A NEW PERSPECTIVE ON HIERARCHICAL SPLINE SPACES FOR ADAPTIVITY

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Abstract. We introduce a framework for spline spaces of hierarchical type, based on a parent-children relation, which is very convenient for the analysis as well as the implementation of adaptive isogeometric methods. Such framework makes it simple to create hierarchical basis with control on the overlapping. Linear independence is always desired for the well posedness of the linear systems, and to avoid redundancy. The control on the overlapping of basis functions from different levels is necessary to close theoretical arguments in the proofs of optimality of adaptive methods. In order to guarantee linear independence, and to control the overlapping of the basis functions, some basis functions additional to those initially marked must be refined. However, with our framework and refinement procedures, the complexity of the resulting bases is under control, i.e., the resulting bases have cardinality bounded by the number of initially marked functions.

1. Introduction. Adaptive methods are a fundamental computational tool in science and engineering to approximate partial differential equations.

For the finite element method (AFEM) there has been a lot a work starting in the 1980s and 1990s with the design of a posteriori error estimators with very successful practical result. In the 2000s adaptive processes have been shown to converge, and to exhibit optimal complexity for several stationary PDE.

The adaptive process for stationary PDE can be described with the classical adaptive step

\[ \text{Solve} \rightarrow \text{Estimate} \rightarrow \text{Mark} \rightarrow \text{Refine}, \]

where Solve computes the solution on a discrete space with basis \( \mathcal{H} \); Estimate computes a posteriori localized error estimators and Mark uses the estimators to indicate where more resolution should be invested in order to obtain maximum benefit. Let \( \mathcal{M} \) be this indication, then from \( \mathcal{H} \) and \( \mathcal{M} \) the procedure Refine constructs a new basis \( \mathcal{H}_* \) and thus a new space. We thus arrive at an adaptive sequence

\[ \mathcal{H}_0 \xrightarrow{\mathcal{M}_0} \mathcal{H}_1 \xrightarrow{\mathcal{M}_1} \ldots \xrightarrow{\mathcal{M}_{R-1}} \mathcal{H}_R \rightarrow \ldots \]  \hspace{1cm} (1.1)

A sound theory of adaptivity in the context of FEM \([11, 5]\) hinges on the adequate design of local estimators and certain combinatorial-geometric properties assignable to the underlying mesh that is intimately related to the refinement procedure. The estimator is usually assigned to the mesh elements, the ones with the larger estimators are collected in \( \mathcal{M} \) and then refined to obtain a new mesh and thereby a new basis with adequate local resolution.

Different alternatives have been proposed to obtain adaptive spline methods, such as hierarchical splines, T-splines, LR-splines or PHT-splines. Among them, hierarchical splines such as those in \([14, 8, 2]\), seem to constitute the simplest approaches to obtain adaptive isogeometric methods.

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The structure of B–splines leads naturally to the idea of assigning estimators to and refining basis functions rather than elements [9, 3, 12]. This idea of assigning the local estimator to basis functions instead of elements has also been studied in the context of finite elements in [11, 10].

In this work we want to address adaptivity under the assumptions that the allowed bases are subsets of B–splines of different levels (hierarchical splines) and that the refinement is done on functions rather than elements. Under this setting we study the question of what are the simplest but yet theoretically adequate spaces to successfully develop a theory of adaptivity.

To be more precise, let $\mathcal{B}$ be the set of B–splines of all levels. We want to find a family $\mathcal{F}$ of subsets of $\mathcal{B}$, and a procedure $\text{Refine}$ such that they simultaneously satisfy the following properties.

**Property 1.1 (About $\text{Refine}$).** Given $\mathcal{H} \in \mathcal{F}$ and $\mathcal{M} \subset \mathcal{H}$, the procedure $\text{Refine}$ returns $\mathcal{H}^* \in \mathcal{F}$ where:

(i) $\mathcal{H}^*$ has more resolution in the places indicated by $\mathcal{M}$;

(ii) when used in the adaptive loop (1.1) there is $C$ (independent of $R$ and $\mathcal{M}_r$'s) such that $\# \mathcal{H}_R - \# \mathcal{H}_0 \leq C \sum_{r=0}^{R-1} \# \mathcal{M}_r$;

(iii) It is simple to implement computationally.

**Property 1.2 (About $\mathcal{F}$).** Given $g \in \mathbb{Z}^+$, $\mathcal{F}$ satisfies the following:

(i) The spaces generated by $\mathcal{H} \in \mathcal{F}$ possess good approximation properties in terms of the number of degrees of freedoms;

(ii) if $\mathcal{H} \in \mathcal{F}$ then $\mathcal{H}$ is linearly independent; and

(iii) if $\mathcal{H} \in \mathcal{F}$ for any two functions in $\mathcal{H}$ that overlap their level difference is at most $g$.

Requirement [12(ii)] is important for the $\text{Solve}$ in the adaptive loops as well as the design of estimators. The constraint on the overlap of functions from different levels given by [12(iii)] is a technical requirement for the proof of a contraction property of adaptive algorithms; at a certain point an inverse inequality is required, which cannot be bounded with a uniform constant unless this assumption is met. When [12(iii)] holds, we say that the gap is bounded by $g$. The complexity bound [1.1(ii)] is key for the optimality results.

The motivational guideline to simultaneously satisfy both sets of properties is to start by considering all possible generators (hierarchical generators) obtainable by refinements; a concept that needs a rigorous definition. We define the refinement relation in $\mathcal{F}$ as a set inclusion of certain function sets, the lineages, which are associated in a one-to-one fashion with the generators in $\mathcal{F}$ (see Lemma 4.4). Thus, the refinement relation induces a partial order on the hierarchical spaces allowing us to rigorously and simply state questions such as

Given a hierarchical basis $\mathcal{H}$, which is the smallest refinement of $\mathcal{H}$ that is a basis, has refined $\varphi \in \mathcal{H}$ and its gap is bounded by a given number $g$?

We start with some technical preliminaries where we set the notation and language to conveniently handle the ancestry and overlapping relation in terms of the multilevel tensor index of the B–splines (sections 2 and 3). We have two notations, one that keeps track of the indexes which is useful for the computational implementation of the actual algorithms and another one which bounds “distances” which is useful for the analysis of complexity. Section 4 introduces the concept of lineage which is key for the definition of refinement in Section 5. In general the generators are not linearly independent, thus in Section 6 we discuss how to restrict the refinement process to
yield bases. Section 7 deals with the important matter of restricting the refinement to yield bases with a uniformly bounded gap. Finally in Section 9 we show that the refinement process we have presented, which yields a refined bases with a given bound on their gaps, satisfies the required complexity bound from Property [11(ii)] when used in the adaptive loop (1.1). More precisely, the dimension of each hierarchical space is linearly bounded by the history of marked functions.

Some auxiliary results have been collected in an Appendix (Section 10) in order not to interrupt the flow of ideas in the core of this article.

The main contributions of the work are:
1. Characterize refinement as set inclusion (of associated lineages) which allows to rigorously talk about the smallest refinement that satisfies certain property.
2. Provide constructive algorithms with rigorous proofs that they perform the required tasks.
3. A thorough analysis of the different ingredients, which are separated showing what implications and properties are linked to each other.

Hierarchical spline spaces were introduced in [14, 8], and some modifications were introduced in [9, 2]. Those definitions, except the one in [9], are stated in terms of sequences of subdomains. Recently, optimality of adaptive isogeometric methods has been proved for elliptic problems [1, 7]. Those results are based on residual-type a posteriori estimators associated to elements or cells, where one notices some difficulties in the handling on the definition of hierarchical spaces in terms of a domain plus the need to have admissible meshes (gap bounded by one), both works are element-oriented. The function-based refinement can be traced to ideas of Krysl [9], further develop by Garau and Buffa [2], where they focus on a positive partition of unity.

Our main goal was to rethink a definition of hierarchical spaces with the concept of refinement based on basis functions, that would yield the simplest language to work, theoretically as well as practically.

2. Preliminaries. This section is mainly intended to set the general notation and introduce basic concepts. We will work in the euclidean space $\mathbb{R}^d$ ($d \in \mathbb{Z}^+$). The tensor product structure intrinsic to the multivariate spline spaces leads to many concepts being isomorphic to the integer lattice $\mathbb{Z}^d$, hence the following notation is convenient. Given two integers $j$ and $k$ we let $[j : k] := \{l \in \mathbb{Z} : j \leq l \leq k\}$ denote the section of integers bounded by them. A multi-index $j$ is an element of the integer lattice $\mathbb{Z}^d$, with its $i$-th component denoted by $j_i$. Given $j$ and $k$ in $\mathbb{Z}^d$, we let $[j : k] := \times_{i=1}^d [j_i : k_i]$ denote the lattice box bounded by them. For $x$ and $y$ in $\mathbb{R}^d$ we let $\text{dist}(x, y) = \|x - y\|_\infty = \max_{1 \leq i \leq d} |x_i - y_i|$, so that, for instance, $\text{diam}[j - k : j + k] = 2\|k\|_\infty$.

Given a finite set $A$, its size or number of elements is denoted by $\#A$, whence $\#[j : k] = \prod_{i=1}^d (k_i - j_i + 1)$. When a scalar value appears in the place where a vector value is expected it means the vector where each component has the value of the scalar. For example if $i \in \mathbb{Z}^d$ then $[i : 2] = [i : (2, 2, 2)]$. When an operation/relation (like inequality) is applied to vectors it means that it is applied to each of their components, i.e., $i < j$ if and only if $i_k < j_k$ for $k \in [1 : d]$.

Finally, the notation $C = A \cup B$ means that $C = A \cup B$ and $A \cap B = \emptyset$, and $[x]$ stands for the floor of a real number $x$, i.e. the largest integer smaller than or equal to $x$.

2.1. Index and Box preserving functions. Whenever a function $P$ goes from $\mathbb{Z}^d$ into $\mathbb{Z}^d$ we call it an index function.
Let \( n \geq 2 \) be a fixed integer. For any \( m \in \mathbb{Z} \) we define the index functions

\[
M_m(i) := ni + m \quad \text{and} \quad D_m(i) := \left\lfloor \frac{i - m}{n} \right\rfloor. \tag{2.1}
\]

For \( k \in \mathbb{Z}^+ \) the function \( D^k_m \) denotes the \( k \)-th iterate of \( D_m \), that is \( D^1_m = D_m \) and \( D^k_m(\cdot) = D_m(D^{k-1}_m(\cdot)) \). For completeness \( D^0_m \) is the identity function. Similarly the same definition is considered for \( M^k_m \). Clearly, \( D^k_m(M^k_m(i)) = i \), but it is not always true that \( M^k_m(D^k_m(i)) = i \). However, the following result, which is an immediate consequence of Lemma 10.4 in the Appendix, holds.

**Lemma 2.1.** Let \( m, m' \in \mathbb{Z} \) and \( k \in \mathbb{Z}^+ \), then \( M^k_m(j) \leq i \leq M^k_{m'}(j) \) if and only if \( D^{k'-k-1}(i) \leq j \leq D^{k'}(i) \).

For any fixed \( g \in \mathbb{Z}^+ \) and \( p \in \mathbb{Z} \), let us define two more index functions

\[
L(i) := D^0_g(i) - p \quad \text{and} \quad R(i) := D^0_g(i + p). \tag{2.2}
\]

and their \( k \)-th iterates \( L^k(i) \) and \( R^k(i) \) for \( k \in \mathbb{Z}^+ \).

A function \( F \) given by \( F(\cdot) = [P(\cdot) : Q(\cdot)] \) where \( P, Q \) are index functions is called a box function, and it is called box preserving if \( F([i : j]) = [P(i) : Q(j)] \).

**Lemma 2.2.** Let \( k \in \mathbb{Z}^+ \). If \( m' - m \geq n - 1 \) then \( F(\cdot) = [M^k_m(\cdot) : M^k_{m'}(\cdot)] \) is box preserving. And if \( m - m' \geq 0 \) then \( F(\cdot) = (D^k_m(\cdot) : D^k_{m'}(\cdot)) \) is box preserving.

The proof of Lemma 2.2 is immediate from Lemma 10.5 and Corollary 10.7.

### 2.2. B-spline basis.

#### 2.2.1. One dimensional splines.

Let \( [a, b] \) be an interval and \( \Delta = \{x_j\}_1^n \) with \( a = x_0 < x_1 < \cdots < x_n < x_{n+1} = b \) a partition into \( n + 1 \) 1-cells (subintervals) \( I_j = [x_j, x_{j+1}) \) for \( j \in \{0 : n - 1\} \) and \( I_n = [x_n, x_{n+1}] \). Let \( m \) be a positive integer and \( M \in \mathbb{Z}^n \) with \( 1 \leq M_k < m \). The spline space \( V = V(\Delta, m, M) \) is the space of piecewise polynomials of order \( m \) that are \( C^{m-M_k-1} \) at \( x_j \) for \( j \in [1 : n] \). The elements of \( \Delta \) are the interior knots and \( M \) is the interior multiplicity vector.

Given an extended partition associated with \( V(\Delta, m, M) \) (dictated by the multiplicity vector with an appropriate selection of additional end knots) there exists a constructive process that produces a basis of \( V \) known as the B-spline basis (see \[13\] Theorem 4.9). It is well known \[13\] that these basis functions have minimal support, are locally linear independent, non-negative and form a partition of unity.

#### 2.2.2. Tensor product splines.

If for each \( k \in [1 : d] \) an interval \([a_k, b_k]\), a knot partition \( \Delta_k \), a positive integer \( m_k \) and multiplicity vector \( M_k \in \mathbb{Z}^{n_k} \) are provided, then following the process of Section 2.2.1 we can define \( d \) one-dimensional B-spline bases \( \mathcal{B}_k \), \( k \in [1 : d] \).

It can be shown \[13\] that the set \( \mathcal{B} = \{ \phi = \otimes_{k=1}^d \phi_k : \phi_k \in \mathcal{B}_k \} \) is a family of linearly independent functions over the box \( \Omega = \times_{k=1}^d [a_k, b_k] \), which are also non-negative and form a partition of unity. The function \( \otimes_{k=1}^d \phi_k : \Omega \to \mathbb{R} \) is the tensor product of the univariate functions \( \phi_k \), i.e., \( (\otimes_{k=1}^d \phi_k)(y) = \prod_{k=1}^d \phi_k(y_k) \).

Under this scenario we define the tensor product spline space

\[
V = V(\{\Delta_k\}, \{m_k\}, \{M_k\})
\]
as the set of linear combinations of the elements of \( \mathcal{B} \).
3. Multilevel B-splines. To each non negative integer $\ell$ we want to associate a set of B-splines where $\ell$ indicates the level of resolution. Roughly speaking, $\ell$ is a measure of the knot density. In this work for the sake of clarity we restrict ourselves to the subclass of spline spaces where the knots of level $\ell+1$ are obtained by adding $s \in \mathbb{Z}^+$ knots uniformly distributed between the knots of level $\ell$. And the knots of level 0 are $\{0,1\}^d$. Thus we take the domain $\Omega$ to be the unit cube $[0,1]^d$ and in going from level $\ell$ to $\ell+1$ each subinterval is divided into $n := s+1$ equal-length subintervals of level $\ell+1$.

We also consider maximum interior regularity for the spline spaces. These restrictions simplify the notation to better concentrate in the new concepts and ideas introduced in this work but there is no essential impediment to extend these ideas to non-uniform and less regular cases.

3.1. Multilevel cells. From now on we fix the value of $s \in \mathbb{Z}^+$ and let $n = s+1$. For $\ell \in \mathbb{Z}^+$, $i \in [0, n^\ell - 2]$ let $I^\ell_i := [\frac{i}{n^\ell} : \frac{i + 1}{n^\ell}]$ and $I^\ell_{i-1} := [1 - \frac{1}{n^\ell}, 1]$. And for a given $i \in [0, n^\ell - 1]^d = [M_0(i) : M_0^+(i)]$ we define $I^\ell_k := \times_{k=1}^d I^\ell_{i_k}$ the $i$-th $n$-adic cell of level $\ell$ in dimension $d$. Moreover, we define $\mathcal{I}^\ell := \{I^\ell_k : i \in [M_0(i) : M_0^+(i)]\}$ as the set of cells of level $\ell$ and $\mathcal{I} := \cup_{\ell=0}^\infty \mathcal{I}^\ell$ the multilevel cells.

Definition 3.1 (Level and index of a cell). Given $I \in \mathcal{I}$ the level of $I$, denoted by $\ell_I$, and the index of $I$, denoted by $i_I$, are the unique integer and index respectively such that $I_{i_I} = I$.

Definition 3.2 (Box of cells). Given $\ell \in \mathbb{Z}^+$, $i$ and $j$ in $\mathbb{Z}^d$ with $0 \leq i \leq j \leq n^\ell - 1$, we define $\mathcal{I}^\ell[i : j] := \{I^\ell_k : k \in [i : j]\}$ as the box of cells of level $\ell$ bounded by the corners $i$ and $j$.

Definition 3.3 (Children of a cell). Given $I = I^\ell_k \in \mathcal{I}$, the the set ch of children of $I$ is defined as $\text{ch} I^\ell_k := \mathcal{I}^{\ell+1}[M_0(i) : M_0^+(i)]$. And if $J \subset I$ then $\text{ch} J = \cup_{J \subset I} \text{ch} I$.

Remark 3.4. Note that $J \subset \text{ch} I$ iff $\ell_J = \ell_I + 1$ and $J \subset I$, and moreover $I = \cup_{J \subset \text{ch} I} J$.

Definition 3.5 (Descendants and ancestors). Given $I \in \mathcal{I}$ and $k \in \mathbb{Z}^+$ we define the set of its $k$-descendants as the set $\text{ch}^k I$, resulting from $k$ successive applications of the children operator. We also define the $k$-ancestors of $I$ by $\text{ch}^{-k} I = \{J \in \mathcal{I} : I \subset ch^k J\}$, and $\text{ch}^0 I = \{I\}$. Finally, if $J \subset I$ then $\text{ch}^k J = \cup_{J \subset \text{ch}^k I} \text{ch}^k I$.

Lemma 3.6 (Box ancestry). For $k \in \mathbb{Z}^+$ and $i, j \in \mathbb{Z}^d$ with $i \leq j$

(i) $\text{ch}^k \mathcal{I}^\ell[i : j] = \mathcal{I}^{\ell+k}[M^k_0(i) : M^k_0(j)]$

(ii) $\text{ch}^{-k} \mathcal{I}^\ell[i : j] = \mathcal{I}^{\ell-k}[D_0^k(i) : D_0^k(j)]$

Proof. Let $k = 1$ using Definition 3.3 we have $\text{ch} \mathcal{I}^\ell[i : j] = \cup_{n \in [i : j]} \mathcal{I}^{\ell+1}[M_0(n) : M_0(n)]$. Let $F(n) = [M_0(n) : M_0(n)]$, from Lemma 2.2 it follows that $F$ is box preserving so we have that $\text{ch} \mathcal{I}^\ell[i : j] = \mathcal{I}^{\ell+1}[M_0(i) : M_0(j)]$. For $k > 1$ the proof follows by induction using the result we have just shown.

For (ii) observe that $\text{ch}^{-k} \mathcal{I}^\ell[i : j] = \cup_{n \in [i : j]} \text{ch}^{-k} I_n^\ell$. From here we have that $J \in \text{ch}^{-k} \mathcal{I}^\ell[i : j]$ if and only if $J = I_n^{\ell-k}$ and there is $n \in [i : j]$ such that $I_n^\ell \in \text{ch}^k I_n^\ell$. And the latter happens if and only if $n \in [M_0^k(r) : M_0^k(r)]$. From Lemma 2.1 this is equivalent to $r = D_0^k(n)$. Thus we conclude that $\text{ch}^{-k} \mathcal{I}^\ell[i : j] = \cup_{r \in [D_0^k(i) : D_0^k(j)]} \mathcal{I}^{\ell-k}[D_0^k(i) : D_0^k(j)]$. $\square$

3.2. Multilevel B-splines. Let $V^{(\ell,m,d)}(\Omega)$ be the space of tensor product splines of order $m$ globally $C^{m-2}$ subordinated to the $n$-adic partition of level $\ell$ of $\Omega$ (see Section 2.2). From this point on, $m$ and $d$ are fixed unless explicitly stated so we drop them from the notation and write for example $V^\ell$ instead of $V^{(\ell,m,d)}(\Omega)$. 

\[\text{Lemma 3.6} \]
It is also convenient to introduce the number \( p := m - 1 \) which is the degree of the B–splines.

The “master” B–spline of order \( m \) is

\[
Q(x) := m \sum_{j=0}^{m} (-1)^j \binom{m}{j} (x-j)^{m-1},
\]

where \( f_+ \) stands for the positive part of \( f \). The function \( Q \) is \( C^{m-2} \), positive in \( (0, m) \) with support equal to \( [0, m] \). For \( \ell \geq 0 \) and \( i \in \mathbb{Z} \) let \( \varphi_i^\ell(x) := Q(n^\ell x - i) \) and if \( i \in \mathbb{Z}^d \) define \( \varphi_i^\ell(x) := \Pi_{k=1}^d \varphi_i^{\ell_k}(x_k) \). With these definitions we make the B–splines sets clearly isomorphic to integer lattices. More precisely, it can be shown \cite{[13]} that for each \( \ell \) a normalized tensor product B-spline basis (of level \( \ell \)) on \( \Omega \) is

\[
\mathcal{B}^\ell = \{ \varphi_i^\ell : i \in [-p : M_\ell^i(0)] \}.
\]

These are the so-called cardinal B-splines, which correspond to extending the uniform partitions beyond the boundaries of \( \Omega \). If we considered the so-called interpolatory B-splines, which correspond to the open-knot vector, the only difference would be that not all the basis functions are dilations and translations of the same master B-spline \( Q \), but of a finite number of master functions. All what follows remains valid. Trivially from the definition, the sets \( \mathcal{B}^\ell \) are pairwise disjoint. Let \( \mathcal{B} := \cup_{\ell=0}^{\infty} \mathcal{B}^\ell \) be the set of multilevel B-splines.

**Definition 3.7** (Level and index of a B-spline). Given \( \varphi \in \mathcal{B} \), the level of \( \varphi \), denoted by \( \ell_\varphi \), and the index of \( \varphi \), denoted by \( i_\varphi \), are the unique integer and index, respectively, such that \( \varphi_{i_\varphi}^{\ell_\varphi} = \varphi \).

**Definition 3.8** (Box of B–splines). Given \( \ell \in \mathbb{Z}_0^+ \), \( i \) and \( j \) in \( \mathbb{Z}^d \) with \( i \leq j \) we define \( \mathcal{B}^\ell[i : j] := \{ \varphi_{k,i}^\ell : k \in [i : j] \} \) as the box of B–splines of level \( \ell \) bounded by the corners \( i \) and \( j \).

**Definition 3.9** (Children of a B-spline). Given \( \varphi = \varphi_i^\ell \in \mathcal{B} \) the set \( \text{ch} \varphi \) of children of \( \varphi \) is defined as \( \varphi \text{ch} \varphi := \mathcal{B}^\ell+1[M_{\ell}(i) : M_{\ell+1}(i)] \). Also, if \( \mathcal{F} \subset \mathcal{B} \), we define \( \text{ch} \mathcal{F} := \cup_{\varphi \in \mathcal{F}} \text{ch} \varphi \).

This definition is motivated by the following Lemma.

**Lemma 3.10** (Children properties).

(i) There exist \( c_k \in \mathbb{R}^+ \) for \( k \in [0 : sm] \) such that \( \varphi_i^\ell(x) = \sum_{k=0}^{[0: sm]} c_k \varphi_{ni+k}^{\ell+1} \), for every \( \ell \in \mathbb{Z}_0^+ \) and \( i \in \mathbb{Z} \).

(ii) Let \( \psi \in \mathcal{B}^\ell \) and \( \psi = \sum_{\varphi \in \mathcal{B}^{\ell+1}} \alpha_\varphi \varphi \) be its unique expansion in \( \mathcal{B}^{\ell+1} \) then \( \varphi \in \text{ch} \psi \) if and only if \( \alpha_\varphi > 0 \).

**Proof.** Part \( (i) \) follows from the standard recurrence relation for B–splines of consecutive order (see \cite{[6]} p.90) and the fact that the result holds for B–splines of order 1. Part \( (ii) \) follows easily from \( (i) \) and the fact that all the B–splines from a fixed level are linearly independent.

**Remark 3.11.** This parent-children relationship holds with the same coefficients \( c_k \) for all functions of all levels in the case of cardinal B-splines. In the case of interpolatory B-splines these are a finite number of different situations, corresponding to the cases when the support of the involved basis functions touch the boundary of \( \Omega \). But this is not essential for the discussion of this article.

**Definition 3.12** (Descendants and ancestors). Given \( \varphi \) in \( \mathcal{B} \) and \( k \in \mathbb{Z}^+ \) we define the set of its \( k \)-descendants as the set \( \text{ch}^k \varphi \), resulting from \( k \) successive applications of the children operator. We also define \( k \)-ancestors of \( \varphi \) by \( \text{ch}^{-k} \varphi = \{ \psi \in \mathcal{B} : \varphi \in \text{ch}^k \psi \} \), and \( \text{ch}^0 \varphi = \{ \varphi \} \). Finally, if \( \mathcal{F} \subset \mathcal{B} \) then \( \text{ch}^k \mathcal{F} = \cup_{\varphi \in \mathcal{F}} \text{ch}^k \varphi \).

**Lemma 3.13** (Box ancestry). For \( \ell \geq 0 \), \( i, j \in \mathbb{Z}^d \) and \( k \in \mathbb{Z}^+ \) we have that
obtained by induction.

\[ ch^k B^i[i : j] = B^{i+k} [M_0^k(i) : M^k_{sm}(j)] \]

(ii) \( ch^{-k} B^i[i : j] = B^{-k} [D_{sp}^k(i) : D^k_0(j)] \)

Proof. Let \( k = 1 \), using Definition 3.9, we have \( ch^1 B^i[i : j] = \cup_{n \in [i,j]} B^{i+1} [M_0(n) : M^1_{sm}(n)] \). From Lemma 2.2, the function \( F(n) = [M_0(n) : M^1_{sm}(n)] \) is box preserving, then it follows that \( ch^k B^i[i : j] = B^{i+k} [M_0(i) : M^k_{sm}(j)] \). The rest of the proof follows by induction on \( k \).

To show (ii) observe that \( ch^{-k} B^i[i : j] = \cup_{n \in [i,j]} ch^{-k} \varphi_n^k \). From here we have that \( \varphi_{r-k}^k \in ch^{-k} B^i[i : j] \) if and only if there is \( j \in [i : j] \) such that \( \varphi_{r-k}^k \in ch^k \varphi_{r-k}^k \).

And from part (i) of the result, the latter happens if and only if \( n \in [M^k_s(r) : M^k_s(r)] \) which by Lemma 2.1 is equivalent to \( D^k_{sm-s} (n) \leq r \leq D^k_0(n) \). We thus get \( ch^{-k} B^i[i : j] = \cup_{n \in [i,j]} B^{i-k} (D^k_{s(m-1)}(n) : D^k_0(n)) \). Now \( F(n) = (D^k_{s(m-1)}(n) : D^k_0(n)) \) is a box preserving operator, thus \( ch^{-k} B^i[i : j] = B^{i-k} (D^k_{s(m-1)}(i) : D^k_0(j)) \).

For the complexity results it will be useful to have a notion of a “ball of functions” which are related to a scaled comparison of the indexes as given in the following definition.

Definition 3.14 (Oriented distance). Let \( \varphi_1 \) and \( \varphi_2 \) in \( B \), we define

\[ \rho(\varphi_1, \varphi_2) := \frac{i_{\varphi_1}}{n^1_{\varphi_1} - n^2_{\varphi_2}} - i_{\varphi_2}. \]

Observe that \( \rho \) is not symmetric, in fact \( n^{\varphi_2} \rho(\varphi_1, \varphi_2) = -n^{\varphi_2} \rho(\varphi_2, \varphi_1) \). Moreover it satisfies the following analogous of the triangle inequality, whose proof is easily obtained by induction.

Lemma 3.15 (Weighted triangular equality). Let \( \varphi_0, \ldots, \varphi_j \in B \) then

\[ \rho(\varphi_j, \varphi_0) = \sum_{i=0}^{j-1} \frac{1}{n^{\varphi_j} - n^{\varphi_0}} \rho(\varphi_{i+1}, \varphi_i). \]

We can use \( \rho \) to express the descendants of a B–spline in the sense of the next Lemma. This will be useful when analyzing implications of the refinement algorithm in Section 8.

Lemma 3.16 (Distance to descendants). Let \( \eta \in B^i \) and \( k \in \mathbb{Z}^+ \), then

\[ ch^k \eta = \{ \psi \in B^{i+k} : 0 \leq \rho(\psi, \eta) \leq m(1 - \frac{1}{n^k}) \}
\]

\[ = \{ \psi \in B^{i+k} : -m(n^k - 1) \leq \rho(\eta, \psi) \leq 0 \}
\]

\[ \subset \{ \psi \in B^{i+k} : 0 \leq \rho(\psi, \eta) \leq m \} = \{ \psi \in B^{i+k} : -m n^k \leq \rho(\eta, \psi) \leq 0 \}. \]

Proof. From Lemma 3.13 \( \psi \in ch^k \eta \) iff \( \psi \in B^{i+k} (M^k_0(i_{\eta}) : M^k_{sm}(i_{\eta})) \), i.e., \( M^k_0(i_{\eta}) \leq i_{\psi} \leq M^k_{sm}(i_{\eta}) \), or \( 0 \leq i_{\psi} - M^k_0(i_{\eta}) \leq M^k_{sm}(i_{\eta}) - M^k_0(i_{\eta}) \). Since \( M^k_0(i_{\eta}) = n^k i_{\eta} \) and \( M^k_{sm}(i_{\eta}) - M^k_0(i_{\eta}) = m(n^k - 1) \), we have, after dividing by \( n^k \), that \( \psi \in ch^k \eta \) iff

\[ 0 \leq \frac{i_{\psi}}{n^k} - i_{\eta} \leq m(1 - \frac{1}{n^k}) < m, \]

and the assertion follows.

Definition 3.17 (Ball of functions). Let \( \varphi \in B, D \in \mathbb{R}^+ \) and \( k \in \mathbb{Z} \) then we define \( B(\varphi, D, k) := \{ \psi \in B^{i+k} : |\rho(\varphi, \psi)| \leq D \} \)

Lemma 3.18 (Uniform bound by level). Let \( \varphi \in B, D \in \mathbb{R}^+ \) and \( k \in \mathbb{Z} \) then \( \# B(\varphi, D, k) \leq (2D + 1)^d \).
Proof. By definition
\[ \#B(\varphi, D, k) = \#\left\{ \psi \in B^{\ell_{\varphi}+k} : \frac{i_\varphi}{n_{\ell_{\varphi}}^\psi} - i_\psi \leq D \right\} \]
\[ \leq \#\left\{ i \in \mathbb{Z}^d : \frac{i_\varphi}{n_{\ell_{\varphi}}} - i \leq D \right\} \]
\[ \leq \#\left\{ i \in \mathbb{Z}^d : -D + i_\varphi n_k \leq i \leq D + i_\varphi n_k \right\} , \]
which immediately implies the claim. \( \square \)

3.3. B–splines overlapping. This section considers the overlapping among B–splines of different levels.

Definition 3.19 (Cells supporting a B-spline). Given \( \varphi_i \in B \), we define the set
\( \mathbb{I}(\varphi_i) := \mathcal{I}^\ell[i : i + p] \) as its cell support.

This definition is motivated from the fact that \( I \in \mathbb{I}(\varphi) \) if and only if \( \varphi > 0 \) in \( \hat{I} \). Moreover \( \text{supp} \varphi_i = \bigcup_{k \in [i,i+p]} \mathcal{I}_k^{i} \). The proof of this fact is immediate from the definition of \( \varphi_i \) (see beginning of Section 3.2). We insist on working algebraically with sets of indices, because it is directly translated into the implementation, and easier to check for correctness.

Definition 3.20 (Cells overlapping a B-spline). Given \( k \in \mathbb{Z} \) and \( \varphi \in B \) we define \( \mathbb{I}^k(\varphi) := \text{ch}^k \mathbb{I}(\varphi) \). If \( F \subset B \) then \( \mathbb{I}^k(F) := \cup_{\varphi \in F} \mathbb{I}^k(\varphi) \).

Note that \( \mathbb{I}^k(\varphi) \subset \mathcal{I}^{\ell_{\varphi}+k} \) and in the case of maximum regularity splines, \( \mathbb{I}^k(\varphi) \) is the set of cells of level \( k + \ell_{\varphi} \) which overlap with the support of \( \varphi \). Note also that the \textit{children} operator \( \text{ch} \) is defined from \( B \) to \( B \) (and from \( \mathcal{L} \) to \( \mathcal{L} \), see Section 3). In the next lemma we explore on this.

Lemma 3.21 (Interchange of \( \mathbb{I} \) and \( \text{ch} \)). For \( \varphi \in B \) and \( k \in \mathbb{Z}_0^+ \) we have

(i) \( \mathbb{I}(\text{ch}^k \varphi) = \text{ch}^k \mathbb{I}(\varphi) = \mathbb{I}^k(\varphi) \)

(ii) \( \mathbb{I}(\text{ch}^{-k} \varphi) \supset \text{ch}^{-k} \mathbb{I}(\varphi) = \mathbb{I}^{-k}(\varphi) \)

Proof. Let \( \varphi = \varphi_i \in B \) and \( k \in \mathbb{Z}_0^+ \). On the one hand, from Lemma 3.18 \( \text{ch}^k \varphi_i = B^{\ell_{\varphi}+k}[M_0^k(i) : M_{sm}(i)], \) so that Definition 3.19
\[ \mathbb{I}(\text{ch}^k \varphi_i) = \mathbb{I}(B^{\ell_{\varphi}+k}[M_0^k(i) : M_{sm}(i)]) = \mathcal{I}^{\ell_{\varphi}+k} \left[ M_0^k(i) : M_{sm}(i) + p \right] . \]

On the other hand, due to Lemma 3.6 \( \text{ch}^k \mathbb{I}(\varphi_i) = \text{ch}^k \mathcal{I}^\ell[i : i + p] = \mathcal{I}^{\ell_{\varphi}+k} \left[ M_0^k(i) : M_{sm}(i) + p \right] , \)
and (i) follows from the fact that \( M_{sm}(i) + p = M_s(i + p) \) as can be proved by induction on \( k, \) using that \( n = s + 1 \) and \( m = p + 1. \)

In order to prove (ii), observe first that, again from Lemma 3.18 and Definition 3.19
\[ \mathbb{I}(\text{ch}^{-k} \varphi_i) = \mathbb{I}(B^{\ell_{\varphi}+k}[D_{sp}^k(i) : D_0^k(i)]) = \mathcal{I}^{\ell_{\varphi}+k} \left[ D_{sp}^k(i) : D_0^k(i) + p \right] . \]

Besides,
\[ \text{ch}^{-k} \mathbb{I}(\varphi_i) = \text{ch}^{-k} \mathcal{I}^\ell[i : i + p] = \mathcal{I}^{\ell_{\varphi}+k} \left[ D_{sp}^k(i) : D_0^k(i) + p \right] . \]

Thus, (ii) follows from the fact that \( D_0^k(i + p) \leq D_0^k(i) + p. \) \( \square \)

Definition 3.22 (B-splines overlapping a cell). Given \( k \in \mathbb{Z} \) and \( I \in \mathcal{I} \) we define \( B^k(I) := \{ \varphi \in B : I \in \mathbb{I}^k(\varphi) \} \). If \( F \subset \mathcal{I} \) then \( B^k(F) := \cup_{I \in F} B^k(I) \).
This last definition is reciprocal to the previous one, \( \mathbb{B}^k(I) \subset \mathcal{B}^{\ell+k} \) is the set of B-splines \( \varphi \) of level \( \ell + k \) whose support overlaps with \( I \). We immediately obtain the following.

**Lemma 3.23.** Let \( I \in \mathcal{I} \), \( \varphi \in \mathcal{B} \) and \( k \in \mathbb{Z} \). Then \( \varphi \in \mathbb{B}^k(I) \) if and only if \( \mathbb{I}^k(\varphi) \cap \text{ch}^k I \neq \emptyset \).

Note that \( \mathbb{I} \) and \( \mathbb{B} \) map B-splines into cells and viceversa, so \( \mathbb{I}^0 = \mathbb{I} \) and \( \mathbb{B}^0 = \mathbb{B} \).

**Lemma 3.24 (Cells overlapping a box of B-splines).** Let \( k \in \mathbb{Z}^+ \) then

(i) \( \mathbb{I}^k[B^j[i : j]] = \mathcal{I}^{\ell+k}[M_0^k(i) : M_0^k(j + p)] \)

(ii) \( \mathbb{I}^{-k}[B^j[i : j]] = \mathcal{I}^{-\ell+k}[D_0^k(i) : D_0^k(j + p)] \)

**Proof.** From Definitions 3.20 and 3.19 we have, for \( i \leq n \leq j \) and \( k \in \mathbb{Z} \)

\[ \mathbb{I}^k(\varphi_n^\ell) = \text{ch}^k \mathbb{I}(\varphi_n^\ell) = \text{ch}^k \mathcal{I}^k[n : n + p]. \]

If \( k > 0 \), Lemma 3.6 (i) yields

\[ \mathbb{I}^k(\varphi_n^\ell) = \mathcal{I}^{\ell+k}[M_0^k(n) : M_0^k(n + p)], \]

and for \( k < 0 \), Lemma 3.6 (ii) leads to

\[ \mathbb{I}^k(\varphi_n^\ell) = \mathcal{I}^{\ell+k}[D_0^k(n) : D_0^k(n + p)], \]

and the assertions follow.

**Lemma 3.25 (B-splines overlapping a box of cells).** Let \( \ell \in \mathbb{Z}_0^+ \) and \( i, j \in \mathbb{Z}^d \) then

(i) \( \mathbb{B}^0[\mathcal{I}^k[i : j]] = B^k[i - p : j] \)

(ii) \( \mathbb{B}^k[\mathcal{I}^k[i : j]] = B^{\ell+k}[M_0^k(i) - p : M_0^k(j)], \) for \( k \in \mathbb{Z}^+ \)

(iii) \( \mathbb{B}^{-k}[\mathcal{I}^k[i : j]] = B^{-\ell-k}[D_0^k(i) - p : D_0^k(j)], \) for \( k \in \mathbb{Z}^+ \).

**Proof.** Due to Definitions 3.20 and 3.19 we have that \( \varphi_n^\ell \in \mathbb{B}^k(I^\ell_r) \) iff \( I^\ell_r \in \mathbb{I}(\varphi_n^\ell) \), which holds iff \( n \leq r \leq n + p \) or \( r - p \leq n \leq r \). This implies (i).

From Definition 3.22 and Lemma 3.24 (i) we have, for \( k \in \mathbb{Z}^+ \) that \( \varphi_n^{\ell+k} \in \mathbb{B}^{\ell+k}(I^\ell_r) \) iff \( I^\ell_r \in \mathbb{I}^{-k}(\varphi_n^{\ell+k}) = \mathcal{I}^{\ell+k}[D_0^k(n) : D_0^k(n + p)], \) which holds iff \( D_0^k(n) \leq r \leq D_0^k(n + p) \). Due to Lemma 10.4 this is equivalent to \( n \leq M_0^k(n - 1) \) and \( M_0^k(n) \leq n + p \). The assertion (ii) thus follows.

Analogously, for \( k \in \mathbb{Z}^+ \), \( \varphi_n^{\ell-k} \in \mathbb{B}^{-k}(I^\ell_r) \) iff \( I^\ell_r \in \mathbb{I}^{k}(\varphi_n^{\ell-k}) = \mathcal{I}^{k}[M_0^k(n) : M_0^k(n + p)], \) which holds iff \( M_0^k(n) \leq r \leq M_0^k(n + p) \). Due to Lemma 10.4 this is equivalent to \( n \leq D_0^k(n) \) and \( D_0^k(n) \leq n + p \). The assertion (iii) thus follows.

Next we state that basically, a B-spline overlaps a cell if and only if it is positive on a sub-cell of it.

The next lemma is an immediate consequence of Definitions 3.19 and 3.22.

**Lemma 3.26.** Let \( k \in \mathbb{Z}^+ \), \( A^\ell \subset \mathcal{I}^\ell \) and \( \varphi \in \mathcal{B}^{\ell+k} \). If \( \varphi \notin \mathbb{B}^k(A^\ell) \) then \( \varphi = 0 \) in the interior of \( \bigcup_{I \in A^\ell} I \).

### 3.4. Overlapping chains.

In the current proofs of optimality for adaptive methods, and for some quasi-interpolants to provide local bounds, it seems necessary to have the *level gap* of overlapping basis functions uniformly bounded. More precisely, whenever a cell is contained in the support of two basis functions, it is desirable that the difference in levels of those basis functions is uniformly bounded. This stems from the necessity of using inverse estimates in some stages of the proof. The difference could be large, but should be uniformly bounded. Some of the constants appearing in the results will depend on this bound, and the constants should be uniform to close the arguments.
That is why in this section we deal with \( B \)-splines overlapping other \( B \)-splines.

**Definition 3.27 (\( B \)-splines overlapping \( B \)-splines).** Let \( H \subset B, F \subset B \) and \( g \in Z \) define \( O(F, g, H) := B^g(\{1\} + H) \). And for simplicity we write \( O(\varphi, g, H) \) to denote \( O(\{\varphi, g, H\}) \) when \( \varphi \in B \).

**Remark 3.28.** Note that \( \psi \in O(\varphi, 0, B) \) if and only if \( \ell_\psi = \ell_\varphi \) and \( |\rho(\varphi, \psi)| < m \), so that \( O(\varphi, 0, B) = B(\varphi, m - 1, 0) \), with \( B(\cdot, \cdot, \cdot) \) as in Definition 3.17.

**Definition 3.29 (Chains of overlapping \( B \)-splines).** Let \( H \subset B, F \subset B, k \in Z^+ \) and \( g \in Z \) define \( O^k(F, g, H) \) as the \( k \)-th fold composition of \( O(\cdot, g, H) \), i.e., \( O^{k+1}(F, g, H) = O(O^k(F, g, H), g, H) \) and \( O^1(F, g, H) = O(F, g, H) \).

**Remark 3.30.** It is worth noticing that the computational implementation of these concepts is very easy. It is just the intersection of sets of indices, which are previously grouped by levels.

**Lemma 3.31 (Properties of overlapping chains).** Let \( g \in Z^+ \) and \( i, j \in Z^d \) and \( k = 1, \ldots, \lfloor \frac{d}{g} \rfloor \) then

(i) \( O^k(B^g[i : j], -g, B) = B^{L_k(i) - R_k(j)}(L^k(i) : R^k(j)) \), where \( L \) and \( R \) are the index functions defined in (2.22).

(ii) \( O^k(\varphi, -g, B) \subset B(\varphi, C, \ell_\varphi - gk) \), with \( B(\cdot, \cdot, \cdot) \) the ball of \( B \)-splines from Definition 3.17 and \( C := p \left( \frac{1-1/n^g}{n^g-1} \right) n^g + 1 \leq \frac{m^g}{n^g-1} + 1 \).

(iii) \#O^k(\varphi, g, B) < (2C + 1)^d$.

**Proof.** Using Definitions 3.27, Lemma 3.24 (ii) and Lemma 3.25 (iii) we have

\[
O(B[i : j], -g, B) = B^{-g}(\{B[i : j]\}) = B^{-g}(I[i : j + p]) = B^{-g}[D_0(i) - p : D_0(j + p)] = B^{-g}(L(i) : R(j)),
\]
due to (2.22). By induction (i) follows.

In order to prove (ii) observe that from (i), \( \psi \in O^k(\varphi, -g, B) \) if and only if \( \psi \in B^{L_k(i) - R_k(i)}(L^k(i) : R^k(i)) \), which holds if and only if \( L^k(i) \leq i_\psi \leq R^k(i) \). Due to Lemma 10.2 this is equivalent to

\[
\frac{i}{n^g} - \frac{p}{n^g - 1} = -A \leq i_\psi \leq \frac{i}{n^g} + \frac{p}{n^g - 1} - B,
\]
for some \( 0 \leq A, B \leq 1 - \frac{1}{n^g} \). This, in turn, is equivalent to

\[
B - \frac{p}{n^g - 1} \leq -\frac{i}{n^g} - i_\psi \leq \frac{p}{n^g - 1} + A,
\]
which implies

\[
|\rho(\varphi, \psi)| = \left| \frac{i}{n^g} - i_\psi \right| \leq \frac{1}{n^g - 1} n^g + 1,
\]
and (ii) holds. The final assertion (iii) is an immediate consequence of (ii) and Lemma 3.18.

**Remark 3.32.** Notice that from Definition 3.27, \( \eta \in O(\varphi, -k, B) \) iff \( \eta \in B^{-k}(I(\varphi)) \) = \( \cup_{I \in I(\varphi) B^{-k}(I)} \), and due to Definition 3.22 this holds iff there exists \( I \in I(\varphi) \) with \( \eta \in B^{-k}(I) \), i.e., iff \( I(\varphi) \cap B^{-k}(\eta) \neq \emptyset \). Summarizing,

\[
\eta \in O(\varphi, -k, B) \iff I(\varphi) \cap B^{-k}(\eta) \neq \emptyset.
\]
Definition 3.33 (Totally overlapped). Let \( \varphi \in \mathcal{B} \) and \( \mathcal{F} \subset \mathcal{B} \), we say that \( \varphi \) is totally overlapped by \( \mathcal{F} \) if there is a partition \( \mathcal{P} \) of \( \mathcal{I}(\varphi) \) such that \( \mathcal{P} \subset \mathcal{I}(\mathcal{F}) \).

Lemma 3.34 (Overlapping of descendants). Let \( \varphi \in \mathcal{B} \) and \( \mathcal{H} \subset \mathcal{B} \) then

\[
\mathcal{O}(\text{ch}^k \varphi, j, \mathcal{H}) = \mathcal{O}(\varphi, j + k, \mathcal{H}), \quad \text{for any } j \in \mathbb{Z} \text{ and } k \in \mathbb{Z}_0^+.
\]

Proof. Let \( \varphi = \varphi^j \in \mathcal{B} \). From Definitions 3.27 and 3.20 and Lemma 3.21,

\[
\mathcal{O}(\text{ch}^k \varphi, j, \mathcal{H}) = \mathbb{B}^j \left( \mathbb{I}(\text{ch}^k \varphi) \right) \cap \mathcal{H} = \mathbb{B}^j \left( \mathbb{I}^k(\varphi) \right) \cap \mathcal{H}, \quad \text{for any } j \in \mathbb{Z} \text{ and } k \in \mathbb{Z}_0^+.
\]

Since by definition \( \mathcal{O}(\varphi, j + k, \mathcal{H}) = \mathbb{B}^{j+k}(\varphi) \cap \mathcal{H} \), the rest of the proof will be devoted to proving that \( \mathbb{B}^j(\mathbb{I}^k(\varphi)) = \mathbb{B}^{j+k}(\varphi) \) for any \( j \in \mathbb{Z} \) and \( k \in \mathbb{Z}_0^+ \).

Observe that Lemma 3.24 yields

\[
\mathbb{B}^j(\mathbb{I}^k(\varphi)) = \mathbb{B}^j \left( \mathbb{I}^k(\mathbb{B}^0[i : i]) \right) = \mathbb{B}^j \left( \mathbb{I}^{k+j}[M_0^k(i) : M_0^k(i + p)] \right),
\]

for every \( k \in \mathbb{Z}_0^+ \) and \( j \in \mathbb{Z} \).

Consider first the case \( j \in \mathbb{Z}_0^+ \). From Lemma 3.25 (i)–(ii),

\[
\mathbb{B}^j(\mathbb{I}^k(\varphi)) = \mathbb{B}^{j+k+j} \left( M_0^k(M_0^k(i)) - p : M_0^j(M_0^k(i + p)) \right)
= \mathbb{B}^{j+k+j} \left( M_0^{j+k}(i) - p : M_0^{j+k}(i + p) \right)
= \mathbb{B}^{j+k}(\mathbb{I}^j(i + p)) = \mathbb{B}^{j+k}(\mathbb{I}(\varphi)).
\]

If \( j \in \mathbb{Z}^- \), Lemma 3.25 (iii) yields

\[
\mathbb{B}^j(\mathbb{I}^k(\varphi)) = \mathbb{B}^{k+j} \left( D_0^{-j}(M_0^0(i)) - p : D_0^{-j}(M_0^k(i + p)) \right).
\]

Consider now \( j < 0 \) fixed and \( k > -j \) \((j + k \geq 0)\), then \( D_0^{-j}(M_0^0(i)) = M_0^{j+k}(i) \) and \( D_0^{-j}(M_0^k(i + p)) = M_0^{k+j}(i + p) \), so that Lemma 3.25 (i)–(ii) leads to

\[
\mathbb{B}^j(\mathbb{I}^k(\varphi)) = \mathbb{B}^{k+j} \left[ M_0^{k+j}(i) - p : M_0^{k+j}(i + p) \right] = \mathbb{B}^{j+k}(\mathbb{I}(\varphi)).
\]

If \( k < -j \) \((k + j < 0)\), then \( D_0^{-j}(M_0^k(i)) = D_0^{-(k+j)}(i) \) and \( D_0^{-j}(M_0^k(i + p)) = D_0^{-(k+j)}(i + p) \), so that,

\[
\mathbb{B}^j(\mathbb{I}^k(\varphi)) = \mathbb{B}^{k+j} \left( D_0^{-(k+j)}(i) - p : D_0^{-(k+j)}(i + p) \right) = \mathbb{B}^{j+k}(\mathbb{I}(\varphi)),
\]

due to Lemma 3.25 (iii).

Summarizing, for each \( j \in \mathbb{Z} \) and any \( k \in \mathbb{Z}_0^+ \), \( \mathbb{B}^j(\mathbb{I}^k(\varphi)) = \mathbb{B}^{j+k}(\mathbb{I}(\varphi)) \), and the assertion thus follows. \( \blacksquare \)

4. Hierarchical Generators and Spaces. We are interested in a class of spaces obtained through an iterative process of function refinement. These spaces are similar to other spaces that have been previously defined in the literature; see Remark 6.11. Our approach is based on functions rather than on subdomains, and yields a particular class of subsets of \( \mathcal{B} \) that we call lineages and are given by the following definition.
**Definition 4.1** (Lineage set). A set \( \mathcal{L} \subset \mathcal{B} \), is called a lineage if it is finite and \( \mathcal{L} \subset \mathcal{B}^0 \cup \text{ch} \mathcal{L} \). Given a lineage set \( \mathcal{L} \) we let \( \mathcal{C} := \mathcal{B}^0 \cup \text{ch} \mathcal{L} \), be the children of \( \mathcal{L} \) plus the coarsest B-splines, that we will call the C-set associated to the lineage.

This definition is very simple, resorting to the operator ch and notation from set theory. It says, essentially, that a set \( \mathcal{L} \) is a lineage if every element of \( \mathcal{L} \) is the child of an element of \( \mathcal{L} \) or is itself an element of level zero (belongs to \( \mathcal{B}^0 \)). The well known tree structure fulfills this assumption, among others. This new framework allows us to deal with a simple implementation, which also makes it very easy to control the overlapping of functions from different levels.

The idea behind a lineage \( \mathcal{L} \) is that \( \mathcal{L} \) is the set of functions that have been refined in an adaptive process, so that the hierarchical space is the one spanned by their children. More precisely.

**Definition 4.2** (Hierarchical generator). Let \( \mathcal{L} \) be a lineage, the set
\[
\mathcal{H} = (\mathcal{B}^0 \cup \text{ch} \mathcal{L}) \setminus \mathcal{L} = \mathcal{C} \setminus \mathcal{L}
\]

is the hierarchical generator corresponding to \( \mathcal{L} \).

Notice that \( \mathcal{L} = \emptyset \) is a valid lineage, and its corresponding generator is \( \mathcal{H} = \mathcal{B}^0 \).

It is convenient to have a notation to arrange these sets by level, so for \( \ell \in \mathbb{Z}_+^n \) let \( \mathcal{C}^\ell := \mathcal{C} \cap \mathcal{B}^\ell \), \( \mathcal{L}^\ell := \mathcal{L} \cap \mathcal{B}^\ell \) and \( \mathcal{H}^\ell := \mathcal{H} \cap \mathcal{B}^\ell \). It is easy to see that for any lineage, \( \mathcal{C}^0 = \mathcal{B}^0 \), \( \mathcal{C} = \cup_{\ell=0}^\infty \mathcal{C}^\ell \) and there exists \( \ell \) such that \( \mathcal{L}^\ell = \emptyset \). If \( \mathcal{L}^\ell = \emptyset \) then \( \mathcal{C}^\ell = \emptyset \) for all \( \ell' > \ell \), and the following is well defined.

**Definition 4.3** (Depth of a lineage). Given the hierarchical generator \( \mathcal{H} \) with lineage \( \mathcal{L} \) we define its depth as \( \text{depth}(\mathcal{L}) = \text{depth}(\mathcal{H}) := \min\{\ell : \mathcal{L}^\ell = \emptyset\} \). Observe that \( \mathcal{L} \) has functions of level \( \text{depth}(\mathcal{L}) - 1 \) and \( \mathcal{H} \) has functions of level \( \text{depth}(\mathcal{H}) \), which is the finest level of functions in \( \mathcal{H} \).

From the definition of hierarchical generator it is clear that for each lineage there is a unique hierarchical generator. The reciprocal is also true.

**Lemma 4.4** (Lineage to generator bijection). There is a bijection between hierarchical generators and lineages.

**Proof.** Let \( \mathcal{L} \) and \( \mathcal{L}' \) be two lineages giving the same hierarchical generator \( \mathcal{H} \). That is to say \( \mathcal{C} \setminus \mathcal{L} = \mathcal{C} \setminus \mathcal{L}' \), or by levels using the symmetric difference \( (\mathcal{L}^\ell \triangle \mathcal{C}^\ell) \triangle (\mathcal{L}'^\ell \triangle \mathcal{C}^\ell) = \emptyset \) for each \( \ell \). For \( \ell = 0 \), \( \mathcal{C}^0 = \mathcal{B}^0 \) so using the symmetric difference property that \( (A \triangle B) \triangle (B \triangle C) = (A \triangle C) \) we get that \( \mathcal{L}^0 = \mathcal{L}'^0 \). Assume that \( \mathcal{L}^\ell = \mathcal{L}'^\ell \) for \( \ell \leq n \). Then \( \mathcal{C}^{n+1} = \mathcal{C}^{n+1} \) and so using the previous argument on the symmetric difference we obtain that \( \mathcal{L}^{n+1} = \mathcal{L}'^{n+1} \). Hence, we have shown by induction that \( \mathcal{L} = \mathcal{L}' \) (and that \( \mathcal{C} = \mathcal{C}' \)), thus proving that there is a unique lineage associated to each hierarchical generator. \( \square \)

**Remark 4.5** (Lineages vs. C-sets). This is subtle but important. One may be tempted to use the C-sets to identify the hierarchical generators instead of the lineages. However, the relation between hierarchical generators and the C-sets is not one-to-one, as is the case between hierarchical generators and lineages. In fact, consider \( n = 3 \) with \( d = 1 \), let \( \mathcal{L} = \{ \varphi^0_{-2}, \varphi^0_0 \} \) and \( \mathcal{L}' = \{ \varphi^0_{-2}, \varphi^0_{-1}, \varphi^0_0 \} \) then the corresponding C-awa are \( \mathcal{C} = \mathcal{C}' = \mathcal{B}^0 \cap \mathcal{B}^1 \), even though the corresponding hierarchical generators differ, i.e., the same set \( \mathcal{C} \) can correspond to different hierarchical generators. Thus a lineage has some built-in information that is missing in the C-sets. Something similar happens with the so called hierarchical grids \( \# \) given by nested domains, where every grid leads to a generator but different grids may lead to the same generator.

**Definition 4.6** (Hierarchical Space). Given a hierarchical generator \( \mathcal{H} \), the linear space \( \mathcal{V} = \text{span} \mathcal{H} \) is called a hierarchical space.

**Lemma 4.7** (More relations between \( \mathcal{H} \) and \( \mathcal{L} \)). Given a hierarchical generator
\( \mathcal{H} \) with the associated lineage \( \mathcal{L} \) it follows that

(i) Each \( \psi \in \mathcal{L} \) can be written as a linear combination of its descendants in \( \mathcal{H} \).

More precisely, \( \psi \in \text{span}(\mathcal{H} \cap \text{dsc} \psi) \), where \( \text{dsc} \psi = \cup_{k \in \mathbb{Z}^+} \text{ch}^k \psi \) is the set of all descendants of \( \psi \).

(ii) Each \( \varphi \in \mathcal{H} \) has an ancestor of every possible level in \( \mathcal{L} \). More precisely, for \( \ell \in \mathbb{Z}_0^+ \), \( k \in \mathbb{Z}^+ \) it holds that \( \mathcal{H}^{\ell+k} \subset \text{ch}^k \mathcal{L}^\ell \).

**Proof.** We thus prove (i) by (backward) induction on the level of \( \psi \). Let \( N = \text{depth} \mathcal{L} \) and \( \psi \in \mathcal{L}^{N-1}, \) i.e., \( \psi \in \mathcal{L} \), with \( \ell_\psi = N - 1 \), then no child of \( \psi \) belongs to \( \mathcal{L} \), because \( \mathcal{L}^N = \emptyset \). Therefore, \( \text{ch} \psi \subset \mathcal{C} \setminus \mathcal{L} = \mathcal{H} \) so \( \psi \in \text{span}(\mathcal{H} \cap \text{dsc} \psi) \) due to Lemma 3.11. Suppose now that the assertion is true for all functions in \( \mathcal{L}^{N-j} \) with \( 0 \leq j < N \). Let \( \psi \in \mathcal{L}^{N-(j+1)} \). Lemma 3.11 yields \( \psi \in \text{span} \psi \). Since \( \mathcal{L} \) is a lineage, Definition 4.11 implies that \( \text{ch} \psi \subset \mathcal{C} \). Then, from Definition 4.12, each child of \( \psi \) either belongs to \( \mathcal{L} \) or to \( \mathcal{L}^{N-j} \). Each of the latter belongs to \( \text{span}(\mathcal{H} \cap \text{dsc} \psi) \) from the inductive assumption, and the assertion follows. In order to prove (ii), we proceed by induction on \( k \). Let \( \ell \in \mathbb{Z}_0^+ \), \( k = 1 \) and \( \psi \in \mathcal{H}^{\ell+1} \), i.e., \( \psi \in \mathcal{H} \) and \( \ell_\psi = \ell_\psi - 1 = \ell \) such that \( \psi \in \text{ch} \phi \subset \text{ch} \mathcal{L}^\ell \). Suppose now that the assertion is true for \( k = m \) and \( \ell \in \mathbb{Z}_0^+ \). Let \( \psi \in \mathcal{H}^{\ell+m} \). Then from the inductive assumption \( \psi \in \text{ch}^m \mathcal{L}^{\ell+1} \subset \text{ch}^m \text{ch} \mathcal{L}^\ell = \text{ch}^{m+1} \mathcal{L}^\ell \), so the assertion follows.

**Remark 4.8.** It is worth noticing that as an immediate consequence of the previous lemma, we always have \( \mathcal{L} \subset \text{span} \mathcal{C} = \text{span} \mathcal{H} \).

**Corollary 4.9 (\( \mathcal{H} \) and \( \mathcal{C} \) cell relations).** For any hierarchical generator \( \mathcal{H} \) and \( \ell \in \mathbb{Z}_0^+ \), \( k \in \mathbb{Z}^+ \), we have

(i) If \( \psi \in \mathcal{L} \) then \( \llbracket \psi \rrbracket \subset \cup_{k > 0} \llbracket \text{ch}^k \psi \cap \mathcal{H} \rrbracket \), thus \( \llbracket \mathcal{L}^\ell \rrbracket \subset \cup_{k > 0} \llbracket \text{ch}^k \mathcal{L}^{\ell+k} \rrbracket \).

(ii) \( \llbracket \mathcal{H}^{\ell+k} \rrbracket \subset \llbracket \text{ch}^k \mathcal{L}^\ell \rrbracket \).

**Proof.** To show part (i), let \( \psi \in \mathcal{L} \) and consider a given \( I \in \llbracket \psi \rrbracket \). Then for any \( x \in I^0 \) as \( \psi(x) \neq 0 \), Lemma 3.14(i) implies that there is \( \varphi \in \text{ch}^k \psi \) such that \( \varphi(x) \neq 0 \). Thus there is \( I' \in \llbracket \varphi \rrbracket \) such that \( \text{ch}^{-k} I' = I \). From here part (i) follows. In order to prove (ii) we use Lemma 4.3(ii) to see that \( \llbracket \mathcal{H}^{\ell+k} \rrbracket \subset \llbracket \text{ch}^k \mathcal{L}^\ell \rrbracket \) and Lemma 4.2(i) to conclude that \( \llbracket \text{ch}^k \mathcal{L}^\ell \rrbracket = \llbracket \text{ch}^k \mathcal{L}^\ell \rrbracket \).

Other definitions of hierarchical spline spaces are given in terms of hierarchical grids, or sequence of nested subdomains. In those definitions it is natural to think of active cells, which we now define.

**Definition 4.10 (Active cells).** Given a hierarchical generator \( \mathcal{H} \) we define the set of active cells as \( \mathcal{A} := \llbracket \mathcal{C} \rrbracket \setminus \llbracket \mathcal{L} \rrbracket \). And \( \mathcal{A}^\ell = \mathcal{A} \cap \mathcal{I}^\ell \), for \( \ell \in \mathbb{Z}_0^+ \).

Observe that \( \mathcal{A} \subset \llbracket \mathcal{H} \rrbracket \) but not the other way around. The next result shows that finer B-splines in the generator do not overlap active cells.

**Lemma 4.11.** Given a hierarchical generator \( \mathcal{H} \) we have \( \mathcal{H}^{\ell+k} \cap \text{B}^k(\mathcal{A}^\ell) = \emptyset \) for \( k \in \mathbb{Z}^+ \) and \( \ell \in \mathbb{Z}_0^+ \). In other words, if \( \varphi \in \mathcal{H}^{\ell+k} \) and \( I \in \mathcal{A}^\ell \) then \( \varphi = 0 \) on \( I \).

**Proof.** Let \( \ell \in \mathbb{Z}_0^+ \), \( k \in \mathbb{Z}^+ \) and \( \varphi \in \mathcal{H}^{\ell+k} \), then by Corollary 4.3(iii) \( \llbracket \varphi \rrbracket \subset \llbracket \text{ch}^k \mathcal{L}^\ell \rrbracket \). From Definition 4.11 we have \( \mathcal{A}^\ell \cap \llbracket \mathcal{L}^\ell \rrbracket = \emptyset \), and also \( \text{ch}^k \mathcal{A}^\ell \cap \llbracket \mathcal{L}^\ell \rrbracket = \emptyset \). Therefore, \( \llbracket \varphi \rrbracket \cap \text{ch}^k \mathcal{A}^\ell = \emptyset \), hence \( \varphi \notin \text{B}^k(\mathcal{A}^\ell) \) owing to Lemma 3.23 and the assertion follows.

**Lemma 4.12 (Positive spanning of the unity).** Let \( \mathcal{H} \) be a hierarchical generator. Then for each \( \varphi \in \mathcal{H} \) there exists a positive coefficient \( c_\varphi \) such that \( \sum_{\varphi \in \mathcal{H}} c_\varphi \varphi = 1 \).

**Proof.** As \( \mathcal{B}^0 \) is a partition of unity over \( \Omega \), it follows that \( 1 = \sum_{\varphi \in \mathcal{B}^0} \varphi \). Also we know that \( \mathcal{B}^0 = \mathcal{H}^0 \cup \mathcal{B}^0 \) so \( 1 = \sum_{\varphi \in \mathcal{H}^0} \varphi + \sum_{\varphi \in \mathcal{B}^0} \varphi \). Suppose we have shown that \( 1 = \sum_{j=0}^{\ell} \sum_{\varphi \in \mathcal{H}^j} c_\varphi \varphi + \sum_{\varphi \in \mathcal{B}^j} \beta_\varphi \varphi \) with \( c_\varphi > 0 \) for each \( \varphi \in \bigcup_{j=0}^{\ell} \mathcal{H}^j \) and \( \beta_\varphi > 0 \) for each \( \varphi \in \mathcal{B}^\ell \), using Lemma 3.14(i) each \( \varphi \in \mathcal{B}^\ell \) can be spanned by its children in \( \mathcal{B}^\ell \).
\[C^{\ell+1} = H^{\ell+1} \cup L^{\ell+1}\] with positive coefficients, so that 
\[\sum_{\phi \in L^\ell} \beta_\phi \phi = \sum_{\phi \in \mathcal{H}^{\ell+1}} c_\phi \phi + \sum_{\phi \in L^{\ell+1}} \beta_\phi \phi, \text{ with } c_\phi, \beta_\phi > 0, \text{ and thus } 1 = \sum_{j=0}^{\ell+1} \sum_{\phi \in \mathcal{H}_j} c_\phi \phi + \sum_{\phi \in L^{\ell+1}} \beta_\phi \phi. \]

5. Refinement. Lineages provide a convenient framework to define a concept of refinement that will allow us to rigorously study the process. The germ is the following definition.

**Definition 5.1 (Refinements and refiner sets).** We say that a lineage \(L_*\) is a refinement of the lineage \(L\) whenever \(L \subset L_*\), and we denote it with \(L_* \succ L\). The set difference \(R = L_* \setminus L\) is called the refiner set of the refinement. Accordingly (in light of Lemma 4.4) we say that a hierarchical generator \(H_*\) is a refinement of \(H\), and denote it with \(H_* \succ H\), whenever \(L_* \succ L\).

**Remark 5.2 (Conventional notation).** From now on, unless explicitly stated, whenever we say that \(L, L_*\) are lineages, without further stating, \(H, V, H_*, V_*\) will denote their corresponding hierarchical generators and spaces, respectively, and vice versa. Moreover, if \(L_* \succ L, R = L_* \setminus L\) will be the refiner set.

As an immediate consequence of Lemma 4.7 and the fact that \(L_* \succ L\) yields \(C_* \supset C\) we have that \(V_* \supset V\) due to Remark 4.8. But notice that \(H_* \succ H\) does not necessarily imply that \(H_* \supset H\).

5.1. Order on Refinements and the Smallest Element. The set of all lineages with the inclusion relation is a partially ordered set (POSET). A minimal element of a subset \(S\) of some POSET is defined as an element of \(S\) that is not greater than any other element in \(S\). The least element is an element of \(S\) that is smaller than every other element of \(S\). A set can have several minimal elements without having a least element. However, if it has a least element, it can’t have any other minimal element. As the family of lineages is a POSET and there is a one to one correspondence with the family of hierarchical generators (\(H_* \succ H\) iff \(L_* \supset L\)) we transfer the partial order from the lineages to the hierarchical generators. More precisely.

**Property 5.3 (Generators are partially ordered by refinement).** The “being refinement of” relation \(\succ\) is a partial order in the family of hierarchical generators.

The family of hierarchical generators has a least element \(H = B^0\), which corresponds to \(L = \emptyset\).

The approach to refinement as a partial order allows us to rigorously pose the problem of finding the smallest (minimal or least) refinement of \(H\) that satisfies some given property. For example, if given a hierarchical generator \(H\) we define \(\mathcal{B}(H) := \{H_* \succ H : H_* \text{ is linearly independent}\}\), we can ask what is \(\min \mathcal{B}(H)\), the set of minimal elements; in some cases of interest it can be a singleton with only the least element.

5.2. Algebra of a refinement. The algebra of set inclusion can be applied to Definition 5.1 and obtain some useful properties with simple proofs.

**Lemma 5.4 (Basic properties).** Let \(H_* \succ H\) with \(L_* \), \(L\) the corresponding lineages and \(R\) the refiner set, and let \(M := R \cap H\). Then

(i) \(H_* \setminus H = \text{ch } R \setminus (C \cup R)\)
(ii) \(H_* \cap H = H \setminus R\)
(iii) \(H \setminus H_* = M\)
(iv) \(R \setminus H = R \setminus C\)
(v) \(R \setminus C \subset \text{ch } R \setminus C\)
(vi) \(R \subset (\text{ch } R \setminus C) \cup M\)
Proof. Using that $H = C \setminus L$, $H_* = C_* \setminus L_*$, and the set identity
\[
(A \setminus B) \setminus (C \setminus D) = (A \setminus (B \cup C)) \cup ((A \cap D) \setminus B)
\] (5.1)
we get
\[
H_* \setminus H = (C_* \setminus L_*) \setminus (C \setminus L) = (C_* \setminus (L_* \cup C)) \cup ((C_* \cap L) \setminus L_*) = C_* \setminus (C \cup R),
\]
where in the last equality we have used that $L_* \cup C = L \cup R \cup C = C \cup R$, because $L \subset C$. Finally, $C_* \setminus (C \cup R) = (\text{ch } L_* \cup B_0) \setminus (\text{ch } L \cup B_0 \cup R) = (\text{ch } L_* \setminus \text{ch } L) \setminus (C \cup R)$
and (i) follows.

Identity (ii) follows from the set identity $(A \setminus B) \cap (C \setminus D) = (A \cap C) \setminus (B \cup D)$ and the fact that $C \subset C_*$. Indeed,
\[
H_* \cap H = (C_* \setminus L_*) \cap (C \setminus L) = (C_* \cap C) \setminus (L_* \cup L) = C \setminus L_* = C \setminus (L \cup R) = (C \setminus L) \setminus R = H \setminus R.
\] (5.2)

For (iii) observe that $H \setminus H_* = H \setminus (H_* \cap H)$ so from (ii)
\[
H \setminus H_* = H \setminus (H \setminus R) = H \setminus R = M.
\]
Using the set identity $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$, we conclude
\[
R \setminus H = R \setminus (C \setminus L) = (R \setminus C) \cup (R \cap L) = R \cap C,
\]
because $L \cap R = \emptyset$ and (iv) follows.

To prove (v) we use (iv) and the fact that $L_*$ is a lineage, to conclude that
\[
R \setminus H = R \setminus C \subset L_* \setminus C \subset (\text{ch } L_* \cup C) \setminus C = \text{ch } L_* \setminus C \subset \text{ch } L_* \setminus (C \cup R \setminus L \setminus C).
\]
In order to prove (vi) observe that $R = (R \setminus H) \cup (R \cap M) = (R \setminus H) \cup M$ and use (v).

5.3. Single refinement. Here we show that Definition 5.1 of refinement is in fact equivalent to the natural one of refining one function at a time.

Refining a function in a hierarchical generator means to find the smallest refinement for which that function is in the refiner set. More precisely,

**Definition 5.5** (Refinement of one function). Let $H$ be a hierarchical generator and $\varphi \in H$, the refinement of $\varphi$ is the least element of the family $\{H : H \succ H \text{ and } \varphi \in L\}$.

The next Lemma shows that the definition is well posed.

**Lemma 5.6** (Good definition). Let $H$ be a hierarchical generator and $\varphi \in H$, the least element of $\{H : H \succ H \text{ and } \varphi \in L\}$ is the hierarchical generator whose lineage is $L_* = L \cup \{\varphi\}$, namely $H_* = H \setminus \{\varphi\} \cup (\text{ch } \varphi \setminus \text{ch } L)$.

**Proof.** All we have to prove here is that if $L$ is a lineage and $\varphi$ is an element of the corresponding hierarchical generator $H$, then $L_* = L \cup \{\varphi\}$ is also a lineage. This is very simple, since $\varphi \in H \subset C \subset L \subset C$ and $L$ is a lineage ($L \subset C$) we have
\[
L_* = L \cup \{\varphi\} \subset C \cup \{\varphi\} = C = L^0 \cup \text{ch } L \subset L^0 \cup \text{ch } L_* = C_*.\]
The minimality is a consequence of the fact that the smallest set that contains $\mathcal{L}$ and $\varphi$ is $\mathcal{L} \cup \{\varphi\}$. To find expressions for $H_*$ we use Lemma 5.4 (i)–(ii) as follows

$$H_* = (H_* \cap H) \cup (H_* \setminus H) = (H \setminus R) \cup (\text{ch } R \setminus (\mathcal{L} \cup R))$$

$$= (H \setminus \{\varphi\}) \cup (\text{ch } \varphi \setminus C) = (H \setminus \{\varphi\}) \cup (\text{ch } \varphi \setminus \text{ch } \mathcal{L}).$$

\[ \square \]

Remark 5.7 (Single refinement = adding more resolution). According to Lemma 5.6 refining a B-spline function $\varphi$ consists in substituting it by its children that are not already in the generator. We have added the smallest number of children that are necessary to span $\varphi$.

A constructive process to build $\mathcal{L}_*$ is called SINGLEREFINE and it is described in Algorithm 1 below, which we include despite its simplicity as it will be called from more complex algorithms later on.

\begin{algorithm}
\caption{Refine one function}
\begin{algorithmic}[1]
\Function{SINGLEREFINE}{$H, \varphi$}
\State $\mathcal{L} \leftarrow \mathcal{L} \cup \{\varphi\}$
\EndFunction
\end{algorithmic}
\end{algorithm}

Lemma 5.8 (Single refinement). Let $H$ be a hierarchical generator with lineage $\mathcal{L}$ and $\varphi \in H$. A call to SINGLEREFINE($H, \varphi$) modifies the set $H$, as $\mathcal{L}$ is modified [see Remark 5.6]. Let the original set before the call be $H_0$, and $H_*$ its modification after Algorithm 1 is executed. Then $H_*$ is a refinement of the original $H$ with refiner $\mathcal{R} = \{\varphi\}$. Furthermore, $H_* = H \setminus \{\varphi\} \cup (\text{ch } \varphi \setminus \mathcal{L}^{\varphi+1})$.

Remark 5.9 (Implementation tip). The modification for $H$ is to remove $\varphi$ and add $\text{ch } \varphi$ that are not in $\text{ch } \mathcal{L}$. But the only children of $\varphi$ that could belong to $\text{ch } \mathcal{L}$ are those who are in $\text{ch}(\mathcal{L}^{-1}(\text{ch } \varphi) \cap \mathcal{L})$. This set looks more complicated than $\text{ch } \mathcal{L}$ but is in fact much smaller.

5.4. Refining a set of functions. The process of refining one function can be naturally extended to refining a subset $\mathcal{M}$ of $H$. This can be defined as finding the smallest refinement of $H$ whose refiner contains $\mathcal{M}$.

\begin{algorithm}
\caption{Refine a set of functions}
\begin{algorithmic}[1]
\Function{REFINE}{$H, \mathcal{M}$}
\For{$\varphi \in \mathcal{M}$}
\State SINGLEREFINE($H, \varphi$)
\EndFor
\EndFunction
\end{algorithmic}
\end{algorithm}

Lemma 5.10 (Refining a subset of $H$). Let $H$ be a hierarchical generator and $\mathcal{M} \subset H$. A call to $\text{REFINE}(H, \mathcal{M})$ of Algorithm 2 finishes modifying the set $H$. Let the original set before the call be $H_0$ and $H_*$ its modification after Algorithm 2 is executed. Then $H_*$ is a refinement of $H$ and its refiner is $\mathcal{R} = \mathcal{M} \cup (\mathcal{L}_* = \mathcal{L} \cup \mathcal{M})$. In particular, the same hierarchical generator $H_*$ is obtained independent of the order in which the functions of $\mathcal{M}$ are passed to SINGLEREFINE. Furthermore this is the smallest refinement of $H$ that refines all functions in $\mathcal{M}$.
Proof. To show that Algorithm 2 finishes successfully we must ensure that the precondition of Algorithm 1 is satisfied. Let us order the elements of \( \mathcal{M} \) in a sequence \( (\varphi_0, \ldots, \varphi_N) \) and call \text{SINGLEREFINE}(\mathcal{H}, \varphi_i) \) following that order. Let \( \mathcal{H}_0 = \mathcal{H} \) before the first call, and \( \mathcal{H}_{i+1} \) the state of \( \mathcal{H} \) after the \( i \)-th call. Now we proceed by induction. Clearly, \( \mathcal{H}_0 \) is a hierarchical generator, \( \mathcal{L}_0 = \mathcal{L} \) and \( \{\varphi_0, \ldots, \varphi_N\} \subset \mathcal{H}_0 \). Assume now that \( \mathcal{H}_i \) is a hierarchical generator, \( \mathcal{L}_i = \mathcal{L} \cup \{\varphi_0, \ldots, \varphi_{i-1}\} \) and \( \{\varphi_i, \ldots, \varphi_N\} \subset \mathcal{H}_i \). Under these conditions Lemma 5.8 states that \( \mathcal{H}_{i+1} \) is a hierarchical generator, \( \mathcal{L}_{i+1} = \mathcal{L} \cup \{\varphi_i\} \) and \( \{\varphi_{i+1}, \ldots, \varphi_N\} \subset \mathcal{H}_{i+1} \).

Thus we have shown by induction that for any order of the function in \( \mathcal{M} \) the sequential execution of \text{SINGLEREFINE} will finish giving a hierarchical generator with lineage \( \mathcal{L}_* = \mathcal{L} \cup \mathcal{M} \). Now this lineage is the same independent of order given to the functions of \( \mathcal{M} \), so by Lemma 4.3 they all give the same and unique hierarchical generator \( \mathcal{H}_* \).

That is, the smallest refinement follows trivially from the fact that \( \mathcal{L} \cup \mathcal{M} \) is a lineage. \( \square \)

The most trivial process to construct a hierarchical generator is by a sequence of single refinements starting from \( \mathcal{B}^0 \). More precisely, let \( N \) be a natural number, \( \mathcal{H}_0 = \mathcal{B}^0 \), and for \( i \in [0 : N-1] \) let \( \varphi_i \in \mathcal{H}_i \) and \( \mathcal{H}_{i+1} \) the output of \text{SINGLEREFINE}(\mathcal{H}_i, \varphi_i). \( \mathcal{B}^0 \) is a hierarchical generator (with an empty lineage) repeated application of Lemma 5.8 implies that \( \mathcal{H}_N \) is a hierarchical generator with lineage \( \mathcal{L} = \{\varphi_0, \ldots, \varphi_{N-1}\} \). What is more interesting is that any hierarchical generator can be obtained in this way, thus justifying the more “abstract” definition of lineages given in Definition 4.4.

**Lemma 5.11 (Lineages and refinements).** A sequence of single refinements starting from \( \mathcal{B}^0 \) yields a hierarchical generator and reciprocally any hierarchical generator can be obtained by a sequence of single refinements.

**Proof.** The first statement of the lemma was shown in the previous paragraph above. For the second statement let \( \mathcal{H} \) be a hierarchical generator and \( \mathcal{L} \) its lineage. Define \( \mathcal{M}_i = \hat{\mathcal{L}}^i \) for \( i \in [0 : \text{depth}(\mathcal{L})-1] \). Let \( \mathcal{H}_0 = \mathcal{B}_0 \) and \( \mathcal{H}_{i+1} \) the output of \text{REFINE}(\mathcal{H}_i, \mathcal{M}_i). Clearly \( \mathcal{H}_0 \) is a hierarchical generator, \( \mathcal{L}_0 = \emptyset \), \( \mathcal{M}_0 \subset \mathcal{H}_0 \). Assume \( \mathcal{H}_n \) is a hierarchical generator, \( \mathcal{L}_n = \bigcup_{i \in [0 : n-1]} \mathcal{M}_i \) and \( \mathcal{M}_n \subset \mathcal{H}_n \). Using Lemma 5.11 it follows that \( \mathcal{H}_{n+1} \) is a hierarchical generator, \( \mathcal{L}_{n+1} = \bigcup_{i \in [0 : n]} \mathcal{M}_i \). Now \( \mathcal{M}_{n+1} = \hat{\mathcal{L}}^{n+1} \subset \mathcal{L}_{n+1} \subset \mathcal{L} \) and \( \mathcal{M}_{n+1} \subset \mathcal{C}_{n+1} \) as \( \mathcal{L}_{n+1} \) it follows that \( \mathcal{M}_{n+1} \subset \mathcal{H}_{n+1} \). Thus we have shown that \( \mathcal{H} \) is a hierarchical generator that was obtained by a sequence of single refinements with lineage \( \mathcal{L} = \bigcup_{i \in [0 : \text{depth}(\mathcal{L})-1]} \mathcal{M}_i = \hat{\mathcal{L}} \), so using Lemma 4.3 \( \mathcal{H} = \mathcal{H}_* \). \( \square \)

**5.5. Origin of new functions.** Given a refinement \( \mathcal{H}_* \) of \( \mathcal{H} \), one intuitively expects that any function in the refiner set \( \mathcal{R} = \mathcal{L}_* \setminus \mathcal{L} \) has been originated by a refined function in \( \mathcal{H} \), i.e., from \( \mathcal{R} \cap \mathcal{H} \). Similarly, if a function is in the refiner set \( \mathcal{R} \) it must have generated a new function in \( \mathcal{H}_* \). This relation is important to obtain complexity results relating the number of marked functions and the dimension of the hierarchical spaces in the context of an adaptive loop, and we elaborate on this below.

**Lemma 5.12 (New function cause).** Let \( \mathcal{H} \) be a hierarchical generator, let \( \mathcal{H}_* \vartriangleleft \mathcal{H} \) with refiner set \( \mathcal{R} \), and \( \mathcal{M} := \mathcal{R} \cap \mathcal{H} \) then

(i) if \( \psi_0 \in \mathcal{R} \) there exists \( k \geq 0 \) and a sequence \( (\psi_0, \ldots, \psi_k) \) with \( \psi_k \in \mathcal{M} \), \( \psi_j \in \mathcal{R} \setminus \mathcal{M} \) and \( \psi_j \in \text{ch} \psi_{j+1} \), for \( j \in [0 : k-1] \)

(ii) if \( \varphi \in \mathcal{H}_* \setminus \mathcal{H} \) then there is \( \psi \in \mathcal{R} \) such that \( \varphi \in \text{ch} \psi \)

(iii) if \( \varphi \in \mathcal{H}_* \setminus \mathcal{H} \) then there is \( k \geq 1 \) and \( \varphi \in \mathcal{M} \) such that \( \varphi \in \text{ch}^k \varphi \).
Proof. To show result (i) we proceed by induction on the level of \( \psi_0 \). If \( \ell_{\psi_0} = 0 \) Lemma 5.2(vi) implies that \( \psi_0 \in \mathcal{M} \) so the result follows with \( k = 0 \). Now assume that statement (i) holds for any function of level \( n \), and let \( \ell_{\psi_0} = n + 1 \). Again using Lemma 5.2(vi) there are two possibilities. Either \( \psi_0 \in \mathcal{M} \) and the result follows with \( k = 0 \) or \( \psi_0 \in \text{ch} \mathcal{R} \). In the latter case, there is \( \psi_1 \in \mathcal{R} \) with \( \psi_0 \in \text{ch} \psi_1 \), whence \( \ell_{\psi_1} = n \) and the inductive assumption yields the desired assertion.

Assertion (ii) is a direct consequence of Lemma 5.4(i), and (iii) follows from (i) and (ii).

6. Linear Independence. The hierarchical generators together with the Refine procedure of Algorithm 2 give a remarkably simple mechanism to obtain spaces with the required local resolution [cf. Requirement in Properties 1.1(iii)]. But, as we can see in the next example it may not give automatically the linear independence stated in Property 1.1(ii).

**Example 6.1 (Generator not linearly independent).** Consider \( m = 3 \) with \( d = 1 \), let \( \mathcal{L} = \{ \varphi_{d-2,0}^0, \varphi_0 \} \) so \( \mathcal{H} = \{ \varphi_{d-1,1}^0, \varphi_{d-2,1}^1, \varphi_1 \} \). Clearly, \( \varphi_{d-1}^0 \) can be spanned as linear combination of \( \{ \varphi_{d-2,1}^1, \varphi_{d-1,1}^0, \varphi_1 \} \), thus \( \mathcal{H} \) is not linearly independent.

In this section we deal with transformations that can be applied to a generator to ensure it is a basis.

**Definition 6.2 (Hierarchical basis).** We say that \( \mathcal{H} \) is a hierarchical basis if it is a linearly independent hierarchical generator.

From Lemma 1.12 we immediately obtain the following.

**Lemma 6.3 (Unique positive partition of unity).** Let \( \mathcal{H} \) be a hierarchical basis, then for each \( \varphi \in \mathcal{H} \), there exists a unique constant \( c_\varphi > 0 \) such that \( \sum_{\varphi \in \mathcal{H}} c_\varphi \varphi = 1 \).

One interesting property of a hierarchical basis which does not hold for arbitrary hierarchical generators is that every function in \( \mathcal{H} \) that is refined has a descendant in \( \mathcal{H}_s \setminus \mathcal{H} \).

**Lemma 6.4 (Refined function effect).** Let \( \mathcal{H} \) be a hierarchical basis, let \( \mathcal{H}_s \supset \mathcal{H} \) with refiner set \( \mathcal{R} = \mathcal{L}_s \setminus \mathcal{L} \). If \( \varphi \in \mathcal{M} = \mathcal{R} \cap \mathcal{H} \), then \( \text{dsc} \varphi \cap (\mathcal{H}_s \setminus \mathcal{H}) \neq \emptyset \), i.e., there is \( k > 0 \) such that there exists \( \varphi_\ast \in (\mathcal{H}_s \setminus \mathcal{H}) \cap \text{ch}^k(\varphi) \).

**Proof.** We prove the result by contradiction. Let \( \varphi \in \mathcal{M} \) and suppose that \( \text{dsc} \varphi \cap (\mathcal{H}_s \setminus \mathcal{H}) = \emptyset \), then \( \text{dsc} \varphi \cap \mathcal{H}_s \subset \text{dsc} \varphi \cap \mathcal{H} \). Besides, from Lemma 1.7(i) as \( \varphi \in \mathcal{L}_s \), we have that \( \varphi \in \text{span}(\text{dsc} \varphi \cap \mathcal{H}_s) \subset \text{span}(\text{dsc} \varphi \cap \mathcal{H}) \subset \text{span}(\mathcal{H} \setminus \{ \varphi \}) \). This implies that \( \mathcal{H} \) is linearly dependent which contradicts the assumption.

6.1. Linearly independent refinement. We want to work with hierarchical spaces, in particular with those appearing in an adaptive process where some functions are selected and refined to add local resolution. It turns out that the refinement procedures defined thus far produce hierarchical generators that may not be linearly independent. Linear independence is desirable in order to fulfill Property 1.1(ii) to avoid redundancy and ill-posedness of the resulting (non-)linear systems. Removing redundant functions may be demanding task and may lead to generators that are not hierarchical. One interesting approach is to consider linearly independent refinements of hierarchical generators while investigating the following questions.

1. Given a hierarchical generator, which is the smallest linearly independent refinement? Does it exist?
2. If it exists, can we characterize it in terms of a property of the lineage?
3. Does it span the same space or a larger one? How much larger?
4. Can we provide a simple constructive procedure to find it?

The first question can be mathematically written as follows: Given a generator
\( \mathcal{H} \), find the smallest element of the family
\[ \mathcal{B}(\mathcal{H}) = \{ \mathcal{H}_* : \mathcal{H}_* \triangleright \mathcal{H} \text{ and } \mathcal{H}_* \text{ is linearly independent} \}. \tag{6.1} \]

It is still an open question whether in general the minimal set of \( \mathcal{B}(\mathcal{H}) \) is a singleton, empty or larger. This matter, which is intimately related with the characterization of linear independence in terms of the lineage properties, is part of an ongoing work.

An important point is that the condition imposed in (6.1) can be replaced by one stronger than just linear independence. The mathematical framework in which to develop a successful theory in the light of Properties 1.1 and 1.2 can be summarized as follows.

(i) State a condition \( \mathfrak{A} \) for the hierarchical generators that implies linear independence
(ii) Consider the family \( \mathfrak{A}(\mathcal{H}) = \{ \mathcal{H}_* : \mathcal{H}_* \triangleright \mathcal{H} \text{ and } \mathcal{H}_* \text{ satisfies condition } \mathfrak{A} \} \).
(iii) Show that the smallest element of \( \mathfrak{A}(\mathcal{H}) \) exists.
(iv) Show that the cardinality of this element is not much larger than \( \dim(\text{span } \mathcal{H}) \).
(v) Provide a simple method to construct this element.

We remark that (i) only asks for a sufficient condition for linear independence, thus the smallest refinement of (iii) may yield a basis bigger than the dimension of \( \mathcal{H} \). Thus (iv) is an important restriction on the condition \( \mathfrak{A} \).

6.2. A sufficient condition. Absorbing Generator. A sufficient condition for linear independence of a generator can be obtained following the intuition that if a function in \( \mathcal{H} \) is totally overlapped by finer functions in \( \mathcal{H} \), then that function is very likely redundant. This idea with a different language can be ascribed to the work of [14]. We now explore this concept in the framework described in Section 6.1, presenting a sufficient condition for a hierarchical generator to be linearly independent.

Definition 6.5 (Absorbing Generator). A hierarchical generator \( \mathcal{H} \) is called absorbing if for any \( \varphi \in C \) such that \( \mathbb{I}(\varphi) \subset \mathbb{I}(\mathcal{L}) \) it holds that \( \varphi \in \mathcal{L} \). In other words, \( \mathcal{H} \) is absorbing if there is no \( \varphi \in \mathcal{H} \) such that \( \mathbb{I}(\varphi) \subset \mathbb{I}(\mathcal{L}) \).

For an absorbing generator we have that each B-spline in \( \mathcal{H} \) overlaps an active cell of its own level. We state this more precisely as follows.

Lemma 6.6 (Overlap of active cells). If \( \mathcal{H} \) is absorbing then \( \mathcal{H} \subset \mathcal{B}(\mathcal{A}) \), where \( \mathcal{A} \) denotes the set of active cells corresponding to \( \mathcal{H} \), according to Definition 4.10.

Proof. Let \( \mathcal{H} \) be an absorbing generator, and let \( \varphi \in \mathcal{H} \). We want to prove that \( \varphi \in \mathcal{B}(\mathcal{A}) \), which is equivalent to \( \mathbb{I}(\varphi) \cap \mathcal{A} \neq \emptyset \). Assume, on the contrary, that \( \mathbb{I}(\varphi) \cap \mathcal{A} = \emptyset \). Since by definition \( \mathcal{A} = \mathbb{I}(\mathcal{C}) \setminus \mathbb{I}(\mathcal{L}) \), this implies that \( \mathbb{I}(\varphi) \subset \mathbb{I}(\mathcal{L}) \), which due to the fact that \( \mathcal{H} \) is absorbing, implies that \( \varphi \in \mathcal{L} \), which contradicts the assumption that \( \varphi \in \mathcal{H} \). The assertion thus follows.

Lemma 6.7 (Linear independence). Every absorbing hierarchical generator is linearly independent, and thus a hierarchical basis.

Proof. Let \( \mathcal{H} \) be a hierarchical generator and assume that \( \sum_{\varphi \in \mathcal{H}} \alpha_{\varphi} \varphi = 0 \). Then this function vanishes in every active cell, i.e.,
\[ \sum_{\varphi \in \mathcal{H}} \alpha_{\varphi} \varphi = 0 \quad \text{in } I, \quad \text{for every } I \in \mathcal{A} = \bigcup_{\ell = 0}^{\text{depth}(\mathcal{H})} \mathcal{A}^\ell. \]

Then, we have \( 0 = \sum_{\varphi \in \mathcal{H}^0} \alpha_{\varphi} \varphi \) in each \( I \in \mathcal{A}^0 \). Due to Lemma 4.11 all functions in \( \bigcup_{\ell = 1}^{\text{depth}(\mathcal{H})} \mathcal{H}^\ell \) vanish in all \( I \in \mathcal{A}^0 \), and since \( \mathcal{B}^0 \) are locally linear independent, it follows that \( \alpha_{\varphi} = 0 \) for all \( \varphi \in \mathcal{H}^0 \cap \mathcal{B}(\mathcal{A}^0) \). From Lemma 6.6 \( \mathcal{H}^0 \cap \mathcal{B}(\mathcal{A}^0) = \mathcal{H}^0 \).
and \( \alpha_\phi = 0 \) for each \( \phi \in \mathcal{H}_0 \). Arguing by induction, we conclude that \( \alpha_\phi = 0 \) for all \( \phi \in \mathcal{H} \), and the assertion follows. \( \Box \)

**Definition 6.8 (Absorbing basis).** An absorbing hierarchical generator is called an absorbing hierarchical basis, or merely an absorbing basis.

**Remark 6.9 (Absorbing for linear independence).** It is worth noticing that the absorbing condition is a sufficient condition for linear independence, but not necessary. In fact, consider \( m = 2 \) with \( d = 1 \), let \( \mathcal{L} = \{ \phi_{-1}^0, \phi_0^0, \phi_{-1}^1, \phi_1^1 \} \) so \( \mathcal{H} = \{ \phi_0^1, \phi_{-2}^1, \phi_{-2}^0, \phi_2^0, \phi_3^0 \} \). This \( \mathcal{H} \) is linearly independent, but not absorbing.

**6.3. The absorbing refinement.** Given a hierarchical generator \( \mathcal{H} \), let us consider the family
\[
\mathfrak{A}(\mathcal{H}) = \{ \mathcal{H}_* : \mathcal{H}_* \succ \mathcal{H}, \mathcal{H}_* \text{ is absorbing} \}.
\] (6.2)

Algorithm \ref{absorbing-refinement} constructs the least element of this family.

**Algorithm 3 Absorbing Refinement Algorithm**

```plaintext
1: function AbsRefine(\mathcal{H})
2:     \hat{\mathcal{C}}^0 = \mathcal{B}^0, \hat{\mathcal{L}} = 0 \quad \triangleright At this point \hat{\mathcal{H}} = \mathcal{B}^0
3:     for \( \ell = 0 \) to depth(\mathcal{H}) - 1 do
4:         \mathcal{M} = \{ \phi \in \hat{\mathcal{C}}^\ell \setminus \mathcal{L}^0 : \mathbb{I}(\phi) \subset \mathbb{I}(\mathcal{L}^\ell) \}
5:     \text{Refine}(\hat{\mathcal{H}}, \mathcal{M} \cup \mathcal{L}^\ell) \quad \triangleright Now \hat{\mathcal{L}}^\ell = \mathcal{L}^\ell \cup \mathcal{M} \text{ and } \hat{\mathcal{L}}^{\ell+1} = \text{ch} \hat{\mathcal{L}}^\ell
6:     end for
7:     return \hat{\mathcal{H}}
8: end function
```

The properties of this algorithm are summarized in the following Lemma.

**Lemma 6.10 (Properties of AbsRefine).** Let \( \mathcal{H} \) be a hierarchical generator, a call to AbsRefine(\( \mathcal{H} \)) returns in \( \hat{\mathcal{H}} \) the least element of the family \( \mathfrak{A}(\mathcal{H}) \) from (6.2), i.e., the smallest absorbing basis which is a refinement of \( \mathcal{H} \). Furthermore, we have that

(i) If \( \psi \in \mathcal{R} := \hat{\mathcal{C}} \setminus \mathcal{L} \), then \( \mathbb{I}(\psi) \subset \mathbb{I}(\mathcal{L}^{\hat{\ell}}) \).

(ii) depth \( \mathcal{H} = \text{depth} \hat{\mathcal{H}} \).

(iii) If \( \psi \in \mathcal{R} \) there is \( k > 0 \) such that \( \mathcal{O}(\psi, k, \mathcal{H}) \neq \emptyset \).

Proof. At the start of the loop \( \hat{\mathcal{H}} = \mathcal{B}^0 \) is a hierarchical generator, inside the loop depth(\( \mathcal{H} \)) valid iterated calls to Refine are made. Thus Lemma \ref{absorbing-refinement} implies that \( \mathcal{H} \) is a refinement of \( \hat{\mathcal{H}} \) and
\[
\mathcal{L}^\ell = \{ \phi \in \hat{\mathcal{C}}^\ell : \mathbb{I}(\phi) \subset \mathbb{I}(\mathcal{L}^\ell) \},
\] (6.3)
so that \( \mathbb{I}(\hat{\mathcal{L}}^\ell) = \mathbb{I}(\mathcal{L}^\ell) \). Therefore, if \( \phi \in \hat{\mathcal{L}}^\ell \) and \( \mathbb{I}(\phi) \subset \mathbb{I}(\hat{\mathcal{L}}^\ell) \) then \( \phi \in \hat{\mathcal{L}}^\ell \) and \( \hat{\mathcal{H}} \) is thus absorbing. Hence \( \hat{\mathcal{H}} \) is an absorbing refinement of \( \mathcal{H} \).

Now we show that it is in fact the smallest of such refinements. To see this take \( \mathcal{H}_* \), another absorbing refinement of \( \mathcal{H} \). If \( \phi \in \hat{\mathcal{L}}^0 \) then \( \phi \in \hat{\mathcal{C}}^0 = \mathcal{B}^0 = \mathcal{C}_0^0 \) and from (6.3) \( \phi \in \mathcal{L}^0 \). Then as \( \mathcal{H}_* \) is an absorbing refinement it follows that \( \phi \in \mathcal{L}_n^0 \), thus \( \mathcal{L}_n^0 \subset \mathcal{L}_n^0 \). We now proceed by induction. Suppose we have shown that \( \hat{\mathcal{L}}^n \subset \mathcal{L}_n^0 \), then \( \hat{\mathcal{C}}^{n+1} \subset \mathcal{L}_n^{n+1} \) and take \( \phi \in \mathcal{L}_n^{n+1} \) then \( \phi \in \mathcal{C}_n^{n+1} \) but also \( \mathbb{I}(\phi) \subset \mathbb{I}(\mathcal{L}_n^{n+1}) \subset \mathbb{I}(\mathcal{L}_n^{n+1}) \). Then, as \( \mathcal{H}_* \) is absorbing it follows that \( \phi \in \mathcal{L}_n^{n+1} \), whence \( \mathcal{L}_n^{n+1} \subset \mathcal{L}_n^{n+1} \). Summarizing, we have shown that \( \mathcal{H}_* \succ \hat{\mathcal{H}} \), so \( \hat{\mathcal{H}} \) is in fact the smallest set in \( \mathfrak{A}(\mathcal{H}) \) from (6.2). Assertions \ref{absorbing-refinement} and \ref{absorbing-refinement} immediately follow. From part \ref{absorbing-refinement}, \( \mathbb{I}(\psi) \subset \mathbb{I}(\mathcal{L}^{\hat{\ell}}) \), thus for any \( I \in \mathbb{I}(\psi) \) there is \( \eta \in \mathcal{L} \) such that \( I \in \mathbb{I}(\eta) \). Using Lemma \ref{absorbing-refinement} \ref{absorbing-refinement} \( I \in \cup_{k>0} \text{ch}^k(\text{ch}^k \eta \cap \mathcal{H}) \), thus there
must exist \( k > 0 \) and \( \phi \in \text{ch}^k \eta \cap \mathcal{H} \) such that \( I \in \mathbb{I}^{-k}(\phi) \), thus \( \varphi \in \mathcal{O}(\psi, k, \mathcal{H}) \) and part (iii) follows. \( \Box \)

**Remark 6.11** (Comparison with other hierarchical basis). Our concept of absorbing hierarchical basis coincides with the concept of hierarchical basis from [2]. In fact, given an absorbing hierarchical basis \( \mathcal{H} \) with corresponding lineage \( \mathcal{L} \), after defining \( \omega_l = \cup_{\psi \in \mathcal{L}} \text{supp} \varphi \), it is straightforward to check that the definition from [2] leads to the same space.

The next result will be important when studying the gap of a hierarchical generator which is the subject of the following section.

**Lemma 6.12** (New function). Let \( \mathcal{H} \) be a hierarchical generator, and \( \mathcal{H} \) the result of a call to AbsRefine(\( \mathcal{H} \)). If \( \varphi \in \mathcal{H} \backslash \mathcal{H} \) and \( k > 0 \), then for each \( \eta \in \mathcal{O}(\varphi, -k, \mathcal{B}) \) there exists \( \zeta \in \mathcal{H} \) and \( k' \geq k \) such that \( \eta \in \mathcal{O}(\zeta, -k', \mathcal{B}) \).

**Proof.** From Lemma 5.11, if \( \varphi \in \mathcal{H} \backslash \mathcal{H} \), there exists \( \psi \in \mathcal{R} \) such that \( \varphi \in \text{ch} \psi \), so that Lemma 6.11 yields \( \mathcal{O}(\varphi, -k, \mathcal{B}) \subset \mathcal{O}(\psi, -k+1, \mathcal{B}) \). Therefore, if \( \eta \in \mathcal{O}(\varphi, -k, \mathcal{B}) \), we have \( \eta \in \mathcal{O}(\psi, -k+1, \mathcal{B}) \). Consequently, there exists \( \psi \in \mathcal{L}^{\psi} \) with \( I \in \mathcal{I}(\psi) \cap \mathbb{I}^{k-1}(\eta) \). By Lemma 4.7(i), \( \psi \in \text{span}(\text{dsc}(\hat{\psi}) \cap \mathcal{H}) \), whence there exists \( \zeta \in \text{dsc}(\hat{\psi}) \cap \mathcal{H} \), and \( j \geq 1 \) such that \( \mathbb{I}(\zeta) \cap \text{ch} I \neq \emptyset \), hence, \( \mathbb{I}(\zeta) \cap \mathbb{I}^{k-1+j}(\eta) \neq \emptyset \), which implies that \( \eta \in \mathcal{O}(\zeta, -k', \mathcal{B}) \) with \( k' = k - 1 + j \geq k \), due to (3.3). \( \Box \)

7. Overlapping and gap constraint. We now deal with Property [2][iii], i.e., we address the issue of controlling the level difference of overlapping functions in a given generator.

To measure the function overlapping in a generator we assign a number, called the gap, to each function in the generator or basis. This number, associated to each function in the generator, measures the level difference with the coarsest overlapping function.

**Definition 7.1** (Gap of a function). Let \( \mathcal{H} \) be a hierarchical generator and \( \varphi \in \mathcal{H} \), then we define the gap of \( \varphi \) in \( \mathcal{H} \) as \( \text{gap}_{\mathcal{H}} \varphi := \text{sup}\{g \in \mathbb{Z} : \mathcal{O}(\varphi, -g, \mathcal{H}) \neq \emptyset\} \).

Computing the gap of a function can be rather expensive, but as we will see below, it will never be necessary to perform such a computation; see Remark 8.4.

**Lemma 7.2** (Properties of the gap). Let \( \mathcal{H} \) be a hierarchical generator and \( \varphi \in \mathcal{H} \) then the following properties are satisfied

(i) \( \text{gap}_{\mathcal{H}} \varphi \in \mathbb{Z}_0^+ \).

(ii) \( \mathcal{O}(\varphi, -\text{gap}_{\mathcal{H}} \varphi, \mathcal{H}) \neq \emptyset \).

(iii) \( \mathcal{O}(\varphi, -g, \mathcal{H}) = \emptyset \) for \( g > \text{gap}_{\mathcal{H}} \varphi \).

(iv) If \( \mathcal{O}(\varphi, -g, \mathcal{H}) \neq \emptyset \) then \( g \leq \text{gap}_{\mathcal{H}} \varphi \).

**Proof.** As \( \varphi \in \mathcal{O}(\varphi, 0, \mathcal{H}) \) the set on which the supremum is taken is not empty and \( \mathcal{O}(\varphi, -g, \mathcal{H}) \) is empty for \( g > \ell_{\varphi} \) so the set is bounded above by \( \ell_{\varphi} \). Then every \( \varphi \) has a non negative gap assigned. The last two items follow directly from the definition of supremum. \( \Box \)

The next result states that the gap of a \( k \)-th descendant of a function \( \varphi \) is bounded by the gap of \( \varphi \) plus \( k \).

**Lemma 7.3** (Gap of a descendant). Let \( \mathcal{H} \) be a hierarchical generator, \( \varphi \in \mathcal{H} \) and \( k \geq 0 \), then for \( \psi \in \text{ch}^k \varphi \cap \mathcal{H} \) we have that \( k \leq \text{gap}_{\mathcal{H}} \psi \leq \text{gap}_{\mathcal{H}} \varphi + k \).

**Proof.** From Lemma 3.3, for \( \psi \in \text{ch}^k \varphi \cap \mathcal{H} \) we get that \( \{j : \mathcal{O}(\psi, -j, \mathcal{H}) \neq \emptyset\} \subset \{j : \mathcal{O}(\varphi, -j + k, \mathcal{H}) \neq \emptyset\} \). Observe that the gap of a \( k \)-th descendant of \( \varphi \) can actually take any value between \( k \) and \( \text{gap}_{\mathcal{H}} \varphi + k \).
Refinement and gap. A refinement process may change the gap of the functions in a generator. The following result states that if a function stays in the generator after refinement, its gap does not increase, and in fact it can actually decrease.

**Lemma 7.4 (Refinement and gap).** Let $\mathcal{H}$ be a hierarchical generator and $\mathcal{H}_* \supset \mathcal{H}$, then:

(i) If $\varphi \in \mathcal{H}_*$ there exists $k \geq \text{gap}_{\mathcal{H}_*} \varphi$, such that $O(\varphi, -k, \mathcal{H}) \neq \emptyset$.

(ii) If $\varphi \in \mathcal{H}_* \cap \mathcal{H}$, then $\text{gap}_{\mathcal{H}_*} \varphi \leq \text{gap}_{\mathcal{H}} \varphi$.

**Proof.** Let $\varphi \in \mathcal{H}_*$, due to Lemma 7.2(ii) there exists $\eta \in O(\varphi, -\text{gap}_{\mathcal{H}} \varphi, \mathcal{H}_*)$, i.e., $\eta \in \mathcal{H}_*$ and $\mathcal{I}(\varphi) \cap \mathcal{I}(\eta) \neq \emptyset$. If $\eta \in \mathcal{H}$ then $\eta \in O(\varphi, -\text{gap}_{\mathcal{H}_*} \varphi, \mathcal{H})$ and $O(\varphi, -k, \mathcal{H}) \neq \emptyset$ for $k = \text{gap}_{\mathcal{H}} \varphi$. If $\eta \in \mathcal{H}_* \setminus \mathcal{H}$ then from Lemma 7.2(ii) there exists an ancestor $\xi \in \mathcal{M} \subset \mathcal{H}$ and $j \geq 1$, such that $\eta \in \text{ch}^j \xi$ so that $\mathcal{I}(\eta) \subset \mathcal{V}^j(\xi)$. Therefore, $\mathcal{I}(\varphi) \cap \mathcal{I}(\eta) \neq \emptyset$ and thus $\xi \in O(\varphi, \text{gap}_{\mathcal{H}_*} \varphi + j, \mathcal{H})$ and the assertions follow.

We now define the gap of a generator.

**Definition 7.5 (Gap of a hierarchical generator).** Given a hierarchical generator $\mathcal{H}$ we define its gap as $\text{gap} \mathcal{H} = \max\{\text{gap}_{\mathcal{H}} \varphi : \varphi \in \mathcal{H}\}$.

In the next result we prove that the process of making a hierarchical generator absorbing, through taking its smallest absorbing refinement (with Algorithm AbsRefine) does not increase its gap.

**Proposition 7.6 (AbsRefine does not increase the gap).** Let $\mathcal{H}$ be a hierarchical generator and let $\mathcal{H}$ be the result of a call to AbsRefine($\mathcal{H}$) from Algorithm 3. Then gap $\mathcal{H} \leq$ gap $\mathcal{H}$.

**Proof.** From Lemma 7.2(ii) if $\varphi \in \mathcal{H} \cap \mathcal{H}$ then gap $\mathcal{H} \varphi \leq \text{gap}_{\mathcal{H}} \varphi \leq \text{gap} \mathcal{H} \varphi$. If $\varphi \in \mathcal{H} \setminus \mathcal{H}$ using Lemma 7.2(ii) there is $k \geq \text{gap} \mathcal{H} \varphi$ and $\eta \in O(\varphi, -k, \mathcal{H})$. Now using Lemma 6.12 there is $\zeta \in \mathcal{H}$ and $k' \geq k$ such that $\eta \in O(\zeta, -k', \mathcal{H})$, whence $\eta \in O(\zeta, -k', \mathcal{H})$. From Definition 7.5 and Lemma 7.2(iv) we get gap $\mathcal{H} \geq \text{gap} \mathcal{H} \eta \geq k' \geq k \geq \text{gap} \mathcal{H} \varphi$. We have shown that for any $\varphi \in \mathcal{H}$, gap $\mathcal{H} \varphi \leq$ gap $\mathcal{H} \mathcal{H}$ and the result follows.

**Remark 7.7 (AbsRefine could decrease the gap).** The gap $\mathcal{H}$ is not necessarily equal to the gap of $\mathcal{H}$, the gap can actually decrease after calling AbsRefine. The hierarchical generator from Remark 6.4 has gap equal to 1, and after AbsRefine we obtain the generator $\mathcal{H} = \{\varphi_2^2, \varphi_0, \varphi_1, \varphi_2^2\}$ which has gap 0.

8. Function refinement with gap constraint. Following the requirement in Property 1.2(iii) the refinement of a hierarchical generator $\mathcal{H}$ should maintain the gap bounded above by a fixed given positive integer $g$. We explore on the effect of refinement on the gap of a generator in the following Lemma.

**Lemma 8.1 (Single refinement and gap).** Let $\mathcal{H}$ be a hierarchical generator, $\varphi \in \mathcal{H}$ and $\mathcal{H}_*$ its refinement after a call to SingleRefine($\mathcal{H}, \varphi$) of Algorithm 7. Then, if $\varphi_* \in \mathcal{H}_*$,

$$\text{gap}_{\mathcal{H}_*} \varphi_* \leq \begin{cases} \text{gap}_{\mathcal{H}} \varphi + 1, & \text{if } \varphi_* \in \mathcal{H}_* \setminus \mathcal{H}, \\ \text{gap}_{\mathcal{H}} \varphi, & \text{if } \varphi_* \in \mathcal{H}_* \cap \mathcal{H}. \end{cases}$$

**Proof.** Recall from Lemma 5.6 that $\mathcal{H}_* = (\mathcal{H} \setminus \{\varphi\}) \cup (\text{ch} \varphi \setminus \mathcal{C})$, from where $\mathcal{H}_* \setminus \mathcal{H} = \text{ch} \varphi \setminus \mathcal{C}$ and $\mathcal{H}_* \cap \mathcal{H} = (\mathcal{H} \setminus \{\varphi\})$.

If $\varphi_* \in \mathcal{H}_* \setminus \mathcal{H} = \text{ch} \varphi \setminus \mathcal{C}$, then $O(\varphi, -k, \mathcal{H}_*) \subset O(\varphi, -k, \mathcal{H}) \setminus \{\varphi\}$ for all $k > 0$. Also, by Lemma 6.13 $O(\varphi, -k, \mathcal{H}) \subset O(\varphi, -k + 1, \mathcal{H})$, then Lemma 7.2(iii) implies that $O(\varphi, -k, \mathcal{H}) = \emptyset$ for any $k > \text{gap}_{\mathcal{H}} \varphi + 1$, so that $\text{gap}_{\mathcal{H}_*} \varphi_* \leq \text{gap}_{\mathcal{H}} \varphi + 1$. The case of $\varphi_* \in \mathcal{H}_* \cap \mathcal{H}$ follows from Lemma 7.2(iii).
It is simple to construct an example of a generator with gap \( g \) and a refinement that increases its gap. Indeed, this can happen by refining a single function. The possibility of \textsc{SingleRefine} to increase the gap of a generator makes it necessary to find a new mechanism to obtain the smallest refinement that ensures the bound on the gap. Thus, given a hierarchical generator \( \mathcal{H} \) and \( \varphi \in \mathcal{H} \), consider the set

\[
\mathcal{R}_g(\mathcal{H}, \varphi) = \{ \mathcal{H}, \mathcal{H} \succ \mathcal{H}, \varphi \in \mathcal{L} \text{ and } \text{gap} \mathcal{H} \leq g \},
\]

(8.1)

where \( g \) is a positive integer that we consider fixed from now on. We would like to find the smallest element of \( \mathcal{R}_g(\mathcal{H}, \varphi) \). In order to do it, we first observe the following.

**Remark 8.2.** Let \( \mathcal{H} \) be a hierarchical generator with gap(\( \mathcal{H} \)) \( \leq g \), let \( \varphi \in \mathcal{H} \) and let \( \mathcal{H}_s \) be the refinement obtained after a call to \textsc{SingleRefine}(\( \mathcal{H}, \varphi \)). Then, as a consequence of Lemmas 8.1 and 8.3, we have:

(i) If \( \mathcal{O}(\varphi, -g, \mathcal{H}) = \emptyset \), then gap(\( \mathcal{H} \)) \( \leq g - 1 \) and thus gap(\( \mathcal{H}_s \)) \( \leq g \);

(ii) If \( \mathcal{O}(\varphi, -g, \mathcal{H}) \neq \emptyset \), then gap(\( \mathcal{H} \)) \( = g \) and thus gap(\( \mathcal{H}_s \)) \( = g + 1 \).

Taking this observation into account, we now propose Algorithm 4, which finds the least element of \( \mathcal{R}_g(\mathcal{H}, \varphi) \), as shown in Lemma 8.3.

**Algorithm 4 Refine one function with gap control**

1: function \textsc{GCSingleRefine}(\( \mathcal{H}, \varphi \))
2: while \( \exists \varphi' \in \mathcal{O}(\varphi, -g, \mathcal{H}) \) do
3: \quad \textsc{GCSingleRefine}(\( \mathcal{H}, \varphi' \))
4: end while
5: \textsc{SingleRefine}(\( \mathcal{H}, \varphi \))
6: end function

**Lemma 8.3 (Properties of \textsc{GCSingleRefine}).** Let \( \mathcal{H} \) be a hierarchical generator with gap(\( \mathcal{H} \)) \( \leq g \) and \( \varphi \in \mathcal{H} \). A call to \textsc{GCSingleRefine}(\( \mathcal{H}, \varphi \)) modifies the set \( \mathcal{H} \), yielding a hierarchical generator \( \mathcal{H} \succ \mathcal{H} \), which is the smallest element of the family \( \mathcal{R}_g(\mathcal{H}, \varphi) \), i.e., \( \mathcal{H} \) is the smallest refinement of \( \mathcal{L} \cup \{ \varphi \} \) with gap bounded by \( g \). Furthermore, for each \( \psi \in \mathcal{R} \setminus \{ \varphi \} \) there exists \( k \in \mathbb{Z} \) with \( 1 \leq k \leq \lfloor \frac{g}{2} \rfloor \) such that \( \psi \in \mathcal{O}^k(\varphi, -g, \mathcal{B}) \).

**Remark 8.4.** It is worth noticing that in order to keep the gap bounded by \( g \) it is never necessary to compute the gap of a hierarchical generator, which would be rather costly. More precisely, if we start with a hierarchical generator with gap bounded by \( g \), such as \( \mathcal{H} = \mathcal{B}^0 \), every hierarchical generator obtained via repeated subsequent calls to \textsc{GCSingleRefine} will have its gap bounded \( g \) automatically. The only thing that must be produced are the sets \( \mathcal{O}_\varphi = \mathcal{O}(\varphi, -g, \mathcal{H}) \), which are mere intersections of index sets (see Remark 8.3).

**Proof.** Notice first that if \( \varphi \in \mathcal{B}_\ell \), then \( \mathcal{O}(\varphi, -g, \mathcal{H}) \) is a subset of \( \mathcal{B}_{\ell - g} \), so that all the calls to \textsc{GCSingleRefine} involved in the recursion will be made to \( \mathcal{B} \)-splines from \( \cup_{k=1}^{\ell/g} \mathcal{B}_{\ell - kg} \subset \cup_{j < \ell} \mathcal{B} \) which is finite. Hence, the algorithm will end in finite time. Moreover, since all the calls to \textsc{GCSingleRefine} and \textsc{SingleRefine} will be made with functions from \( \mathcal{B}^j \) with \( j < \ell \), after the execution of the while loop, \( \varphi \) will still belong to \( \mathcal{H} \) and \textsc{SingleRefine}(\( \mathcal{H}, \varphi \)) is a valid call. Since a call to \textsc{SingleRefine}(\( \mathcal{H}, \varphi \)) will add to \( \mathcal{H} \) functions of level \( \ell_\varphi + 1 \) we immediately obtain the last assertion of the Lemma.

We now prove that all the time gap(\( \mathcal{H} \)) \( \leq g \). Recall that the first call to \textsc{GCSingleRefine} is done with gap(\( \mathcal{H} \)) \( \leq g \) and notice that \( \mathcal{H} \) is only modified
through the execution of line 5 (SingleRefine), from the many calls to the recursive function GCSingleRefine. The assertion will be proved if we show that executing line 5 with gap(H) ≤ g leads to a new hierarchical generator with gap less than or equal to g. Notice that line 5 is reached after the while loop has ended, so that \( O(\varphi, -g, H) = \emptyset \), and thus gap_{H}(\varphi) ≤ g − 1. From Lemma 8.1 the hierarchical generator obtained after executing line 5 has gap bounded by g.

Notice also that if \( O(\varphi, -g, H) \neq \emptyset \), a call to SingleRefine(H,\varphi) would lead to gap(H) > g according to Remark 8.2. It is thus necessary to refine all functions in \( O(\varphi, -g, H) \) to maintain gap(H) ≤ g after executing line 5. This shows that any hierarchical generator in \( R_{g}(H, \varphi) \) must be larger than the one obtained by this algorithm.

**Remark 8.5.** It is not difficult to prove that Algorithm 4 is equivalent to Algorithm 5. The main difference being that in the latter the set \( O_{\varphi} \) is defined before entering the recursive loop. The equivalence relies on the fact that the set \( O(\varphi, -g, H) \) of Algorithm 4 does not increase during the execution of the while loop. This is easy to see if \( g \geq 2 \) because in this case, each call to GCSingleRefine inside the loop would incorporate into H functions of level \( \ell_{\varphi} - k + 1 \), for \( k \in \mathbb{Z}_{+} \), which are never from the same level as those in \( O(\varphi, -g, H) \). The case \( g = 1 \) is also true, but the proof is rather technical.

Observing Algorithm 5 it is easy to conclude that if \( R = H_{s} \setminus H \), with \( H_{s} \) the result of GCSingleRefine(H,\varphi) then for each \( \psi \in R \setminus \{\varphi\} \) there exists \( k \in \mathbb{Z} \) with \( 1 \leq k \leq \lfloor \frac{\ell_{\varphi}}{g} \rfloor \) such that \( \psi \in O^{k}(\varphi, -g, R \cap H) \).

**Algorithm 5** Refine one function with gap control (version 2)

1: function GCSingleRefine(H,\varphi)
2: \( O_{\varphi} = O(\varphi, -g, H) \)
3: for \( \varphi' \in O_{\varphi} \) do
4: GCSingleRefine(H,\varphi')
5: end for
6: SingleRefine(H,\varphi)
7: end function

**Algorithm 6** Refine \( M \) with gap control

1: function GCRefine(H,M)
2: while \( \exists \varphi \in M \cap H \) do
3: GCSingleRefine(H,\varphi)
4: end while
5: end function

**Lemma 8.6 (Properties of GCRefine).** Let \( H \) be a hierarchical generator with gap(H) ≤ g and let \( M \subset H \) be a given set of functions to be refined, Algorithm 6 will refine the functions maintaining the gap under control (≤ g).

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(iv) for each $\psi \in \mathcal{R} \setminus \mathcal{M}$ there is $\varphi \in \mathcal{M}$ and $k > 0$ such that $\psi \in \mathcal{O}^k(\varphi, -g, \mathcal{B})$

(v) if $\eta \in \mathcal{H} \setminus \mathcal{H}$ then there exists $\varphi \in \mathcal{M}$ such that either $\eta \in \text{ch}\varphi$ or $\eta \in \text{ch}\mathcal{O}^k(\varphi, -g, \mathcal{B})$ for some $k > 0$.

Proof. Assertions (i) and (ii) are immediate consequences of Lemma 8.3 and assertion (iii) follows from Remark 8.3. assertion (iv)–(v) are son consecuencias directas de (i)–(iii). Observe that if $\mathcal{M} \neq \emptyset$ the call to GC SINGLE REFINES inside the while loop in Algorithm 6 is executed at least one time and at most $\#\mathcal{M}$ times. Let $\varphi_1, \ldots, \varphi_M$ be the functions in $\mathcal{M}$ that were passed to GC SINGLE REFINES in sequential order inside the loop. Let $\mathcal{H}_0 = \mathcal{H}$ and $\mathcal{H}_1, \ldots, \mathcal{H}_M$ be the hierarchical generators obtained after each iteration of the while loop, then $\mathcal{H}_M = \mathcal{H}$ and $\mathcal{M} \leq \#\mathcal{M}$. From Lemma 8.3 it follows that each $\mathcal{H}_j$ is a hierarchical generator and gap $\mathcal{H}_j \leq g$, so the same holds for $\mathcal{H}$. Moreover, $\mathcal{R} = \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_{M-1}$, with $\mathcal{R}_j = \mathcal{L}_j \setminus \mathcal{L}_{j-1}$. Then, if $\psi \in \mathcal{R}$ it must belong to one of the $\mathcal{R}_j$ and again Lemma 8.3 implies (iii), which immediately implies (iv) and (v). □

Algorithm 7 Refine $\mathcal{M}$ with gap and absorbing constraints

```plaintext
function GARefine($\mathcal{H}, \mathcal{M}$)
    GCRefine($\mathcal{H}, \mathcal{M}$)
    $\mathcal{H} = \text{AbsRefine}(\mathcal{H})$
    return $\mathcal{H}$
end function
```

The following result shows that if a function is refined in the process to make the generator absorbing then its cause can be traced to a function refined in the first step of refinement, i.e. in the gap controlled refinement step.

**Lemma 8.7** (Absorbing refinement to gap control). Let $\mathcal{H}$ be an absorbing basis with gap($\mathcal{H}) \leq g$ and $\mathcal{M} \subset \mathcal{H}$. Let $\mathcal{H}$ be the state of the generator $\mathcal{H}$ right after the call to GCRefine($\mathcal{H}, \mathcal{M}$) in Algorithm 7. Let $\mathcal{R} = \mathcal{L} \setminus \mathcal{L}$ be the refiner of $\mathcal{H}$ with respect to $\mathcal{R}$ and $\mathcal{R} = \mathcal{L} \setminus \mathcal{L}$ the refiner of $\mathcal{H}$ with respect to $\mathcal{H}$. Then, there exists a constant $C$, only depending on $m$, $n$, and $g$, such that for each $\psi \in \mathcal{R}$ there is $\varphi \in \mathcal{M}$ with $|\rho(\varphi, \psi)| \leq C$ and $\ell_\varphi - \ell_\psi \geq -g$.

Proof. Let $\psi \in \mathcal{R}$ then by Lemma 6.1(11) there is $\eta \in \mathcal{H} \setminus \mathcal{H}$ such that $\psi \in \text{ch}^k\eta$ for some $k_0 > 0$. Using Lemma 6.1(11) and Lemma 7.3 there is $k_1 > 0$ such that $\emptyset \neq \mathcal{O}(\psi, k_1, \mathcal{H}) \subset \mathcal{O}(\eta, k_1 + k_0, \mathcal{H})$. From where we get $g \geq \text{gap} \mathcal{H} = \text{gap} \mathcal{H} \geq k_1 + k_0 > k_0$. Now we consider the two possible cases:

**Case 1:** $(\mathcal{E}(\eta) \subset \mathcal{E}(\psi))$. If $\mathcal{E}(\eta) \subset \mathcal{E}(\psi)$, then $\eta \notin \mathcal{H}$ because $\mathcal{H}$ is absorbing, and thus $\eta \in \mathcal{H} \setminus \mathcal{H}$. Now from Lemma 8.4(11) there exist $\varphi \in \mathcal{M}$ and $k_2 \geq 0$ such that $\eta \in \mathcal{O}^k(\varphi, -g, \mathcal{B})$, thus $\psi \in \text{ch}^{k_0+1}\mathcal{O}^{k_2}(\varphi, -g, \mathcal{B})$ and $\ell_\psi - \ell_\varphi = k_0 + 1 - k_2 g \leq g$. Using that $\ell_\psi - \ell_\varphi = k_0$ and Lemma 3.15 we get

$$\rho(\varphi, \psi) = n^{k_0} \rho(\varphi, \eta) + \rho(\eta, \psi). \quad (8.2)$$

On the one hand, since there exists $\xi \in \mathcal{O}^{k_2}(\varphi, -g, \mathcal{B})$ which is a parent of $\eta$, Lemma 3.15 yields $\rho(\varphi, \eta) = n \rho(\varphi, \xi) + \rho(\xi, \eta)$. By Lemma 3.10 $|\rho(\xi, \eta)| \leq m(n-1)$, and from Lemma 3.31(11) $|\rho(\varphi, \xi)| \leq C \leq 2m$ so that $|\rho(\varphi, \eta)| \leq n(C + m) \leq 3nm$. On the other hand, since $\psi \in \text{ch}^{k_0}\eta$, Lemma 3.10 implies

$$|\rho(\eta, \psi)| \leq n^{k_0} m, \quad (8.3)$$

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due to (8.3) and the fact that \( k \).

**Lemma 3.31(ii)** leads to \(|k|\) using Lemma 3.15 and that \( \ell \).

Lemma 8.6(iv) implies there is \( k \) and \( \phi \) generated by subsequent calls of the form \( \phi \).

Theorem 8.8 and \( B \) is given below. And define the “reach” of a function \( \phi \)

as the one obtained by a typical adaptive loop, which we denote as \( H \).

We assume, for simplicity that \( H \) is given by Algorithm 7. From Lemma 8.6, \( \bar{H} \) is a hierarchical generator with gap \( \bar{H} \leq g \). Now \( \bar{H} \) is the output of AbsRefine(\( \bar{H} \)), hence Lemma 8.10 and Proposition 7.9 imply that \( \bar{H} \) is an absorbing basis with gap \( \bar{H} \leq g \).

Let \( \phi \in \bar{H} \). Then from Lemma 8.11 there is \( \phi \in R \) such that \( \phi \in \chi \psi \).

If \( \phi \in M \), then \( \psi \) satisfies the assertion because \( \ell \psi - \ell \phi \leq -1 \).

\(|\rho(\phi, \psi)| \leq m\).

due to (8.3) and the fact that \( h \). The assertion follows from (8.4) and (8.6).

**Theorem 8.8 (Properties of GAREFine).** Let \( H \) be an absorbing hierarchical basis with gap \( H \leq g \) and \( M \subset H \). Let \( \bar{H} \) be the output of GAREFine(\( H, M \)) in Algorithm 7. Then \( \bar{H} \supset H \) is an absorbing hierarchical basis with \( M \subset \bar{H} \subset \bar{H} \), \( \bar{H} \leq g \) and more, for each \( \phi \in \bar{H} \setminus H \) there exists \( \phi \in M \) such that \( |\rho(\phi, \psi)| \leq C \) and \( \ell \psi - \ell \phi \geq -g \).

Proof. Let \( \bar{H} \) be the state of the generator \( H \) right after the call to GAREFine(\( H, M \)) in Algorithm 7. From Lemma 8.6, \( \bar{H} \) is a hierarchical generator with gap \( \bar{H} \leq g \). Now \( \bar{H} \) is the output of AbsRefine(\( \bar{H} \)), hence Lemma 8.10 and Proposition 7.9 imply that \( \bar{H} \) is an absorbing basis with gap \( \bar{H} \leq g \).

Let \( \phi \in \bar{H} \), then from Lemma 8.12 there is \( \psi \in R \) such that \( \phi \in \chi \psi \).

If \( \psi \in M \), then \( \phi \) satisfies the assertion because \( \ell \psi - \ell \phi \leq -1 \).

\(|\rho(\phi, \psi)| \leq m\).

due to Lemma 8.13. If \( \phi \in R \setminus M \) then Lemma 8.7 implies that \( |\rho(\phi, \psi)| \leq C \) and \( \ell \phi - \ell \phi \geq -g \).

**Remark 8.9.** Following the steps of this proof and those of Lemma 8.7 it can be easily seen that the alluded constant \( C \) is bounded by \( 4m^g \).

**9. Complexity of Refinement.** We consider a sequence of refinements \( \{H_r\} \) generated by subsequent calls of the form \( H_{r+1} = \text{GAREFine}(H_r, M_r) \), with \( M_r \subset H_r \),

as the one obtained by a typical adaptive loop, which we denote as

\[
H_0 \xrightarrow[M_0]{} H_1 \xrightarrow[M_1]{} \ldots \xrightarrow[M_{R-1}]{} H_R.
\]

We assume, for simplicity that \( H_0 = \mathcal{B}^0 \). Let \( D = CB \), where \( C \) is the constant in Theorem 8.8 and \( B \) is given below. And define the “reach” of a function \( \phi \in \mathcal{B} \) as

\[
\rho(\phi, \psi) \leq n^g(3n + 1)m,
\]

(8.4) because \( h \).

**Case 2:** \( (I(\eta) \notin \mathcal{L}) \).

If \( I(\eta) \notin \mathcal{L} \), \( \eta \notin \mathcal{L} \) and also \( \eta \notin \bar{H} \) so that \( \eta \notin \bar{H} \) and thus \( \eta \notin \bar{R} \).

By Lemma 8.14 and the fact that \( I(\eta) \notin \mathcal{L} \), it follows that there is \( \mu \in (\mathcal{L}_n \setminus \mathcal{L}) \cap O(\eta, 0, B) = \bar{R}^n \cap O(\eta, 0, B) \). Since \( \mu \in \bar{R} \), Lemma 8.15 implies there is \( \phi \in M \) and \( k_2 \geq 0 \) such that \( \mu \in O^{k_2}(\phi, -g, B) \). Thus using Lemma 8.15 and that \( \ell \psi - \ell \eta = \ell \phi - \ell \phi = k_0 \) we get \( \ell \phi - \ell \phi = \ell \phi - \ell \eta + \ell \eta - \ell \psi = k_g - k_0 \geq -g \) and

\[
\rho(\phi, \psi) = n^{k_2} \rho(\phi, \mu) + n^{k_3} \rho(\mu, \eta) + \rho(\eta, \psi).
\]

Lemma 8.31(1) leads to \( |\rho(\phi, \psi)| \leq 2m \) and Remark 8.28 yields \( |\rho(\mu, \eta)| \leq m \), whence

\[
|\rho(\phi, \psi)| \leq 4mn^{k_2} \leq 4mn^g
\]

(8.6) due to (8.3) and the fact that \( k_0 \leq g \). The assertion follows from (8.4) and (8.6).
\(N(\varphi) := \{ \psi \in B : \ell_\psi \leq \ell_\varphi + g \text{ and } |\rho(\varphi, \psi)| \leq D \} \), which is equivalent to defining it as
\(N(\varphi) = \bigcup_{g \geq 0} B(\varphi, D, -k) \), after recalling from Definition 3.17 that \(B(\varphi, D, -k) := \{ \psi \in B^{\ell_\varphi - k} : |\rho(\varphi, \psi)| \leq D \} \). Let \(H_* = H_R \) and \(M = \bigcup_{r=0}^{R-1} M_r \) and consider the following allocation function \(\lambda : M \times H_* \to \mathbb{R} \) given by
\[
\lambda(\varphi, \varphi_* = \begin{cases} a(\ell_\varphi - \ell_{\varphi_*}) & \text{if } \varphi_* \in N(\varphi) \\ 0 & \text{otherwise.} \end{cases}
\] (9.2)

Where \(a(k)\) is a decreasing sequence such that \(\sum_{k=-n}^{\infty} a(k) = A < \infty\) and there is another increasing sequence \(b(k)\) with \(b(0) \geq 1\), \(\sum_{k=-g}^{\infty} b(k)n^{-k} = B < \infty\) and \(\inf_{k \geq g} b(k) < c_* > 0\). For example consider \(a(k) = (k + (g + 1))^2\) and \(b(k) = n^{k/2}\), which satisfy these assumptions.

**Lemma 9.1** (Upper bound). For any \(\varphi \in M := \bigcup_{r=0}^{R-1} M_r\),
\[
\sum_{\varphi_* \in H_* \setminus H_0} \lambda(\varphi, \varphi_*) \leq C_U := (2D + 1)^d A.
\]

**Proof.** Let \(\varphi \in M\) then, due to (9.2) and the definition of \(N(\varphi)\)
\[
\sum_{\varphi_* \in H_* \setminus H_0} \lambda(\varphi, \varphi_*) \leq \sum_{\varphi_* \in N(\varphi)} \lambda(\varphi, \varphi_*)
= \sum_{k=-g-1}^{\infty} a(k) \#B(\varphi, D, -k)
\leq (2D + 1)^d \sum_{k=g-1}^{\infty} a(k) = (2D + 1)^dA.
\]

where in the last inequality we have used Lemma 3.18

**Lemma 9.2** (Lower bound). For every \(\varphi \in H_* \setminus H_0\),
\[
\sum_{\varphi \in M} \lambda(\varphi, \varphi_*) \geq C_L := \inf_{k \geq g} b(k) a(k), \quad \text{with } M = \bigcup_{r=0}^{R-1} M_r.
\]

**Proof.** Let \(\varphi_0 = \varphi_* \in H_* \setminus H_0\), there must exist an integer \(r_0\) with \(0 < r_0 \leq R\) such that \(\varphi_0 \in H_0 \setminus H_{r_0-1}\). Thus from Theorem 3.13 there is \(\varphi_1 \in M_{r_0-1}\) such that \(|\rho(\varphi_1, \varphi_0)| \leq C \leq D\) and \(\ell_{\varphi_1} - \ell_{\varphi_0} \geq -g\), thus \(\varphi_0 \in N(\varphi_1)\). If now \(\varphi_1 \in H_{r_1-1} \setminus H_0\) we can repeat the process and find \(r_1\) with \(r_1 < r_0 \leq R\) such that \(\varphi_1 \in H_{r_1} \setminus H_{r_1-1}\) and from Theorem 3.13 there is \(\varphi_2 \in M_{r_1-1}\) such that \(|\rho(\varphi_2, \varphi_1)| \leq C \leq D\) and \(\ell_{\varphi_2} - \ell_{\varphi_1} \geq -g\), whence \(\varphi_1 \in N(\varphi_2)\).

This process can be repeated to find a sequence of B–splines \(\varphi_0, \ldots, \varphi_J\) with \(J \geq 1\) and integers \(R \geq r_0 = r_1 > \cdots > r_J = 1\) such that:
- \(\varphi_{j+1} \in M_{r_j-1} \setminus H_0\) for \(j \in [0 : J - 2]\);
- \(\varphi_j \in M_0 \subset H_0\);
- \(|\rho(\varphi_{j+1}, \varphi_j)| \leq C\), and \(\ell_{\varphi_{j+1}} - \ell_{\varphi_j} \geq -g\), for \(j \in [0 : J - 1]\);
- \(\varphi_j \in N(\varphi_{j+1})\), for \(j \in [0 : J - 1]\).

It is worth observing that \(\varphi_j \in H_0\), so that \(\ell_{\varphi_j} = 0\) (and \(J \geq 1\)), and \(\ell_{\varphi_{j+1}} - \ell_{\varphi_j} \in [-g, \infty)\), thus, the level of the \(\varphi_j\)’s, as \(j\) increases in the sequence, can increase in any integer amount, but when it decreases, it will do so in steps smaller than \(g + 1\). Since
\( \ell_{\varphi,j} = 0 \), there exists an integer \( s \) with \( 0 < s \leq J \) such that \( \ell_{\varphi,s} < \ell_{\varphi_0} \) and \( \ell_{\varphi} \leq \ell_{\varphi_j} \) for all \( j = 0, \ldots, s - 1 \).

Let \( k_i = \ell_{\varphi_i} - \ell_{\varphi_0} \), using Lemma \( \text{3.1.15} \) for any \( j = 1, \ldots, s \) we have

\[
\rho(\varphi_j, \varphi_0) = \sum_{i=0}^{j-1} \frac{1}{n^{k_i}} \rho(\varphi_{i+1}, \varphi_i), \quad \text{so that} \quad |\rho(\varphi_j, \varphi_0)| < C \sum_{i=0}^{j-1} \frac{1}{n^{k_i}}.
\]

Consider for any \( k \geq 0 \) and \( 1 \leq j \leq s \) the set

\[ B(k, j) = \{ \varphi \in \{ \varphi_0, \ldots, \varphi_{j-1} \} : \ell_{\varphi} = \ell_{\varphi_0} + k \}, \quad \text{and let} \quad m(k, j) = |B(k, j)|. \]

Clearly for a fixed \( k \) the function \( m(k, j) \) is increasing with \( j \). Then rearranging the terms of the sum we have \( \sum_{j=1}^{j-1} \frac{1}{n^k} = \sum_{k=0}^{\infty} \frac{1}{n^k} m(k, j) \).

Let \( K = \{ k \in \mathbb{Z}_0^+ : m(k, j) > b(k) \} \) for some \( j \in [0 : s] \), and consider two possible cases.

**Case 1:** \( (K = 0) \) In this case \( m(k, s) \leq b(k) \) for all \( k \) so we obtain that, for \( 1 \leq j \leq s \),

\[
\sum_{i=0}^{j-1} \frac{1}{n^{k_i}} = \sum_{k=0}^{\infty} \frac{1}{n^k} m(k, j) \leq \sum_{k=0}^{\infty} \frac{1}{n^k} b(k) = B, \quad \text{hence} \quad |\rho(\varphi_j, \varphi_0)| \leq CB = D,
\]

so that \( \varphi_0 = \varphi_s \in \mathcal{N}(\varphi_s) \) and \( \lambda(\varphi_s, \varphi_s) = a(\ell_{\varphi_s} - \ell_{\varphi_0}) \geq a(0) > 0 \) because \( a(\cdot) \) is decreasing and \( \ell_{\varphi} < \ell_{\varphi_s} \). Therefore, since \( \varphi_s \in \mathcal{M} \), \( \sum_{\varphi \in \mathcal{M}} \lambda(\varphi, \varphi_s) \geq \lambda(\varphi_s, \varphi_0) \geq a(0) > 0 \).

**Case 2:** \( (K \neq \emptyset) \): For each \( k \in K \), let \( j(k) = \min \{ j \in \{ 1, \ldots, s \} : m(k, j) > b(k) \} \), so that \( j(k) \geq 2 \) because \( m(k, 1) \leq 1 = b(0) \leq b(k) \) for any \( k \in \mathbb{Z}_0^+ \), and \( m(k, j') \leq b(k) \) if \( 0 \leq j' < j(k) \). Let now \( \hat{j} = \min \{ j(k) : k \in K \} \) and \( \hat{k} \) the minimum \( k \) that verifies \( j = j(k) \), hence \( m(k, j) \leq b(k) \) for all \( k \in \mathbb{Z}_0^+ \) if \( 1 \leq j < \hat{j} \). Then

\[
\sum_{\varphi \in \mathcal{M}} \lambda(\varphi, \varphi_s) \geq \sum_{\varphi \in B(\hat{k}, \hat{j})} \lambda(\varphi, \varphi_s)
\]

Let now \( \varphi \in B(\hat{k}, \hat{j}) \) and compute \( \lambda(\varphi, \varphi_s) \). The first step is to determine that \( \varphi_s \in \mathcal{N}(\varphi) \). On the one hand, by definition of \( B(\hat{k}, \hat{j}) \), \( \ell_{\varphi} = \ell_{\varphi_0} - \hat{k} \leq \ell_{\varphi} + g \). On the other hand, there exists \( j < \hat{j} \) such that \( \varphi = \varphi_j \) so that, as before

\[
|\rho(\varphi, \varphi_s)| = |\rho(\varphi_j, \varphi_s)| \leq C \sum_{i=0}^{j-1} \frac{1}{n^{k_i}} = C \sum_{k=0}^{\infty} \frac{1}{n^k} m(k, j) \leq C \sum_{k=0}^{\infty} \frac{1}{n^k} b(k) = CB = D,
\]

and thus \( \varphi_s \in \mathcal{N}(\varphi) \). Therefore, \( \lambda(\varphi, \varphi_s) = a(\ell_{\varphi} - \ell_{\varphi_0}) = a(\hat{k}) \) for each \( \varphi \in B(\hat{k}, \hat{j}) \) and thus

\[
\sum_{\varphi \in \mathcal{M}} \lambda(\varphi, \varphi_s) \geq |B(\hat{k}, \hat{j})| a(\hat{k}) \geq b(\hat{k}) a(\hat{k}) \geq \inf_{k \geq g} b(k) a(k) = C_L > 0.
\]

The assertion thus follows. \( \square \)

We are now in position to prove the following complexity estimate.

**Theorem 9.3.** Assume that the sequence of hierarchical gap-controlled absorbing bases \( \{ \mathcal{H}_r \} \) has been generated by subsequent calls of the form

\[
\mathcal{H}_{r+1} = \text{GARRefine}(\mathcal{H}_r, \mathcal{M}_r), \quad \text{with} \quad \mathcal{M}_r \subset \mathcal{H}_r, \quad r = 0, 1, \ldots,
\]

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with $H_0 = B^0$. Then
\[
\#H_R - \#H_0 \leq \frac{C_L}{C_U} \sum_{r=0}^{R-1} \#M_r
\]
for any $R$.

Proof. By the previous two lemmas, letting $M = \cup_{r=0}^{R-1} M_r$ we immediately obtain
\[
\#H_R - \#H_0 \leq \frac{1}{C_L} \sum_{\varphi \in H_R \setminus H_0} C_L \leq \frac{1}{C_L} \sum_{\varphi \in H_R \setminus H_0} \sum_{\varphi' \in M} \lambda(\varphi, \varphi')
\]
\[
= \frac{1}{C_L} \sum_{\varphi \in M} \sum_{\varphi' \in H_R \setminus H_0} \lambda(\varphi, \varphi') \leq \frac{C_U}{C_L} \#M.
\]

We finally make an interesting observation about approximation classes.

Given $s > 0$ and a function space $V$ over $\Omega$ with norm $\| \cdot \|$ we define the best approximation error with complexity $N$ as
\[
\sigma_N(u) = \inf_{\#H - \#H_0 \leq N} \inf_{V \in \text{span } H} \| u - V \|, \quad u \in V.
\]
and the approximation class $A_s$ as
\[
A_s = \{ v \in V : \sigma_N(u) \leq C N^{-s}, N \in \mathbb{N} \}.
\]
We have two definitions, depending if we consider any hierarchical space (generated by hierarchical generators) or absorbing and gap controlled hierarchical spaces, i.e., $\sigma_N, \bar{A}_s$: considering absorbing and gap controlled hierarchical spaces (fixed $g > 0$); $\overline{\sigma}_N, \bar{A}_s$: considering all hierarchical spaces.

Clearly, $\overline{\sigma}_N(u) \leq \sigma_N(u)$, so that
\[
\bar{A}_s \subset \overline{A}_s.
\]
But also
\[
\overline{A}_s \subset \bar{A}_s,
\]
i.e. if a function can be approximated with hierarchical spaces at a rate $N^{-s}$ it can also be approximated at the same rate with absorbing and gap controlled hierarchical spaces.

This is an immediate consequence of the following Proposition.

PROPOSITION 9.4. For each hierarchical generator $H$ there exists an absorbing gap controlled hierarchical basis $\overline{H}$ with
\[
\text{span } \overline{H} \subset \text{span } H \quad \text{and} \quad \#H - \#H_0 \lesssim \#\overline{H} - \#H_0
\]

Proof. Given a hierarchical generator $\overline{H}$ construct an absorbing gap controlled hierarchical basis as follows:
1: $H = B^0$
2: for $\ell = 0, 1, \ldots$ do

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3. \( M_\ell = \mathcal{L}_\ell \cap \mathcal{H} \)
4. \( \text{GAR\textsc{es}refine}(\mathcal{H}, M_\ell) \)
5. \textbf{end for}

Then \( \mathcal{L} \supset \mathcal{L}_\ell \), \( \mathcal{H} \prec \mathcal{H}_\ell \) and

\[ \# \mathcal{H} - \# \mathcal{H}_0 \leq \sum_{\ell=0}^{l} \# M_\ell \leq \# \mathcal{L}_\ell \leq \# \mathcal{H}_\ell - \# \mathcal{H}_0. \]

\[ \square \]

**10. Appendix.** In this section we present some auxiliary results and proofs which are simple, but a little bit technical, and would have obstructed the reading of the previous sections where they were presented there.

Let \( n \) be a fixed integer such that \( n > 1 \), and let \( D_m \) and \( M_m \) be the index functions defined in Section 2.1. For any \( k \in \mathbb{Z}^+ \) we have the following results.

**Lemma 10.1 (Formulas I).**

(i) \( M_m^k(i) = n^k i + m \frac{n^k - 1}{n - 1} \).

(ii) \( D_m^k(i) = \frac{1}{n^k} - \frac{m}{n^k} \frac{n^k - 1}{n - 1} - R, \) with \( R \in [0, 1 - \frac{1}{n^k}]^d \).

\textbf{Proof.} Note that for \( k = 1 \), both (i) and (ii), are just the definition of \( M_m \) and \( D_m \). The case \( k > 1 \) follows immediately for (i). In order to prove (ii) let \( r_j \) be such that \( D_m^k(i) = \frac{D_m^{k-1}(m(i) - m)}{n} + r_j \), for \( j = 1, \ldots, k \). Then \( 0 \leq r_j \leq 1 - 1/n \) and \( R = \frac{1}{n} \sum_{j=0}^{k-1} \frac{r_j}{n^j} \). Therefore \( 0 \leq R \leq 1 - 1/n^k \). \( \square \)

**Lemma 10.2 (Formulas II).** For any \( k \in \mathbb{Z}^+ \) we have that

(i) \( L_k^p(i) = \frac{4}{n^{k-1}} - \frac{p}{n^{k-1}} \frac{n^k - 1}{n - 1} - A, \) with \( 0 \leq A \leq 1 - \frac{1}{n^k} \).

(ii) \( R_k^p(i) = \frac{1}{n^k} + \frac{p}{n^k} \frac{1}{n^k - 1} - B, \) with \( 0 \leq B \leq 1 - \frac{1}{n^k} \).

\textbf{Proof.} The result follows by induction applying Lemma 10.1 (ii). \( \square \)

**Lemma 10.3 (Inverse).** Let \( p, q \in \mathbb{Z} \) then

(i) \( \text{if } 0 \leq p - q < n \text{ then } D_q^p(M_m^k(i)) = i. \)

(ii) \( \text{if } m \in \mathbb{Z} \text{ then } i - (n^k - 1) \leq M_m^k(D_m^k(i)) \leq i. \)

\textbf{Proof.} \( D_q^p(M_m^k(i)) = \lfloor i - \frac{p}{n^k} \rfloor \) which equals \( i \) if \( 0 \leq p - q < n \). For \( k > 1 \) it is a matter of iterating the previous result.

To show (ii) observe that applying Lemma 10.1 (ii) and then Lemma 10.1 (i) we have that \( M_m^k(D_m^k(i)) = i - n^k R_i \), for some \( R_i \) such that \( 0 \leq R_i \leq 1 - \frac{1}{n^i} \). So the result is clear. \( \square \)

**Lemma 10.4 (Inequalities).**

(i) \( M_m^k(j) \leq i \text{ if and only if } j \leq D_m^k(i) \).

(ii) \( M_m^k(j) \geq i \text{ if and only if } j \geq D_m^k(n-i) \).

\textbf{Proof.} First note that \( M_m^k \) and \( D_m^k \) are non decreasing index functions. Hence if \( M_m^k(j) \leq i \) then \( D_m^k(M_m^k(j)) \leq D_m^k(i) \) and by Lemma 10.3 (i) we have that \( j \leq D_m^k(i) \). The other implication follows exactly in the same way applying Lemma 10.3 (ii).

Furthermore, if \( M_m^k(j) \geq i \) taking \( p = m \) and \( q = m - (n - 1) \) in Lemma 10.3 (i) we have \( j \geq D_m^k(n-i) \). On the other hand, by Lemma 10.1 (ii), we can show that \( M_m^k(D_m^k(n-i)) = i + n^k R_i \), for some \( R_i \) such that \( 0 \leq R_i \leq 1 - \frac{1}{n^i} \). Hence if \( j \geq D_m^k(n-i) \) then \( M_m^k(j) \geq M_m^k(D_m^k(n-i)) = i + n^k R_i \geq i. \)

We say that a index function \( P \) \textit{decouples} if there exist functions \( P_j: \mathbb{Z} \to \mathbb{Z} \) such that \( [P(i)]_j = P_j(i) \), for \( i \in [1 : d] \). As well, we say that \( P \) is \textit{non decreasing} if \( P(i) \leq P(j) \) for \( i \leq j \).

**Lemma 10.5.** Let \( F \) be a box function given by \( F(i) = (P(i) : Q(i)) \) such that \( P \)

\[ \]
and \( Q \) are non decreasing and decouple. If \( Q_i(i_{\ell}) - P_i(i_{\ell} + 1) \geq -1 \), for \( j = [1 : d] \). Then \( F \) is box preserving, i.e. \( F([i : j]) = (P([i : j])) \).

Proof. Let \( z \in F([i : j]) \), then there exists \( w \in [i : j] \) such that \( z \in (P(w) : Q(w)) \), i.e. \( P(w) \leq z \leq Q(w) \). Since \( P \) and \( Q \) are non decreasing \( P(i) \leq P(w) \leq z \leq Q(w) \). Hence \( z \in (P(i) : Q(j)) \), therefore \( F([i : j]) \subseteq (P([i : j])) \).

On the other hand, let \( z \in (P(i) : Q(j)) \) then \( P(i) \leq z \leq Q(j) \). Since \( P \) and \( Q \) decouple, for each \( j = [1 : d] \) we have that \( P_i(i_{\ell}) \leq z_{\ell} \leq Q_i(j_{\ell}) \). Besides \( Q_i(i_{\ell}) - P_i(i_{\ell} + 1) \geq -1 \), for \( \ell = [1 : d] \), then there exists \( w_{\ell} \in [i_{\ell} : j_{\ell}] \) such that \( P_i(w_{\ell}) \leq z_{\ell} \leq Q_i(w_{\ell}) \). Hence, \( w = (w_1, \ldots, w_d) \in [i : j] \) satisfies that \( P(w) \leq z \leq Q(w) \). Therefore, \( z \in F([i : j]) \).

Lemma 10.6. If \( F \) and \( G \) are box preserving functions, then the composition \( F \circ G \) is a box function and even more is box preserving.

Proof. Let \( F, Q, P \) and \( G \) index functions such that \( F(i) = (P_F(i), Q_F(i)) \) and \( G(i) = (P_G(i), Q_G(i)) \). Notice that \( F \circ G(i) = F(G(i)) = F(P_G(i) : Q_G(i)) \), since \( F \) is box preserving then \( F \circ G(i) = (P_F \circ P_G(i) : Q_F \circ Q_G(i)) \). Hence, \( F \circ G \) is a box function. Moreover, \( F \circ G([i : j]) = F(G([i : j])) \), since \( F \) and \( G \) are box preserving \( F \circ G([i : j]) = F(P_G(i) : Q_G(j)) = (P_F \circ P_G(i) : Q_F \circ Q_G(j)) \), i.e. \( F \circ G \) is box preserving.

Corollary 10.7. If \( F(i) = (P(i) : Q(i)) \) is box preserving, then the \( k \)-th iterate of \( F \) satisfies \( F^K(i) = (P^K(i), Q^K(i)) \) and is box preserving.

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