Transforming Metastable Memories: 
The Nonequilibrium Thermodynamics of Computation

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Framing computation as the transformation of metastable memories, we explore its fundamental thermodynamic limits. The true power of information follows from a novel decomposition of nonequilibrium free energy derived here, which provides a rigorous thermodynamic description of coarse-grained memory systems. In the nearly-quasistatic limit, logically irreversible operations can be performed with thermodynamic reversibility. Yet, here we show that beyond the reversible work Landauer’s bound requires of computation, dissipation must be incurred both for modular computation and for neglected statistical structure among memory elements used in a computation. The general results are then applied to evaluate the thermodynamic costs of all two-input–one-output logic gates, including the universal NAND gate. Interwoven discussion clarifies the prospects for Maxwellian demons and information engines as well as opportunities for hyper-efficient computers of the future.

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I. INTRODUCTION

Modern scientific understanding suggests that computation can be performed without any dissipation at all—a perplexing result since we still plug in our computers and eat to fuel our brains every day. To reconcile this discrepancy between theoretical possibility and familiar reality requires a nonequilibrium thermodynamics of realistic computation: where nonequilibrium distributions—corresponding to metastable memories—are transformed under practical constraints by controlled driving in finite time. From the perspective of nonequilibrium thermodynamics employed here, logical irreversibility is indeed compatible with thermodynamic reversibility if accompanied by a metastable increase in nonequilibrium-addition to free energy which can later be leveraged to reclaim the original work input. Hence: computation without dissipation. However, our demands for speed and modularity each imply trade-offs that necessitate dissipation, while practical limitations of the controller’s knowledge and dexterity further challenge the attainable thermodynamic efficiency of computation. Here we will develop a few of the fundamental thermodynamic consequences of transforming metastable memories and identify several practical opportunities for greater energetic efficiency.

The following contains several new results, including: (1) A new decomposition of the nonequilibrium free energy that shows under what circumstances a coarse-grained description is sufficient to understand the thermodynamics of metastable memory transformations; (2) Implications for composite memory systems and the role of knowledge in work extraction; (3) The thermodynamic cost of modular computation, which generalizes a recent result by Boyd et al. [1]; and (4) The minimal work expected of any two-input–one-output logic function, and the dissipation incurred when these circuits are not designed for the statistics of the memories they transform. We close with a short tutorial that explicitly calculates the fundamental thermodynamic limits of the universal NAND gate.

II. METASTABLE MEMORY SYSTEMS

We start by considering a memory system, which is simply a physical system meant to store information. During computations, the dynamics of the memory system is driven by an external work reservoir to transform the memory from its initial state to its final state. For the memory system to be of much practical utility, it should be able to store memories robustly between computations. One way to achieve this is with non-volatile memory elements that—through metastability—retain their memories over long timescales without active power consumption, even when the computer is turned off. At each moment, the work reservoir can exert influences according to the vector quantities $x \in \chi$. For example, $x$ may represent the configuration of the applied electromagnetic field, a collection of piston positions, or any other controllable factors that influence the Hamiltonian of the memory system. The instantaneous Hamiltonian $H_x$ of the memory system determines the instantaneous energies $\{E_x(s)\}_{s \in S}$ of the system’s microstates $S$. The control parameter $x$ is held fixed while the memory is to be retained. Changes to the memory system are implemented by a trajectory of time-varying control $x_{0:\tau}$ (often called a ‘protocol’ in the literature) over a duration $\tau$ that drives the system to a new state.

Computations utilizing metastable memories imply a strong separation of timescales in the non-driven dynamics of the memory system, such that the dynamics of the various metastable regions are nearly autonomous with respect to each other and can quickly establish local equilibria. The autonomy within certain regions of state space suggests that we partition the set of microstates $S$ into a set of metastable memory states $\mathcal{M}$. The system is also in contact with an effectively-memoryless heat bath at temperature $T$ with which it exchanges energy, which enables the system’s relaxation to both local and global equilibrium.

Over very long times (times much longer than any computation performed by the system, and much longer even than the waiting times between computations), the memory system—if left undriven, experiencing only the static control setting

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1 Two types of memory are common in practical computers. The first uses a non-volatile metastable memory that does not require energetic upkeep and retains its memory even when the computer is powered off. The second type requires active power to retain the memory, as in CMOS transistor architectures, where the inevitable leakage of currents implies constant power consumption. Without power, the volatile memory is lost. For reasons of both anticipated supremacy in energetic efficiency and clarity of our exposition, we choose to describe transformations of the former non-volatile type of memory in the following. However, we expect our results to maintain at least some relevance in the energetic limits of transformations of active memories, where the work and dissipation discussed here should be roughly additive to the background ‘housekeeping’ power consumption by active circuits.
A deterministic computation $C : \mathcal{M} \to \mathcal{M}$ is an operation mapping the set of memory states $\mathcal{M}$ to itself. In practice, it is implemented by a driving protocol $x_{0,\tau}$ that controls the evolution of the system for a duration $\tau$. Fig. 1 (Left) shows schematically how the protocol can change the energy landscape to induce (Right) a net transition among microstates that corresponds to a (generically stochastic) computation on the coarse-grained memory states (depicted as the different shaded regions). The system should start and end with the same resting influence $x_0 = x_\tau$, as can be seen in the top-left bubble of Fig. 1, if a consistent metastable memory landscape is desired between computations.

The set of all protocols that reliably implement a computation in finite time with the minimal resultant work dissipation.

However, metastability implies that this timescale of global relaxation is much longer than the timescale of computation. On the timescale between computations, all probability density within each memory state $m \in \mathcal{M}$ is assumed to relax approximately to its local-equilibrium distribution $\pi_x^{(m)}$, as discussed in [2] for the case of strong separation of timescales, with:

$$
\pi_x^{(m)}(s) = \frac{e^{-\beta E_x(s)}}{Z_x^{(m)}} = \frac{e^{-\beta E_x(s)} Z_x}{Z_x^{(m)}},
$$

where $Z_x^{(m)}$ is the memory’s local partition function: $Z_x^{(m)} = \sum_{s \in \mathcal{M}} e^{-\beta E_x(s)}$. This quantity strongly suggests defining the local-equilibrium free energy:

$$
F_x^{(m)} = -k_B T \ln Z_x^{(m)},
$$

which turns out to provide useful intuition for the thermodynamics of transformations between metastable states, as we shall soon see.

Between computations, the distribution relaxes quickly to a classical superposition of local equilibria determined by the net probability in each memory state at the end of the last computation. Given a post-computation distribution over memory states $\Pr(M_\tau)$, the distribution over microstates quickly approaches the metastable superposition:

$$
\Pr(S_{\tau+\delta t}) \approx \sum_{m \in \mathcal{M}} \Pr(M_\tau = m) \pi_x^{(m)},
$$

where $S_\tau$ is the random variable for the microstate at time $t$ and $M_\tau$ is the random variable for the memory state at time $t$.

However, during a computation, the dynamic control protocol $x_{0,\tau}$ induces a net state-to-state stochastic transition dynamic $T_{x_0,\tau}$ over $\mathcal{S}$ that can strongly couple and transform memory states, as required of a computation.

**III. DRIVEN DYNAMICS AND COMPUTATIONS**

A deterministic computation $C : \mathcal{M} \to \mathcal{M}$ is an operation mapping the set of memory states $\mathcal{M}$ to itself. In practice, it is implemented by a driving protocol $x_{0,\tau}$ that controls the evolution of the system for a duration $\tau$. Fig. 1 (Left) shows schematically how the protocol can change the energy landscape to induce (Right) a net transition among microstates that corresponds to a (generically stochastic) computation on the coarse-grained memory states (depicted as the different shaded regions). The system should start and end with the same resting influence $x_0 = x_\tau$, as can be seen in the top-left bubble of Fig. 1, if a consistent metastable memory landscape is desired between computations.

The set of all protocols that reliably implement a computation $C$ is:

$$
\chi_c \equiv \left\{ x_{0,\tau} \in \chi^{[0,\tau]} : \Pr \left( S_\tau \notin C(m) \mid S_0 \sim \pi_x^{(m)} \right) < \epsilon \quad \text{for all } m \in \mathcal{M} \right\},
$$

for some error tolerance $\epsilon$. The assumed separation of timescales allows us to employ the local equilibrium distribution $\pi_x^{(m)}$ as the initial distribution in the test for reliable memory evolution.

Different protocols implementing the same computation typically dissipate different amounts of work. The grand challenge for energy-efficient computation is to identify the control protocols, given realistic control restrictions, that reliably implement a computation in finite time with the minimal resultant work dissipation.
Above, the work performed to implement the computation. As the memory system relaxes to a new metastable distribution on any microstates—it can be thought of as the distribution over microstates conditioned on the full driving history \( x_t \), including both the controlled preparation of \( x_0 \) and the protocol implementing the computation. Above, \( \Pr \{ x_t \} \) is the driving-induced probability of the microstate \( x_t \) at time \( t \). Similarly, \( \Pr \{ x_t \} \) is the driving-induced probability distribution over microstates—it can be thought of as the distribution over microstates conditioned on the full driving history \( x_t \), including both the controlled preparation of the system prior to the computation and the protocol implementing the computation, up to time \( t \).

The expected dissipation, given some initial preparation of the memory system and a particular driving protocol, is thus:

\[
\langle W_{\text{diss}} \rangle = \langle W \rangle - \Delta F ,
\]

where the expected nonequilibrium free energy at time \( t \) is:

\[
\mathcal{F} = U - k_B T H \left( \Pr \{ x_t \} \right) = E_{x_t}^{\text{eq}} + k_B T D_{\text{KL}} \left( \Pr \{ x_t \} \| \pi_{x_t} \right)
\]

Above, \( U = \langle E_{x_t}(s_t) \rangle \) is the expected internal energy of the system at time \( t \). \( H(\cdot) \) is the Shannon entropy of

**IV. WORK, NONEQUILIBRIUM FREE ENERGY, AND DISSIPATION**

The work dissipated in a computation is the work \( W \) that is irretrievably lost to the environment:

\[
W_{\text{diss}} = W - \Delta E_{x_t}(s_t) - k_B T \Delta \ln \left( \Pr \{ x_t \} \right) ,
\]

where the right-most term is recognized as the change in the non-averaged precursor to nonequilibrium entropy. Equivalently, \( W_{\text{diss}} \) is the amount of heat that is not offset by a corresponding change in this trial-specific internal entropy. Notably, Eq. (5) is valid over any time interval \( t \in [0, \tau] \), and therefore (by adjusting the considered duration \( \tau \)) also tracks the dynamics of the dissipation associated with the computation as the system relaxes during and after the work performed to implement the computation. As the memory system relaxes to a new metastable distribution on a relatively short timescale relevant to computation, the dissipation should appear to saturate to the total dissipation associated with the computation. Above, \( \Pr \{ x_t \} \) is the driving-induced probability of the microstate \( x_t \) at time \( t \). Similarly, \( \Pr \{ x_t \} \) is the driving-induced probability distribution over microstates—it can be thought of as the distribution over microstates conditioned on the full driving history \( x_t \), including both the controlled preparation of the system prior to the computation and the protocol implementing the computation, up to time \( t \).
its argument, and we will use $H_{x,t}(S_t)$ to denote $H(\Pr_{x,t}(S_t))$. Finally, $D_{\text{KL}}(\cdot)$ is the Kullback–Leibler divergence, which is always non-negative. $D_{\text{KL}}(\Pr_{x,t}(S_t)\|\pi_x)$ is the nonequilibrium addition to free energy—the thermodynamic resource corresponding to the distribution being out of equilibrium. It should be noted that $\Delta F_{x,t}^\text{eq} = 0$ over the full course of a computation since a computation starts and ends with the same resting influence $x_0 = x_f$.

Recent finite-time fluctuation theorems (most directly: Eqs. (38) and (42) of Ref. [3]) guarantee that the average dissipated work, when starting in any (potentially non-equilibrium and non-steady-state) distribution, is always non-negative:

$$\langle W_{\text{diss}} \rangle \geq 0 .$$

Eq. (9) can also be derived by other means, as in Refs. [4, 5]. The work performed in surplus to $\Delta F$ is eventually dissipated and contributes to the entropy production by the computation.

So, how much work is actually dissipated? Surely, the average work dissipated in transforming a distribution depends on the particulars of the protocol, with plenty of room for wastefulness if the protocol is not carefully designed. However, the minimal dissipated work is characterized by the allowed duration $\tau$ to implement the transformation and also by the degree of control one has over the system’s Hamiltonian. Let us briefly consider the case of perfect control, in which the controller can apply any Hamiltonian to the system. By instantaneously changing the initial Hamiltonian—to make any initial distribution the canonical distribution of the new Hamiltonian—before subsequent finite-speed driving of the system, we can immediately apply the recent results of finite-time thermodynamics [6–10] to conclude that the work dissipated by an optimal protocol—meant to transform between two distributions in a finite time $\tau$ with minimal dissipation—is generically (to first order of approximation) inversely proportional to the allowed duration $\tau$. I.e.: $\langle W_{\text{diss}} \rangle_{\text{min}} \sim \tau^{-1}$. In part, the next section will show that this same $\tau^{-1}$ scaling of the dissipation can be achieved at intermediate timescales as long as the distribution stays close to a local-equilibrium distribution, even if it is never close to a global equilibrium distribution. More generally, the next section contains the fundamental thermodynamics of computation through the transformation of metastable memories.

V. NONEQUILIBRIUM THERMODYNAMICS, AT LEVEL OF MEMORY STATES

The above is all now-standard nonequilibrium thermodynamics. However, we seek thermodynamic implications for transformation of memory states rather than microstates. Fortunately, a rigorous hierarchical description can be achieved through a series of decompositions of familiar thermodynamic quantities.

To start, we note that since $\mathcal{M}$ is a coarse-graining of $\mathcal{S}$, we have:

$$H_{x,t}(S_t) = H_{x,t}(\mathcal{M}_t, S_t) = H_{x,t}(\mathcal{M}_t) + H_{x,t}(S_t|\mathcal{M}_t) .$$

We will use the above together with a novel decomposition of the expected internal energy, which is valid at any time given any coarse-graining of microstates $\mathcal{S}$ into the coarse-grained set $\mathcal{M}$. In particular, the expected internal energy of the system can be decomposed as:

$$U = \langle E_x(s_t) \rangle_{\Pr_{x,t}(S_t)} = \langle \langle E_x(s_t) \rangle_{\Pr_{x,t}(S_t, M_t=m)} \rangle_{\Pr_{x,t}(M_t=m)} .$$

Crutially, utilizing the identity $E_x(s) = -k_B T \ln(\pi_x^{(m)}(s)) + F_x^{(m)}$, we find that the expected internal energy, if the system has been driven into memory state $m$ is:

$$\langle E_x(s_t) \rangle_{\Pr_{x,t}(S_t|M_t=m)} = k_B T H_{x,t}(S_t|M_t=m) + F_x^{(m)} + F_x^{(m)}_{x,t,\text{add}} .$$
Above, $F_{x_t}^{(m)}$ is the local-equilibrium free energy and

$$F_{x_t,\text{add}}^{(m)} = k_B T D_{KL}(\Pr(S_t|M_t = m) || \pi_t^{(m)})$$

(13)

is the local nonequilibrium addition to free energy in region $m$. The expected internal energy thus always has the decomposition: $U = k_B T H_{x_t}(S_t|M_t) + (F_{x_t}^{(m)})_{\Pr(M_t)} + (F_{x_t,\text{add}}^{(m)})_{\Pr(M_t)}$. At the same time, we always have that $U = F + k_B T H_{x_t}(S_t)$. Putting these together, we find that the nonequilibrium free energy can always be decomposed according to the contributions commensurate with the coarse-grained description:

$$F = (F_{x_t}^{(m)})_{\Pr(M_t)} + (F_{x_t,\text{add}}^{(m)})_{\Pr(M_t)} - k_B T H_{x_t}(M_t).$$

(14)

Moreover, when the coarse graining is according to well-designed metastable memory states, the separation of timescales implies that $F_{x_t,\text{add}}^{(m)} \to 0$ quickly after any driving.\(^2\) Hence, before and shortly after a computation, we can decompose the nonequilibrium entropy into two very manageable parts:

$$F \approx (F_{x_t}^{(m)})_{\Pr(M_t)} - k_B T H_{x_t}(M_t);$$

(15)

that is: the expected local-equilibrium free energy, less the coarse-grained entropy of the memory states. Any difference from equality is due to work already performed that is expected to soon be dissipated in the relaxation to local equilibria. Eq. (15) was previously highlighted in [11]. However the local nonequilibrium addition to the free energy, as in Eq. (14), is a new finding that offers further insight.

The local nonequilibrium addition to free energy is a thermodynamic resource that in principle can be traded to perform useful work. However, if either (1) the timescale of relaxation within each memory state is faster than the relevant speed of the driving protocol or (2) the control parameters are too coarse or otherwise incapable of influencing the fine degrees of freedom within the memory state, then local nonequilibrium addition to free energy will inevitably be lost to dissipation as the local distributions relax to their local equilibria. Conversely, the coarse-grained memory probabilities are assumed to be metastable and controllable, and so nonequilibrium changes at the coarse-grained level can be implemented thermodynamically reversibly.

With these considerations in mind, Eq. (14) suggests that a driving protocol that keeps the distribution close to a metastable one (i.e., a weighted average of local-equilibrium distributions) at all times, such that $F_{x_t,\text{add}}^{(m)}$ always stays close to zero in each metastable region, can be used to implement thermodynamically-efficient computations with $\langle W_{\text{dim}} \rangle \sim \tau^{-1} \to 0$ as $\tau \to \infty$. In this nearly-quasistatic limit, such processes will be thermodynamically reversible. Finally, coming back to Eq. (6), this implies that the minimal average work necessary to implement a computation on metastable memory states is:

$$\langle W \rangle_{\text{min}} = \Delta (F_{x_t}^{(m)})_{\Pr(M_t)} - k_B T \Delta H_{x_t}(M_t).$$

(16)

In the computational setting, Eq. (16) can be interpreted as a generalization of Landauer’s principle for the minimum work necessary to implement computations that transform metastable memories of different local-equilibrium free energies. In a more general setting, when $\langle W \rangle_{\text{min}}$ is negative, $\langle W_{\text{extracted}}\rangle_{\text{max}} = -\langle W \rangle_{\text{min}}$ can also be interpreted as the maximal work that can be extracted from a metastable system—and the heterogeneity of local free energy then offers an easy explanation of how a single bit of macroscopic information (e.g., "Is the apple in the left or right box?") can carry a macroscopically huge (much larger than $k_B T \ln 2$) energetic gain, as in [12], when interacting with far-from-equilibrium systems. Fig. 2 gives further intuition for the meaning of the local-equilibrium free energies in Eq. (16): larger local-equilibrium free energy can result from either larger average energy or from more certainty in the local-equilibrium microstate distribution.

Clearly, if all local-equilibrium free energies are equal (either through identically-constructed potentials or otherwise through some delicate balance of local energies and entropies), such that $F_{x_0}^{(m)} = F_{x_0}^{(m')}$ for all $m, m' \in M$, then

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\(^2\) It is important to note that this is an assumption about the dynamics which is well-suited to the memory systems typically used in practical computations. The results of the following are only as reliable as this assumption is valid.
In the case where the memory elements are arranged in a 1-dimensional topology, the entropy density has been taken

such that the minimum work depends solely on the change in entropy over memory states. Eq. (17) constitutes the modern understanding of Landauer’s principle for the minimum work necessary to implement computations that transform metastable memories of equal free energy.

If the computation reduces the internal entropy of the memory system, then it will require work (although this work can later be reclaimed and recycled if the computation was performed without dissipation). Conversely, when \( \langle W \rangle_{\text{min}} \) is negative, work can be extracted as a result of the transformation, for example to lift a weight or to fuel other computations. In this latter case, the memory system can perform as an information engine, trading certainty for useful work. In this regime, \( \langle W_{\text{extracted}} \rangle_{\text{max}} = -\langle W \rangle_{\text{min}} = k_B T \Delta H_{\text{max}}(\mathcal{M}_t) \) is the maximal average work that can be extracted from transformations that result in this memory-entropy change.

VI. IMPLICATIONS FOR COMPOSITE MEMORY SYSTEMS

If the system is composed of \( N \) multi-stable memory elements, then it is natural to treat the memory state as a composite state of \( N \) random variables: \( \mathcal{M}_t = \mathcal{M}_t^{(1)} \cdots \mathcal{M}_t^{(N)} = (\mathcal{M}_t^{(1)}, \mathcal{M}_t^{(2)}, \ldots, \mathcal{M}_t^{(N)}) \). In general, the memory elements are correlated, such that \( \text{Pr}_{\mathcal{X}_t}(\mathcal{M}_t^{(1)}, \mathcal{M}_t^{(2)}, \ldots, \mathcal{M}_t^{(N)}) \neq \prod_{n=1}^{N} \text{Pr}_{\mathcal{X}_t}(\mathcal{M}_t^{(n)}) \). Moreover, this correlation has important thermodynamic consequences.

Consider a nonequilibrium system of \( N \) identical memory elements, each having \( K \) identical metastable regions (such that \( F_x^{(m)} = F_x^{(m')} \) for all \( m, m' \in \mathcal{M} \)), and let \( h_{\mathcal{X}_t} \equiv \frac{H_{\mathcal{X}_t}(\mathcal{M}_t^{(1)} \cdots \mathcal{M}_t^{(K)})}{N} \) denote the coarse-grained entropy density of the memory system. If the system is transformed by some driving protocol \( x_{0:t} \) that increases the system’s entropy, then, from Eq. (17), the maximal work (per memory element) that could have been extracted in the process is given by:

\[
\frac{h}{N} \langle W_{\text{extracted}} \rangle_{\text{max}} = h_{\mathcal{X}_t} - h_{\mathcal{X}_0}.
\]

In the case where the memory elements are arranged in a 1-dimensional topology, the entropy density has been taken
to mean the Shannon entropy rate of the sequence as it is scanned spatially [13]. While this is technically correct, it is sufficiently nuanced to require careful interpretation. In particular, the entropy density \( h \equiv \lim_{L \to \infty} \frac{H(p_L)}{L} \) that can be inferred from the frequentist statistics gathered along the sequence of the instantaneous configuration will in
general converge to a value that is not the same as $h_{\tau}$.\footnote{For example, the string of the digits of pi ($3.14159\ldots$) has a Shannon entropy rate of $h = \log_2(K)$ bits-per-symbol in any $K$-ary expansion for $K \in \{2, 3, \ldots\}$. (E.g., $h = \log_2(10)$ in the given decimal expansion of pi, whereas $h = 1$ in its binary expansion.) It would naively seem to provide no thermodynamic fuel as it appears to be a completely ‘random’ sequence. To the contrary, if stored into memory, the sequence contains full thermodynamic fuel (i.e., it can be fully leveraged to do work) because the memory system will be driven by $\tau$ to uniquely hold this sequence, so that $H_{\tau} = 0$ since $\Pr(M_t) = \delta_{M(1),3}\delta_{M(2),1}\delta_{M(3),4}\ldots$. I.e., in the space of sequences, the state of the memory system is delta-distributed. This particular example points to the proper way to think about entropy and about what kind of information can be thermodynamically leveraged in computer memory in general.} Rather:

$$\ln(K) \geq h \geq h_{\tau} \geq 0.$$\hspace{1cm}(19)

Crucially, it is the entropy conditioned on the driving history that matters in the theoretical limit of what orderliness can be thermodynamically leveraged—and this is a priori distinct from anything that could be inferred from the instantaneous configuration, even in the limit of $N \to \infty$.

Moreover, this thermodynamic entropy density is independent of spatial dimension, or even any spatial topology of the memory elements. Indeed, the topology of the memory elements—being arranged in a 1-dimensional string for example—is a priori independent of the topology of the correlations among random variables. And it is only the latter that fundamentally matters for the thermodynamics of information processing. However, spatial locality occasionally does correspond to the correlational structure, especially when correlations develop as a consequence of local physical interactions.

Turing machines and related models of computation require not only a bit string but also a memoryful read–write head that can operate on the tape. Treating these two components inclusively as part of the memory system makes the system self-contained and provides important lessons about the thermodynamics of such functionally segregated systems [13, 14].

This sort of inclusiveness\footnote{Including two subsystems (say, ‘subsystem’ and ‘demon’) explicitly as components of the same memory system means that the coarse-grained memory entropy decomposes as: $H(M_{\text{sub}}, M_{\text{demon}}) = H(M_{\text{sub}}) + H(M_{\text{demon}}) - I(M_{\text{sub}}, M_{\text{demon}})$, where $I$ denotes mutual information. Interpreting the latter quantity as the knowledge the demon has of the system, we can see from substitution into Eq. (16) that knowledge is another thermodynamic resource that can be exchanged for entropy reduction, free-energy gain, or work extraction; but we also see that its origination carries either an energetic or entropic cost of at least what the knowledge was later worth—knowledge is a medium for thermodynamic transactions rather than a free source of energy.} also sheds light on ‘Maxwell’s demon’-type scenarios where the net system is decomposed into a subsystem and a ‘demon’. If the subsystem is initially out of equilibrium, then the demon can simply extract work by extracting the subsystem’s nonequilibrium addition to free energy—no mystery there. However the more complicated story arises when the subsystem is initially in equilibrium, as in Maxwell’s original gedanken-experiment where the subsystem is a two-compartment box of gas starting in equilibrium. The demon can nevertheless decrease the entropy of the subsystem by increasing its knowledge of the subsystem [15]. Work can then be extracted as the subsystem is subsequently brought back to equilibrium. But if this process is to reset to form a full cycle—if the demon’s memory is to be erased for example—then no net work can be extracted on average. Interestingly, it is not necessarily erasure where cost is incurred [16], but whether the cost is incurred in measurement or in erasure can be explained via heterogeneous local-equilibrium free energies of the memory states as an application of Eq. (16).

To close this topic, we note that work can be extracted at a constant rate when an active environment continuously drives the subsystem out of equilibrium, at no cost to the extractor (formerly known as ‘demon’). The extractor can then siphon off the power that it needs to sustain its luxurious nonequilibrium lifestyle—perhaps even to appease its greed for massive speedy computations.

VII. THE THERMODYNAMIC COST OF IGNORANCE AND NEGLECT

If one has sufficient control over the energy levels of a system, and if quasistatically-slow transformations are tolerable, then a transformation between any two distributions is always possible without dissipation. Example methods to implement such dissipationless computations are given, for example, in [1, 11, 17, 18]. We have further argued that the zero-dissipation limit is also approached with the slightly weaker requirement that metastable distributions (rather than strictly the global equilibrium distribution) are maintained throughout the transformation. However, even in this nearly-quasistatic case, if correlations are ignored (or if other features of the distribution are neglected or mis-represented for whatever reason in the manipulation of the distribution), then there is necessarily extra work incurred and dissipated when the driving protocol is run.

Suppose a driving protocol $x_{\tau}^{\omega}$ is chosen to minimize dissipation while implementing a computation $\mathcal{C}$ and starting in the distribution $q_0$. I.e., $x_{\tau}^{\omega} \equiv \arg\min_{x_{\tau}^{\omega}} \{ W_{\text{diss}} \Pr_{x_{\tau}^{\omega}}(S_0) = q_0 \}$. For simplicity, let’s further assume that we
When the modular computations are being performed on metastable memory states, then assuming the memory starts
we find that we can formulate the result in terms of the memory states of the two subsystems:

\[ \beta \langle W_{\text{diss}}(\mu_0) \rangle - \beta \langle W_{\text{diss}}(q_0) \rangle = D_{KL}(\mu_0 \| q_0) - D_{KL}(\mu_\tau \| q_\tau) , \tag{20} \]

where \( \langle W_{\text{diss}}(\mu_0) \rangle \equiv \langle W_{\text{diss}}(\mu_0) \rangle_{\text{Pr}_{\mu_0}(S_0 \sim \mu_0)} \) and \( \langle W_{\text{diss}}(q_0) \rangle \equiv \langle W_{\text{diss}}(q_0) \rangle_{\text{Pr}_{q_0}(S_0 \sim q_0)} \) which is equal to 0 in this case. Above, \( \mu_\tau = \text{Pr}_{x_{0:\tau}^*} (S_\tau | S_0 \sim \mu_0) \) and \( q_\tau = \text{Pr}_{x_{0:\tau}^*} (S_\tau | S_0 \sim q_0) \) are the time-evolved versions of \( \mu_0 \) and \( q_0 \), respectively, under the influence of the driving \( x_{0:\tau}^* \).

There are several immediate novel consequences of Eq. (20) when applied to our framework, that are worth teasing out since they yield important general lessons about dissipation incurred during computation.

### A. Dissipation through modularity and neglected correlation

Let us say that the system is actually in distribution \( \mu_0 \), but the controller thinks—or otherwise acts as if—the distribution is \( q_0 \).

One case in which this happens in practice is when correlations exist among parts of a memory system but computations are implemented only modularly. Modular computing—by implicitly marginalizing over some of the memory elements—necessarily ignores the correlations among modular units.

Suppose for example that we partition the memory system into two composite pieces \( S_t = (S_t^{(1)}, S_t^{(2)}) \) and that the two memory subsystems are correlated: \( \mu_t = \text{Pr}(S_t^{(1)}, S_t^{(2)}) \neq \text{Pr}(S_t^{(1)}) \text{Pr}(S_t^{(2)}) \); but the two memory subsystems are operated on independently (i.e., modularly), which means \( q_t = \text{Pr}(S_t^{(1)}) \text{Pr}(S_t^{(2)}) \). I.e., the distribution, although correlated, is operated on as if the two components were statistically independent. The implications are immediately accessible:

\[ \beta \langle W_{\text{diss}}^{(\text{mod})} \rangle = D_{KL}(\mu_0 \| q_0) - D_{KL}(\mu_\tau \| q_\tau) \tag{21} \]

\[ = -D_{KL}(\text{Pr}(S_t^{(1)}; S_t^{(2)}) \| \text{Pr}(S_t^{(1)}) \text{Pr}(S_t^{(2)})) \tag{22} \]

\[ = I(S_t^{(1)}; S_t^{(2)}) - I(S_\tau^{(1)}; S_\tau^{(2)}) , \tag{23} \]

where \( I(\cdot; \cdot) \) is the mutual information between its arguments. This means that work is necessarily dissipated whenever a modular computation discards information between two subsystems.\(^5\)

When the modular computations are being performed on metastable memory states, then assuming the memory starts and ends in a metastable distribution with microstate probabilities \( \mu_t(s) = \sum_{m \in \mathcal{M}} \mu_t^{(m)}(s) \) for \( t = 0 \) and \( t = \tau \), we find that we can formulate the result in terms of the memory states of the two subsystems:

\[ \beta \langle W_{\text{diss}}^{(\text{mod})} \rangle = I(\mathcal{M}_0^{(1)}; \mathcal{M}_0^{(2)}) - I(\mathcal{M}_\tau^{(1)}; \mathcal{M}_\tau^{(2)}) . \tag{24} \]

Although we have arrived at this result by rather different means, Eq. (24) is essentially the same as the main result of [1] (although there it was assumed that \( \mathcal{M}^{(2)} \) is unchanged by the computation, which led to \( \mathcal{M}_\tau^{(2)} = \mathcal{M}_0^{(2)} \) there, and it was also assumed there that the local-equilibrium free energies were all the same). It is notable that our result does not require any assumption about the local free energies of the memory subsystems—they can be arbitrarily heterogeneous.

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\(^5\) The opposite situation (i.e., mutual information increasing between the two subsystems) does not happen under the current assumption of modularity, and so we are not in danger of deriving \( \langle W_{\text{diss}}(\mu_0) \rangle < 0 \) here, which would be counter to the second law of thermodynamics. If a computation creates correlation between two subsystems, then \( q_\tau \) would not be separable, and the analysis would have proceeded differently.
With modular computations happening on $N$ different subsystems, the result generalizes easily. With $\mu_t = \Pr(S^{(1)}_t, S^{(2)}_t, \ldots S^{(N)}_t)$ and $q_t = \prod_{n=1}^{N} \Pr(S^{(n)}_t)$, we find that:

$$\beta \langle W_{\text{diss}}^{(\text{mod})} \rangle = -\Delta D_{\text{KL}} \left( \Pr(S^{(1)}_t, S^{(2)}_t, \ldots S^{(N)}_t) \left\| \prod_{n=1}^{N} \Pr(S^{(n)}_t) \right. \right)$$

$$= C_{\text{tot}}(S^{(1)}_0, S^{(2)}_0, \ldots S^{(N)}_0) - C_{\text{tot}}(S^{(1)}_t, S^{(2)}_t, \ldots S^{(N)}_t).$$

(25)

(26)

where $C_{\text{tot}}(S^{(1)}_t, S^{(2)}_t, \ldots S^{(N)}_t) = \left( \sum_{n=1}^{N} H(S^{(n)}_t) \right) - H(S^{(1)}_t, S^{(2)}_t, \ldots S^{(N)}_t)$ is the so-called total correlation among its arguments. This generalization is necessary for predicting the dissipation that will be incurred when many modular computations are performed in parallel.

Suppose instead we want to consider the problem at the level of metastable memory states, with the joint distribution over the memory states of the subsystems $\mu'_t = \Pr(M^{(1)}_t, M^{(2)}_t, \ldots M^{(N)}_t)$. If the memory system is assumed to start and end the computation in a classically superposed metastable distribution such that $\mu_t = \sum_{m \in \mathcal{M}} \mu'_t(m) \pi'(m)$ at $t = 0, \tau$, as in Eq. (3), then using $\frac{\mu'_t(s)}{q'_t(s)} = \frac{\mu'_t(m(s))}{q'_t(m(s))}$ and making use of $\pi'(m)(s) = \delta_{s,m} \pi'(m)(s)$ in our calculation, regardless of any heterogeneity among the local-equilibrium free energies, we again find that:

$$\beta \langle W_{\text{diss}}^{(\text{mod})} \rangle = C_{\text{tot}}(M^{(1)}_0, M^{(2)}_0, \ldots M^{(N)}_0) - C_{\text{tot}}(M^{(1)}_t, M^{(2)}_t, \ldots M^{(N)}_t).$$

(27)

Whether framed in terms of microstates or memory states, our general result means that: the minimal extra dissipation incurred by modular computation is exactly $k_B T$ times the reduction in total correlation among all memory subsystems.

### B. Dissipation through failing to model statistics of manipulated memory

Let us consider the implications for the common logic gates that serve as the building blocks for practical computers. Recall that the simple NAND gate is sufficient for universal computation. It is therefore worthwhile to consider what dissipation is commonly incurred in these important logic gates—and to show how this dissipation can be avoided.

It is important to note that even without correlation it is critical to correctly model the input statistics of a computation in order to avoid dissipating work. Modeling correlations is then a requirement on top of this. Since we have already briefly discussed the role of correlations, let us focus here on the more basic point of modeling input statistics whatsoever.

To address this, we will consider a physical instantiation of the memory components of a NAND gate, where we explicitly consider two memory elements whose memory states are to be used as the input for the NAND gate and another memory element that will store the value of the output. We will assume that only the output is over-written during the computation—the input memory states may be kept around for later use.

Note that this is already a particular physical model of the NAND computation—indeed, alternatives exist such as storing the output in the location of one of the former inputs by over-writing one of the inputs. However, we will analyze the proposed two-input–one-output model here since it is arguably the most relevant to the typical desired use of a NAND gate. Other ancillary memory elements may be used in the computation as in [20] but, since they will be returned to their original state by the end of the computation, these ancillary memories do not need to result in any additional dissipation and so can be left implicit in the self-consistency of the current analysis.

Each of the three explicitly-considered memory elements is assumed to be bistable (i.e., each of the three memory elements is assumed to have two metastable regions of state space). Let the microstate of each memory element be specified by its position in the interval $(-\pi, \pi]$. Between computations, including at $t = 0$ and $t = \tau$, the metastable regions for each memory element are $0 \equiv (-\pi, 0]$ and $1 \equiv (0, \pi]$ which gives a natural partition for the memory states.

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6 For example, each memory element may be realized physically by the bistable magnetic moment of a superparamagnetic nanocrystal in the so-called ‘blocked’ regime where the Néel relaxation time between metastable regions is much larger than the timescale of computation in the system. We can assume that there is sufficient uniaxial anisotropy (or sufficiently low temperature) to create a potential barrier many times the thermal energy between the potential wells of the two metastable regions. Although it is nice to have several realistic physical systems in mind, ultimately the physical details of the bistable memory element will be largely irrelevant, and the analysis transcends these specifics.
The microstate of the memory system at any time \( t \) can be treated as a composite random variable \( S_t = (S_t^{(in_1)}, S_t^{(in_2)}, S_t^{(out)}) \) with \( S_t^{(i)} \in (\pi, \pi] \). Similarly, the memory state is the composite random variable \( M_t = (M_t^{(in_1)}, M_t^{(in_2)}, M_t^{(out)}) \) with \( M_t^{(i)} \in \{0, 1\} \) corresponding to the two metastable regions for each memory element. Thus, the joint state-space \( S = \mathbb{R}^{3\pi} \) has eight metastable regions, which we identify as the joint memory system’s eight memory states: \( M = \{m_{0000}, m_{0001}, m_{0010}, \ldots m_{1111}\} \) where each memory state is labeled according to its corresponding region of state-space: \( m_{ijk} = \{s \in S: s^{(in_1)} \in [0 - j\pi, j\pi], s^{(in_2)} \in (0 - k\pi, \pi - k\pi], s^{(out)} \in (0 - \ell\pi, \pi - \ell\pi]\}. \)

I.e., each of the memory states is one of the octants of state space, as shown in Fig. 3 (Left).

As a first analysis of this system, let us suppose that all memory elements are initially uncorrelated: \( \Pr(M_0) = \Pr(M_0^{(in_1)}) \Pr(M_0^{(in_2)}) \Pr(M_0^{(out)}). \) Suppose though that each memory element has an initial bias such that \( \Pr(M_0^{(in_1)}) = 1 = b_1, \ Pr(M_0^{(in_2)}) = 1 = b_2, \) and \( \Pr(M_0^{(out)}) = 1 = b_3. \) Let us call this distribution over memory states \( \mu_0 = \Pr(M_0). \) The corresponding initial distribution over microstates is: \( \Pr(S_0) = \sum_{m \in M} \Pr(M_0 = m) \delta_{S_0 = m} \pi_{x_0}, \) which we will call \( \mu_0. \) This physical setup, and the logical transformation of the output bit, is diagrammed in Fig. 3 (Right).

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**FIG. 3.** Composite state space, memory states, and physical transformation associated with the logical NAND operation.

Suppose that a NAND gate is constructed such that it does not dissipate any work when it transforms the memory system in its equilibrium state: \( q_0 = \sum_{m \in M} \delta_{S_0 = m} \pi_{x_0} / 8. \) That is, the transformation is designed to dissipate no work when the memory states are all initialized in the uniform distribution \( q_0 = \frac{1}{8} [1 \ldots 1]. \)

The minimal extra dissipation incurred by using this NAND transformation (optimized for minimal dissipation in the case of uniform distribution over memory states), given that the initial statistics of the memory elements are actually biased by the \( b_i, \) is:

\[
\beta \langle W_{\text{diss}}^{\text{mismatch}} \rangle = D_{\text{KL}}(\mu_0 \parallel q_0) - D_{\text{KL}}(\mu_{\tau} \parallel q_{\tau}) \\
= D_{\text{KL}}(\mu_0 \parallel q_0^\prime) - D_{\text{KL}}(\mu_{\tau} \parallel q_{\tau}^\prime) \\
= \sum_{m \in M} \mu_0(m) \ln \frac{\mu_0(m) / \pi_{x_0}}{\mu_{\tau}(m) / \pi_{x_0}} - \mu_{\tau}(m) \ln \frac{\mu_{\tau}(m) / \pi_{x_0}}{\mu_0(m) / \pi_{x_0}} \\
= \ln 8 - H(\mu_0) - \ln 4 + H(\mu_{\tau}) \\
= \ln 2 - H_2(b_3),
\]

where \( H_2(b) \equiv -b \ln b - (1 - b) \ln(1 - b). \)

We reflect that the full dissipation of operating the NAND gate (when not optimized for the correct memory biases) is essentially the entropy production from ignoring the single bias of \( b_3 \) associated with the output. The protocol could have been designed to be dissipation free, but the current NAND implementation does not erase \( M_0^{(out)} \) in a way that salvages its original nonequilibrium addition to free energy.

More generally, if we allow any kind of initial correlation among the initial configuration, such that the input and output bits are initially correlated according to \( \mu_0 = \Pr(M_0) \neq \Pr(M_0^{(in_1)}) \Pr(M_0^{(in_2)}) \Pr(M_0^{(out)}), \) then the resulting
dissipation generalizes to:

\[
\langle W_{\text{diss}}^{(\text{mismatch})} \rangle = k_B T \ln 2 - k_B T H_{\mu_0'}(\mathcal{M}_0^{(\text{out})} | \mathcal{M}_0^{(\text{in1})}, \mathcal{M}_0^{(\text{in2})}),
\]

where the right-most term is the conditional Shannon entropy of the \textit{initialized} value of the output bit (i.e., before the NAND operation is implemented), given the initial values of the input.

Again, this \( \beta \langle W_{\text{diss}}^{(\text{mismatch})} \rangle \) turns out to be the irreversible entropy production of ignoring the nonequilibrium distribution of the original output bit. A smarter protocol could have instead leveraged this ordered nonequilibrium addition to free energy to perform the NAND computation with less work. But, since the protocol was not altered to take advantage of this initial nonequilibrium situation, the thermodynamic resource is forever lost through dissipation by the end of the computation.

\[\text{C. Dissipation and minimal work for any two-input–one-output logic gate}\]

Once having gone through this analysis, it should be clear that the NAND function played no essential role in determining the minimal dissipation from neglected initial biases, other than the fact that the NAND operation is a deterministic two-input–one-output function. Hence, Eq. (33) describes the minimal dissipation of all two-input–one-output logic gates—NAND, AND, XOR, etc.—when the output memory element is overwritten by a computation that does not leverage the initial biases of the memory elements it is manipulating.

We focused above on the work \textit{dissipated} since this is the undesirable waste that designers of future hyper-efficient computers should be hyper-aware about. It is noteworthy though that even when no work is dissipated, the minimal \textit{work} to implement the computation will also depend on the initial biases of the memory elements that are to be manipulated. From Eq. (17), we see that the minimal work necessary to implement the NAND gate—and indeed to implement \textit{any} two-input–one-output gate where the output memory element is to be overwritten—is:

\[
\langle W \rangle_{\text{min}} = k_B T H_{\mu_0'}(\mathcal{M}_0^{(\text{out})} | \mathcal{M}_0^{(\text{in1})}, \mathcal{M}_0^{(\text{in2})}) - k_B T H_{\mu'}(\mathcal{M}_0^{(\text{in1})}, \mathcal{M}_0^{(\text{in2})}, \mathcal{M}_\tau^{(\text{out})})
\]

(34)

In the case of biased but uncorrelated initial inputs and outputs, this reduces to: \( \langle W \rangle_{\text{min}} = k_B T H_2(b_3) \). However, as long as this work is not dissipated, it can continue to be salvaged and recycled in future computations.

Looking back at \( \langle W_{\text{diss}} \rangle \), we see that our result for the minimum work puts the dissipated work in a new context. In particular:

\[
\langle W_{\text{diss}}^{(\text{mismatch})} \rangle = k_B T \ln 2 - \langle W \rangle_{\text{min}} \cdot
\]

(36)

We interpret this as the minimum work that would need to be performed given the uniform distribution of initial memory states that the system was designed for, minus the minimum work given the actual biases. In the case of biased but uncorrelated initial memory states this can be framed as:

\[
\langle W_{\text{diss}}^{(\text{mismatch})} \rangle = k_B T H_2(\frac{1}{2}) - k_B T H_2(b_3)
\]

(37)

When all memory states have the same local-equilibrium free energy, biases in the input should be treated as a resource—a nonequilibrium addition to free energy. Ignoring these biases is to ignore and waste this resource, resulting in unnecessary dissipation.

\[\text{VIII. ONWARD}\]

The results of our analysis highlight the fundamental thermodynamics limits of conventional computation—a limit we are steadily approaching but are still quite far from. Constructively, our analysis also point to ways around these limits for future hyper-efficient computers. First, these hypothetical future computers could have implementations that adapt
to the input biases and thus eliminate needless dissipation within each modular computation. Second, we propose that future hyper-efficient computers can compute common composite routines in a single global transformation, to reduce modular dissipation.

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