ON CORRELATION FUNCTIONS OF DRINFELD CURRENTS AND SHUFFLE ALGEBRAS

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Abstract. We express the vanishing conditions satisfied by the correlation functions of Drinfeld currents of quantum affine algebras, imposed by the quantum Serre relations. We discuss the relation of these vanishing conditions with a shuffle algebra description of the algebra of Drinfeld currents.

Introduction. This paper is concerned with the functional properties of the correlation functions of Drinfeld currents of untwisted quantum affine algebras. Let \( \mathfrak{g} \) be a simple finite-dimensional complex Lie algebra, \( \hat{\mathfrak{g}} \) be its affinization and \( U_q \mathfrak{g} \) the associated quantum algebra. We will denote the positive nilpotent currents of \( U_q \mathfrak{g} \) by \( e_\alpha(z) \), where \( z \) is a formal variable and \( \alpha \) is a simple root of \( \hat{\mathfrak{g}} \). The correlation functions of these currents are defined as follows.

For \( V \) a highest weight module over \( U_q \mathfrak{g} \), \( v \) in \( V \) and \( \xi \) in \( V^* \) weight vectors, one considers the series

\[
 f(z_{i_1}^{(\alpha)}) = \langle \xi, \prod_{\alpha=1}^r \prod_{i \in \mathcal{I}_\alpha} e_\alpha(z_{i_1}^{(\alpha)}) v \rangle, \tag{1}
\]

where the \( \mathcal{I}_\alpha \) are the sets \( \{1, \ldots, n_\alpha\} \), and \( r \) is the rank of \( \hat{\mathfrak{g}} \); this series is defined in \( \mathbb{C}[[z_{i_1}^{(\alpha)}, z_{i_1}^{(\alpha)} - 1]] \).

In the classical case \( (q = 1) \), the functional properties of these correlation functions are the following.

**Theorem 0.1.** ([7]) Denote by \((a_{\alpha\beta})_{1 \leq \alpha,\beta \leq r}\) the Cartan matrix of \( \hat{\mathfrak{g}} \). \( f(z_{i_1}^{(\alpha)}) \) belongs to \( \mathbb{C}((z_{1_1}^{(1)}))((z_{2_1}^{(1)})) \cdots ((z_{n_1}^{(r)})) \); it is the expansion of a rational function in the \( (z_{i_1}^{(\alpha)})_{i \in \mathcal{I}_\alpha} \) for each \( \alpha \), regular except for simple poles at the diagonals \( z_{i_1}^{(\alpha)} = z_{j_1}^{(\beta)} \) for \( a_{\alpha\beta} \neq 0 \), and when some \( z_{i_1}^{(\alpha)} \) meets the origin; it satisfies the vanishing conditions

\[
 \operatorname{res}_{z_{i_1}^{(\alpha)} = z_{j_1}^{(\beta)}} \cdots \operatorname{res}_{z_{i_N}^{(\alpha)} = z_{j_N}^{(\beta)}} f(z_{i_1}^{(\alpha)}) dz_{i_1}^{(\alpha)} \cdots dz_{i_N}^{(\alpha)} = 0, \tag{2}
\]

for any simple roots \( \alpha, \beta \), where \( N = 1 - a_{\alpha\beta} \), \( j \) is some index of \( \mathcal{I}_\beta \) and \( i_1, \ldots, i_N \) are \( N \) distinct indices of \( \mathcal{I}_\alpha \). In other words, \( f(z_{i_1}^{(\alpha)}) \) has the form

\[
 f(z_{i_1}^{(\alpha)}) = \frac{1}{\prod_{\alpha<\beta} \prod_{i \in \mathcal{I}_\alpha, j \in \mathcal{I}_\beta} (z_{i_1}^{(\alpha)} - z_{j_1}^{(\beta)})} A(z_{i_1}^{(\alpha)}),
\]

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with $A(z_i^{(a)})$ in $\mathbb{C}[z_i^{(a)}, z_i^{(a)-1}]$ zero whenever $z_i^{(a)} = z_i^{(\beta)} = \cdots = z_{i_{1-a\beta}}^{(a)}$.

The vanishing conditions are consequences of the Serre relations, and of the identity $\langle \xi, [x, y](z)v \rangle = \text{res}_{z=x}(\xi, x(z')y(z)v)dz$, for $x, y$ in $\tilde{g}$ and $x(z)$ the field associated with $x$.

Our goal in this paper is to express vanishing conditions for correlators of Drinfeld currents, analogous to the relations (2). We show

**Theorem 0.2.** Assume that $\tilde{g}$ is not of type $G_2$. Let $q$ be an arbitrary nonzero complex number. Let $d_\alpha$ be the symmetrizing factors of $\tilde{g}$, so that $(d_\alpha a_{\alpha\beta})$ is a symmetric matrix. Set $q_\alpha = q^{d_\alpha}$. The correlation function $f(z_i^{(a)})$ defined by (4) has the following properties:

1) it belongs to $\mathbb{C}((z_1^{(1)}))((z_2^{(1)})) \cdots ((z_n^{(n)}))$;

2) it is the expansion of a rational function on $\prod_\alpha (\mathbb{C}^\times)^{I_\alpha}$, regular everywhere except for simple poles on shifted diagonals $q_\alpha a_{\alpha\beta} z_i^{(a)} = z_j^{(\beta)}$ for $\alpha < \beta$ or $\alpha = \beta$ and $i < j$, or when some $z_i^{(a)}$ meets the origin; it satisfies the twisted symmetry relations

$$\left(q_\alpha^2 z_i^{(a)} - z_j^{(a)}\right)f(z_i^{(a)}) = \left(z_i^{(a)} - q_\alpha^2 z_j^{(a)}\right)(\sigma_{ij}^{(a)} f)(z_i^{(a)}),$$

where $\sigma_{ij}^{(a)}$ is the interchange of variables $z_i^{(a)}$ and $z_j^{(a)}$. In other words, it has the form

$$f(z_i^{(a)}) = \frac{\prod_\alpha \prod_{i,j \in I_\alpha, i < j} \left( z_i^{(a)} - z_j^{(a)} \right)}{\prod_{\alpha \leq \beta} \prod_{i,j \in I_\alpha, j < i, \alpha = \beta} \left( q_\alpha a_{\alpha\beta} z_i^{(a)} - z_j^{(a)} \right)} A(z_i^{(a)}),$$

where $A$ belongs to the space of Laurent polynomials $\mathbb{C}[z_i^{(a)}, z_i^{(a)-1}]$ and is symmetric in the $(z_i^{(a)})_{i \in I_\alpha}$, for each $\alpha$;

3) moreover, $A$ satisfies

$$A(z_i^{(a)}) = 0 \quad \text{when} \quad z_i^{(a)} = q_\alpha^2 z_i^{(a)} = \cdots = q_\alpha^{-2a_{\alpha\beta}} z_{i_{1-a\beta}}^{(a)} = q_\alpha^{-a_{\alpha\beta}} z_j^{(\beta)},$$

for any $\alpha, \beta$, where $j$ belongs to $I_\beta$ and the $i_j$ are pairwise different elements of $I_\alpha$. In other words, $f$ has the form

$$f(z_i^{(a)}) = \frac{\prod_\alpha \prod_{i,j \in I_\alpha, i < j} \left( z_i^{(a)} - z_j^{(a)} \right)}{\prod_{\alpha \leq \beta, a_{\alpha\beta} \neq 0} \prod_{i \in I_\alpha, j, i < j, \alpha = \beta} \left( q_\alpha a_{\alpha\beta} z_i^{(a)} - z_j^{(a)} \right)} B(z_i^{(a)}),$$

where $B(z_i^{(a)})$ is in $\mathbb{C}[z_i^{(a)}, z_i^{(a)-1}]$ and satisfies (2) for all $\alpha, \beta$ such that $a_{\alpha\beta} \neq 0$.

1) and 2) are standard facts that are explained in sect. [4]. The proof of 3) (sect. [3]) rests on some delta-function identities that are established in sect. [2].

We then discuss the relation of this result with the shuffle algebra description of the algebra $U_q n_+$ (sect. [3]). Define $\overline{S H}$ the direct sum $\oplus_{n \in \mathbb{N}} \overline{S H}_n$, where for $n = (n_1, \ldots, n_r)$, $\overline{S H}_n$ is the subspace of $\mathbb{C}((z_1^{(1)})) \cdots ((z_n^{(r)}))$ formed of the elements
of the form (4), where \( A \) belongs to \( \mathbb{C}[z_i^{(\alpha)}, z_i^{(\alpha)-1}] \) and is symmetric in the group of variables \( (z_i^{(\alpha)})_{i \in I_\alpha} \), for each \( \alpha \). \( \overline{Sh} \) is endowed with the a product of shuffle type, which was introduced by B. Feigin and A. Odesskii ([5]). We show

**Theorem 0.3.** There exists an algebra morphism \( \pi \) from \( U_qn_+ \) to \( \overline{Sh} \), sending each \( e_\alpha[n] \) to \( (z^{(\alpha)})^n \in Sh_{\delta_\alpha} \) (here \( \delta_\alpha \) is the vector with coordinates \( (\delta_\alpha)_\beta = \delta_{\alpha\beta} \)).

If \( \mathfrak{g} \) is not of type \( G_2 \), \( \pi \) maps \( U_qn_+ \) to the subspace \( Sh \) of \( \overline{Sh} \), defined as the direct sum \( \oplus_{n \in \mathbb{N}} Sh_n \), where \( Sh_n \) consists of the elements of the form (4), such that \( A \) satisfies the vanishing conditions (4).

It is natural to expect that \( \pi \) is an isomorphism, so that \( Sh \) is actually a subalgebra of \( \overline{Sh} \). This result would probably lead to a simple proof of the PBW theorem for \( U_qn_+ \) (see [1,4]).

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### 1. Properties 1) and 2) of the Correlation Functions

Recall first the presentation of the positive nilpotent part \( U_qn_+ \) of \( U_q\mathfrak{g} \). It has generators \( e_\alpha[n] \), \( \alpha \) simple, \( n \) integer, organized in generating series \( e_\alpha(z) = \sum_{n \in \mathbb{Z}} e_\alpha[n] z^{-n} \), subject to the vertex relations

\[
(qz^{\alpha}\beta z-w)e_\alpha(z)e_\beta(w) = (z-qz^{\alpha\beta})e_\beta(w)e_\alpha(z),
\]

and to the the quantum Serre relations

\[
\sum_{r=0}^{1-a_{\alpha\beta}} (-1)^r \left[ \begin{array}{c} 1-a_{\alpha\beta} \\ r \end{array} \right]_{q_\alpha} \operatorname{Sym}_{z_1 \cdots z_{1-a_{\alpha\beta}}} (e_\alpha(z_1) \cdots e_\alpha(z_r)e_\beta(w)e_\alpha(z_{r+1}) \cdots e_\alpha(z_{1-a_{\alpha\beta}})) = 0,
\]

where \( \left[ \begin{array}{c} n \\ p \end{array} \right]_q = \frac{[n]_q!}{[p]_q!\cdot [n-p]_q!} \), and \( [n]_q! = [1]_q[2]_q \cdots [n]_q \), with \( [n]_q = \frac{q^n-q^{-n}}{q-q^{-1}} \).

Let us pass to the proof of properties 1) and 2) of the \( f(z_i^{(\alpha)}) \).

Recall that \( \mathbb{C}[[z_1, z_1^{-1}, \cdots, z_n, z_n^{-1}]] \) is defined as the space of expansions

\[
\sum_{i_1 \cdots i_n} u_{i_1 \cdots i_n} z_1^{i_1} \cdots z_n^{i_n},
\]

with no restrictions on the \( (i_j) \). On the other hand, for any ring \( R \), \( R((z)) \) is the ring defined as the localization \( R[[z]][z^{-1}] \), that is the set of formal series \( \sum_{i \in \mathbb{Z}} r_i z^i \), where the \( r_i \) belong to \( R \) and vanish if \( i \) is smaller than some integer.

For \( x_i \) a sequence of generating fields, the series

\[
\langle \psi, x_1(z_1) \cdots x_n(z_n)v \rangle
\]
is defined as an element of $\mathbb{C}[[z_1, z_1^{-1}, \cdots, z_n, z_n^{-1}]]$.

Let $V$ be a highest weight module over $U_{qg}$. $V$ has a direct sum decomposition $V = \sum_{i \leq 0} V_i$, which makes it a graded module over $U_{qg}$, endowed with the homogeneous gradation (where each $e^a[n]$ has homogeneous degree $n$).

$V$ is expanded as $\sum_{i_1, \ldots, i_n \in \mathbb{N}} \langle \xi, x_1[i_1] \cdots x_n[i_n]v \rangle z_1^{-i_1} \cdots z_n^{-i_n}$. On the other hand, $v$ belongs to some $V_j$. The coefficient of $z_1^{-i_1} \cdots z_n^{-i_n}$ therefore vanishes if $i_n > N - j$ or if $i_{n-1} > N - (j + i_n)$, or if $i_p > N - (j + i_n + \cdots + i_{p+1})$. This means that $V$ belongs to $\mathbb{C}((z_1)) \cdots ((z_n))$, proving 1) of Thm. (2).

Let now show first that it implies that $f(z_i^{(a)})$ has the form (2), with $A$ in $\mathbb{C}[[z_i^{(a)}]][z_i^{(a)-1}]$. Rename the fields and variables occurring in the definition of $f(z_i^{(a)})$ as

$$f(z_1, \cdots, z_N) = \langle \xi, x_1(z_1) \cdots x_N(z_N)v \rangle.$$

We have the relations $(q^{b_{ij}}z - w)x_i(z)x_j(w) = (z - q^{b_{ij}}w)x_j(w)x_i(z)$. Set for $\sigma$ in $S_n$,

$$f_\sigma(z_1, \cdots, z_N) = \langle \xi, x_\sigma(1)(z_{\sigma(1)}) \cdots x_\sigma(N)(z_{\sigma(N)})v \rangle.$$

Then $f_\sigma(z_1, \cdots, z_N)$ belongs to $\mathbb{C}((z_{\sigma(1)})) \cdots ((z_{\sigma(N)}))$. Since

$$\prod_{i<j} (q^{b_{ij}}z_i - z_j) f(z_1, \cdots, z_N) = \prod_{\sigma(i) > \sigma(j)} (z_i - q^{b_{ij}}z_j) \prod_{\sigma(i) < \sigma(j)} (q^{b_{ij}}z_i - z_j) f_\sigma(z_1, \cdots, z_N),$$

the product $\prod_{i<j} (q^{b_{ij}}z_i - z_j) f(z_1, \cdots, z_N)$ belongs to the intersection of all the $\mathbb{C}((z_{\sigma(1)})) \cdots ((z_{\sigma(N)}))$, with $\sigma$ in $S_n$, which is $\mathbb{C}[[z_1 \cdots z_N]][z_1^{-1} \cdots z_N^{-1}]$. Therefore $f_\sigma(z_1, \cdots, z_N)$ has the form

$$f_\sigma(z_1, \cdots, z_N) = \frac{B(z_1, \cdots, z_N)}{\prod_{i<j} (q^{b_{ij}}z_i - z_j)},$$

with $B(z_1, \cdots, z_N)$ in $\mathbb{C}[[z_1 \cdots z_N]][z_1^{-1} \cdots z_N^{-1}]$ (expansion for $z_N << z_{N-1} << \cdots$).

Now $\prod_{\alpha \leq \beta} \prod_{i \in I_\alpha,j \in I_\beta, i < j} (q^{a_{ij}^{(\alpha)}}z_i^{(\alpha)} - z_j^{(\beta)}) f(z_i^{(\alpha)})$ is totally antisymmetric in each group of variables $(z_i^{(\alpha)})_{i \in I_\alpha}$, for each $\alpha$, which implies that $B$ has the form $\prod_{\alpha} \prod_{i < j} (z_i^{(\alpha)} - z_j^{(\alpha)}) A(z_i^{(\alpha)})$, with $A(z_i^{(\alpha)})$ in $\mathbb{C}[z_i^{(\alpha)}][[z_i^{(\alpha)-1}]]$.

Then the fact that $\xi$ has fixed homogeneous degree implies that $f(z_i^{(a)})$ has a fixed total degree in the variables $z_i^{(\alpha)}$. It follows that $A$ has also a fixed total degree in these variables. Write $A(z_i^{(\alpha)}) = \sum_{i_j^{(\alpha)}} A[i_j^{(\alpha)}][z_i^{(\alpha)}]$, the $A[i_j^{(\alpha)}]$ therefore vanish unless the sum of all $i_j^{(\alpha)}$ is equal to a fixed number; they also vanish unless the $i_j^{(\alpha)}$ are greater than a fixed number, which implies that all but finitely many $A[i_j^{(\alpha)}]$ vanish. This means that $A$ belong to $\mathbb{C}[z_i^{(\alpha)}, z_i^{(\alpha)-1}]$, therefore proving Thm. (2).
2. Delta-function identities

The proof of Thm. 0.2, relies on the following combinatorial identities. Here and below, we use the convention that $\frac{1}{z-w} = \sum_{i \geq 0} w^i z^{-i-1}$ and define $\delta(z, w)$ as $\sum_{i \in \mathbb{Z}} z^i w^{-i-1}$. We have $\delta(z, w) = \frac{1}{z-w} + \frac{1}{w-z}$.

**Proposition 2.1.** We have for $m = 1, 2$ the identity

$$\sum_{k=0}^{m} \binom{m+1}{k} q^{\sum_{l=1}^{k} \frac{1}{z_l-w} + \frac{1}{w-z} + 1} \prod_{i<j} (z_i - z_j) = q^m \sum_{w, z_1, \ldots, z_{m+1}} \delta(w, q^{-m} z_1) \delta(z_1, q^2 z_2) \cdots \delta(z_m, q^2 z_{m+1}),$$

where we set

$$\sum_{w, z_1, \ldots, z_{m+1}} f(z_1, \ldots, z_{m+1}) = \sum_{\sigma \in S_{m+1}} f(z_{\sigma(1)}, \ldots, z_{\sigma(m+1)}).$$

**Proof of Prop. 2.1.** In the case $m = 1$, the left hand side of (2.1) is

$$lhs(w, z_1, z_2) = \frac{1}{q^{-1} w - z_1} \left( \frac{z_1 - z_2}{(q^2 z_1 - z_2)(z_1 - z_2)} - (q + q^{-1}) \frac{z_1 - z_2}{(q^2 z_2 - z_1)(q^{-1} z_2 - q z_1)} \right)$$

$$+ \frac{1}{q^{-1} z_1 - w} \left( \frac{z_1 - z_2}{(q^2 z_1 - z_2)(q^{-1} z_2 - q z_1)} + (q + q^{-1}) \frac{z_1 - z_2}{(q^2 z_1 - z_2)(q^{-2} z_1 - z_2)} \right)$$

$$+ (z_1 \leftrightarrow z_2),$$

that is

$$\frac{1}{q^{-1} w - z_1} \left( \frac{1}{q^2 z_1 - z_2} - (q + q^{-1}) \frac{z_1 - z_2}{(q^2 z_2 - z_1)(q^{-1} z_2 - q z_1)} - \frac{1}{q^2 z_2 - z_1} \right)$$

$$+ \frac{1}{q^{-1} z_1 - w} \left( -q \frac{1}{q^2 z_1 - z_2} + (q + q^{-1}) \frac{z_1 - z_2}{(q^2 z_1 - z_2)(q^{-2} z_1 - z_2)} + q \frac{1}{q^2 z_2 - z_1} \right)$$

$$+ (z_1 \leftrightarrow z_2).$$

Now

$$\frac{z_1 - z_2}{(q^2 z_2 - z_1)(q^{-1} z_2 - q z_1)} = -\frac{q}{q^2 + 1} \left( \frac{1}{q^2 z_2 - z_1} + \frac{1}{z_2 - q^2 z_1} \right),$$
Then \( f \) implies the combinatorial identity (6.1) of [8].

Remark 1. It is natural to expect that identity (2.1) is valid for any \( m \). For \( m = 3 \), this would imply the statement of Thm. 1.2 also for \( g \) of type \( G_2 \). Identity (2.1) implies the combinatorial identity (6.1) of [8].
3. Vanishing properties of the correlation functions

Let us now show that identity (2.1) imply Thm. 0.2, 3).
This statement is nonempty only if \( N_{\alpha} \geq 1 - a_{\alpha \beta} \) and \( N_{\beta} \geq 1 \). If \( a_{\alpha \beta} = 0 \), the commutation of \( e_{\alpha}(z_{i}^{(\alpha)}) \) and \( e_{\beta}(z_{j}^{(\beta)}) \) implies that \( z_{i}^{(\alpha)} - z_{j}^{(\beta)} \) divides \( A(z_{i}^{(\alpha)}) \).

Assume that \( a_{\alpha \beta} \) is equal to \(-1 \) or \(-2 \). Set \( z_{i} = z_{i}^{(\alpha)} \), for \( i = 1, \ldots, 1 - a_{\alpha \beta} \) and \( w = z_{1}^{(\beta)} \). Denote as \( \prod_{i=1}^{N'} e_{\alpha}(z_{i}') \) the product

\[
\prod_{i > 1 - a_{\alpha \beta}} e_{\alpha}(z_{i}^{(\alpha)}) \prod_{i > 1} e_{\beta}(z_{i}^{(\beta)}) \prod_{\gamma \neq \alpha, \beta} \prod_{i \in I_{\gamma}} e_{\gamma}(z_{i}^{(\gamma)})
\]

and set

\[
v' = \prod_{i=1}^{N'} e_{\alpha}(z_{i}') v
\]

and

\[
P = \prod_{i=1}^{N'} \frac{z_{i} - z_{j}'}{q^{(\alpha, \alpha)} z_{i} - z_{j}'} \prod_{i=1}^{N'} \frac{w - z_{i}'}{q^{(\beta, \alpha)} w - z_{i}'} \prod_{1 \leq i < j \leq N'} \frac{z_{i}' - z_{j}'}{q^{(\alpha, \alpha)} z_{i}' - z_{j}'}
\]

where we denote by \( (\alpha, \beta) = d_{\alpha} a_{\alpha \beta} \) the scalar product of two simple roots. \( P \) belongs to \( \mathbb{C}[z_{i}, w](z_{i}' \cdots (z_{N}')) \).

Then we have for any \( k = 0, \ldots, 1 - a_{\alpha \beta} \),

\[
\langle \xi, e_{\alpha}(z_{1}) \cdots e_{\alpha}(z_{k}) e_{\beta}(w) e_{\alpha}(z_{k+1}) \cdots e_{\alpha}(z_{1 - a_{\alpha \beta}}) v' \rangle
\]

\[
= (-1)^{k} \prod_{i=1}^{1 - a_{\alpha \beta}} (w - z_{i}) \cdot P \cdot \frac{A(z_{1}^{(\alpha)})}{\prod_{i=1}^{k} (q^{(\alpha, \beta)} z_{i} - w) \prod_{i=k+1}^{1 - a_{\alpha \beta}} (q^{(\alpha, \alpha)} w - z_{i}) \prod_{1 \leq i < j \leq 1 - a_{\alpha \beta}} z_{i} - z_{j}}
\]

[equality in \( \mathbb{C}((z_{1})) \cdots ((z_{k}))(w)((z_{k+1})) \cdots ((z_{1 - a_{\alpha \beta}}))((z_{1}')) \cdots ((z_{N}')) \)].

The quantum Serre relation (7) implies that \( \langle \xi, \text{left side of relation (7) } v' \rangle \) is equal to zero. (2.1) implies that this relation is written as

\[
P \cdot \left( \text{Sym}_{z_{1}, \ldots, z_{1 - a_{\alpha \beta}}} \delta(w, q_{a_{\alpha \beta}} z_{1}) \delta(z_{1}, q_{a_{\alpha \beta}} z_{2}) \cdots \delta(z_{1 - a_{\alpha \beta}}, q_{a_{\alpha \beta}} z_{1 - a_{\alpha \beta}}) \right) A(z_{1}^{(\alpha)}) = 0.
\]

This implies that the product of the last two terms is itself equal to zero. But the product of \( A(z_{1}^{(\alpha)}) \) with each delta-function is the product of this delta-function and of a function depending only on the \( z_{i}' \) and \( w \). Since the delta-functions are linearly independent, the evaluation of \( A(z_{1}^{(\alpha)}) \) on each variety \( z_{\sigma(1)}^{(\alpha)} = q_{a_{\alpha \beta}}^{2} z_{(1 - a_{\alpha \beta})} = \cdots = q_{a_{\alpha \beta}}^{-2 a_{\alpha \beta}} z_{(1 - a_{\alpha \beta})} = q_{a_{\alpha \beta}}^{-a_{\alpha \beta}} z_{(1 - a_{\alpha \beta})} \) is zero. Since \( A \) is symmetric in the group of variables \( z_{i}^{(\alpha)} \), this is equivalent to the conditions of Thm. 0.2, 3). This completes the proof of Thm. 0.2.

Remark 2. Conditions (1) were obtained in [2] in the case where \( \mathfrak{g} = sl_{n} \) and \( V \) is integrable, using products of Frenkel-Kac realizations. In that case, the
correlation functions (\ref{expansion for \(z \in \mathbb{K}\) by \(q\)\)}) have other functional properties that were studied in that paper.

4. APPLICATION TO FUNCTIONAL DESCRIPTION OF \(U_q n_+\)

In this section, we prove Thm. \ref{thm:0.3}.

Define dual Hopf algebras \((U_q b_\pm, \Delta_\pm)\) as follows. \(U_q b_+\) has generators \(e_\alpha[n]\), \(n \in \mathbb{Z}\) and \(K^+_\alpha[n], n \geq 0, K^+\alpha[0]^{-1}\), generating series \(e_\alpha(z) = \sum_{n \in \mathbb{Z}} e_\alpha[n] z^{-n}\), \(K^+_\alpha(z) = \sum_{n \geq 0} K^+_\alpha[n] z^{-n}\), relations \(\ref{expansion for \(z \in \mathbb{K}\) by \(q\)\)}\) and

\[
[K^+_\alpha[n], K^+_\beta[m]] = 0,
\]

and

\[
K^+_\alpha(z) e_\beta(w) K^+_\alpha(z)^{-1} = \frac{z - q^{(\alpha, \beta)}w}{q^{(\alpha, \beta)}z - w} e_\beta(w);
\]

(expansion for \(w << z\)); the coproduct is defined by

\[
\Delta_+(K^+_\alpha(z)) = K^+_\alpha(z) \otimes K^+_\alpha(z), \quad \Delta_+(e_\alpha(z)) = e_\alpha(z) \otimes K^+_\alpha(z) + 1 \otimes e_\alpha(z).
\]

\(U_q b_-\) has generators \(f_\alpha[n]\), \(n \in \mathbb{Z}\) and \(K^-\alpha[n], n \leq 0, K^-\alpha[0]^{-1}\), generating series \(f_\alpha(z) = \sum_{n \in \mathbb{Z}} f_\alpha[n] z^{-n}\), \(K^-\alpha(z) = \sum_{n \geq 0} K^-\alpha[-n] z^n\), relations \(\ref{expansion for \(z \in \mathbb{K}\) by \(q\)\)}\) with \(q\) replaced by \(q^{-1}\) between the \(f_\alpha(z)\) and

\[
[K^-\alpha[n], K^-\beta[m]] = 0,
\]

and

\[
K^-\alpha(z) f_\beta(w) K^-\alpha(z)^{-1} = \frac{z - q^{(\alpha, \beta)}w}{q^{(\alpha, \beta)}z - w} f_\beta(w);
\]

(expansion for \(z << w\)); the coproduct is defined by

\[
\Delta_-(K^-\alpha(z)) = K^-\alpha(z) \otimes K^-\alpha(z), \quad \Delta_-(f_\alpha(z)) = f_\alpha(z) \otimes 1 + K^-\alpha(z)^{-1} \otimes f_\alpha(z).
\]

So \(U_q \tilde{b}_\pm\) are the usual opposite Hopf subalgebras of the new realizations algebras \(\ref{expansion for \(z \in \mathbb{K}\) by \(q\)\)}, where the quantum Serre conditions are not imposed. The following result can be viewed as an infinite-dimensional analogue of results in \ref{thm:3.2}.\ref{thm:10}.

**Proposition 4.1.** We have a Hopf algebra pairing between \((U_q \tilde{b}_+, \Delta_+)\) and \((U_q \tilde{b}_-, \Delta_-)\), defined by

\[
\langle K^+_\alpha(z), K^-\beta(w) \rangle = \frac{z - q^{(\alpha, \beta)}w}{q^{(\alpha, \beta)}z - w}, \quad \langle e_\alpha[n], f_\beta[m] \rangle = \delta_{\alpha \beta} \delta_{n+m,0},
\]

(expansion for \(w << z\)). The ideals defined by the quantum Serre relations are contained in the radicals of this pairing.

**Proof.** The verification of the first statement is standard. To show the last statement, let us compute the pairing of the Serre relation \(\ref{expansion for \(z \in \mathbb{K}\) by \(q\)\}}\) with any element of \(U_q \tilde{b}_-\). \(U_q \tilde{b}_-\) has a gradation by the root lattice of \(\tilde{g}\), defined by \(\deg(f_\alpha[n]) = \alpha\) and \(\deg K^-\alpha[-n] = 0\). Then the pairing of the left side of \(\ref{expansion for \(z \in \mathbb{K}\) by \(q\)\}}\) can be nontrivial only
against an element of degree \((1 - a_{\alpha})\alpha + \beta\). Translating the Cartan modes \(K^-_n[a]\) to the left of the \(f_{a,\beta}[\phi]\), we have to compute for any \(k\), and \(\phi, \psi \in \mathbb{C}[z, z^{-1}]\),

\[
\langle \text{left side of (7)}, f_{\alpha}[\phi_1] \cdots f_{\alpha}[\phi_k] f_{\beta}[\psi] f_{\alpha}[\phi_{k+1}] \cdots f_{\alpha}[\phi_{1-a_{\alpha}}] \rangle.
\]

(10)

Denote by \(L_m(z_1, \cdots, z_{m+1}, w)\) the left side of identity (2.1), with \(q\) replaced by \(q^{-1}\), viewed as a rational function and expanded for \(z_1 \gg z_2 \gg \cdots \gg z_{m+1} \gg w\). We find that (10) is equal to

\[
\text{Sym}_{z_1 \cdots z_{1-a_{\alpha}}} \left( \phi_1(z_1) \cdots \phi_{1-a_{\alpha}}(z_{1-a_{\alpha}}) \psi(w) \prod_{i=1}^{k} (q^{(\alpha,\beta)}z_i - w) \prod_{i=k+1}^{1-a_{\alpha}} (q^{(\alpha,\beta)}w - z_i) \prod_{j>i} (q_{\alpha}^{-2}z_i)L_{-a_{\alpha}}(z_1, \cdots, z_{1-a_{\alpha}}, w) \right).
\]

On the other hand, from [8] follows that \(L_m(z_1, \cdots, z_{m+1}, w)\) is identically zero (as a rational function, and therefore as a formal series in \(\mathbb{C}(z_1) \cdots (z_{m+1})(w)\)). It follows that (10) is zero. In the same way, one shows that the quantum Serre relations of \(U_qb_-\) are in the radical of the pairing.

\[\square\]

Corollary 4.1. The coproducts \(\Delta_{\pm}\) induce Hopf algebra structures on the quotients \(U_q^{}b_\pm\) of \(U_q\tilde{b}_\pm\) by the ideals generated by the quantum Serre relations.

Then

Proposition 4.2. The direct sum \(\widetilde{\pi}\) of the maps \(\pi_n\) from \(U_q^{}b_+\) to \(\mathbb{C}[z_1^{(\alpha)}], z_2^{(\alpha)-1}]\), defined by \(\pi_n(x) = \langle x, \prod_{i=1}^{n} f_\alpha(z_i^{(\alpha)}) \rangle\) induces a linear map \(\pi\) from \(U_q^{}b_+\) to \(Sh\); if \(\tilde{g}\) is not of type \(G_2\), the image of \(\pi\) is contained in \(Sh\).

Proof. The Hopf pairing rules and the commutation relations imply that \(\pi_n\) defines a linear map from \(U_q^{}b_+\) to \(\mathbb{C}(z_1^{(1)}), \cdots (z_{n+1}^{(r)})\). Crossed vertex relations allow to apply the reasoning of sect. 2 to prove exchange relations, which imply that the image of \(\pi_n\) is contained in subspace of elements of the form (4), with \(A\) in \(\mathbb{C}[z_1^{(\alpha)}, z_2^{(\alpha)-1}]\). A degree argument can be used as in sect. 2 to show that \(A\) belongs to \(\mathbb{C}[z_1^{(\alpha)}, z_2^{(\alpha)-1}]\). Therefore \(\pi\) maps \(U_q^{}b_+\) to \(Sh\). Since \(\pi\) sends the radical of \(\langle, \rangle\) to zero, it induces a map \(\pi\) from \(U_q^{}b_+\) to \(Sh\).

If \(\tilde{g}\) is not of type \(G_2\), we can then follow the reasoning of sect. 3 to show that \(A\) satisfies (3), which shows that the image of \(\pi\) is contained in \(Sh\).

\[\square\]

Define the shuffle product on \(Sh\) by the following rule. Let \(f\) and \(g\) belong to in \(Sh_n\) and \(Sh_m\). Set \(z_1 = z_1^{(1)}, z_2 = z_2^{(1)}\), etc., \(z_{n_1+m_1+\cdots+n_{\alpha-1}+m_{\alpha-1}+i} = z_i^{(\alpha)}\) for \(i = 1, \cdots, n_\alpha+m_\alpha\), \(z_{N+M} = z_{n_\alpha+m_\alpha}\); we set \(N = \sum_{i=1}^{r} n_i, M = \sum_{i=1}^{r} m_i\). We associate “colors” to the variables \(z_i\) by the rule \(\alpha(i) = \alpha i\) if \(i = n_1+m_1+\cdots+n_{\alpha-1}+m_{\alpha-1}+i, i = 1, \cdots, n_\alpha+m_\alpha\). Let \(M_{n,m}\) be the set of bijective maps from \(\{1, \cdots, N+M\}\)
to itself, such that for each $\alpha$, we have
\[
\{\sum_{i=1}^{\alpha-1} (n_i + m_i) + 1, \ldots, \sum_{i=1}^{\alpha} (n_i + m_i)\} = \sigma(\{\sum_{i=1}^{\alpha-1} n_i + 1, \ldots, \sum_{i=1}^{\alpha} n_i\}) \\
\cup \sigma(\{N + \sum_{i=1}^{\alpha-1} m_i + 1, \ldots, N + \sum_{i=1}^{\alpha} m_i\})
\]
and $\sigma(i) < \sigma(j)$ if $i < j$ and $i, j$ both belong to some $\{\sum_{i=1}^{\alpha-1} n_i + 1, \ldots, \sum_{i=1}^{\alpha} n_i\}$ or $\{N + \sum_{i=1}^{\alpha-1} m_i + 1, \ldots, N + \sum_{i=1}^{\alpha} m_i\}$. Then the product $f \ast g$ is defined in $\overline{Sh_{n+m}}$ as
\[
(f \ast g)(z_1, \ldots, z_{N+M}) = \sum_{\sigma \in M_{n,m}} \varepsilon(\sigma) \prod_{i < j, \sigma^{-1}(i) > \sigma^{-1}(j)} q^{(\alpha(i), \alpha(j))} \frac{z_i - z_j}{z_i - q^{(\alpha(i), \alpha(j))} z_j} f(z_{\sigma(1)}, \ldots, z_{\sigma(N)}) g(z_{\sigma(N+1)}, \ldots, z_{\sigma(N+M)}). \tag{11}
\]
Note that each summand in the right side belongs to $\mathbb{C}((z_1)) \cdots ((z_{N+M}))$, because the prefactor cancels poles in $f$ or $g$.

Remark 3. $M_{n,m}$ is isomorphic to the product $\prod_{i=1}^{r} sh_{n_i, m_i}$, where $sh_{n,m}$ is the set of shuffle maps of the pair of sequences $((1, \ldots, n), (n+1, \ldots, n+m))$.

Remark 4. The product (11) is a transcription of the shuffle algebra structure $(FO, \bar{\pi})$ defined in [4]. The isomorphism $i$ from $Sh$ to $FO$ is given by the formula
\[
i(f)(z_1, \ldots, z_N) = \prod_{i < j} \frac{z_i - q^{(\alpha(i), \alpha(j))} z_j}{z_i - z_j} f(z_1, \ldots, z_N).
\]

Proposition 4.3. For $\bar{\pi}$ arbitrary, $\pi$ is a morphism of algebras from $U_q \mathfrak{n}_+$ to $\overline{Sh}$, endowed with the shuffle product (11).

Proof. It suffices to see that $\bar{\pi}$ is an algebra morphism. It is clear that $\bar{\pi}$ maps $e_{a}[n]$ to the element $(z^{(a)})^n$ of $Sh_{\delta_a}$. So we should check that $\bar{\pi}$ maps the crossed vertex relations to zero. This is an easy computation, relying on the equalities
\[
(z^{(\alpha)})^n \ast (z^{(\beta)})^m = (z^{(\alpha)})^n (z^{(\beta)})^m
\]
and
\[
(z^{(\beta)})^m \ast (z^{(\alpha)})^n = \frac{q^{(\alpha, \beta)} z^{(\alpha)} - z^{(\beta)}}{z^{(\alpha)} - q^{(\alpha, \beta)} z^{(\beta)}} (z^{(\alpha)})^n (z^{(\beta)})^m,
\]
for $\alpha < \beta$, and
\[
(z^{(\alpha)})^n \ast (z^{(\alpha)})^m = (z^{(\alpha)})^n (z^{(\alpha)})^m + \frac{q^{2}_{\alpha} z_1 - z_2}{z_1 - q^{2}_{\alpha} z_2} (z^{(\alpha)})^n (z^{(\alpha)})^m.
\]
\[\square\]
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Remark 5. Conditions (3) appeared in [6] in an elliptic situation and the $\mathfrak{sl}_n$ case as a substitute to quantum Serre relations.

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