We consider anisotropic fluids with directional pressures \( p_i = w_i \rho \) (\( \rho \) is the density, \( w_i = \text{const} \), \( i = 1, 2, 3 \)) as sources of gravity in stationary cylindrically symmetric space-times. We describe a general way of obtaining exact solutions with such sources, where the main features are splitting of the Ricci tensor into static and rotational parts and using the harmonic radial coordinate. Depending on the values of \( w_i \), it appears possible to obtain general or special solutions to the Einstein equations, thus recovering some known solutions and finding new ones. Three particular examples of exact solutions are briefly described: with a stiff isotropic perfect fluid (\( p = \rho \)), with a distribution of cosmic strings of azimuthal direction (i.e., forming circles around the \( z \) axis), and with a stationary combination of two opposite radiation flows along the \( z \) axis.

1 Introduction

Studies of stationary cylindrically symmetric configurations in general relativity (GR) have a long history, beginning with the Lanczos (1924) — Lewis (1932) vacuum solution \([1, 2]\) and continuing till now. Among motivations of such studies one can mention the relative simplicity of the gravitational field equations as compared to more realistic axial symmetry (to say nothing on the general non-symmetric case) and the possible existence of linearly extended structures like cosmic strings in the Universe. In addition, cylindrical symmetry is the simplest one that admits rotation and is most suitable for studying rotational phenomena in GR, so not too surprising is the wealth of existing stationary (that is, assuming rotation) exact solutions to the Einstein equations with various sources of gravity: the cosmological constant \( \Lambda \) \([3–5]\); scalar fields with or without self-interaction potentials \([6–7]\), including \( \Lambda \) as a constant potential \([8]\); dust in rigid or differential rotation \([9–11]\), electrically charged dust \([12]\), dust with a scalar field \([13]\), perfect fluids with various equations of state, mostly \( p = w \rho \), \( w = \text{const} \) (in usual notations) \([14–18]\); some examples of anisotropic fluids \([19–21]\) etc, see also references therein as well as reviews \([22, 23]\).

There are also many solutions to the field equations in various extensions of GR most of which come into play at extremely large curvatures or matter densities, at very small or very large length scales etc. If our interest is in macroscopic scales, say, from meters to kiloparsecs, then, in our opinion, it makes sense to adhere to GR which is the simplest and has a brilliant experimental status. Moreover, many solutions of alternative theories can be obtained from those of GR by different solution generating methods, such as the well-known conformal mapping connecting the Jordan and Einstein frames in scalar-tensor and \( f(R) \) theories.

In this paper we undertake a systematic study of the Einstein equations for stationary cylindrically symmetric space-times where the source of gravity is an anisotropic fluid with each principal pressure \( p_i \) equal to density times a constant proportionality factor \( w_i \). This ansatz includes as special cases perfect isotropic fluids with \( p/\rho = w = \text{const} \) and many anisotropic sources of interest such as cosmic string bundles with radial, longitudinal or azimuthal directions, some stationary cases with radiation flows of the same directions, some field configurations, etc. Our purpose here is to present a general method of solving the Einstein equations using the fact that in stationary cylindrically symmetric space-times the Ricci and Einstein tensors split into static and rotational parts \([6]\) and employing the harmonic radial coordinate, first applied to static cylindrical systems in \([24]\). This leads to reproducing some known solutions and obtaining new ones which can be applied to describe configurations with a regular symmetry axis and those
where such an axis is absent (cylindrical wormholes [6,7,25,27]. A thorough analysis of numerous systems of interest is postponed for the future, here we only briefly consider some examples.

2 Stationary cylindrical space-times

2.1 Basic relations

Consider a stationary cylindrically symmetric metric

\[
ds^2 = e^{2\gamma(x)}[dt - E(x) e^{-2\gamma(x)} d\varphi]e^{2\alpha(x)} dx^2 \quad \text{or} \quad e^{2\mu(x)} d\varphi^2 - e^{2\nu(x)} dz^2 - e^{2\beta(x)} d\varphi^2,
\]

where \( x, z \in \mathbb{R} \) and \( \varphi \in [0, 2\pi) \) are the radial, longitudinal and angular coordinates. The coordinate \( x \) is specified up to a reparametrization \( x \to f(x) \), so its range depends on both its choice (the “gauge”) and the geometry itself. The off-diagonal component \( E \) describes rotation, and the corresponding vortex gravitational field in spacetimes with the metric (11) is characterized by the angular velocity \( \omega(x) \) of a congruence of timelike curves (vorticity) [6,28,29],

\[
\omega = \frac{1}{2}(E e^{-2\gamma})' e^{\gamma-\beta-\alpha}.
\]

under an arbitrary choice of the coordinate \( x \) (a prime stands for \( d/dx \)). Furthermore, in the reference frame comoving to matter in its motion in \( \varphi \)-direction, we have the SET component \( T^{\varphi\varphi}_0 = 0 \), hence (via the Einstein equations) we have the Ricci tensor component \( R^0_\varphi \sim (\omega e^{2\gamma+\mu})' = 0 \), so that

\[
\omega = \omega_0 e^{-\mu-2\gamma}, \quad \omega_0 = \text{const}.
\]

Then, according to (2),

\[
E(x) = 2\omega_0 e^{2\gamma(x)} \int e^{\alpha+\beta-\mu-3\gamma} dx.
\]

The nonzero components of the Ricci \( (R^\mu_\nu)_0 \) are

\[
R^0_0 = -e^{-2\alpha}[\gamma'' + \gamma'(\sigma' - \alpha')] - 2\omega^2,
R^1_0 = -e^{-2\alpha}[\sigma'' + \sigma' - 2U - \alpha'\sigma'] + 2\omega^2,
R^2_2 = -e^{-2\alpha}[\mu'' + \mu'\sigma'] + 2\omega^2,
R^3_3 = -e^{-2\alpha}[\beta'' + \beta'\sigma'] + 2\omega^2,
R^0_3 = G^0_3 = E e^{-2\gamma}(R^3_3 - R^0_0),
\]

where we have introduced the notations

\[
\sigma = \beta + \gamma + \mu, \quad U = \beta'\gamma' + \beta'\mu' + \gamma'\mu'.
\]

The Einstein equations may be used in two equivalent forms

\[
G^\nu_\mu = R^\nu_\mu - \frac{1}{2}g^\nu_\mu R = -\kappa T^\nu_\mu, \quad \text{or}
R^\nu_\mu = -\kappa T^\nu_\mu \equiv -\kappa(T^\nu_\mu - \frac{1}{2}g^\nu_\mu T),
\]

where \( \kappa = 8\pi G \) is the gravitational constant, \( R \) the Ricci tensor, and \( T \) the trace of the SET. In what follows we will mostly use the equations in the form (5), but it is also helpful to join the constraint equation from (7), which is the first integral of the others and contains only first-order derivatives of the metric functions:

\[
G^1_1 = e^{-2\alpha} U + \omega^2 = -\kappa T^1_1.
\]

One can see from (5) that the diagonal components of the Ricci \( (R^\nu_\mu)_0 \) and Einstein \( (G^\nu_\mu)_0 \) tensors split into those for the static metric (that is, the metric (11) with \( E = 0 \)) plus a contribution from \( \omega \) [6]:

\[
R^\nu_\mu = sR^\nu_\mu + \omega R^\nu_\mu, \quad \omega R^\nu_\mu = \omega^2 \text{diag}(-2,2,0,2),
\]

\[
G^\nu_\mu = sG^\nu_\mu + \omega G^\nu_\mu, \quad \omega G^\nu_\mu = \omega^2 \text{diag}(-3,1,-1,1),
\]

where \( sR^\nu_\mu \) and \( sG^\nu_\mu \) are the static parts. The tensors \( sG^\nu_\mu \) and \( sG^\nu_\mu \) (each separately) satisfy the conservation law \( \nabla_\alpha G^\alpha_\mu = 0 \) according to this static metric. Thus the tensor \( \omega G^\nu_\mu / \kappa \) acts as an additional SET with exotic properties (e.g., the effective energy density is \(-3\omega^2 / \kappa < 0\)), favorable for obtaining wormhole solutions, as confirmed by a number of examples in Refs [6,7,27,29].

Notably, it is sufficient to solve the diagonal components of the Einstein equations, their single off-diagonal component then holds as well [6].

2.2 Boundary conditions

1. A regular symmetry axis. In addition to the field equations, physical problems generally require imposing some well-motivated boundary conditions. In cylindrical symmetry, it is most often required that the space-time metric should be regular on the symmetry axis, which guarantees that there is no source of gravity (say, a cosmic string
along the axis) other than the one under consideration. A regular axis certainly assumes that at the corresponding value of the radial coordinate, where \( g_{\phi\phi} \to 0 \), all curvature invariants tend to finite limits, which, in terms of the metric \( g_{ab} \) requires, in particular, finite limits of \( \beta, \gamma, \mu, E \). One more requirement (similar to spherical symmetry) is that of a correct circumference to radius ratio \((2\pi)\) for small circles around the axis. This is achieved as long as (see, e.g., \cite{21}):

\[
\frac{1}{4X} e^{-2\alpha} X^2 \to 1, \quad X = |g_{\phi\phi}|.
\]

If \( E = 0 \) on the axis, this condition reduces to \( e^{-2\alpha+2\beta} r^2 \to 1 \) (quite similar to spherical symmetry where \( e^{2\alpha} = -g_{11} \) and \( e^{2\beta} \) is the coefficient near the angular part of the metric \cite{30}). It prevents the occurrence of a conical singularity on the axis and is necessary for the existence of a unique tangent flat space at points located on the axis.

2. The outer boundary of a fluid distribution. A particular solution to the field equations may describe a fluid distribution that occupies the whole space, but in many cases it proves to be necessary to restrict this distribution to a finite region outside which space-time is empty or filled with another kind of matter, such as an electromagnetic or scalar field, a cosmological constant, etc. At a boundary surface \( \Sigma \) between two media, it is required that the metric tensor should be continuous (otherwise it is impossible to say that this boundary as seen “from the left” and “from the right” is the same surface). In addition (see, e.g., \cite{31,33}), the extrinsic curvature tensor \( K_{ab} \) of \( \Sigma \) (the indices \( a, b \) correspond to coordinates on \( \Sigma \)) must be continuous across it. For cylindrically symmetric space-times, in which case \( \Sigma \) is a cylinder, continuity of \( K_{ab} \) implies continuity of the radial pressure \( p_r \), hence, in particular, matching to a vacuum region with \( T^\nu_\mu = 0 \) is only possible on a surface on which \( p_r = 0 \).

If we admit a finite discontinuity of \( K_{ab} \), it means that we admit \( \Sigma \) as a thin shell with a surface SET proportional to this discontinuity.

3. Wormhole geometries. It can happen that a cylindrically symmetric space-time does not contain a symmetry axis, but, instead, the circular radius \( r(x) = e^{\beta(x)} \) has a regular minimum on a certain cylinder \( x = x_0 \), called a throat\(^5\) and is large or infinite far from this minimum. Then it makes sense to discuss the boundary conditions at such large radii or outer boundaries on both sides of the throat.

4. Asymptotic flatness. If an object under consideration is to be observable from a distant, almost flat space-time region, the corresponding metric should be asymptotically flat. A shortcoming of almost all cylindrically symmetric solutions is a lack of asymptotic flatness: even the simplest static vacuum Levi-Civita solution is asymptotically flat only if it is completely flat. This problem is ignored in many studies, in which static cylindrical solutions are matched on some surfaces with a Levi-Civita exterior, and stationary ones with a Lewis exterior. This problem was discussed for wormhole models in \cite{6,7,20,31}, where, following the suggestion of \cite{6}, wormhole space-times were constructed with three regions: a central one containing a throat and two external Minkowski regions matched to this central one at some cylinders \( \Sigma_- \) and \( \Sigma_+ \). These surfaces inevitably contained some densities and pressures calculated from jumps of \( K_{ab} \) across them. A separate problem is to find such models where matter in the central region and on its both boundaries satisfies the Weak and Null Energy Conditions. These issues are beyond the scope of the present paper, for details see \cite{6,7,20,21,31}.

3 Solutions with anisotropic fluids

3.1 Anisotropic fluids: the formalism

The general formalism of nondissipative anisotropic fluids in Riemannian space-time \cite{19,21} assumes matter with a certain energy density \( \rho \) and three principal pressures \( p_i \) in mutually orthogonal directions, so that the SET has the form: \(^6\)

\[
T^{\alpha\beta} = \text{diag}(\rho, p_1, p_2, p_3)
\]

in an orthonormal tetrad corresponding to the fluid’s comoving reference frame (its comoving nature is evident from the zero values of the energy

\(^5\)There can be another definition of a cylindrical throat, related to the behavior of the area function \( a(x) = e^{\beta+\mu} \) instead of \( e^{\beta} \), see a discussion in \cite{6,25}.

\(^6\)Tetrad indices are written in parentheses.
flux components $T^{(0i)}$, $i = 1, 2, 3$)

$$\left( e_\mu^{(a)} \right) = \left( u^\mu, \phi^\mu, \chi^\mu, \psi^\mu \right),$$  \hspace{1cm} (14)

where $u^\mu$ is timelike and has the meaning of the fluid’s 4-velocity while the other tetrad vectors $\phi^\mu, \chi^\mu, \psi^\mu$ are spacelike. The usual coordinate SET components are

$$T^{\mu\nu} = e_\mu^{(\alpha)} e_\nu^{(\beta)} T^{(\alpha\beta)}$$

$$= \rho u^\mu u^\nu + p_1 \phi^\mu \phi^\nu + p_2 \chi^\mu \chi^\nu + p_3 \psi^\mu \psi^\nu.$$  \hspace{1cm} (15)$$

A more conventional form of $T^{\mu\nu}$ is obtained from (15) by adding and subtracting the quantity $p_1 g^{\mu\nu}$, where the choice of $p_1$ among the pressures is arbitrary, and taking into account that

$$g^{\mu\nu} = e_\mu^{(\alpha)} e_\nu^{(\beta)} \eta^{(\alpha\beta)} = u^\mu u^\nu - \phi^\mu \phi^\nu - \chi^\mu \chi^\nu - \psi^\mu \psi^\nu.$$  \hspace{1cm} We thus obtain

$$T^{\mu\nu} = (\rho + p_1) u^\mu u^\nu - p_1 g^{\mu\nu} + (p_2 - p_1) \chi^\mu \chi^\nu + (p_3 - p_1) \psi^\mu \psi^\nu.$$  \hspace{1cm} (16)$$

If $p_1 = p_2 = p_3 = p$, we obtain the usual (isotropic) perfect fluid SET, $T^{\mu\nu} = (\rho + p) u^\mu u^\nu - pg^{\mu\nu}$.

Let us now consider such matter in space-times with the metric (1), assuming that $p_1 = p_z$ corresponds to the radial pressure, $p_2 = p_x$ and $p_3 = p_\varphi$ to pressures in the $z$ and $\varphi$ directions. Then, as easily verified, the conservation law $\nabla_\nu T^{\nu}_\mu = 0$ takes the explicit form:

$$p_z'(\rho + p_x) + (p_2 - p_x) \chi' + (p_3 - p_\varphi) \beta' = 0.$$  \hspace{1cm} (17)$$

It is of interest that $-g_{03} = E$ does not appear in (17), and this relation is the same as with the static metric.

### 3.2 A search for solutions

Consider the simplest class of equations of state for the SET (16):

$$p_x = w_1 \rho, \quad p_z = w_2 \rho, \quad p_\varphi = w_3 \rho,$$  \hspace{1cm} (18)$$

with $w_i = \text{const.}$ Up to now, we worked with an arbitrary radial coordinate $x$. To solve the Einstein equations (8), it is helpful to choose the harmonic “gauge”

$$\alpha = \beta + \gamma + \mu.$$  \hspace{1cm} (19)$$

Then the components $(\alpha^\mu_0), (\alpha^\mu_2), (\alpha^\mu_3)$ of (8) and Eq. (9) can be written in the form

$$e^{-2\alpha} x'' + 2 \omega^2 = \frac{\kappa \rho}{2} (1 + w_1 + w_2 + w_3),$$  \hspace{1cm} (20)$$

$$e^{-2\alpha} y'' = \frac{\kappa \rho}{2} (-1 + w_1 - w_2 + w_3),$$  \hspace{1cm} (21)$$

$$e^{-2\alpha} (\beta' \gamma' + \beta' \mu' + \gamma' \mu') + \omega^2 = w_1 \kappa \rho,$$  \hspace{1cm} (22)$$

respectively. The sum of (20) and (22) gives

$$\beta'' + \gamma'' = \kappa \rho e^{2\alpha} (w_1 + w_2).$$  \hspace{1cm} (24)$$

Equation (21) also does not contain $\omega$, and combining it with (24), we obtain an easily integrable relation involving only second-order derivatives of the metric functions

$$2(w_1 + w_2) \mu'' = (1 + w_1 - w_2 + w_3)(\beta'' + \gamma'').$$  \hspace{1cm} (25)$$

This relation enables us to exclude one of the three unknowns $\beta, \gamma, \mu$ from our equations.

Next, we note that, if we leave $\beta'', \gamma'', \mu''$ in any linear combination of Eqs. (20)–(22), the r.h.s. will contain a linear combination of two terms, one proportional to

$$\omega^2 e^{2\alpha} = \omega_0^2 e^{2\xi}, \quad \xi := \beta - \gamma,$$  \hspace{1cm} (26)$$

and the other proportional to $\rho e^{2\alpha}$. Assuming $w_1 \neq 0$, the conservation equation (17) is integrated giving

$$\rho = \rho_0 \exp \left\{ - \frac{1}{w_1} \left[ (1 + w_1) \gamma + (w_1 - w_2) \mu \right. \right.$$  $$\left. + (w_1 - w_3) \beta \right\}, \quad \rho_0 = \text{const.}$$  \hspace{1cm} (27)$$

Therefore, recalling (17), we can write

$$\rho e^{2\alpha} = \rho_0 e^{2\eta},$$

$$2\eta := \beta \left(1 + \frac{w_3}{w_1}\right) + \gamma \left(1 - \frac{1}{w_1}\right) + \mu \left(1 + \frac{w_2}{w_1}\right).$$  \hspace{1cm} (28)$$

Taking into account the relationship (25) and combining the equations (21)–(22), it is straightforward to obtain coupled equations for $\xi$ and $\eta$:

$$\xi'' = A e^{2\xi} + B e^{2\eta},$$  \hspace{1cm} (29)$$

$$\eta'' = C e^{2\xi} + D e^{2\eta},$$  \hspace{1cm} (30)$$

where the constant factors $A, B, C, D$ depend on the values of $w_i$ as well as $\rho_0$ and $\omega_0$. If $B = C =$
0, we have two separate Liouville-type equations which are solved directly and completely, and it remains to substitute the solution to the first-order equation \(24\) to verify the solution and to obtain a relation between the integration constants.

Apart from the case \(B = C = 0\), in which the matrix of coefficients in Eqs. \((29), (30)\) is diagonal, this set of equations can be integrated if this matrix is degenerate, that is, its determinant \(AD - BC\) is zero. Indeed, in this case we have

\[
A\eta'' = C\xi'' \Leftrightarrow B\eta'' = D\xi'',
\]

so that \(\eta = (C/A)\xi + \eta_1 x + \eta_0\), where \(\eta_1, \eta_0 = \text{const.}\), and at least in the case \(\eta_1 = 0\) we then obtain for \(\xi(x)\) an equation integrable by quadratures

\[
\xi'' = Ae^{2\xi} + B e^{2(C/A)\xi + 2\eta_0}.
\]

A nonzero \(\eta_1\) with explicit \(x\)-dependence may prevent such easy integration. Thus, in general, in this case we can obtain an “almost general” solution with one lacking integration constant. There exist other completely integrable cases related to Toda systems, see, e.g., [35].

If the matrix of coefficients in Eqs. \((29), (30)\) is neither diagonal nor degenerate, it is still often possible to obtain special solutions of interest, as is described below. The case \(w_1 = 0\) should be considered separately.

### 3.3 A general solution

Consider the general case \(w_1 \neq 0\) and assume \(w_1 + w_2 \neq 0\), then Eq. \((25)\) allows us to express \(\mu\) in terms of \(\beta\) and \(\gamma\). Indeed, integration of \((25)\) gives

\[
\mu = \frac{-1 + w_1 - w_2 + w_3}{2(w_1 + w_2)}(\beta + \gamma + mx),
\]

where \(m = \text{const.}\) and another integration constant is suppressed by choosing a scale along the \(z\) axis. Then, after some algebra, we obtain the following coefficients in Eqs. \((29)\) and \((30)\):

\[
A = 4\omega_0^2,
B = -\kappa\rho_0(1 + w_3),
C = \omega_0^2 w_3(1 + w_3),
D = \frac{\kappa\rho_0}{4w_1}[w_1 + w_3)(w_1 + w_2 + 2w_3 - 2) - (1 + w_3)^2],
\]

while the unknown functions \(\xi(x)\) and \(\eta(x)\) are

\[
\xi = \beta - \gamma, \\
\eta = (3w_1 - w_2 + 3w_3 - 1)\frac{\beta}{4w_1} + (3w_1 - w_2 + w_3 - 3)\frac{\gamma}{4w_1} + Km, \\
\text{with } K := \frac{1}{4}(w_1 - w_2 + w_3 - 1).
\]

The set of equations \((29), (30)\) can be completely integrated in the case \(w_3 = -1\). That is, assuming \(w_1 \neq 0, w_1 + w_2 \neq 0, w_3 = -1\), we have the following decoupled equations for \(\xi(x)\) and \(\eta(x)\):

\[
\xi'' = 4\omega_0^2 e^{2\xi}, \quad \eta'' = D e^{2\eta}
\]

with \(D = \kappa\rho_0(w_1 + w_2)(3w_1 - w_2 - 4)/(4w_1)\). These equations are easily solved, and the specific form of the solutions will depend on the sign of \(D\). We leave their full analysis for the future.

The other integrable case, \(AD = BC\), takes place if \(3w_1 - w_2 + 2w_3 = 2\), but the resulting equation \((32)\) for \(\xi(x)\) does not in general lead to solutions in elementary functions.

### 3.4 Special solutions for \(w_1 \neq 0\)

If \(w_3 + 1 \neq 0\), it is in general impossible to solve the coupled equations \((29)\) and \((30)\) completely, but some special solutions can be obtained. Indeed, let us suppose

\[
\eta = \xi + \ln h, \quad h = \text{const.}
\]

so that \(\xi'' = \eta''\), and, comparing Eqs. \((29)\) and \((30)\), we obtain the consistency requirement

\[
A + Bh^2 = C + Dh^2,
\]

or

\[
A - C = (D - B)h^2.
\]

Note that \(A - C\) is proportional to \(\omega_0^2 \kappa\rho_0\) to \(\kappa\rho_0\), but these two parameters are still not directly related due to the existence of the arbitrary factor \(h^2\). Evidently, a solution only exists if and only if the signs of \(A - C\) and \(D - B\) are the same. If it is the case, we have

\[
\xi'' = (A + Bh^2) e^{2\xi},
\]

again a Liouville equation which is easily solved. After that, knowing \(\xi = \beta - \gamma\) and a relation between \(\beta\) and \(\gamma\) that follows from \((37)\), one easily finds \(\beta(x)\) and \(\gamma(x)\) and then \(\mu(x)\) from \((33)\). To
know the metric completely, it remains to substitute the results for $\beta, \gamma, \mu$ to (39) and (41) to find a relation between integration constants and to determine $E(x)$.

3.5 Fluids with $w_1 \neq 0$, special cases of Eq. (25)

1. Fluids with $w_1+w_2 = 0$ but $1+w_2 \neq w_1+w_3$. We return to the set of equations (20)–(22), and (25) as their consequence, but the l.h.s. of the latter is now zero, and the r.h.s then yields

$$\beta'' + \gamma'' = 0 \Rightarrow \beta = -\gamma + bx + \ln r_0,$$

with $b, r_0 = \text{const}$, and both $\xi$ and (eta) are expressed in terms of $\gamma$:

$$\xi = -2\gamma + bx + \ln r_0,$$

$$\eta = -\frac{(1 + w_3)\gamma + (w_1 + w_3)(bx + \ln r_0)}{2w_1}. \quad (41)$$

Therefore, Eq. (20) contains only one unknown function $\gamma(x)$, though, it is uneasy to solve it in the general case $b \neq 0$. Having solved it, it is then easy to find $\mu(x)$ by directly integrating Eq. (21).

In the special case $b = 0$, it makes sense to use the constraint (23), which is then a first integral of (20): we have $\beta' + \gamma' = 0$, so that $\mu'$ is excluded from (23) that now reads

$$\gamma'' = \omega_0^2 r_0^2 e^{2\gamma} - \omega_0^2 x_0 w_1 r_0^{1+w_3/w_1} e^{-(1+w_3)\gamma/w_1} \quad (42)$$

and can be solved by quadratures (though in known functions only for some values of $w_1$ and $w_3$). And, as in other cases, already knowing $\beta, \gamma, m\mu$, it remains to find $E(x)$ from (41).

2. Fluids with $1 + w_2 = w_1 + w_3$ but $w_1 + w_2 \neq 0$. Under these assumptions, instead of (40), we obtain from (25) (or equivalently from (21))

$$\mu'' = 0 \Rightarrow \mu = mx + m_0,$$

$$m, m_0 = \text{const}, \quad (43)$$

where we can get $m_0 = 0$ by rescaling the $z$ axis.

It turns out that in this case the situation with solving our set of equations is practically the same as in the general case. Thus, we have Eqs. (29) and (30) that decouple under the condition $w_3 = -1$, in which case we have $B = C = 0$ and arrive at Eq. (36) with $D = \omega_0^2 r_0 (w_1 - 1)^2/w_1$.

If $w_3 \neq -1$, some special solutions can be obtained in the same manner as with Eqs. (37)–(39).

3. Fluids with $w_1 - w_2 + w_3 - 1 = w_1 + w_2 = 0$. Now both equations (10) and (43) are valid, and there remains only one unknown function $\gamma(x)$ in the set (20)–(22). Moreover, for $w_1$ we have

$$w_2 = -w_1, \quad w_3 = 1 - 2w_1, \quad (44)$$

and the functions $\xi(x)$ and $\eta(x)$ are expressed as

$$\xi = -2\gamma + bx + \ln r_0,$$

$$\eta = \frac{1-w_1}{2w_1} (-2\gamma + bx + \ln r_0). \quad (45)$$

Also, as in the previous case, we have $\mu'' = 0 \Rightarrow \mu = mx$. The function $\gamma(x)$ can be found by quadratures from the equation that follows from (23),

$$\gamma'' - b\gamma' - bm = \omega_0^2 e^{3\gamma} - \omega_0 r_0 w_1 e^{2\gamma}. \quad (46)$$

Thus in this case the problem is solved completely by quadratures.

3.6 Solutions with $w_1 = 0$

We so far assumed $w_1 \neq 0$. What changes if $w_1 = 0$? In such a case Eq. (17) yields (provided $\rho \neq 0$)

$$\gamma' = w_2\mu' + w_3\beta',$$

whence

$$\gamma = w_2\mu + w_3(\beta - \ln r_0), \quad (47)$$

where $r_0 = \text{const}$ has the meaning of a length scale because $e^{3\gamma}$ has the dimension of length. As before, from (20)–(22) we have the relation (25) which now reduces to

$$2w_2\mu'' = (1 - w_2 + w_3)(\beta'' + \gamma''). \quad (48)$$

Taken together, (48) (after integration) and (47) leave only one unknown among $\beta, \gamma, \mu$. For this remaining function, the corresponding equation should be combined again from (20)–(22), depending on the values of $w_2$ and $w_3$, after which it will be straightforward to find $\rho(x)$, and, as before, the last steps in obtaining the solution are a substitution to (6), leading to a relation between the integration constants, and finding $E(x)$ by integration in (11).

It should be noticed that under the condition $w_1 = 0$ the anisotropic fluid distribution may be in general cut at any value of $x$ and matched to the external vacuum (Lewis) solution without any thin shells with nonzero surface stress-energy.
3.7 Isotropic perfect fluids, \( w \neq 0 \)

In the important case of \( p = \omega \rho \), where \( \omega = \omega_1 = \omega_2 = \omega_3 \), the general solution available at \( \omega_3 = -1 \) corresponds to a cosmological constant \( \Lambda = p/\kappa = -p/\kappa \). We thus arrive at the Lewis well-known solution, which is traditionally written in other notations [1,2] and is also discussed in [22, 23]: note also its generalizations with a massless scalar field [6, 8] and scalar fields with exponential potentials [7].

At \( w \neq -1 \) we can hope to obtain special solutions as described in the previous subsection. Assuming \( \eta = \xi + \ln h \), we have

\[
\eta = \frac{1}{4w}[(5w - 1)\beta + 3(w - 1)\gamma + (w - 1)mx],
\]

\[
D = \frac{x_0}{4w}(7w^2 - 6w - 1),
\]

and evident expressions for \( A, B, C \) from [31]. Equation (38) takes the form

\[
4\omega_0^2(3w - 1) = x_0 h^2(11w^2 - 2w - 1).
\]

(50)

It can be easily verified that both sides of this equality are nonzero and have the same sign if

\[
w \in \left(\frac{1 - 2\sqrt{3}}{11}, \frac{1}{3}\right) \quad \text{or} \quad w > \frac{1 + 2\sqrt{3}}{11} \approx 0.406
\]

(51)

(both parts of (50) are negative in the first range and positive in the second one). Only in these ranges of \( w \) we can obtain solutions. Notably, two physically distinguished cases \( w = 1/3 \) (chaotic radiation, or an ultrarelativistic gas) and \( w = -1/3 \) (a chaotic gas of cosmic strings) do not belong to the range (51). In particular, under the assumption \( w = 1/3 \) Eq. (50) leads to \( \rho_0 = 0 \), that is, the absence of a matter distribution.

With (31) and (50), Eq. (39) takes the Liouville form

\[
\xi'' = \frac{4w(2w - 1)}{3w - 1}x_0 h^2 e^{\xi}. \quad (52)
\]

The sign of its r.h.s., determining the form of its solutions, is thus different for different \( w \).

Solutions for arbitrary \( w \) were obtained in [18] using a very unusual coordinate condition for \( x \) and reducing the problem to a special form of the Emden-Fowler equation. The presently described method leads to more special solutions of a much simpler form.

4 Some special solutions

We have described the way of obtaining exact solutions for quite diverse isotropic and anisotropic fluid distributions. Their detailed description would be too long for this paper, and we here restrict ourselves to some simple examples.

Example 1: Stiff perfect fluid, \( w = 1 \)

As is clear from Subsection 3.5, for perfect fluids with \( w \neq -1 \) our approach gives only special solutions and only in the ranges (51) of the parameter \( w \). Such a solution is especially simple for a maximally stiff perfect fluid, \( w = 1 \), for which the speed of sound is equal to the speed of light. Indeed, Eq. (21) leads to \( \mu'' = 0 \), hence we can write \( \mu = mx \), \( m = \text{const} \). The assumption (37) then leads to

\[
\xi'' = 2\omega_0^2 e^{2\xi}, \quad \omega_0^2 = x_0 h^2.
\]

(53)

while since now we have \( \eta = \beta + \mu = \beta + mx \) while \( \xi = \beta - \gamma \), Eq. (37) gives \( \gamma = -mx + \ln h \). Furthermore, adjusting the time scale, we can assume \( h = 1 \), so that

\[
\mu = mx, \quad \gamma = -mx, \quad \beta = \xi + mx. \quad (54)
\]

Already at this stage we can substitute (54) to Eq. (49), where \( \beta' \) is excluded due to \( \mu' + \gamma' = 0 \), and due to the second equality (53) we obtain \( m = 0 \), so that we have simply \( \xi = \beta \). According to (27), in this configuration the density \( \rho \) is constant.

The Liouville equation (53) for \( \xi = \beta \) has the first integral

\[
\beta'^2 = 2\omega_0^2 e^{2\beta} + k^2 \text{sign} k, \quad k = \text{const}, \quad (55)
\]

and its solution splits into three branches according to the sign of \( k \):

\[
e^\beta = \frac{k_1}{\sqrt{2\omega_0^2 \cos(k_1 x)}}, \quad k = -k_1 < 0,
\]

\[
e^\beta = \frac{1}{\sqrt{2\omega_0^2 x}}, \quad k = 0,
\]

\[
e^\beta = \frac{k}{\sqrt{2\omega_0^2 \sinh(k x)}}, \quad k > 0. \quad (56)
\]

where in all cases an integration constant is excluded by choosing the zero point of \( x \). The first branch corresponds to a metric of wormhole nature,
with $e^\beta$ having a minimum at $x = 0$, and $\beta \to \infty$ as $x \to \pm \pi/(2k_1)$. In this case Eq. (4) gives

$$E = \frac{k_1}{\omega_0} \tan(k_1x) + E_0, \quad E_0 = \text{const.} \quad (57)$$

This solution was used in [27] for obtaining a twice asymptotically flat cylindrical wormhole model by cutting it at sufficiently small $|x| < \pi/(2k_1)$ and joining it to flat space regions on both sides from the throat $x = 0$ through thin shells.

The other two branches describe a full range of circular radii $r = e^\beta$ from $r = 0$ at $x = \infty$ to $r \to \infty$ as $x \to 0$. For $E(x)$, Eq. (4) gives

$$E = \frac{1}{\omega_0 x} \quad (k = 0),$$

$$E = \frac{k}{\omega_0} \left[1 - \coth(kx)\right] \quad (k > 0), \quad (58)$$

where we chose the integration constant in (4) so that $E \to 0$ as $x \to \infty$. It can be verified using Eq. (12) that the axis is regular in the only case $k = 1$ belonging to the third branch (56). To our knowledge, this solution was obtained for the first time in other notations in [14].

**Example 2: Azimuthal strings**

A distribution of closed cosmic strings allocated in the $\phi$ direction corresponds to

$$(w_1, w_2, w_3) = (0, 0, -1), \quad (59)$$

so that the tension $-p_\varphi$ in the angular $\varphi$ direction is equal to the energy density $\rho$. For this set of $w_i$, Eq. (47) is applicable, now leading to

$$\beta = -\gamma + \ln r_0, \quad \xi = -2\gamma + \ln r_0, \quad (60)$$

where, as before, $r_0$ is an arbitrary length scale. It turns out that in this case our set of field equations is underdetermined since both (20) and (22) lead to the same equation for $\gamma(x)$,

$$\gamma'' = -2\omega_0^2 r_0^2 e^{-4\gamma}, \quad (61)$$

and Eq. (23) yields their integral,

$$\gamma'^2 = \omega_0^2 r_0^2 e^{-4\gamma}, \quad (62)$$

whence it follows

$$e^{2\gamma} = 2r_0|\omega_0|x, \quad e^{2\beta} = \frac{r_0}{2|\omega_0|x} \quad (63)$$

(as before, we choose the zero point of $x$ as to kill an insignificant integration constant). The density $\rho$ dropped out from both (61) and (62), and there is a single Eq. (64) for the two unknowns $\mu$ and $\rho$:

$$\mu'' = -\omega_0^2 \rho e^{2\mu}. \quad (64)$$

So one can take either $\mu(x)$ or $\rho e^{2\mu}$ as an arbitrary function and calculate the remaining unknowns with Eq. (63). This means, in fact, that a set of rotating azimuthal cosmic strings can have an arbitrary density distribution along the radial direction.

The last unknown $E$ is obtained from (4) as

$$E = r_0(\text{sign } \omega_0)(E_0 x - 1), \quad E_0 = \text{const.} \quad (65)$$

Choosing $E_0 = 0$, we obtain $E = -r_0 \text{sign } \omega_0 = \text{const}$ (which certainly does not mean zero vorticity).

The solution is defined for $x \in \mathbb{R}^+$, $x \to 0$ corresponds to large radii $r = e^\beta$ with an attracting singularity due to $e^\gamma \to 0$, while $x \to \infty$ leads to $r \to 0$ and large positive $g_{33} = -e^\beta + E^2 e^{-2\gamma}$, therefore circles parametrized by $\varphi$ are closed time-like curves violating causality.

**Example 3: Longitudinal radiation flows**

One more situation of interest admitting a description in terms of anisotropic fluids is that there are two opposite radiation flows of equal intensity in the $z$ direction, such that the net flux is zero.

If there are radiation flows with intensities $\Phi_+$ and $\Phi_-$ in the positive and negative $z$ directions, respectively, the SET of each has the form $T'_\mu = \Phi_+ k_\mu k^\nu$, where $k^\mu$ is one of the the null vectors $(e^{-\gamma}, 0, \pm e^{-\mu}, 0)$. Then the nonzero part of the summed SET of these two flows reads

$$T'_\theta = \begin{pmatrix} \Phi_+ + \Phi_- & \Phi_+ - \Phi_- \\ \Phi_+ - \Phi_- & -\Phi_+ + \Phi_- \end{pmatrix}, \quad (66)$$

where $a, b = 0, 2$. (A similar construction was discussed in [36] in relation to generalizations of the Birkhoff theorem.) If $\Phi_+ = \Phi_-$, the summed SET has the form

$$T'_\mu = \rho \text{ diag}(1, 0, -1, 0), \quad \rho = 2\Phi. \quad (67)$$

It may be considered as the one of an anisotropic fluid with

$$(w_1, w_2, w_3) = (0, 1, 0). \quad (68)$$
With such \( w_i \), Eqs. (25) and (47) lead to
\[
\beta'' + \gamma'' + \mu'' = 0, \\
\gamma' = \mu' \Rightarrow \gamma = \mu 
\]
(with a proper choice of scales along the \( z \) and \( t \) axes), and we can write, in the previous manner,
\[
\beta = -2\gamma + bx + \ln r_0, \quad b, r_0 = \text{const.} \quad (70)
\]
Thus \( \beta \) and \( \mu \) are expressed in terms of \( \gamma \), but it is reasonable to use as the remaining unknown
\[
\xi(x) = \beta - \gamma = -3\gamma + bx + \ln r_0, \quad (71)
\]
for which Eq. (23) leads to the easily integrable equation
\[
\xi'' = 3\omega_0^2 e^{2\xi} + b^2. \quad (72)
\]
Its solution is
\[
e^{-\xi} = \frac{1}{|b|}\sqrt{3\omega_0^2}\sinh(|b|x), \quad b \neq 0, \\
e^{-\xi} = \sqrt{3\omega_0^2}x, \quad b = 0. \quad (73)
\]
and without loss of generality we assume \( x > 0 \). Then for \( b \neq 0 \) the metric coefficients are
\[
e^{3\gamma} = e^{3\mu} = \frac{r_0}{2|b|}\sqrt{3\omega_0^2}e^{2bx} - 1, \\
e^{3\beta} = \frac{r_0}{b^2}e^{bx}\sinh^2(bx), \\
E = (\text{sign } \omega_0)(E_0 - \coth(|b|x)) \times \left[ \frac{2r_0^2|b|}{9|\omega_0|} (e^{2bx} - 1)^2 \right]^{1/3}. \quad (74)
\]
In the case \( b = 0 \) the metric coefficients are
\[
e^{3\gamma} = e^{3\mu} = \sqrt{3\omega_0^2}r_0x, \quad e^{3\beta} = \frac{r_0}{3\omega_0^2x^2}, \\
E = \frac{2\omega_0}{(3\omega_0^2)^{1/3}}(r_0x)^{2/3}(E_0 - \frac{1}{x}). \quad (75)
\]
Lastly, the density can be determined from Eq. (21): for any value of \( b \): substituting \( \alpha = \beta + \gamma + \mu \) and Eqs. (70) and (71), we find
\[
x'r_0^2 = 2\omega_0^2 e^{2\xi - 2bx}. \quad (76)
\]
and consequently
\[
x'r_0^2 = \begin{cases} 
\frac{8b^2}{3(e^{2bx} - 1)^2}, & b \neq 0, \\
\frac{2}{3x^2} & b = 0. 
\end{cases} \quad (77)
\]
The solution is defined for \( x \in \mathbb{R}_+ \). We see that in all cases the value \( x = 0 \), corresponding to large radii \( r = e^\beta \), is a singularity with \( \rho \to \infty \).
However, other properties of the solution strongly depend on the sign of \( b \) and, to a smaller extent, on the value of \( E_0 \). For example, as \( x \to \infty \) (small \( r \)), \( \rho \to 0 \) if \( b \geq 0 \), and \( \rho \to 8b^2/3 \) if \( b < 0 \). Closed timelike curves are observed at large \( x \) if \( b = 0 \), \( E_0 \neq 0 \). A more detailed analysis is beyond the scope of this paper.

5 Conclusion

This paper demonstrates the way of obtaining exact solutions for the gravitational fields of rotating anisotropic fluids with pressures \( p_i = w_i\rho \) \((i = 1, 2, 3)\) under the assumption of cylindrical symmetry. It is clear that this framework covers a diversity of physical systems such as perfect fluids, cosmic string distributions, combinations of mutually opposite radiation flows as well as some classical field systems. The three examples briefly considered here show a great diversity of physical properties of the solutions: some of them may describe cylindrical wormholes, others can possess a regular axis, and many of these space-times contain closed timelike curves. We postpone a detailed analysis of such solutions for the near future.

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