Sample-Efficient Low Rank Phase Retrieval

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Abstract

In this paper we obtain an improved guarantee for the Low Rank Phase Retrieval (LRPR) problem: recover an \(n \times q\) rank-\(r\) matrix \(X^*\) from \(y_k := \|A_k^* x_k^*\|, k = 1, 2, \ldots, q\), when each \(y_k\) is an \(m\)-length vector containing the \(m\) phaseless linear projections of column \(x_k^*\). We study the AltMinLowRaP algorithm which is a fast non-convex LRPR solution developed in our previous work. We show that, as long as the right singular vectors of \(X^*\) satisfy the incoherence assumption, and the measurements’ matrix \(A_k\) consists of i.i.d. real or complex standard Gaussian entries, we can recover \(X^*\) to \(\epsilon\) accuracy if \(mq \geq C n r^3 \log(1/\epsilon)\) and \(m \geq C \max(r, \log q, \log n) \log(1/\epsilon)\). Thus the per column (per signal) sample complexity is only \(\max(\frac{n^2}{\mu^3}, r, \log q, \log n) \log(1/\epsilon)\). This improves upon our previous work by a factor of \(r\); moreover our previous result only allowed real-valued Gaussian measurements. Finally, the above result also provides an immediate corollary for the linear (with phase) version of the LRPR problem, often referred to in literature as “PCA via random projections” or “compressive PCA”.

I. INTRODUCTION

The generalized phase retrieval (PR) problem – recover an \(n\)-length signal \(x^*\) from measurements \(y := \|A^* x^*\|\) where \(A\) is a known \(n \times m\) matrix – has been extensively studied in the last decade \[11, 2, 3\]. Here ‘\(\cdot\)’ denotes (conjugate) transpose and \(|\cdot|\) denotes element-wise magnitudes. Recent works \[4, 5, 6, 7\] have developed provably correct and fast recovery algorithms for PR that can both achieve order-optimal sample complexity (require \(m \geq C n\)); and work in near linear time, \(C n m \log(1/\epsilon)\), when \(A\) contains independent identically distributed (i.i.d.) standard Gaussian entries; \[4, 6, 7\] assume real-valued Gaussian entries in \(A\), while \[5\] assumes complex-valued Gaussians. Here and below \(C\) denotes a different numerical constant in each use.

The only way to reduce the sample complexity \(m\) to less than \(n\) is by imposing assumptions on \(x^*\). Sparsity is a commonly used assumption. The sparse PR problem (recover an \(s\)-sparse signal \(x^*\) from \(y := \|A^* x^*\|\)) has received significant attention in recent years \[8, 9, 2, 10, 11, 12\]. Low rank is another common assumption. As explained in \[13, 14\], the practical way to impose it is to consider joint recovery of a set \(q\) of correlated signals, that together form an (exactly or approximately) low rank matrix, from \(m\) different phaseless linear projections of each of the \(q\) signals. This model is useful to enable fast and low-cost dynamic phaseless imaging applications, such as dynamic Fourier ptychography, where measurement acquisition is slow or expensive \[15\].

1) **Problem**: Low Rank PR (LRPR) involves recovering an \(n \times q\) rank-\(r\) matrix \(X^*\) from

\[
y_k := \|A_k^* x_k^*\|, \quad k \in [q],
\]

when the \(A_k\)’s are \(n \times m\) i.i.d. matrices with each containing i.i.d. (real-valued or complex-valued) standard Gaussian entries. Here, \([q] := \{1, 2, \ldots, q\}\) and \(x_k^*\) is the \(k\)-th column of \(X^*\). Thus each scalar measurement \(y_{ik}\) satisfies

\[
y_{ik} := |\langle a_{ik}, x_k^* \rangle|, \quad i \in [m], \quad k \in [q].
\]

The above problem with \(|\cdot|\) removed is commonly referred to as “compressive PCA” or PCA via random projections. Both LRPR and compressive PCA need the right singular vectors of \(X^*\) to satisfy the incoherence assumption defined next \[16, 17\].

**Assumption 1.1** (Right incoherence). Let \(X^* \xrightarrow{\text{SVD}} U^* \Sigma^* B^*\) denote its reduced (rank \(r\)) SVD\(^1\), \(\kappa = \frac{\sigma^*_r}{\sigma^*_\min}\) its condition number. Also let \(B^* := \Sigma^* B^*\).

Assume that \(\max_k \|b_k^*\|^2 \leq \mu^2 \frac{\|x_k^*\|}{q}\), with \(\mu \geq 1\) being a constant. Clearly, this implies that \(\max_k \|x_k^*\|^2 \leq \kappa^2 \mu^2 \frac{\|X^*\|_F^2}{q}\).

\(^1\)This notation is a bit non-standard, if the SVD was \(U^* \Sigma^* V^*\), we are letting \(B^* := V^*\). Thus, columns of \(U^*\) and rows of \(B^*\) are orthonormal.
2) **Notation:** We use \( \text{dist}(x^*, \hat{x}) := \min_{\theta \in \mathbb{R}} |x^* - e^{-i\theta} \hat{x}| \) to denote the standard phase invariant distance. For real-valued data, \( \text{dist}(x^*, \hat{x}) = \min(\|x^* - \hat{x}\|, \|x^* + \hat{x}\|) \). We say \( \hat{X} \) is an estimate of \( X^* \) with \( \epsilon \) accuracy if \( \sum_{k=1}^q \text{dist}(x_k^*, \hat{x}_k)^2 \leq \epsilon \|\mathcal{X}\|^2 \). To quantify the distances between \( r \)-dimensional subspaces of \( \mathbb{R}^n \), represented by their \( n \times r \) basis matrices \( U_i \) (matrices with orthonormal columns), we use \( \text{SE}(U_1, U_2) := \| (I - U_1^* U_1^r) U_2 \| \). This measures the sine of maximum principal angle between the two subspaces. Here and below \( ||.||_F \) denotes the Frobenius norm, \( \|.\| \) denotes the induced \( l_2 \) norm, and \( \dagger \) denotes (conjugate) transpose. For a complex number, \( z, \bar{z} \) denotes the complex conjugate and we misuse the term “phase” to refer to the number divided by its magnitude, \( \text{phase}(z) := z/|z| \) as also done in earlier works on PR, e.g., [2]. An \( n \)-length vector \( a \) is a real-valued standard Gaussian if \( a \sim \mathcal{N}(0, I) \) (the entries are zero mean, unit variance and mutually independent). An \( n \)-length vector \( a \) is a complex-valued standard Gaussian if \( a = a_{\text{real}} + ja_{\text{imag}} \) with \( a_{\text{real}}, a_{\text{imag}} \) being mutually independent, and \( a_{\text{real}} \sim \mathcal{N}(0, 0.5I) \) and \( a_{\text{imag}} \sim \mathcal{N}(0, 0.5I) \). Lastly, we use \( \sum_{ik} \) as short for the double summation \( \sum_{k=1}^q \sum_{i=1}^m \). Similarily \( \sum_{k} \) is short for \( \sum_{k=1}^q \).

3) **Related Work and Our Contributions:** LRPR was first studied in [13] where we introduced an alternating minimization (AltMin) algorithm and analyzed its initialization step. In recent work [14] (and shorter version in [18]), we developed the first provably correct LRPR solution, that we called AltMinLowRaP (AltMin for Low Rank PR). We also showed extensive numerical experiments that demonstrated the practical power of AltMinLowRaP. Our guarantee from [14] showed that, if right incoherence holds, if a new set of samples was used in each iteration, and for each update of \( U \) and \( B^* \) (sample-splitting), and if the total number of samples \( mq \geq C_{\kappa, \mu} n r^4 \cdot \log(1/\epsilon) \), and if \( m \geq C \max(r, \log q, \log n) \), one can recover \( X^* \) to \( \epsilon \) accuracy by using AltMinLowRaP. Here \( C_{\kappa, \mu} = C n^{10} \mu^4 \). Moreover, AltMinLowRaP is also fast, i.e., it converges geometrically.

In this work, (i) we show that the sample complexity of LRPR can be reduced to \( mq \geq C_{\kappa, \mu} n r^3 \cdot \log(1/\epsilon) \). To obtain this new result, we replace our previous upper bound on the phase error term by an improved one that does not rely on the Cauchy-Schwarz inequality. See Lemma 3.7 (ii) Secondly, in [14], we only considered real-valued Gaussian measurements. In this work, we extend our result to also handle complex-valued Gaussian measurements. (iii) As an immediate corollary, our result also provides a similarly improved guarantee for PCA via random projections / “compressive PCA”. To keep this paper compact, we do not show any new simulations here.

Notice that LRPR involves recovery from measurements that only depend on individual columns of \( X^* \) and not on the entire \( X^* \). This non-global measurement setting is what makes LRPR a more difficult problem than sparse PR for which each \( y_i \) is a function of the entire sparse signal \( x^* \). For the same reason, it is also more difficult than low rank matrix sensing [17].

In this sense, low rank matrix completion (LRMC) is the most closely related linear setting to LRPR that is well-studied. LRMC involves recovery from row-wise and column-wise local measurements while LRPR measurements are row-wise local but column-wise global. In order to allow for correct “interpolation” across rows and columns, LRMC needs an incoherence (denseness) assumption on its left and right singular vectors [16, 17]. Since our measurements are column-wise global, we need such an assumption on only the right singular vectors (Assumption 1.1).

The number of degrees of freedom in a rank-\( r \) \( n \times q \) matrix is \( (n+q)r \). Thus, even our current sample complexity is clearly sub-optimal. However, we should point out that, for non-convex solutions to the two related problems – sparse PR (phaseless but global measurements) and LRMC (linear but non-global measurements) – that have been extensively studied for nearly a decade, the best guarantees are still sub-optimal: ignoring log factors, the best non-convex LRMC result [19] requires \( mq \) to be of order \( (n+q)r^2 \); while the best sparse PR one [11] requires \( m \) to be of order \( ns^2 \). Compared to these results, our result for LRPR via AltMinLowRaP (which is also a non-convex solution) is sub-optimal by a factor of only \( r \).

The linear version of LRPR, PCA via random projections or “compressive PCA”, has received little attention until recently. As explained in [14], there have been some older attempts to develop a solution and try to analyze sub-parts of it [20, 21, 22]. AltMinLowRaP [14, 18] can be understood as the first provably correct algorithm for this problem. In more recent work [23], a provable convex optimization approach was developed. In our notation, this approach needs right incoherence and \( mq > \frac{r(n+q) \log(n+q)}{\kappa^2} \) to achieve \( \epsilon \) accuracy. The comparison with our current result is as follows: (i) AltMinLowRaP is significantly faster: its time complexity depends logarithmically on \( 1/\epsilon \) while that of convex approaches depends linearly on \( 1/\epsilon \). (ii) The same is true also for the sample complexity dependence: theirs depends linearly on \( 1/\epsilon^2 \) while ours depends only logarithmically. But their sample complexity has a near optimal dependence on \( r \) while ours is sub-optimal by a factor of \( r^2 \). In summary, our sample complexity is better if \( \epsilon < 1/r \).
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guarantee for noisy standard PR and since it was used in our earlier work [14] from which we borrow some lemmas. RWF

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Algorithm 1 AltMinLowRaP (AltMin for Low Rank PR). PR refers to the algorithm used for solving the standard PR

problem.

1: Parameters: $T$, $T_{PR,t}$, $\omega$.
2: Partition the $m_{tot}$ measurements and design vectors for each $x_k^*$ into one set for initialization and $2T$ disjoint sets for the

main loop.
3: Set $\hat{r}$ as the largest $j$ for which $\lambda_j(Y_U) - \lambda_0(Y_U) \geq \omega$,

$$Y_U = \frac{1}{mq} \sum_{k=1}^{q} \sum_{i=1}^{m} y_{ik}^2 a_{ik} a_{ik}^* \{ y_{ik}^2 \leq \sigma^2 \nu^2 \frac{1}{mq} \sum_{ik} y_{ik}^2 \}.$$ 

4: $U^0 \leftarrow \hat{U}^0 \leftarrow \text{top } \hat{r} \text{ singular vectors of } Y_U$.
5: for $t = 0 : T$
do
6: $\hat{b}_k^t \leftarrow \text{PR}(y_{ik}^{(t)}, (U^t)'A_k^{(t)}, T_{PR,t}), k \in [q]$.
7: $\hat{x}_k^t \leftarrow U^t \hat{b}_k^t, k \in [q]$.
8: $\hat{c}_{ik} \leftarrow \text{phase}(a_{ik}^{(t)} \hat{x}_k^t)$, $i \in [m], k \in [q]$.
9: Get $B^t$ by QR decomp: $\hat{B}^t \leftarrow QR \hat{R}_U^t B^t$.
10: $\hat{U}^{t+1} \leftarrow \text{arg min}_U \sum_{k=1}^{q} \| \hat{C_k} y_{ik}^{(t+1)} - A_k^{(t+1)} \hat{U} \hat{b}_k^t \|^2$.
11: Get $U^{t+1}$ by QR decomp: $\hat{U}^{t+1} \leftarrow QR (U^{t+1} \hat{R}_U^t \hat{B}^t)$.
end for

4) Organization: We briefly explain the AltMinLowRaP algorithm and state and discuss our new result for it in Sec. III.

We prove the result for real-valued measurements in Sec. III. We explain the changes needed for the complex case in Sec.

IV. We conclude in Sec. V

II. MAIN RESULT

Before we state our result, we should point out that the real-valued Gaussian measurements’ case is not directly a special
case of the complex-valued case. But, it can converted into a special case if we use the squared sum of two real measurements as “one” complex measurement squared.

We study the AltMinLowRaP algorithm [14] summarized in Algorithm 1. AltMinLowRaP can be understood as truncated

spectral initialization (line 3), followed by minimizing $\sum_{k=1}^{q} \| y_k - |A_k'Ub_k| \|^2$ alternatively over $U, B$ with the constraint that

$U$ is a basis matrix. Each of the two minimizations involves recovery from phaseless measurements, but the two problems

are quite different (as explained in detail in [14]). A simpler way to understand the approach is to split it into a three-way

AltMin problem over $U^*, \hat{b}_k^*$, and $c_{ik}^*$ where $c_{ik}^* := \text{phase}(a_{ik} x_k^*)$. This discussion assumes “sample-splitting” (line 2). (1) At each new iteration, given an estimate of range($U^*$), denoted $U$, we recover the $\hat{b}_k^*$’s, by solving easy individual $r$-dimensional noisy standard PR problems. To understand this step, let $g_k := U'x_k^* = U'U^* \hat{b}_k^*$. We can rewrite $y_{ik}$ as

$$y_{ik} = |\hat{a}_{ik}' g_k + a_{ik}' (I - UU') x_k^*| := |\hat{a}_{ik}' g_k| + \nu_{ik}$$

where $\hat{a}_{ik} := U' a_{ik}$. Due to sample splitting, $U$ is independent of $a_{ik}$’s and so $\hat{a}_{ik}$’s are still i.i.d. standard Gaussian.

Thus, recovering $g_k$ from $y_{ik}, i = 1, 2, \ldots, m$ is an $r$-dimensional noisy standard PR problem with noise $\nu_{ik}$ satisfying

$|\nu_{ik}| \leq |a_{ik}' (I - UU') x_k^*|$. Using the sub-exponential Bernstein inequality [24], it can be shown that, w.h.p., $|\nu_{ik}| \leq \sqrt{T \text{ TSE}(U, U^*)} x_k^* ||x_k^*||$, i.e., the noise is proportional to the error in $U$.

(2) Given a good estimate of $\hat{b}_k^*$ (or actually of $g_k$) and of $U^*$, we can get an equally good estimate of $x_k^*$ and hence of

the measurement phases $c_{ik}^*$. (3) Finally, we can obtain a new estimate of $U^*$ using the estimates of $\hat{b}_k^*$’s and of the $c_{ik}^*$’s. This is a Least Squares (LS) problem; see line 10. Its output may not have orthonormal columns. So we use QR after it (line 11).

In case of real-valued Gaussian measurements, for the standard PR step for recovering the $\hat{b}_k^*$’s, we can use any algorithm

with order-optimal sample complexity: TWF [4] or RWF [6] or AltMin [5]. We assume RWF is used since it already has a

guarantee for noisy standard PR and since it was used in our earlier work [14] from which we borrow some lemmas. RWF

and TWF guarantees are only for the real-valued case. For the complex-valued case, we assume that the AltMin with truncated
spectral initialization (AltMin-TSI) algorithm from [5] is used. The guarantee for this approach assumes noise-free data, but its extension to the noisy case is straightforward (see Appendix [B]).

We can prove the following for LRPR via AltMinLowRaP.

**Theorem 2.1** (Real or Complex Gaussian measurements). Consider Algorithm [7] Assume that Assumption [7] holds. Set $T := C \log(1/\epsilon)$, $T_{PR,k} = C \log r + r \log \kappa + c t$, $\omega = 1.3 \sigma_{\min}^{2}/q$. Assume that, for the initialization step and for each new update, we use a new set of $m$ measurements with $m \leq C \kappa^{10} \mu^{4} \cdot nm^{3}$ and $m \geq C \max(r, \log q, \log n)$. Then, with probability (w.p.) at least $1 - Cn^{-10}$, in at most $T = C \log(1/\epsilon)$ iterations,

\[
SE(U^{*}, U^{T}) \leq \epsilon, \quad \text{dist}(\hat{x}^{T}, x^{*}) \leq \epsilon\|x^{*}\|, \quad k \in [q].
\]

The time complexity is $mqnr \log^{2}(1/\epsilon)$.

**Corollary 2.2.** Compressive PCA (PCA via random projections) solved using AltMinLowRaP has the same performance and time-complexity guarantees as those given in Theorem 2.1 for LRPR.

Theorem 2.1 implies that one can achieve geometric convergence as long as the sample complexity $m_{\text{tot}} := (2T + 1)m$ satisfies $m_{\text{tot}}q \geq C \kappa^{10} \mu^{4} n^{3} \log(1/\epsilon)$ along with $m_{\text{tot}} \geq C \max(r, \log q, \log n) \log(1/\epsilon)$. The second lower bound is redundant, except when $q \geq C n^{2}$. This result improves the sample complexity by a factor of $r$ compared to [14].

For the linear case (PCA via random projections), for accuracy levels $\epsilon < 1/r$, Corollary 2.2 provides the best sample complexity and computational complexity guarantee. As explained earlier, the only other existing complete guarantee for it is a convex optimization solution [23] that is slower and that needs $mq > (n + q)r/\epsilon^{2}$ times log factors.

**A. Explaining our sample complexity**

Assuming $q = n$, the order-optimal sample complexity for low-rank matrix recovery is $nr$. However, as noted earlier, for iterative (non-convex) solutions to problems that have non-global measurements such as LRMC, the best known sample complexity after more than a decade of research is $nr^{2}$ times log factors. Our problems and solution approach fall in this category too. The reason for the extra factor of $r$ is that, conditioned on $X^{*}$, the measurements are not identically distributed. To analyze each iteration of the algorithm, one needs to exploit the incoherence assumption to show that the distributions are “similar” enough so that the concentration bound (sub-exponential Bernstein in our case) can be applied jointly for the sum over all $mq$ samples (or their functions). This step introduces an extra factor of $1/r$. Thus, at each iteration we need $mq \geq C \frac{nr^{2}}{\delta_{t}}$ where $\delta_{t}$ is the subspace error bound from the previous iteration.

The second extra factor of $r$ in our sample complexity is because of our AltMin step of updating $U^{*}$. We obtain a new estimate of $U^{*}$ by solving a LS problem to recover its vectorized version (the $nr$ length vector $U_{\text{vec}}^{*}$). As a result the recovery error is proportional to the 2-norm of an $nr$ length vector. But this is equal to the Frobenius norm of the corresponding reshaped $n \times r$ matrix. Because of this, eventually when we bound the subspace recovery error at iteration $t + 1$, the numerator contains terms of the form $\delta_{t}^{2}\|X^{*}\|_{F} \leq \delta_{t}^{2}\sqrt{\tau} \sigma_{\min}^{*}$ while the denominator contains $\sigma_{\min}^{*}$. Thus, in order to ensure exponential decay of the subspace error bound, we need $\delta_{t} < c/\sqrt{\tau}$. As a result the sample complexity bound becomes $mq \geq C \frac{nr^{2}}{\delta_{t}} \geq Cnr^{3}$. Here we treated $\kappa$ and $\mu$ as constants.

It may be possible to improve the LRPR, and compressive PCA, sample complexity to $nr^{2}$ if we developed a projected gradient descent approach similar to the one used in [19].

**III. PROOF OF THEOREM 2.1 REAL MEASUREMENTS**

We first prove our result for the case of real measurements. This proof is simpler and hence illustrates the main ideas easily. In this case, the phase uncertainty gets replaced by a simpler sign uncertainty in the measurements, and $\text{dist}(\tilde{x}, x^{*}) = \min(\|x^{*} - \tilde{x}\|, \|x^{*} + \tilde{x}\|)$. Since we can only recover $x^{*}_{k}$ modulo its sign, as is done past works on PR, e.g., [2], without loss of generality, we can assume that $x^{*}_{k}$ is replaced by $\text{sign}(x^{*}_{k}, \tilde{x}_{k})x^{*}_{k}$. With this, $\text{dist}(x^{*}_{k}, \tilde{x}_{k}) = \|x^{*}_{k} - \tilde{x}_{k}\|$. 


A. Main Lemmas

The SE bound of the theorem follows by combining the two claims given next. The bound on \( \text{dist}(\mathbf{x}_k^*, \hat{\mathbf{x}}_k^T) \) follows by Lemma 3.8.

Claim 3.1 (Rank estimation and Initialization of \( U^* \). Claim 3.1 of [14]). Let \( U_{\text{init}} = \hat{U}^0 \). Pick a \( \delta_0 < 0.25 \). Set the rank estimation threshold \( \omega = 1.3\sigma_{\min}^2 / q \). Then, w.p. at least \( 1 - 2 \exp \left( \frac{nr \log(17) - \frac{3\delta_{mq}}{\sqrt{c_0^2 \mu^2}}}{2} \right) \), the rank is correctly estimated and \( \text{SE}(U_{\text{init}}) \leq \delta_0 \).

Claim 3.2 (Error decay). At iteration \( t \), assume that \( \text{SE}(U^*, U^t) \leq \delta_t \). If \( \delta_t \leq \frac{c}{\sqrt{r}} \), then w.p. at least \( 1 - 4 \exp \left( nr \log(17) - \frac{\delta_{mq}}{\sqrt{c_0^2 \mu^2}} \right) - 2 \exp \left( -nr \log(q + r \log(17) - cm) \right) \), \( \text{SE}(U^t + 1, U^*) \leq 0.7 \delta_t := \delta_{t+1} \).

Claim 3.3 follows by combining the five lemmas given next. Of these, the first four are taken from [14] while the fifth, Lemma 3.7, is the new contribution of this paper. We discuss it in Sec. [III-B] and prove it in Sec. [III-C].

Below, we remove the superscript \( ^t \) at most places. Also, we let \( a_{ik} := a_{i(k+1)} \) and \( y_{ik} := y_{i(k+1)} \).

Lemma 3.3 (Lemma 3.9 in [14]). We have

\[
\text{SE}(U^{t+1}, U^*) \leq \frac{\text{Main Term}}{\sigma_{\min}(U^* \Sigma^* B^* B^*)} - \text{Main Term}
\]

where \( \text{Main Term} := \max_{W \in S_\omega} \left| \text{Term1}(W) \right| + \max_{W \in S_\omega} \left| \text{Term2}(W) \right|, \)

\[
\text{Term1}(W) := \sum_{ik} b_k' W' a_{ik} a_{ik}' (\hat{B}^* B' b_k - \hat{b}_k^*),
\]

\[
\text{Term2}(W) := \sum_{ik} (c_{ik}' \hat{c}_{ik} - 1) (b_k' W' a_{ik})(a_{ik}' \hat{x}_k),
\]

\[
\text{Term3}(W) := \sum_{ik} |a_{ik}' W b_k|^2,
\]

\[
\text{S}_\omega := \{ W \in \mathbb{R}^{n \times r} : \|W\|_F = 1 \},
\]

and \( c_{ik}', \hat{c}_{ik} \) are the phases of \( a_{ik}' \hat{x}_k^* \) and \( a_{ik}' \hat{x}_k \).

Lemma 3.4 (Lemma 3.13 in [14]). Under the conditions of Theorem 2.7, if \( \text{SE}(U^*, U) \leq \delta_t \) with \( \delta_t \leq \frac{c}{\sqrt{r}} \), w.p. at least \( 1 - 2 \exp \left( (\log q + r \log(17) - cm) \right) \),

\[
\sigma_{\min}(U^* \Sigma^* B^* B^*) \geq 0.9 \sigma_{\min}.
\]

Lemma 3.5 (Lemma 3.10 in [14]). Under the conditions of Theorem 2.7, for a \( \delta < 0.1 \), w.p. at least \( 1 - 2 \exp \left( nr \log(17) - \frac{\delta_{mq}}{\sqrt{c_0^2 \mu^2}} \right) - 2 \exp \left( (\log q + r \log(17) - cm) \right) \),

\[
\min_{W \in S_\omega} \text{Term3}(W) \geq 0.5(1 - \delta)m
\]

and \( \max_{W \in S_\omega} \text{Term3}(W) \leq 1.5(1 + \delta)m \).

Lemma 3.6 (Lemma 3.11 in [14]). Under the conditions of Theorem 2.7 and assuming that \( \text{SE}(U^*, U) \leq \delta_t \), with \( \delta_t < 0.1 \), w.p. at least \( 1 - 2 \exp \left( nr \log(17) - \frac{\delta_{mq}}{\sqrt{c_0^2 \mu^2}} \right) - 2 \exp \left( (\log q + r \log(17) - cm) \right) \), we have

\[
\max_{W \in S_\omega} \text{Term1}(W) \leq C m \delta_2^2 \|X^*\|_F.
\]

Lemma 3.7 (New lemma). Under the conditions of Theorem 2.7 and assuming \( \text{SE}(U^*, U) \leq \delta_t \) with \( \delta_t < \frac{c}{\sqrt{r \mu}} \), w.p. at least \( 1 - 2 \exp \left( nr \log(17) - \frac{\delta_{mq}}{\mu^2} \right) - 2 \exp \left( (\log q + r \log(17) - cm) \right) \), we have

\[
\max_{W \in S_\omega} \text{Term2}(W) \leq C m \delta_2^2 \|X^*\|_F.
\]

All the above lemmas use the following lemma in their proofs. This analyzes the standard PR step to recover \( \hat{b}_k^* \)’s and uses it to (i) bound \( \text{dist}(\hat{x}_k^*, \mathbf{x}_k^*) \) (recall \( \hat{x}_k = U\hat{b}_k \)) and (ii) to show that the estimates \( \hat{b}_k \) satisfy incoherence as well.
Lemma 3.8 (Lemmas 3.2, 3.3 of [14]). Let \( g_k := U^* U_k \). Pick a \( \delta < 1 \). Under the conditions of Theorem 2.7 and assuming \( \text{SE}(U^*, U) \leq \delta t \), w.p. at least \( 1 - \exp \left( \log q + r \log(17) - c\delta^2 m \right) \), the following is true for each \( k = 1, 2, \ldots, q \): (i) \( \text{dist} \left( \hat{b}_k, g_k \right) \leq C\delta \| x_k^* \| \), (ii) \( \text{dist} \left( \hat{x}_k, x_k^* \right) \leq 2\delta \| x_k^* \| \), and (iii) \( \| b_k \| \leq \mu^2 r/q \) with \( \mu = C\kappa \), i.e., the estimates \( b_k \) are \( \hat{\mu} \)-incoherent.

Proof of Claim 3.2 By Lemmas 3.3, 3.4 if \( \delta_t < \frac{c}{\sqrt{tr}} \),

\[
\text{SE}(U_t^{t+1}, U^*) \leq \frac{0.9\sigma^*_{\min} - \text{MainTerm}}{
}
\]

Set \( \delta = 0.1 \). Combining the bounds on Term1, Term2, Term3 from above, and using \( \| X^* \|_F \leq \sqrt{\sigma^*_{\max}} \), we conclude that

\[
\text{MainTerm} \leq C\delta_t^2 \sqrt{\sigma^*_{\max}}.
\]

To ensure that \( \text{MainTerm} \leq 0.7\delta(\sigma^*_{\min}) \), we need to set \( \delta_t = c/\sqrt{tr} \) for all \( t \) including \( t = 0 \) (thus we also need to set \( \delta_0 = c/\sqrt{tr} \)). This, along with (3), finishes the proof.

B. Discussion of Lemma 3.7

Lemma 3.7 is the new contribution of this paper. It replaces Lemma 3.12 of [14] to bound the phase error term \( \text{Term2}(W) \). In [14 Lemma 3.12], we first showed that \( \sum_{ik} \left| (c_{ik} \hat{c}_{ik} - 1) (a_{ik} x_k^*) \right|^2 \leq C m \delta_t^2 \| X^* \|_F^2 \) and then used Cauchy-Schwarz and the bound on Term3 to obtain \( |\text{Term2}| \leq \sqrt{\text{Term3}} \cdot \sum_{ik} \left| (c_{ik} \hat{c}_{ik} - 1) (a_{ik} x_k^*) \right|^2 \leq C m \sqrt{1 + \delta_t^3} \| X^* \|_F \leq C m \delta_t^3 \sqrt{\sigma^*_{\max}} \). Because of this, our bound on \( \text{SE}(U_t^{t+1}, U^*) \) was \( C\delta_t^{3/2} \sqrt{r} \). Thus to get geometric convergence, we needed to assume \( \delta_t < c/\sqrt{tr} \).

However, in Lemma 3.7, we directly bound \( \text{Term2}(W) \) without using Cauchy-Schwarz as is done above. As a result, we are able to get a bound of the form \( C\delta_t^2 \| X^* \|_F \) on it. This is what ensures that \( \delta_t \leq c/\sqrt{r} \) suffices for ensuring geometric convergence. Since our bounds on Term1 and Term3 hold with probability \( 1 - 4 \exp(nr - cmq\delta_t^2/r) \), this then implies that \( mq \) of order \( nr^3 \) suffices for our main result to hold.

C. Proof of Lemma 3.7

In the real case, phase gets replaced by sign. Since \( c_{ik}^* |a_{ik} x_k^*| = a_{ik} x_k^* \) and since \( (c_{ik}^*)^2 = 1 \),

\[
\text{Term2}(W) = \sum_{ik} (\hat{c}_{ik} - c_{ik}^*) (b_k W' a_{ik}) |a_{ik} x_k^*|.
\]

Recall that \( a_{ik} := a_{ik}^{(T+t)} \) and same for \( y_{ik} \). Thus, these are independent of the current \( \hat{x}_k \)'s and \( b_k \)'s. We have the following bound on the expected value of \( \text{Term2}(W) \).

Lemma 3.9. For a given \( W \) independent of \( a_{ik} \)'s, \( |\mathbb{E}[\text{Term2}(W)]| \leq C m \delta_t^2 \| X^* \|_F \).

This lemma is proved in Sec. III-D. Here we use this lemma and Lemma A.1 from the Appendix (concentration bound for sums of products of sub-Gaussians, follows using [24] Lemma 2.7.7 and Theorem 2.8.1)) to show that, w.h.p., \( \text{Term2}(W) \) satisfies the same bound for any \( W \in \mathcal{S}_W \). We do this first for a fixed \( W \in \mathcal{S}_W \) and then use an epsilon net argument to extend it to the entire \( \mathcal{S}_W \). To apply Lemma A.1 we pick \( X_{ik} = (\hat{c}_{ik} - c_{ik}^*) |a_{ik} x_k^*| \) and \( Y_{ik} = a_{ik} W b_k \). Next we obtain bounds on their sub-Gaussian norms.

Let \( h_k = x_k^* - \hat{x}_k \). It is easy to see that \( \hat{c}_{ik}^* \neq \hat{c}_{ik} \) implies \( |a_{ik} x^*| < |a_{ik} h_k| \) [8]. Thus,

\[
|\hat{c}_{ik}^* - c_{ik}^* |a_{ik} x_k^*| \leq 2|a_{ik} h_k|
\]

\( ^{2} \)Cauchy-Schwarz is still used but in the very last step of bounding \( \mathbb{E}[\text{Term2}(W)] \).

\( ^{3} \) \( (a_{ik} x_k^*) (a_{ik} \hat{x}_k) < 0 \) implies \( (a_{ik} x_k^*) (a_{ik} x_k^*) - (a_{ik} x_k^*) (a_{ik} h_k) < 0 \) and hence \( |a_{ik} x_k^*| < |a_{ik} h_k| \).
From above, $|X_{ik}| \leq 2|a_{ik}'h_k|$ and thus its sub-Gaussian norm, $K_X \leq C\|h_k\|$. Also, $K_{Y_{ik}} = C\|W b_k\|$. Now we apply the lemma with $t = m\delta_t^2\|X^*\|_F$. We have

$$
\frac{t^2}{\sum_{ik} K_{X_{ik}} K_{Y_{ik}}} = \frac{m^2\delta_t^4\|X^*\|_F^2}{\sum_{ik} \|W b_k\|^2 \|h_k\|^2} \\
\geq \frac{m^2\delta_t^4\|X^*\|_F^2}{\max_k \|h_k\|^2 \sum_k \|W b_k\|^2} \\
\geq \frac{\mu^2\kappa^2\delta_t^2 \max_k \|W b_k\|^2}{m^2\delta_t^4 \|X^*\|_F^2} \\
\geq \frac{m\delta_t^2}{\mu \kappa \|W b_k\|} \geq \frac{m\delta_t^2}{\mu \kappa}. 
$$

The second inequality used Lemma 3.8 the third used right incoherence (Assumption 1.1); and the last equality used $\|WB\|_F = 1$ (follows since $\|W\|_F = 1$ and $BB' = I$). Similarly,

$$
\frac{t}{\max_{ik} K_{X_{ik}} K_{Y_{ik}}} = \frac{m^2\delta_t^4\|X^*\|_F}{\max_{ik} \|W b_k\|^2 \|h_k\|^2} \\
\geq \frac{m\sqrt{\mu \delta_t}}{\mu \max_k \|W b_k\|} \geq \frac{m\delta_t}{\mu \kappa \sqrt{\mu \delta_t}}. 
$$

The last inequality used $\|W b_k\| \leq \|b_k\| \leq \kappa \sqrt{\mu \delta_t}$ (by Lemma 3.8). Hence, if $\delta_t \leq C/(\kappa \sqrt{\mu \delta_t})$, the first term is the minimum of the above two terms. Thus, applying Lemma A.1 and using the bound on $\mathbb{E}[\text{Term2}(W)]$ from Lemma 3.9 for a fixed $W \in S_W$,

$$
\Pr \left\{ \left| \text{Term2}(W) \right| \leq Cm\delta_t^2\|X^*\|_F \right\} \geq 1 - 2 \exp \left( -\frac{c m \mu \delta_t^2}{\mu \kappa} \right).
$$

By Lemma 5.2 of [25] there exists a set (called $\epsilon$-net), $\tilde{S}_W \subset S_W$ so that, for any $W \in S_W$, there is a $\tilde{W} \in \tilde{S}_W$ such that $\|\tilde{W} - W\|_F \leq \epsilon$ and $|\tilde{S}_W| \leq (1 + \frac{2}{\mu})^n$. By picking $\epsilon = 1/8$ we have $|\tilde{S}_W| \leq (17)^n$. Define $\Delta W := W - \tilde{W}$ so that $\|\Delta W\|_F \leq \epsilon \leq \frac{1}{8}$. Using union bound for all entries in $\tilde{S}_W$,

$$
\Pr \left\{ \text{Term2}(W) \leq Cm\delta_t^2\|X^*\|_F \text{ for all } W \in \tilde{S}_W \right\} \\
\geq 1 - 2 \exp \left( nr \log(17) - \frac{c m \mu \delta_t^2}{\mu \kappa} \right).
$$

Next we extend this for the entire hyper-sphere, $S_W$. Define $\Gamma_W := \max_{W \in S_W} \{\text{Term2}(W)\}$. Since $\Delta W = \sum_{ik} (\hat{c}_{ik} - c_{ik}) (b_k' \Delta W' a_{ik}) |a_{ik}' x_k'| \leq \Gamma_W \|W\|_F \leq \epsilon \Gamma_W$.

Thus, w.p. at least $1 - 2 \exp \left( nr \log(17) - \frac{c m \mu \delta_t^2}{\mu \kappa} \right)$,

$$
\text{Term2}(W) = \sum_{ik} (\hat{c}_{ik} - c_{ik}) (b_k' (W + \Delta W)' a_{ik}) |a_{ik}' x_k'| \\
\leq Cm\delta_t^2\|X^*\|_F + \epsilon \Gamma_W.
$$

and so $\Gamma_W \leq Cm\delta_t^2\|X^*\|_F + \frac{\epsilon}{2} Cm\delta_t^2\|X^*\|_F$.

D. Proof of Lemma 3.9

Since $c_{ik}', \hat{c}_{ik}$ only take values $+1$ or $-1$, $\mathbb{E}[\text{Term2}] = \sum_{ik} \mathbb{E}[\text{Term2}_{ik}]$ with $\text{Term2}_{ik} = 0$ if $c_{ik}' = \hat{c}_{ik}$, $\text{Term2}_{ik} = 2(\hat{a}_{ik}'W b_k) |a_{ik}' x_k'|$ if $c_{ik}' = 1, \hat{c}_{ik} = -1$, and $\text{Term2}_{ik} = -2(\hat{a}_{ik}'W b_k) |a_{ik}' x_k'|$ if $c_{ik}' = -1, \hat{c}_{ik} = 1$.

For simplicity we will remove subscripts $ik$ from $a_{ik}$. Notice that every quantity in $\text{Term2}_{ik}$ is a function of a quantity of the form $a'z$ where $z$ is either $x_k^*, \hat{x}_k$ or $Wb_k$. All the three vectors are independent of $a$. Since $a$ is a standard
Gaussian, \( \mathbf{a} := \mathbf{Oa} \), with \( \mathbf{O} \) being any unitary matrix independent of \( \mathbf{a} \), is also standard Gaussian. Since \( \mathbf{OO'} = \mathbf{I} \), \( \mathbf{a}'z = \mathbf{a}'\mathbf{O} \mathbf{O}'z = \mathbf{a}'(\mathbf{O}'z) \). By carefully picking the matrix \( \mathbf{O} \) we can show that

\[
\mathbb{E}[\text{Term2}_{ik}'] = 2\mathbb{E}[\text{Term2}_{1ik}] - 2\mathbb{E}[\text{Term2}_{2ik}],
\]

where

\[
\text{Term2}_{1ik} = \mathbbm{1}_{\{a(1)>0, \alpha_k a(1)+\sqrt{1-\alpha_k^2}a(2)<0\}} \times (\beta_{1,W}a(1) + \beta_{2,W}a(2) + \beta_{3,W}a(3))|a(1)| \|x_k^*\| \|\mathbf{Wb}_k\|,
\]

\[
\text{Term2}_{2ik} = \mathbbm{1}_{\{a(1)<0, \alpha_k a(1)+\sqrt{1-\alpha_k^2}a(2)<0\}} \times (\beta_{1,W}a(1) + \beta_{2,W}a(2) + \beta_{3,W}a(3))|a(1)| \|x_k^*\| \|\mathbf{Wb}_k\|,
\]

\( \alpha_k = \frac{\langle \hat{x}_k, x_k^* \rangle}{\|\hat{x}_k\| \|x_k^*\|} \) and \( \beta_{j,W} \) for all \( j = 1, 2, 3 \) are scalars that depend on \( x_k^*, \hat{x}_k, \mathbf{Wb}_k \), and satisfy \( \beta_{j,W} \leq 1 \). Here, \( \mathbbm{1}_{\text{statement}} \) equals 1 if statement is true and equals 0 otherwise (indicator function); and \( a(j) \) refers to the \( j \)-th entry of \( \mathbf{a}_k \) (shortened to \( \mathbf{a} \) for simplicity). Also since we have assumed \( x_k^* \) is replaced by \( \text{sign}(a(1)\hat{x}_k) x_k^* \), we have \( \alpha_k \geq 0 \). The above simplification is motivated by similar ideas used in other works, e.g., [2].

Notice that the only difference between Term21ik and Term22ik is the indicator function. Since the distribution of \( \mathbf{a} \) is equal to that of \(-\mathbf{a}\) (both are standard Gaussians),

\[
\mathbb{E}[\text{Term2}_{2ik}'] = -\mathbb{E}[\text{Term2}_{1ik}].
\]

Using this and the fact that \( a(3) \) is independent of \( a(1), a(2) \), and is zero mean,

\[
\mathbb{E}[\text{Term2}_{2ik}] = 4\mathbb{E}[\text{Term2}_{1ik}] = 4\|\mathbf{Wb}_k\| \|x_k^*\| \times
\]

\[
\left( \beta_{1,W} \mathbb{E} \left[ a(1)^2 \mathbbm{1}_{\{a(1)>0, \alpha_k a(1)+\sqrt{1-\alpha_k^2}a(2)<0\}} \right] + \beta_{2,W} \mathbb{E} \left[ a(2) a(1) \mathbbm{1}_{\{a(1)>0, \alpha_k a(1)+\sqrt{1-\alpha_k^2}a(2)<0\}} \right] \right) .
\]

The rest of the proof needs the following two lemmas (proved in Appendix A).

**Lemma 3.10.** Assume \( a(1), a(2) \) are two independent standard Gaussian scalars and \( 0 \leq \alpha \leq 1 \). Then we have

\[
\mathbb{E} \left[ a(1)^2 \mathbbm{1}_{\{a(1)>0, \alpha a(1)+\sqrt{1-\alpha^2}a(2)<0\}} \right] \leq \frac{(1-\alpha^2)^2}{\alpha^3},
\]

and

\[
\mathbb{E} \left[ a(2) a(1) \mathbbm{1}_{\{a(1)>0, \alpha a(1)+\sqrt{1-\alpha^2}a(2)<0\}} \right] \leq \frac{(1-\alpha^2)}{\alpha^2}.
\]

**Lemma 3.11.** Consider two vectors \( \mathbf{x} \) and \( \hat{\mathbf{x}} \). If \( \text{dist}(\mathbf{x}, \hat{\mathbf{x}}) \leq 0.5\|\mathbf{x}\| \), then,

\[
1 - \frac{\|\mathbf{x} - \hat{\mathbf{x}}\|^2}{\|\mathbf{x}\|^2} \leq 2 \frac{\text{dist}(\mathbf{x}, \hat{\mathbf{x}})^2}{\|\mathbf{x}\|^2}.
\]

Recall that \( \alpha_k = \frac{\langle \hat{x}_k, x_k^* \rangle}{\|\hat{x}_k\| \|x_k^*\|} \). By Lemma 3.11 Lemma 3.8 and \( \delta_t < 0.1 \),

\[
\sqrt{1-\alpha_k^2} \leq 4\delta_t, \quad \text{and} \quad \alpha_k \geq \sqrt{1-16\delta_t^2} > 0.84 .
\]

---

4 Pick \( \mathbf{O} = [\mathbf{O}_1, \mathbf{O}_2, \mathbf{O}_3, \mathbf{O}_{rest}] \) with \( \mathbf{O}_1 = x_k^*/\|x_k^*\| \), \( \mathbf{O}_2 = (I-\mathbf{O}_1\mathbf{O}_1')\hat{x}_k/\|I-\mathbf{O}_1\mathbf{O}_1'\hat{x}_k\| \) and \( \mathbf{O}_3 = (I-\mathbf{O}_1\mathbf{O}_1'-\mathbf{O}_2\mathbf{O}_2')\mathbf{Wb}_k/\|I-\mathbf{O}_1\mathbf{O}_1'-\mathbf{O}_2\mathbf{O}_2'\mathbf{Wb}_k\| \) and \( \mathbf{O}_{rest} \) being an \( n \times (n-3) \) matrix that is such that \( \mathbf{O} \) is a unitary matrix. Then, \( \mathbf{a}'x_k^* = \mathbf{a}'(\|x_k^*\|e_1 = \hat{a}(1))\|x_k^*\| \), \( \mathbf{a}'\hat{x}_k = \hat{a}(1)\alpha_k\|\hat{x}_k\| + \hat{a}(2)\sqrt{1-\alpha_k^2}\|\hat{x}_k\| \) and \( \mathbf{a}'(\mathbf{Wb}_k) = \hat{a}(1)\beta_{1,W} + \hat{a}(2)\beta_{2,W} + \hat{a}(3)\beta_{3,W}\|\mathbf{Wb}_k\| \) where \( \alpha_k = \frac{\|\hat{x}_k\|}{\|x_k^*\|} \), \( \beta_{1,W} = \mathbf{O}_1'(\mathbf{Wb}_k)/\|\mathbf{Wb}_k\| \), \( \beta_{2,W} = \mathbf{O}_2'(\mathbf{Wb}_k)/\|\mathbf{Wb}_k\| \), and \( \beta_{3,W} = \sqrt{1-\beta_{1,W}^2-\beta_{2,W}^2} \). Clearly, \( \beta_{1,W} \leq 1 \), this is the only fact we will use. With the above simplification, all quantities used in Term2ik depend only on the first three entries of the vector \( \mathbf{a} \). Since \( \mathbf{a} \) has the same distribution as \( \mathbf{a} \), \( \mathbb{E}[\hat{f}(\mathbf{a})] = \mathbb{E}[f(\mathbf{a})] \) for any function \( f(.) \). Thus \( \mathbb{E}[\text{Term2}_{ik}] \) satisfies (1).
Using this, Lemma 3.10 and \( \beta_{j,W} \leq 1 \) in (5),

\[
|E\text{Term}2| = \sum_{ik} |E[\text{Term}2,ik]| \leq 4C (\delta_t^3 + \delta_t^2) \sum_k m\|Wb_k\|\|x_k^*\| \\
\leq 4Cm\delta_t^2\|Wb\|_F\|X^*\|_F = 4Cm\delta_t^2\|X^*\|_F.
\]

The last inequality used Cauchy Schwarz and \( \delta_t^3 < \delta_t^2 \). The last equality holds because \( \|Wb\|_F = 1 \).

IV. PROOF OF THEOREM 2.1 COMPLEX MEASUREMENTS

The result follows using the lemmas given in Sec. III-A exactly as explained there. The proofs also remain the same for all but one lemma. The reason is that we use concentration bounds from [25] and these apply (with minor changes to constants) for complex Gaussians as well. Lemma 3.7, which bounds the phase error term, needs a new proof. Besides this, there is also a simple change to the proof of Lemma 3.8 since we use AltMin-TSI [5] instead of RWF for the noisy PR step to recover \( \hat{b}_k \)'s. We explain this in Appendix B. Proof of Lemma 3.7 is provided next.

Similar to the real case, we assume for this proof that \( x_k^* \) is replaced by \( zx_k^* \) where \( z = \text{phase}(\langle x_k^*, \hat{x}_k \rangle) \). With this, \( \text{dist}(x_k^*, \hat{x}_k) = \|x_k^* - \hat{x}_k\| \) and \( \alpha_k \) is real and non-negative.

A. Proof of Lemma 3.7 for complex measurements

Recall that

\[
\text{Term}2(W) = \sum_{ik} \left( b_k W^*a_{ik} \langle \hat{c}_{ik}, c_{ik}^* \rangle - 1 \right) (a_{ik}^* x_k).
\]

We have the following lemma to bound \( E[\text{Term}2(W)]_{ik} \).

**Lemma 4.1.** For a given \( W \) independent of \( a_{ik} \)'s, with \( a_{ik} \in \mathbb{C}^{n \times 1} \) and \( x_k^* \in \mathbb{C}^{n \times 1} \)

\[
|E[\text{Term}2(W)]_{ik}| \leq \|x_k^*\|\|Wb_k\| \times C\delta_t^2 \left( \frac{\delta_t^2}{(1 - \delta_t)^4} + \frac{1}{(1 - \delta_t)^2} \right).
\]

Thus, using \( \delta_t < 0.1 \), Cauchy-Schwarz, and \( \|Wb\|_F = 1 \),

\[
|E[\text{Term}2(W)]| \leq Cm\delta_t^2\|X^*\|_F.
\]

Next we need to show that \( \text{Term}2(W) \) concentrates around its expected value. Consider a fixed \( W \) first. Let \( h_k = \hat{x}_k - x_k^* \). Then, it is easy to see that \( 1 - c_{ik}^* \hat{c}_{ik} = 1 - \text{phase} \left( 1 + \frac{a_{ik}^* h_k}{a_{ik}^* x_k^*} \right) \). By using Lemma A.7 of [2], \( 1 - \text{phase} \left( 1 + \frac{a_{ik}^* h_k}{a_{ik}^* x_k^*} \right) \leq 2\frac{|a_{ik}^* h_k|}{|a_{ik}^* x_k^*|} \).

Thus,

\[
|\langle \hat{c}_{ik}, c_{ik}^* \rangle - 1| \leq 2|a_{ik}^* h_k|.
\]

We can apply Lemma A.1 [24], Lemma 2.7.7 and Theorem 2.8.1 with \( X_{ik} = \langle \hat{c}_{ik}, c_{ik}^* \rangle - 1 \) and \( K_{X,ik} \leq 2\|h_k\| \), and \( Y_{ik} = (b_k^* W^* a_{ik}) \) and \( K_{Y,ik} = \|b_k\| \) exactly as in the real case. After this the epsilon-net argument also follows as before.

B. Proof of Lemma 4.1

The steps are similar to the real case, but the details are much more complicated. Proceeding exactly as before,

\[
|E[\text{Term}2(W)]_{ik}| \leq 4\|x_k^*\|\|Wb_k\| \times \\
\left( |E\left[ \left( 1 - \text{phase} \left( \alpha_k^2 a(1)^2 + \sqrt{1 - \alpha_k^2 a(2)\bar{a}(1)} \right) \right) \right] | \right)
\]

\[
+ |E\left[ \left( 1 - \text{phase} \left( \alpha_k^2 a(1)^2 + \sqrt{1 - \alpha_k^2 a(2)\bar{a}(1)} \right) \right) \right] |.
\]
where \( \alpha_k = \frac{(\|a_k\|, x_k)}{\|a_k\|\|x_k\|} \). The result then follows by combining the following two lemmas (whose proofs are different for the complex case) and finally using Lemma 3.11 and Lemma 3.8 We prove these lemmas in Appendix B.

**Lemma 4.2.** Assume \( a(1), a(2) \) are two independent standard complex Gaussian scalars and \( 0.8 \leq \alpha \leq 1 \). Then we have
\[
\mathbb{E} \left[ \left( 1 - \text{phase} \left( a \left| a(1) \right|^2 + \sqrt{1 - \alpha^2} a(2)\tilde{a}(1) \right) \right) \left| a(1) \right|^2 \right] 
\leq C \left( \frac{1 - \alpha^2}{\alpha^2} \right)^2.
\]

**Lemma 4.3.** Assume \( a(1), a(2) \) are two independent standard complex Gaussian scalars and \( 0.84 \leq \alpha \leq 1 \). Then we have
\[
\mathbb{E} \left[ \left( 1 - \text{phase} \left( a \left| a(1) \right|^2 + \sqrt{1 - \alpha^2} a(2)\tilde{a}(1) \right) \right) \tilde{a}(1)a(2) \right] 
\leq C \left( \frac{1 - \alpha^2}{\alpha^2} \right).
\]

V. Conclusions

In this work we studied an alternating minimization solution, AltMinLowRaP, to Low Rank PR and showed that order \( \max \left( \frac{n^3}{q}, r, \log \max(q, n) \right) \) measurements per column (per signal) suffice to recover a rank-\( r \) \( n \times q \) matrix. We proved this result for both real-valued and complex-valued Gaussian measurements. We also discuss why, for an AltMin solution, it is not possible to improve the sample complexity any further. Of course, it may be possible to go down to \( n^2/q \) by developing a projected gradient descent approach. This result improves upon our earlier guarantee from [14]. AltMinLowRaP was developed in [14] where we also showed extensive empirical experiments to demonstrate its practical utility.

APPENDIX A

**CONCENTRATION BOUND **Lem**A AND PROOF OF LEMMAS 3.10 AND 3.11**

Our proofs use the following concentration bound. This follows by combining Lemma 2.7.7 and Theorem 2.8.1 of [24].

**Lemma A.1.** Let \( X_i, Y_i, i = 1, 2, \ldots, N \), be sub-Gaussian random variables with sub-Gaussian norm \( K_{X_i} \) and \( K_{Y_i} \) respectively and with \( \mathbb{E}[X_iY_i] = 0 \). Assume that \( \{X_i, Y_i\}, i = 1, 2, \ldots, N \) are mutually independent for different \( i \). Then
\[
\Pr \left\{ \left| \sum_{i=1}^{N} X_iY_i \right| \leq t \right\} \leq 2 \exp \left( -c \min \left( \frac{t^2}{\sum_i K_{X_i}^2 K_{Y_i}^2}, \frac{t}{\max_i |K_{X_i}, K_{Y_i}|} \right) \right).
\]

A. Proofs of Lemmas 3.10 and 3.11

**Proof of Lemma 3.10** We first bound the magnitude of the expectation conditioned on \( a(1) = z \).
\[
|\mathbb{E} \left[ \left| a(1) \right|^2 \mathbbm{1}_{\{\alpha(1) > 0, \alpha(a(1)) + \sqrt{1 - \alpha^2} a(2) < 0\}} a(1) = z \right]| \\
= z^2 \Pr \left\{ a(2) < \frac{-\alpha z}{\sqrt{1 - \alpha^2}} \right\} \mathbbm{1}_{\{z > 0\}} \\
= z^2 \Pr \left\{ a(2) > \frac{\alpha z}{\sqrt{1 - \alpha^2}} \right\} \mathbbm{1}_{\{z > 0\}} \\
= z^2 \text{erfc} \left( \frac{\alpha z}{\sqrt{1 - \alpha^2}} \right) \mathbbm{1}_{\{z > 0\}} \\
\leq (1) z^2 \exp \left( -\frac{\alpha z}{\sqrt{1 - \alpha^2}} \right) \mathbbm{1}_{\{z > 0\}} \\
\leq z \mathbbm{1}_{\{z > 0\}} \frac{\sqrt{1 - \alpha^2}}{\alpha} \exp \left( -\frac{\alpha^2 z^2}{1 - \alpha^2} \right).
\]
and hence, letting $f_2(z)$ be the probability density function (PDF) of a standard Gaussian,

$$
\left| E \left[ a(1)^2 I\{a(1) > 0, a(1) + \sqrt{1 - \alpha^2} a(2) < 0\} \right] \right|
\leq \frac{\sqrt{1 - \alpha^2}}{2\sqrt{2\pi\alpha}} \int_0^\infty ze^{-\frac{z^2}{2(1 - \alpha^2)}} \, dz
\leq \frac{\sqrt{1 - \alpha^2}}{2\sqrt{2\pi\alpha}} \int_0^\infty ze^{-\frac{z^2}{2}} \, dz
= \frac{(1 - \alpha^2)^{\frac{3}{2}}}{4\sqrt{2\pi\alpha^3}}.
$$

In (1) we used the fact that $\text{erfc}(x) < \frac{e^{-\frac{x^2}{2}}}{2x}$ and in (2) we used $e^{-\frac{x^2}{2}} \leq 1$.

Now consider the second term. Let $f_W(w)$ be the PDF of a standard Gaussian. Proceeding as before,

$$
\left| E \left[ a(2) |a(1)| I\{a(1) > 0, a(1) + \sqrt{1 - \alpha^2} a(2) < 0\} \right| a(1) = z \right| 
= \left| E \left[ z a(2) I\{z > 0, a(2) < \frac{\sqrt{1 - \alpha^2}}{\sqrt{1 - \alpha^2}}\} \right] \right|
= zI\{z > 0\} \int_{w = -\infty}^{\infty} w f_W(w) \, dw
= zI\{z > 0\} \int_{\alpha = 0}^{\infty} w f_W(w) \, dw \leq zI\{z > 0\} \frac{\sqrt{\alpha^2 + 2}}{\alpha^2}.
$$

Hence we have

$$
\left| E \left[ a(2) |a(1)| I\{a(1) > 0, a(1) + \sqrt{1 - \alpha^2} a(2) < 0\} \right] \right|
\leq \int_0^\infty \frac{1}{\sqrt{2\pi}} ze^{\frac{-z^2}{2(1 - \alpha^2)}} \, dz
\leq (1) \int_0^\infty ze^{-\frac{z^2}{2}} \, dz
= (1 - \alpha^2)^{\frac{3}{2}} \frac{\sqrt{1 - \alpha^2}}{4\sqrt{2\pi\alpha^3}}.
$$

where in (1) we used the fact that $\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \leq 1$. 

**Proof of Lemma 3.11** Define $\gamma^2 = 1 - \frac{\langle x, \hat{a} \rangle^2}{\|x\|^2 \|\hat{a}\|^2}$ and $\eta^2 = \min_{\theta \in [0, 2\pi]} \|x - e^{i\theta} \hat{a}\|^2$. Thus, we just need to show that $\gamma^2 \leq C\eta^2$ for $\eta \leq c\|x\|$. To do this we can write

$$
\min_{\theta \in [0, 2\pi]} \|x - e^{i\theta} \hat{a}\|^2 = \|x\|^2 + \|\hat{a}\|^2 - 2\langle x, \hat{a} \rangle.
\Rightarrow \eta^2 = (\|x\| - \|\hat{a}\|)^2 + 2\|x\| \|\hat{a}\| \left(1 - \sqrt{1 - \gamma^2}\right)
\geq 2\|x\| \|\hat{a}\| \left(1 - \sqrt{1 - \gamma^2}\right)
\geq 2(1 - c)\|x\|^2 \left(1 - \sqrt{1 - \gamma^2}\right),
$$

where in the last line we used the fact that $\|\hat{a}\| \geq \|x\| - \|x - \hat{a}\| \geq (1 - c)\|x\|$. This implies that

$$
\sqrt{1 - \gamma^2} \geq 1 - \frac{\eta^2}{2(1 - c)\|x\|^2}
\Rightarrow 1 - \gamma^2 \geq 1 + \frac{\eta^2}{4(1 - c)^2\|x\|^4} - \frac{\eta^2}{(1 - c)\|x\|^2}
\Rightarrow \gamma^2 \leq \frac{\eta^2}{(1 - c)\|x\|^2} \left(1 - \frac{\eta^2}{4(1 - c)\|x\|^2}\right)
\leq \frac{\eta^2}{(1 - c)\|x\|^2}
$$

where in the last inequality we used the fact that $0 \leq \frac{\eta^2}{4(1 - c)\|x\|^2} \leq 1$. 


APPENDIX B

PROOF OF LEMMAS FOR COMPLEX MEASUREMENTS

We first prove Lemmas 4.2 and 4.3 below. After that, we provide a brief proof of Lemma 3.8 for complex measurements.

A. Proof of Lemmas 4.2 and 4.3

We need the following for both proofs. Let \( \nu = \sqrt{\frac{1-\alpha^2}{\sigma^2}} \). Since \( \alpha \geq 0.8 \), \( \nu < 1 \). Observe that the phase term in both lemmas can be expressed as

\[
\text{phase} \left( \frac{a(1)}{a(2)} \right) = \text{phase} \left( 1 + \frac{a(2)}{a(1)} \right)
\]

Conditioned on \( a(1) = w \), the term inside phase(.) is a complex Gaussian. Letting \( a(2) = a_x + j a_y \) and \( w = w_x + j w_y \), it equals

\[
Z := 1 + \frac{\nu}{|w|^2} (a_x w_x + a_y w_y) + j \frac{\nu}{|w|^2} (a_y w_x - a_x w_y)
\]

It is easy to see that \( \sigma^2_X = \sigma^2_Y = \frac{\nu^2}{|w|^2} \). \( \mathbb{E}[X] = 1 \), \( \mathbb{E}[Y] = 0 \) and \( X \) is uncorrelated with \( Y \), \( \mathbb{E}[(X-1)(Y-0)] = 0 \), so that \( \rho = 0 \). Thus, \( Z \) is a non-zero mean complex Gaussian, with real and imaginary parts being independent and having the same variance \( \sigma^2 := \frac{\nu^2}{|w|^2} \) but different means: \( \mathbb{E}[X] = 1 \) but \( \mathbb{E}[Y] = 0 \). We will use a result from [26] that provides an expression for the PDF of the angle \( \theta \) of such a complex Gaussian, i.e., for \( \theta \), when we write \( Z \) in polar form as \( Z = Re^{j\theta} \). From [26],

\[
f_{\theta}(\theta) = \frac{1}{2\pi} \exp \left\{ -\frac{1}{\sigma^2} \right\} \left\{ \frac{\pi \Omega_{X,Y}}{\Omega(\theta)} \cos(\theta - \phi) \right. \\
\left. \text{erfc} \left( -\frac{\sqrt{\Omega_{X,Y} \cos(\theta - \phi)}}{\sqrt{\Omega(\theta)}} \right) \exp \left( \frac{\Omega_{X,Y} \cos^2(\theta - \phi)}{\Omega(\theta)} \right) + 1 \right\}
\]

where \( \Omega_{X,Y} = 1 \), \( \Omega(\theta) = 2\sigma^2 \), \( \Omega_{X,Y}/\Omega(\theta) = \frac{1}{2\sigma^2} \cos \phi = \frac{1}{\sqrt{\Omega_{X,Y}}} = 1 \Rightarrow \phi = 0 \). Hence we have

\[
f_{\theta}(\theta) = \frac{1}{2\pi} \exp \left\{ -\frac{1}{\sigma^2} \right\} \times
\left\{ \sqrt{\frac{\pi}{2\sigma^2}} \cos(\theta) \text{erfc} \left( -\sqrt{\frac{1}{2\sigma^2}} \cos(\theta) \right) \exp \left( \frac{\cos^2(\theta)}{2\sigma^2} \right) + 1 \right\}
\]

\[
\leq \frac{1}{2\pi} \exp \left\{ -\frac{1}{\sigma^2} \right\} \left\{ \sqrt{\frac{\pi}{2\sigma^2}} + 1 \right\}.
\]

where in (1) we used the fact that \( \text{erfc} \left( -\sqrt{\frac{1}{2\sigma^2}} \cos(\theta) \right) \leq \exp \left( \frac{\cos^2(\theta)}{2\sigma^2} \right) \) along with \( \cos(\theta) \leq 1 \).

We will use (6) in the proofs below. Moreover, we will also frequently use the following: for integers \( n = 1, 2, 3, 4, 5 \ldots \)

\[
\int_{\tau=0}^{\infty} \tau^n \exp(-\tau^2/\nu^2)d\tau \leq C\nu^{n+1}
\]

where \( C \leq 2 \) for \( n \leq 4 \). This follows from the property of Gamma function that \( \Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt = (z-1)! \).

Proof of lemma 4.2 We need to bound

\[
\left| \mathbb{E} \left[ \left( 1 - \text{phase} \left( \frac{a(1)}{a(2)} \right) \right) \right] \right|
\]

\[
= \left| \mathbb{E} \left[ \left( 1 - \text{phase} \left( 1 + \frac{\nu}{a(1)} \right) \right) \right] \right|
\]
First we bound $|\mathbb{E}[\text{Tr}m1|a(1) = w]|$.

$$
|\mathbb{E}[\text{Tr}m1|a(1) = w]| = |w|^2|\mathbb{E}[1 - \text{phase}(Z)]| = |w|^2|\mathbb{E}[1 - e^{i\theta}]| = |w|^2\left|\int_{0}^{2\pi}(1 - e^{i\theta})f_\theta(\theta)d\theta\right|.
$$

In the above $w$ is just a dummy variable that we are using for the conditional expectation as the known value for $a(1)$. It is completely different from matrix $W \in \mathbb{C}^{n \times r}$ or its vectorized version $w$ which was used previously.

Since $f_\theta(\theta) = f_\theta(-\theta)$, $f_\theta^\ast \sin(\theta)f_\theta(\theta)d\theta = 0$. Thus,

$$
\left|\int_{0}^{2\pi}(1 - e^{i\theta})f_\theta(\theta)d\theta\right| = \left|\int_{-\pi}^{\pi}(1 - \cos \theta)f_\theta(\theta)d\theta\right|
\leq \frac{1}{2\pi}\exp\left\{-\frac{1}{\sigma^2}\right\}\left\{\sqrt{\frac{\pi}{2\sigma^2}} + 1\right\}\int_{-\pi}^{\pi}|1 - \cos \theta|d\theta
\leq 2\exp\left\{-\frac{1}{\sigma^2}\right\}\left\{\sqrt{\frac{\pi}{2\sigma^2}} + 1\right\}
\tag{8}
$$

where $\sigma^2 = \frac{\nu^2}{|w|^2}$. The first inequality used the upper bound on $f_\theta(\theta)$ while the second used $|1 - \cos \theta| \leq 2$. Hence

$$
|\mathbb{E}[\text{Tr}m1]| \leq \sqrt{\frac{2\pi}{\nu^2}}\mathbb{E}\left[|a(1)|^4 \exp\left\{-\frac{|a(1)|^2}{\nu^2}\right\}\right]
+ 2\mathbb{E}\left[|a(1)|^2 \exp\left\{-\frac{|a(1)|^2}{\nu^2}\right\}\right].
$$

Since $a(1)$ is a standard complex Gaussian, $f_{|a(1)|}(x) = xe^{-\frac{x^2}{2\nu^2}}, \forall x > 0$. Using this,

$$
|\mathbb{E}[\text{Tr}m1]| \leq \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty}\left(\sqrt{\frac{2\pi}{\nu^2}}x^4 + 2x^3\right)e^{-x^2/\nu^2}e^{-x^2/2}dx
\leq \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty}\left(\sqrt{\frac{2\pi}{\nu^2}}x^4 + 2x^3\right)e^{-x^2/\nu^2}dx
\leq C\nu^4.
$$

The second inequality used $e^{-x^2/2} \leq 1$. The third one follows using $\{7\}$ with $n = 4$ for the first term and $n = 3$ for the second one.

\textbf{Proof of lemma \ref{lem:to prove}} By using Cauchy Schwarz,

$$
|\mathbb{E}\left[\left(1 - \text{phase}\left(\alpha|a(1)|^2 + \sqrt{1 - \alpha^2 a(2)\tilde{a}(1)}\right)\right)\tilde{a}(1)a(2)\right]| = |\mathbb{E}\left[\tilde{a}(1)a(2)\left(1 - \text{phase}\left(1 + \nu\frac{a(2)}{a(1)}\right)\right)\right]| \leq \sqrt{\mathbb{E}\left[|a(1)|^2\right] - \text{phase}(1 + \nu\frac{a(2)}{a(1)})^2}\sqrt{(\mathbb{E}[|a(2)|^2])}
\leq \mathbb{E}\left[|a(1)|^2\left(1 - \text{phase}\left(1 + \nu\frac{a(2)}{a(1)}\right)\right)\right].
\tag{9}
$$
The last equality follows from the fact that \( \mathbb{E} \left[ |a(2)|^2 \right] = 1 \) since \( a(2) \) is standard complex Gaussian random variable. We will first bound \( \text{Trm2} \) conditioned on \( a(1) = w \).

\[
|\mathbb{E}[\text{Trm2}|a(1) = w]| \leq |w|^2 \mathbb{E} \left[ \left| 1 - \text{phase}(1 + \nu a(1)/a(2)) \right|^2 \right] \\
= |w|^2 \int_{0}^{2\pi} \left| 1 - e^{j\theta} \right|^2 f_\Theta(\theta) \, d\theta \\
\leq 4|w|^2 \exp \left\{ -\frac{1}{\sigma^2} \right\} \left\{ \sqrt{\frac{\pi}{2\sigma^2}} + 1 \right\}.
\]  

(10)

The last inequality used \( |1 - e^{j\theta}|^2 \leq 4 \) and the upper bound on \( f_\Theta(\theta) \). Recall that \( \sigma = \nu/|w| \). Thus,

\[
|\mathbb{E}[\text{Trm2}]| \leq 4 \mathbb{E} \left[ |a(1)|^2 \right] \left| 1 - \text{phase}(1 + \nu a(2)/a(1)) \right|^2 \\
= 4 \mathbb{E} \left[ |a(1)|^2 \right] \exp \left\{ -\frac{|a(1)|^2}{\nu^2} \right\} \left\{ \sqrt{\frac{\pi}{2\nu^2}} + 1 \right\} \\
= 4 \left( \frac{\pi}{2} \right)^{1/2} \mathbb{E} \left[ |a(1)|^3 \right] \exp \left\{ -\frac{|a(1)|^2}{\nu^2} \right\} \\
\leq C\nu^4.
\]  

(11)

The last inequality follows since \( f_{a(1)}(x) = xe^{-x^2/2} \), for \( x \geq 0 \), thus the integral in both terms is of the form (7) with \( n = 4 \) and \( n = 3 \) respectively. Finally using (11) in (9) we have

\[
|\mathbb{E} \left[ \left( 1 - \text{phase} \left( a|a(1)|^2 + \sqrt{1 - a^2a(2)}\tilde{a}(1)a(2) \right) \right) \tilde{a}(1)a(2) \right] | \\
\leq C\nu^2.
\]

\( \square \)

B. Proving Lemma 3.8 for complex case

The proof approach is exactly the same as that of Lemmas 3.2 and 3.3 of [14]. Since \( y_{ik} = |\tilde{a}_{ik} g_k| + \nu_{ik} \) with noise \( |\nu_{ik}| \leq a_{ik}(I - U^*U^t)x_k^* \), by the sub-exponential Bernstein inequality [24], w.h.p., the noise vector \( \nu_k \) satisfies

\[
||\nu_k||^2 \leq 1.1m\text{SE}(U^t, U^*)^2 ||x_k^*||^2 \leq 1.1m\delta_k^2 ||x_k^*||^2
\]  

(12)

Since the PR problem is solved using AltMin-TSI from [5], we need the following result to analyze it (this replaces [6, Theorem 2] which was used in [14]). It follows by combining Theorem 2 of [4] and Theorem 3.1 of [5] with a minor change to deal with noise.

Theorem B.1 (Corollary 3.7 of [5]). Consider measurements of the form \( y_i = |a_i g^*| + v_i, i = 1, \ldots, m, \) with \( v \) satisfying \( \frac{\|v\|}{\sqrt{m}} \leq C\|x^*\| \). Here \( g^* \) is an \( r \)-length complex vector and \( a_i \) are i.i.d. complex standard Gaussian vectors of length \( r \). Pick a \( 0 < \rho < 1 \) and a \( 0 < \rho_0 < 1 \). There exists a constant \( C_0 \) that depends on \( \rho, \rho_0 \), such that if \( m > C_0(\rho, \rho_0)r \), then w.p. at least \( 1 - C \exp \{ -\rho m \} \) for numerical constants \( C, C_0 \), the following holds after \( T \) iterations:

\[
\text{dist}(\hat{g}^{T+1}, g^*) \leq \rho^T \text{dist}(\hat{g}^0, g^*) + 1.5 \frac{1}{\sqrt{m}} ||v|| \\
\leq \rho^T \rho_0 ||g^*|| + 3 \frac{1}{\sqrt{m}} ||v||
\]

By picking \( T \) large enough, the first term above can be made smaller than the second; then, \( \text{dist}(\hat{g}^{T+1}, g^*) \leq 6 \frac{1}{\sqrt{m}} ||v||. \)

We apply the above result with \( y \equiv y_k, a_i \equiv \tilde{a}_{ik}, i = 1, \ldots, m, g^* \equiv g_k, \hat{g} \equiv \hat{b}_k, \) and \( v \equiv \nu_k \). Using (12), \( ||v||/\sqrt{m} \leq \sqrt{1.1\delta_k} ||x_k^*||. \) Since we use \( T = T_{PR,t} \), we conclude that \( \text{dist}(\hat{b}_k, g_k) \leq 6\sqrt{1.1\delta_k} ||x_k^*||. \) This completes the proof of the first bound. The other claims follow exactly as in the proof of Lemma 3.3 of [14].
Proof of Theorem 6.7. The initialization step of AltMin-TSI uses the truncated spectral initialization from [4]. Guarantees in [4] are proved for real-valued measurements. However, even with complex Gaussian measurements, there is no change to the analysis of truncated spectral initialization. Thus we can use Theorem 2 of [4] with $t = 0$ (only initialization part) to conclude that, w.p. $1 - C \exp(-\rho_0 m)$,

$$\text{dist}(g^0, g^*) \leq \rho_0 \| g^* \| + 2 \| v \| / \sqrt{m}.$$  

Consider iteration $t + 1$. Since $\hat{g}^{t+1} = A^\dagger (y \circ \text{phase}(A' \hat{g}^t))$ (see Algorithm 1 of [5]), where $\circ$ is the Hadamard product (.* operation in MATLAB), we have

$$\text{dist}(\hat{g}^{t+1}, g^*) = \min_{\hat{\phi}} \| e^{j\hat{\phi}} g^* - \hat{g}^{t+1} \|$$

$$= \min_{\hat{\phi}} \| e^{j\hat{\phi}} g^* - (A')^\dagger \left( (y - v) \circ \text{phase}(A' \hat{g}^t) \right) \|$$

$$+ \| (A')^\dagger \left( v \circ \text{phase}(A' \hat{g}^t) \right) \|$$

$$\leq \rho \text{dist}(\hat{g}^t, g^*) + \| \sqrt{m} (A')^\dagger \| 1 / \sqrt{m} v \|,$$

The last inequality follows using Theorem 3.1 of [5]. By Lemma A.2 of [2] (or directly by using Theorem 4.6.1 of [24]), we have that, w.p. $1 - 2 \exp(-c m)$,

$$\| \sqrt{m} (A')^\dagger \| \leq \| m (AA')^{-1} \| \| A / \sqrt{m} \| \leq \frac{1}{1 - \epsilon} \sqrt{1 + \epsilon} \leq 1.5$$

if we let $\epsilon = 0.1$. Thus, $\text{dist}(\hat{g}^{t+1}, g^*) \leq \rho \text{dist}(\hat{g}^t, g^*) + 1.5 \| v \| / \sqrt{m}$. Using this and the initialization bound, $\text{dist}(\hat{g}^{t+1}, g^*) \leq \rho t \rho_0 + 3 \| v \| / \sqrt{m}$. 

\[ \square \]

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