Online State Estimation for Systems with Asynchronous Sensors

Guido Cavraro, Emiliano Dall’Anese, Joshua Comden, Andrey Bernstein

Abstract

The paper investigates the problem of estimating the state of a network from heterogeneous sensors with different sampling and reporting rates. In lieu of a batch linear least-squares (LS) approach – well-suited for static networks, where a sufficient number of measurements could be collected to obtain a full-rank design matrix – the paper proposes an online algorithm to estimate the possibly time-varying network state by processing measurements as and when available. The design of the algorithm hinges on a generalized LS cost augmented with a proximal-point-type regularization. With the solution of the regularized LS problem available in closed-form, the online algorithm is written as a linear dynamical system where the state is updated based on the previous estimate and based on the new available measurements. Conditions under which the algorithmic steps are in fact a contractive mapping are shown, and bounds on the estimation error are derived for different noise models. Numerical simulations, including a power system case, are provided to corroborate the analytical findings.

Index Terms

State estimation, data fusion, asynchronous sensors, networked systems, sensor networks, stability.

I. INTRODUCTION

State estimation plays a crucial role in large-scale engineering systems – including traffic, energy, and communication networks – because it is essential for monitoring purposes and to support underlying control and optimization tasks. For instance, state estimation in power systems pertains to the reconstruction of voltage profiles given a set of sparse measurements; in traffic networks, traffic flows and vehicle densities in highways and roads are monitored and used for congestion control. Estimating the state of a network may be challenging, since oftentimes key quantities are not constantly measured or are not directly accessible. For example, challenges that system operators face arise because: i) the network may feature heterogeneous sensors with different sampling and reporting rates (and, hence, measurements are gathered asynchronously), and ii) networks have a variable structure, e.g., they may change configurations.

G. Cavraro, J. Comden, and A. Bernstein are with the National Renewable Energy Laboratory (NREL), Golden CO, USA (Email: {guido.cavraro, joshua.comden, andrey.bernstein}@nrel.gov). E. Dall’Anese is with the University of Colorado Boulder, Boulder, CO, USA (Email: emiliano.dallanes@colorado.edu). The work of E. Dall’Anese was supported by NREL award APUP UGA-0-41026-109.
In this paper, we consider a measurement model of the form

$$y(t) = A(t)x(t) + n(t)$$  \quad (1)$$

where $y(t)$ is a vector of available measurements at time $t$, the system state is represented by the vector $x(t)$, $n(t)$ is the vector of noise, and $A(t)$ is a possibly time-varying regression matrix. The state $x(t)$ can be time-varying, as we will explain more concretely shortly. One can think of (1) as the measurement equation for a dynamical system modeling the network’s state. This model can be used for several applications, e.g., in wireless sensor networks or power systems [1], [2].

Data fusion, the process of integrating information from different sensors, has to be performed to obtain the state estimate [3]. When a sufficient number of measurements can be collected before the state $x(t)$ changes and the regression matrix is full rank, state estimation is classically performed via least squares methods [4]; pertinent regularized counterparts can be used to handle underdetermined systems when a prior on the state is available or when the state can be embedded into a lower dimensional space. Alternatively, maximum likelihood or Bayesian approaches [5] can be pursued, in which a priori statistical information is leveraged.

In dynamic settings where the system state evolves in time, streams of measurements are received asynchronously [6], and the time-variability of the state might be such that a sufficient number of measurements to obtain a full-rank regression matrix can not be collected. The fusion of data from multi-rate asynchronous sensors with measurements randomly missing is studied in [7]. Missing data and delays are likely to occur in asynchronous multi-sensor systems. Algorithms suited for this scenario have been proposed in [8], [9]; estimators handling real-time measurements are discussed in, e.g., [6].

The present paper considers a setting in which:

1) heterogeneous reporting rates make the number of available measurements at every time step much smaller than the number of state variables, preventing the use of traditional least squares estimators;
2) the state variability can hardly be captured and modeled and hence a meaningful (deterministic or stochastic) state space description is not available
3) the measured output can be modeled as a time varying, or switching, function of the state.

For this scenario, we propose an online asynchronous state estimator (OASE) that, at each time step, solves a strongly convex optimization problem, aiming at minimizing the sum of a weighted least squares term capturing the available measurements data and a regularization term that introduces “memory” on the estimate by feeding back the previous-step estimation to the optimization problem. This momentum term ensures a consistent and accurate estimate under low-observability conditions. From the optimization perspective, the mathematical formulation is the one of the proximal point method (PPM) [10], [11], [12]. The main difference is that, whereas the PPM is used to find iteratively a solution of a static optimization problem, we are considering the case in which the optimization problem changes at every iteration and the goal is to track the particular solution of the optimization problem which represents the true system state. We show that the state estimate follows a dynamic linear system, with the measurements collected from the field as an input. We then analyze the performance of this system under bounded deterministic and zero-mean stochastic noise assumptions. These two cases are both meaningful: in the first, the
noise can be interpreted as a bounded modeling error, in the second, as measurement noise. We prove bounds on the estimation error under certain conditions on the available measurements; these conditions roughly speaking require that the system is fully observable given the measurements in any \( \tau \) consecutive steps, where \( \tau \) is a constant that can be strictly greater than 1.

It is worth pointing out that, in dynamic settings, state estimation can be performed via Kalman filtering [5]. Customized extensions of the Kalman filter have been tailored to handle dynamic systems where measurements are both taken by measurement devices and collected by the system operators at different times. In [13], the Kalman filter was generalized for the case in which the arrival of an observation is modeled as a random process which depends on the communication channel features. We consider a case where the state space description is not available, preventing the use of Kalman filter-based approaches.

The paper illustrates the application of the proposed method to the state estimation task in a distribution power network [14]. Examples of distribution system state estimation (DSSE) include the Bayesian linear state estimator [15], Kalman filter-based approaches [16], [17], and methods based on machine learning and signal processing tools [18]. Pseudo-measurements derived from historical data are used when the power grid is not observable [19]. Next-generation power systems are good candidates for applying the OASE because:

1) measurements from PMUs, DERs, and smart meters [20], [21] are generally not synchronized, that the difference between measurement times can be significant [22], [23], and that system operator gather data asynchronously.

2) integration of renewables, electric vehicles, and other power-electronics-interfaced distributed energy resources (DERs) are leading to net-loading conditions that are less predictable and highly variable [24]. As a consequence, it is much harder to model the network behavior and obtain meaningful pseudo-measurement.

3) the measurements can be modeled as a time varying function of the state.

The paper is structured as follows. The OASE algorithm is presented in Section II. The estimation error is introduced and studied in Section III and Section IV respectively. Section V reports the application of the OASE to DSSE. Finally, the numerical validation of the OASE is provided in Section VI and Section VII.

*Notation:* lower- (upper-) case boldface letters denote column vectors (matrices). Calligraphic symbols are reserved for sets. Symbol \( \uparrow \) stands for transposition. Vectors 0 and 1 are the all-zero and all-one vectors, while \( e_m \) is the \( m \)-th canonical vector. Symbol \( \| \mathbf{x} \| \) and \( \| \mathbf{X} \| \) denote the 2-norm of the vector \( \mathbf{x} \) and of the matrix \( \mathbf{X} \), respectively; symbol \( \| \mathbf{X} \|_F \) denotes the Frobenius norm of \( \mathbf{X} \), while \( \| \mathbf{x} \|_Q = \mathbf{x}^\top \mathbf{Q} \mathbf{x} \) for a positive definite matrix \( \mathbf{Q} \). The diagonal matrix having the elements of the finite set \( \{ x_i \} = \{ x_1, x_2, \ldots \} \) on its diagonal is denoted as \( \text{dg}(\{x_i\}) \). Given a matrix \( \mathbf{A} \), its kernel, namely the set of all vectors \( \mathbf{x} \) such that \( \mathbf{A} \mathbf{x} = 0 \), is denoted as \( \ker \mathbf{A} \). The expectation operator is defined as \( \mathbb{E}[\cdot] \). The Kronecker product of the vectors \( \mathbf{x} \) and \( \mathbf{x}' \) is \( \mathbf{x} \otimes \mathbf{x}' \), while \( \text{vec}(\mathbf{X}) \) is the vectorization of the matrix \( \mathbf{X} \). Finally, given a sequence of matrices \( \{ \mathbf{X}(t) \}^T_{t=1} \), we have that \( \prod_{t=1}^T \mathbf{X}(t) = \mathbf{X}(T) \mathbf{X}(T-1) \ldots \mathbf{X}(1) \).
II. THE STATE ESTIMATOR

Consider a discrete time system whose state at time $t = 0, 1, \ldots$ is described by the vector $x(t) \in \mathbb{R}^N$ and whose output $y(t)$ is modeled by (1), where $y(t), n(t) \in \mathbb{R}^{M_t}$, $A(t) \in \mathbb{R}^{M_t \times N}$ and where $M_t$ is allowed to vary in time. In the following, the vector $n$ will be referred to as noise vector since it has a straightforward interpretation as the measurement noise affecting the system output $y$; nevertheless, $n$ can also be used to describe model uncertainty. The system state $x$ is assumed to be time varying and the state variation at time $t$ is denoted as

$$\delta(t) := x(t) - x(t - 1). \quad (2)$$

A model describing how $x$ changes in time, e.g., a state space model, is not available. Rather, mild information on the state variation is assumed to be known. Precisely, for every $t$, there exists a real non-negative number $\Delta_x(t)$ such that

$$\|\delta(t)\| \leq \Delta_x(t). \quad (3)$$

Further, let $\Delta_x := \sup\{\Delta_x(t)\}$, and suppose $\Delta_x < \infty$.

This paper proposes an algorithm that provides an estimate $\hat{x}$ of the system state $x$ given the system output $y$ and the sequence of model matrices $\{A(t)\}_{t \geq 1}$. A straightforward way to obtain $\hat{x}(t), t \geq 1$ would be solving the Weighted Least Square (WLS) problem

$$\min_w \|y(t) - A(t)w\|_{Q_{t-1}}^2 \quad (4)$$

where $Q_t \in \mathbb{R}^{M_t \times M_t}$ is a positive definite matrix. Problem (4) has a unique solution only if the number of measurements available is greater or equal to the number of system’s state, namely, $M_t \geq N$. Otherwise, problem (4) is not strictly convex and has infinitely many solutions. Since the focus of this paper is on systems in which possibly $M_t \ll N$, the WLS approach can not be pursued. Rather, we propose to compute the state estimate by solving the following time-varying regularized WLS problem:

$$\hat{x}(t) = \min_w \|y(t) - A(t)w\|_{Q_{t-1}}^2 + \gamma\|w - \hat{x}(t - 1)\|^2 \quad (5)$$

for $t = 1, 2, \ldots$ and given an initial estimate $\hat{x}(0)$. The second term in (5) acts as a regularizer which penalizes the Euclidean distance of the new estimate from the older one and makes (5) a strongly convex problem having unique solution. The real scalar $\gamma > 0$ will be referred to as inertia parameter. The smaller $\gamma$ is, the further the new estimate $\hat{x}(t)$ is allowed to be from $\hat{x}(t - 1)$. In fact, (5) can be viewed as a time-varying proximal point method [12]; see also Remark [3].

After simple computation, it can be shown that the estimate $\hat{x}(t)$ admits the closed form

$$\hat{x}(t) = A(t)\hat{x}(t - 1) + \frac{1}{\gamma}A(t)A(t)^\top Q_{t-1}^{-1}y(t) \quad (6)$$

where

$$A(t) := \gamma(A(t)^\top Q_{t-1}^{-1}A(t) + \gamma I)^{-1}. \quad (7)$$

That is, the new estimate $\hat{x}(t)$ can be computed recursively given the previous estimate $\hat{x}(t - 1)$, the new measurement $y(t)$ and the new $A(t)$. Equation (6) represents the sought online asynchronous state estimator. The inverse on the right hand side of (7) always exists and $A(t) \in \mathbb{R}^{N \times N}$ is a symmetric positive definite matrix for every $t$. 

June 2, 2020 DRAFT
Lemma 1. A vector \( \mathbf{v} \in \mathbb{R}^N \) is in the kernel of \( \mathbf{J}(t) \), \( \mathbf{v} \in \ker \mathbf{J}(t) \), if and only if \( \mathbf{v} \) is in the kernel of \( \mathbf{A}(t) \), \( \mathbf{v} \in \ker \mathbf{A}(t) \).

Proof: If \( \mathbf{v} \in \ker \mathbf{A}(t) \), trivially \( \mathbf{v} \in \ker \mathbf{J}(t) \). Now assume \( \mathbf{v} \in \ker \mathbf{J}(t) \). Then,

\[
\| \mathbf{A}(t) \mathbf{v} \|_{Q_t^{-1}} = \mathbf{v}^\top \mathbf{A}(t)^\top Q_t^{-1} \mathbf{A}(t) \mathbf{v} = \mathbf{v}^\top \mathbf{0} = 0
\]

which yields \( \mathbf{A}(t) \mathbf{v} = 0 \).

Using equation (5), being \( \mathbf{A}(t) := \gamma (\mathbf{J}(t) + \gamma \mathbf{I})^{-1} \), we have

\[
\mathbf{A}(t) = \begin{bmatrix} \mathbf{U}(t) & \mathbf{V}(t) \end{bmatrix} \begin{bmatrix} \text{dg}(\{ \lambda_i(t) \}) & 0 \\ 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{U}^\top(t) \\ \mathbf{V}^\top(t) \end{bmatrix}.
\]

Hence, matrices \( \mathbf{J}(t) \) and \( \mathbf{A}(t) \) share the same eigenvectors and the spectrum of \( \mathbf{A}(t) \) is given by

\[
\text{eig } \mathbf{A}(t) = \left\{ 1, \frac{\gamma}{\gamma + \lambda_1(t)}, \ldots, \frac{\gamma}{\gamma + \lambda_{K_t}(t)} \right\}
\]

where 1 is an eigenvalue with multiplicity \( K_t \) and \( 1 \geq \frac{2}{\gamma + \lambda_1(t)} \geq \frac{2}{\gamma + \lambda_{K_t}(t)} \geq \ldots \). Moreover, \( \| \mathbf{A}(t) \| \leq 1 \) and

\[
\mathbf{x} \mapsto \mathbf{A}(t) \mathbf{x}
\]

is a non-expansive operator in general. Finally, let \( \bar{\lambda} \) denote the smallest non-zero eigenvalue for all \( \mathbf{J}(t) \):

\[
\bar{\lambda} := \min\{ \lambda_1(t), t \geq 1 \}.
\]

Equations (1) and (6) constitute a linear dynamical system, whose block scheme is reported in Figure 1. Furthermore, heed that equation (6) is essentially a classic closed-loop system.

Remark 1. Model (1) covers the scenario depicted in Figure 2 of network of agents. In this case, agent \( i \), whose state is \( \mathbf{x}_i \), measures the quantity

\[
\mathbf{y}_i(t) = \mathbf{A}_i(t) \mathbf{x}(t) + \mathbf{n}_i(t)
\]

and then, once in a while, transmits it to a central entity in charge of estimating the system state \( \mathbf{x} \). In this scenario, \( \mathbf{y}(t) \) and \( \mathbf{A}(t) \) are obtained by stacking the \( \mathbf{y}_i(t) \)'s and the \( \mathbf{A}_i(t) \)'s associated with agents that reported their measurement at time \( t \). An example of such systems, namely, power systems with heterogeneous sensors having different report rates, is discussed in Section V.
Remark 2. The OASE (6) can be adapted for the more general case in which the system output is a noisy version of a linear affine function of the system state, namely,

\[ y(t) = A(t)x(t) + b(t) + n(t), \]

where \( b(t) \in \mathbb{R}^M \). In fact, it is enough to define the variable \( \tilde{y}(t) = y(t) - b(t) \) and then compute the state estimate via

\[ \hat{x}(t) = A(t)\hat{x}(t-1) + \frac{1}{\gamma}A(t)A(t)^T Q_t^{-1} \tilde{y}(t) \]

However, we will consider systems of the form (1) to reduce needed notations.

Remark 3. The proximal point method (PPM) is an algorithm aiming at minimizing a function \( f(x) \) by iteratively solving the problem (12)

\[ \hat{x}(t) = \arg \min_w f(w) + \frac{1}{2\lambda} \| w - \hat{x}(t-1) \|^2. \]  

(13)

After denoting the first term of the cost in (5) as \( f_t(w) \)

\[ f_t(w) = \| y(t) - A(t)w \|^2_{Q_t^{-1}}, \]

we can rewrite (5) as

\[ \hat{x}(t) = \arg \min_w f_t(w) + \gamma \| w - \hat{x}(t-1) \|^2, \]  

(14)

which a time-varying PPM for the sequence of functions \( \{f_t\} \). In particular, while the PPM aims at finding the minimum of \( f \), which is static, the goal of the OASE is to track the time varying state of the system.
III. THE ESTIMATION ERROR

Define the estimation error $\xi$, namely, the difference between the state estimate and the true state for $t \geq 1$, as

$$\xi(t) = \hat{x}(t) - x(t). \quad (15)$$

Like $x$, the error $\xi$ has a closed form expression whose derivation is possible thanks to the following result.

**Lemma 2.** Consider the matrix $\Lambda(t)$ defined in equation (7). It holds

$$\left( I - \Lambda(t) \right)x = \frac{1}{\gamma} \Lambda(t) A(t)^\top Q^{-1} A(t)x. \quad (16)$$

**Proof:** Since $\Lambda(t)$ is a positive definite matrix and its inverse always exists, it holds

$$x = \Lambda(t) \Lambda(t)^{-1} x$$

$$= \frac{1}{\gamma} \Lambda(t) \left( A(t)^\top Q^{-1} A(t) + \gamma I \right)x$$

$$= \Lambda(t)x + \frac{1}{\gamma} \Lambda(t) A(t)^\top Q^{-1} A(t)x$$

from which equation (16) follows. $\blacksquare$

To obtain the estimation error closed form expression, substitute (1) into (6) and use equation (16) to obtain

$$\hat{x}(t) = \Lambda(t)\hat{x}(t-1) + \left( I - \Lambda(t) \right)x(t) + \frac{1}{\gamma} \Lambda(t) A(t)^\top Q^{-1} A(t)x.$$}

Finally, plugging equation (15) into the former equation yields the estimation error update

$$\xi(t) = \Lambda(t)\xi(t-1) - \Lambda(t)\delta(t) + \frac{1}{\gamma} \Lambda(t) A(t)^\top Q^{-1} A(t)x.$$}

By iteratively applying (17), we can find the expression of $\xi(T)$, for every $T \geq 1$, namely

$$\xi(T) = \prod_{t=1}^{T} \Lambda(t)\xi(0) + \sum_{t=1}^{T} \prod_{k=t}^{T} \Lambda(k) \left( \frac{1}{\gamma} \Lambda(t)^\top Q^{-1} A(t)x - \delta(t) \right) \quad (18)$$

where $\xi(0)$ is the initial estimation error.

We next analyze the OASE performance under two conditions on the noise. In the first case, $n$ is assumed to be a vector whose norm is bounded. This corresponds to scenarios in which $n$ represent a modeling error that is known to be finite. In the second case, $n(t)$ is assumed to be a stochastic vector with a certain mean and variance. This case can describe scenarios in which $n$ represents the measurement error.

**Remark 4.** Lemma 2 can be used to provide a familiar interpretation for equation (6). Reminding that we defined $f_t(w) = \|y(t) - A(t)w\|_Q^2$, we have that

$$\nabla f_t(\hat{x}(t-1)) = A(t)^\top Q_t^{-1}(y(t) - A(t)w)$$

June 2, 2020 DRAFT
Equation (16) can be used to rewrite (6) as
\[
\dot{x}(t) = \bar{x}(t-1) - \frac{1}{\gamma} \Lambda(t) \Lambda(t)^\top Q_t^{-1} (y(t) - \Lambda(t)w)
\]
\[
= \bar{x}(t-1) - \frac{1}{\gamma} \Lambda(t) \nabla f_t(\bar{x}(t-1))
\]
Being \(\Lambda(t)\) a positive definite matrix, \(\Lambda(t) \nabla f_t(\bar{x}(t-1))\) is a descent direction for the function \(f_t(w)\), i.e., \(\bar{x}(t)\) is computed, for every \(t\), via a Newton-like descent of \(f_t(w)\)

IV. ESTIMATOR’S PERFORMANCE

In this section, the estimation error is characterized. To that end, we make the following assumption regarding the model matrices \(\{A_t\}\).

Assumption 1. There exists a constant \(\tau > 0\) such that
\[
\bigcap_{k=0}^{\tau-1} \ker A(t+k) = \{0\}, \quad t \geq 1.
\]

Roughly speaking, Assumption 1 means that every \(\tau\) time steps, the system is fully observable; this will be quantified precisely in Propositions 1 and 2 below.

A. Bounded Noise

Next, the case in which \(n\) is a bounded unknown vector will be considered; namely, we make the following assumption.

Assumption 2. The noise vector \(n\) is bounded, i.e., here exists a real non-negative number \(\Delta_n(t)\) such that
\[
\|n(t)\| \leq \Delta_n(t).
\]

Further, let \(\Delta_n := \sup\{\Delta_n(t)\}\), and suppose \(\Delta_n < \infty\).

The results reported hereafter are proved in Appendix A. Assumption 1 has as a direct consequence the next Proposition, which will be used next to prove the main result.
Proposition 1. Consider the system described by
\[ \xi(t) = \Lambda(t)\xi(t-1). \]
and define
\[ \psi = \max_t \left\{ \frac{\gamma}{\gamma + \lambda_1(t)} \right\}. \]
Then, it holds that
\[ \left\| \prod_{k=0}^{t-1} \Lambda(t + k) \right\| \leq \psi < 1 \]
and the system (21) is asymptotically stable
\[ \lim_{T \to \infty} \xi(T) = \prod_{t=1}^{T} \Lambda(t)\xi(0) = 0. \]

Note that Proposition 1 implies that the operator
\[ x \mapsto \Lambda(t+\tau-1)\Lambda(t+\tau-2)\ldots\Lambda(t)x \]
is a contraction even if the map in (11) is not contractive. The estimation error meets the next property.

Theorem 1. Let Assumptions 1 and 2 hold. Define \( c(t) := \|\Lambda(t)^\top Q^{-1}_t\| \). The estimation error at time \( T \) is upper bounded as
\[ \|\xi(T)\| \leq \psi^{\lfloor \frac{T}{\tau} \rfloor} \|\xi(0)\| + \sum_{t=1}^{T} \psi^{\lfloor \frac{T-1}{\tau} \rfloor} \left( \Delta_x(t) + \frac{c(t)}{\gamma} \Delta_n(t) \right). \]
Moreover, define the constant \( c := \sup_t \|\Lambda(t)^\top Q^{-1}_t\| \). The estimation error is asymptotically upper-bounded, i.e.,
\[ \limsup_{t \to \infty} \|\xi(t)\| \leq \tau \left( \Delta_x + \frac{1}{\gamma} \Delta_n \right) \left( 1 + \frac{\gamma}{\lambda} \right). \]
Finally, the error upper bound in (24) is minimized by
\[ \gamma^* = \sqrt{\frac{c\lambda \Delta_n}{\Delta_x}}. \]

B. Stochastic Noise

Here, the case in which \( n \) is a random vector will be considered; namely, we make the following assumption.

Assumption 3. The noise vector \( n \) is an i.i.d. random vector with zero-mean and finite positive definite covariance \( N_t \in \mathbb{R}^{M_t \times M_t}, \ E[n(t)] = 0, \ E[n(t)n(t)^\top] = N_t \).

In this case, a standard choice is to set \( Q_t = N_t \). Denote as \( \mu(t) := E[\xi(t)] \) and \( \Sigma(t) := E[(\xi(t) - \mu(t))(\xi(t) - \mu(t))^\top] \) the mean and the covariance of the estimation error at time \( t \). Given Assumption 3 and by applying the expectation operator to (18), at every time \( T \geq 1 \) we have that
\[ \mu(T) = \prod_{t=1}^{T} \Lambda(t)\xi(0) - \sum_{t=1}^{T} \prod_{k=t}^{T} \Lambda(k)\delta(t). \]
Equation (26) can be used to compute also the error covariance at time $T$:

$$\Sigma(T) = \mathbb{E}[(\xi(T) - \mu(T))(\xi(T) - \mu(T))^\top]$$

$$= \frac{1}{\gamma^2} \mathbb{E} \left[ \left( \sum_{t=1}^{T} \prod_{k=t}^{T} \Lambda(k) A(t) N_t^{-1} n(t) \right) \left( \sum_{t=1}^{T} \prod_{k=t}^{T} \Lambda(k) A(t) N_t^{-1} n(t) \right)^\top \right]$$

$$= \sum_{t=1}^{T} \left( \prod_{k=t}^{T} \Lambda(k) \right) \frac{A(t) N_t^{-1} A(t)}{\gamma^2} \left( \prod_{k=t}^{T} \Lambda(k) \right)^\top.$$

(27)

Similar computations can be used to find $\Sigma(T + 1)$:

$$\Sigma(T + 1) = \sum_{t=1}^{T+1} \left( \prod_{k=t}^{T+1} \Lambda(k) \right) \frac{A(t) N_t^{-1} A(t)}{\gamma^2} \left( \prod_{k=t}^{T+1} \Lambda(k) \right)^\top +$$

$$\Lambda(T + 1) \frac{A(T + 1) N_{T+1}^{-1} A(T + 1)}{\gamma^2} \Lambda(T + 1)^\top.$$

(28)

Comparing equations (27) and (28), it can be shown that the error covariance obeys the linear system

$$\Sigma(t + 1) = \Lambda(t + 1) \Sigma(t) \Lambda(t + 1)^\top + \Lambda(t + 1) \frac{A(t + 1) N_t^{-1} A(t + 1)}{\gamma^2} \Lambda(t + 1)^\top$$

(29)

To conveniently study the estimation error variance, introduce the vector $\sigma(t) := \text{vec}(\Sigma(t))$, $\sigma(t) \in \mathbb{R}^{n^2}$. By exploiting the well known properties of the Kronecker product, the evolution of $\sigma$ can be expressed as

$$\sigma(t) = F(t) \sigma(t - 1) + \frac{1}{\gamma^2} F(t) C(t) m(t)$$

(30)

where $F(t) := \Lambda(t) \otimes \Lambda(t)$, $C(t) := A(t)^\top \otimes A(t)^\top$, and $m(t) := \text{vec}(N_t^{-1})$. Iterating equation (30) yields, for $T \geq 1$,

$$\sigma(T) = \prod_{t=1}^{T} F(t) \sigma(0) + \frac{1}{\gamma^2} \sum_{t=1}^{T} \prod_{k=t}^{T} F(k) C(t) m(t)$$

(31)

The results reported hereafter are proved in Appendix A. Firstly, we provide a direct consequence of Assumption 1.

**Proposition 2.** Consider the system described by

$$\sigma(t) = F(t) \sigma(t - 1).$$

(32)

It holds that

$$\left\| \prod_{k=0}^{T-1} F(t + k) \right\| \leq \psi < 1$$

(33)

and the system (21) is asymptotically stable

$$\lim_{T \to \infty} \sigma(T) = \prod_{t=1}^{T} F(t) \sigma(0) = 0.$$

Proposition 2 is used to prove the next main result.

**Theorem 2.** Let Assumptions 1 and 2 hold, and define $C(t) := \|A^\top(t) \otimes A^\top(t)\|_F$ and $m(t) := \|N_t^{-1}\|_F$. 

June 2, 2020
1) The error mean at time $T$ is such that
\[
\|\mu(T)\| \leq \psi(\frac{T}{\tau}) \|\xi(0)\| + \sum_{t=1}^{T} \psi(\frac{T-t+1}{\tau}) |\Delta_x(t)|
\] (34)

2) The error variance at time $T$ is such that
\[
\|\Sigma(T)\|_F \leq \psi(\frac{T}{\tau}) \|\Sigma(0)\|_F + \sum_{t=1}^{T} \psi(\frac{T-t+1}{\tau}) |C(t)m(t)|
\] (35)

Moreover, set $C := \sup_t \{C(t)\}$ and $m := \sup_t \{m(t)\}$.

1) The error mean is asymptotically upper-bounded by
\[
\limsup_{t \to \infty} \|\mu(t)\| \leq \tau \Delta_x (1 + \frac{\gamma}{\lambda})
\] (36)

2) The error variance is asymptotically upper-bounded by
\[
\limsup_{t \to \infty} \|\Sigma(t)\|_F \leq \frac{\tau C m (1 + \gamma)}{\lambda}.
\] (37)

3) The average distance between the estimate $\hat{x}$ and the true state $x$, namely $\sqrt{\mathbb{E}[(x - \hat{x})^T (x - \hat{x})]} = \sqrt{\mathbb{E}||\xi||^2}$ is asymptotically upper-bounded by
\[
\limsup_{t \to \infty} \sqrt{\mathbb{E}||\xi^2(t)||} \leq \tau \sqrt{\frac{C^2 m^2}{\gamma^4} + \Delta_x^2 (1 + \frac{\gamma}{\lambda})}
\] (38)

Remark 5. Theorem 1 and Theorem 2 have been derived essentially by studying the bounded input-bounded output (BIBO) stability properties of the systems (17), (26), and (30); see the region within the dashed rectangle in Figures 3. In [25], the BIBO stability is proved for linear switching systems which are uniformly exponentially stable. These are systems for which, given an initial condition $x(0)$ and when the input is identically zero, there exists a $\lambda < 1$ and a $c < 1$ such that the norm of the state $x$ can be bounded as
\[
||x(t)|| \leq c e^{\lambda t ||x(0)||}
\]
for any $t \geq 1$ and for any switching path. Unfortunately, this is not the case for the systems (17), (26), and (30), for which a similar property holds but only once every $\tau$ time steps.

Remark 6. Heed that the estimation error is finite for any inertia parameter meeting the condition $\gamma < \infty$. That is, for any finite choice of $\gamma$, the estimation errors upper bounded by (24) and (38) do not diverge.

V. DYNAMIC DISTRIBUTION NETWORK STATE ESTIMATION

In this section, an important application of the DASE, namely, dynamic distribution network state estimation [2], is described. A three-phase power distribution grid having $B + 1$ buses can be modeled by a graph $G = (\mathcal{B}, \mathcal{E})$. Nodes in $\mathcal{B} := \{0, \ldots, B\}$ represent grid buses, and the edges in $\mathcal{L}$ correspond to distribution lines. For simplicity and space limitations we restrict ourselves to only delta connections between phases. Bus $i$ active and reactive power injected from phase $\phi'$ to $\phi$ are respectively denoted by $p_{i}^{\phi\phi'}$ and $q_{i}^{\phi\phi'}$, while its phase $\phi$ to ground complex voltage is denoted by $v_{i}^{\phi}$. Bus $i$ quantities are are collected in the vectors $p_i := [p_i^{ab}, p_i^{bc}, p_i^{ca}]^T$, $q_i := [q_i^{ab}, q_i^{bc}, q_i^{ca}]^T$, and $v_i := [v_i^{a}, v_i^{b}, v_i^{c}]^T$. The substation bus is indexed by $i = 0$ and it is assumed to be an ideal voltage generator.
(slack bus) imposing the nominal voltage \( v_0 = [1, 1 - j2\pi/3, 1 + j2\pi/3]^\top \). The vectors \( v \in \mathbb{C}^3_B \), \( p, q \in \mathbb{R}^3_B \) collect the voltages and power injections at all buses excluding the substation. Let \( Y \) be the three-phase bus admittance matrix.

Power injections are non-linearly related to nodal voltage phasors; however, after linearizing complex power injections around the zero-load voltage profile \( w \), the real part, the imaginary part, and the absolute value of the voltage deviations \( \tilde{v} := v - w \) can be approximated by

\[
\begin{bmatrix}
\Re(\tilde{v}) \\
\Im(\tilde{v}) \\
|\tilde{v}| \\
p \\
q
\end{bmatrix} = \begin{bmatrix}
\Re(M) \\
\Im(M) \\
K \\
I
\end{bmatrix} \begin{bmatrix}
p \\
q
\end{bmatrix}
\]

where \( M, K, \) and \( w \) are derived from the admittance matrix \( Y \); see [26] for the details. Trivially, from (39) it follows that

\[
\begin{bmatrix}
\Re(\tilde{v}) \\
\Im(\tilde{v}) \\
|\tilde{v}| \\
p \\
q
\end{bmatrix} = \begin{bmatrix}
\Re(M) \\
\Im(M) \\
K \\
I
\end{bmatrix} \begin{bmatrix}
p \\
q
\end{bmatrix} := \Phi \begin{bmatrix}
p \\
q
\end{bmatrix}
\]

where we introduce the matrix \( \Phi \in \mathbb{R}^{15B \times 6B} \).

Assume that two kinds of metering devices are used: conventional smart meters, able to measure power injections and voltage magnitudes, and PMUs, able to measure power injections and complex voltages. Buses endowed with smart meters are collected in the set \( M_{SM} \), while buses endowed with PMUs – in the set \( M_{PMU} \). At every time \( t \), the system operator gathers and stacks in the vector \( y(t) \) measurements from a subset of buses, denoted as \( S(t) \). Then,

\[
y(t) = S(t) \begin{bmatrix}
\Re(\tilde{v})^\top (t), \Im(\tilde{v})^\top (t), |\tilde{v}|^\top (t), p^\top (t), q^\top (t)
\end{bmatrix}^\top + \mathbf{n}(t)
\]

where \( \mathbf{n}(t) \) is the measurement noise and \( S(t) \) is a matrix that selects the quantities associated with the buses in \( S(t) \). For simplicity, we assume that \( |S(t)| = S \) for all \( t \). Matrix \( S(t) \) can be written as

\[
S(t) = \begin{bmatrix}
S_{s_1}^\top & \cdots & S_{s_S}^\top
\end{bmatrix}^\top
\]

where every \( S_{s_i} \) can be defined in two ways:

- if \( s_i \in M_{SM} \), then \( S_{s_i} \in \{0, 1\}^{9 \times 15B} \)

\[
S_{s_i} = \begin{bmatrix}
0 & 0 & E_{s_i} & 0 & 0 \\
0 & 0 & 0 & E_{s_i} & 0 \\
0 & 0 & 0 & 0 & E_{s_i}
\end{bmatrix}
\]

June 2, 2020 DRAFT
if \( s_i \in \mathcal{M}_{PMU} \), then \( S_{s_i} \in \{0, 1\}^{15 \times 15}. \)

\[
S_{s_i} = \begin{bmatrix}
E_{s_i} & 0 & 0 & 0 \\
0 & E_{s_i} & 0 & 0 \\
0 & 0 & E_{s_i} & 0 \\
0 & 0 & 0 & E_{s_i}
\end{bmatrix}
\]  \hspace{1cm} (43)

and where \( E_{s_i} = (e_{s_i} \otimes I) \), \( E_{s_i} \in \mathbb{R}^{3 \times 3}. \) The value of \( M_t \), i.e., the size of \( y(t) \), varies as a function of the type of reporting metering devices. For instance, if at time \( t \) the system operator gathers measurements from \( C \) buses in \( \mathcal{M}_{SM} \) and \( S - C \) buses in \( \mathcal{M}_{PMU} \), then \( M_t = 9C + 15(S - C). \) Finally, the measurement noise \( n(t) \) is assumed to be zero-mean with diagonal covariance \( N_t. \)

Set the nodal power injections as the state of the network and denote \( x := [p^\top \ q^\top]^\top, x \in \mathbb{R}^{6B}. \) By combining (40) with (41), we obtain the following linear measurement model

\[
y(t) = S(t)\Phi x(t) + n(t). \]  \hspace{1cm} (44)

Remember that measurements are processed as they come in, and that \( y(t) \) carries information about a limited number of buses. We make the following Assumption.

**Assumption 4.** There exists a constant \( \tau > 0 \) such that the system operator gathers measurements from every bus at least once in the interval \([t, t+1, \ldots, t+\tau]\), for every \( t = 1, 2, \ldots. \)

Assumption 4 ensures that Assumption 1 holds true for the case of interest, as proven in the next result.

**Lemma 3.** Let Assumption 4 holds. Then,

\[
\bigcap_{k=0}^{\tau-1} \ker \left( S(t+k)\Phi \right) = \{0\}. \]  \hspace{1cm} (45)

**Proof:** Suppose equation (45) does not hold, i.e., there exists \( x' \neq 0 \) such that

\[
x' \in \bigcap_{k=0}^{\tau-1} \ker \left( S(t+k)\Phi \right),
\]

and let the \( j \)-th entry of \( x' \) be different from zero, \( x'_j \neq 0. \) Assumption 4 ensures that, for every \( t \), there exists a \( t' \leq \tau \) such that \( j \in S(t+t') \). Let \( y' \) be the vector such that

\[
y' = S(t+t')\Phi x'. \]  \hspace{1cm} (46)

By using equations (40), (42), (43), and (46), it is easy to verify that \( y'_j = x'_j \neq 0, \) meaning that \( x' \notin \ker S(t+t')\Phi. \)

After plugging (44) into (6), the state estimator for the power distribution grid can be written as

\[
\hat{x}(t) = \Lambda(t)\hat{x}(t-1) + \frac{1}{\gamma}\Lambda(t)\Phi^\top S(t)^\top N_t^{-1} y(t). \]  \hspace{1cm} (47)
where we set
\[ \Lambda(t) = \gamma(\Phi^\top S(t)^\top N_t^{-1} S(t)\Phi + \gamma I)^{-1}. \] (48)

**Remark 7.** A similar dynamic state estimator was adopted in [27], where a prediction-correction method is applied to DSSE. The scheme proposed in this paper does not require a prediction step and can handle asynchronous measurements. Another similar approach can be found in [28]. However, authors of [28] are considering systems fully observable and the PPM is used to solve an optimization problem providing the state estimate at a certain time instant rather than to track the state variation.

**Remark 8.** Classically, voltage phasors are considered state in power systems because every other quantity can be computed explicitly given the voltages. In this paper, similarly to [27], we use a broader definition of state, which is the minimum number of variables that is required to obtain a solution of the power-flow equations.

**VI.Numerical Results**

The performance of the DASE are evaluated next. Precisely, we consider a scenario in which

- the state \( x \) has dimension \( N = 15 \). The state variation \( \delta \) is drawn from a uniform distribution \( \mathcal{U}(-\Delta_x/2, \Delta_x/2) \), for every \( t \), with \( \Delta_x = 1 \);
- the measurement vector \( y \) has, for simplicity, fixed dimension \( M_t = M = 3 \);
- at every time step \( t \), the model matrix \( A(t) \) is chosen from a library of 10 matrices. Each matrix in the library is an \( M \) by \( N \) matrix of standard normal random variables; the matrix is then scaled so that Frobenius norm is equal to 1.

Five thousands Monte Carlo simulations are run in both the bounded noise and the stochastic noise case. In every simulation run, the sequence of model matrices \( \{A(t)\}_{t \geq 1} \) is generated by randomly selecting a matrix from the matrices library so that \( \tau = 4 \). Since the same library of model matrices is used, all the Monte Carlo simulations share the same values of \( c, \tau \) and \( \lambda \).

**A. Bounded Noise Case**

Here, the noise vector \( n(t) \) is generated by drawing from a uniform distribution \( \mathcal{U}(-\Delta_n/2, \Delta_n/2) \), with \( \Delta_n = 1 \). For every \( t \), matrix \( Q_t \) is set to \( Q_t = I \). Figure 4 reports the average tracking error over the 5,000 Monte Carlo simulations for different choices of \( \gamma \). According to equation (25), the best inertia parameter should be \( \gamma^* = 0.25 \). Figure 5 reports the bound (24), as a function of the inertia parameter \( \gamma \) and denoted as

\[ H_b(\gamma) := \tau \left( \Delta_x + \frac{1}{\gamma} c \Delta_n \right) \left( 1 + \gamma \right). \]

If the inertia parameter is chosen too small, e.g. see \( \gamma = 0.01 \), the estimation uses almost no past information and becomes very sensitive to the noise. On the other hand, if the inertia parameter is too large, e.g. see \( \gamma = 2 \), the estimation moves very sluggish and does not react much to the recent measurement information. When the inertia parameter strikes a balance between new and old information, it can track the true value relatively closely.

Finally, Figure 6 shows the estimation of one particular state element over time in one particular Monte Carlo run. The curve associated with \( \gamma^* \) is the best in tracking the true state trajectory.
Fig. 4. Mean estimation error vs. time under different inertia parameter $\gamma$ settings, averaged over 5,000 Monte Carlo simulations under the bounded noise case.

Fig. 5. Upper bound on the estimation error with bounded noise vs. the inertia parameter $\gamma$ which is RHS of Equation (24) in Theorem 1.

B. Stochastic Noise Case

Here, the noise vector $\mathbf{n}(t)$ is generated by drawing from a Gaussian distribution with zero mean and diagonal finite covariance, namely, $\mathbf{n}(t) \sim \mathcal{N}(0, \Delta_n \mathbf{I})$ for every $t$, with $\Delta_n = 0.25$. For every $t$, matrix $\mathbf{Q}_t$ is set to $\mathbf{Q}_t = \Delta_n \mathbf{I}$. The estimation error averaged over 5,000 Monte Carlo simulations over time is shown in Figure 8 for different choices of $\gamma$.

Minimizing the upper bound provided in equation (38), denoted as

$$H_s(\gamma) := \tau \sqrt{\frac{\bar{m}^2}{\gamma^2} + \Delta_x^2 \left(1 + \frac{\gamma}{\lambda}\right)}$$

and reported in Figure 7, yields to the theoretical optimal inertia parameter $\gamma^* = 0.4$. The inertia parameter, $\gamma^* = 0.4$ gives the experimental low average-case error. Increasing the inertia parameter to $\gamma = 25\gamma^* = 10$ or decreasing the inertia parameter to $\gamma = \frac{1}{20}\gamma^* = 0.02$ almost doubles the error.

Finally, looking at estimation of one particular state over time in Figure 9, the results are very similar to that of the bounded noise; an inertia parameter that is too large lags and does not respond immediately to changes in the
Fig. 6. Estimation of $x_0(t)$ over time under different inertia parameter settings with bounded noise.

Fig. 7. Upper bound on the expected estimation error with stochastic noise vs. the inertia parameter $\gamma$ which is RHS of Equation (38) in Theorem 2.

state, and an inertia parameter that is too small is very sensitive to the noise. However, the inertia parameter $\gamma^*$ that minimizes the worst-case error strikes a good balance between the two extremes.

VII. A POWER SYSTEM CASE

In this section, the DASE algorithm is used to solve the problem of dynamic state estimation on the 3-phase distribution power system shown in Figure 10, namely, the IEEE 37 bus test feeder [29]. Buses in the network host two types of measurement devices: smart meters, that provide measurements of active power, reactive power, and voltage magnitude, and PMU, providing measurement of active power, reactive power, voltage magnitude and voltage angle.

The network state, i.e., the nodal power injections, is updated every second in the following way. At every time, the state variation $\delta(t)$ components are drawn from a truncated Gaussian distribution with zero mean and with a relative standard deviation of 0.0068; see the statistical analysis of loads in [30]. Hence, the network state change
meets equation (3). Sensors are affected by Gaussian zero-mean measurement noise and have different reporting rates:

- smart meters provide measurements once every hour and introduce noise that is modeled as a zero mean Gaussian random variable with a relative standard deviation $\tilde{\sigma}_{SM}$ and truncated outside $[-3\tilde{\sigma}_{SM}, 3\tilde{\sigma}_{SM}]$ to reflect a maximum error of 0.5% [31];
- PMU provide measurements every minute and introduce noise that is modeled as a zero mean Gaussian random variable with a relative standard deviation $\tilde{\sigma}_{PMU}$ and truncated outside $[-3\tilde{\sigma}_{PMU}, 3\tilde{\sigma}_{PMU}]$ to reflect a maximum error of 0.05% [21].

Moreover, at every time $t$:

- matrix $A(t)$ is built based on the measurement gathered at time $t$ and accordingly to what described in Section V
- matrix $Q$, used by the state estimation algorithms is of the associated variances.

The DASE was compared with a Weighted Least Squares (WLS) estimation algorithm adapted for asynchronous
measurements. In general, the WLS algorithm assumes that there are enough measurements at each timestep for the problem to be over-determined, but this is not necessarily true for asynchronous measurements. Thus, the WLS algorithm was modified by concatenating the current timestep’s measurements with the previous 60 minutes of outdated past measurements, keeping only the most recent one if a measurement appears more than once, to create a full set. In a sense, we are treating the outdated past measurements as pseudo-measurements for state estimation.

The state estimation algorithms are tested on 250 Monte Carlo simulations. Since every sensor reports its measurement at least once every hour and state estimation is performed every minute, we have $\tau \leq 60$. Although our approach relies on the approximate grid model of (39), voltages were calculated using the full ac grid model throughout our tests.

First, the DASE’s performance for different values of $\gamma$ is studied. Figure 11 shows the average state estimation error among the Monte Carlo simulations under various settings of the inertia parameter $\gamma$. Here, $\gamma^\ast$ minimizes the RHS of equation (38), whereas $\gamma^+$, experimentally found, is the value that gives the minimum estimation error. Second, we show how the DASE compares with the adopted WLS. Figure 12 reports the average relative estimation error in the power injections among the Monte Carlo runs. It can be seen that the WLS is very inaccurate for the first 60 minutes before stabilizing to a more accurate level since it has not experienced enough measurements to have system observability. Since the goal of state estimation in power systems is usually to infer the system voltages, Figure 13 reports the average relative error in voltage estimation over time. Notably, the DASE outperforms the WLS. Finally, Figure 14 shows the tracking of the real power injection for one phase at bus 23.

In general, the behavior is similar to the one described in Section VI: the estimate results jumpy from its sensitivity to the measurement error or not very responsive in tracking the true state when $\gamma$ is smaller or $\gamma$ is bigger than the optimal parameter, respectively.

VIII. CONCLUSION

We have proposed a dynamic state estimation algorithm for systems with heterogeneous sensors. The estimator has a recursive expression in which the new estimate is found as a function of the previous estimate and of the
gathered measurements. The estimator is designed to tackle the cases in which the system is not observable, namely, when the measurements do not contain enough information to reconstruct the entire system state. The estimation error was proved to be bounded under mild assumptions. The estimator was applied to an interesting problem, namely, the distribution grid state estimation.

APPENDIX A

PROOFS OF SECTION VI-B

Before proving, we provide the following result, which is a direct consequence of Assumption I. The next proposition ensures that the system enclosed in the dashed line in Figure 3 is asymptotically stable.

Proof of Proposition 7 First, heed that equation (9) implies for every $t$ that

$$\|A(t)\| \leq 1$$

yielding

$$\left\| \prod_{k=0}^{\tau} A(t+k) \right\| \leq 1$$
Now consider any vector $v$ such that $\|x\| = 1$ and assume that

$$\left\| \prod_{k=0}^{r} \Lambda(t + k)x \right\| = 1$$

(51)

For equation (51) to hold, it must be that

$$\|\Lambda(t)x\| = 1$$

$$\|\Lambda(t + 1)\Lambda(t)x\| = 1$$

$$\|\Lambda(t + 2)\Lambda(t + 1)\Lambda(t)x\| = 1$$

and so on. Consider now the decomposition of $\Lambda(t)$, given by (9). Since $\begin{bmatrix} U(t) & V(t) \end{bmatrix}$ spans $\mathbb{R}^N$, the vector $v$ can be written as

$$x = \begin{bmatrix} U(t) & V(t) \end{bmatrix} \begin{bmatrix} \beta_u(t) \\ \beta_v(t) \end{bmatrix}$$

(52)
Consider now the product $\Lambda(t)v$, which can expressed as

$$
\Lambda(t)x = \begin{bmatrix}
U(t)^\top & V(t)^\top
\end{bmatrix}
\begin{bmatrix}
dg \left( \frac{\gamma}{\gamma + \lambda_i(t)} \right) & 0 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
U(t)^\top \\
V(t)^\top
\end{bmatrix}x
= \begin{bmatrix}
U(t) \\
V(t)
\end{bmatrix}dg \left( \frac{\gamma}{\gamma + \lambda_i(t)} \right) \beta_u(t) + \beta_v(t).
$$

It is then easy to see that $\|\Lambda(t)x\| = 1$ if and only if $x = V(t)\beta_v(t)$. Moreover, in this case $\Lambda(t)x = x$, i.e., $x$ is an eigenvector of $\Lambda(t)$ associated with the eigenvalue 1 and $x \in \ker \Lambda(t)$. Hence, it holds

$$
\Lambda(t+1)\Lambda(t)x = \Lambda(t+1)x
$$

Again, $\|\Lambda(t+1)x\| = 1$ if and only if $x$ is an eigenvector of $\Lambda(t+1)$ associated with the eigenvalue 1 and $x \in \ker \Lambda(t+1)$.

By iterating the previous reasoning eventually we can state that equation (51) holds only if $x \in \ker \Lambda(t)$, $\ker \Lambda(t+1), \ldots, \ker \Lambda(t+\tau)$. But this contradicts Assumption [1]. As a consequence,

$$
\max_{0 \leq k \leq \tau-1} \left\{ \frac{\gamma}{\gamma + \lambda_1(k)} \right\} \leq \psi < 1
$$

The asymptotic stability of (21) follows directly from (22).

**Proof of Theorem [7]** Taking the norm on both sides of (18) and using the triangle inequality yields

$$
\|\xi(t)\| \leq \left\| \prod_{t=1}^{T} \Lambda(t)\xi(0) \right\| + \sum_{t=1}^{T} \left\| \prod_{k=t}^{T} \Lambda(k) \left( \frac{\gamma}{\gamma + \lambda_1} Q^{-1}_t n(t) - \delta(t) \right) \right\|
$$

$$
= \left\| \prod_{t=1}^{T} \Lambda(i) \right\| \|\xi(0)\| + \sum_{t=1}^{T} \left\| \prod_{k=t}^{T} \Lambda(k) \right\| \left\| \frac{\gamma}{\gamma + \lambda_1} Q^{-1}_t n(t) - \delta(t) \right\|. \quad (54)
$$

The norm of $\frac{\gamma}{\gamma + \lambda_1} Q^{-1}_t n(t) - \delta(t)$ in the second terms on the right-hand-side of (54) can be bounded as:

$$
\left\| \frac{\gamma}{\gamma + \lambda_1} Q^{-1}_t n(t) - \delta(t) \right\| \leq \Delta_x(t) + \frac{c}{\gamma} \Delta_n(t).
$$

Let $\rho$ and $r$ be scalars such that, for any $t'$, $t' = \rho \tau + r$, with $r < \tau$, namely, $\rho = \left\lfloor \frac{t'}{\tau} \right\rfloor$. Then, it holds

$$
\left\| \prod_{i=1}^{\rho \tau} \Lambda(i) \right\| \leq \prod_{i=1}^{\tau} \Lambda(i) \left\| \prod_{i=\tau+1}^{2\tau} \Lambda(i) \right\| \cdots \left\| \prod_{i=(\rho-1)\tau+1}^{\rho \tau} \Lambda(i) \right\| \leq \psi^{\rho \tau} \cdot \psi \cdot \psi \cdots \psi \cdot 1
$$

where the last step is because of Proposition [1] Equation (23) then follows.
To prove equation (24), express $T$ as $T = \rho \tau + r$, with $r < \tau$. Note that, as $T$ goes to infinity, equation (54) tends to

$$
\sum_{\phi=0}^{\rho} \sum_{t=\phi+1}^{T} \left\| \prod_{k=t}^{T} \Lambda(k) \right\| \left\| \frac{1}{\gamma} A(t)^T Q_{\ell}^{-1} n(t) - \delta(t) \right\| \leq \left( \frac{1}{\gamma} c \Delta_n + \Delta_x \right) \sum_{\phi=0}^{\rho} \tau \phi = \tau \left( \frac{1}{\gamma} c \Delta_n + \Delta_x \right) \left( 1 + \frac{2}{\gamma} \right).
$$

since the first term of (54) vanishes due to Proposition 1 and, for any $t$,

$$
\left\| \frac{1}{\gamma} A(t)^T Q_{\ell}^{-1} n(t) - \delta(t) \right\| \leq \left\| \delta(t) \right\| + \left\| \frac{1}{\gamma} A(t)^T Q_{\ell}^{-1} n(t) \right\| \leq \Delta_x + \frac{\Delta_n}{\gamma} \max \{ \left\| A(t)^T Q_{\ell}^{-1} \right\| \} = \Delta_x + \frac{1}{\gamma} c \Delta_n.
$$

Finally, equation (25) can be easily found by minimizing the left hand side of (24).

**APPENDIX B**

**PROOF OF SECTION VI-B**

**Proof of Proposition 2** First, we characterize the eigenvalues and the eigenvectors of $F(t)$. Let $x_i$ and $x_j$ be two eigenvectors of $\Lambda(t)$ associated with two eigenvalues $\mu_i$ and $\mu_j$, i.e., $\Lambda(t)x_i = \mu_i x_i, \Lambda(t)x_j = \mu_j$. Then, $x_i \otimes x_j$ is an eigenvector of $F(t)$ associated with the eigenvalue $\mu_i \mu_j$, since

$$
F(t)(x_i \otimes x_j) = (\Lambda(t) \otimes \Lambda(t))(x_i \otimes x_j) = (\Lambda(t)x_i) \otimes (\Lambda(t)x_j) = \mu_i \mu_j x_i \otimes x_j.
$$

Hence, the spectrum of $F(t)$ is given by

$$
eig F(t) = \left\{ 1, \frac{\gamma}{\gamma + \lambda_1(t)}, \ldots, \frac{\gamma}{\gamma + \lambda_1(t)}, \frac{\gamma}{\gamma + \lambda_1(t)} \frac{\gamma}{\gamma + \lambda_2(t)} \ldots, \frac{\gamma}{\gamma + \lambda_1(t)} \frac{\gamma}{\gamma + \lambda_1(t)} \lambda_1(t) \lambda_1(t) \ldots \right\}.
$$

where 1 has multiplicity $k_1^2$, each $\frac{\gamma}{\gamma + \lambda_1(t)}$ has multiplicity $2k_1$ and each $\frac{\gamma}{\gamma + \lambda_1(t)} \frac{\gamma}{\gamma + \lambda_1(t)}$ has multiplicity 1. Heed that the biggest eigenvalue of $F(t)$ smaller than 1, similarly to $\Lambda(t)$, is $\frac{\gamma}{\gamma + \lambda_1(t)}$. From (55) it is also clear that the eigenvectors of $F(t)$ associated with the eigenvalue 1 have the form $v_i \otimes v_j$, where $v_i, v_j$ are the $i$-th and the $j$-th column of $V(t)$, respectively. Now consider any vector $x \in \mathbb{R}^{n^2}$, with $\|x\| = 1$ and the product $\|\prod_{k=0}^{\tau} F(t + k)\|$. Retracing the same reasoning used in the proof of Proposition 1 it can be shown that $\|\prod_{k=0}^{\tau} F(t + k)\| = 1$ if and only if $x \in \ker F(t + k) = \text{span} \{ v_i \otimes v_j, v_i, v_j \in \ker \Lambda(t + k) \}$, for every $k = 0, \ldots, \tau$, contradicting Assumption 1. Hence,

$$
\left\| \prod_{k=0}^{\tau-1} F(t + k)x \right\| \leq \max_{0 \leq k \leq \tau} \left\{ \frac{\gamma}{\gamma + \lambda_1(t + k)} \right\} \leq \max_{t} \left\{ \frac{\gamma}{\gamma + \lambda_1(t)} \right\} = \psi < 1
$$

The asymptotic stability of $\|x\|$ follows directly from (33).

**Proof of Theorem 2** Applying the triangle inequality to equations (26) and (31), the norm of $\mu(T)$ and $\sigma(T)$ can be upper bounded by
\[\|\mathbf{\mu}(T)\| \leq \left\| \prod_{t=1}^{T} \Lambda(t)\mathbf{\xi}(0) \right\| \quad (56)\]

\[\|\mathbf{\sigma}(T)\| \leq \left\| \prod_{t=1}^{T} \mathbf{F}(t)\mathbf{\sigma}(0) \right\| + \left\| \frac{1}{\gamma^2} \sum_{t=1}^{T} \prod_{k=t}^{T} \mathbf{F}(k)\mathbf{C}(t)\mathbf{m}(t) \right\|. \quad (57)\]

Note that \(\|\mathbf{\delta}(t)\| \leq \Delta_x(t)\) and that \(\|\mathbf{C}(t)\mathbf{m}(t)\| \leq C(t)\mathbf{m}(t)\). Equations (34) and (35) can be obtained by retracing the same steps used to prove (23).

The first term in the RHS of (56) tends to zero as \(T\) goes to infinity, due to Proposition 1. Consider now the second term and let \(\rho\) and \(r\) be scalars such that \(T = \rho\tau + r\), with \(r < \tau\). It holds

\[
\left\| \sum_{t=1}^{T} \prod_{k=t}^{T} \Lambda(k)\mathbf{\delta}(t) \right\| \leq \Delta_x \sum_{\phi=0}^{\rho \cdot \min\{\phi+1,\tau,T\}} \sum_{t=\phi+1}^{\phi+\tau} \left\| \prod_{k=t}^{T} \Lambda(k) \right\| \leq \Delta_x \sum_{\phi=0}^{\rho} \tau \psi^{\phi} \leq \Delta_x \tau \sum_{\phi=0}^{\infty} \psi^{\phi} \leq \Delta_x \tau \left( 1 + \frac{\gamma}{\lambda} \right).
\]

Similarly, as \(T\) goes to infinity, Proposition 2 ensures that the first term of the right hand side of (57) goes to zero.

Now consider the second term. Then, we have

\[
\left\| \sum_{t=1}^{T} \prod_{k=t}^{T} \frac{\mathbf{F}(k)}{\gamma^2} \mathbf{C}(t)\mathbf{m}(t) \right\| \leq \left\| \sum_{\phi=0}^{\rho} \sum_{t=\phi+1}^{\phi+\tau} \left\| \prod_{k=t}^{T} \frac{\mathbf{F}(k)}{\gamma^2} \mathbf{C}(t)\mathbf{m}(t) \right\| \right. \leq \frac{Cm}{\gamma^2} \sum_{\phi=0}^{\rho \cdot \min\{\phi+1,\tau,T\}} \sum_{t=\phi+1}^{\phi+\tau} \left\| \prod_{k=t}^{T} \mathbf{F}(k) \right\| \leq \frac{\bar{m}}{\gamma^2} \sum_{\phi=0}^{\rho} \tau \psi^{\phi} \leq \frac{mT}{\gamma^2} \sum_{\phi=0}^{\infty} \psi^{\phi} = \frac{mT}{\gamma^2} \left( 1 + \frac{\gamma}{\lambda} \right).
\]

Finally, to prove equation (38), heed that

\[
\mathbb{E}[\mathbf{\xi}^T(t)\mathbf{\xi}(t)] = \mathbb{E}[((\mathbf{\xi}(t) - \mathbf{\mu}(t))^T(\mathbf{\xi}(t) - \mathbf{\mu}(t)))] + \mathbf{\mu}(t)^T\mathbf{\mu}(t) \\
= \mathbb{E}[(\text{Tr}((\mathbf{\xi}(t) - \mathbf{\mu}(t))(\mathbf{\xi}(t) - \mathbf{\mu}(t))^T)) + \mathbf{\mu}(t)^T\mathbf{\mu}(t)] \\
= \left\| \mathbf{\Sigma}(t) \right\|^2 + \left\| \mathbf{\mu}(t) \right\|^2 \\
= \tau^2 \left( 1 + \frac{\gamma}{\lambda} \right)^2 \left( \frac{C^2m^2}{\gamma^4} + \Delta_x^2 \right)
\]

where we used equation (56) and (57).

**REFERENCES**

[1]  L. Schenato and F. Fiorentin, “Average timesynch: A consensus-based protocol for clock synchronization in wireless sensor networks,” *Automatica*, vol. 47, no. 9, pp. 1878 – 1886, 2011. [Online]. Available: [http://www.sciencedirect.com/science/article/pii/S0005109811003116](http://www.sciencedirect.com/science/article/pii/S0005109811003116)
[2] G. Cavraro, E. Dall’Anese, and A. Bernstein, “Dynamic power network state estimation with asynchronous measurements,” in Proc. IEEE Global Conf. on Signal and Inf. Process., Ottawa, Canada, Nov. 2019.

[3] Y. Hu, Z. Jin, S. Qi, and C. Sun, “Estimation fusion for networked systems with multiple asynchronous sensors and stochastic packet dropouts,” Journal of the Franklin Institute, vol. 354, no. 1, pp. 145 – 159, 2017. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0016003216303635

[4] P. Swerling, “Modern state estimation methods from the viewpoint of the method of least squares,” IEEE Trans. Autom. Contr., vol. 16, no. 6, pp. 707–719, 1971.

[5] S. M. Kay, Fundamentals of Statistical Signal Processing, Vol. I: Estimation Theory. Upper Saddle River, NJ: Prentice Hall, 1993.

[6] E. Dall’Anese, A. Simonetto, S. Becker, and L. Madden, “Optimization and learning with information streams: Time-varying algorithms and applications,” IEEE Signal Process. Mag., vol. 37, no. 3, pp. 71–83, 2020.

[7] L. P. Yan, D. H. Zhou, M. Y. Fu, and Y. Q. Xia, “State estimation for asynchronous multirate multisensor dynamic systems with missing measurements,” IET Signal Processing, vol. 4, no. 6, pp. 728–739, 2010.

[8] M. S. Mahmoud and M. F. Emzir, “State estimation with asynchronous multi-rate multi-smart sensors,” Information Sciences, vol. 196, pp. 15 – 27, 2012. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S002002551200062X

[9] A. S. Matveev and A. V. Savkin, “The problem of state estimation via asynchronous communication channels with irregular transmission times,” IEEE Trans. Autom. Contr., vol. 48, no. 4, pp. 670–676, 2003.

[10] R. T. Rockafellar, “Monotone operators and the proximal point algorithm,” SIAM Journal on Control and Optimization, vol. 14, no. 5, pp. 877–898, 1976.

[11] O. Güler, “On the convergence of the proximal point algorithm for convex minimization,” SIAM Journal on Control and Optimization, vol. 29, no. 2, pp. 403–419, 1991.

[12] N. Parikh and S. Boyd, “Proximal algorithms,” Found. Trends Optim., vol. 1, no. 3, p. 127239, Jan. 2014. [Online]. Available: https://doi.org/10.1561/2400000003

[13] B. Sinopoli, L. Schenato, M. Franceschetti, K. Poolla, M. I. Jordan, and S. S. Sastry, “Kalman filtering with intermittent observations,” IEEE Trans. Autom. Contr., vol. 49, no. 9, pp. 1453–1464, 2004.

[14] A. Monticelli, “Electric power system state estimation,” Proceedings of the IEEE, vol. 88, no. 2, pp. 262–282, 2000.

[15] L. Schenato, G. Barchi, D. Macii, R. Arghandeh, K. Poolla, and A. Von Meier, “Bayesian linear state estimation using smart meters and pmus measurements in distribution grids,” in 2014 IEEE International Conference on Smart Grid Communications (SmartGridComm), Nov 2014, pp. 572–577.

[16] C. Carquex, C. Rosenberg, and K. Bhattacharya, “State estimation in power distribution systems based on ensemble kalman filtering,” IEEE Trans. Power Syst., vol. 33, no. 6, pp. 6600–6610, Nov 2018.

[17] M. Netto, J. Zhao, and L. Mili, “A robust extended kalman filter for power system dynamic state estimation using pmu measurements,” in 2016 IEEE Power and Energy Society General Meeting (PESGM), July 2016, pp. 1–5.

[18] P. L. Donti, Y. Liu, A. J. Schmitt, A. Bernstein, R. Yang, and Y. Zhang, “Matrix completion for low-observability voltage estimation,” IEEE Transactions on Smart Grid, vol. 11, no. 3, pp. 2520–2530, 2020.

[19] F. F. Wu and A. Monticelli, “Network observability: Theory,” IEEE Transactions on Power Apparatus and Systems, vol. PAS-104, no. 5, pp. 1042–1048, 1985.

[20] J. Zheng, D. W. Gao, and L. Lin, “Smart meters in smart grid: An overview,” in 2013 IEEE Green Technologies Conference (GreenTech), April 2013, pp. 57–64.

[21] A. von Meier, D. Culler, A. McEachern, and R. Arghandeh, “Micro-synchrophasors for distribution systems,” in Proc. IEEE Conf. on Innovative Smart Grid Technologies, Washington, DC, Feb. 2014.

[22] A. Angioni, C. Muscas, S. Sulis, F. Ponci, and A. Monti, “Impact of heterogeneous measurements in the state estimation of unbalanced distribution networks,” in 2013 IEEE International Instrumentation and Measurement Technology Conference (I2MTC), May 2013, pp. 935–939.

[23] A. Alimardani, F. Therrien, D. Atanackovic, J. Jatskevich, and E. Vahedi, “Distribution system state estimation based on nonsynchronized smart meters,” IEEE Trans. Smart Grid, vol. 6, no. 6, pp. 2919–2928, Nov 2015.

[24] A. Primadianto and C. Lu, “A review on distribution system state estimation,” IEEE Trans. Power Syst., vol. 32, no. 5, pp. 3875–3883, Sep. 2017.

[25] G. Michaletzky and L. Gerencser, “Bibo stability of linear switching systems,” vol. 47, no. 11, pp. 1895–1898, 2002.
[26] A. Bernstein and E. Dall’Anese, “Linear power-flow models in multiphase distribution networks,” in *2017 IEEE PES Innovative Smart Grid Technologies Conference Europe (ISGT-Europe)*. IEEE, 2017, pp. 1–6.

[27] J. Song, E. DallAnese, A. Simonetto, and H. Zhu, “Dynamic distribution state estimation using synchrophasor data,” *IEEE Trans. Smart Grid*, vol. 11, no. 1, pp. 821–831, 2020.

[28] G. Wang, G. B. Giannakis, and J. Chen, “Robust and scalable power system state estimation via composite optimization,” *IEEE Trans. Smart Grid*, vol. 10, no. 6, pp. 6137–6147, 2019.

[29] W. H. Kersting, *Distribution System Modeling and Analysis*. New York, NY: CRC Press, 2001.

[30] G. Cavraro and R. Arghandeh, “Power distribution network topology detection with time-series signature verification method,” *IEEE Trans. Power Syst.*, vol. 33, no. 4, pp. 3500–3509, 2018.

[31] EEI-AEIC-UTC, “Smart meters and smart meter systems: A metering industry perspective,” 2011.