GLOBAL BOUNDEDNESS IN HIGHER DIMENSIONS FOR A FULLY PARABOLIC CHEMOTAXIS SYSTEM WITH SINGULAR SENSITIVITY

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Abstract. In this paper we study the global boundedness of solutions to the fully parabolic chemotaxis system with singular sensitivity:

\[ \begin{align*}
  u_t &= \Delta u - \chi \nabla \cdot \left( \frac{u}{v} \nabla v \right), & (x,t) \in \Omega \times (0,T), \\
  v_t &= k \Delta v - v + u, & (x,t) \in \Omega \times (0,T), \\
  \frac{\partial u}{\partial \nu} &= 0, & (x,t) \in \partial \Omega \times (0,T), \\
  (u(x,0),v(x,0)) &= (u_0(x),v_0(x)), & x \in \Omega,
\end{align*} \]

in a bounded and smooth domain \( \Omega \subset \mathbb{R}^n \) (\( n \geq 2 \)), where \( \chi, k > 0 \). It is shown that the solution is globally bounded provided \( 0 < \chi < \frac{k-1+\sqrt{(k-1)^2+8k}}{2k} \). This result removes the additional restriction of \( n \leq 8 \) in Zhao, Zheng [15] for the global boundedness of solutions.

1. The main result and its proof. This paper is concerned with the global boundedness of solutions to the following chemotaxis system with singular sensitivity

\[ \begin{align*}
  u_t &= \Delta u - \chi \nabla \cdot \left( \frac{u}{v} \nabla v \right), & (x,t) \in \Omega \times (0,T), \\
  v_t &= k \Delta v - v + u, & (x,t) \in \Omega \times (0,T), \\
  \frac{\partial u}{\partial \nu} &= 0, & (x,t) \in \partial \Omega \times (0,T), \\
  (u(x,0),v(x,0)) &= (u_0(x),v_0(x)), & x \in \Omega,
\end{align*} \]

in a bounded and smooth domain \( \Omega \subset \mathbb{R}^n \) (\( n \geq 2 \)), where \( \chi, k > 0 \) are constants, \( \nu \) represents the outer normal vector on \( \partial \Omega \), and the initial data \( (u_0, v_0) \in C(\Omega) \times W^{1,\infty}(\Omega) \) satisfies \( u_0 \geq 0 \) with \( u_0 \neq 0 \) and \( v_0 > 0 \) on \( \Omega \).

The problem (1) with the sensitivity function \( \chi/v \) was first proposed in [9] due to the Weber-Fechner law of stimulus perception in the process of chemotactic response, where the sensitivity function \( \chi/v \) is singular near \( v = 0 \) and reflects an inhibition of chemotactic migration at high signal concentrations.

Some rich understanding to this model with \( k = 1 \) is available.

• For the parabolic-elliptic analogue of (1) with the second equation replaced by \( 0 = \Delta v - v + u \), it was shown in [11] that the radial solutions are globally bounded with \( \chi > 0, n = 2 \), or \( \chi < 2/(n-2), n \geq 3 \), and there exist radial blow-up solutions if \( \chi > 2n/(n-2), n \geq 3 \). Without the requirement for symmetry, Biler [2] obtained...
the global existence of solutions for $\chi \leq 1$, $n = 2$, or $\chi < 2/n$, $n \geq 3$. Furthermore, the global boundedness of solutions was established with $\chi < 2/n$, $n \geq 2$ \[7\]. Lately, Fujie and Senba \[4\] proved that the solutions are globally bounded for arbitrary $\chi > 0$ under the 2-dimensional case, for which it is important to determine the behavior of solutions for large $\chi > 0$.

- As for the parabolic-parabolic case (i.e., problem (1)), the global existence of classical (resp. weak) solutions was obtained when $\chi < \sqrt{2/n}$ (resp. $\chi < \sqrt{(n+2)/(3n-4)}$) and $n \geq 2$ \[13\], and further the global classical solutions for $\chi < 2/n$ with $n \geq 2$ were proven to be bounded \[5\]. Recently, Lankeit \[10\] showed in the two dimensional setting that the classical solutions are globally bounded for $\chi \in (0, \chi_0)$ with some $\chi_0 > 1$, and thereby the value $\chi = 1$ is not critical for the global boundedness of classical solutions. In \[12\], Stinner and Winkler constructed certain global weak solutions for any $\chi > 0$ (regardless of the size of $\chi$) in the radial setting.

As to the problem (1) with the second equation replaced by $\tau v_t = \Delta v - v + u$ in a ball $\Omega \subset \mathbb{R}^2$, it was shown in \[5\] that if $\tau \in (0, 1]$ is properly small, then the classical radially symmetric solutions are globally bounded for all $\chi > 0$. But whether the smallness of $\tau > 0$ and the radial symmetry of solutions can be removed has been left as an open problem. Refer to e.g. \[4, 5, 7, 8, 14\] for general singular sensitivity, and \[1, 6\] for the chemotaxis model with singular sensitivity and logistic source.

Recently, Zhao and Zheng \[15\] showed for general $k > 0$ that if
\[
0 < \chi < \frac{-(k-1) + \sqrt{(k-1)^2 + \frac{8k}{\pi}}}{2},
\]
then (1) has a global classical solution. Moreover, the solution is globally bounded under $n \leq 8$. However, the global boundedness of solutions for $n > 8$ is not discussed therein. So, it is natural to ask whether the solution is globally bounded when $n > 8$. In this paper, we give a positive answer to this question. As the main result, it can be formulated as follows.

**Theorem 1.** Assume that $\chi$ satisfies (2) with $n \geq 2$ and $k > 0$. Then the global classical solution of (1) is bounded in the sense that there exists $C > 0$ such that
\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \text{for any } t > 0.
\]

To prove this main result, we first give several well-understood facts described as the coming three propositions. Throughout this paper, let $(u, v)$ be the global classical solution of (1).

**Proposition 1.** (See Lemma 2.2 in \[3\]) There exists $\eta > 0$ such that
\[
\inf_{x \in \Omega} v(x, t) \geq \eta \quad \text{for all } t \geq 0.
\]

**Proposition 2.** (See Lemma 2.4 in \[3\]) Let $1 \leq p, q \leq \infty$.

(i) If $\frac{q}{2} \left( \frac{1}{p} - \frac{1}{q} \right) < 1$, then there exists $C > 0$ such that
\[
\|v(\cdot, t)\|_{L^q(\Omega)} \leq C(1 + \sup_{s \in (0, t)} \|u(\cdot, s)\|_{L^p(\Omega)}) \quad \text{for each } t > 0.
\]

(ii) Assume that $\frac{1}{2} + \frac{q}{2} \left( \frac{1}{p} - \frac{1}{q} \right) < 1$. Then
\[
\|\nabla v(\cdot, t)\|_{L^q(\Omega)} \leq C(1 + \sup_{s \in (0, t)} \|u(\cdot, s)\|_{L^p(\Omega)}) \quad \text{for all } t > 0
\]
with some $C > 0$. 

Proof. Define the function 

\[ \frac{k}{k/(\chi(\chi-1+k))} \] 

then one can readily check that \( \frac{k}{k/(\chi(\chi-1+k))} > n/2 \geq 1 \). For any \( p \in (1, k/(\chi(\chi-1+k))) \), we denote the two different roots of 

\[ p(k-1)^2 + 4k)p^2 + [2(p-1)\chi(k-1) - 4(p-1)k]r + p(p-1)^2 \chi^2 = 0 \]

by \( r_\pm(p) := \frac{(p-1)p\chi(1-k) + 2k \pm \sqrt{k^2 - p\chi(k-1) - p\chi^2 k}}{p(1-k)^2 + 4k} \) with \( r_-(p) < r_+(p) \).

Proposition 3. (See Lemma 2.2 in [15]) For any \( p \in (1, k/(\chi(\chi-1+k))) \) and \( r \in (r_-(p), r_+(p)) \), if 

\[ \|v(\cdot,t)\|_{L^p(\Omega)} \leq C, \quad t > 0 \]

with some \( c > 0 \), then there exists \( C > 0 \) such that 

\[ \int_\Omega u^p(x,t)\nu^{-r}(x,t)dx \leq C \quad \text{for all } t > 0. \]

To obtain the global boundedness of solutions to (1), as performed in [3], Proposition 2(i) and Proposition 3 with the proper choice of \( r \) allow one to elevate the \( L^1 \)-conservation of \( u \) to the boundedness of \( \|u(\cdot,t)\|_{L^q(\Omega)} \) for some \( q > n/2 \) by an iterative argument. In [15], \( r \) is directly taken as 

\[ r = \frac{r_-(p) + r_+(p)}{2} = \frac{(p-1)\chi(1-k) + 2k}{p(1-k)^2 + 4k} \]

satisfying \( r \in (r_-(p), r_+(p)) \) spontaneously. But such complicated choice for \( r \) makes it possible to carry out the iteration only under the case \( n \leq 8 \). To remove the restriction of \( n \leq 8 \), we alternatively pick \( r \) proportional to \( p-1 \). Moreover, the following lemma guarantees that such selected \( r \) with a suitable ratio to \( p-1 \) does lie in \( (r_-(p), r_+(p)) \), and hence plays a key role in the proof of Theorem 1 although it is elementary.

Lemma 2. There exist constants \( \varepsilon_* \in (0, \frac{k}{\chi(\chi-1+k)}) - \frac{n}{2} \) small and \( a_* \in (0,1) \) such that 

\[ a_*(p-1) \in (r_-(p), r_+(p)) \quad \text{for any } p \in (1, \frac{n}{2} + \varepsilon_*). \]

Proof. Define the function 

\[ h(p) := \frac{\chi(1-k) + 2k}{p(1-k)^2 + 4k}, \quad \text{for } p \geq 0. \]

Clearly, \( h(p) < 1 \) for \( 0 \leq p < \hat{p} \) with 

\[ \hat{p} := \begin{cases} 2k \\ (1-k)(\chi-1+k) \\ \infty \end{cases} \quad \text{when } 0 < k < 1, \quad \text{when } k \geq 1. \]

The hypothesis [2] entails that \( \frac{n}{2} < \hat{p} \), and thereby \( h(\frac{n}{2}) < 1 \). By continuity, there exists \( 0 < \varepsilon_1 < 1 \) sufficiently small such that 

\[ h(\frac{n}{2} + \varepsilon) < 1 \quad \text{for any } 0 < \varepsilon \leq \varepsilon_1. \]

Again by [2], we have 

\[ n\chi(k-1) - 4k < 0 \quad \text{and} \]

\[ [n\chi(k-1) - 4k]^2 - 2n\chi^2 \left(\frac{n}{2}(k-1)^2 + 4k\right) = -8nk\left[\chi^2 + (k-1)\chi - \frac{2k}{n}\right] > 0. \]
Therefore, the continuity results in that there exists \( \varepsilon_2 \in (0, 1) \) small enough such that for all \( 0 < \varepsilon \leq \varepsilon_2 \),

\[
(n + 2 \varepsilon)\chi(k - 1) - 4k < 0 \quad \text{and} \\
[(n + 2 \varepsilon)\chi(k - 1) - 4k]^2 - (2n + 4 \varepsilon)^2[(n + \varepsilon)(k - 1)^2 + 4k] > 0. \tag{3}
\]

Take \( \varepsilon_* = \min\{\varepsilon_1, \varepsilon_2, \frac{k}{\chi(1 + k^*)} - \frac{n}{2}\} \). By \( \text{(3)} \), we know that the quadratic equation

\[
f(a) := \left[\left(\frac{n}{2} + \varepsilon_*\right)(k - 1)^2 + 4k\right]a^2 + \left[(n + 2 \varepsilon_*)\chi(k - 1) - 4k\right]a + \left(\frac{n}{2} + \varepsilon_*\right)^2 = 0
\]

admits two positive roots with the smaller one, written by \( a_* \), satisfying

\[
0 < a_* < \left(\frac{n}{2} + \varepsilon_*\right)(1 - k) + 2k \left(\frac{n}{2} + \varepsilon_*\right)(k - 1)^2 + 4k = h\left(\frac{n}{2} + \varepsilon_*\right) < 1.
\]

With the constants \( \varepsilon_* \) and \( a_* \) determined above, we have for any \( p \in (1, \frac{n}{2} + \varepsilon_*) \) that

\[
[r_-(p) - a_*(p - 1)][a_*(p - 1) - r_+(p)] \\
= a_*(p - 1)(r_-(p) + r_+(p)) - r_-(p)r_+(p) - a_*^2(p - 1)^2 \\
= a_*(p - 1)\frac{4(p - 1)^2(k - 2p)(p - 1)\chi(k - 1)}{p(k - 1)^2 + 4k} - \frac{p(p - 1)^2\chi^2}{p(k - 1)^2 + 4k} - a_*^2(p - 1)^2 \\
= \frac{(p - 1)^2}{p(k - 1)^2 + 4k}\left(4a_*k - [\chi + a_*(k - 1)]^2p - 4a_*^2k\right) \\
> \frac{(p - 1)^2}{p(k - 1)^2 + 4k}\left(4a_*k - [\chi + a_*(k - 1)]^2\left(\frac{n}{2} + \varepsilon_*\right) - 4a_*^2k\right) \\
= -\frac{(p - 1)^2}{p(k - 1)^2 + 4k}f(a_*) = 0.
\]

The proof is complete. \( \square \)

Now we are in a position to prove Theorem 1

Proof of Theorem 1

The argument, similar to that in 3, can be performed via two steps.

Step 1. We prove that there is \( q > n/2 \) such that

\[
\|u(\cdot, t)\|_{L^q(\Omega)} \leq C \quad \text{for all} \ t > 0
\]

with some \( C > 0 \). Let \( \varepsilon_* > 0 \) and \( a_* \in (0, 1) \) be the constants as in Lemma 2. To this end, we first claim the following facts.

(a) Fix

\[
p_0 \in \left(1, \min\left\{\frac{n}{2} + \varepsilon_*\frac{(1 - a_*)n + 2a_*}{(1 - a_*)(n - 2)}\right\}\right).
\]

Then for any \( p_0 \in (1, \min\{p_0, \frac{n((1 - a_*)p_0 + a_*)}{n + 2a_* - 2a_*p_0}\}) \), there exists \( c_0 > 0 \) such that

\[
\int_{\Omega} u^{q_0}(x, t)dx \leq c_0 \quad \text{for all} \ t > 0.
\]
(b) Let
\[ \tilde{p} \in \left(1, \frac{n}{2}\right) \quad \text{and} \quad \bar{p} \in \left(\tilde{p}, \min\left\{ \frac{n}{2} + \varepsilon, \frac{n(1 - a_*) n + 2a_*}{1 - a_* (n - 2\bar{p})} \right\} \right). \]
If for each \( \tilde{\varphi} \in (1, \min\{\tilde{p}, \frac{n((1 - a_*) \tilde{p} + a_*)}{n + 2a_* - 2a_*p}\}) \), there is \( \tilde{c} > 0 \) such that
\[ \int_{\Omega} \tilde{\varphi}(x, t) dx \leq \tilde{c} \quad \text{for all} \quad t > 0, \]
then for any \( \tilde{\varphi} \in (1, \min\{\bar{p}, \frac{n((1 - a_*) \bar{p} + a_*)}{n + 2a_* - 2a_*p}\}) \), there exists \( \tilde{c} > 0 \) such that
\[ \int_{\Omega} \tilde{\varphi}(x, t) dx \leq \tilde{c} \quad \text{for all} \quad t > 0. \]
Let us first confirm assertion (a). Indeed, taking \( r_0 = a_*(p_0 - 1) \), we have
\[ 0 < p_0 - r_0 = (1 - a_*)p_0 + a_* < \frac{(1 - a_*) n + 2a_*}{n - 2} + a_* = \frac{n}{n - 2}, \]
i.e., \( \frac{n}{2}(1 - \frac{1}{p_0 - \frac{a_*}{n - 2}r_0}) < 1 \). By Proposition 2(i) and the \( L^1 \)-conservation of \( u \), we obtain with some \( C_1, C_2 > 0 \) that
\[ \|v(\cdot, t)\|_{L^{\frac{n}{n - 2} - r_0}(\Omega)} \leq C_1 \left(1 + \sup_{s \in (0, \infty)} \|u(\cdot, s)\|_{L^{\frac{n}{n - 2}}(\Omega)} \right) \leq c_1, \quad t > 0. \]
Notice that \( r_0 \in (r_-(p_0), r_+(p_0)) \) due to Lemma 2. Hence it further follows from Proposition 3 that
\[ \int_{\Omega} u^{p_0} v^{-r_0}(x, t) dx \leq c_2 \quad \text{for all} \quad t > 0 \]
with some \( c_2 > 0 \). Let \( q_0 \in (1, \min\{\frac{n((1 - a_*) p_0 + a_*)}{n + 2a_* - 2a_*p_0} \}) \). By the H"{o}lder inequality, we get
\[ \int_{\Omega} u^{q_0} dx \leq \left( \int_{\Omega} u^{p_0} v^{-r_0} dx \right)^{\frac{q_0}{p_0}} \left( \int_{\Omega} v^{\frac{n q_0}{n - 2q_0}} dx \right)^{\frac{p_0 - q_0}{p_0}} \leq \epsilon \left( \int_{\Omega} v^{\frac{n q_0}{n - 2q_0}} dx \right)^{\frac{p_0 - q_0}{p_0}}. \]
Since \( \frac{n}{2}(\frac{1}{q_0} - \frac{q_0}{q_0 - r_0}) < 1 \) due to \( q_0 < \frac{n((1 - a_*) p_0 + a_*)}{n + 2a_* - 2a_*p_0} = \frac{n(p_0 - r_0)}{n - 2r_0} \), Proposition 2(i) implies that
\[ \sup_{t \in (0, T)} \|v(\cdot, t)\|_{L^{\frac{n}{n - 2} - r_0}(\Omega)} \leq C_2 \left(1 + \sup_{t \in (0, T)} \|u(\cdot, t)\|_{L^{\frac{n}{n - 2}}(\Omega)} \right) \quad \text{for any} \quad T > 0, \]
with the constant \( C_2 > 0 \) independent of \( T > 0 \). Combining the foregoing two estimates, we conclude with some \( C_3 > 0 \) that
\[ \sup_{t \in (0, T)} \|u(\cdot, t)\|_{L^{\frac{n}{n - 2}}(\Omega)} \leq C_3 \left(1 + \left( \sup_{t \in (0, T)} \|u(\cdot, t)\|_{L^{\frac{n}{n - 2}}(\Omega)} \right)^{\frac{r_0}{p_0}} \right) \quad \text{for each} \quad T > 0, \]
and thereby \( \sup_{t \in (0, \infty)} \|u(\cdot, t)\|_{L^{\frac{n}{n - 2}}(\Omega)} \leq \infty \) for \( r_0/p_0 < 1 \). This proves assertion (a).
Next, we proceed to verify assertion (b). As above, assertion (b) follows if we can infer with \( \bar{r} := a_*(\bar{p} - 1) \in (r_-(\bar{p}), r_+(\bar{p})) \) that \( \int_{\Omega} \bar{u}^{\bar{r}} \bar{r}^\varepsilon dx \) is bounded in \( (0, \infty) \).
Clearly,
\[ 0 < \bar{p} - \bar{r} = (1 - a_*)\bar{p} + a_* < \frac{\bar{p}(1 - a_*) n + 2a_*}{n - 2\bar{p}} + a_* = \frac{n((1 - a_*) \bar{p} + a_*)}{n + 2a_* - 2a_*p}, \]
\[ \int_{\Omega} \bar{u}^{\bar{r}} \bar{r}^\varepsilon dx \leq n \cdot \frac{n((1 - a_*) \bar{p} + a_*)}{n + 2a_* - 2a_*p}. \]
which allows us to chose \(q \in (1, \frac{n((1-a_\ast)p+a_\ast)}{n+2a_\ast-2a_\ast p})\) fulfilling \(0 < \tilde{p} - \tilde{r} < \frac{nq}{n-2}\), or \(\frac{1}{2} \left(1 - \frac{1}{\tilde{p} - 1}\right) < 1\). Such \(q\) does lie in \((1, \min(\tilde{p}, \frac{n((1-a_\ast)p+a_\ast)}{n+2a_\ast-2a_\ast p}))\) since \(\frac{n((1-a_\ast)p+a_\ast)}{n+2a_\ast-2a_\ast p} \leq \tilde{p}\) with \(\tilde{p} \leq \frac{nq}{n-2}\). By hypothesis, \(\|u(\cdot,t)\|_{L^\tilde{p}(\Omega)}\) is bounded, and hence it follows from Proposition 2(i) that \(\|v(\cdot,t)\|_{L^{\tilde{r}}(\Omega)} \leq c_3\) in \((0, \infty)\) for some \(c_3 > 0\). By Lemma 2 \(\tilde{r} = a_\ast(\tilde{p} - 1) \in (r_-, \bar{r}_+)\). Thus, apply Proposition 3 to obtain with some \(c_4 > 0\) that \(\int_\Omega u^q(x,t)dx \leq c_4\) for all \(t > 0\).

Now we can establish the \(L^q\)-boundedness of \(u\) for some \(q > n/2\) by an iterative argument. Take a sequence \(\{p_k\}_{k=-1}^\infty\) as follows:

\[
p_{k-1} = 1, \quad p_k = \begin{cases} \min\left\{\frac{n+\varepsilon_\ast}{2}, \frac{p_{k-1}((1-a_\ast)n+2a_\ast)}{(1-a_\ast)(n-2p_{k-1})+2a_\ast}\right\} & \text{if } p_{k-1} \leq \frac{n}{2}, \\
\frac{n}{2} + \varepsilon_\ast & \text{if } p_{k-1} > \frac{n}{2}
\end{cases}
\]

for \(k = 0, 1, \ldots\). It is easy to see that \(1 < p_{k-1} \leq p_k\) for all \(k \geq 1\), and hence there is a definitive integer \(k_0 \geq 0\) such that \(p_{k_0} > \frac{n}{2}\) and \(p_{k-1} \leq \frac{n}{2}\) for \(k = 0, 1, \ldots, k_0\). If not, \(1 < p_k \leq \frac{n}{2}\) for all \(k \geq 0\), which implies that \(p_k = \min\left\{\frac{n+\varepsilon_\ast}{2}, \frac{p_{k-1}((1-a_\ast)n+2a_\ast)}{(1-a_\ast)(n-2p_{k-1})+2a_\ast}\right\}\) for any \(\varepsilon_\ast > 0\) with \(p_{k-1} \leq \frac{n}{2}\). We thus gain by taking limits that \(\ell = \min\left\{\frac{n+\varepsilon_\ast}{2}, \frac{p_{k-1}((1-a_\ast)n+2a_\ast)}{(1-a_\ast)(n-2p_{k-1})+2a_\ast}\right\}\), which is absurd.) By the choice of \(\{p_k\}\), we also have

\[
p_0 = \left(1, \min\left\{\frac{n}{2} + \varepsilon_\ast, \frac{(1-a_\ast)n+2a_\ast}{(1-a_\ast)(n-2)}\right\}\right).
\]

and if \(k_0 \geq 1\), then

\[
p_k = \left(p_{k-1}, \min\left\{\frac{n}{2} + \varepsilon_\ast, \frac{p_{k-1}((1-a_\ast)n+2a_\ast)}{(1-a_\ast)(n-2p_{k-1})}\right\}\right) \quad \text{with } p_{k-1} \in \left(1, \frac{n}{2}\right)
\]

for \(k = 1, 2, \ldots, k_0\). Due to assertion (a) only if \(k_0 = 0\), or by assertion (a) with successive applications of assertion (b) to \(\tilde{p} = p_{k-1}\) and \(\tilde{p} = p_k\) when \(k_0 \geq 1\), we obtain that for any \(q \in (1, \min(p_{k_0}, \frac{n((1-a_\ast)p_{k_0}+a_\ast)}{n+2a_\ast-2a_\ast p_{k_0}}))\), there exists \(c_5 > 0\) such that

\[
\int_\Omega u^q(x,t)dx \leq c_5 \quad \text{for all } t > 0.
\]

Because \(\min(p_{k_0}, \frac{n((1-a_\ast)p_{k_0}+a_\ast)}{n+2a_\ast-2a_\ast p_{k_0}}) = p_{k_0} > \frac{n}{2}\), we can pick \(q > \frac{n}{2}\), as desired.

**Step 2.** A standard semigroup technique (see e.g. Lemma 3.4 in [13]) along with Propositions 1 and 2(ii) as well as \(L^q\)-boundedness of \(u\) in Step 1 leads to the \(L^\infty\)-boundedness of \(u\).

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