Extraordinary transition in the two-dimensional $O(n)$ model

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Abstract

The extraordinary transition which occurs in the two-dimensional $O(n)$ model for $n < 1$ at sufficiently enhanced surface couplings is studied by conformal perturbation theory about infinite coupling and by finite-size scaling of the spectrum of the transfer matrix of a simple lattice model. Unlike the case of $n \geq 1$ in higher dimensions, the surface critical behaviour differs from that occurring when fixed boundary conditions are imposed. In fact, all the surface scaling dimensions are equal to those already found for the ordinary transition, with, however, an interesting reshuffling of the corresponding eigenvalues between different sectors of the transfer matrix.
1 Introduction

The behaviour of linear polymers interacting with a surface has been studied extensively [1, 2, 3]. In the dilute limit they are modelled on the lattice by self-avoiding walks, which are themselves related to the $n \to 0$ limit of an $O(n)$ spin model. In two dimensions, in particular, it has been possible, using methods of conformal field theory and integrable models, to obtain much exact information concerning the critical behaviour.

An explicit example is provided by the loop model on the honeycomb lattice, defined as follows. The partition function is a sum over all configurations of closed oriented loops on the lattice. A given site of coordination number 3 is therefore unoccupied (with weight $x$) or has just one oriented bond entering and leaving the site. There are 6 such configurations for a site, each occurring with weight 1. The parameter $x$ is thus the fugacity for empty sites, which controls the overall monomer density in the polymer language. There is also a fugacity $y$ for empty sites on the boundary, which have coordination number 2. In addition, each oriented closed loop carries a factor $n/2$ (so that on summing over both orientations the factor is $n$). This nonlocal weight is equivalent to inserting local weights $e^{\pm i\chi}$ at each occupied site, with the sign chosen according to whether the walk turns left or right as it passes through the site, as long as $n = 2\cos 6\chi$. These oriented loop configurations then correspond to the diagrams of the high temperature expansion in powers of $1/x$ of a complex $O(n/2)$ ($\equiv$ real $O(n)$) model, with complex spins $S_a$ ($a = 1, \ldots, n/2$) at each lattice site.

The schematic phase diagram of this model is shown in Fig. 1. Only the details depend on the specific model. There is a bulk critical point at $x = x_c$, where

$$x_c = \sqrt{2 + \sqrt{2 - n}},$$

(1)

which controls the behaviour of long walks of fixed length $N$ in the bulk. This is because the relation between fixed fugacity and fixed length ensembles is through a discrete Laplace transform: for example the generating function $\sum_N c_N x^N$ for the number of $N$-step walks per site is given by the susceptibility of the $O(n)$ model, which diverges as $x \to x_c$ from above. When $y > y_s$, quantities near the surface become singular at $x = x_c$, but with exponents which differ from those in the bulk. This is the line $SO$ of ordinary transitions. It terminates at the point $S$ where $y = y_s$, the special transi-
Figure 1: Schematic phase diagram of the $O(n)$ model for $n < 1$, with bulk and surface vacancy fugacities $x$ and $y$ respectively. $S$ is the special transition, from which emerge the lines of ordinary ($SO$), extraordinary ($SE$), and surface transitions. The particular model considered is integrable at the points $S$, $I$, and $E$. 
When \( y < y_s \) surface quantities actually become singular at a value of \( x = x'_c(y) > x_c \), while the bulk remains nonsingular. This is termed the surface transition, and is in the universality class of the \((d-1)\)-dimensional \(O(n)\) model. However, for \( y < y_s \), surface quantities then undergo a second transition on the line \( x = x_c \), since they are coupled to the bulk critical behaviour. This is the extraordinary transition. The phase diagram in Fig. [1] is believed to be generic to short-range \(O(n)\) models, both in three dimensions, and in two dimensions when \( n < 1 \). (For \( n \geq 1 \) in \( d = 2 \) the surface and ordinary transitions disappear since the surface cannot order independently of the bulk, being one-dimensional.) This picture is confirmed by exact results on the specific model described above, which turns out to be solvable by Bethe ansatz methods at \( x = x_c \) and for two values of \( y \): \( y = x_c \) [1, 7, 8] which yields values for the surface exponents in agreement with those expected for the ordinary transition on the basis of conformal field theory arguments [9, 10]; and \( y = y_s \), where [11, 7, 8]

\[
y_s = \frac{\sqrt{2 - n}}{\sqrt{2 + \sqrt{2 - n}}},
\]

at which the exponents are different. This latter point is therefore identified with the special transition.

Note that the extraordinary transition is irrelevant for the fixed \( N \) ensemble of dilute polymers, since its large \( N \) behaviour is controlled by the largest singularity in \( x \), which in this case is the surface transition. Physically this is because for \( y < y_s \) a finite length polymer will bind to the surface and therefore be governed by \((d-1)\)-dimensional exponents. Conversely, for \( y > y_s \) it will explore the bulk and have an effective repulsion from the surface. The special transition at \( y = y_s \) is therefore important in that it describes the absorption transition for a large but finite length polymer at the boundary.

Nevertheless, within the fixed fugacity ensemble, which is certainly realisable at least in simulations and transfer matrix calculations, the extraordinary transition is accessible, and one may enquire as to its universal properties. For conventional spin models, including the \(O(n)\) model with \( n \geq 1 \) in \( d > 2 \), it may be argued that the extraordinary transition is identical to that which occurs in the presence of a surface ordering field. This is because for temperatures below that of the surface transition the surface is expected to be ordered, which means that the \(O(n)\) symmetry is spontaneously broken.
However, once it is broken, it should not matter whether this breaking was the result of an explicit surface symmetry-breaking field. For Ising-like systems, there is ample evidence that these two problems are indeed in the same universality class. The limit of infinite surface ordering field is equivalent to the case of fixed boundary conditions.

An analogous line of argument has led the authors of Refs. [12, 13] to analyse the extraordinary transition of the $O(n)$ model in the limit $n \to 0$ in terms of the same model with fixed boundary conditions. However, this is not correct, because in the absence of explicit symmetry breaking fields the $O(n)$ model cannot exhibit surface order for $n < 1$, even when $x < x_c$. For, if the spins near the boundary were to be aligned in some fixed direction, walks in the vicinity of the boundary would be counted with weight 1, rather than $n$. The situation is very analogous to the case when $x < x_c$ in the bulk: in that case, the $O(n)$ spins do not order and the symmetry is not broken. Rather this corresponds to the so-called dense phase, when a single polymer occupies a finite fraction of the sites in a finite volume. Similarly, $x < x'_c$ corresponds to the ‘dense’ phase on the surface (which, being one-dimensional, is rather trivial.) There is, nevertheless, an extraordinary transition in this dense phase when the bulk goes critical.

The critical properties of this transition are in fact identical to those of the ordinary transition. To see this, consider the special point $y = 0$ at the end of the line. Then there are no vacant sites on the boundary, and if there is just one polymer in the system, it will be rigidly bound to the surface. However this will mean that other polymers are in fact repelled from the surface. In the explicit model introduced above, it may be seen that sites one lattice constant into the bulk from the boundary will have an effective surface fugacity precisely $y = x_c$, which is the integrable ordinary point! This is illustrated in Fig. 2. For this reason, one may refer to the point $y = 0$ as the ‘teflon’ point: one polymer coats the boundary and makes it effectively repulsive to other polymers.

The properties at $y = 0$ are therefore very simple: they are those of a lattice with one layer of sites removed, at the ordinary point $y = x_c$. This leaves the remainder of the line $0 < y < y_s$ to be discussed, and this forms the bulk of this paper. We shall argue that all the universal properties along this line are the same as those at $y = 0$, by using two sets of arguments. In Sec. 2 we use perturbation theory in $y$, coupled with various arguments from conformal field theory, to show that the variable $y$ is irrelevant in the sense of
the renormalisation group, so that all the universal properties are the same, at least for sufficiently small \( y \). In order to perform perturbation theory about such a critical point it is necessary to impose an infrared cutoff, and this is most easily realised by considering the system in an infinitely long strip of width \( L \). The universal properties are then encoded in the finite-size scaling behaviour of the spectrum of the transfer matrix of this model. This may be studied within conformal perturbation theory about the teflon point. We find indeed that the scaling part of the spectrum is unaffected by the perturbation. However, there are additional degeneracies at \( y = 0 \) compared to those along the line of the ordinary transition. Some of these are resolved by the perturbation, but not in the scaling part of the spectrum, where the degeneracies persist. We show how this comes about and derive the qualitative nature of the splitting.

This analytic work near \( y = 0 \) is then supplemented in Sec. 3 by a study of the numerical diagonalisation of the transfer matrix for small widths but for all values of \( 0 < y < y_s \). This confirms the analytic results but also shows how the eigenstates cross in a complicated manner as \( y \) is decreased. In particular, the low-lying states occur in different sectors for \( y \ll y_s \) and \( y_s - y \ll y_s \). However, as \( L \) is increased all these crossings move towards \( y = y_s \), indicating that in the limit the scaling theory at all points \( y < y_s \) is the same as at \( y = 0 \), and different from that at the special transition \( y = y_s \).
These numerical results are limited by the small values of $L$ we use, and the fact that, at $y = 0$, a layer of sites is effectively removed at each boundary.

2 Perturbation theory about the teflon point

We first consider the spectrum of the transfer matrix at the point $y = 0$ and how it differs from that at the integrable ordinary point $y = x_c$. The transfer matrix is constructed from the ensemble of oriented self-avoiding loops. The number of up arrows minus the number of down arrows crossing any given ‘time’ slice is therefore conserved. This total ‘charge’ $Q$ therefore commutes with the transfer matrix, and the eigenstates of the latter may be organised into sectors of a given $Q$. All the way along the ordinary line $OS$ and at the special point $S$, the largest eigenvalue $\Lambda_0$ of the transfer matrix is nondegenerate and lies in the $Q = 0$ sector and the corresponding quantity $E_0 \equiv -\ln \Lambda_0$ has the finite-size scaling form

$$E_0 = f_b L + 2 f_s - \frac{\pi \zeta c(n)}{24 L} + o(L^{-1}).$$

Here $\zeta = 2/\sqrt{3}$ is a lattice dependent scale factor, inserted since $L$ counts the number of sites, rather than the linear dimension, across the strip. The bulk and surface free energies $f_b$ and $f_s$ are also nonuniversal. In particular $f_s$ depends on the surface fugacity $y$. However the central charge $c$ is universal, with

$$c(n) = 1 - 6(g - 1)^2 / g,$$

where $n = -2 \cos \pi g$. Similarly the excited state eigenvalues $E_m \equiv -\ln \Lambda_m$ scale according to

$$E_m - E_0 = \frac{\pi \zeta x_m}{L} + o(L^{-1}),$$

where $x_m$ is one of the scaling dimensions of the allowed surface operators. The terms in the $E_m$ which scale like $1/L$ may therefore be referred to as the

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1 Actually at the point $S$ the largest eigenvalues in the $Q = 0$ and $Q = 1$ sectors are equal when $n = 1$, with $x_{1}^{(s)} < 0$ for $n < 1$. Thus for $n < 1$ there is a point on the line $OS$ (below point $I$) where the two eigenvalues cross. The central charge is still measured from the largest eigenvalue in the $Q = 0$ sector.
universal, scaling, part of the spectrum. There is always an excited state in the $Q = 0$ sector corresponding to the energy operator of the $O(n)$ model. In the spin language this is the operator $S^*_a S_a$, and it measures the local density of monomers, that is, the probability that a given site is occupied. Along the ordinary line $OS$, it has been argued that this operator has surface scaling dimension $x_e = 2$, and is in fact not an independent primary operator but is proportional to the component $T_{||\perp}$ of the stress tensor. This has been verified by the exact solution at $y = x_c$. However, at the special point $S$, this is not true, and it is found that $x_e^{(s)} = 2/g - 1$. There is another excited state in the $Q = 0$ sector, degenerate with the ground state in the $Q = 2$ sector. The former corresponds to the operator $S^*_a S_b$ (with $a \neq b$) while the latter corresponds to $S^*_a S^*_b$ (or $S_a S_b$) which acts as a source (or a sink) for a pair of oriented walks. These have the same eigenvalue (and hence the same scaling dimension because they are related by an $O(n)$ rotation (the full symmetry group of the problem.) In fact, at $n = 0$ these states are also degenerate with that corresponding to the energy operator, as may be seen by a study of the form of the transfer matrix. In general the ground state in the charge $Q$ sector corresponds to the operator $S_{a_1}^* S_{a_2}^* \ldots S_{a_Q}^*$, which acts as a source for $Q$ oriented walks. It has scaling dimension

$$x_Q = \frac{1}{2} g Q^2 + \frac{1}{2} (g - 1) Q$$

(6)

at the ordinary transition and

$$x_Q^{(s)} = \frac{1}{4} g (Q + 1)^2 - \frac{3}{2} (Q + 1) + \frac{9 - (g - 1)^2}{4g}$$

(7)

at the special point. In general, these states will be degenerate with states in the sectors $Q - 2, Q - 4, \ldots$ corresponding to reversing some of the arrows.

Now let us consider how this picture changes at the teflon point $y = 0$. Then all the boundary states must be occupied, and so there must be a polymer lying along each boundary. However, these may have either orientation. Once these are in place, the remaining problem is that of a strip of width $L - 2$ at the integrable ordinary point. Apart from the boundary polymers, therefore, the spectrum is exactly that described above. However, when these are included we now see that the ground states in the $Q = 0$ and $Q = \pm 2$ sectors are identical. Moreover, that in the $Q = 0$ sector is doubly degenerate. This is illustrated in Fig. The first excited state, with
finite-size scaling behaviour controlled by the scaling dimension $x_e = 2$, now occurs in all the even sectors with $|Q| \leq 4$, with different degeneracies, as shown in Fig. 4.

When $y \neq 0$, the first effect to $O(y)$ is to allow a single boundary site to be vacant. This immediately allows one of the boundary polymers to wander away from the surface at this point, as illustrated in Fig. 5. Since the bulk is critical, it may in fact wander all the way across to the other side and therefore modify the finite-size scaling spectrum. This may be quantified by observing that if we now ignore the two boundary layers, the effect is of a long polymer loop attached at one point to the boundary. Such configurations are generated by inserting the energy operator $S_a^* S_a$ near the boundary. Since this has scaling dimension $x_e = 2$, the $O(y)$ change in the free energy per unit length will therefore scale like $L^{-2}$. The universal scaling part of the spectrum is therefore unaffected, the change in $E_0$ being $O(y/L^2)$. This also indicates that from the renormalisation group point of view $y$ is irrelevant, with eigenvalue $-1$. Thus flows from the vicinity of $y = 0$ go into the teflon fixed point, and all universal quantities like the scaling part of the eigenvalue spectrum should be identical. Note also that the $O(y)$ corrections will be identical in the $|Q| = 2$ and $Q = 0$ sectors, so they remain exactly degenerate to this order. The higher order corrections are more interesting, however. They correspond to the configurations shown in Figs. (6,7). Those in Fig. (3) are diagonal in the basis of unperturbed ground states, and therefore produce shifts which, since the matrix elements...
Figure 4: Configurations giving rise to the first excited state in the $|Q| = 2$ and $Q = 0$ sectors, at $y = 0$. Orientations are not shown: depending on these, this state occurs in all even sectors with $|Q| \leq 4$.

Figure 5: First order corrections to the ground state. Orientations are not shown, but this contribution is diagonal and therefore independent of the sector in which it occurs.
Figure 6: Second order contributions to the ground state. These are diagonal in arrow space and therefore give equal shifts in all sectors.

Figure 7: Second order contributions to the ground state which are responsible for mixing in the $Q = 0$ sector.
are independent of the orientation, are equal in all three sectors. However some of the diagrams in Fig. 7 produce mixing between the two $Q = 0$ states. Because of the extra $e^{\pm i \chi}$ factors introduced in some of these configurations, the mixing matrix is proportional to

$$
\begin{pmatrix}
e^{6i\chi} & 1 \\
1 & e^{-6i\chi}
\end{pmatrix}
$$

with eigenvalues 0 and $2 \cos 6\chi = n$. The two states in the $Q = 0$ sector are therefore split at $O(y^2)$, but one remains degenerate with those in the $|Q| = 2$ sectors. At $n = 0$ this splitting should vanish. The order of magnitude of these second order shifts is expected on the basis of the renormalisation group to be $O(L^{-3})$. However, if they are computed explicitly in terms of integrals over two-point functions in the unperturbed theory, those corresponding to the first diagram in Fig. 8 diverge at short distances and must be regulated by a cut-off of the order of the lattice spacing, giving rise to contributions $O(y^2/L^2)$. Such non-universal corrections to finite-size scaling from higher order effects of an irrelevant operator are standard. However, they do not affect the splitting coming from the diagrams in Fig. 7 where the two insertions are on opposite sides of the strip.

Next we study the excited states in perturbation theory in $y$. These effects are easiest to see in the odd $Q$ sectors. At $y = 0$ the lowest states in the $Q = \pm 1, \pm 3$ sectors are all degenerate, corresponding to the configurations shown in Fig. 8. To $O(y)$, there are two distinct types of configurations which enter, shown in Fig. 9. The first two are diagonal and therefore shift
all states equally, by an amount $O(y/L^2)$. The second pair induce mixing in between the 3 degenerate states in the $|Q| = 1$ sectors, however. As above, the various $e^{\pm i\chi}$ factors must be taken into account, and when this is done the mixing matrix in the $Q = 1$ sector, for example, is proportional to
\[
\begin{pmatrix}
e^{6i\chi} + e^{-6i\chi} & 1 & 1 \\
1 & e^{6i\chi} & 0 \\
1 & 0 & e^{-6i\chi}
\end{pmatrix}
\]
with eigenvalues $(n - 1,0,n+1)$. Therefore, to $O(y/L^2)$, we expect to find a single state in the $Q = 1$ sector remaining degenerate with that in the $|Q| = 3$ sectors, while the remaining two split away. At $n = 0$ this splitting is equidistant.

\section{Numerical evidence}

In order to supplement the conformal perturbation theory results we have numerically diagonalised the double-row transfer matrix of the vertex model. The detailed investigation of degeneracies and splitting of the eigenvalues as a function of the boundary weight $y$ necessitates calculating many eigenvalues of the transfer matrix. We have thus restricted ourselves to calculating the complete eigenspectrum for small widths of up to $L = 8$. This is sufficient to confirm both the order in $y$ and the finite-size dependence of a given splitting. Our numerical results are suggestive that the corrections to the finite-size scaling spectrum away from the teflon point are $O(y/L^2)$ and thus do not effect the scaling dimensions.
Consider first the teflon point, with \( y = 0 \) on each side of the strip. We confirm for finite \( L \) that, apart from the shuffling of eigenvalues, the eigenspectrum is equivalent to that on a strip of width \( L - 2 \) with \( y = x_c \) at the boundaries, i.e. at the integrable ordinary point. The eigenvalues at \( y = x_c \) appear with a higher degeneracy at \( y = 0 \) (there are \( 3^L \) eigenvalues at \( y = 0 \) compared to \( 3^{L-2} \) at \( y = x_c \)). The known exact results at the ordinary point can thus be used to infer exact results at the teflon point. In contrast with the points \( I \) and \( S \), the largest eigenvalue in the \( Q = 0 \) sector is seen to be 2-fold degenerate. It is also degenerate with the largest eigenvalue in the \( Q = 2 \) sector. At \( n = 0 \) the largest eigenvalue is given by \( \Lambda_0 = (2 + \sqrt{2})^{L-2} \) and thus \( c = 0 \). For general \( n \) the central charge is given by (4), as calculated for the integrable ordinary point. The leading thermal excitation in the \( Q = 0 \) sector at \( y = 0 \) corresponds to the thermal excitation at the integrable ordinary point, and thus \( x_e = 2 \), independent of \( n \). More generally the scaling dimensions (4) also appear at \( y = 0 \). In particular, the magnetic scaling dimension is given by \( x_h = x_1 \).

Consider the largest eigenvalue in the \( Q = 0 \) sector along the line \( SO (y \geq y_s) \) at \( n = 0 \). At \( n = 0 \), this corresponds to the state with no arrows, which, for this value of \( n \), is an exact eigenstate. Its eigenvalue is given exactly by \( \Lambda_0 = (2 + \sqrt{2})^{L-2} a^2 \), where \( a = y/\sqrt{2 + \sqrt{2}} \) is the surface free energy contribution. There are no further correction terms in either \( y \) or \( L \). Clearly for finite \( L \), \( \Lambda_0 \to 0 \) as \( y \to 0 \). Thus, for finite \( L \), there is a point at which this eigenvalue crosses the continuation of the 2-fold degenerate ground state for \( 0 < y < y_s \). For increasing \( L \), the successive crossing points \( y_{cr}(L) \) are expected to approach \( y_s \) at a rate determined by the crossover exponent \( y' \) at the special point:

\[
y_s - y_{cr} \sim L^{-y'},
\]

where \( y' = 1 - x_e^{(s)} = \frac{2}{3} \) at \( n = 0 \). A log-log plot of the above estimates, exhibited in Fig. 10 reveals the value \( y' \simeq 0.67 \), in excellent agreement.

The 2-fold degeneracy is seen to be broken away from \( n = 0 \), with one of the eigenvalues remaining degenerate with the largest eigenvalue in the \( Q = 2 \) sector. According to the conformal perturbation theory, there should be a uniform shift \( O(y/L^2) \) while the splitting should be \( O(y^2/L^2) \). Examination of these eigenvalues for small \( y \) indicates that the \( O(y) \) term tends to a constant as \( L \) increases, being a surface energy contribution in \( y \). For
fixed $y$ the finite-size convergence to this value is consistent with $O(1/L^2)$, however the available data is insufficient to unambiguously determine the exponent. Nevertheless this term cancels in the finite-size estimates of the scaling dimensions. The more important term for the scaling spectra is the $O(y^2)$ term governing the splitting of the degeneracy. The splitting is absent at $n = 0$ and clearly seen to be $O(y^2)$ away from $n = 0$. For finite $y$, the splitting term is seen to vanish at least as fast as $O(1/L^2)$.

The splitting of the 3-fold degenerate largest eigenvalue in the $Q = 1$ sector is clearly seen to be $O(y)$. Moreover, the splitting is precisely proportional to the eigenvalues $(n - 1, 0, n + 1)$ derived from the mixing matrix. Denote the three split eigenvalues by $\Lambda_1^{(1)} < \Lambda_1^{(2)} < \Lambda_1^{(3)}$ and define their differences by $a = \Lambda_1^{(1)} - \Lambda_1^{(2)}$ and $b = \Lambda_1^{(2)} - \Lambda_1^{(3)}$. We observe the equal splitting at $n = 0$ with $b = 0$ at $n = 1$. Finite-size estimates of $a/b$ for the particular value $n = 0.445 \ldots (g = \frac{10}{7})$ are shown in Table 3. They are seen to be in good agreement with the expected value $a/b = -(n + 1)/(n - 1) = 2.60388$. The middle eigenvalue $\Lambda_1^{(2)}$ remains degenerate with the largest eigenvalue in the $Q = 3$ sector.

At fixed $y$ the amplitude of the $O(y)$ splitting is seen to decrease with increasing $L$. Consider, for example, $n = 0$ where the splitting is equidistant,
Table 1: Finite-size estimates of the splitting ratio $a/b$ for the largest eigenvalue in the $Q = 1$ sector. The expected value derived from the mixing matrix is $a/b = 2.60388$.

| $L$ | $y = 0.0001$ | $y = 0.001$ | $y = 0.01$ |
|-----|-------------|-------------|------------|
| 3   | 2.60369     | 2.60202     | 2.58003    |
| 4   | 2.60360     | 2.60107     | 2.57390    |
| 5   | 2.60362     | 2.60133     | 2.57734    |
| 6   | 2.60365     | 2.60163     | 2.58059    |
| 7   | 2.60368     | 2.60187     | 2.58320    |
| 8   | 2.60370     | 2.60207     | 2.58529    |

with

$$E_1(y) - E_0(y) = E_1(y = 0) - E_0(y = 0) + A y + \begin{bmatrix} B \\ 0 \\ -B \end{bmatrix} y + o(y^{-1}).$$  (9)

The amplitudes $A$ and $B$ are constant for given $L$. However, they vanish as $O(1/L^2)$, as can be seen in the log-log plot of Fig. 11.

The corresponding finite-size estimates of the magnetic scaling dimension $x_1$ are shown as a function of $1/L$ in Fig. 12 for increasing values of $y$. In each case the observed trend is again consistent with $O(y/L^2)$ convergence in the eigenvalues, i.e. the scaling dimensions being independent of $y$. We have observed similar behaviour in the estimates of the thermal dimension $x_e = 2$.

4 Conclusion

We have studied the extraordinary transition in the two-dimensional $O(n)$ model for $n < 1$ by conformal perturbation theory and by finite-size scaling of the transfer matrix eigenspectrum of a simple model on the honeycomb lattice. We argued that the universal properties along the line $0 < y < y_s$ in the phase diagram shown in Fig. 4 are independent of $y$, being those at
Figure 11: Log-log plot of the amplitudes $A$ (lower points) and $B$ (upper points) appearing in Eq. (9) vs $1/L$ for $L = 4$ to 8. The solid line indicates the expected $1/L^2$ convergence.

Figure 12: Estimates of the magnetic scaling dimension $x_1$ as a function of $1/L$ for $y = 0$ (●), $y = 0.1$ (□) and $y = 0.2$ (△). For each case the expected asymptotic value is $x_1 = 0.625$. 
the teflon point \( y = 0 \). The surface scaling dimensions at \( y = 0 \) are equal to those for the ordinary transition. There are, however, additional eigenspectra degeneracies at \( y = 0 \) compared to those along the line of the ordinary transition. The result for the magnetic scaling dimension at the extraordinary transition differs from the value \( x_h = x_e = 2 \) obtained previously with fixed boundary conditions \[12, 13\]. As we have explained, this is because the \( O(n) \) model does not order in the absence of a symmetry-breaking field when \( n < 1 \).

Although the extraordinary transition we have discussed in this paper is not relevant to the statistics of a single self-avoiding walk, or linear polymer, of fixed finite length, it may be realised within the fixed fugacity ensemble. In this case a single polymer will simply bind to the wall and not exhibit any interesting scaling behaviour. The exponents we have discussed in the body of this paper, and which are observed in the finite-size scaling spectrum, refer to the behaviour of a second polymer in the vicinity of the surface. Thus our results are relevant to the semi-dilute regime rather than the statistics of a single polymer.

It is also possible to consider mixed boundary conditions, when one side of the strip has \( 0 < y < y_s \) and the other is either at the ordinary or the special point. In that case, at the teflon point, the finite-size spectrum will be that of a strip of width \( L - 1 \) with either ordinary exponents or mixed ordinary-special exponents \[17, 8, 18\]. The degeneracy of these states will be different, however, and may be discussed in the manner as in Sec. \[2\] as may be the splittings which should occur away from \( y = 0 \).

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