EXISTENCE OF NONTRIVIAL SOLUTIONS TO CHERN-SIMONS-SCHRÖDINGER SYSTEM WITH INDEFINITE POTENTIAL

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Abstract. We consider a class of Chern-Simons-Schrödinger system

\[
\begin{align*}
-\Delta u + V(x)u + A_0 u + \sum_{j=1}^{2} A_j^2 u &= g(u), \\
\partial_1 A_0 &= A_2 |u|^2, \quad \partial_2 A_0 = -A_1 |u|^2, \\
\partial_1 A_2 - \partial_2 A_1 &= -\frac{1}{2} u^2, \quad \partial_1 A_1 + \partial_2 A_2 = 0,
\end{align*}
\]

where \( V \) is coercive sign-changing potential and \( g \) satisfies some suitable conditions. Due to lack of the mountain pass geometry and the link geometry for the corresponding variational functional, we obtain the existence of nontrivial solutions via the local link theorem.

1. Introduction and main result. In this paper, we are interested in the following Chern-Simons-Schrödinger system in \( H^1(\mathbb{R}^2) \)

\[
\begin{align*}
-\Delta u + V(x)u + A_0 u + \sum_{j=1}^{2} A_j^2 u &= g(u), \\
\partial_1 A_0 &= A_2 |u|^2, \quad \partial_2 A_0 = -A_1 |u|^2, \\
\partial_1 A_2 - \partial_2 A_1 &= -\frac{1}{2} u^2, \quad \partial_1 A_1 + \partial_2 A_2 = 0,
\end{align*}
\]

where \( V(x) \) is sign-changing and \( g \) is 5-superlinear at infinity. System (1) derives from studying the standing wave solutions of the following nonlinear Schrödinger system

\[
\begin{align*}
(iD_0 \phi + (D_1 D_1 + D_2 D_2) \phi + g(\phi) &= 0, \\
\partial_0 A_1 - \partial_1 A_0 &= -\text{Im} (\bar{\phi} D_2 \phi), \\
\partial_0 A_2 - \partial_2 A_0 &= -\text{Im} (\bar{\phi} D_1 \phi), \\
\partial_1 A_2 - \partial_2 A_1 &= -\frac{1}{2} |\phi|^2,
\end{align*}
\]

where \( i \) denotes the imaginary unit, \( \partial_0 = \frac{\partial}{\partial t} \), \( \partial_1 = \frac{\partial}{\partial x_1} \), \( \partial_2 = \frac{\partial}{\partial x_2} \) for \((t, x_1, x_2) \in \mathbb{R}^{1+2}, \phi : \mathbb{R}^{1+2} \to \mathbb{C}\) denotes the complex scalar field, \( A_\nu : \mathbb{R}^{1+2} \to \mathbb{R}\) denotes the gauge field and \( D_\nu = \partial_\nu + i A_\nu \) denotes the covariant derivative for \( \nu = 0, 1, 2 \).

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System (2) consists of Schrödinger equations augmented by the gauge field, which was firstly proposed in [12, 13], where system (2) is usually called Chern-Simons-Schrödinger system. The two-dimensional Chern-Simons-Schrödinger system describes an external uniform magnetic field which is of great phenomenological interest for applications of Chern-Simons theory to the quantum Hall effect. For more physical backgrounds of system (2), we refer readers to [11, 23, 24, 25].

Let $A_{\nu}(x, t) = A_{\nu}(x), \nu = 0, 1, 2$. Also, if we assume that the standing wave ansatz $\phi = u(x)e^{i\omega t}, g(u)e^{i\omega t} = g(u)e^{i\omega t}, \omega > 0$, system (2) can be simplified as

$$\begin{cases}
-\Delta u + \omega u + A_{0}u + \sum_{j=1}^{2} A_{j}^{2}u = g(u), \\
\partial_{1}A_{0} = A_{2}|u|^{2}, \quad \partial_{2}A_{0} = -A_{1}|u|^{2}, \\
\partial_{1}A_{2} - \partial_{2}A_{1} = -\frac{1}{2}u^{2}, \\
(\partial_{1}A_{1} + \partial_{2}A_{2})u + 2(A_{1}\partial_{1}u + A_{2}\partial_{2}u) = 0.
\end{cases} \tag{3}$$

If the Coulomb gauge condition $\partial_{0}A_{0} + \partial_{1}A_{1} + \partial_{2}A_{2} = 0$ hold. Then we deduce that $A_{1}\partial_{1}u + A_{2}\partial_{2}u = 0$ and system (3) can be rewritten as

$$\begin{cases}
-\Delta u + \omega u + A_{0}u + \sum_{j=1}^{2} A_{j}^{2}u = g(u), \\
\partial_{1}A_{0} = A_{2}|u|^{2}, \quad \partial_{2}A_{0} = -A_{1}|u|^{2}, \\
\partial_{1}A_{2} - \partial_{2}A_{1} = -\frac{1}{2}u^{2}, \quad \partial_{1}A_{1} + \partial_{2}A_{2} = 0.
\end{cases} \tag{4}$$

Here the components $A_{1}$ and $A_{2}$ in system (4) can be represented by solving the elliptic equation

$$\Delta A_{1} = \partial_{2}(\frac{|u|^{2}}{2}) \quad \text{and} \quad \Delta A_{2} = -\partial_{1}(\frac{|u|^{2}}{2}),$$

which provide

$$A_{1} = A_{1}[u](x) = \frac{x_{2}}{2\pi|x|^{2}} \ast \left(\frac{|u|^{2}}{2}\right) = \frac{1}{2\pi} \int_{\mathbb{R}^{2}} \frac{x_{2} - y_{2}}{|x - y|^{2}} \frac{|u(y)|^{2}}{2} dy,$$

$$A_{2} = A_{2}[u](x) = -\frac{x_{1}}{2\pi|x|^{2}} \ast \left(\frac{|u|^{2}}{2}\right) = -\frac{1}{2\pi} \int_{\mathbb{R}^{2}} \frac{x_{1} - y_{1}}{|x - y|^{2}} \frac{|u(y)|^{2}}{2} dy,$$

where $\ast$ denotes the convolution. It follows system (4) that

$$\Delta A_{0} = \partial_{1}(A_{2}|u|^{2} - \partial_{2}(A_{1}|u|^{2}),$$

which gives the following representation of the component $A_{0}$:

$$A_{0} = A_{0}[u](x) = \frac{x_{1}}{2\pi|x|^{2}} \ast (A_{2}|u|^{2}) - \frac{x_{2}}{2\pi|x|^{2}} \ast (A_{1}|u|^{2}).$$

In recent years, many scholars pay attention to system (2) in radial functions space $H_{s}^{1}(\mathbb{R}^{2})$ ( $H_{s}^{1}(\mathbb{R}^{2})$ consists of all radial functions in $H^{1}(\mathbb{R}^{2})$). Via the variational method, Byeon, Huh and Seok in [2] first studied the solutions to system (2) of the form

$$\phi(t, x) = u(|x|)e^{i\omega t}, \quad A_{0}(t, |x|) = k(|x|),$$

$$A_{1}(t, x) = \frac{x_{2}}{|x|^{2}}h(|x|), \quad A_{2}(t, x) = -\frac{x_{1}}{|x|^{2}}h(|x|),$$

where $\omega > 0$ and $u, k, h$ are real value functions depending only $|x|$. The existence and nonexistence results on nontrivial radial solutions have been obtained when
\[ g(u) = \lambda |u|^{q-2}u, \lambda > 0 \text{ and } q > 2. \]

Later, based on the work of [2], system (2) have been studied by many researchers in radial functions space \( H^1_r(\mathbb{R}^2) \), and they obtained lots of results, see [2, 3, 7, 15, 14, 18, 19, 17, 26, 27, 28, 32] and references therein. After these works, mathematicians also began to consider system (2) in \( H^1(\mathbb{R}^2) \). In this case, system (2) can be transformed into the form of system (4).

As far as we know, for the existence of solutions to system (1) or (4) in \( H^1 \), there are few works presented in [10, 21, 30]. More precisely, Wan and Tan [30] investigated the existence of nontrivial solution for system (4) as \( p > 4 \) by the concentration compactness principle. Moreover, they obtained the same result for system (1) when \( g(u) = |u|^{p-2}u \) with \( p > 4 \) by the concentration compactness principle. Moreover, they obtained the same result for system (1) when \( g(u) = |u|^{p-2}u \) with \( p > 4 \) and \( V(x) \) satisfies \((V_1)\) \( V \in C^1(\mathbb{R}^2), 0 < V_0 := \inf_{x \in \mathbb{R}^2} V(x) < V(x) < V_\infty := \lim \inf_{|x| \to \infty} V(x) \) and \((\nabla V(x), x) \geq 0\) for a.e. \( x \in \mathbb{R}^2 \).

Liang and Zhai [21] got the existence of bound state solution when \( g(u) = |u|^{p-2}u \) with \( p > 4 \) for system (4) under the \( L^2 \)-norm constraint

\[ S(c) := \{u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} u^2 dx = c \}. \]

Gou and Zhang [10] studied the existence and the orbital stability of normalized solutions for system (4) when \( g(u) = |u|^{p-2}u \) with \( p > 2 \). Some results of semiclassical solutions for system (4) can be found in [5, 6, 29]. They considered the following Chern-Simons-Schrödinger system

\[
\begin{cases}
-\varepsilon^2 \Delta u + V(x)u + A_0u + \sum_{j=1}^2 A_j^2 u = g(u), \\
\varepsilon \partial_1 A_0 = A_2|u|^2, \\
\varepsilon \partial_2 A_0 = -A_1|u|^2, \\
\varepsilon (\partial_1 A_2 - \partial_2 A_1) = -\frac{1}{2} u^2, \\
\partial_1 A_1 + \partial_2 A_2 = 0,
\end{cases}
\]

where \( \varepsilon > 0 \) is small. In this sense, system (5) can be viewed as a generalized form of system (4). The solutions of system (5) are called semiclassical states. When \( V \) satisfies

\[(V_2)\] \( V \in C(\mathbb{R}^2), 0 < V_0 := \inf_{x \in \mathbb{R}^2} V(x) < V(x) < V_\infty := \lim \inf_{|x| \to \infty} V(x) \); \n
\[(V_3)\] \( V \in C^1(\mathbb{R}^2), (4\alpha - 2) V(x) + \nabla V(x) \cdot x > 0 \) for all \( x \in \mathbb{R}^2 \), where and in the sequel \( \alpha := \frac{2}{p-2} > 1 \).

Chen et al. [5] obtained the existence and concentration of semiclassical ground state solutions for system (5). Deng et al. [6] studied the existence of multi-peak solutions when \( V \) satisfies

\[(V_4)\] \( V \in C(\mathbb{R}^2), \inf_{x \in \mathbb{R}^2} V(x) > 0 \), and there exist positive constants \( L \) and \( \theta \) such that \( |V(x) - V(y)| \leq L|x - y|^{\theta} \) for all \( x, y \in \mathbb{R}^2 \);

\[(V_5)\] There exist \( \delta > 0 \) and \( x^0 \in \mathbb{R}^2 \) such that \( V(x) < V(x^0) \) for \( x \in B_\delta(x^0) \}\{x^0\} \subset \mathbb{R}^2 \).

Wan and Tan [29] proved the existence and concentration of least energy solution when \( g(u) = |u|^{p-2}u \) with \( p > 6 \) for system (5) and \( V \) satisfies

\[(V_6)\] \( V \in C(\mathbb{R}^2), V_0 := \inf_{x \in \mathbb{R}^2} V(x) < V_\infty := \lim \inf_{|x| \to \infty} V(x) \).

We emphasize that in the above results, the Schrödinger operator \( -\Delta + V \) is positive definite, namely, \( \inf_{x \in \mathbb{R}^2} V > 0 \) holds. A natural question is whether the system (1) has a nontrivial solution when \( V \) is a sign-changing potential and the Schrödinger operator \( -\Delta + V \) is indefinite.

To state our result, we give the following assumptions in this work.
(V) $V \in C(\mathbb{R}^2, \mathbb{R})$ is bounded from below and $\inf \sigma(-\Delta + V) \leq 0$, where $\sigma(-\Delta + V)$ means the spectrum of $-\Delta + V$. Moreover, $|\{V \leq l\}| < \infty$ for all $l \in \mathbb{R}$,

$(g_1)$ $g \in C(\mathbb{R}, \mathbb{R})$, $\lim_{t \to 0} \frac{g(t)}{t} = 0$ and

$$\lim_{|t| \to \infty} e^{\mu t^2} = 0$$

for any $\mu > 0$,

$(g_2)$ There is $b > 0$ such that

$$tg(t) - 6G(t) \geq -bt^2$$

for every $t \in \mathbb{R}$,

where $G(t) := \int_0^t g(\tau)d\tau$,

$(g_3)$ $\lim_{|t| \to \infty} \frac{g(t)}{|t|^2} = +\infty$,

$(g_4)$ Either

$(g_{41})$ there exist $C_0 > 0$ and $\nu \in (0, 6)$ satisfying

$$G(t) \geq C_0 |t|^{\nu}$$

for all $t \in \mathbb{R}$

or

$(g_{42})$ for some $\delta > 0$,

$$G(t) \leq 0$$

for all $|t| \leq \delta$.

Our main result is as follows.

**Theorem 1.1.** Assume that (V) and $(g_1) - (g_3)$ are satisfied. If $0 \in \sigma(-\Delta + V)$, we also assume that $(g_4)$ holds. Then system (1) has at least a nontrivial solution.

**Remark 1.** To the best of our knowledge, our result for system (1) seems to be the first work to concern indefinite Schrödinger operator $-\Delta + V$. Additionally, there are many functions satisfying $(g_1) - (g_3)$ and $(g_{41})$ or $(g_1) - (g_3)$ and $(g_{42})$, for example, $g(t) = 8t^7 + 12t^3$ for every $t \in \mathbb{R}$ or $g(t) = t^7 - t^3$ for all $t \in \mathbb{R}$.

**Remark 2.** (V) implies that $V(x)$ is coercive sign-changing potential so that the Schrödinger operator $-\Delta + V$ is indefinite. When $\inf_{x \in \mathbb{R}^2} V > 0$, the Schrödinger operator $-\Delta + V$ is positive definite. Clearly, $u = 0$ is a local minimizer for the energy functional $\Phi$ (see (8)) of system (1). Besides, $\Phi$ satisfied mountain pass geometry when $g$ satisfies $(g_3)$. Hence, it is easy to get the existence of nontrivial solutions by the Mountain Pass Theorem. However, when the Schrödinger operator $-\Delta + V$ is indefinite, the mountain pass geometry is destroyed. For stationary NLS equations $-\Delta u + Vu = g(u)$, the link theorem is usually used to obtain the solutions in this case, see [16, 22]. But, due to the presence of nonlocal term in system (1), the link geometry and the mountain pass geometry does not work, which makes the study of system (1) particularly interesting. In order to overcome this difficulty, we use the local link theorem (see [20]) to obtain nontrivial solutions of system (1). In addition, by $(g_1)$, it is difficult to verify that the Palais-Smale condition and the local link geometry hold. A key inequality, i.e., Trudinger-Moser inequality (see [4, 8]) in this paper can help us to solve this problem.

The rest of this paper is organized as follows. In Section 2, we give some preliminary work. In Section 3, we show some important lemmas and prove Theorem 1.1.
2. Preliminaries. Since \( V(x) \) is bounded from below, there exists \( m > 0 \) satisfying
\[
\tilde{V}(x) := V(x) + m > 1 \quad \text{for all } x \in \mathbb{R}^2.
\]
We introduce the following subspace \( E \) of \( H^1(\mathbb{R}^2) \):
\[
E = \{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} \tilde{V} u^2 dx < +\infty \}.
\]
It is easy to recognize that \( E \) is a Hilbert space with scalar product and norm given by
\[
\langle u, v \rangle = \int_{\mathbb{R}^2} \nabla u \cdot \nabla v + \tilde{V} u v dx \quad \text{and} \quad ||u|| = \sqrt{\langle u, u \rangle}.
\]
By [1], we obtain that the embedding \( E \hookrightarrow L^q(\mathbb{R}^2) \) with \( q \in [2, +\infty) \) is compact. Next, we will give an important inequality, that is, Trudinger-Moser inequality in \( \mathbb{R}^2 \) (see [4, 8]), which is stated as follows:
\[
\int_{\mathbb{R}^2} (e^{\mu |u|^2} - 1) dx < \infty \quad \text{for all } \mu > 0 \text{ and } u \in H^1(\mathbb{R}^2).
\]
Moreover, if \( \mu < 4\pi \) and \( |u|_{L^2(\mathbb{R}^2)} \leq M < \infty \), there exists a constant \( C = C(M, \mu) > 0 \) such that
\[
\sup_{|\nabla u|_{L^2(\mathbb{R}^2)} \leq 1} \int_{\mathbb{R}^2} (e^{\mu |u|^2} - 1) dx \leq C.
\]
Let \( G(t) := \int_0^t g(\tau) d\tau \) be the primitive function of \( g \). Then the energy functional of system (1) is defined as
\[
\Phi(u) = \frac{1}{2} \int_{\mathbb{R}^2} \left( |\nabla u|^2 + V(x)u^2 + A_1^2(u)u^2 + A_2^2(u)u^2 \right) dx - \int_{\mathbb{R}^2} G(u) dx.
\]
Let \( f(t) := g(t) + mt \) and \( F(t) := \int_0^t f(\tau) d\tau \), then the energy functional \( \Phi(u) \) can be rewritten as
\[
\Phi(u) = \frac{1}{2} ||u||^2 + \frac{1}{2} \int_{\mathbb{R}^2} (A_1^2(u)u^2 + A_2^2(u)u^2) dx - \int_{\mathbb{R}^2} F(u) dx.
\]
According to (6), [30, Proposition 2.1] and [31], it is standard to verify that \( \Phi \) is well defined in \( E \) and \( \Phi \in C^1(E, \mathbb{R}) \). Moreover, for any \( u, \varphi \in E \),
\[
\langle \Phi'(u), \varphi \rangle = \langle u, \varphi \rangle + \int_{\mathbb{R}^2} (A_1^2 + A_2^2)u \varphi + A_0 u \varphi dx - \int_{\mathbb{R}^2} f(u) \varphi dx.
\]
Obviously, the critical points of \( \Phi \) are weak solutions of system (1). By simple calculation (also see [5, 30]), we obtain, for any \( u \in E \),
\[
\langle \Phi'(u), u \rangle = ||u||^2 + 3 \int_{\mathbb{R}^2} (A_1^2(u) + A_2^2(u))u^2 dx - \int_{\mathbb{R}^2} f(u) u dx.
\]
Besides, we give some properties of \( f \). It follows from \((g_1) - (g_3)\) that
\[
\begin{align*}
(f_1) \quad & \lim_{|t| \to 0} \frac{f(t)}{t^2} = \lim_{|t| \to 0} (\frac{t^2}{\theta} \cdot \frac{g(t) + mt}{t^2}) = +\infty, \\
(f_2) \quad & F(t) := \int_0^t f(\tau) d\tau \leq \frac{1}{6} tf(t) + \frac{b}{6} t^2, \text{ where } b = b + 2m > 0, \\
(f_3) \quad & \lim_{|t| \to \infty} \frac{f(t)}{t^2} = +\infty.
\end{align*}
\]
We can see that \( \Phi \) does not satisfy the mountain pass geometry, because the last term of \( \Phi \) is not \( o(|u|^2) \) as \( ||u|| \to 0 \). Since the compactness of \( E \hookrightarrow L^2(\mathbb{R}^2) \), we obtain that the bilinear form

\[
Q(u, v) = \frac{1}{2} \int_{\mathbb{R}^2} \nabla u \cdot \nabla v + Vuvdx
\]

for all \( u, v \in E \), is essentially selfadjoint (by Kato’s criterion), semibounded from below on \( E \subseteq L^2(\mathbb{R}^2) \) and the spectrum of the corresponding Schrödinger operator \( \sigma(\Delta + V) \) is discrete (with finite multiplicity) and bounded from below. We consider the case of \( 0 \in \sigma(\Delta + V) \), because the case of \( 0 \not\in \sigma(\Delta + V) \) is similar. If \( (V) \) and \( (g_{41}) \) hold, let \( E^+ \) be the space spanned by the eigenfunctions corresponding to positive eigenvalues of \( -\Delta + V \) and \( E^- = (E^+)^\perp \). Hence,

\[
E = E^- \oplus E^+ \quad \text{and} \quad \dim E^- < +\infty.
\]

Decompose \( E^- \) into \( Z + W \) where \( Z = \ker(-\Delta + V), W = (E^+ + Z)^\perp \). Then, there is a constant \( \kappa > 0 \) such that

\[
Q(u, u) \geq \kappa ||u||^2 \quad \text{for all} \quad u \in E^+,
\]

\[
Q(u, u) \leq -\kappa ||u||^2 \quad \text{for all} \quad u \in W. \tag{12}
\]

Besides, if \( (V) \) and \( (g_{42}) \) hold, let \( E^2 \) be the space spanned by the eigenfunctions corresponding to negative eigenvalues of \( -\Delta + V \) and \( E^1 = (E^2)^\perp \). Hence,

\[
E = E^1 \oplus E^2 \quad \text{and} \quad \dim E^2 < +\infty.
\]

Decompose \( E^1 \) into \( Z + \tilde{W} \) where \( Z = \ker(-\Delta + V), \tilde{W} = (E^2 + Z)^\perp \). Then, there is a constant \( \gamma > 0 \) such that

\[
Q(u, u) \geq \gamma ||u||^2 \quad \text{for all} \quad u \in \tilde{W},
\]

\[
Q(u, u) \leq -\gamma ||u||^2 \quad \text{for all} \quad u \in E^2. \tag{13}
\]

Due to the existence of nonlocal term in system (1), the link geometry structure of \( \Phi \) may not work. In fact, let \( 0 < r < R \) and let \( z \in E^1 \) be such that \( ||z|| = r \). Define

\[
N := \{ u \in E^1 : ||u|| = r \},
\]

\[
M := \{ u = y + \lambda z : ||u|| \leq R, \lambda \geq 0, y \in E^2 \},
\]

\[
M_0 := \{ u = y + \lambda z : y \in E^2, ||u|| = R \text{ and } \lambda \geq 0 \text{ or } ||u|| \leq R \text{ and } \lambda = 0 \}.
\]

\( \Phi \) satisfies the link geometry if \( b_1 := \inf_N \Phi > b_2 := \max_{M_0} \Phi \). We can see that, for \( u \in E^2 \),

\[
\Phi(u) = Q(u, u) + \frac{1}{2} \int_{\mathbb{R}^2} (A_1^2(u) + A_2^2(u))u^2dx - \int_{\mathbb{R}^2} G(u)dx > b_1
\]

may hold for some \( R > r > 0 \) because \( A_1^2(u) + A_2^2(u) \geq 0 \). Therefore, in order to overcome this difficulty and get the critical point of \( \Phi \), we choose the local link theorem [20].

Given a Hilbert space \( X \) with a direct sum decomposition \( X = X^1 \oplus X^2, \Phi \in C^1(X, \mathbb{R}) \) has a local linking at 0 if there holds

\[
\begin{align*}
\Phi(u) &\geq 0, \quad u \in X^1 \text{ and } ||u|| \leq \rho, \\
\Phi(u) &\leq 0, \quad u \in X^2 \text{ and } ||u|| \leq \rho,
\end{align*}
\]

where \( \rho > 0 \) is such that

\[
\int_{\mathbb{R}^2} (A_1^2(u) + A_2^2(u))u^2dx > \frac{ho^2}{2R}.
\]
where $\rho > 0$. Clearly, $u = 0$ is a critical point of $\Phi$. Consider two sequences of subspaces:

$$X_0^2 \subset X_1^2 \subset X_2^2 \subset \cdots \subset X^2,$$

$$X_0^1 \subset X_1^1 \subset X_2^1 \subset \cdots \subset X^1$$

such that

$$X^2 = \bigcup_{n \in \mathbb{N}} X_n^2$$

and

$$X^1 = \bigcup_{n \in \mathbb{N}} X_n^1.$$

For every multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, we set $X_\alpha = X_{\alpha_1}^2 \oplus X_{\alpha_2}^1$ and denote by $\Phi_\alpha$ the functional $\Phi$ restricted to $X_\alpha$. We recall that $\alpha \leq \beta \leftrightarrow \alpha_1 \leq \beta_1, \alpha_2 \leq \beta_2$. A sequence $\{\alpha_n\} \subset \mathbb{N}^2$ is admissible if, for every $\alpha \in \mathbb{N}^2$ there is $k \in \mathbb{N}$ such that $n \geq k \Rightarrow \alpha_n \geq \alpha$. Obviously, if $\alpha_n$ is admissible, then any subsequence of $\alpha_n$ is also admissible. In present paper, we choose an Hilbertian basis $\{e_n\}_{n \geq 0}$ for $E^+$ or $E^1$ and defined

$$E_n^+ := \text{span}\{e_0, e_1, \ldots, e_n\}, \quad n \in \mathbb{N},$$

$$E_n^- := E^-, \quad n \in \mathbb{N}.$$  

or

$$E_n^1 := \text{span}\{e_0, e_1, \ldots, e_n\}, \quad n \in \mathbb{N},$$

$$E_n^2 := E^2, \quad n \in \mathbb{N}.$$  

**Definition 2.1.** ([20]) The functional $\Phi \in C^1(E, \mathbb{R})$ satisfies the $(PS)^*$ condition if every sequence $\{u_{\alpha_n}\}$ such that $\{\alpha_n\}$ is admissible and

$$u_{\alpha_n} \in E_{\alpha_n}, \quad \sup_n \Phi(\alpha_n) < \infty, \quad \Phi'(\alpha_n)(u_{\alpha_n}) \to 0,$$

contains a subsequence which converges to a critical point of $\Phi$.

The following local link theorem can be found in [20, Theorem 2].

**Theorem 2.2.** Suppose that

(i) $\Phi \in C^1(E, \mathbb{R})$ has a local linking at 0,

(ii) $\Phi$ satisfies $(PS)^*$ condition,

(iii) $\Phi$ maps bounded sets into bounded sets,

(iv) for every $k \in \mathbb{N}$ and $u \in E^- \oplus E_k^+$ or $u \in E^2 \oplus E_k^1$,

$$\Phi(u) \to -\infty \quad \text{as } ||u|| \to \infty.$$

Then $\Phi$ has at least a nontrivial critical points.

3. **Proof of Theorem 1.1.** In present paper, we consider only the case when $0 \in \sigma(-\Delta + V)$. Because the other case ($0 \notin \sigma(-\Delta + V)$) can be studied in a similar way and is simpler. Let $|u|_s := \left(\int_{\mathbb{R}^2} |u|^s dx\right)^{\frac{1}{s}}$ be the norm of the usual Lebesgue space $L^s(\mathbb{R}^2)$ for all $s \in [1, +\infty)$. For all $u \in E$, we set, by the Sobolev embedding theorem,

$$|u|_q \leq S_q ||u|| \quad \text{with } q \in [2, +\infty).$$

From [30, Proposition 2.1], there is a constant $a_1 > 0$ such that

$$0 \leq \int_{\mathbb{R}^2} (A_1^2(u) + A_2^2(u)) \leq a_1 ||u||^6 \quad \text{for all } u \in H^1(\mathbb{R}^2).$$

Here we state some important inequalities.
Lemma 3.1. (see [9, Lemma 2.2]) If \( \mu > 0 \) and \( s > 1 \), then for all \( \hat{s} > s \), there is \( C > 0 \) such that
\[
(e^{\mu t^2} - 1)^s \leq C(e^{\hat{s} t^2} - 1) \quad \text{for every } t \in \mathbb{R}.
\]

Lemma 3.2. (see [30, Proposition 2.1]) Let \( 1 < r < 2 \) and \( \frac{1}{r} - \frac{1}{2} = \frac{1}{2} \). If \( u \in H^1(\mathbb{R}^2) \), then
\[
|A_0(u)|_t \leq C|u|_{2r'}^2 |u|_{1}^2,
\]
\[
|A_1(u)|_t \leq C|u|_{2r'}^2,
\]
where \( i = 1, 2 \).

Lemma 3.3. Suppose that \((V)\) and \((g_1) - (g_3)\) hold. Then \( \Phi \) satisfies the \((PS)^*\) condition.

Proof. Let \( \{u_{\alpha_n}\} \) be satisfying (14), where \( \{\alpha_n\} \) is admissible. We claim that \( \{u_{\alpha_n}\} \) is bounded in \( E \). Assume for contradiction that \( \|u_{\alpha_n}\| \to \infty \) in \( E \). Note that
\[
0 = \left\langle \Phi'_\alpha(u_{\alpha_n}), u_{\alpha_n} \right\rangle = \left\langle \Phi'(u_{\alpha_n}), u_{\alpha_n} \right\rangle.
\]
In view of (9), (11) and (f2), we obtain
\[
6 \cdot \sup_n \Phi(u_{\alpha_n}) + \|u_{\alpha_n}\| \geq 6\Phi(u_{\alpha_n}) - \left\langle \Phi'_\alpha(u_{\alpha_n}), u_{\alpha_n} \right\rangle
\]
\[
= 2\|u_{\alpha_n}\|^2 + \int_{\mathbb{R}^2} f(u_{\alpha_n})u_{\alpha_n} - 6F(u_{\alpha_n})dx
\]
\[
\geq 2\|u_{\alpha_n}\|^2 - \hat{b} \int_{\mathbb{R}^2} u_{\alpha_n}^2 dx.
\]
Set \( v_n = \|u_{\alpha_n}\|^{-1}u_{\alpha_n}, \) by the compactness of \( E \hookrightarrow L^2(\mathbb{R}^2) \), there is \( v \in E \) such that, up to a subsequence,
\[
v_n \to v \quad \text{in } E,
\]
\[
v_n \to v \quad \text{in } L^2(\mathbb{R}^2),
\]
\[
v_n(x) \to v(x) \quad \text{a.e. in } \mathbb{R}^2.
\]
Then, it follows from (16) that, as \( n \to \infty, \)
\[
\hat{b} \int_{\mathbb{R}^2} v^2 dx \geq 2.
\]
Hence, \( v \neq 0 \). Applying (f3), there exists \( T > 0 \) such that
\[
f(t)t > 0 \quad \text{for all } t > T.
\]
For some \( \epsilon > 0 \), let \( \Omega := \{x \in \mathbb{R}^2 : |v(x)| \geq \epsilon\} \). Clearly, \( |\Omega| > 0 \). We can take \( \Omega_0 \subset \Omega \) satisfying \( |\Omega_0| \in (0, +\infty) \). Since \( v_n(x) \to v(x) \) a.e. in \( \Omega_0 \) up to a subsequence, Egorov Theorem implies that there exists \( \Omega_1 \subset \Omega_0 \) such that \( |\Omega_1| > \frac{|\Omega_0|}{2} \) and
\[
v_n(x) \to v(x) \quad \text{as } n \to \infty \text{ uniformly for } x \in \Omega_1.
\]
Notice that \( |v(x)| \geq \epsilon \) for all \( x \in \Omega_1 \). Then, there is \( N_1 \in \mathbb{N}_+ \) such that
\[
|v_n(x)| \geq \frac{\epsilon}{2} \quad \text{for any } n \geq N_1 \text{ uniformly for } x \in \Omega_1.
\]
Further,
\[
u_{\alpha_n}(x) \to \infty \quad \text{as } n \to +\infty \text{ uniformly for } x \in \Omega_1.
\]
Hence, there exists \( N_2 \in \mathbb{N}_+ \) with \( N_2 \geq N_1 \) satisfying
\[
|u_{\alpha_n}(x)| \geq T \quad \text{for all } n \geq N_2 \text{ uniformly for } x \in \Omega_1.
\]
This implies that
\[
f(u_{\alpha_n}(x))u_{\alpha_n}(x) \geq 0 \quad \text{for all } n \geq N_2 \text{ uniformly for } x \in \Omega_1.
\]
Therefore, it follows from \((f_3)\) and the Fatou lemma that
\[
\lim_{n \to \infty} \int_{\Omega_1} \frac{f(u_{\alpha_n}(x))u_{\alpha_n}(x)}{\|u_{\alpha_n}\|_6^6} \, dx = \lim_{n \to \infty} \int_{\Omega_1} \frac{f(u_{\alpha_n}(x))u_{\alpha_n}(x)}{u_{\alpha_n}(x)^6} v_n^6(x) \, dx = +\infty. \tag{17}
\]
In virtue of \((f_1)\) and \((f_3)\), there exists \( \eta > 0 \) such that
\[
f(t) t \geq -\eta t^6 \quad \text{for all } t \in \mathbb{R}.
\]
Then,
\[
\int_{\Omega_1} \frac{f(u_{\alpha_n})u_{\alpha_n}}{\|u_{\alpha_n}\|_6^6} \, dx = \int_{\Omega_1} \frac{f(u_{\alpha_n})u_{\alpha_n}}{u_{\alpha_n}^6} v_n^6 \, dx
\geq -\eta \int_{\Omega_1} v_n^6 \, dx
\geq -\eta \int_{\mathbb{R}^2} v_n^6 \, dx
= -\eta |v_n|_6^6
\geq -\eta S^6_6. \tag{18}
\]
Using \((17)\) and \((18)\), one has
\[
\int_{\mathbb{R}^2} \frac{f(u_{\alpha_n})u_{\alpha_n}}{\|u_{\alpha_n}\|_6^6} \, dx \geq \int_{\Omega_1} \frac{f(u_{\alpha_n})u_{\alpha_n}}{\|u_{\alpha_n}\|_6^6} \, dx - \eta S^6_6 \to +\infty \quad \text{as } n \to \infty. \tag{19}
\]
Hence, by \((15)\) we deduce
\[
3a_1 + 1
\geq \frac{1}{\|u_{\alpha_n}\|_6^6} \left(\|u_{\alpha_n}\|^2 + 3 \int_{\mathbb{R}^2} (A_2^2(u_{\alpha_n}) + A_2^2(u_{\alpha_n})) u_{\alpha_n}^2 \, dx - \langle \Phi'(u_{\alpha_n}), u_{\alpha_n} \rangle \right)
\geq \int_{\mathbb{R}^2} \frac{f(u_{\alpha_n})u_{\alpha_n}}{\|u_{\alpha_n}\|_6^6} \, dx \to +\infty \quad \text{as } n \to \infty,
\]
which is absurd. Therefore, \( \{u_{\alpha_n}\} \) is bounded in \( E \). So we can assume that there exists \( u \in E \) such that
\[
u_{\alpha_n} \rightharpoonup u \quad \text{in } E,
\]
\[
u_{\alpha_n} \to u \quad \text{in } L^q(\mathbb{R}^2) \text{ with } q \in [2, +\infty),
\]
\[
u_{\alpha_n}(x) \to u(x) \quad \text{a.e. in } \mathbb{R}^2,
\]
up to a subsequence. From \((g_1)\), there exist \( \varepsilon > 0 \) and \( C_\varepsilon > 0 \) such that
\[
|g(u)| \leq \varepsilon |u| + C_\varepsilon (e^{\mu u^2} - 1) \quad \text{for any } \mu > 0.
\]
Hence,
\[
|f(u)| \leq (m + \varepsilon) |u| + C_\varepsilon (e^{\mu u^2} - 1) \quad \text{for any } \mu > 0.
\]
Then, by (7), Lemma 3.1 and the Hölder inequality we obtain, for any \( \tilde{\sigma} > 2 \) and \( \mu < \min \left\{ \frac{4\pi}{\|u\|} \right\} \),

\[
\int_{\mathbb{R}^2} (f(u_{\alpha_n}) - f(u))(u_{\alpha_n} - u)dx \\
\leq (m + \varepsilon) \int_{\mathbb{R}^2} |u_{\alpha_n}||u_{\alpha_n} - u|dx + C \int_{\mathbb{R}^2} \left( e^{\mu u_{\alpha_n}^2} - 1 \right) |u_{\alpha_n} - u|dx \\
\leq (m + \varepsilon) |u_{\alpha_n}| |u_{\alpha_n} - u| + C \left( \int_{\mathbb{R}^2} \left( e^{\mu u_{\alpha_n}^2} - 1 \right)^2 dx \right)^{\frac{1}{2}} |u_{\alpha_n} - u|_2 \\
\leq o(1) + C \left( \int_{\mathbb{R}^2} \left( e^{\tilde{\sigma} u_{\alpha_n}^2} - 1 \right) dx \right)^{\frac{1}{2}} |u_{\alpha_n} - u|_2 \\
\rightarrow 0 \quad \text{as} \quad n \rightarrow \infty,
\]

where the compactness of \( E \hookrightarrow L^2(\mathbb{R}^2) \) is applied. It follows from (10), Lemma 3.2 and the Hölder inequality that

\[
\|u_{\alpha_n} - u\|^2 \\
= \langle \Phi'(u_{\alpha_n}) - \Phi'(u), u_{\alpha_n} - u \rangle + \int_{\mathbb{R}^2} (A_0(u) u - A_0(u_{\alpha_n}) u_{\alpha_n}) (u_{\alpha_n} - u)dx \\
\leq \left( \int_{\mathbb{R}^2} \left( e^{\mu u_{\alpha_n}^2} - 1 \right)^2 dx \right)^{\frac{1}{2}} |u_{\alpha_n} - u|_2 \\
\rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]

Hence, \( u_{\alpha_n} \rightarrow u \) in \( E \) and \( \Phi'(u) = 0 \). Therefore, we complete the proof. \( \square \)

**Lemma 3.4.** Suppose that (V), (g_1) and (g_{11}) hold. Then \( \Phi \) has a local linking at 0 with respect to the decomposition \( E = E^- \oplus E^+ \).

**Proof.** By (g_1), there exist \( \varepsilon > 0 \) and \( C_\varepsilon > 0 \) such that, for every \( \mu > 0 \),

\[
|G(u)| \leq \varepsilon |u|^2 + C_\varepsilon |u|^l (e^{\mu u^2} - 1) \quad \text{for all} \quad l > 2. \quad (20)
\]

Then, let \( \varepsilon \in (0, \frac{\mu}{2\tilde{\sigma}^2}) \) and \( u \in E^+ \) be such that \( \|u\| \leq k_1 \) for some \( k_1 > 0 \). Choosing \( \mu \in (0, \frac{4\pi}{\|u\|^2}) \) with \( \tilde{\sigma} > 2 \), from (7), (8), (12), Lemma 3.1 and the Hölder inequality we get

\[
\Phi(u) = Q(u) + \frac{1}{2} \int_{\mathbb{R}^2} \left( A_1^2(u) + A_2^2(u) \right) u^2 dx - \int_{\mathbb{R}^2} G(u) dx \\
\geq \frac{k_1}{2} \|u\|^2 - C_\varepsilon \int_{\mathbb{R}^2} |u|^l (e^{\mu u^2} - 1) dx
\]
Hence, taking $0 < \rho < \min\{\rho_1, \rho_2\}$, we have
\[
\Phi(u) \geq 0, \quad u \in E^+ \text{ and } \|u\| \leq \rho_1.
\]
Decompose $E^-$ into $Z + W$ where $Z = \ker(-\Delta + V)$, $W = (E^+ + Z)\perp$. Let $u = z + w \in E^-$ with $z \in Z, w \in W$. Since all norms in $E^-$ are equivalent, by $(g_{41})$, $(12)$ and $(15)$ we infer that, for $u \in E^-$,
\[
\begin{align*}
\Phi(u) &= \frac{1}{2} \int_{R^2} (|\nabla u|^2 + Vu^2)dx + \frac{1}{2} \int_{R^2} (\rho A_1(u) + \lambda A_2(u))u^2dx - \int_{R^2} G(u)dx \\
&= \frac{1}{2} \int_{R^2} (|\nabla w|^2 + Vw^2)dx + \frac{1}{2} \int_{R^2} (\rho A_1(u) + \lambda A_2(u))u^2dx - \int_{R^2} G(u)dx \\
&\leq \int_{R^2} (|\nabla w|^2 + Vw^2)dx + \frac{1}{2} \int_{R^2} (\rho A_1(u) + \lambda A_2(u))u^2dx - C_0 \int_{R^2} |w|^pdx \\
&\leq -\frac{\kappa}{2} \|w\|^2 + \frac{1}{2} \|a_1\|^6 - C\|u\|^p.
\end{align*}
\]
Due to $0 < \nu < 6$, for $\rho_2 > 0$ small enough, one has
\[
\Phi(u) \leq 0, \quad u \in E^- \text{ and } \|u\| \leq \rho_2.
\]
Hence, taking $0 < \rho < \min\{\rho_1, \rho_2\}$, we get
\[
\begin{align*}
\Phi(u) &\geq 0, \quad u \in E^+ \text{ and } \|u\| \leq \rho, \\
\Phi(u) &\leq 0, \quad u \in E^- \text{ and } \|u\| \leq \rho.
\end{align*}
\]
Thus, we complete the proof.

**Lemma 3.5.** Assume that $(V)$, $(g_4)$ and $(g_{42})$ are satisfied. Then $\Phi$ has a local linking at $0$ with respect to the decomposition $E = E^1 \oplus E^2$.

**Proof.** Let $\varepsilon \in (0, \frac{\gamma}{2\tilde{\sigma}_2})$ and $u \in E^2$ with $\|u\| \leq k_2$ for some $k_2 > 0$. By $(7)$, $(13)$, $(15)$, $(20)$, Lemma 3.1 and the Hölder inequality one has, for $\mu \in (0, \frac{\gamma^2}{4\tilde{\sigma}})$ with $\tilde{\sigma} > 2$,
\[
\begin{align*}
\Phi(u) &= Q(u) + \frac{1}{2} \int_{R^2} (\rho A_1(u) + \lambda A_2(u))u^2dx - \int_{R^2} G(u)dx \\
&\leq -\frac{\gamma}{2} \|u\|^2 + \frac{1}{2} \|a_1\|^6 - C\|u\|^p + \frac{1}{2} \|a_1\|^6 - C\|u\|^p.
\end{align*}
\]
Hence, for all $u \in E^2$ with $\|u\| \leq \tilde{\rho}_1$, where $0 < \tilde{\rho}_1 < k_2$ small enough, we have
\[
\Phi(u) \leq 0.
\]
Decompose $E^1$ into $Z + \tilde{\tilde{W}}$ where $Z = \ker(-\Delta + V)$, $\tilde{\tilde{W}} = (E^2 + Z)\perp$. Since $Z$ is a finite dimensional space, there exists $\rho > 0$ such that
\[
\|z\|_{\infty} \leq \rho\|z\| \quad \text{for all } z \in Z.
\]

\[
(21)
\]
Let $u = z + \tilde{w} \in E^1$ with $z \in Z, \tilde{w} \in \tilde{W}$ be such that $\|u\| \leq \delta_{2\sigma}$ and set
\[
\mathcal{A} := \left\{ x \in \mathbb{R}^2 : |\tilde{w}(x)| \leq \frac{\delta}{2} \right\},
\]
\[
\mathcal{B} := \mathbb{R}^2 \setminus \mathcal{A}.
\]
On $\mathcal{A}$, it follows from (21) that
\[
|u(x)| \leq |z(x)| + |\tilde{w}(x)| \leq \|z\|_\infty + |\tilde{w}(x)| \leq \delta.
\]
Thus, we complete the proof.

Therefore, it follows from (8), (13) and (15) that, for $u \in E^1$, as
\[
|u(x)| \leq |z(x)| + |\tilde{w}(x)| \leq \|z\|_\infty + |\tilde{w}(x)| \leq \frac{\delta}{2} |\tilde{w}(x)|.
\]
Using (20) one has
\[
G(u) \leq \varepsilon |u|^2 + C_\varepsilon |u|^4 (e^{\mu u^2} - 1) \\
\leq 4\varepsilon \bar{w}^2 + 2^2 C_\varepsilon |\bar{w}|^4 (e^{4\mu \bar{w}^2} - 1).
\]
Then, in view of (7), Lemma 3.1 and the Hölder inequality, as $\mu \in (0, \frac{4\pi^2}{\delta^2})$ with $\delta > 2$
\[
\int_{\mathcal{B}} G(u) \leq 4\varepsilon \int_{\mathcal{B}} \bar{w}^2 dx + 2^2 C_\varepsilon \int_{\mathcal{B}} |\bar{w}|^4 (e^{4\mu \bar{w}^2} - 1) dx \\
\leq 4\varepsilon \int_{\mathcal{B}} \bar{w}^2 dx + C \|\bar{w}\|^4.
\]
Therefore, it follows from (8), (13) and (15) that, for $u \in E^1$ and $\varepsilon \in (0, \frac{\gamma}{2\sigma})$, we have
\[
\Phi(u) = \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + V u^2) dx + \frac{1}{2} \int_{\mathbb{R}^2} (A_1^2(u) + A_2^2(u)) u^2 dx - \int_{\mathbb{R}^2} G(u) dx \\
= \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla \bar{w}|^2 + V \bar{w}^2) dx + \frac{1}{2} \int_{\mathbb{R}^2} (A_1^2(u) + A_2^2(u)) u^2 dx - \int_{\mathbb{R}^2} G(u) dx \\
\geq \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla \bar{w}|^2 dx + V \bar{w}^2) + \frac{1}{2} \int_{\mathbb{R}^2} (A_1^2(u) + A_2^2(u)) u^2 dx \\
- 4\varepsilon \int_{\mathcal{B}} \bar{w}^2 dx - C \|\bar{w}\|^4 - \int_{\mathcal{A}} G(u) dx \\
\geq \frac{\gamma}{2} \|\bar{w}\|^2 - C \|\bar{w}\|^4.
\]
Choosing $0 < \bar{\rho}_2 < \frac{\delta}{2\sigma}$ small enough, we obtain
\[
\Phi(u) \geq 0 \quad \text{for any } u \in E^1 \text{ with } \|u\| \leq \bar{\rho}_2.
\]
Therefore, let $0 < \bar{\rho} < \min\{\bar{\rho}_1, \bar{\rho}_2\}$, one has
\[
\begin{cases}
\Phi(u) \geq 0, & u \in E^1 \text{ and } \|u\| \leq \bar{\rho}, \\
\Phi(u) \leq 0, & u \in E^2 \text{ and } \|u\| \leq \bar{\rho}.
\end{cases}
\]
Thus, we complete the proof. \qed
Lemma 3.6. Suppose that $Y$ is a subspace of $E$ and $\dim Y < \infty$. Then $\Phi$ satisfies, for $u \in Y$,

$$\Phi(u) \to -\infty \quad \text{as } \|u\| \to \infty.$$  

Proof. For any sequence $\{u_n\} \subset Y$ satisfies $\|u_n\| \to \infty$. Let $v_n = \|u_n\|^{-1}u_n$, there exists $v \in Y$ with $\|v\| = 1$ such that

$$\|v_n - v\| \to 0 \quad \text{in } E,$$

$$v_n(x) \to v(x) \quad \text{a.e. in } \mathbb{R}^2,$$

because all norms in finite dimension are equivalent. Since $v \neq 0$, similar to (19), we have

$$\int_{\mathbb{R}^2} \frac{F(u_n)}{\|u_n\|^6} dx \to +\infty \quad \text{as } n \to \infty.$$

Then we deduce that, as $n \to \infty$,

$$\Phi(u_n) = \|u_n\|^6 \left( \frac{1}{2\|u_n\|^4} + \frac{1}{\|u_n\|^6} \int_{\mathbb{R}^2} (A_1^2(u_n) + A_2^2(u_n))u_n^2 dx - \int_{\mathbb{R}^2} \frac{F(u_n)}{\|u_n\|^6} dx \right)$$

$$\to -\infty.$$  

Thus, we complete the proof.

Proof of Theorem 1.1. In view of (7) and (15), $\Phi$ maps bounded sets into bounded sets. Since $\dim(E_1^+ \oplus E_2^+) < \infty$ or $\dim(E_2 \oplus E_1^+) < \infty$, Lemma 3.4 and Lemma 3.6 or Lemma 3.5 and Lemma 3.6 imply the existence of nontrivial solutions for system (1) according to Theorem 2.2. Consequently, we complete the proof of Theorem 1.1.

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