On the total vertex irregularity strength of comb product of cycle and path with order 3

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Abstract. Let \( G = (V(G),E(G)) \) be a graph and \( k \) be a positive integer. A total \( k \)-labeling of \( G \) is a map \( f: V(G) \cup E(G) \rightarrow \{1, 2, \ldots, k\} \). The vertex weight \( v \) under the labeling \( f \) is denoted by \( w_f(v) \) and defined by \( w_f(v) = f(v) + \sum_{u \in E(G)} f(uv) \). A total \( k \)-labeling of \( G \) is called vertex irregular if there are no two vertices with the same weight. The total vertex irregularity strength of \( G \), denoted by \( tvs(G) \), is the minimum \( k \) such that \( G \) has a vertex irregular total \( k \)-labeling. Let \( G \) and \( H \) be two connected graphs. Let \( o \) be a vertex of \( H \). The comb product between \( G \) and \( H \), denoted by \( G \circ_o H \), is a graph obtained by taking one copy of \( G \) and \( |V(G)| \) copies of \( H \) and grafting the \( i \)-th copy of \( H \) at the vertex \( o \) to the \( i \)-th vertex of \( G \). In this paper, we determine the total vertex irregularity strength of comb product of cycle and path with order 3.

1. Introduction

A total \( k \)-labeling of a graph \( G \) is a map \( f: V(G) \cup E(G) \rightarrow \{1, 2, \ldots, k\} \) for a positive integer \( k \). A kind of total \( k \)-labeling of graph was introduced in 2007 by Baća et al. Name of the labeling is vertex irregular total \( k \)-labeling. [1] The definition of the labeling is given in Definition 1.

Definition 1. [1] Let \( G = (V,E) \) be a graph. A total \( k \)-labeling \( f: V \cup E \rightarrow \{1, 2, \ldots, k\} \), for an integer \( k \), is called a vertex irregular total \( k \)-labelling of \( G \) if every two distinct vertices \( u \) and \( v \) in \( V \) satisfy \( w_f(u) \neq w_f(v) \), where \( w_f(u) \) is defined by \( w_f(u) = f(u) + \sum_{u \in E} f(uv) \). The notation \( w_f(u) \) is called by the weight of \( u \) under the labeling \( f \).

Definition 2. [1] The total vertex irregularity strength of \( G \), denoted by \( tvs(G) \), is the minimum \( k \) for which a graph \( G \) has a vertex irregular total \( k \)-labeling.

In [2], Nurdin et al. determined another lower bound of \( tvs(G) \) for \( G \) a connected graph as follows.

Theorem 1 [2] Let \( G \) be a connected graph having \( n_i \) vertices of degree \( i \) \( (i = \delta, \delta + 1, \delta + 2, \ldots, \Delta) \), where \( \delta \) and \( \Delta \) are the minimum and the maximum degree of \( G \), respectively. Then
\[ \text{tvs}(G) \geq \max \left\{ \left\lceil \frac{\delta + n \delta}{\delta + 1} \right\rceil, \left\lceil \frac{\delta + n \delta + n \delta + 1}{\delta + 2} \right\rceil, \ldots, \left\lceil \frac{\delta + \sum_{i=1}^{n} n_i}{\Delta + 1} \right\rceil \right\}. \]

Rajasing et al obtained a bound for the total vertex irregularity strength of swing graph, triangular graph, and series-triangular graph [3]. Ramdani et al. determined an upper bound on the total vertex irregularity strength of Cartesian product of \( P_2 \) and arbitrary regular graph \( G \) [4]. Ramdani et al. determined an exact value of \( \text{tvs}(G) \) for \( G \) are ladders and books [5,6]. Nurdin et al. gave the exact values of the total vertex irregularity strength for several types of trees and disjoint union of paths [7]. In Nurdin et al determined the total vertex irregularity strength for several types of trees [8]. Przybylo, in, gave a linear bound on \( \text{tvs}(G) \)[9]. In Wijaya and Slamin constructed the total vertex irregular labeling of wheels, fans, suns, and friendship [10].

In this paper, we determine the total vertex irregularity strength of comb product of cycle and path with order 3.

**Definition 3.** Let \( G \) and \( H \) be two connected graphs. Let \( o \) be a vertex of \( H \). The comb product between \( G \) and \( H \), denoted by \( G \bowtie o \ H \), is a graph obtained by taking one copy of \( G \) and \( |V(G)| \) copies of \( H \) and grafting the \( i \)-th copy of \( H \) at the vertex \( o \) to the \( i \)-th vertex of \( G \).

2. **Methodology**

To get the exact value of the total vertex irregularity strength of comb product of cycle and path with order 3, we consider a lower bound and an upper bound on the total vertex irregularity strength of the graph. We use Theorem 1 to get a lower bound on \( \text{tvs}(C_m \bowtie P_3) \). On the other hand, to get an upper bound on \( \text{tvs}(C_m \bowtie P_3) \) we construct an edge irregular total labeling by minimizing the maximum label.

3. **Main result**

The exact value of \( \text{tvs}(C_m \bowtie_o P_3) \) is given by Theorem 2.

Theorem 2 Let \( C_m \) be a cycle with \( m \) vertices and \( P_3 \) be a path with 3 vertices. Let \( o \) be a vertex on \( P_3 \) with degree 2, then for \( m \geq 3 \),

\[ \text{tvs}(C_m \bowtie_o P_3) = m + 1. \]

**Proof.** Let the vertex set of \( C_m \bowtie_o P_3 \) be

\[ \{ v_i^j \mid 1 \leq i \leq m, \ 0 \leq j \leq 2 \} \]

and the edge set be

\[ \{ v_i^{0}v_i^{j}, v_i^{0}v_i^{j+1} \mid 1 \leq i \leq m \} \cup \{ v_i^{0}v_{i+1} \mid 1 \leq i \leq m - 1 \} \cup \{ v_m^{0}v_1^{0} \}. \]

Figure 1 gives an illustration of the notating of vertices in \( C_m \bowtie_o P_3 \) for \( m = 6 \).
Graph $C_m \triangleright_o P_3$ has $2m$ vertices with degree 1 and $m$ vertices with degree 4. By using Theorem 1, we have
\[ \text{tvs}(G) \geq \max \left\{ \left\lceil \frac{1 + 2m}{1 + 1} \right\rceil, \left\lceil \frac{1 + 2m + m}{4 + 1} \right\rceil \right\} = \left\lceil \frac{1 + 2m}{1 + 1} \right\rceil = m + 1. \]
So that, we have a lower bound on $\text{tvs}(C_m \triangleright_o P_3)$ is $m + 1$.

Define a vertex irregular total $(m+1)$-labeling of $C_m \triangleright_o P_3$ as follows.
\[
\begin{align*}
  f(v_i^1) &= f(v_i^2) = f(v_i^0 v_i^1) = i, \text{ for } 1 \leq i \leq m; \\
  f(v_i^0 v_i^1) &= i + 1 \text{ for } 1 \leq i \leq m; \\
  f(v_i^0) &= m + 1, \text{ for } 1 \leq i \leq m; \\
  f(v_{m+2}^0 v_i^0) &= f(v_i^0 v_{i+1}^0) = m + 1, \text{ for } 1 \leq i \leq m - 1.
\end{align*}
\]

From the labeling $f$, there are no two vertices with the same weight. The maximum label used in the labeling $f$ is $m + 1$. So that, $f$ is a vertex irregular total $(m+1)$-labeling of $C_m \triangleright_o P_3$. So, we have an upper bound on $\text{tvs}(C_m \triangleright_o P_3)$ is $m + 1$.

Since a lower bound and an upper bound on $\text{tvs}(C_m \triangleright_o P_3)$ is $m + 1$, we have an equality $\text{tvs}(C_m \triangleright_o P_3) = m + 1$.

An illustration of the labelling $f$ and the weight of vertices under the labelling are given by Figure 2.
Figure 2. The vertex irregular total 7-labelling $f$ of $C_6 \triangleright \!_o P_3$

The weight of each vertex of $C_6 \triangleright \!_o P_3$, under the labeling in Figure 2, can be seen in the Figure 3.

Figure 3. The weight of vertices of $C_6 \triangleright \!_o P_3$ under the labeling $f$ in the Figure 2.

4. Conclusion

By using Theorem 1, we have a lower bound on $tvs(C_m \triangleright \!_o P_3)$ is $m + 1$.

On the other hand, there are the vertex irregular total labeling $f$ with the maximum label is $m+1$. So, we have an upper bound on $tvs(C_m \triangleright \!_o P_3)$ is $m + 1$.

So, we can conclude that the exact value of $tvs(C_m \triangleright \!_o P_3)$ is $m + 1$. 
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