Lower entropy bounds and particle number fluctuations in a Fermi sea

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(Dated: June 2003)

In this Letter we demonstrate, in an elementary manner, that given a partition of the single particle Hilbert space into orthogonal subspaces, a Fermi sea may be factorized into pairs of entangled modes, similar to a BCS state. We derive expressions for the entropy and for the particle number fluctuations of a subspace of a Fermi sea, at zero and finite temperatures, and relate these by a lower bound on the entropy. As an application we investigate analytically and numerically these quantities for electrons in the lowest Landau level of a quantum Hall sample.

The study of quantum many particle states, when measurements are only applied to a given subsystem, are at the heart of many questions in physics. Examples where the entropy of such subsystems is interesting range from the quantum mechanical origins of black hole entropy, where the existence of an event horizon thermalizes the field density matrix inside the black hole, to entanglement structure of spin systems. In this work we address the relation of entanglement entropy of fermions with the fluctuations in the number of fermions.

First we show that given a subspace of the single particle Hilbert space, a Fermi sea may be factorized into pairs of entangled modes in and out of the subspace, thereby writing the state as a BCS state.

While upper bounds on entropy, were a subject of numerous investigations, especially since Bekenstein’s writing the state as a BCS state, lower bounds on entropy are less known.

We show that given a "Fermi sea", the entropy of the ground state, restricted to a particular subspace $A$ of the single particle space, relates to the particle number fluctuations in the subspace via the inequality:

$$S_A \geq 4 \log 2\Delta N_A^2 \geq -8 \log 2 \ll N_A^4 \gg$$  \hspace{1cm} (1)

Where $S_A$ is the entropy associated with the subspace $A$, and $\Delta N_A^2$ are the fluctuations in the particle number in $A$, and $\ll N_A^4 \gg$ is the fourth cummulant of particle number. The importance of this result lies in the fact that particle fluctuations are, in principle, easier to measure, and are fundamentally related to the quantum noise in various systems. On the technical side note that the right-hand side, has the advantage of being easy to calculate analytically in a wider class of problems.

We start by examining the ground state of non-interacting fermions, in arbitrary external potential, when measurements are applied to a given part of the space. The basic example is a Fermi sea or a Dirac sea where we are interested in the relative entropy of a given region of space, and in fluctuations in the number of particles there, but one may also consider entanglement in Fermion traps (Fermi degeneracy of potassium atoms $^{40}K$ has been observed by De Marco and Jin).

The discussion is also relevant for systems which behave like a non-interacting Fermi gas, as in problems of transport at the zero temperature limit, in ideal metals, where transport may be approximated well within a non-interacting theory, due to good screening. In an ideal single channel conductor the analogy is done by mapping excitations that travel at the Fermi velocity to a time-energy coordinate representation in discussion of quantum pumps. The analogy is especially manifest in the problem of switching noise.

The ground state of a noninteracting Fermi gas, containing $N$ particles is obtained by occupying the allowed states $\phi_i \in H$ ($H$ is the single particle Hilbert space) up to energy $E_f$, i.e.:

$$\ket{gs} = \prod_{E(\phi_i) < E_f} \psi^\dagger(\phi_i) \ket{0}$$  \hspace{1cm} (2)

Where $\psi^\dagger$ are creation operators which satisfy the usual canonical anti-commutation relations (CAR):

$$[\psi(\phi_i), \psi^\dagger(\phi_j)]_+ = \delta \phi_i \phi_j$$  \hspace{1cm} (3)

The state $\ket{gs}$ or the "Fermi sea", is a typical ground state for a large class of Hamiltonians (sometimes it is necessary to carry a suitable Bogolubov transformation). It can also describe spin chains via the Jordan-Wigner transformation.

Let $A$ be a subspace of the single particle Hilbert space $H$, so that $H = A \oplus A^\perp$ and let $E = \text{span}\{\phi_i; 1 \leq i \leq N\}$ be the subspace of occupied single particle states of $H$ (the Fermi sea).

Let $P_A$ be the orthogonal projection on $A$. Consider the matrix $M(A)_{ij} = \langle P_A \phi_j, P_A \phi_i \rangle$, $i,j = 1, \ldots, N$. $M_{ij}$ is a hermitian matrix so that it can be diagonalized by a unitary $U$: $M = U^\dagger \text{diag}(d_i) U$.

The new orthonormal modes are defined by

$$A_l = \sum_k U_{lk}^\dagger P_A \phi_k \sqrt{d_l}$$

where the factor $d_l$ is the $l$-th eigenvalue of $M$, and serves to normalize the $A_l$ with the inner product on $H$. Similarly we take $B_l = \sum_k U_{lk}^\dagger P_A^\perp \phi_k \sqrt{1/d_l}$, which are orthonormal since $M(A^\perp) = I - M(A)$. Obviously $A_l \in A$ and $B_l \in A^\perp$. 

Since $U$ is unitary we write $E = \text{span}\{U^\dagger\phi_i\}$. Using $U^\dagger_\nu\phi_i = \sqrt{d_i}A_i + \sqrt{1-d_i}B_i$, the ground state may be written as:

$$|gs> = \prod_{i=1}^{N} \Psi^\dagger(U^\dagger_\nu\phi_i)|0> = \prod_{i=1}^{N}(\sqrt{d_i}\Psi^\dagger(A_i) + \sqrt{1-d_i}\Psi^\dagger(B_i))|0>$$

Note that the $0 \leq d_i \leq 1$ due to the structure of $M$. In the particular case that $|P_E,P_A| = 0$ then the exercise is trivial, as they can be diagonalized simultaneously. In addition, whenever there is a symmetry operator that commutes with $P_A$ and $P_E$, then the resultant eigenmodes are invariant under the symmetry.

The ground state may be written also as a BCS state in the following way. Consider the vector $|A> = \prod \psi^\dagger(A_i)|0>$, defined by "filling" the modes in $A$, and redefine $\psi(A_i) = \psi^\dagger_h(A_i)$, then:

$$|gs> = \prod_i(\sqrt{d_i} - \sqrt{1-d_i}\psi^\dagger(B_i)\psi^\dagger_h(A_i))|A>$$

Similar to a BCS state, where the entanglement structure of the modes pairs $A_i,B_i$ is clearly seen.

If we factor the single particle Hilbert space $H$ into

$$H = \bigoplus_{i=1}^{N} \text{span}\{A_i,B_i\} \bigoplus(\text{Complement}),$$

the fermion Fock space factors into an appropriate tensor product. We can write naturally the density matrix as:

$$\rho = |gs><gs| = \bigotimes_{i=1}^{N}|\nu_i><\nu_i|,$$

where $\nu_i = (\sqrt{d_i}\psi^\dagger(A_i) + \sqrt{1-d_i}\psi^\dagger(B_i))|0>$. After a partial trace over the $B_i$s, the reduced density matrix is given by: $\rho_A = \bigotimes_{i=1}^{N}(1 - d_i 0 0 d_i)$, where $d_i$ is the probability of having a particle in mode $A_i$.

From the density matrix $\rho_A$ we have:

$$<N>_A = \text{Tr}(M),$$

$$\Delta N_A^2 = <N^2> - <N>^2_A = \text{Tr}(M(1 - M)).$$

To study further moments, denote $P(k)$ the probability of having $k$ fermions in $A$ and consider the generating function:

$$\chi(\lambda) = \sum P(k)e^{i\lambda k} = \text{det}(1 + M(e^{i\lambda} - 1)).$$

The cummulants of the number of fermions in $A$ may be extracted from the generating function by differentiating $\log \chi$ and setting $\lambda = 0$. For example, the fourth cumulant is given by:

$$\ll N^4 \gg = \partial^4_{\lambda} \log \chi|\lambda=0 = \text{Tr}(M(1 - M)(1 - 6M + 6M^2))$$

The reduced entropy is obtained from $\rho_A$:

$$S_A = -\text{Tr}(M \log M + (1 - M) \log(1 - M)).$$

This entropy is equivalent to the usual entropy of a Fermi gas with occupation number operator $M$.

Comparing the expressions $8,10$ and $11$, and using the fact that $0 \leq M \leq 1$, we get the inequalities $11$ as promised (Fig. 1).

The difference between the cummulants and the entropy comes from eigenstates with probability away from 1/2 but not exactly 0 or 1. The fourth cumulant may be used to estimate the contribution to the second moments of eigenvalues of $n_A$ away from 1/2, as the function $-2x(1-x)(1-6x + 6x^2)$ is negative outside the interval $(1/2 - \sqrt{3}/4,1/2 + \sqrt{3}/4)$.

The inequalities are valid for finite temperatures too. To see this, consider a general density matrix of the form $\rho = (1 - K_{ij}n_{ij})$. By tracing out the $A^\dagger$ degrees of freedom, the reduced density matrix acquires the form

$$\rho_A = \text{det}(1 - n_A)e^{\log(n_A/n_A)}.$$
In most cases particle fluctuations will dominate the entropy, however one can check that whenever \( n_{BE} < 1 \) (a dilute boson gas) then:

\[
S_{\text{Bosons}} \geq (\log 2)(\Delta N)^2_{\text{Bosons}}
\]  

(16)

For systems in which the single particle energies have a gap \( \Delta \) above the zero energy, as the temperature goes to zero we have \( n_i \rightarrow 0 \), making the inequality above valid.

While the cases of spin chains and free fields, were studied in numerous works, other important examples wait addressing. Here we consider the problem of 2D electrons in a quantum Hall system.

The filled lowest Landau level states are spanned by \(|k\rangle = \frac{1}{\sqrt{2\pi}k!} e^{-|z|^2/2} |k\rangle\).

In [10], the entanglement entropy of a Chern-Simons theory describing a quantum Hall defined on a disk was shown to scale like the radius. Let us now calculate the entropy and the fluctuations in such systems. In particular, if one is interested in particle fluctuations, then the theorem of Lebowitz and Costin [12], ensures a gaussian behavior for the scaled particle number fluctuations as the volume grows large for a large class of determinantal processes.

In the case of a 1D Fermi gas, both entropy and particle fluctuations in a box of size \( L \) scale as \( \log k_F L \).

Particle number fluctuations in 1D conductors may be observed using ultra-fast transistors, however it will not be possible to ignore the Coulomb interaction. We note that for a 1D conductor, there is a pre-factor due to the cross section, which means that the fluctuations may be quite large.
In this Letter we have shown how the ground state of fermions can be factored into sets of pairs of modes - inside and outside a given subsystem. We have outlined the connection between the entanglement entropy and the fluctuations in the particle number in two ways: First, the inequality supplies a lower bound on the available entanglement entropy, and second, noting that in some cases (namely the 1d Fermi sea and the lowest Landau level) both quantities scale in a similar way.

Acknowledgment: I thank J. D. Bekenstein, J. Lebowitz, L. S. Levitov, B. Shapiro, A. Retzker and M. Reznikov for useful remarks.