Color triplet excitations in two dimensional QCD

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ABSTRACT

We present a novel calculation of color triplet excitations in two dimensional QCD with $SU(2)$ colors. It is found that the lowest energy of the color triplet excitations is proportional to the box length $L$, and can be written as $\mathcal{M}_C = \frac{L^2 \pi^2}{2 \pi}$. Therefore, the color triplet excited states go to infinity when the system size becomes infinity. The properties of the color triplet states such as the wave functions are studied for the finite box length.

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1. Introduction

In Quantum Chromodynamics (QCD), physical particles are all color singlet states. The quarks cannot be free, and there is no color excitation. This is a well known fact, and there is no doubt about it.

The confinement mechanism in QCD plays a most important role, and one learns in textbooks that the quark and antiquark cannot be separated since they feel linear confining forces between them when they are apart. Therefore, there are no free quarks in nature, and this must be of course a right picture for QCD.

However, this may not be everything for the confinement mechanism of QCD. For example, bosonic states with color octet in SU(3) color should not exist in nature. But naive questions are in which way they cannot exist and why they should not be found in nature. At least, a simple-minded picture suggests that they have little to do with the linear confining potential since the non-existence of the bosonic states with colors should not necessarily be due to the confining force between the quarks. Instead, the bosonic objects themselves with colors cannot exist.

In order to understand physics in QCD in depth, we study the behavior of the color triplet excitations in QCD\textsubscript{2} with SU(2) color. The calculation of the color triplet excitations in QCD\textsubscript{2} is carried out with the Bogoliubov vacuum state [1, 2, 3], and we find that the lowest state energy of the color triplet excitations $M_C$ can be described as

$$M_C = \frac{L}{2\pi} \cdot \frac{g^2}{\pi}$$

(1.1)

where $L$ denotes the box length, and $g$ is the gauge coupling constant. Therefore, these color triplet excitations are not realized in nature since we should let the box length $L$ infinity at the final stage.

This is in contrast to the singlet boson in QCD\textsubscript{2} with SU($N$) colors [4, 5, 6, 7]. In a recent paper [8], the mass of the boson $M_N$ is expressed for large values of $N$ at the massless fermion limit,

$$M_N = \frac{2}{3} \sqrt{\frac{Ng^2}{3\pi}}.$$  

(1.2)

This boson mass together with the singlet excitations do not depend on the box length $L$ at all, once the box length $L$ is taken to be sufficiently large. Therefore, it
is clear that these states can be realized in nature.

However, as long as the box length is finite, the color triplet excitations have of course some finite excitation energy, and from the study of the behavior of the wave function, we may learn the basic mechanism of the non-existence of the color triplet excitations in QCD$_2$.

In this paper, we clarify the behavior of the wave function of the color triplet state, and show that the color triplet state has a strongly localized component in momentum space, and therefore, in coordinate space, it should be rather flat. Therefore, the color triplet state has a very small kinetic energy, but the potential energy from interactions gives rise to the excitation energy which is proportional to the box length $L$.

Here, we present a schematic explanation of the energy behavior of the color singlet boson and color triplet excitation. The total energy of the bosonic states for QCD$_2$ can be schematically written

$$E = \frac{a}{L} + bLg^2$$

(1.3)

where $a$ and $b$ are simple dimensionless constants. The first term corresponds to the kinetic energy term, and the second term is due to the interactions.

For singlet bosonic states, we minimize the energy $E$, and obtain the boson mass

$$M = 2\sqrt{ab}g.$$  

(1.4)

Obviously, the singlet bosonic states do not depend on the box length $L$. But it should be noted that the minimum energy of eq.(1.4) can be realized only when the two terms are just in the same magnitude.

On the other hand, the color triplet states have very large repulsive energy from the interaction part, and therefore, the two terms cannot become the same order of magnitude. From the calculations, we know that the color triplet states have strong localizations around $p = 0$ in momentum space, and therefore the color triplet states behave like the energy given in eq.(1.1).
2. Mass of color singlet boson in QCD\(_2\)

In this section, we summarize the calculated results of the boson mass and the condensate value of the color singlet states in QCD\(_2\). The detailed discussions are found in [8]. The calculations are carried out with the Bogoliubov vacuum, and it is shown that the calculated values of the condensate agree very well with prediction of the \(1/N\) expansion. Also, the calculated boson mass is finite at the massless fermion limit, and this is quite consistent with the constraint that is inherent to the two dimensional field theory. Namely, the massless boson should not physically exist in nature.

In Table 1, we summarize the results obtained in [8].

| \(SU(N)\) | \(M_N\) | \(C_N\) |
|---|---|---|
| \(N = 2\) | \(0.467 \frac{g}{\sqrt\pi}\) | \(-0.495 \frac{g}{\sqrt\pi}\) |
| \(N = 3\) | \(0.625 \frac{g}{\sqrt\pi}\) | \(-0.995 \frac{g}{\sqrt\pi}\) |
| \(N \gg 1\) | \(\frac{2}{3} \sqrt{\frac{Ng^2}{3\pi}}\) | \(-\frac{N}{\sqrt{12}} \sqrt{\frac{Ng^2}{2\pi}}\) |

3. Bogoliubov transformation in QCD\(_2\)

In this section, we briefly discuss the Bogoliubov transformation in QCD\(_2\). The Lagrangian density for QCD\(_2\) with \(SU(N)\) color is described as

\[
\mathcal{L} = \bar{\psi} \left( i\gamma^\mu \partial_\mu - g \gamma^\mu A_\mu - m_0 \right) \psi - \frac{1}{4} G^{\alpha \beta}_{\mu \nu} G_{\alpha \beta}^{\mu \nu}, \quad (3.1)
\]
where $G_{\mu\nu}$ is written as

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$$

$$A_\mu = A_\mu^a T^a, \quad T^a = \frac{\tau^a}{2}.$$ 

Now, we first fix the gauge by

$$A_1^a = 0.$$ (3.2)

In this case, the Hamiltonian of QCD$_2$ with $SU(N)$ color can be written as

$$H = \sum_{n,\alpha} p_n \left( a_{n,\alpha}^\dagger a_{n,\alpha} - b_{n,\alpha}^\dagger b_{n,\alpha} \right) + m_0 \sum_{n,\alpha} \left( a_{n,\alpha}^\dagger b_{n,\alpha} + b_{n,\alpha}^\dagger a_{n,\alpha} \right)$$

$$- \frac{g^2}{4NL} \sum_{n,\alpha,\beta} \frac{1}{p_n^2} \left( \tilde{j}_{1,n,\alpha\alpha} + \tilde{j}_{2,n,\alpha\alpha} \right) \left( \tilde{j}_{1,-n,\beta\beta} + \tilde{j}_{2,-n,\beta\beta} \right)$$

$$+ \frac{g^2}{4L} \sum_{n,\alpha,\beta} \frac{1}{p_n^2} \left( \tilde{j}_{1,n,\alpha\beta} + \tilde{j}_{2,n,\alpha\beta} \right) \left( \tilde{j}_{1,-n,\beta\alpha} + \tilde{j}_{2,-n,\beta\alpha} \right)$$ (3.3)

where

$$\tilde{j}_{1,n,\alpha\beta} = \sum_m a_{m,\alpha}^\dagger a_{m+n,\beta}$$ (3.4a)

$$\tilde{j}_{2,n,\alpha\beta} = \sum_m b_{m,\alpha}^\dagger b_{m+n,\beta}.$$ (3.4b)

Now, we define new fermion operators by the Bogoliubov transformation,

$$a_{n,\alpha} = \cos \theta_{n,\alpha} c_{n,\alpha} + \sin \theta_{n,\alpha} d_{-n,\alpha}^\dagger$$ (3.5a)

$$b_{n,\alpha} = -\sin \theta_{n,\alpha} c_{n,\alpha} + \cos \theta_{n,\alpha} d_{-n,\alpha}^\dagger$$ (3.5b)

where $\theta_{n,\alpha}$ denotes the Bogoliubov angle.

In this case, the Hamiltonian of QCD$_2$ can be written as

$$H = \sum_{n,\alpha} E_{n,\alpha} \left( c_{n,\alpha}^\dagger c_{n,\alpha} + d_{-n,\alpha}^\dagger d_{-n,\alpha} \right) + H'$$ (3.6)

where

$$E_{n,\alpha}^2 = \left\{ p_n + \frac{g^2}{4NL} \sum_{m,\beta} \frac{N \cos 2\theta_{m,\beta} - \cos 2\theta_{m,\alpha}}{(p_m - p_n)^2} \right\}^2$$

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\[
H' = m_0 + \frac{g^2}{4NL} \sum_{m,\beta} \left( \frac{N \sin 2\theta_{m,\beta} - \sin 2\theta_{m,\alpha}}{(p_m - p_n)^2} \right)^2.
\]

(3.7)

\(H'\) denotes the interaction Hamiltonian in terms of the new operators and is given in Appendix of [8].

Now, we can calculate the boson mass for the \(SU(3)\) color. Here, we define the wave function for the color triplet boson as

\[
|\Psi_0\rangle = \frac{1}{\sqrt{2}} \sum_n f_n (c_{n,1}^{\dagger}d_{-n,1}^{\dagger} - c_{n,2}^{\dagger}d_{-n,2}^{\dagger}) |0\rangle.
\]

(3.8)

In this case, the boson mass \(M_C\) can be described as

\[
M_C = \langle \Psi_0 | H | \Psi_0 \rangle = \sum_{n,\alpha=1,2} E_{n,\alpha} |f_n|^2
\]

\[-\frac{g^2}{8L} \sum_{l,m} \frac{f_l f_m}{(p_l - p_m)^2} \left( \cos(\theta_{l,1} - \theta_{m,1}) \cos(\theta_{l-K,1} - \theta_{m-K,1})
\]

\[+ \cos(\theta_{l,2} - \theta_{m,2}) \cos(\theta_{l-K,2} - \theta_{m-K,2}) \right)
\]

\[+ \frac{g^2}{4L} \sum_{l,m} \frac{f_l f_m}{(p_l - p_m)^2} \left( \cos(\theta_{l,1} - \theta_{m,2}) \cos(\theta_{l-K,1} - \theta_{m-K,2})
\]

\[+ \cos(\theta_{l,2} - \theta_{m,1}) \cos(\theta_{l-K,2} - \theta_{m-K,1}) \right)
\]

\[-\frac{g^2}{8L} \sum_{l,m} \frac{f_l f_m}{K^2} \left( \sin(\theta_{l,1} - \theta_{l-K,1}) \sin(\theta_{m-K,1} - \theta_{m,1})
\]

\[+ \sin(\theta_{l,2} - \theta_{l-K,2}) \sin(\theta_{m-K,2} - \theta_{m,2})
\]

\[+ \sin(\theta_{l,1} - \theta_{l-K,1}) \sin(\theta_{m-K,2} - \theta_{m,2})
\]

\[+ \sin(\theta_{l,2} - \theta_{l-K,2}) \sin(\theta_{m-K,1} - \theta_{m,1}) \right) \]

(3.9)

where the last terms are finite with \(K \to 0\). This equation can be easily diagonalized together with the Bogoliubov angles, and we obtain the boson mass as the function of \(L\).
4. Color triplet excitation

We carry out the calculations of eq.(3.9), and obtain the spectrum of the color triplet excited states with the massless fermion. In Fig. 1, we show the first excited state of the color triplet states as the function of the box length $L$ and the matrix dimension $D$. The solid line is the phenomenological formula of eq.(1.1),

$$\mathcal{M}_C = \frac{L}{D} \frac{g^2}{\pi}. \tag{1.1}$$

This fits to the calculated spectrum quite well.

Eq.(1.1) indicates that the color triplet excitation becomes infinity when the box length $L$ becomes infinity, which is the real nature. Therefore, this state cannot be observed in nature. On the other hand, the singlet boson mass for $SU(2)$ is determined to be [8]

$$\mathcal{M} = 0.467 \frac{g}{\sqrt{\pi}} \tag{4.1}$$

which does not depend on the box length $L$ at all, and therefore this boson can certainly be observed in nature.

However, as long as the box length $L$ is finite, the color triplet excitation energy is finite and the wave function can be determined. In Fig. 2, we show the calculated
The calculated wave function $f_n$ for the first excited state of the color triplet states. Here, $L_0 = \frac{L}{2\pi}$ is taken to be $L_0 = 50/\sqrt{\pi}$, and the matrix dimension is $D = 4000$. The wave function for this case can be phenomenologically described as

$$f_n = f_0 \exp \left( -35 \left( \frac{p_n}{\sqrt{\pi}} \right)^2 + i\pi n \right)$$

Therefore, the wave function plotted in Fig. 2 is found only around $p_n = 0$ area, and it is quite localized in momentum space. Therefore, it should be flat in coordinate space.

In Fig. 3, the energy spectrum of the singlet as well as the triplet states in QCD are shown. Here, the box length $L$ is fixed to $L_0 = 50/\sqrt{\pi}$. As can be seen, the triplet excitation energies are much higher than those of the singlet states. As stated above, these color singlet bosonic states do not depend on the box length $L$ at all. In this respect, the color triplet states are very special.

5. Intuitive interpretation of color triplet excitations

What is the physics behind the difference between the triplet excitations and the singlet bosonic states?
In order to intuitively understand the physics in connection to the confinement mechanism, we carry out the evaluation of the interaction term which becomes dominant to the color triplet state. This dominant part for the color triplet states is the last term of eq.(3.9). It is interesting to note that the last term of eq.(3.9) completely vanishes for the singlet state due to the kinematical constraint of $SU(2)$ group. On the other hand, the color triplet states are quite different, and the last term of eq.(3.9) gives rise to the very large repulsive energy for the color triplet states.

Further, we can easily show that the last term of eq.(3.9) cannot be described as the local potential, but it is a highly nonlocal interaction. In fact, the last part of eq.(3.9) can be approximately described in terms of the separable interaction in momentum space,

$$H_{nm} = \frac{V_0}{(\frac{n}{a_0})^2 + 1} \frac{1}{(\frac{m}{a_0})^2 + 1} f_n f_m$$

where $V_0$ and $a_0$ are some constants which depend on the values of $D$ and $L$.

Due to the nonlocality of the interaction, it gives rise to the energy which is proportional to the box length $L$. Even for the nonrelativistic kinematics, one cannot find a behavior which is expected from a simple-minded potential picture for the interaction term of the last term in eq.(3.9).
6. Conclusions

We have presented a novel calculation of the bosonic excitation energy for the color triplet state in QCD\textsubscript{2} with $SU(2)$. The bosonic excitation energy is proportional to the box length $L$, and therefore it goes to infinity when the box length $L$ becomes infinity. This means that the color triplet excited states cannot exist in nature. This nonexistence of the color excited states seems to have little to do with the confinement potential, but it is the consequence of the $SU(2)$ kinematics rather than the potential shape like the linear rising potential. Further studies on this line may well be a good help to clarify the color confinement mechanism in four dimensional QCD.

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