Complete solution of Altarelli-Parisi evolution equation in next-to-leading order and non-singlet structure function at low-x

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Abstract
We present complete solution of Altarelli-Parisi (AP) evolution equation in next-to-leading order (NLO) and obtain t-evolution of non-singlet structure function at low-x. Results are compared with HERA low-x and low-\(Q^2\) data and also with those of complete solution in leading order (LO) of AP evolution equation.

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1. Introduction:
The Altarelli-Parisi (AP) evolution equations [1-4] are fundamental tools to study the t(\(=\ln(Q^2/\Lambda^2)\)) and x-evolutions of structure functions, where x and \(Q^2\) are Bjorken scaling variable and four momenta transfer in a deep inelastic scattering (DIS) process [5] respectively and \(\Lambda\) is the QCD cut-off parameter. Though numerical solutions are available in the literature [6], the explorations of the possibility of obtaining analytical solutions of AP evolution equations are always interesting. In this connection, complete solutions of AP evolution equations at low-x in leading order (LO) have been obtained [7]. Its natural improvement will be the next-to-leading order (NLO) calculation.

In this paper, we present complete solution of AP evolution equation in NLO at low-x and obtain t-evolution of non-singlet structure function. Results are compared with the HERA low-x low-\(Q^2\) data, and also with those of complete solution in LO. Here Section 1, Section 2 and Section 3 give the introduction, the necessary theory and the results and discussion respectively.

2. Theory:
The AP evolution equation for non-singlet structure function in NLO is [8]

\[
\frac{\partial F_{2NS}^N(x,t)}{\partial t} = \frac{\alpha_s(t)}{2\pi} \left[ \frac{2}{3} \{3 + 4\ln(1 - x)\} F_{2NS}^N(x,t) - \frac{4}{3} \int_x^1 \frac{dw}{1 - w} \left\{ (1 + w^2) F_{2NS}^N\left(\frac{x}{w},t\right) - 2 F_{2NS}^N(x,t) \right\} \right]
\]
- \left( \frac{\alpha_s(t)}{2\pi} \right)^2 \left[ (x-1)F_{2}^{NS}(x, t) \int_{0}^{1} f(w)dw + \int_{x}^{1} f(w)F_{2}^{NS} \left( \frac{x}{w}, t \right) dw \right] = 0 \quad (1)

where,
\[ f(w) = \frac{16}{9} P_F(w) + 2P_G(w) + \frac{2}{3} n_f P_{N_F}(w) + \frac{2}{9} P_A(w). \]

The explicit forms of higher order kernels are [9]
\[ P_F(w) = -\frac{2(1+w^2)}{1-w} \ln w \ln(1-w) - \left( \frac{3}{1-w} + 2w \right) \ln w - \frac{1}{2}(1+w)\ln^2 w - 5(1-w), \]
\[ P_G(w) = \frac{1+w^2}{1-w} \left( \ln^2 w + \frac{11}{3} \ln w + \frac{67}{9} - \frac{\pi^2}{3} \right) + 2(1+w)\ln w + \frac{40}{3}(1-w), \]
\[ P_{N_F}(w) = \frac{2}{3} \left[ \frac{1+w^2}{1-w} \left( -\ln w - \frac{5}{3} \right) - 2(1-w) \right] \]
and
\[ P_A(w) = \frac{2(1+w^2)}{1+w} \int_{w/(1+w)}^{1/(1+w)} \frac{dk}{k} \ln \frac{1-k}{k} + 2(1+w)\ln w + 4(1-w). \]

Running coupling constant in higher order has the form [10,11]
\[ \alpha_s(t) = \frac{4\pi}{\beta_0 t} \left[ 1 - \frac{\beta_1 lnt}{\beta_0^2 t} \right] \]
for one loop with
\[ \beta_o = \frac{33 - 2n_f}{3} \quad \text{and} \quad \beta_1 = \frac{306 - 38n_f}{3}, \]
\( n_f \) being the number of flavours.

Using Taylor expansion method [12] and neglecting higher order terms as discussed in our earlier works [7,13,14], \( F_{2}^{NS}(x/w, t) \) can be approximated for low-x as
\[ F_{2}^{NS} \left( \frac{x}{w}, t \right) \approx F_{2}^{NS}(x, t) + x \sum_{l=1}^{\infty} u_l \frac{\partial F_{2}^{NS}(x, t)}{\partial x} \quad (2) \]
where
\[ x = 1-w \quad \text{and} \quad \frac{x}{1-u} = x \sum_{l=0}^{\infty} u^l. \]

Putting equation (2) in equation (1) and performing u-integrations we get,
\[ \frac{\partial F_{2}^{NS}(x, t)}{\partial t} - \left[ \frac{\alpha_s(t)}{2\pi} A(x) + \left( \frac{\alpha_s(t)}{2\pi} \right)^2 B(x) \right] \frac{\partial F_{2}^{NS}(x, t)}{\partial x} \]
\[ - \left[ \frac{\alpha_s(t)}{2\pi} C(x) + \left( \frac{\alpha_s(t)}{2\pi} \right)^2 D(x) \right] F_{2}^{NS}(x, t) = 0 \quad (3) \]
where,
\[ A(x) = \frac{2}{3}\{-2xlnx + x(1 - x^2)} \],
\[ B(x) = x \int_x^1 \frac{1-w}{w} f(w)dw, \]
\[ C(x) = \frac{2}{3}\{3 + 4ln(1 - x) + (x - 1)(x + 3)} \]
and
\[ D(x) = -\int_0^x f(w)dw + x \int_0^1 f(w)dw. \]

For a possible solution, we assume that
\[ \left( \frac{\alpha_s(t)}{2\pi} \right)^2 = k \left( \frac{\alpha_s(t)}{2\pi} \right) \]
where, \( k \) is a numerical parameter to be obtained from the particular \( Q^2 \)- range under study. By a suitable choice of \( k \) we can reduce the error to a minimum. Now equation (3) can be recast as
\[ \frac{\partial F_{NS}^2(x, t)}{\partial t} - P(x, t) \frac{\partial F_{NS}^2(x, t)}{\partial x} - Q(x, t)F_{NS}^2(x, t) = 0, \] (4)
where,
\[ P(x, t) = \frac{\alpha_s(t)}{2\pi}[A(x) + kB(x)] \]
and
\[ Q(x, t) = \frac{\alpha_s(t)}{2\pi}[C(x) + kD(x)]. \]

The general solution of equation (4) is
\[ F(U, V) = 0 \]
where, \( F \) is an arbitrary function and
\[ U(x, t, F_{NS}^2) = C_1 \text{ and } V(x, t, F_{NS}^2) = C_2, \]
where \( C_1 \) and \( C_2 \) are constants, form a solution of equations
\[ \frac{dx}{P(x, t)} = -\frac{dt}{-1} = \frac{dF_{NS}^2(x, t)}{-Q(x, t)}. \] (5)

Solving equation (5) we obtain,
\[ U(x, t, F_{NS}^2) = t^{(b/t+1) \exp \left[ \frac{b}{t} + \frac{N(x)}{a} \right]} \]
and
\[ V(x, t, F_{NS}^2) = F_{NS}^2(x, t) \exp[M(x)] \]
where

\[ a = \frac{2}{\beta_o}, \quad b = \frac{\beta_1}{\beta_o^2}, \]

\[ N(x) = \int \frac{dx}{A(x) + kB(x)} \]

and

\[ M(x) = \int \frac{C(x) + kD(x)}{A(x) + kB(x)} dx. \]

If \( U \) and \( V \) are two independent solutions of equation (4) and if \( \alpha \) and \( \beta \) are arbitrary constants, then

\[ V = \alpha U + \beta \]

is called a complete solution of equation (4). Then the complete solution [12]

\[ F_{NS}^{2}(x, t) \exp[M(x)] = \alpha \left[ t^{(b/t+1)} \exp \left( \frac{b + N(x)}{a} \right) \right] + \beta \]

is a two-parameter family of planes. The one parameter family determined by taking \( \beta = \alpha^2 \) has equation

\[ F_{NS}^{2}(x, t) \exp[M(x)] = \alpha \left[ t^{(b/t+1)} \exp \left( \frac{b + N(x)}{a} \right) \right] + \alpha^2. \] (6)

Differentiating equation (6) with respect to \( \alpha \), we get

\[ \alpha = -\frac{1}{2} t^{(b/t+1)} \exp \left[ \frac{b + N(x)}{a} \right]. \]

Putting the value of \( \alpha \) again in equation (6), we obtain

\[ F_{NS}^{2}(x, t) \exp[M(x)] = -\frac{1}{4} \left[ t^{(b/t+1)} \exp \left( \frac{b + N(x)}{a} \right) \right]^2. \]

Therefore,

\[ F_{NS}^{2}(x, t) = -\frac{1}{4} t^{2(b/t+1)} \exp \left[ \frac{2b}{t} + \frac{2N(x)}{a} - M(x) \right]. \] (7)

Now, defining

\[ F_{NS}^{2}(x, t_o) = -\frac{1}{4} t_o^{2(b/t_o+1)} \exp \left[ \frac{2b}{t_o} + \frac{2N(x)}{a} - M(x) \right]. \]

at \( t = t_o \), where \( t_o = \ln(Q_o^2/\Lambda^2) \) at any lower value \( Q = Q_o \), we get from equation (7)

\[ F_{NS}^{2}(x, t) = F_{NS}^{2}(x, t_o) \left( \frac{t^{(b/t+1)}}{t_o^{(b/t_o+1)}} \right)^2 \exp \left[ 2b \left( \frac{1}{t} - \frac{1}{t_o} \right) \right]. \] (8)
which gives the t-evolution of non-singlet structure function $F_2^{NS}(x, t)$ in NLO. In an earlier communication [7], we suggested that for low-x in LO

$$F_2^{NS}(x, t) = F_2^{NS}(x, t_0) \left( \frac{t}{t_0} \right)^2.$$  \hspace{1cm} (9)

We observe that if $b$ tends to zero, then equation (8) tends to equation (9), i.e., solution of NLO equation goes to that of LO equation. Physically $b$ tends to zero means number of flavours is high.

Again defining,

$$F_2^{NS}(x_o, t) = -\frac{1}{4} t^{(b/t+1)} e x p \left[ \frac{2b}{t} + \frac{2N(x)}{a} - M(x) \right]_{x=x_o},$$

we obtain from equation (7)

$$F_2^{NS}(x, t) = F_2^{NS}(x_o, t) e x p \int_{x_o}^{x} \left[ \frac{2}{a} \frac{1}{A(x) + k B(x)} - C(x) + k D(x) \right] dx$$ \hspace{1cm} (10)

which gives the x-evolution of non-singlet structure function $F_2^{NS}(x, t)$ in NLO.

Proton and neutron structure functions measured in deep inelastic electro-production can be written in terms of singlet and non-singlet quark distribution functions as

$$F_2^p(x, t) = \frac{5}{18} F_2^S(x, t) + \frac{3}{18} F_2^{NS}(x, t)$$

and

$$F_2^n(x, t) = \frac{5}{18} F_2^S(x, t) - \frac{3}{18} F_2^{NS}(x, t).$$

These equations give

$$F_2^{NS} = 3(F_2^p - F_2^n)$$

from which we can calculate experimental values of $F_2^{NS}$ in $t$ and $x$ ranges given in $F_2^p$ and $F_2^n$.

3. Results and Discussion:

In the present paper, we compare our results of t-evolution of non-singlet structure functions from equation (8) with the HERA low-x and low-$Q^2$ data [15]. Here proton and neutron structure functions are measured in the range $2 \leq Q^2 \leq 50 GeV^2$. Moreover here $P_T \leq 200 MeV$, where $P_T$ is the transverse momentum of the final state baryon. We consider number of flavours $n_f=4$.

In figures 1(a-c) we present our results of t-evolution of non-singlet structure functions $F_2^{NS}$ (solid lines) for the representative values of $x$ given to test the evolution equation (8) in next-to-leading order. Agreement is found to be good. In the same figures we also plot the results of t-evolutions of non-singlet structure functions $F_2^{NS}$ (dashed lines) for the complete
solutions from equation (9) in leading order. Data points at lowest-$Q^2$ values in the figures are taken as inputs. We observe that t-evolutions are slightly steeper in NLO calculations then those of LO. We can also calculate x-evolution of non-singlet structure functions at low-x from equation (10). But it involves complicated triple integrations and we keep it as our subsequent work.

In figure 2 we plot $T(t)^2$ and $kT(t)$ against $Q^2$ in the $Q^2$ range $2 \leq Q^2 \leq 50 \text{ GeV}^2$ as required by our data used[15]. Though the explicit value of $k$ is not necessary in calculating t-evolution of $F_{2}^{NS}$, yet we observe that for $k=0.027$, errors become minimum and varies from $17.68\%$ to $-13.68\%$ in the $Q^2$ range $2 \leq Q^2 \leq 50 \text{ GeV}^2$.

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Figure 1: t-evolutions of non-singlet structure functions $F_{2}^{NS}(x,t)$ (solid lines) for the representative values of $x$ given in the figures. Data points at lowest-$Q^2$ values in the figures are taken as input to test NLO t-evolution of non-singlet structure functions $F_{2}^{NS}$ from equation (8). In the same figures we also plot the results of t-evolutions of non-singlet structure functions $F_{2}^{NS}$ (dashed lines) for LO from equation (9).

Figure 2: $T(t)^2$ and $kT(t)$ against $Q^2$ in the range $2 \leq Q^2 \leq 50$ GeV$^2$ for $k=0.027$. 