ON THE MOMENTS OF A G.C.D. RELATED TO LUCAS SEQUENCES

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Abstract. Let \((u_n)_{n\geq 0}\) be a non-degenerate Lucas sequence. For every \(k \geq 1\), we prove that

\[
\sum_{n \leq x} (\gcd(n,u_n))^k = O_{u,k}(x^{k+1}/\exp(-\frac{1}{12}\sqrt{\log x}(\log \log x)) ),
\]

when \(x\) is sufficiently large. This gives a partial answer to a question posed by C. Sanna. Moreover, we conjecture that a bound of the size \(O_{u,k}(x^{k+2/3+\epsilon})\) holds for every \(\epsilon > 0\).

1. Introduction

Let \((u_n)_{n\geq 0}\) be an integral linear recurrence, that is, \((u_n)_{n\geq 0}\) is a sequence of integers and there exist \(a_1,\ldots,a_k \in \mathbb{Z}\), with \(a_k \neq 0\), such that

\[
u_n = a_1 u_{n-1} + \cdots + a_k u_{n-k},
\]

for all integers \(n \geq k\), with \(k\) a fixed positive integer. We recall that \((u_n)_{n\geq 0}\) is said to be non-degenerate if none of the ratios \(a_i/a_j\) (\(i \neq j\)) is a root of unity, where \(\alpha_1,\ldots,\alpha_r \in \mathbb{C}\) are all the pairwise distinct roots of the characteristic polynomial

\[
f_u(X) = X^k - a_1 X^{k-1} - \cdots - a_k.
\]

Moreover, \((u_n)_{n\geq 0}\) is said to be a Lucas sequence if \(u_0 = 0, u_1 = 1\), and \(k = 2\). We note that the Lucas sequence with \(a_1 = a_2 = 1\) is known as the Fibonacci sequence. We refer the reader to [5, Chapter 1] for the basic terminology and theory of linear recurrences.

The function \(g_u(n) := \gcd(n,u_n)\) has attracted the interest of several authors. For example, the set of fixed points of \(g_u(n)\), or equivalently the set of positive numbers \(n\) such that \(n|u_n\), has been studied by Alba González, Luca, Pomerance, and Shparlinski [1], under the mild hypotheses that \((u_n)_{n\geq 0}\) is non-degenerate and that its characteristic polynomial has only simple roots. Moreover, this problem has been studied also by André-Jeannin [2], Luca and Tron [7], Sanna [11], Smyth [14] and Somer [15], when \((u_n)_{n\geq 0}\) is a Lucas or the Fibonacci sequence.

On the other hand, Sanna and Tron [12, 13] have analysed the fiber \(g_u(y)^{-1}\), when \((u_n)_{n\geq 0}\) is non-degenerate and \(y = 1\), and when \((u_n)_{n\geq 0}\) is the Fibonacci sequence and \(y\) is an arbitrary positive integer. Moreover, the image \(g_u(\mathbb{N})\) has been investigated by Leonetti and Sanna [6], again when \((u_n)_{n\geq 0}\) is the Fibonacci sequence.

Other important questions about the function \(g_u(n)\) are related to its behaviour on average and its distribution as arithmetic function. From now on, we focus on the specific case in which \((u_n)_{n\geq 0}\) is a non-degenerate Lucas sequence with non-zero discriminant \(\Delta_u = a_1^2 + 4a_2\). Otherwise, the sequence reduces to \(u_n = na^n\), for a suitable \(a \in \mathbb{Z}\), and \(g_u(n) = n\), for every positive integer \(n\). Even in this particular situation, it is very difficult to find information on the distribution of \(g_u(n)\), because of its oscillatory behaviour. For this reason, it is natural to consider the flatter function \(\log(g_u(n))\), for which an asymptotic formula for its mean value, and more in general for its moments, has been given by Sanna, who proved the following theorem [10, Theorem 1.1].

Theorem 1.1. Fix a positive integer \(\lambda\) and some \(\epsilon > 0\). Then, for all sufficiently large \(x\), how large depending on \(a_1, a_2, \lambda\) and \(\epsilon\), we have

\[
\sum_{n \leq x} (\log g_u(n))^\lambda = M_{u,\lambda} x + E_{u,\lambda}(x),
\]

where \(M_{u,\lambda} > 0\) is a constant depending on \(a_1, a_2\) and \(\lambda\), and the error term is bounded by

\[
E_{u,\lambda}(x) \ll_{u,\lambda} x^{(1+3\lambda)/(2+3\lambda) + \epsilon}.
\]

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Also, Sanna showed that the constant $M_{u, \lambda}$ can be expressed through a convergent series. An immediate consequence of the previous result is the possibility of finding information about the distribution of $g_u$ [10, Corollary 1.3].

**Corollary 1.2.** For each positive integer $\lambda$, we have
\begin{equation}
\# \{ n \leq x : g_u(n) > y \} \ll_{u, \lambda} \frac{x}{(\log y)^{\lambda}},
\end{equation}
for all $x, y > 1$.

In the same article, Sanna raised the question of finding an asymptotic formula for the moments of the function $g_u(n)$ itself. We are not able to answer to this apparently difficult question, but we can at least give a non-trivial estimate for them. The result is the following.

**Theorem 1.3.** For every integer $k \geq 1$ and $u_n$ a non-degenerate Lucas sequence, we have
\begin{equation}
\sum_{n \leq x} g_u(n)^k \ll_{u, k} x^{k+1} \exp \left( -\frac{1}{12} \sqrt{\frac{x}{(\log x)(\log \log x)}} \right),
\end{equation}
when $x$ is sufficiently large.

It is immediate to deduce from Theorem 1.3 the following improvement on the distribution of $g_u(n)$ at least when $y$ varies uniformly in a certain range.

**Corollary 1.4.** We have
\begin{equation}
\# \{ n \leq x : g_u(n) > y \} \ll_{u, k} \frac{x^2}{y \exp \left( \frac{1}{12} \sqrt{\frac{x}{(\log x)(\log \log x)}} \right)},
\end{equation}
for every $y \geq 1$, when $x$ is sufficiently large.

**Proof.** By using (1.3) with $k = 1$ we obtain
\begin{equation}
\# \{ n \leq x : g_u(n) > y \} \leq \sum_{n \leq x} \frac{g_u(n)}{y} \ll_{u, k} \frac{x^2}{y \exp \left( \frac{1}{12} \sqrt{\frac{x}{(\log x)(\log \log x)}} \right)},
\end{equation}
for every $y \geq 1$. \hfill $\square$

We observe that this is an improvement of (1.2), only for certain values of $y$, like for those satisfying
\begin{equation}
x \exp \left( -\frac{1}{24} \sqrt{\frac{x}{(\log x)(\log \log x)}} \right) \leq y \leq x.
\end{equation}

Using arguments coming from the theory of Dirichlet series of multiplicative functions, we end up with the following conjecture.

**Conjecture 1.5.** For every integer $k \geq 1$ and $u_n$ a non-degenerate Lucas sequence, we have
\begin{equation}
\sum_{n \leq x} g_u(n)^k \ll_{u, k} x^{k+2/3+\varepsilon},
\end{equation}
for every $\varepsilon > 0$, when $x$ is sufficiently large.

Obviously, arguing as in the proof of Corollary 1.4, we can suppose that the following stronger result on the distribution of $g_u(n)$ holds.

**Conjecture 1.6.** We have
\begin{equation}
\# \{ n \leq x : g_u(n) > y \} \ll_{u, k} \frac{x^{5/3+\varepsilon}}{y},
\end{equation}
for every $\varepsilon > 0$ and $y \geq 1$, when $x$ is sufficiently large.

This would be an improvement of (1.2) and (1.4) for values of $y$ satisfying
\begin{equation}
x^{2/3+\varepsilon'} \leq y \leq x,
\end{equation}
for a small $\varepsilon' > 0$. 


2. Notation

For a couple of real functions \( f(x), g(x) \), with \( g(x) > 0 \), we indicate with \( f(x) = O(g(x)) \) or \( f(x) \ll g(x) \) that there exists an absolute constant \( c > 0 \) such that \( |f(x)| \leq cg(x) \), for \( x \) sufficiently large. When the implicit constant \( c \) depends from a parameter \( \alpha \) we indicate the above bound with \( f(x) \ll_{\alpha} g(x) \) or equivalently with \( f(x) = O_{\alpha}(g(x)) \).

Throughout, the letters \( p \) and \( q \) are reserved for prime numbers. We write \((a, b)\) and \([a, b]\) to denote the greatest common divisor and the least common multiple of integers \( a, b \). As usual, we denote with \( \lfloor w \rfloor \) and \( \{w\} \) the integer and the fractional part of a real number \( w \). We indicate with \( \tau(n) \) and \( P(n) \) the number of divisors and the greatest prime factor of a positive integer \( n \), respectively.

For each positive integer \( m \) relatively prime with \( a_2 \), let \( z_u(m) \) be the rank of appearance of \( m \) in the Lucas sequence \((u_n)_{n \geq 0} \), that is, \( z_u(m) \) is the smallest positive integer \( n \) such that \( m \) divides \( u_n \). It is well known that \( z_u(m) \) exists (see, e.g., [9]). Also, put \( \ell_u(m) := [m,z_u(m)] \).

3. Preliminaries

We begin by recalling the definition of the Jordan’s totient function.

**Definition 3.1.** The Jordan’s totient function of degree \( k \) is defined as

\[
J_k(n) = n^k \prod_{p|n} \left(1 - \frac{1}{p^k}\right),
\]

for every \( k \geq 1 \) and natural integers \( n \).

Clearly, \( J_1(n) = \varphi(n) \), the Euler’s totient function, and it is immediate to see that \( J_k(n) \) verifies the following identity.

**Lemma 3.1.** We have

\[
n^k = \sum_{d|n} J_k(d),
\]

for any \( k \geq 1 \) and natural integers \( n \).

The next lemma summarizes some basic properties of \( \ell_u(n) \) and \( z_u(n) \), which we will implicitly use later without further mention.

**Lemma 3.2.** For all positive integers \( m, n \) and all prime numbers \( p \), we have:

1. \( m \mid u_n \) if and only if \( z_u(m) \mid n \) and \((m,a_2) = 1\).
2. \( z_u([m,n]) = [z_u(m), z_u(n)] \), whenever \((mn,a_2) = 1\).
3. \( m \mid \gcd(n, u_n) \) if and only if \((m,a_2) = 1 \) and \( \ell_u(m) \mid n \).
4. \( \ell_u([m,n]) = [\ell_u(m), \ell_u(n)] \), whenever \((mn,a_2) = 1\).
5. \( \ell_u(p^j) = p^j z_u(p) \) if \( p \nmid \Delta_u \), and \( \ell_u(p^j) = p^j \) if \( p \mid \Delta_u \), for every prime and integer \( j \geq 1 \).

For any \( \gamma > 0 \), let us define

\[ Q_u,\gamma := \{p : p \nmid a_2, z_u(p) \leq p^\gamma\}. \]

The following is [1, Lemma 2.1].

**Lemma 3.3.** For all \( x, \gamma > 0 \) and for any non-degenerate Lucas sequence \((u_n)_{n \geq 0} \), we have

\[
Q_u,\gamma \ll_u \frac{x^{2\gamma}}{\gamma \log x}.
\]

It has been proven by Sanna and Tron [13, Lemma 3.2] that the series \( \sum_{(n,a_2)=1} 1/\ell_u(n) \) converges.

In the proof of Theorem 1.3, a non-trivial estimate on the tail of this series will play a fundamental role. In fact, the more we know about this tail the more we can improve the bound on our main sums.

To this aim, we consider the following identity

\[
\sum_{n>x \atop (n,a_2)=1} \frac{1}{\ell_u(n)} = \sum_{n>x \atop P(n)>y \atop (n,a_2)=1} \frac{1}{\ell_u(n)} + \sum_{n>x \atop P(n)\leq y \atop (n,a_2)=1} \frac{1}{\ell_u(n)}.
\]

We note that the first sum in the right hand side of (3.2) has been already investigated by Sanna [10, Lemma 2.5] and we report here the result which he obtained.
Proposition 3.4. We have

\[ (3.3) \sum_{\substack{(m,n) = 1 \\ P(n) > y}} \frac{1}{\ell_u(m)} \ll_y \frac{1}{y^{1/3 - \varepsilon}}, \]

for all \( \varepsilon \in (0, 1/4) \) and \( y \gg_y 1 \).

Regarding the second sum in the right hand side of (3.2) we provide an estimate for it in the next lemma. Such estimate will be crucial to obtain the bound in our theorem, but it could have important applications also in other contexts. For this reason we analyse it carefully, producing a strong bound.

Lemma 3.5. Supposing that \( y > (\log x)^2 \) and \( v = \log x/\log y \) tends to infinite as \( x \) tends to infinite, we have

\[ (3.4) \sum_{\substack{n > x \\ P(n) \leq y}} \frac{1}{\ell_u(n)} \ll_y (\log y) e^{-\sqrt{y}/2 \log y} + \frac{\log y}{\log v} e^{-v \log v}, \]

if \( y \) and \( x \) are sufficiently large.

Proof. Since \( \ell_u(n) \geq n \), we may write

\[ (3.5) \sum_{\substack{n > x \\ P(n) \leq y}} \frac{1}{\ell_u(n)} \leq \int_x^\infty \frac{d\psi(t,y)}{t}, \]

where \( \psi(t,y) \) is the counting function of the \( y \)-smooth numbers less than \( t \). Clearly, we have

\[ (3.6) \int_x^\infty \frac{d\psi(t,y)}{t} = \psi(t,y) \bigg|_x^\infty + \int_x^\infty \frac{\psi(t,y)}{t^2} dt. \]

To estimate the second term on the right hand side of (3.5) we suppose first that \( y > \log^2(x) \) and then we split it into two parts:

\[ \int_x^\infty \frac{\psi(t,y)}{t^2} dt = \int_x^z \frac{\psi(t,y)}{t^2} dt + \int_z^\infty \frac{\psi(t,y)}{t^2} dt, \]

where we put \( z = e\sqrt{y} \). Using the estimate [16, Theorem 1, §5.1, Chapter III]

\[ \psi(t,y) \ll t e^{-\log t/\log y} = t^{1-1/\log y}, \]

valid uniformly for \( t \geq y \geq 2 \), we obtain

\[ (3.7) \int_z^\infty \frac{\psi(t,y)}{t^2} dt \ll \int_z^\infty t^{-1-1/(2\log y)} dt \ll (\log y) z^{-1/(2 \log y)} = (\log y) \exp \left( -\frac{\sqrt{y}}{2 \log y} \right). \]

By the Corollary of the Theorem 3.1 in [4], we know that

\[ \psi(t,y) \leq t \exp \left( - (1 + o(1)) \frac{\log t}{\log y} \log \left( \frac{\log t}{\log y} \right) \right), \]

in the region \( y > \log^2 t \). If \( v = \log x/\log y \) tends to infinite as \( x \) tends to infinite, then we may use the simpler

\[ (3.8) \psi(t,y) \leq t \exp \left( - \frac{\log t}{\log y} \log \left( \frac{\log t}{\log y} \right) \right), \]

for any \( x \leq t \leq z \). Let us suppose to be in this situation. Now, inserting this bound and using the change of variable \( s = \log t \), we get

\[ \int_x^z \frac{\psi(t,y)}{t^2} dt \leq y \log y \int_{\log x}^{\sqrt{y}/\log y} \exp \left( - s \log y \log \left( \frac{s}{\log y} \right) \right) ds, \]

which after another change of variable \( s = w \log y \) it becomes

\[ (\log y) \int_{\log x/\log y}^{\sqrt{y}/\log y} \exp(-w \log w) dw. \]

Using that \( w \geq v \) and putting \( w \log v = r \), we find

\[ (3.9) \int_x^z \frac{\psi(t,y)}{t^2} dt \leq \log y \log v \int_{v \log v}^{\sqrt{y} \log v/\log y} e^{-r} dr \leq \frac{\log y}{\log v} e^{-v \log v}. \]
Regarding the first term on the right hand side of (3.5), we note that
\[
\frac{\psi(t,y)}{t} \bigg|_{x}^{\infty} \leq \lim_{t \to \infty} \frac{\psi(t,y)}{t} \approx \lim_{t \to \infty} t^{-1/2} \log y = 0,
\]
by (3.6). Collecting the results, we obtain the estimate (3.4).

Finally, we can state the result about \(\sum_{n>x} 1/\ell_u(n)\).

**Proposition 3.6.** For every non-degenerate Lucas sequence \((u_n)_{n \geq 0}\), we have
\[
(3.10) \quad \sum_{n>x} \frac{1}{\ell_u(n)} \ll_u \exp \left( -\frac{1}{6} \sqrt{(\log x)(\log \log x)} \right),
\]
if \(x\) is sufficiently large.

**Proof.** By Proposition 3.4 and Lemma 3.5 we conclude that
\[
\sum_{n>x} \frac{1}{\ell_u(n)} \ll_u \frac{1}{y^{1/6}} + \frac{\log y}{\log v} \epsilon^{-v \log v},
\]
choosing for example \(\epsilon = 1/6\) and \(y\) sufficiently large. Letting \(y = \exp(\sqrt{(\log x)(\log \log x)})\), we obtain
\[
\sum_{n>x} \frac{1}{\ell_u(n)} \ll_u \exp \left( -\frac{1}{6} \sqrt{(\log x)(\log \log x)} \right),
\]
if \(x\) is large enough. \(\square\)

**Remark 3.1.** There is no attempt here to find the best explicit constant inside the exponential. Obviously, one may find better constants instead of 1/6.

### 4. Proof of Theorem 1.3

**Proof.** We start inserting equation (3.1) inside our main sums.
\[
(4.1) \quad \sum_{n \leq x} (n, u_n)^k = \sum_{n \leq x} \sum_{d \mid (n, u_n)} J_k(d) = \sum_{d \leq x} J_k(d) \sum_{n \leq x} \frac{1}{\ell_u(n)} \sum_{d \mid (n, u_n)} 1 = \sum_{d \leq x} J_k(d) \sum_{\ell_u(d) \mid n} 1,
\]
by part (3) of Lemma 3.2. Clearly, the last sum in (4.1) is
\[
(4.2) \quad \sum_{d \leq x} J_k(d) \left| \frac{x}{\ell_u(d)} \right| \leq x \sum_{d \leq x} \frac{J_k(d)}{\ell_u(d)} \leq x \sum_{d \leq x} \frac{d^k}{\ell_u(d)}.
\]
But now we observe that
\[
(4.3) \quad \sum_{d \leq x} \frac{d^k}{\ell_u(d)} = \sum_{d \leq \sqrt{x}} \frac{d^k}{\ell_u(d)} + \sum_{\sqrt{x} < d \leq x} \frac{d^k}{\ell_u(d)} \ll x^{k/2} + x^k \sum_{d > \sqrt{x}} \frac{1}{\ell_u(d)} \\
\ll x^k \exp \left( -\frac{1}{12} \sqrt{(\log x)(\log \log x)} \right),
\]
using that the series \(\sum_n 1/\ell_u(n)\) converges and the bound (3.10). Inserting (4.3) in (4.2) and (4.2) in (4.1), the proof is finished. \(\square\)

### 5. Notes

Let us define the multiplicative function \(L_u(n)\) such that \(L_u(p^k) = \ell_u(p^k)\), for every prime numbers \(p\) and power \(k \geq 1\). We expect \(L_u(n)\) and \(\ell_u(n)\) to be very close, by virtue of part (4) of Lemma 3.2. Now, consider the Dirichlet series of the function \(n/L_u(n)\), given by
\[
\alpha(s) = \sum_{n \geq 1} \frac{n}{n^s L_u(n)}.
\]
Suppose that it converges for \(s > \sigma_c\), where \(\sigma_c\) is the abscissa of absolute and ordinary convergence of \(\alpha(s)\). Certainly, since \(\ell_u(n) \leq L_u(n)\), for every \(n\), and since we know that the series of the reciprocals
of \( \ell_u(n) \) converges, we have \( \sigma_c \leq 1 \). Then, for any \( s \in \mathbb{C} \) with \( \Re(s) = \sigma > \sigma_c \) we can consider the Euler product and it converges to the Dirichlet series in this range. Therefore, we can write

\[
\alpha(s) = \prod_{p \nmid \Delta_u} \left( 1 + \sum_{k \geq 1} \frac{f(p^k)}{p^{ks}} \right) \beta(s),
\]

where \( f(n) = n/L_u(n) \) and \( \beta(s) \) is an analytic function in \( \Re(s) > 0 \). Since by property (5) of Lemma 3.2 we find that \( f(p^k) = 1/z_u(p) \), for any \( k \geq 1 \) and prime \( p \nmid \Delta_u \), we have

\[
\alpha(s) = \prod_{p \nmid \Delta_u} \left( 1 + \frac{f(p)}{p^s} \frac{p^s}{p^s - 1} \right) \beta(s) = \prod_{p \nmid \Delta_u} \left( 1 + \frac{1}{z_u(p)(p^s - 1)} \right) \beta(s).
\]

Now, the final product in (5.2) converges if and only if

\[
\sum_{p \mid \Delta_u} \frac{1}{z_u(p)(p^s - 1)}
\]

converges. Therefore, it suffices to prove that

\[
\lim_{x \to \infty} \sum_{p \mid \Delta_u} \frac{1}{z_u(p)(p^s - 1)} = 0.
\]

We estimate the last sum separating between primes \( p \in \mathcal{D}_{u,\gamma} \) or \( p \notin \mathcal{D}_{u,\gamma} \). In the first case we obtain

\[
\sum_{p > x, p \mid \Delta_u} \frac{1}{z_u(p)(p^s - 1)} \ll \int_x^{\infty} \frac{d(\# \mathcal{D}_{u,\gamma}(t))}{t^{s+\gamma}} \ll_u \frac{1}{(\sigma - 2\gamma)t^{\sigma-2\gamma}},
\]

by Lemma 3.3, if we choose \( \sigma > 2\gamma \). On the other hand, in the second case we get

\[
\sum_{p > x, p \notin \mathcal{D}_{u,\gamma}} \frac{1}{z_u(p)(p^s - 1)} \ll \sum_{p > x} \frac{1}{p^{s+\gamma}} \ll \frac{1}{(\sigma + \gamma - 1)x^{\sigma+\gamma-1}},
\]

if we choose \( \sigma + \gamma > 1 \). Therefore, we take \( \gamma = 1/3 \) and we have showed that

\[
\sum_{p \mid \Delta_u} \frac{1}{z_u(p)(p^s - 1)} \ll_u \frac{1}{xe^\varepsilon},
\]

if \( \sigma = 2/3 + \varepsilon \), for every \( \varepsilon > 0 \), and consequently that \( \alpha(s) \) converges for every \( s \) with \( \Re(s) > 2/3 \), or equivalently that \( \sigma_c \leq 2/3 \). An immediate application of this result is the following. Let us define

\[
F(s) = \sum_{n \geq 1} \frac{1}{n^s L_u(n)}.
\]

Then, \( F(s) \) has the abcissa of convergence \( \sigma'_c \leq -1/3 \). This is equivalent to have obtained a strong bound on the tail of \( F(0) \). The intermediate passage is made explicit in the next lemma (see e.g. [3, §11.3, Lemma 1]).

**Lemma 5.1.** Suppose that \( G(s) = \sum_{n \geq 1} a_n n^{-s} \) is a Dirichlet series of a sequence \( (a_n)_{n \geq 1} \) of positive real numbers, with abcissa of convergence \( \sigma'_c \). Suppose that \( G(0) \) converges. Then, we have \( \sigma'_c = \inf \{ \theta : \sum_{n > x} a_n \ll x^{\theta} \} \).

Since \( F(s) \) satisfies the hypotheses of the Lemma 5.1, we deduce that

\[
\sum_{n > x} \frac{1}{L_u(n)} \ll_u x^{-1/3+\varepsilon},
\]

for every \( \varepsilon > 0 \). We conclude guessing a better bound for the moments of the function \( g_u(n) \). Indeed, by (5.7) we expect that

\[
\sum_{n > x} \frac{1}{\ell_u(n)} \ll_u x^{-1/3+\varepsilon},
\]
for every $\varepsilon > 0$. Arguing as in the proof of Theorem 1.3 we deduce that
\[
(5.9) \quad \sum_{n \leq x} g_u(n)^k \ll_{u,k} x^{k+2/3+\varepsilon},
\]
for every $\varepsilon > 0$. In fact, we consider the following estimate
\[
(5.10) \quad \sum_{d \leq x \atop (d,a_2) = 1} J_k(d) \left\lfloor \frac{x}{\ell_u(d)} \right\rfloor \ll \sum_{i=1}^{\lfloor \log x \log 2 \rfloor} 2^{ki} \sum_{2^{i-1} < d \leq 2^i} \ell_u(d),
\]
which by (5.8) is
\[
(5.11) \quad \ll_u x \sum_{i=1}^{\lfloor \log x \log 2 \rfloor} 2^{ki(2i-1)(-1/3+\varepsilon)} \ll_u x \sum_{i=1}^{\lfloor (k-1/3+\varepsilon)\lfloor \frac{\log x}{\log 2} \rfloor + 1 \rfloor} 2^{(k-1/3+\varepsilon)} \ll_u x^2 x^{k+2/3+\varepsilon}.
\]

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