Completion of the Conjecture: Quantum Cohomology of Fano Hypersurfaces

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Abstract

In this paper, we propose the formulas that compute all the rational structural constants of the quantum Kähler sub-ring of Fano hypersurfaces.

1 Introduction and Statement of the Main Results

The main result of this paper is the completion of the conjectures proposed in [4] on the quantum Kähler sub-ring of Fano hypersurfaces.

The quantum cohomology ring of Fano hypersurfaces are studied both in physical and mathematical points of view [13]. One of the most successful approach from mathematics is the use of Gauss-Manin system with the aid of the gravitational descendants [6], [11], [2]. These works showed that the quantum cohomology ring of Fano hypersurfaces are realized as the Gauss-Manin connection of the Gauss-Manin system, whose solution is given by the generating function of a certain type of the gravitational correlation functions (including the gravitational descendants) of the topological sigma model coupled to the topological gravity. Moreover, they also showed that the solutions of the Gauss-Manin system are written as the generalized hyper geometric functions in the case of Fano hypersurfaces. Their result is beautiful, but in compensation for the introduction of gravitational descendants, it is not very clear to see the explicit structural constants and the multiplication table of the quantum cohomology ring from their results.

Here, we construct a procedure that computes the explicit structural constants and the multiplication table of the quantum cohomology ring of Fano hypersurfaces without using the gravitational descendants.

In [4], we have constructed the recursive formulas that describe the rational structural constant \( I^{N,k} \) of the quantum Kähler sub-ring of \( M_N^k \), which is the degree \( k \) hypersurfaces in \( CP^{N-1} \), in terms of those of \( M_{N+1}^k \) when the degree of the rational curves concerned is no more than 5. These recursive formulas determine the structural constants of the quantum cohomology ring of \( M_N^k \) with the aid of the initial conditions given by the following formula in [4]:

\[
\sum_{n=0}^{k-1} I^{N,k,1}_{n} w^n = k \prod_{j=1}^{k-1} (jw + (k-j)),
\]

(1.1)
where $N - k$ must be greater than 1. But in [4], our construction of the recursive formula needs some explicit data of the structural constants coming from the higher degree curves, and the determination of the recursive formulas gets extremely harder as the degree of the rational curves grows.

In this paper, we propose the formulas that give the recursive formula for the rational curves of arbitrary degree $d$ without using any numerical data of the explicit structural constants. Combining this procedure with the formula (1.1), we can compute all the structural constants of the quantum Kähler sub-ring of $M_N^k (N \geq k)$ by use of our previous results in [4].

Here, we summarize the result to be proposed in this paper. First, we introduce the formal variable $x, y, z_1, z_2, \cdots, z_{d-1}, t_1, t_2, \cdots, t_{d-1}$ ($d$ is the degree of the rational curves in the hypersurface). Then we take the following linear combination of the variables $x, y, t_1, t_2, \cdots, t_{d-1}$:

$$
\varphi_i (x, y, t_1, t_2, \cdots, t_{d-1}) = \frac{(d - i) \cdot x + i \cdot y}{d} + \sum_{j=1}^{i-1} \frac{d - i}{d - j} \cdot t_j + \sum_{j=i+1}^{d-1} i \cdot t_j.
$$

(1.2)

Next, consider the rational function $p_i$ in the formal variables,

$$
p_i (x, y, z_1, z_2, \cdots, t_{d-1}) = t_i + \varphi_i + z_i \cdot \frac{t_i + \varphi_i}{t_i + \varphi_i - z_i}.
$$

(1.3)

And we expand $\frac{t_i + \varphi_i}{t_i + \varphi_i - z_i}$ into the form,

$$
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left( \frac{-n}{k} \right) \left( \frac{z_i}{t_i} \right)^n \left( \frac{\varphi_i}{t_i} \right)^k,
$$

(1.4)

where

$$
\left( \frac{-n}{0} \right) := 1, \quad \left( \frac{-n}{k} \right) := \frac{1}{k!} \prod_{j=1}^{k} (-n - j + 1) \quad (k \geq 1).
$$

(1.5)

Then we take the product of these formal series $p_i = t_i + \varphi_i + z_i (\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left( \frac{-n}{k} \right) \left( \frac{z_i}{t_i} \right)^n \left( \frac{\varphi_i}{t_i} \right)^k)$,

$$
R_d (x, y, z_1, z_2, \cdots, z_{d-1}, t_1, t_2, \cdots, t_{d-1}) = \prod_{j=1}^{d-1} p_j (x, y, z_j, t_1, t_2, \cdots, t_{d-1}),
$$

(1.6)

and pick up the terms that are free from (constant terms in) $t_i$'s,

$$
\text{Poly}_{d} (x, y, z_1, z_2, \cdots, z_{d-1}) = \int_{C_1} \cdots \int_{C_{d-1}} \frac{dt_1}{t_1} \cdots \frac{dt_{d-1}}{t_{d-1}} R_d (x, y, z_1, z_2, \cdots, z_{d-1}, t_1, \cdots, t_{d-1}).
$$

(1.7)

In [1.7], we have to choose the path $C_i$ in the domain where the expansion in (1.3) is effective. $\text{Poly}_{d}$ is the polynomial first introduced in the formula (5.96) in [3] (web-site version), and it is possible to reconstruct the recursive formula for the degree $d$ rational curves from it. We prepare some definitions that are needed for the reconstruction of the recursive formula from $\text{Poly}_{d}$. Consider the monomial $x^{d_{i_0}} z_{i_1}^{d_{i_1}} \cdots z_{i_m}^{d_{i_m}} y^{d_{i_{m+1}}} \left( \sum_{j=0}^{m+1} d_{i_j} = d - 1 \right)$, that appear in $\text{Poly}_{d}$, associated with the following “comb type” of a positive integer $d$ [3]:

$$
0 = i_0 < i_1 < i_2 < \cdots < i_m < i_{m+1} = d.
$$

(1.8)

Next, we prepare some elements in (a free abelian group) $\mathbb{Z}^{m+1}$, which are determined for each monomial $x^{d_{i_0}} z_{i_1}^{d_{i_1}} \cdots z_{i_m}^{d_{i_m}} y^{d_{i_{m+1}}}$, as follows:

$$
\alpha := (m + 1 - d, m + 1 - d, \cdots, m + 1 - d),
$$

2
\[ \beta := (0, i_1 - 1, i_2 - 2, \cdots, i_m - m), \]
\[ \gamma := (0, i_1(N - k), i_2(N - k), \cdots, i_m(N - k)), \]
\[ \epsilon_1 := (1, 0, 0, 0, \cdots, 0), \]
\[ \epsilon_2 := (1, 1, 0, 0, \cdots, 0), \]
\[ \epsilon_3 := (1, 1, 1, 0, \cdots, 0), \]
\[ \cdots \]
\[ \epsilon_{m+1} := (1, 1, 1, 1, \cdots, 1). \]  
(1.9)

Now we define \( \delta = (\delta_1, \cdots, \delta_{m+1}) \in \mathbb{Z}^{m+1} \) by the formula:
\[ \delta := \alpha + \beta + \gamma + \sum_{j=1}^{m} (d_{ij} - 1) \epsilon_j + d_{m+1} \epsilon_{m+1}. \]  
(1.10)

Then our main conjecture is the following:

**Conjecture 1** The recursive formula for the degree \( d \) rational curves is obtained from \( \text{Poly}_d \) by the formula:
\[ L_n^{N,k,d} = \phi(\text{Poly}_d), \]  
(1.11)

where \( \phi \) is a \( \mathbb{Q} \)-linear map from the \( \mathbb{Q} \)-vector space of the homogeneous polynomials of degree \( d - 1 \) in \( x, y, z_1, \cdots, z_{d-1} \) to the \( \mathbb{Q} \)-vector space of the weighted homogeneous polynomials of degree \( d \) in \( L_m^{N+1,k,d'} \) defined as follows:
\[ \phi(x^{d_0}y^{d_1}z_1^{d_1} \cdots z_m^{d_m}) = \prod_{j=1}^{m+1} L_n^{N+1,k,i_j-i_j-1}. \]  
(1.12)

This paper is organized as follows. In Section 2, we introduce the notation on the quantum Kähler sub-ring of the hypersurface in \( CP^{N-1} \) and briefly review the results obtained in [4]. In Section 3, we first observe the agreement between the results obtained by Conjecture 1 and the recursive formulas up to the \( d = 5 \) rational curves derived in [4]. Next, we briefly explain the trend of thoughts, that leads us to the formula proposed in Conjecture 1. In Section 4, we test our conjecture in the \( d = 6 \) case. We explicitly write down the recursive formula for the \( d = 6 \) rational curves, which is derived from the direct use of Conjecture 1 and see that this formula has the non-trivial property conjectured in [4]. In Appendix A, we present the table of the recursive formulas derived in [4], that are heavily referred in the body of this paper.

## 2 Quantum Kähler Sub-ring of Projective Hypersurfaces

### 2.1 Notation

In this section, we introduce the quantum Kähler sub-ring of the quantum cohomology ring of a degree \( k \) hypersurface in \( CP^{N-1} \). Let \( M_N^k \) be a hypersurface of degree \( k \) in \( CP^{N-1} \). We denote by \( \text{QH}^*_e(M_N^k) \) the subring of the quantum cohomology ring \( \text{QH}^*(M_N^k) \) generated by \( \mathcal{O}_e \) induced from the Kähler form \( \epsilon \) (or, equivalently the intersection \( H \cap M_N^k \) between a hyperplane class \( H \) of \( CP^{N-1} \) and \( M_N^k \)). The multiplication rule of \( \text{QH}^*_e(M_N^k) \) is determined by the Gromov-Witten invariant of genus 0 \( \langle \mathcal{O}_e \mathcal{O}_{eN-2-m} \mathcal{O}_{e^m-1-(k-N)d} \rangle_{d,M_N^k} \) and it is given as follows:
\[ L_{im}^{N,k,d} := \frac{1}{k} \langle \mathcal{O}_e \mathcal{O}_{eN-2-m} \mathcal{O}_{e^m-1-(k-N)d} \rangle, \]
\[ \mathcal{O}_e \cdot 1 = \mathcal{O}_e, \]
\[ O_e \cdot O_{e^{N-2-m}} = O_{e^{N-1-m}} + \sum_{d=1}^{\infty} L_m^{N,k,d} q^d O_{e^{N-1-m+(k-N)d}}, \]
\[ q := \exp(t), \]  
where the subscript \( d \) counts the degree of the rational curves measured by \( e \). So \( q = \exp(t) \) is the degree counting parameter. Since \( M_k \) is a complex \((N-2)\) dimensional manifold, we see that a structure constant \( L_m^{N,k,d} \) is non-zero only if the following condition is satisfied:
\[ 1 \leq N - 2 - m \leq N - 2, \quad 1 \leq m - 1 + (N - k)d \leq N - 2, \]
\[ \iff \max\{0, 2 - (N - k)d\} \leq m \leq \min\{N - 3, N - 1 - (N - k)d\}. \]  
(2.14)

We rewrite (2.14) into
\[ (N - k \geq 2) \implies 0 \leq m \leq (N - 1) - (N - k)d \]
\[ (N - k = 1, d = 1) \implies 1 \leq m \leq N - 3 \]
\[ (N - k = 1, d \geq 2) \implies 0 \leq m \leq N - 1 - (N - k)d \]
\[ (N - k \leq 0) \implies 2 + (k - N)d \leq m \leq N - 3. \]  
(2.15)

From (2.15), we easily see that the number of the non-zero structure constants \( L_m^{N,k,d} \) is finite except for the case of \( N = k \). Moreover, if \( N \geq 2k \), the non-zero structure constants come only from the \( d = 1 \) part and the non-vanishing \( L_m^{N,k,1} \) is determined by \( k \) and independent of \( N \). The \( N \geq 2k \) region is studied by Beauville [1], and his result plays the role of an initial condition of our discussion later. Explicitly, they are given by the formula (1.1). In the case of \( N = k \), the multiplication rule of \( QH^*_e(M_k^e) \) is given as follows:
\[ O_e \cdot 1 = O_e, \]
\[ O_e \cdot O_{e^{k-2-m}} = (1 + \sum_{d=1}^{\infty} q^d L_m^{k,k,d}) O_{e^{k-1-m}} \quad (m = 2, 3, \cdots, k-3), \]
\[ O_e \cdot O_{e^{k-3}} = O_{e^{k-2}}. \]  
(2.16)

We introduce here the generating function of the structure constants of the Calabi-Yau hypersurface \( M_k^e \):
\[ L_m^{k,k}(e^t) := 1 + \sum_{d=1}^{\infty} L_m^{k,k,d} e^d \quad (m = 2, 3, \cdots, k-3). \]  
(2.17)

### 2.2 Review of Results for Fano and Calabi-Yau Hypersurfaces and Virtual Structure Constants

Let us summarize the results of [1]. In [1], we showed that the structure constants \( L_m^{N,k,d} \) of \( QH^*_e(M_k^e) \) for \( (N - k \geq 2) \) can be obtained by applying the recursive formulas in Appendix A (up to the \( d = 5 \) case), which describe \( L_m^{N,k,d} \) in terms of \( L_{m'}^{N+1,k,d'} \) (\( d' \leq d \)), with the initial conditions \( L_m^{N,k,1} \) and \( L_m^{N,k,d} = 0 \) (\( d \geq 2 \)) in the \( N \geq 2k \) region. These recursive formulas naturally lead us to the relation:
\[ (O_e)^{N-1} - k^k (O_e)^{k-1} q = 0 \]  
(2.18)
of \( QH^*_e(M_k^e) \) \((N - k \geq 2)\) by descending induction using Beauville’s result [1], [1].

In the \( N - k = 1 \) case, the recursive formulas receive modification only in the \( d = 1 \) part:
\[ L_m^{k+1,k,1} = L_m^{k+2,k,1} - L_0^{k+2,k,1} = L_m^{k+2,k,1} - k!. \]  
(2.19)

This leads us to the following relation of \( QH^*_e(M_{k+1}^e) \):
\[ (O_e + klq)^{N-1} - k^k (O_e + klq)^{k-1} q = 0. \]  
(2.20)
The structural constant $L_{m}^{k,k,d}$ for a Calabi-Yau hypersurface does not obey the recursive formulas in appendix A. We introduce here the virtual structure constants $\tilde{L}_{m}^{N,k,d}$ as follows.

**Definition 1** Let $\tilde{L}_{m}^{N,k,d}$ be the rational number obtained by applying the recursion relations of Fano hypersurfaces in Appendix A for arbitrary $N$ and $k$ with the initial condition $L_{1}^{N,k,1}$ ($N \geq 2k$) and $L_{m}^{N,k,d} = 0$ ($d \geq 2$, $N \geq 2k$).

**Remark 1** In the $N - k \geq 2$ region, $\tilde{L}_{m}^{N,k,d} = L_{m}^{N,k,d}$.

We define the generating function of the virtual structure constants of the Calabi-Yau hypersurface $M_{k}$ as follows:

$$\tilde{L}_{n}^{k,k}(e^{x}) := 1 + \sum_{d=1}^{\infty} \tilde{L}_{n}^{k,k,d} e^{dx},$$

$$n = 0, 1, \ldots, k - 1.$$

In [4], we conjectured that $\tilde{L}_{n}^{k,k}(e^{x})$ gives us the information of the B-model of the mirror manifold of $M_{k}$. More explicitly, we conjectured

$$\tilde{L}_{0}^{k,k}(e^{x}) = \sum_{d=0}^{\infty} \frac{(kd)!}{(d!)^{k}} e^{dx},$$

$$\tilde{L}_{1}^{k,k}(e^{x}) = \frac{dt(x)}{dx} := \frac{d}{dx} \left( x + \sum_{d=1}^{\infty} \frac{(kd)!}{(d!)^{k}} \left( \sum_{i=1}^{d} \sum_{m=1}^{k-1} \frac{m}{(ki-m)} e^{dx} \right) / \left( \sum_{d=0}^{\infty} \frac{(kd)!}{(d!)^{k}} e^{dx} \right) \right)$$

where the r.h.s. of (2.22) is derived from the solutions of the ODE for the period integral of the mirror manifold of $M_{k}$,

$$((\frac{d}{dx})^{k-1} - ke^{x}(\frac{d}{dx} + 1)(\frac{d}{dx} + 2)\cdots(\frac{d}{dx} + k - 1))w(x) = 0,$$

that was used in the computation based on the mirror symmetry, [3], [7], [8]. Of course, we can extend the conjecture (2.22) to the general $\tilde{L}_{n}^{k,k}(e^{x})$ if we compare the $\tilde{L}_{n}^{k,k}(e^{x})$ with the B-model three point functions in [7]. Hence we obtain the mirror map $t = t(x)$ without using the mirror conjecture:

$$t(x) = x + \int_{-\infty}^{x} dx' (\tilde{L}_{1}^{k,k}(e^{x'}) - 1) = x + \sum_{d=1}^{\infty} \frac{\tilde{L}_{1}^{k,k,d}}{d} e^{dx}.$$

With the conjecture given by (2.22), we can construct the mirror transformation that transforms the virtual structure constants of the Calabi-Yau hypersurface into the real ones as follows:

$$L_{m}^{k,k}(e^{t}) = \frac{\tilde{L}_{m}^{k,k}(e^{x(t)})}{\tilde{L}_{1}^{k,k}(e^{x(t)})},$$

$$m = 2, \ldots, k - 3.$$

Therefore we can compute all the structural constants of quantum Kähler sub-ring of $M_{k}^{N}$ ($N \geq k$) if we can determine the form of the recursive formulas for the rational curves of arbitrary degree as in Appendix A.

## 3 Agreement with the Known Results

In the first part of this section, we show that our main conjecture reproduces the recursive formulas in the $d \leq 5$ cases obtained in [4]. First, we write down $R_{d}$ in (1.6) up to the $d = 5$ case. Here, we
omit some irrelevant terms that do not appear in $Poly_d$.

$$R_1 = 1$$
$$R_2 = \left( \frac{x+y}{2} + z_1 \right)$$
$$R_3 = \left( \frac{2x+y}{3} + z_1 + \frac{(z_1)^2}{t_1} + \frac{1}{2}t_2 \right) \left( \frac{x+2y}{3} + z_2 + \frac{(z_2)^2}{t_2} + \frac{1}{2}t_1 \right)$$
$$R_4 = \left( \frac{3x+y}{4} + z_1 + \frac{(z_1)^2}{t_1} \right) \left( 1 - \frac{3x+y}{4t_1} \right) + \frac{(z_1)^3}{(t_1)^2} + \frac{1}{2}t_2 + \frac{1}{3}t_3 \right) \times$$
$$\left( \frac{2x+2y}{4} + z_2 + \frac{(z_2)^2}{t_2} \right) \left( 1 - \frac{2x+2y}{4t_2} \right) + \frac{(z_2)^3}{(t_2)^2} + \frac{2}{3}t_1 + \frac{2}{3}t_3 \right) \times$$
$$\left( \frac{x+3y}{4} + z_3 + \frac{(z_3)^2}{t_3} \right) \left( 1 - \frac{x+3y}{4t_3} \right) + \frac{(z_3)^3}{(t_3)^2} + \frac{1}{3}t_1 + \frac{1}{2}t_2 \right)$$

$$R_5 = \frac{4x+y}{5} + \frac{1}{2}t_2 + \frac{1}{3}t_3 + \frac{1}{4}t_4 + z_1 +$$
$$\frac{(z_1)^2}{t_1} \left( \frac{(t_1)(t_4)}{t_1} + \frac{4x+y}{5} + \frac{1}{2}t_2 + \frac{1}{3}t_3 + \frac{1}{4}t_4 \right) + \frac{(z_1)^3}{(t_1)^2} \left( 1 - 2\frac{4x+y}{5t_1} \right) + \frac{(z_1)^4}{(t_1)^3} \times$$
$$\frac{3x+2y}{5} \left( \frac{(t_2)}{t_2} + \frac{3x+2y}{5t_2} + \frac{3}{4}t_1 + \frac{2}{3}t_3 + \frac{1}{4}t_4 \right) + \frac{(z_2)^3}{(t_2)^2} \left( 1 - 2\frac{3x+2y}{5t_2} \right) + \frac{(z_2)^4}{(t_2)^3} \times$$
$$\frac{2x+3y}{4} \left( \frac{(t_3)}{t_3} + \frac{2x+3y}{5t_3} + \frac{1}{2}t_1 + \frac{2}{3}t_2 + \frac{1}{3}t_3 + \frac{1}{4}t_4 \right) + \frac{(z_3)^3}{(t_3)^2} \left( 1 - 2\frac{2x+3y}{5t_3} \right) + \frac{(z_3)^4}{(t_3)^3} \times$$
$$\frac{x+4y}{5} \left( \frac{(t_4)}{t_4} + \frac{x+4y}{5t_4} + \frac{1}{4}t_1 + \frac{1}{3}t_2 + \frac{1}{3}t_3 \right) + \frac{(z_4)^3}{(t_4)^2} \left( 1 - 2\frac{x+4y}{5t_4} \right) + \frac{(z_4)^4}{(t_4)^3} \right) \right) \quad (3.26)$$

In the $d \leq 4$ cases, the derivation of $Poly_d$ is just picking up the constant terms in $t_i$ from the above polynomials.

$$Poly_1 = 1$$
$$Poly_2 = \left( \frac{x+y}{2} + z_1 \right)$$
$$Poly_3 = \left( \frac{2x^2+5xy+2y^2}{9} \right) + \left( \frac{2x+y}{3} + \frac{1}{2}z_2 \right)z_2 + \left( \frac{x+2y}{3} + \frac{1}{2}z_1 \right)z_1 + z_1z_2$$
$$Poly_4 = \left( \frac{3x^3+13x^2y+13xy^2+3y^3}{32} \right)$$

$$+ \left( \frac{x^2+4xy+3y^2}{8} + \frac{3x+11y}{18} \right)z_1 + \frac{2}{9}(z_1)^2z_1$$
$$+ \left( \frac{3x^2+10xy+3y^2}{16} + \frac{3x+3y}{8} \right)z_2 + \frac{1}{4}(z_2)^2z_2$$
$$+ \left( \frac{3x^2+4xy+y^2}{8} + \frac{11x+3y}{18} \right)z_3 + \frac{2}{9}(z_3)^2z_3$$
$$+ \left( \frac{3x+y}{4} + \frac{1}{2}z_2 + \frac{1}{3}z_3 \right)z_2z_3 + \left( \frac{x+y}{2} + \frac{1}{3}z_1 + \frac{2}{3}z_3 \right)z_1z_3$$
\[+\left(\frac{x+3y}{4} + \frac{1}{3}z_1 + \frac{1}{2}z_2\right)z_1z_2
\]
\[+z_1z_2z_3\]  \quad (3.27)

Our main conjecture in Section 1 straightforwardly leads us to the recursive formulas in Appendix A. But in the \(d = 5\) case, we have to take residues of some rational functions in \(t_i\) in evaluating the coefficients of the monomials \((z_i)^2(z_j)^2\) \((i < j)\). For example, we have to evaluate the following residue integral to determine the coefficient of \((z_1)^2(z_4)^2\),

\[
\frac{1}{(2\pi \sqrt{-1})^2} \int_{C_1} dt_1 \int_{C_4} dt_4 \left(\frac{4t_1 + 4t_4}{t_1 t_4(t_1 + t_4)(t_4 + 1/t_1)}\right) = \frac{2}{3}. \quad (3.28)
\]

After these integrations, we can see the complete agreement of \(Poly_4\) derived from \(3.26\) with the recursive formula for the \(d = 5\) rational curves.

Now, we briefly explain the process how we reached the formula \(R_d\). Fundamental idea is the observation in [4] that \(Poly_d\) always includes the factorized polynomial of the form,

\[
\prod_{j=1}^{d-1}(d-j)x + jy + z_j. \quad (3.29)
\]

We try to interpret this formula combinatorially. First, we prepare the array of the following \(d\) circles.

\[
\circ \circ \circ \cdots \circ \circ \quad (3.30)
\]

Then we can see \(d - 1\) gaps, and we label these gaps as \(j\) \((j = 1, \cdots, d - 1)\) from the left to the right. In this set up, we can regard that the formula \(3.29\) is summing up all the configurations whether we insert \(z_j\) or \(\frac{(d-j)x + jy}{d}\) into the \(j\)-th gap. A graphical example is given by the following figure,

\[
\circ | \circ | \circ \cdots \circ \circ \iff z_1z_2\left(\frac{x + 3y}{4}\right), \quad (3.31)
\]

where we insert \(z_1\) and \(z_2\) to the first and the second gaps respectively, and put \(\frac{(x+3y)}{4}\) in the third gap. With this interpretation, we search for the extension of \(3.29\) that includes all the terms appearing in \(Poly_4\). Generally, we cannot expect the factorization of \(Poly_d\) like \(3.29\). Instead, we introduce the \(d - 1\) \(U(1)\) charge variables \(t_1, t_2, \cdots, t_{d-1}\) and expect that factorization characteristics is conserved by including the \(d - 1\) \(U(1)\) charges. But afterwards, we have to pick up the neutral charge portion from the factorized formula, which is no longer factorized. To see if these expectations hold, we carefully look at the \(d = 4\) case. The difference between \(Poly_4\) and \(3.29\) is given by,

\[
Poly_4 - \left(\frac{3x+y}{4} + z_1\right)\left(\frac{x+y}{2} + z_2\right)\left(\frac{x+3y}{4} + z_3\right)
\]
\[= \left(\frac{3x+11y}{18}\right)(z_1)^2 + \frac{2}{9}(z_1)^3 + \left(\frac{3x+3y}{8}\right)(z_2)^2 + \frac{1}{4}(z_2)^3 + \frac{11x+3y}{18}(z_3)^2 + \frac{2}{9}(z_3)^3
\]
\[+\frac{1}{2}(z_2)^2z_3 + \frac{1}{3}z_2(z_3)^2 + \frac{2}{3}(z_1)^2z_3 + \frac{2}{3}z_1(z_3)^2 + \frac{1}{3}(z_1)^2z_2 + \frac{1}{2}z_1(z_2)^2. \quad (3.32)
\]

This formula tells us that we have to admit the insertion of \((z_j)^n\) \((n = 1, 2, 3)\) into the \(j\)-th gap. Moreover, we found the following equalities:

\[
\frac{3x+11y}{18} = \frac{2}{3} \left(\frac{x+3y}{4}\right) + \frac{1}{3} \left(\frac{x+y}{2}\right) - \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{3x+y}{4}
\]
\[
\frac{3x+3y}{8} = \frac{1}{2} \left(\frac{x+y}{4}\right) + \frac{1}{2} \left(\frac{3x+y}{4}\right) - \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{x+y}{2}
\]
\[
\frac{2}{9} = \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2}. \quad (3.33)
\]
Thus, we speculate that the minimal factorization polynomial that includes $Poly_4$ as the neutral charge part is given by,

$$
\begin{align*}
&\left(\frac{3x+y}{4} + z_1 + \frac{(z_1)^2}{t_1}(1 - \frac{3x+y}{4t_1}) + \frac{(z_1)^3}{(t_1)^2} + \frac{1}{2}t_2 + \frac{1}{3}t_3\right) \\
&\left(\frac{2x+2y}{4} + z_2 + \frac{(z_2)^2}{t_2}(1 - \frac{2x+2y}{4t_2}) + \frac{(z_2)^3}{(t_2)^2} + \frac{2}{3}t_1 + \frac{2}{3}t_3\right) \\
&\left(\frac{x+3y}{4} + z_3 + \frac{(z_3)^2}{t_3}(1 - \frac{x+3y}{4t_3}) + \frac{(z_3)^3}{(t_3)^2} + \frac{1}{3}t_1 + \frac{1}{2}t_2\right).
\end{align*}
$$

(3.34)

Using the same observation, we first expect that the desired factorized polynomial including $t_1, \cdots, t_{d-1}$ in the $d = 5$ case is given by,

$$
\begin{align*}
&\left(\frac{4x+y}{5} + \frac{1}{2}t_2 + \frac{1}{3}t_3 + \frac{1}{4}t_4 + z_1 + \frac{(z_1)^2}{t_1}(1 - \frac{4x+y}{5t_1}) + \frac{(z_1)^3}{(t_1)^2} + \frac{2}{3}t_2 + \frac{1}{4}t_4 + z_2 + \frac{(z_2)^2}{t_2}(1 - \frac{3x+2y}{5t_2}) + \frac{(z_2)^3}{(t_2)^2} + \frac{3}{4}t_1 + \frac{1}{2}t_4 + z_3 + \frac{(z_3)^2}{t_3}(1 - \frac{3x+3y}{5t_3}) + \frac{(z_3)^3}{(t_3)^2} + \frac{1}{4}t_1 + \frac{1}{3}t_2 + \frac{1}{2}t_3 + z_4 + \frac{(z_4)^2}{t_4}(1 - \frac{x+4y}{5t_4}) + \frac{(z_4)^3}{(t_4)^2} \right) \\
&\left(\frac{1}{2}t_2 + \frac{2}{3}t_3 + \frac{3}{4}t_4 + z_2 + \frac{(z_2)^2}{t_2}(1 - \frac{3x+2y}{5t_2}) + \frac{(z_2)^3}{(t_2)^2} + \frac{3}{4}t_1 + \frac{1}{2}t_4 + z_3 + \frac{(z_3)^2}{t_3}(1 - \frac{3x+3y}{5t_3}) + \frac{(z_3)^3}{(t_3)^2} + \frac{1}{4}t_1 + \frac{1}{3}t_2 + \frac{1}{2}t_3 + z_4 + \frac{(z_4)^2}{t_4}(1 - \frac{x+4y}{5t_4}) + \frac{(z_4)^3}{(t_4)^2}\right).
\end{align*}
$$

(3.35)

Looking at (3.34) and (3.35), we can imagine that the terms including $z_j$ come from picking up the relevant terms of the universal rational function,

$$
\sum_{n=0}^{\infty} \frac{(z_j)^{n+1}}{(t_j)^n} \sum_{m=0}^{\infty} \frac{(-n)^m}{m!} \frac{(d-j)x+jy}{d} \frac{dt_j}{t_j} = z_j(t_j + \frac{(d-j)x+jy}{d} + \frac{(d-j)x+jy}{d} - z_j).
$$

(3.36)

Hence our first trial of the factorized formula including $Poly_d$ as the neutral charge portion takes the following form,

$$
\prod_{j=1}^{d-1} \left(\frac{(d-j)x+jy}{d} + \sum_{i=1}^{j-1} \frac{d-j}{d-i} t_i + \sum_{i=j+1}^{d-1} \frac{j}{d-i} t_i + \frac{(d-j)x+jy}{d} - z_j\right).
$$

(3.37)

In the $d = 5$ case, (3.37) reproduces almost all the terms in the $d = 5$ recursive formula determined in [1] except for the coefficients of $(z_j)^2(z_j)^2 (i < j)$.

Therefore, let us consider the exceptional terms. For example, we pick up the coefficient of the monomial $(z_1)^2(z_4)^2$, which corresponds to the term :

$$
L_{n_{N+1,k,1}}N_{n_{N+1,k,3}}F_{n_{N+1,k,1}}^{N+1,k,1}.
$$

(3.38)
The prediction of the coefficient of \((z_1)^2(z_4)^2\) by (3.37) is
\[
\frac{3}{4} \cdot \frac{3}{4} + \frac{1}{2} \cdot \frac{1}{2} = \frac{13}{16}. \tag{3.39}
\]

But the true coefficient of \((z_1)^2(z_4)^2\) is \(\frac{2}{3}\) according to Appendix A. So we have to modify the formula (3.37) to reproduce the correct coefficient \(\frac{2}{3}\).

At this stage, we look back at the formula (3.37) and pay attention to the term:
\[
\frac{(d-j)x+jy}{d} + \sum_{i=1}^{j-1} \frac{d-j}{d-i} t_i + \sum_{i=j+1}^{d-1} \frac{j}{i} t_i. \tag{3.40}
\]

In (3.40), we can interpret formally \(x\) and \(y\) as \(t_0\) and \(t_d\) respectively. Moreover, it is natural to add \(t_j\) to (3.40), which is irrelevant in picking up the neutral charge portion. In sum, we guess that if the term \(\frac{(d-j)x+jy}{d}\) appear in the factorized formula, it is always accompanied by \(t_j + \sum_{i=1}^{j-1} \frac{d-j}{d-i} t_i + \sum_{i=j+1}^{d-1} \frac{j}{i} t_i\). In this way, we modify (3.37) into the form:
\[
\prod_{j=1}^{d-1} \left( \frac{(d-j)x+jy}{d} + \sum_{i=1}^{j} \frac{d-j}{d-i} t_i + \sum_{i=j+1}^{d-1} \frac{j}{i} t_i \right) = z_j \left( \frac{(d-j)x+jy}{d} + \sum_{i=1}^{j} \frac{d-j}{d-i} t_i + \sum_{i=j+1}^{d-1} \frac{j}{i} t_i - z_j \right) \tag{3.41}
\]

that is the formula given in (3.3). With the formula (3.41), we compute the coefficient of \((z_1)^2(z_4)^2\). If we expand the \(j\)-th factor of (3.41) in powers of \(\frac{t_j}{t_1}\), we can see, in the case of \((z_1)^2(z_4)^2\), that we have infinite contributions from the terms like \(\frac{(t_j)^{n}}{t_1 \cdot t_4 \cdot (t_1)^{n}} (n \geq 0)\), whose \(j\)-th factor \((j = 1, 2, 3, 4)\) comes from the \(j\)-th factor of (3.41) respectively. And we have to sum up these contributions. This operation is done as follows:

\[
\text{(coefficients of } (z_1)^2(z_4)^2) = \left( \text{constant term of } \frac{3}{4}(t_1 + \frac{1}{2} t_4) \cdot (\frac{1}{2} t_1 + \frac{3}{4} t_4) \cdot \left( \frac{1}{t_1} \sum_{i=0}^{\infty} (-1)^i \left( \frac{t_4}{4 t_1} \right)^i \right) \cdot \left( \frac{1}{t_4} \sum_{j=0}^{\infty} (-1)^j \left( \frac{t_1}{4 t_4} \right)^j \right) \right) = \frac{3}{4} \cdot \frac{3}{4} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{4} = \frac{16}{15} - \frac{1}{2} = \frac{2}{3}. \tag{3.42}
\]

Thus, we can see that (3.41) reproduces the right coefficient of \((z_1)^2(z_4)^2\)! Similar computations lead us to complete agreement between the prediction of (3.41) and the \(d = 5\) recursive formula in Appendix A.

### 4 Test for the \(d = 6\) case

Now, we test our main conjecture for the \(d = 6\) rational curves. First, we write down the relevant part of \(R_6\):

\[
R_6 = \sum_{t_1 \cdot t_4} \left( \frac{5x+y}{6} + \frac{1}{2} t_2 + \frac{1}{3} t_3 + \frac{1}{4} t_4 + \frac{1}{5} t_5 + z_1 + \frac{(z_1)^2}{t_1} (t_1/(t_1 + \frac{5x+y}{6} + \frac{1}{2} t_2 + \frac{1}{3} t_3 + \frac{1}{4} t_4 + \frac{1}{5} t_5)) + \frac{(z_1)^3}{(t_1)^2} (t_1/(t_1 + \frac{5x+y}{6} + \frac{1}{2} t_2 + \frac{1}{3} t_3 + \frac{1}{4} t_4 + \frac{1}{5} t_5)^2) + \frac{(z_1)^4}{(t_1)^3} (1 - 3(\frac{5x+y}{6 t_1}) + (\frac{z_1}{(t_1)^2})) \right) \times \sum_{\text{other terms}}
\]


\[
\left( \frac{4x+2y}{6} + \frac{4}{5}t_1 + \frac{2}{3}t_3 + \frac{2}{4}t_4 + \frac{2}{5}t_5 + z_2 + \frac{(z_2)^2}{t_2} \right) \left( t_2/(t_2 + \frac{4x+2y}{6}) + \frac{4}{5}t_1 + \frac{2}{3}t_3 + \frac{2}{4}t_4 + \frac{2}{5}t_5 \right) + \frac{(z_2)^3}{(t_2)^2} \left( t_2/(t_2 + \frac{4x+2y}{6}) + \frac{4}{5}t_1 + \frac{2}{3}t_3 + \frac{2}{4}t_4 + \frac{2}{5}t_5 \right) \times \left( \frac{3x+3y}{6} + \frac{3}{5}t_1 + \frac{3}{4}t_2 + \frac{4}{5}t_5 + z_3 + \frac{(z_3)^2}{t_3} \right) \left( t_3/(t_3 + \frac{3x+3y}{6}) + \frac{3}{5}t_1 + \frac{3}{4}t_2 + \frac{4}{5}t_5 \right) + \frac{(z_3)^3}{(t_3)^2} \left( t_3/(t_3 + \frac{3x+3y}{6}) + \frac{3}{5}t_1 + \frac{3}{4}t_2 + \frac{4}{5}t_5 \right) \times \left( \frac{2x+4y}{6} + \frac{2}{5}t_1 + \frac{2}{3}t_3 + \frac{2}{4}t_4 + \frac{2}{5}t_5 + z_4 + \frac{(z_4)^2}{t_4} \right) \left( t_4/(t_4 + \frac{2x+4y}{6}) + \frac{2}{5}t_1 + \frac{2}{3}t_3 + \frac{2}{4}t_4 + \frac{2}{5}t_5 \right) + \frac{(z_4)^3}{(t_4)^2} \left( t_4/(t_4 + \frac{2x+4y}{6}) + \frac{2}{5}t_1 + \frac{2}{3}t_3 + \frac{2}{4}t_4 + \frac{2}{5}t_5 \right) \times \left( \frac{x+5y}{6} + \frac{1}{5}t_1 + \frac{1}{4}t_2 + \frac{1}{3}t_3 + \frac{1}{2}t_4 + z_5 + \frac{(z_5)^2}{t_5} \right) \left( t_5/(t_5 + \frac{x+5y}{6}) + \frac{1}{5}t_1 + \frac{1}{4}t_2 + \frac{1}{3}t_3 + \frac{1}{2}t_4 \right) + \frac{(z_5)^3}{(t_5)^2} \left( t_5/(t_5 + \frac{x+5y}{6}) + \frac{1}{5}t_1 + \frac{1}{4}t_2 + \frac{1}{3}t_3 + \frac{1}{2}t_4 \right) \left( t_6/(t_6 + \frac{x+5y}{6}) + \frac{1}{5}t_1 + \frac{1}{4}t_2 + \frac{1}{3}t_3 + \frac{1}{2}t_4 \right) + \frac{(z_6)^3}{(t_6)^2} \left( t_6/(t_6 + \frac{x+5y}{6}) + \frac{1}{5}t_1 + \frac{1}{4}t_2 + \frac{1}{3}t_3 + \frac{1}{2}t_4 \right) \right)
\]

Like the \( d = 5 \) cases, we have to evaluate some non-trivial residue integrals of some rational functions in \( t_1, \ldots, t_5 \). One of the typical examples appears in evaluating the coefficient of \((z_1)^3(z_2)^2\):

\[
\frac{1}{(2\pi i)^2} \int_{C_1} dt_1 \int_{C_2} dt_2 \left( \frac{\frac{1}{5}t_1 + \frac{1}{4}t_2 + \frac{1}{3}t_3 + \frac{1}{2}t_4}{t_1t_2(t_1 + \frac{1}{5}t_1 + \frac{1}{4}t_2 + \frac{1}{3}t_3 + \frac{1}{2}t_4)} \right) = \frac{3}{50}.
\]

After the tedious but elementary calculation like (4.44), we obtain the following formula of \( Poly_6 \).

\[
Poly_6 = (5/324)x^5 + (29/216)x^4y + (227/648)x^3y^2 + (227/648)x^2y^3 + (29/216)xy^4 + (5/324)y^5
\]

\[
+ z_1(1/54)x^4 + (17/108)x^3y + (7/18)x^2y^2 + (37/108)xy^3 + (5/54)y^4
\]

\[
+ (1/45)x^3y^3 + (83/450)x^2y^4 + (967/2250)xy^5 + (1829/5625)y^6)z_1
\]

\[
+ [(2/75)x^2 + (27/125)xy + (886/1875)y^2)(z_1)^2
\]

\[
+ [(4/125)x + (158/625)y)(z_1)^3 + (24/625)(z_1)^4
\]

\[
+ z_2((5/216)x^4 + (41/216)x^3y + (31/72)x^2y^2 + (67/216)xy^3 + (5/108)y^4
\]

\[
+ (5/144)x^3y + (77/288)x^2y^2 + (295/576)xy^3 + (5/36)y^4)z_2
\]

\[
+ [(5/96)x^2 + (3/8)xy + (53/192)y^2)(z_2)^2
\]

\[
+ [(5/64)x + (7/32)y)(z_2)^3 + (3/64)(z_2)^4
\]

\[
+ z_3((5/162)x^4 + (77/324)x^3y + (25/54)x^2y^2 + (77/324)xy^3 + (5/162)y^4
\]

\[
+ (5/81)x^3 + (67/162)x^2y^2 + (67/162)xy^3 + (5/81)y^4)z_3
\]

\[
+ [(10/81)x^2 + (37/81)xy + (10/81)y^2)(z_3)^2
\]

\[
+ [(4/27)x + (4/27)y)(z_3)^3 + (4/81)(z_3)^4
\]

\[
+ z_4((5/108)x^4 + (67/216)x^3y + (31/72)x^2y^2 + (41/216)xy^3 + (5/216)y^4
\]

\[
+ (5/36)x^3 + (295/576)x^2y^2 + (77/288)xy^3 + (5/144)y^4)z_4
\]

\[
+ [(53/192)x^2 + (3/8)xy + (59/96)y^2)(z_4)^2
\]
$\frac{(7/32)x + (5/64)y}{(z_1)^3 + (3/64)(z_4)^4}
\frac{(z_1)((5/54)x^4 + (37/108)x^3y + (7/18)x^2y^2 + (17/108)xy^3 + (1/54)y^4 \left(\frac{(1829/5625)x^3 + (967/2250)x^2y + (83/450)xy^2 + (1/45)y^3}{z_5}\right) + ((886/1875)x^2 + (27/125)xy + (2/75)y^2)(z_5)^2}{+(158/625)x + (4/125)y)(z_5)^3 + (24/625)(z_5)^4})
\frac{+z_1z_2(((1/36)x^3 + (2/9)x^2y + (17/36)xy^2 + (5/18)y^3)}{+(1/30)x^2 + (13/50)xy + (193/375)y^2)z_1 + ((1/24)x^2 + (5/16)xy + (53/96)y^2)z_2
\frac{+((1/25)x + (38/125)y)(z_1)^3 + ((1/16)x + (7/16)xy)(z_2)^2 + ((1/20)x + (73/200)y)z_1z_2}{+(3/50)(z_1)^2z_2 + (3/40)z_1(z_2)^2 + (6/125)(z_1)^3 + (3/32)(z_2)^3})
\frac{+z_1z_2(((1/9)x^3 + (7/18)x^2y + (7/18)xy^2 + (1/9)y^3)}{+(2/15)x^2 + (11/25)xy + (142/375)y^2)z_1 + ((142/375)x^2 + (11/25)xy + (2/15)y^2)z_5
\frac{+((4/25)x + (62/125)y)(z_1)^2 + (25/125)(z_1)^3 + (27)/(z_1)^2z_5 + (24/125)(z_1)^3 + (51/100)(z_1)^3z_5 + (51/100)z_1(z_5)^2)}{+(3/32)(z_4)^3 + (6/125)(z_5)^3 + (3/40)(z_4)^2z_5 + (3/50)z_4(z_5)^2})
\frac{+z_1z_3(((1/27)x^3 + (5/18)x^2y + (1/2)xy^2 + (5/27)y^3)}{+(2/45)x^2 + (73/225)xy + (602/1125)y^2)z_1 + ((2/27)x^2 + (13/27)xy + (10/27)y^2)z_3
\frac{+((4/75)x + (142/375)y)(z_1)^2 + ((4/27)x + (4/9)xy)(z_3)^2 + ((4/45)x + (14/25)y)z_1z_3}{+(8/125)(z_1)^3 + (4/27)(z_3)^3 + (8/75)(z_1)^2z_3 + (8/45)z_1(z_3)^2})
\frac{+z_1z_4(((1/18)x^3 + (13/36)x^2y + (4/9)xy^2 + (5/36)y^3)}{+(1/15)x^2 + (21/50)xy + (337/750)y^2)z_1 + ((1/6)x^2 + (9/16)xy + (5/24)y^2)z_4
\frac{+(2/25)x + (61/125)y)(z_1)^2 + ((5/16)x + (5/16)xy)(z_4)^2 + ((1/5)x + (61/100)y)z_1z_4}{+(12/125)(z_1)^3 + (3/16)(z_4)^3 + (6/25)(z_1)^2z_4 + (7/20)z_1(z_4)^3})
\frac{+z_2z_4(((5/54)x^3 + (19/36)x^2y + (1/3)xy^2 + (5/108)y^3)}{+(5/27)x^2 + (31/54)xy + (5/54)y^2)z_3 + ((5/18)x^2 + (67/144)xy + (5/72)y^2)z_4
\frac{+(2/9)x + (5/27)y)(z_3)^2 + ((13/48)x + (5/48)xy)(z_4)^2 + ((1/3)x + (5/36)y)z_3z_4}{+(2/27)(z_3)^3 + (1/16)(z_4)^3 + (1/9)(z_3)^2z_4 + (1/12)z_3(z_4)^2})
\frac{+z_2z_5(((5/27)x^3 + (1/2)x^2y + (5/18)xy^2 + (1/27)y^3)}{+(10/27)x^2 + (13/27)xy + (2/27)yx^2 + (602/1125)x^2 + (73/225)xy + (2/45)y^2)z_5
\frac{+(4/9)x + (4/27)y)(z_3)^2 + ((142/375)x + (4/75)yx)(z_5)^2 + ((14/25)x + (4/45)y)z_3z_5}{+(4/27)(z_3)^3 + (8/125)(z_5)^3 + (8/45)(z_3)^2z_5 + (8/75)z_3(z_5)^2})
\[+z_2z_3\left(18\right)z_3 + (1/3)x^2y + (19/36)xy^2 + (5/54)y^3\]
\[+\left(5/72\right)x^2 + (67/144)xy + y(18)\right)z_2 + \left((5/54)x^2 + (31/54)xy + (5/27)y^2\right)z_3\]
\[+\left(5/48\right)x + (13/48)y)(z_2^2 + (5/27)x + (2/9)y)^2z_3 + (5/36)x + (1/3)y)z_2z_3\]
\[+ (1/32)(z_2)^3 + (2/27)(z_3)^3 + (1/12)(z_2^2)z_3 + (1/9)(z_2)(z_3^2)\]
\[+z_2z_4((5/72)x^2 + (31/72)xy + (31/72)xy^2 + (5/72)y^3)\]
\[+(5/48)x^2 + (19/32)xy + (5/18)y)z_2 + ((5/24)x^2 + (19/32)xy + (5/48)y^2)z_4\]
\[+(5/32)x + (11/32)y)(z_2^2 + ((11/32)x + (5/32)y)(z_4^2 + (5/16)x + (5/16)y)z_2z_4\]
\[+ (3/32)(z_2)^3 + (3/32)(z_4)^3 + (3/16)(z_2^2)z_4 + (3/16)(z_4)(z_2^2)\]
\[+z_1z_2z_3((1/8)x^2 + (7/18)xy + (5/9)y^2) + ((1/5)x + (15/75)yz_1\]
\[+(1/12)x + (13/24)y)(z_2 + (1/9)x + (2/3)y)z_3\]
\[+(1/25)(z_1)^2 + (1/8)(z_2)^2 + (2/9)(z_3)^2 + ((1/10)z_1z_2 + (1/6)z_2z_3 + (2/15)z_1z_3\]
\[+z_1z_2z_5((1/6)x^2 + (1/2)xy + (1/3)y^2) + (1/5)x + (14/25)y)z_1\]
\[+(1/4)x + (5/8)y)(z_2 + (13/25)x + (2/5)y)z_5\]
\[+(6/25)(z_1)^2 + (3/8)(z_2)^2 + (12/25)(z_5)^2 + (2/9)(z_1z_2 + (7/15)z_2z_5 + (3/5)z_1z_5\]
\[+z_1z_4z_5((1/3)x^2 + (1/2)xy + (1/6)y^2) + (2/5)x + (13/25)y)z_1\]
\[+(1/8)x + (1/4)y)(z_4 + (14/25)x + (1/5)y)z_5\]
\[+(12/25)(z_1)^2 + (3/8)(z_4)^2 + (25/25)(z_5)^2 + (7/10)z_1z_4 + (3/10)z_4z_5 + (3/5)z_1z_5\]
\[+z_2z_4z_5((5/9)x^2 + (7/18)xy + (1/8)y^2) + (2/3)x + (1/9)y)z_3\]
\[+(13/24)x + (1/12)y)(z_4 + (34/75)x + (1/15)y)z_5\]
\[+(2/9)(z_3)^2 + (1/8)(z_4)^2 + (2/25)(z_5)^2 + (6/25)z_4z_5 + (1/10)z_4z_5 + (2/15)z_3z_5\]
\[+z_1z_2z_4((1/12)x^2 + (1/2)xy + (5/12)y^2) + (1/10)x + (29/50)y)z_1\]
\[+(1/8)x + (11/16)y)(z_2 + (1/4)x + (5/8)y)z_4\]
\[+(1/325)(z_1)^2 + (3/16)(z_2)^2 + (3/8)(z_4)^2 + (1/20)z_1z_2 + (3/8)z_2z_4 + (3/10)z_1z_4\]
\[+z_1z_3z_4((1/9)x^2 + (11/18)xy + (5/18)y^2) + (2/15)x + (53/75)y)z_1\]
\[+(2/9)x + (5/9)y)(z_3 + (1/3)x + (5/12)y)z_4\]
\[+(4/25)(z_1)^2 + (2/9)(z_3)^2 + (4/14)(z_4)^2 + (4/15)z_1z_3 + (1/3)z_3z_4 + (2/5)z_1z_4\]
\[+z_2z_3z_4((5/36)x^2 + (13/18)xy + (5/36)y) + (5/24)x + (5/12)yz_2\]
\[+(1/8)(z_2)^2 + (1/9)(z_3)^2 + (1/8)(z_4)^2 + (1/6)z_2z_3 + (1/4)z_2z_4 + (1/6)z_3z_4\]
of the differential equation
Here, i.e., we should have the equalities:

Conjecture 2
The first non-trivial test for this formula is the compatibility with the conjecture proposed in [4].

Formal iteration of the recursive formula of Fano hypersurfaces for descending $N$
down to the case $N = k$ yields the coefficients of the hypergeometric series used in the mirror
calculation, i.e., we should have the equalities:

$$
a_d = \tilde{t}_0^{k,k,d}$$

$$
b_d = \frac{1}{d} \tilde{L}_1^{k,k,d} + \sum_{m=1}^{d-1} \frac{1}{m} \tilde{t}_1^{k,k,m} \cdot \tilde{t}_0^{k,k,d-m}. \quad (4.46)
$$

Here

$$
a_d = \frac{(kd)!}{(d)!^k}, \quad b_d = a_d \sum_{i=1}^{d} \sum_{m=1}^{d-1} \frac{m}{i(ki-m)}
$$

are the coefficients of the hypergeometric series associated to the solutions

$$
w_0(x) = \sum_{d=0}^{\infty} a_d e^{dx}, \quad w_1(x) = \sum_{d=1}^{\infty} b_d e^{dx} + w_0(x)x
$$

of the differential equation

$$
((\frac{d}{dx})^{k-1} - ke^x(k \frac{d}{dx} + 1)(k \frac{d}{dx} + 2) \cdots (k \frac{d}{dx} + k - 1))w_1(x) = 0. \quad (4.47)
$$
We tested the recursive formula for the $d = 6$ rational curves obtained from the formula (4.45) by checking that it reproduces the coefficients $a_6$ and $b_6$ for the $k \leq 20$ cases. Of course, we argue that in the $N > k$ region, this recursive formula predicts the rational structural constant $L_n^{N,k,6}$ for the $d = 6$ rational curves. For example, we predict

$$L_0^{8,7,6} = 1379915335327680772049582771200.$$  

(4.48)

Finally, we discuss the validity of Conjecture 1 for higher degree cases. In these cases, the most non-trivial part of the calculation using Conjecture 1 comes from the terms like $$(z_1)^2(z_2)^2 \cdots (z_m)^2.$$ In evaluating the coefficients of these terms, we have to integrate the rational functions in many variables. For example, we consider the coefficient of $(z_1)^2(z_2)^2(z_3)^2$ in the $d = 7$ case. This coefficient is evaluated by the residue integral of the following rational function:

$$\frac{1}{(2\pi\sqrt{-1})^3} \int_{C_1} \frac{dt_1}{t_1} \int_{C_2} \frac{dt_2}{t_2} \int_{C_3} \frac{dt_3}{t_3} \frac{1}{(t_1 + \frac{1}{3}t_2 + \frac{1}{3}t_3)(\frac{1}{6}t_1 + t_2 + \frac{1}{6}t_3)(\frac{1}{6}t_1 + \frac{1}{3}t_2 + \frac{1}{3}t_3)(\frac{1}{6}t_1 + \frac{1}{6}t_2 + \frac{1}{6}t_3)} = \frac{1}{20}.$$  

(4.49)

The rational number $\frac{1}{20}$ has already appeared as the coefficient of $x(z_1)^2(z_2)^2$ in the $d = 6$ case. This coincidence is compatible with the relation (2.18), which suggests many relations between the coefficients of Polynomials with different $d$'s. Thus, we can rely on our predictions using the residue integral of rational functions in many variables like (4.49). We believe that our main conjecture is valid in the case of the rational curves of arbitrary degree!

### 5 Conclusion

We first discuss applications of our main conjecture. The recursive formulas for descending the dimension of the hypersurface while conserving its degree are first introduced by A.Collino in an attempt to prove the relation (2.18). Now, we have the general form of the recursive formula and it is in principle possible to prove (2.18) along the line proposed in [4] (of course, we have to prove Conjecture 1 before this). We also have to consider the rigorous connection between the recursive formulas and the hypergeometric series. This line of thought also contributes to the analysis of the quantum cohomology ring of the general type hypersurface along the line of [10].

Physically, we can try to interpret our approach as the large $N$ expansion of the gauged linear sigma model. From this point of view, we can regard the recursive formulas as the (discretized) differential equations that describe the behavior of the model in the complex $N$-plane. Since the formula given in our main conjecture is rather simple, it might be possible to derive them through the analysis of the gauged linear sigma model.

Geometrically, it seems to be natural to regard the appearance of $d - 1$ $U(1)$ charge variables $t_1, t_2, \ldots, t_{d-1}$ as the consequence of application of some kind of fixed point theorem like the discussion in [3]. And it is of course interesting to prove the formulas from this line of thought.

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Appendix A: Recursive Formulas for Fano Hypersurfaces with \( c_1(M_N^k) \geq 2 \)

\[
L_{m}^{N,k,1} = L_{m}^{N+1,k,1} := L_{m}^{k}
\]
\[
L_{m}^{N,k,2} = \frac{1}{2}(L_{m-1}^{N+1,k,2} + L_{m}^{N+1,k,2} + 2L_{m}^{N+1,k,1} \cdot L_{m+1}(N-k))
\]  
\[
L_{m}^{N,k,3} = \frac{1}{18}(4L_{m-2}^{N+1,k,3} + 10L_{m-1}^{N+1,k,3} + 4L_{m}^{N+1,k,3} + 12L_{m-1}^{N+1,k,2} \cdot L_{m+2}(N-k) + 9L_{m}^{N+1,k,2} \cdot L_{m+2}(N-k) + 6L_{m}^{N+1,k,1} \cdot L_{m+1+2}(N-k))
\]  
\[
L_{m}^{N,k,4} = \frac{1}{32}(3L_{n-3}^{1} + 13L_{n-2}^{4} + 13L_{n-1}^{4} + 3L_{n}^{4})
\]

In the following, we omit the subscripts \( N +1 \) and \( k \) of \( L_{m}^{N+1,k,4} \) in the r.h.s. for brevity.
\[ L_{n}^{N,k,5} = \frac{24}{625} L_{n-4}^{5} + \frac{154}{625} L_{n-3}^{5} + \frac{269}{625} L_{n-2}^{5} + \frac{154}{625} L_{n-1}^{5} + \frac{24}{625} L_{n}^{5} + \frac{6}{125} L_{n-3}^{4} L_{n-3+N-k}^{1} + \frac{3}{50} L_{n-2}^{4} L_{n-3+N-k}^{3} + \frac{3}{40} L_{n-1}^{4} L_{n-3+N-k}^{4} + \frac{3}{32} r_{n}^{4} L_{n-3+N-k}^{1} + \frac{37}{125} r_{n-2}^{4} L_{n-2+N-k}^{3} + \frac{71}{200} r_{n-1}^{4} L_{n-2+N-k}^{4} + \frac{17}{40} r_{n}^{4} L_{n-2+N-k}^{1} + \frac{58}{125} r_{n-1}^{4} L_{n-1+N-k}^{3} + \frac{393}{800} r_{n}^{4} L_{n-1+N-k}^{4} + \frac{24}{125} L_{n}^{4} L_{n+N-k}^{4} + \frac{8}{125} L_{n-3}^{3} L_{n-2+2(N-k)}^{2} + \frac{8}{75} L_{n-2}^{3} L_{n-2+2(N-k)}^{2} + \frac{8}{45} L_{n-1}^{3} L_{n-2+2(N-k)}^{2} + \frac{1}{9} L_{n}^{3} L_{n-2+2(N-k)}^{2} + \frac{46}{125} L_{n-2}^{3} L_{n-1+2(N-k)}^{2} + \frac{122}{225} L_{n-1}^{3} L_{n-1+2(N-k)}^{2} + \frac{29}{90} L_{n}^{3} L_{n-1+2(N-k)}^{2} + \frac{59}{125} L_{n-1}^{3} L_{n+2(N-k)}^{2} + \frac{66}{25} L_{n}^{3} L_{n+2(N-k)}^{2} + \frac{12}{125} L_{n}^{3} L_{n+1+3(N-k)}^{2} + \frac{6}{25} L_{n-2}^{3} L_{n+1+3(N-k)}^{2} + \frac{29}{90} L_{n-1}^{3} L_{n+1+3(N-k)}^{2} + \frac{12}{125} L_{n}^{3} L_{n+1+3(N-k)}^{2} + \frac{2}{9} L_{n}^{3} L_{n+1+3(N-k)}^{2} + \frac{46}{125} L_{n-1}^{3} L_{n+1+3(N-k)}^{2} + \frac{8}{75} L_{n}^{3} L_{n+1+3(N-k)}^{2} + \frac{8}{25} L_{n}^{3} L_{n+2+3(N-k)}^{2} + \frac{24}{125} L_{n-3}^{3} L_{n+4(N-k)}^{1} + \frac{393}{800} L_{n-2}^{3} L_{n+4(N-k)}^{1} + \frac{17}{40} L_{n-1}^{3} L_{n+4(N-k)}^{1} + \frac{3}{32} r_{n}^{3} L_{n+4(N-k)}^{1} + \frac{58}{125} r_{n-2}^{3} L_{n+1+4(N-k)}^{1} + \frac{71}{200} r_{n-1}^{3} L_{n+1+4(N-k)}^{1} + \frac{3}{40} r_{n}^{3} L_{n+1+4(N-k)}^{1} + \frac{37}{125} r_{n-1}^{3} L_{n+2+4(N-k)}^{1} + \frac{3}{50} r_{n}^{3} L_{n+2+4(N-k)}^{1} + \frac{6}{125} L_{n}^{3} L_{n+3+4(N-k)}^{1} + \frac{2}{25} L_{n-2}^{3} L_{n+2+N-k}^{3} L_{n-2+2(N-k)}^{2} + \frac{1}{10} L_{n-1}^{3} L_{n+1+N-k}^{3} L_{n-2+2(N-k)}^{2} + \frac{1}{8} L_{n}^{3} L_{n+1+N-k}^{3} L_{n-2+2(N-k)}^{2} + \frac{2}{15} L_{n-1}^{3} L_{n-1+N-k}^{3} L_{n-2+2(N-k)}^{2} + \frac{2}{9} L_{n}^{3} L_{n+1+N-k}^{3} L_{n-2+2(N-k)}^{2} + \frac{11}{25} r_{n-1}^{3} L_{n-1+N-k}^{3} L_{n-1+2(N-k)}^{1} + \frac{21}{40} L_{n}^{3} L_{n-1+N-k}^{3} L_{n-1+2(N-k)}^{1} + \frac{29}{45} L_{n}^{3} L_{n+1+N-k}^{3} L_{n-1+2(N-k)}^{1} + \frac{1}{25} L_{n}^{3} L_{n+1+N-k}^{3} L_{n-1+2(N-k)}^{1} + \frac{3}{6} L_{n-2}^{3} L_{n+2+N-k}^{1} L_{n+4(N-k)}^{1} + \frac{3}{10} L_{n-1}^{3} L_{n+2+N-k}^{1} L_{n+4(N-k)}^{1} + \frac{3}{8} L_{n}^{3} L_{n+2+N-k}^{1} L_{n+4(N-k)}^{1} + \frac{23}{40} L_{n-1}^{3} L_{n+1+N-k}^{1} L_{n+4(N-k)}^{1} \]
\[ \begin{align*}
&\quad \frac{2}{3} L_n^3 L_{n+1}^3 L_{n+1}^1 L_{n+4}^1 (N-k) \\
&+ \frac{13}{25} L_n^3 L_{n-1}^3 L_{n+1}^1 L_{n+4}^1 (N-k) \\
&+ \frac{3}{25} L_n^3 L_{n+1}^3 L_{n+1}^1 L_{n+4}^1 (N-k) \\
&+ \frac{6}{25} L_n^3 L_{n+1}^3 L_{n+2}^1 L_{n+4}^1 (N-k) \\
&+ \frac{12}{25} L_n^3 L_{n+2}^1 L_{n+3}^1 L_{n+4}^1 (N-k) \\
&+ \frac{2}{9} L_n^3 L_{n+3}^1 L_{n+1}^3 L_{n+4}^1 (N-k) \\
&+ \frac{23}{40} L_n^3 L_{n-1}^3 L_{n+1}^1 L_{n+4}^1 (N-k) \\
&+ \frac{9}{25} L_n^3 L_{n+1}^3 L_{n+1}^1 L_{n+4}^1 (N-k) \\
&+ \frac{1}{2} L_n^3 L_{n+1}^3 L_{n+1}^1 L_{n+4}^1 (N-k) \\
&+ \frac{1}{8} L_n^3 L_{n+1}^3 L_{n+1}^1 L_{n+4}^1 (N-k)
\end{align*} \]
\[ + \frac{1}{2} L_n^1 L_{n-1+N-k}^1 L_n^2 L_{n-1+2(N-k)}^1 L_{n+4(N-k)}^1 + \frac{2}{3} L_n^1 L_{n+1+N-k}^1 L_n^2 L_{n+2(N-k)}^1 L_n^1 + \frac{3}{5} L_n^1 L_{n+1+N-k}^1 L_n^2 L_{n+2(N-k)}^1 L_n^1 + \frac{3}{4} L_{n+1+N-k}^1 L_n^1 L_n^2 L_{n+2(N-k)}^1 L_n^1 + \frac{2}{3} L_{n+1+N-k}^1 L_n^2 L_{n+3(N-k)}^1 L_n^1 L_n^1 + \frac{1}{2} L_n^1 L_{n+2(N-k)}^1 L_n^1 L_n^2 L_{n+3(N-k)}^1 L_n^1 L_n^1 + \frac{1}{3} L_n^1 L_{n+1+2(N-k)}^1 L_n^1 L_{n+3(N-k)}^1 L_n^1 L_n^1 + \frac{1}{4} L_n^1 L_{n+1+2(N-k)}^1 L_n^1 L_{n+3(N-k)}^1 L_n^1 L_{n+4(N-k)}^1 + L_n^1 L_{n+2(N-k)}^1 L_n^1 L_{n+3(N-k)}^1 L_n^1 L_{n+4(N-k)}^1 \]  

(5.54)
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