Topological Anderson insulating phases in the long-range Su-Schrieffer-Heeger model

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In the long-range Su-Schrieffer-Heeger (SSH) model, in which the next nearest-neighbor hopping is considered, there exists a rich topological phase diagram that contains winding numbers $w = 0, 1,$ and 2. In the presence of disorder, the change in mean winding number with various disorder strengths is numerically calculated. We find that the disorder drives phase transitions: $w = 0 \rightarrow 1, 0 \rightarrow 2, 1 \rightarrow 2$ and $2 \rightarrow 1$, in which the non-zero winding numbers driven by disorder are called the topological Anderson insulating (TAI) phases. The transition mechanisms in this long-range SSH model are investigated by means of localization length and self-energy. We find that the localization length has a drastic change at the transition point, and the corresponding critical disorder strength is obtained by further using Born approximation to the self-energy. The origin for the $w = 0$ to $w = 1, 2$ transitions is attributed to the energy gap renormalization (first Born approximation), while for the transitions from $w = 1$ to $w = 2$ and vice versa are the strong scattering (self-consistent Born approximation). The result of self-consistent Born approximation suggests that in the thermodynamic limit, the gap closing point is not at the Fermi level for the system with localized edge states. This agrees with the topological protection of edge states, such that the pre-existent edge states are protected from being scattered into bulk. Given the experimental feasibility of the long-range SSH model, the predicted TAI phases and transitions could be observed in experiments.

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I. INTRODUCTION

The phenomena of localization of electronic wave function in random potential, now called Anderson localization, was proposed by P. W. Anderson in his pioneering work in 1958 \cite{1}. Anderson localization gives a quantum mechanical description of electronic states in disordered systems \cite{2, 3}. The scaling theory of localization shows that in low dimensions, all states are localized no matter how weak the disorder is. As a result, in the thermodynamic limit, any low dimensional system is an insulator. In Ref. \cite{4}, the Anderson localization is directly observed in one-dimensional matter waves of rubidium-87 Bose-Einstein condensates with controlled disorder. When disorder meets topology, interesting phenomena arise. It is known that topological boundary modes are robust to disorder. In topological insulators with high Fermi level that both bulk and boundary modes transport, disorder plays a role that suppresses the bulk bands and leaves the boundary modes conduct \cite{5}. Furthermore, in a normal insulator, disorder can drive the transition to a topological insulator. This is called the topological Anderson insulating (TAI) phase \cite{6, 7}.

The TAI phase has been theoretically shown in several studies \cite{6, 10}. Li et al. \cite{8} showed that in a two-dimensional BHZ model, disorder can lead to band inversion and topological phase transition from normal to a nontrivial phase that carries quantized conductance. In the same system, Groth et al. \cite{9} applied Born approximation to estimate the renormalization of gap parameter that leads to inverted bands. They conclude that the normal insulating and TAI phase boundary correspond to the crossing of a band edge. Although the name seems to suggest Anderson localization, the phase boundary actually exhibits a weak disorder transition. Guo et al. \cite{7} also found that disorder transforms a normal insulator to a topological insulator in three dimensions. Similar to the conclusion drawn by Groth et al. \cite{9}, the weak-disorder boundary is the crossing of a band edge. Nonetheless, they found the TAI phase extends to a regime where energy broadening becomes significant and localization is the leading factor. Xu et al. \cite{11} showed there are two kinds of TAI, the gapped and ungapped phases, in two-dimensional BHZ model. In the gapped TAI, only edge states exist inside the energy gap. In the ungapped TAI, the bulk and edge states coexist, while the bulk states are localized by disorder. The latter is the counterpart of the extended TAI in three dimension shown by Guo et al. \cite{7}. Altland et al. \cite{10, 11} studied TAI phase transitions in quasi-one-dimension and found that localization can stabilize the topological phases.

It is only until recently that TAI phase has been observed experimentally \cite{12}. A one-dimensional Su-Schrieffer-Heeger (SSH) model that preserves chiral symmetry was simulated in ultracold atoms. It was shown that the winding number ($w$) transitions from $w = 0$ to $w = 1$ as disorder strength increases. The experimental feasibility of the SSH model \cite{12, 14} makes it suitable for studying the interplay between topology and disor-
order. Therefore, it is worthwhile studying the rich phase diagram and the TAI phases of the SSH model. In this study, to explore the nontrivial phases with high winding number in the presence of disorder, the long-range interaction, which is experimentally applicable [13], is included in the SSH model.

In this article, TAI phases are shown in numerical simulations and the mechanisms are explained with Born approximation and theory of localization. In particular, we show that there are two types of TAI transitions, which are named type I and type II in this article. The type I is the transition from normal insulating to non-trivial insulating phases driven by disorder. Besides the \( w = 0 \) to \( w = 1 \) transition, as proposed in several theoretical works, we found that the transition can go directly from \( w = 0 \) to \( w = 2 \) without crossing \( w = 1 \). The type II is the transition between two nontrivial insulating phases with different winding numbers. Type II includes the transition from \( w = 1 \) to \( w = 2 \) and vice versa. The main difference between the two type of transitions is whether the nontrivial phases are driven from normal or nontrivial insulating phases in the clean limit. In this work, we show that type I transition boundary corresponds to crossing a band edge by weak scattering, while the type II transition is a consequence of strong scattering. Moreover, in all TAI phases, bulk and edge states coexist, while the bulk states are localized in the thermodynamic limit.

In the next section, the model Hamiltonian and the methods for characterizing the TAI phases are presented. The findings of the TAI phase and the two types of TAI phase transitions are given in section III. The calculation of self-energy within the Born approximation is addressed in section IV. After presenting the theoretical results, the role of disorder is discussed. Finally, a conclusion is given in section V.

II. THE LONG-RANGE SSH MODEL

The one-dimensional SSH model with long-range hopping that preserves chiral(sublattice) symmetry [15–18] is considered

\[
H_0 = \sum_{i=0}^{N} J_0 C_{i,a}^{\dagger} C_{i,b} + J_1 C_{i+1,a}^{\dagger} C_{i,b} + J_2 C_{i+1,b}^{\dagger} C_{i,a} + h.c.,
\]

where \( i \) is the lattice site, \( N \) is the length of the model, \( C_{i,a}^{\dagger}, C_{i,a} \) are the creation and annihilation operators on sublattices \( a, b \) on the \( i \)th unit cell. There are two types of nearest neighbor coupling. \( J_0 \) denotes the intracell coupling, while \( J_1 \) denotes the intercell coupling. Fourier transform to the momentum space, the Hamiltonian is

\[
H_0(k) = h_x(k)\sigma_x + h_y(k)\sigma_y,
\]

where

\[
h_x(k) = J_0 + J_1 \cos k + J_2 \cos(2k),
\]

\[
h_y(k) = J_1 \sin k + J_2 \sin(2k),
\]

\( \sigma_{x,y} \) acts on the sublattices \( a,b \) and the lattice constant is taken to be 1. The eigen energy is \( E = \pm \sqrt{(J_0 + J_1 \cos k + J_2 \cos(2k))^2 + (J_1 \sin k + J_2 \sin(2k))^2}. \)

Before directly calculating the topological phase, the phase diagram can be inferred by adiabatic connection, see main text for discussion. The red dashed line is in the vicinity of the phase boundary, where the TAI phases are shown in Fig. 1(a). In (b), the length of chain is \( N_x = 400 \) for winding numbers. Comparing the two figures, the vanishing energy gap coincide with phase boundaries given by winding numbers.

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**FIG. 1.** (color online). (a) The energy gap (b) The winding number as a function of \( J_1,J_2 \). In (a), the winding numbers, annotated by the numbers on the plot, are inferred from adiabatic connection, see main text for discussion. The red dashed line is in the vicinity of the phase boundary, where the TAI phases are shown in Fig. 1(c). In (b), the length of chain is \( N_x = 400 \) for winding numbers. Comparing the two figures, the vanishing energy gap coincide with phase boundaries given by winding numbers.
term. The gap closing conditions are
\[ 1 + \frac{J_1}{J_0} \cos k_o + \frac{J_2}{J_0} \cos(2k_o) = 0, \quad (4) \]
\[ \frac{J_1}{J_0} \sin k_o + \frac{J_2}{J_0} \sin(2k_o) = 0. \quad (5) \]

Eq. (4) is satisfied when \( k_o = 0(\pi) \) and \( J_0 + (\pm)J_1 + J_2 = 0 \), giving rise to the straight phase boundary with slope \( = (+)(-1) \) and intercept \( J_2/J_0 = 1 \), as shown in Fig. 1(a). Nonetheless, Eq. (5) is also satisfied by \( J_1/J_2 = -2 \cos k_o \), which is plugged into Eq. (4) to obtain another condition \( J_2/J_0 = 1 \). As a result, these conditions give rise to the horizontal boundary of \( J_2/J_0 = 1 \) for \( J_1/J_0 = [-2, 2] \), as shown in Fig. 1(a).

The geometrical origin of the topology lies in the sublattice pairing. In nontrivial topological phase, the bonding is formed between opposite sublattices from different lattice sites, i.e., sublattice \( a \) bonds with sublattice \( b \) at other site. This is referred to as the singlet pairing [12]. Quantitatively, this pairing is described by winding number [12, 15]. For \( w = 1 \), the average singlet pairing forms between the nearest neighbor, while for \( w = 2 \), the average singlet pairing forms between the next nearest neighbor. In contrast, for trivial topology, the bonding is formed within the same lattice site.

To examine the effect of disorder on the topological phases, the disordered intracell coupling is taken into account

\[ H_w = \sum_{i=1}^{N} U_0 C^\dagger_{i,a} C_{i,b}, \quad (6) \]

where \( U_{n=0} \) are given by the random number in the range \([-L, U] \) with \( U \) the disorder strengths in the unit of \( J_0 \).

For characterizing the topological phases in disordered system, the winding numbers are computed numerically for the tight-binding Hamiltonian. Here, we adopt the method proposed by [12, 15] that applies for the chiral symmetric systems [12, 17]. By defining \( Q = P_+ - P_- \) and the chiral symmetry operators \( S = S_+ - S_- \), where \( P_\pm \) are the projection operators that project to the positive or negative energy bands, \( S_\pm \) are the projection to sublattice \( a \) or \( b \), the winding number is given by

\[ w = -\frac{1}{N} \text{Tr}_{\text{sublat}} [Q_+ [X, Q_-]], \quad (7) \]

where \( Q_+ = S_-QS_+ \), \( X \) is the position operator and \( \text{Tr}_{\text{sublat}} \) is the trace over the sublattices. The winding number as a function of \( J_1/J_0, J_2/J_0 \) for the clean limit is shown in Fig. 1(b). The results coincide with that from adiabatic connection.

However, the finite size effect in the tight-binding Hamiltonian could smooth out the phase boundaries. Therefore, the localization length, which peaks identifies the topological phase transitions [13], is also computed. To compute the localization length, the iterative Green's function method is adopted [21]. The localization length can be extracted from the Green's function [22, 23]

\[ \frac{2}{\lambda} = -\lim_{n \rightarrow \infty} \frac{1}{n} \ln |G_{1n}|^2, \quad (8) \]

where \( n \) is the total number of site of the SSH model, \( G_{1n} \) is the propagator connecting the first and the last slice of the system. A well-known challenge in this method is the vanishing small eigenvalues due to successive matrix multiplication. To overcome the numerical instability, we apply the method proposed in [22] that normalizes the Green’s function regularly.

### III. TAI PHASES AND TRANSITIONS

We study TAI phase driven by the disordered intracell coupling, as in Eq. (3). The TAI phase transition is investigated near the phase boundaries in the clean limit. Fixing \( J_0 = 1 \) and \( J_2 = J_1 - 0.94 \), the mean winding numbers are plotted as a function of disorder strength and \( J_1/J_0 \), as shown in Fig. 2. The winding number in

![FIG. 2. (color online). The topological phases as a function of disorder strength and \( J_1/J_0 \) near the phase boundary, following the red dashed line in Fig. 1 (a). The black crosses denote the peaks of the localization length. The white stars indicate the crossing of band edge according to SCBA.](image)

| \( J_1/J_0 \) | \( U \) | winding number | \( \beta \) | error |
|---------|--------|----------------|------|------|
| a       | -0.8   | 3.5            | 2    | -1.11| 0.08 |
| b       | -0.8   | 8              | 1    | -0.94| 0.07 |
| c       | 0      | 2.5            | 2    | -1.00| 0.05 |
| d       | 1      | 3              | 1    | -1.11| 0.07 |
| e       | 2.4    | 5              | 1    | -1.03| 0.07 |

**TABLE I.** The fitting parameters and fitting errors of scaling function for five parameters labeled by cyan letters in Fig. 2.
the clean limit is retained up to a critical disorder. As disorder strength increases, winding numbers experience a change to another nonzero integer. In Fig. 2, two types of transitions are observed from this numerical result. Type I is the transition from normal insulating to topological insulating phase driven by disorder. Such transition can be seen for \( J_1/J_0 = [0, 2) \) in Fig. 2. Type II is the transition between the two nontrivial topological phases. It occurs for \( J_1/J_0 = [-1, 0) \) and \([2, 3]\) at weak disorder and \([-1, 0) \) at strong disorder limit in Fig. 2.

To better locate the phase transitions, we compute the localization length, which peaks indicate the topological phase transitions \([15, 24]\). The boundaries of the mean winding number match the peaks of localization length as denoted by the black crosses on Fig. 2. Some examples of the localization length and the corresponding winding numbers are given in Fig. 3.

Fig. 3 (a) is a type I transition for \( J_1 = 0 \). Disorder drives the system directly to \( w = 2 \) because there is no singlet pairing between nearest neighbor for this Hamiltonian. When \( J_1 \) is nonzero, disorder drives the formation of single pairing between the nearest neighbor and the transition goes to \( w = 1 \). This result suggests that weak disorder scattering strengthens topology, but not give rise to it. Fig. 3 (b) and (c) are type II transitions. Disorder drives the transition between different topological phases and eventually to normal insulating phase at extremely strong disorder. The change of the mean winding number is 1, meaning only one pair of edge states is removed or formed at a time. The second and third peaks are smaller, indicating that the delocalization at transition is rather weak in the strong disorder regime.

In Fig. 3 the fluctuation of winding number, denoted as \( \Delta w \), is due to the finite size effect. In the thermodynamic limit, \( \Delta w \) should vanish for a nontrivial phase. To see the behaviors of \( \Delta w \) in the TAI phases, we calculate the scaling function for one value of \( (J_1/J_0, U) \) in each TAI phase in the phase diagram. The selected values are denoted by the letters in Fig. 2. The scaling functions are found by fitting the linear relation between \( \ln \Delta w \) and \( \ln L \), where \( L \) is the length of the system. It is found that

\[
\beta = \frac{d \ln(\Delta w)}{d \ln(L)} = -1. \tag{9}
\]

The fitting parameters and errors are listed in Table. The fitting function and its fitting for TAI phase. The scaling function is negative, indicating that \( \Delta w \) indeed vanishes and \( w \) reaches exact quantization in the thermodynamic limit \( (L \to \infty) \).

In topological phases, the bulk-edge correspondence predicts that the numbers of pairs of edge states are the same as winding numbers \([23, 24]\). To study the edge states in TAI phases, we plot the disorder averaged probability density, projected to each sublattice, of the states near the band center. Fig. 5(a) and (b) show the probability density near the band center for TAI phases with \( w = 2 \) via type I and II transitions, respectively.

The probability densities were obtained from direct diagonalization for a chain with length 1000 lattice sites. The 998th to 1001th states are plotted in each row and the corresponding energy is shown in the legend. The disorder strength increases from the left to the right column. Fig 5(a) shows the probability density along the
$w = 0 \rightarrow 2$ transition. The leftmost column shows the probability density in the clean limit, where the states near the band center are bulk states and are away from zero-energy. The center column is the probability density when disorder strength is 1. The electronic states are at zero-energy, but diffusive in the bulk. The rightmost column is the probability density at $U = 3$, which is in the TAI phase with $w = 2$. There are two pairs of edge states at $E = 0$, the same as the winding number. Fig. 4(b) gives the probability density along the $w = 1 \rightarrow 2$ transition. There is one pair of edge states at $E = 0$ in the clean limit, as shown in the leftmost column. As disorder strength increases to 1, as shown in the center column, the edge states are protected against disorder, while the bulk states tend to shift to the edge. The rightmost column is the probability density at $U = 3$, which is in the TAI phase with $w = 2$. There are two pairs of edge states at $E = 0$, as expected by the bulk-edge correspondence. The bulk-edge correspondence for TAI phase with $w = 1$ is also satisfied. The probability density along the $w = 0 \rightarrow 1$ and $w = 2 \rightarrow 1$ transitions, are presented in the Appendix.

For transitions between TAI phases, the leakage of pre-existent edge states to the bulk is prohibited. This implies that in the thermodynamic limit, the Fermi level must be renormalized to a complex value such that the gap closing point is not at zero energy. Therefore, the Feynman diagrams for the impurity scattering must be included in the calculation of self-energy, and thus, the self-consistent Born approximation would be valid in the type II transitions [27]. On the other hand, for type I transitions, because the pre-existent states are not localized, the gap closing point is still at zero energy. This means that the first Born approximation is valid for the type I transition, as we will show in the following section.

IV. BORN APPROXIMATION

To understand the role of disorder, we calculate the self-energy ($\Sigma$) within Born approximation

$$\Sigma = \frac{U^2}{12} \sum_{k \in BZ} (E_f + i\eta - H_0(k) - \Sigma)^{-1}. \tag{10}$$

It gives the information about energy renormalization and energy broadening. The self-energy has only nonzero coefficients for $\sigma_x$ and $\sigma_0$, which are denoted by $\delta J_0$ and $\delta E_f$.

A. First order Born Approximation

For the First order Born approximation (FBA), the parameters on the right-hand side of Eq. (10) are replaced with the bare values.

$$\delta J_0 = -\frac{U^2}{12} \sum_{k \in BZ} \frac{(J_0 + J_1 \cos k + J_2 \cos(2k))}{-(E_f + i\eta)^2 + h_x^2 + h_y^2}. \tag{11}$$

$$\delta E_f = \frac{U^2}{12} \sum_{k \in BZ} \frac{(E_f + i\eta)}{-(E_f + i\eta)^2 + h_x^2 + h_y^2}. \tag{12}$$

In the limit of short wavelength, $(k_f \to \infty)$, the self-energy can be solved analytically. The integration of Eq. (11) gives $\delta J_0 \approx -U^2/24J_0$. The details of the calculation are shown in the appendix. At half-filling, i.e. $E_f = 0$, the renormalization of the Fermi energy is zero at weak disorder. While disorder renormalizes $J_0$ through nonzero $\delta J_0$. Thus, the energy dispersion becomes $E = \pm \sqrt{(h_x(k) + \delta J_0)^2 + h_y(k)^2}$. The condition for topological phase transition is when the spectral gap closes. This is given by

$$J_0 + \delta J_0 + J_1 \cos k_o + J_2 \cos(2k_o) = 0, \tag{13}$$

and

$$J_1 \sin k_o + J_2 \sin(2k_o) = 0. \tag{14}$$

For $k_o = 0$ or multiples of $\pi$, one can obtain the $U_c$ by equating

$$E_{\text{min}} = \frac{U^2}{24J_0}, \tag{15}$$

where $E_{\text{min}}$ is the lowest energy of the conduction band in the clean limit. For the type I transitions shown in Fig. 2, $U_c$ is estimated as 1.2, denoted by white stars in the figure, by substituting $J_0 = 1, k_o = \pi$ and $E_{\text{min}} = 0.06$ into Eq. (15). It agrees well quantitatively with the localization length and winding numbers.

Therefore, type I transition can be understood by the gap renormalization due to the presence of weak disorder.
FIG. 5. (color online). The probability density for states near the band center in the clean limit (leftmost column), during transition (center column) and TAI phases (rightmost column) for $(J_0, J_1, J_2) = (1, 0, -0.96)$ (a) and $(1, -0.8, -1.74)$ (b). The number of disorder configuration is 20. The length of the chain is 1000.
similar to the TAI phases proposed in two- and three-dimensional topological insulators [6,7]. However, FBA fails to predicts $U_c$ for type II transitions. Because the chain is in a topological phase, in which electronic states are already localized at the edge, weak disorder limit doest not apply and higher order scattering needs to be taken into account. Next, we turn to self-consistent Born approximation (SCBA).

B. self-consistent Born approximation

We numerically solve for the self-energy, Eq. (10), self-consistently. At strong disorder, the Fermi energy $\delta E_f$ obtains an imaginary part. The transition occurs when the band gap is renormalized by $\delta J_0$ and smeared out by the Fermi level broadening in the presence of disorder. Therefore, the critical disorder strength is determined when

$$\sqrt{(h_x(k_o) + \delta J_0)^2 + h_y(k_o)^2 - |\delta E_f|} = 0. \quad (16)$$

The critical disorder strength ($U_c^{SCBA}$) found by this criterion is drawn on Fig. 2 with red dots. The results agree with the peaks of localization length within the order of $10^{-1}$ for type II transitions.

It is worth noting that with this method, $U_c^{SCBA}$ is determined with the Fermi level broadening, which implies that the gap closing point is not necessarily at $E_f = 0$. This agrees with the topological protection of the edge states, which would be destroyed in the presence of delocalized bulk states. However, for type I transitions, there is no pre-existent edge states, the transition boundary is given by the crossing of energy band edge without lifetime broadening at $E_f = 0$. The inclusion of the imaginary part of the Fermi energy lower estimates $U_c^{SCBA}$, as shown by the red dots in Fig. 2.

As pointed out by Guo et al. [7] in their study of the three dimension TAI, the TAI regime where the self-energy obtains an imaginary part is the "true" TAI phase. In three dimension TAI, because bulk states do not contribute to conductance, bulk states must be localized. In one dimension, we found that in all TAI phases, the self-energy has an imaginary part. Moreover, the fluctuation in winding numbers vanishes and the winding number reaches exact quantization for large system size, as shown in Fig. 1 and Table I. Therefore, localization is an essential ingredient for the TAI phases found in the long-range SSH model.

V. CONCLUSION

Rich topological phases are determined numerically by the mean winding number and localization length in the long-range SSH model. The phase diagram that identifies the TAI phases is given. Two types of transitions to the TAI phases are found and explained within Born approximation. We also showed that the fluctuation of winding number vanishes in the thermodynamic limit by using the scaling function. This implies that the bulk-edge correspondence can be applied to the system under consideration. Furthermore, we found that the influence of disorder on the long-range SSH model depends on which type of transition occurs in the system. For type I transition, the renormalization of energy band gap gives rise to the crossing of band edge and leads to TAI. For type II transitions, disorder drives a nontrivial topological phase to a TAI phase, the SCBA predicts the transition point where energy gap smears out. In summary, disorder strengthens topology and stabilizes winding numbers by localization, giving rise to TAI phases.

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Appendix A: Analytical calculation of Born Approximation

By regarding the self energy as $\Sigma = \Sigma_0 \sigma_x + \Sigma_0 \sigma_y$ (which is consistent with the numerical result), it can be shown that the self-consistent Born approximation for the self energy ($z = E_f + i\eta$)

$$\Sigma = \frac{U^2}{12} \sum_{k \in BZ} \frac{1}{z - H_0(k) - \Sigma}. \quad (A1)$$

can be written as the renormalized $\tilde{J}_0$ and $\tilde{E}_f$ through the definitions $\tilde{J}_0 = J_0 + \Sigma_0 \sigma_x$ and $\tilde{E}_f = E_f - \Sigma_0$. We have

$$\tilde{J}_0 = J_0 - \frac{U^2}{12} \sum_{k \in BZ} \frac{J_0 + c_k^2}{(J_0 + c_k^2)^2 + s_k^2 - (E_f + i\eta)^2}, \quad (A2)$$

and

$$\tilde{E}_f = E_f + \frac{U^2}{12} \sum_{k \in BZ} \frac{E_f + i\eta}{(J_0 + c_k^2)^2 + s_k^2 - (E_f + i\eta)^2}, \quad (A3)$$

where $c_k = J_1 \cos(k) + J_2 \cos(2k)$ and $s_k = J_1 \sin(k) + J_2 \sin(2k)$, where the lattice constant is taken to be 1. The summation can be replaced by the integral, i.e., $\sum_{k \in BZ} = (1/2\pi) \int d \kappa$. Eq. (A2) and (A3) can be evaluated analytically. The result of Eq. (A2) is given by (let $\eta \to 0$)

$$\tilde{J}_0 = J_0 - \frac{U^2}{12} \frac{1}{2 J_0} \left\{ 1 + (-i) \sqrt{2} \left[ \frac{1}{\Gamma_1} - \frac{1}{\Gamma_2} \right] \right\}, \quad (A4)$$
where

\[
\frac{1}{\Gamma_1} = \frac{J_4}{\sqrt{\mu_1 \sqrt{M_1}}} \left[ 2J_0^3 + J_0(2E_f - J_1^2 - 2J_2^2) + J_1(J_1J_2 + \sqrt{\mu_1}) \right],
\]

\[
\frac{1}{\Gamma_2} = \frac{J_2}{\sqrt{\mu_1 \sqrt{M_2}}} \left[ -2J_0^3 + J_0(-2E_f + J_1^2 + 2J_2^2) + J_1(-J_1J_2 + \sqrt{\mu_1}) \right],
\]

and

\[
\mu_1 = -4J_0^2J_2 + J_1^2J_2^2 - 2J_0J_2(-2E_f + J_1^2 + 2J_2^2) + J_0^2(J_1^2 + 8J_2^2),
\]

\[
M_1 = 2J_0^3J_2 - J_0^2(J_1^2 - 4J_2^2) - J_1J_2(J_1J_2 + \sqrt{\mu_1}) - J_0(2E_fJ_2 - 2J_2^2 + J_1\sqrt{\mu_1}),
\]

\[
M_2 = 2J_0^3J_2 - J_0^2(J_1^2 - 4J_2^2) + J_1J_2(-J_1J_2 + \sqrt{\mu_1}) + J_0(-2E_fJ_2 + 2J_2^2 + J_1\sqrt{\mu_1}).
\]

In the Born approximation (not self consistent), \(J_0\) and \(E_f\) are replaced by the bare values \(\bar{J}_0\) and \(\bar{E}_f\) at the right hand side of the equality of Eq. (A4). In this case, the second term in \(\cdots\) of Eq. (A4) is pure imaginary and we have

\[
\delta J_0 \equiv -|E| = \mathcal{R}[\bar{J}_0] - J_0 = -\frac{U^2}{24\bar{J}_0}. \quad (A7)
\]

On the other hand, Eq. (A3) can also be evaluated analytically, and the result is

\[
\bar{E}_f = E_f + (-i)\frac{U^2}{12\sqrt{2}} \left( \frac{1}{\Gamma_3} + \frac{1}{\Gamma_4} \right), \quad (A8)
\]

where

\[
\frac{1}{\Gamma_3} = \frac{\bar{J}_0J_2}{\sqrt{\mu_1 \sqrt{M_1}}},
\]

\[
\frac{1}{\Gamma_4} = \frac{\bar{J}_0J_2}{\sqrt{\mu_1 \sqrt{M_2}}}. \quad (A9)
\]

In the Born approximation, \(\bar{E}_f\) is replaced by \(E_f\) at the right hand side of Eq. (A8). Furthermore, in this approximation, when \(E_f = 0\), we have \(\bar{E}_f = 0\).

**Appendix B: Bulk-Edge correspondence**

In the subsection, we show more examples of probability density in TAI phases. For TAI phase with \(w = 1\) driven from \(w = 0\), the probability densities are given in Fig. 6(a). There is no edge state in the clean limit, as shown in the leftmost column. In the TAI phase with \(w = 1\), there is one pair of edge states for \(U = 3\), as shown in the rightmost column. Fig. 6(b) is for TAI phase with \(w = 1\) driven from \(w = 2\) in the clean limit. There are two pairs of edge states at \(E = 0\) in the clean limit as shown in the leftmost column. As disorder strength increases to 1, as shown in the center column, the edge states are protected against disorder, while one pair of states start to diffuse into bulk. The rightmost column is the probability density at \(U = 3\), which is in the TAI phase with \(w = 1\). There is only one pairs of edge states at \(E = 0\), as expected by the bulk-edge correspondence.

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FIG. 6. (color online). The probability density for states near the band center in the clean limit, during transition and in TAI phases for \((J_0, J_1, J_2) = (1, 1, 0.06)\) (a) and \((1.2, 1.46)\) (b). The leftmost column is for the clean limit and the rightmost column is for the TAI phase with \(w = 1\). While the middle column is for an intermediate disorder strength in between.
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