CONTINUITY OF THE WEIL-PETERSSON POTENTIAL

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ABSTRACT. Let $\mathcal{M}_{KSB}$ (resp. $\overline{\mathcal{M}}_{KSB}$) be the the moduli space of $n$-dimensional Kähler-Einstein manifolds (resp. varieties) $X$ with $K_X$ ample. We prove that the Weil-Petersson metric on $\mathcal{M}_{KSB}$ extends uniquely to the projective variety $\overline{\mathcal{M}}_{KSB}$, as a closed positive current with continuous local potentials. This generalizes a theorem of Wolpert [26] which treats the case $n = 1$, and also confirms a conjecture of Berman-Guenancia [4]. In addition, we derive uniform estimates for the volumes of sublevel sets of Kähler-Einstein potentials.

1. Introduction

The Deligne-Mumford moduli space $\mathcal{M}_g$ is a quasi-projective variety which parametrizes (isomorphism classes of) smooth curves of genus $g$. If $X \in \mathcal{M}_g$, then the tangent space of $\mathcal{M}_g$ at $X$ is the space of quadratic differentials $H^0(X, 2K_X)$. The length of a tangent vector $\eta$ with respect to $\omega_{WP}$, the Weil-Petersson metric, is its $L^2$ norm with respect to the hyperbolic metric on $X$, and measures the rate at which the complex structure changes in the direction $\eta$. The metric has negative holomorphic sectional curvature and finite diameter. Wolpert [26, 27] proved that $\omega_{WP}$ extends to a Kähler current on $\overline{\mathcal{M}}_g$, a projective variety which parametrizes stable curves of genus $g$, and he showed that locally on $\overline{\mathcal{M}}_g$ we have $\omega_{WP} = \sqrt{-1} \partial \bar{\partial} \varphi$ for some continuous plurisubharmonic function $\varphi$.

In dimension $n > 1$, the generalization of $\mathcal{M}_g$ is $\mathcal{M}_{KSB}$, which parametrizes smooth projective varieties $X$ with $K_X$ ample (i.e. Kähler-Einstein manifolds with $K_X$ ample, by the theorem of Aubin, Yau). The Weil-Petersson metric $\omega_{WP}$ on the tangent space of $\mathcal{M}_{KSB}$ at a point $X \in \mathcal{M}_{KSB}$ is the $L^2$-metric on the space of harmonic $(0, 1)$-forms with coefficients in the tangent bundle of the manifold $X$. Here the theory is not as complete as the Riemann surface case, but much is known. An early key result was the computation of the holomorphic bisectional curvature of $\omega_{WP}$ which was achieved by Siu [19]. A second important advance is the result of Schumacher [18] which can be formulated as follows. Suppose $\pi : \mathcal{X} \to B$ be a smooth family of Kähler-Einstein manifolds $X$ of dimension $n$ and $K_X$ ample. The relative canonical bundle $K_{\mathcal{X}/B}$ has a canonical metric $h$, given by the Kähler-Einstein volume form on each fiber. The main result of [18] shows that $\omega$, the curvature of $h$, is a positive $(1, 1)$ form on $\mathcal{X}$. Moreover, $\pi_* (\omega^{n+1}) = \omega_{WP}$. Let $\overline{\pi} : \overline{\mathcal{X}} \to \overline{B}$ be a flat family of Kähler-Einstein varieties $X$ of dimension $n$ and $K_X$ ample (i.e. “stable” varieties $X$ - cf. [4] for background and precise definitions) extending the smooth family $\pi : \mathcal{X} \to B$. Then $\omega$ can be extended to a closed positive current on $\overline{\mathcal{X}}$ with analytic singularities. In [21], we show that this positive current has vanishing Lelong number and hence $K_{\overline{\mathcal{X}}/\overline{B}}$, the relative canonical bundle, is nef on $\overline{\mathcal{X}}$. As a consequence, the Weil-Petersson volume of $\overline{\mathcal{M}}_{KSB}$ is a finite rational number.

The goal of this paper is generalize Wolpert’s result [26, 27] to higher dimensions, i.e. to establish the continuity of the local Kähler potential of $\omega_{WP}$ on the compactified moduli...
space $\overline{\mathcal{M}}_{\text{KSB}}$. In the process, we establish uniform estimates for the volumes of sublevel sets for the Kähler-Einstein potential, which may be of independent interest.

In order to state our results precisely, we first review some necessary background. By the classical results Aubin and Yau, [3, 28], a Kähler manifold with $K_X$ ample has a unique negatively curved Kähler-Einstein metric. In [21], we prove that the Gromov-Hausdorff completion of the moduli space of $n$-dimensional negatively curved Kähler-Einstein manifolds is canonically identified with the KSB compatification of the moduli space of smooth canonically polarized manifolds. In addition, we show the local potentials of the Weil-Petersson metric are bounded: Let $L_{CM} \to \overline{\mathcal{M}}_{\text{KSB}}$ be the CM line bundle (as described in [16, 21]) and let $h$ be a fixed smooth hermitian metric on $L_{CM}$. Since $L_{CM}$ is ample [16], we can choose $h = h_{FS}$ to be the Fubini-Study metric restricted to $L_{CM}$ and $\omega_{FS} = \text{Ric}(h_{FS})$. Then $\omega_{WP}$, the Weil-Petersson metric on $\overline{\mathcal{M}}_{\text{KSB}}$, is the curvature of a hermitian metric

\begin{equation}
 h_{WP} = h_{FS} e^{-\varphi_{WP}}
\end{equation}

and

\begin{equation}
 \omega_{WP} = \omega_{FS} + \sqrt{-1} \partial \bar{\partial} \varphi_{WP} \in c_1(L_{CM})
\end{equation}

for some $\varphi_{WP} \in \text{PSH}(\overline{\mathcal{M}}_{\text{KSB}}, \omega_{FS}) \cap L^\infty(\overline{\mathcal{M}}_{\text{KSB}})$, as shown in [21]. Our main result is the following.

**Theorem 1.1.** The hermitian metric $h_{WP}$ is continuous and

\[ \varphi_{WP} \in C^0(\overline{\mathcal{M}}_{\text{KSB}}). \]

In particular, the Weil-Petersson metric extends to a closed positive current $\omega_{WP}$ on $\overline{\mathcal{M}}_{\text{KSB}}$, i.e., for any point $p \in \overline{\mathcal{M}}_{\text{KSB}}$, there exist an open neighborhood $U$ of $p$ and $\Phi_{WP} \in \text{PSH}(U) \cap C^0(U)$ such that in $U$,

\[ \omega_{WP} = \sqrt{-1} \partial \bar{\partial} \Phi_{WP}. \]

Theorem 1.1 confirms a conjecture of Berman and Guenancia [4]. The estimates we need use the approach of [21], but sharper bounds are required. Our argument should generalize to the case of stable families of klt pairs whose general fibres are not necessarily smooth. We conjecture that the Weil-Petersson current can still be still defined for the moduli space of canonically polarized projective varieties with klt singularities and it can be extended to a Kähler current with continuous local potentials on the compactified KSB moduli space. Let $(\mathcal{M}_{\text{KSB}})^{\circ}$ be the smooth interior part of $\overline{\mathcal{M}}_{\text{KSB}}$. Then as we conjectured in [21], $((\mathcal{M}_{\text{KSB}})^{\circ}, \omega_{WP})$ should have finite diameter and its metric completion should be homeomorphic to $\overline{\mathcal{M}}_{\text{KSB}}$. The conjecture always holds for a stable family of Kähler-Einstein manifolds over a one-dimensional disk where the central fibre has only singularities of complete simple normal crossings [22, 17].

Next we introduce the notions of stable varieties and stable families. The following equivalent characterizations of “stable variety” are established in [4]. A projective variety $X$ is stable if $K_X$ is ample and it satisfies any one of the following additional conditions.

1. $X^{\text{reg}}$ has a Kähler-Einstein metric whose volume is $c_1(K_X)^n$.
2. $X$ is smooth or $X$ has a worst semi-log canonical singularities.
3. $X$ is K-stable (in the sense of [23, 6]).
Now we can define stable families as follows.

**Definition 1.1.** A flat projective morphism $\pi : X \to B$ between normal varieties a stable family if the following holds.

(1) The relative canonical bundle $K_{X/B}$ is $\mathbb{Q}$-Cartier and is $\pi$-ample.

(2) The fibres of $\pi$ are stable and the generic fiber is smooth.

We let $B^\circ$ be the set of smooth points of $B$ over which $\pi$ is smooth and $X^\circ = \pi^{-1}(B^\circ)$. For each $t \in B^\circ$, there exists a unique Kähler-Einstein metric $\omega_t \in c_1(X_t)$ on the fibre $X_t = \pi^{-1}(t)$ and one can define the hermitian metric on the relative canonical bundle $K_{X^\circ/B^\circ}$ by $h_t = (\omega_t^n)^{-1}$. It is proved by Schumacher [18] and Tsuji [24], using different methods, that $\text{Ric}(h) = -\sqrt{-1} \partial \bar{\partial} \log h$ is nonnegative on $X^\circ$ and it is strictly positive for a nowhere infinitesimally trivial family. It is further shown in [18] that $h$ can be uniquely extended to a non-negatively curved singular hermitian metric on $K_{X/B}$ with analytic singularities. In [21], we prove that $h$ has vanishing Lelong number everywhere on $X$ and it tends $-\infty$ exactly at the non-klt locus of the special fibres of $\pi : X \to B$. In particular, our result gives an analytic proof of Fujino’s theorem [8] that the relative canonical bundle $K_{X/B}$ is nef.

If we let $E_{\text{slc}}$ be the union of semi-log and log canonical locus of special fibres of $X$ (or equivalently, the non-klt locus), then by [4, 20],

$$E_{\text{slc}} = \{ \phi = -\infty \}$$

where $\phi$ is the Kähler-Einstein potential. Our next theorem establishes the continuity of $h$ on $X \setminus E_{\text{slc}}$.

**Theorem 1.2.** Let $\pi : X \to B$ be a stable family of $n$-dimensional canonical models over a normal variety $B$. Let $\omega_t$ be the unique Kähler-Einstein metric on $X_t$ for $t \in B^\circ$ and $h$ be the hermitian metric on the relative canonical sheaf $K_{X^\circ/B^\circ}$ defined by

$$h_t = (\omega_t^n)^{-1}.$$ 

The curvature $\theta = \sqrt{-1} \partial \bar{\partial} \log h$ of $(K_{X^\circ/B^\circ}, h)$ extends uniquely to a closed nonnegative $(1,1)$-current on $X$ satisfying the following.

(1) $\theta$ has vanishing Lelong number everywhere in $X$.

(2) $\theta|_{X_t} = \omega_t$ for $t \in B^\circ$.

(3) $h$ is continuous on $X \setminus E_{\text{slc}}$.

Moreover, if all the fibres of $\pi : X \to B$ have at worst log terminal singularities, then $h$ is continuous on $X$.

The variety $\overline{M}_{\text{KSB}}$ is a coarse moduli space, and as such does not support a universal family, but (cf. [21]) there exists a projective variety $B$, a finite cover $F : B \to \overline{M}_{\text{KSB}}$ and a stable family $X_B \to B$ such that if $t \in B$ then $[X_t] = F(t)$. Let $K(n,V)$ be the set of all negatively curved Kähler-Einstein manifolds of dimension $n$ and volume at most $V$. Matsusaka’s big theorem implies that every $X \in K(n,V)$ lies in one of a finite number of such families. In other words, there is a single projective family $X \subseteq B \times \mathbb{P}^N$,
pluricanonically imbedded, which contains every element of $\mathcal{K}(n, V)$. Let $\chi = \frac{1}{m} \omega_{FS}$ where $\omega_{FS}$ is the Fubini-Study metric on $\mathbb{P}^n$. For $X \in \mathcal{K}(n, V)$ let

$$\omega_X = \chi_X + \sqrt{-1} \partial \bar{\partial} \phi_X$$

denote the unique Kähler-Einstein metric on $X$ ([4, 20, 21]). It is necessary to estimate decay rates of $\phi_X$ in order to understand the asymptotic behavior of the Weil-Petersson metric near the boundary of the KSB moduli space. More precisely, we shall need the following volume estimate for the sublevel sets of the Kähler-Einstein potentials.

**Theorem 1.3.** Fix $n, V > 0$ and $\chi \to \mathcal{B}$ as above. Then there exists $C = C(n, V, \chi) > 0$ such that for all negatively curved Kähler-Einstein manifolds $X$ of dimension $n$ and volume at most $V$ the following holds. For all $0 \leq m \leq n$ and all $K \geq 1$ have

$$\int_{\{\phi_X < -K\}} (\chi_X + \sqrt{-1} \partial \bar{\partial} \phi_X)^m \wedge \chi_X^{n-m} \leq C K^n e^{-K/(4n+2)}.$$

Moreover, the estimate (1.3) holds for all smoothable negatively curved Kähler-Einstein varieties of dimension $n$ and volume at most $V$.

Theorem 1.3 implies a uniform exponential decay for the volumes of sub-level sets of the Kähler-Einstein potentials. We believe that these estimates will be useful in understanding the incompleteness of the Weil-Petersson metric. One would also like to derive a geometric version of Theorem 1.3 by replacing the Kähler-Einstein potential by the distance function.

We give a brief outline of the paper. In section 2, we review semi-stable reduction and in section 3, we establish the $C^0$-estimate for the Kähler-Einstein potentials. In section 4, we prove Theorem 1.3 for the exponential decay of the volume measure of the level sets of the Kähler-Einstein potentials. In section 5, we give an alternative proof for an integral estimate derived in [21]. In section 6 and 7, we prove the continuity of Weil-Petersson potentials for stable families and complete the proof of Theorem 1.1. Finally, we prove Theorem 1.2 in section 8.

### 2. Stable families and semistable reduction

In this section, we will review the semi-stable reduction theorem and follow the discussion in [21] (section 4) with some simplification. Let $\pi : \mathcal{X} \to B$ be a stable family with $\dim B = d \geq 1$. The theorem of Adiprasito-Liu-Temkin [2] says that there is a smooth variety $S'$, a finite base change $B' \to B$, and a birational map $\Psi : \mathcal{X}' \to \mathcal{X} \times_B B'$ such that the projection $\pi' : \mathcal{X}' \to \mathcal{X} \times_B B'$ is a semistable reduction. By abuse of notation, we shall write $B$ for $B'$ and $\mathcal{X}$ for $\mathcal{X}' \to \mathcal{X} \times_B B'$. We obtain a diagram

$$\xymatrix{ \mathcal{X} \ar[r]^{\Psi} \ar[d]_{\pi} & \mathcal{X}' \ar[r]^{\Psi} \ar[dl]_{\hat{\pi}} & \mathcal{X} \ar[dl]_{\pi'} \ar[d]^\pi B }$$

(2.4)
where $\tilde{X}$ and $\Phi$ are to be defined later.

Semistable reduction means that there are local coordinates $t = (t_1, ..., t_d)$ on $B$ and $(x_1, ..., x_{n+d})$ on $X'$ such that the $t_i \circ \pi'$ are multiplicity free monomials in $x$. We introduce some additional notations for more precise statements. For simplicity, we assume that $B$ is a Euclidean ball in $\mathbb{C}^d$ and so we can choose $t = (t_1, ..., t_d)$ as global coordinates on $B$.

Let
$$H_i = \{ t_i \circ \pi = 0 \} \subseteq X.$$

To say that $\pi : X' \to B$ is a semistable reduction means that for $1 \leq i \leq d$, the Cartier divisor $\Psi^* H_i$, the pullback of the Cartier divisor $H_i$, is a divisor of simple normal crossings whose components $F_1, F_2, ...$ all have multiplicity one. Note that these components may not be smooth since they can intersect themselves.

We write
$$\Psi^* H_i = \{ t_i \circ \pi' = 0 \} = \sum_{F \in A_i} F$$
where $A_i$ is the set of irreducible components of $\Psi^* H_i$. Observe that
$$\pi'(F) = \{ t_i = 0 \} \text{ for all } F \in A_i,$$
so in particular, if $i \neq j$,
$$A_i \cap A_j = \phi.$$

There is a disjoint partition $A_i = D_i \sqcup V_i$ such that the elements of $D_i$ are non-exceptional and those of $V_i$ are exceptional satisfying
\begin{equation}
(2.5) \quad \Psi^* H_i = H_i' + V_i' = \sum_{D \in D_i} D + \sum_{E \in V_i} E.
\end{equation}

Here $H_i'$ is the strict transform of $H_i$.

Let $\operatorname{Exc}(\Psi)$ be the exceptional set of $\Psi$ and $\mathcal{F} \subseteq \operatorname{Exc}(\Psi)$ the union of divisorial components of $\operatorname{Exc}(\Psi)$. Then $\mathcal{F}$ can be written as a disjoint union as follows
\begin{equation}
(2.6) \quad \mathcal{F} = \mathcal{V} \sqcup \mathcal{H},
\end{equation}
where $\mathcal{H}$ is the set of horizontal exceptional divisors, that is, $E \in \mathcal{H}$ if and only if $\pi'(E) = B$, and $\mathcal{V}$ is the set of vertical exceptional divisors.

We can write $X' = \cup \alpha U^\alpha$, a finite covering by coordinate neighborhoods and choose local coordinates $x \in U$ with the following properties. Fix $\alpha$ and let $U = U^\alpha$. Let $I = \{1, 2, ..., n+d\}$. Then there exist a disjoint decomposition $I = A_1 \sqcup A_2 \sqcup ... \sqcup A_d \sqcup R$ such that
$$A_i = D_i \sqcup V_i$$
and
$$t_i \circ \pi' = \prod_{j \in A_i} x_j = \prod_{j \in D_i} x_j \cdot \prod_{j \in V_i} x_j.$$  

Here $D_i$ and $V_i$ are the set of indices associated to $D_i$ and $V_i$ given by the following: $j \mapsto \{ x_j = 0 \}$ defines maps $\nu : A_i \to A_i, D_i \to D_i$ and $V_i \to V_i$ which need not be one to one since if $j, k \in D_i$ with $j \neq k$, then $\nu(j) = \nu(k) = D$ if the divisor $D$ intersects itself.

The following lemma gives the construction of for $\tilde{\pi} : \tilde{X} \to B$ in diagram (2.4).
Lemma 2.1. There exist \( \Phi : \tilde{X} \to \mathcal{X}' \), a series of locally toric blow-ups, and local coordinates on open charts \( \{ \tilde{U}^\alpha \}_\alpha \) of \( \tilde{X} \) such that

\[
(\Psi \circ \Phi)^* H_i = \tilde{H}_i + \tilde{V}_i = \sum_{D \in \tilde{D}_i} D + \sum_{E \in \tilde{V}_i} a_E E,
\]

where \( \tilde{H}_i, \tilde{V}_i, \tilde{D}_i \) and \( \tilde{V}_i \) are defined for \( \tilde{\pi} \) similarly as in (2.5), and on each \( \tilde{U} = \tilde{U}^\alpha \),

\[
t_i \circ \tilde{\pi} = \prod_{j \in \tilde{D}_i} x_j \cdot \prod_{j \in \tilde{V}_i} x_j^{a_{ij}}, \quad D_i \sqcup \tilde{V}_i = \tilde{D}_i \sqcup \tilde{V}_i, \text{ and } \# \tilde{D}_i \leq 1,
\]

where \( a_{ij} \in \mathbb{Z}^+ \cup \{0\} \), \( \tilde{D}_i \) and \( \tilde{V}_i \) are the set of indices associated to \( \tilde{D}_i \) and \( \tilde{V}_i \). Moreover, the map \( \tilde{\pi} : \tilde{X} \to B \) is flat.

Proof. We use induction: Fix \( i \) suppose there exists \( \alpha \) such that in \( U = U^\alpha \) we have \( \# D_i^\alpha > 1 \). Choose \( \alpha \) such that \( \# D_i^\alpha \) is maximal. Then we blow up the smooth variety \( \cap_{j \in D_i^\alpha} \nu(j) \) (smoothness is a consequence of the fact that \( \# D_i^\alpha \) is maximal). In other words, if \( D_i = D_i^\alpha = \{ j_1, \ldots, j_m \} \) then we make the changes of variables of the form

\[
x_{j_1} \mapsto x_{j_1} \text{ and } x_{j_p} \mapsto x_{j_1} x_{j_p} \text{ if } p > 1
\]

In these new coordinates, \( j_1 \notin \tilde{D}_i \). Instead, \( j_1 \in \tilde{V}_i \). Thus we have reduced \( \# D_i \) by one. Continuing in this fashion we prove the first part of the lemma. To see that \( \tilde{\pi} : \tilde{X} \to B' \) is flat we observe that both \( \tilde{X} \) and \( B' \) are smooth and \( \tilde{\pi} \) is equi-dimensional. The ‘miracle flatness theorem’ (cf. 26.2.11. of [25]) implies \( \tilde{\pi} \) is flat.

We remark that in Lemma 2.1, \( \nu(j) \subseteq \tilde{X} \) is in the strict transform of \( H_i \) if and only if \( j \in \tilde{D}_i \). In other words, in the local coordinate chart \( \tilde{U} \subseteq \tilde{X} \) at most one component of the total transform of \( H_i \) is non-exceptional. Lemma 2.1 is an improvement of the procedure in [21] and helps to simplify future calculations.

Finally, we recall the following adjunction lemma proved in [21].

Lemma 2.2. There exist \( a_k \leq 1 \) and \( b_i \leq 0 \) such that

\[
K_{\tilde{X}} + \sum_{i=1}^d \tilde{H}_i = (\Psi \circ \Phi)^* \left( K_{\mathcal{X}} + \sum_{i=1}^d H_i \right) - \sum_{E \in \tilde{V} \atop E_k \in \tilde{H}} a_k E_k - \sum_{F \in \tilde{H}} b_i F_i.
\]

3. The \( C^0 \)-estimate for the Kähler-Einstein potential

In this section, we will prove a sharp \( C^0 \)-estimate for the Kähler-Einstein potential. We will follow the notations in section 2 and replace \( B \) (or \( B' \)) by \( B \), a Euclidean ball in \( \mathbb{C}^d \). Let \( \eta_0, \ldots, \eta_M \) be a basis for the plurcanonical system \( |mK_{\mathcal{X}/B}| \) for some sufficiently large \( m \in \mathbb{Z} \) so that \( mK_{\mathcal{X}/B} \) is globally generated. Let

\[
\Omega = \left( \sum_{j=0}^M |\eta_j|^2 \right)^{1/m} \quad \text{and} \quad \chi = \sqrt{-1} \partial \bar{\partial} \log \Omega.
\]

We can immediately translate the algebraic adjunction formula (2.8) into the following analytic local formula with coordinates on \( \tilde{X} \).
Lemma 3.1. There exists $C > 0$ such that on each open chart $\tilde{U} = \tilde{U}^\alpha$,

$$
(\sqrt{-1})^{n+d} (\Psi \circ \Phi)^* \Omega_{t} \wedge_{1} (\sqrt{-1} dt \wedge d\bar{t}_{i}) \leq C \prod_{x_{j} \in \tilde{D}} |x_{j}|^{2n_{j}},
$$

where $\{x_{j} = 0\}$ corresponds to the divisor in $\tilde{D} = \bigcup \tilde{D}_{t}$ or $\tilde{V} = \bigcup \tilde{V}_{i}$.

For $t \in B$, let $\varphi_{t}$ be the solution to the following complex Monge-Ampere equation

$$
(\chi_{t} + \sqrt{-1} \partial \bar{\partial} \varphi_{t})^{n} = e^{\varphi_{t}} \Omega_{t},
$$

where $\chi_{t}$ and $\Omega_{t}$ are the restrictions of $\chi$ and $\Omega$ to $\mathcal{X}_{t} = \pi^{-1}(t)$. By [20], equation (3.11) admits a unique solution $\varphi_{t}$ with vanishing Lelong number along the semi-log and log locus of $\mathcal{X}_{t}$ for each $t \in B$. For $t \in B$, $\omega_{t} = \chi_{t} + \sqrt{-1} \partial \bar{\partial} \varphi_{t}$ is the Kähler-Einstein metric on $\mathcal{X}_{t}$ satisfying

$$
\text{Ric}(\omega_{t}) = -\omega_{t},
$$

including those $t$ for which $\mathcal{X}_{t}$ is a singular semi-log canonical model. We define $\mathcal{X}^{\text{reg}}$ be the union of smooth points of $\mathcal{X}_{t}$ for all $t \in B$ and $\varphi \in L^{\infty}_{\text{loc}}(\mathcal{X}^{\text{reg}})$ such that

$$
\varphi|_{\mathcal{X}_{t}} = \varphi_{t}.
$$

It is proved in [21] that $\varphi$ extends globally to $\mathcal{X}$ in $\text{PSH}(\mathcal{X}, \chi)$ with vanishing Lelong number. In particular, $\varphi$ is bounded above, and locally bounded below away from the non-klt locus of $\mathcal{X}_{t}$ for all $t \in B$. In fact, $\chi + \sqrt{-1} \partial \bar{\partial} \varphi$ is the curvature of the relative canonical line bundle $K_{\mathcal{X}|B}$.

Let $\mathcal{E}$ be any divisor of $\tilde{X}$ containing the exceptional locus of $\Psi \circ \Phi$. Theorem 4.7 of [2] says that we may choose $\Psi$ in (2.4) to be an isomorphism over any open subset of $B$ for which $\pi$ is smooth (in particular in (2.6) we have $\mathcal{H} = \emptyset$). Thus, we may choose $\mathcal{E} \subseteq \tilde{X}$ in such a way that

$$
B \setminus B^{\circ} \subseteq \bar{\pi}(\mathcal{E}) \quad \text{and} \quad \bar{\pi}(\mathcal{E}) \subseteq B \quad \text{has codimension one}
$$

We let $\sigma_{\mathcal{E}}$ be a defining section of $\mathcal{E}$ and $h_{\mathcal{E}}$ be a smooth hermitian metric on the line bundle on $\tilde{X}$ associated to $\mathcal{E}$. Then the function $\log |\sigma_{\mathcal{E}}|_{h_{\mathcal{E}}}^{2}$ is defined on $\tilde{X}$ and by abusing the notations, we identify it with its push-forward onto $\mathcal{X}$. The following is the main result of this section and it is a sharp improvement of the $C^{0}$-estimate in [20, 21].

**Proposition 3.1.** For any $\epsilon > 0$, there exists $C_{\epsilon} > 0$ such that on $\mathcal{X}$, we have

$$
\varphi \geq -(2n + \epsilon) \log \left( - \log |\sigma_{\mathcal{E}}|_{h_{\mathcal{E}}}^{2} \right) - C_{\epsilon}.
$$

Remark: If we choose $\mathcal{E}_{1}, ..., \mathcal{E}_{p}$ as in (3.12) such that $B \setminus B^{\circ} = \bigcap_{i=1}^{p} \bar{\pi}(\mathcal{E}_{i})$ we obtain

$$
\varphi \geq -(2n + \epsilon) \log \left( \sum_{i=1}^{p} |\sigma_{\mathcal{E}_{i}}|_{h_{\mathcal{E}}}^{2} \right) - C_{\epsilon}.
$$

Note that the right side of (3.14) is finite on each smooth fiber.

In the case where the base has dimension 1 and the central fibre is irreducible, a sharper estimate is achieved in [15] with a coefficient of $-(n + 1 + \epsilon)$. However, any constant will suffice for our application to the estimates of Weil-Petersson potentials.
Before proving Proposition 3.13, let us have a quick review on a standard procedure for constructing plurisubharmonic functions from convex functions.

**Definition 3.1.** A continuous convex function $H : \mathbb{R}^n \to \mathbb{R}$ is said to be an admissible function if

$$H' > 0, \ H'' > 0$$
on $\mathbb{R}^n$.

For example, $H(x) = -\log(-x)$ is an admissible function and $H(x) = -(x)^{1-\delta}$ is an admissible function whenever $\delta \in (0, 1)$.

**Lemma 3.2.** Suppose $\phi$ is a negative plurisubharmonic function on a domain in $\mathbb{C}^n$ and $H$ is an admissible function. Then $H \circ \phi$ is also plurisubharmonic.

**Proof.** Straightforward calculations show that

$$\sqrt{-1}\partial \bar{\partial}(H \circ \phi) = H' \sqrt{-1}\partial \bar{\partial} \phi + \sqrt{-1}H'' \partial \phi \wedge \bar{\partial} \phi \geq 0.$$ 

If $v = \log |z|^2$ on the unit ball in $\mathbb{C}$ and $F_\delta(x) = -(x)^{1-\delta}$ for some $0 < \delta < 1$, then

$$F_\delta'(x) = (1 - \delta)(-x)^{-\delta}, \ F_\delta''(x) = \delta(1 - \delta)(-x)^{-1-\delta}.$$ 

On $\mathbb{C}^*$, we have

$$\sqrt{-1}\partial \bar{\partial}( - (\log |z|^2)^{1-\delta} ) = \delta(1 - \delta) \cdot \frac{\sqrt{-1}dz \wedge d\bar{z}}{(-\log (|z|^2)^{1+\delta} \cdot |z|^2)}.$$ 

Let $\varphi$ be a smooth function and $v = \log(|z|^2 e^{-\varphi}) = \log |z|^2 - \varphi$ on the unit ball, then for $|z|$ sufficiently small, we have

$$\sqrt{-1}\partial \bar{\partial}( - v^{1-\delta} ) \geq \frac{1}{2} \delta(1 - \delta) \frac{\sqrt{-1}dz \wedge d\bar{z}}{(-\log (|z|^2 e^{-\varphi}))^{1+\delta} \cdot (|z|^2 e^{-\varphi})}.$$ 

If $\chi + \sqrt{-1}\partial \bar{\partial}u > 0$ for some Kähler form $\chi$ and if $H'(u) \leq 1$ for some admissible function $H$, then

$$(3.15) \quad \chi + \sqrt{-1}\partial \bar{\partial}H(u) \geq \chi + H'(u)\sqrt{-1}\partial \bar{\partial}u \geq H'(u)(\chi + \sqrt{-1}\partial \bar{\partial}u).$$

We now begin to prove Proposition 3.1.

**Proof of Proposition 3.1.** We will follow the same approach in [21] with some simplification. Let $\text{Exc}(\Psi \circ \Phi)$ be the exceptional locus in the diagram (2.4) and Lemma 2.1 for $\Psi \circ \Phi$. By Kodaira’s lemma, there exists an effective $\mathbb{Q}$-Cartier divisor $E$ whose support contains $\text{Exc}(\Psi \circ \Phi)$ and the class $(\Psi \circ \Phi)^* [\chi] - [E]$ is ample on $\tilde{X}$. Furthermore, we let $\sigma_E$ be a defining section of $E$ and $h_E$ be a smooth hermitian metric on $\tilde{X}$ such that

$$|\sigma_E|^2_{h_E} \leq 1/2$$

and

$$(\Psi \circ \Phi)^* \chi + \sqrt{-1}\partial \bar{\partial} \log |\sigma_E|^2_{h_E} > 0$$

is a Kähler form on $\tilde{X}$.

We now define for $\varepsilon > 0$ sufficiently small,

$$u_\varepsilon = \varepsilon \log |\sigma_E|^2_{h_E} - \varepsilon^2 ( - \log |\sigma_E|^2_{h_E} )^{1-\delta} - 3n \leq -3n$$

and

$$(3.16) \quad \chi + \sqrt{-1}\partial \bar{\partial}u_\varepsilon \geq \chi + H'(u_\varepsilon)\sqrt{-1}\partial \bar{\partial}u_\varepsilon \geq H'(u_\varepsilon)(\chi + \sqrt{-1}\partial \bar{\partial}u_\varepsilon).$$

We now proceed to prove Proposition 3.1.
\[ f_\epsilon = H(u_\epsilon) = H \left( \epsilon \log |\sigma_E|_{\mathbb{H}_E}^2 - \epsilon^2 \left( - \log |\sigma_E|_{\mathbb{H}_E}^2 \right)^{1-\delta} - 3n \right). \]

It is straightforward to verify that there exist \( \epsilon_0 \) and \( \delta_0 > 0 \) such that for any \( 0 < \epsilon < \epsilon_0 \) and \( 0 < \delta < \delta_0 \), we have

\[ u_\epsilon \in \text{PSH}(\mathcal{X}, \chi) \]

since \( - \log |\sigma_E|_{\mathbb{H}_E}^2 \) is bounded below by \( \log 2 \).

The admissible function \( H \) will also be chosen later and it has to satisfy the condition that \( H'(u_\epsilon) \leq 1 \).

We let

\[ \varphi_\epsilon = \varphi - f_\epsilon, \quad \varphi_{t,\epsilon} = \varphi_{t} \mid_{x_i} = \varphi_t - f_{t,\epsilon}, \quad f_{t,\epsilon} = f_{\epsilon} \mid_{x_i}. \]

Then the complex Monge-Ampere equation (3.11) can be rewritten as

\[ (\chi_t + \partial \bar{\partial} f_{t,\epsilon} + \sqrt{-1} \partial \bar{\partial} \varphi_{t,\epsilon})^n = e^{\varphi_{t,\epsilon}} e^{f_{t,\epsilon}} \Omega_t \]

Note that since \( \varphi_\epsilon \) tends to \( \infty \) near the exceptional locus of \( \Psi \Phi \) or \( E \), we may conclude that \( \varphi_\epsilon \) achieves its minimum. Suppose \( \varphi_{t,\epsilon} \) achieves its minimum at \( p \), we have at \( p \),

\[ e^{\varphi_{t,\epsilon}} \geq \frac{\left( \sqrt{-1} \right)^d |t|^{-2} e^{-f_{t,\epsilon}} \left( \chi_t + \sqrt{-1} \partial \bar{\partial} f_{t,\epsilon} \right)^n \wedge_{i=1}^d dt_i \wedge d\bar{t}_i}{\left( \sqrt{-1} \right)^d |t|^{-2} \Omega_t \wedge_{i=1}^d dt_i \wedge d\bar{t}_i} \]

Applying (3.15) and (3.16), there exists \( c_1, c_2 > 0 \) such that

\[ \frac{1}{H'(u_\epsilon)} (\chi + \sqrt{-1} \partial \bar{\partial} f_\epsilon) \geq \chi + \sqrt{-1} \partial \bar{\partial} u_\epsilon \]

\[ \geq c_1 \sqrt{-1} \sum_{i=1}^d \left( \sum_{j \in \tilde{D}_i} dx_j \wedge d\bar{x}_j + \sum_{j \in \tilde{V}_i} \frac{dx_j \wedge d\bar{x}_j}{(- \log |x_j|^2)^{1+\delta} |x_j|^2} \right) + c_1 \sqrt{-1} \sum_{j=1}^{n+d} dx_j \wedge d\bar{x}_j \]

\[ \geq c_2 \sqrt{-1} \sum_{i=1}^d \left[ \sum_{j \in \tilde{V}'_i} \frac{dx_j \wedge d\bar{x}_j}{(- \log |x_j|^2)^{1+\delta} |x_j|^2} \right] + c_2 \sqrt{-1} \sum_{j=1}^{n+d} dx_j \wedge d\bar{x}_j \]

where \( \tilde{V}'_i \subseteq \tilde{V}_i \) with equality if and only if \( \tilde{D}_i \neq \phi \), otherwise, \( \tilde{V}'_i \) equals \( \tilde{V}_i \) with one element removed. On the other hand,

\[ \frac{dt_i}{t_i} = \sum_{j \in \tilde{D}_i} \frac{dx_j}{x_j} + \sum_{j \in \tilde{V}_i} \frac{dx_j}{x_j} \]

and

\[ (\chi + \sqrt{-1} \partial \bar{\partial} f_\epsilon)^n \geq (H'(u_\epsilon))^n (\chi + \sqrt{-1} \partial \bar{\partial} u_\epsilon)^n. \]

Therefore,

\[ (\sqrt{-1})^d e^{-f_\epsilon} (\chi + \sqrt{-1} \partial \bar{\partial} f_\epsilon)^n \wedge_{i=1}^d dt_i \wedge d\bar{t}_i \]

\[ \geq e^{-H(u_\epsilon)} \frac{(H'(u_\epsilon))^n}{\prod_{j \in \tilde{V}_i} (- \log |x_j|^2)^{1+\delta}} \frac{(\sqrt{-1})^{n+d} \wedge_{i=1}^{n+d} dx_i \wedge d\bar{x}_i}{\prod_{j \in \tilde{D}_i} |x_j|^2 \prod_{j \in \tilde{V}_i} |x_j|^2}, \]
where $\tilde{V}' = \cup_{i=1}^{d} \tilde{V}'_i$, $\tilde{D}' = \cup_{i=1}^{d} \tilde{D}'_i$. Immediately, there exists $C > 0$ such that
\[
e^{\varphi_\epsilon} \geq \frac{e^{-H(u_\epsilon)H'(u_\epsilon)}}{C|u_\epsilon|^n(1+\delta)}.
\]
If we now let
\[H(x) = -(2n + \epsilon) \log(-x),\]
then
\[H'(u_\epsilon) = \frac{2n + \epsilon}{|u_\epsilon|} < 1\]
and at the minimal point $p$, we have
\[e^{-H(u_\epsilon)H'(u_\epsilon)} \geq |u_\epsilon|^{2n+\epsilon} \geq \frac{1}{|u_\epsilon|} \geq 1,\]
if we choose $\epsilon > n\delta$. Therefore there exists $C_\epsilon > 0$ such that
\[\varphi_\epsilon \geq -C_\epsilon\]
or equivalently,
\[(3.21) \quad \varphi \geq f_\epsilon - C_\epsilon.\]

This completes the proof of Proposition 3.13 by combining (3.21) and the definition of $f_\epsilon$ in (3.18).

\section{Integrability of $\log |\sigma_\epsilon|^2_{h_\epsilon}$}

Let $\sigma_\epsilon$ and $h_\epsilon$ be the holomorphic section and hermitian metric of $E$ in Proposition 3.1. For $t \in B' = B \setminus Z$ we will prove a uniform estimate for the integrals of $\log |\sigma_\epsilon|^2_{h_\epsilon}$. This can be proved by local calculations from the argument in [21]. In this section, we give an alternative proof using a result of Morikawi ([13]).

\begin{proposition}
For $t \in B' = B \setminus Z$, let
\[(4.22) \quad I(t) = \int_{X_t} (-\log |\sigma_\epsilon|^2_{h_\epsilon}) \chi_t^n.\]
Then $I(t)$ extends to a continuous function $B \to \mathbb{R}$. In particular, $I(t)$ is uniformly bounded for all $t \in B$.
\end{proposition}

The proof is an application of the following theorem of Morikawi ([13]).

\begin{theorem}
Let $f_X : X \to B$ be a flat projective morphism of algebraic varieties over $\mathbb{C}$ of relative dimension $n$, let $L_0, ..., L_n$ be line bundles equipped with smooth hermitian metrics $h_0, ..., h_n$. Then $\langle h_0, ..., h_n \rangle$ is a continuous hermitian metric on the Deligne pairing $\langle L_0, ..., L_n \rangle(X/B) \to B$. Thus, if $l_0, ..., l_n$ are generic rational sections of $L_0, ..., L_n$ such that $\cap_{j=0}^n \text{div}(l_j) = \phi$, then $|\langle l_0, ..., l_n \rangle|$ is a positive continuous function on $B$.
\end{theorem}
Note that the fibers $X_t$ are all projective varieties, but $X$ and $B$ need not be projective.

For details on the definition and properties of Deligne pairings see [29] and [13], but let us here recall the basic framework. If $f_X : X \to B$ is as above, and $L_0, \ldots, L_n$ are hermitian line bundles on $X$, then $(L_0, \ldots, L_n)(X/B)$ is a hermitian line bundle on $B$. For example, if $n = 0$ then $\langle L_0 \rangle$ is the “norm” of $L = L_0$. Thus, the fiber of $\langle L \rangle$ at $b \in B$ equals $\otimes_{x \in f_X^{-1}b} L_x$ and if $l$ is a rational section of $L$ then $(l) = \otimes_{x \in f_X^{-1}b} l_x$ is a rational section of $\langle L \rangle$ and $|\langle l \rangle|_b = \otimes_{x \in f_X^{-1}b} |l_x|_b$. In general, $(L_0, \ldots, L_n)(X/B) = (L_0, \ldots, L_{n-1})(\text{div}(l_n)/B)$, and the restriction of $\langle l_0, \ldots, l_n \rangle$ to $\text{div}(l_n)$ is just $\langle l_0, \ldots, l_{n-1} \rangle$. Moreover the formula for the norm of the restriction is given below by (4.23). In fact these properties give an inductive characterization of the Deligne pairing.

Let $f_X : X \to B$ be as in Theorem 4.1, let $B^0 = \{ t \in B : f_X^{-1}(t) \text{ is smooth} \}$ and $B' \subseteq B^0$ a Zariski open subset. We shall need the following lemma.

**Lemma 4.1.** Fix $l_0, \ldots, l_n$ rational sections of $L_0, \ldots, L_n$ such that $\cap_{j=0}^n \text{div}(l_j) = \phi$. Assume that one of the following conditions holds

1. $\text{div}(l_0) \to B$ is flat or
2. $f(\text{div}(l_0)) \cap B' = \phi$ and $\text{div}(l_j) \to B$ is flat for $1 \leq j \leq n$.

Then the map $I' : B' \to \mathbb{R}$ given by

$$t \mapsto \int_{X_t} \log |l_0|_{h_0}^2 c_1(h_1) \wedge \cdots \wedge c_n(h_n)$$

is continuous and extends to a continuous function $B \to \mathbb{R}$.

**Proof.** Assume $\text{div}(l_0) \to B$ is flat. Then the result follows from Theorem 4.1 and the induction formula for Deligne pairings:

(4.23)

$$\log |\langle l_0, \ldots, l_n \rangle(X/B)| = \log |\langle l_1, \ldots, l_n \rangle(\text{div}(l_0)/B)| - \int_{X_t} \log |l_0|_{h_0}^2 c_1(h_1) \wedge \cdots \wedge c_n(h_n)$$

on $B'$.

Now assume $f(\text{div}(l_0)) \cap B' = \phi$ and $\text{div}(l_j) \to B$ is flat for $1 \leq j \leq n$. We proceed using induction on $n$. Let $Y = \text{div}(l_1)$. Then $f_Y : Y \to B$ is flat with fiber dimension $n - 1$. Writing $[L_j] := \text{div}(l_j)$ we compute, for $t \in B'$,

$$\int_{X_t} \log |l_0|_{h_0}^2 c_1(h_1) \wedge \cdots \wedge c_n(h_n) = \int_{X_t} \log |l_0|_{h_0}^2 ([L_1] - \sqrt{-1} \partial \bar{\partial} \log |l_1|_{h_1}^2) \wedge c_1(h_2) \wedge \cdots \wedge c_n(h_n)$$

$$= \int_{X_t} \log |l_0|_{h_0}^2 c_1(h_2) \wedge \cdots \wedge c_n(h_n) - \int_{X_t} \log |l_1|^2 ([L_0] - c_1(h_0)) \wedge c_1(h_2) \wedge \cdots \wedge c_n(h_n)$$

Now the first integral is continuous by induction and the second is continuous by part (1) of the lemma (note that $[L_0] \cap X_t = \phi$ for $t \in B'$ by assumption).
Proof of Proposition 4.1. We wish to apply Lemma 4.1 as follows (see (2.4) for the notation). Let $X = \tilde{X}$, $L_0 = O(E)$, $l_0 = \sigma_E$, and $L_1 = L_2 = \cdots = L_n = (\Psi \circ \Phi)^*(mK_{X/B})$. Then the hypotheses of part (2) of Lemma 4.1 apply by virtue of (3.12). □

5. LEVEL SETS OF THE KÄHLER-EINSTEIN POTENTIALS

Let $X \to B$ be a stable family of canonically polarized manifolds, where $B \subseteq \mathbb{C}^d$ is a Euclidean ball. Let

$$B^o = \{ t \in B : X_t \text{ is smooth} \}$$

and

$$X^o = \pi^{-1}(B^o).$$

For $t \in B^o$, we let $\varphi_t$ be the solution of the complex Monge-Ampère equation induced by the Kähler-Einstein equation satisfying

$$\omega_t^n = (\chi_t + \sqrt{-1} \partial \bar{\partial} \varphi_t)^n = e^{\varphi_t} \Omega_t,$$

where $\chi_t$ and $\Omega_t$ are defined in (3.9). The main goal of this section is to bound the volume of the level sets of the Kähler-Einstein potentials $\varphi_t$.

We first recall the definition of capacity in pluripotential theory.

Definition 5.1. Let $X$ be an $n$-dimensional compact Kähler manifold and let $\omega$ be a smooth closed nonnegative $(1,1)$-form on $X$. The capacity for a Borel subset $K$ of $X$ associated to $\omega$ is defined by

$$(5.24) \quad \text{Cap}_\omega(K) = \sup \left\{ \int_K (\omega + \sqrt{-1} \partial \bar{\partial} u)^n \mid u \in \text{PSH}(X, \omega), \ -1 \leq u \leq 0 \right\}.$$

The following lemma is well-known [12].

Lemma 5.1. Suppose $X$ is a compact complex manifold, $\omega \geq 0$ a closed nonnegative $(1,1)$-form and $\psi \in \text{PSH}(X, \omega) \cap L^\infty(X)$ with $\psi \leq 0$. Then for $K \geq 1$, we have

$$\int_{\{\psi \leq -K\}} (\omega + \sqrt{-1} \partial \bar{\partial} \psi)^n \leq K^n \text{Cap}_\omega(\psi \leq -K)$$

Proof. Let $\psi_K = \max(\psi, -K)$ and $u_K = K^{-1} \psi_K$. Then $u_K \in \text{PSH}(X, \omega)$ and

$$-1 \leq u_K \leq 0.$$

Then

$$\int_{\psi \leq -K} (\omega + \sqrt{-1} \partial \bar{\partial} \psi)^n = \int_X \omega^n - \int_{\psi > -K} (\omega + \sqrt{-1} \partial \bar{\partial} \psi)^n$$

$$= \int_X \omega^n - \int_{\psi > -K} (\omega + \sqrt{-1} \partial \bar{\partial} \psi_K)^n$$

$$= \int_{\psi \leq -K} (\omega + \sqrt{-1} \partial \bar{\partial} \psi_K)^n$$

$$\leq K^n \int_{\psi \leq -K} (\omega + \sqrt{-1} \partial \bar{\partial} u_K)^n$$

$$\leq K^n \text{Cap}_\omega(\psi \leq -K).$$

\[\square\]
The following lemma is proved in [9] (Proposition 2.6).

**Lemma 5.2.** Let $\varphi \in \text{PSH}(X, \omega)$ with $\varphi \leq 0$. Then for $K \geq 1$

$$\text{Cap}_\omega(\varphi < -K) \leq K^{-1} \left( \int_X (-\varphi) \omega^n + n \int_X \omega^n \right).$$

The following proposition is the main result of the section and is equivalent to Theorem 1.3. It implies the exponential decay for the measure of the level set of the potential $\varphi_t$.

**Proposition 5.1.** There exists $C > 0$ such that for $0 \leq m \leq n$ and all $t \in B^\circ$,

$$\int_{\{\varphi_t < -K\}} (\chi_t + \sqrt{-1} \partial \bar{\partial} \varphi_t)^m \wedge \chi_t^{n-m} \leq CK^n e^{-K/(4n+2)}$$

**Proof.** The uniform $C^0$-estimate of Proposition 3.1 implies that there exists $C_1 > 0$ such that for all $t \in B^\circ$,

$$\varphi_t \geq -(2n + 1) \log(-\log |\sigma| h_{\kappa})| x_t - C_1 = -(2n + 1) \log(-\theta_t) - C_1.$$

We define $\psi_t$ by

$$\psi_t = -2(2n + 1) \log(-\theta_t) - 2C_1.$$

By the upper bound estimate for $\varphi_t$ in [21], there exists $A > 0$ such that for all $t \in B$,

$$\sup_{X_t} \varphi_t \leq A.$$

Then $\sup_{X_t} \psi_t \leq A$. We let

$$\psi_{t, M_t} = \max(\psi_t, -2M_t),$$

where $-M_t << \inf_{X_t} \varphi_t$. Obviously, we still have

$$2\varphi_t \geq \psi_{t, M_t}.$$

Thus

$$\int_{\{\psi_t < -K\}} (2\chi_t + \sqrt{-1} \partial \bar{\partial} \varphi_t)^n \leq \int_{\{\psi_{t, M_t} < -K + \psi_t\}} (2\chi_t + \sqrt{-1} \partial \bar{\partial} \varphi_t)^n$$

$$\leq \int_{\{\psi_{t, M_t} < -K + \psi_t\}} (2\chi_t + \sqrt{-1} \partial \bar{\partial} \psi_{t, M_t})^n$$

$$\leq \int_{\{\psi_{t, M_t} < -K\}} (2\chi_t + \sqrt{-1} \partial \bar{\partial} \psi_{t, M_t})^n,$$

where the second inequality follows from the comparison principle for plurisubharmonic functions. We now may replace $\varphi_t$ by $\psi_{t, M_t}$ in proving the proposition. By Lemma 5.1 and Lemma 5.2, we have

$$\int_{\{\psi_{t, M_t} < -K\}} (2\chi_t + \sqrt{-1} \partial \bar{\partial} \psi_t)^n \leq K^n \text{Cap}_{X_t}(\psi_{t, M_t} < -K + A)$$

$$\leq K^n \text{Cap}_{X_t}(\psi_t < -K + A)$$

$$= K^n \text{Cap}_{X_t}(-(4n + 2) \log(-\theta_t) < -K + A + 2C_1)$$

$$= K^n \text{Cap}_{X_t}(-\theta_t < e^{\frac{K-A-2C_1}{4n+2}})$$

$$\leq K^n e^{-\frac{K-A-2C_1}{4n+2}} \left( \int_{X_t} (-\theta_t) \chi_t^n + n \int_{X_t} \chi_t^n \right).$$
By letting $M_t \to \infty$, we have, by Proposition 4.1, that there exists $C_2 > 0$ such that
\[
\int_{\{\varphi_t < -K\}} (2\chi_t + \sqrt{-1} \partial \bar{\partial} \varphi_t)^n \leq C_2 K^n e^{-\frac{K-A-2C_1}{4n+2}}.
\]
The proposition now follows from
\[
\int_{\{\varphi_t < -K\}} (2\chi_t + \sqrt{-1} \partial \bar{\partial} \varphi_t)^n \geq \sum_{m=0}^{n} \int_{\{\varphi_t < -K\}} (\chi_t + \sqrt{-1} \partial \bar{\partial} \varphi_t)^m \wedge \chi_t^{n-m}.
\]

\[\square\]

**Corollary 5.1.** There exists $C > 0$ such that for $0 \leq m \leq n$ and all $t \in B^\circ$,
\[
\int_{\{\varphi_t < -K\}} \vert \varphi_t \vert (\chi_t + \sqrt{-1} \partial \bar{\partial} \varphi_t)^m \wedge \chi_t^{n-m} \leq C e^{-K/(4n+4)}.
\]

**Proof.** We apply Proposition 5.1 and there exist $C_1, C_2 > 0$ such that for any $t \in B^\circ$,
\[
\int_{\{\varphi_t < -K\}} (\varphi_t)(\chi_t + \sqrt{-1} \partial \bar{\partial} \varphi_t)^m \wedge \chi_t^{n-m} = \sum_{j=K}^{\infty} \int_{\varphi_t < -j} (\varphi_t)(\chi_t + \sqrt{-1} \partial \bar{\partial} \varphi_t)^m \wedge \chi_t^{n-m} \leq \sum_{j=K}^{\infty} (j+1) \int_{\varphi_t < -j} (\chi_t + \sqrt{-1} \partial \bar{\partial} \varphi_t)^m \wedge \chi_t^{n-m} \leq C_1 \sum_{j=K}^{\infty} (j+1) j^n e^{-j/(4n+2)} \leq C_2 e^{-K/(4n+4)}.
\]

\[\square\]

Finally, we remark that Proposition 5.1 and Corollary 5.1 also hold uniformly for all $t \in B$. If $t \in B \setminus B^\circ$, we can always approximate $X_t$ by a smooth Kähler manifold of dimension $n$ after normalization and resolution of singularities. The constants in the proposition and corollary are uniformly controlled. This can also be achieved by applying continuity of $\varphi$ on $X^\circ$ from section 8.

6. **CONTINUITY OF THE WEIL-PETERSSON POTENTIALS FOR STABLE FAMILIES**

In this section, we will prove the continuity of the Weil-Petersson potentials for stable families of Kähler-Einstein manifolds. As before we consider a stable family $\pi : X \to B$ of $n$-dimensional Kähler-Einstein manifolds over a Euclidean ball $B \subseteq \mathbb{C}^d$ whose general fibre is smooth. We will use the same notations as in the previous sections.

We define the relative Weil-Petersson potential $\psi_{WP}$ by
\[
\psi_{WP} = \frac{1}{(n+1)V} \sum_{j=0}^{n} \int_{X_t} \varphi_t (\chi_t + \sqrt{-1} \partial \bar{\partial} \varphi_t)^j \wedge \chi_t^{n-j}
\]
The Weil-Petersson metric on $B$ is given by (c.f. [18, 21])
\[
\omega_{WP} = \int_{X_t} \chi^{n+1} + \sqrt{-1} \partial \bar{\partial} \psi_{WP}.
\]

The following is the main result of this section.

**Proposition 6.1.** \(\psi_{WP}\) is continuous in $B$.

It is proved in [14] that $\int_{X_t} \chi^{n+1}$ is a nonnegative closed $(1,1)$-current on $B$ with continuous local potentials. Then Proposition 6.1 immediately implies $\omega_{WP}$ has continuous local potentials.

Let $X_{i\text{reg}}$ be the set of all smooth points of $X_i$ for $t \in B$ and let $X_{i\text{reg}}^t$ be the smooth points of $X_i^t$. Before proving Proposition 6.1, we first prove that the Kähler-Einstein potential $\varphi$ is continuous on $X_{i\text{reg}}$, which is slightly weaker than the conclusion in Theorem 1.2. The following lemma is implicitly proved in [21] (Lemma 5.2 and Lemma 5.3).

**Lemma 6.1.** Suppose $t_j \in B$ and $t_j \to t_{\infty} \in B$. Then $\varphi_{i\text{reg}} \mid X_{i\text{reg}}^t$ converges smoothly to $\varphi_{i\text{reg}} \mid X_{i\text{reg}}^t_{\infty}$.

**Proof.** By Lemma 5.2 in [21], for any $k > 0$ and compact set $K \subset X_{i\text{reg}}$, $\varphi_t$ is uniformly bounded in $C^k(K \cap X^t_i)$ for all $t \in B$. Therefore for all $t_j \to t_{\infty} \in B$, $\varphi_{t_j}$ converges smoothly to $\varphi_{t_{\infty}}$ away from $K$, after taking a subsequence. However, by the uniform $L^\infty$-estimate, $\varphi_{t_{\infty}}$ is bounded above and bounded below by any log poles. By uniqueness of the canonical Kähler-Einstein current on $X_{i\text{reg}}$ (Theorem 1.1. in [20]),
\[
\varphi_{t_{\infty}} = \varphi_{t_{\infty}}.
\]
The lemma follows immediately. \(\square\)

Lemma 6.1 immediately implies continuity of $\varphi$ on $X_{i\text{reg}}$.

**Corollary 6.1.** $\varphi$ is continuous on $X_{i\text{reg}}$.

It suffices to prove continuity of the Weil-Petersson potential at one point. We will fix $\hat{t} \in B$. By Corollary 5.1, there exists sufficiently small $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$ and $t \in B$, we have
\[
\sum_{m=0}^{n} \int_{\{\varphi_t < -\epsilon^{-1}\}} |\varphi_t| (\chi_t + \sqrt{-1} \partial \bar{\partial} \varphi_t)^m \wedge (\chi_t)^{n-m} < \epsilon^2.
\]

**Lemma 6.2.** For any $0 < \epsilon < \epsilon_0$, there exists an open neighborhood $\hat{U} \subset X_i$ of $X_i \setminus X_{i\text{reg}}$ such that
\[
\sum_{m=0}^{n} \int_{\hat{U}} (\chi_t + \sqrt{-1} \partial \bar{\partial} \varphi_t)^m \wedge (\chi_t)^{n-m} < \epsilon^2
\]
and
\[
\{\varphi_t < -\epsilon^{-1}\} \subset \hat{U}.
\]
Proof. \( \varphi_i \) is uniformly bounded on \( \mathcal{X}_t \setminus \{ \varphi_i < -\varepsilon^{-1} \} \). Therefore there exists an open set \( U \) containing \( \mathcal{X}_i^{\text{sing}} \cap \{ \varphi_i \geq -\varepsilon^{-1} \} \) such that
\[
\sum_{m=0}^{n} \int_U (\chi_i + \sqrt{-1} \partial \bar{\partial} \varphi_i)^m \wedge (\chi_i)^{n-m} < \epsilon^4,
\]
where \( \mathcal{X}_i^{\text{sing}} = \mathcal{X}_i \setminus \mathcal{X}_i^{\text{reg}} \) is the singular set of \( \mathcal{X}_t \), since \( \mathcal{X}_i^{\text{sing}} \) is a closed pluriclosed set. The lemma is then proved by choosing \( \hat{U} = U \cup \{ \varphi_i < -\varepsilon^{-1} \} \).

We choose open neighborhoods \( U \) and \( V \) of \( \mathcal{X} \setminus \mathcal{X}^{\text{reg}} \) in \( \mathcal{X} \) such that
\[
(6.27) \quad \{ \varphi < -\varepsilon^{-1} \} \subset \subset U \subset \subset V, \quad V \cap \mathcal{X}_i \subset \hat{U}.
\]
Since \( \mathcal{X}^{\text{reg}} \) is locally a smooth holomorphic product, by partition of unity there exist an open neighborhood \( B_{\delta_0} \) of \( \hat{t} \) in \( B \) and a collection of finitely many smooth functions
\[
(6.28) \quad \{ \rho_\alpha \}_\alpha
\]
on \( \mathcal{X} \) satisfying
1. the support of \( \rho_\alpha \) is biholomorphic to \( \{ |t - \hat{t}| < \delta_0 \} \times \mathbb{B} \subset (\mathcal{X} \setminus U) \), where \( \mathbb{B} \) is a unit ball in \( \mathbb{C}^n \) with \( \{ t \} \times \mathbb{B} \subset \mathcal{X}_t \).
2. \( 0 \leq \rho_\alpha \leq 1 \).
3. For any \( p \in \pi^{-1}(B_{\delta_0}) \setminus V \),
\[
\sum_\alpha \rho_\alpha(p) = 1.
\]
Since \( \varphi_t \) converges smoothly to \( \varphi_i \) as \( t \to \hat{t} \) on \( \mathcal{X}^{\text{reg}} \), by straightforward calculation on local Euclidean spaces, for each \( \alpha \), we have
\[
\lim_{t \to \hat{t}} \int_{\mathcal{X}_t} \rho_\alpha(\chi_t + \sqrt{-1} \partial \bar{\partial} \varphi_t)^m \wedge (\chi_t)^{n-m} = \int_{\mathcal{X}_i} \rho_\alpha(\chi_i + \sqrt{-1} \partial \bar{\partial} \varphi_i)^m \wedge (\chi_i)^{n-m}
\]
uniformly for \( t \in B_{\delta_0} \). Since \( \cup_\alpha \{ \text{supp} \rho_\alpha \} \subset (\mathcal{X} \setminus U) \), we have
\[
\lim_{t \to \hat{t}} \int_{\mathcal{X}_t \setminus U} (\chi_t + \sqrt{-1} \partial \bar{\partial} \varphi_t)^m \wedge (\chi_t)^{n-m}
\]
\[
\geq \int_{\mathcal{X}_t \setminus V} (\chi_t + \sqrt{-1} \partial \bar{\partial} \varphi_t)^m \wedge (\chi_t)^{n-m}
\]
\[
\geq \int_{\mathcal{X}_i} (\chi_i + \sqrt{-1} \partial \bar{\partial} \varphi_i)^m \wedge (\chi_i)^{n-m} - \int_{\hat{U}} (\chi_i + \sqrt{-1} \partial \bar{\partial} \varphi_i)^m \wedge (\chi_i)^{n-m}
\]
\[
\geq |\chi_i|^n - \epsilon^2
\]
by Lemma 6.2. In particular, \( |\chi_i|^n \) is a topological number independent of \( t \) and by Proposition 5.1 or the fact that \( \varphi_i \in \mathcal{E}^1 \) (c.f. [4]), we have
\[
\int_{\mathcal{X}_i} (\chi_t + \sqrt{-1} \partial \bar{\partial} \varphi_t)^m \wedge (\chi_t)^{n-m} = |\chi_i|^n.
\]
We then have the following lemma.
Lemma 6.3. For any $0 < \epsilon < \epsilon_0$, there exists $0 < \delta < \delta_0$, such that for any $t$ with $|t - \hat{t}| < \delta$ and $0 \leq m \leq n$, we have

$$\int_{X_i \cap U} (\chi_t + \sqrt{-1} \partial \overline{\partial} \varphi_t)^m \wedge (\chi_t)^{n-m} < 2\epsilon^2$$

and

$$\int_{X_i \setminus U} (\chi_t + \sqrt{-1} \partial \overline{\partial} \varphi_t)^m \wedge (\chi_t)^{n-m} \geq [\chi_t]^n - 2\epsilon^2,$$

where $U$ is constructed as in (6.27).

Corollary 6.2. For any $0 < \epsilon < \epsilon_0$, there exists $0 < \delta < \delta_0$ such that for any $t$ with $|t - \hat{t}| < \delta$, we have

$$\sum_{m=0}^{n} \int_{X_i \cap U} |\varphi_t|(\chi_t + \sqrt{-1} \partial \overline{\partial} \varphi_t)^m \wedge (\chi_t)^{n-m} < 2(n+1)\epsilon.$$

Proof. By Lemma 6.3, we have

$$\int_{X_i \cap U \cap \{|\varphi_t| > \epsilon^{-1}\}} |\varphi_t|(\chi_t + \sqrt{-1} \partial \overline{\partial} \varphi_t)^m \wedge (\chi_t)^{n-m}$$

$$< \int_{X_i \cap U \cap \{|\varphi_t| > \epsilon^{-1}\}} \epsilon(\chi_t + \sqrt{-1} \partial \overline{\partial} \varphi_t)^m \wedge (\chi_t)^{n-m}$$

$$< 2\epsilon.$$

The corollary follows by combining the above estimate and the assumption (6.26).

Lemma 6.4. For any $0 < \epsilon < \epsilon_0$, there exists $0 < \delta < \delta_0$ such that for $|t - \hat{t}| < \delta$,

$$\sum_{m=0}^{n} \left| \int_{X_i} \varphi_t(\chi_t + \sqrt{-1} \partial \overline{\partial} \varphi_t)^m \wedge (\chi_t)^{n-m} - \int_{X_i} \varphi_t(\chi_t + \sqrt{-1} \partial \overline{\partial} \varphi_t)^m \wedge (\chi_t)^{n-m} \right| < \epsilon,$$

i.e.,

$$|\psi_{WP}(t) - \psi_{WP}(\hat{t})| < \epsilon.$$

Proof. By the same partition of unity as in (6.28), we have

$$\lim_{t \to \hat{t}} \int_{X_i} \rho_\alpha \varphi_t(\chi_t + \sqrt{-1} \partial \overline{\partial} \varphi_t)^m \wedge (\chi_t)^{n-m} = \int_{X_i} \rho_\alpha \varphi_t(\chi_t + \sqrt{-1} \partial \overline{\partial} \varphi_t)^m \wedge (\chi_t)^{n-m}$$

uniformly for $t \in B_{\delta_0}$. Let $U' = X \setminus \cup_\alpha \{\text{supp } \rho_\alpha\}$. Then

$$\lim_{t \to \hat{t}} \int_{X_i \setminus U'} \varphi_t(\chi_t + \sqrt{-1} \partial \overline{\partial} \varphi_t)^m \wedge (\chi_t)^{n-m}$$

$$\geq \int_{X_i \setminus U'} \varphi_t(\chi_t + \sqrt{-1} \partial \overline{\partial} \varphi_t)^m \wedge (\chi_t)^{n-m} - \epsilon^{-1} \lim_{t \to \hat{t}} \int_{X_i \cap U' \cap \{|\varphi_t| > \epsilon^{-1}\}} (\chi_t + \sqrt{-1} \partial \overline{\partial} \varphi_t)^m \wedge (\chi_t)^{n-m}$$

$$+ \lim_{t \to \hat{t}} \int_{X_i \cap \{|\varphi_t| < \epsilon^{-1}\}} \varphi_t(\chi_t + \sqrt{-1} \partial \overline{\partial} \varphi_t)^m \wedge (\chi_t)^{n-m}$$

$$\geq \int_{X_i \setminus U'} \varphi_t(\chi_t + \sqrt{-1} \partial \overline{\partial} \varphi_t)^m \wedge (\chi_t)^{n-m} - 3\epsilon.$$
and by similar argument,
\[
\lim_{t \to \hat{t}} \int_{X_t \setminus U'} \varphi_i(\chi_t + \sqrt{-1} \partial \bar{\partial} \varphi_t)^m \wedge (\chi_t)^{n-m} \\
\leq \int_{X_t \setminus U'} \varphi_i(\chi_t + \sqrt{-1} \partial \bar{\partial} \varphi_t)^m \wedge (\chi_t)^{n-m} + \epsilon^{-1} \lim_{t \to \hat{t}} \int_{X_t \cap U' \cap \{\varphi \geq -\epsilon^{-1}\}} (\chi_t + \sqrt{-1} \partial \bar{\partial} \varphi_t)^m \wedge (\chi_t)^{n-m} \\
- \lim_{t \to \hat{t}} \int_{X_t \cap \{\varphi < -\epsilon^{-1}\}} \varphi_t(\chi_t + \sqrt{-1} \partial \bar{\partial} \varphi_t)^m \wedge (\chi_t)^{n-m} \\
\leq \int_{X_t \setminus U'} \varphi_i(\chi_t + \sqrt{-1} \partial \bar{\partial} \varphi_t)^m \wedge (\chi_t)^{n-m} + 3\epsilon.
\]

On the other hand, by Corollary 6.2, for any \(0 < \epsilon < \epsilon_0\), there exists \(\delta > 0\) such that for \(|t - \hat{t}| < \delta\),
\[
\int_{X_t \cap U} |\varphi_i(\chi_t + \sqrt{-1} \partial \bar{\partial} \varphi_t)^m \wedge (\chi_t)^{n-m} | \leq (n+1)\epsilon.
\]

Combining the above estimates,
\[
\left| \lim_{t \to \hat{t}} \int_{X_t} \varphi_i(\chi_t + \sqrt{-1} \partial \bar{\partial} \varphi_t)^m \wedge (\chi_t)^{n-m} - \int_{X_t} \varphi_i(\chi_t + \sqrt{-1} \partial \bar{\partial} \varphi_t)^m \wedge (\chi_t)^{n-m} \right| \\
\leq 2\epsilon + \int_{X_t \cap U} \left| \varphi_i(\chi_t + \sqrt{-1} \partial \bar{\partial} \varphi_t)^m \wedge (\chi_t)^{n-m} \right| \\
< 3\epsilon.
\]

The lemma is proved by choosing \(t\) sufficiently close to \(\hat{t}\).

\[\square\]

Now Proposition 6.1 immediately follows from Lemma 6.4 and the result in [14] that \(\int_{X_t} \chi^{n+1} = \sqrt{-1} \partial \bar{\partial} \psi_X\) for some \(\psi_X \in \text{PSH}(B) \cap C^0(B)\).

**Corollary 6.3.** There exists \(\Phi_{WP} \in \text{PSH}(B) \cap C^0(B)\) such that
\[
\omega_{WP} = \sqrt{-1} \partial \bar{\partial} \Phi_{WP}.
\]

**7. Proof of Theorem 1.1**

In order to prove Theorem 1.1, we have to replace the smooth base by the compactified KSB moduli space in Corollary 6.3. As in section 7 of [21], we will replace \(\overline{\mathcal{M}}_{\text{KSB}}\) by a smooth model as follows. There exists a finite map
\[
\phi : B \to \overline{\mathcal{M}}_{\text{KSB}}
\]
together with a stable family
\[
\pi : \mathcal{X} \to B
\]
such that for any \(p \in \overline{\mathcal{M}}_{\text{KSB}}\), the fibres \((\phi \circ \pi)^{-1}(p)\) are Kähler-Einstein varieties correspond to the point \(p\) in the moduli space. After normalization and resolution of singularities we can assume \(B\) is smooth and \(\phi\) is generically finite. We can define \(\omega_{WP}\) globally on \(\overline{\mathcal{M}}_{\text{KSB}}\) by pushing forward the Weil-Petersson current \(\omega_{WP,B}\) on \(B\) by \(\phi\). It is shown in [21] that the Weil-Petersson current on \(\overline{\mathcal{M}}_{\text{KSB}}\) has bounded local potentials and is independent of the choice of \(B\). Furthermore, \(\omega_{WP}\) is the curvature of the CM line.
bundle $\mathcal{L}_{CM} \to \overline{\mathcal{M}}_{KSB}$. Let $h$ be the smooth hermitian metric on $\mathcal{L}_{CM}$ with $\omega_{FS} = \text{Ric}(h)$ being the Fubini-Study metric induced by a projective embedding from the linear system of a sufficiently large power of $\mathcal{L}_{CM}$. Then by Corollary 6.3, we have

$$\phi^* \omega_{WP,B} = \phi^* \omega_{FS} + \sqrt{-1} \partial \overline{\partial} \phi_{WP,B}$$

for some $\phi_{WP,B} \in \text{PSH}(B, \phi^* \omega_{FS}) \cap C^0(B)$. In particular, it is proved in [21] that $\phi_{WP,B}$ coincides with the pullback of a bounded function $\phi_{WP} \in \text{PSH}(\overline{\mathcal{M}}_{KSB}, \omega_{FS}) \cap L^\infty(\overline{\mathcal{M}}_{KSB})$ on the regular set of $\pi$. Let $F$ be an irreducible component of a fibre of $\pi : B \to \overline{\mathcal{M}}_{KSB}$ with $\text{dim} F \geq 1$. Obviously, $\phi^* \mathcal{L}_{CM}|_F = 0$ and $\omega_{FS}|_F = 0$. This implies that $\phi_{WP,B}$ is plurisubharmonic and continuous on $F$, hence $\phi_{WP,B}$ must be constant along $F$. Therefore $\phi_{WP,B}$ can descend to a continuous function on $\overline{\mathcal{M}}_{KSB}$. This completes the proof of Theorem 1.1.

8. Proof of Theorem 1.2

In this section, we will complete the proof of Theorem 1.2. Let $\mathcal{X} \to B$ be a stable family of canonically polarized manifolds, where $B \subseteq \mathbb{C}^d$ is a ball. Let $\mathcal{E}_{slc}$ be the set of all non-klt locus of $\mathcal{X}_t$ for $t \in B$.

For $t \in B^o$, let $\varphi_t$ satisfy $\omega_t^n = (\chi + \sqrt{-1} \partial \overline{\partial} \varphi_t)^n = e^{\varphi_t} \Omega_t$. The following is the main result of this section to bound the volume of the level sets of the Kähler-Einstein potentials $\varphi_t$. We let $g_t$ be the Kähler-Einstein metric associated to $\omega_t$.

The following theorem is proved in [21].

**Theorem 8.1.** For any $t_j \in B^o$ with $t_j \to t_{\infty}$, $(\mathcal{X}_{t_j}, g_{t_j})$ converges in pointed Gromov-Hausdorff topology to the metric completion of $(\mathcal{X}_{t_{\infty}}^o, g_{t_{\infty}})$, where $\mathcal{X}_{t_{\infty}}^o$ is the smooth part of $\mathcal{X}_{t_{\infty}}$ and $g_{t_{\infty}}$ is the smooth Kähler-Einstein metric on $\mathcal{X}_{t_{\infty}}^o$ that extends to the unique Kähler-Einstein current on $\mathcal{X}_{t_{\infty}}$. In particular, the metric completion of $(\mathcal{X}_{t_{\infty}}^o, g_{t_{\infty}})$ is homeomorphic to $\mathcal{X}_{t_{\infty}} \setminus \mathcal{E}_{slc}$ and the convergence is smooth on $(\mathcal{X}_{t_{\infty}}^o, g_{t_{\infty}})$.

In fact, it is proved in [20] (that $\varphi_t$ is always bounded away from $\mathcal{E}_{slc}$).

**Lemma 8.1.** Let $p_t \in \mathcal{X}_t$ be a continuous section of $\pi : \mathcal{X} \to B^o$ such that

$$\{p_t\}_{t \in B} \cap \mathcal{E}_{slc} = \phi.$$

Then for any $R > 0$, there exists $C_R > 0$ such that for any $t \in B$,

$$\sup_{B_{g_t}(p_t, R)} |\varphi_t| \leq C_R.$$

**Proof.** We prove by contradiction. By assumption, geodesic unit balls centered at $p_t$ are uniformly non-collapsed, i.e., there exists $c > 0$ such that for all $t \in B^o$,

$$\text{Vol}_{g_t}(B_{g_t}(p_t, 1)) \geq c.$$

Suppose there exist a sequence $t_j \in B^o$ with $t_j \to t_{\infty}$ in $B$ and a sequence $x_j \in \mathcal{X}_{t_j}$ with $x_j \to x_{\infty} \in \mathcal{X}_{t_{\infty}} \setminus \mathcal{E}_{slc}$ in Gromov-Hausdorff distance such that for any $A > 0$, there exists $J > 0$ so that for all $j \geq J$, we have

$$\varphi_{t_j}(x_j) < -A.$$
We can assume that for some fixed sufficiently small $r > 0$, $B_{g_{t_j}}(x_j, r)$ converges in Gromov-Hausdorff topology to $B_{g_{t_\infty}}(x_\infty, r)$. By the local $L^2$-estimate and partial $C^0$-estimate in [21], there exists a sequence of $\sigma_j \in H^0(\mathcal{X}_{t_j}, mK_{\mathcal{X}_{t_j}})$ for some fixed $m >> 1$ satisfying the following.

(1) For each $j$,
\[ \int_{\mathcal{X}_{t_j}} |\sigma_j|^2 h_{t_j}^m dV_{g_{t_j}} = 1, \]
where $h_t = (\omega_t^n)^{-1}$ is the hermitian metric on the relative canonical bundle $K_{\mathcal{X}/B}$ induced by the Kähler-Einstein metrics on the fibres.

(2) There exists $C > 0$ such that for each $j$,
\[ \sup_{B_{g_{t_j}}(x_j, r)} |\nabla \sigma_j|^2 h_{t_j}^m \leq C. \]

(3) There exists $c > 0$ such that for each $j$,
\[ |\sigma_j|^2 h_{t_j}^m(x_j) \geq c. \]

By passing to a subsequence, we can assume that $\sigma_j$ converges to an $L^2$-integrable pluricanonical section $\sigma_{t_\infty}$ on $B_{g_{t_\infty}}(x_\infty, r)$. Since $h_{t_j}$ converges smoothly to $h_{t_\infty}$ on the smooth part of $\mathcal{X}_{t_\infty}$, by the uniform gradient estimate, $|\sigma_{t_\infty}|^2_{h_{t_\infty}^m}$ extends to a continuous function on $B_{g_{t_\infty}}(x_\infty, r)$ and there exist $\epsilon > 0$ and $0 < 2\delta < r$ such that
\[ \inf_{B_{g_{t_\infty}}(x_\infty, 2\delta)} |\sigma_{t_\infty}|^2_{h_{t_\infty}^m} > \epsilon. \]

We fix a projective embedding $\Phi : \mathcal{X} \to \mathbb{CP}^N_m$ by $mK_{\mathcal{X}/U}$ for an affine neighborhood $U$ of $t_\infty$ in $B$ such that $\chi = m^{-1}\sqrt{-1} \partial \bar{\partial} \log(1 + \sum_{i=1}^{N_m} |z_i|^2)$ in an affine and bounded open set of $\mathbb{CP}^N_m$ containing $B_{g_{t_\infty}}(x_\infty, r)$ and $B_{g_{t_j}}(x_j, r)$ for all $j$. Then there exist $a_{j,i} \in \mathbb{C}$ and $C > 0$ such that
\[ \sigma_j = a_{j,0} + \sum_{i=1}^{N_m} a_{j,i} z_i \]
and for all $i, j$, we have
\[ |a_{i,j}| \leq C \]
and
\[ \lim_{j \to \infty} a_{j,i} = a_{\infty,i}. \]
In particular, $a_{\infty,0} \neq 0$. Since
\[ |\sigma_j|^2 h_{t_j}^m = \left( \frac{a_{j,0} + \sum_{i=1}^{N_m} a_{j,i} |z_i|^2}{1 + \sum_{i=1}^{N_m} |z_i|^2} \right) e^{-m\varphi_{t_j}} \]
is uniformly bounded above for in $B_{g_{t_j}}(x_j, r)$, $\varphi_{t_j}$ must be uniformly bounded from below in $B_{g_{t_j}}(x_j, r)$. This leads to contradiction.

□

The proof of Lemma 8.1 already implies the continuity of $\varphi$. However, we include the following two estimates as first and second order control of $\varphi$. 

\[ \int_{\mathcal{X}_{t_j}} |\sigma_j|^2 h_{t_j}^m dV_{g_{t_j}} = 1, \]
where $h_t = (\omega_t^n)^{-1}$ is the hermitian metric on the relative canonical bundle $K_{\mathcal{X}/B}$ induced by the Kähler-Einstein metrics on the fibres.
Lemma 8.2. For any relative compact \( K \subset \subset X \setminus E_{scl} \), there exists \( C_K > 0 \) such that
\[
\sup_K tr_{g_t}(\chi) \leq C_K. \tag{8.29}
\]

Proof. We will pick base points \( p_t \) for \( t \in B^o \) and we can assume that there exists \( c > 0 \) such that
\[
Vol(B_{g_t}(p_t, 1)) \geq c
\]
for all \( t \in B^o \). Let \( r_t(x) \) be the distance function from \( x \) to \( p_t \) on \( X_t \). Immediately we have
\[
|\nabla r_t|_{g_t} = 1, \quad \Delta_t r_t \leq (r_t^{-1} + 1).
\]

We define the cut-off function \( \phi_{t,R}(x) = \rho(r_t(x)) \) satisfying
\[
\rho_R(r) = 1, \text{ if } r \leq R, \quad \rho_R(r) = 0, \text{ if } r \geq 2R,
\]
where smooth nonnegative decreasing function \( \rho \) satisfies
\[
\rho_R \geq 0, \quad 0 \leq \rho_R^{-1}(\rho'_R)^2 \leq CR^{-2}, \quad |\rho''_R| \leq CR^{-2}
\]
for a fixed constant \( C > 0 \).

We let
\[
H_{t,R} = \phi_R \left( \log tr_{g_t}(\chi_t) - 3A\varphi_t \right).
\]

Straightforward calculations show that for some fixed and sufficiently large \( A >> 1 \), we have
\[
\Delta_t H_{t,R} \geq \phi_R \left( -3An - \frac{|\nabla tr_{g_t}(\chi_t)|_{g_t}^2}{tr_{g_t}(\chi_t)} + 2At r_{g_t}(\chi_t) \right) - 2Re \left( \nabla \phi_R \cdot \frac{\nabla tr_{g_t}(\chi_t)}{tr_{g_t}(\chi_t)} \right)
\]
\[
+ (\Delta_t \phi_R)(\log tr_{g_t}(\chi_t) - 3A\varphi_t)
\]
\[
\geq 2A\phi_R tr_{g_t}(\chi_t) - A(\phi_R tr_{g_t}(\chi_t))^{-1} - 4An - A \log tr_{g_t}(\chi_t).
\]

Since \( x \log x \leq e \) for \( x > 0 \) and \( \varphi_t \) is uniformly bounded in the support of \( \phi_R \) for all \( t \), we can apply the maximum principle and conclude that \( H_{t,R} \) is uniformly bounded for all \( t \in B^o \) for any fixed \( R \geq 1 \). The lemma then immediately follows.

We remark that Lemma 8.2 can also be used to prove continuity of the Weil-Petersson potentials in section 6.

Proposition 8.1. For any relatively compact \( K \subset \subset X \setminus E_{scl} \), there exists \( C_K > 0 \) such that
\[
\sup_K |\nabla \varphi_t|_{g_t} \leq C_K. \tag{8.30}
\]

Proof. The proposition can be proved by the partial \( C^0 \)-estimate and \( L^2 \)-estimate. We give an alternative proof using the maximum principle. Let \( \Delta_t \) be the Laplace operator of \( g_t \) on \( X_t \), where \( t \in B^o \). Straightforward calculations show that
\[
\Delta_t \left( |\nabla \varphi_t|_{g_t}^2 \right) = -|\nabla \varphi_t|_{g_t}^2 + |\nabla^2 \varphi_t|_{g_t}^2 + |\nabla \nabla \varphi_t|_{g_t}^2 + 2Re \left( \nabla (tr_{g_t}(\chi_t) \cdot \nabla \varphi_t)_{g_t} \right).
\]
We will pick base points $p_t$ for $t \in B^\circ$ and the cute-off function $\phi_R$ as in the proof of Lemma 8.2. By the $C^0$-estimate of $\varphi_t$, there exists $A_R > 1$ such that

$$2\|\varphi_t\|_{L^\infty(B_{g_t}(p_t, 2R))} \leq A_R$$

for all $t \in B^\circ$. In $B_{g_t}(p_t, 2R)$, we have

$$\Delta_t \left( \frac{|\nabla \varphi_t|^4_{g_t}}{A - \varphi_t} \right) = \Delta_t \left( \frac{|\nabla \varphi_t|^2_{g_t}}{A - \varphi_t} \right) + 2Re \left( \frac{\nabla |\nabla \varphi_t|^2_{g_t}}{A - \varphi_t} \cdot \nabla \varphi_t \right)_{g_t} + 2 \frac{|\nabla \varphi_t|^4_{g_t}}{(A - \varphi_t)^3}

= \frac{|\nabla \varphi_t|^2_{g_t}}{A - \varphi_t} + 2Re \left( \frac{\nabla |\nabla \varphi_t|^2_{g_t}}{A - \varphi_t} \cdot \nabla \varphi_t \right)_{g_t} + 2 \frac{|\nabla \varphi_t|^4_{g_t}}{(A - \varphi_t)^3}.$$

Also we have the following calculations because the Fubini-Study metric

$$A > C$$

for some sufficiently large $C$. By the result in [21], $tr_g(\chi_t)$ is uniformly bounded away from $E_{slc}$ and so there exists $A'_R > 0$ such that in $A'_{g_t}(p_t, 2R)$, we have

$$\Delta_t tr_g(\chi_t) \geq |tr_g(\chi_t)|^2_{g_t} - tr_g(\chi_t) - C (tr_g(\chi_t))^2.$$

By the result in [21], $tr_g(\chi_t)$ is uniformly bounded away from $E_{slc}$ and so there exists $A'_R > 0$ such that in $A'_{g_t}(p_t, 2R)$, we have

$$\Delta_t tr_g(\chi_t) \geq |tr_g(\chi_t)|^2_{g_t} - A'_R.$$

Now we let

$$H_{t,R} = \phi_R \left( \frac{|\nabla \varphi_t|^2_{g_t}}{A - \varphi_t} + B \, tr_g(\chi_t) \right), \quad K_{t,R} = \left( \frac{|\nabla \varphi_t|^2_{g_t}}{A - \varphi_t} + B \, tr_g(\chi_t) \right)$$

for some sufficiently large $C > 0$ to be determined. Then there exist $C_1, C_2, ..., C_5 > 0$ such that on $X_t'$ for all $t \in B^\circ$, we have

$$\Delta_t H_{t,R} \geq (\Delta_t \phi_R) K_{t,R} + 2Re \left( \nabla \phi_R \cdot \nabla \left( \frac{|\nabla \varphi_t|^2_{g_t}}{A - \varphi_t} \right) \right)_{g_t} + \phi_R \Delta_t \left( \frac{|\nabla \varphi_t|^2_{g_t}}{A - \varphi_t} + C tr_g(\chi_t) \right) + \phi_R \left( \frac{|\nabla \varphi_t|^2_{g_t} + |\nabla^2 \varphi_t|^2_{g_t} + |\nabla \varphi_t|^2_{g_t}}{A - \varphi_t} \right)

+ 2\epsilon \phi_R Re \left( \nabla K_{t,R} \cdot \nabla \frac{\varphi_t}{A - \varphi_t} \right)_{g_t} + 2\epsilon \phi_R |\nabla \varphi_t|^4_{g_t} + B \, \phi_R \Delta_t tr_g(\chi_t)

\geq (2 - 2\epsilon) Re \left( \nabla H_{t,R} \cdot \nabla \frac{\varphi_t}{A - \varphi_t} \right)_{g_t} + 2\epsilon \phi_R \left| \nabla \varphi_t \right|^4_{g_t} + 2\epsilon \phi_R \left| \nabla \varphi_t \right|^4_{g_t} - C_1 \left| \nabla \varphi_t \right|^4_{g_t} - C_2 \left| \nabla \varphi_t \right|^4_{g_t} - C_3

\geq (2 - 2\epsilon) Re \left( \nabla H_{t,R} \cdot \nabla \frac{\varphi_t}{A - \varphi_t} \right)_{g_t} + 2\epsilon \frac{H_{t,R}}{A - \varphi_t} - C_4 \frac{H_{t,R}}{A - \varphi_t} - C_5,$$
By applying the maximum principle, there exists $C_R > 0$ such that for all $t \in B^\circ$,
$$\sup_{X_t} H_{t,R} \leq C_R.$$
This completes the proof of the proposition. \qed

The following corollary is an immediate consequence of the uniform gradient estimate in Proposition 8.1 as $\varphi_t$ is uniformly Lipschitz in any compact set $X'_t \cap X^\circ$.

**Corollary 8.1.** $\varphi_t \in C^0(X \setminus E_{sc})$.

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