A Study on Strain Energy of Ellipsoidal Inclusion in Half-space

H P Qian¹, K Zhu²*, R Zhang¹, P Li¹ and X Jin¹,²
¹ State Key Laboratory of Mechanical Transmissions, Chongqing University, Chongqing, 400030, China
² College of Aerospace Engineering, Chongqing University, Chongqing, 400030, China
E-mail: kaychu0523@foxmail.com

Abstract. An inclusion refers to localized eigenstrains appearing in such processes as thermal expansion and plastic deformation. In view of micromechanics, the existence of inclusions may significantly influence the mechanical properties of the engineering materials. A micromechanical model is proposed to determine the variation of the strain energies in the presence of the near-surface inclusions. The corresponding inclusion problem in a half-space is usually difficult to be solved analytically. In this work, the strain energy is evaluated numerically via the method of images, which superposes the counterpart solutions in full-space and eliminates the tractions on the boundary surface of the half-space. The validity of the present work is confirmed by comparing with the published results and the finite element method (FEM).

1. Introduction
The shortage of advanced and high-quality steel has seriously restricted the development of transportation, energy, and other core areas of the economy [1]. On one hand, various manufacturing defects are found in the steel materials and affect the serviceability as well as reliability of machine components. On the other hand, the existence of reinforcing phases, such as carbon fibers, may improve certain properties (e.g. strength, hardness) of steels [2]. According to Mura’s work [3], such defects or reinforcing phases may be recognized as inclusions or inhomogeneities. The in-depth study of inclusions or inhomogeneities provides a feasible method to quantify defects or reinforcing phase features on the materials.

Since the pioneering work of Eshelby [4], full-space inclusion has been extensively studied, but half-space inclusion is awaiting further exploration. This is because the problem of half-space inclusion is rather complex due to the interactions between the boundary surface and near-surface inclusions. Aderogba [5] used Papkovitch-Neuber stress functions to solve a spherical inclusion problem in a half-space, while the perturbation solutions of ellipsoidal inclusions subjected to uniform eigenstrains are derived by Seo and Mura [6]. Nevertheless, derivative terms appear in Seo and Mura’s solutions, so they cannot be regarded as explicit or complete expressions. Lyu et al. [7] derived the explicit analytical solutions of a half-space ellipsoidal inclusion subjected to uniform thermal expansion. The results of the displacements, strains and stresses are presented in explicit tensorial form.

The previous studies of inclusion and impurity problems mainly focus on the elastic fields, and only a few works have analyzed elastic strain energy, which is closely related to the stability of the
inclusion/impurity system [8-10]. Early scholars either studied the strain energy of simple spherical inclusions or only considered the influence of inclusion shape and strain state [10-13]. However, many studies [3,14-15] have found that cracks in mechanical materials usually occur in the vicinity of inclusions, and the size and morphology of inclusions are closely related to their strain energy, which tends to be minimized. This paper intends to give an alternative point of view to enhance material design through corresponding energy considerations.

2. Strain Energy of Ellipsoidal Inclusions in a Half-space
Consider an elastic half-space containing an ellipsoidal inclusion centered at \((0,0,d)\) with its semi-axes \(a_1, a_2, a_3\) and subjected to uniform eigenstrains, \(\varepsilon_{ij}^0\), as shown in figure 1. The boundary surface of the half-space is assumed to be traction-free. The stresses of the interior and exterior points of the ellipsoidal inclusion can be obtained numerically by using the method of images.

![Figure 1](image1.png)

**Figure 1.** An ellipsoidal inclusion is subjected to uniform eigenstrains in a half-space.

The method of images may be implemented in two different ways. The schematic of the first image method is shown in figure 2, and the corresponding solutions of the ellipsoidal inclusion in a half-space can be obtained by the superposition of two sub-solutions. The first part corresponds to the summation of the solution of an ellipsoidal inclusion in full space with its center located at \((0,0,d)\) and that of a image ellipsoidal inclusion centered at \((0,0,−d)\). The second part corresponds to the cancellation of the remaining normal stresses at the boundary surface, i.e.

\[
\sigma^{(I)}_{ij} = T_{ijkl}^0 \varepsilon_{kl}^* + T'_{ijkl} \varepsilon_{kl}^{**} + \sum_{m=1}^{N_r} \sum_{l=1}^{N_r} 2\delta_{j33} \sigma_{33}^0
\]  

(1)

The superscript \((I)\) in Eq. (1) represents the stress solution of the first image method, and \(T_{ijkl}\) and \(T'_{ijkl}\) are Eshelby’s stress tensors [4] corresponding to the original space and the image space, respectively.

![Figure 2](image2.png)

**Figure 2.** The first method of image for inclusion problem in a half-space.
The last term of Eq. (1) indicates that the surface domain (cf. the right-most figure in figure 2) is numerically discretized by rectangular elements, with $N_x$ and $N_y$ representing the number of small rectangular elements in the $x_1$ and $x_2$ directions, and $l$, $m$ denoting the element numbers of an source domain. $\varepsilon_{ij}^*$ and $\varepsilon_{ij}^{**}$ represent the eigenstrains in the original and image space, respectively,

$$
\varepsilon_{ij}^* = \left[ \varepsilon_{11}^*, \varepsilon_{22}^*, \varepsilon_{33}^*, \varepsilon_{23}^*, \varepsilon_{31}^*, \varepsilon_{12}^* \right]^T
$$

(2)

$$
\varepsilon_{ij}^{**} = \left[ \varepsilon_{11}^{**}, \varepsilon_{22}^{**}, \varepsilon_{33}^{**}, -\varepsilon_{23}^{**}, -\varepsilon_{31}^{**}, \varepsilon_{12}^{**} \right]^T
$$

(3)

The normal traction, $\sigma_{33}^0$, can be written as,

$$
\sigma_{33}^0 = 2\sigma_{33} = T_{33ij} \varepsilon_{ij}^*
$$

(4)

Inspired by the first image method, the second image method is accordingly proposed, as illustrated in figure 3.

![Figure 3](image)

**Figure 3.** The second method of image for inclusion problem in half-space.

The second image method also includes two parts, where the first part corresponds to a full-space solution of an ellipsoidal inclusion and the second part cancels the remaining surface tractions in three directions. Therefore, the resultant solutions of an ellipsoidal inclusion in a half-space can be obtained as,

$$
\sigma_{ij}^{(II)} = T_{ijkl} \varepsilon_{kl}^* + \sum_{m=1}^{N_x} \sum_{l=1}^{N_y} \lambda_{ijkl} \sigma_{jk}
$$

(5)

The superscript (II) in Eq. (5) represents the stress solution of the second image method.

After the stresses are obtained, the elastic strain energy can be determined through the following formula [3],

$$
W^* = -\frac{1}{2} \int_{\Omega} \sigma_{ij}^* \varepsilon_{ij}^* dD
$$

(6)

By substituting the previously solved stress solutions into Eq.(6), the elastic strain energy of the half-space ellipsoidal inclusion can be evaluated through a numerical scheme shown in figure 4.
Figure 4. Schematic of a numerical method for surface tractions in solving ellipsoidal inclusion embedded in half-space.

Let $\Delta_1, \Delta_2, \Delta_3$ be the length, width, and height of the discrete elements, respectively, then elastic strain energy may be written as,

$$W^{(\alpha)} = \frac{1}{2} \sum_{n=1}^{N_x} \sum_{m=1}^{N_y} \sum_{l=1}^{N_z} \sigma_{kl}^{(\alpha)} \epsilon_{kl}^{*} \Delta_1 \Delta_2 \Delta_3, \quad \text{(where } \alpha \text{ stands for I or II)} \quad (7)$$

Strain energy may be evaluated according to Eq. (7) by either of the above two image methods as indicated by superscript $(\alpha)$.

3. Comparison and Verification of Elastic Strain Energy

To verify the solution of the half-space ellipsoidal inclusions, the present elastic strain energy is compared with the analytical solution obtained by Seo and Mura [6]. Commercial finite element software, ABAQUS, is also used to study the same model. The surface of the half-space is considered to be traction-free, and the center of the spherical inclusion is $(0,0,d)$, while the radius of the spherical inclusion is denoted by $a$, and the inclusion is subjected to uniform thermal strain.

For thermal inclusion, Young's moduli $E$ and Poisson’s ratio $\nu$ of the matrix and inclusion are identical. In this work, $E$ and $\nu$ are set to be 210 Gpa, and 0.3, respectively. To examine the effect of the burying depth, the distance from the free surface to the spherical center ranges from $0.2d$ to $0.6d$ in the computations. The dimensionless parameter is,

$$W_0 = \frac{8\pi \mu (1+\nu) \left( \epsilon^* \right)^2 r^3}{3(1-\nu)} \quad (8)$$

The final results are shown in figure 5. The solid lines represent the results of the analytical solutions given by Seo and Mura, and the “Image1” and “Image2” refer to solutions deduced by the first and second image methods, respectively. It can be seen that the results of analytical solution, image solutions, and FEM are fully consistent, which verifies the validity of the two image methods and elastic strain energy solved by the current method.
Figure 5. Comparison between analytical, numerical, and FEM solutions of elastic strain energy of spheroidally thermal expansion inclusion in half-space.

The two image methods are also further applied to solve both the prolate and oblate spheroidal inclusions subjected to thermal expansion, for various central locations. The corresponding ABAQUS model is established for verification. The properties of the material are identical to the previous one, and the depth $d$ is the distance from the center of the spheroid to the free surface. The results obtained by the method of images and FEM are shown in figure 6.

The present study shows that the solutions of image methods agree well with the results of FEM. The elastic strain energy, regardless of prolate or oblate spheroid, gradually increases with depth, which is consistent with the trend of the spherical strain energy shown in figure 5.

Figure 6. Comparing two kinds of ellipsoidal thermal expansion inclusion’s elastic strain energy by numerical and FEM solutions in half-space.
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References
[1] Wang X, Hou Y, Sun S, et al. 2021 Chinese Journal of Theoretical and Applied Mechanics 53(1): 19-34
[2] Lui Q 2020 Steel Rolling 37 (04): 70-75+95.
[3] Mura T 1987 Springer, Dordrecht.
[4] Eshelby J D 1957 Proceedings of the Royal Society of London Series A-Mathematical and Physical Sciences 241(1226): 376-396.
[5] Aderogba K 1976 Mathematical Proceedings of the Cambridge Philosophical Society 80: 555-562.
[6] Seo K and Mura T 1979 Journal of Applied Mechanics 46(3).
[7] Lyu D, Zhang X, Li P, et al. 2018 Journal of Applied Mechanics 85(5): 051005-8.
[8] Yang W, Zhou Q, Zhai Y, et al. 2019 International Journal of Thermal Sciences 139: 326-338.
[9] Zhang X, Li P, Lyu D, et al. 2020 Computers & Geosciences 145: 104623.
[10] Li P, Lyu D, Soewardiman H, et al. 2021 Mechanics of Materials 156: 103788.
[11] Böhm H J, Fischer F D, et al. 1997 Scripta Materialia 36(9): 1053-1059.
[12] Fischer F D, Böhm H J, et al. 2006 Acta Materialia 54(1): 151-156.
[13] Shibata M and Ono K 1975 Acta Metallurgica 23(5): 587-597.
[14] Onaka S, Fujii T and Kato M 1995 Mech. Mater. 20(4): 329-336.
[15] Young S Y and Nam T H 2012 Materials Research Bulletin 47(10): 2936-2938.