Examples of Enhanced Quantization:
Bosons, Fermions, and Anyons

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Abstract

Enhanced quantization offers a different classical/quantum connection than that of canonical quantization in which \( \hbar > 0 \) throughout. This result arises when the only allowed Hilbert space vectors allowed in the quantum action functional are coherent states, which leads to the classical action functional augmented by additional terms of order \( \hbar \). Canonical coherent states are defined by unitary transformations of a fixed, fiducial vector. While Gaussian vectors are commonly used as fiducial vectors, they cannot be used for all systems. We focus on choosing fiducial vectors for several systems including bosons, fermions, and anyons.

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1 Introduction

Canonical quantization would seem to be, as its name suggests, a closed subject. However, some attempts at canonical quantization lead to unnatural results—most notably the “Triviality of $\phi^4_n$ for $n \geq 5$”, and possibly for $n = 4$ as well. Rather than accept such an outcome, a natural result may perhaps be obtained with an alternative quantization procedure, which, for the $\phi^4_n$ example, has been presented elsewhere [1]. The basis for an alternative quantization procedure [2] is readily summarized here; for some early motivation, see [3, 4, 5]. For the simple example of a single particle, the classical action functional is given by

$$A_C = \int_0^T [p(t)\dot{q}(t) - H_c(p(t), q(t))] \, dt,$$

while the corresponding quantum action functional is given by

$$A_Q = \int_0^T \langle \psi(t) | [i\hbar(\partial/\partial t) - \mathcal{H}(P, Q)] | \psi(t) \rangle \, dt.$$  (2)

A stationary variation of the paths $p(t)$ and $q(t)$ leads to Hamilton’s equations of motion, while a stationary variation of normalized Hilbert state vectors $\langle \psi(t) |$ (and $| \psi(t) \rangle$) leads to Schrödinger’s equation of motion (and its adjoint). The connection between these two theories—known as “canonical quantization”—comes when we promote $p \rightarrow P$ and $q \rightarrow Q$ as Hermitian operators, and $\mathcal{H}(P, Q)$ is chosen as $H_c(P, Q)$ apart from possible $O(\hbar)$ corrections. This prescription works best when $(p, q)$ are Cartesian coordinates [6]—despite the fact that phase space carries no metric with which to determine Cartesian coordinates!

There are many other proposed connections between classical and quantum expressions. We consider just two examples: (1) The Wigner [7] phase-space representation of a quantum state leads to a distribution function, but it generally suffers from not being a positive distribution; (2) The Husimi [8] phase-space representation provides a positive representation of an associated phase-space distribution function, but its partners in forming expectations are often singular distributions [9]. Further examples are presented in most textbooks dealing with quantum theory, e.g., [10].

Let us examine an alternative quantization procedure which has been called “Enhanced Quantization” [2]. The derivation of Schrödinger’s equation of motion assumes that sufficiently many vectors $| \psi(t) \rangle$ can be varied in (2), but suppose that is not possible. A macroscopic observer can only change a few features of a quantum system, such as its position or velocity, which thanks to Galilean covariance, can be realized by moving the observer rather than disturbing the microscopic system. Thus, choosing a suitable, normalized fiducial vector $| \eta \rangle$, and with two self-adjoint operators $P$ and $Q$, $[Q, P] = i\hbar$, and initially choosing $| \eta \rangle$ as $| 0 \rangle$, an harmonic oscillator ground
state determined by \((Q+iP)|0\rangle = 0\), we generate two, independent unitary transformations of \(|0\rangle\) leading to

\[ |p,q\rangle \equiv e^{-iqP/\hbar} e^{ipQ/\hbar} |0\rangle. \]  

We choose this set of states for all \((p,q) \in \mathbb{R}^2\) as states that can be varied by a macroscopic observer. This set is also well known as a set of canonical coherent states with \(|0\rangle\) serving as its fiducial vector [11]. Using these states, the restricted quantum action functional becomes

\[ A_{Q(R)} = \int_0^T \langle p(t),q(t)|i\hbar (\partial/\partial t) - \mathcal{H}(P,Q)|p(t),q(t)\rangle \, dt \]

\[ = \int_0^T [p(t)\dot{q}(t) - \mathcal{H}(p(t),q(t))] \, dt. \]

Thus a stationary variation of the restricted action leads to Hamilton’s equations with \(H(p,q)\) serving as the enhanced classical Hamiltonian since \(\hbar > 0\) still. Specifically,

\[ H(p,q) \equiv \langle p,q|\mathcal{H}(P,Q)|p,q\rangle \]

\[ = \langle 0|\mathcal{H}(P+p,Q+q)|0\rangle \]

\[ = \mathcal{H}(p,q) + \mathcal{O}(\hbar; p,q; |0\rangle). \]

It is important to note that this classical/quantum relation is exactly what is meant by the relation between the classical and quantum Hamiltonians when \((p,q)\) are the favored Cartesian coordinates. Although the phase space cannot determine Cartesian coordinates, the Hilbert space provides a metric. Since the overall phase of a Hilbert space vector has no physical significance in quantum theory, we use a suitably scaled version of the distance determined by \(D_{\text{ray}}(|\psi\rangle; |\phi\rangle)^2 \equiv (2\hbar) \min_{\alpha} \|\psi - e^{i\alpha} |\phi\rangle\|^2\), which, when applied to two infinitesimally close coherent states, leads to

\[ d\sigma(p,q)^2 \equiv (2\hbar) \left[ \|dp,q\|^2 - |\langle p,q| dp,q \rangle|^2 \right] = dp^2 + dq^2. \]

Thus, it is the Hilbert space that determines that the chosen coordinates are Cartesian, and, if one so desired, that metric could be added to the phase space as well. In that sense the alternative classical/quantum connection used in enhanced quantization has nevertheless yielded the very same result as canonical quantization since the Cartesian coordinates determined by the Hilbert space metric are linked with the very same choice of the Hamiltonian operator as related to the classical Hamiltonian as is chosen by canonical quantization.

There are additional features of enhanced quantization that are covered elsewhere, e.g., [2, 12,
They also include affine quantization, which although not discussed in this article, could also be related to our further discussions. In fact, the main role of this article is to feature the choice of the fiducial vector \(|\eta\rangle\). So far we have chosen \(|0\rangle\) as the fiducial vector, as this is a common choice whenever discussing coherent states. On the other hand, the choice of \(|0\rangle\) is not always optimal or even possible, and in the rest of this paper we discuss the features and requirements of various fiducial vectors \(|\eta\rangle\).

2 Choosing the Fiducial Vector

2.1 General properties

Coherent states are often defined with the aid of a group, and in so doing the action of the group acting on a fixed, fiducial vector defines the coherent states. As an example consider the set of canonical coherent states given by

\[
|p, q; \eta\rangle \equiv e^{-i q P/\hbar} e^{i p Q/\hbar} |\eta\rangle. \tag{7}
\]

Normally, the choice of the normalized fiducial vector \(|\eta\rangle\) is left implicit, but on this occasion, since we are deciding on how to choose a "good"—or the "best"—fiducial vector, we include it explicitly in the previous equation. The transformation of the fiducial vector to make coherent states involves unitary transformations by the given expressions, which means that both P and Q must be self-adjoint operators so they may generate unitary operators. The real variables \(p\) and \(q\) each generate one-parameter groups expressed in so-called canonical group coordinates of the second kind \([14]\). Since such unitary operators are strongly continuous in their parameters, e.g., \(\|e^{i p Q/\hbar} - 1\| |\psi\rangle \rightarrow 0\) as \(p \rightarrow 0\) for all \(|\psi\rangle \in \mathcal{H}\), it follows that the coherent states are strongly continuous in their parameters for any \(|\eta\rangle\). This property ensures the continuity of the coherent state representation for any abstract vector \(|\psi\rangle\) given by \(\psi(p, q; \eta) \equiv \langle p, q; \eta | |\psi\rangle\) for any \(|\eta\rangle\).

Besides continuity, the other basic feature of coherent states is a resolution of unity by an integral over the entire phase space of coherent state projection operators that involves an absolutely continuous measure with a suitable positive weighting, which, for the example under consideration, is given by

\[
\mathbb{1} = \int |p, q; \eta\rangle \langle p, q; \eta| \, d\mu(p, q), \quad d\mu(p, q) \equiv dp \, dq/(2\pi\hbar), \tag{8}
\]

a relation that holds weakly (as well as strongly) for any choice of the normalized fiducial vector.
Remark: For other sets of putative coherent states, it sometimes happens that the resolution of unity fails; in this case we deal with so-called “weak coherent states” \[11\]. When that happens, it is useful to let the inner product of weak coherent states serve as a reproducing kernel and to generate a reproducing kernel Hilbert space \[15\].

**2.2 First look at choosing the fiducial vector**

In enhanced quantization, such as discussed in Sec. 1 of this paper, the restricted quantum action functional is given by

\[
A_{Q(R)} = \int_0^T \langle p, q; \eta | [i\hbar (\partial / \partial t) - \mathcal{H}(P, Q)] | p, q; \eta \rangle \, dt
\]

and thus it is necessary that the coherent states are in the domain of the Hamiltonian operator \(\mathcal{H}\), which, for an unbounded Hamiltonian operator, will already induce a certain restriction on |\(\eta\rangle\).

Additionally, in giving physical meaning to the variables \(p\) and \(q\), it is useful to impose “physical centering”, i.e., \(\langle P \rangle = \langle Q \rangle = 0\), wherein we have introduced the shorthand that \(\langle (\cdot) \rangle \equiv \langle \eta | (\cdot) | \eta \rangle\).

The virtue of physical centering becomes clear when we note that it leads to

\[
\langle p, q; \eta | P | p, q; \eta \rangle = p, \quad \langle p, q; \eta | Q | p, q; \eta \rangle = q,
\]

yielding a natural physical interpretation of the parameters \(p\) and \(q\).

The enhanced classical Hamiltonian \(H(p, q)\) (with \(\hbar > 0\)) differs from the classical Hamiltonian \(H_c(p, q) = \lim_{\hbar \to 0} H(p, q)\) by terms of order \(\hbar\). As an example, let \(\mathcal{H}(P, Q) = P^2 + Q^2 + Q^4\). in which case

\[
H(p, q) = p^2 + q^2 + q^4 + 6q^2\langle Q^2 \rangle + \langle [P^2 + Q^2 + Q^4] \rangle,
\]

assuming for simplicity that odd expectations vanish. Apart from a constant, there is the term \(6q^2\langle Q^2 \rangle\) which will modify the usual equations of motion and their solution. If we choose \(\langle x|\eta\rangle \propto \exp(-\omega x^2 / 2\hbar)\), it follows that \(\langle Q^2 \rangle \propto \hbar\), meaning that the correction is important only when \(p\) and \(q\) are “quantum-sized” themselves. That limitation makes good sense since then the enhanced classical description is effectively unchanged for macroscopic motion, only showing quantum “uncertainty”, i.e., dependence on \(\omega\), for quantum-sized motion. Alternatively, we could in principle choose \(\langle x|\eta\rangle \propto \exp(-ax^2 / 2)\), where, say, \(a = 10^{-137} m^{-2}\) and independent of \(\hbar\), which means that the additional term would modify the quadratic term by a potentially huge amount even for...
macroscopic motions. Such a choice is mathematically possible, just as studying a simple harmonic oscillator with displacements on a planetary scale or energies equivalent to the mass energy of the Earth are mathematically possible. However, they are unphysical applications of the mathematical description of a simple harmonic oscillator. In a similar story, although it is mathematically possible to choose fiducial vectors so that \( \langle Q^2 \rangle \) leads to macroscopic modifications, such a fiducial vector could never be physically realized. Thus we conclude that it is logical to choose the fiducial vector supported largely on a “quantum-sized” region. However, that still leaves open many possibilities. Indeed, Troung [16] has showed that choosing the ground state of a quartic Hamiltonian as the fiducial vector can recast Schrödinger’s equation into a new form that offers novel analysis options.

2.3 Second look at choosing the fiducial vector

It is popular to choose the fiducial vector to be a “quantum-sized” Gaussian; indeed, we have done so in Sec. 1. This choice often leads to fairly simple analytic expressions, and sometimes plausible arguments can be advanced that even help choose the variance parameter for such a vector [17]. However, it is important to understand that such a choice is not suitable in all cases. Let us reexamine the old discussion about “The rest of the universe” [18]. A single system seldom exists in isolation; instead, it is surrounded by other systems. If we can imagine one specific system, then it is possible to imagine \( N \) independent, identical systems, and even infinitely many such systems, i.e., \( N = \infty \). A system may involve several degrees of freedom; however, for clarity our basic system has a single degree of freedom.

For example, consider the Hamiltonian operator \( \mathcal{H}_{(N)} \equiv \sum_{n=1}^{N} \mathcal{H}(P_n, Q_n) \), which represents \( N \) independent, identical copies of the “original” Hamiltonian \( \mathcal{H}(P_1, Q_1) \) involving independent operator pairs \( (P_n, Q_n) \). The first system (for \( P_1 \) and \( Q_1 \)) is chosen as the “physical” one, while the other sub-systems are “spectator” systems. The coherent states we choose are for the physical system only, that is only for the first operator pair; specifically

\[
|p, q; \eta\rangle = e^{-iqP_1/\hbar} e^{ipQ_1/\hbar} |\eta\rangle , \quad |\eta\rangle \equiv \otimes_{n=1}^{N} |\eta_n\rangle .
\]  

(12)

To preserve the equivalence of all subsystems we choose each \( |\eta_n\rangle \) to be identical to one another, i.e., the same single-system fiducial vector. In this case it follows that

\[
A_{Q(R)} = \int_{0}^{T} [p(t)\dot{q}(t) - H_{(N)}(p(t), q(t))] dt ,
\]

(13)
where, in the present case,

$$H_{(N)}(p, q) = H(p, q) + \sum_{n=2}^{N} \langle H(P_n, Q_n) \rangle,$$  \hspace{1cm} (14)

which differs by a constant from the single system story. So long as \(N < \infty\), that constant is finite and not important. But, as \(N \to \infty\), and we eventually deal with an infinite number of spectator systems, it becomes important that that constant must be zero. At this point in the argument we restrict attention to quantum systems that have a non-negative spectrum and a unique ground state, which we choose as \(|\eta\rangle\), with an energy eigenvalue adjusted to be zero. If that is the case, then the added constant in (14) vanishes for all \(N\) including \(N = \infty\). It may be argued that we could choose \(|\eta\rangle\), say, as the first excited state for each subsystem and subtract that energy to obtain a zero. However, that would imply, for \(N = \infty\), that there were infinitely many energy levels with \(-\infty\) for their energy level, which is clearly an unphysical situation. The remedy for that situation is to insist that the fiducial vector be chosen as the unique ground state of each system with an energy level adjusted to vanish. This choice applies to the physical system, and to all of the spectator systems as well. Thus we can imagine any one of the \(N\) identical systems being the physical one and the remaining \(N - 1\) systems as spectators.

Moreover, we can imagine there are many other spectator systems different from the physical one we have chosen. For example, suppose there is another multiple-system type, with a Hamiltonian \(\tilde{H}_{(\tilde{N})} = \sum_{\tilde{n}=1}^{\tilde{N}} \tilde{H}(P_{\tilde{n}}, Q_{\tilde{n}})\), that is also present. This operator, too, is assumed to have a unique ground state with zero energy eigenvalue. Thus this new Hamiltonian could be present in the overall Hamiltonian but it would contribute nothing to the restricted quantum action functional because coherent states for it have not been “turned on”. Indeed there could be many such systems, even infinitely many such new (sub)systems. This argument can be carried to yet new families of spectator systems all of which are there, just “resting”, or “hibernating”, in their own ground state, and contributing nothing to the restricted quantum action functional. In this fashion we have found how to include “the rest of the universe” in such a way that it makes no contribution whatsoever, just as if we had ignored it altogether at the beginning of the story.

This desirable property requires that we choose the fiducial vector as the unique ground state of the system under consideration adjusted to have zero energy, which for this esoteric exercise of dealing with the surroundings proved extremely convenient if not absolutely necessary. This choice of fiducial vector also eliminates any nonsense regarding intrusion of the micro world into the macro world as we argued above. Of course, concerns about the intrusion of the surroundings are not always necessary, and thus it is acceptable to consider other fiducial vectors that are “close” to the ground state in some unspecified way, if one so desires.
Finally, we need to comment on other model systems that have Hamiltonians, which (i), near a lower bound, have a continuous spectrum, or (ii) instead have a spectrum that is unbounded below. These are interesting mathematical models, but it is difficult to find any real physical systems that have such features.

Having shown how we can, if necessary, deal with the rest of the universe, we revert to simple systems without concerning ourselves with such big issues. Thus, in what follows, we allow ourselves to consider a variety of useful fiducial vectors, particularly those where Gaussian form are not appropriate to describe certain physical systems, as in the case of fermions and anyons.

3 Bosons, Fermions, and Anyons

In this section we confine ourselves to $2 + 1$ spacetime dimensions so that anyons can be treated as well as bosons and fermions, although the results and calculations for bosons and fermions are not limited to $2 + 1$ dimensions; generalizations to arbitrary spacetime dimensions for them are straightforward.

Charged particles orbiting around a magnetic flux tube and interacting with it have fractional statistics in $2 + 1$ dimensions [19]. Such composites, known as anyons [19], are basically characterized by their peculiar statistics: under a half rotation centered on a magnetic flux, leading to a particle permutation, the state of the system changes by a complex phase factor,

$$\psi(r_2, r_1) = e^{i\alpha} \psi(r_1, r_2), \quad (0 \leq \alpha < 2\pi),$$

$$r_1 = (x_1, y_1), \quad r_2 = (x_2, y_2),$$

rather than the standard bosons ($\alpha = 0$) or fermions ($\alpha = \pi$) statistics. Here and throughout in this paper $r_\sigma$ designates the two-dimensional cartesian coordinates of the first ($\sigma = 1$) particle and the second ($\sigma = 2$) particle, respectively.

As has been noticed originally by Wilczek [19], this consideration has a very clear mathematical explanation based on the fact that in two space dimensions the rotation group $SO(2)$ is isomorphic with the Abelian unitary group $U(1)$ whose representations are labeled by real numbers. Due to the latter fact and the spin-statistics connection, the wave function of these particles may admit an arbitrary phase under rotations in $2 + 1$ spacetime. More precisely, circling a magnetic flux in the same direction, by two half turns, the state of the system may not necessarily need to return to its original state, meaning that now the system can be described by a multivalued wave function.

Anyons have attracted much attention due to their own richness, both in theoretical treatment
of fundamental concepts, as well as in their physical implications. Particles with fractional statistics were considered in the $O(3)\sigma$ model due to the existence of solitons [20], it also has been shown that anyons can be described as ordinary particles interacting with a Chern-Simons field [28, 31]. A relativistic wave equation for anyons was formulated in [33] and more recently in [34]. Regarding the physical implications, anyons have a central role in the explanation of the quantum Hall effect [21], in high-$T_c$ superconductivity [22], and more recently in topological quantum computation [23, 24, 25]. For a complete review underlying the fundamental theoretical descriptions and applications, we recommend the reviews [25, 31, 32].

All this interest motivates us to study the subject. Particularly, we are interested in the enhanced classical theory of anyons in $2+1$ dimensions. In this respect it should be noted that classical theories for anyons have been considered before [35, 36]. In these papers the authors follow a canonical quantization procedure [6, 37] to derive the corresponding quantum theory for anyons. Here, we adopt a different construction to quantize the theory of particles with fractional statistics by the application of the coherent state quantization and derive, for the first time, the corresponding enhanced classical theory. More precisely we consider a rotationally invariant Hamiltonian operator with a quartic interaction as a basic example.

Beneath the physical consideration behind anyons in $2+1$ dimensions, a coherent state representation [3, 4] of the quantum theory for anyons must exhibit the same property as (15). A direct consequence of such a requirement is the statistical invariance of the corresponding representation since now the fiducial vectors also obey,

$$\langle \mathbf{r}_2, \mathbf{r}_1 | \eta \rangle = \eta(\mathbf{r}_2, \mathbf{r}_1) = e^{i\alpha / \pi} \eta(\mathbf{r}_1, \mathbf{r}_2) .$$

(16)

In choosing a coherent state representation we must, as a starting point, provide a suitable fiducial vector consistent with the property (16). One possible way is to assume the fiducial vector is composed by two parts $\eta(\mathbf{r}_1, \mathbf{r}_2) = A(\mathbf{r}_1, \mathbf{r}_2) S(\mathbf{r}_1, \mathbf{r}_2)$ where $S(\mathbf{r}_1, \mathbf{r}_2)$ is a symmetrical function and $A(\mathbf{r}_1, \mathbf{r}_2)$ has a nonsymmetrical form. In particular,

$$A(\mathbf{r}_2, \mathbf{r}_1) = (-1)^\gamma A(\mathbf{r}_1, \mathbf{r}_2) ,$$

and the condition (16) is recovered for $\gamma = \alpha / \pi$. This construction leads to a proper description of nonrelativistic fermions for $\gamma = 1$, and it is generalized for anyons assuming $0 < \gamma < 2$ ($\gamma \neq 1$).

The wave functions for two or more anyon systems have been discussed before. One of the first proposals was a multivalued function with a complex nonsymmetrical part and a Gaussian-like symmetrical part [26, 27]. Generalizations of wave functions of this form have been considered in several works (e.g. [29, 30, 31, 38]) and in particular represents a bound state for two-anyons system.
in a $(2+1)$-dim. interaction to an external harmonic potential like $(1/2)\Omega r^2$ [38]. Following these constructions, we study a family of fiducial vectors, parametrized by $\lambda$, suitable to discuss bosons, fermions, and anyons in the same framework,

$$\eta_1(\mathbf{r}_1, \mathbf{r}_2) = N_\gamma (z_1 - z_2)^\gamma e^{-\frac{\lambda}{2}(\mathbf{r}_1^2 + \mathbf{r}_2^2)}, \quad 0 \leq \gamma < 2, \quad \gamma = \frac{\alpha}{\pi}, \quad (17)$$

$$z_1 = x_1 + iy_1, \quad z_2 = x_2 + iy_2, \quad \mathbf{r}_\sigma^2 = x_\sigma^2 + y_\sigma^2, \quad \lambda = \frac{\Omega}{\hbar}, \quad \Omega = \text{const.},$$

where $z_1, z_2$ are position of particles in the complex notation, $\Omega$ is an arbitrary real constant and $N_\gamma$ are the corresponding normalization constants. It is clear that interchanging $z_1$ and $z_2$ is equivalent to changing the corresponding positions of the particles, and, as a result, one obtains the desired phase $\eta(\mathbf{r}_2, \mathbf{r}_1) = (-1)^\gamma \eta(\mathbf{r}_1, \mathbf{r}_2)$. We initially study bosons and fermions in the next subsection. Exact solutions and results for anyons are presented afterwards.

### 3.1 General calculations for bosons and fermions

In this subsection we examine, in detail, the choice of fiducial vectors (17) for the particular cases of bosons ($\gamma = 0$) and fermions ($\gamma = 1$). More precisely, we are interested to obtain the enhanced classical description for these particles with bosons and fermions under an influence of a quartic interaction as (11). To do it one has to evaluate expectation values of the self-adjoint operators $Q, P, Q^2, P^2$ and $Q^4$, and the subsequent subsections are reserved to present that. Detailed derivations are placed in the Appendix.

#### 3.1.1 Bosons ($\gamma = 0$)

From (17) the fiducial vectors for bosons are pure Gaussian functions,

$$\eta_1(\mathbf{r}_1, \mathbf{r}_2) = N_0 e^{-\frac{\lambda}{2}(\mathbf{r}_1^2 + \mathbf{r}_2^2)}, \quad N_0 = \frac{\lambda}{\pi}. \quad (18)$$

One of the useful conditions underlying usual coherent state representations lies in the fact that expectation values for the position and momentum operators are zero. Due to parity properties one can see that $\langle Q_{x_1} \rangle = 0$ for the functions (18). Moreover it is straightforward to see that $\langle Q_{x_1}^{2n+1} \rangle = 0$ for $n \in \mathbb{N}$. The expectation value of the momentum operator is equivalently zero since the function (18) is symmetric in these variables. Therefore,

$$\langle Q_{x_1} \rangle = 0 = \langle P_{x_1} \rangle. \quad (19)$$
Intrinsic to the calculation of the enhanced Hamiltonian are the expectation values of $Q^2_{x_i}$, $P^2_{x_i}$ and $Q^4_{x_i}$. For example, the expectation value of $Q^2_{x_1}$ is,

$$
\langle Q^2_{x_1} \rangle = N_0^2 \left( \frac{\pi}{\lambda} \right) \int_0^{2\pi} d\vartheta \cos^2 \vartheta \int_0^\infty dr_1 r_1^2 e^{-\lambda r_1^2} = \frac{1}{2\lambda} = \frac{\hbar}{2\Omega}.
$$

(20)

where $\vartheta$ is the polar angle. [Remark: Here and in what follows we adopt the convention that one integration symbol over a bold letter means a double integration over two coordinates, for example, $\int_{c'}^c d \bf{r} f(x,y) \equiv \int_{c'}^c dx \int_{c'}^c dy f(x,y)$.] In addition the expectation value for $P^2_{x_1}$ is

$$
\langle P^2_{x_1} \rangle = -\hbar^2 \lambda N_0^2 \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \left( \lambda x_1^2 - 1 \right) e^{-\lambda (x_1^2 + x_2^2)} = \frac{\hbar^2 \lambda}{2} = \frac{\hbar \Omega}{2},
$$

(21)

and the remaining expectation values have the same result.

In order to discuss the enhanced classical theory for bosons with a quartic interaction one has to evaluate several expectation values like $\langle Q^4_{x_1} \rangle$, $\langle Q^2_{x_a} Q^2_{x_{a'}} \rangle$ and $\langle Q^2_{x_a} Q^2_{y_{a'}} \rangle$. However, due to the symmetry of the Gaussian fiducial vectors (18), only $\langle Q^4_{x_1} \rangle$ is independent,

$$
\langle Q^4_{x_1} \rangle = N_0^2 \left( \frac{\pi}{\lambda} \right) \int_0^{2\pi} d\vartheta \cos^4 \vartheta \int_0^\infty dr_1 r_1^5 e^{-\lambda r_1^2} = \frac{3}{4\lambda^2} = \frac{3\hbar^2}{4\Omega^2}.
$$

(22)

To show that consider, for example, $\langle Q^2_{x_1} Q^2_{x_2} \rangle$ (which is equivalent to $\langle Q^2_{x_1} Q^2_{y_1} \rangle$ in the present case). From its form it is straightforward to conclude that

$$
\langle Q^2_{x_1} Q^2_{x_2} \rangle = N_0^2 \left( \frac{\pi}{\lambda} \right) \int_{-\infty}^{\infty} dx_1 x_1^2 e^{-\lambda x_1^2} \int_{-\infty}^{\infty} dx_2 x_2^2 e^{-\lambda x_2^2} = \frac{\langle Q^4_{x_1} \rangle}{3}.
$$

(23)

### 3.1.2 Fermions ($\gamma = 1$)

For fermions the fiducial vectors (17) take the form

$$
\eta_1 (\bf{r}_1, \bf{r}_2) = N_1 \left( z_1 - z_2 \right) e^{-\frac{\lambda}{2} (r_1^2 + r_2^2)},
$$

(24)

and the corresponding normalization constant is obtained as usual,

$$
|N_1| = \left( \int_{-\infty}^{\infty} d\bf{r}_1 \int_{-\infty}^{\infty} d\bf{r}_2 \left| \bf{r}_1 - \bf{r}_2 \right|^2 e^{-\lambda (r_1^2 + r_2^2)} \right)^{-\frac{1}{2}} = \sqrt{\frac{\lambda^3}{2\pi^2}},
$$

(25)
The expectation value of the coordinate operators, $Q_{x_1}$, etc, are zero for these states. For example, $\langle Q_{x_1} \rangle$ has the form

$$\langle Q_{x_1} \rangle = N_1^2 \int_0^{2\pi} d\vartheta \cos \vartheta \int_0^\infty dr_1 r_1^2 e^{-\lambda r_1^2} \int_{-\infty}^\infty dr_2 |r_1 - r_2|^2 e^{-\lambda r_2^2} = 0,$$  \hspace{1cm} (26)$$

In addition the expectation values for the momentum operator $\langle P_{x_1} \rangle$ reads,

$$\langle P_{x_1} \rangle = -i\hbar N_1^2 \int_{-\infty}^\infty dr_1 \int_{-\infty}^\infty dr_2 \left\{ (z_1^* - z_2^*) - \lambda x_1 |r_1 - r_2|^2 \right\} e^{-\lambda (r_1^2 + r_2^2)} = 0,$$  \hspace{1cm} (27)$$

and the remaining linear expectation values are also zero. The expectation value for $Q_{x_1}^2$ are

$$\langle Q_{x_1}^2 \rangle = N_1^2 \int_{-\infty}^\infty dr_1 \int_{-\infty}^\infty dr_2 (r_1^2 + r_2^2) x_1^2 e^{-\lambda (r_1^2 + r_2^2)} = \frac{3}{4\lambda} = \frac{3\hbar}{4\Omega},$$  \hspace{1cm} (28)$$

and for $P_{x_1}^2$ are

$$\langle P_{x_1}^2 \rangle = -\hbar^2 \lambda N_1^2 \int_{-\infty}^\infty dr_1 \int_{-\infty}^\infty dr_2 \left\{ |r_1 - r_2|^2 \left( \lambda x_1^2 - 1 \right) - x_1^2 \right\} e^{-\lambda (r_1^2 + r_2^2)}$$

$$= \frac{3\hbar^2 \lambda}{4} = \frac{3\hbar \Omega}{4},$$  \hspace{1cm} (29)$$

In order to discuss rotationally invariant quartic interactions we have to evaluate, for example, expectation values as $\langle Q_{x_1}^4 \rangle$, $\langle Q_{x_1}^2 Q_{x_1}^2 \rangle$ and $\langle Q_{x_1}^2 Q_{y_1}^2 \rangle$. The first one has the form

$$\langle Q_{x_1}^4 \rangle = N_1^2 \int_{-\infty}^\infty dr_1 \int_{-\infty}^\infty dr_2 |r_1 - r_2|^2 x_1^4 e^{-\lambda (r_1^2 + r_2^2)} = \frac{3}{2\lambda^2} = \frac{3\hbar^2}{2\Omega^2},$$  \hspace{1cm} (30)$$

and the same result can be obtained for $\langle Q_{y_1}^4 \rangle$ ... etc. Next it can easily seen that expectation values like $\langle Q_{x_1}^2 Q_{x_2}^2 \rangle$ and $\langle Q_{x_1}^2 Q_{y_1}^2 \rangle$ enjoy the same property as (23). At last, there exists one more interesting expectation value $\langle Q_{x_1} Q_{x_2} \rangle$ which vanishes for bosons but is nonzero for fermions,

$$\langle Q_{x_1} Q_{x_2} \rangle = -2N_1^2 \int_{-\infty}^\infty dr_1 \int_{-\infty}^\infty dr_2 x_1 x_2 (r_1 \cdot r_2) e^{-\lambda (r_1^2 + r_2^2)} = -\frac{1}{4\lambda} = -\frac{\hbar}{4\Omega},$$  \hspace{1cm} (31)$$

which again, due to the symmetry of the problem, this result coincides with $\langle Q_{y_1} Q_{y_2} \rangle$. 

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3.2 General calculations for anyons

In this section we generalize the previous results for arbitrary $\gamma$ within the range $[0, 2)$, with the $(0, 1) \cup (1, 2)$ used for anyons. More precisely, the choice of fiducial vector (17) admits an exact solution for arbitrary $\gamma$ and the results here are an extension to those particular cases with $\gamma = 0$ and 1. The first step is to evaluate the norm of the corresponding fiducial vectors,

$$\|\eta\|^2 = |N_\gamma|^2 \int_{-\infty}^{\infty} dr_1 \int_{-\infty}^{\infty} dr_2 |r_1 - r_2|^{2\gamma} e^{-\lambda(r_1^2 + r_2^2)}.$$  \hspace{1cm} (32)

where again the normalization constant is obtained as usual

$$|N_\gamma| = \left( \int_{-\infty}^{\infty} dr_1 e^{-\lambda r_1^2} \int_{-\infty}^{\infty} dr_2 |r_1 - r_2|^{2\gamma} e^{-\lambda r_2^2} \right)^{-\frac{1}{\gamma}}.$$ \hspace{1cm} (33)

This rotationally-invariant integral can simplified by choosing a convenient coordinate system. For example, in the case where $r_2$ is aligned along the axis $y_1$ (33) has the form,

$$|N_\gamma|^2 = \left( \int_{-\infty}^{\infty} dr_1 e^{-\lambda r_1^2} \int_{-\infty}^{\infty} dr_2 |\rho_\gamma^(-)(r_2)| \right)^{-1}, \quad \rho_\gamma^(-)(r_2) \equiv \int_{0}^{\infty} dr_1 r_1 e^{-\lambda r_1^2 I_\theta^(-)}(r_1, r_2; \gamma),$$

where $\theta$ is the angle between the axis $x_1$ and $r_1$ such that $r_1 \cdot r_2 = r_1 r_2 \cos(\pi/2 - \theta) = r_1 r_2 \sin \theta$. Detailed solution of $\rho_\gamma^(-)(r_2)$ and subsequent calculations can be found at the Appendix (60)-(65).

Using those results the norm (32) has the final form

$$\|\eta\|^2 = |N_\gamma|^2 \frac{\pi^2 \sqrt{\pi}}{2\lambda^{\gamma+2}} \frac{\Gamma(2 + \gamma)}{\Gamma\left(\frac{3}{2}\right)} F_2 \left( \begin{array}{c} 1 - \frac{\gamma}{2}, -\frac{\gamma}{2} \end{array}; \frac{3}{2}; 1 \right),$$ \hspace{1cm} (35)

from which follows that

$$|N_\gamma| = \left[ \frac{\pi^2 \sqrt{\pi}}{2\lambda^{\gamma+2}} \frac{\Gamma(2 + \gamma)}{\Gamma\left(\frac{3}{2}\right)} F_2 \left( \begin{array}{c} 1 - \frac{\gamma}{2}, -\frac{\gamma}{2} \end{array}; \frac{3}{2}; 1 \right) \right]^{-1/2}.$$ \hspace{1cm} (36)

where $F_2 (\alpha, \beta; \delta; z)$ is the hypergeometric function [39]. The expectation value $Q_{x_1}^2$ is conveniently evaluated in the same reference system, whose form is

$$\langle Q_{x_1}^2 \rangle = |N_\gamma|^2 \int_{-\infty}^{\infty} dr_1 e^{-\lambda r_1^2} x_1^2 \rho_\gamma^(-)(r_1),$$ \hspace{1cm} (37)
where $\rho^{(-)}(r_1)$ is defined in (60). Using its solution (64) this expectation value can be written as

$$
\langle Q_{x_1}^2 \rangle = \pi^2 |N_\gamma|^2 \sum_{s=0}^\infty \Lambda_s^{(-)} R_1^{(-)} (\gamma, s) = \left( \frac{\pi \sqrt{\pi}}{8 \lambda^3 + \gamma} \right) |N_\gamma|^2 \frac{\Gamma (3 + \gamma)}{\Gamma (\frac{3}{2})} \frac{1}{2} \left( \frac{1 - \gamma}{2}, -\gamma; \frac{3}{2}; 1 \right),
$$

(38)

and substituting (36) we get a remarkably simple result:

$$
\langle Q_{x_1}^2 \rangle = \frac{1}{4} \frac{\Gamma (3 + \gamma)}{\Gamma (2 + \gamma)} = \frac{\gamma + 2}{4 \lambda} = \frac{\hbar (\gamma + 2)}{4 \Omega}.
$$

(39)

By the same symmetry arguments, which has been discussed in the section on bosons and fermions, one can see that $\langle Q_1 \rangle = 0$ (again, the integral with respect to $x_1$ has an odd integrand within a symmetric interval). This is also true for the remaining coordinates expectation values. By virtue of this fact it follows that the expectation values of the momentum operators are also zero, for example,

$$
\langle P_{x_1} \rangle = |N_\gamma|^2 \int_{-\infty}^{\infty} d\mathbf{r}_1 \int_{-\infty}^{\infty} d\mathbf{r}_2 \left\{ (z_1^* - z_2^*) [\gamma - \lambda x_1 (z_1 - z_2)] \right. \\
\times \left. (r_1^2 + r_2^2 - 2 \mathbf{r}_1 \cdot \mathbf{r}_2)^{\gamma-1} e^{-\lambda (r_1^2 + r_2^2)} \right\} = 0,
$$

(40)

since it only has odd contributions.

The full expression of $\langle P_{x_1}^2 \rangle$ is,

$$
\langle P_{x_1}^2 \rangle = -\hbar^2 |N_\gamma|^2 \int_{-\infty}^{\infty} d\mathbf{r}_1 \int_{-\infty}^{\infty} d\mathbf{r}_2 e^{-\lambda (r_1^2 + r_2^2)} \left\{ |\mathbf{r}_1 - \mathbf{r}_2|^2 (\gamma - 1) (z_1^* - z_2^*)^2 - 2 \lambda \gamma x_1 (z_1^* - z_2^*) |\mathbf{r}_1 - \mathbf{r}_2|^2 \\
+ \lambda (\lambda x_1^2 - 1) |\mathbf{r}_1 - \mathbf{r}_2|^4 \right\}.
$$

Due to the appearance of odd coefficients some terms in this integral vanishes. Using previous results (35), (38) and the definitions (60), (63), (65), this expectation value can be written as (replacing $\lambda$ by $\Omega/\hbar$),

$$
\langle P_{x_1}^2 \rangle = -\hbar^2 |N_\gamma|^2 \sum_{s=0}^\infty \Lambda_s^{(-)} R_1^{(-)} (\gamma - 1, s) + \hbar^2 \lambda (1 - \lambda \langle Q_{x_1}^2 \rangle) \\
= \frac{\Omega \hbar}{2} \left[ 1 + \gamma \left( \frac{2 \sqrt{\gamma}}{2, \frac{1 - \gamma}{2}, \frac{3}{2}; 1} \right) - \frac{\gamma}{2} \right].
$$

(41)

\footnote{For the explicit definition of $\Lambda_s^{(-)}$ and $R_1^{(-)} (\gamma, s)$ see the Appendix (61), (65).}
It should be noted that both results (39) and (41) coincide with the particular considerations (20), (21) and (28), (29), as expected.

Among all possible rotationally-invariant potentials within the problem under consideration are quartic interactions. There is a particular interest in rotational invariant models, which justifies our study. We consider the rotationally invariant potential $V$ given by,

$$V \equiv (Q_1^2 + Q_2^2)^2, \quad Q_i^2 \equiv Q_{x_i}^2 + Q_{y_i}^2, \quad Q_2^2 \equiv Q_{x_2}^2 + Q_{y_2}^2,$$

whose expectation value can separated into a homogenous part $V_H$ and in a nonhomogeneous part $V_N$,

$$\langle V \rangle = \langle V_H \rangle + \langle V_N \rangle, \quad \langle V_H \rangle \equiv \langle Q_1^4 \rangle, \quad \langle V_N \rangle \equiv 2 \langle Q_1^2 Q_2^2 \rangle. \quad (43)$$

From its form it can be seen that the corresponding expectation value contains terms like $\langle Q_4^4 \rangle$ (the same for all other coordinates operators), as well as correlation functions such as $\langle Q_{x_2}^2 Q_{y_2}^2 \rangle$. For example, the expectation value of $Q_{x_1}^4$ is

$$\langle Q_{x_1}^4 \rangle = \left| N_\gamma \right|^2 \int_{-\infty}^{\infty} dr_1 e^{-\lambda r_1^2} \langle Q_{x_1}^4 \rho_{\gamma} (-)(r_1) \rangle = \frac{3\pi}{8\lambda^3} \left| N_\gamma \right|^2 \sum_{s=0}^{\infty} \Lambda_{s}^{(-)} R_{2}^{(-)} (\gamma, s), \quad (44)$$

which, after a few simplifications, can be split into two parts

$$\langle Q_{x_1}^4 \rangle = \left| N_\gamma \right|^2 \left( \frac{3\pi \sqrt{\pi}}{64\lambda^{3\gamma+4}} \frac{\Gamma(\gamma + 4)}{\Gamma(5/2)} \right) \left( \sum_{s=1}^{\infty} \frac{1}{(s-1)!} \left( \frac{1 - \gamma}{2} \right)^s \left( \frac{1}{2} \right)^{-1} \right) + \left( \frac{\gamma(\gamma - 1)}{10} \right) \left( \frac{3 - \gamma}{2} \right) \left( \frac{2 - \gamma}{2} \right) \left( \frac{7}{2} \right) \left( \frac{5}{2} \right) \left( \frac{1}{2} \right)^{s} \right) . \quad (45)$$
Although (45) has an apparently complicated form, one can see that it reduces to (22) and (28) when \( \gamma = 0, \ \gamma = 1 \), respectively. Since \( _2F_1( a, b; c; 1) = 1 \) for \( a = 0 \) or \( b = 0 \), one can readily check that
\[
\langle Q^4_{x_1} \rangle|_{\gamma=0} = \frac{12\hbar^2}{16\Omega^2} = \frac{3\hbar^2}{4\Omega^2}, \quad \langle Q^4_{x_1} \rangle|_{\gamma=1} = \frac{24\hbar^2}{16\Omega^2} = \frac{3\hbar^2}{2\Omega^2},
\]
as expected.

The second kind of interaction from (42) has the form \( \langle Q^2_{x_1} Q^2_{y_1} \rangle \). It can be seen that for this interaction
\[
\langle Q^2_{x_1} Q^2_{y_1} \rangle = |N_\gamma|^2 \int_{\infty}^\infty \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy e^{-(x^2+y^2)} x^2 y^2 f (r),
\]
differs from (44) only in the first integral which, after changing to polar coordinates, one obtains the simple relation
\[
\langle Q^2_{x_1} Q^2_{y_1} \rangle = \frac{\langle Q^4_{x_1} \rangle}{3}.
\]

It is worth noting that this result obeys the same standard property of pure Gaussian states, i.e., (47) is true for Gaussian states and here we see that this same property holds true. The reason behind it is due to the fact that the “non-Gaussian” part of these expectation values is a rotational invariant function which, as a matter of fact, does not spoil such a property. More specifically, the property
\[
\int_{\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dy e^{-(x^2+y^2)} x^2 y^2 f (r) = \frac{1}{3} \int_{\infty}^{\infty} dx \int_{-\infty}^{\infty} dy e^{-(x^2+y^2)} x^4 f (r),
\]
is true for any rotational invariant function \( f (r) \).

At last, the remaining interactions are mixed and nonhomogeneous like \( \langle Q^2_{x_1} Q^2_{x_2} \rangle \). These are slightly different from the previous cases (44), (46) as one can see from its form,
\[
\langle Q^2_{x_1} Q^2_{x_2} \rangle = |N_\gamma|^2 \int_{\infty}^{\infty} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy e^{-(x^2+y^2)} x^2 y^2 f (r),
\]
and the solution of \( \mathcal{I}^{(\gamma)}_{\phi} (r_1, r_2; \gamma) \) can be found at the appendix (61). Using the series representation of the hypergeometric function as (62) the integral \( \rho^{(+)}_{\gamma} (u_1) \) can be written as (63)
\[
\rho^{(+)}_{\gamma} (u_1) = \sum_{s=0}^{\infty} \Lambda^{(+)}_{\gamma} (u_1) u_1^s, \quad \Lambda^{(+)}_{\gamma} = \frac{\pi}{\lambda_{\gamma+2}^2} \frac{4^s}{s!} \left( \frac{1-\gamma}{2} \right)^s \left( -\frac{\gamma}{2} \right)^s (2)_s^{-1},
\]
\[
\phi^{(+)}_{\gamma} (u_1) = \Gamma (s+2) u_1^{\gamma+1} e^{u_1^2} W(\gamma-3s-1)/2,(s-\gamma-2)/2 (u_1),
\]

(49)
where the formula (64) has been used above. After some minor algebraic manipulations and using (65) we get the final result (replacing \( \lambda \) by \( \Omega/\hbar \)),

\[
\langle Q_{x_1}^2 Q_{x_2}^2 \rangle = \left( \frac{\hbar^2 (\gamma + 3) (\gamma + 2)}{24\Omega^2} \right) \frac{2F_1 \left( \frac{1}{2}, \frac{3}{2}; \frac{5}{2}; 1 \right)}{2F_1 \left( \frac{1}{2}, \frac{3}{2}; \frac{5}{2}; 1 \right)}.
\]

(50)

Finally, taking into account the symmetries of the several homogeneous and nonhomogeneous expectation values under the changes \( x_1 \leftrightarrow y_1, x_2 \leftrightarrow y_2, x_1 \leftrightarrow x_2, y_1 \leftrightarrow y_2 \) the full expectation value of the potential (42) has the form (replacing \( \lambda \) by \( \Omega/\hbar \)),

\[
\langle V \rangle = \frac{16}{3} \langle Q_{x_1}^4 \rangle + 4 \langle Q_{x_1}^2 Q_{x_2}^2 \rangle = \frac{\hbar^2 (\gamma + 3) (\gamma + 2)}{6\Omega^2} \left( 2F_1 \left( \frac{1 - \gamma}{2}, -\frac{\gamma}{2}; \frac{3}{2}; 1 \right) \right)^{-1} \times \left\{ 5 \frac{2F_1 \left( \frac{1 - \gamma}{2}, -\frac{\gamma}{2}; \frac{5}{2}; 1 \right)}{\gamma (\gamma - 1)} + \frac{\gamma (\gamma - 1)}{5} \frac{2F_1 \left( \frac{3 - \gamma}{2}, -\frac{\gamma}{2}; \frac{7}{2}; 1 \right)}{\gamma (\gamma - 1)} \right\}.
\]

(51)

### 4 Selected Hamiltonian operators

Traditionally the Hamiltonian operator for particles with fractional statistics is represented by the standard nonrelativistic Hamiltonian for a charged particle interacting with a magnetic flux tube through the minimum coupling with a vector potential [19, 27]. The latter potential may conveniently be removed by a gauge transformation which, in particular, gives the multivalued character to the corresponding wave function [26, 27]. Once we are working with a multivalued wave function (more precisely with multivalued coherent states), the nonrelativistic momentum operator does have such potential vector. Moreover we add to the previous standard descriptions [27] a quartic potential of the form (42). For us this potential has a great importance due to its close relation with rotational invariant models.

The nonrelativistic Hamiltonian operator in consideration has the form

\[
\mathcal{H} (P, Q) = \sum_{\sigma = 1}^{2} \left[ \frac{P_{\sigma x}^2}{2m} + \frac{m\varpi^2}{2} Q_{\sigma x}^2 \right] + gV, \quad V = (Q_{1x}^2 + Q_{2x}^2)^2,
\]

\[
P_{\sigma} = (P_{x\sigma}, P_{y\sigma}), \quad Q_{\sigma} = (Q_{x\sigma}, Q_{y\sigma}),
\]

where \( m \) represent the mass of the particles, \( \varpi \) is the harmonic potential frequency, \( g \) is the quartic

---

2Regarding the gauge transformation there is a frequently used terminology associated with that [28, 31]. The corresponding singular gauge transformation defines the so-called anyons gauge in which the wave function is a multivalued function and the Hamiltonian does not have the potential vector. The other possibility is the so-called CS-gauge where the wave function obeys the standard statistics but the Lagrangian contains a Chern-Simons term. In this paper we do not follow the latter approach.
interaction coupling constant. In all cases the enhanced classical hamiltonian is identified with the expectation value of the Hamiltonian operator (52) with respect to the coherent states,

\[ H(p, q) = \langle p, q | H(P, Q) | p, q \rangle \]

\[ = \sum_{\sigma=1}^{2} \left[ \frac{P_{\sigma}^2}{2m} + \frac{m \omega \nu^2}{2} q_{\sigma}^2 + \frac{P_{\sigma}^2}{2m} + \frac{m \omega \nu^2}{2} Q_{\sigma}^2 \right] + g \langle p, q | V | p, q \rangle, \quad (53) \]

where the coherent states are defined, as in (3), by

\[ | p, q \rangle = \prod_{\sigma=1}^{2} U(q_{\sigma}) V(p_{\sigma}) | \eta \rangle, \quad U(q_{\sigma}) = e^{-i q_{\sigma} \cdot P_{\sigma}/\hbar}, \quad V(p_{\sigma}) = e^{i p_{\sigma} \cdot Q_{\sigma}/\hbar}. \quad (54) \]

As one can see from (53) there is no fundamental\(^{3}\) difference for the enhanced classical Hamiltonian between bosons, fermions or anyons. Nevertheless there does exist quantum differences between them, or more precisely, the numerical values of the coefficients proportional to \(\hbar\) have a different value for each considered particle. By virtue of (5) and symmetries behind the several expectation values of (43), as discussed in the subsection above, the expectation value of the potential \(V\) with respect to the coherent states (54) reads

\[ \langle p, q | V | p, q \rangle = \sum_{\sigma=1}^{2} \left[ (q_{\sigma}^2 + q_{\sigma}^2)^2 + 10 q_{\sigma}^2 \langle Q_{\sigma}^2 \rangle \right] \]

\[ + 2 (q_1 \cdot q_2) (\langle Q_{x_1} Q_{x_2} \rangle + \langle Q_{y_1} Q_{y_2} \rangle) + \frac{16}{3} \langle Q_{x_1}^4 \rangle + 4 \langle Q_{x_1}^2 Q_{x_2}^2 \rangle. \quad (55) \]

Labeling \(H_k(p, q)\) as the enhanced Hamiltonian for bosons \((k = b)\), fermions \((k = f)\), \((k = \gamma)\) for anyons and using the results of the previous subsections we list below the enhanced hamiltonian for bosons and fermions:

\[ H_b(p, q) = H_c(p, q) + \hbar \sum_{\sigma=1}^{2} \left( \frac{3g}{\Omega} \right) q_{\sigma}^2 + \hbar \left( \frac{\Omega}{m} + \frac{m \omega \nu^2}{\Omega} \right) + \hbar^2 \left( \frac{3g}{\Omega^2} \right), \quad (56) \]

\[ H_f(p, q) = H_c(p, q) + 6 \hbar \sum_{\sigma=1}^{2} \left( \frac{3g}{\Omega} \right) q_{\sigma}^2 + \hbar \left( \frac{3\Omega}{2m} + \frac{3m \omega \nu^2}{2\Omega} \right) + 2 \hbar^2 \left( \frac{3g}{\Omega^2} \right), \quad (57) \]

\(^{3}\)More precisely speaking, the functional form of the enhanced Hamiltonian for bosons, fermions and anyons are the same. The difference appears only in the numerical values of the expectation values which, as a matter of fact, depend on the choice of fiducial vectors under consideration.
and for anyons we have

\[
H_{\gamma}(p, q) = H_c(p, q) + \hbar \left[ \frac{\Omega}{m} \left( 1 + \gamma \frac{2}{2F_1 \left( \frac{2-\gamma}{2}, \frac{1-\gamma}{2}, \frac{3}{2}, 1 \right) - \gamma^2}{2F_1 \left( \frac{1-\gamma}{2}, \frac{-\gamma}{2}, \frac{3}{2}, 1 \right)} + \frac{m\omega^2}{2\Omega} (2 + \gamma) \right) \right] + \frac{\hbar g (\gamma + 2)}{\Omega} \left[ \sum_{\sigma=1}^{2} \frac{5q_{\sigma}^2}{2} - (q_1 \cdot q_2) \left( 2F_1 \left( \frac{1-\gamma}{2}, \frac{-\gamma}{2}, \frac{3}{2}, 1 \right) - 1 \right) \right] \nonumber
\]

\[
+ \frac{\hbar^2 (\gamma + 3)(\gamma + 2)}{6\Omega^2} \left( 2F_1 \left( \frac{1-\gamma}{2}, \frac{-\gamma}{2}, \frac{3}{2}, 1 \right) \right)^{-1} \times \left\{ 5 \frac{2F_1 \left( \frac{1-\gamma}{2}, \frac{-\gamma}{2}, \frac{5}{2}, 1 \right)}{2} + \frac{\gamma(\gamma - 1)}{5} \frac{2F_1 \left( 3-\gamma, \frac{2-\gamma}{2}, \frac{7}{2}, 1 \right)}{2F_1 \left( \frac{3-\gamma}{2}, \frac{2-\gamma}{2}, \frac{7}{2}, 1 \right)} \right\} . \tag{58}
\]

where the classical Hamiltonian \(H_c(p, q)\), in which \(\hbar \to 0\), has the same form for all cases,

\[
H_c(p, q) = \lim_{\hbar \to 0} H_k(p, q) = \sum_{\sigma=1}^{2} \left( \frac{p_{\sigma}^2}{2m} + \frac{m\omega^2}{2} q_{\sigma}^2 + g \left( q_{x,\sigma}^2 + q_{y,\sigma}^2 \right) \right). \tag{59}
\]

5 Conclusion

In this paper we have focussed on a central question in enhanced quantization using canonical coherent states, namely, the choice of the fiducial vector and the issues that choice involves. Initially, it was argued that a good choice is largely dictated by the explicit form of the Hamiltonian operator under consideration, and, in many cases the choice of the unique ground state as the fiducial vector has several virtues. However, that choice can also be relaxed to consider other fiducial vectors, and we can illustrate that choice by focussing attention on a Hamiltonian operator with a quartic interaction. One reason behind this choice is basically due to the fact that it is, effectively, the simplest example in which \(O(\hbar)\) coefficients of dynamical terms are involved that modify the classical description. Secondly, this choice can exhibit a problem with symmetry, e.g., rotational invariance, and similar properties can be extended to other models. In particular, fiducial vectors based on a Gaussian form are appropriate for bosons to deal with these Hamiltonian operators and \(\hbar\)-dynamical coefficients can be consistently treated with them. Generally, such coefficients can also be reduced by choosing Gaussian fiducial vectors, although this is not required according to the principles of enhanced canonical quantization [2, 3, 4, 5]. In such cases, the corresponding enhanced Hamiltonian is a symbol [8] of the respective Hamiltonian operator where \(\hbar\)-dependence is included.

Although commonly used, it is a fact that Gaussian fiducial vectors are not always suitable to consistently describe certain physical systems. In this respect, we have chosen two examples where
this form clearly fails: the enhanced quantization of fermions and anyons. In these cases the fiducial vectors cannot be independent Gaussians, and, instead, they must involve cross correlations for fermions or anyons. This latter property has a necessary physical consequence, namely, ensuring that permutations of the variables of the coherent state representation of Hilbert space vectors involve the required change of phase.

In section 3 non-Gaussian fiducial vectors have been used in the coherent states constructed for fermions and anyons. We have calculated several expectation values for both systems and, in this regard, the exact calculations for anyons also include the corresponding results for fermions simply by choosing $\gamma = 1$. Using these results, we have calculated the enhanced classical Hamiltonian and after taking the limit in which $\hbar \to 0$, we have shown that bosons, fermions, and anyons all have the same classical Hamiltonian, despite the fundamental differences between their properties when $\hbar > 0$.

**Appendix**

In dealing with expectation values of coordinate and momentum operators for the anyon case we frequently encounter integrals of the form [recall the convention: $\int_a^b (\cdot) dx = \int_a^b (\cdot) dy$

\[
\int_{-\infty}^{\infty} d\mathbf{r}_{\sigma'} |\mathbf{r}_{\sigma'} - \mathbf{r}_{\sigma}|^{2\gamma} e^{-\lambda r_{\sigma'}^2},
\]

(60)

where the indexes $\sigma, \sigma'$ label particles. Due to rotational symmetry, one can choose a particular reference system to simplify its solution. Choosing, for example, a reference frame where the vector $\mathbf{r}_{\sigma}$ is aligned along the axis $y_{\sigma'}$, we have $\mathbf{r}_{\sigma} \cdot \mathbf{r}_{\sigma'} = r_{\sigma} r_{\sigma'} \cos (\pi/2 - \vartheta) = r_{\sigma} r_{\sigma'} \sin \vartheta$. Here $\vartheta$ is the angle between the axis $x_{\sigma'}$ and $\mathbf{r}_{\sigma'}$ such that (60) admits the form [39],

\[
\rho_\gamma^{(\pm)} (r_{\sigma}) = \int_0^{\infty} dr_{\sigma'} r_{\sigma'}^{2\pm1} e^{-\lambda r_{\sigma'}^2} T_{\theta}^{(\pm)} (r_{\sigma}, r_{\sigma'}; \gamma),
\]

\[
T_{\theta}^{(\pm)} (r_{\sigma}, r_{\sigma'}; \gamma) = \int_0^{2\pi} d\vartheta \cos (\vartheta)^{1\pm1} \left( r_{\sigma}^2 + r_{\sigma'}^2 - 2r_{\sigma} r_{\sigma'} \sin \vartheta \right)^\gamma
\]

\[
= \left( \frac{3 \pm 1}{2} \right) \pi \left( r_{\sigma}^2 + r_{\sigma'}^2 \right)^\gamma \quad _2F_1 \left( \frac{1 - \gamma}{2}, -\frac{\gamma}{2}, \frac{3 \pm 1}{2}; \frac{4r_{\sigma}^2 r_{\sigma'}^2}{(r_{\sigma}^2 + r_{\sigma'}^2)^2} \right).
\]

(61)
To evaluate $\rho_{r_{\sigma}}^{(\pm)}(r_{\sigma})$ it is convenient to make use the series representation of the hypergeometric functions \[39\],

$$2F_1\left(\frac{1-\gamma}{2}, -\frac{\gamma}{2}, \frac{3\pm 1}{2}; \frac{4r_{\sigma}^2r_{\sigma'}^2}{(r_{\sigma}^2 + r_{\sigma'}^2)^2}\right) = \sum_{s=0}^{\infty} \frac{1}{s!} \left(\frac{1-\gamma}{2}\right)_s \left(-\frac{\gamma}{2}\right)_s \left(\frac{3\pm 1}{2}\right)_s^{-1} \left(\frac{4r_{\sigma}r_{\sigma'}}{r_{\sigma}^2 + r_{\sigma'}^2}\right)^{2s}, \tag{62}$$

which, after the change of variables $u_{\sigma} = \lambda r_{\sigma}^2$ and $u_{\sigma'} = \lambda r_{\sigma'}^2$, can be brought into the form

$$\rho_{\gamma}^{(\pm)}(u_{\sigma}) = \sum_{s=0}^{\infty} A_{s}^{(\pm)}(\gamma, s) u_{\sigma}^s,$$

$$g_{\gamma, s}^{(\pm)}(u_{\sigma}) \equiv \int_0^\infty du_{\sigma'} e^{-u_{\sigma'}^s} u_{\sigma'}^{s+1/2} (u_{\sigma'} + u_{\sigma})^{-2s},$$

$$A_{s}^{(\pm)} = \frac{\pi}{\lambda^{\gamma+\frac{3\pm 1}{4}}} \frac{4^s}{s!} \left(\frac{1-\gamma}{2}\right)_s \left(-\frac{\gamma}{2}\right)_s \left(\frac{3\pm 1}{2}\right)_s^{-1}, \tag{63}$$

where $(n)_s = \Gamma(n + s)/\Gamma(s)$ is the Pochhammer symbol \[39\]. The integral above can be analytically solved,

$$g_{\gamma, s}^{(\pm)}(u_{\sigma}) = \Gamma\left(s + \frac{3\pm 1}{2}\right) u_{\sigma}^{2s+1/2} e^{\frac{\mu}{2}} W_{\mu, \nu}(u_{\sigma}),$$

$$\mu = \frac{2(\gamma - 3s) - 1 \mp 1}{4}, \quad \nu = \frac{2(s-\gamma) - 3 \mp 1}{4}, \tag{64}$$

with $W_{\mu, \nu}(u_{\sigma})$ being a Wittaker function \[39\].

To complete the description it is also needed to solve one more integral of the expressions above, namely an integral over the variables $u_{\sigma}$ \[39\],

$$R_n^{(\pm)}(\gamma, s) \equiv \int_0^\infty du_{\sigma} e^{-\frac{u_{\sigma}}{2}} u_{\sigma}^{2s+1/2} W_{\mu, \nu}(u_{\sigma})$$

$$= \frac{\Gamma(n + s + \frac{3\pm 1}{4})}{\Gamma(2s + n + \frac{9+1}{4})} \Gamma(n + \gamma + \frac{9+1}{4}), \quad n > -3 \pm 1.$$ \[4]\)

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