Statistical Origin of Black Hole Entropy in Induced Gravity

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Abstract

The statistical-mechanical origin of the Bekenstein-Hawking entropy $S_{BH}$ in the induced gravity is discussed. In the framework of the induced gravity models the Einstein action arises as the low energy limit of the effective action of quantum fields. The induced gravitational constant is determined by the masses of the heavy constituents. We established the explicit relation between statistical entropy of constituent fields and black hole entropy $S_{BH}$.

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1 Introduction

It is well known [1],[2] that a black hole in the Einstein gravity behaves as a thermodynamical system with the entropy

$$S_{BH} = \frac{1}{4} \frac{\mathcal{A}^H}{G} ,$$

(1.1)

where $\mathcal{A}^H$ is the area of the event horizon and $G$ is the Newton constant. The Bekenstein-Hawking entropy $S_{BH}$ is the physical quantity which can be measured in (gedanken) experiments by making use of the first law of thermodynamics which can be represented in the form [3]:

$$dF^H = -S_{BH} dT^H .$$

(1.2)

Here $M^H$ is mass of a black hole, $T^H = (8\pi M^H)^{-1}$ is its Hawking temperature and $F^H = M^H - T^H S_{BH}$ is the free energy.

Recently a lot of attempts has been made to provide the statistical-mechanical foundation of the black hole thermodynamics and in particular to relate $S_{BH}$ with counting the internal degrees of freedom of a black hole. One of the proposed ideas is to relate the dynamical degrees of freedom of a black hole with its quantum excitations [4]-[7]. However, this approach meets the evident difficulty because the Bekenstein-Hawking entropy arises at the tree-level while the entropy of quantum excitations is a one-loop quantity [8]-[22]. This difficulty might be overcome if the gravity itself arises as the result of quantum effects, so that the metric $g_{\mu\nu}$ is a collective variable and the general relativity is a low-energy effective theory [23].

One of the possible realizations of this idea is the theory of the induced gravity proposed by Sakharov [24]. According to this idea the background fundamental theory is described by the action $I[\varphi_i, g_{\mu\nu}]$ of fields $\varphi_i$ propagating in the external geometry with the metric $g_{\mu\nu}$. By averaging over the constituent fields $\varphi_i$ one gets the dynamical effective action for the metric $g_{\mu\nu}$

$$\exp(-W[g_{\mu\nu}]) = \int \mathcal{D}\varphi_i \exp(-I[\varphi_i, g_{\mu\nu}]) .$$

(1.3)

This approach resembles the well-known approach in the solid state physics when instead of variables describing the oscillations of the atoms of the lattice one uses new collective variables describing the phonon field. It is important that the thermodynamical characteristics of a solid state in the low energy regime, say at the low temperatures, can be expressed by using only the spectrum of the phonon excitations.

The idea of this paper is to relate the Bekenstein-Hawking entropy with the statistical mechanics of the ultraheavy constituents that induce the gravity in the low energy limit of the theory. We shall illustrate it by direct calculations in a simple class of induced gravity models. Our result is an explicit representation for the Bekenstein-Hawking entropy

$$S_{BH} = - \sum_i \text{Tr} \rho_i \ln \rho_i - \sum_s 2\pi \xi_s \int_\Sigma \phi_s^2 > d\sigma$$

(1.4)
in terms of the statistical-mechanical entropies $-\text{Tr}_i \ln \rho_i$ of all heavy constituents and the average of the square of the field operators on the horizon $\Sigma$. The latter quantity appears for the fields which have coupling terms with the Riemann tensor in the Lagrangian. Such interactions are specific, for instance, for gauge fields. In our model we demonstrate an appearance of these terms for the scalar fields $\phi_s$ with nonminimal couplings $\xi_s R \phi_s^2$ with the scalar curvature.

The arguments in favour of the idea that the black hole entropy can be explained in the framework of the induced gravity were first given by T. Jacobson [23]. In our paper we analyse the statistical-mechanical origin between $S^{BH}$ in such theories and establish explicit relation (1.4) of $S^{BH}$ and the entropies of the constituent fields.

2 Induced gravity: a simple model

One of the simplest models that can be used to illustrate the idea of the induced gravity consists of $N_s$ scalar fields $\phi_s$ with the classical actions

$$I[\phi_s, g_{\mu \nu}] = \frac{1}{2} \int_M dV \left[ (\nabla \phi_s)^2 + (m_s^2 + \xi_s R) \phi_s^2 \right] + \xi_s \int_{\partial M} dv K \phi_s^2 \quad ,$$

(2.1)

and $N_d$ fields described by the Dirac fermions with the actions

$$I[\psi_d, g_{\mu \nu}] = \int_M dV \tilde{\psi}_d (i \gamma^\mu \nabla_\mu + m_d) \psi_d \quad .$$

(2.2)

Here, $dV$ and $dv$ are the volume elements of the background space $M$ and its boundary $\partial M$. Indices $s = 1,..,N_s$ and $d = 1,..,N_d$ enumerate the scalar and Dirac fields, correspondingly. It is important for our purpose to introduce into the classical actions (2.1) the non-minimal couplings with the curvature $R$. In order to have a consistent variational principle for the metric $g_{\mu \nu}$ one must also add the boundary term to $I[\phi_s, g_{\mu \nu}]$. The variational procedure, which corresponds to this action, fixes only the metric on $\partial M$ and keeps free the extrinsic curvature $K$ of the boundary.

The effective gravitational action $W[g_{\mu \nu}]$ is defined by Eq. (1.3) where the classical action for the constituent fields is the sum

$$I[\phi_i, g_{\mu \nu}] = \sum_s I[\phi_s, g_{\mu \nu}] + \sum_d I[\psi_d, g_{\mu \nu}] \quad .$$

(2.3)

We assume that the fields (that can have different masses and the coupling constants $\xi_s$) obey the following constraints:

$$c_1 = N_s - 4N_d = 0 \quad ,$$

(2.4)

$$c_2 = N_s + 2N_d - 6 \sum_s \xi_s = 0 \quad ,$$

(2.5)

$$c_3 = \sum_s m_s^2 - 4 \sum_d m_d^2 = 0 \quad ,$$

(2.6)
\[ c_4 = \sum_s m_s^2 (1 - 6\xi_s) + 2 \sum_d m_d^2 = 0 \quad , \] (2.7)

\[ c_5 = \sum_s m_s^4 - 4 \sum_d m_d^4 = 0 \quad , \] (2.8)

\[ c_6 = \sum_s m_s^4 \ln m_s^2 - 4 \sum_d m_d^4 \ln m_d^2 = 0 \quad . \] (2.9)

In 4-dimensional space-time the first five constraints ensure the ultraviolet finiteness of the induced Newton and cosmological constants \( (G, \Lambda) \) respectively. The last constraint \( (2.9) \) is imposed for the \( \Lambda \)-constant to vanish so that the induced gravity theory possesses the vacuum black hole solutions. As it is seen from Eq.\( (2.5) \) the finite gravitational constant is impossible in the models like \( (2.3) \) without curvature couplings, i.e. when all \( \xi_s = 0 \).

The main preposition of the induced gravity is that at least some of the constituent fields possess masses comparable with the Planckian mass \( m_{Pl} \). The low energy limit of the theory corresponds to the regime when the curvature radius \( L \) of the space-time \( \mathcal{M} \) is much greater than the Planck length \( m_{Pl}^{-1} \). In this limit the functional \( W[g_{\mu\nu}] \) can be approximated by the Euclidean Einstein action

\[ W[g_{\mu\nu}] \approx -\frac{1}{16\pi G} \left( \int_\mathcal{M} dV R + 2 \int_{\partial \mathcal{M}} d\nu K \right) \] (2.10)

where the Newton constant \( G \) is the function of the parameters of the constituents

\[ \frac{1}{G} = \frac{1}{12\pi} \left( \sum_s (1 - 6\xi_s) m_s^2 \ln m_s^2 + 2 \sum_d m_d^2 \ln m_d^2 \right) \quad . \] (2.11)

The value of this constant is dominated by the masses of the heavy constituents \( G \sim m_{Pl}^{-2} \).

Now several remarks are in order how to obtain Eq.\( (2.10) \). The one-loop expression for the action \( W[g_{\mu\nu}] \) in the Schwinger-DeWitt representation is

\[ W[g_{\mu\nu}] = \sum_s \left( W_s + \xi_s \int_{\partial \mathcal{M}} d\nu K < \phi_s^2 > \right) + \sum_d W_d \quad , \] (2.12)

where \( W_s \) and \( W_d \) are the effective actions of a single field \( \varphi_i \)

\[ W_i = -\frac{\eta_i}{2} \int_{\delta}^\infty \frac{ds}{s} e^{-m_i^2 s} \text{Tr} e^{-s\Delta_i} \quad , \quad i = s, d \quad , \] (2.13)

and the factors \( \eta_s = 1, \eta_d = -1 \) correspond to the different statistics of the fields \( \phi_s \) and \( \psi_d \). The wave operators read

\[ \Delta_s = -\nabla^2 + \xi_s R \quad , \quad \Delta_d = -(\gamma^\mu \nabla_\mu)^2 = -\nabla^2 + \frac{1}{4} R \quad . \] (2.14)

The boundary term in \( (2.12) \) appears in the one-loop approximation in the perturbation theory (see for the details \[21\]) and the field average is defined as

\[ < \phi_s^2(x) > = \int_{\delta}^\infty ds \ e^{-m_s^2 s} < x | e^{-\Delta_i s} | x > \quad . \] (2.15)
In Eqs. (2.13) and (2.15) the parameter $\delta$ is the ultraviolet regulator that must be introduced in order to make finite the integral for each separate field in the limit $s \to 0$. In intermediate computations it is possible to use other type of the ultraviolet regularizations, but in the induced gravity models after the regularization is removed the physical quantities are well defined and do not depend on the chosen regularization scheme.

In the low energy limit the main contribution in the integrals (2.13) and (2.15) comes from the small values of the proper-time parameter $s \sim m_{Pl}^{-2}$, so that one can use the asymptotic expansion of the heat kernel

$$
\text{Tr } e^{-\Delta_i s} \simeq \frac{1}{(4\pi s)^{D/2}} \left( (A_0)_i + (A_{1/2})_i s^{1/2} + (A_1)_i s + ... \right),
$$

with $D$ standing for the dimension of the background geometry. The first heat coefficients are well-known:

$$(A_0)_s = \int_{\mathcal{M}} dV, \quad (A_0)_d = 4 \int_{\mathcal{M}} dV,$$  

$$(A_1)_s = \left( \frac{1}{6} - \xi_s \right) \int_{\mathcal{M}} dV R + \frac{1}{3} \int_{\partial \mathcal{M}} d\nu K, \quad (A_1)_d = -2 (A_1)_s|_{\xi_s=0}.\quad (2.18)$$

The coefficients $(A_{1/2})_i$ depend on the imposed boundary conditions and they are proportional to the volume of $\partial \mathcal{M}$. For this reason in the variational procedure, where the metric of $\partial \mathcal{M}$ is fixed, the coefficients $(A_{1/2})_i$ give the constant contributions to the effective action, and therefore they can be neglected.

If we approximate now the heat kernels in (2.12), (2.13) and (2.15) by the asymptotic expansions (2.16), then we get the decomposition of $W[g_{\mu\nu}]$ in the curvature with the small expansion parameter $(m_{Pl}L)^{-1}$. Eq. (2.10) is obtained when all terms higher then the first order in curvature $R$ are neglected. The gravitational constant reads:

$$
\frac{1}{G} = \frac{1}{12\pi} \left[ \sum_s m_s^{D-2} (1 - 6\xi_s) \Gamma \left( 1 - D/2, m_s^2 \delta \right) + 2 \sum_d m_d^{D-2} \Gamma \left( 1 - D/2, m_d^2 \delta \right) \right],
$$

where

$$
\Gamma(z, \sigma) = \int_{\sigma}^{\infty} x^{z-1} e^{-x} dx
$$

is incomplete gamma function. In four dimensions if the constraints (2.5), (2.7) hold the $G$ remains finite in $D = 4$ in the limit $\delta = 0$ and is given by Eq.(2.11).

Note that the $R^2$-terms that are neglected here are ultraviolet divergent and so they must be either renormalized or made finite by adding further restrictions and complicating the model. However we will not dwell on this issue because it is not related with the Bekenstein-Hawking entropy (1.1) which appears in the Einstein theory of gravity.
3 Induced entropy: on and off shell

The black hole entropy can be derived for the induced action by standard methods. One may use the York [25],[26] formulation of the canonical ensemble for black holes inside a spherical cavity of a radius \( r_B \) with the fixed temperature \( T \) on its surface. The free energy \( F \) of the system is defined in terms of the effective action \( W[g_{\mu\nu}] \), taken on the Euclidean black hole instanton \( M \), as \( F = T W[g_{\mu\nu}] \). Then the second law of thermodynamics (1.2) gives for the entropy \( S_{BH} \) the Bekenstein-Hawking expression (1.1). Such a definition of the entropy is called thermodynamical. It considers only equilibrium changes of the system. This is equivalent to the requirement that the effective action is always taken on-shell, i.e for the regular black hole instanton that is a solution of the vacuum Euclidean Einstein equations.

Contrary to the thermodynamical approach the statistical-mechanical one, as it was pointed out by many authors [8]-[22], requires the off-shell computations. Let \( \beta \) be the period of the Euclidean time-coordinate \( \tau \) of \( M \), connected with the temperature \( T \) on the boundary \( \partial M \) as \( \beta = g_{\tau\tau}^{-1/2}(r_B)T^{-1} \) where \( g_{\tau\tau} \) is the time component of the metric. For arbitrary \( \beta \) and fixed mass \( M_H \) the black hole instanton has the conical singularity at the horizon \( \Sigma \) with the conical deficit angle \( 2\pi(1 - \beta/\beta_H) \). Here, \( \beta_H^{-1} \) is the Hawking temperature (\( \beta_H = 8\pi M_H \) for the Schwarzschild solution). The off-shell entropy \( S^{CS} \) in the conical-singularity method is defined as

\[
S^{CS}(\beta) = \left( \beta \frac{\partial}{\partial \beta} - 1 \right) W^{CS}[g_{\mu\nu}, \beta] ,
\]

where \( W^{CS}[g_{\mu\nu}, \beta] \) is the action on the singular instanton (\( W^{CS}[g_{\mu\nu}, \beta_H] = W[g_{\mu\nu}] \)) and the derivative with respect to \( \beta \) is taken under fixed \( g_{\mu\nu} \). Let us compute the value of \( S^{CS}(\beta) \) in the induced gravity and show that at the Hawking temperature this quantity coincides with the Bekenstein-Hawking entropy \( S_{BH} \).

First, it is important to note that in a space with the conical singularities the total scalar curvature \( \bar{R} \) in addition to its standard regular part \( R \) has a delta-function contribution concentrated on \( \Sigma \)

\[
\bar{R} = R + 4\pi(1 - \beta/\beta_H)\delta_\Sigma .
\]

Therefore the classical action \( I^{CS}[\phi_s, g_{\mu\nu}] \) for a scalar field differs from the action (2.1) on a regular manifold by a term on the horizon

\[
I^{CS}[\phi_s, g_{\mu\nu}] = I[\phi_s, g_{\mu\nu}] + 2\pi \xi_s(1 - \beta/\beta_H) \int_\Sigma d\sigma \phi_s^2 ,
\]

(\( d\sigma \) is the area element of \( \Sigma \)). The spinor action (2.2) does not change. According to this observation the one-loop effective action in the conical singularity method \( W^{CS} \) can be written as [14]

\[
W^{CS}[g_{\mu\nu}, \beta] = W[g_{\mu\nu}, \beta] + \sum_s 2\pi \xi_s(1 - \beta/\beta_H) \int_\Sigma d\sigma <\phi_s^2 > \beta
\]
where \( W[\mu_\nu, \beta] \) and \(< \hat{\phi}_s^2 >_\beta \) are given by Eqs. (2.13) and (2.13) in terms of the heat kernel operator on the singular manifold. To calculate these quantities one can use the asymptotic heat kernel expansion (2.16) with the modified integer index coefficients. In particular the first scalar \[27],[28] and spinor \[16],[29] coefficients \((A^{CS})_i \) read

\[
(A^{CS}_i) = (A_i)_i + (A_{i,\beta})_i , \quad i = s, d ,
\]

where \( A^H = \int_\Sigma d\sigma \) is the area of the black hole horizon. Note that \((A^{CS})_i \) are given by integrals (2.18) and they are proportional to \( \beta \). Thus, as follows from (3.5), (3.6), in the low energy limit the effective action \( W^{CS}[\mu_\nu, \beta] \) can be written as

\[
W^{CS}[\mu_\nu, \beta] = \frac{\beta}{\beta_H} W[\mu_\nu] + U(\beta) - \frac{\beta}{\beta_H} U(\beta_H) + \sum_s 2\pi\xi_s (1 - \beta/\beta_H) \int_\Sigma d\sigma < \hat{\phi}_s^2 >_\beta.
\]

Both quantities \( U(\beta) \) and \( < \hat{\phi}_s^2 >_\beta \) are ultraviolet divergent. However it is possible to demonstrate that in the induced gravity, when conditions (2.4)-(2.8) are satisfied, the only divergent terms that enter the off-shell action \( W^{CS}[\mu_\nu, \beta] \) are of the second and higher order in the deficit angle. These divergences vanish when \( \beta = \beta_H \), so that the corresponding terms do not contribute to the entropy. The entropy in the conical singularity method (3.1) reads

\[
S^{CS}(\beta_H) = \left( \frac{\beta}{\beta_H} \frac{\partial}{\partial \beta} - 1 \right) U(\beta) \bigg|_{\beta = \beta_H} - \sum_s 2\pi\xi_s \int_\Sigma d\sigma < \hat{\phi}_s^2 > .
\]

Here \( < \hat{\phi}_s^2 > \equiv < \hat{\phi}_s^2 >_{\beta_H} \) is the average in the Hartle-Hawking state. Now it is easy to show by making use of (2.13) and (3.8) that the off-shell induced entropy \( S^{CS} \) is ultraviolet finite at \( \beta = \beta_H \) and coincides exactly to the Bekenstein-Hawking entropy \( S^{BH} \):

\[
S^{CS}(\beta_H) = \frac{1}{4G} A^H = S^{BH}.
\]
4 Relation to statistical mechanics

In the considered model the gravity appears as the collective effect related to averaging over the fields with the planckian masses. Let us find the total statistical-mechanical entropy $S_{SM}$ of these ultra heavy constituents and its relation to $S_{BH}$. The statistical-mechanical entropy

$$S_{SM} = -\text{Tr} \hat{\rho} \ln \hat{\rho}$$

is determined in terms of the thermal density matrix

$$\hat{\rho} = \frac{e^{-\hat{H}/T}}{\text{Tr} e^{-\hat{H}/T}}$$

where $\hat{H}$ is the Hamiltonian of the system. To find $S_{SM}$ for the fields of our model we shall rewrite (4.1) in a more suitable form

$$S_{SM} = \left( \beta \frac{\partial}{\partial \beta} - 1 \right) W_{SM}$$

in terms of the quantity $W_{SM}$, related to the free energy $F$ of the system

$$W_{SM} = T^{-1} F , \quad e^{-F/T} \equiv \text{Tr} e^{-\hat{H}/T} .$$

In (4.3), as before, $\beta = g_{r\tau}^{-1/2}(r_B)T^{-1}$. For quantum fields on a static curved background $W_{SM}$ has the functional integral representation \[13\] similar to that of the covariant effective action $W[g_{\mu\nu}]$:

$$\exp(-W_{SM}[g_{\mu\nu}, \beta]) = \int \mathcal{D}H \phi_i \exp(-I[\phi_i, g_{\mu\nu}]) .$$

We call $W_{SM}$ the statistical-mechanical action. The difference between (1.3) and (4.3) is in the integration measures. The statistical-mechanical action is determined with the help of the non-covariant canonical measure $\mathcal{D}_H \phi$. The equation (4.3) can be obtained from the canonical representation of Tr $e^{-\hat{H}/T}$ after the integration over the canonical momenta, see the details in Ref. \[13\]. The covariant $\mathcal{D}\phi$ and canonical $\mathcal{D}_H \phi$ measures are related to each other in the simple way

$$\mathcal{D}_H \phi_i |_{g_{\mu\nu}} = \mathcal{D}\tilde{\phi}_i |_{\tilde{g}_{\mu\nu}} ,$$

where

$$\tilde{g}_{\mu\nu} = \frac{g_{\mu\nu}}{g_{r\tau}} , \quad \tilde{\phi}_s = (g_{r\tau})^{\frac{D-2}{2}} \phi_s , \quad \tilde{\psi}_d = (g_{r\tau})^{\frac{D-1}{2}} \psi_d .$$

So by making use of the field redefinitions in (1.3) one can also rewrite $W_{SM}$ as a covariant action \[13\],\[15\]

$$W_{SM}[g_{\mu\nu}, \beta] = \tilde{W}[	ilde{g}_{\mu\nu}, \beta] .$$
but for the ultrastatic space \( \widetilde{M} \) with the metric \( \tilde{g}_{\mu\nu} \).

A stationary black hole is a special case of the stationary geometry that possesses the Killing horizon where \( g_{rr} = 0 \). In the Euclidean formulation the Euclidean horizon is formed by the fixed points of the Killing vector field. In the presence of the horizon the standard canonical formulation of the theory meets difficulties. In particular, the canonical integration measure diverges when approaching the horizon. It means that on a black hole background the statistical-mechanical quantities are ill defined. There are different ways to overcome this difficulty. We use here the volume cut-off prescription (see, for instance Ref.[18]). In this method one simply cuts the spatial integrations in the statistical-mechanical quantities near the horizon at some proper distance \( \epsilon \). As the result \( \epsilon \) appears in all physical quantities as an additional volume cut-off parameter.

Let us calculate now \( \tilde{W}[\tilde{g}_{\mu\nu}, \beta] \). This functional has the form

\[
\tilde{W}[\tilde{g}_{\mu\nu}, \beta] = \sum_{s} \tilde{W}_{s}[\tilde{g}_{\mu\nu}, \beta] + \sum_{d} \tilde{W}_{d}[\tilde{g}_{\mu\nu}, \beta] ,
\]

where the actions of the scalar and Dirac particles read

\[
\tilde{W}_{i}[\tilde{g}_{\mu\nu}, \beta] = -\frac{\eta_{i}}{2} \int_{0}^{\infty} \frac{dt}{t} \sum_{l=-\infty}^{\infty} e^{-4\pi^{2}(l+w_{i})^{2}\beta^{-2}t} \operatorname{Tr} e^{-\left(L_{i}+m_{i}^{2}g_{rr}\right)t} , \quad i = s, d .
\]

This formula follows from the structure of the background space \( \tilde{M} = S^{1} \times \tilde{M}' \). The numbers \( w_{s} = 0 \) and \( w_{d} = 1/2 \) are related to the different periods in \( \tau \) for integer and half-odd-integer spins. The operators \( L_{i} \) act on \((D-1)\)-dimensional space \( \tilde{M}' \) and their form can be easily found by applying the transformation (4.7) to the operators (2.14). The functionals \( \tilde{W}_{i}[\tilde{g}_{\mu\nu}, \beta] \) are ultraviolet divergent and for their calculation one must introduce the ultraviolet regulator as well. Let us note that the regularization parameters in the covariant and statistical-mechanical actions can be different in some regularization procedures. We prefer to work further with the dimensional regularization in the parameter \( D \). This prescription is identical for the both functionals and enables us to compute these quantities on the equal footing. This will be important for us later.

The features of this regularization and its alternatives are discussed below.

It is suitable to use the Poisson formula to rewrite the sums

\[
\sum_{l=-\infty}^{\infty} e^{-4\pi^{2}(l+w_{i})^{2}\beta^{-2}t} = \frac{\beta}{\sqrt{4\pi t}} \sum_{k=-\infty}^{\infty} \eta_{i}^{k} e^{-\frac{\beta^{2}k^{2}}{4t}} .
\]

After that the actions can be separated onto the vacuum (\( \tilde{W}_{i}^{V} \)) and thermal (\( \tilde{W}_{i}^{T} \)) parts:

\[
\tilde{W}_{i} = \tilde{W}_{i}^{V} + \tilde{W}_{i}^{T} ,
\]

\[
\tilde{W}_{i}^{V} = -\frac{1}{2} \int_{0}^{\infty} \frac{dt}{t} \frac{\beta}{\sqrt{4\pi t}} \operatorname{Tr} e^{-\left(L_{i}+m_{i}^{2}g_{rr}\right)t} ,
\]

\[
\tilde{W}_{i}^{T} = -\int_{0}^{\infty} \frac{dt}{t} \frac{\beta}{\sqrt{4\pi t}} \sum_{k=1}^{\infty} \eta_{i}^{k+1} e^{-\frac{\beta^{2}k^{2}}{4t}} \operatorname{Tr} e^{-\left(L_{i}+m_{i}^{2}g_{rr}\right)t} .
\]
Obviously, $\tilde{W}'_i$ are proportional to $\beta$ and give a contribution only to the vacuum energy, while the entropy is determined by the thermal parts $\tilde{W}^T_i$.

Our aim is to calculate the functionals $\tilde{W}^T_i$ by decomposing them in powers of the curvature of the physical background $\mathcal{M}$, similar to the decomposition of the covariant action, and then to find from this expansion the entropy $S^{SM}$. To do this we use the asymptotic series of the heat kernels of the operators $L_i + m_i^2 g_{\tau\tau}$. As we will see it is sufficient to consider only the zero-order approximation:

$$\text{Tr} \ e^{-(L_i+m_i^2 g_{\tau\tau})t} \approx \frac{n_i}{(4\pi t)^{D/2}} \int_{\tilde{\mathcal{M}}'} \ d\tilde{V}' e^{-m_i^2 g_{\tau\tau} t} , \tag{4.15}$$

where $n_s = 1$ and $n_d = 4$ are the number of components of a field in question and $d\tilde{V}'$ is the $(D-1)$-volume element on $\tilde{\mathcal{M}}'$. Then by taking into account that

$$\int_0^\beta d\tau \int_{\tilde{\mathcal{M}}'} d\tilde{V}' = \int_{\mathcal{M}} dV (g_{\tau\tau})^{-D/2} , \tag{4.16}$$

we get from Eq. (4.14) the expression

$$\tilde{W}^T_i \approx - \beta \beta_H (4\pi) \frac{D/2+1}{\Gamma(1-D/2)} \left[ \int_0^\infty \frac{dt}{t} \sum_{k=1}^\infty \eta_i^{k+1} e^{-\frac{\beta^2 k^2}{4t}} \int_{\tilde{\mathcal{M}}'} dV (g_{\tau\tau})^{-D/2} e^{-m_i^2 g_{\tau\tau} t} \right] A^H , \tag{4.17}$$

It is written in terms of geometrical characteristics of the physical space $\mathcal{M}$ with the volume cutoff in the integrals at the proper distance $\epsilon$ (in the metric $g_{\mu\nu}$) near the horizon $\Sigma$.

To evaluate the last integral in (4.17) let us point out that the main contribution here comes out from the region near $\Sigma$ where $g_{\tau\tau}$ is small. In this region the black hole metric can be approximated by the Rindler-like metric

$$ds^2 = \left( \frac{2\pi}{\beta_H} \right)^2 y^2 d\tau^2 + dy^2 + dl^2 , \quad 0 \leq \tau \leq \beta , \quad y \geq \epsilon , \tag{4.18}$$

where $dl^2$ is the line element on the horizon. Then the integration over $\mathcal{M}$ is reduced to the surface integral

$$\int_{\mathcal{M}} dV (g_{\tau\tau})^{-D/2} e^{-m_i^2 g_{\tau\tau} t} = \frac{\beta \beta_H}{4\pi} (m_i^2 t) \frac{D}{2}-1 \Gamma \left( 1 - \frac{D}{2} , e^2 m_i^2 \left( \frac{2\pi}{\beta_H} \right)^2 t \right) A^H , \tag{4.19}$$

where $\Gamma(z, \sigma)$ is incomplete gamma function (2.20). Now we use the relation between complete and incomplete gamma functions:

$$\Gamma(z, \sigma) = \Gamma(z) - \sum_{p=0}^{\infty} \frac{(-1)^p \sigma^{z+p}}{p!} \frac{1}{z+p} \tag{4.20}$$

that is valid when $z \neq 0, -1, -2, \ldots$. By substituting (4.19) into (4.17) and using (4.20) one can get

$$\tilde{W}_i^T = - \frac{\beta \beta_H}{(4\pi)^{D/2+1}} \Gamma \left( 1 - \frac{D}{2} , e^2 m_i^2 \left( \frac{2\pi}{\beta_H} \right)^2 t \right) A^H + P_i(\epsilon, D) . \tag{4.21}$$
The function $P_i(\epsilon, D)$ includes all the dependence on the cut-off parameter $\epsilon$ and has the representation

$$P_i(\epsilon, D) = \frac{\epsilon^{2-D}}{(4\pi)^{D/2}} \left( \frac{\beta H}{\pi \beta} \right)^{D-1} Q_i \, A^H , \quad (4.22)$$

where

$$Q_i = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \left( \frac{\pi \beta m_i}{\beta H} \right)^{2p} \zeta_i(D - 2p) \Gamma(D/2 - p) \frac{\zeta_i}{p + 1 - D/2} , \quad (4.23)$$

and

$$\zeta_i(z) = \sum_{k=1}^{\infty} \eta_i^{k+1} k^{-z} . \quad (4.24)$$

Eqs. (4.21)-(4.24) enable one to analyse the behavior of the thermal part $\tilde{W}_i^T$ in two different limits. First one can keep $\epsilon \neq 0$ and remove the ultraviolet regularization by going to $D = 4$. As it should be, the functional $\tilde{W}_i^T$ doesn’t have the ultraviolet divergences because the pole terms $\sim (D - 4)^{-1}$ in the first term in the r.h.s. in (4.21) and in $P_i(\epsilon, D)$ cancel each other. However the part $P_i(\epsilon, D)$ develops another divergence in $D = 4$ when $\epsilon \to 0$:

$$P_S \simeq -\frac{1}{1440 \pi} \left( \frac{\beta H}{\beta} \right)^3 \frac{1}{\epsilon^2} A^H , \quad P_d \simeq n_d \frac{7}{8} P_S . \quad (4.25)$$

It is easy to see that $P_s$ and $P_d$ reproduce the well-known high temperature behavior of the free energy of the relativistic Bose and Fermi gas.

For our purpose we need another way of computations when one takes the limit $\epsilon \to 0$ first but keeps the ultraviolet regularizator fixed. The ultraviolet infinities disappear when $\text{Re}D < 0$. If this condition is satisfied the functions $P_i$ vanish at $\epsilon = 0$. So by integrating the first term in r.h.s. in (4.21) we get when $\text{Re}D < 0$:

$$\tilde{W}_S^T = -\frac{\pi \beta H}{6} \frac{\Gamma(1 - D/2)}{(4\pi)^{D/2}} m_s^{D-2} A^H , \quad (4.26)$$

and

$$\tilde{W}_d^T = -\frac{\pi \beta H}{3} \frac{\Gamma(1 - D/2)}{(4\pi)^{D/2}} m_d^{D-2} A^H . \quad (4.27)$$

Finally substituting (4.26) and (4.27) we get the following expression for the statistical-mechanical action:

$$W^{SM} = \sum_i \tilde{W}_i \simeq \sum_i \tilde{W}_i^V + U(\beta) , \quad \text{Re}D < 0 , \quad \epsilon = 0 , \quad (4.28)$$

where $U(\beta)$ is defined by expressions (3.8) and (3.9) calculated in the dimensional regularization scheme ($\delta = 0$). This equation represents rather interesting result because it demonstrates that statistical-mechanical action includes the same function $U(\beta)$ which
appears in the covariant action on the singular instanton, see (3.7). This is the function which, being regularized according to our prescription, determines the statistical-mechanical entropy (4.3) of the heavy constituents:

\[ S_{SM} = \left( \beta \frac{\partial}{\beta} - 1 \right) U(\beta) \bigg|_{\beta = \beta_H}. \]  

(4.29)

Let us suppose now that the field average (3.10) is calculated in the same regularization. Then, according to (3.12), we can rewrite the Bekenstein-Hawking entropy in the following form

\[ S_{BH} = S_{CS} \bigg|_{\beta = \beta_H} = S_{SM} - \sum_s 2\pi\xi_s \int_{\Sigma} d\sigma \left< \hat{\phi}_s^2 \right> \]

(4.30)

It is assumed that the regularization here should be removed at the end, simultaneously in \( S_{CS} \) and in \( \left< \hat{\phi}_s^2 \right> \). This relation is our main result.

We see therefore that the part of the black hole entropy in the induced gravity is the statistical-mechanical entropy of the heavy constituents with the Planckian masses. The other part, that is expressed in terms of fluctuations of scalar fields on the horizon, is not related (at least in our model) to the counting of states. The role of the horizon term is to remove the divergences in \( S_{SM} \). It is easy to see that \( S_{BH} \) cannot be represented as a pure statistical-mechanical entropy of the thermal gas around a black hole, because all fields, regardless the spin, give the positive divergent contributions into \( S_{SM} \). So the statistical-mechanical representation of \( S_{BH} \) requires a subtraction procedure. There is some similarity between this subtraction procedure and one discussed in Refs. [18, 22].

It should be also emphasized that because of the exponential decrease of the Euclidean heat kernels the main contributions to the integrals (4.19) appearing in \( \tilde{W}_T^i \) are given by the region close to the horizon where \( g_{\tau\tau} \ll 1 \). In other words, in the induced gravity the entropy \( S_{SM} \) comes from the narrow layer located near the horizon and having the depth of order of the Planck length \( \sim m_{Pl}^{-1} \). Hence \( S_{SM} \) depends only on the local properties of the horizon. This conclusion implies that the result (4.30) must be valid for a general static or stationary black hole in the Einstein theory and in more general induced gravity theories.

A remark about the regularization scheme is in order. To write the black hole entropy in the form (4.30) the quantities \( S_{SM} \) and \( \left< \hat{\phi}_s^2 \right> \) must be calculated by the same method. The dimensional regularization used here is the easiest tool to do this. However, as it is known, such regularization does not take into account some of the ultraviolet divergences. An alternative way could be to use the Pauli-Villars regularization based on the introduction of the fictitious particles with the wrong statistics [12]. The Pauli-Villars scheme is more complicated but one can verify that it gives the same result for \( S_{BH} \). This indicates that Eq. (4.30) does not depend on the regularization. Another observation is
that for $\xi_s = 0$ the statistical-mechanical entropy (4.29) obtained for the scalar fields in the Pauili-Villars regularization coincides with the result for the entropy derived in [12] in the WKB approximation directly from the spectrum of the energy operator.

5 Conclusions

In conclusion, we would like to stress that all our calculations were made for a special class of models of induced gravity. The main point of our consideration is construction and comparison of the covariant $W$ and statistical-mechanical $W_{CS}$ effective actions. The functional $W$ generates the Einstein action as the low-energy limit of the induced theory, while the functional $W_{CS}$, being presented in (3+1)-form, allows one to relate the entropy to the statistical properties of the constituents. Moreover, with the help of $W_{CS}$ thermodynamical characteristics of a black hole can be presented in the form of spatial integrals of local quantities, in which only narrow region $\sim m_{Pl}^{-1}$ near the horizon does contribute.

The mechanism relating $S^{BH}$ to the statistical-mechanics of constituents seems to be quite general and it should work for the theories of different types. Any two microscopically different theories that result in the same low-energy limit for induced gravity must predict the same value of the black hole entropy. The details of the statistical-mechanical calculations and the form of the representation of the black hole entropy in terms of constituents may differ, but the final result (at least for black holes of mass much greater that $m_{Pl}$) must be determined only by the form of the low energy effective action for gravity and macroscopic parameters of a black hole. This requirement of consistency of the statistical mechanics of constituents in fundamental theory with the standard low energy gravitational calculations can be formulated as the general principle, that we call the low-energy censorship conjecture.

In particular this conjecture implies that in superstring theory, which induces gravity as the effective low-energy theory, and where the metric $g_{\mu\nu}$ arises as the result of collective string excitations the statistical-mechanical calculations for string constituents must reproduce the Bekenstein-Hawking entropy. The recent calculations of the black hole entropy obtained in the superstring theory for the special type black hole solutions (see, for instance, [30]-[32] and references therein) might be considered as supporting this point of view.

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