DOMINIONS IN FINITELY GENERATED NILPOTENT GROUPS

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Abstract. In the first part, we prove that the dominion (in the sense of Isbell) of a subgroup of a finitely generated nilpotent group is trivial in the category of all nilpotent groups. In the second part, we show that the dominion of a subgroup of a finitely generated nilpotent group of class two is trivial in the category of all metabelian nilpotent groups.

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Section 1. Introduction

Suppose that a nilpotent group $G$ and a subgroup $H$ of $G$ are given. Are there any elements $g \in G \setminus H$ such that any two group morphisms to a nilpotent group which agree on $H$ must also agree on $g$?

To put this question in a more general context, let $\mathcal{C}$ be a full subcategory of the category of all algebras (in the sense of Universal Algebra) of a fixed type, which is closed under passing to subalgebras. Let $A \in \mathcal{C}$, and let $B$ be a subalgebra of $A$. Recall that, in this situation, Isbell [3] defines the *dominion of $B$ in $A$ (in the category $\mathcal{C}$)* to be the intersection of all equalizer subalgebras of $A$ containing $B$. Explicitly,

$$\text{dom}^\mathcal{C}_A(B) = \{ a \in A \mid \forall C \in \mathcal{C} \forall f, g : A \to C, \text{ if } f|_B = g|_B \text{ then } f(a) = g(a) \}.$$ 

Note that the dominion depends on the category of context, that it is a subalgebra of $A$, and that it always contains $B$. If $B \not\subseteq \text{dom}^\mathcal{C}_A(B)$, we say that the dominion of $B$ in $A$ (in the category $\mathcal{C}$) is *nontrivial*, and we say the dominion is *trivial* otherwise.

Therefore, the question above is equivalent to asking whether the dominion of $H$ in $G$ (in the category $\mathcal{N}il$, consisting of all nilpotent groups) is non-trivial.

To provide some context for the results in this paper, we note that in [6] we proved that there exists an infinitely generated nilpotent group $G$ of class two, and a finitely generated subgroup $H$ of $G$, such that

$$H \not\subseteq \text{dom}^{\mathcal{N}_2}_G(H) = \text{dom}^{\mathcal{N}_{\text{nil}}}_G(H)$$

(where $\mathcal{N}_2$ denotes the variety of all nilpotent groups of class at most 2, and $\mathcal{N}_{\text{nil}}$ the category of all nilpotent groups), and the dominion of $H$ in $G$ is not finitely generated. We also proved that there exists a *finitely generated*
nilpotent group $G$ of class two, such that for any fixed $c > 1$ there exists a subgroup $H_c$ of $G$ such that

$$H_c \subseteq \text{dom}^{N_c}_G(H_c) = \text{dom}^{N_c}_G(H_c),$$

where $N_c$ is the category of all nilpotent groups of class at most $c$. We also indicated there, but gave no proof, that dominions of subgroups of finitely generated nilpotent groups are trivial in $\mathcal{Nil}$, filling the gap between the two results quoted above. We will provide a proof of this fact in the present work.

In Section 2 we will review basic facts about dominions which we will use in later parts, as well as establish notation.

The main results of this work are in two parts; in the first part, Section 3, we will prove that if $G$ is a finitely generated nilpotent group (of any class), then for all subgroups $H$ of $G$ we have that $\text{dom}^{\mathcal{Nil}}_G(H) = H$. This section is elementary, and only assumes knowledge of basic facts about finitely generated nilpotent groups, and a result of Higman [2] on amalgams of $p$-groups. We will recall these results below.

In the second part, Section 4, we will prove the analogous result for $G$ a finitely generated nilpotent group of class two, and $\mathcal{Nil}$ replaced by the category of all metabelian nilpotent groups.

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Section 2. Notation and basic facts about dominions

All groups will be written multiplicatively unless otherwise stated, and all maps will be assumed to be group morphisms unless otherwise specified. Given a group $G$, the multiplicative identity of $G$ is denoted $e_G$, although we will omit the subscript if it is understood from context. Given a group $G$, $Z(G)$ denotes the center of $G$.

Given two groups $A$ and $B$, $A \wr B$ denotes the standard wreath product of $A$ by $B$, that is, the semidirect product of $A^B$ by $B$, with $B$ acting on the index set by the regular right action. Elements of $A \wr B$ will be written as $b\varphi$, where $b \in B$ and $\varphi \in A^B$; that is, $\varphi$ is a set map from $B$ into $A$.

We briefly recall some of the basic properties of dominions in categories of groups:

**Lemma 2.1.** Fix a full subcategory $\mathcal{C}$ of $\text{Group}$, and let $G \in \mathcal{C}$. Then, for every subgroups $H$ and $K$ of $G$:

(i) $H \subseteq \text{dom}^\mathcal{C}_G(H)$.

(ii) If $H \subseteq K$, then $\text{dom}^\mathcal{C}_G(H) \subseteq \text{dom}^\mathcal{C}_G(K)$.

(iii) $\text{dom}^\mathcal{C}_G(\text{dom}^\mathcal{C}_G(H)) = \text{dom}^\mathcal{C}_G(H)$.

In particular, $\text{dom}^\mathcal{C}_G(-)$ is a closure operator on the lattice of subgroups of $G$.

**Proof:** (i) follows directly from the definition of dominion. For (ii), note that any two maps $f, g: G \to C$ which agree on $K$ must also agree on $H$, hence on $\text{dom}^\mathcal{C}_G(H)$. Finally, for (iii) note that by the definition of dominion, the equalizer subgroups of $G$ which contain $H$ and those which contain $\text{dom}^\mathcal{C}_G(H)$ are exactly the same. \hfill \qed

**Lemma 2.2.** If $\mathcal{C}$ is closed under taking quotients, then normal subgroups are dominion closed. That is, if $N \triangleleft G$, then $\text{dom}^\mathcal{C}_G(N) = N$.

**Proof:** Consider the pair of maps $\pi, \zeta: G \to G/N$, where $\pi$ is the canonical surjection onto the quotient, and $\zeta$ is the zero map. Their equalizer is $N$, so the dominion of $N$ cannot be any bigger. \hfill \qed
Lemma 2.3. If $G_1$, $G_2$, and $G_1 \times G_2$ are groups in $C$, and if $H_1$ is a subgroup of $G_1$, and $H_2$ a subgroup of $G_2$, then

$$\text{dom}^C_{G_1 \times G_2}(H_1 \times H_2) = \text{dom}^C_{G_1}(H_1) \times \text{dom}^C_{G_2}(H_2).$$

That is, the dominion construction respects finite direct products.

Proof: Identify $G_1$ with the subgroup $G_1 \times \{e\}$ of $G_1 \times G_2$, and analogously with $G_2$. First we claim that $\text{dom}^C_{G_1 \times G_2}(H_1) = \text{dom}^C_{G_1}(H) \times \{e\}$. Indeed, if we compare the canonical projection $\pi_2: G_1 \times G_2 \to G_2$, with the zero map onto $G_2$ we see that the dominion of $H_1$ must be contained in $G_1$. Composing any pair of maps from $G_1$ with the canonical projection $\pi_1: G_1 \times G_2 \to G_1$ on the left, we see that the dominion cannot be any bigger than $\text{dom}^C_{G_1}(H)$. And composing any maps from $G_1 \times G_2$ with the obvious immersion $i: G_1 \to G_1 \times G_2$ on the right we see that it cannot be any smaller. Symmetrically, $\text{dom}^C_{G_1 \times G_2}(H_2) = \{e\} \times \text{dom}^C_{G_2}(H)$.

By Lemma 2.1(ii), we have:

$$\text{dom}^C_{G_1}(H_1) \times \text{dom}^C_{G_2}(H_2) \subseteq \text{dom}^C_{G_1 \times G_2}(H_1 \times H_2).$$

For the reverse inclusion, let $(g_1, g_2) \notin \text{dom}^C_{G_1 \times G_2}(H_1 \times H_2)$. Without loss of generality, assume that $g_1$ is not in the dominion of $H_1$. Therefore there exists a group $K \in C$ and a pair of maps $\phi, \psi: G_1 \to K$ such that $\phi$ and $\psi$ have the same restriction to $H_1$, and $\phi(g_1) \neq \psi(g_1)$. Considering the maps $\phi \circ \pi_1$ and $\psi \circ \pi_1$, we see that they agree on $H_1 \times H_2$, but disagree on $(g_1, g_2)$. This proves the lemma.

Lemma 2.4. If $C$ is closed under taking subgroups, quotients, and finite direct products, then dominions respect quotients. That is, if $G \in C$, $H$ is a subgroup of $G$, and $N$ is a normal subgroup of $G$ such that $N \subseteq H$, then

$$\text{dom}^C_{G/N}(H/N) = \left(\text{dom}^C_G(H)\right) / N.$$
Proof: Trivially, \( \text{dom}_G^C(H)/N \subseteq \text{dom}_{G/N}^C(H/N) \), since we may compose any map \( f: G/N \to K \) with the canonical projection \( \pi: G \to G/N \) to obtain maps from \( G \) to \( K \).

For the converse inclusion, assume that \( x \notin \text{dom}_G^C(H) \). Therefore there exists a group \( K \in \mathcal{C} \) and a pair of maps \( f, g: G \to K \) such that \( f|_H = g|_H \), but \( f(x) \neq g(x) \).

Consider the induced homomorphisms \((f \times f), (f \times g): G \to K \times K\), and let \( L \) be the subgroup of \( K \times K \) generated by the images of \( G \) under these maps. Since \( f \) and \( g \) agree on \( H \), they also agree on \( N \). Since \( N \triangleleft G \), the common image of \( N \) under these two morphisms is normal in \( L \). This image is the subset of the diagonal subgroup of \( K \times K \) given by

\[
(f \times f)(N) = \{(f(n), f(n)) \mid n \in N\}.
\]

However, \((f \times f)(x)\) and \((f \times g)(x)\) are not in the same coset of \((f \times f)(N)\) in \( L \). This will prove the lemma, by moding out by \((f \times f)(N)\) to obtain maps \( G/N \to L/(f \times f)(N) \) which agree on \( H/N \) but not on \( xN \).

Indeed, \((f \times f)(x)(f \times g)(x)^{-1} = (e, f(x)g(x)^{-1})\), and the second coordinate is nontrivial. Hence, this is not a diagonal element and cannot lie in the image of \( N \). This proves the lemma. \( \square \)

Dominions are closely related to group amalgams. Recall that an amalgam of two groups \( A \) and \( C \) with core \( B \) consists of groups \( A, C, \) and \( B \), equipped with one-to-one morphisms \( \Phi_A: B \to A \) and \( \Phi_C: B \to C \). We denote this situation by writing \([A,C;B]\). We say that the amalgam is weakly embeddable in \( \mathcal{C} \) (where it is understood that \( A, B, \) and \( C \) lie in \( \mathcal{C} \)) if there exist a group \( M \in \mathcal{C} \) and one-to-one mappings

\[
\lambda_A: A \to M, \quad \lambda_C: C \to M, \quad \lambda_B: B \to M
\]

such that

\[
\lambda_A \circ \Phi_A = \lambda_B, \quad \lambda_C \circ \Phi_C = \lambda_B.
\]
We say the amalgam is strongly embeddable if, furthermore, there is no identification between elements of $A \setminus B$ and $C \setminus B$. Finally, by a special amalgam we mean an amalgam $[A, C; B]$, where there exists an isomorphism $\alpha : A \to C$ such that $\alpha \circ \Phi_A = \Phi_C$. In this case, we usually write $[A, A; B]$, with $\alpha = \text{id}_A$ being understood.

Note that a special amalgam is always weakly embeddable. Also note that if the amalgam $[A, A; B]$ is strongly embeddable in $C$, then it follows that the dominion of $B$ in $A$ (in $C$) is trivial, by looking at the equalizer of the two embeddings from $A$ into $M$. In general, the dominion of $B$ in $A$ (in $C$) is the smallest subgroup $D$ of $A$, such that $B \subseteq D$, and $[A, A; D]$ is strongly embeddable. We refer the reader to the survey article by Higgins [1] for the details.

Given two elements $x, y \in G$, we write $x^y = y^{-1}xy$, and we will denote their commutator by $[x, y] = x^{-1}y^{-1}xy$. Given two subsets $A, B$ of $G$ (not necessarily subgroups), we denote by $[A, B]$ the subgroup of $G$ generated by all elements $[a, b]$ with $a \in A$ and $b \in B$. We also define inductively the left-normed commutators of weight $c + 1$:

$$[x_1, \ldots, x_c, x_{c+1}] = [[x_1, \ldots, x_c], x_{c+1}]; \ c \geq 2.$$  

**Definition 2.5.** For a group $G$ we define the lower central series of $G$ recursively as follows: $G_1 = G$, and $G_{c+1} = [G_c, G]$ for $c \geq 1$. We call $G_c$ the $c$-th term of the lower central series of $G$; $G_c$ is generated by elements of the form $[x_1, \ldots, x_c]$ for $c \geq 2$, and $x_i$ ranging over the elements of $G$.

For future use we also define the first two terms of the derived series of $G$, $G' = [G, G]$ and $G'' = [G', G']$. Thus, $G_2 = G' = [G, G]$.

A group $G$ is nilpotent of class $c$ if and only if $G_{c+1} = \{e\}$. A group $G$ is metabelian if and only if $G'' = \{e\}$.

We will write $\mathcal{N}_c$ to denote the variety of all nilpotent groups of class at most $c$. We also write $\mathcal{A}$ to denote the variety of all abelian groups, and
we use $\mathcal{A}^2$ to denote the variety of all metabelian groups (that is, groups which are extensions of an abelian group by an abelian group). We write $\mathcal{Nil}$ to denote the category of all nilpotent groups, and $\mathcal{A}^2 \cap \mathcal{Nil}$ to denote the category of all metabelian nilpotent groups. Note that the last two categories are not varieties.

Section 3. Dominions of subgroups of finitely generated groups in $\mathcal{Nil}$

In this section we will prove that if $G$ is a finitely generated nilpotent group (of any class), and $H$ is a subgroup of $G$, then the dominion of $H$ in $G$ in the category of all nilpotent groups is trivial. The idea for the proof is simple: first we will prove that this is the case when $G$ is a finite $p$-group. Using the fact that dominions respect finite direct products, and that a finite nilpotent group is the direct product of its Sylow subgroups, we extend the result to the case when $G$ is a finite nilpotent group. Finally, we use the fact that in a finitely generated nilpotent group every subgroup is closed in the profinite topology to extend the result to all finitely generated nilpotent groups.

Suppose that $G$ is a group and $H$ is a subgroup of $G$. If $(G_i)$ is a chief series of $G$, the distinct terms of the series

$$(H \cap G_0, H \cap G_1, \ldots, H \cap G_n)$$

form a chief series of $H$, which we denote by $H \cap (G_i)$.

**Theorem 3.6.** (G. Higman [2]) Let $[A, C; B]$ be an amalgam, with $A$ and $C$ both finite $p$-groups. The amalgam is strongly embeddable into a finite $p$-group if and only if there exist chief series $(A_i)$ of $A$ and $(C_i)$ of $C$ such that

$$B \cap (A_i) = B \cap (C_i).$$

In particular, a special amalgam of finite $p$-groups is always strongly embeddable into a finite $p$-group.

\[\]
We deduce the following corollary:

**Corollary 3.7.** Let $G$ be a finite $p$-group, and let $H$ be a subgroup of $G$. Let $\mathcal{P}$ be the category of all finite $p$-groups. Then $\text{dom}^\mathcal{P}_G(H) = H$. In particular, since $\mathcal{P} \subset \text{Nil}$, we have $\text{dom}^{\text{Nil}}_G(H) = H$.

**Lemma 3.8.** Let $G$ be a finite nilpotent group, and let $H$ be a subgroup of $G$. Then $\text{dom}^{\text{Nil}}_G(H) = H$.

**Proof:** Since $G$ is finite and nilpotent, $G = \prod G_p$ where $G_p$ is the $p$-Sylow subgroup of $G$. Since $H$ is also nilpotent, we have $H = \prod H_p = \prod (G_p \cap H)$. In particular, $H_p$ is a subgroup of $G_p$.

Since dominions respect finite direct products, we have

$$\text{dom}^{\text{Nil}}_G(H) = \prod \left( \text{dom}^{\text{Nil}}_{G_p}(H_p) \right).$$

By Corollary 3.7, each of the dominions in the right hand side equals $H_p$, so

$$\text{dom}^{\text{Nil}}_G(H) = \prod H_p = H,$$

as claimed.

Finally, to take the step from finite to finitely generated, we recall the definition of the profinite topology of a group.

Given a group $G$, we define the *profinite topology* on $G$ to be the coarsest topology which makes all normal subgroups of finite index open, and makes $G$ into a topological group (so multiplication is a continuous map $G \times G \to G$, where $G \times G$ is given the product topology, and the map $G \to G$ given by $g \mapsto g^{-1}$ is also continuous).

Since the complement of a subgroup $H$ is the union of all cosets $xH$ with $x \notin H$, it follows that the normal subgroups of finite index are both open and closed.

We say that a subgroup $H$ of $G$ is *closed* in the profinite topology if it is a closed subset of the topological space $G$; equivalently, if it is the intersection
of subgroups of $G$ of finite index. Therefore, if the subgroup $H$ is closed, and $x \in G \setminus H$, then there exists a normal subgroup $N \triangleleft G$ such that $G/N$ is finite, and $x \notin HN$.

Recall also that a group $G$ is said to be polycyclic iff it has a normal series

$$\{e\} = G_n \subseteq G_{n-1} \subseteq \cdots \subseteq G_1 = G$$

such that $G_{i+1} \triangleleft G_i$, and $G_i/G_{i+1}$ is cyclic. If furthermore we may find a normal series such that $G_i \triangleleft G$ and $G_i/G_{i+1}$ is cyclic then $G$ is called supersolvable.

**Lemma 3.9.** (Theorem 31.12 in [8]) A finitely generated nilpotent group is supersolvable, hence polycyclic.

**Theorem 3.10.** (Mal’cev [7]) If $G$ is a polycyclic group, then every subgroup of $H$ is closed in the profinite topology.

In particular, every subgroup of a finitely generated nilpotent group is closed.

**Theorem 3.11.** Let $G$ be a finitely generated nilpotent group, and let $H$ be a subgroup of $G$. Then $\text{dom}_{G/N}^{\text{Nil}}(H) = H$.

**Proof:** Let $x \notin H$. By Theorem 3.10 there exists a normal subgroup $N \triangleleft G$, such that $G/N$ is finite, and $xN \notin HN/N$. Since $\text{Nil}$ is closed under quotients, subgroups, and finite direct products, it follows from Lemma 2.4 that the dominion construction in $\text{Nil}$ respects quotients. Therefore,

$$\text{dom}_{G/N}^{\text{Nil}}(HN/N) = \left(\text{dom}_G^{\text{Nil}}(HN)\right)/N.$$  

By Lemma 3.8, the dominion of $HN/N$ in $G/N$ is equal to $HN/N$. Therefore,

$$xN \notin \text{dom}_{G/N}^{\text{Nil}}(HN/N).$$

By (3.12), it follows that $x \notin \text{dom}_G^{\text{Nil}}(HN)$, and hence in particular that $x$ is not in $\text{dom}_G^{\text{Nil}}(H)$. Therefore, the dominion of $H$ in $G$ in $\text{Nil}$ is equal to $H$, as claimed.
Section 4. Dominions of subgroups of finitely generated nilpotent groups of class two in $A^2 \cap \mathbb{Nil}$

In this section we will prove the analogous results to those in the previous section, where the category of context is now $A^2 \cap \mathbb{Nil}$, and the group $G$ is restricted to the variety of nilpotent groups of class at most two.

**Theorem 4.13.** Let $G$ be a finite $p$-group lying in $N_2$, with $p$ a prime, and let $H$ be a subgroup of $G$. Then $\text{dom}_{G}^{A^2 \cap \mathbb{Nil}}(H) = H$.

**Proof:** Let $G$ and $H$ be as in the statement of the lemma. Let $N = [G, G]$ be the commutator of $G$. First note that $\text{dom}_{G}^{A^2 \cap \mathbb{Nil}}(H) \subseteq HN$. Indeed, $HN$ is normal in $G$, hence dominion closed by Lemma 2.2. Since $H \subseteq HN$, this now follows from Lemma 2.1(ii).

Also, note that $N$ is central in $G$, since $G \in N_2$. In particular, $N$ is abelian. First, we define a transversal of $N$ in $G$ (that is, a set of coset representatives). We claim that there is a transversal $\tau : G/N \to G$ (note that $\tau$ is only a set map, not a group morphism), with the following properties:

(4.14) $\tau(N) = e$.

(4.15) For every $h \in H$, $y \in G$, there exists an element $h' \in H$ such that $\tau(yh^{-1}N) = \tau(yN)h'^{-1}$.

To construct such a transversal, consider the left action of $H$ on the set of cosets of $N$, under which $h \in H$ takes $tN$ to $tNh^{-1} = th^{-1}N$. Since $N$ is normal, this is a well defined action. This action partitions the set of cosets of $N$ into orbits. For each $H$-orbit, we first define $\tau$ to take some arbitrary coset $tN$ in that orbit to any representative, which we now choose once and for all, making sure to select $e$ as a representative for $N$. For any other coset $t'N$ in the same orbit, there exists some element $h \in H$ such that

$$t' \equiv \tau(tN)h^{-1} \pmod{N},$$
because this is precisely the $H$-action. Choose such an $h$ for each coset (the choice of $h$ is only determined up to congruence modulo $H \cap N$), and define \( \tau(t'N) = \tau(tN)h^{-1} \).

Let \( \pi: G \to G/N \) be the canonical projection onto the quotient. Note that \( G/N \) is also abelian. For simplicity, we write the cosets using their chosen representatives; that is, whenever we write a coset as $tN$ it will be understood that $\tau(tN) = t$. If we wish to represent the coset of an arbitrary element $y \in G$, where $y$ is not the chosen representative, we will write $\pi(y)$ instead.

Since $G$ is an extension of $N$ by $G/N$, we can embed $G$ into \( N \wr (G/N) \) by a map $\gamma$, given by $\gamma(g) = \pi(g)\varphi_g$, where $\varphi_g \in N^{(G/N)}$, and for each $\pi(y) \in G/N$,

\[
\varphi_g(\pi(y)) = \left( \tau(y\pi(g)^{-1})g\tau(\pi(y))^{-1} \right)
\]

(this is a theorem of Kaloujnine and Krasner [5]).

Note that \( N \wr (G/N) \) is an extension of an abelian group by an abelian group, hence lies in $A^2$. Since it is also a finite $p$-group, it is nilpotent, and therefore lies in $A^2 \cap \text{Nil}$.

Consider the two group morphisms $\eta, \zeta: N \to N/H \cap N$, where $\eta$ is the canonical surjection, and $\zeta$ is the zero map. The equalizer of the two maps is exactly $H \cap N$, and the maps induce two maps

\[
\eta^*, \zeta^*: N \wr (G/N) \to (N/H \cap N) \wr (G/N)
\]

by

\[
\eta^*(\pi(g)\varphi_g) = \pi(g)(\eta \circ \varphi_g)
\]
\[
\zeta^*(\pi(g)\varphi_g) = \pi(g)(\zeta \circ \varphi_g).
\]

Note that \( (N/H \cap N) \wr (G/N) \) is also a finite $p$-group, and metabelian. We now consider the two maps $\eta^* \circ \gamma$ and $\zeta^* \circ \gamma$. 
Let $n \in N$. By definition of $\gamma$, we have $\gamma(n) = \varphi_n$, where $\varphi_n : G/N \to N$ and is given by

$$
\varphi_n(yN) = \tau(yN\pi(n)^{-1})n\tau(yN)^{-1} \\
= \tau(yN)n\tau(yN)^{-1} \\
= yny^{-1} \\
= n
$$

since $n \in N$, and $N$ is central.

Therefore $\eta \circ \varphi_n(yN) = \eta(n)$, and $\zeta \circ \varphi_n(yN) = \zeta(n) = e$. So $\eta^* \circ \gamma(n)$ is equal to $\zeta^* \circ \gamma(n)$ if and only if $\eta(n) = e$, that is if and only if $n \in H \cap N$. In particular, $(\eta^* \circ \gamma)|_N$ and $(\zeta^* \circ \gamma)|_N$ agree exactly on $H \cap N$.

We further claim that $\eta^* \circ \gamma$ and $\zeta^* \circ \gamma$ agree on $H$. Indeed, let $h \in H$. Then $\gamma(h) = \pi(h)\varphi_h$. Since $\eta^*$ and $\zeta^*$ leave the $G/N$ component unchanged, we may concentrate on $\varphi_h$.

Let $yN \in G/N$. By definition of $\gamma$ we have

$$
\varphi_h(yN) = \tau(yN\pi(h)^{-1})h\tau(yN)^{-1} \\
= \tau(yh^{-1}N)h\tau(yN)^{-1} \\
= yh'^{-1}hy^{-1}
$$

where $yh'^{-1} = \tau(yN\pi(h)^{-1})$, with $h' \in H$. This is possible by (4.15).

In particular, we have $h'^{-1} \equiv h^{-1}$ (mod $N$), so $h'^{-1} \in N$. Therefore, since $N$ is central, we have that $\varphi_h(yN) = h'^{-1}h$, where $h'$ depends on $y$ and $h$, lies in $H$, and $h'^{-1}h \in H \cap N$.

Therefore, for every $yN \in G/N$,

$$
\eta \circ \varphi_h(yN) = \eta(h'^{-1}h) = e = \zeta(h'^{-1}h) = \zeta \circ \varphi_h(yN).
$$

Therefore the two maps agree on $h$, and since $h$ was arbitrary, they agree on $H$, as claimed. In particular,

$$
dom_{G}^{A^2 \cap \Nil}(H) \subseteq \Eq(\eta^* \circ \gamma, \zeta^* \circ \gamma).
$$
Therefore, $\text{dom}_{G}^{A^{2} \cap \text{Nil}}(H) \cap N = H \cap N$. 

Now consider $d \in \text{dom}_{G}^{A^{2} \cap \text{Nil}}(H)$. Since the dominion is contained in $HN$, there exists $h \in H$ and $n \in N$ such that $d = hn$. In particular, $dh^{-1} = n$ lies in $N$. Since $dh^{-1}$ is also in the dominion, and is in $N$, it lies in $N \cap H$. In particular, $d = hn$ lies in $H$. This proves the required inclusion, and hence the theorem.

The rest of the argument now proceeds as in the previous section. We pass from finite $p$-groups to finite nilpotent groups of class two by decomposing the group into a direct product of its $p$-Sylow subgroups:

**Theorem 4.16.** Let $G \in \mathcal{N}_{2}$ be a finite group and $H$ a subgroup of $G$. Then $\text{dom}_{G}^{A^{2} \cap \text{Nil}}(H) = H$.

Finally, we use Theorem 3.10 to pass from the finite case to the finitely generated case:

**Theorem 4.17.** Let $G$ be a finitely generated group lying in $\mathcal{N}_{2}$, and let $H$ be a subgroup of $G$. Then $\text{dom}_{G}^{A^{2} \cap \text{Nil}}(H) = H$.

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