EIGENVALUES OF THE DRIFTED LAPLACIAN ON COMPLETE METRIC MEASURE SPACES

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Abstract. Let $(M^n, g, e^{-f} dv)$ be a complete smooth metric measure space with the $(\infty)$-Bakry-Émery Ricci curvature tensor $\text{Ric}_f \geq \frac{a}{2} g$, where constant $a$ is positive. It is known that the spectrum of the drifted Laplacian $\Delta_f$ on $M$ is discrete and the first nonzero eigenvalue of $\Delta_f$ has lower bound $\frac{a}{2}$. In this paper, we proved that if the lower bound $\frac{a}{2}$ is achieved with multiplicity $k$, $1 \leq k \leq n$, then $M$ is isometric to $\Sigma^{n-k} \times \mathbb{R}^k$ for some complete $(n-k)$-dimensional manifold $\Sigma$.

1. Introduction

The well-known Lichnerowicz theorem [13] states that if the Ricci curvature of a closed, i.e., compact and without boundary, Riemannian manifold $(M^n, g)$ of dimension $n \geq 2$ satisfies $\text{Ric} \geq (n-1)a$, where $a$ is a positive constant, then the first nonzero eigenvalue of the Laplacian $\Delta$ satisfies $\lambda_1 \geq na$. Obata’s Theorem [20] states that equality holds if and only if the manifold is an $n$-dimensional sphere with constant sectional curvature $a$. Observe that if a complete Riemannian manifold $M$ has Ricci curvature bounded from below by a positive constant, then $M$ must be compact.

One may ask whether a phenomenon corresponding to Lichnerowicz-Obata’s theorem would happen for complete smooth metric measure spaces $(M^n, g, e^{-f} dv)$ with the $(\infty)$-Bakry-Émery Ricci curvature tensor $\text{Ric}_f = \text{Ric} + \nabla^2 f \geq \frac{a}{2} g$, where $a$ is a positive constant. Recall that a complete smooth metric measure space $(M^n, g, e^{-f} dv)$ is a complete $n$-dimensional Riemannian manifold $(M^n, g)$ together with a weighted volume form $d\mu = e^{-f} dv$ on $M$, where $f$ is a smooth function on $M$ and $dv$ the volume element induced by the metric $g$. For an $(M, g, e^{-f} dv)$, a suitable operator on $M$ is the drifted Laplacian $\Delta_f = \Delta - \langle \nabla f, \nabla \cdot \rangle$ since it is a densely defined self-adjoint operator in the space $L^2(e^{-f} dv)$ of square integrable functions on $M$ with respect to the measure $e^{-f} d\sigma$.

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that is, for \( u, w \in C^\infty_c(M) \),
\[
\int_M u \Delta_f w e^{-f} dv = -\int_M \langle \nabla u, \nabla w \rangle e^{-f} dv.
\]

For \((M, g, e^{-f} dv)\), when \(\text{Ric}_f \geq \frac{a}{2} g\), constant \(a > 0\). It is known that \(M\) is not necessary to be compact. One of examples is the shrinking Gaussian soliton \((\mathbb{R}^n, g_{\text{can}}, |x|^2)\) with the canonical Euclidean metric \(g_{\text{can}}\), \(f = \frac{|x|^2}{4}\). Hence one must deal with complete manifolds including noncompact case, which is different from Lichnerowicz-Obata’s theorem. A basic fact is the discreteness of the spectrum of \(\Delta_f\) for \((M, g, e^{-f} dv)\) with \(\text{Ric}_f \geq \frac{a}{2} g\), \(a > 0\), which was pointed out by Hein-Naber in [10] (see Theorem 4 of the present paper). So the spectrum of \(\Delta_f\) is just the set of points which are both eigenvalues of \(\Delta_f\) in \(L^2(e^{-f} dv)\) with finite multiplicity and the isolated points of the spectrum (cf [9] Definition 10.1). Since the weighted volume \(\int_M e^{-f} dv\) of \(M\) is finite ([16], cf [23]), 0 is the least eigenvalue with multiplicity one and nonzero constant functions are the associated eigenfunctions. Thus the set of all eigenvalues of \(\Delta_f\), counted with multiplicity, is an increasing sequence
\[
0 = \lambda_0(\Delta_f) < \lambda_1(\Delta_f) \leq \cdots
\]
with \(\lambda_i(\Delta_f) \to \infty\) as \(i \to \infty\). Moreover, there exists a countable orthonormal base \(\{\psi_i\}\) of \(L^2(e^{-f} dv)\) so that each \(\psi_i\) is an eigenvector of \(\Delta_f\) associated with the eigenvalue \(\lambda_i\).

On the other hand, it has been known that for such \(M\), a Poincaré inequality holds on it with spectrum gap \(\frac{a}{2}\), cf [13], [22] (see Section 2 for details). Therefore an observation of the above facts leads the following Lichnerowicz type theorem for \(\Delta_f\):

**Theorem 1.** (Barky-Émery [1]) Let \((M^n, g, e^{-f} dv)\) be a complete smooth metric measure space with \(\text{Ric}_f \geq \frac{a}{2} g\), where \(a\) is a positive constant. Then the first nonzero eigenvalue, denoted by \(\lambda_1(\Delta_f)\), of \(\Delta_f\) on \(M\) in \(L^2(e^{-f} dv)\) satisfies
\[
\lambda_1(\Delta_f) \geq \frac{a}{2}.
\]

Observe that the lower bound \(\frac{a}{2}\) can be achieved by some \((M, g, e^{-f} dv)\), for instance:

**Example 1.** Gauss shrinking soliton \((\mathbb{R}^n, g_{\text{can}}, |x|^2)\) with \(\text{Ric}_f = \frac{1}{2}\), \(\lambda_1(\Delta_f) = \frac{1}{2}\) with multiplicity \(n\).
Example 2. Cylinder shrinking solitons: $S^{n-k}(\sqrt{2(n-k-1)}) \times \mathbb{R}^k$, $n-k \geq 2, k \geq 1$ with the product metric and $f = \frac{|t|^2}{4}, t \in \mathbb{R}^k$. Here $S^{n-k}(\sqrt{2(n-k-1)})$ is the $(n-k)$-dimensional round sphere with radius $\sqrt{2(n-k-1)}$. By a direct calculation, $\text{Ric}_f = \frac{1}{2}, \lambda_1(\Delta_f) = \frac{1}{2}$ with multiplicity $k$.

In this paper we study the rigidity of equality in (1) and give the geometric characteristic of the equality. More precisely, we prove that

Theorem 2. Let $(M^n, g, e^{-f}dv)$ be a complete smooth metric measure space with $\text{Ric}_f \geq \frac{a}{2}g$, where $a$ is a positive constant. If the equality in (1) holds and $\lambda_1(\Delta_f) = \frac{a}{2}$ with multiplicity $k$, $1 \leq k \leq n$, then $M$ is a noncompact manifold which is isometric to $\Sigma^{n-k} \times \mathbb{R}^k$ with the product metric for some complete $(n-k)$-dimensional manifold $\Sigma$ satisfying that $\text{Ric}_\Sigma \geq \frac{a}{2}$ and $\lambda_1(\Delta_\Sigma) > \frac{a}{2}$. Also, the function $f$ must be, by an isometry, for $(x_1, \ldots, x_{n-k+1}, t_1, \ldots, t_k) \in \Sigma^{n-k} \times \mathbb{R}^k$,

$$f(x_1, \ldots, x_{n-k+1}, t_1, \ldots, t_k) = f(x_1, \ldots, x_{n-k+1}, 0, \ldots, 0) + \frac{a}{4} \sum_{i=1}^{k} t_i^2.$$

In the above $\text{Ric}_\Sigma$ and $\lambda_1(\Delta_\Sigma)$ denote the Barky-Émery Ricci curvature of $\Sigma$ and the first nonzero eigenvalue of the drifted Laplacian $\Delta_f$ on $\Sigma$ respectively. Here the restriction of $f$ on $\Sigma$ is still denoted by $f$.

From Theorem 2 we see that if the lower bound $\frac{a}{2}$ of the first nonzero eigenvalue is achieved with multiplicity $k$, then $M$ must split off an Euclidean space according to the multiplicity. To prove Theorem 2 we use the Bochner formula, the approach taken to prove the Lichnerowicz-Obata theorem. This method lets us be able to not only prove Theorem 2 but also to give a different proof of Theorem 1. Theorem 2 also gives the rigidity of spectrum gap or Poincaré inequality. It is known that for $(M^n, g, e^{-f}dv)$ with $\text{Ric}_f \geq \frac{a}{2}g$, the Poincaré inequality holds, that is, for all $u \in H^1_f(M)$ with $\int_M u e^{-f}dv = 0$, it holds that

$$\int_M |\nabla u|^2 e^{-f}dv \geq \frac{a}{2} \int_M u^2 e^{-f}dv,$$

where $H^1_f(M)$ denotes the weighted Sobolev space (see its definition in Section 2). Theorem 2 can be written in the following equivalent form:

Theorem 3. Let $(M^n, g, e^{-f}dv)$ be a complete smooth metric measure space with $\text{Ric}_f \geq \frac{a}{2}g$, where $a$ is a positive constant. Let $E$ denote the subspace of the functions in $H^1_f(M)$ so that Poincaré inequality (2) achieves the equality. If the dimension of $E$ is $k$, $1 \leq k \leq n$, then $M$
and $f$ must be the same as the ones respectively in the conclusion in Theorem 2.

Further, in Section 4, we discuss the special case: gradient shrinking Ricci solitons, namely $\text{Ric}_f = \frac{1}{2}$. We obtain an upper bound estimate of $\lambda_1(\Delta_f)$ for noncompact gradient shrinking Ricci solitons (Theorem 5). In addition, in Section 5 we discuss self-shrinkers in the Euclidean space $\mathbb{R}^{n+1}$. We apply the results in Section 2 to self-shrinkers of mean curvature flow in the Euclidean space $\mathbb{R}^{n+1}$. We show that if the maximum of the principal curvatures of an immersed self-shrinker in $\mathbb{R}^{n+1}$ is strictly less than $\frac{1}{2}$, then the immersion is proper and the spectrum of the induced drifted Laplacian is discrete (Corollary 4).

Finally we would like to mention that for compact Riemannian manifolds with $\text{Ric}_f \geq \frac{a}{2}$, Andrews-Ni [2], and Futaki-Li-Li [7] obtained lower bound estimates for $\lambda_1(\Delta_f)$, which depends on the diameter of the manifolds. The authors of [2] stated, by constructing examples, that their estimates are sharp when Bakry-Émery Ricci tensor is not constant (see details in [8], [2] and [7]). On the other hand, the spectrum properties of $\Delta_f$ and the Laplacian $\Delta$ on complete $(M, g, e^{-f}dv)$ with various conditions on $\text{Ric}_f$ have been studied (cf [4], [14], [17], [18], [19], [12] and the references therein). One interesting fact is that contrary to the drifted laplacian, the essential spectrum of the Laplacian for the gradient shrinking Ricci solitons is $[0, +\infty)$ ([14], [6]).

The rest of this paper is organized as follows: In Section 2 as a preparation, we give the proof of Theorem 2. In Section 3 we prove Theorem 2. In Section 4 we prove Theorem 5. In Section 5 we prove Corollary 4. In Appendix, we prove Proposition 1 for the sake of completeness of proof.

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2. Compact embedding of weighted Sobolev space and discreteness of spectrum

In this section, we give some known results as preparation. Let $(M^n, g, e^{-f}dv)$ be an $n$-dimensional complete smooth metric measure space. Denote by $\mu$ the measure induced by the weighted volume element $e^{-f}dv$, i.e., $d\mu = e^{-f}dv$. Suppose that $\mu(M) = \int_M e^{-f}dv < \infty$. Denote by $L^2_f(M)$ and $H^1_f(M)$ the closures of the space $C^\infty_c(M)$ of the smooth compactly-supported functions on $M$, with respect to the
following norms respectively:

\[ \|u\|_{L^2_f(M)} := \left( \int_M u^2 d\mu \right)^{\frac{1}{2}}, \]

\[ \|u\|_{H^1_f(M)} := \left( \int_M (u^2 + |\nabla u|^2) d\mu \right)^{\frac{1}{2}}. \]

\( L^2_f(M) \) and \( H^1_f(M) \) are Hilbert spaces. In particular, \( H^1_f(M) \) is Sobolev space with respect to measure \( d\mu = e^{-f} dv \). Further, suppose that a logarithmic Sobolev inequality holds on \((M, g, e^{-f} dv)\), that is, there exists a positive constant \( C \) such that,

\[ \int_M u^2 \log u^2 d\mu \leq C \int_M |\nabla u|^2 d\mu, \]

for all \( u \in C^\infty_c(M) \) satisfying \( \int_M u^2 d\mu = \mu(M) \).

With the above assumptions, it is known that the embedding of \( H^1_f(M) \hookrightarrow L^2_f(M) \) is compactly embedded and equivalently the spectrum of \( \Delta_f \) on \( L^2_f(M) \) is discrete.

**Proposition 1.** Let \((M, g, e^{-f} dv)\) be a complete smooth metric measure space with finite total measure \( \mu(M) = \int_M e^{-f} dv \). Suppose that the logarithmic Sobolev inequality (3) holds on \((M, g, e^{-f} dv)\). Then the inclusion \( H^1_f(M) \subset L^2_f(M) \) is compactly embedded and equivalently the spectrum of \( \Delta_f \) on \( L^2_f(M) \) is discrete.

When \((M, g, e^{-f} dv)\) has Ric \( \geq \frac{a}{2} \), \( a > 0 \), Bakry-Émery [1] showed that a logarithmic Sobolev inequality (3) with \( C = \frac{4}{a} \) is satisfied if the weighted volume \( \mu(M) = \int_M e^{-f} dv \) is finite. Recently it was obtained by Morgan [16] that its weighted volume \( \int_M e^{-f} dv \) is finite (also see its proof in [23]). Hence the following logarithmic Sobolev inequality holds on \( M \):

\[ \int_M u^2 \log u^2 d\mu \leq \frac{4}{a} \int_M |\nabla u|^2 d\mu, \]

for all \( u \in C^\infty_c(M) \) satisfying \( \int_M u^2 d\mu = \mu(M) \).

These facts together with Proposition 1 imply the following result.

**Theorem 4.** (Hein-Naber) Let \((M, g, e^{-f} dv)\) be a complete smooth metric measure space with Ric \( \geq \frac{a}{2} \), where constant \( a \) is positive. Then the inclusion \( H^1_f(M) \subset L^2_f(M) \) is compactly embedded and equivalently the spectrum of \( \Delta_f \) in \( L^2_f(M) \) is discrete.
3. First nonzero eigenvalue estimate

Before we prove Theorem 2, we include a proof of Theorem 1 using Poincaré inequality. It is known that in general for a complete measure space \((M, g, \mu)\) with finite total measure, a logarithmic Sobolev inequality \((1)\) with constant \(C\) will implies a Poincaré inequality (cf [11] Prop 2.1), that is, for any \(u \in H_1^f(M)\),

\[
\int_M u^2 d\mu \leq \frac{C}{2} \int_M |\nabla u|^2 d\mu, \int_M ud\mu = 0.
\]

Here constant \(\frac{2}{C}\) is so-called spectrum gap.

**Proof of Theorem 1.** We have known that the weighted volume \(\mu(M) = \int_M e^{-f} dv\) is finite and the logarithmic Sobolev inequality \((1)\) holds. So Poincaré inequality holds with spectrum gap \(\frac{a}{2}\). By Theorem 4, the spectrum of \(\Delta_f\) is discrete and hence the variational characterization of \(\lambda_1(\Delta_f)\) states that

\[
\lambda_1(\Delta_f) = \inf_{u \not\equiv 0, u \in H_1^f(M)} \left\{ \frac{\int_M |\nabla u|^2 d\mu}{\int_M u^2 d\mu}; \int_M ud\mu = 0 \right\}.
\]

By Poincaré inequality, we get that the lower bound of \(\lambda_1(\Delta_f)\) is just spectrum gap \(\frac{a}{2}\).

\[ \square \]

Now we prove Theorem 2 whose proof also give a direct proof of Theorem 1 without using Poincaré inequality.

**Proof of Theorem 2.** It suffices to consider, by a scaling of metric \(g\), the case of \((M, g, e^{-f})\) with \(\text{Ric}_f \geq \frac{1}{2}\). Assume that \(u\) is a nonconstant eigenfunction of \(\Delta_f\) corresponding to an eigenvalue \(\lambda\), i.e.,

\[
\Delta_f u + \lambda u = 0, \quad \int_M u^2 d\mu < \infty,
\]

where \(d\mu = e^{-f} dv\). It is known that \(u \in H_1^f \cap C^\infty(M)\). By \(\text{Ric}_f \geq \frac{1}{2}\), \(\square\) and the weighted Bochner formula:

\[
\Delta_f |\nabla u|^2 = |\nabla^2 u|^2 + \langle \nabla u, \nabla (\Delta_f u) \rangle + \text{Ric}_f(\nabla u, \nabla u),
\]

we have

\[
\Delta_f |\nabla u|^2 \geq |\nabla^2 u|^2 + \left( \frac{1}{2} - \lambda \right) |\nabla u|^2.
\]

If \(M\) is compact, integrating \(\square\), we have

\[
\int_M |\nabla^2 u|^2 d\mu + \int_M \left( \frac{1}{2} - \lambda \right) |\nabla u|^2 d\mu \leq 0.
\]
If $M$ is noncompact, we claim that (8) also holds. Given a fixed point $p \in M$, let $B_R$ denote the geodesic sphere of $M$ of radius $R$ centered at $p$. Let $\phi$ be the nonnegative cut-off function satisfying that $\phi$ is 1 on $B_R$, $|\nabla \phi| \leq 1$ on $B_{R+1} \setminus B_R$, and $\phi = 0$ on $\Sigma \setminus B_{R+1}$. Multiplying (7) by $\phi^2$ and then integrating, we get

$$\int_M \phi^2 \Delta f |\nabla u|^2 d\mu \geq \int_M \phi^2 |\nabla^2 u|^2 d\mu + \left( \frac{1}{2} - \lambda \right) \int_M \phi^2 |\nabla u|^2 d\mu. \tag{9}$$

On the other hand, by the weighted Green formula,

$$\int_M \phi^2 \Delta f |\nabla u|^2 d\mu = - \int_M \langle \nabla \phi^2, \nabla |\nabla u|^2 \rangle d\mu$$

$$= 4 \int_M \phi \langle \nabla \nabla \phi \nabla u, \nabla u \rangle d\mu$$

$$= 4 \int_M \phi (\nabla^2 u)(\nabla \phi, \nabla u) d\mu. \tag{10}$$

By $2ab \leq \varepsilon a^2 + \frac{b^2}{\varepsilon}$, for any $\varepsilon > 0$,

$$2\phi(\nabla^2 u)(\nabla \phi, \nabla u) = 2 \sum_{i,j=1}^n \phi(\nabla^2 u)_{ij} \phi_i u_j$$

$$\leq \sum_{i,j=1}^n \left[ \varepsilon \phi^2 (\nabla^2 u)^2_{ij} + \frac{1}{\varepsilon} \phi_i^2 u_j^2 \right]$$

$$= \varepsilon \phi^2 |\nabla^2 u|^2 + \frac{1}{\varepsilon} |\nabla \phi|^2 |\nabla u|^2. \tag{11}$$

In the above, the subscripts $i, j$ denote the covariant derivatives with respect to $e_i, e_j$ respectively, where $\{e_i\}$ denotes a local orthonormal frame on $M$. Substituting (11) into (10), we have

$$\int_M \phi^2 \Delta f |\nabla u|^2 d\mu \leq 2\varepsilon \int_M \phi^2 |\nabla^2 u|^2 d\mu + \frac{2}{\varepsilon} \int_M |\nabla \phi|^2 |\nabla u|^2 d\mu. \tag{12}$$

Combining (9) and (12), it holds that

$$(1 - 2\varepsilon) \int_M \phi^2 |\nabla u|^2 d\mu \leq \frac{2}{\varepsilon} \int_M |\nabla \phi|^2 |\nabla u|^2 d\mu + (\lambda - \frac{1}{2}) \int_M \phi^2 |\nabla u|^2 d\mu.$$

Noting $\int_M |\nabla u|^2 < \infty$ and letting $R \to \infty$ in the above inequality, by the monotone convergence theorem, we have $\int_M |\nabla^2 u|^2 d\mu < \infty.$
Furthermore, observe that

\[ |2\phi(\nabla^2 u)(\nabla \phi, \nabla u)| = 2| \sum_{i,j=1}^{n} \phi(\nabla^2 u)_{ij} \phi_i u_j | \]

\[ \leq \varepsilon \sum_{i,j=1}^{n} \phi^2(\nabla^2 u)_{ij} |\phi_i| + \frac{1}{\varepsilon} \sum_{i,j=1}^{n} |\phi_i| |u_j^2| \]

\[ \leq \varepsilon \sqrt{n\phi^2} |\nabla^2 u| |\nabla \phi| + \frac{\sqrt{n}}{\varepsilon} |\nabla \phi||\nabla u|^2. \]

So (13) implies that

\[ |\int_M \phi^2 \Delta_f |\nabla u|^2 d\mu| = |4 \int_M \phi(\nabla^2 u)(\nabla \phi, \nabla u) d\mu| \]

\[ \leq 2\varepsilon \sqrt{n} \int_M \phi^2 |\nabla^2 u|^2 |\nabla \phi| d\mu + \frac{2\sqrt{n}}{\varepsilon} \int_M |\nabla \phi||\nabla u|^2 d\mu \]

\[ \leq 2\varepsilon \sqrt{n} \int_{M \setminus \bar{B}_{R+1}} |\nabla^2 u|^2 d\mu + \frac{2\sqrt{n}}{\varepsilon} \int_{M \setminus \bar{B}_{R+1}} |\nabla u|^2 d\mu. \]

Letting \( R \to \infty \) in (14), the right side converges to zero. Thus

\[ \lim_{R \to \infty} \int_M \phi^2 \Delta_f |\nabla u|^2 d\mu \to 0. \]

Letting \( R \to \infty \) in (9), using the dominate convergence theorem, we have (8):

\[ \int_M |\nabla^2 u|^2 d\mu + \left( \frac{1}{2} - \lambda \right) \int_M |\nabla u|^2 d\mu \leq 0. \]

So the claim holds. Since \( u \) is not constant, from (8), \( \lambda \geq \frac{1}{2} \). This implies that \( \lambda_1(\Delta_f) \geq \frac{1}{2} \), as in Theorem 1.

Now we consider the case of the equality. From the proof, \( \lambda = \frac{1}{2} \) if and only if

\[ \nabla^2 u = 0, \]

\[ \text{Ric}_f(\nabla u, \nabla u) = \frac{1}{2} |\nabla u|^2 \]

\[ \Delta_f u + \frac{1}{2} u = 0, \quad \int_M u^2 d\mu < \infty. \]

By (15), \( \Delta u = 0 \) and hence by (17),

\[ - \langle \nabla f, \nabla u \rangle + \frac{1}{2} u = 0. \]

Thus \( u \) is a nonconstant harmonic function and \( M \) must be noncompact. Moreover, (15) together with the fact \( u \) is not constant means
that $\nabla u$ is a nontrivial parallel vector field and hence implies that $M$ must be isometric to a product manifold $\Sigma^{n-1} \times \mathbb{R}$ for some complete manifold $\Sigma$. Besides $u$ is constant on the level set $\Sigma \times \{t\}, t \in \mathbb{R}$. Without lost of generality, suppose that $\Sigma := u^{-1}(\{0\})$. By passing an isometry, we may assume that $M = \Sigma^{n-1} \times \mathbb{R}$. Take $(x, t) \in \Sigma^{n-1} \times \mathbb{R}$.

From (18), $\frac{\partial f}{\partial t} = t^2$. So

$$f(x, t) = \frac{t^2}{4} + f(x, 0).$$

From (19), for any vector field $X \in T\Sigma$ and the unit normal $\nu$ to $\Sigma$, it holds that

$$\nabla^2 f(X, \nu) = 0, \quad \nabla^2 f(\nu, \nu) = \frac{1}{2},$$

$$\nabla^2 f(X, X) = (\nabla^\Sigma)^2 f(X, X).$$

Here and thereafter we denote still by $f$ the restriction of $f$ on the corresponding submanifolds, for instance, $f|_{\Sigma}$ by $f$. Also the superscripts $\Sigma, \mathbb{R}$ denote the corresponding geometric quantities of $\Sigma$ and $\mathbb{R}$ respectively, for instance, $\nabla^\Sigma$ denotes the connection of $\Sigma$. By a direct computation, the Ricci curvature satisfies that on $\Sigma$

$$\text{Ric}(X, X) = \text{Ric}^\Sigma(X, X), \quad \text{Ric}(\nu, \nu) = \text{Ric}(\nu, X) = 0.$$ 

Therefore

$$\text{Ric}^\Sigma_f \geq \frac{1}{2}.$$ 

By Theorem 4, the spectrum of $\Delta^\Sigma_f$ on $\Sigma$ for $L^2(\Sigma)$ is also discrete. By direct computation, we have the identity:

$$\Delta_f u(x, t) = \Delta^\Sigma_f u|_{\Sigma \times \{t\}}(x) + \Delta^\mathbb{R}_f u|_{\{x\} \times \mathbb{R}}(t)$$

$$= \Delta^\Sigma_f u|_{\Sigma \times \{t\}}(x) + \left(\frac{d^2}{dt^2} - \frac{t}{2} \frac{d}{dt}\right)u|_{\{x\} \times \mathbb{R}}(t),$$

where by abuse of notations, $\Delta^\Sigma_f$ and $\Delta^\mathbb{R}_f$ denote the drifted Laplacians of $\Sigma \times \{t\}$ and $\{x\} \times \mathbb{R}$, which act on functions $u|_{\Sigma \times \{t\}}$ and $u|_{\{x\} \times \mathbb{R}}$ respectively. By the theory of functional analysis, the discreteness of the spectrum of $\Delta^\Sigma_f$ implies that there exists a complete orthonormal system for space $L^2(\Sigma, e^{-\int f \, d\sigma})$ consisting of eigenfunctions of $\Delta^\Sigma_f$, where $d\sigma$ is the volume element of $\Sigma$ induced by the metric of $\Sigma$. Also, for the operator $\frac{d^2}{dt^2} - \frac{t}{2} \frac{d}{dt}, t \in \mathbb{R}$, it is known that its spectrum on $L^2(\mathbb{R}, e^{-\frac{t^2}{4}} dt)$ is discrete and the so-called Hermite polynomials are orthonormal eigenfunctions, which form a complete orthonormal system for space $L^2(\mathbb{R}, e^{-\frac{t^2}{4}} dt)$. By these facts and (20), one can verify that the products of the orthonormal eigenfunctions of $\Delta^\Sigma_f$ and the
orthonormal eigenfunctions of $\frac{d^2}{dt^2} - \frac{k^2}{4} \frac{\partial}{\partial t}$ are the eigenfunctions of $\Delta_f$ and, by a standard argument, form a complete orthonormal system for space $L^2(M, e^{-f} dv)$. Therefore the eigenvalues $\sigma(\Delta_f)$ of $M$ counted with multiplicity are just the sums of the corresponding eigenvalues $\sigma(\Delta^R_f)$ of $\Sigma$ and $\sigma(\Delta^R_f)$ of $R$ counted with multiplicity. It is known that

$$\sigma(\Delta^R_f) = \{0, \frac{1}{2}, 1, \frac{3}{2}, \cdots\},$$

where the first nonzero eigenvalue $\frac{1}{2}$ has multiplicity one. Hence

$$\sigma(\Delta^M_f) = \{0, \frac{1}{2}, \min\{\lambda_1(\Delta^\Sigma_f), 1\}, \cdots\},$$

where $\lambda_1(\Delta^\Sigma_f)$ is the first nonzero eigenvalue of $\Delta^\Sigma_f$.

To conclude the proof, we claim that if the multiplicity of $\lambda_1(\Delta_f) = \frac{1}{2}$ is $k$, then $M$ is isometric to $\Sigma^{n-k} \times \mathbb{R}^k$ with $\operatorname{Ric}^{\Sigma^{n-k}} \geq \frac{1}{2}$, $\lambda_1(\Delta^{\Sigma^{n-k}}_f) > \frac{1}{2}$, and

$$f(x_1, \ldots, x_{n-k+1}, t_1, \ldots, t_k) = f(x, 0) + \frac{1}{4} \sum_{i=1}^{k} t_i^2,$$

where $(x, t) = (x_1, \ldots, x_{n-k+1}, t_1, \ldots, t_k) \in \Sigma^{n-k} \times \mathbb{R}^k$.

In the following proof, we will use $\Sigma^j$ with superscript $j$ to distinguish different $\Sigma$, whose dimension is $j$. We will prove the claim by induction.

First suppose that the multiplicity $k = 1$. By the proof before, we know that $M = \Sigma^{n-1} \times \mathbb{R}$, $\operatorname{Ric}^{\Sigma^{n-1}} \geq \frac{1}{2}$ and $f$ is as in (19). From (21), we know that $\lambda_1(\Delta^{\Sigma^{n-1}}_f) > \frac{1}{2}$. So the claim holds for $k = 1$.

Next suppose that the conclusion of the claim holds for multiplicity $k - 1$ and $\lambda_1(\Delta_f) = \frac{1}{2}$ has multiplicity $k$. Then we have that $M = \Sigma^{n-1} \times \mathbb{R}$ with $\operatorname{Ric}^{\Sigma^{n-1}} \geq \frac{1}{2}$ and $f = \frac{t_k^2}{4} + f|_{\Sigma^{n-1}}$, where $t_k \in \mathbb{R}$. By (21), $\lambda_1(\Delta^{\Sigma^{n-1}}_f) = \frac{1}{2}$ must have multiplicity $k - 1$. Hence by hypothesis of induction, $\Sigma^{n-1} = \Sigma^{n-k} \times \mathbb{R}^{k-1}$ with $\operatorname{Ric}^{\Sigma^{n-k}} \geq \frac{1}{2}$, $\lambda_1(\Delta^{\Sigma^{n-k}}_f) > \frac{1}{2}$ and

$$f|_{\Sigma^{n-1}}(x_1, \ldots, x_{n-k+1}, t_1, \ldots, t_{k-1}) = f(x, 0) + \frac{1}{4} \sum_{i=1}^{k-1} t_i^2,$$

where $(x_1, \ldots, x_{n-k+1}, t_1, \ldots, t_{k-1}) \in \Sigma^{n-k} \times \mathbb{R}^{k-1}$. Thus $M = \Sigma^{n-k} \times \mathbb{R}^k$ and $f$ is as (22). So the conclusion of the claim holds for $k$. Therefore by induction, the claim is proved. Thus we complete the proof of theorem.

Theorem 2 has the following corollaries.
Corollary 1. Let \((M^n, g, e^{-f} dv)\) be a closed smooth metric measure space with \(\text{Ric}_f \geq \frac{a^2}{2} g\), where constant \(a\) is positive, then the first nonzero eigenvalue \(\lambda_1(\Delta_f)\) of \(\Delta_f\) satisfies
\[
\lambda_1(\Delta_f) > \frac{a^2}{2}.
\]

Corollary 2. Let \((M^n, g, e^{-f} dv)\) be a complete smooth metric measure space with \(\text{Ric}_f \geq \frac{a^2}{2} g\), where constant \(a\) is positive. If \(\frac{a^2}{2}\) is the first nonzero eigenvalue \(\lambda_1(\Delta_f)\) with multiplicity \(n-1\), then \(M\) is isometric to the Euclidean space \(\mathbb{R}^n\) and \(f\) can be expressed as
\[
(23) \quad f(x_1, \ldots, x_n) = \varphi(x_1) + \frac{a(x_2^2 + \cdots + x_n^2)}{4},
\]
where \(\varphi\) is smooth function satisfying \(\varphi'' \geq \frac{a^2}{2}\).

Proof. From Theorem 2, \(M\) is isometric to \(\Sigma \times \mathbb{R}^{n-1}\). \(\Sigma\) has dimension 1. Meanwhile it is known by [16] that the fundamental group \(\pi_1(M)\) is finite. So \(\Sigma\) must be \(\mathbb{R}\), not a circle and \(M\) is isometric to \(\mathbb{R}^n\). In this case, \(\text{Ric}_f = \nabla^2 f\). Using the general expression of \(f\) in Theorem 2 we obtain (23) by direct computation.

With Theorem 2 we may further estimate for other eigenvalues:

Corollary 3. Let \((M, g, e^{-f} dv)\) be a complete smooth metric measure space with \(\text{Ric}_f \geq \frac{1}{2}\). Suppose that the first nonzero eigenvalue \(\lambda_1(\Delta_f) = \frac{1}{2}\) with multiplicity \(k\). If the splitting \(M = \Sigma^{n-k} \times \mathbb{R}^k\), \(k \geq 1\), satisfies that \(\Sigma\) is compact and \(f\) is constant on \(\Sigma\), then the next eigenvalue \(\lambda_2(\Delta_f)\) of \(\Delta_f\) on \(M\) satisfies \(\lambda_2(\Delta_f) \geq \frac{1}{2} \frac{n-k}{n-k-1}\). Moreover the equality holds if and only if \(\Sigma\) is isometric to the round sphere \(S^{n-k}\) in with radius \(\sqrt{2(n-k-1)}\).

Proof. Since \(f\) is constant on \(\Sigma\), \(\text{Ric}^\Sigma = \text{Ric}_f^\Sigma \geq \frac{1}{2}\) on \(\Sigma\) and \(n-k \geq 2\). Analogous to the proof of Theorem 2 the eigenvalues of \(\sigma(\Delta_f)\) of \(M\) are the sums of the corresponding eigenvalues \(\sigma(\Delta_f^\Sigma)\) of \(\Sigma\) and \(\sigma(\Delta_f^{\mathbb{R}^k})\) of \(\mathbb{R}^k\) with restricted \(f\) on \(\Sigma\) and \(\mathbb{R}^k\) respectively. By Theorem 2 we know that
\[
f = \sum_{i=1}^{k} t_i^2 \frac{1}{4} + f|_{\Sigma},
\]
where \((t_1, \ldots, t_k) \in \mathbb{R}^k\). In this case
\[
\sigma(\Delta_f^{\mathbb{R}^k}) = \{0, \frac{1}{2}, \ldots, \frac{1}{2}, 1, \cdots\}.
\]
\[ \sigma(\Delta f^\Sigma) = \{0, \frac{1}{2}, \ldots, \frac{1}{2}, \min\{\lambda_1(\Delta f^\Sigma), 1\}, \ldots\}, \]

where \( \frac{1}{2} \) has multiplicity \( k \). On the other hand, Note that \( \text{Ric}^\Sigma \geq \frac{1}{2} \) and \( f \) is constant on \( \Sigma \). By Lichnerowicz theorem,

\[ \lambda_1(\Delta f^\Sigma) = \lambda_1(\Delta f^\Sigma) \geq \frac{n-k}{2(n-k-1)}. \]

Observe that \( \frac{n-k}{2(n-k-1)} \leq 1 \). (24) implies that \( \lambda_2(\Delta f) \geq \frac{1}{2} \frac{n-k}{n-k-1} \), and by Obata theorem, the equality holds if and only if \( \Sigma^{n-k} \) is isometric to the round sphere \( S^{n-k} \). The radius is determined by the curvature directly.

\[ \Box \]

4. GRADIENT SHRINKING RICCI SOLITON CASE

Let \( (M^n, g) \) be a Riemannain manifold and \( f \) is a smooth function on \( M \). The quadruple \( (M, g, f, \rho) \) is called a \textit{gradient shrinking Ricci soliton} if

\[ \text{Ric}_f = \rho g, \]

where constant \( \rho > 0 \). Theorem 4 states that the spectrum of the drifted Laplacian \( \Delta f \) is discrete and the essential spectrum is empty.

By a scaling \( g \) one can normalize \( \rho = \frac{1}{2} \) so that

\[ \text{Ric}_f = \frac{1}{2} g. \]

It is known that (25) implies the following identities about complete gradient shrinking solitons.

\[ R + \Delta f = \frac{n}{2}, \]

\[ R + |\nabla f|^2 - f = C_0 \]

for some constant \( C_0 \). Here \( R \) denotes the scalar curvature of \( (M, g) \). For gradient shrinking Ricci solitons, we may give the upper bound estimate of the first nonzero eigenvalue of \( \Delta f \) as follows.

\textbf{Theorem 5.} Let the quadruple \( (M, g, f, \frac{1}{2}) \) be a complete noncompact gradient shrinking Ricci soliton. Then 1 is an eigenvalue of \( \Delta f \) and a translation of \( f \) with some constant is an associated eigenfunction. Moreover the first nonzero eigenvalue \( \lambda_1(\Delta f) \) of \( \Delta f \) on \( M \) satisfies \( \frac{1}{2} \leq \lambda_1(\Delta f) \leq 1 \).
Proof. Without loss of generality, by adding the constant $C_0 - \frac{n}{2}$ to $f$, by (27), we can assume that $f$ satisfies
\[
R + |\nabla f|^2 - f = \frac{n}{2}.
\]
Then (26) and (28) imply
\[
\Delta f + f = 0.
\]
From [3], we know that for a fixed point $p \in M$ there exists a constant $C$ such that
\[
\lim_{x \to +\infty} \frac{f(x)}{r^2(x)} = \frac{1}{4},
\]
and the volume $\text{vol}(B_p(r)) \leq Cr^n$, where $r(x)$ is the distance function from $p$ and $B_p(r)$ is the geodesic ball of radius $r$ centered at $p$. These facts together with $R \geq 0$ implies that $f \in H^1_1(M)$. Thus $f$ is an eigenfunction associated the eigenvalue 1. By Theorem 1, we complete the proof.

5. self-shrinkers in Euclidean space

We will give some remarks related to self-shrinkers in Euclidean space. Recall an immersed hypersurface $(M^n, g)$ is called a self-shrinker in the Euclidean space $\mathbb{R}^{n+1}$, if it satisfies that
\[
H = -\frac{\langle x, \nu \rangle}{2},
\]
where $x$ denotes the position vector in $\mathbb{R}^{n+1}$, $\nu$ is the outer unit normal to $M$, and $H$ is the mean curvature of $M$, defined by $H = \text{tr}A = \sum_{i=1}^{n} \langle \nabla e_i \nu, e_i \rangle$. It is known that the self-shrinker $M$ is an $f$-minimal hyper surface in $\mathbb{R}^{n+1}$ with $f = \frac{|x|^2}{4}$ (cf [5]). Directly from the Gauss equations,

\[
\text{Ric}_f = \frac{1}{2} g - A^2,
\]
where $A$ denotes the shape operator of $M$, defined by $AX = \nabla_X \nu, X \in TM$. Theorems 1 and 4 have the following corollary.

Corollary 4. Let $M^n$ be a complete self-shrinker in $\mathbb{R}^{n+1}$. If the principle curvatures $\eta_i, i = 1, \ldots, n$, of $A$ satisfy $\sup_i \eta_i^2 \leq \delta < \frac{1}{2}$, then

- $M$ has finite weighted volume, namely, $\int_M e^{-\frac{|x|^2}{4}} dv$, polynomial volume growth and properly immersed.
The logarithmic Sobolev inequality holds, that is, for all $u \in H^1_f(M)$ satisfying \( \int_M u^2 d\mu = \int_M e^{-f} dv \),

\[
\int_M u^2 \log u^2 e^{-f} dv \leq \frac{4}{1 - 2\delta} \int_M |\nabla u|^2 e^{-f} dv.
\]

The spectrum of $\Delta_f = \Delta - \frac{g}{2}\langle x, \nabla \cdot \rangle$ on $M$ is discrete.

**Proof.** Since all eigenvalues, that is, principle curvatures, of $A$ satisfy $\sup_i \eta_i^2 \leq \delta < \frac{1}{2}$, then

$$\text{Ric}_f \geq \frac{1}{2} - 2\delta g.$$ By [16], $M$ has finite weighted volume. By [6], $M$ is equivalently properly immersed and with the polynomial volume growth. By [1], we known that logarithmic Sobolev inequality (1) holds on $M$ with constant $\frac{4}{1 - 2\delta}$. Moreover by Theorem 4, the spectrum of $\Delta_f$ is discrete. 

**Remark 1.** It is interesting to compare a self-shrinker with a minimal submanifolds $M$ in Euclidean space $\mathbb{R}^{n+p}$. It is well-known that $M$ inherits Sobolev inequalities from $\mathbb{R}^{n+p}$. Since a self-shrinker $\Sigma$ can be considered as an $f$-minimal submanifold in $(\mathbb{R}^{n+p}, g_{can}, e^{-f} dv)$ with $f = \frac{|x|^2}{4}$ which enjoys logarithmic Sobolev inequality (3) with respect the measure $e^{-f} dv$, we ask if $\Sigma$ inherits a logarithmic Sobolev inequality (3) from $\mathbb{R}^{n+p}$.

6. **Appendix**

In this section, we give a proof of Proposition 1. We first recall some needed facts in measure theory. Let $(\Omega, \mathcal{F}, \mu)$ denote measure space with finite total measure, i.e., $\mu(\Omega) < \infty$. Let $L^p(\mu)$ denote the Banach space of classes of measurable, real-valued functions on $\Omega$, whose $p$-th power is $\mu$-integrable. Recall that a subset $K$ of $L^1(\mu)$ is called uniformly integrable if given $\varepsilon > 0$, there is a $\delta > 0$ so that $\sup\{\int_E |f| d\mu : f \in K\} < \varepsilon$ whenever $\mu(E) < \delta$. It is known that

**Lemma 1.** (De La Vallée Poussin theorem, cf [13]) A subset $K$ of $L^1(\mu)$ is uniformly integrable if and only if there exists a non-negative convex function $Q$ with $\lim_{t \to \infty} \frac{Q(t)}{t} = \infty$ so that

$$\sup\{ \int_\Omega Q(|f|) d\mu : f \in K\} < \infty.$$
Now let \((M^n, g, e^{-f}dv)\) be a complete smooth metric measure space. With the same notations as in Section 2, assume that 
\[ \mu(M) = \int_M e^{-f}dv \]
is finite and the logarithmic Sobolev inequality \((\mathbf{3})\) holds on \((M, g, e^{-f}dv)\) for all \(u \in C_c^\infty(M)\) satisfying 
\[ \int_M u^2d\mu = \mu(M), \]
where \(d\mu = e^{-f}dv\).

Remark 2. Logarithmic Sobolev inequality \((\mathbf{3})\) holds for \(u \in H^1_f(M)\) with 
\[ \int_M u^2d\mu = \mu(M). \]
In fact, for such \(u\), there exists a sequence \(\{u_k\}\), 
\(u_k \in C_c^\infty(M)\) satisfying that 
\[ \int_M u_k^2d\mu = \mu(M) \]
and \(u_k \rightarrow u\) in \(H^1_f(M)\). This implies that
\[ \int_M |\nabla u_k|^2d\mu \rightarrow \int_M |\nabla u|^2d\mu, \]
Since \(u_k \rightarrow u\) in \(H^1_f(M)\), there is a subsequence of \(u_k\), still denoted by \(u_k\), satisfies \(u_k\) a.e. converges to \(u\). Note that \(t \log t \geq a_0, t \in [0, \infty)\) for some constant \(a_0\), and \(\mu(M) < \infty\). By Fatou’s lemma and \((\mathbf{3})\),
\[
0 \leq \int_M (u^2 \log u^2 - a_0)d\mu \\
\leq \lim \inf \int_M (u_k^2 \log u_k^2 - a_0)d\mu \\
\leq \lim \inf \left( C \int_M |\nabla u_k|^2d\mu \right) - \int_M a_0d\mu \\
= C \int_M |\nabla u|^2d\mu - a_0\mu(M) \\
< \infty.
\]
So \(\int_M u^2 \log u^2d\mu\) exists and
\[
\int_M u^2 \log u^2d\mu \leq C \int_M |\nabla u|^2d\mu.
\]
Hence \((\mathbf{3})\) holds for \(u \in H^1_f(M)\) with \(\int_M u^2 = \mu(M)\).

Now we prove compact embedding of \(H^1_f(M)\) in \(L^2_f(M)\).

Proof of Proposition 4. It is known that the identical map \(H^1_f(M) \rightarrow L^2_f(M)\) is an embedding (cf. \([9]\), Section 4.1). So it suffices to prove that any sequence of \(\{u_k\}_{k=1}^\infty\) bounded in \(H^1_f(M)\)-norm has a subsequence converging in \(L^2_f(M)\) to a function \(u \in L^2_f(M)\). From the standard Sobolev space theory, it is true when \(M\) is a compact manifold with or without \(C^1\) boundary. So we only assume that \(M\) is noncompact. Let \(\{D_i\}\) be a compact exhaustion of \(M\) with \(C^1\) boundary \(\partial D_i\) for each \(i\). It is known that the embedding \(H^1_f(\Omega_i) \subset L^2_f(\Omega_i)\) is compact, that is,
\{u_k\}, restrict to \(\Omega_i\), has a subsequence converging in \(L^2_{\Omega_i}(\Omega_i)\). Note that an \(L^2\) convergence sequence has an a.e. convergent subsequence. By passing to a diagonal subsequence, there exists a subsequence of \(\{u_k\}\), still denoted by \(\{u_k\}\), and a function \(u\) defined on \(M\) so that \(\{u_k\}\) a.e. converges to \(u\) on each \(D_i\) and hence on \(M\). By Fatou’s lemma, \(\int_M u^2 d\mu \leq \liminf \int_M u_k^2 < \infty\), that is \(u \in L^2_f(M)\). On the other hand, by assumption of theorem and Remark 2 before the theorem, the functions in \(H^1_f(M)\) also satisfies logarithmic Sobolev inequality (3). This fact together with the boundedness of \(H^1_f(M)\)-norm of \(u_k\) implies that there exists a constant \(C\) satisfying

\[\int_M u_k^2 \log u_k^2 d\mu \leq C.\]

Take \(Q(t) = t \log t - a_0, t \in [0, \infty)\). One can see that \(Q(t)\) and \(\{u_k^2\}\) satisfy the conditions of De La Vallée Poussin theorem and thus \(\{u_k^2\}\) is uniformly integrable.

Now with the facts of a.e. convergence of \(\{u_k\}\) to \(u\) and the uniform integrability of \(\{u_k^2\}\), by an argument using Egorov’s theorem, similar to the proof of Vitali convergence theorem, one can prove that \(\int_M |u_k - u|^2 \to 0\), that is, \(u_k \to u\) in \(L^2_f(M)\). Therefore the embedding \(H^1_f(M) \hookrightarrow L^2_f(M)\) is compact.

By the standard theory in PDE (cf [9] Theorem 10.20), the compact embedding of \(H^1_f(M)\) is equivalent to the discreteness of spectrum of \(\Delta_f\). Thus we complete the proof.

\[\square\]

References

[1] D Bakry and M Émery, \textit{Diffusions hypercontractives, Seminaire de probabilites, XIX, 1983/84}, Lecture Notes in Math. \textbf{1123} (1985), 177-206.

[2] Ben Andrews and Lei Ni, \textit{Eigenvalue comparison on Bakry-Emery manifolds}, arXiv:1111.4967v1 [math.AP] 21 Nov 2011.

[3] Huai-Dong Cao and Detang Zhou, \textit{On complete gradient shrinking Ricci solitons}, J. Differential Geom. \textbf{85} (2010), no. 2, 175–185. MR2732975

[4] Nelia Charalambous and Zhiqin Lu, \textit{The essential spectrum of the Laplacian}, arXiv:1211.3225, 2012.

[5] Xu Cheng, Tito Mejia, and Detang Zhou, \textit{Eigenvalue estimate and compactness for closed f-minimal surfaces}, arXiv:1210.8448v1 [math.DG] 31 Oct 2012.

[6] Xu Cheng and Detang Zhou, \textit{Volume estimate about shrinkers}, Proc. AMS.

[7] Akito Futaki, Haizhong Li, and Xiang-Dong Li, \textit{On the first eigenvalue of the Witten–Laplacian and the diameter of compact shrinking solitons}, Ann Glob Anal Geom, online, 2013.

[8] Akito Futaki and Y Sano, \textit{Lower diameter bounds for compact shrinking Ricci solitons}, The Asian Journal of Mathematics \textbf{17} (2013), no. 1, 17–32.
[9] Alexander Grigoryan, Heat Kernel and Analysis on Manifolds, American Mathematical Soc., 2009 (English).
[10] Hans-Joachim Hein and Aaron Naber, New logarithmic Sobolev inequalities and an -regularity theorem for the Ricci flow, arXiv:1205.0380v1 [math.DG] 2 May 2012.
[11] Michel Ledoux, Concentration of measure and logarithmic Sobolev inequalities, Séminaire de probabilités de Strasbourg 33 (1999), 120–216.
[12] Leonardo Silvares, On the essential spectrum of the Laplacian and the drifted Laplacian, arXiv:1302.1834v1 [math.DG] 7 Feb 2013.
[13] André Lichnerowicz, Géométrie des groupes de transformations, Travaux et Recherches Mathématiques, III. Dunod, Paris, 1958 (French). MR0124009 (23 #A1329)
[14] Zhiqin Lu and Detang Zhou, On the essential spectrum of complete non-compact manifolds, J. Funct. Anal. 260 (2011), no. 11, 3283-3298.
[15] P. Meyer, Probability and Potentials, Blaisdell Publishing Co., 1966.
[16] Frank Morgan, Manifolds with Density. 1118.53022., Notices of the Amer. Math. Soc. 52 (2005), no. 8, 853-868. MR2161354
[17] Ovidiu Munteanu and Jiaping Wang, Smooth metric measure spaces with non-negative curvature, Comm. Anal. Geom. 19 (2011), no. 3, 451–486. MR2843238
[18] ______, Analysis of weighted Laplacian and applications to Ricci solitons, preprint [arXiv:1112.3027], 2011.
[19] ______, Geometry of manifolds with densities, preprint [arXiv:1211.3996], 2012.
[20] Morio Obata, Certain conditions for a Riemannian manifold to be isometric with a sphere, J. Math. Soc. Japan 14 (1962), 333–340. MR0142086 (25 #5479)
[21] Guiseppe Da Prato and Jerzy Zabczyk, Stochastic Equations in Infinite Dimensions, Cambridge University Press, 2008 (English).
[22] C Villani, Optimal Transportation: Old and New, Springer, 2009.
[23] Guofang Wei and Will Wylie, Comparison geometry for the Bakry-Emery Ricci tensor, J. Differential Geom. 83 (2009), no. 2, 377–405. MR2577473 (2011a:53064)

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