Richardson-Gaudin models: the hyperbolic family

Jorge Dukelsky¹, Stefan Rombouts¹,² and Gerardo Ortiz³

¹ Instituto de Estructura de la Materia. CSIC. Serrano 123. 28006 Madrid. Spain
² Departamento de Física Aplicada, Universidad de Huelva, 21071 Huelva, Spain
³ Department of Physics, Indiana University, Bloomington IN 47405, USA

E-mail: j.dukelsky@csic.es

Abstract.
We show that the hyperbolic family of Richardson-Gaudin models could describe p-wave superfluids in two dimensional lattices. The analysis of the exact solution reveals an exotic phase diagram with a third order quantum phase transition. Moreover, we propose a separable integrable pairing Hamiltonian within the same family that could be useful to describe exactly superconductivity in heavy nuclei.

1. Introduction
The celebrated paper of Bardeen, Cooper and Schrieffer (BCS) of 1957 [1] gave, within a variational approach, the first microscopic description of the superconducting phenomenon. The following year, Bohr, Mottelson and Pines [2] noted that similar physics may underly the large gaps seen in the spectra of even-even atomic nuclei, emphasizing however that finite-size effects would be critical for a proper description of such systems. At roughly the same time that number projected BCS (PBCS) [3] theory was developed, Richardson [4] showed that for a pure pairing Hamiltonian it is possible to exactly solve the Schrödinger equation by following closely Cooper’s original idea. Years later and from a different perspective, Gaudin introduced an integrable spin model having striking similarities with the Richardson exact solution [5]. However, he couldn’t find the explicit relation between both integrable models. In an intensive collaboration with Peter Schuck, we were able to find this relation through a generalization of the Gaudin integrals of motion giving rise to three classes of pairing-like models that we called the Richardson-Gaudin (RG) integrable models, all of which were integrable and all of which could be solved exactly for both fermion and boson systems [6, 7]. During the last decade, the rational family of the RG models was extensively used to describe ultrasmall superconduction grains, heavy nuclei, quantum dots, ultracold atomic gases, etc [8]. More recently, we have found two physical realizations of the hyperbolic models, one for p-wave polarized atomic gases in two dimensional lattices [9, 10], and the other as a potentially useful realistic pairing Hamiltonian for heavy nuclei. In this contribution we will briefly describe the exact solution for $p_x + ip_y$ superfluids, and we will present preliminary results showing how the integrable hyperbolic Hamiltonian could reproduce Gogni Hartree-Fock-Bogoliubov (HFB) gaps in heavy nuclei.
2. The Hyperbolic Model
We start with the integrals of motion of the hyperbolic RG model [7], which can be written in a compact form [11] by making the replacements \( \sinh(x) = \frac{e^x - 1}{2\sqrt{\eta}} \) and \( \coth(x) = \frac{e^x + 1}{\eta} \) as

\[
R_i = S_i^z - 2\gamma \sum_{j \neq i} \left( \frac{\sqrt{\eta_i / \eta_j}}{\eta_i - \eta_j} \left(S_i^+ S_j^- + S_i^- S_j^+ \right) \right),
\]

where \( S_i^+, S_i^- \) are the three generators of the \( SU(2) \) algebra of mode \( i \), \( i = 1, \ldots, L \), with spin representation \( s_i \) such that \( \langle S_i^2 \rangle = s_i(s_i + 1) \). We assume that there are \( L \) copies of the \( SU(2) \) algebra or equivalently \( L \) modes. Therefore, the \( L \) operators \( R_i \) contain \( L \) free parameters \( \eta_i \) plus the strength of the quadratic term \( \gamma \). The integrals of motion (2) commute with the \( z \) component of the total spin \( S_z = \sum_{i=1}^L S_i^z \).

It is worthwhile to verify that the set of operators \( R_i \) commute among themselves, conforming a complete set of integrals of motion. Therefore, they have a complete set of common eigenstates which are parametrized by the ansatz

\[
|\Psi\rangle = \prod_{\alpha=1}^M S_\alpha^+ |\nu\rangle, \quad S_\alpha^+ = \sum_i \frac{\sqrt{\eta_i}}{\eta_i - E_\alpha} S_i^+, \tag{2}
\]

where the \( E_\alpha \) are the pair energies or pairons which are to be determined such that the ansatz (2) satisfies the eigenvalue equations \( R_i |\Psi\rangle = r_j |\Psi\rangle \).

In the pairing representations each \( SU(2) \) copy is associated with a single particle level \( i \) and \( M \) is the number of active pairs. The vacuum \( |\nu\rangle \) is defined by a set of seniorities, \( |\nu\rangle = |\nu_1, \nu_2, \ldots, \nu_L\rangle \), where the seniority \( \nu_i \) is the number of unpaired particles in level \( i \) with single particle degeneracy \( \Omega_i \), such that \( s_i = (\Omega_i - 2\nu_i)/4 \).

Although any function of the integrals of motion generates an exactly solvable Hamiltonian, we will restrict ourselves in this presentation to the simple linear combination \( H = \sum_i \eta_i R_i \) that after some algebraic manipulations reduces to

\[
H = \sum_i \eta_i S_i^z - G \sum_{i,j} \sqrt{\eta_i \eta_j} S_i^+ S_j^-.
\]

This separable Hamiltonian has the eigenvectors (2) and the eigenvalues \( E = \sum_i \nu |S_i^z| \nu > + \sum_{\alpha} E_\alpha \), where the pairons \( E_\alpha \) are a solution of the set of non-linear Richardson equations

\[
\sum_i \frac{s_i}{\eta_i - E_\alpha} - \sum_{\alpha' \neq \alpha} \frac{1}{E_{\alpha'} - E_\alpha} = \frac{Q}{E_\alpha}, \tag{4}
\]

with \( Q = \frac{1}{\gamma^2} - \frac{L_c}{\gamma} + \frac{M - 1}{\gamma^2} \), \( L_c = 2 \sum_j s_i \), and \( M \) is the number of pairons.

Each particular solution of Eq. 4 defines a unique eigenstate. For the remaining discussion we will assume that \( \langle \nu | H_k | \nu > = 0 \), which amounts to a simple shift in the energy scale, without loss of generality.

3. The \( px + ipy \) pairing Hamiltonian
In recent years \( p \)-wave paired superfluids have attracted a lot of attention, in part due to their exotic properties [12]. Of particular interest is the chiral two-dimensional (2D) \( px + ipy \) superfluid of spinless fermions, that supports a topological phase with zero energy Majorana modes [13] and, unlike the \( s \)-wave superfluid, it has a quantum phase transition in the crossover from BCS to BEC whose properties are not yet well understood. Therefore, the derivation of an exactly solvable model could be essential for the understanding of this exotic superfluid.
In two spatial dimensions, one can define a representation of the $SU(2)$ algebra in terms of creation (annihilation) spinless fermions operators in momentum space, $c^\dagger_k$ ($c_k$). Each pair of states $(k, -k)$ is associated to a single-particle level $\eta_k$, where the index $k$ now refers to the momentum in 2D (in order to avoid double counting we select $k_x > 0$ to label the levels). Furthermore, one can include a phase factor in the definition of $SU(2)$ generators:

$$S^z_k = \frac{1}{2} (c^\dagger_k c_k + c^\dagger_{-k} c_{-k} - 1), S^+_k = \frac{k_x + ik_y}{|k|} c^\dagger_k c_{-k}, S^-_k = \frac{k_x - ik_y}{|k|} c^\dagger_{-k} c_k.$$  \hspace{1cm} (5)

By taking $\eta_k = k^2$, one obtains the exactly solvable $p_x + ip_y$ model first introduced Ibañez et. al [9]:

$$H_{p_x+ip_y} = \sum_{k, k_x > 0} \frac{k^2}{2} (c^\dagger_k c_k + c^\dagger_{-k} c_{-k}) - G \sum_{k, k_x > 0, k'_x > 0} (k_x + ik_y)(k'_x - ik'_y)c^\dagger_k c_{-k} c^\dagger_{-k} c_{-k}. \hspace{1cm} (6)$$

Coming back to the Richardson equations (4) that solves the Hamiltonian (6), we recognize two special cases: case (i) all are real and negative if $\frac{1}{G} \leq L - 2M + 1$; we will see that the boundary coincides with the phase transition line. case (ii) All pairons converge to zero for $\frac{1}{G} = L - M + 1$; this situation determines the so called Moore-Read line [14, 9], with interesting properties associated with the fractional quantum Hall effect. Between these two regimes, a fraction of the pairons can converge to zero at integer values of $G^{-1}$. The phase diagram of the $p_x + ip_y$ Hamiltonian (6) depicted in Fig. 1 is characterized by the density $\rho = M/L$ and the scaled pairing strength $g = GL$. The transition between the strong pairing region (BEC) with all pairons real and negative and the weak pairing region (BCS) takes place when one pairon change sign implying that one of the bound molecules in the BEC gets unbounded. For this reason we characterized the transition as a confinement-deconfinement quantum phase transition.

In order to get a more quantitative picture of the pairon distribution in the three regions of the quantum phase diagram, we plot in Fig. 2 the pairon distributions for three representative values of the coupling strength, $g = 0.5, 1.5, 2.5$, at quarter filling for a disk of radius 18 corresponding to a total pair degeneracy $L = 504$. The positions of these points in the quantum phase diagram

**Figure 1.** Phase diagram of the $p_x + ip_y$ model in terms of the density $\rho$ and the pairing strength $g$. The three circles at quarter filling indicate the configurations studied in the following figure.
of Fig. 1 is indicated by the three filled circles. In weak coupling BCS part of the pairons stick to the lower part of the real positive axis, while the remaining pairons form an arc in the complex plane. Approaching the Moore-Read line it looks like the arc is going to close around the origin, but just at the Moore-Read line all pairons collapse to zero, and then a first real negative pairon emerges. In the intermediate weak pairing region a successive series of collapses ensues, at integer values of \( Q \), each time producing one more real negative pairon and reducing the size of the arc around the origin. When the last pairon turns real and negative, the system enters the strong pairing phase. From then on the most negative pairon diverges proportional to the interaction strength \( G \), while the least negative pairon converges to a finite value.

In order characterize the quantum phase transition, we study the energy density derivatives as describe by the BCS theory which is exact in the thermodynamic limit. As can be seen in Fig. 3, the third derivative shows a discontinuity confirming that the phase transition is third order in the Erhenfest classification.

4. The integrable nuclear pairing Hamiltonian
Let us come back to the separable pairing Hamiltonian (3) to note that if we interpret the parameters \( \eta_k \) as single particle energies corresponding to a nuclear mean field potential, the pairing interaction has the unphysical behavior of increasing in strength with energy. In order to reverse this unwanted effect we define \( \eta_k = 2(\varepsilon_k - \alpha) \), where the free parameter \( \alpha \) plays the role of an energy cutoff and \( \varepsilon_k \) is the single particle energy in the mean field level \( k \). Making

\[
\eta_k = 2(\varepsilon_k - \alpha)
\]
Figure 3. Higher order derivatives of the energy density as a function of $g$ for various densities. The open circles mark the transition point at $g = (1 - 2\rho)^{-1}$.

Our aim is to compare at the BCS level of approximation the results coming from the integrable Hamiltonian (7) with those from a Gogni HFB calculation. As a first step in ascertain the quality of the Hamiltonian (7) to reproduce the superfluid features of heavy nuclei we compare the trend of the energy dependent gaps in both treatments. The energy dependent gaps of the Gogni self-consistent mean field in the canonical basis and those of the integrable pairing Hamiltonian in the BCS approximation are

$$\Delta_{Gogni}^k = \sum_{kk',>0} V_{kk',kk'} \langle c_{k'}^\dagger c_k \rangle$$

$$\Delta_{Exact}^k = G\sqrt{\alpha - \varepsilon_k} \sum_{kk',>0} \sqrt{\alpha - \varepsilon_{k'}} \langle c_{k'}^\dagger c_k \rangle$$

where the single particle energies $\varepsilon_k$, the pair interaction $V_{kk',kk'}$, and the pair wavefunction $\langle c_{k'}^\dagger c_k \rangle$ are obtained from a self-consistent Gogni calculations in the canonical basis. The integrable gaps can be expressed as $\Delta_{Exact}^k = C\sqrt{\alpha - \varepsilon_k}$, having a square root dependence on the single particle energies. In figure 4 we plot the pairing gaps of $^{154}Sm$ for protons and neutrons using $\alpha = 35 MeV$ as a cutoff and fitting the parameter $C$. In spite of the significant dispersion of the Gogni gaps, it is clear that the integrable gaps follow correctly the global trend. It is interesting to note that a constant pairing interaction, extensively used in the past and also exactly solvable within the rational family of RG models, would give a non reliable constant gap (horizontal line).

These preliminary results suggest that the hyperbolic model could be extremely useful in nuclear structure calculations as a benchmark realistic exactly solvable to test approximations beyond HFB. On a more ambitious respect, it might be possible to fit the pairing strength $G$ as a function of $N$ and $Z$ to the whole table of nuclides and to set up a program of self-consistent Hartree-Fock plus exact pairing. Work along these lines is in progress.
**Figure 4.** Energy dependent gaps of $^{154}$Sm for protons and neutrons. Open circles are Gogni HFB calculations in the canonical basis while the continuous lines are the integrable pairing gaps with $\alpha = 35\,\text{MeV}$ and $C$ fitted to Gogni gaps.

**Acknowledgments**

JD would like to thank Peter Schuck for being his teacher, collaborator and friend in the last 23 years. Furthermore, Peter has been the main collaborator at the beginning of this program on RG integrable models. We are grateful to Rayner Rodríguez-Guzmán for providing us results of the Gogni mean field calculation of $^{154}$Sm. We acknowledge support from a Marie Curie Action of the European Community Project No. 220335 and the Spanish Ministry for Science and Innovation Project No. FIS2009-07277

5. References

[1] Bardeen J, Cooper L N and Schrieffer J R 1957 Phys. Rev. 108 1175
[2] Bohr A, Mottelson BR and Pines D 1958 Phys. Rev. 110 936
[3] Dietrich K, Mang H J and Pradal J H 1964 Phys. Rev. 135 B22
[4] Richardson R W 1963 Phys. Lett. 3 277
[5] Gaudin M 1976 J. Physique 37 1087
[6] Dukelsky J and Schuck P 2001 Phys. Rev. Lett. 86 4207
[7] Dukelsky J, Esebbag C and Schuck P 2001 Phys. Rev. Lett. 87 066403
[8] Dukelsky J, Pittel S and Sierra G 2004 Rev. Mod. Phys. 76 643
[9] Ibañez M, Links J, Sierra G and Zhao S-Y 2009 Phys. Rev. B 79 180501
[10] Rombouts S M A, Dukelsky J and Ortiz G 2010 Phys. Rev. B 82 224510
[11] Ortiz G, Somma R, Dukelsky J and Rombouts S M A 2005 Nucl. Phys. B 707 421
[12] Gurarie V and Radzihovsky L 2007 Ann. Phys. 322 2
[13] Read N and Green D 2000 Phys. Rev. B 61 10267
[14] Moore G and Read N 1991 Nucl. Phys. B 360 362