ON NONSHEARING MAGNETIC CONFIGURATIONS IN DIFFERENTIALLY ROTATING DISKS

STEVEN A. BALBUS
Virginia Institute of Theoretical Astronomy, Department of Astronomy, University of Virginia, Charlottesville, VA 22903-0818; sb@virginia.edu

AND

MASSIMO RICOTTI
Center for Astrophysics and Space Astronomy, University of Colorado, Campus Box 389, Boulder, CO 80309-0389; ricotti@casa.colorado.edu

Received 1998 September 29; accepted 1999 January 25

ABSTRACT

A new class of disk MHD equilibrium solutions is described, which is valid within the standard level (shearing sheet) approximation scheme. These solutions have the following remarkable property: velocity streamlines and magnetic lines of force rotate rigidly, even in the presence of differential rotation. This situation comes about because the Lorentz forces acting upon modified epicycles compel fluid elements to follow magnetic lines of force. Field line (and streamline) configurations may be elliptical or hyperbolic, prograde or retrograde. These structures have previously known hydrodynamical analogs: the planet solutions described by Goodman, Narayan, & Goldreich. The primary focus of this investigation is configurations in the disk plane. A related family of solutions lying in a vertical plane is briefly discussed; other families of solutions may exist. Whether these MHD structures are stable is not yet known, but could readily be determined by three-dimensional simulations. If stable or quasi-stable, these simple structures may find important applications in both accretion and galactic disks.

Subject headings: hydrodynamics — ISM: clouds — MHD — planetary systems — stars: formation

1. INTRODUCTION

The essential role of magnetohydrodynamics (MHD) in understanding the behavior of accretion disks is now widely recognized (e.g., Papaloizou & Lin 1995; Balbus & Hawley 1998). Usually, however, detailed solutions to the dynamical equations are quite complicated: one must contend with turbulence in weak field systems, and exact equilibrium solutions, when they can be found at all, are highly technical and rather special. Generally, differential rotation will complicate matters by producing a time-dependent toroidal magnetic field when a radial field is present.

Given the above, it might appear that any true equilibrium MHD configuration would require at the very least a vanishing radial field component. This is not so. In this paper, we present exact, two-dimensional solutions to the MHD equations using only the approximation of the standard shearing sheet model. (In essence, curvature effects are neglected.) The solutions are noteworthy for the fact that the field lines, which do indeed have radial components, rotate rigidly, even if the disk is not rotating uniformly. Furthermore, the field configurations have very simple geometries: hyperbolae and ellipses. The dynamical and induction equations are mutually consistent because magnetic forces cause departures from the standard epicyclic paths, departures that in these solutions lead fluid elements directly along the field lines. Gas pressure gradients are unimportant.

Of dynamical interest on their own, these solutions may also be astrophysically relevant. Since only the shearing sheet approximation is used, both galactic as well as Keplerian disks are possible venues. The crucial question is whether the solutions are stable—or more generously, whether they are stable enough. We do not attempt to answer the stability question in this paper, which can be best addressed by three-dimensional numerical simulations. It should be noted, however, that the configurations are not obviously unstable to the weak field instability discussed by Balbus & Hawley (1991), since the field strengths involved fall at the edge of the stability domain. If they are long lived, the elliptical solutions (which can be both prograde and retrograde), would represent coherent disk structures. One might then speculate that in galactic disks these magnetically pened regions become natural sites for molecular cloud complex formation. In a more energetic venue, the localized elliptical solutions may represent the nonlinear resolution of tearing mode instabilities in a strongly magnetized disk.1 But whether nature makes use of these solutions or not is at present unknown.

Although very different in their detailed physics, these MHD solutions are kin to the shearing sheet planet solutions discovered numerically by Hawley (1987) and elucidated analytically by Goodman, Narayan, & Goldreich (1987, hereafter GNG). (The GNG solutions allowed for the presence of significant pressure gradients in the underlying equilibrium disk; the appropriate limit for comparison with this work is restricted to Keplerian rotation profiles.) To understand the similarity, note that the gravitational tidal force varies linearly with radius (to lowest retained order). This drives a velocity response also linear in spatial coordinates. The associated streamlines are in both cases familiar conic sections. What is surprising, however, is that the presence of Lorentz forces can be so easily accommodated by this remarkably simple scaling.

In § 2 we present a detailed derivation of the MHD structures, and in § 3 a brief discussion of the astrophysical implications of these solutions is presented.

2. COHERENT MHD STRUCTURES

2.1. Basic Equations

Consider a differentially rotating disk with a cylindrical coordinate system (R, φ, z) centered on the origin. The disk’s angular velocity is given by \( \Omega(R) \). The fundamental

1 We are grateful to our anonymous referee for suggesting this possibility.
MHD equations are mass conservation
\[ \frac{d \ln \rho}{dt} + \nabla \cdot \mathbf{w} = 0 , \]
the dynamical equation
\[ \rho \frac{d \mathbf{w}}{dt} = -\nabla \left( P + \frac{B^2}{8\pi} \right) - \rho \nabla \Phi + \left( \frac{B}{4\pi} \cdot \nabla \right) \mathbf{B} , \]
and the induction equation
\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left( \mathbf{w} \times \mathbf{B} \right) . \]
Here, \( d/dt \) is a Lagrangian time derivative:
\[ \frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{w} \cdot \nabla , \]
\( \mathbf{w} \) is the velocity in the inertial frame, \( P \) represents the gas pressure, \( \Phi \) is the central gravitational potential, and the other symbols have their standard meanings. We shall not require an internal energy equation in our analysis.

We work in the local (or shearing sheet) approximation. This consists of choosing a fiducial radius \( R_0 \) rotating at angular velocity \( \Omega_0 \) and erecting a local Cartesian system:
\[ x = R - R_0 , \quad y = R_0 (\phi - \Omega_0 t) , \quad z = z . \]
In the local approximation, \( x \ll R_0 \), and \( v_x, v_y \ll R_0 \Omega_0 \), where the \( v_i \) are velocities relative to a corotating origin. The Alfvén velocity is defined by
\[ v_A = \frac{B}{\sqrt{4\pi \rho}} . \]
We shall initially assume that the Alfvén velocity components \( v_{Ax} \) and \( v_{Ay} \) are \( \ll R_0 \Omega_0 \), a result that will prove to be self-consistent. When present, \( v_z \) and \( v_Az \) will also be assumed to satisfy this inequality.

Let the global scale angular velocity given by a power law be of the form \( \Omega(R) = \Omega_0 (R/R_0)^g \). Keplerian flow corresponds to \( g = 3/2 \), a flat galactic rotation curve to \( g = 1 \). Effecting our coordinate transformation (eq. [5]) and letting \( v \) now represent the velocity relative to our new axes leads to the Hill equations (e.g., GNG; Hawley, Gammie, & Balbus 1995):
\[ \frac{d \ln \rho}{dt} + \nabla \cdot \mathbf{v} = 0 , \]
\[ \frac{dv}{dt} = -\frac{1}{\rho} \nabla \left( P + \frac{B^2}{8\pi} \right) + \frac{(\mathbf{B} \cdot \nabla) \mathbf{B}}{4\pi \rho} - 2\Omega \times \mathbf{v} + 2q \Omega^2 x \hat{\mathbf{e}}_x - \Omega^2 z \hat{\mathbf{e}}_z , \]
\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left( \mathbf{v} \times \mathbf{B} \right) , \]
where \( v_A \) is now measured relative to \( R_0 \), and all differential operators may be taken to be Cartesian. We have dropped the 0 subscript from \( \Omega_0 \) in equation (8).

2.2. Solutions

As our first case, let \( \rho \) and \( P \) be functions of \( z \) only, which is certainly consistent with the local approximation. We seek solutions in which divergence-free fluid velocities and the magnetic field are confined to the \( x-y \) plane, and \( v_A \) depends upon \( x \) and \( y \) only. In component form, the steady-state mass and induction equations may be written
\[ \partial_i v_i = 0 , \]
\[ v_i \partial_i v_{Ax} = v_{Ax} \partial_i v_j , \]
with \( i \) and \( j \) taking on the values \( x \) and \( y \). The summation convention on repeated indices is used unless otherwise stated. The notation \( \partial_i \) denotes the partial derivative with respect to the \( i \)th Cartesian coordinate. The dynamical equations become
\[ v_i \partial_i v_x - 2\Omega v_y = -\partial_x \left( \frac{v_x^2}{2} \right) + v_{Ax} \partial_i v_{Ax} + 2q\Omega^2 x , \]
\[ v_i \partial_i v_y + 2\Omega v_x = -\partial_y \left( \frac{v_y^2}{2} \right) + v_{Ay} \partial_i v_{Ay} , \]
\[ \partial_i \left( \frac{H + \Omega^2 z^2}{2} \right) = 0 . \]
where the enthalpy \( H \) is defined by \( dH = dP/\rho \).

The \( z \) equation may be immediately integrated,
\[ H - H_c = \frac{\Omega^2 z^2}{2} , \]
where \( H_c \) is the central midplane value of the enthalpy. The vertical structure decouples completely from the planar dynamics of this problem, and we need not pursue it further.

2.2.1. Hydrodynamic Limit

To orient ourselves, let us recover some familiar solutions for nonmagnetized disks from the system (eqs. [10]–[14]). Setting \( v_x = Ay \), \( v_y = Bx \) (\( A \) and \( B \) are constants to be determined), and \( v_A = 0 \), equations (12) and (13) lead immediately to
\[ B(A - 2\Omega) = 2q\Omega^2 , \quad A(B + 2\Omega) = 0 . \]

There are two distinct branches of solutions. One possibility is \( A = 0 \), \( B = -q\Omega \), which corresponds to simple differential rotation of the background flow. The other possibility is \( B = -2\Omega \), \( A = \Omega(2 - q) \). This is epicyclic motion about the guiding center \( R = R_0 \). The streamline are retrograde ellipses with minor-to-major axis ratio \( (1 - q^2)^{1/2} \). We emphasize the simple but important point that fluid elements in the neighborhood of the corotation point do not separate from one another as time goes on, but instead interact about the local origin. Therefore, in the absence of Lorentz forces, magnetic field lines placed along the epicyclic ellipses would remain undisturbed.

When \( q = 2 \), the ellipses become degenerate and shrink to straight lines. If \( q > 2 \), the streamlines become hyperbolic and correspond to Rayleigh unstable flow (angular momentum decreases outward).

2.2.2. Elliptical Configurations

When magnetic fields and Lorentz forces are self-consistently included, it is a remarkable fact that equilibrium solutions are still possible. Equations (10)–(14) have the following exact solution:
\[ v_x = Ay , \quad v_y = Bx , \quad v_{Ax} = \alpha y , \quad v_{Ay} = \beta x , \]
where the constants \( A, B, \alpha, \beta \) are given by
\[ B = \pm \Omega \sqrt{\frac{2q}{1 - \epsilon^2}} , \quad \frac{A}{B} = \frac{\alpha}{\beta} = -\epsilon^2 , \]
\[ \alpha^2 = \frac{2q\Omega^2\epsilon^4}{1 - \epsilon^4} \left( 1 \pm \sqrt{\frac{1 - \epsilon^2}{q/2}} \right). \] (19)

The physical significance of \( \epsilon < 1 \) becomes evident from consideration of the equations for the flow streamlines that coincide with the magnetic field lines:

\[ y^2 + \frac{x^2}{\epsilon^2} = \text{constant}. \] (20)

Thus, \( \epsilon \) is the minor-to-major axis ratio of the elliptical streamlines. Unlike the hydrodynamical case, its value is not fixed by the background \( \Omega(R) \). Furthermore, both retrograde and prograde motion is possible; the former (latter) corresponds to taking the minus (plus) sign in equations (18) and (19). The sign of \( \alpha \) has no dynamical consequence (it determines the sense of the current flow), but once chosen, \( \beta \) must be of the opposite sign. Note that the magnetic field lines do not shear, even though a radial component is present and there is background (global) differential rotation. Rather, the field lines precisely track the velocity streamlines and rigidly rotate with the modified epicycle. The miracle is that this behavior is dynamically fully self-consistent.

Let us next consider the conditions under which retrograde or prograde ellipses are obtained. Retrograde motion corresponds to \( A > 0, B < 0 \), i.e., to taking the minus sign in equations (18) and (19). The requirement that \( \alpha^2 > 0 \) means that

\[ 1 - q/2 \leq \epsilon^2 < 1 \quad \text{(retrograde)}. \] (21)

For a Keplerian disk, \( \epsilon \) must therefore exceed \( \frac{1}{2} \); for a galactic \( q = 1 \) law, \( \epsilon \) must exceed \( \sqrt{2}/2 \). The Keplerian planet solutions of GNG, which were also retrograde ellipses, had a complementary restriction on the domain \( \epsilon : 0 \leq \epsilon \leq \frac{1}{2} \). What is inaccessible hydrodynamically becomes available magnetohydrodynamically, and vice versa. Note that when \( 1 - \epsilon^2 \) approaches \( q/2 \), we recover the hydrodynamical solution of the previous section. This is what ultimately limits the domain of \( \epsilon \) for retrograde ellipses. The inclusion of magnetic hoop stresses plumps retrograde ellipses into more circular structures.

Prograde motion corresponds to \( A < 0, B > 0 \) in equation (18). The prograde branch is not restricted by the condition \( \alpha^2 > 0 \), since this is obviously guaranteed by choosing the plus sign in equation (19). Hence, the domain of \( \epsilon \) for prograde ellipses,

\[ 0 \leq \epsilon^2 < 1 \quad \text{(prograde)}, \] (22)

extends beyond the retrograde domain.

The vorticity of the ellipses is given by

\[ B - A + 2\Omega = \Omega \left[ 2 \pm (1 + \epsilon^2) \sqrt{\frac{2q}{1 - \epsilon^2}} \right]. \] (23)

Equation (21) implies that retrograde ellipses always have negative vorticity; prograde ellipses have positive vorticity.

The Lorentz forces are \(-\beta^2(1 + \epsilon^2)\) in the \( x \) direction and \(-\epsilon^2\beta^2(1 + \epsilon^2)\) in the \( y \) direction. This is an inwardly directed force (confining) for both prograde and retrograde ellipses, for a given \( \epsilon \) larger in magnitude for the prograde solutions. Prograde rotation has greater associated vorticity than retrograde rotation, and larger confining forces are required.

2.2.3. Hyperbolic Configurations

Hyperbolic configurations are another possible local solution in a disk. Following the \( AB2\beta \) parameterization of equation (17) we obtain

\[ B = \pm \sqrt{\frac{2q\Omega^2}{1 + M^2}} \quad A = \frac{\alpha}{\beta} = M^2, \] (24)

and

\[ \alpha^2 = \frac{2q\Omega^2 M^4}{1 - M^4} \left( 1 \pm \sqrt{\frac{1 + M^2}{q/2}} \right). \]

As before, the sign of \( \alpha \) is not dynamically significant, but once chosen, \( \alpha \) and \( \beta \) must have the same sign. These solutions have flow streamlines and magnetic field configurations given by the hyperbolae

\[ y^2 - \frac{x^2}{M^2} = \text{constant}. \] (25)

Prograde hyperbolic flow corresponds to \( B > 0, M^2 < 1 \); retrograde flow corresponds to \( B < 0, M^2 > 1 \). (Note that \( \alpha^2 \) is always positive under these restrictions.) Lorentz forces for hyperbolic flow are directed neither radially inward nor outward (relative to our local origin), but vary with quadrant.

2.2.4. Constant Density Solutions

The planar field configurations of the previous section are fully compatible with vertical stratification in the disk. If one further restricts the disk structure to the case of \( \rho \) and \( P \) being independent of \( z \), a greater variety of possible field configurations exists. This is not an uninteresting limit because numerical simulations are often performed ignoring the vertical disk structure, and because the local midplane structure of astrophysical disks is approximately one of constant density. [More precisely, \( \rho = \rho_0 - O(z^2) \), and we shall work to \( O(z) \).]

We may first note that if \( \rho \) and \( P \) are assumed to be constant, nothing in our original solution would change if a constant vertical magnetic field were present. This is one possible generalization. But it is also possible to find qualitatively new solutions.

If \( z \) components of the velocity and magnetic field are present, but vertical structure is absent, instead of equation (14) our new \( z \) equation is

\[ v_x \partial_z v_x = -\partial_z (v_x^2/2) + v_{Ay} \partial_z v_Az. \] (26)

Consider now a velocity of the form

\[ v_x = Az + D, \quad v_y = Bx, \quad v_z = Cx, \] (27)

where \( A, B, C, \) and \( D \) are constants, and an Alfvén velocity of the form

\[ v_A = \gamma \psi \] (28)

with \( \gamma \) constant. Equations (10) and (11) are automatically satisfied, while equations (12), (13), and (26) lead to elliptical and hyperbolic streamline solutions, as before. Elliptical solutions take the form

\[ C^2 = \frac{2\Omega^2}{1 - \epsilon^2} [2(1 + \epsilon^2)^2 - q], \quad \frac{C}{A} = -\epsilon^2, \quad \gamma^2 = \frac{\epsilon^2}{1 + \epsilon^2}, \] (29)
and

\[ B = -2\alpha(1 + \epsilon^2) \].

(30)

These solutions have the interesting property that \( D \) is completely unconstrained, and the entire ellipse can stream, in bulk, in the radial direction! Unusual streaming behavior is also characteristic of the magnetorotational streaming solutions in two-dimensional axisymmetric simulations (Hawley & Balbus 1992). However, these streaming solutions are known to be unstable to Kelvin-Helmholtz types of instabilities when three-dimensional structure is permitted (Goodman & Xu 1994; Hawley et al. 1995). The ordered flow breaks down into MHD turbulence.

3. DISCUSSION

The solutions presented in this paper represent both \( O \)-type (elliptical) and \( X \)-type (hyperbolic) neutral points in the local magnetic field topology. In essence, we have shown that each of these topologies has a range of well-defined field strengths that leads to exact static equilibrium solutions to the local MHD equations in a shearing disk. It is remarkable that such solutions exist in the presence of differential rotation.

Unless \( \epsilon^2 - 1 \) is very small, the local Alfvén velocities of the solutions are of order \( r\Omega \), where \( r \) is the radial distance from the local origin. If \( r \) is less than or of the order of the disk scale height, the magnetic field strengths are comparable to or less than thermal values. As \( \epsilon \rightarrow 1 \), field strengths becomes unbounded, and the configurations are probably unstable. At more nominal field strengths, the question of stability is unclear. Both the Parker (1966) and magnetorotational (Balbus & Hawley 1991) instabilities are potentially disruptive, but in either case the length scale and magnitude of the magnetic field make direct application of the relevant criterion marginal. The best option at this point is direct numerical simulation.

It should also be noted that the hyperbolic solutions presented in this paper, though treated on the same footing as the elliptical solutions, require flow streamlines and field lines to escape to infinity. Since our treatment is local, these solutions are not fully self-consistent. In the purely hydrodynamical case, hyperbolic solutions are not stable configurations at all, but represent the breakdown of the flow into turbulence due to the violation of the Rayleigh criterion. The stability of the analogous magnetic configurations is at present unclear, as are the details of how a local solution is incorporated into a global structure.

Even if these structures prove unstable on rotational timescales, however, they may still be astrophysically interesting. \( X \)-type neutral points facilitate reconnection, and their presence as an equilibrium configuration at thermal field strengths may be an important saturation mechanism for accretion disk dynamos. Reconnection at the interface between a disk and central magnetosphere is another problem of interest. Magnetic field lines enforce rigid rotation within a corotation radius, while differential rotation obtains outside corotation. The results discussed here suggest the possibility that magnetic fields at the interface need not be continuously sheared, if it is possible for the open field lines of the hyperbolic solutions to join the magnetospheric fields. This would facilitate reconnection in the presence of shear, without generating large off-diagonal Maxwell stresses.

\( O \)-type elliptical structures offer interesting possibilities for galactic disks. If we identify these large scale vortices with sites of molecular cloud complexes, there are a number of straightforward predictions. We should see, of course, at least a vaguely elliptical morphology, with the major axis oriented azimuthally, and magnetic fields following this morphology. Both retrograde and prograde rotation of the complexes is possible. The minor-to-major axis ratio of retrograde systems must exceed 0.7; however, prograde systems with considerably smaller values should exist.

Why have numerical simulations done to date not shown these structures? Three-dimensional MHD disk simulations have yet to reveal the midplane elliptical and hyperbolic configurations. There are several possible reasons for this. First, unlike the GNG solutions, the density does not fall to zero at the edge of an ellipse (or at some convenient point in an hyperbola). This means these solutions must be actively confined by the ambient disk, and conditions for this need not arise spontaneously. Second, numerical simulations have concentrated on subthermal magnetic fields because their presence leads to disk turbulence. If the field saturates at subthermal values, and it generally does, the \( O \) and \( X \) neutral points may not be able to form. (In the simulation where the traveling ellipses were seen, the field strength grew to suprathermal values.) The best strategy might be to start with thermal fields and allow some combination of differential rotation, internal dynamical instabilities, and external driving to a new field configuration. Finally, there is always the possibility that the structures never form spontaneously because they are too unstable.

Undue pessimism, however, is not yet warranted. Properly crafted three-dimensional simulations should be able to clarify the most important uncertainties.

Part of this work was completed when one of us (S. A. B.) was a visitor with the radio astronomy group at the École Normale Superieure, and he would like to thank E. Falgarone and M. Perault for their generous hospitality and advice. We also thank J. Goodman for very useful and thoughtful comments. This work has been supported by NASA grants NAG5-4600, NAG5-3058, and by NSF grant AST-9423187.

REFERENCES

Balbus, S. A., & Hawley, J. F. 1991, ApJ, 376, 214
———. 1998, Rev. Mod. Phys., 70, 1
Goodman, J., Narayan, R., & Goldreich, P. 1987, MNRAS, 225, 695
(GNG)
Goodman, J., & Xu, G. 1994, ApJ, 432, 213
Hawley, J. F. 1987, MNRAS, 225, 677
Hawley, J. F., & Balbus, S. A. 1992, ApJ, 400, 595
Hawley, J. F., Gammie, C. F., & Balbus, S. A. 1995, ApJ, 440, 742
Papaloizou, J. C. B., & Lin, D. N. C. 1995, ARA&A, 33, 505
Parker, E. N. 1966, ApJ, 145, 811