Two-Dimensional Problems of Minimal Resistance in a Medium of Positive Temperature*

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Abstract

We study the Newton-like problem of minimal resistance for a two-dimensional body moving with constant velocity in a homogeneous rarified medium of moving particles. The distribution of the particles over velocities is centrally symmetric. The problem is solved analytically; the minimizers are shown to be of four different types. Numerical results are obtained for the physically significant case of gaussian circular distribution of velocities, which corresponds to a homogeneous ideal gas of positive temperature.

Keywords. Two-dimensional Newton-type problems of minimal resistance, temperature motion, gaussian distribution of velocities, calculus of variations, optimal control.

1 Introduction

In 1686, in his *Principia Mathematica*, Newton considered one of the oldest problems of optimal control. The problem consists of finding the shape of a body, moving with constant velocity in a homogeneous medium consisting of infinitesimal particles, such that the total resistance of the medium to the body would be minimal. Newton assumed that the collisions of the particles with the body are absolutely elastic, the medium is very rare, so that the particles do not mutually interact, and that the particles are immovable, i.e., there is no temperature motion of particles. Problems of this kind may appear in construction of high-speed and high-altitude flying vehicles, such as missiles and artificial satellites.

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Newton solved this minimization problem in the class of convex axially symmetric bodies with the axis parallel to the velocity of the body, of fixed length along the axis and fixed maximal cross section orthogonal to the axis. Convexity implies that each particle hits the body at most once, and this fact allows one to write down the explicit formula for resistance. An account of the problem originally considered by Newton can be found, e.g., in the book [1].

Since the early days of the calculus of variations, many modifications of Newton’s problem of minimal resistance have been studied in the literature [8]. In 1990th the interest to Newton’s problem revived. Interesting results were obtained when dealing with the minimization problem in various classes of bodies obtained by withdrawing or relaxing the conditions initially imposed by Newton: axial symmetry [2, 4, 9]; convexity (still maintaining the single shock assumption) [5, 6]; single shock condition [10, 11].

On the other hand, the problem was studied under the more realistic assumptions of presence of friction at the moment of collision (so that the collisions are not absolutely elastic) [7], and of mutual interaction of particles [14]. In the present paper, the case of temperature noise of particles is considered. First, we obtain general formulas in the $d$-dimensional case, $d \geq 2$, and then study the case $d = 2$ in more detail.

We suppose that a convex and axisymmetric body is moving in $\mathbb{R}^d$ with a constant velocity $V$, in a homogeneous medium of moving particles; the distribution of particles over velocities is the same at every point. In fact, it is more convenient to assume that the body is immovable, and there is a flux of particles falling upon it. This picture will be taken in the sequel. The length $h$ of the body along the axis is fixed, and the maximal cross section by a hyperplane orthogonal to the axis is supposed to be a $(d - 1)$-dimensional ball of radius 1.

In section 2 the preliminary analysis of the problem in the $d$-dimensional case is made. In section 3 the minimization problem for $d = 2$ is solved. In section 4 we consider the physically relevant special case of gaussian circular distribution of velocities. Analytical formulas for resistance are given, and results of numerical simulations are presented.

## 2 Calculation of Pressure and Resistance in the General Case

Consider a flux of infinitesimal particles in $\mathbb{R}^d$, $d \geq 2$. The density of flux and the distribution of particles over velocities are the same at each point, the distribution being given by a density function $\rho(v)$. The pressure of the flux at a regular point $x \in \partial B$ of the boundary of a convex body $B \subset \mathbb{R}^d$ equals $\pi(n_x)$, where $n_x$ is the outer normal to $\partial B$ at $x$,

$$\pi(n) = -\int (v \cdot n) \rho(v) dv \cdot n; \quad (1)$$

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here $x_- := \min\{x, 0\}$ and $\langle \cdot | \cdot \rangle$ means scalar product; and resistance of the body to the flux equals
\[
R(B) = \int_{\partial B} \pi(n_x) \, dH^{d-1}(x),
\]
where $H^{d-1}$ means $(d-1)$-dimensional Hausdorff measure. The formulas (1) and (2) are obvious modifications of the corresponding formulas from [3].

In what follows, we shall assume that
(i) the function $\rho$ is spherically symmetric with the center $-V e_d$, where $V > 0$ and $e_d$ is the $d$th coordinate vector, i.e. $\rho(v) = \sigma(|v + V e_d|)$; besides
(ii) the function $\sigma$ is continuous and monotone decreasing, and
\[
\int_0^\infty r^2 \sigma(r) \, r^{d-1} \, dr < \infty.
\]

The body $B$ is supposed to be convex, open, and symmetric with respect to the $d$th coordinate axis. By translation along this axis and by subsequent homothety, it can be reduced to the form
\[
B = \{ (x', x_d) : |x'| < 1, f_-(|x'|) < x_d < -f_+ (|x'|) \},
\]
where $x' = (x_1, \ldots, x_{d-1})$, $f_+$ and $f_-$ are convex negative non-decreasing functions defined on $[0, 1)$. We shall imagine that the $d$th coordinate axis is directed vertically upwards. The height of the body is
\[
h = -f_+(0) - f_-(0).
\]
We shall suppose that $h$ is constant.

At a point $x_+ = (x', -f_+ (|x'|))$ of the upper part of the boundary $\partial B$, the outer normal equals
\[
n_{x_+} = \frac{1}{\sqrt{f'_+ (|x'|)^2 + 1}} \left( f'_+ (|x'|) \frac{x'}{|x'|}, 1 \right),
\]
and due to the property (ii) of $\rho$, from (1) one finds that pressure of the flux at this point equals $\pi(n_{x_+}) = -p_+ (f'_+ (|x'|)) \cdot n_{x_+}$, where
\[
p_+ (u) := \left| \pi \left( \left( \frac{u}{\sqrt{u^2 + 1}}, 0, \ldots, 0, \frac{1}{\sqrt{u^2 + 1}} \right) \right) \right|.
\]
Similarly, the outer normal to $\partial B$ at a point $x_- = (x', f_- (|x'|))$ of the lower part of $\partial B$ equals $\pi(n_{x_-}) = p_- (f'_- (|x'|)) \cdot n_{x_-}$, where
\[
p_- (u) := -\left| \pi \left( \left( \frac{u}{\sqrt{u^2 + 1}}, 0, \ldots, 0, -\frac{1}{\sqrt{u^2 + 1}} \right) \right) \right|.
\]
From (6), (7), and (1) one obtains
\[
p_\pm (u) = \pm \int \frac{(v_1 u \pm v_{d})^2}{1 + u^2} \, \rho(v) \, dv.
\]

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Using the assumptions (i) and (ii), one can show that the functions $p_+$ and $p_-$ satisfy the following properties:

(a) $p_\pm \in C^1[0, +\infty)$;
(b) there exist $\lim_{u\to +\infty} p_\pm(u)$;
(c) $p'_\pm(0) = \lim_{u\to +\infty} p'_\pm(u) = 0$;
(d) $p'_\pm(u) < p'_\pm(\infty) \leq 0$.
(e) There exist $\bar{u}_\pm > 0$ such that $p'_\pm$ is strictly monotone decreasing on $[0, \bar{u}_\pm]$ and strictly monotone increasing on $[\bar{u}_\pm, +\infty)$. The proof of these properties is not very difficult, but rather technical and long, and will be presented elsewhere.

Let us calculate the resistance $R(B)$, using the formula (2). The integral in the right hand side of (2) is the sum of two integrals corresponding to the upper and lower parts of $\partial B$. Change the variable in each of these integrals, substituting $dH^{d-1}(x)$ for $\int f_+^+(|x'|)^2 + 1 d\alpha'$ and for $\int f_-^-(|x'|)^2 + 1 d\alpha'$, and substituting $\pi(n_x)$ for $\pi(n_{x_+})$ and for $\pi(n_{x_-})$, respectively. One obtains

$$R(B) = \int_{|x'| < 1} p_+(f_+^+(|x'|)) \left( f_+^+(|x'|) \frac{x'}{|x'|} \right) dx'$$

$$+ \int_{|x'| < 1} p_-(f_-^-(|x'|)) \left( -f_-^-(|x'|) \frac{x'}{|x'|} \right) dx',$$

and as a result of integrating, using the fact that the functions $p_\pm(f_\pm^+(|x'|))$ and $\pm f_\pm^+(|x'|) \frac{x'}{|x'|}$ are symmetric and antisymmetric, respectively, with respect to $x'$, one gets

$$R(B) = \left( \int_{|x'| < 1} p_+(f_+^+(|x'|)) dx' + \int_{|x'| < 1} p_-(f_-^-(|x'|)) dx' \right) e_d.$$

Therefore

$$R(B) = a_{d-1} \left( R_+(f_+) + R_-(f_-) \right) : c_d,$$  
(9)

where $a_{d-1}$ is the volume of a unit ball in $\mathbb{R}^{d-1}$, and

$$R_\pm(f) = \int_0^1 p_\pm(f'(t)) dt^{d-1}.$$  
(10)

From the formulas (9), (10), and (12) it is seen that the problem of finding the functions $f_+$, $f_-$, minimizing $R(B)$, can be solved in two steps. First, find the values

$$\inf_{f \in \mathcal{M}(h_\pm)} R_\pm(f)$$

in the class $\mathcal{M}(h_\pm)$ of negative convex functions $f$ such that $f(0) = -h_\pm$. Second, minimize the sum $\inf_{\mathcal{M}(h_+)} R_+(f) + \inf_{\mathcal{M}(h_-)} R_-(f)$ provided that $h_+ + h_- = h$.

Let us fix the sign "+" or "−", and introduce shorthand notation $p_\pm = p$, $h_\pm = h$, $R_\pm = R$.

The following auxiliary lemma is a consequence of the Pontryagin Maximum Principle (12).
Lemma 2.1. Let $\lambda > 0$, $f_h \in \mathcal{M}(h)$, $f_h(1) = 0$, and let for any point $t$ of differentiability of $f_h$, $u = f'_h(t)$ be a solution of the problem
\[
t^{d-2} p(u) + \lambda u \to \min.
\] (11)
Then $f_h$ is a unique solution of the minimization problem
\[
\inf_{f \in \mathcal{M}(h)} \mathcal{R}(f), \quad \mathcal{R}(f) = \int_0^1 p(f'(t)) \, dt^{d-1}.
\] (12)
Proof. (cf. [13]). Indeed, for any $f \in \mathcal{M}(h)$, $f \neq f_h$ one has
\[
t^{d-2} p(f'(t)) + \lambda f'(t) \geq t^{d-2} p(f'_h(t)) + \lambda f'_h(t).
\]
Moreover, this inequality is strict on a set of $t$ of positive measure. Integrating both sides of this inequality over $t \in [0, 1]$, one gets
\[
\frac{1}{d-1} \int_0^1 p(f'(t)) \, dt^{d-1} + \lambda (f(1) - f(0)) > \frac{1}{d-1} \int_0^1 p(f'_h(t)) \, dt^{d-1} + \lambda (f_h(1) - f_h(0)),
\]
and using that $f(1) \leq f_h(1) = 0$ and $f(0) = f_h(0) = -h$, one obtains that $\mathcal{R}(f) > \mathcal{R}(f_h)$.

3 Solution of the Problem in the Two-Dimensional Case

From now on, we shall assume that $d = 2$.

Using the property (e) of the function $p$, it is easy to prove that the problem
\[
\frac{p(0) - p(u)}{u} \to \max
\]
has a unique solution; denote it by $u^0$.

Also, denote
\[
B := \frac{p(0) - p(u^0)}{u^0} = -p'(u^0).
\]

There may appear three different cases:

(a) As $\lambda > B$, the unique solution of (11) is $u = 0$.

(b) As $\lambda = B$, there are two solutions: $u = 0$ and $u = u^0$.

(c) As $\lambda < B$, the solution $\hat{u}$ is unique, besides $\hat{u} > u^0$, and $p'(\hat{u}) = -\lambda$.

Consider these cases separately.

(a) $\lambda > B$. The unique solution of (11) is $u = 0$, hence, according to lemma 2.1, $f \equiv 0$ is the solution of (12) for $h = 0$, and $\mathcal{R}(f) = p(0)$. 5
(b) $\lambda = B$. There are two solutions: $u = 0$ and $u = u^0$, hence a function $f_h \in M(h)$, whose derivative takes the values 0 and $u^0$, is the solution of (12). By virtue of convexity of $f_h$, there exists $t_0 \in [0, 1]$ such that $f'_h(t) = 0$ as $t \in [0, t_0]$, and $f'_h(t) = u^0$ as $t \in [t_0, 1]$. Thus,

$$f_h(t) = \begin{cases} -h & \text{as } t \in [0, t_0] \\ -h + u^0 \cdot (t - t_0) & \text{as } t \in [t_0, 1]. \end{cases} \tag{13}$$

Taking into account that $f_h(1) = 0$, one concludes that $h \leq u^0$ and $t_0 = 1 - h/u^0$. Further, one has

$$R(f_h) = \int_0^{t_0} p(0) \, dt + \int_{t_0}^1 p(u^0) \, dt = p(0) + \frac{h}{u^0} (p(u^0) - p(0)).$$

Using the definition of $B$, one gets

$$R(f_h) = p(0) - B h.$$  

(c) $\lambda < B$. There is a unique solution $\tilde{u}$ of (11), hence the function

$$f_h(t) = -h + \tilde{u} t \tag{14}$$

solves (12), where $h = \tilde{u}$. Here the resistance equals

$$R(f_h) = p(h).$$

Define the function

$$\bar{p}(u) = \begin{cases} p(0) - B u & \text{if } 0 \leq u \leq u^0, \\ p(u) & \text{if } u \geq u^0. \end{cases}$$

Thus, one comes to

**Lemma 3.1.** The solution $f_h$ of (12) is given by (13), if $h < u^0$, and by (14), if $h \geq u^0$; moreover, $R(f_h) = \bar{p}(h)$.

Reverting to the subscripts “+” and “−” and using lemma 3.1 one concludes that the problem of finding

$$\inf_{h_+ + h_- = h} (R_+(f_{h_+}) + R_-(f_{h_-})) =: R(h)$$

amounts to the problem

$$\min_{0 \leq z \leq h} p_h(z), \quad \text{where } p_h(z) = \bar{p}_+(z) + \bar{p}_-(h - z). \tag{15}$$

The functions $\bar{p}_\pm(u)$ are continuously differentiable on $[0, +\infty)$, and their derivatives are monotone increasing, hence $\bar{p}'_\pm(z)$, $0 \leq z \leq h$ is also monotone increasing.
Using the property (d), one concludes that $B_+ > B_-$, hence there exists a unique value $u_* > u_0^+$ such that $\tilde{p}'_+(u_*) = -B_-$. Consider four cases:

1) $h < u_0^+$;
2) $u_0^+ \leq h \leq u_*$; 
3) $u_* < h < u_* + u_0^-$;
4) $h \geq u_* + u_0^-$.

In the cases 1) and 2) one has $p_h'(z) \leq p_h'(h) = \tilde{p}'_+(h) + B_- \leq 0$ as $0 \leq z \leq h$, hence $z = h$ is the solution of $|B_-|$. Therefore, the optimal value of $B_- \leq 0$, and $f_{h^+}(t) \equiv 0$.

1) $h < u_0^+$. One has $h_+ = h < u_0^+$, and $f_{h^+}$ is given by (13), with $t_0 = 1 - h/u_0^+$. The optimal body is a trapezium, with tangent of slope of its lateral sides equal to $u_0^+$. The minimal resistance equals $R(h) = p_+(0) - B_+ h + p_-(0)$.

2) $u_0^+ \leq h \leq u_*$. One has $f_{h^+}(t) = -h + h t$, hence the optimal body is an isosceles triangle. Here $R(h) = p_+(h) + p_-(0)$.

In the cases 3) and 4) one has $\tilde{p}'_+(h) > -B_- > -B_+ = \tilde{p}'_+(u_0^+)$, hence $h > u_0^+$. Further, one has $p_h'(h) = \tilde{p}'_+(h) - B_- > 0$; on the other hand, $p_h'(u_0^+) = \tilde{p}'_+(u_0^+) - \tilde{p}'_- (h - u_0^+) \leq B_- - B_+ < 0$. It follows that the minimum of $p_h$ is achieved at an interior point of $[u_0^+, h]$, so $u_0^+ < h_+ \leq h$, and $f_{h^+}(t) = (t - 1)h_+$.

3) $u_* < h < u_* + u_0^-$. Denoting $\tilde{h} = \max\{0, h - u_0^\}$, one has $\tilde{h} < u_*$, hence

$$p_h'(\tilde{h}) = \tilde{p}'_+(\tilde{h}) - \tilde{p}'_- (h - \tilde{h}) \leq \tilde{p}'_+(\tilde{h}) + B_- < 0,$$

therefore the minimum of $p_h$ is reached at an interior point of $[\tilde{h}, h]$. Thus, $0 < h_- < h - \tilde{h} \leq u_-$, and $f_{h^+}(t) = -h_-$ if $t \in [0, 1 - h_-/u_0^-]$; $f_{h^+}(t) = -h_+ + u_0^- (t - 1 + h_-/u_0^-)$ if $t \in [1 - h_-/u_0^-, 1]$. The optimal body here is the union of a triangle and a trapezium turned over. The tangent of slope of lateral sides of the trapezium equals $-u_0^-$. The minimal resistance equals $R(h) = p_+(u_*) + p_-(0) - B_- (h - u_*)$.

4) $h \geq u_* + u_0^-$. One has $p_h'(h - u_0^-) = \tilde{p}'_+(h - u_0^-) + B_- \geq 0$, hence the minimum of $p_h$ is reached at a point of $[u_0^+, h - u_0^-]$. Thus, $h_- > u_0^-$, and $f_{h^+}(t) = t h_-$. The optimal body is a union of two isosceles triangles with common base and of heights $h_+$ and $h_-$ defined from the relations $h_+ + h_- = h$, $p'_{h_+}(h_+) = p'_{h_-}(h_-)$, $h_+ \geq u_0^+$, $h_- \geq u_0^-$. The minimal resistance here equals $R(h) = p_+(h_+) + p_-(h_-)$.

4 Gaussian Distribution of Velocities – Exact Solutions

Suppose that the density $\rho$ is gaussian circular, with the mean $-V e_d$ and variance 1, i.e.

$$\rho_V(v) = \frac{1}{2\pi} e^{-\frac{1}{2}(v + V e_d)^2}.$$ (16)

Here, the value $V$ is allowed to vary, so we shall denote the pressure functions by $p_\pm(u, V)$ instead of $p_\pm(u)$. Fixing the sign "+" and passing to polar coordinates
\[ v = (-r \sin \varphi, -r \cos \varphi) \] in the formula (8), one obtains

\[ p_+ (u, V) = \int \int \frac{r^2 (\cos \varphi + u \sin \varphi)}{1 + u^2} \rho_+ (r, \varphi, V) r dr d\varphi, \] (17)

where \( x_+ := \max\{x, 0\} \), and \( \rho_+ (r, \varphi, V) \) is the gaussian density (16) written in the introduced polar coordinates,

\[ \rho_+ (r, \varphi, V) = \frac{1}{2\pi} e^{-\frac{1}{2} (r^2 - 2r \cos \varphi + V^2)}. \] (18)

Next, fixing the sign "-" and introducing polar coordinates in a slightly different way, \( v = (-r \sin \varphi, r \cos \varphi) \), one obtains

\[ p_- (u, V) = -\int \int \frac{r^2 (\cos \varphi + u \sin \varphi)}{1 + u^2} \rho_- (r, \varphi, V) r dr d\varphi. \] (19)

Here \( \rho_- (r, \varphi, V) \) is the same density (16) written in these coordinates,

\[ \rho_- (r, \varphi, V) = \frac{1}{2\pi} e^{-\frac{1}{2} (r^2 + 2r \cos \varphi + V^2)}. \] (20)

Combining the formulas (17), (18), (19), and (20), one comes to the more general expression

\[ p_{\pm} (u, V) = \pm e^{-V^2/2} \int \int \left\{ \frac{(\cos \varphi + u \sin \varphi)^2}{1 + u^2} e^{-\frac{1}{2} (r^2 \pm 2r V \cos \varphi + V^2)} \right\} r dr d\varphi. \] (21)

Passing to the iterated integral and integrating over \( r \), one obtains

\[ p_{\pm} (u, V) = \pm \frac{e^{-V^2/2}}{\pi} \int \frac{\left( \cos \varphi + u \sin \varphi \right)^2}{1 + u^2} l(\pm V \cos \varphi) d\varphi, \]

where

\[ l(z) = 1 + \frac{z^2}{2} + \frac{\sqrt{\pi}}{2\sqrt{2}} e^{z^2/2} (2z + z^3) \left( 1 + \mathrm{erf} \left( \frac{z\sqrt{2}}{\sqrt{\pi}} \right) \right). \] (22)

Changing the variable \( \tau = \varphi - \arcsin (u/\sqrt{1 + u^2}) \), one finally comes to

\[ p_{\pm} (u, V) = \pm \frac{e^{-V^2/2}}{\pi} \int_{-\pi/2}^{\pi/2} \cos^2 \tau l(\pm z(\tau, u, V)) d\tau, \] (23)

where \( z(\tau, u, V) = V (\cos \tau - u \sin \tau)/\sqrt{1 + u^2} \).

Using (23), one comes to the asymptotic formula

\[ p_{\pm} (u, V) = \pm \frac{1}{2} + \frac{V}{\sqrt{1 + u^2}} + O(V^2), \quad V \to 0 + . \]

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On the other hand, using (22), in the limit $V \to \infty$ one obtains the asymptotic formula for pressure

$$p_+(u, V) = \frac{V^2}{1 + u^2} (1 + o(1)), \quad p_-(u, V) = o(V^2);$$

here the main term coincides, up to the factor $V^2$, with the pressure in Newton (i.e. zero-temperature) case.

Numerical simulations were done using Maple. The results are presented on figures 1, 2 and 3.

Graphs of the three functions, shown on figure 1, separate the $V$-$h$ plane into four regions, which correspond to the four possible solutions. The lower function tends to 1 as $V \to \infty$. As $V = 0$, the lower, the middle, and the upper functions take the values $a$, $a$, and $2a$, respectively, where $a = \sqrt{(1 + \sqrt{5})/2} \approx 1.27$.

Let $R(V, h)$ be the minimal value of resistance at given values $V$ and $h$; define reduced resistance by $\tilde{R}(V, h) = R(V, h)/V^2$. The graphs of $\tilde{R}(V, h)$ versus $h$ are...
Figure 2: Reduced resistance $\tilde{R}(V, h)$ versus height $h$ of the body
Figure 3: Reduced resistance $\tilde{R}(V, h)$ versus height $h$ of the body
shown on figures 2 and 3 for different values of $V$. As $V \to \infty$, $\bar{R}(V, h)$ tends to

\[
\bar{R}(\infty, h) = \begin{cases} 
    1 - h/2 & \text{if } h \leq 1 \\
    1/(1 + h^2) & \text{if } h \geq 1.
\end{cases}
\]

As $V \to 0$, $\bar{R}(V, h)$ goes to infinity, besides

\[
\sqrt{2/\pi} \cdot \lim_{v \to \infty} (V \bar{R}(V, h)) = \begin{cases} 
    2 - h/a^5 & \text{if } h \leq 2a \\
    4/(4 + h^2) & \text{if } h \geq 2a.
\end{cases}
\]

5 Conclusions and Final Comments

In this paper, we treat one of the earliest problems in optimal control: Newton’s problem of minimal resistance. We have obtained a full analytical description of the case when a two-dimensional body moves through a rarefied medium of infinitesimal particles, whose velocities are distributed according to the gaussian law. From the physical viewpoint, such a medium is just a homogeneous gas of positive temperature, while the case of immovable particles is related to a zero-temperature gas.

The analytical formulas of the two-dimensional problem considered in this work are quite complicated, even for Maple, but using the current computational power, numerical simulations are made within a few days of calculations. The study of the tree-dimensional case would be a natural next step. However, calculation of the resistance functions and the whole analysis are more involved. This question is under development, and will be addressed elsewhere.

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