The $\beta$-function in duality-covariant noncommutative $\phi^4$-theory

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Abstract

We compute the one-loop $\beta$-functions describing the renormalisation of the coupling constant $\lambda$ and the frequency parameter $\Omega$ for the real four-dimensional duality-covariant noncommutative $\phi^4$-model, which is renormalisable to all orders. The contribution from the one-loop four-point function is reduced by the one-loop wavefunction renormalisation, but the $\beta_\lambda$-function remains non-negative. Both $\beta_\lambda$ and $\beta_\Omega$ vanish at the one-loop level for the duality-invariant model characterised by $\Omega = 1$. Moreover, $\beta_\Omega$ also vanishes in the limit $\Omega \to 0$, which defines the standard noncommutative $\phi^4$-quantum field theory. Thus, the limit $\Omega \to 0$ exists at least at the one-loop level.
1 Introduction

For many years, the renormalisation of quantum field theories on noncommutative $\mathbb{R}^4$ has been an open problem [1]. Recently, we have proven in [2] that the real duality-covariant $\phi^4$-model on noncommutative $\mathbb{R}^4$ is renormalisable to all orders. The duality transformation exchanges positions and momenta [3],

$$\hat{\phi}(p) \leftrightarrow \pi^2 \sqrt{|\det \theta|} \phi(x), \quad p_\mu \leftrightarrow \tilde{x}_\mu := 2(\theta^{-1})_{\mu
u}x^\nu,$$  \hspace{1cm} (1)

where $\hat{\phi}(p_a) = \int d^4x \ e^{(-1)^{a\mu} p_a, \mu \phi(x_a)}$. The subscript $a$ refers to the cyclic order in the $\star$-product. The duality-covariant noncommutative $\phi^4$-action is given by

$$S[\phi; \mu_0, \lambda, \Omega] := \int d^4x \ (\frac{1}{2}(\partial_\mu \phi) \star (\partial^\mu \phi) + \frac{\Omega^2}{2}(\tilde{x}_\mu \phi) \star (\tilde{x}^\mu \phi) + \frac{\mu_0^2}{2}\phi \star \phi$$
$$\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad

Under the transformation (1) one has

$$S[\phi; \mu_0, \lambda, \Omega] \mapsto S[\hat{\phi}; \mu_0, \lambda, \Omega] = \Omega^2 S[\phi; \frac{\mu_0}{\Omega}, \frac{\lambda}{\Omega^2}, \frac{1}{\Omega}].$$ \hspace{1cm} (3)

In the special case $\Omega = 1$ the action $S[\phi; \mu_0, \lambda, 1]$ is invariant under the duality (1). Moreover, $S[\phi; \mu_0, \lambda, 1]$ can be written as a standard matrix model which is closely related to an exactly solvable model [4].

Knowing that the action (2) gives rise to a renormalisable quantum field theory [2], it is interesting to compute the $\beta_\lambda$ and $\beta_\Omega$ functions which describe the renormalisation of the coupling constant $\lambda$ and of the oscillator frequency $\Omega$. Whereas we have proven the renormalisability in the Wilson-Polchinski approach [5, 6] adapted to non-local matrix models [7], we compute the one-loop $\beta_\lambda$ and $\beta_\Omega$ functions by standard Feynman graph calculations. Of course, these are Feynman graphs parametrised by matrix indices instead of momenta. We rely heavily on the power-counting behaviour proven in [2], which allows us to ignore in the $\beta$-functions all non-planar graphs and the detailed index dependence of the planar two- and four-point graphs. Thus, only the lowest-order (discrete) Taylor expansion of the planar two- and four-point graphs can contribute to the $\beta$-functions. This means that we cannot refer to the usual symmetry factors of commutative $\phi^4$-theory so that we have to carefully recompute the graphs.

We obtain interesting consequences for the limiting cases $\Omega = 1$ and $\Omega = 0$ as discussed in Section 3.

2 Definition of the model

The noncommutative $\mathbb{R}^4$ is defined as the algebra $\mathbb{R}_n^4$ which as a vector space is given by the space $\mathcal{S}(\mathbb{R}^4)$ of (complex-valued) Schwartz class functions of rapid decay, equipped
with the multiplication rule

$$(a * b)(x) = \int \frac{d^4k}{(2\pi)^4} \int d^4y \ a(x+\frac{1}{2} \theta \cdot k) b(x+y) e^{ik \cdot y} ,$$  \hspace{1cm} (4)

$$\theta \cdot k = \theta^\mu k_\mu , \quad k \cdot y = k_\mu y^\mu , \quad \theta^{\mu\nu} = -\theta^{\nu\mu} .$$

We place ourselves into a coordinate system in which the only non-vanishing components \(\theta_{\mu\nu}\) are \(\theta_{12} = -\theta_{21} = \theta_{34} = -\theta_{43} = \theta\). We use an adapted base

$$b_{mn}(x) = f_{m^1n^1}(x^1, x^2) f_{m^2n^2}(x^3, x^4) , \quad m = \frac{m^1}{m^2} \in \mathbb{N}^2 , \quad n = \frac{n^1}{n^2} \in \mathbb{N}^2 ,$$  \hspace{1cm} (5)

where the base \(f_{m^1n^1}(x^1, x^2) \in \mathbb{R}_f^2\) is given in [8]. This base satisfies

$$(b_{mn} \ast b_{kl})(x) = \delta_{nk} b_{ml}(x) , \quad \int d^4x \ b_{mn}(x) = 4\pi^2 \theta^2 \delta_{mn} .$$  \hspace{1cm} (6)

According to [2], the duality-covariant \(\phi^4\)-action (2) expands as follows in the matrix base (1):

$$S[\phi; \mu_0, \lambda, \Omega] = 4\pi^2 \theta^2 \sum_{m,n,k,l \in \mathbb{N}^2} \left( \frac{1}{2} G_{mn;kl} \phi_{mn} \phi_{kl} + \frac{\lambda}{4!} \phi_{mn} \phi_{nk} \phi_{kl} \phi_{lm} \right) ,$$  \hspace{1cm} (7)

where \(\phi(x) = \sum_{m,n,} \phi_{mn} b_{mn}(x)\) and

$$G_{mn;kl} = \left( \mu_0^2 + \frac{2}{\theta} (1+\Omega^2)(m^1+n^1+m^2+n^2+2) \right) \delta_{n^1,k^1} \delta_{m^1+l^1} \delta_{n^2,k^2} \delta_{m^2+l^2}$$

$$- \frac{2}{\theta} (1-\Omega^2) \left( (\sqrt{(n^1+1)(m^1+1)} \delta_{n^1+1,k^1} \delta_{m^1+1,l^1} + \sqrt{n^1m^1} \delta_{n^1-1,k^1} \delta_{m^1-1,l^1}) \delta_{n^2,k^2} \delta_{m^2+l^2} + (\sqrt{(n^2+1)(m^2+1)} \delta_{n^2+1,k^2} \delta_{m^2+1,l^2} + \sqrt{n^2m^2} \delta_{n^2-1,k^2} \delta_{m^2-1,l^2}) \delta_{n^1,k^1} \delta_{m^1+l^1} \right) .$$  \hspace{1cm} (8)

The quantum field theory is defined by the partition function

$$Z[J] = \int \left( \prod_{a,b \in \mathbb{N}^2} d\phi_{ab} \right) \exp \left( - S[\phi] - 4\pi^2 \theta^2 \sum_{m,n,} \phi_{mn} J_{mn} \right) .$$  \hspace{1cm} (9)

For the free theory defined by \(\lambda = 0\) in (7), the solution of (8) is given by

$$Z[J]_{|\lambda=0} = Z[0] \exp \left( 4\pi^2 \theta^2 \sum_{m,n,k,l} \frac{1}{2} J_{mn} \Delta_{mn;kl} J_{kl} \right) ,$$  \hspace{1cm} (10)

where the propagator \(\Delta\) is defined as the inverse of the kinetic matrix \(G\):

$$\sum_{k,l \in \mathbb{N}^2} G_{mn;kl} \Delta_{lk;sr} = \sum_{r \in \mathbb{N}^2} \Delta_{nm;lk} G_{kl;rs} = \delta_{mr} \delta_{ns} .$$  \hspace{1cm} (11)
We have derived the propagator in [4]:

\[
\Delta_{m_1, n_1, k_1, l_1}^{m_2, n_2, k_2, l_2} = \frac{\theta}{2(1+\Omega)^2} \delta^{m_1+k_1, n_1+l_1} \delta^{m_2+k_2, n_2+l_2} \times \sum_{v^1 = \frac{|m_1 - i|}{2}} \sum_{v^2 = \frac{|n_2 - |v^1|}{2}} B \left( 1 + \frac{\nu^2 \theta}{8 \Omega} + \frac{1}{2} (m^1 + m^2 + k^1 + k^2) - v^1 - v^2, 1 + 2v^1 + 2v^2 \right) 
\]

\[
\times \, _2F_1 \left( \begin{array}{c} 1 + 2v^1 + 2v^2, \frac{\nu^2 \theta}{8 \Omega} - \frac{1}{2} (m^1 + m^2 + k^1 + k^2) + v^1 + v^2 \end{array} ; \frac{(1-\Omega)^2}{(1+\Omega)^2} \right) 
\times \prod_{i=1}^{2} \sqrt{\frac{n^i}{v^i + n^i - k^i}} \left( v^i + k^i - n^i \right) \left( v^i + m^i - l^i \right) \left( v^i + l^i - m^i \right) \left( \frac{(1-\Omega)^2}{(1+\Omega)^2} \right)^{v^i} . \tag{12}
\]

Here, \( B(a, b) \) is the Beta-function and \( _2F_1(a, b \mid z) \) the hypergeometric function.

As usual we solve the interacting theory perturbatively:

\[
Z[J] = Z[0] \exp \left( - V \left[ \frac{\partial}{\partial J} \right] \right) \exp \left( 4\pi^2 \theta^2 \sum_{m,n,k,l \in \mathbb{N}^2} \frac{1}{2} J_{mn} \Delta_{mn; kl} J_{kl} \right) ,
\]

\[
V \left[ \frac{\partial}{\partial J} \right] := \frac{\lambda}{4! (4\pi^2 \theta^2)^3} \sum_{m,n,k,l \in \mathbb{N}^2} \partial^4 \frac{\partial J_{ml} \partial J_{lk} \partial J_{kn} \partial J_{nm}}{\partial J_{mn}} . \tag{13}
\]

It is convenient to pass to the generating functional of connected Green’s functions, \( W[J] = \ln Z[J] \):

\[
W[J] = \ln Z[0] + W_{\text{free}}[J] + \ln \left( 1 + e^{-W_{\text{free}}[J]} \left( \exp \left( - V \left[ \frac{\partial}{\partial J} \right] \right) - 1 \right) e^{W_{\text{free}}[J]} \right) ,
\]

\[
W_{\text{free}}[J] := 4\pi^2 \theta^2 \sum_{m,n,k,l \in \mathbb{N}^2} \frac{1}{2} J_{mn} \Delta_{mn; kl} J_{kl} . \tag{14}
\]

In order to obtain the expansion in \( \lambda \) one has to expand \( \ln(1+x) \) as a power series in \( x \) and \( \exp(-V) \) as a power series in \( V \). By Legendre transformation we pass to the generating functional of one-particle irreducible (1PI) Green’s functions:

\[
\Gamma[\phi^{\text{cl}}] := 4\pi^2 \theta^2 \sum_{m,n \in \mathbb{N}^2} \phi^{\text{cl}}_{mn} J_{mn} - W[J] , \tag{15}
\]

where \( J \) has to be replaced by the inverse solution of

\[
\phi^{\text{cl}}_{mn} := \frac{1}{4\pi^2 \theta^2} \frac{\partial W[J]}{\partial J_{nm}} . \tag{16}
\]
3 Renormalisation group equation

The computation of the expansion coefficients

\[ \Gamma_{m_1n_1;\ldots;m_Nn_N} := \frac{1}{N!} \frac{\partial^N \Gamma[\phi^{cf}]}{\partial \phi^{cf}_{m_1n_1} \ldots \partial \phi^{cf}_{m_Nn_N}} \]  

of the effective action involves possibly divergent sums over undetermined loop indices. Therefore, we have to introduce a cut-off \( \mathcal{N} \) for all loop indices. According to \( \text{[2]} \), the expansion coefficients (17) can be decomposed into a relevant/marginal and an irrelevant piece. As a result of the renormalisation proof, the relevant/marginal parts have—after a rescaling of the field amplitude—the same form as the initial action (2), (7) and (8), now parametrised by the “physical” mass, coupling constant and oscillator frequency:

\[ \Gamma_{\text{rel/marg}}[Z\phi^{cf}] = S[\phi^{cf}; \mu_{\text{phys}}, \lambda_{\text{phys}}, \Omega_{\text{phys}}] . \]  

In the renormalisation process, the physical quantities \( \mu^2_{\text{phys}}, \lambda_{\text{phys}} \) and \( \Omega_{\text{phys}} \) are kept constant with respect to the cut-off \( \mathcal{N} \). This is achieved by starting from a carefully adjusted initial action \( S[Z[\mathcal{N}]\phi, \mu_0[\mathcal{N}], \lambda[\mathcal{N}], \Omega[\mathcal{N}]] \), which gives rise to the bare effective action \( \Gamma[\phi^{cf}; \mu_0[\mathcal{N}], \lambda[\mathcal{N}], \Omega[\mathcal{N}], \mathcal{N}] \). Expressing the bare parameters \( \mu_0, \lambda, \Omega \) as a function of the physical quantities and the cut-off, the expansion coefficients of the renormalised effective action

\[ \Gamma^R[\phi^{cf}; \mu_{\text{phys}}, \lambda_{\text{phys}}, \Omega_{\text{phys}}] := \Gamma[Z[\mathcal{N}]\phi^{cf}, \mu_0[\mathcal{N}], \lambda[\mathcal{N}], \Omega[\mathcal{N}], \mathcal{N}]}|_{\mu_{\text{phys}}, \lambda_{\text{phys}}, \Omega_{\text{phys}}=\text{const}} \]  

are finite and convergent in the limit \( \mathcal{N} \to \infty \). In other words,

\[ \lim_{\mathcal{N} \to \infty} \mathcal{N} \frac{d}{d\mathcal{N}} \left( Z^N[\mathcal{N}] \Gamma_{m_1n_1;\ldots;m_Nn_N}[\mu_0[\mathcal{N}], \lambda[\mathcal{N}], \Omega[\mathcal{N}], \mathcal{N}] \right) = 0 . \]  

This implies the renormalisation group equation

\[ \lim_{\mathcal{N} \to \infty} \left( N \frac{\partial}{\partial \mathcal{N}} + N \gamma + \mu_0^2 \beta_{\mu_0} \frac{\partial}{\partial \mu_0} + \beta_\lambda \frac{\partial}{\partial \lambda} + \beta_\Omega \frac{\partial}{\partial \Omega} \right) \Gamma_{m_1n_1;\ldots;m_Nn_N}[\mu_0, \lambda, \Omega, \mathcal{N}] = 0 , \]

where

\[ \beta_{\mu_0} = \frac{1}{\mu_0^2} \mathcal{N} \frac{\partial}{\partial \mathcal{N}} \left( \mu_0^2[\mu_{\text{phys}}, \lambda_{\text{phys}}, \Omega_{\text{phys}}, \mathcal{N}] \right) , \]  

\[ \beta_\lambda = \mathcal{N} \frac{\partial}{\partial \mathcal{N}} \left( \lambda[\mu_{\text{phys}}, \lambda_{\text{phys}}, \Omega_{\text{phys}}, \mathcal{N}] \right) , \]

\[ \beta_\Omega = \mathcal{N} \frac{\partial}{\partial \mathcal{N}} \left( \Omega[\mu_{\text{phys}}, \lambda_{\text{phys}}, \Omega_{\text{phys}}, \mathcal{N}] \right) , \]

\[ \gamma = \mathcal{N} \frac{\partial}{\partial \mathcal{N}} \left( \ln Z[\mu_{\text{phys}}, \lambda_{\text{phys}}, \Omega_{\text{phys}}, \mathcal{N}] \right) . \]
4 One-loop computations

Defining \((\Delta J)_{mn} := \sum_{p,q \in \mathbb{N}^2} \Delta_{mn;pq} J_{pq}\) we write (parts of) the generating functional of connected Green’s functions up to second order in \(\lambda\):

\[
W[J] = \ln Z[0] + 4\pi^2 \theta^2 \sum_{m,n,k,l \in \mathbb{N}^2} \frac{1}{2} J_{mn} \Delta_{mn,kl} J_{kl} - (4\pi^2 \theta^2)^\frac{\lambda}{4} \sum_{m,n,k,l \in \mathbb{N}^2} \left\{ (\Delta J)_{ml}(\Delta J)_{lk}(\Delta J)_{kn}(\Delta J)_{nm} + \frac{1}{4\pi^2 \theta^2} \left( (\Delta_{nm;kn}(\Delta J)_{ml}(\Delta J)_{lk} + \Delta_{kn;lk}(\Delta J)_{nm}(\Delta J)_{ml} \\
+ \Delta_{nm;ml}(\Delta J)_{lk}(\Delta J)_{kn} + \Delta_{lk;mn}(\Delta J)_{kn}(\Delta J)_{nm} \right) + \frac{1}{4\pi^2 \theta^2} \left( (\Delta_{nm;lk}(\Delta J)_{kn}(\Delta J)_{ml} + \Delta_{kn;ml}(\Delta J)_{nm}(\Delta J)_{lk} \right) + \frac{1}{2(4\pi^2 \theta^2)^2} \left\{ \left( (\Delta_{nm;kn}(\Delta J)_{ml} + \Delta_{lk;mn}(\Delta J)_{kl} \right) \right. \\
\left. \times (\Delta J)_{ru}(\Delta J)_{ut} \right\} + 5 \text{ permutations of}_{ts, sr, ru, ut} \right\} + \mathcal{O}(\lambda^3) \tag{26}
\]

In second order in \(\lambda\) we get a huge number of terms so that we display only the 1PI contribution with four \(J\)'s.

For the classical field \((16)\) we get \(\phi_{\text{cl}} = \sum_{p,q \in \mathbb{N}^2} \Delta_{nm;pq} J_{pq} + \mathcal{O}(\lambda)\) so that

\[
J_{pq} = \sum_{r,s \in \mathbb{N}^2} G_{qp;rs} \phi_{\text{cl}}^{rs} + \mathcal{O}(\lambda) \tag{27}
\]

The remaining part not displayed in \((27)\) removes the 1PR-contributions when passing to
\[ \Gamma[\phi^{cl}] \]. We thus obtain

\[ \Gamma[\phi^{cl}] = \Gamma[0] \]

\[ + 4\pi^2 \theta^2 \sum_{m,n,k,l \in \mathbb{N}^2} \frac{1}{2} \left\{ G_{mn;kl} + \frac{\lambda}{6(4\pi^2 \theta^2)} \left( \delta_{ml} \sum_{p \in \mathbb{N}^2} \Delta_{pn;kp} + \delta_{kn} \sum_{p \in \mathbb{N}^2} \Delta_{mp;pl} \right) \right\} \] (28a)

\[ + \frac{\lambda}{6(4\pi^2 \theta^2)} \Delta_{ml;kn} + O(\lambda^2) \} \phi_{mn}^{cl} \phi_{kl}^{cl} \] (28b)

\[ + 4\pi^2 \theta^2 \sum_{m,n,k,l,r,s,t,u \in \mathbb{N}^2} \frac{\lambda}{4!} \left\{ \delta_{nk} \delta_{lr} \delta_{st} \delta_{um} \right\} \] (28c)

\[ - \frac{\lambda}{2(4\pi^2 \theta^2)} \sum_{p,q \in \mathbb{N}^2} \left( 4\Delta_{mp;qs} \Delta_{pl;rq} \delta_{kn} \delta_{ur} + 4\Delta_{kp;qs} \Delta_{pm;rq} \delta_{ml} \delta_{st} \right) \] (28d)

\[ + \sum_{p \in \mathbb{N}^2} \left( 4\Delta_{ml;ps} \Delta_{kn;tp} \delta_{ur} + 4\Delta_{kn;ps} \Delta_{ml;tp} \delta_{ur} + 4\Delta_{mp;ts} \Delta_{pl;ru} \delta_{nk} \right) \] (28e)

\[ + \sum_{p,q \in \mathbb{N}^2} \left( 4\Delta_{pl;qs} \Delta_{mp;rq} \delta_{nk} \delta_{ur} + 4\Delta_{kp;qs} \Delta_{mp;rq} \delta_{ml} \delta_{st} \right) \] (28f)

\[ + 4\Delta_{ml;ts} \Delta_{kn;ru} + 4\Delta_{kn;ts} \Delta_{ml;ru} \right) + O(\lambda^2) \} \phi_{mn}^{cl} \phi_{kl}^{cl} \phi_{st}^{cl} \phi_{tu}^{cl} \] (28g)

\[ + O(\lambda^2) \] .

Here, (28a) contains the contribution to the planar two-point function and (28b) the contribution to the non-planar two-point function. Next, (28c) and (28d) contribute to the planar four-point function, whereas (28e), (28f) and (28g) constitute three different types of non-planar four-point functions.

Introducing the cut-off \( p, q \in \mathbb{N}^2 \) in the internal sums over \( p, q \), we split the effective action according to [2] as follows into a relevant/marginal and an irrelevant piece (\( \Gamma[0] \) can be ignored):

\[ \Gamma[\phi^{cl}] \equiv \Gamma_{\text{rel/marg}}[\phi^{cl}] + \Gamma_{\text{irrel}}[\phi^{cl}] \] (29)

\[ \Gamma_{\text{rel/marg}}[\phi^{cl}] = 4\pi^2 \theta^2 \sum_{m,n,k,l \in \mathbb{N}^2} \frac{1}{2} \left\{ G_{mn;kl} + \frac{\lambda}{6(4\pi^2 \theta^2)} \delta_{ml} \delta_{kn} \left( 2 \sum_{p^1,p^2 = 0}^{N} \Delta_{0 p^1, p^2 0} \right) \right\} \phi_{mn}^{cl} \phi_{kl}^{cl} \]

\[ + \frac{\lambda}{4!} \left( 1 + m^1 + m^2 + n^2 \right) \sum_{p^1,p^2 = 0}^{N} \left( \Delta_{0 p^1, p^2 0} \delta_{p^1, 0}^1 \delta_{p^2, 0}^1 - \Delta_{0 p^1, p^2 0} \right) + O(\lambda^2) \} \phi_{mn}^{cl} \phi_{kl}^{cl} \]

\[ + 4\pi^2 \theta^2 \sum_{m,n,k,l \in \mathbb{N}^2} \frac{\lambda}{4!} \left\{ 1 - \frac{\lambda}{3(4\pi^2 \theta^2)} \sum_{p^1,p^2 = 0}^{N} \left( \Delta_{0 p^1, p^2 0} \right)^2 \right\} + O(\lambda^2) \} \phi_{mn}^{cl} \phi_{nk}^{cl} \phi_{kl}^{cl} \phi_{lm}^{cl} \] (30)
To the marginal four-point function and the relevant two-point function there contribute only the projections to planar graphs with vanishing external indices. The marginal two-point function is given by the next-to-leading term in the discrete Taylor expansion around vanishing external indices.

In a regime where $\lambda[\mathcal{N}]$ is so small that the perturbative expansion is valid in $\mathcal{O}(\theta)$, the irrelevant part $\Gamma_{\text{irrel}}$ can be completely ignored. Comparing (30) with the initial action according to (32), (33) and (34), we have $\Gamma_{\text{rel/marg}}[\mathcal{Z}\phi^c]\mathcal{Z} = S[\phi^c, \mu_{\text{phys}}, \lambda_{\text{phys}}, \Omega_{\text{phys}}]$ with

$$Z = 1 - \frac{\lambda}{192\pi^2\theta} \sum_{p^1, p^2=0}^N (\Delta_{0, p^1, p^1 0} - \Delta_{0, p^1, p^1 0}) + \mathcal{O}(\lambda^2), \quad (31)$$

$$\mu^2_{\text{phys}} = \mu^2_0 \left( 1 + \frac{\lambda}{12\pi^2\theta^2 \mu_{\text{phys}}} \sum_{p^1, p^2=0}^N (2\Delta_{0, p^1, p^1 0} - \Delta_{0, p^1, p^1 0}), \quad (32)$$

$$\lambda_{\text{phys}} = \lambda \left( 1 - \frac{\lambda}{12\pi^2\theta^2} \sum_{p^1, p^2=0}^N (\Delta_{0, p^1, p^1 0})^2 \right.$$\n
$$\quad - \frac{\lambda}{48\pi^2\theta} \sum_{p^1, p^2=0}^N (\Delta_{0, p^1, p^1 0}) + \mathcal{O}(\lambda^2)), \quad (33)$$

$$\Omega_{\text{phys}} = \Omega \left( 1 + \frac{\lambda(1-\Omega^2)}{192\pi^2\theta^2} \sum_{p^1, p^2=0}^N (\Delta_{0, p^1, p^1 0}) + \mathcal{O}(\lambda^2) \right). \quad (34)$$

Solving (32), (33) and (34) for the bare quantities, we obtain to one-loop order

$$\mu^2_0[\mu_{\text{phys}}, \lambda_{\text{phys}}, \Omega_{\text{phys}}, \mathcal{N}]$$

$$= \mu^2_{\text{phys}} \left( 1 - \frac{\lambda_{\text{phys}}}{12\pi^2\theta^2 \mu_{\text{phys}}} \sum_{p^1, p^2=0}^N \Delta_{0, p^1, p^1 0} \right.$$\n
$$\quad + \frac{\lambda_{\text{phys}}}{96\pi^2\theta} \left( 1 + \frac{8}{\theta \mu_{\text{phys}}} \sum_{p^1, p^2=0}^N (\Delta_{0, p^1, p^1 0})^2 \quad (35)$$

$$\lambda[\mu_{\text{phys}}, \lambda_{\text{phys}}, \Omega_{\text{phys}}, \mathcal{N}]$$

$$= \lambda_{\text{phys}} \left( 1 + \frac{\lambda_{\text{phys}}}{12\pi^2\theta^2} \sum_{p^1, p^2=0}^N (\Delta_{0, p^1, p^1 0})^2 \quad (36)$$

$$\quad + \mathcal{O}(\lambda_{\text{phys}}^2) \right).$$
\[\Omega[\mu_{\text{phys}}, \lambda_{\text{phys}}, \Omega_{\text{phys}}, \mathcal{N}] = \Omega_{\text{phys}} \left(1 - \frac{\lambda_{\text{phys}}(1-\Omega_{\text{phys}}^2)}{192\pi^2 \theta \Omega_{\text{phys}}^2} \sum_{p^1, p^2 = 0}^{N} \left(\Delta_{p^1_{\mu} p^2_{\nu}} - \Delta_{p^2_{\mu} p^1_{\nu}}\right) + \mathcal{O}(\lambda_{\text{phys}}^2)\right).\] (37)

Inserting (12) into (34) we can now compute the \( \beta_{\lambda} \)-function (23) up to one-loop order, omitting the index \( \text{phys} \) on \( \mu^2 \) and \( \Omega \) for simplicity:

\[\beta_{\lambda} = \frac{\lambda_{\text{phys}}^2}{48\pi^2} \mathcal{N} \frac{\partial}{\partial \mathcal{N}} \sum_{p^1, p^2 = 0}^{N} \left\{ \left(2 F_1 \frac{1}{(1+\Omega)^2(1 + \frac{\mu_{\text{phys}}^2}{8\Omega_{\text{phys}}} + \frac{1}{2}(p^1+p^2))} \right)^2 \right.\]
\[\left. + \frac{p^1(1-\Omega)^2}{(1+\Omega)^4(\frac{1}{2} + \frac{\mu_{\text{phys}}^2}{8\Omega_{\text{phys}}} + \frac{1}{2}(p^1+p^2))} \left(\frac{3}{2} + \frac{\mu_{\text{phys}}^2}{8\Omega_{\text{phys}}} + \frac{1}{2}(p^1+p^2)\right) \frac{(5 \frac{1}{2} + \mu_{\text{phys}}^2 + \frac{1}{2}(p^1+p^2))}{2(1+\Omega)^2(1 + \frac{\mu_{\text{phys}}^2}{8\Omega_{\text{phys}}} + \frac{1}{2}(p^1+p^2))} \right\} + \mathcal{O}(\lambda_{\text{phys}})\right\}.\] (38)

Symmetrising the numerator in the second line \( p^1 \mapsto \frac{1}{2}(p^1+p^2) \) and using the expansions

\[2 F_1 \left(1, \frac{a-p}{b+p} \bigg| z\right) = \frac{1}{1+z} + \frac{z(a+b) + z^2(a+b-2)}{p(1+z)^3} + \mathcal{O}(p^{-2}),\]
\[2 F_1 \left(3, \frac{a-p}{b+p} \bigg| z\right) = \frac{1}{(1+z)^3} + \mathcal{O}(p^{-1}),\] (39)

which are valid for large \( p \), we obtain up to irrelevant contributions vanishing in the limit \( N \to \infty \)

\[\beta_{\lambda} = \frac{\lambda_{\text{phys}}^2}{48\pi^2} \mathcal{N} \frac{\partial}{\partial \mathcal{N}} \sum_{p^1, p^2 = 0}^{N} \left\{ \frac{1}{(1+\Omega_{\text{phys}}^2)^2} \frac{1}{1 + p^1+p^2} \left(1 + \frac{(1-\Omega_{\text{phys}}^2)^2}{2(1+\Omega_{\text{phys}}^2)^2} - \frac{(1+\Omega_{\text{phys}}^2)}{2}\right) \right.\]
\[\left. + \mathcal{O}(\lambda_{\text{phys}}^3) + \mathcal{O}(\mathcal{N}^{-1})\right\} + \mathcal{O}(\mathcal{N}^{-1}),\] (40)

Similarly, one obtains

\[\beta_{\Omega} = \frac{\lambda_{\text{phys}}}{96\pi^2} \Omega_{\text{phys}} \left(1 - \frac{\Omega_{\text{phys}}^2}{1+\Omega_{\text{phys}}^2}\right) + \mathcal{O}(\lambda_{\text{phys}}^3) + \mathcal{O}(\mathcal{N}^{-1}),\] (41)

\[\beta_{\mu_0} = -\frac{\lambda_{\text{phys}}}{48\pi^2 \theta \mu_{\text{phys}}^2} \left(4 \mathcal{N} \log(2) + \frac{(8+\theta \mu_{\text{phys}}^2) \Omega_{\text{phys}}^2}{(1+\Omega_{\text{phys}}^2)^2}\right) + \mathcal{O}(\lambda_{\text{phys}}^3) + \mathcal{O}(\mathcal{N}^{-1}),\] (42)
\[ \gamma = \frac{\lambda_{\text{phys}}^2}{96\pi^2} \frac{\Omega_{\text{phys}}^2}{(1+\Omega_{\text{phys}}^2)^3} + \mathcal{O}(\lambda_{\text{phys}}^2) + \mathcal{O}(\mathcal{N}^{-1}) \]  

5 Discussion

We have computed the one-loop \( \beta \)- and \( \gamma \)-functions in real four-dimensional duality-covariant noncommutative \( \phi^4 \)-theory. Remarkably, this model has a one-loop contribution to the wavefunction renormalisation which compensates partly the contribution from the planar one-loop four-point function to the \( \beta_\lambda \)-function. The one-loop \( \beta_\lambda \)-function is non-negative and vanishes in the distinguished case \( \Omega = 1 \) of the duality-invariant model, see (3). At \( \Omega = 1 \) also the \( \beta_\Omega \)-function vanishes. This is of course expected (to all orders), because for \( \Omega = 1 \) the propagator (12) is diagonal, \( \Delta_{m_1 n_1, k_1 l_1}^{m_2 n_2, k_2 l_2} |_{\Omega=1} = \delta_{m_1 k_1} \delta_{n_1 l_1} \delta_{m_2 k_2} \delta_{n_2 l_2} \), so that the Feynman graphs never generate terms with \(|m_i - l_i| = |n_i - k_i| = 1\) in (8).

The similarity of the duality-invariant theory with the exactly solvable models discussed in [4] suggests that also the \( \beta_\lambda \)-function vanishes to all orders for \( \Omega = 1 \). The crucial differences between our model with \( \Omega = 1 \) and [4] is that we are using real fields, for which it is not so clear that the construction of [4] can be applied. But the planar graphs of a real and a complex \( \phi^4 \)-model are very similar so that we expect identical \( \beta_\lambda \)-functions (possibly up to a global factor) for the complex and the real model. Since a main feature of [4] was the independence on the dimension of the space, the model with \( \Omega = 1 \) and matrix cut-off \( \mathcal{N} \) should be (more or less) equivalent to a two-dimensional model, which has a mass renormalisation only [8]. Therefore, we conjecture a vanishing \( \beta_\lambda \)-function in four-dimensional duality-invariant noncommutative \( \phi^4 \)-theory to all orders.

The most surprising result is that the one-loop \( \beta_\Omega \)-function also vanishes for \( \Omega \to 0 \). We cannot directly set \( \Omega = 0 \), because the hypergeometric functions in (38) become singular and the expansions (39) are not valid. Moreover, the power-counting theorems of [2], which we used to project to the relevant/marginal part of the effective action (30), also require \( \Omega > 0 \). However, in the same way as in the renormalisation of two-dimensional noncommutative \( \phi^4 \)-theory [8], it is possible to switch off \( \Omega \) very weakly with the cut-off \( \mathcal{N} \), e.g. with

\[ \Omega = e^{-\left(\ln(1+\ln(1+\mathcal{N}))\right)^2}. \]  

The decay (44) for large \( \mathcal{N} \) over-compensates the growth of any polynomial in \( \ln \mathcal{N} \), which according to [2] is the bound for the graphs contributing to a renormalisation of \( \Omega \). On the other hand, (14) does not modify the expansions (13). Thus, in the limit \( \mathcal{N} \to \infty \), we have constructed the usual noncommutative \( \phi^4 \)-theory given by \( \Omega = 0 \) in (2) at the one-loop level. It would be very interesting to know whether this construction of the noncommutative \( \phi^4 \)-theory as the limit of a sequence (14) of duality-covariant \( \phi^4 \)-models can be extended to higher loop order.

We also notice that the one-loop \( \beta_\lambda \)- and \( \beta_\Omega \)-functions are independent of the noncommutativity scale \( \theta \). There is, however a contribution to the one-loop mass renormalisation via the dimensionless quantity \( \mu_{\text{phys}}^2 \theta \), see (12).
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