0. Introduction

The purpose of this article is to relate non-critical special values of the $p$-adic $L$-functions associated to algebraic Hecke characters of an imaginary quadratic field with class number one to $p$-adic Coleman function called the $p$-adic Eisenstein-Kronecker series, when the conductors of the algebraic Hecke characters are divisible by $p$.

We expect to use this result in the future to consider the $p$-adic Beilinson conjecture for the corresponding Hecke character. The Beilinson conjectures about special values of $L$-functions are a vast generalization of the class number formula for Dedekind zeta function (see [5]), which state that non-critical values of $L$-functions of an algebraic variety can be expressed using invariants arising from the Beilinson regulator map. More generally, these conjectures can be formulated for motives. The $p$-adic Beilinson conjectures are the $p$-adic analogues of the Beilinson conjectures, which state that non-critical values of $p$-adic $L$-functions of an algebraic variety may be concretely expressed by invariants arising from the syntomic regulator. The $p$-adic Beilinson conjectures were formulated and proved by Gros in the case of Dirichlet motives ([17], [18]) and were generalized to $p$-adic $L$-functions of motives by Perrin-Riou (see [25] §4.2). For $p$-adic $L$-functions of Abelian Artin motives, Coleman related special values of these $p$-adic $L$-functions to $p$-adic polylogarithms defined by using his theory of $p$-adic integration ([13]). The polylogarithms have a motivic...
Then we define $\psi$ choice of $O$ where $\pi$ is unramified, i.e. $O$ is the integer ring of an imaginary quadratic field. Then the reduced at a prime above $p$ is ordinary at a prime above $p$.

Let $K$ be an imaginary quadratic field and $E$ be an elliptic curve defined over $K$ with complex multiplication by the integer ring $O_K$ of $K$. Then $K$ has class number 1 since the class number of $K$ equals to $[K(j(E)) : K]$ for $j$-invariant $j(E)$ of $E$ by ([27] II §4 Theorem 4.3). We fix a Weierstrass model $E$ by

$$E : y^2 = 4x^3 - g_2x - g_3 \quad (g_2, g_3 \in O_K)$$

defined over $O_K$. We assume that $E$ has a good ordinary reduction at a prime above $p \geq 5$.

Let $\Gamma$ be the period lattice corresponding to the invariant differential $\omega = dx/y$ obtained by the uniformization theorem. Then we have $E(\mathbb{C}) \cong \mathbb{C}/\Gamma$. By ([26] VI Theorem 4.1(a)) we have $\text{End}(E(\mathbb{C})) \cong \{\alpha \in \mathbb{C} | \alpha \Gamma \subset \Gamma\}$. In addition, since the elliptic curve $E$ has complex multiplication by $O_K$, we have $O_K \Gamma = \Gamma$. Hence there exists a complex period $\Omega \in \mathbb{C}^\times$ satisfying $\Gamma = \Omega O_K$.

Let $\psi := \psi_{E/K} : \mathbb{A}^\infty_K / K^\times \rightarrow \mathbb{C}^\times$ be the Hecke character of $K$ associated to $E$, where $\mathbb{A}^\infty_K$ is the idele group of $K$ ([27] II Theorem 9.2). Let $p$ be a prime ideal of $O_K$ above $p$. Since $E$ has a good reduction at a prime above $p$, by ([27] II Theorem 9.2), the Hecke character $\psi$ of $K$ associated to $E$ is unramified, i.e. $\psi(O_{K\mathfrak{p}}) = 1$, where $K_{\mathfrak{p}}$ is the completion at $p$ and $O_{K\mathfrak{p}}$ is its integer ring. Then we define $\psi(p)$ to be

$$\psi(p) := \psi(..., 1, 1, \tau_p^{-1}, 1, 1, ...).$$

where $\pi$ is a uniformizer at $p$. Since $\psi$ is unramified at $p$, $\psi(p)$ is well-defined independent of the choice of $p$. From the assumption that $E$ is ordinary at a prime above $p$, $p$ splits ($p = \mathfrak{p}\mathfrak{p}$ by
Chapter 10. §4 Theorem 10), where \( \overline{\pi} \) is the complex conjugation of \( \pi \). Since \( \psi(p) \) is the value of \( \psi \) at an idèle with 1's in its archimedean components, we have \( \psi(p) \in \mathcal{O}_K \). Then we have \( \pi = \psi(p) \) and \( p = \pi \overline{\pi} \) (\( \overline{\pi} \) is the complex conjugation of \( \pi \)). Now we take an immersion \( \overline{K} \hookrightarrow \mathbb{C}_p \) satisfying \( |\pi| < 1 \) in \( \mathbb{C}_p \).

For a natural number \( N \), we let \( \mathfrak{g}' := \mathfrak{g}p^N \) be an integral ideal of \( \mathcal{O}_K \) which is divisible by the conductor \( f \) of \( \psi \), where \( \mathfrak{g} = (g) \) is the integral ideal of \( \mathcal{O}_K \) prime to \( p \).

Let \( I(\mathfrak{g}') \) is a group of fractional ideals of \( \mathcal{O}_K \) prime to \( \mathfrak{g}' \). Let \( \varphi : I(\mathfrak{g}') \rightarrow \overline{K}^\times \) be an algebraic Hecke character of infinite type \( (m, n) \in \mathbb{Z}^2 \) whose conductor divides \( \mathfrak{g}' \): In other words, \( \varphi \) is the group homomorphism satisfying

\[
\varphi(\alpha) := \chi(\alpha)\alpha^m\overline{\alpha}^n \quad \text{for any } \alpha \in \mathcal{O}_K \text{ prime to } \mathfrak{g}'
\]

for some finite character \( \chi : (\mathcal{O}_K/\mathfrak{g}')^\times \rightarrow \overline{K}^\times \). The classical complex Hecke \( L \)-function of \( \varphi \) is defined by

\[
L_{\mathfrak{g}'}(s, \varphi) := \sum_{(a, \mathfrak{g}) = 1} \frac{\varphi(a)}{N(a)^s} \quad \text{for } \Re(s) > 0
\]

where the sum runs over all integral ideals \( a \) of \( \mathcal{O}_K \) prime to \( \mathfrak{g}' \). By defining \( \varphi(a) \) by 0 if \( a \) is not prime to \( \mathfrak{g}' \), we can consider \( \varphi \) as a function on the group of all fractional ideals of \( \mathcal{O}_K \). If \( \Re(s) > (m + n)/2 + 1 \), the Hecke \( L \)-function \( L_{\mathfrak{g}'}(s, \varphi) \) converges absolutely. The analytic continuation and functional equation of \( L_{\mathfrak{g}'}(s, \varphi) \) is well known. If we put

\[
\hat{L}_{\mathfrak{g}'}(s, \varphi) := (d_KN(\mathfrak{g}'))^{s/2}\Gamma(s - \min\{m, n\})L_{\mathfrak{g}'}(s, \varphi)
\]

for \( \pi = 3.1415\cdots \) and discriminant \( d_K \) of \( K \), we have

\[
\hat{L}_{\mathfrak{g}'}(s, \varphi) = W \cdot \hat{L}_{\mathfrak{g}'}(1 + m + n - s, \overline{\varphi}),
\]

where \( W \) is a constant of absolute value 1, called the Artin root number. Since \( \Gamma \)-functions of both sides of (2) have no poles on \( \{(m, n) \in \mathbb{Z}^2 \mid m < 0, n \geq 0\} \) or \( \{(m, n) \in \mathbb{Z}^2 \mid n < 0, m \geq 0\} \), following Deligne ([14] Définition 1.3), these two sets are critical domains. In addition, for integers \( m, n \) with \( m < 0 \) and \( n \geq 0 \) (resp. \( n < 0 \) and \( m \geq 0 \), by Damerell’s theorem ([1] Corollary 2.12), we have

\[
\frac{L_{\mathfrak{g}'}(0, \varphi)}{\Omega_{m-n}} \in \mathbb{Q} \quad \text{resp. } \frac{L_{\mathfrak{g}'}(0, \overline{\varphi})}{\Omega_{m-n}} \in \mathbb{Q}.
\]

[3] asserts that Deligne’s conjecture ([13] Conjecture 1.8) holds with respect to \( L \)-functions associated to algebraic Hecke characters. Main theorem of this article is the theorem that the \( p \)-adic analogue of \( L_{\mathfrak{g}'}(0, \varphi)/\Omega_{m-n} \) can be expressed by using the \( p \)-adic Eisenstein-Kronecker series in the non-critical domain. In order to achieve our purpose, we use the \( p \)-adic Eisenstein-Kronecker series as Coleman functions constructed by [4], which we call the Coleman Eisenstein-Kronecker series.

By the conditions \( (g, p) = 1 \) and \( (\pi, \overline{\pi}) = 1 \), by the Chinese remainder theorem, we have

\[
(\mathcal{O}_K/\mathfrak{g}')^\times \cong (\mathcal{O}_K/\mathfrak{g})^\times \times (\mathcal{O}_K/(\pi)^n)^\times \times (\mathcal{O}_K/(\overline{\pi}^n)^\times
\]

Therefore if we restrict the finite character \( \chi \) on \( (\mathcal{O}_K/\mathfrak{g}')^\times \) respectively to \( \chi_0 : (\mathcal{O}_K/\mathfrak{g})^\times \rightarrow \overline{K}^\times \), \( \chi_1 : (\mathcal{O}_K/(\pi)^n)^\times \rightarrow \overline{K}^\times \), and \( \chi_2 : (\mathcal{O}_K/(\overline{\pi}^n)^\times \rightarrow \overline{K}^\times \), we can decompose \( \chi \) by

\[
\chi(\alpha) = \chi_0(\alpha)\chi_1(\alpha)\chi_2(\alpha).
\]

We extend \( \chi_0, \chi_1, \chi_2 \) respectively into characters with \( \mathbb{C}_p \)-values by using an inclusion map \( i : \overline{K}^\times \hookrightarrow \mathbb{C}_p^\times \).
Now we put \( X := \lim_{n \to \infty} (O_K / \mathfrak{p}^n O_K)^{\times} \). We define the \( p \)-adic character \( \phi_p : X \to \mathbb{C}_p^{\times} \) by (15) Chapter II §1 (5)). Similarly to ([15] Chapter II §4.16 (49)) or ([2] §2.4), for a measure \( \mu_g \) on \( X \) (for details, see (22)), we define the value of the \( p \)-adic \textit{L-function} at the \( p \)-adic character \( \phi_p : X \to \mathbb{C}_p^{\times} \) by

\[
L_p(\phi_p) := \int_X \phi_p(\alpha) d\mu_g(\alpha).
\]

Now for the \( p \)-adic character \( \varphi_p : X \to \mathbb{C}_p^{\times} \) defined by \( \varphi_p(\alpha) := \varphi((\alpha)) \) for any \( \alpha \in O_K \) prime to \( \mathfrak{p} \), since

\[
X \cong (O_K / \mathfrak{p})^{\times} \times (O_K \otimes \mathbb{Z} \mathfrak{p})^{\times},
\]

we have \( \varphi_p = \chi_\mathfrak{p} \chi_1 \chi_2 \kappa_1 \kappa_2 \) as \( p \)-adic characters on \( X \), where \( \kappa_1, \kappa_2 \) are the projections to the first and second factors of the following isomorphism:

\[
(O_K \otimes \mathbb{Z} \mathfrak{p})^{\times} \cong O_{Kp}^{\times} \times O_{Kp}^{\times} \xrightarrow{\cong} \mathbb{Z}_p^{\times} \times \mathbb{Z}_p^{\times} \xrightarrow{\sim} (\kappa_1(\alpha), \kappa_2(\alpha)).
\]

Note that \( \chi_1, \chi_2 \) which are components of \( \varphi_p \) can be regarded as characters on \( X \) by the following liftings of the natural projections:

\[
\begin{align*}
\mathbb{K} &\longrightarrow \lim_{n \to \infty} (O_K / \mathfrak{p}^n O_K)^{\times} \cong (O_K \otimes \mathbb{Z} \mathfrak{p})^{\times} \xrightarrow{\pi_1} O_{Kp}^{\times} \xrightarrow{\chi_1} \mathbb{K}^{\times} \xrightarrow{\iota} \mathbb{C}_p^{\times} \\
\mathbb{K} &\longrightarrow \lim_{n \to \infty} (O_K / \mathfrak{p}^n O_K)^{\times} \cong (O_K \otimes \mathbb{Z} \mathfrak{p})^{\times} \xrightarrow{\pi_2} O_{Kp}^{\times} \xrightarrow{\chi_2} \mathbb{K}^{\times} \xrightarrow{\iota} \mathbb{C}_p^{\times}.
\end{align*}
\]

Let \( \Omega_\mathfrak{p} \in \mathcal{O}_{\mathbb{C}_p}^{\times} \) be the \( p \)-adic \textit{period} obtained from the isomorphism between the formal group of the elliptic curve and the multiplicative formal group (for details, see (22)). By calculation, we know that for integers \( m, n \) with \( m < 0 \) and \( n \geq 0 \),

\[
\frac{L_p(\varphi_p)}{\Omega_\mathfrak{p}^{n-m}} = (\text{interpolation factor}) \times \frac{L_{\mathfrak{p}'}(0, \varphi)}{\Omega_\mathfrak{p}^{n-m}},
\]

holds. Hence \( L_p(\varphi_p)/\Omega_\mathfrak{p}^{n-m} \) is the \( p \)-adic analogue of \( L_{\mathfrak{p}'}(0, \varphi)/\Omega_\mathfrak{p}^{n-m} \).

K. Bannai, S. Kobayashi, and T. Tsuji related the non-critical values of the \( p \)-adic \( L \)-function and \( p \)-adic Eisenstein-Kronecker series constructed by the measure when the conductor of algebraic Hecke character is not divisible by \( p \). However when the conductor of the algebraic Hecke character is divisible by \( p \), we cannot express special values of the \( p \)-adic \( L \)-function by using the \( p \)-adic Eisenstein-Kronecker series constructed by the measure. So we use the \( p \)-adic Eisenstein-Kronecker series constructed by the Coleman integration established in [H]. For any integers \( m, n \) with \( n \geq 0 \), let \( E_{m,n}^c(z) \) be the Coleman Eisenstein-Kronecker series on \( E(\mathbb{C}_p) \setminus \{0\} \). We expressed non-critical values of the \( p \)-adic \( L \)-function by using the Coleman Eisenstein-Kronecker series as follows:

\textbf{Theorem 0.1} (=Theorem 3.11). Let \( m, n \) be any integers with \( n \geq 0 \). For a primitive \( g' \)-torsion point \( \xi_{g'} := i_*(\Omega C / g'n) \) that \( C \) is a special constant of \( z \) contained in the following equation, we have

\[
\frac{L_p(\varphi_p)}{\Omega_\mathfrak{p}^{n-m}} = \frac{g^{-1}n!(-1)^{m+n+1}}{\tau(\chi_1)\pi_n} \sum_{z \in (O_K / g')^{\times}} \chi(z)E_{m+1,n+1}^c(\xi_{g'}z),
\]

where \( \tau(\chi_1) \) is the Gauss sum defined by Lemma 3.3 and \( i_* : E(\mathbb{K}) \hookrightarrow E(\mathbb{C}_p) \) is the inclusion map induced by the inclusion map \( i : \mathbb{K} \hookrightarrow \mathbb{C}_p \). Note that \( \Omega C / g'n \) is the element in \( E(\mathbb{K}) \) through the isomorphism \( g^{-1} \Gamma / \Gamma \cong E(\mathbb{C})(g') \cong E(\mathbb{C})(g') \). Here \( E(\mathbb{K})(g') \) (resp. \( E(\mathbb{C})(g') \)) is an abelian extension of \( E(\mathbb{K}) \) (resp. \( E(\mathbb{C}) \)), which consists of \( g' \)-torsion points.
1. Coleman integration theory and its applications

1.1. A brief review of Coleman integration theory.

In this chapter, we give a brief review of Coleman integration theory. Coleman constructed $p$-adic integration theory by using rigid analysis which Tate introduced in [29] and defined in the sense of Monsky-Washnitzer. In other words, we can regard

of 1-forms. These are respectively rings of overconvergent functions and overconvergent 1-forms

In this chapter, we give a brief review of Coleman integration theory. Coleman constructed

A brief review of Coleman integration theory.

Let $\mathcal{O}_{\mathbb{C}_p}$ be an integer ring of the completion $\mathbb{C}_p$ of algebraic closure of $\mathbb{Q}_p$. Then a residue field of $\mathbb{C}_p$ is $\mathbb{F}_p$. Let $X$ be a proper, smooth, and connected scheme of locally finite type of relative dimension 1 defined over $\mathcal{O}_{\mathbb{C}_p}$ with a good reduction. Let $X(\mathbb{C}_p)$ be a generic fiber and $X(\mathbb{F}_p)$ be a special fiber. According to [6] Proposition 0.3.5, $X(\mathbb{C}_p)$ is isomorphic to a rigid analytic $\mathbb{C}_p$-space $X^{an}$ obtained by a rigid analysis of $X$, so we may also denote $X(\mathbb{C}_p)$ by $X(\mathbb{C}_p)^{an}$. Let $Y \subset X$ be an open affine subscheme defined over $\mathcal{O}_{\mathbb{C}_p}$ which is proper, smooth, connected and has a good reduction. Let $Y(\mathbb{F}_p)$ be a special fiber of $Y$. Then we can take finite points $e_1, \ldots, e_n$ such that $X(\mathbb{F}_p) \setminus Y(\mathbb{F}_p) = \{e_1, \ldots, e_n\}$. For a $\mathbb{F}_p$-subscheme $S \subset X(\mathbb{F}_p)$, let $|S| := \text{sp}^{-1}(S) \subset X(\mathbb{C}_p)^{an}$ be a tube of $S$, where sp is the specialization map

$$\text{sp} : X(\mathbb{C}_p)^{an} \xrightarrow{\text{reduction}} X(\mathbb{F}_p).$$

In particular, for a closed point $x \in X(\mathbb{F}_p)$, $|x| := \text{sp}^{-1}(x)$ is a unit open disk by [6] Proposition 1.1.1. In other words, $|x| \equiv \{z \in \mathbb{C}_p \mid |z| < 1\}$. We denote a local parameter $z$ around $x$ by $z_x$. For $0 < r \leq 1$, we put $U_r := X(\mathbb{C}_p)^{an} \setminus \cup_{i=1}^n D(\tilde{e}_i, r)^{-}$. Here, $\tilde{e}_i \in X(\mathbb{C}_p)$ is a lift of $e_i$ and $D(\tilde{e}_i, r)^-$ is a closed disk centered at $\tilde{e}_i$ with radius $r$.

Let $\mathcal{O}_{X^{an}}$ be a structure sheaf of the rigid analytic $\mathbb{C}_p$-space $X^{an} = X(\mathbb{C}_p)^{an}$. Let $U := \text{sp}^{-1}(X(\mathbb{F}_p)) = \lim_{r \to 1} U_r$. Let $A(U)$ be a subset of a locally rigid analytic function $\mathcal{L}(U)$ on $U$ such that

$$A(U) := \{f \in \mathcal{L}(U) \mid f|_{X^{an}} \in \mathcal{O}_{X^{an}}(X^{an})\}$$

where $\mathcal{O}_{X^{an}}(X^{an}) = \Gamma(X^{an}, \mathcal{O}_{X^{an}})$ is an algebra of rigid analytic functions. Let $\Omega^1(U)$ be a space of 1-forms. These are respectively rings of overconvergent functions and overconvergent 1-forms in the sense of Monsky-Washnitzer. In other words, we can regard $A(U)$ as $\Gamma(|Y(\mathbb{F}_p)|, j^! \mathcal{O}_{|Y(\mathbb{F}_p)|})$ and $\Omega^1(U)$ as $\Gamma(|Y(\mathbb{F}_p)|, j^! \Omega^1_{|Y(\mathbb{F}_p)|})$, where for an open immersion $j : S \to \overline{S}$ corresponding to a closed subscheme $S \subset X(\mathbb{F}_p)$ and an open $S$ of $\overline{S}$, $j^!$ is a functor defined by [6] §2.1 (2.1.1.1)).

A branch of $p$-adic logarithms is any locally analytic homomorphism $\log : \mathbb{C}_p^\times \to \mathbb{C}_p^+$ with the
usual expansion for \( \log \) around 1. Such a function is determined by choosing \( \pi \in \mathbb{C}_p \) such that \(|\pi| < 1\) and declaring \( \log(\pi) = 0 \). Coleman’s \( p \)-adic integration theory depends on the choice of the branch of the \( p \)-adic logarithms. We choose such a branch “\( \log \)” of \( p \)-adic logarithms. We define

\[
A_{\log}(\lfloor x \rfloor) := \begin{cases} A(\lfloor x \rfloor) & \text{if } x \in Y(\overline{\mathbb{F}}_p) \\
\lim_{r \to 1} A(\lfloor x \rfloor \cap U_r)[\log(z_x)] & \text{if } x \in X(\overline{\mathbb{F}}_p) \setminus Y(\overline{\mathbb{F}}_p) \end{cases}
\]

\[
\Omega_{\log}(\lfloor x \rfloor) := A_{\log}(\lfloor x \rfloor)dz_x
\]

Here, note that if \( x \in Y(\overline{\mathbb{F}}_p) \), then \( A(\lfloor x \rfloor) \) is the ring \( \mathcal{O}_{\lfloor x \rfloor} \) consisting of formal power series \( f(z_x) = \sum_{n=0}^{\infty} a_n z_x^n \) which converges on \( \{ z_x \in \mathbb{C}_p \mid |z_x| < 1 \} \), and if \( x \in X(\overline{\mathbb{F}}_p) \setminus Y(\overline{\mathbb{F}}_p) \), then formal power series \( f(z_x) = \sum_{n=-\infty}^{\infty} a_n z_x^n \) which converges on \( \{ z_x \in \mathbb{C}_p \mid r < |z_x| < 1 \} \) for some \( r < 1 \). We define rings of locally analytic functions and 1-forms on \( U \) by

\[
A_{\log}(U) := \prod_{x \in X(\overline{\mathbb{F}}_p)} A_{\log}(\lfloor x \rfloor), \quad \Omega_{\log}(U) := \prod_{x \in X(\overline{\mathbb{F}}_p)} \Omega_{\log}(\lfloor x \rfloor).
\]

These are independent of the choice of \( z_x \). We can define a differential \( d : A_{\log}(U) \to \Omega_{\log}(U)^1 \) in the natural way. Then \( d : A_{\log}(U) \to \Omega_{\log}(U)^1 \) is surjective. The point is that we are able to integrate \( dz/z \) by adding logarithms. So we can integrate any elements in \( \Omega_{\log}(U)^1 \) i.e. (B) holds. But since \( \ker(d) = \prod_{x \in X(\overline{\mathbb{F}}_p)} \mathbb{C}_p \), we do not have the notion of analytic continuation yet i.e. (A) does not hold.

**Definition 1.1** (Coleman function). Coleman defined a subalgebra \( M(U) \subset A_{\log}(U) \) equipped with an integration map

\[
\int : M(U) \otimes_A(U) \Omega^1(U) \to M(U)/\mathbb{C}_p \quad \omega \mapsto F_\omega := \int \omega
\]

which is one of \( \mathbb{C}_p \)-linear maps, in order to obtain the notion of analytic continuation, with the surjectivity of \( d \) keeping as follows. A map \( \int \) is characterized by three properties:

i) (The existence of a primitive function) \( dF_\omega = \omega \)

ii) (Frobenius invariance) For a Frobenius automorphism \( \phi : U \to U \), we have

\[
\int (\phi^*(\omega)) = \phi^* \left( \int \omega \right)
\]

iii) \( \int dg = g + \mathbb{C}_p \) for \( g \in A(U) \).

We call \( M(U) \) a space of Coleman functions on \( U \). As for the construction of such a space \( M(U) \), see (4) §2.

In summary, when \( f \) is a function in \( A_{\log}(U) \) and \( P(x) \) is a polynomial with \( \mathbb{C}_p \)-coefficients whose roots do not contain the roots of 1, if \( df \in M(U) \otimes_A(U) \Omega_{\log}(U)^1 \) and \( P(\phi^*)f \in M(U) \), then we have \( f \in M(U) \). Note that we extend the classes of integrable differential forms so that the integration is unique up to a constant in \( \mathbb{C}_p \), not in \( \prod_{x \in X(\overline{\mathbb{F}}_p)} \mathbb{C}_p \). In other words, we have an exact sequence

\[
0 \to \mathbb{C}_p \to M(U) \xrightarrow{d} M(U) \otimes_A(U) \Omega^1(U) \to 0.
\]
The entire theory turns out to be independent of the choice of $\phi$. The important idea is to extend the classes of the integrable differential forms from $d(A(U))$ by using Frobenius invariance.

1.2. Applications of Coleman integrations.

Put $U := \mathbb{P}^1(\mathbb{C}_p) \setminus \{0, 1, \infty\}$. Coleman defined $p$-adic polylogarithms recursively as follows:

**Definition 1.2 (p-adic polylogarithm).** Let $k$ be an integer. If $k \geq 0$, we define a locally analytic function $\ell_k \in M(U)$ ($k \geq 0$) satisfying

i) $\ell_0(z) = \frac{z}{1 - z}$

ii) $d\ell_k(z) = \frac{\ell_{k-1}(z)}{z} dz$

iii) $\lim_{z \to 0} \ell_k(z) = 0$.

$\ell_k(z)$ is an analytic function $\ell_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}$ on $|z| < 1$. The existence and uniqueness of $\ell_k$ is insured by ([11] Corollaire 2.2.2.1). If $k \leq 0$, we take $\ell_k \in A(U)$ satisfying i), ii), iii). This $\ell_k$ is the $p$-adic polylogarithm defined by Coleman [13].

The important application of the $p$-adic polylogarithm is that Kubota-Leopoldt $p$-adic $L$-function $L_p(s, \chi)$ associated to a non-trivial Dirichlet character $\chi$ can be written as the sum of the $p$-adic polylogarithms at positive integers (i.e. non-critical values). It is well known that Kubota-Leopoldt $p$-adic $L$-function $L_p(s, \chi)$ is obtained by interpolating at negative integers (i.e. critical points) of the complex Dirichlet $L$-function (see [20] §3 Theorem 3 ii)).

**Theorem 1.3 (Coleman [13] §7).** Let $p$ be an odd prime. Let $\chi : (\mathbb{Z}/d\mathbb{Z})^\times \to \mathbb{C}_p^\times$ be a Dirichlet character with a conductor $d > 1$ such that $d$ is prime to $p$, and $\omega : \mathbb{Z}_p^\times \to (\mathbb{Z}/p\mathbb{Z})^\times$ be a Teichm"uller character. For all integers $k \geq 1$, we have

$$L_p(k, \chi \omega^{1-k}) = \left(1 - \frac{\chi(p)}{p^k}\right) \frac{g(\chi, \zeta)}{d} \sum_{a=1}^{d-1} \chi(a) \ell_k(\zeta^{-a}),$$

where $\zeta$ is a primitive $d$-th root of 1 and $g(\chi, \zeta)$ is the Gauss sum defined by $g(\chi, \zeta) = \sum_{a=0}^{d-1} \chi(a) \zeta^a$.

Substituted for $k = 1$, Theorem 1.3 is reduced to ([20] §5 Theorem 3). Note that Theorem 1.3 is the $p$-adic analogue of the classical formula

$$L(k, \chi) = \frac{g(\chi, \zeta)}{d} \sum_{a=1}^{d-1} \chi(a) \ell_k(\zeta^{-a}).$$

We can show this formula by using two properties of the Gauss sum

$$\sum_{a=0}^{d-1} \chi(a) \zeta^{-an} = \chi(n) g(\chi, \zeta)$$

and

$$g(\chi, \zeta) g(\chi, \overline{\zeta}) = d.$$ 

Main theorem of this article is the elliptic analogue of Theorem 1.3.
2. Review of the classical Eisenstein-Kronecker series and its $p$-adic analogue

In this section, we review the definition of the classical Eisenstein-Kronecker series by A. Weil [30] and of the $p$-adic Eisenstein-Kronecker series as the Coleman function by K. Bannai, H. Furusho, and S. Kobayashi in [4]. The classical Eisenstein-Kronecker series is defined by constructing with generating functions appeared (see [32] Theorem 1). The $p$-adic Eisenstein-Kronecker series as the Coleman function i.e. Coleman Eisenstein-Kronecker series is defined by constructing with generating functions appeared in Laurent coefficients of the Kronecker theta function.

2.1. Review of the classical Eisenstein-Kronecker series.
Recall that the definition of the classical Eisenstein-Kronecker series ([30] VIII §12).
Let $\Gamma \subset \mathbb{C}$ be a lattice, $\varpi = 3.1415\cdots$ be the circular constant, $A(\Gamma) = (\text{Area of } \mathbb{C}/\Gamma)/\varpi$, $\chi_w(z) := \exp((z\varpi - w\varpi)/A(\Gamma))$ for any $z, w \in \mathbb{C}$.
In particular, if we can write $\Gamma := \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2 \subset \mathbb{C}$ with $\text{Im}(\omega_2/\omega_1) > 0$, note that

$$A(\Gamma) = \frac{1}{\varpi} \text{Im}(\omega_2/\omega_1) = \frac{1}{2\varpi}(\omega_2\overline{\omega_1} - \omega_1\overline{\omega_2}).$$

In addition, by direct calculations, we know the following properties.

i) $\chi_w(z)\Gamma = \chi_w(-w)\Gamma = \chi_w(w)\Gamma^{-1}$

ii) $\chi_w(az)\Gamma = \chi_w(z)\Gamma$ for any $a \in \mathbb{C}$

iii) $z \in \Gamma \iff \chi_{\gamma}(z)\Gamma = 1$ for any $\gamma \in \Gamma$.

Definition 2.1 (Eisenstein-Kronecker-Lerch series).
Let $a$ be an integer and $z_0, w_0 \in \mathbb{C}$ be complex numbers. The Eisenstein-Kronecker-Lerch series is defined by

$$K^*_a(z_0, w_0, s; \Gamma) := \sum_{\gamma \in \Gamma \setminus \{z_0\}} \frac{(\overline{z_0} + \gamma)^a}{|z_0 + \gamma|^2s} \chi_{w_0}(\gamma)\Gamma \quad (s \in \mathbb{C}).$$

This series converges absolutely for $\text{Re}(s) > a/2 + 1$.

Hereafter, by abuse of notations, we omit “$\Gamma$” except the case where we want to express the lattice clearly. $K^*_a(z_0, w_0, s)$ has the following important properties.

Proposition 2.2. Let $a$ be an integer and $z_0, w_0 \in \mathbb{C}$ be complex numbers.

i) $K^*_a(z_0, w_0, s)$ can be continued meromorphically on $\mathbb{C}$ as a function of $s$. Moreover, if $a = 0$ and $w_0 \in \Gamma$, $K^*_0(z_0, w_0, s)$ has a simple pole at $s = 1$.

ii) $K^*_a(z_0, w_0, s)$ has a functional equation:

$$\Gamma(s)K^*_a(z_0, w_0, s) = A^{a+1-2s}\Gamma(a + 1 - s)K^*_a(w_0, z_0, a + 1 - s)\chi_{z_0}(w_0),$$

where $\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt \quad (\text{Re}(s) > 0)$ is a Gamma function.

Proof. If $a \geq 0$, see ([30] VIII §13). If $a \leq 0$, see ([2] Proposition 2.4.).

□

Definition 2.3 (Eisenstein-Kronecker number).
Let $z_0, w_0 \in \mathbb{C}$ and we take $a, b \in \mathbb{Z}$ as $(a, b) \neq (1, -1)$ if $w_0 \in \Gamma$. The Eisenstein-Kronecker numbers $e^*_a(z_0, w_0)$ is defined by

$$e^*_a(z_0, w_0) := K^*_a(z_0, w_0, b) = \sum_{\gamma \in \Gamma \setminus \{z_0\}} \frac{(\overline{z_0} + \gamma)^a}{(z_0 + \gamma)^b}\chi_{w_0}(\gamma).$$
For \((a, b) = (0, 0)\), we have \(e_{a,0}^*(z_0, w_0) := K_0^*(z_0, w_0, 0) = -\chi_{z_0}(w_0)\).

We define the Kronecker theta function. The Kronecker theta function was defined by using the reduced theta function associated to the divisor \([0]\) (i.e. the holomorphic pseudo-periodic function with the Appell-Humbert data) by using that a group of isomorphism classes of invertible sheaves on the torus \(\mathbb{C}/\Gamma\) is classified by Appell-Humbert’s theorem. For details, see ([1] Example 1.9).

**Definition 2.4** (Kronecker theta function). Let \(\theta(z)\) be a reduced theta function associated to the divisor \([0]\) defined by ([1] Example 1.9). \(\theta(z)\) is characterized by \(\theta'(0) = 1\).

By using this \(\theta(z)\), the Kronecker theta function is defined as follows. For any \(z, w \in \mathbb{C}\), we define the **Kronecker theta function** \(\Theta(z, w)\) by

\[
\Theta(z, w) := \frac{\theta(z + w)}{\theta(z)\theta(w)}.
\]

In addition, for \(z_0, w_0 \in \mathbb{C}\), we define

\[
\Theta_{z_0, w_0}(z, w) := \exp\left(-\frac{z_0 w_0}{A}\right) \exp\left(-\frac{z w_0 + w z_0}{A}\right) \Theta(z + z_0, w + w_0).
\]

According to ([1] Proposition 1.16), \(\Theta_{z_0, w_0}(z, w)\) has the following distribution relation: Let \(c, c' \in \Gamma\), \(n\) be a natural number, and \(z_0, w_0 \in \mathbb{C}\). Then we have

\[
\sum_{w_n \in \pi^{-n}\Gamma/\Gamma} \chi_{w_n}(c) \Theta_{z_0, w_0+w_n}(z, w) = \pi^n \chi_c(w_0) \Theta(z_0-c, \pi^{-n}w_0, \pi^n w)
\]

\[
\sum_{z_m \in \pi^{-m}\Gamma/\Gamma} \chi_{c'}(z_m) \Theta_{z_0+z_m, w_0}(z, w) = \pi^m \Theta_{\pi^m z_0, c', \pi^m}(\pi^m z, \pi^m w)
\]

\(\Theta_{z_0, w_0}(z, w)\) can be expanded to Laurent series of \(z, w\) as the generating function of the following Eisenstein-Kronecker number.

**Theorem 2.5.** \(\Theta_{z_0, w_0}(z, w)\) has a Laurent expansion in the neighborhood of \((z, w) = (0, 0)\), that is,

\[
\Theta_{z_0, w_0}(z, w) = \chi_{z_0}(w_0) \frac{\delta_{z_0}}{z} + \frac{\delta_{w_0}}{w} + \sum_{a,b \geq 0} (-1)^{a+b} \frac{e_{a,b+1}^*(z_0, w_0)}{a! A^a} z^b w^a,
\]

where \(\delta_x\) is defined by

\[
\delta_x = \begin{cases} 
1 & (x \in \Gamma) \\
0 & \text{(otherwise)}
\end{cases}
\]

**Proof.** See ([1] §1.14 Theorem 1.17). \(\square\)

Substituting \(w_0 = 0\) for the formula \([\mathbb{S}]\), we define a function \(F_{z_0,b}(z)\) as follows.

**Definition 2.6.** For any \(z_0 \in \mathbb{C}\), we define \(F_{z_0,b}\) by a function satisfying

\[
\Theta_{z_0,0}(z, w) = \sum_{b \geq 0} F_{z_0,b}(z) w^{b-1}.
\]

If \(z_0 = 0\), we define \(F_b(z) := F_{0,b}(z)\). We observe \(F_b(z)\) if \(b = 0, 1\).

\(F_0(z) = 1\). Noting that \(\Theta_{0,0}(z, w) = \Theta(z, w) := \theta(z + w)/\theta(z)\theta(w)\) and observing coefficients of \(w^0\) in the formula \([\mathbb{S}]\) and \([\mathbb{M}]\), we find that \(F_1(z)\) satisfies

\[
F_1(z) = \lim_{w \to 0} (\Theta(z, w) - w^{-1}) = \frac{\theta'(z)}{\theta(z)}
\]
$F_{z_0,b}(z)$ is dependent only on a choice of $z_0$ modulo $\Gamma$ because we have

$$\Theta_{z_0 + \gamma, 0}(z, w) = \exp\left[-\frac{w(z_0 + \gamma)}{A}\right] \Theta(z + z_0 + \gamma, w) = \Theta_{z_0, 0}(z, w).$$

As we define later, the $p$-adic analogue of $F_{z_0,b}$ for a variable $z_0$ is constructed as the Coleman function by glueing together each unit open disk. By using the $p$-adic analogue of $F_{z_0,b}$, we construct the $p$-adic analogue of the Eisenstein-Kronecker series $E_{m,n}$. We have the Laurent expansion of $F_{z_0,b}(z)$ from Theorem 2.6.

**Corollary 2.7** (Generating function). For any $b \geq 0$, the Laurent series of $F_{z_0,b}(z)$ at $z = 0$ can be written as

$$F_{z_0,b}(z) = \frac{\delta_{z_0,b}}{z} + \sum_{a \geq 0} (-1)^{a+b-1} \frac{\delta_{a,b}(0,z_0)}{a! A^a} z^a,$$

where

$$\delta_{x,b} = \begin{cases} 1 & (b = 0 \text{ and } x \in \Gamma) \\ 0 & \text{(otherwise)} \end{cases}$$

**Proof.** See ([2] Corollary 2.11). \qed

When we define the $p$-adic Eisenstein-Kronecker series later, we use the connection function $L_n(z)$ of $F_b(z)$ for $b \geq 0$. We define the connection function $L_n(z)$ by

$$\Xi(z, w) := \exp(-F_1(z)w)\Theta(z, w) = \sum_{n \geq 0} L_n(z)w^{n-1}.$$ 

Since $F_b(z) := F_{0,b}(z)$, $\Theta(z, w) = \Theta_{0,0}(z, w) = \sum_{b \geq 0} F_b(z)w^{b-1}$, and by giving “exp” Taylor expansion, we have

$$L_n(z) = \sum_{a+b=n \atop a \geq 0, b \geq 0} \frac{(-F_1(z))^a F_b(z)}{a!} = \sum_{b=0}^{n} \frac{(-F_1(z))^{n-b}}{(n-b)!} F_b(z).$$

By the translation by $\gamma \in \Gamma$ of $\Xi(z, w)$, $L_n(z)$ is the periodic function on $\mathbb{C}/\Gamma$, i.e. the elliptic function and the holomorphic function on $\mathbb{C} \setminus \Gamma$. If $n = 0, 1$, by the formula (10), we have

$$L_0(z) = F_0(z) = 1, \quad L_1(z) = -F_1(z)F_0(z) + F_1(z) = 0.$$

Since $\Theta_{z_0,0}(z, w) = \exp(F_{z_0,1}(z)w)\Xi(z + z_0, w)$, we have a relation

$$F_{z_0,b}(z) = \sum_{n=0}^{b} \frac{F_{z_0,1}(z)^{b-n}}{(b-n)!} L_n(z + z_0)$$

between $F_{z_0,b}(z)$ and $L_n(z)$.

Now we assume that a complex torus has an algebraic model. Let $K$ be an imaginary quadratic field and we fix an immersion $K \hookrightarrow \mathbb{C}$. We define an elliptic curve $E$ over $K$ by a Weierstrass equation

$$y^2 = 4x^3 - g_2x - g_3, \quad g_2, g_3 \in K,$$

and its invariant $\omega$ by $\omega = dx/y$. By the uniformization theorem, there exists a period lattice $\Gamma \subset \mathbb{C}$ of $E$ satisfying an isomorphism

$$\xi : \mathbb{C}/\Gamma \xrightarrow{\sim} E(\mathbb{C}), \quad z \mapsto (\wp(z), \wp'(z)),$$

where $\wp(z)$ is the Weierstrass $\wp$-function. Then we have $\omega = d\wp(z)/\wp'(z) = dz$. According to ([2] Proposition 1.12), the connection function $L_n(z)$ is algebraic since we have $L_n(z) \in K[\wp(z), \wp'(z)]$ as the rational function on $E$ over $K$. In addition, we assume that the elliptic curve $E$ has complex
multiplication by the integer ring $\mathcal{O}_K$ of $K$. Then by Damerell’s theorem, $\Theta_{z_0,w_0}(z,w)$ and $F_{z_0,b}(z)$ are algebraic in the following sense:

- If $z_0, w_0$ correspond to torsion points in $\mathbb{C}/\Gamma \cong E(\mathbb{C})$, then

$$\Theta_{z_0,w_0}(z,w) - \chi_{z_0}(w_0)\frac{\delta_{z_0}}{z} - \frac{\delta_{w_0}}{w} \in \overline{\mathbb{Q}}[[z,w]],$$

where if $x \in \Gamma$ then $\delta_x = 1$, and otherwise $\delta_x = 0$ (see [1] Theorem 2.13).

- If $z_0 \in \mathbb{C}$ corresponds to a torsion point in $\mathbb{C}/\Gamma \cong E(\mathbb{C})$, then

$$F_{z_0,b}(z) - \frac{\delta_{z_0,b}}{z} \in \overline{\mathbb{Q}}[[z]],$$

where if $b = 1$ and $x \in \Gamma$ then $\delta_{x,b} = 1$, and otherwise $\delta_{x,b} = 0$ (see [2] Corollary 2.14).

These algebraicities allow us to view this value as an element in $\mathbb{C}_p$ through the immersion $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. In addition, since $E$ has complex multiplication by the integer ring $\mathcal{O}_K$, by ([27] II §1 Proposition 1.1), there is a unique isomorphism

$$[\cdot] : \mathcal{O}_K \xrightarrow{\sim} \text{End}(E)$$

such that for any invariant differential $\omega = dx/y$ on $E$,

$$[\alpha]^* (\omega) = \alpha \omega \quad \text{for all } \alpha \in \mathcal{O}_K.$$

For any $0 \neq \alpha \in \mathcal{O}_K$, let $E[\alpha]$ be a subgroup of $E(\overline{\mathbb{Q}})$ such that $E[\alpha] := \{ P \in E(\overline{\mathbb{Q}}) | [\alpha] P = 0 \}$. According to ([3] Proposition 2.15), the function $F_{z_0,b}(z)$ is known to satisfy the following distribution relation with respect to $E[\alpha]$:

$$\sum_{z_0 \in E[\alpha]} F_{z_0+b,z_0,b}(z) = \alpha \pi^{1-b} F_{az_0,b}(az) \quad \text{for any } 0 \neq \alpha \in \mathcal{O}_K.$$

2.2. Review of the $p$-adic analogue of the Eisenstein-Kronecker series.

For an integer $b \geq 0$, we review that the $p$-adic analogue of $F_{z_0,b}$ is constructed as the Coleman function on an elliptic curve along [4].

Let $E$ be an elliptic curve in $\mathbb{C}$. Let $t := -2x/y$ be a formal parameter of $E$ at the origin. Let $\hat{E}$ be a formal group of $E$ for $t$ equipped with a maximal ideal of a complete local ring of $\mathcal{O}_K$ and group operations $\oplus$. Let $\lambda : \hat{E} \xrightarrow{\sim} \hat{G}_a$ be a normalized formal logarithm for an additive formal group $\hat{G}_a$. For a torsion point $z_0 \in E(\overline{\mathbb{Q}})_{\text{tors}}$, we define $\hat{F}_{z_0,b}(t)$ by

$$\hat{F}_{z_0,b}(t) := F_{z_0,b}(z)|_{z = \lambda(t)} = F_{z_0,b}(\lambda(t))$$

By formula (14) and $\lambda(z_0) = 0$, we have

$$\hat{F}_{z_0,b}(t) = \sum_{n=0}^{b} \frac{\hat{F}_{z_0,1}(t)^{b-n}}{(b-n)!} \hat{F}_{z_0,n}(t)$$

where $\hat{F}_{z_0,n}(t) := L_n(z + z_0)|_{z = \lambda(t)}$.

By formula (11), we consider $\hat{F}_{z_0,b}(t)$ as power series with $\mathbb{C}_p$-coefficients through the immersion $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. In other words, if $z_0 \in E(\overline{\mathbb{Q}})$ is a torsion point with an order prime to $p$, according to ([2] Proposition 2.16), the following series converges

$$\hat{F}_{z_0,b}(t) - \frac{\delta_{z_0,b}}{t} = \sum_{a \geq 0} (-1)^{a+b-1} \frac{\varepsilon_{a,b}(0,z_0)}{a!A^a} z^a \bigg|_{z = \lambda(t)} \in \mathbb{C}_p[[t]]$$

converges on $B(0,1) := \{ t \in \mathbb{C}_p | |t| < 1 \}$ if $b \neq 1$ or $z_0 \neq 0$. In particular, this series is a rigid analytic function on $B(0,1)$. Moreover $\hat{F}_{1}(t) := \hat{F}_{0,1}(t)$ converges on $\{ t \in \mathbb{C}_p | 0 < |t| < 1 \}$. In addition, we have a formula for translation by $\pi^n$-torsion points.
Proposition 2.8 (Translation). Let $z_0 \in E(\overline{Q})$ be a torsion point with an order prime to $p$. Then we have
\[ \hat{F}_{z_0,b}(t \oplus t_n) = \hat{F}_{z_0+z_n,b}(t), \]
where $t_n \in \hat{E}[\pi^n]$ is a $\pi^n$-torsion point and $z_n \in E(\overline{Q})_{\text{tors}}$ is the image of $t_n$ through an immersion map $\hat{E}(\mathfrak{m}_{p})_{\text{tors}} \rightarrow E(\overline{Q})_{\text{tors}} \rightarrow \mathbb{C}/\Gamma$. Here $\mathfrak{m}_p$ is a maximal ideal of an integer ring $\mathcal{O}_p$ of $\mathbb{C}_p$.

Proof. See ([4] Lemma 2.17).

Let $F \subset \mathbb{C}_p$ be a finite extension field of $K_p$. By abuse of notations, we denote the extension of $E$ into an integer ring $\mathcal{O}_F$ of $F$ by again $E$. For $\pi := \psi(p)$, let
\[ \phi : E \rightarrow E \]
be a Frobenius automorphism induced by a multiplication by $[\pi]$.

Let $E(\mathbb{C}_p) := E(\mathbb{C}_p)_{\text{an}}$ be an extension into $\mathbb{C}_p$ of a rigid analytic $F$-space $E(F)_{\text{an}}$, and we fix a variable $z$ on $E(\mathbb{C}_p)$.

Each residue disk of $E(\mathbb{C}_p)$ contains a Teichmüller representative, where a Teichmüller representative is a unique element in the residue disk fixed by some power of proper Frobenius. By choice of Frobenius morphism $\phi$, a Teichmüller representative is a torsion point $z_0$ with an order prime to $p$.

For $t = -2r/y$, a unit open disk $\{t \in \mathbb{C}_p \, | \, |t| < 1\}$ expresses the residue disk $\{0\} := \mathfrak{sp}^{-1}(0) \subset E(\mathbb{C}_p)$ containing a unit element with respect to group operations of the elliptic curve $E$, where $\mathfrak{sp} : E(\mathbb{C}_p)_{\text{an}} \rightarrow E(\mathbb{C}_p)$ is a specialization map.

Let $z_0 \in E(\mathbb{C}_p)$ be a torsion point with an order prime to $p$ and $\tau_{z_0} : E \rightarrow E$ be $\tau_{z_0}(z) := z + z_0$. For this $\tau_{z_0}$, we define $\{z_0\}$ by
\[ \{z_0\} := \tau_{z_0}(\{0\}). \]

Then $\{z_0\}$ is a residue disk containing $z_0$.

Let $U := E(\mathbb{C}_p) \setminus \{0\}$, where $\{0\}$ is a unit element in group laws of the elliptic curve. If $t_{z_0}$ is a local parameter of $E$ at a point $z_0$, by the formula ([10], $\hat{F}_{z_0,b}(t)$ defines an element in $A(\{z_0\})$ via $\{z_0\} \equiv \{t_{z_0} \in \mathbb{C}_p \, | \, |t_{z_0}| < 1\}$.

Lemma 2.9. We define $F^\text{col}_{1} \in A_{\text{loc}}(U)$ by
\[ F^\text{col}_{1}(z)|_{z_0} := \hat{F}_{z_0,1}(t) \in A(\{z_0\}) \subset A_{\text{log}}(\{z_0\}) \]
on each residue disk $\{z_0\}$, where $z_0 \in E(\overline{Q})$ is a torsion point with an order prime to $p$. Then $F^\text{col}_{1}$ is a Coleman function on $U$.

Proof. See ([4] Lemma 3.4.)

The formula ([10]) indicates that $L_n$ is a rational function on $E$ with poles only at $\{0\}$ in $E$, hence is in particular a Coleman function on $U$. The set of Coleman functions is a ring, and we define $F^\text{col}_b$ as follows.

For $b \geq 1$, we define $F^\text{col}_b$ by if $b = 1$ $F^\text{col}_1(z)|_{z_0} := \hat{F}_{z_0,1}(t)$, and if $b > 1$
\[ F^\text{col}_b := \sum_{n=0}^{b} \frac{(F^\text{col}_1)^{b-n}}{(b-n)!} L_n. \]

According to ([4] Proposition 3.6), $F^\text{col}_b(z)$ interpolates $F_{z_0,b}(z)$ on each unit open disk as follows: For a torsion point $z_0 \in E(\overline{Q})_{\text{tors}}$ with an order prime to $p$, we have
\[ F^\text{col}_b(z)|_{z_0} = \hat{F}_{z_0,b}(t) \in A_{\text{log}}(\{z_0\}). \]
In addition, $F_b^\text{col}$ has the following distribution relation by (4) Proposition 3.7:

$$
\sum_{z_\alpha \in E[\alpha]} F_b^\text{col}(z + z_\alpha) = \alpha \overline{\alpha}^{-b} F_b^\text{col}(\alpha z) \quad \text{for all } 0 \neq \alpha \in \mathcal{O}_K.
$$

In (4) §3.3, K. Bannai, H. Furusho, and S. Kobayashi constructed the $p$-adic Eisenstein-Kronecker series as the Coleman function by using $F_b^\text{col}$ whose constant term is chosen to satisfy the distribution relation. The distribution relation plays an important role to resolve the ambiguity of integration constants.

**Definition 2.10.** (Coleman Eisenstein-Kronecker series) Let $m, b$ be integers with $b \geq 0$. We define the Coleman Eisenstein-Kronecker series $E_{m,b}^\text{col}$ on $U := E(\mathbb{C}_p) \setminus \{0\}$ recursively as follows:

i) $E_{0,b}^\text{col} := (-1)^{b-1} F_b^\text{col}$.

This function satisfies the distribution relation by the formula (18).

ii) If $m > 0$, we define $E_{m,b}^\text{col}$ by the Coleman function

$$
E_{m,b}^\text{col} := \int E_{m-1,b}^\text{col} \omega
$$

with a constant term normalized by satisfying the distribution relation

$$
\sum_{z_\alpha \in E[\alpha]} E_{m,b}^\text{col}(z + z_\alpha) = \alpha^{-m} \overline{\alpha}^{-b} E_{m,b}^\text{col}(\alpha z) \quad \text{for any } 0 \neq \alpha \in \mathcal{O}_K.
$$

iii) If $m < 0$, we define $E_{m,b}^\text{col}$ by

$$
dE_{m+1,b}^\text{col} := -E_{m,b}^\text{col} \omega,
$$

where $\omega$ is the invariant differential of the elliptic curve.

There exists uniquely such a Coleman function $E_{m,b}^\text{col}$ on $U$ defined by the iterated integration

$$
E_{m+1,b}^\text{col} := \int E_{m,b}^\text{col} \omega
$$

satisfying the distribution relation

$$
\sum_{z_\alpha \in E[\alpha]} E_{m+1,b}^\text{col}(z + z_\alpha) = \alpha^{-m} \overline{\alpha}^{-b} E_{m+1,b}^\text{col}(\alpha z) \quad \text{for any } 0 \neq \alpha \in \mathcal{O}_K.
$$

The existence and uniqueness are insured by (4) Proposition 3.9).

**Remark 2.11.** The distribution relation (19) is the $p$-adic analogue of the distribution relation of the classical complex Eisenstein-Kronecker series

$$
\sum_{z_\alpha \in E[\alpha]} E_{m,b}(z + z_\alpha) = \alpha^{-m} \overline{\alpha}^{-b} E_{m,b}(\alpha z) \quad \text{for any } 0 \neq \alpha \in \mathcal{O}_K.
$$

We can show this by using the following orthogonality of character

$$
\sum_{z_\alpha \in E[\alpha]} \exp \left( \frac{z_\alpha \gamma - \overline{z_\alpha} \gamma}{A} \right) = \begin{cases} 
N(\alpha)(= \alpha \overline{\alpha}) & \text{if } \gamma \in \pi \Gamma \\
0 & \text{if } \gamma \notin \pi \Gamma
\end{cases}
$$

The construction of $p$-adic Eisenstein-Kronecker series in Definition 2.10 allows us to choose constant term when $m > 0$. By the convergence property of $F_1$ in (16), $E_{m,1}^\text{col}$ is defined at any point in $U := E(\mathbb{C}_p) \setminus \{0\}$, and if $b > 1$ then $E_{m,b}^\text{col}$ is defined on $E(\mathbb{C}_p)$. When $b = 0$, since $F_0 = 1$ and the definition of $E_{0,b}^\text{col}$, we have $E_{0,0}^\text{col} = -F_0^\text{col} = -1$. This show that we have $E_{a,0}^\text{col} = 0$ for $a < 0$. Note that the values of $E_{m,b}^\text{col}(z)$ are independent of the choice of the branch of the $p$-adic logarithm (see 4 Lemma 3.12).

According to (4) Proposition 3.11, the $p$-adic Eisenstein-Kronecker series $E_{m,b}^\text{col}$ interpolates the
classical complex Eisenstein-Kronecker series $E_{m,b}$ for $m \leq 0$.

For $m, b$ be integers with $b \geq 0$, we define

$$E_{m,b}^{(p)}(z) := E_{m,b}^{\text{col}}(z) - \frac{1}{\pi_m \pi_b} L_{m,b}^{\text{col}}(\pi z).$$

3. Main result

In previous results, the construction of the $p$-adic measure interpolating special values of algebraic Hecke characters was established by Manin and Vishik [24], N. Katz [21], R. I. Yager [31], and de Shalit [15] etc. In [2], when the conductor of an algebraic Hecke character is prime to $p$, K. Bannai, S. Kobayashi, T. Tsuji related $p$-adic Eisenstein-Kronecker numbers and non-critical values of the $p$-adic $L$-function associated to algebraic Hecke characters under the another construction of the measure with the Kronecker theta function.

In this paper, by the method of $p$-adic analogue as Coleman functions, we related $p$-adic Eisenstein-Kronecker series and the non-critical values of $p$-adic $L$-function of an imaginary quadratic field with class number 1 associated to the algebraic Hecke character whose conductor is divisible by $p$.

3.1. Preludes for main results.

We keep the notation in §0. The $p$-adic $L$-function interpolates the classical $L$-function at critical points in the meaning of Deligne [14]. Let $\varphi : I(\mathfrak{g}) \to \overline{K}^\times$ be an algebraic Hecke character of infinite type $(m,n) \in \mathbb{Z}^2$ whose conductor divides $\mathfrak{g}$ and $\varphi_p : \mathfrak{g} \to \mathbb{C}_p^\times$ be a $p$-adic character. We fix a pair $(\Omega, \Omega_p) \in \mathbb{C}^\times \times \mathcal{O}^\times_{\mathbb{C}_p}$ of a complex period and a $p$-adic period. There exists a $p$-adic function $L_p(\varphi_p)$ such that

$$L_p(\varphi_p) \bigg|_{\Omega_p \cdot (-1)^m} = (m-1)! \left( \frac{2\pi}{\sqrt{d_K}} \right)^n \left( 1 - \frac{\varphi^{-1}(p)}{p} \right) \left( 1 - \varphi(\mathfrak{p}) \right) \frac{L_g(0, \varphi)}{\Omega_{n-m}}$$

for $m < 0$ and $n \geq 0$, both of which lies in $\overline{\mathbb{Q}}$ (see [2] Proposition 2.26). We call $L_p(\varphi_p)$ the $p$-adic $L$-function at the $p$-adic character $\varphi_p$.

We give the construction of this $p$-adic $L$-function along [2] §2.4. If $\hat{E}$ is a formal group associated to $E \otimes \mathcal{O}_K$ for $t = -2x/y$, $\hat{E}$ is the Lubin-Tate formal group over $\mathcal{O}_K$. Let a formal group law of $\hat{E}$ be $\oplus$. Let $\lambda : \hat{E} \xrightarrow{\sim} \hat{G}_a$ be a normalized formal logarithm by $\lambda(0) = 1$. We have $\mathcal{O}_K$-linear isomorphisms

$$\text{Hom}_{\mathcal{O}_p}(\hat{E}, \hat{G}_m) \cong \text{Hom}_{\mathbb{Z}_p}(T_p(\hat{E}), T_p(\hat{G}_m)) \cong \mathcal{O}_K,$$

the first isomorphism holds by ([25] §4.2 Corollary 1) and the second isomorphism holds since $\text{Hom}_{\mathbb{Z}_p}(T_p(\hat{E}), T_p(\hat{G}_m))$ is a free $\mathcal{O}_K$-module of rank 1 by ([25] §4.2 Proposition 12), where $T_p(\hat{E}) := \lim \frac{\hat{E}[p^n]}{\mathfrak{m}_p^n}$ (resp. $T_p(\hat{G}_m) := \lim \frac{\hat{G}_m[p^n]}{\mathfrak{m}_p^n}$) is a Tate module and $\hat{G}_m$ is a multiplicative formal group. $T_p(\hat{G}_m)$ is a $\mathbb{Z}_p$-module of rank 1 (see [25] §2.1 Examples (b)). Then there exists a homomorphism $\eta_p : \hat{E} \to \hat{G}_m$ such that if $t \in \hat{E}(\mathcal{O}_p)[1]$, then $1 + \eta_p(t)$ is a $p$-th power root of 1 and that the image of $\hat{E}(\mathcal{O}_p)[1]$ does not contain power roots of 1. If $\eta_p(t) = \Omega^{-1} t + \cdots \in \mathcal{O}_p[[t]]$, by the uniqueness of the formal logarithm $\lambda$, we have the commutative diagram:
where the isomorphism $\widehat{\Gamma} \cong \widehat{\mathbb{G}}_m$ follows from that a characteristic of $\mathcal{O}_{C_p}$ is 0. 
Since $E$ is ordinary, $\widehat{E}$ has a height 1 ([20, V, Theorem 3.1 (b)]). Then $\eta_p : E \to \widehat{\mathbb{G}}_m$ is isomorphism (see [24, §4, Corollary 4.3.3]). Then we can take the isomorphism $\eta_p : \widehat{E} \cong \widehat{\mathbb{G}}_m$ over $\mathcal{O}_{C_p}$ by

$$
\eta_p(t) = \exp \left( \frac{\lambda(t)}{1_p} \right) - 1.
$$

We call $\Omega_p \in \mathcal{O}_{C_p}^\times$ a $p$-adic period, which is regarded as the $p$-adic analogue of the complex period $\Omega$. The second isomorphism of ([21]) is given by associating to any $x \in \mathcal{O}_{K_p}$ the homomorphism of formal groups defined by $\exp(x \lambda(s)/\Omega_p)$, and depends on the choice of $\Omega_p$. In addition, since $E$ is ordinary, we have $(p) = p\mathbb{P}$. Therefore we have $\mathcal{O}_{K_p} \cong \mathbb{Z}_p$.

Now we fix a natural number $N$. Let $s_N \in \widehat{E}(\mathcal{O}_{C_p})[p^N]$ be any $p^N$-torsion point. Let $z_N$ be the image of $s_N$ by an inclusion map $\widehat{E}(\mathcal{O}_{C_p})[p^N] \hookrightarrow E(\overline{\mathbb{Q}}) \hookrightarrow E(\mathbb{C}) \cong \mathbb{C}/\Gamma$. Then $z_N$ is a $p^N$-torsion point in $E(\mathbb{C})[p^N]$. Now we put $z := \Omega/p \in p^{-N}\mathbb{Z}/\Gamma \cong E(\mathbb{C})[p^N]$. Then we can take an isomorphism $\eta_p : \widehat{E}(\mathcal{O}_{C_p})[p^N] \cong \widehat{\mathbb{G}}_m(\mathcal{O}_{C_p})[p^N]$ of formal groups over $\mathcal{O}_{C_p}$ satisfying

$$
1 + \eta_p(s_N) = \chi_{z_N}(\Omega) := \exp \left( \frac{\Omega z_N - \overline{\Omega} z_N}{A(\Gamma)} \right).
$$

This $\eta_p$ is induced by $\eta_p(t) = \exp(\lambda(t)/\Omega_p) - 1$.

Let $W$ be the integer ring of the completion field at $p$ of a maximal unramified extension of $\mathbb{Q}_p$. According to ([11, Corollary 2.18]), the Kronecker theta function has the property of $p$-adic integrality as follows: For torsion points $z_0, w_0 \in E(K)_\text{tors}$ with orders prime to $p$, we have

$$
\widehat{\Theta}_{20, w_0}(s, t) := \Theta_{20, w_0}(z, w)|_{z = \lambda(s), w = \lambda(t)} \in W[s^{-1}, t^{-1}][[s, t]].
$$

In particular,

$$
\widehat{\Theta}^*_{20, w_0}(s, t) := \widehat{\Theta}_{20, w_0}(s, t) - \chi_{z_0}(w_0)\delta_{x_0} - \delta_{w_0}^{-1} \in W[[s, t]],
$$

where $\delta_x = 1$ if $x \in \Gamma$ and $\delta_x = 0$ otherwise. Therefore for non-zero torsion points $z_0, w_0 \in E(K)_\text{tors}$ with orders prime to $p$, we can characterize a $p$-adic measure $\mu_{z_0, w_0} : \mathbb{Z}_p \times \mathbb{Z}_p \to W$ by

$$
\int_{\mathbb{Z}_p \times \mathbb{Z}_p} (1 + s)^x(1 + t)^yd\mu_{z_0, w_0}(x, y) = \widehat{\Theta}^*_{20, w_0}(s, t).
$$

In particular, if $z_0, w_0 \notin \Gamma$, we have $\widehat{\Theta}^*_{20, w_0}(s, t) = \widehat{\Theta}_{20, w_0}(s, t)$. The existence and uniqueness of this $p$-adic measure $\mu_{z_0, w_0} : \mathbb{Z}_p \times \mathbb{Z}_p \to W$ follows from $\widehat{\Theta}^*_{20, w_0}(s, t) \in W[[s, t]]$ (see [31, §6]). By the isomorphism $\eta_p : \widehat{E} \cong \widehat{\mathbb{G}}_m$, we can rewrite the formula ([24]) as

$$
\int_{\mathbb{Z}_p \times \mathbb{Z}_p} \exp \left( \frac{x\lambda(s)}{\Omega_p} \right) \exp \left( \frac{y\lambda(t)}{\Omega_p} \right) d\mu_{z_0, w_0}(x, y) = \widehat{\Theta}^*_{20, w_0}(s, t).
$$

For a non-zero torsion point $z_0 \in E(K)_\text{tors}$ with an order prime to $p$ and $g$ as above, we define a variant measure $\mu^{(g)}_{z_0, 0}$ on $\mathbb{Z}_p \times \mathbb{Z}_p$ of the measure $\mu_{z_0, 0}$ in ([23]) by the formula

$$
\int_{\mathbb{Z}_p \times \mathbb{Z}_p} \exp \left( \frac{x\lambda(s)}{\Omega_p} \right) \exp \left( \frac{y\lambda(t)}{\Omega_p} \right) d\mu^{(g)}_{z_0, 0}(x, y) = \widehat{\Theta}^*_{z_0, 0}(g^{-1}s, [g]t)
$$

For $\alpha_0 \in (\mathcal{O}_K/\mathfrak{g})^\times$, let $\alpha_0\Omega/g \in g^{-1}\Gamma/\Gamma \cong E(\mathbb{C})[\mathfrak{g}]$ be a primitive $\mathfrak{g}$-torsion point and $\mu^{(g)}_{\alpha_0\Omega/\mathfrak{g}, 0}$ induces a measure on $(\mathcal{O}_K \otimes \mathbb{Z}_p)^\times$ through the isomorphism ([4]). Similarly to ([22, §2.4]), for a continuous function $f : \mathbb{X} \to \mathbb{C}$, we define the measure $\mu_g$ on $\mathbb{X}$ by

$$
\int_{\mathbb{X}} f(\alpha)d\mu_g(\alpha) := g^{-1}\Omega_p \sum_{\alpha_0 \in (\mathcal{O}_K/\mathfrak{g})} \int_{(\mathbb{O}_K \otimes \mathbb{Z}_p)^\times} f(\alpha_0\alpha)\kappa_1(\alpha)d\mu^{(g)}_{\alpha_0\Omega/\mathfrak{g}, 0}(\alpha).$$
Definition 3.1 (p-adic L-function). Let \( \mathfrak{f} \) be an integral ideal of \( \mathcal{O}_K \) prime to \( p \). The p-adic L-function of \( K \) is the function whose domain is the set of all p-adic continuous characters \( \phi_p : \mathbb{X} \to \mathbb{C}_p^* \) on \( \mathbb{X} \), and is defined at the character \( \phi_p \) by

\[
L_p(\phi_p) := \int_{\mathbb{X}} \phi_p(\alpha) d\mu_\alpha(\alpha).
\]

Here the measure \( \mu_\alpha \) corresponds to the measure denoted by \( \mu \) in (15) Theorem 4.14 through the canonical isomorphism \( \text{Gal}(K(g_0^\infty)/K) \cong \mathbb{X} \).

By using the formula (27), we can rewrite the p-adic L-function (28) at the p-adic character \( \varphi_p \) in §0 as

\[
L_p(\varphi_p) = g^{-1} \Omega_p \sum_{\alpha_0 \in (\mathcal{O}_K/\mathfrak{g})^\times} \chi_\delta(\alpha_0) \int_{(\mathcal{O}_K \otimes \mathbb{Z}_p)^\times} \chi_1(\alpha) \chi_2(\alpha) \kappa_1(\alpha)^{m_1} \kappa_2(\alpha)^{n_1} d\mu_{\alpha_0 \mathcal{O}_K/\mathfrak{g},0}(\alpha),
\]

where \( \chi_1, \chi_2 \) in (29) are characters in §0. In addition by the following isomorphism

\[
\mathcal{O}_K^\times \times \mathcal{O}_\mathfrak{g}^\times \overset{\sim}{\longrightarrow} \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times \quad (\alpha_1, \alpha_2) \mapsto (\alpha_1^{-1}, \alpha_2),
\]

and putting \( x := \kappa_1(\alpha)^{-1} \) and \( y := \kappa_2(\alpha) \), we rewrite the formula (29) as

\[
L_p(\varphi_p) = g^{-1} \Omega_p \sum_{\alpha_0 \in (\mathcal{O}_K/\mathfrak{g})^\times} \chi_\delta(\alpha_0) \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} \chi_1(x)^{-1} \chi_2(y)x^{-m_1}y^{n_1} d\mu_{\alpha_0 \mathcal{O}_K/\mathfrak{g},0}(x, y).
\]

Since \( p = \pi \bar{\pi} \) and \( \chi_1 \) (resp. \( \chi_2 \)) is defined by extending to 0 at all values not prime to \( \pi \) (resp. \( \bar{\pi} \)), \( \chi_1 \) (resp. \( \chi_2 \)) is 0 on \( p \mathbb{Z}_p \). Therefore if we replace the integration domain of (30) by \( \mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times \), or \( \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times \), all integrals of the right side of the formula (30) have the same value on \( \mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times, \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times, \) or \( \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times \).

Definition 3.2 (Gauss sum).

a) For \( x \in \mathcal{O}_K/(\pi^N) \), we define the Gauss sum on \( \mathcal{O}_K/(\pi^N) \) by

\[
\tau(\chi_1, x) := \sum_{u \in (\mathcal{O}_K/(\pi^N))^\times} \chi_1(u) \langle \Omega, xu \bar{z}_N \rangle = \sum_{u \in (\mathcal{O}_K/(\pi^N))^\times} \chi_1(u) \exp \left( \frac{\Omega \bar{u} z_N - \Omega uz_N}{A} \right),
\]

where \( z_N := \Omega/\pi^N \in \pi^{-N} \Gamma/\Gamma \cong E(\mathbb{C})[\pi^N] \) is a \( \pi^N \)-torsion point.

b) For \( y \in \mathcal{O}_K/(\bar{\pi}^N) \), we define the Gauss sum on \( \mathcal{O}_K/(\bar{\pi}^N) \) by

\[
\tau(\chi_2, y) := \sum_{v \in (\mathcal{O}_K/(\bar{\pi}^N))^\times} \chi_2(v) \langle \Omega, \bar{y} \bar{v} w_N \rangle = \sum_{v \in (\mathcal{O}_K/(\bar{\pi}^N))^\times} \chi_2(v) \exp \left( \frac{\Omega \bar{y} \bar{v} w_N - \Omega \bar{y} \bar{v} w_N}{A} \right),
\]

where \( w_N := \Omega/\pi^N \in \pi^{-N} \Gamma/\Gamma \cong E(\mathbb{C})[\pi^N] \) is a \( \pi^N \)-torsion point.

Lemma 3.3 (Properties of Gauss sum). Under the assumption of Definition 3.2, we have

a) \( \tau(\chi_1, x) = \chi_1(x) \tau(\chi_1), \) where \( \tau(\chi_1) := \tau(\chi_1, 1) \). In addition, if \( x \) is not prime to \( \pi \), we have \( \tau(\chi_1, x) = 0 \), and if \( x \) is prime to \( \pi \), we have \( |\tau(\chi_1, x)| = \sqrt{N(p^N)} = \sqrt{p^N} \).

b) \( \tau(\chi_2, y) = \chi_2(\bar{y}) \tau(\chi_2), \) where \( \tau(\chi_2) := \tau(\chi_2, 1) \). In addition, if \( y \) is not prime to \( \bar{\pi} \), we have \( \tau(\chi_2, y) = 0 \) and if \( y \) is prime to \( \bar{\pi} \), we have \( |\tau(\chi_2, y)| = \sqrt{N(p^N)} = \sqrt{p^N} \).

Proof. For the proof of b), if we replace \( y \) for \( \bar{y} \) and \( v \) for \( \bar{v} \), we can reduce to a). So we have only to prove a).

If \( x \) is prime to \( \pi \), since \((\mathcal{O}_K/(\pi^N))^\times \to (\mathcal{O}_K/(\pi^N))^\times, u \mapsto xu \) is bijective and \( \chi_1^{-1} = \chi_1 \) because \( \chi_1 \) is a finite character, then we have

\[
\tau(\chi_1, x) = \chi_1(x) \tau(\chi_1).
\]
Next we consider when \( x \) is not prime to \( \pi \). Let \( d \neq 1 \) be a greatest common divisor of \( x \) and \( \pi \). Since \( \chi_1 \) is primitive, there exists \( a \in (O_K/(\pi^N))^\times \) such that
\[
\chi_1(a) \neq 1 \quad \text{and} \quad a \equiv 1 \mod \frac{\pi^N}{d}.
\]
So we have \( xa \equiv x \pmod{\pi^N} \). Therefore we have
\[
\overline{\chi_1(a)} \tau(\chi_1, x) = \tau(\chi_1, x).
\]
Since \( \chi_1(a) \neq 1 \), we have \( \tau(\chi_1, x) = 0 \). On the other hand, since \( \chi_1(x) = 0 \), we have \( \chi_1(x) \tau(\chi_1) = 0 \). Hence \( \tau(\chi_1, x) = \chi_1(x) \tau(\chi_1) \).

We show the latter part of a). Since \( \Gamma = \Omega \), we have \( A := A(\Gamma) = \sqrt{d_K \Omega} \), where \( d_K \) is a discriminant of \( K \). Then
\[
\tau(\chi_1, x) = \sum_{u \in (O_K/(\pi^N))^\times} \overline{\chi_1(u)} \exp \left[ \frac{2\omega}{\sqrt{d_K}} \left( \frac{\overline{z}}{\pi^N} - \frac{z}{\pi^N} \right) \right].
\]
Now for \( z \in O_K/(\pi^N) \), we have
\[
\tau(\chi_1, z) \tau(\chi_1, z) = \sum_{u, v \in (O_K/(\pi^N))^\times} \chi_1(u) \chi_1(v) \exp \left[ \frac{2\omega}{\sqrt{d_K}} \left( \overline{\frac{\overline{z}}{\pi^N} - \frac{z}{\pi^N}} \right) \right].
\]
If we take the sum of both sides of (31) over \( z \in O_K/(\pi^N) \), since \( \tau(\chi_1, z) = 0 \) holds if \( (z, \pi) \neq 1 \), so the sum over \( z \in O_K/(\pi^N) \) of the left side of (31) and the sum over \( z \in (O_K/(\pi^N))^\times \) have the same value. Then giving the sum over \( z \in (O_K/(\pi^N))^\times \) to the left side of (31), we have
\[
\sum_{z \in (O_K/(\pi^N))^\times} \tau(\chi_1, z) \tau(\chi_1, z) = \sum_{u, v \in (O_K/(\pi^N))^\times} \chi_1(u) \chi_1(v) \exp \left[ \frac{2\omega}{\sqrt{d_K}} \left( \overline{\frac{\overline{z}}{\pi^N}} - \frac{z}{\pi^N} \right) \right].
\]
Since \( z \) is prime to \( \pi \), \( (O_K/(\pi^N))^\times \to (O_K/(\pi^N))^\times \), \( u \mapsto z^{-1} u, v \mapsto z^{-1} v \) is bijective, we have
\[
\sum_{z \in (O_K/(\pi^N))^\times} \tau(\chi_1, z) \tau(\chi_1, z) = \varepsilon(\pi^N) \sum_{u, v \in (O_K/(\pi^N))^\times} \chi_1(u) \chi_1(v) \langle \Omega, uz \rangle \langle \Omega, u \rangle \overline{\langle \overline{z}, \overline{u} \rangle} = \varepsilon(\pi^N) |\tau(\chi_1, x)|^2,
\]
where \( \varepsilon(\pi^N) := \#(O_K/(\pi^N))^\times \) is an Euler function and we used \( \chi_1(z) \overline{\chi_1(z)} = |\chi_1(z)|^2 = 1 \). Next we give the sum over \( z \in (O_K/(\pi^N))^\times \) to the right side of (31). Since
\[
\sum_{z \in (O_K/(\pi^N))^\times} \exp \left[ \frac{2\omega}{\sqrt{d_K}} \left( \overline{\frac{\overline{z}}{\pi^N}} - \frac{z}{\pi^N} \right) \right] = \begin{cases} \varepsilon(\pi^N) N(p^N) & \text{if } u \equiv v \pmod{\pi^N} \\ 0 & \text{otherwise} \end{cases}
\]
the sum over \( z \in (O_K/(\pi^N))^\times \) of the right side of (31) is \( \varepsilon(\pi^N)p^N \). Therefore we have
\[
|\tau(\chi_1, x)|^2 = N(p^N) = p^{2N}.
\]
\[
\square
\]
By Lemma 3.3 and using the formula (33), we can rewrite the \( p \)-adic \( L \)-function (30) at the \( p \)-adic character \( \varphi_p \) as
\[
\hat{L}_p(\varphi_p) = \frac{g^{-1} \Omega_p}{\tau(\chi_1) \tau(\chi_2)} \sum_{\alpha_0 \in (O_K/(\pi^N))^\times} \chi_0(\alpha_0) \sum_{u \in (O_K/(\pi^N))^\times} \sum_{v \in (O_K/(\pi^N))^\times} \chi_1(u) \chi_2(v) \times
\int_{\mathbb{Z}_p^2 \times \mathbb{Z}_p^2} \exp \left( \frac{x \lambda(u \lambda(s \delta))}{\zeta_p} \right) \exp \left( \frac{y \lambda(\tilde{\psi}\delta \delta_N)}{\zeta_p} \right) x^{-m - 1} y^N d \mu_d^{(\varphi_p)}(x, y),
\]
where \( x \in \mathbb{Z}_p \) (resp. \( y \in \mathbb{Z}_p \)) in the formula (32) is an element obtained by the lifting via the natural projection \( \mathbb{Z}_p \cong O_K \to O_K/(\pi^N) \) (resp. \( \mathbb{Z}_p \cong O_K \to O_K/(\pi^N) \)) and \( s_N \in \hat{E}(m_c_p)[\pi^N] \).
For any \( \chi \in \hat{E}(\mathcal{M}_p)[\pi^N] \) is the \( \pi^N \)-torsion point in the formal group corresponding to the \( \pi^N \)-torsion point \( z_N = \Omega/\pi^N \in \pi^{-N}\Gamma/\Gamma \cong E(\mathbb{C})[\pi^N] \) (resp. \( w_N = \Omega/\pi^N \in \pi^{-N}\Gamma/\Gamma \cong E(\mathbb{C})[\pi^N] \)) through the inclusion map \( \hat{E}(\mathcal{M}_p)[\pi^N] \hookrightarrow E(\mathbb{Q})[\pi^N] \hookrightarrow \mathbb{C}/\Gamma \).

Since the \( p \)-adic \( L \)-function \((32)\) is rewritten by using Lemma 3.3 similarly to \((30)\), the \( p \)-adic \( L \)-function \((32)\) has the same value on \( \mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_p^\times \times \mathbb{Z}_p, \mathbb{Z}_p \times \mathbb{Z}_p^\times, \) or \( \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times \).

3.2. The non-critical values of the \( p \)-adic Hecke \( L \)-function whose conductor is divisible by \( p \) and \( p \)-adic Eisenstein-Kronecker series.

We calculate the rewritten \( p \)-adic \( L \)-function \((32)\), focusing on each summand of the \( p \)-adic \( L \)-function. The method of calculation of the \( p \)-adic \( L \)-function \((32)\) is as follows: We first rewrite the \( p \)-adic \( L \)-function \((32)\) constructed using the two-variable measure in terms of the one-variable measure (see Proposition 3.6). Then we express the integration in Proposition 3.6 using the formula \((20)\) (see Lemma 3.7). Finally, by using Lemma 3.10 we express the \( p \)-adic \( L \)-function \((32)\) using the Coleman Eisenstein-Kronecker series (see Proposition 3.9).

**Lemma 3.4.** For any \( s, t \in \hat{E}(\mathcal{M}_p) \), we have

\[
\sum_{v \in (\mathcal{O}_K/(\pi^N))^\times} \chi_2(v) \int_{\mathbb{Z}_p \times \mathbb{Z}_p} \exp \left( \frac{x \lambda(s + [u]s_N)}{\Omega_p} \right) \exp \left( \frac{y \lambda(t + [v]t_N)}{\Omega_p} \right) \mu_{\alpha_0/\gamma}(x, y) = \frac{\pi^N}{\tau(\chi_2)} \sum_{\gamma \in (\mathcal{O}_K/(\pi^N))^\times} \chi_2(\gamma) \hat{\Theta}^{* g}_{\alpha_0/\gamma}([g^{-1}])(s + [u]s_N), [\gamma t]).
\]

**Proof.**

\[
\text{(Left side)} = \sum_{v \in (\mathcal{O}_K/(\pi^N))^\times} \chi_2(v) \hat{\Theta}^{* g}_{\alpha_0/\gamma}([g^{-1}])(s + [u]s_N), [\gamma t])
\]

\[
= \frac{1}{\tau(\chi_2)} \sum_{\gamma \in (\mathcal{O}_K/(\pi^N))^\times} \chi_2(\gamma) \sum_{v \in (\mathcal{O}_K/(\pi^N))^\times} \langle \Omega, v \gamma w_N \rangle \hat{\Theta}^{* g}_{\alpha_0/\gamma}([g^{-1}])(s + [u]s_N), [\gamma t]).
\]

Here the first equation can be calculated by the formula \((26)\) and the \( p \)-adic translation of the Kronecker theta function (see [H] Corollary 2.22) and the second equation can be calculated by \( \tau(\chi_2, v) = \chi_2(v) \tau(\chi_2) \) by Lemma 3.3b. The result of this lemma can be showed by the following lemma 3.5.

**Lemma 3.5.**

\[
\sum_{v \in (\mathcal{O}_K/(\pi^N))^\times} \langle \gamma g^{-1} \Omega, w'_N \rangle \hat{\Theta}^{* g}_{\alpha_0/\gamma}([g^{-1}])(s + [u]s_N), [\gamma t])
\]

\[
= \sum_{w''_N \in \pi^{-N}\Gamma/\Gamma} \langle \gamma g^{-1} \Omega, w'_N \rangle \hat{\Theta}^{* g}_{\alpha_0/\gamma}([g^{-1}])(s + [u]s_N), [\gamma t])
\]

**Proof.** Since the formula \((33)\) is 0 when \( v \) is not prime to \( \pi \), we have only consider the above formula when \( v \) is prime to \( \pi \). Since \( \mathcal{O}_K/(\pi^N) \xrightarrow{\cong} \pi^{-N}\Gamma/\Gamma, v \mapsto \gamma w_N \) is bijective, if we put
\( w_N' := \overline{v}w_N \), we have
\[
\sum_{v \in (\mathcal{O}_K/(\pi N))^\times} \langle \Omega, \overline{\gamma} w_N \rangle \hat{\Theta}_{\alpha_0/\mathcal{O}, g, g \overline{v}w_N}(\{g^{-1}\}(s + [u]s_N), [g]t) = \sum_{w'_N \in \pi^{-N} \Gamma/\Gamma} \langle \gamma \Omega, w'_N \rangle \hat{\Theta}_{\alpha_0/\mathcal{O}, g, g \overline{w}'_N}(\{g^{-1}\}(s + [u]s_N), [g]t)
\]

Since \( g \) is prime to \( p \) by the assumption, \( g \) is prime to \( \pi \). Then \( \pi^{-N} \Gamma/\Gamma \rightarrow \pi^{-N} \Gamma/\Gamma, w'_N \mapsto \overline{g} w'_N \) is bijective. If we put \( w''_N := \overline{g} w'_N \), then we have
\[
\sum_{v \in (\mathcal{O}_K/(\pi N))^\times} \langle \Omega, \overline{\gamma} w_N \rangle \hat{\Theta}_{\alpha_0/\mathcal{O}, g, g \overline{v}w_N}(\{g^{-1}\}(s + [u]s_N), [g]t) = \sum_{w''_N \in \pi^{-N} \Gamma/\Gamma} \langle \gamma g^{-1} \Omega, w''_N \rangle \hat{\Theta}_{\alpha_0/\mathcal{O}, g, g \overline{w}''_N}(\{g^{-1}\}(s + [u]s_N), [g]t)
\]

Since \( g \) is prime to \( p \), \( g \) is prime to \( \pi \). Hence \( \gamma g^{-1} \in (\mathcal{O}_K/(\pi N))^\times \). By the lifting \( \gamma g^{-1} \) into \( \mathcal{O}_K \), we have \( \gamma g^{-1} \Omega \in \mathcal{O}_K \Omega = \Gamma \). So we can use the distribution relation of the Kronecker theta function (6). Then we obtain this lemma. \( \square \)

We show that the p-adic L-function \( \mathbb{L} \) can be rewritten as follows:

**Proposition 3.6.** Let \( m, n \) be any integers with \( n \geq 0 \). Then we have

\[
L_p(\varphi_p) = \frac{g^{-1} \Omega_p \pi^N}{\tau(\Omega) \tau(\pi^N)} \int_{\mathbb{Z}_p \times \mathbb{Z}_p} \exp \left( \frac{x\lambda \left( \left\lfloor g^{-1} \pi^N \right\rfloor s \right)}{\Omega_p} \right) \exp \left( \frac{y\lambda \left( \left\lfloor g^{-1} \pi^N \right\rfloor t \right)}{\Omega_p} \right) (x, y)
\]

**Proof.** Since \( \alpha_0/\mathcal{O}/g \in \mathfrak{g}^{-1} \Gamma/\Gamma \) and \( g^{-1} \Omega \in \Gamma \), this value \( (\alpha_0/\mathcal{O}/g - \gamma/\mathcal{O}/g)/\pi^N \in g^{-1} \pi^{-N} \Gamma/\Gamma \cong E(\mathbb{C})[\mathfrak{g} \pi^N] \) is a \( g \pi^N \)-torsion point. Since \( (\alpha_0/\mathcal{O}/g - \gamma/\mathcal{O}/g)/\pi^N \) is a torsion point with an order prime to \( \pi \), by the formula (25), Lemma 3.4 becomes

\[
\int_{\mathbb{Z}_p \times \mathbb{Z}_p} \exp \left( \frac{x\lambda \left( \left\lfloor g^{-1} \pi^N u \right\rfloor s \right)}{\Omega_p} \right) \exp \left( \frac{y\lambda \left( \left\lfloor g^{-1} \pi^N t \right\rfloor N \right)}{\Omega_p} \right) (x, y)
\]

Here for simplicity, we put

\[
d\mu_1(x,y) := \sum_{v \in (\mathcal{O}_K/(\pi N))^\times} \chi_2(v) \exp \left( \frac{x\lambda \left( \left\lfloor s \right\rfloor s_N \right)}{\Omega_p} \right) \exp \left( \frac{y\lambda \left( \left\lfloor t \right\rfloor N \right)}{\Omega_p} \right) d\mu_{\alpha_0/\mathcal{O}, g, 0}(x, y)
\]

\[
d\mu_2(x,y) := \frac{\pi^N}{\tau(\pi)} \sum_{\gamma \in (\mathcal{O}_K/(\pi N))^\times} \chi_2(\gamma) \exp \left( \frac{x\lambda \left( \left\lfloor g^{-1} \pi^N \right\rfloor s \right)}{\Omega_p} \right) \exp \left( \frac{y\lambda \left( \left\lfloor g^{-1} \pi^N \right\rfloor t \right)}{\Omega_p} \right) d\mu_{\alpha_0/\mathcal{O}, g, 0}(x, y).
\]

By the isomorphism of formal groups \( \eta : \widehat{E}(m_{\mathbb{C}_p}) \rightarrow \widehat{G}(m_{\mathbb{C}_p}) \) the formula (31) is

\[
\int_{\mathbb{Z}_p \times \mathbb{Z}_p} (1 + \eta_p(s))^r (1 + \eta_p(t))^y d\mu_1(x, y)
\]

\[
= \int_{\mathbb{Z}_p \times \mathbb{Z}_p} (1 + \eta_p(\left\lfloor g^{-1} \pi^N \right\rfloor s)) (1 + \eta_p(\left\lfloor g^{-1} \pi^N \right\rfloor t))^{y} d\mu_2(x, y)
\]
By the binomial expansion, we have
\[
\sum_{m,n=0}^{\infty} \eta_p(s)^m \eta_p(t)^n \int_{\mathbb{Z}_p} \left( \frac{x}{m} \right) \left( \frac{y}{n} \right) d\mu_1(x, y)
= \sum_{m,n=0}^{\infty} \eta_p([g^{-1} \pi^{-N}]^m \eta_p([\pi^N \mathcal{F}]^n t)^n \int_{\mathbb{Z}_p} \left( \frac{x}{m} \right) \left( \frac{y}{n} \right) d\mu_2(x, y)
\]
Now we put
\[
f(x, y) := \sum_{m,n=0}^{\infty} \eta_p(s)^m \eta_p(t)^n \left( \frac{x}{m} \right) \left( \frac{y}{n} \right),
\]
\[
g(x, y) := \sum_{m,n=0}^{\infty} \eta_p([g^{-1} \pi^{-N}]^m \eta_p([\pi^N \mathcal{F}]^n t)^n \left( \frac{x}{m} \right) \left( \frac{y}{n} \right).
\]
Since \(|\eta_p(s)| < 1, |\eta_p(t)| < 1\) by \(s, t \in \widehat{E}(m_{C_p})\), then \(\lim_{m,n \to \infty} \eta_p(s)^m \eta_p(t)^n = 0\). By Mahler’s theorem, \(f(x, y)\) is a continuous function on \(\mathbb{Z}_p \times \mathbb{Z}_p\). On the other hand, since \(g\) and \(\pi\) are prime to \(\pi\), \([g^{-1} \pi^{-N}] \in \widehat{E}(m_{C_p})\). Then \(\lim_{m,n \to \infty} \eta_p([g^{-1} \pi^{-N}]^m \eta_p([\pi^N \mathcal{F}]^n t)^n = 0\). By Mahler’s theorem, \(g(x, y)\) is a continuous function on \(\mathbb{Z}_p \times \mathbb{Z}_p\). By arbitrariness of \(s, t \in \widehat{E}(m_{C_p})\), \(f(x, y)\) and \(g(x, y)\) are arbitrary continuous function on \(\mathbb{Z}_p \times \mathbb{Z}_p\). Since we pick up a part of the \(p\)-adic L-function \([22]\), we can regard \(f(x, y)\) and \(g(x, y)\) as continuous functions of \((x, y)\) on \(\mathbb{Z}_p^\times \times \mathbb{Z}_p\). Therefore we can replace \(f(x, y)\) by \(x^{-m} y^n f(x, y)\) and \(g(x, y)\) by \(x^{-m} y^n g(x, y)\). Then we obtain this proposition.

Next we show that
\[
\int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} \exp \left( \frac{x \lambda([g^{-1} \pi^{-N}] u s N)}{\Omega_p} \right) x^{-m} y^n d\mu_{(a_0 \Omega/g – \gamma \Omega/g)/\pi N,0}(x, y)
\]
can be expressed as a Coleman function on \(E(C_p)\). Note that the value \((a_0 \Omega/g – \gamma \Omega/g)/\pi N\) is an algebraic element since \((a_0 \Omega/g – \gamma \Omega/g)/\pi N \in g^{-1} \pi^{-N} \Gamma/\Gamma \cong E(\mathbb{C})[\pi^N]\) \(\cong E(\mathbb{K})[\pi^N]\), so that \((a_0 \Omega/g – \gamma \Omega/g)/\pi N\) can be embedded into \(E(C_p)\) by an inclusion \(i : \mathbb{K} \hookrightarrow C_p\). For an inclusion map \(i_* : E(\mathbb{K}) \hookrightarrow E(C_p)\) induced by the inclusion map \(i\) through the lifting \(E(\mathbb{K})[\pi^N] \xrightarrow{\times \pi^N} E(\mathbb{K})[\pi^N]\), we put \(\tilde{\omega}_0 := i_* (a_0 \Omega/g \pi N)\) and \(\tilde{\gamma} := i_* (\gamma / \pi^N)\).

**Lemma 3.7.** The formula \([35]\) can be expressed as a Coleman function on \(E(C_p)\) as follows:
\[
\int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} \exp \left( \frac{x \lambda([g^{-1} \pi^{-N}] u s N)}{\Omega_p} \right) x^{-m} y^n d\mu_{(a_0 \Omega/g – \gamma \Omega/g)/\pi N,0}(x, y)
= n! (-1)^m + n + 1 \Omega_p^{-m-1} \mathcal{E}_{m+1, n+1}^{(p)}(\lambda([g^{-1} \pi^{-N}] u s N) + \pi N(\tilde{\omega}_0 – \tilde{\gamma})).
\]

Before proving Lemma \([3.7]\) we introduce a new definition.

**Definition 3.8** (= [2] Definition 2.18). For a non-zero torsion point \(z_0 \in E(\mathbb{Q})_{\text{tors}}\) with an order to prime \(p\) and any integer \(b \geq 0\), we define the \(p\)-adic measure \(\mu_{z_0,b}\) on the set \(\mathcal{C}^{\text{an}}(O_{K_p}, C_p)\) consisting of locally \(K_p\)-analytic functions on \(O_{K_p}\):
\[
\int_{O_{K_p}} \exp \left( \frac{x \lambda(s)}{\Omega_p} \right) d\mu_{z_0,b}(x) := \widehat{F}_{z_0,b}(s).
\]
Then according to (2) §2.4, for any integer \(a, b\) with \(b \geq 0\), we have
\[
\int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} x^n y^b d\mu(\alpha_0/g - \gamma_0/g)/\pi^N, 0(x, y) = b! \Omega_p^b \int_{\mathcal{O}_{K_p}^\times} x^a d\mu(\alpha_0/g - \gamma_0/g)/\pi^N, b+1(x)
\]
Since \(x^a\) is a continuous function on \(\mathbb{Z}_p^\times \cong \mathcal{O}_{K_p}^\times\) and \(a\) is arbitrary, \(x^a\) is arbitrary continuous function on \(\mathbb{Z}_p^\times \cong \mathcal{O}_{K_p}^\times\). So we can replace by
\[
x^a \mapsto x^a \exp \left( \frac{x \lambda([g^{-1}] \pi^{-N} u]\pi N)}{\Omega_p} \right).
\]
If we substitute \(a = -m - 1, b = n\), then we have
\[
(36) \quad \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} x^{-m-1} y^n \exp \left( \frac{x \lambda([g^{-1}] \pi^{-N} u]\pi N)}{\Omega_p} \right) d\mu(\alpha_0/g - \gamma_0/g)/\pi^N, 0(x, y)
\]
\[
= n! \Omega_p^n \int_{\mathcal{O}_{K_p}^\times} x^{-m-1} \exp \left( \frac{x \lambda([g^{-1}] \pi^{-N} u]\pi N)}{\Omega_p} \right) d\mu(\alpha_0/g - \gamma_0/g)/\pi^N, n+1(x)
\]
**Proof of Lemma 3.7** For a torsion point \(z_0 \in E(K)_{\text{tors}} \xrightarrow{\iota} E(\mathbb{C}_p)_{\text{tors}}\) with a prime to \(\pi\) and any integer \(b \geq 0\), by using the induction on \(m\) and the distribution relation of the \(p\)-adic Eisenstein-Kronecker series (19), we can show the following relation (for details, see 4 Remark 3.17.):
\[
(37) \quad E_{m, b}(z) \bigg|_{\mu_{z_0, b}} = (-1)^b (-\Omega_p)^m \int_{\mathcal{O}_{K_p}^\times} x^{-m} \exp \left( \frac{x \lambda(t)}{\Omega_p} \right) d\mu_{z_0, b}(x)
\]
where \(t\) and \(z\) are variables related by \(z = \lambda(t)\) with the normalized formal logarithm \(\lambda : \hat{E} \xrightarrow{\cong} \hat{G}_a\).
Now \((\alpha_0/g - \gamma_0/g)/\pi^N\) has an order prime to \(\pi\), we can use (37). Then (36) becomes
\[
\int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} x^{-m-1} y^n \exp \left( \frac{x \lambda([g^{-1}] \pi^{-N} u]\pi N)}{\Omega_p} \right) d\mu(\alpha_0/g - \gamma_0/g)/\pi^N, 0(x, y)
\]
By using (37), we have
\[
= n! (-1)^{m+n+1} \Omega_p^{n-m-1} \int_{\mathcal{O}_{K_p}^\times} \lambda([g^{-1}] \pi^{-N} u]\pi N)(\alpha_0 - \gamma) \bigg|_{\pi^N(\alpha_0 - \gamma)}
\]
\[
= n! (-1)^{m+n+1} \Omega_p^{n-m-1} \int_{\mathcal{O}_{K_p}^\times} \lambda([g^{-1}] \pi^{-N} u]\pi N)(\alpha_0 - \gamma) \bigg|_{\pi^N(\alpha_0 - \gamma)}
\]
By the translation of the residue disk (17), we have
\[
= n! (-1)^{m+n+1} \Omega_p^{n-m-1} \int_{\mathcal{O}_{K_p}^\times} \lambda([g^{-1}] \pi^{-N} u]\pi N)(\alpha_0 - \gamma) \bigg|_{\pi^N(\alpha_0 - \gamma)}
\]
Since \(z = \lambda([g^{-1}] \pi^{-N} u]\pi N) \in [0,\lambda(\mathbb{C}_p)]\), we have
\[
= n! (-1)^{m+n+1} \Omega_p^{n-m-1} \int_{\mathcal{O}_{K_p}^\times} \lambda([g^{-1}] \pi^{-N} u]\pi N)(\alpha_0 - \gamma) + \pi^N(\alpha_0 - \gamma)).
\]
Then we obtain this lemma. \(\square\)

Here for \(u \in (\mathcal{O}_K/\pi^N)^\times\), the value \(\lambda([g^{-1}] \pi^{-N} u]\pi N) = g^{-1} \pi^{-N} u\pi N = u\Omega/\mathfrak{p}^{N}\) defines a primitive \(\mathfrak{p}^N\)-torsion point. Since \(u\Omega/\mathfrak{p}^{N}\) is the element in \(E(\mathbb{C})[\mathfrak{p}^{N}]\) through the isomorphism \(g^{-1} \mathfrak{p}^{-N}\Gamma/\Gamma \cong E(\mathbb{C})[\mathfrak{p}^{N}] \cong E(\mathbb{C})[\mathfrak{p}^{N}])\), the value \(u\Omega/\mathfrak{p}^{N}\) can be embedded in \(E(\mathbb{C}_p)\). For an inclusion map \(i_* : E(\mathbb{K}) \hookrightarrow E(\mathbb{C}_p)\) induced by the inclusion map \(i : \mathbb{K} \hookrightarrow \mathbb{C}_p\), we put \(\tilde{u} := i_*(u\Omega/\mathfrak{p}^{N})\). Then the value of the \(p\)-adic \(L\)-function (32) can be calculated as follows:
Proposition 3.9. Let $m, n$ be any integers with $n \geq 0$. Then we have
\[
\frac{L_p(\varphi_p)}{\Omega_p^{n-m}} = \frac{g^{-1}n!(1)^{m+n+1}}{\tau(\chi)p^N} \sum_{\alpha \in (\mathcal{O}_K/g)}\chi_{\mathcal{O}}(\alpha_0) \sum_{u \in (\mathcal{O}_K/(pN))^\times} \chi_1(u) \sum_{\gamma \in (\mathcal{O}_K/(pN))^\times} \chi_2(\gamma) \times E_{m+1,n+1}(\bar{u} + \pi^N(\bar{\alpha}_0 + \bar{\gamma})).
\]

Proof. First, we combine Lemma 3.6 with Lemma 3.7. Second, by Lemma 3.8 b), we have $\tau(\chi_2) = \chi_2(-1)p^N$. Third, we use the fact that $\chi_2(-1)^{-1} = \chi_2(-1)$ holds since $\chi_2(-1) = \pm 1$ and that $\gamma \mapsto -\gamma$ is bijective on $(\mathcal{O}_K/(pN))^\times$. Finally, if we recall 20 and use the following Lemma 3.10 we obtain this proposition.

Lemma 3.10.
\[
\sum_{u \in (\mathcal{O}_K/(pN))^\times} \chi_1(u)E_{m+1,n+1}(\pi(\bar{u} + \pi^N(\bar{\alpha}_0 + \bar{\gamma}))) = 0.
\]

Proof. Since $(\pi, \pi^N) = \pi$ and $\chi_1$ is a primitive character on $(\mathcal{O}_K/(pN))^\times$, there exists $a \in (\mathcal{O}_K/(pN))^\times$ such that $\chi_1(a) \neq 1$ and $a \equiv 1 \pmod{\pi^{N-1}}$. Therefore
\[
\chi_1(a) \sum_{u \in (\mathcal{O}_K/(pN))^\times} \chi_1(u)E_{m+1,n+1}(\pi(\bar{u} + \pi^N(\bar{\alpha}_0 + \bar{\gamma}))) = \sum_{u \in (\mathcal{O}_K/(pN))^\times} \chi_1(au)E_{m+1,n+1}(\pi(\bar{u} + \pi^N(\bar{\alpha}_0 + \bar{\gamma})))
\]
since $a$ is prime to $\pi$, $u \mapsto a^{-1}u$ is bijective on $(\mathcal{O}_K/(pN))^\times$, then we have
\[
\sum_{u \in (\mathcal{O}_K/(pN))^\times} \chi_1(u)E_{m+1,n+1}(\pi(a^{-1}\bar{u} + \pi^N(\bar{\alpha}_0 + \bar{\gamma})))
\]
since $a \equiv 1 \pmod{\pi^{N-1}}$ and the value $a^{-1}\bar{u} = i_x(a^{-1}u\Omega/\pi^N)$ is the image of $\pi^N$-torsion point in $\mathfrak{g}^{-1}p^{-N}\Gamma/\Gamma \cong \mathcal{E}(\mathbb{C})[\pi^N]$, we have
\[
\sum_{u \in (\mathcal{O}_K/(pN))^\times} \chi_1(u)E_{m+1,n+1}(\pi(\bar{u} + \pi^N(\bar{\alpha}_0 + \bar{\gamma}))).
\]
Since $\chi_1(a) \neq 1$, this lemma holds.

We order Proposition 3.9 by gathering toward a character on $(\mathcal{O}_K/g)^\times$. First we gather $\chi_{\mathcal{O}}, \chi_1, \chi_2$ by solving the following simultaneous congruences of first degree:
\[
z \equiv \alpha_0 \pmod{\mathfrak{g}}, \ z \equiv u \pmod{p^N}, \ z \equiv \gamma \pmod{\pi^N}.
\]
We assume that $z = s_1 \in \mathcal{O}_K$ is a solution of $p^Nz = \pi^N\pi^Nz \equiv 1 \pmod{\mathfrak{g}}$. The existence of this solution follows from that $\mathfrak{g}$ is prime to $p$. Similarly, we assume that $z = s_2 \in \mathcal{O}_K$ (resp. $z = s_3 \in \mathcal{O}_K$) is a solution of $g^{p^N}z \equiv 1 \pmod{p^N}$ (resp. $g^p\pi^Nz \equiv 1 \pmod{\pi^N}$). If we put $z = \alpha_0p^Ns_1 + u\pi^Ns_2 + \gamma\pi^Ns_3$, $z$ satisfies simultaneously $z \equiv \alpha_0 \pmod{\mathfrak{g}}$, $z \equiv u \pmod{p^N}$, $z \equiv \gamma \pmod{\pi^N}$. Therefore the solution of the given simultaneous congruences of first degree is $z \equiv \alpha_0p^Ns_1 + u\pi^Ns_2 + \gamma\pi^Ns_3 \pmod{\mathfrak{g}p^N}$. 

Recalling $\chi = \chi_0 \chi_2$, and $g' = g p^N$, Proposition 3.9 is

$$\frac{L_p(\varphi_p)}{\Omega_p^{-m}} = g^{-1} n! (-1)^{m+n+1} \frac{\sum_{z \in (O_K/q')^\times} \chi(z) E^{\text{col}}_{m+1,n+1}(\tilde{u} + \pi^N(\tilde{a}_0 + \tilde{\gamma})).}{\tau(\chi) / \pi^N}$$

Next, we express the contents $\tilde{u} + \pi^N(\tilde{a}_0 + \tilde{\gamma})$ of $E^{\text{col}}_{m+1,n+1}$ by the formula of $\tilde{z}$. We find a constant $C$ satisfying

$$u + \pi^N(\alpha_0 + \gamma) = C \tilde{z} = C(\alpha_0 p^N s_1 + u g \pi^N s_2 + \gamma g \pi^N s_3)$$

in $O_K/q'$. Since $u$, $\alpha_0$, and $\gamma$ are arbitrary, comparing respectively coefficients of $u$, $\alpha_0$, and $\gamma$, we have

$$1 = C g \pi^N s_2, \quad \pi^N = C p^N s_1, \quad \pi^N = C g \pi^N s_3$$

in $O_K/q'$. Since $s_1$, $s_2$, and $s_3$ respectively satisfies $p^N s_1 \equiv 1(\text{mod } g)$, $g \pi^N s_2 \equiv 1(\text{mod } \pi^N)$, and $g \pi^N s_3 \equiv 1(\text{mod } \pi^N)$, we have the following simultaneous congruences of first degree:

$$C \equiv 1(\text{mod } \pi^N), \quad C \equiv \pi^N(\text{mod } g), \quad C \equiv \pi^N(\text{mod } \pi^N).$$

Solving these simultaneous congruences of first degree, for the above $s_1$, $s_2$, and $s_3$, we have

$$C \equiv \pi^N p^N s_1 + g \pi^N s_2 + g^{2N} s_3(\text{mod } g p^N).$$

Now if we put $\xi_{g'} := i_s(O(\pi^N p^N s_1 + g \pi^N s_2 + g^{2N} s_3)/g p^N) = i_s(OC/gp^N)$, $\xi_{g'}$ is a primitive $g'$-torsion point. $\xi_{g'}$ is the primitive $g'$-torsion point in $E(C_p)$, which is the special case when we substitute $\alpha_0 = \pi^N$, $u = 1$, and $\gamma = \pi^N$ for $z \equiv \alpha_0 p^N s_1 + u g \pi^N s_2 + \gamma g \pi^N s_3(\text{mod } g p^N)$. Then we have

**Theorem 3.11** (Main Theorem). Let $m, n$ be any integers with $n \geq 0$. For the above symbols, we have

$$\frac{L_p(\varphi_p)}{\Omega_p^{-m}} = g^{-1} n! (-1)^{m+n+1} \sum_{z \in (O_K/q')^\times} \chi(z) E^{\text{col}}_{m+1,n+1}(\xi_{g'}z),$$

where $\tau(\chi)$ is the Gauss sum defined by Lemma 3.3.

Main Theorem is the $p$-adic analogue of the relation of the complex Hecke $L$-function and the classical Eisenstein-Kronecker series

$$L_{g'}(s, \varphi) = \frac{1}{w_{g'}} K_{[m-n]}^s(\alpha, 0, s - \min\{m, n\}; ((\alpha)^{-1} g')^d),$$

where $w_{g'}$ is the number of roots of 1 in $O_K^\times$ congruent to 1 modulo $g'$, $\alpha$ is any element of $\mathfrak{a}^{-1}$ in 11 such that $\alpha \equiv 1(\text{mod } g')$ and $\delta \in \text{Gal} (\mathbb{C}/\mathbb{R})$ is an element satisfying that $\delta$ is trivial if and only if $m - n > 0$ (see 11 Proposition 1.6).

**Acknowledgements**

The result of this article is an extended version of the result of my master’s thesis 19. This research was supported by KAKENHI 21674001. Thanks to this KAKENHI, I could promote this research smoothly. I wish to thank my professor K. Bannai and other post doctors for having sincerely advised and led when I was in trouble.
REFERENCES

[1] K. Bannai, S. Kobayashi, Algebraic theta functions and the $p$-adic interpolation of Eisenstein-Kronecker numbers, Duke Math Journal, 153. (2010), 229-295.
[2] K. Bannai, S. Kobayashi, T. Tsuji, On the de Rham and $p$-adic realizations of the elliptic polylogarithm for CM elliptic curves, Annales scientifiques de l’ENS 43, fascicule 2 (2010), 185-234
[3] K. Bannai, G. Kings, $p$-adic Beilinson conjecture for ordinary Hecke motives associated to imaginary quadratic fields, RIMS Kokyuroku Bessatsu B25: Algebraic Number Theory and Related Topics 2009, eds. T. Ichikawa, M. Kida, T. Yamazaki, June (2011), 9-30.
[4] K. Bannai, H. Furusho, S. Kobayashi, $p$-adic Eisenstein-Kronecker function and the elliptic polylogarithm for CM elliptic curves, preprint, 2008, [arXiv:0807.4007v1 [math.NT]]
[5] A. A. Beilinson, Higher regulators and values of $L$-functions, J. Sov. Math., 30, 2036-2070, 1985.
[6] P. Berthelot. Cohomologie rigide et cohomologie rigide à supports propres, première partie. Institut de recherche math. de Rennes, 1996
[7] A. Besser, Syntomic Regulators and $p$-adic Integration II: $K2$ of Curves, Israel Journal of Mathematics December 2000, Volume 120, Issue 1, pp. 335-359
[8] A. Besser, A generalization of Coleman’s $p$-adic integration theory, Inv. Math. 142, pp.397-434, 2000
[9] S. Bloch, Higher Regulators, Algebraic $K$-theory, and Zeta functions of Elliptic Curves, CRM MONOGRAPH SERIES Vol 11, American Math. Society, 2000
[10] S. Bosch, U. Güntzer, R. Remmert, Non-archimedean analysis, Grundlehren der Wissenschaften 261, Springer Verlag,1984
[11] C. Breuil, Intégration sur les variétés $p$-adiques [d’après Coleman, Colmez], Séminaire N. Bourbaki, pp.319-350,1998-1999
[12] R. Coleman and E. de Shalit, $p$-adic regulators on curves and special values on $p$-adic $L$-function, Inv. Math. 93, pp.239-266, 1988
[13] R. Coleman, Dilogarithms, regulators, and $p$-adic $L$-functions, Inv. Math. 69, pp.171-208, 1982
[14] P. Deligne, Valeurs de Fonctions $L$ et Périodes D’intégrales, Proceedings of Symposia in Pure Mathematics, Vol 33. (1979), part 2, pp. 313-346.
[15] E. de Shalit, Iwasawa theory of elliptic curves with complex multiplication, Perspectives in Math, Vol 3, 1986
[16] J. Fresnel, M. van der Put, Rigid analytic geometry and its applications, Boston Birkhäuser, 2003
[17] M. Gros. Régulateurs syntomiques et valeurs de fonctions $L$ $p$-adiques. I. Invent. Math., 99(2):293-320, 1990. With an appendix by Masato Kurihara.
[18] M. Gros. Régulateurs syntomiques et valeurs de fonctions $L$ $p$-adiques. II. Invent. Math., 115(1):61-79, 1994.
[19] T. Hirotsune, A Study on the Relation of Special Values of $p$-adic $L$-functions to $p$-adic Eisenstein-Kronecker Series, Master’s Thesis, Keio University, March 2012.
[20] K. Iwasawa. Lectures on $p$-adic $L$-functions, Annals of Math. Studies 74, Princeton University Press, 1972
[21] N. Katz, $p$-adic interpolation of real analytic Eisenstein series, Ann. of Math., pp. 459-571, 1976
[22] S. Lang, Elliptic Functions, Graduate Texts in Math. 112, Springer-Verlag, 1973
[23] J. Lubin, One-parameter formal Lie groups over $p$-adic integer rings, Annals of Mathematics 80 (1964), 464-484.
[24] Ju. I. Manin, S. Višik, $p$-adic Hecke series for quadratic imaginary fields, Math. Sbornik, V. 95(137), No.3(11), 1974
[25] B. Perrin-Riou. Fonctions $L$ $p$-adiques des representations $p$-adiques. Astérisque, (229), 1995.
[26] J. Silverman, The Arithmetic of Elliptic Curves, Graduate Texts in Math. 106 2nd edition, Springer, 2008
[27] J. Silverman, Advanced topics in the Arithmetic of elliptic curves, Graduate Texts in Math. 151, Springer, 1994
[28] J. Tate, $p$-divisible groups, Proceedings of a Conference on Local Fields, Springer, 1967, pp. 158-183.
[29] J. Tate, Rigid analytic space, Inv. Math. 12, pp.257-289, 1971
[30] A. Weil, Elliptic functions according to Eisenstein and Kronecker, Ergebnisse der Mathematik Und Ihrer Grenzgebiete 88,
[31] R. I. Yager, On two variable $p$-adic $L$-functions, Ann. of Math. 115, 411-449, 1982
[32] D. Zagier, The Bloch-Wigner-Ramakrishnan polylogarithm function, Math. Ann. 286, 613-624 (1990)

Department of Mathematics, Keio University, 3-14-1 Kouhoku-ku, Hiyoshi, Yokohama, 223-8522
Japan

E-mail address: bibun.to.sekibun@gmail.com