Hierarchical Bayesian analysis of the velocity power spectrum in subsonic and supersonic turbulence

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We analyse the statistical properties of turbulence using a suite of three-dimensional numerical simulations. We model driven, compressible, isothermal, turbulence with r.m.s. Mach numbers ranging from the subsonic to the highly supersonic regime. We focus on the extreme cases of purely solenoidal (divergence-free) and purely compressive (curl-free) forcing. By employing a hierarchical Bayesian fitting method, we estimate the parameters describing the scaling relationships of the velocity power spectra. The method explicitly treats uncertainties and time-dependent fluctuations through Markov Chain Monte Carlo sampling. We find that the scaling exponents of the decomposed spectra strongly depend on the forcing mechanism, due to the energy transfer between the transverse and longitudinal components. Accordingly, we derive a phenomenological model describing this behaviour. The scaling exponents are in agreement with a Kolmogorov $-5/3$ spectrum in a tiny range of $k \in [12 : 15]$ in the case of the transverse velocity spectrum driven with solenoidal forcing. With compressive forcing and with high Mach number, both longitudinal and transverse spectra show an almost universal behaviour and resemble the Burgers case with a slope of $-2$. We also analyse the spectra in the bottleneck regime and show that the bottleneck bump decreases with increasing r.m.s. Mach number. It is also more evident on smaller scales in the transverse spectrum in comparison with the longitudinal spectrum.

Key words:

1. Introduction

Turbulence is a critical component of gaseous flows on nearly all scales, as it is intimately related to many physical properties of the medium, such as the morphology, mixing characteristics, and thermal structure. Turbulence is known to play a strong if not dominant role in a variety of systems, from terrestrial incompressible flows (e.g. combustion engines, aerodynamics) to highly supersonic compressible flows (e.g. astrophysics). Accurately characterising the statistical properties of turbulence is necessary for developing a comprehensive understanding of fluid dynamics across a range of environments.

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The statistical properties of turbulence, such as the power spectrum, may serve as diagnostics for distinguishing between different models. In the astrophysical context, for instance, these are analytical and numerical models describing accretion disks in protoplanetary systems (see e.g. Meschiari [2012]), the dynamics of the interstellar medium relevant for star formation (see e.g. Mac Low & Klessen [2004], McKee & Ostriker [2007], Federrath & Klessen [2012], and references therein), the formation of star clusters and galaxies (Hopkins [2012]) and galaxy evolution (Iannuzzi & Dolag [2012]). Turbulence theory is also important in the description of the diffuse interstellar medium (Elmegreen & Scalo [2004]) and for galactic or protogalactic dynamos (Brandenburg & Subramanian [2005], Schober et al. [2012]). Despite the impact of compressible turbulence across a range of disciplines, we still lack a comprehensive theoretical understanding of it.

For incompressible turbulence many studies evaluate the energy distribution and the transfer of energy flux across the spatial scales (see e.g. Toschi & Bodenschatz [2009]). The kinetic energy cascades to smaller scales through nonlinear coupling, until viscous effects become important with respect to the advective terms. At this “Kolmogorov” scale $\eta$ the size of turbulent eddies becomes comparable to the mean free path and the kinetic energy is converted into heat (i.e. internal energy). This description has to be extended for compressible turbulence. Scale locality is crucial for the Richardson-Kolmogorov picture of a cascade with constant energy flux through the scales. The non-local, inter-scale processes of compressible turbulence via shock fronts have to be taken into account. Also the complex interplay between a varying density/pressure distribution with different components of the velocity field may be the dominant processes in a supersonic, compressible flow.

In a compressible fluid the kinetic energy is distributed in many kinds of motions across a large range of scales, all of which strongly interact with one another. With the Helmholtz decomposition theorem the velocity field of any flow separates into solenoidal (rotational) modes $v_s$ and compressible (longitudinal) modes $v_c$, defined by $\nabla \cdot v_s = 0$ and $\nabla \times v_c = 0$. The solenoidal modes $v_s$ lead to vortex structures, whereas the compressive modes $v_c$ result in shocks, compressions and rarefactions in the density field. The description of an incompressible turbulent flow by Kolmogorov (1941) uses the assumptions of a constant density and $v_c = 0$. This yields a flow with constant energy flux through the scales, which is dominated by local processes. This description has to be extended for supersonic turbulence. The complex coupling of the two components, $v_s$ and $v_c$ of the velocity field by the advection term and the influence on and of the density field need to be included. Pressure-dilation interaction and viscous dissipation terms cause the transfer of kinetic into internal energy [Kida & Orszag 1991]. If the medium is isothermal, vorticity generation is suppressed, as the baroclinic term $\nabla p \times \nabla (1/\rho)$, which generates vorticity, vanishes. This leads to an asymmetric transfer of energy between the solenoidal and compressive modes. However, even in the absence of the baroclinic term, vorticity is generated by viscous interactions in the presence of density gradients (Mee & Brandenburg [2006], Federrath et al. [2011a]).

In this paper, we focus on analysing the power spectrum of the pure velocity field and measure its scaling behaviour using a hierarchical Bayesian technique. Bayesian inference has the advantage that uncertainties in the data are rigorously and self-consistently treated (e.g. Kelly [2007], Gelman et al. [2004]). Additionally, Bayesian methods are well suited for hierarchical problems, where different datasets, such as individual snapshots, can be analysed simultaneously resulting in parameter estimates of both the individuals as well as for the whole population. In astrophysics, Bayesian methods have been developed for analysing observational data, such as turbulence in the ISM (Shetty et al. [2012]), analysis of dust extinction (Foster et al. [2013]) and spectral energy distributions (Kelly et al. [2012]). Here, we apply a general hierarchical model for the statistical analyses
of turbulence in numerical simulations. We demonstrate that the Bayesian method has important advantages, including accurate parameter estimation, over traditional non-hierarchical $\chi^2$-based methods.

The paper is organised as follows: Section 2 describes the simulations, the calculation of the spectra, the hierarchical Bayesian inference and the test of this method on synthetic data. We discuss the relevant scales and the scaling behaviour of the velocity field in section 3. Section 4 provides an interpretation of the scaling behaviour and describes a simple model considering the energy flux between the two components of the velocity field. In the last section we summaries our findings.

2. Simulations and Methods

2.1. The simulations

To model the dynamics of a turbulent gaseous flow, we solve the equations of hydrodynamics

$$\frac{\partial \rho}{\partial t} + (\mathbf{v} \cdot \nabla) \rho = -\rho \nabla \cdot \mathbf{v}, \quad (2.1)$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{\nabla p}{\rho} + \mathbf{F}, \quad (2.2)$$

$$p = \kappa \rho^{\Gamma}, \quad (2.3)$$

the continuity equation, the Euler equation with a stochastic forcing term $\mathbf{F}$ per unit mass, and the equation of state. Here, $\rho$ denotes the mass density, $\mathbf{v}$ the velocity field, $p$ the pressure, $\kappa$ the polytropic constant, and $\Gamma$ the polytropic index. Since we simulate an isothermal medium throughout this study $\Gamma = 1$, $\kappa = c_s^2$ and therefore $p = \rho c_s^2$, with the sound speed $c_s$.

We employ the FLASH4 code [Fryxell et al. 2000; Dubey et al. 2008] to solve the set of partial differential equations 2.1 – 2.3. We use the HLL5R solver [Waagan et al. 2011] implemented into FLASH on a uniform three-dimensional grid. To ensure that our models are well resolved, we run all simulations with $256^3$, $512^3$, and $1024^3$ grid cells and study the resolution dependence below.

We adjust the amplitude of the forcing, such that we have root mean square (r.m.s.) Mach numbers $\mathcal{M} \cong 0.1, 0.5, 2, 5, 17$ in the stationary state of fully developed turbulence. As one of our goals is to analyse the influence of the forcing scheme, we focus on the extreme cases of purely solenoidal and purely compressive forcing schemes [Kida & Orszag 1990; Federrath et al. 2008, 2010]. The gas initially has uniform density, and evolves for $\approx 10T$ dynamical time scales $T = L/(2c_s\mathcal{M})$, where $L$ is the box size. The physical quantities in the simulations are scale-free so that we define $L = 1$, the mean mass-density $\langle \rho \rangle = 1$ and $c_s = 1$.

Figure 1 (left panel) shows the time evolution of $\mathcal{M}$ for the simulations with $1024^3$ grid cells. The fluid reaches the equilibrium state after about two turbulent crossing times $t \approx 2T$. As the energy flux towards the internal energy is not fully converged by $t = 2T$ in solenoidal driven subsonic cases, we start analysing all simulations at $t \geq 4T$, when the energy dissipation rate is also fully converged, such that we can analyse all simulations consistently and compare them directly with one another.

Table 1 gives an overview of the properties of all simulations with the highest resolution ($1024^3$ grid cells). The second column of Table 1 states the time-averaged Mach number $\overline{\mathcal{M}}$. The solenoidal and compressive Mach numbers are shown in columns three and
Table 1. Time averaged quantities of the simulations with $1024^3$ grid points for both types of forcing. The first column states the name of the simulations. The total, solenoidal $M_{sol.}=\langle v^2_s/c_s^2 \rangle$ and compressive $M_{comp.}=\langle v^2_c/c_s^2 \rangle$ r.m.s. Mach numbers are listed in column 2–4.

| Simulation | $M$  | $M_{sol.}$ | $M_{comp.}$ |
|-----------|-----|-----------|------------|
| $M0.1sol$ | 0.096 | 0.096 | $3.2 \cdot 10^{-4}$ |
| $M0.5sol$ | 0.508 | 0.505 | 0.048 |
| $M2sol$   | 2.5  | 2.26     | 1.01       |
| $M5sol$   | 5.4  | 4.86     | 2.38       |
| $M17sol$  | 17.6 | 15.7     | 7.7        |
| $M0.1comp$| 0.11 | $8.5 \cdot 10^{-3}$ | 0.11 |
| $M0.5comp$| 0.418 | 0.147 | 0.39 |
| $M2comp$  | 2.72 | 1.50     | 2.14       |
| $M5comp$  | 5.73 | 3.22     | 4.38       |
| $M17comp$ | 16.6 | 9.55     | 12.2       |

Figure 1. Left panel: R.m.s. Mach number as a function of the dynamical time scale, calculated by averaging over all grid cells. Shown are the results of the simulations with $1024^3$ grid cells for solenoidal (solid lines) and compressive forcing (dotted lines). Right panel: Real scaling exponent of the synthetic dataset against values of the scaling parameter measured using the Bayesian model (green-circles), a normal linear regression (LR NOT-averaged) applied to each time snapshot (grey-triangles), and on the data time-averaged in linear space (brown-red-diamonds) and in log space (blue crosses). The data points of the LR LOG-averaged and NOT-averaged are slightly shifted to the left and right, respectively, for a better overview.

2.2. The Forcing algorithm

A detailed description of the forcing algorithm is provided in Schmidt et al. (2009), Federrath et al. (2010), and Konstandin et al. (2012a). We briefly describe its main properties here and refer the reader to those publications and references therein for a detailed explanation.

The random forcing $\mathbf{F}$ is derived from a stochastic Ornstein-Uhlenbeck process in Fourier space ($k$-space). It has finite autocorrelation time scale $T_{ac}$, so that it is smooth in space and time. The forcing amplitude is a three-dimensional paraboloid function of...
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$k$. The forcing only occurs on the large (integral) scales $1 < |k| < 3$, peaking at $k = 2$, which corresponds to half of the box size $L/2$, as we measure $k$ in units of $2\pi/L$. The amplitude is exactly zero at the $k = 1$ and $3$ mode. The autocorrelation timescale of the forcing algorithm is set equal to the dynamical timescale of each simulation $T_{ac} = T(M)$.

We use the projection tensor in Fourier space to get a purely solenoidal (divergence-free, $\nabla \cdot F = 0$) or a purely compressive (curl-free, $\nabla \times F = 0$) vector field. Therefore, in this statistical study we are only considering the extreme cases of the forcing mechanism.

Federrath et al. (2010) additionally studied intermediate cases with mixed forcing and Girichidis et al. (2012) studied initial velocity fields with solenoidal, mixed and compressive turbulence. In natural flows there are no obvious criteria for identifying the ratio of solenoidal and compressive modes of the forcing field. The compressive component likely dominates a shock-driven turbulent medium. The rotational component likely dominates when turbulence is generated in mixing layers or caused by magnetic phenomena. A discussion of the mode mixture of turbulence drives in astrophysical systems is provided in Federrath & Klessen (2012).

2.3. The Fourier spectra

The Fourier spectrum of the velocity field is defined as

$$P(k)dk = \int 4\pi k^2 \hat{v}(k) \cdot \hat{v}^*(k) dk,$$

(2.4)

where $\hat{v}$ is the Fourier-transformed velocity field. We denote the Fourier spectra of the decomposed velocity field as $P_\perp$ and $P_\parallel$ for the transverse and longitudinal component of the velocity field, respectively. With this definition the integral over the whole $k$-range corresponds to the square of the Mach number

$$\mathcal{M}^2 = \int_0^\infty P(k)dk$$

(2.5)

and the zeroth mode contains the averaged velocity field for velocity components $i \in x, y, z$

$$\langle v_i \rangle = P_i(0) = \frac{1}{L^3} \int_0^\infty v_i(r) d^3r,$$

(2.6)

which follows from Parseval’s theorem.

The Fourier spectrum can be used to describe the transition between Kolmogorov and Burgers turbulence. Kolmogorov turbulence occurs in incompressible flows with constant density and vanishing dilatational modes of the velocity field $\nabla \cdot \mathbf{v} = 0$. Alternatively, for Burgers turbulence shocks, strong gradients in the density field, discontinuities, and large dilatational modes in the velocity field dominate in the flow. Kolmogorov theory predicts a Fourier spectrum of the velocity field $P(k)$ following a power law, $P(k) \propto k^{-5/3}$, with scaling exponent $-5/3$ (Kolmogorov 1941). In the Burgers case we expect a scaling exponent of $-2$ (Burgers 1948).

In practise, the scaling exponent is often measured by linear regression in a log-log plot of the time-averaged power spectrum, or on a $k^{5/3}$ or $k^2$ compensated spectrum (e.g. Kaneda et al. 2003; Kritsuk et al. 2007; Lemaster & Stone 2009; Federrath et al. 2010).

We describe in the following four common assumptions/methods that lead either to inaccurate scaling parameter estimates, or to difficulties in interpreting the results.

First, in a doubly logarithmic plot it is often difficult to see if the best-fit regression line accurately reproduces the data. Many functions may appear to follow a power law in a doubly logarithmic plot. For example, if the scaling exponent varies slightly with $k$,
a simple linear regression in log-space often does not reveal such fluctuations. Hence, a qualitative validation if the fit can reproduce the measured data is needed.

Second, as the \( k \) extent of the inertial range is not known exactly it has to be estimated. To demonstrate the influence of the chosen fitting range, we perform several ‘normal’ linear-regression fits in different ranges of the time-averaged total velocity spectrum with \( \mathcal{M} = 2 \) and solenoidal forcing in log-log space. The results are summarised in Table 2.

The table shows that the estimated slopes strongly depend on the chosen extent in \( k \), because the power-spectrum slope of the data is not necessarily constant in \( k \). It indicates a Burgers spectrum at low \( k \) and roughly a Kolmogorov spectrum at high \( k \). Note also the measured error is small and does not describe the intrinsic uncertainty of the data.

In this case, an unbiased estimate of the inertial range is very difficult to obtain.

Third, a key assumption in a \( \chi^2 \) linear regression is that the uncertainties are independently and normally distributed. The common practise of fitting in log space implicitly assumes that the uncertainties are normally distributed in log space. Usually the power spectra are averaged to diminish their time-dependence and to reduce the uncertainties and the scatter. However, averaging data also assumes that the uncertainties of the data are Gaussian or at least symmetrically distributed. Hence, both methods are based on the assumption of a Gaussian/symmetric scatter, but for the normal space and also for the log space. Therefore, performing the averaging in normal space and the \( \chi^2 \) fitting in log space is not consistent and violates their intrinsic assumptions. To obtain reasonable measurements of the slope and the uncertainty estimates with linear regression thus requires averaging log \( P \) instead of \( P \), as done in e.g., Federrath (2013).

Finally, when analysing just averaged data, information about the intrinsic variations in the data is lost. In general, information contained in the data may be discarded when averaging data. Hierarchical models exploit all the information in the data, simultaneously estimating model parameters on multiple levels. In the next section we introduce a hierarchical Bayesian method to account for these issues for analysing the turbulent power spectra of numerical simulations.

Table 2. The measured slope in different fit-ranges performed with a normal linear regression on the time-averaged spectrum of the \( \mathcal{M} = 2 \) solenoidal forced simulation.

| fit range \( k \) | Slope | Error |
|------------------|-------|-------|
| [4 : 12]         | −2.28 | 0.06  |
| [6 : 14]         | −2.10 | 0.09  |
| [8 : 16]         | −1.79 | 0.04  |

2.4. Hierarchical Bayesian inference

To mitigate the effects mentioned above, we develop a hierarchical Bayesian method. Hierarchical modelling provides significant advantages when the dataset is naturally structured into two or more groups. For instance, the hydrodynamic simulations provide spatial information of all relevant quantities, such as the fluid densities and velocities, at different snapshots in time. The data is therefore structured into temporal groups. For our analysis, our goal is to estimate the power spectrum. We can assess the variation in the spectrum by analysing the datasets on the individual time-level, as well as estimate the parameters of the mean spectrum. This process is naturally hierarchical.

† Hierarchical modelling is sometimes referred to as “multi-level” or “random-effects” modelling (Gelman & Hill [2007]).
Bayesian methods are well suited for estimating model parameters on multiple levels in a hierarchical model.

With Bayes' theorem the probability $P$ of a set of parameters $\theta$ given the observed data $D$ can be calculated

$$P(\theta|D) \propto P(D|\theta)P(\theta),$$

(2.7)

where $P(D|\theta)$ is the likelihood, which is the probability of the set of data $D$ given the set of parameters $\theta$. $P(\theta)$ is called the prior and is the probability of the set of parameters. We will define the set of parameters $\theta$ in detail below. The outcome of Bayesian inference is the probability of the model parameters $\theta$ given the data $D$ and is called posterior distribution. We perform a Markov Chain Monte Carlo method sampling the prior probability distributions for all parameters computing the posterior distributions. It uses the likelihood to find the way through parameter space to calculate the most probable values for the set of parameters yielding the observed data. The result of the Bayesian inference is the joint probability posterior distribution of the regression parameters. This method has the advantage that variations and uncertainties of measured data can be treated self-consistently. Therefore the errors in each measured quantity are also assumed to be drawn from some a priori defined distributions described by one of the parameters. For a detailed description of the Bayesian inference method, we refer the reader to the standard textbooks about statistical methods (Gelman et al. 2004; Kruschke 2011; Wakefield 2013).

In the following we will describe the construction of the Bayesian model, where we use the standard statistical notation. We describe how quantities are conditionally related and use $y/x$ for a variable $y$ given a value of $x$. Characterising values and their distribution, like $y|\mu, \sigma^2 \sim \mathcal{N}(\mu, \sigma^2)$ denotes that $y$ is drawn from a normal distribution $\mathcal{N}$, given the mean value $\mu$ and the variance $\sigma^2$. We also use gamma functions $\Gamma(s, r)$ for the inverse of the variance with $s$ and $r$ being the shape and rate parameters, respectively. The mean value of a dataset $x$ is denoted by $\bar{x}$. Before performing the fit we standardise the data, i.e. we transform it with

$$\tilde{y} \equiv \frac{y - \overline{y}}{\sigma_y}, \quad \tilde{x} \equiv \frac{x - \overline{x}}{\sigma_x},$$

(2.8)

to decouple the measurement of the intercept and the slope.

In a Bayesian model all quantities are drawn from some prior defined distributions. Therefore, we assume that the velocity power spectrum $P(k, t)$ follows a power law, i.e. a linear function in log-log space

$$\log P(k_i, t_j) = A(t_j) + \log k_i \ast \zeta(t_j) + \delta_s(k_i, t_j),$$

(2.9)

for 61 snapshots in time. This equation describes the relationship between the parameters on the individual level in the hierarchy. The intercept $A(t_j)$, the power-law index $\zeta(t_j)$ and the scatter $\delta_s(k_i, t_j)$ of each individual time snapshot $t_j$ must be drawn from the prior conditional probability distributions

$$A(t_j)|\overline{A}, \overline{\sigma_A^2} \sim \mathcal{N}\left(\overline{A}, \overline{\sigma_A^2}\right),$$

(2.10)

$$\zeta(t_j)|\overline{\zeta}, \overline{\sigma_\zeta}^2 \sim \mathcal{N}\left(\overline{\zeta}, \overline{\sigma_\zeta}^2\right),$$

(2.11)

$$\delta_s(k_i, t_j)|\overline{\sigma_s^2(t_j)} \sim \mathcal{N}\left(0, \overline{\sigma_s^2(t_j)}\right),$$

(2.12)

$$\frac{1}{\overline{\sigma_s^2(t_j)}}|\overline{\sigma}, \overline{\tau} \sim \Gamma(\overline{\sigma}, \overline{\tau}).$$

(2.13)

Our model uses normal distributions for the slope, intercept and the scatter and a gamma
distribution for the the inverse of the variance of the scatter term. The inverse of the variance is also called precision. We chose a gamma distribution for the precision of the scatter to have a really broad prior, which is positive definite. Using a gamma distribution for the prior of the precision is commonly done (e.g. Gelman et al. 2004; Kruschke 2011; Shetty et al. 2013).

Those quantities that depend on $t_j$ refer to individual time frames. For instance, $\zeta(t_j)$ is the slope of the time snapshot $t_j$ whereas $\zeta$ refers to the group slope of the whole dataset. The fitting results of each relationship above depend on quantities from the higher group level of the hierarchy, i.e. describe the time-averaged behaviour of the power spectrum. The prior assumptions for this final level, which are called “hyper priors”, are

$$\mathcal{A} \sim \mathcal{N}(0, 100) ,$$

$$\zeta \sim \mathcal{N}(−2, 100) ,$$

$$1/\sigma^2_\mathcal{A} \sim \Gamma(1, 0.1) ,$$

$$1/\sigma^2_\zeta \sim \Gamma(1, 0.1) ,$$

$$\pi m, d = m^2/d^2 ,$$

$$\pi m, d = m/d^2 ,$$

$$m \sim \Gamma(0.01, 0.1) ,$$

$$d \sim \Gamma(0.1, 0.1) .$$

We construct broad “hyper priors” in our model, as we would like to rely on the data to estimate the group parameters.

With this Bayesian model we avoid some of the difficulties mentioned in the last section. Variations of the scaling exponents with time yield a larger variance of the group slope. Fluctuations of the scaling exponents with $k$ increase the group scatter $\sigma^2_\Delta(t_j)$.

Variations and uncertainties of the measured data are also treated self-consistently. Both individual and also the global parameters are estimated simultaneously, avoiding the need for averaging. However, we still have to define a fitting range, which introduces the largest uncertainty. Therefore, we test in the next section our Bayesian model on synthetic data, where we fit just three data points to obtain the “local” slope of the power spectrum.

2.4.1. Test with synthetic data

We verify the Bayesian model with synthetic data and compare with some methods using normal linear regression (LR). To test the Bayesian model measuring the local slope of the power spectrum as a function of $k$, we apply the extreme case fitting just three data points. We create a synthetic dataset with 61 realisations according to the Bayesian model (equations 2.9–2.13), where the group intercepts, the slopes and the scatter-standard deviations (SD) follow distributions with mean values of $(1, \zeta, 0.3)$ and SDs of 30%, so $(0.3, 0.3\zeta, 0.1)$.

Figure 1 (right panel) shows the measured group slope as a function of the real group slope.

The linear regression (LR, NOT-averaged) performed on each individual time realisation results in a histogram with 61 measured slopes. This mimics a hierarchical modelling with a normal linear regression. Additionally, we perform the LR on the data

† The mean $\mu$ and variance $\sigma$ of a gamma distribution with shape $s$ and rate $r$ is defined as, $\mu = s/r$ and $\sigma = s/r^2$, respectively.
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time-averaged in normal space and in log space. The error bars in figure (right panel) correspond to the 2σ uncertainty of the resulting group slope for the Bayesian method. The error bars for the LR (not averaged) measure the 2σ−variation in time and those of the averaged LR correspond to the output of the normal linear regression.

The hierarchical model rigorously accounts for a number of uncertainties. The posterior contains the resulting fit parameters, for both the group and the individuals, which are PDFs that provide well defined error estimates. For example, the width of the PDF, or highest density interval (HDI), of the group slope and intercepts yields the range in plausible parameters, considering the measurement uncertainty or insufficient statistics. The standard deviation of the group slope values indicates the variation between individuals. The error bars in figure (right panel) and all subsequent figures show the 2σ-HDI of the measured quantity, (and therefore are not representative of the variation in time).

Drawing 61 data points from a normal distribution yields statistical uncertainties for the slope \( \sigma_\zeta / \sqrt{N} = 0.04 \zeta \) and the amplitude \( \sigma_A / \sqrt{N} = 0.04 \). The scatter term \( \delta_s(k_i, t_j) \), which vanishes for an infinite number of time realisations, still causes a mean influence on each data point of \( \delta_s / \sqrt{N} = 0.04 \) in each bin of the resulting spectra. All these fluctuations can add up for measuring the scaling exponent with an inaccurate method.

The linear regression (NOT-averaged) shows large error bars, indicating that this method does not provide slope estimates with high precision, whereas the Bayesian model (equation 2.9) fits the synthetic data and 95% of the true values lie within the 2σ intervals. The true value is just for one point outside the error bars, which corresponds to 5% of the data points and is in agreement with the definition of 2σ. The LR applied data that are averaged in linear space shows systematic deviations from the real slope for steeper spectra. In contrast, the LR averaged in log space has a comparable accuracy to the Bayesian model, but without a proper error estimate. The error estimates are rather small, sometimes completely missing true slope.

Figure (right panel) indicates that the choice of the regression method can have a major influence on the results and should be chosen carefully.

3. Results: The slopes of the velocity power spectra

We can now apply the hierarchical Bayesian fitting method to identify and investigate the different regimes of the power spectra from the suite of numerical simulations presented in Section 2. We begin the discussion by reviewing some properties of the power spectrum of isothermal, compressible turbulence. Figure illustrates the different ranges and summarises the corresponding scales. We focus our discussion on the simulations with 1024^3 grid points, which yield power down to scales reaching the Nyquist value \( k_N \ast \frac{L}{2\pi} = \frac{N}{2} = 512 \) for a simulation with resolution \( N^3 \). Kitsionas et al. (2009), Federrath et al. (2010), and Federrath et al. (2011) show that at scales less than \( \approx 30 \) grid cells, the statistical properties are strongly influenced by numerical effects (range I), and therefore poorly resolved. Therefore, for the 1024^3 simulations the largest resolved wave number is \( k_{Num} = 34 \). Towards lower wave numbers, the first physically interpretable effect outside the numerical dissipation range is the bottleneck (BN) effect (section 3.1), ending at the lower limit of the inertial range \( k_{BN} \) (range II). The inertial range is expected to yield a Kolmogorov scaling behaviour (range III) for subsonic flows, or Burgers scaling (range IV) for gas with supersonic motions, or possibly both separated by the sonic scale \( k_s \) (section 3.2). Range V starts at \( k_F \) (section 4.2), where the forcing has an indirect influence on the statistics of the large scale flow and therefore also on the power spectrum. The smallest wave number of this range corresponds to the driving scale \( k_D \), where the forcing has a direct influence and which is \( k = 3 \) for our numerical setups.
Figure 2. Schematic view of the spectrum for isothermal, compressible turbulence. The different scaling ranges and their scales are discussed in section 3.

Figure 3. The velocity power spectra for different Mach numbers $\mathcal{M}$, resolutions, and solenoidal (upper panels) and compressive forcing (bottom panels). Shown are the total (left), the transverse (middle) and longitudinal spectra (right). Some spectra are multiplied with a factor $10^x$ as indicated in the figure.

We will continuously refer to the simple illustration of the various regimes in figure 2 as we analyse the power spectrum of the numerical simulations.

Figure 3 shows the velocity power spectrum for solenoidal (upper panels) and compressive forcing (bottom panels). The left, middle, and right panels show the total, decomposed transverse, and decomposed longitudinal modes of the velocity spectrum, respectively. Some spectra are multiplied by a scaling factor of $10^x$, such that they can be easily compared with the others (the factors are shown in the figure).

Figure 3 indicates that the effect of the resolution on the spectra decreases with increasing $\mathcal{M}$. The longitudinal spectra seem to be better converged with resolution than the transverse ones. An exception is the subsonic simulation with solenoidal forcing, where the longitudinal spectrum is not converged at all resolutions considered. The $\mathcal{M}0.1$ sol
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\[ P_\perp(k) \propto k^{5/3} \int_0^\infty P_\perp(k') dk' \]

\[ P_\parallel(k) \propto k^{5/3} \int_0^\infty P_\parallel(k') dk' \]

Figure 4. The normalised and compensated transverse (upper panels) and longitudinal (bottom panels) velocity power spectra for different Mach numbers \( M \) for solenoidal (left panels) and compressive forcing (right panels).

Simulation shows a bump in the longitudinal spectrum at a scale that corresponds to 8 grid cells for all resolutions. The resolution has also a strong influence on the transverse spectrum of the \( M = 0.1 \text{comp} \) simulation. Comparing the amount of energy in the subsonic regime the different components of the velocity field contain, show that they are nearly decoupled for \( M < 1 \). Most of the energy in either component is transferred, without passing through the other component, to internal energy (see also Kida & Orszag 1990).

In the subsonic regime, the dependence of resolution is due to an interplay of two factors. First, the components of the velocity field are nearly decoupled. Second, the viscous dissipation is weaker for these simulations in comparison with the supersonic case. The presence of shocks increases the energy dissipation yielding a non-local energy transfer. Hence, numerical errors, with such as spurious pressure and velocity waves (Dellacherie 2010) or slowly moving shocks (Jin 1996) are not sufficiently damped away. These numerical effects may affect the statistical properties in the component of the velocity field containing a small fraction of the energy. These issues will be further investigated in future work.

3.1. The bottleneck effect

We now turn our attention to the bottleneck effect, which delimits the inertial range from the smallest scales. Accordingly, if not properly identified it can lead to inaccurate interpretations about the slope of the spectra in the inertial range. With finite Reynolds number in the “near dissipation” range around \( k\eta = O(1) \), the velocity power spectrum does not decrease monotonically with increasing wave number. Rather, there is a “bump” of energy, known as the bottleneck effect (BNE). Many studies measured the influence of the BNE in simulations and experiments (see e.g. Saddoughi & Veeravalli 1994, Kaneda et al. 2003, Tsui 2004, Schmidt et al. 2006) and various analytic models describe its behaviour (see e.g. Qian 1984, Yakhot & Zakharov 1993, Falkovich 1994, Bershadskii 2008, Dobler et al. 2003, Kurien et al. 2004). It is well known for incompressible turbulence that the position of the peak of the BNE and its width are independent
of the Reynolds number (see e.g. She & Jackson 1993, Donzis & Sreenivasan 2010) and its amplitude \( \mathcal{P} \propto 1/Re^\gamma \) decreases with the Reynolds number with \( \gamma \approx 0.04 \) (see e.g. Meyers & Meneveau 2008, Donzis & Sreenivasan 2010).

Figure 4 shows normalised transverse and longitudinal power spectra compensated with \( k^{5/3} \) for solenoidal and compressive forcing. The normalisation is calculated such that the integrals of the spectra are unity before compensating. The transverse spectra show a clear bump of the BNE at scales \( k \in [25,150] \) for both types of forcing. With increasing \( M \) the influence of the BNE decreases, until the bump vanishes for the \( M=17 \) simulations. The longitudinal spectra do not have a BNE for any Mach number and both types of forcing, which is in agreement with the studies of Saddoughi & Veeravalli (1994). They found a small influence of the BNE at much larger scales in the longitudinal spectra in comparison with the transverse spectra in the subsonic case.

Figure 4 also shows that the range with a Kolmogorov scaling behaviour is small even with a resolution of 1024\(^3\) for all simulations.

To measure which scales are influenced by the BNE we use the method of measuring the Kolmogorov constant \( C_K \) proposed by Donzis & Sreenivasan (2010). They find that the position of the peak of the BNE \( k_{\text{peak}} \eta \approx 0.13 \) and the width of the BNE bump are independent of the Reynolds number, if \( Re_\lambda > 240 \). They use the constant width of the BNE to estimate the scale, where the BNE has no influence anymore \( k_d \eta \approx 0.029 \), such that \( k_d \) lies in the inertial range. We use \( C_K = 1.58 \) (Donzis & Sreenivasan 2010) to normalise the amplitude of the spectra and measure the scale \( k_d \) by taking the minimum in the compensated spectra. This method ensures that the spectra are steeper than the Kolmogorov scaling on scales \( k > k_d \) and have the Kolmogorov scaling behaviour for \( k < k_d \). We use the estimates of Donzis & Sreenivasan (2010) for the position of the peak \( k_{\text{peak}} \eta \approx 0.13 \) to calculate \( \eta \) of our simulations and check our normalisation with \( k_d \eta \approx 0.029 \).

Assuming that \( \mathcal{P}(k_d) = C_K \epsilon^{2/3} k_d^{-5/3} \), we can normalise the spectra and measure the mean energy dissipation rate. Figure 4 shows the normalised and compensated transverse spectra for the simulations, where we get the same \( \eta \) for both estimates, so that we can be sure that we measure the BNE. We get \( k_d = 12, 12, 14 \) and \( k_{\text{peak}} = 54, 61, 44 \) for the \( M=0.1, M=0.5, M=2 \) simulations and 1024\(^3\) grid cells, respectively. The simulations with low \( M \) and compressive forcing do not have enough energy in transverse modes on the largest scales, so that we cannot find a minimum for measuring \( k_d \) (see figure 4).
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Both the calculated values of $\eta$ for the $M_{2\text{comp}}$ simulation deviate from each other. This effect gets stronger with increasing $\mathcal{M}$, so that this normalisation does not work for the compressive forcing (blue line in figure 5).

Hence, the universal behaviour of the BNE proposed by [Donzis & Sreenivasan 2011] appropriately describes the subsonic regime for the simulations with solenoidal forcing and breaks down for supersonic flows with $\mathcal{M} > 2$. With $\mathcal{M} < 1$ and compressive forcing the coupling of the different components of the velocity is too weak, so that the spectra do not have enough energy in transverse modes on the largest scales. In the supersonic regime the processes yielding the BNE change with increasing $\mathcal{M}$ and the non-local energy transfer through the scales is fast enough to prevent an accumulation of energy on the smallest scales.

Before closing this section, we mention an important caveat about the BNE. Our hydrodynamic simulations rely on numerical dissipation on the smallest scales, with the precise BNE scale likely depending on the resolution, the solver [Porter & Woodward 1994] and other quantities of the simulation setup. As all the power spectra analysed in this work emerge from numerical simulations with the same setup, we can compare the results between the subsonic and supersonic regimes, for which we do indeed find a clear difference. But, due to numerical dissipation we should not over-interpret the results with respect to the specific range of the BNE.

3.2. Scaling behaviour of the decomposed power spectra

For analysing the slope of the power spectra, we begin by estimating the local slope and compare the fit with the data directly. Figure 6 shows the transverse spectrum of the $M_{0.5\text{sol}}$ simulation and two functions with the local slope describing the power spectrum in the ranges II and V. The Bayesian method provides models that can reproduce the spectral behaviour.

Figure 6 shows the local scaling exponents of the transverse (top panels) and longitudinal (bottom panels) velocity power spectrum for solenoidal (left) and compressive (right) forcing at different $\mathcal{M}$. According to our discussion in section 3 we distinguish the four different ranges in the spectrum (we do not discuss range I and VI, which are dominated by numerical effects and the forcing, respectively).

Range II is dominated by the BNE, which has an influence on the scaling exponents of the transverse spectra in the range $18 \approx k_{\text{BN}} \lesssim k$. In the range (III) $12 \lesssim k \lesssim 18$, the transverse spectra with solenoidal forcing indicate a universal behaviour independent of $\mathcal{M}$. Changes of some spectra from values close to the Burgers one with a shallower spectrum indicate the sonic scale $k_s$, which separates the ranges III and IV.

For $k \lesssim 7$ we find the transition range, where the statistical properties are still strongly influenced by the forcing routine. The values of the above mentioned scales show in some cases a dependence on $\mathcal{M}$, the mode of the forcing, or the mode of the spectra. They should not be interpreted as a fixed measurement, as we will discuss further. We simply indicate the regimes in figure 6 to guide the reader in the analysis that follows.

The sonic scale $k_s$ mentioned above is defined as the scale, where the local slopes start deviating from the Burgers $-2$ scaling behaviour. This differs from the definition of the sonic scale using the integral of the total velocity power spectrum (e.g. Schmidt et al. 2003, Federrath et al. 2010) or using the r.m.s. Mach number $k_s \approx k_{\text{inj}} \mathcal{M}^2$ (e.g. Federrath 2013). The latter estimate results in sonic scales of $k_s \approx 13, 61, 610$ for the $\mathcal{M} \approx 2.5, 5.5, 17$ simulations. Hence, with this estimate of the sonic scale only the simulations with $\mathcal{M} = 2$ should yield spectra with the sonic scale. Our measurements of the sonic scale deviate strongly from the theoretical estimates. The transversal spectra with solenoidal forcing lead to sonic scales that are independent of $\mathcal{M}$ and strongly depend on
the resolution. Hence, interpreting the measurements as the scale separating the shock dominated and the incompressible case is questionable. As we do not want to introduce a new scale, also defined as the transition scale between Burgers and Kolmogorov scaling behaviour, we will call it $k_s$ also in the following.

We will now describe the behaviour of the scaling exponents and interpret the results in the next section.

In the transverse spectrum with solenoidal forcing (top left panel) the influence of the Mach number on the BNE is clearly visible on scales $k \gtrsim 18$. In the subsonic case the scaling exponents stay roughly constant $\approx -1.1$ in the range $25 \leq k \leq 35$, which is in agreement with the theories of Yakhot & Zakharov (1993) and Bershads'ki (2008), who predict a constant scaling behaviour for the BNE with a scaling exponent of $-1$. The scaling exponents in the range of the BNE decrease with increasing $M$. It reaches the Burgers scaling exponent of $-2$ for the largest $M$. This indicates that the non-local energy transfer becomes stronger with $M$. This reduces the BNE and prevents the accumulation of energy on the smallest scales. The transverse scaling exponents of the simulations with solenoidal forcing are scattering around the Kolmogorov $-5/3$ value in the small range $12 \leq k \leq 15$ independent of $M$. The simulation with $M_5\text{sol}$ is an exception and has larger scaling exponents $\approx -1.55$ in this range. On larger scales $k \lesssim 12$ the scaling exponents are smaller than the Kolmogorov scaling ($-1.9 \leq \zeta \leq -1.7$). At scales $k \lesssim 7$ the spectral slopes have large scatter $\zeta \approx -1.8$. The transient regime does show a dependence of the scaling exponents on $M$, but without a clear trend.

The transverse spectra of the simulations with compressive forcing (top right panel)
The scaling behaviour of velocity power spectrum in a turbulent flow also show an influence of $\mathcal{M}$ on the scaling exponents in the BNE regime $k \gtrsim 18$. The scaling exponents get smaller with increasing $\mathcal{M}$. Also the subsonic simulation $\mathcal{M}_{0.5\text{comp}}$ shows a roughly constant behaviour of the scaling exponents in the transverse spectrum, which scatter around $-1.1$. On scales where we expect the inertial range this set of simulations does not show a universal behaviour with constant scaling exponent. The scaling exponents decrease with increasing $\mathcal{M}$, until the $\mathcal{M}_{17\text{comp}}$ simulation reaches the Burgers value with scaling exponent of $-2$. We note that this simulation has a constant scaling exponent over the very large range of $6 \leq k \leq 35$, indicating an efficient coupling between the two components of the flow. The $\mathcal{M}_{2\text{comp}}$ and $\mathcal{M}_{5\text{comp}}$ simulations show scaling exponents on the largest scales close to the Burgers value. As the energy is injected on the largest scales the transfer of longitudinal energy towards transverse energy happens in a highly supersonic medium and is efficient. As the supply of energy is smaller for the $\mathcal{M}_{2\text{comp}}$ simulation it has a shallower spectrum already at smaller scales $k_s \approx 8$ than the $\mathcal{M}_{5\text{comp}}$ simulation, where $k_s \approx 11$. The subsonic simulations are shallow in the regime with scaling exponents smaller than the Kolmogorov value $\zeta = -1.5$.

The scaling exponents of both longitudinal spectra (bottom panels) stay roughly constant in the range $k \gtrsim 12$, as they are not influenced by the BNE. They do, however, depend on the Mach number.

The longitudinal spectra of the simulation with solenoidal forcing (bottom left) can be split up in two regimes. On large scales they have a transient range, where the scaling exponents change their behaviour strongly with $\mathcal{M}$. The $\mathcal{M}_{2\text{sol}}$ simulation has a constant scaling exponent scattered about $\approx -1.7$ over the whole range (the Kolmogorov $-5/3$ scaling always lies inside the error bars). The $\mathcal{M}_{5\text{sol}}$ and $\mathcal{M}_{17\text{sol}}$ lead to scaling exponents that increase continuously with $k$ until they saturate. This transition from a steep Burgers spectrum to a shallower one occurs in the low $\mathcal{M}$ simulations on larger scales ($k_s = 5$ and $k_s = 11$ for the $\mathcal{M}_{5\text{sol}}$ and $\mathcal{M}_{17\text{sol}}$ cases, respectively). On smaller scales the longitudinal spectra of the simulation with solenoidal forcing yield scaling exponents that stay roughly constant, but with a large scatter. Depending on $\mathcal{M}$ they scatter around values between $-1.7$ and $-1.9$ with smaller values for larger $\mathcal{M}$. The $\mathcal{M}_{0.5\text{sol}}$ simulation shows a completely different behaviour. The spectrum is shallow on the largest scales ($\zeta \approx -1.6$) and quickly becomes steeper towards small scales ($\zeta \approx -2.2$ at $k = 13$). As the $\mathcal{M} = 0.1$ is strongly resolution dependent, we do not interpret the results of the hierarchical fit.

The longitudinal spectra of the simulation with compressive forcing (bottom right panel) converge towards the Burgers scaling of $-2$ for increasing $\mathcal{M}$. Especially the simulations in the subsonic regime have smaller scaling exponents on both the large scales with the transient regime of the forcing and on smaller scales, where the scaling exponents stay roughly constant for the $\mathcal{M}_{0.5\text{comp}}$ simulation. The transient range vanishes with increasing $\mathcal{M}$ so that the $\mathcal{M}_{17\text{comp}}$ simulation has a constant scaling exponent over the whole range.

4. Discussion: Understanding the Turbulent Power Spectra

4.1. Interpretation of the results

Clearly, there are differences in the spectra of the different simulations. One of our goals is to understand the resulting scaling exponents.

The energy transfer between the longitudinal and transverse spectra plays a major role. The effects of transferring energy between the two mode families of the velocity field effectively correspond to additional energy sources or sinks for the spectrum under
consideration. For example, for energy transfer from the longitudinal to the transverse modes, the longitudinal spectrum is a source and the sink is the transverse spectrum (and vice versa). Introducing an additional energy transfer in the Kolmogorov picture results in that source spectra becoming steeper and the sink spectra becoming shallower in this range. As the coupling between modes is more efficient at higher velocities, the transfer of energy also has a $k$-dependence, where most energy is transferred on the large scales. Similarly, the transfer of energy is more efficient for flows with larger $M$. We provide an analytical description of the transfer processes in the next section.

Before we discuss the local scaling exponents as measured in Section 3.2, let us briefly return to Table 1. The relative energy distribution in the different modes remain roughly constant in the supersonic regime, whereas it changes dramatically in the subsonic regime (see also Figure 3, bottom panel, in [Federrath et al. 2011]). The discrepancy is due to the variable efficiency of the transfer processes. In the subsonic regime the modes are nearly decoupled, so that the energy remains in the injection mode. Table 1 also shows that energy transfer from compressive towards solenoidal modes is always larger than that from solenoidal towards compressive modes (for the same $M$). Note also that the energy injection yielding the same $M$ is always larger for the compressive driven case than for the solenoidal one. This occurs because the coupling with the internal energy is more efficient for the longitudinal mode.

The subsonic simulations show a distinct behaviour from the supersonic ones for all cases. The largest differences are in the sink spectra, where they have the largest influence of the resolution (see figure 3). Hence, we are not interpreting these results and focus on the supersonic cases.

Figure 6 shows that the sink spectra have a stronger $M$ dependence than the source spectra. The transfer processes and therefore the energy flux varies strongly with $M$. The scale $k_s$ gets an additional dependence on $M$ from the energy transfer and is more clearly identifiable in the sink spectra. On the large scale there is a Burgers scaling, followed by a transition regime at the sonic scale $k_s$. As the longitudinal spectra are not affected by the BNE the spectra saturate at smaller scales. It is interesting that the $M5sol$ and $M2sol$ simulations are in agreement with a $-5/3$ scaling over this extended range. Comparing the longitudinal spectra with solenoidal forcing with the transverse spectra with compressive forcing shows that the sink spectra are similar in the supersonic cases. However, the transverse spectra with compressive forcing have larger $k_s$. This occurs because of the asymmetric energy distribution (see Table 1). The compressively driven simulations always have more energy in the transverse mode compared to the energy in the longitudinal mode of the corresponding solenoidal simulation.

In the Kolmogorov theory the inertial range is defined as the range where the statistical properties of the flow are independent of the driving mechanism. Therefore, it is noteworthy that the statistical properties of the sink spectra seem to be influenced by the scaling behaviour of the source spectra in the high Mach number limit. The transverse spectra of the $M17sol/comp$ simulations show obvious differences in their scaling behaviour. On the other hand the transverse and longitudinal spectra and the same kind of driving have nearly the same scaling behaviour. One open question is why the transverse spectrum driven with compressive forcing does not result in Kolmogorov-like scaling exponents, whereas the solenoidal driven simulations do. The only difference between the two cases is that the former one receives energy by coupling processes from the longitudinal modes and the latter one obtains energy directly from the forcing. Table 1 shows that the $M5sol$ and $M2comp$ simulations have nearly the same compressive Mach numbers. Yet, the simulations show a significant difference in the scaling behaviour of the longitudinal spectrum. The full answer to this problem requires detailed studies.
of the scale-dependent coupling between the two components. Our results indicate that a theory describing the scaling behaviour of a compressible turbulent fluid cannot be described solely by the parameter $M$ (and $M_{\text{comp}}$ or $M_{\text{sol}}$). Rather, one or more parameters quantifying the flux of energy between the two components of the flow are likely required. We explore a simple model considering the energy flux to explain the spectra we measure in Section 3.

4.2. A model for the energy transfer

In this section we construct a simple model describing the transfer of energy between the mode families of the velocity field. We start with a description of the model and its assumptions. We then compare it with the results from the simulations.

Figure 3 shows that both spectra have their peaks at large scales although the forcing algorithm injects the energy only in one component of the velocity field. This requires an energy transfer $\tau(k)$, much of which takes place on the large scales. We will refer to the two spectra as source ($S_o$) and sink spectra ($S_i$) according to the net energy flux. We assume here, that the energy transfer term $\tau(k)$ preserves the scale $k$. Hence, following energy conservation

$$\epsilon_{S_o}(k + dk) = \epsilon_{S_o}(k) - \tau(k)dk,$$  \hspace{1cm} (4.1)

$$\epsilon_{S_i}(k + dk) = \epsilon_{S_i}(k) + \tau(k)dk,$$  \hspace{1cm} (4.2)

where $\epsilon$ indicates the energy flux through the scales and $\tau$ the energy flux between the two components. We further assume that $\tau(k)$ is to first order proportional to the difference in energy between the two spectra $\tau(k) \propto \mathcal{P}_{S_o}(k) - \mathcal{P}_{S_i}(k)$, so the net energy transfer can be described as

$$\tau(k) = \vartheta_{S_o}(M, k)\mathcal{P}_{S_o}(k) - \vartheta_{S_i}(M, k)\mathcal{P}_{S_i}(k),$$  \hspace{1cm} (4.3)

where both $\vartheta(M, k)$ contain the proportionality constants that determine the energy distribution between the two components in the final state. It also describes the efficiency of the transfer processes, which depend strongly on $M$, and introduces an additional $k$ dependence through e.g. the sonic scale. From the discussion in section 3.1 we can conclude that $\vartheta \to 0$ for $M \to 0$, as both components are nearly decoupled in the subsonic limit. For simplicity we assume $d\vartheta(M, k)/dk = 0$, which can be interpreted as the high Mach number limit where the coupling of the spectra is efficient and independent of $k$, so $\vartheta \to \text{const}$ for $M \to \infty$ (see M17sol/comp spectra and discussion in section 4.1).

Taking the $k$-derivative of equation (4.3), along with the Kolmogorov law,

$$\mathcal{P}(k) = C_0\epsilon(k)^{-\frac{1+\zeta}{2}}k^\zeta$$  \hspace{1cm} (4.4)

and equations (4.1) and (4.2), we obtain

$$\frac{d\tau}{dk} = \tau(k) \left( \vartheta_{S_o}\alpha_{S_o}\xi_{S_o}(\alpha_{S_o}^{-1})k^\xi_{S_o} - \vartheta_{S_i}\alpha_{S_i}\xi_{S_i}(\alpha_{S_i}^{-1})k^\xi_{S_i} \right) - \vartheta_{S_o}\mathcal{P}_{S_o}(k)\xi_{S_o}/k - \vartheta_{S_i}\mathcal{P}_{S_i}(k)\xi_{S_i}/k,$$  \hspace{1cm} (4.5a)

with $\alpha_x = -(1+\zeta_x)$. Equations (4.1), (4.2) and (4.5) form a coupled system of first-order differential equations. Though the solution of this equation is beyond the scope of this work we consider the limiting case of Burgers turbulence.

To mimic the M17comp simulation, which produces Burgers scaling in both the source

† If we assume instead of equation (4.3) that the transfer is proportional to the flux through the scales $\tau = \vartheta_{S_o}\xi_{S_o} - \vartheta_{S_i}\xi_{S_i}$, equation (4.5a) simplifies to $d\tau(k)/dk = -\tau(k)(\vartheta_{S_o} + \vartheta_{S_i})$ resulting in $\tau(k) = C^* \exp(-((\vartheta_{S_o} + \vartheta_{S_i})k)$, independent of the scaling behaviour of the source and the sink spectrum.
and the sink spectrum (see figure 6), \( \alpha_{So} = \alpha_{Si} = 1 \) and \( \zeta_{So} = \zeta_{Si} = -2 \). Then equation (4.3) simplifies to

\[
\frac{d\tau(k)}{dk} = -\tau(k) \left[ (\vartheta_{So} + \vartheta_{Si}) k^{-2} - 2k^{-1} \right],
\]

(4.6)

and results in

\[
\tau(k) = C_1 \exp \left( -\frac{\vartheta_{So} + \vartheta_{Si}}{k} - 2 \log(k) \right) + C_2.
\]

(4.7)

Hence, a spectrum with a different energy distribution on the largest scales as the 'natural' mixing will yield an energy transfer. This transfer is than also confined to the largest scales, as it drops off exponentially. In this context the integral constant \( C_2 \) plays an important role as it determines whether the transfer term vanishes or approaches a constant. \( C_2 = 0 \) results in the normal energy cascade with constant energy flux through both spectra behind the transfer regime. When \( C_2 \neq 0 \) there is a linear increasing/decreasing energy flux through the scales for the source/sink spectra, respectively, which should dominate the scaling behaviour on the smaller scales. Neglecting the first term in equation (4.7) for \( k > k_0 \) and substituting the result of the integral of (4.1) and (4.2) in (4.4) results in

\[
P(k) \propto \epsilon_0 (k - k_0)^{-2} \pm C_2 (k - k_0)^{-1},
\]

(4.8)

for the source and the sink spectrum, where \( \epsilon_0 \) is the energy flux from Burgers theory.

This model of the energy transfer (equation 4.7) can explain the behaviour of the scaling exponent (figure 4) on the large scales. For example, the subsonic and supersonic transverse spectra with solenoidal driving show a steeper scaling behaviour than the Kolmogorov \(-5/3\) scaling on scales \( k \lesssim 8 \). An additional transfer of energy between the modes makes these source spectra stepper on the large scales. This could also explains, why the scale \( k_s \) of the transition between Burgers and Kolmogorov scaling behaviour occurs also for the subsonic simulations and is independent of \( M \), what we do not expect for the sonic scale.

4.3. Measuring the energy transfer

The Bayesian model (equation 2.9) includes estimates of the deviations from a perfect power law through the scatter term. Figure 7 shows the group standard deviation of the scatter as a function of \( k \) for solenoidal (left panel) and compressive (right panel) forcing, for the different components of the velocity field and different \( M \). To make the different SDs of the scatter directly comparable for different \( M \) and \( k \), the results in the standardised system described by equation (2.8) are shown. As the scatter term measures the intrinsic deviations from the power law, we can associate it with the energy transfer, which in turn influences the scaling behaviour. However, as many processes influences the power spectra, it will also affect the scatter term. We therefore employ the scatter as a rough estimate of the energy transfer.

Figure 7 shows that the deviations from a perfect power law increases steeply towards larger scales, where the leading term is an exponential function. This is in agreement with the calculations above, resulting in an exponential function as the leading term, independent of the assumption of \( \tau \propto \epsilon \) or \( \tau \propto P \). To illustrate this we perform a fit of equation (4.7) to the data from the \( M17sol/comp \) simulations (black straight line). It shows that the energy transfer is confined to scales \( k \lesssim 7 \) for the supersonic simulations. On larger scales the deviations of the spectra from a power law increase by an order of

† For instance, the scatter does not vanish for \( \tau(k) = 0 \) and so we cannot deduce the behaviour of \( C_2 \) from it.
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The standard deviation of the scatter term in the Bayesian model $\sigma_\Delta$ as a function of $k$ for solenoidal (left panels) and compressive (right panels) forcing. The results are shown for the transverse (top panels) and longitudinal (bottom panels) with different Mach numbers. The lines are the fit of equation 4.7 for the $M_{17}$ simulations.

Figure 7. The standard deviation of the scatter term in the Bayesian model $\sigma_\Delta$ as a function of $k$ for solenoidal (left panels) and compressive (right panels) forcing. The results are shown for the transverse (top panels) and longitudinal (bottom panels) with different Mach numbers. The lines are the fit of equation 4.7 for the $M_{17}$ simulations.

magnitude. The sink spectra have smaller deviations from the power law on the larger scales.

We note here that with the ansatz (4.3) the transfer term only vanishes ($\tau = 0$) if the relation between both spectra remains constant

$$\frac{P_{So}}{P_{Si}} = \frac{\theta_{Si}}{\theta_{So}}.$$  (4.9)

Consequently they must have the same scaling exponents $\zeta$. This is in agreement with the measurements from the $M_{17}sol/comp$ simulations, where there is a similar scaling behaviour for the transverse and longitudinal spectra, indicative of the correlation between the source and sink spectra.

Figure 7 also shows that the $M_{0.1}comp$ simulation has a larger $\sigma_\Delta(k)$ than the other simulations. Namely, the SD does not decrease exponentially. This indicates a higher energy transfer term over an extended $k$-range, which confirms the estimates and interpretation of the spectra in section 4.1. The $M_{0.5}comp$ simulation does not have a large scatter although the scaling exponents in the last section also indicate an energy transfer on all scales. This is reflective of the inaccuracy in measuring the energy transfer solely with the scatter term.

4.4. The total velocity power-spectrum

For completeness figure 8 shows the local scaling exponents of the total velocity power spectrum for both types of forcing (left solenoidal and right compressive). The scaling exponents of the total spectra are nearly equivalent to those from the transverse spectra
for solenoidal forcing. On the other hand, the scaling exponents of the total spectra change significantly for the compressive forcing case. The reason for this behaviour is that the longitudinal spectra are much steeper than the transverse ones. Although the longitudinal modes dominate in a global statistical sense, the influence of the transverse modes increases on the small scales. So the scaling exponents of the total spectra are an average of the transverse and longitudinal spectra on the smaller scales.

It is noteworthy that the scaling exponents of the simulation with compressive forcing converge towards the Kolmogorov scaling exponent for intermediate $k$ and decreasing $M$. An exception is the $M0.1\,comp$ simulation, where there are large deviations from a power law behaviour as discussed in the last section, due to numerical effects.

Finally, we mention here that the model for the transfer term $\tau(k)$ described in section 4.2 does not influence the scaling behaviour of the total spectra. One of our assumptions is that $\tau(k)$ transfers the energy between the decomposed spectra without changing the scale $k$, which ensures that the total spectrum is not affected. However, Figure 8 shows that the total spectra in the highly supersonic case $M17$ have a distinct scaling behaviour on all $k$ depending on the forcing. This contradicts Kolmogorov’s definition of an inertial range. If this is caused by the resolution of the simulation or by a physical effect can not be answered here.

4.5. Comparison with previous works

Numerous other investigations have focused on the velocity power spectra of isothermal turbulent simulations. The discussion above indicates that analysing the local scaling properties of the velocity power spectrum is a complex problem, where many processes play a role. Nevertheless, Figure 8 shows that the global spectra can be well described with a power law over an extended range. These scaling exponents fitted in some extended range in $k$ may identify whether the energy cascade is more Kolmogorov-like or shock dominated. Table 3 shows the scaling exponents of recent studies arranged by $M$. A comparison with Figure 8 shows that all authors average the scaling exponents in a range, where our results vary strongly. Nevertheless these results are in agreement with our spectra and we can reproduce the scaling exponents, apart from small variations. One reason for the slightly distinct results are the different setups of the simulations e.g. the forcing routine, $M$ or the solver, which makes a direct comparison difficult.
5. Summary and conclusions

We investigated a set of three-dimensional numerical simulations of driven, compressible, isothermal turbulence with Mach numbers ranging from the subsonic to the highly supersonic regime. We focused our analysis on the influence of the limiting cases of purely solenoidal (divergence-free) and purely compressive (curl-free) forcing. We measure the local scaling exponents of the velocity power spectra with a hierarchical Bayesian power law model, which employs a Markov Chain Monte Carlo method to probabilistically estimate the model parameters. This method can handle uncertainties and time-dependent fluctuations in the spectra and is consequently applicable for observational and numerical datasets. With this method we investigated the local scaling behaviour of the decomposed velocity power spectrum. Our main results are as follows:

- The scaling exponents vary significantly in the simulations, depending on Mach number, wave number range, resolution, and velocity mode considered. However, the longitudinal spectra obtained with compressive driving have a steep, almost universal Burgers slope of $-2$. Our models does not show a Kolmogorov $-5/3$ scaling for these cases and there is no sign of a sonic scale or a bottleneck effect (BNE) in these simulation.

- In the decomposed velocity spectra there is just a small range $12 \leq k \leq 15$, where the measured scaling exponents of the transverse spectra driven with solenoidal forcing are in agreement with the Kolmogorov value and are independent of the Mach number.

- We develop a model where the transfer of energy between the two components lead to the shapes of the spectra. The scaling exponents seem to be dominated by the statistical behaviour of the spectra providing the energy in the $M = 17$ simulations. This indicates that a theory describing the scaling behaviour of supersonic, compressible turbulence must have more than $\mathcal{M}$ and $\mathcal{M}_{\text{comp}}$ as basic parameters. A parameter describing the flux of energy between the transverse and longitudinal mode families should be included.

- The provided phenomenological model describing the energy transport between the transverse and longitudinal velocity components of the flow explains the scaling behaviour in the high Mach number limit in the transient regime. This model shows that the energy transfer can have a major influence on the scaling exponents, if the source and sink spectra have intrinsically different scaling behaviours. In this model we distinguish between a source spectrum providing the energy and a sink spectrum accumulating the

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| author               | resolution | $M$ | fit-range $[\ell]$ | $\zeta$ | simulation | $\zeta$-HDI min | $\zeta$-HDI max | $\sigma$-HDI min | $\sigma$-HDI max |
|----------------------|------------|-----|-------------------|--------|------------|----------------|----------------|----------------|----------------|
| Schmidt et al. (2009) | 768$^3$    | 5.5 | 5 – 15            | -1.87  | $\mathcal{M}_{\text{comp}}$ | -1.96          | -2.02          | 0.05           | 0.07           |
| Federrath et al. (2010) | 1024$^3$  | 5.5 | 5 – 15            | -1.86 $\pm$ 0.05 | $\mathcal{M}_{\text{sol}}$ | -1.82          | -1.88          | 0.05           | 0.08           |
| Federrath et al. (2010) | 1024$^3$  | 5.5 | 5 – 15            | -1.94 $\pm$ 0.05 | $\mathcal{M}_{\text{comp}}$ | -1.98          | -2.04          | 0.05           | 0.07           |
| Krueger et al. (2007)  | 1024$^3$  | 6   | 4 – 26            | -1.95 $\pm$ 0.02 | $\mathcal{M}_{\text{sol}}$ | -1.78          | -1.84          | 0.05           | 0.07           |
| Lemaster & Stone (2009) | 1024$^3$  | 7   | 6 – 15            | -2     | $\mathcal{M}_{\text{sol}}$ | -1.82          | -1.90          | 0.06           | 0.08           |
| Lemaster & Stone (2009) | 1024$^3$  | 7   | 11 – 36           | -1.7   | $\mathcal{M}_{\text{sol}}$ | -1.75          | -1.80          | 0.05           | 0.07           |

Table 3. Comparison of different scaling exponents of the total spectrum measured in recent studies with the results of the Bayesian method presented in this work. The column indicates the study (column 1), their resolution (column 2), the Mach number in the state of fully developed turbulence (column 3), the wave number range over which a fit is applied (column 4), the measured scaling exponent (column 5). Column 6 indicates the simulation in our set with which we want to compare the results and column 7 and 8 state 2$\sigma$-HDI minimum and maximum value of the resulting group slope of the total velocity power spectrum in this fit range. Column 9 an 10 show the 2$\sigma$-HDI minimum and maximum value of the group standard deviation of the slope. Note, that a direct comparison of the results is difficult as the numerical setup differ slightly in e.g. the forcing routines, $\mathcal{M}$, the used solver.
energy. We analysed a special case of the set of coupled differential equations. With the assumption of the same scaling behaviour for the sink and source spectra we found an exponential decreasing energy transfer between the spectra as a function of the scale $k$ in a transient regime.

- Our simulations show that the BNE is pronounced in the transverse spectra and is weaker in the longitudinal spectra. It also influences much smaller scales in the longitudinal spectra and only has a universal behaviour in the subsonic case, as proposed by Donzis & Sreenivasan (2010), and breaks down in the supersonic regime. For $M > 2$ the influence of the non-local energy transfer through the scales increases with $M$ yielding a decreasing amplitude and width of the BNE. We found that the subsonic simulations with solenoidal forcing produce a scaling exponent $\approx -1.1$ in the BNE region, which is in agreement with the model of Yakhot & Zakharov (1993) and Bershadskii (2008).

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REFERENCES

Bershadskii, A. 2008 Near-dissipation range in nonlocal turbulence. Physics of Fluids 20 (8), 085103.

Brandenburg, A. & Subramanian, K. 2005 Astrophysical magnetic fields and nonlinear dynamo theory. Physics Report 417, 1–209.

Burgers, J.M. 1948 Advances in Applied Mechanics I.

Dellacherie, S. 2010 Analysis of Godunov type schemes applied to the compressible Euler system at low Mach number. Journal of Computational Physics 229, 978–1016.

Dobler, W., Haugen, N. E., Yousef, T. A. & Brandenburg, A. 2003 Bottleneck effect in three-dimensional turbulence simulations. Phys. Rev. E 68 (2), 026304.

Donzis, D. A. & Sreenivasan, K. R. 2010 The bottleneck effect and the Kolmogorov constant in isotropic turbulence. Journal of Fluid Mechanics 657, 171–188.

Dubey, A., Fisher, R., Graziani, C., Jordan, IV, G. C., Lamb, D. Q., Reid, L. B., Rich, P., Sheeler, D., Townsley, D. & Weide, K. 2008 Challenges of Extreme Computing
using the FLASH code. In Numerical Modeling of Space Plasma Flows (ed. N. V. Pogorelov, E. Audit & G. P. Zank), Astronomical Society of the Pacific Conference Series, vol. 385, p. 145.

Elmegreen, B. G. & Scalo, J. 2004 Interstellar Turbulence I: Observations and Processes. In Numerical Modeling of Space Plasma Flows (ed. N. V. Pogorelov, E. Audit & G. P. Zank), Astronomical Society of the Pacific Conference Series, vol. 385, p. 145.

Falkovich, G. 1994 Bottleneck phenomenon in developed turbulence. Physics of Fluids 6, 1411–1414.

Federrath, C. 2013 On the universality of supersonic turbulence. ArXiv e-prints.

Federrath, C., Chabrier, G., Schober, J., Banerjee, R., Klessen, R. S. & Schleicher, D. R. G. 2011a Mach Number Dependence of Turbulent Magnetic Field Amplification: Solenoidal versus Compressive Flows. Physical Review Letters 107 (11), 114504.

Federrath, C. & Klessen, R. S. 2012 The Star Formation Rate of Turbulent Magnetized Clouds: Comparing Theory, Simulations, and Observations. The Astrophysical Journal 761, 154.

Federrath, C., Klessen, R. S. & Schmidt, W. 2008 The Density Probability Distribution in Compressible Isothermal Turbulence: Solenoidal versus Compressive Forcing. The Astrophysical Journal, Letters 688, L79–L82.

Federrath, C., Roman-Duval, J., Klessen, R. S., Schmidt, W. & Mac Low, M.-M. 2010 Comparing the statistics of interstellar turbulence in simulations and observations. Solenoidal versus compressive turbulence forcing. Astronomy and Astrophysics 512, A81.

Federrath, C., Sur, S., Schleicher, D. R. G., Banerjee, R. & Klessen, R. S. 2011b A New Jeans Resolution Criterion for (M)HD Simulations of Self-gravitating Gas: Application to Magnetic Field Amplification by Gravity-driven Turbulence. The Astrophysical Journal 731, 62.

Foster, J. B., Mandel, K. S., Pineda, J. E., Covey, K. R., Arce, H. G. & Goodman, A. A. 2013 Evidence for grain growth in molecular clouds: A Bayesian examination of the extinction law in Perseus. MNRAS 428, 1606–1622.

Fryxell, B., Olson, K., Ricker, P., Timmes, F. X., Zingale, M., Lamb, D. Q., MacNeice, P., Rosner, R., Truran, J. W. & Tufo, H. 2000 FLASH: An Adaptive Mesh Hydrodynamics Code for Modeling Astrophysical Thermonuclear Flashes. Astronomy and Astrophysics, Supplement 131, 273–334.

Gelman, A., Carlin, J. B., Stern, H. S. & Rubin, D. B. 2004 Bayesian Data Analysis, 2nd edn. Chapman & Hall.

Gelman, Andrew & Hill, Jennifer 2007 Data analysis using regression and multilevel/hierarchical models, vol. Analytical methods for social research. New York: Cambridge University Press.

Girichidis, P., Federrath, C., Banerjee, R. & Klessen, R. S. 2012 Importance of the initial conditions for star formation - II. Fragmentation-induced starvation and accretion shielding. MNRAS 420, 613–626.

Hopkins, P. F. 2012 An excursion-set model for the structure of giant molecular clouds and the interstellar medium. MNRAS 423, 2016–2036.

Iannuzzi, F. & Dolag, K. 2012 On the orbital and internal evolution of cluster galaxies. MNRAS 427, 1024–1033.

Jin, S. 1996 The Effects of Numerical Viscosities I. Slowly Moving Shocks. Journal of Computational Physics 126, 373–389.

Kaneda, Y., Ishihara, T., Yokokawa, M., Itakura, K. & Uno, A. 2003 Energy dissipation rate and energy spectrum in high resolution direct numerical simulations of turbulence in a periodic box. Physics of Fluids 15, L21–L24.

Kelly, B. C. 2007 Some Aspects of Measurement Error in Linear Regression of Astronomical Data. The Astrophysical Journal 665, 1489–1506.

Kelly, B. C., Shetty, R., Stutz, A. M., Kaufmann, J., Goodman, A. A. & Laughardt, R. 2012 Dust Spectral Energy Distributions in the Era of Herschel and Planck: A Hierarchical Bayesian-Fitting Technique. The Astrophysical Journal 752, 55.

Kida, S. & Orszag, S. A. 1990 Energy and spectral dynamics in forced compressible turbulence. Journal of Scientific Computing 5, 85–125.

Kitsionas, S., Federrath, C., Klessen, R. S., Schmidt, W., Price, D. J., Dursi, L. J., Gritschneider, M., Walch, S., Piontek, R., Kim, J., Jappsen, A.-K., Ciecielag, P.
L. Konstandin, R. Shetty, C. Federrath, P. Girichidis and R. S. Klessen & Mac Low, M.-M. 2009 Algorithmic comparisons of decaying, isothermal, supersonic turbulence. aap 508, 541–560.

Kolmogorov, A. 1941 The Local Structure of Turbulence in Incompressible Viscous Fluid for Very Large Reynolds’ Numbers. Akademiia Nauk SSSR Doklady 30, 301–305.

Konstandin, L., Federrath, C., Klessen, R. S. & Schmidt, W. 2012a Statistical properties of supersonic turbulence in the Lagrangian and Eulerian frameworks. Journal of Fluid Mechanics 692, 183–206.

Konstandin, L., Girichidis, P., Federrath, C. & Klessen, R. S. 2012b A New Density Variance-Mach Number Relation for Subsonic and Supersonic Isothermal Turbulence. The Astrophysical Journal 761, 149.

Kritsuk, A. G., Norman, M. L., Padoan, P. & Wagner, R. 2007 The Statistics of Supersonic Isothermal Turbulence. The Astrophysical Journal 665, 416–431.

Kruschke, John K 2011 Doing bayesian data analysis: a tutorial with R and BUGS. Amsterdam: Elsevier.

Kurien, S., Taylor, M. A. & Matsumoto, T. 2004 Cascade time scales for energy and helicity in homogeneous isotropic turbulence. pre 69 (6), 066313.

Lemaster, M. N. & Stone, J. M. 2009 Dissipation and Heating in Supersonic Hydrodynamic and MHD Turbulence. The Astrophysical Journal 691, 1092–1108.

Mac Low, M.-M. & Klessen, R. S. 2004 Control of star formation by supersonic turbulence. Reviews of Modern Physics 76, 125–194.

McKee, C. F. & Ostriker, E. C. 2007 Theory of Star Formation. Annual Review of Astronomy and Astrophysics 45, 565–687.

Mee, A. J. & Brandenburg, A. 2006 Turbulence from localized random expansion waves. MNRAS 370, 415–419.

Meschiari, S. 2012 Planet Formation in Circumbinary Configurations: Turbulence Inhibits Planetesimal Accretion. The Astrophysical Journal 761, L7.

Meyers, J. & Meneveau, C. 2008 A functional form for the energy spectrum parametrizing bottleneck and intermittency effects. Physics of Fluids 20 (6), 065109.

Porter, D. H. & Woodward, P. R. 1994 High-resolution simulations of compressible convection using the piecewise-parabolic method. The Astrophysical Journals 93, 309–349.

Qian, J. 1984 Universal equilibrium range of turbulence. Physics of Fluids 27, 2229–2233.

Saddoughi, S. G. & Veeravalli, S. V. 1994 Local isotropy in turbulent boundary layers at high Reynolds number. Journal of Fluid Mechanics 268, 333–372.

Schmidt, W., Federrath, C., Hupp, M., Kern, S. & Niemeyer, J. C. 2009 Numerical simulations of compressively driven interstellar turbulence. I. Isothermal gas. Astronomy and Astrophysics 494, 127–145.

Schmidt, W., Hillebrandt, W. & Niemeyer, J. C. 2006 Numerical dissipation and the bottleneck effect in simulations of compressible isotropic turbulence. Comp. Fluids 35, 353–357.

Schober, J., Schleicher, D., Federrath, C., Klessen, R. & Banerjee, R. 2012 Magnetic field amplification by small-scale dynamo action: Dependence on turbulence models and Prandtl numbers. pre 85 (2), 026303.

She, Z.-S. & Jackson, E. 1993 On the universal form of energy spectra in fully developed turbulence. Physics of Fluids 5, 1526–1528.

Shetty, R., Beaumont, C. N., Burton, M. G., Kelly, B. C. & Klessen, R. S. 2012 The linewidth-size relationship in the dense interstellar medium of the Central Molecular Zone. MNRAS 425, 720–729.

Shetty, R., Kelly, B. C. & Bigiel, F. 2013 Evidence for a non-universal Kennicutt-Schmidt relationship using hierarchical Bayesian linear regression. MNRAS 430, 288–304.

Toschi, F. & Bodenschatz, E. 2009 Lagrangian Properties of Particles in Turbulence. Annual Review of Fluid Mechanics 41, 375–404.

Tsui, Y. 2004 Intermittency effect on energy spectrum in high-Reynolds number turbulence. Physics of Fluids 16, L43–L46.

Waagan, K., Federrath, C. & Klessen, R. 2011 A robust numerical scheme for highly compressible magnetohydrodynamics: Nonlinear stability, implementation and tests. Journal of Computational Physics 230, 3331–3351.
The scaling behaviour of velocity power spectrum in a turbulent flow

Wakefield, Jon 2013 Bayesian and frequentist regression methods. Springer Series in Statistics.

Yakhot, V. & Zakharov, V. 1993 Hidden conservation laws in hydrodynamics; energy and dissipation rate fluctuation spectra in strong turbulence. Physica D Nonlinear Phenomena 64, 379–394.