Integral equations of a rectilinear ribbon vibrator near a perfectly conducting infinite screen

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Abstract. The problem of excitation of a rectilinear ribbon vibrator near a perfectly conducting infinite screen is considered. A general equation, a two-dimensional integral equation, and a one-dimensional integral equation with respect to the current density are obtained. The results of numerical calculations are presented.

1. Introduction
The strip vibrator is widely used in various antenna devices, both as an independent radiator and as part of antenna arrays. In connection with the development of the technology of printed (strip and microstrip) antennas, the analysis of a strip vibrator located near the screen is of considerable interest. This work is devoted to a rigorous electrodynamic approach to solving this problem.

2. General Equation
Let us consider a vibrator in the form of a rectilinear tape near an ideally conducting body. Under the action of the primary field \( E^1, H^1 \), surface electric currents will be induced on the surface \( S \) of the vibrator. These currents are the source of the secondary electromagnetic field \( E^2, H^2 \). The secondary field is reflected from a perfectly conducting body located near the vibrator. Let us denote the reflected field by \( E^3, H^3 \). The tangential component of the total electric field should vanish on the surface of an ideally conducting vibrator

\[
\left[ \bar{n}, E^3 \right]_S + \left[ \bar{n}, E^2 \right]_S = - \left[ \bar{n}, E^1 \right]_S.
\]

This equation is a general equation. In the case when there are no bodies near the vibrator, \( E^3 \) is absent, and an equation for the vibrator in free space is obtained.

3. Two-dimensional integral equation
Let us consider a narrow rectilinear belt vibrator that is in a plane \( y = 0 \). The vibrator surface is described in Cartesian coordinates

\[-l \leq z \leq l, \quad -\frac{d}{2} \leq x \leq \frac{d}{2}.\]

Vibrator is narrow vibrator when

\[d \ll l, \quad d \ll \lambda.\]
where $\lambda$ - is the wave length.

Let us place a perfectly conducting screen parallel to the vibrator in the plane $y = h$, Figure 1.

![Figure 1. Belt vibrator near the screen.](image)

Under the action of the primary field $E_1$, surface currents with density $j_z(z, x)$ will be induced on both sides of the vibrator. We introduce the vector potential

$$\vec{A} = \vec{A}_z \vec{t}_z = \mu \int_S j_z(z', x') \frac{e^{-ikR}}{4\pi R} dS,$$

(1)

where $\vec{t}_z$ - is the unit vector along the axis $z, i = \sqrt{-1}, \mu$ - is the absolute magnetic permeability, $k = \frac{2\pi}{\lambda}$ - is the wave number, $R$ - is the distance between emission point and integration point.

Let us express the vectors of the secondary field in terms of the vector potential

$$E_2 = -\frac{i}{\omega \varepsilon \mu} \text{grad} \left( \text{div} \vec{A} \right),$$

(2)

$$H_2 = \frac{1}{\mu} \text{rot} \vec{A},$$

(3)

where $\varepsilon$ - is the absolute dielectric constant, $\omega$ - cyclic frequency.

Let us determine the components of the electric field from (2)

$$E_{2z} = -\frac{i}{\omega \varepsilon \mu} \left( \frac{\partial^2}{\partial z^2} + k^2 \right) A_z,$$

(4)

$$E_{2x} = -\frac{i}{\omega \varepsilon \mu} \frac{\partial^2}{\partial z \partial x} A_z.$$

(5)
The Green’s function enters the right-hand side of expression (1) in closed form. We use the representation of the Green’s function in the form of a double integral [1]

$$
\frac{\exp(-ikR)}{4\pi R} = \frac{1}{8\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-ik_1(z-z')-ik_2(x-x') \pm \beta(y-y')\right) d\kappa_1 d\kappa_2,
$$

where $\beta = \sqrt{\kappa_1^2 + \kappa_2^2 - k^2}$.

Let us represent the components of the electric field considering (6)

$$
E_z^2 = -\frac{i}{\omega e} \left( \frac{\partial^2}{\partial z^2} + k^2 \right) \int j_z(z',x') \frac{\exp(-ikR)}{4\pi R} dS =
$$

$$
= -\frac{i}{\omega e} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-ik_1z - ik_2x \pm \beta y\right) \left(-\kappa_1^2 + k^2\right) f(\kappa_1, \kappa_2) d\kappa_1 d\kappa_2
$$

$$
E_z^2 = -\frac{i}{\omega e} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-ik_1z - ik_2x \pm \beta y\right) \left(-\kappa_1^2 + k^2\right) f(\kappa_1, \kappa_2) d\kappa_1 d\kappa_2,
$$

where $f(\kappa_1, \kappa_2) = \frac{1}{8\pi^2} \int j_z(z',x') \exp\left(-ik_1z' + ik_2x'\right) dz' dx'$.

The secondary field is reflected off the screen. This field has no sources in the area of its definition, for this reason, it is sufficient to specify only the longitudinal components. We represent the longitudinal components in the following form:

$$
E_y^3 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-ik_1z - ik_2x + \beta(y-h)\right) \frac{A(\kappa_1, \kappa_2)}{\beta} d\kappa_1 d\kappa_2,
$$

$$
H_y^3 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-ik_1z - ik_2x + \beta(y-h)\right) \frac{B(\kappa_1, \kappa_2)}{\beta} d\kappa_1 d\kappa_2.
$$

Spectral densities and are unknown and must be determined. First, it is necessary to determine the tangential component of the reflected field through the components to do this [1]

$$
E_z^3 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-ik_1z - ik_2x\right) \exp\left(\beta(y-h)\frac{-i\kappa_\beta\beta}{\kappa_1^2 + \kappa_2^2} A + \frac{\omega \mu \kappa_1}{\kappa_1^2 + \kappa_2^2} B\right) d\kappa_1 d\kappa_2,
$$

$$
E_x^3 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-ik_1z - ik_2x\right) \exp\left(\beta(y-h)\frac{-i\kappa_\beta\beta}{\kappa_1^2 + \kappa_2^2} A + \frac{\omega \mu \kappa_1}{\kappa_1^2 + \kappa_2^2} B\right) d\kappa_1 d\kappa_2.
$$

The total tangential component of the electric field on the surface of the screen must vanish

$$
E_z^3 \big|_{y=h} = -E_z^2 \big|_{y=h}
$$

$$
E_x^3 \big|_{y=h} = -E_x^2 \big|_{y=h}.
$$

Let us introduce expressions for the secondary and reflected fields into equations (13) and (14). Next, let us apply the inverse Fourier transform. As a result, we obtain a system of two equations for determining the unknown spectral densities
\[-i \kappa_1 \beta A - \omega \mu \kappa_2 B = \frac{i}{\varepsilon \mu} \left( -\kappa_1^2 + k^2 \right) \left( \kappa_1^2 + \kappa_2^2 \right) f \left( \kappa_1, \kappa_2 \right) \exp(-\beta h), \tag{15} \]

\[-i \kappa_2 \beta A - \omega \mu \kappa_1 B = \frac{i}{\varepsilon \mu} \left( -\kappa_1 \kappa_2 \right) \left( \kappa_1^2 + \kappa_2^2 \right) f \left( \kappa_1, \kappa_2 \right) \exp(-\beta h). \tag{16} \]

Having solved this system, we find \( A(\kappa_1, \kappa_2) \) and \( B(\kappa_1, \kappa_2) \), and then from (11) and (12) we can find the components of the reflected field. Let us give the final expression for the longitudinal component of the field

\[ E_3^z(z, y, x) = \frac{i}{\varepsilon \mu} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\exp\left(-i \kappa_1 z - i \kappa_2 x \right)}{\beta} \left( -\kappa_1^2 + k^2 \right) \exp(\beta h) \exp\left( \beta (y - h) \right) f \left( \kappa_1, \kappa_2 \right) d\kappa_1 d\kappa_2. \tag{17} \]

Now let us write a two-dimensional integral equation for the current density flowing on both sides of the vibrator surface. For this, we substitute the expressions from (7) and (17) into (1)

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} j_z(z', x') \exp\left(-i \kappa_1 (z - z') - i \kappa_2 (x - x') \right) \left( -\kappa_1^2 + k^2 \right) \ast \left( 1 - \exp(-2 \beta h) \right) d\kappa_1 d\kappa_2 dz' dx' = -i k \sqrt{\frac{\varepsilon}{\mu}} E_3^z(z, x). \tag{18} \]

Thus, we have obtained a two-dimensional integral equation.

4. One-dimensional integral equation for current density

Let us represent the current density in the following form

\[ j_z(z, x) = J(z) \Psi(x), \tag{19} \]

where the function \( \Psi(x) \) will be given as

\[ \Psi(x) = \frac{1}{\pi} \left( \frac{d}{2} \right)^2 - x^2 \right)^{\frac{1}{2}}. \]

This function satisfies the Meixner condition on the edge; moreover,

\[ \int_{-d/2}^{+d/2} \Psi(x) dx = 1. \tag{20} \]

From (19 and (20)) it follows that it has the meaning of the current flowing along the vibrator. Let us apply the formula [2]

\[ \int_{-d/2}^{+d/2} \Psi(x) \exp(-i \kappa_2 x) dx = J_0 \left( \kappa_2 \frac{d}{2} \right), \]

where \( J_0(\cdot) \) - is the zero-order Bessel function.

In equation (18), we can assume that it is justified since the vibrator is narrow. As a result, we get
\[
\frac{1}{8\pi^2} \int_{-\infty}^{+\infty} J(z')dz' \int_{-\infty}^{+\infty} \frac{\kappa_1^2 + k^2}{\beta} \exp\left(-i\kappa_1(z-z')\right) J_0\left(\kappa_2 \frac{d}{2}\right) \ast (1 - \exp(-2\beta h)) d\kappa_1 d\kappa_2 dz' = -ik \sqrt{\frac{E}{\mu}} E'_i(z). \tag{21}
\]

Next, we apply the integrals [3]

\[
\int_{0}^{+\infty} \frac{1}{\beta} \exp\left(-2h\sqrt{\kappa_1^2 + \kappa_2^2}\right) J_0\left(\kappa_2 \frac{d}{2}\right) d\kappa_2 =
= I_0\left(\kappa_1\left(\sqrt{h^2 + \left(\frac{d}{4}\right)^2} - h\right)\right) K_0\left(\kappa_1\left(\sqrt{h^2 + \left(\frac{d}{4}\right)^2} + h\right)\right), \tag{22}
\]

\[
\int_{0}^{+\infty} \frac{1}{\beta} \exp\left(-2h\sqrt{\kappa_1^2 - \kappa_2^2}\right) J_0\left(\kappa_2 \frac{d}{2}\right) d\kappa_2 =
= I_0\left(i\kappa_1\left(\sqrt{h^2 + \left(\frac{d}{4}\right)^2} - h\right)\right) K_0\left(i\kappa_1\left(\sqrt{h^2 + \left(\frac{d}{4}\right)^2} + h\right)\right), \tag{23}
\]

where \(I_0(\cdot)\) - is the modified zero-order Bessel function, \(K_0(\cdot)\) - is the zero-order Macdonald function.

Taking into account (22) and (23), we obtain the one-dimensional integral equation

\[
\frac{1}{8\pi^2} \int_{-\infty}^{+\infty} J(z')dz' \int_{-\infty}^{+\infty} (-\kappa_1^2 + k^2) \exp\left(-ih(z-z')\right) F(\kappa_i) d\kappa_i dx = -ik \sqrt{\frac{E}{\mu}} E'_i(z), \tag{24}
\]

where \(F(\kappa_i) = G(\kappa_i, 0) - G(\kappa_i, h)\).

\[
G(\kappa_1, h) = I_0\left(\sqrt{\kappa_1^2 - k^2}\left(\sqrt{h^2 + \left(\frac{d}{4}\right)^2} - h\right)\right) \ast
*K_0\left(\sqrt{\kappa_1^2 - k^2}\left(\sqrt{h^2 + \left(\frac{d}{4}\right)^2} + h\right)\right), \text{ at } \kappa_1^2 > k^2; \tag{25}
\]

\[
G(\kappa_1, h) = I_0\left(i\sqrt{\kappa_1^2 - k^2}\left(\sqrt{h^2 + \left(\frac{d}{4}\right)^2} - h\right)\right) \ast
*K_0\left(i\sqrt{\kappa_1^2 - k^2}\left(\sqrt{h^2 + \left(\frac{d}{4}\right)^2} + h\right)\right), \text{ at } \kappa_1^2 < k^2. \tag{26}
\]
Integral equation (24) is one-dimensional. It should be noted that the function $F(\kappa)$ at $\kappa \to +\infty$ behaves in the same way as in the case when the vibrator is in free space.

The methods for solving the integral equation for a vibrator in free space are completely applicable to the considered problem.

5. Results of numerical calculations

Calculations of the dependence of the input resistance $Z = R + iX$ on the parameters of the vibrator and the distance to the screen were carried out by the numerical-analytical method [4]. It also considered the fact that the belt vibrator with the width $d$ is equivalent in terms of parameters to a cylindrical vibrator with a radius $a = \frac{d}{4}$. The primary field was taken to be the field created by the annular magnetic current

$$E^0_z(z) = \frac{1}{2 \ln \left( \frac{b}{a} \right)} \frac{\exp \left( -ik \sqrt{z^2 + a^2} \right)}{\sqrt{z^2 + a^2}} \frac{\exp \left( -ik \sqrt{z^2 + b^2} \right)}{\sqrt{z^2 + b^2}},$$

where $a$ is the equivalent radius of the vibrator, $b$ is the radius of the annular magnetic current. In the calculations below, it is assumed that $\frac{b}{a} = 2$. The ring current covers the vibrator in the middle in the $x0y$ plane, in the section $z=0$. Figure 2 and Figure 3 show the dependences of the active and reactive components of the input resistance of the strip vibrator length and width on the distance to the screen.

![Figure 2](image.png)

**Figure 2.** Dependences of the active $R$ and reactive $X$ components of the input resistance of the ribbon vibrator with the length $2l = \frac{\lambda}{2}$ and width $kd = \frac{\pi}{50}$ from the distance to the screen.
Figure 3. Dependences of the active and reactive components of the input resistance of the ribbon vibrator with the length $2l = \frac{\lambda}{2}$ and width $kd = \frac{\pi}{10}$ from the distance to the screen.

The obtained dependences of the input resistance are of an oscillatory nature. The amplitude of the oscillations decreases with distance from the screen and, at a considerable distance from the screen, approaches the values of the input resistance characteristic of free space. The input impedance changes especially sharply at a short distance from the screen.

6. Conclusion
The paper proposes a method for calculating flat dipole antennas near the screen. The original two-dimensional integral equation is reduced to a one-dimensional equation. This was achieved thanks to the analytical determination of integrals containing a special Bessel function. The obtained one-dimensional integral equation is solved by a numerical-analytical method. An example of calculation is given, the effect of the screen on the input resistances is shown. The approach developed in this work can be used to calculate antenna arrays, both linear and two-dimensional.

References
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