Schwarzschild Solution of the Generally Covariant Quaternionic Field Equations of Sachs

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Abstract

Sachs has derived quaternion field equations that fully exploit the underlying symmetry of the principle of general relativity, one in which the fundamental 10 component metric field is replaced by a 16 component four-vector quaternion. Instead of the 10 field equations of Einstein’s tensor formulation, these equations are 16 in number corresponding to the 16 analytic parametric functions \(\partial x^\mu \left/ \partial x^\nu\right.\) of the Einstein Lie Group. The difference from the Einstein equations is that these equations are not covariant with respect to reflections in space-time, as a consequence of their underlying quaternionic structure. These equations can be combined into a part that is even and a part that is odd with respect to spatial or temporal reflections. This paper constructs a four-vector quaternion solution of the quaternionic field equation of Sachs that corresponds to a spherically symmetric static metric. We show that the equations for this four-vector quaternion corresponding to a vacuum solution lead to differential equations that are identical to the corresponding Schwarzschild equations for the metric tensor components.

1 Introduction

This paper develops a solution of the quaternionic metrical field equations of Sachs\(^1\), \(^2\), \(^3\) corresponding to the Schwarzschild solution in ordinary general relativity. In analogy to Dirac’s idea of taking the matrix square root of the Klein Gordon equation and with it important predictions for the electron, Sachs used quaternions to take the square root of the metric condition\(^4\)

\[
d\tau^2 = -g^{\mu\nu} dx_\mu dx_\nu. \tag{1}
\]

In a similar manner, Sachs uses quaternions to factorize the Einstein equation, in effect, taking its matrix square root. We begin with a review of the key assumptions and intermediate steps he took in deriving his quaternionic metrical
field equations before presenting our new result. Our review is taken from his books and key articles. In Sec. 1, we review the connection between what he calls the Einstein group and the role of quaternions. In Sec. 2 we give the connection between the metric and four-vector quaternion functions. In Sect. 3 the derivation of the spin affine connection is reviewed and in Sec. 4 the connections between the spin curvature tensor and Riemann curvature tensors are established. Then in Sec. 5 Sachs’ quaternion field equation is developed. In Sec. 6 we construct a four-vector quaternion that corresponds to a spherically symmetric static metric. We then show that the equations for this quaternion corresponding to a vacuum solution lead to differential equations that are identical to the corresponding Schwarzschild equations for the metric tensor components.

2 The Einstein Group and Quaternions

As with Einstein’s original form of the general theory of relativity, Sachs’ metric field equations are based on the fundamental axiom of the principle of relativity which is ‘general covariance’. General covariance is the assertion that all laws of nature must be independent of the frame of reference in which they may be represented. Sachs’ ‘Einstein group’ refers to the group of all analytic transformations between the space and time coordinates of all possible frames of reference\(^5\). The space-time transformations of the Einstein group are characterized by the set of continuously distributed derivatives \(\partial x'^\mu / \partial x^\nu = x'^{\mu}_{\nu}\). These 16 parametric functions are the rate of change of the space-time coordinates in one frame, \(x'^\mu\), with respect to those of another, \(x^\nu\), where \(\mu, \nu = 0, 1, 2, 3\) are the temporal and three spatial coordinates. He states that the significance of this number is that there must be 16 independent field equations to prescribe the space-time. He introduces four vector quaternion functions \(q^\mu(x)\), with each of the four vector components being a quaternion rather than a real number field to embody the 16 component metrical field\(^5\). He then argues that the corresponding independent field equations should also be 16 in number.

The Einstein group is a symmetry group of general relativity and is defined as the set of proper transformations that leave invariant the metric condition Eq. \(^1\), excluding time reversal and parity inversion. This set of continuous and analytic transformations also preserves the forms of the laws of nature. Sachs asks the question why the Einstein equations are 10 in number rather than 16. His answer is that the form of these equations are more symmetric than they need be in accordance with the 16 parameter Einstein group. They are not only covariant with respect to continuous transformation, but are also covariant with respect to discrete reflections in space and time. The latter is not an absolute requirement. Sachs demonstrated with the use of the four-vector quaternion functions \(q^\mu(x)\), how, in effect, the Einstein equation can factorize into two equations, neither of which by itself is reflection symmetric or antisymmetric\(^6\). How does this come about?
First one recalls that the irreducible representations of the proper Poincaré group of special relativity obey the algebra of quaternions. (Related to this is the well known fact that one cannot produce parity inversions, reflections, or time reversal by using the Pauli matrices as generators). Sachs points out that the irreducible representations of the Einstein group of general relativity also obey the algebra of quaternions.

Let us recall some elementary properties of quaternions. Recall that Hamilton introduced them as generalizations of complex numbers from a two dimensional space to a four dimensional space.

\[
Q = x^4 + ix^1 + jx^2 + lx^3, \quad x^4, x^1, x^2, x^3 \text{ real,}
\]

(2)

Their conjugates are

\[
\bar{Q} = x^4 - ix^1 - jx^2 - lx^3, \quad (3)
\]

so

\[
i = -\tilde{i}, \quad j = -\tilde{j}, \quad l = -\tilde{l}.
\]

(4)

and requiring

\[
QQ = (x^4)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2,
\]

(5)

implies

\[
i^2 = -1, \quad j^2 = -1, \quad l^2 = -1,
\]

(6)

and

\[
ij = -ji, \quad jk = -kj, \quad li = -il.
\]

(7)

Closure implies

\[
ij = l = -ji, \quad \tilde{li} = j = -\tilde{lj}, \quad \tilde{j} = i = -\tilde{lj}.
\]

(8)

Since Pauli matrices satisfy

\[
\sigma_i \sigma_j = \delta_{ij} \sigma_0 + i \varepsilon_{ijk} \sigma_k,
\]

(9)
they can be used to represent quaternions if one chooses
\[
1 = \sigma_0, \\
i = -i\sigma_1, \\
j = -i\sigma_2, \\
k = -i\sigma_3.
\]
(10)

In compact form
\[
Q = -i\sigma_\mu x^\mu = \left(\sigma_0 x^4 - i\sigma \cdot r\right), \\
\]
(11)
with a *space conjugate* form
\[
\bar{Q} = \sigma_0 x^4 + i\sigma \cdot r,
\]
(12)
and
\[
Q\bar{Q} = (x^4)^2 + r^2.
\]
(13)

Hamilton, of course, had no empirical reason to choose
\[
x^4 = -ix^0 = -ict.
\]
(14)

With that choice, motivated of course by special relativity,
\[
Q = (\sigma_0 x^4 - i\sigma \cdot r) = -i(\sigma_0 x^0 + \sigma \cdot r) = -i\sigma_\mu x^\mu,
\]
and the invariant metric is
\[
(x^0)^2 - r^2 = -\bar{Q}Q = -x^2 = -\eta_\mu\nu x^\mu x^\nu,
\]
(16)
where
\[
Q = -ix^\mu\sigma_\mu = -i \begin{bmatrix} x^0 + x^3 \\ x^1 + ix^2 \\ x^1 - ix^2 \\ x^0 - x^3 \end{bmatrix}.
\]
(17)

Another way of writing this quaternion is to introduce the *time conjugate* operation
\[
Q \rightarrow \tilde{Q} = \varepsilon Q^* \varepsilon,
\]
\[
\varepsilon = i\sigma_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
\]
(18)

In that case
\[
\tilde{Q} = ix^\mu\tilde{\sigma}_\mu, \\
\tilde{\sigma}_0 = -\sigma_0, \\
\tilde{\sigma}_i = \sigma_i.
\]
(19)

\footnote{Our Minkowski metric is \(\eta_{00} = -1\), \(\eta_{11} = \eta_{22} = \eta_{33} = 1\), \(\eta_{\mu\nu} = 0\), \(\mu \neq \nu\).}
Then
\[ \tilde{Q}Q = x^\nu x^\mu \tilde{\sigma}_\nu \sigma_\mu = x^\nu x^\mu \eta_{\mu\nu}. \] (20)

The simplest quaternionic four vector is the set of four constant matrices
\[ q_\mu = \sigma_\mu, \mu = 0, 1, 2, 3. \]
\[ \tilde{q}_\mu = \tilde{\sigma}_\mu. \] (21)

The Lorentz transformation matrix coefficients \( \alpha_{\kappa'}^\mu \), are restricted by
\[ \eta_{\mu\nu} \alpha_{\kappa'}^\mu \alpha_{\lambda'}^\nu = \eta_{\kappa'\lambda'}, \] (22)
which gives 10 conditions on otherwise 16 independent space-time independent elements \( \alpha_{\kappa'}^\mu \), so that, including four space-time translations the Poincaré’ group has just 6+4=10 independents elements (parametrized additionally by three Euler angles and three boost velocities). Thus
\[ d\tau^2 = -\eta_{\mu\nu} dx^\mu dx^\nu \rightarrow d\tau'^2 = -\eta_{\mu\nu} \alpha_{\kappa'}^\mu \alpha_{\lambda'}^\nu dx^\kappa dx^\lambda = -\eta_{\kappa'\lambda'} dx^\kappa dx^\lambda = d\tau^2. \] (23)

In contrast, the Einstein group entails 16 instead of 10 independent spacetime dependent parametric functions since in general
\[ g_{\mu\nu} x_{\mu'} x_{\nu'} = g_{\kappa'\lambda'} x_{\kappa'} x_{\lambda'} \neq g_{\kappa\lambda}, \] (24)
and therefore, unlike Eq. (22), does not restrict the \( x_{\mu'} \) (in case of Lorentz transformations one has \( x_{\mu'} = \alpha_{\mu'}^\mu \)) and of course
\[ d\tau^2 = -g^{\mu\nu}(x) dx_\mu dx_\nu = -g_{\mu\nu}(x) x_{\mu'} x_{\nu'} dx^\kappa dx^\lambda = -g'_{\kappa\lambda}(x) dx^\kappa dx^\lambda = d\tau'^2. \] (25)

In analogy with Dirac’s idea of taking the square root of the Klein-Gordon equation by introducing matrices, Sachs came upon the idea of taking the square root of the metric and, in a sense, ultimately of the Einstein equations themselves by using quaternions[1-3]. He does this by introducing
\[ dS = q_\mu(x) dx^\mu, \] (26)
as a matrix square root of the squared line element instead of \( \pm \sqrt{-d\tau^2} \).

The quaternionic function \( q^\mu(x) \) has both a vector character and a second rank spinor character. That is, one can view it in terms of its transformation properties as the outer product of two two-component spinors \( \sim (\eta \gamma^1)^\mu \) and so it transforms as a combination of a four vector (first rank tensor) and as a second rank spinor under the Einstein group,
\[ q'_{\lambda'}(x') = x'^{\nu'} \lambda' S(x) q_\nu S^{-1}(x), \] (27)
where \( S(x) \) are the spinor transformation matrices for the Einstein group.
3 Square root of the metric condition.

In addition to Eq. (26) Sachs introduces the time conjugate line element
\[ d\tilde{S} = \tilde{q}_\mu(x)dx^\mu, \] (28)
where, as in Eq. (18), the quaternionic conjugate is defined by
\[ \tilde{q}_\mu(x) = \varepsilon q^*_\mu(x)\varepsilon. \] (29)
Then
\[ ds^2 = -dSd\tilde{S} = -q_\mu(x)\tilde{q}_\nu(x)dx^\mu dx^\nu = -\sigma_0 g_{\mu\nu}(x)dx^\mu dx^\nu, \] (30)
implies [4], because of the symmetry in the differential indices,
\[ g_{\mu\nu}(x)\sigma_0 = \frac{1}{2}[q_\mu(x)\tilde{q}_\nu(x) + q_\nu(x)\tilde{q}_\mu(x)]. \] (31)

For Minkowski space, or in the local limit, \[ g_{\mu\nu} \rightarrow \eta_{\mu\nu} \] and \[ q_\mu(x) \rightarrow \sigma_\mu, \] as one can readily verify,
\[ \eta_{\mu\nu}\sigma_0 = \frac{1}{2}[^\sigma_\mu \tilde{\sigma}_\nu + \sigma_\nu \tilde{\sigma}_\mu]. \] (32)

One can see how the 16 independent quaternion components can be explicitly labeled by introducing the 16 real tetrads \[ v^\mu_a(x) \] with one Greek index in general space time and one Latin index in Minkowski space,
\[ q^\mu(x) = \eta^{ab}\sigma_b v^\mu_a(x) = q^\mu(x)^\dagger, \] (33)
with
\[ v^\mu_a(x)v^\nu_b(x)g_{\mu\nu}(x) = \eta_{ab}, \] (34)
and
\[ \eta^{ab}v^\mu_a(x)v^\nu_b(x) = g_{\mu\nu}. \] (35)
As with Eq. (31), this can be viewed as giving 10 conditions on the 16 functions embodied in the 4 tetrads. The conjugate quaternion is given by
\[ \tilde{q}^\mu(x) = \varepsilon q^{\mu}(x)^*\varepsilon = \eta^{ab}\tilde{\sigma}_b v^\mu_a(x) = \tilde{q}^\mu(x)^\dagger. \] (36)
The Minkowski limit is defined by
\[ v^\mu_a(x) \rightarrow \delta^\mu_a, \]
\[ v_{\alpha\mu}(x) \rightarrow \eta_{\alpha\mu}. \] (37)
Using Eqs. (33), (36), and
\[ Tr\sigma_a\tilde{\sigma}_b = 2\eta_{ab} \] (38)
one can show that
\[ Trq^\mu(x)\tilde{q}^\nu(x) = 2g^{\mu\nu}(x). \] (39)
4 The Spin Affine Connection

Just as a vector field’s covariant derivative requires the introduction of the affine connection $\Gamma^\kappa_{\mu\nu}$

$$\Gamma^\kappa_{\mu\nu} = g^{\kappa\sigma} \Gamma_{\sigma\mu\nu} = \frac{g^{\kappa\sigma}}{2} (g_{\nu\sigma,\mu} + g_{\mu\sigma,\nu} - g_{\mu\nu,\sigma}) = \Gamma_{\kappa\mu\nu},$$ \hspace{1cm} (40)

so a spinor field requires the introduction of the spin-affine connection. Both affine connections are due to the nonlinear space-time. It is important to note that the introduction of the 2x2 matrix structure of the quaternion $q_\mu$ implies a spinor vector space upon which it can act, with two component spinors as elements,

$$\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}.$$

(41)

Just as a four vector $V^\mu(x)$ under the Einstein group transforms as a first rank vector,

$$V^\mu(x) \rightarrow V^\mu(x') = x'^\mu V^\mu(x),$$

so the spinor $\eta(x)$ transforms as a first rank spinor,

$$\eta(x) \rightarrow \eta'(x') = S(x)\eta(x).$$

(43)

One thus anticipates a covariant derivative of the form

$$\eta_{;\mu} = \eta_{,\mu} + \Omega_{\mu\eta}.$$

(44)

This becomes more clear by making explicit the spinor index

$$\eta^\mu_{;\mu} = \eta^\alpha_{;\mu} + \Omega_{\mu\beta} \eta^\beta,$$

in analogy with the way in which the ordinary affine connection modifies the gradient of a vector to produce a covariant derivative,

$$V^\nu_{;\mu} = V_{;\mu}^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda.$$

(46)

One finds $\Omega_{\mu}$ by noting that just as the metric has a zero covariant derivative, $g_{\mu\nu,\lambda} = 0$, so $q_{\mu,\lambda} = 0 = \tilde{q}_{\mu,\lambda}$ because of the connection (31) between $q$ and $g$ or that in the local limit $q_{\mu,\lambda} \rightarrow \sigma_{\mu,\lambda} = 0$. That leaves open the question of how to define the covariant derivative of an object that is at the same time a vector and a second rank spinor.

One expects that, in analogy to the expression for the covariant derivative of an ordinary third rank tensor,

$$T^{\mu\nu\kappa}_{;\lambda} = T^{\mu\nu\kappa}_{,\lambda} + \Gamma^\mu_{\lambda\eta} T^{\nu\eta\kappa} + \Gamma^\nu_{\lambda\eta} T^{\mu\eta\kappa} + \Gamma^\kappa_{\lambda\eta} T^{\mu\nu\eta},$$

(47)

that

$$q^{\mu\alpha\beta}_{;\lambda} = q^{\mu\alpha\beta}_{,\lambda} + \Gamma^{\mu}_{\tau\lambda} q^{\alpha\beta}_{,\tau} + \Omega^{\alpha}_{\gamma\lambda} q^{\mu\gamma\beta} + q^{\mu\alpha\gamma} \Omega^{\beta}_{\gamma\lambda},$$

(48)
or \[\Omega^\dagger \lambda = \frac{1}{4} \tilde{q}_\mu (q^{\mu \lambda} + \Gamma^\mu_{\tau \lambda} \tilde{q}^\tau), \]

\[\Omega \lambda = \varepsilon \left( \Omega^\dagger \right) \ast \varepsilon = \frac{1}{4} \varepsilon \tilde{q}_\mu \varepsilon \varepsilon (q^{\mu \lambda} \varepsilon + \Gamma^\mu_{\tau \lambda} \tilde{q}^\tau \varepsilon) = \frac{1}{4} q_\mu (\tilde{q}^{\mu \lambda} + \Gamma^\mu_{\tau \lambda} \tilde{q}^\tau), \] (57)

Taking the adjoint, gives us the two additional forms

\[\Omega^\dagger \lambda = \frac{1}{4} (\tilde{q}^{\mu \lambda} + \Gamma^\mu_{\tau \lambda} \tilde{q}^\tau) q_\mu, \]

\[\Omega \lambda = -\frac{1}{4} (q^{\mu \lambda} + \Gamma^\mu_{\tau \lambda} q^\tau) \tilde{q}_\mu. \] (58)

5 The Riemann Curvature Tensor, the Spin Curvature Tensor and Their Relation

For an arbitrary first rank tensor \(A_\nu\) the mixed second covariant derivatives do not commute, with their difference

\[A_{\nu ; \rho ; \sigma} - A_{\nu ; \sigma ; \rho} = (\Gamma^\kappa_{\nu \sigma ; \rho} - \Gamma^\kappa_{\nu \rho ; \sigma}) A_\kappa + (\Gamma^\kappa_{\nu \sigma} \Gamma^\lambda_{\kappa \rho} - \Gamma^\kappa_{\nu \rho} \Gamma^\lambda_{\kappa \sigma}) A_\lambda \]

\[= R^\lambda_{\nu \rho \sigma} A_\lambda = R_{\lambda \nu \rho \sigma} A^\lambda, \]

defining the fourth rank mixed Riemann Christoffel curvature tensor.

\[R^\lambda_{\nu \rho \sigma} = \Gamma^\lambda_{\nu \sigma ; \rho} - \Gamma^\lambda_{\nu \rho ; \sigma} + \Gamma^\kappa_{\nu \sigma} \Gamma^\lambda_{\kappa \rho} - \Gamma^\kappa_{\nu \rho} \Gamma^\lambda_{\kappa \sigma}, \] (59)
In analogy to this use the fact that \( \eta_{;\rho} = \Omega_{\rho \eta} \) is both a first rank tensor and a first rank spinor from which one obtains

\[
\eta_{;\rho;\lambda} = \left( \eta_{;\rho} + \Omega_{\rho \eta} \right)_{;\lambda} - \Gamma^\nu_{\rho \lambda} \left( \eta_{;\nu} + \Omega_{\nu \eta} \right) + \Omega_{\lambda \eta} \left( \eta_{;\rho} + \Omega_{\rho \eta} \right), \tag{60}
\]

and so

\[
\eta_{;\rho;\lambda} - \eta_{;\lambda;\rho} = \left[ \Omega_{\rho \lambda} - \Omega_{\lambda \rho} \right] \eta_{;\rho} + \Omega_{\lambda \rho} \eta - \Omega_{\rho \lambda} \eta + \Omega_{\lambda \rho} \eta - \Omega_{\rho \lambda} \eta + \Omega_{\lambda \rho} \eta = \Omega_{\rho \lambda} \eta + \Omega_{\lambda \rho} \eta - \Omega_{\rho \lambda} \eta + \Omega_{\lambda \rho} \eta.
\]

\[
K_{\lambda \rho} \equiv \Omega_{\rho \lambda} \eta + \Omega_{\lambda \rho} \eta - \Omega_{\rho \lambda} \eta + \Omega_{\lambda \rho} \eta.
\]  \( 61 \)

\( K_{\lambda \rho} \) denotes the spin curvature tensor. Similarly

\[
\eta_{;\rho;\lambda} - \eta_{;\lambda;\rho} = \eta_{;\lambda} K^\dagger_{\lambda \rho},
\]

\[
K^\dagger_{\lambda \rho} = \Omega_{\rho \lambda} + \Omega_{\lambda \rho} - \Omega_{\rho \lambda} - \Omega_{\lambda \rho}.
\]  \( 62 \)

Note that from Eq. (57)

\[
\varepsilon K^\dagger_{\lambda \rho} \varepsilon = \varepsilon \left( \Omega_{\rho \lambda} \varepsilon - \Omega_{\lambda \rho} \varepsilon \right) - \varepsilon \Omega_{\rho \lambda} \varepsilon + \varepsilon \Omega_{\lambda \rho} \varepsilon = K^\dagger_{\lambda \rho}.
\]  \( 63 \)

Appendix A demonstrates, by using the above two connections between mixed covariant derivatives and the spin curvature tensor, that

\[
K_{\rho \lambda \eta \mu} + q_{\mu} K^\dagger_{\rho \lambda} = -R_{\kappa \mu \rho \lambda} q^\kappa,
\]

\[
K^\dagger_{\rho \lambda \eta \mu} + \tilde{q}_{\mu} K_{\rho \lambda} = R_{\kappa \mu \rho \lambda} \tilde{q}^\kappa.
\]  \( 64 \)

Given the forms in Eq. (61), (62) and (67, 58) on the one hand and Eq. (57) on the other, this equation is plausible because of the connection between the metric and the quaternions given in Eq. (31) and the fact that the curvature tensor on the right hand side involves first and second derivatives of the metric tensor through the affine connection while the spin curvature tensor on the left hand side involves first and second derivative of the quaternions \( q_{\mu} \) and \( \tilde{q}_{\mu} \) through the spin affine connection. As far as we have been able to determine, however, there has been no published proof of Eq. (64) by manipulations involving traces say of the sort

\[
- \frac{1}{2} Tr [(K_{\rho \lambda \eta \mu} + q_{\mu} K^\dagger_{\rho \lambda}) \tilde{q}^\kappa] = \frac{1}{2} R_{\kappa \mu \rho \lambda} Tr q^\kappa \tilde{q}^\eta = R_{\eta \mu \rho \lambda}.
\]  \( 65 \)

\( ^2 \)The missing technology appears to be the analogue of the traces involving 4 and 6 gamma matrices when derivatives are involved. One could mimic the gamma matrix proofs by use of the tetrad relations in Eq. (33) and Eq. (36), but the expressions involving the derivatives of the \( q \)’s in the expression for \( K \) complicates attempts to show explicitly that evaluation of the left hand side of Eq. (65) yields the right hand side.
Let us multiply the first of Eqs. (64) by $\tilde{q}^\mu$ on the right and the second by $q^\mu$ on the left and add the two expressions. One obtains

$$K_{\rho\lambda}q_\mu \tilde{q}^\mu + q_\mu K_{\rho\lambda}^\dagger \tilde{q}^\mu + q^\mu q_\mu K_{\rho\lambda} = R_{\kappa\mu\rho\lambda}(q^\mu \tilde{q}^\kappa - q^\kappa \tilde{q}^\mu). \quad (66)$$

To simplify this one uses $q^\mu A q_\mu = 2 \text{Tr} A$, $q^\mu \tilde{q}^\mu = 2 \text{Tr} 1 = 4$, $\text{Tr} \Omega = 0$ and

$$\text{Tr} K_{\rho\lambda} = \text{Tr}[\partial_\rho \Omega_\lambda + \Omega_\rho \Omega_\lambda - \partial_\lambda \Omega_\rho] = \text{Tr} \partial_\rho \text{Tr} \Omega_\lambda - \partial_\lambda \text{Tr} \Omega_\rho = 0. \quad (67)$$

Hence one finds that the following simple connection between the spin and Riemann curvature tensor,\[8\]

$$K_{\rho\lambda} = \frac{1}{4} R_{\lambda\rho\mu\kappa} q^\mu \tilde{q}^\kappa,$$

$$K_{\rho\lambda}^\dagger = \frac{1}{4} R_{\lambda\rho\mu\kappa} \tilde{q}^\mu q^\kappa. \quad (68)$$

Even though these two equations as the two in (64) demonstrate formally the connection between the spin and Riemann curvature tensors there has been no verification of their equivalence by using just the definitions of $K$ in terms of $\Omega$ and ultimately $q$ and its derivatives. We will not attempt this here. Instead, we shall concern ourselves with finding a solution to the quaternionic field equations.

### 6 Sachs’ Quaternionic Field Equation

The Einstein equation is written in terms of the symmetric Ricci tensor

$$R_{\nu\kappa} = R_{\kappa\nu} = R^\mu_{\nu\lambda} g_{\mu\lambda} = R^\mu_{\nu\mu}, \quad (69)$$

and the scalar curvature,

$$R = g^{\nu\kappa} R_{\nu\kappa} = R^\kappa_{\kappa}. \quad (70)$$

In the presence of matter or electromagnetic fields, it is

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = -8 \pi T^{\mu\nu}. \quad (71)$$

The source term $T^{\mu\nu}$, corresponding to the density and flux of nongravitational energy and momentum, must satisfy

$$T^{\mu\nu}_{;\mu} = 0, \quad (72)$$

as does

$$G^{\mu\nu}_{;\mu} = 0 \quad (73)$$

corresponding to the contracted Bianchi identity.
As is well known, the Einstein equation also follows from the action principle applied to
\[ I = \int L_E d^4x, \quad (74) \]
in which the Lagrange function is
\[ L_E = g_{\mu\nu} R^{\mu\nu} \sqrt{-g} \quad (75) \]
where \( g \) is the determinant of the metric tensor. Including matter terms \( L_M \) and applying the action principle to \( L_E + L_M \) gives Eq. (71). Sachs derives his quaternionic field equation from a similar action, but with a quaternionic version of Eq. (75).

To find the quaternionic version of \( R \) he shows first (see Appendix A) that
\[
\sigma_0 R_{\gamma\mu\rho\lambda} = -\frac{1}{2} \left[ \bar{q}_\gamma K_{\rho\lambda} q_\mu + \bar{q}_\gamma q_\mu K^\dagger_{\rho\lambda} - K^\dagger_{\rho\lambda} \bar{q}_\mu q_\gamma - \bar{q}_\mu K_{\rho\lambda} q_\gamma \right] \equiv \mathcal{R}_{\gamma\mu\rho\lambda} \\
\sigma_0 R_{\mu\rho} = -\frac{1}{2} [\bar{q}^\gamma K_{\rho\lambda} q^{\mu} + \bar{q}^\gamma q^{\mu} K^\dagger_{\rho\lambda} - K^\dagger_{\rho\lambda} \bar{q}^{\mu} q^\gamma - \bar{q}^{\mu} K_{\rho\lambda} q^\gamma] \equiv \mathcal{R}_{\mu\rho}, \\
\sigma_0 R = g^{\mu\rho} \sigma_0 R_{\mu\rho} = -\frac{1}{2} [\bar{q}^\gamma K_{\rho\lambda} q^{\mu} + \bar{q}^\gamma q^{\mu} K^\dagger_{\rho\lambda} - K^\dagger_{\rho\lambda} \bar{q}^{\mu} q^\gamma - \bar{q}^{\mu} K_{\rho\lambda} q^\gamma] \equiv \mathcal{R}. \quad (76)
\]

We stress the difference between the number field forms on the left hand side and the quaternion field forms on the right hand side by the use of Roman and script variables. Taking the trace of the scalar curvature,
\[
R = \frac{1}{2} \text{Tr} \mathcal{R} = \frac{1}{4} \text{Tr} [\bar{q}^\gamma K_{\rho\lambda} q^{\mu} + h.c.].
\]

For the Lagrangian density Sachs thus uses, in analogy to Eq. (76),
\[
L_E = (\text{Tr} \mathcal{R})(-g)^{1/2} = \frac{1}{2} \text{Tr} [\bar{q}^\gamma K_{\rho\lambda} q^{\mu} + h.c.][-g]^{1/2}. \quad (77)
\]

Sachs derives his quaternionic form of the metrical field equations in which \( \mathcal{R} \) is regarded as a function of \( q^\mu \) and \( \bar{q}^\mu \) by way of the right hand side from
\[
\delta \int \{ \text{Tr} [\mathcal{R}(q^\mu, \bar{q}^\mu, \Omega_\mu, \Omega^\dagger_\mu)] + L_M \} (-g)^{1/2} d^4x,
\]
\[
\mathcal{L}_M = \text{ (matter and electromagnetic contributions)}. \quad (78)
\]

As with the method devised by Palatini in which the form of the affine connection is not assumed but instead an outcome of the equations of motion, so the spin affine connection \( \Omega_\mu \) and its relation to the derivatives of \( q \) is an outcome of the equations of motion. This is accomplished by regarding \( \Omega \) as an independent variable. Using the Palatini-like method, in which \( L_E \) depends on the spin curvature \( K_{\mu\nu} \) only through \( \Omega_{\mu;\nu} \) one finds
\[
\delta \int L_E d^4x = \int \left[ \frac{\partial L_E}{\partial K_{\mu\nu}} \left( \frac{\partial K_{\mu\nu}}{\partial \Omega_{\mu;\nu}} \right) \right]_{\Omega} \delta \Omega_\mu d^4x = 0,
\]
\[
\Rightarrow \left( \frac{\partial L_E}{\partial K_{\mu\nu}} \right)_{\Omega} = (q^\nu \bar{q}^\mu - q^{\mu} \bar{q}^\nu)_{\Omega} = 0, \quad (79)
\]
which in turn leads to the relation derived above
\[ \Omega^{\lambda} = -\frac{1}{4}(q^{\mu},_{\lambda} + \Gamma^{\mu}_{\gamma\lambda}q^{\gamma})\tilde{q}_{\mu}, \] (80)

between \( \Omega^{\mu} \) and the quaternion \( q^{\mu} \), its derivatives and the affine connection. Having established this, using the general relativistic Lagrange equations of motion,
\[ \frac{\partial L}{\partial \Lambda^{(i)}} = \left[ \frac{\partial L}{\partial \Lambda^{(j)}} \right]_{;_{[i}^{j]}}^{\mu}, \] (81)

and \( K^{\rho\lambda} = -K^{\lambda\rho} \) leads to [6]
\[ \frac{\partial L_{E}}{\partial q^{\rho}} = \frac{1}{4}\left[-(K^{\lambda}_{\rho\lambda}\tilde{q}^{\lambda} + \tilde{q}^{\lambda}K^{\lambda}_{\rho\lambda}) - \frac{1}{2}R\tilde{q}_{\rho}\right]^{*}(-g)^{1/2}, \]
\[ \frac{\partial L_{E}}{\partial \tilde{q}^{\rho}} = \frac{1}{4}\left[(K^{\rho\lambda}q^{\lambda} + q^{\lambda}K^{\rho\lambda}) - \frac{1}{2}Rq_{\rho}\right]^{*}(-g)^{1/2} \] (82)

and so
\[ \frac{1}{4}(K^{\rho\lambda}q^{\lambda} + q^{\lambda}K^{\rho\lambda}) - \frac{1}{8}Rq_{\rho} = \frac{\partial L_{M}}{\partial \tilde{q}^{\rho}} = k\mathcal{F}_{\rho}, \]
\[ -\frac{1}{4}(K^{\rho\lambda}\tilde{q}^{\lambda} + \tilde{q}^{\lambda}K^{\rho\lambda}) - \frac{1}{8}R\tilde{q}_{\rho} = k\tilde{\mathcal{F}}_{\rho} = k\varepsilon\mathcal{F}^{*}_{\rho}\varepsilon. \] (83)

Using Eq. (83), one sees that these two equations are quaternionic conjugates of one another. Each of these equations transform as a vector quaternion with 16 independent components. Either of these nonlinear second order partial differential equations is sufficient to determine fully the 16 independent parts of the quaternion \( q^{\mu}(x) \) given appropriate boundary conditions. It is appropriate to call this equation and its conjugate the Sachs equations. But before we go on to construct a quaternionic solution to these Sachs equations let us make some remarks about their intrinsic lack of either time reversal symmetry or space inversion.

Recall that quaternionic conjugation Eq. (29) is equivalent to time reversal. The two equations in (83) are therefore temporal reflections of each other but are distinct and independent. One could likewise show that in Eq. (83) the two equations are spatial reflections of one another. The separation into conjugated field equations appears because of the lack of reflection symmetry in the Einstein group. These two equations, from a mathematical point of view, are analogous to two complex equations, say \( f(z^{*}, z) = 0 \) and \( f(z, z^{*}) = 0 \), which are complex conjugates of one another assuming \( f \) is a real function. These are similar to what Sachs has with the two equations in (83) which are the quaternionic conjugates of each other. Since quaternionic conjugation is the same as either spatial or temporal inversion, his equations go into each other under either parity or time reversal. One could obtain an even parity equation from Eq. (83) by adding the two equations, in analogy to using the real equation \( f(z^{*}, z) + f(z, z^{*}) \) instead of the separate complex equations. Likewise one could obtain an odd
parity equation from Eq. (83) by subtracting the two equations, in analogy to using the imaginary equation \( f(z^*, z) - f(z, z^*) \) instead of the separate complex equations. Note this absence of reflection symmetry of each of the two equations in (83) is, however, not the same as parity or time reversal violation in a single equation because the physics involves not just one of the equations, but both.

7 An Exact Solution to the Vacuum Sachs Equation.

The most well known solution of the Einstein equation is the Schwarzschild solution. It is an exact solution. In this section we will demonstrate a similar exact solution to the Sachs equation (83). But first, we review the standard form of the Schwarzschild solution of the Einstein equation in a vacuum with spherical symmetry and static conditions. Let

\[
\begin{align*}
x^0 &= t, \\
x^1 &= r, \\
x^2 &= \theta, \\
x^3 &= \phi.
\end{align*}
\]  

(84)

Using Dirac’s form of the metric,[9]

\[
\begin{align*}
d\tau^2 &= e^{2\nu}dt^2 - e^{2\lambda}dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2), \\
g_{00} &= -e^{2\nu} = 1/g_{00} \\
g_{11} &= e^{2\lambda} = 1/g_{11}, \\
g_{22} &= r^2 = 1/g_{22}, \\
g_{33} &= r^2 \sin^2 \theta = 1/g_{33}, \\
g_{\mu\nu} &= 0, \quad \mu \neq \nu.
\end{align*}
\]  

(85)

Recall that

\[
\Gamma^\kappa_{\mu\nu} = \frac{g^{\kappa\sigma}}{2}(g_{\nu\sigma,\mu} + g_{\mu\sigma,\nu} - g_{\mu\nu,\sigma}) = \Gamma^\kappa_{\nu\mu}.
\]  

(86)

The only nonzero \( \Gamma^s \)s are [9]

\[
\begin{align*}
\Gamma^0_{10} &= \nu'e^{2\nu-2\lambda}, & \Gamma^0_{10} &= \nu', \\
\Gamma^1_{11} &= \lambda', & \Gamma^2_{12} &= \Gamma^3_{13} = r^{-1}, \\
\Gamma^2_{22} &= -re^{-2\lambda}, & \Gamma^3_{23} &= \cot \theta, \\
\Gamma^3_{33} &= -r \sin^2 \theta e^{-2\lambda}, & \Gamma^2_{33} &= -\sin \theta \cos \theta.
\end{align*}
\]  

(87)

With

\[
R_{\nu\sigma} = \Gamma^\lambda_{\nu\lambda,\sigma} - \Gamma^\lambda_{\nu\sigma,\lambda} + \Gamma^\kappa_{\nu\lambda} \Gamma^\lambda_{\kappa\lambda} - \Gamma^\kappa_{\nu\kappa} \Gamma^\lambda_{\sigma\lambda},
\]  

(88)
the vacuum Einstein equation $R_{\nu\sigma} = 0$ are the Schwarzschild equations:

\[
\begin{align*}
R_{00} &= \left(-\nu'' + \lambda' \nu' - \nu'^2 - \frac{2\nu'}{r}\right)e^{2\nu-2\lambda} = 0, \\
R_{11} &= \nu'' - \lambda' \nu' + \nu'^2 - \frac{2\lambda'}{r} = 0, \\
R_{22} &= (1 + r\nu' - r\lambda')e^{-2\lambda} - 1 = 0, \\
R_{33} &= R_{22}\sin^2 \theta = 0.
\end{align*}
\]

(89)

These lead to

\[
\begin{align*}
\lambda' &= -\nu', \\
\nu'' + 2\nu'^2 + \frac{2\nu'}{r} &= 0, \\
(1 + 2r\nu')e^{2\nu} - 1 &= 0.
\end{align*}
\]

(90)

They are sufficient, together with matching at large $r$ onto the Newtonian form to determine the solutions,

\[
\begin{align*}
g_{00} &= -1 + \frac{r_s}{r} = 1/g_{00} \\
g_{11} &= e^{2\lambda} = 1/(1 - \frac{r_s}{r}) = 1/g_{11}, \\
g_{22} &= r^2 = 1/g_{22}, \\
g_{33} &= r^2 \sin^2 \theta = 1/g_{33}.
\end{align*}
\]

(91)

in which $r_s = 2GM$ is the Schwarzschild radius with $G$ the gravitational constant and $M$ the gravitating mass (we use units with $c = 1$).

We consider the quaternionic four-vector equation given in Eq. (83), the Sachs equation. The vacuum form of this equation is

\[
\frac{1}{4}(K_{\rho\lambda}q^\lambda + q^\lambda K^\dagger_{\rho\lambda}) - \frac{1}{8}Rq_\rho = 0
\]

(92)

Now, we seek to find the quaternionic solution of the Sachs equation corresponding to the above metric. Consider the ansatz

\[
\begin{align*}
q_0 &= e^\nu, \\
q_1 &= \hat{x} \cdot \sigma e^\lambda, \\
q_2 &= r\hat{\theta} \cdot \sigma, \\
q_3 &= \sin \theta r \hat{\phi} \cdot \sigma; \\
q_4 &= \frac{\hat{x} \cdot \sigma}{\sin \theta r}.
\end{align*}
\]

(93)

\[\text{Strictly speaking we are solving the equation for } r \neq 0 \text{ or outside the source and matching the solution to Newton’s for large } r. \text{ Sachs points out [3] that the Schwarzschild solution should be a valid approximation for a bona fide nonlinear solution of the full non-homogeneous Einstein equation, though this has yet to be demonstrated analytically.}\]
They are plausible since they satisfy
\[ q^\mu q^\nu + q^\nu q^\mu = 2g^{\mu\nu}\sigma_0. \]  
(94)

However, this does not necessarily imply that these satisfy the Sachs metrical field equation (83). Eq. (93) parametrizes the tetrads of Eq. (33) by
\begin{align*}
\nu_0^0(x) &= e^{-\nu}, \\
\nu_1(x) &= e^{-\lambda}, \\
\nu_2^2(x) &= \frac{1}{r}, \\
\nu_3^3(x) &= \frac{1}{r \sin \theta}, \\
\nu_\mu^\mu(x) &= 0, \quad \mu \neq a.
\end{align*}
(95)

To compute the terms of the Sachs equation (83) consider first
\[ K_{\rho\lambda} = \Omega_{\rho\lambda} + \Omega_{\lambda\rho} - \Omega_{\rho\lambda} \Omega_{\lambda\rho}, \]  
(96)
where
\[ \Omega_{\lambda} = -\frac{1}{4}(q^{\mu,\lambda} + \Gamma^\mu_{\tau\lambda}q^\tau)\tilde{q}_\mu. \]  
(97)

From this we have
\[ \Omega_{\rho\lambda} = -\frac{1}{4}\partial_\lambda \left[ (q^{\mu,\rho} + \Gamma^\mu_{\tau\rho}q^\tau)\tilde{q}_\mu \right] \\
= -\frac{1}{4}((q^{\mu,\rho} + \Gamma^\mu_{\tau\rho}q^\tau + \Gamma^\mu_{\tau\rho}q^{\tau,\tau})\tilde{q}_\mu \\
+ (q^{\mu,\rho} + \Gamma^\mu_{\tau\rho}q^\tau)\tilde{q}_{\mu,\lambda}]. \]  
(98)

Now, for the static case, there is no time dependence so that
\[ K_{0l} = \Omega_{0,l} + \Omega_l \Omega_0 - \Omega_0 \Omega_l, \]
\[ \Omega_0 = -\frac{1}{4}\Gamma^\mu_{0\rho}q^\rho\tilde{q}_\mu, \]
\[ \Omega_l = -\frac{1}{4}(q^{\mu,l} + \Gamma^\mu_{\tau l}q^\tau)\tilde{q}_\mu, \]
(99)

and using the only nonzero components of the affine connection from Eq. (87) and the ansatz Eq. (93) we find
\begin{align*}
\Omega_0 &= -\frac{1}{4}\Gamma^\mu_{\tau 0}q^\tau q_\mu = \frac{\nu' \hat{k} \cdot \sigma e^{\nu - \lambda}}{2}, \\
\Omega_1 &= -\frac{1}{4}(q^{\mu,1} + \Gamma^\mu_{\tau 1}q^\tau)\tilde{q}_\mu = 0, \\
\Omega_2 &= -\frac{1}{4}(q^{\mu,2} + \Gamma^\mu_{\tau 2}q^\tau)\tilde{q}_\mu = \frac{i}{2}(1 - e^{-\lambda})\phi \cdot \sigma, \\
\Omega_3 &= -\frac{1}{4}(q^{\mu,3} + \Gamma^\mu_{\tau 3}q^\tau)\tilde{q}_\mu = -\frac{i}{2}(1 - e^{-\lambda}) \sin \theta \cdot \sigma.
\end{align*}
Thus
\[ K_{01} = \Omega_{0,1} + \Omega_{1,0} - \Omega_{0,0} = \frac{\left(\nu'' + \nu'^2 - \nu'\lambda\right) \hat{x} \cdot \sigma e^{\nu - \lambda}}{2}, \]
\[ K_{02} = \Omega_{0,2} + \Omega_{2,0} - \Omega_{0,0} = \frac{\nu' \hat{\theta} \cdot \sigma e^{\nu - 2\lambda}}{2}, \]
\[ K_{03} = \Omega_{0,3} + \Omega_{3,0} - \Omega_{0,0} = \frac{\nu' \sin \theta e^{\nu - 2\lambda}}{2}, \]
\[ K_{12} = \Omega_{1,2} + \Omega_{2,1} - \Omega_{1,1} = -\frac{i\lambda}{2} e^{-\lambda} \hat{\phi} \cdot \sigma, \]
\[ K_{13} = \Omega_{1,3} + \Omega_{3,1} - \Omega_{1,1} = \frac{i\lambda}{2} e^{-\lambda} \sin \theta \hat{\phi} \cdot \sigma, \]
\[ K_{23} = \Omega_{2,3} - \Omega_{3,2} + \Omega_{3,2} - \Omega_{2,2} = -\frac{i \sin \theta \hat{x} \cdot \sigma}{2} (1 - e^{-2\lambda}). \] (100)

For our quaternionic ansatz, we use these expressions for the spin curvature tensors. Omitting details we obtain
\[ K_{0\lambda} q^\lambda = K_{01} q^1 + K_{02} q^2 + K_{03} q^3 = \frac{e^{\nu - 2\lambda}}{2} \left[\nu'' - \lambda' \nu' + \nu'^2 + \frac{2\nu'}{r}\right] = (K_{0\lambda} q^\lambda)^\dagger = q^\lambda K_{0\lambda}^\dagger, \] (101)
and
\[ K_{1\lambda} q^\lambda = K_{10} q^0 + K_{12} q^2 + K_{13} q^3 = \frac{e^{-\lambda}}{2} \left[\nu'' - \lambda' \nu' + \nu'^2 - \frac{2\lambda}{r}\right] \hat{x} \cdot \sigma = (K_{1\lambda} q^\lambda)^\dagger = q^\lambda K_{1\lambda}^\dagger, \] (102)
and
\[ K_{2\lambda} q^\lambda = K_{20} q^0 + K_{21} q^1 + K_{23} q^3 = \frac{e^{-2\lambda}}{2} \left[\nu' - \lambda' - \frac{(e^{2\lambda} - 1)}{r}\right] \hat{\theta} \cdot \sigma = (K_{2\lambda} q^\lambda)^\dagger = q^\lambda K_{2\lambda}^\dagger, \] (103)
and finally
\[ K_{3\lambda} q^\lambda = K_{30} q^0 + K_{31} q^1 + K_{32} q^2 = \frac{\sin \theta e^{-2\lambda}}{2} \left[\nu' - \lambda' - \frac{(e^{2\lambda} - 1)}{r}\right] \hat{\phi} \cdot \sigma = (K_{3\lambda} q^\lambda)^\dagger = q^\lambda K_{3\lambda}^\dagger. \] (104)

To complete the remainder of the Sachs equation we need to evaluate the quaternionic version of the scalar curvature, that is,
\[ R \equiv -\frac{1}{2} [\tilde{q}^\lambda K_{\rho \lambda} q^\rho + \tilde{q}^\lambda q^\rho K_{\rho \lambda}^\dagger - K_{\rho \lambda}^\dagger \tilde{q}^\rho q^\lambda - \tilde{q}^\rho K_{\rho \lambda} q^\lambda]. \] (105)
Consider
\[ \tilde{q}^\lambda K_{\rho \lambda} q^\rho = -\tilde{q}^\rho K_{\rho \lambda} q^\lambda = \tilde{q}^0 K_{\rho 0} q^\rho + \tilde{q}^1 K_{\rho 1} q^\rho + \tilde{q}^2 K_{\rho 2} q^\rho + \tilde{q}^3 K_{\rho 3} q^\rho. \] (106)
Using Eqs. (102) - Eq. (104) we find

\[ \tilde{q}^0 K_{\rho 0} q^0 + \tilde{q}^1 K_{\rho 1} q^0 + \tilde{q}^2 K_{\rho 2} q^0 + \tilde{q}^3 K_{\rho 3} q^0 \]

\[ = e^{-2\lambda}[ -\nu'' + \lambda' \nu' - \nu'^2 - \frac{2(\nu' - \lambda')}{r} + \frac{1}{r^2}(e^{2\lambda} - 1) ] \quad (107) \]

Next, consider

\[ \tilde{q}^\lambda q^0 K_{\rho \lambda}^\dagger = (K_{\rho \lambda} q^0 \tilde{q}^\lambda)^\dagger, \]

\[ K_{\rho \lambda} q^0 \tilde{q}^\lambda = K_{\rho 0} q^0 \tilde{q}^0 + K_{\rho 1} q^0 \tilde{q}^1 + K_{\rho 2} q^0 \tilde{q}^2 + K_{\rho 3} q^0 \tilde{q}^3 \]

\[ = -\frac{e^{\nu' - 2\lambda}}{2}[\nu'' - \chi \nu' + \nu'^2 + \frac{2(\nu' - \lambda')}{r}] \tilde{q}^0 - \frac{e^{-\nu}}{2}[\nu'' - \lambda' \nu' + \nu'^2 - \frac{2\lambda'}{r}] \tilde{q}^1 \]

\[ = e^{-2\lambda}[ -\nu'' + \lambda' \nu' - \nu'^2 - \frac{2(\nu' - \lambda')}{r} + \frac{(e^{2\lambda} - 1)}{2} \tilde{q}^3 \]

\[ = \tilde{q}^\lambda q^0 K_{\rho \lambda}^\dagger = \tilde{q}^\lambda K_{\rho \lambda} q^0, \quad (108) \]

and similarly

\[ K_{\rho \lambda}^\dagger \tilde{q}^\rho q^\lambda = K_{\rho 0}^\dagger \tilde{q}^\rho q^0 + K_{\rho 1}^\dagger \tilde{q}^\rho q^1 + K_{\rho 2}^\dagger \tilde{q}^\rho q^2 + K_{\rho 3}^\dagger \tilde{q}^\rho q^3 \]

\[ = K_{\rho 0} q^0 \tilde{q}^\rho q^0 + K_{\rho 1} q^0 \tilde{q}^\rho q^1 + K_{\rho 2} q^0 \tilde{q}^\rho q^2 + K_{\rho 3} q^0 \tilde{q}^\rho q^3. \quad (109) \]

Now

\[ K_{\rho 0}^\dagger \tilde{q}^\rho = K_{10}^\dagger q^1 + K_{20}^\dagger q^2 + K_{30}^\dagger q^3 \]

\[ = -\frac{e^{\nu' - 2\lambda}}{2}[\nu'' - \chi \nu' + \nu'^2 + \frac{2(\nu' - \lambda')}{r}], \]

\[ K_{\rho 1}^\dagger \tilde{q}^\rho = K_{10}^\dagger q^0 - K_{12}^\dagger q^2 - K_{13}^\dagger q^3 \]

\[ = K_{10} q^0 + K_{12} q^2 + K_{13} q^3 \]

\[ = -\frac{e^{-\nu}}{2}[\nu'' - \lambda' \nu' + \nu'^2 - \frac{2\lambda'}{r}] \tilde{q}^1 \]

\[ K_{\rho 2}^\dagger \tilde{q}^\rho = K_{10}^\dagger q^0 - K_{12}^\dagger q^2 - K_{13}^\dagger q^3 \]

\[ = K_{20} q^0 + K_{21} q^2 + K_{32} q^3 \]

\[ = -\frac{e^{2\lambda}}{2}[\nu' - \lambda' - \frac{1}{r}(e^{2\lambda} - 1)] \tilde{q}^2 \]

\[ K_{\rho 3}^\dagger \tilde{q}^\rho = K_{10}^\dagger q^0 - K_{12}^\dagger q^2 - K_{13}^\dagger q^3 \]

\[ = K_{30} q^0 + K_{31} q^1 + K_{32} q^2 \]

\[ = \frac{\sin \theta e^{-2\lambda}}{2}[\nu' - \lambda' - \frac{1}{r}(e^{2\lambda} - 1)] \tilde{q}^3. \quad (110) \]
and so
\[ K_{\rho\lambda}^\dagger \tilde{q}^\rho q^\lambda = K_{\rho\lambda}^\dagger \tilde{q}^\rho q^0 + K_{\rho\lambda}^\dagger \tilde{q}^\rho q^3 + K_{\rho\lambda}^\dagger \tilde{q}^\rho q^2 + K_{\rho\lambda}^\dagger \tilde{q}^\rho q^3 \]
\[ = e^{-2\lambda} \left[ \frac{\nu'' - \nu'' + \nu'^2}{r^2} + \frac{2\nu'}{r} \right] + e^{-2\lambda} \left[ \frac{\nu'' - \nu'' + \nu'^2 - 2\lambda'}{r} \right] \]
\[ + e^{-2\lambda} \left[ \frac{\nu' - \nu' - 1}{r} (e^{2\lambda} - 1) \right] + \frac{e^{-2\lambda}}{2r} \left[ \nu' - \nu' - 1 \right] \hat{\phi} \cdot \sigma \]
\[ = e^{-2\lambda} \left[ \frac{\nu'' - \nu'' + \nu'^2}{r^2} + \frac{2\nu'}{r} - \frac{2\lambda'}{r} - \frac{1}{r} (e^{2\lambda} - 1) \right] = -\tilde{q}^\rho q^\rho K_{\rho\lambda}^\dagger. \]

Thus, we have
\[ R \equiv -\frac{1}{2} \left[ \frac{e^{-2\lambda}}{r^2} \left[ 3\nu'' + 3\nu' - 3\nu'^2 - \frac{6(\nu' - \nu' - 1)}{r} \right] \right] + \frac{3}{r^2} (e^{2\lambda} - 1) \]
\[ -e^{-2\lambda} \left[ \frac{\nu' - \nu' + \nu'^2}{r^2} - \frac{2\nu'}{r} - \frac{2\lambda'}{r} - \frac{1}{r} (e^{2\lambda} - 1) \right] \]
\[ = -2\left[ \frac{e^{-2\lambda}}{r^2} \left[ -\nu'' + \nu' + \nu'^2 - \frac{2\nu' - 2\lambda'}{r} + \frac{1}{r^2} (e^{2\lambda} - 1) \right] \right]. \] (112)

Note this agrees with the scalar curvature obtained from Eq. (88) as could be anticipated from Eq. (77). So, now we are in a position to consider each term in the Sachs equation for \( \rho = 0, 1, 2, 3 \). We obtain
\[ \frac{1}{4}(K_{00}q^\lambda + q^\lambda K_{00}^\dagger) - \frac{1}{8} R q_0 = \frac{1}{8} (4K_{0\lambda}q^\lambda - R q_0) \]
\[ = e^{\nu - 2\lambda} \left[ \frac{4\nu'}{r} + \frac{2}{r^2} (e^{2\lambda} - 1) \right] = 0, \]
\[ \frac{1}{4}(K_{11}q^\lambda + q^\lambda K_{11}^\dagger) - \frac{1}{8} R q_1 = \frac{1}{8} (4K_{1\lambda}q^\lambda - R q_1) \]
\[ = e^{-\lambda} \hat{\phi} \cdot \sigma \left[ -\frac{4\nu'}{r} + \frac{2}{r^2} (e^{2\lambda} - 1) \right] = 0, \]
\[ \frac{1}{4}(K_{22}q^\lambda + q^\lambda K_{22}^\dagger) - \frac{1}{8} R q_2 = \frac{1}{8} (4K_{2\lambda}q^\lambda - R q_2) \]
\[ = e^{\nu - 2\lambda} \left[ -2\nu'' + 2\nu' - 2\nu'^2 - \frac{2(\nu' - \nu' - 1)}{r} \right] = 0, \]
\[ \frac{1}{4}(K_{33}q^\lambda + q^\lambda K_{33}^\dagger) - \frac{1}{8} R q_3 = \frac{1}{8} (4K_{3\lambda}q^\lambda - R q_3) \]
\[ = e^{\nu - 2\lambda} \left[ -2\nu'' + 2\nu' - 2\nu'^2 - \frac{2(\nu' - \nu' - 1)}{r} \right] = 0. \] (113)

The general case, in which each of the four quaternions \( q^\mu(x) \) has four components is a much more complicated set of 16 coupled highly nonlinear equations. In the case of Eq. (113) each of the four quaternionic Sachs equations has only
one component, not four. This simplicity is not surprising since our quaternionic ansatz Eq. (93) has a similar property, that is each of the quaternions has only one component. In summary these four equations give the three independent equations

\[
\begin{align*}
\frac{2\lambda'}{r} + \frac{1}{r^2}(e^{2\lambda} - 1) &= 0, \\
-\frac{2\nu'}{r} + \frac{1}{r^2}(e^{2\lambda} - 1) &= 0, \\
-\nu'' + \lambda'\nu' - \nu'^2 - \frac{(\nu' - \lambda')}{r} &= 0.
\end{align*}
\]

Subtracting the first two equations in the above set and substituting in the last two we obtain

\[
\begin{align*}
\nu' + \lambda' &= 0, \\
-2\nu' e^{2\nu} + (1 - e^{2\nu}) &= 0, \\
-\nu'' - 2\nu'^2 - \frac{2\nu'}{r} &= 0.
\end{align*}
\]

Thus we get agreement between the Schwarzschild metric derived directly from the standard Einstein equation and from Eq. (93) together with the Sachs equation (83).

### 8 Summary and Conclusion

We have presented an in depth review of Sachs’ early work [1] on his quaternionic field equations of general relativity, culminating in the Sachs equation and its quaternionic conjugate given in Eq. (83). Since this equation is of the form of a four-vector quaternion, it is a 16 component equation with the correct number of components to solve uniquely, given the appropriate boundary conditions, for the 16 components of the basic quaternionic four vector \(q^\mu(x)\). The new result we present in this paper is an exact solution of this equation corresponding to the static, spherically symmetric conditions that Schwarzschild used in his derivation. The quaternionic four vector \(q^\mu(x)\) that we found in Eq. (93) constructs the metric via Eq. (31) and using the Sachs equation (83) leads to differential equations in terms of the parametric functions \(\nu(r)\) and \(\lambda(r)\) that agree with those obtained in Schwarzschild’s treatment of the Einstein equation. The Sachs equation (83) is, as mentioned in the introduction, a factorized version of the Einstein equation. In [1]-[3] Sachs claims that 10 of the 16 equations can be brought to symmetric tensor form and therefore identified with with gravity (Einstein’s equations) and that 6 of the 16 equations can be brought to an antisymmetric tensor form and thus be identified with electromagnetism (Maxwell’s equations). (See appendix B). However, in our view,
solving the Sachs equation, which has the same number of components as \( q^\mu(x) \), is more direct and economical than solving either of the above two factorized combinations.

\[ \text{A The Spin Curvature Tensor and the Riemann Curvature Tensor} \]

Here we review the Sachs derivation \([1]-[3], [8]\) of the results given in Eq. (76). Recall \( g^{\mu\nu}_{\lambda\rho} = 0 = [q^\mu(x)q^\nu(x)]_{\lambda\rho} \) implies \( q^\mu(x)_{;\lambda} = 0 \). Also, recall that in terms of its transformation properties, the quaternion \( q^\mu \sim (\eta\eta^\dagger)^\mu \), so

\[ 0 = q_{\mu;\rho\lambda} - q_{\mu;\lambda\rho} = ((\eta_{\rho;\lambda} - \eta_{\lambda;\rho})\eta^\dagger + \eta(\eta^\dagger_{\rho;\lambda} - \eta^\dagger_{\lambda;\rho})) \]

in which the last term refers to difference between covariant derivatives of \( q^\mu \) as a four vector only. That should be given in terms of the Riemann curvature tensor by

\[ R^\kappa_{\mu\rho\lambda}q^\kappa. \]

Hence

\[ \left[(\eta_{\rho;\lambda} - \eta_{\lambda;\rho})\eta^\dagger\right]_{\mu} + \left[\eta(\eta^\dagger_{\rho;\lambda} - \eta^\dagger_{\lambda;\rho})\right]_{\mu} = -R^\kappa_{\mu\rho\lambda}q^\kappa. \]

Similarly we find that

\[ K^\dagger_{\rho\lambda}q^\mu + q^\mu K^\dagger_{\rho\lambda} = R^\kappa_{\mu\rho\lambda}q^\kappa. \]

Now

\[ \tilde{q}^\dagger_{\mu;\lambda} - \tilde{q}^\dagger_{\mu;\lambda} = \tilde{q}^\dagger_{\gamma}K^\dagger_{\rho\lambda}q^\gamma + \tilde{q}^\dagger_{\gamma}q^\mu K^\dagger_{\rho\lambda} - \tilde{q}^\dagger_{\gamma}q^\mu K^\dagger_{\rho\lambda}q^\gamma = -R^\kappa_{\mu\rho\lambda}(\tilde{q}^\dagger q^\kappa + \tilde{q}^\kappa q^\gamma) \]

and so we have the following relation between the Riemann curvature tensor and the spin curvature tensor,

\[ \sigma_0 R^\gamma_{\mu\rho\lambda} = \frac{1}{2} \left[ \tilde{q}_\gamma K^\dagger_{\rho\lambda}q^\gamma - \tilde{q}_\gamma K^\dagger_{\rho\lambda}q^\mu - \tilde{q}_\gamma q^\mu K^\dagger_{\rho\lambda}K^\dagger_{\rho\lambda} \tilde{q}_\gamma. \]

We distinguish between the right and left hand side by defining the quaternionic Riemann curvature tensor

\[ \mathcal{R}^\gamma_{\mu\rho\lambda} = \frac{1}{2} \left[ \tilde{q}_\gamma K^\dagger_{\rho\lambda}q^\gamma - \tilde{q}_\gamma K^\dagger_{\rho\lambda}q^\mu - \tilde{q}_\gamma q^\mu K^\dagger_{\rho\lambda} + K^\dagger^\gamma_{\rho\lambda} \tilde{q}_\gamma q^\gamma. \]

20
Although formally the two fourth rank tensors $R_{\gamma \mu \rho \lambda}$ and $\mathcal{R}_{\gamma \mu \rho \lambda}$ should be equivalent based on Eqs. 6.1-6.3, direct verification with use of just Eq. (6.1), (6.4), (6.5) and connections has not been determined in published papers. The Ricci tensor is obtained by

$$
\sigma_0 g^{\gamma \lambda} R_{\gamma \mu \rho \lambda} = \sigma_0 R_{\mu \rho} = \frac{1}{2} [\tilde{q}_\mu K_{\rho \lambda} q^\lambda - \tilde{q}^\lambda K_{\rho \lambda} q_\mu - \tilde{q}^\lambda q_\mu K^\dagger_{\rho \lambda} + K^\dagger_{\rho \lambda} \tilde{q}_\mu q^\lambda] \equiv R_{\mu \rho},
$$

(A.8)

and the curvature scalar by

$$
\sigma_0 R = g^{\mu \rho} \sigma_0 R_{\mu \rho} = \frac{1}{2} [\tilde{q}^\rho K_{\rho \lambda} q^\lambda - \tilde{q}^\lambda K_{\rho \lambda} q^\rho - \tilde{q}^\lambda q^\rho K^\dagger_{\rho \lambda} + K^\dagger_{\rho \lambda} \tilde{q}^\rho q^\lambda] \equiv \mathcal{R},
$$

(A.9)

with $\mathcal{R}_{\mu \rho}$ called the quaternionic Ricci tensor and $\mathcal{R}$ the quaternionic scalar curvature.

### B The Relation Between the Sachs equations and the Einstein and Maxwell Equations.

The Einstein equations and the structure of the Maxwell equations can be reproduced from the Sachs equations by a process that begins with multiplying the first of Eq. 6.3 on the right by $\tilde{q}_\gamma$ and the second on the left by $q_\gamma$ giving

$$
\frac{1}{4} (K_{\rho \lambda} q^\lambda \tilde{q}_\gamma + q^\lambda K^\dagger_{\rho \lambda} \tilde{q}_\gamma) - \frac{1}{8} q_\rho \tilde{q}_\gamma \mathcal{R} = k \tilde{F}_\rho \tilde{q}_\gamma,
$$

$$
\frac{1}{4} (-q_\gamma \tilde{q}^\lambda K_{\rho \lambda} - q^\lambda K^\dagger_{\rho \lambda} \tilde{q}_\gamma) - \frac{1}{8} q_\gamma \tilde{q}_\rho \mathcal{R} = k q_\rho \tilde{F}_\rho.
$$

(B.1)

Adding and subtracting these produces

$$
\frac{1}{4} (K_{\rho \lambda} q^\lambda \tilde{q}_\gamma - q_\gamma \tilde{q}^\lambda K_{\rho \lambda} + q^\lambda K^\dagger_{\rho \lambda} \tilde{q}_\gamma - q_\gamma K^\dagger_{\rho \lambda} \tilde{q}^\lambda) - \frac{1}{8} (q_\rho \tilde{q}_\gamma + q_\gamma \tilde{q}_\rho) \mathcal{R} = k (\mathcal{F}_\rho \tilde{q}_\gamma + q_\gamma \tilde{F}_\rho),
$$

$$
\frac{1}{4} (K_{\rho \lambda} \tilde{q}^\lambda \tilde{q}_\gamma + q_\gamma \tilde{q}^\lambda K_{\rho \lambda} + q_\gamma K^\dagger_{\rho \lambda} \tilde{q}_\gamma + q^\lambda K^\dagger_{\rho \lambda} \tilde{q}^\lambda) - \frac{1}{8} (q_\gamma \tilde{q}_\gamma - q_\rho \tilde{q}_\rho) \mathcal{R} = k (\mathcal{F}_\rho \tilde{q}_\gamma - q_\gamma \tilde{F}_\rho).
$$

(B.2)

Taking the trace of both equations

$$
Tr [K_{\rho \lambda} (q^\lambda \tilde{q}_\gamma - q_\gamma \tilde{q}^\lambda) + K^\dagger_{\rho \lambda} (\tilde{q}^\lambda \tilde{q}_\gamma - \tilde{q}^\lambda q_\gamma)] - \frac{1}{2} Tr (q_\rho \tilde{q}_\gamma + q_\gamma \tilde{q}_\rho) \mathcal{R} = 4k Tr (\mathcal{F}_\rho \tilde{q}_\gamma + q_\gamma \tilde{F}_\rho),
$$

$$
Tr [K_{\rho \lambda} (q^\lambda \tilde{q}_\gamma + q_\gamma \tilde{q}^\lambda) + K^\dagger_{\rho \lambda} (\tilde{q}^\lambda \tilde{q}_\gamma + \tilde{q}^\lambda q_\gamma)] - \frac{1}{2} Tr (q_\rho \tilde{q}_\gamma - q_\gamma \tilde{q}_\rho) \mathcal{R} = 4k (\mathcal{F}_\rho \tilde{q}_\gamma - q_\gamma \tilde{F}_\rho).
$$

(B.3)

Using Eq. 6.3 leads to

$$
Tr [K_{\rho \lambda} (q^\lambda \tilde{q}_\gamma - q_\gamma \tilde{q}^\lambda) + K^\dagger_{\rho \lambda} (\tilde{q}^\lambda \tilde{q}_\gamma - \tilde{q}^\lambda q_\gamma)] = g_{\rho \gamma} Tr \mathcal{R} = 4k Tr (\mathcal{F}_\rho \tilde{q}_\gamma + q_\gamma \tilde{F}_\rho),
$$

$$
2Tr [K_{\rho \gamma} + K^\dagger_{\rho \gamma}] - \frac{1}{2} Tr (q_\rho \tilde{q}_\gamma - q_\gamma \tilde{q}_\rho) \mathcal{R} = 4k (\mathcal{F}_\rho \tilde{q}_\gamma - q_\gamma \tilde{F}_\rho).
$$

(B.4)
Using, from Eqs. \((76)\)

\[
R_{\mu\rho} = -\frac{1}{2}\left[\tilde{q}^{\lambda}K_{\rho\lambda}q_{\mu} + \tilde{q}^{\lambda}q_{\mu}K_{\rho\lambda}^\dagger - K_{\rho\lambda}^\dagger\tilde{q}_{\mu}q^{\lambda} - \tilde{q}_{\mu}K_{\rho\lambda}q^{\lambda}\right],
\]

\[
TrR_{\gamma\rho} = -\frac{1}{2}Tr[K_{\rho\lambda}(q_{\gamma}\tilde{q}^{\lambda} - \tilde{q}^{\lambda}q_{\gamma}) + K_{\rho\lambda}^\dagger(\tilde{q}^{\lambda}q_{\gamma} - \tilde{q}_{\gamma}q^{\lambda})] = TrR_{\gamma\rho}
\]

\[
R = \sigma_0 R,
\]

and that \(Tr(q_{\mu}\tilde{q}_{\gamma} - q_{\gamma}\tilde{q}_{\mu}) = 0\), we obtain

\[
R_{\rho\gamma} - \frac{1}{2}q_{\rho\gamma}R = kTr(F_{\rho\gamma}q_{\gamma} + q_{\gamma}F_{\rho\gamma}),
\]

\[
Tr[K_{\rho\gamma} + K_{\rho\gamma}^\dagger] = 2kTr(F_{\rho\gamma}q_{\gamma} - q_{\gamma}F_{\rho\gamma}).
\]

The first equation is equivalent to Einstein equation. What is the structure of second equation? Define the antisymmetric tensor

\[
F_{\rho\gamma} = Tr[K_{\rho\gamma} + K_{\rho\gamma}^\dagger] = -F_{\gamma\rho}
\]

It has 6 independent components. It is also the curl of a four vector

\[
F_{\rho\gamma} = Tr[K_{\rho\gamma} + h.c.] = Tr[\Omega_{\rho\gamma\gamma\gamma\rho} - \Omega_{\gamma\rho\gamma\rho\gamma} + h.c.]
\]

\[
= A_{\rho\gamma\gamma\rho} - A_{\gamma\rho\gamma\rho}
\]

(B.7)

Defining a current by

\[
F^{\rho\gamma}\gamma = 2kTr(F^{\rho\gamma}q_{\gamma} - q_{\gamma}F^{\rho\gamma}), \gamma \equiv j_{\rho}^\gamma.
\]

As a consequence of the antisymmetry we have that this current is covariantly conserved

\[
F^{\rho\gamma}_{\gamma;\rho} = 0 = j_{\rho}^\gamma = 0.
\]

And, since it also the curl of a four vector

\[
F_{\rho\gamma;\lambda} + F_{\gamma\lambda;\rho} + F_{\lambda\rho;\gamma} = F_{\rho\gamma,\lambda} + F_{\gamma\lambda,\rho} + F_{\lambda\rho,\gamma} = 0
\]

(B.9)

These equations have the derived structure of the Maxwell equations including the no magnetic charge and Faraday laws. However, the tensor \(F_{\rho\gamma}\) is not equivalent to the Faradaw tensor since from Eq. \((67)\) we have that

\[
F_{\rho\gamma} = 0.
\]

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