GEOMETRY OF SOLUTIONS OF N=2 SYM-THEORY IN HARMONIC SUPERSPACE

B.M. Zupnik

Joint Institute for Nuclear Research, Bogoliubov Laboratory of Theoretical Physics, Dubna, Moscow Region, 141980, Russia. E-mail: zupnik@thsun1.jinr.ru

Abstract

In harmonic superspace, the classical equations of motion of $D = 4, N = 2$ supersymmetric Yang-Mills theory for Minkowski and Euclidean spaces are analyzed. We study dual superfield representations of equations and subsidiary conditions corresponding to classical SYM-solutions with different symmetries. In particular, alternative superfield constructions of self-dual and static solutions are described in the framework of the harmonic approach.

1. Introduction

The off-shell superfield constraints of the $N = 2$ super-Yang-Mills theory in the Minkowski $D = (3, 1)$ space has been solved in the framework of harmonic superspace (HSS) using the auxiliary coordinates of the coset space $S_2 \sim SU_A(2)/U_A(1)$ where $SU_A(2)$ is the corresponding automorphism group [1]-[3]. An analogous harmonic formalism can be applied to reformulate the off-shell $N = 2$ supersymmetric models in the Euclidean $D = 4$ space. We shall use the notation with the subscripts $HSS_M$ or $HSS_E$ for harmonic-superspace structures over Minkowski or Euclidean spaces, respectively. Auxiliary harmonic coordinates are used in covariant conditions of the Grassmann (G-) analyticity. The unconstrained superfields of the $N = 2$ theories live in the G-analytic superspaces with the reduced odd dimension 4.

Harmonic variables connected with coset $SU_R(2)/U_R(1)$ of the subgroup of the Euclidean rotation group have been used to study self-dual solutions of the YM- and SYM-theories [4, 5, 6]. These harmonic superspaces have the reduced values of even and odd dimensions, and the corresponding analytic superfields parametrize moduli spaces of self-dual solutions.

The superfield equations of motion in the most general harmonic formalism can be effectively used for the analysis of the geometry of classical $N = 2$ SYM-solutions. In particular, one can consider dual transformations of the $N = 2$ SYM-superfield variables in the harmonic superspace which allow us to formulate unusual representations of the equations of motion and simple gauge conditions [7, 8]. It is important to stress that the dual
change of variables transforms the part of SYM-constraints and equations in the harmonic formalism to linear restrictions on HSS-connection $V^{--}$, then the zero-curvature relation between the harmonic connections can be interpreted as a dynamical equation. Dual representations of the HSS-superfield equations allow us to describe symmetry properties of different classes of partial solutions.

It is interesting to compare the alternative HSS-descriptions of 4D self-dual SYM-solutions or solutions of $D < 4$ SYM-equations. We analyze the $(t = 0)$ reduction of the $HSS_M$-formalism and formulate the static 3-dimensional superfield BPS-conditions. It is shown that the supersymmetry of the static self-dual SYM-equations is equivalent to the corresponding 3-dimensional subgroup of the Euclidean supersymmetry with 8 supercharges.

The $HSS_M$-equations of the $N = 2$ SYM-theory are considered in Sect. 2. We discuss the off-shell formalism with independent harmonic connections and dually-equivalent constructions of the superfield equations of motion. One analyzes the conditions of the G-analyticity for connections in this formalism. The superfield equations of motion can be transformed to relations between the G-analytic functions arising in decompositions of the non-analytic connection $V^{--}$.

Sect. 3 is devoted to the discussion of the static 3D reduction of the $N = 2$ superfield SYM-equations. The time component of the 4D superfield connection $A_t$ becomes the new 3D-scalar superfield in this limit. We consider the 3D-superfield BPS-type relation between $A_t$ and the superfield strengths $W$ and $\bar{W}$ which is equivalent to the 2nd order differential constraint for connection $V^{--}$. It is shown that this 2nd order constraint generates the standard 4th order constraint for the same connection and all solutions of the superfield BPS-equation satisfy the SYM-equation.

In Sect. 4 we study the 4D Euclidean version of the harmonic $HSS_E$ formalism. The chiral superfield self-duality condition in this approach can also be interpreted as the 2nd order constraint on $V^{--}$. The alternative bridge representation and the nilpotent gauge condition for the bridge superfield are used to analyze the $HSS_E$ self-dual solutions.

The comparison of these harmonic constructions with the analogous self-dual $N = 2$ solutions in the alternative $SU_\mu(2)/U_\mu(1)$ formalism is discussed in Sect. 5. This version of the harmonic approach transforms the super-self-duality condition to the specific Grassmann-bosonic analyticity of the on-shell harmonic connection. We consider the identification of the harmonic variables for different $SU(2)$ subgroups and the simple Ansatz for the self-dual harmonic connection in the gauge group $SU(2)$ which yield the explicit construction of solutions in quadratures [11].

The static self-dual solutions are discussed in Sect. 6. We discuss the relations between the conjugated spinor coordinates in the static limit of the 4D superspace $HSS_M$ and the corresponding pseudoreal spinor coordinates of the 3D reduction of the Euclidean superspace $HSS_E$.

The basic formulae of the harmonic-superspace approach are reviewed in Appendix. We describe conjugation rules for different versions of harmonic superspaces which are very important in the SYM-theory.
2. Harmonic-superspace representations of N=2 SYM-equations

The harmonic SU(2)/U(1) transform has been used first to solve the off-shell constraints for superfield connections $A^a_k(z)$ of the $N = 2, D = (3, 1)$ SYM-theory [1, 2]

$$u^+_k(D^+_a + A^+_a) \equiv \nabla^+_a = e^{-v}D^+_a e^v$$

(2.1)

where $a = (\alpha, \dot{\alpha})$ are the $SL(2, C)$ indices, $k$ is the 2-component index of automorphism group $SU(2)_A$, $u^+_k$ are the auxiliary harmonic variables (see Appendix) and $v(z, u)$ is the bridge matrix. This transform connects formally different off-shell representations of the gauge group in the central basis (CB) and the analytic basis (AB).

The basic off-shell harmonic superfield of the SYM-theory is the connection $V^{++} = e^v D^{++} e^{-v}$ for the harmonic covariant derivative of the analytic basis

$$\nabla^{++} = D^{++} + V^{++} , \quad \delta V^{++} = -D^{++} \lambda - [V^{++}, \lambda] .$$

(2.2)

The prepotential $V^{++}$ and the AB-gauge parameters satisfy the G-analyticity conditions

$$(D^+_a, \bar{D}^+_a)(V^{++}, \lambda) = 0 .$$

(2.3)

The G-analytic superfields are described by the unconstrained functions of the analytic coordinates $\zeta$ (A.1) and harmonics $u^\pm_i$. The second harmonic connection $V^{--}$ satisfies the harmonic zero-curvature equation

$$D^{++} V^{--} - D^{--} V^{++} + [V^{++}, V^{--}] = 0 ,$$

(2.4)

which can be solved explicitly in terms of $V^{--}$-serias.

In the analytic basis, the spinor and vector connections $A_M$ as well as the superfield strengths $W$ and $\bar{W}$ can be written directly via spinor derivatives of connection $V^{--}$ [3]

$$A^+_a = \bar{A}^+_a = 0 , \quad A^-_a = -D^+_a V^{--} , \quad \bar{A}^-_a = -\bar{D}^+_a V^{--} ,$$

$$A^a_{\alpha \dot{\beta}} \equiv i D^+_a \bar{A}^-_{\alpha \dot{\beta}} \equiv -i \bar{D}^+_a A^-_{a \beta} = -i D^+_a \bar{D}^+_a V^{--} ,$$

$$\bar{W} = -\frac{1}{2} D^a A^-_a = -(D^+)^2 V^{--} , \quad W = -\frac{1}{2} \bar{D}^+_a \bar{A}^a = (\bar{D}^+)^2 V^{--} .$$

(2.5)

(2.6)

(2.7)

The nonlinear harmonic-superfield equation of motion for the prepotential $V^{++}$ can be reformulated as a linear differential condition on the composed connection $V^{--}(V^{++})$

$$(D^+)^2 (\bar{D}^+)^2 V^{--} = 0 .$$

(2.8)

Eq.(2.4) is treated as a kinematic solvable equation in the standard SYM-formalism with basic superfield $V^{++}$. The alternative HSS$_M$ formalism with the independent off-shell superfield $V^{--}$ has been considered in Refs.[7]. This approach is analogous to the first-order formalism for the ordinary Yang-Mills theory, since the independent superfield $V^{--}$ contains off-shell an infinite number of auxiliary fields including the fields of dimension $-2 : F, F_{mn}$ and $F_m$. (Note that the changes of basic HSS variables $V^{++} \rightarrow V^{--}$ or $V^{++} \rightarrow v$ can be interpreted as dual transformations in the SYM-theory. We hope
that the rich structure of the dual transformations in HSS will be useful for the explicit constructions of different classical solutions.)

The G-analyticity of $V^{++}$ is equivalent to the nonlinear analyticity equation for $V^{--}$

$$[\nabla^{-}, \nabla^{--}] = D^{-}V^{--} + D^{-}D_{a}^{+}V^{--} + [V^{--}, D_{a}^{+}V^{--}] = 0.$$  \hfill (2.9)

The linear constraint (2.8) for the independent superfield $V^{--}$ can be readily solved

$$V^{--} = D_{a}^{+}B^{\alpha(-3)} - \bar{D}_{a}^{\bar{\alpha}}\bar{B}^{\bar{\alpha}(-3)}.$$  \hfill (2.10)

In the picture with independent SYM superfield variables $B^{\alpha(-3)}$ and $V^{++}$, the basic dynamical equation is

$$\nabla^{++}(D_{a}^{+}B^{\alpha(-3)} - \bar{D}_{a}^{\bar{\alpha}}\bar{B}^{\bar{\alpha}(-3)}) = D^{-}V^{++}. \hfill (2.11)$$

Acting by operators $D_{a}^{+}, \bar{D}_{a}^{\bar{\alpha}}$ on both sides of this equation one can obtain the following relations:

$$\nabla^{++}(\bar{D}^{+})^{2}D_{a}^{+}B^{\alpha(-3)} = 0,$$  \hfill (2.12)

$$\nabla^{++}[((\bar{D}^{+})^{2}D_{a}^{+}B^{\bar{\alpha}(-3)} - (D^{+})^{2}\bar{D}_{a}^{\bar{\alpha}}B^{\bar{\alpha}(-3)}] = i\partial_{a\bar{\alpha}}V^{++}. \hfill (2.13)$$

Decomposition of Eq.(2.11) in terms of all spinor coordinates gives the component SYM-equations; however, it is also useful to analyze the partial decomposition of this superfield equation in terms of $\theta^{\alpha-}$ and $\bar{\theta}^{\bar{\alpha}-}$. Let us define the G-analytic components of superfield $V^{--}$

$$\tilde{\phi} = -(D^{+})^{2}V^{--} \left| \right., \quad \phi = (\bar{D}^{+})^{2}V^{--} \left| \right., \quad B_{\alpha\bar{\beta}} = \bar{D}_{\bar{\alpha}}^{\bar{\beta}}D_{a}^{+}V^{--} \left| \right., \quad (2.14)$$

$$\lambda_{a}^{+} = -D_{a}^{+}(D^{+})^{2}V^{--} \left| \right., \quad \bar{\lambda}_{a}^{+} = -\bar{D}_{a}^{\bar{\alpha}}(D^{+})^{2}V^{--} \left| \right., \quad F^{++} = (D^{+})^{2}(\bar{D}^{+})^{2}V^{++} \left| \right..$$

where the symbol $\left| \right.$ means $\theta^{--} = \bar{\theta}^{--} = 0$.

The full supersymmetry transformation $\delta V^{--} = 0$ yields the corresponding transformations of the analytic components

$$\delta_{\epsilon}\phi = u_{k}^{--}\epsilon^{k}\lambda_{a}^{+}, \quad \delta_{\epsilon}B_{\alpha\bar{\beta}} = u_{k}^{--}\epsilon^{k}\lambda_{\bar{\beta}}^{+} + u_{k}^{--}\epsilon_{\alpha}^{k}\lambda_{\bar{\beta}}^{+}, \quad (2.15)$$

$$\delta_{\epsilon}\lambda_{\alpha} = -u_{k}^{--}\epsilon_{\alpha}^{k}F^{++}, \quad \delta_{\epsilon}F^{++} = 0.$$  \hfill (2.16)

Using the gauge transformation $\delta V^{--} = -D^{-}\lambda - [V^{--}, \lambda]$, one can obtain the nilpotent gauge condition for the harmonic connection

$$V^{--} = (\theta^{--})^{2}\tilde{\phi} - (\bar{\theta}^{--})^{2}\phi + \theta^{\alpha-}\bar{\theta}^{\bar{\alpha}--}B_{\alpha\bar{\beta}} + (\theta^{--})^{2}\bar{\theta}^{\alpha-}\bar{\lambda}_{\bar{\beta}}^{+} + (\bar{\theta}^{--})^{2}\theta^{\alpha-}\lambda_{\alpha}^{+} + (\bar{\theta}^{--})^{2}(\bar{\theta}^{--})^{2}F^{++}.$$  \hfill (2.17)

The geometric superfields have the following form in this gauge:

$$W = \phi - \theta^{\alpha-}\lambda_{\alpha}^{+} - (\theta^{--})^{2}F^{++}, \quad (2.18)$$

$$W = \tilde{\phi} + \bar{\theta}^{\bar{\alpha}-}\bar{\lambda}_{\bar{\alpha}}^{+} + (\bar{\theta}^{--})^{2}F^{++}, \quad (2.19)$$

$$A_{a\bar{\beta}} = -iB_{a\bar{\beta}} + i\theta_{a}^{\alpha-}\bar{\lambda}_{\bar{\beta}}^{+} + i\theta_{\alpha}^{\bar{\beta}-}\lambda_{\alpha}^{+} - i\theta_{\alpha}^{\bar{\beta}-}\bar{\theta}_{\bar{\alpha}}^{--}F^{++}.$$  \hfill (2.20)
The constraint $F^{++} = 0$ gives the following analytic equations of motion equivalent to Eq. (2.11)

\[
\begin{align*}
\nabla^{++} \phi &= \theta^{\alpha+} \lambda_\alpha^{+} , \quad (\nabla^{++})^2 \phi = 0 , \\
\nabla^{++} B_{\alpha \beta} &= -i \partial_{\alpha \beta} \nu^{++} + \theta^{\alpha+} \lambda_\beta^{+} + \bar{\theta}^{\beta+} \lambda_\alpha^{+} , \\
\nabla^{++} \lambda_\alpha^{+} &= 0 .
\end{align*}
\]

The component solutions for the on-shell $N = 2$ analytic superfields are

\[
\begin{align*}
\phi &= \varphi(x) + \theta^{\alpha+} u_k^\alpha \bar{\psi}_\alpha^k(x) , \\
B_{\alpha \beta} &= a_{\alpha \beta}(x) + \theta^{\alpha+} u_k^\alpha \bar{\psi}_\beta^k(x) + \bar{\theta}^{\beta+} u_k^\beta \psi_\alpha^k(x) , \\
\nu^{++} &= (\theta^+)^2 \bar{\varphi}(x) - (\bar{\theta}^+)^2 \varphi(x) + \theta^{\alpha+} \bar{\theta}^{\beta+} a_{\alpha \beta}(x) - \theta^{\alpha+} (\bar{\theta}^+)^2 u_k^\alpha \bar{\psi}_\alpha^k(x) \\
&- \bar{\theta}^{\alpha+} (\theta^+)^2 u_k^\beta \psi_\beta^k(x) .
\end{align*}
\]

The $N = 2$ equations of motion for these components follow from $G$-analytic superfield equations. Of course, the main purpose of the harmonic approach is to find nonstandard superfield methods of solving the SYM-equations which cannot be completely reduced to the analysis of component equations. We hope that a comparison of dual superfield representations of SYM-solutions will be useful for the search of unusual symmetry properties which cannot be seen directly in the component representation.

3. Static superfield equations

Three-dimensional monopole and dyon solutions play an important role in modern nonperturbative methods of the quantum $N = 2$ SYM-theory [17, 18]. The harmonic analysis of the monopole Yang-Mills solutions has been considered in Ref. [5]. We shall analyze the alternative harmonic-superfield constructions of the static $N = 2$ SYM-solutions and their relations with the $4D$ self-dual solutions.

Let us consider the nonrelativistic representation of the coordinates in the $D = (3,1), N = 2$ harmonic superspace based on the static group $SO(3)$

\[
\begin{align*}
x^{\alpha \beta}_A &\rightarrow iy^{\alpha \beta}_A + \delta^\alpha_\gamma t_A , \quad y^{\alpha \beta}_A = 0 , \quad y^{\alpha \beta} = \varepsilon^{\alpha \rho} y^\rho_{\beta} , \\
\partial_{\alpha \beta} &\rightarrow -i \partial_{\alpha \beta} + \delta^\beta_\gamma \partial_t , \quad \partial^\sigma y^\alpha_{\beta} = 2 \delta^\sigma_\beta \delta^\alpha_\gamma - \delta^\beta_\gamma \delta^\sigma_\alpha , \\
\bar{\theta}^{\alpha \pm} &\rightarrow \bar{\theta}^{\alpha \pm} , \quad \bar{\theta}^{\alpha \pm} = \varepsilon^{\alpha \beta} \bar{\theta}^{\beta \pm} , \\
\bar{\theta}^{\pm} &\rightarrow \bar{\theta}^{\alpha \pm} , \quad \bar{\theta}^{\alpha \pm} \bar{\theta}^{\beta \pm} = \delta^\alpha_\beta .
\end{align*}
\]

We shall use the following conjugation rules:

\[
\begin{align*}
(y^{\alpha \beta}_A)^\dagger &= -y^{\alpha \beta}_A , \quad (y^{\alpha \beta}_A)^\dagger = y_{\alpha \beta} , \quad (t_A)^\dagger = t_A , \\
(\theta^{\alpha \pm})^\dagger &= \bar{\theta}^{\alpha \pm} , \quad (\theta^{\pm})^\dagger = -\theta^{\alpha \pm} , \quad (\bar{\gamma}^{\alpha \beta})^\dagger = -\varepsilon^{\alpha \beta} , \\
(\bar{\theta}^{\alpha \pm})^\dagger &= \bar{\theta}^{\alpha \pm} , \quad (\bar{\theta}^{\pm})^\dagger = -\bar{\theta}^{\alpha \pm} , \\
(D^{\alpha \pm}_A)^\dagger &= \bar{D}^{\alpha \pm} , \quad (\bar{D}^{\alpha \pm})^\dagger = -D^{\alpha \pm} , \quad (\bar{D}^{\alpha \pm})^\dagger = D^{\alpha \pm} .
\end{align*}
\]
Note that changes of the position of dotted $SL(2, C)$ indices after the static reduction is connected with a convention of the conjugation of the $SU(2)$-spinors which transforms upper indices to lower ones. The time reduction ($t = 0$) transforms 4-dimensional harmonic superfields to the 3-dimensional Euclidean superfields which are covariant with respect to the reduced supersymmetry with 8 supercharges.

The corresponding representation of the 4-vector connection is

$$A_{\alpha \beta} \Rightarrow A_{\alpha}^{\beta} + \delta_{\alpha}^{\beta} A_{t} , \quad (A_{\alpha}^{\beta})^\dagger = -A_{\beta}^{\alpha} , \quad (A_{t})^\dagger = -A_{t} ,$$

(3.9)

where the time component of the connection becomes covariant with respect to residual gauge static transformations

$$A_{\alpha}^{\beta} = i(\bar{D}_{\alpha}^{\beta}D_{\alpha}^{i} + \frac{1}{2}\delta_{\alpha}^{\beta}(\bar{D}_{i}^{D} + D_{i}^{D}))V^{--} ,$$

(3.10)

$$A_{t} = \frac{i}{2}(\bar{D}_{t}^{D} + D_{t}^{D})V^{--} , \quad \delta_{\lambda}A_{t} = [\lambda , A_{t}] .$$

(3.11)

It seems useful to construct the manifestly supersymmetric generalization of the Bogomolnyi equation for the $N = 2$ gauge theory in order to analyze the symmetry properties of the corresponding monopole solutions. In addition to the $t = 0$ reduction of the HSS-equation (2.4), we propose to consider the following superfield BPS-type relation:

$$S \equiv -2irA_{t} + p(W + \bar{W}) = 0 ,$$

(3.12)

where $r$ and $p$ are some real parameters. Note that this condition breaks the $R$-symmetry $W \rightarrow e^{i\theta}W$.

We shall interpret (3.12) as a linear constraint on $V^{--}$

$$S = [r(\bar{D}^{+}D^{+}) + p(\bar{D}^{+})^{2} - p(D^{+})^{2}]V^{--} = 0 ,$$

(3.13)

which gives also the spinor condition

$$(-r(D^{+})^{2}\bar{D}_{\alpha}^{+} + p(\bar{D}^{+})^{2}D_{\alpha}^{+})V^{--} = 0 .$$

(3.14)

For the case $r \neq p$ , the last spinor relation is not self-adjoint, so one can obtain the strong restrictions

$$D_{\alpha}^{+}(\bar{D}^{+})^{2}V^{--} = D_{\alpha}^{+}W = 0 \Rightarrow \nabla^{--}D_{\alpha}^{+}W = \nabla_{\alpha}W = 0 .$$

(3.15)

It is evident that the corresponding covariantly constant solution $W = const$ preserves all 8 supercharges [9]. The case $r = p$ corresponds to the self-adjoint relation for 3D spinors

$$\lambda^{\alpha}_{k} = \bar{\lambda}^{\alpha}_{k} = \varepsilon^{\alpha\beta}\varepsilon_{kl}(\lambda_{l}^{\beta})^\dagger .$$

(3.16)

The corresponding self-dual static solutions will be discussed in Sect. 6.
4. Euclidean $N = 2$ SYM-equations

We shall analyze the superfield constraints of the $N = 2$ SYM-theory in the Euclidean superspace

\begin{align}
\{\nabla_{\alpha}^k, \nabla_{\beta}^l\} &= \varepsilon_{\alpha\beta}\varepsilon^{kl} W, \\
\{\nabla_{\dot{\alpha}}^k, \nabla_{\dot{\beta}}^l\} &= \varepsilon_{\dot{\alpha}\dot{\beta}}\varepsilon^{kl} \hat{W}, \\
\{\nabla_{\alpha}^k, \nabla_{\dot{\beta}}^l\} &= \varepsilon^{kl}\nabla_{\alpha\dot{\beta}} ,
\end{align}

where $W$ and $\hat{W}$ are independent real superfield strengths and indices $\alpha, \beta; \dot{\alpha}, \dot{\beta}$ and $k, l$ describe, respectively, the 2-spinor representations of the Euclidean group $SU_L(2) \times SU_R(2)$ and the automorphism group $SU_A(2)$.

Superspace analogues of the Euclidean self-duality equations have been considered in Refs. [13]-[16]. We shall analyze the $N = 2$ self-dual solutions in the framework of the Euclidean version of the harmonic superspace $HSS_E$ (see Appendix). Underline that conjugation rules of the $N = 2$ superspace coordinates and derivatives are essentially different for the cases of the Euclidean group $SO(4)$ and the Lorentz group $SO(3,1)$.

The $SU_A(2)/U_A(1)$ harmonic-superspace description of the Euclidean $N = 2$ SYM-theory is similar to the corresponding formalism in $HSS_M$, in particular, one can use the same notation $V^{++}, V^{--}$ for the Euclidean harmonic connections. The Euclidean spinor and vector connections of the analytic basic have the following form:

\begin{align}
A_{\alpha}^+ = \hat{A}_{\dot{\alpha}}^+ = 0, & \quad A_{\alpha}^- = -D_{\alpha}^+ V^{--}, \quad \hat{A}_{\dot{\alpha}}^- = -\hat{D}_{\dot{\alpha}}^+ V^{--}, \\
A_{\alpha\dot{\beta}} \equiv D_{\alpha}^+ \hat{A}_{\dot{\beta}}^+ \equiv D_{\dot{\alpha}}^+ \hat{D}_{\dot{\beta}}^+ V^{--} .
\end{align}

The reality of the Euclidean superfield strengths is evident in $HSS_E$

\begin{align}
W = (D^+)^2 V^{--}, \quad \hat{W} = (\hat{D}^+)^2 V^{--} .
\end{align}

The system of self-duality equations has the following form in the harmonic-superspace:

\begin{align}
D^{++} V^{--} + [V^{++}, V^{--}] = D^{--} V^{++}, \quad (D^+)^2 V^{--} = 0 .
\end{align}

The manifest solution of the 2nd constraint

\begin{align}
V^{--} = D_{\alpha}^+ B^{\alpha(-3)},
\end{align}

yields the simple relation between $B^{\alpha(-3)}$ and $V^{++}$

\begin{align}
D_{\alpha}^+ \nabla^{++} B^{\alpha(-3)} = D^{--} V^{++} .
\end{align}

The self-duality equation is equivalent to the relation

\begin{align}
D_{\alpha}^+ A_{\dot{\beta}}^+ = 0 .
\end{align}

Using the equation $[\nabla^{--}, \nabla_{\dot{\alpha}}^\dot{\beta}] = 0$ one can derive the chirality relations

\begin{align}
\{\nabla_{\alpha}^\pm, \nabla_{\dot{\beta}}^\pm\} = 0 .
\end{align}
Let us reformulate now the self-duality condition in the bridge representation of the harmonic connection \([1]\)

\[
V^- \equiv e^v D^-- e^{-v}
\]  

(4.12)

The condition \((1.10)\) for \(V^-=\) corresponds to the partial analyticity condition for the self-dual bridge matrix

\[
\mathcal{D}^+ v = 0 \Rightarrow -\mathcal{D}^+ \left(e^v D^- e^{-v}\right) = e^v \mathcal{D}^- e^{-v}.
\]  

(4.13)

Comparing representations \((4.8)\) and \((1.12)\) one can obtain the relation

\[
\mathcal{B}^{\alpha(-3)} = \frac{1}{2} \Theta^\alpha e^v D^{--} e^{-v} + \frac{1}{2} (\Theta^-)^2 e^v \mathcal{D}^{\alpha--} e^{-v}.
\]  

(4.14)

Harmonic projections of the central-basis connections have the following form in this representation:

\[
A^\pm_\alpha = 0, \quad \hat{A}^+_\alpha(v) = e^{-v} \mathcal{D}^\alpha e^v, \quad \hat{A}^-_\alpha = \mathcal{D}^{--} \hat{A}^+_\alpha(v), \quad A_{\alpha\beta}(v) = \mathcal{D}^- \hat{A}^+_\alpha(v).
\]  

(4.15)

In the bridge representation, the system of self-duality equations \((4.7)\) is equivalent to the single equation

\[
\hat{D}^+_\alpha \left(e^v D^{++} e^{-v}\right) \equiv \mathcal{D}_\alpha^+ V^+(v) = 0.
\]  

(4.16)

Let us introduce the nilpotent gauge condition for the bridge matrix

\[
v = \hat{\Theta}^\alpha b^+_\alpha + (\hat{\Theta}^-)^2 b^{++}, \quad v^2 = -(\hat{\Theta}^-)^2 b^{\alpha+} b^+_\alpha,
\]  

(4.17)

where \(b^+_\alpha\) and \(b^{++}\) are G-analytic matrix coefficients. Note that the analogous nilpotent bridge representation has been used recently in the \(N = 3\) SYM-theory \([12]\).

Let us compose harmonic connection \(V^{++}(v)\)

\[
e^v D^{++} e^{-v} = -\hat{\Theta}^{\hat{\beta}+} b^+_\alpha + \hat{\Theta}^{\hat{\alpha}-} \left(-D^{++} b^+_\alpha - \hat{\Theta}^{\hat{\beta}+} b^{++} + \frac{1}{2} \hat{\Theta}^{\hat{\beta}+} \{b^+_\alpha, b^+_\beta\}\right)
\]  

\[
+ (\hat{\Theta}^-)^2 \left(D^{++} b^{++} + \frac{1}{2} \{b^{\hat{\beta}+}, D^{++} b^+_\alpha\} + \hat{\Theta}^{\hat{\alpha}+} [b^+_\alpha, b^{++}] + \frac{1}{2} \hat{\Theta}^{\hat{\alpha}+} \{b^+_\alpha, b^{\beta+} b^+_\beta\}\right),
\]  

(4.18)

where we have used the identity

\[
\{b^+_\alpha, b^{\beta+} b^+_\beta\} = \frac{1}{3} [b^{\hat{\beta}+}, \{b^+_\alpha, b^+_\beta\}].
\]  

(4.19)

The dynamical analyticity equation

\[
V^{++}(v) = -\hat{\Theta}^{\hat{\alpha}+} b^+_\alpha
\]  

(4.20)

yields the following analytic relations:

\[
D^{++} b^+_\alpha + \hat{\Theta}^{\hat{\beta}+} b^{++} - \frac{1}{2} \hat{\Theta}^{\hat{\beta}+} \{b^+_\alpha, b^+_\beta\} = 0,
\]  

(4.21)

\[
D^{++} b^{++} + \frac{1}{2} \{b^{\hat{\beta}+}, D^{++} b^+_\alpha\} + \hat{\Theta}^{\hat{\alpha}+} [b^+_\alpha, b^{++}] + \frac{1}{6} \hat{\Theta}^{\hat{\alpha}+} [b^{\hat{\beta}+}, \{b^+_\alpha, b^+_\beta\}] = 0.
\]
Let us use the intermediate decomposition of (2,2)-analytic superfields in $\hat{\Theta}^{\dot{\alpha}+}$

\[
\begin{align*}
  b^{\dot{\alpha}}_\alpha &= \beta^{\dot{\alpha}}_\alpha + \hat{\Theta}^{\dot{\beta}+} C_{\dot{\alpha}\dot{\beta}} + (\hat{\Theta}^+)^2 \eta_{\dot{\alpha}}, \\
  b^{++} &= B^{++} + \hat{\Theta}^{\dot{\alpha}+} \gamma^{\dot{\alpha}}_\alpha + (\hat{\Theta}^+)^2 B,
\end{align*}
\]

where all coefficients are polynomials in $\Theta^{\alpha+}$. The (2,2)-equations for $b^{\dot{\alpha}}_\alpha$ and $b^{++}$ yield (2,0)-equations for these coefficients, for instance,

\[
\begin{align*}
  \partial^{++} \beta^{\dot{\alpha}}_\alpha &= 0, \\
  \partial^{++} B^{++} &= 0, \\
  \partial^{++} C_{\dot{\alpha}\dot{\beta}} + \hat{\Theta}^{\dot{\beta}+} \partial_{\dot{\beta}} \beta^{\dot{\alpha}}_\alpha + \varepsilon_{\dot{\alpha}\dot{\beta}} B^{++} - \frac{1}{2} \{ \beta^{\dot{\alpha}}_\alpha, \beta^{\dot{\beta}}_\beta \} &= 0.
\end{align*}
\]

The linear harmonic equations for $\beta^{\dot{\alpha}}_\alpha$ and $B^{++}$ can be solved explicitly. The inhomogeneous linear harmonic equation for $C_{\dot{\alpha}\dot{\beta}}$ contains the composed source calculated at the previous stage, so it is also solvable, in principle. The inhomogeneous harmonic equations for other (2,0)-coefficients $\eta_{\dot{\alpha}}$, $\gamma^{\dot{\alpha}}_\alpha$ and $B$ can be derived and solved analogously. It should be stressed that self-dual equations (4.7) and (4.21) are convenient for the search of dimensionally reduced solutions.

5. Harmonic constructions of self-dual solutions

The alternative harmonic formalism for the self-dual SYM-solutions has been considered in Ref.\[6\]. This formalism harmonizes one of the space groups: $SU_R(2)$ acting on the indices $\dot{\alpha}$.

In order to compare alternative harmonic approaches to the $N = 2$ self-dual equations we shall consider the non-covariant procedure of identification $SU_R(2) = SU_A(2)$ for solutions of these equations. Let us identify also the corresponding spinor indices $\dot{\alpha} \equiv k$, $\dot{\beta} \equiv l \ldots$ The $N = 2$ superspace coordinates have the following form in this notation:

\[
z^M = (y^{\alpha k}, \Theta^{\alpha k}, \hat{\Theta}^k).
\]

Using the single set of the $SU(2)/U(1)$ harmonics

\[
u_l^A \equiv \varepsilon^{AB} u_{Bl}, \quad A, B = \pm
\]

one can consider the harmonic projections of central, analytic and chiral coordinates, correspondingly:

\[
\begin{align*}
  y^{\alpha A} &= u^A_l y^{\alpha k}, & Y^{\alpha A} &= y^{\alpha A} - \Theta^{+\alpha} \hat{\Theta}^{-A} - \Theta^{-\alpha} \hat{\Theta}^{+A}, \\
  y^{\alpha A} &= y^{\alpha A} + \Theta^{+\alpha} \hat{\Theta}^{-A} - \Theta^{-\alpha} \hat{\Theta}^{+A}, \\
  \Theta^A &= \varepsilon_{AB} \Theta^{B\alpha} u^{k\alpha}_k, \\
  \hat{\Theta}^A &= \varepsilon_{AC} \hat{\Theta}^{BC} = u^I_A \hat{u}^B_k \hat{\Theta}^k_I.
\end{align*}
\]

The corresponding projections of the flat derivatives are

\[
\begin{align*}
  \partial_{\alpha A}, & \quad D^A_{\alpha}, & \quad \hat{D}^A_B, \\
  \{ D^A_{\alpha}, \hat{D}^C_B \} &= -\varepsilon^{AC} \partial_{\alpha B}.
\end{align*}
\]
It the central basis, one can obtain the following representations of the $N = 2$ covariant self-dual derivatives
\[
\nabla^\pm_\alpha = \mathcal{D}^\pm_\alpha , \quad \hat{\nabla}^\pm_\alpha = h^{-1}\hat{\mathcal{D}}^\pm h , \quad h^{-1}\partial_\alpha h ,
\]
where $h(z, u)$ is the chiral self-dual bridge
\[
\mathcal{D}^\pm_\alpha h = 0 .
\]

The harmonic connection for the self-dual bridge is the basic chiral-analytic potential of this formalism
\[
h\mathcal{D}^{++}_\alpha h^{-1} = v^{++} ,
\]
\[
(\partial_{\alpha-}, \mathcal{D}^\pm_\alpha, \hat{\mathcal{D}}^\pm) v^{++} = 0 .
\]

Stress that self-dual solutions possess the combined analyticity. The self-dual prepotential $v^{++}$ can be treated as unconstrained matrix function of coordinates $y^{\alpha+}, u^{\pm}$ and $\hat{\Theta}^{\pm}$ which parametrizes the general $N = 2$ self-dual solution.

In the gauge group $SU(2)$, the simple solvable Ansatz for the self-dual prepotential can be choosen
\[
\begin{align*}
(v^{++})^k_i &= u^{+k}u^+_ib^0 + (u^{+k}u^-_i + u^{-k}u^+_i)b^{++} ,
\end{align*}
\]
where $b^0$ and $b^{++}$ are real chiral-analytic functions. The bridge matrix for this Ansatz can be calculated via harmonic quadratures \[11\]. In the simple example of this parametrization with $b^0 = 0$ and
\[
b^{++} = Y^{\alpha+}Y^{\beta+}\rho_{\alpha\beta} + \hat{\Theta}^{+}_{\alpha}Y^{\alpha+}u^+_i\lambda^l
\]
the solution depends on constant tensor and spinor coefficients.

It is not difficult to relate self-dual representations for different methods of harmonization of superfield equations
\[
e^{-v}\hat{\mathcal{D}}^-_\alpha e^v = h^{-1}\hat{\mathcal{D}}^- h , \quad \mathcal{D}^-_\alpha (e^{-v}\hat{\mathcal{D}}^-_\alpha e^v) = h^{-1}\partial_\alpha h .
\]

6. **Self-dual static $N = 2$ solutions**

Consider now the subsidiary conditions in BPS-equations \(3.12\) $r = p$ then these equations can be transformed to the following simple constraint:
\[
-2iA_\alpha + W + \bar{W} \equiv ((\bar{D}^+D^+) + (\bar{D}^+)^2 - (D^+)^2) V^{--}
\]
\[
= -\frac{1}{2}(D^{\alpha+} - \bar{D}^{\alpha+})(D^+_{\alpha} - \bar{D}^+_{\alpha})V^{--} = 0 .
\]

One can consider also the equivalent 2nd order constraints on $V^{--}$ using the transformation $D^+_{\alpha} \rightarrow e^{i\rho/2}D^+_{\alpha}$.

It is convenient to introduce the new pseudoreal spinor coordinates of the static harmonic superspace
\[
\Theta^{\alpha\pm} \equiv \frac{1}{\sqrt{2}}(\theta^{\alpha\pm} + \bar{\theta}^{\alpha\pm}) , \quad \hat{\Theta}^{\alpha\pm} \equiv \frac{1}{\sqrt{2}}(\theta^{\alpha\pm} - \bar{\theta}^{\alpha\pm}) ,
\]
\[
(\Theta^{\alpha\pm})^\dagger = \Theta^\dagger_\alpha , \quad (\hat{\Theta}^{\alpha\pm})^\dagger = -\hat{\Theta}^\dagger_\alpha .
\]
The corresponding transformed spinor derivatives

\[ D^\pm_\alpha \equiv \frac{1}{\sqrt{2}}(D^\pm_\alpha - \bar{D}^\pm_\alpha) \], \quad (D^\pm_\alpha)^\dagger = -D^{\alpha\pm}, \quad (6.4) \]

\[ \hat{D}^\pm_\alpha = -\frac{1}{\sqrt{2}}(D^\pm_\alpha + \bar{D}^\pm_\alpha) \], \quad (\hat{D}^\pm_\alpha)^\dagger = \hat{D}^{\alpha\pm}. \quad (6.5) \]

have the following algebra:

\[ \{D^+_\alpha, \hat{D}^-_\beta\} = -\{\hat{D}^+_\alpha, D^-_\beta\} = -\partial_{\alpha\beta}, \quad (6.6) \]

\[ \{D^+_\alpha, D^-_\beta\} = \{\hat{D}^+_\alpha, \hat{D}^-_\beta\} = 0. \quad (6.7) \]

Consider the 3D-covariant representation of the Euclidean superspace coordinates (6.21)

\[ y^{\alpha\beta} \Rightarrow y^{\alpha\beta} + \varepsilon^{\alpha\beta}y_4. \quad (6.8) \]

Stress that the algebra of spinor derivatives (6.6) arises also in the dimensional reduction \( y_4 = 0 \) of the Euclidean \( N = 2 \) harmonic superspace (see Appendix). It is clear that the transformation (6.2) connects the equivalent 3D subspaces of the Minkowski and Euclidean types of harmonic superspaces. Thus, the (3,1)-equation (6.1) is equivalent to the 3D limit of the Euclidean \( N = 2 \) self-duality equation (4.7)

\[ (D^+)^2V^{--} = 0. \quad (6.9) \]

The component static self-dual solutions can be obtained in the following gauge:

\[ \mathcal{V}^{++} = \frac{1}{2} \Theta^{\alpha+}\hat{\Theta}^+_\alpha c(x) + i \frac{1}{2} \Theta^{\alpha+}\hat{\Theta}^+_\alpha a_t(x) + i \Theta^{\alpha+}\hat{\Theta}^{\beta+}a_{\alpha\beta}(x) \]

\[ -\hat{\Theta}^{\alpha+}\hat{\Theta}^{\beta+}u^\alpha_\beta \Psi^k(x), \quad (6.10) \]

where all fields are Hermitian

\[ (c, a_t, a_{\alpha\beta}, \Psi^k)^\dagger = (c, a_t, a^{\alpha\beta}, \Psi^k_\alpha). \quad (6.11) \]

One can derive the component 3D self-dual equations

\[ F_{\alpha\beta} = -\nabla_{\alpha\beta}a_t, \quad (6.12) \]

\[ \nabla^{\alpha\beta}\nabla_{\alpha\beta}c = 2[a_t, [a_t, c]] - \{\Psi^k, \Psi^k_\alpha\}, \quad (6.13) \]

\[ \nabla_{\beta\gamma}\Psi^{\gamma k} + \frac{i}{2}[a_t, \Psi^k_\beta] = 0, \quad (6.14) \]

where

\[ \nabla_{\alpha\beta} \equiv \partial_{\alpha\beta} - i[a_{\alpha\beta}], \quad F_{\alpha\beta} = \partial_{\alpha\rho}a^\rho_\beta + \partial_{\beta\rho}a^\rho_\alpha - i[a_{\alpha\rho}, a^\rho_\beta]. \quad (6.15) \]

Note that the static gauge field strength and field \( a_t \) are connected by the self-dual Bogomolnyi equation.

The superfield analysis of the static self-duality equations can be made by analogy with the analysis of Euclidean 4D self-dual equations in Sect. 4. The static limit of the bridge representation is

\[ A^\pm_\alpha = 0, \quad \hat{A}^+_\alpha(v) = e^{-v\hat{D}^+_\alpha}v, \quad (6.16) \]
where \( v \) is the static self-dual bridge \( D_\alpha^+ v = 0 \).

By analogy with the 4D self-duality equation one can use the formal identification of all groups \( SU(2) \) in the 3D Euclidean harmonic superspace

\[
y^{\alpha\beta} \rightarrow y^{jk}, \quad \Theta_i^\alpha \rightarrow \Theta^k_i, \quad \hat{\Theta}_i^\alpha \rightarrow \hat{\Theta}_i^k.
\]

It is clear that one can use harmonic projections of the superspace coordinates

\[
y_{r}^{AB} = u_A^k u_B^l y_{kl}^r, \quad \Theta^{AB}_A = u_A^k u_B^l \Theta_k^l, \quad \hat{\Theta}^{AB}_A = u_A^k u_B^l \hat{\Theta}_k^l.
\]

The covariant derivative \( \nabla^i_k \) are flat in the chiral self-dual representation. The corresponding bridge representation of the 3D self-dual covariant derivatives has the following form:

\[
h^{-1} \frac{\partial}{\partial y_{r}^{-}} h, \quad h^{-1} \frac{\partial}{\partial \hat{\Theta}_r^+} h.
\]

The prepotential \( v^{++} = hD^{++} h^{-1} \) for these solutions depends on the (1+2) analytic coordinates \( y_{r}^{++} \) and \( \hat{\Theta}_r^+ \) only. It is clear that these solutions can be interpreted as the dimensionally reduced self-dual 4D solutions (5.12).

**Acknowledgement.** Author is grateful to E. Ivanov and J. Niederle for discussions. This work is supported in part by grants RFBR 99-02-18417, RFBR-DGF-99-02-04022, INTAS-2000-254 and NATO PST.CLG 974874, and by the Votruba-Blokhintsev program in LTP JINR.

### A Appendix

**Harmonic-superspace coordinates in \( D = (3,1), N = 2 \) superspace**

The \( SU(2)/U(1) \) harmonics \([4]\) parametrize the sphere \( S^2 \). They form an \( SU(2) \) matrix \( u_i^\pm \) and are defined modulo \( U(1) \).

The \( SU(2) \)-invariant harmonic derivatives act on the harmonics

\[
[\partial^{++}, \partial^{-}] = \partial^0, \quad [\partial^0, \partial^{\pm\pm}] = \pm 2 \partial^{\pm\pm}.
\]

The special \( SU(2) \)-covariant conjugation of harmonics preserves the \( U(1) \)-charges

\[
\tilde{u}_i^\pm = u_i^{\mp}, \quad \tilde{u}^{\pm i} = -u_i^{\pm}.
\]

On the harmonic derivatives of an arbitrary harmonic function \( f(u) \) this conjugation acts as follows

\[
\tilde{\partial}^{\pm\pm} f = \partial^{\pm\pm} \tilde{f}.
\]

Let us consider the coordinates of the \( N = 2 \) superspace \( M(3,1|8) \) over the \( D = (3,1) \) Minkowski space

\[
z^M = (x^{\alpha\beta}, \theta^\alpha_k, \bar{\theta}^{\dot{\alpha}k}) ,
\]

\[
(x^{\alpha\beta})^\dagger = x^{\beta\dot{\alpha}}, \quad (\theta^\alpha_k)^\dagger = \bar{\theta}^{\dot{\alpha}k},
\]
where $\alpha, \dot{\alpha}$ are the $SL(2, C)$ indices.

One can define the Minkowski analytic harmonic superspace with 2 coset harmonic dimensions $u^i_\alpha$ and the following set of 4 even and $(2+2)$ odd coordinates:

$$\zeta = (x^\alpha, \theta^\alpha, \bar{\theta}^\dot{\alpha}),$$
$$x^\alpha = x^\alpha - i(u_k^+ u_k^- + u_k^- u_k^+)\theta^\alpha \bar{\theta}^\dot{\alpha},$$
$$\theta^\alpha \equiv \theta^\alpha u_k^+ \bar{\theta}^\dot{\alpha} u_k^+. \quad \text{(A.6)}$$

This superspace is covariant with respect to the $N = 2$ supersymmetry transformations

$$\delta x^\alpha = 2i\theta^\alpha \epsilon^{\beta \dot{\alpha}} u^-_\beta + 2i\bar{\theta}^{\dot{\alpha}} \epsilon^{\beta \alpha} u^-_\beta, \quad \delta \theta^\alpha = \epsilon^{\alpha \beta} u^+_\beta, \quad \delta \bar{\theta}^{\dot{\alpha}} = \epsilon^{\beta \dot{\alpha}} u^+_\beta. \quad \text{(A.7)}$$

The conjugation of the odd analytic coordinates has the following form:

$$\theta^\alpha \to \bar{\theta}^{\dot{\alpha}}, \quad \bar{\theta}^{\dot{\alpha}} \to - \theta^\alpha \quad \text{(A.8)}$$

and coordinates $x^\alpha$ are real.

The corresponding CR-structure involves the derivatives

$$D^+_\alpha, \bar{D}^+_\dot{\alpha}, D^{++} \quad \text{(A.9)}$$

which have the following explicit form in these coordinates:

$$D^+_\alpha = \partial^+_\alpha, \quad \bar{D}^+_\dot{\alpha} = \bar{\partial}^+_{\dot{\alpha}},$$
$$D^{++} = \partial^{++} + i\theta^\alpha \bar{\theta}^{\dot{\alpha}} + \theta^\alpha \partial^+_{\dot{\alpha}} + \bar{\theta}^{\dot{\alpha}} \bar{\partial}^+_{\dot{\alpha}}, \quad \text{(A.10, A.11)}$$

where the partial derivatives satisfy the following relations:

$$\partial_{\alpha \dot{\alpha}} x^{\beta \dot{\beta}} = 2\delta^{\beta \dot{\beta}} \delta_{\alpha \dot{\alpha}}, \quad \partial^+_\alpha \theta^\beta = \delta^\beta_\alpha, \quad \bar{\partial}^+_{\dot{\alpha}} \bar{\theta}^{\dot{\beta}} = \delta^{\dot{\beta}}_{\dot{\alpha}}. \quad \text{(A.12)}$$

One can construct also all harmonic and Grassmann derivatives in these coordinates

$$D^{--} = \partial^{--} - i\theta^\alpha \bar{\theta}^{\dot{\alpha}} \partial^\alpha_{\dot{\alpha}} + \theta^\alpha \partial^-_{\dot{\alpha}} + \bar{\theta}^{\dot{\alpha}} \partial^-_{\dot{\alpha}},$$
$$D^-_{\alpha} = -\partial^-_{\alpha} + i\theta^\alpha \partial^-_{\dot{\alpha}}, \quad D^-_{\dot{\alpha}} = -\partial^-_{\dot{\alpha}} - i\theta^\alpha \partial^\alpha_{\dot{\alpha}}, \quad \text{(A.13, A.14)}$$
$$\partial^-_{\alpha} \theta^\beta = \delta^\beta_\alpha, \quad \partial^-_{\dot{\alpha}} \bar{\theta}^{\dot{\beta}} = \delta^{\dot{\beta}}_{\dot{\alpha}}. \quad \text{(A.15)}$$

It is useful to write down the properties of these derivatives with respect to Hermitian conjugation

$$(D^{++}, D^{--}) \to -(D^{++}, D^{--}), \quad D^\pm_{\alpha} \to D^\pm_{\alpha}, \quad D^\pm_{\dot{\alpha}} \to -D^\pm_{\dot{\alpha}}. \quad \text{(A.16)}$$

In the text, we use the following conventions:

$$\varepsilon^{ik}\varepsilon_{kl} = \delta^i_l, \quad \varepsilon^{\alpha\gamma}\varepsilon_{\gamma\dot{\beta}} = \delta^\alpha_{\dot{\beta}}, \quad \text{(A.17)}$$
$$\theta^\alpha \theta^\alpha = \frac{1}{2} \theta^\alpha \bar{\theta}^\alpha, \quad \bar{\theta}^{\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} = \frac{1}{2} \bar{\theta}^{\dot{\alpha}} \bar{\theta}^{\dot{\alpha}}, \quad \text{(A.18)}$$
$$\partial^+_{\alpha} \partial^+_{\alpha} = \frac{1}{2} \partial^+_{\alpha} \partial^+_{\alpha}, \quad \bar{\partial}^+_{\dot{\alpha}} \bar{\partial}^+_{\dot{\alpha}} = \frac{1}{2} \bar{\partial}^+_{\dot{\alpha}} \bar{\partial}^+_{\dot{\alpha}}. \quad \text{(A.19)}$$
Harmonic-superspace coordinates in Euclidean superspace

The Euclidean $N = 2$ superspace $E(4|8)$ has the coordinates

$$z^M = (y^{\alpha \dot{\beta}}, \Theta^\alpha_k, \dot{\Theta}^{\dot{\alpha} k}) \quad (A.20)$$

with the alternative conjugation rules

$$(y^{\alpha \dot{\alpha}})^\dagger = y_{\alpha \dot{\alpha}}, \quad (\Theta^\alpha_k)^\dagger = \Theta^k_\alpha, \quad (\dot{\Theta}^{\dot{\alpha} k})^\dagger = \dot{\Theta}^{\dot{\alpha} k}, \quad (A.21)$$

where $\alpha, \dot{\alpha}$ and $k$ are the indices of the direct product of three $SU(2)$ groups.

The Euclidean $N = 2$ analytic-superspace coordinates can be defined as follows

$$\zeta^E = (y^{\alpha \dot{\beta}}, \Theta^\alpha_k + \dot{\Theta}^{\dot{\alpha} k}, \Theta^\alpha_{\pm} + \Theta^\alpha_{\mp}) \quad (A.22)$$

The Euclidean spinor and harmonic derivatives have the following form in these coordinates:

$$D^+\alpha = \partial / \partial \Theta^\alpha - \Theta^\alpha_d, \quad \dot{D}^+\dot{\alpha} = \partial / \partial \dot{\Theta}^{\dot{\alpha}}, \quad (A.23)$$

$$D^{\pm}\alpha = \partial^{\pm} - \Theta^\alpha d^\pm, \quad \dot{D}^{\pm}\dot{\alpha} = \partial^{\pm} + \dot{\Theta}^{\dot{\alpha} d^\pm}, \quad (A.24)$$

$$D^-\alpha = -d^- - \dot{\Theta}^{\dot{\alpha} \beta} \partial^\beta, \quad \dot{D}^-\dot{\alpha} = -d^- - \Theta^{\alpha \beta} \partial_\beta \quad (A.25)$$

We shall use the following rules of conjugation for the Euclidean harmonized odd coordinates and derivatives:

$$(\Theta^\alpha)^\dagger = \Theta^\alpha, \quad (D^\alpha)^\dagger = -D^\alpha, \quad (A.26)$$

$$(\dot{\Theta}^{\dot{\alpha}})^\dagger = -\dot{\Theta}^{\dot{\alpha}}, \quad (\dot{\dot{D}}^{\dot{\alpha}})^\dagger = \dot{\dot{D}}^{\dot{\alpha}}. \quad (A.27)$$

Our conventions for the bilinear combinations of the Euclidean spinors are

$$\Theta^\pm = \frac{1}{2} \Theta^\alpha \Theta^\pm, \quad (\dot{\Theta}^{\dot{\alpha}})^\pm = \frac{1}{2} \dot{\Theta}^{\dot{\alpha}} \Theta^{\pm \dot{\alpha}}, \quad (A.28)$$

References

[1] A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky and E. Sokatchev, Class. Quant. Grav. 1 (1984) 469.

[2] A. Galperin, E. Ivanov, V. Ogievetsky and E. Sokatchev, Class. Quant. Grav. 2 (1985) 601.

[3] B.M. Zupnik, Phys. Lett. B 183 (1987) 175.
[4] A. Galperin, E. Ivanov, V. Ogievetsky and E. Sokatchev, Ann. Phys. 185 (1988) 1.

[5] O. Ogievetsky, Lect. Notes in Physics, v. 313, p.548, Springer-Verlag, 1988.

[6] C. Devchand and V. Ogievetsky, Nucl. Phys. B 414 (1994) 763; Erratum, Nucl. Phys. B 451 (1995) 768.

[7] B.M. Zupnik, Phys. Lett. B 375 (1996) 170; Lecture Notes in Physics, v. 509, p.157-160. Springer-Verlag, 1998.

[8] B.M. Zupnik, Short harmonic superfields and light-cone gauge in super-Yang-Mills equations, Proceedings of the International conference "Quantization, gauge theory and strings" devoted to the memory of prof. E.S. Fradkin, eds. A. Semikhatov, M. Vasiliev and V. Zaikin, p. 277, Scientific World, Moscow, 2001; hep-th/0011013.

[9] E.A. Ivanov, S.V. Ketov and B.M. Zupnik, Nucl. Phys. B 509 (1998) 53.

[10] B.M. Zupnik, Nucl. Phys. B 554 (1999) 365.

[11] B.M. Zupnik, Phys. Lett. B 209 (1988) 513.

[12] J. Niederle and B. Zupnik, Nucl. Phys. B 598 (2001) 645; hep-th/0012114.

[13] A. Semikhatov, Phys. Lett. B 120 (1983) 171.

[14] I.V. Volovich, Phys. Lett. 129 (1983) 429; Lett. Math. Phys. 7 (1983) 517.

[15] Yu.I. Manin, Gauge fields and complex geometry, Nauka, Moscow, 1984; English version, Springer-Verlag, Berlin, 1988.

[16] W. Siegel, Phys. Rev. D 52 (1995) 1042.

[17] N.Seiberg and E.Witten, Nucl.Phys., B426 (1994) 19

[18] J. Gauntlett, Nucl. Phys. B 411 (1994) 443.