On the relation between quantum theory and probability

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Abstract

The theory of probability and the quantum theory, the one mathematical and the other physical, are related in that each admits a number of very different interpretations. It has been proposed that the conceptual problems of the quantum theory could be, if not resolved, at least mitigated by a proper interpretation of probability. We rather show, through a historical and analytical overview of probability and quantum theory, that if some interpretations of the one and the other go along particularly well, none follows in a unique way.

Keywords: Determinism, indeterminism, interpretation, probability, quantum theory.

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The popularity of gambling dates back to ancient times, as evidenced by bones and ivory artifacts found by archaeologists and clearly used for this purpose. Yet although it gives the gambler an undeniable advantage, the calculus of probability was not developed until much later. In the Renaissance, a probable opinion was not understood as one backed by evidence, but rather as one that came from a recognized authority.

The concept of probability as we know it crystallized in a few places in Europe around 1660, in particular through the correspondence between Blaise Pascal and Pierre de Fermat. Hacking (2006) notes that from the beginning probability has been associated with two distinct notions: one, subjective, consisting in a degree of belief and the other, objective, referring to random processes displaying in the long run stable relative frequencies.

Beyond games of chance, probability theory quickly found applications in the calculation of premiums for life insurance and life annuities. In physics, however, and in spite of major contributions by Pierre Simon de Laplace, probability theory did not become truly important until the middle of the nineteenth century, with the rise of statistical physics.

Developed in the second half of the nineteenth century by James Clerk Maxwell, Ludwig Boltzmann and Josiah Willard Gibbs, statistical physics aims to explain the thermodynamic properties of a gas (for example) on the basis of atomic and molecular theory. The theory assumes that the gas is made up of a huge number of molecules (typically, $10^{25}$) that move and collide according to the deterministic laws of Newtonian mechanics. Since, however, it is impossible to solve the equations of motion of all these molecules, one must resort to approximations. They consist in associating to thermodynamic parameters (e.g. temperature, pressure, energy and entropy) averages of microscopic parameters computed under the assumption of a statistical

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1Hacking (1990, 2006) and Galavotti (2005) recall key episodes in the history of probability.
model. Several features of the model (for example, the conservation of energy of an isolated system) can be justified objectively. Yet the introduction of probabilities is in this case only a second-best choice, linked to a lack of knowledge of the exact conditions of the molecules. Paradoxically, this compromise often allows to predict the values of certain macroscopic parameters with remarkable accuracy.

Hacking (1990) points out that the concept of chance, associated with a purely random process irreducible to deterministic laws, really took off only at the end of the nineteenth century, with Charles Sanders Peirce in particular. In physics, it was first brought to light through the phenomenon of radioactivity.

Discovered in 1896 by Henri Becquerel, radioactivity was quickly studied by Marie and Pierre Curie and by Ernest Rutherford. The latter noted that what he called “thorium emanation” (our radon 220) displays an exponentially decreasing radioactivity (Rutherford, 1900). Specifically, if there are \( N_0 \) active atoms at time \( t = 0 \), then the number of active atoms at time \( t > 0 \) is given by

\[
N(t) = N_0 \exp\{-\lambda t\}
\]

(1)

where \( \lambda \) is a positive constant that characterizes the radioactive substance. The exponential law implies that the probability of a given atom emitting radiation does not depend on time. How comes that the probability of radioactive decay does not depend on the age of the atom? Why does one atom decay now, and another much later? Here is how James Jeans retrospectively summarized the situation:

In a milligram of radium, about 500 million atoms disintegrate every second […] Interesting but difficult questions arise when we discuss which atoms will disintegrate first, and which will survive longest without disintegration […] It seemed to remove causality from a large part of our picture of the physical world […] If we are told the position and the speed of motion of every atom of radium at any moment, we might expect that Laplace’s supermathematician would be able to predict the future of every atom. And so he would if their motion had conformed to the classical mechanics. But the new laws merely tell him that one of his atoms is destined to disintegrate today, another tomorrow, and so on. No amount of calculation will tell him which atoms will do this. (Jeans, 1943, pp. 148–150)
Although the question arose early on, most investigators did not immediately conclude that the principle of causality had to be abandoned. For Rutherford and Marie Curie, for example, complex atomic processes, impossible to specify at the time, could be at the origin of the apparent violation of causality\(^2\) in the same way, perhaps, that the randomness of the tossing of a die comes from not knowing the exact conditions of the throw. This twofold way of understanding radioactivity would be carried to the quantum theory, which can be construed in a deterministic or indeterministic way.

The purpose of this paper is to explore and clarify the link between quantum theory and probability theory. We will first recall, in section 2, different ways of interpreting probability. Section 3 will be devoted to Born’s rule, the key to the introduction of probability in quantum theory, and to the quantum measurement problem, the origin of the theory’s different interpretations. These interpretations can be indeterministic or deterministic. We will examine, in sections 4 and 5, how the interpretation of probability can be adapted to them, before concluding in a more general way.

### 2 The interpretation of probability

The distinction between objective and subjective probability, already entertained by the founders of the theory, as well as the gradual realization that natural phenomena can in principle be necessary, contingent or purely random, led to the development of different interpretations of probability. Mellor (2005) classifies them in three types: (i) physical probability, i.e. chance (e.g. the probability that such and such radioactive nucleus will decay in the next hour); (ii) epistemic probability (in statistical physics, in particular, where it is related to the lack of knowledge of the exact initial conditions); and (iii) an agent’s belief with respect to a contingent situation (which leads him, for example, to bet 2 to 1 on the victory of a sports team). For her part, Galavotti (2005) proposes another classification, articulated historically under five types, which is well suited to our analysis\(^3\).

\(^2\)See Pais (1986), p. 123.

\(^3\)Galavotti (2005) provides many references to the contributions of the various investigators we will mention.
2.1 The classical interpretation

From the beginning, probability theory has been concerned with games of chance, such as the tossing of a die or the throw of a coin. As long as the die is made properly, none of the six faces is favored. We then say that the probability of getting a specific outcome, say three, is equal to $1/6$, since three is one of the six possible outcomes. This is the classical conception of probability, formulated by Laplace in his *Philosophical Essay on Probabilities*:

\[ \text{The theory of chance consists in reducing all the events of the} \]
\[ \text{same kind to a certain number of cases equally possible, that is} \]
\[ \text{to say, to such as we may be equally undecided about in regard to} \]
\[ \text{their existence, and in determining the number of cases favorable} \]
\[ \text{to the event whose probability is sought. The ratio of this number} \]
\[ \text{to that of all the cases possible is the measure of this probability,} \]
\[ \text{which is thus simply a fraction whose numerator is the number of} \]
\[ \text{favorable cases and whose denominator is the number of all the} \]
\[ \text{cases possible.} \quad (\text{Laplace, 1814, pp. 6–7}) \]

Thus, Laplace considers that two cases are equally possible if one is equally undecided about them. For him this indecision, or lack of knowledge, is the only justification for the use of probabilities. Indeed, Laplace considers that “[a]ll events, even those which on account of their insignificance do not seem to follow the great laws of nature, are a result of it just as necessarily as the revolutions of the sun” (Laplace, 1814, p. 3). The universe obeys a rigorous determinism. Probabilities are justified by our limited knowledge of the laws of nature and of initial conditions, and are therefore epistemic.

The classical interpretation of probability has been the object of several criticisms. It can be very difficult, for example, to enumerate all the cases. *A fortiori*, dividing them into “equally possible” cases already implies, in a circular way, a prior notion of probability. If, on the other hand, the number of cases is infinite, the fraction that Laplace speaks of is indeterminate. Subsequent interpretations will attempt to answer these criticisms.

2.2 The frequency interpretation

Several names are associated with the frequency interpretation of probability, in particular those of Robert Leslie Ellis, John Venn, Richard von Mises, Hans
Reichenbach and Ernest Nagel. For these investigators, the probability of a type of event refers to its relative frequency in an arbitrarily long and, in the limit, infinite sequence.

Not all sequences necessarily give rise to probabilities. For von Mises, for example, the sequence must correspond to a specific ensemble of events, such as the repeated toss of a coin. It should also have a random character, a concept of which we naturally have an intuition but which is very difficult to define rigorously.

Among the objections made to the frequency interpretation, two particularly stand out. The first consists in observing that to identify a probability with a relative frequency, one should strictly speaking consider an infinite sequence, which is in practice impossible. The second is the difficulty of associating, in terms of frequency, a probability to a single event. One solution consists in considering the event as belonging to a class. To evaluate, for example, the probability that an individual will die in the coming year, we consider the class of people of the same age, same sex and comparable health. The specification of such classes (“comparable health”), however, creates new problems.

Many consider these objections serious and try to define probability differently. Yet whatever the preferred way of defining probability, testing a specific statistical hypothesis can only be done by observing relative frequencies.

2.3 Propensity

Anticipated by Charles Sanders Peirce, the propensity interpretation of probability was really developed by Karl Popper. He proposed it (i) to solve the problem of the interpretation of quantum theory (Popper, 1959, 1982) and (ii) to define the probability of single events (Popper, 1959).

Popper begins his analysis on the basis of the frequency interpretation, which he had previously advocated. From this point of view, asserting that the probability of obtaining three, when throwing a die, is equal to 1/6 means that the relative frequency of a three in a virtually infinite sequence of throws is 1/6. Popper notes that the sequence is not arbitrary, but corresponds to the specification of experimental conditions that are repeated from one throw to the next. The probability of getting three in a particular throw is thus construed as being related to these experimental conditions. Popper then proposes
a new physical hypothesis} (or perhaps a metaphysical hypothesis) analogous to the hypothesis of Newtonian forces. It is the hypothesis that every experimental arrangement (and therefore every state of a system) generates physical propensities which can be tested by frequencies. (Popper, 1959, p. 38).

Although not directly observable, propensities are nonetheless, according to Popper, objective.

### Bayes’ Theorem

Let $A$ and $B$ be two random variables and let $a$ and $b$ be values of these variables. The conditional probability $P(a|b)$, i.e. the probability that $A$ has value $a$ if $B$ has value $b$, is defined as $P(a,b)/P(b)$, where $P(a,b)$ is the probability that $A$ has value $a$ and $B$ has value $b$. Because $P(a,b) = P(b,a)$, we immediately obtain

$$P(b|a) = P(a|b)P(b)/P(a)$$

This is Bayes’ theorem. It is especially useful to evaluate the probability of a hypothesis conditional to an observation.

As an example, suppose that $A$ is associated with the outcome of the throw and $B$ with the choice of die (see below). Let $+$ represent the fair die and $-$ the rigged die. Then

$$P(-|3) = P(3|-)P(-)/P(3)$$

By assumption, $P(-) = 1/2 = P(+).$ This means that $P(3) = 5/24$, the average of $1/4$ and $1/6$. Since $P(3|-) = 1/4$, we find that $P(-|3) = 3/5$.

It is important to note that, although propensity can be used to shed light on the notion of probability, the two concepts cannot be identified. Consider two dice, the first fair and the second rigged so that the probability of obtaining three by throwing it is $1/4$. It is natural to quantify the propensity of getting three by throwing each die. On the other hand, Bayes’ theorem tells us that if the \textit{a priori} probability of throwing each die is equal, and we get three, then the probability of having thrown the rigged die is equal to $3/5$ (see box). In this case, it would be highly artificial to speak of the propensity
to use the rigged die. To associate probability with propensity, there must be a causal link between the experimental conditions and the outcome.

2.4 The logical interpretation

Anticipated by Auguste de Morgan and George Boole, this point of view was developed in the twentieth century, especially by John Maynard Keynes and Rudolf Carnap.

The basic idea of the logical interpretation is to associate probability with a degree of belief. Unlike the subjective interpretation that we shall introduce in the next section, we do not have in mind here the belief of specific agents, but rather the belief of an ideal rational agent. Probability is thus a logical relation between propositions, referring to arguments that do not lead to a unique conclusion.

The abstract nature of the logical interpretation makes it difficult to apply it directly to real situations. This led Carnap to propose two distinct notions of probability. Probability$_1$, a logical concept, represents the degree of confirmation brought to a given hypothesis by different clues. Probability$_2$, on the other hand, refers to relative frequencies. The two concepts are related inasmuch as the first kind of probability can be used to estimate the second one.

2.5 The subjective interpretation

The identification of probability with a degree of belief characterizes the subjective interpretation, developed in the last century mainly by Frank Ramsey and Bruno de Finetti. That belief, according to these investigators, is in no way unique, nor is it determined by considerations of rationality. Two distinct agents can have distinct degrees of belief, with the only restriction that the beliefs of each must be consistent.

De Finetti proposes an operational definition of probability in terms of bets. Believing that an event will occur once in every four times, for example, means that the agent is willing to bet one dollar against three on its occurrence. Consistency consists in restricting the type of bets so that no combination leads to a sure loss.

Of course, observation and experience imply that an agent’s beliefs can change. For example, suppose I believe that a die is fair, and therefore that each of the six possible outcomes of a throw has the same probability. If I
observe that in 100 subsequent throws, the outcome three occurs 25 times, my initial belief will be altered. Bayes’ rule allows me to refresh my belief. Thus two agents with different initial beliefs may end up with closer beliefs. Nevertheless this does not mean, according to de Finetti, that the beliefs converge to a probability that would be objective.

3 Born’s rule and measurement

The formalism of quantum theory was developed in 1925–1926 by Werner Heisenberg, Erwin Schrödinger and Paul Dirac. As early as 1926, Max Born brought to light the probabilistic nature of the theory.

The formalism of the theory can be interpreted in several ways. We will see that each interpretation of the theory naturally adapts to an interpretation of probability.

In this section, we will point out elements of the quantum theory that do not depend on the theory’s interpretation. Such is, in particular, Born’s rule, a statistical postulate that introduces probability in a purely operational way.

Any physical system adequately described by quantum theory will be called a quantum system. Many believe that quantum theory has universal scope, and therefore that all physical systems are quantum systems. It is not necessary at this time to commit oneself on this issue. Everyone agrees, however, that atoms and simple molecules are quantum systems. In what follows, the term atomic system will denote any microscopic physical system that obeys the laws of quantum theory.

The description of a quantum system is carried out by means of a mathematical object called a state vector. The most general description is done by means of a density operator (or density matrix), which doesn’t need to be introduced at this stage. The Schrödinger wave function is an example of state vector. It is typically denoted by means of a ket like $|\phi\rangle$. The interpretation of the state vector is a controversial issue. Some believe that it describes an individual system, others a statistical ensemble of systems, while for others still it represents an agent’s information about the system. Whatever the preferred interpretation, however, the state vector corresponds to a preparation procedure. An example of preparation consists in aiming photons of a given wavelength toward a linear polarizer. All photons from the polarizer are then associated with a specific state vector. The preparation process constitutes an operational definition of the state vector, which
somehow isolates its uncontroversial features.

The state vector is an element of a vector space \( \mathcal{V} \) that is called the *state space*. Although the dimension of the state space of many quantum systems is infinite, we can restrict ourselves to finite-dimensional spaces. In this case, \( \mathcal{V} \) coincides with \( \mathbb{C}^N \), the complex vector space of dimension \( N \), in which the usual scalar product is defined. All identical quantum systems (for example, all helium atoms) have isomorphic state spaces. And each vector of the state space corresponds, in principle, to a possible preparation procedure. The correspondence, however, is not one-to-one. Two vectors which are multiples of each other correspond to the same preparation, i.e. to the same physical situation (to the same state of the system, if the vector is so interpreted).

Let \( \{ |a_i\rangle, i = 1 \ldots N \} \) be an orthonormal basis of the state space. It is always possible, in principle, to construct a macroscopic measuring apparatus that has the following property. If the quantum system is prepared in the state \( |a_i\rangle \), then at the end of the measurement the apparatus indicates the value \( \alpha_i \), where these values are all distinct real numbers. Many interpret this situation by stating that the apparatus then measures a physical quantity \( A \) associated with the atomic system, whose possible values are the \( \alpha_i \).

The vectors \( |a_i\rangle \) make up a basis of the state space. Therefore, any normalized vector \( |\phi\rangle \) can be expressed as a linear combination of the \( |a_i\rangle \), that is,

\[
|\phi\rangle = \sum_{i=1}^{N} c_i |a_i\rangle
\]

where the \( c_i \) are complex numbers. Operationally, *Born’s rule* is then formulated as follows. If one prepares a large number of identical systems in state \( |\phi\rangle \), and performs on each system the measurement specified above, then one will obtain the value \( \alpha_j \) with relative frequency \( |c_j|^2 \) (in the limit where the number of systems tends to infinity). Formulated in this way, the rule depends neither on the interpretation of quantum theory nor on that of probability.

Born’s rule refers to the notions of measurement and apparatus. From a phenomenological point of view, these notions seem fairly clear. If, however, we wish to describe the measurement process in more detail, we have to be more precise.

\[\text{It may also happen that two operationally distinct procedures prepare the same quantum state.}\]
In the remainder of this section, we will assume that the state vector of a quantum system describes an individual system (rather than, for example, a statistical ensemble of systems). From a fundamental point of view, an apparatus consists of a very large number of atoms, arranged in a complex way. Individually as well as in restricted aggregates, these atoms obey the laws of quantum theory. Let us assume that quantum theory is truly fundamental and that the scope of its applications has no limit. In this case, the measuring apparatus itself constitutes a quantum system, to which we can associate a state space. It is, of course, a very complicated space, but the sole hypothesis of its existence leads to unexpected conclusions.

To perform a measurement of the physical quantity $A$, the apparatus must be able to display $N$ different values $\alpha_i$. To do this, the state space of the apparatus must contain at least $N$ orthogonal vectors $|\alpha_i\rangle$, each vector corresponding to a state where the apparatus displays the corresponding value. We can also assume that there is a vector $|\alpha_0\rangle$ that corresponds to an initial state where the apparatus does not indicate any value.

Consider a situation where the apparatus is prepared in state $|\alpha_0\rangle$ and the atomic system is prepared in state $|a_i\rangle$. We then say that the global system (consisting of the atomic system and the apparatus) is prepared in state $|a_i\rangle|\alpha_0\rangle$. If the atomic system is aimed toward the apparatus, the latter, at the completion of the measurement, will indicate the value $\alpha_i$. This implies (assuming the state vector of the atomic system does not change) that the state vector of the global system will be given by $|a_i\rangle|\alpha_i\rangle$.

What will happen now if we prepare the atomic system in the state $|\phi\rangle$, represented by equation (2)? The global system’s initial state will then be given by $|\phi\rangle|\alpha_0\rangle$. What will be the global system’s state at the end of the measurement? The dynamics of a quantum system can be complicated but, for all microscopic systems it is governed by a linear equation (the Schrödinger equation, in the nonrelativistic case). Assuming that this property is universal, the evolution equation of the global system should also be linear. In

\footnote{As a matter of fact, there is a very large number of state vectors associated with each value $\alpha_i$, but this will not change our conclusion.}

\footnote{The object $|a_i\rangle|\alpha_0\rangle$ is what is called a tensor product. This notion, and others succinctly introduced here, are specified in quantum mechanics textbooks, e.g. Marchildon (2000).}
other words, the initial state vector

\[ |\phi\rangle|\alpha_0\rangle = \sum_{i=1}^{N} c_i |a_i\rangle|\alpha_i\rangle \] (3)

should evolve in such a way that at the end of measurement it becomes

\[ \sum_{i=1}^{N} c_i |a_i\rangle|\alpha_i\rangle \] (4)

At first glance, this result seems quite different from what was expected, i.e. a situation where the apparatus displays a well-defined outcome, e.g. \( \alpha_j \), with relative frequency \( |c_j|^2 \). Nevertheless, this result is an inescapable consequence of associating the state vector with an individual system and of assuming that quantum theory has universal scope. The incompatibility, apparent at least, of result (4) with what we would like to obtain is called the measurement problem. The various interpretations of quantum theory aim primarily at solving this problem.

4 Indeterministic interpretations of quantum theory

From its beginning, and during the two or three decades that followed, quantum theory was almost unanimously interpreted in an indeterministic way. The Copenhagen interpretation, a constellation of ideas proposed mostly by Bohr and also by Heisenberg, ruled virtually unchallenged for a long time (Freire Jr., 2015). The expositions of quantum theory grafted onto it the notion of state vector collapse, now recognized as largely alien to Bohr’s ideas. From the Copenhagen interpretation the idea more recently developed that the state vector does not represent the state of an atomic system, but the information of a more or less ideal observer.

4.1 The collapse of the state vector

Considered very early by Dirac, the collapse of the state vector was formalized by John von Neumann (1932). The idea is to assume that the evolution of an atomic system does not always follow a linear equation.
Specifically, von Neumann assumed that an atomic system can evolve in two ways, which he called Process 1 and Process 2. Process 2 applies to every circumstance other than a measurement, and it simply consists in the linear evolution of the state vector, governed by the Schrödinger equation. Process 1, on the other hand, applies exclusively to a measurement situation, and occurs as a result of the transformation of vector (3) into vector (4). It consists in the transformation of vector (4) into one of its terms, in a purely random way. Von Neumann postulates, however, that the probability that vector (4) transforms into the term $|\alpha_j\rangle c_j$ is equal to $|c_j|^2$.

It is easy to see that von Neumann’s hypothesis implies Born’s rule. In both formulations, we take for granted that the notions of measurement and apparatus are clear enough.

In the collapse theory, the vector $|\alpha_j\rangle c_j$ into which the superposition (4) transforms is completely random. The statistical distribution of the outcomes, however, depends entirely (through the coefficients $c_i$ and the normalized vectors $|a_i\rangle$) on the preparation of the atomic system and the nature of the apparatus. In other words, it leads to objective probabilities that depend entirely on the experimental setup. This is precisely the context for which Popper proposed the propensity interpretation.

Originally, von Neumann did not suggest any specific mechanism through which collapse could occur. Such mechanisms came later, one of the best known being spontaneous localization (Ghirardi, Rimini, & Weber, 1986; Ghirardi, Pearle, & Rimini, 1990). This is a random process which, from time to time, reduces the spatial extension of the wave function of a particle. The theory is designed in such a way that the localization of an atomic object is extremely rare, whereas a macroscopic object (such as the pointer of an apparatus) is localized in a few nanoseconds. As in von Neumann’s approach, probability is objective and neatly fits in the propensity interpretation.

The collapse of the state vector aims at solving the measurement problem. This has been formulated under the assumption that the state vector describes an individual quantum system. What happens if we do away with this assumption? Suppose, for example, that the state vector describes not an individual system, but a statistical ensemble of similarly prepared systems (Ballentine, 1970). It is tempting then to assume that, at the end of the measurement, the vector (4) represents not a superposition of macroscopically distinct states, but a statistical ensemble of systems in each of which the apparatus displays a specific value. And could it not be that this value corresponds to the one that the physical quantity would have had just before
the measurement, so that the latter would consist in a process of separation of the systems of the statistical ensemble according to the initial value of the physical quantity? Unfortunately, this last hypothesis is untenable. It can indeed be shown (Kochen & Specker, 1967) that it is impossible to assign precise values (even unknown ones) to all the quantities of a physical system, if they satisfy the algebraic relations prescribed by quantum theory. Nevertheless, it is possible to assign precise values to some quantities, as we will see in section 5.1 with hidden variables.

4.2 The Copenhagen interpretation

Bohr’s and Heisenberg’s ideas about the interpretation of quantum theory were developed in the 30 years following the advent of the theory.

According to Bohr and Heisenberg, an atomic system has well-defined properties only in the context of a measurement, carried out by means of an apparatus necessarily described by the classical theory. Thus the problem of measurement stated above does not arise, since one cannot attribute a state vector to the apparatus. The problem of measurement gives way to the problem of the distinction between the classical and the quantum: for example, under what circumstances, or from what size, mass or level of complexity does an aggregate of atoms or molecules satisfy the laws of the classical theory rather than those of the quantum theory?

The Heisenberg uncertainty principle, a pillar of the Copenhagen interpretation, asserts that no state vector allows to predict both the result of the measurement of the position and the result of the measurement of the momentum of a particle such as an electron. Moreover, no apparatus can simultaneously measure both quantities. According to Bohr, this in no way implies that the quantum theory is incomplete:

Although the phenomena in quantum physics can no longer be combined in the customary manner, they can be said to be complementary in that sense that only together do they exhaust the evidence regarding the objects, which is unambiguously definable. (Bohr, 1998, p. 130).

Thanks to complementarity, the quantum theory gives a complete and objective description of an atomic system.

The objective character of the description was also emphasized by Heisenberg:
The probability function [...] contains statements about possibilities or better tendencies ("potentia" in Aristotelian philosophy), and these statements are completely objective, they do not depend on any observer. (Heisenberg, 1958, p. 27).

As Popper himself noted, Heisenberg’s “tendencies” are akin to propensity. The probabilities depend only on the state vector and the experimental configuration. Thus, the Copenhagen interpretation fits well within the propensity interpretation of probability.

4.3 The observer

In our presentation, state vector collapse occurs when a physical quantity associated with an atomic system is measured with a macroscopic apparatus. Several investigators (London & Bauer, 1939; Wigner, 1961) have suggested that the collapse occurs at the moment a conscious subject becomes aware of the outcome. This idea is less popular today, but the intuition that an observer is necessary, if not for physical collapse, at least for the formulation of the theory, is alive and well. It was formulated in a precise way by Rudolph Peierls:

> In my view the most fundamental statement of quantum mechanics is that the wavefunction, or more generally the density matrix, represents our knowledge of the system we are trying to describe. (Peierls, 1991, p. 19).

Peierls believes that this assumption allows to view collapse in a completely different light. Collapse is no longer a physical process, but corresponds rather to a change in information. There is therefore no reason for it to obey the Schrödinger equation.

What is the nature of the observer we have in mind here? Of course, different concrete individuals can have more or less complete knowledge about

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7It should be pointed out here that Heisenberg’s text quoted above goes on as follows: “[The probability function also contains] statements about our knowledge of the system, which of course are subjective in so far as they may be different for different observers. In ideal cases the subjective element in the probability function may be practically negligible as compared with the objective one. The physicists then speak of a ‘pure case’.” This remark can apply to situations where the observer might have only a partial knowledge of the preparation of the state of the atomic system, which he must then describe by a density operator. Heisenberg’s remark can also anticipate the discussion in the next section.
the atomic system. Is there an ideal observer, whose knowledge would be maximal and more correct than that of all the others? Although Peierls does not answer this question precisely, his argument seems to call more for a positive answer. If so, the logical interpretation of probability seems to best represent Peierls’ view.

Others, however, do not accept the idea of an ideal observer. This is the case of QBism, or quantum Bayesianism. According to this point of view, quantum mechanics is a tool anyone can use to evaluate, on the basis of one’s past experience, one’s probabilistic expectations for one’s subsequent experience. (Fuchs, Mermin, & Schack, 2014, p. 749).

Unlike in the Copenhagen interpretation, there is no distinction here between the classical and the quantum. The distinction lies between the agent and the rest of the world. Everything outside an agent A (including other agents) constitutes a quantum system for agent A. It is the same for any other agent. Different agents generally have different beliefs about a given atomic system. Thus probabilities are assigned to an event by an agent […] and are particular to that agent. The agent’s probability assignments express her own personal degrees of belief about the event. (Fuchs, Mermin, & Schack, 2014, p. 749).

Therefore, the QBist position, that quantum states are personal judgments of an agent, is an inevitable consequence of the subjective view of probability expressed so eloquently by Bruno de Finetti. (Fuchs, Mermin, & Schack, 2014, p. 753).

Note in closing the pragmatic, or instrumentalist, character of Peierls’ and QBism’s views. Their critics believe that these approaches do not provide a sufficiently realistic description of quantum systems and therefore fail to solve the measurement problem.
5 Deterministic interpretations of quantum theory

5.1 The theory of Bohm and de Broglie

The formalism of quantum theory, based on the Schrödinger equation and the Born rule, does not allow to predict with certainty the result of the measurement of a physical quantity. Yet Bohr and Heisenberg have consistently argued that the formalism is complete. If one cannot predict the result of the measurement of a quantity it is, they claimed, because the quantity does not have a well-defined value before the measurement.

Quite early, however, other investigators (Louis de Broglie and Albert Einstein in particular) adopted a different attitude. According to them, the reason why quantum theory does not make unique predictions is that it gives only an incomplete description of an atomic system. In principle, so they say, a more complete description of the system is possible. The complete description would include, in addition to the state vector, various parameters whose knowledge would allow to make unique predictions. These are usually called hidden variables, since their values do not follow from those of the state vector.

As early as 1927, de Broglie proposed a theory of hidden variables. It really took off in 1952, when David Bohm answered the objections to which it had given rise. This theory postulates the existence of a preferred physical quantity, in this case position. An electron, for example, always has a well-defined position which, however, cannot be known accurately. Position is the theory’s hidden variable and, together with the state vector, it gives a complete description of the quantum system.

Let us see briefly how, for a particle like the electron, the theory of Bohm and de Broglie is formulated. The dynamics of the electron is governed by two equations: on the one hand the Schrödinger equation, which determines the temporal evolution of the wave function \( \psi(\vec{r}, t) \); on the other hand the equation of motion, which is given by

\[
m\vec{v} = \hbar \vec{\nabla} S
\]

Here \( m \) is the mass and \( \vec{v} \) is the velocity of the electron, while \( S \) is the phase.

\[8\text{Strictly speaking, the theory of a single particle must be derived from the theory of all particles in the universe, as explained in detail by Dürr, Goldstein, & Zanghí (1992).}\]
of the complex wave function. The trajectory of the electron is entirely determined by the wave function, which de Broglie called the pilot wave.

How can we, by means of a perfectly deterministic theory, recover the statistical predictions of the quantum theory? The answer is related to the state preparation process. The Bohm and de Broglie theory assumes that the preparation of an electron in the $\psi$ state is incompatible with the specification of its position. Specifically, the theory assumes that of the position of an electron in the $\psi$ state, we only know the probability density, equal to $|\psi(\vec{r}, t)|^2$. Clearly, we are dealing here with a purely subjective probability: the theory is deterministic, but our lack of knowledge of the initial conditions compels us to use probabilities. These probabilities, however, do not depend on the subjectivity of a particular agent. They are related to an ideal agent who knows the wave function exactly. They are thus associated with the logical interpretation of probability.

It can be shown that the hypothesis that the position is distributed according to the absolute square of the wave function exactly reproduces all the statistical predictions of the quantum theory. The hypothesis is also true at any time if it is true at a given time. Specifically, if the position is distributed according to $|\psi(\vec{r}, t_0)|^2$ at time $t_0$, then equation (5) and the Schrödinger equation imply that it will be distributed according to $|\psi(\vec{r}, t)|^2$ at time $t$.

Note finally that Bohm’s and de Broglie’s theory solves the measurement problem without appealing to an assumption such as collapse. At the end of the interaction between the atomic system and the apparatus, the state vector is well represented by equation (4). Nevertheless, the position of the atomic system and those of the particles of the apparatus are always well-defined, and are concentrated in only one of the terms of (4). For all practical purposes, one can account for the subsequent evolution of the global system by retaining only this one term.

5.2 Many worlds

In the theory of Bohm and de Broglie, the wave function of a quantum system always evolves unitarily. At the end of a measurement, however, all but one of the terms of the superposition vanish where the particles are located.

In a very different approach, Hugh Everett also proposed, in 1957, that every quantum system always evolves unitarily. Thus, there is no physical process that, just like collapse, would transform a system described by (4)
into a system described by only one term of the sum. Rather, Everett boldly assumes that all the terms of the sum correspond to real systems.

Let’s see more precisely what this means. An apparatus measures a physical quantity associated with an atomic system, a quantity that can take $N$ distinct values. Just before the measurement, there is a system, an apparatus and, say, a human observer, all of which, according to Everett, are described by quantum theory. At the end of the measurement there will be $N$ systems, $N$ apparatus (each pointing to a distinct outcome) and $N$ observers (each observing a distinct outcome). In other words, the initial world has split into $N$ different worlds, any one as real as any other.

What we have just described is called the many-worlds theory. Not everyone interprets Everett’s hypothesis in such a radical way. Some consider rather a splitting of the observer’s consciousness, others the formation of decoherent sectors of the wave function. In fact, the nature of multiplicity is a significant problem in Everett’s approach (Marchildon, 2015). We will focus on the many-worlds theory because it is particularly clear, and lends itself well to probability analysis.

At first glance, probability seems completely foreign to Everett’s approach. Everything, in fact, seems certain. Before the measurement, the observer notices that the apparatus is in the neutral position. If she knows quantum theory, and believes in Everett’s approach, she is certain that she will split into $N$ copies of herself, each one recording that the apparatus indicates a specific value. After measurement, each copy of the observer (say $O_j$) will note that the apparatus indicates the corresponding value (here $\alpha_j$).

Nevertheless, probability can be introduced by the following argument, proposed by Lev Vaidman (1998). Suppose that throughout the measurement process the observer is in a deep slumber. She wakes up only when the measurement is completed, and does not immediately take note of the value indicated by the apparatus. At this moment, she does not know in which of the many worlds she is. She doesn’t know what is the value indicated by the apparatus located in the same world as her. To the question “What value does your apparatus indicate?” she cannot give a categorical answer. She can only say (if she knows quantum theory) that the probability that the apparatus indicates the value $\alpha_j$ is equal to $|c_j|^2$. When she later notes the result, she will of course have to correct her judgment.

Thus, probability also finds a place in Everett’s approach. Just as in the theory of Bohm and de Broglie, we are dealing with a subjective probability. As it is the judgment of an ideal observer, the logical interpretation of
probability is also appropriate here.

6 Conclusion

There is no doubt that, among all physical theories, quantum theory is the one where problems of interpretation are the most acute. Probability theory is also characterized, in the field of mathematics, by the diversity of its interpretations. The problem, however, arises in different ways in the two cases.

The main purpose of an interpretation of quantum theory is to clarify how the formalism can account for the measurement of properties of an atomic system. These interpretations all make distinct assumptions and in that sense are mutually contradictory. If the state vector collapses in the von Neumann manner, it cannot always evolve in a unitary way as in the case of the Bohm and de Broglie theory. And if all the results of a measurement coexist, there can be no univocal quantity like position. The interpretations can all try to answer the question “How can the world be for the quantum theory to be true?”, but these answers are mutually exclusive, and only one can correspond to the real world. By contrast, there is no need to give a single interpretation of probability theory.

| Quantum theory       | Probability     |
|----------------------|-----------------|
| Collapse             | Propensity      |
| Copenhagen           | Propensity      |
| Epistemic (Peierls)  | Logical         |
| QBism                | Subjective      |
| Bohm and de Broglie  | Logical         |
| Many worlds          | Logical         |

Table 1: The relation between the interpretation of quantum theory and the interpretation of probability

In this article, we have examined different interpretations of quantum theory and have pointed out, in each case, the interpretation of probability that seems most adequate. These relationships are summarized in Table 1. We will not claim that the interpretation of probability solves the conceptual
problems of quantum theory. However, the clarification of the links between one and the other allows to state them in a clearer way.

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