POLYNOMIAL INEQUALITIES ON THE $\pi/4$–CIRCLE SECTOR

G. ARAÚJO*, P. JIMÉNEZ-RODRÍGUEZ**, G. A. MUÑOZ-FERNÁNDEZ***, AND J. B. SEOANE-SEPÚLVEDA**

Abstract. A number of sharp inequalities are proved for the space $\mathcal{P}(\mathcal{D}(\frac{\pi}{4}))$ of 2-homogeneous polynomials on $\mathbb{R}^2$ endowed with the supremum norm on the sector $\mathcal{D}(\frac{\pi}{4}) := \{e^{\theta} : \theta \in [0, \frac{\pi}{4}]\}$. Among the main results we can find sharp Bernstein and Markov inequalities and the calculation of the polarization constant and the unconditional constant of the canonical basis of the space $\mathcal{P}(\mathcal{D}(\frac{\pi}{4}))$.

1. Preliminaries

The study of low dimensional spaces of polynomials can be an interesting source of examples and counterexamples related to more general questions. In this paper we mind 2-variable, real 2-homogeneous polynomials endowed with the supremum norm on the sector $\mathcal{D}(\frac{\pi}{4}) := \{e^{\theta} : \theta \in [0, \frac{\pi}{4}]\}$. The space of such polynomials is represented by $\mathcal{P}(\mathcal{D}(\frac{\pi}{4}))$. This paper can be seen as a continuation of [15] and [20].

Other publications in the same spirit can be found in [11, 12, 21, 22, 24, 25].

In order to obtain sharp polynomial inequalities in $\mathcal{P}(\mathcal{D}(\frac{\pi}{4}))$ we will use the Krein-Milman approach, which is based on the fact that norm attaining convex functions attain their norm at an extreme point of their domain. Hence, an explicit description of the norm $\|\cdot\|_{\mathcal{D}(\frac{\pi}{4})}$ and the extreme points of the unit ball $B_{\mathcal{D}(\frac{\pi}{4})}$, denoted by $\text{ext}\left(B_{\mathcal{D}(\frac{\pi}{4})}\right)$, will be required. Both are presented below:

Lemma 1.1. [20] Theorem 3.1 If $P(x,y) = ax^2 + by^2 + cxy$, then

$$\|P\|_{\mathcal{D}(\frac{\pi}{4})} = \max \left\{ |a|, \frac{1}{2} |a + b + c|, \frac{1}{2} a + b + \text{sign}(c) \sqrt{(a-b)^2 + c^2} \right\}$$

if $c(a-b) \geq 0$,

$$\max \left\{ |a|, \frac{1}{2} |a + b + c| \right\}$$

if $c(a-b) \leq 0$,

Lemma 1.2. [20] Theorem 4.4 The extreme points of the unit ball of $\mathcal{P}(\mathcal{D}(\frac{\pi}{4}))$ are given by

$$\text{ext}\left(B_{\mathcal{D}(\frac{\pi}{4})}\right) = \left\{ \pm P_t, \pm Q_s, \pm (1,1,0) : -1 \leq t \leq 1 \text{ and } 1 \leq s \leq 5 + 4\sqrt{2} \right\},$$

where

$$P_t : = (t, 4 + t + 4\sqrt{1 + t}, -2 - 2t - 4\sqrt{1 + t}),$$

$$Q_s : = (1, s, -2\sqrt{2(1 + s)}).$$

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Let us describe now the three inequalities that will be studied in this paper. Section 2 is devoted to obtain a Bernstein type inequality for polynomials in $\mathcal{P}(2D(\frac{x}{4}))$. Namely, for a fixed $(x,y) \in D(\frac{x}{4})$, we find the best (smallest) constant $\Phi(x,y)$ in the inequality

$$\|\nabla P(x,y)\|_2 \leq \Phi(x,y)\|P\|_{D(\frac{x}{4})},$$

for all $P \in \mathcal{P}(2D(\frac{x}{4}))$, where $\|\cdot\|_2$ denotes the euclidean norm in $\mathbb{R}^2$. Similarly, we also obtain a Markov global estimate on the gradient of polynomials in $\mathcal{P}(2D(\frac{x}{4}))$, or in other words, the smallest constant $M > 0$ in the inequality

$$\|\nabla P(x,y)\|_2 \leq M\|P\|_{D(\frac{x}{4})},$$

for all $P \in \mathcal{P}(2D(\frac{x}{4}))$ and $(x,y) \in D(\frac{x}{4})$. It is necessary to mention that the study of Bernstein and Markov type inequalities has a longstanding tradition. The interested reader can find further information on this classical topic in [2, 13, 14, 16, 18, 19, 20, 21, 23, 24, 25, 26, 27, 29, 30, 31].

In Section 3 we find the smallest constant $K > 0$ in the inequality

$$\|L\|_{D(\frac{x}{4})} \leq K\|P\|_{D(\frac{x}{4})},$$

where $P$ is an arbitrary polynomial in $\mathcal{P}(2D(\frac{x}{4}))$ and $L \in \mathcal{L}(2D(\frac{x}{4}))$ is the polar of $P$. Observe that here $\|L\|_{D(\frac{x}{4})}$ stands for the sup norm of $L$ over $D(\frac{x}{4})$. Hence, what we do is to provide the polarization constant of the space $\mathcal{P}(2D(\frac{x}{4}))$. The calculation of polarization constants in various polynomial spaces is largely motivated as the extensive, existing bibliography on the topic shows (see for instance [10, 18, 17, 27]).

Finally, in Section 4 we investigate the smallest constant $C > 0$ in the inequality

$$(1.1) \quad \|\nabla P\|_{D(\frac{x}{4})} \leq C\|P\|_{D(\frac{x}{4})},$$

for all $P \in \mathcal{P}(2D(\frac{x}{4}))$, where $|P|$ is the modulus of $P$, i.e., if $P(x,y) = ax^2 + by^2 + cxy$, then $|P|(x,y) = |a|x^2 + |b|y^2 + |c|x y$. The constant $C$ turns out to be the unconditional constant of the canonical basis of $\mathcal{P}(2D(\frac{x}{4}))$. It is interesting to note that already in 1914, H. Bohr [11] studied this type of inequalities for infinite complex power series. Actually, the study of Bohr radii is nowadays a fruitful field (see for instance [11, 12, 13, 14, 15, 17]). Observe that the relationship between unconditional constants in polynomial spaces and inequalities of the type (1.1) was already noticed in [7].

2. Bernstein and Markov-type inequalities for polynomials on sectors

In this section we provide sharp estimates on the Euclidean length of the gradient $\nabla P$ of a polynomial $P$ in $\mathcal{P}(2D(\frac{x}{4}))$.

Theorem 2.1. For every $(x,y) \in D(\frac{x}{4})$ and $P \in \mathcal{P}(2D(\frac{x}{4}))$ we have

$$\|\nabla P\|_2 \leq \Phi(x,y)\|P\|_{D(\frac{x}{4})},$$

where

$$\Phi(x,y) = \begin{cases} 4 \left(13 + 8\sqrt{2}\right) x^2 + (69 + 48\sqrt{2}) y^2 - 2 (28 + 20\sqrt{2}) xy \quad & \text{if } 0 \leq y \leq \frac{\sqrt{2} - 1}{4} x 	ext{ or } (4\sqrt{2} - 5) x \leq y \leq x, \\ \frac{x^2}{2} + 4(x^2 + y^2) \quad & \text{if } \frac{\sqrt{2} - 1}{4} x \leq y \leq (\sqrt{2} - 1) x, \\ \frac{(3x^2 - 2xy + 3y^2)}{2(x^2 - y^2)} \quad & \text{if } (\sqrt{2} - 1) x \leq y \leq (4\sqrt{2} - 5) x. \end{cases}$$

Proof. In order to calculate $\Phi(x,y) := \sup\{\|\nabla P(x,y)\|_2 : \|P\|_{D(\frac{x}{4})} \leq 1\}$, by the Krein-Milman approach, it is sufficient to calculate

$$\sup\{\|\nabla P(x,y)\|_2 : P \in \text{ext}(B_{D(\frac{x}{4})})\}.$$

By symmetry, we may just study the polynomials of Lemma 1.2 with positive sign. Let us start first with $P_1(x,y) = tx^2 + (4 + t + 4\sqrt{1 + t}) y^2 - 2 (1 + t + 2\sqrt{1 + t}) xy$, $t \in [-1, 1]$. Then,

$$\nabla P_1(x,y) = (2tx - 2(1 + t + 2\sqrt{1 + t}) y, 2(4 + t + 4\sqrt{1 + t}) y - 2(1 + t + 2\sqrt{1 + t}) x),$$
so that
\[
\| \nabla P_t(x, y) \|_2^2 = 4 t^2 x^2 + 4 \left( 1 + t + 2 \sqrt{1 + t} \right)^2 y^2 - 8 t \left( 1 + t + 2 \sqrt{1 + t} \right) x y \\
+ 4 \left( 4 + t + 4 \sqrt{1 + t} \right)^2 y^2 + 4 \left( 1 + t + 2 \sqrt{1 + t} \right)^2 x^2 \\
- 8 \left( 4 + t + 4 \sqrt{1 + t} \right) \left( 1 + t + 2 \sqrt{1 + t} \right) x y
\]

Make now the change \( u = \sqrt{1 + t} \in [0, \sqrt{2}] \), so that
\[
\| \nabla P_u(x, y) \|_2^2 = 8(x - y)^2 u^2 + 16 \left( x^2 - 4 x y + 3 y^2 \right) u^3 \\
+ 8 \left( x^2 - 10 x y + 13 y^2 \right) u^2 + 32 \left( 3 y^2 - x y \right) u + 4 \left( x^2 + 9 y^2 \right) .
\]

Since
\[
\frac{\partial}{\partial u} \| \nabla P_u(x, y) \|_2^2 = 16 \left( 2(x - y)^2 u^2 + \left( x^2 - 8 x y + 7 y^2 \right) u + 2 y \left( 3 y - x \right) \right) (u + 1),
\]
it follows that the critical points of \( \| DP_u(x, y) \|_2^2 \) are \( u = \frac{2y}{x-y} \), \( u = \frac{3y-x}{2(x-y)} \) and \( u = -1 \) if \( x \neq y \) and \( u = 4 \) and \( u = -1 \) if \( x = y \). Since we need to consider \( 0 \leq u \leq \sqrt{2} \), we can directly omit the case \( x = y \).

Therefore, we can write
\[
\frac{\partial}{\partial u} \| \nabla P_u(x, y) \|_2^2 = 32(x - y)^2 \left( u - \frac{2y}{x-y} \right) \left( u - \frac{3y-x}{2(x-y)} \right) (u + 1).
\]

Let \( u_1 = \frac{2y}{x-y} \) and \( u_2 = \frac{3y-x}{2(x-y)} \) (Again, since we need to consider \( 0 \leq u \leq \sqrt{2} \), we can omit the solution \( u = -1 \)). Also, we have the extra conditions \( u_1 \in [0, \sqrt{2}] \) whenever \( 0 \leq y \leq (\sqrt{2} - 1) x \) and \( u_2 \in [0, \sqrt{2}] \) whenever \( \frac{1}{3} x \leq y \leq (4\sqrt{2} - 5) x \). Considering all these facts, we need to compare the quantities
\[
C_1(x, y) := \| \nabla P_{u_1}(x, y) \|_2^2 = \| \nabla P_{u_1} \|_2^2 = 4 \frac{x^6 - 4x^5 y + 7x^4 y^2 - 8x^3 y^3 + 7x^2 y^4 + 4x y^5 + y^6}{(x-y)^4}
\]
\[
= 4 \left( x^2 + y^2 \right) ,
\]
for \( 0 \leq y \leq (\sqrt{2} - 1) x \) and \( t_1 = \frac{2y^2 + 2xy - y^2}{(x-y)^2} \),
\[
C_2(x, y) := \| \nabla P_{u_2}(x, y) \|_2^2 = \| \nabla P_{u_2} \|_2^2 = 9 \frac{x^6 - 30x^5 y + 55x^4 y^2 - 68x^3 y^3 + 55x^2 y^4 - 30x y^5 + 9y^6}{(x-y)^4}
\]
\[
= \frac{\left( 3x^2 - 2xy + 3y^2 \right)^2}{2(x-y)^2},
\]
for \( \frac{1}{3} x \leq y \leq (4\sqrt{2} - 5) x \) and \( t_2 = \frac{5x^2 + 2xy - 3x^2}{4(x-y)^2} \),
\[
C_3(x, y) := \| \nabla P_{s=-1} \|_2^2 = 4 \left( x^2 + 9y^2 \right) ,
\]
and
\[
C_4(x, y) := \| \nabla P_{s=1} \|_2^2 = 4 \left[ \left( 13 + 8\sqrt{2} \right) x^2 + \left( 69 + 48\sqrt{2} \right) y^2 - 2 \left( 28 + 20\sqrt{2} \right) xy \right] .
\]

Let us focus now on \( Q_s = \left( 1, s, -2\sqrt{2(1+s)} \right), 1 \leq s \leq 5 + 4\sqrt{2} \). Then, we have
\[
\| \nabla Q_s(x, y) \|_2^2 = 4x^2 + 4s^2 y^2 + 8(1+s)(x^2 + y^2) - 8(1+s)\sqrt{2(1+s)} xy
\]
Making the change \( v = \sqrt{2(1+s)} \in [2, 2 + 2\sqrt{2}] \), we need to study the function
\[
\| \nabla Q_v(x, y) \|_2^2 = v^2 \left( y^2 v^2 - 4xyv + 4x^2 \right) + 4 \left( x^2 + y^2 \right) .
\]
If \( x = y = 0 \) we have \( \| \nabla Q_v(x, y) \|_2^2 = 0 \), so we will assume both \( x \neq 0 \) and \( y \neq 0 \). The critical points of \( \| \nabla Q_v(x, y) \|_2^2 \) are \( v = \frac{x}{y}, v = \frac{2x}{y} \) and \( v = 0 \) (but \( 0 \notin [2, 2 + 2\sqrt{2}] \)). Observe that \( v_1 = \frac{x}{y} \in [2, 2 + 2\sqrt{2}] \) whenever \( \frac{\sqrt{2} - 1}{x} \leq y \leq \frac{1}{x} \) and \( v_2 = \frac{2x}{y} \in [2, 2 + 2\sqrt{2}] \) whenever \( y \geq (\sqrt{2} - 1) x \). Thus, we also need to compare the quantities
\[
C_5(x, y) := \| \nabla Q_{v_1}(x, y) \|_2^2 = \| \nabla Q_{s_1}(x, y) \|_2^2 = \frac{x^4}{y^2} + 4 \left( x^2 + y^2 \right) ,
\]
for $\frac{\sqrt{2} - 1}{2} x \leq y \leq \frac{1}{2}x$ and $s_1 = \frac{x^2 - 2y^2}{2y^2}$,

$$C_6(x, y) := \|\nabla Q_{v_2}(x, y)\|_2^2 = \|\nabla Q_{s_2}(x, y)\|_2^2 = 4 \left( x^2 + y^2 \right),$$

for $(\sqrt{2} - 1) x \leq y \leq x$ and $s_2 = \frac{2x^2 - y^2}{y^2}$, and also

$$C_7(x, y) := \|\nabla Q_{s_1=1}\|_2^2 = 4 \left( x^2 + y^2 \right) + 16(x - y)^2,$$

and

$$C_8(x, y) := \|\nabla Q_{s_4=5+4\sqrt{2}}\|_2^2$$

$$= \left( 12 + 8\sqrt{2} \right) \left[ 4x^2 + \left( 12 + 8\sqrt{2} \right) y^2 - \left( 8 + 8\sqrt{2} \right) xy \right] + 4 \left( x^2 + y^2 \right)$$

$$= 4 \left[ \left( 13 + 8\sqrt{2} \right) x^2 + \left( 69 + 48\sqrt{2} \right) y^2 - 2 \left( 28 + 20\sqrt{2} \right) xy \right].$$

Note that (the reader can take a look at Figures 1, 2 and 3)

$$C_1(x, y), C_6(x, y) \leq C_7(x, y) \leq \begin{cases} C_4(x, y) & \text{if } 0 \leq y \leq \frac{2-x^2}{2} \text{ or } \frac{1}{2} x \leq y \leq x, \\
C_5(x, y) & \text{if } \frac{2-x^2}{2} \leq y \leq \frac{1}{2} x, \end{cases}$$

$$C_3(x, y) \leq \begin{cases} C_2(x, y) & \text{if } \frac{1}{2} x \leq y \leq (4\sqrt{2} - 5) x, \\
C_4(x, y) & \text{if } 0 \leq y \leq \frac{1}{2} x \text{ or } (4\sqrt{2} - 5) x \leq y \leq x, \\
C_5(x, y) & \text{if } (\sqrt{2} - 1)x \leq y \leq (4\sqrt{2} - 5)x. \end{cases}$$

Hence, for $(x, y) \in D \left( \frac{\pi}{4} \right)$,

$$\Phi(x, y) = \sup \left\{ \|\nabla P(x, y)\|_2 : P \in \text{ext} \left( B_D \left( \frac{\pi}{4} \right) \right) \right\}$$

$$= \begin{cases} C_4(x, y) & \text{if } 0 \leq y \leq \frac{\sqrt{2}-1}{2} x \text{ or } (4\sqrt{2} - 5)x \leq y \leq x, \\
C_5(x, y) & \text{if } \frac{\sqrt{2}-1}{2} x \leq y \leq (\sqrt{2} - 1)x, \\
C_2(x, y) & \text{if } (\sqrt{2} - 1)x \leq y \leq (4\sqrt{2} - 5)x. \end{cases}$$

In order to illustrate the previous step, the reader can take a look at Figure 4. \hfill \Box

**Corollary 2.2.** If $P \in \mathcal{P} \left( D \left( \frac{\pi}{4} \right) \right)$, then

$$\sup \left\{ \|\nabla P(x, y)\|_2 : (x, y) \in D \left( \frac{\pi}{4} \right) \right\} \leq 4(13 + 8\sqrt{2}) \|P\|_{D \left( \frac{\pi}{4} \right)},$$

with equality for the polynomials $P_1(x, y) = \pm \left( x^2 + (5 + 4\sqrt{2})y^2 - 2(2 + 2\sqrt{2})xy \right)$. 

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**Figure 1.** Graphs of the mappings $C_1(1, \lambda)$, $C_6(1, \lambda)$, $C_7(1, \lambda)$. 

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with equality for the polynomials
3. Polarization constants for polynomials on sectors

In this section we find the exact value of the polarization constant of the space $P(2D(\pi/4))$. In order to do that, we prove a Bernstein type inequality for polynomials in $P(2D(\pi/4))$. Observe that if $P \in P(2D(\pi/4))$ and $(x,y) \in D(\pi/4)$ then the differential $DP(x,y)$ of $P$ at $(x,y)$ can be viewed as a linear
form. What we shall do is to find the best estimate for \( \|DP(x, y)\|_{D(\pi)} \) (the sup norm of \( DP(x, y) \) over the sector \( D(\pi) \)) in terms of \((x, y)\) and \( \|P\|_{D(\pi)} \). First, we state a lemma that will be useful in the future:

**Lemma 3.1.** Let \( a, b \in \mathbb{R} \). Then,

\[
\sup_{\theta \in [0, \pi]} |a \cos \theta + b \sin \theta| = \begin{cases} 
\max \left\{ |a|, \frac{\sqrt{a^2 + b^2}}{2y} |a + b| \right\} & \text{if } \frac{b}{a} > 1 \text{ or } \frac{b}{a} < 0, \\
\sqrt{a^2 + b^2} & \text{otherwise.}
\end{cases}
\]

By symmetry, we may just study the polynomials of Lemma 1.2 with positive sign. Let us start first with

\[\Psi(x, y) = \begin{cases} 
\sqrt{2} \left[ (1 + 2\sqrt{2}) x - (3 + 2\sqrt{2}) y \right] & \text{if } 0 \leq y < \frac{2\sqrt{2} - 1}{2} x, \\
\sqrt{2} x^2 + 3y^2 & \text{if } \frac{2\sqrt{2} - 1}{2} x \leq y < (\sqrt{2} - 1) x, \\
2 \left( x + \frac{y^2}{x - y} \right) & \text{if } (\sqrt{2} - 1) x \leq y < (2 - \sqrt{2}) x, \\
4 (1 + \sqrt{2}) y - 2x & \text{if } (2 - \sqrt{2}) x \leq y \leq x
\end{cases}\]

Moreover, inequality (3.1) is optimal for each \((x, y) \in D(\pi)\).

**Proof.** In order to calculate \( \Psi(x, y) = \sup \{ \|DP(x, y)\|_{D(\pi)} : \|P\|_{D(\pi)} \leq 1 \} \), by the Krein-Milman approach, it suffices to calculate

\[
\sup \{ \|DP(x, y)\|_{D(\pi)} : P \in \text{ext}(B_{D(\pi)}) \}.
\]

By symmetry, we may just study the polynomials of Lemma 1.2 with positive sign. Let us start first with

\[P_t(x, y) = tx^2 + (4 + t + 4\sqrt{1 + t}) y - (2t + 4\sqrt{1 + t}) xy.
\]

So we may write

\[
\nabla P_t(x, y) = (2tx - (2t + 4\sqrt{1 + t}) y, 2 (4 + t + 4\sqrt{1 + t}) y - (2t + 4\sqrt{1 + t}) x),
\]

from which

\[
\|DP_t(x, y)\|_{D(\pi)} = \sup_{0 \leq \theta \leq \pi} \left| 2 \left[ tx - (1 + t + 2\sqrt{1 + t}) y \right] \cos \theta + 2 \left[ (4 + t + 4\sqrt{1 + t}) y - (1 + t + 2\sqrt{1 + t}) x \right] \sin \theta \right|
\]

\[
= 2x \sup_{0 \leq \theta \leq \pi} |f_{\lambda}(t, \theta)|,
\]

for \( f_{\lambda}(t, \theta) = \left[ t - (1 + t + 2\sqrt{1 + t}) \lambda \right] \cos \theta + \left[ (4 + t + 4\sqrt{1 + t}) \lambda - (1 + t + 2\sqrt{1 + t}) \right] \sin \theta \),

where \( \lambda = \frac{y}{x}, x \neq 0 \) (the case \( x = 0 \) is trivial, since the only point in \( D(\pi) \) where \( x = 0 \) is \((0, 0)\), in which case \( P_t(0, 0) = \|DP_t(0, 0)\|_{D(\pi)} = 0 \)).

We need to calculate

\[
\sup_{-1 \leq t \leq 1} \|DP_t(x, y)\|_{D(\pi)} = 2x \sup_{0 \leq \theta \leq \pi} |f_{\lambda}(t, \theta)|.
\]

Let us define \( C_1 = [-1, 1] \times [0, \frac{\pi}{4}] \). We will analyze 5 cases.

1. \((t, \theta) \in (-1, 1) \times (0, \frac{\pi}{4})\).

We are interested just in critical points. Hence,
\[
\frac{\partial f_\lambda}{\partial \theta}(t, \theta) = \left[ \left( 1 + \frac{2}{\sqrt{1+t}} \right) \lambda - \left( 1 + \frac{1}{\sqrt{1+t}} \right) \right] \sin \theta \\
+ \left[ 1 - \left( 1 + \frac{1}{\sqrt{1+t}} \right) \lambda \right] \cos \theta = 0,
\]

(3.2)

\[
\frac{\partial f_\lambda}{\partial \theta}(t, \theta) = \left[ \left( 1 + t + 2\sqrt{1+t} \right) \lambda - t \right] \sin \theta \\
+ \left[ \left( 4 + t + 4\sqrt{1+t} \right) \lambda - \left( 1 + t + 2\sqrt{1+t} \right) \right] \cos \theta = 0
\]

(3.3)

Equation (3.3) tells us that

\[
\sin \theta = \frac{(4 + t + 4\sqrt{1+t}) \lambda - \left( 1 + t + 2\sqrt{1+t} \right)}{t - \left( 1 + t + 2\sqrt{1+t} \right) \lambda} \cos \theta.
\]

(3.4)

If we now plug (3.4) in equation (3.2), we obtain

\[
0 = \left\{ \left[ 1 - \left( 1 + \frac{1}{\sqrt{1+t}} \right) \lambda \right] + \left[ \left( 1 + \frac{2}{\sqrt{1+t}} \right) \lambda - \left( 1 + \frac{1}{\sqrt{1+t}} \right) \right] \right\} \times \frac{(4 + t + 4\sqrt{1+t}) \lambda - \left( 1 + t + 2\sqrt{1+t} \right)}{t - \left( 1 + t + 2\sqrt{1+t} \right) \lambda} \cos \theta.
\]

Using that \(0 < \theta < \frac{\pi}{2}\), we can conclude

\[
0 = \left[ 1 - \left( 1 + \frac{1}{\sqrt{1+t}} \right) \lambda \right] \cdot \left[ t - \left( 1 + t + 2\sqrt{1+t} \right) \lambda \right] \\
+ \left[ \left( 1 + \frac{2}{\sqrt{1+t}} \right) \lambda - \left( 1 + \frac{1}{\sqrt{1+t}} \right) \right] \cdot \left[ (4 + t + 4\sqrt{1+t}) \lambda - (1 + t + 2\sqrt{1+t}) \right]
\]

\[
= t \left( 1 + t + 2\sqrt{1+t} \right) \lambda - t \lambda + (1 + t + 2\sqrt{1+t}) \lambda^2 - \frac{\lambda t}{\sqrt{1+t}} \\
+ \frac{\lambda^2 t}{\sqrt{1+t}} \left( 1 + t + 2\sqrt{1+t} \right) + \left( 1 + \frac{2}{\sqrt{1+t}} \right) (4 + t + 4\sqrt{1+t}) \lambda^2 \\
- \left( 1 + \frac{2}{\sqrt{1+t}} \right) (1 + t + 2\sqrt{1+t}) \lambda - \left( 1 + \frac{1}{\sqrt{1+t}} \right) (4 + t + 4\sqrt{1+t}) \lambda \\
+ \left( 1 + \frac{1}{\sqrt{1+t}} \right) (1 + t + 2\sqrt{1+t})
\]

\[
= t (1 - 2\lambda + 2\lambda^2 - 2\lambda + 1) + (-2\lambda + 2\lambda^2 + 4\lambda^2 - 2\lambda - 4\lambda + 2) \sqrt{1+t} \\
+ \frac{t}{\sqrt{1+t}} (-\lambda + \lambda^2 + 2\lambda^3 - 2\lambda - \lambda + 1) + \frac{1}{\sqrt{1+t}} (\lambda^3 + 8\lambda^2 - 2\lambda - 4\lambda + 1) \\
+ (-\lambda + \lambda^2 + 2\lambda^3 + 4\lambda^2 - \lambda - 4\lambda + 2 + 2\lambda^2 - 8\lambda) \\
= 2t(\lambda - 1)^2 + 6\sqrt{1+t}(\lambda - 1) \left( \lambda - \frac{1}{3} \right) + 3\frac{t}{\sqrt{1+t}}(\lambda - 1) \left( \lambda - \frac{1}{3} \right) \\
+ \frac{1}{\sqrt{1+t}}(3\lambda - 1)^2 + 15 \left( \lambda - \frac{1}{3} \right) \left( \lambda - \frac{3}{5} \right).
\]
Working with this last expression, we get
\[ 0 = 2t\sqrt{1+t}(\lambda - 1)^2 + 6(1+t)(\lambda - 1) \left( \lambda - \frac{1}{3} \right) + 3t(\lambda - 1) \left( \lambda - \frac{1}{3} \right) \]
\[ + (3\lambda - 1)^2 + 15\sqrt{1+t} \left( \lambda - \frac{1}{3} \right) \left( \lambda - \frac{3}{5} \right) \]
and hence, rearranging terms,
\[ \sqrt{1+t} \left[ 15 \left( \lambda - \frac{1}{3} \right) \left( \lambda - \frac{2}{5} \right) + 2t(\lambda - 1)^2 \right] = -9t(\lambda - 1) \left( \lambda - \frac{1}{3} \right) - 15 \left( \lambda - \frac{1}{3} \right) \left( \lambda - \frac{3}{5} \right). \]
If \( \lambda = 1 \), we obtain
\[ \sqrt{1+t} + 1 = 0 \]
and so, in particular, we have \( \lambda \neq 1 \). Equation (3.5) has two solutions,
\[ t_1(\lambda) = \frac{-1 + 2\lambda + 3\lambda^2}{(\lambda - 1)^2} \quad \text{and} \quad t_2(\lambda) = \frac{5\lambda^2 + 2\lambda - 3}{4(\lambda - 1)^2}. \]
Using equation (3.2), we may see
\[ \tan \theta = \frac{\left( 1 + \frac{\lambda}{\sqrt{1+t}} \right) \lambda - 1}{\left( 1 + \frac{2}{\sqrt{1+t}} \right) \lambda - \left( 1 + \frac{1}{\sqrt{1+t}} \right)}. \]
In particular, evaluating in \( t_1(\lambda) \) we obtain
\[ \tan \theta_1 = \frac{\left( 1 + \frac{\lambda}{\lambda - 1} \right) \lambda - 1}{\left( 1 + \frac{1}{\lambda - 1} \right) \lambda - \left( 1 + \frac{1}{\lambda - 1} \right)} = \lambda, \]
in which case we have
\[ D_{1,1}(\lambda) := |f_\lambda(t_1, \theta_1)| = \left| -\sqrt{1 + \lambda^2} \right| = \sqrt{1 + \lambda^2}. \]
Regarding \( t_2(\lambda) \), we obtain
\[ \tan \theta_2 = \frac{\left( 1 + \frac{1}{4(\lambda - 1)^2} \right) \lambda - 1}{\left( 1 + 2\sqrt{4(\lambda - 1)^2} \right) \lambda - \left( 1 + \frac{4(\lambda - 1)^2}{(\lambda - 1)^2} \right)}. \]
Since \( \theta_2 \in (0, \frac{\pi}{4}) \), we need to guarantee \( 0 < \tan \theta_2 < 1 \), and for this we need \( 0 < \lambda < \frac{1}{5} \). Therefore
\[ \tan \theta_2 = \frac{5\lambda - 1}{7\lambda - 3} \]
and in this case,
\[ D_{1,2}(\lambda) := |f_\lambda(t_2, \theta_2)| \]
\[ = \left| \frac{5\lambda^2 + 2\lambda - 3}{4(\lambda - 1)^2} - \frac{9\lambda^2 - 6\lambda + 1}{4(\lambda - 1)^2} + \frac{3\lambda - 1}{\lambda - 1} \right| \frac{3 - 7\lambda}{\sqrt{74\lambda^2 - 52\lambda + 10}} \]
\[ + \left| \frac{3 + 9\lambda^2 - 6\lambda + 1}{4(\lambda - 1)^2} + \frac{6\lambda - 2}{\lambda - 1} \right| \lambda \left( 9\lambda^2 - 6\lambda + 1 \right) - \frac{3\lambda - 1}{\lambda - 1} \right| \frac{1 - 5\lambda}{\sqrt{74\lambda^2 - 52\lambda + 10}} \]
\[ = \left| \frac{-78\lambda^4 - 208\lambda^3 + 196\lambda^2 - 80\lambda + 14}{4(\lambda - 1)^2\sqrt{74\lambda^2 - 52\lambda + 10}} \right| \frac{1}{\sqrt{74\lambda^2 - 52\lambda + 10}} \]
\[ = \frac{39\lambda^2 - 26\lambda + 7}{2\sqrt{74\lambda^2 - 52\lambda + 10}}. \]
(2) \( \theta = 0, -1 \leq t \leq 1. \)
We have
\[ f_\lambda(t, 0) = t - (1 + t + 2\sqrt{1 + t}) \lambda. \]

Then,
\[ f_\lambda(-1, 0) = -1, \]
\[ f_\lambda(1, 0) = 1 - 2 \left( 1 + \sqrt{2} \right) \lambda, \]

and hence
\[
|f_\lambda(1, 0)| = \begin{cases} 
1 - 2(1 + \sqrt{2})\lambda & \text{if } 0 \leq \lambda < \frac{\sqrt{2} - 1}{2} \\
2(1 + \sqrt{2})\lambda - 1 & \text{if } \frac{\sqrt{2} - 1}{2} \leq \lambda \leq 1.
\end{cases}
\]

Working now on \((-1, 1),\) since
\[ f'_\lambda(t, 0) = 1 - \left( 1 + \frac{1}{\sqrt{1 + t}} \right) \lambda, \]

the critical point of \(f_\lambda(t, 0)\) is
\[ t = \frac{\lambda^2}{(1 - \lambda)^2} - 1. \]

Recall that we need to make sure that \(-1 < t < 1.\) Therefore, in this case we also need to ask
\[ \lambda < \frac{\sqrt{2}}{1 + \sqrt{2}} = 2 - \sqrt{2}. \]

Plugging the critical point of \(f_\lambda(t, 0)\) into \(f_\lambda(t, 0),\) we obtain
\[ f_\lambda \left( \frac{\lambda^2}{(\lambda - 1)^2} - 1, 0 \right) = \frac{\lambda^2}{(\lambda - 1)^2} - 1 - \left[ \frac{\lambda^2}{(\lambda - 1)^2} + \frac{2\lambda}{1 - \lambda} \right] \lambda = \frac{\lambda^2}{\lambda - 1} - 1, \]

and hence
\[
\left| f_\lambda \left( \frac{\lambda^2}{(\lambda - 1)^2} - 1, 0 \right) \right| = 1 + \frac{\lambda^2}{1 - \lambda}.
\]

- Assume first \(0 \leq \lambda < \frac{\sqrt{2} - 1}{2}.\) Then,
\[
\sup_{-1 \leq t \leq 1} |f_\lambda(t, 0)| = \max \left\{ 1, 1 - 2 \left( 1 + \sqrt{2} \right) \lambda, 1 + \frac{\lambda^2}{1 - \lambda} \right\} = 1 + \frac{\lambda^2}{1 - \lambda}.
\]

- Assume now \(\frac{\sqrt{2} - 1}{2} \leq \lambda < 2 - \sqrt{2}.\) Then,
\[
\sup_{-1 \leq t \leq 1} |f_\lambda(t, 0)| = \max \left\{ 1, 2 \left( 1 + \sqrt{2} \right) \lambda - 1, 1 + \frac{\lambda^2}{1 - \lambda} \right\} = 1 + \frac{\lambda^2}{1 - \lambda}.
\]

- Assume finally \(2 - \sqrt{2} \leq \lambda \leq 1.\) Then,
\[
\sup_{-1 \leq t \leq 1} |f_\lambda(t, 0)| = \max \left\{ 1, 2 \left( 1 + \sqrt{2} \right) \lambda - 1 \right\} = 2 \left( 1 + \sqrt{2} \right) \lambda - 1.
\]

So, in conclusion,
\[
\sup_{-1 \leq t \leq 1} |f_\lambda(t, 0)| = \begin{cases} 
1 + \frac{\lambda^2}{1 - \lambda} & \text{if } 0 \leq \lambda < 2 - \sqrt{2}, \\
(2 + 2\sqrt{2})\lambda - 1 & \text{if } 2 - \sqrt{2} \leq \lambda \leq 1,
\end{cases}
\]

=:
\[
D_{2,1}(\lambda) \quad \text{if } 0 \leq \lambda < 2 - \sqrt{2},
\]

\[
D_{2,2}(\lambda) \quad \text{if } 2 - \sqrt{2} \leq \lambda \leq 1.
\]

(3) \(\theta = \frac{\pi}{4}\) and \(-1 \leq t \leq 1.\)

We have
\[ f_\lambda \left( t, \frac{\pi}{4} \right) = \frac{\sqrt{2}}{2} \left[ t - (1 + t + 2\sqrt{1 + t}) \lambda + (4 + t + 4\sqrt{1 + t}) \lambda - (1 + t + 2\sqrt{1 + t}) \right] = \frac{\sqrt{2}}{2} \left[ (3 + 2\sqrt{1 + t}) \lambda - (1 + 2\sqrt{1 + t}) \right].\]
Again, we have
\[
\begin{align*}
f_\lambda \left( -1, \frac{\pi}{4} \right) &= \frac{\sqrt{2}}{2} (3\lambda - 1), \\
f_\lambda \left( 1, \frac{\pi}{4} \right) &= \frac{\sqrt{2}}{2} \left( (3 + 2\sqrt{2}) \lambda - (1 + 2\sqrt{2}) \right), \\
f_\lambda' \left( t, \frac{\pi}{4} \right) &= \frac{\sqrt{2}}{2} \left( \frac{\lambda}{\sqrt{1+t}} - \frac{1}{\sqrt{1+t}} \right).
\end{align*}
\]
and \( f_\lambda'(t, \frac{\pi}{4}) = 0 \) implies \( \lambda = 1 \) (in which case \( f_\lambda(t, \frac{\pi}{4}) = \sqrt{2} \) for every \( t \)).

- Assume first \( 0 \leq \lambda < \frac{1}{4} \). Then,
  \[
  \sup_{-1 \leq t \leq 1} |f_\lambda \left( t, \frac{\pi}{4} \right)| = \frac{\sqrt{2}}{2} \max \left\{ (1 + 2\sqrt{2}) - (3 + 2\sqrt{2}) \lambda, 1 - 3\lambda \right\} 
  = \frac{\sqrt{2}}{2} \left[ (1 + 2\sqrt{2}) - (3 + 2\sqrt{2}) \lambda \right]
  \]

- Assume now \( \frac{1}{4} \leq \lambda < 4\sqrt{2} - 5 \). Then,
  \[
  \sup_{-1 \leq t \leq 1} |f_\lambda \left( t, \frac{\pi}{4} \right)| = \frac{\sqrt{2}}{2} \max \left\{ (1 + 2\sqrt{2}) - (3 + 2\sqrt{2}) \lambda, 3\lambda - 1 \right\} 
  = \left\{ \begin{array}{ll}
  \frac{\sqrt{2}}{2} \left[ (1 + 2\sqrt{2}) - (3 + 2\sqrt{2}) \lambda \right] & \text{if } \frac{1}{4} \leq \lambda < \frac{2\sqrt{2}+1}{\sqrt{2}} \\
  \frac{\sqrt{2}}{2} (3\lambda - 1) & \text{if } \frac{2\sqrt{2}+1}{\sqrt{2}} \leq \lambda < 4\sqrt{2} - 5.
  \end{array} \right.
  \]

- Assume finally \( 4\sqrt{2} - 5 \leq \lambda \leq 1 \). Then,
  \[
  \sup_{-1 \leq t \leq 1} |f_\lambda \left( t, \frac{\pi}{4} \right)| = \frac{\sqrt{2}}{2} \max \left\{ 3\lambda - 1, (3 + 2\sqrt{2}) \lambda - (1 + 2\sqrt{2}) \right\} = \frac{\sqrt{2}}{2} (3\lambda - 1).
  \]

Hence, we can say that
\[
\sup_{-1 \leq t \leq 1} |f_\lambda (t, \frac{\pi}{4})| = \left\{ \begin{array}{ll}
  \frac{\sqrt{2}}{2} (1 + 2\sqrt{2} - (3 + 2\sqrt{2}) \lambda) & \text{if } 0 \leq \lambda < \frac{2\sqrt{2}+1}{\sqrt{2}} \\
  \frac{\sqrt{2}}{2} (3\lambda - 1) & \text{if } \frac{2\sqrt{2}+1}{\sqrt{2}} \leq \lambda \leq 1.
  \end{array} \right.
\]

(4) \( t = -1, 0 \leq \theta \leq \frac{\pi}{4} \).

Applying lemma 3.1 we obtain
\[
\sup_{0 \leq \theta \leq \frac{\pi}{4}} f_\lambda (-1, \theta) = \left\{ \begin{array}{ll}
  1 & \text{if } 0 \leq \lambda < \frac{1+\sqrt{2}}{4}, \\
  \frac{\sqrt{2}}{2} (3\lambda - 1) & \text{if } \frac{1+\sqrt{2}}{4} \leq \lambda \leq 1.
  \end{array} \right.
\]

(5) \( t = 1, 0 \leq \theta \leq \frac{\pi}{4} \).

We use again lemma 3.1, with \( a = 1 - (2 + 2\sqrt{2}) \lambda \) and \( b = (5 + 4\sqrt{2}) \lambda - (2 + 2\sqrt{2}) \). Through standard calculations, we see that \( \frac{b}{a} < 0 \) if and only if \( \lambda \in \left( 0, \frac{\sqrt{2}-1}{2} \right) \cup \left( \frac{9-4\sqrt{2}}{4}, 1 \right] \) and \( \frac{b}{a} > 1 \) if and only
we can directly rule out case (3). Since (see Figures 5 and 7)

$$\sup_{0 \leq \theta \leq \frac{\pi}{4}} |f_\lambda(1, \theta)|$$

$$= \begin{cases} 
\max \left\{ \frac{\sqrt{2}}{2} \left| (1 - (2 + 2\sqrt{2}) \lambda) \right|, \frac{\sqrt{2}}{2} \left| (3 + 2\sqrt{2}) \lambda - (1 + 2\sqrt{2}) \right| \right\} & \text{if } 0 \leq \lambda < \frac{3 + 4\sqrt{2}}{23}, \\
\sqrt{(1 - (2 + 2\sqrt{2}) \lambda)^2 + \left( \frac{2 + 4\sqrt{2}}{23} \lambda - (2 + 2\sqrt{2}) \right)^2} & \text{if } \frac{3 + 4\sqrt{2}}{23} \leq \lambda < \frac{5 - 2\sqrt{2}}{6}, \\
\max \left\{ \frac{\sqrt{2}}{2} \left| (1 - (2 + 2\sqrt{2}) \lambda) \right|, \frac{\sqrt{2}}{2} \left| (3 + 2\sqrt{2}) \lambda - (1 + 2\sqrt{2}) \right| \right\} & \text{if } \frac{5 - 2\sqrt{2}}{6} \leq \lambda \leq 1.
\end{cases}$$

Since $0 \leq \lambda < \sqrt{2} - 1$ implies $\left| 1 - (2 + 2\sqrt{2}) \lambda \right| < \frac{\sqrt{2}}{2} \left| (3 + 2\sqrt{2}) \lambda - (1 + 2\sqrt{2}) \right|$, it follows that

$$\sup_{0 \leq \theta \leq \frac{\pi}{4}} |f_\lambda(1, \theta)| = \begin{cases} 
\frac{\sqrt{2}}{2} \left| (1 + 2\sqrt{2}) \lambda - (1 + 2\sqrt{2}) \right| & \text{if } 0 \leq \lambda < \frac{3 + 4\sqrt{2}}{23}, \\
\sqrt{48\sqrt{2}\lambda^2 - 56\lambda + 69\lambda^2 - 40\sqrt{2}\lambda + 8\sqrt{2} + 13} & \text{if } \frac{3 + 4\sqrt{2}}{23} \leq \lambda < \frac{5 - 2\sqrt{2}}{6}, \\
\frac{\sqrt{2}}{2} \left| (1 + 2\sqrt{2}) \lambda - (3 + 2\sqrt{2}) \lambda \right| & \text{if } \frac{5 - 2\sqrt{2}}{6} \leq \lambda \leq 1.
\end{cases}$$

Since (see Figures 5 and 6)

$$D_{1,1}(\lambda) \leq \begin{cases} 
D_{2,1}(\lambda) & \text{if } 0 \leq \lambda < 2 - \sqrt{2}, \\
D_{2,2}(\lambda) & \text{if } 2 - \sqrt{2} \leq \lambda \leq 1,
\end{cases}$$

$$D_{1,2}(\lambda) \leq D_{3,1}(\lambda) \text{ for } 0 < \lambda < \frac{1}{2},$$

we can rule out case (1). Since

$$D_{3,1}(\lambda) = D_{5,1}(\lambda) \text{ for } 0 \leq \lambda \leq \frac{3 + 4\sqrt{2}}{23},$$

$$D_{3,2}(\lambda) = D_{4,2}(\lambda) \text{ for } \frac{1 + \sqrt{2}}{3} \leq \lambda \leq 1,$$

we can directly rule out case (3). Since (see Figures 5 and 7)

$$D_{4,1}(\lambda) = 1 \leq \begin{cases} 
D_{2,1}(\lambda) & \text{if } 0 \leq \lambda < 2 - \sqrt{2}, \\
D_{2,2}(\lambda) & \text{if } 2 - \sqrt{2} \leq \lambda < \frac{1 + \sqrt{2}}{3},
\end{cases}$$

$$D_{4,2}(\lambda) \leq D_{2,2} \text{ for } \frac{1 + \sqrt{2}}{3} \leq \lambda \leq 1,$$

we can rule out case (4). Finally, since (see Figure 8)

$$D_{5,2}(\lambda) \leq D_{2,1}(\lambda) \text{ for } \frac{3 + 4\sqrt{2}}{23} \leq \lambda < \frac{5 - 2\sqrt{2}}{6},$$

$$D_{5,3}(\lambda) = D_{2,2}(\lambda) \text{ for } 2 - \sqrt{2} \leq \lambda \leq 1,$$

we can rule out the expressions $D_{5,2}(\lambda)$ and $D_{5,3}(\lambda)$ of case (5).
Thus, putting all the above cases together, we may reach the conclusion

\[
\sup_{(t, \theta) \in C_1} |f_\lambda(t, \theta)| = \begin{cases} 
D_{5,1}(\lambda) & \text{if } 0 \leq \lambda < \frac{(2-3\sqrt{2})\sqrt{4\sqrt{2}+7+5\sqrt{2}+6}}{14}, \\
D_{2,1}(\lambda) & \text{if } \frac{(2-3\sqrt{2})\sqrt{4\sqrt{2}+7+5\sqrt{2}+6}}{14} \leq \lambda < 2 - \sqrt{2}, \\
D_{2,2}(\lambda) & \text{if } 2 - \sqrt{2} \leq \lambda \leq 1,
\end{cases}
\]

and hence

\[
\sup_{-1 \leq t \leq 1} \|DP_t(x, y)\|_{D(\xi)} = 2x \sup_{(t, \theta) \in C_1} |f_\lambda(t, \theta)|
\]

\[
= \begin{cases} 
\sqrt{2} \left[ (1 + 2\sqrt{2}) x - (3 + 2\sqrt{2}) y \right] & \text{if } 0 \leq y < \frac{(2-3\sqrt{2})\sqrt{4\sqrt{2}+7+5\sqrt{2}+6} x}{14}, \\
2 \left( x + \frac{y^2}{x-y} \right) & \text{if } \frac{(2-3\sqrt{2})\sqrt{4\sqrt{2}+7+5\sqrt{2}+6} x}{14} \leq y < (2 - \sqrt{2}) x, \\
4 \left( 1 + \sqrt{2} \right) y - 2x & \text{if } (2 - \sqrt{2}) x \leq y \leq x,
\end{cases}
\]

assuming in every moment \( x \neq 0 \) (in order to illustrate the previous step, the reader can take a look at Figure 9).
Let us deal now with the polynomials

\[ Q_s(x, y) = x^2 + sy^2 - 2\sqrt{2(1 + s)}xy, \quad 1 \leq s \leq 5 + 4\sqrt{2}. \]
Then,
\[ \nabla Q_s(x, y) = \left( 2x - 2\sqrt{2(1 + s)}y, 2sy - 2\sqrt{2(1 + s)}x \right), \]
\[ \|DQ_s(x, y)\|_{D(\xi)} = \sup_{0 \leq \xi \leq \frac{\pi}{4}} \left| 2x \left[ 1 - \sqrt{2(1 + s)} \right] \cos \theta + \left( s\lambda - \sqrt{2(1 + s)} \right) \sin \theta \right|, \]
and thus
\[ \sup_{1 \leq s \leq 5 + 4\sqrt{2}} \|DQ_s(x, y)\|_{D(\xi)} = 2x \sup_{(s, \theta) \in C_2} |g_\lambda(s, \theta)|, \]
with
\[ g_\lambda(s, \theta) = \left( 1 - \sqrt{2(1 + s)} \right) \cos \theta + \left( s\lambda - \sqrt{2(1 + s)} \right) \sin \theta \]
and \( C_2 = [1, 5 + 4\sqrt{2}] \times [0, \frac{\pi}{4}] \). Again, we have several cases:

(6) \((s, \theta) \in (1, 5 + 4\sqrt{2}) \times (0, \frac{\pi}{4})\).

Let us first calculate the critical points of \( g_\lambda \) over \( C_2 \).

\[ \frac{\partial g_\lambda}{\partial s}(s_0, \theta_0) = \frac{-\lambda}{\sqrt{2(1 + s_0)}} \cos \theta_0 + \left( \frac{\lambda - 1}{\sqrt{2(1 + s_0)}} \right) \sin \theta_0, \]
\[ \frac{\partial g_\lambda}{\partial \theta}(s_0, \theta_0) = \left( s_0\lambda - \sqrt{2(1 + s_0)} \right) \cos \theta_0 - \left( 1 - \sqrt{2(1 + s_0)} \lambda \right) \sin \theta_0, \]
so, if \( Dg_\lambda(s_0, \theta_0) = 0 \), using the first expression, we obtain \( \tan \theta_0 = \frac{\lambda}{\sqrt{2(1 + s_0)}\lambda - 1} \), and, using the second one, we obtain \( \tan \theta_0 = \frac{s_0\lambda - \sqrt{2(1 + s_0)}}{1 - \sqrt{2(1 + s_0)}\lambda} \).
Hence, we may say
\[ \frac{s_0\lambda - \sqrt{2(1 + s_0)}}{1 - \sqrt{2(1 + s_0)}\lambda} = \frac{\lambda}{\sqrt{2(1 + s_0)}\lambda - 1}, \]
and thus
\[ s_0 = \frac{2 - \lambda^2}{\lambda^2}. \]
Then, \( \tan \theta_0 = \lambda \) and also, if we want to guarantee that \( 1 < s_0 < 5 + 4\sqrt{2} \), we need \( \sqrt{2} - 1 < \lambda < 1 \).
In that case, \( \sin \theta_0 = \frac{\lambda}{\sqrt{1 + \lambda^2}} \), and \( \cos \theta_0 = \frac{1}{\sqrt{1 + \lambda^2}} \), and then
\[ g_\lambda(s_0, \theta_0) = \frac{-1}{\sqrt{1 + \lambda^2}} + \frac{-\lambda^2}{\sqrt{1 + \lambda^2}} = -\sqrt{1 + \lambda^2}, \]
so
\[ D_0(\lambda) := |g_\lambda(s_0, \theta_0)| = \sqrt{1 + \lambda^2}. \]

(7) \( s = 1, 0 \leq \theta \leq \frac{\pi}{4} \).

Apply lemma 3.1 with \( a = 1 - 2\lambda \) and \( b = \lambda - 2 \). Using \( 0 \leq \lambda \leq 1 \), observe that we always have \( b < 0 \) and \( b \leq a \). Also, \( a < (1 - \sqrt{2}) b \) if and only if \( \lambda > \frac{5 - 3\sqrt{2}}{4} \).
Putting everything together, we can say
\[ \sup_{0 \leq \theta \leq \frac{\pi}{4}} |g_\lambda(1, \theta)| = \begin{cases} 1 - 2\lambda & \text{if } 0 \leq \lambda < \frac{5 - 3\sqrt{2}}{4}, \\ \frac{5 - 3\sqrt{2}}{4}(1 + \lambda) & \text{if } \frac{5 - 3\sqrt{2}}{4} \leq \lambda \leq 1, \end{cases} \]
\[ =: \begin{cases} D_{7,1}(\lambda) & \text{if } 0 \leq \lambda < \frac{5 - 3\sqrt{2}}{4}, \\ D_{7,2}(\lambda) & \text{if } \frac{5 - 3\sqrt{2}}{4} \leq \lambda \leq 1. \end{cases} \]

(8) \( s = 5 + 4\sqrt{2}, 0 \leq \theta \leq \frac{\pi}{4} \).
Apply again lemma 3.1, this time to \( a = 1 - 2 \left( 1 + \sqrt{2} \right) \lambda \) and \( b = (5 + 4 \sqrt{2}) \lambda - 2 \left( 1 + \sqrt{2} \right) \). As usual, we notice that \( a < 0 \) if and only if \( \lambda > \frac{\sqrt{2} - 1}{2} \), \( b < 0 \) if and only if \( \lambda < \frac{2 - \sqrt{2}}{2} \) and \( a < b \) if and only if \( \lambda > \frac{3 + 4 \sqrt{2}}{23} \). All together, we can say that, for \( \frac{3 + 4 \sqrt{2}}{23} < \lambda < \frac{6 - 2 \sqrt{2}}{7} \), we have

\[
\sup_{0 \leq \theta \leq \frac{\pi}{4}} |g_\lambda(5 + 4 \sqrt{2}, \theta)| = \sqrt{a^2 + b^2} = \sqrt{13 + 8 \sqrt{2} - (56 + 40 \sqrt{2}) \lambda + (69 + 48 \sqrt{2}) \lambda^2}.
\]

Also, notice that, for any \( \lambda \in [0, 1] \), we are going to have \( b < - \left( 1 + \sqrt{2} \right) a \) and \( a < \left( 1 - \sqrt{2} \right) b \). Hence,

\[
\sup_{0 \leq \theta \leq \frac{\pi}{4}} |g_\lambda(5 + 4 \sqrt{2}, \theta)|
= \begin{cases} 
\frac{\sqrt{2}}{2} \left[ \left( 1 + 2 \sqrt{2} \right) - (3 + 2 \sqrt{2}) \lambda \right] & \text{if } 0 \leq \lambda < \frac{3 + 4 \sqrt{2}}{23}, \\
\sqrt{13 + 8 \sqrt{2} - (56 + 40 \sqrt{2}) \lambda + (69 + 48 \sqrt{2}) \lambda^2} & \text{if } \frac{3 + 4 \sqrt{2}}{23} \leq \lambda < \frac{6 - 2 \sqrt{2}}{7}, \\
2 (1 + \sqrt{2}) |\lambda| - 1 & \text{if } \frac{6 - 2 \sqrt{2}}{7} \leq \lambda \leq 1,
\end{cases}
\]

As usual, \( g_\lambda(s, \pi) = 0 \) if and only if \( \lambda = 0 \) for \( \lambda \neq 0 \).

Thus,

\[
\sup_{1 \leq s \leq 5 + 4 \sqrt{2}} |g_\lambda(s, 0)| = \max \left\{ |1 - 2 \lambda|, |1 - 2 \left( 1 + \sqrt{2} \right) \lambda| \right\}
= \begin{cases} 
1 - 2 \lambda & \text{if } 0 \leq \lambda < \frac{2 - \sqrt{2}}{2}, \\
2 (1 + \sqrt{2}) |\lambda| - 1 & \text{if } \frac{2 - \sqrt{2}}{2} \leq \lambda \leq 1,
\end{cases}
\]

As usual, \( g_\lambda \left( s, \frac{\pi}{4} \right) = 0 \) if and only if \( s = \frac{(1 + \lambda)^2}{2 \lambda^2} - 1 \)
and since we need to ensure that \( 1 < s_0 < 5 + 4 \sqrt{2} \), we need \( \frac{2 \sqrt{2} - 1}{7} < \lambda < 1 \). In that case,

\[
g_\lambda \left( s_0, \frac{\pi}{4} \right) = -\frac{\sqrt{2}(1 + 3 \lambda^2)}{4 \lambda}.
\]
Hence,
\[
\sup_{1 \leq s \leq 5 + 4\sqrt{2}} \left| g_\lambda \left( \frac{s}{4} \right) \right| = \begin{cases} 
\frac{\sqrt{s}}{4} \left[ (1 + 2\sqrt{2}) - (3 + 2\sqrt{2}) \lambda \right] & \text{if } 0 \leq \lambda < \frac{2\sqrt{2}-1}{7}, \\
\frac{\sqrt{2}}{4\lambda} \left( 1 + 3\lambda^2 \right) & \text{if } \frac{2\sqrt{2}-1}{7} \leq \lambda \leq 1,
\end{cases}
\]
\[
= \begin{cases} 
D_{10,1}(\lambda) & \text{if } 0 \leq \lambda < \frac{2\sqrt{2}-1}{7}, \\
D_{10,2}(\lambda) & \text{if } \frac{2\sqrt{2}-1}{7} \leq \lambda \leq 1.
\end{cases}
\]

Since (the reader can take a look at Figure 10)
\[
D_0(\lambda) \leq \begin{cases} 
D_{8,2}(\lambda) & \text{if } \sqrt{2} - 1 < \lambda < \frac{6-2\sqrt{2}}{7}, \\
D_{8,3}(\lambda) & \text{if } \frac{6-2\sqrt{2}}{7} \leq \lambda < 1,
\end{cases}
\]
we can rule out case (6). Since (see Figures 11 and 12)
\[
D_{7,1}(\lambda) \leq D_{10,1}(\lambda) \text{ for } 0 \leq \lambda < \frac{5-3\sqrt{2}}{7},
\]
\[
D_{7,2}(\lambda) \leq \begin{cases} 
D_{10,1}(\lambda) & \text{if } \frac{5-3\sqrt{2}}{7} \leq \lambda < \frac{2\sqrt{2}-1}{7}, \\
D_{10,2}(\lambda) & \text{if } \frac{2\sqrt{2}-1}{7} \leq \lambda \leq 1,
\end{cases}
\]
we can rule out case (7). Since
\[
D_{8,1}(\lambda) = D_{10,1}(\lambda) \text{ for } 0 \leq \lambda < \frac{2\sqrt{2} - 1}{7},
\]
we can rule out the expression \( D_{8,1}(\lambda) \) of case (8). Since
\[
D_{9,1}(\lambda) = D_{7,1}(\lambda) \text{ for } 0 \leq \lambda < \frac{5-3\sqrt{2}}{7},
\]
\[
D_{9,2}(\lambda) = D_{8,3}(\lambda) \text{ for } \frac{6-2\sqrt{2}}{7} \leq \lambda \leq 1,
\]
we can directly rule out case (9). Furthermore, since (see Figure 13)
\[
D_{8,2}(\lambda) \leq D_{10,2}(\lambda) \text{ for } \frac{5+4\sqrt{2}}{7} \leq \lambda < \frac{6-2\sqrt{2}}{7},
\]
\[
D_{8,3}(\lambda) \leq D_{10,2}(\lambda) \text{ for } \frac{6-2\sqrt{2}}{7} \leq \lambda \leq \frac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7}+8-5\sqrt{2}}{7},
\]
we can conclude that
\[
\sup_{(s, \theta) \in C_2} |g_\lambda(s, \theta)| = \begin{cases} 
D_{10,1}(\lambda) & \text{if } 0 \leq \lambda < \frac{2\sqrt{2}-1}{7}, \\
D_{10,2}(\lambda) & \text{if } \frac{2\sqrt{2}-1}{7} \leq \lambda < \frac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7}+8-5\sqrt{2}}{7}, \\
D_{8,3}(\lambda) & \text{if } \frac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7}+8-5\sqrt{2}}{7} \leq \lambda \leq 1. 
\end{cases}
\]
\[
= \begin{cases} 
\frac{\sqrt{2}}{4\lambda} \left[ (1 + 2\sqrt{2}) - (3 + 2\sqrt{2}) \lambda \right] & \text{if } 0 \leq \lambda < \frac{2\sqrt{2}-1}{7}, \\
\frac{\sqrt{2}}{4\lambda} \left( 1 + 3\lambda^2 \right) & \text{if } \frac{2\sqrt{2}-1}{7} \leq \lambda < \frac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7}+8-5\sqrt{2}}{7}, \\
2 \left( 1 + \sqrt{2} \right) \lambda - 1 & \text{if } \frac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7}+8-5\sqrt{2}}{7} \leq \lambda \leq 1.
\end{cases}
\]
and hence
\[
\sup_{1 \leq s \leq 5 + 4\sqrt{2}} \|DQ_s(x, y)\|_{D(\frac{1}{4})} = \begin{cases} 
\frac{\sqrt{2}}{4\lambda} \left[ (1 + 2\sqrt{2}) x - (3 + 2\sqrt{2}) y \right] & \text{if } 0 \leq y < \frac{2\sqrt{2}-1}{7} x, \\
\frac{\sqrt{2}}{4\lambda} \left( x^2 + 3y^2 \right) & \text{if } \frac{2\sqrt{2}-1}{7} x \leq y < \frac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7}+8-5\sqrt{2}}{7} x, \\
4 \left( 1 + \sqrt{2} \right) y - 2x & \text{if } \frac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7}+8-5\sqrt{2}}{7} x \leq y \leq x.
\end{cases}
\]
Finally, if we compare the results obtained with $P_t$ and $Q_s$, since $\frac{\sqrt{2}(1+3\lambda^2)}{4\lambda} \geq 1 + \frac{\lambda^2}{1-\lambda}$ whenever $\lambda \leq \sqrt{2} - 1$, we obtain

$$\Phi(x, y) = \begin{cases} 
\sqrt{2} \left[ (1+2\sqrt{2})x - (3+2\sqrt{2})y \right] & \text{if } 0 \leq y < \frac{2\sqrt{2}-1}{2}x, \\
\frac{\sqrt{2}}{2y} \left[ x + \frac{x^2}{y} \right] & \text{if } \frac{2\sqrt{2}-1}{2}x \leq y < \left( \sqrt{2} - 1 \right)x, \\
4 \left( 1 + \sqrt{2} \right) y - 2x & \text{if } \left( \sqrt{2} - 1 \right)x \leq y < (2 - \sqrt{2})x, \\
4 \left( 1 + \sqrt{2} \right) y - 2x & \text{if } (2 - \sqrt{2})x \leq y \leq x.
\end{cases}$$
Proof. We just need to calculate words, the inequality

The unconditional constant of the canonical basis of Theorem 4.1.

Consider first the polynomials \( P \) for all \( P \in \mathcal{P}(^2 D (\frac{\pi}{4})) \), then

\[ \|L\|_{D(\frac{\pi}{4})} \leq \left( 2 + \frac{\sqrt{2}}{2} \right) \|P\|_{D(\frac{\pi}{4})}. \]

Moreover, equality is achieved for \( P(x, y) = Q_{5+4\sqrt{2}}(x, y) = x^2 + (5 + 4\sqrt{2}) y^2 - (4 + 4\sqrt{2}) xy \). Hence, the polarization constant of the polynomial space \( \mathcal{P}(^2 D (\frac{\pi}{4})) \) is \( 2 + \frac{\sqrt{2}}{2} \).

4. Unconditional constants for polynomials on sectors

Here, we obtain a sharp estimate on the norm of the modulus of a polynomial in \( \mathcal{P}(^2 D (\frac{\pi}{4})) \) in terms of its norm. That sharp estimate turns out to be the unconditional constant of the canonical basis of \( \mathcal{P}(^2 D (\frac{\pi}{4})) \).

Theorem 4.1. The unconditional constant of the canonical basis of \( \mathcal{P}(^2 D (\frac{\pi}{4})) \) is \( 5 + 4\sqrt{2} \). In other words, the inequality

\[ \|P\|_{D(\frac{\pi}{4})} \leq (5 + 4\sqrt{2}) \|P\|_{D(\frac{\pi}{4})}, \]

for all \( P \in \mathcal{P}(^2 D (\frac{\pi}{4})) \). Furthermore, the previous inequality is sharp and equality is attained for the polynomials \( \pm P_1(x, y) = \pm Q_{5+4\sqrt{2}}(x, y) = \pm [x^2 + (5 + 4\sqrt{2}) y^2 - (4 + 4\sqrt{2}) xy] \).

Proof. We just need to calculate

\[ \sup \left\{ \|P\|_{D(\frac{\pi}{4})} : P \in \text{ext} \left( B_{D(\frac{\pi}{4})} \right) \right\}. \]

In order to calculate the above supremum we use the extreme polynomials described in Lemma 1.2. If we consider first the polynomials \( P_t \), then \( |P_t| = (|t|, 4 + t + 4\sqrt{1 + t}, 2 + 2t + 4\sqrt{1 + t}) \). Now, using Lemma 1.1, we have

\[
\sup_{-1 \leq t \leq 1} \|P_t\|_{D(\frac{\pi}{4})} = \sup_{-1 \leq t \leq 1} \max \left\{ |t|, \frac{1}{2} (|t| + 4 + t + 4\sqrt{1 + t} + 2 + 2t + 4\sqrt{1 + t}) \right\} \\
= \sup_{-1 \leq t \leq 1} \frac{1}{2} (|t| + 6 + 3t + 8\sqrt{1 + t}) = 5 + 4\sqrt{2}.
\]

\[ \square \]

Figure 13. Graphs of the mappings \( D_{8,2}(\lambda), D_{8,3}(\lambda) \) and \( D_{10,2}(\lambda) \).
Notice that the above supremum is attained at $t = 1$. On the other hand, if we consider the polynomials $Q_s$, we have $|Q_s| = (1, s, 2\sqrt{2(1+s)})$. Now, using Lemma 1.1 we have

$$
\sup_{1 \leq s \leq 5+4\sqrt{2}} \|Q_s\|_{D(\pi/4)} = \sup_{1 \leq s \leq 5+4\sqrt{2}} \max \left\{ \frac{1}{2} \left( 1 + s + 2\sqrt{2(1+s)} \right) \right\} = \sup_{1 \leq s \leq 5+4\sqrt{2}} \frac{1}{2} \left( 1 + s + 2\sqrt{2(1+s)} \right) = 5 + 4\sqrt{2}.
$$

Observe that the last supremum is now attained at $s = 5 + 4\sqrt{2}$. □

5. Conclusions

Comparing the results obtained in [11] and [25] for polynomials on the simplex $\Delta$, in [12] for polynomials on the unit square $\Box$, in [15] for polynomials on the sector $D\left(\frac{\pi}{2}\right)$ and the results obtained in the previous sections, we have the following:

|                  | $P(2\Delta)$ | $P(2D\left(\frac{\pi}{2}\right))$ | $P(2D\left(\frac{\pi}{4}\right))$ | $P(2\Box)$ |
|------------------|--------------|------------------------------------|-----------------------------------|------------|
| Markov constants | $2\sqrt{10}$ | $2\sqrt{5}$                        | $4(13 + 8\sqrt{2})$              | $\sqrt{13}$|
| Polarization constants | $3$          | $2$                                | $2 + \frac{\sqrt{2}}{2}$        | $\frac{3}{2}$|
| Unconditional Constants | $2$          | $3$                                | $5 + 4\sqrt{2}$                 | $5$        |

Furthermore, all the constants appearing in the previous table are sharp. Actually, the extreme polynomials where the constants are attained are the following:

1. $\pm(x^2 + y^2 - 6xy)$ for the simplex.
2. $\pm(x^2 + y^2 - 4xy)$ for the sector $D\left(\frac{\pi}{2}\right)$.
3. $\pm(x^2 + (5 + 4\sqrt{2})y^2 - (4 + 4\sqrt{2})xy)$ for the sector $D\left(\frac{\pi}{4}\right)$.
4. $\pm(x^2 + y^2 - 3xy)$ for the unit square.

Compare the previous table with similar results that hold for $2$-homogeneous polynomials on the Banach spaces $\ell^2_1$, $\ell^2_2$ and $\ell^2_\infty$:

|                  | $P(2\ell^2_1)$ | $P(2\ell^2_2)$ | $P(2\ell^2_\infty)$ |
|------------------|----------------|----------------|----------------------|
| Markov constants | $4$            | $2$            | $2\sqrt{2}$          |
| Polarization constants | $2$          | $1$            | $2$                  |
| Unconditional Constants | $\frac{1+\sqrt{2}}{2}$ | $\sqrt{2}$        | $1 + \sqrt{2}$        |

Observe that the Markov constants of the spaces $P(2\ell^2_1)$ and $P(2\ell^2_\infty)$ can be calculated taking into consideration the description of the geometry of those spaces given in [5]. Also, the Markov constant of $P(2\ell^2_2)$ is twice its polarization constant, or in other words, $2$.

On the other hand, the constants appearing in the second line of the previous table are well-known results (see for instance [27]).

Finally, the unconditional constants corresponding to the third line of the previous table were calculated in Theorem 3.5, Theorem 3.19 and Theorem 3.6 of [11].
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POLYNOMIAL INEQUALITIES ON THE $\pi/4$-CIRCLE SECTOR

Departamento de Análisis Matemático,
Facultad de Ciencias Matemáticas,
Plaza de Ciencias 3,
Universidad Complutense de Madrid,
Madrid, 28040 (Spain).

E-mail address:
gdasaraajo@gmail.com
pablo.jimenez.rod@gmail.com
gustavo.fernandez@mat.ucm.es
jseoane@mat.ucm.es