STABILITY CONDITIONS AND QUANTUM DILOGARITHM IDENTITIES FOR DYNKIN QUIVERS

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Abstract. We study fundamental group of the exchange graphs for the bounded derived category \( D(Q) \) of a Dynkin quiver \( Q \) and the finite-dimensional derived category \( D(\Gamma_N Q) \) of the Calabi-Yau-N Ginzburg algebra associated to \( Q \). In the case of \( D(Q) \), we prove that its space of stability conditions (in the sense of Bridgeland) is simply connected; as applications, we show that its Donaldson-Thomas invariants can be calculated via a quantum dilogarithm function on exchange graphs. In the case of \( D(\Gamma_N Q) \), we show that faithfulness of the Seidel-Thomas braid group action (which is known for \( Q \) of type \( A \) or \( N = 2 \)) implies the simply connectedness of its space of stability conditions; moreover we provide a topological realization of almost completed cluster tilting objects.

Key words: space of stability conditions, Calabi-Yau-N Ginzburg algebra, higher cluster category, Donaldson-Thomas invariant, quantum dilogarithm identity

1. Introduction

1.1. Overall. The notion of a stability condition on a triangulated category was defined by Bridgeland [6] (c.f. Section 2.8). The idea was inspired from physics by studying D-branes in string theory. Nevertheless, the notion itself is interesting purely mathematically. A stability condition on a triangulated category \( \mathcal{D} \) consists of a collection of full additive subcategories of \( \mathcal{D} \), known as the slicing, and a group homomorphism from the Grothendieck group \( K(\mathcal{D}) \) to the complex plane, known as the central charge. Bridgeland [6] showed a key result that the space \( \text{Stab}(\mathcal{D}) \) of stability conditions on \( \mathcal{D} \) is a finite dimensional complex manifold. Moreover, these spaces carry interesting geometric/topological structure which shade light on the properties of the original triangulated categories. Most interesting examples of triangulated categories are derived categories. They are weak homological invariants arising in both algebraic geometry and representation theory, and indeed different manifolds and quivers (usually with relation) might share the same derived category (say complex projective line and Kronecker quiver). Also note that the space of stability conditions are related to Kontsevich’s homological mirror symmetry, that the (quotient) space of stability conditions of the Fukaya categories of Lagrangian submanifolds of certain symplectic manifolds are supposed to be some Kähler moduli space. We will study the spaces of stability conditions of the bounded derived category \( \mathcal{D}(Q) \) of a Dynkin quiver \( Q \) and the finite-dimensional derived category \( \mathcal{D}(\Gamma_N Q) \) of the Calabi-Yau-N Ginzburg algebra associated to \( Q \). Noticing that when \( Q \) is of Dynkin type, \( \mathcal{D}(\Gamma_N Q) \) was studied by Khovanov-Seidel-Thomas [27]/[37]

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via the derived Fukaya category of Lagrangian submanifolds of the Milnor fibres of the
singularities of type $A_n$.

In understanding stability conditions and triangulated categories, t-structures play
an important role. In fact, we can view a t-structure as a ‘discrete’ (integer) structure
while a stability condition (resp. a slicing) is its ‘complex’ (resp. ‘real’) refinement.
Every t-structure carries an abelian category sitting inside it, known as its heart. Note
that an abelian category is a canonical heart in its derived category, e.g. $\mathcal{H}_Q = \text{mod} kQ$
is the canonical heart of $\mathcal{D}(Q)$. The classical way to understand relations between
different hearts is via HRS-tilting (c.f. Section 2.7), in the sense of Happel-Reiten-
Smalø. To give a stability condition is equivalent to giving a t-structure and a stability
function on its heart with the Harder-Narashimhan (HN) property. This implies that
a finite heart (i.e. has $n$ simples and has finite length) corresponds to a (complex)
$n$-cell in the space of stability conditions. Moreover, Woolf [41] shows that the tilting
between finite hearts corresponds to the tiling of these $n$-cells. More precisely, two $n$-
cells meet if and only if the corresponding hearts differ by a HRS-tilting; and they meet
in codimension one if and only if the hearts differ by a simple tilting. Thus, our main
method to study a space of stability conditions of a triangulated category $\mathcal{D}$ is via its
‘skeleton’ – the exchange graph $\text{EG}(\mathcal{D})$, that is, the oriented graphs whose vertices are
hearts in $\mathcal{D}$ and whose edges correspond to simple (forward) tiltings between them (c.f.
[35]). Figure 1 (taken from [35], which in fact, the quotient graph of $\text{EG}^\gamma(\mathcal{D}(\Gamma_3 A_2))/[1]$ and $\text{Stab}^\delta(\mathcal{D}(\Gamma_3 A_2))/\mathbb{C}$) demonstrates the duality between the exchange graph and the
tiling of the space of stability conditions by many cells like the shaded area, so that
each vertex in the exchange graph corresponds to a cell and each edge corresponds to
a common edge (codimension one face) of two neighboring cells. We will prove certain
simply connectedness of spaces of stability conditions via exchange graph.

Stability conditions naturally link to Donaldson-Thomas (DT) invariants, which was
originally defined as the weighted Euler characteristics (By Behrend function) of moduli
spaces for Calabi-Yau 3-folds (c.f. [32]). Reineke [38] (c.f. Section 7.1) realized that
the DT-invariant for a Dynkin quiver can be calculated as a product of quantum dilog-
arithms, indexing by any HN-stratum of $\mathcal{H}_Q$, which is a ‘maximal refined version’ of
torsion pairs on an abelian category. His approach was integrating certain identities in
Hall algebras to show the stratum-independence of the product. We will apply exchange
graphs to give a combinatorial proof of such quantum dilogarhythm identities.

1.2. Contents. We will collect related background in Section 2.

In Section 3 and Section 4, we first make a key observation (Proposition 3.5) that the
fundamental group of the exchange graphs generates by squares and pentagons. Then
we prove (Theorem 3.7) the simply connectedness of the space of stability conditions
$\text{Stab}(\mathcal{D}(Q))$ and show that (Corollary 4.6) the faithfulness of the Seidel-Thomas braid
group action (which is known for $Q$ of type $A$ or $N = 2$) implies the simply connectedness
of its space of stability conditions. Moreover, the quotient space of $\text{Stab}^\delta(\Gamma_n Q)$ by
the Seidel-Thomas braid group $\text{Br}(\Gamma_n Q)$ is the ‘right’ space of stability conditions for
the higher cluster category $\mathcal{C}_{N-1}(Q)$ (see Remark 4.7). In fact, the generators of its
fundamental group provide a topological realization of almost completed cluster tilting
objects in $\mathcal{C}_{N-1}(Q)$ (Theorem 4.5).
In Section 5, we present (Theorem 5.2) a limit formula of spaces of stability conditions

\[ \text{Stab}(Q) \cong \lim_{N \to \infty} \text{Stab}^\circ(\Gamma_N Q) / \text{Br}(\Gamma_N Q), \]

which reflects a philosophical point of view that, in a suitable sense,

\[ Q = \lim_{N \to \infty} \Gamma_N Q. \quad (1.1) \]

In Section 6, we study directed paths in exchange graphs. We will first show (Theorem 6.9) that HN-strata of \( \mathcal{H}_Q \) can be naturally interpreted as directed paths connecting \( \mathcal{H}_Q \) and \( \mathcal{H}_Q[1] \) in \( \text{EG}(Q) \). Then we discuss total stability of stability functions (c.f. Conjecture 6.13) and the path-inducing problem. We will provide explicit examples and a conjecture.

In Section 7, we observe that the existence of DT-invariant of \( Q \) is equivalent to the path-independence of the quantum dilogarithm product over certain directed paths. Then we give a slight generalization (Theorem 7.3) of this path-independence, to all paths (not necessarily directed) whose vertices lie between \( \mathcal{H}_Q \) and \( \mathcal{H}_Q[1] \). The point is that this path-independence reduces to the cases of squares and pentagons in Proposition 3.5; therefore such type of quantum dilogarithm identities are just compositions of the classical Pentagon Identities. We will also discuss the wall-crossing formula for APR-tilting (c.f. \([31]\)). Note that Keller \([20]\) also spotted this phenomenon and proved a more remarkable quantum dilogarithm identities via mutation of quivers with potential. In fact, his formula can also be rephrased as quantum dilogarithm product over paths in the exchange graph of the corresponding Calabi-Yau-3 categories.
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2. Preliminaries

2.1. Dynkin Quivers. A (simply laced) Dynkin quiver $Q = (Q_0, Q_1)$ is a quiver whose underlying unoriented graph is one of the following unoriented graphs:

\[
\begin{align*}
A_n : & \quad 1 \quad 2 \quad \cdots \quad n \\
D_n : & \quad 1 \quad 2 \quad \cdots \quad n - 1 \\
E_{6,7,8} : & \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \\
\end{align*}
\]  

For a Dynkin quiver $Q$, we denote by $kQ$ the path algebra; denote by $\text{mod } kQ$ the category of finite dimensional $kQ$-modules, which can be identified with $\text{Rep}_k(Q)$, the category of representations of $Q$ (c.f. [2]). We will not distinguish between $\text{mod } kQ$ and $\text{Rep}_k(Q)$. Recall that the Euler form $\langle -, - \rangle : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \to \mathbb{Z}$

is defined by

\[\langle \alpha, \beta \rangle = \sum_{i \in Q_0} \alpha_i \beta_i - \sum_{(i \rightarrow j) \in Q_1} \alpha_i \beta_j.\]

Moreover for $M, L \in \text{mod } kQ$, we have

\[\langle \dim M, \dim L \rangle = \dim \text{Hom}(M, L) - \dim \text{Ext}^1(M, L),\]

where $\dim E \in \mathbb{N}^{Q_0}$ is the dimension vector of any $E \in \text{mod } kQ$.

2.2. Hearts and t-structures. Let $\mathcal{D}(Q) = \mathcal{D}^b(\text{mod } kQ)$ be the bounded derived category of $Q$, which is a triangulated category.

Recall (e.g. from [6]) that a t-structure on a triangulated category $\mathcal{D}$ is a full subcategory $\mathcal{P} \subset \mathcal{D}$ with $\mathcal{P}[1] \subset \mathcal{P}$ and such that, if one defines

\[\mathcal{P}^\perp = \{G \in \mathcal{D} : \text{Hom}_\mathcal{D}(F, G) = 0, \forall F \in \mathcal{P}\},\]

then, for every object $E \in \mathcal{D}$, there is a unique triangle $F \to E \to G \to F[1]$ in $\mathcal{D}$ with $F \in \mathcal{P}$ and $G \in \mathcal{P}^\perp$. Any t-structure is closed under sums and summands and hence it is determined by the indecomposables in it. Note also that $\mathcal{P}^- [-1] \subset \mathcal{P}^-$. A t-structure $\mathcal{P}$ is bounded if for every object $M$, the shifts $M[k]$ are in $\mathcal{P}$ for $k \gg 0$ and in $\mathcal{P}^\perp$ for $k \ll 0$. The heart of a t-structure $\mathcal{P}$ is the full subcategory

\[\mathcal{H} = \mathcal{P}^\perp[1] \cap \mathcal{P}\]
and any bounded t-structure is determined by its heart. More precisely, any bounded t-structure \( \mathcal{P} \) with heart \( \mathcal{H} \) determines, for each \( M \) in \( \mathcal{D} \), a canonical filtration (\cite[Lemma 3.2]{6})

\[
0 = M_0 \hookrightarrow M_1 \hookrightarrow \cdots \hookrightarrow M_{m-1} \hookrightarrow M_m = M
\]

where \( H_i \in \mathcal{H} \) and \( k_1 > \ldots > k_m \) are integers. Using this filtration, one can define the \( k \)-th homology of \( M \), with respect to \( \mathcal{H} \), to be

\[
H_k(M) = \begin{cases} 
H_i & \text{if } k = k_i \\
0 & \text{otherwise.}
\end{cases}
\]

Then \( \mathcal{P} \) consists of those objects with no negative homology and \( \mathcal{P}^\perp \) those with only negative homology. For any object \( M \) in \( \mathcal{D} \), define the (homological) width \( \text{Wid}_H(M) \) to be the difference \( k_1 - k_m \) of the maximum and minimum degrees of its non-zero homology as in (2.3). It is clear that the width is an invariable under shifts.

In this paper, we only consider bounded t-structures and their hearts and all categories will be implicitly assumed to be \( k \)-linear. Note that a heart is always abelian. For instance, \( \mathcal{D}(Q) \) has a canonical heart mod \( kQ \), which we will write as \( \mathcal{H}_Q \) from now on.

We recall an object in an abelian category is simple if it has no proper sub objects, or equivalently it is not the middle term of any (non-trivial) short exact sequence. We will denote a complete set of simples of an abelian category \( \mathcal{C} \) by \( \text{Sim}(\mathcal{C}) \). Denote by \( \langle T_1, \ldots, T_m \rangle \) the smallest full subcategory containing \( T_1, \ldots, T_m \) and closed under extensions.

2.3. Sections in AR-quiver. For quivers, a convenient way to picture the categories \( \mathcal{H}_Q \) and \( \mathcal{D}(Q) \) is by drawing their Auslander-Reiten (AR) quivers. (c.f. \cite[Chapter II,IV]{2}). Let \( \Lambda(\mathcal{C}) \) be the AR-quiver of a (\( k \)-linear) category \( \mathcal{C} \) with AR-functor \( \tau \). Note that, we have the AR-formula

\[
\text{Ext}^1(Y, X) \cong \text{Hom}(X, \tau Y)^*.
\]

When \( Q \) is of Dynkin type, \( \Lambda(\mathcal{D}(Q)) \) is isomorphic to the translation quiver \( \mathbb{Z}Q \). In particular, we have \cite{15}

\[
\text{Ind} \mathcal{D}(Q) = \bigcup_{j \in \mathbb{Z}} \text{Ind} \mathcal{H}_Q[j].
\]

We will give several characterization of standard hearts in \( \mathcal{D}(Q) \) in this subsection. Following \cite[Chapter IX]{2}, we introduce several notions:

- A path in \( \Lambda(\mathcal{C}) \) is a sequence

\[
M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \longrightarrow \cdots \xrightarrow{f_i} M_{t-1} \xrightarrow{f_t} M_t
\]
of irreducible maps $f_i$ between indecomposable modules $M_i$ with $t \geq 1$. When such a path exists, we say that $M_0$ is a predecessor of $M_t$ or $M_t$ is a successor of $M_0$.

- A path $M_0 \rightarrow \ldots \rightarrow M_t$ in $\Lambda(C)$ is called sectional if, for all $1 < i \leq t$, $\tau M_i \not\cong M_{i-2}$.
- Let $Ps(M)$ be the set of objects that lie in some sectional path starting from $M$ and $Ps^{-1}(M)$ be the set of objects that lie in some sectional path ending at $M$.

We have the following elementary lemma.

**Lemma 2.1** ([2]). We have

1. Any section in $ZQ$ is isomorphic to some orientation of $\Delta$.
2. For any object $M$ in $ZQ$, $Ps(M)$ and $Ps^{-1}(M)$ are sections.
3. The projectives of $H_Q$ together with the irreducible maps between them are a section in $\Lambda(D(Q))$. Moreover the section has the exactly the opposite orientation of $Q$.

For a section $P$ in $\Lambda(D(Q)) \cong ZQ$, define $[P, \infty) = \bigcup_{m \geq 0} \tau^{-m}P$. Similarly for $(-\infty, P]$ define $[P_1, P_2] = [P_1, \infty) \cap (-\infty, P_2]$.

The following lemmas characterize such type of intervals.

**Lemma 2.2.** The interval $[Ps(M), \infty)$ consists precisely of all the successors of $M$. Similarly, $(-\infty, Ps^{-1}(M)]$ consists precisely all the predecessors of $M$.

**Proof.** We only prove the first assertion. The second is similar.

By the local property of the translation quiver $ZQ$, any object in $[Ps(M), \infty)$ is a successor of $M$. On the other hand, let $L$ be any successor of $M$ with path

$$M = M_0 \xrightarrow{f_1} M_1 \rightarrow \ldots \xrightarrow{f_i} M_j = L.$$ 

If $\tau M_i = M_{i-2}$ for some $3 \leq i \leq j$, then consider $\tau L$ with path

$$M = M_0 \xrightarrow{f_1} \ldots \xrightarrow{f_{i-2}} M_{i-2} = \tau M_i \xrightarrow{\tau f_i} \tau M_{i+1} \xrightarrow{\tau f_{i+1}} \ldots \xrightarrow{\tau f_j} \tau M_j = \tau L.$$ 

we can repeat the process until the path is sectional, i.e. until we obtain $\tau^k L \in Ps(M)$ for some $k \geq 0$. Thus $L \in [Ps(M), \infty)$.

**Lemma 2.3.** Let $M, L \in \text{Ind} D(Q)$. If $\text{Hom}(M, L) \neq 0$ then

$$L \in \left[Ps(M), Ps^{-1}(\tau(M[1]))\right], \quad M \in \left[Ps(\tau^{-1}(L[-1])), Ps^{-1}(L)\right].$$

**Proof.** By the Auslander-Reiten formula, we have

$$\text{Hom}(M, L)^* = \text{Hom}(\tau^{-1}(L), M[1]).$$

The lemma now follows from Lemma 2.2.

For later use, we define the position function as follows.
Definition/Lemma 2.4. There is a position function \( \text{pf} : \Lambda(D(Q)) \to \mathbb{Z} \), unique up to an additive constant, such that \( \text{pf}(M) - \text{pf}(\tau M) = 2 \) for any \( M \in \Lambda(D(Q)) \) and \( \text{pf}(M) < \text{pf}(L) \) for any successor \( L \) of \( M \). For a heart \( \mathcal{H} \) in \( D(Q) \), define

\[
\text{pf}(\mathcal{H}) = \sum_{S \in \text{Sim} \mathcal{H}} \text{pf}(S).
\]

2.4. Standard hearts in \( D(Q) \).

Proposition 2.5. A section \( P \) in \( D(Q) \) will induce a unique t-structure \( \mathcal{P} \) on \( D(Q) \) such that \( \text{Ind} \mathcal{P} = [P, \infty) \). For any t-structure \( \mathcal{P} \) on \( D(Q) \), the followings are equivalent:

1. \( \mathcal{P} \) is induced by some section \( P \).
2. \( \text{Ind} D(Q) = \text{Ind} \mathcal{P} \cup \text{Ind} \mathcal{P}^\perp. \)
3. The corresponding heart \( \mathcal{H} \) is isomorphic to \( \mathcal{H}'_Q \), where \( Q' \) has the same underlying diagram of \( Q \).
4. \( \text{Wid}_\mathcal{H} M = 0 \) for any \( M \in \text{Ind} D(Q) \), where \( \mathcal{H} \) is the corresponding heart.

Proof. For a section \( P \), let \( \mathcal{P} \) be the subcategory which is generated by the elements in \( \text{Ind} \mathcal{P} = [P, \infty) \). Notice that \( \text{Ind} \mathcal{P}^\perp = (\infty, \tau^{-1}P] \) which implies \( \mathcal{P} \) is a t-structure. Thus \( 1 \Rightarrow 2 \). Since \( \mathcal{H} = [P, P[1]) \), \( 1 \Rightarrow 3 \).

If \( \mathcal{H} \) is isomorphic to \( \mathcal{H}'_Q \) for some quiver \( Q' \), then \( \text{Ind} \mathcal{P} = \bigcup_{j \geq 0} \mathcal{H}[j] = [P', \infty) \), where \( P' \) is the sub-quiver in \( \Lambda(D(Q)) \) consists of the projectives. Thus \( 3 \Rightarrow 4 \). Since for any \( M \in \text{Ind} D(Q) \), \( \text{Wid}_\mathcal{H} M = 0 \) if and only if \( M \in \mathcal{H}[k] \) for some integer \( k \), we have \( 4 \Rightarrow 3 \). Noticing that \( \mathcal{H}[k] \) is either in \( \mathcal{P} \) or \( \mathcal{P}^\perp \), we have \( 4 \Rightarrow 2 \).

Now we only need to prove \( 2 \Rightarrow 1 \). If an indecomposable \( M \) is in \( \mathcal{P} \) (resp. \( \mathcal{P}^\perp \)), then, inductively, all of its successors (resp. predecessors) are in \( \mathcal{P} \) (resp. \( \mathcal{P}^\perp \)). By the local property, \( \tau^m(M) \) is a successor of \( M \) if \( m \geq 0 \) and a predecessor if \( m \leq 0 \). Hence, in any row \( \pi^{-1}(v) \in \mathbb{Z}Q \cong \Lambda(D(Q)) \), for any vertex \( v \in Q_0 \), there is a unique integer \( m_v \), such that \( \tau^j(v) \in \mathcal{P} \), for \( j \geq m_v \), while \( \tau^j(v) \in \mathcal{P}^\perp \), for \( j < m_v \). Furthermore, for a neighboring vertex \( w \) of \( v \), the local picture looks like this

\[
\begin{array}{ccc}
\square & \rightarrow & \square \\
\square & \rightarrow & ? \\
\square & \rightarrow & \square \\
\end{array}
\]

where \( \square \in \mathcal{P} \) and \( \square \in \mathcal{P}^\perp \). Hence \( v_{m_v} \) and \( w_{m_w} \) must be connected in \( \mathbb{Z}Q \) and so the full sub-quiver of \( \mathbb{Z}Q \) consisting of all vertices \( \{v_{m_v}\}_{v \in Q_0} \) is a section and furthermore it induces \( \mathcal{P} \).

We call a heart on \( D(Q) \) is standard if the corresponding t-structure is induced by a section.

2.5. Calabi-Yau categories. Let \( N > 1 \) be an integer. Denote by \( \Gamma_N Q \) the Calabi-Yau-N Ginzburg (dg) algebra associated to \( Q \), that is, the dg algebra

\[
kQ_0(x, x^*, e^* \mid x \in Q_1, e \in Q_0)
\]

with degrees

\[
\deg e = \deg x = 0, \quad \deg x^* = N - 2, \quad \deg e^* = N - 1
\]
and only nontrivial differentials
\[ d \sum_{e \in Q_0} e^* = \sum_{x \in Q_1} [x, x^*]. \]

Write \( D(\Gamma N Q) \) for \( D_{\text{fd}}(\text{mod} \Gamma N Q) \).

Recall that a triangulated category \( C \) is called Calabi-Yau-N if, for any objects \( L, M \) in \( C \) we have a natural isomorphism
\[ S : \text{Hom}^\bullet_C(L, M) \simto \text{Hom}^\bullet_C(M, L)^\vee[N]. \] (2.6)

An object \( S \) is \( N \)-spherical if \( \text{Hom}^\bullet(S, S) = k \oplus k[-N] \).

By \([24]\) (see also \([27],[37],[40]\)), we know that \( D(\Gamma N Q) \) is a Calabi-Yau-N category which admits a standard heart \( H \) generated by simple \( \Gamma N Q \)-modules \( S_e, e \in Q_0 \), each of which is \( N \)-spherical. Denote by \( EG(\Gamma N Q) \) the principal component of the exchange graph \( EG(D(\Gamma N Q)) \), that is, the component containing \( H \).

2.6. Twist functors and braid groups. We recall (c.f. \([27],[37],[40]\)) a distinguished family of auto-equivalences in \( \text{Aut} D(\Gamma N Q) \), for the CY-N category \( D(\Gamma N Q) \).

**Definition 2.6.** The twist functor \( \phi \) of a spherical object \( S \) is defined by
\[ \phi_S(X) = \text{Cone}(S \otimes \text{Hom}^\bullet(S, X) \to X). \] (2.7)

with inverse
\[ \phi_S^{-1}(X) = \text{Cone}(X \to S \otimes \text{Hom}^\bullet(X, S)^\vee)[-1] \] (2.8)

The Seidel-Thomas braid group, denoted by \( \text{Br}(\Gamma N Q) \), is the subgroup of \( \text{Aut} D(\Gamma N Q) \) generating by the twist functors of the simples in \( \text{Sim} H \).

2.7. Exchange graphs. A similar notion to a t-structure on a triangulated category is a torsion pair on an abelian category. Tilting with respect to a torsion pair in the heart of a t-structure provides a way to pass between different t-structures.

**Definition 2.7.** A torsion pair in an abelian category \( C \) is a pair of full subcategories \( \langle F, T \rangle \) of \( C \), such that \( \text{Hom}(T, F) = 0 \) and furthermore every object \( E \in C \) fits into a short exact sequence \( 0 \to T \to E \to F \to 0 \) for some objects \( T \in T \) and \( F \in F \).

We will use the notation \( H = \langle F, T \rangle \) to indicate an abelian category with a torsion pair.

**Proposition 2.8** (Happel, Reiten, Smalø). Let \( H = \langle F, T \rangle \) be a heart in a triangulated category \( D \). Then there exists the following two hearts with torsion pairs
\[ H^\sharp = \langle T, F[1] \rangle, \quad H^\flat = \langle T[-1], F \rangle. \]

We call \( H^\sharp \) the forward tilt of \( H \) with respect to the torsion pair \( \langle F, T \rangle \), and \( H^\flat \) the backward tilt of \( H \). Clearly \( H^\flat = H^\sharp[-1] \).

For two hearts \( H_i, i = 1, 2 \) in \( D \) with corresponding t-structure \( P_i \), we say \( H_1 \leq H_2 \) if and only if \( P_1 \supset P_2 \), or equivalently, \( P_1^+ \subset P_2^+ \), with equality if and only if the two hearts are the same.

The following lemma collect several well-known facts about tilting.

**Lemma 2.9** ([16],[35]). In any triangulated category, we have
\[\begin{align*}
&\bullet \mathcal{H}[-1] \leq \mathcal{H}^c \leq \mathcal{H} \leq \mathcal{H}^t \leq \mathcal{H}[1]. \\
&\mathcal{H} \leq \mathcal{H} \leq \mathcal{H}[1] \text{ if and only if } \mathcal{H}' = \mathcal{H}^t. \text{ Moreover, in such case, the tilting is with respect to the torsion pair } (\mathcal{F}, \mathcal{T}) \text{ in } \mathcal{H}, \text{ where } \mathcal{T} = \mathcal{H} \cap \mathcal{H}' \text{ and } \mathcal{F} = \mathcal{H} \cap \mathcal{H}'[-1].
\end{align*}\]

We say a forward tilting is simple, if the corresponding torsion free part is generated by a single simple object \(S\), and denote the heart by \(\mathcal{H}_S^c\). Similarly, a backward tilting is simple if the corresponding torsion part is generated by a single simple object \(S\), and denote the heart by \(\mathcal{H}_S^b\).

**Definition 2.10.** Define the exchange graph \(\text{EG}(\mathcal{D})\) of a triangulated category \(\mathcal{D}\) to be the oriented graph whose vertices are all hearts in \(\mathcal{D}\) and whose edges correspond to simple forward tiltings between them.

We will label an edge of \(\text{EG}(\mathcal{D})\) by the simple object of the corresponding tilting, i.e. the edge with end points \(\mathcal{H}\) and \(\mathcal{H}_S^c\) will be labeled by \(S\). For \(S \in \text{Sim }\mathcal{H}\), inductively define

\[\mathcal{H}_S^{mb} = \left(\mathcal{H}_S^{(m-1)b}\right)^{♭}\]

for \(m \geq 1\) and similarly for \(\mathcal{H}_S^{mb}, m \geq 1\). We will write \(\mathcal{H}_S^{mb} = \mathcal{H}_S^{-mb}\) for \(m < 0\).

**Definition 2.11.** A line \(l = l(\mathcal{H}, S)\) in \(\text{EG}(\mathcal{D})\), for some triangulated category \(\mathcal{D}\), is the full subgraph consisting of the vertices \(\{\mathcal{H}_S^{mb}\}_{m \in \mathbb{Z}}\), for some heart \(\mathcal{H}\) and a simple \(S \in \text{Sim }\mathcal{H}\). We say an edge in \(\text{EG}(\mathcal{D})\) has direction \(T\) if its label is \(T[m]\) for some integer \(m\); we say a line \(l\) has direction-T if some (and hence every) edge in \(l\) has direction \(T\).

By [25], we know that \(\text{EG}(\mathcal{D}(Q))\) is connected when \(Q\) is of Dynkin type, which will be wrote as \(\text{EG}(Q)\). For an alternate proof, see Appendix A. Denote by \(\text{EG}^o(\Gamma_N Q)\) the principal component of the exchange graph \(\text{EG}(\mathcal{D}(\Gamma_N Q))\), that is, the component containing \(\mathcal{H}_T\).

Recall some notation and results from [35]. There is special kind of exact functors from \(\mathcal{D}(Q)\) to \(\mathcal{D}(\Gamma_N Q)\), known as the Lagrangian immersions (L-immersions), see [35, Definition 6.2]. Let \(\mathcal{H}\) be a heart in \(\mathcal{D}(\Gamma_N Q)\) with \(\text{Sim }\mathcal{H} = \{S_1, ..., S_n\}\). If there is a L- immersion \(f : \mathcal{D}(Q) \to \mathcal{D}(\Gamma_N Q)\) and a heart \(\mathcal{H} \in \text{EG}^o(Q)\) with \(\text{Sim }\mathcal{H} = \{\hat{S}_1, ..., \hat{S}_n\}\), such that \(f(\hat{S}_i) = S_i\), then we say that \(\mathcal{H}\) is induced via \(f\) from \(\mathcal{H}\) and write \(f_*(\mathcal{H}) = \mathcal{H}\).

Further, let \(\mathcal{H}\) be a heart in some exchange graph \(\text{EG}^o(Q)\). Define the exchange graph \(\text{EG}_N(Q, \mathcal{H})\) with base \(\mathcal{H}\) to be the full subgraph of \(\text{EG}(Q)\) induced by

\[\{\mathcal{H}_0 \mid \mathcal{H}_0 \in \text{EG}(Q), \mathcal{H}[1] \leq \mathcal{H}_0 \leq \mathcal{H}[N - 1]\}\]

and \(\text{EG}^o_N(Q, \mathcal{H})\) its principal component (that is, the connected component containing \(\mathcal{H}[1]\)). Similarly for \(\text{EG}_N(\Gamma_N Q, \mathcal{H})\) and \(\text{EG}^o_N(\Gamma_N Q, \mathcal{H})\).

**Theorem 2.12 ([35]).** For an acyclic quiver \(Q\), we have the following:

\(1^o\). there is a canonical L-immersion \(I : \mathcal{D}(Q) \to \mathcal{D}(\Gamma_N Q)\) that induces an isomorphism

\[I_* : \text{EG}^o_N(Q, \mathcal{H}_Q) \to \text{EG}^o(\Gamma_N Q, \mathcal{H}_T). \quad (2.9)\]

\(2^o\). as vertex set, we have

\[\text{EG}(Q, \mathcal{H}_Q) \cong \text{EG}^o_N(\Gamma_N Q, \mathcal{H}_T) \cong \text{EG}^o(\Gamma_N Q) / \text{Br}; \quad (2.10)\]
3. for any heart \( \mathcal{H} \) in \( \text{EG}^0(\Gamma_N Q) \), \( \text{Sim} \mathcal{H} \) has \( n \) elements and \( \{ \phi_S \}_{S \in \text{Sim} \mathcal{H}} \) is a generating set for \( \text{Br}(\Gamma_N Q) \);

4. for any line \( l \) in \( \text{EG}^0(\Gamma_N Q) \), its orbit in \( \text{EG}^0(\Gamma_N Q)/\text{Br} \) is \( (N-1) \) cycle.

Besides, we have

**Proposition 2.13.** Let \( Q \) be a Dynkin quiver. \( \text{EG}_N(Q, \mathcal{H}_Q) \) is finite for any \( N > 1 \) and we have

\[
\text{EG}(Q) = \lim_{N \to \infty} \text{EG}_2N(Q, \mathcal{H}_Q[-N]).
\]

**Proof.** Notice that there are only finitely many indecomposables in \( \bigcup_{j=1}^{N-1} \mathcal{H}_Q[j] \) and hence only finitely many hearts in \( \text{EG}_N(Q, \mathcal{H}_Q) \).

Let \( \mathcal{H} \in \text{EG}(Q) \). Consider the homology \( \mathcal{H}_r \), with respect to \( \mathcal{H}_Q \), of any simple \( S \) of \( \mathcal{H} \). Then we know that if \( N > 1 \), then \( S \in \bigcup_{j=1}^{N-1} \mathcal{H}_Q[j] \) which implies \( \mathcal{H}_Q[-N+1] \leq \mathcal{H} \leq \mathcal{H}_Q[N-1] \). Then \( \mathcal{H} \in \text{EG}_2N(Q, \mathcal{H}_Q[-N]) \) which implies (2.11).

---

2.8. **Stability conditions.** This section (following [6]) collects the basic definitions of stability conditions. Denote \( \mathcal{D} \) a triangulated category and \( K(\mathcal{D}) \) its Grothendieck group.

**Definition 2.14** ([7] Definition 3.1). A stability condition \( \sigma = (Z, \mathcal{P}) \) on \( \mathcal{D} \) consists of a group homomorphism \( Z : K(\mathcal{D}) \to \mathbb{C} \) called the central charge and full additive subcategories \( \mathcal{P}(\varphi) \subset \mathcal{D} \) for each \( \varphi \in \mathbb{R} \), satisfying the following axioms:

1. if \( 0 \neq E \in \mathcal{P}(\varphi) \) then \( Z(E) = m(E) \exp(\varphi n_1) \) for some \( m(E) \in \mathbb{R}_{>0} \),
2. for all \( \varphi \in \mathbb{R} \), \( \mathcal{P}(\varphi + 1) = \mathcal{P}(\varphi)[1] \),
3. if \( \varphi_1 > \varphi_2 \) and \( A_i \in \mathcal{P}(\varphi_i) \) then \( \text{Hom}_\mathcal{D}(A_1, A_2) = 0 \),
4. for each nonzero object \( E \in \mathcal{D} \) there is a finite sequence of real numbers

\[
\varphi_1 > \varphi_2 > \ldots > \varphi_m
\]

and a collection of triangles

\[
0 \to E_0 \to E_1 \to E_2 \to \ldots \to E_{m-1} \to E_m = E ,
\]

with \( A_j \in \mathcal{P}(\varphi_j) \) for all \( j \).

We call the collection of subcategories \( \{ \mathcal{P}(\varphi) \} \), satisfying 2\(^o\)-4\(^o\) in Definition 2.14, the slicing and the collection of triangles in 4\(^o\) the Harder-Narashimhan (HN) filtration.

For any nonzero object \( E \in \mathcal{D} \) with HN-filtration above, define its upper phase to be \( \Psi^u(E) = \varphi_1 \) and lower phase to be \( \Psi^l(E) = \varphi_n \). By [7, Lemma 5.2], \( \mathcal{P}(\varphi) \) is abelian. An object \( E \in \mathcal{P}(\varphi) \) for some \( \varphi \in \mathbb{R} \) is said to be semistable; in which case, \( \varphi = \Psi^u(E) \).

Moreover, if \( E \) is simple in \( \mathcal{P}(\varphi) \), then it is said to be stable. Let \( I \) be an interval in \( \mathbb{R} \) and define

\[
\mathcal{P}(I) = \{ E \in \mathcal{D} \mid \Psi^u(E) \in I \} .
\]

Then for any \( \varphi \in \mathbb{R} \), \( \mathcal{P}(\{ \varphi \}) \) and \( \mathcal{P}(\{ \varphi, \infty \}) \) are t-structures in \( \mathcal{D} \).

There is a natural \( \mathbb{C} \) action on the set \( \text{Stab}(\mathcal{D}) \) of all stability conditions on \( \mathcal{D} \), namely:

\[
\Theta \cdot (Z, \mathcal{P}) = (Z \cdot z, \mathcal{P}_z),
\]
where $z = \exp(\Theta \pi i), \Theta = x + yi$ and $P_x(m) = P(x + m)$ for $x, y, m \in \mathbb{R}$. There is also a natural action on $\text{Stab}(\mathcal{D})$ induced by $\text{Aut}(\mathcal{D})$, namely:

$$\xi \circ (Z, P) = (Z \circ \xi, \xi \circ P).$$

Similarly to stability condition on triangulated categories, we have the notation of stability function on abelian categories.

**Definition 2.15 ([6]).** A stability function on an abelian category $\mathcal{C}$ is a group homomorphism $Z : K(\mathcal{C}) \to \mathbb{C}$ such that for any object $0 \neq M \in \mathcal{C}$, we have $Z(M) = m(M) \exp(\mu Z(M) i\pi)$ for some $m(M) \in \mathbb{R}^+ > 0$ and $\mu Z(M) \in [0, 1)$, i.e. $Z(M)$ lies in the upper half-plane

$$H = \{r \exp(i\pi \theta) \mid r \in \mathbb{R}^+, 0 \leq \theta < 1\} \subset \mathbb{C}.$$  \hfill (2.12)

Call $\mu Z(M)$ the phase of $M$. We say an object $0 \neq M \in \mathcal{C}$ is semistable (with respect to $Z$) if every subobject $0 \neq L$ of $M$ satisfies $\mu Z(L) \leq \mu Z(M)$. Further, we say a stability function $Z$ on $\mathcal{C}$ satisfies HN-property, if for an object $0 \neq M \in \mathcal{C}$, there is a collection of short exact sequences

$$0 = M_0 \rightarrowtail M_1 \rightarrowtail \cdots \rightarrowtail M_{m-1} \rightarrowtail M_k = M$$

in $\mathcal{C}$ such that $L_1, \ldots, L_k$ are semistable objects (with respect to $Z$) and their phases are in decreasing order, i.e. $\phi(L_1) > \cdots > \phi(L_k)$.

Note that we have a different convention $0 \leq \theta < 1$ for the upper half plane $H$ in (2.12) as Bridgeland’s $0 < \theta \leq 1$.

Then we have another way to give a stability condition on triangulated categories.

**Proposition 2.16 ([6], [7]).** To give a stability condition on a triangulated category $\mathcal{D}$ is equivalent to giving a bounded t-structure on $\mathcal{D}$ and a stability function on its heart with the HN-property. Further, to give a stability condition on $\mathcal{D}$ with a finite heart $\mathcal{H}$ is equivalent to giving a function $\text{Sim} \mathcal{H} \to H$, where $H$ is the upper half plane as in (2.12).

Recall a crucial result of spaces of stability conditions.

**Theorem 2.17 (Bridgeland [6]).** The space of stability conditions on a triangulated category $\mathcal{D}$ is a complex manifold, denoted by $\text{Stab}(\mathcal{D})$.

Therefore every finite heart $\mathcal{H}$ corresponds to a (complex, half closed and half open) $n$-cell $U(\mathcal{H}) \simeq H^n$ inside $\text{Stab}(\mathcal{D})$.

3. SIMPLY CONNECTEDNESS OF $\text{Stab}(\mathcal{Q})$

Let $\text{Stab}(\mathcal{Q}) = \text{Stab}(\mathcal{D}(\mathcal{Q}))$. The connectedness of $\text{Stab}(\mathcal{Q})$ follows from the connectedness of $\text{EG}(\mathcal{Q})$. 


3.1. A canonical embedding. By connectedness of \( \text{EG}(Q) \), we have a disjoint union \( \text{Stab}(Q) = \bigcup_{\mathcal{H} \in \text{EG}(Q)} U(\mathcal{H}) \). Moreover, by the results in [41, Section 2], we have

\[
\overline{U(\mathcal{H})} - U(\mathcal{H}) = \bigcup_{\mathcal{H}[-1] \leq \mathcal{H'} < \mathcal{H}} \left( \overline{U(\mathcal{H})} \cap U(\mathcal{H'}) \right),
\]

and hence the gluing structure of \( \text{Stab}(Q) \) follows.

\[
\partial U(\mathcal{H}) = \bigcup_{\mathcal{H}[-1] \leq \mathcal{H'} < \mathcal{H}} \left( \overline{U(\mathcal{H})} \cap U(\mathcal{H'}) \right) \bigcup \bigcup_{\mathcal{H} < \mathcal{H'} \leq \mathcal{H}[1]} \left( \overline{U(\mathcal{H})} \cap U(\mathcal{H'}) \right),
\]

Call a term in the RHS in (3.2) a face of the n-cell \( U(\mathcal{H}) \). Further, by [7, Lemma 5.5], codimension one faces of \( \mathcal{H} \) corresponds to its simple tilts. More precisely, \( \dim \overline{U(\mathcal{H})} \cap U(\mathcal{H}) = n - 1 \) if and only if \( \mathcal{H}' = \mathcal{H}'_S \) or \( \mathcal{H} = \mathcal{H}'_S \) for some \( S \in \text{Sim} \mathcal{H} \). Therefore, we have the following lemma.

**Lemma 3.1.** There is a canonical embedding (unique up to homotopy)
\[
i : \text{EG}(Q) \to \text{Stab}(Q)
\]

such that

1. for each vertex (heart) \( \mathcal{H} \), its image is the center of the n-cell \( U(\mathcal{H}) \), i.e. \( i(\mathcal{H}) = (Z_H, \mathcal{P}_H) \) with heart \( H \) satisfying \( Z_H(S_j) = \exp(\frac{1}{2} \pi i) \).
2. for each edge \( S_i : \mathcal{H} \to \mathcal{H}'_{S_i} \), its image \( \sigma_{(0,1)} = \{ \sigma_t = (Z_t, \mathcal{P}_t) \mid t \in (0,1) \} \) is contained in \( (U(\mathcal{H}) \cup U(\mathcal{H}'_{S_i}))^0 \) and intersects \( (U(\mathcal{H}) \cap U(\mathcal{H}'_{S_i}))^0 \) exactly once.

Now we fix a canonical embedding \( i \) and will identify the exchange graph with the image of this embedding.

**Lemma 3.2.** We have a surjection \( \pi_1(\text{EG}(Q)) \to \pi_1(\text{Stab}(Q)) \).

*Proof.* Let \( Y \) be the union of all faces, with codimension bigger than one, of some heart in \( \text{EG}(Q) \). We can slightly perturb any path in \( \text{Stab}(Q) \), without changing its class in \( \pi_1(\text{Stab}(Q)) \), such that it misses \( Y \). Since \( \text{Stab}^y(Q) - Y \) contracts onto \( \text{EG}(Q) \), the lemma follows.

\[ \square \]

3.2. Simply connectedness. First, we prove two elementary but important lemmas.

**Lemma 3.3.** Let \( \mathcal{H} \) be a heart of \( \text{D}(Q) \) with Sim \( \mathcal{H} = \{ S_1, ..., S_n \} \) and \( \mathcal{E}_{ij} = \text{Hom}^\bullet(S_i, S_j) \).

Then for \( i \neq j, j \neq k \),

1. \( \dim \mathcal{E}_{ij} + \dim \mathcal{E}_{ji} \leq 1 \).
2. If \( \mathcal{E}_{ij}, \mathcal{E}_{jk}, \mathcal{E}_{ik} \neq 0 \), then the multiplication \( \mathcal{E}_{ij} \otimes \mathcal{E}_{jk} \hookrightarrow \mathcal{E}_{ik} \) is an isomorphism.

*Proof.* Suppose that \( \mathcal{E}_{ij}^{\delta_1} \neq 0 \) for some \( \delta_1 > 0 \). Let \( A = S_i \) and \( B = S_j[\delta_1] \). By Lemma 2.3, we have

\[
B \in \left[ \text{Ps}(A), \text{Ps}^{-1}(\tau(A[1])) \right].
\]

Thus \( \mathcal{E}_{ij}^m = 0 \) for \( m \neq \delta_1 \) and \( \mathcal{E}_{ij}^m = 0 \) for \( m > 1 - \delta_1 \). But \( \mathcal{E}_{ji} \) is also concentrated in positive degrees and hence \( \mathcal{E}_{ji} = 0 \).
By Proposition 2.5, there is a quiver $Q'$ such that, $P(Q)$ consists precisely the projectives in $\text{mod} \, kQ'$. Moreover, we have $B \in \text{mod} \, kQ'$. Let $b = \dim B$ and $a = \dim A$, then we have

$$
\begin{align*}
\dim \text{Hom}(A, B) - \dim \text{Ext}^1(A, B) &= \langle a, b \rangle = \dim E_{ij}^\delta, \\
\dim \text{Hom}(B, A) - \dim \text{Ext}^1(B, A) &= \langle b, a \rangle = \dim E_{ij}^\delta = 0.
\end{align*}
$$

(3.4)

Since $Q'$ is of Dynkin type, the quadratic form $q(x) = \langle x, x \rangle$ is positive definite and, furthermore, since $A \not\cong B$, we have $a \neq b$. Hence

$$
0 < \langle a - b, a - b \rangle = 2 - \langle a, b \rangle
$$

i.e. $\dim E_{ij}^\delta \leq 1$. Thus 1$^\circ$ follows.

For 2$^\circ$, suppose that $E_{jk}^\delta \neq 0$. Since $B \in H'_Q$, Lemma 2.3 implies that

$$S_k[\delta_1 + \delta_2] \in (H'_Q)[1] \cup H'_Q.
$$

Suppose that $E_{ik}^\delta \neq 0$ and we have $C = S_k[\delta_3]$ is also in $H'_Q$. Thus either $\delta_3 = \delta_1 + \delta_2$ or $\delta_3 = \delta_1 + \delta_2 - 1$.

Suppose that $\delta_3 = \delta_1 + \delta_2 - 1$. Let $c = \dim C$. As in (3.4), we have

$$
\begin{align*}
\langle a, b \rangle &= 1, \quad \langle a, c \rangle = 1, \quad \langle b, c \rangle = -1, \\
\langle b, a \rangle &= 0, \quad \langle c, a \rangle = 0, \quad \langle c, b \rangle = 0.
\end{align*}
$$

Because $A$ is simple, $a \neq b + c$. But $\langle b + c - a, b + c - a \rangle = 0$, which is a contradiction. Therefore $\delta_3 = \delta_1 + \delta_2$.

Since $A$ is a simple, any non-zero $f \in \text{Hom}(A, B)$ is injective and so gives a short exact sequence $0 \to A \to B \to D \to 0$ in $\text{mod} \, kQ'$. Applying $\text{Hom}(\cdot, C)$ to it, we get an exact sequence

$$
0 \to \text{Hom}(D, C) \to \text{Hom}(B, C) \xrightarrow{f^*} \text{Hom}(A, C) \to \\
\to \text{Hom}(D, C[1]) \to \text{Hom}(B, C[1]) = 0
$$

If $f^*$ is not an isomorphism, then $\text{Hom}(D, C) \neq 0$ and $\text{Hom}(D, C[1]) \neq 0$, contradicting Lemma 2.3. Hence multiplication $E_{ij} \otimes E_{jk} \to E_{ik}$, i.e. composition $\text{Hom}(A, B) \otimes \text{Hom}(B, C) \to \text{Hom}(A, C)$, is an isomorphism, as required. □

**Lemma 3.4.** Let $H$ be a heart in $D(Q)$ and $S_i, S_j$ be two simples in $\text{Sim} \, H$. Suppose that $\text{Hom}^1(S_i, S_j) = 0$. Let $H_i = H^S_i, H_j = H^S_j$, and $H_{ij} = (H_j)^S_i$.

1$^\circ$. If $\text{Hom}^1(S_j, S_i) = 0$, then $(H_i)^S_j = H_{ij}$.

2$^\circ$. If $\text{Hom}^1(S_j, S_i) \neq 0$, let $T_j = \phi^{-1}_{S_j}(S_i)$. Then we have $H_{ij} = (H_i)^S_j$, where $H_i = (H_i)^S_j$. 

Proof. We have \( \dim \text{Hom}^*(S_j, S_i) \leq 1 \) by Lemma 3.3. Applying [35, Proposition 5.5], the lemma follows by a direct calculation.

Proposition 3.5. If \( Q \) is of Dynkin type, then \( \pi_1(\text{EG}_N(Q, \mathcal{H}_Q)) \) is generated by squares and pentagons as in (3.5) for any \( N \geq 2 \). Further, \( \pi_1(\text{EG}(Q)) \) is generated by such squares and pentagons.

Proof. Choose any cycle \( c \) in \( \text{EG}_N(Q, \mathcal{H}_Q) \). By Proposition 2.13,

\[
B(c) = \{ \mathcal{H} \mid \exists \mathcal{H}' \in c, \mathcal{H}' \leq \mathcal{H} \leq \mathcal{H}_Q[N - 1] \}
\]

is finite. We use induction on \( \# B(c) \) to prove the first statement. If \( \# B(c) = 1 \), then \( c \) is trivial. Suppose that \( \# B(c) > 1 \) and any cycle \( c' \subset \text{EG}_N^N(Q, \mathcal{H}_Q) \) with \( \# B(c') < \# B(c) \) is generated by the squares and pentagons. Choose a source \( \mathcal{H} \in c \) such that \( \mathcal{H}' \neq \mathcal{H} \) for any other source \( \mathcal{H}' \) in \( c \). Let \( S_i \) and \( S_j \) be the arrows coming out at \( \mathcal{H} \). If \( i = j \) we can delete them in \( c \) to get a new cycle \( c' \). If \( i \neq j \), we know that \( S_i : \mathcal{H} \to \mathcal{H}_i \) and \( S_j : \mathcal{H} \to \mathcal{H}_j \) are either in a square or a pentagon as in (3.5). By the second part of [35, Lemma 5.8], we know that \( \mathcal{H}_{N-1}(S_i) = 0 \) and hence \( \mathcal{H}_{ij} = (\mathcal{H}_j)_{S_i} \in \text{EG}_N^N(Q, \mathcal{H}_Q) \).

Thus this square/pentagon are in \( \text{EG}_N^N(Q, \mathcal{H}_Q) \) and we can replace \( S_i \) and \( S_j \) in \( c \) by other edges in this square/pentagon to get a new cycle \( c' \subset \text{EG}_N^N(Q, \mathcal{H}_Q) \). Either way, we have \( B(c') \subset (B(c) - \{ \mathcal{H} \}) \) for the new cycle \( c' \) and we are done.

Now choose any cycle \( c \in \text{EG}(Q) \). By (2.11), we can choose \( N \gg 1 \) such that all hearts in \( c[k] \) are in \( \text{EG}_N^N(Q, \mathcal{H}_Q) \) for some integer \( k \). Then the second statement follows from the first one.

Lemma 3.6. Any square or pentagon as in (3.5) is trivial in \( \pi_1(\text{Stab}(Q)) \).

Proof. Recall that we embed \( \text{EG}(Q) \) into \( \text{Stab}(Q) \). Suppose in case 2° of Lemma 3.4 and consider the path \( \mathcal{L}_p : \mathcal{H} \to \mathcal{H}_i \to \mathcal{H}_a \to \mathcal{H}_{ij} \) in \( \text{EG}(Q) \). Let \( \text{Sim} \mathcal{H} = \{ S_1, \ldots, S_n \} \).

Consider the stability condition \( \sigma \) whose heart is \( \mathcal{H} \) satisfying

\[
\begin{align*}
Z(S_k) &= \exp(\frac{k}{2} \pi i) \quad k \neq i, j, \\
Z(S_i) &= \exp(\delta \pi i), \\
Z(S_j) &= \exp(3\delta \pi i),
\end{align*}
\]

for some small \( \delta > 0 \). Notice that \( \dim \text{Hom}^*(S_j, S_i) = 1 \), hence there are only three indecomposables in \( \mathcal{H} \) generated by \( S_i \) and \( S_j \), i.e. \( S_i, T_j \) and \( S_j \), where \( T_j \) is the unique extension of \( S_j \) and \( S_i \) (with phase \( 2\delta \)). Thus we can choose \( \delta \) so small that any stable object other than \( S_i, T_j \) and \( S_j \) has phase larger than \( 4\delta \).
Consider the interval $L_0 = \{ \sigma, \epsilon \} \epsilon \in (-4 \delta, 0]$, where $\sigma, \epsilon \in U(H)$, $\epsilon \in (-\delta, 0]$. We have

$$
\begin{cases}
\sigma, \epsilon \in U(H), & \epsilon \in (-\delta, 0], \\
\sigma, \epsilon \in U(H_i), & \epsilon \in (-2 \delta, -\delta), \\
\sigma, \epsilon \in U(H_s), & \epsilon \in (-3 \delta, -2 \delta), \\
\sigma, \epsilon \in U(H_{ij}), & \epsilon \in (-4 \delta, -3 \delta).
\end{cases}
$$

Therefore $L_0$ is homotopy to $L_p$. Notice that $L_0$ is contained in the contractible ‘prism’ $P = \mathbb{C} \cdot U(H) \cong \mathbb{C} \cdot H^n$, where $H$ is the upper half plane in (2.12). Similarly, the path $\mathcal{H} \to \mathcal{H}_j \to \mathcal{H}_{ij}$ is homotopy to some interval $L'_0 = \{ \sigma', \epsilon \} \epsilon \in (-4 \delta', 0]$ in $P$, where $\sigma'$ is the stability condition whose heart is $\mathcal{H}$ satisfying

$$
\begin{cases}
Z'(S_k) = \exp\left(\frac{1}{2} \pi i\right) & k \neq i, j, \\
Z'(S_i) = \exp(3 \delta' \pi i), \\
Z'(S_j) = \exp(\delta' \pi i),
\end{cases}
$$

for some small $\delta' > 0$. Hence such pentagon is trivial. Same argument for the square. □

**Theorem 3.7.** If $Q$ is of Dynkin type, then $\text{Stab}(Q)$ is simply connected.

**Proof.** By Proposition 3.5 and Lemma 3.6 we know that $\pi_1(\text{EG}(Q))$ is trivial in $\text{Stab}(Q)$. Then the theorem follows from the surjection in Lemma 3.2. □

4. Simply connectedness of Calabi-Yau Dynkin case

4.1. The principal component. In this subsection, we show that $\text{EG}^\circ(\Gamma_N Q)$ induces a connected component in the space of stability conditions $\text{Stab}(\mathcal{D}(\Gamma_N Q))$.

**Lemma 4.1.** $\text{EG}^\circ_0(\Gamma_N Q, \mathcal{H})$ is finite, for any heart $\mathcal{H} \in \text{EG}^\circ(\Gamma_N Q)$.

**Figure 2.**
Proof. By (2.10), we can assume that $\mathcal{H} \in \text{EG}_3^N(\Gamma_N Q, \mathcal{H}_I)$ without lose of generality. By Theorem 2.12, we have isomorphism (2.9) and hence $\text{EG}_3^N(\Gamma_N Q, \mathcal{H}_I)$ is finite by Proposition 2.13.

Now we claim that, for $\mathcal{H} \in \text{EG}_3^N(\Gamma_N Q, \mathcal{H}_I)$, if $\text{EG}_3^N(\Gamma_N Q, \mathcal{H}_0)$ is finite for any $H\mathcal{H}_Q[1] \leq \mathcal{H}_0 < \mathcal{H}$, then $\text{EG}_3^N(\Gamma_N Q, \mathcal{H})$ is also finite.

If $\mathcal{H} \in \text{EG}_3^{N-1}(\Gamma_N Q, \mathcal{H}_I)$, then $\text{EG}_3^N(\Gamma_N Q, \mathcal{H}) \subset \text{EG}_N^N(\Gamma_N Q, \mathcal{H}_I)$, which implies that $\text{EG}_3^N(\Gamma_N Q, \mathcal{H})$ is finite. Now suppose that $\mathcal{H} \notin \text{EG}_3^{N-1}(\Gamma_N Q, \mathcal{H}_I)$. Let $\mathcal{H}$ is induced from $\hat{\mathcal{H}} \in \text{EG}_3^N(\Gamma_N Q, \mathcal{H}_I)$ via $I$, and we have $\hat{\mathcal{H}} \notin \text{EG}_3^N(Q, \mathcal{H}_Q)$ by (2.9).

By (2.5), for any simple $\hat{S} \in \text{Sim} \hat{\mathcal{H}}$, there is some integer $m$ such that $\hat{S} \in \mathcal{H}_Q[m]$, and we have $1 \leq m \leq N - 1$ by [35, Lemma 5.8]. Since $\hat{H} \notin \text{EG}_3^N(Q, \mathcal{H}_Q)$, there exists a simple $\hat{S} \in \text{Sim} \hat{\mathcal{H}}$ such that $H_{N-1}(\hat{S}) \neq 0$, where $H_0$ is with respect to $\mathcal{H}_Q$. By (2.5), $\hat{S} \in \mathcal{H}_Q[N - 1]$. Then $S = I(\hat{S}) \in \mathcal{H}_I[N - 1]$. By [35, Lemma 5.8], we have

$$l(H, S) \cap \text{EG}_3^N(\Gamma_N Q, \mathcal{H}_I) = \{H_{N-1}^S\}_{t=0}^{N-2}.$$ 

By the inductive assumption, we know that $\text{EG}_3^N(\Gamma_N Q, \mathcal{H}_S)$ and $\text{EG}_3^N(\Gamma_N Q, \mathcal{H}_S^{(N-2)})$ is finite; hence, so is

$$\text{EG}_3^N(\Gamma_N Q, \mathcal{H}_S^t) = \phi_S^{-1} \text{EG}_3^N(\Gamma_N Q, \mathcal{H}_S^{(N-2)}).$$

By [35, Lemma 8.1], we have

$$\text{EG}_3^N(\Gamma_N Q, \mathcal{H}) \subset \left(\text{EG}_3^N(\Gamma_N Q, \mathcal{H}_S) \cup \text{EG}_3^N(\Gamma_N Q, \mathcal{H}_S^t)\right)$$

which implies the finiteness of $\text{EG}_3^N(\Gamma_N Q, \mathcal{H})$. Thus the lemma follows by induction. 

Proposition 4.2. $\text{EG}_3^N(\Gamma_N Q, \mathcal{H}) = \text{EG}_3^N(\Gamma_N Q, \mathcal{H}_I)$, for any heart $\mathcal{H} \in \text{EG}_3^N(\Gamma_N Q)$. 

Proof. Suppose that there exists a heart $\mathcal{H}' \in \text{EG}_3^N(\Gamma_N Q, \mathcal{H}_I) - \text{EG}_3^N(\Gamma_N Q, \mathcal{H})$, we claim that there is an infinite directed path

$$\mathcal{H}_1 \xrightarrow{S_1} S_2 \xrightarrow{S_2} \mathcal{H}_3 \rightarrow \cdots$$

in $\text{EG}_3^N(\Gamma_N Q, \mathcal{H})$ satisfying $\mathcal{H}_j \prec \mathcal{H}'$ for any $j \in \mathbb{N}$.

Use induction starting from $\mathcal{H}_1 = \mathcal{H}[1]$. Suppose we have $\mathcal{H}_j \in \text{EG}_3^N(\Gamma_N Q, \mathcal{H})$ such that $\mathcal{H}_j \prec \mathcal{H}'$. If for any simple $S \in \mathcal{H}_j$, we have $S \in H'$, then $\mathcal{H}' \supset H_j$ which implies $\mathcal{P}_j \supset \mathcal{P}_j$, or $\mathcal{H}' \leq \mathcal{H}_j$; this contradicts to $\mathcal{H}_j \prec \mathcal{H}'$. Thus there is a simple $S_j \in \mathcal{H}_j$ such that $S_j \notin \mathcal{H}'$. Notice that $\mathcal{H}_j \prec \mathcal{H}' \leq \mathcal{H}[2] \leq \mathcal{H}_j[1]$, then by [35, Lemma 8.1], we have $\mathcal{H}_{j+1} = \mathcal{H}_{j}^{-1} \mathcal{H}_j \leq \mathcal{H}'(\leq \mathcal{H}[2])$. Notice that $\mathcal{H}' \notin \text{EG}_3^N(\Gamma_N Q, \mathcal{H})$, therefore $\mathcal{H}_{j+1} \neq \mathcal{H}'$, which implies the claim.

Then we have that $\text{EG}_3^N(\Gamma_N Q, \mathcal{H})$ is infinite, which contradicts to the finiteness in Lemma 4.1.

Similar to Section 3.1, we have the following results.

Theorem 4.3. We have the formula (3.2). Moreover, there is a principal component

$$\text{Stab}(\Gamma_N Q) = \bigcup_{\mathcal{H} \in \text{EG}_3^N(\Gamma_N Q)} \text{U}(\mathcal{H})$$

in $\text{Stab}(\mathcal{D}(\Gamma_N Q))$, which is the connected component containing $\text{U}(\mathcal{H}_I)$. 


Proof. By Proposition 4.2, we have the finiteness condition (∗∗) in [41, Section 2], and hence [41, Proposition 2.15 and Theorem 2.17] apply which implies the theorem. □

We will also call a term in the RHS in (3.2) a face of the n-cell U(ℋ), for any ℋ ∈ EG°(Γ_N Q). Similarly, codimension one faces of ℋ corresponds to its simple tilts and, as in Section 3.1, we have the corresponding canonical embedding and surjection as below.

**Proposition 4.4.** There is a canonical embedding (unique up to homotopy)
\[ \iota : EG°(Γ_N Q) \hookrightarrow \text{Stab}°(Γ_N Q) \] (4.1)
such that the conditions 1° and 2° in Lemma 3.1. Moreover, we have a surjection \( \pi_1(\text{EG}°(Q)) \to \pi_1(\text{Stab}°(Q)) \).

**4.2. Simply connectedness.** Define the basic cycles in \( \text{Stab}°(Γ_N Q)/\text{Br} \) to be braid group orbits of lines in \( \text{Stab}°(Γ_N Q) \).

**Theorem 4.5.** Suppose that Q is of Dynkin type and let ℋ ∈ EG°(Γ_N Q). Then
\[ \pi_1(\text{Stab}°(Γ_N Q)/\text{Br},[ℋ]) \]
is generated by basic cycles containing [ℋ] and it is a quotient group of the braid group Br_Q.

Proof. Let Sim ℋ = {S_1,...S_n}, \( \phi_k = \phi_{S_k} \) and let \( c_k \) be the basic cycle corresponding to \( l(ℋ, S_k) \), for \( k = 1,...,n \). Denote by \( p \) the quotient map
\[ p : \text{Stab}°(Γ_N Q) \to \text{Stab}°(Γ_N Q)/\text{Br} . \]
We will drop \( Y \in \{ \text{Stab}°(Γ_N Q)/\text{Br}, \text{Stab}°(Γ_N Q) \} \) in \( \pi_1(Y, y) \) if there is no ambiguity. By [11, Theorem 13.11], we have a short exact sequence
\[ 0 \to p_*(\pi_1(ℋ)) \to \pi_1([ℋ]) \xrightarrow{\rho} \text{Br}(Γ_N Q) \to 0, \] (4.2)
where \( \rho \) sends \( c_k \) to \( \phi_k^{-1} \). What we only need to show is that \( \{c_k\} \) satisfies the braid group relation and generates \( \pi_1([ℋ]) \).

For \( i,j \) satisfying \( \text{Hom}^°(S_i, S_j) = 0 \), consider the lifting \( L_1 \) of \( c_ic_j^{-1}c_i^{-1} \) in \( \pi_1(ℋ) \) starting at ℋ. Let
\[ \mathcal{H}^i = \phi_i^{-1}(ℋ), \quad \mathcal{H}^{ij} = \phi_j^{-1} \circ \phi_i^{-1}(ℋ), \]
\[ \mathcal{H}^j = \phi_j^{-1}(ℋ), \quad \mathcal{H}^{ij} = \phi_i^{-1} \circ \phi_j^{-1}(ℋ). \]
We have \( \mathcal{H}^{ij} = \mathcal{H}^{ji} \) in this case. Then \( L_1 \in \pi_1(ℋ) \) is the boundary in Figure 3 with clockwise orientation. Notice that \( \dim \text{Hom}^°(S_j, S_i) \leq 1 \) by Lemma 3.3. By the iterated use [35, Proposition 5.5] we can use \( (N - 1)^2 \) squares, as in (3.5), to cover \( L_1 \). For instance, Figure 3 is the CY-4 case, where the blue (resp. red) edges have direction \( S_i \) (resp. \( S_j \)) and the hearts are uniquely determined by these edges. Notice that using the same argument in Lemma 3.6, one can show any squares covering \( L_1 \) is trivial in \( \pi_1(ℋ) \). Thus \( L_1 \) is trivial in \( \pi_1(ℋ) \) which implies \( c_ic_j = c_jc_i \) in \( \pi_1([ℋ]) \).
For $i, j$ satisfying $\text{Hom}^\bullet(S_j, S_i) = k[-1]$, consider the lifting $L_2$ of $c_i c_j c_i^{-1} c_j^{-1}$ in $\pi_1(\mathcal{H})$ that stating at $\mathcal{H}$. Let $\mathcal{H}', \mathcal{H}^j$ as before and

$$T = \phi_i^{-1}(S_j), \quad R = \phi_j^{-1}(S_i),$$

$$\mathcal{H}' = \phi_j^{-1} \circ \phi_T^{-1} \circ \phi_i^{-1}(\mathcal{H}).$$

By [37, Lemma 2.11], we have

$$\phi_j^{-1} \circ \phi_T^{-1} \circ \phi_i^{-1} = \phi_i^{-1} \circ \phi_R^{-1} \circ \phi_j^{-1}.$$ 

Then $L_2 \in \pi_1(\mathcal{H})$ is the boundary in Figure 4 with clockwise orientation.

Similarly, we can use $(N - 1)(2N - 3)$ squares/pentagons, as in (3.5), to cover $L_2$. For instance, Figure 4 is the CY-4 case, where the blue (resp. red, dashed, dotted)
edges have direction $S_i$ (resp. $S_j$, $T$, $R$). Then we deduce that $L_2$ is trivial in $\pi_1(\mathcal{H})$ as before. Thus $c_i c_j c_i = c_j c_i c_j$ in $\pi_1(\mathcal{H})$ as required.

To finish, we only need to show that $\{c_k\}_{k=1}^n$ generates $\phi_1([\mathcal{H}])$. By Theorem 2.12, we have $\text{EG}(\Gamma N Q)/\text{Br} \cong \text{EG}_{\mathcal{H}}(\Gamma N Q, \mathcal{H}_T)$ and hence $\pi_1(\text{EG}(\Gamma N Q)/\text{Br})$ is generated by all squares and pentagons in $\text{EG}_{\mathcal{H}}(\Gamma N Q, \mathcal{H}_T)$ and basic cycles (c.f. [35]). These squares and pentagons are trivial as in Lemma 3.6. Therefore, it is essential to show that another basic cycle that does not contain $[\mathcal{H}]$ is generated by $\{c_k\}_{k=1}^n$.

Let $\mathcal{H}_i = \mathcal{H}_{S_i}^2$, $T = \phi_i^{-1}(S_j)$, $c_T$ be the basic cycle induced by the line $l(\mathcal{H}_i, T)$ and $s_i$ be the path from $\mathcal{H}$ to $\mathcal{H}_i$ in the line $l(\mathcal{H}, S)$. Consider the basic cycle $s_i c_T s_i^{-1}$. If $\text{Hom}^*(S_j, S_i) = 0$, let $L_3$ be the lifting of $(s_i c_T s_i^{-1})c_i^{-1}$ in $\pi_1(\mathcal{H})$ stating at $\mathcal{H}$. As the gray area in Figure 3, we can cover $L_3$ using part of the covering for $L_1$ which implies $s_i c_T s_i^{-1} = c_i$. If $\text{Hom}^*(S_j, S_i) = k[-1]$, let $L_4$ be the lifting of $c_j(s_i c_T s_i^{-1})c_j^{-1}c_i^{-1} \in \pi_1([\mathcal{H}])$ in $\pi_1(\mathcal{H})$ stating at $\mathcal{H}$. Similarly, we can cover $L_4$ using part of the covering for $L_2$ (as the gray area in Figure 4) which implies $s_i c_T s_i^{-1} = c_j^{-1}c_i c_j$. Either way, $s_i c_T s_i^{-1}$ is generated by $\{c_k\}_{k=1}^n$ as required. □

**Corollary 4.6.** Let $Q$ be a Dynkin quiver. If the braid group action on $D(\Gamma N Q)$ is faithful, i.e. $\text{Br}(\Gamma N Q) \cong \text{Br}_Q$, then $\text{Stab}^0(\Gamma N Q)$ is simply connected. In particular, this is true for $Q$ of type $A_n$ or $N = 2$.

**Proof.** If $\text{Br}(\Gamma N Q) \cong \text{Br}_Q$, then by Theorem 4.5 and (4.2) we deduce that $\varphi$ is an isomorphism. Hence $\pi_1(\text{Stab}^0(\Gamma N Q)) = 1$ which implies the simply connectedness. The faithfulness for $Q$ of type $A_n$ follows from [37] and faithfulness for $N = 2$ follows from [5]. □

**Remark 4.7.** By Theorem 4.5, basic cycles in $\text{Stab}^0(\Gamma N Q)/\text{Br}$ are the generators of its fundamental group, which provide a topological realization of almost completed cluster tilting objects (c.f. [35, Remark 7.8]). In fact, our philosophy is that $\text{Stab}^0(\Gamma N Q)/\text{Br}$ is the ‘complexification’ of the dual of cluster complex and provides the ‘right’ space of stability conditions for the higher cluster category

$$C_{N-1}(Q) = D(Q)/\Sigma_{N-1},$$

where $\Sigma_{N-1} = \tau^{-1} \circ [N - 2] \in \text{Aut} D(Q)$. Notice that there are no hearts in $C_{N-1}(Q)$ and thus the space of stability conditions $\text{Stab}(C_{N-1}(Q))$ is empty in the standard sense.

Here are two sensible conjectures.

**Conjecture 4.8.** For any acyclic quiver $Q$, $\text{Br}(\Gamma N Q) \cong \text{Br}_Q$.

**Conjecture 4.9.** For a Dynkin quiver $Q$, $\text{Stab}(D(Q))$ and $\text{Stab}^0(D(\Gamma N Q))$ are contractible.

5. A LIMIT FORMULA

In this section, we provide a limit formula for spaces of stability conditions.

**Lemma 5.1.** If $\mathcal{H} = F\ast(\hat{\mathcal{H}})$ for some heart $\hat{\mathcal{H}} \in \text{EG}^0(Q)$, then a stability condition $\hat{\sigma} = (\hat{Z}, \hat{P})$ on $D(Q)$ with heart $\hat{\mathcal{H}}$ canonically induces a stability condition $\sigma = (Z, P)$ with heart $\mathcal{H}$ and such that $Z(F(\hat{S})) = \hat{Z}(\hat{S})$ for any $\hat{S} \in \text{Sim} \hat{\mathcal{H}}$. Thus we have a homeomorphism $F\ast : \text{U}(\mathcal{H}) \rightarrow \text{U}(\mathcal{H})$. 

**Proof.** The heart $\hat{\mathcal{H}}$ and $\mathcal{H}$ are both good by Theorem 2.12. Thus the lemma follows by Proposition 2.16. \hfill \Box

**Theorem 5.2.** We have

$$\text{Stab}(Q) \cong \lim_{N \to \infty} \text{Stab}^\circ(\Gamma_N Q)/\text{Br}(\Gamma_N Q)$$

in the following sense:

1°. There exists a family of open subspaces $\{S_N\}_{N \geq 2}$ in $\text{Stab}^\circ(Q)$ satisfying $S_N \subset S_{N+1}$ and $\text{Stab}(Q) \cong \lim_{N \to \infty} S_N$.

2°. $S_N$ is homemorphic to a fundamental domain for $\text{Stab}^\circ(\Gamma_N Q)/\text{Br}$.

**Proof.** Let $\text{Stab}^\circ_N(Q)$ and $\text{Stab}^\circ_N(\Gamma_N Q)$ be the interior of

$$\bigcup_{\mathcal{H} \in \text{EG}^\circ_N(Q, \mathcal{H}_Q)} \overline{U(\mathcal{H})} \quad \text{and} \quad \bigcup_{\mathcal{H} \in \text{EG}^\circ_N(\Gamma_N Q, \mathcal{H}_T)} \overline{U(\mathcal{H})}$$

respectively, By (3.2), we know that a face $F_Q$ of some cell $U(\hat{\mathcal{H}})$ is in $\text{Stab}^\circ_N(Q)$ if and only if $F_Q = U(\mathcal{H}) \cap U(\hat{\mathcal{H}})$ for some $\mathcal{H}, \mathcal{H}' \in \text{EG}^\circ_N(Q, \mathcal{H}_Q)$ satisfying $\hat{\mathcal{H}}[1] \leq \hat{\mathcal{H}}' < \hat{\mathcal{H}}$. Similarly, a face $F_T$ of some cell $U(\mathcal{H})$ is in $\text{Stab}^\circ_N(Q)$ if and only if $F_T = U(\mathcal{H}) \cap U(\mathcal{H}')$ in $\text{Stab}^\circ_N(\Gamma_N Q)$ for some $\mathcal{H}, \mathcal{H}' \in \text{EG}^\circ_N(\Gamma_N Q, \mathcal{H}_T)$ satisfying $\hat{\mathcal{H}}[1] \leq \hat{\mathcal{H}}' < \hat{\mathcal{H}}$.

Notice that we have isomorphism (2.9) and formulae (3.2) (for $\text{Stab}^\circ_N(Q)$ as well as $\text{Stab}^\circ_N(\Gamma_N Q)$). Then by Lemma 5.1, we know that any such face $F_T$ in $\text{Stab}^\circ_N(\Gamma_N Q)$ is induced from some face $F_Q$ in $\text{Stab}^\circ_N(Q)$ via $\mathcal{I}$, in the sense that we have

$$\mathcal{I}_*(F_Q) = \mathcal{I}((U(\mathcal{H}) \cap U(\hat{\mathcal{H}}))) = \mathcal{I}_*(U(\mathcal{H}')) \cap \mathcal{I}_*(U(\hat{\mathcal{H}}')) = U(\mathcal{H}) \cap U(\mathcal{H}') = F_T$$

Thus we can glue the homomorphisms in Lemma 5.1 to a homomorphism $\mathcal{I}_*: \text{Stab}^\circ_N(Q) \to \text{Stab}^\circ_N(\Gamma_N Q)$.

Let $S_N = \exp(-m\pi i) \cdot \text{Stab}^\circ_N(Q)$, for $m = \lfloor -\frac{N}{2} \rfloor$, where $\cdot$ is the $\mathbb{C}$-action. Then $1^\circ$ follows from the limit formula in Proposition 2.13 and we have $S_N \cong \text{Stab}^\circ_N(Q) \cong \text{Stab}^\circ_N(\Gamma_N Q)$, which completes the proof. \hfill \Box

**Example 5.3.** The calculation of $\text{Stab}(A_2)$ and $\text{Stab}^\circ(\Gamma_N A_2)$ in [36, Section 7.5 and 7.6] (c.f. [1, p16, Figure 2]) illustrate the idea of the limit in Theorem 5.2 in $A_2$ case.

6. **Directed paths and HN-strata**

In this section, we will study the relations between directed paths in the exchange graph $\text{EG}(Q)$, HN-strata for $\mathcal{H}_Q$, slicings on $\mathcal{D}(Q)$ and stability functions on $\mathcal{H}_Q$.

6.1. **Directed paths.** Let $\text{EG}(Q; \mathcal{H}_1, \mathcal{H}_2)$ be the full subgraph of $\text{EG}(Q)$ consisting of hearts $\mathcal{H}_1 \leq \mathcal{H} \leq \mathcal{H}_2$. Denote by $\overline{\mathcal{P}}(\mathcal{H}_1, \mathcal{H}_2)$ the set of all directed paths from $\mathcal{H}_1$ to $\mathcal{H}_2$ in $\text{EG}(Q; \mathcal{H}_1, \mathcal{H}_2)$.

**Lemma 6.1.** Suppose $\mathcal{H}_1 \leq \mathcal{H}_2$. Then $\overline{\mathcal{P}}(\mathcal{H}_1, \mathcal{H}_2) \neq \emptyset$ if at least one of $\mathcal{H}_1$ and $\mathcal{H}_2$ is standard. In particular, we have

$$\text{EG}(Q; \mathcal{H}[1], \mathcal{H}[N-1]) = \text{EG}(Q, \mathcal{H}) = \text{EG}^\circ_N(Q, \mathcal{H}),$$

for any standard heart $\mathcal{H} \in \text{EG}(Q)$. 
Proof. Without lose of generality, suppose that \( \mathcal{H}_1 = \mathcal{H}_Q[1] \) which is standard. For any simple \( S_i \in \text{Sim} \mathcal{H}_2, S_i \in \mathcal{H}_Q[m_i] \) for some integer \( m_i \) by (2.5). Since \( \mathcal{H}_1 \leq \mathcal{H}_2 \), we have \( m_i \geq 1 \). Choose \( N \gg 1 \) such that \( \mathcal{H}_2 \in \text{EG}_N^\circ(Q, \mathcal{H}_Q) \) and then \( \# \text{Ind}(\mathcal{P}_1 - \mathcal{P}_2) \) is finite. If \( \mathcal{H}_1 < \mathcal{H}_2 \), there exists \( j \) such that \( m_j > 1 \). By [35, Lemma 5.8], we can backward tilt \( \mathcal{H}_2 \) to \((\mathcal{H}_2)_j\) within \( \text{EG}_N^\circ(Q, \mathcal{H}_Q) \) which reduces \( \# \text{Ind}(\mathcal{P}_1 - \mathcal{P}_2) \). Thus we can iterated backward tilt \( \mathcal{H}_2 \) to \( \mathcal{H}_1 \) by induction, which implies the lemma.

Define the directed distance \( \text{dis}(\mathcal{H}_1, \mathcal{H}_2) \) and diameter \( \text{diam}(\mathcal{H}_1, \mathcal{H}_2) \) between \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) to be the minimum and respectively maximum over the lengths of the paths in \( \overrightarrow{\text{P}}(\mathcal{H}_1, \mathcal{H}_2) \). Recall the position function \( \text{pf} \) defined in Definition/Lemma 2.4. Since \( \tau h_Q = [-2] \), we have

\[
\text{pf}(M[1]) - \text{pf}(M) = h_Q, \quad \forall M \in \Lambda(\mathcal{D}(Q)).
\]

Here \( h_Q \) is the Coxeter number, which equals \( n + 1, 2(n - 1), 12, 18, 30 \) for \( Q \) of type \( A_n, D_n, E_6, E_7, E_8 \) respectively. There are the following easy estimations.

**Lemma 6.2.** Suppose that \( \overrightarrow{\text{P}}(\mathcal{H}_1, \mathcal{H}_2) \neq 0 \). Let \( \mathcal{P}_i \) be the t-structure corresponding to \( \mathcal{H}_i \). We have

\[
\begin{align*}
\text{diam}(\mathcal{H}_1, \mathcal{H}_2) &\leq \# \text{Ind}(\mathcal{P}_1 - \mathcal{P}_2) \quad (6.1) \\
\text{diam}(\mathcal{H}_1, \mathcal{H}_2) &\leq \# \text{Ind}(\mathcal{P}_2^+ - \mathcal{P}_1^+) \quad (6.2) \\
\text{dis}(\mathcal{H}_1, \mathcal{H}_2) &\geq \frac{\text{pf}(\mathcal{H}_2) - \text{pf}(\mathcal{H}_1)}{h_Q} \quad (6.3)
\end{align*}
\]

In particular dis\((\mathcal{H}, \mathcal{H}[m]) \geq mn \) with equality for standard heart \( \mathcal{H} \).

**Proof.** For any edge \( \mathcal{H} \rightarrow \mathcal{H}_Q^\circ \), we have \( \text{Ind} \mathcal{P} \supsetneq \text{Ind} \mathcal{P}_S^\circ \) by Lemma 2.9 and hence (6.1) follows. Similarly for (6.2).

By [35, Proposition 5.5] we have formula [35, (5.8)]. Notice that \( T_j \) is a successor of \( S_j \) and hence \( \text{pf}(T_j) > \text{pf}(S_j) \). We have

\[
\text{pf}(\mathcal{H}_S^\circ) - \text{pf}(\mathcal{H}) = \text{pf}(S[1]) - \text{pf}(S) + \sum_{j \in J} (\text{pf}(T_j) - \text{pf}(S_j)) \geq \text{pf}(S[1]) - \text{pf}(S) = h_Q
\]

which implies the inequality (6.3). In particular, if \( \mathcal{H}_1 = \mathcal{H}, \mathcal{H}_2 = \mathcal{H}[m] \), the RHS of (6.3) equals \( mn \).

Now suppose \( \mathcal{H} \) is standard, without loss of generality let \( \mathcal{H} = \mathcal{H}_Q \). Label the simples \( S_1, ..., S_n \) such that \( \text{pf}(S_1) \leq \text{pf}(S_2) \leq ... \leq \text{pf}(S_n) \). By Lemma 2.3, \( \text{Hom}(M, L) \neq 0 \) implies \( L \) is a successor of \( M \) and hence \( \text{pf}(M) < \text{pf}(L) \). Thus \( \text{Hom}^1(S_i, S_j) = 0 \) for \( i > j \). By [35, Proposition 5.5] we can tilt from \( \mathcal{H} \) to \( \mathcal{H}[1] \) with respect to the simples \( S_n, ..., S_1 \) in order, which implies dis\((\mathcal{H}, \mathcal{H}[m]) = mn \).

We can give a characterization of the longest paths in \( \overrightarrow{\text{P}}(\mathcal{H}_Q, \mathcal{H}_Q[1]) \).

**Proposition 6.3.** Let \( \mathcal{H} \) be a standard heart, then we have

\[
\text{diam}(\mathcal{H}, \mathcal{H}[1]) = \# \text{Ind} \mathcal{H}_Q = n \cdot \frac{h_Q}{2} \quad (6.4)
\]
Moreover, a path $p$ in $\overrightarrow{P}(\mathcal{H}, \mathcal{H}[1])$ has the longest length if and only if all vertices of $p$ are standard hearts.

**Proof.** We can tilt from $\mathcal{H}$ to $\mathcal{H}[1]$ by a sequence of APR-tiltings, which are L-tiltings. By Corollary A.2, such a path consisting of L-tiltings has length

$$\# \text{Ind}(\mathcal{P} - \mathcal{P}[1]) = \# \text{Ind}(\mathcal{P}[1] - \mathcal{P}^\perp) = \# \text{Ind} \mathcal{H}_Q.$$  

Then the first claim follows from (6.1).

Suppose $p$ is a longest path and use induction starting from $\mathcal{H}_Q$ which is standard. Consider an edge $\mathcal{H} \to \mathcal{H}^S$ in $p$ with $\mathcal{H}$ is standard. Since $p$ is longest, by (6.1), we have

$$\# \text{Ind}(\mathcal{P} - \mathcal{P}^S) = 1.$$  

Notice that $S \in (\mathcal{P} - \mathcal{P}^S)$, we have

$$\text{Ind} \mathcal{P}^S = \text{Ind} \mathcal{P} - \{S\}.$$  

Similarly, we have

$$\text{Ind} \left( \mathcal{P}^S \right)^\perp = \text{Ind}(\mathcal{P})^\perp \cup \{S\}.$$  

and hence

$$\text{Ind} \mathcal{P} \cup \text{Ind} \mathcal{P}^\perp = \text{Ind} \mathcal{P}^S \cup \text{Ind} \left( \mathcal{P}^S \right)^\perp.$$  

(6.5)

By Proposition 2.5, the fact that a heart $\mathcal{H}'$ is standard is equivalent to

$$\text{Ind} \mathcal{D}(Q) = \text{Ind} \mathcal{P}' \cup \text{Ind}(\mathcal{P}')^\perp.$$  

Therefore, by (6.5), the standardness of $\mathcal{H}$ implies the standardness of $\mathcal{H}^S$. Thus the necessity follows.

On the other hand, if $\mathcal{H}$ and its simple forward tilts $\mathcal{H}^S$ are standard, we claim that it is an APR-tilting at a sink. Suppose not, that the vertex $V \in Q_0$ corresponding to $S$ is not a sink. Then there is an edge $(V \to V') \in Q_1$ which corresponds to a nonzero map in $\text{Ext}^1(S, S')$, where $S'$ is the simple corresponding to $V'$. Then $S \notin (\mathcal{P}^S)^\perp$ since $S[1] \in \mathcal{P}[1] \subset \mathcal{P}^S$ by Lemma 2.9. Notice that $S \notin \mathcal{P}_S$, we know that $\mathcal{H}_S$ is not standard by Proposition 2.5, which is a contradiction. Thus if all the vertices of a path $p$ are standard then it consisting of APR-tiltings, which are L-tiltings. By Corollary A.2, we know that the length of $p$ is $\# \text{Ind} \mathcal{H}_Q$ which implies $p$ is longest. \qed

### 6.2. HN-strata

In this subsection, we use Reineke’s notion of HN-strata to give an algebraic interpretation of

$$\overrightarrow{P}(Q) := \overrightarrow{P}(\mathcal{H}_Q, \mathcal{H}_Q[1]).$$

**Definition 6.4.** A (discrete) HN-stratum $[T_1, \ldots, T_1]_{\text{HN}}$ in an abelian category $\mathcal{C}$ is an ordered collection of objects $T_1, \ldots, T_1$ in $\text{Ind} \mathcal{C}$, satisfying the HN-property:

- $\text{Hom}(T_i, T_j) = 0$ for $i > j$.
For any nonzero object $M$ in $\mathcal{C}$, there is an HN-filtration by short exact sequences

$$0 = M_0 \longrightarrow M_1 \longrightarrow \ldots \longrightarrow M_{m-1} \longrightarrow M_m = M \quad (6.6)$$

with $A_{j_i}$ is in $\langle T_{j_i} \rangle$ and $1 \leq j_m < \ldots < j_1 \leq l$.

Notice that the uniqueness of HN-filtration follows from the first condition in HN-property. Denote by HN($Q$) the set of all HN-strata of $\mathcal{H}_Q$. We claim that there is a bijection between $\widehat{\mathcal{P}}(Q)$ and HN($Q$).

Let $p = T_l \cdots T_1$ be a path in $\widehat{\mathcal{P}}(Q)$

$$p : \mathcal{H}_Q = \mathcal{H}_0 \xrightarrow{T_1} \mathcal{H}_1 \xrightarrow{T_2} \ldots \xrightarrow{T_l} \mathcal{H}_l = \mathcal{H}_Q[1]$$

with corresponding t-structures $\mathcal{P}_0 \supset \mathcal{P}_1 \supset \ldots \supset \mathcal{P}_l$. We have the following lemmas.

**Lemma 6.5.** For any indecomposable $M$ in $\mathcal{H}_Q$, there is a filtration as (6.6) such that $A_j$ is in $\langle T_{j} \rangle$ and $1 \leq j_m < \ldots < j_1 \leq l$.

**Proof.** We construct such a filtration as follows. Since

$$M \in \mathcal{P}_0 - \mathcal{P}_l = \bigcup_{i=1}^{l} (\mathcal{P}_{i-1} - \mathcal{P}_i),$$

there exists an integer $0 < j \leq l$ such that $M \in \mathcal{P}_{j-1} - \mathcal{P}_j$. Since $\mathcal{H}_j = (\mathcal{H}_{j-1})^{\sharp}_{T_j}$, we have a short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow A_j \longrightarrow 0$$

such that $A_j$ in $\langle T_j \rangle$. This is the last short exact sequence in the required filtration. Since $M'$ is in the torsion part corresponding to $(\mathcal{H}_{j-1})^{\sharp}_{T_j}$, we have

$$M' \in \mathcal{P}_j - \mathcal{P}_l = \bigcup_{i=j}^{l} (\mathcal{P}_{i-1} - \mathcal{P}_i).$$

Therefore we can repeat the procedure above and the lemma follows by induction. □

**Lemma 6.6.** Let $0 \leq j \leq l$. Let $\mathcal{F}_j = \langle T_1, \ldots, T_j \rangle$ and $\mathcal{T}_j = \langle T_{j+1}, \ldots, T_l \rangle$. Then $(\mathcal{F}_j, \mathcal{T}_j)$ is a torsion pair in $\mathcal{H}_Q$ and $\mathcal{H}_j = (\mathcal{H}_Q)^{\sharp}$ with respect to this torsion pair.

**Proof.** Use induction on $j$ starting from the trivial case when $j = 0$. Now suppose that $\mathcal{H}_j = (\mathcal{H}_Q)^{\sharp}$ with respect to $(\mathcal{F}_j, \mathcal{T}_j)$. Since $T_{j+1}$ is a simple in $\mathcal{H}_{j+1}$ and $T_k \in \mathcal{P}_{j+1}$ for $k > j + 1$, we have $\text{Hom}(T_k, T_{j+1}) = 0$ which implies $\text{Hom}(A, B) = 0$ for $A \in \mathcal{T}_{j+1}, B \in \mathcal{F}_{j+1}$. By Lemma 6.5 we know that for any object $M$ in Ind $\mathcal{H}_Q$, there is a short exact sequence $0 \rightarrow A \rightarrow M \rightarrow B \rightarrow 0$ such that $A \in \mathcal{T}_{j+1}$ and $B \in \mathcal{F}_{j+1}$. Therefore $(\mathcal{F}_{j+1}, \mathcal{T}_{j+1})$ is a torsion pair in $\mathcal{H}_Q$. By Lemma 2.9, we have $\mathcal{H}_j \cap \mathcal{H}_Q = \mathcal{T}_j$. To finish we only need to show that $\mathcal{H}_{j+1} \cap \mathcal{H}_Q = \mathcal{T}_{j+1}$. This follows from $\mathcal{H}_{j+1} = (\mathcal{H}_j)^{\sharp}_{T_j}$. □

Now we have an injection $\widehat{\mathcal{P}}(Q) \rightarrow \text{HN}(Q)$ as follows.
Corollary 6.7. Any directed path \( p = p = T_1 \cdots T_1 \) in \( \overrightarrow{P}(Q) \) induces an HN-stratum \([T_1, \ldots, T_1]_{HN} \) in \( \operatorname{HN}(Q) \).

Proof. Since \( T_i \in \mathcal{F}_j \) and \( T_j \in \mathcal{F}_j \) for \( j > i \), \( \operatorname{Hom}(T_j, T_i) = 0 \) follows from Lemma 6.6. Together with Lemma 6.5, the corollary follows. \( \square \)

For the converse construction, we have the following lemma.

Lemma 6.8. Let \([T_i, \ldots, T_1]_{HN} \) be an HN-stratum. For \( 0 \leq j \leq l \), let \( \mathcal{F}_j = \langle T_1, \ldots, T_j \rangle \) and \( T_j = \langle T_{j+1}, \ldots, T_l \rangle \). Then \((\mathcal{F}_j, T_j)\) is a torsion pair in \( \mathcal{H}_Q \). Let \( \mathcal{H}_j = (\mathcal{H}_Q)^{\perp} \) with respect to this torsion pair. Then \( T_{j+1} \) is a simple in \( \mathcal{H}_j \) and \( \mathcal{H}_{j+1} = (\mathcal{H}_j)^{\perp} \).

Proof. Similar to Lemma 6.6. \( \square \)

Combine the lemmas above, we have the following theorem.

Theorem 6.9. The HN-stratas in \( \operatorname{HN}(Q) \) are precisely the directed paths in \( \overrightarrow{P}(Q) \).

We will not distinguish \( \operatorname{HN}(Q) \) and \( \overrightarrow{P}(Q) \) from now on.

Corollary 6.10. For any shortest path \( p \) in \( \overrightarrow{P}(Q) \), the set of labels of its edges are precisely \( \operatorname{Sim} \mathcal{H}_Q \).

Proof. The HN-filtration of a simple in \( \mathcal{H}_Q \) (with respect to \( p \)) can only have one factor, i.e. itself. Hence any simple of \( \mathcal{H}_Q \) appears in an HN-stratum, and in particular, the labels of edges of \( p \). Thus the length of \( p \) is at least \( n \). By Lemma 6.2, the length of a shortest path \( p \) is exactly \( n \) and hence the corollary follows. \( \square \)

6.3. Slicing interpretation. We say a slicing \( S \) of \( \mathcal{D}(Q) \) is discrete if the abelian category \( S(\phi) \) is either zero or contains exactly one simple for any \( \phi \in \mathbb{R} \). We say a heart \( \mathcal{H} \) is in a slicing \( S \) if \( \mathcal{H} = S[\phi, \phi + 1) \) or \( \mathcal{H} = S(\phi, \phi + 1] \) for some \( \phi \in \mathbb{R} \). Let \( \operatorname{Sli}^*(\mathcal{D}(Q), \mathcal{H}) \) be the set of all discrete slicings of \( \mathcal{D}(Q) \) that contain \( \mathcal{H} \).

Definition 6.11. Let \( S_1 \) and \( S_2 \) in \( \operatorname{Sli}(\mathcal{D}) \). If there is a monotonic function \( \mathbb{R} \to \mathbb{R} \) such that \( S_1(\phi) = S_2(f(\phi)) \), then we say that the slicing \( S_1 \) is homotopic \((\sim)\) to \( S_2 \).

Now we can describe the relation between directed paths and slicings.

Proposition 6.12. There is a canonical bijection \( \operatorname{Sli}^*(\mathcal{D}(Q), \mathcal{H}_Q)/\sim \to \operatorname{HN}(Q) \).

Proof. Let \( S \in \operatorname{Sli}^*(\mathcal{D}(Q), \mathcal{H}_Q) \) and suppose \( \mathcal{H}_Q = S(I) \) for some interval \( I \) with \( |I| = 1 \). Then it induces an HN-stratum by taking the collection of objects which are simple in \( S(\phi) \) for \( \phi \in I \) with decreasing order. On the other hand, an HN stratum \([K_1, \ldots, K_1]_{HN} \) is induced by the slicing

\[
\{ P(m + \frac{j}{l}) = \langle K_j[m] \rangle \mid j = 1, \ldots, l, m \in \mathbb{Z}, \}.
\]

Hence we have a surjection \( \operatorname{Sli}^*(\mathcal{D}(Q), \mathcal{H}_Q) \to \operatorname{HN}(Q) \) while the condition that \( S_1 \) and \( S_2 \) maps to one HN-stratum is exactly the homotopy condition. \( \square \)
6.4. **Total stability.** Recall that we have the notion of a stability function on an abelian category (Definition 2.15). We call a stability function on $\mathcal{A}$ **totally stable** if every indecomposable is stable. Reineke made the following conjecture.

**Conjecture 6.13 ([39]).** Let $Q$ be a Dynkin quiver. There exists a totally stable stability function on $\mathcal{H}_Q$.

This was first proved by Hille-Juteau (unpublished, see the comments after [20, Corollary 1.7]).

We say a stability condition on a triangulated category is **totally stable** if any indecomposable is stable. Let $\sigma = (Z, P)$ be a totally stable stability condition. Then it will induce a totally stable stability function $Z$ on any abelian category $\mathcal{P}(I)$, for any half open half closed interval $I \subset \mathbb{R}$ with length 1; in particular, on its heart. On the other hand, a totally stable stability function on $\mathcal{H}_Q$ will induce a stability condition on $\mathcal{D}(Q)$, which is also totally stable.

Now we give explicit examples to prove the existence of the totally stable stability condition on $\mathcal{D}(Q)$, which is a slightly weak version of Conjecture 6.13 because orientation matters.

**Proposition 6.14.** Let $Q$ be a Dynkin quiver. There exists a totally stable stability condition on $\mathcal{D}(Q)$.

**Proof.** We treat the cases $A, D$ and $E$ separately.

For $A_n$-type, we use [38, Example A, Section 2]. Choose the orientation of $Q$ as

\[
\begin{array}{c}
n \rightarrow n-1 \rightarrow \cdots \rightarrow 1 \\
\end{array}
\]

Let the stability function $Z$ on $\mathcal{H}_Q$ be defined by $Z(S_j) = -j + i$. Then $Z$ induces a totally stable stability condition on $\mathcal{D}(Q)$.

For $D_n$-type, choose the orientation of $Q$ as

\[
\begin{array}{c}
n-2 \rightarrow n-3 \rightarrow \cdots \rightarrow 1 \rightarrow n-1 \\
\end{array}
\]

Let the stability function $Z$ on $\mathcal{H}_Q$ be defined by

\[
\left\{
\begin{array}{l}
Z(S_1) = \frac{n-3n}{2} + i, \\
Z(S_j) = -j + i, \quad j = 2, \ldots, n-2, \\
Z(S_{n-1}) = Z(S_n) = \frac{6+3n-n^2}{4} + i.
\end{array}
\right.
\]

Notice that the $\tau$-orbit of $S_{n-2}$ in $\Lambda(\mathcal{H}_Q)$ is

\[
P_{n-2} - M - S_2 - S_3 - \cdots - S_{n-2}
\]

with central charges

\[
i, -1 + i, -2 + i, -3 + i, \ldots, -(n-2) + i
\]

it is easy to check that $Z$ induces a totally stable stability condition on $\mathcal{D}(Q)$. 


For the exceptional cases, we use Keller’s quiver mutation program [23] to produce explicit examples of totally stable stability conditions for $E_{6,7,8}$. Choose the orientation of $E_{6,7,8}$ as follows

and we have following totally stable stability functions respectively:

\[
\begin{align*}
Z(S_1) &= 258 + 9i \\
Z(S_2) &= -53 + 32i \\
Z(S_3) &= -150 + 36i \\
Z(S_4) &= -75 + 33i \\
Z(S_5) &= -99 + 64i \\
Z(S_6) &= -101 + 10i
\end{align*}
\]

where $S_i$ is the simple corresponding to vertex $i$. Figure 5 is the AR-quiver of the $E_6$ quiver under such a totally stable function, where the bullets are simples or origins, and the stars are other indecomposables.

6.5. Inducing directed paths. We call a stability function *discrete*, if $\mu_Z$ is injective when restricted to the stable indecomposables.

**Proposition 6.15.** [28] Let $Z : \mathcal{K}(\mathcal{H}_Q) \to \mathbb{C}$ be a discrete stability function. Then the collection of its stable indecomposables in the order of decreasing phase is an HN-stratum of $\mathcal{H}_Q$.

We say that a directed path in $\overrightarrow{P}(Q)$ is *induced* if the corresponding HN-stratum is induced by some discrete stability function on $\mathcal{H}_Q$. Notice that, a totally stable stability function on $\mathcal{H}_Q$ induced a directed path $p_s$ in $\overrightarrow{P}(Q)$ such that there is an edge $M$ in $p_s$ for any $M \in \text{Ind} \mathcal{H}_Q$. By (6.4), we know that $p_s$ is the longest path in $\overrightarrow{P}(Q)$. Thus, in the language of exchange graphs, Reineke’s conjecture translates to, that there exists a longest path in $\overrightarrow{P}(Q)$ which is induced.

It is natural to make a very strong generalization of Reineke’s conjecture, that any path in $\overrightarrow{P}(Q)$ is induced. However, this is not true, even for some longest path as below.
We claim that the following longest path is not induced. Suppose not, that \( p \) is induced by some stability function \( Z \). The phase function \( \mu_Z \) is decreasing on the edges in \( p \) from left to right in (6.7). Then \( Z(I_3), Z(I_4), Z(M_3), Z(M_4) \) are in the parallelogram \( \Phi \) with vertices \( Z(I_2), Z(I_1), Z(M_2) \) and 0, as shown in Figure 6. Let \( Z_V \) be the intersection of the line passing through points \( Z(I_1), Z(M_3) \) and the line passing through points \( Z(M_4), 0 \). Notice that
\[
\mu_Z(P_3), \mu_Z(P_4) \in [0, \mu_Z(I_2)],
\]
we have
\[
\mu_{Z(P_3)\pi} = \arg(Z(M_4) - Z(I_2)) < \arg(Z(M_4) - Z_V) < \arg(Z(M_3) - Z(I_2)) = \mu_{Z(P_4)\pi},
\]
which is a contradiction.

This suggests another generalization of Reineke’s conjecture as follows. We say two directed paths in $\overrightarrow{P}(Q)$ are *weakly equivalent* if the unordered sets of their edges coincide.

**Conjecture 6.17.** There is an induced path in each weak equivalence class in $\overrightarrow{P}(Q)$.

Notice that by (6.4), all longest paths in $\overrightarrow{P}(Q)$ form a weak equivalent class $E$. Thus Reineke’s conjecture can be stated as: there is a path in the weak equivalence class $E$ which is induced.

7. **Quantum dilogarithms via exchange graph**

In this section, we define a DT-function on paths in exchange graphs, which provides another proof of Reineke’s identities (see Theorem 7.1) and the existence of DT-type invariants for a Dynkin quiver.

7.1. **DT-invariant for a Dynkin quiver.** Let $q^{1/2}$ be an indeterminate and $\mathbb{A}_Q$ be the quantum affine space
\[
\mathbb{Q}(q^{1/2})\{y^\alpha | \alpha \in \mathbb{N}_Q^0, y^\alpha y^\beta = q^{\frac{1}{2}(\langle \beta, \alpha \rangle - \langle \alpha, \beta \rangle)} y^{\alpha + \beta}\},
\]

Figure 6. The parallelogram $\mathfrak{P}$
where $\langle -, - \rangle$ is the Euler form associated to $Q$ (see Section 2.1). Denote $y^\dim M$ by $y^M$ for $M \in \mathcal{H}_Q$. Notice that $\mathbb{A}_Q$ can be also written as

$$Q(q^{1/2})|y^S | S \in \text{Sim} \mathcal{H}_Q | (y^S, y^{S_j} - q^{\lambda_Q(i,j)} y^{S_i} y^{S_j}),$$

where $\lambda_Q(i,j) = \langle S_j, S_i \rangle - \langle S_i, S_j \rangle$. Let $\hat{\mathbb{A}}_Q$ be the completion of $\mathbb{A}_Q$ with respect to the ideal generated by $y^S, S \in \text{Sim} \mathcal{H}_Q$.

The DT-invariant $DT(Q)$ of the quiver $Q$ can be calculated by the product (7.2) as follows.

**Theorem 7.1** (Reineke [38], c.f. [20]). For any HN-stratum $\varsigma = [K_1, \ldots, K_l]_{HN}$ in $\text{HN}(Q)$, the product

$$DT(Q; \varsigma) = \prod_{j=1}^l E(y^{K_j})$$

(7.2)

in $\mathbb{A}_Q$ is actually independent of $\varsigma$, where $E(X)$ is the quantum dilogarithm defined as the formal series

$$E(X) = \sum_{j=0}^{\infty} q^{j^2/2} X^j \prod_{k=0}^{j-1} (q^j - q^k).$$

In this subsection, we will review Reineke’s approach to Theorem 7.1, via identities in the Hall algebra (closely following [20]).

Let $k_0$ be a finite field with $q_0 = |k_0|$ and consider $\mathcal{H}_Q(k_0) = \text{mod} k_0 Q$. Recall that the completed (non twisted, opposite) Hall algebra $\hat{\mathcal{H}}_{k_0}(Q)$ consists of formal series with rational coefficients

$$\sum_{[M] \in \mathcal{H}_Q} a_m[M],$$

where the sum is over all isomorphism classes $[M]$ in $\mathcal{H}_Q$. The product in $\hat{\mathcal{H}}_{k_0}(Q)$ is given by the formula

$$[L][M] = \sum c^K_M(q_0)[K]$$

where $c^K_M(q_0)$ is the number of submodules $L'$ of $K$ such that $L' \cong L$ and $K/L' \cong M$ in $\mathcal{H}_Q(k_0)$. Then the HN-property of an HN-stratum $\varsigma = [K_1, \ldots, K_l]_{HN}$ translates into the identity (in Hall algebra) as

$$\sum_{[M] \in \mathcal{H}_Q} [M] = \prod_{j=1}^l \sum_{[M] \in (K_j)} [M]$$

(7.3)

Reineke showed that there is an algebra homomorphism (called integration)

$$\int : \hat{\mathcal{H}}_{k_0}(Q) \to \hat{\mathbb{A}}_{Q,q=q_0}$$

$$[M] \mapsto q^{\dim M \dim M} y^M \frac{y^M}{\text{Aut } M}.$$
Example 7.2. [20, Corollary 2.7] By the proof of Lemma 6.2, we know that \( \prod_{S \in \text{Sim} \mathcal{H}} S \) is a shortest path in \( \overrightarrow{P}(Q) \), where the product is with respect to the increasing order of the position function (if two objects have the same position function, then their order does not matter). Moreover, by direct checking, we know that \( \prod_{M \in \text{Ind} \mathcal{H}} M \) is a longest path in \( \overrightarrow{P}(Q) \) consisting of APR tiltings, where the product is with respect to the decreasing order of the position function. Then these two paths (or the corresponding HN-strata) give the equality

\[
\prod_{M \in \text{Ind} \mathcal{H}} E(y^M) = \prod_{S \in \text{Sim} \mathcal{H}} E(y^S). \tag{7.4}
\]

7.2. Generalized DT-invariants for a Dynkin quiver. We will give a combinatorial proof of Theorem 7.1, which provides a slightly stronger statement.

Let \( p = \prod_{j=1}^{l} K_j^{\varepsilon_j} : \mathcal{H} \to \mathcal{H}' \) be a path (not necessarily directed) in \( \text{EG}(Q; \mathcal{H}_Q, \mathcal{H}_Q[1]) \), where \( K_i \) are edges in \( \text{EG}(Q) \) and the sign \( \varepsilon_j = \pm 1 \) indicates the direction of \( K_j \) in \( p \). Define the DT-function of \( p \) to be

\[
\text{DT}(Q; p) = \prod_{j=1}^{l} E(y^{K_j})^{\varepsilon_j}.
\]

Since we identify HN-strata with directed paths in Theorem 6.9, thus Theorem 7.1 can be rephrased as: the quantum dilogarithm of a directed path connecting \( \mathcal{H}_Q \) and \( \mathcal{H}_Q[1] \) is independent of the choice of the path. It is natural to ask if the path-independence holds for more general paths (not necessary directed). The answer is yes within the subgraph \( \text{EG}(Q; \mathcal{H}_Q, \mathcal{H}_Q[1]) \).

Theorem 7.3. If \( p \) is a path in \( \text{EG}(Q; \mathcal{H}_Q, \mathcal{H}_Q[1]) \), then \( \text{DT}(Q; p) \) only depends on the head \( \mathcal{H} \) and tail \( \mathcal{H}' \) of \( p \).

Proof. We give a combinatorial proof. By Proposition 3.5, \( \pi_1(\text{EG}(Q; \mathcal{H}_Q, \mathcal{H}_Q[1])) \) is generated by the squares and pentagons as in (3.5). Thus we only need to check these two cases for the path-independence.

Notice that in the square or pentagon, we have \( \text{Hom}(S_i, S_j) = \text{Hom}(S_j, S_i) = 0 \) and \( S_i, S_j \in \mathcal{H}_Q \). In the square case we have

\[
\text{Hom}^1(S_i, S_j) = \text{Hom}^1(S_j, S_i) = 0
\]

and hence \( \langle \dim S_i, \dim S_j \rangle = \langle \dim S_j, \dim S_i \rangle = 0 \) by (2.2), which implies

\[
y^{S_i} \cdot y^{S_j} = y^{S_j} \cdot y^{S_i}
\]

and

\[
E(y^{S_i}) \cdot E(y^{S_j}) = E(y^{S_j}) \cdot E(y^{S_i}) \tag{7.5}
\]

as required. In the pentagon case we have a triangle \( S_i \to T_j \to S_j \to S_i[1] \) and \( \dim S_i + \dim S_j = \dim T_j \). Then

\[
\text{Hom}^1(S_i, S_j) = 0, \quad \dim \text{Hom}^1(S_j, S_i) = 1
\]
and hence $\langle \dim S_i, \dim S_j \rangle = 0$ and $\langle \dim S_j, \dim S_i \rangle = -1$ by (2.2). By the relations of the quantum affine space we have

$$y^{S_i} \cdot y^{S_j} = q \cdot y^{S_j} \cdot y^{S_i},$$

$$y^{T_j} = q^{-\frac{1}{2}} \cdot y^{S_i} \cdot y^{S_j}.$$  

By the Pentagon Identity (see for example [20, Theorem 1.2]) we have

$$\mathbb{E}(y^{S_i}) \cdot \mathbb{E}(y^{S_j}) = \mathbb{E}(y^{S_j}) \cdot \mathbb{E}(y^{S_i} \cdot y^{T_j})$$  

(7.6) as required.

Therefore for any two heart $\mathcal{H}, \mathcal{H}'$ in $\text{EG}(Q; H, H[1])$, we have a \textit{generalized DT-invariant}

$$\text{DT}(Q; \mathcal{H}, \mathcal{H}') := \text{DT}(Q; p)$$  

(7.7)

where $p$ is any path connecting $\mathcal{H}$ and $\mathcal{H}'$. In particular, we have

$$\text{DT}(Q) = \text{DT}(Q; \mathcal{H}_Q, \mathcal{H}_Q[1]).$$  

(7.8)

7.3. \textbf{Wall crossing formula for APR-tilting}. Let $i$ be a sink in $Q$ and Sim $\mathcal{H}_Q = \{S_j\}_{j=1}^n$. Then the APR-tilt $\mathcal{H}_Q' = (\mathcal{H}_Q)'$ is also a standard hearts in $D(Q)$, where $Q'$ is obtained from $Q$ by reversing the arrow at $i$. By [35, Proposition 5.5], we have Sim $\mathcal{H}_Q' = \{T_j\}_{j=1}^n$, where $T_i = S_i[1]$,

$$T_j = \text{Cone}(S_j \to S_i[1] \otimes \text{Ext}^1(S_j, S_i^*)[-1]$$

for $j \neq i$. Let $\dim'$ and $\langle -, - \rangle'$ be the dimension vector and the Euler form, respectively, associated to $Q'$. Consider the quantum affine space $\mathfrak{h}_Q'$

$$\mathbb{Q}(q^{1/2}) \langle z^T | T \in \text{Sim} \mathcal{H}_Q' \rangle = \langle z^{T_i} z^{T_j} = q^{\lambda_Q(i,j)} z^{T_i} z^{T_j} \rangle$$

where $z^S = z'^{\dim'}$ and $\lambda_Q(i,j) = \langle T_j, T_i \rangle' - \langle T_i, T_j \rangle'$. By Theorem 7.3, we can also define DT-invariants $\text{DT}(Q'; \mathcal{H}_1, \mathcal{H}_2)$ in $\mathfrak{h}_Q'$ for any $\mathcal{H}_1, \mathcal{H}_2 \in \text{EG}(Q; \mathcal{H}_Q', \mathcal{H}_Q'[1])$. Notice that the labels of edges in $\text{EG}(Q; \mathcal{H}_Q', \mathcal{H}_Q[1])$ are in

$$\text{Ind}(\mathcal{H}_Q \cap \mathcal{H}_Q') \cap \text{Ind} \mathcal{H}_Q - \{S_i\} = \text{Ind} \mathcal{H}_Q' - \{S_i[1]\}.$$  

It is straightforward to check that the following conditions are equivalent

1°. for any hearts $\mathcal{H}_1, \mathcal{H}_2 \in \text{EG}(Q; \mathcal{H}_Q, \mathcal{H}_Q[1]),$

$$\text{DT}(Q; \mathcal{H}_1, \mathcal{H}_2) = \text{DT}(Q'; \mathcal{H}_1, \mathcal{H}_2),$$

2°. we have $z^{T_i} = (y^{S_i})^{-1}$ and $z^M = y^M$ for any $M \in \text{Ind}(\mathcal{H}_Q \cap \mathcal{H}_Q')$.

3°. we have $z^{T_j} = (y^{S_j})^{-1}$ and $z^{T_j} = y^{T_j}$ for $j \neq i$.

4°. we have $z^{T_i} = (y^{S_i})^{-1}$ and $z^{S_j} = y^{S_j}$ for $j \neq i$.

Further, if the conditions above hold, the \textit{wall crossing formula}

$$\text{DT}(Q) \cdot \mathbb{E}(y^{S_i})^{-1} = \mathbb{E}(y^{S_i})^{-1} \cdot \text{DT}(Q')$$  

(7.9)

comes for free because both sides equal to $\text{DT}(Q; \mathcal{H}_Q, \mathcal{H}_Q[1]).$
Remark 7.4. One can rephrase Keller’s green mutation formula (to calculate DT-invariants for quivers with potential) as DT-functions on the corresponding exchange graphs in the same way, while wall-crossing formula comes for free. In fact, exchange graphs are simplified (homological) version of Keller’s cluster groupoids in [20], c.f. [35].

Appendix A. Connectedness of $\mathcal{D}(Q)$

We give two proofs of the connectedness of the exchange graph for $\mathcal{D}(Q)$, which was a result of Keller-Vossieck [25].

We say an indecomposable object $L$ in a subcategory $C \subset \mathcal{D}(Q)$ is leftmost if there is no path from any other indecomposable in $C$ to $L$, or equivalently that no predecessor of $L$ is in $C$. In particular, a leftmost object in a heart is simple. If in a simple forward tilting, the simple object is leftmost, we call it a $L$-tilting. Similarly, an indecomposable object $R$ is rightmost if there is no path from any other indecomposable to $L$.

Lemma A.1. Let $S$ be leftmost in $\mathcal{H}$ and $\mathcal{H}^2 = \mathcal{H}^2_S$. We have

1. $(\text{Ind } \mathcal{H} \setminus \{S\}) \subset \mathcal{H}^2$.
2. Follow the notation of [35, Proposition 3.3]. If $m > 1$, then $H^F_m = 0$.
3. For any $M \in \text{Ind } \mathcal{D}(Q)$, $\text{Wid}_{\mathcal{H}^2} M \leq \text{Wid}_H M$.

Proof. Since $S$ is a leftmost object, then $\text{Ind } \mathcal{F} = \{S\}$ and $\mathcal{F} = \{S^i \mid i \in \mathbb{Z}^+\}$. For any indecomposable in $\mathcal{H}$ other than $S$, we have $\text{Hom}(M,S) = 0$ which implies $(\text{Ind } \mathcal{H} \setminus \{S\}) \subset \mathcal{T} \subset \mathcal{H}^2$.

For 2, suppose $H^F_m = S^j \neq 0$, then $M[-k_m]$ is the predecessor of $S$. Consider an indecomposable summand $L$ of $H_1$. If $L = S$, then $S[k_1]$ is the predecessor of $M$. Since $k_1 > k_m$, $S$ is the predecessor of $S[k_1 - k_m]$, hence the predecessor of $M[-k_m]$. Then $M$ and $S$ are predecessors to each other which is a contradiction. If $L \neq S$, then $L \in \mathcal{T}$. $L$ is the predecessor of $M[-k_1]$, hence the predecessor of $M[-k_m]$. Then $L$ is the predecessor of $S$ which is also a contradiction.

For 3, if $\text{Wid}_H M > 0$, then $m > 1$. By 2, $H^F_m = 0$. Then by the filtration (3.2) of [35], $\text{Wid}_{\mathcal{H}^2} M \leq k_1 - k_m = \text{Wid}_H M$. If $\text{Wid}_H M = 0$, or equivalently $m = 1$, then by the filtration (3.2) of [35] again, $\text{Wid}_{\mathcal{H}^2} M = 0 = \text{Wid}_H M$. \qed

By the same argument in the proof of Lemma A.1 2, we know that an object $S$ is a leftmost object in a heart $\mathcal{H}$ in $\mathcal{D}(Q)$, if and only if $S$ is a leftmost object in the corresponding t-structure $\mathcal{P}$.

Corollary A.2. For a $L$-tilting with respect to a leftmost object $S$, we have $\text{Ind } \mathcal{P}^S = \text{Ind } \mathcal{P} - \{S\}$.

Proof. Consider the filtration (3.1) of [35], we have $M \notin \mathcal{P}$ if and only if $k_m < 0$. If so, since $H^F_m$ or $H^T_m$ is not 0 in the filtration (3.1) of [35], then $M \notin \mathcal{P}^S$. Thus $\text{Ind } \mathcal{P}^S \subset \text{Ind } \mathcal{P}$. On the other hand, $M \in \text{Ind } \mathcal{P} - \text{Ind } \mathcal{P}^S$ if and only if $H^F_m \neq 0$ and $k_m = 0$. In which case, $m = 1$ by Lemma A.1, and hence $M = S$. \qed

Lemma A.3. For any object $M \in \text{Ind } \mathcal{D}(Q)$, if $\text{Wid}_H M > 0$, then applying any sequence of $L$-tiltings to $\mathcal{H}$ must reduce the width of $M$ after finitely many steps.
**Proof.** Suppose not, let $\text{Wid}_H M > 0$ is the minimal width of $M$ under any $L$-tilting. We have $m > 1$ in filtration (3.1) of $[35]$. Then $H^F_m = 0$ by Lemma A.1. In the filtration (3.2) of $[35]$, if $H^T_1$ vanishes, then $\text{Wid}_H M \leq (k_t - k_m)$. But $\text{Wid}_H M$ is minimal, thus $H^T_1 \neq 0$ after any $L$-tilting.

Consider the size of $H^T_1$. Let $H^T_1 = \bigoplus_{j=1}^{l} T^s_j$, where $T_j$ are different indecomposables in $T$ and $l$ is a positive integer. By the filtration (3.2) of $[35]$ we know $H^T_1$ will not change if we do $L$-tilting that is not with respect to any $T_j$. And if we do $L$-tilting with respect to some $T_j$, then $H^T_1$ will lose the summand $T_j$. Since $H^T_1$ cannot vanish, we can assume after many $L$-tilting, $l$ reaches the minimum, and we can not do $L$-tilting that is with respect to any $T_i$.

On the other hand, for any object $M \in \text{Ind} \mathcal{D}(Q)$, while $M[m]$ is the successor of some $T_j$ when $m$ is large enough, it can not be leftmost in any heart that contains $T_j$. Besides we can only do $L$-tilting with respect to any object once. Thus, we will eventually have to tilt $T_i$, which will reduce $l$ and it is a contradiction. □

Now we have a proposition about how one can do $L$-tilting.

**Proposition A.4.** Applying any sequence of $L$-tiltings to any heart, will make it standard after finitely many steps.

**Proof.** By Lemma A.3, the width of any particular indecomposable will become zero after finitely steps in the sequence. But, up to shift, there are only finitely many indecomposables in $\text{Ind} \mathcal{D}(Q)$. Thus, after finitely steps, we must reach a heart with respect to which all indecomposables have width zero and which is therefore standard, by Proposition 2.5. □

Now we can prove the connectedness:

**Theorem A.5** (Keller-Vossieck [25]). $\text{EG}(Q)$ is connected.

**Proof.** Since $t$-structure and heart are one-one correspondent, any heart is connected to a standard heart by Proposition A.4. One the other hand, using the equivalent definition $3^0$ in Proposition 2.5 for ‘standard’, the set of all standard hearts is connected by APR-tilting (c.f.[2, page 201]). So the theorem follows. □

**Appendix B. Stability Space of $\mathcal{D}(A_2)$**

Let $Q$ be the quiver of type $A_2$ with orientation $2 \rightarrow 1$ and $\text{Ind} \mathcal{H}_Q = \{C_1, C_2, C_3\}$ such that $\text{Ext}^1(C_3, C_1) \neq 0$. Write $C_{3n+1}$ for $C_i[m]$. We have $\text{Aut} \mathcal{D}(A_2) \cong \mathbb{Z}(\xi)$, where the generator $\xi = \tau \circ [1]$ satisfying $\xi(C_j) = C_{j+1}$ and $\xi^3 = [1]$.

**Lemma B.1.** Let $\sigma = (Z, P)$ be a stability condition in $\mathcal{D}(A_2)$. There exists an element $\xi \in \text{Aut} \mathcal{D}(A_2)$ and an nonnegative integer $m$ such that the simples in the heart of $\xi \circ \sigma$ are $C_1$ and $C_3[m]$. In particular, there are three types of stability conditions on $\mathcal{D}(A_2)$:

- Every indecomposable object is stable.
- Up to shift, two indecomposables are stable and one is semistable (but not stable).
- Up to shift, two indecomposables are stable and one is not semistable.
Proof. Notice that $\text{EG}(Q)$ is connected. By [35, Proposition 5.5], we know the changes of simple during tilting. Then the first assertion follows by direct calculating. By comparing the phases of $C_1$ and $C_3$ with respect to the stability condition $\xi \circ \sigma$, we get the three cases. □

Let

\[
\begin{align*}
\tilde{U} &= \{(Z, P) \in \text{Stab}(A_2) \mid C_j \text{ are stable for } j = 1, 2, 3\}, \\
\tilde{W}_j &= \{(Z, P) \in \text{Stab}(A_2) \mid C_j \text{ is not semistable}\}, \quad j = 1, 2, 3.
\end{align*}
\]

A straightforward calculation shows that

\[
\begin{align*}
\partial \tilde{W}_j &= \{(Z, P) \in \text{Stab}(A_2) \mid C_j \text{ is semistable but not stable}\}, \\
\partial \tilde{U} &= \partial \tilde{W}_1 \cup \partial \tilde{W}_2 \cup \partial \tilde{W}_3, \\
\text{Stab}(A_2) &= \tilde{U} \cup \partial \tilde{U} \cup \tilde{W}_1 \cup \tilde{W}_2 \cup \tilde{W}_3.
\end{align*}
\]

Notice that the intersection of $C$-actions and $\text{Aut} \mathcal{D}(A_2)$ is $\mathbb{Z}$ with generator $-1 \in \mathbb{C}$ or $[1] \in \text{Aut} \mathcal{D}(A_2)$. Therefore we have a commutative diagram:

\[
\begin{array}{ccc}
\text{Stab}(A_2) & \xrightarrow{/\text{Aut}} & \mathcal{M}_A \\
/\mathbb{C} & /\mathbb{C} & /\mathbb{C} \\
\mathcal{M}_C & \xleftarrow{/\mathbb{C}^*} & \mathcal{M}
\end{array}
\]

where $\mathbb{C}_3 = \text{Aut} / \mathbb{Z}[1], \mathbb{C}^* = \mathbb{C} / \mathbb{Z}$ and $\mathcal{M} = \text{Aut} \mathcal{D}(A_2) \backslash \text{Stab}(A_2) / \mathbb{C}$. Let $U, W_j \subset \mathcal{M}_C$ be the quotient spaces of $\tilde{U}$ and $\tilde{W}_j$ in $\mathcal{M}_C$ respectively. We have a conformal isomorphism $f : R \rightarrow W_2 \cup U$ (see Figure 7), where

\[
R = \{\Theta = x + yi \mid x < 1\} \subset \mathbb{C}
\]

such that $f(\Theta) = [\sigma]$ in $\mathcal{M}_C$ and the stability condition $\sigma = (Z, P)$ is determined by the following conditions:

- $Z(C_1) = 1$ and $Z(C_3) = \exp(\pi i \Theta)$;
- The simples in the heart of $\sigma$ are $C_1$ and $C_3[m]$, where $m = -|\text{Im } \Theta|$.

Let $V = f^{-1}(U)$ and $V_2 = f^{-1}(W_2)$. Denote $T$ the triangle with vertices $T_1 = 1, T_2 = 0$ and $T_3 = -Z(C_3)$. The $C_3$-action on $U$ will identify the stability conditions whose corresponding triangles $T$ are similar to each other. The red lines $l_i$ in Figure 7 correspond to the case when $T$ is an isosceles triangle (with vertex angle at $T_i$), where

\[
\begin{align*}
l_1 &= \{\Theta = x + yi \mid x \in \left(\frac{1}{2}, \frac{2}{3}\right), y \pi = -\ln(-2 \cos x \pi)\}, \\
l_2 &= \{\Theta = x + yi \mid y = 0, x \in \left[\frac{2}{3}, 1\right)\}, \\
l_3 &= \{\Theta = x + yi \mid x \in \left(\frac{2}{3}, \frac{2}{1}\right), y \pi = \ln(-2 \cos x \pi)\}.
\end{align*}
\]

Moreover let $\omega_0 : \mathcal{M}_C \rightarrow \mathcal{M}_C$ be the conformal map with order 3 corresponding to the $C_3$-action and sending $W_j$ to $W_{j+1}$. Also denote by $\omega_0$, the induced $C_3$-action on $V$. Denote $\mathcal{M}_2'$ the region strictly right bounded by $l_1 \cup l_3$ in Figure 7. Then $\mathcal{M}_2 = f(\mathcal{M}_2')$ is a fundamental domain for the quotient map $\mathcal{M}_C \rightarrow \mathcal{M}$.
Figure 7. The conformal isomorphism $f : R \rightarrow W_2 \cup U$
Lemma B.2. \( M \) can be obtained from \( \overline{M_2} \) by identifying the points on the boundary \( l_1 \cup l_3 \) with respect to the reflection of \( x \)-axis. Moreover, \( z_0 = l_1 \cap l_2 \cap l_3 = \frac{1}{2} \) is the only orbitifold point in \( \partial M \), which is with order \( \frac{1}{3} \).

Proof. The lemma follows from the facts that \( \omega_0(l_j) = l_{j+1} \) and \( l_1 \cap l_2 \cap l_3 = \{0\} \). \( \square \)

Let \( M_3 = \omega_0(M_2) \) and \( M_1 = \omega_0(M_3) \). By Lemma B.2, we have

\[
\mathcal{M}_C = \overline{M_1} \cup \overline{M_2} \cup \overline{M_3} \quad \text{and} \quad \overline{M_j} \cap \overline{M_j} = f(l_{j+1}).
\]

Lemma B.3. We have a conformal isomorphism \( g : \mathcal{M}_C \xrightarrow{\sim} \mathbb{C} \).

Proof. Let \( l(j) = \{z \in \mathbb{C} \mid \arg z = \frac{2\pi j}{3}\} \). Using Riemann mapping theorem and Reflection Principle (as in [33, Lemma 4.4]), we have a map \( g_2' \) sending \( \overline{M_2} \) conformally isomorphic to

\[
\mathcal{M}(2) = \{z \in \mathbb{C} \mid \arg z \in \left[\frac{2\pi}{3}, \frac{4\pi}{3}\right]\},
\]

such that \( g_2'(z) = g_2'(z) \). Let \( g_2 = g_2' \circ f^{-1} \), then we have \( g_2 : \mathcal{M}_2 \xrightarrow{\sim} \mathcal{M}(2) \). Define \( \omega : \mathbb{C} \to \mathbb{C} \) by \( \omega(z) = z \cdot \exp(\frac{2\pi i}{3}) \) and let

\[
\mathcal{M}(3) = \omega(\mathcal{M}(2)), \quad \mathcal{M}(1) = \omega(\mathcal{M}(3)).
\]

Then we have two conformal isomorphisms

\[
\begin{align*}
g_1(z) &= \omega^{-1} \circ g_2 \circ \omega : \mathcal{M}_1 \xrightarrow{\sim} \mathcal{M}(1), \\
g_3(z) &= \omega \circ g_2 \circ \omega^{-1} : \mathcal{M}_3 \xrightarrow{\sim} \mathcal{M}(3).
\end{align*}
\]

By [10, Theorem 11-8], we can conformally extend \( g_j \) to the smooth boundary

\[
f(l_{j-1} \cup l_{j+1} - \{z_0\})
\]

such that \( g \circ f(l_j \pm 1) = l(j \pm 1) \). Notice that the extended maps \( g_1, g_2 \) and \( g_3 \) agree on

\[
f(l_1 \cup l_2 \cup l_3 - \{z_0\})
\]

by a direct calculation, thus we obtain a conformally isomorphism

\[
g : \mathcal{M}_C - \{f(z_0)\} \to \mathbb{C} - \{0\}.
\]

Then by [10, Theorem 11-8] again, we can conformally extend \( g \) to the boundary \( \{f(z_0)\} \) which implies the lemma. \( \square \)

Theorem B.4. \( \text{Stab}(A_2) \) is isomorphic to \( \mathbb{C}^2 \) as complex manifold.

Proof. The theorem follows from \( \text{Stab}(A_2) / \mathbb{C} \simeq \mathcal{M}_C \simeq \mathbb{C} \) and \( H^1(\mathbb{C}, \mathcal{O}) = 0 \). \( \square \)

Appendix C. Stability Space of Calabi-Yau-N A2-type

C.1. Autequivalences and the universal cover. Let \( S_1, S_2 \) be the simples in the standard heart \( \mathcal{H}_1 \) in \( \mathcal{D}(\Gamma_N A_2) \) such that \( \text{Ext}^1(S_1, S_2) \neq 0 \). Then the braid group \( \text{Br}(\Gamma_N A_2) \cong \text{Br}_3 \) has a set of generators \( \phi_{S_1}, \phi_{S_2} \). By [9], \( \xi^3 = [3N - 4] \) generates the center of \( \text{Br}_3 \), where \( \xi = \phi_{S_2}^{-1} \circ \phi_{S_1}^{-1} \). Let \( \text{Aut}_0(\Gamma_N A_2) \) be the subgroup of \( \text{Aut} \mathcal{D}(\Gamma_N A_2) \)
which is generated by $\phi_{S_1}, \phi_{S_2}$ and $[1]$. By [35, Proposition 8.8], we have the following commutative diagram of short exact sequences

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}[3N - 4] & \longrightarrow & \text{Br}(\Gamma_N A_2) & \longrightarrow & P_2 & \longrightarrow & 0 \\
\phantom{0} & \phantom{\longrightarrow} & \phantom{\mathbb{Z}[3N - 4]} & \phantom{\longrightarrow} & \phantom{\text{Br}(\Gamma_N A_2)} & \phantom{\longrightarrow} & \phantom{P_2} & \phantom{\longrightarrow} & \phantom{0} \\
0 & \longrightarrow & \mathbb{Z}[1] & \longrightarrow & \text{Aut}_0(\Gamma_N A_2) & \longrightarrow & P_2 & \longrightarrow & 0 \\
\end{array}
\]

where $\text{Br}(\Gamma_N A_2) = \text{Br}_3$, $P_2 = \text{PSL}_2(\mathbb{Z})$, and hence $\text{Aut}_0(\Gamma_N A_2) \cong \text{Br}_3$. Therefore we have the following commutative diagram:

Moreover, let $\Delta = \{\alpha_1, \alpha_2, \alpha_3 \mid \alpha_1 = \alpha_2, \alpha_2 = \alpha_3 \text{ or } \alpha_3 = \alpha_1\}$, $W = S_3$ and $\Delta_0 = \Delta/W$. We have (c.f. [7])

\[
b^{reg} = \left\{ f(x) = \prod_{j=1}^{3}(x - \alpha_j) \mid \sum_{j=1}^{3} \alpha_j = 0, \alpha_j \in \mathbb{C} \right\} \backslash \Delta \quad (C.3)
\]

Write $\Omega = b^{reg}/W$ and denote by $C^U$ the universal cover of $\Omega$. Thus we have the following commutative diagram:

where $J = H/P_2$ is the $j$-line. Recall that $H$ is the upper half plane in $\mathbb{C}$ and the $j$-line is an orbifold surface with two orbifold points (of orders 2 and 3).

If $N = 2$, we can identify (see [8]) (C.4) with the right square of (C.2). We will show that this identification works for $N > 2$ in the following subsection.
C.2. Deformations. Let $N \geq 2$. Let $\mathcal{N}|_l$ be the area right bounded by $l_1 \cup l_3$ and left bounded by $b_l = \{x = -t\}$ (see Figure 8). We have the following lemma.

**Lemma C.1.** The orbitifold $\mathcal{L}^N$ can be obtained from $\mathcal{N}|_{(N-2)/2}$ by gluing its boundary $l_1 \cup l_3 \cup b_{(N-2)/2}$ with respect to the reflection of $x$-axis.

**Proof.** Recall that $\Sim \mathcal{H}_T = \{S_1, S_2\}$ with $\Ext^1(S_2, S_1) \neq 0$. By Lemma 5.1, we have a conformal map

$$\alpha : V \cup \mathcal{N}|_{(N-2)/2} \to \mathcal{L}^N$$

sending $\Theta$ to $[\sigma]$, where the stability condition $\sigma = (Z, \mathcal{P})$ is determined by the following conditions

- $Z(S_1) = 1$ and $Z(S_2) = \exp(i\pi \Theta)$;
- The simples in the heart of $\sigma$ are $S_1$ and $S_3[m]$, where $m = -\lfloor \Im \Theta \rfloor$.

The surjectivity of $\alpha$ follows by Theorem 2.12. To complete the proof, it is essential to show that for $\Theta_1 \neq \Theta_2 \in V \cup \mathcal{N}|_{(N-2)/2}$ satisfying $\alpha(\Theta_1) = \alpha(\Theta_2)$, we have

1°. either $\Theta_1, \Theta_2 \in b_{(N-2)/2}$ such that $\Theta_1 + \Theta_2 = 2 - N$.

2°. or $\Theta_1, \Theta_2 \in V$ such that $\omega_i^b(\Theta_1) = \Theta_2$, where $k \in \{\pm 1\}$ and $\omega_0$ is the $C_3$-action on $V$ sending $l_i$ to $l_{i+1}$.

Let $\sigma_1$ and $\sigma_2$ be the corresponding stability conditions. Notice that for any $\sigma$ in the orbit of $\alpha(z)$, if $z \in \mathcal{N}|_0$, then there are three (up to shift) indecomposables are semistable; otherwise there are two. Therefore $\alpha(\Theta_1) = \alpha(\Theta_2)$ implies $\Theta_1$ are both in $\mathcal{N}|_0$ or neither.

Suppose that $\Theta_1, \Theta_2 \in \mathcal{N}|_{(N-2)/2} - V$. Notice that the two stable objects (up to shift) are $S_1$ and $S_2$. Consider the central charges and phases of them with respect to $\sigma_i$. Since $\Ext^1(S_2, S_1) = \Ext^1(S_1, S_2[N - 2])$, either we have

$$\begin{cases}
Z_1(S_1) = Z_2(S_1) = Z_2(S_2), \\
\varphi_1(S_1) - \varphi_1(S_2) = \varphi_2(S_1) - \varphi_2(S_2).
\end{cases}
$$

(C.5)

or

$$\begin{cases}
Z_1(S_1) = \frac{Z_2(S_1[N - 2])}{Z_2(S_1)}, \\
\varphi_1(S_1) - \varphi_1(S_2) = \varphi_2(S_2[N - 2]) - \varphi_2(S_1).
\end{cases}
$$

(C.6)

where $\phi_i$ is the phase function with respect to $\sigma_i$, for $i = 1, 2$. Equation (C.5) implies $\sigma_1 = \sigma_2$ which is a contradiction. Hence equation (C.6) holds, which implies $\Theta_1 + \Theta_2 = 2 - N$ as required in 1°.

Now let $\Theta_1, \Theta_2 \in U$. Then up to shift, there are three semistable objects $S_1, S_2$ and $\phi_{S_3}(S_1)$. Consider their central charges and we know that the triangles $T_1$ and $T_2$ are similar, where $T_i$ has vertices $0, Z_i(S_1)$ and $Z_i(S_2))$. This condition exactly means that $\sigma_i$ differs by a $C_3$-action (c.f. Section 5) as required in 2°.

**Lemma C.2.** We have a conformal isomorphism $\mathcal{L}^N \cong \mathcal{L}^2$ for any $N > 1$.

**Proof.** Let $X(N) = g(\mathcal{N}|_{(N-2)/2})$ where $g$ is the map in Lemma B.3. We only need to prove that $X(N)$ is conformally isomorphic to $X(2)$. Consider $Y(N) = \bigcup_{j=1}^{3} \omega^j X(N)$. By Riemann mapping theorem, there is a conformal isomorphism $h : Y(N) \to Y(2)$ such that $h(0) = 0$ and $h'(0) = 1$ (see Figure 9). Let $h_j = \omega^{-j} \circ h \circ \omega^j$ for any $j \in \mathbb{Z}$. Since
Figure 8. $j$-line for $\text{Stab}^0(\Gamma_N A_2)/\text{Aut}_0(\Gamma_N A_2)$

Figure 9. Deformation
\( h_j(0) = 0 \) and \( h_j'(0) = 1 \), we have \( h \equiv h_j \) by the uniqueness of Riemann mapping theorem. Notice that \( Y(N) \) and \( Y(2) \) are symmetry with respect to \( x \)-axis by construction, hence \( h(l(0)) = l(0) \) by Reflection Principle. Then \( h = h_j \) implies \( h(l(j)) = l(j) \) for any \( j \in \mathbb{Z} \) and hence \( h \mid_{X(N)} : X(N) \to X(2) \) is a conformal isomorphism as required.

**Theorem C.3.** We have \( \text{Stab}^\sigma(\Gamma_N A_2) \cong \text{Stab}^\sigma(\Gamma_2 A_2) \cong C^U \) as complex manifold.

**Proof.** By Lemma C.2, we have \( L^N \cong L^2 \cong J \). Since the \( \mathbb{C}^*_A \)-bundle \( L^N_A \) is the principal bundle over \( L_N \), we have \( L^N_A \cong L^2_A \cong \Omega \). Finally, we have \( \text{Aut}_0(\Gamma_N A_2) \cong \text{Aut}_0(\Gamma_2 A_2) \cong \text{Br}_3 \). Hence \( \text{Stab}^\sigma(\Gamma_N A_2) \) and \( \text{Stab}^\sigma(\Gamma_2 A_2) \) are both the universal cover of \( \Omega \) which implies the assertion.

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