The geometry of quantum spin networks

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Abstract

The discrete picture of geometry arising from the loop representation of quantum
gravity can be extended by a quantum deformation. The operators for area and volume
defined in the q-deformation of the theory are partly diagonalized. The eigenstates are
expressed in terms of q-deformed spin networks. The q-deformation breaks some of
the degeneracy of the volume operator so that trivalent spin-networks have non-zero
volume. These computations are facilitated by use of a technique based on the recoin-
pling theory of $SU(2)_q$, which simplifies the construction of these and other operators
through diffeomorphism invariant regularization procedures.

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1 Introduction.

Topological quantum field theory arose from a study of invariants of three and four dimensional manifolds (1 - 13). This led to a new realm of mathematical structure, which ties together topology, representation theory and category theory. In three dimensions, the Chern-Simons topological quantum field theory is closely related to invariants of knots, links and networks. Three dimensional topological quantum field theories also provide models of quantum gravity in 2 + 1 dimensions which teach us about the construction and interpretation of diffeomorphism invariant quantum field theories [14]. However, from the beginning, there have been reasons to believe that three dimensional topological quantum field theories might play a direct role in quantum gravity in 3 + 1 dimensions. One reason for this is that the Kodama state [15], which is the exponential of the Chern-Simons invariant, is the only state known to be an exact physical state of quantum gravity and, in the same time, to have a semi-classical interpretation as the vacuum state associated to DeSitter spacetime [16]. Another reason is that in the presence of a boundary, conditions may be chosen so that Chern-Simon theory is induced from quantum gravity in the same way that, one dimension lower, Wess-Zumino-Witten theory is induced on the boundary of Chern-Simons theory [17, 18]. Still other arguments for a role for Chern-Simons theory in quantum gravity come from the interpretation problem in quantum cosmology [19], from the holographic hypothesis [20] and from the attempts to construct quantum gravity algebraically as an extension (or categorification) of it [8, 10, 19].

Recently another reason for suspecting a close relation between quantum gravity and topological quantum field theory has emerged. Spin networks play a key role in the states in both formalisms. (A spin network is a closed graph with edges labeled by the representations of SU(2) and vertices labeled by the ways the representations joined at the vertex can be combined into a singlet [21].) In canonical quantum gravity (22 - 27) it has been discovered that there is a basis of spatially diffeomorphism invariant states of the gravitational field which are labeled by spin networks (28 - 30). This construction proved to be an important step in the search for physical states. It is also an essential ingredient in the development of the measure theory on the space of connections modulo gauge transformations [31]. A closely related structure, quantum spin networks, play a basic role in topological quantum field theory [32, 33]. In a quantum spin network the edges are labeled by representations of some quantum group and the vertices are labeled by the corresponding intertwiners. It is possible that spin networks provide a bridge from quantum gravity to topological quantum field theory and conformal field theory, and perhaps even to string theory. But if this is to happen, quantum gravity itself must be expressed in terms of quantum spin networks. It turns out that this can be done directly [34]. In this alternative quantization, framing factors arise at order $\hbar$ that modify the naive algebra of products of holonomies. One result is that one can no longer multiply operators associated to Wilson loops at will, as one can in the classical theory. For example, there is an $n$ such that for any loop $\alpha$, \[ (\hat{T}[\alpha])^n = 0. \] It is expected that this alternate quantization is appropriate to the

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1 We may also mention an interesting paper in which it seems possible also to derive four dimensional quantum gravity as the boundary theory of a five dimensional TQFT [13].

2 Note that the elements of the basis must be differentiated by labels attached to vertices of valence higher than three to resolve the degeneracy.
case of non-vanishing cosmological constant, $\Lambda$, in which case the deformation parameter $q$ of the quantum group is given by

$$q = e^{i\pi r} = e^{i\hbar^2 G^2 \Lambda / 6}. \quad (1)$$

More generally, we may note that any quantization of a non-canonical algebra such as the loop algebra involves a deformation of the classical algebra by factors proportional to $\hbar$. When we consider the “higher” loop operators, with more than one insertions of the frame field $\tilde{E}^{ai}$ then the quantum algebra is necessarily deformed by such factors. The alternative quantization involves simultaneously deforming in the factor $q$ and $\hbar$ providing an alternative to the usual quantization such that the quantum observables’ algebra reduces to the classical one as $\hbar \to 0$.

Using this alternative quantization, we may express physically interesting observables directly in terms of operators in this q-deformed quantum theory. Among these are kinematical observables such as volume and area. If we label these regions and surfaces by the values of physical fields these may be promoted to diffeomorphism invariant observables.

The primary purpose of this work is to construct several observables for the alternate, q-deformed quantization. The result may be considered to endow quantum spin networks with a new physical interpretation in terms of three dimensional quantum geometry. For given any quantum spin network we are able to associate areas to its edges and volumes to its vertices. In related papers, these results will be extended to dynamical operators such as the hamiltonian and hamiltonian constraint of quantum gravity.

We may note several advantages of the formulation we present here. First, we use techniques from the Temperly-Lieb recoupling theory. These have strong practical advantages as they provide an elegant and efficient way to do calculations involved in the construction and action of operators in non-perturbative quantum gravity. Second, some problems associated with ordinary spin network states in non-perturbative quantum gravity seem to be ameliorated by the quantum deformation. In particular, the volume operator has a great deal of degeneracy in the ordinary representation, in that trivalent vertices all contribute zero volume. We find that the quantum deformation lifts this degeneracy. This may make possible several developments which we mention in the conclusion.

The next three sections summarize the construction of the space of states in q-deformed quantum gravity as well as the basic hypothesis that underlies the regularization procedure in this theory. This is followed by a short summary of the recoupling theory of quantum spin networks. After these preliminaries, we show in Sections 6 to 8 how to define, regulate and compute the area and volume operators in q-deformed quantum gravity. We close with some comments on the applications of these results and techniques which are currently under study.

## 2 The state space of q-deformed quantum gravity

We first briefly describe the structure of the space $\mathcal{H}_q$ of quantum states of the gravitational field after the $q$-deformation. More details may be found in [34]. The space has an orthonormal basis of states $| \alpha \rangle$ labeled by distinct quantum spin networks $\alpha$. A quantum spin network (or q-spin net) consists of the embedding of closed graph into the three
manifold \( \Sigma \) (with fixed topology) with edges labeled by the representations of \( SU(2)_q \) and vertices labeled by distinct ways to decompose the incoming representations into a singlet (trivalent vertices are unique and are thus unlabeled.)

The deformation parameter \( q = e^{i\pi/r} \) will be taken to be at a root of unity, in which case the representations are labeled by integers, \( j \), denoting twice a spin. We will also find it convenient to parameterize the deformation by \( A \) such that \( A^2 = q \). The usual loop representation is then recovered in the limit \( A \to -1 \) (in the binor convention in which \(-1\) has been inserted into each \( SU(2) \) trace). To avoid confusion we will call this the “ordinary” case.

The edges of the spin network may be thought of as being decomposable into framed loops. This is described in [34], where it is shown that straight intersections may be decomposed into linear combinations of two linearly independent intersections, called the “over” and “under” touch, and are denoted \( \hat{\otimes} \) and \( \hat{\otimes} \), respectively. To have consistency with the Kauffman bracket [37] the Mandelstam identity is extended to the two relations

\[
\begin{align*}
\hat{\otimes} & = A^{-1} \otimes + A \otimes, \\
\hat{\otimes} & = A \otimes + A^{-1} \otimes.
\end{align*}
\]

The resulting structure is a graph with labeled edges represented diagrammatically as

\[
\begin{array}{c}
n \\
\hline
\end{array} = \frac{A^{2n-2}}{[n]!} \sum_{\sigma \in S_n} (A^{-3})^{t(\sigma)} \sigma
\]

where \( \sigma \) refers to the right handed braidings [33] and the “quantum integer” \([n]\) is defined by

\[
[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = \frac{A^{2n} - A^{-2n}}{A^2 - A^{-2}}.
\]

Intertwining operators at vertices of the \( q \)-spin net label the way the different loops pass through the vertex. In the case of trivalent vertex the intertwining operator is trivial because there is a unique way for the loops to pass through the vertex. This can be seen on Fig. (1b) where, for instance, \( a \) is the number of loops shared by the \( j \)-edge to the \( k \)-edge. In the case of vertices joining more than three edges, this decomposition is not available and we need additional information for the structure of the vertex. One of the ways to describe such a vertex is to first form a set of trivalent vertices joined with “internal” lines as in Fig. (1c). All these “internal” lines have zero length. Then the vertex is labeled by the color of the “internal” lines as well as by topological factors such as those illustrated in Fig. (2) associated with how the four vertex is defined.

As the \( q \)-spin networks comprise a linearly independent basis, they will also label a set of bras, \( \langle \Gamma \mid \) such that a general state is given by

\[
\Psi[\Gamma] = \langle \Gamma \mid \Psi \rangle.
\]
Figure 1: The trivalent vertex (a.) is decomposed into three projectors as in (b.) with
\[ a = (j + k - l)/2, \quad b = (k + l - j)/2, \quad \text{and} \quad c = (j + l - k)/2. \]
Higher valent intersections may be decomposed in terms of trivalent ones as in (c).

\[ \mathcal{H}^q \] may be endowed with an inner product so that
\[ \langle \Gamma | \Gamma' \rangle = \delta_{\Gamma \Gamma'} . \] (7)

Following the procedure for ordinary spin network states \[28\], we may also define a space \( \mathcal{H}^q_{\text{diff}} \subset \mathcal{H}^q \) of diffeomorphism invariant states. These are labeled by an orthonormal basis \( | \{ \Gamma \} \rangle \) where \( \{ ... \} \) stands for diffeomorphism equivalence class defined by
\[ < \Gamma' | \{ \Gamma \} > = \delta_{\{ \Gamma' \} \{ \Gamma \}}. \] (8)

3 Operators in \( q \)-deformed quantum gravity

In the \( q \)-deformed quantum gravity the basic operators \( \hat{T}^q_{\alpha} \) are associated with framed loops \( \alpha^f \). The framing is necessary to represent the behavior of Wilson loops in the presence of the Chern-Simon measure \[34\]. These act as,
\[ \langle \Gamma | \hat{T}^q_{\alpha} = \langle \Gamma \cup \alpha \rangle . \] (9)

where \( \cup \) is a commutative product among quantum spin networks defined by decomposing using the edge addition formula \[33\],
\[ \begin{array}{c}
\text{old} \\
\begin{array}{c}
\text{n} \\
\end{array}
\end{array}
= \begin{array}{c}
\text{new} \\
\begin{array}{c}
\text{n+1} \\
\end{array}
\end{array}
- \begin{array}{c}
\text{old} \\
\begin{array}{c}
\text{n} \\
\end{array}
\end{array}
\begin{array}{c}
\text{new} \\
\begin{array}{c}
\text{n-1} \\
\end{array}
\end{array} . \] (10)

There is as well an extension of the \( \hat{T}^1_{\alpha} \) operators, denoted \( \hat{T}^q_{\alpha} \alpha(s) \), where \( s \) is a point on the framed loop \( \alpha \). Their action is defined by,
\[ \langle \Gamma | \hat{T}^q_{\alpha} \alpha(s) \rangle = l^2 \sum_j j j l \Delta^q_{\alpha}[e_l, \alpha(s)] \langle e_l \# \alpha(s) \rangle \] (11)

where the \( e_l \) are the edges of the spin network, \( j j l \) are the corresponding colors and \( \langle e_l \# \alpha(s) \rangle \) denotes the action of grasping of the “hand” of the \( T \)-operator on the link \( e_l \). The action amounts to creating a new four-valent vertex. (Unless the point of coincidence is already a vertex, in which case the valence of the vertex is increased by two.)
There is an ambiguity in the definition of the four-valent vertex corresponding to the action of the “hand” of the $\hat{T}^a_q[\alpha](s)$. This is illustrated by Fig.(2) which shows possible ways to decompose the vertex in Fig.(b) and (c). These correspond to additional operator ordering ambiguities which arise due to the phase factors that can appear in quantum spin networks from twistings and braidings of edges. In the construction of operators through regularization procedures these ambiguities must be resolved so as to give well defined operators. As in the case of operator ordering ambiguities that appear elsewhere in quantum theory we know of no general prescription for resolving them. However, in particular cases natural orderings arise, as we will see below.

Physically interesting operators in quantum gravity are constructed by higher loop operators that have more than one site on the loop at which the action we have just defined takes place. For example, the action of a $\hat{T}^{ab}_q[\alpha](s,t)$ is given by the expression:

$$\langle \Gamma | \hat{T}^{ab}_q[\alpha](s,t) = t_{\delta} \sum_{I,J} j_{I,J} \Delta^a[e_I, \alpha(s)] \Delta^b[e_J, \alpha(t)] \langle \Gamma | \# I | \# J | \alpha \rangle$$

where $\langle \Gamma | \# I | \# J | \alpha \rangle$ represents the graphical action of grasping. Each four-valent vertex defined by the action of a hand must be defined by decomposing the vertex into a pair of trivalent vertices, joined with an “internal” line of color two as shown on Fig.(3).

The $\hat{T}$ operators with three and more hands are then defined by extending the definition of the $\hat{T}^2$ so that the action of each hand is defined according to Fig.(2). Again, when the operator is used in the regularization of a particular observable choices must be made of additional phases coming from the operator ordering ambiguities associated with the $q$-deformation.

4 Regularization and recoupling

Operator products may be dealt with in this formulation of quantum gravity just as they are in the ordinary loop representation [27, 39, 29]. A classical diffeomorphism invariant observable $\mathcal{O}$, is rewritten as a limit, in which one or more parameters, $\delta$, are taken to zero, of regulated observables $\mathcal{O}_\delta$. These are not diffeomorphism invariant as they depend explicitly on an arbitrary background metric $h_{\mu}^0$ which is used to define the scale of the point splitting. This regulated observable is represented as a quantum operator $\hat{\mathcal{O}}_\delta$, which is constructed from loop operators. For example, the area, hamiltonian constraint $\mathcal{H}$ and
\[ H = \int \sqrt{-\mathcal{H}} \] are all represented in terms of \( \hat{T}^2 \)'s while the volume is represented in terms of \( \hat{T}^3 \) as in the ordinary case.

The operators are then defined in terms of the limits \( \delta \to 0 \). The \( c \)-number factors must be shown to assemble into finite factors independent of \( h_{\mu\nu}^0 \), while the limits in the dependence of the loop functional must be evaluated in a topology inherited from the manifold topology, as described in detail in \( \text{[27]} \). After these limits are taken one has combinatorial expressions involving loops at a single point. The basic hypothesis we make is that the Chern-Simon measure determines the behavior of the operators in the limit of short distances. This means that the Kauffman bracket relations, satisfied by loop factors in the presence of the Chern-Simon measure, rather than the naive Mandelstam relations, must be used to evaluate the combinatorics of loops that are shrunk to a point in a regularization procedure.

We may note that as the theory is not defined by a loop transform from a measure on the space of connections, such a hypothesis is necessary to complete the definition of the regularization procedure.

5 Excerpts from recoupling theory.

The Kauffman bracket relations are very compactly expressed by the recoupling theory for the quantum group \( SU(2)_q \), which extends the classical theory of recoupling of angular momentum\( \text{[33]} \). The basic relation in this theory expresses the relation between the different ways in which three angular momenta, say \( j_1, j_2, \) and \( j_3 \) can couple to form a fourth one, \( j_4 \). The two possible recouplings are related by the formula:

\[
\sum_I \{ j_1 \quad j_2 \quad j_3 \quad j_4 \} q_I = \sum_I \{ j_1 \quad j_2 \quad J \} q^J_I
\]

where on the right hand side is the \( q \)-6j symbol, as defined in \( \text{[33]} \). Closed loops which have been shrunk to a point may be replaced by their loop value, which is (for a single loop with zero-self-linking) equal to \(-A^2 - A^{-2}\). This extends Penrose's notion of the evaluation of a closed spin network. The evaluation of a single unknotted \( q \)-spin loop with color \( n \) is \( \text{[33]} \):

\[
[n] = (-1)^n [n + 1]
\]

where \([n + 1]\) is a quantum integer.

To see how the given formulae can be used in the evaluation of a \( q \)-spin net let us consider a "bubble" diagram. Upon shrinking of the "bubble" this diagram will reduce to a single edge so the evaluation will be different from zero only if the colors of both ends of the "bubble" are the same. Thus we expect that the "bubble" diagram is equal to some function of the deformation parameter times a single edge. By closing the free ends of the

\[\text{[31]}\]
diagram it is straightforward to show that

\[
\frac{n}{n'} = \delta_{nn'} \frac{(-1)^n}{n+1} \theta(a, b, n)
\]

in which the function \(\theta(a, b, n)\) is given, in general, by

\[
\theta(m, n, l) = \frac{(-1)^{a+b+c}}{[a+b][b+c][c+a]}
\]

where \(a + b = m\), \(a + c = n\), \(b + c = l\) and \([a] = [a][a−1]...[2][1]\).

A main ingredient in the derivation of the \(q\)-6j symbol is the tetrahedral net, which will be written as \(\text{Tet}[a, b, c; d, e]\). The lengthy expression in terms of quantum factorials can be found in Kauffman and Lins’ book [33].

6 The \(q\)-deformed area operator

We consider first the regularization of the \(q\)-deformation of the area operator [27, 41, 29], which was discussed briefly in [34]. We then consider a smooth 2-surface \(S\) in the 3-manifold \(\Sigma\) whose area, \(A\) is classically given by

\[
A(S) = \int_S d^2\sigma \sqrt{\mathcal{E}^{ai} \mathcal{E}^{bi} n_a n_b}
\]

Using the auxiliary background metric \(h^0_{\mu\nu}\), we partition the surface into small squares \(S_I\) of size \(L\), so that we have

\[
A(S) = \lim_{L \to 0} \sum_I A_I = \lim_{L \to 0} \sum_I \sqrt{A_I^2}.
\]

For small surfaces \(A_I^2\) can be approximated by

\[
A_I^2 = \int_{S_I} d^2\sigma \int_0^\epsilon \frac{ds}{2\epsilon} \int_{-\epsilon}^{\epsilon} \frac{d\tau}{2\epsilon} \int_0^\epsilon \frac{dt}{2\epsilon} \left| \frac{1}{8} n_a(\sigma, s) n_b(\tau, t) T_q^{ab}[\alpha](\sigma, s; \tau, t) \right|.
\]

Here, we have a one parameter family of surfaces \(S_I(s)\) displaced in the background metric normaly to the surface \(S\) by a coordinate distance \(s\). The coordinates on each of these surfaces are labled by \(\sigma\) or \(\tau\), and \(n_a(\sigma, s)\) are the normals at each \(\sigma\) and \(s\). (The addition of the integration in the normal direction, which was not part of the regularization procedures used before in [27, 11, 28, 34] is added here to resolve the ambiguities in the definition of the operator.) \(T_q^{ab}[\alpha](\sigma, s; \tau, t)\) is the two-handed loop variable based on the q-spin net \(\alpha\) passing through the points \(\sigma, s\) and \(\tau, t\). \(\alpha\) can be taken to be a single q-symmetrized edge with its ends at \(\sigma, s\) and \(\tau, t\). (In spin network language it is a 2-line.) With the above construction we can define a quantum area operator by

\[
\hat{A}(S) = \lim_{L \to 0} \sum_I \sqrt{A_I^2}.
\]
where \( \hat{A}_I^2 \) is obtained by replacing \( T_q^{ab} [\alpha_{\sigma,s;\tau,t}] (\sigma, s; \tau, t) \) by the corresponding loop operator. In this loop operator the ambiguities about the choices of the new four valent vertices and the framing of the new internal line are resolved by choosing \( \alpha_{\sigma,s;\tau,t} \) to be the straight untwisted line between its two end points.

We compute the action of this operator on a state \( \langle \Gamma | \) whose edges intersect the surface \( S \), but are never tangent to it. The result of the action is

\[
\langle \Gamma | \hat{A}_I^2 | \rangle = \int_{S_I} d^2 \sigma \int_{-\epsilon}^\epsilon ds \int_{S_I} d^2 \tau \int_{-\epsilon}^\epsilon dt \frac{1}{8} n_{\alpha}(\sigma, s)n_{\beta}(\tau, t)|\langle \Gamma | \rangle
\]

where \( n_I \) is the color of the \( I \)th edge.

Because of the presence of \( \Delta^a[e_I, \alpha(\sigma, s)] \) and \( \Delta^b[e_J, \alpha(\tau, t)] \) in the last expression, it is different from zero only if there is an edge from the \( q \)-spin net crossing the \( I \)th square. We consider only the case in which there is a single edge of color \( n_I \) crossing the \( S_I \). (The more general case in which there are nodes of the spin network in the surface can be treated but for simplicity we do not carry out the computation here.) The only nonzero terms are those in which two new four valent vertices are formed on the edge at \( \epsilon \) above and below the plane, to which the lines \( \alpha \) is attached. Because of the prescription we have given \( \alpha \) runs between the two points without self-linking and, for sufficiently small \( \epsilon \) without linking any of the edges of the graph. The next step will be to take \( \epsilon \to 0 \) so that the line \( \alpha \) shrinks and the two new vertices coincide in the limit. In this case we will have,

\[
\langle \Gamma | \hat{A}_I^2 | \rangle = c_q(n_I)|\langle \Gamma | \rangle
\]

where \( c_q(n_I) \) is the result from the grasping which we can calculate using the recoupling theory. The sum in Eq. (20) reduces to a sum only over the intersections between the surface \( S \) and the \( q \)-spin net \( \Gamma \) so

\[
\langle \Gamma | \hat{A}(S) = l_{Pl}^4 \frac{1}{8} n_I^2 c_q(n_I)|\langle \Gamma | \rangle
\]

To calculate \( c_q(n_I) \) we will note that the graphical action of grasping of \( T_q^{ab} [\alpha] \) reduces to the creation of two new trivalent vertices and thus a “bubble” on the edge \( n_I \). Upon shrinking of the loop \( \alpha \) we get for the factor \( c_q(n_I) \)

\[
c_q(n_I) = \frac{\theta(n_I, 2, n_I)}{(-1)^{n_I} [n_I + 1]} = \frac{[n_I + 2]}{[2[n_I]]} = \frac{[n_I + 2]}{[2[n_I]]}.
\]

The last step in the above equation follows from the use of (16). Thus finally we get

\[
\langle \Gamma | \hat{A}(S) = l_{Pl}^4 \frac{1}{8} n_I^2 [n_I + 2] |\langle \Gamma | \rangle
\]

It is a simple exercise for one to show that this result coincides with the result obtained in [34] where a direct calculation was used.
7 The q-deformed volume operator

Classically, the volume of a 3-dimensional region $R$ is given by

$$V = \int_R d^3 x \sqrt{g}. \quad (26)$$

We regularize this expression, following a procedure used in the ordinary case [24]. We divide the region $R$ into cubes of size $L$ (using some background metric) so the classical expression for the volume becomes

$$V = \lim_{L \to 0} \sum_I L^3 \sqrt{|\det \tilde{E}(x_I)|} \quad (27)$$

This expression can be expressed in terms of the limit of regulated observables as

$$\hat{V} = \lim_{L \to 0} \sum_I \frac{1}{\sqrt{2^3 3!}} \sqrt{\hat{W}_I} \quad (28)$$

where $\hat{W}_I$ is given by the integral

$$\hat{W}_I = \int_{\partial I} d^2 \sigma \int_{\partial I} d^2 \tau \int_{\partial I} d^2 \rho \left| n_a(\sigma)n_b(\tau)n_c(\rho)\hat{T}^{abc}[\alpha](\sigma, \tau, \rho) \right|. \quad (29)$$

We take the framed loop $\alpha$ to be be the triangle formed by the straight (with respect to the background metric) lines between the points $\sigma, \tau, \text{ and } \rho$.

The action of the operator $\hat{W}_I$ obtained in this way on a q-spin net will be different from zero only when there is a vertex in the $I$-th cube. We will consider here only q-spin nets with trivalent vertices. The higher valence vertices are treated in [35]. We first compute the action of $\hat{W}_I$ on a q-spin net graph $\Gamma$ with a trivalent vertex in the $I$-th box, with edges $m, n, l$ The result is given symbolically by

$$\langle \Gamma | \hat{W}_I = \frac{1}{P_{mnnl}} \sum_i c_i \langle (\Gamma^{##}{\alpha}_{\sigma\tau\rho})_i | \quad (30)$$

here $(\Gamma^{##}{\alpha}_{\sigma\tau\rho})_i$ are a finite set of q-spin nets in which each vertex of the triangle $(\alpha_{\sigma\tau\rho})_i$ is attached to one of the three edges of the vertex at the points they intersected the box. There is a sum over possibilities because a choice of ordering must be made to resolve the ambiguity illustrated in Fig. (2) in the definition of the three new four-valent vertices in the action of the operator.

The only principle we have to guide this choice is that the operator should be hermitian as it corresponds to a real quantity. This means that we must choose the definitions of the vertices so that the eigenvalues are real. The simplest choice that realizes this is to average over two spin networks

$$\hat{W}_I \begin{array}{c} \includegraphics[width=0.3\textwidth]{triangle} \end{array} = \frac{1}{2} \left[ \begin{array}{c} \includegraphics[width=0.3\textwidth]{triangle} + \includegraphics[width=0.3\textwidth]{triangle} \end{array} \right]. \quad (31)$$

where we have shown only the graphical part of the expression.
In each of these the triangle $\alpha$ has been deformed smoothly to three edges meeting at a trivalent node, either to the front or the back of the vertex of $\Gamma$, without changing the evaluation of the Kauffman bracket. (This is an illustration of how loops that are to be shrunk down may be deformed subject to preserving the Kauffman bracket relations.)

In the limit that $L \to 0$ we will then have

$$\langle \Gamma | \hat{V} = \frac{l^3 \pi}{4} \sum_{I} \left[ m_1 n_1 l_1 \frac{1}{2} \sum_{i=1}^{2} w_i(m_1, n_1, l_1) \right]^{1/2} \langle \Gamma | \right)$$

(32)

where the sum $I$ is over the vertices of the graph (which again are assumed to be all trivalent) and the sum $i$ in each case is a sum over the two q-spin nets illustrated in Eq. (31). The quantities, $w_i(m_1, n_1, l_1)$ are the result of the evaluation of the parts of the q-spin net around each vertex containing the additional edges coming from the volume operator which shrink to the vertex. They depend only on $m_1$, $n_1$, and $l_1$ which are the colors of the edges joined at the $I$-th vertex.

The two diagrams in Eq. (31) are related to each other by a parity operation. But with $q$ at a root of unity, the action of parity in the evaluation of a Kauffman bracket corresponds to complex conjugation. Hence $w_1(m_1, n_1, l_1) = \overline{w_2(m_1, n_1, l_1)}$, so that the average is real.

Thus, we have defined a diffeomorphism invariant prescription for the action of the volume operator on q-spin nets having only trivalent vertices. In the next section we will compute $w_i(m_1, n_1, l_1)$ using the recoupling theory.

8 Eigenvalues of the volume operator for trivalent vertices.

We can now evaluate the graphs of Eq. (31) in order to extract the eigenvalues of the volume operator [27, 29, 38] corresponding to trivalent vertices of quantum spin nets. The graphical part of the action can be calculated with the use of the recoupling theory of the angular momentum [33].

We will work out the result for the first term on Eq. (31) and will discuss on the differences for the second term. Let us define $w(m, n, l)$ as a sum, $w(m, n, l) = w_1(m, n, l) + w_2(m, n, l)$, representing the contributions from the two diagrams in Eq. (31). Because the routing of the loops through the trivalent vertex is unique the trivalent vertex will then represent an eigenstate of the volume operator:

$$\hat{W}_1 \left( \begin{array}{c} n \\ m \\ l \end{array} \right) = w_1(m, n, l) \left( \begin{array}{c} n \\ m \\ l \end{array} \right)$$

(33)

where $w_1(m, n, l)$ is the corresponding eigenvalue, which is to be determined. The next step for us is to view the vertex cut from the spin network and closed to form a new spin network. By Eq. (33) it also should be true that:

$$\left( \begin{array}{c} n \\ m \\ l \end{array} \right) = w_1(m, n, l) \left( \begin{array}{c} n \\ m \\ l \end{array} \right)$$

(34)
(We no longer draw dotted circles around the networks to indicate that they are being evaluated at a point.) Thus, in diagrammatic form the \( w_1(m, n, l) \) is given by:

\[
 w_1(l, m, n) = \left[ \begin{array}{c}
 \theta_{m, n, l} \\
 m \\
 n \\
l 
\end{array} \right]^{-1}
\]

The graph in the denominator is the \( \theta \)-net. We will evaluate the numerator by using the basic formula of recoupling theory. We apply first the identity of Eq. (13) to one of the \( m \)-edges to get:

\[
 n_m l_2 = \sum_j \left\{ \frac{2}{n} \begin{array}{ccc}
 m & j & m \\
 l & m & n \\
 n & l & m \\
\end{array} \right\}^{n_m l_2 j} \] (36)

Next we use the following identity [33],

\[
 \lambda_{lmn} = \lambda_{lm}^{n_m l_2} \]

where \( \lambda \) is

\[
 \lambda_{nm}^{lm} = (-1)^{(l+m-n)/2}A^{(l+1)+m(m+2)-n(n+2)}/2. \] (38)

We may note that this step is the only place where the difference between the two terms in Eq. (31) shows up. At the corresponding step, the second term in Eq. (31) will pick up \( \lambda^{-1} = \bar{\lambda} \) instead of \( \lambda \). We then apply the basic recoupling formula (36) two more times in each term to get finally:

\[
 w(l, m, n) = \frac{1}{2} \sum_{j=|l-2,l+2|} \left( \lambda_{lm}^{n_m l_2 j} + (\lambda_{lm}^{n_m l_2 j})^{-1} \right) (-1)^j [j + 1] \text{Tet}[2, m, j; n, l, m] \text{Tet}[2, 2, l; j, 2] \text{Tet}[2, n, j; m, l, n] \] (39)

\[
 \theta(m, n, j) \theta(m, n, l) \theta(2, l, j)^2 \]

We may note that the eigenvalue is generally real, as we expected. Further evaluation is tedious due to the factorials of quantum integers, but it is easily done on a computer, using Mathematica. As an example, the eigenvalues of the vertex with the lowest admissible colors \( w(2, 1, 1) \) is, from this formula or worked out directly,

\[
 w(2, 1, 1) = \frac{(1 - A^4)^2(1 + A^8)}{2A^4(1 + A^4)^2}. \] (40)

We see that this vanishes in the ordinary case when \( A = -1 \).

There are general arguments that, in the ordinary case, the volume of trivalent vertices vanishes [38, 12]. Recoupling theory provides a simple argument for this. First, note that the general expression for the volume Eq. (39) is invariant under switching \( m \) and \( n \); the only effect is to switch the first and third tetrahedra. This agrees with the fact that the labeling of the graphs is arbitrary. Then, performing a third Reidemeister move and a twist we have

\[
 \theta_{m, n, l} = \lambda_{lm}^{n_m l_2} \big|_{A=-1} \]

\[
 (41)
\]
However, $\lambda_{1}^{2}|_{A=-1} = -1$, so that, with the invariant property, the result gives

$$\begin{aligned}
\begin{array}{c}
\begin{array}{c}
\lambda_1 \\
\end{array}
\end{array}
&= - \begin{array}{c}
\begin{array}{c}
\lambda_2 \\
\end{array}
\end{array}
= - \begin{array}{c}
\begin{array}{c}
\lambda_3 \\
\end{array}
\end{array}
= - \begin{array}{c}
\begin{array}{c}
\lambda_4 \\
\end{array}
\end{array}.
\end{aligned}
$$

(42)

Thus, an the evaluation is equivalent to the negative of itself; the volume must vanish.

9 Concluding remarks.

We expect that the results and techniques that we have described here may be useful for several directions of further work.

First, the recoupling theory provides an efficient means of computation that may be readily extended to the computation of the action of the volume, Hamiltonian constraint and hamiltonian for general spin networks of arbitrary valence [13], both in the ordinary and the q-deformed case. The fact that the degeneracy of the volume is at least partially lifted in the q-deformed case may make possible the construction of a variety of interesting operators that involve powers of the inverse of $\text{det} \bar{E}^{ai}$. This may make possible the construction of a strong-coupling expansion for quantum gravity [38], the evaluation of Thiemann’s Wick rotation operator [44, 45] and the construction of Hamiltonians corresponding to interesting gauge choices [16]. For this reason, even if one is not interested in the hypothesis that the q-deformation is required to quantize gravity nonperturbatively when the cosmological constant is non-vanishing, it may still be useful to regard the quantum deformation as a kind of diffeomorphism invariant infrared regularization.

Beyond these we may note that the formulation we have defined here suggests the existence of a class of diffeomorphism invariant quantum field theories whose basis of states is spanned by the diffeomorphism invariant classes of embeddings of the spin networks of an arbitrary quantum group. These are a large class of theories that most generally are defined in terms of modular tensor categories [7, 4, 8, 19, 10]. It is natural to extend the definitions of area and volume operators to these theories. By doing so we can interpret each of them as diffeomorphism invariant quantum theories which unify spatial geometry with other degrees of freedom. Dynamics may be postulated for such theories combinatorially, by generalizing the action of the hamiltonian constraint of quantum gravity on quantum spin networks [38] to these cases. It will then be sufficient to discover the connection to general relativity only in the limit in which the volume of space becomes large. The possibility of recovering general relativity from such a limit of a discrete theory is suggested also by the recent result of Jacobson; which requires only that there be a relationship between area and information content [19]. However, this relationship may be preserved in these extensions, given the results of [17, 18]. Finally, given that the language of such a theory is closely connected to that of the minimal conformal field theories [17], it is tempting to speculate that such a formulation might provide a link between non-perturbative formulations of quantum gravity and string theory.

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