Möbius structures and Ptolemy spaces: boundary at infinity of complex hyperbolic spaces

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Abstract

The paper initiates a systematic study of Möbius structures and Ptolemy spaces. We conjecture that every compact Ptolemy space with circles and many space inversions is Möbius equivalent to the boundary at infinity of a rank one symmetric space $\mathbb{H}^n$ of noncompact type. We prove this conjecture for the class of complex hyperbolic spaces $\mathbb{C}H^n$ as our main result.

1 Introduction

The paper initiates a systematic study of Möbius structures and Ptolemy spaces. A Möbius structure on a set $X$ is a class of Möbius equivalent metrics. If a Möbius structure is fixed then $X$ is called a Möbius space. Ptolemy spaces are Möbius spaces with property that the inversion operation preserves the Möbius structure. A classical example of a Ptolemy space is an extended $\hat{\mathbb{R}}^n = \mathbb{R}^n \cup \infty = S^n$, $n \geq 0$, where the Möbius structure is generated by an Euclidean metric on $\mathbb{R}^n$, and $\mathbb{R}^n \cup \infty$ is identified with the unit sphere $S^n \subset \mathbb{R}^{n+1}$ via the stereographic projection. For more detail see Section 2.

Our motivation is to find a Möbius characterization of the boundary at infinity of rank one symmetric spaces $Y$ of noncompact type. In the case $Y = H^n$ this problem is solved in [FS2] for every $n \geq 1$: every compact Ptolemy space such that through any three points there is a Ptolemy circle is Möbius equivalent to $\hat{\mathbb{R}} = \partial_\infty \mathbb{H}^{n+1}$. Here a Ptolemy circle is a subspace Möbius equivalent to the Ptolemy space $\hat{\mathbb{R}} = S^{1}$.

Given distinct points $\omega, \omega'$ in a Ptolemy space $X$, there is a well defined notion of a metric sphere $S$ between $\omega, \omega'$, and a notion of a space inversion w.r.t. $\omega, \omega', S$, which is a Möbius involution $\varphi_{\omega, \omega'} : X \to X$, see sect. 4.1.

We consider a Ptolemy space $X$ with the following basic properties.

(E) Existence: there is at least one Ptolemy circle in $X$.

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(I) Inversion: for each distinct \( \omega, \omega' \in X \) and every metric sphere \( S \subset X \) between \( \omega, \omega' \) there is a unique space inversion \( \varphi_{\omega,\omega',S} : X \to X \) w.r.t. \( \omega, \omega' \) and \( S \).

**Conjecture 1.1.** Let \( X \) be a compact Ptolemy space with properties (E) and (I). Then \( X \) is Möbius equivalent to the boundary at infinity of a rank one symmetric space \( \mathbb{R} H^k \) of noncompact type taken with a canonical Möbius structure.

Our main result is the proof of the following particular case of Conjecture 1.1, which gives a Möbius characterization of the boundary at infinity \( \partial_\infty H^n \) of a real hyperbolic space \( H^n \) as well as of the boundary at infinity \( \partial_\infty C H^k \) of a complex hyperbolic space \( C H^k \).

**Theorem 1.2.** Let \( X \) be a compact Ptolemy space with properties (E) and (I). Then \( X \) is homeomorphic to a sphere \( S^n, n \geq 1 \), and for every \( \omega \in X \) there is a fibration \( \pi_\omega \) of \( X_\omega = X \setminus \omega \) with fibers homeomorphic to \( \mathbb{R}^p \) for some \( p, 0 \leq p < n \), such that for \( \omega, \omega' \) the fibrations \( \pi_\omega, \pi_\omega' \) are transformed to each other by any space inversion \( \varphi : X \to X \) with \( \varphi(\omega) = \omega' \). So the number \( p \) is a Möbius invariant of \( X \). In the case \( p = 0 \) the space \( X \) is Möbius equivalent to \( \partial_\infty H^{n+1} = \hat{\mathbb{R}}^n \). In the case \( p = 1 \) the space \( X \) is Möbius equivalent to \( \partial_\infty C H^k \) with \( n = 2k - 1, k \geq 2 \), taken with a canonical Möbius structure.

In sections 2 and 3 we give an introduction to Möbius structures and Ptolemy spaces. We emphasize that a Ptolemy space is not just a metric space, and there is no distinguished metric in its Möbius structure. This is a source of a duality phenomenon between Busemann and distance functions which takes place in any Ptolemy space and which cannot be even formulated for an individual metric space, see Section 3.1. The duality plays an important role in our paper.

The proof of Theorem 1.2 consists of two parts. In the first part, which occupies sections 4 – 7, we prove Theorem 4.5. That theorem gives a much more detailed information about Ptolemy spaces discussed in Conjecture 1.1 than it is formulated in Theorem 1.2. The second part of the proof occupies sections 8 – 14 and it is dedicated to a particular case when fibers of fibrations \( \pi_\omega \) are homeomorphic to \( \mathbb{R} \).

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2 Möbius structures and Ptolemy spaces

In this section we discuss basic notions of Möbius geometry.

2.1 Möbius structures

A quadruple $Q = (x, y, z, u)$ of points in a set $X$ is said to be admissible if no entry occurs three or four times in $Q$. Two metrics $d$, $d'$ on $X$ are Möbius equivalent if for any admissible quadruple $Q = (x, y, z, u) \subset X$ the respective cross-ratio triples coincide, $	ext{crt}_d(Q) = \text{crt}_{d'}(Q)$, where

$$\text{crt}_d(Q) = (d(x, y)d(z, u) : d(x, z)d(y, u) : d(x, u)d(y, z)) \in \mathbb{R}P^2.$$ 

We actually consider extended metrics on $X$ for which existence of an infinitely remote point $\omega \in X$ is allowed, that is, $d(x, \omega) = \infty$ for all $x \in X$, $x \neq \omega$. We always assume that such a point is unique if exists, and that $d(\omega, \omega) = 0$. We use notation $X_\omega := X \setminus \omega$ and the standard conventions for the calculation with $\omega = \infty$. If $\infty$ occurs once in $Q$, say $u = \infty$, then $\text{crt}_d(x, y, z, \infty) = (d(x, y) : d(x, z) : d(y, z))$. If $\infty$ occurs twice, say $z = u = \infty$, then $\text{crt}_d(x, y, \infty, \infty) = (0 : 1 : 1)$.

A Möbius structure on a set $X$ is a class $\mathcal{M} = \mathcal{M}(X)$ of metrics on $X$ which are pairwise Möbius equivalent.
The topology considered on \((X, d)\) is the topology with the basis consisting of all open distance balls \(B_r(x)\) around points in \(x \in X_\omega\) and the complements \(X \setminus D\) of all closed distance balls \(D = \overline{B}_r(x)\). Möbius equivalent metrics define the same topology on \(X\). When a Möbius structure \(\mathcal{M}\) on \(X\) is fixed, we say that \((X, \mathcal{M})\) or simply \(X\) is a Möbius space.

A map \(f : X \to X'\) between two Möbius spaces is called Möbius, if \(f\) is injective and for all admissible quadruples \(Q \subset X\)

\[
\text{crt}(f(Q)) = \text{crt}(Q),
\]

where the cross-ratio triples are taken with respect to some (and hence any) metric of the Möbius structures of \(X, X'\). Möbius maps are continuous. If a Möbius map \(f : X \to X'\) is bijective, then \(f^{-1}\) is Möbius, \(f\) is homeomorphism, and the Möbius spaces \(X, X'\) are said to be Möbius equivalent.

In general different metrics in a Möbius structure \(\mathcal{M}\) can look very different. However if two metrics have the same infinitely remote point, then they are homothetic. Since this result is crucial for our considerations, we state it as a lemma.

**Lemma 2.1.** Let \(\mathcal{M}\) be a Möbius structure on a set \(X\), and let \(d, d' \in \mathcal{M}\) have the same infinitely remote point \(\omega \in X\). Then there exists \(\lambda > 0\), such that \(d'(x,y) = \lambda d(x,y)\) for all \(x, y \in X\).

*Proof.* Since otherwise the result is trivial, we can assume that there are distinct points \(x, y \in X_\omega\). Take \(\lambda > 0\) such that \(d'(x,y) = \lambda d(x,y)\). If \(z \in X_\omega\), then \(\text{crt}_d(x,y,z,\omega) = \text{crt}_{d'}(x,y,z,\omega)\), hence \((d'(x,y) : d'(x,z) : d'(y,z)) = (d(x,y) : d(x,z) : d(y,z))\). Since \(d'(x,y) = \lambda d(x,y)\) we therefore obtain \(d'(x,z) = \lambda d(x,z)\) and \(d'(y,z) = \lambda d(y,x)\).

In what follows we always consider \(X_\omega = X \setminus \omega\) as a metric space with a metric from the Möbius structure for which the point \(\omega\) is infinitely remote.

A classical example of a Möbius space is the extended \(\hat{\mathbb{R}}^n = \mathbb{R}^n \cup \infty = S^n\), \(n \geq 1\), where the Möbius structure is generated by some extended Euclidean metric on \(\hat{\mathbb{R}}^n\), and \(\mathbb{R}^n \cup \infty\) is identified with the unit sphere \(S^n \subset \mathbb{R}^{n+1}\) via the stereographic projection. Note that Euclidean metrics which are not homothetic to each other generate different Möbius structures by the lemma above, which however are Möbius equivalent.

### 2.2 Ptolemy spaces

A Möbius space \(X\) is called a Ptolemy space, if it satisfies the Ptolemy property, that is, for all admissible quadruples \(Q \subset X\) the entries of the respective cross-ratio triple \(\text{crt}(Q) \in \mathbb{R}P^2\) satisfies the triangle inequality. We can reformulate this property as follows.

Let \(\Sigma\) be the subset of points \((a : b : c) \in \mathbb{R}P^2\), where all entries \(a, b, c\) are nonnegative or all entries are nonpositive. Note that \(\Sigma\) can be identified
with the standard 2-simplex, \( \{(a, b, c) \in \mathbb{R}^3 : a, b, c \geq 0, a + b + c = 1\} \). Let \( \Delta \subset \Sigma \) be the set of points \((a : b : c) \in \Sigma\) such that the entries \(a, b, c\) satisfy the triangle inequality. This is obviously well defined. If we identify \( \Sigma \subset \mathbb{R}^2 \) with the standard 2-simplex, i.e. the convex hull of the unit vectors \( e_1, e_2, e_3 \), then \( \Delta \) is the convex subset spanned by \((0, \frac{1}{2}, \frac{1}{2})\), \((\frac{1}{2}, 0, \frac{1}{2})\) and \((\frac{1}{2}, \frac{1}{2}, 0)\).

The importance of the Ptolemy property comes from the following fact. Given a metric \( d \in M(X) \) possibly with infinitely remote point \( \omega \in X \) and a point \( z \in X \), the reverse Ptolemy property, or \( m\)-inversion for brevity, of \( d \) of radius \( r > 0 \) with respect to \( z \) is a function \( d_z(x, y) = \frac{r^2 d(x, y)}{d(z, x) d(z, y)} \) for all \( x, y \in X \) distinct from \( z \), \( d_z(x, z) = \infty \) for all \( x \in X \setminus \{z\} \) and \( d_z(z, z) = 0 \). Using the standard convention we also have \( d_z(x, \omega) = \frac{r^2 d(x, \omega)}{d(x, z)} \). A direct computation shows that \( d_z \) is Möbius equivalent to \( d \).

Remark 2.2. When saying about an \( m\)-inversion of a metric without specifying its radius, we mean that the radius is 1.

In general \( d_z \) is not a metric because the triangle inequality may not be satisfied. However, we have

**Proposition 2.3.** A Möbius structure \( M \) on a set \( X \) is Ptolemy if and only if for all \( z \in X \) there exists a metric \( d_z \in M \) with infinitely remote point \( z \).

**Proof.** Assume that \( M \) is Ptolemy and that \( z \in X \). Choose some \( d \in M \). If \( z \) is infinitely remote with respect to \( d \) then \( d \) is our desired metric. If not we define \( d_z \) as the \( m\)-inversion (of radius \( r = 1 \)) of \( d \) with respect to \( z \).

Since for \( x, y, u \in X \setminus z \)

\[ (d_z(x, y) : d_z(y, u) : d_z(x, u)) = \text{crt}_{d_z}(x, y, z, u) = \text{crt}_{d}(x, y, z, u) \in \Delta \]

we see that \( d_z \) satisfies the triangle inequality and hence \( d_z \in M \).

If on the other hand for every \( z \in X \) there is a metric \( d_z \in M \) with infinitely remote point \( z \), then for all \( x, y, u \in X \setminus z \) and all \( d \in M \)

\[ \text{crt}_{d}(x, y, z, u) = \text{crt}_{d_z}(x, y, z, u) = (d_z(x, y) : d_z(y, u) : d_z(x, u)) \in \Delta, \]

which implies the Ptolemy property.

The classical example of Ptolemy space is \( \mathbb{R}^n \) with a standard Möbius structure as it follows from the proposition above. Here is the list some known results on metric spaces with Ptolemy property. A real normed vector space, which is ptolemaic, is an inner product space (Schoenberg, 1952, [Sch]); a Riemannian locally ptolemaic space is nonpositively curved (Kay, 1963, [Kay]); all Bourdon and Hamenstädter metrics on \( \partial_{\infty} Y \), where \( Y \) is \( \text{CAT}(1) \), generate a Ptolemy space (Faerts-Schroeder, 2006, [FeSt]); a geodesic metric space is \( \text{CAT}(0) \) if and only if it is ptolemaic and Busemann convex, a ptolemaic proper geodesic metric space is uniquely geodesic.
any Hadamard space ptolemaic, a complete Riemannian manifold is ptolemaic if and only if it is a Hadamard manifold, a Finsler ptolemaic manifold is Riemannian (Buckley-Falk-Wraith, 2009, [BFW]). These results allow to suggest that the Ptolemy property is a sort of a Möbius invariant nonpositive curvature condition.

2.3 Circles in Ptolemy spaces

A Ptolemy circle in a Ptolemy space $X$ is a subset $\sigma \subset X$ homeomorphic to $S^1$ such that for every quadruple $(x, y, z, u) \in \sigma$ of distinct points the equality

$$d(x, z)d(y, u) = d(x, y)d(z, u) + d(x, u)d(y, z)$$

(1)

holds for some and hence for any metric $d$ of the Möbius structure, where it is supposed that the pair $(x, z)$ separates the pair $(y, u)$, i.e. $y$ and $u$ are in different components of $\sigma \setminus \{x, z\}$. Recall the classical Ptolemy theorem that four points $x, y, z, u$ of the Euclidean plane lie on a circle (in this order) if and only if their distances satisfy the Ptolemy equality (1). One can reformulate this via the cross ratio triple. A subset $\sigma$ homeomorphic to $S^1$ is a Ptolemy circle, if and only if for all admissible quadruples $(x, y, z, u)$ of points in $\sigma$ we have $\text{crt}(x, y, z, u) \in \partial \Delta$, where the set $\Delta$ is defined in sect. 2.2.

Let $\sigma$ be a Ptolemy circle passing through the infinitely remote point $\omega$ for some metric $d \in M$ and let $\sigma_\omega = \sigma \setminus \omega$. Then $\text{crt}(x, y, z, \omega) \in \partial \Delta$ says that for $x, y, z \in \sigma_\omega$ (in this order) $d(x, y) + d(y, z) = d(x, z)$, i.e. it implies that $\sigma_\omega$ is a geodesic, actually a complete geodesic isometric to $\mathbb{R}$.

We recall the following facts from [FS2]. Let $\sigma$ be a Ptolemy circle in a Ptolemy space and let $x_1, x_2, x_3 \in \sigma$ be distinct points, then the map $\sigma \to \partial \Delta, t \mapsto \text{crt}(x_1, x_2, x_3, t)$ is a homeomorphism. The inverse of this map gives a canonical parametrization of $\sigma$ (for given points $x_1, x_2, x_3 \in \sigma$). By composing two of these canonical parameterizations we have:

**Proposition 2.4.** Let $\sigma$ and $\sigma'$ be Ptolemy circles. Let $x_1, x_2, x_3$ and $x'_1, x'_2, x'_3$ be distinct points on $\sigma$ respectively on $\sigma'$. Then there exists a unique Möbius homeomorphism $\varphi : \sigma \to \sigma'$ with $\varphi(x_i) = x'_i$. □

In particular all Ptolemy circles are Möbius equivalent. The standard metric models of a circle are $(\mathbb{R}, d)$, where $d$ is the standard Euclidean metric, or $(S^1, d_c)$, where $d_c$ is the chordal metric on $S^1$, i.e. the metric induced by the standard embedding $S^1 \subset \mathbb{R}^2$ as a unit circle. These two standard realizations of a circle are Möbius equivalent via the stereographic projection. Note that by Lemma 2.1 there is up to homothety only one metric on a circle with a infinitely remote point, while there are plenty of bounded metrics (for a description of all Ptolemy metrics on $S^1$ see [FS2]).
3 Properties of Ptolemy spaces

In this section we discuss various properties of Ptolemy spaces which include duality between Busemann and distance functions, and Busemann flat Ptolemy spaces.

3.1 Duality between Busemann and distance functions

Let $X$ be a Ptolemy space, $d$ a metric of the Möbius structure with infinitely remote point $\omega$, $X_\omega = X \setminus \omega$. Recall that every Ptolemy circle $\sigma \subset X$ that passes through $\omega$ is isometric w.r.t. $d$ to a geodesic line. Such a line $l = \sigma_\omega$ is called a Ptolemy line. We fix $\omega' \in l$, and let $d'$ be the m-inversion of $d$ w.r.t. $\omega'$. Then $d'$ is a metric of the Möbius structure with infinitely remote point $\omega'$. In particular, $l' = \sigma_{\omega'}$ is a Ptolemy line in $X_{\omega'}$. One easily checks that $d$ is the m-inversion of $d'$ w.r.t. $\omega$, that is, the inversion operation (of radius 1) is involutive.

With every oriented Ptolemy line $l \subset X_\omega$ and every point $\omega' \in l$ we associate a function $b : X_\omega \to \mathbb{R}$, called a Busemann function of $l$, as follows. Given $x \in X_\omega$, the difference $|xy| - |\omega'y|$ is nonincreasing by triangle inequality as $y \in l$ goes to infinity according the orientation of $l$, $y > \omega'$. Thus the limit $b(x) = \lim_{y \to \omega'}(|xy| - |\omega'y|)$ exists. Note that $b(\omega') = 0$ and $b(x) = -|\omega'x|$ for all $l \ni x > \omega'$.

For any Ptolemy space $X$ there is a remarkable duality between Busemann and distance functions which is described as follows. Let $c : \mathbb{R} \to X_{\omega'}$ be a unit speed parameterization of $l'$ with $c(0) = \omega$, $b^\pm : X_\omega \to \mathbb{R}$ the opposite Busemann functions of $l$, that is, associated with opposite ends of $l$, which are normalized by $b^\pm(\omega') = 0$ and $b^\pm \circ c(t) < 0$ for $t > 0$, $b^- \circ c(t) < 0$ for $t < 0$. Since $d(x, \omega') \cdot d'(x, \omega) = 1$ for every $x \in X \setminus \{\omega, \omega'\}$, we have $b^\pm \circ c(t) = \mp 1/t$ for all $t \neq 0$.

**Proposition 3.1.** For all $x \in X \setminus \{\omega, \omega'\}$ we have

\[
b^\pm(x) = \frac{d^\pm}{dt} \ln d'(x, c(t))|_{t=0},
\]

where $\frac{d^\pm}{dt}$ is the right/-left derivative.

**Proof.** We first note that the function $t \mapsto d'(x, c(t))$ is convex by the Ptolemy condition, and thus it has the right and the left derivatives at every point. Hence, the right hand side of Equation (2) is well defined. By definition, $d(x, y) = \frac{d'(x, y)}{d'(x, x)d'(y, y)}$ and $d(x, \omega') = \frac{1}{d'(x, \omega)}$ for all $x, y \in X_\omega$. Now, we compute

\[
d(x, c(t)) - d(\omega', c(t)) = \frac{d'(x, c(t))}{d'(\omega, x)d'(\omega, c(t))} - \frac{1}{d'(\omega, c(t))} = \frac{1}{|t|d'(x, c(0))} \left( d'(x, c(t)) - d'(x, c(0)) \right)
\]
for all $t \neq 0$, because $d'(x, \omega) = d'(x, c(0))$ and $d'(\omega, c(t)) = |t|$. Since $b^\pm(x) = \lim_{t \to \pm 0}(d(x, c(t)) - d(\omega', c(t)))$, we obtain

$$b^\pm(x) = \frac{d^\pm}{dt} \ln d'(x, c(t))|_{t=0}.$$  

\[ \Box \]

Given a Ptolemy circle $\sigma \subset X$ and distinct points $\omega, \omega' \in \sigma$, we denote with $D_{\sigma, \omega}'$ the subset in $X_{\omega'}$ which consists of all $x$ such that $\omega$ is a closest to $x$ point in the geodesic line $\sigma_{\omega'}$ (w.r.t. the metric of $X_{\omega'}$).

**Lemma 3.2.** Let $X$ be a Ptolemy space. Then for every Ptolemy circle $\sigma \subset X$ and each pair of distinct points $\omega, \omega' \in \sigma$ we have

$$D_{\sigma, \omega}' \cup \omega' = B_{\sigma, \omega}' \cup \omega,$$  

where $B_{\sigma, \omega}' = \{x \in X_{\omega} : b^+(x) \geq 0 \text{ and } b^-(x) \geq 0\}$, $b^\pm : X_{\omega} \to \mathbb{R}$ are the opposite Busemann functions of the Ptolemy line $\sigma_{\omega'} \subset X_{\omega}$ with $b^\pm(\omega') = 0$.

**Proof.** Denote with $d'$ the metric of $X_{\omega'}$ and let $c : \mathbb{R} \to X_{\omega'}$ be the unit speed parameterization of the Ptolemy line $\sigma_{\omega'} \subset X_{\omega'}$ such that $c(0) = \omega$ and $b^\pm \circ c(t) = \mp 1/t$, see the paragraph preceding Proposition 3.1. For every $x \in D_{\sigma, \omega}'$ we have $\frac{d^\pm}{dt}d'(x, c(t))|_{t=0} \geq 0$ for the right derivative, and $-\frac{d^\pm}{dt}d'(x, c(t))|_{t=0} \leq 0$ for left derivative because $t = 0$ is a minimum point of the convex function $t \mapsto d'(x, c(t))$. Equation (2) implies that $x \in B_{\sigma, \omega}'$.

Assume that $b^+(x) \geq 0$ and $b^-(x) \geq 0$ for some $x \in X \setminus \{\omega, \omega'\}$. Equation (2) implies that the right derivative $\frac{d^\pm}{dt}d'(x, c(t))|_{t=0} \geq 0$ and the left derivative $-\frac{d^\pm}{dt}d'(x, c(t))|_{t=0} \leq 0$. Thus $t = 0$ is a minimum point of the convex function $t \mapsto d'(x, c(t))$ and hence $x \in D_{\sigma, \omega}'$. \[ \Box \]

### 3.2 Busemann flat Ptolemy spaces

A Ptolemy space $X$ is said to be (Busemann) flat if for every Ptolemy circle $\sigma \subset X$ and every point $\omega \in \sigma$, we have

$$b^+ + b^- \equiv \text{const}$$  

for opposite Busemann functions $b^\pm : X_{\omega} \to \mathbb{R}$ associated with Ptolemy line $\sigma_{\omega}$. This property is equivalent to that any horospheres of $b^+, b^-$ coincide whenever they have a common point. Thus the horosphere $H_{\sigma, \omega}' \subset X_{\omega}$ of $\sigma_{\omega}$ through $\omega' \in \sigma_{\omega}$ is well defined in a flat Ptolemy space.

**Proposition 3.3.** A Ptolemy space $X$ is flat if and only for every $\omega \in X$ and every $x \in X_{\omega}$ the distance function $d(x, \cdot)$ is $C^1$-smooth along any Ptolemy line $l \subset X_{\omega}$, $l \not\equiv x$.  

9
Proof. Assume that distance functions are $C^1$-smooth along Ptolemy lines. We fix $\omega \in X$, a Ptolemy line $l \subset X_\omega$, and let $b^\pm$ be opposite Busemann functions of $l$. We suppose W.L.G. that $b^\pm(\omega') = 0$ for some point $\omega' \in l$. Then $b^+ + b^- = 0$ along $l$. Equation (2) implies that in fact $b^+(x) + b^-(x) = 0$ for every $x \in X_\omega$. Thus $X$ is flat.

Conversely, assume that $X$ is flat. Given $\omega' \in X$, a Ptolemy line $l' \subset X_{\omega'}$ and $x \in X_{\omega'} \setminus l'$, we show that the distance function $d'(x, \cdot)$ in $X_{\omega'}$ is $C^1$-smooth along $l'$ at every point $\omega \in l'$.

Let $c : \mathbb{R} \to X_{\omega'}$ be a unit speed parameterization of $l'$ with $c(0) = \omega$, $b^\pm : X_\omega \to \mathbb{R}$ the opposite Busemann function associated with the Ptolemy line $l = (l' \cup \omega') \setminus \omega \subset X_\omega$ such that $b^\pm(\omega') = 0$, $b^+ \circ c(t) < 0$ for all $t > 0$. Then $b^+ + b^- \equiv 0$ by the assumption, and by Proposition 3.1 we have

$$
\frac{d^+}{dt}d'(x, c(t))|_{t=0} = -\frac{d^-}{dt}d'(x, c(t))|_{t=0}, \quad \text{where } \frac{d^+}{dt} \text{ is the right derivative and } -\frac{d^-}{dt} \text{ is the left derivative. Hence } d'(x, \cdot) \text{ is } C^1\text{-smooth.}
$$

By Proposition 3.3 the duality equation (2) in a flat Ptolemy space $X$ takes the following form

$$
b^\pm(x) = \pm \frac{d}{dt} \ln d'(x, c(t))|_{t=0}. \quad (5)
$$

Example 3.4. The Ptolemy space $\hat{\mathbb{H}}^n$, $n \geq 2$, generated by the real hyperbolic space $\mathbb{H}^n$, is not flat because the equality $b^+ + b^- \equiv \text{const}$ is violated in $\mathbb{H}^n$. (Recall that $\mathbb{H}^n$ possesses the Ptolemy property and thus it generates a Ptolemy space by taking all metrics on $\hat{\mathbb{H}}^n$ which are Möbius equivalent to the metric of $\mathbb{H}^n$.) Note that the distance function $d(x, \cdot) : \mathbb{H}^n \to \mathbb{R}$ is smooth for every $x \in \mathbb{H}^n$ along any geodesic line $l, x \not\in l \subset \mathbb{H}^n$. This does not contradict Proposition 3.3 because the m-inversion of $d$ with respect to any point $x \in \mathbb{H}^n$ has a singularity at the infinity point of $\hat{\mathbb{H}}^n$.

In flat Ptolemy spaces, the duality between distance and Busemann functions is as follows.

Lemma 3.5. Let $X$ be a flat Ptolemy space, $\sigma \subset X$ a Ptolemy circle, and $\omega, \omega' \subset \sigma$ distinct points. Let $H_\omega^\sigma \subset X_\omega$ be the horosphere through $\omega'$ of the Ptolemy line $\sigma_\omega \subset X_\omega$, $D_\omega^\sigma \subset X_{\omega'}$ the set of all $x \in X_{\omega'}$ such that $\omega$ is the closest to $x$ point in the Ptolemy line $\sigma_{\omega'}$. Then

$$
H_\omega^\sigma \cup \omega = D_\omega^\sigma \cup \omega'. \quad (6)
$$

Proof. In a flat Ptolemy space we have $H_\omega^\sigma = B_\omega^\sigma$ because level sets of opposite Busemann functions associated with a Ptolemy line coincide when they have a common point. On the other hand, by duality, Lemma 3.2 we have $B_\omega^\sigma \cup \omega = D_\omega^\sigma \cup \omega'$. □
4 Ptolemy spaces with circles and many space inversions

We begin this section with discussion of what is a space inversion of an arbitrary Ptolemy space.

4.1 Space inversions

A M"obius automorphism $\varphi : X \to X$ of a Ptolemy space induces a map $\varphi^* : M \to M$, $(\varphi^*d)(x,y) = d(\varphi(x),\varphi(y))$ for every metric $d \in M$ and each $x, y \in X$, where $M$ is the M"obius structure of $X$. Note that a metric inversion of a bounded metric cannot be induced by any M"obius automorphism $X \to X$, because a metric inversion w.r.t. $\omega \in X$ has $\omega$ as the infinitely remote point.

Given distinct $\omega, \omega' \in X$, we say that a subset $S \subset X$ is a metric sphere between $\omega, \omega'$, if

$$S = \{x \in X : d(x,\omega) = r\} = S^d_r(\omega)$$

for some metric $d \in M$ with infinitely remote point $\omega'$ and some $r > 0$. We define a space inversion, or s-inversion for brevity, w.r.t. distinct $\omega, \omega' \in X$ and a metric sphere $S \subset X$ between $\omega, \omega'$ as a M"obius automorphism $\varphi = \varphi_{\omega,\omega'} : X \to X$ such that

1. $\varphi$ is an involution, $\varphi^2 = \text{id}$, without fixed points;
2. $\varphi(\omega) = \omega'$ (and thus $\varphi(\omega') = \omega$);
3. $\varphi$ preserves $S$, $\varphi(S) = S$;
4. $\varphi(\sigma) = \sigma$ for any Ptolemy circle $\sigma \subset X$ through $\omega, \omega'$.

Remark 4.1. Motivation of this definition comes from the fact that in the case $X = \partial_\infty M$, where $M$ is a symmetric rank one space of noncompact type, any central symmetry $f : M \to M$ with a center $o \in M$, $f(o) = o$, induces a space inversion $\partial_\infty f = \varphi_{\omega,\omega'} : X \to X$, where a geodesic line $l = \omega \omega' \subset Y$ with the end points $\omega, \omega'$ passes through $o$, and $S \subset X$ is a metric sphere between $\omega, \omega'$, see Proposition 14.2.
More precisely, if \( \varphi = \varphi_{\omega, \omega', S} \) is uniquely determined by its data \( \omega, \omega', S \). However, if \( \varphi' \) is another s-inversion with the same data, then it coincides with \( \varphi \) along any Ptolemy circle through \( \omega, \omega' \) because any Möbius automorphism of a Ptolemy circle is uniquely determined by values at three distinct points, see Proposition \( \ref{p4} \).

**Remark 4.2.** In general, there is no reason that an s-inversion \( \varphi = \varphi_{\omega, \omega', S} \) is uniquely determined by its data \( \omega, \omega', S \). However, if \( \varphi' \) is another s-inversion with the same data, then it coincides with \( \varphi \) along any Ptolemy circle through \( \omega, \omega' \) because any Möbius automorphism of a Ptolemy circle is uniquely determined by values at three distinct points, see Proposition \( \ref{p4} \).

**Lemma 4.3.** Given distinct \( \omega, \omega' \in X \) and a metric sphere \( S \subset X \) between \( \omega, \omega' \), for any metric \( d \in M \) with infinitely remote point \( \omega' \), an s-inversion \( \varphi = \varphi_{\omega, \omega', S} \) induces the m-inversion of \( d \) w.r.t. \( \omega \) of radius \( r = r(d) > 0 \), \((\varphi^*d)(x, y) = \frac{r^2d(x, y)}{d(\varphi(x), \varphi(y))} \), where \( r \) is determined by \( S = S^d_\omega(\omega) \), and \( x, y \in X \) are not equal to \( \omega \) simultaneously. The similar property holds true for any metric \( d' \in M \) with the infinitely remote point \( \omega \).

**Proof.** Since \( \varphi(\omega) = \omega' \), the point \( \omega \) is infinitely remote for the metric \( \varphi^*d \). Thus \( \varphi^*d = \lambda d' \) for some \( \lambda > 0 \), where \( d' \) is the m-inversion of \( d \) w.r.t. \( \omega \),

\[
(\varphi^*d)(x, y) = \frac{\lambda d(x, y)}{d(x, \omega)d(y, \omega)}
\]

for each \( x, y \in X \) which are not equal to \( \omega \) simultaneously. We compute \( \lambda \) by taking \( x \in S \), \( y = \varphi(x) \). Then \( \varphi(y) = x \) by \( \ref{def1} \), and since \( (\varphi^*d)(x, y) = d(x, y), d(x, \omega) = r = d(y, \omega) \), we have \( \lambda = r^2 \).

Contrary to metric inversions which always exist, in general there is no reason for a space inversion to exist. If however an s-inversion \( \varphi = \varphi_{\omega, \omega', S} : X \to X \) exists, and \( S = S^d_\omega(\omega) \) for a metric \( d \in M \) with infinitely remote point \( \omega' \), then \( S = S^d_{\varphi^*d}(\omega') \). This follows from Lemma \( \ref{lem4} \). Moreover, Lemma \( \ref{lem4} \) implies that \( \varphi^*(\varphi^*d) = d \) for any metric \( d \in M \) with infinitely remote point \( \omega \) or \( \omega' \). Thus the property \( \ref{def1} \) agrees with Lemma \( \ref{lem4} \) and actually \( \ref{def1} \) refines the property \( \varphi^*(\varphi^*d) = d \).

**Lemma 4.4.** For any metric sphere \( S' \subset X \) between \( \omega, \omega' \), we have \( \varphi(S') \) is a metric sphere between \( \omega, \omega' \) for every s-inversion \( \varphi = \varphi_{\omega, \omega', S} : X \to X \). More precisely, if \( S = S^d_\omega(\omega), S' = S^d_{\omega'}(\omega), \) then \( \varphi(S') = S^{d_{\varphi^*d}}_{\varphi(S')}(\omega) \).

**Proof.** For every \( x \in S' \), by Lemma \( \ref{lem4} \) we have

\[
d(\varphi(x), \omega) = (\varphi^*d)(x, \omega') = \frac{r^2}{d(x, \omega)} = r^2/r',
\]

hence the claim.

In support of Conjecture \( \ref{conjecture1} \) we prove the following theorem which recovers some basic features of \( \partial_{\infty} \mathbb{H}^n \). To formulate it, we introduce another important property which is useful for many things.

(E2) Extension: any Möbius map between any Ptolemy circles in \( X \) extends to a Möbius automorphism of \( X \).
**Theorem 4.5.** Let \( X \) be a compact Ptolemy space with properties (E) and (I) (see sect. [7]). Then \( X \) is homeomorphic to a sphere \( S^n \) for some \( n \geq 1 \), possesses the extension property \((E_2)\), and for every \( \omega \in X \) there is a 1-Lipschitz submetry \( \pi_\omega : X_\omega \to B_\omega \) with the base \( B_\omega \) isometric to an Euclidean space \( \mathbb{R}^k \), \( 0 < k \leq n \), such that any Möbius automorphism \( \varphi : X \to X \) with \( \varphi(\omega) = \omega' \) induces a homothety \( \varphi_\omega : B_\omega \to B_{\omega'} \) with \( \pi_{\omega'} \circ \varphi = \varphi \circ \pi_\omega \).

The fibers of \( \pi_\omega \) also called \( \mathbb{K} \)-lines are homeomorphic to \( \mathbb{R}^p \) for some \( p \geq 0 \) if \( k + p = n \), and for them the following properties hold

1. \( \mathbb{K} \)-line \( F \subset X_\omega \) and \( x \in X \setminus F \), there is a unique Ptolemy line \( l \subset X_\omega \) through \( x \) that intersects \( F \);
2. given distinct \( \mathbb{K} \)-lines \( F, F' \subset X_\omega \) and two Ptolemy line that intersect both \( F, F' \), if any other \( \mathbb{K} \)-line \( F'' \subset X_\omega \) intersects one of the Ptolemy lines, then it necessarily intersects the other.

Furthermore, if \( k = 1 \), then \( X = \hat{\mathbb{R}} \) is a Ptolemy circle.

**Remark 4.6.** In the case \( p = 0 \) the space \( X \) from Theorem 4.5 is Möbius equivalent to \( \hat{\mathbb{R}}^n = \partial_\infty \mathbb{H}^{n+1} \) with \( n = \dim X \). This proves Conjecture [1.1] for real hyperbolic spaces.

**Remark 4.7.** Recall that a map \( f : X \to Y \) between metric spaces is called a submetry if for every ball \( B_r(x) \subset X \) of radius \( r > 0 \) centered at \( x \) its image \( f(B_r(x)) \) coincides with the ball \( B_r(f(x)) \subset Y \).

In what follows, we always consider the weak topology on the group \( \text{Aut} X \) of Möbius automorphisms of \( X \), i.e. a sequence \( \varphi_i \in \text{Aut} X \) converges to \( \varphi \in \text{Aut} X \), \( \varphi_i \to \varphi \), if and only if \( \varphi_i(x) \to \varphi(x) \) for every \( x \in X \).

### 4.2 Möbius automorphisms of \( X \)

In this section we establish some important additional properties of a Ptolemy space \( X \) which follow from (E) and (I).

Given two distinct points \( \omega, \omega' \in X \), we denote with \( C_{\omega,\omega'} \) the set of all the Ptolemy circles \( \sigma \subset X \) through \( \omega, \omega' \), and with \( \Gamma_{\omega,\omega'} \) the group of Möbius automorphisms \( \varphi : X \to X \) such that \( \varphi(\omega) = \omega, \varphi(\omega') = \omega', \varphi(\sigma) = \sigma \) and \( \varphi \) preserves an orientation of \( \sigma \) for every \( \sigma \in C_{\omega,\omega'} \).

**Proposition 4.8.** Any Ptolemy space \( X \) with properties (E) and (I) possesses the following property (H) Homothety: for each distinct \( \omega, \omega' \in X \) the group \( \Gamma_{\omega,\omega'} \) acts transitively on every arc of \( \sigma \setminus \{\omega, \omega'\} \) for every circle \( \sigma \in C_{\omega,\omega'} \).

**Remark 4.9.** If one of the points \( \omega, \omega' \) is infinitely remote for a metric \( d \) of the Möbius structure, then every \( \gamma \in \Gamma_{\omega,\omega'} \) is a homothety w.r.t. \( d \). This is why we use (H) for the notation of the property above.
Proof. We assume that $\omega'$ is infinitely remote for a metric $d \in M$. Then for any $\sigma \in C_{\omega',\omega}$ the curve $\sigma_\omega = \sigma \setminus \omega'$ is a Ptolemy line w.r.t. $d$, and any $\gamma \in \Gamma_{\omega',\omega}$ acts on $\sigma_\omega$ as a homothety preserving an orientation.

Composing $s$-inversions $\varphi = \varphi_{\omega',\omega}, \varphi' = \varphi_{\omega',\omega'}$ of $X$, where $S, S' \subset X$ are spheres between $\omega, \omega'$, we obtain a Möbius automorphism $\gamma(\omega) = \omega, \gamma(\omega') = \omega'$ and $\gamma(\sigma) = \sigma$ for any Ptolemy circle $\sigma \in C_{\omega',\omega}$. Having no fixed point, both $\varphi, \varphi'$ preserve orientations of $\sigma$. Hence, $\gamma$ preserves its orientations, thus $\gamma$ acts on every arc of $\{\omega, \omega'\}$ as a homothety. That is, $\gamma \in \Gamma_{\omega,\omega'}$.

Let $r, r' > 0$ be the radii of $S, S'$ respectively w.r.t. the metric $d$, $S = S^d(\omega), S' = S^d(\omega')$. Then for every $x \in X \setminus \{\omega, \omega'\}$ we have $d(\varphi(x), \omega) = \frac{r^2}{d(x, \omega)}$ and $d(\gamma(x), \omega) = d(\varphi'(x \circ \varphi(x)), \varphi'(\omega')) = d(\varphi(x), \varphi'(\omega')) = r'^2/d(x, \omega)$. Therefore, the dilatation coefficient of $\gamma$ equals $\lambda := (r'/r)^2$, and it can be chosen arbitrarily by changing $S, S'$ appropriately. 

**Corollary 4.10.** Any two distinct Ptolemy circles in a Ptolemy space with properties (E) and (I) have in common at most two points.

**Proof.** Assume $\omega, \omega', x \in \sigma \cap \sigma'$ are distinct common points of Ptolemy circles $\sigma, \sigma' \subset X$. We have $\gamma(x) \in \sigma \cap \sigma'$ for every $\gamma \in \Gamma_{\omega',\omega}$. Then by property (H), the arcs of $\sigma$ and $\sigma'$ between $\omega, \omega'$ which contain $x$ coincide. Taking $\omega''$ inside of this common arc and applying the same argument to $\omega', \omega''$, $x = \omega$, we obtain $\sigma = \sigma'$.

### 4.3 Busemann parallel lines, pure homotheties and shifts

In this section we assume that the Ptolemy space $X$ possesses the properties (E) an (I). A point, we also assume that $X$ is compact.

We say that Ptolemy lines $l, l' \subset X_\omega$ are **Busemann parallel** if $l, l'$ share Busemann functions, that is, any Busemann function associated with $l$ is also a Busemann function associated with $l'$ and vice versa.

**Lemma 4.11.** Let $l, l' \subset X_\omega$ be Ptolemy lines with a common point, $o \in l \cap l'$, $b : X_\omega \to \mathbb{R}$ a Busemann function of $l$ with $b(o) = 0$. Assume $b \circ c(t) = -t = b \circ c'(t)$ for all $t \geq 0$ and for appropriate unit speed parameterizations $c, c' : \mathbb{R} \to X_\omega$ of $l, l'$ respectively with $c(0) = o = c'(0)$. Then $l = l'$. In particular, Busemann parallel Ptolemy lines coincide if they have a common point.

**Proof.** We show that the concatenation of $c|(-\infty,0]$ with $c'|[0,\infty)$ is also a Ptolemy line. Then $l = l'$ by Corollary 4.10. It suffices to show that for $s, t \geq 0$ we have $|c(-s)c'(t)| = t + s$. By triangle inequality we have $|c(-s)c'(t)| \leq t + s$. Letting $t_i \to \infty$ we have $|c'(t)c(t_i)| - t_i \to b \circ c'(t) = -t$. Thus by triangle inequality again, we have

$$|c(-s)c'(t)| \geq |c(-s)c(t_i)| - |c'(t)c(t_i)| = (t_i + s) - |c'(t)c(t_i)| \to t + s.$$
Thus $|c(-s)c'(t)| = t + s$. □

Next, we show that a sublinear divergence of Ptolemy lines is equivalent for them to be Busemann parallel.

**Lemma 4.12.** If two Ptolemy lines $l, l' \subset X_\omega$ are Busemann parallel, then they diverge at most sublinearly, that is $|c(t)c'(t)|/|t| \to 0$ as $|t| \to \infty$ for appropriate unit speed parameterizations $c, c'$ of $l, l'$.

Conversely, if $|c(t_i)c'(t_i)|/|t_i| \to 0$ for some sequences $t_i \to \pm \infty$, then the lines $l, l'$ are Busemann parallel.

**Proof.** Let $c, c' : \mathbb{R} \to X_\omega$ be unit speed parameterizations of Busemann parallel lines $l, l' \subset X_\omega$ respectively, and a common Busemann function $b : X_\omega \to \mathbb{R}$ such that $b \circ c(t) = b \circ c'(t) = -t$ for all $t \in \mathbb{R}$. Let $\mu(t) := |c(t)c'(t)|$. We claim that $\mu(t)/|t| \to 0$ for $t \to \pm \infty$. Assume to the contrary, that W.L.G. there exists a sequence $t_i \to \infty$ with $\mu(t_i)/t_i \geq a > 0$.

By the homothety property (H) there exists a homothety $\varphi_i$ of $X_\omega$ with factor $1/t_i$ such that $\varphi_i \circ c(s) = c(s/t_i)$ for all $s \in \mathbb{R}$. Note that $c'_i(s) = \varphi_i \circ c'(t_is)$ is a unit speed parameterization of the Ptolemy line $\varphi_i(l')$. For fixed $i$ we calculate

$$b \circ c'_i(t) = \lim_{s \to \infty} (|c'_i(t)c(s)| - s) = \lim_{s \to \infty} (|\varphi_i(c'(tt_i))c(s)| - s)$$

$$= \lim_{s \to \infty} (|\varphi_i(c'(tt_i))c(s/t_i)| - s/t_i) = \lim_{s \to \infty} (|\varphi_i(c'(tt_i))\varphi_i(c(s))| - s/t_i)$$

$$= \lim_{s \to \infty} \frac{1}{t_i}(|c'(tt_i)c(s)| - s) = \frac{1}{t_i}b(c'(tt_i)) = \frac{1}{t_i}(-tt_i) = -t$$

for all $t \in \mathbb{R}$. The Ptolemy lines $\varphi_i(l')$ subconverge to a Ptolemy line $l''$ through $c(0)$. If $c'' : \mathbb{R} \to X$ is the limit unit speed parameterization of $l''$, then $b \circ c''(t) = -t$ for all $t \in \mathbb{R}$, and $|c''(1)c(1)| \geq a > 0$. This contradicts Lemma 2.11 by which $l = l'$ and thus $c''(t) = c(t)$ for all $t \in \mathbb{R}$.

Conversely, assume $c, c' : \mathbb{R} \to X_\omega$ are unit speed parameterizations of Ptolemy lines $l, l' \subset X_\omega$ with $c(0) = o, c'(0) = o'$ such that $b(o) = b(o') = 0$ for the Busemann function $b : X_\omega \to \mathbb{R}$ of $l$ with $b \circ c(t) = -t, t \in \mathbb{R}$, and $\mu(t_i)/t_i \to 0$ for some sequence $t_i \to \infty$, where $\mu(t) = |c(t)c'(t)|$. Let $b' : X_\omega \to \mathbb{R}$ be the Busemann function of $l'$ with $b' \circ c'(t) = -t$. Applying the Ptolemy inequality to the cross-ratio triple $crt(Q_i)$ of the quadruple $Q_i = (o, c(t_i), c'(t_i), o')$, we obtain

$$|ac'(t_i)||o'c(t_i)| - |oc(t_i)||o'c'(t_i)| \leq |oo'||c(t_i)c'(t_i)|.$$  

Using $|oc(t_i)| = t_i = |o'c'(t_i)|$, $|o'c(t_i)| = b(o') + t_i + o(1)$, $|ac(t_i)| = b'(o) + t_i + o(1)$, and $|c(t_i)c'(t_i)| = \mu(t_i) = o(1)t_i$, we obtain

$$|b'(o) + t_i + o(1)) - (b(o') + t_i + o(1)) - t_i^2| \leq |oo'||o(1)t_i,$$

thus $|b'(o)| \leq o(1)$ and hence $b'(o) = b(o') = 0$. 15
Finally, for an arbitrary \( x \in X_\omega \) consider the quadruple \( Q_{x,t} = (x, c(t), c'(t), o) \). By the same argument as above, we have

\[
||xc'(t_i)|t_i - |xc(t_i)||oc'(t_i)|| \leq |ox|\mu(t_i).
\]

Using \( |oc'(t_i)| = b'(o) + t_i + o(1) = t_i + o(1), |xc'(t_i)| = b'(x) + t_i + o(1), |xc(t_i)| = b(x) + t_i + o(1) \), we finally obtain \( |b'(x) - b(x)| \leq o(1) \) and hence \( b(x) = b'(x) \). Therefore, the lines \( l, l' \) are Busemann parallel.

Now, we assume that our Ptolemy space \( X \) is compact. Given \( x, x' \in X_\omega \), we construct an isometry \( \eta_{xx'} : X_\omega \to X_\omega \) called a shift as follows. We take a sequence \( \lambda_i \to \infty \) and using the homothety property (H) for every \( i \) consider homotheties \( \varphi_i \in \Gamma_{\omega,x}, \psi_i \in \Gamma_{\omega,x'} \) with coefficient \( \lambda_i \). Then \( \eta_i = \psi_i^{-1} \circ \varphi_i \) is an isometry of \( X_\omega \) for every \( i \) because the coefficient of the homothety \( \eta_i \) is 1. Furthermore, we have \( |\eta_i(x)x'| = \lambda_i^{-1}|xx'| \to 0 \) as \( i \to \infty \). Since \( X \) is compact, the sequence \( \eta_i \) subconverges to an isometry \( \eta = \eta_{xx'} \) with \( \eta(x) = x' \). The term shift for \( \eta \) is justified by the following

**Lemma 4.13.** A shift \( \eta_{xx'} \) moves any Ptolemy line \( l \) through \( x \) to a Busemann parallel Ptolemy line \( \eta_{xx'}(l) \) through \( x' \).

**Proof.** We show that the line \( l' = \eta_{xx'}(l) \) cannot have a linear divergence with \( l \). Assume to the contrary that \( \mu(t) \geq at \) for some \( a > 0 \) and all \( t > 0 \), where \( \mu(t) = |c(t)c'(t)|, c : \mathbb{R} \to X_\omega \) is a unit speed parameterization of \( l \) with \( c(0) = x, c' = \eta_{xx'} \circ c \).

Recall that \( \eta_{xx'} = \lim \eta_i \), where \( \eta_i = \psi_i^{-1} \circ \varphi_i \), and \( \varphi_i \in \Gamma_{\omega,x}, \psi_i \in \Gamma_{\omega,x'} \) are homotheties with the same coefficient \( \lambda_i \to \infty \). By definition of the groups \( \Gamma_{\omega,x}, \Gamma_{\omega,x'} \), we have \( \varphi_i(l) = l, \psi_i(l') = l' \). We take \( y = c(1), y' = c'(1) \). Then for \( y_i = \varphi_i(y) = c(\lambda_i) \) we have \( |y_i c'(\lambda_i)| = \mu(\lambda_i) \geq a \lambda_i \). Thus for \( y'_i = \psi_i^{-1}(y_i) \) the estimate \( |y_i y'_i| = |\psi_i^{-1}(y_i)\psi_i^{-1} \circ c'(\lambda_i)| \geq a \) holds for all \( i \) in contradiction with \( y'_i \to y' \) as \( i \to \infty \).

Therefore, there are sequences \( t_i \to \pm \infty \), with \( \mu(t_i) = o(1)|t_i| \). By Lemma 4.12 the lines \( l, l' \) are Busemann parallel.

From Lemma 4.11 and Lemma 4.13, we immediately obtain

**Corollary 4.14.** Given a Ptolemy line \( l \subset X_\omega \), through any point \( x \in X_\omega \) there is a unique Ptolemy line \( l(x) \) Busemann parallel to \( l \).

Recall that any Möbius map \( \varphi : X \to X \) with \( \varphi(\omega) = \omega \) for \( \omega \in X \) acts on \( X_\omega \) as a homothety. A homothety \( \varphi : X_\omega \to X_\omega \) is said to be pure if it preserves any foliation of \( X_\omega \) by Busemann parallel Ptolemy lines.

**Lemma 4.15.** For every \( o \in X_\omega \) the group \( \Gamma_{\omega,o} \) consists of pure homotheties. In particular, every shift of \( X_\omega \) preserves any foliation of \( X_\omega \) by Busemann parallel Ptolemy lines.

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Proof. Let $l \subset X_\omega$ be a Ptolemy line through $o$, $b : X_\omega \to \mathbb{R}$ a Busemann function of $l$ with $b(o) = 0$. Then $b \circ \varphi = \lambda b$ for every homothety $\varphi \in \Gamma_{\omega,o}$, where $\lambda > 0$ is the coefficient of $\varphi$. By Corollary 4.13 any Busemann function of any Ptolemy line $l(x)$ through $x \in X_\omega$ is a Busemann function of a line $l$ through $o$. Therefore, every $\varphi \in \Gamma_{\omega,o}$ preserves any Busemann function $b$ of $l(x)$ with $b(o) = 0$ in the sense that $\lambda^{-1} b \circ \varphi = b$, where $\lambda > 0$ is the coefficient of $\varphi$. Since $\lambda^{-1} b \circ \varphi$ is a Busemann function of the Ptolemy line $\varphi^{-1}(l(x))$, we see that this line is Busemann parallel to $l(x)$. Thus $\varphi$ preserves the foliation $l(x), x \in X_\omega$ by Busemann parallel Ptolemy lines.

A construction of a homothety from the group $\Gamma_{\omega,\omega'}$ given in Proposition 4.16 is not uniquely determined because to obtain a homothety with the same coefficient $\lambda$ one can take a composition of different pairs of $s$-inversions. Thus for given $x, x' \in X_\omega$ a shift $\eta_{xx'}$ is not uniquely determined. We give a refined construction of shifts with property $\eta_{xx'} \to \text{id}$ as $x \to x'$ which will be used in the proof of Lemma 4.17 below.

Lemma 4.16. For $x, x' \in X_\omega$ there is a shift $\eta_{xx'} : X_\omega \to X_\omega$ with $\eta_{xx'}(x) = x'$ such that $\eta_{xx'} \to \text{id}$ as $x \to x'$.

Proof. For $\lambda_i \to \infty$ we denote with $S = S_1(x), S_i = S_{\lambda_i}(x)$ the metric spheres in $X_\omega$ centered at $x$ of radius 1 and $\lambda_i$ respectively. Similarly we put $S'_i = S_1(x'), S'_i = S_{\lambda_i}(x')$. Then $\psi_i = \varphi_{x',\omega}^{-1} \circ \varphi_{x,\omega} \circ S_i \in \Gamma_{\omega,x}, \psi_i = \varphi_{x',\omega}^{-1} \circ \varphi_{x,\omega} \\ S_i \in \Gamma_{\omega,x'}$ are homotheties with the same coefficient $\lambda_i^2$, see the proof of Proposition 4.16. The sequence of isometries $\eta_i = \psi_i^{-1} \circ \varphi_i : X_\omega \to X_\omega$ converges to a shift $\eta : X_\omega \to X_\omega$ with $\eta(x) = x'$. We have

$\eta_i = \varphi_{x',\omega} \circ S_i \circ \varphi_{x',\omega} \circ S_i \circ \varphi_{x,\omega} \circ S_i \circ \varphi_{x,\omega} \circ S_i$ and $\varphi_{x',\omega} \circ S_i \circ \varphi_{x,\omega} \circ S_i \to \text{id},$ $\varphi_{x',\omega} \circ S_i \circ \varphi_{x,\omega} \circ S_i \to \text{id}$ as $x \to x'$ because $S_i \to S'_i$, $S \to S'$ in the Hausdorff metric, every $s$-inversion is uniquely determined by its data according to our assumption, and by Lemma 4.14 $s$-inversions preserve the family of metric spheres between data points. Thus $\eta_i \to \text{id}$ for every $i$ as $x \to x'$. Moreover, the convergence of metric spheres around $x$ to metric spheres around $x'$ in the Hausdorff metric is uniform in radius as $x \to x'$, $\text{Hd}(S_{r'}^d(x), S_{r'}^d(x')) \leq |xx'|$ for every $r > 0$. Therefore, $\eta_i \to \text{id}$ uniformly in $i$ as $x \to x'$. This implies $\eta \to \text{id}$ as $x \to x'$.

4.4 Enhancing the existence property (E)

By property (E) formulated in sect. 4 we know that the space $X$ contains at least one Ptolemy circle.

Proposition 4.17. Any compact Ptolemy space with the inversion property (I) is two-point homogeneous, that is, for each (ordered) pairs $(x, y), (x', y')$ of distinct points in $X$ there is a Möbius automorphism $f : X \to X$ with $f(x) = x', f(y) = y'$.
Proof. Applying an inversion, we can map \(x\) to \(x'\). Let \(y''\) be the image of \(y\) under the inversion. Then \(y'' \neq x'\) by the assumption. We consider a metric of the Möbius structure with infinitely remote point \(x'\). By discussion above, there is a shift w.r.t. that metric which maps \(y''\) to \(y'\). The resulting composition gives a required Möbius automorphism.

It immediately follows from Proposition 4.17 that the property (E) in any compact Ptolemy space with (I) is promoted to (E) Enhanced existence: through any two points in \(X\) there is a Ptolemy circle.

In what follow, we use this property under the name (E).

4.5 Busemann functions on \(X_\omega\)

The proof of Theorem 4.5 is based on study of Busemann functions on \(X_\omega\). In this section we assume that a compact Ptolemy space \(X\) possesses the properties (E) and (I).

Lemma 4.18. Assume that \(x_i \to x\) in \(X\), and a point \(\omega \in X\) distinct from \(x\) is fixed. Then any Ptolemy circle \(l \subset X\) through \(\omega, x\) is the (pointwise) limit of a sequence of Ptolemy circles \(l_i \subset X\) through \(\omega, x_i\).

Proof. In the space \(X_\omega\) the circle \(l\) is a Ptolemy line (with infinitely remote point \(\omega\)) through \(x\). Then the sequence \(l_i = \eta_i(l)\) of Ptolemy lines with \(x_i \in l_i\) converges to \(l\), where \(\eta_i : X_\omega \to X_\omega\) is a shift with \(\eta_i(x) = x_i\), because the lines \(l_i\) are Busemann parallel to \(l\) by Lemma 4.15, and any sublimit of the sequence \(\{l_i\}\) coincides with \(l\) by Lemma 4.11.

We fix \(\omega \in X\) and a metric \(d\) of the Möbius structure such that \(\omega\) is the infinitely remote point. Then every Ptolemy circle in \(X\) through \(\omega\) is a Ptolemy line with respect to that metric. It immediately follows from the Ptolemy inequality that the distance function \(d(z, \cdot)\) to a point \(z \in X_\omega\) is convex along any Ptolemy line, see [FS2]. Under the homothety property (H) we prove that in fact \(d(z, \cdot)\) is \(C^1\)-smooth.

Lemma 4.19. In any compact Ptolemy space \(X\) with the homothety property (H), the distance function \(d_z = d(z, \cdot) : X_\omega \to \mathbb{R}\) is convex and \(C^1\)-smooth along any Ptolemy line \(l \subset X_\omega\) for any \(\omega \in X, z \in X_\omega \setminus l\). Therefore, \(X\) is Busemann flat.

Proof. Assume that \(d_z\) is not \(C^1\)-smooth at some point \(x \in l\). We fix an arclength parameterization \(c : \mathbb{R} \to X_\omega\) of \(l\) such that \(x = c(0)\). Since \(f = d_z \circ c\) is convex, it has the left and the right derivatives at every point. By assumption, these derivatives are different at \(t = 0\). It follows that

\[
\liminf_{t \to 0} \frac{f(t) - 2f(0) + f(-t)}{t} > 0.
\]
Now, using property (H), we find for every $\lambda > 0$ a homothety $h_\lambda : X_\omega \to X_\omega$ with coefficient $\lambda$ that preserves the point $x$ and the Ptolemy line $l$, $h_\lambda(x) = x$, $h_\lambda(l) = l$. Then $d(x, h_\lambda(z)) \to 0$ as $\lambda \to \infty$ and thus $\omega_\lambda = h_\lambda(z) \to \omega$. By Lemma 4.19 there is a Ptolemy circle $l_\lambda$ through $x$, $\omega_\lambda$ such that $l_\lambda \to l$ as $\lambda \to \infty$ (maybe after passing to a subsequence). We put $\lambda = 1/t$ and consider points $x_\lambda^+$, $x_\lambda^-$ in $l_\lambda$ separated by $x$ with $d(x, x_\lambda^+) = 1$. Then, W.L.G., $x_\lambda^+ \to c(\pm 1)$ as $t \to 0$. The points $x_\lambda^+$, $x$, $x_\lambda^-$, $\omega_\lambda$ lie on the Ptolemy circle $l_\lambda$ (in this order), thus

$$2d(x, \omega_\lambda) \geq d(x, \omega_\lambda)d(x_\lambda^+, x_\lambda^-) = d(x_\lambda^+, \omega_\lambda) + d(x_\lambda^-, \omega_\lambda)$$

by the Ptolemy equality. On the other hand, $|f(0)/t = d(x, \omega_\lambda)$ and $f(\pm t)/t = d(c(\pm 1), \omega_\lambda)$. Thus $|d(x_\lambda^+, \omega_\lambda) - f(\pm t)/t| \leq d(x_\lambda^+, c(\pm 1)) \to 0$ as $t \to 0$. Therefore, $(f(t) - 2f(0) + f(-t))/t \to 0$ as $t \to 0$ in contradiction with our assumption. Now, $X$ is flat by Proposition 3.3.
is a common horosphere for $b^+, b^-$. Since horoballs, i.e. sublevel sets of Busemann functions, are convex, the geodesic segment $zz' \subset l$ lies in $H_0$.

By Proposition 1.20, the function $b$ is affine along $l'$, that is, $b \circ c(t) = \alpha t + \beta$ for any arclength parameterization $c : \mathbb{R} \to l'$ of $l'$ and some $\alpha, \beta \in \mathbb{R}$, $|\alpha| \leq 1$. We choose $c$ so that $c(0) = z$, $c(|zz'|) = z'$. Then $\beta = 0$ by the assumption $b(z) = 0$, and $0 = b(z') = b \circ c(|zz'|) = \alpha |zz'|$. Hence $\alpha = 0$ and $b|l' \equiv 0$. This shows that $l' \subset H_0$.

4.6 Slope of two Ptolemy lines

By Proposition 1.20 a Busemann function associated with a Ptolemy line is affine along any other Ptolemy line. We introduce a quantity which measures a mutual position of Ptolemy lines in the space.

Let $l, l' \subset X_\omega$ be oriented Ptolemy lines. We define the slope of $l'$ w.r.t. $l$ as the coefficient of a Busemann function $b$ associated with $l$ when restricted to $l'$, $\text{slope}(l'; l) = \alpha$ if and only if $b \circ c'(t) = \alpha t + \beta$ for some $\beta \in \mathbb{R}$ and all $t \in \mathbb{R}$, where $c' : \mathbb{R} \to X_\omega$ is a unit speed parameterization of $l'$ compatible with its orientation. The quantity $\text{slope}(l'; l) \in [-1, 1]$ is well defined, i.e. it depends of the choice neither the Busemann function $b$ nor the parameterization $c'$ (we assume that $b$ is defined via a parameterization of $l$ compatible with its orientation). Note that the slope changes the sign when the orientation of $l$ or $l'$ is changed,

$$\text{slope}(-l'; l) = -\text{slope}(l'; l) \quad \text{and} \quad \text{slope}(l'; -l) = -\text{slope}(l'; l).$$

The first equality is obvious, while the second one holds because $X$ is Busemann flat by Lemma 4.19.

By definition, we have $\text{slope}(l; l) = -1$ for any oriented Ptolemy line $l \subset X_\omega$. More generally, let $l, l' \subset X_\omega$ be Busemann parallel Ptolemy lines. If an orientation of $l$ is fixed, then a compatible orientation of $l'$ is well defined. Indeed, we take a Busemann function $b$ of $l$ such that $b \to -\infty$ along $l$ in the chosen direction. Since $b$ is also a Busemann function of $l'$, the respective direction of $l'$ such that $b \to -\infty$ along $l'$ is well defined, and it is independent of the choice of $b$.

Now, if orientations of $l, l'$ are compatible, then $\text{slope}(l'; l) = -1 = \text{slope}(l; l')$.

Lemma 4.22. Let $l, l' \subset X_\omega$ be Busemann parallel Ptolemy lines with compatible orientations. Then for any oriented Ptolemy line $l'' \subset X_\omega$ we have $\text{slope}(l; l'') = \text{slope}(l'; l'')$.

Proof. Let $b : X_\omega \to \mathbb{R}$ be a Busemann function associated with $l''$. Since $b$ is affine along Ptolemy lines, there are unit speed parameterizations $c : \mathbb{R} \to l$, $c' : \mathbb{R} \to l'$ compatible with the orientations of $l, l'$ such that $b \circ c(0) = b \circ c'(0) =: \beta$. Then $b \circ c(t) = \alpha t + \beta$, $b \circ c'(t) = \alpha' t + \beta$ for some $|\alpha|, |\alpha'| \leq 1$ and all $t \in \mathbb{R}$. We show that $\alpha = \alpha'$.
Since the orientations of \( l, l' \) are compatible, we have \(|c(t)c'(t)| = o(1)|t|\) as \(|t| \to \infty\) by Lemma 4.12. Let \( c'' : \mathbb{R} \to l'' \) be a unit speed parameterization such that \( b(x) = \lim_{s \to \infty} |c''(s)x| - s, x \in X_\omega. \) Since \(||c''(s)c'(t)|| \leq |c(t)c'(t)| = o(1)t\) as \( t \to \infty\), we have \(|b \circ c(t) - b \circ c'(t)| = o(1)t\) and hence
\[
\alpha = \lim_{t \to \infty} b \circ c(t)/t = \lim_{t \to \infty} b \circ c'(t)/t = \alpha'.
\]

Proposition 4.20 combined with duality gives rise to a first variation formula to describe which we use the following agreement. Let \( \sigma, \sigma' \subset X \) be Ptolemy circles meeting each other at two distinct points \( \omega, \omega' \), \( \sigma \cap \sigma' = \{\omega, \omega'\} \), which decompose the circles into (closed) arcs \( \sigma = \sigma_+ \cup \sigma_- \) and \( \sigma' = \sigma'_+ \cup \sigma'_- \). The choice of \( \omega \) as an infinitely remote point automatically introduces the orientation of \( \sigma \) as well as of \( \sigma' \) such that \( \omega' \) is the initial point of the arcs \( \sigma_+, \sigma'_+ \), while \( \omega \) the final point of \( \sigma_-, \sigma'_- \), and the similar agreement holds true for the choice of \( \omega' \) as an infinitely remote point. Then the \( \text{slope}(\sigma'_+; \sigma_+) \) of the Ptolemy line \( \sigma'_+ \subset X_\omega \) w.r.t. the Ptolemy line \( \sigma_\omega \subset X_\omega \) is well defined.

**Lemma 4.23.** Let \( \sigma, \sigma' \subset X \) be Ptolemy circles meeting each other at two distinct points \( \omega \) and \( \omega' \), \( \sigma \cap \sigma' = \{\omega, \omega'\} \), which decompose the circles into (closed) arcs \( \sigma = \sigma_+ \cup \sigma_- \) and \( \sigma' = \sigma'_+ \cup \sigma'_- \). Let \( c : \mathbb{R} \to X_{\omega'}, c' : \mathbb{R} \to X_\omega \) be the unit speed parameterizations of the oriented Ptolemy lines \( \sigma_{\omega'} \subset X_{\omega'} \), \( \sigma'_\omega \subset X_\omega \) respectively compatible with the orientations such that \( c(0) = \omega, c'(0) = \omega' \). Then
\[
\frac{d}{dt}d'(c'(s), c(t))_{t=0} = \alpha \text{ sign } s
\]
for all \( s \neq 0 \), where \( d' \) is the metric of \( X_{\omega'} \), and \( \alpha = \text{slope}(\sigma'_+; \sigma_+) \).

**Remark 4.24.** We emphasize that (7) is a typical duality equality where the left hand side is computed in the space \( X_{\omega'} \), while the right hand side is computed in the opposite space \( X_\omega \).

**Proof.** By Proposition 4.20 the Busemann function \( b \) of \( \sigma_\omega \) with \( b(\omega') = 0 \) is affine along the Ptolemy line \( \sigma'_\omega \subset X_\omega \). Thus \( b \circ c'(s) = \alpha s \) for \( \alpha = \text{slope}(\sigma'_+; \sigma_+) \) and all \( s \in \mathbb{R} \) because \( b \circ c'(0) = b(\omega') = 0 \). Since \( X \) is Busemann flat by Lemma 4.19 Equation (5) applied to \( b^+ = b \) gives
\[
\alpha s = b \circ c'(s) = \frac{d}{dt} \ln d'(c'(s), c(t))_{t=0} = \frac{1}{d'(c'(s), \omega)} \frac{d}{dt} d'(c'(s), c(t))_{t=0}
\]
for all \( s \neq 0 \). Using \( d'(c'(s), \omega) = 1/|s| \), we obtain the required equality. \( \square \)

In the situation with two Ptolemy circles intersecting at two distinct points as in Lemma 4.23, we have four a priori different slopes. The duality and existence of s-inversions allows to reduce this number to one.
Lemma 4.25. Let $\sigma, \sigma' \subset X$ be Ptolemy circles meeting each other at two distinct points $\omega$ and $\omega'$, $\sigma \cap \sigma' = \{\omega, \omega'\}$, which decompose the circles into (closed) arcs $\sigma = \sigma_+ \cup \sigma_-$ and $\sigma' = \sigma'_+ \cup \sigma'_-$. Then

$$\text{slope}(\sigma'_\omega; \sigma_\omega) = \text{slope}(\sigma_\omega; \sigma'_\omega).$$

Proof. Denote $\alpha = \text{slope}(\sigma'_\omega; \sigma_\omega)$ and $\alpha' = \text{slope}(\sigma_\omega; \sigma'_\omega)$. As in Lemma 4.23 let $c : \mathbb{R} \to X_{\omega'}, c' : \mathbb{R} \to X_{\omega}$ be the unit speed parameterizations of the oriented Ptolemy lines $\sigma_{\omega'} \subset X_{\omega'}$, $\sigma_{\omega} \subset X_{\omega}$ respectively compatible with the orientations such that $c(0) = \omega$, $c'(0) = \omega'$. Let $b' : X_{\omega'} \to \mathbb{R}$ be the Busemann function associated with $\sigma_{\omega'}$ such that $b'(\omega) = 0$ (according to our agreement, $b'$ is computed via a parameterization of $\sigma_{\omega'}$ which is opposite in orientation to that of the parameterization $c'$). Then $b' \circ c(t) = \alpha t$ for all $t \in \mathbb{R}$ by the definition of $\alpha' = \text{slope}(\sigma_{\omega'}; \sigma_{\omega}).$ Thus

$$\alpha' t = b' \circ c(t) \leq d'(c'(s), c(t)) - d'(c'(s), \omega)$$

for all $s > 0$ and all (sufficiently small) $t \in \mathbb{R}$, where $d'$ is the metric of $X_{\omega'}$. The last inequality holds because the right hand side decreases monotonically as $s \to 0$. Applying Equality (7) we obtain $\alpha' \leq \alpha$. Interchanging $\omega$ with $\omega'$ and $\sigma$ with $\sigma'$, we have $\alpha \leq \alpha'$ by the same argument. Hence, the claim.

Using Lemma 4.25 the first variation formula (7) can be rewritten as follows

$$\frac{d}{dt} d'(c'(s), c(t))|_{t=0} = \alpha' \text{sign } s$$

for all $s \neq 0$, where $\alpha' = \text{slope}(\sigma_{\omega'}; \sigma_{\omega}).$ Now, the both sides of (8) are computed in the same space $X_{\omega'}$.

Lemma 4.25 implies the symmetry of the slope w.r.t. the arguments.

Lemma 4.26. For any oriented Ptolemy lines $l, l' \subset X_{\omega}$ we have $\text{slope}(l'; l) = \text{slope}(l; l')$.

Proof. We assume W.L.G. that $l \cap l' = \omega'$ and represent $l = \sigma_{\omega}$, $l' = \sigma'_{\omega}$ for Ptolemy circles $\sigma = l \cup \sigma$, $\sigma' = l' \cup \omega$. Then $\sigma \cap \sigma' = \{\omega, \omega'\}$. Let $S \subset X$ be a sphere between $\omega$, $\omega'$, $\varphi = \varphi_{\omega, \omega', S} : X \to X$ the s-inversion w.r.t. $\omega, \omega'$, $S$. Then $\varphi$ preserves any Ptolemy circle though $\omega, \omega'$ and its orientations. In particular, $\varphi(\sigma_{\omega}) = \sigma_{\omega'}$ and $\varphi(\sigma'_{\omega}) = \sigma'_\omega$. We assume that $S = S_{d}(\omega')$, where $d \in M$ is the metric of $X_{\omega}$. Then the metric $d' = \varphi^\ast d$ is the m-inversion of $d$, and vice versa, see Lemma 4.3. It follows that the map $\varphi : (X_{\omega}, d) \to (X_{\omega'}, d')$ is an isometry.

Now, we have

$$\text{slope}(\sigma'_{\omega}; \sigma_{\omega}) = \text{slope}(\varphi(\sigma'_{\omega}); \varphi(\sigma_{\omega})) = \text{slope}(\sigma'_{\omega}; \sigma_{\omega}) = \text{slope}(l'; l).$$

Using Lemma 4.25 we obtain $\text{slope}(l; l') = \text{slope}(\sigma_{\omega}; \sigma'_{\omega}) = \text{slope}(\sigma'_{\omega}; \sigma_{\omega}) = \text{slope}(l'; l)$. 

□

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From now on, we use notation \( \text{slope}(l, l') \) for the slope instead of \( \text{slope}(l'; l) \). We say that Ptolemy lines \( l, l' \subset X_\omega \) are orthogonal if \( \text{slope}(l', l) = 0 \). By Lemma 4.26 this is a symmetric relation. For orthogonal lines we also use notation \( l \perp l' \).

### 4.7 Tangent lines to a Ptolemy circle

A Ptolemy line \( l \subset X_\omega \) is tangent to a Ptolemy circle \( \sigma \subset X_\omega \) at a point \( x \in \sigma \) if for every \( y \in \sigma \) sufficiently close to \( x \) we have \( \text{dist}(y, l) = o(|xy|) \).

**Proposition 4.27.** Every Ptolemy circle \( \sigma \subset X_\omega \) possesses a unique tangent Ptolemy line \( l \) at every point \( x \in \sigma \).

**Proof.** Let \( d \) be the metric of \( X_\omega \), \( d' \) the metric inversion of \( d \) w.r.t. \( x \), \( d'(y, z) = \frac{d(y, z)}{d(y, x)d(z, x)} \). Then \( x \) is infinitely remote for \( d' \), and \( \sigma \setminus x \subset X_x \) is a Ptolemy line w.r.t. the metric \( d' \) on \( X_x \). By Corollary 4.14 there is a unique Ptolemy line \( \tilde{l} \subset X_x \) through \( \omega \) which is Busemann parallel to the line \( \sigma \setminus x \). Then \( l = (\tilde{l} \cup x) \setminus \omega \subset X_\omega \) is a Ptolemy line through \( x \). We show that \( l \) is tangent to \( \sigma \) at \( x \).

We fix on \( \sigma \setminus x \) and \( \tilde{l} \) compatible orientations, see sect. 4.6 and choose \( y \in \sigma, y' \in l \) with sufficiently small positive \( t = d(x, y) = d(x, y') \) according the orientations. Recall that \( d \) is also the metric inversion of \( d' \) w.r.t. \( \omega \), and that \( d(x, z) = 1/d'(\omega, z) \) for every \( z \in X \setminus \{x, \omega\} \). Then

\[
d(y, y') = \frac{d'(y, y')}{d'(\omega, y)d'(\omega, y')} = t \frac{d'(y, y')}{1/t}.
\]

By Lemma 4.12 \( \frac{d'(y, y')}{1/t} \rightarrow 0 \) as \( t \rightarrow 0 \), hence \( d(y, y') = o(t) \), and thus \( l \) is tangent to \( \sigma \) at \( x \).

If \( l' \subset X_\omega \) is another tangent line to \( \sigma \) at \( x \), then reversing the argument above we observe that the Ptolemy lines \( \tilde{l}, \tilde{l}' = (l' \cup \omega) \setminus x \subset X_x \) through \( \omega \) diverge sublinearly and thus they are Busemann parallel again by Lemma 4.12. It follows that \( \tilde{l} = \tilde{l}' \) and \( l = l' \).

Now, we reformulate Corollary 4.14 as follows.

**Corollary 4.28.** Given a Ptolemy line \( l \subset X_\omega \) and a point \( x \in l \), for any other point \( y \in X_\omega \) there exists a unique Ptolemy circle \( \sigma \subset X_\omega \) through \( y \) tangent to \( l \) at \( x \). In particular, if \( y \in l \), then \( \sigma = l \).

**Proof.** Consider a metric of the Möbius structure on \( X \) with the infinitely remote point \( x \) and apply Corollary 4.14. \( \square \)
5 Fibration $\pi_\omega : X_\omega \to B_\omega$

Given $x \in X_\omega$, we define

$$F_x = \bigcap_{l \ni x} H_l,$$

where the intersection is taken over all the Ptolemy lines $l \subset X_\omega$ through $x$, $H_l$ is the horosphere through $x$ of a Busemann function associated with $l$ (since $X$ is Busemann flat, $H_l$ is independent of choice of a Busemann function).

Lemma 5.1. For any $y \in F_x$ we have $F_y = F_x$.

Proof. By Corollary 4.14 for every Ptolemy line $l$ through $x$ there is a unique Ptolemy line $l'$ through $y$ such that $l$, $l'$ are Busemann parallel. Let $b$ be a Busemann function of $l$ such that $b(x) = 0$. Then $b(y) = 0$ because $y$ lies in the horosphere through $x$ of $b$. Hence $H_l = H_{l'}$ because $b$ is a Busemann function also of $l'$ and thus $F_y = F_x$.

By Lemma 5.1 the sets $F_x$, $F_{x'}$ coincide or are disjoint for any $x, x' \in X_\omega$. We let $B_\omega = \{ F_x : x \in X_\omega \}$ and define $\pi_\omega : X_\omega \to B_\omega$ by $\pi_\omega(x) = F_x$. Therefore, the fibers $F_b = \pi_\omega^{-1}(b)$, $b \in B_\omega$, form a partition of $X_\omega$, $B_\omega$ is the factor-space of this partition, and $\pi_\omega$ is the respective factor-map. A fiber $F$ of $\pi_\omega$ is also called a $\mathbb{K}$-line.

Lemma 5.2. For any $\omega$, $\omega' \in X$, any Möbius automorphism $\varphi : X \to X$ with $\varphi(\omega) = \omega'$ induces a bijection $\overline{\varphi} : B_\omega \to B_{\omega'}$ such that $\pi_{\omega'} \circ \varphi = \overline{\varphi} \circ \pi_\omega$.

Proof. It follows from Lemma 2.1 that for any metrics $d, d'$ of the Möbius structure with infinitely remote points $\omega, \omega'$ respectively, the map $\varphi : (X_\omega, d) \to (X_{\omega'}, d')$ is a homothety. Thus $\varphi$ maps any Ptolemy line $l \subset X_\omega$ to the Ptolemy line $l' = \varphi(l) \subset X_{\omega'}$, and $b \circ \varphi^{-1}$ is proportional a Busemann function of $l'$ for any Busemann function $b$ of $l$. It follows that $\varphi$ induces a bijection $\overline{\varphi} : B_\omega \to B_{\omega'}$ such that $\overline{\varphi} \circ \pi_\omega = \pi_{\omega'} \circ \varphi$.

Proof of property (1$\mathbb{K}$): uniqueness. Let $F \subset X_\omega$ be a $\mathbb{K}$-line, and let $x \in X \setminus F$. We show that there is at most one Ptolemy line in $X_\omega$ through $x$ that meets $F$. Assume that there are Ptolemy lines $l, l' \in X_\omega$ through $x$ that intersect $F$. Let $c : \mathbb{R} \to l, c' : \mathbb{R} \to l'$ be unit speed parameterizations such that $c(0) = x = c'(0)$, and $c(s) \in F, c'(s') \in F$ for some $s, s' > 0$. For the Busemann function $b : X_\omega \to \mathbb{R}$, $b(y) = \lim_{t \to -\infty} |yc(t)| - |t|$, of $l$ we have $b(c(s)) = s = b(c'(s'))$ because $c(s), c'(s') \in F$ and $F$ is a fiber of the Busemann function $b : X_\omega \to \mathbb{R}$. The function $t \mapsto |c'(s')c(t)| - (|t| + s)$ is nonincreasing and it converges to $b(c'(s')) - s = 0$ as $t \to -\infty$, thus $s' = |c'(s')c(0)| \geq s = |c(s)x|$. Interchanging $l$ and $l'$ we obtain $s \geq s'$ by the same reason. Hence $s = s'$. Since the Busemann function $b$ is affine along $l'$ by Proposition 4.20 and it takes the equal values $0 = b \circ c(0)$ and
\(b \circ c(s) = s = b \circ c'(s)\) along \(l, l'\) at two different parameter points, we have \(b \circ c(t) = b \circ c'(t)\) for every \(t \in \mathbb{R}\). By Lemma 4.11 \(l = l'\).

5.1 Semi-\(\mathbb{K}\)-planes

We fix \(\omega \in X\) and a metric from the Möbius structure with infinitely remote point \(\omega\). For a Ptolemy line \(l \subset X_\omega\) we put \(M = M_l := \bigcup F\), where the union is taken over all the fibers \(F \subset X_\omega\) of the fibration \(\pi_\omega : X_\omega \to B_\omega\) which intersect \(l\), \(F \cap l \neq \emptyset\) (the Ptolemy line \(l\) has at most one point in common with any fiber \(F \subset X_\omega\) because it intersects only once any its horosphere). The set \(M_l \subset X_\omega\) is called a semi-\(\mathbb{K}\)-plane over \(l\). Since different fibers of \(\pi_\omega\) are disjoint, we have if \(M_l \cap F \neq \emptyset\) for some fiber \(F\) of \(\pi_\omega\), then \(F \subset M_l\) and there is a uniquely determined point \(x \in l\) such that \(x \in F\), i.e. \(F\) is a member of the family of fibers that form \(M_l\).

Lemma 5.3. Through every point \(x\) of a semi-\(\mathbb{K}\)-line \(M_l\) there is a uniquely determined Ptolemy line \(l'\) that meets every \(\mathbb{K}\)-line of \(M_l\) and moreover \(l' \subset M_l\) is Busemann parallel to \(l\). Furthermore, any two \(\mathbb{K}\)-lines \(F, F'\) of \(M_l\) are equidistant in the sense that the segments of any two Ptolemy lines \(l, l' \subset M_l\) between \(F, F'\) have equal lengths.

**Proof.** By Corollary 4.11 there is a unique Ptolemy line \(l'\) through \(x\) which is Busemann parallel to \(l\). Consider compatible unit speed parameterizations \(c: \mathbb{R} \to l, c': \mathbb{R} \to l'\) such that \(c(0) \in F_x, c'(0) = x\), where \(F_x \subset M_l\) is the \(\mathbb{K}\)-line through \(x\).

Let \(F\) be a \(\mathbb{K}\)-line of \(M_l\). Then by definition \(c(t) \in F\) for some \(t \in \mathbb{R}\). We show that \(c'(t) \in F\). Let \(l''\) be a Ptolemy line through \(c(t), b''\) a Busemann function of \(l''\) with \(b''(c(t)) = 0\). We show that \(c'(t)\) lies in the zero level set of \(b'', b''(c'(t)) = 0\).

By Corollary 4.11, there is a Ptolemy line through \(c(0)\) for which \(b''\) is a Busemann function. Then the \(\mathbb{K}\)-line \(F_x\) lies in a level set of \(b''\), in particular, \(b''(c(0)) = b''(c'(0)) =: \beta\). By Lemma 4.22 we have \(b'' \circ c(s) = \alpha s + \beta = b'' \circ c'(s)\) for all \(s \in \mathbb{R}\). In particular, \(b''(c'(t)) = b''(c(t)) = 0\), hence \(c'(t) \in F\). This also shows that the \(\mathbb{K}\)-lines \(F, F_x\) are equidistant. Moreover, this argument shows that for every \(s \in \mathbb{R}\), the point \(c'(s)\) lies in the \(\mathbb{K}\)-line \(F_x\) through \(c(s)\), thus \(l' \subset M_l\).

Lemma 5.4. Every semi-\(\mathbb{K}\)-plane \(M \subset X_\omega\) is geodesically convex, i.e., every Ptolemy line \(l' \subset X_\omega\) that meets \(M\) in two different points is contained in \(M, l' \subset M\).

**Proof.** Let \(x, x' \in l' \cap M\) be different points. By Lemma 5.3 there is a Ptolemy line \(l \subset M\) through \(x\). Both \(l, l'\) meet the fiber \(F_x \subset M\) of the fibration \(\pi_\omega\) through \(x'\). Then \(l = l'\) by the uniqueness part of property (1\(\mathbb{K}\)).
Proof of property (2\(_E\)). Let \( F, F' \subset X_\omega \) be distinct fibers of the fibration \( \pi_\omega : X_\omega \to B_\omega \). Assume that Ptolemy lines \( l, l' \subset X_\omega \) intersect both \( F, F' \), and let \( F'' \) be a \( K \)-line that intersects \( l \). We show that \( F'' \) intersects also \( l' \).

Let \( M_l \) be the semi-\( K \)-plane over \( l \). Then \( F, F', F'' \subset M_l \) by our assumption. We have \( l' \subset M_l \) by Lemma 5.4. Hence \( l' \) intersects \( F'' \) by Lemma 5.3.

### 5.2 Zigzag curves

We fix \( \omega \in X \) and consider a metric on \( X_\omega \) with infinitely remote point \( \omega \). Let \( l \subset X_\omega \) be an oriented Ptolemy line. By Corollary 4.14, there is a foliation \( l(x), x \in X_\omega \) of the space \( X_\omega \) by Ptolemy lines, which are Busemann parallel to \( l \). Moreover, every member \( l(x) \) of the foliation has a well defined orientation compatible with that of \( l \), see sect. 4.6.

**Lemma 5.5.** Let \( l_1, l_2 \subset X_\omega \) be oriented Ptolemy lines which induce respective foliations of \( X_\omega \). We start moving from \( x \in X_\omega \) along \( l_1(x) \) by some distance \( s_1 \geq 0 \) up to a point \( y \), and then switch to \( l_2(y) \) and move along it by some distance \( s_2 \geq 0 \) up to a point \( z \). Next, we move from \( x' \in X_\omega \) along \( l_2(x') \) by the distance \( s_2 \) up to a point \( y' \), and then switch to \( l_1(y') \) and move along it by the distance \( s_1 \) up to a point \( z' \), where we always move in the directions prescribed by the orientations. If \( x, x' \) lie in a \( K \)-line \( F \subset X_\omega \), then \( z, z' \) also lie in one and the same \( K \)-line \( F' \subset X_\omega \).

**Proof.** Let \( c_1, c_2 : \mathbb{R} \to X_\omega \) be the unit speed parameterizations of \( l_1(x), l_2(x') \) respectively compatible with the orientations such that \( c_1(0) = x, c_2(0) = x' \). We also consider the unit speed parameterizations \( c_1', c_2' : \mathbb{R} \to X_\omega \) of \( l_1(y'), l_2(y) \) respectively compatible with the orientations such that \( c_1'(0) = y', c_2'(0) = y \). Then \( c_1(s_1) = y = c_2'(0), c_2(s_2) = y' = c_1'(0) \) and \( c_2'(s_2) = z, c_1'(s_1) = z' \).

Let \( b \) be a Busemann function of a Ptolemy line \( l \subset X_\omega \) which vanishes along \( F \), in particular, \( b(x) = 0 = b(x') \). By Proposition 4.20, \( b \) is affine along any Ptolemy line in \( X_\omega \), in particular, \( b \circ c_1(t) = c_1(t), b \circ c_2(t) = c_2(t) \) for some \( \alpha_i \) which by Lemma 4.22 only depends on \( l_i, i = 1, 2 \), and for all \( t \in \mathbb{R} \). Thus we have \( b(z') = b \circ c_1'(s_1) = \alpha_1 s_1 + \alpha_2 s_2 \) and similarly \( b(z) = b \circ c_2'(s_2) = \alpha_2 s_2 + \alpha_1 s_1 \). Hence any Busemann function on \( X_\omega \) takes the same value at the points \( z \) and \( z' \), i.e. these points lie in a common \( K \)-line \( F' \).

Given a base point \( o \in X_\omega \), a finite ordered collection \( \mathcal{L} = \{l_1, \ldots, l_k\} \) of oriented Ptolemy lines in \( X_\omega \), and a collection \( S = \{s_1, \ldots, s_k\} \) of non-negative numbers with \( s_1 + \cdots + s_k > 0 \), we construct a sequence \( \gamma_p = \gamma_p(o, \mathcal{L}, S) \subset X_\omega, p \geq 1 \), of piecewise geodesic curves through \( o \) as follows. Recall that we have \( k \) foliations of \( X_\omega \) by oriented Ptolemy lines
The curve $\gamma_p$ starts at $o = v^0_p$ for every $p \geq 1$. We move along $l_1(o)$ by the distance $s_1/2^{p-1}$ up to the point $v^1_p \in l_1(v^0_p)$, then switch to the line $l_2(v^1_p)$ and move along it by the distance $s_2/2^{p-1}$ up to the point $v^2_p$ etc. On the $i$th step, for $1 \leq i \leq k$, we move along the line $l_i(v^{i-1}_p)$ by the distance $s_i/2^{p-1}$ in the direction prescribed by the orientation of the line up to the point $v^i_p \in l_i(v^{i-1}_p)$. Starting with the point $v^k_p$ we then repeat this procedure only taking the subindices for $l_i$, $s_i$ modulo $k$ for all integer $i \geq k + 1$.

This produces the sequence $v^n_p$ of vertices of $\gamma_p$ for all $n \geq 0$. For integer $n < 0$ the vertices $v^n_p$ are determined in the same way with all the orientations of the lines $l_1, \ldots, l_k$ reversed, with the starting line $l_k(o)$, and with the ordered collections $\bar{L} = \{l_k, \ldots, l_1\}$ of lines, and $\bar{S} = \{s_k, \ldots, s_1\}$ of numbers.

Every curve $\gamma_p$ receives the arclength parameterization, for which we use the same notation $\gamma_p : \mathbb{R} \to X_\omega$, with $\gamma_p(0) = o$. Then for every $m \in \mathbb{Z}$, $1 \leq i \leq k$, we have $\gamma_p(t^n_p) = v^n_p$ is a vertex of $\gamma_p$, where $n = k(m-1)+i$, $t^n_p = [(s_1 + \cdots + s_i)m + (s_{i+1} + \cdots + s_k)(m-1)]/2^{p-1}$ (the sum $(s_1 + \cdots + s_k)$ is assumed to be zero for $i = k$).

It follows from Lemma 5.5 by induction that for every $n = km \in \mathbb{Z}$, the vertices $v^n_p = \gamma_p(t^n_p)$ of $\gamma_p$ and $v^{kn+1}_p = \gamma_{p+1}(t^{kn+1}_p)$ of $\gamma_{p+1}$ lie in a common $K$-line in $X_\omega$ for every $p \geq 1$. From this one easily concludes that the sequence of the projected curves $\pi_\omega(\gamma_p) \subset B_\omega$ converges (pointwise in the induced topology). At this stage, we do not have tools to prove that the sequence $\gamma_p$ itself converges in $X_\omega$. However, we need a limiting object of $\gamma_p$. Thus, for instance, we fix a nonprincipal ultra-filter on $\mathbb{Z}$ and say that $\gamma = \lim \gamma_p$ w.r.t. that ultra-filter. By this we mean that $\gamma(t) = \lim \gamma_p(t)$ for every $t \in \mathbb{R}$. The curve $\gamma = \gamma(o, L, S)$ is called a zigzag curve, and it is obtained together with the limiting parameterization $\gamma : \mathbb{R} \to X_\omega$, $\gamma(0) = o$, which in general is not an arclength parameterization.

**Lemma 5.6.** Every Busemann function $b : X_\omega \to \mathbb{R}$ is affine along any zigzag curve $\gamma$, that is, the function $b \circ \gamma : \mathbb{R} \to \mathbb{R}$ is affine. Moreover, if $\gamma = \gamma(o, L, S)$ for a base point $o \in X_\omega$, some ordered collection $L = \{l_1, \ldots, l_k\}$ of oriented Ptolemy lines in $X_\omega$, and a collection $S = \{s_1, \ldots, s_k\}$ of nonnegative numbers with $s_1 + \cdots + s_k > 0$, and $b(o) = 0$, then $b \circ \gamma(t) = \beta t$ for all $t \in \mathbb{R}$, where $\beta = \sum_i \alpha_i s_i / \sum_i s_i$, $\alpha_i = \text{slope}(l_i, l)$, $i = 1, \ldots, k$, and $l \in X_\omega$ is the oriented Ptolemy line for which the function $b$ is associated.

**Proof.** We assume that $\gamma = \lim \gamma_p$. Since $\gamma_p$ is piecewise geodesic for every $p \geq 1$, the function $b \circ \gamma_p : \mathbb{R} \to \mathbb{R}$ is piecewise affine. Recall that the points $v^n_p = \gamma_p(t^n_p)$ are vertices of $\gamma_p$, where $t^n_p = [(s_1 + \cdots + s_i)m + (s_{i+1} + \cdots + s_k)(m-1)]/2^{p-1}$ for $n = k(m-1)+i \in \mathbb{Z}$. Thus we have by induction

$$b \circ \gamma_p(t^n_p) = [(\alpha_1 s_1 + \cdots + \alpha_i s_i)m + (\alpha_{i+1} s_{i+1} + \cdots + \alpha_k s_k)(m-1)]/2^{p-1}$$
for \( n = k(m - 1) + i \in \mathbb{Z} \). Hence, \( b \circ \gamma_p(t^n_p) = \beta t^n_p + o(1) \) as \( p \to \infty \). Since the step \( t^{n+1}_p - t^n_p \leq \max s_i/2^{p-1} \to 0 \) as \( p \to \infty \), we conclude that \( b \circ \gamma_p \to b \circ \gamma \) pointwise as \( p \to \infty \), and \( b \circ \gamma(t) = \beta t \) for all \( t \in \mathbb{R} \).

Lemma 5.6 gives a strong evidence in support of the expectation that a zigzag curve under natural assumptions actually is a Ptolemy line. However we need additional arguments for the proof of this.

For \( \omega, o \in X \), the group \( \Gamma_{\omega,o} \) consists of homotheties \( \varphi : X_\omega \to X_\omega \) with \( \varphi(o) = o \) such that \( \varphi(l) = l \) for every Ptolemy line \( l \subset X_\omega \) through \( o \) preserving an orientation of \( l \), and moreover by property (H), \( \Gamma_{\omega,o} \) acts transitively on the open rays of \( l \) with the vertex \( o \), see Proposition 4.8.

**Lemma 5.7.** The homothety \( \varphi \in \Gamma_{\omega,o} \) with the coefficient \( \lambda = 1/2 \) leaves invariant a zigzag curve \( \gamma = \gamma(o, L, S) \) for any base point \( o \in X_\omega \), any ordered collection of oriented Ptolemy lines \( L = \{l_1, \ldots, l_k\} \) in \( X_\omega \), and any collection \( S = \{s_1, \ldots, s_k\} \) of nonnegative numbers with \( s_1 + \cdots + s_k > 0 \).

**Proof.** Let \( \gamma_p, p \geq 1 \), be the sequence of piecewise geodesic curves in \( X_\omega \) used in the construction of \( \gamma = \lim \gamma_p \). For \( p \geq 1 \), we let \( \nu^{km}_p, m \in \mathbb{Z} \), be the sequence of vertices of \( \gamma_p, \nu^{km}_p = \pi_\omega(\nu^{km}_p) \) the sequence of respective fibers of the fibration \( \pi_\omega : X_\omega \to B_\omega \). Recall that the sequences \( \{\nu^{km}_p : m \in \mathbb{Z}\} \), \( p \geq 1 \), approximate the projection \( \pi_\omega(\gamma) \) of \( \gamma \), that is, \( \pi_\omega(\gamma) \) coincides with the closure of the union \( \cup_p \{\nu^{km}_p : m \in \mathbb{Z}\} \).

We have \( \varphi(\gamma_p) = \gamma_{p+1} \) and \( \varphi(\nu^{km}_p) = \nu^{km}_{p+1} \) by the construction of \( \gamma_p \) and Lemma 4.13. For every dyadic number \( q = m/2^r, m \in \mathbb{Z}, r \geq 0 \), the sequence \( \nu(q) = \{\nu^q_p : p \geq r + 1\} \), where \( q_p = 2^{p-(r+1)} \cdot km \), lies in a common fiber \( F = F(q) \) of \( \pi_\omega \). Thus \( \varphi \) maps this sequence into the sequence \( \nu'(q) = \{\nu'^q_{p+1} : p \geq r + 1\} \subset \varphi(F) = F(q/2) \), where \( q'_p = 2^{p-r}(km/2) \), shrinking the mutual distances by the factor \( 1/2 \). Hence for the limit point \( x = \lim v(q) \in \gamma \) of any limiting procedure giving \( \gamma = \lim \gamma_p \) we have \( \varphi(x) = \lim v'(q) \in \gamma \). The points of type \( x = \lim v(q) \) with dyadic \( q \) are dense in \( \gamma \), thus \( \varphi \) preserves \( \gamma, \varphi(\gamma) = \gamma \).

**Lemma 5.8.** Let \( \gamma = \gamma(o, L, S) \subset X_\omega \) be a zigzag curve with base point \( o \in X_\omega, \) where \( L = \{l_1, \ldots, l_k\} \), \( S = \{s_1, \ldots, s_k\}, s_1 + \cdots + s_k > 0 \). Then for any \( \delta \in \gamma \) we have \( \gamma(\delta, L, S) = \gamma \), i.e. any zigzag curve \( \gamma \) is independent of a choice of its base point \( o \).

**Proof.** We first consider the case \( \delta = \gamma(t_q) \) is a dyadic point with dyadic \( q = m/2^r, m \in \mathbb{Z}, r \geq 0 \), and \( t_q = (s_1 + \cdots + s_k)q \), for the canonical parameterization \( t \mapsto \gamma(t) \) of \( \gamma \). Then \( \delta \) is an accumulation point of the vertices \( \nu_p = \nu^p_q = \gamma_p(t^p_q), p \geq r + 1 \), where \( q_p = 2^{p-(r+1)} \cdot km \) and \( t^p_q = (s_1 + \cdots + s_k)(q_p/k)/2^{p-1} = t_q \), of approximating piecewise geodesic curves \( \gamma_p, \gamma = \lim \gamma_p \) (recall that the sequence \( v(q) = \{v^q_p : p \geq r + 1\} \) lies
in a fiber $F(q) \subset X_\omega$ of the projection $\pi_\omega$, see the proof of Lemma 5.7). That is, $o' = \lim v_q$ for our limiting procedure.

By Lemma 5.10 there is a shift $\eta_p = \eta_{p, o'} : X_\omega \to X_\omega$ with $\eta_p(v_p) = o'$ and $\lim \eta_p = \text{id}$. Then $\eta_p(\gamma_p) = \gamma_p'$, where $\gamma_p' = \gamma_p(o', \mathcal{L}, \mathcal{S})$ is the piecewise geodesic curve with the base point $o'$ approximating the zigzag curve $\gamma' = \gamma(o', \mathcal{L}, \mathcal{S})$, $\gamma' = \lim \gamma_p'$. Now for an arbitrary point $x \in \gamma$, $x = \gamma(t)$, we have $x = \lim \gamma_p(t)$. We put $x' = \gamma'(t) = \lim \gamma_p'(t)$. Then for an arbitrary $\varepsilon > 0$ we have $|x - \gamma_p(t)| < \varepsilon$, $|x' - \gamma_p'(t)| < \varepsilon$, and $|\eta_p(x)| < \varepsilon$ for sufficiently large $p$. The last estimate holds since $\lim \eta_p = \text{id}$. Using $|\eta_p(x)\gamma_p'(t)| = |\eta_p(x)\eta_p \circ \gamma_p(t)| = |x\gamma_p(t)|$, we obtain

$$|xx'| \leq |x\eta_p(x)| + |\eta_p(x)\gamma_p'(t)| + |\gamma_p(t)x'| \leq 3\varepsilon,$$

thus $x = x'$, that is, $\gamma = \gamma'$.

For a general case, the point $o' = \gamma(t)$ can be approximated by dyadic ones, $t_q \to t$. Then respective piecewise geodesic curves $\gamma_p'\gamma_q$ with dyadic base points $(t_q)$ approximate pointwise the curve $\gamma_p'$ with the base point $o'$ for every $p \geq 1$. Thus $\gamma' = \gamma$ also in that case.

**Proposition 5.9.** Every zigzag curve $\gamma \subset X_\omega$ is either a geodesic and hence a Ptolemy line, or it degenerates to a point. More precisely, if $\gamma = \gamma(o, \mathcal{L}, \mathcal{S})$ for some base point $o \in X_\omega$ and collections $\mathcal{L} = \{l_1, \ldots, l_k\}$ of oriented Ptolemy lines in $X_\omega$, $\mathcal{S} = \{s_1, \ldots, s_k\}$ of nonnegative numbers with $s_1 + \cdots + s_k > 0$, then $\gamma$ is degenerate if and only if $\sum \alpha_is_i = 0$ for every oriented Ptolemy line $l \subset X_\omega$, where $\alpha_i = \text{slope}(l_i, l)$, $i = 1, \ldots, k$.

**Proof.** We first show that for each $o, o' \in \gamma$ there is a midpoint $x \in \gamma$. By Lemma 5.8 and Lemma 5.7 homotheties $\varphi \in \Gamma_{\omega, o}$, $\varphi' \in \Gamma_{\omega, o'}$ with coefficient $\lambda = 1/2$ both preserve $\gamma$, $\varphi(\gamma) = \gamma = \varphi'(\gamma)$. Then for $x = \varphi(o') \in \gamma$ we have $|ox| = |oo'|/2$, and similarly for $x' = \varphi'(o) \in \gamma$ we have $|x'o'| = |oo'|/2$. Furthermore, the length of the segment of $\gamma$ between $o, x$ is half of the length of the segment between $o, o'$, $L([ox], \gamma) = L([oo'], \gamma)/2$. Thus $L([ox], \gamma) = L([oo'], \gamma)/2$ by additivity of the length. Then $L([x'o'], \gamma) = L([oo'], \gamma)/2$ and hence $x' = x$ by monotonicity of the length, and $x$ is the required midpoint.

It follows that the segment of $\gamma$ between its any two points is geodesic. Since $\gamma$ is invariant under the nontrivial homothety $\varphi \in \Gamma_{\omega, o}$, we see that $\gamma$ is a Ptolemy line unless it degenerates to a point.

If $\gamma$ is degenerate, then any Busemann function $b : X_\omega \to \mathbb{R}$ is constant along $\gamma$. By Lemma 5.9 we have $\sum \alpha_is_i = 0$ for every oriented Ptolemy line $l \subset X_\omega$, where $\alpha_i = \text{slope}(l_i, l)$, $i = 1, \ldots, k$. Conversely, if $\gamma$ is non-degenerate, then $l = \gamma(R)$ is a Ptolemy line in $X_\omega$ by the first part of the proof, and the associated Busemann function $b : X_\omega \to \mathbb{R}$ is nonconstant along $l$. By Lemma 5.6 we have $b \circ \gamma(t) = \beta t$ for the canonical parameterization of $\gamma$ with $\beta = \sum \alpha_is_i/\sum s_i$, $\alpha_i = \text{slope}(l_i, l)$, $i = 1, \ldots, k$. Thus $\sum \alpha_is_i \neq 0$. \qed
Remark 5.10. Assume \( \gamma_1 \) is a piecewise geodesic curve (with finite number of edges) between different fibers in \( X_\omega \), \( \gamma_1(0) \in F', \gamma_1(s_1 + \cdots + s_k) \in F'' \), where \( s_1, \ldots, s_k \) are the lengths of its edges. Then the respective zigzag curve \( \gamma = \lim \gamma_\nu \) is not degenerate. This follows from \( \gamma(0) \in F', \gamma(s_1 + \cdots + s_k) \in F'' \) by construction of the approximating sequence \( \gamma_1, \gamma_\nu, \ldots \rightarrow \gamma \).

Now, we compute a unit speed parameterization a zigzag curve \( \gamma = \gamma(o, L, S) \) assuming for simplicity that the collection \( L \) consists of mutually orthogonal Ptolemy lines.

Lemma 5.11. Let \( \gamma = \gamma(o, L, S) \) be a zigzag curve in \( X_\omega \), where the collection \( L = \{l_1, \ldots, l_k\} \) consists of mutually orthogonal oriented Ptolemy lines, \( l_i \perp l_j \) for \( i \neq j \). Then

\[
|\sigma_\gamma(t)| = \sqrt{\sum_i s_i^2} \frac{|t|}{\sum_i s_i}
\]

for all \( t \in \mathbb{R} \), where, we recall, \( t \mapsto \gamma(t) \) is the canonical parameterization, and \( S = \{s_1, \ldots, s_k\} \). In particular, \( \gamma \) is nondegenerate.

Proof. By Proposition 5.9, \( l = \gamma(\mathbb{R}) \) is a Ptolemy line or it degenerates to a point, in particular, \( l \) is invariant for every pure homothety \( \varphi : X_\omega \rightarrow X_\omega \) with \( \varphi(o) = o \). From this one easily finds that there is \( \lambda \in [0, 1] \) such that \( |\sigma_\gamma(t)| = \lambda |t| \) for all \( t \in \mathbb{R} \). Let \( b \) be the Busemann function of \( l \) normalized by \( b(o) = 0 \) and \( b(t) < 0 \) for \( t > 0 \) (if \( l \) is degenerate, then \( b \equiv 0 \) by definition). By Lemma 5.6, \( -|\sigma_\gamma(t)| = b \circ \gamma(t) = \beta t \) with \( \beta = \sum_i \alpha_i s_i / \sum_i s_i \) for all \( t \geq 0 \), where \( \alpha_i = \text{slope}(l_i, l) \), \( i = 1, \ldots, k \). Let \( \beta_i, i = 1, \ldots, k \), be the Busemann function of \( l_i \) normalized by \( b_i(o) = 0 \). Using symmetry of the slope (Lemma 4.26), \( \alpha_i = \text{slope}(l_i, l) \), we obtain \( \beta_i t = b_i \circ \gamma(t) = \alpha_i \lambda t \) for all \( t \geq 0 \), where by Lemma 5.9 again,

\[
\beta_i = \sum_j \text{slope}(l_j, l_i) s_j / \sum_j s_j = -s_i / \sum_j s_j
\]

for \( i = 1, \ldots, k \). Thus \( \lambda \beta = -\sum_i \beta_i^2 / (\sum_i s_i)^2 \) and \( \lambda^2 t = \lambda |\sigma_\gamma(t)| = t \sum_i s_i^2 / (\sum_i s_i)^2 \) for all \( t \geq 0 \). Therefore, \( \lambda^2 = \sum_i s_i^2 / (\sum_i s_i)^2 > 0 \). In particular, \( l \) is nondegenerate. \( \square \)

5.3 Orthogonalization procedure

As usual, we fix \( \omega \in X \) and a metric \( d \) of the Möbius structure with infinitely remote point \( \omega \).

Proposition 5.12. There is a finite collection \( L_\perp \) of mutually orthogonal Ptolemy lines such that for every \( x \in X_\omega \), the fiber \( F \subset X_\omega \) through \( x \) of the fibration \( \pi_\omega : X_\omega \rightarrow B_\omega \) is represented as \( F = \cap_{L \in L_\perp} H_L \), where \( H_L \) is the horosphere of \( l \) through \( x \).
We first note that the cardinality of any collection of mutually orthogonal Ptolemy lines in $X_\omega$ is uniformly bounded above.

**Lemma 5.13.** There is $N \in \mathbb{N}$ such that the cardinality of any collection $\mathcal{L}$ of mutually orthogonal Ptolemy lines in $X_\omega$ is bounded by $N$, $|\mathcal{L}| \leq N$.

**Proof.** We fix $x \in X_\omega$ and assume W.L.G. that all the lines of $\mathcal{L}$ pass through $x$. For every line $l \in \mathcal{L}$ we fix a point on $l$ at the distance 1 from $x$. Let $A \subset X_\omega$ be the set of obtained points. By compactness of $X$ and homogeneity of $X_\omega$ it suffices to show that the distance $|a a'| \geq 1$ for each distinct $a, a' \in A$. We have $a \in l$, $a' \in l'$ for some distinct lines $l, l' \in \mathcal{L}$. Since slope$(l', l) = 0$, the lines $l, l'$ are also orthogonal at the infinite remote point $\omega$ according Lemma 1.25 and Lemma 1.26. Thus in the space $X_\omega$ with infinitely remote point $x$, the Ptolemy line $l' \setminus x \subset X_\omega$ lies in the horosphere $H$ of the line $l \setminus x \subset X_\omega$ through $\omega$. By duality, see Lemma 3.5, the point $x \in l$ is closest on the line $l$ to any fixed point of $l' \subset X_\omega$, in particular $|a a'| \geq |a' x| = 1$.

Next, we describe an orthogonalization procedure.

**Lemma 5.14.** Let $l_1, \ldots, l_k$ be a collection of mutually orthogonal Ptolemy lines in $X_\omega$, $l_i \perp l_j$ for $i \neq j$. Given a Ptolemy line $l \subset X_\omega$, through any $o \in X_\omega$ there is a zigzag curve $\gamma = \gamma(o, \mathcal{L}, S)$, where $\mathcal{L} = \{l_1, \ldots, l_k, l\}$ is an ordered collection of oriented Ptolemy lines, $S = \{s_1, \ldots, s_{k + 1}\}$ a collection of nonnegative numbers with $s_1 + \cdots + s_{k + 1} > 0$, which is orthogonal to $l_1, \ldots, l_k$, $\gamma(\mathbb{R}) = l_{k + 1} \perp l_i$ for $i = 1, \ldots, k$. Furthermore, if $\sum_i \alpha_i^2 \neq 1$, where $\alpha_i = \text{slope}(l, l_i)$, then $\gamma$ is nondegenerate, and $l_{k + 1}$ is a Ptolemy line.

**Proof.** We fix an orientation of $l$ and for every $i = 1, \ldots, k$ we choose an orientation of $l_i$ so that $\alpha_i = \text{slope}(l, l_i) \geq 0$, and put $\alpha := \sum_i \alpha_i \geq 0$. For any zigzag curve $\gamma = \gamma(o, \mathcal{L}, S)$ in $X_\omega$, where $\mathcal{L} = \{l_1, \ldots, l_k, l\}$, $S = \{s_1, \ldots, s_{k + 1}\}$, for $i = 1, \ldots, k$ and for the Busemann function $b_i$ of $l_i$ with $b_i(o) = 0$, by Lemma 5.6 we have $b_i \circ \gamma(t) = \beta_i t$ for all $t \in \mathbb{R}$, where $\beta_i = (-s_i + \alpha_i s_{k + 1})/(s_1 + \cdots + s_{k + 1})$. Then putting $s_i = \frac{\alpha_i}{1 + \alpha}, i = 1, \ldots, k$, $s_{k + 1} = \frac{1}{1 + \alpha}$, we have $s_1 + \cdots + s_{k + 1} = 1$ and $\beta_i = 0$ for every $i = 1, \ldots, k$. Thus $\gamma$ is orthogonal to $l_1, \ldots, l_k$, and it gives us a required Ptolemy line $l_{k + 1} = \gamma(\mathbb{R})$ unless $\gamma$ degenerates.

Let $b$ be the Busemann function of $l$ with $b(o) = 0$. By Lemma 5.6 we have $b \circ \gamma(t) = \beta t$ for all $t \in \mathbb{R}$ with $\beta = \sum_i \text{slope}(l, l_i) s_i - s_{k + 1}$. Using the symmetry of the slope, we see that slope$(l, l_i) = \text{slope}(l, l_i) - \alpha_i$ and thus $\beta = (\sum_i \alpha_i^2 - 1)/(1 + \alpha)$. If $\sum_i \alpha_i^2 \neq 1$, then this shows that $\gamma$ is nondegenerate.

We say the a collection $\{l_1, \ldots, l_k\}$ mutually orthogonal Ptolemy lines in $X_\omega$ is **maximal** if there is no Ptolemy line in $X_\omega$ which is orthogonal to every $l_1, \ldots, l_k$. By Lemma 5.13 such a collection exists.
Lemma 5.15. Let \( \{l_1, \ldots, l_k\} \) be a maximal collection of mutually orthogonal Ptolemy lines in \( X_\omega \). Then every Ptolemy line \( l \subset X_\omega \) can be represented as a zigzag curve \( \gamma(o, \mathcal{L}, S) \) with \( o \in l \) for a collection \( \mathcal{L} = \{l_1, \ldots, l_k\} \) of oriented Ptolemy lines and a collection \( S = \{s_1, \ldots, s_k\} \) of nonnegative numbers with \( s_1 + \cdots + s_k > 0 \). Furthermore, we have \( \sum_i a_i^2 = 1 \), where \( a_i = \text{slope}(l, l_i), i = 1, \ldots, k \).

Proof. We apply the orthogonalization procedure described in Lemma 5.14 to the collection \( \{l_1, \ldots, l_k, l\} \) and construct a zigzag curve \( \gamma_l = \gamma_l(o, \mathcal{L}_l, S_l) \), where \( \mathcal{L}_l = \{l_1, \ldots, l_k, l\} \) is the collection above of oriented Ptolemy lines, \( S = \{s_1, \ldots, s_{k+1}\} \) for an appropriate choice of entries described there. Since \( \gamma_l \) of orthogonal to \( l_1, \ldots, l_k \), we conclude from maximality of \( \{l_1, \ldots, l_k\} \) that \( \gamma_l \) degenerates and moreover \( \sum_i a_i^2 = 1 \) by Lemma 5.14.

According to Remark 5.10, the ends of the piecewise geodesic curve \( \gamma_{l,1} \) with \( k + 1 \) edges \( \sigma_1, \ldots, \sigma_k, \sigma \) on Ptolemy lines Busemann parallel to \( l_1, \ldots, l_k, l \) respectively with \( |\sigma_i| = s_i, i = 1, \ldots, k, |\sigma| = s_{k+1} \), lie in one and the same fiber (K-line) \( F \subset X_\omega \), that is, \( o = \gamma_{l,1}(0) \) and \( x = \gamma_{l,1}(s + s_{k+1}) \in F \), where \( s = s_1 + \cdots + s_k \). Thus the reduced piecewise geodesic curve \( \gamma_1 = \sigma_1 \cup \cdots \cup \sigma_k \) and the last edge \( \sigma \) of \( \gamma_{l,1} \) have the ends in the same fibers \( F, F' \), where \( F' \) is the fiber through \( \gamma_{l,1}(s) = \gamma_1(s) \).

Then the zigzag curve \( \gamma = \gamma(o, \mathcal{L}, S) \) is nondegenerate, where \( \mathcal{L} = \{l_1, \ldots, l_k\}, S = \{s_1, \ldots, s_k\} \), and it gives a Ptolemy line \( \gamma(\mathbb{R}) \subset X_\omega \) though \( o \) which hits the fiber \( F' \). Since the Ptolemy line \( l' \) containing the segment \( \sigma \) is Busemann parallel to \( l \) and intersects the fibers \( F, F' \), the line \( \gamma(\mathbb{R}) \) is Busemann parallel to \( l \), see Lemma 5.3. Hence \( \gamma(\mathbb{R}) = l \) by uniqueness, see Lemma 4.11. □

Lemma 5.16. Let \( \mathcal{L} = \{l_1, \ldots, l_k\} \) be a maximal collection of mutually orthogonal oriented Ptolemy lines in \( X_\omega \), \( S = \{s_1, \ldots, s_k\} \) a collection of nonnegative numbers with \( s_1 + \cdots + s_k = 1 \), \( b, b_i : X_\omega \rightarrow \mathbb{R} \) the Busemann functions of the zigzag curve \( \gamma = \gamma(o, \mathcal{L}, S) \), Ptolemy line \( l_i \) with \( b(o) = 0 = b_i(o) \) respectively, \( i = 1, \ldots, k \). Then \( \lambda b = \sum_i s_i b_i \), where \( \lambda = \sqrt{\sum_i s_i^2} \).

Proof. We denote by \( h = \lambda b - \sum_i s_i b_i \) the function \( X_\omega \rightarrow \mathbb{R} \) with \( h(o) = 0 \), which is an affine function on every Ptolemy line on \( X_\omega \). First, we check that \( h \) vanishes along \( l_1, \ldots, l_k \) (assuming that these lines pass through \( o \)). Indeed, \( \sum_i s_i b_i(z) = s_j b_j(z) \) for every \( z \in l_j \) because \( l_i \perp l_j \) for \( i \neq j \). Since \( b \) is a Busemann function of \( l \), it is affine on \( l_j \) with the coefficient \( \text{slope}(l_j, l) \), \( b(z) = -\text{slope}(l_j, l) b_j(z) \). Using symmetry of the slope, we obtain

\[
\text{slope}(l_j, l) = \text{slope}(l, l_j) = \frac{1}{\lambda} \sum_i \text{slope}(l_i, l_j) s_i = -s_j / \lambda,
\]

see the proof of Lemma 5.11. Thus \( \lambda b(z) = s_j b_j(z) \), and \( h(z) = 0 \).
Next, we show that if \( h \) is constant of a Ptolemy line \( l \), then it is constant on every Ptolemy line \( l' \) that is Busemann parallel to \( l \) (with maybe a different value). By Lemma 4.12 we know that \( l, l' \) diverge at most sublinearly, and also that \( h \) is affine on \( l' \). Thus \( h|l' \) cannot be nonconstant because \( h \) is a Lipschitz function on \( X_\omega \).

It follows that \( h \) vanishes on every piecewise geodesic curve with origin \( o \) and with edges Busemann parallel to the lines \( l_1, \ldots, l_k \). Hence, \( h \) vanishes along any zigzag curve of type \( \gamma(o, \mathcal{L}, S) \). Using Lemma 5.15 we conclude that \( h \) is constant along any Ptolemy line in \( X_\omega \).

By Proposition 4.27, every Ptolemy circle possesses a unique tangent line, which is certainly a Ptolemy line, at every point. Using standard approximation arguments, we see that \( h \) is constant along any Ptolemy circle in \( X_\omega \). By the existence property (E), every \( x \in X_\omega \) is connected with \( o \) by a Ptolemy circle. Thus \( h(x) = 0 \) and \( \lambda b = \sum_i b_i \).

**Proof of Proposition 5.12.** Let \( \mathcal{L}_\perp = \{l_1, \ldots, l_k\} \) be a maximal collection of mutually orthogonal Ptolemy lines in \( X_\omega \). This means that we actually consider respective foliations of \( X_\omega \) by Busemann parallel Ptolemy lines. For any \( x \in X_\omega \), for the fiber \( F \) and for the respective lines from \( \mathcal{L}_\perp \) through \( x \), we have by definition \( F \subset \cap_j H_j \), where \( H_j \subset X_\omega \) is the horosphere of \( l_j \) through \( x \). It follows from Lemma 5.15 and Lemma 5.16 that any Busemann function \( b : X_\omega \to \mathbb{R} \) with \( b(x) = 0 \) is a linear combination of the Busemann functions \( b_1, \ldots, b_k \) of the lines \( l_1, \ldots, l_k \) which vanish at \( x \). Thus \( b \) vanishes on \( \cap_j H_j \), and therefore \( F = \cap_j H_j \).

**Proof the property (1K).** Given a fiber (\( \mathbb{R} \)-line) \( F \subset X_\omega \) and \( x \in X_\omega \setminus F \), we show that there is a Ptolemy line \( l \subset X_\omega \) through \( x \) that hits \( F \). Uniqueness of \( l \) is proved at the end of sect. [5].

Using Proposition 5.12 we represent \( F = \cap_{l \in \mathcal{L}_\perp} H_l \), where \( \mathcal{L}_\perp \) is a finite collection of mutually orthogonal Ptolemy lines, \( H_l \) a horosphere of \( l \). Choosing appropriate orientations of the members of \( \mathcal{L}_\perp \), we can assume that \( H_l = b_l^{-1}(0) \) and \( b_l(x) > 0 \) for every \( l \in \mathcal{L}_\perp \), where \( b_l : X_\omega \to \mathbb{R} \) is a Busemann function of \( l \). Moving from \( x \) in an appropriate direction along a Ptolemy line, which is Busemann parallel to \( l \in \mathcal{L}_\perp \) with \( b_l(x) > 0 \), we reduce the value of \( b_l \) to zero keeping up every other Busemann function \( b_{l'}, l' \in \mathcal{L}_\perp \), constant. Repeating this procedure, we connect \( x \) with \( F \) by a piecewise geodesic curve with at most \( |\mathcal{L}_\perp| \) edges. Now, the zigzag construction produces a required Ptolemy line through \( x \) that hits \( F \). \( \square \)

### 5.4 Properties of the base \( B_\omega \)

We fix \( \omega \in X \) and a metric \( d \) from the Möbius structure for which \( \omega \) is infinitely remote. We also use notation \( |xy| = d(x, y) \) for the distance between \( x, y \in X_\omega \).
Lemma 5.17. Let \( \varphi : X_\omega \to X_\omega \) be a pure homothety with \( \varphi(o) = o \), \( o \in X_\omega \). Then \( \varphi \) preserves the fiber (\( \mathbb{K} \)-line) \( F \) through \( o \), \( \varphi(F) = F \). In particular, every shift \( \eta_{xx'} : X_\omega \to X_\omega \) with \( x, x' \in F \) preserves \( F \).

Proof. As in Lemma 4.15 we have \( \lambda b \circ \varphi = b \) for any Busemann function \( b : X_\omega \to \mathbb{R} \) with \( b(o) = 0 \), where \( \lambda \) is the coefficient of the homothety \( \varphi \). Since \( b(x) = 0 \) for every \( x \in F \), we see that \( b \circ \varphi(x) = 0 \), and thus \( \varphi(F) = F \), that is, \( \varphi(F) = F \). The assertion about a shift \( \eta_{xx'} \) follows now from the definition of \( \eta_{xx'} \). \( \square \)

Lemma 5.18. Let \( \eta : X_\omega \to X_\omega \) be a shift that preserves a \( \mathbb{K} \)-line \( F \subset X_\omega \). Then \( \eta \) preserves any other \( \mathbb{K} \)-line \( F' \subset X_\omega \).

Proof. Let \( b : X_\omega \to \mathbb{R} \) be a Busemann function associated with an (oriented) Ptolemy line \( l \subset X_\omega \). Then for any isometry \( \eta : X_\omega \to X_\omega \) the function \( b \circ \eta \) is a Busemann function associated with Ptolemy line \( \eta^{-1}(l) \). Thus for an arbitrary shift \( \eta : X_\omega \to X_\omega \), we have \( b \circ \eta = b + c_b \), where \( c_b \in \mathbb{R} \) is a constant depending on \( b \), because the line \( \eta^{-1}(l) \) is Busemann parallel to \( l \), hence the function \( b \circ \eta \) is also a Busemann function of \( l \), and the functions \( b, b \circ \eta \) differ by a constant.

In our case, when \( \eta \) preserves a \( \mathbb{K} \)-line, this constant is zero, \( c_b = 0 \), thus \( b \circ \eta = b \) for any Busemann function \( b : X_\omega \to X_\omega \). Therefore, \( \eta \) preserves any \( \mathbb{K} \)-line. \( \square \)

We define

\[
d(F, F') = \inf \{|xx'| : x \in F, x' \in F'\}
\]

for \( \mathbb{K} \)-lines \( F, F' \subset X_\omega \).

Lemma 5.19. Given \( \mathbb{K} \)-lines \( F, F' \subset X_\omega \), and \( x \in F \), there is \( x' \in F' \) such that \( d(F, F') = |xx'| \).

Proof. Let \( x_i \in F, x'_i \in F' \) be sequences with \( |x_i x'_i| \to d(F, F') \). Using Lemma 5.18 we can assume that \( x_i = x \) for all \( i \). Then the sequence \( x'_i \) is bounded, and by compactness of \( X \) it subconverges to \( x' \in F' \) with \( |xx'| = d(F, F') \). \( \square \)

Lemma 5.20. For any \( \mathbb{K} \)-lines \( F, F' \subset X_\omega \), we have \( d(F, F') = |xx'| \), where \( x = l \cap F, x' = l \cap F' \), and \( l \) is any Ptolemy line in \( X_\omega \) that meets both \( F, F' \).

Proof. By Lemma 5.13 the distance \( |xx'| \) is independent of the choice of \( l \). By definition \( |xx'| \geq d(F, F') \). By Lemma 5.19 there is \( x'' \in F' \) with \( |xx''| = d(F, F') \). The horosphere \( H \) of (a Busemann function associated with) \( l \) through \( x' \) contains \( F' \), in particular, \( x'' \in H \). Then \( |xx''| \geq |xx'| \) and hence, \( d(F, F') = |xx'| \). \( \square \)
Let \( \pi_\omega : X_\omega \to B_\omega \) be the canonical fibration, see sect. 5. For \( b \in B_\omega \), we denote with \( F_b = \pi_\omega^{-1}(b) \) the \( K \)-line over \( b \). For \( b, b' \in B_\omega \) we put \( |bb'| := |xx'| \), where \( x = l \cap F_b, x' = l \cap F_{b'} \), and \( l \subset X_\omega \) is any Ptolemy line that meets both \( F_b \) and \( F_{b'} \). By property \((1_K)\), such a line \( l \) exists, by Lemma 5.20 the number \( |bb'| \) is well defined, and the function \( (b, b') \mapsto |bb'| \) is a metric on \( B_\omega \). This metric is said to be canonical.

Proposition 5.21. The canonical projection \( \pi_\omega : X_\omega \to B_\omega \) is a 1-Lipschitz submetry with respect to the canonical metric on \( B_\omega \). Furthermore, \( B_\omega \) is a geodesic metric space with the property that through any two distinct points \( b, b' \in B_\omega \) there is a unique geodesic line in \( B_\omega \).

Proof. It follows from Lemma 5.20 that the map \( \pi_\omega \) is 1-Lipschitz. Let \( D = D_r(o) \) be the metric ball in \( X_\omega \) of radius \( r > 0 \) centered at a point \( o \in X_\omega \), \( D' \subset B_\omega \) the metric ball of the same radius \( r \) centered at \( \pi_\omega(o) \).

The inclusion \( D' \subset \pi_\omega(D) \) follows from the definition of the metric of \( B_\omega \). The opposite inclusion \( D' \supset \pi_\omega(D) \) holds because \( \pi_\omega \) is 1-Lipschitz. Thus \( \pi_\omega : X_\omega \to B_\omega \) is a 1-Lipschitz submetry.

Furthermore, by Lemma 5.20 the projection \( \pi_\omega \) restricted to every Ptolemy line in \( X_\omega \) is isometric, and thus by property \((1_K)\), the base \( B_\omega \) is a geodesic metric space. Moreover, it follows from \((1_K)\) that through any two distinct points \( b, b' \in B_\omega \) there is a unique geodesic line in \( B_\omega \).

Corollary 5.22. For any homothety \( \varphi : X_\omega \to X_\omega \), the induced map \( \pi_\omega(\varphi) : B_\omega \to B_\omega \) is a homothety with the same dilatation coefficient.

Proposition 5.23. The base \( B_\omega \) is isometric to an Euclidean \( \mathbb{R}^k \) for some \( k \leq \dim X \).

Proof. Any Busemann function \( b : X_\omega \to \mathbb{R} \) is affine on Ptolemy lines by Proposition 4.20. By definition, \( b \) is constant on the fibers of \( \pi_\omega \), thus it determines a function \( \overline{b} : B_\omega \to \mathbb{R} \) such that \( \overline{b} \circ \pi_\omega = b \). This function is affine on geodesic lines in \( B_\omega \) because every geodesic line \( \overline{l} \subset B_\omega \) is of the form \( \overline{l} = \pi_\omega(l) \) for some Ptolemy line \( l \subset X_\omega \), and each unit speed parameterization \( c : \mathbb{R} \to X_\omega \) of \( l \) induces the unit speed parameterization \( \overline{c} = \pi_\omega \circ c \circ \overline{l} \). Then \( \overline{b} \circ \overline{c} = b \circ c = b \circ c \) is an affine function on \( \mathbb{R} \).

We fix a base point \( o \in X_\omega \) and a maximal collection \( \mathcal{L} = \{l_1, \ldots, l_k\} \) of mutually orthogonal Ptolemy lines of \( X_\omega \) through \( o \). Let \( b_1, \ldots, b_k \) be Busemann functions of the lines \( l_1, \ldots, l_k \) respectively which vanish at \( o \). We denote with \( \overline{l}_i \) the projection of \( l_i \) to \( B_\omega \), and with \( \overline{b}_i : B_\omega \to \mathbb{R} \) the function corresponding to \( b_i, i = 1, \ldots, k \). By Proposition 5.12 the functions \( b_1, \ldots, b_k \) separates fibers in \( X_\omega \). Thus the functions \( \overline{b}_1, \ldots, \overline{b}_k \) separates points of \( B_\omega \), that is, for each \( z, z' \in B_\omega \) there is \( i \) with \( \overline{b}_i(z) \neq \overline{b}_i(z') \). Therefore, the continuous map \( h : B_\omega \to \mathbb{R}^k, h(z) = (\overline{b}_1(z), \ldots, \overline{b}_k(z)) \) is injective. This map is surjective by the same argument as in the proof of the property \((1_K)\), and it introduces coordinates on \( B_\omega \). We compute
the distance on $B_\omega$ in these coordinates. Applying a shift if necessary, see Corollary 5.22 we consider W.L.G. the distance $|\bar{z}|$ for every $z \in X_\omega$, where $\bar{z} = \pi_\omega(z)$. By (1) there is a unique Ptolemy line $l \subset X_\omega$ through $o$ that hits the fiber $F_z$ through $z$. It follows from our definitions that for $\alpha_i = \text{slope}(l,l_i)$ we have $b_i(z) = \alpha_i |o_i z|$, $i = 1, \ldots, k$. By Lemma 5.15, $\sum_i \alpha_i^2 = 1$, thus $|o z| = \sum_i b_i^2(z)$. This shows that $B_\omega$ is isometric to an Euclidean $\mathbb{R}^k$. We have $k \leq \dim X$, because $\pi_\omega : X_\omega \to B_\omega$ is a 1-Lipschitz submetry.

6 Extension of Möbius automorphisms of circles

6.1 Distance and arclength parameterizations of a circle

In this section, we establish existence of a distance parameterization in a Ptolemy circle and study its relationship with an arclength parameterization. A distance parameterization is convenient to obtain an important estimate (9) below. On the other hand, in an application of this estimate we use computation of slopes, which are most convenient to do in an arclength parameterization.

In what follows, we consider a (bounded) Ptolemy circle $\sigma \subset X_\omega$ and points $x, y \in \sigma$ with $a := |xy| > 0$.

Lemma 6.1. Let $\sigma_+, \sigma_-$ be the two components of $\sigma \setminus \{x, y\}$. Then for all $0 < t < a$ there exists exactly one point $x_t^+ \in \sigma_+$ (resp. $x_t^- \in \sigma_-$) with $|xx_t^+| = |xx_t^-| = t$. Therefore $\gamma : (-a, a) \to \sigma$ with $\gamma(0) = x$, $\gamma(t) = x_t^+$ for $t > 0$, and $\gamma(t) = x_t^-$ for $t < 0$ parameterizes a neighborhood of $x$ in $\sigma$.

Proof. The existence of a point $x_t^+ \in \sigma_+$ with $|xx_t^+| = t$ is clear by continuity. To prove uniqueness consider points $x < p < q < y$ in this order on $\sigma_+$ and assume $b := |xp| < a = |xy|$. Let $c := |xq|$, $\lambda_a := |pq|$, $\lambda_b := |qy|$, $\lambda_c := |py|$.

The Ptolemy equality and the triangle inequality give

$$a \lambda_a + b \lambda_b = c \lambda_c \leq c(\lambda_a + \lambda_b).$$

Therefore

$$c \geq \frac{\lambda_a}{\lambda_a + \lambda_b} a + \frac{\lambda_b}{\lambda_a + \lambda_b} b > b$$

where the last equality holds, since $a > b$. In particular $|xp| \neq |xq|$. □

In what follows, we use the parameterization $\gamma : (-a, a) \to \sigma$ of a neighborhood of $x \in \sigma$, and call it a distance parameterization.

Lemma 6.2. The function $g(t) := |\gamma(t)y|$ is concave and $C^1$-smooth on $(-a, a)$.
Proof. For \(-a \leq t_1 < t_2 \leq a\) the Ptolemy equality for the points \(x, \gamma(t_1), \gamma(t_2), y\) implies
\[ t_{2g}(t_1) - t_{1g}(t_2) = a|\gamma(t_1)\gamma(t_2)|. \]
Thus for \(-a \leq t_1 < t_2 < t_3 \leq a\) the triangle inequality \(|\gamma(t_1)\gamma(t_3)| \leq |\gamma(t_1)\gamma(t_2)| + |\gamma(t_2)\gamma(t_3)|\) implies
\[ t_{3g}(t_2) - t_{2g}(t_3) + t_{2g}(t_1) - t_{1g}(t_2) \geq t_{3g}(t_1) - t_{1g}(t_3) \]
which is equivalent to
\[ \frac{g(t_2) - g(t_1)}{t_2 - t_1} \geq \frac{g(t_3) - g(t_2)}{t_3 - t_2}. \]
Therefore, \(g\) is concave. It follows, in particular, that \(g\) has the left \(g'_-(t)\) and the right derivative \(g'_+(t)\) at every \(t \in (-a,a)\), \(g'_-(t) \geq g'_+(t)\) and these derivatives are nonincreasing, \(g'_+(t) \geq g'_+(t')\) for \(t < t'\). Furthermore, \(g'_-(t) \rightarrow g'_-(t')\) as \(t \nearrow t'\), \(g'_+(t') \rightarrow g'_+(t)\) as \(t' \searrow t\). These are standard well known facts about concave functions, see e.g. [H-U1].

We fix \(t_0 \in (-a,a)\) and consider the Ptolemy line \(l \subset X_{\omega}\) tangent to \(\sigma\) at \(x_0 = \gamma(t_0)\). We assume that \(l\) is oriented and that its orientation is compatible with the orientation of \(\sigma\) given by the distance parameterization \(\gamma\). Let \(c : \mathbb{R} \rightarrow X_{\omega}\) be the unit speed parameterization of \(l\) compatible with the orientation, \(c(0) = x_0\). By Corollary 4.19 \(g \notin l\). By Lemma 8.1 the function \(\tilde{g}(s) = [c(s)]y\), \(s \in \mathbb{R}\), is \(C^1\)-smooth. If \(t_0 = 0\), then \(g'_-(t_0) = \frac{d\omega}{dt}(0) = g'_+(t_0)\), because \(l\) is tangent to \(\sigma\) at \(x_0 = x\), and thus \(g\) is differentiable at \(t_0 = 0\).

Consider now the case \(t_0 \neq 0\). Then again by Corollary 4.28 \(x \notin l\), thus the function \(\tilde{f}(s) = [c(s)]x\), \(s \in \mathbb{R}\), is \(C^1\)-smooth. We show that \(\frac{d\tilde{f}}{dt}(0) \neq 0\). We suppose W.L.G. that \(t_0 > 0\). Then for all \(t_1 \in (0,t_0)\) sufficiently close to \(t_0\) we have \(\frac{d\tilde{h}}{dt}(0) \neq 0\), where \(\tilde{h}(s) = [x_1c(s)], x_1 = \gamma(t_1)\). We fix such a point \(t_1\), and using Lemma 6.1 consider the distance parameterization of a neighborhood of \(x_0 = \gamma(t_0)\) in \(\sigma\), \(|z(\tau)x_1| = \tau\) for all \(z \in \sigma\) sufficiently close to \(x_0\). Then \(t = t(\tau)\) and the function \(f(\tau) = |x\gamma o t(\tau)|\) is concave by the first part of the proof. Since the functions \(\tilde{f}(s) = [xc(s)], \tilde{h}(s) = [x_1c(s)]\) are \(C^1\)-smooth, and \(\frac{d\tilde{h}}{dt}(0) \neq 0\), the function \(\tilde{f}(\tau) = \tilde{f} \circ \tilde{h}^{-1}(\tau)\) is \(C^1\)-smooth in a neighborhood of \(\gamma(t_0) = [x_1x_0]\) by the inverse function theorem. Therefore, \(\tilde{f}'(\tau_0) = \frac{d\tilde{f}}{dt}(0) = f'_+(\gamma(t_0))\) because \(l\) is tangent to \(\sigma\) at \(x_0\). The assumption \(\frac{d\tilde{h}}{dt}(0) = 0\) implies \(\frac{d\tilde{h}}{dt}(\gamma(t_0)) = 0\). By concavity, \(\gamma(t_0)\) is a maximum point of the function \(f(\tau)\), and there are different \(\tau, \tau'\) arbitrarily close to \(\gamma(t_0)\) with \(f(\tau) = f(\tau')\). This contradicts properties of the parameterization \(\gamma(\tau) = \gamma o t(\tau)\). Hence, \(\frac{d\tilde{f}}{dt}(0) \neq 0\).

Again, by the inverse function theorem, the function \(\tilde{g}(t) = \tilde{g} \circ \tilde{f}^{-1}(t)\) is \(C^1\)-smooth in a neighborhood of \(t_0\). However, \(\frac{d\tilde{g}}{dt}(t_0)\) coincides with the left as well as with the right derivative of the function \(g\) at \(t_0\) because \(l\)
is tangent to \( \sigma \) at \( x_0 \). Therefore, \( g \) is differentiable at \( t_0 \). It follows from continuity properties of one-sided derivatives of concave functions that the derivative \( g' \) is continuous, i.e., \( g \) is \( C^1 \)-smooth.

**Lemma 6.3.** Every Ptolemy circle \( \sigma \subset X_\omega \) is rectifiable and

\[
L(xx') = |xx'| + o(|xx'|^2)
\]

as \( x' \to x \) in \( \sigma \), where \( L(xx') \) is the length of the (smallest) arc \( xx' \subset \sigma \).

**Proof.** We fix \( y \in \sigma \), \( y \neq x \), and introduce a distance parameterization \( \gamma : (-a,a) \to \sigma \) of a neighborhood of \( x = \gamma(0) \) in \( \sigma \). Rescaling the metric of \( X_\omega \) we assume that \( a = |xy| = 1 \) for simplicity of computations. We use notation \( d(z,z') = |zz'| \) for the distance in \( X_\omega \), and \( |zz'|_y \) for the distance in \( X_y \), assuming that

\[
|zz'|_y = \frac{|zz'|}{|zy||z'y|}
\]

is the metric inversion of the metric \( d \).

Recall that \( \sigma \setminus y \) is a Ptolemy line in \( X_y \). Thus for a given \( r \in (0,1) \), and for every partition \( 0 = t_0 \leq \cdots \leq t_n = r \) we have

\[
\sum_i |\gamma(t_i)\gamma(t_{i+1})| \leq \Lambda |xx_r|_y,
\]

where \( \Lambda = \max\{g^2(t) : 0 \leq t \leq r \}, g(t) = |\gamma(t)|_y, x_r = \gamma(r) \). Hence \( \sigma \) is rectifiable and \( L(xx_r) \leq \Lambda |xx_r|_y \). Moreover, using \( |\gamma(t_i)\gamma(t_{i+1})| = g(t_i)g(t_{i+1})|\gamma(t_i)\gamma(t_{i+1})|_y \), we actually have

\[
L(r) = L(xx_r) = \int_0^{r/g(r)} g^2(s)ds,
\]

where \( g(s) = g \circ t(s) \) with \( s = \frac{t(t_0)}{g(t_0)} = t/g(t) \). Recall that the function \( g(t) \) is \( C^1 \)-smooth by Lemma 6.2. Then \( ds = \frac{g(t)-tg'(t)}{g^2(t)} \) and \( \frac{dg}{ds} = \frac{g^2(t)}{g(t)-tg'(t)} \), in particular, \( \frac{dg}{ds}(0) = 1 \).

Using developments \( g(s) = 1 + \frac{dg}{ds}(0)s + o(s) \), \( g^2(s) = 1 + 2\frac{dg}{ds}(0)s + o(s) \), we obtain

\[
L(r) = \frac{r}{g(r)} + \frac{dg}{ds}(0) \frac{r^2}{g^2(r)} + o(r^2),
\]

where \( \frac{dg}{ds}(0) = g'(0)\frac{dg}{ds}(0) = g'(0) \). Since \( g(r) = 1 + g'(0)r + o(r) \), \( g^2(r) = 1 + 2g'(0)r + o(r) \), we finally have

\[
L(r) = r(1 - g'(0)r) + g'(0)r^2(1 - 2g'(0)r) + o(r^2) = r + o(r^2).
\]

Hence \( L(xx') = |xx'| + o(|xx'|^2) \) as \( x' \to x \) in \( \sigma \). \( \square \)
6.2 Slope of Ptolemy circles

If oriented Ptolemy circles $\sigma, \sigma' \subset X$ are disjoint, then their slope is not determined. Assume now that $\omega \in \sigma \cap \sigma'$. Then $slope_\sigma(\sigma, \sigma') \in [-1, 1]$ is defined as the slope of oriented Ptolemy lines $\sigma \setminus x, \sigma' \setminus x \subset X_\omega$. This is well defined and symmetric, $\text{slope}_\omega(\sigma, \sigma') = \text{slope}_\omega(\sigma', \sigma)$, by Lemma 4.20. In the case $\text{slope}_\omega(\sigma, \sigma') = \pm 1$, the Ptolemy circles are tangent to each other at $\omega$, having compatible ($-1$) or opposite ($+1$) orientations. This means that in any space $X_\omega$ with $\omega' \neq \omega$, the Ptolemy circles $\sigma \setminus \omega, \sigma' \setminus \omega' \subset X_\omega$ have a common tangent line at $\omega$.

More generally, let $l, l' \subset X_\omega$ be the tangent lines to $\sigma, \sigma'$ respectively at $\omega$, oriented according to the orientations of $\sigma, \sigma'$. Then $\text{slope}_\omega(\sigma, \sigma') = \text{slope}(l, l')$ because in the space $X_\omega$ the Ptolemy line $\sigma \setminus \omega$ is Busemann parallel to $l \setminus \omega$, and $\sigma' \setminus \omega$ is Busemann parallel to $l' \setminus \omega$, and we can apply Lemma 4.22.

Furthermore, if $\sigma, \sigma'$ have two distinct common points, $\omega, \omega' \in \sigma \cap \sigma'$, then $\text{slope}_\omega(\sigma, \sigma') = \text{slope}_\omega(\sigma, \sigma')$. This follows from Lemma 4.25.

**Lemma 6.4.** Assume that a (bounded) oriented Ptolemy circle $\sigma \subset X_\omega$ has two points in common with a Ptolemy line $l \subset X_\omega, x, y \in \sigma \cap l$, and the line $l$ is oriented from $y$ to $x$. Let $\sigma_+ \subset \sigma$ be the arc of $\sigma$ from $x$ to $y$ chosen according to the orientation of $\sigma$. Let $x_t, y_t$ be the distance parameterizations of neighborhoods of $x, y$ respectively such that $x_t, y_t \in \sigma_+$ for $t > 0, |x_t x| = t = |y_t y|$. Furthermore, let $b^\pm : X_\omega \rightarrow \mathbb{R}$ be the opposite Busemann functions of $l$ normalized by $b^+(x) = 0, b^+(y) = -a, b^-(x) = -a, b^-(y) = 0$, where $a = |xy|$. Then

$$b^+(x_t) + b^-(y_t) \leq 2\alpha t - \frac{1}{a} (1 - \alpha^2) t^2$$  \hspace{1cm} (9)

for all $t > 0$ in the domain of the parameterizations, where $\alpha = \text{slope}_\omega(\sigma, l) = \text{slope}_\omega(\sigma, l)$.

**Proof.** By Lemma 6.2 the functions $g(t) = |x_t y|, f(t) = |y_t x|$ are $C^1$-smooth and concave. Furthermore, their first derivatives at 0, $g'(0)$ and $f'(0)$, coincide with first derivatives of the distance functions to the respective tangent lines, $g'(0) = g'(0)$ and $f'(0) = f'(0)$, where $g(s) = |c_x(s) y|$, $f(s) = |c_y(s) x|$, and the unit speed parameterizations of the tangent lines $l_x$ to $\sigma$ at $x$ and $l_y$ to $\sigma$ at $y$ are chosen compatible with the distance parameterizations $x_t, y_t$ so that $c_x(0) = x, c_y(0) = y$.

Using that $l$ is oriented from $y$ to $x$, and $\sigma_+$ from $x$ to $y$ and applying equation (8), we find $g'(0) = \text{slope}(l_x, l)$. By the same equation (8) we have $f'(0) = \text{slope}(l_y, -l) = -\text{slope}(l_y, l)$. The sign $-1$ appears because the orientation of $l$ from $x$ to $y$ is opposite to the chosen orientation. Note that the orientation of $l_x$ is compatible with that of $\sigma$, while the orientation of $l_y$ is opposite to that of $\sigma$. Therefore, $g'(0) = \alpha = \text{slope}(\sigma, l)$ and $f'(0) = -\text{slope}(l_y, l) = \text{slope}(\sigma, l) = \alpha$.  

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Using concavity we obtain $g(t) \leq g(0) + g'(0)t = a + \alpha t$ and similarly $f(t) \leq a + \alpha t$ for all $0 \leq t < a$. The Ptolemy equality applied to the ordered quadruple $(x, x_t, y_t, y) \subset \sigma$ gives $g(t)f(t) = t^2 + a|x_ty_t|$, hence

$$|x_ty_t| \leq a + 2\alpha t - \frac{1}{a}(1 - \alpha^2)t^2.$$  

Let $H^+_t, H^-_t$ be the horospheres of $b^+, b^-$ through $x_t, y_t$ respectively, $x_t \in H^+_t, y_t \in H^-_t$. Since $X$ is Busemann flat, see Lemma 4.19 $H^+_t$ is also a horosphere of $b^-$ and $H^-_t$ is a horosphere of $b^+$. Thus $|x_ty_t| \geq \xi$, where $\xi$ is the distance between the points $l \cap H^+_t, l \cap H^-_t, \xi = a + b^+(x_t) + b^-(y_t)$. Therefore,

$$b^+(x_t) + b^-(y_t) \leq 2\alpha t - \frac{1}{a}(1 - \alpha^2)t^2.$$  

\[\square\]

### 6.3 Extension property

Surprisingly, the proof of the extension property (E2) is based on study of second order properties of Ptolemy circles like Lemma 6.4.

**Proposition 6.5.** Any compact Ptolemy space with properties (E) and (I) possesses the extension property (E2), see sect. 4.1.

**Lemma 6.6.** Any Möbius automorphism of any Ptolemy circle $\sigma \subset X$ preserving orientations extends to a Möbius automorphism of $X$.

**Proof.** We represent $\sigma$ as the boundary at infinity of the real hyperbolic plane, $\sigma = \partial_\infty H^2$. Then the group $G$ of preserving orientations Möbius automorphisms of $\sigma$ is identified with the group of preserving orientations isometries of $H^2$. The last is generated by central symmetries, and any central symmetry of $H^2$ induces an s-inversion of $\sigma$. Thus $G$ is generated by s-inversions of $\sigma$.

Now, any s-inversion of $\sigma$ can be obtained as follows. Take distinct $\omega, \omega' \in \sigma$ and a metric sphere $S \subset X$ between $\omega, \omega'$. Then an s-inversion $\varphi = \varphi_{\omega, \omega', S} : X \to X$ restricts to an s-inversion of $\sigma$. Thus any Möbius automorphism of $\sigma$ from $G$ extends to a Möbius automorphism of $X$.

The group Aut$X$ of Möbius automorphisms of $X$ is noncompact: a sequence of homotheties of $X_\omega$ with coefficients $\lambda_i \to \infty$ and with the same fixed point has no converging subsequences. However, we have the following standard compactness result.

**Lemma 6.7.** Assume that for a nondegenerate triple $T = (x, y, z) \subset X$ and for a sequence $\varphi_i \in$ Aut$X$ the sequence $T_i = \varphi_i(T)$ converges to a nondegenerate triple $T' = (x', y', z') \subset X$. Then there exists $\varphi \in$ Aut$X$ with $\varphi(T) = T'$.
Proof. For every $u \in X \setminus T$ the quadruple $Q = (T, u)$ is nondegenerate in the sense that its cross-ratio triple $\text{crt}(Q) = (a : b : c)$ has no zero entry. Since $\text{crt}(\varphi_i(Q)) = \text{crt}(Q),$ any accumulation point $u'$ of the sequence $u_i = \varphi_i(u)$ is not in $T'.$ Thus for the nondegenerate triple $S = (x, y, u)$ any sublimit $S' = (x', y', u')$ of the sequence $S_i = \varphi(S)$ is nondegenerate. Applying the same argument to any $v \in X \setminus S,$ we observe that the sequences $u_i, v_i = \varphi_i(v)$ have no common accumulation point. This shows that any limiting map $\varphi$ of the sequence $\varphi_i,$ obtained e.g. by taking a nonprincipal ultra-filter limit, is injective, and hence it is a Möbius automorphism of $X$ with $\varphi(T) = T'.$ □

**Proposition 6.8.** The group of Möbius automorphisms of $X$ acts transitively on the set of the oriented Ptolemy circles in $X.$ In particular, for any oriented circle $\sigma \subset X$ there is a Möbius automorphism $\varphi : X \rightarrow X$ such that $\varphi(\sigma) = \sigma$ and $\varphi$ reverses the orientation of $\sigma.$

Proof. We fix an oriented Ptolemy circle $\sigma \subset X$ and distinct points $x, y \in \sigma.$ For an oriented circle $\sigma_0 \subset X$ we denote with $A$ the set of all the circles $\varphi(\sigma_0), \varphi \in \text{Aut } X,$ with the induced orientation which pass also through $x$ and $y.$ Let

$$\alpha = \inf \{\text{slope}(\sigma, \sigma') : \sigma' \in A\}.$$ 

By two-point homogeneity property, see Proposition 4.17, $A \neq \emptyset.$ Applying Lemma 6.7 we find $\sigma' \in A$ with slope$(\sigma, \sigma') = \alpha.$ Next, we show that $\alpha < 1.$ Applying a shift we first make $\sigma_0$ disjoint with $\sigma.$ Taking a point $\omega \in \sigma_0$ as infinitely remote, we consider all Ptolemy lines in $X_\omega$ which are Busemann parallel to $\sigma_0 \setminus \omega$ and intersect $\sigma.$ Since $\sigma$ is bounded in $X_\omega,$ at least one of them, $l,$ is not tangent to $\sigma.$ Then slope$(\sigma, l) < 1.$ This $l$ can be obtained from $\sigma_0 \setminus \omega$ by a shift. Applying another shift to $l$ in the space $X_{\omega'}$ with $\omega' \in \sigma \cap l$ (this does not change the slope), we can assume that $x \in \sigma \cap l.$ Repeating this in the space $X_x,$ we find $\tilde{\sigma} \in A$ with slope$(\sigma, \tilde{\sigma}) < 1.$ Thus $\alpha < 1.$

We show that $\alpha = -1.$ Then $\sigma = \varphi(\sigma_0)$ as oriented Ptolemy circles for some $\varphi \in \text{Aut } X,$ which would complete the proof.

Assume that $\alpha > -1.$ The points $x, y$ subdivide each of the circles $\sigma,$ $\sigma'$ into two arcs. We choose an arc $\sigma_+ \subset \sigma$ leading from $x$ to $y$ according to the orientation of $\sigma,$ and an arc $\sigma'_+ \subset \sigma'$ leading from $y$ to $x$ according to the orientation of $\sigma'.$ Taking a point $\omega \in \sigma'$ inside of the opposite to $\sigma'_+$ arc, we see that $l = \sigma' \setminus \omega$ is a Ptolemy line in the space $X_\omega$ oriented from $y$ to $x.$

Given $x' \in \sigma_+,$ for every Ptolemy line $l_{x'} \subset X_\omega$ through $x'$, which is Busemann parallel to $l$ and is oriented as $l,$ we have slope$(\sigma, l_{x'}) \geq \alpha$ by the definition of $\alpha,$ because by the same argument as above $l_{x'}$ can be put in the set $A$ without changing the slope.

Let $b^ \pm : X_\omega \rightarrow \mathbb{R}$ be the opposite Busemann functions of $l$ normalized by $b^+(x) = 0, b^+(y) = -a, b^-(x) = -a, b^-(y) = 0,$ where $a = |xy|.$ Using
Lemma \([6.3]\) we consider for a sufficiently small \(\varepsilon > 0\) arclength parameterizations \(c_x, c_y : (-\varepsilon, \varepsilon) \to \sigma\) with \(c_x(0) = x, c_y(0) = y, c_x(s), c_y(s) \in \sigma_+\) for \(s > 0\), of neighborhoods of \(x, y\), respectively in \(\sigma\). Since Busemann functions on \(X_\omega\) are affine and hence differentiable along Ptolemy lines, and since the derivative \(\frac{db^+ c_x}{ds}(s)\) coincides with the derivative of \(b^+\) along the tangent line to \(\sigma\) at \(c_x(s)\), we have

\[
\frac{db^+ c_x}{ds}(s) = \text{slope}(\sigma, l_{c(s)}) \geq \alpha.
\]

For a sufficiently small \(t > 0\) let \(x_t \in \sigma_+\) be a point at the distance \(t\) from \(x\), \([x_t x] = t, x_t = c_x(\tau)\) for some \(\tau = \tau(t)\). By integrating we obtain \(b^+(x_t) \geq \alpha L(x(t)) \geq \alpha t\). A similar argument shows that \(b^-(y_t) \geq \alpha t\), where \(y_t = c_y(\tau')\) for some \(\tau' = \tau'(t), |y_t| = t\). Therefore, \(b^+(x_t) + b^-(y_t) \geq 2\alpha t\). This contradicts the estimate \([9]\) of Lemma \([6.4]\) Thus \(\alpha = -1\). \(\square\)

**Proof of Proposition \([6.7]\)** Given a Möbius map \(\psi : \sigma \to \sigma'\) between Ptolemy circles \(\sigma, \sigma' \subset X\), we choose orientations of \(\sigma, \sigma'\) so that \(\psi\) preserves the orientations. By Proposition \([6.8]\) there is \(\varphi \in \text{Aut } X\) with \(\varphi(\sigma) = \sigma'\) preserving the orientations. Then \(\varphi^{-1} \circ \psi : \sigma \to \sigma\) preserves the orientation of \(\sigma\), and hence it extends by Lemma \([8.4]\) to \(\varphi' \in \text{Aut } X\), \(\varphi'|\sigma = \varphi^{-1} \circ \psi\). Then \(\varphi \circ \varphi' \in \text{Aut } X\) is a required Möbius automorphism. \(\square\)

### 7 Topology of the space \(X\) and of \(\mathbb{K}\)-lines

#### 7.1 Groups of shifts

Recall that by Lemma \([4.15]\) a shift \(\eta : X_\omega \to X_\omega\) is an isometry that preserves every foliation of \(X_\omega\) by (oriented) Busemann parallel Ptolemy lines. Clearly, the shifts of \(X_\omega\) form a group which we denote with \(N_\omega\). Then \(N_\omega\) is a subgroup of the group \(\text{Aut } X\) of the Möbius automorphisms of \(X\).

**Lemma 7.1.** The group \(N_\omega\) acts simply transitively on \(X_\omega\).

**Proof.** Given \(x, x' \in X_\omega\), the shift \(\eta_{xx'}\) moves \(x\) to \(x'\), \(\eta_{xx'}(x) = x'\), by construction, see sect. \([4.3]\). Thus \(N_\omega\) acts transitively on \(X_\omega\).

Assume that \(\eta(x) = x\) for some shift \(\eta : X_\omega \to X_\omega\) and some \(x \in X_\omega\). We denote with \(V\) the fixed point set of \(\eta, \eta(y) = y\) for every \(y \in V\). We show that \(V = X\). Note that every Ptolemy line \(l \subset X_\omega\), which meets \(V\), is contained in \(V\) because the isometry \(\eta\) preserves every foliation of \(X_\omega\) by oriented Busemann parallel Ptolemy lines. Next, we note that every Ptolemy circle \(\sigma \subset X_\omega\), which meets \(V\) at two different points, is contained in \(V\). Indeed, the tangent to \(\sigma\) lines at these points are contained in \(V\). However, \(\sigma\) is uniquely determined by its tangent line and any other its point, see Corollary \([4.28]\).
Assume that \( V \neq X \), and let \( \omega' \in X \setminus V \). Since \( V \) is closed, \( \omega' \) is contained in \( X \setminus V \) together with a neighborhood \( U \) of \( \omega' \). Let \( \varphi : X \to X \) is a space inversion with \( \varphi(\omega') = \omega \), \( W = \varphi(V) \). Then \( W \) misses the neighborhood \( \varphi(U) \) of \( \omega \), and thus \( W \) is compact in \( X_\omega \). Furthermore, \( W \) contains together with any two different points every Ptolemy circle through these points. The image \( \mathcal{W} = \pi_\omega(W) \subset B_\omega \) under the canonical projection \( \pi_\omega : X_\omega \to B_\omega \) is compact since \( W \) is compact in \( X_\omega \). On the other hand, given \( z \in W \) and a Ptolemy line \( l \subset X_\omega \) through \( z \), there is a Ptolemy circle \( \sigma \subset W \) through \( z \) with the tangent line \( l \). Indeed, for every \( z' \in W \), \( z' \neq z \), by Corollary 1.28 there is a unique circle through \( z \), \( z' \) that is tangent to \( l \). This circle is contained in \( W \) by properties of \( W \). It follows that \( \mathcal{W} \) is open in \( B_\omega \) and thus \( \mathcal{W} = B_\omega \). This contradicts the fact that \( \mathcal{W} \) is compact.

Therefore, \( V = X \), and thus \( \eta = \text{id} \), i.e. the group \( N_\omega \) acts simply transitively on \( X_\omega \).

We fix \( o \in X_\omega \) and using Lemma 7.1 identify \( N_\omega \) with \( X_\omega \) by \( \eta \mapsto \eta(o) \).

Then \( N_\omega \) is a locally compact topological group.

An automorphism \( \tau : N_\omega \to N_\omega \) is said to be contractible if for every \( \eta \in N_\omega \) we have \( \lim_{n \to \infty} \tau^n(\eta) = \text{id} \). If \( N_\omega \) admits a contractible automorphism, then \( N_\omega \) is also said to be contractible.

**Lemma 7.2.** There is a contractible automorphism \( \tau : N_\omega \to N_\omega \).

**Proof.** We take any pure homothety \( \varphi : X_\omega \to X_\omega \) with \( \varphi(o) = o \) and with the coefficient \( \lambda \in (0, 1) \). Then we define \( \tau(\eta) = \varphi \circ \eta \circ \varphi^{-1} \). The map \( \eta' = \tau(\eta) : X_\omega \to X_\omega \) is an isometry preserving every foliation of \( X_\omega \) by Busemann parallel Ptolemy lines, i.e. \( \eta' \) is a shift, and it is clear that \( \tau \) is an automorphism of \( N_\omega \).

For the sequence of shifts \( \eta_n = \tau^n(\eta) \) we have \( \eta_n(o) = \varphi^n \circ \eta(o) \to o \) as \( n \to \infty \). Thus \( \eta_n \) converges to a shift \( \eta_\infty \) with \( \eta_\infty(o) = o \), hence, \( \eta_\infty = \text{id} \).

**Corollary 7.3.** The group \( N_\omega \) is a simply connected nilpotent Lie group. In particular, the space \( X_\omega \) is homeomorphic to \( \mathbb{R}^n \), and the space \( X \) is homeomorphic to the sphere \( S^n \) with \( n = \dim X \).

**Proof.** The group \( N_\omega \) is connected and locally compact because the space \( X_\omega \) is. By Lemma 7.2 \( N_\omega \) is contractible. Then by [Sieb] Corollary 2.4] \( N_\omega \) is a simply connected nilpotent Lie group.

We denote with \( Z_\omega \) a subgroup in \( N_\omega \) which consists of all shifts \( \eta \in N_\omega \) acting identically on the base \( B_\omega \), \( \pi_*(\eta) = \text{id} \), where \( \pi_*(\eta) : B_\omega \to B_\omega \) is the shift induced by the projection \( \pi_\omega : X_\omega \to B_\omega \), see Corollary 5.22. Every \( \eta \in Z_\omega \) preserves every fiber (K-line) of \( \pi_\omega \), see Lemma 5.17 and Lemma 5.18.
Proposition 7.4. The group $Z_\omega$ acts simply transitively on every $K$-line $F \subset X_\omega$, and thus it is a contractible, connected, locally compact topological group. Therefore, $Z_\omega$ is a simply connected nilpotent Lie group, and $F$ is homeomorphic to $\mathbb{R}^p$ for some $0 \leq p < n$.

Proof. The group $Z_\omega$ acts transitively on $F$ by Lemma [5.17]. The action is simply transitive by Lemma [7.1]. We fix $o \in F$ and identify $Z_\omega$ with $F$ by $\eta \mapsto \eta(o)$. By the same argument as in Lemma [7.2] we see that the group $Z_\omega$ is contractible. Furthermore, $F$ is locally compact. Given $x$, $x' \in F$, there is a Ptolemy circle $\sigma \subset X_\omega$ through $x$, $x'$. By (1$_K$), through any point $z \in \sigma$ there is a uniquely determined Ptolemy line that hits $F$. This defines a continuous map $\sigma \to F$. Thus $F$ is linearly connected. Hence, $Z_\omega$ is a contractible, locally compact, connected topological group. By [Sieh Corollary 2.4], $Z_\omega$ is a simply connected nilpotent Lie group, and thus $F$ is homeomorphic to $\mathbb{R}^p$ for some $0 \leq p \leq n$. In fact $p < n$ because $X$ contains Ptolemy circles and thus $k = \dim B_o > 0$, while $n = k + p$. \qed

To complete the proof of Theorem [4.5] it remains to show that if $k = 1$, then $X = \mathbb{R}$. In the following section, we establish a more general fact from which this property follows immediately.

7.2 Non-integrability of the canonical distribution

Given $o \in X_\omega$, $\pi \in B_\omega$, by the property (1$_K$) there is a unique $x \in F_\pi = \pi_\omega^{-1}(\pi)$ that is connected with $o$ by a geodesic segment $ox$. The point $x$ is called the lift of $\pi$ with respect to $o$, and we use notation $x = \text{lift}_o(\pi)$. This defines an embedding lift$_o : B_\omega \to X_\omega$ with $\pi_\omega \circ \text{lift}_o = \text{id}$ for every $o \in X_\omega$. We denote $D_o = \text{lift}_o(B_o)$. The embedding lift$_o$ is radially isometric, $|\text{lift}_o(\pi)| = |\pi(o)|\pi$ for every $\pi \in B_\omega$. Though there is no reason for lift$_o$ as well as for the projection $\pi_\omega|D_o$ to be isometric, the map lift$_o$ is continuous which follows the uniqueness property of (1$_K$) and compactness of $X$.

The family of subspaces $D_o$, $o \in X_\omega$, is called the (canonical) distribution on $X_\omega$. We say that the canonical distribution $\mathcal{D} = \{D_o : o \in X_\omega\}$ on $X_\omega$ is integrable if for any $o \in X_\omega$ and any $o' \in D_o$, the subspaces $D_o$ and $D_{o'}$ of $X_\omega$ coincide, $D_o = D_{o'}$. For example, if the base $B_o$ is one-dimensional, then the canonical distribution $\mathcal{D}$ is obviously integrable.

**Proposition 7.5.** Assume that the canonical distribution on $X_\omega$ is integrable for every $\omega \in X$. Then $p = 0$, i.e. every fiber of every projection $\pi_\omega$ is a point, and the space $X$ is Möbius equivalent to $\mathbb{R}^n$ with $n = \dim X$.

Proof. We first show that $\pi_\omega : D_o \to B_\omega$ is an isometry for every $o \in X_\omega$. Recall that the map $\pi_\omega : D_o \to B_\omega$ is radially isometric. For any $o'$, $x \in D_o$ we have $x \in D_{o'}$ because $D_o = D_{o'}$. Thus $|\pi_\omega(o')\pi_\omega(x)| = |o'x|$, and the map $\pi_\omega : D_o \to B_\omega$ is an isometry.
It follows that through any two distinct points \( \omega, o \in X \) there is a uniquely determined subspace \( B \subset X \), the induced Möbius structure of which is the canonical Möbius structure of the sphere \( S^k \), where \( k \) is the dimension of any base \( B_\omega, \omega \in X \). Any such a sphere is called a foliating sphere.

Next we show that two different foliating spheres \( B, B' \subset X \) have at most one point in common. For \( \omega \in B \cap B' \) consider a metric of the Möbius structure such that \( \omega \) is the infinitely remote point. Then \( B \setminus \{\omega\}, B' \setminus \{\omega\} \) are disjoint being the members of the foliation of \( X_\omega \) by foliating spheres.

Now, we exploit the same idea as in the proof of Lemma 4.21. Assume \( p > 0 \). Then there are different \( x, x' \in F \), and let \( B \subset X \) the foliating sphere through \( x, x' \). We have \( \omega \notin B \) since otherwise \( B \setminus \omega \) is a member of the foliation of \( X_\omega \), and \( B \setminus \omega \) is covered by Ptolemy lines \( l \) through \( x, x' \). Then by Lemma 4.21 \( B \setminus \omega \) must lie in \( F \). However \( F \) contains no Ptolemy line by construction.

Thus \( B \subset X_\omega \) is compact, and its projection \( \overline{B} = \pi_\omega(B) \subset B_\omega \) is compact. On the other hand, given \( z \in B \) and a Ptolemy line \( l \subset X_\omega \) through \( z \), there is a Ptolemy circle \( \sigma \subset B \) through \( z \) with the tangent line \( l \). Indeed, for every \( z' \in B, z' \neq z \), by Corollary 4.28 there is a unique circle through \( z, z' \) that is tangent to \( l \). This circle is contained in \( B \) because \( B \) is Möbius equivalent to \( \hat{\mathbb{R}}^k \) and it contains with any two points every Ptolemy circle in \( X \) through these points. It follows that \( \overline{B} \) is open in \( B_\omega \) and thus \( \overline{B} = B_\omega \).

This contradicts the fact that \( \overline{B} \) is compact in \( B_\omega \).

Hence \( p = 0, n = k \), and \( X \) is Möbius equivalent to \( \hat{\mathbb{R}}^n \) with \( n = \dim X \).

**Corollary 7.6.** Assume \( p > 0 \), that is, fibers of the canonical projections \( \pi_\omega, \omega \in X \), are nondegenerate. Then the canonical distribution on \( X_\omega \) is non-integrable for every \( \omega \in X \).

**Proof.** By Proposition 4.17 the space \( X \) is 2-point homogeneous. It follows that if the canonical distribution on \( X_\omega \) is integrable for some \( \omega \), then this is true for every \( \omega \in X \). Then \( p = 0 \) by Proposition 7.5. This contradicts our assumption.

The next corollary follows immediately from Proposition 7.5 and it completes the proof of Theorem 4.5.

**Corollary 7.7.** If the base \( B_\omega \) of \( X_\omega \) is one-dimensional (this is independent of \( \omega \in X \)), then \( p = 0 \) and \( X = \hat{\mathbb{R}} \).

### 8 Semi-C-planes

Starting from this place, we consider in what follows a compact Ptolemy space \( X \) that satisfies the assumption of Theorem 4.3 and its conclusion for
the case \( p = 1 \), i.e. when fibers of the canonical projection \( \pi_\omega : X_\omega \to B_\omega \)
are one-dimensional for every \( \omega \in X \), and therefore they are homeomorphic to \( \mathbb{R} \). This also means that the notation \( K \) is replaced for \( X \) by \( C \), e.g. fibers are \( C \)-lines, the properties (1\( C \)), (2\( K \)) become (1\( C \)), (2\( C \)) respectively, semi-\( K \)-planes are called semi-\( C \)-planes, etc. It follows from Corollary 7.7 that \( \dim B_\omega = k \geq 2 \), \( \omega \in X \).

### 8.1 Foliations of semi-\( C \)-planes

Let \( l \subset X_\omega \) be a Ptolemy line, \( M = M \) the respective semi-\( C \)-plane, see sect. 5.1. Any fiber (\( C \)-line) \( F \subset X_\omega \) of \( \pi_\omega \), that meets \( l \), is contained in \( M \) by definition of \( M \). We fix such an \( F \subset M \). By Lemma 5.3 every Ptolemy line in \( X_\omega \) through any point of \( F \), which is Busemann parallel to \( l \), is contained in \( M \) and hits every other \( C \)-line \( F' \subset M \). Furthermore, any two \( C \)-lines \( F, F' \subset M \) are equidistant in the sense that the segments of any two Ptolemy lines \( l, l' \subset M \) between \( F, F' \) have equal lengths.

**Proposition 8.1.** The map \( \psi : l \times F \to M \), \( \psi(x, y) = F_x \cap l_y \), where \( F_x \subset M \) is the \( C \)-line through \( x \in l \), \( l_y \subset M \) is the Ptolemy line through \( y \in F \), is a homeomorphism, in particular \( M \) is homeomorphic to \( \mathbb{R}^2 \).

**Proof.** The map \( \psi \) is well defined and bijective by properties of the projection \( \pi_\omega : X_\omega \to B_\omega \), of the semi-\( C \)-plane \( M \), and by (1\( C \)). Since \( X \) is Hausdorff, to show that \( \psi \) is a homeomorphism, it suffices to show that \( \psi \) is continuous.

Assume that \( F \ni y_i \to y \in F \), \( z = l_y \cap F_x \), \( z_i = l_i \cap F_x \), where \( l_y \), \( l_i = l_{y_i} \subset M \) are Ptolemy lines through \( y, y_i \) respectively, \( F_x \subset M \) is the \( C \)-line through \( x \in l \). Since \( |y_i z_i| = |y z| \) by the equidistant property, the sequence \( |z z_i| \) is bounded, thus \( z_i \in F_x \) subconverges to some \( z' \in F_x \). Since a pointwise limit of geodesics in any metric space is a geodesic, we see that \( l_i \to l' \) pointwise, where \( l' \subset X_\omega \) is a Ptolemy line through \( y, z' \). By (1\( C \)), \( l' = l_y \subset M \), hence \( z' = z = \lim z_i \). This shows that the map \( \psi_x : F \to F_x \), \( \psi_x(y) = \psi(x, y) \), is continuous for every \( x \in l \).

The map \( \psi_y : l \to l_y \) given by \( \psi_y(x) = \psi(x, y) \) coincides with the isometry \( \text{lift}_y \circ \pi_\omega | l : l \to l_y \). Now the map \( \psi \) is continuous at every point \((x, y) \in l \times F \) because

\[
|\psi(x', y') \psi(x, y)| \leq |\psi(x', y')\psi(x, y')| + |\psi(x, y')\psi(x, y)|
= |x'x| + |\psi(x, y')\psi_x(y)| \to 0
\]
as \((x', y') \to (x, y)\). \( \square \)

It follows from Proposition 8.1 that every semi-\( C \)-plane \( M \subset X_\omega \) carries two foliations, one by \( C \)-lines and the other by Ptolemy lines. Thus fixing an order on a \( C \)-line \( F \subset M \), we have a well defined order on any other
C-line $F' \subset M$ compatible with the Ptolemy line foliation of $M$. Therefore, given a Ptolemy line $l \subset M$, the notion of a half-plane of $M$ bounded by $l$ is well defined, and there are two half-planes bounded by $l$ whose union is $M$. Any homothety $\varphi : M \to M$ with $\varphi(l) = l$ either preserves each of the two half-plane bounded by $l$ or it permutes them.

### 8.2 Flips of semi-C-planes and self-duality

By the extension property $(E_2)$, every flip $\varphi : l \to l$ of a Ptolemy line $l \subset X_\omega$ extends to an isometry of $X_\omega$ for which we use the same notation $\varphi$. Since $\varphi$ preserves the line $l$, the semi-C-plane $M$ that contains $l$ is also preserved by $\varphi$ and $\varphi : M \to M$ is an isometry with a fixed point $o \in l \subset M$. If $\varphi$ preserves each of the half-planes bounded by $l$, then we say that $\varphi : M \to M$ is a flip.

**Proposition 8.2.** For every semi-C-plane $M \subset X_\omega$ every flip $\varphi : l \to l$ of any Ptolemy line $l \subset M$ extends to an isometry of $X_\omega$, which is a flip on $M$.

We begin the proof with following

**Lemma 8.3.** Let $M \subset X_\omega$ be a semi-C-plane, $l \subset M$ a Ptolemy line. Given distinct $p, p' \in l$, there exists a continuous path $\sigma : [0, 1] \to X_\omega$ between $p, p'$, $\sigma(0) = p, \sigma(1) = p'$, such that $\sigma([0, 1)) \subset X_\omega \setminus M$, and $o \sigma(t)$ is a geodesic segment in $X_\omega$ of length $|pp'|/2$ for the midpoint $o \in pp'$ and for all $t \in [0, 1]$.

**Proof.** It follows from the property (1$_C$) that for each distinct C-lines $F$, $F' \subset X_\omega$ there is a unique semi-C-plane $M \subset X_\omega$ that contains $F$, $F'$, and if distinct semi-C-planes $M, M' \subset X_\omega$ intersect, then their intersection $M \cap M'$ is a unique common C-line.

Let $F, F' \subset X_\omega$ be the C-lines through $p, p'$ respectively. Since the dimension of the base $B_\omega$ is at least 2, and every semi-C-plane in $X_\omega$ projects down to a line in the base, there is a point $x \in X_\omega \setminus M$. Let $F''$ be the C-line through $x$. We denote with $M', M''$ the semi-C-planes that contain the pairs $F, F''$ and $F', F''$ respectively. Then $F, F' \subset M$ bound a strip $FF' \subset M$ foliated by C-lines. Similarly, $FF'' \subset M', F'F'' \subset M''$ are respective strips foliated by C-lines.

We parameterize the set of the C-lines of $FF'' \cup F'F''$ by the segment $[0, 1]$ in the natural way, $t \mapsto F_t$, so that $F_0 = F, F_1/2 = F'', F_1 = F'$. Then the C-line $F_t$ is disjoint with the semi-C-plane $M$ for all $0 < t < 1$ by our construction. By the property (1$_C$), for each $F_t, 0 \leq t \leq 1$, there is a unique Ptolemy line $l_t \subset X_\omega$ connecting $o$ with $F_t$ (note that $l_0 = l = l_1$).

Now, the point $\sigma(t) \in l_t, 0 \leq t \leq 1$, is uniquely determined by conditions $|o\sigma(t)| = |pp'|/2$, and $\sigma(t)$ lies on the subray of $l_t$ with the vertex $o$ that intersects $F_t$. In particular, $\sigma(0) = p, \sigma(1) = p'$ and $\sigma(0, 1)) \subset X_\omega \setminus M$. Furthermore, $o\sigma(t)$ is a subsegment of the Ptolemy line $l_t$ in $X_\omega$. 

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Note that by Proposition \[8.1\] the map \( t \mapsto F_t \) is continuous in the sense that if \( t_i \to t \), then \( F_{t_i} \to F_t \) pointwise. Thus continuity of the map \( \sigma : [0,1] \to X_\omega \) follows from compactness of \( X \) and the fact that \( \sigma(t) \) is the unique geodesic segment in \( X_\omega \) between \( o \) and \( F_t \) for every \( t \in [0,1] \).

**Proof of Proposition \[8.2\]** Let \( o \in l \) be the fixed point of the flip \( \varphi : l \to l \), \( F \subset M \) the \( \mathbb{C} \)-line through \( o \). Since \( \varphi : l \to l \) is an isometry, every its extension \( \overline{\varphi} : X_\omega \to X_\omega \) is an isometry, the semi-\( \mathbb{C} \)-plane \( M \) is invariant under \( \overline{\varphi} \), \( \overline{\varphi}(M) = M \), by definition of \( M \), and \( \overline{\varphi}(F) = F \) because there is only one \( \mathbb{C} \)-line through \( o \). If the restriction \( \overline{\varphi}|F \) is the identity of \( F \) for some extension \( \overline{\varphi} \) of \( \varphi \), then \( \overline{\varphi} : M \to M \) is a flip.

We fix \( p \in l \), \( p \neq o \), and put \( p' = \varphi(p) \). Let \( \sigma : [0,1] \to X_\omega \) be the path between \( p = \sigma(0) \) and \( p' = \sigma(1) \) constructed in Lemma \[8.3\]. We denote by \( l_t \) the Ptolemy line in \( X_\omega \) that contains the segment \( o\sigma(t) \), and by \( \varphi_t : l \to l_t \) the isometry with \( \varphi_t(o) = o \), \( \varphi_t(p') = \sigma(t), t \in [0,1] \). Then \( \varphi_0 = \text{id}_l \) and \( \varphi_1 = \varphi \).

Assume that an isometry \( \alpha_t : X_\omega \to X_\omega \) fixes the Ptolemy line \( l_t \) pointwise for some \( t \in [0,1] \) and the restriction \( \alpha_t|F : F \to F \) is a flip. Using the same notation \( \varphi_t : X_\omega \to X_\omega \) for an extension of \( \varphi_t : l \to l_t \) that exists by \((E_2)\), we note that \( \beta_t = \varphi_t^{-1} \circ \alpha_t \circ \varphi_t : X_\omega \to X_\omega \) preserves \( M \), fixes the Ptolemy line \( l \) pointwise and \( \beta_t : F \to F \) is a flip. Then for every extension \( \overline{\varphi} \) of \( \varphi \) that is not a flip on \( M \), the isometry \( \overline{\varphi} \circ \beta_t : M \to M \) is a flip that extends \( \varphi \).

Thus we assume that for every \( t \in [0,1] \), every isometry \( \alpha_t : X_\omega \to X_\omega \), that fixes the Ptolemy line \( l_t \) pointwise, restricts to the identity of \( F \), \( \alpha_t|F = \text{id}_F \). The set

\[
A = \{ t \in [0,1] : \overline{\varphi}_t|F = \text{id}_F \text{ for every extension } \overline{\varphi}_t : X_\omega \to X_\omega \text{ of } \varphi_t \}
\]

is closed in \([0,1]\) by continuity and \( 0 \in A \) by our assumption. We show that \( A \) is open in \([0,1]\). If not, then there is a sequence \( t_i \in [0,1] \setminus A \) converging to some \( t \in A \), and for every \( i \) there is an extension \( \overline{\varphi}_i : X_\omega \to X_\omega \) of \( \varphi_{t_i} \), such that \( \overline{\varphi}_i|F : F \to F \) is a flip. The sequence \( \{\overline{\varphi}_i\} \) converges to an isometry \( \psi : X_\omega \to X_\omega \) such that \( \psi|l = \varphi_l \) and \( \psi|F : F \to F \) is a flip. This contradicts the condition \( t \in A \). Thus \( A = [0,1] \) and \( \varphi = \varphi_1 \) extends to the required flip \( \overline{\varphi} : M \to M \).

Given a Ptolemy circle \( \sigma \subset X \) and distinct \( \omega, \omega' \in \sigma \), recall that the set \( B_{\sigma,\omega}^\omega \) consists of all \( x \in X_\omega \) with \( b^\pm(x) \geq 0 \), where \( b^\pm : X_\omega \to \mathbb{R} \) are the opposite Busemann functions of the Ptolemy line \( \sigma \setminus \omega \subset X_\omega \) with \( b^\pm(\omega') = 0 \), see sect. \[3.1\]. In a Busemann flat Ptolemy space, \( B_{\sigma,\omega}^\omega = H_{\sigma,\omega}^\omega \) is the horosphere of \( \sigma \setminus \omega \) through \( \omega' \). Furthermore, \( D_{\sigma,\omega}^\omega \) is the subset in \( X_\omega \) which consists of all \( x \) such that \( \omega \) is a closest to \( x \) point in the geodesic line \( \sigma \setminus \omega' \) (w.r.t. the metric of \( X_\omega \)). By duality, see Lemma \[3.2\], we have

\[
B_{\sigma,\omega}^\omega \cup \omega = D_{\sigma,\omega}^\omega \cup \omega',
\]

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or in the case of a flat space by Lemma 3.3

\[ H_{\omega}^\sigma \cup \omega = D_{\omega}^\sigma \cup \omega'. \]

It follows that

\[ B_{\omega}^\sigma \cup \omega = D_{\omega}^\sigma \cup \omega', \]

where \( B_{\omega}^\sigma \) is the intersection of \( B_{\omega}^\sigma \) taken over all the Ptolemy circles \( \sigma \) in \( X \) containing both \( \omega, \omega' \). \( D_{\omega}^\sigma \) is the intersection of \( D_{\omega}^\sigma \) taken over all the Ptolemy circles \( \sigma \subset X \) containing both \( \omega, \omega' \). In the case of a flat space this equality takes the form

\[ H_{\omega}^\sigma \cup \omega = D_{\omega}^\sigma \cup \omega', \]

where \( H_{\omega}^\sigma = F \) is the fiber through \( \omega' \) of the canonical projection \( \pi_\omega : X_\omega \to B_{\omega} \). In other words, for the fiber (C-line) \( F \subset X_\omega \) through \( \omega' \), every point \( x \in F \setminus \omega' \) has the property that \( \omega \) is a closest point to \( x \) on the Ptolemy line \( \sigma \setminus \omega' \subset X_\omega \) in the space \( X_\omega \) for every Ptolemy circle \( \sigma \subset X \) through \( \omega, \omega' \).

Corollary 8.4. The space \( X \) is self-dual in the sense that \( H_{\omega}^\sigma = D_{\omega}^\sigma \) for each distinct \( \omega, \omega' \in X \). That is, every point \( x \) of the C-line \( F \subset X_\omega \) through \( \omega' \) has the property that \( \omega' \) is a closest to \( x \) point on every Ptolemy line in \( X_\omega \) through \( x \), and vice versa.

Proof. We show that \( H_{\omega}^\sigma \subset D_{\omega}^\sigma \). We fix \( x \in H_{\omega}^\sigma \) and let \( l \subset X_\omega \) be a Ptolemy line through \( \omega' \). There is \( z \in l \) closest to \( x \) among all the points of \( l \). By Proposition 8.2 there is an isometry \( \varphi : X_\omega \to X_\omega \) with \( \varphi(x) = x \) that flips \( l \) at \( \omega' \). Then \( z' = \varphi(z) \in l \) is also closest to \( x \). By convexity of the distance function from \( x \) along \( l \), the segment \( zz' \subset l \) consists of points closest to \( x \). However, \( \omega' \in zz' \). Hence \( H_{\omega}^\sigma \subset D_{\omega}^\sigma \). By duality, \( D_{\omega}^\sigma \cup \omega = H_{\omega}^\sigma \cup \omega' \). Thus \( H_{\omega}^\sigma \cup \omega \subset H_{\omega}^\sigma \cup \omega' \). Interchanging \( \omega \) with \( \omega' \), we obtain \( H_{\omega}^\sigma \cup \omega = H_{\omega}^\sigma \cup \omega' = D_{\omega}^\sigma \cup \omega, \) i.e. \( H_{\omega}^\sigma = D_{\omega}^\sigma \), and \( X \) is self-dual. \( \square \)

We put \( \hat{F} = F \cup \omega \) for every fiber \( F \) of \( \pi_\omega \). Then \( \hat{F} \) is a compact subset of \( X \) called a C-circle. Since \( F \) is homeomorphic to \( \mathbb{R} \), \( \hat{F} \) is homeomorphic to \( S^1 \). It follows from Lemma 3.2 that the collection of all the C-circles in \( X \) is invariant under any Möbius automorphism of \( X \).

Lemma 8.5. For every C-circle \( \hat{F} \subset X \) and every point \( \omega \in \hat{F} \), the set \( F = \hat{F} \setminus \omega \) is a fiber of the fibration \( \pi_\omega : X_\omega \to B_\omega \).

Proof. We have \( \hat{F} = F' \cup \omega' \) for some \( \omega' \in X \), where \( F' \) is a fiber of the fibration \( \pi_{\omega'} : X_{\omega'} \to B_{\omega'} \). We suppose that \( \omega' \neq \omega \) since otherwise the assertion is trivial. Then \( \omega \in F' \) and \( F = H_{\omega}^\omega \). By self-duality, see Corollary 8.4, \( H_{\omega}^\omega = D_{\omega}^\omega \). Using duality we obtain \( \hat{F} = F' \cup \omega' = H_{\omega}^\omega \cup \omega \). Thus \( F = \hat{F} \setminus \omega = H_{\omega}^\omega \) is the fiber of \( \pi_\omega \) through \( \omega' \). \( \square \)
Corollary 8.6. In addition to (1C) and (2C), see Theorem 4.5, the space $X$ has the following basic properties

(3C) through any two distinct points in $X$ there is a unique $C$-circle;

(4C) any $C$-circle and any Ptolemy circle in $X$ have at most two points in common.

Proof. (3C). Given distinct $w, w' \in X$, let $F \subset X_w$ be the fiber of $\pi_w : X_w \to B_w$ through $w'$. Then $\tilde{F} = F \cup w$ is a $C$-circle through $w, w'$. By Lemma 8.3, $F' = \tilde{F} \setminus w'$ is the fiber of $\pi_{w'} : X_{w'} \to B_{w'}$ through $w$. Since the fibers of $\pi_w, \pi_{w'}$ through given points are uniquely determined, it follows that $\tilde{F}$ is a unique $C$-circle through $w, w'$.

(4C). Let $w \in \tilde{F} \cap \sigma$ be a common point of a $C$-circle $\tilde{F} \subset X$ and a Ptolemy circle $\sigma \subset X$. Then $l = \sigma \setminus w \subset X_w$ is a Ptolemy line, and by Lemma 8.3, $F = \tilde{F} \setminus w$ is a fiber of $\pi_w : X_w \to B_w$. It follows from the definition of fibers of $\pi_w$ and Lemma 12A that $F$ and $l$ have at most one point in common. Hence, the claim. \qed

8.3 Both foliations of semi-$C$-planes are equidistant

We already know that the foliation of a semi-$C$-plane $M \subset X_w$ by $C$-lines is equidistant. Now, Ptolemy lines $l, l'$ in $M$ are called equidistant if the distance between $x = l \cap F$ and $x' = l' \cap F$ is independent of a $C$-line $F \subset M$.

Lemma 8.7. All the Ptolemy lines in every semi-$C$-plane $M \subset X$ are pairwise equidistant.

Proof. Let $l, l' \subset M$ be Ptolemy lines, $c, c' : \mathbb{R} \to M$ unit speed parameterizations of $l, l'$ respectively with $c(0), c'(0) \in F$, where $F \subset M$ is a $C$-line. We assume using equidistant property of $C$-lines that $c, c'$ are compatible in the sense that the points $c(t), c'(t)$ lie in one and the same $C$-line $F_t \subset M$ for every $t \in \mathbb{R}$. We put $\mu(t) = |c(t)c'(t)|$. By self-duality, $c(t)$ is a closest to $c(t)$ point on $l'$, and vice versa, $c(t)$ is a closest to $c'(t)$ point on $l$ for every $t \in \mathbb{R}$. Thus $|c(t)c'(t)|, |c'(t)c(t)| \geq \max\{\mu(t), \mu(t')\}$ for each $t, t' \in \mathbb{R}$. Applying the Ptolemy inequality to the quadruple $(c(t), c(t'), c'(t'), c'(t))$, we obtain

$$\max\{\mu(t), \mu(t')\}^2 \leq |c(t)c'(t')||c'(t)c(t')| \leq \mu(t)\mu(t') + (t-t')^2.$$ 

We show that $\mu(a) = \mu(0)$ for every $a \in \mathbb{R}$. Assume W.I.G. that $a > 0$ and put $m = 1/\min_{0 \leq s \leq a} \mu(s)$. Then $|\mu(t) - \mu(t')| \leq m(t-t')^2$ for each $0 \leq t, t' \leq a$. Now

$$\mu(a) - \mu(0) = \mu(s) - \mu(0) + \mu(2s) - \mu(s) + \cdots + \mu(a) - \mu((k-1)s),$$

where $s = a/k$ for $k \in \mathbb{N}$. It follows $|\mu(a) - \mu(0)| \leq mks^2 = ma^2/k \to 0$ as $k \to \infty$. Hence, $\mu(a) = \mu(0).$ \qed

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Lemma 8.8. For any \( x, y \in F \), \( x', y' \in F' \), where \( F, F' \) are \( \mathbb{C} \)-lines in a semi-\( \mathbb{C} \)-plane \( M \), such that \( x, x' \in l \), \( y, y' \in l' \), where \( l, l' \subset M \) are Ptolemy lines, we have \( |xy| = |x'y'| =: a, |xx'| = |yy'| =: b, |xy'| = |x'y| =: c \), and \( c^2 \leq a^2 + b^2 \).

Proof. We have \( |xx'| = |yy'| \) because \( \mathbb{C} \)-lines in \( M \) are equidistant, and \( |xy| = |x'y'| \) because Ptolemy lines in \( M \) are also equidistant. By Proposition 8.2, there exists a flip \( \varphi : M \to M \) that permutes \( x, y \) and \( x', y' \) respectively, \( \varphi (x) = x', \varphi (y) = y' \). Thus \( |xy'| = |x'y| \). Applying the Ptolemy inequality, we obtain \( c^2 \leq a^2 + b^2 \). \( \square \)

For any two fibers \( F_b, F_{b'} \) of \( \pi_\omega \), by (1\( \mathbb{C} \)) and Lemma 8.7, we have the canonically determined isometry \( \mu_{b'} : F_b \to F_{b'} \).

Lemma 8.9. The isometries \( \mu_{b'} : F_b \to F_{b'} \) depend continuously of \( b, b' \in B_\omega \), that is, for \( b_i \to b' \) and for any \( x \in F_b \), we have \( \mu_{b_i}(x) \to \mu_{b'}(x) \).

Proof. If a sequence of geodesic segments in a metric space pointwise converges, then the limit is also a geodesic segment. Together with uniqueness of Ptolemy lines in \( \omega \) and compactness of \( X_\omega \), this implies the claim. \( \square \)

Now, we fix an order of \( F_b \) and define the order of \( F_{b'} \) via the isometry \( \mu_{b'} \). This gives a simultaneously determined order \( O \) on all the fibers of \( \pi \).

Lemma 8.10. The order \( O \) is well defined and independent of the choice of \( b \in B_\omega \).

Proof. The base \( B_\omega \) is contractible. Using Lemma 8.9, we see that the order of \( F_{b'} \) induced by \( \mu_{b'b'} \) coincides with the order induced by \( \mu_{b'b'} \circ \mu_{b'} \) for each \( b', b'' \in B_\omega \). Hence, the claim.

Let \( G_\omega \) be the group of all Möbius automorphisms of \( X \) fixing \( \omega \) and preserving the order \( O \). Every \( \varphi \in G_\omega \) acts on \( \omega \) as a homothety. By Corollary 8.22, we have a homomorphism \( \pi_* \) of \( G_\omega \) into the group of homotheties of \( B_\omega \). We denote \( H = \pi_*(G_\omega) \).

9 Isometry group of \( X_\omega \)

Fix \( \omega \in X \) and a metric from the Möbius structure of \( X \) such that \( \omega \) is the infinitely remote point. Recall that \( N_\omega \subset \text{Aut} \ X \) is the group of shifts of \( X_\omega \), see sect. 7.4.

Lemma 9.1. Every shift \( \eta \in N_\omega \) preserves the order \( O \) on \( \mathbb{C} \)-lines in \( X_\omega \).

Proof. Let \( \varphi : X \to X \) be a space inversion that permutes \( \omega, \omega' \in X \). By Corollary 8.11 (3\( \mathbb{C} \)), \( \varphi \) preserves the (unique) \( \mathbb{C} \)-circle \( \hat{F} \) through \( \omega, \omega' \), \( \varphi(\hat{F}) = \hat{F} \). Since \( \varphi \) has no fixed point, \( \varphi \) also preserves orientations of
\(\hat{F}\). Let \(\gamma : X_\omega \to X_\omega\) be a pure homothety with \(\gamma(\omega') = \omega'\). Recall that 
\(\gamma = \varphi' \circ \varphi\) for some space inversions \(\varphi, \varphi'\) which both permute \(\omega, \omega'\). It follows that 
\(\gamma(\hat{F}) = \hat{F}\) and \(\gamma\) preserves the order \(O\) of the \(C\)-line \(F = \hat{F}\setminus \omega\) through \(\omega'\).

By Lemma 8.10, \(\gamma\) preserves every foliation of \(X_\omega\) by Busemann parallel Ptolemy lines, thus \(\gamma(F')\) is a \(C\)-line for every \(C\)-line \(F' \subset X_\omega\). Now, by continuity \(\gamma\) preserves the order \(O\) on \(C\)-lines in \(X_\omega\). Then it follows from the definition of shifts that every shift \(\eta : X_\omega \to X_\omega\) preserves the order \(O\).

Lemma 9.2. Given a Ptolemy line \(l \subset X_\omega\), \(x, x' \in l\), the shift \(\eta : X_\omega \to X_\omega\) with \(\eta(x) = x'\) preserves every Ptolemy line \(l' \subset M\), \(\eta(l') = l'\), where \(M \subset X_\omega\) is the semi-\(C\)-plane containing \(l\). In particular, every isometry \(\mu_{bb'} : F_b \to F_{b'}\) between \(C\)-lines \(F_b, F_{b'} \subset X_\omega\) extends to shift \(\eta : X_\omega \to X_\omega\).

Proof. Let \(F, F' \subset X_\omega\) be the \(C\)-lines through \(x, x'\) respectively. Then \(F, F' \subset M\). We put \(y = l' \cap F, y' = l' \cap F'\) and note that \(|xy| = |x'y'|\) by Lemma 8.7.

We have \(\eta(M) = M\), thus \(\eta(l') \subset M\). By Lemma 9.1, \(\eta\) preserves the order \(O\), thus \(\eta(y) = y'\) and \(y' \in \eta(l')\). Since the Ptolemy line in \(M\) through \(y'\) is unique, we obtain \(\eta(l') = l'\).

9.1 Isometries of \(C\)-lines

A shift \(\eta : X_\omega \to X_\omega\) is said to be vertical if it induces the identity of the base \(B_\omega, \pi_\ast(\eta) = \text{id}\).

Proposition 9.3. The displacement function of every vertical shift \(\eta : X_\omega \to X_\omega\) is constant.

Let \(T \subset B_\omega\) be a pointed oriented triangle, i.e., we assume that an orientation and a vertex \(o\) of \(T\) are fixed. Then \(T\) determines a map \(\tau_T : F \to F\), where \(F \subset X_\omega\) is the fiber of \(\pi_\omega\) over \(o\), \(F = \pi^{-1}_\omega(o)\). Given \(x \in F\), we lift to \(X_\omega\) the sides of \(T\) in the cyclic order according to the orientation and starting with \(o\) which is initially lifted to \(x\). Then \(\tau_T(x) \in F\) is the resulting lift of the triangle sides.

Lemma 9.4. The map \(\tau_T : F \to F\) is an isometry that preserves the order and has the constant displacement function.

Proof. The map \(\tau_T\) is obtained as a composition of three \(C\)-line isometries of type \(\mu_{bb'}\), see sect. 8.3. Any isometry \(\mu_{bb'}\) preserves the order \(O\), see Lemma 8.10. Thus \(\tau_T : F \to F\) is an isometry preserving the order.

Given \(x, x' \in F\), we let \(y = \tau_T(x), y' = \tau_T(x')\). There is a shift \(\eta : X_\omega \to X_\omega\) with \(\eta(x) = x'\). By Lemma 8.11 \(\eta\) is vertical, and by Lemma 9.1 it preserves the order \(O\). Thus \(\eta\) preserves \(F\), every semi-\(C\)-plane \(M \subset X_\omega\).
and maps isometrically every Ptolemy line \( l \subset M \) to an equidistant Ptolemy line \( l' \subset M \). Then it follows from definition of \( \tau_T \) that \( \eta(y) = y' \). Therefore \(|x'y'| = |xy|\), and the displacement function of \( \tau_T \) is constant. \( \square \)

**Lemma 9.5.** For every \( b \in B_\omega \) there is a pointed oriented triangle \( T \subset B_\omega \), for which the map \( \tau_T : F \to F, F = \pi_\omega^{-1}(b) \), is not identical.

**Proof.** By Corollary 7.6 the canonical distribution \( D \) on \( X_\omega \) is not integrable. This means that there are \( o \in X_\omega, x \in D_o \) such that \( x \notin D_{o'} \). Then the points \( b' = \pi_\omega(o), \pi_\omega(o'), \pi_\omega(x) \in B_\omega \) are vertices of a pointed oriented triangle \( T' \) for which the map \( \tau_{T'} : F' \to F' \) is not identical, where \( F' = \pi_\omega^{-1}(b') \). There is a shift \( \eta : X_\omega \to X_\omega \) with \( \eta(F') = F \). The shift \( \eta \) induces a shift \( \pi_*(\eta) : B_\omega \to B_\omega \) of the base. Then \( T = \pi_*(\eta)(T') \) is a required triangle in \( B_\omega \). \( \square \)

**Proof of Proposition 9.3.** We fix a \( \mathbb{C} \)-line \( F \subset X_\omega \) and first show that the displacement function of \( \eta \) is constant along \( F \).

Let \( \lambda(T) \) be the perimeter of a pointed oriented triangle \( T \subset B_\omega \) with the base point \( b = \pi_\omega(F) \). By triangle inequality, we have \(|x\tau_T(x)| \leq \lambda(T)\) for every \( x \in F \). Note that \( \tau_T^k = \tau_{Tk} \) for every \( k \in \mathbb{Z} \), where \( T^k \) means that the triangle \( T \) is passed around \( |k| \) times in the direction prescribed by the sign of \( k \). Thus the displacement function of \( \tau_T^k \) is also constant by Lemma 9.4.

Using Lemma 9.5 and applying appropriate homotheties of \( X_\omega \), we can find a pointed oriented triangle \( T \) with the base point \( b \), arbitrarily small perimeter and non-identical isometry \( \tau_T : F \to F \). By construction, \( \tau_T \) is a composition of three isometries of type \( \mu_{ib} : F_0 \to F_Y \). Then by Lemma 9.2 \( \tau_T \) extends to an isometry from the group \( N_\omega \), which is a vertical shift. We use the same notation \( \tau_T \) for this extension, \( \tau_T : X_\omega \to X_\omega \).

Given \( x \in F \), there is a sequence \( T_i \) of pointed oriented triangles with the base point \( b \) and \( \lambda(T_i) \to 0 \), and a sequence \( k_i \in \mathbb{Z} \) such that \( \tau_i(x) = \tau_T^{k_i}(x) \to \eta(x) \). Then \( \tau_i : X_\omega \to X_\omega \) subconverges to a vertical shift \( \tau : X_\omega \to X_\omega \) with \( \tau(x) = \eta(x) \) that preserves the order and has the constant displacement along \( F \). Using Lemma 7.4 we conclude that \( \eta = \tau \) has constant displacement along \( F \).

Now, the displacement of \( \eta \) is constant because \( \eta \) preserves every \( \mathbb{C} \)-line in \( X_\omega \) and Ptolemy lines in semi-\( \mathbb{C} \)-planes are equidistant by Lemma 5.7. \( \square \)

**9.2 Existence of an unclosed parallelogram**

Let \( P \subset B_\omega \) be a pointed oriented parallelogram, i.e., we assume that an orientation and a vertex \( o \) of \( P \) are fixed. Similarly to the map \( \tau_T \) discussed in sect. 9.1, we have a map \( \tau_P : F \to F \), where \( F \subset X_\omega \) is the fiber of \( \pi \) over \( o \), \( F = \pi^{-1}(o) \). Namely, given \( x \in F \), we lift the sides of \( P \) to \( X_\omega \) in the cyclic order according the orientation and starting with \( o \) which is
initially lifted to $x$. Then $\tau_P(x) \in F$ is the resulting lift of the parallelogram sides. We say that an oriented parallelogram $P \subset B_\omega$ is closed, if the map $\tau_P : F \to F$ is identical, $\tau_P = \text{id}_F$. This property depends of the choice neither of the base vertex nor of the orientation of $P$. Thus we can speak about closed or unclosed parallelograms as well as closed or unclosed triangles in $B_\omega$. By Lemma 9.5 there exists an unclosed triangle.

We need the following fact.

**Lemma 9.6.** Let $G$ be a closed subgroup of the orthogonal group $\mathcal{O}(n)$, $n \geq 2$, which acts transitively on the sphere $S^{n-1} \subset \mathbb{R}^n$. Then for any 2-dimensional subspace $L \subset \mathbb{R}^n$ there is $g \in G$ such that $g(v) = -v$ for every $v \in L$.

**Proof.** A list of all compact connected Lie groups acting transitively and effectively on the sphere $S^{n-1}$ is obtained in [MS], [Bor]. It consists of $SO(n), U(m), SU(m)$ with $n = 2m$, $Sp(m)Sp(1)$, $Sp(m)U(1)$, $Sp(m)$ with $n = 4m$, $G_2$ with $n = 7$, Spin$(7)$ with $n = 8$, Spin$(9)$ with $n = 16$, see [Bes], sect. 7.13. The required property is obvious for $SO(n)$. The groups $U(m)$, $SU(m)$ with $n = 2m$ include the element $-\text{id}$. This is also true for $Sp(m)$ with $n = 4m$ and Spin$(7)$, Spin$(9)$.

The group $G_2$ is the automorphism group of the octonions $\mathcal{O}$ which acts on imaginary octonions $\text{Im } \mathcal{O} = \mathcal{R}^7$ in a way that any basic triple $e_1, e_2, e_3 \in \text{Im } \mathcal{O}$ is moved to any other basic triple by uniquely determined automorphism $g \in G_2$, see [Ba]. Here $e_1$ can be chosen arbitrarily with $e_1^2 = -1$, $e_2$ with $e_2^2 = -1$ anticommutes with $e_1$, $e_1e_2 = -e_2e_1$, and $e_3$ with $e_3^2 = -1$ anticommutes with $e_1, e_2$ and $e_1e_2$. We can take basic triples $(e_1,e_2,e_3)$ and $(e_1',e_2',e_3')$ so that $e_1, e_2 \in L \cap S^6 \subset \text{Im } \mathcal{O}, e_1' = -e_1, e_2' = -e_2, e_3' = e_3$ and put $g(e_1) = e_1', g(e_2) = e_2', g(e_3) = e_3'$. This defines a required $g \in G_2$ with $g|L = -\text{id}_L$.

**Lemma 9.7.** There exists a unclosed parallelogram in $B_\omega$.

**Proof.** Using property (E2) and argue as in the proof of Proposition 8.2 one shows that the stabilizer of any $o \in X_\omega$ in the isometry group of $X_\omega$ preserving the order acts transitively on the set of the directed Ptolemy lines through $o$. Let $T \subset B_\omega$ be an unclosed triangle. By Lemma 9.6 there is an isometry $g : X_\omega \to X_\omega$ preserving the order such that $g = \pi_*(\bar{g}) : B_\omega \to B_\omega$ is the central symmetry of the plane $L \subset B_\omega$ containing $T$ with respect to the midpoint $\bar{o}$ of some side of $T$.

Let $b$ be a common vertex of the triangles $T, g(T)$ and the parallelogram $P = T \cup g(T), F = F_b \subset X_\omega$ the fiber of $\pi_\omega$ over $b$. The isometries $\tau_T$ and $\tau_{g(T)}$ of $F$ coincide, $\tau_{g(T)} = \tau_T$, because the isometry $\bar{g} : X_\omega \to X_\omega$ preserves the order. Then one easily sees that the isometry $\tau_P : F \to F$ satisfies $\tau_P = \tau_T^2 \neq \text{id}_F$. Therefore, the parallelogram $P$ is unclosed.
9.3 The maximal unipotent subgroup

Recall that the group $N_\omega$ of the shifts $X_\omega \to X_\omega$ acts on $X_\omega$ simply transitively (Lemma 9.1) and that the subgroup $Z_\omega \subset N_\omega$ of vertical shifts acts simply transitively of every $\mathbb{C}$-line $F \subset X_\omega$ (Proposition 7.4). In context of rank one symmetric spaces $Y = \mathbb{K}H^n$ of noncompact type the group $N_\omega$ is called a maximal unipotent subgroup of the isometry group of the space $Y$.

We have $[N_\omega, N_\omega] \subset Z_\omega$ because any two shifts of the base $B_\omega$ commute.

Lemma 9.8. The group $Z_\omega$ lies in the center of $N_\omega$.

Proof. By Proposition 9.3, the displacement function $\delta_\zeta$ of $\zeta$ is constant for every $\zeta \in Z_\omega$. Thus for every $\alpha \in N_\omega, x \in X_\omega$, we have $\delta_\zeta(x) = \delta_\zeta(\alpha(x))$. The points $x, \alpha(x), \zeta\alpha(x), \alpha\zeta(x)$ lie in one and the same semi-$\mathbb{C}$-plane, thus $\zeta\alpha(x) = \alpha\zeta(x)$. It follows that $\left[\zeta, \alpha\right] = \text{id}_{X_\omega}$, i.e., $\zeta$ lies in the center of $N_\omega$. \hfill $\Box$

Lemma 9.9. Given $a > 0$ and a Ptolemy line $l \subset X_\omega$, there are $\alpha, \beta \in N_\omega$ such that $\alpha(l) = l$ and the displacement of $\gamma = [\alpha, \beta]$ equals $a$.

Proof. By Lemma 9.7, there is an unclosed parallelogram $P \subset B_\omega$. Applying an appropriate homothety $\varphi : X_\omega \to X_\omega$ if necessary, we can assume that the Ptolemy line $l \subset X_\omega$ projects down to the line $\pi_\omega(l) \subset B_\omega$ that contains a side $b'b \subset P$, and the displacement of $\tau_P : F \to F$ is $a$, $|x\tau_P(x)| = a$ for every $x \in F$, where $F \subset X_\omega$ is the fiber of $\pi_\omega$ over a vertex of $P$, cp. Lemma 9.4.

Let $b' \in P$ be the vertex adjacent to $b''$ and opposite to $b$. We consider the Ptolemy line $l' \subset X_\omega$ through $o = l \cap F_{b''}$ that project down to the line $\pi_\omega(l') \subset B_\omega$ containing the side $b'b''$ of $P$. There are shifts $\alpha, \beta \in N_\omega$, which leave invariant the Ptolemy lines $l, l'$ respectively, such that $\pi_\omega(\alpha)(b'') = b, \pi_\omega(\beta)(b'') = b'$.

Since the parallelogram $P$ is unclosed, we have $\alpha\beta \neq \beta\alpha$, and $\delta_\gamma(o) = a$, where $\delta_\gamma : X_\omega \to \mathbb{R}$ is the displacement of $\gamma = [\alpha, \beta]$. By Proposition 9.3, the displacement $\delta_\gamma$ is constant, thus $\delta_\gamma(x) = a$ for every $x \in X_\omega$. \hfill $\Box$

Proposition 9.10. The group $N_\omega$ is nilpotent and the group $Z_\omega$ is the center of $N_\omega$. Moreover $Z_\omega = [N_\omega, N_\omega]$.

Proof. Assume $\alpha' \in N_\omega$ commutes with every $\beta \in N_\omega$. We show that $\alpha' \in Z_\omega$. Together with Lemma 9.8, this implies that $Z_\omega$ is the center of $N_\omega$.

Composing $\alpha'$ with an appropriate $\nu \in Z_\omega$ if necessary, we can assume that $\alpha'(l) = l$ for some Ptolemy line $l \subset X_\omega$. It suffices to show that $\alpha'(o) = o$ for some $o \in l$. By Lemma 9.9, there are $\alpha, \beta \in N_\omega$ such that $\alpha(l) = l$ and the displacement $\delta_\gamma = a$ for a given $a > 0$, where $\gamma = [\alpha, \beta]$. Suppose that $|\alpha\alpha'(o)| \neq 0$. Conjugating $\alpha'$ by an appropriate pure homothety of $X_\omega$ preserving $o$, we can assume that $|\alpha\alpha'(o)| = |\alpha\alpha'(o)|$. Replacing $\alpha'$ with
$(\alpha')^{-1}$ if necessary, we can also assume $\alpha'(o) = \alpha(o)$. Then $\alpha' = \alpha$. But this contradicts the assumption that $\alpha'$ commutes with every $\beta \in N_\omega$. Hence $\alpha'(o) = o$.

Let $a > 0$ be the displacement of some $\zeta \in Z_\omega$, $\zeta \neq \id$. Again by Lemma 7.1 there are $\alpha, \beta \in N_\omega$ such that the displacement of $\gamma = [\alpha, \beta]$ equals $a$. Then $\zeta$ coincides with $\gamma$ or $\gamma^{-1}$. Thus $Z_\omega = [N_\omega, N_\omega]$. \hfill $\square$

10 Area law of lifting and metrics of $\mathbb{C}$-lines

10.1 Lifting of polygons

Let $P \subset B_\omega$ be an oriented parallelogram. Fixing a vertex $b \in P$, we obtain a preserving the order isometry $\tau_b : F \to F$, where $F \subset X_\omega$ is the fiber of $\pi_\omega : X_\omega \to B$ over $b$, see sect. 9.2. It is convenient to associate with $P$ an extension of $\tau_b$ to $X_\omega$ which is defined as $\tau_b(x) = \nu(x)$ for every $x \in X_\omega$, where $\nu : X_\omega \to X_\omega$ is a vertical shift, $\pi_\omega(\nu) = \id_{B_\omega}$, while restricted to $F$ coincides with $\tau_b$. The isometry $\nu$ exists by Lemma 9.2 and it is unique by Lemma 7.1. We use the same notation for the extension $\tau_b : X_\omega \to X_\omega$ and call it a lifting isometry.

Furthermore, for every isometry $\phi \in G_\omega$ (recall that such an isometry preserves $\omega$ and the order $O$) we have according to Proposition 9.3 $\tau_{b} \circ \phi = \tau_{b}$, where $P = \pi_\omega(\phi)(P)$. In particular, the map $\tau_P$ is not changed if we replace the parallelogram $P$ by any its shifted copy $P' \subset B$.

The vertical shift $\tau_P : X_\omega \to X_\omega$ lies in the group $Z_\omega$, see sect. 9.3 for every parallelogram $P \subset B_\omega$. Since $Z_\omega$ is commutative, we have $\tau_P \circ \tau_{P'} = \tau_{P'} \circ \tau_P$ for any parallelograms and even for any closed oriented polygons $P, P' \subset B_\omega$.

Let $Q \subset B_\omega$ be a closed, oriented polygon. Adding a segment $qq' \subset B_\omega$ between points $q, q' \in Q$ we obtain closed, oriented polygons $P, P'$ such that $Q \cup qq' = P \cup P'$, the orientations of $P, P'$ coincide with that of $Q$ along $Q$, and the segment $qq' = P \cap P'$ receives from $P, P'$ opposite orientations. In this case we use notation $Q = P \cup P'$.

**Lemma 10.1.** In the notation above we have $\tau_Q = \tau_{P'} \circ \tau_P$.

**Proof.** We fix $q \in Q \cap P \cap P'$ as the base point. Moving from $q$ along $Q$ in the direction prescribed by the orientation of $Q$, we also move along one of $P, P'$ according to the induced orientation. We assume W.L.G. that this is the polygon $P$. In that way, we first lift $P$ to $X_\omega$ starting with some point $o \in F$, where $F$ is the fiber of the projection $\pi_\omega : X_\omega \to B_\omega$ over $q$, such that the side $q'q \subset P$ is the last one while lifting $P$. Now, we lift $P'$ to $X_\omega$ starting with $o' = \tau_P(o) \in F$ moving first along the side $qq' \subset P'$. Then clearly the resulting lift of $P'$ gives $\tau_P(o) = \tau_{P'}(o') \in F$. Thus $\tau_Q = \tau_{P'} \circ \tau_P$. \hfill $\square$
10.2 Area law of lifting

Let $P_0 \subset B_\omega$ be a parallelogram. Applying if necessary a homothety from the group $H = \pi_*(G_\omega)$, we assume W.L.G. that area $P_0 = 1$. Furthermore, cutting $P_0$ by a line in the plane of $P_0$ and gluing back the obtained pieces shifted appropriately, one easily transforms $P_0$ to a rectangle $P_1$, which therefore satisfies $\tau_{P_1} = \tau_{P_0}$ by Lemma 10.1.

Let $L \subset B_\omega$ be the 2-dimensional subspace containing $P_1$. We fix two mutually orthogonal directions in $L$ such that the sides of $P_1$ are parallel to them, and call one of them the horizontal direction and the other one vertical direction.

We denote by $\delta(P)$ the displacement of any parallelogram $P \subset B_\omega$, $\delta(P) = |x\tau_P(x)|$ for every $x \in X_\omega$, see Proposition 9.3 and put $c_0 := \delta(P_0) = \delta(P_1) \geq 0$.

We denote by $P_1$ the class of all the rectangles in $L$ of area 1 with horizontal and vertical sides such that $\delta(P) = c_0$ for every $P \in P_1$. We write this equality as

$$\delta(P)^2 = c_0^2 \cdot \text{area } P \quad (10)$$

and call $c_0$ the lifting constant of the class $P_1$. Equality (10) is called the area law of lifting. Note that $P_1 \in P_1$.

Lemma 10.2. The class $P_1$ is closed under the following operations with rectangles:

(a) a shift in $L$;

(b) cutting by finitely many parallel horizontal or vertical lines and gluing back the shifted pieces;

(c) taking the limit of a convergent sequence of rectangles.

Proof. Operation (a) preserves the class $P_1$ because every shift $\gamma : B_\omega \to B_\omega$ is of the form $\gamma = \pi_*(\zeta)$, where $\zeta : X_\omega \to X_\omega$ is a shift, and thus $\tau_P = \tau_\gamma(P)$ for every parallelogram $P \subset B_\omega$. Using Lemma 10.1 we see that operation (b) preserves the class $P_1$. Operation (c) preserves the class $P_1$ because the map $\tau_P : X_\omega \to X_\omega$ depends continuously on $P$. □

Lemma 10.3. The class $P_1$ includes a unit square $Q_0 \subset L$.

Proof. Given a rectangle $P \subset L$ with horizontal and vertical sides, and integer $m, n \geq 1$, we construct a new rectangle $P' \subset L$ using operation (b) as follows. First, we subdivide $P$ into $m$ pairwise congruent rectangles cutting it by $(m - 1)$ horizontal lines and gluing back the shifted pieces into a horizontal row. Second, we subdivide the obtained rectangle $\tilde{P}$ into $n$ pairwise congruent rectangles cutting $\tilde{P}$ by $(n - 1)$ vertical lines and gluing back the shifted pieces into a vertical column. This gives the resulting
rectangle $P' = \Psi_{m,n}(P)$. Note that if $P \in \mathcal{P}_1$, then $\Psi_{m,n}(P) \in \mathcal{P}_1$ for each integer $m, n \geq 1$ by Lemma 10.2. Furthermore, if $a = a(P)$ is the length of the horizontal sides of $P$ and $b = b(P)$ is the length of the vertical sides of $P$, then
\[ a' = a(P') = \frac{m}{n}a, \quad b' = b(P) = \frac{n}{m}b. \]

Assume W.L.G. that $a < b$. Then there are integer $m > n \geq 1$ such that
\[ \frac{1}{2} \left( 1 + \frac{b}{a} \right) \leq \frac{m^2}{n^2} < \frac{b}{a}. \]
Thus
\[ 1 \leq \frac{b'}{a'} \leq \frac{b}{a}, \]
where $\lambda^{-1} = \frac{1}{2}(1 + \frac{b}{a}) > 1$. This generates a sequence of rectangles $P_i$, $P_{i+1} = \Psi_{m_i,n_i}(P_i)$. By the choice of $m_i, n_i$ this sequence cannot have accumulation points different from $Q_0$, thus $P_i \to Q_0$. Therefore $Q_0 \in \mathcal{P}_1$ by operation (c).

**Corollary 10.4.** Every rectangle $P \subset L$ of area 1 with horizontal and vertical sides is in the class $\mathcal{P}_1$, $P \in \mathcal{P}_1$.

**Proof.** According to Lemma 10.2, it suffices to show that $P$ can be obtained from $Q_0$ by operations (a)–(c). By the proof of Lemma 10.3, there is a sequence of integer pairs $(m_i, n_i)$ such that the sequence of rectangles $P_i$, where $P_1 = P$, $P_{i+1} = \Psi_i(P_i)$ and $\Psi_i = \Psi_{m_i,n_i}$, converges to $Q_0$, $P_i \to Q_0$. For every integer $i \geq 1$ we inductively define a rectangle $Q_i$ such that $Q_i = \Phi_i(Q_{i-1})$, where $\Phi_i = (\Psi_i \circ \cdots \circ \Psi_1)^{-1}$. Since $\Phi_i^{-1}(P) = P_i \to Q_0$, we have $Q_i \to P$. Furthermore, $Q_i \in \mathcal{P}_1$ for every $i \geq 1$ because $Q_0 \in \mathcal{P}_1$ by Lemma 10.3. Therefore $P \in \mathcal{P}_1$.

We denote with $\mathcal{P}$ the class of all the rectangles $P \subset L$ with horizontal and vertical sides that satisfy the area law of lifting, $\delta(P)^2 = c_0^2 \cdot \text{area } P$, with the lifting constant $c_0$.

**Proposition 10.5.** Every rectangle $P \subset L$ with horizontal and vertical sides is in the class $\mathcal{P}$.

**Proof.** By property (H) and Lemma 4.15 for every $o \in X_\omega$ and every $\lambda > 0$ there is a pure homothety $h_\lambda : X_\omega \to X_\omega$ with $h_\lambda(o) = o$ and with coefficient $\lambda$. Then $h_\lambda$ preserves every foliation of $X_\omega$ by Busemann parallel Ptolemy lines. Thus its projection to the base, $h_\lambda = \pi_\omega(h_\lambda) : B_\omega \to B_\omega$, is a homothety with coefficient $\lambda$ that fixes $\overline{\sigma} = \pi_\omega(o)$, and $h_\lambda(l) = \overline{l}$ for every (geodesic) line $l \subset B_\omega$ through $\overline{o}$.
We take $o \in \Omega$ with $\bar{\pi} \in L$. Then $\tilde{h}_\lambda$ preserves $L$ and horizontal and vertical directions on it. Furthermore, for every $P' = \tilde{h}_\lambda(P)$, $P \subset L$ is a polygon, we have $\delta(P') = \lambda \delta(P)$. Taking $P \in \mathcal{P}_1$ we obtain

$$\delta(P')^2 = \lambda^2 \delta(P)^2 = c_0^2 \cdot \text{area}(P').$$

Thus $P' \in \mathcal{P}$. However, by Corollary 10.4 every rectangle $P' \subset L$ with horizontal and vertical sides can be represented as $P' = \tilde{h}_\lambda(P)$ for some $P \in \mathcal{P}_1$ and some $\lambda > 0$.

10.3 Metrics on $\mathbb{C}$-lines

Recall that every $\mathbb{C}$-line $F \subset \Omega$ is a fiber of the projection $\pi_\omega : \Omega \rightarrow B_\omega$.

**Proposition 10.6.** Given $x, y, z \in F$, $y$ lies between $x$ and $z$ with respect to the order on a $\mathbb{C}$-line $F$, we have

$$|xz|^2 = |xy|^2 + |yz|^2.$$

**Proof.** By Lemma 9.7 there is an unclosed parallelogram $P \subset B_\omega$. Thus by Proposition 10.5 there is a two-dimensional subspace $L \subset B_\omega$ with horizontal and vertical directions satisfying the area law of lifting with a lifting constant $c_0 > 0$. W.L.G. we can assume that $F = \pi_\omega^{-1}(o)$ for some point $o \in L$. By Proposition 10.5 for every $a > 0$ a rectangle $P_a \subset L$ with the horizontal sides of length 1 and the vertical sides of length $a$ belongs to the class $\mathcal{P}$, $\delta(P_a)^2 = c_0^2 \cdot a$. We put $a = |xy|^2/c_0^2$, $b = |yz|^2/c_0^2$ and represent the rectangle $P_{a+b}$ as the union of $P_a$ and $P_b$ with a common horizontal side. Then $\tau P_{a+b} = \tau P_a \circ \tau P_b$ by Lemma 11.1. Since $\delta(P_a) = c_0 \sqrt{a} = |xy|$, we have $\tau P_a(x) = y$ and similarly $\tau P_b(y) = z$. Therefore $\tau P_{a+b}(x) = z$, and we obtain

$$|xz|^2 = \delta(P_{a+b})^2 = c_0^2(a + b) = \delta(P_a)^2 + \delta(P_b)^2 = |xy|^2 + |yz|^2.$$

11 Canonical complex structure on the base

In this section we show that the base $B_\omega$ possesses a complex structure uniquely determined by the geometry of the space $\Omega$. This complex structure is said to be *canonical*.

11.1 Functional $\xi_u$

We fix a base point $o \in B_\omega$ and regard $B_\omega$ as an Euclidean vector space identifying $u \in B_\omega$ with the vector $\overrightarrow{o}$. Given a unit vector $u \in B_\omega$, $|u| := |ou| = 1$, we let $u^\perp \subset B_\omega$ be the orthogonal complement to $u$. For every vector $v \in u^\perp$ we denote with $u \wedge v \subset B_\omega$ the oriented rectangle spanned by
Lemma 11.1. Let $x$, and hence any $gles$ in $B$ the closed oriented polygon $Q$ the closed oriented polygon $Q$ thus their contributions cancel out, and we have $\tau$

Proof. We define the sign of $v$ with respect to $u$ as

$$\text{sign}_u(v) = \begin{cases} +1, & x < \tau_{u \wedge v}(x) \\ 0, & x = \tau_{u \wedge v}(x) \\ -1, & \tau_{u \wedge v}(x) < x \end{cases}$$

for some and hence any $x \in X_\omega$. Now, we define a function $\xi_u : u^\perp \to \mathbb{R}$ by

$$\xi_u(v) = \text{sign}_u(v) \cdot \delta(u \wedge v)^2,$$

where $\delta(u \wedge v)$ is the displacement of $\tau_{u \wedge v}$, $\delta(u \wedge v) = |x\tau_{u \wedge v}(x)|$ for some and hence any $x \in X_\omega$.

The following lemma will be used in the proof of additivity of $\xi_u$.

Lemma 11.1. Let $P = xyzu$, $Q = xyz'w'$, $Q' = u'z'u'w'$ be oriented rectangles in $B_\omega$ such that $z'u'w' = Q \cap Q'$ is the common side of $Q$ and $Q'$. Then for the closed oriented polygon $Q \cup Q' = xyz'w'x \in B_\omega$ the lifting isometries $\tau_P$, $\tau_{Q \cup Q'} : X_\omega \to X_\omega$ coincide, $\tau_P = \tau_{Q \cup Q'}$.

Proof. The lifting isometries $\tau_P$ and $\tau_{Q \cup Q'}$ differ by lifting isometries $\tau_\Delta$, $\tau'_{\Delta'}$,

$$\tau_{Q \cup Q'}^{-1} \circ \tau_P = \tau'_{\Delta'} \circ \tau_\Delta,$$

where $\Delta = uxy'$, $\Delta' = z'y'$. However, the triangle $\Delta'$ is a shifted copy of the triangle $\Delta$, and $\Delta$, $\Delta'$ enter the formula above with opposite orientations. Thus their contributions cancel out, and we have $\tau_P = \tau_{Q \cup Q'}$.

Proposition 11.2. The function $\xi_u : u^\perp \to \mathbb{R}$ is a nonzero linear functional for every unit vector $u \in B_\omega$.

Proof. First, we show that $\xi_u$ is additive, $\xi_u(v + v') = \xi_u(v) + \xi_u(v')$ for all $v, v' \in u^\perp$. We denote $w = v + v'$. It follows from Lemma 11.1 and Lemma 11.1 that $\tau_{u \wedge w} = \tau_{u \wedge v} \circ \tau_{u \wedge v}$. We fix a fiber $F \subset X_\omega$, a point $x \in F$, and assume W.L.G. that $x < \tau_{u \wedge v}(x)$ (this assumption depends neither on $F$ nor on $x \in F$). If $x < \tau_{u \wedge v}(x) < \tau_{u \wedge v}(x)$, then $\text{sign}_u(w) = \text{sign}_u(v) = \text{sign}_u(v') = 1$ by definition and $\delta(u \wedge w)^2 = \delta(u \wedge v)^2 + \delta(u \wedge v')^2$ by Proposition 10.9. In the opposite case we have W.L.G. that $\tau_{u \wedge v}(x) < x < \tau_{u \wedge v}(x) < \tau_{u \wedge v}(x)$ and thus $\text{sign}_u(w) = \text{sign}_u(v) = -\text{sign}_u(v')$, $\delta(u \wedge w)^2 = \delta(u \wedge v)^2 - \delta(u \wedge v')^2$. In both cases this gives $\xi_u(w) = \xi_u(v) + \xi_u(v')$.

Next, we show that $\xi_u$ is homogeneous. We have $\text{sign}_u(\lambda v) = \text{sign}(\lambda) \cdot \text{sign}_u(v)$ for each $v \in u^\perp$, $\lambda \in \mathbb{R}$, because the orientation of the rectangle $u \wedge (\lambda v)$ depends on the sign of $\lambda$. Thus using the area law of lifting, which holds by Proposition 10.5 in the plane $L$ spanned by $u, v$, with some constant $c_0 \geq 0$, we have

$$\xi_u(\lambda v) = \text{sign}(\lambda) \text{sign}_u(v)c_0^2|\lambda||v| = \lambda \text{sign}_u(v)c_0^2|v| = \lambda \xi_u(v)$$

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for every \( v \in u^\perp \) and every \( \lambda \in \mathbb{R} \). Thus \( \xi_u : u^\perp \to \mathbb{R} \) is linear.

Finally, there are unclosed rectangles in \( B_\omega \) and the subgroup of isometries in \( H = \pi_s(G_\omega) \) preserving \( o \) acts transitively on the unit sphere \( S(o) \).

Then there is a unit vector \( v \in u^\perp \) such that \( \text{sign}_u(v) = 1 \). Thus \( \xi_u \neq 0 \).

We put \( c = \sup_Q \delta(Q) \), where the supremum is taken over all the unit squares \( Q \subset B \), and call \( c \) the lifting constant of \( B \).

**Lemma 11.3.** For every unit vector \( u \in B_\omega \) the norm of the linear functional \( \xi_u : u^\perp \to \mathbb{R} \) is \( |\xi_u| = c^2 \).

**Proof.** By Proposition 11.2 \( \xi_u \neq 0 \). Thus there is a unique unit vector \( v \in u^\perp \) with \( \xi_u(v) = |\xi_u| \leq c^2 \).

The lifting constant \( c \) can be computed by taking the supremum \( \sup_Q \delta(Q) \) over the compact set of all unit squares \( Q \subset B_\omega \) having the base point \( o \) as a vertex. By continuity of the lift \( Q \to \tau_Q \), there is a unit square \( Q_0 \) with \( \delta(Q_0) = c \). Applying to \( Q_0 \) an isometry of type \( \pi_s(\varphi) : B_\omega \to B_\omega \) if necessary, we can assume that \( Q_0 = u \land v' \), where \( v' \in u^\perp \), \( \text{sign}_u(v') = 1 \). Then \( \xi_u(v') = c^2 \). It follows that \( v' = v \) and \( |\xi_u| = c^2 \).

11.2 Complex structure on the base

By Lemma 11.3 for every unit vector \( u \in B_\omega \) there is a unique \( v \in u^\perp \), \( |v| = 1 \), \( \text{sign}_u(v) = 1 \), such that \( \xi_u(v) = c^2 \). This define a map \( J : S(o) \to S(o) \), \( J(u) = v \), where \( S(o) \subset B_\omega \) is the unit sphere centered at \( o \). We put \( J(o) = o \) and extend \( J \) on \( B \) by homogeneity, \( J(u) = |u|J(u/|u|) \) for every \( u \in B_\omega \).

**Lemma 11.4.** Let \( h : S(o) \to S(o) \) be a map commuting with a subgroup \( G \subset \mathbb{O}(k) \), \( k = \dim B_\omega \), acting transitively on \( S(o) \) (we do not require that \( h \) is an isometry). Then the displacement of \( h \) is constant.

**Proof.** Given \( x, y \in S(o) \), there is \( g \in G \) with \( g(x) = y \). Then \( |y h(y)| = |g(x) h(g(x))| = |g(x) g(h(x))| = |x h(x)| \).

**Proposition 11.5.** The map \( J : B_\omega \to B_\omega \) is a complex structure on \( B_\omega \), that is, \( J \) is a linear isometry with \( J^2 = -\text{id} \). Moreover, every Möbius automorphism \( \varphi : X_\omega \to X_\omega \) respecting the order of the \( \mathbb{C} \)-lines in \( X_\omega \) preserves \( J \), \( \pi_s(\varphi) \circ J = J \circ \pi_s(\varphi) \).

**Proof.** For \( u \in S(o) \) let \( K_u \subset u^\perp \) be the kernel of \( \xi_u \), \( K_u = \ker \xi_u \). By definition, \( v = J(u) = \text{grad} \xi_u/|\text{grad} \xi_u| \), thus \( K_u \subset u^\perp \cap v^\perp \). Since \( \dim K_u = \dim u^\perp - 1 = \dim (u^\perp \cap v^\perp) \), we have \( K_u = u^\perp \cap v^\perp = K_v \), that is, the kernel \( K_u \) is invariant under \( J \), \( J(K_u) = K_v = K_u \). Furthermore, \( \text{sign}_v(u) = -\text{sign}_u(v) \) because the rectangles \( u \land v \) and \( v \land u \) have opposite orientations. We conclude that \( J(v) = -u \), i.e., \( J^2 = -\text{id} \).
Let $\varphi : X_\omega \to X_\omega$ be a M"obius automorphism that respects the order $O$ of the $\mathbb{C}$-lines in $X_\omega$. Recall that then $\varphi$ is a homothety, because $\varphi$ preserves the infinitely remote point $\omega$. Applying if necessary a shift, we can assume W.L.G. that the homothety $\varphi = \pi_* (\varphi) : B_\omega \to B_\omega$ also preserves the base point $o$, $\varphi (o) = o$. Given $u, v \in S(o), v \in u^\perp$, we denote by $u', v' \in S(o)$ the unit vectors $u' = \varphi (u)/|\varphi (u)|, v' = \varphi (v)/|\varphi (v)|$. Then $v' \in u'^\perp$. Furthermore, for the displacements $\delta(u \wedge v), \delta(u' \wedge v')$ we have

$$|\varphi (u)| \cdot |\varphi (v)| \cdot \delta(u' \wedge v') = \delta(\varphi (u) \wedge \varphi (v)) = \lambda^2 \delta(u \wedge v),$$

where $\lambda > 0$ is the homothety coefficient of $\varphi$, the first equality follows from Proposition 10.3 and the second one follows from the fact that the homothety $\varphi : B_\omega \to B_\omega$ is the projection of the homothety $\varphi : X_\omega \to X_\omega$. Thus $\delta(u' \wedge v') = \delta(u \wedge v)$. Since $\varphi$ preserves the order $O$ of the $\mathbb{C}$-lines in $X_\omega$, we obtain that the isometry $J$ on the basis $\varphi$ is identical on every complex line above. These complex lines span $O$ order maximal value over all 2-dimensional subspaces.

Remark 11.6. A complex line in $B_\omega$ is a 2-dimensional subspace $L$ invariant under $J$, $J(L) = L$. By definition of $J$, the lifting constant for $L$ takes the maximal value over all 2-dimensional subspaces.
12 Coordinates in $X_\omega$

For a given $\omega \in X$ we fix as usual a metric from the Möbius structure with infinitely remote point $\omega$. We also fix an order $O$ on $X_\omega$, a base point $o \in X_\omega$, and identify the base $B_\omega$ with Euclidean space $\mathbb{R}^k$, $k = \dim B_\omega$, with origin $\pi_\omega(o)$. With $\mu : X_\omega \to F_\omega$ we denote the projection onto the fiber $F_\omega = \pi_\omega^{-1}(\pi_\omega(o))$ of $\pi_\omega$. It follows from Lemma 8.9 that $\mu$ is continuous. We define the standard coordinates of every point $x \in X_\omega$ as $(z, h) \in \mathbb{R}^k \times \mathbb{R} = \mathbb{R}^{k+1}$, where $z = \pi_\omega(x)$, $h = \frac{1}{2} \sign \mu(x) |\mu(x)|^2$.

$$\sign \mu(x) = \begin{cases} 
+1 & o < \mu(x) \\
0 & o = \mu(x) \\
-1 & \mu(x) < o.
\end{cases}$$

The coefficient $\frac{1}{2}$ in front of the expression for the coordinate $h$ is introduced to provide the property that the standard generator $c = [a, b]$ of the center of the classical Heisenberg group $\mathbb{H} = \mathbb{H}^1$ has coordinates $c = (0,1)$.

12.1 Multiplication law in coordinates

Since the group $N_\omega$ acts on $X_\omega$ simply transitively, see Lemma 7.1, every isometry $g \in N_\omega$ can be written as $g = (z, h)$, where $(z, h)$ are the coordinates of $g(o)$.

Lemma 12.1. For $g = (z, h)$, $g' = (z', h') \in N_\omega$ we have

$$g \cdot g' = (z + z', h + h' + h \circ \tau_T(o)),$$

where $T = \frac{1}{2}(z \wedge z') = \pi_\omega(o)z(z + z') \subset B_\omega$ is the oriented triangle, $\tau_T : X_\omega \to X_\omega$ the respective lifting isometry, see sect. 10.1.

Proof. The center $Z_\omega$ of $N_\omega$ acts on $X_\omega$ by vertical shifts, and by Proposition 10.6 we have $g \cdot g' = (0, h + h')$ for $g = (0, h)$, $g' = (0, h') \in Z_\omega$.

The group $\pi_\omega(N_\omega) = N_\omega/Z_\omega$ acts on the base $B_\omega$ by shifts, and we have $\pi_\omega(g) \cdot \pi_\omega(g') = z + z'$ for $g = (z, h)$, $g' = (z', h') \in N$.

Let $l \subset X_\omega$ be the Ptolemy line through $o$ such that $z : B_\omega \to B_\omega$ is the shift along the line $\pi_\omega(l)$. There is an isometry $\tilde{g} \in N_\omega$ which preserves $l$ and projects down to $\tilde{z} = \pi_\omega(\tilde{g})$. Then $\tilde{g} = (z, 0)$, and $g = (z, h)$ can be written as $(z, h) = (z, 0) \cdot (0, h) = \tilde{g} \cdot (0, h)$. Similarly we have $g' = (z', h') = (z', 0) \cdot (0, h') = \tilde{g}' \cdot (0, h')$, and the isometry $g' \in N_\omega$ preserves a line $l' \subset X_\omega$ through $o$. Then $l'' = \tilde{g}(l')$ is the Ptolemy line in $X_\omega$ through $\tilde{g}(o) = (z, 0)$ and $\tilde{g} \cdot \tilde{g}'(o)$. The $z$-coordinate of the point $\tilde{g} \cdot \tilde{g}'(o)$ is $z + z'$.

Now, we compute the $h$-coordinate of $\tilde{g} \cdot \tilde{g}'(o)$. Let $z \wedge z' \subset B_\omega$ be the oriented parallelogram spanned by $z$, $z'$, $T = \frac{1}{2}(z \wedge z')$ the oriented triangle $\pi_\omega(o)z(z + z')$. Then $\mu(\tilde{g} \cdot \tilde{g}'(o)) = \tau_T(o)$. Thus $\tilde{g} \cdot \tilde{g}' = (z + z', h \circ \tau_T(o))$, and we obtain $g \cdot g' = \tilde{g} \cdot (0, h) \cdot \tilde{g}' \cdot (0, h') = (z + z', h + h' + h \circ \tau_T(o))$.  

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Lemma 12.2. For any triangle \( T = \frac{1}{2} z \land z' \subset B_\omega \), we have
\[
h \circ \tau_T(o) = \frac{c^2}{8} \langle J(z), z' \rangle,
\]
where \( c > 0 \) is the lifting constant of \( B_\omega \).

Proof. By Lemma 11.1 \( \tau_{z \land z'} = \tau^2_T \). Thus by Proposition 11.6 \( |o \tau_T(o)|^2 = \frac{1}{2} \delta(z \land z')^2 \), where \( \delta(z \land z') \) is the displacement of the isometry \( \tau_{z \land z'} \). By results of section 10.2 the lifting isometry \( \tau_T \) does not change when we replace the parallelogram \( z \land z' \) by a rectangle of equal area that has a side of length one in the same 2-dimensional subspace. Thus we can assume that \( |z| = 1 \) and \( z \perp z' \). Since \( \tau_T(o) \in F_o \), we have \( \mu \circ \tau_T = \tau_T \) and sign \( \mu \circ \tau_T = \text{sign}_z(z') \), see section 11. It follows that \( h \circ \tau_T(o) = \frac{1}{8} \xi_z(z') \), where the linear functional \( \xi_z : z^\perp \rightarrow \mathbb{R} \) is used in the definition of the complex structure \( J \) (for the definition of \( \xi_z \) see sect. 11).

By Lemma 11.3 \( |\xi_z| = c^2 \). Then \( \xi_z(z') = c^2 \langle J(z), z' \rangle \) and \( h \circ \tau_T(o) = \frac{c^2}{8} \langle J(z), z' \rangle \). \( \square \)

### 12.2 The distance function \( D \)

For two points \( x, y \in X_\omega \) let \( F_x \) and \( F_y \) be the \( C \)-lines through \( x, y \), and let \( \mu_{xy} : F_x \rightarrow F_y \) be the projection, see sect. S.3. We denote with
\[
|xy|^R_\omega = |\pi_\omega(x)\pi_\omega(y)|, \quad |xy|^C_\omega = |\mu_{xy}(x)y|.
\]
Note that \( |xy|^C_\omega = |xy|^C_{\omega'} \) because Ptolemy lines are equidistant on every semi-\( C \)-plane in \( X_\omega \), see Lemma S.7

Lemma 12.3. There exists some function \( D : [0, \infty) \times [0, \infty) \rightarrow [0, \infty) \), such that for all \( \omega \in X \) and all \( x, y \in X_\omega \)
\[
|xy|_\omega = D(|xy|^R_\omega, |xy|^C_\omega).
\]

Proof. To prove this, we have to show, that given triples of points \( \omega, x, y \) and \( \omega', x', y' \) in \( X \) with \( |xy|^R_\omega = |x'y'|^R_{\omega'} \) and \( |xy|^C_\omega = |x'y'|^C_{\omega'} \), we have \( |xy|_\omega = |x'y'|_{\omega'} \).

We can assume that \( |xy|^R_\omega \neq 0 \), since otherwise the claim is trivial. Let \( z = \mu_{xy}(x) \) and \( z' = \mu_{x'y'}(x') \). By assumption \( |xz|_\omega = |x'z'|_{\omega'} \neq 0 \) and \( |zy|_\omega = |z'y'|_{\omega'} \). By (E2) there exists a Möbius map \( \varphi : X \rightarrow X \) which maps \( \omega \rightarrow \omega' \), \( x \rightarrow x' \), \( z \rightarrow z' \) because each triple of points \( (\omega, x, z) \) and \( (\omega', x', z') \) belongs to a respective Ptolemy circle in \( X \). Note that \( \varphi : X_\omega \rightarrow X_{\omega'} \) is an isometry that maps the fibers \( F_x, F_y \) to \( F_{x'}, F_{y'} \). Then \( \varphi \) maps \( y \) either to \( y' \) or to \( y'' \in F_{y'} \), the points symmetric to \( y \) with \( |y''z'|_{\omega'} = |z'y'|_{\omega'} \). In the last case, applying an isometry of \( X_{\omega'} \) that order preserves the \( C \)-lines \( F_{x'}, F_{y'} \) and maps \( y \) to \( z' \), and Lemma S.8, we have \( |xy|_\omega = |x'y''|_{\omega'} = |x'y'|_{\omega'} \). This proves our claim. \( \square \)
Lemma 12.4. The distance function $D$ is homogeneous, $D(\lambda a, \lambda b) = \lambda D(a, b)$ for every $a, b \geq 0, \lambda > 0$.

Proof. Let $\varphi_\lambda : X_\omega \to X_\omega$ be a homothety with coefficient $\lambda$. Then for each $x, x' \in X_\omega$ we have $|\varphi_\lambda(x)\varphi_\lambda(x')|^2_\omega = \lambda |xx'|_\omega^2$ and $|\varphi_\lambda(x)\varphi_\lambda(x')|^2 = \lambda |xx'|^2$. Together with $|\varphi_\lambda(x)\varphi_\lambda(x')| = \lambda |xx'|$ this implies the claim. \(\square\)

12.3 Existence of a vertical flip

Results of this section have important applications in sect. 13.

A vertical flip w.r.t. $o \in X_\omega$ is an isometry $j : X_\omega \to X_\omega$ that reverses the order $O$ and preserves $o$, $j(o) = o$. In particular, $j(F) = F$ for the $\mathbb{C}$-line $F$ through $o$ and $j|F$ reverses the order of $F$.

Let $J : B_\omega \to B_\omega$ be the canonical complex structure on $B_\omega$. The orthonormal basis $b = \{u_1, v_1, \ldots, u_k, v_k\}$, where $v_i = J(u_i)$, $i = 1, \ldots, k$, is called a canonical basis of $B_\omega$ for the complex structure $J$. We define a linear map $\text{Conj} : B_\omega \to B_\omega$ by $\text{Conj}(u_i) = u_i$, $\text{Conj}(v_i) = -v_i$ for $i = 1, \ldots, k$. Then $\text{Conj}$ is an isometry that anticommutes with $J$, $J \circ \text{Conj} = -\text{Conj} \circ J$. In particular, we have

$$\langle J \circ \text{Conj}(z), \text{Conj}(z') \rangle = -\langle J(z), z' \rangle$$ (11)

for each $z, z' \in B_\omega$, where $\langle z, z' \rangle$ is the inner product of $z, z'$ of the Euclidean space $B_\omega$.

Proposition 12.5. For every $o \in X_\omega$ and a Ptolemy line $l \subset X_\omega$ through $o$ there exists a vertical flip of $X_\omega$ with respect to $o$ that fixes $l$ pointwise.

Proof. We introduce standard coordinates $x = (z, h)$ in $X_\omega$ with the origin $o$ and take a canonical basis $b = \{u_1, v_1, \ldots, u_k, v_k\}$ of the complex structure $J$ such that the vector $u_1$ generates $1$-dimensional subspace $\pi_o(l) \subset B_\omega$. Let $\text{Conj} : B_\omega \to B_\omega$ be the conjugation isometry associated with $b$. We define $j : X_\omega \to X_\omega$ as

$$j(x) = j(z, h) = (\text{Conj}(z), -h).$$

Then $j(o) = o$, $j$ fixes $l$ pointwise by the choice of $b$, $j(F) = F$, where $F$ is the $\mathbb{C}$-line through $o$, and $j|F$ reverses the order. We only have to show that $j$ is an isometry.

Recall that $X_\omega$ is identified with the group $N_\omega$ by $x = g(o), x \in X_\omega, g \in N_\omega$. For $g \in N_\omega, g = (z, h)$, we put $\|g\| := |og(o)| = D(|z|, 2\sqrt{|h|})$, where we used Lemma 12.3 in the last equality. Since $N_\omega$ acts on $X_\omega$ by isometries, for $x = g(o), x' = g'(o)$, we have $|xx'| = |g(o)g'(o)| = |og^{-1} \cdot g'(o)| = \|g^{-1} \cdot g'\|$. Using Lemma 12.1 we obtain for $g = (z, h), g' = (z', h')$

$$g^{-1} \cdot g' = (-z, -h) \cdot (z', h') = (z' - z, h' - h + h \circ \tau_T(o)), \quad \|g^{-1} \cdot g'\| = \sqrt{|z'|^2 + |h'|^2}$$
where \( T = \frac{1}{2} (z \wedge z') = \pi(o) (-z) (z' - z) \subset B_\omega \) is the oriented triangle. By Lemma \[2.2\] \( h \circ \tau(o) = \frac{c^2}{8} (J(z), z') = -\frac{c^2}{8} (J(z), z'). \) Therefore

\[
|xx'| = D \left( |z - z'|, 2|h - h'| + \frac{c^2}{8} (J(z), z')^{1/2} \right). 
\]

Similarly for \( j(x) = (\text{Conj}(z), -h), j(x') = (\text{Conj}(z'), -h') \) we have

\[
|j(x)j(x')| = \| (\text{Conj}(z), -h)^{-1} \cdot (\text{Conj}(z'), -h') \|
\]

\[
= \| (\text{Conj}(z') - \text{Conj}(z), h - h' + h \circ \tau(o)) \|
\]

where \( T' = \frac{1}{2}( - \text{Conj}(z) \wedge \text{Conj}(z')) = \pi(o) (-\text{Conj}(z))(\text{Conj}(z' - z)) \) is the oriented triangle in \( B_\omega. \) Then \( h \circ \tau(o) = -\frac{c^2}{8} (J \circ \text{Conj}(z), \text{Conj}(z')) = \frac{c^2}{8} (J(z), z') \) by Equality \[111\]. Using that \( \text{Conj} : B_\omega \rightarrow B_\omega \) is an isometry, we obtain

\[
|j(x)j(x')| = D \left( |z - z'|, 2|h - h'| + \frac{c^2}{8} (J(z), z')^{1/2} \right).
\]

Comparing the formulae for the distances \( |xx'|, |j(x)j(x')| \) we see that \( j \) is an isometry.

\[\square\]

### 13 Ptolemy circles in \( X_\omega \)

Here we study shape of Ptolemy circles in \( X_\omega. \) Results of this section are used to compute the lifting constant \( c. \)

#### 13.1 Ptolemy circles meeting a \( \mathbb{C}\)-line twice

Let now \( \omega \in X, \) and let \( F \subset X_\omega \) be a \( \mathbb{C}\)-line. With \( \mu = \mu_\omega : X_\omega \rightarrow F \) we denote the projection onto \( F. \)

**Lemma 13.1.** Let \( z \in F \) and assume \( \mu(u) = z \) and \( \mu(v) = z \) for \( u, v \in X_\omega \) such that \( |uv| = |uz| + |zv|. \) Then \( u, v, z \) lie on a common Ptolemy line in \( X_\omega. \)

**Proof.** We can assume that both \( u, v \) are not on \( F, \) since otherwise the claim is obvious. Let \( l \) be the Ptolemy line through \( z \) and \( u, \) and let \( l' \) be the Ptolemy line through \( z \) and \( v. \) Let \( c, c' : \mathbb{R} \rightarrow X_\omega \) be unit speed parameterizations of \( l, l', \) such that \( c(0) = c'(0) = z, c(t_0) = u, \) for \( t_0 = |zu| \) and \( c'(s_0) = v, \) for \( s_0 = -|vz|. \) By assumption \( |c(t_0)c'(s_0)| = t_0 - s_0. \) For every \( \lambda > 0 \) there is a pure homothety \( \varphi_\lambda : X_\omega \rightarrow X_\omega, \) \( \varphi_\lambda(z) = z, \) with coefficient \( \lambda. \) Then \( \varphi_\lambda \circ c(t) = c(\lambda t) \) for every \( t \in \mathbb{R}, \) and \( \varphi_\lambda \circ c'(s) = c'(\lambda s) \) for every \( s \in \mathbb{R}. \) Thus \( |c(t)c'(s)| = t - s \) for all \( t \geq 0, s \leq 0. \) This implies \( b(c'(s)) = -s \) for all \( s \leq 0, \) where \( b : X_\omega \rightarrow \mathbb{R} \) is a Busemann function, associated with \( l, \) \( b(y) = \lim_{t \rightarrow \infty} (|c(t)y| - |c(t)z|), \) \( y \in X_\omega. \) Thus \( b(c'(s)) = -s \) for all \( s \in \mathbb{R}, \) since \( b \) is affine. Therefore \( b \circ c' = b \circ c. \) Then \( l' = l \) by Lemma \[4.11\].

\[\square\]
Lemma 13.2. Let $\sigma \subset X_\omega$ be a $\mathbb{R}$-circle intersecting the $\mathbb{C}$-line $F$ in two points $x$ and $y$. Let $u, v \in \sigma$ be points in different components of $\sigma \setminus \{x, y\}$ such that $\mu(u) = \mu(v) = z \in F$. Then the points $u, z, v$ are contained in a Ptolemy line.

Proof. Let $l$ be the Ptolemy line through the points $u$ and $z = \mu(u)$. Consider the point $v'$ on $l$ with order $v' < z < u$ such that $|v'z| = |vz|$. We show that $v = v'$. By the Ptolemy equality on $\sigma$ we have

$$|vu||xy| = |xu||vy| + |uy||vx|,$$

where we use that the pair $u, v \in \sigma$ is separated by the pair $x, y \in \sigma$. On the other hand, we have

$$|v'u||xy| \leq |xu||vy'| + |uy||v'x|.$$

Note also that $|xy| = |v'y|$, $|vx| = |v'x|$ by Lemma 13.3. Then $|v'u| \leq |vu|$. On the other hand, $|vu| \leq |v'u|$ by definition of $v'$. Thus $|vu| = |v'u|$ and $v' = v$ by Lemma 13.1.

Corollary 13.3. Let $F \subset X_\omega$ be a $\mathbb{C}$-line, and $\sigma \subset X_\omega$ a Ptolemy circle intersecting $F$ in two distinct points $x, y \in F$. Let $z \in F$ a point between $x$ and $y$, $x < z < y$. Then $|\mu^{-1}(z) \cap \sigma| = 2$.

Proof. Note that $\omega \notin \sigma$ since otherwise $\sigma$ is a Ptolemy line in $X_\omega$ intersecting $F$ in $x, y$ in contradiction with Corollary 8.6(4$\mathbb{C}$). By the continuity of $\mu$ we have $|\mu^{-1}(z) \cap \sigma| \geq 2$, since at least one point of every arc of $\sigma$ from $x$ to $y$ projects to $z$. Assume that there are 3 points projecting to $z$, $u$ from one arc, and $v, v'$ from the other, then by Lemma 13.2 the points $u, v, v'$ are on the Ptolemy line through $\mu(u)$ and $u$. But they are also on $\sigma$. Thus $\sigma$ and this Ptolemy line coincide by Corollary 4.10. This is a contradiction.

Lemma 13.4. Given a $\mathbb{C}$-line $F \subset X_\omega$, points $x, y, z \in F$ such that $x < z < y$ and a Ptolemy line $l \subset X_\omega$ through $z$, there is at most one Ptolemy circle through $x, y$ that meets $l$.

Proof. It follows from Lemma 13.2 and Corollary 13.3 that every Ptolemy circle $\sigma$ through $x, y$ that meets $l$ intersects $l$ in two points $u, v$ separated by $z, u < z < v \subset l$. If $\sigma' \neq \sigma$ is another Ptolemy circle through $x, y$ that meets $l$ in points $u', v', u' < z < v' \subset l$, then the points $u, v, u', v' \subset l$ are pairwise distinct by Corollary 4.10. Consider now the situation in a metric of the Möbius structure with infinitely remote point $y$. In this metric $l$ is a Ptolemy circle intersecting $F$ in $z, \omega$, and $x \in F$ is a point between $z$ and $\omega$. The Ptolemy circles $\sigma, \sigma'$ are Ptolemy lines through $x$. Let $\mu_y : X_y \rightarrow F$ be the projection in this metric. Then $\mu_y(u) = \mu_y(v) = \mu_y(u') = \mu_y(v') = x$, in contradiction to Corollary 13.3. Thus $\sigma = \sigma'$.

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Lemma 13.5. Under the assumptions of Lemma 13.3 we have $|uz| = |zv|$.

Proof. We denote by $l$ the Ptolemy line containing $u, z, v$. By Proposition 8.2, there exists an isometry $\varphi$ of $X_{\omega}$ preserving the order of $C$-lines that extends the flip of $l$ at $z$. Thus $\varphi(x) = x, \varphi(y) = y, \varphi(z) = z, \varphi(\omega) = \omega$. Then $\sigma' = \varphi(\sigma)$ is a Ptolemy circle through $x, y$ that meets $l$ at the points $\varphi(u), \varphi(v)$. By Lemma 13.4, $\sigma' = \sigma$, hence $\varphi(u) = v, \varphi(v) = u$ and thus $|uz| = |zv|$.

Corollary 13.6. The projection $\mu(\sigma)$ is the segment in $F$ from $x$ to $y$. Let $\sigma_1$ and $\sigma_2$ be the two segments of $\sigma$ with endpoints $x$ and $y$, then $\mu|\sigma_i : \sigma_i \rightarrow \mu(\sigma_i)$ is a homeomorphism for $i = 1, 2$.

Proof. Let $z \in F$ be a point between $x$ and $y$, $x < z < y$. There are $u_i \in \sigma_i$ with $\mu(u_i) = z, i = 1, 2$, and by Corollary 13.3 the points $u_1, u_2$ are uniquely determined. By Lemma 13.3 we have $|u_1z| = |u_2z|$. Then $|u_1z| = |u_2z| \rightarrow 0$ as $z \rightarrow x$ or $y$. Indeed otherwise we find $u_1 \in \sigma_1, u_2 \in \sigma_2$ distinct from $x, y$ such that $\mu(u_1) = \mu(u_2) = x$ or $y$. Then by Lemma 13.3 there is a Ptolemy line $l \subset X_{\omega}$ through $u_1, u_2, i = 1, 2$. These points are common for $l$ and $\sigma$, hence $l = \sigma$ by Corollary 13.10, a contradiction since $\omega \notin \sigma$.

If follows that $\mu^{-1}(x) \cap \sigma = x$ and $\mu^{-1}(y) \cap \sigma = y$. Furthermore $\mu^{-1}(w) \cap \sigma = \emptyset$, if $w \in F$ is not in the segment from $x$ to $y$, since $\sigma \subset X_{\omega}$ is an embedded circle. Therefore both the restrictions $\mu|\sigma_i : \sigma_i \rightarrow \mu(\sigma_i)$, $i = 1, 2$, are continuous bijections and thus homeomorphisms.

Lemma 13.7. Given points $x < z < y$ in the $C$-line $F$ and a Ptolemy circle $\sigma \subset X_{\omega}$ through $x, y$, we have $|xz| \cdot |zy| = |zu|^2$, where $u \in \sigma$ is a point with $\mu(u) = z$.

Proof. We denote $|xz| = a, |zy| = d$. In a first step we show that the distance $r = |uz|$ depends only on $a, d$. Let $F'$ be a $C$-line with infinitely remote point $\omega', z' < z' < y'$ be points in $F'$ with $|x'z'| = a, |z'y'| = d, \sigma' \subset X_{\omega'}$ a Ptolemy circle through $x', y'$, a point $u' \in \sigma'$ is projected to $z'$ by the respective projection $\mu' : X_{\omega'} \rightarrow F'$, where the distances are taken in a metric $d_{\omega'}$ of the Möbius structure.

There are Ptolemy circles $l$ and $l'$ in $X$ through $\omega, z, u$ and $\omega', z', u'$ respectively. By (E2), a Möbius map $\varphi : l \rightarrow l'$ with $\varphi(\omega) = \omega', \varphi(z) = z', \varphi(u) = u'$, is extended to a Möbius automorphism $X \rightarrow X$ for which we use the same notation $\varphi$. Then $\varphi : (X, d_{\omega}) \rightarrow (X, d_{\omega'})$ is a homothety with $\varphi(F) = F'$. Thus applying if necessary a homothety w.r.t. the metric $d_{\omega'}$ that fixes $z'$ and leaves invariant $l'$, we can assume that $|z'\varphi(x)| = |z'x'| = a$ and $|z'\varphi(y)| = |z'y'| = d$. Then $\varphi : (X, d_{\omega}) \rightarrow (X, d_{\omega'})$ is an isometry. Next, applying if necessary a vertical flip $j : X_{\omega'} \rightarrow X_{\omega'}$ which fixes the Ptolemy line $l'$ pointwise, we can assume that $\varphi(x) = x', \varphi(y) = y'$. Then by Lemma 13.4 we have $\varphi(\sigma) = \sigma'$. Hence using Lemma 13.5 we obtain $|z'u'| = |zu| = r$.
In a second step we show that \( r^2 = ad \). Let \( d_z \) be the inversion of the metric \( \lambda d_\omega(p,q) = \lambda|pq| \) with respect to \( z \), where \( \lambda = 1/ad \), \( d_z(p,q) = \frac{|pq|}{|zpq| zq} \). Then \( d_z(\omega, x) = d_z(\omega, y) = a \), and \( \sigma \) is still a Ptolemy circle through the points \( x, y \) in the \( \mathbb{C} \)-line \( F \) with infinitely remote point \( z \). The Ptolemy line \( l \) in the metric \( d_\omega \) through \( z, u \) with infinitely remote point \( \omega \) is the Ptolemy line in the metric \( d_z \) through \( \omega, u \) with infinitely remote point \( z \). Hence \( d_z(\omega, u) = r \) by the first step. On the other hand, we have \( d_z(\omega, u) = ad/r \). Thus \( ad = r^2 \).

**Proposition 13.8.** Let \( \sigma \subset X_\omega \) be a Ptolemy circle that intersects the \( \mathbb{C} \)-line \( F \) in distinct points \( x, y \) and meets a Ptolemy line \( l \subset X_\omega \) in points \( u, v \) with \( |uz| = |zw| = 1 \), where \( z \in l \cap F \) is the midpoint between \( x, y \). Then \( |zw| = 1 \) for every \( w \in \sigma \).

**Proof.** Let \( \varphi : l \to l \) be a Möbius involution with \( \varphi(u) = u \), \( \varphi(v) = v \) and \( \varphi(z) = \omega \), \( \varphi(\omega) = z \). By (E2), \( \varphi \) extends to a Möbius automorphism of \( X \) for which we use the same notation, \( \varphi : X \to X \). Then \( \varphi \) preserves the \( \mathbb{C} \)-circle \( F \), \( \varphi(F) = F \), because by Corollary 8.6(3C), \( F \) is the unique \( \mathbb{C} \)-circle through \( z, \omega \).

Let \( d_z \) be the metric inversion of \( d_\omega \) w.r.t. \( z \), where \( d_\omega(p,q) = |pq| \) for \( p, q \in X \), that is, \( d_z(p,q) = \frac{|pq|}{|zp| |zq|} \). Then \( \varphi : (X,d_\omega) \to (X,d_z) \) is an isometry because

\[
d_z(\varphi(u), \varphi(z)) = d_z(u,\omega) = \frac{1}{d_\omega(z,u)} = 1.
\]

We have \( |xz| = |zy| \), and \( |xz| \cdot |zy| = 1 \) by Lemma 13.7. Thus \( |xz| = |zy| = 1 \). Then \( d_z(x,\omega) = 1/|xz| = 1 \) and similarly \( d_z(y,\omega) = 1 \). It follows that \( \varphi \) preserves the set \( \{x,y\} \), \( \varphi(\{x,y\}) = \{x,y\} \) and hence the Ptolemy circle \( \sigma \) is invariant under \( \varphi \), \( \varphi(\sigma) = \sigma \).

By Proposition 12.3 there exists a vertical flip \( j : X_\omega \to X_\omega \) that fixes the Ptolemy line \( l \) pointwise. Then \( j \) flips \( F \), \( j(x) = y \), \( j(y) = x \) and thus \( j(\sigma) = \sigma \). Applying \( j \) if necessary, we can assume that \( \varphi(x) = x \), \( \varphi(y) = y \). Then \( \varphi \) fixes \( \sigma \) pointwise, \( \varphi(w) = w \) for every \( w \in \sigma \). Now, we have

\[
|zw| = d_z(\varphi(x),\varphi(w)) = d_z(x,w) = \frac{|zw|}{|zw| \cdot |xz|} = \frac{|zw|}{|zw|},
\]

which implies \( |zw| = 1 \) for every \( w \in \sigma \).

**13.2 Computing the distance function** \( D \)

Now, we are able to find the distance function \( D \), see section 12.2.

**Proposition 13.9.** We have \( D(a,b)^4 = a^4 + b^4 \) for each \( a, b \geq 0 \).

**Proof.** Let \( \sigma \subset X_\omega \) be a Ptolemy circle through points \( x, y \) in the \( \mathbb{C} \)-line \( F \), \( \sigma \in F \) the midpoint between \( x, y \), \( |xo| = |oy| = 1 \). By rescaling with
coefficient $\lambda > 0$, which has to be found, we assume that $|oz| = \lambda b$, $|zu| = \lambda a$ for some $u \in \sigma$ with $\mu(u) = z$, where $z \in F$ is a point between $o$ and $y$. Using Proposition 10.6 and notations $|xz| = e$, $|zy| = d$, we obtain $d^2 = 1 - \lambda^2 b^2$, $c^2 = 1 + \lambda^2 b^2$. By Lemma 13.7, we have $c^2 d^2 = \lambda^4 a^4$. Hence $\lambda^4 = 1/(a^4 + b^4)$.

By Proposition 13.8 $|ou| = 1 = \lambda D(a,b)$. Therefore, $D(a,b)^4 = 1/\lambda^4 = a^4 + b^4$.

13.3 The lifting constant

Lemma 13.10. Let $\sigma \subset \mathcal{X}_\omega$ be a Ptolemy circle centered at $z = l \cap F$, where the Ptolemy line $l \subset \mathcal{X}_\omega$ intersects $\sigma$ in $u,v$, the $\mathbb{C}$-line $F \subset \mathcal{X}_\omega$ intersects $\sigma$ in $x,y$. Let $l' \subset \mathcal{X}_\omega$ be the tangent Ptolemy line to $\sigma$ at the point $u$. Then the lines $\overline{l} = \pi_\omega(l)$, $\overline{l'} = \pi_\omega(l') \subset B_\omega$ span a complex line, that is, $J(\overline{l}) \parallel \overline{l'}$ for the canonical complex structure $J : B_\omega \rightarrow B_\omega$.

Proof. We regard $B_\omega$ as an Euclidean space with origin $o = \pi_\omega(z) = \pi_\omega(x) = \pi_\omega(y)$, and $J$ as an isometry of $B_\omega$ with $J(o) = o$.

By Proposition 12.3 there is a vertical flip $j : \mathcal{X}_\omega \rightarrow \mathcal{X}_\omega$ with respect to $z$ that fixes $l$ pointwise. Then $j(u) = u$, $j(v) = v$, $j(x) = y$ and $j(y) = x$. Thus $j$ preserves $\sigma$, $j(\sigma) = \sigma$. Moreover $j$ preserves $l'$, $j(l') = l'$, by uniqueness the tangent Ptolemy line, see Proposition 4.27, and it acts on $l'$ as a flip because $j$ flips the Ptolemy circle $\sigma$.

Let $\overline{l'} \subset B_\omega$ be the line through the origin $o$ parallel to $\overline{l}$. Recall that $\pi_\omega(j) = \text{Conj}$, see sect. 12.3. Since $j$ preserves $l$ pointwise and flips $l'$, the line $\overline{l'}$ and thus the line $\overline{l''}$ is orthogonal to $\overline{l}$. It suffices to show that $J(\overline{l}) = \overline{l''}$. To this end, we use a freedom in the construction of the vertical flip $j$. Namely, assume that $J(\overline{l}) \neq \overline{l''}$. Since $J(\overline{l}) \perp \overline{l}$, the projection $m \subset B_\omega$ of $\overline{l''}$ to the orthogonal complement $L^\perp \subset B_\omega$ of the subspace $L$ spanned by $u_1$, $v_1 = J(u_1)$ is nontrivial. We construct a canonical basis $b = \{v_1, v_2, \ldots, v_k\}$ of $B_\omega$ for the complex structure $J$, see sect. 12.3, so that $u_1$ generates $\overline{l}$ and $u_2$ generates $m$. Then for the respective conjugation isometry $\text{Conj} : B_\omega \rightarrow B_\omega$ we have $\text{Conj}(\overline{l''}) \neq \overline{l''}$ unless $\overline{l''} = m$ because by definition $\text{Conj}(u_1) = u_1$, $\text{Conj}(v_1) = -v_1$ for $i = 1, \ldots, k$.

Since $j(l') = l'$, it follows from definition of the respective flip $j : \mathcal{X}_\omega \rightarrow \mathcal{X}_\omega$, see Proposition 12.3, that $\text{Conj}(\overline{l''}) = \overline{l''}$ and hence $\overline{l''} = m$. But then the isometry $\text{Conj}$ preserves $\overline{l''}$ pointwise and thus $j$ preserves the tangent Ptolemy line $l'$ pointwise. This contradicts the property that $j$ acts on $l'$ as a flip. Therefore $J(\overline{l}) = \overline{l''}$, and $\overline{l}, \overline{l'}$ span the complex line $L \subset B_\omega$.

Lemma 13.11. Let $\mu = \mu_F : \mathcal{X}_\omega \rightarrow \mathcal{X}_\omega$ be the projection onto a $\mathbb{C}$-line $F \subset \mathcal{X}_\omega$. Then for each $u,v \in \mathcal{X}_\omega$ we have

$$|\mu(u)\mu(v)|^2 \leq |uv|^2 + \delta(T)^2,$$

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where $T = \tau_{\tau_T}$ is the (oriented) triangle in the base $B_\omega$ with vertices $\tau = \pi_\omega(F)$, $\overline{u} = \pi_\omega(u)$, $\overline{v} = \pi_\omega(v)$, and $\delta(T)$ is the displacement of the lifting isometry $\tau_T : X_\omega \to X_\omega$.

Proof. Let $F' = \pi^{-1}(\overline{u})$ be the $C$-line in $X_\omega$ through $u$, $u' = \tau_F(u) \in F'$, $v' = \mu_F(v) \in F'$. By Proposition 10.6 we have $|u'v'|^2 = |uv|^2 \pm |uu'|^2$, and $|uu'| = \delta(T)$. It follows from Proposition 13.9 that $|uv'| \leq |uv|$. Thus

$$|\mu(u)\mu(v)|^2 = |u'v'|^2 \leq |uv|^2 + \delta(T)^2.$$

}\[
\]

Proposition 13.12. Let $c > 0$ be the supremum of the lifting constants, taken over all the unit squares $Q$ in $B_\omega$, $c = \sup_Q \delta(Q)$, see sect. 12.3. Then $c = 2$.

Proof. Let $\sigma \subset X_\omega$ be a Ptolemy circle of radius 1 centered at $z \in F$, where $F \subset X_\omega$ is a $C$-line, $x, y \in \sigma \cap F$ such that $x < z < y$ with respect to the order $O$, $l$ the Ptolemy line through $z$ that intersects $\sigma$. We denote with $j : X_\omega \to X_\omega$ a vertical flip which preserves $l$ pointwise, see sect. 12.3. Then recall $j(\sigma) = \sigma$.

Given $t \in F$, $x < t < y$, we denote with $a = a(t) = |xt|$, $d = d(t) = |ty|$. Then by Proposition 10.6 we have $a^2 + d^2 = 2$. For $v \in \sigma$ with $\mu(v) = t$, we let $r = r(t) = |tv|$. Then by Lemma 13.7, $r^2 = a \cdot d$. Denote with $b^+ = b^+(t) = |xv|$ and with $b^- = b^-(t) = |yv|$. Then by Proposition 13.9

$$(b^+)^4 = a^4 + r^4 = a^2(a^2 + d^2) = 2a^2$$

and similarly $(b^-)^4 = 2d^2$. Thus $(b^+)^2 = a\sqrt{2}$ and $(b^-)^2 = d\sqrt{2}$.

Now, we assume that $a < d$ and take a point $w \in \sigma$ on the arc $xvy$ of $\sigma$ with $\mu(w) = s \in F$ such that $|yw| = a = |xt|$, that is, $w = j(v)$. Then $|yw| = |xv| = b^+$, $|xw| = |yw| = b^-$, and using the Ptolemy equality applied to the quadruple $(x, v, w, y) \subset \sigma$, we obtain $(b^-)^2 - (b^+)^2 = |uw|\sqrt{2}$. This gives

$$d - a = |uw|.$$

Furthermore, $|st|^2 = d^2 - a^2$ again by Proposition 10.6.

The Ptolemy line $l$ intersects the Ptolemy circle $\sigma$ in two points, and we denote with $u \in l \cap \sigma$ the point between $v$, $w$ on the arc $xwvy$ of $\sigma$. Then $j(u) = u$ and $j$ preserves the tangent Ptolemy line $l'$ to $\sigma$ through $u$. We denote with $v'$, $w' \in l'$ points closest in $l'$ to $v$, $w$ respectively. Then $|vv'| = o(|uv|)$, $|ww'| = o(|uw|)$. Note that $|uv| = |uw| \geq 1$ by $j$-symmetry. Thus $|v'w'| = |uw| + o(|uw|) = d - a + o(|uw|)$.

Since $\overline{v'}\overline{w'} \subset \overline{l'}$, by Lemma 13.10 the triangle $T = \overline{v'}\overline{w'}\overline{l'}$ lies a complex line in $B_\omega$, where $\overline{v} = \pi_\omega(z) = \pi_\omega(s) = \pi_\omega(t)$, $\overline{w} = \pi_\omega(v')$, $\overline{w'} = \pi_\omega(w')$, $\overline{l'} = \pi_\omega(l')$, and thus the canonical complex structure $J$ with $J(\overline{z}) = \overline{z}$
preserves the 2-subspace of $B_{\omega}$ that contains $T$. Hence the area law of lifting applied to $T$ gives
\[ \delta(T)^2 = c^2 \cdot \text{area } T, \]
see Remark 11.6 where $\delta(T)$ is the displacement of the lifting isometry $\tau_T : X_{\omega} \to X_{\omega}$. We have $|\overline{v'}w'| = |v'w'| = d - a + o(|uv|)$, $|\overline{v}w'| = |\pi_\omega(t)\pi_\omega(v)| + o(|uv|) = |tv| + o(|uv|) = r + o(|uv|)$ and similarly $|\overline{w'}v'| = r + o(|uv|)$. Thus area $T = \frac{1}{2}r(d - a) + o(|uv|)$.

Denote with $t' = \mu(v')$ and $s' = \mu(w')$ points in the $C$-line $F$. Then $\delta(T) = |t's'|$. By Lemma 13.11 $|tt'|^2 \leq |vv'|^2 + \delta(T_t)^2$, $|ss'|^2 \leq |ww'|^2 + \delta(T_s)^2$, where $T_t = \overline{v}w', T_s = \overline{w}v'$ are triangles in $B_{\omega}$. We have $|\overline{v}w'| = o(|uv|)$ and $|\overline{v}v', \overline{w}w'| \leq 1$. Thus area $T_t$, area $T_s = o(|uv|)$, and hence $\delta(T_t)^2$, $\delta(T_s)^2 = o(|uv|)$ by the area law of lifting. Therefore $|tt'|^2$, $|ss'|^2 = o(|uv|)$, and using Proposition 11.6 we obtain
\[ |tt'\pi - ts| \leq |tt'| + |ss'| = o(|uv|). \]
Then
\[ \delta(T)^2 = |tt'|^2 = |ts|^2 + o(|uv|) = d^2 - a^2 + o(|uv|) \]
and therefore
\[ d^2 - a^2 = \frac{c^2}{2}r(d - a) + o(|uv|). \]
Since $\sigma$ is a Ptolemy circle of radius one, we have $a, d \to 1$ and
\[ c^2 = \frac{2(a + d)}{\sqrt{ad}} + o(1) \to 4 \]
as $|uv| \to 0$. Thus $c = 2$.

\section{The model space $\partial \infty KH^k$}

Every rank one symmetric space $M$ of non-compact type is a hyperbolic space $KH^k$ over a normed division algebra $K$. The only possibilities are the real numbers $K = \mathbb{R}$, dim $K = 1$; the complex numbers $K = \mathbb{C}$, dim $K = 2$; the quaternions $K = \mathbb{H}$, dim $K = 4$; and the octonions $K = \mathbb{O}$, dim $K = 8$. Then dim $M = k \cdot $ dim $K$, where $k \geq 2$ and $k = 2$ in the case $K = \mathbb{O}$. We choose a normalization of the metric so that in the case $K = \mathbb{R}$ the sectional curvatures of $M$ are $K_\sigma \equiv -1$ and in the case $K \neq \mathbb{R}$ the sectional curvatures of $M$ are pinched as $-4 \leq K_\sigma \leq -1$.

We use the standard notation $TM$ for the tangent bundle of $M$ and $UM$ for the subbundle of the unit vectors. For every unit vector $u \in U_0 M$, where $o \in M$, the eigenspaces $E_u(\lambda)$ of the curvature operator $\mathcal{R}(\cdot, u)u : u^\perp \to u^\perp$, where $u^\perp \subset T_o M$ is the subspace orthogonal to $u$, are parallel along the geodesic $\gamma(t) = \exp_o(tu)$, $t \in \mathbb{R}$, and the respective eigenvalues $\lambda = -1, -4$ are constant along $\gamma$. The dimensions of the eigenspaces are $\dim E_u(-1) = (k - 1) \dim K$, dim $E_u(-4) = \dim K - 1$, $u^\perp = E_u(-1) \oplus E_u(-4)$.
14.1 The Möbius structure of $\partial_{\infty} \mathbb{H}^k$

We let $Y = \partial_{\infty} M$ be the geodesic boundary at infinity of $M$. For every $o \in M$ the function $d_o(\xi, \xi') = e^{-\langle \xi | \xi' \rangle} o$ for $\xi, \xi' \in Y$ is a (bounded) metric on $Y$, where $(\xi | \xi')_o$ is the Gromov product based at $o$. For every $\omega \in Y$ and every Busemann function $b : M \to \mathbb{R}$ centered at $\omega$, the function $d_b(\omega, \omega) := 0$ and $d_b(\xi, \xi') = e^{-\langle \xi | \xi' \rangle} b$, except the case $\xi = \xi' = \omega$, is an (unbounded) metric on $Y$, where $(\xi | \xi')_b$ is the Gromov product with respect to $b$. Since $M$ is a CAT($-1$)-space, the metrics $d_o, d_b$ satisfy the Ptolemy inequality and furthermore all these metrics are pairwise Möbius equivalent, see [FS1]. We let $\mathcal{M}$ be the canonical Möbius structure on $Y$ generated by the metrics of type $d_o, o \in M$. Recall that $\mathcal{M}$ is the class of metrics on $Y$ each of which is Möbius equivalent to any $d_o$. Then $Y$ endowed with $\mathcal{M}$ is a compact Ptolemy space. Every metric $d \in \mathcal{M}$ is of type $d = d_o$ for some Busemann function $b : M \to \mathbb{R}$, or $d = \lambda d_o$, for some $o \in M$ and $\lambda > 0$. We emphasize that in general metrics of $\mathcal{M}$ are neither Carnot-Carathéodory metrics nor length metrics.

The following lemma is a modification of Lemma 113.

**Lemma 14.1.** Let $\varphi : X \to X$ be a Möbius involution, $\varphi^2 = \text{id}$, of a Ptolemy space $X$ with $\varphi(\omega) = \omega'$ for distinct $\omega, \omega' \in X$. Then there is a unique metric sphere $S \subset X$ between $\omega, \omega'$ invariant for $\varphi$, $\varphi(S) = S$.

**Proof.** Let $d$ be a metric of the Möbius structure with infinitely remote point $\omega'$. Since $\varphi(\omega) = \omega'$, the point $\omega$ is infinitely remote for the induced metric $\varphi^*d$. Thus for some $\lambda > 0$ we have

$$(\varphi^*d)(x, y) = \frac{\lambda d(x, y)}{d(x, \omega) d(y, \omega)}$$

for each $x, y \in X$ which are not equal to $\omega$ simultaneously. We let $S = S^d(\omega) \subset X$ be a metric sphere between $\omega, \omega'$ with $r^2 = \lambda$, $d(x, \omega) = r$ for every $x \in S$. Then

$$d(\varphi(x), \omega) = d(\varphi(x), \varphi(\omega')) = (\varphi^*d)(x, \omega') = \lambda d(x, \omega) = r$$

for every $x \in S$. Hence $\varphi(S) = S$. For any $x \in X$ with $d(x, \omega) \leq r$ the same argument shows that $d(\varphi(x), \omega) \geq r$, thus an invariant sphere between $\omega, \omega'$ is unique. \hfill $\square$

**Proposition 14.2.** Let $Y = \partial_{\infty} M$ be the boundary at infinity of a rank one symmetric space $M$ of noncompact type. Then regarded as a compact Ptolemy space with respective Möbius structure $\mathcal{M}$, $Y$ satisfies properties (E) and (I).

**Proof.** Every rank one symmetric space $M$ of noncompact type contains geodesic subspaces $L$ isometric to $\mathbb{H}^2$. Then $\sigma = \partial_{\infty} L \subset Y$ is a Ptolemy circle. Hence, the property (E) is fulfilled for $Y$. 

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Given distinct $\omega, \omega' \in Y$ and a metric sphere $S \subset Y$ between $\omega, \omega'$, we show that there is a unique space inversion $\varphi = \varphi_{\omega,\omega',S} : Y \to Y$ w.r.t. $\omega, \omega', S$.

Let $b : M \to \mathbb{R}$ be a Busemann function of the geodesic ray $x\omega \subset M$, $x \in l = \omega \omega' \subset M$. Then the Gromov product $(\xi|\xi')_b \in \mathbb{R}$ w.r.t. $b$ is well defined for $\xi, \xi' \in Y \setminus \omega$, see e.g. [BS sect.3.2], [BK], and the function $d(\xi, \xi') = e^{-(\xi|\xi')_b}$ is a metric of the Möbius structure with infinitely remote point $\omega$. Let $r > 0$ be the radius of $S$, $S = S^b_0(\omega')$. We take $o \in l$ with $b(o) = -\ln r$. Let $u \in U_o M$ be a tangent vector to $l$. For every $\xi \in Y$ such that the direction $v_\xi \in U_o M \cap u^\perp$ of the ray $o \xi \subset M$ is an eigenvector of the curvature operator, $v_\xi \in E_u(-1) \cup E_u(-4)$, the ideal triangle $\omega \xi \omega'$ lies in a geodesic subspace of $M$ isometric to $H^2$ (the case $v_\xi \in E_u(-1)$) or to $\frac{1}{2}H^2$ (the case $v_\xi \in E_u(-4)$). By symmetry, $o$ is an equiradial point of $\omega \xi \omega'$, see [BK] Proposition 2.5], thus $(\xi|\omega')_b = b(o) = -\ln r$ and $d(\xi, \omega') = r$, that is, $\xi \in S$.

The central symmetry $f : M \to M$ at $o$, $d_of = -\text{id}_{T_o M}$, is an isometry that induces a Möbius involution $\varphi = \partial_\infty f : Y \to Y$ without fixed points and with $\varphi(\omega) = \omega'$. Then for every $\xi \in S$ with $v_\xi \in E_u(-i^2)$ we have $v_\varphi(\xi) = -v_\xi \in E_u(-i^2), i = 1, 2$. Thus $\varphi(\xi) \in S$. It follows from Lemma 14.1 that the sphere $S$ is invariant for $\varphi, \varphi(S) = S$. Furthermore, every Ptolemy circle $\sigma \subset Y$ through $\omega, \omega'$ is the boundary at infinity of a geodesic subspace $L \subset M$ containing $l$ and isometric to $H^2$, see [FS1]. Since $L$ is invariant for $f$, we have $\varphi(\sigma) = \sigma$. Therefore, $\varphi = \varphi_{\omega,\omega',S} : Y \to Y$ is a space inversion w.r.t. $\omega, \omega', S$.

Assume there is a Möbius involution $\psi : Y \to Y$, $\psi^2 = \text{id}$, without fixed points and with $\psi(\omega) = \omega'$, $\psi(S) = S$, such that $\psi(\sigma) = \sigma$ for every Ptolemy circle $\sigma \subset Y$ through $\omega, \omega'$. We show that $\psi = \varphi$.

Let $B \subset Y$ be the union of all the Ptolemy circles through $\omega, \omega'$. Then $\psi|B = \varphi|B$ because $S$ intersects every arc between $\omega, \omega'$ of a Ptolemy circle through $\omega, \omega'$ at a unique point, and any Möbius automorphism of $\sigma$ is uniquely determined by values at three distinct points.

Note that the existence of the fibration $\pi_\omega : Y_\omega \to B_\omega$ with the base $B_\omega$ isometric to $\mathbb{R}^{(k-1)\dim K}$, $\dim M = k \dim K$, is well known, and it follows from consideration of a model, see [GO] for the case $M = \mathbb{C}H^k$.

We put $\eta = \psi^{-1} \circ \varphi$ and note that $\eta|B = \text{id}_B$. Then $\eta : Y_\omega \to Y_\omega$ is an isometry w.r.t. the metric $d$ that induces the identity map of the base $B_\omega$. Thus $\eta$ preserves any foliation of $Y_\omega$ by Busemann parallel Ptolemy line. Then $\eta = \text{id}$ by the same argument as in the proof of Lemma 14.1. Hence, $\psi = \varphi$, and the property (I) is fulfilled for $Y$.

### 14.2 Isomorphism with the model space $\partial_\infty \mathbb{C}H^k$

For a given $\omega \in X$ we fix as usual a metric from the Möbius structure with infinitely remote point $\omega$. As in sect. 12 we introduce standard coordinates
in $X_\omega$ with origin $o \in X_\omega$ identifying $X_\omega$ with $\mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$ by $x = (z, h)$. Moreover, we regard $X_\omega$ as the nilpotent group $N_\omega$, see sect. 9.3, identifying every $x \in X_\omega$ with $g \in N_\omega$ via the rule $x = g(o)$. Then the multiplication law of $N_\omega$ in the coordinates for $g = (z, h)$, $g' = (z', h')$ is given by
\[
g \cdot g' = \left( z + z', h + h' + \frac{1}{2} \langle J(z), z' \rangle \right),\]
where we used Lemmas 12.1 and Proposition 13.12. Next, we fix a canonical basis $b = \{u_1, v_1, \ldots, u_k, v_k\}$ of $B_\omega = \mathbb{R}^n$, $n = 2k$, for the complex structure $J$, where $v_i = J(u_i)$, $i = 1, \ldots, k$. Then $z = (z_1, \ldots, z_k)$ for every $z \in B_\omega$, where $z_i = x_i u_i + y_i v_i$ with $x_i, y_i \in \mathbb{R}$ and $z = \sum_i z_i$, that is, we identify $B_\omega$ with the complex coordinate space $\mathbb{C}^k$. Note that $\text{Conj} z_i = x_i u_i - y_i v_i$ for every $z_i = x_i u_i + y_i v_i$, and we use the standard notation $\text{Conj} z = \overline{z}$ for the complex conjugation.

Let $(z, z') = \sum_i z_i z_i'$ be the standard Hermitian form on $\mathbb{C}^k$.

**Lemma 14.3.** For each $g = (z, h)$, $g' = (z', h') \in N_\omega$ we have
\[
g \cdot g' = \left( z + z', h + h' - \frac{1}{2} \text{Im}(z, z') \right).
\]

**Proof.** This follows from the equality $\langle J(z), z' \rangle = -\text{Im}(z, z')$. \hfill \Box

It follows that $N_\omega$ is the higher-dimensional Heisenberg group $H^k$, see CDPT sect. 2.1.2.

**Proof of Theorem 1.2.** We only need to consider the case $p = 1$ and to show that $X$ is Möbius equivalent to $\partial_\infty \mathbb{C}H^k$ taken with the canonical Möbius structure, $\dim X = n = 2k - 1$.

It follows from Proposition 11.3 that the dimension of the base $B_\omega$ is even, $\dim B_\omega = n - 1 = 2(k - 1)$. By Corollary 7.3, $k \geq 2$. In a given dimension a compact Ptolemy space that satisfies (E), (I) with $p = 1$ is uniquely determined up to a Möbius isomorphism by the lifting constant $c$, and this constant itself is uniquely determined by these properties, $c = 2$.

We give more detail for this argument, which are however straightforward. Let $Y = \partial_\infty \mathbb{C}H^k$ be the boundary at infinity of the complex hyperbolic space $\mathbb{C}H^k$ taken with the canonical Möbius structure. Since $\mathbb{C}H^k$ is a CAT$(-1)$-space, the Möbius structure of $Y$ is a compact Ptolemy space, and $\dim Y = 2k - 1$. By Propositions 14.2, $Y$ satisfies properties (E) and (I).

We fix metrics from the Möbius structures of $X$, $Y$ with infinitely remote points $\omega \in X$ and $\varsigma \in Y$ respectively and introduce standard coordinates in $X_\omega, Y_\varsigma$ identifying these spaces with respective nilpotent groups of isometries $N_\omega, N_\varsigma$. By Lemma 14.3, the multiplications laws in $N_\omega, N_\varsigma$ are identical, thus the groups are isomorphic via the isomorphism which associates to each
other the elements with equal coordinates. Moreover, this isomorphism is an isometry between $X_\omega, Y_\zeta$ because for any $x = (z,h), x' = (z',h') \in X_\omega$ we have
\[ |xx'|^4 = |z - z'|^4 + 16|h - h'| - \frac{1}{2} \text{Im}(z\bar{z}')^2 \] 
by the distance formula (12) from Proposition 12.5 together with Proposition 13.9.

**Remark 14.4.** The distance on a Heisenberg group $\mathbb{H}^k$ given by Eq. (13) is called the *Korányi gauge*, see [CDPT]. In our approach, this metric is a member of the canonical Möbius structure on the boundary at infinity $Y = \partial_\infty C\mathbb{H}^k$, cp. [CDPT] sect. 3.4.5.

**References**

[Ba] J. Baez, The octonions, Bull. Amer. Math. Soc. (N.S.) 39, no.2 (2002), 145–205 (electronic)

[Bes] A. Besse, Einstein manifolds, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 10. Springer-Verlag, Berlin, 1987. xii+510 pp. ISBN: 3-540-15279-2.

[Bor] A. Borel, Some remarks about Lie groups transitive on spheres and tori, Bull. AMS, 55 (1949), 580–587.

[BFW] S. Buckley, K. Falk, D. Wraith, Ptolemaic spaces and CAT(0), Glasg. Math. J. 51 (2009), no. 2, 301–314.

[BK] S. Buyalo, A. Kuznetsov, Boundary at infinity of symmetric rank one spaces, St. Petersbourg Math. J. 21 (2010), no.5, 681–691; arXive:math.DG/0906.0779.

[BS] S. Buyalo, V. Schroeder, Elements of asymptotic geometry, EMS Monographs in Mathematics, 2007, 209 pages.

[CDPT] L. Capogna, D. Danielli, S. Pauls, J. Tyson, An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem. Progress in Mathematics, 259. Birkhäuser Verlag, Basel, 2007. xvi+223 pp. ISBN: 978-3-7643-8132-5; 3-7643-8132-9.

[H-UL] J.-B. Hiriart-Urruty and C. Lemaréchal, Fundamentals of Convex analysis. Berlin: Springer, 2001.

[Kay] D. Kay, Ptolemaic metric spaces and the characterization of geodesics by vanishing metric curvature, Ph.D. thesis, Michigan State Univ., East Lansing, MI, 1963.
[FLS] T. Foertsch, A. Lytchak, V. Schroeder, Nonpositive curvature and the Ptolemy inequality, Int. Math. Res. Not. IMRN 2007, no. 22, Art. ID rnm100, 15 pp.

[FS1] T. Foertsch, V. Schroeder, Hyperbolicity, CAT(−1)-spaces and Ptolemy inequality, arXiv:math/0605418v2, 2006.

[FS2] T. Foertsch, V. Schroeder, Ptolemy spaces with many circles, arXiv:math/1008.3250, 2010.

[Gol] W. Goldman, Complex hyperbolic geometry, Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1999. xx+316 pp. ISBN: 0-19-853793-X

[MS] D. Montgomery and H. Samelson, Transformation groups of spheres, Annals of Math., 44 (1943), 454–470.

[Sch] I. Schoenberg, A remark on M. M. Day’s characterization of inner-product spaces and a conjecture of L. M. Blumenthal, Proc. Amer. Math. Soc. 3 (1952) 961–964.

[Sieb] E. Siebert, Contractive automorphisms on locally compact groups, Math.Z.191 (1986), 73–90.