An initial-boundary value problem for the integrable spin-1 Gross-Pitaevskii equations with a $4 \times 4$ Lax pair on the half-line

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We investigate the initial-boundary value problem for the integrable spin-1 Gross-Pitaevskii (GP) equations with $4 \times 4$ Lax pair on the half-line. The solution of this system can be obtained in terms of the solution of a $4 \times 4$ matrix Riemann-Hilbert (RH) problem formulated in the complex $k$-plane. The relevant jump matrices of the RH problem can be explicitly found using the two spectral functions $s(k)$ and $S(k)$, which can be defined by the initial data, the Dirichlet-Neumann boundary data at $x = 0$. The global relation is established between the two dependent spectral functions. The general mappings between Dirichlet and Neumann boundary values are analyzed in terms of the global relation.

In 1967, Gardner, Greene, Kruskal, and Miura presented a powerful inverse scattering transformation (IST) to investigate solitons of the KdV equation with an initial value problem. After that this method was used to solve the initial value problems for many integrable nonlinear evolution partial differential equations (PDEs) with the Lax pairs. Moreover, the IST method was further extended such as the Fokas' unified transformation method. The Fokas unified method can be used to study the initial-boundary value problems for some integrable nonlinear integrable evolution PDEs with $2 \times 2$ and $3 \times 3$ Lax pairs on the half-line and the finite interval. To the best of our knowledge, so far there is no work on the IBV problems of integrable equations with $4 \times 4$ Lax pairs on the half-line. In this paper, We investigate the initial-boundary value problem for the integrable spin-1 Gross-Pitaevskii (GP) equations with $4 \times 4$ Lax pair on the half-line. The solution of this system can be obtained in terms of the solution of a $4 \times 4$ matrix Riemann-Hilbert problem formulated in the complex $k$-plane. The relevant jump matrices of the RH problem can be explicitly found using the two spectral functions $s(k)$ and $S(k)$, which can be defined by the initial data, the Dirichlet-Neumann boundary data at $x = 0$. The global relation is established between the two dependent spectral functions. The general mappings between Dirichlet and Neumann boundary values are analyzed in terms of the global relation.

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I. INTRODUCTION

The initial value problems for many integrable nonlinear evolution partial differential equations (PDEs) with the Lax pairs can be solved in terms of the inverse scattering transform (IST) [1–3]. After that, there exist some important extensions of the IST such as the Deift-Zhou nonlinear steepest descent method [4] and the Fokas unified method [5–7]. Particularly, the Fokas unified method can be used to study the initial-boundary value problems for both linear and nonlinear integrable evolution PDEs with 2 × 2 Lax pairs on the half-line and the finite interval, such as the nonlinear Schrödinger equation [6, 8–11], the sine-Gordon equation [12, 13], the KdV equation [14], the mKdV equation [15, 16], the derivative nonlinear Schrödinger equation [17], Ernst equations [18], and etc. (see Refs. [19–21] and references therein). Recently, Lenells extended the Fokas method to study the initial-boundary value (IBV) problems for integrable nonlinear evolution equations with 3 × 3 Lax pairs on the half-line [22]. After that, the idea was extended to study IBV problems of some integrable nonlinear evolution equations with 3 × 3 Lax pairs on the half-line or the finite interval, such as the Degasperis-Procesi equation [23], the Sasa-Satsuma equation [24], the coupled nonlinear Schrödinger equations [25–28], and the Ostrovsky-Vakhnenko equation [29]. To the best of our knowledge, so far there is no work on the IBV problems of integrable equations with 4 × 4 Lax pairs on the half-line.

The aim of this paper is to develop a methodology for analyzing the IBV problems for integrable nonlinear evolution equations with 4 × 4 Lax pairs on the half-line by extending the method [5–7, 22] for the integrable nonlinear PDEs with 2 × 2 and 3 × 3 Lax pairs. In this paper, we will study the IVB problem of the integrable spin-1 GP equations

\[
\begin{align*}
    iq_{1t} + q_{1xx} - 2\alpha (|q_1|^2 + 2|q_0|^2) q_1 - 2\alpha |q_0|^2 q_1 q_{-1} &= 0, \\
    iq_{0t} + q_{0xx} - 2\alpha (|q_1|^2 + |q_0|^2 + |q_{-1}|^2) q_0 - 2\alpha q_1 q_{-1} q_0 &= 0, \\
    iq_{-1t} + q_{-1xx} - 2\alpha (2|q_0|^2 + |q_{-1}|^2) q_{-1} - 2\alpha |q_0|^2 q_{-1} &= 0,
\end{align*}
\]

with the initial-boundary value conditions

Initial conditions: \( q_j(x, t = 0) = q_0j(x) \in S(\mathbb{R}^+), \ j = 1, 0, -1, \ 0 < x < \infty, \)

Dirichlet boundary conditions: \( q_j(x = 0, t) = u_{0j}(t), \ j = 1, 0, -1, \ 0 < t < T, \)

Neumann boundary conditions: \( q_{jx}(x = 0, t) = u_{1j}(t), \ j = 1, 0, -1, \ 0 < t < T, \)

where the complex-valued spinor condensate wave functions \( q_j = q_j(x, t), j = 1, 0, -1 \) are the sufficiently smooth functions defined in the finite region \( \Omega = \{(x, t) | x \in [0, \infty), t \in [0, T]\} \) with \( T > 0 \) being the fixed finite time, the overbar denotes the complex conjugate, \( S(\mathbb{R}^+) \) denotes the space of Schwartz functions, the initial data \( q_0j(x), j = 1, 0, -1 \) and boundary data \( u_{0j}(t), u_{1j}(t), j = 1, 0, -1 \) are sufficiently smooth and compatible at points \( (x, t) = (0, 0) \).

The spin-1 GP system (1) can describe soliton dynamics of an \( F = 1 \) spinor Bose-Einstein condensates [30]. The four types of parameters: \( (\alpha, \beta) = \{(1, 1), (1, -1), (-1, 1), (-1, -1)\} \) in the spin-1 GP system (1) correspond to the four roles of the self-cross-phase modulation (nonlinearity) and spin-exchange modulation, respectively, that is, (attractive, attractive), (attractive, repulsive), (repulsive, attractive), and ((repulsive, repulsive). In particular, Eq. (1) with the attractive mean-field nonlinearity and ferromagnetic spin-exchange
modulation was shown to possess multi-bright soliton solutions [31]. Eq. (1) with the repulsive mean-field nonlinearity and ferromagnetic spin-exchange modulation was shown to possess multi-dark soliton solutions [32]. Moreover, double-periodic wave solutions of Eq. (1) were also found [33]. System (1) is associated with a variational principle

\[
i q_{jt}(x, t) = \frac{\delta \mathcal{E}_{GP}}{\delta q_j(x, t)}, \quad \bar{q}_1(x, t) = \bar{q}_1(x, t), \quad \bar{q}_0(x, t) = 2\bar{q}_0(x, t), \quad \bar{q}_{-1}(x, t) = \bar{q}_{-1}(x, t),
\]

with the energy functional being of the form

\[
\mathcal{E}_{GP} = \int dx \left\{ \sum_{j=1,0,-1} |q_{jx}|^2 + \alpha \left[ |q_1|^4 + |q_{-1}|^4 + 2|q_0|^4 + 4(|q_1|^2 + |q_{-1}|^2)|q_0|^2 \right] + 2\alpha \bar{\beta} \text{Re}(\bar{q}_0^2 \bar{q}_1 \bar{q}_{-1}) \right\}.
\]

The rest of this paper is organized as follows. In Sec. 2, we study the spectral analysis of the associated 4 × 4 Lax pair of Eq. (1). Sec. 3 presents the corresponding 4 × 4 matrix RH problem in terms of the jump matrices found in Sec. 2. The global relation is used to establish the map between the Dirichlet and Neumann boundary values in Sec. 4.

II. THE SPECTRAL ANALYSIS OF THE LAX PAIR

In this subsection, we will simultaneously consider the spectral analysis of the Lax pair (4) to present sectionally its analytic eigenfunctions in order to formulate a 4 × 4 matrix RH problem defined in the complex k-plane.

(a) The closed one-form for the Lax pair

The spin-1 GP equations (1) admits the 4 × 4 Lax pair [30]

\[
\begin{cases}
\psi_x + ik\sigma_4 \psi = U(x, t)\psi, \\
\psi_t + 2ik^2 \sigma_4 \psi = V(x, t, k)\psi,
\end{cases}
\]

where \( \psi = \psi(x, t, k) \) is a 4×4 matrix-valued or 4 × 1 column vector-valued spectral function, \( k \in \mathbb{C} \) is an isospectral parameter, \( \sigma_4 = \text{diag}(1, 1, -1, -1) \). and the 4 × 4 matrix-valued functions \( U(x, t) \) and \( V(x, t, k) \) are defined by

\[
U(x, t) = \begin{pmatrix}
0 & 0 & q_1 & q_0 \\
0 & 0 & \beta q_0 & q_{-1} \\
\alpha \bar{q}_1 & \alpha \bar{\beta} \bar{q}_0 & 0 & 0 \\
\alpha \bar{q}_0 & \alpha \bar{q}_{-1} & 0 & 0
\end{pmatrix}, \quad V(x, t, k) = 2kU + V_0, \quad V_0 = i\sigma_4(U_x - U^2).
\]

A new eigenfunction \( \mu = \mu(x, t, k) \) is defined by the transform

\[
\mu(x, t, k) = \psi(x, t, k)e^{(kx + 2k^2 t)\sigma_4},
\]

such that the Lax pair (4) is changed into an equivalent form

\[
\begin{cases}
\mu_x + ik\sigma_4 \mu = U(x, t)\mu, \\
\mu_t + 2ik^2 \sigma_4 \mu = V(x, t, k)\mu,
\end{cases}
\]
FIG. 1: (a) the region $\Omega$; (b)-(d) three contours $\gamma_j$ ($j = 1, 2, 3$) in the $(x, t)$-plane.

where $\hat{\sigma}_4 \mu = [\sigma_4, \mu]$, $\hat{\sigma}_4$ denote the commutator with respect to $\sigma_4$ and the operator acting on a $4 \times 4$ matrix $X$ by $\hat{\sigma}_4 X = [\sigma_4, X]$ such that $e^{x \hat{\sigma}_4 X} = e^{x \sigma_4} X e^{-x \sigma_4}$. The Lax pair (7) leads to a full derivative form

$$
d \left[ e^{i(kx+2k^2t)\hat{\sigma}_4} \mu(x, t, k) \right] = W(x, t, k),$$

where the closed one-form $W(x, t, k)$ is

$$W(x, t, k) = e^{i(kx+2k^2t)\hat{\sigma}_4} [U(x, t)\mu(x, t, k)dx + V(x, t, k)\mu(x, t, k)dt].$$

(b) The basic eigenfunctions $\mu_j$'s

For any point $(x, t)$ in the considered region $\Omega = \{(x, t)|0 < x < \infty, 0 < t < T\}$ (see Fig. 1(a)), $\{\gamma_j\}_{1}^{3}$ denote the three contours in the domain $\Omega$ connecting $(x_j, t_j)$ to $(x, t)$, respectively, where $(x_1, t_1) = (0, T)$, $(x_2, t_2) = (0, 0)$, $(x_3, t_3) = (\infty, 0)$ (see Figs. 1(b)-(d)). Thus for the point $(\xi, \tau)$ on the each contour, we have

$$\gamma_1: x - \xi \geq 0, \quad t - \tau \leq 0,$$

$$\gamma_2: x - \xi \geq 0, \quad t - \tau \geq 0,$$

$$\gamma_3: x - \xi \leq 0, \quad t - \tau = 0.$$  \hspace{1cm} (10)

It follows from the one-form (8) that we can use the Volterra integral equations to define its three eigenfunctions $\{\mu_j\}_{1}^{3}$ on the above-mentioned three contours $\{\gamma_j\}_{1}^{3}$

$$\mu_j(x, t, k) = \mathbb{I} + \int_{(x_j, t_j)}^{(x, t)} e^{-i(kx+2k^2t)\hat{\sigma}_4} W_j(\xi, \tau, k), \quad j = 1, 2, 3, \quad (x, t) \in \Omega, \hspace{1cm} (11)$$

where $\mathbb{I} = \text{diag}(1, 1, 1, 1)$, the integral is over a piecewise smooth curve from $(x_j, t_j)$ to $(x, t)$, $W_j(x, t, k)$ is given by Eq. (9) with $\mu(x, t, k)$ replaced by $\mu_j(x, t, k)$. Since the one-form $W_j$ is closed, thus $\mu_j$ is independent of the path of integration. If we choose the paths of integration to be parallel to the $x$ and $t$ axes, then the integral Eq. (11) becomes ($j = 12, 3$)

$$\mu_j = \mathbb{I} + \int_{x_j}^{x} e^{-ik(x-\xi)\hat{\sigma}_4} (U\mu_j)(\xi, t, k)d\xi + e^{-ik(x-x_j)\hat{\sigma}_4} \int_{t_j}^{t} e^{-2ik^2(t-\tau)\hat{\sigma}_4} (V\mu_j)(x_j, \tau, k)d\tau, \hspace{1cm} (12)$$

Eq. (12) implies that the first, second, third, and fourth columns of the matrices $\mu_j(x, t, k)$'s contain these
where the complex $\mu$ planes are bounded and analytic in the complex $\kappa$-

\[ x = e^{2ik(x-\xi)+4ik^2(t-\tau)} \quad e^{2ik(x-\xi)+4ik^2(t-\tau)}, \]

\[ \mu_2 : (f_-(k) \cap g_+(k), f_-(k) \cap g_-(k), f_+(k) \cap g_-(k), f_+(k) \cap g_-(k)) =: (D_2, D_2, D_3, D_3), \]

\[ \mu_3 : (f_+(k), f_+(k), f_-(k), f_-(k)) =: (C^-, C^-, C^+, C^+), \]

where $C^- = D_3 \cup D_4$, $C^+ = D_1 \cup D_2$, $f_+(k) := \text{Re} f(k) = -\text{Im} k > 0$, $f_-(k) := \text{Re} f(k) = -\text{Im} k < 0$, $g_+(k) := \text{Re} g(k) = -2 \text{Re} k \text{Im} k > 0$, and $g_-(k) := \text{Re} g(k) = -2 \text{Re} k \text{Im} k < 0.$

(c) **Symmetries of eigenfunctions**

For the convenience, we write a $4 \times 4$ matrix $X = (X_{ij})_{4 \times 4}$ as

\[ X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}, \quad \tilde{X}_{11} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}, \quad \tilde{X}_{12} = \begin{pmatrix} X_{13} & X_{14} \\ X_{23} & X_{24} \end{pmatrix}, \]

\[ \tilde{X}_{21} = \begin{pmatrix} X_{31} & X_{32} \\ X_{41} & X_{42} \end{pmatrix}, \quad \tilde{X}_{22} = \begin{pmatrix} X_{33} & X_{34} \\ X_{43} & X_{44} \end{pmatrix}. \]
Let \( U(x, t, k) = -ik\sigma_4 + U(x, t) \), \( V(x, t, k) = -2ik^2\sigma_4 + V(x, t, k) \). Then the symmetry properties of \( U(x, t, k) \) and \( V(x, t, k) \) imply that the eigenfunction \( \mu(x, t, k) \) have the symmetries

\[
(\tilde{\mu}(x, t, k))_{11} = P^\delta (\overline{\mu(x, t, k)})_{22} P^\delta, \quad (\tilde{\mu}(x, t, k))_{12} = \alpha (\overline{\mu(x, t, k)})_{21},
\]

where \( P^\delta = \text{diag}(1, \beta) \), \( \beta^2 = 1 \).

Since

\[
P^\alpha \overline{U(x, t, k)} P^\alpha = -U(x, t, k)^T, \quad P^\alpha \overline{V(x, t, k)} P^\alpha = -V(x, t, k)^T,
\]

where \( P^\alpha = \text{diag}(\pm \alpha, \pm \alpha, \mp 1, \mp 1) \), \( \alpha^2 = 1 \).

According to Eq. (21) (see the similar proof in Ref. [11]), we know that the eigenfunction \( \psi(x, t, k) \) of the Lax pair (4) and \( \mu(x, t, k) \) of the Lax pair (7) are of the same symmetric relation

\[
\psi^{-1}(x, t, k) = P^\alpha \overline{\psi(x, t, k)} P^\alpha, \quad \mu^{-1}(x, t, k) = P^\alpha \overline{\mu(x, t, k)} P^\alpha,
\]

Moreover, In the domains where \( \mu \) is bounded, we have

\[
\mu(x, t, k) = \mathbb{I} + O\left(\frac{1}{k}\right), \quad k \to \infty,
\]

and \( \det[\mu(x, t, k)] = 1 \) since \( \text{tr}(U(x, t, k)) = \text{tr}(V(x, t, k)) = 0 \).

(d) The minors of eigenfunctions

The cofactor matrix \( X^A \) (or the transpose of the adjugate) of a \( 4 \times 4 \) matrix \( X \) is given by

\[
\text{adj}(X)^T = X^A = \begin{pmatrix}
m_{11}(X) & -m_{12}(X) & m_{13}(X) & -m_{14}(X) \\
-m_{21}(X) & m_{22}(X) & -m_{23}(X) & m_{24}(X) \\
m_{31}(X) & -m_{32}(X) & m_{33}(X) & -m_{34}(X) \\
-m_{41}(X) & m_{42}(X) & -m_{43}(X) & m_{44}(X)
\end{pmatrix},
\]

where \( m_{ij}(X) \) denote the \((ij)\)th minor of \( X \) and \( (X^A)^T X = \text{adj}(X) X = \det X \).

It follows from Eq. (7) that be shown that the matrix-valued functions \( \mu_j^A \)'s satisfy the Lax pair

\[
\begin{cases}
\mu_{j,x}^A - ik\tilde{\sigma}_4 \mu_j^A = -U^T \mu_j^d, \\
\mu_{j,t}^A - 2ik^2\tilde{\sigma}_4 \mu_j^A = -V^T \mu_j^d,
\end{cases}
\]

whose solutions can be expressed as

\[
\mu_j^A(x, t, k) = \mathbb{I} - \int_{x_j}^x e^{ik(x-\xi)\tilde{\sigma}_4} (U \mu_j^A)(\xi, t, k) d\xi - e^{ik(x-x_j)\tilde{\sigma}_4} \int_{t_j}^t e^{2ik^2(1-\tau)\tilde{\sigma}_4} (V \mu_j^A)(x_j, \tau, k) d\tau,
\]

by using the Volterra integral equations, where \( U^T \) and \( V^T \) denote the transposes of \( U \) and \( V \), respectively.

It is easy to check that the regions of boundedness of \( \mu_j^A \):

\[
\begin{cases}
\mu_j^A(x, t, k) \text{ is bounded for } k \in (D_3, D_3, D_2, D_2), \\
\mu_j^A(x, t, k) \text{ is bounded for } k \in (D_4, D_4, D_1, D_1), \\
\mu_j^A(x, t, k) \text{ is bounded for } k \in (C^+, C^+, C^-, C^-),
\end{cases}
\]

which are symmetric ones of \( \mu_j \) about the Re \( k \)-axis (cf. Eq. (15)).
(e) The spectral functions and the global relation

Let us introduce the $4 \times 4$ matrix-valued functions $S(k)$, $s(k)$, and $\mathcal{S}(k)$ by $\mu_j$, $j = 1, 2, 3$

$$
\begin{align*}
\mu_1(x, t, k) &= \mu_2(x, t, k) e^{-i(kx+2kt)\hat{\sigma}_3} S(k), \\
\mu_3(x, t, k) &= \mu_2(x, t, k) e^{-i(kx+2kt)\hat{\sigma}_3} s(k), \\
\mu_3(x, t, k) &= \mu_1(x, t, k) e^{-i(kx+2kt)\hat{\sigma}_3} \mathcal{S}(k),
\end{align*}
$$

(Eq. 23)

Evaluating system (23) at $(x, t) = (0, 0)$ and $(x, t) = (0, T)$, respectively, we have

$$
\begin{align*}
S(k) &= \mu_1(0, 0, k) = e^{2ik^2T\hat{\sigma}_3} \mu_2^{-1}(0, T, k), \\
s(k) &= \mu_3(0, 0, k), \\
\mathcal{S}(k) &= \mu_1^{-1}(0, 0, k) \mu_3(0, 0, k) = S^{-1}(k) s(k) = e^{2ik^2T\hat{\sigma}_3} \mu_3(0, T, k),
\end{align*}
$$

(Eq. 24)

These relations among $\mu_j$ are displayed in Fig. 3. Thus these three functions $S(k)$, $s(k)$, and $\mathcal{S}(k)$ are dependent such that we only consider two of them, e.g., $S(k)$ and $s(k)$.

According to the definition (12) of $\mu_j$, Eq. (24) implies that

$$
\begin{align*}
S(k) &= I - \int_0^\infty e^{ik\xi \hat{\sigma}_3} (U\mu_3)(\xi, 0, k) d\xi \\
s(k) &= I - \int_0^T e^{2ik^2\tau \hat{\sigma}_3} (V\mu_1)(0, \tau, k) d\tau = \left[ I + \int_0^T e^{2ik^2\tau \hat{\sigma}_3} (V\mu_2)(0, \tau, k) d\tau \right]^{-1},
\end{align*}
$$

(Eq. 25)

where $\mu_j(x, 0, k)$, $j = 1, 2$ and $\mu_3(x, 0, k)$, $0 < x < \infty$, $0 < t < T$ satisfy the Volterra integral equations

$$
\begin{align*}
\mu_3(x, 0, k) &= I - \int_x^\infty e^{-ik(x-\xi)\hat{\sigma}_3} (U\mu_3)(\xi, 0, k) d\xi, \\
\mu_1(0, t, k) &= I - \int_t^\infty e^{-2ik^2(\tau-t)\hat{\sigma}_3} (V\mu_1)(0, \tau, k) d\tau, \\
\mu_2(0, t, k) &= I + \int_0^t e^{-2ik^2(t-\tau)\hat{\sigma}_3} (V\mu_2)(0, \tau, k) d\tau,
\end{align*}
$$

(Eq. 26)

Thus, it follows from Eqs. (25) and (26) that $s(k)$ and $S(k)$ are determined by $U(x, 0, k)$ and $V(0, t, k)$, i.e., by the initial data $q_3(x, t = 0)$ and the Dirichlet-Neumann boundary data $q_3(x = 0, t)$ and $q_{3x}(x = 0, t)$, $j = 1, 0, -1$, respectively. In fact, $\mu_3(x, 0, k)$ and $\mu_{1, 2}(0, t, k)$ satisfy the $x$-part and $t$-part of the Lax pair (7) at
for $k$ via the Volterra integral equations, where

$$
\begin{align*}
W &\equiv \{ (\gamma) \}, \quad \text{and the definition of the contours } (\gamma) \in \mathbb{D}, \\
\gamma_1 &\equiv \frac{\gamma_3 + \gamma_3 + \gamma_3 + \gamma_3}{\gamma_4 + \gamma_4 + \gamma_4 + \gamma_4}, \\
\gamma_2 &\equiv \frac{\gamma_3 + \gamma_3 + \gamma_3 + \gamma_3}{\gamma_3 + \gamma_3 + \gamma_3 + \gamma_3}, \\
\gamma_3 &\equiv \frac{\gamma_3 + \gamma_3 + \gamma_3 + \gamma_3}{\gamma_3 + \gamma_3 + \gamma_3 + \gamma_3}, \\
\gamma_4 &\equiv \frac{\gamma_3 + \gamma_3 + \gamma_3 + \gamma_3}{\gamma_3 + \gamma_3 + \gamma_3 + \gamma_3}.
\end{align*}
$$

Moreover, the functions $\{S(k), s(k)\}$ and $\{S^4(k), s^4(k)\}$ have the following boundedness:

$$
\begin{align*}
S(k) &\text{ is bounded for } k \in (D_2 \cup D_4, D_2 \cup D_4, D_1 \cup D_3, D_1 \cup D_3), \\
s(k) &\text{ is bounded for } k \in (C^-, C^-, C^+, C^+), \\
S^4(k) &\text{ is bounded for } k \in (D_1 \cup D_3, D_1 \cup D_3, D_2 \cup D_4, D_21 \cup D_4), \\
s^4(k) &\text{ is bounded for } k \in (C^+, C^+, C^-, C^-).
\end{align*}
$$

It follows from the third one in Eq. (24) that we have the so-called global relation

$$
c(T, k) = \mu_3(0, T, k) = e^{-2ik^2T\delta_4} [S^{-1}(k)s(k)],
$$

where $\mu_3(0, t, k), 0 < t < T$ satisfies the Volterra integral equation

$$
\mu_3(0, t, k) = 1 - \int_0^\infty e^{ik\xi T}(U\mu_3)(\xi, t, k) d\xi, \quad 0 < t < T, \quad k \in (C^-, C^-, C^+, C^+),
$$

(f) The definition of matrix-valued functions $M_n$’s

In each domain $D_n, n = 1, 2, 3, 4$ of the complex $k$-plane, the solution $M_n(x, t, k)$ of Eq. (7) is

$$
(M_n(x, t, k))_{ij} = \delta_{ij} + \int_{(\gamma^n)_{ij}} (e^{-i(kx + 2k^2t)\delta_4} W_n(\xi, \tau, k))_{ij}, \quad k \in D_n, \quad l, j = 1, 2, 3, 4.
$$

via the Volterra integral equations, where $W_n(x, t, k)$ is given by Eq. (9) with $\mu(x, t, k)$ replaced with $M_n(x, t, k)$, and the definition of the contours $(\gamma^n)_{ij}$’s is given by

$$
(\gamma^n)_{ij} = \begin{cases} \\
\gamma_1, & \text{if Re } f_i(k) < \text{ Re } f_j(k) \text{ and Re } g_i(k) \geq \text{ Re } g_j(k), \\
\gamma_2, & \text{if Re } f_i(k) < \text{ Re } f_j(k) \text{ and Re } g_i(k) < \text{ Re } g_j(k), \\
\gamma_3, & \text{if Re } f_i(k) \geq \text{ Re } f_j(k),
\end{cases}
$$

for $k \in D_n$, where $f_{1,2}(k) = -f_{3,4}(k) = -ik, g_{1,2}(k) = -g_{3,4}(k) = -ik^2$.

The definition (32) of $(\gamma^n)_{ij}$ implies that the matrices $\gamma^n$ $(n = 1, 2, 3, 4)$ are of the forms

$$
\begin{align*}
\gamma^1 &\equiv \begin{pmatrix} \gamma_3 & \gamma_3 & \gamma_3 & \gamma_3 \\
\gamma_3 & \gamma_3 & \gamma_3 & \gamma_3 \\
\gamma_2 & \gamma_2 & \gamma_3 & \gamma_3 \\
\gamma_2 & \gamma_2 & \gamma_3 & \gamma_3 \\
\end{pmatrix}, \\
\gamma^2 &\equiv \begin{pmatrix} \gamma_3 & \gamma_3 & \gamma_3 & \gamma_3 \\
\gamma_3 & \gamma_3 & \gamma_3 & \gamma_3 \\
\gamma_1 & \gamma_1 & \gamma_3 & \gamma_3 \\
\gamma_1 & \gamma_1 & \gamma_3 & \gamma_3 \\
\end{pmatrix}, \\
\gamma^3 &\equiv \begin{pmatrix} \gamma_3 & \gamma_3 & \gamma_1 & \gamma_1 \\
\gamma_3 & \gamma_3 & \gamma_1 & \gamma_1 \\
\gamma_3 & \gamma_3 & \gamma_1 & \gamma_1 \\
\gamma_3 & \gamma_3 & \gamma_1 & \gamma_1 \\
\end{pmatrix}, \\
\gamma^4 &\equiv \begin{pmatrix} \gamma_3 & \gamma_3 & \gamma_2 & \gamma_2 \\
\gamma_3 & \gamma_3 & \gamma_2 & \gamma_2 \\
\gamma_3 & \gamma_3 & \gamma_2 & \gamma_2 \\
\gamma_3 & \gamma_3 & \gamma_2 & \gamma_2 \\
\end{pmatrix}.
\end{align*}
$$
According to the similar proof for the $3 \times 3$ Lax pair in [22] and the above-mentioned properties of $\mu(x,t,k)$, we have the boundedness and analyticity of $M_n$:

**Proposition 2.1.** The matrix-valued functions $M_n(x,t,k)$, $n = 1, 2, 3, 4$ are well defined by Eq. (31) for \( k \in \bar{D}_n \) and \( (x,t) \in \Omega \). For any fixed point \( (x,t) \), $M_n$’s are the bounded and analytic function of $k \in D_n$ away from a possible discrete set of singularity \( \{k_j\} \) at which the Fredholm determinants vanish. $M_n(x,t,k)$ also admits the bounded and continuous extensions to $\bar{D}_n$ and

\[
M_n(x,t,k) = 1 + O \left( \frac{1}{k} \right), \quad k \in D_n, \quad k \to \infty, \quad n = 1, 2, 3, 4.
\]

**Proposition 2.2.** The matrix-valued functions $S_n(x,t,k)$ ($n = 1, 2, 3, 4$) defined by

\[
M_n(x,t,k) = \mu_2(x,t,k)e^{-i(kx+2k^2t)\delta_4}S_n(k), \quad k \in D_n,
\]

can be determined by the entries of $s(k) = (s_{ij})_{4 \times 4}$, $S(k) = (S_{ij})_{4 \times 4}$ (cf. Eq. (24)) as follows:

\[
S_1(k) = \begin{pmatrix} m_{11}(s) & m_{12}(s) & s_{13} & s_{14} \\ m_{21}(s) & m_{22}(s) & s_{23} & s_{24} \\ 0 & 0 & s_{33} & s_{34} \\ 0 & 0 & s_{43} & s_{44} \end{pmatrix}, \quad S_2(k) = \begin{pmatrix} s_{11}^{(11)} & s_{12}^{(12)} & s_{13} & s_{14} \\ s_{21}^{(21)} & s_{22}^{(22)} & s_{23} & s_{24} \\ s_{31}^{(31)} & s_{32}^{(32)} & s_{33} & s_{34} \\ s_{41}^{(41)} & s_{42}^{(42)} & s_{43} & s_{44} \end{pmatrix},
\]

\[
S_3(k) = \begin{pmatrix} s_{11} & s_{12} & s_{13}^{(13)} & s_{14}^{(14)} \\ s_{21} & s_{22} & s_{23}^{(23)} & s_{24}^{(24)} \\ s_{31} & s_{32} & s_{33}^{(33)} & s_{34}^{(34)} \\ s_{41} & s_{42} & s_{43}^{(43)} & s_{44}^{(44)} \end{pmatrix}, \quad S_4(k) = \begin{pmatrix} s_{11} & s_{12} & 0 & 0 \\ s_{21} & s_{22} & 0 & 0 \\ s_{31} & s_{32} & m_{33}(s) & m_{34}(s) \\ s_{41} & s_{42} & m_{43}(s) & m_{44}(s) \end{pmatrix}.
\]
where $n_{i_1,j_1,i_2,j_2}(X)$ denotes the determinant of the sub-matrix generated by taking the cross elements of $i_1, i_2$th rows and $j_1, j_2$th columns of the $4 \times 4$ matrix $X$ and

$$
\begin{align*}
S_2^{(1j)} &= \frac{n_{1j,2(3-j)}(S)m_{2(3-j)}(s) + n_{1j,3(3-j)}(S)m_{3(3-j)}(s) + n_{1j,4(3-j)}(S)m_{4(3-j)}(s)}{N([S]_1[S]_2[S]_3[S]_4)} , \\
S_2^{(2j)} &= \frac{n_{2j,1(3-j)}(S)m_{1(3-j)}(s) + n_{2j,2(3-j)}(S)m_{2(3-j)}(s) + n_{2j,4(3-j)}(S)m_{4(3-j)}(s)}{N([S]_1[S]_2[S]_3[S]_4)} , \\
S_2^{(3j)} &= \frac{n_{3j,1(3-j)}(S)m_{1(3-j)}(s) + n_{3j,2(3-j)}(S)m_{2(3-j)}(s) + n_{3j,4(3-j)}(S)m_{4(3-j)}(s)}{N([S]_1[S]_2[S]_3[S]_4)} , \\
S_2^{(4j)} &= \frac{n_{4j,1(3-j)}(S)m_{1(3-j)}(s) + n_{4j,2(3-j)}(S)m_{2(3-j)}(s) + n_{4j,3(3-j)}(S)m_{3(3-j)}(s)}{N([S]_1[S]_2[S]_3[S]_4)} ,
\end{align*}
$$

$$
\begin{align*}
S_3^{(1j)} &= \frac{n_{1j,2(7-j)}(S)m_{2(7-j)}(s) + n_{1j,3(7-j)}(S)m_{3(7-j)}(s) + n_{1j,4(7-j)}(S)m_{4(7-j)}(s)}{N([s]_1[s]_2[S]_3[S]_4)} , \\
S_3^{(2j)} &= \frac{n_{2j,1(7-j)}(S)m_{1(7-j)}(s) + n_{2j,3(7-j)}(S)m_{3(7-j)}(s) + n_{2j,4(7-j)}(S)m_{4(7-j)}(s)}{N([s]_1[s]_2[S]_3[S]_4)} , \\
S_3^{(3j)} &= \frac{n_{3j,1(7-j)}(S)m_{1(7-j)}(s) + n_{3j,2(7-j)}(S)m_{2(7-j)}(s) + n_{3j,4(7-j)}(S)m_{4(7-j)}(s)}{N([s]_1[s]_2[S]_3[S]_4)} , \\
S_3^{(4j)} &= \frac{n_{4j,1(7-j)}(S)m_{1(7-j)}(s) + n_{4j,2(7-j)}(S)m_{2(7-j)}(s) + n_{4j,3(7-j)}(S)m_{3(7-j)}(s)}{N([s]_1[s]_2[S]_3[S]_4)} ,
\end{align*}
$$

$$
\begin{align*}
&\text{where } N([S]_1[S]_2[S]_3[S]_4) = \det(n([S]_1,[S]_2,[s]_3,[s]_4)) \text{ denotes the determinant of the matrix generated by choosing the first and second columns of } S(k) \text{ and the third and fourth columns of } s(k), \text{ and } N([s]_1[s]_2[S]_3[S]_4) = \det(n([s]_1,[s]_2,[S]_3,[S]_4)).
\end{align*}
$$

**Proof.** Let $\gamma_3^{x_0}$ with $x_0 > 0$ denote the contour $(x_0,0) \to (x,t)$ in the $(x,t)$-plane and $\mu_3(x,t,k;x_0)$ be determined by Eq. (11) with $j = 3$ and the contour $\gamma_3$ replaced by $\gamma_3^{x_0}$. $M_n(x,t,k;x_0)$ is defined by Eq. (31) with the contour $\gamma_3$ replaced by $\gamma_3^{x_0}$.

We introduce the functions $R_n(k;x_0)$, $S_n(k;x_0)$, and $T_n(k;x_0)$ in the form

$$
\begin{align*}
M_n(x,t,k;x_0) &= \mu_1(x,t,k)e^{-i(kx+2k^2t)\hat{s}_4}R_n(k;x_0), \\
M_n(x,t,k;x_0) &= \mu_2(x,t,k)e^{-i(kx+2k^2t)\hat{s}_4}S_n(k;x_0), \\
M_n(x,t,k;x_0) &= \mu_3(x,t,k;x_0)e^{-i(kx+2k^2t)\hat{s}_4}T_n(k;x_0),
\end{align*}
$$

It follows from Eq. (40) that we have the relations

$$
\begin{align*}
R_n(k;x_0) &= e^{2ik^2T\hat{s}_4}M_n(0,T,k;x_0), \\
S_n(k;x_0) &= M_n(0,0,k;x_0),
\end{align*}
$$

$$
\begin{align*}
T_n(k;x_0) &= e^{ikx_0\hat{s}_4}[\mu_3^{-1}(x_0,0,k;x_0)M_n(x_0,0,k;x_0)],
\end{align*}
$$

and

$$
\begin{align*}
S(k) &= \mu_1(0,0,k) = S_n(k;x_0)R_n^{-1}(k;x_0), \\
s(k;x_0) &= \mu_3(0,0,k;x_0) = S_n(k;x_0)T_n^{-1}(k;x_0),
\end{align*}
$$

(42)
Proposition 2.4. Let \( \{M_n(x,t,k)\}_{n}^{4} \) be the eigenfunctions given by Eq. (31) and suppose that the set \( \{k_j\}_{1}^{N} \) of singularities are as the above-mentioned Assumption 2.3. Then we have the following residue conditions:

\[
\text{Res}_{k=k_j}[M_1(x,t,k)]=\begin{cases} 
\frac{m_{2(3-l)}(s)(k_j)s_{24}(k_j) - m_{1(3-l)}(s)(k_j)s_{14}(k_j)}{\hat{n}_{33,44}(s)(k_j)n_{13,24}(s)(k_j)}[M_1(x,t,k)]_{3}e^{-20(k_j)} & \text{for } 1 \leq j \leq n_1, \text{ } k \in D_1, \text{ } l = 1, 2, \\
\frac{m_{1(3-l)}(s)(k_j)s_{14}(k_j) - m_{2(3-l)}(s)(k_j)s_{23}(k_j)}{\hat{n}_{33,44}(s)(k_j)n_{13,24}(s)(k_j)}[M_1(x,t,k)]_{4}e^{-20(k_j)}, & \text{for } 1 \leq j \leq n_2, \text{ } k \in D_2, \text{ } l = 1, 2, \\
\frac{m_{1(3-l)}(s)(k_j)s_{13}(k_j) - m_{2(3-l)}(s)(k_j)s_{23}(k_j)}{\hat{n}_{33,44}(s)(k_j)n_{13,24}(s)(k_j)}[M_1(x,t,k)]_{4}e^{-20(k_j)}, & \text{for } 1 \leq j \leq n_3, \text{ } k \in D_3, \text{ } l = 1, 2, \\
\frac{m_{1(3-l)}(s)(k_j)s_{12}(k_j) - m_{2(3-l)}(s)(k_j)s_{22}(k_j)}{\hat{n}_{33,44}(s)(k_j)n_{13,24}(s)(k_j)}[M_1(x,t,k)]_{4}e^{-20(k_j)}, & \text{for } 1 \leq j \leq n_4, \text{ } k \in D_4, \text{ } l = 1, 2.
\end{cases}
\]

(h) The residue conditions

Since \( \mu_2(x,t,k) \) is an entire function, it follows from Eq. (38) that \( M(x,t,k) \) only has the singularities at the points where the \( S_n(k) \)'s have the singularities. The \( S_n(k) \)'s given by Eq. (39) imply that the possible singularities of \( M(x,t,k) \) are as follows:

- \( [M]_{j}, j = 1, 2 \) could have poles in \( D_1 \) at the zeros of \( n_{33,44}(s)(k) \);
- \( [M]_{j}, j = 1, 2 \) could have poles in \( D_2 \) at the zeros of \( N([S]_1[S]_2[S]_3[S]_4)(k) \);
- \( [M]_{j}, j = 3, 4 \) could have poles in \( D_3 \) at the zeros of \( N([S]_1[S]_2[S]_3[S]_4)(k) \);
- \( [M]_{j}, j = 3, 4 \) could have poles in \( D_4 \) at the zeros of \( n_{11,22}(s)(k) \).

We use \( \{k_j\}_{1}^{N} \) to denote the above-mentioned possible zeros and suppose that they satisfy the following assumption.

Assumption 2.3. We suppose that

- \( n_{33,44}(s)(k) \) admits \( n_1 \) possible simple zeros in \( D_1 \) denoted by \( \{k_j\}_{1}^{n_1} \);
- \( N([S]_1[S]_2[S]_3[S]_4)(k) \) admits \( n_2 - n_1 \) possible simple zeros in \( D_2 \) denoted by \( \{k_j\}_{n_1+1}^{n_2} \);
- \( N([S]_1[S]_2[S]_3[S]_4)(k) \) admits \( n_3 - n_2 \) possible simple zeros in \( D_3 \) denoted by \( \{k_j\}_{n_2+1}^{n_3} \);
- \( n_{11,22}(s)(k) \) admits \( N - n_3 \) possible simple zeros in \( D_4 \) denoted by \( \{k_j\}_{n_3+1}^{N} \);

and that none of these simple zeros coincide. Moreover, none of these functions are assumed to have zeros on the boundaries of the \( D_n \)’s (\( n = 1, 2, 3, 4 \)).
where the overdot denotes the derivative with respect to the parameter $k$.

Proof. It follows from Eqs. (38) and (39) that we find the four columns of $M_1(x,t,k)$ as

$$[M_1]_1 = [\mu_2]_1 \frac{m_{22}(s)}{n_{33,44}(s)} + [\mu_2]_2 \frac{m_{12}(s)}{n_{33,44}(s)},$$  \hspace{1cm} (48a) $$[M_1]_2 = [\mu_2]_1 \frac{m_{21}(s)}{n_{33,44}(s)} + [\mu_2]_2 \frac{m_{11}(s)}{n_{33,44}(s)},$$  \hspace{1cm} (48b) $$[M_1]_3 = [\mu_2]_1 s_{13} e^{\theta} + [\mu_2]_2 s_{23} e^{\theta} + [\mu_2]_3 s_{33} + [\mu_2]_4 s_{43},$$  \hspace{1cm} (48c) $$[M_1]_4 = [\mu_2]_1 s_{14} e^{\theta} + [\mu_2]_2 s_{24} e^{\theta} + [\mu_2]_3 s_{34} + [\mu_2]_4 s_{44},$$  \hspace{1cm} (48d)

For the case that $k \in D_1$ is a simple zero of $n_{33,44}(s)(k)$, it follows from Eqs. (48c) and (48d) that we obtain $[\mu_2]_1$ and $[\mu_2]_2$ and then substitute them into Eqs. (48a) and (48b) to yield

$$[M_1]_1 = \frac{m_{22}(s)s_{24} - m_{12}(s)s_{14}}{n_{33,44}(s)n_{13,24}(s)} [M_1]_3 e^{-\theta} + \frac{m_{12}(s)s_{13} - m_{22}(s)s_{23}}{n_{33,44}(s)n_{13,24}(s)} [M_1]_4 e^{-\theta}$$ 
$$+ \frac{m_{24}(s)[\mu_2]_3 + m_{32}(s)[\mu_2]_4}{n_{13,24}(s)} e^{-\theta},$$  \hspace{1cm} (49a) $$[M_1]_2 = \frac{m_{21}(s)s_{24} - m_{11}(s)s_{14}}{n_{33,44}(s)n_{13,24}(s)} [M_1]_3 e^{-\theta} + \frac{m_{11}(s)s_{13} - m_{21}(s)s_{23}}{n_{33,44}(s)n_{13,24}(s)} [M_1]_4 e^{-\theta}$$ 
$$+ \frac{m_{41}(s)[\mu_2]_3 + m_{31}(s)[\mu_2]_4}{n_{13,24}(s)} e^{-\theta},$$  \hspace{1cm} (49b)

whose residues at $k = k_j, k_j \in D_1$ yield Eq. (44).

Similarly, we can show Eq. (45) for $k_j \in D_2$, Eq. (46) for $k_j \in D_3$, and Eq. (47) for $k_j \in D_4$ by studying Eqs. (38) and (39) for $n = 2, 3, 4$. □
III. THE 4 × 4 MATRIX RIEMANN-HILBERT PROBLEM

By using the district contours γj (j = 1, 2, 3, 4), the integral solutions of the revised Lax pair (7), and Sn due to \{S(k), s(k)\}, we have defined the sectionally analytic function \(M_n(x, t, k)\), n = 1, 2, 3, 4, which solves a 4 × 4 matrix Riemann-Hilbert (RH) problem. This RH problem can be formulated on basis of the initial conditions of the Schwartz class \(q_j(x, t = 0)\) and Dirichlet-Neumann boundary data \(q_j(x = 0, t)\) and \(q_{jx}(x = 0, t)\), j = 1, 0, −1. Thus the solution of Eq. (1) for all values of x, t can be found by solving the RH problem.

Theorem 3.1. Suppose that \((q_1(x, t), q_0(x, t), q_{−1}(x, t))\) is a solution of Eq. (1) in the domain \(Ω = \{(x, t)|0 < x < ∞, t ∈ [0, T]\}\) with sufficient smoothness and decay as \(x → ∞\). Then it can be reconstructed from the initial data defined by \(q_j(x, t = 0) = q_{0j}(x)\), j = 1, 0, −1 and Dirichlet and Neumann boundary values defined by \(q_j(x = 0, t) = w_0(t)\) and \(q_{jx}(x = 0, t) = u_{1j}(t)\), j = 1, 0, −1.

We use the initial and boundary data to define the jump matrices \(J_{mn}(x, t, k)\), n, m = 1, ..., 4, by Eq. (37) as well as the spectral functions \(S(k), s(k)\) given by Eq. (24). Assume that the possible zeros \(k_j\) of the functions \(n_{33, 44}(s)(k), N([S]_1[S]_2[S]_3[S]_4)(k), N([s]_1[s]_2[s]_3[s]_4)(k)\) and \(n_{11, 22}(s)(k)\) are as in Assumption 2.4. Then the solution \((q_1(x, t), q_0(x, t), q_{−1}(x, t))\) of Eq. (1) is given by \(M(x, t, k)\) in the form

\[
\begin{align*}
q_1(x, t) &= 2i \lim_{k \to ∞} (kM(x, t, k))_{13}, \\
q_0(x, t) &= 2i \lim_{k \to ∞} (kM(x, t, k))_{14} = 2iβ \lim_{k \to ∞} (kM(x, t, k))_{23}, \\
q_{−1}(x, t) &= 2i \lim_{k \to ∞} (kM(x, t, k))_{24},
\end{align*}
\]

where \(M(x, t, k)\) satisfies the following 4 × 4 matrix Riemann-Hilbert problem:

- \(M(x, t, k)\) is sectionally meromorphic on the Riemann k-sphere with jumps across the contours \(\bar{D}_n \cup \bar{D}_m\), (n, m = 1, 2, 3, 4) (see Fig. 2a).
- Across the contours \(\bar{D}_n \cup \bar{D}_m\) (n, m = 1, 2, 3, 4), \(M(x, t, k)\) satisfies the jump condition (36).
- The residue conditions of \(M(x, t, k)\) are satisfied in Proposition 2.4.
- \(M(x, t, k) = I + O(1/k)\) as \(k → ∞\).

Proof. System (50) can be deduced from the large k asymptotics of the eigenfunctions. We can follow the similar one in Refs. [6, 11] to show the rest proof of the Theorem. □

IV. NONLINEARIZABLE BOUNDARY CONDITIONS

The main difficulty of the initial-boundary value problems is to find the boundary values for a well-posed problem. All boundary conditions are required for the definition of \(S(k)\), and hence for the formulate the 4 × 4 matrix RH problem. Our main conclusion exhibits the unknown boundary condition on basis of the prescribed boundary condition and the initial condition in terms of the solution of a system of nonlinear integral equations.
(a) The time evolution of the global relation

By evaluating Eq. (23) at \((x, t) = (0, t)\) and considering the global relation (29), we have

\[
c(t, k) = \mu_2(0, t, k)e^{-2ik^2t\beta_4}s(k), \quad 0 < t < T, \quad k \in (C^-, C^-, C^+, C^+),
\]

which can be written as

\[
[c(t, k)]_l = \sum_{j=1}^{2} [\mu_2(0, t, k)]_j s_{j l}(k) + \sum_{j=3}^{4} [\mu_2(0, t, k)]_j s_{j l}(k)e^{-4ik^2t}, \quad l = 1, 2,
\]

\[
[c(t, k)]_l = \sum_{j=1}^{2} [\mu_2(0, t, k)]_j s_{j l}(k)e^{4ik^2t} + \sum_{j=3}^{4} [\mu_2(0, t, k)]_j s_{j l}(k), \quad l = 3, 4,
\]

Thus, the column vectors \([c(t, k)]_l, l = 1, 2\) are analytic and bounded in \(C^-\) away from the possible zeros of \(n_{11,22}(s)(k)\) and of order \(O(1/k)\) as \(k \to \infty\), and the column vectors \([c(t, k)]_l, l = 3, 4\) are analytic and bounded in \(C^+\) away from the possible zeros of \(n_{33,44}(s)(k)\) and of order \(O(1/k)\) as \(k \to \infty\).

(b) Asymptotic behaviors of eigenfunctions

It follows from Eq. (7) that we have the asymptotics of eigenfunctions \(\{\mu_j\}^3\) as \(k \to \infty\)

\[
\mu_j(x, t, k) = 1 + \sum_{s=1}^{2} \frac{1}{k^s} \left( \begin{array}{cccc}
\mu_{j,11}^{(s)} & \mu_{j,12}^{(s)} & \mu_{j,13}^{(s)} & \mu_{j,14}^{(s)} \\
\mu_{j,21}^{(s)} & \mu_{j,22}^{(s)} & \mu_{j,23}^{(s)} & \mu_{j,24}^{(s)} \\
\mu_{j,31}^{(s)} & \mu_{j,32}^{(s)} & \mu_{j,33}^{(s)} & \mu_{j,34}^{(s)} \\
\mu_{j,41}^{(s)} & \mu_{j,42}^{(s)} & \mu_{j,43}^{(s)} & \mu_{j,44}^{(s)}
\end{array} \right) + O\left(\frac{1}{k^7}\right)
\]

\[
= 1 + \frac{1}{k} \left( \begin{array}{cccc}
\int_{(x, t)}^{(x, t)} \Delta_{11}^{(1)} & \int_{(x, t)}^{(x, t)} \Delta_{12}^{(1)} & -\frac{i}{2} q_1 & -\frac{i}{2} q_0 \\
\int_{(x, t)}^{(x, t)} \Delta_{21}^{(1)} & \int_{(x, t)}^{(x, t)} \Delta_{22}^{(1)} & -\frac{i}{2} q_0 & -\frac{i}{2} q_0 \\
\frac{i\alpha}{2} q_0 & -\frac{i\alpha}{2} q_0 & f_{(x, t)}^{(x, t)} \Delta_{33}^{(1)} & f_{(x, t)}^{(x, t)} \Delta_{34}^{(1)} \\
\frac{i\beta}{2} q_0 & -\frac{i\beta}{2} q_0 & f_{(x, t)}^{(x, t)} \Delta_{43}^{(1)} & f_{(x, t)}^{(x, t)} \Delta_{44}^{(1)}
\end{array} \right)
\]

\[
+ \frac{1}{k^2} \left( \begin{array}{cccc}
\int_{(x, t)}^{(x, t)} \Delta_{11}^{(2)} & \int_{(x, t)}^{(x, t)} \Delta_{12}^{(2)} & \mu_{j,13}^{(2)} & \mu_{j,14}^{(2)} \\
\int_{(x, t)}^{(x, t)} \Delta_{21}^{(2)} & \int_{(x, t)}^{(x, t)} \Delta_{22}^{(2)} & \mu_{j,23}^{(2)} & \mu_{j,24}^{(2)} \\
\mu_{j,31}^{(2)} & \mu_{j,32}^{(2)} & f_{(x, t)}^{(x, t)} \Delta_{33}^{(2)} & f_{(x, t)}^{(x, t)} \Delta_{34}^{(2)} \\
\mu_{j,41}^{(2)} & \mu_{j,42}^{(2)} & f_{(x, t)}^{(x, t)} \Delta_{43}^{(2)} & f_{(x, t)}^{(x, t)} \Delta_{44}^{(2)}
\end{array} \right) + O\left(\frac{1}{k^7}\right),
\]
where we have introduced the following functions

\[
\begin{align*}
\Delta_{11}^{(1)} &= -\Delta_{33}^{(1)} = \frac{i\alpha}{2}(|q_1|^2 + |q_0|^2)\,dx + \frac{\alpha}{2} \sum_{j=0,1} (q_j\bar{q}_j - q_j\bar{q}_j)\,dt, \\
\Delta_{22}^{(1)} &= -\Delta_{44}^{(1)} = \frac{i\alpha}{2}(|q_{-1}|^2 + |q_0|^2)\,dx + \frac{\alpha}{2} \sum_{j=-1,0} (q_j\bar{q}_j - q_j\bar{q}_j)\,dt, \\
\Delta_{12}^{(1)} &= -\Delta_{21}^{(1)} = -\Delta_{34}^{(1)} = \frac{\mu}{2}(\beta q_1\bar{q}_0 + q_0\bar{q}_{-1})\,dx + \frac{\mu}{2}(\beta q_1\bar{q}_0 + q_0\bar{q}_{-1} - q_{-1}\bar{q}_1 - q_0\bar{q}_{-1})\,dt,
\end{align*}
\]

\[
\begin{align*}
\mu_{j,33}^{(2)} &= \frac{1}{4}q_{1x} + \frac{i\alpha}{2} \left( q_1\mu_{j,33}^{(1)} + q_0\mu_{j,43}^{(1)} \right) = \frac{1}{4}q_{1x} + \frac{1}{2i} \left[ q_1 \int_{(x,t)}^{(x,t)} \Delta_{33}^{(1)} + q_0 \int_{(x,t)}^{(x,t)} \Delta_{43}^{(1)} \right], \\
\mu_{j,44}^{(2)} &= \frac{1}{4}q_{0x} + \frac{i\alpha}{2} \left( q_1\mu_{j,34}^{(1)} + q_0\mu_{j,44}^{(1)} \right) = \frac{1}{4}q_{0x} + \frac{1}{2i} \left[ q_1 \int_{(x,t)}^{(x,t)} \Delta_{34}^{(1)} + q_0 \int_{(x,t)}^{(x,t)} \Delta_{44}^{(1)} \right], \\
\mu_{j,23}^{(2)} &= \frac{\beta}{4}q_{0x} + \frac{i\alpha}{2} \left( \beta q_0\mu_{j,33}^{(1)} + q_{-1}\mu_{j,43}^{(1)} \right) = \frac{\beta}{4}q_{0x} + \frac{1}{2i} \left[ \beta q_0 \int_{(x,t)}^{(x,t)} \Delta_{33}^{(1)} + q_{-1} \int_{(x,t)}^{(x,t)} \Delta_{43}^{(1)} \right], \\
\mu_{j,24}^{(2)} &= \frac{1}{4}q_{-1x} + \frac{i\alpha}{2} \left( \beta q_0\mu_{j,34}^{(1)} + q_{-1}\mu_{j,44}^{(1)} \right) = \frac{1}{4}q_{-1x} + \frac{1}{2i} \left[ \beta q_0 \int_{(x,t)}^{(x,t)} \Delta_{34}^{(1)} + q_{-1} \int_{(x,t)}^{(x,t)} \Delta_{44}^{(1)} \right], \\
\mu_{j,31}^{(2)} &= \frac{\alpha}{4}q_{1x} + \frac{i\alpha}{2} \left( \bar{q}_1\mu_{j,11}^{(1)} + \bar{q}_0\mu_{j,21}^{(1)} \right) = \frac{\alpha}{4}q_{1x} + \frac{1}{2i} \left[ \bar{q}_1 \int_{(x,t)}^{(x,t)} \Delta_{11}^{(1)} + \bar{q}_0 \int_{(x,t)}^{(x,t)} \Delta_{21}^{(1)} \right], \\
\mu_{j,32}^{(2)} &= \frac{\alpha\beta}{4}q_{0x} + \frac{i\alpha}{2} \left( q_1\mu_{j,12}^{(1)} + \bar{q}_0\mu_{j,22}^{(1)} \right) = \frac{\alpha\beta}{4}q_{0x} + \frac{1}{2i} \left[ q_1 \int_{(x,t)}^{(x,t)} \Delta_{12}^{(1)} + \bar{q}_0 \int_{(x,t)}^{(x,t)} \Delta_{22}^{(1)} \right], \\
\mu_{j,41}^{(2)} &= \frac{\alpha}{4}\bar{q}_{0x} + \frac{i\alpha}{2} \left( q_0\mu_{j,11}^{(1)} + \bar{q}_{-1}\mu_{j,21}^{(1)} \right) = \frac{\alpha}{4}\bar{q}_{0x} + \frac{1}{2i} \left[ q_0 \int_{(x,t)}^{(x,t)} \Delta_{11}^{(1)} + \bar{q}_{-1} \int_{(x,t)}^{(x,t)} \Delta_{21}^{(1)} \right], \\
\mu_{j,42}^{(2)} &= \frac{\alpha}{4}\bar{q}_{-1x} + \frac{i\alpha}{2} \left( q_0\mu_{j,12}^{(1)} + \bar{q}_{-1}\mu_{j,22}^{(1)} \right) = \frac{\alpha}{4}\bar{q}_{-1x} + \frac{1}{2i} \left[ q_0 \int_{(x,t)}^{(x,t)} \Delta_{12}^{(1)} + \bar{q}_{-1} \int_{(x,t)}^{(x,t)} \Delta_{22}^{(1)} \right],
\end{align*}
\]
\[ \Delta_{11}^{(2)} = \left\{ \frac{\alpha}{4}(q_1 \bar{q}_1 x + q_0 \bar{q}_0 x) + \frac{i \alpha}{2} \left[ (|q_1|^2 + |q_0|^2) \mu_{j,11}^{(1)} + (\beta q_1 \bar{q}_0 + q_0 \bar{q}_1 - 1) \mu_{j,21}^{(1)} \right] \right\} dx \\
+ \left\{ \frac{\alpha}{4}(q_1 \bar{q}_1 t + q_0 \bar{q}_0) + \frac{i \alpha}{4}(q_1 x \bar{q}_1 x + q_0 x \bar{q}_0 x) - \frac{i}{4}(|q_1|^2 + |q_0|^2)^2 \\
+ (\beta q_1 \bar{q}_0 + q_0 \bar{q}_1 - 1)(\beta q_0 \bar{q}_1 + q_1 \bar{q}_0) + \frac{\alpha}{2}(q_1 \bar{q}_1 x - q_1 x \bar{q}_1 + q_0 \bar{q}_0 x - q_0 x \bar{q}_0) \mu_{j,11}^{(1)} \\
+ \frac{\alpha}{2}(\beta q_1 \bar{q}_0 x - \beta q_1 x \bar{q}_0 + q_0 \bar{q}_1 - 1 - q_0 x \bar{q}_1 - 1) \mu_{j,21}^{(1)} \right\} dt, \tag{54} \]

\[ \Delta_{12}^{(2)} = \left\{ \frac{\alpha}{4}(\beta q_1 \bar{q}_0 x + q_0 \bar{q}_1 - 1) + \frac{i \alpha}{2} \left[ (|q_1|^2 + |q_0|^2) \mu_{j,12}^{(1)} + (\beta q_1 \bar{q}_0 + q_0 \bar{q}_1 - 1) \mu_{j,22}^{(1)} \right] \right\} dx \\
+ \left\{ \frac{\alpha}{4}(\beta q_1 \bar{q}_0 x + q_0 \bar{q}_1 - 1) - \frac{i}{4}(\beta q_1 \bar{q}_0 + q_0 \bar{q}_1 - 1)(|q_1|^2 + 2|q_0|^2 + |q_1|^2) \\
+ \frac{i \alpha}{4}(\beta q_1 x \bar{q}_0 x + q_0 x \bar{q}_1 - 1) + \frac{\alpha}{2}(q_1 \bar{q}_1 x - q_1 x \bar{q}_1 + q_0 \bar{q}_0 x - q_0 x \bar{q}_0) \mu_{j,12}^{(1)} \\
+ \frac{\alpha}{2}(\beta q_1 \bar{q}_0 x - \beta q_1 x \bar{q}_0 + q_0 \bar{q}_1 - 1 - q_0 x \bar{q}_1 - 1) \mu_{j,22}^{(1)} \right\} dt, \tag{55} \]

\[ \Delta_{21}^{(2)} = \left\{ \frac{\alpha}{4}(\beta q_0 \bar{q}_1 x + q_1 \bar{q}_0 x) + \frac{i \alpha}{2} \left[ (|q_0|^2 + |q_1|^2) \mu_{j,11}^{(1)} + (|q_1|^2 + |q_0|^2) \mu_{j,21}^{(1)} \right] \right\} dx \\
+ \left\{ \frac{\alpha}{4}(\beta q_0 \bar{q}_1 t + q_0 \bar{q}_1) - \frac{i}{4}(\beta q_0 \bar{q}_1 + q_0 \bar{q}_1)(|q_1|^2 + 2|q_0|^2 + |q_1|^2) \\
+ \frac{i \alpha}{4}(\beta q_0 x \bar{q}_1 x + q_0 x \bar{q}_1 - 1) + \frac{\alpha}{2}(q_0 \bar{q}_1 x - q_0 x \bar{q}_1 + q_0 \bar{q}_0 x - q_0 x \bar{q}_0) \mu_{j,21}^{(1)} \\
+ \frac{\alpha}{2}(\beta q_0 \bar{q}_1 x - \beta q_0 x \bar{q}_1 + q_0 \bar{q}_1 - 1 - q_0 x \bar{q}_1 - 1) \mu_{j,11}^{(1)} \right\} dt, \tag{56} \]

\[ \Delta_{22}^{(2)} = \left\{ \frac{\alpha}{4}(q_0 \bar{q}_1 x + q_0 \bar{q}_0 x) + \frac{i \alpha}{2} \left[ (|q_0|^2 + |q_1|^2) \mu_{j,11}^{(1)} + (|q_1|^2 + |q_0|^2) \mu_{j,22}^{(1)} \right] \right\} dx \\
+ \left\{ \frac{\alpha}{4}(q_0 \bar{q}_1 t + q_0 \bar{q}_1) + \frac{i \alpha}{4}(q_0 x \bar{q}_1 x + q_0 x \bar{q}_0 x) - \frac{i}{4}|q_0|^2 + |q_1|^2)^2 \\
+ (\beta q_0 \bar{q}_1 + q_0 \bar{q}_0)(\beta q_1 \bar{q}_0 + q_0 \bar{q}_1 - 1) + \frac{\alpha}{2}(q_1 \bar{q}_1 x - q_1 x \bar{q}_1 + q_0 \bar{q}_0 x - q_0 x \bar{q}_0) \mu_{j,22}^{(1)} \\
+ \frac{\alpha}{2}(\beta q_0 \bar{q}_1 x - \beta q_0 x \bar{q}_1 + q_0 \bar{q}_1 - 1 - q_0 x \bar{q}_1 - 1) \mu_{j,12}^{(1)} \right\} dt. \tag{57} \]
\[
\Delta^{(2)}_{33} = \left\{ \frac{\alpha}{4} (q_{1x} \bar{q}_1 + q_{0x} \bar{q}_0) - \frac{i \alpha}{2} \left[ (|q_1|^2 + |q_0|^2) \mu_{j,33}^{(1)} + (\beta q_{-1} \bar{q}_0 + q_0 \bar{q}_1) \mu_{j,43}^{(1)} \right] \right\} dx \\
+ \left\{ \frac{\alpha}{4} (q_{1x} \bar{q}_1 + q_{0x} \bar{q}_0) - \frac{i \alpha}{4} (q_{1x} \bar{q}_{1x} + q_{0x} \bar{q}_{0x}) + i \frac{i}{4} (|q_1|^2 + |q_0|^2)^2 \\
+ (\beta q_{-1} \bar{q}_0 + q_0 \bar{q}_1)(\beta q_{0} \bar{q}_{-1} + q_1 \bar{q}_0) + \frac{\alpha}{2} (q_{1x} \bar{q}_1 - q_{1} \bar{q}_{1x} + q_{0x} \bar{q}_{x} - q_0 \bar{q}_{0x}) \mu_{j,33}^{(1)} \\
+ \frac{\alpha}{2} (\beta q_{-1} \bar{q}_0 - \beta q_{-1} \bar{q}_{0x} + q_{0x} \bar{q}_1 - q_0 \bar{q}_{1x}) \mu_{j,44}^{(1)} \right\} dt,
\]
\[
\Delta^{(2)}_{34} = \left\{ \frac{\alpha}{4} (\beta q_{-1} \bar{q}_0 + q_0 \bar{q}_{1x}) - \frac{i \alpha}{2} \left[ (|q_1|^2 + |q_0|^2) \mu_{j,34}^{(1)} + (\beta q_{-1} \bar{q}_0 + q_0 \bar{q}_1) \mu_{j,44}^{(1)} \right] \right\} dx \\
+ \left\{ \frac{\alpha}{4} (\beta q_{-1} \bar{q}_0 + q_0 \bar{q}_{1x}) + \frac{i}{4} (\beta q_{-1} \bar{q}_0 + q_0 \bar{q}_1)(|q_1|^2 + |q_0|^2)^2 \\
- \frac{i \alpha}{4} (q_{-1} \bar{q}_{0x} + q_{0x} \bar{q}_{x}) + \frac{\alpha}{2} (q_{1x} \bar{q}_1 - q_{1} \bar{q}_{1x} + q_{0x} \bar{q}_{x} - q_0 \bar{q}_{0x}) \mu_{j,34}^{(1)} \\
+ \frac{\alpha}{2} (\beta q_{-1} \bar{q}_0 - \beta q_{-1} \bar{q}_{0x} + q_{0x} \bar{q}_1 - q_0 \bar{q}_{1x}) \mu_{j,44}^{(1)} \right\} dt,
\]
\[
\Delta^{(2)}_{43} = \left\{ \frac{\alpha}{4} (\beta q_{0x} \bar{q}_1 + q_1 \bar{q}_0) - \frac{i \alpha}{2} \left[ (\beta q_{00} \bar{q}_{-1} + q_1 \bar{q}_0) \mu_{j,33}^{(1)} + (|q_{-1}|^2 + |q_0|^2) \mu_{j,43}^{(1)} \right] \right\} dx \\
+ \left\{ \frac{\alpha}{4} (\beta q_{0x} \bar{q}_1 + q_1 \bar{q}_0) + \frac{i}{4} (\beta q_{00} \bar{q}_{-1} + q_1 \bar{q}_0)(|q_{-1}|^2 + |q_0|^2)^2 \\
- \frac{i \alpha}{4} (\beta q_{0x} \bar{q}_{1x} + q_{1x} \bar{q}_{0x}) + \frac{\alpha}{2} (q_{-1} \bar{q}_{1} - q_{-1} \bar{q}_{1x} + q_{0x} \bar{q}_{x} - q_0 \bar{q}_{0x}) \mu_{j,33}^{(1)} \\
+ \frac{\alpha}{2} (\beta q_{0x} \bar{q}_{-1} - \beta q_{00} \bar{q}_{-1} + q_{1x} \bar{q}_0 - q_1 \bar{q}_{0x}) \mu_{j,43}^{(1)} \right\} dt,
\]
\[
\Delta^{(2)}_{44} = \left\{ \frac{\alpha}{4} (q_{-1} \bar{q}_{-1} + q_{0x} \bar{q}_0) - \frac{i \alpha}{2} \left[ (\beta q_{00} \bar{q}_{-1} + q_1 \bar{q}_0) \mu_{j,33}^{(1)} + (|q_{-1}|^2 + |q_0|^2) \mu_{j,44}^{(1)} \right] \right\} dx \\
+ \left\{ \frac{\alpha}{4} (q_{-1} \bar{q}_{-1} + q_{0x} \bar{q}_0) + \frac{i}{4} (\beta q_{00} \bar{q}_{-1} + q_1 \bar{q}_0)(|q_{-1}|^2 + |q_0|^2)^2 \\
- \frac{i \alpha}{4} (q_{-1} \bar{q}_{1x} + q_{0x} \bar{q}_{0x}) + \frac{\alpha}{2} (q_{-1} \bar{q}_{-1} - q_{-1} \bar{q}_{-1x} + q_{0x} \bar{q}_{x} - q_0 \bar{q}_{0x}) \mu_{j,44}^{(1)} \\
+ \frac{\alpha}{2} (\beta q_{0x} \bar{q}_{-1} - \beta q_{00} \bar{q}_{-1} + q_{1x} \bar{q}_0 - q_1 \bar{q}_{0x}) \mu_{j,44}^{(1)} \right\} dt,
\]

where the functions \{ \mu_{j,l}^{(i)} = \mu_{j,l}^{(i)}(x,t) \}_{i=1,2} \text{ are independent of } k.

We define the matrix-valued function \Psi(t,k) = (\Psi_{ij}(t,k))_{4 \times 4} as

\[
\mu_2(0,t,k) = \Psi(t,k) = I + \sum_{s=1}^{2} \frac{1}{k^s} \begin{pmatrix}
\Psi^{(s)}_{11}(t) & \Psi^{(s)}_{12}(t) & \Psi^{(s)}_{13}(t) & \Psi^{(s)}_{14}(t) \\
\Psi^{(s)}_{21}(t) & \Psi^{(s)}_{22}(t) & \Psi^{(s)}_{23}(t) & \Psi^{(s)}_{24}(t) \\
\Psi^{(s)}_{31}(t) & \Psi^{(s)}_{32}(t) & \Psi^{(s)}_{33}(t) & \Psi^{(s)}_{34}(t) \\
\Psi^{(s)}_{41}(t) & \Psi^{(s)}_{42}(t) & \Psi^{(s)}_{43}(t) & \Psi^{(s)}_{44}(t)
\end{pmatrix} + O\left(\frac{1}{k^3}\right),
\]
By using the asymptotic of Eq. (53) and the boundary data at $x = 0$, we find

$$
\begin{align*}
\Psi^{(1)}_{13} (t) &= - \frac{i}{2} u_{01} (t), \quad \Psi^{(1)}_{14} (t) = \beta \Psi^{(1)}_{23} (t) = - \frac{i}{2} u_{00} (t), \quad \Psi^{(1)}_{24} (t) = - \frac{i}{2} u_{0-1} (t), \\
\Psi^{(2)}_{13} (t) &= \frac{1}{4} u_{11} (t) - \frac{i}{2} \left[ u_{01} (t) \Psi^{(1)}_{33} (t) + u_{00} (t) \Psi^{(1)}_{43} (t) \right], \\
\Psi^{(2)}_{14} (t) &= \frac{1}{4} u_{10} (t) - \frac{i}{2} \left[ u_{01} (t) \Psi^{(1)}_{34} (t) + u_{00} (t) \Psi^{(1)}_{44} (t) \right], \\
\Psi^{(2)}_{23} (t) &= \beta u_{01} (t) - \frac{i}{2} \left[ \beta u_{00} (t) \Psi^{(1)}_{33} (t) + u_{0-1} (t) \Psi^{(1)}_{43} (t) \right], \\
\Psi^{(2)}_{24} (t) &= \frac{1}{4} u_{1-1} (t) - \frac{i}{2} \left[ \beta u_{00} (t) \Psi^{(1)}_{34} (t) + u_{0-1} (t) \Psi^{(1)}_{44} (t) \right], \\
\Psi^{(3)}_{33} (t) &= \frac{\alpha}{2} \int_0^t \sum_{j=0,1} \left[ \bar{u}_{0j} (t) u_{1j} (t) - u_{0j} (t) \bar{u}_{1j} (t) \right] dt, \\
\Psi^{(3)}_{44} (t) &= \frac{\alpha}{2} \int_0^t \sum_{j=-1,0} \left[ \bar{u}_{0j} (t) u_{1j} (t) - u_{0j} (t) \bar{u}_{1j} (t) \right] dt, \\
\Psi^{(3)}_{34} (t) &= \alpha \int_0^t \left[ \beta u_{11} (t) \bar{u}_{00} (t) - \beta u_{01} (t) \bar{u}_{10} (t) + u_{10} (t) \bar{u}_{0-1} (t) - u_{00} (t) \bar{u}_{1-1} (t) \right] dt, \\
\Psi^{(3)}_{43} (t) &= \alpha \int_0^t \left[ \beta u_{10} (t) \bar{u}_{01} (t) - \beta u_{00} (t) \bar{u}_{11} (t) + u_{1-1} (t) \bar{u}_{00} (t) - u_{0-1} (t) \bar{u}_{10} (t) \right] dt,
\end{align*}
$$

Thus we have the the Dirichlet-Neumann boundary data at $x = 0$: 

$$
\begin{align*}
u_{01} (t) &= 2i \Psi^{(1)}_{13} (t), \quad u_{00} (t) = 2i \Psi^{(1)}_{14} (t) = 2i \beta \Psi^{(1)}_{23} (t), \quad u_{0-1} (t) = 2i \Psi^{(1)}_{24} (t), \\
u_{11} (t) &= 4 \Psi^{(2)}_{13} (t) + 2i \left[ u_{01} (t) \Psi^{(1)}_{33} (t) + u_{00} (t) \Psi^{(1)}_{43} (t) \right], \\
u_{1-1} (t) &= 4 \Psi^{(2)}_{24} (t) + 2i \beta \left[ u_{00} (t) \Psi^{(1)}_{34} (t) + u_{0-1} (t) \Psi^{(1)}_{44} (t) \right], \\
u_{10} (t) &= 4 \Psi^{(2)}_{14} (t) + 2i \left[ u_{01} (t) \Psi^{(1)}_{34} (t) + u_{00} (t) \Psi^{(1)}_{44} (t) \right] \\
&= 4 \beta \Psi^{(2)}_{23} (t) + 2i \beta \left[ u_{00} (t) \Psi^{(1)}_{33} (t) + u_{0-1} (t) \Psi^{(1)}_{43} (t) \right],
\end{align*}
$$

For the vanishing initial values, it follows from Eqs. (52a) and (52b) that we have the following asymptotic of $c_{24} (t, k)$ and $c_{13} (t, k)$, $j = 3, 4$.

**Proposition 4.1.** The global relation (51) implies that the large $k$ behavior of $c_{13} (t, k)$, $j = 3, 4$ and $c_{24} (t, k)$ is of the form

$$
\begin{align*}
c_{13} (t, k) &= \frac{\Psi^{(1)}_{13}}{k} + \frac{\Psi^{(2)}_{13}}{k^2} + O \left( \frac{1}{k^3} \right), \\
c_{14} (t, k) &= \frac{\Psi^{(1)}_{14}}{k} + \frac{\Psi^{(2)}_{14}}{k^2} + O \left( \frac{1}{k^3} \right), \\
c_{24} (t, k) &= \frac{\Psi^{(1)}_{24}}{k} + \frac{\Psi^{(2)}_{24}}{k^2} + O \left( \frac{1}{k^3} \right),
\end{align*}
$$
**Proof.** The global relation (51) can be written as

\[
\begin{align*}
c_{13}(t, k) &= [\Psi_{11}(t, k)s_{13} + \Psi_{12}(t, k)s_{23}]e^{-4ik^2t} + \Psi_{13}(t, k)s_{33} + \Psi_{14}(t, k)s_{43}, \\
c_{14}(t, k) &= [\Psi_{11}(t, k)s_{14} + \Psi_{12}(t, k)s_{24}]e^{-4ik^2t} + \Psi_{13}(t, k)s_{34} + \Psi_{14}(t, k)s_{44}, \\
c_{24}(t, k) &= [\Psi_{21}(t, k)s_{14} + \Psi_{22}(t, k)s_{24}]e^{-4ik^2t} + \Psi_{23}(t, k)s_{34} + \Psi_{24}(t, k)s_{44}.
\end{align*}
\]

(66a) (66b) (66c)

According to the asymptotics (53), we have

\[
\begin{pmatrix}
s_{13} \\
s_{23} \\
s_{33} \\
s_{43}
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{2ik} \begin{pmatrix} q_1(0, 0) \\ \beta q_0(0, 0) \\ 2i \int_{(\infty, 0)} (\Delta_{33}^{(1)}(0, 0)) \\ 2i \int_{(\infty, 0)} (\Delta_{44}^{(1)}(0, 0)) \end{pmatrix} + O\left(\frac{1}{k^2}\right),
\]

(67)

and

\[
\begin{pmatrix}
s_{14} \\
s_{24} \\
s_{34} \\
s_{44}
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{2ik} \begin{pmatrix} q_1(0, 0) \\ \beta q_0(0, 0) \\ 2i \int_{(\infty, 0)} (\Delta_{34}^{(1)}(0, 0)) \\ 2i \int_{(\infty, 0)} (\Delta_{44}^{(1)}(0, 0)) \end{pmatrix} + O\left(\frac{1}{k^2}\right),
\]

(68)

Recalling the time-part of the Lax pair (7)

\[
\mu_t + 2ik^2[\sigma_4, \mu] = V(x, t, k)\mu,
\]

(69)

It follows from the first column of Eq. (69) with \(\mu = \mu_2(0, t, k) = \Psi(t, k)\) that we have

\[
\begin{align*}
\Psi_{11}(t, k) &= 2k(u_0\Psi_{31} + u_0\Psi_{41}) + i(u_{11}\Psi_{31} + u_{10}\Psi_{41}) \\
&\quad - i\alpha([|u_0|^2 + |u_0|^2])\Psi_{11} + (\beta u_{01}\bar{u}_{02} + u_{00}\bar{u}_{00-1})\Psi_{21}, \\
\Psi_{21}(t, k) &= 2k(\beta u_{00}\Psi_{31} + u_{0-1}\Psi_{41}) + i(\beta u_{10}\Psi_{31} + u_{1-1}\Psi_{41}) \\
&\quad - i\alpha([\beta u_{00}\bar{u}_{01} + u_{0-1}\bar{u}_{00}]\Psi_{11} + (|u_{0-1}|^2 + |u_{00}|^2)\Psi_{21}], \\
\Psi_{31}(t, k) &= 4ik^2\Psi_{31} + 2ak(u_0\Psi_{11} + \beta u_0\Psi_{21}) - i\alpha(\bar{u}_{11}\Psi_{11} + \beta \bar{u}_{10}\Psi_{21}) \\
&\quad + i\alpha([|u_0|^2 + |u_0|^2])\Psi_{31} + (\beta u_{0-1}\bar{u}_{00} + u_{00}\bar{u}_{01})\Psi_{41}, \\
\Psi_{41}(t, k) &= 4ik^2\Psi_{41} + 2ak(\bar{u}_0\Psi_{11} + \bar{u}_{0-1}\Psi_{21}) - i\alpha(\bar{u}_{10}\Psi_{11} + \bar{u}_{1-1}\Psi_{21}) \\
&\quad + i\alpha([\beta u_{00}\bar{u}_{0-1} + u_{01}\bar{u}_{00}]\Psi_{31} + (|u_{0-1}|^2 + |u_{00}|^2)\Psi_{41}],
\end{align*}
\]

(70)
The second column of Eq. (69) with $\mu = \mu_2(0, t, k) = \Psi(t, k)$ yields

$$
\begin{align*}
\Psi_{12,t}(t, k) &= 2k(u_{01}\Psi_{32} + u_{00}\Psi_{42}) + i(u_{11}\Psi_{32} + u_{10}\Psi_{42}) \\
&\quad - i\alpha([u_{01}]^2 + [u_{00}]^2)\Psi_{12} + (\beta u_{01}\bar{u}_{02} + u_{00}\bar{u}_{0-1})\Psi_{22}, \\
\Psi_{22,t}(t, k) &= 2k(\beta u_{00}\Psi_{32} + u_{0-1}\Psi_{42}) + i(\beta u_{10}\Psi_{32} + u_{1-1}\Psi_{42}) \\
&\quad - i\alpha([u_{00}]^2 + [u_{00}]^2)\Psi_{32} + ([u_{0-1}]^2 + [u_{00}]^2)\Psi_{22}, \\
\Psi_{32,t}(t, k) &= 4ik^2\Psi_{32} + 2ak(\bar{u}_{01}\Psi_{12} + \beta \bar{u}_{00}\Psi_{22}) - i\alpha(\bar{u}_{11}\Psi_{12} + \beta \bar{u}_{10}\Psi_{22}) \\
&\quad + i\alpha([u_{01}]^2 + [u_{00}]^2)\Psi_{32} + (\beta u_{0-1}\bar{u}_{00} + u_{00}\bar{u}_{01})\Psi_{42}, \\
\Psi_{42,t}(t, k) &= 4ik^2\Psi_{42} + 2ak(\bar{u}_{00}\Psi_{12} + \bar{u}_{0-1}\Psi_{22}) - i\alpha(\bar{u}_{10}\Psi_{12} + \bar{u}_{1-1}\Psi_{22}) \\
&\quad + i\alpha((\beta u_{00}\bar{u}_{0-1} + u_{01}\bar{u}_{00})\Psi_{32} + ([u_{0-1}]^2 + [u_{00}]^2)\Psi_{42},
\end{align*}
$$

(71)

The third column of Eq. (69) with $\mu = \mu_2(0, t, k) = \Psi(t, k)$ yields

$$
\begin{align*}
\Psi_{13,t}(t, k) &= -4ik^2\Psi_{13} + 2k(u_{01}\Psi_{33} + u_{00}\Psi_{43}) + i(u_{11}\Psi_{33} + u_{10}\Psi_{43}) \\
&\quad - i\alpha([u_{01}]^2 + [u_{00}]^2)\Psi_{13} + (\beta u_{01}\bar{u}_{02} + u_{00}\bar{u}_{0-1})\Psi_{23}, \\
\Psi_{23,t}(t, k) &= -4ik^2\Psi_{23} + 2k(\beta u_{00}\Psi_{33} + u_{0-1}\Psi_{43}) + i(\beta u_{10}\Psi_{33} + u_{1-1}\Psi_{43}) \\
&\quad - i\alpha((\beta u_{00}\bar{u}_{01} + u_{0-1}\bar{u}_{00})\Psi_{33} + ([u_{0-1}]^2 + [u_{00}]^2)\Psi_{23}, \\
\Psi_{33,t}(t, k) &= 2ak(\bar{u}_{01}\Psi_{13} + \beta \bar{u}_{00}\Psi_{23}) - i\alpha(\bar{u}_{11}\Psi_{13} + \beta \bar{u}_{10}\Psi_{23}) \\
&\quad + i\alpha([u_{01}]^2 + [u_{00}]^2)\Psi_{33} + (\beta u_{0-1}\bar{u}_{00} + u_{00}\bar{u}_{01})\Psi_{43}, \\
\Psi_{43,t}(t, k) &= 2ak(u_{00}\Psi_{13} + \bar{u}_{0-1}\Psi_{23}) - i\alpha(u_{10}\Psi_{13} + \bar{u}_{1-1}\Psi_{23}) \\
&\quad + i\alpha((\beta u_{00}\bar{u}_{0-1} + u_{01}\bar{u}_{00})\Psi_{33} + ([u_{0-1}]^2 + [u_{00}]^2)\Psi_{43},
\end{align*}
$$

(72)

and the fourth column of Eq. (69) with $\mu = \mu_2(0, t, k) = \Psi(t, k)$ yields

$$
\begin{align*}
\Psi_{14,t}(t, k) &= -4ik^2\Psi_{14} + 2k(u_{01}\Psi_{34} + u_{00}\Psi_{44}) + i(u_{11}\Psi_{34} + u_{10}\Psi_{44}) \\
&\quad - i\alpha([u_{01}]^2 + [u_{00}]^2)\Psi_{14} + (\beta u_{01}\bar{u}_{02} + u_{00}\bar{u}_{0-1})\Psi_{24}, \\
\Psi_{24,t}(t, k) &= -4ik^2\Psi_{24} + 2k(\beta u_{00}\Psi_{34} + u_{0-1}\Psi_{44}) + i(\beta u_{10}\Psi_{34} + u_{1-1}\Psi_{44}) \\
&\quad - i\alpha((\beta u_{00}\bar{u}_{01} + u_{0-1}\bar{u}_{00})\Psi_{14} + ([u_{0-1}]^2 + [u_{00}]^2)\Psi_{24}, \\
\Psi_{34,t}(t, k) &= 2ak(\bar{u}_{01}\Psi_{14} + \beta \bar{u}_{00}\Psi_{24}) - i\alpha(\bar{u}_{11}\Psi_{14} + \beta \bar{u}_{10}\Psi_{24}) \\
&\quad + i\alpha([u_{01}]^2 + [u_{00}]^2)\Psi_{34} + (\beta u_{0-1}\bar{u}_{00} + u_{00}\bar{u}_{01})\Psi_{44}, \\
\Psi_{44,t}(t, k) &= 2ak(u_{00}\Psi_{14} + \bar{u}_{0-1}\Psi_{24}) - i\alpha(u_{10}\Psi_{14} + \bar{u}_{1-1}\Psi_{24}) \\
&\quad + i\alpha((\beta u_{00}\bar{u}_{0-1} + u_{01}\bar{u}_{00})\Psi_{34} + ([u_{0-1}]^2 + [u_{00}]^2)\Psi_{44}.
\end{align*}
$$

(73)

Suppose that $\Psi_{1j}$'s, $j = 1, 2, 3, 4$ are of the form

$$
\begin{pmatrix}
\Psi_{11} \\
\Psi_{21} \\
\Psi_{31} \\
\Psi_{41}
\end{pmatrix}
= \left(\begin{array}{c}
a_{10}(t) + \frac{a_{11}(t)}{k} + \frac{a_{12}(t)}{k^2} + \cdots \\
b_{10}(t) + \frac{b_{11}(t)}{k} + \frac{b_{12}(t)}{k^2} + \cdots 
\end{array}\right) e^{4ik^2t},
$$

(74)

where the $4 \times 1$ column vector functions $a_{1j}(t), b_{1j}(t) (j = 0, 1, \ldots)$ are independent of $k.$
By substituting Eq. (74) into Eq. (70) and using the initial conditions $a_{10}(0)+b_{10}(0) = (1, 0, 0, 0)^T$, $a_{11}(0)+b_{11}(0) = (0, 0, 0, 0)^T$, we have

\[
\begin{pmatrix}
    \Psi_{11} \\
    \Psi_{21} \\
    \Psi_{31} \\
    \Psi_{41}
\end{pmatrix}
= \frac{1}{k} \begin{pmatrix}
    1 \\
    0 \\
    0 \\
    0
\end{pmatrix}
+ \frac{1}{k^2} \begin{pmatrix}
    \Psi_{11}^{(2)} \\
    \Psi_{21}^{(2)} \\
    \Psi_{31}^{(2)} \\
    \Psi_{41}^{(2)}
\end{pmatrix}
+ O \left( \frac{1}{k^3} \right) + \left[ \frac{1}{k} \begin{pmatrix}
    0 \\
    0 \\
    -\frac{\alpha}{2} \bar{u}_{01}(0) \\
    -\frac{\alpha}{2} \bar{u}_{00}(0)
\end{pmatrix} + O \left( \frac{1}{k^2} \right) \right] e^{4ik^2t}, \quad (75)
\]

Similarly, it follows from Eqs. (71)-(73) that we have the asymptotic formulae for $\Psi_{ij}, i = 1, 2, 3, 4; j = 2, 3, 4$ in the form

\[
\begin{pmatrix}
    \Psi_{12} \\
    \Psi_{22} \\
    \Psi_{32} \\
    \Psi_{42}
\end{pmatrix}
= \frac{1}{k} \begin{pmatrix}
    1 \\
    0 \\
    0 \\
    0
\end{pmatrix}
+ \frac{1}{k^2} \begin{pmatrix}
    \Psi_{12}^{(2)} \\
    \Psi_{22}^{(2)} \\
    \Psi_{32}^{(2)} \\
    \Psi_{42}^{(2)}
\end{pmatrix}
+ O \left( \frac{1}{k^3} \right) + \left[ \frac{1}{k} \begin{pmatrix}
    0 \\
    0 \\
    -\frac{i\alpha^2}{2} \bar{u}_{00}(0) \\
    -\frac{i\alpha^2}{2} \bar{u}_{0-1}(0)
\end{pmatrix} + O \left( \frac{1}{k^2} \right) \right] e^{4ik^2t}, \quad (76)
\]

\[
\begin{pmatrix}
    \Psi_{13} \\
    \Psi_{23} \\
    \Psi_{33} \\
    \Psi_{43}
\end{pmatrix}
= \frac{1}{k} \begin{pmatrix}
    0 \\
    1 \\
    0 \\
    0
\end{pmatrix}
+ \frac{1}{k^2} \begin{pmatrix}
    \Psi_{13}^{(2)} \\
    \Psi_{23}^{(2)} \\
    \Psi_{33}^{(2)} \\
    \Psi_{43}^{(2)}
\end{pmatrix}
+ O \left( \frac{1}{k^3} \right) + \left[ \frac{1}{k} \begin{pmatrix}
    \frac{i}{2} \bar{u}_{01}(0) \\
    \frac{i}{2} \bar{u}_{00}(0) \\
    0 \\
    0
\end{pmatrix} + O \left( \frac{1}{k^2} \right) \right] e^{-4ik^2t}, \quad (77)
\]

and

\[
\begin{pmatrix}
    \Psi_{14} \\
    \Psi_{24} \\
    \Psi_{34} \\
    \Psi_{44}
\end{pmatrix}
= \frac{1}{k} \begin{pmatrix}
    0 \\
    0 \\
    1 \\
    0
\end{pmatrix}
+ \frac{1}{k^2} \begin{pmatrix}
    \Psi_{14}^{(2)} \\
    \Psi_{24}^{(2)} \\
    \Psi_{34}^{(2)} \\
    \Psi_{44}^{(2)}
\end{pmatrix}
+ O \left( \frac{1}{k^3} \right) + \left[ \frac{1}{k} \begin{pmatrix}
    \frac{i}{2} \bar{u}_{00}(0) \\
    \frac{i}{2} \bar{u}_{0-1}(0) \\
    0 \\
    0
\end{pmatrix} + O \left( \frac{1}{k^2} \right) \right] e^{-4ik^2t}, \quad (78)
\]

The substitution of Eqs. (67) and (75)-(78) into Eq. (66a) yields Eq. (65a). Similarly, we can also get Eqs. (65b) and (65c). □

(c) The map between Dirichlet and Neumann problems

In the following we mainly show that the spectral functions $S(k)$ and $S_L(k)$ can be expressed in terms of the prescribed Dirichlet and Neumann boundary data and the initial data using the solution of a system of integral equations.

Define the new notations as

\[
F_{\pm}(t, k) = F(t, k) \pm F(t, -k), \quad \Sigma_{\pm}(k) = e^{2ikL} \pm e^{-2ikL}.
\]

The sign $\partial D_j, j = 1, ..., 4$ stands for the boundary of the $j$th quadrant $D_j$, oriented so that $D_j$ lies to the left of $\partial D_j$. $\partial D_3^0$ denotes the boundary contour which has not contain the zeros of $\Sigma_-(k)$ and $\partial D_1^0 = -\partial D_3^0$. 
Theorem 4.2. Let the initial data of Eq. (1) \( q_j(x, t = 0) = q_0 j(x) \), \( j = 1, 0, -1 \) be the functions of Schwartz class on the domain \( x \in [0, \infty) \) and \( 0 < t < T < \infty \). For the Dirichlet problem, the boundary data \( u_j(t), (j = 1, 0, -1) \) on the interval \( t \in [0, T] \) are sufficiently smooth and compatible with the initial data \( q_j(x), (j = 1, 0, -1) \) at the point \( (x_2, t_2) = (0, 0) \), i.e., \( u_j(0) = q_j(0), j = 1, 0, -1 \). Similarly, for the Neumann problem, the boundary data \( u_1(t), (j = 1, 0, -1) \) on the interval \( t \in [0, T] \) are sufficiently smooth and compatible with the initial data \( q_j(x), j = 1, 0, -1 \) at the origin \( (x_2, t_2) = (0, 0) \). For simplicity, let \( n_{33,44}(s)(k) \) have no zeros in the domain \( D_1 \). Then the matrix-valued spectral function \( S(k) \) is defined by

\[
S(k) = \begin{pmatrix}
  m_{11}(\Psi(T, k)) & -m_{21}(\Psi(T, k)) & m_{31}(\Psi(T, k))e^{4ik^2T} & -m_{41}(\Psi(T, k))e^{4ik^2T} \\
  -m_{12}(\Psi(T, k)) & m_{22}(\Psi(T, k)) & -m_{32}(\Psi(T, k))e^{4ik^2T} & m_{42}(\Psi(T, k))e^{4ik^2T} \\
  m_{13}(\Psi(T, k))e^{-4ik^2T} & -m_{23}(\Psi(T, k))e^{-4ik^2T} & m_{33}(\Psi(T, k)) & -m_{34}(\Psi(T, k)) \\
  -m_{14}(\Psi(T, k))e^{-4ik^2T} & m_{24}(\Psi(T, k))e^{-4ik^2T} & -m_{34}(\Psi(T, k)) & m_{44}(\Psi(T, k))
\end{pmatrix}, \quad (79)
\]

and the complex-valued functions \( \{\Psi_{ij}(t, k)\}_{i,j=1}^{4} \) have the following system of integral equations

\[
\Psi_{11}(t, k) = 1 + \int_{0}^{t} \left\{ -i\alpha([|u_{01}|^2 + |u_{00}|^2]\Psi_{11} + (\beta u_{01}\bar{u}_{02} + u_{00}\bar{u}_{01})\Psi_{21} \right\} \Psi_{31} + (2ku_{00} + i\alpha)\Psi_{41} \} \Psi_{11} \]

\[
\Psi_{21}(t, k) = \int_{0}^{t} \left\{ -i\alpha([\beta u_{00}\bar{u}_{01} + u_{01}\bar{u}_{00}]\Psi_{11} + (|u_{01}|^2 + |u_{00}|^2)\Psi_{21} \right\} \Psi_{31} + (2k\beta u_{00} + i\beta)\Psi_{41} \} \Psi_{21} + (2k\bar{u}_{00} + i\alpha)\Psi_{41} \} \Psi_{21} \]

\[
\Psi_{31}(t, k) = \int_{0}^{t} e^{4ik^2(t-t')} \left\{ 2\alpha k(u_{00}\bar{u}_{01} + u_{01}\bar{u}_{00})\Psi_{11} - i\alpha(u_{01}\bar{u}_{01} + \beta u_{00}\Psi_{21}) \right\} \Psi_{31} + (\beta u_{00}\bar{u}_{01} + u_{01}\bar{u}_{00})\Psi_{41} \} \Psi_{31} \]

\[
\Psi_{41}(t, k) = \int_{0}^{t} e^{4ik^2(t-t')} \left\{ [\beta u_{00}\bar{u}_{01} + u_{01}\bar{u}_{00}]\Psi_{11} + (u_{01}|^2 + |u_{00}|^2)\Psi_{21} \right\} \Psi_{31} + (\beta u_{00}\bar{u}_{01} + u_{01}\bar{u}_{00})\Psi_{41} \} \Psi_{41} \}
\]

\[
\Psi_{12}(t, k) = \int_{0}^{t} \left\{ -i\alpha([|u_{01}|^2 + |u_{00}|^2]\Psi_{12} + (\beta u_{01}\bar{u}_{02} + u_{00}\bar{u}_{01})\Psi_{22} \right\} \Psi_{32} + (2ku_{01}\Psi_{32} + u_{00}\Psi_{42}) \} \Psi_{12} \]

\[
\Psi_{22}(t, k) = 1 + \int_{0}^{t} \left\{ -i\alpha([\beta u_{00}\bar{u}_{01} + u_{01}\bar{u}_{00}]\Psi_{12} + (|u_{01}|^2 + |u_{00}|^2)\Psi_{22} \right\} \Psi_{32} + (2k\beta u_{00}\Psi_{32} + u_{01}\Psi_{42}) \} \Psi_{22} + (2k\bar{u}_{00}\Psi_{32} + u_{01}\Psi_{42}) \} \Psi_{22} \]

\[
\Psi_{32}(t, k) = \int_{0}^{t} e^{4ik^2(t-t')} \left\{ 2\alpha k(u_{00}\bar{u}_{12} + \beta u_{00}\Psi_{22}) - i\alpha(u_{01}\bar{u}_{12} + \beta u_{10}\Psi_{22}) \right\} \Psi_{32} + (2k\bar{u}_{00}\Psi_{32} + u_{01}\Psi_{42}) \} \Psi_{32} \]

\[
\Psi_{42}(t, k) = \int_{0}^{t} e^{4ik^2(t-t')} \left\{ [\beta u_{00}\bar{u}_{12} + u_{01}\bar{u}_{00}]\Psi_{12} + (u_{01}|^2 + |u_{00}|^2)\Psi_{22} \right\} \Psi_{32} + (\beta u_{00}\bar{u}_{12} + u_{01}\bar{u}_{00})\Psi_{42} \} \Psi_{42} \}
\]
\[
\begin{align*}
\Psi_{13}(t, k) &= \int_0^t e^{-4ik^2(t-t')} \{ -i\alpha[|u_{01}|^2 + |u_{00}|^2]\Psi_{13} + (\beta u_{01} \bar{u}_{02} + u_{00}\bar{u}_{0-1})\Psi_{23} \}, \\
&\quad + 2k(u_{01}\Psi_{33} + u_{00}\Psi_{43}) + i(u_{11}\Psi_{33} + u_{10}\Psi_{43}) \} (t', k)dt', \\
\Psi_{23}(t, k) &= \int_0^t e^{-4ik^2(t-t')} \{ -i\alpha[(\beta u_{00}\bar{u}_{01} + u_{0-1}\bar{u}_{00})\Psi_{13} + (|u_{0-1}|^2 + |u_{00}|^2)\Psi_{23} \}, \\
&\quad + 2k(\beta u_{00}\Psi_{33} + u_{0-1}\Psi_{43}) + i(\beta u_{10}\Psi_{33} + u_{1-1}\Psi_{43}) \} (t', k)dt', \\
\Psi_{33}(t, k) &= 1 + \int_0^t \{ 2\alpha k(\bar{u}_{01}\Psi_{13} + \beta \bar{u}_{00}\Psi_{23}) - i\alpha(\bar{u}_{11}\Psi_{13} + \beta \bar{u}_{10}\Psi_{23}) \\
&\quad + i\alpha[(|u_{01}|^2 + |u_{00}|^2)\Psi_{33} + (\beta u_{0-1}\bar{u}_{00} + u_{00}\bar{u}_{01})\Psi_{43}] \} (t', k)dt', \\
\Psi_{43}(t, k) &= \int_0^t \{ 2\alpha k(\bar{u}_{00}\Psi_{13} + \bar{u}_{0-1}\Psi_{23}) - i\alpha(\bar{u}_{10}\Psi_{13} + \bar{u}_{1-1}\Psi_{23}) \\
&\quad + i\alpha[(\beta u_{00}\bar{u}_{0-1} + u_{01}\bar{u}_{00})\Psi_{33} + (|u_{0-1}|^2 + |u_{00}|^2)\Psi_{43}] \} (t', k)dt',
\end{align*}
\]

\begin{align*}
(82) \quad &
\end{align*}

\[
\begin{align*}
\Psi_{14}(t, k) &= \int_0^t e^{-4ik^2(t-t')} \{ -i\alpha[|u_{01}|^2 + |u_{00}|^2]\Psi_{14} + (\beta u_{01} \bar{u}_{02} + u_{00}\bar{u}_{0-1})\Psi_{24} \}, \\
&\quad + 2k(u_{01}\Psi_{34} + u_{00}\Psi_{44}) + i(u_{11}\Psi_{34} + u_{10}\Psi_{44}) \} (t', k)dt', \\
\Psi_{24}(t, k) &= \int_0^t e^{-4ik^2(t-t')} \{ -i\alpha[(\beta u_{00}\bar{u}_{01} + u_{0-1}\bar{u}_{00})\Psi_{14} + (|u_{0-1}|^2 + |u_{00}|^2)\Psi_{24} \}, \\
&\quad + 2k(\beta u_{00}\Psi_{34} + u_{0-1}\Psi_{44}) + i(\beta u_{10}\Psi_{34} + u_{1-1}\Psi_{44}) \} (t', k)dt', \\
\Psi_{34}(t, k) &= \int_0^t \{ 2\alpha k(\bar{u}_{01}\Psi_{14} + \beta \bar{u}_{00}\Psi_{24}) - i\alpha(\bar{u}_{11}\Psi_{14} + \beta \bar{u}_{10}\Psi_{24}) \\
&\quad + i\alpha[(|u_{01}|^2 + |u_{00}|^2)\Psi_{34} + (\beta u_{0-1}\bar{u}_{00} + u_{00}\bar{u}_{01})\Psi_{44}] \} (t', k)dt', \\
\Psi_{44}(t, k) &= 1 + \int_0^t \{ 2\alpha k(\bar{u}_{00}\Psi_{14} + \bar{u}_{0-1}\Psi_{24}) - i\alpha(\bar{u}_{10}\Psi_{14} + \bar{u}_{1-1}\Psi_{24}) \\
&\quad + i\alpha[(\beta u_{00}\bar{u}_{0-1} + u_{01}\bar{u}_{00})\Psi_{34} + (|u_{0-1}|^2 + |u_{00}|^2)\Psi_{44}] \} (t', k)dt',
\end{align*}
\]

\begin{align*}
(83) \quad &
\end{align*}

(i) For the known Dirichlet problem, the unknown Neumann boundary conditions \(u_{1j}(t), j = 1, 0, -1, 0 < \)
\[ t < T \text{ can be found by} \]
\[
u_{11}(t) = \int_{\partial D_3} \frac{2}{\pi} [k \Psi_{13}(t, k) - i u_{01}(t)] dk + \frac{4i}{\pi} \int_{\partial D_3} k \left\{ [\Psi_{11}(t, k) s_{13} + \Psi_{12}(t, k) s_{23}] e^{-4ik^2t} + \Psi_{13}(t, k)(s_{33} - 1) + \Psi_{14}(t, k) s_{43} \right\} dk,
\]
\[ (84a) \]
\[
u_{10}(t) = \int_{\partial D_3} \frac{2}{\pi} [k \Psi_{10}(t, k) - i \nu_{00}(t)] dk + \beta u_{00}(t) \Psi_{44}(t, k) dk + \frac{4i}{\pi} \int_{\partial D_3} k \left\{ [\Psi_{11}(t, k) s_{14} + \Psi_{12}(t, k) s_{24}] e^{-4ik^2t} + \Psi_{13}(t, k) s_{34} + \Psi_{14}(t, k)(s_{44} - 1) \right\} dk,
\]
\[ (84b) \]
\[
u_{11}(t) = \int_{\partial D_3} \frac{2}{\pi} [k \Psi_{24}(t, k) - i \nu_{01}(t)] dk + \beta u_{01}(t) \Psi_{44}(t, k) dk + \frac{4i}{\pi} \int_{\partial D_3} k \left\{ [\Psi_{21}(t, k) s_{14} + \Psi_{22}(t, k) s_{24}] e^{-4ik^2t} + \Psi_{23}(t, k) s_{34} + \Psi_{24}(t, k)(s_{44} - 1) \right\} dk,
\]
\[ (84c) \]
(ii) For the known Neumann problem, the unknown Dirichlet boundary conditions \( u_0(t) \), \( j = 1, 0, -1, 0 < t < T \) can be found by
\[
u_{01}(t) = \frac{1}{\pi} \int_{\partial D_3} \Psi_{13}(t, k) dk - \frac{2}{\pi} \int_{\partial D_3} \left\{ [\Psi_{11}(t, k) s_{13} + \Psi_{12}(t, k) s_{23}] e^{-4ik^2t}
\]
\[ + \Psi_{13}(t, k)(s_{33} - 1) + \Psi_{14}(t, k) s_{43} \right\} dk,
\]
\[ (85a) \]
\[
u_{00}(t) = \frac{1}{\pi} \int_{\partial D_3} \Psi_{14}(t, k) dk - \frac{2}{\pi} \int_{\partial D_3} \left\{ [\Psi_{11}(t, k) s_{14} + \Psi_{12}(t, k) s_{24}] e^{-4ik^2t}
\]
\[ + \Psi_{13}(t, k) s_{34} + \Psi_{14}(t, k)(s_{44} - 1) \right\} dk,
\]
\[ (85b) \]
\[
u_{11}(t) = \frac{1}{\pi} \int_{\partial D_3} \Psi_{24}(t, k) dk - \frac{2}{\pi} \int_{\partial D_3} \left\{ [\Psi_{21}(t, k) s_{14} + \Psi_{22}(t, k) s_{24}] e^{-4ik^2t}
\]
\[ + \Psi_{23}(t, k) s_{34} + \Psi_{24}(t, k)(s_{44} - 1) \right\} dk,
\]
\[ (85c) \]
where \( s_{ij} = s_{ij}(k), i, j = 1, 2, 3, 4. \)

**Proof.** We can show Eq. (79) by means of Eq. (24), that is,
\[
S(k) = e^{-2ik^2T \tau} \mu_2^{-1}(0, T, k) = e^{-2ik^2T \tau} \left( \mu_2(T, k) \right)^T = e^{-2ik^2T \tau} \left( \Psi^T(T, k) \right)^T,
\]
Moreover, Eqs. (80)-(83) for \( \Psi_{ij}(t, k), i, j = 1, 2, 3, 4 \) can be obtained by using the Volterra integral equations of \( \mu_2(0, t, k) \).

(i) In the following we show Eqs. (84a)-(84c). Applying the Cauchy’s theorem to Eq. (62), we have
\[
-\frac{i}{2} \Psi^{(1)}_{33}(t) = \int_{\partial D_2} [\Psi_{33}(t, k) - 1] dk = \int_{\partial D_3} [\Psi_{33}(t, k) - 1] dk,
\]
\[ (87) \]
\[
-\frac{i}{2} \Psi^{(1)}_{43}(t) = \int_{\partial D_2} \Psi_{43}(t, k) dk = \int_{\partial D_3} \Psi_{43}(t, k) dk,
\]
\[
-\frac{i}{2} \Psi^{(2)}_{13}(t) = \int_{\partial D_2} \left[ k \Psi_{13}(t, k) + \frac{i}{2} u_{01}(t) \right] dk = \int_{\partial D_3} \left[ k \Psi_{13}(t, k) + \frac{i}{2} u_{01}(t) \right] dk,
\]
From Eq. (87), we further find

\[ i\pi \Psi_{33}^{(1)}(t) = -\left( \int_{\partial D_2} + \int_{\partial D_1} \right) [\Psi_{33}(t, k) - 1]dk = \left( \int_{\partial D_1} + \int_{\partial D_3} \right) [\Psi_{33}(t, k) - 1]dk \]

\[ = \int_{\partial D_2} [\Psi_{33}(t, k) - 1]dk - \int_{\partial D_3} [\Psi_{33}(t, -k) - 1]dk = \int_{\partial D_2} \Psi_{33}(t, k)dk, \]

\[ i\pi \Psi_{43}^{(1)}(t) = \int_{\partial D_3} \Psi_{43}(-t, k)dk, \]

\[ i\pi \Psi_{13}^{(2)}(t) = \left( \int_{\partial D_1} - \int_{\partial D_3} \right) \left[ k\Psi_{13}(t, k) + \frac{i}{2}u_{01}(t) \right]dk + C_1(t) = \int_{\partial D_3} [k\Psi_{13}(-t, -k) - i u_{01}(t)]dk + C_1(t) \]

where we have introduced the function \( C_1(t) \) in the form

\[ C_1(t) = 2 \int_{\partial D_3} \left[ k\Psi_{13}(t, k) + \frac{i}{2}u_{01}(t) \right]dk, \]

We use the global relation (66a), the Cauchy’s theorem and asymptotic (65a) to further reduce \( C_1(t) \) to be

\[ C_1(t) = 2 \int_{\partial D_3} \left[ k\Psi_{13}(t, k) + \frac{i}{2}u_{01}(t) \right]dk - 2 \int_{\partial D_3} k [\Psi_{13}(t, k) - \Psi_{13}(t, k)]dk, \]

\[ = -i\pi \Psi_{13}^{(2)} - 2 \int_{\partial D_3} k \left\{ [\Psi_{11}(t, k)s_{13}(k) + \Psi_{12}(t, k)s_{23}(k)]e^{-4ik^2t} + \Psi_{13}(t, k)(s_{33}(k) - 1) + \Psi_{14}(t, k)s_{43}(k) \right\}dk, \]

It follows from Eqs. (89) and (90) that we have

\[ 2i\pi \Psi_{13}^{(2)}(t) = \int_{\partial D_3} \left[ k\Psi_{13}(-t, -k) - i u_{01}(t) \right]dk - 2 \int_{\partial D_3} k \left\{ [\Psi_{11}(t, k)s_{13}(k) + \Psi_{12}(t, k)s_{23}(k)]e^{-4ik^2t} \right. \]

\[ + \Psi_{13}(t, k)(s_{33}(k) - 1) + \Psi_{14}(t, k)s_{43}(k) \right\}dk, \]

Thus substituting Eqs. (88a), (88b) and (91) into the fourth one of system (64), we can get Eq. (84a). Similarly, we can also show that Eqs. (84b) and (84c) hold.

(ii) We now derive the Dirichlet boundary value conditions (85a)-(85c) at \( x = 0 \) from the given Neumann boundary value problems. It follows from the first one of Eq. (64) that \( u_{01}(t) \) can be expressed by means of \( \Psi_{13}^{(1)} \). Applying the Cauchy’s theorem to Eq. (62) yields

\[ i\pi \Psi_{13}^{(1)}(t) = \left( \int_{\partial D_1} + \int_{\partial D_3} \right) \Psi_{13}(t, k)dk = \left( \int_{\partial D_1} - \int_{\partial D_3} \right) \Psi_{13}(t, k)dk + C_2(t) \]

\[ = \int_{\partial D_1} \Psi_{13}(t, k)dk + C_2(t) = \int_{\partial D_3} \Psi_{13}(t, -k)dk + C_2(t), \]

where we have introduced the function \( C_2(t) \) in the form

\[ C_2(t) = 2 \int_{\partial D_3} \Psi_{13}(t, k)dk, \]
We use the global relation (66a), the Cauchy’s theorem and asymptotics (65a) to further reduce $C_2(t)$ to be

$$C_2(t) = 2 \int_{\partial D_3} c_{13}(t,k) dk - 2 \int_{\partial D_3} [c_{13}(t,k) - \Psi_{13}(t,k)] dk,$$

$$= -i\pi \Psi_{13}^{(1)} - 2 \int_{\partial D_3} \left\{ [\Psi_{11}(t,k)s_{13}(k) + \Psi_{12}(t,k)s_{23}(k)]e^{-4ik^2t} + \Psi_{13}(t,k)(s_{33}(k) - 1) + \Psi_{14}(t,k)s_{43}(k) \right\} dk,$$

Eqs. (92) and (93) imply that

$$2i\pi \Psi_{13}^{(1)}(t) = \int_{\partial D_3} \Psi_{13} + (t,-k) dk - 2 \int_{\partial D_3} \left\{ [\Psi_{11}(t,k)s_{13}(k) + \Psi_{12}(t,k)s_{23}(k)]e^{-4ik^2t} + \Psi_{13}(t,k)(s_{33}(k) - 1) + \Psi_{14}(t,k)s_{43}(k) \right\} dk,$$

Thus, the substitution of Eq. (94) into the first one of Eq. (64) yields Eq. (85a). Similarly, in terms of the global relation (66b) and (66c), we can also show Eqs. (85b) and (85c) by using the second and third ones of Eq. (64) and $\Psi_{14}^{(1)}(t)$ and $\Psi_{24}^{(1)}(t)$. □

(d) Effective characterizations

Substituting the perturbed expressions of the eigenfunction and initial-boundary data

$$\Psi_{ij}(t,k) = \Psi_{ij}^{[0]} + \epsilon \Psi_{ij}^{[1]} + \epsilon^2 \Psi_{ij}^{[2]} + \cdots, \quad i, j = 1, 2, 3, 4,$$

$$u_{0j}(t) = \epsilon u_{0j}^{[1]}(t) + \epsilon^2 u_{0j}^{[2]}(t) + \cdots, \quad j = 1, 0, -1,$$

$$u_{1j}(t) = \epsilon u_{1j}^{[1]}(t) + \epsilon^2 u_{1j}^{[2]}(t) + \cdots, \quad j = 1, 0, -1,$$

where $\epsilon > 0$ is a small parameter, into Eqs. (70)-(73), we have these terms of $O(1)$ and $O(\epsilon)$ as

$$O(1) : \begin{cases} 
\Psi_{ij}^{[0]}(t,k) = 1, & j = 1, 2, 3, 4, \\
\Psi_{ij}^{[0]}(t,k) = 0, & i, j = 1, 2, 3, 4, i \neq j,
\end{cases}$$

$$O(\epsilon) : \begin{cases} 
\Psi_{11}^{[1]}(t,k) = \Psi_{12}^{[1]} = \Psi_{21}^{[1]} = \Psi_{33}^{[1]}(t,k) = \Psi_{34}^{[1]} = \Psi_{43}^{[1]} = \Psi_{44}^{[1]} = 0, \\
\Psi_{13}^{[1]}(t,k) = \int_{0}^{t} e^{-4ik^2(t-t')} \left( 2ku_{01}^{[1]} + iu_{11}^{[1]} \right) (t') dt', \\
\Psi_{14}^{[1]}(t,k) = \int_{0}^{t} e^{-4ik^2(t-t')} \left( 2ku_{00}^{[1]} + iu_{10}^{[1]} \right) (t') dt', \\
\Psi_{23}^{[1]}(t,k) = \beta \int_{0}^{t} e^{-4ik^2(t-t')} \left( 2ku_{00}^{[1]} + iu_{10}^{[1]} \right) (t') dt', \\
\Psi_{31}^{[1]}(t,k) = \alpha \int_{0}^{t} e^{4ik^2(t-t')} \left( 2ku_{01}^{[1]} - iu_{11}^{[1]} \right) (t') dt', \\
\Psi_{41}^{[1]}(t,k) = \alpha \int_{0}^{t} e^{4ik^2(t-t')} \left( 2ku_{00}^{[1]} - iu_{10}^{[1]} \right) (t') dt',
\end{cases}$$

(97)
If we assume that \( n_{33,44}(s) \) has no zeros, then we substitute the fourth one in Eq. (95) into Eqs. (84a)-(84c) to find

\[
\begin{align*}
\Psi_{13}^{[1]}(t, -k) &= -4k \int_0^t e^{-4ikt(t-t')} u_{01}^{[1]}(t') \, dt', \\
\Psi_{14}^{[1]}(t, -k) &= -4k \int_0^t e^{-4ikt(t-t')} u_{00}^{[1]}(t') \, dt', \\
\Psi_{24}^{[1]}(t, -k) &= -4k \int_0^t e^{-4ikt(t-t')} u_{0-1}^{[1]}(t') \, dt',
\end{align*}
\]  

(99)

where \( s_{13} = \varepsilon_{13}^{[1]}(t) + \varepsilon^2 s_{13}^{[2]}(t) + O(\varepsilon^3), \ s_{14} = \varepsilon_{14}^{[1]}(t) + \varepsilon^2 s_{14}^{[2]}(t) + O(\varepsilon^3), \) and \( s_{24} = \varepsilon s_{24}^{[1]}(t) + \varepsilon^2 s_{24}^{[2]}(t) + O(\varepsilon^3). \)

It further follows from Eq. (97) that we have

\[
\begin{align*}
\Psi_{13}^{[1]}(t, -k) &= -4k \int_0^t e^{-4ikt(t-t')} u_{01}^{[1]}(t') \, dt', \\
\Psi_{14}^{[1]}(t, -k) &= -4k \int_0^t e^{-4ikt(t-t')} u_{00}^{[1]}(t') \, dt', \\
\Psi_{24}^{[1]}(t, -k) &= -4k \int_0^t e^{-4ikt(t-t')} u_{0-1}^{[1]}(t') \, dt',
\end{align*}
\]  

(99)

Thus, the Dirichlet problem can now be solved perturbatively as follows: for \( n_{33,44}(s) \) having no zeros and given \( u_{ij}^{[1]}, \ j = 1, 0, -1, \) we can obtain \( \{\Psi_{ij}^{[1]}\}, \ i = 12; \ j = 3, 4 \) from Eq. (99) and further find \( u_{ij}^{[1]}, \ j = 1, 0, -1 \) from Eq. (98). Finally, we can have \( \Psi_{ij}^{[1]} \) from Eq. (97). In fact, these arguments for \( \Psi_{ij} \) can be extended to all orders such that we can determine all orders of \( S(k) \).

In fact, the above recursive formulae can be continued indefinitely. We assume that they hold for all \( 0 \leq j \leq n-1 \), then for \( n > 0 \), the substitution of Eq. (95) into Eqs. (84a)-(84c) yields the terms of \( O(\varepsilon^n) \) as

\[
\begin{align*}
u_{11}^{[n]}(t) &= \int_{\partial D_3} \left[ \frac{2}{i\pi} \left( k\Psi_{13}^{[n]}(t, -k) - i u_{01}^{[n]} \right) + \frac{4ik}{\pi} s_{13}^{[n]} \right] \, dk + \text{lower order terms}, \\
u_{10}^{[n]}(t) &= \int_{\partial D_3} \left[ \frac{2}{i\pi} \left( k\Psi_{14}^{[n]}(t, -k) - i u_{00}^{[n]} \right) + \frac{4ik}{\pi} s_{14}^{[n]} \right] \, dk + \text{lower order terms}, \\
u_{1-1}^{[n]}(t) &= \int_{\partial D_3} \left[ \frac{2}{i\pi} \left( k\Psi_{24}^{[n]}(t, -k) - i u_{0-1}^{[n]} \right) + \frac{4ik}{\pi} s_{24}^{[n]} \right] \, dk + \text{lower order terms},
\end{align*}
\]  

(100a)-(100c)

where ‘lower order terms’ stands for the result involving known terms of lower order.

The terms of \( O(\varepsilon^n) \) in Eqs. (80)-(83) yield

\[
\begin{align*}
\Psi_{13}^{[n]}(t, k) &= \int_0^t e^{-4ikt(t-t')} \left( 2k u_{01}^{[n]} + i u_{11}^{[n]} \right) (t') \, dt' + \text{lower order terms}, \\
\Psi_{14}^{[n]}(t, k) &= \int_0^t e^{-4ikt(t-t')} \left( 2k u_{00}^{[n]} + i u_{10}^{[n]} \right) (t') \, dt' + \text{lower order terms}, \\
\Psi_{24}^{[n]}(t, k) &= \int_0^t e^{-4ikt(t-t')} \left( 2k u_{0-1}^{[n]} + i u_{1-1}^{[n]} \right) (t') \, dt' + \text{lower order terms},
\end{align*}
\]  

(101)
which leads to

\[
\begin{align*}
\Psi_{13}^{[n]}(t, -k) &= -4k \int_0^t e^{-4ik^2(t-t')} u_{01}^{[n]}(t') dt' + \text{lower order terms,} \\
\Psi_{14}^{[n]}(t, -k) &= -4k \int_0^t e^{-4ik^2(t-t')} u_{00}^{[n]}(t') dt' + \text{lower order terms,} \\
\Psi_{24}^{[n]}(t, -k) &= -4k \int_0^t e^{-4ik^2(t-t')} u_{0-1}^{[n]}(t') dt' + \text{lower order terms,}
\end{align*}
\] (102)

It follows from system (102) that \( \Psi_{13}^{[n]}(t, -k) \), \( \Psi_{14}^{[n]}(t, -k) \), and \( \Psi_{24}^{[n]}(t, -k) \) can be generated at each step from the known Dirichlet boundary data \( u_{01}^{[n]}(t) \), \( j = 1, 0, -1 \) such that we know that the Neumann boundary data \( u_{1j}^{[n]}(t) \), \( j = 1, 0, -1 \) can be generated by Eqs. (100a)-(100c) and then \( \Psi_{13}^{[n]}(t, k) \), \( \Psi_{14}^{[n]}(t, k) \), and \( \Psi_{24}^{[n]}(t, k) \) can be determined by Eq. (101) and other \( \Psi_{ij}^{[n]}(t, k) \) can also be found.

Similarly, it follows from Eqs. (85a)-(85c) that we have

\[
\begin{align*}
\Psi_{13}^{[1]}(t, -k) &= 2i \int_{\partial D_3} \left[ \Psi_{13+}^{[1]}(t, -k) - 2s_{13}^{[1]} \right] dk, \\
\Psi_{14}^{[1]}(t, -k) &= 2i \int_{\partial D_3} \left[ \Psi_{14+}^{[1]}(t, -k) - 2s_{14}^{[1]} \right] dk, \\
\Psi_{24}^{[1]}(t, -k) &= 2i \int_{\partial D_3} \left[ \Psi_{24+}^{[1]}(t, -k) - 2s_{24}^{[1]} \right] dk,
\end{align*}
\]
(103)

It further follows from Eq. (97) that we have

\[
\begin{align*}
\Psi_{13+}^{[1]}(t, -k) &= 2i \int_0^t e^{-4ik^2(t-t')} u_{11}^{[1]}(t') dt', \\
\Psi_{14+}^{[1]}(t, -k) &= 2i \int_0^t e^{-4ik^2(t-t')} u_{10}^{[1]}(t') dt', \\
\Psi_{24+}^{[1]}(t, -k) &= 2i \int_0^t e^{-4ik^2(t-t')} u_{1-1}^{[1]}(t') dt',
\end{align*}
\] (104)

Thus, the Neumann problem can now be solved perturbatively as follows: for \( n_{33,44}(s) \) having no zeros and given \( u_{ij}^{[1]}(t) \), \( j = 1, 0, -1 \), we can obtain \( \Psi_{13+}^{[1]} \), \( \Psi_{14+}^{[1]} \), \( \Psi_{24+}^{[1]} \) from Eq. (104) and further find \( u_{0j}^{[1]} \), \( j = 1, 0, -1 \) from Eq. (103). Finally, we can have \( \Psi_{ij}^{[1]} \) from Eq. (97). In fact, these arguments for \( \Psi_{ij} \) can be extended to all orders such that we can determine all orders of \( S(k) \).

Similarly, the substitution of Eq. (95) into Eqs. (85a)-(85c) yields the terms of \( O(\epsilon^n) \) as

\[
\begin{align*}
u_{01}^{[n]}(t) &= \int_{\partial D_3} \left[ \Psi_{13}^{[n]}(t, -k) - 2s_{13}^{[n]} \right] dk + \text{lower order terms,} \\
u_{00}^{[n]}(t) &= \int_{\partial D_3} \left[ \Psi_{14}^{[n]}(t, -k) - 2s_{14}^{[n]} \right] dk + \text{lower order terms,} \\
u_{0-1}^{[n]}(t) &= \int_{\partial D_3} \left[ \Psi_{24}^{[n]}(t, -k) - 2s_{24}^{[n]} \right] dk + \text{lower order terms,}
\end{align*}
\] (105a) (105b) (105c)
Eq. (101) implies that
\[
\begin{align*}
    \Psi^{[n]}_{13+}(t, -k) &= 2i \int_{0}^{t} e^{-4ik^2(t-t')} u^{[n]}_{11}(t') dt' + \text{lower order terms}, \\
    \Psi^{[n]}_{14+}(t, -k) &= 2i \int_{0}^{t} e^{-4ik^2(t-t')} u^{[n]}_{10}(t') dt' + \text{lower order terms}, \\
    \Psi^{[n]}_{24+}(t, -k) &= 2i \int_{0}^{t} e^{-4ik^2(t-t')} u^{[n]}_{1-1}(t') dt' + \text{lower order terms},
\end{align*}
\]

It follows from system (106) that \( \Psi^{[n]}_{13+}, \Psi^{[n]}_{14+}, \Psi^{[n]}_{24+} \) can be generated at each step from the known Neumann boundary data \( u^{[n]}_{1j}, j = 1, 0, -1 \) such that we know that the Dirichlet boundary data \( u^{[n]}_{0j}, j = 1, 0, -1 \) can then be given by Eqs. (105a)-(105c).

Remark 4.3. We can also give the corresponding Gelfand-Levitan-Marchenko representations for the Dirichlet and Neumann boundary value problems and the IVB of spin-1 GP equation on the finite interval, which will be studied in another paper. The analogous analysis of the Fokas unified method can also be extended to study the IBV problems for other integrable nonlinear evolution PDEs with \( 4 \times 4 \) Lax pairs on the half-line or the finite interval.

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