Remarks on the calculus of variations on time scales

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ABSTRACT

The calculus of variations is a classical subject which has gain throughout the last three hundred years a level of rigor and elegance that only time can give. In this note we show that, contrary to the classical field, available formulations and results on the recent calculus of variations on time scales are still at the heuristic level.

Keywords: time scales, calculus of variations, fundamental lemma of the calculus of variations, Euler-Lagrange equations, multiple integrals.

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1 Introduction

The calculus on time scales was introduced by Stefan Hilger during his PhD project, carried out under the scientific supervision of Bernd Aulbach. The new theory unifies continuous and discrete analysis and, at the same time, can be applied to other closed subsets of the real numbers.

The calculus of variations on time scales is in its infancy, still being possible to give reference to all the works on the subject: [1, 2, 3, 5, 6, 7, 8].

The calculus on time scales has a very similar notation with the differential calculus. When one is not carefully enough, this leads to “results” without any meaning. This means that extra care must be taken when dealing with problems on time scales.

In this note we show and highlight some weaknesses that may arise when proving results on a general time scale. Such weaknesses can be of various kinds and we hope that by the end of this note they can be better understood.

Unless the contrary, throughout the text the notation conforms to that used in the references.
2 Basics on the time scale calculus

A nonempty closed subset of $\mathbb{R}$ is called a time scale and is denoted by $\mathbb{T}$.

The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) = \inf \{s \in \mathbb{T} : s > t\}, \text{ for all } t \in \mathbb{T},$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup \{s \in \mathbb{T} : s < t\}, \text{ for all } t \in \mathbb{T},$$

with $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(M) = M$ if $\mathbb{T}$ has a maximum $M$) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(m) = m$ if $\mathbb{T}$ has a minimum $m$).

A point $t \in \mathbb{T}$ is called right-dense, right-scattered, left-dense and left-scattered if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$ and $\rho(t) < t$, respectively.

We define $\mathbb{T}^k = \mathbb{T} \setminus (\rho(b), b]$ and $\mathbb{T}^{k^2} = (\mathbb{T}^k)^k$ and whenever we write $[a, b]$, we mean $[a, b] \cap \mathbb{T}$ for a given time scale $\mathbb{T}$.

The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) = \sigma(t) - t, \text{ for all } t \in \mathbb{T}.$$  

We say that a function $f : \mathbb{T} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{T}^k$ if there is a number $f^\Delta(t)$ such that for all $\varepsilon > 0$ there exists a neighborhood $U$ of $t$ (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|, \text{ for all } s \in U.$$  

We call $f^\Delta(t)$ the delta derivative of $f$ at $t$.

For delta differentiable $f$ and $g$, the next formulas hold:

$$f^\sigma(t) = f(t) + \mu(t)f^\Delta(t) \tag{2.1}$$

$$(fg)^\Delta(t) = f^\Delta(t)g^\sigma(t) + f(t)g^\Delta(t)$$

$$= f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t), \tag{2.2}$$

where we abbreviate $f \circ \sigma$ by $f^\sigma$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous if it is continuous at right-dense points and if the left-sided limit exists at left-dense points. We denote the set of all rd-continuous functions by $C^\text{rd}$ or $C^\text{rd}[\mathbb{T}]$, and the set of all delta differentiable functions with rd-continuous derivative by $C^1_\text{rd}$ or $C^1_\text{rd}[\mathbb{T}]$.

It is known that rd-continuous functions possess an antiderivative, i.e., there exists a function $F$ with $F^\Delta = f$, and in this case an integral is defined by $\int_a^b f(t) \Delta t = F(b) - F(a)$. It satisfies

$$\int_t^{\sigma(t)} f(\tau) \Delta \tau = \mu(t)f(t). \tag{2.3}$$

We now present the integration by parts formulas of the delta integral:

**Lemma 2.1.** If $a, b \in \mathbb{T}$ and $f, g \in C^\text{rd}_\mathbb{T}$, then

1. $\int_a^b f(\sigma(t))g^\Delta(t) \Delta t = [(fg)(t)]_{t=a}^{t=b} - \int_a^b f^\Delta(t)g(t) \Delta t$;

2. $\int_a^b f(t)g^\Delta(t) \Delta t = [(fg)(t)]_{t=a}^{t=b} - \int_a^b f^\Delta(t)g(\sigma(t)) \Delta t$. 

2
3 On the calculus of variations on time scales

We point out some of the issues which are not completely clear in the results available in the literature. We do not claim the main results to be wrong, we just comment on things which are not clear to us. We hope our remarks to be of some usefulness for the interested reader on the calculus of variations on time scales, and we look forward to readers comments and insights.

3.1 Critical reading of [8]

We start with some comments on [8]. There, the basic problem of the calculus of variations on time scales with variable endpoints, in the class of weak local piecewise $C^1_{rd}$ functions, is studied. Among other things, the following transversality condition is obtained:

$$ (\hat{L}_v(a), -\hat{L}_v(b)) = \nabla K(\hat{y}(a), \hat{y}(b)) + \gamma^T M. \tag{3.1} $$

As the authors point out, the delta derivative of a function is not well defined at a left-scattered maximum point of $T$. In virtue of this, since $L$ depends on $y^\Delta$, the variable $t$ in $L$ is defined “only” in $[a, \rho(b)]$. The Euler-Lagrange equation (in integral form) is

$$ \hat{L}_v(t) = \int_a^t \hat{L}_y(\tau) \Delta \tau + c^T, \ t \in [a, \rho(b)]. \tag{3.2} $$

Proof of (3.1) in [8] uses the fact that $\hat{L}_v(b) = \int_a^b \hat{L}_y(\tau) \Delta \tau + c^T$, which is not included in (3.2) if $b$ is a left-scattered point.

3.2 Critical reading of [2]

Now we discuss [2, Lemma 2.1]. We first enunciate it:

**Lemma 3.1.** ([2, Lemma 2.1]) If $f(t)$ is continuous on $[\rho(a), b]$, where $\rho(a) < b$, and if

$$ \int_{\rho(a)}^b f(t)g(t) \nabla t = 0 $$

for every function $g(t) \in C[\rho(a), b]$ with $g(\rho(a)) = g(b) = 0$, then $f(t) = 0$ for $t \in [\rho(a), b]$.

We claim that the lemma is not proved for $t = \rho(a)$ and $t = b$. To see this let us consider the time scale $T = \{1, 2, 3, 4, 5\}$. Then, every function is continuous. Let $g$ be an arbitrary function such that $g(1) = g(5) = 0$. Define $f$ in $T$ by

$$ f(t) = \begin{cases} 
1 & \text{if } t = 1 \text{ or } t = 5; \\
0 & \text{otherwise}.
\end{cases} $$

Then,

$$ \int_{\rho(a)}^b f(t)g(t) \nabla t = \sum_{t=2}^5 f(t)g(t) = 0, $$

but $f(\rho(a))$ and $f(b)$ are not zero.
3.3 Critical reading of [1]

In what follows, we will make some observations about the proof of the fundamental lemma of the calculus of variations and the derivation of the Euler-Lagrange equation in [1]. The fundamental lemma for the one independent variable problem of the calculus of variations on time scales, as stated in [1], is (here we do not use the notation of [1]):

**Lemma 3.2.** ([1, Lemma 8]) If \( M \) is a continuous function on \([a, b]^{k^2}\), and if
\[
\int_a^b M(t)\eta''(t)\Delta t = 0
\]
for all \( \eta \in C^1 \) with \( \eta(a) = \eta(b) = 0 \), then \( M(t) = 0 \) on \([a, b]^{k^2}\).

We first note that in differential calculus, a \( C^1 \) function is (by definition) a function that possess a derivative and this derivative is continuous. Therefore, it is natural that in the context of time scales, the notation \( C^1 \) should mean that a function has a delta derivative and that the delta derivative is continuous.

The proof of the lemma is made in various steps (cases). Essentially, it is made by contradiction, i.e., assuming that \( M(t_0) > 0 \) for some \( t_0 \in [a, b]^{k^2} \). It is constructed a function \( \eta \) that is positive in the same points of \( M \) and zero at the other points. In case A of the proof in [1], the authors intend to show that the lemma is true for left-dense points (independently of the points being right-dense or right-scattered). The point we want to emphasize now is that the function constructed in the proof does not necessarily belong to \( C^1 \). The above mentioned function is
\[
\eta(t) = \begin{cases} 
(t - \sigma(u_1))^2(\sigma(t_0) - t)^2 & \text{if } t \in [\sigma(u_1), \sigma(t_0)], \\
0 & \text{otherwise,}
\end{cases}
\]
with \( \sigma(u_1) < t_0 \) (\( \sigma(u_1) \in [t_0 - \delta, t_0] \), for some \( \delta > 0 \)). We now assume that \( t_0 \) is right-scattered. Function \( \eta \) is delta-differentiable in \((\sigma(u_1), \sigma(t_0))\) since it is the product of two delta differentiable functions, and by (2.2)
\[
\eta^\Delta(t) = [(t - \sigma(u_1))^2]^{\Delta}(\sigma(t_0) - t)^2 + (\sigma(t) - \sigma(u_1))^2[(\sigma(t_0) - t)^2]^{\Delta} = (t + \sigma(t) - 2\sigma(u_1))\sigma(t_0) - t)^2 + (\sigma(t) - \sigma(u_1))^2(t + \sigma(t) - 2\sigma(t_0)).
\] (3.3)

What is important to note here is that \( \sigma \) is not continuous at left-dense right-scattered points. This is due to the fact that if \( t \) is such a point, then
\[
\lim_{s \to t^-} \sigma(s) = t < \sigma(t).
\]

By (3.3), it immediately follows that \( \eta^\Delta \) is not continuous at \( t_0 \). Clearly, analogous observations to those above can be made to the double integral problem.

Now we turn our attention to [1, page 48]. There, after applying the integration by parts formula with respect to \( x \), it appears the following term within a formula (see [1, page 48] for a better comprehension of the notation):

\[
\int_a^b M(t)\eta''(t)\Delta t = 0
\]
\[
\int_c^d L_p(b, y, \dot{z}(b), \tau(y)), \dot{z}(b, \tau(y)), \dot{z}(\sigma(b), y))\zeta(b, y)\Delta y.
\]

By definition, \(b = \max X\), so by the same reasoning as we did before, we may say that \(\dot{z}(b, \tau(y))\) doesn’t make sense. However, it is possible to bypass this problem and we shall show how.

Multiple integration on time scales was introduced in [4]. There, it is defined the double Riemann integral. We use this concept and we define the problem of minimizing the functional

\[
J(u) = \int_{a_1}^{b_1} \int_{a_2}^{b_2} L(t_1, t_2, u, \sigma_1(t), \sigma_2(t_2)), u^{\Delta_1}(t_1, \sigma_2(t_2)), u^{\Delta_2}(\sigma_1(t_1), t_2))\Delta t_2\Delta t_1,
\]

among all the functions \(u\) that have partial derivatives of the second order with respect to its arguments, and that become a given continuous function on the boundary of \(R = [a_1, b_1] \times [a_2, b_2] \subset T_1 \times T_2\). We assume that \(\sigma_1\) and \(\sigma_2\) are delta differentiable. This immediately implies that \(\au(t_1, t_2)\) is a continuous function for \((t_1, t_2) \in R\). Further, let \(L(t_1, t_2, y_0, y_1, y_2) : T_1^k \times T_2^k \times \mathbb{R}^3 \rightarrow \mathbb{R}\) have the necessary smoothness properties in order for the calculations made below to make sense.

**Remark 3.1.** If \(f(t_1, t_2)\) is continuous in a “rectangle” \(R\), then we can interchange the order of integration in

\[
\int \int_R f(t_1, t_2)\Delta t_1\Delta t_2.
\]

We only want to show how to eliminate the problem that appears in [1]; so we start with

\[
\int_{a_1}^{b_1} \int_{a_2}^{b_2} \left[L_{y_0}(\cdot)\eta(\sigma_1(t_1), \sigma_2(t_2)) + L_{y_1}(\cdot)\eta^{\Delta_1}(t_1, \sigma_2(t_2)) + L_{y_2}(\cdot)\eta^{\Delta_2}(\sigma_1(t_1), t_2)\right]\Delta t_2\Delta t_1,
\]

where \((\cdot) = (t_1, t_2, \tilde{u}(\sigma_1(t_1), \sigma_2(t_2)), \tilde{u}^{\Delta_1}(t_1, \sigma_2(t_2)), \tilde{u}^{\Delta_2}(\sigma_1(t_1), t_2))\) and \(\eta\) is a function that have partial derivatives of the second order with respect to its arguments and is zero on the boundary of \(R\). We proceed as follows,

\[
\int_{a_1}^{b_1} \int_{a_2}^{b_2} \left[L_{y_0}(\cdot)\eta(\sigma_1(t_1), \sigma_2(t_2)) + L_{y_1}(\cdot)\eta^{\Delta_1}(t_1, \sigma_2(t_2)) + L_{y_2}(\cdot)\eta^{\Delta_2}(\sigma_1(t_1), t_2)\right]\Delta t_2\Delta t_1
\]

\[
= \int_{a_1}^{b_1} \int_{a_2}^{b_2} \left[L_{y_0}(\cdot)\eta(\sigma_1(t_1), \sigma_2(t_2)) + L_{y_1}(\cdot)\eta^{\Delta_1}(t_1, \sigma_2(t_2)) + L_{y_2}(\cdot)\eta^{\Delta_2}(\sigma_1(t_1), t_2)\right]\Delta t_2\Delta t_1
\]

\[
+ \int_{a_1}^{b_1} \int_{a_2}^{b_2} L_{y_0}(\cdot)\eta(\sigma_1(t_1), \sigma_2(t_2)) + L_{y_1}(\cdot)\eta^{\Delta_1}(t_1, \sigma_2(t_2)) + L_{y_2}(\cdot)\eta^{\Delta_2}(\sigma_1(t_1), t_2)\Delta t_2\Delta t_1
\]

\[
= \int_{a_2}^{b_2} \int_{a_1}^{b_1} \left[L_{y_0}(\cdot)\eta(\sigma_1(t_1), \sigma_2(t_2)) + L_{y_1}(\cdot)\eta^{\Delta_1}(t_1, \sigma_2(t_2)) + L_{y_2}(\cdot)\eta^{\Delta_2}(\sigma_1(t_1), t_2)\right]\Delta t_1\Delta t_2
\]

\[
+ \int_{a_2}^{b_2} \int_{a_1}^{b_1} \left[L_{y_0}(\cdot)\eta(\sigma_1(t_1), \sigma_2(t_2)) + L_{y_1}(\cdot)\eta^{\Delta_1}(t_1, \sigma_2(t_2)) + L_{y_2}(\cdot)\eta^{\Delta_2}(\sigma_1(t_1), t_2)\right]\Delta t_1\Delta t_2
\]

\[
+ \int_{a_2}^{b_2} \int_{a_1}^{b_1} \left[L_{y_0}(\cdot)\eta(\sigma_1(t_1), \sigma_2(t_2)) + L_{y_1}(\cdot)\eta^{\Delta_1}(t_1, \sigma_2(t_2)) + L_{y_2}(\cdot)\eta^{\Delta_2}(\sigma_1(t_1), t_2)\right]\Delta t_2\Delta t_1.
\]

(3.6)
The first double integral in the last equality becomes

\[
\int_{a_2}^{\rho_2(b_2)} \int_{a_1}^{\rho_1(b_1)} \left[ L_{y_0} (\cdot) \eta(\sigma_1(t_1), \sigma_2(t_2)) + L_{y_1} (\cdot) \eta^{\Delta_1}(t_1, \sigma_2(t_2)) + L_{y_2} (\cdot) \eta^{\Delta_2}(\sigma_1(t_1), t_2) \right] \Delta t_1 \Delta t_2
\]

\[
= \int_{a_2}^{\rho_2(b_2)} \int_{a_1}^{\rho_1(b_1)} \left[ L_{y_0} (\cdot) \eta(\sigma_1(t_1), \sigma_2(t_2)) - L_{y_1} (\cdot) \eta^{\Delta_1}(t_1, \sigma_2(t_2)) - L_{y_2} (\cdot) \eta^{\Delta_2}(\sigma_1(t_1), t_2) \right] \Delta t_1 \Delta t_2
\]

\[
+ \int_{a_2}^{\rho_2(b_2)} \int_{a_1}^{\rho_1(b_1)} \left\{ \left[ L_{y_1} (\cdot) \eta(t_1, \sigma_2(t_2)) \right]^{\Delta_1} + \left[ L_{y_2} (\cdot) \eta(\sigma_1(t_1), t_2) \right]^{\Delta_2} \right\} \Delta t_1 \Delta t_2
\]

\[
= \int_{a_2}^{\rho_2(b_2)} \int_{a_1}^{\rho_1(b_1)} \left[ L_{y_0} (\cdot) - L_{y_1} (\cdot) \eta^{\Delta_1}(\cdot) - L_{y_2} (\cdot) \eta^{\Delta_2}(\cdot) \right] \eta(\sigma_1(t_1), \sigma_2(t_2)) \Delta t_1 \Delta t_2
\]

\[
+ \int_{a_2}^{\rho_2(b_2)} \int_{a_1}^{\rho_1(b_1)} \left[ L_{y_1} (\cdot) \eta(t_1, \sigma_2(t_2)) \right]^{t_1=\rho_1(b_1)}_{t_1=a_1} \Delta t_2 + \int_{a_1}^{\rho_1(b_1)} \left[ L_{y_2} (\cdot) \eta(\sigma_1(t_1), t_2) \right]^{t_2=\rho_2(b_2)}_{t_2=a_2} \Delta t_1
\]

\[
= \int_{a_2}^{\rho_2(b_2)} \int_{a_1}^{\rho_1(b_1)} \left[ L_{y_0} (\cdot) - L_{y_1} (\cdot) \eta^{\Delta_1}(\cdot) - L_{y_2} (\cdot) \eta^{\Delta_2}(\cdot) \right] \eta(\sigma_1(t_1), \sigma_2(t_2)) \Delta t_1 \Delta t_2
\]

\[
+ \int_{a_2}^{\rho_2(b_2)} \int_{a_1}^{\rho_1(b_1)} \left[ L_{y_1} (\cdot) \eta(\sigma_1(t_1), \sigma_2(t_2)) \right]^{t_2=\rho_2(b_2)}_{t_2=a_2} \Delta t_2 + \int_{a_1}^{\rho_1(b_1)} \left[ L_{y_2} (\cdot) \eta(\sigma_1(t_1), \sigma_2(t_2)) \right] \Delta t_1,
\]

(3.7)

where \((\rho_1(b_1))\cdot, (\rho_2(t_2))\) is equal to \(\cdot\) with \(t_1 = \rho_1(b_1), t_2 = \rho_2(b_2)\), respectively.

The second double integral in (3.6) becomes

\[
\int_{a_2}^{\rho_2(b_2)} \int_{a_1}^{\rho_1(b_1)} \left[ L_{y_0} (\cdot) \eta(\sigma_1(t_1), \sigma_2(t_2)) + L_{y_1} (\cdot) \eta^{\Delta_1}(t_1, \sigma_2(t_2)) + L_{y_2} (\cdot) \eta^{\Delta_2}(\sigma_1(t_1), t_2) \right] \Delta t_1 \Delta t_2
\]

\[
= \int_{a_2}^{\rho_2(b_2)} \mu_1(\rho_1(b_1)) \left\{ L_{y_1} (\rho_1(b_1)) \eta^{\Delta_1}(\rho_1(b_1), \sigma_2(t_2)) + L_{y_2} (\rho_1(b_1)) \eta^{\Delta_2}(\rho_1(b_1), t_2) \right\} \Delta t_2,
\]

(3.8)

because \(L_{y_0}(\rho_1(b_1)) \eta(\sigma_1(\rho_1(b_1)), \sigma_2(t_2)) = L_{y_0}(\rho_1(b_1)) \eta(\rho_1(b_1), \sigma_2(t_2)) = 0\). Now, note that

\[
\eta(b_1, \sigma_2(t_2)) = \eta(\rho_1(b_1), \sigma_2(t_2)) + \mu_1(\rho_1(b_1)) \eta^{\Delta_1}(\rho_1(b_1), \sigma_2(t_2)),
\]

hence \(\mu_1(\rho_1(b_1)) \eta^{\Delta_1}(\rho_1(b_1), \sigma_2(t_2)) = -\eta(\rho_1(b_1), \sigma_2(t_2))\). Therefore, (3.8) becomes

\[
\int_{a_2}^{\rho_2(b_2)} \left\{ -L_{y_1}(\rho_1(b_1)) \eta(\rho_1(b_1), \sigma_2(t_2)) + \mu_1(\rho_1(b_1)) L_{y_2}(\rho_1(b_1)) \eta^{\Delta_2}(\rho_1(b_1), t_2) \right\} \Delta t_2.
\]

(3.9)

Now,

\[
\int_{a_2}^{\rho_2(b_2)} L_{y_2}(\rho_1(b_1)) \eta^{\Delta_2}(\rho_1(b_1), t_2) \Delta t_2
\]

\[
= \int_{a_2}^{\rho_2(b_2)} L_{y_2}(\rho_1(b_1)) \eta^{\Delta_2}(b_1, t_2) \Delta t_2
\]

\[
= [L_{y_2}(\rho_1(b_1)) \eta(b_1, t_2)]^{t_2=\rho_2(b_2)}_{t_2=a_2} - \int_{a_2}^{\rho_2(b_2)} L_{y_2}^{\Delta_2}(b_1) \eta(b_1, \sigma_2(t_2)) \Delta t_2
\]

\[
= 0,
\]

hence (3.9) becomes

\[
\int_{a_2}^{\rho_2(b_2)} -L_{y_1}(\rho_1(b_1)) \eta(\rho_1(b_1), \sigma_2(t_2)) \Delta t_2.
\]

(3.10)
Finally, the third integral in (3.6) becomes (repeating analogous steps as above)

\[
\int_{a_1}^{b_1} \int_{\rho_2(b_2)}^{\rho_1(b_1)} \left[ L_{y_0}(\cdot)\eta(\sigma_1(t_1), \sigma_2(t_2)) + L_{y_1}(\cdot)\eta^{\Delta_1}(t_1, \sigma_2(t_2)) + L_{y_2}(\cdot)\eta^{\Delta_2}(\sigma_1(t_1), t_2) \right] \Delta t_2 \Delta t_1
\]

\[
= \int_{a_1}^{b_1} \mu_2(\rho_2(b_2)) \left[ L_{y_1}(\cdot)\rho_2(b_2))\eta^{\Delta_1}(t_1, b_2) + L_{y_2}(\cdot)\rho_2(b_2))\eta^{\Delta_2}(\sigma_1(t_1), \rho_2(b_2)) \right] \Delta t_1
\]

\[
= \int_{a_1}^{b_1} \mu_2(\rho_2(b_2)) \left[ L_{y_1}(\cdot)\rho_2(b_2))\eta^{\Delta_1}(t_1, b_2) + L_{y_2}(\cdot)\rho_2(b_2))\eta^{\Delta_2}(\sigma_1(t_1), \rho_2(b_2)) \right] \Delta t_1
\]

\[
+ \mu_1(\rho_1(b_1))\mu_2(\rho_2(b_2)) \left[ L_{y_1}(\rho_1(b_1), \rho_2(b_2))\eta^{\Delta_1}(\rho_1(b_1), b_2) + L_{y_2}(\rho_1(b_1), \rho_2(b_2))\eta^{\Delta_2}(b_1, \rho_2(b_2)) \right]
\]

\[
= \int_{a_1}^{b_1} -L_{y_1}(\cdot)\rho_2(t_2))\eta(\sigma_1(t_1), \rho_2(t_2)) \Delta t_1,
\]

(3.11)

where \((\rho_1(b_1), \rho_2(b_2))\) are the arguments of \((t_1, t_2)\) on \(\cdot\). Combining (3.5), (3.6), (3.7), (3.10) and (3.11), we obtain

\[
\int_{a_2}^{b_2} \int_{a_1}^{\rho_1(b_1)} \left[ L_{y_0}(\cdot) - L_{y_1}(\cdot) - L_{y_2}(\cdot) \right] \eta(\sigma_1(t_1), \sigma_2(t_2)) \Delta t_1 \Delta t_2.
\]

(3.12)

**Remark 3.2.** A version of the fundamental lemma of the calculus of variations on time scales can be proved in order to deduce the Euler-Lagrange equation from (3.12). Note that, since we assumed that \(\sigma_i (i = 1, 2)\) are \(\Delta_i\)-differentiable, it is not possible to have left-dense, right-scattered points at the same time within the time scales \(\mathbb{T}_i\).

### 3.4 Critical reading of [5]

Finally, we want to make some observations regarding the paper [5].

**Lemma 3.3.** ([5] Lemma 5.1) If \(M(x, y)\) is continuous on \(E \cup \Gamma\) with

\[
\int \int_E M(x, y)\eta(\sigma_1(x), \sigma_2(y)) \Delta_1 x \Delta_2 y = 0
\]

for every admissible variation \(\eta\), then

\[
M(x, y) = 0, \text{ for all } (x, y) \in E^0.
\]

We claim that this lemma is not proved. To see this, consider

\[
E = [0, 5] \cap \mathbb{Z} \times [0, 5] \cap \mathbb{Z}.
\]

Suppose that we want to prove the lemma for the point \((1, 1) \in E^0\). Continuity of \(M\) ensures that \(M\) is positive in \(\Omega = [1, \sigma_1(1)] \times [1, \sigma_2(1)]\). To get a contradiction, the authors created the function

\[
\eta(x, y) = \begin{cases} 
(x-x_0)^2(x-\sigma_1(x_1)]^2(y-y_0)^2[y-\sigma_2(y_1)]^2 & \text{for } (x, y) \in \Omega, \\
0 & \text{for } (x, y) \in E \setminus \Omega,
\end{cases}
\]

and concluded that

\[
\int \int_E M(x, y)\eta(\sigma_1(x), \sigma_2(y)) \Delta_1 x \Delta_2 y = \int \int_{\Omega} M(x, y)\eta(\sigma_1(x), \sigma_2(y)) \Delta_1 x \Delta_2 y > 0.
\]
With the notation used, we have $x_0 = y_0 = 1$. Now note that the set $\Omega$ consists of only the point $(x_0, y_0)$, and hence $\eta$ is zero in all its domain. Therefore,

$$\int \int_E M(x, y)\eta(\sigma_1(x), \sigma_2(y))\Delta_1 x \Delta_2 y = 0,$$

which is not a contradiction.

There is another point that we would like to mention. The authors define an *admissible variation* to be a function in $C^{(1)}_{rd}(E \cup \Gamma)$ that satisfies $\eta = 0$ on $\Gamma$. The set $C^{(1)}_{rd}$ consists of all continuous functions for which both the $\Delta_1$-partial derivative and the $\Delta_2$-partial derivative exist and are of class $C_{rd}$.

When the authors are deducing the Euler-Lagrange equation, a crucial step is the application of Green’s formula to the expression

$$\int \int_E \left\{ \frac{\partial}{\Delta_1 x}[L_p(\cdot)\eta(x, \sigma_2(y))] + \frac{\partial}{\Delta_2 y}[L_q(\cdot)\eta(\sigma_1(x), y)] \right\} \Delta_1 x \Delta_2 y.$$

In order to apply Green’s formula, we must have the partial delta derivatives in the previous integral, continuous (see [5, Theorem 2.25]). If $\eta \in C^{(1)}_{rd}(E \cup \Gamma)$, why should those partial derivatives be continuous? A justification to this fact could be that the authors, when enunciating Theorem 5.2 (Euler’s necessary condition), assume that the admissible functions have continuous partial delta derivatives of the second order, hence it would be implicit that the admissible variations should belong to the same class. However, if this would be the case, why can Lemma 5.1 be applied to formula

$$\int \int_E \left\{ L_u(\cdot) - \frac{\partial}{\Delta_1 x}L_p(\cdot) - \frac{\partial}{\Delta_2 y}L_q(\cdot) \right\} \eta(\sigma_1(x), \sigma_2(y))\Delta_1 x \Delta_2 y = 0 \ ? \quad (3.13)$$

It is well known that if a function $f$ is continuous then $f \in C_{rd}$, but the converse is not necessarily true. So, in order to apply (a version of) Lemma 5.1 to (3.13) it would be necessary to prove it for the class of functions $\eta$ that have continuous partial delta derivatives of the second order. In particular, Lemma 5.1 would be a corollary from this previous one.

More could be said, in particular about our own work [7] which is being rewritten in order to make it correct!

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