Noisy Grover’s search algorithm

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Abstract

External environment influences on Grover’s search algorithm modeled by quantum noise are investigated. The algorithm is shown to be robust under that external dissipation. Explicitly we prove that the resulting search positive maps acting on unsorted N-dimensional database made of projective density matrices depend on the strength of the environment, and that there are infinitely many $x$ values for which search is successful after $O(\sqrt{N})$ queries. These algorithms are quantum entropy increasing.

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1 Introduction

Let us consider Grover’s algorithm with searching matrix $U_G = -UI_uU^\dagger I_w$, used to investigate for a single item $|w\rangle$ among $N$ orthonormal others, that span the complex Hilbert space of an unsorted quantum database $D = \{ |i\rangle \}_{i=1}^{N}$, where $U$ is a general U(2) unitary matrix, and $I_u = 1 - 2|s\rangle\langle s|$, $I_w = 1 - 2|w\rangle\langle w|$ are reflection operators wrt the vectors $\{ |s\rangle, |w\rangle \}$, while $|s\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |i\rangle$ is the uniform superposition state of the database [1-4]. The remarkable fact is that while classical search requires on the average $N$ trials for finding the target item $w$, the quantum algorithm is quadratically faster since it determines $|w\rangle$ after only $O(\sqrt{N})$ queries. Our aim here is to reconsider the algorithm and to extend the concept of database searching to more realistic conditions where we take into account the presence of an external environment in the form of quantum noise. Indeed external influences on quantum processes and their induced decoherence and dissipation are unavoidable, so their quantitative study is of fundamental importance for the field of quantum information [5]. Such a study will be carried out here utilizing the U(2) symmetry of Grover’s algorithm [3, 4] by first selecting $U = U_{\pi/4}$ to be $\pi/4$-rotation which is corrupted by external influences modeled as an interaction with a two dimensional quantum environment of strength $\chi \geq 0$.

2 Construction of noisy search algorithm

Let the environment be described by a quantum system with 2D- Hilbert space that perturbates the Hamiltonian of the $\pi/4$-rotation by the following interaction: $H_R = H_{\pi/4} + H_{\text{env}} = \frac{\pi}{4} \sigma_y \otimes 1 + \frac{\pi}{4} (1 - \sigma_z) \otimes \sigma_y$ [6]. From this Hamiltonian we obtain the evolution operator $U_R = \exp(iH_R)$, where $x$ is the strength of interaction with the environment. This operator signifies a noisy $\pi/4$-rotation, where phase damping has been incorporated into it. The net effect of noise on the rotation operator is to replace it by the completely positive trace preserving map (CP) $\rho \rightarrow \epsilon'(\rho) = \frac{1}{2} \sum_{i=0}^{2} R_i \rho R_i^\dagger$, where $R_i = Tr_{\text{env}}(U_R |i\rangle\langle 0|U_R^\dagger) = |i\rangle\langle 0|$, are the so called Kraus operators of the CP map satisfying the completeness relation $\sum_{i=0}^{2} R_i^\dagger R_i = 1$. [7].

**Proposition 1** The noisy $\pi/4$-rotation map $\rho \rightarrow \epsilon'(\rho) = \frac{1}{2} \sum_{i=0}^{2} R_i \rho R_i^\dagger$, is determined by the operators

\[ R_0 = (\cos \mu(\chi) + \frac{i\pi}{4} \delta(\chi) \sigma_y) e^{-\frac{i\pi}{4} \sigma_y}, \quad R_1 = \frac{i\pi}{4} \delta(\chi) e^{-\frac{i\pi}{4} \sigma_y}, \]

where

\[ \delta(\chi) = \frac{\sin \mu(\chi)}{\mu(\chi)}, \quad \mu(\chi) = \sqrt{\frac{\chi^2}{4} + \frac{\pi^2}{16}}, \chi \geq 0. \]

**Lemma 2** Self-composition of unitary CP map gives unitary CP map.
At this point we perform an optimal unitary preconditioning on \( \varepsilon'(\rho) \). We recall that the nearest wrt Euclidean metric unitary matrix to a given square one, is the one involved in the so called polar decomposition, namely in the decomposition given by the product of a hermitian times a unitary matrix \([8]\).

Then due to Lemma 2 we replace the generators of the CP map \( \varepsilon'(\rho) \) by their respective nearest unitary ones and in this way we obtain the following unitary CP map \( \varepsilon(\rho) \).

**Proposition 3** The unitary CP \( \varepsilon \) is generated by the unitaries \( \{ \frac{1}{\sqrt{2}} V_0, \frac{1}{\sqrt{2}} V_1 \} \) with \( V_0 = e^{i(\psi(\chi) - \frac{i}{2})} \sigma_y \) and \( V_1 = e^{-i\frac{\phi}{2}} \sigma_y \), where

\[
\cos^2 \mu(\chi) + \frac{\chi^2}{4} \delta^2(\chi) = \cos^2 \psi(\chi).
\]

We now introduce the following extension of Grover’s algorithm. Let the database \( \Pi = \{|i\rangle\langle i|\}_{i=1}^{N} = \{\rho_i\}_{i=1}^{N} \), consisting from a collection of \( N \) pure density matrices obtained by the state vector database \( D \). For unitaries \( X,Y \) let the adjoin map \( AdX : \Pi \to \Pi : \rho \to AdX(\rho) = X\rho X^\dagger \), with the property \( AdXY(\rho) = AdX(AdY(\rho)) \). Then \( AdU_G(\rho_s) = AdU AdI_s AdU^\dagger AdI_w = U_G\rho U_G^\dagger \), is the implementation of Grover’s search map in the database \( \Pi \).

Let e.g \( U = U_\frac{\pi}{4} \), then the effect of the environment will amount to replace the \( \frac{\pi}{4} \)-rotation by the unitary CP map \( \varepsilon \) as was shown before. This in turn will cause the embedding of \( AdU_G \) into the unitary CP map \( t : \Pi \to hull(\Pi) \), that maps pure density matrices of the database \( \Pi \), to mixtures of states that form the convex hull of elements of \( \Pi \). Explicitly we obtain \( t = \frac{1}{\sqrt{2}} AdV_0 I_s V_0^\dagger I_w + \frac{1}{\sqrt{2}} AdV_1 I_s V_1^\dagger I_w \).

### 3 Grover’s algorithm is robust under quantum noise

We now proceed employing the unitary CP search map \( t^m(\rho_s) = t^m(|s\rangle\langle s|) \), acting \( m \) times on the initial pure density matrix \( \rho_s \). To quantify the complexity of searching we note that the Bloch vector associated to the density matrices of the database gives a more clear picture of searching. Indeed we can show that for general noise parameter \( x \) the search map \( t \) induces an exponential, wrt the number of queries, damping in the norm of Bloch vectors. Therefore to evaluate the efficiency of the algorithm we need two figures of merit, the radial fidelity giving the projection between the Bloch vectors of target and final density operators \( f = \langle t^m(\rho_s), \rho_w \rangle = \frac{1}{2} Tr (t^m(\rho_s) \rho_w) \), and the cosine of angular fidelity between the same vectors \( \cos \gamma = \frac{|\langle t^m(\rho_s), \rho_w \rangle|}{||t^m(\rho_s)||\cdot||\rho_w||} \). The quadratic overhead in the efficiency of the algorithm occurs if \( f = \cos \gamma = 1 \) for \( m = O(\sqrt{N}) \). Indeed this is the case.

**Proposition 4** The radial and angular fidelities are respectively \( f = \frac{1}{4} [1 + \cos^m(2\psi(x)) \cos \phi(\chi)] \), and \( \cos \gamma = \cos^2 \frac{\phi(\chi)}{2} \).
where $\frac{d(\chi)}{dx} = n\psi(\chi) - n\theta(x) + \alpha$, with $\theta(\chi) = \pi + \chi + \sin^{-1}\left(\frac{\chi N}{N}-1\right)$, and $\cos \alpha = \frac{1}{\sqrt{N}}$. For these figures of merit there exist environments with parameters $\chi > 0$ such that, $\psi(\chi) = 0$, for which $f = \cos \gamma = 1$, when $m=O(\sqrt{N})$. These values of $\chi$ are: $\chi = \pi (2n^2 - \frac{1}{4})^\frac{1}{2}$, for $n \in \mathbb{Z}_+$. 

### 4 Majorization and entropy increase in quantum searching

**Proposition 5** The map $\rho \rightarrow \rho' = t^m(\rho), m \in \mathbb{N}$, majorizes [9,10] the vector $\lambda' \rho'$ of eigenvalues of $\rho'$, by the vector $\lambda' \rho$ of eigenvalues of $\rho$ i.e. $\lambda' \rho < \lambda' \rho$ and renders the dissipative search algorithm an entropy (disorder) increasing one, in the sense that for the quantum entropy $S(k) = -Tr(k \log k)$ is valid that $S(\rho') > S(\rho)$.

### 5 Discussion

Grover’s search algorithm together with its various extensions and applications hold a prominent role in the flourishing field of quantum algorithms, complexity and information. Here we have made a crucial test to the algorithm. We question its efficiency when the omnipresent quantum noise corrupts some of the ideal operation constituting the search map. If e. g. quantum phase damping is incorporated in the search operation, the ensuing dissipative algorithm may exponentially fail in its speed of finding and the accuracy of determining the target quantum state. Still there exist a countable infinity of values of the damping parameter for which the noisy algorithm is robust and performs its task quadratically faster than any classical rival does. Finally, noisy searching creates in every step density matrices that are majorized by the initial matrix, a thing that implies that searching increases entropy if environment influences are taken into account. 

Robustness of search algorithm for other types of quantum noise, as well as entropy production, majorization and information aspects of quantum searching are topics worth of future studying. Some of them are taken up in [11] where detailed proofs of the statements of this paper will be given.

### References

[1] L. Grover , ”Quantum mechanics helps in searching for needle in a haystack,” Phys. Rev. Lett. 78, 325-328 (1997);

[2] ACM: L. Grover, ”A fast quantum mechanical algorithm for database search,” in Proc. 28th Ann. ACM Symp. on the Theory of Computing, , (ACM Press, New York, 1996), pp. 212-218.
[3] PhysComp’96 : M. Boyer, G. Brassard, P. Hoyer and A. Tapp, "Tight bounds on quantum searching," in Proc. of 4th Workshop on Physics and Computation, pp. 36-43.

[4] L. Grover, "Quantum computers can search rapidly by using almost any transformation," Phys. Rev. Lett. 80, 4329-4332 (1998).

[5] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information, (Cambridge Univ. Press. 2000).

[6] I. L. Chuang and Y. Yamamoto, "Creation of a persistent bit using error correction," Phys. Rev. A. 55, 114-127 (1997).

[7] K. Kraus, States, Effects and Operations (Springer, 1983).

[8] A. W. Marshall and I. Olkin, Inequalities: Theory of majorization and applications (Academic Press, 1979).

[9] A. Uhlmann, Wiss. Z. Karl-Marx-Univ. Leipzig 20, 633 (1971).

[10] D. Ellinas and Ch. Konstadakis, "Fast dissipative quantum search", to appear.