Entanglement detection from channel parameter estimation problem

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We derive a general criterion to detect entangled states in multi-partite systems based on the symmetric logarithmic derivative quantum Fisher information. This criterion is a direct consequence of the fact that separable states do not improve the accuracy upon estimating one-parameter family of quantum channels. Our result is a generalization of the previously known criterion for one-parameter unitary channel to any one-parameter quantum channel. We discuss several examples to illustrate our criterion. The proposed criterion is extended to the case of open quantum systems and we briefly discuss how to detect entangled states in the presence of decoherence.

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I. INTRODUCTION

In this paper we address a problem of general relationship between entanglement in multi-partite systems and quantum Fisher information from channel-parameter estimation perspective. This problem has gained a great interest in the field of research so called quantum metrology, that is, quantum mechanically enhanced precision measurements [1–4]. Our work is motivated by the theoretical work of Pezzé and Smirzi and the experimental verification of their criterion [5, 6]. In Ref. [5], they observed that the symmetric logarithmic derivative (SLD) quantum Fisher information for any separable state cannot be greater than the total number of qubits when qubit states undergo a global rotation along some axis. Their result was further generalized to detect \( k \)-producible states and to derive other criteria by taking averages with respect to rotation axes [7–10], see also updated references cited in review articles [2–4]. In the recent paper [6], a beautiful experimental result has been reported showing that a global rotation of atomic spin states was used to detect non-gaussian entangled states. The main objective of this paper is to generalize their criterion to any quantum channel.

The second motivation of our work is to examine whether entanglement brings benefit upon estimating parameters for non-unitary channels, i.e., general quantum channels. To answer this question, we shall use a standard language of quantum parameter estimation theory developed by Helstrom, Yuen-Lax, Holevo, and others [11–15]. A formal channel-parameter estimation problem in quantum systems was initiated by Fujiwara and his collaborators [16, 17], where they utilize basic tools developed in quantum parameter estimation theory mentioned above.

There exist at least four known no-go theorems regarding the observation of quantum metrologically enhanced measurement upon estimating one-parameter family of quantum channels. Ji et al. [18] showed a rather remarkable result that any programmable channels cannot be estimated with quantum metrological enhancement. Here a given channel is programmable or not is defined by Ref. [19]. Fujiwara and Imai [20] provided another no-go theorem stating that quantum metrological enhancement cannot occur for any full-rank channels changing smoothly with the parameter. Their result is very general and implies that almost all realistic quantum channels do not exhibit such quantum metrological enhancement. In the unpublished work, Matsumoto gave a simple criterion where any classically simulated channel cannot be estimated with quantum metrological enhancement [21]. Two results in Refs. [20, 21] are well summarized in the paper [22], where authors applied these two criteria for physically important quantum channels. Lastly, Hayashi [23] gave a very powerful argument; no quantum metrological enhancement occurs when a given channel admits finite amount of the right logarithmic derivative (RLD) quantum Fisher information.

These no-go theorems state that there are quantum channels in which we cannot utilize quantum entanglement to go beyond the standard quantum limit [24]. This poses a question whether or not non-unitary channels which satisfy the above no-go criteria can be used to detect entanglement. In this paper, we answer this question first by showing that all separable states do not bring any benefit upon estimating one-parameter channels. This is then translated into a simple yet general criterion: If the amount of SLD quantum Fisher information of the output state for a given family of one-parameter quantum channel is above a certain threshold, then the input state must be entangled. We then examine several examples to demonstrate the obtained criterion, such as the unitary channel, depolarizing channel, and transpose channel. We show that detection of entanglement is possible for certain range of parameter of these channels at least bi-partite case.

This paper is organized as follows. In Sec. II we summarize notations and discuss relationship between classical information quantities and quantum Fisher information. In Sec. III we prove the main result of this paper. In Sec. IV we apply our criterion to the bi-partite case and compare result for different quantum channels. In
Sec. V we extend our criterion from the i.i.d. setting to a more general setting in order to apply it for open quantum systems. In the last section, we summarize our results.

II. CHANNEL-PARAMETER ESTIMATION IN QUANTUM SYSTEMS

We provide basic terminologies and notations used in this paper. We then summarize the basic result known for one-parameter channel estimation problems in quantum systems.

A. Preliminaries

Let $\mathcal{H}$ be a finite dimensional Hilbert space and $S(\mathcal{H})$ be the set of all density operators on $\mathcal{H}$, which are semi-definite positive. Let $\Gamma_\theta$ be a trace-preserving and completely-positive (TP-CP) map (also called a quantum channel, a quantum map, etc.) from $S(\mathcal{H})$ to itself that is parametrized by a single parameter $\theta$:

$$\Gamma_\theta : S(\mathcal{H}) \rightarrow S(\mathcal{H}) \text{ (TP - CP).}$$

Assume that the parameter $\theta$ takes values in an open subset of real numbers, $\Theta \subset \mathbb{R}$, then the output state $\rho_\theta = \Gamma_\theta (\rho)$ for a given input state $\rho \in S(\mathcal{H})$ can be regarded as a quantum statistical model parametrized this parameter $\theta \in \Theta$:

$$\mathcal{M} = \{ \rho_\theta = \Gamma_\theta (\rho) \mid \theta \in \Theta \subset \mathbb{R} \}. \quad (2)$$

Depending on the channel and the given input state, the rank of the output states $\Gamma_\theta (\rho)$ may vary with respect to the parameter $\theta$ in general. For mathematical convenience, we further assume that the rank of the quantum statistical model $\mathcal{M}$ does not change for all values $\theta \in \Theta$ at least for each fixed input quantum state.

The SLD operator about $\rho_\theta \in \mathcal{M}$ is defined by an hermite operator $L_\theta$ satisfying the equation:

$$\frac{d}{d\theta} \rho_\theta = \frac{1}{2} (\rho_\theta L_\theta + L_\theta \rho_\theta). \quad (3)$$

The SLD quantum Fisher information about $\rho_\theta$ is defined by

$$g_\theta [\rho_\theta] := \text{tr} (\rho_\theta L_\theta^2). \quad (4)$$

By definition, it also holds that $g_\theta [\rho_\theta] = \text{tr} (L_\theta \frac{d}{d\theta} \rho_\theta)$. For full-rank states, the solution to the above operator equation (3) is unique, i.e., the SLD operator is uniquely defined. For low-rank states such as pure states, on the other hand, SLD operator is not uniquely determined from the above equation. In this case, one has to consider equivalent classes to define a proper inner product first and then to define SLD operator resulting in the unique SLD quantum Fisher information [26].

There are several important properties of the SLD quantum Fisher information to be listed below. First, it is non-negative, i.e., $g_\theta [\rho_\theta] \geq 0$.

Second, quantum Fisher information is additive, i.e., for any product state $\rho_\theta = \rho_1^\theta \otimes \rho_2^\theta \in S(\mathcal{H}_1 \otimes \mathcal{H}_2)$,

$$g_\theta [\rho_1^\theta \otimes \rho_2^\theta] = g_\theta [\rho_1^\theta] + g_\theta [\rho_2^\theta]. \quad (5)$$

Third, it cannot increase when a CP-TP $\Gamma$ is applied to the state, i.e. the following inequality holds.

$$g_\theta [\Gamma(\rho)] \leq g_\theta [\rho]. \quad (6)$$

This property is usually referred to as the monotonicity of SLD quantum Fisher information [27]. As a special case of quantum channels, let us consider a general measurement, described by a positive operator-valued measure (POVM), $\Pi = \{ \Pi_x \mid x \in \mathcal{X} \}$. The map from a state $\rho$ to a probability distribution $p_\theta (x) = \text{tr} (\rho E_x)$ is regarded as quantum to classical channel since the state $p_{\Pi} = \text{diag}(p_\theta (x_1), p_\theta (x_2), \ldots)$ describes the probability distribution for measurement outcomes. The SLD quantum Fisher information about this (classical) state is equal to the (classical) Fisher information $G_{\theta}^c [\rho_\theta]$. Since the probability distribution for measurement outcomes is determined by a given POVM $\Pi$, we also write it as $G_{\theta}^c [\Pi, \rho_\theta]$. Thus, the following inequality for any POVM holds.

$$g_\theta [\rho_\theta] \geq G_{\theta}^c [\Pi, \rho_\theta]. \quad (7)$$

We call this property as the q-c (quantum to classical) monotonicity of the SLD quantum Fisher information.

Last, it is convex with respect to quantum states. Let $\rho_\theta^1, \rho_\theta^2 \in S(\mathcal{H}) (j = 1, 2)$ be two families of states with the same parameter set $\Theta$. The convex property states

$$g_\theta [\lambda \rho_\theta^1 + (1 - \lambda) \rho_\theta^2] \leq \lambda g_\theta [\rho_\theta^1] + (1 - \lambda) g_\theta [\rho_\theta^2], \quad (8)$$

for any $\lambda \in [0, 1]$. This convexity can be proven by many ways. The simplest is given in Ref. [16] using the monotonicity of the SLD quantum Fisher information. It seems that the equality condition for the above convex inequality is in general complicated. Since this condition is important, we examine it for a simple unitary model, which is given at the end of Sec. III B.

The main objective of channel-parameter estimation in quantum systems is to find the ultimate precision bound and the optimal strategy upon estimating the value of parameters describing a given channel. Here we stress that there is no unique way to define the optimality and one has to analyze a given problem according to a suitable figure of merit. A strategy upon estimating the value of the given quantum channel consists of three elements: An input state, a measurement, and an estimator. One way to get the optimal strategy is as follows. For a fixed input state $\rho$, we optimize over all possible quantum measurements described by a POVM $\Pi$ and an estimator $\hat{\theta}$ which is a classical data processing. The set $(\Pi, \theta)$ is called a quantum estimator or simply estimator in this
paper. With this optimal estimator, we optimize over all possible input states available. A triplet \((\rho, \Pi, \theta)\) is called an estimation strategy for the quantum channel. For a one-parameter problem, this procedure gives at least asymptotically optimal one [28].

When one concerns the mean-square error (MSE) as a figure of merit for the channel estimation, one can derive the lower bounds for the MSE depending upon resources and estimation schemes under consideration. Let \(\mathcal{H}^\otimes N\) and \(\mathcal{S}(\mathcal{H}^\otimes N)\) be \(N\) tensor product of Hilbert space and totality of positive density operators on it, respectively. Consider an \(N\)th i.i.d. extension of the given channel and denote it as

\[
\Gamma^N_{\theta} := \Gamma_{\theta} \otimes \Gamma_{\theta} \otimes \cdots \otimes \Gamma_{\theta} : \mathcal{S}(\mathcal{H}^\otimes N) \rightarrow \mathcal{S}(\mathcal{H}^\otimes N). \tag{9}
\]

When one only uses \(N\)th i.i.d. extension of input states \(\rho^\otimes N\) to estimate the channel, the problem is to find an optimal input state maximizing the SLD quantum Fisher information for the channel \(\Gamma_{\theta}\). Let \(\rho^*\) be one of such an optimal input state and \(g_{\theta}^\rho\) be the maximum of the SLD quantum Fisher information;

\[
g_{\theta}^\rho(\Gamma_{\theta}) := \max_{\rho \in \mathcal{S}(\mathcal{H})} \{g_{\theta}[\Gamma_{\theta}(\rho)]\}. \tag{10}
\]

Importantly, the convexity property of the SLD Fisher information guarantees that the optimal input state attaining \(g_{\theta}^\rho(\Gamma_{\theta})\) can be a pure state [16].

The additivity of SLD quantum Fisher information concludes \(g_{\theta}(\Gamma^N_{\theta}(\rho^\otimes N)) = Ng_{\theta}(\Gamma_{\theta}(\rho))\). For any locally unbiased estimators \((\Pi, \theta)\), the MSE is equal to the variance of estimating the value of parameter and is bounded as

\[
\operatorname{Var}_{\theta}[\Pi, \hat{\theta}] \geq \frac{1}{N} (g_{\theta}^\rho(\Gamma_{\theta}))^{-1}. \tag{11}
\]

In general, this bound is attained adaptively in the \(N\) infinite limit unless the channel possesses a special symmetry. See the discussion given in Ref. [30] and an experimental demonstration of the adaptive estimation [31]. Alternatively, one can use the two-step method proposed in Refs. [32, 33].

When one estimates the \(N\)th i.i.d. extension of the channel \(\Gamma^N_{\theta}\), one can also use other resources such as entangled states \(\rho \in \mathcal{S}(\mathcal{H}^\otimes N)\) for input states or ancillary states. In this case, the variance for estimation can be further lowered. This enhancement effect, known as quantum metrology, is of importance for quantum information processing protocols and has been investigated actively [1–3].

**B. Experimental detection of SLD quantum Fisher information**

In this subsection, we discuss a general strategy how to detect the amount of quantum Fisher information about the output state of a given family of quantum channels in experiment. We assume that the parameter for the quantum channel can be tuned at will and there are identical resources to repeat the same experiment sufficiently many times. A prominent step was already reported in Ref. [6]. In this paper we shall present a more general framework to supplement their result.

For a given one-parameter family of quantum channels \(\Gamma_{\theta}\), let us fix the input state \(\rho\) and consider a fixed measurement \(\Pi\) on the output state \(\rho_0 = \Gamma_{\theta}(\rho)\). Then, the family of probability distributions for the measurement outcomes is regarded as a classical statistical model:

\[
\mathcal{M}(\Pi, \Gamma_{\theta}, \rho_0) = \{p_0[\Pi] | \rho \in \Theta\}, \tag{12}
\]

By performing sufficiently large repetition of the same measurement for a fixed value of the parameter \(\theta\), we can obtain experimental data according to the classical probability distribution \(p_0[\Pi]\). We next change the channel parameter \(\theta\) and redo the same step as before. After sufficiently many observations with respect the changes in \(\theta\), say \(M\) different choices, we can obtain the set of classical probability distributions \(\{p_\theta[\Pi] | \theta \in \{\theta_1, \theta_2, \ldots, \theta_M\}\}\). If we choose the parameter set \(\{\theta_1, \theta_2, \ldots, \theta_M\}\) \((\theta_{k+1} > \theta_k)\) such that the differences \(\Delta_k = \theta_{k+1} - \theta_k\) is sufficiently small, then one can directly calculate the classical Fisher information \(G_{\theta}^\rho[p_\theta]\) approximately from the definition:

\[
G_{\theta}^\rho[p_\theta] := \sum_{x \in \mathcal{X}} \frac{|\frac{\partial x}{\partial \theta} p_\theta(x) - \frac{\partial x}{\partial \theta} p_\theta(x)|^2}{p_\theta(x)}. \tag{13}
\]

Alternatively, one can estimate other information quantities first and then to calculate the classical Fisher information as follows. In classical information theory, the general information quantity is the \(f\)-divergence [34]. This family of information quantity is a measure of “distance” between two probability distributions. The formal definition of \(f\)-divergence for two probability distributions \(p, q\) on \(\mathcal{X}\) is

\[
D_f(p || q) := \sum_{x \in \mathcal{X}} p(x) f\left(\frac{q(x)}{p(x)}\right), \tag{14}
\]

where \(f : \mathbb{R}^+ \rightarrow \mathbb{R}^+\) is a monotonically decreasing and convex function and \(f(1) = 0\) is a standard convention. Familiar examples are: \(f(t) = -\log(t)\) (the relative entropy), \(f(t) = 1 - \sqrt{t}\) (the Hellinger distance), \(f(t) = t^\alpha\) (the relative Rényi entropy). One of important properties of the \(f\)-divergence is that the following relation to Fisher information:

\[
G_{\theta}^\rho[p_\theta] = 2 \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^2} D_f(p_\theta || p_{\theta + \epsilon}). \tag{15}
\]

From experimental data obtained after many repetitions, we can construct a curve for \(f\)-divergence \(D_f(p_\theta || p_{\theta'})\) for various different values of \(\theta, \theta'\). It is easy to see that the formula (15) provides an approximated value for the classical Fisher information.
We next show that this experimentally obtained Fisher information can attain the SLD quantum Fisher information by the optimal measurement. As remarked in the previous subsection, q-c monotonicity of the SLD quantum Fisher information implies

$$g_0[\Gamma_\theta(\rho)] \geq G_0^\theta[\Pi],$$

(16)

where the equality holds if and only if $\Pi$ is optimal one and it is given by the projection measurement about the SLD operator $L_0$ [35–37]. By choosing the optimal measurement, the classical Fisher information obtained from the above described method yields the approximated value of the quantum Fisher information.

III. RESULT

A. Separability criterion

The main result of this paper is the following theorem:

**Theorem III.1** For a given channel $\Gamma_\theta$ parametrized by a single parameter $\theta$, let $\Gamma_\theta^N$ be an $N$th i.i.d. extension of $\Gamma_\theta$ and $g_\theta^\theta(\Gamma_\theta)$ be the largest value of SLD quantum Fisher information, which is given by Eq. (10). For each value $\theta$, if a density operator $\rho$ on $\mathcal{S}(\mathcal{H}^\otimes N)$ is separable, then the SLD quantum Fisher information $g_\theta[\Gamma_\theta^N(\rho)]$ is smaller or equal to the value $Ng_\theta^\theta(\Gamma_\theta)$.

Several remarks are in order. First, taking the contrapositive of this theorem, it is equivalent to state if the value of SLD quantum Fisher information for the output states $\Gamma_\theta^N(\rho)$ is larger than $Ng_\theta^\theta(\Gamma_\theta)$, then the input state $\rho$ on $\mathcal{S}(\mathcal{H}^\otimes N)$ is entangled. Second, the special case of this separability criterion was stated by Pezzé and Smerzi [5], where the channel is given by a rotation along a given axis on $N$ qubits in the context of quantum metrology. In this case, $g_\theta^\theta = 1$ holds for all values of $\theta$ due to symmetry of a shift-parameter model. This special case will be examined in the next subsection. Thus, our contribution is to first prove their criterion in a general setting and to provide more general criterion.

Third, since the MSE for estimation of the value $\theta$ is bounded by the inverse of SLD quantum Fisher information, Theorem III.1 states that separable states are not efficient upon the usage of $N$th extension of a given channel. But, of course, this theorem does not tell if all entangled states are more efficient than separable ones or not.

Fourth, it is straightforward to see this theorem can be extended to more general channels parameterized by several parameters. In this case, the SLD quantum Fisher information becomes a matrix and the corresponding inequality is given by a matrix inequality. It is also not difficult to see from the proof that SLD quantum Fisher information can be replaced by other quantum Fisher information.

Lastly, the most important is that the parameter $\theta$ is arbitrary value in Theorem III.1. Since we can vary it as an arbitrary value, one take a union of all possible parameter regions of entangled states. Let $r_{\text{ent}}(\theta)$ be the entangled region of the states derived from the inequality $g_\theta[\Gamma_\theta^N(\rho)] > Ng_\theta^\theta(\Gamma_\theta)$, i.e.,

$$r_{\text{ent}}(\theta) = \{\rho \in \mathcal{S}(\mathcal{H}^\otimes N) | g_\theta[\Gamma_\theta^N(\rho)] > Ng_\theta^\theta(\Gamma_\theta)\},$$

(17)

then, the union

$$R_{\text{ent}} := \bigcup_{\theta \in \Theta} r_{\text{ent}}(\theta),$$

(18)

provides the most powerful criterion. Since the subset of states $R_{\text{ent}} \subset \mathcal{S}(\mathcal{H}^\otimes N)$ is solely determined by the given quantum channel $\Gamma_\theta$, we denote it as $\rho \in R_{\text{ent}}[\Gamma_\theta]$. With these notations, our contribution is to derive the new criterion:

$$\rho \in R_{\text{ent}}[\Gamma_\theta] \Rightarrow \rho \text{ is entangled.}$$

(19)

This point will be illustrated by several examples in Sec. IV.

Proof for Theorem III.1 is straightforward and is given as follows:

**Proof:** Consider an arbitrary separable states on $\mathcal{S}(\mathcal{H}^\otimes N)$ of the form

$$\rho_{\text{sep}} = \sum_j p_j \rho_j^{(1)} \otimes \rho_j^{(2)} \otimes \cdots \otimes \rho_j^{(N)},$$

(20)

where $\sum_j p_j = 1$, $\forall p_j \geq 0$ and $\rho_j^{(k)}$ are states on the $k$th Hilbert space. Then, the following sequence of inequalities holds,

$$g_\theta[\Gamma_\theta^N(\rho_{\text{sep}})] = g_\theta[\sum_j p_j \Gamma_\theta^N(\rho_j^{(1)} \otimes \cdots \otimes \rho_j^{(N)})]$$

$$\leq \sum_j p_j g_\theta[\Gamma_\theta^N(\rho_j^{(1)} \otimes \cdots \otimes \rho_j^{(N)})]$$

(22)

$$= \sum_j p_j \sum_{k=1}^N g_\theta[\Gamma_\theta(\rho_j^{(k)})]$$

(24)

$$\leq \sum_j p_j \sum_{k=1}^N g_\theta^\theta[\Gamma_\theta]$$

(25)

$$= Ng_\theta^\theta[\Gamma_\theta].$$

(26)

The inequality in (22) follows from the convexity of SLD quantum Fisher information with respect to states and the inequality (25) does from the definition of $g_\theta^\theta[\Gamma_\theta]$. □

B. Shift-parameter model

As noted in the remarks, the above theorem is simplified when the channel is given by a unitary transformation of the form:

$$\Gamma_\theta(\rho) = e^{i\theta A} \rho e^{-i\theta A},$$

(27)
where an Hermite operator $A$ on $\mathcal{H}$ is called a generator of the unitary transformation. The parameter region is any $2\pi$ interval of real numbers, e.g., $\Theta = [0, 2\pi)$. The quantum statistical model about the output states is given by

$$\mathcal{M}_A = \left\{ \rho_0 = e^{i\theta A} \rho_0 e^{-i\theta A} | \theta \in \mathbb{R} \right\}. \quad (28)$$

Here, $\rho_0$ is called as a reference state. This model was referred to as a shift-parameter model or a displacement model in Refs. [11, 13].

The following lemma is fundamental for the unitary model.

**Lemma III.2** For a shift-parameter model, the SLD quantum Fisher information is independent of the parameter $\theta$ and is bounded from above as

$$g_\theta = g_{\theta=0} \leq 4\Delta_{\rho_0} A, \quad (29)$$

where $\Delta_{\rho_0} A := \text{tr}(\rho A^2) - (\text{tr}(\rho A))^2$ is the square of variance of the operator $A$ with respect to the state $\rho_0$.

This lemma can be proven in different manners, here we sketch the most transparent one due to Holevo [13].

**Proof:** For a given state $\rho$, let $D_\rho$ be a super-operator acting on Hermite operators $X$ on $\mathcal{H}$, which is formally defined by the solution to the following operator equation:

$$\rho D_\rho(X) + D_\rho(X) \rho = \frac{1}{2} [\rho, X]. \quad (30)$$

It follows from the definition that the SLD operator is expressed as

$$L_\theta = 2D_{\rho_0}(A) = e^{-i\theta A} L_0 e^{i\theta A}, \quad (31)$$

$$L_0 = 2D_{\rho_0}(A). \quad (32)$$

This relation proves the first equality in Eq. (29).

We define a symmetric inner product for linear operators $X, Y$ on $\mathcal{H}$ by

$$\langle X, Y \rangle_\rho := \frac{1}{2} \text{tr} (\rho (Y X^\dagger + X^\dagger Y)), \quad (33)$$

then the SLD quantum Fisher for the shift-parameter model (28) is written as

$$g_\theta = \langle L_0, L_0 \rangle_{\rho_0} = 4 \langle D_{\rho_0}(A), D_{\rho_0}(A) \rangle_{\rho_0}. \quad (34)$$

Next, we note that the relation

$$\langle X, Y \rangle_{\rho} - \langle D_\rho(X), D_\rho(X) \rangle_{\rho} = \langle X, (1 + D_\rho^2)(X) \rangle_{\rho} \geq 0, \quad (35)$$

holds for any Hermite operators $X$ and a state $\rho$ on $\mathcal{H}$, since the super-operator $1 + D_\rho^2$ is positive with respect to the inner product. Writing the variance as $\Delta_{\rho_0} A = \langle A - \bar{A}, A - \bar{A} \rangle_{\rho_0}$ with $\bar{A} = \text{tr}(\rho A)$ and using the relation $D_{\rho_0}(A) = D_{\rho_0}(A - \bar{A})$, we prove the inequality in Eq. (29).

We note that equality condition for the inequality in Eq. (29) is equivalent to the condition [13]:

$$(1 + D_\rho^2)(A - \bar{A}) = 0 \Leftrightarrow \rho_0 A \rho_0 = \bar{A} \rho_0^2. \quad (36)$$

In some literature it is stated that “the equality in Eq. (29) is satisfied if and only if $\rho_0$ is a pure state.” This is true if the dimension of Hilbert space is 2, i.e., qubit. However, we remark that the condition $\rho_0$ is pure is just a sufficient condition in general. The sufficiency is immediate if we uses the second condition in Eq. (36). A simple counter example of mixed states satisfying the upper bound is given by a rank 2 state in dim $\mathcal{H}=3$ as follows.

$$\rho_0 = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 - \lambda & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} a & 0 & c^* \\ 0 & a & d^* \\ c & d & b \end{pmatrix}. \quad (37)$$

where $\lambda \in (0, 1)$, and $a, b$ and $c, d$ are real and complex numbers, respectively.

The variance of all possible states $\rho \in \mathcal{S}(\mathcal{H})$ is maximized when we take an equal weighted superposition of the eigenstates whose eigenvalues are maximum and minimum [38]. With this observation, theorem III.1 and lemma III.2 can be combined to give the following corollary:

**Corollary III.3** For a given shift-parameter model, let $U_N^X = \bigotimes_{j=1}^N e^{i\theta j A}$ be a global unitary for $\mathcal{S}(\mathcal{H}^\otimes N)$. If a density operator $\rho$ on $\mathcal{S}(\mathcal{H}^\otimes N)$ is separable, then the SLD quantum Fisher information $g_\theta [U_N^X \rho (U_N^X)^\dagger]$ is smaller or equal to the value $N(a_{max} - a_{min})^2$, where $a_{max}$ ($a_{min}$) is the maximum (minimum) of the eigenvalues of $A$.

Since we have an achievable bound for unitary channels, one can consider various extensions of the criterion. An immediate one is to consider a set of rotations around certain axes and to take average SLD Fisher information. Another one is to consider $k$-productive states rather than completely separable state of $N$ qubits. These extensions seem to work quite well as reported in Refs. [8–10].

Before moving to the next section to apply our criterion to examples, we shall analyze the equality condition for the convexity of SLD quantum Fisher information, i.e., the equality condition in the inequality (8), in the case of this simple unitary model. Consider mixed qubit states generated by the following unitary:

$$\rho_\theta^j = e^{-i\theta \sigma_x/2} \rho_0^j e^{i\theta \cdot \sigma_x/2} \quad (j = 1, 2), \quad (38)$$

where $\sigma$ is a given unit vector and $\theta$ is the rotation angle and their convex mixture

$$\rho_\theta = \lambda \rho_0^j + (1 - \lambda) \rho_\theta^j. \quad (39)$$

Let $g_\theta^j = g_\theta [\rho_\theta^j]$ and $g_\theta^\lambda = g_\theta [\rho_\theta^\lambda]$ be the SLD quantum Fisher information about the state $\rho_\theta^j$ and $\rho_\theta^\lambda$, respectively. A straightforward calculation shows that

$$g_\theta^j = g_{\theta=0}^j = |n \times s_j|^2, \quad (40)$$

where $n$ is a given unit vector and $\theta$ is the rotation angle and their convex mixture.
when is expressed in terms of the Bloch vector of the state: \( \rho^{\Delta}_{\theta} = (I + s \cdot \sigma)/2. \) Let us define the difference \( \Delta g^\lambda_{\theta} = \lambda g^\lambda_{\theta} + (1 - \lambda)g^\lambda_{\theta} - g^\lambda_{\theta}, \) then it reads

\[
\Delta g^\lambda_{\theta} = \lambda(1 - \lambda)|\mathbf{n} \times (s_1 - s_2)|^2.
\] (41)

Therefore, the equality in the convexity inequality (8) holds if and only if the difference of the two Bloch vectors \( s_1 - s_2 \) is parallel to the rotation direction \( \mathbf{n}. \) This is equivalent to satisfying the condition \( s_2 = s_1 - 2(\mathbf{n} \cdot s_1)\mathbf{n}. \)

### IV. EXAMPLES

In this section we analyze several examples to illustrate the proposed criterion to detect entanglement, in particular, the criterion (19). To get analytical results, we simplify the setting to the two-qubit case, that is \( N = 2 \) and \( \dim \mathcal{H} = 2. \) The input states analyzed in this section are the Bell-diagonal states defined by

\[
\rho_{BD}(c_1, c_2, c_3) = \frac{1}{4}(I + \sum_{j=1}^{3} c_j \sigma_j \otimes \sigma_j).
\] (42)

Here \( \sigma_j \) are usual Pauli spin operators and the coefficients are restricted from the positivity condition as

\[
1 - c_1 - c_2 - c_3 \geq 0,
1 - c_1 + c_2 + c_3 \geq 0,
1 + c_1 - c_2 + c_3 \geq 0,
1 + c_1 + c_2 - c_3 \geq 0.
\] (43)

Among the Bell-diagonal states, we focus on the two subfamilies,

\[
\rho^+_\lambda := \rho_{BD}(\lambda, \lambda, -\lambda),
\rho^-_\lambda := \rho_{BD}(\lambda, -\lambda, -\lambda).
\] (44, 45)

The states \( \rho^\pm_\lambda \) are also written as

\[
\rho^\pm_\lambda := \lambda|\psi_\pm\rangle\langle \psi_\pm| + \frac{1}{4}(1 - \lambda)I,
\] (46)

with \( |\psi_\pm\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2} \) are the Bell states. For \( \lambda \) in \( \Lambda := (-1/3, 1), \) both states \( \rho^\pm_\lambda \) are strictly positive. Further, \( \rho^+_\lambda \) are entangled if and only if \( \lambda \in \Lambda_{\text{ent}} := (1/3, 1]. \) The difference between \( \rho^+_\lambda \) is that \( \rho^+_\lambda \) is rotationally invariant state (spin singlet state), whereas \( \rho^-_\lambda \) is only invariant around the \( z \)-axis.

Our concern is to find a set of entangled states which can be detected by a given quantum channel. This quantity is represented by Eq. (17) or Eq. (18). For the above family of states, states are uniquely specified by a parameter \( \lambda, \) detected entangled regions are expressed by some interval which is a subset of \( \Lambda_{\text{ent}}. \)

### A. Unitary channel

We first consider a rotation around the \( z \)-axis on a single qubit system as

\[
U_z(\theta) = e^{i\theta \sigma_z}/2,
\] (47)

with \( \theta \in \Theta. \) The maximum variance of the generator \( \sigma_z/2i = 1, \) and Theorem III.1 reduces to the Pezzé-Smerzi criterion as discussed before. It compare the value of SLD quantum Fisher information about the state \( U_z^N(\theta)\rho(U_z^N(\theta))^\dagger \) with the total number of qubit systems, i.e., \( N. \) Here, \( U_z^N(\theta) = \bigotimes_{k=1}^{N} e^{i\theta \sigma_z/2} = e^{i\theta J_z} \) with \( J_z \) the \( z \) component of the total angular momentum operator. As noted before, the SLD quantum Fisher information for the state \( \rho^\pm_\lambda \) is zero due to the fact that \( J_z \) commutes with \( \rho^\pm_\lambda. \) Thus, one cannot get any useful information about \( \rho^\pm_\lambda \) by applying any global rotation around the \( z \)-axis.

We next consider a rotation around the \( x \)-axis as

\[
U_x(\theta) = e^{i\theta \sigma_x}/2.
\] (48)

The SLD quantum Fisher information about \( \rho^\pm_\lambda \) is calculated as

\[
g_\theta[U_x^2 \rho (U_x^2)^\dagger] = \frac{8\lambda^2}{1 + \lambda}.
\] (49)

Since the maximum SLD quantum Fisher information for the single system is 1 as discussed in Sec. III.B, the Pezzé-Smerzi criterion states the state \( \rho^\pm_\lambda \) is entangled if the following inequality holds.

\[
\frac{8\lambda^2}{1 + \lambda} > 2 \iff 8\lambda^2 - 2\lambda - 2 > 0.
\] (50)

Solving this inequality leads to the sufficient condition for the entangled region:

\[
R_{\text{ent}} = \left(\frac{\sqrt{17} + 1}{8}, 1\right).
\] (51)

The numerical value \( (\sqrt{17} + 1)/8 \simeq 0.64 \) is larger than the true boundary \( 1/3 \) as it should be.

This example shows that entanglement in the state \( \rho^-_\lambda \) cannot be detected by a rotation around any axis. For the state \( \rho^+_\lambda, \) on the other hand, a rotation around the \( x \)-axis can detect entanglement.

### B. Depolarizing channel

The depolarizing channel for a two-dimensional quantum system is defined by

\[
\Gamma_\theta(\rho) := \theta \rho + \frac{1 - \theta}{2} \text{tr}(\rho) I.
\] (52)

Here, the channel parameter \( \theta \) represents a probability of errors taking values in \( \Theta = (0, 1), \) e.g., no error \( \iff \theta = 1 \) \[39\].
Optimal parameter estimation strategies for this channel were studied based on various figures of merit, for example Refs. [16, 40]. It was shown that this channel is programmable and hence \( \theta \) cannot be estimated with quantum metrological enhancement [18]. The maximum value of the SLD quantum Fisher information for a single input state is given by an arbitrary pure state as

\[
    g_0[\Gamma_\theta] = \frac{1}{1-\theta^2}.
\]

The SLD quantum Fisher information are same for the two input states \( \rho_\lambda^\pm \) and are calculated as

\[
    g_0[\Gamma_0^2(\rho_\lambda^\pm)] = \frac{12\theta^2\lambda^2}{(1-\theta^2\lambda)(1+3\theta^2\lambda)}. \tag{54}
\]

Thus, a sufficient condition for entanglement obtained from Theorem III.1 is \( g_0[\Gamma_0^2(\rho_\lambda^\pm)] > 2g_0[\Gamma_\theta] \), equivalently,

\[
    3\theta^2(2-\theta^2)\lambda^2 - 2\theta^2\lambda - 1 > 0. \tag{55}
\]

This inequality then gives the entangled region for \( \lambda \) as

\[
    r_{\text{ent}}(\theta) = \left( \frac{\theta + \sqrt{3(2-\theta^2)}}{3\theta(2-\theta)}, 1 \right), \tag{56}
\]

which depends explicitly on the value of the channel parameter \( \theta \). An important remark is that the parameter \( \theta \) needs to satisfy \( \theta \in [\theta_c, 1] \) in order for the depolarizing channel to detect entanglement successfully. Otherwise, the criterion cannot tell if the states are entangled or not. Here the threshold is found numerically as \( \theta_c \approx 0.551 \). Physically speaking, the channel cannot be too noisy to detect entangled states.

Since this sufficient condition holds for any \( \theta \in (\theta_c, 1) \), the most useful one is given by the union of

\[
    R_{\text{ent}} = \bigcup_{\theta \in \Theta} r_{\text{ent}}(\theta) = (\lambda_s, 1), \tag{57}
\]

where \( \lambda_{DPC} \) is the minimum of the function appearing in the expression (56), and is given by

\[
    \lambda_{DPC} := \min_{\theta \in \Theta} \frac{\theta + \sqrt{3(2-\theta^2)}}{3\theta(2-\theta^2)} = 1 + \omega - \omega^{-1}, \tag{58}
\]

with \( \omega = (\sqrt{2} - 1)^{1/3}/2 \). The numerical value \( \lambda_{DPC} \approx 0.837 \) is larger than the one from a rotation around the \( x \)-axis.

We note that Ref. [41] analyzed a parameter estimation problem of the depolarizing channel based on a specific measurement and an estimator. They observed that entangled states are superior to separable states for a certain sub-family of the Werner state. The numerical value found in Ref. [41] is close to the value reported in this paper, yet they are different by nature of problem.

### C. Transpose channel

In this last example, we shall analyze a rather unusual channel defined in terms of a transpose operation. It is known that transposition operations are not completely positive, but only 1-positive. Here, a key point is that the convexity of the SLD quantum Fisher information follows from monotonicity of SLD quantum Fisher information about arbitrary 1-positive maps. Thus, trace-preserving and 1-positive map is also capable of detecting entanglement.

We consider the following channel from \( \mathcal{S}(\mathbb{C}^2) \) to itself:

\[
    \Gamma_\theta(\rho) := \theta \rho + (1-\theta)\rho^T, \tag{59}
\]

with \( T \) the transpose operation. Here the parameter \( \theta \) takes values in \( \Theta = (0,1) \). The maximum value of the SLD quantum Fisher information when one uses a single qubit input state is

\[
    g_0[\Gamma_\theta] = \frac{1}{\theta(1-\theta)}, \tag{60}
\]

which is attained by the eigenstates of \( \sigma_y \). A straightforward calculation gives the same values of the SLD quantum Fisher information for the input states \( \rho_\lambda^\pm \) and is given by

\[
    g_0[\Gamma_0^2(\rho_\lambda^\pm)] = 8f_\theta\lambda^2 \left[ \frac{1}{1-f_\theta\lambda} + \frac{1 + f_\theta\lambda}{(1 + f_\theta + 2\lambda)^2 - 4\lambda^2} \right], \tag{61}
\]

with \( f_\theta = (1-2\theta)^2 \).

A sufficient condition for entanglement is provided by Theorem III.1: \( g_0[\Gamma_0^2(\rho_\lambda^\pm)] > 2g_0[\Gamma_\theta] \). This is equivalently expressed as

\[
    F_\theta(\lambda) := 4f_\theta(1-f_\theta)\lambda^4 + f_\theta(f_\theta^3 - 2f_\theta + 4)\lambda^3 \\
    + (f_\theta^3 - 2f_\theta - 4)\lambda^2 + f_\theta\lambda + 1 < 0. \tag{62}
\]

Detail analysis on this quartic equation \( F_\theta(\lambda) = 0 \) shows that there are four real roots for all value of \( \theta \in \Theta \). The relevant entangled region is then found as

\[
    r_{\text{ent}}(\theta) = \left( \lambda_2(\theta), \frac{1}{2-f_\theta} \right). \tag{63}
\]

Here, \( \lambda_2(\theta) \) is the second largest solution to the quartic equation \( F_\theta(\lambda) = 0 \). Numerically, \( \lambda_2(\theta) \) varies from 1/2 to 1 depending on the value of \( \theta \). As in the depolarizing channel, we take union of \( r_{\text{ent}}(\theta) \) to get the most useful criterion:

\[
    R_{\text{ent}} = \bigcup_{\theta \in \Theta} r_{\text{ent}}(\theta) = (\lambda_{TPC}, 1), \tag{64}
\]

where \( \lambda_{TPC} = \min_{\theta \in \Theta} \lambda_2(\theta) = 1/2 \).
D. Comparison and discussion

In this last section, we compare four different channels studied in the previous sections and discuss our result. Rotation around $z$-axis; $U_z(\theta) = e^{i\theta \sigma_z / 2}$,

Rotation around $x$-axis; $U_x(\theta) = e^{i\theta \sigma_x / 2}$,

Depolarizing channel (DPC); $\Gamma_\theta(\rho) := \theta \rho + \frac{1 - \theta}{2} \text{tr}(\rho) I$,

Transpose channel (TPC); $\Gamma_\theta(\rho) := \theta \rho + (1 - \theta) \rho^T$.

The result for $\rho^\pm_\lambda$ are summarized in Table 1. Results for $\rho^\pm_\lambda$ are same for the depolarizing and transpose channels. In Table 1, “No” is indicated if a channel cannot be used to detect entangled states. $\lambda_2(\theta)$ is the second largest solution to the quartic equation $F_\theta(\lambda) = 0$ (Eq. (62)). Numerically, $\lambda_2(\theta)$ varies from 1/2 to 1 depends on the value of $\theta$.

| $\theta$ dependence | $r_{\text{ent}}(\theta)$ | $R_{\text{ent}}$ |
|----------------------|-------------------------|----------------|
| $U_z$                | No                      | No             |
| $U_x$                | No                      | $\frac{\sqrt{17} + 1}{8}, 1$ |
| DPC                  | Yes                     | $\frac{\theta + \frac{3}{2} \sqrt{2} - \theta^2}{20(2 - \theta^2)}, 1$ |
| TPC                  | Yes                     | $\frac{\lambda_2(\theta)}{2 - (1 - 20^2)}, \frac{1}{2}, 1$ |

TABLE I. Summary of entanglement detection for the state $\rho^\pm_\lambda$ from four different channels; two unitary channels, depolarizing channel (DPC), and transpose channel (TPC). Entanglement region $r_{\text{ent}}(\theta)$ is defined in Eq. (17) and its union is denoted by $R_{\text{ent}}$ defined by Eq. (18). Numerical values are $\lambda_{\text{DPC}} \approx 0.837 > (\sqrt{17} + 1)/8 \approx 0.64 > 1/2$.

As noted before, the rotation around any axis is not useful for the Werner state $\rho^\pm_\lambda$ since SLD quantum Fisher information about the output state is always zero. As we can see from Table 1, for $\rho^\pm_\lambda$, the rotation around the $x$-axis can be used to detect entanglement which performs better than the depolarizing channel. Interestingly, the (unphysical) transpose channel can detect entangled states better than the depolarizing channel.

The main difference between unitary channels and non-unitary channels is that SLD quantum Fisher information is $\theta$ independent for the unitary case. This might be an advantage in realistic situation if one wishes to detect entangled states with unknown unitary channel. For our point of view, however, this is not a problem, since we are willing to detect entangled states by engineering appropriate quantum channels.

Experimentally, we prepare a family of quantum channels $\Gamma_\theta$ with a controllable parameter $\theta$. We next send an unknown multi-partite state and perform a good measurement on the output state. The measurement results then give probability distributions depending on the value of the parameter $\theta$. We can then calculate classical Fisher information, which coincides with the SLD quantum Fisher information if the measurement is chosen as the optimal one. By comparing the value of Fisher information for multi-partite states with the optimal Fisher information for single input state, which is exploit in advance, one can tell if the states are entangled or not based on the criterion given in Theorem III.1.

Lastly, we show that for a certain parameter range (low-noise regime), the depolarizing channel can be estimated more efficiently if we use entangled input states. Although we cannot get a full benefit from entanglement to attain quantum metrological enhancement, entanglement indeed enables the estimation error lower than the separable input states. Whether this effect is significantly important depends on how accurate one wishes to estimate the value of a parameter of a given channel. More analysis on other quantum channels as well as various entangled input states are needed to make any general statement.

V. EXTENSION TO OPEN QUANTUM SYSTEM

So far we have concerned with the i.i.d. extension of quantum channels only. In this section, we shall extend the proposed criterion for the non i.i.d. case and discuss how to apply it to open quantum systems briefly. Let $\Gamma_\theta$ be a quantum channel from quantum states on $\mathcal{H}^N := \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$ to itself. We say that a quantum channel is separable if all separable states remain separable under the action of this channel. We consider a further restricted class of separable channels such that all product states remain product states. We call these channels completely separable meaning that they do not create any classical correlation. Mathematically, a completely separable quantum channel $\Gamma^{\text{sep}}$ satisfies the condition: For all possible states $\rho^{(j)} \in \mathcal{S}(\mathcal{H}_i)$, there exist some output states $\sigma^{(j)} \in \mathcal{S}(\mathcal{H}_i)$ such that

$$\Gamma^{\text{sep}}(\rho^{(1)} \otimes \rho^{(2)} \otimes \cdots \otimes \rho^{(N)}) = \sigma^{(1)} \otimes \sigma^{(2)} \otimes \cdots \otimes \sigma^{(N)},$$

holds.

When considering completely separable channels, we have the following theorem:

**Theorem V.1** Consider a completely separable channel $\Gamma_\theta : \mathcal{S}(\mathcal{H}^N) \rightarrow \mathcal{S}(\mathcal{H}^N)$ parametrized by a single parameter $\theta$. Let $g^*_\theta(\Gamma_\theta)$ be the largest value of SLD quantum Fisher information defined by

$$g^*_\theta(\Gamma_\theta) := \max_{\theta} \max_{\rho^{(i)} \in \mathcal{S}(\mathcal{H}_i), i \neq i} \text{Tr}\{\Gamma_\theta(\rho^{(1)}_{\text{cm}} \otimes \cdots \otimes \rho^{(i)}_{\text{cm}} \otimes \cdots \otimes \rho^{(N)}_{\text{cm}})\},$$

with $\rho^{(i)}_{\text{cm}}$ the completely mixed state on $\mathcal{H}_i$. For each value of $\theta$, if a density operator $\rho$ on $\mathcal{S}(\mathcal{H}^N)$ is separable, then the SLD quantum Fisher information $g^*_\theta[\Gamma_\theta(\rho)]$ is smaller or equal to the value $N g^*_\theta(\Gamma_\theta)$.

Proof of this theorem goes exactly in the same as Theorem III.1 as follows.
The master equation (70) can be solved analytically, but the resulting SLD quantum Fisher information gets complicated in general, in particular when the damping coefficients depend on \(\theta\). Below we consider an isotropic noise \(\gamma_1 = \gamma_2 = \gamma_3 = \gamma\) and \(\gamma\) is independent of \(\theta\) to simplify the result. In this case, the obtained maximum SLD quantum Fisher information over all possible initial state is

\[
\rho_0[\Gamma_\theta] = t^2 e^{-2\gamma t},
\]

at some later time \(t\). Thus, the proposed criterion to detect entangled states is as follows. For a given initial state \(\rho_0\) on \(S(H^{\otimes N})\) with \(H = \mathbb{C}^2\), \(\rho_0\) is entangled if the inequality

\[
g_0[\rho_0(t)] > N t^2 e^{-2\gamma t},
\]

holds for later time \(t\). Here \(\rho_0(t)\) is the solution to the master equation (68) with the initial state \(\rho_0\). Here, two remarks on this result. First, the above criterion seems counterintuitive at first sight. Since the right hand becomes exponentially small for fixed \(N\) as the time \(t\) increases, this criterion states that almost all states having non-zero SLD quantum Fisher information at later time are entangled. A simple explanation for this observation is that as time grows the solution to the master equation (70) approaches to \(\theta\)-independent state, typically to the completely mixed state, for any initial state. It is then clear that the amount of SLD quantum Fisher information decreases in time as well. Therefore, the inequality (72) still provides useful information to detect entangled states.

Second remark is that the above criterion can be weakened by replacing \(\exp(-2\gamma t)\) by 1. That is, if, the simplified inequality

\[
g_0[\rho_0(t)] > N t^2,
\]

holds, then the state \(\rho_0\) is entangled. This later criterion (73) is certainly simple, in particular, it is independent of the external noise parameter \(\gamma\). However, it is obvious that this weaker version becomes useless for the large \(t\) regime.

In the experiment reported in Ref. [6], authors apply the weaker version of entanglement criterion even though non-negligible decoherence effects are present. The above simple example implies that a more sharpened criterion can be applied to their experimental data by analyzing the effects of quantum noises and to detect entangled states faithfully.

VI. CONCLUSION

We have derived a general criterion to detect entanglement based on the SLD quantum Fisher information for any one-parameter family of quantum channels. This criterion includes previously known criteria based on unitary channels as a special case. We then apply our criterion to detect entanglement in the Bell-diagonal states.
based on the unitary channel, depolarizing channel, and transpose channel. Our result shows that even the depolarizing channel can be used to detect entangled states for a certain parameter range. To put it differently, entanglement is still useful to lower the estimation errors even though channels cannot be estimated with quantum metrological enhancement. Lastly, we have derived a more general criterion that can be applied to the estimation of channel parameter in open quantum systems. We briefly discussed how to apply it for the phase estimation in presence of a coupling to an environment, which is described by some master equation. A more detail discussion on the entanglement detection in open quantum systems deserves further studies and it will be analyzed in due course.

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