Hermann Hankel’s
“On the general theory of motion of fluids”
An essay including an English translation of the complete Preisschrift from 1861

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The present is a companion paper to “A contemporary look at Hermann Hankel’s 1861 pioneering work on Lagrangian fluid dynamics” by Frisch, Grimberg and Villone (2017). Here we present the English translation of the 1861 prize manuscript from Göttingen University “Zur allgemeinen Theorie der Bewegung der Flüssigkeiten” (On the general theory of the motion of the fluids) of Hermann Hankel (1839–1873), which was originally submitted in Latin and then translated into German by the Author for publication. We also provide the English translation of two important reports on the manuscript, one written by Bernhard Riemann and the other by Wilhelm Eduard Weber, during the assessment process for the prize. Finally we give a short biography of Hermann Hankel with his complete bibliography.

I. INTRODUCTORY NOTES

Here we present, with some supplementary documents, a full translation from German of the winning essay, originally written in Latin, by Hermann Hankel, in response to the “extraordinary mathematical prize” launched on 4th June 1860 by the Philosophical Faculty of the University of Göttingen with a deadline of end of March 1861. By request of the Prize Committee to Hankel, the essay was then revised by him and finally published in 1861, in German, as a Preisschrift. The Latin manuscript has been very probably returned to the Author and now appears to be irretrievably lost. The Göttingen University Library possesses two copies of the 1861 German original edition of the Preisschrift. Hankel participated in this prize competition shortly after his arrival in the Spring 1860 in Göttingen as a 21-years old student in Mathematics, from the University of Leipzig. The formulation of the prize (see section II) highlighted the problem of the equations of fluid motion in Lagrangian coordinates and was presented in memory of Peter Gustav Lejeune-Dirichlet. A post mortem paper of his had outlined the advantages of the use of the Lagrangian approach, compared to the Eulerian one, for the description of the fluid motion.

During the decision process for the winning essay, there was an exchange of German-written letters among the committee members. Notably, among all of these letters, the ones from Wilhelm Eduard Weber (1804–1891) and from Bernhard Riemann (1826–1866)
have been decisive for the evaluation of the essay. Hankel had been the only one to sub-
mmit an essay, and the discussion among the committee members was devoted to deciding
whether or not the essay written by Hankel deserved to win the prize. For completeness,
we provide an English translation of two of these letters, one signed by Bernhard Riemann,
the other by Wilhelm Eduard Weber. As is thoroughly discussed in the companion paper
(Frisch, Grimberg and Villone, 2017), Hankel’s Preisschrift reveals indeed a truly deep un-
derstanding of Lagrangian fluid mechanics, with innovative use of variational methods and
differential geometry. Until these days, this innovative work of Hankel has remained appar-
ently poorly known among scholars, with some exceptions. For details and references, see
the companion paper.

The paper is organised as follows. In section II, we present the translation of the Preis-
schrift; this section begins with a preface signed by Hankel, containing the stated prize
question together with the decision by the committee members. For our translation we used
the digitized copy of the Preisschrift from the HathyTrust Digital Library (indicated as a
link in the reference). It has been verified by the Göttingen University Library that the
text that we used is indeed the digitized copy of the 1861 original printed copy. Section III
contains the translation of two written judgements on Hankel’s essay in the procedure of
assessment of the prize, one by Weber and the other by Riemann. In section IV we provide
some biographical notes, together with a full publication list of Hankel.

Let us elucidate our conventions for author/translator footnotes, comments and equation
numbering. Footnotes by Hankel are denoted by an “A.” followed by a number (A stands
for Author), enclosed in square brackets; translator footnotes are treated identically except
that the letter “A” is replaced with “T” (standing for translator). For both author and
translator footnotes, we apply a single number count, e.g., [T.1], [A.2], [A.3]. Very short
translator comments are added directly in the text, and such comments are surrounded by
square brackets. Only a few equations have been numbered by Hankel (denoted by numbers
in round brackets). To be able to refer to all equations, especially relevant for the companion
paper, we have added additional equation numbers in the format [p.n], which means the nth
equation of §p. Finally, we note that the abbreviations “S.” and “Bd.”, which occur in the
Preisschrift, refer respectively to the German words for “page” and “volume”.

II. HANKEL’S PREISSCHRIFT TRANSLATION

About the prize question set by the philosophical Faculty of Georgia Augusta on 4th June
1860: [T.6]

The most useful equations for determining fluid motion may be presented in
two ways, one of which is Eulerian, the other one is Lagrangian. The illustrious Dirichlet
pointed out in the posthumous unpublished paper “On a problem of hydrodynamics” the
almost completely overlooked advantages of the Lagrangian way, but he was prevented from
unfolding this way further by a fatal illness. So, this institution asks for a theory of fluid
motion based on the equations of Lagrange, yielding, at least, the laws of vortex motion
already derived in another way by the illustrious Helmholtz.

Decision of the philosophical Faculty on the present manuscript:

The extraordinary mathematical-physical prize question about the derivation of
laws of fluid motion and, in particular, of vortical motion, described by the so-called

[T.6] The prize question is in Latin in the Preisschrift: “Aequationes generales motui fluidorum determinando
inservientes duobus modis exhiberi possunt, quorum alter Eulerio, alter Lagrangio debetur. Lagrangiani
modi utilitates adhuc fere penitus neglecti clarissimus Dirichlet indicavit in commentatione postuma
’de problemate quodam hydrodynamicum’ inscripta; sed ab explicatione carum ubiore morbo supremo
impeditus esse videtur. Itaque postulat ordo theoriam motus fluidorum aequationibus Lagrangianis
superstructam eamque eo saltem perductam, ut leges motus rotatorii a clarissimo Helmholtz allo modo
erutae inde redudent.” At that time it was common to state prizes in Latin and to have the submitted
essays in the same language.
Lagrangian equations, was answered by an essay carrying the motto: *The more signs express relationships in nature, the more useful they are.*[^T.7] This manuscript gives commendable evidence of the Author’s diligence, of his knowledge and ability in using the methods of computation recently developed by contemporary mathematicians. In particular §6[^T.8] contains an elegant method to establish the equations of motion for a flow in a fully arbitrary coordinate system, from a point of view which is commonly referred to as Lagrangian. However, when developing the general laws of vortex motion, the Lagrangian approach is unnecessarily left aside, and, as a consequence, the various laws have to be found by quite independent means. Also the relation between the vortex laws and the investigations of Clebsch, reported in §.14.15, is omitted by the Author. Nonetheless, as his derivation actually begins from the Lagrangian equations, one may consider the prize-question as fully answered by this manuscript. Amongst the many good things to be found in this essay, the evoked incompleteness and some mistakes due to rushing which are easy to improve, do not prevent this Faculty from assigning the prize to the manuscript, but the Author would be obliged to submit a revised copy of the manuscript, improved according to the suggestions made above before it goes into print.

On my request, I have been permitted by the philosophical Faculty to have the manuscript, originally submitted in Latin, printed in German. —

The above mentioned §.6 coincides with §.5 of the present essay. The §.§.14.15 included what is now in the note of §.45 of the text [now page 33]; these §.§. were left out, on the one hand, because of lack of space, and on the other hand because, in the present view, these §.§. are not anymore connected to the rest of the essay.

Leipzig, September 1861.

Hermann Hankel.

§.1.

The conditions and forces which underlie most of the natural phenomena are so complicated and various, that it is rarely possible to take them into account by fully analytical means. Therefore, one should, for the time being, discard those forces and properties which evidently have little impact on the motion or the changes; this is done by just retaining the forces that are of essential and of fundamental importance. Only then, once this first approximation has been made, one may reconsider the previously disregarded forces and properties, and modify the underlying hypotheses. In the case of the general theory of motion of liquid fluids, it seems therefore advisable to take into account solely the continuity [of the fluid] and the constancy of volume, and to disregard both viscosity and internal friction which are also present as suggested by experience; it is advisable to take into account just the continuity and the elasticity, and consider the pressure determined as a function of the density. Even if, in specific cases, the analytical methods have improved and may apply as well to more realistic hypotheses, these hypotheses are not yet suitable for a fundamental general theory.

The hypotheses of the general hydrodynamical equations, as given by Euler, are the following. The fluid is considered as being composed of an aggregate of molecules, which are so small that one can find an infinitely large number of them in an arbitrarily small space.

[^T.7] This motto is in Latin in the *Preisschrift*: “Tanto utiliores sunt notae, quanto magis exprimunt rerum relationes”. At that time it was common that an Author submitting his work for a prize signed it anonymously with a motto.

[^T.8] A number such as “§ 6” refers to a section number in the (lost) Latin version of the manuscript. Numbers are different in the revised German translation.
Therefore, the fluid is considered as divided into infinitely small parts of dimensionality of the first order; each of these parts is filled with an infinite number of molecules, whose sizes have to be considered as infinitely small quantities of the second order. These molecules fill space continuously and move without friction against each other.

The flow can be set in motion either by accelerating forces from the individual molecules or from external pressure forces. Considering the nature of fluids, one easily comes to the conclusion, also confirmed by experience, that the pressure on each fluid element of the external surface acts normally and proportionally to the size of that element. In order to have also a clear definition of the pressure at a given point within the fluid, let us think of an element of an arbitrary surface through this point: the pressure will be normal to this surface element and proportional to its size, but independent on its direction. The difference between liquid and elastic flows is that, for the former, the density is a constant and, for the latter, the density depends on the pressure, and, conversely, the pressure depends on the density in a special way.

We shall not insist here on a detailed discussion of these properties as they are usually discussed in the better textbooks on mechanics.

In order to study the above properties analytically, two methods have been so far applied, both owed to Euler. The first method considers the velocity of each point of the fluid as a function of position and time. If \( u, v, w \) are the velocity components in the orthogonal coordinates \( x, y, z \), then \( u, v, w \) are functions of position \( x, y, z \) and time \( t \). The velocity of the fluid in a given point is thus the velocity of the fluid particles flowing through that point. This method was exclusively used for the study of motion of fluids until Dirichlet observed that this method had necessarily the drawback that the absolute space filled by the flow in general changes over time and, as a consequence, the coordinates \( x, y, z \) are not entirely independent variables. The method just discussed seems appropriate if the flow is always filling the same space, i.e., in the case when the flow is filling the infinite time.

The second, ingenious method of Euler considers the coordinates \( x, y, z \) of a flow particle, in any reference system, as a function of time \( t \) and of its position \( a, b, c \) at initial time \( t = 0 \). This method, by which the same fluid particle is followed during its motion, was reproduced, indeed in a slightly more elegant way, by Lagrange without giving any reference. Since it appears that nowadays Euler’s work is rarely read in detail, this method is considered due to Lagrange. However, the method was already present in its full completeness in Euler’s work, 29 years before. I owe this interesting, historical note to my honoured Professor B. Riemann. According to this method, one has thus

\[
x = \varphi_1(a, b, c, t), \quad y = \varphi_2(a, b, c, t), \quad z = \varphi_3(a, b, c, t),
\]

where \( \varphi_1, \varphi_2, \varphi_3 \) are continuous functions of \( a, b, c, t \). We have at initial time \( t = 0 \) that

\[
x = a, \quad y = b, \quad z = c
\]

and, thus, the following conditions are valid for \( \varphi_1, \varphi_2, \varphi_3 \):

\[
a = \varphi_1(a, b, c, 0), \quad b = \varphi_2(a, b, c, 0), \quad c = \varphi_3(a, b, c, 0).
\]
Obviously, one can take values of \( t \) so small, that, according to Taylor’s theorem, \( x, y, z \) can be expanded in powers of \( t \). Since at time \( t = 0 \) we have \( x = a, y = b, z = c \), it follows evidently that

\[
\begin{align*}
x &= a + A_1 t + A_2 t^2 + \ldots \\
y &= b + B_1 t + B_2 t^2 + \ldots \\
z &= c + C_1 t + C_2 t^2 + \ldots
\end{align*}
\]

From these equations, one easily finds that at time \( t = 0 \):

\[
\begin{align*}
\frac{dx}{da} &= 1, & \frac{dx}{db} &= 0, & \frac{dx}{dc} &= 0 \\
\frac{dy}{da} &= 0, & \frac{dy}{db} &= 1, & \frac{dy}{dc} &= 0 \\
\frac{dz}{da} &= 0, & \frac{dz}{db} &= 0, & \frac{dz}{dc} &= 1
\end{align*}
\]

In the following, we will try to give a presentation of the general theory of hydrodynamics based on the second method. It will turn out that the second [Lagrangian] form also merits to be preferred over the first one in some cases, because the fundamental equations of the former are more closely connected to the customary forms in mechanics.

§ 2.

The two infinitely near particles

\( a, b, c \)

and

\( a + da, b + db, c + dc \)

after some time \( t \), will be at the two points

\( x, y, z \)

and

\( x + dx, y + dy, z + dz, \)

where one needs to put

\[
\begin{align*}
dx &= \frac{dx}{da}da + \frac{dx}{db}db + \frac{dx}{dc}dc \\
dy &= \frac{dy}{da}da + \frac{dy}{db}db + \frac{dy}{dc}dc \\
dz &= \frac{dz}{da}da + \frac{dz}{db}db + \frac{dz}{dc}dc
\end{align*}
\]

Let us think of \( a, b, c \) as linear — in general non-orthogonal — coordinates; then, we have that \( da, db, dc \) are the coordinates of a point with respect to a congruent system of coordinates \( S_0 \), whose origin is at \( a, b, c \). As \( x, y, z \) refer to the same coordinate system as \( a, b, c \) do, then \( dx, dy, dz \) are the analogous coordinates with respect to a coordinate system, whose origin is at \( x, y, z \). Now, let us think of an infinitely small surface

\[
F(da, db, dc) = 0
\]

with reference to \( S_0 \), then, at time \( t \), the surface will have been moved into another surface

\[
F(dx, dy, dz) = 0
\]

with reference to \( S \), or,

\[
F\left(\frac{dx}{da}da + \frac{dx}{db}db + \frac{dx}{dc}dc, \frac{dy}{da}da + \frac{dy}{db}db + \frac{dy}{dc}dc, \frac{dz}{da}da + \frac{dz}{db}db + \frac{dz}{dc}dc\right) = 0
\]
We see from this that an infinitely small algebraic surface of degree \( n \) always remains of the same degree.

Hence, at any time, infinitely close points on a plane, always stay on a plane; and since a straight line may be thought of as an intersection of two planes, infinitely close points on a line at a certain time always stay on a line.

The points within an infinitely small ellipsoid will always stay in such an ellipsoid, because a closed surface cannot transform into a non-closed surface and, with the exception of the ellipsoid, all the surfaces of second degree are not closed. The section of a plane and of an infinitely small ellipsoid will thus have to remain always the same. Since that section always constitutes an ellipse, so an infinitely small ellipse will always remain the same.

The four points:

\[
\begin{align*}
 a &+ m \, da & b &+ n \, db & c &+ p \, dc \\
 a &+ m' \, da & b &+ n' \, db & c &+ p' \, dc \\
 a &+ m'' \, da & b &+ n'' \, db & c &+ p'' \, dc \\
\end{align*}
\]

where \( m, n, p, m', n', p', m'', n'', p'' \) are finite numbers, at time \( t \) will be at the position

\[
\begin{align*}
 x + \frac{dx}{da} m \, da + \frac{dx}{db} n \, db + \frac{dx}{dc} p \, dc, & \quad y + \frac{dy}{da} m \, da + \frac{dy}{db} n \, db + \frac{dy}{dc} p \, dc, & \quad z + \frac{dz}{da} m \, da + \frac{dz}{db} n \, db + \frac{dz}{dc} p \, dc \\
 x + \frac{dx}{da} m' \, da + \frac{dx}{db} n' \, db + \frac{dx}{dc} p' \, dc, & \quad y + \frac{dy}{da} m' \, da + \frac{dy}{db} n' \, db + \frac{dy}{dc} p' \, dc, & \quad z + \frac{dz}{da} m' \, da + \frac{dz}{db} n' \, db + \frac{dz}{dc} p' \, dc \\
 x + \frac{dx}{da} m'' \, da + \frac{dx}{db} n'' \, db + \frac{dx}{dc} p'' \, dc, & \quad y + \frac{dy}{da} m'' \, da + \frac{dy}{db} n'' \, db + \frac{dy}{dc} p'' \, dc, & \quad z + \frac{dz}{da} m'' \, da + \frac{dz}{db} n'' \, db + \frac{dz}{dc} p'' \, dc \\
\end{align*}
\]

at time \( t \).

The volume of the tetrahedron \( T_0 \) whose vertices are those points at time \( t = 0 \), is expressed by the determinant:

\[
6T_0 = \left| \begin{array}{ccc}
 m & n & p \\
 m' & n' & p' \\
 m'' & n'' & p'' \\
\end{array} \right| \, da \, db \, dc \quad [2.5]
\]

The volume of the tetrahedron \( T \) whose vertices at time \( t \) are formed by the same particles, is:

\[
6T = \left| \begin{array}{ccc}
 \frac{dx}{da} m \, da + \frac{dx}{db} n \, db + \frac{dx}{dc} p \, dc, & \frac{dy}{da} m \, da + \frac{dy}{db} n \, db + \frac{dy}{dc} p \, dc, & \frac{dz}{da} m \, da + \frac{dz}{db} n \, db + \frac{dz}{dc} p \, dc \\
 \frac{dx}{da} m' \, da + \frac{dx}{db} n' \, db + \frac{dx}{dc} p' \, dc, & \frac{dy}{da} m' \, da + \frac{dy}{db} n' \, db + \frac{dy}{dc} p' \, dc, & \frac{dz}{da} m' \, da + \frac{dz}{db} n' \, db + \frac{dz}{dc} p' \, dc \\
 \frac{dx}{da} m'' \, da + \frac{dx}{db} n'' \, db + \frac{dx}{dc} p'' \, dc, & \frac{dy}{da} m'' \, da + \frac{dy}{db} n'' \, db + \frac{dy}{dc} p'' \, dc, & \frac{dz}{da} m'' \, da + \frac{dz}{db} n'' \, db + \frac{dz}{dc} p'' \, dc \\
\end{array} \right| \quad [2.6]
\]

By known theorems, this determinant can however be written as a product of two determinants:

\[
6T = \left| \begin{array}{ccc}
 \frac{dx}{da} & \frac{dx}{db} & \frac{dx}{dc} \\
 \frac{dy}{da} & \frac{dy}{db} & \frac{dy}{dc} \\
 \frac{dz}{da} & \frac{dz}{db} & \frac{dz}{dc} \\
\end{array} \right| \left| \begin{array}{ccc}
 m & n & p \\
 m' & n' & p' \\
 m'' & n'' & p'' \\
\end{array} \right| \, da \, db \, dc \quad [2.7]
\]
or

\[
T = T_0 \begin{vmatrix}
\frac{dx}{da} & \frac{dx}{db} & \frac{dx}{dc} \\
\frac{dy}{da} & \frac{dy}{db} & \frac{dy}{dc} \\
\frac{dz}{da} & \frac{dz}{db} & \frac{dz}{dc}
\end{vmatrix}
\]  \hspace{1cm} [2.8]

It results from the preceding considerations, that all particles which at time \( t = 0 \) are in the tetrahedron \( T_0 \), will be also in the tetrahedron \( T \) at time \( t \). Let \( \rho_0 \) be the mean density of the tetrahedron \( T_0 \) at time \( t = 0 \), and \( \rho \) the mean density in \( T \) at time \( t \), so one has \( T : T_0 = \rho_0 : \rho \) and hence

\[
\begin{vmatrix}
\frac{dx}{da} & \frac{dx}{db} & \frac{dx}{dc} \\
\frac{dy}{da} & \frac{dy}{db} & \frac{dy}{dc} \\
\frac{dz}{da} & \frac{dz}{db} & \frac{dz}{dc}
\end{vmatrix}
= \frac{\rho_0}{\rho}.
\]  \hspace{4cm} (1), [2.9]

If the density is constant, the fluid is a liquid flow and thus

\[
\begin{vmatrix}
\frac{dx}{da} & \frac{dx}{db} & \frac{dx}{dc} \\
\frac{dy}{da} & \frac{dy}{db} & \frac{dy}{dc} \\
\frac{dz}{da} & \frac{dz}{db} & \frac{dz}{dc}
\end{vmatrix}
= 1.
\]  \hspace{4cm} (2), [2.10]

One can reasonably refer to these equations as the density equations, more particularly the last one as the equation of the constancy of the volume.

The values of the functional determinant of \( x, y, z \) with respect to \( a, b, c \) may be used to develop a set of relationships, which are often needed to pass from the Eulerian representation to the other [i.e., to the Lagrangian representation; or reciprocally]. Indeed, if one solves the system of equations:

\[
\begin{align*}
\frac{dx}{da} &= \frac{dx}{db} + \frac{dx}{dc} \\
\frac{dy}{da} &= \frac{dy}{db} + \frac{dy}{dc} \\
\frac{dz}{da} &= \frac{dz}{db} + \frac{dz}{dc}
\end{align*}
\]  \hspace{3cm} [2.11]

one has:

\[
\begin{align*}
\left( \frac{dy}{db} \frac{dz}{dc} - \frac{dy}{dc} \frac{dz}{db} \right) dx + \left( \frac{dz}{db} \frac{dx}{dc} + \frac{dz}{dc} \frac{dx}{db} \right) dy + \left( \frac{dx}{db} \frac{dy}{dc} + \frac{dx}{dc} \frac{dy}{db} \right) dz &= \frac{\rho_0}{\rho} da \\
\left( \frac{dy}{dc} \frac{dz}{da} - \frac{dy}{da} \frac{dz}{dc} \right) dx + \left( \frac{dz}{dc} \frac{dx}{da} + \frac{dz}{da} \frac{dx}{dc} \right) dy + \left( \frac{dx}{dc} \frac{dy}{da} + \frac{dx}{da} \frac{dy}{dc} \right) dz &= \frac{\rho_0}{\rho} db \\
\left( \frac{dy}{da} \frac{dz}{db} - \frac{dy}{db} \frac{dz}{da} \right) dx + \left( \frac{dz}{da} \frac{dx}{db} + \frac{dz}{db} \frac{dx}{da} \right) dy + \left( \frac{dx}{da} \frac{dy}{db} + \frac{dx}{db} \frac{dy}{da} \right) dz &= \frac{\rho_0}{\rho} dc
\end{align*}
\]  \hspace{3cm} [2.12]

where we have substituted the value \( \rho_0/\rho \) for the functional determinant. The comparison
of these equations with:

\[
\frac{da}{dx} dx + \frac{da}{dy} dy + \frac{da}{dz} dz = da
\]

\[
\frac{db}{dx} dx + \frac{db}{dy} dy + \frac{db}{dz} dz = db
\]

\[
\frac{dc}{dx} dx + \frac{dc}{dy} dy + \frac{dc}{dz} dz = dc
\]

[2.13]

gives this equation system:

\[
\begin{align*}
\frac{\varrho_0 da}{dx} &= \frac{dy}{db} \frac{dz}{dc} - \frac{dy}{dc} \frac{dz}{db}, \\
\frac{\varrho_0 db}{dx} &= \frac{dz}{dc} \frac{dx}{da} - \frac{dz}{da} \frac{dx}{dc}, \\
\frac{\varrho_0 dc}{dx} &= \frac{dx}{db} \frac{dy}{da} - \frac{dx}{da} \frac{dy}{db}
\end{align*}
\]

(3), [2.14]

If the equation (1) is differentiated with respect to one of the independent variables \(a, b, c\), and these differentiations are indicated with \(\delta\), one obtains

\[
\begin{align*}
\frac{\delta x}{dx} &= \frac{dy}{db} \frac{dz}{dc} - \frac{dy}{dc} \frac{dz}{db}, \\
\frac{\delta y}{dy} &= \frac{dz}{dc} \frac{dx}{da} - \frac{dz}{da} \frac{dx}{dc}, \\
\frac{\delta z}{dz} &= \frac{dx}{db} \frac{dy}{da} - \frac{dx}{da} \frac{dy}{db}
\end{align*}
\]

[2.15]

and, by equations (3):

\[
\begin{align*}
\frac{d\delta x}{dx} \frac{da}{dx} + \frac{d\delta y}{dy} \frac{db}{dy} + \frac{d\delta z}{dz} \frac{dc}{dz} + \frac{\delta \varrho}{\varrho} = 0
\end{align*}
\]

[2.16]

or

\[
\frac{d\delta x}{dx} + \frac{d\delta y}{dy} + \frac{d\delta z}{dz} + \frac{\delta \varrho}{\varrho} = 0
\]

(4), [2.17]

If by \(\delta\), one understands the differentiation with respect to \(t\), one has:

\[
\varphi \left( \frac{d\varphi}{dx} + \frac{d\varphi}{dy} + \frac{d\varphi}{dz} \right) + \frac{d\varphi}{dt} = 0
\]

[2.18]

Since \(t\) appears not only explicitly in \(\varphi\), but also implicitly in \(\varphi\) through its dependence on \(x, y, z\), one has

\[
\frac{d\varphi}{dt} = \frac{dx}{dt} \frac{d\varphi}{dx} + \frac{dy}{dt} \frac{d\varphi}{dy} + \frac{dz}{dt} \frac{d\varphi}{dz}
\]

[2.19]

If one sets

\[
u = \frac{dx}{dt}, \quad v = \frac{dy}{dt}, \quad w = \frac{dz}{dt}
\]

[2.20]
one thus has:

\[ \varrho \left( \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \right) + \frac{d\varrho}{dt} + \frac{d\varrho}{dx} u + \frac{d\varrho}{dy} v + \frac{d\varrho}{dz} w = 0 \]  

[2.21]

or

\[ \frac{d\varrho}{dt} + \frac{d(\varrho u)}{dx} + \frac{d(\varrho v)}{dy} + \frac{d(\varrho w)}{dz} = 0 \]  

(5), [2.22]

This is the form in which Euler first presented the density equation. For the case when \( \varrho \) is constant in time, in this first form of dependence, one obtains as the constancy-of-volume equation:

\[ \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0 \]  

(6), [2.23]

Lagrange\[^{[A.16]}\] treats the relation of these equations with (2) quite extensively; but this connection seems to be a special case of a theorem by Jacobi,\[^{[T.17]}\] which for three variables is [actually] fully included in equations (1) and (5).

If an integral:

\[ \iiint f(x,y,z) \varrho \, dx \, dy \, dz \]

which is extended over the whole fluid mass, is transformed into an integral over \( a, b, c \), one has:

\[ \iiint f(x,y,z) \varrho \, dx \, dy \, dz = \iiint f(a,b,c) \varrho_0 \, da \, db \, dc \]

[2.24]

where also the second integral has to be extended over all particles, and \( f(a, b, c) \) is the function into which \( f(x, y, z) \) transforms by substituting \( a, b, c \) for \( x, y, z \). Thus, from the density equation (1) follows:

\[ \iiint f(x,y,z) \varrho \, dx \, dy \, dz = \iiint f(a,b,c) \varrho_0 \, da \, db \, dc \]  

[2.25]

— an important transformation.

§. 3.

Despite the complete analogy of these equations for the equilibrium and motion of liquid fluids with the equations for elastic fluids, there is still an essential difference with regard to their derivation. For this reason, these cases have to be treated separately.

Before developing the equations of motion, it will be convenient to study more in detail the equilibrium conditions, at first for liquid fluids.

If \( X, Y, Z \) are the accelerating forces in the direction of the coordinates axes,
acting on the point \(x, y, z\), then it follows easily from the principle of virtual velocities that:

\[
\iiint \left[ \rho \left( X \delta x + Y \delta y + Z \delta z \right) + p \delta L \right] \, dx \, dy \, dz = 0
\]  

[3.1]

in which \(L = 0\) gives the density equation for incompressible fluids; \(\delta L\) is its relative variation corresponding to the variations of coordinates \(\delta x, \delta y, \delta z\), and \(p\) is a not yet determined quantity. The integral has to be extended over all parts of the continuous flow. From (4) in § 2, since \(\delta \rho = 0\), we have

\[
\delta L = \frac{d \delta x}{dx} + \frac{d \delta y}{dy} + \frac{d \delta z}{dz}
\]  

[3.2]

And thus the previous integral becomes:

\[
\iiint \left[ \rho \left( X \delta x + Y \delta y + Z \delta z \right) + p \left( \frac{d \delta x}{dx} + \frac{d \delta y}{dy} + \frac{d \delta z}{dz} \right) \right] \, dx \, dy \, dz = 0 \quad (1), \quad [3.3]
\]

Integrating by parts, one finds:

\[
\iiint p \frac{d \delta x}{dx} \, dy \, dz = \int \int p \delta x \, dy \, dz - \iiint \frac{\delta x}{dx} \frac{dp}{dx} \, dx \, dy \, dz
\]

\[
\iiint p \frac{d \delta y}{dy} \, dx \, dz = \int \int p \delta y \, dx \, dz - \iiint \frac{\delta y}{dy} \frac{dp}{dy} \, dx \, dy \, dz
\]

\[
\iiint p \frac{d \delta z}{dz} \, dx \, dy = \int \int p \delta z \, dx \, dy - \iiint \frac{\delta z}{dz} \frac{dp}{dz} \, dx \, dy \, dz
\]

[3.4]

where the double integrals extend over the surface of the fluid mass. Thus, one has for the equation of the principle of virtual velocity :

\[
0 = \iiint \left[ \rho X \delta x + (\rho Y - \frac{dp}{dx}) \delta y + (\rho Z - \frac{dp}{dy}) \delta z \right] \, dx \, dy \, dz + \int p \left( dx \cos \alpha + dy \cos \beta + dz \cos \gamma \right) \, d\omega
\]

[3.5]

where \(d\omega\) is an element of the external surface, and \(\alpha, \beta, \gamma\) indicate the angles between the normal to the element \(d\omega\) and the coordinates axes. From these equations, it follows that the equilibrium conditions are :

\[
\int p \left( \delta x \cos \alpha + \delta y \cos \beta + \delta z \cos \gamma \right) \, d\omega = 0 \quad (2), \quad [3.6]
\]

\[
\frac{dp}{dx} = \rho X, \quad \frac{dp}{dy} = \rho Y, \quad \frac{dp}{dz} = \rho Z
\]

[3.7]

The last three equations require that the components \(X, Y, Z\) be the differential quotients of an arbitrary function \(V\) with respect to \(x, y, z\); thus

\[
X = \frac{dV}{dx}, \quad Y = \frac{dV}{dy}, \quad Z = \frac{dV}{dz}
\]

[3.8]

so that one has:

\[
p = \rho V + c
\]

[3.9]

where \(p\) is determined up to an arbitrary constant \(c\).

Instead of equations (3), one can also write:

\[
(p_{x+dx} - p_x) \, dy \, dz - \rho X \, dx \, dy \, dz = 0
\]

\[
(p_{y+dy} - p_y) \, dz \, dx - \rho Y \, dx \, dy \, dz = 0
\]

\[
(p_{z+dz} - p_z) \, dx \, dy - \rho Z \, dx \, dy \, dz = 0
\]

[3.10]
from which it is apparent that $p$ is the pressure at the point $x, y, z$, which acts against the given accelerating forces. This pressure is determined up to an additive constant by $p = \varrho V + c$; it is fully determined if its value is given at any point. Suppose now that in addition to the acceleration of the fluid particles there are also pressure forces acting on the external surface. We then find as an equilibrium condition that at each point of the external surface, these pressure forces must be equal and opposite to the pressure $p = \varrho V + c$.

In equation (2), the variations $\delta x, \delta y, \delta z$ depend on certain conditions which result from the nature of the walls. If in special cases the actual meaning of equation (2) is specified more precisely, then its mechanical necessity becomes manifest. Here we want to discuss just a few such cases.

Let us assume that a part of the external surface be free and that the same forces act on all parts of it. One can set the pressure to zero in the points of that free surface, so that $p$, the difference between the pressure in a certain point and the pressure in a point of the free surface, be exactly determined in all other remaining points of the fluid.

However, since $\delta x, \delta y, \delta z$ are evidently arbitrary in a free surface, it follows that, if (2) has to be satisfied, we must have $p = 0$.

For fluid parts that are lying on a fixed wall, it is evident that no motion normal to the wall surface can take place. The normal component of the motion is obviously:

$$\delta x \cos \alpha + \delta y \cos \beta + \delta z \cos \gamma$$

and, since this must vanish, equation (2) is indeed satisfied.

In the points which are on a moving wall, one can set

$$\delta x = \delta x' + \delta \xi, \quad \delta y = \delta y' + \delta \eta, \quad \delta z = \delta z' + \delta \zeta$$

where $\delta x', \delta y', \delta z'$ are the motions relative to the wall and $\delta \xi, \delta \eta, \delta \zeta$ are the motions of the fluid particles simultaneously in motion with the wall. Therefore, one has instead of equation (2)

$$0 = \int p(\delta x' \cos \alpha + \delta y' \cos \beta + \delta z' \cos \gamma) d\omega + \int p(\delta \xi \cos \alpha + \delta \eta \cos \beta + \delta \zeta \cos \gamma) d\omega$$

but, since no motion may happen against the wall, one has:

$$\delta x' \cos \alpha + \delta y' \cos \beta + \delta z' \cos \gamma = 0$$

and therefore

$$\int p(\delta \xi \cos \alpha + \delta \eta \cos \beta + \delta \zeta \cos \gamma) d\omega = 0$$

which amounts to setting

$$\int (p_x \delta \xi + p_y \delta \eta + p_z \delta \zeta) d\omega = 0$$

provided that $p_x, p_y, p_z$ are the components of the pressure with respect to the coordinates axes. However this integral is the equilibrium condition of a body, on which act external surface forces $p_x, p_y, p_z$, where $\delta \xi, \delta \eta, \delta \zeta$ indicate the variations, that the body can have, under the given circumstances. Actually, also in this case, equation (2) is needed by the nature of things.[A.18]

[A.18] Cf. Méc. analy. Bd. I, S. 193–201. [First edition, 1788]
We now discuss the case of elastic fluids; here the relation $L = 0$ is not valid anymore. In that case one has to consider also the forces due to elasticity of the fluid in addition to the accelerating and external pressure forces. Let $p$ be the pressure at a point $x, y, z$, then this tends to reduce the volume of the element $dx, dy, dz$: the momentum by this force is thus $p\delta(dx\,dy\,dz)$; in order to get a different expression for $\delta(dx\,dy\,dz)$, we note that $\rho\,dx\,dy\,dz$, being the mass of an element, is always the same; thus we have $\delta(\rho\,dx\,dy\,dz) = 0$; herefrom it follows that

$$\rho\delta(dx\,dy\,dz) + dx\,dy\,dz\,\delta\rho = 0 \quad [3.16]$$

and thus

$$\delta(dx\,dy\,dz) = -\frac{\delta\rho}{\rho}dx\,dy\,dz \quad [3.17]$$

or, by equation (4) in §.2,

$$\delta(dx\,dy\,dz) = \left(\frac{d\delta x}{dx} + \frac{d\delta y}{dy} + \frac{d\delta z}{dz}\right)dx\,dy\,dz; \quad [3.18]$$

Therefore the equilibrium conditions are:

$$\iiint \left\{\rho\left(X\delta x + Y\delta y + Z\delta z\right) + p\left(\frac{d\delta x}{dx} + \frac{d\delta y}{dy} + \frac{d\delta z}{dz}\right)\right\}dx\,dy\,dz = 0 \quad [3.19]$$

Since this equation is identical with (1), the equilibrium equations for elastic and liquid flows are formally the same. Also here, in accordance with equation (2), we have:

$$\frac{dp}{dx} = \rho X, \quad \frac{dp}{dy} = \rho Y, \quad \frac{dp}{dz} = \rho Z \quad [3.20]$$

For elastic flows, $\rho$ is a given function of $p$, say, $\rho = \varphi(p)$. Let us put

$$f(p) = \int \frac{dp}{\varphi(p)} \quad [3.21]$$

from which it follows obviously that $\frac{1}{\varphi(p)}\frac{dp}{dx} = \frac{1}{\varphi(p)}\frac{dp}{dx} = \frac{df(p)}{dx} = \frac{df(p)}{dx} = \frac{df(p)}{dx}$, therefore, the three equations for the equilibrium condition become:

$$\frac{df(p)}{dx} = X, \quad \frac{df(p)}{dy} = Y, \quad \frac{df(p)}{dz} = Z, \quad [3.22]$$

so that, also for elastic fluids in equilibrium, $X, Y, Z$ have to be partial differential quotients of the same function with respect to $x, y, z$. As $\varphi(p)$, and consequently also $f(p)$ is known, $p$ can always be expressed through $X, Y, Z$.

§. 4.

As follows from the considerations of §. 3, the principle of virtual velocities and lost forces for the motion of liquid and elastic fluids, implies:

$$0 = \iiint \left\{\rho\left[(X - \frac{\delta^2 x}{\delta x^2})\delta x + (Y - \frac{\delta^2 y}{\delta y^2})\delta y + (Z - \frac{\delta^2 z}{\delta z^2})\delta z\right] + p\left[\frac{\delta^2 x}{\delta x^2} + \frac{\delta^2 y}{\delta y^2} + \frac{\delta^2 z}{\delta z^2}\right]\right\}dx\,dy\,dz \quad [4.1]$$

from which follows firstly equation (2) of §. 3:

$$\int p(\delta x \cos \alpha + \delta y \cos \beta + \delta z \cos \gamma)\,d\omega = 0, \quad [4.2]$$
which concerns only the external surface; secondly, we have for the fundamental equations of liquid or elastic fluids,

\[ p\left( \frac{d^2 x}{dt^2} - X \right) + \frac{dp}{dx} = 0, \quad p\left( \frac{d^2 y}{dt^2} - Y \right) + \frac{dp}{dy} = 0, \quad p\left( \frac{d^2 z}{dt^2} - Z \right) + \frac{dp}{dz} = 0, \quad [4.3] \]

where \( p \) indicates the pressure in each point.

According to the first Eulerian method, the components \( u, v, \omega \) are considered as function of time and space \( x, y, z \). Therefore,

\[
\frac{d^2 x}{dt^2} = \frac{du}{dt} + \frac{du}{dx} u + \frac{du}{dy} v + \frac{du}{dz} w
\]

\[
\frac{d^2 y}{dt^2} = \frac{dv}{dt} + \frac{dv}{dx} u + \frac{dv}{dy} v + \frac{dv}{dz} w
\]

\[
\frac{d^2 z}{dt^2} = \frac{dw}{dt} + \frac{dw}{dx} u + \frac{dw}{dy} v + \frac{dw}{dz} w
\]

so that the fundamental equations in this form are the following:

\[
\begin{align*}
\frac{du}{dt} + \frac{du}{dx} u + \frac{du}{dy} v + \frac{du}{dz} w - X + \frac{1}{\rho} \frac{dp}{dx} &= 0 \\
\frac{dv}{dt} + \frac{dv}{dx} u + \frac{dv}{dy} v + \frac{dv}{dz} w - Y + \frac{1}{\rho} \frac{dp}{dy} &= 0 \\
\frac{dw}{dt} + \frac{dw}{dx} u + \frac{dw}{dy} v + \frac{dw}{dz} w - Z + \frac{1}{\rho} \frac{dp}{dz} &= 0
\end{align*}
\]

which may be also written in such a way that each equation is obtained from the other by cyclic permutation, namely:

\[
\begin{align*}
\frac{du}{dt} + \frac{du}{dx} u + \frac{du}{dy} v + \frac{du}{dz} w - X + \frac{1}{\rho} \frac{dp}{dx} &= 0 \\
\frac{dv}{dt} + \frac{dv}{dy} v + \frac{dv}{dz} w + \frac{dv}{dx} u - Y + \frac{1}{\rho} \frac{dp}{dy} &= 0 \\
\frac{dw}{dt} + \frac{dw}{dz} w + \frac{dw}{dx} u + \frac{dw}{dy} v - Z + \frac{1}{\rho} \frac{dp}{dz} &= 0
\end{align*}
\]

(4), [4.5]

In addition to these formulae, there is also the density equation (5) of the § 2:

\[ \frac{d\rho}{dt} + \frac{d(\rho u)}{dx} + \frac{d(\rho v)}{dy} + \frac{d(\rho w)}{dz} = 0 \quad [4.7] \]

or, in particular, for liquid flows

\[ \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0 \quad [4.8] \]

We see that these four equations are sufficient to determine the four unknowns \( u, v, w \) and \( p \) as functions of \( x, y, z \) and \( t \); \( \rho \) is either a known function of \( p \) or a constant.

By the second Eulerian representation [Lagrangian representation], equations (3) are respectively multiplied by

\[
\frac{dx}{da}, \quad \frac{dy}{da}, \quad \frac{dz}{da}
\]
and summed, then, similarly, by
\[ \frac{dx}{db}, \frac{dy}{db}, \frac{dz}{db} \]
and
\[ \frac{dx}{dc}, \frac{dy}{dc}, \frac{dz}{dc} \]
Thus, one obtains for the three fundamental equations:
\[
\begin{align*}
\left( \frac{d^2 x}{dt^2} - X \right) \frac{dx}{da} + \left( \frac{d^2 y}{dt^2} - Y \right) \frac{dy}{da} + \left( \frac{d^2 z}{dt^2} - Z \right) \frac{dz}{da} + \frac{1}{\varrho} \frac{dp}{da} &= 0 \\
\left( \frac{d^2 x}{dt^2} - X \right) \frac{dx}{db} + \left( \frac{d^2 y}{dt^2} - Y \right) \frac{dy}{db} + \left( \frac{d^2 z}{dt^2} - Z \right) \frac{dz}{db} + \frac{1}{\varrho} \frac{dp}{db} &= 0 \\
\left( \frac{d^2 x}{dt^2} - X \right) \frac{dx}{dc} + \left( \frac{d^2 y}{dt^2} - Y \right) \frac{dy}{dc} + \left( \frac{d^2 z}{dt^2} - Z \right) \frac{dz}{dc} + \frac{1}{\varrho} \frac{dp}{dc} &= 0
\end{align*}
\]
and, in addition, there is the density equation (1) of § 2
\[
\begin{vmatrix}
\frac{dx}{da} & \frac{dx}{db} & \frac{dx}{dc} \\
\frac{dy}{da} & \frac{dy}{db} & \frac{dy}{dc} \\
\frac{dz}{da} & \frac{dz}{db} & \frac{dz}{dc}
\end{vmatrix} = \frac{\varrho}{\varrho_0}
\]
[4.10]
From these four equations \( x, y, z \) and \( p \) are found as functions of the initial location \( a, b, c \) and time \( t \).

Evidently, the solutions of these partial differential equations must contain arbitrary functions, which have to be determined from initial conditions and are in accordance with the nature of the walls and the flow boundaries.

These last equations, which are usually called after Lagrange, significantly simplify their form by setting:
\[
X = \frac{dV}{dx}, \quad Y = \frac{dV}{dy}, \quad Z = \frac{dV}{dz}
\]
[4.11]
so they become:
\[
\begin{align*}
\frac{d^2 x}{dt^2} \frac{dx}{da} + \frac{d^2 y}{dt^2} \frac{dy}{da} + \frac{d^2 z}{dt^2} \frac{dz}{da} - \frac{dV}{da} + \frac{1}{\varrho} \frac{dp}{da} &= 0 \\
\frac{d^2 x}{dt^2} \frac{dx}{db} + \frac{d^2 y}{dt^2} \frac{dy}{db} + \frac{d^2 z}{dt^2} \frac{dz}{db} - \frac{dV}{db} + \frac{1}{\varrho} \frac{dp}{db} &= 0 \\
\frac{d^2 x}{dt^2} \frac{dx}{dc} + \frac{d^2 y}{dt^2} \frac{dy}{dc} + \frac{d^2 z}{dt^2} \frac{dz}{dc} - \frac{dV}{dc} + \frac{1}{\varrho} \frac{dp}{dc} &= 0
\end{align*}
\]
[4.12]
We will limit ourselves to this assumption about \( X, Y, Z \) which, apart from the boundary conditions, coincides with the one that necessarily has to hold in the equilibrium state of the fluid.

§ 5.

Partially integrating the last three terms of equation (1) of § 2, ignoring the boundary contributions to the double integrals and using as already stated
\[
X = \frac{dV}{dx}, \quad Y = \frac{dV}{dy}, \quad Z = \frac{dV}{dz},
\]
[5.1]
one obtains the following equation:

$$0 = \iiint \rho \, dx \, dy \, dz \left\{ \left( \frac{d^2 x}{dt^2} - \frac{dV}{dx} + \frac{1}{\rho} \frac{dp}{dx} \right) \delta x + \left( \frac{d^2 y}{dt^2} - \frac{dV}{dy} + \frac{1}{\rho} \frac{dp}{dy} \right) \delta y + \left( \frac{d^2 z}{dt^2} - \frac{dV}{dz} + \frac{1}{\rho} \frac{dp}{dz} \right) \delta z \right\}$$

If one puts, as in § 3, $\rho = \varphi(p)$ and:

$$f(p) = \int \frac{dp}{\varphi(p)}$$

one has,

$$0 = \iiint \rho_0 \, da \, db \, dc \left\{ \left( \frac{d^2 x}{dt^2} - \frac{dV}{dx} + \frac{df(p)}{dx} \right) \delta x + \left( \frac{d^2 y}{dt^2} - \frac{dV}{dy} + \frac{df(p)}{dy} \right) \delta y + \left( \frac{d^2 z}{dt^2} - \frac{dV}{dz} + \frac{df(p)}{dz} \right) \delta z \right\}$$

provided that the transformation of the integral is made according to § 2. Now, if one sets

$$V - f(p) = \Omega,$$

one can write:

$$0 = \iiint \rho_0 \, da \, db \, dc \left[ \frac{d^2 x}{dt^2} \delta x + \frac{d^2 y}{dt^2} \delta y + \frac{d^2 z}{dt^2} \delta z - \delta \Omega \right]$$

If one now integrates under the triple integral with respect to the variable $t$ which is independent of $a, b, c$, one obtains

$$\int \frac{d^2 x}{dt^2} \delta x \, dt = \left[ \frac{dx}{dt} \delta x \right] - \int \frac{dx}{dt} \delta x \, dt$$

$$\int \frac{d^2 y}{dt^2} \delta y \, dt = \left[ \frac{dy}{dt} \delta y \right] - \int \frac{dy}{dt} \delta y \, dt$$

$$\int \frac{d^2 z}{dt^2} \delta z \, dt = \left[ \frac{dz}{dt} \delta z \right] - \int \frac{dz}{dt} \delta z \, dt$$

Dropping the triple integrals, one has the equation:

$$0 = \iiint \rho_0 \, da \, db \, dc \int dt \left\{ \frac{dx}{dt} \delta x + \frac{dy}{dt} \delta y + \frac{dz}{dt} \delta z + \delta \Omega \right\}$$

which coincides with the following:

$$0 = \delta \iiint \rho_0 \, da \, db \, dc \int dt \left[ \left( \frac{ds}{dt} \right)^2 + 2\Omega \right].$$

(1), [5.9]

The first three hydrodynamical fundamental equations are satisfied. Hence

$$\iiint \rho_0 \, da \, db \, dc \int dt \left[ \left( \frac{ds}{dt} \right)^2 + 2\Omega \right]$$

disappears; conversely, if the first variation of this integral with respect to $x, y, z$ is set to zero, then one obtains these first three equations.

This theorem, which can be easily considered, in view of the meaning of $\frac{ds}{dt}$ and $\Omega$, as a mechanical principle, has a certain analogy with the principle of least action. For us it possesses an analytical importance, since it gives an extremely simple tool for transforming the hydrodynamical equations. Indeed, in order to introduce in these equations new coordinates, instead of $x, y, z$, it is only necessary to write the arc element

$$ds^2 = dx^2 + dy^2 + dz^2$$

[5.10]
in terms of the new coordinates and, then, apply the simple operation of variation, using the integral in the new coordinates:

\[
\iiint \varrho_0 \, da \, db \, dc \int dt \left[ \left( \frac{ds}{dt} \right)^2 + 2\Omega \right]
\]

By setting the coefficients of the three variations to zero, we thereby obtain three equations in a similar form as equations (3) in § 4: in order to write them into the first or second Eulerian form, one has to apply analogous procedures as was done in § 4.

If the new coordinates:

\[ \varrho_1 = f_1(x, y, z), \quad \varrho_2 = f_2(x, y, z), \quad \varrho_3 = f_3(x, y, z) \]  \[\text{[5.11]}\]

are used instead of \( x, y, z \), one obtains:

\[
\begin{align*}
\frac{dx}{\varrho_1} &= \frac{dx}{\varrho_1} + \frac{dx}{\varrho_2} + \frac{dx}{\varrho_3} \\
\frac{dy}{\varrho_1} &= \frac{dy}{\varrho_2} + \frac{dy}{\varrho_3} \\
\frac{dz}{\varrho_1} &= \frac{dz}{\varrho_2} + \frac{dz}{\varrho_3}
\end{align*}
\]

therefore

\[
\int \int \int \varrho_0 \, da \, db \, dc \int dt \left[ \left( \frac{d\varphi}{dt} \right)^2 \right]
\]

\[\text{[5.12]}\]

\[\text{[5.13]}\]

where

\[
\begin{align*}
N_1 &= \left( \frac{dx}{\varrho_1} \right)^2 + \left( \frac{dy}{\varrho_1} \right)^2 + \left( \frac{dz}{\varrho_1} \right)^2 \\
N_2 &= \left( \frac{dx}{\varrho_2} \right)^2 + \left( \frac{dy}{\varrho_2} \right)^2 + \left( \frac{dz}{\varrho_2} \right)^2 \\
N_3 &= \left( \frac{dx}{\varrho_3} \right)^2 + \left( \frac{dy}{\varrho_3} \right)^2 + \left( \frac{dz}{\varrho_3} \right)^2 \\
n_3 &= \frac{dx}{\varrho_1} \frac{dx}{\varrho_2} + \frac{dy}{\varrho_1} \frac{dy}{\varrho_2} + \frac{dz}{\varrho_1} \frac{dz}{\varrho_2} \\
n_1 &= \frac{dx}{\varrho_2} \frac{dx}{\varrho_3} + \frac{dy}{\varrho_2} \frac{dy}{\varrho_3} + \frac{dz}{\varrho_2} \frac{dz}{\varrho_3} \\
n_2 &= \frac{dx}{\varrho_3} \frac{dx}{\varrho_1} + \frac{dy}{\varrho_3} \frac{dy}{\varrho_1} + \frac{dz}{\varrho_3} \frac{dz}{\varrho_1}
\end{align*}
\]

\[\text{[5.14]}\]

\[\text{[5.15]}\]

Once \( N_1, N_2, N_3, n_1, n_2, n_3 \) are expressed in terms of the new variables \( \varrho_1, \varrho_2, \varrho_3 \), one has to vary the integral

\[
\int \int \int \varrho_0 \, da \, db \, dc \int dt \left[ N_1 \left( \frac{d\varphi}{dt} \right)^2 + N_2 \left( \frac{d\varphi}{dt} \right)^2 + N_3 \left( \frac{d\varphi}{dt} \right)^2 + 2n_3 \frac{d\varphi}{dt} \frac{d\varphi}{dt} + 2n_1 \frac{d\varphi}{dt} \frac{d\varphi}{dt} + 2n_2 \frac{d\varphi}{dt} \frac{d\varphi}{dt} + 2\Omega \right]
\]

with respect to these new variables. Then, after integration by parts with respect to \( t \), one removes from the quadruple integral the quantities in which appear the time derivatives of the variations \( \delta \varrho_1, \delta \varrho_2, \delta \varrho_3 \). Then one has to set the coefficients of \( \delta \varrho_1, \delta \varrho_2, \delta \varrho_3 \) equal to zero. After that one obtains the first three hydrodynamical fundamental equations, which are completed by a fourth one, the density equation. In order to express also the density equation in the new coordinates, we notice that the volume element \( dx \, dy \, dz \) may be expressed in such coordinates very easily, namely:

\[
\begin{align*}
dx \, dy \, dz &= \varrho_1 \, d\varrho_2 \, d\varrho_3 \\
\frac{dx}{d\varrho_1} &= \frac{dx}{d\varrho_2} + \frac{dx}{d\varrho_3}
\end{align*}
\]

\[\text{[5.16]}\]
herefrom follows

\[(dx \, dy \, dz)^2 = (d\theta_1 \, d\theta_2 \, d\theta_3)^2 \times \]

\[
\begin{align*}
\left| \begin{array}{c}
\left( \frac{dx}{d\theta_1} \right)^2 + \left( \frac{dy}{d\theta_1} \right)^2 + \left( \frac{dz}{d\theta_1} \right)^2, \\
\frac{dx}{d\theta_1} \frac{dx}{d\theta_2} + \frac{dy}{d\theta_1} \frac{dy}{d\theta_2} + \frac{dz}{d\theta_1} \frac{dz}{d\theta_2}, \\
\frac{dx}{d\theta_1} \frac{dx}{d\theta_3} + \frac{dy}{d\theta_1} \frac{dy}{d\theta_3} + \frac{dz}{d\theta_1} \frac{dz}{d\theta_3}, \\
\frac{dx}{d\theta_2} \frac{dx}{d\theta_3} + \frac{dy}{d\theta_2} \frac{dy}{d\theta_3} + \frac{dz}{d\theta_2} \frac{dz}{d\theta_3}
\end{array} \right|
\end{align*}
\]

\[= \frac{|N_1 \, n_3 \, n_2|}{n_3 \, N_2 \, n_1} \begin{cases} n_2 \, n_1 \, N_3 \end{cases} \tag{5.17} \]

or, with the notation as defined above:

\[(dx \, dy \, dz)^2 = (d\theta_1 \, d\theta_2 \, d\theta_3)^2 \begin{cases} n_3 \, N_2 \, n_1 \\
N_1 \, n_3 \, n_2 \end{cases} \tag{5.18} \]

Let us denote the values of \(\theta_1, \theta_2, \theta_3, N_1, N_2, N_3, n_1, n_2, n_3\) at time \(t = 0\) with \(\theta_1^0, \theta_2^0, \theta_3^0, N_1^0, N_2^0, N_3^0, n_1^0, n_2^0, n_3^0\), then we get from the previous equation for \(t = 0\)

\[
(\, \text{da db dc})^2 = (d\theta_1^0 \, d\theta_2^0 \, d\theta_3^0)^2 \begin{cases} n_3^0 \, N_2^0 \, n_1^0 \\
n_2^0 \, n_1^0 \, N_3^0 \end{cases} \tag{5.19} \]

One also has to think of the ensuing value of \(\text{da db dc}\) as substituted into the integral to be varied. But now the density equation is, for the general case:

\[
\begin{align*}
\frac{dx \, dy \, dz}{\text{da db dc}} &= \frac{\varrho_0}{\varrho}.
\end{align*} \tag{5.20}
\]

Then, dividing \((dx \, dy \, dz)^2\) by \((\, \text{da db dc})^2\), one obtains the density equation in the new variables:

\[
\left( \frac{d\theta_1 \, d\theta_2 \, d\theta_3}{d\theta_1^0 \, d\theta_2^0 \, d\theta_3^0} \right)^2 \left( \frac{\varrho}{\varrho_0} \right)^2 = \begin{cases} N_1^0 \, n_3^0 \, n_2^0 \\
n_3^0 \, N_2^0 \, n_1^0 \\
n_2^0 \, n_1^0 \, N_3^0 \end{cases} : \begin{cases} N_1 \, n_3 \, n_2 \\
n_3 \, N_2 \, n_1 \\
n_2 \, n_1 \, N_3 \end{cases} \tag{5.21}
\]

or, as it is well-known

\[
d\theta_1 \, d\theta_2 \, d\theta_3 = d\theta_1^0 \, d\theta_2^0 \, d\theta_3^0 \begin{cases} \frac{\varrho_0}{\varrho} \\
\varrho_0 \end{cases} \tag{5.22}
\]

one finally obtains:

\[
\begin{align*}
\frac{d\theta_1}{d\theta_1^0} & \quad \frac{d\theta_1}{d\theta_2^0} & \quad \frac{d\theta_1}{d\theta_3^0} \\
\frac{d\theta_2}{d\theta_1^0} & \quad \frac{d\theta_2}{d\theta_2^0} & \quad \frac{d\theta_2}{d\theta_3^0} \\
\frac{d\theta_3}{d\theta_1^0} & \quad \frac{d\theta_3}{d\theta_2^0} & \quad \frac{d\theta_3}{d\theta_3^0}
\end{align*}
\implies \frac{\varrho}{\varrho_0} = \sqrt{\begin{cases} N_1^0 \, n_3^0 \, n_2^0 \\
n_3^0 \, N_2^0 \, n_1^0 \\
n_2^0 \, n_1^0 \, N_3^0 \end{cases} : \begin{cases} N_1 \, n_3 \, n_2 \\
n_3 \, N_2 \, n_1 \\
n_2 \, n_1 \, N_3 \end{cases}} \tag{5.23}
\]

All quantities appearing in this transformed density equation are already known through the transformation of the arc element. Here, the problem of the transformation of the four
hydrodynamical equations in an arbitrary coordinate system is reduced to the problem of the transformation of the arc element.

It is obvious that the applicability of this procedure does not depend on the number of variables and that, in the same way, by variation of the integral with respect to \( x_1, x_2, \ldots, x_n \):

\[
\int \int \int g_0 \, da_1 \, da_2 \ldots \, da_n \int dt \left\{ \left( \frac{ds}{dt} \right)^2 + 2\Omega \right\}
\]

one obtains

\[
\frac{d^2 x_1}{dt^2} \frac{dx_1}{da_1} + \frac{d^2 x_2}{dt^2} \frac{dx_2}{da_1} + \ldots + \frac{d^2 x_n}{dt^2} \frac{dx_n}{da_1} - \frac{dV}{da_1} + \frac{1}{\varrho} \frac{dp}{da_1} = 0
\]

\[
\frac{d^2 x_1}{dt^2} \frac{dx_1}{da_n} + \frac{d^2 x_2}{dt^2} \frac{dx_2}{da_n} + \ldots + \frac{d^2 x_n}{dt^2} \frac{dx_n}{da_n} - \frac{dV}{da_n} + \frac{1}{\varrho} \frac{dp}{da_n} = 0
\]

where \( a_1, a_2, \ldots, a_n \) are the values of \( x_1, x_2, \ldots, x_n \) at time \( t = 0 \). Then, the transformation of these equations happens exactly in the same way. For the sake of brevity, we here limit ourselves to three variables.

This transformation happens to be very simple when the new variables form an orthogonal system, a case, which apart from this simplification, is also very interesting since all frequently used coordinate systems are included therein.

The points where \( q_1 \) takes a given value will in general form a surface, whose equation with respect to the axes of \( x, y, z \) is

\[
q_1 = f_1(x, y, z)
\]

The cosines of the angles formed by the normal to the point \( x, y, z \) of this surface and the coordinates axes, are:

\[
\frac{1}{\Delta_1} \frac{dq_1}{dx}, \frac{1}{\Delta_1} \frac{dq_1}{dy}, \frac{1}{\Delta_1} \frac{dq_1}{dz}, \quad \Delta_1^2 = \left( \frac{dq_1}{dx} \right)^2 + \left( \frac{dq_1}{dy} \right)^2 + \left( \frac{dq_1}{dz} \right)^2
\]

The analogous cosines for the normal to the surface

\[
q_2 = f_2(x, y, z)
\]

are:

\[
\frac{1}{\Delta_2} \frac{dq_2}{dx}, \frac{1}{\Delta_2} \frac{dq_2}{dy}, \frac{1}{\Delta_2} \frac{dq_2}{dz}, \quad \Delta_2^2 = \left( \frac{dq_2}{dx} \right)^2 + \left( \frac{dq_2}{dy} \right)^2 + \left( \frac{dq_2}{dz} \right)^2
\]

for the surface

\[
q_3 = f_3(x, y, z)
\]

the analogous cosines will be:

\[
\frac{1}{\Delta_3} \frac{dq_3}{dx}, \frac{1}{\Delta_3} \frac{dq_3}{dy}, \frac{1}{\Delta_3} \frac{dq_3}{dz}, \quad \Delta_3^2 = \left( \frac{dq_3}{dx} \right)^2 + \left( \frac{dq_3}{dy} \right)^2 + \left( \frac{dq_3}{dz} \right)^2
\]

Suppose now that \( q_1, q_2, q_3 \) form an orthogonal system; then, in the point of intersection the normals to the three surfaces \( q_1, q_2, q_3 \) are mutually orthogonal to each other. The conditions that the axes of \( x, y, z \) form an orthogonal system, when referring their positions to the normals of the surfaces \( q_1, q_2, q_3 \), will be the following:

\[
\frac{1}{\Delta_1^2} \left( \frac{dq_1}{dx} \right)^2 + \frac{1}{\Delta_2^2} \left( \frac{dq_2}{dx} \right)^2 + \frac{1}{\Delta_3^2} \left( \frac{dq_3}{dx} \right)^2 = 1
\]

\[
\frac{1}{\Delta_1^2} \left( \frac{dq_1}{dy} \right)^2 + \frac{1}{\Delta_2^2} \left( \frac{dq_2}{dy} \right)^2 + \frac{1}{\Delta_3^2} \left( \frac{dq_3}{dy} \right)^2 = 1
\]

\[
\frac{1}{\Delta_1^2} \left( \frac{dq_1}{dz} \right)^2 + \frac{1}{\Delta_2^2} \left( \frac{dq_2}{dz} \right)^2 + \frac{1}{\Delta_3^2} \left( \frac{dq_3}{dz} \right)^2 = 1
\]
\[
\frac{1}{\Delta_1^2} \frac{d\varphi_1}{d\varphi_1} \frac{d\varphi_1}{d\varphi_1} + \frac{1}{\Delta_2^2} \frac{d\varphi_2}{d\varphi_2} \frac{d\varphi_2}{d\varphi_2} + \frac{1}{\Delta_3^2} \frac{d\varphi_3}{d\varphi_3} \frac{d\varphi_3}{d\varphi_3} = 0
\]

\[
\frac{1}{\Delta_1^2} \frac{d\varphi_1}{d\varphi_1} \frac{d\varphi_1}{d\varphi_1} + \frac{1}{\Delta_2^2} \frac{d\varphi_2}{d\varphi_2} \frac{d\varphi_2}{d\varphi_2} + \frac{1}{\Delta_3^2} \frac{d\varphi_3}{d\varphi_3} \frac{d\varphi_3}{d\varphi_3} = 0
\] [5.32]

\[
\frac{1}{\Delta_1^2} \frac{d\varphi_1}{d\varphi_1} \frac{d\varphi_1}{d\varphi_1} + \frac{1}{\Delta_2^2} \frac{d\varphi_2}{d\varphi_2} \frac{d\varphi_2}{d\varphi_2} + \frac{1}{\Delta_3^2} \frac{d\varphi_3}{d\varphi_3} \frac{d\varphi_3}{d\varphi_3} = 0
\]

Now we have:
\[
\frac{d\varphi_1}{\Delta_1} = \frac{1}{\Delta_1} \frac{d\varphi_1}{dx} dx + \frac{1}{\Delta_1} \frac{d\varphi_1}{dy} dy + \frac{1}{\Delta_1} \frac{d\varphi_1}{dz} dz
\]

\[
\frac{d\varphi_2}{\Delta_2} = \frac{1}{\Delta_2} \frac{d\varphi_2}{dx} dx + \frac{1}{\Delta_2} \frac{d\varphi_2}{dy} dy + \frac{1}{\Delta_2} \frac{d\varphi_2}{dz} dz
\] [5.33]

\[
\frac{d\varphi_3}{\Delta_3} = \frac{1}{\Delta_3} \frac{d\varphi_3}{dx} dx + \frac{1}{\Delta_3} \frac{d\varphi_3}{dy} dy + \frac{1}{\Delta_3} \frac{d\varphi_3}{dz} dz
\]

By squaring and adding these equations, using the given relations, one obtains:
\[
\left(\frac{d\varphi_1}{\Delta_1}\right)^2 + \left(\frac{d\varphi_2}{\Delta_2}\right)^2 + \left(\frac{d\varphi_3}{\Delta_3}\right)^2 = dx^2 + dy^2 + dz^2 = ds^2
\] [5.34]

The comparison of this expression with the general one yields:
\[
N_1 = \frac{1}{\Delta_1}, \quad N_2 = \frac{1}{\Delta_2}, \quad N_3 = \frac{1}{\Delta_3}, \quad n_1 = n_2 = n_3 = 0
\] [5.35]

If one sets:
\[
N_1 = \left(\frac{dx}{d\varphi_1}\right)^2 + \left(\frac{dy}{d\varphi_1}\right)^2 + \left(\frac{dz}{d\varphi_1}\right)^2 = \frac{1}{\left(\frac{d\varphi_1}{dx}\right)^2 + \left(\frac{d\varphi_1}{dy}\right)^2 + \left(\frac{d\varphi_1}{dz}\right)^2}
\]

\[
N_2 = \left(\frac{dx}{d\varphi_2}\right)^2 + \left(\frac{dy}{d\varphi_2}\right)^2 + \left(\frac{dz}{d\varphi_2}\right)^2 = \frac{1}{\left(\frac{d\varphi_2}{dx}\right)^2 + \left(\frac{d\varphi_2}{dy}\right)^2 + \left(\frac{d\varphi_2}{dz}\right)^2}
\] [5.36]

\[
N_3 = \left(\frac{dx}{d\varphi_3}\right)^2 + \left(\frac{dy}{d\varphi_3}\right)^2 + \left(\frac{dz}{d\varphi_3}\right)^2 = \frac{1}{\left(\frac{d\varphi_3}{dx}\right)^2 + \left(\frac{d\varphi_3}{dy}\right)^2 + \left(\frac{d\varphi_3}{dz}\right)^2}
\]

so the equation for the density becomes:
\[
\begin{bmatrix}
\frac{d\varphi_1}{d\varphi_1} & \frac{d\varphi_1}{d\varphi_2} & \frac{d\varphi_1}{d\varphi_3} \\
\frac{d\varphi_2}{d\varphi_1} & \frac{d\varphi_2}{d\varphi_2} & \frac{d\varphi_2}{d\varphi_3} \\
\frac{d\varphi_3}{d\varphi_1} & \frac{d\varphi_3}{d\varphi_2} & \frac{d\varphi_3}{d\varphi_3}
\end{bmatrix}
\begin{bmatrix}
\varphi_0 \\
\varphi_0 \\
\varphi_0
\end{bmatrix} = \begin{bmatrix}
N_1^0 & N_2^0 & N_3^0 \\
N_1^0 & N_2^0 & N_3^0 \\
N_1^0 & N_2^0 & N_3^0
\end{bmatrix}
\]

and the equation [“expression” is here meant] to be varied is:
\[
\frac{d}{d\tau} \int dT \left\{ N_1 \left(\frac{d\varphi_0}{d\tau}\right)^2 + N_2 \left(\frac{d\varphi_2}{d\tau}\right)^2 + N_3 \left(\frac{d\varphi_3}{d\tau}\right)^2 + 2\Omega\right\}
\]

where \(d\tau\) indicates the new element of mass \(d\varphi_0 = d\varphi_1 d\varphi_2 d\varphi_3\) written in the new coordinates.

The part dependent on \(d\varphi_1\) of the variation of this integral is:
\[
\frac{d}{d\tau} \int dT \left\{ 2N_1 \left(\frac{d\varphi_1}{d\tau}\right)^2 + \left(\frac{d\varphi_1}{d\varphi_1}\right)^2 \frac{d\varphi_1}{d\varphi_1} + \left(\frac{d\varphi_2}{d\varphi_1}\right)^2 \frac{d\varphi_2}{d\varphi_1} + \left(\frac{d\varphi_3}{d\varphi_1}\right)^2 \frac{d\varphi_3}{d\varphi_1} + 2 \frac{d\varphi_1}{d\varphi_1} \frac{d\varphi_1}{d\varphi_1}\right\}.
\]
When the first member of this expression is integrated by parts in $t$, all members have the factor $\delta q_1$. After it is set to zero, one obtains:

\[
2 \frac{d\Omega}{d\varrho_1} = 2 \left( \frac{dN_1}{dt} \frac{d\varrho_1}{d\varrho_1} \right) - \left( \frac{d\varrho_1}{dt} \right)^2 \frac{dN_1}{d\varrho_1} - \left( \frac{d\varrho_2}{dt} \right)^2 \frac{dN_2}{d\varrho_1} - \left( \frac{d\varrho_3}{dt} \right)^2 \frac{dN_3}{d\varrho_1}, \quad [5.38]
\]

similarly, one has:

\[
2 \frac{d\Omega}{d\varrho_2} = 2 \left( \frac{dN_2}{dt} \frac{d\varrho_2}{d\varrho_2} \right) - \left( \frac{d\varrho_1}{dt} \right)^2 \frac{dN_1}{d\varrho_2} - \left( \frac{d\varrho_2}{dt} \right)^2 \frac{dN_2}{d\varrho_2} - \left( \frac{d\varrho_3}{dt} \right)^2 \frac{dN_3}{d\varrho_2} \quad (3), \quad [5.39]
\]

\[
2 \frac{d\Omega}{d\varrho_3} = 2 \left( \frac{dN_3}{dt} \frac{d\varrho_3}{d\varrho_3} \right) - \left( \frac{d\varrho_1}{dt} \right)^2 \frac{dN_1}{d\varrho_3} - \left( \frac{d\varrho_2}{dt} \right)^2 \frac{dN_2}{d\varrho_3} - \left( \frac{d\varrho_3}{dt} \right)^2 \frac{dN_3}{d\varrho_3} \quad [5.40]
\]

These are equations which are built analogously to (3) of § 4; in order for these equations to take the second Eulerian form, the so-called Lagrangian form, we multiply in turn the previous equations by:

\[
\frac{d\varrho_1}{d\varrho_1'}, \quad \frac{d\varrho_2}{d\varrho_2'}, \quad \frac{d\varrho_3}{d\varrho_3'}
\]

and we add them; then we multiply by

\[
\frac{d\varrho_1}{d\varrho_1''}, \quad \frac{d\varrho_2}{d\varrho_2''}, \quad \frac{d\varrho_3}{d\varrho_3''}
\]

and we add them as well; finally, we multiply by

\[
\frac{d\varrho_1}{d\varrho_1'''}, \quad \frac{d\varrho_2}{d\varrho_2'''}, \quad \frac{d\varrho_3}{d\varrho_3'''}
\]

and we add them too. In this way, we get the following equations:

\[
2 \frac{d\Omega}{d\varrho_1^0} = 2 \left( \frac{dN_1}{dt} \frac{d\varrho_1}{d\varrho_1^0} \right) + 2 \left( \frac{dN_2}{dt} \frac{d\varrho_2}{d\varrho_1^0} \right) + 2 \left( \frac{dN_3}{dt} \frac{d\varrho_3}{d\varrho_1^0} \right)
\]

\[
- \left( \frac{d\varrho_1}{dt} \right)^2 \frac{dN_1}{d\varrho_1^0} - \left( \frac{d\varrho_2}{dt} \right)^2 \frac{dN_2}{d\varrho_1^0} - \left( \frac{d\varrho_3}{dt} \right)^2 \frac{dN_3}{d\varrho_1^0} \quad (4), \quad [5.41]
\]

\[
2 \frac{d\Omega}{d\varrho_2^0} = 2 \left( \frac{dN_1}{dt} \frac{d\varrho_1}{d\varrho_2^0} \right) + 2 \left( \frac{dN_2}{dt} \frac{d\varrho_2}{d\varrho_2^0} \right) + 2 \left( \frac{dN_3}{dt} \frac{d\varrho_3}{d\varrho_2^0} \right)
\]

\[
- \left( \frac{d\varrho_1}{dt} \right)^2 \frac{dN_1}{d\varrho_2^0} - \left( \frac{d\varrho_2}{dt} \right)^2 \frac{dN_2}{d\varrho_2^0} - \left( \frac{d\varrho_3}{dt} \right)^2 \frac{dN_3}{d\varrho_2^0}
\]

\[
2 \frac{d\Omega}{d\varrho_3^0} = 2 \left( \frac{dN_1}{dt} \frac{d\varrho_1}{d\varrho_3^0} \right) + 2 \left( \frac{dN_2}{dt} \frac{d\varrho_2}{d\varrho_3^0} \right) + 2 \left( \frac{dN_3}{dt} \frac{d\varrho_3}{d\varrho_3^0} \right)
\]

\[
- \left( \frac{d\varrho_1}{dt} \right)^2 \frac{dN_1}{d\varrho_3^0} - \left( \frac{d\varrho_2}{dt} \right)^2 \frac{dN_2}{d\varrho_3^0} - \left( \frac{d\varrho_3}{dt} \right)^2 \frac{dN_3}{d\varrho_3^0}
\]

A very elegant example of an orthogonal system are the elliptical coordinates $\varrho_1, \varrho_2, \varrho_3$ which can be defined as the roots of the equation with respect to $\varepsilon$:

\[
\frac{x^2}{a^2 - \varepsilon^2} + \frac{y^2}{b^2 - \varepsilon^2} + \frac{z^2}{c^2 - \varepsilon^2} = 1 \quad [5.42]
\]
and so taken that one has:

\[ \alpha > \varrho_1 > \beta > \varrho_2 > \gamma > \varrho_3 > 0 \tag{5.43} \]

From the identity obtained after partial fraction decomposition:

\[
\frac{(\varepsilon^2 - \varrho_1^2) (\varepsilon^2 - \varrho_2^2) (\varepsilon^2 - \varrho_3^2)}{(\varepsilon^2 - \alpha^2) (\varepsilon^2 - \beta^2) (\varepsilon^2 - \gamma^2)} = 1 - \frac{(\alpha^2 - \varrho_1^2)}{(\alpha^2 - \beta^2) (\alpha^2 - \gamma^2)} \cdot \frac{1}{\alpha^2 - \varepsilon^2} \cdot \frac{1}{(\beta^2 - \varrho_2^2) (\beta^2 - \varrho_3^2)} \cdot \frac{1}{\beta^2 - \varepsilon^2} \cdot \frac{1}{(\gamma^2 - \varrho_3^2)} \cdot \frac{1}{\gamma^2 - \varepsilon^2}.
\tag{5.44}

When \( \varepsilon \) is put equal to one of the roots \( \varrho_1, \varrho_2, \varrho_3 \) of the equation:

\[
\frac{x^2}{\alpha^2 - \varepsilon^2} + \frac{y^2}{\beta^2 - \varepsilon^2} + \frac{z^2}{\gamma^2 - \varepsilon^2} = 1
\tag{5.45}

one obtains:

\[
\frac{(\alpha^2 - \varrho_1^2) (\alpha^2 - \varrho_2^2) (\alpha^2 - \varrho_3^2)}{(\alpha^2 - \beta^2) (\alpha^2 - \gamma^2)} 1 \cdot \frac{1}{\alpha^2 - \varepsilon^2} + \frac{(\beta^2 - \varrho_2^2) (\beta^2 - \varrho_3^2)}{(\beta^2 - \alpha^2) (\beta^2 - \gamma^2)} 1 \cdot \frac{1}{\beta^2 - \varepsilon^2} + \frac{(\varrho_3^2 - \alpha^2)}{(\varrho_3^2 - \beta^2) (\varrho_3^2 - \gamma^2)} 1 \cdot \frac{1}{\varrho_3^2 - \varepsilon^2} = 1
\tag{5.46}

which compared with the previous equations gives the relations:

\[
x^2 = \frac{(\alpha^2 - \varrho_1^2) (\alpha^2 - \varrho_2^2) (\alpha^2 - \varrho_3^2)}{(\alpha^2 - \beta^2) (\alpha^2 - \gamma^2)}
\tag{5.47}
\]

\[
y^2 = \frac{(\beta^2 - \varrho_2^2) (\beta^2 - \varrho_3^2)}{(\beta^2 - \alpha^2) (\beta^2 - \gamma^2)}
\tag{5.48}
\]

\[
z^2 = \frac{(\gamma^2 - \varrho_3^2)}{(\gamma^2 - \alpha^2) (\gamma^2 - \beta^2)}
\tag{5.47}
\]

so that, if \( \varepsilon \) is an arbitrary variable, one has:

\[
\frac{(\varepsilon^2 - \varrho_1^2) (\varepsilon^2 - \varrho_2^2) (\varepsilon^2 - \varrho_3^2)}{(\varepsilon^2 - \alpha^2) (\varepsilon^2 - \beta^2) (\varepsilon^2 - \gamma^2)} = 1 - \frac{x^2}{\alpha^2 - \varepsilon^2} - \frac{y^2}{\beta^2 - \varepsilon^2} - \frac{z^2}{\gamma^2 - \varepsilon^2}
\tag{5.48}
\]

When this equation is differentiated with respect to \( \varepsilon \) and after setting \( \varepsilon = \varrho_1, \varrho_2, \varrho_3 \), one finds:

\[
- \frac{(\varrho_1^2 - \alpha^2)}{(\varrho_1^2 - \beta^2) (\varrho_1^2 - \gamma^2)} = \left( \frac{x}{\alpha^2 - \varrho_1^2} \right)^2 + \left( \frac{y}{\beta^2 - \varrho_1^2} \right)^2 + \left( \frac{z}{\gamma^2 - \varrho_1^2} \right)^2
\tag{5.49}
\]

Logarithmical differentiation of the equations by which \( x^2, y^2, z^2 \) are represented as functions of \( \varrho_1^2, \varrho_2^2, \varrho_3^2 \) gives:

\[
- \frac{dx}{d\varrho_1} = \frac{x \varrho_1}{\alpha^2 - \varrho_1^2}, \quad - \frac{dx}{d\varrho_2} = \frac{x \varrho_2}{\alpha^2 - \varrho_2^2}, \quad - \frac{dx}{d\varrho_3} = \frac{x \varrho_3}{\alpha^2 - \varrho_3^2}
\tag{5.50}
\]
so that one has:

\[
N_1 = \varrho_1^2 \left\{ \left( \frac{x}{\alpha^2 - \varrho_1^2} \right)^2 + \left( \frac{y}{\beta^2 - \varrho_1^2} \right)^2 + \left( \frac{z}{\gamma^2 - \varrho_1^2} \right)^2 \right\}
\]

\[
N_2 = \varrho_2^2 \left\{ \left( \frac{x}{\alpha^2 - \varrho_2^2} \right)^2 + \left( \frac{y}{\beta^2 - \varrho_2^2} \right)^2 + \left( \frac{z}{\gamma^2 - \varrho_2^2} \right)^2 \right\}
\]

\[
N_3 = \varrho_3^2 \left\{ \left( \frac{x}{\alpha^2 - \varrho_3^2} \right)^2 + \left( \frac{y}{\beta^2 - \varrho_3^2} \right)^2 + \left( \frac{z}{\gamma^2 - \varrho_3^2} \right)^2 \right\}
\]

From these relations, one can therefore write:

\[
N_1 = -\varrho_1^2 \frac{(\varrho_1^2 - \varrho_2^2)(\varrho_1^2 - \varrho_3^2)}{\varrho_1^2 - \alpha^2} \frac{\varrho_2^2 - \varrho_1^2}{\varrho_2^2 - \beta^2} \frac{\varrho_3^2 - \varrho_1^2}{\varrho_3^2 - \gamma^2}
\]

\[
N_2 = -\varrho_2^2 \frac{(\varrho_2^2 - \varrho_1^2)(\varrho_2^2 - \varrho_3^2)}{\varrho_2^2 - \alpha^2} \frac{\varrho_1^2 - \varrho_2^2}{\varrho_1^2 - \beta^2} \frac{\varrho_3^2 - \varrho_2^2}{\varrho_3^2 - \gamma^2}
\]

\[
N_3 = -\varrho_3^2 \frac{(\varrho_3^2 - \varrho_1^2)(\varrho_3^2 - \varrho_2^2)}{\varrho_3^2 - \alpha^2} \frac{\varrho_1^2 - \varrho_3^2}{\varrho_1^2 - \beta^2} \frac{\varrho_2^2 - \varrho_3^2}{\varrho_2^2 - \gamma^2}
\]

These three quantities are recognised as positive because of

\[
\alpha > \varrho_1 > \beta > \varrho_2 > \gamma > \varrho_3 > 0
\]

have only to be substituted into equations (4) to obtain the hydrodynamical fundamental equations for elliptical coordinates.

The polar coordinate system \(r, \theta, \varphi\), which is determined by:

\[
x = r \cos \theta, \quad y = r \sin \theta \cos \varphi, \quad z = r \sin \theta \sin \varphi
\]

is orthogonal; in fact one has:

\[
ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi
\]

so that one has

\[
N_1 = 1, \quad N_2 = r^2, \quad N_3 = r^2 \sin^2 \theta.
\]

The density equation (2) becomes:

\[
\begin{vmatrix}
\frac{dr}{dr_0} & \frac{d\theta}{dr_0} & \frac{d\varphi}{dr_0} \\
\frac{dr}{d\theta_0} & \frac{d\theta}{d\theta_0} & \frac{d\varphi}{d\theta_0} \\
\frac{dr}{d\varphi_0} & \frac{d\theta}{d\varphi_0} & \frac{d\varphi}{d\varphi_0}
\end{vmatrix}
= \frac{r^2_0 \sin \theta_0}{r^2 \sin \theta}
\]

In this case, by setting

\[
\Phi_1 = \frac{d^2r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2 - r \sin^2 \theta \left( \frac{d\varphi}{dt} \right)^2
\]

\[
\Phi_2 = \frac{d}{dt} \left( r^2 \frac{d\theta}{dt} \right) - r^2 \sin \theta \cos \theta
\]

\[
\Phi_3 = \frac{d}{dt} \left( r^2 \sin^2 \theta \frac{d\varphi}{dt} \right)
\]
equations (3) become very simply:

\[
\frac{d\Omega}{dr} = \Phi_1, \quad \frac{d\Omega}{d\theta} = \Phi_2, \quad \frac{d\Omega}{d\varphi} = \Phi_3, \quad [5.59]
\]

and equations (4) become:

\[
\begin{aligned}
\Phi_1 \frac{dr}{dr_0} + \Phi_2 \frac{d\theta}{d\theta_0} + \Phi_3 \frac{d\varphi}{d\varphi_0} - \frac{d\Omega}{dr_0} &= 0 \\
\Phi_1 \frac{d\theta}{d\theta_0} + \Phi_2 \frac{d\varphi}{d\varphi_0} &- \frac{d\Omega}{d\theta_0} = 0 \\
\Phi_1 \frac{d\varphi}{d\varphi_0} &- \frac{d\Omega}{d\varphi_0} = 0 \quad [5.60]
\end{aligned}
\]

The same transformation can be directly performed in the following way. One observes that:

\[
\begin{aligned}
\frac{d^2 x}{dt^2} &= \cos \theta \frac{d^2 r}{dt^2} - r \sin \theta \frac{d^3 \theta}{dt^3} - r \cos \theta \left( \frac{d\theta}{dt} \right)^2 - 2 \sin \theta \frac{dr}{dt} \frac{d\theta}{dt} \\
\frac{d^2 y}{dt^2} &= \sin \theta \cos \varphi \frac{d^2 r}{dt^2} + r \cos \theta \cos \varphi \frac{d^2 \theta}{dt^2} - r \sin \theta \sin \varphi \frac{d^2 \varphi}{dt^2} - r \sin \theta \sin \varphi \left( \frac{d\varphi}{dt} \right)^2 \\
&\quad - r \sin \theta \cos \varphi \left( \frac{d\varphi}{dt} \right)^2 + 2 \cos \theta \cos \varphi \frac{dr}{dt} \frac{d\theta}{dt} - 2 \sin \theta \sin \varphi \frac{d\varphi}{dt} \frac{d\varphi}{dt} - 2 r \cos \theta \sin \varphi \frac{d\theta}{dt} \frac{d\varphi}{dt} \\
\frac{d^2 z}{dt^2} &= \sin \theta \sin \varphi \frac{d^2 r}{dt^2} + r \cos \theta \sin \varphi \frac{d^2 \theta}{dt^2} + r \sin \theta \cos \varphi \frac{d^2 \varphi}{dt^2} - r \sin \theta \sin \varphi \left( \frac{d\theta}{dt} \right)^2 \\
&\quad - r \sin \theta \sin \varphi \left( \frac{d\varphi}{dt} \right)^2 + 2 \cos \theta \sin \varphi \frac{dr}{dt} \frac{d\theta}{dt} + 2 \sin \theta \cos \varphi \frac{dr}{dt} \frac{d\varphi}{dt} + 2 r \cos \theta \sin \varphi \frac{d\theta}{dt} \frac{d\varphi}{dt} \quad [5.61]
\end{aligned}
\]

Furthermore one has:

\[
\begin{aligned}
\frac{dx}{da} &= \cos \theta \frac{dr}{da} - r \sin \theta \frac{d\theta}{da} \\
\frac{dy}{da} &= \sin \theta \cos \varphi \frac{dr}{da} + r \cos \theta \cos \varphi \frac{d\theta}{da} - r \sin \theta \sin \varphi \frac{d\varphi}{da} \quad [5.62] \\
\frac{dz}{da} &= \sin \theta \sin \varphi \frac{dr}{da} + r \cos \theta \sin \varphi \frac{d\theta}{da} + r \sin \theta \sin \varphi \frac{d\varphi}{da}
\end{aligned}
\]

then the equation:

\[
\frac{d^2 x}{dt^2} \frac{dx}{da} + \frac{d^2 y}{dt^2} \frac{dy}{da} + \frac{d^2 z}{dt^2} \frac{dz}{da} - \frac{d\Omega}{da} = 0 \quad [5.63]
\]

becomes:

\[
\Phi_1 \frac{dr}{da} + \Phi_2 \frac{d\theta}{da} + \Phi_3 \frac{d\varphi}{da} - \frac{d\Omega}{da} = 0 \quad [5.64]
\]

where \( \Phi_1, \Phi_2, \Phi_3 \) have the meaning as written above. In addition to this equation, there are two others:

\[
\begin{aligned}
\Phi_1 \frac{dr}{db} + \Phi_2 \frac{d\theta}{db} + \Phi_3 \frac{d\varphi}{db} - \frac{d\Omega}{db} &= 0 \\
\Phi_1 \frac{dr}{dc} + \Phi_2 \frac{d\theta}{dc} + \Phi_3 \frac{d\varphi}{dc} - \frac{d\Omega}{dc} &= 0 \quad [5.65]
\end{aligned}
\]
where \(a, b, c\) depend on \(r_0, \theta_0, \varphi_0\) through the relations:

\[
a = r_0 \cos \theta_0, \quad b = r_0 \sin \theta_0 \cos \varphi_0, \quad c = r_0 \sin \theta_0 \sin \varphi_0
\]  \[5.66\]

The change of variables from \(a, b, c\) to \(r_0, \theta_0, \varphi_0\) into the hydrodynamical equations can be easily carried out by multiplying with the appropriate factors and adding the equations. Then, one arrives at the above formulae in a different way.

The transformation of the fundamental equations into cylindrical coordinates is extremely simple. Namely, if one sets:

\[
x = r \cos \theta, \quad y = r \sin \theta, \quad z = z
\]  \[5.67\]

then one has:

\[
ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + dz^2
\]  \[5.68\]

so that

\[
N_1 = 1, \quad N_2 = r^2, \quad N_3 = 1
\]  \[5.69\]

hence:

\[
\frac{d^2 r}{dt^2} - r \left(\frac{d\theta}{dt}\right)^2 = \frac{d\Omega}{dr}
\]  \[5.70\]

\[
\frac{d}{dt} \left(\frac{r^2 \frac{d\theta}{dt}}{dr_0}\right) = \frac{d\Omega}{d\theta}
\]

\[
\frac{d^2 z}{dt^2} = \frac{d\Omega}{dz}
\]

In case \(r, \theta, z\) need to be expressed as functions of the initial values \(r_0, \theta_0, z_0\), one obtains the equations:

\[
\left(\frac{d^2 r}{dt^2} - r \left(\frac{d\theta}{dt}\right)^2\right) \frac{dr}{dr_0} + \frac{d}{dt} \left(\frac{r^2 \frac{d\theta}{dt}}{dr_0}\right) \frac{d\theta}{dr_0} + \frac{d^2 z}{dt^2} \frac{dz}{dz_0} = \frac{d\Omega}{dr_0}
\]

\[
\left(\frac{d^2 r}{dt^2} - r \left(\frac{d\theta}{dt}\right)^2\right) \frac{dr}{d\theta_0} + \frac{d}{dt} \left(\frac{r^2 \frac{d\theta}{dt}}{d\theta_0}\right) \frac{d\theta}{d\theta_0} + \frac{d^2 z}{dt^2} \frac{dz}{d\theta_0} = \frac{d\Omega}{d\theta_0}
\]

\[
\left(\frac{d^2 r}{dt^2} - r \left(\frac{d\theta}{dt}\right)^2\right) \frac{dr}{d\varphi_0} + \frac{d}{dt} \left(\frac{r^2 \frac{d\theta}{dt}}{d\varphi_0}\right) \frac{d\theta}{d\varphi_0} + \frac{d^2 z}{dt^2} \frac{dz}{d\varphi_0} = \frac{d\Omega}{d\varphi_0}
\]

The density equation becomes:

\[
\frac{dr}{dr_0} \frac{dr}{d\theta_0} \frac{dr}{d\varphi_0} = \frac{r_0}{r}
\]  \[5.72\]

If we take the initial conditions and the accelerating forces to be symmetric with respect to the \(z\) axis, these equations become advantageous; then we have \(\frac{d\Omega}{d\theta} = 0\), therefore \(\frac{d}{dt} \left(\frac{r^2 \frac{d\theta}{dt}}{dr_0}\right) = 0\), and thus:

\[
\frac{d\theta}{dt} = \frac{H}{r^2}
\]  \[5.73\]
where \( H \) is a time-independent constant which has always the same value for a certain particle, but varies from particle to particle and has to be determined by the initial conditions. Since \( \frac{d\theta}{dt} \) is the rotational velocity of a particle around the \( z \) axis, the rotational velocity of one and the same particle around the symmetry axis is inversely proportional to the relative distance squared of the particle to the axis. We see from this that no particle initially rotating ceases to rotate under the influence of forces generated by a potential and, conversely, no particle begins to rotate if it is not initially in rotation. This elegant theorem is thanks to Svanberg, who obtained it in cylindrical coordinates from the first Eulerian equations which, actually, are for this purpose slightly less convenient than the second Eulerian equations.

\[ \text{[A.19]} \]

§ 6.

Using the \( \Omega \) function introduced above (cf. S. 18), the hydrodynamical fundamental equations can be written as:

\[
\begin{align*}
\frac{du}{dt} \frac{dx}{da} + \frac{dv}{dt} \frac{dy}{da} + \frac{dw}{dt} \frac{dz}{da} - \frac{d\Omega}{da} &= 0 \\
\frac{du}{dt} \frac{dx}{db} + \frac{dv}{dt} \frac{dy}{db} + \frac{dw}{dt} \frac{dz}{db} - \frac{d\Omega}{db} &= 0 \\
\frac{du}{dt} \frac{dx}{dc} + \frac{dv}{dt} \frac{dy}{dc} + \frac{dw}{dt} \frac{dz}{dc} - \frac{d\Omega}{dc} &= 0
\end{align*}
\]

From these equations, one can easily eliminate the function \( \Omega \) and obtain equations which represent all possible fluid motions under the influence of potential forces. The elimination is easily done through differentiations with respect to \( a, b, c \) and subtractions from which one obtains:

\[
\begin{align*}
\frac{d^2 u}{dt dc} \frac{dx}{db} - \frac{d^2 u}{dt db} \frac{dx}{dc} + \frac{d^2 v}{dt dc} \frac{dy}{db} - \frac{d^2 v}{dt db} \frac{dy}{dc} + \frac{d^2 w}{dt dc} \frac{dz}{db} - \frac{d^2 w}{dt db} \frac{dz}{dc} &= 0 \\
\frac{d^2 u}{dt da} \frac{dx}{db} - \frac{d^2 u}{dt db} \frac{dx}{da} + \frac{d^2 v}{dt da} \frac{dy}{db} - \frac{d^2 v}{dt db} \frac{dy}{da} + \frac{d^2 w}{dt da} \frac{dz}{db} - \frac{d^2 w}{dt db} \frac{dz}{da} &= 0 \\
\frac{d^2 u}{dt da} \frac{dx}{dc} - \frac{d^2 u}{dt dc} \frac{dx}{da} + \frac{d^2 v}{dt da} \frac{dy}{dc} - \frac{d^2 v}{dt dc} \frac{dy}{da} + \frac{d^2 w}{dt da} \frac{dz}{dc} - \frac{d^2 w}{dt dc} \frac{dz}{da} &= 0
\end{align*}
\]

One can readily integrate these equations with respect to time by writing each of the three differences in these equations as an exact time derivative. Denoting the time-independent integration constants as \( 2A, 2B, 2C \), one finds:

\[
\begin{align*}
\frac{du}{dc} \frac{dx}{db} - \frac{du}{da} \frac{dx}{dc} + \frac{dv}{dc} \frac{dy}{db} - \frac{dv}{db} \frac{dy}{dc} + \frac{dw}{dc} \frac{dz}{db} - \frac{dw}{db} \frac{dz}{dc} &= 2A \\
\frac{du}{da} \frac{dx}{dc} - \frac{du}{da} \frac{dx}{db} + \frac{dv}{da} \frac{dy}{dc} - \frac{dv}{dc} \frac{dy}{db} + \frac{dw}{da} \frac{dz}{dc} - \frac{dw}{dc} \frac{dz}{db} &= 2B \\
\frac{du}{db} \frac{dx}{da} - \frac{du}{db} \frac{dx}{dc} + \frac{dv}{db} \frac{dy}{da} - \frac{dv}{da} \frac{dy}{dc} + \frac{dw}{db} \frac{dz}{da} - \frac{dw}{da} \frac{dz}{dc} &= 2C
\end{align*}
\]

\[ \text{[6.1]} \]

\[ \text{[6.2]} \]

\[ \text{[6.3]} \]
The left sides of these integral equations can be seen as the difference of two differential quotients [a curl is meant]. Namely, defining:

\[
\begin{align*}
\alpha &= u \frac{dx}{da} + v \frac{dy}{da} + w \frac{dz}{da} \\
\beta &= u \frac{dx}{db} + v \frac{dy}{db} + w \frac{dz}{db} \\
\gamma &= u \frac{dx}{dc} + v \frac{dy}{dc} + w \frac{dz}{dc}
\end{align*}
\]

one obtains, instead of (1):

\[
\frac{d\beta}{dc} - \frac{d\gamma}{db} = 2A, \quad \frac{d\gamma}{da} - \frac{d\alpha}{dc} = 2B, \quad \frac{d\alpha}{db} - \frac{d\beta}{da} = 2C
\]

These interesting relations are the analogues of the equations that Cauchy\[^{A.20}\] found already in 1816 [actually, already in 1815] for the first Eulerian dependence. They attained their actual importance only when Helmholtz\[^{A.21}\] realised their mechanical significance\[^{T.22}\] of these equations, and thereby laid the foundations for a peculiar treatment of hydrodynamics. It will be our next task to investigate this significance using a method adapted to the second Euler dependence; for this we first need to develop an appropriate theorem.

\[\text{§. 7.}\]

For this purpose we start from the known theorem that if \(\xi\) and \(\eta\) are arbitrary continuous functions of \(x\) and \(y\), one has the relation:

\[
\int (\xi dx + \eta dy) = \iint \left( \frac{d\xi}{dy} - \frac{d\eta}{dx} \right) dx \, dy
\]

where the double integral has to be extended over all elements of a domain on the \(xy\) plane, and the simple integral is over the boundary of the domain suitably oriented.\[^{A.23}\]

One can generalise this theorem in the following way: Let there be an arbitrary closed curve in space and consider the path integral over all elements of this curve:

\[
\int (\xi \, dx + \eta \, dy + \zeta \, dz)
\]

where \(\xi, \eta, \chi\) are arbitrary continuous functions of \(x, y, z\). Let us think of an arbitrary connected surface limited by this curve, so that one can consider on this whole surface \(z\) as a function of \(x\) and \(y\) and set:

\[
dz = \frac{dz}{dx} \, dx + \frac{dz}{dy} \, dy
\]

\[^{A.20}\] In an essay prized by the Paris Academy: Mémoire sur la théorie de la propagation des ondes à la surface d’un fluide pesant d’une profondeur infinie (Mém. sav. étran. Bd. 1) [1827].

\[^{A.21}\] Ueber Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen, Crelle’s Journal, Bd. 55, S.105, [1858]. \[English translation: On Integrals of the hydrodynamic equations that correspond to vortex motions. Philos. Mag. 4, Vol. 33, 485–511, [1868].\]

\[^{T.22}\] It appears likely that Helmholtz was not aware of Cauchy’s equations but stressed the importance of vortex dynamics.

\[^{A.23}\] Cf. B. Riemann, Lehrsätze aus der analysis situs für die Theorie der Integrale von zweigliedrigen vollständigen Differentialen. Crelle’s Journal Bd. 54, S. 105, [1857].
from which the given integral transforms into:

$$\int \left\{ \left( \xi + \frac{dz}{dx} \zeta \right) dx + \left( \eta + \frac{dz}{dy} \zeta \right) dy \right\}$$

By the stated theorem for [the case of] two independent variables, this integral becomes:

$$\iint \left\{ \frac{d}{dy} \left( \xi + \frac{dz}{dx} \zeta \right) - \frac{d}{dx} \left( \eta + \frac{dz}{dy} \zeta \right) \right\} dxdy$$

Since $z$ is a function of $x$ and $y$, one has:

$$\frac{dz}{dx} = -\cos \lambda \frac{dz}{dy} \quad \text{and} \quad \frac{dz}{dy} = -\cos \mu \frac{dz}{dx}$$

where $\cos \nu = \frac{dz}{dy}$ denotes the surface element.

For our purposes, we can bring both of the above integrals into more convenient forms:

If one determines three angles $\lambda'$, $\mu'$, $\nu'$ so that:

$$\cos \lambda' : \cos \mu' : \cos \nu' = \left( \frac{dn}{dz} - \frac{d\zeta}{dy} \right) : \left( \frac{d\zeta}{dx} - \frac{dn}{dz} \right) : \left( \frac{dn}{dy} - \frac{d\zeta}{dx} \right)$$

and, at the same time, assumes that they are the angles of a definite direction with the coordinates axes, so that:

$$\cos^2 \lambda' + \cos^2 \mu' + \cos^2 \nu' = 1$$

one finds:

$$2\Delta \cos \lambda' = \frac{dn}{dz} - \frac{d\zeta}{dy} \quad 2\Delta \cos \mu' = \frac{d\zeta}{dx} - \frac{dn}{dz} \quad 2\Delta \cos \nu' = \frac{dn}{dy} - \frac{d\zeta}{dx}$$

where:

$$4\Delta^2 = \left( \frac{dn}{dz} - \frac{d\zeta}{dy} \right)^2 + \left( \frac{d\zeta}{dx} - \frac{dn}{dz} \right)^2 + \left( \frac{dn}{dy} - \frac{d\zeta}{dx} \right)^2$$
The integral above, whose element is $d\sigma$, now goes into

$$2 \int \Delta (\cos \lambda \cos \lambda' + \cos \mu \cos \mu' + \cos \nu \cos \nu') d\sigma$$

and one obtains:

$$\int \left\{ \left( \frac{d\eta}{ds} - \frac{d\zeta}{dz} \right) \cos \lambda + \left( \frac{d\xi}{dx} - \frac{d\zeta}{dz} \right) \cos \mu + \left( \frac{d\xi}{dy} - \frac{d\eta}{dx} \right) \cos \nu \right\} d\sigma = 2 \int \Delta \cos \theta d\sigma,$$

(2), [7.11]

where $\theta$ is the angle between the directions determined by $\lambda, \mu, r$ and $\lambda', \mu', r'$.

Indicating by $\varrho, \sigma, \tau$ the angles of the coordinate axes with the tangent to the curve at the point $x, y, z$, one has:

$$dx = \cos \varrho ds, \quad dy = \cos \sigma ds, \quad dz = \cos \tau ds,$$

[7.12]

where $ds$ indicates the arc element. Let now $\varrho', \sigma', \tau'$ be the angles of a direction with the coordinate axes, so that:

$$\cos^2 \varrho' + \cos^2 \sigma' + \cos^2 \tau' = 1,$$

[7.13]

and furthermore [require that]

$$\cos \varrho' : \cos \sigma' : \cos \tau' = \xi : \eta : \zeta,$$

[7.14]

so one has:

$$U^2 = \xi^2 + \eta^2 + \zeta^2$$

[7.15]

and

$$U \cos \varrho' = \xi, \quad U \cos \sigma' = \eta, \quad U \cos \tau' = \zeta.$$

[7.16]

These values of $\xi, \eta, \zeta$ and $dx, dy, dz$ substituted into the simple integral, give:

$$\int (\xi dx + \eta dy + \zeta dz) = \int U(\cos \varrho \cos \varrho' + \cos \sigma \cos \sigma' + \cos \tau \cos \tau')ds$$

[7.17]

or

$$\int (\xi dx + \eta dy + \zeta dz) = \int U \cos \theta' ds$$

(3), [7.18]

where $\theta'$ indicates the angle between the directions determined by $\varrho, \sigma, \tau$ and $\varrho', \sigma', \tau'$.

§ 8.

After development of this lemma we come back to the task described at the end of § 6:

According to (1) of § 7., one can write the equation:

$$\int (\alpha da + \beta db + \gamma dc) = \int \left\{ \left( \frac{d\eta}{da} - \frac{d\zeta}{ac} \right) \cos \lambda + \left( \frac{d\xi}{ab} - \frac{d\zeta}{ac} \right) \cos \mu + \left( \frac{d\xi}{bc} - \frac{d\eta}{ac} \right) \cos \nu \right\} d\sigma_0$$

[8.1]

where the first [l.h.s.] integral is over a closed curve, the second [r.h.s.] over an arbitrary simply-connected surface bounded by that curve, and where $\lambda, \mu, \nu$, are the angles of the normal to that surface with the coordinate axes, $d\sigma_0$ is the surface element and $\alpha, \beta, \gamma$ are arbitrary functions of $a, b, c$. According to (2) of the § 7, the second integral can be transformed, so that:

$$\int (\alpha da + \beta db + \gamma dc) = 2 \int \Delta_0 \cos \theta_0 d\sigma_0.$$

(1), [8.2]
If one now takes $\alpha, \beta, \gamma$ to be the quantities defined in (2) of § 6 and makes use of equations (3) of § 6, one obtains:

$$\Delta_0 = \sqrt{A^2 + B^2 + C^2}.$$  \[8.3\]

As to $\theta_0$, it is the angle between the directions determined by $\lambda, \mu, \nu$ and $\lambda', \mu', \nu'$, the latter being determined by:

$$\Delta_0 \cos \lambda' = \Delta, \Delta_0 \cos \mu' = B, \Delta_0 \cos \nu' = C.$$ \[8.4\]

The integral on the right-hand side of the preceding equation (1) does not depend on time, thus one may choose an arbitrary value of $t$ on its left-hand side; if one sets $t = 0$, then $\alpha, \beta, \gamma$ go into the initial values of $u, v, w$ that we denote with $u_0, v_0, w_0$. One then obtains:

$$\int (u_0da + v_0db + w_0dc) = 2\int \Delta_0 \cos \theta_0 d\sigma_0$$ \[8.5\]

According to (3) of the § 7., the first integral may be transformed and thus one finds:

$$\int U_0 \cos \theta_0' ds_0 = 2\int \Delta_0 \cos \theta_0 d\sigma_0$$  \[8.6\]

where

$$U_0 = \sqrt{u_0^2 + v_0^2 + w_0^2}$$ \[8.7\]

is the initial velocity, $\theta_0'$ is the angle between $U_0$ and the element $ds_0$ of the closed curve.

Let us assume, that the given closed curve is a circle with an infinitely small radius $r$ and the surface spanned by the curve is the disk of the circle. So it is clear that $\Delta_0 \cos \theta_0$ for different points of the circle surface changes only by infinitely small quantities. Then, we have $\int \Delta_0 \cos \theta_0 d\sigma = \pi r^2 \cdot \Delta_0 \cos \theta_0$. The component of the initial velocities with respect to the tangent to this circle is $U_0 \cos \theta_0$, a quantity, that for different points of the circumference will take different values. However, $U_0 \cos \theta_0'$ in each point may be considered as the sum of two velocities $T_0 + T_0'$, where $T_0'$ is the progressive motion projected to a tangent, which is common to all points of the circle and $T_0$ the tangential velocity by the rotation around the center of the infinitely small circle. The progressive motion of all particles is the same; as a consequence its component $T_0'$ with respect to the tangent of the circle will be the same for two points diametrically opposite, but of opposite signs, so that $\int T_0' ds_0 = 0$, when one integrates over the whole circle. The relative velocity of the particles will be the same except for infinitely small quantities provided the velocities are continuous functions of the position, which we always suppose. So, it follows that $\int T_0 ds_0 = T_0 \cdot \int ds_0 = 2\pi r \cdot T_0$. From the equation (2) it follows now $2\pi r \cdot T_0 = 2 \cdot \pi r^2 \cdot \Delta_0 \cos \theta_0$ or:

$$\Delta_0 \cos \theta_0 = \frac{T_0}{r}$$ \[8.8\]

Since $T_0$ is the tangential velocity, so $T_0 : r$ is the rotational velocity around the infinitely small distant center of the circle, or — as one can say, in order to take into account, also the location and orientation of the circle — is the rotational velocity around the normal to the surface of the circle taken as axis. Since $\theta_0$ is the angle between the normal to the surface element and the direction determined by the angles $\lambda', \mu', \nu'$, then $\Delta_0$ in a point becomes the rotational velocity around an axis passing by this point and oriented in the direction $\lambda', \mu', \nu'$.

Furthermore $A, B, C$ are the components of the rotational velocity of a particle $a, b, c$ around axes, which are parallel to the coordinates axes through the point $a, b, c$.

§ 9.

Analogously to (2) of § 8, one has at each time $t$:

$$\int U \cos \theta' ds = 2\int \Delta \cos \theta d\sigma$$ \[9.1\]
where now $U \cos \theta'$ is the component of the velocity of a particle $x, y, z$ with respect to the
tangent to the closed curve, over whose element $ds$ the first integral has to be extended,
and where $\Delta \cos \theta$ is the component of the rotational velocity expressed with respect to the
normal to the element $d\sigma$ of the surface, which is bounded by that curve.

According to (3) of § 7, one has

$$\int U \cos \theta' \, ds = \int (udx + vdy + wdz)$$

[9.2]

but, since from (2) of § 6 one easily infers that

$$\alpha da + \beta db + \gamma dc = udx + vdy + wdz,$$

[9.3]

one has :

$$\int U \cos \theta' \, ds = \int (\alpha da + \beta db + \gamma dc)$$

[9.4]

The second integral is calculated according to (1) of § 8 :

$$\int U \cos \theta' \, ds = 2 \int \Delta_0 \cos \theta_0 d\sigma_0 .$$

[9.5]

Hence, because of equation (1), one obtains :

$$\int \Delta \cos \theta d\sigma = \int \Delta_0 \cos \theta_0 d\sigma_0$$

(2), [9.6]

Herefrom follows that $\int \Delta \cos \theta' d\sigma$ is constant with respect to time, provided the integral is
always extended over a surface moving with the flow and consisting of the same particles.
Following Helmholtz, if one designates by rotational intensity the product of the ro-
tational velocity around the normal to the surface as an axis, times the size of the surface
element, one obtains the following result :

The integral of the rotational intensity over a surface always formed by the same particles
remains unchanged in time.

Since this theorem is valid, irrespective of how small the surface may be, it is also valid
for each single surface element : the rotational intensity of a surface element always stays the
same. Because such a particle cannot spread out infinitely, its rotational velocity cannot
decrease infinitely. It follows herefrom that no particle once put into rotational motion can
stop rotating ; and on the other hand, one easily sees that no particle which at initial time
is not rotating, may ever begin to rotate.

One must remark that these results are obtained under the assumption that the accel-
erating forces acting on the fluid are partial derivatives of a potential function. If, however,
the accelerating forces do not possess this property, these theorems do not apply. Here-
from one obtains a criterion to know whether accelerating forces acting on a fluid without
pressure forces, have or do not have a potential. In the latter case, the rotational intensity
of single particles is not conserved; in general new particles begin rotating, and rotating
particles will lose this characteristic motion.

So far we have considered only rotating surface elements, but let us take into account a
mass element which is contained in a cylinder whose axis is the rotation axis. Its constant
mass is the product of its transverse section, its length and its density. Since the product
of the rotational velocity by the transverse section is constant, one sees that for each element
the ratio of its rotational velocity to the product of the distance measured in the direction of
its rotation axis by the density is constant. Therefore, if the fluid is liquid, i.e., its density
considered as constant, the ratio of the rotational velocity to the length of the particle is
constant.

§ 10.

In his theory of rotational motion, Helmholtz introduced this following important prin-
ciple: instead of considering the whole rotating mass, one should fragment it into vortex
lines. Here, a vortex line is a line lying in the flow so that its direction will stay always parallel to the instantaneous rotation axis. By vortex filament, we understand the infinitely thin cylinder which, wrapped around the vortex line, includes the rotating particles.

Denoting by $da, db, dc$ [the three components of] an element of such a vortex line at time $t = 0$, one obviously has:

$$da : db : dc = A : B : C$$  \hspace{1cm} (1), [10.1]

Let $\varphi$ and $\psi$ be functions of $a, b, c$, such that the vortex lines at time $t = 0$ are obtained by setting these two functions to constant values. Then the following conditions must hold:

$$\frac{d\varphi}{da}da + \frac{d\varphi}{db}db + \frac{d\varphi}{dc}dc = 0$$  \hspace{1cm} [10.2]

or, according to (1):

$$\frac{d\varphi}{da}A + \frac{d\varphi}{db}B + \frac{d\varphi}{dc}C = 0$$

$$\frac{d\psi}{da}A + \frac{d\psi}{db}B + \frac{d\psi}{dc}C = 0$$  \hspace{1cm} [10.3]

If $A, B, C$ were known, one could find $\varphi$ and $\psi$ from these equations by integration. One can easily observe that $\varphi, \psi$ must be such that, one may write:

$$-2A = \begin{vmatrix} \frac{d\varphi}{dx} & \frac{d\varphi}{dy} \\ \frac{d\psi}{dx} & \frac{d\psi}{dy} \end{vmatrix}$$

$$-2B = \begin{vmatrix} \frac{d\varphi}{dx} & \frac{d\varphi}{dz} \\ \frac{d\psi}{dx} & \frac{d\psi}{dz} \end{vmatrix}$$

$$-2C = \begin{vmatrix} \frac{d\varphi}{dt} & \frac{d\varphi}{dz} \\ \frac{d\psi}{dt} & \frac{d\psi}{dz} \end{vmatrix}$$  \hspace{1cm} (2), [10.4]

Indeed, from the preceding equations follows that $A, B, C$ are proportional to these determinants; however, by substitution of (2) into (3) of § 6, the following relation is identically satisfied:

$$\frac{dA}{da} + \frac{dB}{db} + \frac{dC}{dc} = 0$$  \hspace{1cm} (3), [10.5]

Thus, $\varphi, \psi$ are always determined by integration in such a way that equations (2) are satisfied.

From these considerations it follows that rotating particles may always be considered as arranged in vortex filaments. When such a vortex filament moves along with the fluid, always the same particles will belong to it, as no particle belonging to the vortex filament may lose its rotational motion and, furthermore, each element parallel to the rotation axis of the vortex filament always remains parallel. Namely, a surface element, which, at some time, is parallel to the rotation axis, will have a vanishing rotational intensity with respect to the direction of its normal; since the rotational intensity always stays the same, the normal to the surface element always stays perpendicular to the rotation axis, because the particles themselves always maintain their rotational motion around the axis of the vortex filament. Therefore, one obtains the equations of the vortex lines at time $t$, if one expresses the values of $a, b, c$ in $\varphi$ and $\psi$ through $x, y, z, t$. The equations of the vortex lines at time $t$ must be such that $\frac{d\varphi}{dt} = 0$ and $\frac{d\psi}{dt} = 0$ are satisfied. Here one has to differentiate with respect to $t$, whether it appears explicitly or implicitly, through $x, y, z$. In motion, the vortex filament will sometimes grow in density and size and sometimes decrease, namely in such a way that the length of a vortex-tube element multiplied by the density remains proportional to the rotational velocity. (Cf. S. 40 [now page 30].)

The rotational intensity in each section of this vortex filament will remain unchanged over time. Moreover, it is also the same for all sections. Obviously we have $\int \Delta \cos \theta d\sigma = 0$ if the integral extends over a closed surface. Namely, let us think of a closed curve drawn on this surface. In order to extend $\int \Delta \cos \theta d\sigma$ over the two portions of the surface separated
by the curve, one has to integrate \(\int U \cos \theta \, ds\) twice but in the opposite direction over this curve, so that one has \(\int \Delta \cos \theta \, d\sigma = 0\). Let this closed surface be formed by two sections of a vortex filament and the part of the filament surface between them, then \(\int \Delta \cos \theta \, d\sigma\) will disappear for the latter part, and so will the sum of the integrals for both sections. Indeed, since \(\cos \theta\), in a section, is +1 [and] −1 in the other one, the rotational intensity with regard to the axis of the vortex filament is the same for each section.

The results of the preceding investigations about these particular rotational motions are essentially due to Helmholtz, who, by restricting himself to liquid flows, derived them in a classical essay\(^{[A.24]}\) from the Eulerian first form of the equations.

A special case of the theorem on the constancy of the rotational velocity has been already given by Svanberg:

Namely, when all motions happen symmetrically around an axis and circular vortex filaments centered on the symmetry axis are given in the flow with radius \(r\) and infinitely small section \(\omega\), such a vortex filament may be considered as a cylinder of height \(2\pi r\) and basis \(\omega\) so that its content will be \(2\pi r\omega\). Since the rotational motion of the particles is conserved, the content \(2\pi r\omega\) for each vortex filament must be constant in time. Then \(\Delta \omega r\) must be proportional to the rotational velocity \(\Delta\). Since \(\Delta \omega\), as rotational intensity, is constant, so the rotational velocity \(\Delta\) for each vortex ring at different times must be proportional to its radius.

This is the meaning of the formulae derived by Svanberg from the first Eulerian form\(^{[A.25]}\) of the fundamental equations.

§ 11.

The concept of vortex lines gives rise to an interesting transformation of the hydrodynamical equations in their first form: Namely, by introducing as dependent variables, functions which are in a certain relation to \(u, v, w\).

From equations (2) of § 10 follows indeed:

\[
\frac{d\gamma}{db} - \frac{d\beta}{dc} = \frac{d\varphi}{db} \frac{d\psi}{dc} - \frac{d\varphi}{dc} \frac{d\psi}{db},
\]

\[
\frac{d\alpha}{dc} - \frac{d\gamma}{da} = \frac{d\varphi}{dc} \frac{d\psi}{da} - \frac{d\varphi}{da} \frac{d\psi}{dc},
\]

\[
\frac{d\beta}{da} - \frac{d\alpha}{db} = \frac{d\varphi}{da} \frac{d\psi}{db} - \frac{d\varphi}{db} \frac{d\psi}{da},
\]

provided that \(\varphi\) and \(\psi\) are functions of \(a, b, c\) that represent the vortex line, when set equal to constants. Herefrom follows that one can represent \(\alpha, \beta, \gamma\) in the form:

\[
\alpha = \frac{dF}{da} + \varphi \frac{d\psi}{da}, \quad \beta = \frac{dF}{db} + \varphi \frac{d\psi}{db}, \quad \gamma = \frac{dF}{dc} + \varphi \frac{d\psi}{dc}
\]

(1), \([11.2]\)

where in general \(F\) will be function of \(a, b, c, t\). Herefrom one finds:

\[
\alpha \frac{da}{dx} + \beta \frac{db}{dx} + \gamma \frac{dc}{dx} = \frac{dF}{dx} + \varphi \frac{d\psi}{dx},
\]

\[
\alpha \frac{da}{dy} + \beta \frac{db}{dy} + \gamma \frac{dc}{dy} = \frac{dF}{dy} + \varphi \frac{d\psi}{dy},
\]

\[
\alpha \frac{da}{dz} + \beta \frac{db}{dz} + \gamma \frac{dc}{dz} = \frac{dF}{dz} + \varphi \frac{d\psi}{dz}
\]

([11.3])

\[\text{[A.24]}\] Crelle’s Journal, Bd. 55, S. 33 and ff [1858].

\[\text{[A.25]}\] Crelle’s Journal Bd. 24. S. 159. Nro. 31 [1842].
By solving the system (2) of § 6 using the relations (3) of § 2, one finds:

\[
\begin{align*}
\alpha \frac{da}{dx} + \beta \frac{db}{dx} + \gamma \frac{dc}{dx} &= u \\
\alpha \frac{da}{dy} + \beta \frac{db}{dy} + \gamma \frac{dc}{dy} &= v \\
\alpha \frac{da}{dz} + \beta \frac{db}{dz} + \gamma \frac{dc}{dz} &= w
\end{align*}
\]  

so that one can always set:

\[
\begin{align*}
u &= \frac{dF}{dx} + \varphi \frac{d\psi}{dx}, \\
v &= \frac{dF}{dy} + \varphi \frac{d\psi}{dy}, \\
w &= \frac{dF}{dz} + \varphi \frac{d\psi}{dz}
\end{align*}
\]  

where \( \varphi = \text{Const} \) and \( \psi = \text{Const} \) are the equations of the vortex lines. If one replaces the quantities \( a, b, c \) in \( \varphi \) and \( \psi \) in terms of \( x, y, z \) [at time] \( t \), one obtains:

\[
\begin{align*}
\frac{d\varphi}{dt} &= 0, \\
\frac{d\psi}{dt} &= 0
\end{align*}
\]

whereby differentiation has to be performed when \( t \) appears explicitly as well as implicitly in \( x, y, z \). Hence we have:

\[
\begin{align*}
\frac{d\varphi}{dt} + \frac{d\varphi}{dx} u + \frac{d\varphi}{dy} v + \frac{d\varphi}{dz} w &= 0 \\
\frac{d\psi}{dt} + \frac{d\psi}{dx} u + \frac{d\psi}{dy} v + \frac{d\psi}{dz} w &= 0
\end{align*}
\]

Substituting the values of \( u, v, w \) [taken from (2)], we obtain:

\[
\begin{align*}
\frac{d\varphi}{dt} + \frac{d\varphi}{dx} \left( \frac{dF}{dx} + \varphi \frac{d\psi}{dx} \right) + \frac{d\varphi}{dy} \left( \frac{dF}{dy} + \varphi \frac{d\psi}{dy} \right) + \frac{d\varphi}{dz} \left( \frac{dF}{dz} + \varphi \frac{d\psi}{dz} \right) &= 0 \\
\frac{d\psi}{dt} + \frac{d\psi}{dx} \left( \frac{dF}{dx} + \varphi \frac{d\psi}{dx} \right) + \frac{d\psi}{dy} \left( \frac{dF}{dy} + \varphi \frac{d\psi}{dy} \right) + \frac{d\psi}{dz} \left( \frac{dF}{dz} + \varphi \frac{d\psi}{dz} \right) &= 0
\end{align*}
\]

In addition to these relations there is also the density equation (5) of § 2.

\[
\begin{align*}
\frac{d\varrho}{dt} + \frac{d\varrho}{dx} \left( \frac{dF}{dx} + \varphi \frac{d\psi}{dx} \right) + \frac{d\varrho}{dy} \left( \frac{dF}{dy} + \varphi \frac{d\psi}{dy} \right) + \frac{d\varrho}{dz} \left( \frac{dF}{dz} + \varphi \frac{d\psi}{dz} \right) &= 0
\end{align*}
\]

We observe that, in general, these three equations are not sufficient to determine the four unknown functions \( F, \varphi, \psi \) and \( \varrho \). Only in the case of liquid fluids, where \( \varrho \) is constant, are these sufficient, insofar as the latter expression transforms into:

\[
\begin{align*}
\frac{d}{dx} \left( \frac{dF}{dx} + \varphi \frac{d\psi}{dx} \right) + \frac{d}{dy} \left( \frac{dF}{dy} + \varphi \frac{d\psi}{dy} \right) + \frac{d}{dz} \left( \frac{dF}{dz} + \varphi \frac{d\psi}{dz} \right) &= 0
\end{align*}
\]

These are the transformed equations in the elegant form that Clebsch found by another way, without giving the meaning of \( \varphi \) and \( \psi \).[A.26]

[A.26] Ueber die Integration der hydrodynamischen Gleichungen (Creelle’s Journal Bd. 56, S. 1 [1859]). In this essay Clebsch starts from the first form of the Euler equation to perform the above transformation. On
§. 12

If one puts into the equations (3) of § 6, \( t = 0 \), one finds:

\[
\frac{dv_0}{dc} - \frac{dw_0}{db} = 2A, \quad \frac{dw_0}{da} - \frac{du_0}{dc} = 2B, \quad \frac{du_0}{da} - \frac{dv_0}{db} = 2C \quad [12.1]
\]

where \( A, B, C \) are the rotational velocities with respect to the coordinates axes in the point \( a, b, c \) at time \( t = 0 \). Quite similarly, at time \( t \) we have:

\[
\frac{dv}{dz} - \frac{dw}{dy} = 2X, \quad \frac{dw}{dx} - \frac{du}{dz} = 2Y, \quad \frac{du}{dy} - \frac{dv}{dz} = 2Z \quad \text{(1), [12.2]}
\]

These are the equations cited at page 34 [now page 26] found by Cauchy, where \( X, Y, Z \) mean the rotational velocities with respect to the coordinates \( x, y, z \) at time \( t \) taken as axes. If \( A, B, C \) are zero overall, i.e. at the beginning of the motion, there are no rotating particles and thus \( X, Y, Z \) will stay zero and hence, one can write

\[
u = \frac{dF}{dx}, \quad \phi = \frac{dF}{dy}, \quad \psi = \frac{dF}{dz} \quad \text{[2], [12.3]}
\]

Restricting ourselves to liquid flows, by this substitution the density equation (6) of § 2. becomes:

\[
\frac{d^2F}{dx^2} + \frac{d^2F}{dy^2} + \frac{d^2F}{dz^2} = 0; \quad \text{(3), [12.4]}
\]

therefore, \( F \) satisfies Laplace’s differential equation for all fluid particles; for this reason the function \( F \) has been designated by Helmholtz as the velocity potential. From the equation (3) derives an interesting analogy between the fluid motions in a simply connected space and the effects of magnetic masses: the velocities are equal and aligned to the forces exerted by a certain distribution of magnetic masses on the surface on a magnetic particle in the interior.

If the function \( F \) is determined by (3) by suitable boundary conditions on the surface, it is still necessary to determine the pressure \( p \) in order to get a complete solution to the problem. The required equation for that is easily obtained, by using the values of \( u, v, w \) from (2) in (4) of § 4. Then, one obtains:

\[
\left\{\begin{aligned}
\frac{d^2F}{dt dx} + \frac{d^2F}{dx^2} + \frac{d^2F}{dx dy} + \frac{d^2F}{dx dz} - \frac{dV}{dx} \frac{1}{\rho} \frac{dp}{dz} & = 0 \\
\frac{d^2F}{dt dy} + \frac{d^2F}{dx dy} + \frac{d^2F}{dy^2} + \frac{d^2F}{dy dz} - \frac{dV}{dy} \frac{1}{\rho} \frac{dp}{dy} & = 0 \\
\frac{d^2F}{dt dz} + \frac{d^2F}{dx dz} + \frac{d^2F}{dx dy} + \frac{d^2F}{dy^2} + \frac{d^2F}{dz^2} - \frac{dV}{dz} \frac{1}{\rho} \frac{dp}{dz} & = 0
\end{aligned}\right. \quad [12.5]
\]

the one hand, the method applied by him is susceptible of a bigger simplification; on the other hand, one can give a procedure quite analogous to that used above to prove that \( \frac{d\phi}{dt} = 0 \) and \( \frac{d\psi}{dt} = 0 \). Clebsch showed (in Theorem 2) that the equations transformed by the introduction of \( F, \phi, \psi \) are the conditions for the variation of a quadruple integral to disappear; this is somewhat similar to what Clebsch did for steady-state flows in his paper: Ueber eine allgemeine Transformation der hydrodynamischen Gleichungen (Crelle’s Journal Bd. 54, S. 301 [1857]). In another form, this is the Theorem, I derived directly from the principle of the virtual velocities § 5, eq. (1). One can easily show that from this result one arrives at the introduction of the functions \( a \) for the case of the stationary motion in a simpler way than Clebsch does in the latter essays; for a lack of space, we are not giving a more accurate argumentation here.
Through multiplication with \(dx, dy, dz\) and summation, one gets:

\[
\frac{dF}{dt} + \frac{1}{2} \left( \frac{dF}{dx} \right)^2 + \frac{1}{2} \left( \frac{dF}{dy} \right)^2 + \frac{1}{2} \left( \frac{dF}{dz} \right)^2 - d\Omega = 0 \quad [12.6]
\]

where \(\Omega\) is the function defined on \(S.18\) [eq. [5.5], now on page 15]. It follows herefrom by integration:

\[
\frac{dF}{dt} + \frac{1}{2} \left( \frac{dF}{dx} \right)^2 + \frac{1}{2} \left( \frac{dF}{dy} \right)^2 + \frac{1}{2} \left( \frac{dF}{dz} \right)^2 - \Omega = 0 \quad [12.7]
\]

where the additive integration constant, which is a function of \(t\), can be included in \(F\). Apart from the unknown \(p\) the known function \(V\) is contained in \(\Omega\), from which one can easily determine \(p\) once \(F\) is obtained.

§ 13.

From the relations (1) of § 12,

\[
\frac{dv}{dz} - \frac{dw}{dy} = 2X, \quad \frac{dw}{dx} - \frac{du}{dz} = 2Y, \quad \frac{du}{dy} - \frac{dv}{dz} = 2Z \quad (1), \quad [13.1]
\]

and from the density equation (6) of § 2 for liquid fluids

\[
\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0, \quad [13.2]
\]

\(u, v, w\) can be determined as functions of \(X, Y, Z\). Indeed, one finds easily from these equations:

\[
\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} = 2 \left( \frac{dZ}{dy} - \frac{dY}{dz} \right)
\]

\[
\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} = 2 \left( \frac{dX}{dz} - \frac{dZ}{dx} \right) \quad [13.3]
\]

\[
\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2} = 2 \left( \frac{dY}{dx} - \frac{dX}{dy} \right)
\]

The integration of these partial differential equations follows from known theorems: \(u, v, w\) appear as potential functions of fictitious, attracting masses, which are distributed through the fluid-filled space with the density,

\[
-\frac{1}{2\pi} \left( \frac{dZ}{dy} - \frac{dY}{dz} \right), \quad -\frac{1}{2\pi} \left( \frac{dX}{dz} - \frac{dZ}{dx} \right), \quad -\frac{1}{2\pi} \left( \frac{dY}{dx} - \frac{dX}{dy} \right). \quad [13.4]
\]

Denoting by \(u_1, v_1, w_1\) the velocity components at point \(x_1, y_1, z_1\), and by \(r\) the distance of this point to \(x, y, z\), one has:

\[
u_1 = \frac{1}{2\pi} \iiint \left( \frac{dZ}{dy} - \frac{dY}{dz} \right) \frac{dxdydz}{r} \quad [13.5]
\]

\[
v_1 = \frac{1}{2\pi} \iiint \left( \frac{dX}{dz} - \frac{dZ}{dx} \right) \frac{dxdydz}{r} \quad [13.5]
\]

\[
w_1 = \frac{1}{2\pi} \iiint \left( \frac{dY}{dx} - \frac{dX}{dy} \right) \frac{dxdydz}{r}
\]
where the integrals have to be extended over all points of the continuous fluid. These are not yet the most general values of \( u_1, v_1, w_1 \): since these values \( u_1, v_1, w_1 \) are determined, one can always add to them successively

\[
\frac{dP_1}{dx_1}, \quad \frac{dP_1}{dy_1}, \quad \frac{dP_1}{dz_1},
\]

while equations (1) are still satisfied as well as the density equation, provided that:

\[
\frac{d^2 P}{dx^2} + \frac{d^2 P}{dy^2} + \frac{d^2 P}{dz^2} = 0
\]

[13.6]
is assumed for all points of the fluid. One can consider here \( P \) as the potential function of attracting masses which are outside of the space filled with the fluid, and must be determined so that the conditions for \( u_1, v_1, w_1 \) on the fluid’s surface are satisfied.

The values found for \( u_1, v_1, w_1 \) in this way can be transformed through integration by parts. Since:

\[
r^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2
\]

one finds:

\[
\iiint \frac{dY}{dz} \frac{dx}{r} \frac{dy}{dz} = \iiint Y \frac{dx}{r} + \iiint Y \frac{z - z_1}{r^3} \frac{dx}{dz} \frac{dy}{dz}
\]

[13.8]

\[
\iiint \frac{dZ}{dy} \frac{dx}{r} \frac{dz}{dy} = \iiint Z \frac{dx}{r} + \iiint Z \frac{y - y_1}{r^3} \frac{dx}{dy} \frac{dz}{dy}
\]

and thus

\[
u_1 = \frac{dP_1}{dx_1} + \frac{1}{2\pi} \iiint (Y \cos \gamma - Z \cos \beta) \frac{d\omega}{r} + \frac{1}{2\pi} \iiint \{Y(z - z_1) - Z(y - y_1)\} \frac{dx}{r^3} \frac{dy}{dz} \frac{dz}{r^3}
\]

and in the same way:

\[
v_1 = \frac{dP_1}{dy_1} + \frac{1}{2\pi} \iiint (Z \cos \alpha - X \cos \gamma) \frac{d\omega}{r} + \frac{1}{2\pi} \iiint \{Z(x - x_1) - X(z - z_1)\} \frac{dx}{r^3} \frac{dy}{dz} \frac{dz}{r^3}
\]

\[
w_1 = \frac{dP_1}{dz_1} + \frac{1}{2\pi} \iiint (X \cos \beta - Y \cos \alpha) \frac{d\omega}{r} + \frac{1}{2\pi} \iiint \{X(y - y_1) - Y(x - x_1)\} \frac{dx}{r^3} \frac{dy}{dz} \frac{dz}{r^3}
\]

[13.9]

where \( d\omega \) denotes a surface element and \( \alpha, \beta, \gamma \) the angle formed by the normal to it with the coordinates axes.

These analytical formulas for the representation of \( u_1, v_1, w_1 \) in terms of \( X, Y, Z \) allow for an interesting interpretation leading to a striking analogy between the effect of vortex filaments and that of electrical currents. Namely, if we indicate the parts of \( u_1, v_1, w_1 \) which originate from the elements \( dx \, dy \, dz \) of the triple integrals with \( du_1 du_1 dv_1 dw_1 \) then one has:

\[
\begin{align*}
&du_1 = \frac{1}{2\pi} \left\{ Y(z - z_1) - Z(y - y_1) \right\} \frac{dx}{r^3} \frac{dy}{dz} \frac{dz}{r^3} \\
&dv_1 = \frac{1}{2\pi} \left\{ Z(x - x_1) - X(z - z_1) \right\} \frac{dx}{r^3} \frac{dy}{dz} \frac{dz}{r^3} \\
&dw_1 = \frac{1}{2\pi} \left\{ X(y - y_1) - Y(x - x_1) \right\} \frac{dx}{r^3} \frac{dy}{dz} \frac{dz}{r^3}
\end{align*}
\]

[13.10]
By known theorems, from these equations one derives the relations:

\[(x - x_1) \, du_1 + (y - y_1) \, dv_1 + (z - z_1) \, dw_1 = 0, \tag{13.11}\]

\[X \, du_1 + Y \, dv_1 + Z \, dw_1 = 0, \tag{13.12}\]

\[du_1^2 + dv_1^2 + dw_1^2 = \left( \frac{dx \, dy \, dz}{2\pi r^3} \right)^2 \left\{ (X^2 + Y^2 + Z^2)(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 \right\} \]

\[- [X(x - x_1) + Y(y - y_1) + Z(z - z_1)]^2]. \tag{13.13}\]

The first two equations show that the velocity with the components \(du_1, dv_1, dw_1\)

\[dU_1 = \sqrt{du_1^2 + dv_1^2 + dw_1^2} \tag{13.14}\]

is normal to the plane containing the point \(x_1, y_1, z_1\) and the rotation axis of the fluid particles \(x, y, z\). The previous equation can also be written:

\[dU_1^2 = \left( \frac{dx \, dy \, dz}{2\pi r^2} \Delta \right)^2 \left\{ 1 - \left[ \frac{X \, x - x_1}{\Delta} + \frac{Y \, y - y_1}{\Delta} + \frac{Z \, z - z_1}{\Delta} \right]^2 \right\} \tag{13.15}\]

where \(\Delta\) means the rotational velocity. Herefrom one finds:

\[dU_1 = \frac{dx \, dy \, dz}{2\pi r^2} \Delta \sin \varepsilon \tag{13.16}\]

where \(\varepsilon\) indicates the angle between the rotation axis of the particle \(x, y, z\) and the connecting line \(r\) between this point and \(x_1, y_1, z_1\). Each rotating particle \(x, y, z\) generates, then, into another particle of the fluid \(x_1, y_1, z_1\) a velocity which is normal to the plane passing through the rotation axis of the particle \(x, y, z\) and \(x_1, y_1, z_1\). This velocity is directly proportional to the volume \(dx \, dy \, dz\) of the particle \(x, y, z\), to its rotational velocity \(\Delta\) and to the sinus of the angle \(\varepsilon\) between the rotation axis of \(x, y, z\) and the connecting line \(r\) of both particles, is inversely proportional to the square of the distance \(r\) between both particles.

However, according to Ampère’s Law, this is the same force that an electrical particle of intensity \(\Delta\) located at \(x, y, z\) in a current aligned parallelly to the rotational axis, would exert on a little magnet located at \(x_1, y_1, z_1\).

This highly remarkable analogy, whose discovery is due to Helmholtz, is firstly of great importance for the theory of the vortex filaments in liquid fluids — since equations (2) are only applicable to such liquid fluids. Indeed, it allows to apply theorems developed in electrodynamics, with minor modifications, to hydrodynamics, making the visualisation significantly simpler. Furthermore, this analogy is also of a certain value for electrodynamics, since it allows to analyse the electrodynamical processes not based on the mutual elementary interactions between two particles — as is it is usually done following Ampère’s procedure — but by considering infinitely thin closed currents as a basis for the whole theory, in analogy with vortex filaments.

§ 14.

The principle of virtual velocities for the motion of liquid fluids

\[
\frac{d\delta x}{dx} + \frac{d\delta y}{dy} + \frac{d\delta z}{dz} = 0, \tag{14.1}\]

gives the next equation, following from (1) of § 4:

\[
\int \int \left[ \left( \frac{d^2 x}{dt^2} - X \right) \delta x + \left( \frac{d^2 y}{dt^2} - Y \right) \delta y + \left( \frac{d^2 z}{dt^2} - Z \right) \delta z \right] dx \, dy \, dz = 0. \tag{14.2}\]
If one takes, instead of the virtual velocities, the actual velocities, one has to put, instead of $\delta x, \delta y, \delta z$:

$$dx = \frac{dx}{dt}, \quad dy = \frac{dy}{dt}, \quad dz = \frac{dz}{dt},$$  \[14.3\]

and, since (1), turns into the continuity equation, when limiting oneself to liquid flow, one finds, by the usual assumptions about $X, Y, Z$:

$$\int \int \int \left( \frac{d^2 x}{dt^2} \frac{dx}{dt} + \frac{d^2 y}{dt^2} \frac{dy}{dt} + \frac{d^2 z}{dt^2} \frac{dz}{dt} \right) dx \, dy \, dz = \int \int \int \left( \frac{dV}{dx} \frac{dx}{dt} + \frac{dV}{dy} \frac{dy}{dt} + \frac{dV}{dz} \frac{dz}{dt} \right) dx \, dy \, dz$$  \[14.4\]

From this equation, after integration in time, denoting by

$$K = \frac{1}{2} \int \int \int \left\{ \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2 \right\} dx \, dy \, dz$$  \[14.5\]

the living force

$$K = \text{Const.} + \int \int \int V \, dx \, dy \, dz$$  \[14.6\]

where the constant is independent of $t$. This equation of the living force has been presumably given for the first time by Lejeune-Dirichlet.[A.27]

Consequently, the living force will be independent of time, provided that the integral $\int \int \int V \, dx \, dy \, dz$ is constant in time, in other words, since $V$ for each point in space is independent of $t$. The condition that the fluid always fills the same absolute space may be also expressed by the fact that the normal velocity components of the particles at the surface is zero. One may show directly that, in this case, $K$ is constant. Namely, one has:

$$\frac{dK}{dt} = \int \int \int \left( \frac{dV}{dx} u + \frac{dV}{dy} v + \frac{dV}{dz} w \right) dx \, dy \, dz$$  \[14.7\]

and because of

$$\int \int \int \frac{dV}{dx} u \, dx \, dy \, dz = \int \int V \, dy \, dz - \int \int \int \frac{dV}{dx} \frac{du}{dx} \, dx \, dy \, dz$$
$$\int \int \int \frac{dV}{dy} v \, dx \, dy \, dz = \int \int V \, dz \, dx - \int \int \int \frac{dV}{dy} \frac{dv}{dy} \, dx \, dy \, dz$$  \[14.8\]
$$\int \int \int \frac{dV}{dz} w \, dx \, dy \, dz = \int \int V \, dx \, dy - \int \int \int \frac{dV}{dz} \frac{dw}{dz} \, dx \, dy \, dz$$

one has:

$$\frac{dK}{dt} = \int \int V \, udz + \int \int V \, vdx + \int \int V \, wdy - \int \int \int V \left( \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \right) dx \, dy \, dz$$  \[14.9\]

The density equation cancels out the last integral and one finds that

$$\frac{dK}{dt} = \int V \, U_n \, d\omega,$$  \[14.10\]

where $U_n$ denotes the normal velocity of a particle at the external surface, and where $d\omega$ is the surface element. If $U_n = 0$ on the whole surface, we find indeed that $\frac{dK}{dt} = 0$.

For every stationary motion the normal velocity component at the surface vanishes; the same happens when the fluid is surrounded by motionless rigid walls; this

[A.27] Untersuchungen über ein Problem der Hydrodynamik. Crelle’s Journal, Bd. 58, S. 202, [1860].
includes also the case when the liquid mass extends to infinity in all directions, since this amounts to having the flow contained in an infinitely large sphere. In all these cases the living forces are constant in time.

When there is no rotational motion in the flow, according to § 12, one can set:

\[ u = \frac{dF}{dx}, \quad v = \frac{dF}{dy}, \quad w = \frac{dF}{dz} \]  \[14.11\]

so that we obtain

\[ K = \frac{1}{2} \iiint \left\{ \left( \frac{dF}{dx} \right)^2 + \left( \frac{dF}{dy} \right)^2 + \left( \frac{dF}{dz} \right)^2 \right\} dx \, dy \, dz \]  \[14.12\]

By a well-known theorem, one finds:

\[ \iiint \left\{ \left( \frac{dF}{dx} \right)^2 + \left( \frac{dF}{dy} \right)^2 + \left( \frac{dF}{dz} \right)^2 \right\} dx \, dy \, dz = - \int F \frac{dF}{dn} \, d\omega \]  \[14.13\]

But, by (3) of § 12, the last integral vanishes, and thus one has:

\[ K = - \int F \frac{dF}{dn} \, d\omega \]  \[14.14\]

Furthermore, if the flow is in a stationary motion, then \( \frac{dF}{dn} \), being the velocity component normal to the external surface, must vanish, and thus \( K = 0 \) from which follows:

\[ \frac{dF}{dx} = 0, \quad \frac{dF}{dy} = 0, \quad \frac{dF}{dz} = 0 \]  \[14.15\]

so that there is no motion at all. Thus, a motion driven by a velocity potential can never become stationary and conversely, a stationary motion will never have a velocity potential — an interesting theorem discovered by Helmholtz.

Corrections [already implemented]

p. 20. l. 7 must be \( \left( \frac{dx}{d\rho} \right)^2 \) instead of \( \left( \frac{dx}{d\rho} \right)^3 \)

p. 30. l. 10 in front of \( 2r \cos \theta \cos \phi \frac{d\phi}{dt} \frac{dz}{dt} \) there must be a + sign

III. LETTERS EXCHANGED FOR THE PRIZE ASSIGNMENT: JUDGEMENTS OF BERNHARD RIEMANN AND WILHELM EDUARD WEBER

Judgement on Hankel’s manuscript by Bernhard Riemann

Decision on the received manuscript on the mathematical prize question carrying the motto: *The more signs express relationships in nature, the more useful they are*:[7.29]

The manuscript gives commendable evidence of the Author’s diligence, of his knowledge and ability in using the methods of computation recently developed by contemporary mathematicians. This is particularly shown in § 6 of the manuscript, which contains an

[7.28] The factor \( F \) in front of the Laplacian in the right-most term was omitted by Hankel.

[7.29] This motto is in Latin in the *Preisschrift*: “Tanto utiliores sunt notae, quanto magis exprimunt rerum relationes”.

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[T.28] The factor \( F \) in front of the Laplacian in the right-most term was omitted by Hankel.

[T.29] This motto is in Latin in the *Preisschrift*: “Tanto utiliores sunt notae, quanto magis exprimunt rerum relationes”.
FIG. 1 Scanned image of the letters from Riemann and Weber, taken from the *Göttingen University Archive*. The image is the result of merging two consecutive pages of the full text of the exchanged letters amongst the commission members for the prize assignment. Riemann’s letter begins on the left page and ends halfway on the right page. Weber’s letter follows after Riemann’s letter.

...elegant method for building the equations of motion for a flow in a fully arbitrary coordinate system, from a point of view which is commonly referred to as Lagrangian. However, when further developing the general laws of vortex motion, the Lagrangian approach is unnecessarily left aside, and, as a consequence, the various laws had to be found by other and quite independent means. Also the relation between the equations of motion and the investigations of Clebsch, reported in § 14.15, is omitted by the Author. Nonetheless, as his derivation actually begins from the Lagrangian equations, one may consider the requirement posed by the prize-question as fulfilled by this part of the manuscript (if one substitutes the wrong text of a given proof of a theorem used by the Author with the right one contained in his note and leaves out of consideration the mistakes due to rushing in § 9).[T.30] In the opinion of the referee, the imperfectness just evoked in handling this part of the question did not give any sufficient reason to deny the prize to the manuscript. However, the Author will have to consolidate this part of the manuscript by a reworking, after which the same would gain in shortness and uniformity. A more important reason against the assignment of the prize could be the incorrectnesses occurring in several places. These [incorrectnesses] do not get to the core of the argument, except [in] two [paragraphs] (§3 and §8) in which the obscurity of another writer may as well serve as an excuse for the Author. [These incorrectnesses] may still be passed over, if our highly esteemed Faculty would like to assign the prize to this manuscript, in view of all manner of good things it contains.

Göttingen, 7th Mai 1861

B. Riemann

[T.30] The § 9 refers to the Latin manuscript.
Judgement on Hankel’s manuscript by Wilhelm Eduard Weber

With the Faculty approval, I have consulted Prof. Riemann, whose judgement I fully agree with, both for the launching of a pure mathematical prize and for the evaluation of the received manuscript. In any case, the work deserves praiseful recognition, and since it needs just a few corrections indicated by my colleague Riemann, thereby easy to do, in order to meet the task requirements, it seems to me that the prize assignment does not give rise to any concern. Nevertheless, the Author will have to consign again his revised work before it is going to print, according to the given suggestions. In my consideration, some incorrectnesses and hastinesses find a fair excuse, in the indeed sparsely proportionated time for such a task, given that only the autumn holiday could be used for the scope.

Wilhelm Weber

IV. HANKEL’S BIOGRAPHICAL NOTES AND LIST OF PUBLISHED PAPERS

Biographical notes about Hermann Hankel

Hermann Hankel was born in Halle, near Leipzig, on 14th February 1839. His father was Gottfried Wilhelm Hankel, a renowned physicist. Hermann Hankel was a brilliant student already in high school, with a particular interest in mathematics and its history. From 1857, he studied Mathematics at the Leipzig University under the mathematicians Scheibner, Moebius and Drobisch. Then, he continued his studies in Göttingen, where, arriving in April 1860, he could attend, among others, Riemann’s lectures. In Göttingen he won in 1861 the extraordnary mathematical prize launched in June 1860 by the Faculty of Göttingen with an essay on the fluid motion theory to be elaborated in a Lagrangian framework. Also in 1861, he obtained his Doctor degree in Leipzig with the dissertation: “Über eine besondere Classe der symmetrischen Determinanten”. Then, in the autumn of the same year, he went to Berlin, where he could attend courses of Weierstrass and Kronecker. In 1862 he returned to Leipzig and, in 1863, at the same place, he habilitated as a Privatdozent with a thesis on the Euler integrals with an unlimited variability of the argument. The writing of the

[T.31] “On a particular class of symmetric determinants”.
habilitation thesis was probably firstly induced by the lectures of Riemann about functions of complex variables. In the spring 1867, he became extraordinary Professor at Leipzig University and, in the same year, ordinary Professor in Erlangen, then, in Tübingen in 1869. He was married to Marie Dippe, who much later became a very important Esperantist. During his life, Hankel was advisor for doctoral dissertations in mechanics, real functions and geometry. He died prematurely on 29th August 1873. Hermann Hankel is known for his Hankel functions, a type of cylindrical functions, Hankel transforms, integral transformations whose kernels are Bessel functions of the first kind, and Hankel matrices, with constant skew diagonals. Hankel was the first to recognise the significance of Grassmann’s extension theory (“Ausdehnungslehre”). Hankel had a passion for research in history of mathematics and published meaningful writings also in this domain (his inaugural lesson in Tübingen was about the development of Mathematics in the last centuries). Curiously, his prized work on the fluid-dynamic theory in Lagrangian coordinates written as a student, is little known.\[^{[T.32]}\]

**List of papers of Hermann Hankel**

1) Hankel, Hermann. 1861. *Zur allgemeinen Theorie der Bewegung der Flüssigkeiten*. Eine von der philosophischen Facultät der Georgia Augusta am 4. Juni 1861 gekrönte Preisschrift, Göttingen. Druck der Dieterichschen Univ.-Buchdruckerei. W.FR.Kaestner, Göttingen.

2) Hankel, Hermann. 1861. *Über eine besondere Classe der symmetrischen Determinanten*. Inaugural-Dissertation zur Erlangung der philosophischen Doktorwürde an der Universität Leipzig von Hermann Hankel.

3) Hankel, Hermann. 1862. Über die Transformation von Reihen in Kettenbrüche. *Zeitschrift für Mathematik und Physik*, 7, 338–343. Also in *Berichte über die Verhandlungen der königlich sächsischen Gesellschaft der Wissenschaften zu Leipzig, mathematisch-physische Classe*, 14, 17-22, 1862. Verlag der Sächsischen Akademie der Wissenschaften zu Leipzig.

4) Hankel, Hermann, (signed as Hl.). 1863. Aufsatz über *Ein Beitrag zu den Untersuchungen über die Bewegung eines flüssigen gleichartigen Ellipsoides* by B.Riemann. *Die Fortschritte der Physik* im Jahre 1861, 17, 50–57.

5) Hankel, Hermann, (signed as Hl.). 1863. Aufsatz über *Zur allgemeinen Theorie der Bewegung der Flüssigkeiten. Eine von der philosophischen Facultät der Georgia Augusta am 4. Juni 1861 gekrönte Preisschrift*, Göttingen by H. Hankel. *Die Fortschritte der Physik* im Jahre 1861, 17, 57–61.

6) Hankel, Hermann, (signed as Hl.). 1863. Aufsatz über *Développements relatifs au §3 de recherches de Dirichlet sur un problème d’hydrodynamique* by F. Brioschi. *Die Fortschritte der Physik* im Jahre 1861, 17, 61–62.

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8) Hankel, Hermann. 1864. Die Zerlegung algebraischer Functionen in Partialbrüche nach den Prinzipien der complexen Functionentheorie. *Zeitschrift für Mathematik und Physik*, 9, 425–433.

9) Hankel, Hermann. 1864. Mathematische Bestimmung des Horopters. *Annalen der Physik und Chemie*, 122, 575–588.

\[^{[T.32]}\] For Hankel’s biography see: Cantor, 1879; Crowe, 2008 Monna, 1973, von Zahn, 1874.
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