A note on the gauge symmetries of pure Chern-Simons theories with $p$-form gauge fields

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Abstract

The gauge symmetries of pure Chern-Simons theories with $p$-form gauge fields are analyzed. It is shown that the number of independent gauge symmetries depends crucially on the parity of $p$. The case where $p$ is odd appears to be a direct generalization of the $p = 1$ case and presents the remarkable feature that the timelike diffeomorphisms can be expressed in terms of the spatial diffeomorphisms and the internal gauge symmetries. By contrast, the timelike diffeomorphisms may be an independent gauge symmetry when $p$ is even. This happens when the number of fields and the spacetime dimension fulfills an algebraic condition which is explicitly written.


1 Introduction

Pure Chern-Simons theory in 3 dimensions is one of the most studied examples of a topological field theory. It is a model which does not involve the spacetime metric but is yet generally covariant. Furthermore, as noted by many authors [1], the spacetime diffeomorphisms

$$\delta_\xi A^a_\mu = \mathcal{L}_\xi A^a_\mu = \xi^\rho F^a_{\rho\mu} + D_\mu (\xi^\rho A^a_\rho),$$

(1.1)

are not independent from the internal gauge symmetries

$$\delta_\xi A^a_\mu = D_\mu \epsilon^a$$

(1.2)

since (1.1) reduces to (1.2) with $\epsilon^a = \xi^\rho A^a_\rho$ when the equations of motion hold ($F^a_{\rho\mu} = 0$). In (1.1) and (1.2), the $A^a_\mu$ are the components of the connection, while the $F^a_{\rho\mu}$ are those of the curvature 2-form, respectively.

Due to the non-independence of the spacetime diffeomorphisms, the only first class-constraints in the Hamiltonian formulation are the constraints associated with the internal gauge symmetry (1.2). There is no independent first class constraint associated with (1.1). Thus, by finding the most general state invariant under (1.2), one may automatically produce states which are invariant under (1.1). Implementing diffeomorphism invariance is then a mere byproduct of implementing invariance under internal gauge symmetries.

It was shown in recent publications [2] that this crucial property of three-dimensional Chern-Simons theory does not survive in the higher dimensional generalizations. In those models, described by the Lagrangian $(2k+1)$-form

$$\mathcal{L}_{C.S.}^{(2k+1)} = g_{a_1...a_{k+1}} F^{a_1} \wedge ... \wedge F^{a_k} \wedge A^{a_{k+1}} + \text{"more"} (k \geq 2)$$

(1.3)

where $g_{a_1...a_{k+1}}$ is an invariant tensor and where "more" completes (1.3) so as to make it invariant under (1.2) up to an exact term. It is generically no longer true that the diffeomorphisms (1.1), which are still symmetries of (1.3), can be expressed in terms of the internal gauge transformations (1.2).

The reason is that the equations of motion no longer imply the vanishing of the curvature two-form $F^a_{\mu\nu}$. At the same time, the theory described by (1.3) does possess degrees of freedom for $k > 1$ while it does not for $k = 1$. 
However, the higher-dimensional Chern-Simons models still possess a remarkable feature. Namely, not all the spacetime diffeomorphisms are independent gauge transformations, but only those defined by spatial reparameterizations. As established in [2], the timelike diffeomorphisms can be expressed in terms of the internal gauge symmetries and the spacelike diffeomorphisms. This implies that, in the Hamiltonian formalism, there are first class constraints $G_a \approx 0$ generating the internal gauge symmetries (1.2) and first class constraints $\mathcal{H}_i \approx 0$ generating the spatial diffeomorphisms, but there are no independent first class constraints associated with the timelike diffeomorphisms. In order to find the most general gauge invariant state, it suffices to solve the conditions $\mathcal{H}_i |\psi> = 0$ and $G_a |\psi> = 0$. Since these conditions are purely kinematical, they are in principle more easily amenable to an exact resolution (using e.g. the loop representation [3]). From this point of view, Chern-Simons theories in higher dimensions are of great interest to relativists since they provide non trivial models (models with a finite number of local degrees of freedom per space point) in which the problem of implementing quantum-mechanically invariance under the full diffeomorphism group is reduced to the kinematical problem of finding quantum states invariant under spatial diffeomorphisms. They are in that respect comparable to the toy models analyzed in [4].

In the search for other generally covariant theories with the same property, we have analyzed the dynamics of pure Chern-Simons theories with forms of higher degree. These are described by the Lagrangian $n$-form

$$\mathcal{L} = g_{a_1...a_{k+1}} B^{a_1} \wedge H^{a_2} \wedge \ldots \wedge H^{a_{k+1}}$$

(1.4)

where $B^a(a = 1, \ldots, N)$ are $p$-forms and $H^a$ are the corresponding (Abelian) curvatures,

$$H^a = dB^a.$$ 

(1.5)

The spacetime dimension $n$ is equal to $p + k(p + 1)$. We shall call $k$ the order of the Chern-Simons theory. The $(p + 1)$-forms $H^a$ are commuting when $p$ is odd and anticommuting when $p$ is even. If $p$ is odd, the not totally symmetric parts of $g_{a_1...a_{k+1}}$ contribute a total derivative to $\mathcal{L}$, and one may then assume that $g_{a_1...a_{k+1}}$ is totally symmetric. Similarly, if $p$ is even, we will assume that $g_{a_1...a_{k+1}}$ is totally antisymmetric. The Chern-Simons Lagrangian is such that in $(n+1)$ dimensions, $\int d\mathcal{L}$ is the topological invariant $\int g_{a_1...a_{k+1}} H^{a_1} \wedge \ldots \wedge H^{a_{k+1}}$. The Lagrangian $\mathcal{L}$ is invariant under
the usual $p$-form Abelian gauge transformations

$$\delta_{\Lambda} B^a = d\Lambda^a$$

(1.6)

where $\Lambda^a$ are $N(p - 1)$-forms, as well as under spacetime diffeomorphisms

$$\delta_\xi B^a = \mathcal{L}_\xi B^a.$$  (1.7)

The first gauge symmetry (1.6) is reducible since if one takes $\Lambda^a = d\mu^a$, one gets $\delta_{\Lambda} B^a = 0$ (identically). The second gauge symmetry (1.7) can be rewritten equivalently as

$$\delta'_\xi B^a = i_\xi H^a$$  (1.8)

since $\mathcal{L}_\xi = i_\xi d + di_\xi$.

The equations of motion following from (1.4) are explicitly

$$g_{a_1a_2...a_{k+1}} H^{a_2} \wedge \ldots \wedge H^{a_{k+1}} = 0$$

(1.9)

and do not imply that the curvatures $H^a$ vanish, unless the order $k$ is equal to one.

$P$-form gauge fields appear systematically in supergravity theories in higher dimensions [5]. A Chern-Simons-like term appears in the Lagrangian of $n = 11$ supergravity [6], besides the kinetic term proportional to $H^{\mu\nu\lambda\rho} H_{\mu\nu\lambda\rho}$.

We have found that the pure Chern-Simons theories with Lagrangian density (1.4) generically have local degrees of freedom when $k \geq 2$. This is not surprising in view of the analysis of [2]. The new result, however, is that the timelike diffeomorphisms may now be independent gauge symmetries.

This will occur only when $p$ is even, and the number $\kappa$ defined by

$$\kappa \equiv N \left( \frac{n - 1}{p} \right) - (n - 1)$$

(1.10)

is an odd integer.

The number $\kappa$ is the difference between the total number of spatial components of the set of $p$-forms, and the number of spatial dimensions in the theory.

The above condition can be fulfilled by choosing appropriately $N$ and $k$ (for a given even $p$). There is thus a striking difference between pure Chern-Simons theories with $p$-forms of even degree and pure Chern-Simons theories
with $p$-forms of odd degree. This difference could not have been anticipated by a mere look at the gauge symmetries (1.6), (1.7), (1.8) - which take the same form in all cases - and requires a more detailed investigation of the dynamics. Our results imply that the dependence of the timelike diffeomorphisms on the other gauge symmetries found for $p = 1$ in [2] is not a remnant of the topological construction leading to these models. Indeed this construction is identical for all values of $p$, $N$ or $k$. The phenomenon has a different origin which is yet to be uncovered.

2 The case $p = 2$, $k = 2$

2.1 Constraints

We begin the discussion with $p = 2$. The condition $\kappa = \text{odd integer}$ becomes, using $n = k(p + 1) + p$:

$$(3k + 1)[\frac{3kN}{2} - 1] = \text{odd integer}$$  \hspace{1cm} (2.1)$$

and cannot be realized for $k = 1$ for which the theory has actually no degrees of freedom. We thus take the simplest case, namely $k = 2$, which yields

$$7[3N - 1] = \text{odd integer},$$  \hspace{1cm} (2.2)$$

a condition that is fulfilled if and only if $N$ is an even integer.

The goal of this section is to show that indeed, the timelike diffeomorphisms can be expressed in terms of the internal gauge transformations and the spatial diffeomorphisms for odd values of $N$, while they are independent gauge symmetries for even values of $N$ (except for low values of $N$ ($N < 8$) where there are accidental degeneracies[1]). The proof of this property turns out to be of mathematical interest in its own right since it involves the discrete projective plane with 7 points related to the octonion algebra.

\footnote{For $N = 1$ and $N = 2$, the Lagrangian vanishes identically and those theories are trivial. For $N = 4$, one field identically drops out because any antisymmetric tensor $g_{a b c}$ admits a “zero vector” $\lambda^a \neq 0$, solution of the equation $g_{a b c} \lambda^c = 0$. So, one is effectively reduced to the $N = 3$ case plus one field with zero Lagrangian. Finally, the case $N = 6$ appears to be equivalent to two independent $N = 3$ theories and thus has twice the number of gauge symmetries as the $N = 3$ case.}
In order to analyze the gauge symmetries, we rewrite the Lagrangian (1.4) in Hamiltonian form. It is well appreciated by now that the concept of gauge symmetry cannot be thought of independently from the dynamics. This appears quite strikingly when discussing the (in)dependence of a given set of a gauge transformations, since the form of the “on-shell trivial” gauge symmetries explicitly involves the dynamics. It is for this reason that the Hamiltonian formulation, where the dynamics takes a transparent form, is convenient in the analysis of the gauge symmetries - see for instance [7]. With \( p = 2 \) and \( k = 2 \), the Lagrangian (1.4) reads

\[
\mathcal{L} = g_{abc} B^a \wedge H^b \wedge H^c \quad \text{(2.3)}
\]

where \( H^b \) is the 3-form \( dB^a \), and where \( g_{abc} \) is completely antisymmetric.

When split into space and time, the action (2.3) is up to a boundary term equal to

\[
S = \int dx^0 d^7x \left[ l_{ij} a \dot{B}_{ij} - B_{0i} K_a^i \right] \quad \text{(2.4)}
\]

with

\[
l_{ij} a = \epsilon^{ijk_1 \ldots k_5} g_{abc} B_{k_1 k_2}^b H_{k_3 k_4 k_5}^c \quad \text{(2.5)}
\]

and

\[
K_a^i = \epsilon^{ij_1 j_2 j_3 j_4 j_5 j_6} g_{abc} H_{j_1 j_2 j_3}^b H_{j_4 j_5 j_6}^c \quad \text{(2.6)}
\]

It follows from (2.4) that the temporal components \( B_{0i} \) are not true dynamical degrees of freedom but rather are Lagrange multipliers for the constraints

\[
K_a^i \approx 0. \quad \text{(2.7)}
\]

These are reducible since

\[
K_a^i = 0 \quad \text{identically} \quad \text{(2.8)}
\]

Although (2.4) is linear in the time derivatives, it is not yet in canonical form because the exterior derivative (in field space) of the one-form \( l_{ij} a \delta B_{ij}^a \) is degenerate and thus does not define a symplectic structure. To deal with this, we follow the standard Dirac procedure and define the conjugate momenta through

\[
p_{ij} a = \frac{\partial \mathcal{L}}{\partial \dot{B}_{ij}^a} = l_{ij} a \quad \text{(2.9)}
\]
These momenta are subject to $21N$ primary constraints

$$\Phi_{ij}^a \equiv p_{ij}^a - l_{ij}^a \approx 0 \quad i, j = 1, \ldots, 7; a = 1, \ldots, N$$

(2.10)

($\Phi_{ij}^a = -\Phi_{ji}^a$). It turns out to be more convenient to replace the constraints

$$G_{ij}^a = K_{ij}^a - \partial_j \Phi_{ij}^a \approx 0$$

(2.11)

because the new constraints generate the internal gauge transformations (1.6) in the Poisson brackets,

$$[B_{ij}^a, \int_{\Sigma} d^7 x \Lambda_b^k G_{kl}^b] = (d\Lambda)_{ij}$$

(2.12)

The constraints (2.11) are also clearly reducible since $\partial_i G_{ij}^a = 0$ (identically).

The Hamiltonian action takes the form

$$I_H = \int dx^0 \int d^7 x [p_{ij}^a \dot{B}_{ij}^a - B_{0i}^a G_{ij}^a - u_{ij}^a \Phi_{ij}^a]$$

(2.13)

The Poisson brackets among the constraints are given by

$$[\Phi_{ij}^a(x), \Phi_{kl}^{b'}(x')] = \Omega_{ab}^{ijkl} \delta(x, x')$$

(2.14)

$$[\Phi_{ij}^a(x), G_{kl}^b] = 0$$

(2.15)

$$[G_{ij}^a, G_{kl}^b] = 0$$

(2.16)

where $\Omega_{ab}^{ijkl}$ is an antisymmetric matrix given by

$$\Omega_{ab}^{ijkl} = g_{abc} \epsilon^{ijklm_1m_2m_3} H_{m_1m_2m_3}^c$$

(2.17)

$$\Omega_{ab}^{ijkl} = -\Omega_{ba}^{kl ij}$$

(2.18)

It follows from the constraint algebra that there are no further constraints. It is also clear that the constraints $G_{ij}^a \approx 0$ are first class, as they should since they generate the internal gauge transformations (1.6).
2.2 Rank of $\Omega_{ijkl}^{ab}$ and projective plane $\Pi_7$

2.2.1 Strategy for computing the rank of $\Omega_{ijkl}^{ab}$

To complete the canonical analysis, it is necessary to determine the nature of the constraints $\Phi_{ij}^a \approx 0$. To that end, one must determine the number of zero eigenvectors possessed by the matrix $\Omega_{ijkl}^{ab}$ of the brackets $[\Phi_{ij}^a, \Phi_{kl}^b]$. This number turns out to depend on the number $N$ of fields, on the antisymmetric tensor $g_{abc}$ defining the theory and, for a given choice of $N$ and $g_{abc}$, it depends also on the phase space location of the dynamical system. Indeed, the matrix $\Omega_{ijkl}^{ab}$ involves both $g_{abc}$ and $H_{ijk}^a$ (the latter being constrained by $K_i^a \approx 0$).

To determine all the possible ranks that the matrix $\Omega_{ijkl}^{ab}$ can achieve is a rather complicated algebraic task and we shall not attempt to pursue it here. We will describe what happens in the generic situation in which $\Omega_{ijkl}^{ab}$ has the maximum possible rank compatible with the constraints $K_i^a \approx 0$. We call it generic because maximum rank conditions define open regions in the space of theories (space of the $g$’s) and of allowed configurations (space of the $H$’s) and are thus stable under small deformations. This is not true for lower ranks, which are associated with equations rather than inequalities.

It is easy to see that the matrix $\Omega_{ijkl}^{ab}$ has at least seven zero eigenvectors $H_{klm}^b (m = 1, 2, \ldots, 7)$, since $\Omega_{ijkl}^{ab} H_{klm}^b \approx 0$. These zero eigenvectors are associated with the spatial diffeomorphisms in the sense that the first class constraints

$$\mathcal{H}_i \equiv H_{ik}^a \Phi_{kl}^a \approx 0 \quad (2.19)$$

are the generators of the spatial diffeomorphisms in the “improved” form (1.8). In the generic case, the zero eigenvectors $H_{klm}^b$ are independent because the equations

$$H_{ikl}^a \xi^i = 0 \quad (2.20)$$

imply $\xi^i = 0$ (more on this in subsection 2.4 below).

The study parallels so far quite closely the discussion of pure Chern-Simons theories with 1-form gauge fields. New features arise when one addresses the question as to whether there are further independent first class constraints among the $\Phi$ ’s. These first class constraints (if any) would correspond to additional independent gauge symmetries.

The total number of constraints is the size of $\Omega$, that is to say $21N$. There are as many second class constraints as there are non-vanishing eigenvalues.
of $\Omega$. Therefore, there has to be an even number of second class constraints.
This means that the number of first class constraints among the $\Phi$'s is odd for odd $N$ and even for even $N$. We already know from (2.19) seven of the first class constraints. If $N$ is even, there must be at least one additional first class constraint still to be identified. This does not need to be the case for $N$ odd.

By computer-assisted investigation of the first values of $N$ ($\leq 20$), we have reached the following conclusions. If $N$ is odd (and greater than or equal to three), then there is generically no further first class constraint among the $\Phi$'s; while if $N$ is even (and greater than or equal to eight in order to avoid the accidental degeneracies), then there is generically one and only one additional first class constraint among the $\Phi$'s.

We have come to this conclusion by constructing examples in which the antisymmetric matrix $\Omega_{ij}^{kl}$ has exactly rank $\kappa = 21N - 7$ for $N$ odd, and exactly rank $\kappa - 1 = 21N - 8$ for $N$ even. Since these values of the rank correspond to the maximum rank that $\Omega$ can achieve (the first class constraints (2.19) are always present), they are stable under small deformations and thus generic.

In order to analyse the equations, we proceed as follows:

i. First, we pick at random an arbitrary set of completely antisymmetric tensors $g_{abc};$

ii. Second, we construct the general solution of the constraints $K_{ia}^{l} \equiv g_{abc}H_{ijk}^{b}H_{mnp}^{c}^{l} \equiv 0$ for these given $g_{abc};$

iii. Knowing the curvatures $H_{ijk}^{a}$ (which are subject to the sole condition $K_{ia}^{l} \approx 0$), we compute the matrix $\Omega_{ab}^{ijkl} = \epsilon_{ijklmnp}g_{abc}H_{mnp}^{a}$ and determine its rank.

Of these three steps, the first one and the last one are direct. Only the second one needs further discussion because the equations $K_{a}^{l} = 0$ are quadratic in the unknown $H_{ijk}^{b}$. It is here that we shall use the properties of the finite projective plane $\Pi_7$.

### 2.2.2 Projective plane $\Pi_7$

The projective plane $\Pi_7$ has seven points and seven lines containing three points each, with the properties that any pair of points determines one and
only one line, while any pair of lines intersects in one and only one point. The points and lines of $\Pi_7$ are drawn in figure 2.1, with an arbitrary choice of indices $(1, 2, \ldots, N)$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.1.png}
\caption{The projective plane $\Pi_7$ has seven points $(1,2,3,4,5,6,7)$ and seven lines containing three points each. The set of these lines is the set of triplets $T = \{\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{2, 4, 7\}, \{2, 5, 6\}, \{3, 4, 6\}, \{3, 5, 7\}\}$.}
\end{figure}

\textbf{Caption:} The projective plane $\Pi_7$ has seven points $(1,2,3,4,5,6,7)$ and seven lines containing three points each. The set of these lines is the set of triplets $T = \{\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{2, 4, 7\}, \{2, 5, 6\}, \{3, 4, 6\}, \{3, 5, 7\}\}$.

To the line $(i, j, k) \in T$, one associates the three-form $dx^i \wedge dx^j \wedge dx^k$. Because $\Pi_7$ arises in the description of the octonion algebra (the “product” of two points is the third point on the same line, with a sign fixed by the orientation), we shall call these seven 3-forms the “octonionic 3-forms” and shall denote them by $\omega_{(3)}^\alpha (\alpha = 1, 2, \ldots, 7)$. The rest of the basis 3-forms, formed with triplets not in $T$, shall be denoted by $\varphi_A^{(3)} (A = 1, \ldots, 28)$.

The main property of the octonionic 3-forms is

$$\omega_{(3)}^\alpha \wedge \omega_{(3)}^\beta = 0 \quad (2.21)$$

for any $\alpha, \beta = (1, 2, \ldots, 7)$. Indeed, any pair of lines of $\Pi_7$ have one (and only one if the lines are distinct) point in common, so that $\omega_{(3)}^\alpha$ and $\omega_{(3)}^\beta$ have one $dx^i$ in common, which implies (2.21).
In what follows we shall use differential form notation in the spatial manifold. The constraint equations

$$K^i_a = g_{abc} \epsilon^{ijklmnp} H^b_{jkl} H^c_{mnp} = 0 \quad (2.22)$$

are rewritten as

$$K_a = g_{abc} H^b \wedge H^c = 0 \quad (2.23)$$

where the $H^a$ are regarded as three forms in seven dimensions. The rank of $\Omega$ is determined by the number of solutions of the equation

$$g_{abc} H^b \wedge V^c = 0 \quad (2.24)$$

where the 3-form curvature $H$ satisfies the constraint equation (2.23) and $V^a$ is a 2-form. The zero eigenvectors $H^a_{klm} \xi^m$ described above are simply $i \xi H^a$.

### 2.2.3 The $N = 3$ case

We begin with the $N = 3$ case (the smallest number of fields for which the antisymmetric tensor $g_{abc}$ is non-zero) because it is particularly straightforward. A simple solution for the constraint is given by a linear combination of the octonionic monomials $H^a = \lambda^a_{\alpha} \omega^\alpha_{(3)}$ because $\omega^\alpha \wedge \omega^\beta = 0$.

This solution has maximum rank when none of the coefficient $\lambda^a_{\alpha}$ is zero. Indeed, since $H^a$ is a 3-form, its dual in seven dimensions can be viewed as the symmetric $21 \times 21$ matrix $*H^a_{ij} = \epsilon^{ijklmnp} H^a_{mnp}$, for each value of $a = 1, 2, 3$. We can thus write Eq. (2.24) in matrix form

$$*H^1 V^2 = *H^2 V^1, \quad *H^1 V^3 = *H^3 V^1, \quad *H^2 V^3 = *H^3 V^2. \quad (2.25)$$

The determinant of the matrix $*H^a$, for a given $a$, is easily expressed in terms of the expansion $H^a = \lambda^a_{\alpha} \omega^\alpha_{(3)}$. Let $*H^3$ be equal to

$$*H^3 = a_1 *\omega^1 + a_2 *\omega^2 + \cdots + a_7 *\omega^7 \quad (2.26)$$

where $*\omega^\alpha$ represents the dual of $\omega^\alpha$. The determinant of $*H^3$ (as a $21 \times 21$ matrix) is equal to $2^7 a_1 a_2 \cdots a_7$. Thus, if all the monomials $\omega^\alpha$ are present in (2.26), $*H^3$ is invertible. In that case, one can solve $V^1$ and $V^2$ in terms of $V^3$ from the last two equations in (2.25). Replacing $V^1$ and $V^2$ back in the first equation we find an equation for $V^3$

$$(*H^1 J *H^2 - *H^2 J *H^1)V^3 = 0 \quad (2.27)$$
where $J$ is the inverse of $\ast H^3$. Equation (2.27) can be investigated numerically and one finds that it has (generically) only 7 solutions for $V^3$. Since $V^1$ and $V^2$ are completely determined in terms of $V^3$, this implies that the matrix $\Omega_{\alpha\beta}^{ijkl}$ for the $N = 3$ case has no other zero eigenvalues besides those associated with spatial diffeomorphisms and has thus maximum rank $21 \times 3 - 7 = 56$.

### 2.2.4 The general case

We now consider the general case. For even $N$, the assumption that the $H'$s are combinations of the octonionic 3-forms does not lead to a maximum rank for $\Omega$. Thus one needs to look for solutions in which the $H'$s have a more general expression.

Any 3-form $F$ can be decomposed as

$$F = F_1 + F_2$$

(2.28)

where $F_1$ is the “octonionic part” of $F$,

$$F_1 = \sum_{\alpha=1}^{7} F_\alpha \omega^{(3)}_\alpha$$

(2.29)

and $F_2$ is the “non octonionic part” of $F$,

$$F_2 = \sum_{A=1}^{28} F_{A\varphi^{(3)}}$$

(2.30)

If $G$ is another 3-form with decomposition $G = G_1 + G_2$, the exterior product $F \wedge G$ reads

$$F \wedge G = F_1 \wedge G_2 + F_2 \wedge G_1 + F_2 \wedge G_2$$

(2.31)

There is no term $F_1 \wedge G_1$ because the product of any two octonionic 3-form is zero [see (2.21)].

If one decomposes the 3-forms $H^a$ as in (2.28), the constraints (2.23) become

$$2g_{abc}H_1^b \wedge H_2^c + g_{abc}H_2^b \wedge H_2^c = 0.$$  

(2.32)

These equations are actually $7N$ linear equations for the $7N$ octonionic components $H^a_\alpha$ of the curvatures. They can be solved by taking arbitrarily the
non octonionic components $H^a_A$ of the curvatures and determining then the octonionic components $H^a_A$ through (2.32). [The linear, inhomogeneous system (2.32) is easily verified to have one and only one solution for generic $H^a_A$ 's]. Consequently, even though the constraints are quadratic in $H^a$ one can produce an explicit rational solution by using the octonionic decomposition.

We have carried out the task of solving the constraint along the lines just described for random choices of the constants $g_{abc}$ and of the components $H^a_A$. We have then computed numerically the dimension of the kernel of $\Omega^{ijkl}_{ab}$ and found in each case that the zero eigenvalue was degenerate exactly eight times for $N$ even ($\geq 8$) and seven times for $N$ odd. What we got from these calculations for the first values of $N$ is thus that the rank of $\Omega^{ijkl}_{ab}$ is generically equal to $21N - 8$ for $N$ even ($\geq 8$) and $21N - 7$ for $N$ odd. We have established this result for $N \leq 20$ and shall take it for granted for higher values of $N$.

2.3 Geometrical interpretation of the eighth first class constraint

For $N$ odd (and $N \geq 3$), the first class constraints $G^i_a \approx 0$ and $H^i_i \approx 0$ are the only (independent) first class constraints. The other constraints are second class. This implies that the internal gauge symmetries (1.6), generated by $G^i_a$, and the spatial diffeomorphisms, generated by $H^i_i$, form a complete set of gauge symmetries. Any gauge symmetry of the system, including the timelike diffeomorphisms, can be expressed in terms of them.

For $N$ even (and $\geq 8$) there is, by contrast, an eighth first class constraint among the $\Phi^{ij}_a$ given by $\mu^a_{ij} \Phi^{ij}_a$ where $\mu^a_{ij}$ is a zero eigenvector of $\Omega^{ijkl}_{ab}$,

$$\Omega^{ijkl}_{ab} \mu^b_{kl} = 0,$$

independent of the seven eigenvectors $H^a_{ijm}$ associated with the spatial diffeomorphisms. One may relate the transformation generated by $\Phi^{ij}_a \mu^a_{ij} \equiv H$

$$\delta_\rho B^o_{ij}(\vec{x}) = [B^o_{ij}(\vec{x}), \int \rho(\vec{x}') H(\vec{x}') dx']$$

and

$$\rho \mu^o_{ij}(\vec{x})$$

to the timelike diffeomorphisms as follows.

In the improved form (1.8), the timelike diffeomorphisms read
Now, the equations of motion $g_{abc} H^b \wedge H^c = 0$ imply

$$\Omega_{ijkl}^b H_{0kl}^b = 0$$

(2.37)

Thus, $H_{0kl}^b$ is a linear combination of the zero eigenvectors of $\Omega_{ijkl}^b$,

$$H_{0ij}^a = \lambda \mu_i^a + \lambda^k H_{kij}^a$$

(2.38)

for some $\lambda$ and $\lambda^k$, which are completely determined by $H_{0ij}^a$, $H_{ijk}^a$ and $\mu_i^a$. As a consequence, one has

$$\delta \xi B_{ij}^a = \xi^0 \lambda \mu_i^a + \xi^0 \lambda^k H_{kij}^a$$

(2.39)

which shows that the timelike diffeomorphisms can be expressed in terms of the spacelike diffeomorphisms and the eighth gauge symmetry (2.35). In the generic case, $\lambda$ does not vanish. Accordingly, one can conversely express the symmetry (2.35) in terms of the spatial and timelike diffeomorphisms.

$$\mu_{ij}^a = \frac{1}{\lambda} H_{0ij}^a - \frac{\lambda^k}{\lambda} H_{kij}^a$$

(2.40)

In that sense, one may say that the eighth first class constraints present among the $\Phi$'s when $N$ is even (and $N \geq 8$) generates (a redefined version of) the timelike diffeomorphisms.

### 2.4 Independence of the $\mathcal{H}_i$'s

We have asserted above that the $\mathcal{H}_i$'s defined by (2.19) were independent in the generic case, i.e., that for generic case $H_{ij}^a$'s solutions of $K_i^a = 0$, the system

$$H_{ijk}^a \xi^i = 0$$

(2.41)

has $\xi^i = 0$ as only solution. This can be straightforwardly verified as follows. It is easy to check that the curvatures automatically solve the constraints $K_i^a = 0$ when they have only octonionic components. This was already used for $N = 3$ but clearly holds for any value of $N$. Furthermore, if for each $\alpha$ there is at least one value of the index such that $H_{ij}^a \neq 0$, then, the equations
(2.41) imply $\xi^i = 0$. In other words, for such curvatures, the $21N \times 7$ matrix $H^{(a)}_{ij,k}$ has maximum rank 7. Using again the argument that maximum rank conditions are stable against small deformations, one infers that the system $H^a_{ijk}\xi^i = 0$ implies generically that $\xi^i$ is zero.

### 2.5 Number of degrees of freedom

We conclude the discussion of the 2-form case by counting the number of degrees of freedom. There are $21N$ conjugate pairs $(B^a_{ij}, p^a_{ij})$. These pairs are constrained by the $6N$ first class constraints $G^a_i = 0$ (there are actually $7N$ such constraints, but they are subject to the differential identity $G^a_{i,a} = 0$), as well as by the $21N$ constraints $\Phi^a_{ij} = 0$. Of these, 7 are first class and $21N - 7$ are second class for $N$ odd (and $N \geq 3$); while 8 are first class and $21N - 8$ are second class for $N$ even (and $N \geq 8$). According to the general rule for counting the degrees of freedom (= number $M$ of physical conjugate pairs), one finds

$$M = \frac{1}{2}[2 \times 21N - 2 \times 6N - 2 \times 7 - (21N - 7)]$$

$$= \frac{1}{2}[9N - 7] \quad (N \text{ odd, } N \geq 3)$$

and

$$M = \frac{1}{2}[2 \times 21N - 2 \times 6N - 2 \times 8 - (21N - 8)]$$

$$= \frac{1}{2}[9N - 8] \quad (N \text{ even, } N \geq 8)$$

For $N = 4$, one has the same number of degrees of freedom as for $N = 3$, i.e., 10, while for $N = 6$, one has twice as many degrees of freedom as for $N = 3$, i.e., 20. [For $N = 1$ and 2, the theory is pure gauge and $M = 0$.] In all cases but $N = 1$ or 2, the theory has local degrees of freedom ($M > 0$), as for 1-forms [2]. This concludes the analysis of the case $p = 2, k = 2$.

### 3 Discussion of the general case ($p \geq 2, \ k \geq 2$)
3.1 Constraints

We now turn to the general case $p \geq 2$. If $k = 1$, the Lagrangian is quadratic, $\mathcal{L} = g_{ab} B^a \wedge H^b$, and the equations of motion are just equivalent to $H^b = 0$ (assuming $g_{ab}$ to be invertible). The theory has no local degrees of freedom and one sees from (1.8) that the diffeomorphisms (both temporal and spatial) are not independent gauge symmetries. The case $k \geq 2$ is, however, much richer. Its analysis proceeds as that of the case $p = 2$, $k = 2$. We shall use throughout differential form notations. The number of spatial dimensions is equal to $n = k(p + 1) + p - 1$.

By following the same method as above, one finds that the spatial field strengths $H^a$ are subject to the algebraic constraints

$$K^{i_1 \ldots i_{p-1}}_a = g_{a b_1 \ldots b_k} [^* (H^{b_1} \wedge \ldots \wedge H^{b_k})]^{i_1 \ldots i_{p-1}} \approx 0,$$

which can be written as

$$K_a = g_{a a_1 \ldots a_k} H^{a_1} \wedge H^{a_2} \wedge \ldots \wedge H^{a_k} \approx 0.$$

The $2N \left( \frac{n - 1}{p} \right)$ phase space variables $B^a_{1 \ldots i_p}$ and $p^a_{i_1 \ldots i_p}$ are restricted not only by (3.1), but also by the primary constraints.

$$\Phi^{i_1 \ldots i_p} = p^a_{i_1 \ldots i_p} - p^a_{i_1 \ldots i_p} (B) \approx 0$$

analogous to (2.10), where $p^a_{i_1 \ldots i_p}$ is given by

$$p^a_{i_1 \ldots i_p} = g_{abc_{1 \ldots k-1}} [^* (B^b \wedge H^{c_1} \wedge \ldots \wedge H^{c_{k-1}})]^{i_1 \ldots i_p}.$$

There are no further constraints. One can replace (3.1) by $G^{i_1 \ldots i_{p-1}} \approx 0$, where

$$G^{i_1 \ldots i_{p-1}}_a = K^{i_1 \ldots i_{p-1}}_a - \partial_i \Phi^{i_1 \ldots i_{p-1}} \approx 0.$$

These constraints generate the internal gauge symmetry (1.4) and are first class. The constraints (3.3) have brackets given by

$$[\Phi^{i_1 \ldots i_p}(\vec{x}), \Phi^{j_1 \ldots j_p}(\vec{x}')] = \Omega^{i_1 \ldots i_p j_1 \ldots j_p}_{ab} \delta(\vec{x}, \vec{x}')$$

where the antisymmetric matrix $\Omega$ is

$$\Omega^{i_1 \ldots i_p j_1 \ldots j_p}_{ab} = g_{abc_{1 \ldots k-1}} [^* (H^{c_1} \wedge \ldots \wedge H^{c_{k-1}})]^{i_1 \ldots i_p j_1 \ldots j_p}$$
and satisfies
\[ \Omega_{ab}^{i_1 \ldots i_p j_1 \ldots j_p} = -\Omega_{ba}^{j_1 \ldots j_p i_1 \ldots i_p}. \]  
(3.8)

The number of independent gauge symmetries is equal to the number of independent first class constraints. To determine this number, we have to find how the constraints \( \Phi^{i_1 \ldots i_p} \) split into first class constraints and second class constraints. To that end, we must compute the number of zero eigenvalues of \( \Omega_{ab}^{i_1 \ldots i_p j_1 \ldots j_p} \).

Now, just as in the \( p = 2, k = 2 \) case of the previous section, the \( H^b_{j_1 \ldots j_p l} \) are \( n - 1 \) zero eigenvalues of \( \Omega_{ab}^{i_1 \ldots i_p j_1 \ldots j_p} \),

\[ \Omega_{ab}^{i_1 \ldots i_p j_1 \ldots j_p} H_{j_1 \ldots j_p l}^b = 0 \]  
(3.9)

This is direct consequence of (3.1) and is most easily verified observing that the zero eigenvalues of \( \Omega \) can be viewed as \( p \)-forms \( \mu \) satisfying

\[ g_{ab_1 \ldots b_k} H^{b_1} \wedge \ldots \wedge H^{b_{k-1}} \wedge \mu^b \wedge = 0. \]  
(3.10)

Since \( i_\xi \) is an antiderivation, one obtains from (3.2) that \( i_\xi H^a \) is a solution of (3.10) for any spatial vector \( \xi^i \).

The corresponding first class constraints

\[ H_l^b = H_{j_1 \ldots j_p l}^b \Phi_{b}^{j_1 \ldots j_p} \]  
(3.11)

generate the spatial diffeomorphisms. The question is: Are there further first class constraints among the \( \Phi \)'s? This depends on whether the number

\[ \kappa \equiv N \left( \frac{n-1}{p} \right) - (n-1) \]

is even or odd (we remind that \( n = k(p+1) + p \)).

The antisymmetric matrix \( \Omega_{ab}^{i_1 \ldots i_p j_1 \ldots j_p} \) is necessarily of even rank. Accordingly, if \( \kappa \) is even, there is no a priori reason that the rank of \( \Omega \), known to be \( \leq \kappa \), could not be precisely equal to \( \kappa \). By constrast, if \( \kappa \) is odd, there is definitely one extra zero eigenvector implying the existence of a further independent gauge symmetry, and the rank of \( \Omega \) is at most equal to \( \kappa - 1 \).

In the generic case, the rank of \( \Omega \) is expected to be equal to the maximum value compatible with the existence of the known zero eigenvalues \( i_\xi H \), i.e. \( \kappa \) (\( \kappa \) even) or \( \kappa - 1 \) (\( \kappa \) odd). This is supported by the results found for
Thus, there will be no further independent gauge symmetries besides the internal gauge symmetries and the spatial diffeomorphisms (κ even), or the internal gauge symmetries, the spatial diffeomorphisms and the timelike diffeomorphisms (κ odd). Other independent gauge symmetries could arise for particular choices of field configurations or tensor \( g_{a_1...a_{k+1}} \), but these should be thought of as accidental.

Of course, the above comments would be somewhat empty if κ was always even. But we have seen that at least for \( p = 2 \), one may have an odd κ. This is not the only case. The number κ may be odd whenever \( p \) is even. To prove this statement, we must investigate the parity of κ as a function of the form-degree \( p \), the number \( N \) of fields and the order \( k \) of the Chern-Simons theory.

### 3.2 Case \( p \) odd

If the form degree \( p \) is odd, the spatial dimension \( n - 1 = k(p + 1) + p - 1 \) is even since both \( p + 1 \) and \( p - 1 \) are even. It is straightforward to see that in a space with an even number of dimensions, a form of odd degree has an even number of components,

\[
\binom{u}{p} = \text{even number} \quad (u \text{ even, } p \text{ odd}) \tag{3.12}
\]

\((u = n - 1)\). More precisely,

\[
\binom{u}{p} = \frac{u(u-1)(u-2)...(u-p+1)}{1.2.3...p} \tag{3.13}
\]

\[
= \frac{uK}{p} \tag{3.14}
\]

where \( K \) is the integer

\[
K = \binom{u - 1}{p - 1} = \frac{(u-1)(u-2)...(u-p+1)}{1.2.3...(p-1)}.
\]

Since the left-hand side of (3.14) is an integer, \( uK \) is a multiple of \( p \). Since \( u \) is even, the product \( uK \) is even. Consequently, \( uK \) is a multiple of \( 2p \) since \( p \) is odd. Hence \( uK/p \) is even.
It follows that the integer $\kappa$, given by the difference between two even integers, is also an even integer. Thus, whenever $p$ is odd, the rank of $\Omega$ should be exactly equal to $\kappa$ (barring accidental degeneracies) and the temporal diffeomorphisms should not be independent gauge symmetries. This is exactly as in the $p = 1$ case studied in \cite{1161.05074}.

### 3.3 The eleven dimensional $H \wedge H \wedge B$ theory

Another interesting illustration of the results just stated is given by the case $N = 1$, $p = 3$ and $k = 2$ (implying $n = 11$) for which the Chern-Simons action reads

$$I = \int H \wedge H \wedge B$$

(3.15)

This term arises in supergravity in eleven dimensions \cite{6161.05074} which has recently received much attention in the context of $M$-theory.

To show that $\Omega$ has generically maximum rank 120 (number of primary constraints) - 10 (number of spatial diffeomorphisms) = 110, it is enough to exhibit one 4-form $H$, solution of the constraint equation

$$H \wedge H = 0,$$

(3.16)

for which this property is fulfilled.

A solution for (3.16) with maximum rank can be constructed as follows. We consider the following 4-form,

$$H = f \, dx^1 \wedge dx^3 \wedge dx^5 \wedge dx^7 + 2dx^2 \wedge dx^4 \wedge dx^6 \wedge dx^8 - \alpha \, dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 - \beta \, dx^1 \wedge dx^2 \wedge dx^5 \wedge dx^6 + a \, dx^1 \wedge dx^2 \wedge dx^7 \wedge dx^8 + b \, dx^1 \wedge dx^2 \wedge dx^9 \wedge dx^{10} - \gamma \, dx^3 \wedge dx^4 \wedge dx^5 \wedge dx^6 + c \, dx^3 \wedge dx^4 \wedge dx^7 \wedge dx^8 + d \, dx^3 \wedge dx^4 \wedge dx^9 \wedge dx^{10} + A \, dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8 + B \, dx^5 \wedge dx^6 \wedge dx^9 \wedge dx^{10}$$

where $a, b, c, d, \alpha, \beta, \gamma, A, B, f$ are constants. This form of $H$ has been found by trial and error, by successive complications of the initial attempt that involved only products of the binomials $dx^1 \wedge dx^2, dx^3 \wedge dx^4, dx^5 \wedge dx^6, dx^7 \wedge dx^8, dx^9 \wedge dx^{10}$. 

19
The equation $H \land H = 0$ imposes the following restrictions among the coefficients

$$\alpha A + \beta c + \gamma a = 2f \quad (3.17)$$
$$\alpha B + \beta d + \gamma b = 0 \quad (3.18)$$

and

$$\alpha = ad + bc$$
$$\beta = aB + bA$$
$$\gamma = cB + dA \quad (3.19)$$

Replacing (3.19) in (3.17) and (3.18) we obtain a system of equations for $A$ and $B$ whose solution is

$$A = -\frac{(ad + bc)f}{acbd - (ad + bc)^2}, \quad B = \frac{bdf}{acbd - (ad + bc)^2} \quad (3.20)$$

Therefore, $\alpha, \beta, \gamma$ and $A, B$ are determined in terms of the parameters $a, b, c, d$ and $f$, which are left arbitrary in the solution.

We have computed the rank of $\Omega$ for generic values of the coefficients $a, b, c, d$ and $f$ and found that the maximum rank is achieved. The $120 \times 120$ matrix $\Omega$ has only 10 zero eigenvalues which correspond to the 10 independent diffeomorphisms of the spatial surface. Note that this theory has 19 local degrees of freedom, as easily checked by using the rule for counting degrees of freedom recalled above.

### 3.4 Case $p$ even

If the form degree $p$ is even, $\kappa$ can be even or odd. As a function of the number of fields $N$ for fixed $p$ and $k$, various possibilities may actually arise:

i) $\kappa$ is even no matter $N$ is;

ii) $\kappa$ is even for odd $N$ and odd for even $N$;

iii) $\kappa$ is even for odd $N$ and odd for odd $N$;

iv) $\kappa$ is odd no matter what $N$ is.
We have found an instance of (ii) in the previous case $p = 2$, $k = 2$. To illustrate the other possibilities, we just consider other values of $k$ while keeping $p$ fixed and equal to two.

i) $k = 5$: the number of spatial dimensions is equal to $5 \times 3 + 1 = 16$. A 2-form has $\binom{16}{2} = \frac{16 \times 15}{2} = 120$ spatial components. Thus $\kappa = 120N - 16$ is even no matter what the integer $N$ is.

iii) $k = 3$: $n - 1 = 10$ and $\binom{n - 1}{2} = 45$. Thus $\kappa = 45N - 10$ is even for $N$ even and odd for $N$ odd.

iv) $k = 4$: $n - 1 = 13$ and $\binom{n - 1}{2} = 78$. Thus $\kappa = 78N - 13$ is odd no matter what the integer $N$ is.

4 Conclusions

We have established that the nature of the independent gauge symmetries of pure Chern-Simons theory based on $p$-form gauge fields crucially depends on the parity of $p$. While the spacelike diffeomorphisms and the internal gauge symmetries are the only independent gauge symmetries for odd $p$ (in the absence of, non-generic, extra gauge symmetries), the timelike diffeomorphisms are independent gauge symmetries when $p$ is even, if, in addition, the number $\kappa \equiv N \binom{n - 1}{p} - (n - 1)$ is an odd integer. The difference between odd $p$ and even $p$ persists even in the presence of accidental gauge symmetries, which must necessarily come in pairs since the rank of $\Omega$ is necessarily even. For odd values of $p$, the timelike diffeomorphisms can be expressed in terms of the spacelike diffeomorphisms, the internal gauge symmetries and the (even number of) accidental gauge symmetries. This is not true for even $p$ (and odd $\kappa$).

This result is somehow unexpected since the construction of the Chern-Simons Lagrangian is identical in all cases. Furthermore, for fixed (even) $p$ and fixed spacetime dimension $n$, the parity of $\kappa$ may depend - again somewhat surprisingly - on the number $N$ of $p$-forms involved. We have
not provided a deep explanation of why the distinction between the cases of even or odd $p$ arises, but we have provided explicit examples illustrating both situations. Perhaps an investigation of Chern-Simons theories involving simultaneously forms of different degrees could shed further light on the question.

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References

[1] See for instance R. Jackiw, *Phys. Rev. Lett.* **41** (1978) 1635.

[2] M. Bañados, L. J. Garay and M. Henneaux, *Phys. Rev.* **D53**, R593 (1996); *Nucl. Phys.* **B476**, 611 (1996)

[3] R. Gambini and A. Trias, *Nucl.Phys.* **B278**, 436 (1986). C. Rovelli and L. Smolin, *Phys.Rev.Lett.* **61**, 115 (1988); *Nucl.Phys* **B331**, 80 (1990)

[4] V. Hussain and K. Kuchař, *Phys. Rev.* **D42**, (1990), 4070. G. Barnich and V. Hussain, [gr-qc/9611030](https://arxiv.org/abs/gr-qc/9611030)

[5] See e.g., P. van Nieuwenhuizen, in *Relativity, Groups and Topology II*, Les Houches lectures 1983, pp 823-932.

[6] E. Cremmer, B. Julia and J. Scherk, *Phys.Lett.* **B76**, 409 (1978)

[7] M. Henneaux and C. Teitelboim, *Quantization of gauge systems*, (Princeton University Press, Princeton, 1992).