Non-commutative lattice-modified Gel’fand–Dikii systems

Adam Doliwa
Faculty of Mathematics and Computer Science, University of Warmia and Mazury, ul. Słoneczna 54, 10-710 Olsztyn, Poland
E-mail: doliwa@matman.uwm.edu.pl

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Abstract
We introduce integrable multicomponent non-commutative lattice systems, which can be considered as analogues of the modified Gel’fand–Dikii hierarchy. We present the corresponding systems of Lax pairs and show directly the multidimensional consistency of these Gel’fand–Dikii-type equations. We demonstrate how the systems can be obtained as periodic reductions of the non-commutative lattice Kadomtsev–Petviashvili hierarchy. The geometric description of the hierarchy in terms of Desargues maps helps to derive a non-isospectral generalization of the non-commutative lattice-modified Gel’fand–Dikii systems. We show also how arbitrary functions of single arguments appear naturally in our approach when making commutative reductions, which we illustrate on the non-isospectral non-autonomous versions of the lattice-modified Korteweg–de Vries and Boussinesq systems.

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1. Introduction

The so-called Gel’fand–Dikii (GD) hierarchy [42,66] generalizes the Korteweg–de Vries (KdV) hierarchy [21,88] to higher-order spectral problems. The lattice GD hierarchy, obtained via the direct linearization approach and given in [75], is a multicomponent system which goes to the ‘continuous’ hierarchy under suitable limits. The simplest systems are the lattice potential KdV equation [74] and a lattice version of the Boussinesq equation. In [75] it was also indicated how to obtain a ‘modified’ version of the lattice GD hierarchy and its two simplest members were given explicitly.

The origins of soliton theory can be traced back to classical developments in the theory of submanifolds and their transformations [9,20,19,37,91]. The analogous geometric interpretation of integrable partial difference systems was developed recently starting from [11,12,23,32,13,54,36]. The integrability of discrete (Adler)–GD equations was investigated recently using so-called pentagram maps [87,80,1,67]. From the author’s personal viewpoint...
the key objects in the geometric theory of integrable discrete systems are the notions of Desargues maps [29, 30] (see section 3) and the closely related lattices of planar quadrilaterals [84, 23, 32] (quadrilateral lattices for short), which are difference geometry counterparts of conjugate nets investigated thoroughly by Darboux, Bianchi and those that followed.

In recent studies on discrete integrable systems, the property of multidimensional consistency [3, 72] is considered as the main concept of the theory. Roughly speaking, it is the possibility of extending the number of independent variables of a given nonlinear system by adding its copies in different directions without creating inconsistency or multivaluedness. The multidimensional consistency is considered as ‘the precise analogue of the hierarchy of nonlinear evolution equations in the case of continuous systems’ [76]. Such an approach lies also at the roots of an earlier ‘principle’ of discretization of integrable differential equations via iterated applications of Bäcklund transformations [60]; for its application to systems of a geometric origin see also [78, 41, 36]. Recently, the multidimensional consistency of the GD hierarchy and related multicomponent systems was studied in [64, 48, 4].

Hirota’s discrete Kadomtsev–Petviashvili (KP) equation [51] plays a dominant role in the whole theory of integrable systems both classical and quantum [57]. Its direct geometric counterparts are Desargues maps, while multidimensional quadrilateral lattices give rise to the so-called discrete Darboux equations [15]. The multidimensional consistency of the geometric construction of quadrilateral lattices and of the discrete Darboux equations was discussed already in [32]. Subsequently, certain geometric restrictions compatible with such a multidimensionally consistent construction scheme were used [18, 24, 33, 25, 27] to isolate integrable reductions of the quadrilateral lattices. The important relation of four-dimensional consistency of the Hirota equation (in its commutative Schwarzian form) and the Desargues configuration has been observed by Wolfgang Schief (the author has learned about that relation from the talk of Alexander Bobenko [10]). Multidimensional consistency of both systems survives the transfer to their non-commutative versions [28, 29], whereas the ambient space of the lattice submanifold takes the projective space over an arbitrary division ring [8]. One of the motivations to study non-commutative versions of integrable discrete systems [70, 73, 58, 14, 77] is their relevance to integrable lattice field theories. In particular, as shown by [6, 89], the four-dimensional consistency of the geometric construction of a multidimensional lattice of planar quadrilaterals [32] is related to Zamolodchikov’s tetrahedron equation [94], which is a multidimensional analogue of the quantum Yang–Baxter equation [5, 56, 52]. Recently, the four-dimensional consistency of Desargues maps has been related [34, 31] to certain solutions of the functional pentagon equation [93] and of its quantum reduction, which is of fundamental importance in the modern analytic theory of quantum groups [90].

The non-autonomous and/or non-isospectral versions of integrable equations contain an additional functional freedom which, from the physical point of view, allows to study solitary waves in non-uniform media with relaxation effects and therefore provide more realistic models. The relevant mathematical tools are an interesting generalization of the standard techniques of integrable systems theory both for differential and difference equations, see for example [16, 40, 43, 61, 63, 65, 17]. It is remarkable that certain non-isospectral equations relevant in gravity theory [38] were known in the purely geometric context of surface theory [9, 62], see also [85, 35] where a corresponding discrete system was investigated. Recently a non-commutative non-isospectral KP equation was studied in [95].

In making the dimensional symmetry reduction [71, 76] of a given integrable partial difference system, to achieve full generality of the resulting equation the non-autonomous version of the system is indispensable, see for example [45] for a description of this feature in the case of discrete Painlevé equations. There are various mechanisms of deautonomization, for example using the singularity confinement [44] or an appropriate reduction condition starting
from a non-autonomous version of the Hirota system [92]. In the geometric approach to
discrete systems there are known cases where non-autonomous factors result from reductions
of autonomous integrable systems of higher dimensionality, see for example [35, 26]. In
the literature there are known examples of non-commutative or quantum Painlevé equations
[69, 82, 7, 46, 70]. It is desirable to apply a similar reduction procedure starting from non-
commutative or quantum non-autonomous and/or non-isospectral integrable systems.

The construction of the paper is as follows. We concentrate on the non-commutative
lattice-modified GD systems and start in section 2 by introducing such a system. We present
its linear problem and also show directly the multidimensional consistency of the system.
Moreover, in making a commutative reduction we discover that arbitrary functions of single
arguments show up here naturally. In the simplest case we recover the non-autonomous
lattice-modified GD system and start in section 2 by introducing such a system. We present
its linear problem and also show directly the multidimensional consistency of the system.
Moreover, in making a commutative reduction we discover that arbitrary functions of single
arguments show up here naturally. In the simplest case we recover the non-autonomous
lattice-modified Boussinesq equation. Then in section 3 we recall the relevant facts from the geometric
theory of the non-commutative Hirota system in order to present the lattice non-commutative-
modified Boussinesq equation. We also make the connection
to a non-commutative version of the KP hierarchy and show an auxiliary result on allowed
modified GD system as a reduction of the Desargues maps. We also make the connection
to a non-commutative version of the KP hierarchy and show an auxiliary result on allowed
gauge transformations of the linear problem for the non-commutative Hirota system. Such
considerations turn out to be useful in section 4 where we consider the periodic reduction
of Desargues maps and of the non-commutative KP hierarchy. In doing that we derive a
non-isospectral generalization of the non-commutative lattice-modified GD system. We show
its integrability by presenting the corresponding Lax system and by proving directly its
multidimensional consistency. We also discuss the commutative reduction of the system and
derive non-isospectral non-autonomous versions of the lattice-modified KdV and Boussinesq
systems. The concluding section discusses some related points of current and future research.

2. Lattice-modified Gel’fand–Dikii systems

For a function $f$ defined on an $N$-dimensional integer lattice $\mathbb{Z}^N$ we denote by $f_{(i)}$ its
translation in the variable $n_i \mapsto n_i + 1$, $i = 1, \ldots, N$, i.e. $f_{(i)}(n_1, \ldots, n_i, \ldots, n_N) = f(n_1, \ldots, n_i + 1, \ldots, n_N)$. Consider the following system of partial difference
equations

$$
\begin{align*}
(r_k^{-1} - r_k^{-1})r_{k(i)} &= r_k^{-1}(r_{k+1(i)} - r_{k+1(i)}), & k = 1, \ldots, K - 1, \\
(r_k^{-1} - r_k^{-1})r_{k(i)} &= r_k^{-1}(r_{i(i)} - r_{i(i)}), & i \neq j,
\end{align*}
$$

(2.1)

where $r_k : \mathbb{Z}^N \to \mathbb{D}$, $k = 1, \ldots, K$ are unknown functions from $N \geq 2$ dimensional integer
lattice $\mathbb{Z}^N \ni (n_1, \ldots, n_N) = n$ taking values in a division ring (skew field) $\mathbb{D}$. To contract
notation, in this section we consider the index $k$ modulo $K$ (starting from $k = 1$).

Equations (2.1) provide compatibility of the linear system

$$
(\psi_1, \ldots, \psi_K)_{(i)} = (\psi_1, \ldots, \psi_K),
$$

(2.2)

$$
\begin{pmatrix}
1 & 0 & \cdots & 0 & \lambda r_1^{-1}r_{K(i)}^{-1} \\
\lambda r_2^{-1}r_{1(i)}^{-1} & 1 & 0 & \cdots & 0 \\
0 & \lambda r_3^{-1}r_{2(i)} & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & 1 & 0 \\
0 & 0 & \cdots & \lambda r_K^{-1}r_{K-1(i)} & 1
\end{pmatrix},
$$

where $\lambda$ is a central (spectral) parameter. We remark that due to their geometric meaning (see
section 3) we can treat $\psi_k$ as column vectors.
2.1. Relation to the lattice-modified Korteweg–de Vries and Boussinesq equations

The linear problem (2.2) in the simplest case \( K = 2 \) and the corresponding nonlinear system (in the commutative case) were studied in [47] as the double lattice-modified KdV system. In this section we show how to approach to this system from a general perspective. When the division ring \( \mathbb{D} \) is commutative then functions of single variables naturally show up, which will play the role of non-autonomous factors.

**Lemma 2.1.** Given the solution \( r_k, k = 1, \ldots, K \) of the GD system (2.1), if the division ring \( \mathbb{D} \) is commutative then the function \( R = r_1 r_2 \ldots r_k \) splits into the product of functions of single variables \( n_i, i = 1, \ldots, N \).

**Proof.** Multiply all the \( K \) equations and use the commutativity assumption to obtain that in the generic case (i.e. when the differences \( r_k(i) - r_k(j) \) do not vanish for \( i \neq j \))

\[
R_{ij} R = R_{i} R_{j}, \quad i \neq j.
\]

which implies the statement. \( \square \)

Let us consider in more detail the simplest case \( K = 2 \). Denote by \( G(n) = G_1(n_1) \ldots G_N(n_N) \) the product of functions of single variables such that the function \( R \) in lemma 2.1 reads \( R = G^2 \) (we assume that square roots exist—we take such a form for convenience to match known results and to make the resulting equations more symmetric). Moreover, define the functions \( F_i \) of a single variable \( n_i \) as 'discrete logarithmic derivatives' of \( G_i \), i.e.

\[
G_i(n_i + 1) = F_i(n_i) G_i(n_i), \quad i = 1, \ldots, N.
\]

We parametrize the unknown functions \( r_1 \) and \( r_2 \) in terms of a single function \( x \) as follows

\[
r_1 = x G, \quad r_2 = \frac{G}{x}.
\]

Then the system (2.1) takes the form of the well-known non-autonomous lattice-modified KdV system \([49, 50, 74, 81]\)

\[
x_{ij} = x \frac{x_{ij} F_j - x_{ij} F_i}{x_{ij} F_j - x_{ij} F_i}, \quad i \neq j.
\]

In such a form the functions \( F_i \) are arbitrary and depend on the model under consideration. Therefore the \( K = 2 \) field version of equations (2.1) can be called the non-commutative (and in a sense also non-autonomous) lattice-modified KdV system. Actually, for \( N > 2 \) it is fair to call it the non-commutative lattice-modified KdV hierarchy; see also section 2.2.

**Remark.** In [14] a different (one-component and with central non-autonomous factors/parameters) non-commutative version of the lattice-modified KdV system (2.5) was studied.

Consider next the case \( K = 3 \), still assuming commutativity of \( \mathbb{D} \) and this time setting \( R = G^3 \) keeping the definition of \( F_i \) unchanged. Then we parametrize the unknown functions \( r_1, r_2 \) and \( r_3 \) in terms of two functions \( x \) and \( y \) as follows

\[
r_1 = x G, \quad r_2 = \frac{y G}{x}, \quad r_3 = \frac{G}{y},
\]

which leads to a new system (the non-autonomous version of that studied in [71, 4])

\[
x_{ij} = x \frac{y_{ij} x_{ij} F_j - y_{ij} x_{ij} F_i}{x_{ij} F_j - x_{ij} F_i}, \quad y_{ij} = x \frac{y_{ij} F_j - y_{ij} F_i}{x_{ij} F_j - x_{ij} F_i}, \quad i \neq j.
\]
which could be rewritten as the non-autonomous lattice-modified Boussinesq equation

\[
\left( \frac{y_{ij} F_j - y_{ij} F_i}{y_{ij}} \right)_{(j)} - y \left( \frac{F_j}{y_{ij}} - \frac{F_i}{y_{ij}} \right)_{(j)} = \left( \frac{y_{ij} y_{ij} F_j^2 - y_{ij} F_i^2}{y} \frac{y_{ij} F_j - y_{ij} F_i}{y} \right)_{(j)}, \quad i \neq j,
\]

which is a direct generalization (the functions of single variables replace parameters) of the equation considered in [71].

A similar procedure can be applied, in principle, for arbitrary \( K \). It is convenient to start from the function \( R \) of the form \( R = G^K \) (we assume the existence of roots of appropriate orders, eventually we can consider the field extensions) and parametrize the functions \( r_k \) as follows

\[
r_1 = \tau_1 G, \quad r_2 = \frac{\tau_2}{\tau_1} G, \quad \ldots, \quad r_{k-1} = \frac{\tau_{k-1}}{\tau_{k-2}} G, \quad r_k = \frac{G}{\tau_{k-1}}.
\]

If we put \( \tau_0 = 1 \) and \( \tau_{k+K} = \tau_k \) then in the autonomous case we recover the approach to the lattice-modified GD systems described in [4] as reductions of the Hirota equation [51]. Our case can be recovered from the non-autonomous version of the Hirota equation [92] for an appropriate choice of the non-autonomous factors; see section 4. We remark that the Lax matrices (2.2) differ from those considered in [4]. Our linear system is chosen to match the Lax pair for the double lattice-modified KdV system [47]. There exists a simple transition between the two Lax systems which will be given in section 3.

2.2. Multidimensional consistency of the system

Equations (2.1) form closed systems for \( N = 2 \) of independent variables. However writing it for arbitrary \( N \geq 2 \) we implicitly considered it as multidimensionally consistent, which we are going to demonstrate. In section 3 we show that equations (2.1) can be obtained as a periodic reduction of the lattice non-commutative KP system, which is itself multidimensionally consistent [29]. From that point of view the following result can be considered as the proof of integrability of the periodic reduction.

**Theorem 2.2.** The non-commutative lattice-modified GD system (2.1) is three-dimensionally consistent.

**Proof.** To show that the two expressions below are equal (below we assume that the indices \( i, j, l \) are distinct)

\[
[r_{kl}^1]_{(j)} = (r_{klj}^{-1} - r_{klij}^{-1})^{-1} r_{k+1lj}^{-1} (r_{k+1lj} - r_{k+1lj}), \quad (2.8)
\]

\[
[r_{kl}^1]_{(j)} = (r_{klj}^{-1} - r_{klij}^{-1})^{-1} r_{k+1lj}^{-1} (r_{k+1lj} - r_{k+1lj}), \quad (2.9)
\]

it is convenient to note first the following identity

\[
(r_{klj}^{-1} - r_{klij}^{-1}) r_{klj}^{-1} (r_{klj}^{-1} - r_{klij}^{-1}) = (r_{klij}^{-1} - r_{klij}^{-1}) r_{klij}^{-1} (r_{klij}^{-1} - r_{klij}^{-1}). \quad (2.10)
\]

which can be verified directly by simple algebraic manipulation taking into account equations (2.1). Notice that generically the identity (2.10) is non-trivial meaning that its sides do not vanish. Then we consider the expression \( E = LR_1 - R R_2 \), where \( L/R \) denotes the
left-/right-hand side of identity (2.10) and \( R_1 / R_2 \) is the right-hand side of equation (2.8)/(2.9).

We show below the main steps of the calculation

\[
E = r^{-1}_{k+1} \left[ (r_{k+1(i)} - r_{k+1(j)}) r^{-1}_{k+1(j)} (r_{k+1(j)} - r_{k+1(i)}) \right]
- \left[ (r_{k+1(i)} - r_{k+1(j)}) r^{-1}_{k+1(j)} (r_{k+1(d)} - r_{k+1(j)}) \right]
- r^{-1}_{k+2} (r_{k+2(i)} - r_{k+2(j)}) = 0,
\]

which concludes the proof. We only remark that periodicity in the index \( k \) was used in the proof but it was not as crucial as in the proof of lemma 2.1.

\( \Box \)

3. Desargues maps, the non-commutative Hirota system and the lattice KP hierarchy

In this section we recapitulate first the main elements of the geometric approach to the non-commutative Hirota system [77] via so-called Desargues maps [29]; see also [86, 55] to compare with earlier related works. We also derive other properties of the equations which will be useful in performing periodic reduction to the (non-isospectral and non-commutative) lattice-modified GD system.

3.1. Desargues maps and the Hirota system

Consider the Desargues maps \( \Phi : \mathbb{Z}^N \to \mathbb{P}^M(\mathbb{D}) \), which are characterized by the condition that for arbitrary \( \hat{n} \in \mathbb{Z}^N \) the points \( \Phi(\hat{n}), \Phi(i)(\hat{n}) \) and \( \Phi(j)(\hat{n}) \), for \( i \neq j \), are collinear (see figure 1). In the homogeneous coordinates \( \Phi : \mathbb{Z}^N \to \mathbb{P}^{M+1} \) the defining condition of Desargues maps can be described in terms of a linear relation between \( \Phi \), \( \Phi(i) \) and \( \Phi(j) \). As was shown in [29] there exists a particular normalization of the homogeneous coordinates (gauge) in which the linear relation takes the form [22, 77] (we consider right vector spaces over division rings)

\[
\Phi(i) - \Phi(j) = \Phi U_{ij}, \quad i \neq j \leq N \tag{3.1}
\]

and the functions \( U_{ij} : \mathbb{Z}^N \to \mathbb{D} \) satisfy to the non-commutative Hirota (or the non-Abelian Hirota–Miwa [77]) system:

\[
U_{ij} + U_{ji} + U_{li} = 0, \quad U_{ij} + U_{ji} = 0, \tag{3.2}
\]

\[
U_{ij} U_{l(ij)} = U_{li} U_{lj(i)}, \tag{3.3}
\]
for distinct indices $i, j, l$. Equation (3.3) allows us to introduce the potentials $\rho_i : \mathbb{Z}^N \to \mathbb{D}^\times$ such that

$$U_{ij} = \rho_i^{-1} \rho_{i(j)}.$$  

(3.4)

One may ask how big the class of gauge functions $F : \mathbb{Z}^N \to \mathbb{D}^\times$ is, such that after the transformation $\phi \to \hat{\phi} = \phi F^{-1}$ the form (3.1) of the linear system remains unchanged. The following results can be demonstrated by direct calculation.

**Proposition 3.1.** The gauge functions of the allowed class are characterized by the condition $F_{(i)} = F_{(j)}$ for all pairs of indices, which means that $F$ is a function of the sum $n_k = n_1 + n_2 + \cdots + n_N$ of the discrete variables. Denote $F_{(\tilde{n})}(n_{\tilde{n}}) = F(n_{\tilde{n}} + 1)$ then the transformed vector-function $\hat{\phi}$ satisfies the equation

$$\hat{\phi}_{(i)} - \hat{\phi}_{(j)} = \hat{\phi} \hat{U}_{ij}, \quad i \neq j \leq \tilde{N}, \quad \text{with} \quad \hat{U}_{ij} = F \hat{U}_{ij} F^{-1}.$$  

(3.5)

**Corollary 3.2.** The resulting transformation $U_{ij} \to \hat{U}_{ij}$ provides a symmetry of the non-commutative Hirota system (3.2)–(3.3). The corresponding transformation of the potentials reads $\hat{\rho}_i = \rho_i F^{-1}$.

We remark that when $\mathbb{D}$ is commutative then the simpler part of the algebraic relations (3.2) implies that the functions $\rho_i$ can be parametrized in terms of a single potential $\tau$ (the tau-function)

$$\rho_i = (-1)^{\sum_{p \neq i} n_p} \frac{\tau_{(ij)}}{\tau}.$$  

(3.6)

Then the remaining equations (3.2) reduce to the celebrated Hirota system [51]

$$\tau_{(i)} \tau_{(j)} - \tau_{(j)} \tau_{(i)} + \tau_{(i)} \tau_{(ij)} = 0, \quad 1 \leq i < j < l \leq \tilde{N}.$$  

(3.7)

**Corollary 3.3.** With still $\mathbb{D}$ commutative, denote by $M$ the ‘discrete logarithmic integral’ of the function $F$, i.e. $M_{(\tilde{n})} = F \mathcal{M}$. Then during transformation (3.5) the $\tau$-function changes to $\tilde{\tau} = \tau / M$, which provides symmetry of the Hirota system (3.7).

In section 4 we will need the following non-autonomous version [92] of the Hirota system (3.7)

$$(A_j - A_l) \tau_{(i)} \tau_{(j)} + (A_l - A_i) \tau_{(j)} \tau_{(l)} + (A_i - A_j) \tau_{(i)} \tau_{(ij)} = 0, \quad 1 \leq i < j < l \leq \tilde{N},$$  

(3.8)

where $A_i$ is an arbitrary function of the variable $n_i$, $i = 1, \ldots, \tilde{N}$. As was discussed in [92] both equations (3.7) and (3.8) are equivalent when considered prior to imposing reductions.

**Remark.** We remark that the ‘Hirota equation’ studied in [39, 14] is the (autonomous version of the) lattice-modified KdV equation (2.5), see also [49, 50]. Equation (3.7) in the basic case of $\tilde{N} = 3$ discrete variables was originally described as the ‘discrete analogue of a generalized Toda equation’. It is also called the ‘Hirota–Miwa’ system after Miwa showed [68] its fundamental role in the whole KP hierarchy.

### 3.2. Non-commutative lattice Kadomtsev–Petviashvili systems

Let $\tilde{N} = N + 1$ and we distinguish the last variable $k = n_{N+1}$. To match the notation of [53] we denote also $n = (n_1, \ldots, n_N)$ and

$$\Phi(n, k) = \phi_k(n), \quad U_{N+1,i}(n, k) = u_{i,k}(n), \quad \rho_{N+1}(n, k) = r_k(n),$$  

$$\tau(n, k) = \tau_k(n), \quad F(n, k) = f_k(n).$$  


which allows us to rewrite a part (that with the distinguished variable) of the linear problem (3.1)
in the form

$$\phi_{k+1} - \phi_{k(i)} = \phi_k u_{i,k}, \quad i = 1, \ldots, N. \tag{3.9}$$

The compatibility of the above linear system reads

$$u_{i,k} u_{k(j)} = u_{i,k} u_{j,k(i)}, \quad i \neq j, \tag{3.10}$$

$$u_{i,k(j)} + u_{j,k+1} = u_{j,k(i)} + u_{i,k+1}. \tag{3.11}$$

Equations (3.10) allow (what we knew already) to define potentials $r_k$ such that $u_{i,k} = r_k^{-1} r_{k(i)}$, while equations (3.11) give the system

$$(r_k^{-1} - r_{k(i)}^{-1}) r_{k(i)} = r_{k+1}^{-1} (r_{k+1(i)} - r_{k+1(j)}), \quad i \neq j. \tag{3.12}$$

Following the terminology of [53] we can call the system (3.10)–(3.11) (or equations (3.12))
the non-commutative lattice KP hierarchy. In fact, from (3.9) and (3.10)–(3.11) we can recover
all other ingredients of the non-commutative Hirota system

$$U_{i,N+1}(n,k) = -u_{i,k}(n) = -r_k(n)^{-1} r_{k(i)}(n),$$

$$U_{i,j}(n,k) = -U_{j,N+1}(n,k) - U_{N+1,i}(n,k) = u_{j,k}(n) - u_{i,k}(n) = r_k(n)^{-1} (r_{k(j)}(n) - r_{k(i)}(n)).$$

One can check that the remaining equations of the system (3.2)–(3.3) are either trivially satisfied or are equivalent to (3.10)–(3.11) (or (3.12) without the periodicity assumption). We remark that the identity (2.10), which turned out to be helpful in the proof of theorem 2.2, was suggested by equation $U_{i,j} U_{i(j)} = U_{i,j} U_{i(j)}$, Notice that the proof of theorem 2.2 goes over to
the ‘infinite-component’ case (3.12).

Let us present the following consequence of proposition 3.1.

**Corollary 3.4.** A system of gauge functions $f_k : \mathbb{Z}^N \rightarrow \mathbb{D}^\times$, $k \in \mathbb{Z},$ leaving the form of
the linear system (3.9) unchanged is characterized by the condition $f_{k(i)} = f_{k+1}$. Then the
transformation

$$\phi_k \rightarrow \Phi_k = \phi_k f_k^{-1}, \quad u_{i,k} \rightarrow \hat{u}_{i,k} = f_k u_{i,k} f_{k+1}^{-1}, \quad r_k \rightarrow \hat{r}_k = r_k f_k^{-1}, \tag{3.13}$$

provides symmetry of equations (3.10)–(3.11) and of the system (3.12), correspondingly.

4. Periodic reductions of Desargues maps

4.1. Non-commutative lattice-modified Gel’fand–Dikii systems

At this point we can impose the periodic reduction $\Phi(n, k + K) = \Phi(n, K)$ on the level of
Desargues maps and study its implications on the level of the functions $\phi_k$ and $u_{i,k}.$

**Remark.** Without entering into details we mention that interpretation [29] of Desargues maps
in terms of quadrilateral lattices [32] and their Laplace transformations [23, 36] provides
a geometric indication of the integrability of the periodic reduction. This follows from
the observation that the Laplace transforms of a periodic quadrilateral lattice preserve the
periodicity condition.

Then, by definition of the homogeneous coordinates, there exist functions (the monodromy
factors) $\mu_k : \mathbb{Z}^N \rightarrow \mathbb{D}^\times$ such that $\phi_k^{+K} = \phi_k \mu_k.$

**Proposition 4.1.** The monodromy factors satisfy the condition $\mu_{k+1} = \mu_{k(i)}, i = 1, \ldots, N.$

**Proof.** The monodromy factors do not change the structure of the linear problem (3.9), therefore
the conclusion follows from corollary 3.4. ∎
Corollary 4.2. The corresponding transformation of the potentials $u_{i,k}$ and $r_k$ is given by

$$u_{i,k+K} = \mu_1^{-1} u_{i,k} \mu_{k(i)}, \quad r_{k+K} = r_k \mu_k. \quad (4.1)$$

The periodicity condition on the geometric level then implies the following linear system

$$(\phi_1, \phi_2, \ldots, \phi_K)_{i_1} = \begin{pmatrix} -u_{i,1} & 0 & \cdots & 0 & \mu_1 \\ 1 & -u_{i,2} & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 1 & -u_{i,K} \end{pmatrix}, \quad (4.2)$$

where $\mu_1$ is a function of the variable $n_\sigma = n_1 + \cdots + n_N$. The corresponding system of nonlinear equations takes the form

$$(r_{k+1} - r_k)_{k(i)} = r_{k+1}^{-1} (r_{k+1(i)} - r_{k+1(j)}), \quad k = 1, \ldots, K-1, \quad (4.3)$$

$$\left( -r_{k+1}^{-1} - r_k^{-1} \right)_{k(j)} = \mu_1^{-1} r_k^{-1} (r_{k(i)} - r_{k(j)}) \mu_{1(\sigma)} \quad i \neq j.$$ 

When $\mu_1$ is a constant from the center of $D$ then the nonlinear system (4.3) takes the form (2.1). In such a case we may introduce for convenience a new central parameter $\lambda$ by $\mu_1 = (-\lambda)^K$ and define

$$\psi_k = (-\lambda)^{-k} (-1)^{n_k} \phi_k r_k^{-1}, \quad k = 1, \ldots, K, \quad (4.4)$$

then the linear problem (4.2) is replaced by (2.2).

4.2. Non-isospectral lattice-modified Gel’fand–Dikii systems

The system (4.3) can be considered as a ‘non-isospectral’ version of the non-commutative lattice-modified GD hierarchy (2.1). In this section we concentrate on the properties of the system (4.3) and of its commutative reduction.

Proposition 4.3. The non-commutative non-isospectral lattice-modified GD system is three-dimensionally consistent.

Proof. In view of the proof of theorem 2.2 it is enough to examine the consistency for $k = K - 1, K$ checking before that identity (2.10) holds for $k = K$. \Box

As we have already mentioned, the multicomponent reductions of the non-commutative Hirota system we consider are meaningful in the case of $N = 2$ independent variables. Other independent variables in the countable number can be considered as symmetries of the original equation. The same applies to the non-isospectral monodromy factor where the sum of the remaining variables gives an additive parameter within the argument of the $\mu_1$ function.

Let us examine the non-isospectral analogue of lemma 2.1.

Lemma 4.4. Given solution $r_k$, $k = 1, \ldots, K$ of the non-isospectral GD system (4.3), if the division ring $\mathbb{D}$ is commutative then the function $R = r_1 r_2 \ldots r_K$, splits into the product of functions of single variables $n_i$, $i = 1, \ldots, N$ and a function of the sum of independent variables.

Proof. By multiplying all the $K$ equations of (4.3) and using the commutativity assumption we obtain that in the generic case

$$R_{(i)(j)} = R_{(i)} R_{(j)} \frac{\mu_{1(\sigma)}}{\mu_1}, \quad i \neq j. \quad (4.5)$$
Introduce the function $m_1$ of the variable $n_\sigma$ as the ‘discrete logarithmic integral’ of the monodromy factor: $m_{1(i\sigma)} = \mu_1 m_1$. Then the function $\tilde{R} = Rm_1^{-1}$ satisfies equation (2.3), which gives the rest of the factorization.

To find the integrable non-isospectral modification of the non-autonomous lattice-modified KdV system (2.5) we repeat the considerations of section 2.1 for $K = 2$ with

$$r_1 = xG, \quad r_2 = \frac{Gm_1}{x}.$$  

Then equations (4.3) reduce to

$$x_{(ij)} = x\mu_1 \frac{x_{(i)}F_j - x_{(j)}F_i}{x_{(i)}F_j - x_{(j)}F_i}, \quad i \neq j. \quad (4.6)$$

In the case $K = 3$ of the non-isospectral lattice-modified Boussinesq equation we parametrize the unknown functions $r_1, r_2$ and $r_3$ in terms of two functions $x$ and $y$ as follows

$$r_1 = xG, \quad r_2 = \frac{yG}{x}, \quad r_3 = \frac{Gm_1}{y},$$

which leads to the system

$$x_{(ij)} = \frac{x}{y} \frac{y_{(i)}x_{(i)}F_j - y_{(j)}x_{(j)}F_i}{x_{(i)}F_j - x_{(j)}F_i}, \quad y_{(ij)} = \mu_1 x \frac{y_{(i)}F_j - y_{(j)}F_i}{x_{(i)}F_j - x_{(j)}F_i}, \quad i \neq j. \quad (4.7)$$

It can be rewritten as a scalar equation

$$\left( \frac{\mu_1}{y_{(ij)}} (y_{(i)}F_j - y_{(j)}F_i) \right)_{(ij)} - \mu_1 y \left( \frac{F_j}{y_{(i)}} - \frac{F_i}{y_{(j)}} \right) = \left( \frac{y_{(i)}y_{(j)}F_j^2 - y_{(j)}F_i^2}{\mu_1 y} y_{(j)}F_j - y_{(i)}F_i \right)_{(ij)}, \quad i \neq j. \quad (4.8)$$

We finally remark, that in the commutative case the non-autonomous Hirota system (3.8) with the distinguished last variable and the specification

$$A_i = F_i + 1, \quad 1 \leq i \leq N, \quad A_N + 1 = 1,$$

gives the $\tau$-function formulation of the (non-autonomous and commutative) KP hierarchy

$$F_i\tau_{k(i)i} - F_i\tau_{k(i)+1} + (F_i - F_j)\tau_{k+1(i)} = 0, \quad i \neq j. \quad (4.9)$$

Its consequence

$$\left( \frac{\tau_k}{\tau_{k+1}} \right)_{(j)} F_i - \left( \frac{\tau_k}{\tau_{k+1}} \right)_{(i)} F_j \left( \frac{\tau_{k+1}}{\tau_{k+2}} \right)_{(i)} = \frac{\tau_{k+1}}{\tau_{k+2}} \left( \frac{\tau_{k+2}}{\tau_{k+1}} \right)_{(i)} F_i - \left( \frac{\tau_{k+2}}{\tau_{k+1}} \right)_{(j)} F_j, \quad (4.10)$$

after identification

$$r_k = \frac{\tau_{k+1}}{\tau_k} G, \quad G_{(i)} = F_k G,$$

gives equations (3.12). By imposing the periodicity condition

$$\tau_{k+K} = \tau_k m_k, \quad m_{k+1} = m_{k(i)} = \mu_k m_k, \quad (4.11)$$

we obtain (the commutative version of) equations (4.3), while trivial monodromy gives equations (2.1).
5. Conclusions

In this article we introduced a non-commutative and integrable (in the sense of multidimensional consistency) version of the lattice-modified Gel’fand–Dikii (GD) equations as periodic reductions of the non-commutative Hirota system. In the commutative specification our approach gives rise to the presence of single argument functions in the lattice GD equations. The simplest case of period two leads to the well-known non-autonomous lattice-modified Korteweg–de Vries equation, but the period three case gives a new non-autonomous version of the lattice-modified Boussinesq equation. We mention that such a type of deautonomization, where parameters present in the integrable equation are replaced by functions of single variables, can be encountered in previous works [3] where it was related to the locality of the multidimensional consistency principle. We presented yet another (non-isospectral) generalization of the lattice GD equations which involves an additional function of a single variable. The origin of that generalization is related to the geometric meaning of the Hirota system in terms of the Desargues maps. A more standard way of deriving the (commutative but non-isospectral and non-autonomous) lattice GD equations from the τ-function of the non-autonomous Hirota system has also been presented.

In recent works [34, 31] we investigated the transition from Desargues maps and the non-commutative Hirota system to the corresponding quantum case, which allowed us to rederive the algebraic properties of the quantum plane. It would be interesting to study from that point of view the non-commutative lattice GD equations and to find a corresponding reduction of the relevant algebraic structures. The present paper is a part of a general program of deriving integrable systems from Desargues maps (or from the Hirota system). In particular it would be desirable to exploit the free functions of single arguments in the non-isospectral and non-autonomous lattice-modified GD equations in order to obtain in this way [45, 53, 79] distinguished systems such as the $Q_4$ equation [2] or more advanced lattice Painlevé equations [83].

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