Off-shell Gauge Fields from BRST Quantization

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Abstract. We propose a construction for nonlinear off-shell gauge field theories based on a constrained system quantized in the sense of deformation quantization. The key idea is to consider the star-product BFV–BRST master equation as an equation of motion. The construction is formulated in terms of the BRST extension of the unfolded formalism that can also be understood as an appropriate generalization of the AKSZ procedure. As an application, we consider a very simple constrained system, a quantized scalar particle, and show that it gives rise to an off-shell higher-spin gauge theory that automatically appears in the parent form and properly takes the familiar trace constraint into account. In particular, we derive a geometrically transparent form of the off-shell higher-spin theory on the AdS background.
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## 1. Introduction

An off-shell field theory is by definition a theory whose equations of motion are equivalent to algebraic constraints. This implies that one can in principle solve the equations of motion and eliminate pure gauge degrees of freedom in order to obtain the unconstrained fields and gauge symmetries. Although the off-shell formulation does not actually describe dynamical equations, it can be useful from various standpoints.

In particular, an off-shell theory can encode all the information on the field content and gauge symmetries in the form adapted for introducing consistent interactions. For example, this applies to the higher-spin (HS) theories, where the relevance of the off-shell formulation in constructing nonlinear equations \[1\] (see also \[2\] for a review) has been recently realized in \[3\]. More recently, a compact and geometrically transparent form of
the nonlinear off-shell HS theory on Minkowski space was constructed by M. Vasiliev in [4].

An off-shell theory can also be regarded as an intermediate step in constructing a Lagrangian formulation. Indeed, being algebraic, the off-shell equations of motion can always be made Lagrangian by introducing Lagrange multipliers (we refer the reader to [4] for a more extensive discussion). Moreover, under some regularity assumptions, any off-shell theory can be equivalently formulated in the Lagrangian Batalin–Vilkovisky formalism [5, 6] by introducing the so-called generalized Lagrange multipliers [7].

In this paper, we propose the generating procedure for constructing off-shell gauge theories starting from a (quantized) constrained system. The underlying idea is to identify the \( \ast \)-product version of the quantum Batalin–Fradkin–Vilkovisky (BFV) [8, 9] master equation

\[
\frac{1}{2\hbar} [\Omega, \Omega]_\ast = 0
\]

for a constrained Hamiltonian system as a dynamical equation for some gauge field theory. Namely, fields are identified as coefficients in the expansion of \( \Omega \) with respect to ghost variables and possibly some extra variables present in the formalism.

The idea to interpret constraints as dynamical fields does not seem completely new. Similar approaches have been developed, e.g., in [10, 11, 12, 4, 3, 13] without using the BRST technique but by analyzing the consistency of the constraint algebra for the particular constrained systems. Somewhat analogous ideas underly the Lagrangian considerations in [14]. The general construction developed in this paper in terms of BRST theory provides a unified framework that allows considering more general constrained systems such that the symmetries present in the model are manifest.

In more technical terms, we use the BRST extention [15] of the Vasiliev unfolded formalism [16, 17, 11] that can also be understood as the non-Lagrangian version of the Alexandrov–Kontsevich–Schwartz–Zaboronsky (AKSZ) sigma model [18] with the target space being the quantum (in the sense of a \( \ast \)-product) extended phase space of a constrained system. Equations of motion for such a system can be identified as a component form of the BFV quantum master equation for an appropriately extended constrained system. The extension is the one used in the Fedosov quantization [19, 20] (or its version adapted to the case of cotangent bundles [21]) and generalized to the case of systems with constraints [22, 23, 24] (see also [25, 7] for the particular constrained systems relevant in the present context).

Expanding the theory around a given solution of the master equation gives, in particular, the linearized theory that can be naturally identified as a parent form [25, 7] of the field theory associated with some BRST first-quantized model. In the two examples explicitly considered in the paper, this BRST first-quantized system is the off-shell version of a parent system for HS fields on the flat [25] and AdS space [7] respectively.
The paper is organized as follows. In Section 2 we propose an elementary construction for off-shell gauge theories from quantum constrained systems and discuss their linearizations around particular solutions. The simplest example of a system with just one constraint is then explicitly considered.

In Section 3 we review the BRST extension of the nonlinear unfolded formalism and consistent reductions in this framework, originally described in [15]. The linearization of such theories is then briefly discussed. The general construction for off-shell gauge theories starting from quantum constrained systems is given in Section 3.4 within the BRST-extended unfolded formulation.

Section 4 is devoted to explicit examples leading to the off-shell HS gauge theories on various backgrounds. In Section 5 we propose a compact form of the off-shell HS theory on AdS space in terms of the embedding space and discuss its relation to the Vasiliev unfolded formulation.

2. Master Equation as an Equation of Motion

2.1. Off-shell gauge theories from quantum constrained systems. An interesting class of gauge theories can be obtained starting from a quantum phase space of a quantum constrained system. Suppose we are given with a constrained system quantized in the sense of deformation quantization. This implies the associative $*$-product algebra $\mathcal{A}$ depending on the quantization parameter $\hbar$. In what follows we restrict ourselves to the formal deformation quantization and therefore allow $\mathcal{A}$ to be the algebra of formal power series in $\hbar$ with the coefficients being functions on the extended phase space of the system (i.e. functions in phase space coordinates and ghost variables). The associative $*$-product on $\mathcal{A}$ reduces to the pointwise multiplication and the Poisson bracket in the $\hbar \to 0$ limit:

\[
\begin{align*}
    f * g &= fg + O(\hbar), \\
    f * g - (-1)^{|f||g|} g * f &= \hbar \{f, g\} + O(\hbar^2),
\end{align*}
\]

where $\{\cdot, \cdot\}$ denotes the Poisson bracket on the extended phase space and $|\cdot|$ denotes the Grassmann parity. $\mathcal{A}$ is also equipped with the ghost number grading denoted by $\text{gh}(\cdot)$. For simplicity we suppose that no physical fermions are present so that $|\phi| = \text{gh}(\phi) \mod 2$ for any homogeneous $\phi \in \mathcal{A}$.

Let us assume in addition that the extended phase space of the system is a bundle over a manifold $\mathcal{X}_0$ identified as the space-time manifold in what follows. Algebra $\mathcal{A}$ can be then considered as that of sections of the appropriate associated vector bundle $\mathcal{H}$ over $\mathcal{X}_0$ with the fiber being the linear space $\mathcal{H}$ of functions on the fiber (here and below we denote by $\mathcal{H}$ the vector bundle with the fiber isomorphic to a vector space $\mathcal{H}$). In what
follows we assume $\mathcal{H}$ to be (graded) finite-dimensional, i.e. that one can always find a suitable degree such that each homogeneous component is finite-dimensional.\footnote{We do not give a precise definitions here. However, in the concrete systems discussed in what follows $\mathcal{H}$ is usually an algebra of polynomials or formal power series so that it suitably decomposes into the finite-dimensional subspaces of elements with fixed homogeneity.}

Let $e_A$ be a basis in $\mathcal{H}$. Any element of $\mathcal{A}$ can be (locally on $X_0$) represented as $\chi = e_A \chi^A(x)$ where $x^\mu$ are local coordinates on $X_0$. In this representation the $\ast$-product is a bilinear bidifferential operation on sections of $\mathcal{H}$. In addition we assume that $[x^\mu, x^\nu]_\ast = 0$ where $x^\mu, x^\nu$ are considered as elements of $\mathcal{A}$. Note that relaxing this condition corresponds to the interesting possibility to describe noncommutative gauge field theories. To simplify the exposition we also assume all the bundles to be trivial unless otherwise specified or, which is the same, restrict ourselves to the local analysis. In particular, $\mathcal{A}$ can be then identified with the $\mathcal{H}$-valued functions on $X_0$.

A quantum BRST charge (more precisely, a symbol of the BRST operator) is an element $\Phi = e_A \Phi^A(x) \in \mathcal{A}$ satisfying the master equation along with the ghost number and the Grassmann parity assignments:

\begin{equation}
\left. \frac{1}{2\hbar} [\Phi, \Phi]_\ast = 0, \quad \text{gh}(\Phi) = 1, \quad |\Phi| = 1. \right. 
\end{equation}

Note that it follows from $\text{gh}(\Phi) = 1$ that $\Phi^A = 0$ for $e_A$ with $\text{gh}(e_A) \neq 1$. Under the standard regularity assumptions on the constraints the quantum BRST charge properly describes the quantum constrained system, at least at the level of deformation quantization (i.e. the algebra of quantum observables). The specification of the correct space of quantum states equipped with the inner product is a separate question which we do not discuss here.

Instead of considering the master equation as a generating equation for the constraint algebra of the quantum constrained system we are going to interpret it in terms of some classical gauge field theory. The key idea is to consider components $\Phi^A(x)$ as dynamical fields defined on the space-time manifold $X_0$ and the master equation as the equation of motion for $\Phi^A$. The field theory defined in this way is invariant under the natural gauge symmetries determined by the master equation itself:

\begin{equation}
\delta_\Lambda \Phi = \frac{1}{\hbar} [\Phi, \Lambda]_\ast, 
\end{equation}

where gauge parameter $\Lambda$ is an arbitrary element from $\mathcal{A}$ with $\text{gh}(\Lambda) = 0$ and $|\Lambda| = 0$.

The physical interpretation of the constructed gauge field theory has to do with the off-shell description of the background fields. The constraints of the system can be identified with (the generating functions for) the background fields determining e.g. configuration space geometry, background Maxwell field etc. The master equation ensuring the consistency of the constraints imposes the equations (usually equivalent to algebraic i.e. not containing derivatives with respect to $x^\mu$) on the background fields and determines their
gauge symmetries. In Section 2.2 we consider the scalar-particle system (see also the discussion in [26, 12]) where this interpretation is transparent.

Suppose $\Phi_0$ be a particular solution to (2.2). The equations of motion and gauge transformations expanded around $\Phi_0$ read as

\[
\frac{1}{\hbar}[\Phi_0, \Phi]_* + \frac{1}{2\hbar}[\Phi, \Phi]_* = 0, \quad \delta_\Lambda \Phi = \frac{1}{\hbar}[\Phi_0, \Lambda]_* + \frac{1}{\hbar}[\Phi, \Lambda]_* ,
\]

where by slight abusing notations the fluctuation around $\Phi_0$ is again denoted by $\Phi$. The terms linear in $\Phi$ determines the linearized equations of motion and gauge symmetries.

The linearized theory can be naturally interpreted as a field theory associated to a BRST first-quantized system $(\Omega, \Gamma(\mathcal{H}, \mathcal{X}_0))$ with the “space of states” being $\Gamma(\mathcal{H}, \mathcal{X}_0) \cong \mathcal{A}$ and nilpotent BRST operator $\Omega$ defined by $\Omega \phi = \hbar^{-1}[\Phi_0, \phi]_*$ for any $\phi \in \mathcal{A}$. Here and in what follows we use notation $(\Omega, \Gamma(\mathcal{H}, \mathcal{X}_0))$ for the first-quantized BRST system specified by the “space of states” $\Gamma(\mathcal{H}, \mathcal{X}_0)$ (the space of sections of the vector bundle $\mathcal{H}$ over $\mathcal{X}_0$) and the BRST operator $\Omega : \Gamma(\mathcal{H}, \mathcal{X}_0) \to \Gamma(\mathcal{H}, \mathcal{X}_0)$. Equations (2.4) can be then rewritten as

\[
\Omega \Phi + \frac{1}{2\hbar}[\Phi, \Phi]_* = 0, \quad \delta_\Lambda \Phi = \Omega \Lambda + \frac{1}{\hbar}[\Phi, \Lambda]_* ,
\]

so that their linear parts indeed take the familiar form $\Omega \Phi = 0$ and $\delta_\Lambda \Phi = \Omega \Lambda$ (see [27, 25, 28] and references therein for more details on field theories associated to the first-quantized BRST systems).

Suppose $\Omega$ be an odd nilpotent operator $\Omega : \mathcal{A} \to \mathcal{A}$ not necessarily generated by some $\Phi_0$. In this case equations (2.5) still determines a consistent gauge field theory provided $\text{gh}(\Omega) = 1$ and $\Omega$ satisfies Leinbitz rule $\Omega(\phi * \chi) = (\Omega \phi) * \chi + (-1)^{|\phi|+|\chi|} \phi * (\Omega \chi)$. This possibility along with further generalizations of the construction are discussed in more details in Section 2.3.

Note that in our approach the space of states appears to be the space $\mathcal{A}$ of functions on the extended phase space. It does not therefore coincide with the space of quantum states (if it was specified) of the starting point constrained system because the later can be (at least formally) identified with the suitable functions on the Lagrangian submanifold of the phase space (i.e. functions in only one half of the phase space coordinates). In order to describe quantum states in terms of $\mathcal{A}$ one needs additional factorization procedure which we do not discuss in this paper.

Let us also note that instead of the $*$-product description one can use the standard language of operators. Moreover, all the constructions can be also reformulated in these terms provided one is given with the suitable representation space. In particular, if variables $x^\mu$ are quantized in the coordinate representation the operators can be identified as differential operators on $\mathcal{X}_0$ with coefficients in linear operators on the “internal” space of states. Such a representation for a constraint operator of a scalar particle has been used in [12] in the related context.
All the considerations of this section remain true if one takes the classical limit by replacing $\mathcal{A}$ with the commutative algebra of the phase space functions equipped with the Poisson bracket (i.e. the classical limit of the $\ast$-product algebra). This can also be understood as replacing the quantum constrained system with the classical one (its classical limit).

2.2. The basic example. To illustrate the construction let us consider nearly the simplest constrained system, a “scalar particle” on the flat Minkowski space $\mathcal{X}_0$, with only one constraint. Let $p_\mu$ be the momenta conjugated to coordinates $x^\mu$ on $\mathcal{X}_0$ and $F(x, p)$ the constraint. In order to handle the constraint in the BRST approach we introduce Grassmann odd ghost variables $c, \pi$ with $\text{gh}(c) = 1$, $\text{gh}(\pi) = -1$ and $[c, \pi]_\ast = -\hbar$. Variables $x^\mu, p_\nu$ are assumed to carry vanishing ghost degree. The quantum BRST charge is then given by

(2.6) $\Phi = cF(x, p)$

and automatically satisfies $[\Phi, \Phi]_\ast = 0$ because $F$ is the only constraint present in the model.

According to the general strategy the dynamical fields are coefficients in the expansion of $F$ with respect to momenta $p_\mu$

(2.7) $F(x, p) = \phi_0(x) + \phi_1^\mu(x)p_\mu + \phi_2^{\mu\nu}p_\mu p_\nu + \phi_3^{\mu\nu\rho}(x) + \ldots$

and can be identified with the symmetric tensor fields on $\mathcal{X}_0$. The gauge symmetries are determined by (2.3) where in this case $\Lambda = \lambda(x;p) + c\pi\chi(x;p)$ (note that terms with nonzero ghost structure do not enter as one can easily see by counting the ghost degree).

Explicitly one gets

(2.8) $\delta_\Lambda F = \frac{1}{\hbar} [F, \lambda]_\ast + \frac{1}{2} \{F, \chi\}_\ast$.

where $\{a, b\}_\ast = a \ast b + (-1)^{|a||b|} b \ast a$ denotes the graded symmetric $\ast$-anticommutator.

Let us expand the theory around a particular solution $\Phi_0 = c\left(\frac{1}{2}p^2 + F(x;p)\right)$ describing a scalar particle on Minkowski space. In this case the gauge transformations take the form

(2.9) $\delta_\Lambda F = -p_\mu \frac{\partial}{\partial x^\mu} \lambda + \frac{1}{2} p^2 \chi + \frac{\hbar^2}{8} \frac{\partial^2}{\partial x^\mu \partial x^\nu} \chi + \frac{1}{\hbar} [F, \lambda]_\ast + \frac{1}{2} \{F, \chi\}_\ast$,

where the first three terms give a linearized gauge transformation. Note that this gauge symmetry has been originally considered in [12] from a slightly different perspective.

In order to identify the off-shell gauge field theory described by $\Phi = c(\frac{1}{2}p^2 + F(x;p))$ let us concentrate on the linearized theory. Using the linearized gauge transformation given by the first three terms in (2.9) one can always achieve $\frac{\partial}{\partial p_\mu} \frac{\partial}{\partial p_\nu} F = 0$, i.e., the tensor fields entering $F(x;p)$ can be assumed traceless. This restricts the gauge transformations to those with $\chi = 0$ and modifies the remaining gauge transformations by the appropriate projector to the traceless component.
There are two interpretations of the resulting off-shell theory. The traceless symmetric tensor fields subjected to the gauge transformation above provides the off-shell definition of the conformal HS theory \[29, 30\]. It then follows that the off-shell theory determined by (2.9) can be also considered as an off-shell description of the conformal HS fields. We will not discuss conformal HS theory and refer instead to [12], where, in particular, conformal HS theory was constructed in the analogous terms. Although in this case we only reproduced the description from [12] the advantage of our approach is that it can be uniformly extended to more general quantum constrained systems.

Another interpretation of the off-shell theory just constructed has to do with the Fronsdal HS gauge theory \[31, 32\]. Namely, we show (see Section 4.1) that the off-shell theory determined by (2.9) is equivalent through the elimination of generalized auxiliary fields to the off-shell theory for the Fronsdal HS fields in the parent form [25] (see section 3.3 for definition of generalized auxiliary fields). More precisely, in 4.1 we construct the appropriate extension of the model (2.9), which provides a geometrically transparent formulation of the (off-shell) Fronsdal HS theory. Note that the extended model can also serve as an off-shell theory for conformal HS fields.

To directly see the relation with the conventional formulation of the Fronsdal theory suppose that the following equations have been in addition imposed on $F$:

\[
(2.10) \quad \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x_\mu} F(x; p) = 0, \quad \frac{\partial}{\partial p^\mu} \frac{\partial}{\partial p_\mu} F(x; p) = 0.
\]

Together with the condition $\frac{\partial}{\partial p_\mu} \frac{\partial}{\partial p^\mu} F = 0$ this coincides with the equations of motion and the partial gauge fixing conditions for Fronsdal HS fields identified in [33]. It is important to note, however, that this does not imply that the theory determined by (2.10) along with $\frac{\partial}{\partial p_\mu} \frac{\partial}{\partial p^\mu} F = 0$ and the remaining gauge symmetry is equivalent to the on-shell Fronsdal HS theory in the strong sense (i.e. through the elimination/addition of generalized auxiliary fields).

2.3. The field theory BRST differential. It is useful to reformulate the procedure in the BRST theory terms. Here, we closely follow the non-Lagrangian BRST formulation from [25] (see also [15, 7] and references therein). Let $e_A$ be a basis in $H$. We then associate a supermanifold $M$ to the superspace $H$. To this end we assign a variable $\psi^A$ to each basis element $e_A$ and prescribe $\text{gh}(\psi^A) = 1 - \text{gh}(e_A), |\psi^A| = 1 + |e_A| \mod 2$. One then defines $M$ to be a supermanifold\(^2\) with coordinates $\psi^A$. In order to define $M$ one also

\(^2\)Note that one can either take the real basis in $H$ or the complex one. Independent fields $\psi^A$ are then to be taken real or complex accordingly. In any case $M$ is the complex supermanifold described either in terms of complex coordinates or the real coordinates and the complex structure. In order to end up with e.g. Fronsdal model with the real fields one needs to impose in addition an appropriate reality condition. We do not discuss reality conditions in the general setting assuming that we are working with the complexified versions of the respective theories. Note, however, that in the examples considered in the paper the required reality conditions are rather obvious (see e.g. [25] for the case of Fronsdal HS theory).
needs to fix the class of functions in $\psi^A$. Although most of our present considerations do not really depend on this choice, for definiteness we take smooth functions. In what follows we call $\mathcal{M}$ the supermanifold associated to $\mathcal{H}$.

Consider the field theory with fields $\psi^A$ defined on the space-time manifold $X_0$. The interpretation of $\psi^A$ depends on its ghost number. In particular, physical fields are those with vanishing ghost number. If $gh(\psi^A) \neq 0$ then $\psi^A$ should be considered as a ghost field or an antifield. The BRST differential determining the theory is given by

$$s\psi^A = \frac{1}{2\hbar}[e_B, e_C]^A_* \psi^C \psi^B (-1)^{|B|},$$

where $|B| = |\psi^B|$ and $[\phi, \chi]^A_*$ denotes the $e_A$ component of $[\phi, \chi]_*$, i.e., $[\phi, \chi]^A_*= [\phi, \chi]_* e_A$. In what follows we mainly utilize the jet-space formulation of local gauge field theories. In this formulation fields $\psi^A$ and their space-time derivatives are treated as independent coordinates on the jet space. The BRST differential is the vector field on the jet space determined by (2.11) and the condition $[s, \partial_\mu] = 0$ where $\partial_\mu$ is a total derivative (see e.g. [34] for more details).

It is useful to introduce the so-called string field $\Psi = e_A \otimes \psi^A$ which is understood as an element of $\mathcal{H} \otimes C^\infty(\mathcal{M})$ (in what follows we simply write $\Psi = e_A \psi^A$, see [25] for more details). In terms of the string field the definition of the BRST differential $s$ takes the form

$$s\Psi = \frac{1}{2\hbar}[\Psi, \Psi]_* .$$

It is also useful to expand $\Psi$ into components containing fields at given ghost degree: $\Psi = \sum_k \Psi^{(k)}$ where $\Psi^{(k)} = e_A^k \psi^{A^k}$ with $gh(\psi^{A^k}) = k$. Note that contrary to the conventional string field associated with the space of states of the first-quantized system, $\Psi$ is associated to the algebra of functions on the entire extended phase space. However, as are going to see, $\Psi$ can be naturally interpreted as a conventional string field (but associated with the different quantum constrained system) if one considers the linearized theory.

The BRST differential determines the equations of motion, the gauge transformation, and the reducibility conditions along with higher order structures of the gauge algebra. In particular, if $\psi^{A_k}$ denote component fields entering $\Psi^{(k)}$ (i.e. $gh(\psi^{A_k}) = k$) then equations of motion and the gauge transformations have the form

$$(s\psi^{A-1})|_{\psi^{A_k}=0, k \neq 0} = 0, \quad \delta \psi^{A_0} = (s\psi^{A_0})|_{\psi^{A_k}=0, k \neq 0, 1} ,$$

with ghost-number-1 fields $\psi^{A_1}$ replaced by gauge parameters $\lambda^{A_1}$ with $|\lambda^{A_1}| = |\psi^{A_1}| + 1 \mod 2$.

Note that if the theory does not contain physical fermionic fields (as we have assumed) then all ghost-number-zero fields $\psi^{A_0}$ are bosonic and can be identified with coefficients
\( \Phi^{A_0} \) in the expansion of \( \Phi \) with respect to the basis \( e_{A_0} \) of the ghost-number-one subspace in \( \mathcal{H} \). Analogously, the gauge parameters correspond to ghost-number-one fields \( \psi^{A_1} \) associated to the basis elements of the ghost-number-zero subspace. However, if one wants to consider theories containing physical fermionic fields or build the complete BV-BRST description one needs to replace the coefficients \( \Phi^{A}(x) \) in the expansion of a generic element from \( \mathcal{A} \) with the fields \( \psi^A \) and to prescribe \( gh(\psi^A) = 1 - gh(e_A) \) and \( |\psi^A| = 1 - |e_A| \) mod 2, i.e. to replace superspace \( \mathcal{H} \) with the supermanifold \( \mathcal{M} \).

The theory expanded around a particular solution to the equations of motion can also be compactly formulated in the BV-BRST terms. Namely, let \( \Psi_0 \) be a particular solution to \( d\Psi_0 + (2\hbar)^{-1}[\Psi_0, \Psi_0]_s = 0 \). The BRST differential of the theory expanded around \( \Psi_0 \) is then given by

\[
(2.14) \quad s\Psi = \Omega \Psi + \frac{1}{2\hbar}[\Psi, \Psi]_s ,
\]
where BRST operator \( \Omega \) is defined by \( \Omega \phi = \hbar^{-1}[\Psi_0, \phi]_s \) for \( \phi \in \mathcal{A} \).

As we have already seen this linearized theory can be considered as the BRST field theory associated to the first-quantized system \( (\Omega, \Gamma(\mathcal{H}, X_0)) \). Indeed, the linearized BRST differential has the familiar form \( s_0 \Psi = \Omega \Psi \). From this point of view \( \Psi \) is to be identified with the string field associated with the space of states \( \Gamma(\mathcal{H}, X_0) \). It is in this sense \( \Psi \) is related to the conventional string field used in the context of string field theory (see e.g. [28]). To be precise one also needs to adjust the ghost number grading in \( \mathcal{A} \) in order to fit the standard convention \( gh(\Psi) = 0 \).

The construction of this section can be naturally extended to a more general class of quantum constrained system. To see this let us consider the theory expanded around a particular solution to the master equation \( \Psi_0 \) which induces the map \( \Omega : \mathcal{A} \rightarrow \mathcal{A} \) by \( \Omega \phi = [\Psi_0, \phi]_s \). Together with the bilinear map induced by the \( \ast \)-commutator \( [\cdot, \cdot] : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A} \) these two maps define the differential graded Lie algebra structure on \( \mathcal{A} \). Even if \( \Omega \) can not be represented as \( \Omega = [\Phi_0, \cdot]_s \) for some \( \Psi_0 \) one can still define a consistent gauge theory determined by \( (2.5) \) or, equivalently, by the BRST differential \( (2.14) \). In particular, for the BRST field theory we construct in Section 3.4 the differential \( s\Psi = \Omega \Psi + (2\hbar)^{-1}[\Psi, \Psi]_s \) is precisely of this type, i.e. \( \Omega \) can not be represented as \( \Omega = [\Psi_0, \cdot]_s \).

More generally, one can replace the differential graded Lie algebra with the more general structure known as \( L_\infty \) algebra [35] which is specified by a collection of polylinear graded-antisymmetric operations:

\[
(2.15) \quad \Omega : \mathcal{A} \rightarrow \mathcal{A} , \quad [\cdot, \cdot]_s : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A} , \quad [\cdot, \cdot, \cdot]_3 : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A} , \quad \ldots
\]
that satisfy certain compatibility conditions generalizing those of the differential graded Lie algebra. In this case one can still consider the consistent gauge field theory determined by the following BRST differential:

\[
(2.16) \quad s\Psi = \Omega \Psi + \frac{1}{2\hbar}[\Psi, \Psi]_s + \frac{1}{6\hbar^2}[\Psi, \Psi, \Psi]_3 + \ldots
\]
Note that the nilpotency of $s$ is equivalent to the defining relations of $L_\infty$-algebra. Moreover, $L_\infty$-algebra structure is usually defined in terms of an odd nilpotent odd vector field on the associated supermanifold.

Equation (2.16) generalizes the construction to the case of quantum constrained systems described by the $A_\infty$ algebra [36]. In this case the $A_\infty$-structure is determined by the nilpotent $\Omega: A \to A$, the homotopy associative $*$-product (i.e. associative only in the $\Omega$-cohomology), and the higher order polylinear maps. The $L_\infty$-structure is then obtained from the $A_\infty$-structure by taking the graded antisymmetrization of the polylinear maps. Quantum constrained systems of this type naturally arise in quantization of some classical constrained systems (see e.g. [37] [24]).

3. Off-shell gauge theories in the BRST extended unfolded formulation

3.1. BRST extension of the non-linear unfolded formalism. In this section we briefly recall the BRST extension [15] of the Vasiliev non-linear unfolded formalism [16] [17] [1], proposed recently by G. Barnich and the present author.

Consider two supermanifolds: a supermanifold $X$ and $M$ playing the roles of (appropriately extended) space-time and the target space manifolds respectively. Let $X$ be a supermanifold equipped with a degree $gh_X(\cdot)$, an odd nilpotent vector field $d$, $gh_X(d) = 1$, and a volume form $d\mu$ preserved by $d$. Let the supermanifold $M$ be equipped with a degree $gh_M$, an odd nilpotent vector field $Q$, $gh_M(Q) = 1$. In the literature a supermanifold equipped with an odd nilpotent vector field is often called $Q$-manifold while the vector field itself is referred to as $Q$-structure [38].

The basic example for $X$ is the odd tangent bundle $\Pi T X_0$ which has a natural volume form and is equipped with the De Rham differential $d$. If $x^\mu$ are local coordinates on $X_0$ and $\theta^\mu$ are associated coordinates on the fibers of $\Pi T X_0$ then $d = \theta^\mu \frac{\partial}{\partial x^\mu}$ and one assumes in addition $gh_X(x^\mu) = 0$ and $gh_X(\theta^\mu) = 1$. Functions on $X$ are then identified with the differential forms on $X_0$ via $dx^\mu = \theta^\mu$ while the degree $gh_X$ is just a standard form degree.

Consider the manifold of smooth maps from $X$ to $M$. This space is naturally equipped with the total degree denoted by $gh(\cdot)$ and an odd nilpotent vector field $s$, $gh(s) = 1$. Indeed, it is a standard geometrical fact that any vector field on the “space-time” manifold or the “target space” manifold determines a vector field on the space of maps. More precisely, if $z^\alpha$ are local coordinates on $X$ (in the case where $X = \Pi T X_0$ coordinates $z^\alpha$ split into $x^\mu$ and $\theta^\mu$) and $\psi^A$ are local coordinates on $M$, the expression for $s$ reads

\begin{equation}
(3.1) \quad s = \int_X d\mu (-1)^{|d\mu|} \left[ d\psi^A(z) + Q^A(\psi(z)) \right] \frac{\delta}{\delta \psi^A(z)} .
\end{equation}
Vector field $s$ can be considered as a BRST differential of a field theory on $\mathcal{X}$. Indeed, the basic properties $s^2 = 0$ and $\text{gh}(s) = 1$ hold. In what follows we refer to this system as a quadruple $(\mathcal{X}, d, M, Q)$.

For the system $(\mathcal{X}, d, M, Q)$ it is easy to check using the explicit form (3.1) that
\begin{equation}
(3.2)
\nonumber
s^2 \psi^A = d\psi^A + Q^A(\psi).
\end{equation}

This equation can be taken as a definition of the BRST differential in the jet-bundle description of the theory. In this approach component fields $(\psi_p)_{\mu_1...\mu_p}$ entering $\psi^A(x, \theta) = (\psi_0)^A(x) + \theta^\mu (\psi_1)^A_{\mu}(x) + \ldots$ and their derivatives with respect to $x^\mu$ are treated as independent coordinates on the jet space.

Let $\psi^{A_k}$ denote component fields with $\text{gh}(\psi^{A_k}) = k$. Using the explicit form (3.2) of the BRST differential one finds the component form of the equations of motion and gauge symmetries
\begin{equation}
(3.3)
\nonumber
(d\psi^A + Q^A(\psi))_{\mid \psi^{A_k = 0, k\neq 0}} = 0,
\end{equation}
and
\begin{equation}
(3.4)
\nonumber
\delta^x \psi^A = (d\psi^A + Q^A(\psi))_{\mid \psi^{A_1 = \lambda A_1, \psi^{A_k = 0, k\neq 0, 1}}},
\end{equation}

where $\delta^x \psi^A$ denotes variation of $\psi^A$ under the gauge variation of its physical component fields $\psi^{A_0}$. In particular, if $\text{gh}(\psi^A) \geq 0$ for all fields then the equations of motion determine the so-called free differential algebra [39]. If one does not require $\text{gh}(\psi^A) \geq 0$ then the equations of motion can also contain some constraints.

In general, instead of $(\mathcal{X}, d, M, Q)$ one can similarly consider a fibered bundle with $\mathcal{X}$ being a base manifold, a fiber isomorphic to $M$, and the transition functions preserving $Q$-structure. In this case the field space is the space of sections of the bundle instead of $M$-valued functions. However, for the sake of simplicity we do not consider here non-trivial bundles unless otherwise specified. Note that the general construction anyway reduces to $(\mathcal{X}, d, M, Q)$ locally.

In the case where the “target” supermanifold $M$ is in addition equipped with a compatible Poisson bracket (antibracket) $\{ \cdot , \cdot \}$ and $Q = \{ S, \cdot \}$ is generated by a “master action” (“BRST charge”) $S$ satisfying the classical master equation $\{ S, S \} = 0$, one can construct a field theory master action $S$ on the space of maps. This procedure was proposed in [18] as an approach for constructing BV-BRST formulations of topological sigma models (see [40] [41] [42] [43] [44] [27] [45] [46] and references therein for further developments and applications). A generalization that also includes the Hamiltonian BRST formulation has been proposed in [47] and covers the case where $S$ is Grassmann odd and is to be interpreted as a BRST charge of the BFV-BRST formulation of the theory.

A simplest but characteristic example is provided by taking $M$ to be $\Pi \mathfrak{g}$ with $\mathfrak{g}$ being a Lie algebra and $\mathcal{X} = \Pi T\mathcal{X}_0$ with $\mathcal{X}_0$ being a space-time manifold. If $e_i$ is a basis in
the Lie algebra then $c^i$ with $|c^i| = 1$ are coordinates on $\mathcal{M}$. In addition we prescribe $gh(c^i) = 1$ and define

$$Qc^i = \frac{1}{2}[e_j, e_k]^i c^j c^k,$$

which is the standard cohomology differential for a Lie algebra $\mathfrak{g}$ with trivial coefficients. The variables $c^i$ are identified with the ghost variables in the BRST formulation of the Lie algebra cohomology. The BRST field theory described by $s$ is then a non-Lagrangian version of the Chern-Simons theory. In particular, the equations of motion are zero curvature equations for $\mathfrak{g}$-valued connection 1-form. If in addition $X_0$ is a 3-dimensional manifold and $\mathfrak{g}$ is equipped with a nondegenerate invariant inner product then the system is naturally Lagrangian and coincides with the AKSZ formulation of the standard Chern-Simons theory.

### 3.2. Linearization.

Suppose we are given with a map $\mathcal{X} \to \mathcal{M}$ defined by $\psi^A = \psi^A_0(x, \theta)$ in terms of local coordinates. Let also this map be such that $d\psi^A_0 + Q^A(\psi_0) = 0$ i.e. it determines a point on the zero locus of the differential $s$. At ghost number zero the configuration $\psi_0$ is a particular solution to the equations of motion determined by $s$. Expanding the BRST differential around a particular solution one gets

$$s\psi^A = d\psi^A + \frac{\partial R Q^A}{\partial \psi^B} \bigg|_{\psi = \psi_0} \psi^B + \frac{1}{2} \frac{\partial^2 R Q^A}{\partial \psi^B \partial \psi^C} \bigg|_{\psi = \psi_0} \psi^C \psi^B + \ldots,$$

where $\ldots$ denote terms of higher orders in $\psi^A$. In particular, the linearized theory is determined by the following linear differential

$$s_0\psi^A = d\psi^A + \frac{\partial R Q^A}{\partial \psi^B} \bigg|_{\psi = \psi_0} \psi^B.$$

The linearized theory determined by $s_0$ can be identified with the BRST field theory associated to the first-quantized system described by the BRST operator

$$\Omega \phi = d\phi + e_A^B \frac{\partial R Q^A}{\partial \psi^B} \bigg|_{\psi = \psi_0} \phi^B,$$

where $\phi = e_A^B \phi^A(x, \theta)$ is a general element of the “space of states” which is the space of functions on $\mathcal{X}$ with values in the linear space $\mathcal{H}$ identified with the tangent space to $\mathcal{M}$. More precisely, $e_A^B \phi^A(x, \theta)$ can be considered as a section of the tangent bundle to $\mathcal{M}$ pulled back by the map $\psi_0$. From this point of view the BRST differential (3.6) can be naturally understood as that of a non-linear deformation of the linear theory determined by the first-quantized BRST operator $\Omega$.

The BRST operator (3.8) has the same structure as that of a parent systems constructed in [25, 7]. More generally, one can consider a linear gauge field theory on $X_0$ whose BRST differential have the form

$$s_0\psi^A = d\psi^A + \tilde{\Omega}_B^A \psi^B,$$
where $\bar{\Omega}^A_B = \Omega^A_B(x, \theta)$ satisfies “generalized zero curvature” condition
\begin{equation}
(3.10) \quad d\bar{\Omega}^A_B + (-1)^{|A|+|C|} \bar{\Omega}^A_C \bar{\Omega}^C_B = 0,
\end{equation}
needed for nilpotency.

The formulation where the BRST differential has the form (3.9) with $\Omega^A_B$ satisfying (3.10) can be considered as a BRST extension of the linear unfolded formulation [48, 49]. Indeed, if $\bar{\Omega}$ is a 1-form (i.e. is linear in $\theta^\mu$) and $\text{gh}(\psi^A) \geq 0$ then $\bar{\Omega}$ can be considered a connection 1-form and equations of motion determined by $s_0$ take the form of a covariant constancy condition
\begin{equation}
(3.11) \quad d(\psi_p)^A + \bar{\Omega}^A_B(\psi_p)^B = 0.
\end{equation}
Here $\psi_p$ is a ghost-number-zero field entering $\psi^A = (\psi_0)^A + \theta^\mu (\psi_1)^A + \theta^\mu \theta^\nu (\psi_2)^A + \ldots$, which is identified with a $p$-form on $X_0$ with $p = \text{gh}(\psi^A)$. From this perspective parent theories constructed in [25, 7] are particular examples of theories naturally emerging in the BRST extended unfolded form.

### 3.3. Consistent reductions.
Two local gauge field theories formulated within BRST framework are naturally considered equivalent if they are related by elimination/addition of generalized auxiliary fields. Suppose that after an invertible change of coordinates, possibly involving derivatives, the set of fields $\psi^A$ splits into $\varphi^\alpha, w^\alpha, v^\alpha$ such that equations $s w^\alpha|_{w^\alpha=0} = 0$ (understood as algebraic equations in the space of fields and their derivatives) are equivalent to $v^\alpha = V^\alpha[\varphi^\alpha]$, i.e., can be algebraically solved for fields $v^\alpha$. Fields $w, v$ are then generalized auxiliary fields. The field theory described by $s$ is equivalent to that described by the reduced differential $\tilde{s}$ acting on the space of fields $\varphi^\alpha$ and their derivatives and defined by $\tilde{s}\varphi^\alpha = s\varphi^\alpha|_{w^\alpha=0, v^\alpha=V^\alpha[\varphi]}$ (see [25] for more details). In the Lagrangian framework, fields $w, v$ are in addition required to be second-class constraints in the antibracket sense. In this context, generalized auxiliary fields were originally proposed in [50]. Generalized auxiliary fields comprise both standard auxiliary fields and pure gauge degrees of freedom as well as their associated ghosts and antifields.

For BRST field theory $(X, d, M, Q)$ one easily finds generalized auxiliary fields as originating from contractible pairs for $Q$. Namely, let $w^\alpha$ be such that $w^\alpha, Qw^\alpha$ are independent constraints on $M$ determining the submanifold $\tilde{M} \subset M$. The theory $(X, d, M, Q)$ is then equivalent via elimination of generalized auxiliary fields to $(X, d, \tilde{M}, \tilde{Q})$ with $\tilde{Q} = Q|_{\tilde{M}}$. In order to see that $Q$ indeed restricts to $\tilde{M}$ it is enough to observe that $(Qw^\alpha)|_{\tilde{M}} = 0$ and $Q(Qw^\alpha) = 0$. For more details we refer to [15].

Analogously, one can consider contractible pairs $t^\alpha$ and $dt^\alpha$ in the extended space-time manifold $\tilde{X}$. Namely, suppose that $t^\alpha$ and $dt^\alpha$ are independent regular constraints determining a submanifold $\tilde{X}$. One can address the question on the relation of $(X, d, M, Q)$ and $(\tilde{X}, \tilde{d}, \tilde{M}, Q)$ where $\tilde{d} = d|_{\tilde{X}}$. These theories can not be considered equivalent as local field theories because they live on different space-time manifolds. However, if the
coordinates transversal to $\widetilde{X} \subset X$ are considered as internal degrees of freedom rather than space-time coordinates one can indeed show that respective theories are equivalent. For more details we again refer to [15]. In particular, if $X = \Pi^*X_0$ with coordinates $x^\mu, \theta^\mu$ one can consistently eliminate any pair $x^\nu, \theta^\nu$. Note that the auxiliary role of space-time coordinates was observed in [51, 52] in the context of HS theories formulated within unfolded framework.

If one is given with a particular solution $\Psi_0$ satisfying $d\psi_A^0 + Q^A(\psi_0) = 0$ then the system expanded around $\Psi_0$ can be reduced using the reduction machinery developed in [25, 7] for the free theories associated to first-quantized systems. Indeed, the linearized theory can be identified with the free field theory associated to the first-quantized system described by (3.8). Under the standard assumptions it then follows that the generalized auxiliary fields for the linearized theory are also generalized auxiliary fields for its non-linear deformation (see e.g. [7]).

3.4. Putting a quantum constrained system to a fiber. Let us consider again a constrained system quantized in the sense of deformation quantization, i.e. the associative $*$-product algebra $A$ of the extended phase space functions depending formally on $\hbar$ and equipped with the ghost number grading and the Grassmann parity. Contrary to the construction of Section 2.1 now we are going to achieve a generally covariant (in the sense of $X_0$) description of the theory. To this end we construct an AKSZ-type sigma-model by, roughly speaking, putting the quantum constrained system to the target space. Moreover, we need to change the class of the phase space functions. Namely, we replace the space-time coordinates $x^\mu$ with the formal variables $y^a$ so that $A$ consist of formal power series in $y^a$ and $\hbar$ with coefficients depending on the remaining variables. In addition, we also assume $A$ to be graded-finite dimensional.

More technically, we first consider a supermanifold associated to $A$. Let $e_A$ be a basis in $A$ and $\psi^A$ coordinates on the associated supermanifold $M$. The string field is then given by $\Psi = e_A\psi^A$. Similarly to the considerations in 2.1 the $Q$-structure on $M$ is given by

$$Q \psi^A = \bar{\Omega}^A_B \psi^B + \frac{1}{2\hbar} [e_B, e_C]^A \psi^C \psi^B (-1)^{|B|}.$$  

(3.12)

where $[f, g]_* = f \ast g - (-1)^{|f||g|} g \ast f$ is a $*$-commutator in $A$ and $\bar{\Omega} : A \rightarrow A$ a “fiber” BRST operator $\bar{\Omega} e_A = e_B \bar{\Omega}_B^A$ satisfying $\bar{\Omega}^2 = 0$, $\hbar \delta (\bar{\Omega}) = 1$, $|\bar{\Omega}| = 1$ and $\bar{\Omega} (\psi^A \chi) = (\bar{\Omega} \psi^A) \chi + (-1)^{|\psi^A|} \psi^A (\bar{\Omega} \chi)$. In terms of the string field the definition of $Q$ takes the form

$$Q \Psi = \bar{\Omega} \Psi + \frac{1}{2\hbar} [\Psi, \Psi]_*.$$  

(3.13)

Note that the construction can be naturally generalized to involve $L_\infty$-structure instead of a differential graded Lie algebra. This corresponds to taking odd nilpotent vector field $Q$ not necessarily quadratic in $\psi^A$ (see [15] for more details).
Given a supermanifold $\mathcal{X}$ equipped with the differential $d$ one can then build a BRST system $(\mathcal{X}, d, M, Q)$ whose BRST differential is determined by

$$s\Psi = d\Psi + \bar{\Omega}\Psi + \frac{1}{2\hbar} [\Psi, \Psi]_*,$$

For definiteness, we take $\mathcal{X} = \Pi T\mathcal{X}_0$ and $d$ to be de Rham differential $\theta^\mu \frac{\partial}{\partial x^\mu}$ with $x, \theta$ being coordinates on $\mathcal{X}_0$ and fibers of $\Pi T\mathcal{X}_0$. But the general considerations remain the same in the case where $\mathcal{X}$ is a general supermanifold equipped with the ghost degree, the odd nilpotent vector field $d$, $\text{gh}(d) = 1$, and the volume form. As was discussed above this also concerns the generalization to the case where $\mathcal{A}$ is identified with the fiber of a nontrivial vector bundle over $\mathcal{X}$ with the transition function preserving the $Q$-structure.

Similarly to the Chern-Simons theory example, the $Q$ structure on $M$ is nothing but the standard cohomology differential for $\mathcal{A}$ considered as a differential graded Lie algebra. Variables $\psi^A$ can also be identified with the respective “ghost” variables. However, contrary to the case of Chern-Simons theory $\psi^A$ can have arbitrary ghost degree depending on $\text{gh}(e_A)$. In particular, it follows from counting the ghost degree that the physical fields entering $\psi^A(x, \theta)$ appear to be differential forms of form degree depending on $\text{gh}(\psi^A)$ and not necessarily 1-forms as in the case of Chern-Simons theory.

Suppose we are interested in a particular $\Psi_0 = e_A\psi_0^A(x, \theta)$ satisfying $(s\Psi)|_{\Psi = \Psi_0} = 0$. Among possible particular solutions there is a class of solutions which are in some sense natural. These can be identified if one observes that equation $(s\Psi)|_{\Psi = \Psi_0} = 0$ for this system can be considered as the quantum master equation (in the $*$-product sense) for the Fedosov-like extension of the constrained system $\mathcal{A}$. Indeed, let us introduce momenta $\bar{p}_\mu, \mathcal{P}_\mu$ conjugates to $x^\mu, \theta^\mu$ so that at the quantum level

$$[x^\mu, \bar{p}_\nu]_* = \hbar\delta^\mu_\nu, \quad [\theta^\mu, \mathcal{P}_\nu]_* = -\hbar\delta^\mu_\nu,$$

and extend $\mathcal{A}$ by the $*$-product algebra generated by $x^\mu, \theta^\mu, \bar{p}_\mu, \mathcal{P}_\mu$ (in the case where $\mathcal{X}_0$ is a curved manifold one needs to consider the star product algebra arising in quantization of $T^*(\Pi T\mathcal{X}_0)$). Identifying $\Psi_0$ as a ghost-number-one phase space function one observes that the quantum master equation $\bar{\Omega}\Psi'_0 + (2\hbar)^{-1}[\Psi'_0, \Psi'_0]_* = 0$ for a quantum BRST charge $\Psi'_0 = \Psi_0 - \theta^\mu \bar{p}_\mu$ is indeed equivalent to $(s\Psi)|_{\Psi = \Psi_0} = 0$. Note that if $\bar{\Omega} = \hbar^{-1}[\Psi_0, -]_*$ for some $\Psi_0 \in \mathcal{A}$ then the master equation for $\Psi''_0 = \Psi'_0 + \bar{\Psi}_0 - \theta^\mu \bar{p}_\mu$ takes the standard form $[\Psi''_0, \Psi'_0]_* = 0$. Under the appropriate regularity assumptions one can also show that all physical quantities (representatives of the BRST cohomology) can be assumed independent on $\bar{p}_\mu$ and $\mathcal{P}_\mu$ so that it is legitimate to eliminate them from the formulation.

Given a particular solution $\Psi_0$ the BRST differential of the theory expanded around $\Psi_0$ is determined by

$$s\Psi = (s_0 + s')\Psi = d\Psi + \bar{\Omega}\Psi + \frac{1}{\hbar}[\Psi_0, \Psi]_* + \frac{1}{2\hbar}[\Psi, \Psi]_*,$$
where \( s_0 \Psi = d\Psi + \bar{\Omega} \Psi + \hbar^{-1}[\Psi_0, \Psi] \) is the BRST differential of the linearized theory. As in the general case considered above the BRST field theory determined by \( s_0 \) can be identified with that associated with the first-quantized system \( (\Omega, \Gamma(A, \mathcal{X})) \) on \( \mathcal{X} \) with the space of states being \( \Gamma(A, \mathcal{X}) \) (i.e. sections of \( A \) considered as a bundle over \( \mathcal{X} \)) and the BRST operator \( \Omega \) determined by \( \Omega \phi = d\phi + \bar{\Omega} \phi + \hbar^{-1}[\Psi_0, \phi] \). It can be more natural to interpret this system as \( (\Omega, \Gamma(A \otimes \Lambda, \mathcal{X}_0)) \) defined on \( \mathcal{X}_0 \). In this case the space of states becomes \( \Gamma(A \otimes \Lambda, \mathcal{X}_0) \), where \( \Lambda \) denotes the Grassmann algebra generated by \( \theta^\mu \).

Indeed, elements of \( \Gamma(A \otimes \Lambda, \mathcal{X}_0) \) over \( \mathcal{X}_0 \) are identified with elements of \( \Gamma(A, \mathcal{X}) \) over \( \mathcal{X} \), which, in turn, are naturally considered as differential forms on \( \mathcal{X}_0 \) with values in \( A \).

Note, that similarly to Section 2 all the present considerations also remain true if one takes the classical limit \( \hbar \rightarrow 0 \) in the expressions for \( s_0 \) and \( \Omega \) and considers \( A \) as a commutative algebra equipped with the Poisson bracket.

It is important to stress that a priori one can take arbitrary \( Q \)-manifolds \( \mathcal{M} \) and \( \mathcal{X} \) in order to construct some BRST field theory \( (\mathcal{X}, d, \mathcal{M}, Q) \). However, this theory can usually be interpreted in terms of some meaningful model if one specifies (a class of) \( \bar{\Omega} \) operators and/or particular solutions \( \Psi_0 \) involving e.g. some geometrical structures on \( \mathcal{X}_0 \). In particular, this can fix the geometry and dimension of \( \mathcal{X}_0 \) (see e.g. [51, 52] where a similar ideas have been utilized in the context of HS theory). The advantage of the BRST theory approach developed here is that it provides a guiding rule by interpreting the particular solutions for the system \( (\mathcal{X}, d, \mathcal{M}, Q) \) as the solution to the master equation for some (quantum) constrained system.

The following example provides a simple illustration and at the same time demonstrates the flexibility of the construction: let \( \mathcal{X}_0 \) be a symplectic manifold and \( A \) be the algebra of formal power series in \( y^a \) identified with the coordinates on the symplectic vector space isomorphic to a tangent space \( T_p \mathcal{X}_0 \) at \( p \in \mathcal{X}_0 \). Equation \( d\Psi_0 + (2\hbar)^{-1}[\Psi_0, \Psi_0] = 0 \) then coincides with the zero curvature condition in the Fedosov quantization [19] provided one imposes suitable additional conditions on \( \Psi_0 \). This is not surprising because as it was shown in [53] the Fedosov quantization itself can be understood as a BRST quantization of some specially prepared constrained system. The case where \( A \) also contains ghost variables corresponds to the extension of the Fedosov quantization to the case of systems with constraints.

4. Off-shell higher spin fields

As an illustration let us consider the simplest constrained system from Section 2.2 but contrary to 2.2 we do not embed the space-time manifold \( \mathcal{X}_0 \) into the extended phase space. Namely, we take the phase space to be \( T^*V \), where \( V \) is a linear \( d \)-dimensional space. The coordinates on \( T^*V \) are \( y^a, p_a, a = 1, \ldots, d \), with \( y^a \) being coordinates on \( V \).
and \( p_a \) its conjugate momenta. The Weyl star product on the phase space is determined by \([y^a, p_b]_* = \hbar \delta^a_b\).

As before the phase space and the star product is further extended by the appropriate Grassmann odd ghost variables \( c, \pi \) with \([\pi, c]_* = -\hbar \) and \( \text{gh}(c) = 1, \text{gh}(\pi) = -1 \).

According to the general construction described in 3.4 we now take \( A \) to be the algebra of formal power series in \( y^a \) with coefficients being polynomials in \( p_a, c, \pi \). One then associates supermanifold \( M \) with \( A \) and introduces the component fields so that the string field \( \Psi \) takes the form

\[
\Psi = A(x, \theta; y, p) + cF(x, \theta; y, p) + \pi R(x, \theta; y, p) + c\pi B(x, \theta; y, p).
\]

It follows from \( \text{gh}(\Psi) = |\Psi| = 1 \) that

\[
\text{gh}(A) = \text{gh}(B) = 1, \quad \text{gh}(R) = 2, \quad \text{gh}(F) = 0
\]

while the Grassmann parity is just a ghost number modulo 2.

As a supermanifold \( X \) we take \( \Pi T\mathcal{X}_0 \), with \( \mathcal{X}_0 \) being the manifold with the tangent space isomorphic \( V \). Throughout this Section we do not assume that \( T\mathcal{X}_0 \) is a trivial bundle (i.e. \( \mathcal{X}_0 \) is parallelizable) because the required generalization of the construction is straightforward. Indeed, in this case the bundle with the fiber \( \mathcal{A} \) is obviously just a vector bundle associated with \( T\mathcal{X}_0 \) (sections of \( \mathcal{A} \) are identified with tensor fields on \( \mathcal{X}_0 \)).

Consider then the BRST field theory \( (\mathcal{X}, d, \mathcal{M}, Q) \) determined by the data above. The BRST differential encoding the equations of motion and gauge symmetries is determined by \( s\Psi = d\Psi + Q\Psi = d\Psi + (2\hbar)^{-1}[\Psi, \Psi]_* \). Note that in this example we took \( \bar{\Omega} = 0 \).

However, expanding the theory around a particular solution \( \Psi_0 \) one can always introduce \( \bar{\Omega} = [\bar{\Psi}_0, \cdot]_* \) by replacing \( \Psi_0 \) with \( \Psi_0 + \bar{\Psi}_0 \).

In order to demonstrate explicitly the structure of equations of motion and gauge symmetries let us spell them out in terms of the component fields. It is useful to represent the string field as \( \Psi = \Psi(x, \theta^\mu; y, p, c, \pi) \). Physical fields are identified with ghost-number-zero component fields entering \( \Psi \) and are given by zero form \( F(x; y, p) \), 1-form \( A_\mu(x; y, p) \), 2-form \( R_{\mu\nu}(x; y, p) \), and 1-form \( B_\mu(x; y, p) \). These are particular component fields entering \( F, A, R \) and \( B \) respectively. In terms of component fields the equations of motion determined by the BRST differential read as

\[
\begin{align*}
DA + \frac{1}{2\hbar}[A, A]_* + \frac{\hbar}{8}[B, B]_* - \frac{1}{2}\{F, R\}_* &= 0, \\
DB + \frac{1}{\hbar}[A, B]_* + \frac{1}{2\hbar}[F, R]_* &= 0, \\
DF + \frac{1}{\hbar}[A, F]_* - \frac{1}{2}\{F, B\}_* &= 0, \\
DR + \frac{1}{\hbar}[A, R]_* + \frac{1}{2}\{R, B\}_* &= 0,
\end{align*}
\]

where \( \{a, b\}_* = a \ast b + (-1)^{|a||b|} b \ast a \) denotes the graded anticommutator.
The gauge symmetries determined by the BRST differential have the form \( \delta \Psi^{(0)} = d\Lambda + \hbar^{-1}[\Psi^{(0)}, \Lambda]_* \), where \( \Lambda \) is \( \Psi^{(1)} \) with the component fields (which are of ghost number one) replaced with the gauge parameters depending arbitrary on \( x \). The gauge transformations take the form

\[
\delta \Lambda A = d\lambda + \frac{1}{\hbar}[A, \lambda]_* + \frac{1}{2}\{F, \xi\}_* + \frac{\hbar}{4}[B, \chi]_* ,
\]

\[
\delta \Lambda B = d\chi + \frac{1}{\hbar}[A, \chi]_* + \frac{1}{\hbar}[B, \lambda]_* - \frac{1}{2}\{F, \xi\}_* ,
\]

\[
\delta \Lambda R = d\xi + \frac{1}{\hbar}[A, \xi]_* + \frac{1}{\hbar}[R, \lambda]_* - \frac{1}{2}\{R, \chi\}_* + \frac{1}{2}\{B, \xi\}_* ,
\]

\[
\delta \Lambda F = \frac{1}{\hbar}[F, \lambda]_* + \frac{1}{2}\{F, \chi\}_* ,
\]

where \( \Lambda = \lambda - b_0\xi + c_0b_0\chi \) (note that the term proportional to just \( c_0 \) is missing because the respective term in the string field does not enter \( \Psi^{(1)} \) as it follows from counting ghost degree). It also follows from counting ghost degree that \( \lambda, \xi \) and \( \chi \) are respectively 0, 1, and 0-forms.

If one puts \( R = B = 0 \) and restricts the gauge parameter such that \( \xi = \chi = 0 \) the system described by (4.3) and (4.4) reduces to that recently considered by M. Vasiliev in [4]. The equations of motion and gauge symmetries take the form

\[
dA + \frac{1}{2\hbar}[A, A]_* = 0 ,
\]

\[
dF + \frac{1}{\hbar}[A, F]_* = 0 ,
\]

and

\[
\delta \lambda A = d\lambda + \frac{1}{\hbar}[A, \lambda]_* , \quad \delta \lambda F = \frac{1}{\hbar}[F, \lambda]_* ,
\]

respectively.

It was shown in [4] that when expanded around the particular solution corresponding to the Minkowski space-time this system gives a non-linear off-shell description of the HS gauge fields without the trace constraint. As we are going to see, the analogous expansion of the BRST field theory \( (\mathcal{X}, d, \mathcal{M}, Q) \) properly describes off-shell HS fields with the trace constraint taken into account. In this description additional fields and gauge symmetries entering component equations (4.3) and (4.4) effectively eliminate the traces at the non-linear level.

Equations (4.5)-(4.6) can also be understood as the basic equations of the particular version of the Fedosov quantization [19]. More precisely, the version [21] adapted to the case where the phase space is a cotangent bundle. In particular, (4.5) is nothing but the vanishing curvature condition for a nonlinear connection while (4.6) is a covariant constancy condition for observable \( F \).\(^3\) As we are going to see, a more general point of

\(^3\) The author wants to thank A. Sharapov for sending his unpublished work with A. Segal where the analogous equations have been also considered in the context of conformal HS theory.
view is to consider the system (4.5)–(4.6) as a truncated version of the bigger system (4.3). In its turn this system can be understood as a component form of the BFV-BRST quantum master equation for the particular quantum constrained system.

4.1. Linearization around Minkowski space. As a simplest case to begin with we take $\mathcal{X}_0$ to be flat Minkowski space and study the linearization of the system $(\mathcal{X}, d, \mathcal{M}, Q)$ around a particular solution that describes a scalar particle on $\mathcal{X}_0$. A useful choice of the particular solution to $d\Psi_0 + (2\hbar)^{-1}[\Psi_0, \Psi_0] = 0$ is then $\Psi_0 = A_0 + cF_0$, where

$$A_0 = \theta^\mu e^a_{\mu} p_a + \omega^a_{\mu b} y^b p_a, \quad F_0 = \frac{1}{2} \eta^{ab} p_a p_b, \quad R_0 = 0, \quad B_0 = 0,$$

all the fields in nonzero ghost degree vanish, and $e, \omega, \eta$ (identified with the coefficients of the flat vielbein, the Lorentz connection, and the flat Minkowski metric on $V$) satisfy

$$de^a + \omega^a_{\mu b} e^b = 0, \quad d\omega^a_{\mu b} + \omega^a_{\nu c} \omega^\nu_{\mu b} = 0, \quad \omega^a_{\mu b} \eta^{cb} + \eta^{ac} \omega^b_c = 0.$$ 

Indeed, $\Psi_0$ coincides with the quantum BRST charge of the parent system [25] for the scalar particle on the Minkowski space (of course, one also needs to add $-\theta^\mu \bar{p}_\mu$ to take $d$ into account, see Section 3.4). Although this observation looks too obvious in the case at hand it can be helpful in identifying particular solutions in more involved situations.

The BRST differential of the linearized theory is given by $s_0 \Psi = d\Psi + \hbar^{-1}[\Psi_0, \Psi]$. Using the explicit expression for $\Psi_0$ one finds

$$s_0 \Psi = \Omega \Psi, \quad \Omega = \nabla + \sigma - cp^a \frac{\partial}{\partial y^a} - \frac{1}{2} p^2 \frac{\partial}{\partial \pi} - \frac{\hbar^2}{8} \frac{\partial^2}{\partial y^a \partial y_a} \frac{\partial}{\partial \pi},$$

where we have introduced the following notations:

$$\nabla = d - \theta^\mu \omega^a_{\mu b} y^b \frac{\partial}{\partial y^a} + \omega^a_{\mu b} p_a \frac{\partial}{\partial p_b}, \quad \sigma = -\theta^\mu e^a_{\mu} \frac{\partial}{\partial y^a}.$$ 

We consider the linearized theory described by $s_0$ as the field theory associated to the first-quantized system $(\Omega, \Gamma(A, \mathcal{X}))$. Recall that in the case at hand $\mathcal{A}$ is the algebra of formal power series in $\hbar$ with coefficients in functions in $y, p, c, \pi$ which are formal power series in $y$ and polynomials in the rest of variables.

To see that $s_0$ indeed describes off-shell version of Fronsdal HS theory let us reduce the system $(\Omega, \Gamma(\mathcal{A}, \mathcal{X}))$ to that with the traceless fields. Namely, we take as a degree homogeneity in $\pi$ so that $\Omega = \Omega_{-1} + \Omega_0$ with $\Omega_{-1} = -(\frac{1}{2} p^2 + \frac{\hbar^2}{8} \Box) \frac{\partial}{\partial \pi}$ (see [25, 7] for details on the consistent reductions in homological terms). Cohomology of $\Omega_{-1}$ in $\mathcal{A}$ is concentrated in degree zero and can be identified with the subspace $\mathcal{E} \subset \mathcal{A}$ of $\pi$-independent elements annihilated by $T = \frac{\partial}{\partial y^a} \frac{\partial}{\partial y_a}$ (i.e. traceless elements). This is the standard fact in zeroth degree in $\hbar$. In order to show this to all orders in $\hbar$ one needs to subtract traces order by order in $\hbar$.

Because the cohomology is concentrated in one degree only the reduction is straightforward and one immediately arrives at the reduced system $(\tilde{\Omega}, \Gamma(\mathcal{E}, \mathcal{X}))$ with $\tilde{\Omega}$ being $\Omega_0$.
projected to $\mathcal{E}$

\begin{equation}
\widehat{\Omega} = \nabla + \sigma - c {\mathcal{P}}'_T p^a \frac{\partial}{\partial p^a}.
\end{equation}

Here $\mathcal{P}'_T$ denotes the projector to $\mathcal{E}$ defined as follows: for $\phi = \phi_0 + (p^2 + \frac{\hbar^2}{8} \Box)\phi'$ with $T\phi_0 = 0$ one defines $\mathcal{P}'_T\phi = \phi_0$. Note that in the classical limit $\hbar \to 0$ this $\mathcal{P}'_T$ reduces to the standard projector $\mathcal{P}_T$ to the traceless component.

If one takes $\Psi(0) = \theta^\mu A_\mu + cF$, the equations of motion $\widehat{\Omega}\Psi(0) = 0$ of the associated field theory take the form

\begin{equation}
(\nabla + \sigma)A = 0, \quad (\nabla + \sigma)F = -\mathcal{P}'_T p^a \frac{\partial}{\partial y^a} A.
\end{equation}

while the gauge symmetry $\delta\lambda \Psi(0) = \widehat{\Omega}\Psi^{(1)}$ (with ghost-number-one fields in $\Psi^{(1)}$ replaced with the gauge parameters) read as

\begin{equation}
\delta A = (\nabla + \sigma)\lambda, \quad \delta F = -\mathcal{P}'_T p^a \frac{\partial}{\partial y^a} \lambda,
\end{equation}

where $\lambda = \lambda(x; y, a)$ is a gauge parameter satisfying $T\lambda = 0$.

The system can be consistently restricted to $(\Omega_{\text{on-shell}}, \Gamma(\mathcal{E}_{\text{on-shell}}, \mathcal{X}))$, with

\begin{equation}
\Omega_{\text{on-shell}} = \nabla + \sigma - c p^a \frac{\partial}{\partial y^a}
\end{equation}

and $\mathcal{E}_{\text{on-shell}} \subseteq \mathcal{E}$ being the subspace of elements annihilated by

\begin{equation}
S = \frac{\partial^2}{\partial p_a \partial y^a}, \quad \Box = \frac{\partial^2}{\partial y_a \partial y^a}.
\end{equation}

Note that projector is not anymore needed because $p^a \frac{\partial}{\partial y^a} \phi$ belongs to $\mathcal{E}_{\text{on-shell}}$ provided $\phi \in \mathcal{E}_{\text{on-shell}}$.

Contrary to the equivalent reduction considered above this restriction imposes dynamical equations. Namely, the system $(\Omega_{\text{on-shell}}, \Gamma(\mathcal{E}_{\text{on-shell}}, \mathcal{X}))$ explicitly coincides with so-called intermediate system originally constructed in [25], where it was shown to properly describe Fronsdal HS theory on the Minkowski space. One then concludes that the theory determined by $s\Psi = \Omega\Psi + (2\hbar)^{-1}[\Psi, \Psi]_{\ast}$ indeed describes the non-linear deformation of the linear off-shell HS theory on the Minkowski space. At the level of the associated field theory this can be easily observed as follows: if $A, F, \lambda$ in (4.13) and (4.14) in addition satisfy $\Box A = SA = 0, \Box F = SF = 0,$ and $\Box \lambda = S\lambda = 0$ then (4.13) and (4.14) are just equations of motion and gauge symmetries of the Fronsdal theory in the intermediate form [25].

The off-shell theory described by $s_0$ is equivalent to the off-shell theory constructed in Section 2.2 through the elimination of generalized auxiliary fields. Indeed, it was shown in [25] that if the BRST operator has the form (4.10) then it describes the parent system
constructed for the first-quantized BRST system with
\begin{equation}
\Omega_{\text{non-extended}} = -cp^\mu \frac{\partial}{\partial x^\mu} - \left( p^2 + \frac{\hbar^2}{8} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x_\mu} \right) \frac{\partial}{\partial \pi}
\end{equation}
(here, \(x^\mu\) are standard flat coordinates on Minkowski space) which is the BRST operator describing the linearized off-shell theory considered in \[2.2\]. More precisely, the BRST operator encodes the linearization of gauge transformations \(2.9\).

Several comments are in order: one can equally consider the classical version of the system \((\mathcal{X}, d, M, Q)\). This gives in general different linearized off-shell system which can be obtained by taking \(\hbar \to 0\) limit. In fact both linearized systems coincide on-shell because the quantum corrections to \(\Omega\) involve operator \(\Box\) that vanishes on-shell.

Another way to see that \((\Omega, \Gamma(\mathcal{A}, \mathcal{X}))\) describes off-shell HS fields in Minkowski space is to incorporate the constraints \(\Box, S\) enforcing dynamical equations in the BRST operator. More precisely, one needs to introduce new ghost pairs \(c_0, b_0\) and \(c^\dagger, b\) (extending therefore \(\mathcal{A}\) to \(\mathcal{A}^T\)) and to consider extended BRST operator
\begin{equation}
\Omega^T = \Omega + c_0 \Box - c^\dagger S + \text{terms cubic in ghosts}.
\end{equation}

One then finds that \(\Omega^T\) is the BRST operator of the parent system \[25\] provided one redefines constraints in order to get rid of the term proportional to \(\hbar^2 \Box\) and identify \(p\) with \(a\) and \(\frac{\partial}{\partial p}\) with \(-a^\dagger\). The only difference is that in \[25\] oscillators \(a, a^\dagger\) are represented on polynomials in \(a^\dagger\) and the representation for some ghost pairs is also chosen differently.\(^4\) Nevertheless, one can directly check that the system \((\Omega^T, \Gamma(\mathcal{A}^T, \mathcal{X}))\) is just a different realization of the same parent system. In fact, the relation between these realizations can be understood in terms of the appropriate automorphism of the \(sp(4)\) algebra (see \[25\]) inducing the exchange \(a \to \frac{2}{\partial a}, \frac{\partial}{\partial a} \to -a\) in the representation. We then conclude that the system \((\Omega, \Gamma(\mathcal{A}, \mathcal{X}))\) is a truncated version of the parent system for Fronsdal HS fields. In particular, \(\Omega\) can be directly obtained from the parent BRST operator by redefining constraints and dropping the terms with constraints \(\Box, S\) and the respective ghost variables.

The off-shell theory determined by \(s_0\) can be also understood as an off-shell formulation not only for the Fronsdal model. Indeed, since it is related through the elimination of generalized auxiliary to the off-shell theory from Section \[2.2\] it can be also used to describe conformal HS fields.

4.2. Linearization around arbitrary background. Let us now turn to the case where \(\mathcal{X}_0\) is not necessarily a Minkowski space and assume for the moment that \(\mathcal{X}_0\) is a general (pseudo)Riemannian manifold. As before local coordinates on \(\mathcal{X}_0\) are denoted by \(x^\mu\). In this more general setting one can also address the question on particular solutions
\(^4\)The choice of representation for Grassmann odd ghost variables is not really important because all “good” ones are equivalent.
to (4.3). It is again enough to restrict to the class of solutions with $R = B = 0$ so that the remaining equations are (4.5) and (4.6).

In order to analyze equations (4.5) and (4.6) for general $X_0$ it is useful to prescribe the following degrees to the variables

\[(4.19) \quad \deg x = \deg \theta = \deg p = 0, \quad \deg y = 1, \quad \deg \hbar = 1,\]

so that the $*$-product carries vanishing degree. Let $A = \sum_n A[n]$ with $\deg A[n] = n$. Suppose that the following boundary conditions are imposed on $A$

\[(4.20) \quad A[0] = \theta^\mu e^a_\mu p_a, \quad A[1] = \theta^\mu \omega^b_{\mu a} y^a p_b,\]

where $e^a_\mu$ is assumed nondegenerate and $e, \omega$ are compatible, i.e. $de^a + \omega^b_{\mu a} e^a_\mu = 0$. The geometric interpretation of $e, \omega$ is standard: $e$ is a vielbein (i.e. it determines an isomorphism between the tangent bundle and the bundle with the fiber $V$) and $\omega$ is a connection 1-form. One has the following

**Proposition 4.1.** [19, 21] Given a nondegenerate vielbein $e$ and a connection $\omega$ there exists $A = \theta^\mu A_\mu(x; y, p)$ satisfying $dA + (2\hbar)^{-1}[A, A]_* = 0$ and boundary conditions (4.20). Under the additional condition $y^a e^b_\mu \frac{\partial}{\partial y^a} A[n] = 0$ for $n \geq 2$ the solution is unique. Any other solution satisfying the same boundary condition can be obtained by a gauge transformation.

**Remark 4.1.** For any $e, \omega$ the solution can always be taken linear in $p$. In particular such a solution also satisfies the classical version of equation (4.5), i.e., with the $*$-commutator replaced with the Poisson bracket.

A geometrical meaning of this statement is that any background geometry described by a vielbein and a (not necessary flat) torsion-free linear connection can be described by a flat non-linear connection.

Suppose that we have a particular solution $A$ satisfying (4.5) and (4.20). Let us now turn to the equation (4.6).

**Proposition 4.2.** [19, 21] Let $F[0](x; p)$ be an arbitrary polynomial in $p$ with coefficients being tensor fields on $X_0$. There exists unique $F(x; y, p)$ satisfying $dF + \hbar^{-1}[A, F]_* = 0$ and the boundary condition

\[(4.21) \quad F(x; 0, p) = F[0](x; p).\]

**Remark 4.2.** If $A$ is linear in $p_a$ then the Proposition also applies to the Poisson bracket counterpart of equation (4.6). Moreover, in this case if $F[0]$ is homogeneous in $p$ then $F$ satisfying the Poisson bracket version of the Proposition is also homogeneous in $p$ of the same order.
Suppose $A_0, F_0$ be a particular solution to \((4.5)-(4.6)\) subjected to the boundary condition
\[(4.22)\]
\[
A_0 = e^a p_a + \omega_0^a y^b p_a + \ldots, \quad F_0 = \frac{1}{2} \eta_0^{ab} p_a p_b + \ldots,
\]
where $\ldots$ denote higher degree terms and $\omega, \eta$ in addition satisfy $\omega_0^a \eta_{cb} + \eta_{ac} \omega_0^b = 0$. By expanding the theory around $\Psi_0 = A_0 + cF_0$ one arrives at the theory described by the BRST differential $s\Psi = d\Psi + h^{-1}[\Psi_0, \Psi] + (2h)^{-1}[\Psi, \Psi]$. This theory can be considered as a natural generalization of the off-shell HS theory constructed in Section 4.1 to the case where the background manifold is an arbitrary (pseudo)Riemannian manifold. However, it is important to note that even a linearization of this theory has, in general, nothing to do with the conventional Fronsdal HS theory because Fronsdal HS fields can live only on constant curvature spaces, i.e. there should be obstructions in putting the theory on-shell.

In the case where $\mathcal{X}_0$ is the constant curvature space one can easily construct a particular solution and consider the respective linearized theory. However, we do not take this way in the present paper. Instead, in Section 5 we describe the off-shell HS theories on constant curvature backgrounds in terms of the embedding space.

5. Off-shell HS fields on AdS in terms of the embedding space

5.1. Non-linear off-shell HS fields on AdS. Following [7] let $Y^A$ be coordinates on $d + 1$-dimensional pseudo-Euclidean embedding space. In the standard basis let the embedding space metric have the form $\eta_{AB} = diag(- + + \ldots -)$. We then consider the phase space with coordinates $Y^A, P_A$ extended by ghost variables $c, \mu$ and their associated ghost momenta $\pi, \rho$. The ghost degree is introduced by prescribing $\text{gh}(c) = \text{gh}(\mu) = 1$, $\text{gh}(\pi) = \text{gh}(\rho) = -1$, and vanishing ghost degree to $Y, P$.

Let $\mathcal{A}$ be the algebra of functions on the extended phase space equipped with the Weyl star product determined by the basic commutation relations:
\[(5.1)\]
\[
[P_A, Y^B] = -\hbar \delta_A^B, \quad [\pi, c] = -\hbar, \quad [\rho, \mu] = -\hbar.
\]

The quantum BRST charge describing the scalar particle on AdS is given by
\[(5.2)\]
\[
\bar{\Psi}_0 = c \frac{1}{2} P^A P_A + \mu Y^A P_A + 2\mu c \pi,
\]

Note that the constraint $YP$ is defined up to adding a constant terms without affecting the constraint algebra and hence the nilpotency of $\bar{\Psi}_0$. This quantum BRST charge and its generalization for HS particles and tensionless strings on AdS are known in the literature (see e.g. [54]). Note that if at the quantum level one directly impose the constraints on wave functions one reproduces the well-known approach from [32].

Now we are going to put the constrained system to the target space. To this end we follow the construction of Section 3.4 with $\mathcal{A}$ and $\bar{\Psi}_0$ described above. More specifically, we take $\mathcal{A}$ to be the $\ast$-product algebra of functions in $Y, P, c, \pi, \mu, \rho$ that are formal power
series in $Y$ and polynomials in the rest of the variables. Besides the $*$-product we also
equip $A$ with the BRST operator determined by $\bar{\Psi}_0$. The associated supermanifold $M$ is
then equipped with the following $Q$-structure:

$$Q\Psi = \bar{\Omega}\Psi + \frac{1}{2\hbar}[\Psi, \Psi]_s, \quad \bar{\Omega}\phi = \frac{1}{\hbar}[\bar{\Psi}_0 + \phi]_s. \quad (5.3)$$

As a $Q$-manifold $X$ we take an odd tangent bundle over the $AdS_d$ space $X_0$ equipped
with the De Rham differential. As before local coordinates on $X_0$ and the fibers of $\Pi T X_0$
are denoted by $x^\mu$ and $\theta^\mu$ so that $d = \theta^\mu \frac{\partial}{\partial x^\mu}$. We then consider a BRST field theory
$(X, d, M, Q)$. As we are going to see it describes the off-shell HS fields on $AdS_d$ space
in terms of the $d + 1$-dimensional embedding space.

5.2. Linearization. Similarly to the flat case considered in Section 4.1 equation of motion for physical fields can be identified with the master equation of the parent theory for
a scalar particle on $AdS_d$. A natural choice for a particular solution to $d\Psi_0 + \bar{\Omega}\Psi_0 +$
$(2\hbar)^{-1}[\Psi_0, \Psi_0]_s = 0$ is then to take $\Psi_0$ describing the parent system \[7\] for a particle on
AdS space. Namely, a particular solution reads as:

$$\Psi_0 = \theta^\mu e^A_\mu P_A + \theta^\mu \omega^A_\mu (Y^B + V^B)P_A + \mu V^A P_A \quad (5.4)$$

so that $\Psi_0 + \bar{\Psi}_0$ indeed coincides with the quantum BRST charge of the parent system
from \[7\] (strictly speaking one should also add $-\theta^\mu \bar{\rho}_\mu$, see Section 3.4). It satisfies $d\Psi_0 +$
$\bar{\Omega}\Psi_0 + (2\hbar)^{-1}[\Psi_0, \Psi_0]_s = 0$ provided $e, \omega, V$ (identified respectively with a vielbein, AdS
connection, and a given section of the vector bundle with $d + 1$-dimensional fiber) satisfy

$$d\Psi^A + \omega^A_B V^B = e^A, \quad de^A + \omega^A_B e^B = 0, \quad d\omega^A_B + \omega^A_C \omega^C = 0, \quad V^A V_A + l^2 = 0, \quad (5.5)$$

where $l$ is the radius of $AdS_d$. In addition one has to require $e$ to have a maximal rank,
i.e. rank$(e) = d$.

Note that both $\bar{\Psi}_0$ and $\bar{\Psi}$ are frame-independent provided the components of $e, \omega, V$
transform as those of a vielbein, a connection, and a section respectively. In particular
one can alway chose the frame where $V^A = \text{const}^A$. In this case one can redefine $\bar{\Psi}_0$, $\Psi_0$
according to $\bar{\Psi}_0 \to \bar{\Psi}_0 + \mu V^A P_A$ and $\Psi_0 \to \Psi_0 - \mu V^A P_A$ in order to completely
distinguish the “space-time” and the “target-space” parts of the BRST differential.

Linearizing $(X, d, M, Q)$ around $\Psi_0$ one gets the following linear BRST differential

$$s_0 \Psi = \left[\nabla + \sigma - c P^A \frac{\partial}{\partial Y^A} + \mu(P_A \frac{\partial}{\partial P_A} - (Y^A + V^A)\frac{\partial}{\partial Y^A} - \frac{1}{2}(P_A P^A)\frac{\partial}{\partial \pi} - (Y^A + V^A)P_A \frac{\partial}{\partial \rho} - 2\mu c \frac{\partial}{\partial c} + 2\mu \pi \frac{\partial}{\partial \pi} - 2c \pi \frac{\partial}{\partial \rho} + \hbar^2(\ldots)\right] \Psi, \quad (5.6)$$

where

$$\nabla = d - \omega^A_B Y^A \frac{\partial}{\partial Y^B} + \omega^A_B P_B \frac{\partial}{\partial P_A}, \quad \sigma = -e^A \frac{\partial}{\partial Y^A}. \quad (5.7)$$
and $(\ldots)$ denote some operator whose explicit form is not needed for the moment.

This differential can be identified as that of the field theory associated to the first-quantized system $(\Omega, \Gamma(A, X))$ with the string field $\Psi$ and the BRST operator determined by $s_0 \Psi = \Omega \Psi$. Note that the nilpotency of $\Omega$ is guaranteed by the nilpotency of $s_0$ which is in turn a consequence of the nilpotency of the original non-linear BRST differential. As we are going to see this first-quantized system can be considered as the parent system (in the sense of [25, 7]) describing off-shell HS fields on AdS space.

Similarly to the flat case the easiest way to see this is to reduce the system described by (5.6) further. We do the necessary reduction in two steps. First we take as a degree homogeneity in $\mu$ so that $\Omega$ decomposes as $\Omega = \Omega_{-1} + \Omega_0$ with

$$\Omega_{-1} = \mu(h - 2c \frac{\partial}{\partial c} + 2\pi \frac{\partial}{\partial \pi}), \quad h = P^A \frac{\partial}{\partial P^A} - (Y^A + V^A) \frac{\partial}{\partial Y^A}. \quad (5.8)$$

It was shown in [7] that the cohomology of operators of this type in the space of formal power series in $Y$ is concentrated in degree zero and hence is given by a subspace $\hat{E} \subset A$ of elements annihilated by $h - 2c \frac{\partial}{\partial c} + 2\pi \frac{\partial}{\partial \pi}$. The reduction is then straightforward and gives the system $(\hat{\Omega}, \Gamma(\hat{E}, X))$ with

$$\hat{\Omega} = \nabla + \sigma - c P^A \frac{\partial}{\partial Y^A} - \frac{1}{2} (P^A P^A) \frac{\partial}{\partial \pi} - (Y^A + V^A) P_A \frac{\partial}{\partial \rho} - 2c \pi \frac{\partial}{\partial \rho} + h^2 (\ldots). \quad (5.9)$$

The next step is to take as a degree homogeneity in $\pi$ and $\rho$. BRST operator $\hat{\Omega}$ then decomposes as $\hat{\Omega} = \hat{\Omega}_{-1} + \hat{\Omega}_0$ with

$$\hat{\Omega}_{-1} = -\frac{1}{2} (P^A P^A) \frac{\partial}{\partial \pi} - (Y^A + V^A) P_A \frac{\partial}{\partial \rho} + h^2 (\ldots). \quad (5.10)$$

It is not difficult to compute cohomology of $\hat{\Omega}_{-1}$. Indeed, the first term factors out traces (elements proportional to $P^2$) and the second one allows to completely control the dependence of representatives on $P^A V_A$ (i.e. on a component of $P$ parallel along $V$). Let us introduce the following notations

$$\Box = \frac{\partial}{\partial Y^A} \frac{\partial}{\partial Y^A}, \quad T = \frac{\partial}{\partial P^A} \frac{\partial}{\partial P^A}, \quad S = \frac{\partial}{\partial P^A} \frac{\partial}{\partial Y^A}, \quad \bar{S}^\dagger = (Y^A + V^A) \frac{\partial}{\partial P^A}, \quad S^\dagger = P^A \frac{\partial}{\partial Y^A}. \quad (5.11)$$

for some of generators of $sp(4)$ algebra identified in [25, 7]. Leaving details of the proof to the Appendix [A] we have

**Proposition 5.1.** Cohomology of $\hat{\Omega}_{-1}$ in the space of formal power series in $Y$ and polynomials in variables $P$ and ghosts is given by

$$H^0(\hat{\Omega}_{-1}, \mathcal{E}) \cong \mathcal{E}, \quad H^n(\hat{\Omega}_{-1}, \mathcal{E}) = 0 \quad n \neq 0, \quad (5.12)$$

where $\mathcal{E} \subset \ker \hat{\Omega}_{-1}$ is the subspace of $\mu, \pi, \rho$-independent elements from $A$ satisfying

$$T \phi = 0, \quad S^\dagger \phi = 0, \quad (h - 2c \frac{\partial}{\partial c}) \phi = 0. \quad (5.13)$$
Note that the subspace \((5.13)\) is particularly convenient for our purposes but, in general, one can use different identification of cohomology as a subspace in \(\ker \tilde{\Omega}_{-1}\). Using the reduction technique from \([25]\) one then finds that the system can be reduced to \((\tilde{\Omega}, \Gamma(\mathcal{E}, X))\). Because \(\Omega_{-1}\)-cohomology is concentrated in one degree only the construction of \(\tilde{\Omega}\) is straightforward and gives:

\[
(5.14) \quad \tilde{\Omega} = \nabla + \sigma - P_E S^\dagger.
\]

Here, \(P_E\) denotes the projector to \(E\) defined as follows: for \(\phi = \phi_0 + \phi'\) with \(\phi_0 \in E\) and \(\phi' \in \text{Im} \tilde{\Omega}_{-1}\) one defines \(P_E \phi = \phi_0\) (see Appendix A for more details).

The physical component of the string field is \(\Psi^{(0)} = \theta^\mu A_\mu(x; Y, A) + cF(x; Y, A)\), where we have introduced physical fields \(A\) and \(F\) identified with the 1-form and 0-form on \(X_0\) respectively. The equation of motion and gauge symmetries of the associated field theory read as

\[
(5.15) \quad (\nabla + \sigma)A = 0, \quad (\nabla + \sigma)F = -P_E S^\dagger A,
\]

\[
(5.16) \quad \delta_\lambda A = (\nabla + \sigma)\lambda, \quad \delta_\lambda F = -P_E S^\dagger \lambda,
\]

where \(\lambda\) is the gauge parameter.

In fact it can also be useful to consider this first-quantized system without the trace constraint, i.e. with \(\mathcal{E}\) being the subspace of \(\pi, \rho\)-independent elements from \(\tilde{\mathcal{H}}\) satisfying \(S^\dagger \phi = (h - 2c\frac{\partial}{\partial c})\phi = 0\) only and the BRST operator given by \(\tilde{\Omega} = \nabla + \sigma - cS^\dagger\). The advantage is that no projectors are needed but at the same time the system does not appear naturally as a linearization of some consistent non-linear system. Reducing this system further to cohomology of \(cS^\dagger\) one arrives at the first-quantized system describing the linearized unfolded off-shell HS theory equivalent to that from \([3]\).

The off-shell system \((\tilde{\Omega}, \Gamma(\mathcal{E}, X))\) can obviously be restricted to the first-quantized system \((\Omega_{\text{on-shell}}, \Gamma(\mathcal{E}_{\text{on-shell}}, X))\), with \(\mathcal{E}_{\text{on-shell}} \subset \mathcal{E}\) being the subspace of elements satisfying the additional conditions \(\Box \phi = S \phi = 0\). Note that in the restricted \(\Omega_{\text{on-shell}}\) the projector is not anymore needed because for \(\phi\) satisfying \(\phi \in \mathcal{E}_{\text{on-shell}}\) one has \(S^\dagger \phi \in \mathcal{E}_{\text{on-shell}}\). The resulting system then explicitly coincides with the so-called intermediate system \([7]\) describing Fronsdal HS fields on AdS space. It was shown in \([7]\) that the field theory associated to \((\Omega_{\text{on-shell}}, \Gamma(\mathcal{E}_{\text{on-shell}}, X))\) is indeed equivalent to the Fronsdal HS gauge theory via elimination/addition of generalized auxiliary fields. In particular, reducing further to the cohomology of \(cS^\dagger\) one arrives \([7]\) at the familiar unfolded formulation \([55, 56]\).

5.3. Topological HS theory. In Section 4.1 and 5.2 we have seen how the linearized off-shell HS theory can be put on-shell in the case of flat and AdS space respectively. Putting the non-linear HS theory on-shell implies constructing the interacting HS theory. At the level of equations of motion this problem was solved by M. Vasiliev \([16, 11]\) for HS fields on AdS space. Here, we do not discuss the entire Vasiliev construction. Instead,
we restrict to the sector of HS connections only and show that the truncated system can be easily put on-shell giving the “topological” HS theory whose equations of motion are zero-curvature equations for the HS connection taking values in the HS algebra.

To this end consider the linearized off-shell theory determined by the BRST differential (5.6) but replace $A$ with the algebra of polynomials in all the variables including $Y$ in contrast to the formal power series in $Y$ considered in the previous section (see also [7]). It was shown in [7] that in the on-shell version, only “one half” of the fields survives if one restricts to polynomials. Indeed, the HS connections are represented by polynomials while the HS curvatures are represented by the formal power series in $Y$-variables so that dropping formal power series indeed corresponds to putting HS curvatures to zero.

In the space of polynomials it is legitimate to redefine $Y$-variables according to $Y^A A + V^A → Y^A$. The expression for the BRST operator then takes the form:

$$\Omega = \nabla - c^A P^A \frac{∂}{∂Y^A} + \mu(P^A \frac{∂}{∂P^A} - Y^A \frac{∂}{∂Y^A}) +$$

$$- \frac{1}{2}(P^A P^A) \frac{∂}{∂π} - (Y^A P^A) \frac{∂}{∂ρ} - 2\mu(c \frac{∂}{∂c} - π \frac{∂}{∂π}) - 2cπ \frac{∂}{∂ρ} + \hbar^2(\ldots).$$

If one restricts to polynomials the system can be put on-shell in a different way. Namely, let $c_0, b_0$ be the new Grassmann odd ghost variables with $gh(c_0) = 1$ and $gh(b_0) = -1$ and let

$$\Omega' = \Omega + c_0 Y^A \frac{∂}{∂P^A} - \frac{1}{2}(Y^A Y^A) \frac{∂}{∂b_0} + \text{ghost terms}$$

where “ghost terms” are terms cubic in ghosts and derivatives with respect to ghosts, needed to maintain nilpotency. The adapted version of the arguments given in [7] immediately show that the first-quantized system determined by $\Omega'$ reduces to $(\Omega', \Gamma(h ⊕ \bar{h}, X))$ with $\Omega' = \nabla$, $h$ being the space of totally traceless polynomials in $Y, A$ described by the rectangular Young tableaux, and $\bar{h}$ being the space of elements of the form $\bar{φ} = μcc_0φ$ with $φ \in h$.

To see this let us reduce the theory to the cohomology of the “fiber” part $Ω' = Ω' - \nabla$. First we reduce to the cohomology of $Y^2 \frac{∂}{∂b_0} + A^2 \frac{∂}{∂π} + AY \frac{∂}{∂ρ}$, which can be identified with $ρ, π, b_0$-independent totally traceless elements, and then to the cohomology of the remaining operators. Because the remaining operators $Y^A \frac{∂}{∂P^A}, P^A \frac{∂}{∂P^A} - Y^A \frac{∂}{∂Y^A}$, and $P^A \frac{∂}{∂Y^A}$ form the standard representation of $sl(2)$ on polynomials the cohomology can indeed be identified with the subspace $h ⊕ \bar{h}$, i.e. $sl(2)$-invariants in vanishing and top ghost degrees. Because physical fields are associated to elements of vanishing ghost number only it follows that there are no physical fields associated to $\bar{h}$ and therefore $(Ω', Γ(h ⊕ \bar{h}, X))$ indeed properly describes linearized HS connections on AdS.

One then observes that the linearized BRST differential determined by $Ω'$ can also be identified as a linearization of some non-linear BRST differential. Indeed, let us extend the $A$ and the Weyl $*$-product (5.1) with the additional ghost variables $c_0, b_0$ such that
\[ [b_0, c_0]_* = -\hbar. \] One then builds the BRST field theory \((\mathcal{X}, d, \mathcal{M}', Q')\), where \(\mathcal{M}'\) and \(Q'\) are, respectively, the associated supermanifold and the \(Q\)-structure determined by

\[
Q\Psi' = \Omega'\Psi' + \frac{1}{2\hbar} [\Psi', \Psi']_*, \quad \Omega'\phi = \frac{1}{\hbar} [\Psi', \phi]_* ,
\]

where

\[
\Psi'_0 = c_2 P^A P_A + \mu Y^A P_A + c_0 \frac{1}{2} Y^A Y_A + 2\mu c\pi - 2\mu c_0 b_0 - c c_0 \rho .
\]

The BRST operator \(\Omega'\) corresponds then to the following choice of the particular solution \(\Psi'_0 = \Theta^\mu \omega_{\mu B} Y^B P_A\) in the sense that \(\Omega' = d + \Omega' + \hbar^{-1} [\Psi'_0, \cdot]_*\).

The “fiber” part \(\Psi'_0\) given by (5.20) is the standard quantum BRST charge for the system with three constraints forming the \(sl(2)\) algebra. In the adapted notations \(Y^A = Y^A\) and \(Y^2 = P^A\) the constraints take the form \(T_{ij} = Y^A Y^B\). Introducing the Lagrange multipliers \(\Lambda^{ij} = \Lambda_{ji}\), the extended Hamiltonian action for a systems with these constraints takes the form

\[
S = \frac{1}{2} \int dt (Y^A \frac{\partial}{\partial t} Y^A + \Lambda^{ij} Y^i Y^j ) ,
\]

familiar from [10, 57] (see also [13] and references therein). Note that \(S\) possesses the gauge symmetry with the gauge algebra \(sl(2)\) if one identifies \(\Lambda\) as a \(sl(2) \cong sp(2)\)-valued gauge field (in the literature this gauge algebra is usually identified with \(sp(2)\), which is perhaps a more appropriate notation in this context). One then concludes that the constrained system determined by the quantum BRST charge \(\Psi'_0 + \bar{\Psi}'_0\) (as before one also needs to add \(-\Theta^\mu \bar{p}_\mu\) to take \(d\) into account) is a version of the parent system constructed for the model (5.21).

To see explicitly the gauge field theory described by \((\mathcal{X}, d, \mathcal{M}', Q')\) let us reduce the “fiber” system \((\mathcal{M}', Q')\) to the cohomology of the “fiber” part \(\Omega'\) of the BRST operator. It then follows that BRST field theory can be reduced accordingly (see Section 3.3). In its turn the cohomology of \(\Omega'\) can be identified with the subspace \(\mathfrak{h} \oplus \bar{\mathfrak{h}}\) defined above. The \(*\)-product determines the associative product (also denoted by \(*\)) in \(\mathfrak{h}\) by identifying elements of \(\mathfrak{h}\) with representatives of the BRST cohomology. One then concludes that \(\mathfrak{h}\) is precisely the HS algebra described in [58, 1]. Note that \(\mathfrak{h}\) can be considered as the algebra of quantum observables of the system determined by the quantum BRST charge \(\Psi'_0\).

Expanding the theory around a particular solution \(\Psi'_0 = \Theta^\mu \omega_{\mu B} Y^B P_A\) one finds the equations of motion and gauge symmetries

\[
d A = \nabla A + \frac{1}{2\hbar} [A, A]_* , \quad \delta_\lambda A = d\lambda + \frac{1}{\hbar} [A, \lambda]_* .
\]

5The author is grateful to Itzhak Bars for the illuminating discussion on this model and its relation to the two-time physics approach.
Here $\lambda = \lambda(x; Y, P)$ is the $\hbar$-valued gauge parameter and $A = \theta^\mu A_\mu(x; Y, P)$ is the $\hbar$-valued 1-form. Note that it follows form counting the ghost degree that $A_\mu$ is the only physical (ghost-number-zero) field. The equation of motion is then the standard zero curvature equation for $\hbar$-valued connection 1-form as it should be if one puts to zero the HS curvatures in the full nonlinear system from \cite{1}.

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**Appendix A. Proof of the Proposition**

First we compute the cohomology of the “classical” part $\tilde{\Omega}_{-1}^c = [\tilde{\Omega}_{-1}]_{\hbar=0}$ given explicitly by

$$\tilde{\Omega}_{-1}^c = -\frac{1}{2} (P_A P^A) \frac{\partial}{\partial \pi} - (Y^A + V^A) P_A \frac{\partial}{\partial \rho}.$$  \hfill (A.1)

It is useful to divide the computation in two steps. First we reduce to the cohomology of $P^2 \frac{\partial}{\partial \rho}$, which can be identified with the traceless $\pi$-independent elements. In this subspace the reduced operator acts as follows

$$\mathcal{P}_T \tilde{\Omega}_{-1}^c = -\mathcal{P}_T (Y^A + V^A) P_A \frac{\partial}{\partial \rho} =$$

$$= -(Y^A + V^A) P_A \frac{\partial}{\partial \rho} + (P^A P_A) \frac{1}{(d+1) + 2} P^A \frac{\partial}{\partial P_A} \bar{S}^\dagger \frac{\partial}{\partial \rho},$$

where $\mathcal{P}_T$ denotes the standard projection to the $T$-traceless (i.e. annihilated by $T$) component.

Any $\rho$-independent traceless element is a representative of a cohomology class. Let us show that by using a coboundary freedom one can always assume that a representative also satisfy $\bar{S}^\dagger \phi = 0$. To this end let $\phi_0 \in \mathcal{H}$ satisfy $T \phi_0 = 0$ and $(\hbar - 2c_0 \frac{\partial}{\partial c_0}) \phi_0 = 0$. Let us assume also that we are in the frame where $V^A = l \delta^A_d$ and use the notations $Y^d = l z, P^d = l w$ and $Y^a = y^a, P^a = p^a$. Consider the following sequence

$$\phi_{n+1} = \phi_n - (Y^A + V^A) P_A \frac{1}{(z+1)^2} M \bar{S}^\dagger \phi_n - P^2 \frac{1}{(1+z)^2} \bar{S}^\dagger N \bar{S}^\dagger \phi_n,$$  \hfill (A.3)

where $M, N$ are some coefficients depending on the dimension $d$, operators $p^a \frac{\partial}{\partial p^a}$ and $w \frac{\partial}{\partial w}$ counting the homogeneity in $p, w$. The requirement $T \phi_{n+1} = 0$ fixes one coefficients in terms of another if one assumes $T \phi_n = 0$. In particular, $\phi_{n+1} - \phi_n$ is in the image of $\mathcal{A.2}$.
Let us introduce the filtration $\mathcal{H} = \mathcal{E}_0 \supset \mathcal{E}_1 \supset \mathcal{E}_2 \supset \ldots \supset \mathcal{E}_n \supset \ldots$ where $\mathcal{E}_n$ consists of elements of degree greater or equal than $n$, with the degree introduced according to $\deg p = \deg y = \deg z = 2$ and $\deg w = 1$. Using the remaining freedom in the coefficients one can always achieve that $\tilde{S}_1 \phi_{n+1} \in \mathcal{E}_{n+1}$ provided $\tilde{S}_1 \phi_n \in \mathcal{E}_n$.

To show that cohomology of $\hat{\Omega}_{-1} = \hat{\Omega}_{-1}^{cl} + \hbar^2(\ldots)$ is the same we need an explicit form of the $\hbar^2$-term:

\[(A.4)\]

\[
\hat{\Omega}_{-1} = \hat{\Omega}_{-1}^{cl} - \frac{\hbar^2}{8} (\Box \partial_x - 2S \partial_\rho).
\]

It then follows that any $\rho, \pi$-independent element from $\mathcal{H}$ is a cocycle of $\hat{\Omega}_{-1}$ as well. By iterating the argument above order by order in $\hbar$ one shows that by adding a coboundary one can always assume a representative to be annihilated by $\tilde{S}_1$ and $T$. The decomposition of $\mathcal{H}$ then reads

\[(A.5)\]

\[
\mathcal{H} = \mathcal{E} \oplus \text{Im} \hat{\Omega}_{-1} \oplus \mathcal{F},
\]

where $\mathcal{F}$ denote the complementary subspace (note that this subspace contains $\pi, \rho$-dependent elements only). This determines the explicit form of the projector $P_\mathcal{E}$ which we need in the main text.

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