On the quantum structure of space-time, gravity, and higher spin

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Abstract

In this introductory review, we argue that a quantum structure of space-time naturally entails a higher-spin theory, to avoid significant Lorentz violation. A suitable framework is provided by Yang-Mills matrix models, which allow to consider space-time as a physical system, which is treated on the same footing as the fields that live on it. We discuss a specific quantum space-time solution, whose internal structure leads to a consistent and ghost-free higher-spin gauge theory. The spin 2 modes give rise to metric perturbations, which include the standard gravitons as well as the linearized Schwarzschild solution.
1 Introduction

Our current understanding of fundamental physics is based on the concept of space-time, which is assumed to be a pseudo-Riemannian manifold. This provides the stage for quantum field theory which describes all fundamental interactions except gravity, while gravity is described through the metric tensor on space-time which is governed by the Einstein equations.

However, there are good reasons to question the classical notion of space-time. We know that quantum mechanics governs all physical matter and fields, and the quantum structure becomes more important at short distances. On the other hand, space-time and its metric is coupled to this quantum matter through the Einstein equations, and the coupling also becomes stronger at short distances. It then seems unreasonable to insist that space-time remains classical at all scales. It is more plausible that both space-time and fields should have a unified quantum description in a fundamental theory.

This suggests the idea that space-time is not just a manifold which provides the stage for physics. Rather, space-time ought to be a dynamical physical system with intrinsic quantum structure, which
should be treated at the same footing as the fields that live on it. This idea will be realized through a simple matrix model, where space-time arises as solution, and gauge fields arise as fluctuations of the space-time structure. In other words, space-time along with physical fields emerge from the basic matrix degrees of freedom. This takes the idea of unification in physics one step further, and the new description is indeed simpler than the previous one(s), as in all successful unification steps in physics.

Of course this proposal is speculative and certainly not new, but it is natural, given the special role of gravity as mediator between space-time and matter. The present specific model shows that it is in fact feasible, and it might be the key to solve some big puzzles in this context.

We will also argue that an algebraic realization of such a space-time structure quite generally leads to some kind of higher-spin theory, if explicit breaking of rotational symmetry should be avoided. The reason is that a noncommutative structure of space-time amounts to some antisymmetric tensor field $\theta^{\mu\nu}$. Avoiding symmetry-breaking through some non-vanishing VEV $\langle \theta^{\mu\nu} \rangle \neq 0$ means that there should be a non-trivial variety of such objects at each point. This amounts to an internal structure of space-time, whose fluctuations lead to a higher-spin theory. A related idea was already discussed in [1], without making the connection to higher spin. Such a structure is realized in a class of covariant quantum spaces, whose prototype is fuzzy $S^4_N$ in the Euclidean case. The model under consideration here [7] provides a minimal realization of a covariant quantum space-time with Minkowski signature, and leads to a consistent higher-spin gauge theory.

The present approach uses ideas of noncommutative (NC) geometry, but it goes beyond it. NC geometry provides the insight that space-time does not need to be a classical manifold, but can be described by some algebra (of quantized “functions”) which need not be commutative. This is well-known by now and will be used throughout. However, NC geometry does not offer a model for a dynamical NC space-time. This is exactly what matrix models provide in an extremely simple way, leading naturally to a gauge theory. Upon closer inspection, such matrix models turn out to display stringy features. This connection with string theory is not imposed by hand, but it is useful to understand the UV properties of models [8,9]. In particular, loops involving such string-like modes generically lead to a strong non-locality known as UV/IR mixing [10]. This is avoided only in a very specific and simple supersymmetric matrix model of the form

$$S[Y, \Psi] = \text{Tr} \left( \{ Y^a, Y^b \} [Y_a, Y_b] + \bar{\Psi} \Gamma_a [Y^a, \Psi] \right)$$

(1.1)

involving 10 hermitian matrices $Y^a$, $a = 0, \ldots, 9$ and Majorana-Weyl fermionic matrices $\psi$. This is the well-known IKKT or IIB model [11] which was proposed some 20 years ago as a constructive definition of string theory. Having a preferred model is of course welcome, and analogous (and almost tantamount) to the statement that among all conceivable string theories, there is basically only one anomaly-free model, which is critical (super)string theory in 10 dimensions. Moreover, this model seems to provide just enough structure to possibly recover particle physics. However that is very much aspirational, and we refer to [12–15] for work towards this goal.

Finally and perhaps most importantly, matrix models peg to be put on a computer. This is not easy in the case of Minkowski signature, but large-scale efforts are underway to implement and simulate the full IKKT model [11], with tantalizing evidence towards an expanding 3+1-dimensional space-time structure. The long-term goal must then be to relate and test analytical investigations as discussed here with such numerical results. This goal should be achievable, which provides a great motivation for more work in this direction.

This article is basically a conceptual introduction and summary of the papers [7] [16] [18], em-
phasizing the ideas behind it and minimizing the technicalities. All the details can be found in the above papers, and further supplementary literature is suggested in section 6.8. There are also some new technical results, such as the regularization in (2.9) and (2.11), and the treatment of diffeomorphism in section 6.1.

2 The regularized IKKT or IIB model & matrix geometry

The starting point of our considerations is the so-called IKKT or IIB matrix model (1.1), which is an action for hermitian matrices $Y^a \in \text{End}(\mathcal{H})$, $a = 0, ..., 9$ acting on a Hilbert space $\mathcal{H}$, and spinors $\Psi$ whose entries are (Grassmann-valued) matrices. This model has a manifest $SO(9,1)$ symmetry, and is invariant under gauge transformations

$$Y^a \rightarrow U Y^a U^{-1}$$

by arbitrary unitary matrices $U$ acting on $\mathcal{H}$. The model also enjoys maximal supersymmetry [11], which is important for the quantization but will not be used explicitly here.

From matrices to geometry. A priori, the model knows nothing about geometry, except for the $SO(9,1)$-invariant tensor $\eta_{ab}$ which enters the action. Obviously there is no way to assign any geometric significance to generic matrix configurations $\{Y^a\}$, and most of the configuration space is basically “white noise”. However, since the action is the square of commutators, it prefers matrix configurations which are almost-commutative. This would be obvious in the Euclidean signature case where $\eta_{ab}$ is replaced by $\delta_{ab}$; consider this case for the moment. Then the dominant contributions are matrices which are almost-commutative, which means that the $Y^a$ can “almost” be simultaneously diagonalized. More specifically, one can define quasi-coherent states $|y\rangle \in \mathcal{H}$ which are optimally localized in a suitable sense [19–21]. These are the approximate common eigenstates of the $Y^a$, localized at some point in target space

$$y^a = \langle y | Y^a | y \rangle \in \mathbb{R}^{9,1} .$$

These $y^a$ sweep out some variety in target space. This can even be implemented on a computer [20]. One can thus associate classical functions to the matrices,

$$Y^a \sim y^a : \quad M \mapsto \mathbb{R}^{9,1}$$

which is interpreted geometrically as quantized embedding map of some “brane” $M$ in target space $\mathbb{R}^{9,1}$. In this way, a fuzzy notion of geometry is extracted from nearly-commuting matrix configurations. In the irreducible case the $Y^a$ will generate the full matrix algebra

$$\text{End}(\mathcal{H}) \sim \mathcal{C}(M) ,$$

which is interpreted as algebra of function on $M$ in the spirit of noncommutative geometry [23]. The brane may be a sharply defined submanifold, or it may be fuzzy in all directions. It can also

1basically by minimizing the uncertainly $\sum_a (Y^a - y^a 1)^2$.
2More precisely, one should consider the variety of coherent states $|y\rangle$, which may be degenerately embedded in target space via [22]. This is indeed what happens in section 6 leading to an internal bundle structure, cf. [16,22],
carry internal extra dimensions which are not resolved by the $Y^a$. In any case, one can then view the commutator

$$-i[Y^a, Y^b] =: \Theta^{ab} \sim \{y^a, y^b\} = \theta^{ab}$$  \hspace{1cm} (2.5)$$
as an antisymmetric tensor field on $M$. More generally, $[\ldots, \ldots] \sim i\{\ldots\}$ defines an anti-symmetric bracket on the space of “fuzzy functions” $\operatorname{End}(\mathcal{H}) \sim \mathcal{C}(M)$, which is a derivation and satisfies the Jacobi identity. In other words, one can typically extract a Poisson bracket on $M$ in a low-energy limit, where the functions are approximately commuting. An effective metric is encoded in the matrix Laplacian or d’Alembertian

$$\Box_Y \phi = [Y^a, [Y^a, \phi]], \quad \phi \in \operatorname{End}(\mathcal{H}).$$  \hspace{1cm} (2.6)$$
This is close in spirit to noncommutative geometry, however the extra “embedding” information $[2.3]$ provides a useful and more direct access to the geometry, e.g. via coherent states. A priori, it is not evident whether $y^a$ should be interpreted as space(time) coordinates, or as some other coordinate functions on a higher-dimensional fuzzy phase space as in section $6$. This can be determined from the effective action describing the physics of the fluctuations in the matrix model.

Now we return to the IKKT model. Due to the Minkowski signature, the above argument seems to have a loop-hole, since there can be configurations whose space-like commutators $[Y^i, Y^j]$ are large but canceled by the equally large space-time contributions $[Y^0, Y^i]$. However, these are then solutions with high energy, as measured by

$$E = T^{00} = 2[Y^0, Y^i][Y^0, Y_i] + \frac{1}{2}[Y^a, Y^b][Y_a, Y_b] = [Y^0, Y^i][Y^0, Y_i] + \frac{1}{2}[Y^i, Y^j][Y_i, Y_j]$$  \hspace{1cm} (2.7)$$
for $i, j = 1, \ldots, 9$. Here $T^{ab}$ is the matrix energy momentum tensor $[24]$, which satisfies $[Y_a, T^{ab}] = 0$. As usual, the most fundamental and significant solutions of the model should be those with lowest energy. These are then almost-commutative configurations according to the above discussion, which can be interpreted as quantized branes embedded in target space with a semi-classical description as Poisson manifolds $[2.5]$. 

Quantization and path integral, IR regularization. Perhaps the most important aspect of matrix models is that they provide a natural notion of quantization, by integrating over the space of all matrices. For finite-dimensional hermitian matrices, the measure is the obvious one, which is invariant under the transformations $[2.1]$. Then in the Euclidean case, the “matrix path integral”

$$Z = \int dY e^{-S_E[Y]}$$  \hspace{1cm} (2.8)$$
is well-defined for traceless $Y^a$ $[25][26]$. In the case of Minkowski signature, the analogous integral is oscillating and not well-defined a priori. However, it can be regularized. One possibility is to put an IR cutoff in both space-like and time-like directions as in $[27]$. A similar but more elegant regularization is by adding a mass term $\operatorname{Tr}(m^2 Y^a Y^b \eta_{ab})$ to the model, and giving the mass a suitable imaginary part. We thus define

$$S_E[Y] = \frac{1}{g^2} \operatorname{Tr} \left( [Y^a, Y^b][Y_a, Y_b] - 2m^2 (- e^{i\varepsilon}(Y^0)^2 + e^{-i\varepsilon}(Y^j)^2) \right)$$

$$\varepsilon \approx 0 \quad \frac{1}{g^2} \operatorname{Tr} \left( [Y^a, Y^b][Y_a, Y_b] - 2m^2 Y^a Y_a \right)$$  \hspace{1cm} (2.9)$$
where \( j = 1, \ldots, D \) and \( a, b = 0, \ldots, D \), which for \( \varepsilon = 0 \) leads to the equations of motion (eom)

\[
\Box Y^a + m^2 Y^a = 0 .
\]

Then the integral

\[
Z_\varepsilon = \int dY e^{i S_\varepsilon[Y]}
\]

(and similarly with fermions) is absolutely convergent for \( \varepsilon \in (0, \pi/2) \), at least for finite-dimensional matrices. It turns out that this imposes at the same time Feynman’s \( i\varepsilon \) - prescription, so that (2.11) provides a solid definition for the quantized model. However, we will restrict ourselves to classical level here, and focus on the bosonic action (2.9) henceforth. Note that \( m^2 \) simply sets the scale of the theory, and there would be no scale in the model without mass term.

3 Unification of space-time and gauge fields

The cubic matrix equation (2.10) has many different solutions with very different significance. Finding the “dominant” one(s) is a non-perturbative problem which we will not address here, and we will simply choose some solution which leads to interesting physics. Whatever background we choose, the fluctuations in the matrix model automatically defines a gauge theory. Indeed if \( X^a \) is some background solution, then the fluctuations

\[
Y^a = X^a + A^a, \quad A^a \in \text{End}(\mathcal{H})
\]

around this background are parametrized in terms of tangential (and possibly transversal) modes \( A^a \), interpreted as vector fields on \( \mathcal{M} \). They transform under gauge transformations as

\[
A_a \rightarrow U^{-1} A_a U + U^{-1} [X_a, U] .
\]

Since \([X_a, .] \) is a derivation, this clearly corresponds to the inhomogeneous transformation law for gauge fields in a Yang-Mills-type gauge theory, whose precise form depends on the background. This will be discussed briefly for the Moyal-Weyl solution \( \mathbb{R}^{3,1}_9 \) below, and worked out in more detail for the fuzzy space-time \( \mathcal{M}_n^{3,1} \) in section 6.3.

Note that (3.1) suggests an interpretation of the gauge fields \( A^a \) as Goldstone bosons of the spontaneously broken translational symmetry \( Y^a \rightarrow Y^a + c^a I \) of the matrix model. These modes are accordingly massless, as they should be.

4 Examples of matrix geometries

In this section we discuss two basic examples of embedded noncommutative spaces described by finite or infinite matrix algebras. The salient feature is that the geometry is defined by a specific set of matrices \( X^a \), interpreted as quantized embedding maps of a sub-manifold in \( \mathbb{R}^D \). We will learn to freely switch between the noncommutative matrix setting and the semi-classical picture of Poisson manifolds.

\(^3\)In the presence of some potential such as \( Y_a Y^a \), the \( SO(9, 1) \) rotations play the role of this symmetry.
4.1 Prototype: the fuzzy sphere $S^2_N$

As a first example, we recall the fuzzy sphere $S^2_N$ [28,29]. This is a quantization or matrix approximation of the usual sphere $S^2$, with a cutoff in angular momentum. The starting point is the observation that the algebra of functions $C(S^2)$ on $S^2$ is generated by the Cartesian coordinate functions $x^a$ of $\mathbb{R}^3$ modulo the relation $\sum_{a=1}^3 x^a x^a = 1$. Similarly, $S^2_N$ is a non-commutative space defined in terms of three $N \times N$ functions $S^a$.

This means that $S^2_N$ is a non-commutative space defined in terms of three $N \times N$ hermitian matrices $X^a$, $a = 1, 2, 3$ subject to the relations

$$[X^a, X^b] = \frac{i}{\sqrt{C_N}} \varepsilon^{abc} X^c, \quad \sum_{a=1}^3 X^a X^a = 1 \quad (4.1)$$

where $C_N = \frac{1}{4} (N^2 - 1)$ is the value of the quadratic Casimir of $\mathfrak{su}(2)$ on $\mathcal{H} = \mathbb{C}^N$. They are realized by the generators of the $N$-dimensional irrep $(N)$ of $\mathfrak{su}(2)$. The matrices $X^a$ should be interpreted as quantized embedding functions in the Euclidean target space $\mathbb{R}^3$.

$$X^a \sim x^a : \quad S^2 \hookrightarrow \mathbb{R}^3. \quad (4.2)$$

They generate the matrix algebra $\text{End}(\mathcal{H})$, which is viewed as quantized algebra of functions on the symplectic space $(S^2, \omega_N)$. Here $\omega_N$ is the $SU(2)$-invariant symplectic form on $S^2$ with $\int \omega_N = 2\pi N$. This is best understood by decomposing $\mathcal{A}$ into irreps of the adjoint action of $SU(2)$,

$$S^2_N \cong (N) \circ (\bar{N}) = (1) \oplus (3) \oplus \cdots \oplus (2N - 1) = \{ \hat{Y}^0_m \} \oplus \cdots \oplus \{ \hat{Y}^{N-1}_m \}. \quad (4.3)$$

This provides the definition of the fuzzy spherical harmonics $\hat{Y}^l_m$, which are symmetric traceless polynomials in $X^a$ of degree $l$. It also provides the $SO(3)$-invariant quantization map [30]

$$\mathcal{Q} : \quad C(S^2) \to \text{End}(\mathcal{H}), \quad Y^l_m \mapsto \begin{cases} \hat{Y}^l_m, & l < N \\ 0, & l \geq N \end{cases}. \quad (4.4)$$

One can easily verify $\mathcal{Q}(x^a) = X^a$ and $\mathcal{Q}(i\{x^a, x^b\}) = [X^a, X^b]$ where $\{\cdot, \cdot\}$ denotes the Poisson brackets corresponding to the symplectic form $\omega_N = \frac{N}{2} \varepsilon_{abc} x^a dx^b dy^c$ on $S^2$, and more generally

$$\frac{1}{N} (\mathcal{Q}(i\{f, g\}) - [\mathcal{Q}(f), \mathcal{Q}(g)]) \mathop{\rightarrow}_{N \to \infty} 0. \quad (4.5)$$

Furthermore, the following integral relation holds

$$\text{Tr}(\mathcal{Q}(f)) = \int_{S^2} \frac{\omega_N}{2\pi} f. \quad (4.6)$$

This means that $S^2_N$ is the quantization of $(S^2, \omega_N)$. The correspondence is summarized in table 4 and an analogous dictionary applies to all fuzzy spaces considered here. Optimally localized coherent states $|x\rangle$ are given by $SU(2)$ rotations of the lowest weight state of $\mathcal{H}$. The (round) metric is encoded in the $SU(2)$-invariant Laplacian

$$\Box = [X^a, [X^b, \cdot]] \delta_{ab}, \quad (4.7)$$

which is nothing but the quadratic $SU(2)$ Casimir on $S^2$. It is easy to see that its spectrum coincides with the spectrum of the classical Laplace operator on $S^2$ up to the cutoff, and the eigenvectors are given by the fuzzy spherical harmonics $\hat{Y}^l_m$. Finally, $S^2_N$ is a solution of (2.10) with $\Box X^a = 2X^a$.

\footnote{The normalization can be fixed by requiring that $\mathcal{Q}$ respects the norm defined via (4.6).}
noncommutative/fuzzy space $\mathcal{M}_n$ | semi-classical space $\mathcal{M}$
|---|---|
| $\text{End}(\mathcal{H}) \ni \phi(X) \rightarrow \hat{\phi}(X)$ | $\mathcal{C}(\mathcal{M}) \ni \phi(x)$ |
| $X^a$ | $x^a$ |
| $[\phi, \psi]$ | $i\{\phi, \psi\}$ |
| $\text{Tr}\phi(X)$ | $\int d\Omega \phi(x)$ |

Table 1: Schematic correspondence between matrices (operators) in $\text{End}(\mathcal{H})$ and functions on $\mathcal{M}$. $d\Omega$ indicates the symplectic volume form. The metric structure is encoded in the Laplacian $\Box$.

### 4.2 The Moyal-Weyl solution $\mathbb{R}^{3,1}_\theta$ and $\mathcal{N} = 4$ SYM

If we allow infinite-dimensional matrices, the model can also describe non-compact brane solutions, including the much-studied Moyal-Weyl quantum plane $\mathbb{R}^{3,1}_\theta$. This is defined in terms of operators $X^\mu \in \text{End}(\mathcal{H})$ which satisfy

$$[X^\mu, X^\nu] = i\theta^{\mu\nu} 1, \quad \mu = 0, \ldots, 3 \quad (4.8)$$

where $\theta^{\mu\nu}$ is a constant anti-symmetric tensor. The analog of the quantization map (4.4) is given by Weyl quantization, or equivalently as

$$Q: \mathcal{C}(\mathbb{R}^{3,1}) \rightarrow \text{End}(\mathcal{H})$$

$$\phi(x) \mapsto \hat{\phi} := \int_{\mathbb{R}^{3,1}} \phi(x) |x\rangle \langle x| \quad . \quad (4.9)$$

with inverse map $\hat{\phi} \rightarrow \phi(x) = \langle x|\hat{\phi}|x\rangle$, where $|x\rangle$ are the coherent states. This is a solution of the massless model (2.10) with $\Box X^\mu = 0$. Parametrizing the fluctuations around this background as $Y^\mu = X^\mu + A^\mu(X)$ and $Y^i = \phi^i(X)$, $i = 4, \ldots, 9$ and including the fermions of the IKKT model, one recovers $\mathcal{N} = 4$ NC super-Yang-Mills theory (SYM) on $\mathbb{R}^{3,1}_\theta$ [31]. This is best seen by recalling that usual $\mathcal{N} = 4$ SYM is obtained by dimensional reduction of $\mathcal{N} = 1$ SYM on $\mathbb{R}^{9,1}$ to $\mathbb{R}^{3,1}$, while the IKKT model is nothing but $\mathcal{N} = 1$ SYM on $\mathbb{R}^{9,1}$ dimensionally reduced to a point; hence the basic structure is the same, and the covariant derivatives on $\mathbb{R}^{3,1}_\theta$ arise via $[X^\mu + A^\mu, .] = iD^\mu$.

An important feature is that the UV divergences are canceled due to the extended supersymmetry, which entails that also the notorious IR divergences are absent, which usually plague NC field theory. Note that this is a property of the model rather than the background; the cancellation of divergences happens on any 4-dimensional background, as discussed in [8] and in section 6.7. In this respect the model is pretty much unique [32,33], and that is the reason for focusing on it.

However, there is an issue: the explicit $\theta^{\mu\nu}$ breaks Lorentz invariance. Even though it is not manifest in the action, loop corrections may lead to significant Lorentz-violating effects. For example, the induced gravity action will contain unwanted terms such as $\int R_{\mu\nu\alpha\beta}\theta^{\mu\nu}\theta^{\alpha\beta}$, where $R_{\mu\nu\alpha\beta}$ is the Riemann tensor of the effective metric [34], which is dynamical [35,36]. This issue is resolved on covariant quantum spaces, where the fixed $\theta^{\mu\nu}$ is replaced by a bundle-like variety of different $\{\theta^{\mu\nu}\}$ over space-time, which is effectively averaged. Thus space-time acquires a non-trivial internal structure, which restores Lorentz invariance at least partially. It also means that the fluctuation modes will involve harmonics on this internal space, leading to a higher-spin theory.

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7 These fluctuating noncommutative coordinates are aptly called “covariant coordinates” in NC field theory [30].
5 Fuzzy $H^4_n$ and higher spin

As a basis for the quantum space-time discussed in section 6, we now discuss a more sophisticated solution of the model known as fuzzy 4-hyperboloid $H^4_n$. This is a prototype of a covariant quantum space with an interesting internal structure, which nicely illustrates the ideas discussed in the introduction leading to a higher-spin gauge theory. The starting point is the Lie algebra $\mathfrak{so}(4,2)$ generated by $M^{ab}$,

$$[M_{ab}, M_{cd}] = i (\eta_{ac}M_{bd} - \eta_{ad}M_{bc} - \eta_{bc}M_{ad} + \eta_{bd}M_{ac})$$

(5.1)

where $\eta_{ab} = \text{diag}(-1,1,1,1,-1)$ and $a,b = 0,\ldots,5$, and a specific class of unitary representations $\mathcal{H}_n$ known as doubletons or minireps [37, 38], labeled by $n \in \mathbb{N}$. These are short discrete series unitary irreps of $\mathfrak{so}(4,2)$, which are lowest weight representations that are multiplicity-free and remain irreducible if restricted to $SO(4,1) \subset SO(4,2)$. In particular, unitarity means that all $M^{ab}$ are hermitian operators. We first define the fuzzy hyperboloid $H^4_n$ in terms of hermitian operators

$$X^a = r M^{a5}, \quad a = 0,\ldots,4$$

(5.2)

Here $r$ has dimension length. Since $\mathcal{H}_n$ is irreducible under $SO(4,1)$, they satisfy the relations of a 4-dimensional Euclidean hyperboloid

$$\eta_{ab}X^a X^b = -R^2 \mathbb{1}, \quad R^2 = \frac{r^2}{4} (n^2 - 4).$$

(5.3)

Even though this suggests that the noncommutative space described by the $X^a$ is just a hyperboloid, this is not quite correct. To understand this construction, we first note that $X^a$ generate the full algebra $\text{End}(\mathcal{H}_n)$, because their commutators

$$[X^a, X^b] = -i r^2 M^{ab} =: i \Theta^{ab}$$

(5.4)

are nothing but the generator of $\mathfrak{so}(4,1)$ on $\mathcal{H}_n$. This is analogous to the case of Snyder space \[.\]

Since the fluctuations $X^a + A^a$ in (3.1) are the most general elements in $\text{End}(\mathcal{H}_n)$, we must find the proper interpretation of $\text{End}(\mathcal{H}_n)$ as a quantized algebra of functions on some space. To understand this, we note that $\text{End}(\mathcal{H}_n)$ transforms under $SO(4,2)$ via

$$M^{ab} \triangleright \phi = [M_{ab}, \phi], \quad \phi \in \text{End}(\mathcal{H}_n), \quad a,b = 0,\ldots,5$$

(5.5)

while the $X^a$ transform covariantly as vector operators of $SO(4,1) \subset SO(4,2)$,

$$M^{ab} \triangleright X^c = [M_{ab}, X^c] = i(\eta^{ac}X^b - \eta^{bc}X^a), \quad a,b = 0,\ldots,4$$

(5.6)

This strongly suggests that $\text{End}(\mathcal{H}_n)$ is the quantized algebra of functions on some coadjoint orbit of $SO(4,2)$ or $SU(2,2)$, which turns out to be quantized $\mathbb{C}P^1$. This is an $S^2$-bundle over $H^4$, whose fiber is given by the space of selfdual 2-forms $\theta^{ab}$ on $H^4$. Covariance means that the bundle is an $SO(4,1)$-equivariant bundle, which means that the action on the bundle is compatible with the action on the base manifold. Thus the local stabilizer $SO(4)$ of any point $p \in H^4$ acts nontrivially on the $S^2$ fiber over $p$. This will imply that the “would-be Kaluza Klein” modes on $S^2$ will transmute into higher spin modes on $H^4$, leading to a higher-spin gauge theory on $H^4$ in the matrix model. Because this is a crucial aspect, we briefly sketch how this bundle arises:
**Bundle construction.** This bundle is obtained explicitly via an oscillator construction, which at the same time provides the doubleton representations $\mathcal{H}_n$. Consider 4 bosonic oscillators $\psi^\alpha$, $\alpha = 1, \ldots, 4$ which satisfy

$$[\psi^\alpha, \overline{\psi}^\beta] = \delta^\beta_\alpha$$

(5.7)

where $\overline{\psi} = \psi^\dagger \gamma_0$ with $\gamma_0 = \text{diag}(1,1,-1,-1)$. They transform in the fundamental representations $\mathbb{C}^4$ of $SU(2,2)$. Then consider the Jordan-Schwinger realization of $\mathfrak{su}(2,2) \cong \mathfrak{so}(4,2)$ via

$$M^{ab} = \overline{\psi} \Sigma^{ab} \psi, \quad a, b = 0,\ldots, 5$$

(5.8)

$$X^a = \frac{r}{2} \overline{\psi}^\gamma a \psi, \quad a = 0,\ldots, 4$$

(5.9)

where $\Sigma^{ab}$ are the generators of $\mathfrak{so}(4,2)$ on $\mathbb{C}^4$, noting that $\Sigma^{a5} = \frac{1}{2} \gamma^a$. The $M^{ab}$ are hermitian, and thus implement unitary representations of $SU(2,2)$ on the Fock space of the bosonic oscillators. More precisely, one defines

$$\psi_\alpha = (a_1^\dagger, a_2^\dagger, b_1, b_2)^T,$$  

(5.10)

and considers the Fock vacuum $a_i |0\rangle = b_i |0\rangle$. Then the doubleton minireps $\mathcal{H}_n$ arise on the lowest weight vectors

$$|\Omega\rangle := \left| E, \frac{n}{2}, 0 \right\rangle := a_1^\dagger \cdots a_n^\dagger |0\rangle, \quad E = 1 + \frac{n}{2}, \; j_L = \frac{n}{2}, \; j_R = 0$$

(5.11)

which satisfy $a_i b_j |\Omega\rangle = 0$. Similarly, exchanging $a$ with $b$ exchanges $j_L \leftrightarrow j_R$. Since the “norm”

$$\overline{\psi} \psi = -N_a + N_b - 2$$

(5.12)

is invariant under $SU(2,2)$, the $\psi_\alpha$ clearly form a noncompact version of $\mathbb{C}P^3$ in the (semi)classical limit, which is called $\mathbb{C}P^{1,2}$. This space is in fact a coadjoint orbit of $SU(2,2)$, and thus carries a canonical symplectic form corresponding to the Poisson structure

$$\{ m^{ab}, m^{cd} \} = \eta^{ac} m^{bd} - \eta^{ad} m^{bc} - \eta^{bc} m^{ad} + \eta^{bd} m^{ac}$$

$$\{ x^a, x^b \} = \theta^{ab} = -r^2 m^{ab}$$

(5.13)

considering $m^{ab} \sim M^{ab}$ as embedding functions

$$m^{ab} : \mathbb{C}P^{1,2} \hookrightarrow \mathfrak{su}(2,2) \cong \mathbb{R}^{15}.$$

(5.14)

The semi-classical limit is indicated by $\sim$. In particular, we note that the definition of $X^a$ in (5.9) amounts to a quantized Hopf map from the bundle space to $H^4$, which is compatible with $SO(4,1)$. The radius of $H^4$ is related to the quantum number $n \in \mathbb{N}$ (5.11) via

$$X_a X^a = -\frac{r^2}{4} (n^2 - 4) =: -R^2.$$

(5.15)
It can also be checked that these $\theta^{ab}$ are self-dual 2-forms on $H^4$ in the semi-classical limit, which sweep out the $S^2$ fiber over $H^4$. For more details we refer to [16, 40]. One can also see that the local $SO(4) \cong SU(2)_L \times SU(2)_R$ stabilizer e.g. over $x^a = (r, 0, 0, 0, 0) \in H^4$ acts via $SU(2)_L$ on the 2-component spinor $(a_1, a_2) \in \mathbb{C}^2$, sweeping out $\mathbb{C}P^1 \cong S^2$. In the noncommutative case, this realizes precisely the oscillator realization of fuzzy 2-sphere $S^2_n$, which will lead to a truncation of the higher spin modes in (5.26).

In particular, we have gained a crucial understanding of $\text{End}(\mathcal{H}_n)$ as quantized algebra of functions on $\mathbb{C}P^{1,2}$, which is an $S^2$ bundle over $H^4$. This can be made more explicit in terms of coherent states $|p\rangle \in \mathcal{H}_n$, which are optimally localized states at $p \in \mathbb{C}P^{1,2}$ given by $SO(4,2)$-rotations of $|\Omega\rangle$, cf. [8]. Then one can write down an $SO(4,2)$-covariant quantization map analogous to (4.9),

$$Q : \mathcal{C}(\mathbb{C}P^{1,2}) \to \text{End}(\mathcal{H}_n)$$

$$\phi(p) \mapsto \hat{\phi} := \int_{\mathbb{C}P^{1,2}} \phi(p) |p\rangle \langle p| .$$

(5.16)

with inverse map $\hat{\phi} \mapsto \langle p|\hat{\phi}|p\rangle = \phi(p)$ below some cutoff depending on $n$. In other words,

$$\text{End}(\mathcal{H}_n) \cong \mathcal{C}(\mathbb{C}P^{1,2}) \quad \text{up to some cutoff}$$

(5.17)

as vector spaces and $SO(4,2)$ modules. A more precise statement would be that the Hilbert-Schmidt operators in $\text{End}(\mathcal{H}_n)$ correspond to square-integrable functions on $\mathbb{C}P^{1,2}$, and both sides decompose into principal series irreps of $SO(4,2)$. However (5.16) applies more generally e.g. also to polynomial functions, which of course do not form unitary representations.

Finally, it is interesting to note that $\mathbb{C}P^{1,2}$ and its quantization can be considered as (quantized) twistor space. However its present use is quite distinct from the usual applications of twistors.

### 5.1 Algebraic description

In the semi-classical limit, the above generators satisfy the following constraints

$$x_ax^a = -R^2,$$  

(5.18)

$$\theta^{ab}x_b = 0 ,$$  

(5.19)

$$\epsilon_{abcd} \theta^{ab}x^c = nr \theta_{de} \sim 2R \theta_{de} ,$$  

(5.20)

$$\eta_{ac} \theta^{ab} \theta^{ab} = \frac{L_{NC}^4}{4} P^{bb'},$$  

(5.21)

where the scale of non-commutativity is

$$L_{NC}^4 := \theta^{ab} \theta_{ab} = 4r^2 R^2 .$$  

(5.22)

Here

$$P^{ab} = \eta^{ab} + \frac{1}{R^2} x^a x^b \quad \text{with} \quad P^{ab} x_b = 0$$  

(5.23)
is the Euclidean projector on $H^4$ (recall that $H^4$ is a Euclidean space). Hence functions on fuzzy $H^4_n$ can be identified for large $n$ with functions on $\mathbb{C}P^{1,2}$, and written in the form

$$ C(\mathbb{C}P^{1,2}) = \bigoplus_{s=0}^{\infty} C^s \ni \phi^{s}_{a_1, \ldots , a_s; b_1, \ldots , b_s}(x) \theta^{a_1 b_1} \ldots \theta^{a_s b_s}. \quad (5.24) $$

This can be viewed as function on $H^4$ taking values in Young diagrams. (5.24) gives a decomposition of $C = C(\mathbb{C}P^{1,2})$ into modules $C^s$ over the algebra of functions $C_0$ on $H^4$, which correspond to bundles over $H^4$ whose structure is determined by the above constraints. In the NC case, this decomposition is defined as

$$ \text{End}(\mathcal{H}_n) = C = C^0 \oplus C^1 \oplus \ldots \oplus C^n \quad \text{with} \quad S^2|_{C^s} = 2s(s + 1) \quad (5.26) $$

in terms of the Casimir

$$ S^2 := \frac{1}{2} \sum_{a, b \neq \bar{5}} [M_{ab}, [M^{ab}, \cdot]] + r^{-2}[X_a, [X^a, \cdot]] = 2C^2[so(4, 1)] - C^2[so(4, 2)] \quad (5.27) $$

which can be interpreted as a spin observable on $H^4$. It satisfies

$$ [S^2, \Box_H] = 0, \quad \Box_H = [X^a, [X_a, \cdot]], \quad (5.28) $$

and it is easy to check that $X^a$ is a solution of the eom

$$ \Box_H X^a = -4r^2 X^a. \quad (5.29) $$

Note that (5.26) has a cutoff at spin $s = n$, which results from the fact that the fiber on fuzzy $\mathbb{C}P^{1,2}$ is really a fuzzy sphere $S^2_n$, which supports only harmonics up to spin $n$. This is discussed briefly below and shown in detail in [16]. This cutoff disappears in the semi-classical limit, where the $C^s$ are modules over $C_0 \cong C(H^4)$, interpreted as sections of higher spin bundles over $H^4$.

Even though this $H^4$ solution is ultimately unphysical, the common features with Vasiliev’s higher spin theory [41, 42] are most visible in this background, which is nothing but Euclidean AdS$^4$. Moreover it is the starting point for the quantum space-time $\mathcal{M}^{3,1}_n$, which is discussed next.

6 Cosmological space-time $\mathcal{M}^{3,1}_n$

Now we make a big step towards real physics, and discuss a solution of the model that describes a quantized cosmological FLRW space-time $\mathcal{M}^{3,1}_n$. The fluctuation modes on this background lead to a consistent higher-spin gauge theory, which is very interesting from the physics point of view. We will review the basic definition of this solution [7] and recent results, including the no-ghost theorem [18] and the linearized Schwarzschild solution [17].

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6 Another description is given by the one-to-one map

$$ \Gamma^{(s)}: H^4 \to C^s \quad \phi^{(s)}_{a_1, \ldots , a_s}(x) \mapsto \phi^{(s)} = \{x^{a_1}, \ldots , x^{a_s}, \phi^{(s)}_{a_1, \ldots , a_s}\} \ldots. \quad (5.25) $$

Here $\Gamma^{(s)}: H^4$ denotes the space of totally symmetric, traceless, divergence-free rank $s$ tensor fields on $H^4$, which are identified with (symmetric tangential divergence-free traceless) tensor fields $\phi^{(s)}_{a_1, \ldots , a_s}$ with $SO(4, 1)$ indices.
The construction is based on the fuzzy $H^4_n$ as discussed above. $M^{3,1}_n$ is generated and embedded in $\mathbb{R}^{3,1}$ by the $X^\mu$ generators of $H^4_n$,

$$X^\mu \sim x^\mu : \quad M^{3,1}_n \hookrightarrow \mathbb{R}^{3,1}, \quad \mu = 0, ..., 3 , \quad (6.1)$$

dropping $X^4$. Greek indices $\mu, \nu$ etc. will run from 0 to 3 from now on. Then (5.15) becomes

$$\eta_{\mu\nu}X^\mu X^\nu = -R^2 \mathbf{1} - X^4 . \quad (6.2)$$

Dropping the $X^4$ generator corresponds to a projection of $H^4$ to $\mathbb{R}^{3,1}$, so that $M^{3,1}_n$ should be interpreted as squashed hyperboloid, as sketched in figure 1. This interpretation is substantiated by the matrix d’Alembertian

$$\Box \phi \equiv \Box_T \phi = [T^\mu, [T^\mu, \phi]] = (C^2[so(4,1)] - C^2[so(3,1)]) \phi , \quad (6.3)$$

which defines an $SO(3,1)$-invariant d’Alembertian for $M^{3,1}$ with Lorentzian structure, where

$$T^\mu = \frac{1}{R} M^{\mu 4} . \quad (6.4)$$

It is easy to see that the $X^\mu$ alone generate the full algebra $End(H_n)$, which is now interpreted as quantized algebra of functions on an $S^2$-bundle over $M^{3,1}$. Hence we have two vector generators, which satisfy the $SO(3,1)$-covariant commutation relations

$$[X^\mu, X^\nu] =: i\Theta^{\mu\nu} = -i\rho^2 M^{\mu\nu} , \quad (6.5)$$

$$[T^\mu, T^\nu] = -\frac{i}{r^2 R^2} \Theta^{\mu\nu} , \quad [T^\mu, X^\nu] = \frac{i}{R} \eta^{\mu\nu} X^4 .$$

Both $X^\mu$ and $T^\mu$ are solutions of the eom (2.10) for different signs of the mass term,

$$\Box_X X^\mu = -3\rho^2 X^\mu , \quad \Box_T T^\mu = 3R^{-2} T^\mu . \quad (6.6)$$

These generators satisfy further constraints due to the special representation $H_n$, which are crucial for the consistency of the resulting gauge theory. To simplify these relations we will focus on the semi-classical (Poisson) limit $n \to \infty$ from now on, working with commutative functions of $x^\mu \sim X^\mu$ and $t^\mu \sim T^\mu$, but keeping the Poisson structure $[.,.] \sim i\{.,.\}$ encoded in $\Theta^{\mu\nu}$. 

Figure 1: Projection $\Pi$ from $H^4$ to $M^{3,1}$ with Minkowski signature.
Semi-classical structure. In the semi-classical limit, the generators \( x^\mu \) and \( t^\mu \) satisfy the following constraints \([16]\)

\[
x_{\mu} x^\mu = - R^2 - x_4^2 = - R^2 \cosh^2(\eta), \quad R \sim \frac{r}{2n}
\]

(6.7a)

\[
t_{\mu} t^\mu = r^{-2} \cosh^2(\eta)
\]

(6.7b)

\[
t_{\mu} x^\mu = 0
\]

(6.7c)

which arise from the special properties of \( H_n \). Here \( x^\mu : \mathcal{M}^{3,1} \hookrightarrow \mathbb{R}^{3,1} \) is interpreted as Cartesian coordinate functions, and \( \eta \) plays the role of a time parameter, defined via

\[
x^4 = R \sinh(\eta).
\]

(6.8)

Hence \( \eta = \text{const} \) defines a foliation of \( \mathcal{M}^{3,1} \) into space-like surfaces \( H^3 \); this will be related to the scale parameter of a FLRW cosmology \((6.33)\) with \( k = -1 \). Note that \( \eta \) distinguishes the two degenerate sheets of \( \mathcal{M}^{3,1} \), cf. figure 1. The \( t^\mu \) generators clearly describe the \( S^2 \) fiber over \( \mathcal{M}^{3,1} \), which is space-like due to (6.7c). These generators satisfy the Poisson brackets

\[
\{ x^\mu, x^\nu \} = \theta^{\mu\nu} = - r^2 R^2 \{ t^\mu, t^\nu \},
\]

\[
\{ t^\mu, x^\nu \} = \frac{x^4}{R} \eta^{\mu\nu},
\]

and the Poisson tensor \( \theta^{\mu\nu} \) satisfies the constraints

\[
t_{\mu} \theta^{\mu\alpha} = - \sinh(\eta) x^\alpha,
\]

(6.10a)

\[
x_{\mu} \theta^{\mu\alpha} = - r^2 R^2 \sinh(\eta) t^\alpha,
\]

(6.10b)

\[
\eta_{\mu\nu} \theta^{\mu\alpha} \theta^{\nu\beta} = R^2 r^2 \eta^{\alpha\beta} - R^2 r^4 t^\alpha t^\beta - r^2 x^\alpha x^\beta
\]

(6.10c)

Due to the self-duality relations of \( \theta^{ab} \) on \( H^4 \), \( \theta^{\mu\nu} \) can be expressed in terms of \( t^\mu \) as \([16]\)

\[
\theta^{\mu\nu} = \frac{r^2}{\cosh^2(\eta)} \left( \sinh(\eta) (x^\mu t^\nu - x^\nu t^\mu) + \epsilon^{\mu\nu\alpha\beta} x_\alpha t_\beta \right),
\]

(6.11)

The above commutation relations imply

\[
\{ t_{\mu}, \phi \} = \sinh(\eta) \partial_{\mu} \phi
\]

(6.12)

for \( \phi = \phi(x) \), which suggest that \( T^\mu \sim t^\mu \) can be viewed as momentum generators on \( \mathcal{M}^{3,1} \).

6.1 Algebra of functions and higher-spin

The full algebra of functions \( \text{End}(H_n) = \oplus C^s \) still decomposes into sectors \( C^s \) which correspond to spin \( s \) harmonics on the \( S^2 \) fiber, which is respected by the d’Alembertian on \( \mathcal{M}^{3,1} \) because

\[
[S^2, \square] = 0.
\]

(6.13)

In particular, this decomposition is compatible with the kinematics defined through \( \square \). We can write the most general \( \phi \in C^s \subset \text{End}(H) \) as a function \( \phi = \phi(x,t) \) on \( \mathbb{C} P^{1,2} \), which is identified with a totally symmetric traceless space-like rank \( s \) tensor fields on \( \mathcal{M}^{3,1} \) via

\[
\phi^{(s)} = \phi_{\mu_1 \ldots \mu_s} (x) t^{\mu_1} \ldots t^{\mu_s}, \quad \phi_{\mu_1 \ldots \mu_s} x^{\mu_i} = 0.
\]

(6.14)
The restriction to space-like and traceless tensors follows from the constraints (6.7). However this just amounts to a gauge, and the physical tensor fields (e.g. the metric perturbation $h_{\mu\nu}$ (6.63), (6.71)) can arise from $\phi^{(s)}$ in a different way, via
\[ \tilde{\phi}_{\mu_1...\mu_s}(x) := \left\{ \{ x_{\mu_1}, \ldots, \{ x_{\mu_s}, \phi^{(s)} \} \} \ldots \right\}_0. \] (6.15)

Here $\left\{ \right\}_0$ denotes the projection to $C^0$. These are totally symmetric tensor fields which are no longer space-like but satisfy other constraints.

To demonstrate consistency of the theory, we must show that the time-like components in tensor fields do not lead to negative-norm states, i.e. ghosts. Basically the only known way to define such a consistent quantum theory is via gauge theory, where the unphysical negative norm modes can be removed while effectively preserving Lorentz invariance. The prime examples are Yang-Mills gauge theory for spin 1 and diffeomorphism-invariant general relativity for spin 2. For higher spin this is achieved by Vasiliev’s higher spin theory [41,42], however no action formulation is known.

The present framework provides a slightly different solution to this problem. A key ingredient is the space-like constraint (6.14), which is built in a priori \(^7\). This comes of course at the expense of manifest local Lorentz invariance, and we will see that only the space-like isometries $SO(3,1)$ are manifest, while boosts are not. However, the model appears to have sufficient extra symmetries to make up for this deficiency. Indeed there is a large higher-spin-like gauge invariance, which includes diffeomorphisms that preserve a volume form, albeit acting in a non-standard way involving the NC structure. It seems that this is sufficient to protect the theory from significant Lorentz violations, and we will see explicitly that all modes propagate in the same standard relativistic way. However, this is an issue which needs further clarification.

One might contemplate the idea of preserving manifest local Lorentz invariance in an extended model where $End(\mathcal{H})$ describes some local geometry $\mathbb{R}^{3,1} \times \mathcal{K}$. This includes many attractive examples such as fuzzy de Sitter spaces and others [43,44], see also [1]. However, then $\mathcal{K}$ must be a homogeneous space of $SO(3,1)$, which is necessarily non-compact. Then an expansion in discrete harmonics on $\mathcal{K}$ no longer makes sense, and the theory would display a higher-dimensional behavior. Thus it seems quite hopeless to obtain a consistent 4-dimensional theory from the fluctuations. That issue is avoided in the the present model, at the price of only partially manifest local Lorentz invariance.

**Higher-spin gauge transformations.** The origin of the higher-spin gauge symmetry can already be understood at this stage: They arise from gauge transformations (2.1) $Y^a \toUY^aU^{-1}$, or infinitesimally for $U = e^{i\Lambda}$

\[ Y^a \to i[\Lambda, Y^a], \quad \Lambda = \Lambda^I \in End(\mathcal{H}) \] (6.16)

We can parametrize the generator $\Lambda \in C^s \subset End(\mathcal{H})$ in two equivalent ways

\[ \Lambda = \Lambda_{\mu_1...\mu_s}(x)t^{\mu_1}...t^{\mu_s} = \tilde{\Lambda}_{\mu_1...\mu_s,\nu_1...\nu_s}(x)\theta^{\mu_1\nu_1}...\theta^{\mu_s\nu_s} \in C^s \]. (6.17)

The first form is reminiscent of a frame-like higher spin generator identifying $t^\mu$ as momentum generator (6.12), while the second form can be viewed as a gauge transformation taking values in

\(^7\)This is not the case for the tangential fluctuation modes $A^\nu$, which do have time-like components a priori. However these are then removed as in Yang-Mills theory, as explained in section 6.4.
$U(\mathfrak{so}(3,1))$ corresponding to Young diagrams $\begin{array}{c}
\end{array}$, as $\theta^{\mu\nu}$ are generators of $\mathfrak{so}(3,1)$ \([5.13]\). Both are equivalent due to the implicit constraints. Using the first form, e.g. the spin 1 gauge transformations
\[
\Lambda = \Lambda_\mu(x)t^\mu
\] (6.18)
are identified with space-like vector fields $\Lambda_\mu(x)$. Even though $\Lambda_\mu(x)$ is space-like, the resulting gauge transformation does include time-like directions, because it acts via the commutator or the Poisson bracket. This also leads to a mixing of the different spin sectors, and the specific transformation depends on the object under consideration. For example, functions $\phi \in \mathbb{C}^0$ transform as
\[
\delta_\Lambda \phi = \{\Lambda, \phi\} = \xi^\mu \partial_\mu \phi, \quad \xi^\mu = \{\Lambda, x^\mu\}.
\] (6.19)
This looks like a diffeomorphism generated by a vector field $\xi^\mu$, however things are more tricky: $\xi^\mu$ consists of a vector field and an associated 2-form,
\[
\xi^\mu = \xi^\mu(x) + \xi^\mu(2) \in \mathbb{C}^0 \oplus \mathbb{C}^2.
\] (6.20)
We can understand the resulting vector field $\xi^\mu(x)$ on $M^{3,1}$ better by decomposing the space-like $\Lambda_\mu(x)$ into divergence-free modes and pure divergence mode,
\[
\Lambda = \Lambda^{(1,0)} + \Lambda^{(1,1)}, \quad \partial^\mu \Lambda^{(1,0)} = 0, \quad \Lambda^{(1,1)} = \partial_\mu \Lambda(x).
\] (6.21)
Then $\xi_\mu(x)$ decomposes into the space-like divergence-free field (see section 9.2 in [18])
\[
\xi_\mu[\Lambda^{(1,0)}] = -\frac{1}{3} \sinh(\eta)(\tau + 3)\Lambda^{(1,0)} + \frac{1}{3} x_\beta \varepsilon^{\beta\mu\alpha\nu} \partial_\nu \Lambda^{(1,0)},
\]
\[
\partial^\mu \xi_\mu = 0 = x^\mu \xi_\mu
\] (6.22)
(in radiation gauge with 2 d.o.f.), and a scalar mode
\[
\xi_\mu[\Lambda^{(1,1)}] = \frac{r^2 R}{3} \sinh(\eta)(x_\mu \partial^\alpha \partial_\alpha \Lambda - (\tau + 3) \partial_\mu \Lambda),
\]
\[
\partial^\mu \xi_\mu = -\frac{1}{R^2 \sinh^2(\eta)} x^\mu \xi_\mu
\] (6.23)
which is neither space-like nor divergence-free. The constraint can be written covariantly as
\[
\nabla_\mu (\beta^3 \xi^\mu) = 0 , \tag{6.24}
\]
where $\nabla$ is the covariant derivative associated to the effective metric \([6.31]\). Therefore $\xi^\mu$ corresponds to a volume-preserving diffeomorphism, up to the factor $\beta^3$. The component $\xi^\mu(2) = -\partial_\nu \phi_\alpha [\theta^{\mu\nu} t^\alpha]_2 \in \mathbb{C}^2$ in \([6.20]\) is some derivative contribution which we will not pursue any further. To put this into context, we note that these spin 1 gauge transformations will indeed induce the usual transformations of graviton modes \([6.70]\)
\[
\delta_\Lambda G^{\mu\nu} = \nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu .
\] (6.25)
In any case, \([6.19]\) provides a nice geometrical interpretation of higher-spin gauge transformations: they are simply Hamiltonian vector fields on $\mathbb{C}P^{1,2}$ which generate the symplectomorphism group. Since these preserve the symplectic volume on $\mathbb{C}P^{1,2}$, the resulting diffeomorphisms on $M^{3,1}$ are volume-preserving in the sense of \([6.24]\), and encode only 3 rather than 4 degrees of freedom (dof).
SO(3,1) substructure. The decomposition of diffeomorphisms found in (6.21) illustrates the sub-structure of tensor fields resulting from the reduced SO(3,1) covariance, indicated by the superscripts of $\Lambda$. All “fundamental” tensor fields are space-like, and decompose further into divergence-free and pure divergence modes. There is a systematic organization developed in [7], which is based on the underlying $\mathfrak{so}(4,2)$ structure. Consider the $SO(3,1)$ -invariant derivation

$$D\phi := \{x^4, \phi\} = r^2 R^2 \frac{1}{x^4} t^\mu \{t_\mu, \phi\} = -\frac{1}{x^4} x_\mu \{x^\mu, \phi\}$$

$$= r^2 R t^{z_1} \ldots t^{z_\mu} \nabla^{(3)}_\mu \phi_{\mu_1 \ldots \mu_\mu}(x)$$

(6.26)

where $\nabla^{(3)}$ is the covariant derivative along the space-like (3

$$\mathcal{C} = \mathcal{C}^{s-1} \oplus \mathcal{C}^{s+1}, \quad D^\pm \phi(s) = [D\phi(s)]_{s\pm 1}$$

(6.27)

where $[,]_s$ denotes the projection to $\mathcal{C}^s$ defined through (5.26). It is easy to see that $(D^+)\dagger = -D^-$ w.r.t. the canonical invariant inner product. Explicitly, $Dx^\mu = r^2 R t^\mu$ and $Dt^\mu = R^{-1} x^\mu$. In particular, $\mathcal{C}^{(s,0)} \subset \mathcal{C}^s$ is the space of divergence-free traceless space-like rank s tensor fields on $\mathcal{M}^{3,1}$, in radiation gauge. We can then organize the $\mathcal{C}^s$ modes into primals and descendants [16,18]

$$\mathcal{C}^{(s,0)} = \{\phi \in \mathcal{C}^s; D^- \phi = 0\}$$

... primal fields

$$\mathcal{C}^{(s+k,k)} = (D^+)^k \mathcal{C}^{(s,0)}$$

... descendants

(6.28)

An important technical step is to work out explicitly the algebraic relation of $D^+$ and $D^-$, which can be shown to satisfy ladder-type commutation relations

$$D^+ D^- \phi^{(s,k)} = (a_{s,k} D^- + b_{s,k}) \phi^{(s,k)}, \quad \phi^{(s,k)} = (D^+)^k \phi^{(s-k,0)}.$$ 

(6.29)

with constants $a_{s,k}, b_{s,k}$ given in [18]. This is possible by exploiting the $\mathfrak{so}(4,2)$ structure as well as the special properties of the minireps $\mathcal{H}_m$. This sub-structure encodes two different concepts on the FRW background, which arise from the presence of a space-like foliation: $S^2 = 2s(s+1)$ measures the 4-dimensional spin on $H^4$, while $s - k$ measures the 3-dimensional spin of $\mathcal{C}^{(s,k)}$ on $H^3$. Nevertheless, local Lorentz invariance should be largely restored through gauge invariance.

Effective metric and d’Alembertian. In the matrix model framework, the effective metric on any given background is obtained by rewriting the kinetic term in covariant form [7,36]. Consider e.g. a transversal fluctuation $\phi = Y^4$ in the model [2.9], on the $\mathcal{M}^{5,1}$ background $Y^\mu = T^\mu$ under consideration. It suffices to consider scalar fluctuations $\phi = \phi(x)$. Then the action for $\phi$ is

$$S[\phi] = -\text{Tr}[T^\mu, \phi] [T_\mu, \phi] \sim \int d^4 x \sqrt{|G|} G^\mu\nu \partial_\mu \phi \partial_\nu \phi$$

(6.30)

where [7]

$$G^\mu\nu = \sinh^{-3}(\eta) \gamma^\mu\nu, \quad \gamma^{\alpha\beta} = \eta_{\mu\nu} \theta^{\mu\alpha} \theta^{\nu\beta} = \sinh^2(\eta) \eta^{\alpha\beta}$$

(6.31)

up to an irrelevant constant. This is a $SO(3,1)$-invariant FLRW metric with signature $(-+++)$,

$$ds^2_G = G_{\mu\nu} dx^\mu dx^\nu = -R^2 \sinh^3(\eta) d\eta^2 + R^2 \sinh(\eta) \cosh^2(\eta) \, d\Sigma^2$$

$$= -dt^2 + a^2(t) d\Sigma^2.$$ 

(6.32)
where $d\Sigma^2 = d\chi^2 + \sinh^2(\chi)d\Omega^2$ is the metric on the unit hyperboloid $H^3$. We can read off the cosmic scale parameter $a(t)$

$$a(t)^2 = R^2 \sinh(\eta) \cosh^2(\eta) \frac{t}{R^2} \sinh^3(\eta),$$

$$dt = R \sinh(\eta)^{\frac{3}{2}} d\eta$$

which leads to $a(t) \sim \frac{3}{7} t$ for late times, and $a(t) \sim t^{1/5}$ near the Big Bounce. This metric can also be extracted from the matrix d’Alembertian $\Box$ acting on $\phi \in C^0$, where $\Box_G = -\frac{1}{\sqrt{|G|}} \partial_{\mu} (\sqrt{|G|} G^{\mu\nu} \partial_{\nu})$.

### 6.2 Matrix model fluctuations and higher-spin Yang-Mills theory

Now we return to the noncommutative setting, and define a dynamical model for the fuzzy $M^{3,1}$ space-time. Consider again the Yang-Mills matrix model (2.9) with specific mass term,

$$S[Y] = \frac{1}{g^2} \text{Tr} \left( [Y^\mu, Y^\nu] [Y^\mu, Y^\nu] + 6 \frac{R^2}{R^2} Y^\mu Y^\mu \right).$$

(6.36)

As observed in [7], $M^{3,1}$ is indeed a solution of this model through $Y^\mu = T^\mu$. Now consider tangential deformations of this background solution, i.e.

$$Y^\mu = T^\mu + A^\mu,$$

(6.37)

where $A^\mu \in \text{End}(\mathcal{H}_n) \otimes \mathbb{R}^4$ is an arbitrary Hermitian fluctuation. The Yang-Mills action (6.36) can be expanded around the solution as

$$S[Y] = S[T] + S_2[A] + O(A^3),$$

(6.38)

and the quadratic fluctuations are governed by

$$S_2[A] = -\frac{2}{g^2} \text{Tr} \left( A^\mu \left(D^2 - \frac{3}{R^2} \right) A^\mu + G(A)^2 \right).$$

(6.39)

This involves the vector d’Alembertian on $M^{3,1}$

$$D^2 A = (\Box - 2 I) A$$

(6.40)

(cf. (6.35)) which is an $SO(3,1)$ intertwiner, as well as

$$\mathcal{I}(A)^\mu := -[[Y^\mu, Y^\nu], A_\nu] = \frac{i}{R^2} \left[ \Theta^{\mu\nu}, A_\nu \right] =: -\frac{1}{R^2} \tilde{I}(A)^\mu$$

(6.41)

This "momentum" embedding via $T^\mu$ has some similarity with the ideas in [45] but avoids excessive dof and the associated ghost issues, cf. [46]. The positive mass parameter in (6.36) simply sets the scale of the background. For negative mass parameter, $X^\mu$ would be a solution [47], but the fluctuation analysis would be less clear.
using (6.5). As usual in Yang-Mills theories, $\mathcal{A}$ transforms under gauge transformations as

$$\delta_\Lambda \mathcal{A} = -i[T^\mu + \mathcal{A}^\mu, \Lambda] \sim \{t^\mu, \Lambda\} + \{\mathcal{A}^\mu, \Lambda\}$$

(6.42)

for any $\Lambda \in \mathcal{C}$, and the scalar ghost mode

$$\mathcal{G}(\mathcal{A}) = -i[T^\mu, \mathcal{A}_\mu] \sim \{t^\mu, \mathcal{A}_\mu\}$$

(6.43)

should be removed. This is achieved by adding a gauge-fixing term $-\mathcal{G}(\mathcal{A})^2$ to the action as well as the corresponding Faddeev-Popov (or BRST) ghost. Then the quadratic action becomes

$$S_2[\mathcal{A}] + S_{g.f} + S_{\text{ghost}} = -\frac{2}{g^2} \text{Tr} \left( \mathcal{A}_\mu \left( D^2 - \frac{3}{R^2} \right) \mathcal{A}^\mu + 2\bar{c}c \right)$$

(6.44)

where $c$ denotes the BRST ghost; see e.g. [48] for more details.

### 6.3 Fluctuation modes

We now expand the vector modes into higher spin modes according to (5.26), (6.14)

$$\mathcal{A}^\mu = A^\mu(x) + A^\mu_a(x) t^a + A^\mu_{a\beta}(x) t^a t^\beta + \ldots \in \mathcal{C}^0 \oplus \mathcal{C}^1 \oplus \mathcal{C}^2 \oplus \ldots$$

(6.45)

We need to find explicitly all eigenmodes of $D^2$. This can be achieved using the $\mathfrak{so}(4,2)$ structure and suitable intertwiners. The result is as follows [18]. First, for any given $\phi^{(s)} \in \mathcal{C}^s$ we define the fluctuation modes

$$\mathcal{A}^{(g)}_i[\phi^{(s)}] = \{t_{\mu}, \phi^{(s)}\} \in \mathcal{C}^s,$$

$$\mathcal{A}^{(+)}_i[\phi^{(s)}] = \{x_{\mu}, \phi^{(s)}\}_{s+1} \equiv \{x_{\mu}, \phi^{(s)}\}_+ \in \mathcal{C}^{s+1},$$

$$\mathcal{A}^{(-)}_i[\phi^{(s)}] = \{x_{\mu}, \phi^{(s)}\}_{s-1} \equiv \{x_{\mu}, \phi^{(s)}\}_- \in \mathcal{C}^{s-1},$$

$$\mathcal{A}^{(n)}_i[\phi^{(s)}] = D^n A^{(-)}_i[\phi^{(s)}] \in \mathcal{C}^s.$$  

(6.46)\(6.47)\(6.48)

Then for any eigenmode of $\Box \phi^{(s)} = m^2 \phi^{(s)}$ we obtain 4-tuples of "regular" eigenmodes $\mathcal{A}^{(i)}_i[\phi^{(s)}] \in \mathcal{C}^s \otimes \mathbb{R}^4$ of $D^2$

$$\mathcal{A}^{(i)}[\phi] = \begin{pmatrix} \mathcal{A}^{(+)}[D^+ \phi] \\ \mathcal{A}^{(-)}[D^+ \phi] \\ \mathcal{A}^{(n)}[\phi] \\ r^2 R \mathcal{A}^{(g)}[\phi] \end{pmatrix}, \quad i, j \in \{+,-,n,g\}$$

(6.49)

for $\phi = \phi^{(s)}$ dropping the index $\mu$, with the same eigenvalue

\[
\begin{align*}
D^2 \mathcal{A}^{(+)}[\phi] &= (m^2 + \frac{3}{R^2}) \mathcal{A}^{(+)}[\phi] \\
D^2 \mathcal{A}^{(-)}[\phi] &= (m^2 + \frac{3}{R^2}) \mathcal{A}^{(-)}[\phi] \\
D^2 \mathcal{A}^{(g)}[\phi] &= (m^2 + \frac{3}{R^2}) \mathcal{A}^{(g)}[\phi] \\
D^2 \mathcal{A}^{(n)}[\phi] &= (m^2 + \frac{3}{R^2}) \mathcal{A}^{(n)}[\phi]
\end{align*}
\]

(6.50)
There is one “special” mode which is not covered by the regular $\tilde{\mathcal{A}}^{(i)}$, namely $\mathcal{A}^{(-)}[\phi^{(s,0)}]$ with
\begin{equation}
D^2 \mathcal{A}^{(-)}[\phi^{(s,0)}] = (\Box + \frac{-2s + 3}{R^2}) \mathcal{A}^{(-)}[\phi^{(s,0)}] .
\end{equation}

We will see that it is orthogonal to all regular modes, and altogether these modes are complete. Hence diagonalizing $D^2$ is reduced to diagonalizing $\Box$ on $C^s$. In particular, we obtain the following on-shell modes $(D^2 - \frac{3}{R^2}) \mathcal{A} = 0$
\begin{equation}
\{\tilde{\mathcal{A}}^{(+)\rho}[\phi^{(s)}], \tilde{\mathcal{A}}^{(-)\rho}[\phi^{(s)}], \tilde{\mathcal{A}}^{(g)}[\phi^{(s)}], \tilde{\mathcal{A}}^{(n)}[\phi^{(s)}]\} \quad \text{for } \Box \phi^{(s)} = 0
\end{equation}
\begin{equation}
\mathcal{A}^{(-)}[\phi^{(s,0)}] \quad \text{for } (\Box - \frac{2s}{R^2}) \phi^{(s,0)} = 0 .
\end{equation}

These modes are not orthogonal yet, but it is possible to find a basis of orthogonal eigenmodes by diagonalizing the intertwiner $I$ (6.41), which commutes with $D^2$
\begin{equation}[I, D^2] = 0 .
\end{equation}

This turns out to be rather tedious, and requires the sub-structure of $\phi^{(s,k)}$ modes as defined in (6.28). The relations (6.29) then allows to compute all the inner products and eigenvalues explicitly, which is carried out in [18]. It turns out that $\tilde{\mathcal{A}}^{(n)}[\phi^{(s,s)}]$ is redundant, and $\tilde{\mathcal{A}}^{(+)}[\phi^{(s,0)}] \equiv 0$. One can then show the following completeness statement:

**Theorem 6.1.** The $\tilde{\mathcal{A}}^{(i)}[\phi^{(s)}]$ modes (6.49) along with the $\mathcal{A}^{(-)}[\phi^{(s,0)}]$ for all $s \geq 0$ span the space of all fluctuations $\mathcal{A}$. A basis is obtained by dropping $\tilde{\mathcal{A}}^{(n)}[\phi^{(s,s)}]$ and $\tilde{\mathcal{A}}^{(+)\rho}[\phi^{(s,0)}]$.

This completes the classification of off-shell modes.

### 6.4 Physical constraint, Hilbert space and no-ghost theorem

Now consider the on-shell modes. We first observe that an (admissible, i.e. square-integrable) fluctuation mode $\mathcal{A}$ satisfies the gauge-fixing condition $\{t^\mu, \mathcal{A}_\mu\} = 0$ if and only if it is orthogonal to all pure gauge modes,
\begin{equation}
\langle \mathcal{A}^{(g)}, \mathcal{A} \rangle \equiv \int \mathcal{A}^{(g)\mu} \mathcal{A}_\mu = 0 .
\end{equation}

where the $SO(4,2)$ - invariant integral arises from the trace on End($H_n$). Now consider an on-shell mode $\mathcal{A} \in C^s$ in some 4-dimensional mode space $\tilde{\mathcal{A}}^{(i)}[\phi]$, $i \in \{-ng\}$ determined by some $\phi \in C^{(s,k)}$ with $\Box \phi = 0$ and $s > k > 0$. One can show that this 4-dimensional space of modes has signature $(+++)$ and $\tilde{\mathcal{A}}^{(g)}$ is null. Then the gauge-fixing constraint (6.54) leads to a 3-dimensional subspace with signature $(+0)$, which contains $\mathcal{A}^{(9)}$. Then the usual definition
\begin{equation}
H_{\text{phys}} = \{\text{gauge-fixed on-shell modes}\} / \{\text{pure gauge modes}\}
\end{equation}
leads to 2 modes with positive norm. By orthogonalizing all the eigenmodes (6.50) of $D^2$, one can similarly establish the no-ghost theorem [18]

**Theorem 6.2.** The space $H_{\text{phys}}$ (6.55) of admissible solutions of $(D^2 - \frac{3}{R^2}) \mathcal{A} = 0$ which are gauge-fixed $\{t^\mu, \mathcal{A}_\mu\} = 0$ modulo pure gauge modes inherits a positive-definite inner product, and forms a Hilbert space.
"Admissible" means that the modes are square-integrable on \( H^4 \) or equivalently on \( \mathcal{M}^{3,1} \), more precisely that they live in principal series unitary representations of \( SO(4,1) \). On-shell, this is essentially equivalent to the requirement that they are square-integrable on space-like slices \( H^3 \). Indeed, the on-shell relation \( \Box \phi = 0 \) determines \( C^2[SO(4,1)] \) via (6.3) from \( C^2[SO(3,1)] \), correspond to an irreducible tensor field on the space-like \( H^3 \). In other words, the state at any given time-slice \( H^3 \) completely determines the time evolution (up to time direction). Hence one obtains the standard picture of time evolution even though time does not commute, and the time evolution is completely captured by \( SO(4,1) \) group theory, even though \( \mathcal{M}^{3,1} \) admits only space-like \( SO(3,1) \) isometries.

Explicitly, this gives the following physical modes:

**The physical modes** \( \mathcal{A}_\mu \in \mathcal{C}^0 \). The off-shell modes \( \mathcal{A}_\mu \in \mathcal{C}^0 \) comprise the spin 1 mode \( \mathcal{A}^{(-)}[\phi^{(1)}] \) and the spin 0 modes \( \mathcal{A}^{(0)}[\phi^{(0)}] \) are in \( \mathcal{C}^0 \otimes \mathbb{R}^4 \). Among these, only the spin 1 modes \( \mathcal{A}^{(-)}[\phi^{(1,0)}] \) are physical, and

\[
\mathcal{H}_{\text{phys}} \cap \mathcal{C}^0 = \{ \mathcal{A}^{(-)}[\phi] \text{ for } \phi \in \mathcal{C}^{(1,0)}, (\Box - \frac{2}{R^2})\phi = 0 \}. \tag{6.56}
\]

These modes satisfy \( \partial^\mu \mathcal{A}_\mu = 0 = x^\mu \mathcal{A}_\mu \), and describe a spin 1 Yang-Mills (or Maxwell) field.

**The physical modes** \( \mathcal{A}_\mu \in \mathcal{C}^1 \). This space comprises 12 off-shell modes \( \mathcal{A}^{(-)}[\phi^{(2)}] \), \( \mathcal{A}^{(n)}[\phi^{(1)}] \), \( \mathcal{A}^{(g)}[\phi^{(1)}] \) and \( \mathcal{A}^{(+)}[\phi^{(0)}] \). Among these, all \( \mathcal{A}^{(-)}[\phi^{(2)}] \) are physical, and there are no further physical states in this sector:

\[
\mathcal{H}_{\text{phys}} \cap \mathcal{C}^1 = \{ \mathcal{A}^{(-)}[\phi] \text{ for } \phi \in \mathcal{C}^{(2,s)}, (\Box - \frac{4}{R^2})\phi = 0 \}. \tag{6.57}
\]

They satisfy \( \{ t^\mu, \mathcal{A}_\mu \} = 0 \), and \( x^\mu \mathcal{A}_\mu[\phi^{(2,0)}] = 0 \). These modes govern the linearized gravity sector, as discussed below.

**Generic physical modes** \( \mathcal{A}_\mu \in \mathcal{C}^s \) with \( s \geq 2 \). Finally in the generic case \( s \geq 2 \), the physical constraints \( \{ t^\mu, \mathcal{A}_\mu \} = 0 \) must be solved directly. This leads to the following modes: There is one physical mode determined by \( \phi^{(s,0)} \), which we can choose to be a linear combination

\[
\{ c_- \mathcal{A}^{(-)}[\phi^{(s,0)}] + \mathcal{A}^{(n)}[\phi^{(s,0)}] \text{ for } \Box \phi^{(s,0)} = 0 \} \subset \mathcal{H}_{\text{phys}} \cap \mathcal{C}^s \tag{6.58}
\]

where \( c_- \) is determined by solving the above constraint. Next, there is one physical scalar mode determined by \( \phi^{(s,s)} \) for each \( s \neq 0 \), which we can choose to be

\[
\{ \mathcal{A}^{(-)}[\phi^{(s,s)}] + c_+ \mathcal{A}^{(s)}[\phi^{(s,s)}] \text{ for } \Box \phi^{(s,s)} = 0 \} \subset \mathcal{H}_{\text{phys}} \cap \mathcal{C}^s. \tag{6.59}
\]

Finally for \( s \neq k \neq 0 \), there are two physical modes determined by \( \phi^{(s,k)} \), which we can choose to be

\[
\{ \mathcal{A}^{(-)}[\phi^{(s,k)}] + c_+ \mathcal{A}^{(s)}[\phi^{(s,k)}] \text{ for } \Box \phi^{(s,k)} = 0 \} \subset \mathcal{H}_{\text{phys}} \cap \mathcal{C}^s \tag{6.60}
\]

This completes the list of physical modes.
To summarize, the model contains generically 2 physical modes parametrized by \( \phi^{(s)} \in C^s \) with \( \Box \phi^{(s)} = 0 \) for each spin \( s \geq 2 \), up to the exceptional cases discussed above. These are "would-be massive" modes, i.e. they contain the \( 2s + 1 \) dof of massive spin \( s \) multiplets with vanishing mass parameter, and decompose further into a series of irreducible massless spin \( s \) modes in radiation gauge for \( k \leq s \) as described above. All modes transform and mix under a higher-spin gauge invariance. It is hence plausible that some of these modes become massive in the interacting theory, but this remains to be clarified.

6.5 Metric fluctuation modes

Now we discuss how metric fluctuations arise from the above modes. The effective metric for functions of \( \mathcal{M}^{3,1} \) on a perturbed background \( Y = T + \mathcal{A} \) can be extracted from the kinetic term in (6.30), which defines the bi-derivation

\[
\gamma : \quad C \times C \to C \\
(\phi, \phi') \mapsto \{Y^\alpha, \phi\} \{Y_\alpha, \phi'\}.
\]

Specializing to \( \phi = x^\mu, \phi' = x'^\mu \) we obtain the coordinate form

\[
\gamma^{\mu\nu} = \eta^{\mu\nu} + \delta A \gamma^{\mu\nu} + \{A^{\alpha}, x^\mu\} \{A_\alpha, x'^\nu\} |_{0}
\]

whose linearized contribution in \( \delta A \) is given by

\[
\delta A \gamma^{\mu\nu} = \sinh(\eta) h^{\mu\nu}[A], \quad h^{\mu\nu}[A] := \{A^\mu, x_\nu\} |_{0} + (\mu \leftrightarrow \nu).
\]

and \( h[A] = 2\{A^\mu, x_\mu\} |_{0} \). The projection on \( C^0 \) ensures that this is the metric for functions on \( \mathcal{M}^{3,1} \). Clearly only \( A \in C^1 \) can contribute, which we assume henceforth. Including the conformal factor in (6.31), this leads to the effective metric fluctuation

\[
\delta G^{\mu\nu} = \beta^2 \tilde{h}^{\mu\nu},
\]

where

\[
\tilde{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} h, \quad \beta = \frac{1}{\sinh(\eta)}.
\]

Let us discuss the mode content of \( h^{\mu\nu}[A] \). Recall that the 12 off-shell dof in \( A_{\mu} = A_{\mu}^{\alpha} t^\alpha \in C^1 \) are realized by \( \mathcal{A}^{(-)}[\phi^{(2)}], \mathcal{A}^{(n)}[\phi^{(1)}], \mathcal{A}^{(g)}[\phi^{(1)}] \) and \( \mathcal{A}^{(+)}[\phi^{(0)}] \). Hence the 10 dof of the most general off-shell metric fluctuations are provided by \( \mathcal{A}^{(-)}[\phi^{(2)}], \mathcal{A}^{(g)}[\phi^{(1)}], \) and the scalar modes \( \mathcal{A}^{(+)}[\phi^{(0)}] \) and \( \mathcal{A}^{(n)}[D^+ \phi^{(0)}] \). According to the results of section 6.4 the physical modes among these are the 5 would-be massive spin 2 modes \( \mathcal{A}^{(-)}[\phi^{(2)}] \), which decompose into the massless graviton \( \mathcal{A}^{(-)}[\phi^{(2,0)}] \), one massless vector mode \( \mathcal{A}^{(-)}[\phi^{(2,1)}] \), and one scalar mode \( \mathcal{A}^{(-)}[\phi^{(2,2)}] \). The vector field can be extracted by

\[
\{t_\mu, h^{\mu\nu}\} \equiv -\frac{2}{R} D^- A^\nu
\]

as shown in [18], which vanishes for the \( \mathcal{A}^{(-)}[\phi^{(2,0)}] \) graviton mode.
Linearized Ricci tensor. To understand the significance of the metric modes, we consider the linearized Ricci tensor
\[ 2R_{(\text{lin})}^{\mu\nu}[G] = -\nabla^\alpha \nabla_\alpha \delta G^{\mu\nu} + \nabla^\mu \nabla_\rho \delta G^{\rho\nu} + \nabla^\nu \nabla_\rho \delta G^{\mu\rho} - \nabla^\mu \nabla^\nu \delta G \] (6.67)
for a metric fluctuation \( \delta G^{\mu\nu} = \beta^2 \tilde{h}^{\mu\nu} \) around the background \( \bar{G}^{\mu\nu} = \beta \eta^{\mu\nu} \). For simplicity, we neglect contributions of the order of the cosmic background curvature. Then we can replace \( \nabla \) by \( \partial \) in Cartesian coordinates, and
\[ 2R_{(\text{lin})}^{\mu\nu}[G] \eta \to \infty \approx \beta^2 \left( \partial^\alpha \partial_\alpha \tilde{h}^{\mu\nu} + \partial^\mu \partial_\rho \tilde{h}^{\rho\nu} + \partial^\nu \partial_\rho \tilde{h}^{\mu\rho} - \partial^\mu \partial^\rho \tilde{h} \right) \] on-shell \( \approx \beta^2 \left( \partial^\rho \partial_\rho h^{\mu\nu} + \partial^\nu \partial_\rho h^{\mu\rho} + O\left( \frac{\partial h^{\mu\nu}}{x^4} \right) \right) \). (6.68)
Here the on-shell relation \( (D^2 - \frac{3}{x^2}) A = 0 \) is used as well as \( \partial \gg \frac{1}{x^4} \) at late times \( \eta \to \infty \), focusing on scales much shorter than the cosmic curvature scale.

Pure gauge modes. Consider first the metric fluctuation corresponding to the pure gauge fields \( A^{(g)}[\phi] \), where \( \phi \in C^1 \) is a spin 1 field. It is not hard to see that (6.64) then takes the form
\[ h^{\mu\nu}[A^{(g)}] = -\{ t^\mu , A^{(\bar{\mu})} \} - \{ t^\nu , A^{(\bar{\nu})} \} + 2\eta^{\mu\nu} \{ t^\alpha , A^{(\bar{\alpha})} \} , \] (6.69)
where \( A^{(-)} = A^{(-)}[\phi] \). Then the pure gauge contribution to the effective metric (6.64) is
\[ \delta G^{\mu\nu}_{(g)} = \nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu , \quad \xi^\mu = -A^{(\bar{\mu})} \] (6.70)
Hence the pure gauge metric modes in the present framework can be identified with diffeomorphisms generated by \( \xi = -A^{(\bar{\mu})} \). This provides a non-trivial consistency check for the correct identification of the effective metric. These diffeomorphisms are essentially volume-preserving due to the constraint \( \nabla_\alpha (\beta^3 \xi^\alpha) = 0 \) [6.24], leaving only 3 rather than 4 diffeomorphism d.o.f., unlike in GR. This reflects the presence of a dynamical scalar metric degree of freedom, which we will discuss in more detail below.

The \( A^{(-)} \) metric modes. Now consider the \( A^{(-)}[\phi^{(s)}] \) modes. Among these, only the ones with spin \( s = 2 \) can contribute to the metric, and these are precisely the physical degrees of freedom as shown above. The corresponding linearized metric fluctuation is
\[ h^{\mu\nu}[A^{(-)}] = -2\{ x^\mu , \{ x^\nu , \phi \} \} = -2\{ x^\nu , \{ x^\mu , \phi \} \} - \eta_{\mu\nu} h^{\mu\nu} = 2D_- D_- \phi = -\frac{1}{x^4} x_\mu x_\nu h^{\mu\nu} \] (6.71)
for \( \phi \in C^2 \). They satisfy the identity
\[ \partial_\mu (\beta h^{\mu\nu}) = 0 , \] (6.72)
which together with (6.68) implies that all these on-shell (would-be massive) spin 2 modes are Ricci-flat up to cosmic scales,
\[ 2R_{(\text{lin})}^{\mu\nu} = 0 + O\left( \frac{\partial G^{\mu\nu}}{x^4} \right) . \] (6.73)
The $\mathcal{A}^{-}[\phi^{(2,0)}]$ modes clearly reproduce the two Ricci-flat graviton modes from GR. The modes arising from $\phi^{(2,1)}$ and $\phi^{(2,2)}$ essentially complete a massive spin 2 multiplet due to (6.72) and (6.71), however the statement (6.73) is largely empty, because these modes are generically dominated by diffeos which are trivially flat. Nevertheless, we will see that the quasi-static $\mathcal{A}^{-}[\phi^{(2,2)}]$ solution leads to a non-trivial Ricci-flat metric perturbation, which is nothing but the linearized Schwarzschild metric. This is consistent with the identity (6.71) for the effective metric (6.64),

$$\eta_{\mu\nu} \delta G^{\mu\nu} = \frac{2}{x^4} x_\mu x_\nu \delta G^{\mu\nu}.$$  

(6.74)

### 6.6 The linearized Schwarzschild solution

Now we work out the metric perturbation arising from the physical $\mathcal{A}^{-}[\phi^{(2,2)}]$ mode, where $\phi^{(2,2)} = D^+ D \phi$ for a scalar field $\phi \in \mathcal{C}^0$. This is part of the would-be massless spin 2 multiplet $\mathcal{A}^{-}[\phi^{(2)}]$, and the associated metric perturbation is guaranteed to be Ricci-flat (on-shell) due to (6.73). We will see that this includes a quasi-static Schwarzschild metric, as well as dynamical solutions which might be related to dark matter. We will use the on-shell condition $\Box \phi = -\frac{2}{R^2} \phi$, and focus on the late-time limit $\eta \to \infty$. Then the trace-reversed metric fluctuation $\tilde{h}^{\mu\nu}$ is found to be [17]

$$\tilde{h}^{\mu\nu} = \frac{4r^4}{45} \left( -\frac{1}{2} (5\tau + 13) \eta^{\mu\nu} + \frac{\beta^2}{R^2} x^\mu x^\nu (\tau - 1) - (x^\nu \partial^\mu + x^\mu \partial^\nu) (\tau + 2) \right) (\tau + 1)(\tau + 2) \phi \right.$$

$$- \frac{4r^4}{45} R^2 \partial^\nu \partial^\mu (\tau + 4)(\tau + 2) \phi.$$  

(6.75)

Observe that for $\tau \neq -2$, the term $(x^\nu \partial^\mu + x^\mu \partial^\nu) \phi$ is dominant at late times, since $x^0 \sim R \cosh(\eta)$. This can be removed using a suitable diffeomorphism contribution $\nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu$, with the result

$$\tilde{h}^{\mu\nu} \sim \frac{4r^4}{45} \left( \frac{1}{2} (\tau - 1) \eta^{\mu\nu} + \frac{3\beta^2}{R^2} x^\mu x^\nu (\tau + 1) \right) (\tau + 1)(\tau + 2) \phi.$$  

(6.76)

for large $\eta$. Then

$$\tilde{h}^{\mu\nu} \, dx^\mu dx^\nu = \frac{2r^4 R^2}{45} \sinh^2(\eta) \left( d\eta^2 (5\tau + 7) + d\Sigma^2 (\tau - 1) \right) (\tau + 1)(\tau + 2) \phi \right.$$

$$\tau \to \infty - 4\phi'(dt^2 + a(t)^2 d\Sigma^2)$$  

(6.77)

using the explicit form (6.32) of the scale parameter $a(t)$ for large $\eta$, where

$$\phi' = -\frac{r^4}{30} \beta (\tau + 2) \phi.$$  

(6.78)

For $\tau \neq -2$ this metric is not Ricci-flat, which seems inconsistent with (6.73). However then the diffeo contribution in (6.75) is very large at late times, invalidating the linearized approximation. Therefore we restrict ourselves to the "quasi-static" case $\tau \approx -2$. Then the full perturbed metric can be written in the form

$$ds^2 = (G_{\mu\nu} - \delta G_{\mu\nu}) dx^\mu dx^\nu = -dt^2 + a(t)^2 d\Sigma^2 + 4\phi'(dt^2 + a(t)^2 d\Sigma^2).$$  

(6.79)
The on-shell condition reduces to $\Delta^{(3)}\phi = 0$ for $\tau = -2$, and in the spherically symmetric case the Newton potential on a $k = -1$ geometry is recovered, more precisely

$$\phi = \frac{e^{-\chi}}{\sinh(\chi) \cosh^2(\eta)} \sim \frac{1}{\rho} e^{-\chi - 2\eta}, \quad \rho = \sinh(\chi).$$

Strictly speaking we should use $\phi'$ rather than $\phi$ in the $(\tau + 2)\phi = 0$ case; then the quasi-static condition becomes $(\tau + 3 + \beta^2)\phi' = 0$ and the on-shell condition is $(\Delta^{(3)} - 4\beta^4)\phi' = 0$, which reduces again to $\Delta^{(3)}\phi' = 0$ in the large $\eta$ limit. This gives

$$\phi' \sim \frac{1}{\rho} e^{-\chi - 3\eta} \sim \frac{e^{-\chi}}{\rho a(t)^2},$$

for large $\eta$, recalling $a(t) \sim e^{-\frac{3}{2}\eta}$ (6.33). This metric is very close to the Vittie solution [49] for the Schwarzschild metric for a point mass $M$ in a FRW spacetime, whose linearization for $k = -1$ is

$$ds^2 = -dt^2 + a(t)^2 d\Sigma^2 + 4\mu(dt^2 + a(t)^2 d\Sigma^2) + O(\mu^2).$$

Here

$$\mu = \mu(t, \chi) = \frac{M}{2\rho a(t)}$$

is the mass parameter, which is not constant but decays during the cosmic expansion; this is as it should be, because local gravitational systems do not participate in the expansion of the universe. Comparing with (6.81) we have

$$\phi' \sim \mu(t, \chi) \frac{e^{-\chi}}{a(t)}. $$

Since $\mu$ (6.83) looks like a constant mass for a comoving observer [49], the effective mass parameter in our solution effectively decreases like $a(t)^{-1}$ during the cosmic evolution. This suggests a time-dependent Newton constant, however this is premature since the coupling to matter is not fully understood, and quantum effects may modify the result. Also, while both metrics have the characteristic $\frac{1}{\rho}$ dependence of the Newton potential, the present solution has an extra $e^{-\chi}$ factor, which reduces its range at cosmic scales. Both effects are irrelevant at solar system scales, but they will be important for cosmological considerations, reducing the gravitational attraction at long scales.

It is interesting that the scalar on-shell modes provide a Ricci-flat metric perturbation only for the quasi-static case $\tau = -2$. Of course it should be expected that a dynamical scalar metric mode, which does not exist in GR, is not Ricci-flat in general. From a GR point of view, such non-Ricci-flat perturbations would be interpreted as dark matter. One can argue [17] that they should be more important at very long wavelengths, however a more detailed examination at the non-linear level is needed to clarify the significance of the extra metric modes.

Finally, note that only vacuum solutions were considered so far. While the metric fluctuations couple to matter as usual, it is not evident that the standard inhomogeneous Einstein equations arise; in fact some higher-derivative contributions of the type $(1 + \Box_H)T^{\mu\nu}$ are expected at the classical level, cf. section 6.3 in [7]. On the other hand, quantum effects are bound to induce an Einstein-Hilbert-like term in the effective action [50], and due to the (partial) local Lorentz invariance it is plausible that the resulting gravity theory will be reasonably close to Einstein gravity. However, further work is needed to corroborate that claim.
6.7 Towards quantization

We briefly comment on the quantization of the model via the matrix integral (2.11). As pointed out before, the maximal supersymmetry of the IKKT model leads to important cancellations in the loops, and the resulting gauge theory is very similar to \( \mathcal{N} = 4 \) SYM with higher-spin-type gauge invariance. It is thus reasonable to expect that the model is well-defined at the quantum level. Nevertheless there are issues to be considered. First, the \( M^3,1 \) background under consideration is non-compact and described by matrices which are infinite-dimensional. Furthermore, the explicit mass term in (2.9) amounts to a soft SUSY breaking term, and will reduce the degree of UV cancellations. There is in fact a background-independent (albeit implicit) formula for the one-loop effective action \[ \Gamma_{\text{loop}}[X] = \frac{1}{2} \text{Tr} \left( \frac{1}{4} \Box^{-1} (M^{(A)}_{ab}[\Theta^{ab},.])^4 - \frac{1}{8} (\Box^{-1} M^{(\psi)}_{ab}[\Theta^{ab},.])^4 + \mathcal{O}(\Box^{-1}[\Theta^{ab},.])^5 \right) + \frac{1}{2} m^2 \text{Tr} \Box^{-1} + \mathcal{O}(m^4) \] \( (6.85) \) with \( a, b = 0, ..., 9 \). Here \( M^{(\psi)}_{ab} \) and \( M^{(A)}_{ab} \) are the \( \mathfrak{so}(9,1) \) generators acting on the spinor and vector representation, respectively. This is basically the standard \( \text{det} = \exp \text{Tr} \log \) formula, taking into account cancellations due to SUSY. Since the formula is background-independent, it provides the full one-loop effective action by including fluctuations \( A \) to the background (3.1). The trace can be evaluated efficiently using the formalism of string states, cf. [8, 52]. In the absence of the mass \( m^2 \), this is manifestly finite on \( M^3,1 \), as on any 4-dimensional background\(^9\). However the last term leads to a divergence for \( m^2 \neq 0 \). Although this is just an irrelevant constant on the unperturbed background, the cancellations are less effective, and it would be better to find a finite-dimensional version of the solution which allows to scale \( m^2 \) with \( N \) to obtain a well-defined large \( N \) limit. This is also required to make contact with numerical simulations. For a possible ansatz see [47], but there may be better ones. This is one of the open problems to be addressed in future work.

6.8 Further literature

We provide a selection of related literature which may be useful to understand the present framework and its broader context. Fuzzy spaces were introduced in [28,29,53], and useful discussions from a field theoretical point of view can be found e.g. in [54,57]. More mathematical details on quantized symplectic spaces can be found in [58,60], and for quantized coadjoint orbits see e.g. [61] or section 4.2 in [62]. Coherent states on fuzzy spaces are very useful [8,19,63,64], and provide the basis for a visualization tool [65]. Fuzzy spaces as solutions of Yang-Mills matrix models and the associated NC gauge theory have been studied e.g. in [6,66,74], and for NC field theory more generally see [75,76] and references therein. The role of fuzzy spaces as D-branes in string theory is discussed in [62,66,77,79], and in the context of matrix quantum mechanics e.g. in [3,80,84]. A relation of NC gauge theory and (emergent) gravity has long been suspected, and the effective metric in matrix models and its dynamical aspects are discussed in [24,35,36,85]. Fuzzy extra dimensions are based on similar structures and can provide a relation with particle physics, see e.g. [12,13].

\(^9\)The internal \( S^2 \) fiber does not cause any complications because it is compact, and in fact \( S^2_N \) admits only finitely many harmonics. Therefore the background is effectively 4-dimensional in the UV.
Covariant higher-dimensional fuzzy spaces were studied e.g. in \[2,4,6,52,86–91\], which are similar in spirit to Snyder space \[92,93\], see also \[45\] for a somewhat related ansatz. In particular, the relation of fuzzy \(S^4\) to fuzzy \(\mathbb{C}P^2\) was pointed out in \[5,94,95\], which is analogous to the bundle structure discussed in section \(\text{5}\). For numerical investigations we refer to \[27,96–98\] and references therein.

7 Conclusion and outlook

The short summary of this review article is that an explicit model for 3+1-dimensional quantum space-time has been established, which is a solution of the mass-deformed IKKT-type matrix model, and leads to a consistent higher-spin gauge theory without ghosts. While local Lorentz invariance is only partially manifest, it appears to be respected effectively. This theory includes a dynamical metric leading to an emergent gravity model, which includes the standard propagating massless spin 2 gravitons, as well as the linearized Schwarzschild solution. However the metric also includes extra physical dof which can be viewed as arising from a would-be massive spin 2 mode. While the full dynamics of the emergent gravity and its dependence on matter is yet to be understood, the results so far make it plausible that it will be reasonably close to Einstein gravity, at least upon taking into account quantum effects.

The matrix model framework thus provides a unified treatment of space-time and field theory, and the present solution is arguably the most satisfactory one so far. However its study is just at the beginning, and many things need to be worked out and clarified.

An important question which should be addressed is the following: what are the observable signature of this scenario, and are there any clear-cut signals which would distinguish it from other approaches? Of course this question can only be answered reliably once the resulting gravitational physics is worked out, which is the most urgent open problem. This is not trivial because it may be necessary to include quantum effects (in the sense of induced gravity a la Sakharov \[50\]), and/or to find a way to break the higher-spin gauge invariance beyond spin 2. Another, perhaps more clear-cut open problem is to find an exact non-linear analog of the (classical) Schwarzschild solution extending the present linearized solution. This would give important information about the formation of horizons in the present framework, and hints about the resolution of singularities. Similarly, the presence of a classical Big Bounce solution is very intriguing, and could certainly be explored further in the present stage. Modifications of the time evolution should be possible by slightly relaxing the Lie algebra structure, and even compactified solutions are conceivable. Finally, closer links with Vasiliev’s higher spin gravity as well string theory realizations of the present brane solutions should be studied. These are only some of the possible directions for further work.

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