1. INTRODUCTION

1.1. It is well known that a general plane curve of degree $d$ has $\frac{1}{2}d(d-2)(d^2-9)$ distinct bitangent lines. The first (and most) interesting case is that of a smooth plane quartic $X$, whose configuration of 28 bitangents we shall denote by $\theta(X)$, to highlight the correspondence with the odd theta-characteristics of $X$.

The properties of $\theta(X)$ have been extensively studied by the classical geometers since the time of Riemann (Aronhold, Cayley, Hesse, Plücker, Schottky, Steiner, Weber just to quote a few). Such an exceptional interest is not only due to its rich geometry, but also and especially to the connection of this configuration with the classical theory of theta functions.

Some of the features of $\theta(X)$ discovered classically have to do with the possibility of recovering the curve $X$ from various data related to $\theta(X)$. The most celebrated of these results is due to Aronhold: he discovered for every nonsingular quartic $X$ the existence of 288 7-tuples of bitangents, called Aronhold systems, characterized by the condition that for any three lines of such a system the six points of contact are not on a conic; he showed that the full configuration $\theta(X)$ and the curve $X$ can be reconstructed from any one of these Aronhold systems. For an account of his construction we refer to [E-Ch], p. 319, or [K-W], p. 783.

Notice that the definition of Aronhold system requires knowing not only $\theta(X)$, but also the contact points between $X$ and its bitangents. And since there is no known way to recognize the Aronhold systems only knowing $\theta(X)$, the result of Aronhold leaves still unanswered the question below, to which this paper is devoted:

**Question (1st form):** Can the curve $X$ be recovered from its bitangent lines?

Or: if $X$ and $Y$ are two nonsingular plane quartics such that $\theta(X) = \theta(Y)$, does it follow that $X = Y$?

Interest in this question is also related to the classical Torelli theorem. Consider the jacobian variety $J(X)$ of $X$, and let $\Theta_X \subset J(X)$ be a (symmetric) theta divisor. Then the bitangent lines are the images, under the Gauss map $\Theta_X \rightarrow \mathbb{P}^2$, of the points of $\Theta_X$ which are 2-torsion points of $J(X)$ (see [ACGH] for details); a positive answer to the above question would therefore imply the possibility of recovering $X$ from the Gauss images of finitely many points of $\Theta_X$, thus giving a refined and sharper version of the classical Torelli theorem.

Recalling that a nonsingular plane quartic is a canonical curve, our question can be therefore more fruitfully restated as follows:

**Question (2nd form):** Can a nonsingular, non hyperelliptic curve $X$ of genus 3 be re-
constructed from the Gauss images of the (smooth) 2-torsion points of the theta-divisor \( \Theta_X \subset J(X) \)?

The question, in the second form, can be asked for curves of any genus \( g \geq 3 \). This was in fact the main motivation for this work: we plan to come back to such a generalization in a subsequent paper.

1.2. On the other hand the question, in the first form, can be asked for plane curves of any degree \( d \geq 4 \). It is a remarkable fact that for curves of degree \( d \geq 5 \) the answer is affirmative for simple reasons; in other words, every general plane curve of degree \( d \geq 5 \) is uniquely determined by its bitangent lines. In fact each of the \( \frac{1}{2}d(d-2)(d^2-9) \) bitangents corresponds to a node of the dual curve (which has degree \( d(d-1) \)). Thus, if two nonsingular curves \( Y_1 \) and \( Y_2 \) of degree \( d \) have the same bitangents, their dual curves have all the corresponding double points in common, and these contribute with \( 4\left(\frac{1}{2}d(d-2)(d^2-9)\right) \) to the degree of the intersection of the dual curves. This contradicts Bézout if \( d \geq 5 \).

This argument fails for \( d = 4 \), and no other argument of an elementary nature seems to work. Our approach has therefore been different. It is not too difficult to show (see Section 4) that certain singular quartics are uniquely determined by their generalized bitangent lines, i.e. they have the theta-property (see 2.3. for the definitions). Then, combining this with a degeneration argument, we have been able to give a positive answer to our Questions at least in a weak form. Our main theorem is the following (5.2.1).

**THEOREM.** The general plane quartic is uniquely determined by its 28 bitangent lines.

We can actually prove that the same is true for a singular curve, so that a general quartic having \( \delta \) nodes (\( \delta = 0, \ldots, 4 \)) has the theta-property (see sections 4 and 5 for details).

We use a degeneration technique; after having generalized in a natural way the definition of \( \theta(X) \) to certain singular quartics (see sections 2 and 3), we show that every irreducible quartic \( X \) having exactly 3 nodes is uniquely determined by its own \( \theta(X) \) (4.2.1). Then we consider suitable families of smooth curves specializing to trinodal curves. To obtain our result we need to eliminate “bad” limits (such as non-reduced curves). To do that we apply Geometric Invariant Theory to the action of \( PGL(3) \) on two different spaces: the space of all plane quartics and the space \( Sym^{28}(\mathbb{P}^2^*) \), where the configurations of bitangents live.

2. PRELIMINARIES.

2.1. We work over \( \mathbb{C} \).

Let us here recall some fundamental facts of Geometric Invariant Theory, including the so-called GIT-semistable replacement property. Given a projective scheme \( H \) (smooth, irreducible for simplicity) over which a reductive group \( G \) (for our purposes, \( G = PGL(3) \) or \( G = SL(3) \)) acts in a linear way, let \( H^{ss} \) be the open subset of GIT-semistable points.
Then there exists a quotient (in a suitable sense) morphism
\[ q : H^{SS} \rightarrow H^{SS}/\!\!/G \]
with \( H^{SS}/\!\!/G \) a projective scheme. The complement of \( H^{SS} \) in \( H \) is the locus of GIT-unstable points. If \( x \) and \( y \) are GIT-semistable points in \( H \), then \( q(x) = q(y) \) if and only if
\[ \overline{O_G(x)} \cap \overline{O_G(y)} \cap H^{SS} \neq \emptyset \]
where \( O_G(x) \) denotes the orbit of \( x \) via \( G \). The GIT-semistable replacement property is just an application of the existence of such a projective quotient. Let \( T \) be a smooth curve and \( t_0 \in T \) be a point. Let \( \phi : T - \{t_0\} \rightarrow H^{SS} \) be a regular morphism. Then there exists a finite covering \( \rho : T' \rightarrow T \), ramified only over \( t_0 \) (let \( t'_0 = \rho^{-1}(t_0) \)) such that there exist morphisms \( \psi : T' \rightarrow H^{SS} \) and \( \gamma : T' - \{t'_0\} \rightarrow G \) with the property that \( \forall t' \neq t'_0 \) we have \( \phi(\rho(t')) \gamma(t') = \psi(t') \). Moreover, \( \psi(t'_0) \) can be chosen arbitrarily in a (uniquely determined) fiber of \( q \), in particular, one can assume that it has finite stabilizer.

2.2. The space of all plane quartics will be identified with \( \mathbb{P}^{14} \). The group \( G = PGL(3) \) of automorphisms of \( \mathbb{P}^2 \) acts on it in a natural way. Such an action is linear in the sense of GIT; we shall denote by \( V' \subset \mathbb{P}^{14} \) the open subset of quartics that are semistable with respect to the action of \( G \). It is well known ([GIT]) that \( V' \) is made of quartics having at most double points as singularities (not all of them, see below). Thus the only non-reduced quartics that are GIT-semistable are double conics. The (closure of the) set of all double conics in \( \mathbb{P}^{14} \) is a closed subset of dimension 5; we let \( V \) be the complement in \( V' \) of such closed subset. Recall also that \( X \in V \) is GIT-stable if and only if \( X \) is reduced and has no tacnodes. In particular, if \( X \) has only ordinary nodes and ordinary cusps it is GIT-stable (we refer to [GIT], chapter 4, section 2). Notice that \( V \) is open and contains the set \( V^0 \) of all smooth plane quartics.

2.3. Let \( X \) be a plane quartic. We say that a line \( L \) is a bitangent of \( X \) if \( L \) intersects \( X \) in smooth points and if the scheme \( X \cap L \) is everywhere nonreduced.

Recall that every smooth plane quartic has exactly 28 distinct bitangent lines. Denote \( P_n := Sym^n(\mathbb{P}^{2*}) \); we can define a map
\[ \theta : V^0 \rightarrow P_{28} \]
such that for every \( X \in V^0 \), \( \theta(X) \) is the set of 28 bitangents of \( X \). We shall use the notation \( \theta(X) \) also to indicate the corresponding plane curve of degree 28. We call \( \theta(X) \) the theta-curve of \( X \).

Our goal now is to extend such a definition to all curves in \( V \). First we need the following

**Definition.** Let \( X \) be a plane quartic. A line \( L \) is called a theta-line of \( X \) if either \( L \subset X \), or if \( X \cap L \) is everywhere non-reduced. Let \( X \) be reduced and not containing \( L \); if \( i \) is a non-negative integer and \( L \) contains exactly \( i \) singular points of \( X \), we shall say that \( L \) is a theta-line of type \( i \).
Some examples: the bitangent lines of a smooth curve are theta-lines of type zero. If \( X \) has one node (one cusp) and no other singularities, then (by the Hurwitz formula applied to the projection from the singular point) there exist exactly 6 distinct lines passing through the node (cusp) and tangent to \( X \). These 6 lines are theta-lines of type 1.

We shall see more examples later. Now we prove the following simple:

**Lemma 2.3.1.** There exists a natural extension of the morphism \( \theta \) to the whole of \( V \), such that for every \( X \in V \), all components of \( \theta(X) \) are theta-lines of \( X \).

*Remark.* In particular, if \( X_0 \) is any singular curve in \( V \) and \( X' \to T \) is any deformation of \( X_0 \) to smooth curves, then the limit \( \Theta_0 \) of the theta curves \( \theta(X_t) \) of the smooth fibers does not depend on the choice of the deformation.

*Proof.* Let \( X \in V \); then (see 2.2.) \( X \) is reduced and has no point of multiplicity greater than 2. Thus, it is easy to see that the set of all theta-lines of \( X \) is finite.

Now consider the incidence correspondence

\[
J^0 \subset V^0 \times P_{28}
\]

defined by \( J^0 = \{(X; \theta(X)) : X \in V^0\} \), or

\[
J^0 = \{(X; L_1, \ldots, L_{28}) : L_i \neq L_j, L_i \cap X \text{ is everywhere nonreduced } \forall i = 1, \ldots, 28\}.
\]

Let \( J \) be the closure of \( J^0 \) in \( V \times P_{28} \) and let \( \rho : J \to V \) be the projection. If \( X \) is any curve in \( V \) and \( \Theta \in \rho^{-1}(X) \), then \( \Theta \) is a set of 28 lines (not necessarily distinct) and such that if \( L \) is a line appearing in \( \Theta \), \( L \) is a theta-line of \( X \). Since there are only finitely many theta-lines for any given \( X \in V \) and since \( \deg \Theta = 28 \) we conclude that \( \rho \) has finite fibers. Of course, \( \rho \) is one-to-one on the open subset \( V^0 \) of smooth curves; moreover \( J \) is irreducible (because \( V^0 \), and hence \( J^0 \), is irreducible). Therefore, by the Zariski connectedness principle (\( V \) is smooth) we conclude that \( \rho \) is one-to-one everywhere. Finally, for every \( X \in V \) we can define \( \theta(X) \) by the rule \( \rho^{-1}(X) = \{(X, \theta(X))\} \) and we are done. \( \square \)

**Definition.** For any curve \( X \) in \( V \), the curve represented by \( \theta(X) \) (defined in the above Lemma) will be called the *theta-curve* of \( X \).

Section 3 is devoted to a precise description of the theta-curve of a singular quartic.

**Definition.** Let \( X \) be a curve in \( V \), and let \( \Theta = \theta(X) \) be its theta-curve. If \( X \) is the only curve in \( V \) having \( \Theta \) as theta-curve (i.e. if \( Y \in V \) is such that \( \theta(Y) = \Theta \), then \( Y = X \)), then we shall say that \( X \) has the *theta-property*.

2.4. Our goal is to show that a general quartic has the theta-property. To do that we shall consider the action of \( G \) on \( P_n := \text{Sym}^n(\mathbb{P}^2) \). David Mumford, in [GIT] 4.4, gives necessary and sufficient conditions for a point in \( P_n \) to be GIT-stable, semistable and
unstable with respect to the natural action of \( G \). His criterion is the following: \( \Sigma \in P_n \) is semistable if and only if the plane curve of degree \( n \) corresponding to \( \Sigma \) contains no point of multiplicity greater than \( 2n/3 \) and no line of multiplicity greater than \( n/3 \). In particular a point \( \Sigma \) in \( P_{28} \) is GIT-unstable if and only if the corresponding plane curve of degree 28 has either a point of multiplicity at least 19 or a line of multiplicity at least 10. Otherwise, if \( \Sigma \) has at most points of multiplicity 18 and lines of multiplicity 9, \( \Sigma \) is GIT-stable (there are no strictly semistable points in this situation).

3. THE STRUCTURE OF THE THETA CURVE OF A SINGULAR QUARTIC

3.1. Let \( X \) be an irreducible quartic in \( V \). Then \( \theta(X) \) consists of 28 lines of which \( b_i \) are of type \( i \), \( i = 0, 1, 2 \), and each of them appears with a multiplicity to be computed.

Recall that for an irreducible plane curve \( C \) having only nodes, ordinary cusps and ordinary tacnodes as singularities, and only ordinary flexes, the degree \( d \), the class (i.e. the degree of the dual curve) \( m \), and the numbers \( \delta \), \( \kappa \), \( \tau \), \( b \) of nodes, cusps, tacnodes, flexes and bitangents (respectively) are related by the classical Plücker formulas:

\[
\begin{align*}
    m &= d(d - 1) - 2\delta - 3\kappa - 4\tau \\
    d &= m(m - 1) - 2b - 3f - 4\tau \\
    f &= 3d(d - 2) - 6\delta - 8\kappa - 12\tau
\end{align*}
\]

(see [W]). From these formulas one easily computes the following expression for \( b \) in terms of \( d \), \( \delta \), \( \kappa \) and \( \tau \):

\[
b = N_d - (d + 2)(d - 3)(2\delta + 3\kappa + 4\tau) + 2\delta(\delta - 1 + 4\tau) + 6\kappa(\delta + 2\tau) + \frac{9\kappa(\kappa - 1)}{2} + 2\tau(4\tau - 3)
\]

where

\[
N_d = \frac{1}{2}d(d - 2)(d^2 - 9)
\]

is the number of bitangents of a nonsingular curve of degree \( d \).

Let’s specialize to the case of irreducible quartics. The above formula computes the number \( b_0 \) of theta-lines of type 0, giving:

\[
b_0 = 28 - 6(2\delta + 3\kappa + 4\tau) + 2\delta(\delta - 1 + 4\tau) + 6\kappa(\delta + 2\tau) + \frac{9\kappa(\kappa - 1)}{2} + 2\tau(4\tau - 3)
\]

Obviously

\[
b_2 = \binom{\delta + \kappa + \tau}{2}
\]

The number \( b_1 \) can be computed as follows. From the Hurwitz formula it follows that through each double point the number of theta-lines is \( 2(3 - \delta - \kappa - 2\tau) + 2 \) (just consider the degree-2 map to \( \mathbb{P}^1 \) given by projecting from the double point). Among these one has
the theta-lines joining the double point with the cusps different from it, if any, which are of type 2, and therefore already counted. This simple rule allows us to compute \( b_1 \) case by case.

Notice that these computations are valid even if we allow the irreducible quartic \( X \) to have hyperflexes (i.e. nonsingular points where the tangent line has multiplicity of intersection equal to four). In fact each such hyperflex diminishes by two the number of ordinary flexes. At the same time it corresponds to a triple point of the dual curve which diminishes by eight its class (see [S], §21, for details). All this accounts for an obvious modification in the second of the above Plücker formulas. Since the first and third formula obviously remain unchanged, the final expression of \( b_0 \) stays the same.

3.2. The following table lists the resulting numbers for all possible cases of irreducible quartics in \( V \):

\[
\begin{array}{cccc}
(\delta, \kappa, \tau) & b_0 & b_1 & b_2 \\
(0, 0, 0) & 28 & 0 & 0 \\
(1, 0, 0) & 16 & 6 & 0 \\
(2, 0, 0) & 8 & 8 & 1 \\
(3, 0, 0) & 4 & 6 & 3 \\
(0, 1, 0) & 10 & 6 & 0 \\
(0, 2, 0) & 1 & 6 & 1 \\
(0, 3, 0) & 1 & 0 & 3 \\
(1, 1, 0) & 4 & 7 & 1 \\
(2, 1, 0) & 2 & 4 & 3 \\
(1, 2, 0) & 1 & 2 & 3 \\
(0, 0, 1) & 6 & 5 & 0 \\
(1, 0, 1) & 2 & 5 & 1 \\
(0, 1, 1) & 0 & 4 & 1 \\
\end{array}
\]

Our values for \( b_0 \) agree with Joe Harris’s Theorem 3.7 of [H] (wherever ours and his table intersect).

3.3. The computation of the multiplicities of the theta-lines in all the above cases is contained in the following lemma.

**Lemma 3.3.1.** Let \( X \) be an irreducible quartic in \( V \). Then:

a) A theta-line of type 0 appears in \( \theta(X) \) with multiplicity 1.
b) A theta-line of type 1 containing a node (resp. a cusp) appears in \( \theta(X) \) with multiplicity 2 (resp. 3).
c) A theta-line containing two nodes appears in \( \theta(X) \) with multiplicity 4.
d) A theta-line containing two cusps appears in \( \theta(X) \) with multiplicity 9.
e) A theta-line containing a node and a cusp appears in \( \theta(X) \) with multiplicity 6.
f) A theta-line of type 1 containing a tacnode appears in \( \theta(X) \) with multiplicity 4, unless it is the tacnodal tangent, which instead appears with multiplicity 6.
g) A theta-line of type 2 containing a tacnode appears in \( \theta(X) \) with multiplicity 8 if it contains a node, and 12 if it contains a cusp.

Proof. Let us suppose first that \( X \) has no hyperflexes. Let \( \text{Sing}(X) \subset X \) be its singular locus, and let \( X^0 = X \setminus \text{Sing}(X) \). Consider in \( X^0 \times X^0 \) the “tangential correspondence”:

\[
Z^0 = \{(x, y): x \neq y \text{ and } y \text{ belongs to the tangent line to } X \text{ at } x\}
\]

The first projection \( Z^0 \to X^0 \) is a generically 2 to 1 map; we let \( Z = \overline{Z^0} \subset X \times X \) and \( \pi : Z \to X \) the first projection.

Now consider a 1-parameter family of plane quartics \( \mathcal{X} \to T \) such that \( X_0 = X \) and \( X_t \) is nonsingular for all \( t \neq t_0 \). We have a corresponding family of double covers:

\[
\begin{array}{ccc}
Z & \xrightarrow{\Pi} & X^0 \\
\downarrow & & \\
\mathcal{X} & \to & T
\end{array}
\]

such that \( Z_t \to X_t \) is the tangential correspondence for all \( t \in T \). Let \( x_0 \in X \) be on a theta-line. In order to compute the multiplicity of this theta-line in \( \theta(X) \) we have to compute the number of branch points of \( \Pi_t \) which tend to \( x_0 \) as \( t \) tends to \( t_0 \). Let \( v \) be a local coordinate in \( T \) around \( t_0 \). We have the following cases:

\( x_0 \in X^0 \) is on a theta-line of type 0. Then \( x_0 \) is an ordinary branch point of \( \pi \) and therefore it is a limit of just one branch point of \( \Pi_t \). This proves (a).

\( x_0 \in X^0 \) is on a theta-line of type 1 containing a node.
The surface \( Z \) has equation \( xy - v = 0 \) locally at \( \Pi^{-1}(x_0) \), and \( \Pi(x, y, v) = (x + y, v) \); the map \( \mathcal{X} \to T \) is locally \((u, v) \mapsto v \) \((u \) being a local coordinate in \( X \) around \( x_0 \)). Then for a given \( v \neq 0 \) the branch points of \( \Pi_t \) are \((\pm2\sqrt{v}, v) \) so there are two branch points tending to \( x_0 \) and this proves (b) in the nodal case.

\( x_0 \in X^0 \) is on a theta-line of type 1 containing a cusp.
Then the surface \( Z \) has equation \( x^2 - y^3 + v = 0 \) locally at \( \Pi^{-1}(x_0) \), and \( \Pi(x, y, v) = (y, v) \). For a given \( v \neq 0 \) the branch points of \( \Pi_t \) are \((\zeta_1, v), (\zeta_2, v), (\zeta_3, v) \), where \( \zeta_1, \zeta_2, \zeta_3 \) are the cubic roots of \( v \); this proves (b) in the cuspidal case.

\( x_0 \in X^0 \) is on a theta-line of type 1 containing a tacnode and different from the tacnodal tangent.
The surface $\mathcal{Z}$ has equation $x^2 - y^4 + v = 0$ locally at $\Pi^{-1}(x_0)$, and $\Pi(x, y, v) = (y, v)$. For a given $v \neq 0$ the branch points of $\Pi_t$ are $(\eta_i, v)$, $i = 1 \ldots 4$, where $\eta_1, \ldots, \eta_4$ are the quartic roots of $v$. This proves the first part of (f).

$x_0 \in X$ is on a theta-line of type 2.

Assume first that $x_0$ is a node and call the other singular point on the theta-line $p$. Let $\mathcal{X} \to T$ be a family of quartics such that $X_0 = X$ and for each $t \neq t_0$ $X_t$ has a node $x_t$ specializing to $x_0$, and no other singularity. $\mathcal{X}$ has a double curve generated by the varying node $x_t$. Let $\nu : \mathcal{X}' \to \mathcal{X}$ be the normalization. Then $\mathcal{X}' \to T$ is a family of generically smooth curves. Let $\sigma : Z' = \mathcal{X}' \times_T Z \to X'$ and let

$$
\sigma_0 : \quad Z' \quad \to \quad X' \\
\downarrow \nu_0 \\
X
$$

be the central fiber. Let $\nu_0^{-1}(x_0) = \{y, y'\}$. Then, arguing as before, we deduce that $y$ and $y'$ are each limit of the appropriate number $\ell$ (two, three, four) of branch points of $\sigma_t$. Thus $x_0$ counts with multiplicity $2\ell$. This takes care of theta-lines of type 2 containing a node.

The case of a theta-line joining a tacnode and a cusp is treated similarly, by considering a degeneration of binodal curves whose nodes specialize to the tacnode of $X$.

There are two remaining cases: a theta-line joining two cusps and the tacnodal tangent line. Of course they cannot occur on the same curve. If $X$ has two cusps and no other singularities and $L$ is the line joining them, then the multiplicity $\text{mult}_{\theta(X)}(L)$ of $L$ in $\theta(X)$ is given by the formula

$$\text{mult}_{\theta(X)}(L) = 28 - 1 - 6 \cdot 3 = 9$$

since $X$ has one theta-line of type 0 and six of type 1 (see the table above).

Similarly, if $X$ has a tacnode and no other singularities and $L$ is the tacnodal tangent

$$\text{mult}_{\theta(X)}(L) = 28 - 6 - 4 \cdot 4 = 6$$

since such a tacnodal quartic has six theta-lines of type 0 and four of type 1 (other than the tacnodal tangent).

The careful reader can verify that the same conclusion can be achieved by the same analysis applied to other examples.

Finally notice that if $X$ has hyperflexes, then it can be obtained as a limit of curves without hyperflexes, and having the same type of singularities. Since the multiplicities of the components of the theta-curves can only increase by specialization and the number of theta-lines of type $b_i$ is constant, such multiplicities must remain unchanged. The conclusion follows. \[\Box\]

Remark. The proof shows that theta-lines of the same type, passing through the same type of singularity, all have the same multiplicity.
Notice also that most of the argument does not require $X$ to be irreducible.

3.4. Now we describe the theta curve for those reducible curves in $V$ having finite stabilizer (which is all we need); we refer to [AF1] for a study of quartics with infinite stabilizer. In this case, the support of the theta-curve is easy to find, by the Hurwitz formula, or looking at the dual curves (when $X$ is a union of conics). The multiplicities of those components of $\theta(X)$, which are not also components of $X$, are computed just like in 3.3.1 (see the above remark). If $X$ contains a line $L$, then $L \subset \theta(X)$; its multiplicity is easily obtained in each case (see below).

Union of two smooth conics: $X = C_1 \cup C_2$
1. If $X$ has 4 nodes, then $X$ is a so-called Split curve and it is treated in detail in section 4.1. below.
2. If $X$ has 2 nodes $n_1$ and $n_2$ and a tacnode $t$, that is, the two conics are tangent in $t$, then
   $$\theta(X) = L_1 \cup L_2 \cup 4E \cup 6L \cup 8(M_1 \cup M_2)$$
where $L_1$ and $L_2$ are the two lines of type 0 (corresponding to the two nodes of the dual curve of $X$, which is also the union of two conics, tangent at one point); $E$ is the line through the two nodes, $L$ is the tangent line to the conics at $t$ (which is a tacnodal tangent) and $M_i$ joins $t$ and $n_i$ (see 3.3.1).

Union of a line $L$ and an irreducible cubic $C$ meeting transversally at $n_1, n_2, n_3$
3. If $X$ has only 3 nodes and no other singularity, then by Hurwitz formula
   $$\theta(X) = 2(\cup_{i=1}^3 (M_1^i \cup M_2^i \cup M_3^i)) \cup 4L$$
where $M_i^j$ is a theta-line of type 1 passing through the node $n_i$ and tangent to $C$.
4. If $C$ has a node $n$, then
   $$\theta(X) = 2(\cup_{i=1}^3 (M_1^i \cup M_2^i)) \cup 4(\cup_{i=1}^3 (E_i)) \cup 4L$$
where $E_i$ joins $n$ with the node $n_i$.
5. If $C$ has a cusp $c$, then
   $$\theta(X) = 2(\cup_{i=1}^3 (M_i^c)) \cup 6(\cup_{i=1}^3 (E_i)) \cup 4L$$
where $E_i$ joins $c$ with the node $n_i$.

Union of a line $L$ and an irreducible cubic $C$ tangent to $L$ at $t$
Here we call $n$ the remaining point of intersection of $L$ and $C$; recall that, for $X$ to be GIT-semistable, it is necessary that $t$ is not an inflectionary point of $C$ ([GIT] 4.2). Clearly $t$ is a tacnode for $X$.
6. If $C$ is smooth, then
   $$\theta(X) = 2(M_1 \cup M_2 \cup M_3) \cup 4(\cup_{i=1}^3 (K_i)) \cup 6L$$
where \( n \in M_i \) and \( t \in K_j \) (all these lines are of type 1)

7. If \( C \) has a node \( m \)

\[
\theta(X) = 2M \cup 4E \cup 8F \cup 4(\cup_{i=1}^{2}(K_i)) \cup 6L
\]

where \( E \) is the line through \( n \) and \( m \) and \( F \) the line through \( m \) and \( t \).

8. If \( C \) has a cusp \( c \)

\[
\theta(X) = 4K \cup 6E \cup 12F \cup 6L
\]

where \( t \in K \) and \( E \) is the line through \( n \) and \( c \) and \( F \) the line through \( c \) and \( t \).

**X contains two distinct lines L and M**

Call \( n = L \cap M \).

9. \( X = L \cup M \cup C \), with \( C \) smooth and transverse to both lines, then

\[
\theta(X) = 2(L_1 \cup L_2) \cup 4(E_1 \cup E_2 \cup E_3 \cup E_4 \cup L \cup M)
\]

where \( L_i \) contains \( n \) and is tangent to \( C \), and the remaining are the 6 lines joining the 4 pts of \( C \cap (L_1 \cup L_2) \).

10. \( X = L \cup M \cup C \), with \( C \) smooth, tangent to \( M \) at a point \( t \) and meeting \( L \) at \( n_1 \) and \( n_2 \).

\[
\theta(X) = 2L_1 \cup 4L \cup 6M \cup 8(E_1 \cup E_2)
\]

where \( L_1 \) contains \( n \) and is tangent to \( C \), \( E_i \) joins \( t \) with \( n_i \).

11. \( X \) is the union of 4 general lines \( L_1, ..., L_4 \), then

\[
\theta(X) = 4(L_1 \cup .... \cup L_7)
\]

in fact, of the 6 nodes of \( X \) there are 3 pairs not lying on any line of \( X \); \( L_5, L_6, L_7 \) are the lines joining such pairs.

**3.5.** The behaviour of theta-curves of quartics in \( V \) with respect to the action of \( G = PGL(3) \) on \( P_{28} \) is described by the following

**Lemma 3.5.1.** Let \( X \in V \) be such that \( Stab_G(X) \) is finite; let \( \Theta = \theta(X) \). Then either \( \Theta \) is GIT-stable (as a point in \( P_{28} \), acted on by \( G \)) or \( X \) is a tacnodal curve, more precisely, \( X \) is one of the following types

1. irreducible with one tacnode and no other singularity;
2. irreducible with one tacnode and one node;
3. irreducible with one tacnode and one cusp;
4. the union of two smooth conics tangent at one point and intersecting transversally in 2 other points;
5. the union of a line \( L \) and a smooth cubic \( C \), with \( C \) tangent to \( L \) at a non-inflectionary point;
6. the union of a line \( L \) and an irreducible, nodal cubic \( C \), with \( C \) tangent to \( L \) at a non-inflectionary point;
7. the union of a line $L$ and an irreducible, cuspidal cubic $C$, with $C$ tangent to $L$ at a non-inflectionary point.

8. the union of a smooth conic $C$ and two lines $L$ and $M$ such that $M$ is tangent to $C$ and $L$ is transverse.

Proof. By 2.4. all semistable points in $P_{28}$ are actually stable, and, if $\Theta$ is not stable, then it has either a line of multiplicity at least 10 or a point of multiplicity at least 19. We refer to [AF1] for the list of quartics with infinite stabilizer. Looking at the description in 3.3. and 3.4., we see that the only quartics in $V$ with finite stabilizer, having a theta-line of multiplicity at least 10 are those with a tacnode and a cusp, i.e. cases 3 and 7 above. Analogously, the only theta-curves having a point of multiplicity at least 19 are the theta-curves of tacnodal curves, where the tacnode of $X$ becomes a point of multiplicity 22 for $\theta(X)$.

Remark. From the results of this section, it follows easily that if two quartics in $V$ have the same theta-curve, then they have the same singularities and decomposition in irreducible components of the same type. Here are some relevant examples, for all of which we restrict our attention to curves in $V$.

(a) $X$ is smooth if and only if $\theta(X)$ is made of 28 distinct lines.

Notice now that a line of multiplicity 2 is a theta-line of type 1 passing through a node, thus

(b) $X$ has one node and no other singularity if and only if $\theta(X)$ contains 6 incident lines of multiplicity 2 and no line of higher multiplicity. The node is determined by the incidence point of such 6 lines.

If $\theta(X)$ contains lines of multiplicity 1, then $X$ does not contain any line as a component (a cubic does not have any bitangent line!). In such a case, the multiplicity of a line of $\theta(X)$ uniquely determines the type of singularities that it contains. We can apply this to the following examples

(c) $X$ has 2 nodes and no other singularity if and only if $\theta(X)$ contains some line of multiplicity 1, one line of multiplicity 4 and two sets of 4 incident lines of multiplicity 2. The two nodes are determined by the 2 incidence points of the two sets above.

(d) $X$ has 3 nodes (and no other singularity) if and only if $\theta(X)$ contains some line of multiplicity 1 and 3 non-concurrent lines of multiplicity 4. The nodes are the 3 vertices of the triangle formed by such 3 lines.

(e) $X$ is the union of 2 smooth conics meeting transversally ($X$ is a “split” curve, see below) if and only if $X$ has 4 non-colinear nodes and no other singularity, if and only if $\theta(X)$ contains some line of multiplicity 1 and a set of 6 lines of multiplicity 4 containing 4 triples of incident lines. The nodes are the intersection points of such 4 triples.
4. SPECIAL CASES

4.1. We prove here the theta-property for two types of singular quartics.

**Definition.** Let $X$ be a plane quartic having two irreducible components: $X = C_1 \cup C_2$ such that $C_1$ and $C_2$ are smooth conics meeting transversally. Then $X$ is called a *split curve*. Thus a split curve $X = C_1 \cup C_2$ has 4 nodes and no other singularities, in particular it is GIT-stable. Denote by $\{n_1, n_2, n_3, n_4\} = C_1 \cap C_2$ the nodes of $X$. Let $E_{i,j}$ be the line through $n_i$ and $n_j$, it is a theta-line of type 2 for $X$. The theta-curve $\Theta$ of $X$ is the following

$$\Theta = L_1 \cup L_2 \cup L_3 \cup L_4 \cup 4\left(\bigcup_{i,j} E_{i,j}\right)$$

where each of the 6 lines $E_{i,j}$ counts with multiplicity 4 by 3.3.1; the remaining 4 lines $L_i$ are of type 0, and they are determined by the intersection of the dual curves $C_1^*$ and $C_2^*$ in the dual plane.

We shall now prove that a split curve is determined, among all curves in $V$, by its theta-curve.

**Proposition 4.1.1.** *Split curves have the theta-property.*

*Proof.* We keep using the notation above. Let $Y \in V$ be a curve such that $\theta(Y) = \Theta$, where $\Theta$ is the theta-curve of a split-curve $X$. By case (e) of the remark at the end of the previous section, $Y$ is also a split curve and has the same nodes of $X$.

Let $P \cong \mathbb{P}^1$ be the pencil of all conics passing through $n_1, ..., n_4$. Of course $Y$ and $X$ are union of two conics belonging to $P$. Now consider the line $L_1$; in $P$ there are exactly two conics that are tangent to $L_1$, in fact $P$ cuts on $L_1 \cong \mathbb{P}^1$ a linear series of degree 2 and dimension 1 which has 2 ramification points, corresponding to those conics of $P$ that intersect $L_1$ in a unique point. We conclude that such conics are the original $C_1$ and $C_2$ and that $X = Y$. $\square$

4.2. Now we treat a second special case: that of irreducible quartics with three nodes. We call such curves *trinodal*. If $X$ is a trinodal curve, denote by $n_1, n_2, n_3$ its nodes; let $\Theta$ be its theta-curve, then $\Theta$ contains the three lines $\overline{n_i, n_j}$ with multiplicity 4, three pairs of theta-lines of type 1, $L_i, M_i$ with $i = 1, 2, 3$ such that $L_i \cap M_i = n_i$ (hence each of these six lines appears with multiplicity 2 in $\Theta$) and four theta-lines of type 0.

**Proposition 4.2.1.** *A trinodal curve has the theta-property.*

*Proof.* Let $X$ and $\Theta$ be as above. Let $Y \in V$ be such that $\theta(Y) = \Theta$; then, by the remark at the end of 3.5. $Y$ is also a trinodal curve with the same nodes of $X$. Let $\gamma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the quadratic transformation with base points $n_1, n_2, n_3$ (that is, $\gamma$ is the birational map associated to the linear system of all conics passing through $n_1, n_2, n_3$). Then $\gamma$ maps the set of all quartics that are singular at $n_1, n_2, n_3$ bijectively to the set of all conics in the image $\mathbb{P}^2$. Thus $X$ and $Y$ are mapped to smooth conics $C$ and $D$, and $X$ and $Y$ are
distinct if and only if $C$ and $D$ are distinct. Consider now the six theta-lines of type 1, $L_i, M_i$ for $i = 1, 2, 3$; their images under $\gamma$ are six lines that are tangent to $C$ and $D$. But now given 6 lines in $\mathbb{P}^2$, there exists at most one smooth conic that is tangent to all of them (this is seen easily by passing to the dual plane $\mathbb{P}^2^*$, where the 6 lines correspond to 6 points and the dual of a smooth conic tangent to those 6 lines is a smooth conic passing through those 6 points). We hence conclude that $C = D$ and therefore $X = Y$.  

5. THE MAIN RESULT

5.1. Our main goal is to prove Theorem 5.2.1. The two next statements, 5.1.1 and 5.1.2 are completely independent but they are proved in a very similar fashion. We include them for the sake of completeness, and to follow the pattern started in the previous section, proving the theta property for nodal quartics with decreasing number of nodes.

We shall say that a plane curve is binodal if it has two ordinary double points and no other singularity.

**Proposition 5.1.1.** A general binodal quartic has the theta-property.

*Proof.* By contradiction, assume that the statement is false. Then we can construct a family of curves $X \rightarrow T$ over a smooth curve $T$ with a marked point $t_0$, with the following properties: there are two fixed points $n_1$ and $n_2$ in $\mathbb{P}^2$ such that if $t \neq t_0$, the fiber $X_t$ has two nodes at $n_1$ and $n_2$ and no other singularities; the special fiber $X_0$ is trinodal so that necessarily two of the nodes of $X_0$ are in $n_1$ and $n_2$. We also assume that each theta-line of $X_0$ meets it in two distinct points (this condition is satisfied for a general trinodal curve). Finally, we require that $X_t$ does not have the theta-property, so that if we denote by $\Theta_t = \Theta(X_t)$, there exists a second family $\mathcal{Y} \rightarrow T$ with $Y_t \in V$ if $t \neq t_0$, $X_t \neq Y_t$, and such that $\theta(Y_t) = \Theta_t$ for every $t$.

Notice that $Y_t$ is necessarily a binodal curve with nodes in $n_1$ and $n_2$, by the analysis in section 3. Consider now $Y_0$; there are of course two possibilities: (a) $Y_0 \in V$; (b) $Y_0 \notin V$.

Case (a).

By construction $\theta(Y_0) = \theta(X_0)$; this implies that $Y_0 = X_0$ because $X_0$ has the theta-property, by 4.2.1. Thus the two families $X$ and $\mathcal{Y}$ have the same special fiber. We shall now prove that this forces $X_t = Y_t$ for every $t$. Denote by $W$ the 8-dimensional linear subspace of $\mathbb{P}^{14}$ consisting of all quartics that have a singular point at $n_1$ and $n_2$. Our $X$ and $\mathcal{Y}$ are thus families of curves in $W \cap V$. Let $\theta_W : W \cap V \rightarrow P_{28}$ be the restriction of $\theta$, we shall prove the following claim:

$\theta_W$ is an immersion at $X_0$.

It is clear that this implies that $X_t = Y_t$, which is a contradiction.

To prove the claim, let us first describe, à la Zariski ([Z]), the tangent space to $W$ at any point $X$. This is identified to the linear series cut on $X$ by curves of degree 4, doubled at the nodes of $X$, that is, let $\nu : X^\nu \rightarrow X$ be the normalization of $X$, then

$$T_X W \cong H^0(X^\nu, \nu^*(\mathcal{O}_X(4) \otimes \mathcal{O}_X(-2n_1 - 2n_2)))$$
let $L := \nu^*(\mathcal{O}_X(4) \otimes \mathcal{O}_X(-2n_1 - 2n_2))$, so that $\deg L = 8$. Now, let $L_1, ..., L_4$ be the 4 lines of type 0 contained in $\theta(X_0)$, and let $M$ be any theta-line of type 1; denote by $W_{L_1, ..., L_4}$ the variety of curves in $W$ that are bitangent to $L_1, ..., L_4$, and tangent to $M$. Then we have (again using Zariski’s theory)

$$T_{X_0}W_{L_1, ..., L_4} \cong H^0(X_0^\nu, \mathcal{L} \otimes \nu^*\mathcal{O}(-p_1 - p_2 - ... - p_8 - q))$$

where $p_1, ..., p_8$ are the 8 points of intersection of $X_0$ with $L_1 \cup ... \cup L_4$ (with $p_i \neq p_j$ by our assumption that $X_0$ has no hyperflexes), and $q$ is the smooth point of intersection of $M$ and $X$. Therefore, since $\deg \mathcal{L} \otimes \mathcal{O}(-p_1 - p_2 - ... - p_8 - q) < 0$, we obtain that $T_{X_0}W_{L_1, ..., L_4} = 0$.

Now notice that $T_{X_0}\theta_W^{-1}(\theta(X_0)) \subset T_{X_0}W_{L_1, ..., L_4}$ hence a fortiori $T_{X_0}\theta_W^{-1}(\theta(X_0)) = 0$, so that $\theta_W$ is an immersion at $X_0$.

This concludes the proof of our proposition in case (a).

Case (b).

Suppose now that $Y_0 \notin V$. Thus $Y_0$ is either GIT-unstable, or it is a double conic (and hence it has infinite stabilizer). We now apply the GIT semistable replacement property (see 2.1.); this gives, up to replacing $T$ with a finite covering ramified only over $t_0$, a new family $Z \to T$ of plane quartics, and a morphism $\gamma : T - \{t_0\} \to G$ such that away from $t_0$ we have

$$Z_t = Y_t^{\gamma(t)},$$

and we can assume that $Z_0$ is GIT-semistable with finite stabilizer.

We thus have that $\theta(Z_t) = \Theta_t^{\gamma(t)}$ away from $t_0$.

Recall now that by 3.5.1 we have that if $\Theta \in P_{28}$ is the theta-curve of a reduced quartic having at most ordinary nodes or ordinary cusps as singularities, then $\Theta$ is GIT-stable. And the curves in $V$ having GIT-unstable theta curve are exactly the tacnodal curves.

Consider now the (rational) quotient map

$$q : P_{28} \to P_{28}/G$$

regular on the set of theta-curves of smooth, nodal and cuspidal quartics.

We have, by construction, $q(\Theta_t) = q(\theta(Z_t))$ for every $t \neq t_0$. Hence $\theta(Z_0)$ lies in the closure of the orbit of $\Theta_0$. Denoting $\Theta'_0 = \theta(Z_0)$ we write:

$$\Theta'_0 \in \overline{O_G(\Theta_0)}.$$ 

There are, of course, two possibilities: $\Theta'_0$ may or may not be in $O_G(\Theta_0)$.

If $\Theta'_0 = \Theta_0^g$ for some $g \in G$, then $Z_0$ is a trinodal curve (by the analysis in Section 3) and, as such, it has the theta-property (see 4.2.1). In particular, since $\theta(X_0^g) = \Theta'_0$ we obtain that $Z_0 = X_0^g$. But then, when applying the GIT-semistable replacement property to $Y \to T$, we can choose $X_0$, rather than $Z_0$, as a semistable replacement for $Y_0$ (2.1.). Thus, after acting again with $G$, we are back to case (a).
Finally, suppose that $\Theta'_{0}$ is not in the $G$-orbit of $\Theta_{0}$. Since $\Theta_{0}$ is GIT-stable (by 3.5.1), $O_{G}(\Theta_{0}) - O_{G}(\Theta_{0})$ only contains GIT-unstable points. Therefore $Z_{0}$ must be a tacnodal curve with finite stabilizer. $Z_{0}$ must then be one of the eight types listed in 3.5.1.

Let us start with case 1 (of Lemma 3.5.1) and show that it cannot occur. Recall that the theta-curve $\Theta'_{0}$ of such a tacnodal quartic contains 6 distinct lines of multiplicity 1 (the 6 theta-lines of type 0). We are saying that $\Theta'_{0}$ is in the orbit closure of the theta curve $\Theta_{0}$ of a trinodal quartic. We know that $\Theta_{0}$ contains only 4 lines of multiplicity 1 (the 4 theta-lines of type 0), and hence any configuration of lines in its orbit closure has at most 4 lines of multiplicity 1. Thus case 1 does not occur.

Instead of using a similar (even though slightly more complicated) argument to show that cases 2, 3, 4 cannot occur, we observe that all such curves, and curves of type 1 as well, contain in their orbit closure reducible quartics that are union of two bitangent conics (that is, quartics $C_{1} \cup C_{2}$ with $C_{1}$ and $C_{2}$ smooth conics having contact of order 2 at exactly two points; see [AF2]). Such reducible quartics are GIT-semistable; moreover for each curve of type 2, 3, or 4 there is a curve of type 1 with the same pair of bitangent conics in the orbit closure. In other words, every curve of type 2, 3, or 4 is identified with one of type 1 in the GIT-quotient of $\mathbb{P}^{14}$. In particular, when we applied the GIT-semistable replacement property to obtain $Z_{0}$ we were free to assume that $Z_{0}$ were a tacnodal curve of type 1, rather than one of type 2, 3 or 4. Since we just showed that $Z_{0}$ cannot be of type 1, we also get that $Z_{0}$ cannot be of type 2, 3 or 4.

Consider now case 5: $Z_{0} = L \cup C$. Then there are no lines of multiplicity 1, since $Z_{0}$ has no theta-lines of type 0; $\Theta'_{0}$ contains $L$ with multiplicity 6, one set of 4 lines of multiplicity 4 all meeting $L$ at the same point $t$ (the tacnode, where $C$ is tangent to $L$) and 3 lines of multiplicity 2 all meeting $L$ at the same point $n$ (the remaining point of intersection of $C$ and $L$). Therefore $t$ is a point of multiplicity 22 and $n$ of multiplicity 12; every other point of $\Theta'_{0}$ has multiplicity less than 12. But now consider $\Theta_{0}$, which contains 3 points of multiplicity 12 (corresponding to the 3 nodes of $X_{0}$). Then a curve in the orbit closure of $\Theta_{0}$ must contain either at least 3 points of multiplicity at least 12, or at least one point of multiplicity at least 24. Neither condition is satisfied by the theta-curve of our $Z_{0}$, hence case 5 does not occur. (Notice that the same argument would work to show that cases 1,2,3 are not possible).

We treat the three remaining cases by noticing that they are both identified in the quotient $\mathbb{P}^{14}/G$ to curves of type 5, since all such curves contain in their orbit closure the GIT-semistable quartics given by a smooth conic union 2 tangent lines (see [AF2]). Hence, arguing as above, when applying the GIT-semistable replacement property to obtain $Z_{0}$ we could assume that $Z_{0}$ was a curve of type 5, rather than one of type 6, 7 or 8.

\[\square\]

**Proposition 5.1.2.** A general nodal quartic has the theta-property.

The argument is completely similar, we just need to replace $W$ with the linear space of quartics having a node in a fixed point $N_{1}$. This amounts to making a few obvious changes to the proof of Case (a) (consisting in replacing the line $M$ with all theta-lines of type 1 of $X_{0}$). The rest remains the same.

5.2. We now prove that a general quartic is uniquely recovered from its 28 bitangents.
**Theorem 5.2.1.** A general smooth plane quartic has the theta-property.

*Proof.* We proceed like in the proof of 5.1.1. By contradiction, assume that for $X$ varying in an open subset of $V^0$ there exists $Y \in V^0$ such that $\theta(X) = \theta(Y)$ and $X \neq Y$.

Then we can construct a family of quartics $\mathcal{X} \rightarrow T$ over a smooth curve $T$ with a marked point $t_0 \in T$ satisfying the following assumptions: we fix two distinct lines $E$ and $F$ and we assume that, for every $t \neq t_0$, the fiber $X_t$ is smooth and has $E$ and $F$ as bitangents; the fiber over $t_0$, $X_0$, is a trinodal curve, having all nodes on $E$ and $F$, with one node at the point of intersection of $E$ and $F$ (so that $E$ and $F$ are theta-lines of type 2 for $X_0$). Just like in the proof of 5.1, it is convenient to assume that each theta-line of $X_0$ meets it in two distinct points (this condition is satisfied for a general trinodal curve).

Such degenerations do exist: in fact denote by $V_{E,F}$ the subvariety of $V$ parametrizing curves having $E$ and $F$ as theta-lines. Then $V_{E,F}$ is irreducible and its general element is a smooth quartic. To see that, consider the dominant, rational map

$$c : V_{E,F} \rightarrow Sym^2E \times Sym^2F$$

mapping a curve $X$ to its two pairs of points of intersection with $E$ and $F$. The general fiber of $c$ is irreducible, being just the intersection of $V$ with a linear subspace of dimension 6 of $\mathbb{P}^{14}$. Hence $V_{E,F}$ is irreducible of dimension $6 + 4 = 10$. The fact that the general element of $V_{E,F}$ is a smooth curve can be seen by a standard tangent space argument (see below).

Finally, we assume that there exists another family $\mathcal{Y} \rightarrow T$ such that for every $t \neq t_0$ $Y_t$ is in $V$, $X_t \neq Y_t$ and $\theta(X_t) = \theta(Y_t)$.

Notice that if $t \neq t_0$, $Y_t$ must itself be smooth, since its theta-curve is made of 28 distinct lines.

Our $X_t$, $X_0$ and $Y_t$ are in $V_{E,F}$. What about $Y_0$? There are two possibilities:

(a) $Y_0 \in V$ and hence $Y_0 \in V_{E,F}$

(b) $Y_0 \notin V$

Case (a).

Then $X_0 = Y_0$, because $X_0$ has the theta-property, by 4.2. Just as in 5.1.1, we will be done after showing that the restriction

$$\theta_{E,F} : V_{E,F} \rightarrow P_{28}$$

of $\theta$ to $V_{E,F}$, is an immersion at $X_0$.

Notice that, if $X \in V_{E,F}$ is smooth and $X$ intersects $E$ and $F$ in $p_1, ..., p_4$, then

$$T_X V_{E,F} \cong H^0(X, \mathcal{O}_X(4) \otimes \mathcal{O}_X(-p_1 - \ldots - p_4))$$

which has dimension 10 by Riemann-Roch. If, instead, we consider our $X_0$, then we have that

$$T_{X_0} V_{E,F} \subset H^0(X_0^\nu, \nu^* \mathcal{O}_{X_0}(4) \otimes \mathcal{O}_{X_0}(-n_1 - n_2 - n_3))$$

where $\nu : X_0^\nu \rightarrow X_0$ is the normalization. Denote by $\mathcal{L} = \nu^* \mathcal{O}_{X_0}(4) \otimes \mathcal{O}_{X_0}(-n_1 - n_2 - n_3)$.

Now, the theta-curve of $X_0$ contains 4 lines $L_1, ..., L_4$ of type 0, meeting $X_0$ in 8 smooth points $p_1, ..., p_8$ (2 distinct on each line, since we picked $X_0$ without hyperflexes).
and 6 lines $M_1, ..., M_6$ of type 1 meeting $X_0$ in 6 smooth points $q_1, ..., q_6$ (1 on each line). Denote by $V_{E,F,L_1,L_2,L_3,L_4}^{M_1, ..., M_6}$ the locus of curves in $V_{E,F}$ that are bitangent to all $L_i$'s and tangent to all $M_j$'s, we have

$$T_{X_0} V_{E,F,L_1,L_2,L_3,L_4}^{M_1, ..., M_6} \subset H^0(X_0^\nu, \mathcal{L} \otimes \nu^* \mathcal{O}_{X_0}(-\sum_{i=1}^{8} p_i - \sum_{j=1}^{6} q_j))$$

and the right hand side vanishes, since the line bundle has negative degree. On the other hand it is clear that

$$T_{X_0} \theta_{E,F}^{-1}(\theta(X_0)) \subset T_{X_0} V_{E,F,L_1,L_2,L_3,L_4}^{M_1, ..., M_6} = 0$$

hence $\theta_{E,F}$ is an immersion at $X_0$ and we are done.

Case (b). Here we just repeat, word by word, the argument for Case (b) in 5.1.1!

\[\square\]

\textbf{Acknowledgements.} We benefitted from conversations with Igor Dolgachev, Lawrence Ein, Joe Harris and Sandro Verra. We are very grateful to Paolo Aluffi for promptly and very clearly answering, by e-mail, all of our many questions.
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