Why Does Stagewise Training Accelerate Convergence of Testing Error Over SGD?

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Abstract

Stagewise training strategy is commonly used for learning neural networks, which uses a stochastic algorithm (e.g., SGD) starting with a relatively large step size (aka learning rate) and geometrically decreasing the step size after a number of iterations. It has been observed that the stagewise SGD has much faster convergence than the vanilla SGD with a continuously decreasing step size in terms of both training error and testing error. But how to explain this phenomenon has been largely ignored by existing studies. This paper provides theoretical evidence for explaining this faster convergence. In particular, we consider the stagewise training strategy for minimizing empirical risk that satisfies the Polyak-Łojasiewicz condition, which has been observed/proved for neural networks and also holds for a broad family of convex functions. For convex loss functions and “nice-behaved” non-convex loss functions that are close to a convex function (namely weakly convex functions), we establish that the faster convergence of stagewise training than the vanilla SGD under the same condition on both training error and testing error lies on better dependence on the condition number of the problem. Indeed, the proposed algorithm has additional favorable features that come with theoretical guarantee for the considered non-convex optimization problems, including using explicit algorithmic regularization at each stage, using stagewise averaged solution for restarting, and returning the last stage-wise averaged solution as the final solution. To differentiate from commonly used stagewise SGD, we refer to our algorithm as stagewise regularized training algorithm or START. Of independent interest, the proved testing error bound of START for a family of non-convex loss functions is dimensionality and norm independent.

1. Introduction

In this paper, we consider learning a prediction model by using a stochastic algorithm to minimize the expected risk via solving the empirical risk problem:

\[
\min_{w \in \Omega} F_S(w) := \frac{1}{n} \sum_{i=1}^{n} f(w, z_i),
\]

where \( f(w, z) \) is a smooth loss function of the model \( w \) on the data \( z \), \( \Omega \) is a closed convex set, and \( S = \{z_1, \ldots, z_n\} \) denotes the set of \( n \) observed data points that are sampled from an underlying distribution \( P_z \) with support on \( \mathcal{Z} \).

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There are tremendous studies devoted to solving this empirical risk minimization (ERM) problem in machine learning and related fields. Most of them are concerned with the convergence of optimization error measured by $F_S(w) - \min_w F_S(w)$ for an iterative optimization algorithm. Among all existing algorithms, stochastic gradient descent (SGD) is probably the simplest and attracts most attention that takes the following update:

$$w_{t+1} = w_t - \eta_t \nabla f(w_t, z_{i_t}),$$

where $i_t \in \{1, \ldots, n\}$ is randomly sampled, and $\eta_t$ is the step size that is usually decreasing to 0. Convergence theories have been extensively studied of SGD for an objective that satisfies various assumptions, e.g., convexity (Nemirovski et al., 2009), non-convexity (Ghadimi and Lan, 2013a), strong convexity (Hazan et al., 2007), local strong convexity (Qu et al., 2016), Polyak-Łojasiewicz inequality (Karimi et al., 2016), Kurdyka-Łojasiewicz inequality (Xu et al., 2017), etc. The list of papers about SGD is so long that can not be exhausted here.

The success of deep learning is mostly driven by stochastic algorithms as simple as SGD and its variants running on big data sets (Krizhevsky et al., 2012; He et al., 2016). However, an interesting phenomenon that has been observed in practice for deep learning is that no one is actually using the vanilla SGD with a continuously decreasing step size that is well studied in theory. Instead, a common trick used to speed up the convergence of SGD is by using a stagewise step size strategy, i.e., starting from a relatively large step size and decreasing it geometrically after a number of iterations. Not only the convergence of training error is accelerated but also is the convergence of testing error. However, there is still lack of theory for explaining this phenomenon. Although a stagewise step size strategy has been considered in some studies (Hazan and Kale, 2011; Xu et al., 2017; Karimi et al., 2016; Kleinberg et al., 2018; Chen et al., 2018), none of them explains the benefit of stagewise training used in practice compared with standard SGD with a decreasing step size, especially on the convergence of testing error for non-convex problems.

### 1.1. Our Contributions

This paper is motivated by providing some theoretical evidence showing that an appropriate stagewise training algorithm can have faster convergence than the vanilla SGD with a decreasing step size under some condition. In particular, we will analyze a stagewise regularized training (START) algorithm under the so-called Polyak-Łojasiewicz inequality (or the gradient dominance property) (Polyak, 1963):

$$2\mu(F_S(w) - \min_w F_S(w)) \leq \|\nabla F_S(w)\|^2.$$

This property has been recently observed/proved for learning deep and shallow neural networks (Hardt and Ma, 2016; Xie et al., 2016; Kleinberg et al., 2018; Li and Yuan, 2017; Zhou and Liang, 2017; Charles and Papailiopoulos, 2018), and it holds for a broad family of convex functions (Xu et al., 2017). We will focus on the scenario that $\mu$ is a **very small positive value**, which corresponds to ill-conditioned problems and is indeed the case for many problems (Hardt and Ma, 2016; Charles and Papailiopoulos, 2018). We will show that the considered stagewise training algorithm has a better dependence on $\mu$ than the existing
results of vanilla SGD for both the training error (with the same number of iterations) and
the testing error (with the same number of data and a less number of iterations), while keep-
ing the same dependence on the number of data for the testing error bound. We will consider
both the convex loss and non-convex loss that makes the objective close to a convex function.
To be fair for comparison between two algorithms, we adopt a unified approach to analyze
both the optimization error and the generalization error, which together with algorithm-
independent optimal training error constitute the testing error. In addition, we use the
same tool for analysis of the generalization error - a key component in the testing error.
We would like to point out that the techniques for us to prove the convergence of optimiza-
tion error and testing error are simple and standard. In particular, the optimization error
analysis is built on existing convergence results for solving convex problems, and the testing
error analysis is built on the uniform stability analysis of a stochastic algorithm introduced
by Hardt et al. (2016). We also notice that some recent studies (Kuzborskij and Lampert,
2018; Zhou et al., 2018; Charles and Papailiopoulos, 2018) have used other techniques (e.g.,
data-dependent bound, average stability, point-wise stability) to analyze the generalization
error of a stochastic algorithm. Nevertheless, we believe similar techniques can be also used
for analyzing stagewise learning algorithm, which is beyond the scope of this paper. It is of
great interest to us that simple analysis for the commonly used stochastic learning strategy
can possibly explain its greater success in practice than using the standard SGD method
with a continuously decreasing step size.

Besides theoretical contributions, the proposed algorithm also has additional favorable
features that come with theoretical guarantee for the considered non-convex problems and
help improve the generalization performance, including using explicit algorithmic regular-
ization at each stage, using stagewise averaged solution for restarting, and returning the
last stagewise averaged solution as the final solution. To differentiate from commonly used
stagewise SGD, we refer to our algorithm as stagewise regularized training algorithm or
Start.

1.2. Other Related Works on Analyzing Algorithms under PL conditions

It is also notable that many papers have proposed and analyzed deterministic/stochastic
optimization algorithms under the PL condition, e.g., (Karimi et al., 2016; Lei et al., 2017;
Reddi et al., 2016b; Bassily et al., 2018). This list could be long if we consider its equivalent
condition in the convex case. However, none of them exhibits the benefit of stagewise learning
strategy used in practice. One may also notice that linear convergence for the optimization
error can be proved by a stochastic variance reduction gradient method (Reddi et al., 2016b).
Nevertheless, its uniform stability bound remains unclear for making a fair comparison.
Two recent works (Zhou et al., 2018; Charles and Papailiopoulos, 2018) have analyzed the
generalization error (or stability) of stochastic algorithms (e.g., the vanilla SGD with a
decreasing step size or small constant step size) under the PL condition and other conditions.
We emphasize that their results are not directly comparable to the results presented in this
work. Zhou et al. (2018) consider the generalization error of SGD with a decreasing step
size in the form $\Theta(c/t \log t)$ with $2/\mu < c < 1/L$ and $L$ being smoothness parameter, which
corresponds to a good conditioned setting \( L/\mu < 1/2 \) \(^1\). Charles and Papailiopoulos (2018) make a strong technical assumption (e.g., the global minimizer is unique) for deriving their uniform stability results, which is unlikely to hold in the real-world and is avoided in this work for establishing a generalization error bound for the standard SGD.

Finally, it was brought to our attention \(^2\) when this paper is almost done that an independent anonymous work (Anonymous, 2019) observes a similar advantage of stagewise SGD over SGD with a polynomial decaying step size lying at the better dependence on the condition number. However, they only analyze the strongly convex quadratic case and the training error of ERM.

2. Preliminaries and Notations

Let \( \mathcal{A} \) denote a randomized algorithm, which returns a randomized solution \( w_S = \mathcal{A}(S) \) based on the given data set \( S \). Denote by \( E_A \) expectation over the randomness in the algorithm and by \( E_S \) expectation over the randomness in the data set. Where it is clear from the context, we will omit the subscript \( S \) and \( A \) in the expectation notations. Let \( w^*_S \in \arg\min_{w \in \Omega} F_S(w) \) denote an empirical risk minimizer, and \( F(w) = E_z[f(w, z)] \) denote the true risk of \( w \) (also called testing error in this paper). We use \( \| \cdot \| \) to denote the Euclidean norm, and use \( [n] = \{1, \ldots, n\} \).

In order to analyze the testing error convergence of an iterative algorithm, we use the following decomposition of testing error.

\[
E_{A,S}[F(w_S)] \leq E_S[F_S(w^*_S)] + E_S,E_A[F_S(w_S) - F_S(w^*_S)] + E_{A,S}[F(w_S) - F_S(w_S)],
\]

where \( \epsilon_{opt} \) measures the optimization error, i.e., the difference between empirical risk (or called training error) of the returned solution \( w_S \) and the optimal value of the empirical risk, and \( \epsilon_{gen} \) measures the generalization error, i.e., the difference between the true risk of the returned solution and the empirical risk of the returned solution. The difference \( E_{A,S}[F(w_S)] - E_S[F_S(w^*_S)] \) is an upper bound bound on the so-called excess risk bound in the literature, which is defined as \( E_{A,S}[F(w_S)] - \min_{w \in \Omega} F(w) \). It is notable that the first term \( E_S[F_S(w^*_S)] \) in the above bound is independent of the choice of randomized algorithms. Hence, in order to compare the performance of different randomized algorithms, we can focus on analyzing \( \epsilon_{opt} \) and \( \epsilon_{gen} \). For analyzing the generalization error, we will leverage the uniform stability tool (Bousquet and Elisseeff, 2002). The definition of uniform stability is given below.

**Definition 1** A randomized algorithm \( \mathcal{A} \) is called \( \epsilon \)-uniformly stable if for all data sets \( S, S' \in \Omega^n \) that differs at most one example the following holds:

\[
\sup_z[f(\mathcal{A}(S), z) - f(\mathcal{A}(S'), z)] \leq \epsilon.
\]

A well-known result is that if \( \mathcal{A} \) is \( \epsilon \)-uniformly stable, then its generalization error is bounded by \( \epsilon \) (Bousquet and Elisseeff, 2002), i.e.,

\(^1\) Indeed this could never happen in unconstrained convex optimization where \( \|\nabla F_S(x)\|^2 \leq 2L(F_S(x) - \min F_S(x)) \) (Nesterov, 2004)[Theorem 2.1.5]. Together with the PL condition, it implies \( L \geq \mu \).

\(^2\) personal communication with Jason D. Lee at NeurIPS 2018.
Lemma 2 If $A$ is $\epsilon$-uniformly stable, we have $\varepsilon_{\text{gen}} \leq \epsilon$.

In light of the above results, in order to compare the convergence of testing error of different randomized algorithms, it suffices to analyze their convergence in terms of optimization error and their uniform stability.

A function $f(w)$ is $L$-smooth if it is differentiable and its gradient is $L$-Lipschitz continuous, i.e., $\|\nabla f(w) - \nabla f(u)\| \leq L\|w - u\|$, $\forall w, u$. A function $f(w)$ is $G$-Lipschitz continuous if $\|\nabla f(w)\| \leq G$. Throughout the paper, we will make the following assumptions.

**Assumption 1** Assume that

(i) $f(w, z)$ is $L$-smooth in terms of $w \in \Omega$ for every $z \in Z$.

(ii) $f(w, z)$ is finite-valued and $G$-Lipschitz continuous in terms of $w \in \Omega$ for every $z \in Z$.

(iii) there exists $\sigma > 0$ such that $E_i[\|\nabla f(w, z_i) - \nabla F_S(w)\|^2] \leq \sigma^2$ for $w \in \Omega$.

(iv) $F_S(w)$ satisfies the PL condition, i.e., there exists $\mu > 0$

$$2\mu(F_S(w) - F_S(w^*_S)) \leq \|\nabla F_S(w)\|^2, \forall w \in \Omega.$$ 

(v) For an initial solution $w_0$, there exists $\epsilon_0 > 0$ such that $F_S(w_0) - F_S(w^*_S) \leq \epsilon_0$.

**Remark:** The second assumption is imposed for the analysis of uniform stability of a randomized algorithm. W.l.o.g we assume $|f(w, z)| \leq 1, \forall w \in \Omega$. The third assumption is for the purpose of analyzing optimization error. It is notable that $\sigma^2 \leq 4G^2$. (iv) assumes the PL condition is global. In the last section, we will consider extensions to a local PL condition. If $F_S$ is a strongly convex function, $\mu$ corresponds to the strong convexity parameter. In this paper, we are particular interested in the case when $\mu$ is small, i.e. the condition number $L/\mu$ is large.

### 3. Review: SGD For Functions Satisfying PL Condition

In this section, we review the training error convergence and generalization error of SGD with a decreasing step size for functions satisfying the PL condition in order to derive its testing error bound. We would like to emphasize the results presented in this section are mainly from existing works (Karimi et al., 2016; Hardt et al., 2016). The optimization error and the uniform stability of SGD have been studied in these two papers separately. Since we are not aware of any studies that piece them together, it is of our interest to summarize these results here for comparing with our new results established later in this paper. The update of SGD takes the following simple form assuming $\Omega = \mathbb{R}^d$:

$$w_{t+1} = w_t - \eta_t \nabla f(w_t, z_{i_t}), t \geq 0$$ \hspace{1cm} (2)

where $i_t \in [n]$ is uniformly sampled.

Let us first consider the optimization error convergence, which has been analyzed in (Karimi et al., 2016) and is summarized below.
Theorem 1  Suppose $\Omega = \mathbb{R}^d$. Under Assumption 1 (i), (iv) and $E_i[\|\nabla f(w, z_i)\|^2] \leq G^2$, by setting $\eta_t = \frac{2^{t+1}}{2\mu(t+1)}$, we have

$$E[F_S(w_t) - F_S(w_0^*)] \leq \frac{LG^2}{2t\mu^2},$$

and by setting $\eta_t = \eta$, we have

$$E[F_S(w_t) - F_S(w_0^*)] \leq (1 - 2\eta\mu)^t(F_S(w_0) - F_S(w_0^*)) + \frac{LG^2\eta}{4\mu}. \quad (4)$$

Remark: It is notable that the above result needs $\Omega = \mathbb{R}^d$. It remains unclear how to extend the analysis in (Karimi et al., 2016) to the constrained case. In order to have an $\epsilon$ optimization error, we can set $t = \frac{LG^2}{2\mu\epsilon}$ in the decreasing step size setting. In the constant step size setting, we can set $\eta = \frac{2\mu}{LG^2}$ and $t = \frac{LG^2}{4\mu\epsilon} \log(2\epsilon_0/\epsilon)$, where $\epsilon_0 \geq F_S(w_0) - F_S(w_0^*)$ is the initial optimization error bound. Karimi et al. (2016) also mentioned a stagewise step size strategy based on the second result above. By starting with $\eta_1 = \frac{4\mu}{LG^2}$ and running for $t_1 = \frac{LG^2\log 4}{2\mu^2\epsilon_0}$ iterations, and restarting the second stage with $\eta_2 = \eta_1$ and $t_2 = 2t_1$, then after $K = \log(\epsilon_0/\epsilon)$ stages, we have optimization error less than $\epsilon$, and the total iteration complexity is $O(\frac{LG^2\log 4}{\mu^2\epsilon})$. We can see that SGD with this stagewise optimization strategy does not bring any improvement compared SGD with a decreasing step size. No matter which step size strategy is used among the ones discussed above, the total iteration complexity is $O(\frac{L}{\mu^2\epsilon})$. It is also interesting to know that the above convergence result does not require the convexity of $f(w, z)$. On the other hand, it is unclear how to directly analyze SGD for a convex loss to obtain a better convergence rate than (3).

Next, we present the generalization error bound by the uniform stability. We will consider convex loss and non-convex loss separately because they will lead to different generalization error. Both have been analyzed in (Hardt et al., 2016). We just need to plug the step size of SGD in Theorem 2 into their results (Theorem 3.8 and Theorem 3.12) and obtain the following.

Theorem 2  Suppose Assumption 1 holds and $n > L/\mu$ is sufficiently large. If $f(\cdot, z)$ is convex for any $z \in Z$, then SGD with step size $\eta_t = \frac{2^{t+1}}{2\mu(t+1)}$ satisfies uniform stability with

$$\epsilon_{\text{stab}} \leq \frac{L}{n\mu} + \frac{2G^2}{n\mu} \sum_{t=1}^{T} \frac{1}{t+1} \leq \frac{L + 2G^2 \log(T + 1)}{n\mu}.$$

If $f(\cdot, z)$ is non-convex for any $z \in Z$, then SGD with step size $\eta_t = \frac{2^{t+1}}{2\mu(t+1)}$ satisfies uniform stability with

$$\epsilon_{\text{stab}} \leq \frac{1 + \mu/L}{n - 1} \left(2G^2/\mu\right)^{1/(L/\mu + 1)} T^{\frac{L/\mu}{L/\mu + 1}}.$$

Remark: We are mostly interested in a large condition number setting $L/\mu \gg 1$. With above results, we obtain the convergence of testing error of SGD for smooth loss functions under the PL condition.
Theorem 3 Suppose $\Omega = \mathbb{R}^d$, Assumption 1 holds and let $\hat{G} = G^2/L$. If $f(\cdot,z)$ is convex for any $z \in Z$, with step size $\eta_t = \frac{2t+1}{2\mu(t+1)^2}$ and $T$ iterations SGD returns a solution $w_T$ satisfying

$$E_{A,S}[F(w_T)] \leq E_S[F_S(w_S^*)] + \frac{LG^2}{2T\mu^2} + \frac{(L + 2G^2)\log(T + 1)}{n\mu}.$$  

If $f(\cdot,z)$ is non-convex for any $z \in Z$, with the same setting SGD returns a solution $w_T$ satisfying

$$E_{A,S}[F(w_T)] \leq E_S[F_S(w_S^*)] + \frac{LG^2}{2T\mu^2} + \frac{2T\min(2G/\sqrt{\mu}, e^{2\hat{G}})}{n - 1}.$$ 

By optimizing the value of $T$ in the above bounds, we obtain the risk bound dependent on $n$ only. The results are summarized in the following corollary.

Corollary 4 Suppose Assumption 1 holds. If $f(\cdot,z)$ is convex for any $z \in Z$, with step size $\eta_t = \frac{2t+1}{2\mu(t+1)^2}$ and $T = \frac{nLG^2}{4(L + 2G^2)\mu}$ iterations SGD returns a solution $w_T$ satisfying

$$E_{A,S}[F(w_T)] \leq E_S[F_S(w_S^*)] + \frac{2(L + 2G^2)}{n\mu} + \frac{(L + 2G^2)\log(T + 1)}{n\mu}.$$ 

If $f(\cdot,z)$ is non-convex for any $z \in Z$, with step size $\eta_t = \frac{2t+1}{2\mu(t+1)^2}$ and $T = \max\{\sqrt{\frac{(n-1)LG}{8\mu^{3/4}}}, \frac{\sqrt{L(n-1)G}}{2\mu^G}\}$ iterations SGD returns a solution $w_T$ satisfying

$$E_{A,S}[F(w_T)] \leq E_S[F_S(w_S^*)] + 2\min\left\{\sqrt{\frac{2L^{1/2}G^{3/2}}{\mu}}, \sqrt{\frac{L^{1/2}G^{3/2}}{\mu}}, \frac{\sqrt{L^{1/2}G^{1/2}}}{\sqrt{n-1}\mu^{5/4}}, \frac{\sqrt{L^{1/2}G^{1/2}}}{\sqrt{n-1}\mu}\right\}.$$ 

Remark 1: If the loss is convex, the excess risk bound is in the order of $O\left(\frac{\log(nL/\mu)}{n\mu}\right)$ by running SGD with $T = O(nL/\mu)$ iterations. It notable that an $O(1/n)$ excess risk bound is called the fast rate in the literature. If the loss is non-convex and $2G/\sqrt{\mu} \leq e^{2\hat{G}}$, the excess risk bound is in the order of $O\left(\frac{L^{1/2}G^{3/2}}{\sqrt{\mu}^{3/4}}\right)$ by running SGD with $T = O\left(\frac{\sqrt{nL}}{\sqrt{\mu}^{3/4}}\right)$ iterations. If the loss is non-convex and $2G/\sqrt{\mu} > e^{2\hat{G}}$, the excess risk bound is in the order of $O\left(\frac{\sqrt{L^{1/2}G^{3/2}}}{\sqrt{\mu}}\right)$ by running SGD with $T = O\left(\frac{\sqrt{nL}}{\mu}\right)$ iterations. When $\mu$ is very small, the convergence of testing error is very slow. In addition, the number of iterations is also scaled by $1/\mu$ in the convex case and $1/\sqrt{\mu}$ or $1/\mu$ in the non-convex case for achieving a minimal excess risk bound.

Remark 2: Another possible choice of decreasing step size is $O(1/\sqrt{T})$ (Ghadimi and Lan, 2013a), which yields an $O(1/\sqrt{T})$ convergence rate for $F_S(w_T) - F_S(w_S^*)$ in the convex case or for $\|\nabla F_S(w_t)\|^2$ in the non-convex case with a randomly sampled $t$. In the latter case, it also implies a convergence rate of $O(1/(\mu\sqrt{T}))$ for $F_S(w_t) - \min_w F_S(w)$ under the PL condition. It will lead to a worse dependence on $n$ for the testing error. Therefore, we do not compare with this result theoretically.
Algorithm 1 STAgewise Regularized Training (START) Algorithm: \textit{START}(F_S, w_0, \gamma, K)

1: \textbf{Input:} \( w_0, \gamma \) and \( K \)
2: \textbf{for} \( k = 1, \ldots, K \) \textbf{do}
3: \hspace{1em} Let \( F_{w_{k-1}}(w) = F_S(w) + \frac{1}{2\gamma} \| w - w_{k-1} \|^2 \)
4: \hspace{1em} \( w_k = \text{SGD}(F_{w_{k-1}}(w_{k-1}), \eta_k, T_k) \)
5: \textbf{end for}
6: \textbf{Return:} \( w_K \)

Algorithm 2 SGD\((F_{w_1}(w_1), \eta, T)\)

1: \textbf{for} \( t = 1, \ldots, T \) \textbf{do}
2: \hspace{1em} Sample a random data \( z_{t} \in S \)
3: \hspace{1em} \( w_{t+1} = \min_{w \in \Omega} \nabla f(w_t, z_t)^\top w + \frac{1}{2\eta} \| w - w_t \|^2 + \frac{1}{2\gamma} \| w - w_1 \|^2 \)
4: \textbf{end for}
5: \textbf{Output:} \( \hat{w}_T = \sum_{t=1}^{T} w_{t+1}/T \)

4. \textbf{START} for a Convex Function Satisfying the PL condition

In this section, we will analyze a \textbf{START}agewise regularized training (\textbf{START}) algorithm for a convex function under the PL condition. First, let us present the algorithm that we intend to analyze in Algorithm 1. At the \( k \)-th stage, a regularized function \( F_{w_{k-1}}(w) \) is constructed that consists of the original objective \( F_S(w) \) and a quadratic regularizer \( \frac{1}{2\gamma} \| w - w_{k-1} \|^2 \).

The reference point \( w_{k-1} \) is the averaged solution from the previous stage, which is also used for an initial solution for the current stage. \( \gamma \) is a regularization parameter whose value will be revealed later. Adding the strongly convex regularizer at each stage is helpful for reducing the generalization error and is also important for non-convex loss considered in next section. For each regularized problem, the SGD with a constant step size is employed for a number of iterations, whose values will be revealed later.

We would like to point out that similar algorithms have been proposed and analyzed in (Hazan and Kale, 2011; Xu et al., 2017). They focus on analyzing the convergence of optimization error for convex problems under a quadratic growth condition or more general local error bound condition. In the following, we will show that the PL condition implies a local error bound condition. Hence, their algorithms can be used for optimizing \( F_S \) as well enjoying a similar convergence rate in terms of optimization error. However, there is still slight difference between the analyzed algorithm from their considered algorithms. In particular, the regularization term \( \frac{1}{2\gamma} \| w - w_{k-1} \|^2 \) is absent in (Hazan and Kale, 2011), which corresponds to \( \gamma = \infty \) in our case. However, adding a small regularization (with not too large \( \gamma \)) can help improve the generalization error. In addition, their initial step size is scaled by \( 1/\mu \). The initial step size of our algorithm depends on the quality of initial solution that seems more natural and practical. A similar regularization at each stage is also used in (Xu et al., 2017). But their algorithm will suffer from a large generalization error, which is due to the key difference between \textbf{START} and their algorithm (ASSG-r). In particular, they use a geometrically decreasing the parameter \( \gamma_k \) starting from a relatively large value in the order of \( O(1/(\mu \epsilon)) \) with a total iteration number \( T = O(1/(\mu \epsilon)) \). According to our
4.1. Convergence of Optimization Error

In this subsection, we analyze the convergence of optimization error for START. We need the following lemma for our analysis.

Lemma 3 If $F_S(w)$ satisfies the PL condition, then for any $w \in \Omega$ we have

$$\|w - w^*_S\|^2 \leq \frac{1}{2\mu} (F_S(w) - F_S(w^*_S)),$$

where $w^*_S \in \arg\min_w F_S(w)$ is the closest optimal solution to $w$.

Remark: The above result does not require the convexity of $F_S$. For a proof, please refer to (Bolte et al., 2015; Karimi et al., 2016). Indeed, this error bound condition instead of the PL condition is enough to derive the results in Section 4 and Section 5.

The following lemma is a standard convergence result of SGD, which can be found in the literature (Zhao and Zhang, 2015).

Lemma 4 Suppose Assumption 1(i) and (iii) hold, and $f(w,z)$ is a convex function of $w$. By applying SGD to $F_k = F_{w_{k-1}}$ with $w_k = \hat{w}_T$ and $\eta \leq 1/L$, for any $w \in \Omega$, we have

$$E_k[F_k(w_k) - F_k(w)] \leq \sigma^2 \eta_k + \frac{\|w_{k-1} - w\|^2}{2\eta_k T_k}.$$

The above convergence result can be boosted for showing the faster convergence of START under the PL condition.

Theorem 5 Suppose Assumption 1, and $f(w,z)$ is a convex function of $w$. Then by setting $\gamma \geq 1.5/\mu$ and $T_k = \frac{9\sigma^2}{2\mu \epsilon_k^2}$, $\eta_k = \frac{\epsilon_k}{\gamma \sigma_k}$, where $\alpha \leq \min(1, \frac{3\sigma^2}{\alpha L})$, after $K = \log(\epsilon_k/\epsilon)$ stages we have

$$E[F(w_K) - F(w_*)] \leq \epsilon.$$

The total iteration complexity is $O(\frac{\mu}{\epsilon\kappa})$.

Remark: Compared to the result in Theorem 1, the convergence rate of START is faster by a factor of $O(1/\mu)$.

Proof We will prove by induction that $E[F_S(w_k) - F_S(w^*_S)] \leq \epsilon_k$, where $\epsilon_k = \epsilon_0 / 2^k$, which is true for $k = 0$ by the assumption. By applying Lemma 4 to the $k$-th stage, for any $w$

$$E_k[F_S(w_k) - F_S(w)] \leq \frac{\|w_{k-1} - w\|^2}{2\gamma} + \eta_k \sigma^2 + \frac{\|w_{k-1} - w\|^2}{2\eta_k T_k} \quad (5)$$

By plugging $w = w^*_S$ into the above inequality we have

$$E[F_S(w_k) - F(w^*_S)] \leq \frac{E[\|w_{k-1} - w^*_S\|^2]}{2\gamma} + \eta_k \sigma^2 + \frac{E[\|w_{k-1} - w^*_S\|^2]}{2\eta_k T_k} \leq \frac{\epsilon_k - 1}{4\mu \gamma} + \eta_k \sigma^2 + \frac{\epsilon_k - 1}{4\mu \eta_k T_k}.$$
where we use the result in Lemma 3. Since \( \eta_k \leq \frac{\gamma_0 \alpha}{2\sigma} \) and \( T_k \eta_h \geq 1.5/\mu \) and \( \gamma_k \geq 1.5/\mu \), we have

\[
E[F(w_k) - F(w_*)] \leq \epsilon_k
\]

By induction, after \( K = \lceil \log(\epsilon_0/\epsilon) \rceil \) stages, we have

\[
E[F(w_K) - F(w_*)] \leq \epsilon
\]

The total iteration complexity is \( \sum_{k=1}^{K} T_k = O(1/(\mu \epsilon)). \)

\[\Box\]

### 4.2. Analysis of Generalization Error

In this subsection, we analyze the uniform stability of \textsc{Start}. By showing \( \sup_z E_A[f(w_K, z) - f(w'_K, z)] \leq \epsilon \), we can show the generalization error is bounded by \( \epsilon \), where \( w_K \) is learned on a data set \( S \) and \( w'_K \) is learned a different data set \( S' \) that only differs from \( S \) at most one example. Our analysis is closely following the route in (Hardt et al., 2016). The difference is that we have to consider the difference on the reference points \( w_{k-1} \) of two copies of our algorithm on two data sets \( S, S' \). We first give the following lemma regarding the growth of stability within one stage of \textsc{Start}.

**Lemma 5** Assume \( f \) is convex. Let \( w_t \) denote the sequence learned on \( S \) and \( w'_t \) be the sequence learned on \( S' \) by \textsc{Start} at one stage, \( \delta_t = \|w_t - w'_t\| \). If \( \eta \leq 2/L \), then

\[
\delta_{t+1} \leq \left\{ \begin{array}{ll}
\frac{\eta}{\eta + \gamma} \delta_1 + \frac{\gamma}{\eta + \gamma} \delta_t & f_t = f'_t \\
\frac{\eta}{\eta + \gamma} \delta_1 + \frac{\gamma}{\eta + \gamma} \delta_t + \frac{2 \eta_0}{\eta + \gamma} & \text{otherwise}
\end{array} \right.
\]

**Proof** Let us define

\[
G(u; f, w_1) = \frac{\gamma u + \eta w_1 - \eta \gamma \nabla f(u)}{\eta + \gamma}.
\]

It is not difficult to show that \( w_{t+1} = \text{Proj}_Q[G(w_t; f_t, w_1)] \), where \( \text{Proj}_Q[\cdot] \) denotes the projection operator. Due to non-expansive of the projection operator, it suffices to bound \( \|G(w_t; f_t, w_1) - G(w'_t; f'_t, w'_1)\| \). Let us consider two scenarios. The first scenario is \( f_t = f'_t = f \) (using the same data). Then

\[
\|G(w_t; f, w_1) - G(w'_t; f', w'_1)\| = \left\| \frac{\gamma w_t + \eta w_1 - \eta \gamma \nabla f(w_t)}{\eta + \gamma} - \frac{\gamma w'_t + \eta w'_1 - \eta \gamma \nabla f(w'_t)}{\eta + \gamma} \right\|
\]

\[
\leq \frac{\eta}{\eta + \gamma} \|w_1 - w'_1\| + \frac{\gamma}{\eta + \gamma} \|w_t - \eta \nabla f(w_t) - w'_t + \eta \nabla f(w'_t)\|
\]

\[
\leq \frac{\eta}{\eta + \gamma} \|w_1 - w'_1\| + \frac{\gamma}{\eta + \gamma} \|w_t - w'_t\| = \frac{\eta}{\eta + \gamma} \delta_1 + \frac{\gamma}{\eta + \gamma} \delta_t,
\]

where we use the result in Lemma 3. Since \( \eta_k \leq \frac{\gamma_0 \alpha}{2\sigma} \) and \( T_k \eta_h \geq 1.5/\mu \) and \( \gamma_k \geq 1.5/\mu \), we have

\[
E[F(w_k) - F(w_*)] \leq \epsilon_k
\]

By induction, after \( K = \lceil \log(\epsilon_0/\epsilon) \rceil \) stages, we have

\[
E[F(w_K) - F(w_*)] \leq \epsilon
\]

The total iteration complexity is \( \sum_{k=1}^{K} T_k = O(1/(\mu \epsilon)). \)

\[\Box\]
where last inequality is due to 1-expansive of GD update with \( \eta \leq 2/L \) for a convex function (Hardt et al., 2016). Next, let us consider the second scenario \( f_t \neq f_t' \). Then

\[
\| G(w_t; f, w_1) - G(w_t'; f', w_1') \| = \left\| \frac{\gamma w_t + \eta w_1 - \eta \gamma \nabla f(w_t)}{\eta + \gamma} - \frac{\gamma w_t' + \eta w_1' - \eta \gamma \nabla f'(w_t')}{\eta + \gamma} \right\|
\]

\[
\leq \frac{\eta}{\eta + \gamma} \| w_1 - w_1' \| + \frac{\gamma}{\eta + \gamma} \| w_t - \eta \nabla f(w_t) - w_t' + \eta \nabla f'(w_t') \|
\]

\[
\leq \frac{\eta}{\eta + \gamma} \| w_1 - w_1' \| + \frac{\gamma}{\eta + \gamma} \| w_t - w_t' \| + \frac{2\eta \gamma G}{\eta + \gamma} = \frac{\eta}{\eta + \gamma} \delta_1 + \frac{\gamma}{\eta + \gamma} \delta_t + \frac{2\eta \gamma G}{\eta + \gamma}.
\]

Based on the above result, we can establish the uniform stability of START.

**Theorem 6** After \( K \) stages, START satisfies uniform stability with

\[
\varepsilon_{stab} \leq \frac{2\gamma G^2 \sum_{k=1}^{K} (1 - \frac{\eta}{\eta + \gamma})^T_k}{n}.
\]

**Proof** By applying the result in Lemma 5 to the \( k \)-th stage, omitting \( k \) in the notation, for \( t \geq 1 \) we have

\[
E[\delta_{t+1}] \leq (1 - 1/n) \left( \frac{\eta}{\eta + \gamma} E[\delta_1] + \frac{\gamma}{\eta + \gamma} E[\delta_t] \right) + \frac{1}{n} \left( \frac{\eta}{\eta + \gamma} E[\delta_1] + \frac{\gamma}{\eta + \gamma} E[\delta_t] + \frac{2\eta \gamma G}{\eta + \gamma} \right)
\]

\[
= \frac{\eta}{\eta + \gamma} E[\delta_1] + \frac{\gamma}{\eta + \gamma} E[\delta_t] + \frac{1}{n} \frac{2\eta \gamma G}{\eta + \gamma}
\]

\[
= \frac{\eta}{\eta + \gamma} E[\delta_1] \sum_{\tau=0}^{t-1} \left( \frac{\gamma}{\eta + \gamma} \right)^\tau + \left( \frac{\gamma}{\eta + \gamma} \right)^t E[\delta_t] + \frac{1}{n} \frac{2\eta \gamma G}{\eta + \gamma} \sum_{\tau=0}^{t-1} \left( \frac{\gamma}{\eta + \gamma} \right)^\tau
\]

\[
= \frac{\eta}{\eta + \gamma} E[\delta_1] \left( 1 - \left( \frac{\gamma}{\eta + \gamma} \right)^t \right) + \left( \frac{\gamma}{\eta + \gamma} \right)^t E[\delta_t] + \frac{1}{n} \frac{2\eta \gamma G}{\eta + \gamma} \left( 1 - \left( \frac{\gamma}{\eta + \gamma} \right)^t \right)
\]

\[
= E[\delta_1] \left( \frac{2\gamma G(1 - (\frac{\gamma}{\eta + \gamma})^t)}{n} \right).
\]

Then,

\[
E \left[ \sum_{t=1}^{T} \delta_{t+1}/T \right] \leq E[\delta_1] + \frac{2\gamma G(1 - (\frac{\gamma}{\eta + \gamma})^T)}{n}.
\]

For the \( k \)-stage, we have \( w_k = \sum_{t=1}^{T} w_{t+1}/T \) and \( w_{k-1} = w_1 \). Then

\[
E[\delta_k] \leq E[\delta_{k-1}] + \frac{2\gamma G(1 - (\frac{\gamma}{\eta + \gamma})^T_k)}{n},
\]

where \( \delta_k = \| w_k - w_k' \| \). By summing the above inequality for \( K \) stages and noting that \( \sup_z E[A(f(w_K, z) - f(w_K', z)) \leq G \| w_K - w_K' \| \), we prove the theorem.
4.3. Put them Together

Finally, we have the following testing error bound of $w_K$ returned by START.

**Theorem 7** After $K = \log(\epsilon_0/\epsilon)$ stages with a total number of iterations $T = \frac{18\sigma^2}{\alpha\mu}$. The testing error of $w_K$ is bounded by

$$
E_{A,S}[F(w_K)] \leq E[F_S(w^*_S)] + \epsilon + \frac{3G^2\log(\epsilon_0/\epsilon)}{n\mu}.
$$

**Remark:** Let $\epsilon = \frac{1}{n\mu}$, the excess risk bound becomes $O(\log(n\mu)/(n\mu))$ and the total iteration complexity is $T = O(nL)$. This improves the convergence of testing error of SGD stated in Corollary 4 for the convex case when $\mu \ll 1$, which needs $T = O(nL/\mu)$ iterations and has a testing error bound of $O(L\log(nL/\mu)/(n\mu))$.

5. **START for a Non-Convex Function Satisfying PL Condition**

Next, we will establish faster convergence of START than SGD for “nice-behaved” non-convex functions. In particular, we will consider a class of non-convex functions that is almost convex, namely weakly convex. One might also extend our analysis to other nice-behaved non-convex functions, e.g., one-point convexity with respect to a global minimum (Kleinberg et al., 2018; Li and Yuan, 2017) under a slightly stronger condition that the inequality in Lemma 3 holds for the global minimum in the one-point convexity condition.

**Definition 6 (Weakly Convex)** A non-convex function $F$ is called $\rho$-weakly quasi convex for $\rho > 0$ if $F(w) + \frac{\rho}{2}\|w\|^2$ is convex.

The considered “nice-behaved” non-convex function belongs to the class of weakly convex functions with $\rho \leq \mu/4$.

5.1. Convergence of Optimization Error

**Lemma 7** Assume $F_S$ is $\rho$-weakly convex. By applying SGD to $F_k = F_{k-1}^S$ with $\gamma \leq 1/\rho$, $\eta \leq 1/L$ and $w_k = \hat{w}_T$, for any $w \in \Omega$, we have

$$
E_k[F_k(w_k) - F_k(w)] \leq \sigma^2\eta_k + \frac{\|w_{k-1} - w\|^2}{2\eta_kT_k} + \frac{\|w_{k-1} - w\|^2}{2\gamma T_k}.
$$

**Remark:** From the above result, we can see that $\gamma$ can not be too large.

**Theorem 8** Suppose Assumption 1 holds, and $F_S(w)$ is $\rho$-weakly convex with $\rho \leq \mu/4$. Then by setting $\eta_k = \frac{\alpha_k\sigma}{2\sigma^2} \leq 1/L$ and $T_k = \frac{4\sigma^2}{\mu\sigma^2}$ and $\gamma = 4/\mu$, where $\alpha \leq \min(1, \frac{2\sigma^2}{\epsilon_0L})$, and after $K = \log(\epsilon_0/\epsilon)$ stages we have

$$
E[F(w_K) - F(w^*_S)] \leq \epsilon.
$$

The total iteration complexity is $O(\frac{1}{\mu})$. 

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Proof We will prove by induction that $E[F(w_k) - F(w_*)] \leq \epsilon_k$, where $\epsilon_k = \epsilon_0/2^k$, which is true for $k = 0$ by the assumption. By applying Lemma 4 to the $k$-th stage, for any $w \in \Omega$

$$E_k[F(w_k) - F(w)] \leq \frac{||w_{k-1} - w||^2}{2\gamma} + \eta_k \sigma^2 + \frac{||w_{k-1} - w||^2}{2\eta k T_k} + \frac{||w_{k-1} - w||^2}{2\gamma T_k}. \quad (6)$$

By plugging $w = w_*$ into the above inequality we have

$$E[F(w_k) - F(w_*))] \leq \frac{E(||w_{k-1} - w_*||^2)}{2\gamma k} + \eta_k \sigma^2 + \frac{||w_{k-1} - w_*||^2}{2\eta k T_k} + \frac{||w_{k-1} - w_*||^2}{2\gamma T_k} \leq \frac{E((F(w_{k-1}) - F(w_*)))}{4\mu \gamma} + \eta_k \sigma^2 + \frac{E((F(w_{k-1}) - F(w_*)))}{4\mu T_k} \leq \frac{\epsilon_{k-1}}{4\mu \gamma} + \eta_k \sigma^2 + \frac{1}{1/\eta k + 1/\gamma} \frac{\epsilon_{k-1}}{4\mu T_k},$$

where we use Lemma 3. By the setting $\eta_k = \frac{1}{\mu \sigma^2}$ and $T_k \eta_k = 1/\mu$ and $\gamma = 4/\mu$, we have

$$E[F(w_k) - F(w_*)] \leq \epsilon_k.$$

By induction, after $K = [\log(\epsilon_0/\epsilon)]$ stages, we have

$$E[F(w_K) - F(w_*)] \leq \epsilon.$$

The total iteration complexity is $\sum_{k=1}^K T_k = O(1/(\mu \epsilon)).$ 

5.2. Generalization Error

Similarly, we will first establish the recurrence of stability within one stage.

Lemma 8 Assume $f$ is $L$-smooth. Let $w_t$ denote the sequence learned on $S$ and $w'_t$ be the sequence learned on $S'$ by START at one stage, $\delta_t = ||w_t - w'_t||$. Then

$$\delta_{t+1} \leq \left\{ \begin{array}{ll} \frac{n}{n + \gamma} \delta_1 + \frac{\gamma (1+\eta L)}{n + \gamma} \delta_t & f_t = f'_t \\ \frac{n}{n + \gamma} \delta_1 + \frac{\gamma}{n + \gamma} \delta_t + \frac{2\eta G}{n + \gamma} & \text{otherwise} \end{array} \right.$$  

Proof Let us consider two scenarios. The first scenario is $f = f'$. Then

$$\|G(w_t; f, w_1) - G(w'_t; f', w'_1)\| = \left\| \gamma w_t + \eta w_1 - \eta \gamma \nabla f(w_t) - \gamma w'_t + \eta w'_1 - \eta \gamma \nabla f(w'_t) \right\| = \frac{\gamma (1+\eta L)}{n + \gamma} \delta_t.$$  

Next, let us consider the second scenario $f \neq f'$. Then

$$\|G(w_t; f, w_1) - G(w'_t; f', w'_1)\| = \left\| \gamma w_t + \eta w_1 - \eta \gamma \nabla f(w_t) - \gamma w'_t + \eta w'_1 - \eta \gamma \nabla f'_t \right\| = \frac{2\gamma G}{n + \gamma} \delta_t.$$  

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Next, we will apply the same analysis for the non-convex loss as in (Hardt et al., 2016). In particular, we will condition on \( w_{k-1} = w'_{k-1} \), i.e., the different example will be used within the last stage, and prove the bound for \( \|w_K - w'_K\| \), which leads to the following stability bound.

**Theorem 9** Let \( S_{K-1} = \sum_{k=1}^{K-1} T_k = \frac{4\sigma^2}{\mu e} \) and \( \eta_K \leq 1/(\mu T_K) \) Then we have

\[
\varepsilon_{\text{stab}} \leq \frac{S_{K-1}}{n} + \frac{1 + \mu/L}{n - 1} (2G^2/\mu)^{1/(1+L/\mu)} T_k^{\mu/L+1}/\mu.
\]

By putting the optimization error and generalization error together, we have the following testing error bound.

**Theorem 10** Under the same assumptions as in Theorem 8. After \( K = \log(\varepsilon_0/\epsilon) \) stages with a total number of iterations \( T = \frac{8\sigma^2}{\alpha \mu e} \). The testing error of \( w_K \) is bounded by

\[
E_{A,S}[F(w_K)] \leq E[F_S(w_*^S)] + \epsilon + \frac{4\sigma^2}{\eta \mu e} + \min \left\{ \frac{16G^2}{(n-1) \mu^{3/2} \alpha e}, \frac{8G^2 \sigma^2}{\eta \mu e}, \frac{8G^2 \sigma^2}{(n-1) \alpha \mu e} \right\}
\]

where \( \hat{G} = G^2/L \).

**Remark:** For simplicity of discussion, we consider \( \alpha = 1 \) due to small \( \mu \). When \( 2G/\sqrt{\mu} \leq e^{2\hat{G}} \), we can let \( \epsilon = \frac{G}{\sqrt{\mu} \alpha} \), the excess risk bound becomes \( O(1/(\sqrt{\mu} \mu^{3/4})) \) and the total iteration complexity is \( T = O(\sqrt{\mu} \mu^{1/4}) \). This improves the convergence of testing error of SGD stated in Corollary 4 for the non-convex case when \( \mu \ll 1 \), which needs \( T = O(\sqrt{\mu} \mu^{3/4}) \) iterations and suffers a testing error bound of \( O(1/(\sqrt{\mu} \mu^{5/4})) \). Similarly when \( 2L^2/\mu > e^{2\hat{G}} \), we can let \( \epsilon = \frac{G}{\sqrt{\mu} \alpha} \), the excess risk bound becomes \( O(1/(\sqrt{\mu} \mu^{3/4})) \) and the total iteration complexity is \( T = O(\sqrt{\mu} \mu^{1/4}) \). This improves the testing error bound of SGD stated in Corollary 4 for the non-convex case when \( \mu \ll 1 \), which needs \( T = O(\sqrt{\mu} \mu^{3/4}) \) iterations and suffers a testing error bound of \( O(1/(\sqrt{\mu} \mu^{3/4})) \).

### 5.3. Even Better Bound Under Individual Weak Convexity

We can establish a better testing error bound when individual loss function \( f(w,z) \) is \( \rho \)-weakly convex with \( \rho \leq \mu/4 \). The optimization error convergence remains the same as in Theorem 8. The improvement lies on the generalization error. Let us first consider the growth of stability within one stage.

**Lemma 9** Assume \( f \) is \( \rho \)-weakly convex with \( \rho \leq \gamma \). Let \( w_t \) denote the sequence learned on \( S \) and \( w'_t \) be the sequence learned on \( S' \) by START at one stage, \( \delta_t = \|w_t - w'_t\| \). Let \( \zeta = \frac{\gamma}{\eta + \gamma} (\gamma^{-1} + L)(\gamma^{-1} - \rho) \). If \( \eta \zeta \leq 1 \), then

\[
\delta_{t+1} \leq \left\{ \begin{array}{ll} \frac{\eta}{\eta + \gamma} \delta_t + (1 - \eta \zeta) \delta_t & f_t = f'_t \\ \frac{\eta}{\eta + \gamma} \delta_t + (1 - \eta \zeta) \delta_t + \frac{2\rho G}{\eta \gamma} & \text{otherwise} \end{array} \right.
\]
**Proof** Let us consider two scenarios. The first scenario is $f_t = f'_t = f$ (using the same data). Then

$$
\|G(w_t; f, w_1) - G(w'_t; f', w'_1)\| = \left\| \frac{\gamma w_t + \eta w_1 - \eta \gamma \nabla f(w_t)}{\eta + \gamma} - \frac{\gamma w'_t + \eta w'_1 - \eta \gamma \nabla f(w'_t)}{\eta + \gamma} \right\|
$$

$$
\leq \frac{\eta}{\eta + \gamma} \|w_1 - w'_1\| + \left\| w_t - \frac{\eta \gamma}{\eta + \gamma} (\nabla f(w_t) + w_t/\gamma) - w'_t + \frac{\eta \gamma}{\eta + \gamma} (\nabla f(w'_t) + w'_t/\gamma) \right\|
$$

Let $f(w) = f(w) + \frac{1}{2\kappa} \|w\|^2$, which is $L + \gamma^{-1}$-smooth and $(\gamma^{-1} - \rho)$-strongly convex due to the weak convexity of $f$. The term $G_{w,f} = w - \frac{\eta \gamma}{\eta + \gamma} (\nabla f(w) + w/\gamma)$ can be considered as an update of GD for function $\hat{f}(w)$ with step size $\frac{\eta \gamma}{\eta + \gamma}$. According to Lemma 3.7 in (Hardt et al., 2016), it is $(1 - \frac{\eta \gamma}{\eta + \gamma} (\gamma^{-1} + L)(\gamma^{-1} - \rho))$-expansive when $\frac{\eta \gamma}{\eta + \gamma} \leq \frac{2}{2\gamma^{-1} + L - \rho}$. Let $\zeta = \frac{\gamma}{\eta + \gamma} \frac{(\gamma^{-1} + L)(\gamma^{-1} - \rho)}{2\gamma^{-1} + L - \rho}$. Then we have

$$
\|G(w_t; f, w_1) - G(w'_t; f', w'_1)\| = \left\| \frac{\gamma w_t + \eta w_1 - \eta \gamma \nabla f(w_t)}{\eta + \gamma} - \frac{\gamma w'_t + \eta w'_1 - \eta \gamma \nabla f(w'_t)}{\eta + \gamma} \right\|
$$

$$
\leq \frac{\eta}{\eta + \gamma} \|w_1 - w'_1\| + (1 - \eta \zeta) \|w_t - w'_t\| \leq \frac{\eta}{\eta + \gamma} \delta_1 + (1 - \eta \zeta) \delta_t.
$$

Similarly when $f_t \neq f'_t$, we have

$$
\|G(w_t; f, w_1) - G(w'_t; f', w'_1)\| = \left\| \frac{\gamma w_t + \eta w_1 - \eta \gamma \nabla f(w_t)}{\eta + \gamma} - \frac{\gamma w'_t + \eta w'_1 - \eta \gamma \nabla f(w'_t)}{\eta + \gamma} \right\|
$$

$$
\leq \frac{\eta}{\eta + \gamma} \|w_1 - w'_1\| + \left\| w_t - \frac{\eta \gamma}{\eta + \gamma} (\nabla f(w_t) + w_t/\gamma) - w'_t + \frac{\eta \gamma}{\eta + \gamma} (\nabla f'(w'_t) + w'_t/\gamma) \right\|
$$

$$
\leq \frac{\eta}{\eta + \gamma} \|w_1 - w'_1\| + (1 - \eta \zeta) \|w_t - w'_t\| + \frac{2 \eta \gamma G}{\eta + \gamma} = \frac{\eta}{\eta + \gamma} \delta_1 + (1 - \eta \zeta) \delta_t + \frac{2 \eta \gamma G}{\eta + \gamma}.
$$

\]

Next, we prove the growth of stability across stages.

**Theorem 11** Under the same assumptions as in Theorem 8. By running START with $\gamma = 4/\mu$, $\eta_k = \frac{\epsilon_0 \alpha}{2\sigma^2}$ and $T_k = \frac{2\sigma^2}{\mu \epsilon_0}$ with $\alpha \leq \frac{4\sigma^2}{\epsilon_0(2\gamma^{-1} + L - \rho)}$. After $K = \log(\epsilon_0/\epsilon)$ stages with a total number of iterations $T = O(\frac{1}{\mu \epsilon})$. The testing error of $w_K$ is bounded by

$$
E_{A,S}[F(w_K)] \leq E[F_S(w_0)] + \epsilon + \frac{2 \epsilon_0 G^2}{\eta \mu \epsilon}.
$$

**Remark:** By setting $\epsilon = \frac{G^2}{\sqrt{n \mu}}$, the excess risk bound becomes $O(1/\sqrt{n \mu})$ with a total iteration complexity of $T = O(\sqrt{n})$. Th above testing error bound becomes better than that in Theorem 10 when $1/\mu$ and $\epsilon G$ are very large.
Proof By applying the result in Lemma 5 to the $k$-th stage, omitting $k$ in the notation, for $t \geq 1$ we have

\[
E[\delta_{t+1}] \leq (1 - 1/n) \left( \frac{\eta}{\eta + \gamma} E[\delta_1] + (1 - \eta \zeta) E[\delta_t] \right) + \frac{1}{n} \left( \frac{\eta}{\eta + \gamma} E[\delta_1] + (1 - \eta \zeta) E[\delta_t] + \frac{2\eta \gamma G}{\eta + \gamma} \right)
\]

\[
= \frac{\eta}{\eta + \gamma} E[\delta_1] + (1 - \eta \zeta) E[\delta_t] + \frac{1}{n} \frac{2\eta \gamma G}{\eta + \gamma}
\]

\[
= \frac{\eta}{\eta + \gamma} E[\delta_1] + (1 - \eta \zeta)^t E[\delta_t] + \frac{1}{n} \frac{2\eta \gamma G}{\eta + \gamma} \sum_{\tau=0}^{t-1} (1 - \eta \zeta)^\tau
\]

\[
= \frac{\eta t}{\eta + \gamma} E[\delta_1] + (1 - \eta \zeta)^t E[\delta_t] + \frac{1}{n} \frac{2\eta \gamma G}{\eta + \gamma}
\]

Since $\eta t \leq 1/\mu$ and $\gamma = 4/\mu$, we have

\[
E[\delta_{t+1}] \leq 2E[\delta_1] + \frac{1}{n} \frac{2G}{\mu}.
\]

By applying the above result to the $k$-th stage, we have

\[
E[\delta_k] \leq 2E[\delta_{k-1}] + \frac{1}{n} \frac{2G}{\mu} \leq \frac{1}{n} \frac{2G}{\mu} \sum_{j=0}^{k-1} 2^j \leq \frac{1}{n} \frac{2^{k+1}G}{\mu},
\]

where we use the fact $\delta_0 = 0$. Hence, after $K = \log(\epsilon_0/\epsilon)$ stages, we have

\[
E[\delta_K] \leq \frac{1}{n} \frac{2\epsilon_0 G}{\mu \epsilon}.
\]

6. Results under a Local PL Condition for Non-convex loss

In this section, we extend the results to the case that the PL condition only holds in a local region of a global minimizer. In particular, we consider unconstrained ERM problem:

\[
\min_{w \in \mathbb{R}^d} F_S(w) := \frac{1}{n} \sum_{i=1}^{n} f(w, z_i), \quad (7)
\]

All assumptions in Assumption 1 are still imposed expect that the global PL condition is replaced by the following local PL condition.

**Definition 10** $F_S$ is said to satisfy a local PL condition, if there exists $\theta > 0$ such that for all $w$ satisfying $\|\nabla F_S(w)\|^2 \leq \theta$ the following inequality holds

\[
2\mu (F_S(w) - \min_w F_S(w)) \leq \|\nabla F_S(w)\|^2.
\]
Stagewise Training Accelerates Convergence of Testing Error Over SGD

Algorithm 3 Two-phase Start under a Local PL Condition

1: \[ w^{(1)} = \text{SGD}(F_{w_0}^\infty, w_0, \eta, T_1) \] with \( w^{(1)}, \eta, T_1 \) specified in Proposition 1

2: \[ w^{(2)} = \text{Start}(F_S, w^{(1)}, 1/(4\mu), K_2) \] with \( w^{(2)}, \eta_k, T_k \) specified in Theorem 8

By Lemma 3, we can see that all points in this region satisfying \( \| w - w^*_S \| \leq \frac{\sqrt{\theta}}{2\mu} \). We will also assume the \( \rho \)-weak-convexity only holds in this local region with \( \rho \leq \mu/4 \).

Under the local PL condition, we can consider a two-phase algorithm, i.e., in the first phase the algorithm will drive the gradient of \( F_S(\cdot) \) to be small enough so that the solution enters into the local region. Once the solution enters into the local region satisfying the PL condition, the algorithm presented in previous section can be employed. How to drive the gradient to be small for the first stage has been considered in many previous works (Ghadimi and Lan, 2013a; Yang et al., 2016; Davis and Drusvyatskiy, 2018; Chen et al., 2018; Lan and Yang, 2018; Allen-Zhu, 2017; Chen and Yang, 2018; Allen-Zhu and Hazan, 2016; Reddi et al., 2016c,a). We consider the simple SGD with a constant step size (Ghadimi and Lan, 2013a)

**Proposition 1** Under Assumption 1(i) and (ii), by running Algorithm 2 with \( \gamma = \infty \) and \( \eta \leq 1/L \)

\[
\| \nabla F_S(x_\tau) \| ^2 \leq \frac{2\epsilon_0}{\eta T} + L\sigma^2 \eta,
\]

where \( \tau \) is randomly selected from \( \{1, \ldots, T\} \). By setting \( \eta = \min\left(\frac{\theta \delta}{2L\sigma^2}, \frac{1}{T}\right) \), \( T = \frac{4\epsilon_0}{\eta \theta} \), then with probability \( 1 - \delta \) we have

\[
\| \nabla F_S(x_\tau) \| ^2 \leq \theta.
\]

The above result implies that with a total number of iterations \( T = O(1/\theta^2) \), we will have \( \| \nabla F_S(x_\tau) \| ^2 \leq \theta \) with large probability (e.g., 0.9). It is notable that we can promote the above result into a high probability result with a logarithmic dependence on \( 1/\delta \) by using the boosting technique presented in (Ghadimi and Lan, 2013b), which requires a light-tail assumption about the stochastic gradient. Finally, we have the following testing error bound of the two-phase Algorithm 1.

**Theorem 12** Suppose Assumption 1(i) and (ii) and a local PL condition holds such that \( F_S \) is \( \rho \)-weakly convex with \( \rho < \mu/4 \) in the local region. Let Algorithm 3 run with \( T_1 = C/\theta^2 \) iterations for the first-phase SGD where \( C \) is a large constant, and \( K_2 = \Theta(\log(1/\epsilon)) \) stages for the second-phase Start, and an overall total number of iterations \( T = O\left(\frac{1}{\theta^2} + \frac{1}{\mu^3}\right) \). When \( n \geq T \) is sufficiently large, with large probability (over randomness in the algorithm) the testing error of \( w^{(2)} \) is bounded by

\[
E_{\mathcal{A}, S}[F(w^{(2)})] \leq E[F_S(w^*_S)] + \epsilon + O\left(\frac{1}{n\mu\epsilon} + \frac{1}{n\theta^2} + \min \left\{ \frac{16G\sigma^2}{(n-1)\mu^3/2\epsilon}, \frac{8e^{2G}\sigma^2}{(n-1)\mu\epsilon} \right\} \right),
\]

where \( \hat{G} = G^2/L \).
Remark: Consider the large condition number setting $2G/\sqrt{\mu} \geq e^{2\hat{G}}$, we can set $\epsilon = 1/(\sqrt{n\mu}^{3/4})$. The excess risk bound becomes $O(\frac{1}{m\sigma^2} + \frac{1}{\sqrt{n\mu}^{3/4}})$ with a total iteration complexity of $O(\frac{1}{\sigma^2} + \frac{\sqrt{n}}{\mu \gamma^2})$.

7. Experiments

In this section, we present some preliminary results on deep learning. We compare four algorithms, SGD with a decreasing step size proportional to $1/\sqrt{t}$, SGD with a decreasing step size proportional to $1/t$, SGD with stagewise geometric decreasing step size (the heuristic approach used in practice), and our START. For all algorithms, we tune their initial step size to obtain the best performance. We also tune the regularization parameter $1/\gamma$ in the range $0.0001 \sim 0.1$ to obtain the best performance on the testing error. We conduct experiments on two datasets CIFAR-10 and CIFAR-100 using two neural network structures, namely ResNet20 and ResNet56 (He et al., 2016). For stagewise SGD and START, we use the same stagewise step size strategy as in (He et al., 2016), i.e., the step size is decreased by 10 at 40k, 60k iterations. The training error, testing error and generalization error are shown in Figure 1. We can see that SGD with a decreasing step size converges slowly, especially SGD with a step size proportional to $1/t$. It is because that the initial step size of SGD ($c/t$) is selected as a small value. We observe that using a large step size it cannot lead to convergence. It is suspected that the PL condition might not hold at the beginning. Hence, we also implement another variant of SGD. We first run SGD with a small constant step size $c$ (the same as the initial step size used in stagewise SGD) and the same number of iterations as that for the first stage of stagewise SGD to obtain a better solution and then switch to the decreasing step size $c/t$, as denoted by SGD ($c\rightarrow c/t$) in the legend. We can see that it still converges slower than stagewise SGD and START. The proposed START performs closely to the stagewise SGD used in practice, but has a slight improvement on the convergence of testing error, which justifies the small regularization added at each stage in our algorithm.

8. Conclusion

In this paper, we have analyzed the convergence of training error and testing error of a stagewise regularized training algorithm for solving empirical risk minimization under the Polyak-Łojasiewicz condition. Our theoretical analysis exhibits why stagewise learning usually yields faster convergence than vanilla SGD with a continuously decreasing step size on both training and testing error for an empirical risk that is close to a convex function. One might also extend our analysis to other nice-behaved non-convex functions, e.g., one-point convexity.

References

Zeyuan Allen-Zhu. Natasha: Faster non-convex stochastic optimization via strongly non-convex parameter. In Proceedings of the 34th International Conference on Machine Learning (ICML), pages 89–97, 2017.

3. 0.5 for ResNet20, 0.5 for ResNet56 on CIFAR-100, and 0.5 for ResNet56 for CIFAR-10.
Figure 1: Training error, testing error and their absolute difference (i.e., \(|\text{training error} - \text{testing error}|\)) for learning ResNets on CIFAR-10 and CIFAR-100 by different algorithms.

Zeyuan Allen-Zhu and Elad Hazan. Variance reduction for faster non-convex optimization. In Proceedings of the 33rd International Conference on Machine Learning (ICML), pages 699–707, 2016. URL http://jmlr.org/proceedings/papers/v48/allen-zhua16.html.

Anonymous. Rethinking learning rate schedules for stochastic optimization. In Submitted to International Conference on Learning Representations, 2019. URL https://openreview.net/forum?id=HJePy3RcF7. under review.

Raef Bassily, Mikhail Belkin, and Siyuan Ma. On exponential convergence of sgd in non-convex over-parametrized learning. CoRR, abs/1811.02564, 2018.

Jerome Bolte, Trong Phong Nguyen, Juan Peypouquet, and Bruce Suter. From error bounds to the complexity of first-order descent methods for convex functions. CoRR,
abs/1510.08234, 2015.

Olivier Bousquet and André Elisseeff. Stability and generalization. J. Mach. Learn. Res., 2:499–526, March 2002. ISSN 1532-4435. doi: 10.1162/153244302760200704. URL https://doi.org/10.1162/153244302760200704.

Zachary Charles and Dimitris Papailiopoulos. Stability and generalization of learning algorithms that converge to global optima. In Jennifer Dy and Andreas Krause, editors, Proceedings of the 35th International Conference on Machine Learning (ICML), volume 80 of Proceedings of Machine Learning Research, pages 745–754, Stockholmsmässan, Stockholm Sweden, 10–15 Jul 2018.

Zaiyi Chen and Tianbao Yang. A variance reduction method for non-convex optimization with improved convergence under large condition number. CoRR, abs/1809.06754, 2018.

Zaiyi Chen, Tianbao Yang, Jinfeng Yi, Bowen Zhou, and Enhong Chen. Universal stage-wise learning for non-convex problems with convergence on averaged solutions. CoRR, /abs/1808.06296, 2018.

Damek Davis and Dmitriy Drusvyatskiy. Stochastic subgradient method converges at the rate $o(k^{-1/4})$ on weakly convex functions. CoRR, /abs/1802.02988, 2018.

Saeed Ghadimi and Guanghui Lan. Stochastic first- and zeroth-order methods for nonconvex stochastic programming. SIAM Journal on Optimization, 23(4):2341–2368, 2013a.

Saeed Ghadimi and Guanghui Lan. Stochastic first-and zeroth-order methods for nonconvex stochastic programming. SIAM Journal on Optimization, 23(4):2341–2368, 2013b.

Moritz Hardt and Tengyu Ma. Identity matters in deep learning. CoRR, abs/1611.04231, 2016.

Moritz Hardt, Ben Recht, and Yoram Singer. Train faster, generalize better: Stability of stochastic gradient descent. In Proceedings of the 33rd International Conference on Machine Learning (ICML), pages 1225–1234, 2016.

Elad Hazan and Satyen Kale. Beyond the regret minimization barrier: an optimal algorithm for stochastic strongly-convex optimization. In Proceedings of the 24th Annual Conference on Learning Theory (COLT), pages 421–436, 2011.

Elad Hazan, Amit Agarwal, and Satyen Kale. Logarithmic regret algorithms for online convex optimization. Machine Learning, 69(2-3):169–192, 2007.

Kaiming He, Xiangyu Zhang, Shaoqing Ren, and Jian Sun. Deep residual learning for image recognition. In CVPR, pages 770–778. IEEE Computer Society, 2016.

Hamed Karimi, Julie Nutini, and Mark W. Schmidt. Linear convergence of gradient and proximal-gradient methods under the Polyak-Łojasiewicz condition. In Machine Learning and Knowledge Discovery in Databases - European Conference (ECML-PKDD), pages 795–811, 2016.
Stagewise Training Accelerates Convergence of Testing Error Over SGD

Bobby Kleinberg, Yuanzhi Li, and Yang Yuan. An alternative view: When does SGD escape local minima? In Proceedings of the 35th International Conference on Machine Learning, pages 2698–2707, 2018.

Alex Krizhevsky, Ilya Sutskever, and Geoffrey E. Hinton. Imagenet classification with deep convolutional neural networks. In Advances in Neural Information Processing Systems (NIPS), pages 1106–1114, 2012.

Ilja Kuzborskij and Christoph H. Lampert. Data-dependent stability of stochastic gradient descent. In Proceedings of the 35nd International Conference on Machine Learning (ICML), volume 80 of JMLR Workshop and Conference Proceedings, pages 2820–2829. JMLR.org, 2018.

Guanghui Lan and Yu Yang. Accelerated stochastic algorithms for nonconvex finite-sum and multi-block optimization. CoRR, abs/1805.05411, 2018.

Lihua Lei, Cheng Ju, Jianbo Chen, and Michael I. Jordan. Non-convex finite-sum optimization via SCG methods. In Advances in Neural Information Processing Systems 30 (NIPS), pages 2345–2355, 2017.

Yuanzhi Li and Yang Yuan. Convergence analysis of two-layer neural networks with relu activation. In Advances in Neural Information Processing Systems 30 (NIPS), pages 597–607, 2017.

Arkadi Nemirovski, Anatoli Juditsky, Guanghui Lan, and Alexander Shapiro. Robust stochastic approximation approach to stochastic programming. SIAM Journal on Optimization, 19:1574–1609, 2009. URL http://dx.doi.org/10.1137/070704277.

Yurii Nesterov. Introductory lectures on convex optimization : a basic course. Applied optimization. Kluwer Academic Publ., 2004. ISBN 1-4020-7553-7.

B. T. Polyak. Gradient methods for minimizing functionals. Zh. Vychisl. Mat. Mat. Fiz., 3:4:864?878, 1963.

Chao Qu, Huan Xu, and Chong Ong. Fast rate analysis of some stochastic optimization algorithms. In Maria Florina Balcan and Kilian Q. Weinberger, editors, Proceedings of The 33rd International Conference on Machine Learning, volume 48 of Proceedings of Machine Learning Research, pages 662–670, New York, New York, USA, 20–22 Jun 2016. PMLR. URL http://proceedings.mlr.press/v48/qua16.html.

Sashank J. Reddi, Ahmed Hefny, Suvrit Sra, Barnabás Póczós, and Alex Smola. Stochastic variance reduction for nonconvex optimization. In Proceedings of the 33rd International Conference on International Conference on Machine Learning (ICML), pages 314–323. JMLR.org, 2016a.

Sashank J. Reddi, Ahmed Hefny, Suvrit Sra, Barnabas Poczos, and Alex Smola. Stochastic variance reduction for nonconvex optimization. In Proceedings of The 33rd International Conference on Machine Learning (ICML), volume 48, pages 314–323, 2016b.
Sashank J. Reddi, Suvrit Sra, Barnabás Póczos, and Alexander J. Smola. Fast incremental method for smooth nonconvex optimization. In 55th IEEE Conference on Decision and Control (CDC), pages 1971–1977, 2016c.

Bo Xie, Yingyu Liang, and Le Song. Diversity leads to generalization in neural networks. CoRR, abs/1611.03131, 2016. URL http://arxiv.org/abs/1611.03131.

Yi Xu, Qihang Lin, and Tianbao Yang. Stochastic convex optimization: Faster local growth implies faster global convergence. In Proceedings of the 34th International Conference on Machine Learning (ICML), pages 3821 – 3830, 2017.

Yi Xu, Qi Qi, Qihang Lin, Rong Jin, and Tianbao Yang. Stochastic optimization for dc functions and non-smooth non-convex regularizers with non-asymptotic convergence. arXiv preprint arXiv:1811.11829, 2018.

Tianbao Yang, Qihang Lin, and Zhe Li. Unified convergence analysis of stochastic momentum methods for convex and non-convex optimization. volume abs/1604.03257, 2016.

Peilin Zhao and Tong Zhang. Stochastic optimization with importance sampling for regularized loss minimization. In Proceedings of the 32nd International Conference on Machine Learning (ICML), pages 1–9, 2015.

Yi Zhou and Yingbin Liang. Characterization of gradient dominance and regularity conditions for neural networks. CoRR, abs/1710.06910, 2017.

Yi Zhou, Yingbin Liang, and Huishuai Zhang. Generalization error bounds with probabilistic guarantee for SGD in nonconvex optimization. CoRR, abs/1802.06903, 2018.

Proof of Theorem 2

We combine the proof of Theorem 3.8 and the result of Lemma 3.11 in Hardt et al. (2016). For applying Theorem 3.8, we need to have \( \eta_t \leq 1/L \), i.e., \( \frac{2t+1}{2\mu(t+1)^2} \leq 1/L \). Let us define \( t_0 = \frac{L}{\mu} \). Then \( \eta_t \leq 1/L, \forall t \geq t_0 \). Then conditioned on \( \delta_{t_0} = 0 \), we apply Lemma 3.11 in Hardt et al. (2016) and have

\[
\varepsilon_{stab} \leq \frac{t_0}{n} + GE[\delta_T | \delta_{t_0} = 0] \leq \frac{t_0}{n} + \frac{2G^2}{n} \sum_{t=t_0}^T \eta_t \\ \leq \frac{t_0}{n} + \frac{2G^2}{n} \sum_{t=t_0}^T \frac{2t+1}{2\mu(t+1)^2} \leq \frac{L}{n\mu} + \frac{2G^2}{n\mu} \log(T+1). \tag{8}
\]

Next we consider the case when \( f(\cdot, z) \) is non-convex. By noting \( \eta_t \leq \frac{1/\mu}{T} \), we can directly applying their Theorem 3.12 of Hardt et al. (2016) and get

\[
\varepsilon_{stab} \leq \frac{1 + \frac{\mu}{T}}{n-1} \left( \frac{2G^2}{\mu} \right)^{\frac{1}{1+\mu}} \cdot T^{\frac{\mu/\mu - 1}{1+\mu}}.
\]
Proof of Theorem 4

Proof Based on the decomposition of testing error, the result of Theorem 1 and Theorem 2, we could upper bound the testing error by combining optimization error and generalization error together. For convex problems, we have

\[
E_{A,S}[F(w_T)] \leq E_S[F_S(w^*_S)] + \frac{LG^2}{2T\mu} + \frac{(L + 2G^2) \log(T + 1)}{n\mu}.
\]

For non-convex problems, we have

\[
E_{A,S}[F(w_T)] \leq E_S[F_S(w^*_S)] + \frac{1 + \frac{\mu}{n-1}}{2} \left( \frac{2G^2}{\mu} \right)^\frac{1}{\frac{n}{n+1}} T^\frac{n}{n+1}
\]

\[
\leq E_S[F_S(w^*_S)] + \frac{2}{n-1} \left( \frac{2G^2}{\mu} \right)^\frac{1}{\frac{n}{n+1}} T
\]

Let \( X = \frac{2G^2}{\mu} - 1 \), which is positive when \( \mu \) is very small. Given \((1 + X)^{1/X} \leq e\), we have

\[
\left( \frac{2G^2}{\mu} \right)^\frac{1}{\frac{n}{n+1}} = \left( \frac{2G^2}{\mu} \right)^{\frac{n}{n+1}} \leq e^{\frac{2G^2}{\mu}} \leq e^{\frac{2G^2}{\mu}}.
\]

We also have \( \left( \frac{2G^2}{\mu} \right)^\frac{1}{\frac{n}{n+1}} \leq \frac{2G}{\sqrt{n}} \) given that \( 2G^2 / \mu \geq 1 \) and \( L/\mu \geq 1 \) for small \( \mu \). Thus, we complete the proof.

Proof of Lemma 4

The proof of Lemma 4 follows the one of Lemma 2 in Xu et al. (2018). For completeness, we prove our result.

Recall that \( F_k = F_S(w) + \frac{1}{2\gamma} ||w - w_{k-1}||^2 \). Let \( r_k(w) = \frac{1}{2\gamma} ||w - w_{k-1}||^2 + \delta_{\Omega}(w) \), so \( F_k(w) = F_S(w) + r_k(w) \), where \( \delta_{\Omega}(\cdot) \) is the indicator function of \( \Omega \). Due to the convexity of \( F_S(w) \), the \( \frac{1}{\gamma} \)-strong convexity of \( r_k(w) \) and the \( L \)-smoothness of \( f(w; z) \), we have the following three inequalities

\[
F_S(w) \geq F_S(w_t) + \langle \nabla F_S(w_t), (w - w_t) \rangle \tag{9}
\]

\[
r_k(w) \geq r_k(w_{t+1}) + \langle \partial r_k(w_{t+1}), w - w_{t+1} \rangle + \frac{1}{2\gamma} ||w - w_{t+1}||^2
\]

\[
F_S(w_t) \geq F_S(w_{t+1}) - \langle \nabla F_S(w_t), w_{t+1} - w_t \rangle - \frac{L}{2} ||w_t - w_{t+1}||^2. \tag{10}
\]

Combining them together, we have

\[
F_S(w_{t+1}) + r_k(w_{t+1}) - (F_S(w) + r_k(w))
\]

\[
\leq \langle \nabla F_S(w_t) + \partial r_k(w_{t+1}), w_{t+1} - w \rangle + \frac{L}{2} ||w_t - w_{t+1}||^2 - \frac{1}{2\gamma} ||w - w_{t+1}||^2. \tag{11}
\]
Recall Line 3 of Algorithm 2, we update $w_{t+1}$ as follows

$$w_{t+1} = \arg\min_{w \in \mathbb{R}^d} \nabla f(w_t, z_{i_t})^\top w + \frac{1}{2\eta} ||w - w_t||^2 + r_k(w),$$

where $w_1$ is the initial point of the current stage, so the last term is in fact $r_k(w)$. If we set the gradient of the above problem in $w_{t+1}$ to 0, there exists $\partial r_k(w_{t+1})$ such that

$$\partial r_k(w_{t+1}) = -\nabla f(w_t, z_{i_t}) + \frac{1}{\eta}(w_t - w_{t+1}).$$

Plugging the above equation to (11), we have

$$F_S(w_{t+1}) + r_k(w_{t+1}) - (F_S(w) + r_k(w))$$

$$\leq \langle \nabla F_S(w_t) - \nabla f(w_t, z_{i_t}), w_{t+1} - w \rangle + \frac{1}{\eta}(w_t - w_{t+1})^\top(w_t - w_{t+1})$$

$$+ \frac{L}{2} ||w_t - w_{t+1}||^2 - \frac{1}{2\gamma} ||w - w_{t+1}||^2$$

$$= \langle \nabla F_S(w_t) - \nabla f(w_t, z_{i_t}), w_{t+1} - w \rangle + \frac{1}{\eta} ||w_t - w||^2$$

$$- \frac{1}{2\eta} ||w_t - w_{t+1}||^2 - \frac{1}{2\gamma} ||w - w_{t+1}||^2 + \frac{L}{2} ||w_t - w_{t+1}||^2 - \frac{1}{2\gamma} ||w - w_{t+1}||^2$$

$$\leq \langle \nabla F_S(w_t) - \nabla f(w_t, z_{i_t}), w_{t+1} - w \rangle + (\nabla F_S(w_t) - \nabla f(w_t, z_{i_t}), \hat{w}_{t+1} - w)$$

$$+ \frac{1}{\eta} ||w_t - w||^2 - \frac{1}{2\eta} ||w_{t+1} - w||^2 - \frac{1}{2\gamma} ||w - w_{t+1}||^2$$

$$\leq \eta ||\nabla F_S(w_t) - \nabla f(w_t, z_{i_t})||^2 + (\nabla F_S(w_t) - \nabla f(w_t, z_{i_t}), \hat{w}_{t+1} - w)$$

$$+ \frac{1}{\eta} ||w_t - w||^2 - \frac{1}{2\eta} ||w_{t+1} - w||^2 - \frac{1}{2\gamma} ||w - w_{t+1}||^2.$$

The first equality is due to

$$2(x - y, y - z) = ||x - z||^2 - ||x - y||^2 - ||y - z||^2$$

and

$$\hat{w}_{t+1} = \arg\min_{w \in \Omega} w^\top \nabla F_S(w) + \frac{1}{2\eta} ||w - w_t||^2 + \frac{1}{2\gamma} ||w - w_1||^2.$$ The second inequality is due to Cauchy-Schwarz inequality and setting $\eta \leq \frac{1}{2\gamma}$. The third inequality is due to Lemma 3 of Xu et al. (2018).

Taking expectation on both sides, we have

$$E[F_k(w_{t+1}) - F_k(w)] \leq \eta \sigma^2 + \frac{1}{2\eta} E[||w_t - w||^2] - \frac{1}{2\gamma} E[||w_{t+1} - w||^2] - \frac{1}{2\gamma} E[||w - w_{t+1}||^2],$$

where $E[||\nabla f(w, z_{i_t}) - \nabla F_S(w)||^2] \leq \sigma^2$ by assumption.

Taking summation of the above inequality from $t = 1$ to $T$, we have

$$\sum_{t=1}^{T} F_k(w_{t+1}) - F_k(w)$$

$$\leq \eta \sigma^2 T + \frac{1}{2\eta} E[||w_1 - w||^2] - \frac{1}{2\gamma} E[||w_{T+1} - w||^2] - \frac{1}{2\gamma} \sum_{t=1}^{T} E[||w - w_{t+1}||^2].$$
By employing Jensens’ inequality on LHS, denoting the output of the \(s\)-th stage by \(w_k = \hat{w}_T = \frac{1}{T} \sum_{t=1}^{T} w_t\) and taking expectation, we have

\[
E[F_k(\hat{w}_T) - F_k(w)] \leq \sigma^2 \eta + \frac{||w_1 - w||^2}{2\eta T}.
\]

**Proof of Lemma 7**

The proof of Lemma 7 follows the one of Lemma 4. The only difference lies on the weak convexity of \(F_S(w)\).

We could replace the first inequality in (9) by the following \(\rho\)-weak convexity condition of \(F_S(\cdot)\):

\[
F_S(w) \geq F_S(w_t) + \langle \nabla F_S(w_t), (w - w_t) \rangle - \frac{\rho}{2} ||w_t - w||^2.
\]

Then we combine it with other two inequalities as follows

\[
F_S(w_{t+1}) + r_k(w_{t+1}) - (F_S(w) + r_k(w)) \\
\leq \langle \nabla F_S(w_t) + \partial r_k(w_{t+1}), w_{t+1} - w \rangle + \frac{L}{2} ||w_t - w_{t+1}||^2 - \frac{1}{2\gamma} ||w - w_{t+1}||^2 + \frac{\rho}{2} ||w - w_t||^2
\]

Then following the proof of Lemma 4 under the condition \(\eta \leq 1/L\) we have

\[
F_k(w_{t+1}) - F_k(w) \\
\leq \langle \nabla F_S(w_t) - \nabla f(w_t, z_{it}), w_{t+1} - \hat{w}_{t+1} \rangle + \eta ||\nabla F_S(w_t) - \nabla f(w_t, z_{it})||^2 \\
+ \frac{1}{2\eta} ||w_t - w||^2 - \frac{1}{2\eta} ||w_{t+1} - w||^2 - \frac{1}{2\gamma} ||w - w_{t+1}||^2 + \frac{\rho}{2} ||w - w_t||^2
\]

Taking expectation on both sides, summing from \(t = 1\) to \(T\) and applying Jensen’s inequality, we have

\[
E[F_k(\hat{w}_T) - F_k(w)] \leq \eta \sigma^2 + \frac{1}{2T\eta} ||w - w_1||^2 + \frac{1}{2T\gamma} ||w - w_1||^2
\]