’s \( L^2 \)-space is functorial with respect to finite homomorphisms \([1]\). The map \( L^2(f) \) is bounded and \( A-A \)-bilinear, but usually not isometric.

A VII. Two-sided fusion on \( L^2 \)-spaces. Let \( M \) be a von Neumann algebra, and let \( M_0 \) and \( A \) be two commuting subalgebras such that \( M_0 \vee A = M \). Let \( H_A \) be a faithful right \( A \)-module, and let \( B \) be its commutant, acting on \( H \) on the left.

Then \( H \) is naturally a \( B-A \)-bimodule, and its conjugate \( \hat{H} \) is an \( A-B \)-bimodule.

Consider the Hilbert space
\[
\hat{H} := H \hat{\otimes} A L^2(M) \hat{\otimes} \bar{A} H,
\]
which is a completion of \( \text{hom}(L^2A_A, H_A) \hat{\otimes} A L^2(M) \hat{\otimes} \bar{A} \text{hom}(A L^2A_A, \bar{A} H) \). There is then an antilinear involution \( \hat{J}: \hat{H} \to \hat{H} \) given by
\[
\hat{J}(\varphi \otimes \xi \otimes \psi) = \bar{\psi} \otimes J_M(\xi) \otimes \varphi,
\]
where \( J_A \) and \( J_M \) are the modular conjugations on \( L^2A \) and \( L^2M \), \( \xi \in L^2M \) is a vector, and for \( \varphi \in \text{hom}(L^2A_A, H_A) \) and \( \psi \in \text{hom}(A L^2A_A, \bar{A} H) \), the maps
\[
\varphi \in \text{hom}(L^2A_A, \bar{A} H), \quad \bar{\psi} \in \text{hom}(A L^2A_A, H_A)
\]
are given by \( \bar{\varphi} = I \circ \varphi \circ J_A \) and \( \bar{\psi} = I \circ \psi \circ J_A \), where \( I \) is the identity map between \( H \) and \( \hat{H} \). There are natural left and right actions of \( B \) on \( H \) coming from its actions on \( H \) and \( \hat{H} \). Moreover, the left and right actions of \( M \) on \( L^2(M) \) induce actions of \( M_0 \) on \( \hat{H} \). The left and right actions are interchanged (up to a star) by \( \hat{J} \), and so the algebra \( \hat{M} := M_0 \vee B \) generated by \( M_0 \) and \( B \) in their left action on \( \hat{H} \) is isomorphic to the algebra generated by \( M_0 \) and \( B \) in their right action on \( \hat{H} \).

From the above discussion, we see that \( \hat{H} \) is an \( \hat{M}-\hat{M} \)-bimodule with an involution \( \hat{J} \) that satisfies \( \hat{J}(a \varphi b) = b^* \hat{J}(\varphi) a^* \).

Proposition A.13. In the above situation, there is a canonical positive cone \( \hat{P} \) in \( \hat{H} := H \hat{\otimes} A L^2M \hat{\otimes} A H \) such that \( (\hat{H}, \hat{J}, \hat{P}) \) is a standard form for \( \hat{M} = M_0 \vee B \).

In the following proof, \( \ell^2 \) stands for \( \ell^2(\mathbb{N}) \), or maybe \( \ell^2(X) \) for a set \( X \) of sufficiently large cardinality. If \( H \) admits a cyclic vector for \( A \) then we can replace \( \ell^2 \) by \( \mathbb{C} \), and the proof simplifies.

Proof. Pick an \( A \)-linear isometry \( u: H_A \to \ell^2 \otimes L^2(A)_A \) (Lemma A.25) and let
\[
\bar{u} := (1 \otimes J_A) \circ u \circ I: A \hat{H} \to \ell^2 \otimes L^2(A) \cong A L^2(A) \otimes \ell^2.
\]
The endomorphisms of \( \ell^2 \otimes L^2(A)_A \) can be identified with \( B(\ell^2) \otimes A \). In particular, the range projection \( p := uu^* \) is in \( B(\ell^2) \otimes A \subset B(\ell^2) \otimes M \).

Let us define \( M_1 := B(\ell^2) \otimes M \), with associated standard form \((L^2M_1, J_{M_1}, P_{M_1})\) and let \( q := p J_{M_1} p J_{M_1} \in B(L^2M_1) \) or, equivalently, \( q(\xi) := p \xi p \). Composing \( u \hat{id}_{L^2M} \hat{\otimes} \bar{u} \) with the obvious identifications \((\ell^2 \otimes L^2A) \hat{\otimes} A L^2M \hat{\otimes} A (L^2A \otimes \ell^2) \cong \ell^2 \otimes L^2M \otimes \ell^2 \cong L^2M_1 \), we get an isometry
\[
v: \hat{H} = H \hat{\otimes} A L^2M \hat{\otimes} A H \to L^2M_1
\]
\[
v(\varphi \otimes \xi \otimes \psi) = (u \varphi) \cdot \xi \cdot (\bar{u} \psi)
\]
with range projection $vv^* = q$. The resulting isomorphism $\hat{H} \cong q(L^2M_1)$ intertwines $J$ and $qJ_M = J_Mq$, as can be seen from the commutative diagram

$$
\begin{array}{ccc}
\varphi \otimes \xi \otimes \psi & \xrightarrow{J} & \psi \otimes J_M(\xi) \otimes \varphi \\
\downarrow v & & \downarrow v \\
(u\psi) \cdot J_M(\xi) \cdot (\bar{u}\varphi) & \xrightarrow{J_{M_1}} & (\bar{u}\psi)^* \cdot J_M(\xi) \cdot (u\varphi)^*
\end{array}
$$

where the last equality holds because the preimage of $u\bar{\psi}$ under the left action map $\ell^2 \otimes A \to \text{hom}(L^2A, \ell^2 \otimes L^2A)$ agrees with the preimage of $(u\psi)^*$ under the right action map $\ell^2 \otimes A \to \text{hom}(A L^2A \otimes \ell^2, A L^2A)$, and the preimage of $\bar{u}\varphi$ under the right action map agrees with the preimage of $(u\varphi)^*$ under the left action map.

Recall that $B$ is the commutant of $A$ on $H$. In its action on $L^2M_1 = \ell^2 \otimes L^2M \otimes \ell^2$, we have $B \cong vBv^* = q(B(\ell^2) \otimes A)q$, and so it follows that

$$
\hat{M} \cong v\hat{M}v^* = v(M_0 \lor B)v^*
$$

(A.14)

$$= qM_0 \lor q(B(\ell^2) \otimes A)q = q(B(\ell^2) \otimes M)q = qM_1q.
$$

Now by [11, Lemma 2.6], we know that $(q(L^2M_1), qJ_M, q(P_{M_1}))$ is a standard form for $qM_1q$. Therefore, by letting $\hat{P} := v^{-1}(q(P_{M_1})), we get that $(\hat{H}, \hat{J}, \hat{P})$ is a standard form for $\hat{M}$.

**Lemma A.15.** Let $M, M_0, A$ be as above. Let $H_A$ be a non-zero $A$-module, and let $\tilde{M}$ be the von Neumann algebra generated by the left actions of $M_0$ and $A'$ on $H \otimes_A L^2(M)$. Assuming $M$ is a factor, then $\tilde{M}$ is a factor.

**Proof.** We have seen in (A.14) that $\tilde{M} = qM_1q = q(B(\ell^2) \otimes M)q$. The result follows since corners of factors are factors. \qed

The map constructed in Proposition A.13 satisfies the following version of associativity. Let $M = M_0 \lor A_1 \lor A_2$ be a von Neumann algebra, where $M_0$, $A_1$, and $A_2$ are commuting subalgebras of $M$. Let $H_l$ be faithful right $A_i$-modules, and let $B_l$ be their commutants. Then we can form the Hilbert spaces

$$H_1 := H_1 \otimes_{A_1} L^2 M \otimes_{A_1} H_1 \quad \text{and} \quad H_2 := H_2 \otimes_{A_2} L^2 M \otimes_{A_2} H_2$$

on which the algebras $\hat{M}_1 := M_0 \lor B_1 \lor A_2$ and $\hat{M}_2 := M_0 \lor A_1 \lor B_2$ act. By Proposition A.13, we get canonical isomorphisms $\hat{H}_1 \cong L^2\hat{M}_1$ and $\hat{H}_2 \cong L^2\hat{M}_2$.

Furthermore, we can form the Hilbert spaces

$$\hat{H}_1 := H_2 \otimes_{A_2} L^2 \hat{M}_1 \otimes_{A_1} H_2 \quad \text{and} \quad \hat{H}_2 := H_1 \otimes_{A_1} L^2 \hat{M}_2 \otimes_{A_2} H_1,$$

on which the algebra $\hat{\tilde{M}} := M_0 \lor B_1 \lor B_2$ acts. By Proposition A.13, we then get canonical isomorphisms $\hat{H}_1 \cong L^2\hat{\tilde{M}} \cong \hat{H}_2$.\[\square\]
Proposition A.16. In the above situation, the diagram
\[
\begin{array}{ccc}
H_1 \boxtimes \hat{H}_2 \boxtimes \hat{H}_1 & \rightarrow & H_1 \boxtimes L^2 \hat{M} \boxtimes \hat{H}_1 = \hat{H}_2 \\
\cong & & L^2 \hat{M} \\
H_2 \boxtimes \hat{H}_1 \boxtimes \hat{H}_2 & \rightarrow & H_2 \boxtimes L^2 \hat{M} \boxtimes \hat{H}_2 = \hat{H}_1
\end{array}
\]
is commutative.

Proof. Let \( \ell_1 \) and \( \ell_2 \) be two copies of \( \ell^2 \). Pick isometries \( u_i : (\hat{H}_i)_{H_i} \hookrightarrow (\ell_i \otimes L^2 A_i)_{A_i} \), so as to identify \( \hat{H}_1 \) with \( L^2(p_1(B(\ell_1) \otimes M)p_1) \), and \( \hat{H}_2 \) with \( L^2(p_2(M \otimes B(\ell_2))p_2) \), for \( p_i := u_i u_i^* \). Here, we have \( p_1 \in B(\ell_1) \otimes M \) and \( p_2 \in M \otimes B(\ell_2) \). Let us also define the projections \( q_1 \) on \( L^2(B(\ell_1) \otimes M) \cong \ell_1 \otimes \ell_1 \otimes L^2 M \) and \( q_2 \) on \( L^2(M \otimes B(\ell_2)) \cong L^2 M \otimes \ell_2 \otimes \ell_2 \) by \( q_i(\xi) = p_i \xi p_i \).

Given the above notations, the proof consists in a careful examination the following commutative diagram:
\[
\begin{array}{c}
\begin{array}{cccccc}
H_1 \boxtimes L^2(p_2(M \otimes B(\ell_2))p_2) & \xrightarrow{\ell_1 \otimes \ell_1 \otimes \hat{L}^2(p_2(M \otimes B(\ell_2))p_2)} & H_1 \boxtimes L^2((p_1 \otimes 1)(1 \otimes p_2)(B(\ell_1) \otimes M)B(\ell_2)(1 \otimes p_2)(1 \otimes 1)) \\
\cong & & L^2((p_1 \otimes 1)(1 \otimes p_2)(B(\ell_1) \otimes M)B(\ell_2)(1 \otimes p_2)(1 \otimes 1)) \\
H_2 \boxtimes L^2(M \otimes B(\ell_2)) & \xrightarrow{\ell_1 \otimes \ell_1 \otimes \hat{L}^2(M \otimes B(\ell_2))} & H_2 \boxtimes L^2(B(\ell_1) \otimes M \otimes B(\ell_2)) \\
\end{array}
\end{array}
\]

in which arrows denote inclusions, and lines denote isomorphisms. \( \square \)

A.VIII. Statistical dimension and minimal index. The statistical dimension of a dualizable bimodule \( \hat{A}_H \) is given by
\[
\dim(\hat{A}_H) := R^* R = S^* S \in \mathbb{R}_{>0}
\]
where \( R \) and \( S \) are as in (A.12). For non-dualizable bimodules, one declares \( \dim(\hat{A}_H) \) to be \( \infty \). If \( A = \oplus A_i \) and \( B = \oplus B_j \) are finite direct sums of factors, then we can decompose \( \hat{H} = \oplus \hat{H}_{ij} \) as a direct sum of \( A_i - B_j \)-bimodules and use the matrix-valued statistical dimension
\[
\dim(\hat{A}_H)_{ij} := \dim(A_i, H_{ij}, B_j).
\]
This matrix-valued dimension is additive
\[
(A.17) \quad \dim(\hat{A}_H \oplus \hat{A}_K) = \dim(\hat{A}_H) + \dim(\hat{A}_K)
\]
with respect to addition of modules and multiplicative
\[ \dim(AH_B \otimes_B K_C) = \dim(AH_B) \cdot \dim(K_C) \]
with respect to Connes fusion \cite{1} Prop. 5.2. Given a finite homomorphism \( f : A \to B \) between von Neumann algebras with finite-dimensional center, we let
\[ \|B : A\| := \dim(AL^2B) \]
denote the matrix of statistical dimensions of \( AL^2B \). If \( AH_B \) is a bimodule where \( B \) acts faithfully, then by \cite{1} Lem. 5.16
\[ (A.19) \quad \dim(AH_B) = \|B' : A\| \]
where \( B' \supset A \) is the commutant of \( B \) on \( H \). If \( A \) also acts faithfully, then
\[ (A.20) \quad \|B : A\| = \|A' : B'\|^T, \]
where \( T \) denotes the transposed matrix \cite{1} Cor. 5.17. The minimal index \( [B : A] \) of an inclusion of factors \( \iota : A \to B \) is the square of the statistical dimension of \( AL^2B \) \cite{1} Definition 5.10. For inclusions \( A \subseteq B \subseteq C \) of von Neumann algebras with finite-dimensional center we have
\[ (A.21) \quad [C : A] = [B : A] \cdot [C : B], \]
by \cite{1} Eq. (5.14). Moreover, the inclusion map \( A \to B \) is an isomorphism if and only if \( [B : A] \) is a permutation matrix \cite{1} Prop. 5.2. If \( C \) is a factor, then
\[ (A.22) \quad \|B \otimes C : A \otimes C\| = \|B : A\|. \]

We recall two further results \cite{1} Cor. 7.26, Cor. 7.27 that are crucial for the proof of Theorem 5.2. Let \( \| \cdot \|_2 \) stand for the \( l^2 \)-norm of a vector. Let \( N \subseteq M \subseteq B(H) \) be factors such that the inclusion \( N \subseteq M \) has finite index. If \( M \subseteq A \subseteq B(H) \) is such that one of the two relative commutants \( N' \cap A \) or \( M' \cap A \) is a factor and the other has finite-dimensional center, then
\[ (A.23) \quad \|N' \cap A : M' \cap A\|_2 \leq \|M : N\|. \]
Similarly, if \( A \subseteq M' \subseteq B(H) \) is such that one of the algebras \( N \vee A \) and \( M \vee A \) is a factor and the other has finite-dimensional center, then
\[ (A.24) \quad \|M \vee A : N \vee A\|_2 \leq \|M : N\|. \]

A. IX. Functors between module categories. This first lemma is well known \cite{12} Remark 2.1.3. (iii)). It is the main distinguishing feature of the representation theory of von Neumann algebras. Recall that \( \ell^2 \) stands for \( \ell^2(\mathbb{N}) \), or possibly \( \ell^2(X) \) with \( X \) a set of sufficiently large cardinality if the Hilbert spaces we deal with are not separable.

Lemma A.25. Let \( A \) be a von Neumann algebra and let \( H \) and \( K \) be two faithful left \( A \)-modules. Then \( H \otimes \ell^2 \cong K \otimes \ell^2 \). In particular, any \( A \)-module is isomorphic to a direct summand of \( H \otimes \ell^2 \).

Let \( A \) and \( B \) be von Neumann algebras. We call a functor \( F : A \text{-modules} \to B \text{-modules} \) normal if it is continuous with respect to the ultra-weak topology on hom-spaces, preserves adjoints \( F(f^*) = F(f)^* \) and is additive in the following sense: for \( A \)-modules \( M_i \) the map \( \oplus F(\iota_i) : \oplus F(M_i) \to F(\oplus M_i) \) induced by the inclusions \( \iota_i : M_i \to \oplus_k M_k \) is a unitary isomorphism. Such functors are uniquely determined by their value on a faithful \( A \)-module.
Lemma A.26. Let $A$ and $B$ be von Neumann algebras. Let $M$ be a faithful $A$-module, let $N$ be an arbitrary $B$-module, and let

$$F_1 : \text{End}_A(M) \to \text{End}_B(N)$$

be a morphism of von Neumann algebras. Then the assignment $F(M) := N$, $F(f) := F_1(f)$ extends uniquely (up to unique unitary isomorphism) to a normal functor $F$ from the category of $A$-modules to the category of $B$-modules.

Proof. We prove existence and leave uniqueness to the reader. Given an $A$-module $H$, then, by Lemma A.25, we may pick an isomorphism

$$H \cong \text{im}\{p : M \otimes \ell^2 \to M \otimes \ell^2\}$$

of $H$ with the image of a projection $p \in \text{End}_A(M) \otimes B(\ell^2)$. We can then define

$$F(H) := \text{im}\{(F_1 \otimes \text{Id}_\ell)(p) : N \otimes \ell^2 \to N \otimes \ell^2\},$$

At the level of arrows, if $H \cong \text{im}(p)$ and $K \cong \text{im}(q)$ are $A$-modules given to us as above, then the image under $F$ of an $A$-linear map $r : H \to K$ is the unique map $F(r) : F(H) \to F(K)$ for which the composite

$$N \otimes \ell^2 \to F(H) \xrightarrow{F(r)} F(K) \hookrightarrow N \otimes \ell^2$$

is the image under $F_1 \otimes \text{Id}_\ell$ of the map $M \otimes \ell^2 \to H \xrightarrow{r} K \hookrightarrow M \otimes \ell^2$. \qed

A similar result holds for natural transformations.

Lemma A.28. Let $F, G : A\text{-modules} \to B\text{-modules}$ be two normal functors and let $M$ be a faithful $A$-module. Then, in order to uniquely define a natural transformation $\alpha : F \to G$, it is enough to specify its value on $M$, and to check that for each $r \in \text{End}_A(M)$, the diagram

$$
\begin{array}{ccc}
F(M) & \xrightarrow{F(r)} & F(M) \\
\downarrow a_M & & \downarrow a_M \\
G(M) & \xrightarrow{G(r)} & G(M)
\end{array}
$$

commutes.

Proof. Given an $A$-module $H$ along with an isomorphism (A.24), one uses the natural inclusion $F(H) \subset F(M \otimes \ell^2) \cong F(M) \otimes \ell^2$ to define

$$a_H := (a_M \otimes \text{Id}_{\ell^2})|_{F(H)}.$$ 

This prescription is independent of the choice of isomorphism. \qed

A.X. The split property.

Definition A.29. Given two commuting von Neumann algebras $A$ and $B$ acting on a Hilbert space $H$, we say that $A$ and $B$ are split on $H$ if the natural map $A \otimes_{\text{alg}} B \to B(H)$ extends to a homomorphism $A \otimes B \to B(H)$. We also say that an inclusion $A \hookrightarrow B$ is split if there exists a (equivalently, for any) faithful $B$-module $H$ such that $A$ and $B'$ are split on $H$.

Lemma A.30. Let $A_0 \subseteq A$ and $B \subseteq A'$ be von Neumann algebras defined on a Hilbert space $H$. If the inclusion $A_0 \hookrightarrow A$ is split, then we have

$$B \vee (A \cap A_0') = (B \vee A) \cap A_0'.$$

(A.31)
Proof. Consider $H \otimes L^2(A_0)$ as a $A' \otimes A_0$-module, where $A'$ acts on the first factor and $A_0$ acts on the second factor. Both $H$ and $H \otimes L^2(A_0)$ are faithful modules. So we may pick a $A' \otimes A_0$-module isomorphism between $H \otimes \ell^2$ and $H \otimes L^2(A_0) \otimes \ell^2$. Let $K := L^2(A_0) \otimes \ell^2$. Our isomorphism then becomes $H \otimes \ell^2 \cong H \otimes K$. Under the above identification, the subalgebra

$$(B \vee (A \cap A_0')) \boxtimes B(\ell^2) = (B \otimes 1) \vee ((A \otimes B(\ell^2)) \cap (A_0 \otimes 1'))$$

of $B(H \otimes \ell^2)$ corresponds to

$$(B \otimes 1) \vee ((A \otimes B(K)) \cap (1 \otimes A_0')) = (B \otimes 1) \vee (A \otimes A_0') = (B \vee A) \bar{\otimes} A_0'$$

in $B(H \otimes K)$. Similarly,

$$( (B \vee A) \cap A_0' ) \boxtimes B(\ell^2) = ( (B \otimes 1) \vee (A \otimes B(\ell^2)) ) \cap (A_0 \otimes 1')$$

corresponds to

$$((B \otimes 1)(A \otimes B(K)) \cap (1 \otimes A_0')) = ((B \vee A) \otimes B(K)) \cap (B(H) \otimes A_0') = (B \vee A) \bar{\otimes} A_0'.$$

The algebras (A.34) agree after tensoring with $B(\ell^2)$. So they are equal. $\square$

Lemma A.32. Let $A \subset B(H)$ be a factor, and let $A_0$ be a subalgebra of $A$. If the inclusion $A_0 \hookrightarrow A$ is split, then

$$A_0 \vee A' \cap A = A_0.$$

Proof. As in the previous lemma, we pick an isomorphism $H \otimes \ell^2 \cong H \otimes K$ of $A' \otimes A_0$-modules, where $K = L^2(A_0) \otimes \ell^2$. Under that isomorphism, the algebras $A_0 \otimes 1$ and

$$((A_0 \vee A') \cap A) \bar{\otimes} 1 = ((A_0 \otimes 1) \vee (A' \otimes 1)) \cap (A \otimes B(\ell^2))$$

correspond to $1 \bar{\otimes} A_0'$ and

$$(1 \bar{\otimes} A_0) \vee (A' \otimes 1) \cap (A \otimes B(K)) = (A' \bar{\otimes} A_0) \cap (A \otimes B(K)) = 1 \bar{\otimes} A_0'.$$

Since their images in $B(H \otimes K)$ agree, the two algebras (A.33) are equal. $\square$

Recall the fiber product operation $*$ from Definition A.10.

Lemma A.34. Let $A_0$ and $A_1$ be commuting subalgebras of $B(H)$, and let $B_0$ and $B_1$ be commuting subalgebras of $B(K)$. Let $C \to A_0$ and $C^{op} \to B_0$ be homomorphisms. If $C$ and $A_0'$ are split on $H$, then we have

$$A_1 \vee (A_0 *_C B_0) \vee B_1 = (A_1 \vee A_0) *_C (B_0 \vee B_1)$$

on $H \boxtimes^C K$.

Proof. Since $A_0'$ and $C$ are split on $H$, the actions of $A_0'$ and $C'$ on $H \boxtimes^C K$ induce an action of $A_0' \otimes C'$. In particular, the actions of $A_0'$ on $H$ and of $B_0'$ on $K$ induce an action of $A_0' \otimes B_0'$ on $H \boxtimes^C K$.

Consider $H \otimes K$ as a $A_0' \otimes B_0'$-module, where $A_0'$ acts on $H$ and $B_0'$ acts on $K$. Since this is a faithful module, we can find an $A_0' \otimes B_0'$-linear isometry

$$H \boxtimes^C K \hookrightarrow H \otimes K \otimes \ell^2.$$

Let $p \in B(H \otimes K \otimes \ell^2)$ be its range projection. Under the induced isomorphism

$$\alpha : B(H \boxtimes^C K) \xrightarrow{\cong} p(B(H) \otimes B(K) \otimes B(\ell^2))p,$$

we have

$$\alpha(A_0') = (A_0' \otimes C \otimes C) p, \quad \alpha(B_0') = (C \otimes B_0' \otimes C) p, \quad \alpha(A_1) = (A_1 \otimes C \otimes C) p, \quad \alpha(B_1) = (C \otimes B_1 \otimes C) p.$$

Recalling the definition $A_0 *_C B_0 := (A_0' \vee B_0')'$, we then see that
\(\alpha(A_0 \ast_C B_0) = p (A_0 \hat{\otimes} B_0 \hat{\otimes} B(\ell^2)) p.\)

Similarly, \(\alpha((A_1 \vee A_0) \ast_C (B_0 \vee B_1)) = p ((A_1 \vee A_0) \hat{\otimes} (B_0 \vee B_1) \hat{\otimes} B(\ell^2)) p,\) and equation (A.35) follows since
\[ (A_1 \hat{\otimes} C \otimes C) \vee (A_0 \hat{\otimes} B_0 \hat{\otimes} B(\ell^2)) \vee (C \otimes B_1 \otimes C) = (A_1 \vee A_0) \hat{\otimes} (B_0 \vee B_1) \hat{\otimes} B(\ell^2).\]

\(\square\)

Lemma A.36. Let \(A, B,\) and \(C\) be factors, and let \(A^{\text{op}} \leftrightarrow C \to B\) be homomorphisms. Let \(B \subset \hat{B}\) be a subfactor of finite index. Assuming that the inclusion \(C \to A^{\text{op}}\) is split, then \(A \ast_C \hat{B}\) and \(A \ast_C B\) are factors, and we have
\[
(A \ast_C \hat{B} : A \ast_C B) = [\hat{B} : B].
\]

Proof. Let \(H\) be a faithful \(A\)-module and \(K\) a faithful \(\hat{B}\)-module. Let \(A'\) be the commutant of \(A\) on \(H\), and let \(B'\) and \(\hat{B}'\) be the commutants of \(B\) and \(\hat{B}\) on \(K\). Finally, let \(C'\) be the commutant of \(C\) on \(K\), and let \(C\) be the commutant of \(C^{\text{op}}\) on \(H\).

Since the inclusion of \(C\) into \(A^{\text{op}}\) is split, so is the inclusion \(A' \hookrightarrow C'.\) The algebra \(C'\) is \(C\)'s commutant on \(H \hat{\otimes}_C K\), and so \(A'\) and \(C'\) are split on \(H \hat{\otimes}_C K\). Finally, \(B'\) and \(\hat{B}'\) being a subalgebras of \(C'\), we conclude that \(A'\) and \(B'\) and that \(A'\) and \(\hat{B}'\) are split on \(H \hat{\otimes}_C K\). It follows that the algebras
\[
(A \ast_C \hat{B})' = A' \vee B' = A' \hat{\otimes} B' \quad \text{and} \quad (A \ast_C \hat{B})' = A' \vee \hat{B}' = A' \hat{\otimes} \hat{B}'
\]
are factors, and that so are \(A \ast_C \hat{B}\) and \(A \ast_C B\). Moreover, we have
\[
[A \ast_C \hat{B} : A \ast_C B] = [A \ast_C (A')' : (A \ast_C (A')')] = [A' \otimes B' : A' \hat{\otimes} \hat{B}'] = [B' : \hat{B}'] = [\hat{B} : B]
\]
by (A.20) and (A.18). \(\square\)

Appendix B. Conformal nets

B.1. Axioms for conformal nets. Let \(\text{VN}\) be the category whose objects are von Neumann algebras with separable preduals, and whose morphisms are \(\mathbb{C}\)-linear homomorphisms, and \(\mathbb{C}\)-linear antihomomorphisms. A net is a covariant functor \(\mathcal{A} : \text{INT} \to \text{VN}\) taking orientation-preserving embeddings to injective homomorphisms and orientation-reversing embeddings to injective antihomomorphisms. It is said to be continuous if for any intervals \(I\) and \(J\), the map \(\text{Hom}_{\text{INT}}(I, J) \to \text{Hom}_{\text{VN}}(\mathcal{A}(I), \mathcal{A}(J)), \varphi \mapsto \mathcal{A}(\varphi)\) is continuous for the \(C^\infty\) topology on \(\text{Hom}_{\text{INT}}(I, J)\) and Haagerup’s \(\text{u}\)-topology on \(\text{Hom}_{\text{VN}}(\mathcal{A}(I), \mathcal{A}(J))\) [2] Appendix].

Given a subinterval \(I \subseteq K\), we will often not distinguish between \(\mathcal{A}(I)\) and its image in \(\mathcal{A}(K)\).

A conformal net is a continuous net \(\mathcal{A}\) subject to the following conditions. Here, \(I\) and \(J\) are subintervals of an interval \(K\):

(i) **Locality**: If \(I \subset K\) and \(J \subset K\) have disjoint interiors, then \(\mathcal{A}(I)\) and \(\mathcal{A}(J)\) are commuting subalgebras of \(\mathcal{A}(K)\).

(ii) **Strong additivity**: If \(I = I \cup J\), then \(\mathcal{A}(K)\) is generated as a von Neumann algebra by its two subalgebras: \(\mathcal{A}(K) = \mathcal{A}(I) \vee \mathcal{A}(J)\).

(iii) **Split property**: If \(I \subset K\) and \(J \subset K\) are disjoint, then the map from the algebraic tensor product \(\mathcal{A}(I) \otimes_{\text{alg}} \mathcal{A}(J) \to \mathcal{A}(K)\) extends to a map from the spatial tensor product \(\mathcal{A}(I) \otimes \mathcal{A}(J) \to \mathcal{A}(K)\).

(iv) **Inner covariance**: If \(\varphi \in \text{Diff}_+(I)\) restricts to the identity in a neighborhood of \(\partial I\), then \(\mathcal{A}(\varphi)\) is an inner automorphism of \(\mathcal{A}(I)\). (A unitary \(u \in \mathcal{A}(I)\) with \(\text{Ad}(u) = \mathcal{A}(\varphi)\) is said to implement \(\varphi\).

(v) **Vacuum sector**: Suppose that \(J \subset I\) contains the boundary point \(p \in \partial I\), and let \(J\) denote \(J\) with the reversed orientation; \(\mathcal{A}(J)\) acts on \(L^2(\mathcal{A}(I))\) via the left action of \(\mathcal{A}(I)\), and \(\mathcal{A}(J) \cong \mathcal{A}(J)^{\text{op}}\) acts on \(L^2(\mathcal{A}(I))\) via the right
action of $\mathcal{A}(I)$. In that case, we require that the action of $\mathcal{A}(J) \otimes_{\text{alg}} \mathcal{A}(\bar{J})$ on $L^2(\mathcal{A}(I))$ extends to an action of $\mathcal{A}(J \cup_p \bar{J})$:

$$\begin{align*}
\mathcal{A}(J) \otimes_{\text{alg}} \mathcal{A}(\bar{J}) &\longrightarrow \mathcal{B}(L^2(\mathcal{A}(I))) \\
\mathcal{A}(J \cup_p \bar{J}) &\longrightarrow \\
\end{align*}$$

Here, $J \cup_p \bar{J}$ is equipped with any smooth structure extending the given smooth structures on $J$ and $\bar{J}$, and for which the orientation-reversing involution that exchanges $J$ and $\bar{J}$ is smooth.

A conformal net $\mathcal{A}$ is called irreducible if the algebras $\mathcal{A}(I)$ are factors. Any conformal net decomposes as a direct integral of irreducible ones [2 Sec. 1.D]. A conformal net is called semisimple if it is a finite direct sum of irreducible conformal nets. We will denote by $\mathcal{CN}_0$ the symmetric monoidal category of semisimple conformal nets and their natural transformations. The tensor product of nets $\mathcal{A}$ and $\mathcal{B}$ is defined using the spatial tensor product of von Neumann algebras: $(\mathcal{A} \otimes \mathcal{B})(I) := \mathcal{A}(I) \otimes \mathcal{B}(I)$. A natural transformation $\tau: \mathcal{A} \rightarrow \mathcal{B}$ between semisimple conformal nets is called finite if for all intervals $I$, the map $\tau_I: \mathcal{A}(I) \rightarrow \mathcal{B}(I)$ is a finite homomorphism (Appendix A.V). We will denote by $\mathcal{CN}_0^f \subset \mathcal{CN}_0$ the symmetric monoidal category of semisimple conformal nets all of whose reducible summands have finite index, with their finite natural transformations.

B.II. The vacuum sector. A conformal circle $S$ is a circle $S$ together with a diffeomorphism $S \rightarrow S^1$ that is specified up to a (not necessarily orientation preserving) Möbius transformation of $S^1$ [2 Def. 2.12]. The set of conformal maps of $S \rightarrow S'$ is denoted by $\text{Conf}(S, S')$. If $S$ and $S'$ are oriented, then we denote by $\text{Conf}^+(S, S')$ and $\text{Conf}^-(S, S')$ the subsets of orientation preserving and orientation reversing maps.

For a conformal net $\mathcal{A}$ there is a functor [2 Thm. 2.13]

$$(B.1) \quad S \mapsto H_0(S, \mathcal{A})$$

from the category of oriented conformal circles to the category of Hilbert spaces. It sends orientation preserving conformal maps to unitary operators and orientation reversing conformal maps to anti-unitary operators. The Hilbert space $H_0(S, \mathcal{A})$ is called the vacuum sector of $\mathcal{A}$ on $S$, and comes equipped with compatible actions of the algebras $\mathcal{A}(I)$ for any subinterval $I$ of $S$.

For $\varphi \in \text{Conf}(S, S')$, the operator $H_0(\varphi, \mathcal{A})$ implements the diffeomorphism $\varphi$, that is:

$$\begin{align*}
\mathcal{A}(\varphi)(a) &= H_0(\varphi, \mathcal{A}) a H_0(\varphi, \mathcal{A})^* \quad \text{if } \varphi \in \text{Conf}^+(S, S') \\
\mathcal{A}(\varphi)(a) &= H_0(\varphi, \mathcal{A}) a^* H_0(\varphi, \mathcal{A})^* \quad \text{if } \varphi \in \text{Conf}^-(S, S')
\end{align*}$$

for any interval $I \subset S$ and any element $a \in \mathcal{A}(I)$.

Moreover, for every interval $I \subset S$, there is a canonical unitary identification

$$(B.2) \quad \psi_I: H_0(\mathcal{A}, S) \rightarrow L^2(\mathcal{A}(I)).$$

These unitaries are such that for $\varphi \in \text{Conf}^+(S, S')$ and $\psi \in \text{Conf}^-(S, S')$, the diagrams

$$(B.3) \quad \begin{array}{ccc}
H_0(S, \mathcal{A}) & \xrightarrow{\psi_I} & L^2(\mathcal{A}(I)) \\
\downarrow H_0(\varphi, \mathcal{A}) & & \downarrow H_0(\psi, \mathcal{A}) \\
H_0(S', \mathcal{A}) & \xrightarrow{\psi_I(\varphi)} & L^2(\mathcal{A}(\varphi(I))) \\
\end{array} \quad \begin{array}{ccc}
H_0(S, \mathcal{A}) & \xrightarrow{\psi_I} & L^2(\mathcal{A}(I)) \\
\downarrow H_0(\varphi, \mathcal{A}) & & \downarrow H_0(\psi, \mathcal{A}) \\
H_0(S', \mathcal{A}) & \xrightarrow{\psi_I(\varphi)} & L^2(\mathcal{A}(\varphi(I))) \\
\end{array}$$
commute, where $J$ is the modular conjugation on $L^2(\mathcal{A}(I))$, and $j_I \in \mathrm{Conf}_-(S)$ is the unique involution that fixes $\partial I$. Note also that we should really have written $L^2(\mathcal{A}(\varphi_I))$ in place of $L^2(\mathcal{A}(\varphi))$, and similarly for $L^2(\mathcal{A}(\psi_{j_I}))$. Taking $\psi := j_I$ in the second diagram, we recover the modular conjugation as $J = \psi H_0(j_I, \mathcal{A})\psi_I$.

If $S$ is a circle without a conformal structure, then it is still possible to define $H_0(S, \mathcal{A})$ as $L^2(\mathcal{A}(I))$ of some interval $I \subset S$, but this only defines $H_0(S, \mathcal{A})$ up to non-canonical unitary isomorphism \cite[Def. 1.16]{Haag}. We recall \cite[Prop. 1.17]{Haag}.

**Proposition B.4** (Haag duality for conformal nets). Let $\mathcal{A}$ be a conformal net, and $S$ be a circle. Then for any $I \subset S$, the algebra $\mathcal{A}(I')$ is the commutant of $\mathcal{A}(I)$ on $H_0(S, \mathcal{A})$.

Given intervals $J \subset K$ such that $J^c$, the closure of $K \setminus J$, is itself an interval, the commutant of $\mathcal{A}(J)$ in $\mathcal{A}(K)$ is $\mathcal{A}(J^c)$.

**B.III. Glueing vacuum sectors.** Consider a theta-graph $\Theta$, and let $S_1, S_2, S_3$ be its three circle subgraphs with orientations as drawn below:

\[ \Theta : \quad \includegraphics[scale=0.5]{theta_graph.png} \]

(Elsewhere in this paper, we more often depict circles as squares:

\[ \Theta : \quad \includegraphics[scale=0.5]{theta_graph_squares.png} \]

We equip $\Theta$ with a smooth structure in the sense of \cite[Definition 1.4]{Bar} and let

\[ I := S_1 \cap S_2, \quad K := S_1 \cap S_3, \quad L := S_2 \cap S_3. \]

Let us give $K$ the orientation coming from $S_1$, and let us give $I$ and $L$ the orientations coming from $S_2$. Given a conformal net $\mathcal{A}$, then there is a non-canonical isomorphism \cite[Cor. 1.33]{Haag}

\[ H_0(S_1, \mathcal{A}) \boxtimes_{\mathcal{A}(I)} H_0(S_2, \mathcal{A}) \cong H_0(S_3, \mathcal{A}), \]

compatible with the actions of $\mathcal{A}(K)$ and $\mathcal{A}(L)$.

Moreover, in the presence of suitable conformal structures, this isomorphism can be constructed canonically: equip $S_1$ and $S_2$ with conformal structures, and let $j_1 \in \mathrm{Conf}_-(S_1)$, $j_2 \in \mathrm{Conf}_-(S_2)$ be the unique involutions fixing $\partial I$. Then there is a unique conformal structure on $S_3$ for which $j_1|_I \cup \mathrm{Id}_K : S_1 \to S_3$ and $j_2|_I \cup \mathrm{Id}_L : S_2 \to S_3$ are conformal. We can then use \cite[Cor. 2.20]{Bar} to obtain the canonical isomorphism \cite[Cor. 2.20]{Haag}

\[ \Upsilon : H_0(S_1, \mathcal{A}) \boxtimes_{\mathcal{A}(I)} H_0(S_2, \mathcal{A}) \xrightarrow{\psi_K \boxtimes \psi_{j_1}} L^2(\mathcal{A}(K)) \boxtimes_{\mathcal{A}(I)} L^2(\mathcal{A}(I)) \xrightarrow{\cong} L^2(\mathcal{A}(K)) \xrightarrow{\psi_K} H_0(S_3, \mathcal{A}). \]
B.IV. Finite-index conformal nets. Let $S$ be a circle, split into four intervals $I_1, I_2, I_3, I_4$ such that each $I_i$ intersects $I_{i+1}$ (cyclic numbering) in a single point:

\[
\begin{array}{c}
\hat{I}_2 \\
\hat{I}_3 \\
\hat{I}_4 \\
\hat{I}_1
\end{array}
\]

(B.7)

Let $A$ be an irreducible conformal net. The algebras $A(I_1 \cup I_3) = A(I_1) \otimes_A A(I_3)$ and $A(I_2 \cup I_4) = A(I_2) \otimes A(I_4)$ act on $H_0(S, A)$ and commute with each other. The index $\mu(A)$ of the conformal net $A$ is the minimal index (see Appendix A.VIII) of the inclusion $A(I_1 \cup I_3) \subseteq A(I_2 \cup I_4)$ [16, 31]:

\[\mu(A) := [A(I_2 \cup I_4) : A(I_1 \cup I_3)],\]

where the commutant is computed in $B(H_0(S, A))$. A natural transformation $A \to B$ between finite conformal nets is said to be finite if for every interval $I$, the map $A(I) \to B(I)$ is finite (Appendix A.VI). The subcategory of $CN_0$ whose objects are finite irreducible conformal nets and whose morphisms are finite natural transformation is denoted $CN_0^f$.

B.V. Sectors and the Hilbert space of the annulus. Let $A$ be an irreducible conformal net and $S$ be a circle (that is, an oriented compact connected closed 1-manifold). An $S$-sector of $A$ is a Hilbert space $H$ together with homomorphisms

\[\rho_I : A(I) \to B(H), \quad I \subset S\]

subject to the compatibility condition $\rho_{IJ} |_{A(IJ)} = \rho_J$ whenever $J \subset I$. We write $\Delta$ for the collection of unitary isomorphism classes of irreducible $S$-sectors of $A$. The vacuum sector discussed before is an example of a sector and we write $0$ for the corresponding element of $\Delta$. As all circles are diffeomorphic $\Delta$ does not depend on the specific circle $S$. There is an involution $\lambda \mapsto \bar{\lambda}$ given by sending an $S$-sector to its pull back along an orientation reversing diffeomorphism of $S$, as defined in [2, (1.12)]. For $\lambda \in \Delta$ we write $H_\lambda(S, A)$ for a representative of $\lambda$ as an $S$-sector. Of course, $H_\lambda(S, A)$ is only determined up to non-canonical isomorphism.

Let $S_l$ be a circle, decomposed into four intervals $I_1, \ldots, I_4$ as in (B.7), and let $S_r$ be another circle, similarly decomposed into four intervals $I_5, \ldots, I_8$. Let $\varphi : I_5 \to I_1$ and $\psi : I_7 \to I_3$ be orientation-reversing diffeomorphisms. These diffeomorphisms equip $H_0(S_l)$ with the structure of a right $A(I_5) \otimes A(I_7)$-module. We are interested in the Hilbert space

\[H_\Sigma := H_0(S_l) \otimes_{A(I_5) \otimes A(I_7)} H_0(S_r) \cong \gamma H_0(S_l) \otimes_{A(I_5) \otimes A(I_3)} H_0(S_r) \otimes_{A(I_3)} \gamma,\]

This space is associated to the annulus $\Sigma = \mathbb{D}_l \cup \mathbb{D}_r \cup \mathbb{D}_r$, where $\mathbb{D}_l$ and $\mathbb{D}_r$ are disks bounding $S_l$ and $S_r$. (As $H_0(S_l)$ and $H_0(S_r)$ are only determined up to non-canonical isometric isomorphism the same is true for $H_\Sigma$ at this point.) Let $S_b := I_2 \cup I_8$ and $S_m := I_4 \cup I_6$ be the two boundary circles of this annulus.
The Hilbert space $H_\Sigma$ is an $S_m$-$S_b$-sector, which means that it is equipped with compatible actions of the algebras $\mathcal{A}(J)$ associated to all subintervals of $S_m$ and $S_b$ [2 Sec. 3.B].

We finish by stating an important result which, formulated in a different language, is due to [10]:

**Theorem B.8** ([2 Thm. 3.23, Thm. 3.14]). *If the conformal net $\mathcal{A}$ has finite index, then there is a unitary isomorphism of $S_m$-$S_l$-sectors $H_\Sigma \cong \bigoplus_{\lambda \in \Delta} H_\lambda(S_m) \otimes H_\lambda(S_b)$.*

**B.VI. Extension of conformal nets to 1-manifolds.** A priori, the only manifolds on which a conformal net $\mathcal{A}$: $\text{INT} \to \text{VN}$ can be evaluated are intervals. However, the functor $\mathcal{A}$ can be extended in a canonical way to the larger category $\text{1MAN}$ of compact oriented 1-manifolds [3, Thm. 1.3]. We denote the extension $\text{1MAN} \to \text{VN}$ by the same letter $\mathcal{A}$.

For $S$ a circle, the construction of $\mathcal{A}(S)$ depends on the Hilbert space $H_\Sigma$ associated to the annulus $\Sigma := S \times [0,1]$—see [3 (1.6) and Thm. 1.38] for a refinement of the construction presented in the previous section. The algebra $\mathcal{A}(S)$ is defined to be the subalgebra of $B(H_\Sigma)$ generated by $\mathcal{A}(I \times \{0\})$ for all $I \subset S$.

**Theorem B.9** ([3 Thm. 1.19]). *Let $\mathcal{A}$ be a conformal net with finite index and let $S$ be a circle. Then there is a canonical isomorphism (B.10) $\mathcal{A}(S) \cong \bigoplus_{\lambda \in \Delta} B(H_\lambda(S,\mathcal{A}))$.*

Note that every though $H_\lambda(S,\mathcal{A})$ is only defined up to non-canonical isomorphism, its algebra of bounded operators is defined up to canonical isomorphism. It therefore makes sense for the isomorphism (B.10) to be canonical.

**References**

[1] A. Bartels, C. L. Douglas, and A. Henriques. Dualizability and index of subfactors. arXiv:1110.5671, 2011.
[2] A. Bartels, C. L. Douglas, and A. Henriques. Conformal nets I: Coordinate free nets. arXiv:1302.2604v2, 2013.
[3] A. Bartels, C. L. Douglas, and A. Henriques. Conformal nets II: conformal blocks. In preparation, preliminary version available at http://www.staff.science.uu.nl/~henri105/PDF/CNII_ConfBlocks.pdf, 2013.
[4] A. Bartels, C. L. Douglas, and A. Henriques. Conformal nets IV: the 3-category. In preparation, 2013.
[5] J. Böckenhauer and D. E. Evans. Modular invariants, graphs and $\alpha$-induction for nets of subfactors. I. Comm. Math. Phys., 197(2):361–386, 1998.
[6] D. Buchholz, G. Mack, and I. Todorov. The current algebra on the circle as a germ of local field theories. Nuclear Phys. B Proc. Suppl., 5B:20–56, 1988. Conformal field theories and related topics (Annecy-le-Vieux, 1988).
[7] A. Connes. Géométrie non commutative. InterEditions, Paris, 1990.
[8] C. L. Douglas and A. Henriques. Internal bicategories. arXiv:1206:4284, 2012.
[9] J. Fuchs, C. Schweigert, and A. Valentino. Bicategories for boundary conditions and for surface defects in 3-d TFT. arXiv:1203.4568, 2013.
[10] F. Gabbiani and J. Fröhlich. Operator algebras and conformal field theory. Comm. Math. Phys., 155(3):569–640, 1993.
[11] U. Haagerup. The standard form of von Neumann algebras. Math. Scand., 37(2):271–283, 1975.
[12] V. Jones and V. S. Sunder. Introduction to subfactors, volume 234 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1997.
[13] A. Kapustin and N. Saulina. Surface operators in 3d topological field theory and 2d rational conformal field theory. In Mathematical foundations of quantum field theory and perturbative string theory, volume 83 of Proc. Sympos. Pure Math., pages 175–198. Amer. Math. Soc., Providence, RI, 2011.
[14] Y. Kawahigashi. Generalized Longo-Rehren subfactors and α-induction. *Comm. Math. Phys.*, 226(2):269–287, 2002.
[15] Y. Kawahigashi and R. Longo. Classification of local conformal nets. Case $c < 1$. *Ann. of Math. (2)*, 160(2):493–522, 2004.
[16] Y. Kawahigashi, R. Longo, and M. Müger. Multi-interval subfactors and modularity of representations in conformal field theory. *Comm. Math. Phys.*, 219(3):631–669, 2001.
[17] H. Kosaki. Canonical $L^p$-spaces associated with an arbitrary abstract von Neumann algebra. *Ph.D. thesis, UCLA*, 1980.
[18] R. Longo and K.-H. Rehren. Nets of subfactors. *Rev. Math. Phys.*, 7(4):567–597, 1995. Workshop on Algebraic Quantum Field Theory and Jones Theory (Berlin, 1994).
[19] R. Longo and F. Xu. Topological sectors and a dichotomy in conformal field theory. *Comm. Math. Phys.*, 251(2):321–364, 2004.
[20] Y. Miura and J. Tomiyama. On a characterization of the tensor product of self-dual cones associated to the standard von Neumann algebras. *Sci. Rep. Niigata Univ. Ser. A*, (20):1–11, 1984.
[21] T. Quella, I. Runkel, and G. M. T. Watts. Reflection and transmission for conformal defects. *J. High Energy Phys.*, (4):095, 45 pp. (electronic), 2007.
[22] J.-L. Sauvageot. Sur le produit tensoriel relatif d’espaces de Hilbert. *J. Operator Theory*, 9(2):237–252, 1983.
[23] J.-L. Sauvageot. Produits tensoriels de $Z$-modules et applications. In *Operator algebras and their connections with topology and ergodic theory (Bütschi, 1983)*, volume 1132 of Lecture Notes in Math., pages 468–485. Springer, Berlin, 1985.
[24] L. M. Schmitt and G. Wittstock. Kernel representation of completely positive Hilbert-Schmidt operators on standard forms. *Arch. Math. (Basel)*, 38(5):453–458, 1982.
[25] C. Schweigert, J. Fuchs, and I. Runkel. Categorification and correlation functions in conformal field theory. In *International Congress of Mathematicians. Vol. III*, pages 443–458. Eur. Math. Soc., Zürich, 2006.
[26] G. Segal. The definition of conformal field theory. In *Topology, geometry and quantum field theory*, volume 308 of London Math. Soc. Lecture Note Ser., pages 421–577. Cambridge Univ. Press, Cambridge, 2004.
[27] M. Takesaki. *Tomita’s theory of modular Hilbert algebras and its applications*. Lecture Notes in Mathematics, Vol. 128. Springer-Verlag, Berlin, 1970.
[28] T. Timmermann. *An invitation to quantum groups and duality*. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2008. From Hopf algebras to multiplicative unitaries and beyond.
[29] A. Wassermann. Operator algebras and conformal field theory. III. Fusion of positive energy representations of $LSU(N)$ using bounded operators. *Invent. Math.*, 133(3):467–538, 1998.
[30] A. J. Wassermann. Operator algebras and conformal field theory. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 966–979, Basel, 1995. Birkhäuser.
[31] F. Xu. Jones-Wassermann subfactors for disconnected intervals. *Commun. Contemp. Math.*, 2(3):397–347, 2000.

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