Black-Litterman in Continuous Time: 
The Case for Filtering

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In this article, we extend the Black-Litterman approach to a continuous time setting. We model analyst views jointly with asset prices to estimate the unobservable factors driving asset returns. The key in our approach is that the filtering problem and the stochastic control problem are effectively separable. We use this insight to incorporate analyst views and non-investable assets as observations in our filter even though they are not present in the portfolio optimization.

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1. Introduction

Black and Litterman (1991, 1992) propose a one-period mean-variance optimisation in which the expected risk premia of the assets incorporate views formulated by securities and market analysts. This model can be viewed as the Bayesian updating of a prior distribution of risk premia (without views) into a posterior distribution reflecting the views. Black and Litterman derive a prior distribution through a reverse optimisation of the market portfolio under the assumption that it corresponds to the equilibrium portfolio. Then they derive the posterior distribution of the risk premia using Theil’s mixed estimation model (see Theil (1971)). Finally, they use the expected (posterior) risk premia in a mean-variance optimisation to obtain an updated asset allocation.

The Black Litterman model is very popular. It has been extended and reformulated by He and Litterman (1999), Litterman and the Quantitative Research Group (2003), Idzorek (2004), Walters (2011), among others. A search on JSTOR for the terms “Black” and “Litterman” generated 184 results and a search on Econlit returned 26 entries between 2007 and 2012. SSRN lists 57 papers for the more restrictive search on “Black-Litterman.” However, none of these articles...
or papers propose an implementation of the Black-Litterman model in a continuous time setting.

In this article we show how standard filtering arguments can be used to incorporate views in a continuous time asset allocation. Filtering theory has developed considerably since the seminal work by Kalman (1960) and Kalman and Bucy (1961). Readers should refer to Chapter 6 in Øksendal (2003) for a concise introduction to filtering and to Bain and Crisan (2009) for an excellent treatment of filtering theory and applications. Filtering techniques quickly gained acceptance in stochastic control, as evidenced by Bucy and Joseph (1987), Davis (1977) or Bensoussan (2004). In a portfolio management context, Brennan (1998) and Xia (2001) used filtering to estimate the parameters of their models. Nagai and Peng (2002) and Davis and Lleo (2011) also applied filtering techniques to risk-sensitive asset management models.

The key is that the filtering problem and the stochastic control problem are effectively separable. We use this insight to incorporate analyst views and non-investable assets as observations in our filter even though they are not present in the portfolio optimization. The paper is organised as follows. We introduce the asset market in section 2. The financial market is comprised of investable and non-investable assets and we assume that unobservable factors drive the evolution of asset prices. Asset managers can hold and trade investable assets, but not non-investable assets. We treat the latter as an additional source of observation to estimate the factors as part of the filtering. In Section 3 we propose a model for analyst views and demonstrate how to estimate the factors via a Kalman filter on the assets and views. Then we show how to incorporate these estimates in a risk-sensitive asset management model.

2. The Financial Market: Asset Prices Are Driven By Unobservable Factors

We start by considering an asset market comprising $M = m_1 + m_2, m_1 \geq 1, m_2 \geq 0$ risky securities $S_i, i = 1, \ldots, M$ and a money market account process $S_0$. The growth rates of the assets depend on $n$ unobservable factors $X_1(t), \ldots, X_n(t)$ which follow the affine dynamics given in the equation (1) below.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the underlying probability space. On this space is defined an $\mathbb{R}^N$-valued $(\mathcal{F}_t)$-Brownian motion $W(t)$ with components $W_j(t), j = 1, \ldots, N$, and $N := n + M + k$. We are in an incomplete market setting with $n$ sources of risks corresponding to the factors, $M$ sources of risk related to the assets and $k$ sources of uncertainty about the analyst views.

The asset returns and risk premia are subject to the evolution of the $n$-dimensional vector of unobservable factors $X(t)$ modelled using an affine dynamics:

$$dX(t) = (b + BX(t))dt + \Lambda dW(t), \quad X(0) = x. \quad (1)$$

We derive an estimate $\hat{X}(t)$ for the factor process $X(t)$ using filtering in the next section. Once we have obtained the estimate, we can solve the optimisation problem.

The dynamics of the money market asset $S_0$ is given by

$$\frac{dS_0(t)}{S_0(t)} = (a_0 + A'_0X(t)) dt, \quad S_0(0) = s_0, \quad (2)$$
and that of the \( M \) risky assets follows the SDEs
\[
\frac{dS_i(t)}{S_i(t)} = (a + AX(t))_i dt + \sum_{k=1}^{N} \sigma_{ik} dW_k(t) \quad S_i(0) = s_i, \quad i = 1, \ldots, M \tag{3}
\]
or alternatively the SDE
\[
dS(t) = D(S(t))(a + AX(t))dt + D(S(t)) \Sigma dW(t), \quad S_i(0) = s_i, \quad i = 1, \ldots, M.
\]

We also assume that no two assets have the same risk profile:
\textbf{Assumption 2.1}: The matrix \( \Sigma \Sigma' \) is positive definite.

The investor is allowed to trade in the first \( m_1 \) securities but not in the next \( m_2 \) securities.
In the case when \( m_2 = 0 \) then \( M = m_1 \) which means that the investor can trade on the entire market. When \( m_2 \neq 0 \) we can decompose the asset price vector \( S(t) \) as:
\[
S(t) = \begin{pmatrix} S_1(t) \\ S_2(t) \end{pmatrix}
\]
where \( S_1(t) \) is the \( m_1 \)-dimensional process of investable asset prices and \( S_2(t) \) is the \( m_2 \)-dimensional process of non-investable asset prices. We further define vectors and matrices of parameters for each segment of the market:
\[
a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_1 \\ \Sigma_2 \end{pmatrix}
\]

We focus on the discounted prices and the risk premia. The money market rate is the base rate or break-even rate for the portfolio and it does not affect the optimisation problem. This observation, which probably dates back to the development of the CAPM in the 1960s, is at the heart of the Black and Litterman (1992) model. Therefore the portfolio construction process is akin to selecting an optimal exposure to various risk premia.

The discounted asset price \( \tilde{S}_i(t) \) are
\[
\frac{\tilde{S}_i(t)}{\tilde{S}_0(t)} = \frac{S_i(t)}{S_0(t)}, \quad i = 1, \ldots, M, \quad \tilde{S}_i(0) = \frac{s_i}{s_0}
\]
and the dynamics of \( \tilde{S}(t) \) are
\[
\frac{d\tilde{S}_i(t)}{\tilde{S}_i(t)} = (\tilde{a} + \tilde{A}X(t))_i dt + \sum_{k=1}^{N} \sigma_{ik} dW_k(t) \quad \tilde{S}_i(0) = \frac{s_i}{s_0}, \quad i = 1, \ldots, M, \tag{4}
\]
where \( \tilde{a} = a - a_0 \mathbf{1}, \tilde{A} = A - A_0 \mathbf{1} \) and \( \mathbf{1} \in \mathbb{R}^M \) denotes the \( M \)-element column vector with entries equal to 1. Alternatively, we could express the asset market dynamics more synthetically as
\[
dS(t) = D(S(t))(a + AX(t))dt + D(S(t)) \Sigma dW(t), \quad S_i(0) = s_i, \quad i = 1, \ldots, M
\]
where $D(S(t))$ denotes the diagonal matrix with $S_1(t), \ldots, S_M(t)$ on the diagonal.

The relation between discounted prices $\tilde{S}_i(t)$ and risk premium $\pi_i(t)$ is simply

$$\pi_i(t) = \log(\tilde{S}_i(t)), \quad i = 1, \ldots, M$$

and hence

$$d\pi_i(t) = \left[ (\bar{a} + \bar{A}X(t))_i - \frac{1}{2} \sum_{ij} \sigma_{ij} \right] dt + \sum_{k=1}^N \sigma_{ik} dW_k(t), \quad \pi_i(0) = \log \frac{s_i}{s_0}, \quad i = 1, \ldots, M \quad (5)$$

Note that the dynamics of the risk premia is Gaussian (conditional on $X_t$) which enables us to use linear filtering. Nagai and Peng (2002) took a similar road: they defined the log returns $\log S_i(t), i = 1, \ldots, m_1$ as their observation vector.

3. The Asset Allocation Model: Incorporating Analyst Views and Market Information Into The Estimation Process

The continuous time asset allocation model we propose follows three steps which we details in the next subsections:

(i) Express the analyst views;
(ii) Filter the views and asset prices to estimate the factors;
(iii) Solve the stochastic control problem.

3.1. Express the Analyst Views

We ask analysts to formulate today views about risk premia or the spread between risk premia over a time horizon. A typical analyst statement would be: “my research leads me to believe that the spread between 10-year Treasury Notes and 3-month Treasury Bills will remain low over the next year before gradually widening over the following 2 years to 200 basis points in response to improving macroeconomic conditions.” Mathematically, we can translate the $k$ views $Z(t)$ expressed by the analysts into a system of ordinary differential equations:

$$dZ(t) = (a_Z + A_Z X(t))dt, \quad Z(0) = z \quad (6)$$

Because analyst predictions are not fully accurate, we introduce a white noise term to construct a dynamic confidence interval around the views:

$$dZ(t) = (a_Z + A_Z X(t))dt + \varphi \mathbb{W} dt, \quad Z(0) = z \quad (7)$$

where $\mathbb{W}$ is a $k$-dimensional white noise process and $\varphi$ is a $k \times k$ matrix. If we assume that the analysts formulate their views independently from each other then $\varphi$ will be a diagonal matrix. If we chose to model herding behaviour among the analysts $\varphi$ will reflect the correlation between the forecasting error terms.
Finally, we express (7) as a stochastic differential equation

\[ dZ(t) = (a_Z + A_Z X(t)) dt + \Psi dW(t), \quad Z(0) = z \]  

where \( W(t) \) is the \( N \)-dimensional Brownian motion and \( \Psi \) is a \( k \times N \) matrix with zeros on its first \( (n + M) \) rows.

This entire construction takes place at initial time \( t = 0 \). We are neither modelling the arrival of new opinions nor the evolution of the views over time. Indeed, we do not believe that either can be predicted: any attempt to model these aspects would defeat the purpose of the analysis. Rather, we are modelling the view formulated today about the evolution of a the risk premia.

### 3.2. Filter the Views and Asset Prices to Estimate the Factors

The money market rate \( r(t) = a_0 + A_0 X(t) \) is observed directly, which generates some complication for the estimation process. We start by solving the case \( A_0 = 0 \) before sketching the argument required in the case when \( A_0 \neq 0 \)

#### 3.2.1. The Case of a Globally Risk Free Money Market Rate: \( A_0 = 0 \)

We have three sources of observations for the risk premia:

(i) \( m_1 \) investable risky assets \( S_1(t), \ldots, S_{m_1}(t) \);
(ii) \( m_2 \) non-investable risky assets \( S_{m_1+1}(t), \ldots, S_M(t) \);
(iii) \( k \) analyst views \( Z_1(t), \ldots, Z_k(t) \).

The pair of processes \((X(t), Y(t))\), where

\[
Y_i(t) = \begin{cases}
\pi_i(t) = \log \frac{S_i(t)}{S_0(t)}, & i = 1, \ldots, M, \\
Z_{i-M}(t), & i = M + 1, \ldots, M + k
\end{cases}
\]

takes the form of the ‘signal’ and ‘observation’ processes in a Kalman filter system, and consequently the conditional distribution of \( X(t) \) is normal \( N(\hat{X}(t), P(t)) \) where \( \hat{X}(t) = E[X(t)|F_Y t] \) satisfies the Kalman filter equation and \( P(t) \) is a deterministic matrix-valued function.

The dynamics of the elements \( Y_i(t), i = 1, \ldots, M \) of the observation vector \( Y(t) \) satisfy

\[ dY_i(t) = (a + \bar{A} X(t)) dt - \frac{1}{2} \Sigma \Sigma_i \]  

\[ dt + \sum_{j=1}^{N} \sigma_{ij} dW_j(t), \quad i = 1, \ldots, M \]

\[ dY_{i+M}(t) = (a_Z + A_Z X(t)) dt + \sum_{j=1}^{N} \psi_{ij} dW_j(t), \quad i = 1, \ldots, k. \]

We express the dynamics of \( Y(t) \) succinctly as

\[ dY(t) = (a_Y + A_Y X(t)) dt + \Xi dW(t), \quad Y(0) = y_0, \]
where the \((M+k)\)-element vector \(a_Y\), \((M+k) \times n\) matrix \(A_Y\) and the \((M+k) \times N\) matrix \(\Xi\) are given by

\[
\begin{align*}
a_Y &= \left( \tilde{a} - \frac{1}{2} \Sigma Z \right), \\
A_Y &= \begin{pmatrix} \tilde{A} \\ A_Z \end{pmatrix}, \\
\Xi &= \begin{pmatrix} \Sigma \\ \Psi \end{pmatrix}.
\end{align*}
\]

Next, we define processes \(Y^1(t), Y^2(t) \in \mathbb{R}^m\) as follows.

\[
Y^1(t) = A_Y X(t) dt + \Psi dW(t), \quad Y^1(0) = 0 
\]

\[
Y^2(t) = a_Y \cdot dt, \quad Y^2(0) = y_0 
\]

so that \(Y(t) = Y^1(t) + Y^2(t)\).

In the present case we need to assume that \(X_0\) is a normal random vector \(N(m_0, P_0)\) with known mean \(m_0\) and covariance \(P_0\), and that \(X_0\) is independent of the Brownian motion \(W\). The processes \((X(t), Y^1(t))\) satisfying (1) and (12) and the filtering equations, which are standard, are stated in the following proposition.

**Proposition 3.1 Kalman Filter:** The conditional distribution of \(X(t)\) given \(\mathcal{F}_t\) is \(N(\hat{X}(t), P(t))\), calculated as follows.

(i) The innovations process \(U(t) \in \mathbb{R}^{M+k}\) defined by

\[
dU(t) = (\Xi \Xi')^{-1/2} (dY(t) - A_Y \hat{X}(t) dt), \quad U(0) = 0 
\]

is a vector Brownian motion.

(ii) \(\hat{X}(t)\) is the unique solution of the SDE

\[
d\hat{X}(t) = (b + B \hat{X}(t)) dt + \dot{\Lambda}(t) dU(t), \quad \hat{X}(0) = m_0, 
\]

where

\[
\dot{\Lambda}(t) = (\Lambda \Xi' + P(t) A_Y') (\Xi \Xi')^{-1/2}.
\]

(iii) \(P(t)\) is the unique non-negative definite symmetric solution of the matrix Riccati equation

\[
\dot{P}(t) = \Lambda P(t) (\Xi')^{-1} A_Y + A_Y P(t) \left( B - \Lambda \Xi' (\Xi \Xi')^{-1} A_Y \right) P(t) \\
+ P(t) \left( B' - A_Y' (\Xi \Xi')^{-1} \Xi A_Y \right), \quad P(0) = P_0,
\]

where \(\Pi^{-1} := I - \Xi' (\Xi \Xi')^{-1} \Xi\).

Now the Kalman filter has replaced our initial state process \(X(t)\) by an estimate \(\hat{X}(t)\) with dynamics given in (15). To recover the asset price process, we use (9)-(10) together with (14) to obtain the dynamics of \(Y(t)\):

\[
dY(t) = dY_1(t) + dY_2(t) \\
= (a_Y + A_Y X(t)) dt + (\Xi \Xi')^{1/2} dU(t), \quad Y(0) = y_0.
\]
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and from there, we recover the dynamics of $Z(t)$, $\pi(t)$, $\tilde{S}(t)$ and finally $S(t)$.

We observe that

$$\Xi' := \begin{pmatrix} \Sigma' \Sigma' + \Psi' \\ \Psi' \Sigma' \Psi' \end{pmatrix}$$

and define the $(M + k) \times (M + k)$ matrix $(\Xi')^{1/2}$ as

$$(\Xi')^{1/2} := \begin{pmatrix} \tilde{\Sigma} \\ \tilde{\Psi} \end{pmatrix}$$

This implies that

$$\Xi' := \begin{pmatrix} \Sigma' \Sigma' + \Psi' \\ \Psi' \Sigma' \Psi' \end{pmatrix} = \begin{pmatrix} \tilde{\Sigma} \tilde{\Psi} \\ \tilde{\Psi} \tilde{\Sigma} \tilde{\Psi} \end{pmatrix}$$

and as a result

$$dZ(t) = \left( a_Z + A_Z X(t) \right) dt + \tilde{\Psi} dU(t), \quad Z(0) = z$$

$$d\pi_i(t) = \left( \tilde{a} + \tilde{A} X(t) \right)_i dt + \sum_{k=1}^{M+k} \tilde{\sigma}_{ik} dU_k(t), \quad \pi_i(0) = \log \frac{s_i}{s_0},$$

$$d\tilde{S}(t) = \left( \tilde{a} + \tilde{A} X(t) \right)_i dt + \sum_{k=1}^{M+k} \tilde{\sigma}_{ik} dU_k(t), \quad \pi_i(0) = \frac{s_i}{s_0},$$

$$dS_i(t) = \left( a + \tilde{A} X(t) \right)_i dt + \sum_{k=1}^{M+k} \tilde{\sigma}_{ik} dU_k(t), \quad S_i(0) = s_i.$$  \hfill (17)

The filtering problem is unrelated to the subsequent stochastic control problem: the dynamics of $\tilde{X}(t)$ will be the same for all investors regardless of their risk aversion or time horizon.

3.2.2. The General Case: $A_0 \neq 0$

We have four sources of observations for the risk premia:

(i) $m_1$ investable risky assets $S_1(t), \ldots, S_{m_1}(t)$;
(ii) $m_2$ non-investable risky assets $S_{m_1+1}(t), \ldots, S_M(t)$;
(iii) $k$ analyst views $Z_1(t), \ldots, Z_k(t)$.
(iv) the money market asset $S_0(t)$.

We solve this case similarly to our earlier article Davis and Lleo (2011). We observe the short rate $r(t) = a_0 + A'_0 X(t)$, and hence the 1-dimensional statistic $Y_0(t) \equiv A'_0 X(t)$, exactly. We need to assume that this observation contains positive ‘noise’, i.e. $A'_0 \Lambda \Lambda' A_0 > 0$. Changing coordinates if necessary, we can assume that $A'_0 = (0, 0, \ldots, 1)$ and hence $Y_0(t) = X_n(t)$.

Our ‘observation’ is now the $(M + k + 1)$-dimensional process $\tilde{Y} = (Y_0, \ldots, Y_M)$ and we can set up a Kalman filter system to estimate the unobserved states $\tilde{X} = (X_1, \ldots, X_{n-1})' \in \mathbb{R}^{n-1}$. 
Ultimately, our optimal strategy will take the form \( h(t, \hat{X}(t), X_n(t)) \), where \( \hat{X}(t) \) is the Kalman filter estimate for \( \bar{X}(t) \) given \( \{ \bar{Y}(u), u \leq t \} \). The details are left to the reader.

### 3.3. Solve the Stochastic Control Problem

By using the idea developed in the previous steps we will express and solve the stochastic control problem in which \( X(t) \) is replaced by \( \hat{X}(t) \) and the dynamic equation (1) by the Kalman filter. Optimal strategies take the form \( h(t, \hat{X}(t)) \). This very old idea in stochastic control goes back at least to Wonham (1968). To illustrate this point, we solve a risk-sensitive asset management problem (see for instance Bielecki and Pliska (1999) or Nagai and Peng (2002)), although the estimation method presented in this article would apply to any continuous time investment problem including the Merton model with consumption and HARA utility.

The factor process \( X(t) \) is not directly observed and the asset allocation strategy \( h_t \) must be adapted to the filtration \( F_Y^t = \sigma \{ S_i(u), Z_j, 0 \leq u \leq t, i = 0, \ldots, M, j = 1, \ldots, k \} \) generated by the asset price processes and the views.

**Definition 3.2:** An \( \mathbb{R}^m \)-valued control process \( h(t) \) is in class \( C(T) \) if the following conditions are satisfied:

(i) \( h(t) \) is progressively measurable with respect to \( \{ \mathcal{B}([0, t]) \otimes F_Y^t \}_{t \geq 0} \) and is càdlàg;
(ii) \( \mathbb{P} \left( \int_0^T |h(s)|^2 \, ds < +\infty \right) = 1, \quad \forall T > 0. \)

The wealth, \( V(t) \) of the investor in response to an investment strategy \( h(t) \in C(T) \), follows the dynamics

\[
\frac{dV(t)}{V(t)} = (a_0 + A_0' \hat{X}_t) dt + h'(t) \left( \tilde{a}_1 + \tilde{A}_1 \hat{X}(t) \right) dt + h'(t) \tilde{\Sigma} dU(t) \tag{18}
\]

with initial endowment \( V(0) = v \). Without loss of generality we will assume that \( v = 1 \).

**Definition 3.3:** An \( \mathbb{R}^m \)-valued control process \( h(t) \) is in class \( A(T) \) if the following conditions are satisfied:

(i) \( h(t) \in C(T) \);
(ii) the Doléans exponential \( \chi^h_t \), given by

\[
\chi^h_t := \exp \left\{ -\theta \int_0^t h(s)^* \tilde{\Sigma} dU_s - \frac{1}{2} \gamma^2 \int_0^t h(s)^* \tilde{\Sigma}^2_1 h(s) ds \right\} \tag{19}
\]

is an exponential martingale, i.e. \( \mathbb{E} \left[ \chi^h_T \right] = 1 \).

The investor’s objective is to maximise the risk-sensitive asset management criterion \( J(t, x, h; T, \theta) \)

\[
J(t, x, h; T, \theta) = -\frac{1}{\theta} \ln \mathbb{E} \left[ e^{-\theta \ln V_T} \right] = -\frac{1}{\theta} \ln \mathbb{E} \left[ V_T^{-\theta} \right] \tag{20}
\]
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with risk sensitivity $\theta \in (-1, 0) \cup (0, \infty)$. In the following we concentrate on the risk averse case, that is $\theta > 0$.

By Itô’s lemma,

$$
e^{-\theta \ln V(t)} = v^{-\theta} \exp \left\{ \theta \int_0^t g(\hat{X}_s, h(s); \theta)ds \right\} \chi_{t}^h.
$$

(21)

where

$$
g(x, h; \theta) = \frac{1}{2} (1 + \theta) h' \Sigma_1 \Sigma_1' h - h'(\hat{a}_1 + \hat{A}_1 x) - a_0 - A'_0 x
$$

(22)

and the exponential martingale $\chi_{t}^h$ is given by (19).

We solve the stochastic control problem associated with (20) by a change of measure argument (see Kuroda and Nagai (2002) for the detailed argument). Let $\mathbb{P}_h$ be the measure on $(\Omega, \mathcal{F}_T)$ defined via the Radon-Nikodým derivative

$$
d\mathbb{P}_h d\mathbb{P} := \chi_{T}^h.
$$

(23)

For $h \in A(T)$,

$$
W_t^h = W_t + \theta \int_0^t \Sigma_1' h(s)ds
$$

is a standard Brownian motion under the measure $\mathbb{P}_h$. Moreover the control criterion under the measure $\mathbb{P}_h$ is

$$
I(t, x, h; T, \gamma) = -\frac{1}{\theta} \ln \mathbb{E}_{t,x}^h \left[ \exp \left\{ \theta \int_t^T g(\hat{X}_s, h(s); \theta)ds \right\} \right],
$$

(24)

where $\mathbb{E}_{t,x}^h [\cdot]$ denotes the expectation taken with respect to the measure $\mathbb{P}_h$ and with initial conditions $(t, x)$. Finally, the dynamics of the state variable estate $\hat{X}(t)$ under the new measure is

$$
d\hat{X}(t) = \left( b + B\hat{X}(t) - \theta \hat{A}(t) \Sigma_1' h(t) \right) dt + \hat{A}(t)dU(t) h,
$$

$$(25)

Let $\Phi$ be the value function for the auxiliary criterion function $I(t, x; h; T, \theta)$. Then $\Phi$ is defined as

$$
\Phi(t, x) = \sup_{h \in A(T)} I(t, x; h; T, \theta).
$$

(26)

The HJB PDE associated with the control problem is

$$
\frac{\partial \Phi}{\partial t}(t, x) + \sup_{h \in \mathbb{R}^m} L_t^h(t, x, D\Phi, D^2\Phi) = 0,
$$

(27)
where
\[ L_t^h(t, x, p, M) = (b + Bx - \theta \hat{\Lambda}(t)^h) p + \frac{1}{2} \text{tr} (\hat{\Lambda}^\prime(t) M) - \frac{\theta}{2} \hat{\Lambda}^\prime(t) p - g(x, h; \theta) \] (28)
for \( r \in \mathbb{R} \) and \( p \in \mathbb{R}^n \) and subject to terminal condition \( \Phi(T, x) = 0 \).

The term inside the sup is quadratic in \( h \). It is globally convex, and as a result it admits a unique maximiser corresponding to the candidate optimal control:

\[ \hat{h}(t, x, p) = \frac{1}{1 + \theta} (\Sigma^\prime)^{-1} \left[ \tilde{a}_1 + \tilde{A}_1 \hat{X}(t) - \theta \Sigma \hat{\Lambda}(t) p \right] \]
\[ = \frac{1}{1 + \theta} (\Sigma^\prime)^{-1} \left[ \tilde{a}_1 + \tilde{A}_1 \hat{X}(t) - \theta \Sigma \hat{\Lambda}(t) p \right] \] (29)

The value function has the form
\[ \tilde{\Phi}(t, x) = \frac{1}{2} x^\prime Q(t) x + x^\prime q(t) + k(t), \] (30)
where \( Q(t) \) satisfies a Riccati equation, \( q(t) \) solves a system of linear ODEs and \( k(t) \) is obtained by direct integration. The verification argument proposed by Kuroda and Nagai (2002) concludes the resolution of the control problem.

4. Conclusion

In this article we showed how to use standard filtering arguments to incorporate analyst views in a continuous time asset allocation. We assumed that the growth rate of asset prices depends on a number of unobservable factors and used analysts views together with asset prices to estimate the factors. Analyst express views on the evolution of the risk premia at initial time only. We added a confidence interval around the forecasts and convert them into stochastic differential equations. The key in our approach is that we effectively solve the filtering problem and the stochastic control problem separately. The views and the entire asset market only appear in the filtering step to estimate the dynamics of the unobserved factor. This estimated dynamics is then used as a state in a stochastic control problem over the set of investable assets.

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