THE DISCRETE GELFAND TRANSFORM AND ITS DUAL

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ABSTRACT. We consider the transformation $ev$ which associates to any element in a $K$-algebra $A$ a function on the set of its $K$-points. This is the analogue of the fundamental Gelfand transform. Both $ev$ and its dual $ev^*$ are the maps from a discrete $K$-module to a topological $K$-module and we investigate in which case the image of each map is dense. The answer is nontrivial for various choices of $K$ and $A$ already for $A = K[x]$, the polynomial ring in one variable. Applications to the structure of algebras of cohomology operations are given.

1. INTRODUCTION

Let $K$ be a commutative ring and $A$ be a commutative $K$-algebra. The set of $K$-points $A(K)$ is the set of $K$-algebra homomorphisms $A \rightarrow K$. The evaluation map is the canonical map associating to any element $a \in A$ the function $ev(a) : A(K) \rightarrow K$. For a $K$-point $x : A \rightarrow K$ the value of $ev(a)$ at $x$ is $x(a)$. Thus we get a map $ev : A \rightarrow \text{Map}(A(K), K)$. Its $K$-dual is a map $ev^* : K[A(K)] \rightarrow A^*$ where $K[A(K)]$ is the free $K$-module on the set $A(K)$. The study of the evaluation map and its dual (defined in a slightly more general way) is the main purpose of this paper. Note that the evaluation map is a direct analogue of the well-known Gelfand transform which identifies a commutative $C^*$-algebra with the algebra of functions on the set of its maximal ideals. We do not know if the analogue of the dual evaluation map has been considered in the context of Banach algebras.

Both $ev$ and $ev^*$ are maps from a discrete $K$-module to a topological $K$-module and we investigate in which case the image of each map is dense. The answer is nontrivial already for $A = K[x]$, the polynomial ring in one variable.

Informally speaking, there are obstructions to $ev$ and $ev^*$ having a dense image related to the existence of numerical functions and numerical functionals. In the simplest case $K = \mathbb{Z}$, $A = K[x]$ numerical functions are known by the name of numerical polynomials. A polynomial in one variable with rational coefficients is called numerical if it takes integer values evaluated at integers. Numerical polynomials appeared in algebro-topological literature [1], [7] and [6]. There is also a large body of literature in which numerical polynomials (called usually integer-valued polynomials) are studied from the purely algebraic point of view. The standard reference is the monograph [5].

The dual notion is that of a numerical functional. For $K = L = \mathbb{Z}$ a numerical functional is a linear combination of $\mathbb{Z}$-points of $A$ with rational coefficients which determines an integer functional on $A$. As far as we know the concept of a numerical functional is new.

Theorem 4.1 gives a rather complete answer when the map $ev$ has a dense image. The corresponding question of the map $ev^*$ is more subtle and a partial answer is provided by Theorem 4.4.

We also give some applications of our results to the structure of algebras of cohomology operations in mod $p$ homology and complex cobordisms. Recall a well-known theorem of Morava, cf. [11], Theorem 6.2.3 which states that the Morava stabilizer algebra $S(n)$ is isomorphic to the dual of a certain group ring after a suitable extension of scalars. The algebra $S(n)$ is essentially the even part of the cooperation algebra for $K(n)$, the $n$th Morava $K$-theory. This theorem reduces the cohomology of $S(n)$ to that of the continuous cohomology of the corresponding group, the so-called Morava stabilizer group. This is of great importance because of the connection with the stable homotopy groups of spheres via certain generalized Adams-Novikov spectral sequences.

Key words and phrases. linear topology, rings of divided powers, numerical polynomials, Landweber-Novikov algebra, Steenrod algebra.

This research was partially supported by the EPSRC grant No. GR/R84276/01.
It is natural to ask if there are other contexts in which algebras of cohomology operations are related with group rings. In the context of usual cohomology theory mod $p$ the analogue of $S(n)$ is $A$, the full Steenrod algebra if $p = 2$ and the algebra of reduced powers if $p > 2$. In the context of complex cobordisms the relevant analogue is the so-called Landweber-Novikov algebra $S$.

The above mentioned theorem of Morava turns on the fact that the Morava stabilizer algebra $S(n) = \mathbb{F}_p[t_1, t_2, \ldots]/(t_i^{p^n} - t_i; i > 0)$ is semisimple i.e. isomorphic to the infinite sum of copies of $\mathbb{F}_p$. This is not true for either $S^*$ or $A^*$ and therefore we cannot expect these algebras to be isomorphic to the duals of some group rings.

Instead, our result states that after an appropriate extension of scalars and completion the algebras $A$ and $S$ contain dense Hopf subalgebras isomorphic to group rings of certain groups which we explicitly identify.

Note that an asymptotically split monomorphism is clearly a monomorphism. In addition if $L$ is a $K$-algebra then a $K$-module homomorphism $W \to V$ is called an asymptotically split monomorphism if for any admissible submodule $U \subset V$ the composition $U \to V \to W$ admits a continuous left inverse $L$-module homomorphism $W \to U$.

Remark 2.2. Note that an asymptotically split monomorphism is clearly a monomorphism. In addition if $V$ is a free $L$-module of finite rank then an asymptotically split monomorphism is simply a (continuously) split monomorphism.

Now let $V, W$ be free $L$-modules and $f : W \to V$ be a $L$-homomorphism. Its $L$-dual homomorphism $f^* : V \to W^*$ is defined as usual by the formula $(f^*(v))(w) = (f(w))(v)$ where $v \in V$ and $w \in W$.

Lemma 2.3. Let $V, W$ be free $L$-modules. Then a $L$-homomorphism $f : W \to V$ has a dense image if and only if the dual homomorphism $f^* : V \to W^*$ is an asymptotically split monomorphism.

Proof. Suppose that $f$ has a dense image and let $U$ be any admissible submodule in $V$. Then the definition of the linear topology in $V^*$ directly implies that the composite map

$$W \to V^* \to V^*/U$$

is an epimorphism. Since $U$ is admissible the map $V^* \to U^*$ dual to the inclusion $U \subset V$ is onto and therefore $V^*/U \cong U^*$. Since $U^*$ is a free $L$-module, the epimorphism $W \to V^*/U \cong U^*$ is split by a
map \( U^* \rightarrow W \). The dual to the latter map is a continuous map \( W^* \rightarrow U \) which is a left inverse to the composition \( U \hookrightarrow V \rightarrow W^* \).

Conversely, suppose that \( f^* \) be an asymptotically split monomorphism and \( U \) is an admissible submodule in \( V \). The \( L \)-dual to the composite map

\[
U \hookrightarrow V \rightarrow W^*
\]

will be the split epimorphism \( W \rightarrow U^* \cong V^*/V_U \) from which it follows that \( f \) has a dense image. \( \square \)

**Remark 2.4.** If \( L \) is a field then any monomorphism \( V \rightarrow W^* \) is asymptotically split. Indeed, let \( U \subset V \) be an admissible submodule in \( V \). Since the composition \( U \hookrightarrow V \rightarrow W^* \) is a monomorphism there exists a subspace \( W_U \subset W^* \) for which the composition

\[
U \hookrightarrow V \rightarrow W^* \rightarrow W^*/W_U,
\]

is also a monomorphism. This last monomorphism necessarily splits and determines a continuous splitting of the inclusion \( U \hookrightarrow W^* \).

Recall that \( K \) is a fixed infinite subring in \( L \) and let \( A \) be a \( K \)-algebra which is free as a \( K \)-module. Define the evaluation homomorphism \( ev_{L/K} \) as follows:

\[
(2.1) \quad ev_{L/K} : A \otimes L \rightarrow \text{Map}(A(K), L) : ev(a \otimes l)(x) = lx(a).
\]

Here \( x : A \rightarrow K \) is an element in \( A(K) \) and \( l \in L \). Clearly, \( ev_{L/K} \) is a homomorphism of \( L \)-modules.

The \( L \)-algebra \( \text{Map}(A(K), L) \) can be supplied with an \( L \)-linear topology where the fundamental system of neighborhoods of 0 is formed by \( L \)-modules

\[
A_X = \{ g \in \text{Map}(A(K), L) : g(x) = 0 \text{ for all } x \in X \}.
\]

Here \( X \) runs through all finite subsets in \( A(K) \).

Let us denote by \( L[A(K)] \) the set of all continuous \( L \)-linear homomorphisms from \( \text{Map}(A(K), L) \) into \( L \). It follows that \( L[A(K)] \) is a free \( L \)-module generated by the set \( A(K) \). Moreover, the canonical homomorphism

\[
\gamma : L[A(K)] \rightarrow \text{Map}(A(K), L) : \gamma(s)(x) = s(x); \ s \in L[A(K)],
\]

is a homeomorphism.

Let us consider the homomorphism

\[
(2.2) \quad ev_{L/K}^* : L[A(K)] \rightarrow (A \otimes L)^* \cong \text{Hom}_K(A, L),
\]

dual to \( ev_{L/K} \). The study of the maps \( (2.1) \) and \( (2.2) \) is our main goal.

**Remark 2.5.** In the case when \( A \) is a Hopf algebra over \( K \) with comultiplication \( \Delta : A \rightarrow A \otimes A \) the set \( A(K) \) becomes a group under the convolution product:

\[
[x][y](a) = (x \otimes y)(\Delta(a)); \ x, y \in A(K).
\]

Therefore \( L[A(K)] \) is in this case the group algebra of \( A(K) \) over \( L \) and also a Hopf algebra over \( L \). Furthermore, \( A^* \) is a topological Hopf algebra. That means that \( A^* \) is a topological algebra and the diagonal in \( A^* \) is a continuous map \( A^* \rightarrow A^* \otimes A^* := (A \otimes A)^* \) from \( A^* \) into the completed tensor square of \( A^* \). Similarly \( \text{Map}(A(K), L) \) is a topological Hopf algebra over \( L \) dual to \( L[A(K)] \). Observe that the \( A \) and \( L[A(K)] \) could also be considered as topological Hopf algebras with the discrete topology.

The following well-known fact is easily checked by direct inspection of definitions.

**Lemma 2.6.** If \( A \) is a Hopf algebra over \( K \) then \( ev_{L/K} \) and \( ev_{L/K}^* \) are maps of topological Hopf algebras over \( L \).

The \( L \)-module \( \text{Hom}_K(A, L) \) has a structure of an \( A \)-module. Indeed, let \( a \in A \) and \( t \in \text{Hom}_K(A, L) \). Then \( at \in \text{Hom}_K(A, L) \) is defined by \( (at)b = t(ab) \) for \( b \in A \).

Likewise the \( L \)-module \( L[A(K)] \) admits a structure of an \( A \)-module via the map

\[
A \otimes L[A(K)] \rightarrow K[A(K)] : a \otimes [x] = x(a)[x].
\]

Here \( x \) is a \( K \)-algebra map \( A \rightarrow K \) and \( [x] \) is the corresponding basis element in \( L[A(K)] \).

**Remark 2.7.** Of course, \( A^* \) and \( L[A(K)] \) are naturally right \( A \)-modules, but since \( A \) is commutative there is no difference between left and right \( A \)-modules.
Example 2.8. Let $A = K[z]$, the ring of polynomials on one generator $z$. Then $\text{Hom}_K(A, L) = \Gamma_L$ is the so-called divided power ring with coefficients in $L$. It can be identified with a topological vector space over $L$ with topological basis $u_n$, where $$u_n(z^k) = \delta^n_k.$$ Here $\delta^n_k$ is the Kronecker symbol. In this basis the action of $A$ on $\Gamma_L$ is specified by the formula $z^k \cdot u_n = u_{n-k}$ where we assume that $u_i = 0$ for $i < 0$. Clearly the set $A(K)$ is identified with $K$ and this gives an isomorphism of $L$-modules $L[A(K)] \longrightarrow L[K]$ where $L[K]$ is the free $L$-module generated by the elements of $K$.

The $L$-algebra map $ev_{L/K} : L[z] \longrightarrow \text{Map}(K, L)$ is given by the formula $ev_{L/K}(z)(k) = k$. The dual map $ev^*_{L/K} : L[K] \longrightarrow \Gamma_L$ is given by the formula $$ev^*_{L/K}[k] = \sum_{i \geq 0} k^i u_i.$$ We now have the following lemma whose proof is a direct inspection of definitions:

Lemma 2.9. The map $ev^*_{L/K} : L[A(K)] \longrightarrow \text{Hom}_K(A, L)$ is a map of $A$-modules.

Next we have the following proposition whose proof is similar to Artin’s proof on linear independence of characters, cf. [3].

Proposition 2.10. The homomorphism $ev^*_{L/K}$ is a monomorphism.

Proof. Let $M = \ker ev^*_{L/K}$. It is an $L$-submodule in $L[A(K)]$. According to the previous lemma $M$ is in fact an $A$-submodule. Assume that $M \neq 0$.

As before we denote by $[x]$ the element in $L[A(K)]$ corresponding to a $K$-algebra map $x : A \longrightarrow K$. We will define the map $\xi : M \longrightarrow \mathbb{Z}$ as follows. Write an arbitrary element $s \in M \subseteq L[A(K)]$ as $s = \sum_{i=1}^{N} \alpha_i [x_i]$ where all $x_i$ are pairwise distinct and all $\alpha_i$ are nonzero. Then $\xi(s) = N$. Further set $m = \min_{s \in M} \xi(s)$. Clearly, $m > 1$. Consider an element $s \in M$ of the form $$s = \alpha_1 [x_1] + \alpha_2 [x_2] + \ldots + \alpha_m [x_m].$$ Choose an element $a \in A$ for which $x_1(a) \neq x_2(a)$. Then $$\hat{s} := x_1(a)s - as = \sum_{i=2}^{m} \alpha_i (x_1(a) - x_i(a))[x_i]$$ is a nonzero element in $M$ such that $\xi(\hat{s}) = m - 1$. (Note that here we used the fact that elements in $K$ are not zero divisors in $L$.) This is a contradiction and our proposition is proved. \hfill $\square$

Remark 2.11. Without the condition that $K$ have no zero divisors in $L$ the above result is no longer true. Indeed, let $A = K[x]$. Then for any $k \in K$ we have $$ev^*_{L/K}([k] - 0) = \sum_{i=1}^{\infty} k^{i-1} u_i \in A^* = \Gamma_L.$$ Suppose that there exist elements $k \in K$ and $l \in L$ for which $kl = 0 \in L$. It follows that the element $l([k] - 0)$ belongs to the kernel of $ev^*_{L/K}$.

Combining the above result with Lemma 2.8 we get the following result:

Corollary 2.12. If $L$ is a field then the homomorphism $ev_{L/K} : A \longrightarrow \text{Map}(A(K), L)$ has a dense image.

Remark 2.13. If, in addition, the field $K$ is algebraically closed and the $K$-algebra $A$ is finitely generated and has no nilpotent elements then Hilbert’s Nullstelsatz says that the $ev_{L/K}$ is a monomorphism. In this case we conclude that $ev^*_{L/K}$ is a monomorphism with a dense image.

Later on we will consider the case $A = L[x]$ in some detail. It is well-known and easy to see that in this case the map $ev_{L/K} : L[x] \longrightarrow \text{Map}(L, K)$ is monomorphic. It also clear that the condition that $K$ have no zero divisors cannot be avoided.
We finish this section by considering the evaluation map and its dual for tensor products of algebras. Let $A_i$ be $K$-algebras where $i$ runs through an indexing set $I$, not necessarily finite. Denote by $ev_{L/K}(i)$ the evaluation map $A_i \otimes L \to \text{Map}(A_i(K), L)$. Let $A$ be the tensor product of $A_i$; more precisely

$$A = \bigotimes_{i \in I} A_i = \lim_{\rightarrow} \bigotimes_{k=1}^{N} A_{i_k}$$

where the limit is taken over all finite collections $(i_1, \ldots, i_N)$ in $I$. We will keep the notation $ev_{L/K}$ for the evaluation map $A \otimes L \to \text{Map}(A(K), L)$.

Similarly we denote the dual evaluation map $L[A_i(K)] \to \text{Hom}_K(A_i, L)$ by $ev_{L/K}^*(i)$. The dual evaluation map for $A$ will still be denoted by $ev_{L/K}^*$. Then we have the following result.

**Proposition 2.14.** The map $ev_{L/K}$ is asymptotically split if and only if $ev_{L/K}(i)$ asymptotically split for any $i \in I$. Similarly $ev_{L/K}^*$ is asymptotically split if and only if $ev_{L/K}^*(i)$ is asymptotically split for any $i \in I$.

**Proof.** We will only give a proof for $ev_{L/K}$; the dual case is treated completely analogously. Assume that each of the maps $ev_{L/K}(i)$ is asymptotically split and let $W \in A \otimes L$ be a finite dimensional $L$-submodule in $A$. We want to show that the composite map

$$W \to (\bigotimes_{i} A_i \otimes L) = A \otimes L \xrightarrow{ev_{L/K}} \text{Map}(\prod_{i} A_i(K), L)$$

splits. Without loss of generality we could assume that $W$ is of the form $\bigotimes_{i}^{n} W_i \otimes L$ where $W_i$ is a finite dimensional free $k$-submodule in $A_i$ for $i = 1, 2, \ldots, n$.

Note that

$$\text{Map}(A_i(K), L) \cong \lim_{\leftarrow} \text{Map}(X_i, L) \cong$$

where the inverse limit is taken over all finite subsets $X_i \in A_i(K)$ for $i = 1, 2$. Since there exists a continuous splitting $\text{Map}(A_i(K), L) \to W_i$ and $W_i$ is finite dimensional it follows that this splitting map factors through $\text{Map}(X_i, L)$ for some finite subset $X_i \in \text{Map}(A_i(K), L)$. So we showed that there exist maps $g_i : \text{Map}(X_i, L) \to W_i$ which split the compositions

$$W_i \to A_i \otimes L \to \text{Map}(A_i(K), L) \to \text{Map}(X_i, L).$$

Then the composition

$$\text{Map}(\prod_{i} A_i(K), L) \to \text{Map}(\prod_{i} X_i, L) \cong \bigotimes_{i} \text{Map}(X_i, L) \xrightarrow{\otimes g_i} \bigotimes_{i} W_i = W$$

is the desired splitting.

Conversely, assume that $ev_{L/K}$ is asymptotically split and let $W_i \in A_i \otimes L$ be an arbitrary finite dimensional $L$-submodule. The inclusion

$$W_i \hookrightarrow A_i \otimes L \to \bigotimes_{i} A_i \otimes L \to \text{Map}(\prod_{i} A_i(K), L)$$

splits and it follows that the inclusion

$$W_i \hookrightarrow A_i \otimes L \to \text{Map}(A_i(K), L)$$

also splits. 

\[\square\]

3. **Numerical functions and functionals**

**Definition 3.1.** Let $\hat{L}$ be the field of fractions of $L$. The ring of numerical functions $\text{Num}_{L/K}(A)$ is defined from the pullback diagram of rings:

$$\begin{array}{ccc}
\text{Num}_{L/K}(A) & \longrightarrow & \text{Map}(A(K), L) \\
\downarrow & & \downarrow \\
A \otimes \hat{L} & \xrightarrow{ev_{L/K}} & \text{Map}(A(K), \hat{L})
\end{array}$$

Here the right vertical map is induced by the inclusion $L \to \hat{L}$. The $L$-module of numerical functionals is defined in a dual fashion as a pullback of the following diagram:

\[
\begin{array}{c}
\text{Num}_{L/K}(A) \downarrow \downarrow \downarrow \\
\downarrow \downarrow \downarrow \\
\hat{L}[A(K)] \to \text{Hom}(A, \hat{L})
\end{array}
\]

**Remark 3.2.** If $A$ is a Hopf algebra over $K$ then $\text{Num}_{L/K}(A)$ is an $L$-algebra.

It follows directly from the definition that the evaluation map $ev_{L/K}$ factors through numerical functions so that the following diagram of algebras is commutative:

(3.1)

\[
\begin{array}{c}
A \otimes L \downarrow e_1 \downarrow e_2 \\
\downarrow \downarrow \downarrow \\
\text{Num}_{L/K}(A) \to \text{Map}(A(K), L)
\end{array}
\]

where the $e_1$ and $e_2$ are natural inclusions. Dually we have the following commutative diagram of $L$-modules:

(3.2)

\[
\begin{array}{c}
L[A(K)] \downarrow e_1 \downarrow e_2 \\
\downarrow \downarrow \downarrow \\
\text{Num}_{L/K}(A)
\end{array}
\]

Finally, taking the (continuous) $L$-dual to the diagram (3.2) and splicing it with (3.1) we get the following two factorizations of the map $ev_{L/K}$:

(3.3)

\[
\begin{array}{c}
\text{Num}_{L/K}^*(A) \downarrow e_1 \downarrow e_2 \\
\downarrow \downarrow \downarrow \\
A \otimes L \to \text{Map}(A(K), L)
\end{array}
\]

The similar factorization of $ev_{L/K}^*$ has the following form:

(3.4)

\[
\begin{array}{c}
\text{Num}_{L/K}^*(A) \downarrow e_1 \downarrow e_2 \\
\downarrow \downarrow \downarrow \\
L[A(K)] \to \text{Hom}(A, L)
\end{array}
\]

**Example 3.3.** Let $A = \mathbb{Z}[x]$. In this important case we will shorten the notation $\text{Num}_{L/K}(A)$ to $\text{Num}_{L/K}$ and call it the ring of $L/K$-numerical polynomials. Furthermore assume that $K = L = \mathbb{Z}$. In this case $\text{Num}_{\mathbb{Z}/\mathbb{Z}}$ is the well-known ring of numerical polynomials which consists of those polynomials with rational coefficients which assume integer values at integers. It is additively generated by the polynomials

\[
P_k(x) = \binom{x}{k} = \frac{x(x-1) \cdots (x-k+1)}{k!}
\]
for \( k = 0, 1, 2, \ldots \).

Furthermore, the ring \( \text{Num}_{\mathbb{Z}/\mathbb{Z}} \) is a Hopf subalgebra inside \( \mathbb{Q}[x] \). Its integral dual \( \text{Num}_{\mathbb{Z}/\mathbb{Z}}^* \) is isomorphic as a ring to the ring of formal power series \( \mathbb{Z}[y] \). For topologists we offer the following explanation: recall that \( \text{Num}_{\mathbb{Z}/\mathbb{Z}} \) is isomorphic to \( KU_*(\mathbb{CP}^\infty) \otimes_{KU} \mathbb{Z} \), cf. [1]. Then by the universal coefficients formula

\[
\text{Num}_{\mathbb{Z}/\mathbb{Z}}^* \cong \text{Hom}_{KU}(KU_*(\mathbb{CP}^\infty), \mathbb{Z}) \cong KU^*(\mathbb{CP}^\infty) \otimes_{KU} \mathbb{Z} \cong \mathbb{Z}[y].
\]

The upper part of the diagram \( \text{ZZ} \) takes the following form:

\[
\begin{array}{ccc}
\mathbb{Z}[y] & \cong & \text{Num}_{\mathbb{Z}/\mathbb{Z}}^* \\
\mathbb{Z}[t, t^{-1}] & \cong & \mathbb{Z}[\mathbb{Z}] \\
\end{array}
\]

Direct inspection shows that \( e_2^*(t) = y + 1 \) and \( e_1^*(y) = \sum_{i=1}^{\infty} u_i \in \Gamma_{\mathbb{Z}} \).

The following result is a useful criterion for (non)existence of nontrivial numerical polynomials. Related results could be found in [5].

**Proposition 3.4.** If there exists an element \( a \in \mathcal{L} \) such that \( a \) is not invertible in \( \mathcal{L} \) and for which the image of the composite map

\[
\pi_a : K \rightarrow \mathcal{L} \rightarrow \mathcal{L}/a
\]

is finite then \( \text{Num}_{\mathcal{L}/K} \neq \mathcal{L}[x] \). Conversely, suppose \( \text{Num}_{\mathcal{L}/K} \neq \mathcal{L}[x] \) and, in addition, \( \mathcal{L} \) is a unique factorization domain (UFD). Then there exists a noninvertible element \( a \in \mathcal{L} \) for which \( \pi_a \) has a finite image.

**Proof.** Suppose that there exists \( a \in \mathcal{L} \) for which \( \text{im}(\pi_a) \) consists of \( m \) elements \( e_1, \ldots, e_m \). Choose a collection of representatives \( k_i \in K \) of these residue classes and consider the polynomial

\[
P(x) = \frac{\prod_{i=1}^{m} (x - k_i)}{a} \in \mathcal{L}[x].
\]

Note that since \( a \) is not invertible in \( \mathcal{L} \) the polynomial \( P(x) \) is not contained in \( \mathcal{L}[x] \). Let \( k \in \mathcal{K} \). Then the collection \( \{ k - k_i \} \) contains an element congruent to zero modulo \( a \). Therefore \( P(k) \in \mathcal{L} \).

 Conversely, suppose that \( P(x) \in \text{Num}_{\mathcal{L}/K} \) and \( P(x) \) is not contained in \( \mathcal{L}[x] \). Then \( P(x) = \frac{Q(x)}{a} \) with \( Q(x) \in \mathcal{L}[x] \), the greatest common divisor of all coefficients of \( Q(x) \) is a unit in \( \mathcal{L} \) and \( a \in \mathcal{L} \) is noninvertible in \( \mathcal{L} \). Without loss of generality we assume that \( a \) is prime in \( \mathcal{L} \). We claim that \( \text{im} \pi_a \) is finite. Indeed, if this is not the case there exists an infinite collection \( \{ k_i, i = 1, 2, \ldots \} \in K \) of representatives of \( \text{im} \pi_a \) in \( K \) which are pairwise distinct.

Denote by \( \tilde{Q}(x) \) the image of \( Q(x) \) under the map \( \mathcal{L}[x] \rightarrow \mathcal{L}/a[x] \). Then \( \tilde{Q}(x) \) is not identically zero and \( \tilde{Q}(k_i) = 0 \) for all \( i \). In other words \( \tilde{Q}(x) \) has infinitely many roots which is impossible since \( \mathcal{L}/a \) is an integral domain (recall that \( a \) is a prime element in \( \mathcal{L} \)). This contradiction finishes the proof. \( \square \)

Rings of integers in algebraic number fields and algebraic function fields provide natural examples of nontrivial (i.e. having nontrivial denominators) numerical polynomials. We restrict ourselves with giving two examples.

**Example 3.5.** \( \mathcal{K} = \mathcal{L} = \mathbb{F}_p[q] \). Then the polynomials

\[
f_n(x) = \frac{x^p^n - x}{q^p^n - q}
\]

belong to \( \text{Num}_{\mathcal{L}/K} \).

**Example 3.6.** \( \mathcal{K} = \mathcal{L} = \mathbb{Z}[i] \), the ring of Gaussian integers. A nontrivial example of \( L/K \)-numerical polynomials is given by

\[
f_n(x) = \frac{1}{n!} \prod_{0 \leq a, b < n} (x - a - ib).
\]
Now we will consider the notion of numerical functionals in more detail. Even in the case $K = L = \mathbb{Z}$, $A = \mathbb{Z}[x]$ they have not been studied to the best of our knowledge. To give nontrivial examples of $\text{Num}_{L/K}(A)$ suppose, as before that $A = K[x]$ in which case we will use the notation $\text{Num}_{L/K}$. Recall that $\text{Num}_{K/K}$ is a subring in the group ring $\hat{L}[K]$.

Let $\omega := (k_1, \ldots, k_n)$ be a collection of pairwise distinct elements in $K$. Let $A_\omega$ be the classical Vandermonde matrix:

\begin{equation}
A_\omega = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
k_1 & k_2 & \cdots & k_n \\
\vdots & \vdots & \ddots & \vdots \\
k_1^{n-1} & k_2^{n-1} & \cdots & k_n^{n-1}
\end{pmatrix}
\end{equation}

Set $d_\omega := \det A_\omega$. Further denote by $e_{l,\omega}$ the element in $\hat{L}[K]$ defined as $\frac{1}{d_\omega} \det \hat{A}$ where $\hat{A}$ is the matrix obtained from the matrix $A_\omega$ by replacing formally the $l$th row by $([k_1], [k_2], \ldots, [k_n])$.

**Example 3.7.** Let $\omega = (k_1, k_2)$. Then $d_\omega = k_2 - k_1$ and

\begin{align*}
e_{1,\omega} &= \frac{1}{k_2 - k_1} \det \begin{pmatrix} [k_1] \\ [k_2] \end{pmatrix} = \left( \frac{k_2}{k_2 - k_1} \right) [k_1] + \left( \frac{k_1}{k_1 - k_2} \right) [k_2]; \\
e_{2,\omega} &= \frac{1}{k_2 - k_1} \det \begin{pmatrix} 1 \\ [k_1] \end{pmatrix} = \left( \frac{1}{k_2 - k_1} \right) [k_1] + \left( \frac{1}{k_2 - k_1} \right) [k_2].
\end{align*}

**Theorem 3.8.**

1. The elements $e_{l,\omega}$ belong to $\text{Num}_{L/K}$ for any $\omega, l$. Moreover, $\text{Num}_{L/K}$ is generated by $e_{l,\omega}$ as an $L$-module.
2. The canonical inclusion $\iota^* : \text{Num}_{L/K} \rightarrow \Gamma_L$ has a dense image.

**Proof.** Denote by $A_{l,\omega}$ the matrix obtained from $A_\omega$ by replacing the $l$th row with $(k_1^m, k_2^m, \ldots, k_n^m)$. Note that $A_{l-1,\omega} = A_\omega$ for all $l = 1, \ldots, n$. Set $d_{l,\omega} := \det A_{l,\omega}$.

The straightforward computation using elementary properties of determinants yields the following key formula.

\[ e_{l,\omega} \left( e_{l,\omega} \right) = \sum_{m=0}^{\infty} d_{l,m,\omega} u_m. \]

Since $d_{l,m,\omega} = 0$ for $0 \leq m \leq n - 1, m \neq l$ and $d_{l,l-1,\omega} = d_\omega$ we see that

\begin{equation}
(3.6) e_{l,\omega} \left( e_{l,\omega} \right) = u_{l-1} + \sum_{m=n}^{\infty} d_{l,m,\omega} u_m,
\end{equation}

The expression $d_{l,m,\omega}$, $m \geq n$, is (up to sign) the classical Schur function in $k_1, \ldots, k_n$ (cf. [9]) corresponding to the partition $(m - n + 1, 0, \ldots, 0)$ of length $n$. The Schur functions are symmetric polynomials with integer coefficients. That shows that $e_{l,\omega} \in \text{Num}_{L/K}$ for all $l, \omega$.

Furthermore, formula (3.6) also implies that the image of $\iota^*$ is dense in $\Gamma_L$. We still need to prove that $e_{l,\omega}$ generate $\text{Num}_{L/K}$. Let $\Gamma_L^n := \Gamma_L/(u_i, i \geq n)$, the quotient of $\Gamma_L$ by the submodule spanned by $u_n, u_{n+1}, \ldots$.

Set $V_\omega := \hat{L}([k_1], \ldots, [k_n]) \cap \text{Num}_{L/K}$. Composing $\iota^*$ with the projection $\Gamma_L \rightarrow \Gamma_L^n$ we get the map $\phi^n_\omega : V_\omega \rightarrow \Gamma_L^n$. Let us prove the following lemma.

**Lemma 3.9.** The map $\phi^n_\omega$ is a monomorphism.

**Proof.** Let $f = \sum_{i=1}^{n} \xi_i [k_i]$ where $\xi_i \in \hat{L}$. Then the condition $\phi^n_\omega(f) = 0$ implies that $\sum_{i=1}^{n} \xi_i k_i^m = 0$ for all $m = 0, 1, \ldots, n - 1$. Since the relevant Vandermonde matrix is nondegenerate we conclude that $\xi_i = 0, i = 1, 2, \ldots, n$. 

Consider the $L$-module $L(e_{1,\omega}, e_{2,\omega}, \ldots, e_{n,\omega})$, the $L$-submodule in $V_\omega$ spanned by $e_{1,\omega}, e_{2,\omega}, \ldots, e_{n,\omega}$. From formula (3.6) we conclude that $\phi^n_\omega$ maps $L(e_{1,\omega}, e_{2,\omega}, \ldots, e_{n,\omega})$ isomorphically onto $\Gamma_L^n$. Combining this with Lemma 3.9 we see that $\phi^n_\omega : V_\omega \rightarrow \Gamma_L^n$ is an isomorphism and therefore $\hat{L}(e_{1,\omega}, e_{2,\omega}, \ldots, e_{n,\omega}) = V_\omega$. Thus, the elements $e_{l,\omega}$ span the whole $\text{Num}_{L/K}$ as claimed. 

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Remark 3.10. Note that our proof shows that $\text{Num}_{L/K}$ is a union of the submodules $L(e_1,\omega, e_2,\omega, \ldots, e_n,\omega)$ and each of these submodules maps isomorphically onto $\Gamma_L^n = \Gamma_L/(u_i, i \geq n)$ under the composition $\text{Num}_{L/K} \rightarrow \Gamma_L \rightarrow \Gamma_L^n$.

Corollary 3.11. The ring of numerical functionals $\text{Num}_{L/K}$ coincides with $L[K]$ if and only if $L$ contains $K$, the field of fractions of $K$.

Proof. By Theorem 3.8 the element $[k]-[0] \in \text{Num}_{L/K}$ for any $k \in K$. Assume that $\text{Num}_{L/K} = L[K]$. Since $[k] - [0]$ could be taken to be one of the basis elements in the free $L$-module $L[K]$ we conclude that $k$ is invertible in $L$.

Conversely, suppose that $L$ contains $K$. From the construction of the basis elements $e_i, \omega \in \text{Num}_{L/K}$ it follows that $e_i, \omega \in L[K]$. Therefore $\text{Num}_{L/K} = L[K]$ in this case.

Remark 3.12. As an aside mention that numerical functionals could be considered as difference operators with constant coefficients. Namely, suppose for simplicity that $L$ is finite dimensional over $\hat{K}$ with constant coefficients. Namely, suppose for simplicity that $L$ contains $K$. From the construction of the basis elements $e_i, \omega \in \text{Num}_{L/K}$ it follows that $e_i, \omega \in L[K]$. Therefore $\text{Num}_{L/K} = L[K]$ in this case.

4. Properties of the evaluation map and its dual for polynomial rings

We will now consider the properties of the evaluation map and its dual in the case $A = K[x]$ from the point of view of numerical functions and numerical functionals. Informally speaking, the existence of nontrivial numerical polynomials (i.e. those not having coefficients in $L$) is an obstruction to the map $ev^*_{L/K} : L[x] \rightarrow \text{Map}(K,L)$ being an asymptotically split monomorphism. Dually, the existence of nontrivial numerical functionals (i.e. polynomials in $\text{Num}_{L/K}$ but not in $\Gamma_L$ implies that $ev_{L/K} : L[K] \rightarrow \Gamma_L$ is not asymptotically split.

More precisely, we have the following result.

Theorem 4.1. The following conditions are equivalent.

1. There exist no nontrivial numerical functionals, i.e. $\text{Num}_{L/K} = L[K]$.
2. The map $ev^*_{L/K} : L[K] \rightarrow \Gamma_L$ is asymptotically split (equivalently the map $ev_{L/K} : L[x] \rightarrow \text{Map}(K,L)$ has a dense image).
3. The ring $L$ contains $K$, the ring of fractions of $K$.

Proof. First observe that the equivalence of (1) and (3) is the statement of Corollary 3.11. Further assume that $ev^*_{L/K} : L[K] \rightarrow \Gamma_L$ is asymptotically split and consider $f \in \text{Num}_{L/K} \subset L[K]$. There exists $l \in L$ for which $lf \in L[K]$. It follows that the inclusion $L(lf) \hookrightarrow \text{Num}_{L/K}$ splits where $L(lf)$ is a one-dimensional $L$-submodule in $L[K]$ spanned by $lf$. Therefore $f \in L[K]$. Thus we proved (2)$\Rightarrow$(1).

Let us prove (3)$\Rightarrow$(2). Suppose that $L$ contains $K$ and let $W$ be a finite dimensional free $L$-submodule in $L[K]$. Without loss of generality we assume that $W$ is spanned by the finite collection of elements $[k] \in K$ and denote by $W_K$ the $K$-submodule in $K[K]$ spanned over $K$ by $[k]$. Recall that $\Gamma_l$ is the quotient $\Gamma_l/(u_k, k \leq i)$ and similarly denote $\Gamma^i_K := \Gamma_K/(u_k, k \leq i))$. Since $W_K$ is finite dimensional over $\hat{K}$ there exists a positive integer $i$ for which the composition $f : W_K \rightarrow \hat{K}[K] \rightarrow \Gamma_K \rightarrow \Gamma^i_K$ is an inclusion. Then there exists a splitting map $g : \Gamma^i_K \rightarrow W_K$ such that $g \circ f = id_{W_K}$. Since $\Gamma^i_K$ is finite dimensional over $\hat{K}$ we have $\Gamma^i_L \cong \Gamma^i_K \otimes_K L$. Therefore tensoring the splitting map $g$ with $L$ over $\hat{K}$ we obtain a map $\hat{g} : \Gamma^i_L \rightarrow W$ which splits the composite map $W \rightarrow L[K] \rightarrow \Gamma_L \rightarrow \Gamma^i_L$. 
Precomposing \( \hat{g} \) with the projection \( \Gamma_L \rightarrow \Gamma_L^* \) we obtain a continuous map \( \Gamma_L \rightarrow W \) which splits the inclusion \( W \hookrightarrow L[K] \rightarrow \Gamma_L \) as desired.

\[ \Box \]

**Example 4.2.** Let \( K = \mathbb{Z} \) and \( L = \mathbb{Q} \). The map

\[ ev_{Q/\mathbb{Z}}^* : \mathbb{Q}[\mathbb{Z}] \rightarrow \Gamma_Q \]

is an asymptotically split monomorphism.

We will now consider the question when the map \( ev_{L/K}^* \) is asymptotically split. In preparation for this let us introduce the notion of the ring of **DS-extension**. Here DS is an abbreviation for ‘divisible sequence’.

**Definition 4.3.** Let \( Q \) be a subring of a commutative ring \( R \) with no zero divisors. We say that \( R \) is a **DS-extension** of \( Q \) if there exists an infinite sequence of elements \( q_i \in Q, i = 1, 2, \ldots \) such that all differences \( q_i - q_j \) are invertible in \( R \) for \( i \neq j \).

We will now give various examples of DS-extensions.

1. If \( Q \) is an infinite ring and \( R \) contains the field of fractions of \( Q \), then \( R \) is a DS-extension of \( Q \).
2. A local ring with infinite residue field is a DS-extension of itself.
3. The so-called Novikov ring \( \mathbb{Z}((q)) \) consisting of Laurent power series with integer coefficients is a DS-extension of its subring \( \mathbb{Z}[q] \).
4. A more economical variant of the previous example is \( \mathbb{Z}[x^\pm 1, \{(x^n - 1)^{-1}\}] \), \( n = 1, 2, \ldots \), the ring of polynomials where the elements \( x \) and all cyclotomic polynomials are inverted.
5. Any DS-extension of \( Z \) must contain \( Q \). Indeed, suppose that \( k_i, i = 1, 2, \ldots \) are integers such that \( k_i - k_j \) is invertible in some ring \( R \supseteq Z \) for \( i \neq j \). Since the sequence \( \{k_i\} \) is infinite for any prime number \( p \) there exists a pair \( i, j \) with \( i \neq j \) and \( k_i \equiv k_j \mod p \). Therefore \( p \) must be invertible in \( R \) and \( R \supseteq Z \). Similarly if \( Q \) is a principal ideal domain such that for any maximal ideal \( (q) \in Q \) the residue field \( Q/(q) \) is finite then any DS-extension of \( Q \) must contain \( Q \), the field of fractions of \( Q \).

We can now formulate our second main theorem which is a partial dualization of Theorem 4.1.

**Theorem 4.4.** We have the following chain of implications for the conditions (1), (2), (3), (4) below: 1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) \( \Rightarrow \) (4). If \( L \) is a UFD then (4) implies (3). If, further, \( L \) is a discrete valuation ring (DVR) then (4) implies (1).

1. \( L \) is a DS-extension of the ring \( K \).
2. The map \( ev_{L/K}^* : L[x] \rightarrow \text{Map}(K, L) \) is asymptotically split (equivalently map \( ev_{L/K}^* : L[K] \rightarrow \Gamma_L \) has a dense image)
3. There exist no nontrivial numerical polynomials, i.e. \( \text{Num}_{L/K} = L[x] \).
4. For any element \( a \in L \) the composite map \( L \rightarrow L/(a) \) has an infinite image.

**Proof.** (1)\( \Rightarrow \) (2). Let \( L \) be a DS-extension of \( K \). Recall that \( \Gamma_L \) has a topological basis \( \{u_n\} \). It is clear that the image of \( ev_{L/K}^* \) is dense in \( \Gamma_L \) if in \( L[K] \) there exists a collection of elements \( a_1, \ldots, a_n, \ldots \) such that

\[ ev_{L/K}^*(a_n) = u_n + \sum_{i \geq 1} b_{n,i} u_{n+i}. \]

Set \( a_n = \sum_{j=1}^{n} \alpha_{n,j}[k_j] \). Then

\[ ev_{L/K}^*(a_n) = \sum_{i=0}^{\infty} \left( \sum_{j=1}^{n} \alpha_{n,j} k_j^i \right) u_i. \]

Therefore, one needs to find a collection of elements \( \alpha_{n,j} \in L \) and \( k_j \in K \) where \( j = 1, \ldots, n \) such that the following equations hold for \( l = 0, \ldots, n - 1 \).

\[ \sum_{j=1}^{n} \alpha_{n,j} k_j^l = 0; \]

\[ \sum_{j=1}^{n} \alpha_{n,j} k_j^n = 1. \]
Since $L$ is a $DS$-extension of $K$ there exists a system of elements $k_j$ where $j = 0, \ldots, n$ and such that all differences $k_i - k_j$ are invertible in $L$ for $i \neq j$. Then the matrix of $101$ considered as a linear system with respect to the unknowns $\alpha_{n,j}$ is the well-known Vandermonde matrix and its determinant equals $\prod_{i<j}(k_i - k_j) \neq 0$. Therefore this linear system admits a solution.

$(2) \Rightarrow (3)$. This implication is almost obvious and proved along the same lines as $(2) \Rightarrow (1)$ in the proof of Theorem 4.1.

$(3) \Rightarrow (4)$. This is proved in Proposition 3.2. The same proposition provides a partial inverse $(4) \Rightarrow (3)$ provided that $L$ is a UFD.

Finally let us prove the implication $(4) \Rightarrow (1)$ assuming that $L$ is a DVR. Let $\pi \in L$ be the generator of the maximal ideal of $L$. Choose the infinite sequence $k_i, i = 1, 2, \ldots$ of distinct elements in the image of the composite map $K \to L \to L/(\pi)$. Let $k_i \in L$ be an arbitrary representative of residue class $k_i \in L(\pi)$. Then for any $i \neq j$ the difference $\tilde{k}_i - \tilde{k}_j$ does not belong to the ideal $(\pi)$ and therefore is invertible. Thus, $L$ is a $DS$-extension of $K$.

$\Box$

**Remark 4.5.** Theorems 4.1 and 4.4 give criteria for the evaluation map and its dual to be asymptotically split (or to have dense images) in the case of the polynomial ring in one variable. Using Proposition 2.14 we see that the same criteria can be used for the rings of polynomials in several variables.

5. **Applications to algebras of cohomology operations**

In this section we show that the Landweber-Novikov and Steenrod algebras after a certain extension of scalars and completion become isomorphic to the (completions of) some group rings. This is analogous to the Theorem 6.2.3 of [14] saying that the continuous dual to the Morava stabilizer algebra over $\mathbb{F}_p$ is isomorphic to a group algebra. Related results in the context of the classical Steenrod algebra are contained in the recent preprint of J. Palmieri, [11].

Let $R$ be a commutative ring. The set of formal power series of the form

$$x(t) = t + x_1t^2 + \ldots + x_k t^{k+1} + \ldots$$

forms a group under composition of formal power series. This group plays an important role in symplectic geometry, algebraic topology, singularity theory and other fields. In group theory it is known by the name Nottingham group cf. [4]. It will be denoted by $N(R)$.

The group $N(R)$ is the group of $R$-points of a certain Hopf algebra $N$. This Hopf algebra is isomorphic to the ring $\mathbb{Z}[x_1, x_2, \ldots]$ as an algebra. The diagonal $\Delta$ is dual to the composition of the power series and is given by the following formula:

$$\Delta(x_k) = \sum_{i+j=k} x_i \otimes (t + x_1 t^2 + x_2 t^3 + \ldots)^{j+1}_i$$

where $(t + x_1 t^2 + x_2 t^3 + \ldots)^{j+1}_i$ denotes the coefficient at the $i$th power of $t$ in the series $(t + x_1 t^2 + x_2 t^3 + \ldots)^{j+1}$. An important property of the Hopf algebra $N$ is that it is graded. Indeed, letting $|x_i| = i$ we see that the diagonal in $N$ preserves grading. Moreover, each graded component of $N$ is a free abelian group of finite rank.

The Hopf algebra $N$ is closely related to complex cobordism theory $MU^*$. Namely, the algebra of formal differential operators on $N$ is isomorphic to the algebra of cohomology operations in the $MU^*$. Under this isomorphism the subalgebra of left-invariant differential operators corresponds to the so-called Landweber-Novikov algebra $S$. We refer the reader to the paper by the first author [2] for an exposition of these results. We would like to take another approach, not using differential operators.

**Definition 5.1.** The Landweber-Novikov algebra $S$ is the graded $\mathbb{Z}$-dual of the Hopf algebra $N$. The complete Landweber-Novikov algebra $\hat{S}$ is the ungraded $\mathbb{Z}$-dual of the Hopf algebra $N$.

Observe that $\hat{S}$ is a topological Hopf algebra over $\mathbb{Z}$. For a ring $R$ the algebra $\hat{S} \otimes R$ has a linear topology inherited from $\hat{S}$ and we denote by $\hat{S} \hat{\otimes} R$ the completion of $\hat{S} \otimes R$ with respect to this topology. Note that $\hat{S} \hat{\otimes} R \cong \text{Hom}(N, R)$.

We have the following result.
Proposition 5.2. Let $R$ be a DS-extension of a ring $Q$. Then the topological Hopf $R$-algebra $\hat{S} \hat{\otimes} R$, has a dense Hopf $R$-subalgebra isomorphic to $R[N(Q)]$.

There is an analogue of this result in the context of the Steenrod algebra. Namely, consider for any $F_p$-algebra $R$ the group $P(R)$ consisting of all formal power series of the form $\sum_{i=0}^{\infty} a_i x^{p^i}$ where $a_i \in R$ and $a_0 = 1$. The group operation is the composition of power series. (Incidentally, $P(R)$ is the group of strict automorphisms of the additive formal group law over $R$.) Then the functor $R \mapsto P(R)$ is represented by a Hopf algebra $P = \mathbb{F}_p[x_1, x_2, \ldots]$ where the diagonal map is given by the formula

$$\Delta x_n = x_n \otimes 1 + 1 \otimes x_n + \sum_{k=1}^{\infty} x_k \otimes y_{p^i-k}.$$

The Hopf algebra $P$ is in fact graded with $\deg(x_i) = p^i - 1$. The graded $F_p$-dual to $P$ will be denoted by $\mathcal{A}$. The algebra $\mathcal{A}$ is isomorphic to the Steenrod algebra if $p = 2$ and to the algebra of reduced powers if $p$ is odd. We will use the symbol $\hat{\mathcal{A}}$ to denote the ungraded dual to $\mathcal{A}$; it is a topological Hopf algebra over $\mathbb{F}_p$.

For any ring $F$ denote by $\hat{\mathcal{A}} \hat{\otimes} F$ the completed tensor product of $\hat{\mathcal{A}}$ and $F$, clearly $\hat{\mathcal{A}} \hat{\otimes} F \cong \text{Hom}(P, F)$.

Then the analogue of Proposition 5.2 reads as follows:

Proposition 5.3. Let $R$ be a DS-extension of a ring $Q$ and suppose that $Q$ is an $F_p$-algebra. Then the topological Hopf $R$-algebra $\hat{\mathcal{A}} \hat{\otimes} R$, has a dense Hopf $R$-subalgebra isomorphic to $R[N(P(Q))]$

The statements of propositions 5.2 and 5.3 follow from Theorem 4.4, taking into account Remark 4.5.

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