The exponential tracking and disturbance rejection for the unstable Burgers’ equation with general references and disturbances

Weijiu Liu
Department of Mathematics
University of Central Arkansas
201 Donaghey Avenue, Conway, AR 72035, USA

Abstract

Asymptotic tracking and disturbance rejection, it has been long always assumed that the reference to be tracked and the disturbance to be rejected must be generated by an exosystem such as a finite dimensional exosystem with pure imaginary eigenvalues. The objective of this paper is to solve such a tracking problem for the unstable Burgers’ equation without this assumption. Our treatment of this problem is straightforward. Using the method of variable transform, the tracking problem is split into two separate problems: a simple Neumann boundary stabilization problem and a dynamical Neumann boundary regulator problem. Unlike the existing literature where the regulator problem is always kept independent, the stabilization problem here is simplified to an independent linear diffusion equation by moving the instability term and the nonlinear term to the dynamical regulator problem, whereas the dynamical regulator problem does depend on the stabilization problem. Then we can first easily handle the stabilization problem and then solve the dynamical regulator problem by using the fundamental theory of partial differential equations. The boundary feedback controller is explicitly constructed by using the reference, the disturbance, and the solution of the stabilization problem while the boundary feedback controller is easily designed for the linear diffusion equation without using a complex method such as the backstepping method. It is proved that, under the designed feedback and feedforward controllers, the tracking error converges to zero exponentially. This theoretical result is confirmed by a numerical example.

Key Words: Burgers’ equation, Feedback and feedforward boundary control, Dynamical regulator equation, Exponential tracking, Disturbance rejection.

1 Introduction

Asymptotic tracking and disturbance rejection is one of fundamental problems in control theory. In solving this problem, it has been long always assumed that the reference to be tracked and the disturbance to be rejected must be generated by an exosystem such as a finite dimensional exosystem with pure imaginary eigenvalues (see, e.g., [2, 9, 10, 11, 12, 14, 19, 25, 26, 27, 28, 29, 31, 32, 33, 34, 35, 36, 37, 39, 41]). In fact, this assumption is sufficient, but not necessary for making the problem solvable.

The objective of this paper is to solve the tracking problem for the unstable Burgers’ equation without this assumption:

\[
\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} + au + u_d, \tag{1}
\]

\[
\nu \frac{\partial u}{\partial x}(0,t) = f_0, \quad \nu \frac{\partial u}{\partial x}(1,t) = f_1, \tag{2}
\]

\[
u u(0,0) = u_0(x). \tag{3}
\]

In the above equations, \( \nu > 0 \) is a viscosity parameter, \( a = a(x,t) \) is a continuous function, \( u_d = u_d(x,t) \) is a source disturbance, \( f_0 \) and \( f_1 \) are control inputs, and the function \( u_0 = u_0(x) \) is an initial state in an appropriate function space. When \( a \) is positively large enough, the equilibrium 0 of the uncontrolled and undisturbed Burgers’ equation is unstable.

Let \( r(t) \) be a desired reference. We introduce the tracking error
\[
\epsilon(t) = \int_0^1 u(x,t) dx - r(t). \tag{4}
\]

Then the problem of exponential tracking and disturbance rejection for the Burgers’ equation is to design a feedback and feedforward controllers \( f_0, f_1 \) such that
\[
|\epsilon(t)| \leq Ce^{-\lambda t}. \tag{5}
\]

where \( C \) and \( \lambda \) are positive constants.

Mathematical theory on the tracking problem has been well developed for finite dimensional control systems (see, e.g., [14]) and recently extended to partial differential equations (PDEs). Aulisa et al. [2], Byrnes et al. [4], and Natarajan et al. [37] developed an abstract theory on the problem in Hilbert spaces and applied it to partial differential equations (PDEs) such as the heat equation and the wave equation. The flatness method was developed to handle the trajectory planning and feedforward control design (see, e.g.,
Wagner et al. \[10\] studied the tracking problem for the one-dimensional semilinear wave equation by using the flatness method. The asymptotic tracking and disturbance rejection of the one-dimensional parabolic partial differential equations were studied by Deutscher \[9, 10\] and the case with a long time delay was investigated by Gu et al. \[13\]. The same problem for the one-dimensional hyperbolic partial differential equations and Schrödinger equation were studied by Deutscher \[11\] and Zhou et al \[41\], respectively. In all of these important researches, the references and disturbances were assumed to be governed by a finite dimensional exosystem with pure imaginary eigenvalues.

Although the problem of feedback stabilization and tracking for the Burgers' equation has received extensive attention (see, e.g., \[2, 14\]), the stabilization problem here is simplified to an independent linear diffusion equation by moving the instability term and the nonlinear term to the dynamical regulator problem, whereas the dynamical regulator problem does depend on the stabilization problem. Thus we can first easily handle the stabilization problem and then solve the dynamical regulator problem by using the fundamental theory of partial differential equations (see, e.g., \[21\]). The feedforward controller is explicitly constructed by using the reference \(r\), the disturbance \(u_d\) and the solution of the stabilization problem while the feedback controller is easily designed for the linear diffusion equation without using a complex method such as the backstepping method. It is proved that, under the designed feedback and feedforward controllers, the tracking error converges to zero exponentially. This theoretical result is confirmed by a numerical example.

\section{Exponential tracking}

In what follows, \(H^s((0,1))\) denotes the usual Sobolev space (see \[11, 22\]) for any \(s \in \mathbb{R}\). For \(s \geq 0\), \(H^s_0(0,1)\) denotes the completion of \(C_0^\infty((0,1))\) in \(H^s((0,1))\), where \(C_0^\infty((0,1))\) denotes the space of all infinitely differentiable functions on \((0,1)\) with compact support in \((0,1)\). We use the following \(H^1\) norm of \(H^1((0,1))\)

\[
\|u\|_{H^1} = \left[ u(0)^2 + \int_0^1 \left( \frac{\partial u}{\partial x} \right)^2 \, dx \right]^{1/2}, \quad u \in H^1((0,1)),
\]

which is equivalent to the usual one. The norm on \(L^2((0,1))\) is denoted by \(\| \cdot \|_2\).

Let \(X\) be a Banach space and \(T > 0\). We denote by \(C^n([0,T];X)\) the space of \(n\) times continuously differentiable functions defined on \([0,T]\) with values in \(X\), and write \(C([0,T];X)\) for \(C^0([0,T];X)\). In what follows, for simplicity, we omit the indication of the varying range of \(x\) and \(t\) in equations and we understand that \(x\) varies from 0 to 1 and \(t\) from 0 to \(\infty\).

To split the tracking problem into a stabilization problem and a dynamical regulator problem, we introduce the variable transform

\[
u \frac{\partial u}{\partial t} + \nu \frac{\partial U}{\partial x} (0,t) = \hat{f}_0 + F_0, \\nu \frac{\partial u}{\partial t} + \nu \frac{\partial U}{\partial x} (1,t) = \hat{f}_1 + F_1, \\
\hat{u}(0,t) + U(x,0) = u_0(x), \\
e(t) = \int_0^1 (\hat{u}(x,t) + U(x,t))dx - r(t).
\]

This problem can be split into a stabilization problem

\[
\frac{\partial \hat{u}}{\partial t} = \nu \frac{\partial^2 \hat{u}}{\partial x^2} - \hat{u} \frac{\partial \hat{u}}{\partial x},
\]

\[
\nu \frac{\partial \hat{u}}{\partial x} (0,t) = \hat{f}_0, \quad \nu \frac{\partial \hat{u}}{\partial x} (1,t) = \hat{f}_1,
\]

\[
\hat{u}(x,0) = u_0(x),
\]

\[
e(t) = \int_0^1 \hat{u}(x,t)dx.
\]

and a dynamical regulator problem

\[
\frac{\partial U}{\partial t} = \nu \frac{\partial^2 U}{\partial x^2} - U \frac{\partial U}{\partial x} + a(\hat{u} + U) - \left( \frac{\partial U}{\partial x} + U \frac{\partial \hat{u}}{\partial x} \right) + u_d,
\]

\[
\nu \frac{\partial U}{\partial x} (0,t) = F_0, \quad \nu \frac{\partial U}{\partial x} (1,t) = F_1
\]

\[
U(x,0) = r(0),
\]

\[
\int_0^1 U(x,t)dx = r(t).
\]
In the control design (see, e.g., [2 14]), the terms \(a(\hat{u} + U)\) and \(\hat{u} \frac{\partial \nu}{\partial x} + U \frac{\partial \hat{u}}{\partial x}\) are usually put in the equation (8) in the stabilization problem such that the regulator problem is independent from the stabilization problem. However, if we do so for the Burgers’ equation, then the feedback controllers \(\hat{f}_0\) and \(\hat{f}_1\) for the stabilization problem (8) - (10) is difficult or impossible to design. In fact, we can further simplify the stabilization problem by moving the term \(\hat{u} \frac{\partial \nu}{\partial x}\) from the stabilization problem to the regulator problem and then obtain the following linear stabilization problem and the regulator problem:

\[
\begin{align*}
\frac{\partial \hat{u}}{\partial t} &= \nu \frac{\partial^2 \hat{u}}{\partial x^2}, \\
\nu \frac{\partial \hat{u}}{\partial x}(0,t) &= \hat{f}_0, \quad \hat{u}(x,0) = u_0(x) - r(0), \\
e(t) &= \int_0^1 \hat{u}(x,t) dx,
\end{align*}
\]

and

\[
\begin{align*}
\frac{\partial U}{\partial t} &= \nu \frac{\partial^2 U}{\partial x^2} - U \frac{\partial \hat{u}}{\partial x} + a(\hat{u} + U) \\
-\hat{u} \frac{\partial U}{\partial x} - \left( \frac{\partial U}{\partial x} + U \frac{\partial \hat{u}}{\partial x} \right) + u_d, \\
\frac{\partial U}{\partial x}(0,t) &= F_0, \quad \nu \frac{\partial U}{\partial x}(1,t) = F_1, \\
U(x,0) &= r(0), \\
\int_0^1 U(x,t) dx &= r(t).
\end{align*}
\]

Because the regulator problem (12) - (15) is not dissipative, its well-posedness is challenging and open. Since \(\int_0^1 U(x,t) dx = r(t)\) exists for all times, we could conjecture that it has a unique global solution. We will use this conjecture in the following theorem.

Using the stabilization problem (8) - (10) and the regulator problem (12) - (15), we can design the feedback and feedforward controllers as stated in the following theorem.

**Theorem 2.1.** Assume that \(k > 1/6\) and the initial condition \(u_0 \in H^2(0,1)\). Suppose that \(a(x,t)\) and \(u_d(x,t)\) are continuous and \(r(t)\) is continuously differentiable. Let \(\hat{u}\) be the solution of the stabilization problem (8) - (10) and assume that the regulator problem (12) - (15) has a unique global classical solution. Then, under the feedback and feedforward controllers:

\[
\begin{align*}
\hat{f}_0(t) &= k \left[ \hat{u}(0,t) + [\hat{u}(0,t)]^2 \right], \\
\hat{f}_1(t) &= -k \left[ \hat{u}(1,t) + [\hat{u}(1,t)]^2 \right], \\
F_0(t) &= \hat{u}(0,t) U(0,t) + \frac{1}{2} [U(0,t)]^2, \\
F_1(t) &= \hat{u}(1,t) U(1,t) + \frac{1}{2} [U(1,t)]^2 + r'(t) - \int_0^1 a(x,t)(\hat{u}(x,t) + U(x,t)) + u_d(x,t) dx.
\end{align*}
\]

the problem (1) - (3) has a unique solution satisfying

\[
|e(t)| \leq \|u_0 - r(0)\| e^{-\lambda t/2},
\]

where \(\lambda = \min(\nu, k - \frac{1}{4})\).

**Proof.** If \(k > 1/6\) and the initial condition \(u_0 \in H^2(0,1)\), then it was proved in [18 23] that the stabilization problem (8) - (10) with the feedback controllers (24) and (25) has a unique solution satisfying

\[
\hat{u} \in C([0, \infty); H^2(0,1)).
\]

Moreover, multiplying the equation (8) by \(\hat{u}\) and integrating it from 0 to 1, we obtain

\[
\begin{align*}
\frac{1}{2} \int_0^1 [\hat{u}(x,t)]^2 dx &= -k \int_0^1 \left[ \hat{u}(0,t) + [\hat{u}(0,t)]^2 \right] \hat{u}(1,t) \\
&\quad - k \int_0^1 \left[ \hat{u}(0,t) + [\hat{u}(0,t)]^2 \right] \hat{u}(1,t) \\
&\quad - \nu \int_0^1 \left[ \frac{\partial \hat{u}}{\partial x}(x,t) \right]^2 dx - \frac{1}{3} \int_0^1 \left[ [\hat{u}(0,t)]^3 - [\hat{u}(0,t)]^3 \right] \\
&\quad \leq \left( k - \frac{1}{6} \right) \int_0^1 \left[ \hat{u}(0,t) + [\hat{u}(0,t)]^2 + r(1,t) \right] \\
&\quad + \int_0^1 \left[ \hat{u}(0,t) + [\hat{u}(0,t)]^4 \right] - \nu \int_0^1 \left[ \frac{\partial \hat{u}}{\partial x}(x,t) \right]^2 dx \\
&\quad \leq \left( k - \frac{1}{6} \right) \int_0^1 \left[ \hat{u}(0,t) + [\hat{u}(0,t)]^4 \right] - \nu \int_0^1 \left[ \frac{\partial \hat{u}}{\partial x}(x,t) \right]^2 dx \\
&\quad \leq -\lambda \left( \left[ \hat{u}(0,t) + [\hat{u}(0,t)]^2 \right] + \int_0^1 \left[ \frac{\partial \hat{u}}{\partial x}(x,t) \right]^2 dx \right) \\
&\quad \leq -\frac{\lambda}{2} \int_0^1 [\hat{u}(x,t)]^2 dx. \quad \text{(use (1))}
\end{align*}
\]

Solving this inequality, we obtain

\[
\|\hat{u}(t)\| \leq \|u_0 - r(0)\| e^{-\lambda t/2}.
\]

Integrating the equation (12) over \([0,1]\), we obtain

\[
\begin{align*}
\frac{d}{dt} \int_0^1 U(x,t) dx &= \nu \frac{\partial U}{\partial x}(1,t) - \nu \frac{\partial U}{\partial x}(0,t) + \hat{u}(0,t) U(0,t) + \frac{1}{2} U^2(0,t) \\
&\quad - \hat{u}(1,t) U(1,t) - \frac{1}{2} U^2(1,t) + \int_0^1 u_d(x,t) dx \\
&\quad + \int_0^1 a(x,t)(\hat{u}(x,t) + U(x,t)) dx.
\end{align*}
\]

It then follows from the boundary condition (13) and the feedforward controllers (24) and (25) that

\[
\frac{d}{dt} \int_0^1 U(x,t) dx = r'(t).
\]
Integrating this equation and using the initial condition, we obtain
\[ \int_0^1 U(x,t)dx = r(t). \]  
(30)

So \( U \) satisfies the equation. Since \( \int_0^1 U(x,t)dx \) exists for all \( t \geq 0 \), the problem \( (12) - (14) \) has a unique solution \( U \) for all \( t > 0 \).

Finally it follows from the equations \( (11), (14), \) and \( (29) \) that
\[ |e(t)| = \left| \int_0^1 u(x,t)dx - r(t) \right| \]
\[ = \left| \int_0^1 u(x,t)dx - \int_0^1 U(x,t)dx \right| \]
\[ = \left| \int_0^1 \hat{u}(x,t)dx \right| \]
\[ \leq \left( \int_0^1 dx \right)^{1/2} \left( \int_0^1 [\hat{u}(x,t)]^2 dx \right)^{1/2} \]
\[ \leq \|u_0 - r(0)\|e^{-\lambda t/2}. \]

This completes the proof.

Using the stabilization problem \( (16) - (18) \) and the regulator problem \( (20) - (23) \), we can design the feedback and feedforward controllers as stated in the following theorem.

**Theorem 2.2.** Assume that \( k > 0 \) and the initial condition \( u_0 \in H^2(0,1) \). Suppose that \( a(x,t) \) and \( u_d(x,t) \) are continuous and \( r(t) \) is continuously differentiable. Let \( \hat{u} \) be the solution of the stabilization problem \( (16) - (18) \) and assume that the regulator problem \( (20) - (23) \) has a unique global classical solution. Then, under the feedback and feedforward controllers:
\[ \hat{f}_0(t) = k\hat{u}(0,t), \]
\[ \hat{f}_1(t) = -k\hat{u}(1,t), \]
\[ F_0(t) = \hat{u}(0,t)U(0,t) + \frac{1}{2} [\hat{u}(0,t)]^2 + [U(0,t)]^2, \]
\[ F_1(t) = \hat{u}(1,t)U(1,t) + \frac{1}{2} [\hat{u}(1,t)]^2 + [U(1,t)]^2 + r'(t) \]
\[ - \int_0^1 [a(x,t)(\hat{u}(x,t) + U(x,t))]dx. \]

the problem \( (11) - (33) \) has a unique solution satisfying
\[ |e(t)| \leq \|u_0 - r(0)\|e^{-\lambda t/2}, \]
where \( \lambda = \min(\nu, k) \).

The proof of this theorem is the same as the proof of Theorem 2.1.

![Figure 1: The average of \( u \), \( u_0(t) = \int_0^1 u(x,t)dx \), quickly tracks the reference \( r(t) = 2 + 4\cos(\pi t) - 3\sin(\pi t) \) under the feedback and feedforward controllers either \( (24) - (27) \) (left figure) or \( (31) - (34) \) (right figure).](image)

3 A numerical example

We conduct a numerical simulation to confirm the above theoretical result. In the numerical computations, we take \( \nu = 5 \), \( a = 20 \), \( k = 15 \), \( u_0(x) = 0 \), \( r(t) = 2 + 4\cos(\pi t) - 3\sin(\pi t) \), and \( u_d(x,t) = 3 + 5\cos(\pi x)\sin(\pi t) - 2\sin(\pi x)\cos(\pi t) \). Then the problem \( (1) - (3) \) and the problem \( (8) - (10) \) are solved numerically by the difference method. The Figure 1 shows that the average of \( u \), \( u_0(t) = \int_0^1 u(x,t)dx \), quickly tracks the reference \( r(t) = 2 + \cos(\pi t) - 3\sin(\pi t) \) under the feedback and feedforward controllers either \( (24) - (27) \) or \( (31) - (34) \).

4 Discussion

Other tracking errors can be considered. Let \( r_0(t) \) and \( r_1(t) \) be two desired references and define the tracking error by
\[ e_1(t) = u(0,t) - r_0(t) + u(1,t) - r_1(t) \]
(36)

Using the variable transform \( \hat{u} \) and this tracking error, we can obtain the stabilization problem
\[ \frac{\partial \hat{u}}{\partial t} = \nu \frac{\partial^2 \hat{u}}{\partial x^2} - \hat{u} \frac{\partial \hat{u}}{\partial x}, \]
\[ \nu \frac{\partial \hat{u}}{\partial x}(0,t) = \hat{f}_0, \quad \nu \frac{\partial \hat{u}}{\partial x}(1,t) = \hat{f}_1, \]
\[ \hat{u}(x,0) = u_0(x) - r_0(0)(1-x) - r_1(0)x, \]
\[ e_1(t) = \hat{u}(0,t) + \hat{u}(1,t), \]
and the dynamical regulator problem
\[ \frac{\partial U}{\partial t} = \nu \frac{\partial^2 U}{\partial x^2} - U \frac{\partial U}{\partial x} + a(\hat{u} + U) \]
\[ - \left( \hat{u} \frac{\partial U}{\partial x} + U \frac{\partial \hat{u}}{\partial x} \right) + u_d, \]
\[ U(0,t) = r_0(t), \quad U(1,t) = r_1(t), \]
\[ U(x,0) = r_0(0)(1-x) + r_1(0)x, \]
\[ \nu \frac{\partial U}{\partial x}(0,t) = F_0, \quad \nu \frac{\partial U}{\partial x}(1,t) = F_1, \]
If the problem (11) - (13) has a global solution, then the feedforward controllers are given by the equation (14). However, it seems challenging to show it has a global solution even though it has a unique classical solution within some time from 0 to T (see, e.g., [21]).

We can also consider the tracking error

$$e_2(t) = u(0,t) - r_0(t) + \int_0^1 u(x,t)dx - r(t). \quad (45)$$

Using the variable transform (7) and this tracking error, we can obtain the stabilization problem

$$\frac{\partial \hat{u}}{\partial t} = \nu \frac{\partial^2 \hat{u}}{\partial x^2} - \hat{u} \frac{\partial \hat{u}}{\partial x} + (\hat{u} + U)$$

$$+ \left( \frac{\partial U}{\partial x} + U \frac{\partial \hat{u}}{\partial x} \right) + u_d, \quad (46)$$

$$U(0,t) = r_0(t), \quad \nu \frac{\partial U}{\partial x}(1,t) = F_1, \quad (47)$$

$$U(x,0) = r(0), \quad \nu \frac{\partial U}{\partial x}(0,t) = F_0. \quad (48)$$

To find $F_1$, we integrate the equation (50) from 0 to 1 and use the equation (53) to obtain

$$F_1 = \nu \left( \frac{\partial U}{\partial x}(0,t) + \frac{1}{2}[(U(1,t))^2 - (U(0,t))^2] \right)$$

$$+ \hat{u}(1,t)U(1,t) - \hat{u}(0,t)U(0,t) + r'(t)$$

$$- \int_0^1 a(x,t)(\hat{u}(x,t) + U(x,t) + u_d(x,t))dx. \quad (55)$$

This results the following complex boundary value problem

$$\frac{\partial U}{\partial t} = \nu \frac{\partial^2 U}{\partial x^2} - (\hat{u} + U) \frac{\partial U}{\partial x} - U \frac{\partial \hat{u}}{\partial x} + u_d, \quad (56)$$

$$U(0,t) = r_0(t), \quad (57)$$

$$\nu \frac{\partial U}{\partial x}(0,t) = \nu \left( \frac{\partial U}{\partial x}(0,t) + \frac{1}{2}[(U(1,t))^2 - (U(0,t))^2] \right)$$

$$+ \hat{u}(1,t)U(1,t) - \hat{u}(0,t)U(0,t) + r'(t)$$

$$- \int_0^1 a(x,t)(\hat{u}(x,t) + U(x,t) + u_d(x,t))dx \quad (58)$$

$$U(x,0) = r(0). \quad (59)$$

It seems that the proof of the solution existence of the problem is challenging.

References

[1] R. Adams, Sobolev Spaces. Academic Press, New York (1975).

[2] E. Aulisa and D. Gilliam, A Practical Guide to Geometric Regulation for Distributed Parameter Systems, Chapman and Hall/CRC, Boca Raton, FL, 2015.

[3] Brandon Ashley and Weijiu Liu, Asymptotic tracking and disturbance rejection of blood glucose regulation system, Mathematical Biosciences, 289, 2017, 78-88.

[4] A. Balogh and M. Krstić, Burgers’ Equation with Nonlinear Boundary Feedback: $H^1$ Stability, Well-Posedness and Simulation. Mathematical Problems in Engineering, vol. 6, 2000, Article ID 649242, https://doi.org/10.1155/S1024123X00001320

[5] J.A. Burns and S. Kang, A control problem for Burgers’ equation with bounded input/output. Nonlinear Dynamics 2 (1992) 235-262.

[6] C. I. Byrnes, I. G. Lauko, D. S. Gilliam, V. I. Shubov, Output regulation for linear distributed parameter systems, IEEE Trans. Autom. Control, 45, no. 12, pp. 2236-2252, 2000.

[7] C.I. Byrnes, D.S. Gilliam and V.I. Shubov, Boundary control for a viscous Burgers’ equation, in Identification Control for Systems Governed by Partial Differential Equations, H.T. Banks, R.H. Fabiano and K. Ito Eds., SIAM (1993) 171-185.

[8] H. Choi, R. Temam, P. Moin and J. Kim, Feedback control for unsteady flow and its application to the stochastic Burgers’ equation. J. Fluid Mech. 253 (1993) 509-543.

[9] J. Deutscher, A backstepping approach to the output regulation of boundary controlled parabolic PDEs, Automatica, 57 (2015), 56-64.

[10] J. Deutscher, Backstepping design of robust output feedback regulators for boundary controlled parabolic PDEs, IEEE Trans. Autom. Control, 61 (2016), 2288-2294.

[11] J. Deutscher, Finite-time output regulation for linear 2 × 2 hyperbolic systems using backstepping, Automatica, 75 (2017), 54-62.

[12] G. Freudenthaler and T. Meurer, PDE-based multi-agent formation control using flatness and backstepping: Analysis, design and robot experiments, Automatica, Vol. 115 (2020), 108897
[13] J. Gu and J. Wang, Backstepping state feedback regulator design for an unstable reaction-diffusion PDE with long time delay, J. Dyn. Control Syst., 24(2018), 563-576.

[14] J. Huang, Nonlinear Output Regulation, Theory and Applications. Society for Industrial and Applied Mathematics, Philadelphia (2004)

[15] K. Ito and S. Kang, A dissipative feedback control for systems arising in fluid dynamics. SIAM J. Control Optim. 32 (1994) 831-854.

[16] K. Ito and Y. Yan, Viscous scalar conservation law with nonlinear flux feedback and global attractors. J. Math. Anal. Appl. 227 (1998) 271-299.

[17] H. K. Khalil, Nonlinear Systems. Prentice-Hall, Inc., New Jersey (1996).

[18] M. Krstić, On global stabilization of Burgers’ equation by boundary control. Systems & Control Letters 37 (1999) 123-141.

[19] M. Krstić, L. Magnis, and R. Vazquez, Nonlinear control of the viscous Burgers equation: trajectory generation, tracking, and observer design, Journal of Dynamic Systems, Measurement, and Control, vol. 131 (2009), 021012 -1-8.

[20] M. Krstić, I. Kanellakopoulos and P. Kokotović, Nonlinear and Adaptive Control Design. John Wiley & Sons, Inc., New York (1995).

[21] O.A. Ladyzenskaja, V.A. Solonnikov and N.N. Uralceva, Linear and Quasi-linear Equations of Parabolic Type. American Mathematical Society, Providence, Rhode Island, 1968.

[22] J.L. Lions and E. Magenes, Non-homogeneous Boundary value Problems and Applications, Vol.I. Springer-Verlag, Berlin (1972).

[23] W. Liu and M. Krstić, Backstepping Boundary Control of Burgers’ Equation with Actuator Dynamics. Systems and Control Letters, 41 (4) 2000, 291 - 303.

[24] W. Liu and M. Krstić, Adaptive control of Burgers’ equation with unknown viscosity. International Journal on Adaptive Control and Signal Processing 15(7), 2001, 745-766.

[25] W. Liu, Elementary Feedback Stabilization of the Linear Reaction Diffusion Equation and the Wave Equation, Mathematiques et Applications, Vol. 66, Springer, 2010.

[26] W. Liu, Boundary feedforward and feedback control for the exponential tracking of the unstable high-dimensional wave equation, Journal of Mathematical Analysis and Applications, vol 499, issue 1, July, 2021, https://doi.org/10.1016/j.jmaa.2021.125010

[27] W. Liu, Independence of convergence rate of the wave tracking error on structures of feedforward controllers, Automatica, https://doi.org/10.1016/j.automatica.2020.109264

[28] W. Liu, Feedforward boundary control for the regulation of a passive and diffusive scalar in 2-D unsteady flows, IEEE Transactions on Automatic Control, vol. 65, no. 11, pp. 4882 - 4886, 2020.

[29] W. Liu, A mathematical model for the robust blood glucose tracking. Mathematical Biosciences and Engineering, 16 (2), 2019, 759 - 781.

[30] H. V. Ly, K. D. Mease and E.S. Titi, Distributed and boundary control of the viscous Burgers’ equation. Numer. Funct. Anal. Optim. 18 (1997) 143-188.

[31] Florian Malchow and Oliver Sawodny, Feedforward Control of Inhomogeneous Linear First Order Distributed Parameter Systems. 2011 American Control Conference, San Francisco, CA, USA, 2011, 3597 - 3602.

[32] T. Meurer and M. Zeitz, Feedforward and Feedback Tracking Control of Nonlinear Diffusion-Convection-Reaction Systems Using Summability Methods. Ind. Eng. Chem. Res. 44, 2532 - 2548, 2005.

[33] Thomas Meurer and Andreas Kugi, Trajectory Planning and Feedforward Control Design for the Temperature Distribution in a Cuboid. Proc. Appl. Math. Mech. 6, 825 - 826, (2006)

[34] Thomas Meurer and M. Krstic, Finite-time multi-agent deployment: A nonlinear PDE motion planning approach, Automatica 47 (2011) 2534 - 2542.

[35] Thomas Meurer, Control of Higher–Dimensional PDEs: Flatness and Backstepping Designs. Springer, New York, 2013

[36] Thomas Meurer, Flatness-based motion planning and tracking. Lecture Notes for the Workshop ”New Trends in Control of Distributed Parameter Systems” at the 2016 IEEE CDC, Las Vegas (NV), USA.

[37] V. Natarajan, D. S. Gilliam, and G. Weiss, The State Feedback Regulator Problem for Regular Linear Systems, IEEE Trans. Autom. Control 59, pp. 2708-2722 (2014)

[38] A. Pisano, Y. Orlov, and E. Usai, Tracking control of the uncertain heat and wave equation via power-fractional and sliding-model techniques. SIAM J. Control. Optim. 49, No. 2, 2011, 363 - 382.

[39] Tilman Utz and Andreas Kugi, Flatness-based feedforward control design of a system of parabolic PDEs based on finite difference semi-discretization. Proc. Appl. Math. Mech. 12, 731 – 732 (2012).
[40] M. Wagner, T. Meurer, and A. Kugi. Feedforward control design for a semilinear wave equation, Proc. Appl. Math. Mech., 9, pp. 7-10, 2000.

[41] H.-C. Zhou and G. Weiss: Solving the regulator problem for a 1-D Schrödinger equation via backstepping, IFAC PapersOnLine, 50-1 (2017), 4516-4521.