GENERAL BALANCE FUNCTIONS
IN THE THEORY OF INTEREST

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ABSTRACT. We develop an axiomatic theory of balance functions (future value functions) in
the theory of interest that is derived from financial considerations and which applies to general
regulated payment streams, including continuous payment streams. Balance functions exist
and are unique up to an initial choice of deposit and investment accumulation functions. In
terms of these balance functions we also construct a unique internal rate of return for each
regulated payment stream that is an investment project. This theory subsumes and clarifies
previous theories of internal rate of return functions for more specialized classes of investment
projects.

KEY WORDS: balance function; investment project; internal rate of return

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1. Introduction

1.1. A historical problem in mathematical finance is the determination of an internal rate of return (IRR) for a general investment project. The financial motivation is that portfolios with a high IRR make for more attractive investment choices. The special case of determining the IRR of a loan contract, i.e., an initial cash outflow followed by a finite sequence of cash inflows, is a well-known computation in the financial mathematics literature. Briefly, the present value function is a strictly decreasing function of the rate per period $i$ whose unique root is the IRR of the loan contract, cf. Donald (1970), Kellison (1991). Indeed, the computation of the IRR in the case of loan contracts (mortgages, bonds, etc.) constitutes one of the main applications historically of financial mathematics to problems in the business world. However general investment projects are not loan contracts since they typically involve positive cash flows (inflows) interspersed with negative cash flows (outflows); it is the occurrence of large swings in the sequence of inflows and outflows that is responsible for the failure of the present value method to determine a unique IRR for a general investment project: in general multiple roots must occur. This rather awkward situation has led over the years to a search for more general methods for computing the IRR of investment projects that include as a special case the classical IRR computation for loan contracts. Notable among these generalizations are the following:

(i) Arrow and Levhari (1969) define a unique IRR $r_f$ for an investment project $f$ that is defined by a continuous payment stream with finite time horizon (duration) that also is differentiable and changes sign only a finite number of times. For a constant rate of discount, these authors consider the maximum of the present values of the project $f$ calculated over all truncated time periods, i.e., over all initial time intervals of the project $f$. The authors’ key observation is that this maximized present value is a monotone decreasing function of the rate of discount. The unique root of this decreasing function is defined to be the IRR $r_f$ of the investment project $f$. The IRR $r_f$ coincides with the classical IRR in case the investment project $f$ is a discrete loan contract.
The decreasing property of the above maximized present value function is a consequence of the well-known argument of “positive tails,” perhaps first employed in the economic literature by Wright (1959) in the case of finite cash flow sequences; further details are provided in Promislow and Spring (1996, Appendix). Refinements of the approach of Arrow and Levhari (1969) are developed in Fleming and Wright (1971), Sen (1975).

(ii) A completely different solution to the IRR problem, of importance to this paper, was proposed by Teichroew et al., (1965a,b) in the context only of investment projects $f$ defined by finite cash flow sequences. These authors begin with the financial observation that there are two types of interest rates: a deposit rate of interest that applies to current balances that are positive, i.e., current surpluses; an investment rate of interest that applies to current balances that are negative, i.e., current debts. In practice the current deposit rate is less than the current investment rate. According to these authors the current balance $B_j(f)$ of an investment project $f$, at the time $t_j$ of the $j$th cash flow $C_j$ of the project $f$, should be calculated inductively by applying the deposit interest rate, respectively the investment interest rate, to the previous current balance $B_{j-1}(f)$ during the $j$th period $[t_{j-1}, t_j]$, according to whether the balance $B_{j-1}(f) \geq 0$ (a surplus), respectively $B_{j-1}(f) \leq 0$ (a debt). In the special case that the deposit and investment rates of interest coincide then the successive balances $B_j(f)$ are just the classical accumulated value functions at times $t_j$ determined by this common interest rate. For a fixed deposit rate $i$ per period during the life of the project, these authors crucially observe that each current balance $B_j(f)$, calculated inductively as above, is a monotone decreasing function of the investment rate of interest. The IRR $i_f$ of an investment project $f$ is then defined to be the unique root of the monotone decreasing balance function $B_n(f)$ calculated at end of the investment project (at the time $t_n$ of the last cash flow $C_n$ of the project). The IRR $i_f$ also coincides with the classical IRR in case the investment project $f$ is a discrete loan contract.

This “two-interest-rate” theory for calculating current balances circumvents the problem of multiple roots that occur in the present value method for determining the IRR. Multiple
roots occur because of the implicit assumption, false for economic reasons, that the above deposit and investment rates of interest coincide. Standard texts, Kellison (1991), discuss with examples the IRR due to Teichroew et al., (1965a,b).

Despite the different financial points of view outlined briefly in (i), (ii) above, Promislow and Spring (1996) prove somewhat surprisingly that the IRR \( r_f \) due to Arrow and Levhari (1969), in the context of finite cash flow sequences, is a special case of the IRR \( i_f \) due to Teichroew et al., (1965a,b) when the deposit rate per period \( i \) tends to infinity: \( r_f = \lim_{i \to \infty} i_f \). In this sense the IRR \( i_f \) is more general than the IRR \( r_f \), and depends on an analysis of balance functions in terms of deposit and investment rates of interest.

1.2. An open question in the literature, related to the above issues, is whether the IRR of Teichroew et al., (1965a) can be extended to the general case of investment projects defined by continuous payment streams, and if so whether the IRR of Arrow and Levhari (1969) will again be a limiting case in this general context. This generalization requires a suitable theory of balance functions for continuous payment streams. A major obstacle to this generalization is that the theory of Teichroew et al., (1965a) treats only finite cash flow sequences for which balance functions are defined inductively and depend importantly on the sign of the previously defined balance function. This inductive procedure to define balance functions cannot apply to continuous payment streams. In addition the work of Promislow and Spring (1996) suggests that the IRR of Teichroew et al., (1965a) should be universal with respect to IRR functions defined in terms of balance functions. These questions form the subject matter of this paper. We remark here that the question of an expectation value for the IRR, in the stochastic setting, has yet to be addressed in this burgeoning new area of financial mathematics. This presupposes a clearer understanding in the financial literature of the determination of the IRR in the classical setting of general investment projects subject to deterministic accumulation functions. The results of our paper contribute towards a better understanding of this classical situation.

1.3. In this paper we solve the balance function problem posed above in §1.2 by developing
a new theory of balance functions (§4), expressed in axiomatic terms, that is sufficiently
general to treat payment streams that are regulated functions of “finite time horizon”

i.e., regulated payment streams supported on compact time intervals (§3.2). Continuous
payment streams and step function payment streams are important special cases. Mathematically, a function \( f \) on a compact interval \([a, b]\) is \textit{regulated} if \( f = \lim_{n \to \infty} f_n \) where each

\( f_n \) is a step function on \([a, b]\) (step function payment streams correspond to finite cash flow
sequences), and where the limit is taken in the topology of uniform convergence of functions
on the compact interval \([a, b]\). Let \( B_t(f) \in \mathbb{R} \) denote the balance of the payment stream

\( f \) at time \( t \). The continuity axiom \( A_5 \) states that the balance \( B_t(f) = \lim_{n \to \infty} B_t(f_n) \),

where \( f = \lim_{n \to \infty} f_n \) as above. In this way balance functions on the space of step func-
tion payment streams extend by the continuity axiom to balance functions on the space of
regulated payment streams. This is the essence of our topological approach to the theory
of balance functions on general regulated payment streams. Implicit in this topological
approach is the development of analytic estimates that ensure the convergence properties
of the limit in axiom \( A_5 \). Some of these analytic estimates are rather lengthy, as in the
proof of Theorem 5.3, and therefore are relegated to the Appendix.

We note here that the axioms for balance functions (§4) allow one to reconstruct deposit
and investment accumulation functions in the spirit of Teichroew et al.,(1965a). Briefly, let \( a(s, t) \geq 0 \) be the balance at time \( t \) of a single cash flow of 1 (deposit of 1 unit) at time

\( s, s \leq t \); similarly let \( b(s, t) \geq 0 \) be the negative of the balance at time \( t \) of a single cash
flow of \(-1 \) (debt of 1 unit) at time \( s, s \leq t \). It follows from Lemma 4.1 that \( a(s, t), b(s, t) \)
are accumulation functions (§2), denoted as deposit, respectively, investment accumulated
functions. In general \( a(s, t), b(s, t) \) are distinct accumulation functions, in conformity
with theory developed by Teichroew et al., (1965a,b). Indeed, a special case is \( a(s, t) = (1+\alpha)^{t-s}, b(s, t) = (1+\beta)^{t-s} \), where \( \alpha \), respectively \( \beta \), is the deposit rate, respectively the
investment rate, per period that was introduced in Teichroew et al.,(1965a). In this way
the axioms code for a “two-interest-rate” general theory of balance functions that applies
for example to continuous payment streams.

Theorem 4.5 proves that balance functions that satisfy the axioms do exist and are unique up to initial choices of deposit and investment accumulation functions. Furthermore, in terms of these balance functions, there is a natural way to define the IRR of all investment projects, including as a special case the IRR of Teichroew et al., (1965a); indeed, our construction of IRR functions is inspired by this special case, thus solving the IRR problem posed in §1.2. In this sense the IRR of Teichroew et al., (1965a) is seen to have a universal character since it occurs naturally in the context of balance functions that themselves are uniquely determined axiomatically by a priori financial considerations.

In somewhat more detail, let \( a(s, t) \) be a fixed positive (deposit) accumulation function of bounded variation (§2.2); in our theory \( a(s, t) \) applies to current balances that are \( \geq 0 \) (surpluses) at time \( s \). Let \( x^{t-s}, x \geq 0 \), be a variable (investment) accumulation function; in our theory \( x^{t-s} \) applies to current balances that are \( \leq 0 \) (debt) at time \( s \). The main result Theorem 5.3 proves that for each investment project \( f \) (§3.3) the balance function \( B_{d}^{x}(f) \), calculated at the time \( d \) at the end of the investment project \( f \), (the dependence on \( a(s, t) \) is omitted), is a strictly decreasing function of \( x \) such that \( \lim_{x \to \infty} B_{d}^{x}(f) = -\infty \), and therefore has at most one root \( x = 1 + i_{f} \geq 0 \). The IRR of \( f \) is defined to be the parameter \( i_{f} \geq -1 \). If there is no root then \( i_{f} = -1 \). (cf. §5.2 for precise details). Although not shown here, if for example \( a(s, t) = (1 + i)^{t-s}, i \geq -1 \), then again \( \lim_{i \to \infty}(i_{f}) = r_{f} \), the IRR defined by Arrow and Levhari (1969), in case \( f \) is also a continuous payment stream.

The IRR \( i_{f} \) coincides with the IRR of Teichroew et al., (1965) in case \( f \) corresponds to a finite number of cash flows (the unit of time is 1 period). In particular, \( i_{f} \) equals the classical IRR in case \( f \) corresponds to a loan contract (mortgages, bonds etc.). To summarize, we propose in this paper an axiomatic theory of balance functions, in terms of which we define an IRR that provides a comprehensive solution to the historical problem of defining an IRR for general investment projects which occur in mathematical economics and finance.
The generality of our approach requires the development \textit{ab initio} of the theory of accumulation functions (§2) and of payment streams (§3). While economic arguments are indicated where appropriate, the estimates developed in §4, §5 to justify our topological approach for proving the main results are presented there in detail since there is no convenient reference to the economic and mathematical literature for these types of calculations.

2. Accumulation Functions

2.1. Accumulation functions are basic to the theory of interest since they relate, in mathematical terms, the value of invested capital at any one date to its value at any subsequent date. In this section we develop the theory of accumulation functions in a more general setting than appears in the economic and financial literature.

Let $H = \{(s, t) \in \mathbb{R}^2 \mid s \leq t\}$, the half-space above the line $y = x$ in $\mathbb{R}^2$. A non-negative function $a: H \to [0, \infty)$, denoted $a \geq 0$, is an accumulation function if $a(t, t) = 1$ for all $t \in \mathbb{R}$, and if the following multiplicative property is satisfied:

$$a(r, s) \cdot a(s, t) = a(r, t) \text{ for all } r \leq s \leq t. \quad (2.1)$$

Accumulation functions are not assumed to be continuous, and the value $a(s, t) = 0$ is allowed. An extreme example is the zero accumulation function: $0(t, t) = 1$ and $0(s, t) = 0$ for all $s < t$. In financial terms $a(s, t)$ is the accumulated (future) value at time $t$ of one monetary unit invested at time $s$, for all $s \leq t$ (throughout, unless specified to the contrary, the conventional time unit is 1 year). If the accumulation function $a(s, t)$ is positive i.e., $a: H \to (0, \infty)$, denoted $a > 0$, then $a(s, t)$ extends naturally to all of $\mathbb{R}^2$ (same notation) by requiring (2.1) to hold universally:

$$a(r, s) \cdot a(s, t) = a(r, t) \text{ for all } r, s, t. \quad (2.2)$$

In particular, $a(s, t) \cdot a(t, s) = a(s, s) = 1$; hence $a(t, s) = 1/a(s, t)$ for all $s, t$. Let $f: \mathbb{R} \to (0, \infty)$ be the positive function $f(t) = a(x_0, t)$ ($x_0$ is an arbitrary reference point). Setting $r = x_0$ in (2), it follows that,

$$a(s, t) = f(t)/f(s) \text{ for all } s, t. \quad (2.3)$$
Furthermore, writing \( f(t) = e^{g(t)} \), we obtain the standard representation of positive accumulation functions,

\[
(2.4) \quad a(s, t) = e^{g(t) - g(s)} \quad \text{for all } s, t.
\]

Note that \( g : \mathbb{R} \to \mathbb{R} \) is unique up to addition of a constant. In the classical theory of interest

\[
g(t) = \int_0^t \delta(u) \, du,
\]

where the continuous function \( \delta(t) \) is called the force of interest; in measure theoretic terms \( \delta(t) \) is the density function associated to \( g(t) \). The classical example is \( g(t) = rt \), for which \( a(s, t) = e^{rt - s} \), where \( r \) is the rate of continuously compounded interest.

If \( a(s, t) > 0 \) then \( a(t, s) = 1/a(s, t) \) is the present value (price) at time \( s \) of one monetary unit at time \( t \geq s \). If \( a(s, t) = 0 \) for \( s < t \), then one monetary unit at time \( s \) becomes worthless (value 0) at time \( t \). The multiplicative property (2.1) ensures coherence of monetary values at all intermediate times. In practical examples \( a(s, t) = (1 + i)^{t-s} \), where \( i > -1 \) is a constant rate per period (the time unit is 1 period). The value at time \( t \) of one monetary unit at time \( s \) is \( (1 + i)^{t-s} \), a basic computation in financial mathematics.

A convenient equivalent formulation of accumulation functions is in terms of real-valued functions defined on the set of all compact intervals in \( \mathbb{R} \): if \( J = [s, t] \), \( s \leq t \), then \( a(J) = a(s, t) \). The multiplicative property (2.1) is then expressed as follows.

\[
(2.5) \quad a(J \cup K) = a(J) \cdot a(K),
\]

where \( J = [r, s] \), \( K = [s, t] \) are adjacent compact intervals. Accumulation functions are partially ordered in the obvious way: \( a \leq b \) if and only if \( a(J) \leq b(J) \) for all compact intervals \( J = [s, t], s \leq t \). Evidently, the product \( a(J) \cdot b(J) \) of accumulation functions is an accumulation function.

**2.2. Monotone Accumulation Functions.** An accumulation function \( a(J) \) is monotone increasing (decreasing) if \( a(J) \leq a(K) \) (\( a(J) \geq a(K) \)) for all nested compact intervals \( J \subseteq
Since $a(t,t) = 1$ for all $t \in \mathbb{R}$, it follows that if $a(J)$ is monotone increasing, respectively decreasing, then $a \geq 1$, hence positive, respectively $a \leq 1$. Evidently, if $y_1(J), y_2(J)$ are monotone increasing, respectively decreasing, accumulation functions then the product accumulation function $y(J) = y_1(J) \cdot y_2(J)$ is also monotone increasing, and $y \geq y_1, y \geq y_2$ (a common upper bound), respectively $y \leq y_1, y \leq y_2$ (a monotone decreasing common lower bound). A positive accumulation function $a(s,t) = \exp(g(t) - g(s))$ is monotone increasing if and only if the function $g: \mathbb{R} \to \mathbb{R}$ is monotone increasing.

An accumulation function is positive on an interval $I$ if $a(s,t) > 0$ for all $s, t \in I$, written $a > 0$ on $I$. As in (2.4), $a(s,t) = e^{g(t)-g(s)}$ for all $s, t \in I$, where $g: I \to \mathbb{R}$. The positive accumulation function $a(s,t)$ is defined to be of bounded variation on the compact interval $I$ if the corresponding function $g: I \to \mathbb{R}$ is of bounded variation. For example, monotone increasing accumulation functions are of bounded variation.

Let $a(s,t) = e^{g(t)-g(s)}$ be an accumulation function of bounded variation on a compact interval $I$. Let $V_g([s,t])$ be the variation function associated to $g$, defined on all intervals $[s,t] \subseteq I$. Thus $V_g$ is a finitely additive interval function and is monotone increasing: $V_g(J) \leq V_g(K)$ for all subintervals $J \subseteq K \subseteq I$. Note that $|g(t) - g(s)| \leq V_g(s,t)$ for all subintervals $[s,t] \subseteq I$. Let $h: I \to \mathbb{R}$ be the monotone increasing function $h(t) = V(c,t)$, $t \in I$. Then $y(s,t) = e^{h(t)-h(s)} = e^{V_g(s,t)}$ is a monotone increasing accumulation function on $I$ such that $a \leq y$ on $I$. Similarly, $x(s,t) = e^{h(s)-h(t)} = e^{-V(s,t)}$ is monotone decreasing and $a \geq x$ on $I$. Conversely one can prove that if $a > 0$ and $a \leq y$ where $y$ is monotone increasing, then $a$ is of bounded variation on all compact intervals.

3. Payment Streams

3.1. Regulated functions. As discussed in the introduction, the classical theory of interest has no framework for defining balance functions of type Teichroew et al., (1965) in the case of continuous payment streams. Our general theory of balance functions applies most naturally to payment streams that are regulated functions. These include all payment
streams of theoretical and of practical interest, such as continuous payment streams and step function payment streams associated to finite cash flow sequences.

A function \( f: \mathbb{R} \rightarrow \mathbb{R} \) is regulated if \( f \) has finite right-hand and left-hand limits at each \( t \in \mathbb{R} \). It is well-known, Bourbaki (1949, Ch.II, §1.3), Dieudonné (1960), that a function \( f: \mathbb{R} \rightarrow \mathbb{R} \) is regulated if and only if on each compact interval \([a, b] \subset \mathbb{R}\), \( f \) is the limit of step functions in the topology of uniform convergence on \([a, b]\). Employing pointwise right- and left-hand limits, it is clear that if \( f, g \) are regulated then the functions \( f + g, \ f \cdot g \) are regulated. The set of regulated functions on a compact interval strictly includes the sets of step functions, continuous functions, monotone functions and hence also the set of functions of bounded variation. (A function of bounded variation can be expressed as a difference of monotone functions.)

Regulated functions occur in the classical theory of interest in the special case of step function payment streams associated to finite cash flow sequences of the form \( \mathcal{C} = (C_i)_{0 \leq i \leq n} \) such that the cash flow \( C_i \) occurs at time \( t_i \in \mathbb{R} \), \( t_0 < t_1 \cdots < t_n \). Associated to \( \mathcal{C} \) is the step function \( f_\mathcal{C}: \mathbb{R} \rightarrow \mathbb{R} \), continuous on the right,

\[
(3.1) \quad f_\mathcal{C}(t) = \sum_{t_i \leq t} C_i.
\]

Hence \( f_\mathcal{C}(t) = 0 \) for all \( t < t_0 \) and \( f_\mathcal{C}(t) \) is constant = \( \sum_i C_i \) for all \( t \geq t_n \). Thus \( C_0 = f_\mathcal{C}(t_0) \), and if \( i \geq 1 \), the cash flow \( C_i \) is the difference,

\[
(3.2) \quad C_i = f_\mathcal{C}(t_i) - f_\mathcal{C}(t_{i-1}), \quad 1 \leq i \leq n.
\]

The cash flow \( C_i \) represents the jump of the step function \( f_\mathcal{C} \) at time \( t_i \), \( 0 \leq i \leq n \). Conversely, let \( f: \mathbb{R} \rightarrow \mathbb{R} \) be a step function, continuous on the right, such that \( f(t) = 0 \) on some interval \((-\infty, a)\). Evidently there is a finite cash flow sequence \( \mathcal{C} = (C_i)_{0 \leq i \leq n} \) such that \( f = f_\mathcal{C} \). Continuity on the right is the standard convention in financial mathematics which implies that a cash flow payment is at the receivers disposal immediately as it falls due and thereafter.
If $s < t$ note that $f_C(t) - f_C(s)$ is the sum of the cash flows in the interval $(s, t]$; hence the terminology that $f_C$ is a \textit{distribution function}: for each $t$, $f_C(t)$ is the sum of all the cash flows on the interval $(-\infty, t]$.

In this paper we develop the theory of interest based on regulated payment stream functions $f : \mathbb{R} \to \mathbb{R}$ which have compact support, §3.2. Our strategy is to prove general theorems in the case of step function payment streams of the type (3.1) above. An important feature is our topological approach: The main constructs (balance functions, internal rates of return etc.) are defined first on the space of step functions. Since step functions are dense in the space of compactly supported regulated functions (in the topology of uniform convergence), the corresponding constructs in the case of regulated payment stream functions are defined topologically by passing to the uniform limit.

3.2. Regulated Payment Streams

A regulated \textit{payment stream}, or flow function, is a regulated function $f : \mathbb{R} \to \mathbb{R}$, continuous on the right, which is supported in a compact interval in the following sense: there is a compact interval $[a, b] \subset \mathbb{R}$, $a \leq b$ (which depends on $f$) such that $f = 0$ on the interval $(-\infty, a)$ and $f$ is constant on the interval $[b, \infty)$. The intersection over all compact intervals on which $f$ is supported is the \textit{minimal support} of the regulated payment stream $f$.

Note that the minimal support is empty only in the extreme case that $f = 0$ on $\mathbb{R}$. The canonical example of a regulated payment stream is the step function (3.1), $f_C : \mathbb{R} \to \mathbb{R}$, associated to a finite cash flow sequence $C = (C_i)_{0 \leq i \leq n}$, supported in the interval $[t_0, t_n]$; this interval is the minimal support if and only if $C_0, C_n$ are both non-zero. As explained above, continuity on the right is the conventional requirement for payment streams in financial mathematics. Let $\mathcal{R}$ be the set of all regulated payment streams $f : \mathbb{R} \to \mathbb{R}$. For each compact interval $K \subset \mathbb{R}$, let $\mathcal{R}_K \subset \mathcal{R}$ be the subset of regulated payment streams whose minimal support is contained in $K$. If $K \subseteq L$, then $\mathcal{R}_K \subseteq \mathcal{R}_L$, and $\mathcal{R} = \bigcup_K \mathcal{R}_K$.

Let $\mathcal{S} \subset \mathcal{R}$ be the subset of step functions, and define $\mathcal{S}_K = \mathcal{S} \cap \mathcal{R}_K$.

For each $K$, $\mathcal{S}_K$ is dense in $\mathcal{R}_K$ in the topology of uniform convergence. To see this, let
$g \in \mathcal{R}_K$ and let $\epsilon > 0$. Since $g$ has left and right hand limits at each point we observe that for each $t$ there is function $h: I = (t - \eta, t + \eta) \rightarrow \mathbb{R}$ such that: $h(t) = g(t)$; $h$ is constant on each interval $(t - \eta, t)$, $[t, t + \eta)$ (in particular $h$ is right continuous); $|h - g|_I < \epsilon$. Since $K$ is compact, employing the observation, there is a partition of the interval $K = [u, v]$, $t_0 = u < t_1 < \cdots < t_n = v$, and a step function $f \in \mathcal{S}_K$ such that:

(i) $f(t_i) = g(t_i), \ 0 \leq i \leq n$; $f(t) = 0, \ t < u$; $f(t) = g(v), \ t \geq v$.

(ii) $f(t)$ is constant $= g(t_i)$ on the interval $[t_i, t_{i+1})$, $1 \leq i \leq n - 1$.

(iii) $|f(t) - g(t)| < \epsilon$ for all $t \in \mathbb{R}$.

Employing (3.1), (3.2), $f = f_C \in \mathcal{S}_K$, where $C = (C_i)_{0 \leq i \leq n}$, is defined as follows:

(3.3) \[ C_0 = g(t_0); \quad C_i = g(t_i) - g(t_{i-1}) \quad 1 \leq i \leq n. \]

Property (iii) shows that $\mathcal{S}_K$ is dense in $\mathcal{R}_K$. Employing (3.3) we remark also that all the cash flows of $C$, for the approximating step function $f_C$, are $\leq 0$, respectively $\geq 0$, if $g(t_0) \leq 0$ and $g$ is monotone decreasing, respectively $g(t_0) \geq 0$ and $g$ is monotone increasing.

Let $f \in \mathcal{R}_K$ and let $(f_n)_{n \geq 1}$ be a Cauchy sequence in $\mathcal{S}_K$ such that $f = \lim_{n \rightarrow \infty} f_n$ uniformly on $K$. For each $n$ the distribution function $f_n(t)$ is the sum of all the corresponding cash flows in the interval $(-\infty, t]$. Consequently, in the uniform limit, the payment stream $f(t)$ also is viewed as a distribution function which for each $t \in \mathbb{R}$ is the “total cash flow” in the interval $(-\infty, t]$. To explain this, suppose in addition $f \in \mathcal{R}_K$ is of bounded variation on $K$, hence of bounded variation on each compact interval in $\mathbb{R}$. For each $t \in \mathbb{R}$,

$$f(t) = \int_{-\infty}^{t} df,$$

where the Stieltjes integral is employed (cf. Promislow (1980)). Furthermore if $f \in \mathcal{R}_K$ is a continuous payment stream of class $C^1$, then,

(3.4) \[ f(t) = \int_{-\infty}^{t} df = \int_{-\infty}^{t} f' dt. \]
In this context \( f'(t) \) is the signed density function of the function \( f(t) \). Following the common practice in applied mathematics for interpreting Riemann integrals, it is still current in the financial and economics literature, Arrow and Levhari (1969), Kellison (1991, §4.8), to view \( df(t) = f'(t) \, dt \) as the payment or cash flow in the interval \([t, t + dt]\) at the density \( f'(t) \).

Consequently, employing the integral (3.4), \( f(t) \) is the total cash flow in the interval \((-\infty, t]\).

**Remark 3.1.** Let \( C \) be a finite cash flow sequence and let \( D \) be the cash flow sequence obtained from \( C \) by introducing cash flows of 0 at a finite number of additional partition points. Employing (3.1), it is clear that \( f_C = f_D \in S \), i.e., the addition of a finite number of 0 cash flows leaves invariant the corresponding step function. Conversely, if \( f_C = f_D \), then the cash flow sequences \( C, D \), differ at most by cash flows of 0 at a finite number of additional partition points.

In view of the above remark, we assume implicitly throughout this paper that step functions \( f_C, f_D \in S \) satisfy the additional property that the cash flow sequences \( C, D \) have a common set of partition points. In particular,

\[
f_C \pm f_D = f_{C \pm D}, \quad \text{where} \quad C \pm D = (C_i \pm D_i)_{0 \leq i \leq n}.
\]

For each \( C = (C_i)_{0 \leq i \leq n} \), let \( \| f_C \| = \sup_{0 \leq p \leq n} \{ |f_C(t_p)| = |C_0 + \cdots + C_p| \} \). Then the step function \( f_{C - D} \in S \) satisfies the following estimates.

\[
-\| f_{C - D} \| \leq \sum_{i=0}^{i=p} (C_i - D_i) \leq \| f_{C - D} \| \quad 0 \leq p \leq n.
\]

Since \( \sum_{i=p}^{i=q} (C_i - D_i) = \sum_{i=0}^{i=q} (C_i - D_i) - \sum_{i=0}^{i=p} (C_i - D_i) \) for all \( p \leq q \) it follows that

\[
(3.5) \quad -2\| f_{C - D} \| \leq \sum_{i=p}^{i=q} (C_i - D_i) \leq 2\| f_{C - D} \| \quad \text{for all} \ 0 \leq p \leq q \leq n.
\]

Since \( f_C \) is a step function then also \( \| f_C \| = \sup \{|f_C(t)| \mid t \in \mathbb{R}\} \), i.e., \( \| f_C \| \) is the sup-norm of \( f_C \), interpreted in terms of the sum of the associated cash flows of \( f_C \in S \).
3.3. Investment Projects

A regulated payment stream \( f : \mathbb{R} \to \mathbb{R} \), minimally supported on the interval \([a, b]\), is an \textit{investment project} if either (i) \( f(a) < 0 \), or (ii) \( f(a) = 0 \) and there is a \( \delta > 0 \) such that the restriction of the function \( f \) to the interval \( (a, a+\delta] \) is negative and is non-increasing. \( \mathcal{I} \subset \mathcal{R} \) is the subset of investment projects. \( \mathcal{I}_K = \mathcal{I} \cap \mathcal{R}_K \) is the subset of investment projects with minimal support in \( K \).

The investment project condition is interpreted to mean that either \( f(a) < 0 \) represents the initial outflow (start-up funds) for the project, or \( f(a) = 0 \) and there is an initial stream of outflows which constitutes these start-up funds. Employing (3.1), a step function \( f_C \in \mathcal{S}_K \) is an investment project, i.e., \( f_C \in \mathcal{S}_K \cap \mathcal{I}_K \), if and only if the initial cash flow \( C_0 < 0 \) (an initial outflow).

4. Axioms For Balance Functions

4.1. In this section we state the axioms for balance functions and we prove a classification Theorem 4.5 for the existence and uniqueness of balance functions. The axioms for balance functions are stated in terms of the space \( \mathcal{R} \) of regulated payment streams, §3.2.

A map \( B : \mathbb{R} \times \mathcal{R} \to \mathbb{R} \) is a \textit{balance function}, or \textit{future value function}, if it satisfies the 5 axioms stated below. We introduce the following preliminary notation.

(i) \( B(t, f) \equiv B_t(f) \in \mathbb{R} \) is the balance (future value) of the regulated payment stream \( f \) at time \( t \in \mathbb{R} \). In financial terms, \( B_t(f) \) is the balance, or future value, of \( f \) at time \( t \) due to market forces, including prevailing interest rates, that act on the payment stream \( f \) over the truncated time interval \((-\infty, t]\), i.e., up until the time \( t \).

(ii) For each \( s \in \mathbb{R} \) let \( c_s \in \mathcal{S} \) be the step function payment stream which corresponds to the single cash flow of 1 at time \( s \): \( c_s(t) = 0 \) if \( t < s \); \( c_s(t) = 1 \) if \( t \geq s \). For example, let \( f_C \) be the step function payment stream associated to a finite cash flow sequence \( C = (A_i)_{0 \leq i \leq n} \), as in (3.1) above. Then \( f_C = A_0 c_{t_0} + \cdots + A_n c_{t_n} \).
(iii) A balance function $B$ induces an “update map” $U \equiv U(B) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$,

\[
U(s, f)(t) \equiv U_s(f)(t) = \begin{cases} 
0 & \text{if } t < s \\
B_s(f)c_s + f(t) - f(s) & \text{if } t \geq s.
\end{cases}
\]

For each $s \in \mathbb{R}$ the payment stream $U_s(f) \in \mathbb{R}$ has the property that its cash flow at time $s$ is $B_s(f)$, the “updated” balance at time $s$ of the payment stream $f$ on the truncated interval $(-\infty, s]$; on the time interval $(s, \infty)$ the payment streams $U_s(f)$, $f$ coincide: $U_s(f)(t) - U_s(f)(s) = f(t) - f(s)$. In particular, let $f_C = A_0c_0 + \cdots + A_nc_n$ be the step function payment stream as in (ii) above. Then at each time $t_k$,

\[
U_{tk}(f_C) = B_{tk}(f_C)c_{tk} + A_{k+1}c_{tk+1} + \cdots + A_nc_{tn}, \quad 0 \leq k \leq n.
\]

We motivate the updated payment stream, and the replacement axiom $A_4$ below, in the case of a discrete loan contract, such as a mortgage contract: the initial debt is $A_0 < 0$, with $n$ constant repayments of $R > 0$ at the end of each period, calculated at the rate of interest $i$ per period, i.e., the payment stream $f = A_0c_0 + Rc_1 + \cdots + Rc_n$. Classically, the current balance of the debt after $k$ periods at the rate per period $i$ is,

\[
B_k(f) = A_0(1 + i)^k + R(1 + i)^{k-1} + \cdots + R(1 + i) + R.
\]

The updated payment stream after $k$ periods is, $U_k(f) = B_k(f)c_k + Rc_{k+1} + \cdots + Rcn$, whose first cash flow is the current balance $B_k(f)$ of the debt at time $k$, and whose remaining $(n - k)$-cash flows are the future unpaid payments of $R$. After $k$ payment periods, the payment stream $f$ can be replaced with the updated payment stream $U_k(f)$. The current balance after $\ell$ periods for the updated payment stream $U_k(f)$ is, $B_\ell(U_k(f)) = B_{k+\ell}(f)$, which is easily verified algebraically. This relation ensures the consistency of calculations of the current balance of the debt, using either $f$ or the updated payment stream $U_k(f)$, at the constant rate per period $i$.

With these preliminaries, the five axioms for balance functions are as follows:
A1. Let $f \in \mathcal{R}$ and let $g \in \mathcal{R}$ be the payment stream $g = f + \lambda c_s$ (the addition of a single cash flow of $\lambda \in \mathbb{R}$ at time $s$). For all $r < s$, $\mathcal{B}_r(g) = \mathcal{B}_r(f)$. Thus cash flows introduced at times later than $r$ do not contribute to the balance $\mathcal{B}_r(f)$ at time $r$.

A2. Linearity in the Final Cash Flow: For each time $t \in \mathbb{R}$, $\mathcal{B}_t(f + \lambda c_t) = \mathcal{B}_t(f) + \lambda$ for all $\lambda \in \mathbb{R}$ and all payment streams $f \in \mathcal{R}$. Informally, market forces in place up until time $t$ do not affect a cash flow that takes place at the instant $t$. The intended interpretation of axioms A1, A2 is that the balances $\mathcal{B}_t(f)$ depend only on the cash flows of the payment stream $f$ on the time interval $(-\infty, t]$.

A3. Scale: For all $t \in \mathbb{R}$, $f \in \mathcal{R}$, $\mathcal{B}_t(\lambda f) = \lambda \mathcal{B}_t(f)$ for all $\lambda \geq 0$. Furthermore for all $s \leq t$, $\mathcal{B}_t(c_s) \geq 0$; $\mathcal{B}_t(-c_s) \leq 0$.

In particular, for all $t$, $\mathcal{B}_t(\overline{0}) = 0$, where $\overline{0}$ is the zero payment stream. Also the balance at $t \geq s$ of a single cash flow at $s$ does not change sign (but could be 0). This corresponds to the economic fact that a single deposit, respectively a single debt, at time $s$ can be reduced to zero over time but cannot change sign into a debt, respectively a deposit. Furthermore Axiom A3 states informally that if all the cash values of a payment stream $f$ are rescaled by a constant factor $\lambda \geq 0$ then all the future values of $f$ are rescaled by $\lambda$, i.e., the balance functions are invariant under a change of monetary unit, a reasonable financial requirement. We do not assume in general that $\mathcal{B}_t(-f) = -\mathcal{B}_t(f)$, which is equivalent to the linearity of $\mathcal{B}_t(f)$ in the payment stream $f \in \mathcal{R}$ (cf. Remark 4.4). However from A3, $\mathcal{B}_t(uf) = \mathcal{B}_t(-u(-f)) = -u \mathcal{B}_t(-f)$ for all $u \leq 0$.

A4. Replacement: For each $f \in \mathcal{R}$, $\mathcal{B}_t(f) = \mathcal{B}_t(U_s(f))$ for all $s \leq t$ in $\mathbb{R}$.

The Replacement axiom ensures time consistency of balance functions: For all $s \leq t$, the balance $\mathcal{B}_t(f)$ is equal to the balance at time $t$ of the updated payment stream $U_s(f) \in \mathcal{R}$ whose cash flow at time $s$ is the balance $\mathcal{B}_s(f)$ and is such that the payment streams $U_s(f)$, $f$ coincide on $(s, \infty)$. This axiom is motivated by the discussion above
on current balances of a loan contract.

\textit{A5. Continuity:} Let \( f \in \mathcal{R}_K \) and let \( f = \lim_{n \to \infty} f_n \), where \((f_n)_{n \geq 1}\) is a Cauchy sequence in \( \mathcal{S}_K \) (topology of uniform convergence). For all \( t \in \mathbb{R} \), \( B_t(f) = \lim_{n \to \infty} B_t(f_n) \in \mathbb{R} \).

In general a balance map \( B_t(f) \) is \textit{non-linear} in \( f \in \mathbb{R} \). As explained in (4.2) below, this non-linearity derives from the difference in general between "deposit" and "investment" accumulation functions discussed in Lemma 4.3. The linear case is discussed in §4.3 and also Remark 4.4.

\section*{4.2. Accumulation Functions}

Let \( B : \mathbb{R} \times \mathcal{R} \to \mathbb{R} \) be a balance function. For all \( s \leq t \) define \( a(s, t) = B_t(c_s) \), the balance at \( t \) of a cash flow of 1 at \( s \). Similarly, for all \( s \leq t \) define \( b(s, t) = -B_t(-c_s) \), the negative of the balance at \( t \) of a cash flow of \(-1\) at \( s \). Employing axiom \( A_3 \), \( a(s, t) \geq 0, b(s, t) \geq 0 \) for all \( s \leq t \).

\textbf{Lemma 4.1.} \( a(s, t), b(s, t) \) are accumulation functions, called the deposit, respectively the investment accumulation function for the balance map \( B : \mathbb{R} \times \mathcal{R} \to \mathbb{R} \).

\textbf{Proof.} Let \( r \leq s \leq t \). We prove that \( a(s, t), b(s, t) \) satisfy the multiplicative property (2.1) for accumulation functions. Employing the replacement axiom \( A_4 \), \( a(r, t) = B_t(c_r) = B_t(U_s(c_r)) \). From (4.1) the payment stream \( U_s(c_r)(u) = 0 \) for all \( u < s \) and is the constant \( B_s(c_r) \geq 0 \) for all \( u \geq s \). Hence \( U_s(c_r) = \lambda c_s, \lambda = B_s(c_r) \geq 0 \). Consequently,

\[ a(r, t) = B_t(U_s(c_r)) = B_t(B_s(c_r) \cdot c_s) \]

\[ = B_s(c_r) \cdot B_t(c_s) \quad \text{by } A_3. \]

Hence \( a(r, t) = a(r, s) \cdot a(s, t) \).

Similarly, employing the replacement axiom \( A_4 \), \( b(r, t) = -B_t(U_s(-c_r)) \). Employing (4.1), the payment stream \( U_s(-c_r)(u) = 0 \) for all \( u < s \) and is the constant \( B_s(-c_r) \leq 0 \) for all
$u \geq s$. Hence $U_s(-c_r) = \lambda c_s$, $\lambda = B_s(-c_r) \leq 0$. Consequently,

$$b(r, t) = -B_t(U_s(-c_r)) = -B_t(B_s(-c_r) \cdot c_s)$$

$$= +B_s(-c_r) \cdot B_t(-c_s) \text{ by } A_3.$$ 

Hence $b(r, t) = b(r, s) \cdot b(s, t)$.

Thus $a(s, t), b(s, t)$ both satisfy the multiplicative property for accumulation functions. Furthermore, for all $t \in \mathbb{R}$, $a(t, t) = b(t, t) = 1$. Indeed, employing axiom $A_2$, for all $t \in \mathbb{R}$,

$$a(t, t) = B_t(c_t) = B_t(0 + 1c_t) = B_t(0) + 1 = 1.$$ 

Similarly employing $A_2$, $B_t(-c_t) = B_t(0 - c_t) = B_t(0) - 1 = -1$. Hence for all $t \in \mathbb{R}$, $b(t, t) = -B_t(-c_t) = 1$, which completes the proof of the lemma. \(\square\)

**Lemma 4.2.** Let $u \in \mathbb{R}$ and suppose $f \in \mathbb{R}$ satisfies $f(t) = 0$ for all $t < u$. Then $B_u(f) = f(u)$.

**Proof.** One may suppose $f = \lim_{n \to \infty} f_n$, where $(f_n)$ is a sequence of step function payment streams such that $f_n(t) = 0$ for all $t < u$. If $f_n(t) = \sum_{i=0}^{m} A_i^n c_{t_i}$, $t_0 = u$, then by Axioms $A_1, A_2$, $B_u(f_n) = B_u(A_0^n c_u) = A_0^n = f_n(u)$. Employing axiom $A_5$ it follows that $B_u(f) = \lim_{n \to \infty} B_u(f_n) = \lim_{n \to \infty} f_n(u) = f(u)$ \(\square\)

**Lemma 4.3.** The Basic Computation. Let $r \leq s$ and suppose $f = xc_r + yc_s$ (thus $f$ represents a cash flow of $x$ at $r$ and a cash flow of $y$ at $s$). Then,

$$B_s(f) = \begin{cases} 
  xa(r, s) + y & \text{if } x \geq 0 \\
  xb(r, s) + y & \text{if } x \leq 0.
\end{cases}$$

**Proof.** Employing axiom $A_2$, $B_s(f) = B_s(xc_r + yc_s) = B_s(xc_r) + y$. In case $x \geq 0$, employing the Scale axiom $A_3$, $B_s(xc_r) = xB_s(c_r) = xa(r, s)$. In case $x \leq 0$, employing axiom $A_3$, $B_s(xc_r) = -xB_s(-c_r) = xb(r, s)$. These two cases prove the lemma. \(\square\)

We remark that Lemma 4.3 shows that the balance $B_s(f)$ is governed by the deposit accumulation function $a(r, s)$ in case the previous balance $B_r(f) = x \geq 0$, or by the
investment accumulation function \( b(r, s) \) in case the previous balance \( B_r(f) = x \leq 0 \). This distinction between deposit and investment accumulation functions derives from our axioms. In financial terms, the investment accumulation function applies to the current debt, and the deposit accumulation function applies to the current surplus, a distinction first employed by Teichroew et al., (1965a) in their study of IRR functions.

Employing Lemma 4.3, we show below that the balance map, when restricted to step function payment streams, \( B: \mathbb{R} \times S \rightarrow \mathbb{R} \), is uniquely determined by the deposit and investment accumulation functions \( a(s, t), b(s, t) \). Uniqueness of balance functions, \( B: \mathbb{R} \times \mathcal{R} \rightarrow \mathbb{R} \), then follows from the continuity axiom \( A_5 \). The existence of balance functions that satisfy all of the axioms is proved in Theorem 4.5.

Let \( f = D_0 c_{t_0} + \cdots + D_n c_{t_n} \in S \) be a step function payment stream whose successive cash flows \( D_j \) occur at times \( t_j, 0 \leq j \leq n \). Let \( B_j(f) \equiv B_{t_j}(f) \) denote the current balance at time \( t_j, 0 \leq j \leq n \). Employing (4.1), the updated payment stream at time \( t_k \) is \( U_k(f) = B_k(f)c_k + D_{k+1}c_{k+1} + \cdots + D_nc_n, 0 \leq k \leq n \). Applying Axiom \( A_1 \) and the replacement axiom \( A_4 \) it follows that the balance at time \( t_j+1 \) is,

\[
B_{j+1}(f) = B_{j+1}(U_j(f)) = B_{j+1} \left( B_j(f)c_j + D_{j+1}c_{j+1} + \cdots + D_nc_n \right)
\]

\[
= B_{j+1}(B_j(f)c_j + D_{j+1}c_{j+1}) \quad 0 \leq j \leq n - 1.
\]

Applying Lemma 4.3 to the times \( t_j, t_{j+1} \), and to the step function payment stream (two cash flows), \( B_j(f)c_{t_j} + D_{j+1}c_{j+1} \), it follows that

(4.2) \( B_{j+1}(f) = \begin{cases} a(t_j, t_{j+1}) B_j(f) + D_{j+1} & \text{if } B_j(f) \geq 0, \\ b(t_j, t_{j+1}) B_j(f) + D_{j+1} & \text{if } B_j(f) \leq 0, \end{cases} \quad 0 \leq j \leq n - 1. \)

Thus on the interval \([t_j, t_{j+1}]\), the balance \( B_j(f) \) at time \( t_j \) accumulates with respect to the accumulation function \( a(s, t) \) in case the balance \( B_j(f) \geq 0 \), or with respect to the accumulation function \( b(s, t) \) in case the balance \( B_j(f) \leq 0 \). Only the cash flows \( D_0, \ldots, D_j \) of \( f \) enter into the computation of \( B_j(f) \). The iteration scheme (4.2) is non-linear in the payment stream \( f \in S \); in general there is no closed form expression for the balances \( B_j(f) \).

This type of iteration scheme for balance functions was first considered by Teichroew et al.,
(1965a,b), in the special case that \(a(s, t) = d^{t-s}, \ b(s, t) = x^{t-s}\), where \(d > 0\) is a constant “deposit” compounding factor and \(x > 0\) is a constant “investment” compounding factor; in addition, these authors assume a constant period, i.e., the intervals \([t_i, t_{i-1}]\) have equal length, \(1 \leq i \leq n\). These balances \(B_j(f) \equiv B_j(f)(a, b)\) are therefore designated throughout this paper as T.R.M. balances, with respect to the deposit and investment accumulation functions \(a(s, t), b(s, t)\).

**Remark 4.4.** A special case of interest for T.R.M. balances occurs in the case \(a = b\), i.e., the deposit and investment accumulation functions are equal. In this case the iteration scheme (4.2) simplifies:

\[
B_{j+1}(f) = a(t_j, t_{j+1})B_j(f) + D_{j+1}.
\]

From (2.1), (4.3) one obtains closed form expressions for the successive T.R.M. balances:

\[
B_j(f) = D_0a(t_0, t_j) + D_1a(t_1, t_j) + \cdots + D_j, \quad 0 \leq j \leq n.
\]

Thus in case \(a = b\) the balances \(B_j(f)\) are **linear** in the payment stream \(f \in \mathcal{S}\); hence \(B_t(\lambda f) = \lambda B_t(f)\) for all \(\lambda \in \mathbb{R}\). If \(a(t, s) = (1+i)^{t-s} (i \geq -1)\), at constant rate per period \(i\) (the time period in 1 unit), then one recovers the classical balance (future value) at the end of the project \((n\) periods)

\[
B_n(f) = \sum_{k=0}^{n} D_k(1+i)^{n-k}.
\]

Thus the T.R.M. balances (4.2) include, as a special case, the classical future value calculations in financial mathematics with respect to a constant rate \(i\) per period.

**4.2.** In this section we prove that balance functions exist, subject to some mild restrictions on the deposit and investment accumulation functions.

**Theorem 4.5.** Let \(a \geq 0, \ b \geq 0\) be accumulation functions which are bounded above by a monotone increasing accumulation function \(y(s, t)\): \(a \leq y, \ b \leq y\). There is a unique
balance function \( B : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) whose corresponding deposit and investment accumulation functions are respectively \( a(s, t) \), \( b(s, t) \).

**Proof.** Note that the hypothesis of the theorem is satisfied if both \( a, b \) are positive and of bounded variation on all compact intervals \( I \subset \mathbb{R} \) (cf. §2.2). For arbitrary accumulation functions \( a(s, t), b(s, t) \) the iteration scheme (4.2) for T.R.M. balances defines a balance function, \( B : \mathbb{R} \times S \to \mathbb{R} \), on the subset of step function payment streams. Clearly these T.R.M. balances satisfy axioms \( A_1, A_2, A_3 \), applied to payment streams \( f \in S \). Furthermore, with respect to these T.R.M. balances, it follows from (4.1) that the update map \( U : \mathbb{R} \times S \to S \). Indeed, \( U_s(f) \in S \) is a step function payment stream whose first cash flow is the balance \( B_s(f) \), itself defined by iteration as in (4.2), and such that \( U_s(f),f \) have the same cash flow sequence in \((s, \infty)\). Consequently continuing the iteration scheme (4.2) for all \( t \geq s \), the replacement axiom \( A_4 \) is satisfied for all \( f \in S \). To complete the existence proof we extend this balance map \( B : \mathbb{R} \times S \to \mathbb{R} \) to a balance map on all regulated payment streams \( f \in \mathcal{R} \), based on the limit process in axiom \( A_5 \). Thus if \( f = \lim_{n \to \infty} f_n \in \mathcal{R}_K \) (topology of uniform convergence) where for all \( n \), \( f_n \in S_K \) is a step function payment stream, then the analytic problem is to prove that, for all \( t \), the sequence of balances \((B_t(f_n))_{n \geq 1} \) is Cauchy, hence \( \lim_{n \to \infty} B_t(f_n) \) exists. It is here that the hypothesis \( a \leq y, b \leq d \), is employed to establish the estimates needed to carry out the limiting process defined by axiom \( A_5 \). The key estimate is Proposition 4.8.

To emphasize the dependence on the accumulation functions \( a(s, t), b(s, t) \) the balance map \( B : \mathbb{R} \times S \to \mathbb{R} \) will be written \( B_t(f)(a, b) \), or \( B_t(f_C)(a, b) \), to include also the dependence on the cash flow sequence \( C = (C_j)_{0 \leq j \leq n} \).

**Lemma 4.6.** Let \( a(s, t), b(s, t), c(s, t), d(s, t) \) be accumulation functions such that \( a \leq c, b \leq d \). For all step functions \( f_C \in S \) \( (B_j(f_C) \equiv B_j(f_C)) \),

\[
B_j(f_C)(a, b) \leq B_j(f_C)(c, b); \quad B_j(f_C)(a, b) \geq B_j(f_C)(a, d), \quad 0 \leq j \leq n.
\]

**Proof.** The intuitive financial content of the lemma may be expressed as follows, and
is the main idea underlying Teichroew et al., (1965a). $B_j(f_C)(a, b)$ is the balance at time $t_j$ in a financial account which credits interest in the case of positive balances according to the deposit accumulation function $a(s, t)$ and which charges interest in the case of negative balances (overdrafts) according to the investment (for the financial institution) accumulation function $b(s, t)$. Evidently, for a given cash flow sequence $C$, the balance at time $t_j$ increases if positive balances at previous times are credited interest at a higher rate, and decreases if overdrafts at previous times are charged interest at a higher rate.

Formally, the proof is by induction. Assuming $B_j(f_C)(a, b) \leq B_j(f_C)(c, b)$, $B_j(f_C)(a, b) \geq B_j(f_C)(a, d)$ (note that $B_0(f_C) = C_0$ for all choices of accumulation functions), the inductive step is proved from (4.2), taking into account the sign of the balance $B_j(f_C)$ at time $j$. The details are trivial and are left to the reader. □

**Corollary 4.7.** Let $a(s, t), b(s, t)$ be accumulation functions and suppose $y(s, t)$ is an accumulation function which is a common upper bound: $a \leq y$, $b \leq y$. For all step functions $f_C \in S_K$ ($0$ is the zero accumulation function),

$$B_j(f_C)(0, y) \leq B_j(f_C)(a, b) \leq B_j(f_C)(y, 0), \quad 0 \leq j \leq n.$$ 

**Proposition 4.8.** Let $f_C, f_D \in S$ be step function payment streams; $C = (C_j)_{0 \leq j \leq n}$, $D = (D_j)_{0 \leq j \leq n}$. Let $a \geq 0$, $b \geq 0$ be accumulation functions that are bounded above by a monotone increasing accumulation function $y(s, t)$: $a \leq y$, $b \leq y$. Then,

$$|B_j(f_C)(a, b) - B_j(f_D)(a, b)| \leq 2y([t_0, t_j]) \cdot ||f_C - D||, \quad 0 \leq j \leq n.$$ 

**Proof.** Employing Remark 3.1 we assume that the cash flows of $C, D$ occur at a common set of partition points, $t_0 < t_1 < \cdots < t_n$. The Proposition is proved by induction, based on the following two lemmas and the iteration scheme (4.2).

**Lemma 4.9.** $B_j(f_C)(a, b) - B_j(f_D)(a, b) \leq B_j(f_{C-D})(y, 0), \quad 0 \leq j \leq n$ (the index $j$ indicates the balance at time $t_j$).
Lemma 4.10. \( B_j(f_C)(a, b) - B_j(f_D)(a, b) \geq B_j(f_{C-D})(0, y), 0 \leq j \leq n. \)

Note that the lemmas are both true with equality at the index \( j = 0 \) (for all accumulation functions) since \( B_0(f_C)(\cdot, \cdot) = C_0; B_0(f_D)(\cdot, \cdot) = D_0; B_0(f_{C-D})(\cdot, \cdot) = C_0 - D_0. \)

Proof of Lemma 4.9. Let \( \Delta_j = B_j(f_{C-D})(y, 0) - (B_j(f_C)(a, b) - B_j(f_D)(a, b)), 0 \leq j \leq n. \) Inductively, we assume \( \Delta_j \geq 0 \) and we prove \( \Delta_{j+1} \geq 0. \) There are four cases, depending on the signs of \( B_j(f_C)(a, b), B_j(f_D)(a, b). \) Employing the iteration scheme (4.2), \( \Delta_{j+1} \) is computed from \( \Delta_j \) by calculating the change in the balance functions over the interval \([t_j, t_{j+1}]. \) The occurrences of the cash flows \( C_{j+1}, D_{j+1} \) at time \( t_{j+1} \) in \( \Delta_{j+1} \) cancel out, hence are omitted in the computations below for \( \Delta_{j+1}. \) For notational convenience let \( B_j(f_C) = B_j(f_C)(a, b), B_j(f_D) = B_j(f_D)(a, b), B_j(f_{C-D}) = B_j(f_{C-D})(y, 0). \)

Case 1: \( B_j(f_C) \geq 0, B_j(f_D) \geq 0. \)

\[
\Delta_{j+1} = \sup \{0, B_j(f_{C-D})y(t_j, t_{j+1})\} - (B_j(f_C) - B_j(f_D))a(t_j, t_{j+1}) \\
\geq B_j(f_{C-D})a(t_j, t_{j+1}) - (B_j(f_C) - B_j(f_D))a(t_j, t_{j+1}) \\
= \Delta_j a(t_j, t_{j+1}) \geq 0.
\]

Case 2: \( B_j(f_C) \geq 0, B_j(f_D) \leq 0. \)

\[
\Delta_{j+1} = \sup \{0, B_j(f_{C-D})y(t_j, t_{j+1})\} - B_j(f_C)a(t_j, t_{j+1}) + B_j(f_D)b(t_j, t_{j+1}) \\
\geq B_j(f_{C-D})y(t_j, t_{j+1}) - B_j(f_C)y(t_j, t_{j+1}) + B_j(f_D)y(t_j, t_{j+1}) \\
= \Delta_j y(t_j, t_{j+1}) \geq 0.
\]

Case 3: \( B_j(f_C) \leq 0, B_j(f_D) \geq 0. \)

\[
\Delta_{j+1} = \sup \{0, B_j(f_{C-D})y(t_j, t_{j+1})\} - B_j(f_C)b(t_j, t_{j+1}) + B_j(f_D)a(t_j, t_{j+1}) \\
\geq 0 \quad \text{(each term is} \geq 0). \]

Case 4: \( B_j(f_C) \leq 0, B_j(f_D) \leq 0. \)

\[
\Delta_{j+1} = \sup \{0, B_j(f_{C-D})y(t_j, t_{j+1})\} - (B_j(f_C) - B_j(f_D))b(t_j, t_{j+1}) \\
\geq B_j(f_{C-D})b(t_j, t_{j+1}) - (B_j(f_C) - B_j(f_D))b(t_j, t_{j+1}) \\
= \Delta_j b(t_j, t_{j+1}) \geq 0.
\]
The above four cases prove the inductive step and hence the lemma is proved. □

**Proof of Lemma 4.10.** Let \( \Delta_j = B_j(f_{c-D})(0, y) - (B_j(f_c)(a, b) - B_j(f_D)(a, b)) \). Inductively, employing the iteration scheme (4.2), we assume \( \Delta_j \leq 0 \) and we prove \( \Delta_{j+1} \leq 0 \). Again, the occurrences of the cash flows \( C_{j+1}, D_{j+1} \) at time \( t_{j+1} \) in \( \Delta_{j+1} \) cancel out, hence are omitted. For notational convenience let \( B_j(f_c) = B_j(f_c)(a, b), B_j(f_D) = B_j(f_D)(a, b), \)
\( B_j(f_{c-D}) = B_j(f_{c-D})(0, y). \)

**Case 1:** \( B_j(f_c) \geq 0, B_j(f_D) \geq 0. \)

\[
\Delta_{j+1} = \inf \{0, B_j(f_{c-D})y(t_j, t_{j+1})\} - (B_j(f_c) - B_j(f_D))a(t_j, t_{j+1}) \\
\leq B_j(f_{c-D})a(t_j, t_{j+1}) - (B_j(f_c) - B_j(f_D))a(t_j, t_{j+1}) \\
= \Delta_j a(t_j, t_{j+1}) \leq 0.
\]

**Case 2:** \( B_j(f_c) \geq 0, B_j(f_D) \leq 0. \)

\[
\Delta_{j+1} = \inf \{0, B_j(f_{c-D})y(t_j, t_{j+1})\} - B_j(f_c)a(t_j, t_{j+1}) + B_j(f_D)b(t_j, t_{j+1}) \\
\leq 0 \quad \text{(each term is \( \leq 0 \)).}
\]

**Case 3:** \( B_j(f_c) \leq 0, B_j(f_D) \geq 0. \)

\[
\Delta_{j+1} = \inf \{0, B_j(f_{c-D})y(t_j, t_{j+1})\} - B_j(f_c)b(t_j, t_{j+1}) + B_j(f_D)a(t_j, t_{j+1}) \\
\leq B_j(f_{c-D})y(t_j, t_{j+1}) - B_j(f_c)y(t_j, t_{j+1}) + B_j(f_D)y(t_j, t_{j+1}) \\
= \Delta_j y(t_j, t_{j+1}) \leq 0.
\]

**Case 4:** \( B_j(f_c) \leq 0, B_j(f_D) \leq 0. \)

\[
\Delta_{j+1} = \inf \{0, B_j(f_{c-D})y(t_j, t_{j+1})\} - (B_j(f_c) - B_j(f_D))b(t_j, t_{j+1}) \\
\leq B_j(f_{c-D})b(t_j, t_{j+1}) - (B_j(f_c) - B_j(f_D))b(t_j, t_{j+1}) \\
= \Delta_j b(t_j, t_{j+1}) \leq 0.
\]

The above four cases prove the inductive step and hence the lemma is proved. □

Returning to the proof of Proposition 4.8, for all \( j, 0 \leq j \leq n \), it follows from Lemma 4.9, Lemma 4.10 that,

\[ (4.6) \quad B_j(f_{c-D})(0, y) \leq B_j(f_c)(a, b) - B_j(f_D)(a, b) \leq B_j(f_{c-D})(y, 0). \]
We now estimate the end terms of (4.6). Employing the iteration scheme (4.2), note that if the deposit, respectively investment, accumulation function is 0 then a balance $B_p(f_{C-D})(y,0) \leq 0$ at time $t_p$ implies that the next balance $B_{p+1}(f_{C-D})(y,0) = C_{p+1} - D_{p+1}$ at time $t_{p+1}$, respectively a balance $B_p(f_{C-D})(0,y) \geq 0$ at time $t_p$ implies that the next balance $B_{p+1}(f_{C-D})(0,y) = C_{p+1} - D_{p+1}$ at time $t_{p+1}$.

There is a largest $k$, $1 \leq k \leq j$, such that $B_{k-1}(f_{C-D})(y,0) \leq 0$; hence $B_k(f_{C-D})(y,0) = C_k - D_k$, and if $k < j$, then $B_r(f_{C-D}) \geq 0$, $k \leq r < j$. Consequently, employing the iteration scheme (4.2), one computes the balance at time $t_j$

$$B_j(f_{C-D})(y,0) = (C_k - D_k)y(t_k, t_j) + (C_{k+1} - D_{k+1})y(t_{k+1}, t_j) + \cdots + (C_j - D_j).$$

Since the sequence $(y(t_i, t_j))_{i \leq j}$ is monotone decreasing, it follows from the classical Abel’s lemma for finite series, cf. Spivak (1980, p. 368), that

$$B_j(f_{C-D})(y,0) \leq y(t_k, t_j) \cdot \Sigma$$

where $\Sigma = \sup\{(C_k - D_k) + \cdots + (C_p - D_p) \mid k \leq p \leq j\}$. Employing (3.5), it follows that $|\Sigma| \leq 2 \|f_{C-D}\|$. Since $y(s,t)$ is monotone increasing, $y(t_k, t_j) \leq y(t_0, t_j)$. Hence

$$B_j(f_{C-D})(y,0) \leq 2 y(t_0, t_j) \cdot \|f_{C-D}\|.$$ 

Similarly, for the other end term $B_j(f_{C-D})(0,y)$ of (4.6), there is a largest $k$ such that $B_{k-1}(f_{C-D})(0,y) \geq 0$; hence $B_k(f_{C-D})(0,y) = C_k - D_k$, and if $k < j$, $B_r(f_{C-D}) \leq 0$, $k \leq r < j$. Consequently, employing the iteration scheme (4.2), one computes the balance at time $t_j$

$$B_j(f_{C-D})(0,y) = (C_k - D_k)y(t_k, t_j) + (C_{k+1} - D_{k+1})y(t_{k+1}, t_j) + \cdots + (C_j - D_j).$$

Since the sequence $(y(t_i, t_j))_{i \leq j}$ is monotone decreasing, it follows from Abel’s lemma that,

$$B_j(f_{C-D})(0,y) \geq y(t_k, t_j) \cdot \sigma,$$
where $\sigma = \inf\{(C_k - D_k) + \cdots + (C_p - D_p) \mid k \leq p \leq j\}$. Employing (3.5), it follows that $|\sigma| \geq -2\|f_{C-D}\|$. Since $y(s,t)$ is monotone increasing, $y(t_k,t_j) \leq y(t_0,t_j)$. Hence

\[(4.12)\quad B_j(f_{C-D})(0,y) \geq -2y(t_0,t_j)\cdot f_{C-D}.\]

Employing (4.6) and the estimates (4.9), (4.12), one obtains the inequality,

\[(4.13)\quad -2y(t_0,t_j)\cdot f_{C-D} \leq B_j(f_D)(a,b) - B_j(f_C)(a,b) \leq 2y(t_0,t_j)\cdot f_{C-D}, 0 \leq j \leq n.\]

Consequently, $|B_j(f_C)(a,b) - B_j(f_D)(a,b)| \leq 2y(t_0,t_j)\cdot f_{C-D}, 0 \leq j \leq n$, which completes the proof of Proposition 4.8. □

Returning to the proof of Theorem 4.5, let $f \in R_K$ and let $(f_n = f_{C_n})_{n \geq 1}$ be a Cauchy sequence of step function payment streams in $S_K$ such that $f = \lim_{n \to \infty} f_n$. From (4.13)

\[(4.14)\quad |B_j(f_n)(a,b) - B_j(f_m)(a,b)| \leq 2y(t_0,t_j)\cdot f_n - f_m, 0 \leq j \leq n.\]

From the remarks following (3.5), $\|f_n - f_m\|$ is the sup-norm. Since $S_K$ is dense in $R_K$ in the sup-norm topology, it follows from (4.14) that at time $t$ ($t = t_j$ for some $j$) the sequence of balances $(B_t(f_m) \in R)_{m \geq 1}$ is Cauchy. Hence there is a balance map $B: R \times R \to R$ such that $B_t(f)(a,b) = \lim_{m \to \infty} B_t(f_m)(a,b)$ for all $t \in R$. Since the balance map $B: R \times S \to R$ satisfies all of the axioms $A_i, 1 \leq i \leq 4$, it follows in the limit that the balance map $B: R \times R \to R$ satisfies these axioms and, by construction, also the continuity axiom $A_5$, which proves Theorem 4.5. □

**4.3. Linear Balance maps.** We consider now the special case that $a = b$ in Theorem 4.5, i.e., the deposit and investment accumulation functions are equal. Employing (4.4) the condition $a = b$ is equivalent to an additional axiom that a balance map $B: R \times S \to R$ is linear in the payment streams $f \in S$; hence by the continuity axiom $A_5$, the balance map $B: R \times R \to R$ is linear in the regulated payment stream $f \in R$. Suppose $a(s,t) = b(s,t) = e^{g(t)-g(s)}$, where $g(t)$ is of bounded variation on all compact intervals. Applying Theorem 4.5, for each $f \in R_K, K = [c,d]$, we write the corresponding balance map

\[(4.15)\quad B_t(f) = \int_{-\infty}^{t} e^{g(t)-g(s)} df(s).\]
This “generalized” integral (4.15) is linear in \( f \in \mathcal{R}_K \) and coincides with the classical balance map (future value map) in case \( f \in \mathcal{R}_K \) is also of bounded variation, i.e., (4.15) specializes to a Lebesgue-Stieltjes integral in this case. Thus (4.15) reduces to the future value calculations (4.4), (4.5), if \( f = f_C \in \mathcal{S}_K \) is a step function payment stream, and 
\[ a(s, t) = (1 + i)^{t-s}. \]

In this respect, Theorem 4.5 generalizes the work of Norberg (1990), Promislow (1994), who propose axioms, including a linearity axiom, for balance functions, denoted by these authors as valuation functions, on payment streams that in their theory are Borel measures \( \mu \) on \( \mathbb{R} \). Thus \( \mu([a, b]) \) is the total cash flow in the interval \([a, b] \subset \mathbb{R} \). In Norberg (1990), payment measures are non-negative; Promislow (1994) generalizes this measure-theoretic approach to include payment streams that are signed Borel measures on \( \mathbb{R} \), i.e., that reflect transactions that may have both positive and negative payments. Since these Borel measures can be represented by functions of bounded variation on compact intervals it follows that the valuation functions of these authors, when calculated at the time of the final cash flow of the payment stream, is a special case of the integral (4.15) where \( f \in \mathcal{R}_K \) is of bounded variation.

### 5. Internal Rate of Return

**5.1.** Let \( a(s, t) \) be a positive accumulation function of bounded variation. As explained in §2.2 there is a positive monotone increasing accumulation function \( y(s, t) \), respectively a positive monotone decreasing accumulation function \( b(s, t) \) such that \( 0 < a \leq y \), respectively \( a \geq b > 0 \).

The accumulation function \( x(s, t) = x^{t-s}, t \leq s \), is positive if \( x > 0 \) and is the zero accumulation function if \( x = 0 \). Let \( x_1 = \sup\{x, 1\}, x_2 = \inf\{x, 1\} \). The accumulation function \( x_1^{t-s} \), respectively \( x_2^{t-s} \), is monotone increasing, respectively monotone decreasing. Consequently, the product accumulation function \( z(s, t) = y(s, t) \cdot x_1^{t-s} \) is a positive monotone increasing common upper bound, respectively if \( x > 0 \), \( c(s, t) = b(s, t) \cdot x_2^{t-s} \) is
a positive monotone decreasing lower bound: for all \( s \leq t \) (cf. §2.2)
\[
(a(s,t) \leq z(s,t), \quad x^{t-s} \leq z(s,t); \quad 0 < c(s,t) \leq a(s,t), \quad 0 < c(s,t) \leq x^{t-s}).
\]

In what follows we let \( a > 0 \) be a fixed deposit accumulation function of bounded variation and we let \( x^{t-s}, x \geq 0 \), be a variable investment accumulation function. With respect to these accumulation functions, and also the common upper bound \( z(s,t) \) in (5.1), it follows from Theorem 4.5 that there is a unique balance map \( B: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) that satisfies all of the axioms \( A_i, 1 \leq i \leq 5 \). Throughout we employ the simplified notation \( B_{x^{t}}(f) = B_{t}(f)(a,x), f \in \mathcal{R} \), to indicate the dependence on the variable investment accumulation function \( x^{t-s} \).

Let \( \mathcal{D} \subset C^0([0,\infty), \mathbb{R}) \), in the compact-open topology, be the subspace of continuous functions \( f: [0,\infty) \to \mathbb{R} \) such that either \( f \) is strictly decreasing with a unique root \( f(x) = 0 \), or \( f \) is negative and non-increasing (for example a constant function < 0). Let \( \nu: \mathcal{D} \to [0,\infty) \) be the Lebesgue measure, \( \nu(f) = m(X_f), X_f = \{x \in [0,\infty) \mid f(x) \geq 0\} \).

In particular \( \nu(f) = 0 \) if \( f \) is negative, and \( \nu(f) = x_0 \) if \( f(x_0) = 0 \). Promislow and Spring (1996, Theorem 4.3) prove that the measure \( \nu: \mathcal{D} \to [0,\infty) \) is continuous.

Let \( g = g_C \in \mathcal{S} \cap \mathcal{I} \) be a step function payment stream that is an investment project: \( \mathcal{C} = (C_i)_{0 \leq i \leq n} \) is a finite cash flow sequence such that \( C_0 < 0 \) since \( g \in \mathcal{I} \). The central point, proved in the next lemma, is that for each \( t \geq t_0 \), the function \( B_{x^{t}}(g) \), as a function of \( x \), lies in \( \mathcal{D} \). For example, at the initial time \( t_0 \), the balance function \( B_{x^{t_0}}(g) = C_0 < 0 \) (a constant function of \( x \)), hence \( B_{x^{t_0}}(g) \in \mathcal{D} \). The IRR of investment projects will be defined in §5.2 in terms of the measure \( \nu(f) \) above on the space \( \mathcal{D} \).

**Lemma 5.1.** As a function of the variable \( x \in [0,\infty) \), for each \( t \geq t_0 \) the balance function \( B_{x^{t}}: \mathcal{S} \cap \mathcal{I} \to \mathcal{D} \). If \( t > t_0 \) then for each \( g \in \mathcal{S} \cap \mathcal{I} \) the function \( B_{x^{t}}(g) \in \mathcal{D} \) is a continuous, strictly decreasing function of \( x \in [0,\infty) \) such that \( \lim_{x \to \infty} B_{x^{t}}(g) = -\infty \).

**Proof.** let \( B_{x^{t_j}}(g) = B_{x^{t_j}}(g) \), where \( t_j \) is the time of the cash flow \( C_j, 0 \leq j \leq n \). By introducing a cash flow of 0 at \( t \) if necessary, we may assume that \( t = t_j \) for some \( j \geq 1 \).

As noted above \( B_{x^{t_0}}(g) \in \mathcal{D} \) is the constant function \( C_0 < 0 \). Inductively on \( j \), suppose
Let \( r_j = \nu(B^x_j(g)) \in [0, \infty) \). Thus if \( r_j > 0 \) then \( B^x_{j+1}(g) = 0 \); \( B^x_j(g) > 0 \) if \( x < r_j; B^x_j(g) < 0 \) if \( x > r_j \). Applying (4.2) with respect to the deposit accumulation function \( a > 0 \) and investment accumulation function \( x^{t-s} \),

\[
B^x_{j+1}(g) = \begin{cases} 
  a(t_j, t_{j+1})B^x_j(g) + C_{j+1} & \text{if } x \in [0, r_j) \ (B^x_j(g) > 0) \\
  x^{t_{j+1} - t_j}B^x_j(g) + C_{j+1} & \text{if } x \in [r_j, \infty) \ (B^x_j(g) \leq 0).
\end{cases}
\]

If \( r_j = 0 \) then only the second alternative in (5.2) applies. One easily checks that \( B^x_{j+1}(g) \) is a continuous function of \( x \) (if \( r_j > 0 \) then \( B^x_{j+1}(g) = 0 \)). Applying (5.2) at \( j = 0 \ (C_0 < 0; r_0 = 0) \), \( B^x_1(g) = C_0 x^{t_1 - t_0} + C_1 \) for all \( x \geq 0 \); hence \( B_1(g) \in \mathcal{D} \) is a strictly decreasing function such that \( \lim_{x \to \infty} B^x_1(g) = -\infty \). Suppose inductively in addition that \( B^x_j(g) \in \mathcal{D} \) is a strictly decreasing function of \( x \) such that \( \lim_{x \to \infty} B^x_j(g) = -\infty, j \geq 1 \). Since \( a > 0 \), employing (5.2), if \( r_j > 0 \) then \( B^x_{j+1}(g) \) is strictly decreasing on \( [0, r_j) \). If \( 0 \leq r_j \leq x < u \) then

\[
x^{t_{j+1} - t_j}B^x_j(g) \geq x^{t_{j+1} - t_j}B^u_j(g) > u^{t_{j+1} - t_j}B^u_j(g),
\]

where the latter inequality obtains since \( B^u_s(g) < 0 \) for all \( s \in (r_j, \infty) \). Consequently the function \( B^x_{j+1}(g) \in \mathcal{D} \) is strictly decreasing. Employing (5.2) for \( x \in [r_j, \infty) \) it follows that \( \lim_{x \to \infty} B^x_{j+1}(g) = -\infty \), which completes the inductive step and the lemma is proved. □

**Lemma 5.2.** Let \( g = g_C \in \mathcal{S} \cap \mathcal{I} \) be a step function investment project, \( C = (C_i)_{0 \leq i \leq n} \). Let \( t_0 < s < t \). For all \( 0 < x < y \),

\[
B^y_t(g) - B^x_t(g) \leq c(s, t)(B^y_s(g) - B^x_s(g)) < 0.
\]

**Proof.** Let \( c(u, v) \) be a positive decreasing accumulation function which is a common lower bound (cf. (5.1)): \( 0 < c(u, v) \leq a(u, v); x^{v-u} \); hence also \( c(u, v) \leq y^{v-u} \). By introducing cash flows of 0 at times \( s, t \) if necessary, one may assume \( s = t_m > t_0 \), and \( t = t_p \). From Lemma 5.1, \( B^x_j(g) = B^x_{j+1}(g) \) is a strictly decreasing function of \( x \in [0, \infty), 1 \leq j \leq n \). The proof of the lemma is by induction on \( j \) and consists of 3 cases, based on the iteration scheme (4.2) with respect to the deposit accumulation function \( a(u, v) > 0 \), the investment accumulation functions \( y^{t-s}, x^{t-s} > 0 \).
I: $B^y_j(g) \geq 0; B^x_j(g) \geq 0$.

\[ B^y_{j+1}(g) - B^x_{j+1}(g) = a(t_j, t_{j+1}) (B^y_j(g) - B^x_j(g)) \leq c(t_j, t_{j+1}) (B^y_j(g) - B^x_j(g)) < 0. \]

II: $B^y_j(g) \leq 0; B^x_j(g) \leq 0$.

\[ B^y_{j+1}(g) - B^x_{j+1}(g) = y^{t_j+1-t_j} B^y_j(g) - x^{t_j+1-t_j} B^x_j(g) \leq c(t_j, t_{j+1}) (B^y_j(g) - B^x_j(g)) < 0. \]

III: $B^y_j(g) \leq 0; B^x_j(g) \geq 0$.

\[ B^y_{j+1}(g) - B^x_{j+1}(g) = y^{t_j+1-t_j} B^y_j(g) - a(t_j, t_{j+1}) B^x_j(g) \leq c(t_j, t_{j+1}) (B^y_j(g) - B^x_j(g)) < 0. \]

Note that the case, $B^y_j(g) > 0, B^x_j(g) < 0$, cannot occur since for all $j \geq 1$ the balances $B^x_j(g)$ are strictly decreasing as a function of $x$.

Concatenating the inequalities I, II, III for $m \leq j \leq p-1 \ (s = t_m, t = t_p)$, one obtains the inequality $B^y_s(g) - B^x_s(g) \leq c(s, t) (B^y_s(g) - B^x_s(g)) < 0$, which completes the proof of the lemma. □

**Theorem 5.3.** Let $f \in \mathcal{I}$ be an investment project, minimally supported on $K = [c, d]$. For each $t > c$ the function $B^x_t(f) \in \mathcal{D}$. More precisely, $B^x_t(f)$ is a continuous, strictly decreasing function of $x \in [0, \infty)$ such that $\lim_{x \to \infty} B^x_t(f) = -\infty$.

Theorem 5.3 is the main result on general investment projects. It generalizes the corresponding Lemma 5.1 which treats the restricted case of step function payment streams that are investment projects. The proof of Theorem 5.3 involves several delicate estimates and is given in the Appendix. We now proceed directly in §5.2 to the construction of the IRR for general investment projects.

### 5.2. IRR of an Investment Project

Let $f \in \mathcal{I}$ be an investment project, minimally supported on $K = [c, d]$. Fix a positive (deposit) accumulation function $a(s, t)$ which is of bounded variation on all compact intervals.
GENERAL BALANCE FUNCTIONS IN THE THEORY OF INTEREST

$I \subset \mathbb{R}$: there is a monotone increasing accumulation function $y(s, t)$ such that $a \leq y$. In practice, $a(s, t) = (1+r)^{t-s}$ where $r$ is an estimated effective interest rate/year on bank deposits during the life of the investment project. Employing Theorem 5.3, the balance function $B^x_d(f) \in C^0([0, \infty), \mathbb{R})$, calculated at the end of the investment project $f$ at time $t = d$, is a continuous, strictly decreasing function of $x \in [0, \infty)$ such that $\lim_{x \to \infty} B^x_d(f) = -\infty$. In particular the measure $\nu(f) = m(X_f) < \infty$, where $X_f = \{x \in [0, \infty) \mid B^x_d(f) \geq 0\}$. Thus $\nu(f) = 0$ if the function $B^x_d(f)$ is negative, and $\nu(f) = x_0$ if $B^{x_0}_d(f) = 0$ i.e., $x_0 \in [0, \infty)$ is the unique root of the strictly decreasing function $B^x_d(f)$; if $x < x_0 \ (x > x_0)$ then the balance $B^x_d(f) > 0$ ($B^x_d(f) < 0$).

The internal rate of return (IRR) of the investment project $f$ is defined to be the effective interest rate/year, $i_f = \nu(f) - 1 \in [-1, \infty)$. If $B^{x_0}_d(f) = 0$ as above, then $1 + i_f = x_0$ and the corresponding investment accumulation function at the IRR $i_f$ is $x_0^{t-s} = (1 + i_f)^{t-s}$. Note that $i_f$ depends on the deposit accumulation function $a(s, t)$. The measure $\nu(f)$ is the relevant parameter, the accumulation factor, for computing the IRR/period in the case of investment projects $f = f_C$, where $C = (C_j)_{0 \leq j \leq n}$ is a finite cash flow sequence such that $C_0 < 0$, cf. Promislow and Spring (1996). The IRR $i_f$, interpreted as a rate per period (the time unit is 1 period), coincides with the IRR defined by Teichroew et al., (1965a) in the special case of discrete investment projects of constant period whose deposit accumulation function is $a(s, t) = (1 + \alpha)^{t-s}$, where $\alpha > -1$ is a fixed deposit interest rate per period. In particular, Promislow and Spring (1996, §4.2), the rate $i_f$ per period coincides with the classical IRR in case $f = f_C$ is a loan contract. In this way, the IRR $i_f$ considerably generalizes the IRR function defined by Teichroew et al., (1965a) in the case of discrete investment projects, to the general case of investment projects defined by payment streams $f \in \mathcal{I}_K$, including the case of continuous payment streams, under the weak assumption that the deposit accumulation function is $a(s, t) = e^{g(t)-g(s)}$, where $g(t)$ can be any function of bounded variation.

**Remark 5.6.** Note that the IRR of an investment project $f \in \mathcal{I}_K$ is robust in the
following sense. Let $f = \lim_{n \to \infty} f_n$, where $(f_n)_{n \geq 1}$ is a Cauchy sequence of step function payment streams in $S_K$. Since the sequence of strictly decreasing balance functions of $x$, $B^x_d(f_n), n \geq 1$, converges uniformly to $B^x_d(f)$ on all compact subsets of $\mathbb{R}$, it follows that $i_f$ is uniformly approximated by $i_{f_n}$ for all $n$ sufficiently large. Furthermore, employing the iteration scheme (4.2), the balance function of $x, B^x_d(f_n)$, can be computed in practice as a finite iteration of T.R.M. balance functions; hence $i_{f_n}$ is a computable IRR approximation to $i_f$ for sufficiently large $n$.

**Remark 5.7.** Let $f \in \mathcal{I}$ be an investment project. The rescaled investment project $\lambda f$, for each $\lambda > 0$, has the same IRR: $i_{\lambda f} = i_f$. Indeed, if in addition $f \in S \cap \mathcal{I}$ is a step function investment project then the scale axiom $A_3$ and (5.2) prove that $i_{\lambda f} = i_f$. The general case follows from Remark 5.6 above. In fact the IRR $i_f$, defined for all investment projects $f \in \mathcal{I}$, satisfies all of the corresponding postulates for IRR functions that are presented in Promislow and Spring (1996).

**Appendix**

**Proof of Theorem 5.3.** Let $f \in \mathcal{I}_K$ be an regulated investment project minimally supported in $K = [c,d]$, and let $t > c$. Thus $f = \lim_{n \to \infty} f_n$, where $(f_n)_{n \geq 1}$ is a Cauchy sequence in $S_K$. Employing (3.3) one may assume that for all $n, f_n \in S_K \cap \mathcal{I}_K$ is a step function investment project: the first non-zero cash flow of $f_n$ is $< 0$. Applying Lemma 5.1, for all $n$ sufficiently large (so that the first non-zero cash flow of $f_n$ occurs in $[c,t]$ and is $< 0$), it follows that the balance $B^x_t(f_n)$ is a continuous, strictly decreasing function of $x \in [0,\infty)$ such that $\lim_{x \to \infty} B^x_t(f_n) = \infty$.

Let $L = [c_1,d_1] \subset [0,\infty)$ be a compact interval, $d_1 \geq 1$. Thus $d_1^{v-u}$ is a positive monotone increasing accumulation function such that $d_1^{v-u} \geq x^{v-u}$ for all $x \in L$. Recall that $a(u,v) \leq y(u,v)$ where $y$ is a monotone increasing accumulation function. Applying (4.14) to the common upper bound $y_1(s,t) = y(s,t) \cdot d_1(s,t)$ ($a(u,v) \leq y_1(u,v); x^{v-u} \leq y_1(u,v)$ for all $x \in L$), it follows that for each $t > c$ the sequence of functions, $(B^x_t(f_n))_{n \geq 1}$,
$x \in L$, is a Cauchy sequence in the space of continuous functions $C^0(L, \mathbb{R})$. Consequently the sequence of functions $(B_t^x(f_n))_{n \geq 1}$ converges to $B_t^x(f)$ in the compact-open topology on $C^0([0, \infty), \mathbb{R})$. It follows that the limit function $B_t^x(f)$ is a continuous, decreasing (i.e., non-increasing) function of $x \in [0, \infty)$. However it is not a formal consequence of convergence in the compact-open topology that the limit function $B_t^x(f)$ either is strictly decreasing or is unbounded below. We prove below additional estimates to show that in fact $B_t^x(f)$ is a strictly decreasing function of $x$, assuming $f \in I$ is an investment project.

Recall $f \in I_K,\ K = [c, d]$. Let $0 < x < y$, and let $t > c$. We prove that $B_t^y(f) < B_t^x(f)$. The proof divides into two (lengthy) cases. Let $h(u) = y^u - x^u,\ u \geq 0;\ h(0) = 0$. If $y \geq 1$ then $h(u)$ is strictly increasing; if $y < 1$ then $h(u)$ is strictly increasing on $[0, \delta_1]$ where $h'(\delta_1) = 0 [\delta_1 = \ln (\ln x/\ln y)/\ln(y/x) > 0;\ \lim_{y \to 1^-} \delta_1 = \infty]$.

**Case I:** $f(c) = 0$. Since $f \in I$ there is a $\delta > 0$, chosen so that also $\delta \leq \delta_1$ if $y < 1$, such that $f$ is negative and non-increasing on the interval $(c, c + \delta]$. One may assume $c + \delta \leq t$.

Let $C^n$ be a cash flow sequence such that $f_n = f_n(C^n) \in S_K$. For all sufficiently large $n$ let the cash flows of the sequence $C^n$ in the interval $[c, c + \delta]$ be $C^n_0 = 0, C^n_1, \ldots, C^n_p,\ p = p(n)$, occurring at times $t_r \equiv t_r^n \in [c, c + \delta],\ t_0 = c,\ t_p = c + \delta$. Since $C^n_r = f(t_r) - f(t_{r-1})$, it follows from (3.3) that for all $n$, $C^n_r \leq 0$ for all $r,\ 0 \leq r \leq p$. In particular the partial sums,

\[(A.1)\quad S_r = f(t_r) = C^n_0 + \cdots + C^n_r < 0,\quad 1 \leq r \leq p;\quad 0 = S_0 \geq S_1 \geq \cdots \geq S_p.\]

Since all the cash flows in the interval $[c, c + \delta]$ are non-positive, employing the iteration scheme (4.2), it follows that for all sufficiently large $n$ the balance functions $B_{t_r}^{u}(f_n) = \sum_{j=0}^r C^n_j u^{t_r-t_j};\ 0 \leq r \leq p,\ u \in [0, \infty)$. In particular at the end point $t_p = c + \delta \in [c, c + \delta]$,

\[(A.2)\quad B_{t_p}^{y}(f_n) - B_{t_p}^{x}(f_n) = \sum_{j=0}^p C^n_j (y^{t_p-t_j} - x^{t_p-t_j}).\]

We prove that $B_{t_p}^{y}(f_n) - B_{t_p}^{x}(f_n) < 0$ and is uniformly bounded away from 0 as $n \to \infty$. 
Lemma 5.4. Let \( S = a_0u_0 + a_1u_1 + \cdots + a_nu_n \), where \( a_0 \geq a_1 \geq \cdots \geq a_n \geq 0 \), and 
\( 0 \geq S_0 \geq S_1 \geq \cdots \geq S_n \), where \( S_r = u_0 + u_1 + \cdots + u_r, 0 \leq r \leq n \). Then \( S \leq a_mS_m \leq 0 \) for all \( m, 0 \leq m \leq n \).

Proof. By a rearrangement of the terms,

\[
S = S_0(a_0 - a_1) + S_1(a_1 - a_2) + \cdots + S_{n-1}(a_{n-1} - a_n) + a_nS_n.
\]

Since \( 0 \geq S_m \geq S_r \) for all \( r \geq m \), and also the successive differences \( a_j - a_{j-1} \geq 0 \), it follows that one can replace each \( S_r \) with \( S_m, r \geq m \), to obtain the inequality,

\[
S \leq S_0(a_0 - a_1) + \cdots + S_{m-1}(a_{m-1} - a_m) + S_m \sum_{j=m}^{n-1} (a_j - a_{j+1}) + a_nS_m
\]

\[= S_0(a_0 - a_1) + \cdots + S_{m-1}(a_{m-1} - a_m) + S_ma_m \leq S_m a_m. \quad \square
\]

Since \( h(s) = y^s - x^s, s \in [0, \delta] \), is strictly increasing it follows that the sequence, \((y^{t_p-t_j} - x^{t_p-t_j})_{0 \leq j < p},\) is strictly decreasing and positive. Applying Lemma 5.4 to the sum (A.2), employing also (A.1), it follows that for each \( m, 1 \leq m < p \),

\[
B_{t_p}^y(f_n) - B_{t_p}^x(f_n) \leq (y^{t_p-t_m} - x^{t_p-t_m}) (C_m^m + \cdots + C_m^m)
\]

(A.3)

\[
= (y^{t_p-t_m} - x^{t_p-t_m}) f(t_m) < 0.
\]

One may assume that for some \( m \geq 1 \), the cash flow \( C_m \) occurs at time \( t_m = c + \delta/2 \in (c, c+\delta) \). Employing (A.3) at time \( t_m \), one has the uniform estimate: for all \( n \),

(A.4) \[ B_{t_p}^y(f_n) - B_{t_p}^x(f_n) \leq (y^{\delta/2} - x^{\delta/2}) f(c + \delta/2) < 0.
\]

Let the cash flows of the sequence \( C^n \) on the complementary interval \([c+\delta, t]\) occur at times \( t_p = c + \delta, t_{p+1}, \ldots, t_q = t \). Applying Lemma 5.2 \((t = t_q, s = t_p)\) and the inequality (A.4) it follows that for all sufficiently large \( n \) (recall that the accumulation function \( c(u, v) \) is positive and monotone decreasing),

\[
B_{t_p}^y(f_n) - B_{t_p}^x(f_n) \leq c(t_p, t) (B_{t_p}^y(f_n) - B_{t_p}^x(f_n))
\]

\[\leq c(L)(y^{\delta/2} - x^{\delta/2}) f(c + \delta/2) < 0, \quad L = [c, t].\]
Passing to the limit as \( n \to \infty \),

\[
B_t^y(f) - B_t^x(f) \leq c(L)(y^{\delta/2} - x^{\delta/2})f(c + \delta/2) < 0
\]

Thus (A.5) proves that \( B_t^y(f) < B_t^x(f) \) if \( 0 < x < y \); hence the decreasing continuous function \( B_t^x(f) \) is a strictly decreasing function of \( x \in [0, \infty) \). Furthermore, if \( y \geq 1 \) then \( h(s) = y^s - x^s, \ s \geq 0 \), is an increasing function; hence \( \delta \) is fixed, independent of \( y \geq 1 \).

From (A.5), \( \lim_{y \to \infty} B_t^y(f) = -\infty \), which completes the proof of Theorem 5.3 in Case I.

**Case II:** \( f(c) < 0 \). Let \( f = \lim_{n \to \infty} f_n \) where \( (f_n)_{n \geq 1} \) is a Cauchy sequence in \( S_K \), \( K = [c, d] \). Employing (3.3) one may assume that for all \( n \), \( f_n(c) = f(c) < 0 \), hence \( f_n \in S_K \cap \mathcal{I}_K \). In what follows we develop an estimate analogous to (A.5) in order to prove that \( B_t^y(f) < B_t^x(f) \), where \( t > c \) and \( 0 < x < y \).

Let \( w = \inf\{1, x\} \) and let \( \epsilon \in (0, \frac{w|f(c)|}{2}] \). Since \( f \) is continuous on the right, there is a \( \delta \equiv \delta(w) \in (0, 1] \), chosen so that also \( \delta \leq \delta_1 \) if \( y < 1 \), such that for all \( u, v \in [c, c + \delta] \): (i) \( f(u) \leq f(c)/2 < 0 \); (ii) \( |f(u) - f(v)| \leq \epsilon \). One may assume also \( c + \delta \leq t \). The construction in \( \S 3.2 \), property (i), for the sequence \( (f_n)_{n \geq 1} \) of step function approximations to \( f \) shows that for all \( n \) one may assume \( f_n(t) \in \{f(t_i)\} \) (a finite set), for all \( t \in \mathbb{R} \), where the cash flows of \( f_n \) occur at times \( t_i \equiv t_i^n \in [c, d], c = t_0 < t_1 < \cdots < t_q(n) = d \). In particular for all \( n \),

(iii) \( f_n(u) \leq f(c)/2 < 0 \) for all \( u \in [c, c + \delta] \).

Let \( f_n = f_n(C^n) \). For all sufficiently large \( n \) let the cash flows of the sequence \( C^n \) in the interval \( [c, c + \delta] \) be \( C^n_0 = f(c), C^n_1, \ldots, C^n_p, p = p(n) \), occurring at times \( t_r \equiv t^n_r \in [c, c + \delta], t_0 = c, t_p = c + \delta \). Employing (3.3), \( C^n_i = f(t_i) - f(t_{i-1}) \); hence applying (ii) \( |f(t_i) - f(t_j)| \leq \epsilon \) one obtains the useful cash flow estimate in the interval \( [c, c + \delta] \):

\[
(A.6) \quad -\epsilon \leq \sum_{j=k}^{r} C^n_j = f(t_r) - f(t_{k-1}) \leq \epsilon, \quad 1 \leq k \leq r \leq p.
\]

Since \( f_n(c) = f(c) < 0 \) it follows from Lemma 5.1 that for all \( n \) the balance functions \( B_j^u(f_n) \equiv B_{t_j}^u(f_n), j \geq 1, \) are strictly decreasing functions of \( u \in [0, \infty) \). With respect to
the above data on the interval \([t_0, t_p] = [c, c + \delta]\), one now proves in addition that these balances are negative on \([c, c + \delta]\), provided \(u \geq w\).

**Lemma 5.5.** Restricted to the interval \([c, c + \delta]\), for all \(n\), the balances \(B_r^u(f_n) < 0\) for all \(u \in [w, \infty)\), \(0 \leq r \leq p\).

**Proof.** Since these balance functions are decreasing it is sufficient to prove the lemma for \(u = w\). Assume inductively on \(r\) that all the balances \(B_j^w(f_n) < 0\), \(0 \leq j \leq r\). The initial balances \(B_0^w(f_n) = C_0^n < 0\), hence the inductive hypothesis is true at \(r = 0\). Since all the balances \(B_j^w(f_n)\) are negative, \(0 \leq j \leq r\), employing the iteration scheme (4.2), it follows that for all \(n\),

\[(A.7)\]
\[B_{r+1}^w(f_n) = C_0^n w^{r+1-t_0} + \cdots + C_r^n w^{r+1-t_r} + C_{r+1}^n.\]

Since \(w \in (0, 1]\) the function \(w^t\) is decreasing as function of \(t\). Also for all \(r\), the differences \(t_r - t_0 \leq t_p - t_0 = \delta \leq 1\). Hence \(w^{t_r-t_0} \geq w, 1 \leq r \leq p\); consequently the first term in the sum (A.7), \(C_0^n w^{t_r+1-t_0} \leq C_0^n w (C_0^n = f(c) < 0)\). Thus

\[(A.8)\]
\[B_{r+1}^w(f_n) \leq C_0^n w + \sum_{j=1}^{r+1} C_j^n w^{t_r+1-t_j}.\]

Since the sequence \((w^{t_r+1-t_j})_{1 \leq j \leq r+1}\) is strictly increasing (last term is \(w^0 = 1\), Abel’s Lemma applies to the sum (A.8) to obtain, employing also (A.6), the estimate: for all \(n\),

\[
B_{r+1}^w(f_n) \leq C_0^n w + \sup_{1 \leq k \leq r+1} (C_k^n + C_{k+1}^n + \cdots + C_{r+1}^n) \\
\leq C_0^n w + \varepsilon \leq C_0^n w + \frac{w|C_0^n|}{2} = \frac{C_0^n w}{2} = \frac{f(c)w}{2} < 0,
\]

which completes the inductive step and the lemma is proved. \(\square\)

Applying Lemma 5.2 and also the iteration scheme (4.2) to the balances \(B_r^u(f_n) < 0\) for all \(u \geq w\), \(0 \leq r \leq p\), one has the explicit computation for all \(n\),

\[(A.9)\]
\[B_r^u(f_n) = B_r(f_n)(a, u) = \sum_{j=0}^{r} C_j^n u^{t_r-t_j}, \ 0 \leq r \leq p.\]
Returning to the proof of Theorem 5.3 in Case II, since \( w \leq x < y \) it follows from (A.9) that at time \( t_p = c + \delta \), for all \( n \)

\[
\mathcal{B}_p^y(f_n) - \mathcal{B}_p^x(f_n) = \sum_{j=0}^{p} C^n_j (y^{t_p-t_j} - x^{t_p-t_j}).
\]

The function \( h(s) = y^s - x^s \); \( h(0) = 0 \), \( s \in [0, \delta] \) is strictly increasing. Consequently the sequence \( (y^{t_p-t_j} - x^{t_p-t_j})_{0 \leq j \leq p} \) is non-negative and strictly decreasing. Applying Abel’s Lemma to the sum (A.10), for all \( n \) \( (\delta = t_p - t_0) \),

\[
\mathcal{B}_p^y(f_n) - \mathcal{B}_p^x(f_n) \leq (y^{\delta} - x^{\delta}) \sup_{0 \leq j \leq p} (C^n_0 + \cdots + C^n_j).
\]

Now \( f_n(t_j) = C^n_0 + \cdots + C^n_j \), where \( t_j \in [c, c + \delta], 0 \leq j \leq p \). In particular employing property (iii) above, for all \( n \), \( f_n(t_j) \leq f(c)/2 < 0, 0 \leq j \leq p \), and hence one obtains the uniform estimate: for all \( n \) and all \( w \leq x < y \),

\[
\mathcal{B}_p^y(f_n) - \mathcal{B}_p^x(f_n) \leq (y^{\delta} - x^{\delta})f(c)/2 < 0.
\]

Again as in Case I, let the cash flows of the sequence \( C^n \) on the complementary interval \( [c + \delta, t] \) occur at times \( t_p = c + \delta, t_{p+1}, \ldots, t_q = t \). Applying Lemma 5.2 \( (t = t_q, s = t_p) \) and the inequality (A.11) it follows that for all \( n \) the final balances at time \( t \) satisfy the uniform estimate: for all \( n \), if \( w \leq x < y \) \( (c(u, v) \) is positive and monotone decreasing)

\[
\mathcal{B}_t^y(f_n) - \mathcal{B}_t^x(f_n) \leq c(t_p, t)(\mathcal{B}_p^y(f_n) - \mathcal{B}_p^x(f_n)) \\
\leq c(L)(y^{\delta} - x^{\delta})f(c)/2 < 0, \quad L = [c, t].
\]

Passing to the limit as \( n \to \infty \),

\[
\mathcal{B}_t^y(f) - \mathcal{B}_t^x(f) \leq c(L)(y^{\delta} - x^{\delta})f(c)/2 < 0
\]

Thus (A.12) proves that \( \mathcal{B}_t^y(f) < \mathcal{B}_t^x(f) \) if \( 0 < x < y \); hence the decreasing continuous function \( \mathcal{B}_t^x(f) \) is a strictly decreasing function of \( x \in [0, \infty) \). Furthermore, if \( y \geq 1 \) then \( h(s) = y^s - x^s, s \geq 0, \) is an increasing function; hence \( \delta \) is fixed, independent of \( y \geq 1 \).

From (A.12), \( \lim_{y \to \infty} \mathcal{B}_t^y(f) = -\infty \), which completes the proof of Theorem 5.3 in Case II.

The proof of Theorem 5.3 is now complete □
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