IN-MEDIUM PION-PION INTERACTION 
AND CHIRAL SYMMETRY RESTORATION

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Abstract

We discuss medium modifications of the unitarized pion-pion interaction in the nuclear medium. We incorporate both the effects of chiral symmetry restoration and the influence of collective nuclear pionic modes, originating from the p-wave coupling of the pion to delta-hole configurations. We show how the resulting strong enhancement of the sigma-meson spectral function is related to large fluctuations of the condensate associated with the partial restoration of chiral symmetry.

1 Introduction

There are at least two excellent reasons to study the in-medium modifications of the pion-pion interaction in the scalar-isoscalar channel both being related to fundamental questions in present day nuclear physics. The first refers the binding energy of nuclear matter since a modification of the correlated two-pion exchange may have some deep consequences on the saturation mechanism. The second one is the direct connection with chiral symmetry restoration. From very general arguments based on QCD and model calculations, partial chiral symmetry restoration is expected to occur in nuclear matter. Hence, there must be a softening of a collective scalar-isoscalar mode, usually called the sigma meson, which becomes degenerate with its chiral partner \textit{i.e.} the pion at full restoration density. This also implies that at some density the sigma-meson spectral function should exhibit a significant enhancement near the two-pion threshold. This effect can be seen as a precursor of chiral symmetry restoration associated with large fluctuations of the quark condensate near the chiral phase transition \cite{1}.
However, the first proposed medium effect was the modification of the two-pion propagator and the unitarized $\pi\pi$ interaction from the softening of the pion dispersion relation by p-wave coupling to $p - h$ and $\Delta - h$ states. The existence of collective pionic modes produces a strong accumulation of strength near the two-pion threshold in the scalar-isoscalar channel [2]. According to recent calculations [3, 4], this reshaping of the strength may provide a partial explanation of the $\pi - 2\pi$ data obtained on various nuclei by the CHAOS collaboration at TRIUMF [5]. These results have been questioned in a recent paper where it is found that pion absorption forces the reaction to occur in the nuclear surface, i.e. at very low density [6]. It is clear that the effect of chiral symmetry restoration has to be included on top of p-wave pionic effects to get a better explanation of the data. One attempt to combine both effects is based on the linear sigma model (implemented with a form factor fitted to phase shifts) in which the sigma mass is dropped through a Brown-Rho scaling relation [7]. It is found that chiral symmetry restoration increases the strength of the threshold enhancement by about a factor four as reported in these proceedings [7]. Here we will discuss another attempt based on the Nambu-Jona-Lasinio model [8]. From this NJL model we derive an in-medium pion-pion potential, formally equivalent to the linear sigma model, but with parameters ($m_\sigma, m_\pi, f_\pi$) replaced by their in-medium values. Hence, at variance with a pure dropping sigma mass scenario, the basic $\sigma\pi\pi$ and $4\pi$ couplings are also modified in a chirally consistent framework. P-wave coupling of the pion is incorporated within a standard nuclear-matter approach since the NJL model completely misses (at least in its present treatment) the phenomenologically well-established strong screening effect from short-range correlations ($g'$ parameter). The underlying philosophy can be summarized in stating that the medium-modified soft physics linked to chiral symmetry ($m_\pi, f_\pi$, low energy $\pi - \pi$ potential) is calculated within the NJL model while p-wave physics yielding pionic nuclear collective modes is described through standard nuclear phenomenology.

2 NJL and Density Dependent Linear Sigma Model

We start with the $SU(2)$ version of the NJL model: $\mathcal{L} = \bar{\psi}(i\partial - m)\psi + g[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\tau_j\psi)^2]$. At finite density, the gap equation which determines the constituent quark $M$ mass reads:

$$M = m + 4N_cN_fg\int_{k_F}^{\Lambda} \frac{k^2dk}{2\pi^2} \frac{M}{E}$$  \hspace{1cm} (1)
where \( \Lambda \) is the (three-momentum) cutoff. The pion and the sigma meson can be constructed within the standard RPA approximation. Limiting ourselves to zero momentum, the meson propagators are obtained according to:

\[
\begin{align*}
g_{\pi qq}^2 D_\pi(\omega) &= \left( \omega^2 I(\omega) - m_\pi^2 I(m_\pi) \right)^{-1} \\
g_{\sigma qq}^2 D_\sigma(\omega) &= \left( (\omega^2 - 4M^2) I(\omega) - m_\sigma^2 I(m_\sigma) \right)^{-1} \\
\text{with } I(\omega) &= 2N_c N_f \int_{k_\rho}^\Lambda \frac{p^2 dp}{2\pi^2} \frac{1}{E_p (4E_p^2 - \omega^2)}
\end{align*}
\]

The in-medium masses of the pion and the sigma meson are found as the pole of the above propagators. It is important to notice that they are pure collective \( q\bar{q} \) states since, at zero momentum, there is no contribution from particle-hole excitations (Fermi sea). According to what is said above, the particle-hole sector is better treated within a standard nuclear physics approach. In addition, for reason of simplicity, we use in the following a simplified scheme where \( I(\omega) \) is “frozen” at \( \omega = 0 \). We have verified numerically that this approximation gives almost the same result than the exact calculation for the pion mass, the sigma-meson mass and also for the pion decay constant (see [8]). The parameters of the model \( (m, \Lambda, g) \) have been adjusted to obtain the vacuum values \( f_\pi = 93 \, \text{MeV} \), \( m_\pi = 139 \, \text{MeV} \) and \( m_\sigma = 1 \, \text{GeV} \) which is precisely the value of the bare sigma-meson mass, systematically used in our previous works [2, 3, 7]. The s-wave optical potential is somewhat too small (4 \( \text{MeV} \) at \( \rho_0 \)) but the incorporation of a phenomenologically more appropriate s-wave optical potential (10 \( \text{MeV} \)) has only minor influences on the medium effects presented in this paper.

We are now in position to construct vacuum and in-medium \( \pi\pi \) potentials following the method developed in [9] but adapted to finite density. The basic diagrams are box diagrams with four internal quark lines for the direct \( 4\pi \) interaction and three internal quark lines for the \( \pi\pi\sigma \) coupling from which one obtains the sigma exchange diagram (Fig. 1). Keeping the full momentum dependence makes the problem of subsequent unitarization hopelessly complex. As in [9], we first limit the calculation to the case where the four pions have momenta \( p_i \) such that \( p_i^2 = m_\pi^2 \) (what we call the exact scheme in [8]). However to make the unitarization tractable we further simplify the calculation by freezing the momentum to \( p = 0 \) in the quark-loop integral (simplified scheme). We have checked numerically that the resulting low-energy \( \pi\pi \) amplitudes are almost identical in the simplified and exact schemes [8]. We obtain for the coupling constants \( \lambda_{4\pi} \equiv \lambda = (m_\sigma^2 - m_\pi^2)/2f_\pi^2 \) and \( \lambda_{\sigma\pi\pi} = \lambda f_\pi \). This
is the well-known result of the linear sigma model but with medium-modified parameters \((m_\sigma, m_\pi, f_\pi)\). The linear sigma model, seen as an \(O(N + 1)\) model with \(N = 3\), is treated to leading order in the \(1/N\) expansion which fulfills all the constraints (Ward identities) of chiral symmetry \([3]\). We supplement the model by adding a one-parameter form factor \(v(k)\) to fit the phase shifts in vacuum once the scattering amplitude is unitarized. The (in-medium) unitarized scalar-isoscalar \(\pi\pi\) \(T\) matrix (in the CM frame and for total energy \(E\) of the pion pair) is \([3, 8]\):

\[
\langle k, -k| T(E) | k', -k' \rangle = v(k)v(k') \frac{6\lambda(E^2 - m_\pi^2)}{1 - 3\lambda\Sigma(E)} D_\sigma(E) \tag{3}
\]

where the unitarized sigma propagator (i.e with two-pion loop) \(D_\sigma(E)\) is:

\[
D_\sigma(E) = \left( E^2 - m_\sigma^2 - \frac{6\lambda^2 f_\pi^2 \Sigma(E)}{1 - 3\lambda\Sigma(E)} \right)^{-1} \tag{4}
\]

Chiral symmetry restoration is accounted for by the in-medium pion- as well as sigma-meson masses and \(f_\pi\). The p-wave collective effects are embedded in the two-pion loop:

\[
\Sigma(E) = \int \frac{dq}{(2\pi)^3} v(q) \int \frac{idq_0}{2\pi} D_\pi(q, q_0) D_\pi(-q, E - q_0). \tag{5}
\]

The pion propagator \(D_\pi(q, q_0)\) is calculated in a standard nuclear matter approach \([8]\) and incorporates the p-wave coupling of the pion to delta-hole states with short-range screening described by the usual \(g'_{\Delta\Delta} = 0.5\) parameter. It is particularly interesting to inspect the sigma-meson spectral function. The result of the calculation is shown on Fig. 2. At twice normal nuclear-matter, which is close to the density where the quasi-pole in \(D_\sigma\) is \(E = 2m_\pi\), we find a very sharp peak which can be understood as a precursor effect of chiral symmetry restoration. At normal nuclear-matter density the threshold peak is enhanced by a factor three as compared to a pure p-wave calculation. This is in qualitative agreement with the dropping sigma-meson mass calculation where a factor four is obtained \([7]\). At density \(0.5\rho_0\), more relevant for the CHAOS experiment, we still have a sizable low-energy reshaping which could help to explain the data.

3 Discussion and Interpretation

Four-quark condensates are fundamental quantities of non-perturbative QCD. They are known to play an important role in QCD sum rule analyses of hadron
Figure 1: Box and $\sigma$-meson exchange diagrams for $\pi\pi$ scattering ($a,b,c$ and $d$ are the isospin indices).

Figure 2: The spectral function for the sigma meson at various densities ($0, 0.5, 1, 1.5, 2 \rho/\rho_0$). Left: only $p$-wave pionic effects. Right: with chiral symmetry restoration on top of $p$-wave effects.

Table 1: $m_\sigma$, $m_\pi$, $f_\pi$ (in MeV), the scalar four-quark condensate and the 'kappa factor' versus density. The definitions are given in the text.

| $\rho/\rho_0$ | $m_\sigma$ | $m_\pi$ | $f_\pi$ | $\Delta Q_1$ | $\Delta Q_2$ | $\Delta \kappa_1$ | $\Delta \kappa_2$ |
|---------------|------------|---------|---------|-------------|-------------|----------------|----------------|
| 0             | 1000       | 139     | 93      | 0           | 0           | 0              | 0              |
| 0.5           | 890        | 140.3   | 87      | -0.2        | 0.03        | 0.24           | 0.33           |
| 1             | 795        | 143.1   | 79      | -0.39       | 0.13        | 0.55           | 0.88           |
| 1.5           | 649        | 148.7   | 69      | -0.57       | 0.29        | 0.96           | 1.79           |
| 2             | 510        | 161     | 56      | -0.75       | 0.62        | 1.51           | 3.74           |
spectral functions in the vacuum and in matter. As an example the density evolution of the four-quark condensate appearing in the rho-meson case is an important input for the medium modification of this vector meson in connection with dilepton production in relativistic heavy ion collisions. Here we will concentrate on the scalar four-quark condensate \( \langle \bar{q}q \rangle^2 \) and study how much it deviates from \( \langle \bar{q}q \rangle^2 \) to assess the evolution of quantum fluctuations with increasing density. Inserting a complete set of states, the scalar four-quark condensate is given as:

\[
\langle 0 | \bar{q}q | 0 \rangle = \langle 0 | \bar{q}q | 0 \rangle \langle 0 | \bar{q}q | 0 \rangle + \sum_n \langle 0 | \bar{q}q | n \rangle \langle n | \bar{q}q | 0 \rangle
\]

The first contribution to the sum are scalar-isoscalar two-pion states, able to build a collective sigma meson. To evaluate (at least qualitatively) this quantity one can use low-energy effective theories.

**Linear sigma model.** In the linear sigma model with chiral-symmetry breaking piece \( \mathcal{L}_{\chi SB} = -f_\pi m_\pi^2 \sigma \), \( \langle \sigma \rangle \) plays the role of the condensate and \( \langle \sigma^2 \rangle \) relates to the four-quark condensate. Here we do not aim to estimate the absolute value of this four-quark condensate but restrict our study to its evolution with density. We thus define the quantity \( Q_1(\rho) = \langle 0 | (\bar{q}q)^2 | 0 \rangle / (\bar{q}q)^2_{\text{vac}} \). Introducing the fluctuating part of the sigma field \( s = \sigma - f_\pi \), one obtains:

\[
Q_1(\rho) = \frac{\langle (\bar{q}q)^2 \rangle (\rho)}{(\bar{q}q)^2_{\text{vac}}} = \frac{\langle \sigma^2 \rangle (\rho)}{f_\pi^2} = 1 + 2 \frac{\langle s \rangle (\rho)}{f_\pi} + \frac{\langle s^2 \rangle (\rho)}{f_\pi^2}
\]

A better measure of the fluctuations of the condensate is provided by the ‘kappa factor’ defined in QCD sum rule analyses:

\[
\kappa_1(\rho) = \frac{\langle (\bar{q}q)^2 \rangle (\rho)}{(\bar{q}q)^2_{\text{vac}}} = \frac{\langle \sigma^2 \rangle (\rho)}{\langle \sigma \rangle^2 (\rho)} = 1 + 2 \frac{\langle s \rangle (\rho)}{f_\pi} + \frac{\langle s^2 \rangle (\rho)}{f_\pi^2} + \frac{\langle s^2 \rangle (\rho)}{f_\pi^2} + \frac{\langle s^2 \rangle (\rho)}{f_\pi^2} + \frac{\langle s^2 \rangle (\rho)}{f_\pi^2}
\]

To first order in the density \( \langle s \rangle (\rho) \) is governed by the pion-nucleon sigma term. Using a value compatible with the model (ignoring the pionic contribution), we take \( \langle s \rangle (\rho) = -0.18 \rho / \rho_0 \). From a standard dispersive analysis, \( \langle s^2 \rangle \) can be expressed in terms of a phase-space integral of the sigma-meson spectral function as:

\[
\langle s^2 \rangle (\rho) = \int_0^{\Lambda_P} \frac{dP}{(2\pi)^3} \int_0^\infty dE \left( -\frac{1}{\pi} \right) \text{Im} D_\sigma(E, P)
\]

where we have introduced a momentum cutoff \( \Lambda_P \) defining the range of validity of the effective approach. Taking \( \Lambda_P \sim 1 \text{ GeV} \), and making a simple estimate
with a sharp sigma meson of mass 1 GeV, we obtain in the vacuum a kappa factor of the order or larger than 2 which is very close (most probably by accident) to $\kappa = 2.36$, generally used in the rho-meson channel. Finally, since we do not know the full momentum dependence of the sigma spectral function, we assume covariance for $D_\sigma$. We have checked that, using another extreme assumption (static approximation), the results are qualitatively similar. To reduce the uncertainty on the cutoff we prefer to present the results in Tabl. 1 for the quantities $\Delta Q_1(\rho) = Q_1(\rho) - (Q_1)_{vac}$ and $\Delta \kappa_1(\rho) = \kappa_1(\rho) - (\kappa_1)_{vac}$.

For the sigma-meson propagator we use the form (4) but with vacuum values for $m_\sigma$, $m_\pi$ and $f_\pi$, hence keeping only medium effects from p-wave pionic collective modes. In the actual calculation we have adopted the cutoff parameter $\Lambda_P = 1.2 GeV$ to (arbitrarily) fix the kappa factor in the vacuum to $\kappa_{vac} = 2.36$ keeping in mind that the density evolution should not be very sensitive to this particular choice.

**NJL model.** In the NJL model the four-quark condensate can be directly calculated. It can be expressed in terms of a phase-space integral of the full scalar-isoscalar response function and subsequently in terms of the $q\bar{q}$ sigma meson spectral function:

$$\kappa_2(\rho) = \frac{\langle (\bar{q}q)^2 \rangle(\rho)}{\langle \bar{q}q \rangle^2(\rho)} = 1 + \frac{1}{f^2_\pi(\rho)} \int_0^{\Lambda_P} \frac{dP}{(2\pi)^3} \int_{-1}^{\infty} dE \left( -\frac{1}{\pi} \right) ImD_\sigma(E, P)$$

$$Q_2(\rho) = \frac{\langle (\bar{q}q)^2 \rangle(\rho)}{\langle \bar{q}q \rangle^2_{vac}} = \frac{(f^2_\pi m^2_\pi)(\rho)}{(f^2_\pi m^2_\pi)_{vac}} \kappa_2(\rho)$$

The sigma propagator is then unitarized by incorporating a dressed pion loop (eq. 4). Notice that, in the vacuum, we exactly recover the linear sigma model results. We show in Tab. 1 the quantities $\Delta Q_2(\rho)$ and $\Delta \kappa_2(\rho)$ which now contain the effect of chiral symmetry restoration on top of the p-wave pionic collective modes. One sees that the four-quark condensate $Q_2$ increases with density when chiral symmetry restoration is incorporated, contrary to the pure p-wave case ($Q_1$). We also see, by looking at the kappa factor, that chiral symmetry restoration considerably increases the fluctuations of the condensate. This demonstrates that the sharp structure near $2m_\pi$, which is mainly associated with the dropping sigma-meson mass, is intimately related to precritical effects with a strong enhancement of chiral fluctuations. In a second-order phase transition these would actually diverge in the chiral limit, $m_\pi \rightarrow 0$. 
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