Abstract. The symplectic blob algebra is a physically motivated quotient of the Hecke algebra $H(\tilde{C}_n)$ with a diagram calculus. We find the blocks for the symplectic blob algebra for all specialisations of its parameters over the complex numbers. We determine Gram determinants for the cell modules with respect to a canonical contravariant form. We show in particular that the algebra is semisimple over the complex numbers unless at least one of the “quantisation” parameters, or the sum or difference of two of these parameters is integral, or the bulk parameter $q$ is a root of unity. We find decomposition numbers in many of the $q$-generic cases.

Introduction

The symplectic blob algebra, $b_n^x$, introduced in [16], is a quotient of the Hecke algebra $H(\tilde{C}_n)$ (see for instance [20], or Definition 6.1 below). It is of interest from a representation theory perspective both formally (i.e. in representation Theory, in the sense of [37, §5.1]), and combinatorially. In statistical mechanics it controls boundary conditions in computation for various important lattice models (see e.g. [11, 16] and cf. [2, 27, 29]). It links to established objects of current study such as the blob algebra [33, 4, 24], Hecke algebras [10, 20], KLR algebras [23, 36, 6], and Lie theory [14, 22, 30]. Our focus in this paper is the computation of fundamental invariants and the role of alcove geometry (confer [22, 9, 30, 7, 39, 32]).

We may define $b_n^x$ using a basis of diagrams which can be thought of loosely as type-$\tilde{C}_n$ Temperley-Lieb diagrams. These are obtained by suitably stacking the ‘decorated’ generators shown in Figure 1 (see [16, §6] or §1 below for details).

![Figure 1. Generating diagrams for the symplectic blob algebra.](image-url)
The algebra $b_n^*$ is defined over any commutative ring $k$ containing a 6-tuple $\delta = (\delta, \delta_L, \delta_R, \kappa_L, \kappa_R, \kappa_{LR})$ of ‘straightening parameters’. Since we study representation theory, our aim is primarily the Artinian cases, and indeed the cases where $k$ is an algebraically closed field. Thus for each such $k$ there is an algebra $b_n^*$ for each point in $k^6$. One knows \[16\] (and see \[7\]) that the non-semisimple cases lie on certain algebraic sets. The generic semisimple cases are well-understood \[16\], so it is the points on the algebraic sets that are of interest.

It turns out that the dependence of representation theory on position in the variety is more easily described in terms of alternative variety-specific parameterisations. In these the non-semisimple sub-varieties correspond to integrality of some or all of the new parameters, as we shall see in section \[8\] (see also \[29, 11\]). In particular we introduce the bulk parameter $q$ (where $\delta = q + q^{-1}$) and quantisation parameters $w_1, w_2$ (derived from $\delta_L$ and $\delta_R$).

In \[16\] various general properties of the algebra $b_n^*$ are established. For instance a cellular basis is constructed; its generic semisimple structure over $\mathbb{C}$ is determined; and it is shown to be quasi-hereditary on an open subvariety of the non-semisimple variety. Full tilting modules are constructed in \[34\]. An efficient presentation is found in \[17\]; and in \[11\] a closely related algebra is studied, leading to useful alternative bases for certain cell modules, which are crucial to our calculation of the action of a certain special central element.

It follows from comparison with the ordinary blob algebra case \[7\] that the programme of study of the non-generic non-semisimple representation theory of $b_n^*$ is a considerably harder challenge. As in \[7\], however, a key component is to construct ‘enough’ standard module morphisms; and these were constructed in \[18\]. This paper, using the morphisms from \[18\] and also using \[11\], investigates the sufficiency of this set.

Quite generally, if there is a non-zero homomorphism between two standard modules, then the two modules belong to the same block. Indeed, determination of all homomorphisms between standard modules in a quasi-hereditary structure allows a complete description of the blocks (see the appendix). Our main block result in this paper, Theorem \[10.2\] is a complete description of blocks over the complex numbers.

The homomorphisms found in \[18\] are not shown there to be a complete set, so only give a lower bound on the size of blocks. However these results combined with a result about the action of certain central elements on the standard modules allow us to obtain an upper bound on the size of blocks. The homomorphisms (along with some restriction results to the blob algebra) then allow a complete characterisation of the blocks.

Algebras related to towers of recollement \[8\] often have a geometric linkage principle, describing their blocks in terms of an alcove geometry on some Euclidean ‘weight’ space, similar to that seen in Lie theory \[22, 11\]. In some cases the link with Lie theory is direct Ringel duality \[13, 35, 28\], and in others it can be intriguingly less direct (cf. e.g. \[31, 30, 9, 14\]). A uniform recipe for this
Figure 2. Graphical depiction of morphisms and reflection orbits for the cell modules of $b_{13}^c$ with quantisation parameters $w_1 = 3$ and $w_2 = 1$.

is not yet known. In such characterisations there are two challenges that vary in difficulty. A fundamental one is the complexity of the underlying weight space and arrangement of reflection hyperplanes. Then there is the ‘(parabolic) Kazhdan-Lusztig polynomial aspect’ \cite{39, 38, 12}: determining which reflections between weights correspond to homomorphisms between modules. To illustrate these points, we refer first to the Temperley-Lieb and blob algebras over $\mathbb{C}$. Both have $\mathbb{R}$ as the underlying space. The reflection hyperplanes are also easily described (see \cite{27, Ch.7} and \cite{31} respectively for the in-depth results). However in the case of the former only reflections of weights through the adjacent hyperplanes correspond to non-zero homomorphisms, whereas in the latter we have homomorphisms coming from all reflections. At the other end of the spectrum we have the Brauer algebra, where the underlying space and alcove geometry is much more complicated, as well as the correspondence between the reflections and the representation theory of the algebra. In the process of studying the symplectic blob algebra, we hope to obtain an example which sheds light on the general phenomenon indicated by these two extremes. As indicated by Figure 2 taken from Section 9.5 this paper does indeed report progress on this front.

The paper is structured as follows. We give a brief review of notation together with an index in section 1 and of the construction of cell modules in \S 2. In \S 3 we discuss the role of the ground ring. In \S 4 we review the De Gier-Nichols path basis of cell modules. The first main theorems are in \S 6 which gives conditions for two cell modules to be in the same block. In \S 7 we compute
Gram determinants, and in [33,34] the main theorems on decomposition matrices and geometric characterisation of blocks are given.

1. Notation and preliminary definitions

Let $k$ be a field and

$$\delta = (\delta, \delta_L, \delta_R, \kappa_L, \kappa_R, \kappa_{LR}) \in k^6.$$  \hspace{1cm} (1)

Let $\mathbb{N}_0$ denote the natural numbers including 0. Let $n, m \in \mathbb{N}_0$.

The set $B^x_n$ of left-right blob pseudo-diagrams [18] may be defined as follows. Consider the set of decorated Temperley–Lieb diagrams on $n$ strings in Figure 1 (as usual for Temperley–Lieb diagrams, isotopic pictures are identified). Then $B^x_n$ is the set of pictures (up to isotopy) obtained by stacking such pictures. Write $d|d'$ for diagram $d$ stacked over diagram $d'$.

Let $B^x_n$ denote the subset of $B^x_n$ excluding diagrams with features as in Table 1. Given $d \in B^x_n$, an element $f(d)$ of $kB^x_n$ is obtained by applying the straightening relations encoded in Table 2 (the feature on the top is replaced by the given scalar multiple of the feature beneath) and the “topological relation”:

$$\kappa_{LR}$$

(2)

(\text{where each labelled shaded area is a subdiagram without propagating lines) until such operations are exhausted. It is shown in [18] that $f(d)$ does not depend on the details (i.e. we have confluence in a Bergman diamond sense). Thus we have in particular a well-defined map $B^x_n \times B^x_n \rightarrow kB^x_n$ given by

$$(d, d') \mapsto f(d|d').$$  \hspace{1cm} (3)
Definition 1.1. Fix $k$ and $\delta \in k^6$. Then the symplectic blob algebra $b_n^x = b_n^x(\delta)$ is the $k$-algebra with basis $B_n^x$, and multiplication as in (3).

For example, consider the poset $(\Lambda_n, \prec)$ given in Figure 3(a) and the elements $d_l$ ($l \in \Lambda_n$) of $b_n^x$ as indicated in Figure 4, where in particular

\[ d_0 = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \cdots & \text{if } n \text{ is odd.} \end{cases} \]

Verification of the following is a simple exercise in $b_n^x$ arithmetic.

Proposition 1.2 ([16]). The ideals of $b_n^x$ generated by the elements $d_l$ in Figure 4 include according to the poset structure indicated by the Hasse diagram in Figure 3.

Let $I_n(0)$ denote the ideal generated by $d_0$. Define $b'_n$ as the quotient algebra by this ideal.

2. Review of construction of $b_n^x$ cell modules

Consider the poset $(\Lambda_n, \prec)$ given in Figure 3. A set $\{S_n(l)\}_{l \in \Lambda_n}$ of $b_n^x$-modules is constructed over arbitrary $k$ in [16]. In this section we review the construction. One should start by thinking of $k$ not as a field but rather as the commutative ring $\mathbb{Z}[\delta]$ here. Then one can pass to any case by base change. These modules pass to simple modules in the semisimple cases (see [16]), so they can be thought of as the integral forms of the ‘ordinary’ irreducibles in a Brauer-modular system [5, 3].
Figure 4. Representative diagrams in the cell ideal poset.
The left $b^x_n$-module $S_n(l)$ has a basis of half-diagrams constructed similarly to the blob algebra case (cf. [7, p. 593], [16, Section 8]). See Figure 5 for an example. Note that by (3) the left action corresponds to stacking a diagram on top of the basis element.

Consider $l \in \Lambda_n$. To construct a basis $\beta_n(l)$ for $b^x_n$-module $S_n(l)$ in general we proceed as follows. Consider the subset of $B^x_n$ of diagrams with $|l|$ undecorated propagating lines. If $l$ is positive, then further restrict to diagrams with a left blob on the first propagating line. Otherwise, if $l$ is negative, then there must be no such blob. Now pick any one of the remaining diagrams $d$, and take the subset of diagrams agreeing with $d$ in the lower half. Finally, as the lower half is the same in all diagrams, and does not affect multiplication, we omit it. (As another example, half-diagram bases for the cell modules for low rank $b^x_n$ are listed in [16, Figure 3]. There cut lines are used in place of blobs.) The algebra action is by diagram stacking, except that diagrams arising that lie outside the basis (necessarily with higher weight in the sense of Figure 3) are zero.

The case with no decorated propagating lines is easiest to explain. In this case, as a left $b^x_n$-module, the 2-sided ideal $b^x_n d_0 b^x_n$ is a direct sum of isomorphic copies of the cell module $S_n(0)$ where the number of such copies is the same as the number of possible lower half diagrams. We have $S_n(0) \cong b^x_n d_0$. (This is the formulation used in Green et al’s original analysis of the representation theory of $b^x_n$ [16], but not that used in the subsequent crucial work of De Gier and Nichols [11]. When helpful, we colloquially refer to this as the “blob-theoretic” definition to distinguish from other formulations.)

Recall:

**Proposition 2.1** ([16]). The algebra $b^x_n$ is a cellular algebra, in the sense of [15]. The modules $\{S_n(l)\}_{l \in \Lambda_n}$ are the cell modules. The labelling poset $\Lambda_n$ for the cell modules is as in Figure 3.

When all parameters are invertible, $\Lambda_n$ also labels the simple modules, in which case the algebra is also quasi-hereditary with the above poset and the cell modules are standard modules.
As an aid to the reader we include the following index of notation.

\[ a_{\varepsilon_1, \varepsilon_2}^{(n,m)} = [n]^{\frac{2(-m+n(1+\varepsilon_1+\varepsilon_2))}{|m+n+\varepsilon_1+n_1+\varepsilon_2|}} \]

scalar for the action of \( Z_n \) (theorem 6.7)

**ASTL**

Affine Symmetric Temperley-Lieb

**b**

De Gier-Nichols parameter (plays same role as \( \kappa_{LR} \))

\( b^\sigma_n \)

symplectic blob algebra

\( B_x \)

left-right blob pseudo-diagrams

\( B'_x \)

left-right blob pseudo-diagrams

\( \beta_n(l) \)

basis for the cell module \( S_n(l) \)

\( B_{n,m}^{\varepsilon_1, \varepsilon_2} \)

basis of \( W_{\varepsilon_1, \varepsilon_2}^{(n,m)} \)

\( C \)

complex numbers

\( d | d' \)

diagram \( d \) stacked over \( d' \)

\( d_i \)

element of \( b^\sigma_n \), as defined in figure 4

\( \delta \delta = (\delta, \delta_L, \delta_R, \kappa_L, \kappa_R, \kappa_{LR}) \)

6-tuple of parameters (cf. table 2)

**DN**

De Gier-Nichols parameterisation (table 3)

\( e \)

left blob (cf. figure 1)

\( e_i \)

TL generator (cf. figure 1)

\( E'_n = d_0 \)

element of \( b^\sigma_n \) (section 4)

\( \varepsilon_i \in \{\pm 1\} \)

sign parameters for cell modules

\( f \)

right blob (cf. figure 1)

\( F \)

a localisation functor (proposition 8.2)

\( F' \)

a localisation functor (proposition 8.2)

\( f(h) \)

definition 7.4

\( g_0, g_1, g_2, \ldots, g_n \)

generators for the Hecke algebra of type \( \tilde{C} \)

\( G \)

a globalisation functor (proposition 8.2)

\( G' \)

a globalisation functor (proposition 8.2)

\( G_{n,m}^{\varepsilon_1, \varepsilon_2} \)

the Gram matrix (section 7)

\( \Gamma_{n,m}^{\varepsilon_1, \varepsilon_2} \)

Gram determinant (section 7)

\( g(h) \)

definition 7.4

**GMP1**

A Green-Martin-Parker parametrisation (table 3)

**GMP2**

A Green-Martin-Parker parametrisation (table 3)

\( H(\tilde{C}_n) \)

Hecke algebra of type affine-C

\( J_i \)

‘Jucys-Murphy’ elements of \( H(\tilde{C}_n) \) (definition 6.3)

\( k \)

an algebraically closed field

\( k(u) = \frac{(u-w_1+\theta)/2+1)((u-w_1-\theta)/2)}{[n][w_2+1]} \)

an element of \( k[\delta] \)

\( \lambda_p \)

eigenvalue associated to path \( p \) of Gram matrix (Proposition 7.5)
(\(\Lambda_n, \prec\)) labelling poset = \{-n, -n+1, \ldots, 0, 1, \ldots, n-1\} with order as in figure 3

(\(\Lambda_n^+, \prec\)) De Gier-Nichols labelling poset see figure 3

\([m]\) quantum integer (section 3.1)

\(\mathbb{N}_0\) natural numbers (including 0)

\(p_0\) fundamental path (section 4)

\(\mathcal{P}_n\) set of paths (section 4)

\(\pi_n\) set of paths \(w_p\) giving a diagram basis for \(W^{(n)}(b)\) (section 4)

\(\Pi_n\) set of paths \(v_p\) giving a diagram basis for \(W^{(n)}(b)\) (section 4)

\(q = [2] = \delta + \delta^{-1}\) “bulk” parameter

\(q, Q_1, Q_2\) indeterminates for the Hecke algebra of type \(\tilde{C}\)

\(r(u) = \frac{[u+1]}{[u]}\) an element of \(k[\delta]\)

\(\{S_n(l)\}_{l \in \Lambda_n}\) the cell modules for \(b_n^\circ\)

\(T_n\) Tchebychev recursion (section 3.1)

\(\theta\) De Gier-Nichols parameter that reparametrises \(b\)

\(\text{TL}\) Temperley-Lieb

\(2\text{BTL}\) two boundary Temperley-Lieb

\(u_{r_1}(d)\) the number of lines crossing the right wall

\(u_{r_0}(d)\) the number of lines crossing the left wall

\(v_p\) an element of \(b_n^\circ E_n'\) associated to the path \(p\) (section 4)

\(w_1\) a quantisation parameter

\(w_2\) a quantisation parameter

\(w_p\) an element of \(b_n^\circ E_n'\) associated to the path \(p\) (section 4)

\(W^{n,m}_{\varepsilon_1, \varepsilon_2}\) De Gier-Nichols Cell module

\(Z_n = \sum_{i=1}^{n-1} (J_i + J_i^{-1})\) a central element

\(\langle -, - \rangle\) inner product on the cell module (section 7)

2.1. On standard and De Gier–Nichols weight labelling. In [11] there is a useful reformulation of \((\Lambda_n, \prec)\) as follows. The basis \(B_n^\circ\) is equivalent to a basis of affine-symmetric TL (ASTL) diagrams (see [16] for the equivalence). In an ASTL diagram “blobs” are indicated by paired lines that touch the left (for a left blob) or the right (a right blob) side of a diagram. A corresponding half-diagram can in principle have any number of lines touching the left or right side, but the parity of each number is preserved in the (ASTL version of the) basis of a cell module. Thus for \(\varepsilon_i \in \{\pm 1\}\) the module \(W^{n,m}_{\varepsilon_1, \varepsilon_2}\) is the cell module with ASTL half-diagram basis with \(\varepsilon_1\) parity on the left side, \(\varepsilon_2\) parity on the right side and \(m + \frac{1}{2}(\varepsilon_1 + \varepsilon_2)\) propagating lines (here \(\varepsilon_i = +1\) for even, written as +; and \(\varepsilon_i = -1\) for odd, written as \(-\)). We write \(\Lambda_n^+\) for the new labelling scheme — see Figure 3(b). (Note in [11] they have brackets on the cell modules. We have dropped the brackets as the notation is already complicated enough. So our \(W^{n,m}_{\varepsilon_1, \varepsilon_2}\) is their \(W^{(n,m)}_{\varepsilon_1, \varepsilon_2}.\))
The correspondence (in both directions) is given as follows:

\[
S_n(l) = \begin{cases} 
W_n,|l|, \text{sgn } l, \text{sgn } l & \text{if } n \text{ and } l \text{ have opposite parity, } l \neq 0, \\
W_n,|l+1|, -\text{sgn } l, \text{sgn } l & \text{if } n \text{ and } l \text{ have the same parity, } l \neq 0, \\
W_n(b) & \text{if } l = 0 
\end{cases}
\]

where \(\text{sgn } l\) is the sign of \(l\).

Remark: The argument \(b\) used for the cell module with no propagating lines indicates that the structure of this module depends on a parameter \(b\). This is essentially the same as \(\kappa_{LR}\) (see §3.2).

3. On \(\delta\) parameter conventions and reparameterisation

3.1. Ground ring arithmetic. In the modular system \([5]\) one works largely in the integral ground ring, passing to a specific modular case (to address specific Artinian representation theory) as late as possible. However for reasons of arithmetic manipulation it may be expedient to perform computations as if in a different ground ring. This looks like base-change away from the generality of the integral ring. But provided the change is arithmetically reversible back to the integral ring, it is not restrictive.

An example is as follows. The substitution homomorphism \(\mathbb{Z}[\delta] \rightarrow \mathbb{Z}[q, q^{-1}]\) given by \(\delta \mapsto q+q^{-1}\) is not an isomorphism. However it is an injection, so the map can be inverted on any element of the image. Thus one can do arithmetic on elements of \(\mathbb{Z}[\delta]\) working in the image, and then recover identities that hold in \(\mathbb{Z}[\delta]\).

In the example, a merit of the substitution is if one works with elements of \(\mathbb{Z}[\delta]\) satisfying the recursion \(T_n = |2|T_{n-1} - T_{n-2}\), with \(T_0 = 0\) and \(T_1 = 1\) (for example certain Gram determinants of the Temperley–Lieb algebra satisfy this recursion \([29]\)). This is the Tchebychev recursion \([27, \S 6.3.3]\). The complex roots of \(T_n\) are the so-called Beraha numbers \([2]\) — but factorisation is not obvious. However working in the image these elements take the simple form \(T_n = [n]\), where

\[
[m] := q^{-m+1} + q^{-m+3} + \cdots + q^{m-3} + q^{m-1}.
\]

This formulation has manifest factorisation properties. In particular \([n] = 0\) requires \(q\) to be a root of unity.

3.2. Parameterisation by exponents \(w_1, w_2\). In order to determine the representation theory of \(b_\mu^\rho(\delta)\) it is useful to reparameterise as discussed in \([13, \S 2]\). We recall the key parameterisations in Table 3. Generator scaling “1” in Table 3 induces an isomorphism with the algebra with parameters rescaled as shown, reducing from 6 parameters to 4 \([17]\). “GMP1”, “GMP2” and “DN” reparameterise with parameters \(q, w_1, w_2\) (cf. \([2, 25, 26, 40, 29]\)). DN is the parameter choice of De Gier–Nichols in \([11]\). GMP1 and GMP2 are the parameter choices that were most
blocks for the symplectic blob algebra over the complex field

| generator scaling | parameter scaling / reparameterisation |
|-------------------|----------------------------------------|
| label             | e \mapsto e, f \mapsto f, \delta \mapsto \delta, \kappa_L \mapsto \kappa_L, \kappa_R \mapsto \kappa_R, \kappa_{LR} \mapsto \kappa_{LR} |
| 1                 | e \mapsto e, f \mapsto f, \delta \mapsto \delta, \kappa_L \mapsto \kappa_L, \kappa_R \mapsto \kappa_R, \kappa_{LR} \mapsto \kappa_{LR} |
| DN                | e \mapsto -e, f \mapsto -f, \delta \mapsto -\delta, \kappa_L \mapsto \kappa_L, \kappa_R \mapsto \kappa_R, \kappa_{LR} \mapsto \kappa_{LR} |
| GMP1              | [2] \mapsto [w_1], [w_2] \mapsto [w_1 + 1], [w_2 + 1] \mapsto \kappa_{LR} |
| GMP2              | [-2] \mapsto -[w_1], -[w_2] \mapsto -[w_1 + 1], [w_2 + 1] \mapsto \kappa_{LR} |

Table 3. Alternative parameterisations for $b_n^x$.

useful for [18]. GMP1 and GMP2 can be converted from one to another by taking the isomorphic algebra with generators multiplied by $-1$, i.e. using $"2"$ to rescale. GMP2 turns out to be the most convenient for presenting the results about general families of homomorphisms in [18]. Then $"1"$ converts from DN to GMP1 and then to GMP2 via $"2"$.

De Gier–Nichols further reparameterise $b$ in terms of a new parameter $\theta$:

$$
\begin{align*}
\frac{\left[w_1 + w_2 + \theta + 1\right]}{2} & \text{ if } n \text{ even,} \\
-\frac{\left[w_1 - w_2 - \theta\right]}{2} & \text{ if } n \text{ odd.}
\end{align*}
$$

(5)

4. BASES OF THE $b_n^x$-MODULE $W^n(b) = S_n(0)$

**Definition 4.1.** For $n \in \mathbb{N}$ a *Pascal path* $p$ is an element of the subset of $\mathbb{Z}^{n+1}$ given by:

$$
\mathcal{P}_n = \{ p = (h_0, h_1, \ldots, h_n) \mid h_0 = 0; \text{ and } |h_{i+1} - h_i| = 1 \text{ for } 0 \leq i \leq n - 1 \}
$$

In particular, define the *fundamental path* $p_0 = (0, -1, 0, -1, 0, \ldots)$. We can move from $p_0$ to $p$ through a sequence of intermediate paths $p_i$ by ‘adding tiles’ (or half tiles on the right) within each envelope. In particular note that if $p_i \neq p$ then there is always a lowest numbered position (from left to right) at which a tile can be added. Define $\mathcal{P}(p)$ as the ordered set passing from $p_0$ to $p$ in this way.

Define $E_n'$ in $b_n^x$ by $E_n' = d_0$ (to make the dependence on $n$ manifest). That is:

$$
E_n' = \begin{cases} 
\cdots & \text{if } n \text{ even,} \\
\cdots & \text{if } n \text{ odd.}
\end{cases}
$$
Note from (4) that
\[ W_n(b) = S_n(0) = b_n^x d_0 = b_n^x E_n'. \]
Define a subset $\pi_n = \{ w_p \mid p \in \mathcal{P}_n \}$ of $W^n(b)$ as follows. To a path $p \in \mathcal{P}_n$ we associate an element $w_p$ defined recursively through $\mathcal{P}(p)$: firstly $w_{p_0} = E'_n$; then $w_{p_j+1} = e_i w_{p_j}$ if $p_{j+1}$ obtained from $p_j$ by adding a tile in position $i$.

To determine if $\pi_n$ is spanning the argument is essentially analogous to the TL case (cf. [27]). We partially order paths by $p > q$ if $q$ lies in the envelope of $p$, and work by induction on $p$ with $p_0$ as base. We aim to show that $e_i w_p$ lies in the span of $\pi_n$ for every $i$ if this holds for each $q < p$. (This holds for the base case since $e_i p_0$ is in the span by construction.) We need to consider the action of elements $e_i$ on each $w_p$ when $p$ does not have a max or min at $i$ (since otherwise $e_i w_p$ is clearly in the span by construction of $\pi_n$). Note that if $p$ is straight at $i$ (consider e.g. $i = 5$ in Fig 9 or 10) then $w_p = w e_{i+1} e_i w_{p'}$ (or similar) for some $w = e_{i+a} \cdots e_{i+2}$ commuting with $e_i$ (in our example it is simply $w = e_7$) and some $w_{p'}$, whereupon we have $e_i w e_{i+1} e_i w_{p'} = w e_i e_{i+1} e_i w_{p'} = w e_i w_{p'}$. Note that $e_i w_{p'}$ is a $w_q$ for a $q < p$ and straight at $i + 2$. Thus either $w = 1$ and we are done or we may iterate to an even lower path, until the inductive step is completed. Thus $\pi_n$ is spanning.

And then, comparing with the dimension of $W^n(b)$ we have:
Theorem 4.2. The subset $\pi_n$ is a basis for $W^n(b)$ as a left $b_n^\varepsilon$-module. □

4.1. Path basis for $W^n(b)$ for generic $\delta$.

Here we will use a notion of generic $\delta \in k^6$. A point is generic if it lies in the (Zariski) open subset excluding a certain variety (in our case the variety given by the collection of denominators in a construction below — see (6)). The utility is that every $\delta$ in $\mathbb{C}^6$ is the limit of a set of generic points, so that certain identities $f(\delta) = 0$ that hold generically will hold at every point where $f$ makes sense.

We define a formal subset of the $b_n^\varepsilon$-module $W^n(b) = b_n^\varepsilon E_n'$ for generic $\delta$. To a path $p$ we associate an element $v_p$, defined recursively through $P(p)$ as follows:

$$v_{p_0} = E_n',$$
$$v_{p'} = Y_i v_p$$

if $p'$ is obtained from $p$ by adding a tile at position $i$, where $Y_i$ is one of the following operators:

- $X_i = e_i - r(w_1 - h_{i-1})1$ if a full tile is added from above at position $i$;
- $X_i = e_n - k(w_1 - h_{n-1})1$ if a half tile is added from above at the right boundary;
- $X_i' = e_i - r(-w_1 + h_{i-1})1$ if a full tile is added from below at position $i$;
- $X_i' = e_n - k(-w_1 + h_{n-1})1$ if a half tile is added from below at the right boundary,

where

$$r(u) = \frac{[u+1]}{[u]}, \quad k(u) = -\frac{[(u - w_2 + \theta)/2][(u - w_2 - \theta)/2]}{[u][w_2 + 1]}.$$  \hspace{1cm} (6)

Define

$$\Pi_n = \{v_p | p \in P_n\}$$

Comparing the constructions for $\pi_n$ and $\Pi_n$ we have immediately from Theorem 4.2:

Theorem 4.3. When defined, the set $\Pi_n$ can be obtained from $\pi_n$ by an upper-unitriangular transformation; and hence is a basis for $W^n(b)$. □

Theorem 4.4. In general there are other ways of adding tiles to pass from $p_0$ to each $p$ (cf. the ordered sequence $P(p)$). The construction does not depend on the choice of route.

Proof. For each $p$ note that if there are two routes to $p$ from some lower path then the different sequences of multiplications involve pairwise commuting factors. □

Remark. If the scalar term is omitted in $X_i$, this construction builds the diagram basis, up to the DN rescaling factors. We shall later keep track of these scalars explicitly, and hence recover ‘integral-valued’ Gram matrices from certain nominal Gram matrix calculations. It is interesting to contrast this with the path basis for the Temperley–Lieb case in [27]. There the orthogonal basis is orthonormal, so the nominal Gram matrix is the identity matrix, and one only has to work out the basis scaling factor.
Theorem 4.5 ([11 Theorem 5.9]). Let $p = (h_0, h_1, \ldots, h_n) \in \mathcal{P}_n$. Then the generators $e = e_0, e_1, \ldots, e_n = f$ have the following action on $v_p$:

- Each $v_p$ is an eigenvector for the left blob generator $e_0$:
  1. If $h_1 = -1$ then $e_0 v_p = \frac{w_1}{w_1 + 1} v_p$.
  2. If $h_1 = 1$ then $e_0 v_p = 0$.

- The action of $e_i$ ($1 \leq i \leq n$) on $v_p$ is zero if $p$ has positive or negative slope at position $i$, i.e. $|h_i - h_{i+1}| = 2$. If this is not the case, then let $p'$ be the path obtained by adding a tile to $p$ at position $i$. Then $e_i$ acts on the pair $\{v_p, v_{p'}\}$ in the following way:
  1. If $h_{i-1} \geq 0$ then
     $$e_i v_p = v_{p'} + r(w_1 - h_{i-1})v_p$$
     $$e_i v_{p'} = r(-w_1 + h_{i-1})v_{p'} + r(-w_1 + h_{i-1})r(w_1 - h_{i-1})v_p.$$  
  2. If $h_{i-1} < 0$ then
     $$e_i v_p = v_{p'} + r(-w_1 + h_{i-1})v_p$$
     $$e_i v_{p'} = r(w_1 - h_{i-1})v_{p'} + r(-w_1 + h_{i-1})r(w_1 - h_{i-1})v_p.$$ 

- Let $p'$ be the path obtained by adding a half tile to $p$ at the right boundary. Then $e_n$ acts on the pair $\{v_p, v_{p'}\}$ in the following way:
  1. If $h_{n-1} \geq 0$ then
     $$e_n v_p = v_{p'} + k(w_1 - h_{n-1})v_p$$
     $$e_n v_{p'} = k(-w_1 + h_{n-1})v_{p'} + k(-w_1 + h_{n-1})k(w_1 - h_{n-1})v_p.$$  
  2. If $h_{n-1} < 0$ then
     $$e_n v_p = v_{p'} + k(-w_1 + h_{n-1})v_p$$
     $$e_n v_{p'} = k(w_1 - h_{n-1})v_{p'} + k(-w_1 + h_{n-1})k(w_1 - h_{n-1})v_p.$$ 

5. Restricting standard modules to the blob algebra

The (left) blob algebra $b_n$ is the subalgebra of $b_n^+$ generated by $\{e_0, e_1, \ldots, e_{n-1}\}$ [29]. The generators $\{e_1, \ldots, e_{n-1}, e_n\}$ generate another copy of $b_n$ which we will call the right blob algebra.

In [16] §8 the restriction to $b_n$ is used to determine the dimensions of the standard modules, $W_{m,n}^{e_1,e_2}$. There it is shown that each restricted $W_{m,n}^{e_1,e_2}$ is filtered by standard $b_n$-modules (as defined in [29] — the construction is analogous to [22]. We follow the notation of [2] and use $W_{\pm t}(n)$ for the standard $b_n$-modules. Recall that $W_t(n)$ is the standard blob module with half diagram basis that has $n$ northern nodes and $t$ (undecorated) propagating lines. $W_{-t}(n)$ is the standard blob module with half diagram basis that has $n$ northern nodes and $t - 1$ undecorated propagating lines and one decorated propagating line.
Table 4. The standard content of $W_{n,m}^{e_1,e_2}$ as a left (resp. right) blob module.

| $e_1 = 1$ | $e_2 = 1$ | $e_2 = -1$ | $e_2 = 1$ | $e_2 = -1$ |
|-----------|-----------|------------|-----------|------------|
| $W_n(n)$  | $W_n(n)$  | $W_n(n)$   | $W_{-n}(n)$ |
| $W_{m+3}(n)$ | $W_{m+3}(n)$ | $W_{m+3}(n)$ | $W_{-(m+3)}(n)$ |
| $W_{m+1}(n)$ | $W_{m+1}(n)$ | $W_{m+1}(n)$ | $W_{-(m+1)}(n)$ |
| $e_1 = -1$ | $W_{-n}(n)$ | $W_{-n}(n)$ | $W_n(n)$ | $W_{-n}(n)$ |
| $W_{-(m+3)}(n)$ | $W_{-(m+3)}(n)$ | $W_{m+3}(n)$ | $W_{-(m+3)}(n)$ |
| $W_{-(m+1)}(n)$ | $W_{-(m+1)}(n)$ | $W_{m+1}(n)$ | $W_{-(m+1)}(n)$ |

The restriction will again be useful here. Any $b_n^x$-homomorphism is also a left (right) blob homomorphism upon restriction, and thus must respect any left (right) blob structure.

Let $d$ be a half diagram that generates some $W_{n,m}^{e_1,e_2}$ as in §2. We define $u_{r1}$ to be the number of lines crossing the 1-wall (in the sense of [16], i.e. the right wall), not counting any lines that are part of non-contractible loops. We similarly define $u_{r0}$ as the number of lines crossing the left wall.

When we restrict $W_{n,m}^{e_1,e_2}$ to the left blob algebra then it is filtered by $u_{r1}$ and each section is isomorphic to a standard blob module. A similar situation occurs when we restrict to the right blob algebra. We have the following.

**Proposition 5.1** ([16]). The $b_n$-standard content of $W_{n,m}^{e_1,e_2}$ is as in Table 4. □

When the left (or right) blob algebra is semi-simple, then every standard module is simple.

### 6. A NECESSARY BLOCK CONDITION

In this section we recall a central element $Z_n$ (see [19] below) of $b_n^x$. We prove Conjecture 6.5 from [11] and deduce the action of $Z_n$ on cell modules. We use this to investigate the block structure.

6.1. **The central element $Z_n$**. We shall need a surjection from $H(\tilde{C}_n)$ to $b_n^x$ as in [16] Proposition 6.3.2]. Further details can be found in [11 §2] (caveat: there are typos in [11]; cf. e.g. [25]).
Definition 6.1. Let $g, Q_1$ and $Q_2$ be indeterminates. The Hecke algebra $H(\tilde{C}_n)$ of type $\tilde{C}_n$ over $\mathbb{Z}[q^{\pm 1}, Q_1^{\pm 1}, Q_2^{\pm 1}]$ is the associative algebra with generators $g_0, g_1, \ldots, g_n$ and relations:

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \quad 1 \leq i \leq n - 2,$$

$$g_0 g_1 g_0 g_1 = g_1 g_0 g_1 g_0,$$

$$g_n g_{n-1} g_n = g_{n-1} g_n g_{n-1},$$

$$g_i g_j = g_j g_i, \quad |i - j| > 1,$$

$$(g_i - q)(g_i + q^{-1}) = 0, \quad 1 \leq i \leq n - 2,$$

$$(g_0 - Q_1)(g_0 + Q_1^{-1}) = 0,$$

$$(g_n - Q_2)(g_n + Q_2^{-1}) = 0.$$  

For suitable base change and choices of the parameters we have successive quotients:

$$H(\tilde{C}_n) \to 2\text{BTL} \to b_n^\pi \to \frac{b_n^\pi}{I_n(0)}$$

where 2BTL is defined in [17].

The algebra $b_n^\pi$ is defined over a ring $k$ with parameters $\delta = (\delta_L, \delta_R, \kappa_L, \kappa_R, \kappa_{LR}) \in k^6$. For any three units in $k$ we can view $k$ as a $\mathbb{Z}[q^{\pm 1}, Q_1^{\pm 1}, Q_2^{\pm 1}]$-algebra by making $q, Q_1$ and $Q_2$ act as these units. For each such triple we understand $H(\tilde{C}_n)$ as a $k$-algebra by base change.

Note that we are using the Saleur normalisation [29] for generators.

Proposition 6.2. By abuse of notation let us write $q, Q_1, Q_2$ for the actions of these three scalars in $k$ defining $H(\tilde{C}_n)$ as a $k$-algebra as described above. If they satisfy

$$\delta = [2], \quad (q Q_1 - q^{-1} Q_1^{-1}) \delta_L = Q_1 - Q_1^{-1}, \quad (q Q_2 - q^{-1} Q_2^{-1}) \delta_R = Q_2 - Q_2^{-1}.$$  

then there is a surjective $k$-algebra homomorphism $\pi : H(\tilde{C}_n) \to b_n^\pi$, given by

$$\pi(g_i^{\pm 1}) = e_i - q^{\mp 1},$$

$$\pi(g_0^{\pm 1}) = Q_1^{\pm 1} - (q^{\pm 1} Q_1^{\mp 1} - q^{\mp 1} Q_1^{\pm 1}) e_0,$$

$$\pi(g_n^{\pm 1}) = Q_2^{\pm 1} - (q^{\pm 1} Q_2^{\mp 1} - q^{\mp 1} Q_2^{\pm 1}) e_n.$$  

(Note that there is no dependence on $\kappa_{LR}$.)

Proof. (Outline) Consider (11):

$$\pi(g_i) \pi(g_i) = (e_i - q^{-1})(e_i - q^{-1}) = \delta e_i - 2q^{-1} e_i + q^{-2} = (q - q^{-1}) e_i + q^{-2}$$

$$\pi(g_0^2) \pi((q - q^{-1}) g_0 + 1) = (q - q^{-1})(e_i - q^{-1}) + 1 = (q - q^{-1}) e_i + q^{-2}$$

Alternatively here note that by (11) $g_i$ has eigenvalues $q$ and $-q^{-1}$; while $e_i$ has eigenvalues $\delta, 0$. Then $e_i - q^{-1}$ has eigenvalues $q, -q^{-1}$, by (12), as required. Similarly by (12) $g_0$ has eigenvalues...
$Q_1, -Q_1^{-1}$; while $e_0$ has eigenvalues $\delta_L, 0$. Then $Q_1 - (qQ_1 - q^{-1}Q_1^{-1})e_0$ has eigenvalues $Q_1, -Q_1^{-1}$ as required provided that (15) holds. The verification for $g_n$ is directly analogous. 

The homomorphism $\pi$ allows elements of $H(\tilde{C}_n)$ to act on $b_n^\ast$. In particular,

**Definition 6.3 ([III, Definition 2.8]).** The ‘Jucys-Murphy elements’ for $H(\tilde{C}_n)$ are:

- $J_0 = g_1^{-1}g_2^{-1} \cdots g_{n-1}^{-1}g_n g_{n-1} \cdots g_2 g_1 g_0$
- $J_i = g_i J_{i-1} g_i, \quad 1 \leq i \leq n-1.$

**Proposition 6.4 ([III, Proposition 2.10]).** The Jucys-Murphy elements $J_i$ are pairwise commuting and obey the following relations:

- $[g_0, J_j] = 0, \quad j \neq 0$,
- $[g_i, J_j] = 0, \quad 1 \leq i \leq n-1, \quad j \neq i-1, i$,  
- $[g_i, J_{i-1} J_i] = 0, \quad 1 \leq i \leq n-1$,  
- $[g_i, J_{i-1} + J_i] = 0, \quad 1 \leq i \leq n-1$,  
- $[g_0, J_0 + J_0^{-1}] = 0$.  

In particular, the symmetric polynomials in $J_i, J_i^{-1} (0 \leq i \leq n-1)$ are central in $H(\tilde{C}_n)$.

We hence let $Z_n$ be the central element

$$Z_n = \sum_{i=0}^{n-1} (J_i + J_i^{-1}).$$

(19)

6.2. **Aside on substitutions.** We can interpret $[w_1 + a] (a \in \mathbb{Z})$ in the following way:

$$[w_1 + a] = \frac{q^{w_1+a} - q^{-w_1-a}}{q - q^{-1}} = \frac{Q_1 q^a - Q_1^{-1} q^{-a}}{q - q^{-1}}.$$

Similarly we have

$$[w_2 + a] = \frac{Q_2 q^a - Q_2^{-1} q^{-a}}{q - q^{-1}} \quad \text{and} \quad [w_1 + w_2 + a] = \frac{Q_1 Q_2 q^a - Q_1^{-1} Q_2^{-1} q^{-a}}{q - q^{-1}}.$$

6.3. **The $Z_n$-action theorem.** The following lemma is mostly a restatement of [III, Proposition 5.19]. However we have also included the labels of the irreducible modules.

**Lemma 6.5.** The generic $b_n$-module with basis $\{v_p \mid p \text{ of fixed final height } h_n\}$ in $W^n(b)$ is isomorphic to the generic irreducible $b_n$-module $W_{h_n}(n)$.

**Proof.** Note first that this is indeed a module for the blob algebra, as the only elements of the symplectic blob algebra that change the final height of a path involve the generator $e_n$, which is not present when we consider the restricted action.

Now by [III, Proposition 5.19] these modules are the generic irreducibles for the blob algebra, so it suffices to show that the labelling matches up.
First consider the case when $n$ is even. From [11] (3.2) we have a maximal heredity chain of idempotents

\[
\left( \frac{1}{\delta n/2} e_1 e_2 \ldots e_{n-1}, \frac{1}{\delta L}, \frac{1}{L} e_0 e_1 e_2 \ldots e_{n-1}, \ldots, \frac{1}{\delta e_{n-1}}, \frac{1}{\delta L} e_0, 1 \right)
\]

(20)
corresponding to the standard modules $W_0(n), W_{-2}(n), \ldots, W_{n-2}(n), W_{-n}(n), W_n(n)$ respectively.

We must therefore show that the module with basis \{\(v_p | p \) of fixed final height $h_n$\} is associated to the correct heredity idempotent. Suppose first that the final height is $h_n = 0$, then this module contains the element $v_{p_0}$, where $p_0$ is the fundamental path. By Theorem 4.5 none of the idempotents in (20) kill $v_{p_0}$, therefore this module must be isomorphic to $W_0(n)$.

If now the final height is $h_n > 0$, then all paths must either have a slope at at least $h_n$ points, or start with $h_1 = 1$ and have a slope at at least $h_n - 1$ points. Since the $e_i$ in the heredity idempotents commute, we therefore see that any idempotent containing a product of at least $(n-h_n)/2 + 1$ of the $e_i$ will kill the basis elements obtained from these paths, but those containing $(n-h_n)/2$ will not. Therefore the first heredity idempotent that does not annihilate this module is

\[
\frac{1}{\delta (n-h_n)/2} e_{h_n+1} \ldots e_{n-1},
\]

which corresponds to the left blob module $W_{h_n}(n)$.

If the final height is $h_n < 0$, then again all paths must have a slope at at least $|h_n|$ points, or start with $h_1 = -1$ and have a slope at at least $h_n - 1$ points. In this case, any idempotent containing a product of at least $(n+h_n)/2 + 1$ of the $e_i$ for $i \neq 0$ will kill the basis elements obtained from these paths, but those containing a product of $(n+h_n)/2$ of the $e_i$ ($i \neq 0$) and $e_0$ will not. Therefore the first heredity idempotent that does not annihilate this module is

\[
\frac{1}{\delta L \delta (n+h_n)/2} e_0 e_{h_n+1} \ldots e_{n-1},
\]

which corresponds to the left blob module $W_{h_n}(n)$.

The proof for $n$ odd is similar. \(\square\)

We now use the path basis to determine submodules of $W^n(b)$ for specific parameter choices.

**Proposition 6.6** ([11 Proposition 6.3]). Fix $m, n \in \mathbb{N}_0$ and $\varepsilon_2 \in \{\pm 1\}$. Fix $\delta \in k^6$ except for $k_{LR}$, generic, but so that $w_1, w_2$ are defined.

(i) Choose $\theta$ so that $([(-m+w_1+\varepsilon_2 w_2 \pm \theta)/2] = 0$. Then the $b_1^+ b_2^-$ module $W^n(b)$ has a submodule $V_{\varepsilon_2}(m)$ with basis $\pi_{\varepsilon_2}^+(m) = \{v_p | p = (h_0, h_1, \ldots, h_n) \text{ with } h_n \geq m+1\}$.

(ii) Choose $\theta$ so that $([(-m-w_1+\varepsilon_2 w_2 \pm \theta)/2] = 0$. Then $W^n(b)$ has a submodule $V_{-\varepsilon_2}(m)$ with basis $\pi_{-\varepsilon_2}^-(m) = \{v_p | p = (h_0, h_1, \ldots, h_n) \text{ with } h_n \leq -m-1\}$.

This statement is slightly modified from [11]. The key point is that $k(\pm (w_1 - m))$ is zero, and this is equivalent to requiring $[(-m \pm w_1 \pm w_2 \pm \theta)/2] = 0$ for appropriate signs.
Theorem 6.7 ([32] Theorem 6.4)]. Fix $\delta \in k^d$ except for $\kappa_{LR}$, generic, but so that $w_1, w_2$ are defined. Fix $\frac{n}{\varepsilon_1, \varepsilon_2} \in \Lambda^+_n$. Let $\theta = -m + \varepsilon_1 w_1 + \varepsilon_2 w_2$. The action of the central element $Z_n$ as defined in [32] on $V^{(n,m)}_{\varepsilon_1, \varepsilon_2}$ as defined in Proposition 6.6 is given by

$$Z_n V^{(n,m)}_{\varepsilon_1, \varepsilon_2} = \alpha^{(n,m)}_{\varepsilon_1, \varepsilon_2} V^{(n,m)}_{\varepsilon_1, \varepsilon_2}$$

where

$$\alpha^{(n,m)}_{\varepsilon_1, \varepsilon_2} = [n] \frac{2(-m + \varepsilon_1 w_1 + \varepsilon_2 w_2)}{[-m + \varepsilon_1 w_1 + \varepsilon_2 w_2]}.$$

Theorem 6.8. Let $\theta = -m + \varepsilon_1 w_1 + \varepsilon_2 w_2$. Then the generic $b^*_n$-module $W^{n,m}_{\varepsilon_1, \varepsilon_2}$ is isomorphic to the submodule $V^{(n,m)}_{\varepsilon_1, \varepsilon_2}$ of $W^n(b)$.

Proof. We first note that the dimensions of the modules are equal [32] Theorem 6.4]. We also note that both modules are generically irreducible $b^*_n$-modules. Our strategy will be to compare their left blob content when restricted to the left blob algebra and then to further distinguish using the action of the right blob generator, $e_n = f$.

We know that the modules $W^{n,m}_{\varepsilon_1, \varepsilon_2}$ and $V^{(n,m)}_{\varepsilon_1, \varepsilon_2}$ are both generically irreducible for $b^*_n$. We also know that upon restriction to the left blob subalgebra of $b^*_n$ that they have the same irreducible content as left blob modules. Thus we can say that $V^{(n,m)}_{\varepsilon_1, \varepsilon_2}$ is either $W^{n,m}_{\varepsilon_1, \varepsilon_2}$ or $W^{n,m}_{\varepsilon_1, -\varepsilon_2}$. I.e. the left blob structure doesn’t distinguish between $\pm \varepsilon_2$. We now consider the action of the right blob generator $e_n = f$.

We work out the trace of the action of $f$ on these two modules which is an easy calculation. As $f$ has a monic action, (i.e. maps a basis element to another element), we need only write down those elements which map to the same basis element times a scalar. (We only need the diagonal entries in the matrix representing the action of $f$.) If $\varepsilon_2 = 1$ then the trace is $\delta_R$ times number of basis elements with arcs with right blob ending on node $n$. If $\varepsilon_2 = -1$ then the trace of $f$ is $\delta_R$ times number of basis elements with arcs with right blob ending on node $n$ plus $\delta_R$ times number of basis elements with propagating lines starting at node $n$ and decorated with a right blob.

Now note there is a set bijection on the basis of the module $W^{n,m}_{\varepsilon_1, \varepsilon_2}$ to the module $W^{n,m}_{\varepsilon_1, -\varepsilon_2}$ given by putting a right blob on the right most propagating line. So these modules have the same dimension. Moreover, while the first term in the two cases of the trace is clearly equal for either value of $\varepsilon_2$, the second term in the second sum is non-zero and thus these sums are clearly different from each other. (In principle, this is a number that could be made explicit by combinatorics. This won’t be needed though.) This means that the action of $f$ is enough to determine the sign $\varepsilon_2$.

Now consider the module $V^{(n,m)}_{\varepsilon_1, \varepsilon_2}$. Here we need to consider the paths in pairs and use the action defined before.

Let $p'$ be the path obtained by adding a half tile to $p$ at the right boundary. The path $p'$ is “further away” from the fundamental path than $p$. In particular, the absolute value of the last entry in $p'$ is bigger than $p$. Then $e_n$ acts on the pair $\{v_p, v_{p'}\}$ in the following way:
(1) If $h_{n-1} \geq 0$ then

$$e_n v_p = v_{p'} + k(w_1 - h_{n-1})v_p$$
$$e_n v_{p'} = k(-w_1 + h_{n-1})v_p + k(-w_1 + h_{n-1})k(w_1 - h_{n-1})v_p.$$ 

(2) If $h_{n-1} < 0$ then

$$e_n v_p = v_{p'} + k(-w_1 + h_{n-1})v_p$$
$$e_n v_{p'} = k(w_1 - h_{n-1})v_p + k(-w_1 + h_{n-1})k(w_1 - h_{n-1})v_p.$$ 

So the trace of this action on the 2 by 2 matrix given by \{v_p, v_{p'}\} is in both cases

$$k(w_1 - h_{n-1}) + k(-w_1 + h_{n-1})$$

Using the formula for $k$ we get:

$$k(w_1 - h_{n-1}) + k(-w_1 + h_{n-1})$$

$$= -\left(\frac{[w_1 - h_{n-1} - w_2 + \theta]/2}{w_1 - h_{n-1}[w_2 + 1]}\right)$$

$$- \left(\frac{[-w_1 + h_{n-1} - w_2 + \theta]/2}{-w_1 + h_{n-1}[w_2 + 1]}\right)$$

with $\theta$ such that $[-m + \epsilon_1 w_1 + \epsilon_2 w_2 \pm \theta] = 0$.

Set $h := h_{n-1}$ and let $\theta = -m + \epsilon_1 w_1 + \epsilon_2 w_2$. Then $k(w_1 - h) + k(-w_1 + h)$

$$= -\left(\frac{[(1 + \epsilon_1)w_1 - h - (1 - \epsilon_2)w_2 - m]/2}{w_1 - h[w_2 + 1]}\right)$$

$$+ \left(\frac{[(1 - \epsilon_1)w_1 + h - (1 + \epsilon_2)w_2 - m]/2}{w_1 - h[w_2 + 1]}\right)$$

Set $u = \frac{h + m}{2}$ and $v = \frac{h - m}{2}$ so $h = u + v$ and $m = u - v$. Also set $\alpha_1 = \frac{1 + \epsilon_2}{2}$, $\alpha_2 = \frac{1 - \epsilon_2}{2}$, $\bar{\alpha}_1 = \frac{1 - \epsilon_1}{2}$, and $\bar{\alpha}_2 = \frac{1 + \epsilon_1}{2}$. Note that these numbers are all 0 or 1, $\alpha_i + \bar{\alpha}_i = 1$, and $\alpha_i - \bar{\alpha}_i = \epsilon_i$.

$$= -\left[\frac{[\alpha_1 w_1 - u - \bar{\alpha}_2 w_2][\alpha_1 w_1 - v - \alpha_2 w_2] + [\bar{\alpha}_1 w_1 - v + \bar{\alpha}_2 w_2][\alpha_1 w_1 - u + \alpha_2 w_2]}{w_1 - h[w_2 + 1]}\right]$$

We now expand the quantum integer products.

$$[\alpha_1 w_1 - u - \bar{\alpha}_2 w_2][\alpha_1 w_1 - v - \alpha_2 w_2]$$

$$= [w_1 - w_2 - h - 1] + [w_1 - w_2 - h - 3] + \cdots + [\epsilon_1 w_1 + \epsilon_2 w_2 - m + 3] + [\epsilon_1 w_1 + \epsilon_2 w_2 - m + 1]$$

$$[\bar{\alpha}_1 w_1 - v - \bar{\alpha}_2 w_2][\alpha_1 w_1 - u + \alpha_2 w_2]$$

$$= [w_1 + w_2 - h - 1] + [w_1 + w_2 - h - 3] + \cdots + [\epsilon_1 w_1 + \epsilon_2 w_2 - m + 3] + [\epsilon_1 w_1 + \epsilon_2 w_2 - m + 1]$$
The numerator is
\[ [w_1 + w_2 - h - 1] + [w_1 + w_2 - h - 3] + \cdots + [w_1 - w_2 - h + 3] + [w_1 - w_2 - h + 1] \]

\[ = [w_1 - h][w_2] \]

Thus we get:
\[ k(w_1 - h) + k(-w_1 + h) = \frac{[w_1 - h][w_2]}{[w_1 - h][w_2 + 1]} = \frac{[w_2]}{[w_2 + 1]} = \delta_R \]

if both elements of the pair \( \{ v_p, v_{p'} \} \) are in \( V_{r_1, r_2}^{n, m} \).

But it may be that the \( v_p \)'s don't always occur in pairs like this inside \( V_{r_1, r_2}^{n, m} \).

Now we always get a pair \( \{ v_p, v_{p'} \} \) in \( V_{r_1, r_2}^{n, m} \) if the last entry of \( p \) is at least \( |m| \). (Since the last entry of \( p' \) has absolute value \( |m| + 2 \).)

Let's consider the case where \( v_{p'} \) is in \( V_{r_1, r_2}^{n, m} \) but \( v_p \) is not.

In this case: the second last entry of \( p \) and \( p' \) is \( -m + 1 \) (\( \varepsilon_1 = -1 \)) or \( m - 1 \) (\( \varepsilon_1 = 1 \)).

So we check, noting that:
- \( k(-w_1 + m) = 0 \) if \( \varepsilon = 1 + w_1 + w_2 - m \) (\( \varepsilon_1 = \varepsilon_2 = 1 \)).
- \( k(-w_1 + m) = \delta_R \) and \( k(w_1 - m) = 0 \) if \( \varepsilon = 1 - w_1 + w_2 - m \) (\( \varepsilon_1 = 1 \) and \( \varepsilon_2 = -1 \)).
- \( k(w_1 + m) = 0 \) if \( \varepsilon = -w_1 + w_2 - m \) (\( \varepsilon_1 = -1 \) and \( \varepsilon_2 = 1 \)).
- \( k(w_1 + m) = \delta_R \) and \( k(-w_1 - m) = 0 \) if \( \varepsilon = -w_1 - w_2 - m \) (\( \varepsilon_1 = -1 \) and \( \varepsilon_2 = -1 \)).

Thus, putting \( p^+ \) for the \( p' \) with last entry \(+|m|\) and putting \( p^- \) for the \( p' \) with last entry \(-|m|\).

\[ f v_{p^+} = k(-w_1 + m)v_{p^+} + k(-w_1 + m)k(w_1 - m)v_p \]
\[ f v_{p^-} = k(w_1 + m)v_{p^-} + k(-w_1 - m)k(w_1 + m)v_p \]

For \( v_{p^+} \) we see that the coefficient of \( v_p \) is zero for either value of \( \varepsilon_2 \) as it needs to be for \( f v_{p^+} \in V_{r_1, r_2}^{n, m} \). The trace is \( k(-w_1 + m) \).

For \( v_{p^-} \) we see that the coefficient of \( v_p \) is zero for either value of \( \varepsilon_2 \) as it needs to be for \( f v_{p^-} \in V_{r_1, r_2}^{n, m} \). The trace is \( k(w_1 + m) \).

Thus in both cases, the trace is one-dimensional with value \( 0 \) if \( \varepsilon_2 = 1 \) and \( \delta_R \) if \( \varepsilon_2 = -1 \).

Thus we see at that the traces of \( f \) are different on these modules and allow us to distinguish as claimed.

\[ \square \]

**Corollary 6.9.** Fix \( \delta \) and hence \( b_\delta \). Whenever \( \alpha^{(n, m)}_{r_1, r_2} \in k \), the central element \( Z_n \) acts by \( \alpha^{(n, m)}_{r_1, r_2} \) on \( W_{r_1, r_2}^{n, m} \).

**Proof.** Use [11] Theorem 6.4 and the isomorphism from Theorem 6.8 for the generic case. The other cases follow by analytic continuity — \( Z_n \) acts by some scalar, so the limit of generic actions approaching \( \delta \) exists and is this scalar.

An important and immediate consequence is the following corollary.
Corollary 6.10. Fix δ and hence $b^n_\delta$. A necessary condition for two cell modules $W^{n,m}_{\varepsilon_1,\varepsilon_2}$ and $W^{n,t}_{\eta_1,\eta_2}$ to be in the same block is that $\alpha^{(n,m)}_{\varepsilon_1,\varepsilon_2} = \alpha^{(n,t)}_{\eta_1,\eta_2}$.

7. Gram determinants for cell modules

We recall from [16, §8.2] that each cell module $W^{n,m}_{\varepsilon_1,\varepsilon_2}$ has a basis $B^{n,m}_{\varepsilon_1,\varepsilon_2}$ of half diagrams with arcs on the northern edge (hereafter referred to as the standard diagram basis).

Example 7.1. The cell module $W^{5,2}_{-,-}$ has the following basis of half diagrams:

Consider $u, v \in B^{n,m}_{\varepsilon_1,\varepsilon_2}$. We define a scalar $\langle u, v \rangle$ as follows. We first flip $v$ vertically and identify the southern nodes of this diagram with the respective northern nodes of $u$. After applying the straightening rules for $b^n_\delta$, we obtain a diagram with a number of (possibly decorated) strings. The value of $\langle u, v \rangle$ is the coefficient of this diagram if the strings match the number and decorations needed for the cell module, and is zero otherwise. For instance

\[
\langle \bullet \bullet \bullet \rangle = \delta^2 \delta_R \quad \text{and} \quad \langle \bullet \bullet \bullet \rangle = \delta_L \delta_R \delta_L \delta_R = 0,
\]

giving $\langle u, v \rangle = \delta^2 \delta_R$ and $= 0$ respectively. (This is $\langle u, v \rangle$ as defined in [21, §6] with the “semimander” convention.)

Proposition 7.2. Let $\sigma : b^n_\delta \to b^n_\delta$ be the involution defined by flipping diagrams vertically. Then the inner product defined by $\langle -,- \rangle$, together with $\sigma$, defines a contravariant bilinear form on $W^{n,m}_{\varepsilon_1,\varepsilon_2}$. That is, for $d \in b^n_\delta$, we have

\[
\langle du,v \rangle = \langle u, \sigma(d)v \rangle.
\]

Proof. This follows from the definition of $\langle u, v \rangle$ and the action of the algebra on half diagrams. □

While this form can be used over the integral ring $\mathbb{Z}[q^\pm, Q^+_1, Q^+_2]$, we will need to specialise in order to use results from the parameterisation of De Gier–Nichols.

We define the Gram matrix $G^{n,m}_{\varepsilon_1,\varepsilon_2}$ to be the matrix of entries $\langle u, v \rangle_{u,v}$, where $u, v$ runs over the basis $B^{n,m}_{\varepsilon_1,\varepsilon_2}$ of $W^{n,m}_{\varepsilon_1,\varepsilon_2}$. We also define the Gram determinant

\[
\Gamma^{n,m}_{\varepsilon_1,\varepsilon_2} = \det G^{n,m}_{\varepsilon_1,\varepsilon_2}. \tag{21}
\]

When we base change to a field, the rank of $G^{n,m}_{\varepsilon_1,\varepsilon_2}$ is also the rank of a corresponding map from the module to its contravariant dual. The module is thus simple if and only if the matrix is non-singular.
Example 7.3. With the ordering of basis elements as in example 7.1, the Gram matrix is therefore

\[
G^{5.2}_{-,\text{L}} = \begin{pmatrix}
\delta^2_{\text{L}} \delta_{K\text{L}} & \delta_{\text{L}} \delta_{R\text{K}} & \delta^2_{\text{L}} \delta_R & 0 & 0 & 0 \\
\delta_{L \text{K}} \delta_{R \text{L}} & \delta_{L \text{R}} \delta_{\text{L}} & \delta_{L \text{R}} \delta_{\text{R}} & 0 & 0 & 0 \\
\delta^2_{\text{L}} \delta_R & \delta_{L \text{R}} \delta_{\text{L}} & \delta_{L \text{R}} \delta_{\text{R}} & 0 & 0 & 0 \\
0 & 0 & \delta_{L \text{R}} \delta_{\text{R}} & \delta_{L \text{R}} \delta_{\text{R}} & \delta_{L \text{R}} \delta_{\text{R}} & \delta_{L \text{R}} \delta_{\text{R}} \\
0 & 0 & 0 & \delta_{L \text{R}} \delta_{\text{R}} & \delta_{L \text{R}} \delta_{\text{R}} & \delta_{L \text{R}} \delta_{\text{R}} \\
0 & 0 & 0 & 0 & \delta_{L \text{R}} \delta_{\text{R}} & \delta_{L \text{R}} \delta_{\text{R}} & \delta_{L \text{R}} \delta_{\text{R}}
\end{pmatrix}
\]

We wish to calculate the determinant of this matrix. By Laplace expansion, we obtain

\[
(\delta_{L \text{R}})^{-6} \Gamma^{5.2}_{-,\text{L}} = \delta_{L \text{K}} \left| \begin{array}{cccc}
\delta & 1 & 0 & 0 \\
1 & \delta & 1 & 0 \\
0 & 1 & \delta & 1 \\
0 & 0 & \delta & \kappa_R \kappa_{\text{R}}
\end{array} \right| - \kappa^2 \left| \begin{array}{cccc}
\delta & 1 & 0 & 0 \\
1 & \delta & 1 & \delta_R \\
0 & 1 & \delta & \kappa_R \\
0 & 0 & \delta & \kappa_{\text{R}} \kappa_{\text{R}}
\end{array} \right| + (2 \delta_{L \text{K}}L - \delta^2_{L \text{L}} \delta) \left| \begin{array}{cc}
\delta & 1 \\
\delta_R & \kappa_R \kappa_{\text{R}}
\end{array} \right|.
\]

Laplace expanding the first of these determinants results in the following:

\[
\delta_{R \text{K}} \kappa_{\text{R}} \left| \begin{array}{cccc}
\delta & 1 & 0 & 0 \\
1 & \delta & 1 & 0 \\
0 & 1 & \delta & 1 \\
0 & 0 & \delta & \kappa_{\text{R}} \kappa_{\text{R}}
\end{array} \right| - \kappa^2 \left| \begin{array}{cccc}
\delta & 1 & 0 & 0 \\
1 & \delta & 1 & \delta_R \\
0 & 1 & \delta & \kappa_R \\
0 & 0 & \delta & \kappa_{\text{R}} \kappa_{\text{R}}
\end{array} \right| + (2 \delta_{R \text{K}} \kappa_{\text{R}} - \delta^2_{R \text{R}}) \left| \begin{array}{cc}
\delta & 1 \\
\delta_R & \kappa_R \kappa_{\text{R}}
\end{array} \right|.
\]

Since \(\delta = [2]\), we can use the identities for quantum integers to show that each of these determinants is equal to \([n+1]\), where \(n\) is the size of the matrix. Using our parameterisation for \(\delta_{L}, \delta_{R}, \kappa_{L}, \kappa_{R}\), we then see that (22) is equal to

\[
[w_2 + 1]^{-2} ([w_2][w_2 + 1][5] - [w_2 + 1]^2[4] + 2[w_2][w_2 + 1] - [2][w_2]^2)[3])
= [w_2 + 1]^{-2} ([w_2][w_2 + 1][5] - [w_2 + 1]^2[4] + [w_2]([w_2 + 1] - [w_2 - 1])[3])
= [w_2 + 1]^{-2} ([w_2][w_2 + 1][5] - [w_2 - 1][3])
= [w_2 + 1]^{-2} ([w_2][w_2 + 1] - [w_2 + 1][w_2 + 4])
= [w_2 + 1]^{-2} [w_2 + 4][w_2 - 1].
\]

Expanding the other matrices in the same way, we see that

\[
\left( \frac{[w_1][w_2]}{[w_1 + 1][w_2 + 1]} \right)^{-6} [w_1 + 1]^2 [w_2 + 1]^2 G^{5.2}_{-,\text{L}}
= [w_1][w_1 + 1][w_2 + 4][w_2 - 1] - [w_1 + 1]^2[w_2 + 3][w_2 - 1]
+ (2[w_1][w_1 + 1] - [w_1]^2[2]) [w_2 + 2][w_2 - 1]
= [w_2 - 1]([w_1][w_1 + 1][w_2 + 4] - [w_1 + 1]^2[w_2 + 3]
+ [w_1]([w_1 + 1] - [w_1 - 1])[w_2 + 2]).
\]

Note that the last expression in the brackets takes the same form as the second line in our evaluation of (22) above. Hence we finally arrive at

\[
\Gamma^{5.2}_{-,\text{L}} = \frac{[w_1]^6[w_2]^6}{[w_1 + 1]^8[w_2 + 1]^8} ([w_1 - 1][w_2 - 1][w_1 + w_2 + 3]).
\]
As demonstrated by this example, calculating \(\Gamma_{x_1,x_2}^{n,m}\) is non-trivial. However we can apply results of [11] to calculate it with respect to the path basis, which we will see is easier.

**Proposition 7.4 ([11, Proposition 5.13]).** In the path basis, \(G_{x_1,x_2}^{n,m}\) is diagonal.

**Definition 7.5 ([11, Definition 5.14]).** We define the functions \(f(h)\) and \(g(h)\) to be:

\[
\begin{align*}
  f(h) &= r(w_1 - h)r(-w_1 + h) \\
  g(h) &= k(w_1 - h)k(-w_1 + h)
\end{align*}
\]

where \(r(u)\) and \(k(u)\) are as in (6).

**Proposition 7.6 ([11, Proposition 5.15]).** The eigenvalue \(\lambda_p\) of the Gram matrix, \(G_{x_1,x_2}^{n,m}\), for each path \(p\) is given by the following recursive procedure. Let \(p_0\) be the fundamental path, and let \(p'\) be a path obtained from another path \(p\) by the addition of a tile (or half tile) at point \(i\). The following hold:

- \(\lambda_{p_0} = 1\).
- If \(p'\) and \(p\) differ by a full tile we have \(\lambda_{p'} = f(h_i - 1)\lambda_p\).
- If \(p'\) and \(p\) differ by a half tile we have \(\lambda_{p'} = g(h_{n-1})\lambda_p\).

Thus to find \(\Gamma_{x_1,x_2}^{n,m}\) we take the product of the eigenvalues corresponding to the paths that form a basis of that cell module (having chosen \(\theta\) appropriately).

To illustrate, we return to Example 7.1 above, and recalculate the Gram matrix with respect to the path basis.

**Example 7.7.** The basis of \(W_{-3,-3}^{5,2}\) here consists of all paths of a final height \(-3\) or lower, and our value of \(\theta\) is \(-2 - w_1 - w_2\). These paths are given below, along with the tiles that are needed to construct them.

\[
\begin{array}{cccc}
  & & \square & \\
  & & \triangle & \\
  & & \circ & \\
 0 & -1 & -2 & -3 & -4
\end{array}
\]

The eigenvalues of the Gram matrix for these paths are

\[
1, f(0), f(-1), f(-1)f(-2), f(-1)f(-2)f(-3), f(-1)f(-2)f(-3)g(-4)
\]

respectively. The Gram determinant is the product of these, which we evaluate to be

\[
\Gamma_{-3,-3}^{5,2} = \frac{[w_1]^2}{[w_1 + 1][w_1 + 2][w_2 - 1][w_2 - 1][w_1 + w_2 + 3]}.
\]

Note that this is the same result as Example 7.1 up to rescaling by a power of the parameters.
Note this is easier than the calculation in Example 6.1. However in general $\Gamma_{\epsilon_1,\epsilon_2}^{n,m}$ may still be difficult to calculate, due to the large number of paths and tiles as the cell modules increase in size. We next appeal to results about changing bases and the effect on the Gram determinant.

**Theorem 7.8.** With respect to the path basis above, the Gram determinant of $W_{\epsilon_1,\epsilon_2}^{n,m}$ is

$$
\Gamma_{\epsilon_1,\epsilon_2}^{n,m} = \left(\delta_L^\frac{1}{2}(1-\epsilon_1)\delta_R^\frac{1}{2}(1-\epsilon_2)\right) \dim W_{\epsilon_1,\epsilon_2}^{n,m} \prod_{k=0}^{\frac{1}{2}(n-m-3)} \left[\frac{1}{2}(n-m-2k-1)\right]
$$

$$
\times [\epsilon_1 w_1 - \frac{1}{2}(-n+m+2k+1)] [\epsilon_2 w_2 - \frac{1}{2}(-n+m+2k+1)]
$$

$$
\times [\epsilon_1 w_1 + \epsilon_2 w_2 - \frac{1}{2}(n+m-2k-1)] [w_1+1]^{-2} [w_2+1]^{-2} \dim W_{\epsilon_1,\epsilon_2}^{n,n-1-2k}
$$

**Proof.** From the definitions of $f(h)$ and $g(h)$, we see that $\Gamma_{\epsilon_1,\epsilon_2}^{n,m}$ is a product of box numbers of the form $[a_1w_1+a_2w_2-b]$, where $a_i \in \{-1,0,1\}$ and $b,c \in \mathbb{Z}$. Note that all such terms with either $a_1 = 0$ or $a_1,a_2 \neq 0$ arise from the contributions of some $g(h)$ at the right boundary. Moreover, apart from $[w_2+1]$ they all appear to a positive power. Therefore when calculating the product over all permitted paths, there can be no cancellation of these terms. To determine the power of $[w_2+1]$, we multiply the above product by $[w_2+1]^{-2}$ for every factor $g(h)$. The power to which $g(h)$ appears in the product is the number of paths of final height $h'$ for $|h'| \geq |h|$, which in turn is the dimension of the cell module defined by paths of such height. Therefore we see that

$$
\mu \prod_{k=0}^{\frac{1}{2}(n-m-3)} \left[\frac{1}{2}(n-m-2k-1)\right][\epsilon_2 w_2 - \frac{1}{2}(-n+m+2k+1)]
$$

$$
\times [\epsilon_1 w_1 + \epsilon_2 w_2 - \frac{1}{2}(n+m-2k-1)] [w_1+1]^{-2} [w_2+1]^{-2} \dim W_{\epsilon_1,\epsilon_2}^{n,n-1-2k}
$$

is a factor of $\Gamma_{\epsilon_1,\epsilon_2}^{n,m}$, where $\mu$ is a product of box numbers of the form $[w_1-a]$ for $a \in \mathbb{Z}$.

In order to determine the other factors, we will change basis and recalculate the Gram determinant. First, note from the proof of Theorem 6.8 that the change of basis matrix between the standard and path bases is upper triangular, with diagonal entries equal to powers of the parameters for the symplectic blob algebra. Note also that these diagonal entries do not contain the parameter $\delta$. Indeed, the relations that could result in a factor of $\delta$ must be the standard Temperley-Lieb relations, i.e.

$$
e_i e_{i\pm 1} = e_i \text{ and } e_i^2 = \delta e_i,$$

but these cannot appear in the leading term of the path basis as we can never add tiles in position $i$, followed by $i \pm 1$, then in $i$ again, nor can we add tiles in position $i$ twice in a row. We also cannot obtain a $\delta$ by adding to the initial diagram $d_{m+1}$ (or $d_{-m-1}$, $d_{m+1}$, $d_{-m-1}$).

From the standard diagram basis, we change to an alternative path basis, which we obtain by replacing $e_i$ by $e_{n-i}$, $w_1$ by $w_2$ and $\epsilon_1$ by $\epsilon_2$ in the above. In other words, we are working with the path basis defined by the right blob as opposed to the left. For the same reasons as in Theorem 6.8...
the change of basis matrix is again upper triangular. Therefore the change of basis matrix between the first and second path bases is upper triangular, and has determinant equal to a product of powers of the parameters (except $\delta$, as before). Moreover, by considering the contribution at the half tile boundary in the second path basis, we see that

$$
\mu' \prod_{k=0}^{\frac{1}{2}(n-m-3)} \left( \frac{1}{2}(n-m-2k-1) \right) \left[ \varepsilon_1 w_1 - \frac{1}{2}(-n+m+2k+1) \right] \\
\times \left[ \varepsilon_1 w_1 + \varepsilon_2 w_2 - \frac{1}{2}(n+m-2k-1) \right] \left[ w_1 + 1 \right]^{-2} \left[ w_2 + 1 \right]^{-2} \dim W_{\varepsilon_1, \varepsilon_2}^{n,n-1-2k}
$$

is a factor of $\Gamma_{\varepsilon_1, \varepsilon_2}^{n,m}$, where $\mu'$ is a product of powers of the parameters and box numbers of the form $[w_2 - a]$ for $a \in \mathbb{Z}$. When we combine these two results we have

$$
\Gamma_{\varepsilon_1, \varepsilon_2}^{n,m} = \mu' \prod_{k=0}^{\frac{1}{2}(n-m-3)} \left( \frac{1}{2}(n-m-2k-1) \right) \left[ \varepsilon_1 w_1 - \frac{1}{2}(-n+m+2k+1) \right] \\
\times \left[ \varepsilon_2 w_2 - \frac{1}{2}(-n+m+2k+1) \right] \left[ \varepsilon_1 w_1 + \varepsilon_2 w_2 - \frac{1}{2}(n+m-2k-1) \right] \left[ w_1 + 1 \right]^{-2} \left[ w_2 + 1 \right]^{-2} \dim W_{\varepsilon_1, \varepsilon_2}^{n,n-1-2k},
$$

where $\lambda$ is a product of powers of the parameters $\delta_L, \delta_R$ (since our parameterisation has $\kappa_L = \kappa_R = 1$). To determine $\lambda$, we return to the Gram matrix of the standard diagram basis and determine the highest powers of $\delta_L$ and $\delta_R$ which divide the determinant. Since propagating lines cannot cross, any non-zero entry in the Gram matrix must have a factor of $\delta_L$ (resp. $\delta_R$) if there is a left (resp. right) blob on propagating lines. Therefore we can extract a factor of $(\delta_L^{1-\varepsilon_1} \delta_R^{1-\varepsilon_2})^{\dim W_{\varepsilon_1, \varepsilon_2}^{n,m}}$ from the matrix. In fact, this is the largest power of $\delta_L$ and $\delta_R$ we can extract from any row. We deal with factors that may arise from further left blobs, those arising from the right follow by symmetry. Suppose a diagram has a horizontal arc with a left blob. Then this must be the outermost arc of a left-exposed nest of arcs. We construct a diagram which, when taking the inner product with the first, does not add any factors of $\delta_L$ with this nested set of arcs. Indeed, we simply place undecorated arcs in the leftmost side of the diagram so that the bloomed arc forms a closed loop after taking the inner product. This results in a factor of $\kappa_L = 1$ appearing, and no $\delta_L$.

Finally, the range of values over which we take the product ensure that neither $[w_1]$ nor $[w_2]$ can appear. Therefore $\lambda$ must be the greatest factor of $\delta_L$ and $\delta_R$, and the result follows. \( \square \)

This final example returns to the cell module $W_{\varepsilon_1, \varepsilon_2}^{5,2}$.

**Example 7.9.** For $n = 5$, $m = 2$, we have $\frac{1}{2}(n-m-3) = 0$. Therefore by Theorem 7.5 we have

$$
\Gamma_{\varepsilon_1, \varepsilon_2}^{5,2} = ([w_1][w_1 + 1]^{-1}[w_2][w_2 + 1]^{-1})^6[w_1 + 1]^{-2}[w_2 + 1]^{-2}[1][-w_1 + 1][-w_2 + 1][-w_1 - w_2 - 3] \\
= [w_1]^6[w_1 + 1]^{-8}[w_2]^6[w_2 + 1]^{-8}[w_1 - 1][w_2 - 1][w_1 + w_2 + 3].
$$

We can compare this with Example 7.4 to see that we indeed have the Gram determinant.
8. Homological tools for decomposition matrices and blocks of $b_n$

In this section, we will use the constants $\alpha_{\varepsilon_1, \varepsilon_2}^{(n, m)}$ and homomorphisms from \[18\] to determine the block structure of $b_n^{\varepsilon}$ for $\delta \in \mathbb{C}$. Fixing $\delta$ is done by choosing values for $q, w_1, w_2$. We will here restrict those values such that none of $[w_1], [w_2], [w_1 + 1]$ or $[w_2 + 1]$ are zero. We now change our parameterisation to that of GMP2 in §1 above, in order to use the results of \[18\]. This is achieved by rescaling generators in the following way:

\[
e_0 \mapsto -[w_1 + 1]e_0,
\]

\[
e_i \mapsto -e_i \quad \text{for } 1 \leq i \leq n - 1,
\]

\[
e_n \mapsto -[w_2 + 1]e_n.
\]

8.1. Globalisation functors. We will also use the globalisation functors to work in a “large $n$ limit” symplectic blob algebra where both parameters are positive. Having determined blocks in this limit, we will then localise back to the original algebra with original parameter values. The following proposition taken from \[18\, §3\] justifies this.

**Proposition 8.1** (\[18\, §3\]). There exist right exact globalisation functors

\[ G : b_n^{\varepsilon} \text{ mod} \rightarrow b_{n+1}^{\varepsilon} \text{ mod}, \quad G' : b_n^{\varepsilon} \text{ mod} \rightarrow b_{n+1}^{\varepsilon} \text{ mod} \]

with the following properties:

1. There is a parameter change from $b_n^{\varepsilon}$ to $b_{n+1}^{\varepsilon}$ under $G$ which sends $w_1 \mapsto -w_1 - 1$;

2. There is a parameter change under $G'$ which sends $w_2 \mapsto -w_2 - 1$;

3. $GW_{\varepsilon_1, \varepsilon_2}^{n, m} = W_{-\varepsilon_1, \varepsilon_2}^{n+1, m+1}$;

4. $G'W_{\varepsilon_1, \varepsilon_2}^{n, m} = W_{\varepsilon_1, -\varepsilon_2}^{n+1, m+1}$.

There are also exact localisation functors

\[ F : b_n^{\varepsilon} \text{ mod} \rightarrow b_{n-1}^{\varepsilon} \text{ mod}, \quad F' : b_n^{\varepsilon} \text{ mod} \rightarrow b_{n-1}^{\varepsilon} \text{ mod} \]

such that $F \circ G = \text{id}$ and $F' \circ G' = \text{id}$, and also

1. \[ FW_{\varepsilon_1, \varepsilon_2}^{n, m} = \begin{cases} W_{-\varepsilon_1, \varepsilon_2}^{n-1, m+\varepsilon_1} & \text{if } m \neq n - 1 \text{ or } \varepsilon_1 = -1 \\ 0 & \text{if } m = n - 1 \text{ and } \varepsilon_1 = 1; \end{cases} \]

2. \[ FW_{\varepsilon_1, \varepsilon_2}^{n, m} = \begin{cases} W_{\varepsilon_1, -\varepsilon_2}^{n-1, m+\varepsilon_2} & \text{if } m \neq n - 1 \text{ or } \varepsilon_2 = -1 \\ 0 & \text{if } m = n - 1 \text{ and } \varepsilon_2 = 1. \end{cases} \]

Note that the localisation functor can annihilate modules, and therefore it is possible for a block to “break up” when localising. We will address this on a case by case basis when determining the blocks below. Also, since we will always localise back after globalising, we need only consider in the arguments below cell modules $W_{\varepsilon_1, \varepsilon_2}^{N, m}$ with $m \ll N$. 


8.2. On standard module homomorphisms. We now recall the homomorphisms from [IS] and reformulate them into the notation consistent with this paper.

**Theorem 8.2** ([IS] Theorem 4.3.5). Let \( q \) be a primitive \( 2\ell \)-th root of unity and \( w_1, w_2 \not\in \mathbb{Z} \). Suppose that \( m - 2\ell \geq 0 \) (with equality if and only if \( \varepsilon_1 = \varepsilon_2 = 1 \)). Then there exists a non-zero homomorphism

\[
\psi : W^{n,m}_{\varepsilon_1, \varepsilon_2} \rightarrow W^{n,m-2\ell}_{\varepsilon_1, \varepsilon_2}.
\]

**Theorem 8.3** ([IS] Theorem 4.1.4). Let \( q \) be a primitive \( 2\ell \)-th root of unity and \( w_1 \in \mathbb{Z} \). Suppose for \( r \in \mathbb{Z} \) that \( m > m - 2(\varepsilon_1w_1 + r\ell) > 0 \). Then there exists a non-zero homomorphism

\[
\psi : W^{n,m}_{\varepsilon_1, \varepsilon_2} \rightarrow W^{n,m-2(\varepsilon_1w_1 + r\ell)}_{-\varepsilon_1, \varepsilon_2}.
\]

If \( q \) is not a root of unity, then set \( \ell = 0 \) in the above.

**Theorem 8.4** ([IS] Theorems 4.1.6 and 4.1.7). Let \( q \) be a primitive \( 2\ell \)-th root of unity and \( w_2 \in \mathbb{Z} \). Suppose for \( r \in \mathbb{Z} \) that \( m > m - 2(\varepsilon_2w_2 + r\ell) > 0 \). Then there exists a non-zero homomorphism

\[
\psi : W^{n,m}_{\varepsilon_1, \varepsilon_2} \rightarrow W^{n,m-2(\varepsilon_2w_2 + r\ell)}_{\varepsilon_1, -\varepsilon_2}.
\]

If \( q \) is not a root of unity then set \( \ell = 0 \) in the above.

**Theorem 8.5** ([IS] Theorems 4.2.11 and 4.2.12). Let \( q \) be a primitive \( 2\ell \)-th root of unity and \( \varepsilon_1w_1 + \varepsilon_2w_2 \in \mathbb{Z} \). Suppose for \( r \in \mathbb{Z} \) that \( m > 2(\varepsilon_1w_1 + \varepsilon_2w_2 + r\ell) - m \geq 0 \) (with equality only if \( \varepsilon_1 = \varepsilon_2 = 1 \)). Then there exists a non-zero homomorphism

\[
\psi : W^{n,m}_{\varepsilon_1, \varepsilon_2} \rightarrow W^{n,2(\varepsilon_1w_1 + \varepsilon_2w_2 + r\ell) - m}_{\varepsilon_1, \varepsilon_2}.
\]

If \( q \) is not a root of unity then set \( \ell = 0 \) in the above.

8.3. Block master equations. By Proposition 6.3, a necessary condition for any two cell modules to be in the same block is that \( Z_n \) acts by the same constant on both modules. Notice that

\[
\alpha^{(n,m)}_{\varepsilon_1, \varepsilon_2} = [n] \frac{2(-m + \varepsilon_1w_1 + \varepsilon_2w_2)}{[m - \varepsilon_1w_1 + \varepsilon_2w_2]} = [n] \left( q^{-m+\varepsilon_1w_1+\varepsilon_2w_2} + q^{m-\varepsilon_1w_1-\varepsilon_2w_2} \right)
\]

Therefore if \( \alpha^{(n,m)}_{\varepsilon_1, \varepsilon_2} = \alpha^{(n,t)}_{\eta_1, \eta_2} \) and \( [n] \neq 0 \), then

\[
q^{-m+\varepsilon_1w_1+\varepsilon_2w_2} + q^{m-\varepsilon_1w_1-\varepsilon_2w_2} = q^{-\varepsilon_1w_1+\eta_1w_1+\eta_2w_2} + q^{\varepsilon_1w_1-\eta_1w_1-\eta_2w_2}.
\]

Thus we have \( q^{-m+\varepsilon_1w_1+\varepsilon_2w_2} = q^{\pm(-\varepsilon_1w_1+\eta_2w_2)} \). This can only be satisfied if either

\[-(m-t) + (\varepsilon_1 - \eta_1)w_1 + (\varepsilon_2 - \eta_2)w_2 \equiv 0 \pmod{2\ell},
\]

or

\[-(m+t) + (\varepsilon_1 + \eta_1)w_1 + (\varepsilon_2 + \eta_2)w_2 \equiv 0 \pmod{2\ell}.
\]
In the first case, the allowed values of $\eta_1, \eta_2$ lead to the following possibilities:

$$
\epsilon_1 \neq \eta_1, \epsilon_2 \neq \eta_2 \Rightarrow m - t \equiv 2\epsilon_1 w_1 + 2\epsilon_2 w_2 \pmod{2\ell}
$$

(23)

$$
\epsilon_1 \neq \eta_1, \epsilon_2 = \eta_2 \Rightarrow m - t \equiv 2\epsilon_1 w_1 \pmod{2\ell}
$$

(24)

$$
\epsilon_1 = \eta_1, \epsilon_2 \neq \eta_2 \Rightarrow m - t \equiv 2\epsilon_2 w_2 \pmod{2\ell}
$$

(25)

$$
\epsilon_1 = \eta_1, \epsilon_2 = \eta_2 \Rightarrow m - t \equiv 0 \pmod{2\ell},
$$

(26)

and in the second case we have:

$$
\epsilon_1 = \eta_1, \epsilon_2 = \eta_2 \Rightarrow m + t \equiv 2\epsilon_1 w_1 + 2\epsilon_2 w_2 \pmod{2\ell}
$$

(27)

$$
\epsilon_1 = \eta_1, \epsilon_2 \neq \eta_2 \Rightarrow m + t \equiv 2\epsilon_1 w_1 \pmod{2\ell}
$$

(28)

$$
\epsilon_1 \neq \eta_1, \epsilon_2 = \eta_2 \Rightarrow m + t \equiv 2\epsilon_2 w_2 \pmod{2\ell}
$$

(29)

$$
\epsilon_1 \neq \eta_1, \epsilon_2 \neq \eta_2 \Rightarrow m + t \equiv 0 \pmod{2\ell}.
$$

(30)

If $q$ is not a root of unity, then all of the congruences modulo $2\ell$ in the above become equalities.

If $[n] = 0$, then we can still use equations (23)–(30) by first globalising to $b'_N$ where $[N] \neq 0$, determining the blocks there, and then localising again.

9. Decomposition Matrices and Blocks of $b'_n$

In the following subsections we will consider separately various cases relating to whether or not certain linear combinations of $w_1$ and $w_2$ are integers.

To visualise solutions to the master equations (23), we will plot points in the plane corresponding to cell modules, in such a way that solutions are manifested geometrically. (Remark: this indicates the potential for a geometric linkage principle, cf. [22], to describe the representation theory of the algebra.) The cell module $W^{n,m}_{\epsilon_1,\epsilon_2}$ is given ‘weight’ coordinates

$$
W^{n,m}_{\epsilon_1,\epsilon_2} \mapsto (\epsilon_1 (m - \epsilon_1 w_1 - \epsilon_2 w_2), \epsilon_2 (m - \epsilon_1 w_1 - \epsilon_2 w_2)),
$$

— see e.g. Figure 11 Figure 14. In this geometry (in the $q$ not a root of unity case) two cell modules have the same $Z_n$-eigenvalue if and only if one can be reached from the other by successive reflections in the coordinate axes. As a guide to the eye, the cell module $W^{n,m}_{\epsilon_1,\epsilon_2}$ is plotted on the ‘arm’ labelled by $\epsilon_1, \epsilon_2$.

9.1. Cases with none of $w_1, w_2, w_1 + w_2, w_1 - w_2$ integral. Suppose first that $q$ is not a root of unity. Since $m$ and $t$ are positive integers, it is only possible for at most one of (23) and (27) to be satisfied (similarly for (24),(28); (25),(29); and (26),(30)). The case (30) is impossible as both $m$ and $t$ are positive integers, and at most one can be zero. The case (26) is trivial, as the two modules are equal here. Also since $\epsilon_1$ and $\epsilon_2$ take values $\pm 1$, we can have non-trivial coincidences of the eigenvalues of $Z_n$ if and only if \{w_1, w_2, w_1 + w_2, w_1 - w_2\} \cap \mathbb{Z} \neq \emptyset. This leads to the first main theorem of this paper:
Theorem 9.1. Suppose $q$ is not a root of unity and \( \{ w_1, w_2, w_1 + w_2, w_1 - w_2 \} \cap \mathbb{Z} = \emptyset \). Then the algebra $b'_n$ is semisimple. If in addition, $\theta \neq \pm (-m \pm w_1 \pm w_2)$ for any $m \in \mathbb{Z}$ then symplectic blob algebra $b'_n$ is semisimple.

Proof. To prove the first statement, it suffices to show that the eigenvalues of $Z_n$ are all distinct. Indeed, since none of $w_1, w_2, w_1 + w_2$ or $w_1 - w_2$ are integral the only possible solution to equations (23)–(30) is the trivial one in (26). Therefore each cell module is alone in its block and the algebra is semisimple.

To prove the second, the only additional information needed is that $W_n(b)$ is simple. This is guaranteed as for our chosen value of $\theta$, the Gram determinant of $W_n(b)$ is non-zero by [11, Theorem 5.17]. □

If now $q$ is a $2\ell$-th root of unity, then we must consider equations (26) and (30). Note that the left- and right-blob algebras are semisimple, so if $\eta_1 \neq \varepsilon_1$ and $\eta_2 \neq \varepsilon_2$ then by restricting to either algebra and considering the standard contents of Table 4 we see that there can be no homomorphisms between any modules satisfying (30). It therefore remains to consider (26). Suppose without loss of generality that $t < m$. Then $0 \leq m - 2\ell$ (with equality only if $\varepsilon_1 = \varepsilon_2 = 1$), so by Theorem 8.2 we have a non-zero homomorphism

$$W^{n,m}_{\varepsilon_1, \varepsilon_2} \rightarrow W^{n,m-2\ell}_{\varepsilon_1, \varepsilon_2} \quad (31)$$

Thus $W^{n,m}_{\varepsilon_1, \varepsilon_2}$ and $W^{n,m-2\ell}_{\varepsilon_1, \varepsilon_2}$ are in the same block. Similarly we have $W^{n,m-2\ell}_{\varepsilon_1, \varepsilon_2}$ and $W^{n,m-4\ell}_{\varepsilon_1, \varepsilon_2}$ in the same block, and so on. By transitivity, we therefore see that $W^{n,m}_{\varepsilon_1, \varepsilon_2}$ and $W^{n,t}_{\eta_1, \eta_2}$ are in the same block.

Theorem 9.2. Suppose $q^{2\ell} = 1$ and $\{ w_1, w_2, w_1 + w_2, w_1 - w_2 \} \cap \mathbb{Z} = \emptyset$. Then two cell modules $W^{n,m}_{\varepsilon_1, \varepsilon_2}$ and $W^{n,t}_{\eta_1, \eta_2}$ are in the same block if and only if $\varepsilon_1 = \eta_1$, $\varepsilon_2 = \eta_2$ and $m \equiv t \pmod{2\ell}$. 

Figure 11. Graphical depiction of the cell modules of $b'_8$ with $w_1 = \frac{1}{2}$ and $w_2 = \frac{3}{4}$. 
The combinatorial-geometric expression of linkage in this case is as in Figure 12. See Figure 13 for the truncation to $n = 8$.

9.2. Either $w_1$ or $w_2$ integral. We will determine the blocks when precisely one of $w_1$ and $w_2$ is integral. We begin with the case $w_1 \in \mathbb{Z}, w_2 \notin \mathbb{Z}$, and first assume that $q$ is not a root of unity. Now the only equations from (23)–(30) with non-trivial solutions are (24) and (28), and by fixing $m, \varepsilon_1$ and $\varepsilon_2$ we see that blocks have size at most two. Note that if $w_2 \in \frac{1}{2}\mathbb{Z}$ then we still do not obtain extra solutions since $m \pm t$ is always even.
Consider first the case (28), where we have \( \varepsilon_1 = \eta_1 \) and \( \varepsilon_2 \neq \eta_2 \). We will show that although these two modules have the same eigenvalue, they are not in the same block. As right-blob modules, they have a filtration as in Table 4. However since the parameter \( w_2 \) is not integral the right-blob algebra is semisimple, and thus it is not possible to have a non-zero homomorphism between the modules. Since the block has size at most two, we deduce that these modules are not in the same block.

Now consider (24). Here, we have \( \varepsilon_1 \neq \eta_1 \) and \( \varepsilon_2 = \eta_2 \). By swapping labels if necessary we may assume that \( m > t \) (equality is not possible due to [18 Proposition 3.4.1]). Since both \( m \) and \( t \) are non-negative integers we must have \( \varepsilon_1 = \text{sgn}(w_1) \), thus \( m > t = m - 2\varepsilon_1 w_1 > 0 \). Therefore the conditions of Theorem 8.3 are satisfied and we have a homomorphism \( W^{n,m}_{\varepsilon_1,\varepsilon_2} \to W^{n,t}_{-\varepsilon_1,\varepsilon_2} \).

We will now consider the case when \( q \) is a 2\( \ell \)-th root of unity and \( w_1 \in \mathbb{Z} \), \( w_2 \notin \mathbb{Z} \). In this case, the only equations from (23)–(30) with solutions are (24), (26), (28) and (30). We still do not obtain extra solutions if \( w_2 \in \frac{1}{2}\mathbb{Z} \) by parity considerations in the same way as above.

Begin by fixing the cell module with labels \( m, \varepsilon_1 \) and \( \varepsilon_2 \). By restricting to the right-blob algebra as before, any other cell module \( W^{n,t}_{\eta_1,\eta_2} \) in this block has \( \eta_2 = \varepsilon_2 \). We can therefore rule out equations (28) and (30). In the case of equation (24) we again see that the conditions of Theorem 8.3 are satisfied (this time with \( \ell \neq 0 \)), and so these cell modules are in the same block. So it remains to consider the case of (26).

We will begin by showing that \( W^{n,m}_{\varepsilon_1,\varepsilon_2} \) and \( W^{n,m+2\ell}_{\varepsilon_1,\varepsilon_2} \) are in the same block, and the general result will follow. Indeed, we choose \( r \in \mathbb{Z} \) such that \( 0 < \varepsilon_1 w_1 + r\ell < \ell \), then by Theorem 8.3 we have non-zero homomorphisms \( W^{n,m+2\ell}_{\varepsilon_1,\varepsilon_2} \to W^{n,m+2\ell-2(\varepsilon_1 w_1 + r\ell)}_{-\varepsilon_1,\varepsilon_2} \) and \( W^{n,m+2\ell-2(\varepsilon_1 w_1 + r\ell)}_{-\varepsilon_1,\varepsilon_2} \to W^{n,m}_{-\varepsilon_1,\varepsilon_2} \). Therefore our original pair of cell modules are in the same block.

The proof for \( w_2 \in \mathbb{Z} \), \( w_1 \notin \mathbb{Z} \) is similar, except we must consider cases (25) and (29), and use Theorem 8.4 in place of Theorem 8.3. We therefore have the following theorem:

**Theorem 9.3.** Suppose \( q \) is a 2\( \ell \)-th root of unity and \( w_1 \in \mathbb{Z} \), \( w_2 \notin \mathbb{Z} \). Then two cell modules \( W^{n,m}_{\varepsilon_1,\varepsilon_2} \) and \( W^{n,t}_{\eta_1,\eta_2} \) are in the same block if and only if \( \varepsilon_2 = \eta_2 \) and \( |m - \varepsilon_1 w_1 - \varepsilon_2 w_2| \equiv |t - \eta_1 w_1 - \varepsilon_2 w_2| \) (mod 2\( \ell \)).

If now \( w_2 \in \mathbb{Z} \), \( w_1 \notin \mathbb{Z} \), then two cell modules \( W^{n,m}_{\varepsilon_1,\varepsilon_2} \) and \( W^{n,t}_{\eta_1,\eta_2} \) are in the same block if and only if \( \varepsilon_1 = \eta_1 \) and \( |m - \varepsilon_1 w_1 - \varepsilon_2 w_2| \equiv |t - \eta_1 w_1 - \varepsilon_2 w_2| \) (mod 2\( \ell \)).

If \( q \) is not a root of unity, then replace the above two congruences modulo 2\( \ell \) by equalities.

Figure 14 shows two plots of the cell modules, when just \( w_1 \) and just \( w_2 \) are integral respectively. The arrows indicate a homomorphism between the corresponding modules.

### 9.3. Either \( w_1 + w_2 \) or \( w_1 - w_2 \) Integral

We first turn to the case \( w_1 + w_2 \in \mathbb{Z} \) but \( w_1 - w_2 \notin \mathbb{Z} \). Again, we begin by taking \( q \) to not be a root of unity. Here, we are looking to satisfy equations (23) and (27) with \( \varepsilon_1 = \varepsilon_2 \). Once more, we see that blocks have size at most two. Now if we have a solution to equation (23), then for these modules to be in the same block we must have a non-zero
by Theorem 8.5 we have non-zero homomorphisms \( W \) follow. Indeed, we choose Theorem 9.4.

will begin by showing that between the cell modules in the same way as above. It remains, therefore, to deal with (26). We can show that there exists a non-zero homomorphism since neither cell modules have different standard contents. Thus, since \( w_1 \not\in \mathbb{Z} \), we deduce that there can be no such homomorphism.

In the case of equation (27), we can again assume that \( m > t \). Then since \( m \) and \( t \) are both non-negative integers we can only have a solution if \( \varepsilon_1 = \varepsilon_2 = \text{sgn}(w_1 + w_2) \). Therefore we have \( m > t = 2(\varepsilon_1 w_1 + \varepsilon_2 w_2) - m \geq 0 \) (with equality only if \( \varepsilon_1 = \varepsilon_2 = 1 \)), and thus we can use Theorem 8.5 to show that the cell modules \( W_{+,+}^{n,m} \) and \( W_{+,+}^{n,t} \) are in the same block.

If \( q \) is a \( 2\ell \)-th root of unity, then we must consider equations (23), (26), (27) and (30). Again, since neither \( w_1 \) nor \( w_2 \) are integers we can rule out (23) and (30) by restricting to either the left- or right-blob algebra. In the case of (27), we can show that there exists a non-zero homomorphism between the cell modules in the same way as above. It remains, therefore, to deal with (26). We will begin by showing that \( W_{\varepsilon_1,\varepsilon_2}^{n,m} \) and \( W_{\varepsilon_1,\varepsilon_2}^{n,m+2\ell} \) are in the same block, and the general result will follow. Indeed, we choose \( r \in \mathbb{Z} \) such that \( m + 2\ell > 2(\varepsilon_1 w_1 + \varepsilon_2 w_2 + r\ell) - (m + 2\ell) > m \), then by Theorem 8.5 we have non-zero homomorphisms \( W_{\varepsilon_1,\varepsilon_2}^{n,m+2\ell} \rightarrow W_{\varepsilon_1,\varepsilon_2}^{n,2(\varepsilon_1 w_1 + \varepsilon_2 w_2 + r\ell) - (m + 2\ell)} \) and \( W_{\varepsilon_1,\varepsilon_2}^{n,2(\varepsilon_1 w_1 + \varepsilon_2 w_2 + r\ell) - (m + 2\ell)} \rightarrow W_{\varepsilon_1,\varepsilon_2}^{n,m} \). Therefore our original pair of cell modules are in the same block.

**Theorem 9.4.** Suppose \( q \) is a \( 2\ell \)-th root of unity and \( w_1 + w_2 \in \mathbb{Z}, w_1 - w_2 \not\in \mathbb{Z} \). Then two cell modules \( W_{\varepsilon_1,\varepsilon_2}^{(n,m)} \) and \( W_{\eta_1,\eta_2}^{(n,t)} \) are in the same block if and only if \( \varepsilon_1 = \eta_1 = \varepsilon_2 = \eta_2 \) and \( |m - \varepsilon_1 w_1 - \varepsilon_2 w_2| \equiv |t - \eta_1 w_1 - \eta_2 w_2| \mod 2\ell \).

If \( q \) is not a root of unity then replace the above congruence modulo \( 2\ell \) by an equality.

The case \( w_1 - w_2 \in \mathbb{Z}_{>0} \) but \( w_1, w_2, w_1 + w_2 \not\in \mathbb{Z} \) is proved similarly.

![Graphical depiction of the cell modules](image-url)
Both only consider cell modules $W$ of unity. As explained in the beginning of this section, we can globalise appropriately so that we

$$w = \frac{1}{2};$$

and

$$w = -\frac{7}{4},$$

Figure 15 shows two plots of the cell modules, when just $w_1 + w_1$ and just $w_1 - w_2$ are integral respectively. The arrows indicate a homomorphism between the corresponding modules.

### Theorem 9.5

Suppose $q$ is a $2\ell$-th root of unity and $w_1 - w_2 \in \mathbb{Z}, w_1 + w_2 \notin \mathbb{Z}$. Then two cell modules $W^{n,m}_{\xi_1,\xi_2}$ and $W^{n,t}_{\eta_1,\eta_2}$ are in the same block if and only if $\varepsilon_1 = \eta_1 = -\varepsilon_2 = -\eta_2$ and

$$|m - \varepsilon_1 w_1 - \varepsilon_2 w_2| \equiv |t - \eta_1 w_1 - \eta_2 w_2| \pmod{2\ell}.$$

If $q$ is not a root of unity then replace the above congruence modulo $2\ell$ by an equality.

Figure 15 shows two plots of the cell modules, when both $w_1 + w_1$ and $w_1 - w_2$ are integral respectively. The arrows indicate a homomorphism between the corresponding modules.

### Theorem 9.6

Suppose $q$ is a $2\ell$-th root of unity and $w_1 + w_2 \in \mathbb{Z}, w_1 - w_2 \in \mathbb{Z}$ but $w_1, w_2 \notin \mathbb{Z}$.

Then two cell modules $W^{n,m}_{\xi_1,\xi_2}$ and $W^{n,t}_{\eta_1,\eta_2}$ are in the same block if and only if $|m - \varepsilon_1 w_1 - \varepsilon_2 w_2| \equiv |t - \eta_1 w_1 - \eta_2 w_2| \pmod{2\ell}$ and either

1. $\varepsilon_1 = \eta_1 = \varepsilon_2 = \eta_2$, or
2. $\varepsilon_1 = \eta_1 = -\varepsilon_2 = -\eta_2$.

If $q$ is not a root of unity then replace the above congruence modulo $2\ell$ by an equality.

Figure 15 shows a plot the cell modules when both $w_1 + w_2$ and $w_1 - w_2$ are integral, but not $w_1$ nor $w_2$. The arrows indicate a homomorphism between the corresponding modules.

### 9.5 Both $w_1$ and $w_2$ integral

Finally, we consider the case when $w_1, w_2 \in \mathbb{Z}$ and $q$ not a root of unity. As explained in the beginning of this section, we can globalise appropriately so that we only consider cell modules $W^{n,m}_{\xi_1,\xi_2}$ with $m \ll N$ and $w_1, w_2 > 0$. We may also assume that $w_1 \leq w_2$, and...
Figure 16. Graphical depiction of the cell modules of $b'_8$ with $w_1 = \frac{5}{2}$ and $w_2 = -\frac{1}{2}$.

since we can swap blobs and flip diagrams horizontally to make this the case. By fixing $\epsilon_1$ and $\epsilon_2$ and considering the equations (23)–(30) we see that each block has size at most 4 as $q$ is not a root of unity. By repeated globalisation we will determine the blocks containing modules of the form $W_{+,+}^N$ where $N \gg m$. By then repeated localising, we see that we will in fact deal with each cell module in $b'_u$. Care must be taken when localising, as we will encounter blocks of size 4 in the large $N$ limit which may break up into singleton blocks when localising back to $n$.

Consider first the case $m < w_1 + w_2$. Then we see by Theorem 8.5 that we have a homomorphism

$$W_{+,+}^{N,2w_1+2w_2-m} \rightarrow W_{+,+}^{N,m}.$$ 

Now we are assuming that $0 < w_1 \leq w_2$. Thus $2w_1 + 2w_2 - m > w_1 + w_2 \geq 2w_1$, therefore $2w_1 + 2w_2 - m - 2w_1 = 2w_2 - m > 0$ and we can use Theorem 8.3 to obtain a homomorphism

$$W_{+,+}^{N,2w_1+2w_2-m} \rightarrow W_{+,+}^{N,2w_2-m}.$$ 

When localising it is possible that the module $W_{+,+}^{N,2w_1+2w_2-m}$ will be annihilated, in which case the remaining two modules may not be linked via homomorphisms. We now show that this is the case:

**Lemma 9.7.** Suppose $q \in \mathbb{C}^\times$ is not a root of unity and $w_1, w_2 \in \mathbb{Z}_{>0}$ be as above. Let $m < w_1 + w_2$ be a non-negative integer. Then we have

$$\text{Hom}(W_{+,+}^{n,m}, W_{+,+}^{n,2w_2-m}) = \text{Hom}(W_{+,+}^{n,2w_2-m}, W_{+,+}^{n,m}) = 0.$$ 

**Proof.** If we have a non-zero homomorphism between the cell modules, then this must restrict to a homomorphism between cell modules for the left blob algebra. We will show that there can be no such homomorphism.
Suppose first that we have a non-zero homomorphism $W_{+,+}^{n,m} \longrightarrow W_{-,+}^{n,2w_2-m}$. By applying the localisation functor $F$ an even number of times (so that no parameter change occurs), we may assume that $n = m + 1$. Now restricting to the left blob algebra, we see from Table 3 that $W_{+,+}^{m+1,m}$ has standard content $W_{m+1}(m+1)$ and $W_{+,+}^{m+1,2w_2-m}$ has standard content $W_{-(m+1)}(m+1), W_{-(m+1)}(m+1), \ldots, W_{-(m+1)}(m+1)$. As such, we must have a left blob homomorphism from the trivial module to one of the latter standard modules. However since $q$ is not a root of unity, the module $W_{m+1}(m+1)$ is mapped only to $W_{2w_1-m-1}(m+1)$. We then note that $2w_1 - m - 1 > w_1 - w_2 - 1$, whereas $-(2w_2 - m + 1) < w_1 - w_2 - 1$. Therefore there is no left-blob homomorphism in the restriction, and thus no symplectic blob homomorphism.

If we now assume that we have a homomorphism $W_{-,+}^{n,2w_2-m} \longrightarrow W_{+,+}^{n,m}$, then by applying the localisation functor $F$ an odd number of times we may assume that $n = 2w_2 - m$. Then by restricting to the left blob algebra we see that $W_{+,+}^{2w_2-m,2w_2-m-1}$ has standard content $W_{2w_2-m}(2w_2 - m)$, whereas $W_{-,+}^{2w_2-m,m+1}$ has standard content $W_{-(2w_2-m)}(2w_2 - m), \ldots, W_{-(m+2)}(2w_2 - m)$. Again by considering left blob homomorphisms with $q$ not a root of unity, and taking the parameter change $w_1 \mapsto -w_1 - 1$ into account, we see that the trivial module is mapped only to $2(-w_1 - 1) - 2w_2 + m = -2(w_1 + 2w_2) + m - 2$. Note now that this is less than $-2w_2 + m$, which is the least label in the latter set of standard contents. Thus there can be no left blob homomorphism, and hence no symplectic blob homomorphism.

So far we have found three distinct modules with the same eigenvalue, all linked via homomorphisms.

If also $2w_1 + 2w_2 - m > 2w_2$ (so that $m < 2w_1$), then we can use Theorem 8.4 to obtain a homomorphism

$$W_{+,+}^{N,2w_1+2w_2-m} \longrightarrow W_{+,+}^{N,2w_1-m}.$$ 

In a manner similar to Lemma 9.4, we can show that there are no homomorphisms between $W_{+,+}^{N,2w_1-m}$ and $W_{+,+}^{N,2w_2-m}$ or $W_{+,+}^{N,m}$. Since blocks have size at most four, this block has the structure as in Figure 17 where arrows indicate the existence of a homomorphism and lack of arrows indicates the non-existence of a homomorphism.

![Figure 17. The block structure for $m < 2w_1$.](image)

Note that if \( n \leq 2w_1 + 2w_2 - m \), then the module \( W_{n,2w_1+2w_2-m} \) does not exist. Since there are no homomorphisms between the remaining three modules, the block breaks up into singleton blocks.

If now \( 2w_1 + 2w_2 - m = 2w_2 \) (so that \( m = 2w_1 \)), then we claim that there are at most three possible solutions to equations (23)–(30). In particular we can satisfy neither (24) nor (28) and so the block has size at most three, and we have found enough homomorphisms. Since we have fixed \( \varepsilon_1 = 1 \), equations (24) and (28) reduce to \( m \pm t = m \), so that \( t = 0 \). However the only module that exists when \( t = 0 \) also must have \( \eta_1 = \eta_2 = 1 \), which is not valid when considering this pair of equations. We thus have the block structure as in Figure 15 with the same convention for arrows (or lack thereof).

![Figure 18](image-url)  
*Figure 18. The block structure for \( m = 2w_1 \).*

Again we see that if \( n \leq 2w_2 \) then the module \( W_{n,2w_2} \) does not exist. Therefore the block once more breaks up into singleton blocks.

Finally, if \( 2w_1 + 2w_2 - m < 2w_2 \), then we have \( m > 2w_1 \) and so we can again use Theorem 8.3 to obtain

\[
W_{n,m}^+ \rightarrow W_{n,m-2w_1}^-.
\]

We also see that the modules \( W_{n,2w_2-m}^- \) and \( W_{n,m-2w_1}^- \) satisfy the conditions of Theorem 8.5 and thus have a homomorphism

\[
W_{n,2w_2-m}^- \rightarrow W_{n,m-2w_1}^-.
\]

Again, since blocks have size at most four, this block has the structure as in Figure 19 where again the arrows indicate the existence of a homomorphism, the lack of arrows the non-existence, and a dotted line indicates an unknown (which will not matter when considering the block structure).

As in the previous two cases, if \( n \leq 2w_1 + 2w_2 - m \) then the module \( W_{n,2w_1+2w_2-m} \) no longer exists. However since we have homomorphisms between the remaining modules, the block does not decompose further. The same is true if \( n \leq m \) or \( n < 2w_2 - m \).

We have now accounted for all modules \( W_{n,m}^+ \) with \( 0 \leq m \leq 2w_1 + 2w_2 \) \( (m \neq w_1 + w_2) \), \( W_{n,m}^- \) with \( 0 < m \leq 2w_2 \) \( (m \neq w_2 - w_1) \), and \( W_{n,m}^\pm \) with \( 0 < m \leq 2w_1 \).

Suppose now \( m = w_1 + w_2 \), again with \( w_1 \leq w_2 \). In this case, we show that there are at most two modules in the block and find a homomorphism between them. Indeed, from equations (23)–(30)
the only non-trivial solution is \( l = w_2 - w_1, \varepsilon_1 \neq \eta_1, \varepsilon_2 = \eta_2 \). But then this satisfies the conditions of Theorem 8.3, and we have a homomorphism

\[ W_{N,w_1+w_2}^{N,m} \rightarrow W_{N,w_2-w_1}^{N,m}. \]

This gives a block structure as in Figure 20.

\[ W_{N,w_1+w_2}^{N,m+} \rightarrow W_{N,w_2-w_1}^{N,m+} \]

\[ W_{N,w_1+w_2}^{N,m-} \rightarrow W_{N,w_2-w_1}^{N,m-} \]

**Figure 20.** The block structure for \( m = w_1 + w_2 \).

This means that now we have covered all modules \( W_{N,m}^{n,m} \) with \( 0 \leq m \leq 2w_1 + 2w_2 \), \( W_{N,m}^{n,m} \) with \( 0 < m \leq 2w_2 \), and \( W_{N,m}^{n,m} \) with \( 0 < m \leq 2w_1 \).

So it remains to consider the case \( m > 2w_1 + 2w_2 \). Here we have both \( 2w_1 < m \) and \( 2w_2 < m \), so both Theorems 8.3 and 8.4 are satisfied and we have homomorphisms

\[ W_{N,m}^{n,m} \rightarrow W_{N,m-2w_1}^{n,m}, \text{ and} \]

\[ W_{N,m}^{n,m} \rightarrow W_{N,m-2w_2}^{n,m}. \]

But in this case we also have \( m - 2w_2 > 2w_1 \) and \( m - 2w_1 > 2w_2 \), so we can again use Theorems 8.3 and 8.4 to obtain homomorphisms

\[ W_{N,m}^{n,m} \rightarrow W_{N,m-2w_1-2w_2}^{n,m}, \text{ and} \]

\[ W_{N,m}^{n,m} \rightarrow W_{N,m-2w_1-2w_2}^{n,m}. \]

Note that this final case deals with all modules \( W_{N,m}^{n,m} \) with \( m > 2w_1 + 2w_2 \), \( W_{N,m}^{n,m} \) with \( m > 2w_2 \), \( W_{N,m}^{n,m} \) with \( m > 2w_1 \) and \( W_{N,m}^{n,m} \) with \( m > 0 \). In this case, the block structure is shown in Figure 21.
Figure 21. The block structure for large $m$.

The cases displayed in Figures 17–21 above exhaust the list of modules and we are able to give the final main result.

**Theorem 9.8.** Suppose $q$ is not a root of unity and $w_1, w_2 \in \mathbb{Z}$. Let $\sigma_1 = \text{sgn}(w_1), \sigma_2 = \text{sgn}(w_2)$. Then for $n \geq 2|w_1| + 2|w_2| + \frac{1}{2}(\sigma_1 + \sigma_2)$, two cell modules $W_{\varepsilon_1, \varepsilon_2}^{n,m}$ and $W_{\eta_1, \eta_2}^{n,l}$ are in the same block if and only if $|m - \varepsilon_1 w_1 - \varepsilon_2 w_2| = |l - \eta_1 w_1 - \eta_2 w_2|$.

**Proof.** In the case $n \geq 2|w_1| + 2|w_2| + \frac{1}{2}(\sigma_1 + \sigma_2)$, none of the blocks described by Figures 17 and 18 break up into singleton blocks. Therefore in the globalised case with $w_1, w_2 > 0$, the Theorem follows. Notice that

$$|m - \varepsilon_1 w_1 - \varepsilon_2 w_2| = |(m + \varepsilon_1) - (-\varepsilon_1)(-w_1 - 1) - \varepsilon_2 w_2|$$

$$= |(m + \varepsilon_2) - \varepsilon_1 w_1 - (-\varepsilon_2)(-w_2 - 1)|$$

$$= |(m + \varepsilon_1 + \varepsilon_2) - (-\varepsilon_1)(-w_1 - 1) - (-\varepsilon_2)(-w_2 - 1)|,$$

and so the result holds after localising.

In the case $n < 2|w_1| + 2|w_2| + \frac{1}{2}(\sigma_1 + \sigma_2)$, we do not have a succinct characterisation of the blocks of $b'_n$. We therefore make the following statement.

**Theorem 9.9.** Suppose $q$ is not a root of unity and $w_1, w_2 \in \mathbb{Z}$. Let $\sigma_1 = \text{sgn}(w_1), \sigma_2 = \text{sgn}(w_2)$. Then for $n < 2|w_1| + 2|w_2| + \frac{1}{2}(\sigma_1 + \sigma_2)$, the blocks of $b'_n$ are obtained by considering the blocks of the algebra $b'_N$ with $N \gg n, w_1, w_2 > 0$ as in Figures 17–21 then localising back to $b'_n$. The blocks are then given by removing any annihilated modules and associated arrows from the appropriate figure and taking the connected components of what remains.

Turning now to the case when $q$ is a $2\ell$-th root of unity, we will determine blocks in the large $N$ limit. Here, we will show that any pair of cell modules satisfying equations (23)–(30) are in the same block. Therefore in what follows, assume that $W_{\varepsilon_1, \varepsilon_2}^{n,m}$ and $W_{\eta_1, \eta_2}^{n,l}$ are two cell modules satisfying the current equation in question. This will determine $\eta_1, \eta_2$ in terms of $\varepsilon_1, \varepsilon_2$, and also the congruence class of $t$ modulo $2\ell$. 
We deal with equations (24), (25), (26) and (27) simultaneously. By applying the arguments from Sections 9.2 and 9.3 we see that $W_{\varepsilon_1,\varepsilon_2}^{N,m}$ and $W_{\eta_1,\eta_2}^{N,t}$ are in the same block.

Now consider equation (23). In this case we choose $r \in \mathbb{Z}$ such that $m - 2(\varepsilon_1 w_1 + r \ell) > 0$ and consider the cell module $W_{\varepsilon_1,\varepsilon_2}^{N,m-2(\varepsilon_1 w_1 + r \ell)}$. We then see that $W_{\varepsilon_1,\varepsilon_2}^{N,m-2(\varepsilon_1 w_1 + r \ell)}$ and $W_{\varepsilon_1,\varepsilon_2}^{N,m}$ satisfy (24), so are in the same block by previous arguments. Moreover $W_{\varepsilon_1,\varepsilon_2}^{N,m-2(\varepsilon_1 w_1 + r \ell)}$ and $W_{\eta_1,\eta_2}^{N,t}$ satisfy (25) (due to our assumptions on $\eta_1$, $\eta_2$ and $t$), and are therefore in the same block. By transitivity, we see that the original two modules are in the same block.

If the two modules satisfy (28), then then we will again link these modules via a third. In particular we consider $W_{\varepsilon_1,\varepsilon_2}^{N,2(\varepsilon_1 w_1 + \varepsilon_2 w_2 + r \ell) - m}$, where $r \in \mathbb{Z}$ is chosen so that $2(\varepsilon_1 w_1 + \varepsilon_2 w_2 + r \ell) - m > 0$. Then this module and $W_{\varepsilon_1,\varepsilon_2}^{N,m}$ satisfy (27) and are therefore in the same block. Moreover, $W_{\varepsilon_1,\varepsilon_2}^{N,2(\varepsilon_1 w_1 + \varepsilon_2 w_2 + r \ell) - m}$ and $W_{\eta_1,\eta_2}^{N,t}$ satisfy (25) and are in the same block. Therefore the original pair of modules are in the same block.

If the two modules satisfy (29), then we use an argument analogous to the previous paragraph.

Finally, if two modules satisfy (30), then we again use a third module to show that the original two are in the same block. In particular we choose $r \in \mathbb{Z}$ so that $m - 2(\varepsilon_1 w_1 + r \ell) > 0$, then we see that $W_{\varepsilon_1,\varepsilon_2}^{N,m}$ and $W_{\varepsilon_1,\varepsilon_2}^{N,m-2(\varepsilon_1 w_1 + r \ell)}$ are in the same block (as they satisfy (24)), and also $W_{\varepsilon_1,\varepsilon_2}^{N,m-2(\varepsilon_1 w_1 + r \ell)}$ and $W_{\eta_1,\eta_2}^{N,t}$ are in the same block (as they satisfy (28)).

We therefore arrive at the following theorem:

**Theorem 9.10.** Suppose $q$ is a $2\ell$-th root of unity and $w_1, w_2 \in \mathbb{Z}$. Then for $N \gg \max\{m, t\}$, two cell modules $W_{\varepsilon_1,\varepsilon_2}^{N,m}$ and $W_{\eta_1,\eta_2}^{N,t}$ are in the same block if and only if $|m - \varepsilon_1 w_1 - \varepsilon_2 w_2| \equiv |t - \eta_1 w_1 - \eta_2 w_2| \pmod{2\ell}$.

Figure 22 shows a plots of the cell modules when both $w_1$ and $w_2$ are integral. The arrows indicate a homomorphism between the corresponding modules, and the dashed square indicate that these modules are in the same block. Note that away from the extremes of each arm, there is a uniform pattern of concentric squares.

10. Linkage via the Module $W^n(b)$

In the above we have been working with cell modules for $b_n' := b_n^c/I_n(0)$. This precisely excludes the $2^n$-dimensional module $W^n(b)$. We will now deal with linkage via this module in $b_n^c$ and thus complete our investigation into the block structure. We begin with the following theorem:

**Theorem 10.1** ([11, Theorem 5.17]). The Gram determinant $\Gamma_b^n$ of $W^n(b)$ (with respect to the path basis, as defined in (21)) is given by:

- for $n$ even:
  \[
  \Gamma_b^n \equiv \alpha_n \prod_{m=0}^{(n-2)/2} \left( \prod_{\varepsilon_1,\varepsilon_2,\varepsilon_3 = \pm 1} [(1 + 2m + \varepsilon_1 w_1 + \varepsilon_2 w_2 + \varepsilon_3 \theta)/2] \right)^{\sum_{i=1}^{\frac{n}{2}} \sum_{j=1}^{\frac{n}{2}} \varepsilon_i \varepsilon_j} ;
  \]

- for $n$ odd:
  \[
  \Gamma_b^n \equiv \beta_n \prod_{m=0}^{(n-1)/2} \left( \prod_{\varepsilon_1,\varepsilon_2,\varepsilon_3 = \pm 1} [(1 + 2m + \varepsilon_1 w_1 + \varepsilon_2 w_2 + \varepsilon_3 \theta)/2] \right)^{\sum_{i=1}^{\frac{n-1}{2}} \sum_{j=1}^{\frac{n-1}{2}} \varepsilon_i \varepsilon_j} ;
  \]
• for $n$ odd:

$$
\Gamma^n_b = \alpha_n \left( \prod_{\varepsilon_2, \varepsilon_3 = \pm 1} [(w_1 + \varepsilon_2 w_2 + \varepsilon_3 \theta)/2] \right)^{2^{n-1}} \times \prod_{m=1}^{(n-1)/2} \left( \prod_{\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1} [(2m + \varepsilon_1 w_1 + \varepsilon_2 w_2 + \varepsilon_3 \theta)/2] \right)^{\sum_{i=1}^{n-2m+1}(\varepsilon_1 w_1 + \varepsilon_2 w_2 + \varepsilon_3 \theta)/2},
$$

where $\alpha_n$ is given in both cases by

$$
\alpha_n = ([w_1][w_2 + 1])^{-2 \sum_{m=0}^{n-1}(\varepsilon_1 w_1 + \varepsilon_2 w_2 + \varepsilon_3 \theta)/2},
$$

up to factors that are units under our standing assumptions.

Assuming $\alpha_n \neq 0$, we therefore see that the module $W^n(b)$ is irreducible unless $\theta$ is congruent to $\pm(-m + \varepsilon_1 w_1 + \varepsilon_2 w_2)$ modulo $2\ell$ for some integer $m$. Since this module has label larger than all the other cell modules in the poset ordering, being irreducible implies that it is alone in its block. (This remains true even in the non-quasi-hereditary case as this module has dimension larger than all the other cell modules.) We will say that if $\theta \equiv \pm(-m + \varepsilon_1 w_1 + \varepsilon_2 w_2)$ (mod $2\ell$) then $\theta$ is critical. If $\theta$ is not critical, then the module $W^n(b)$ is always a singleton block and the algebra $b^n_x$ has the same blocks as $b^n_x$ with the added singleton block. Therefore we henceforth suppose that $\theta$ is critical.

Suppose that we have two non-isomorphic submodules $W^n_{\varepsilon_1, \varepsilon_2}$, $W^n_{\varepsilon_1, \varepsilon_2}$ of $W^n(b)$. These each correspond to a singular factor of the gram determinant, $\Gamma^n_b$. The condition for two of the factors in the gram determinant to be equal to zero is the same condition for the central element $Z^n$. 

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**Figure 22.** Graphical depiction of the cell modules of $b'_1$ with $w_1 = 3$ and $w_2 = 1$. 

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Theorem 10.2. The block of the symplectic blob algebra, \( b_n^\theta \), containing \( W^n(b) \) when \( \theta \) has a critical value, \( \theta = -m + \epsilon_1 w_1 + \epsilon_2 w_2 \), is as follows:

(i) If \( q \) is not a root of unity and none of \( w_1, w_2, w_1 \pm w_2 \) are integral, then the only non-singleton block is the one containing \( W^n(b) \) and then \( W^n(b) \) and \( W^{n,m} \) are in the same block.

(ii) If \( q \) is a primitive \( 2\ell \)-th root of unity and none of \( w_1, w_2, w_1 \pm w_2 \) are integral, then the module \( W^n(b) \) is in the same block as all \( W^{n,t}_{\eta_1,\eta_2} \) with \( \epsilon_1 = \eta_1, \epsilon_2 = \eta_2 \) and \( m \equiv t \pmod{2\ell} \) and all \( W^{n,t'}_{\eta_1,\eta_2} \) with \( \epsilon_1 = -\eta_1', \epsilon_2 = -\eta_2' \) and \( m \equiv -t' \pmod{2\ell} \).

(iii) If \( w_1 + w_2 \in \mathbb{Z} \) and none of \( w_1, w_2, w_1 - w_2 \) are integral, then the module \( W^n(b) \) is in the same block as all \( W^{n,t}_{\eta_1,\eta_2} \) with \( \epsilon_1 = \eta_1 = \epsilon_2 = \eta_2 \) and \( |m - \epsilon_1 w_1 - \epsilon_2 w_2| \equiv |t - \eta_1 w_1 - \eta_2 w_2| \pmod{2\ell} \), and all \( W^{n,t'}_{\eta_1',\eta_2'} \) with \( \epsilon_1 = -\eta_1' = \epsilon_2 = -\eta_2' \) and \( |m - \epsilon_1 w_1 - \epsilon_2 w_2| \equiv |t - \eta_1' w_1 - \eta_2' w_2| \pmod{2\ell} \), where the \( q \) is a primitive \( 2\ell \)-th root of unity, or the congruence is an equality if \( q \) is not an \( 2\ell \)-th root of unity.

(iv) If \( w_1 - w_2 \in \mathbb{Z} \) and none of \( w_1, w_2, w_1 + w_2 \) are integral, then the module \( W^n(b) \) is in the same block as all \( W^{n,t}_{\eta_1,\eta_2} \) with \( \epsilon_1 = -\eta_1 = -\epsilon_2 = \eta_2 \) and \( |m - \epsilon_1 w_1 - \epsilon_2 w_2| \equiv |t - \eta_1 w_1 - \eta_2 w_2| \pmod{2\ell} \), and all \( W^{n,t'}_{\eta_1',\eta_2'} \) with \( \epsilon_1 = -\eta_1' = -\epsilon_2 = \eta_2' \) and \( |m - \epsilon_1 w_1 - \epsilon_2 w_2| \equiv |t - \eta_1' w_1 - \eta_2' w_2| \pmod{2\ell} \), where the \( q \) is a primitive \( 2\ell \)-th root of unity, or the congruence is an equality if \( q \) is not an \( 2\ell \)-th root of unity.

(v) If \( w_1 + w_2 \) and \( w_1 - w_2 \in \mathbb{Z} \) and none of \( w_1, w_2, \) are integral, then the module \( W^n(b) \) is in the same block as all \( W^{n,t}_{\eta_1,\eta_2} \) with \( |m - \epsilon_1 w_1 - \epsilon_2 w_2| \equiv |t - \eta_1 w_1 - \eta_2 w_2| \pmod{2\ell} \), where the \( q \) is a primitive \( 2\ell \)-th root of unity, or the congruence is an equality if \( q \) is not an \( 2\ell \)-th root of unity.

(vi) If \( w_1 \in \mathbb{Z} \) or \( w_2 \in \mathbb{Z} \) (but not both) and none of \( w_2, w_1 \pm w_2 \) are integral, then the module \( W^n(b) \) is in the same block as all \( W^{n,t}_{\eta_1,\eta_2} \) with \( |m - \epsilon_1 w_1 - \epsilon_2 w_2| \equiv |t - \eta_1 w_1 - \eta_2 w_2| \pmod{2\ell} \), where the \( q \) is a primitive \( 2\ell \)-th root of unity, or the congruence is an equality if \( q \) is not an \( 2\ell \)-th root of unity.
2ℓ), where the q is a primitive 2ℓ-th root of unity, or the congruence is an equality if q is not an 2ℓ-th root of unity.

(vii) If both \( w_1, w_2 \in \mathbb{Z} \) then the module \( W^n(b) \) is in the same block as all \( W_{m,n,t}^{n',m'} \) with \( |m - \varepsilon_1 w_1 - \varepsilon_2 w_2| \equiv |t - \eta_1 w_1 - \eta_2 w_2| \pmod{2\ell} \), where the q is a primitive 2ℓ-th root of unity, or the congruence is an equality if q is not an 2ℓ-th root of unity. (This is the only case where the module \( W^n(b) \) does not join two blocks from \( b_n \).)

Proof. (i). Clear as all \(-m + \varepsilon_1 w_1 + \varepsilon_2 w_2 - \theta\) are distinct in this case and considering the Gram determinant of \( W^n(b) \).

(ii). The module \( W_{n,m}^{n',m'} \) embeds into \( W^n(b) \), by the assumption on \( \theta \). Thus \( W^n(b) \) is in the same block as all \( W_{m,n,t}^{n',m'} \) that are linked to \( W_{n,m}^{n',m'} \). Pick \( a \in \mathbb{Z} \) such that \( 0 \leq 2a \ell - m \leq n \). Then \([2a \ell - m + \varepsilon_1 w_1 + \varepsilon_2 w_2 - \theta]/2] = 0\), so \( W_{n,m}^{2a \ell - m} \) is a submodule of \( W^n(b) \). Hence \( W^n(b) \) is in the same block as all \( W_{m,n}^{n',m'} \) that are linked to \( W_{n,m}^{2a \ell - m} \).

Thus using the proof of Theorem 9.2, the block containing \( W^n(b) \) is the same as the solutions to equations (20) and (30).

(iii). As in (ii), \( W^n(b) \) is in the same block as \( W_{n,m}^{n',m'} \) and this gives the first condition. Pick \( a \in \mathbb{Z} \) such that \( 0 \leq 2a \ell + m - 2\varepsilon_1 (w_1 + w_2) \leq n \). (NB: \( \ell \) is taken to be zero if q is not a primitive 2ℓ-th root of unity. Of course, then such an a may not exist, but then there are no further solutions to the equations that need to be considered.) Consider \( t = m + 2\varepsilon_1 (w_1 + w_2) \) and the module \( W_{n,m}^{n,t} \). For this module \([t + \varepsilon_1 w_1 + \varepsilon_2 w_2 + \theta]/2] = 0\), so \( W_{n,m}^{n,t} \) is a submodule of \( W^n(b) \) and thus they are in the same block. This gives the second condition using Theorem 9.2. These two conditions combined give all solutions to equations (23), (26), (27) and (30), thus this is the whole block.

(iv). This is similar to (iii).

(v). This merges (iii) and (iv) and says that the block is determined by the action of the central element.

(vi). As in (ii), \( W^n(b) \) is in the same block as \( W_{n,m}^{n',m'} \). Pick \( a \in \mathbb{Z} \) such that \( 0 \leq 2a \ell - m + 2\varepsilon_1 w_1 \leq n \). (NB: \( \ell \) is taken to be zero if q is not a primitive 2ℓ-th root of unity. Of course, then such an a may not exist, but then there are no further solutions to the equations that need to be considered.) Consider \( t = -m + 2\varepsilon_1 w_1 \) and the module \( W_{n,m}^{n,t} \). For this module \([-t + \varepsilon_1 w_1 - \varepsilon_2 w_2 + \theta]/2] = 0\), so \( W_{n,m}^{n,t} \) is a submodule of \( W^n(b) \) and thus they are in the same block. This gives the condition as stated using Theorem 9.3, which is the same as the condition for the central element to act by zero.

(vii). Clear as the block is already determined by the action of the central element. □
Appendix A. Reduction to Hom spaces

It is well known that we may identify the blocks of a finite dimensional algebra with the connected components of the Ext\(^1\) quiver between simple modules. Here we prove that finding these connected components is equivalent to determining the connected components of the Hom quiver between standards. Thus in determining the blocks, it is only necessary to compute enough homomorphisms between standard modules in order to find these connected components.

(Here by the Hom quiver between standards, we mean take the quiver whose vertices are labelled by standard modules, ∆(µ) and with the number of arrows from λ to µ equal to the dimension of Hom(∆(λ), ∆(µ)).)

Proposition A.1. Let \( A \) be a quasi-hereditary algebra with a simple preserving duality and poset \((Λ, ≤)\). For \( λ \in Λ \), let the standard modules be denoted by ∆(λ), costandards by ∇(λ), the principal indecomposable modules by \( P(λ) \), the irreducible head of this module by \( L(λ) \) and the indecomposable injective hulls by \( I(λ) \).

The blocks of \( A \) may be identified with the connected components of the Hom quiver between standards.

Proof. Now if Hom(∆(λ), ∆(µ)) is non-zero then \( L(λ) \) must be a composition factor of ∆(µ) as ∆(λ) has simple head \( L(λ) \). Thus \( L(λ) \) and and \( L(µ) \) are in the same block. Thus the connected components of the Ext\(^1\) quiver on simples are disjoint unions of the connected components of the Hom quiver on standards.

We now prove the converse, that the connected components of the Hom quiver on standards are disjoint unions of the of the Ext\(^1\) quiver on simples. Let λ and µ ∈ Λ with Ext\(^1\)(L(λ), L(µ)). Without loss of generality we may assume that \( λ ≥ µ \) as \( A \) has a simple preserving duality (and hence Ext\(^1\)(L(λ), L(µ)) = Ext\(^1\)(L(µ), L(λ))). Since the extension of \( L(λ) \) by \( L(µ) \) has simple socle and \( λ ≥ µ \), this extension must be a quotient of ∆(λ) and in particular [∆(λ) : L(µ)] ≠ 0. Thus Hom(\( P(µ) \), ∆(λ)) is non-zero, as the dimension of Hom(\( P(µ) \), ∆(λ)) is equal to [∆(λ) : L(µ)].

Now if Hom(∆(µ), ∆(λ)) is non-zero then we are done, so suppose that Hom(∆(µ), ∆(λ)) = 0. As Hom(\( P(µ) \), ∆(λ)) is non-zero, there is a submodule of ∆(λ), \( M \), which is a quotient of \( P(µ) \) and hence has simple head \( L(µ) \). As Hom(∆(µ), ∆(λ)) is zero, this quotient \( M \) must contain a composition factor, \( L(ν) \) say, for which \( ν ≤ µ \). Let \( L(ν) \) be a largest composition factor in \( M \).

Now as \( M \) is a submodule of ∆(λ) we must have that \( ν < λ \). (It cannot be equal as then \( M \) would be the whole of ∆(λ) contradicting Hom(∆(µ), ∆(λ)) = 0.) Now take the largest submodule of \( M \) with \( L(ν) \) as its simple head. As \( ν \) was maximal, this submodule has composition factors strictly less than \( ν \) and hence is a quotient of ∆(ν) and so Hom(Δ(ν), ∆(λ)) ≠ 0.

Also, as \( M \) is not a quotient of ∆(µ), it must contain a proper submodule \( N \) which is is a quotient of the kernel \( Q \) of the projection map from \( P(µ) \) to ∆(µ). This proper submodule \( N \)
must contain the $L(\nu)$ as $L(\nu)$ is not a composition factor of $\Delta(\mu)$. As $\nu$ is maximal, this $\nu$ must be the head of some $\Delta$ appearing in a $\Delta$-filtration of $Q$. I.e. $\Delta(\nu)$ is a section of $Q$. Thus $\text{Hom}(P(\mu), \nabla(\nu))$ is nonzero. Using the duality we then have $\text{Hom}(\Delta(\nu), I(\mu)) \neq 0$, which implies that $L(\mu)$ is a composition factor of $\Delta(\mu)$ and thus that $\text{Hom}(P(\mu), \Delta(\nu))$ is non-zero.

If $\text{Hom}(\Delta(\mu), \Delta(\nu))$ is non-zero we may stop, otherwise we repeat the argument until we have a chain of $\nu_i$’s with $\lambda > \nu_1 > \nu_2 > \cdots > \nu_m$ and $\text{Hom}(\Delta(\nu_i), \Delta(\nu_{i+1})) \neq 0$. Since $\Lambda$ is finite this chain must stop eventually with $\text{Hom}(\Delta(\nu_m), \Delta(\mu)) \neq 0$. We may thus conclude that $\lambda$ and $\mu$ are in the same connected component of the Hom quiver on standards. □

References

[1] H. H. Andersen, The strong linkage principle, J. reine angew. Math. 315 (1980), 53–59.
[2] R. J. Baxter, Exactly solved models in statistical mechanics, Academic, 1981.
[3] D. J. Benson, Representations and Cohomology I, Cambridge Studies in Advanced Mathematics, no. 30, Cambridge University Press, 1995.
[4] Christopher Bowman, Anton Cox, and Liron Speyer, A family of graded decomposition numbers for diagrammatic Cherednik algebras, Int. Math. Res. Not. IMRN (2017), no. 9, 2686–2734. MR 3658213
[5] R. Brauer, On modular and $p$-adic representations of algebras, Proceedings of the National Academy of Sciences 25 (1939), no. 5, 252–258.
[6] Jonathan Brundan and Alexander Kleshchev, Blocks of cyclotomic Hecke algebras and Khovanov-Lauda algebras, Invent. Math. 178 (2009), no. 3, 451–484. MR 2551762
[7] A. G. Cox, J. J. Graham, and P. P. Martin, The blob algebra in positive characteristic, J. Algebra 266 (2003), 584–635.
[8] A. G. Cox, P. P. Martin, A. E. Parker, and C. Xi, Representation theory of towers of recollement: theory, notes, and examples, J. Algebra 302 (2006), 340–360.
[9] A G Cox, M De Visscher, and P P Martin, A geometric characterisation of the blocks of the Brauer algebra, JLMS 80 (2009), 471–494, [math.RT/0612584].
[10] Z. Daugherty and A. Ram, Two boundary Hecke algebras and combinatorics of type C, [arXiv:1803.10296] (2018).
[11] J. de Gier and A. Nichols, The two-boundary Temperley-Lieb algebra, J. Algebra 321 (2009), no. 4, 1132–1167.
[12] V. Deodhar, A brief survey of Kazhdan–Lusztig theory and related topics, Proceedings of the Summer Research Institute on Algebraic Groups and their generalisations, July 6-26, 1991, AMS, 1994, pp. 105–124.
[13] V. Drabl and C. M. Ringel, The module theoretical approach to quasi-hereditary algebras, Representations of algebras and related topics (Kyoto, 1990) (H. Tachikawa and S. Brenner, eds.), London Math. Soc. Lecture Note Ser., vol. 168, Cambridge Univ. Press, Cambridge, 1992, pp. 200–224.
[14] M. Ehrig and C. Stroppel, Schur-Weyl duality for the Brauer algebra and the ortho-symplectic Lie superalgebra, [arXiv:1412.7853] (2014).
[15] J. Graham and G. Lehrer, Cellular algebras, Inventiones Math. 123 (1996), 1–34.
[16] R. M. Green, P. P. Martin, and A. E. Parker, Towers of recollement and bases for diagram algebras: planarity and beyond, J. Algebra 316 (2007), 392–452.
[17] , A presentation for the symplectic blob algebra, J. Algebra Appl. 11 (2012), no. 3, 1250060, 22.
[18] , On quasi-heredity and cell module homomorphisms in the symplectic blob algebra, [arXiv:1707.06520] (2017).
[19] R. Hartshorne, Algebraic geometry, Springer, 1977.
[20] J. E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge Studies in Advanced Mathematics, no. 30, Cambridge University Press, 1990.

[21] J. L. Jacobsen and H. Saleur, *Combinatorial aspects of boundary loop models*, J. Stat. Mech. (2008).

[22] J. C. Jantzen, *Representations of Algebraic Groups*, Pure Appl. Math., vol. 131, Academic Press, San Diego, 1987.

[23] M. Khovanov and A. D. Lauda, *A diagrammatic approach to categorification of quantum groups. I*, Represent. Theory 13 (2009), 309–347.

[24] N Libedinsky and D Plaza, *Blob algebra approach to modular representation theory*, arXiv:1801.07200 (2018).

[25] G. Lusztig, *Left cells in Weyl groups*, LNM 1024, Springer (1983).

[26] P. P. Martin, *Potts models and related problems in statistical mechanics*, World Scientific, Singapore, 1991.

[27] P P Martin, *On Schur-Weyl duality, \(A_n\) Hecke algebras and quantum \(sl(N)\)*, Int J Mod Phys A 7 suppl.1B (1992), 645–674.

[28] P. P. Martin and D. Woodcock, *The partition algebras and a new deformation of the Schur algebras*, J. Algebra 203 (1998), 91–124.

[29] P. P. Martin and D. Woodcock, *Generalized blob algebras and alcove geometry*, LMS Journal of Computation and Mathematics 6 (2003), 249–296.

[30] D. Plaza and S. Ryom-Hansen, *Graded cellular bases for Temperley-Lieb algebras of type A and B*, J. Algebraic Combin. 40 (2014), no. 1, 137–177.

[31] A. Reeves, *Tilting modules for the symplectic blob algebra*, arXiv:1111.0146 (2012).

[32] C. M. Ringel, *The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences*, Math. Z. 208 (1991), 209–225.

[33] R. Rouquier, *q-Schur algebras and complex reflection groups*, Mosc. Math. J. 8 (2008), no. 1, 119–158, 184.

[34] S. Ryom-Hansen, *Kazhdan-Lusztig polynomials and a combinatoric for tilting modules*, Representation Theory 1 (1997), 83–114.

[35] H. N. V. Temperley and E. H. Lieb, *Relations between the “percolation” and “colouring” problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the “percolation” problem*, Proc. Roy. Soc. London Ser. A 322 (1971), no. 1549, 251–280.

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