Behavioral Metrics via Functor Lifting

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We study behavioral metrics in an abstract coalgebraic setting. Given a coalgebra \( \alpha : X \rightarrow FX \) in \( \text{Set} \), where the functor \( F \) specifies the branching type, we define a framework for deriving pseudometrics on \( X \) which measure the behavioral distance of states.

A first crucial step is the lifting of the functor \( F \) on \( \text{Set} \) to a functor \( F \) in the category \( \text{PMet} \) of pseudometric spaces. We present two different approaches which can be viewed as generalizations of the Kantorovich and Wasserstein pseudometrics for probability measures. We show that the pseudometrics provided by the two approaches coincide on several natural examples, but in general they differ.

Then a final coalgebra for \( F \) in \( \text{Set} \) can be endowed with a behavioral distance resulting as the smallest solution of a fixed-point equation, yielding the final \( F \)-coalgebra in \( \text{PMet} \). The same technique, applied to an arbitrary coalgebra \( \alpha : X \rightarrow FX \) in \( \text{Set} \), provides the behavioral distance on \( X \). Under some constraints we can prove that two states are at distance 0 if and only if they are behaviorally equivalent.

Keywords: behavioral metric, functor lifting, pseudometric, coalgebra

1. Introduction

Increasingly, modelling formalisms are equipped with quantitative information, such as probability, time or weight. Such quantitative information should be taken into account when reasoning about behavioral equivalence of system states, such as bisimilarity. In this setting the appropriate notion is not necessarily equivalence, but a behavioral metric that measures the distance of the behavior of two states. In a quantitative setting, it is often unreasonable to assume that two states have exactly the same behavior, but it makes sense to express that their behavior differs by some (small) value \( \varepsilon \).

The above considerations led to the study of behavioral metrics which aims at quantifying the the distance between the behavior of states. Since different states can have exactly the same behavior it is quite natural to consider pseudometrics, which allow different elements to be at zero distance.

Earlier contributions defined behavioral metrics in the setting of probabilistic systems [DGJP04, vBW06] and of metric transition systems [dAFS09]. Our aim is to generalize these ideas and to study behavioral metrics in a general coalgebraic setting. The theory of coalgebra [Rut00] is nowadays a well-established tool for defining and reasoning about various state based transition systems such as deterministic, nondeterministic, weighted

*This is an extended version of [BBKK14].
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or probabilistic automata. Hence, it is the appropriate setting to ask and answer general questions about behavioral metrics.

▶ How can we define behavioral metrics for transition systems with different branching types? We provide a coalgebraic framework in the category of pseudometric spaces PMet that allows to define and reason about such metrics.

▶ Are the behavioral metrics canonical in some way? We provide a natural way to define metrics by lifting functors from Set to the category of pseudometric spaces. In fact, we study two liftings: the Kantorovich and the Wasserstein lifting and observe that they coincide in many cases. This provides us with a notion of canonicity and justification for the choice of metrics.

▶ Does the measurement of distances affect behavioral equivalence? If we start by considering coalgebras in PMet (as, e.g., in [vBW06]), it is not entirely clear a priori whether the richer categorical structure influences the notion of behavioral equivalence. In our setting we start with coalgebras in Set and put distance measurements “on top”, showing that, under some mild constraints, the original notion of behavioral equivalence is not compromised, in the sense that two states are behaviorally equivalent iff their distance is 0.

▶ Are there generic algorithms to compute metrics? Coalgebra is a valuable tool to define generic methods that can be instantiated to concrete cases in order to obtain prototype algorithms. In our case we give a (high-level) procedure for computing behavioral distances on a given coalgebra, based on determining the smallest solution of a fixed-point equation.

A central contribution of this paper is the lifting of a functor \( F \) from Set to PMet. Given a pseudometric space \((X, d)\), the goal is to define a suitable pseudometric on \( FX \). Such liftings of metrics have been extensively studied in transportation theory [Vil09], e.g. for the case of the (discrete) probability distribution functor, which comes with a nice analogy: assume several cities (with fixed distances between them) and two probability distributions \( s, t \) on cities, representing supply and demand (in units of mass). The distance between \( s, t \) can be measured in two ways: the first is to set up an optimal transportation plan with minimal costs (in the following also called coupling) to transport goods from cities with excess supply to cities with excess demand. The cost of transport is determined by the product of mass and distance. In this way we obtain the Wasserstein distance. A different view is to imagine a logistics firm that is commissioned to handle the transport. It sets prices for each city and buys and sells for this price at every location. However, it has to ensure that the price function is nonexpansive, i.e., the difference of prices between two cities is smaller than the distance of the cities, otherwise it will not be worthwhile to outsource this task. This firm will attempt to maximize its profit, which can be considered as the Kantorovich distance of \( s, t \). The Kantorovich-Rubinstein duality informs us that these two views lead to the exactly same result, a very good argument for the canonicity of this notion of distance.

It is our observation that these two notions of distance lifting can analogously be defined for arbitrary functors \( F \), leading to a rich general theory. The lifting has an evaluation function as parameter. As concrete examples, besides the probability distribution functor, we study the (finite) powerset functor (resulting in the Hausdorff metric) and the coproduct and product bifunctors. In the case of the product bifunctor we consider different evaluation functions, each leading to a well-known product metric. The Kantorovich-Rubinstein duality holds for these functors, but it does not hold in general (we provide a counterexample).
After discussing functor liftings, we define coalgebraic behavioral pseudometrics and answer the questions above. Specifically we show how to compute distances on the final coalgebra as well as on arbitrary coalgebras via fixed-point iteration and we prove that the pseudometric obtained on the final coalgebra is indeed a metric. In [Appendix A] we discuss a fibrational perspective on our work and we compare with [HCKJ13]. All proofs for our results are in [Appendix F].

2. Preliminaries, Notation & Evaluation Functions

We assume that the reader is familiar with the basic notions of category theory, especially with the definitions of functor, product, coproduct and weak pullbacks.

For a function \( f : X \to Y \) and sets \( A \subseteq X, B \subseteq Y \) we write \( f[A] := \{ f(a) \mid a \in A \} \) for the image of \( A \) and \( f^{-1}[B] = \{ a \in A \mid f(x) \in B \} \) for the preimage of \( B \). If \( Y \subseteq [0, \infty] \) and \( f, g: X \to Y \) are functions we write \( f \leq g \) when \( \forall x \in X : f(x) \leq g(x) \).

Given a natural number \( n \in \mathbb{N} \) and a family \( (X_i)_{i=1}^{n} \) of sets \( X_i \) we denote the projections of the (cartesian) product of the \( X_i \) by \( \pi_i : \prod_{i=1}^{n} X_i \to X_i \), or just by \( \pi_i \) if \( n \) is clear from the context. For a source \( (f_i : X \to X_i)_{i=1}^{n} \) we denote the unique mediating arrow to the product by \( (f_1, \ldots, f_n) : X \to \prod_{i=1}^{n} X_i \). Similarly, given a family of arrows \( (f_i : X_i \to Y_i)_{i=1}^{n} \), we write \( f_1 \times \cdots \times f_n = (f_1 \circ \pi_1, \ldots, f_n \circ \pi_n) : \prod_{i=1}^{n} X_i \to \prod_{i=1}^{n} Y_i \).

We quickly recap the basic ideas of coalgebras. Let \( F \) be an endofunctor on the category Set of sets and functions. An \( F \)-coalgebra is just a function \( \alpha : X \to FX \). Given another \( F \)-coalgebra \( \beta : Y \to FY \) a coalgebra homomorphism from \( \alpha \) to \( \beta \) is a function \( f : A \to B \) such that \( \beta \circ f = Ff \circ \alpha \). We call an \( F \)-coalgebra \( \kappa : \Omega \to F\Omega \) final if for any other coalgebra \( \alpha : X \to FX \) there is a unique coalgebra homomorphism \( \llbracket \alpha \rrbracket : X \to \Omega \). The final coalgebra need not exist but if it does it is unique up to isomorphism. It can be considered as the universe of all possible behaviors. If we have an endofunctor \( F \) such that a final coalgebra \( \kappa : \Omega \to F\Omega \) exists then for any coalgebra \( \alpha : X \to FX \) two states \( x_1, x_2 \in X \) are said to be behaviorally equivalent if and only if \( [x_1] = [x_2] \).

We now introduce some preliminaries about (pseudo)metric spaces. Our (pseudo)metrics assume values in a closed interval \([0, T]\), where \( T \in (0, \infty] \) is a fixed maximal element (for our examples we will use \( T = 1 \) or \( T = \infty \)). In this way the set of (pseudo)metrics over a fixed set with pointwise order is a complete lattice (since \([0, T]\) is) and the resulting category of pseudometric spaces is complete and cocomplete.

**Definition 2.1 (Pseudometric, Pseudometric Space).** Given a set \( X \), a pseudometric on \( X \) is a function \( d : X \times X \to [0, T] \) such that for all \( x, y, z \in X \), the following axioms hold: \( d(x, x) = 0 \) (reflexivity), \( d(x, y) = d(y, x) \) (symmetry), \( d(x, z) \leq d(x, y) + d(y, z) \) (triangle inequality). If additionally \( d(x, y) = 0 \) implies \( x = y \), \( d \) is called a metric. A (pseudo)metric space is a pair \( (X, d) \) where \( X \) is a set and \( d \) is a (pseudo)metric on \( X \).

By \( d_c : [0, T]^2 \to [0, T] \) we denote the ordinary Euclidean distance on \([0, T]\), i.e., \( d_c(x, y) = |x - y| \) for \( x, y \in [0, T] \backslash \{\infty\} \), and – where appropriate – \( d_c(x, \infty) = \infty \) if \( x \neq \infty \) and \( d_c(\infty, \infty) = 0 \). Addition is defined in the usual way, in particular \( x + \infty = \infty \) for \( x \in [0, \infty] \).

Hereafter, we only consider those functions between two pseudometric spaces that do not increase the distances.

**Definition 2.2 (Nonexpansive Function, Isometry).** Let \( (X, d_X), (Y, d_Y) \) be pseudometric spaces. A function \( f : X \to Y \) is called nonexpansive if \( d_Y \circ (f \times f) \leq d_X \). In this case we write \( \boxed{f : (X, d_X) \xrightarrow{\text{d}} (Y, d_Y)} \). If equality holds, \( f \) is called an isometry.

\(^1\)The nonexpansive functions correspond exactly to the Lipschitz-continuous functions with Lipschitz constant \( L \leq 1 \), which is the reason for writing \( f : (X, d_X) \xrightarrow{\text{d}} (Y, d_Y) \).
For our purposes it will turn out to be useful to consider the following alternative characterization of the triangle inequality using the concept of nonexpansive functions.

**Lemma 2.3.** A symmetric function \( d: X^2 \to [0, \top] \) with \( d(x, x) = 0 \) for all \( x \in X \) satisfies the triangle inequality iff for all \( x \in X \) the function \( d(x, \_): X \to [0, \top] \) is nonexpansive.

As stated before, our definition of a pseudometric gives rise to a suitably rich category.

**Definition 2.4 (Category of Pseudometric Spaces).** For a fixed \( \top \in (0, \infty] \) we denote by \( \operatorname{P} \operatorname{M} \operatorname{e} \operatorname{t} \) the category of all pseudometric spaces and nonexpansive functions.

This category is complete and cocomplete (see Proposition P.2.2) and, in particular, it has products and coproducts as we will see in Examples 5.1 and 5.2.

We now introduce two motivating examples borrowed from [vBW06] and [dAFS09].

**Example 2.5 (Probabilistic Transition Systems and Behavioral Distance).** We regard probabilistic transition systems as coalgebras of the form \( \alpha: X \to D(X + 1) \), where \( D \) is the probability distribution functor (with finite support) which maps a set \( X \) to the set \( D X = \{ P: X \to [0, 1] \mid \sum_{x \in X} P(x) = 1, P \) has finite support\). and a function \( f: X \to Y \) to the function \( D f: D X \to D Y, P \mapsto \lambda x. \sum_{y \in f^{-1}(\{ y \})} P(x) \) of \( \operatorname{P} \operatorname{M} \operatorname{e} \operatorname{t} \). Here \( \alpha(x)(y) \), for \( x, y \in X \), denotes the probability of a transition from a state \( x \) to \( y \) and \( \alpha(x)(\check{\_}) \) stands for the probability of terminating from \( x \) (we use \( \check{\_} \) for the single element of the set \( 1 \)).

In [vBW06] a metric for the continuous version of these systems is introduced, by considering a discount factor \( c \in (0, 1) \). In the discrete case we obtain the behavioral distance \( d: X^2 \to [0, 1], \) defined as the least solution of the equation \( d(x, y) = \overline{d}(\alpha(x), \alpha(y)) \), where \( x, y \in X \) and \( \overline{d}: (D(X + 1))^2 \to [0, 1] \) is defined in two steps: First, \( \overline{d}: (X + 1)^2 \to [0, 1] \) is defined as \( \overline{d}(x, y) = c \cdot d(x, y) \) if \( x, y \in X \), \( \overline{d}(\check{x}, \check{y}) = 0 \) and 1 otherwise. Then, for all \( P_1, P_2 \in D(X + 1) \), \( \overline{d}(P_1, P_2) \) is defined as the supremum of all values \( \sum_{x \in X + 1} f(x) \cdot (P_1(x) - P_2(x)) \), with \( f: (X + 1, \overline{d}) \to ([0, 1], d_c) \) being an arbitrary nonexpansive function. As we will further discuss in Example 3.3, \( d \) is the Kantorovich pseudometric given by the space \( (X + 1, \overline{d}) \).

We consider a concrete example from [vBW06], illustrated on the left of Figure 1. The behavioral distance of \( u \) and \( z \) is \( d(u, z) = 1 \) and hence \( d(x, y) = c \cdot \varepsilon \).

**Example 2.6 (Metric Transition Systems and Propositional Distances).** We give another example based on the notions of [dAFS09]. A finite set \( \Sigma = \{ r_1, \ldots, r_n \} \) of propositions is given and each proposition \( r \in \Sigma \) is associated with a pseudometric space \( (M_r, d_r) \).

A valuation \( v \) is a function with domain \( \Sigma \) that assigns to each \( r \in \Sigma \) an element of \( M_r \). We denote the set of all valuations by \( \mathcal{U}(\Sigma) \). A metric transition system is a tuple \( M = (S, \tau, \Sigma, \cdot) \) with a set \( S \) of states, a transition relation \( \tau \subseteq S \times S \), a finite set \( \Sigma \) of propositions and a valuation \( [s] \) for each state \( s \in S \). We write \( \tau(s) \) for \( \{ s' \in S \mid (s, s') \in \tau \} \) and require that \( \tau(s) \) is finite.

In [dAFS09] the propositional distance between two valuations is given by \( \overline{d}(u, v) = \max_{r \in \Sigma} d_r(u(r), v(r)) \) for \( u, v \in \mathcal{U}(\Sigma) \). The (undirected) branching distance \( d: S \times S \to \mathbb{R}_0^+ \) is defined as the smallest fixed-point of the following equation, where \( s, t \in S \):

\[
    d(s, t) = \max\{ \overline{d}([s], [t]), \max_{s' \in \tau(s)} \min_{t' \in \tau(t)} d(s', t'), \max_{t' \in \tau(t)} \min_{s' \in \tau(s)} d(s', t') \} \tag{1}
\]

Note that, apart from the first argument, this coincides with the Hausdorff distance.

We consider an example which appears similarly in [dAFS09] (see Figure 1, right) with a single proposition \( r \in \Sigma \), where \( M_r = [0, 1] \) is equipped with the Euclidean distance \( d_c \).

According to (1), \( d(x_1, y_1) \) equals the Hausdorff distance of the reals associated with the sets of successors, which is 0.3 (since this is the maximal distance of any successor to the closest successor in the other set of successors, here: the distance from \( y_3 \) to \( x_3 \)).
In order to model such transition systems as coalgebras we define the following \( n \)-ary auxiliary functor: \( G(X_1, \ldots, X_n) = \{ u : \Sigma \to X_1 + \cdots + X_n \mid u(r_i) \in X_i \} \). Then coalgebras are of the form \( \alpha : S \to G(M_{r_1}, \ldots, M_{r_n}) \times \mathcal{P}_{\text{fin}}(S) \), where \( \mathcal{P}_{\text{fin}} \) is the finite powerset functor and \( \alpha(s) = ([s], \tau(s)) \). As we will see later in Example 6.7, the right-hand side of (1) can be seen as lifting a metric \( d \) on \( X \) to a metric on \( G(M_{r_1}, \ldots, M_{r_n}) \times \mathcal{P}_{\text{fin}}(X) \).

Generalizing from the examples, we now establish a general framework for deriving such behavioral distances. In both cases, the crucial step is to find, for a functor \( F \), a way to lift a pseudometric on \( X \) to a pseudometric on \( FX \). Based on this, one can set up a fixed-point equation and define behavioral distance as its smallest solution. Hence, in the next sections we describe how to lift an endofunctor \( F \) on \( \text{Set} \) to an endofunctor on \( \text{PMet} \).

**Definition 2.7 (Lifting).** Let \( U : \text{PMet} \to \text{Set} \) be the forgetful functor which maps every pseudometric space to its underlying set. A functor \( F : \text{PMet} \to \text{PMet} \) is called a lifting of a functor \( F : \text{Set} \to \text{Set} \) if it satisfies \( U \circ F = F \circ U \).

It is not difficult to prove that such a lifting is always monotone on pseudometrics over a common set, i.e. for any two pseudometrics \( d_1 \leq d_2 \) on the same set \( X \), we also have \( d_1^F \leq d_2^F \) where \( d_i^F \) are the pseudometrics on \( FX \) obtained by applying \( F \) to \( (X, d_i) \) (see Proposition P.2.3). Similarly to predicate lifting of coalgebraic modal logic [Sch08], liftings on \( \text{PMet} \) can be conveniently defined via an evaluation function.

**Definition 2.8 (Evaluation Function & Evaluation Functor).** Let \( F \) be an endofunctor on \( \text{Set} \). An evaluation function for \( F \) is a function \( \text{ev}_F : F[0, \top] \to [0, \top] \). Given such a function, we define the evaluation functor to be the endofunctor \( \tilde{F} \) on \( \text{Set}/[0, \top] \), the slice category over \([0, \top]\), via \( \tilde{F}(g) = ev_F \circ Fg \) for all \( g \in \text{Set}/[0, \top] \). On arrows \( \tilde{F} \) is defined as \( F \).

3. **Lifting Functors to Pseudometric Spaces à la Kantorovich**

Let us now consider an endofunctor \( F \) on \( \text{Set} \) with an evaluation function \( \text{ev}_F \). Given a pseudometric space \((X, d)\), our first approach will be to take the smallest possible pseudometric \( d^F \) on \( FX \) such that, for all nonexpansive functions \( f : (X, d) \xrightarrow{\lambda} ([0, \top], d_e) \), the relation \( Ff : (FX, d^F) \xrightarrow{\lambda} ([0, \top], d_e) \) is nonexpansive again, i.e. we want to ensure that for all \( t_1, t_2 \in FX \) we have \( d_e(Ff(t_1), Ff(t_2)) \leq d^F(t_1, t_2) \). This idea immediately leads us to the next definition.

**Definition 3.1 (Kantorovich Pseudometric & Kantorovich Lifting).** Let \( F : \text{Set} \to \text{Set} \) be a functor with an evaluation function \( \text{ev}_F \). For every pseudometric space \((X, d)\) the Kantorovich pseudometric on \( FX \) is the function \( d^F : FX \times FX \to [0, \top] \), where for all \( t_1, t_2 \in FX \):

\[
\begin{align*}
d^F(t_1, t_2) &:= \sup \{ d_e(\tilde{F}f(t_1), \tilde{F}f(t_2)) \mid f : (X, d) \xrightarrow{\lambda} ([0, \top], d_e) \}. \\
\end{align*}
\]

\(\text{The slice category } \text{Set}/[0, \top] \text{ has as objects all functions } g : X \to [0, \top] \text{ where } X \text{ is an arbitrary set. Given } g \text{ as before and } h : Y \to [0, \top], \text{ an arrow from } g \text{ to } h \text{ is a function } f : X \to Y \text{ satisfying } h \circ f = g.\)
The Kantorovich lifting of the functor \( F \) is the functor \( \mathcal{F} : \text{PMet} \rightarrow \text{PMet} \) defined as \( \mathcal{F}(X, d) = (FX, d^\mathcal{F}) \) and \( \mathcal{F}f = Ff \).

It is easy to show that \( d^\mathcal{F} \) is indeed a pseudometric. Since \( \mathcal{F} \) inherits the preservation of identities and composition of morphisms from \( F \) we can prove that nonexpansive functions are mapped to nonexpansive functions and isometries to isometries.

**Proposition 3.2.** The Kantorovich lifting \( \mathcal{F} \) of a functor \( F \) preserves isometries.

We chose the name Kantorovich because our definition is reminiscent of the Kantorovich pseudometric in probability theory. If we take the proper combination of functor and evaluation function, we can recover that pseudometric (in the discrete case) as the first instance for our framework.

**Example 3.3 (Probability Distribution Functor).** We take \( \mathbb{T} = 1 \) and the probability distribution functor \( \mathcal{D} \) from [Example 2.5]. As evaluation function we take the expected value of the identity on \([0, 1]\), i.e. for any \( P \in \mathcal{D}[0, 1] \) we have \( ev_P(P) = \mathbb{E}_P[\text{id}_{[0,1]}] = \sum_{x \in [0,1]} x \cdot P(x) \) yielding \( \mathcal{D}g(P) = \mathbb{E}_P[g] = \sum_{x \in [0,1]} g(x) \cdot P(x) \) for all \( g : X \rightarrow [0,1] \).

For every pseudometric space \((X,d)\) we obtain the discrete Kantorovich pseudometric \( d^{\mathcal{D}} : \mathcal{D}X \times \mathcal{D}X \rightarrow [0,1] \), defined as, for all \( P, P_1, P_2 \in \mathcal{D}X \), \( d^{\mathcal{D}}(P_1, P_2) = \sup\{\sum_{x \in X} f(x) \cdot (P_1(x) - P_2(x)) \mid f : (X,d) \rightarrow ([0,1],d_c)\} \).

In general Kantorovich liftings do not preserve metrics, as shown by the following example.

**Example 3.4.** Let \( F : \text{Set} \rightarrow \text{Set} \) be given as \( FX = X \times X \) on sets and \( Ff = f \times f \) on functions and take \( \mathbb{T} = \infty \), \( ev_F : F[0,\infty] \rightarrow [0,\infty], ev_F(r_1, r_2) = r_1 + r_2 \). For a metric space \((X,d)\) with \(|X| \geq 2\) let \( t_1 = (x_1, x_2) \in FX \) with \( x_1 \neq x_2 \) and define \( t_2 := (x_2, x_1) \). Clearly \( t_1 \neq t_2 \) but for every nonexpansive function \( f : (X,d) \rightarrow ([0,\infty],d_c) \) we have \( Ff(t_1) = f(x_1) + f(x_2) = f(x_2) + f(x_1) = Ff(t_2) \) and thus \( d^{\mathcal{F}}(t_1, t_2) = 0 \).

**4. Wasserstein Pseudometric and Kantorovich-Rubinstein Duality**

We have seen that our first lifting approach bears close resemblance to the original Kantorovich pseudometric on probability measures. In that context there exists another pseudometric, the Wasserstein pseudometric, which under certain conditions coincides with the Kantorovich pseudometric. We will define a generalized version of the Wasserstein pseudometric and compare it with our generalized Kantorovich pseudometric. To do that we first need to define how we can couple elements of \( FX \).

**Definition 4.1 (Coupling).** Let \( F : \text{Set} \rightarrow \text{Set} \) be a functor and \( n \in \mathbb{N} \). Given a set \( X \) and \( t_i \in FX \) for \( 1 \leq i \leq n \) we call an element \( t \in F(X^n) \) such that \( F \pi_i(t) = t_i \) a coupling of the \( t_i \) (with respect to \( F \)). We write \( \Gamma_F(t_1, t_2, \ldots, t_n) \) for the set of all these couplings.

If \( F \) preserves weak pullbacks, we can define new couplings based on given ones.

**Lemma 4.2 (Gluing Lemma).** Let \( F : \text{Set} \rightarrow \text{Set} \) be a weak pullback preserving functor, \( X \) a set, \( t_1, t_2, t_3 \in FX, t_{12} \in \Gamma_F(t_1, t_2), \text{ and } t_{23} \in \Gamma_F(t_2, t_3) \) be couplings. Then there is a coupling \( t_{123} \in \Gamma_F(t_1, t_2, t_3) \) such that \( F((\pi_1, \pi_2))(t_{123}) = t_{12} \) and \( F((\pi_2, \pi_3))(t_{123}) = t_{23} \).

This lemma already hints at the fact that our new lifting will only work for weak pullback preserving functors, which is a standard requirement in coalgebra. In addition to that we have to impose three extra conditions on the evaluation functions.

**Definition 4.3 (Well-Behaved Evaluation Function).** Let \( ev_F \) be an evaluation function for a functor \( F : \text{Set} \rightarrow \text{Set} \). We call \( ev_F \) well-behaved if it satisfies the following conditions:

1. \( \tilde{F} \) is monotone, i.e., for \( f, g : X \rightarrow [0,\infty] \) with \( f \leq g \), we have \( \tilde{F}f \leq \tilde{F}g \).
2. For each \( t \in F([0, \infty]^2) \) it holds that \( d_e(ev_F(t_1), ev_F(t_2)) \leq \tilde{F}d_e(t) \) for \( t_i := F\pi_i(t) \).

3. \( ev_F^{-1}\{0\} = F_i(F\{0\}) \) where \( i: \{0\} \rightarrow [0, \infty) \) is the inclusion map.

While the first condition of this definition is quite natural, the other two need to be explained. Condition 2 is needed to ensure that \( F\pi_{[0, \infty]} = ev_F: F[0, \infty] \rightarrow [0, \infty] \) is nonexpansive once \( d_e \) is lifted to \( F[0, \infty] \) (cf. the intuition behind the Kantorovich lifting, where we ensure that \( \tilde{F}f \) is nonexpansive whenever \( f \) is nonexpansive). Furthermore Condition 3 intuitively says that exactly the elements of \( F\{0\} \) are mapped to 0 via \( ev_F \). Before we define the Wasserstein pseudometric and the corresponding lifting, we take a look at an example of a functor together with a well-behaved evaluation function.

**Example 4.4 (Finite Powerset Functor).** Let \( \infty = \infty \). We take the finite powerset functor \( \mathcal{P}_{\mathbb{N}} \) with evaluation function \( \max: \mathcal{P}_{\mathbb{N}}([0, \infty]) \rightarrow [0, \infty] \) with \( \max 0 = 0 \). This evaluation function is well-behaved whereas \( \min: \mathcal{P}_{\mathbb{N}}([0, \infty]) \rightarrow [0, \infty] \) is not well-behaved.

**Definition 4.5 (Wasserstein Pseudometric & Wasserstein Lifting).** Let \( F: \text{Set} \rightarrow \text{Set} \) be a weak-pullback preserving functor with well-behaved evaluation function \( ev_F \). For every pseudometric space \( (X, d) \) the Wasserstein pseudometric on \( FX \) is the function \( d^{IF}: FX \times FX \rightarrow [0, \infty] \) given by, for all \( t_1, t_2 \in FX \),

\[
d^{IF}(t_1, t_2) := \inf \{ \tilde{F}d(t) \mid t \in \Gamma_F(t_1, t_2) \}.
\]

We define the Wasserstein lifting of \( F \) to be the functor \( \overline{F}: \text{PMet} \rightarrow \text{PMet}, \overline{F}(X, d) = (FX, d^{IF}), \overline{F}f = Ff \).

This time it is not straightforward to prove that \( d^{IF} \) is a pseudometric, so we explicitly provide the following result. Its proof relies on all properties of well-behavedness of \( ev_F \) and uses Lemma 4.2 which explains why we need a weak pullback preserving functor.

**Proposition 4.6.** The Wasserstein pseudometric is a well-defined pseudometric on \( FX \).

In contrast to that, it is not hard to show functoriality of \( \overline{F} \) and, as in the Kantorovich case, the lifted functor preserves isometries.

**Proposition 4.7.** The Wasserstein lifting \( \overline{F} \) of a functor \( F \) preserves isometries.

In contrast to our previous approach, metrics are preserved in certain situations.

**Proposition 4.8 (Preservation of Metrics).** Let \( (X, d) \) be a metric space and \( F \) be a functor. If the infimum in \( \text{Definition 4.5} \) is a minimum for all \( t_1, t_2 \in FX \) where \( d^{IF}(t_1, t_2) = 0 \) then \( d^{IF} \) is a metric, thus also \( \overline{F}(X, d) = (FX, d^{IF}) \) is a metric space.

Please note that a similar restriction for the Kantorovich lifting (i.e. requiring that the supremum in \( \text{Definition 3.1} \) is a maximum) does not yield preservation of metrics: In Example 3.4 the supremum is always a maximum but we do not get a metric.

Let us now compare both lifting approaches. Whenever it is defined, the Wasserstein pseudometric is an upper bound for the Kantorovich pseudometric.

**Proposition 4.9.** Let \( F \) be a weak pullback preserving functor with well-behaved evaluation function. Then for all pseudometric spaces \( (X, d) \) it holds that \( d^{IF} \leq d^{IF} \).

In general this inequality may be strict in general, as the following example shows.

**Example 4.10.** The functor of [Example 3.4] preserves weak pullbacks and the evaluation function is well-behaved. We continue the example and take \( t_1 = (x_1, x_2), t_2 = (x_2, x_1) \). The unique coupling \( t \in \Gamma_F(t_1, t_2) \) is \( t = (x_1, x_2, x_2, x_1) \). Using that \( d \) is a metric we conclude that \( d^{IF}(t_1, t_2) = \tilde{F}d(t) = d(x_1, x_2) + d(x_2, x_1) = 2d(x_1, x_2) > 0 = d^{IF}(t_1, t_2). \)

When the inequality can be replaced by an equality we will in the following say that the Kantorovich-Rubinstein duality holds. In this case we obtain a canonical notion of distance.
on $FX$, given a pseudometric space $(X, d)$. To calculate the distance of $t_1, t_2 \in FX$ it is then enough to find a nonexpansive function $f : (X, d) \xrightarrow{1} ([0, \top], d_e)$ and a coupling $t \in \Gamma_F(t_1, t_2)$ such that $d_e(\tilde{F} f(t), \tilde{F} f(t_2)) = F d_e(t)$. Then, due to Proposition 4.9, this value equals $d^{1F}(t_1, t_2) = d^{1F}(t_1, t_2)$. We will now take a look at some examples where the duality holds.

**Example 4.11 (Identity Functor).** Take $F = \text{Id}$ with the identity evaluation map $ev_{\text{Id}} = \text{id}_{[0, \top]}$. For any $t_1, t_2 \in X$, $t := (t_1, t_2)$ is the unique coupling of $t_1, t_2$. Hence, $d^{1F}(t_1, t_2) = d(t_1, t_2)$. With the function $d(t_1, :) : (X, d) \xrightarrow{1} ([0, \top], d_e)$ we obtain reflexivity because we have $d(t_1, t_2) = d_e(d(t_1, t_1), d(t_1, t_2)) \leq d^{1F}(t_1, t_2) \leq d^{1F}(t_1, t_2) = d(t_1, t_2)$ and thus equality. Similarly, if we define $ev_{\text{Id}}(r) = c \cdot r$ for $r \in [0, \top]$, $0 < c \leq 1$, the Kantorovich and Wasserstein liftings coincide and we obtain the discounted distance $d^{1F}(t_1, t_2) = d^{1F}(t_1, t_2) = c \cdot d(t_1, t_2)$.

**Example 4.12 (Probability Distribution Functor).** The functor $D$ of Example 3.3 preserves weak pullbacks [Sok11] and the evaluation function $ev_D$ is well-behaved. We recover the usual Wasserstein pseudometric $d^{1D}(P1, P2) = \inf \{ \sum x_1 x_2 \in X d(x_1, x_2) \cdot P(x_1, x_2) \mid P \in \Gamma_D(P1, P2) \}$ and the Kantorovich-Rubinstein duality [Vil09] from transportation theory for the discrete case.

**Example 4.13 (Finite Powerset Functor & Hausdorff Pseudometric).** Let $\top = \infty$, $F = \mathcal{P}^m_X$ with evaluation map $ev_{\mathcal{P}^m_X} : \mathcal{P}^m_X([0, \infty]) \to [0, \infty]$, $ev_{\mathcal{P}^m_X}(R) = \max R$ with $\max \emptyset = 0$ (as in Example 4.4). In this setting we obtain duality and both pseudometrics are equal to the Hausdorff pseudometric $d_H$ on $\mathcal{P}^m_X(X)$ which is defined as, for all $X_1, X_2 \in \mathcal{P}^m_X(X)$,

$$d_H(X_1, X_2) = \max \left\{ \max_{x_1 \in X_1} \min_{x_2 \in X_2} d(x_1, x_2), \max_{x_2 \in X_2} \min_{x_1 \in X_1} d(x_1, x_2) \right\}.$$ 

Note that the distance is $\infty$, if either $X_1$ or $X_2$ is empty.

It is also illustrative to consider the countable powerset functor. Using the supremum as evaluation function, one obtains again the Hausdorff pseudometric (with supremum/infimum replacing maximum/minimum). However, in this case the Hausdorff distance of different countable sets might be $0$, even if we lift a metric. This shows that in general the Wasserstein lifting does not preserve metrics but we need an extra condition, e.g. the one in Proposition 4.8.

## 5. Lifting Multifunctors

Our two approaches can easily be generalized\(^3\) to lift a multifunctor $F : \text{Set}^n \to \text{Set}$ (for $n \in \mathbb{N}$) in a similar sense as given by Definition 2.7 to a multifunctor $\tilde{F} : \text{PMet}^n \to \text{Set}$. The only difference is that we start with $n$ pseudometric spaces instead of one. Now we need an evaluation function $ev_F : F([0, \top], \ldots, [0, \top]) \to [0, \top]$ which we call well-behaved if it satisfies conditions similar to Definition 4.3 and which gives rise to an evaluation multifunctor $\tilde{F} : (\text{Set} / [0, \top])^n \to \text{Set} / [0, \top]$. Given $t_1, t_2 \in F(X_1, \ldots, X_n)$ we write again $\Gamma_F(t_1, t_2) \subseteq F(X_1^2, \ldots, X_n^2)$ for the set of couplings which is defined analogously to Definition 4.1. For pseudometrics $d_i : X_i^2 \to [0, \top]$, we can then define the Kantorovich/Wasserstein pseudometric $d_i^{1F}(t_1, t_2) := \sup \{ d_e(\tilde{F} f_1, \ldots, \tilde{F} f_n)(t_1), \tilde{F} f_1, \ldots, \tilde{F} f_n)(t_2) \mid f_i : (X_i, d_i) \xrightarrow{1} ([0, \top], d_e) \}$ and $d_i^{1F}(t_1, t_2) := \inf \{ F(d_1, \ldots, d_n)(t) \mid t \in \Gamma_F(t_1, t_2) \}$. This setting grants us access to new examples such as the product and the coproduct bifunctors.

\(^3\)The details are spelled out in Appendix P.5 here we provide just the basic ideas.
Example 5.1 (Product Bifunctor). For the product bifunctor $F: \text{Set}^2 \to \text{Set}$ where $F(X_1, X_2) = X_1 \times X_2$ and $F(f_1, f_2) = f_1 \times f_2$ we consider the evaluation function $\max: [0, \top]^2 \to [0, \top]$ and for fixed parameters $c_1, c_2 \in (0, 1)$ and $p \in \mathbb{N}$ the function $\rho: [0, \top]^2 \to [0, \top]$, $\rho(x_1, x_2) = (c_1 x_1^p + c_2 x_2^p)^{1/p}$. These functions are well-behaved, the Kantorovich-Rubinstein duality holds and the supremum [infimum] of the Kantorovich [Wasserstein] pseudometrics is always a maximum [minimum]. For the first function we obtain the $\infty$-product pseudometric $d_\infty((x_1, x_2), (y_1, y_2)) = \max(d_1(x_1, y_1), d_2(x_2, y_2))$ and for the other function the weighted $p$-product pseudometric $d_p((x_1, x_2), (y_1, y_2)) = (c_1 d_1^p(x_1, y_1) + c_2 d_2^p(x_2, y_2))^{1/p}$.

Note that the pseudometric space $(X_1 \times X_2, d_\infty)$ is the usual binary (category theoretic) product of $(X_1, d_1)$ and $(X_2, d_2)$. Similarly, we can also obtain the binary coproduct.

Example 5.2 (Coproduct Bifunctor). For the coproduct bifunctor $F: \text{Set}^2 \to \text{Set}$, where $F(X_1, X_2) = X_1 + X_2 = X_1 \cup X_2 \cup \{\top\}$ and $F(f_1, f_2) = f_1 + f_2$ we take the evaluation function $\ev_F: [0, \top] + [0, \top] \to [0, \top]$, $\ev_F(x, i) = x$. This function is well-behaved, the Kantorovich-Rubinstein duality holds and the supremum of the Kantorovich pseudometric is always a maximum whereas the infimum of the Wasserstein pseudometric is a minimum if and only if any coupling of the two elements exists. We obtain the coproduct pseudometric $d_+$ where $d_+((x_1, i_1), (x_2, i_2))$ is equal to $d_i(x_1, x_2)$ if $i_1 = i_2 = i$ and equal to $\top$ otherwise.

6. Final Coalgebra & Coalgebraic Behavioral Pseudometrics

In this section we assume an arbitrary lifting $\mathcal{F}: \text{PMet} \to \text{PMet}$ of an endofunctor $F$ on Set. For any pseudometric space $(X, d)$ we write $d^\mathcal{F}$ for the pseudometric obtained by applying $\mathcal{F}$ to $(X, d)$. Such a lifting can be obtained as described earlier, but also by taking a lifted multifunctor and fixing all parameters apart from one, or by the composition of such functors. The following result ensures that if $\kappa: \Omega \to F\Omega$ is a final $F$-coalgebra, then there is also a final $\mathcal{F}$-coalgebra which is constructed by simply enriching $\Omega$ with a pseudometric $d_0$.

Theorem 6.1. Let $\mathcal{F}: \text{PMet} \to \text{PMet}$ be a lifting of a functor $F: \text{Set} \to \text{Set}$ which has a final coalgebra $\kappa: \Omega \to F\Omega$. For every ordinal $i$ we construct a pseudometric $d_i: \Omega \times \Omega \to [0, \top]$ as follows: $d_0 := 0$ is the zero pseudometric, $d_{i+1} := d_i^\mathcal{F} \circ (\kappa \times \kappa)$ for all ordinals $i$ and $d_i = \sup_{i < j} d_i$ for all limit ordinals $j$. This sequence converges for some ordinal $\theta$, i.e $d_\theta = d_\theta^\mathcal{F} \circ (\kappa \times \kappa)$. Moreover $\kappa: (\Omega, d_\theta) \to (F\Omega, d_\theta^\mathcal{F})$ is the final $\mathcal{F}$-coalgebra. We noted that for any set $X$, the set of pseudometrics over $X$, with pointwise order, is a complete lattice. Moreover the lifting $\mathcal{F}$ induces a monotone function $\mathcal{F}$ which maps any pseudometric $d$ on $X$ to $d^\mathcal{F}$ on $FX$. If, additionally, such function is $\omega$-continuous, i.e., it preserves the supremum of $\omega$-chains, the construction in Theorem 6.1 will converge in at most $\omega$ steps, i.e., $d_0 = d_\omega$. We show in Proposition P.6.1 that the liftings induced by the finite powerset functor and the probability distribution functor with finite support are $\omega$-continuous. The arguments used for convergence here suggests a connection with the work in [vBHMW07], which provides fixed-point results for metric functors which are not locally contractive.

Beyond equivalences of states, in PMet we can measure the distance of behaviors in the final coalgebra. More precisely, the behavioral distance of two states $x, y \in X$ of some coalgebra $\alpha: X \to FX$ is defined via the pseudometric $\beta d(x, y) = d_\alpha([x], [y])$. Such distances can be computed analogously to $d_\theta$ above, replacing $\kappa: \Omega \to F\Omega$ by $\alpha$. This way we do not need to explore the entire final coalgebra (which might be too large) but can restrict to the interesting part.
Theorem 6.2. Let the chain of the $d_i$ converge in $\theta$ steps and $F$ preserve isometries. Let furthermore $\alpha: X \to FX$ be an arbitrary coalgebra. For all ordinals $i$ we define a pseudometric $e_i: X \times X \to [0, \infty]$ as follows: $e_0$ is the zero pseudometric, $e_{i+1} = e_i \circ (\alpha \times \alpha)$ for all ordinals $i$ and $e_j = \sup_{i<j} e_i$ for all limit ordinals $j$. Then we reach a fixed point after $\zeta \leq \theta$ steps, i.e. $e_\zeta = e_\zeta \circ (\alpha \times \alpha)$, such that $bd = e_\zeta$.

Since $d_0$ is a pseudometric, we have that if $[x]_i = [y]$ then $bd(x, y) = 0$. The other direction does not hold in general: for this $d_0$ has to be a proper metric. [Theorem 6.5]

To this aim, we proceed by recalling the final coalgebra construction via the final chain which was first presented in the dual setting (free/initial algebra).

Definition 6.3 (Final Coalgebra Construction [Adâ74]). Let $C$ be a category with terminal object $1$ and limits of ordinal-indexed cochains. For any functor $F: C \to C$ the final chain consists of objects $W_i$ for all ordinals $i$ and connection morphisms $p_{i,j}: W_j \to W_i$ for all ordinals $i \leq j$. The objects are defined as $W_0 := 1$, $W_{i+1} := FW_i$ for all ordinals $i$, and $W_j := \lim_{i<j} W_i$ for all limit ordinals $j$. The morphisms are determined by $p_{0,i} := !: W_i \to 1$, $p_{i,i} = \text{id}_{W_i}$ for all ordinals $i$, $p_{i+1,j+1} := FP_{i,j}$ for all ordinals $i < j$ and if $j$ is a limit ordinal the $p_{i,j}$ are the morphisms of the limit cone. They satisfy $p_{i,k} = p_{i,j} \circ p_{j,k}$ for all ordinals $i \leq j \leq k$. We say that the chain converges in $\lambda$ steps if $p_{\lambda,\lambda+1}: W_{\lambda+1} \to W_\lambda$ is an iso.

This construction does not necessarily converge, but if it does, we get a final coalgebra.

Proposition 6.4 ([Adâ74]). Let $C$ be a category with terminal object $1$ and limits of ordinal-indexed cochains. If the final chain of a functor $F: C \to C$ converges in $\lambda$ steps then $p_{\lambda,\lambda+1}^{-1}: W_\lambda \to FW_\lambda$ is the final coalgebra.

We now show under which circumstances $d_0$ is a metric and how our construction relates to the construction of the final coalgebra.

Theorem 6.5. Let $\overline{F}: PMet \to PMet$ be a lifting of a functor $F: \text{Set} \to \text{Set}$ which has a final coalgebra $\kappa: \Omega \to F\Omega$. Assume that $\overline{F}$ preserves isometries and metrics, that the final chain for $F$ converges and the chain of the $d_i$ converges in $\theta$ steps. Then $d_0$ is a metric, i.e. for $x, y \in \Omega$ we have $d_0(x, y) = 0 \iff x = y$.

We will now get back to the examples studied at the beginning of the paper [Example 2.5] and [Example 2.6] and discuss in which sense they are instances of our framework.

Example 6.6 (Probabilistic Transition System, revisited). To model the behavioral distance from [Example 2.5] in our framework, we set $\mathcal{T} = 1$ and proceed to lift the following three functors: we first consider the identity functor $\text{id}$ with evaluation map $ev_{\text{id}}: [0, 1] \to [0, 1]$, $ev_{\text{id}}(z) = c \cdot z$ in order to integrate the discount [Example 4.11]. Then, we take the coproduct with the singleton metric space [Example 5.2]. The combination of the two functors yields the discrete version of the refusal functor of [vBW06], namely $\overline{R}(X, d) = (X + 1, d)$ where $\overline{d}$ is taken from [Example 2.5]. Finally, we lift the probability distribution functor $\mathcal{D}$ to obtain $\overline{\mathcal{D}}$ [Example 3.3]. All functors satisfy the Kantorovich-Rubinstein duality and preserve metrics.

It is readily seen that $\overline{D}(\overline{R}(X, d)) = (\mathcal{D}(X + 1), \overline{d})$, where $\overline{d}$ is defined as in [Example 2.5]. Then, the least solution of $d(x, y) = \overline{d}(\alpha(x), \alpha(y))$ can be computed as in [Theorem 6.2].

Example 6.7 (Metric Transition Systems, revisited). To obtain propositional distances in metric transition systems we set $\mathcal{T} = \infty$. We also define, for the auxiliary functor $G$, an evaluation function $ev_G: G([0, \infty], \ldots, [0, \infty]) \to [0, \infty]$ with $ev_G(u) = \max_{r \in \mathcal{G}} u(r)$. Let $\overline{\mathcal{G}}$ be the corresponding lifted functor. It can be shown, similarly to [Example 5.1], that the Kantorovich-Rubinstein duality holds and metrics are preserved. We instantiate the given pseudometric spaces $(M_r, d_r)$ as parameters and obtain the functor...
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\[ F(X, d) = G((M_{r_1}, d_{r_1}), \ldots, (M_{r_n}, d_{r_n})) × F_{PM}(X, d) \] (for the lifting of the powerset functor see [Example 4.13]). Then, via [Theorem 6.2] we obtain exactly the least solution of (1) in Example 2.6.

7. Related and Future Work

The ideas for our framework are heavily influenced by work on quantitative variants of (bisimulation) equivalence of probabilistic systems. In that context at first Giacalone et al. [GJS90] observed that probabilistic bisimulation [LS89] is too strong and therefore introduced a metric based on the notion of \( \varepsilon \)-bisimulations.

Using a logical characterization of bisimulation for labelled Markov processes (LMP) [DEP02], Desharnais et al. defined a family of metrics between these LMPs [DGJP04] via functional expressions: if evaluated on a state of an LMP, such a functional expression measures the extent to which a formula is satisfied in that state. A different, coalgebraic approach, which inspired ours, is used by van Breugel et al. [vBW06]. As presented in more detail in the examples above, they define a pseudometric on probabilistic systems via the Kantorovich pseudometric for probability measures. Moreover, they show in [vBW05] that this metric is related to the logical pseudometric by Desharnais et al.

Our framework provides a toolbox to determine behavioral distances for different types of transition systems modeled as coalgebras. Moreover, the liftings introduced in this paper pave the way to extend several coalgebraic methods to reason about quantitative properties of systems. For instance the bisimulation proof principle, which allows to check behavioral equivalence, assumes a specific meaning in PMet: every coalgebra \( \alpha: (X, d) \to F(X, d) \) coinductively proves that the behavioral distance \( bd \) of the underlying \( F \)-coalgebra on Set is smaller or equal than \( d \). Indeed, since \( [\cdot] \) is nonexpansive, \( d ≥ d_0([\cdot], [\cdot]) = bd \). This principle, which has already been stated in different formulations (see e.g. [DCPP06, DJGP02, vBSW08]), can now be enhanced via up-to techniques by exploiting the liftings introduced in this paper and the coalgebraic understanding of such enhancements given in [BPPR14].

Since up-to techniques can exponentially improve algorithms for equivalence-checking, we hope that they can also optimize some of the algorithms for computing (or approximating) behavioral distances [vBW06, vBSW08, CvBW12, BBLM13]. At this point, it is worth recalling that the Kantorovich-Rubinstein duality has been exploited in [vBW06] for defining one of these algorithms: the characterization given by the Wasserstein metric allows to reduce to linear programming.

Another line of research potentially stemming from our work concerns the so-called abstract GSOS [Khi11] which provides abstract coalgebraic conditions ensuring compositionality of behavioral equivalence (with respect to some operators). By taking our lifting to PMet, abstract GSOS guarantees the nonexpansiveness of behavioral distance, a property that has captured the interest of several researchers [DGJP04, GT13]. The main technical challenge would be to lift to PMet not only functors, but also distributive laws. Lifting of distributive laws would also be needed for defining linear behavioral distances, exploiting the coalgebraic account of trace semantics based on Kleisli categories [HJS07].

We finally observe that the chains of Theorems 6.1 and 6.2 can be understood in terms of fibrations along the lines of [HCKJ13]. A detailed comparison with [HCKJ13] can be found in Appendix A.

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A. A Fibrational Perspective

A recent work [HCKJ13] studies final chains in fibrations. This is interesting for our work, since the forgetful functor $U : \text{PMet} \to \text{Set}$ is a poset fibration: the fibre above a set $X$ (denoted by $\text{PMet}_X$) is the poset of all pseudometric spaces over $X$ with the order $d_1 \sqsubseteq d_2$ iff $d_2 \leq d_1$; for a function $f : X \to Y$, the reindexing functor $f^* : \text{PMet}_Y \to \text{PMet}_X$ maps a space $(Y, d)$ into $(X, d^*)$ where $d^*(x_1, x_2) = d(f(x_1), f(x_2))$, namely the cartesian lifting $\tilde{f}_d : (X, d^*) \to (Y, d)$ is an isometry. By virtue of Lemma 3.5 in [HCKJ13], one can readily check that this fibration has fiberwise limits. Indeed, $\text{PMet}$ is complete and $U$ preserves limits since it has a left adjoint (mapping a set $X$ into the space $(X, d^\top)$ where $d^\top(x_1, x_2) = \top$ for all $x_1 \neq x_2$).

In this setting, a pair of functors $F : \text{Set} \to \text{Set}$, $F : \text{PMet} \to \text{PMet}$ is a map of fibrations iff $\overline{F}$ is a lifting of $F$ and, additionally, $\overline{F}$ preserves isometries, a property enjoyed by both Kantorovich and Wasserstein liftings (Propositions 3.2 and 4.7). By Proposition 4.1 in [HCKJ13], $U$ lifts to a fibration $\overline{U} : \text{Coalg}(\overline{F}) \to \text{Coalg}(F)$ from $F$-coalgebras to $F$-coalgebras and, again by virtue of Lemma 3.5 in [HCKJ13], the final object of $\text{Coalg}(\overline{F})$ is the final object of $\text{Coalg}(F)$, the fibre above the final $F$-coalgebra $\kappa : \Omega \to F\Omega$.

For any coalgebra $\alpha : X \to FX$, the fibre above $\alpha$ is isomorphic to the category $\text{Coalg}(\alpha^* \circ \overline{F})$ of coalgebras for the endofunctor $\alpha^* \circ \overline{F} : \text{PMet}_X \to \text{PMet}_X$ (Proposition 4.2 in [HCKJ13]). Therefore the final $\overline{F}$-coalgebra is just the final coalgebra for the endofunctor $\kappa^* \circ \overline{F} : \text{PMet}_\Omega \to \text{PMet}_\Omega$. By unfolding the definitions, one can see that the sequence of $d_i$ in Theorem 6.1 is indeed the final chain for $\kappa^* \circ \overline{F}$. Similarly, the sequence of $e_i$ in Theorem 6.2 is just the final chain for $\alpha^* \circ \overline{F}$.

Theorem 3.7 in [HCKJ13] provides sufficient conditions for a fibration to ensure the convergence of these chains in $\omega$ steps. Unfortunately, $U$ does not fulfill these conditions, as it is not “well-founded”. 

13 Behavioral Metrics via Functor Lifting
P. Proofs

Here we provide proofs for the soundness of our definitions (where needed), the stated theorems, propositions, lemmas, examples and also for all claims made in the in-between texts. If a theorem environment starts with ■ (orange square) it has been stated in the main text and is repeated here for convenience of the reader (using the numbering from the main text). Otherwise it is a new statement which clarifies/justifies claims made in the main text and its number starts with P.

P.2. Preliminaries, Notation & Evaluation Functions

In the following lemma we rephrase the well-known fact that for \( a, b, c \in [0, \infty) \) we have \( |a - b| \leq c \iff a - b \leq c \land b - a \leq c \) to include the cases where \( a, b, c \) might be \( \infty \).

Lemma P.2.1. For \( a, b, c \in [0, \infty) \) we have \( d_e(a, b) \leq c \iff (a \leq b + c) \land (b \leq a + c) \).

Proof. This equivalence is obvious for \( a, b, c \in [0, \infty) \). In the cases \( a, b \in [0, \infty] \), \( c = \infty \) or \( a = b = \infty \), \( c \in [0, \infty] \) both sides of the equivalence are true whereas for the cases \( a = \infty \), \( b, c \in [0, \infty) \) or \( b = \infty \), \( a, c \in [0, \infty) \) both sides are false.

Lemma 2.3. A symmetric function \( d : X^2 \to [0, \top] \) with \( d(x, x) = 0 \) for all \( x \in X \) satisfies the triangle inequality iff for all \( x \in X \) the function \( d(x, \_): X \to [0, \top] \) is nonexpansive.

Proof. We show both implications for all \( x, y, z \in X \).

\( \Rightarrow \) Using the triangle equation and symmetry we know that \( d(x, y) \leq d(x, z) + d(y, z) \) and \( d(x, z) \leq d(x, y) + d(y, z) \). With Lemma P.2.1 we conclude that \( d_e(d(x, y), d(x, z)) \leq d(y, z) \).

\( \Leftarrow \) Using \( d(x, x) = 0 \), the triangle inequality for \( d_e \) and nonexpansiveness of \( d(x, \_\_\_) \) we get \( d(x, z) = d_e(d(x, x), d(x, z)) \leq d_e(d(x, x), d(x, y)) + d_e(d(x, y), d(x, z)) \leq d(x, y) + d(y, z) \).

We will now show the claimed properties of \( \text{PMet} \) by providing the following, general result which encompasses the existence of all small products (including the empty product, i.e. the terminal object) and all small coproducts (including the empty coproduct, i.e. the initial object).

Proposition P.2.2. \( \text{PMet} \) is a bicomplete category, i.e. it is complete and cocomplete.

Proof. Let \( D : I \to \text{PMet} \) be a small diagram, \( (X_i, d_i) := Di \) for each object \( i \in I \). Obviously \( UD : I \to \text{Set} \) is also a small diagram. We show completeness and cocompleteness separately.

Completeness: Let \( (f_i : X \to X_i)_{i \in I} \) be the limit cone to \( UD \). We define the function \( d := \sup_{i \in I} d_i \circ (f_i \times f_i) : X^2 \to [0, \top] \) and claim that this is a pseudometric on \( X \). Since all \( d_i \) are pseudometrics, we immediately can derive that \( d(x, x) = 0 \) and \( d(x, y) = d(y, x) \) for all \( x, y \in X \). Moreover, for all \( x, y, z \in X \):

\[
\begin{align*}
    d(x, y) + d(y, z) &= \sup_{i \in I} d_i(f_i(x), f_i(y)) + \sup_{i \in I} d_i(f_i(y), f_i(z)) \\
    &\geq \sup_{i \in I} \left( d_i(f_i(x), f_i(y)) + d_i(f_i(y), f_i(z)) \right) \\
    &\geq \sup_{i \in I} d_i(f_i(x), f_i(z)) = d(x, z).
\end{align*}
\]

With this pseudometric all \( f_j \) are nonexpansive functions \( (X, d) \xrightarrow{j} (X_j, d_j) \). Indeed we have for all \( j \in I \) and all \( x, y \in X \)

\[
d_j(f_j(x), f_j(y)) \leq \sup_{i \in I} d_i(f_i(x), f_i(y)) = d(x, y).
\]
Moreover, if \( (f_i': (X', d') \overset{\rightarrow}{\rightarrow} (X_i, d_i))_{i \in I} \) is a cone to \( D \), \( (f'_i': X' \to X_i)_{i \in I} \) is a cone to \( UD \) and hence there is a unique function \( g: X' \to X \) in \( \text{Set} \) satisfying \( f_i \circ g = f'_i \) for all \( i \in I \). We finish our proof by showing that this is a nonexpansive function \( (X', d') \overset{\rightarrow}{\rightarrow} (X, d) \). By nonexpansiveness of the \( f'_i \) we have for all \( i \in I \) and all \( x, y \in X' \) that \( d_i(f'_i(x), f'_i(y)) \leq d'(x, y) \) and thus also
\[
d(g(x), g(y)) = \sup_{i \in I} d_i(f_i(g(x)), f_i(g(y))) = \sup_{i \in I} d_i(f'_i(x), f'_i(y)) \leq \sup_{i \in I} d'(x, y) = d'(x, y).
\]

We conclude that \( (f_i: (X, d) \overset{\rightarrow}{\rightarrow} (X_i, d_i))_{i \in I} \) is a limit cone to \( D \).

**Cocompleteness:** Let \( (f_i: X_i \to X)_{i \in I} \) be the colimit co-cone from \( UD \) and \( M_X \) be the set of all pseudometrics \( d_X: X^2 \to [0, \top] \) on \( X \) such that the \( f_i \) are nonexpansive functions \( (X_i, d_i) \overset{\rightarrow}{\rightarrow} (X, d_X) \). We define \( d := \sup_{x \in M_X} d_X \) and claim that this is a pseudometric. Since all \( d_X \) are pseudometrics, we can derive immediately that \( d(x, x) = 0 \) and \( d(x, y) = d(y, x) \) for all \( x, y \in X \). Moreover, for all \( x, y, z \in X \) we have:
\[
d(x, y) + d(y, z) = \sup_{d_X \in M_X} d_X(x, y) + \sup_{d_X \in M_X} d_X(y, z) \\
\geq \sup_{d_X \in M_X} \left( d_X(x, y) + d_X(y, z) \right) \geq \sup_{d_X \in M_X} d_X(x, z) = d(x, z).
\]

With this pseudometric all \( f_j \) are nonexpansive functions \( (X_j, d_j) \overset{\rightarrow}{\rightarrow} (X, d) \). Indeed we have for all \( j \in I \) and all \( x, y \in X_j \)
\[
d(f_j(x), f_j(y)) = \sup_{d_X \in M_X} d_X(f_j(x), f_j(y)) \leq \sup_{d_X \in M_X} d_j(x, y) = d_j(x, y).
\]

Moreover, if \( (f_i': (X_i, d_i) \overset{\rightarrow}{\rightarrow} (X', d'))_{i \in I} \) is a co-cone from \( D \), \( (f'_i: X_i \to X')_{i \in I} \) is a co-cone from \( UD \) and hence there is a unique function \( g: X' \to X' \) in \( \text{Set} \) satisfying \( g \circ f_i = f'_i \) for all \( i \in I \). We finish our proof by showing that this is a nonexpansive function \( (X, d) \overset{\rightarrow}{\rightarrow} (X', d') \). Let \( d_g := d' \circ (g \times g): X^2 \to [0, \top] \), then it is easy to see that \( d_g \) is a pseudometric on \( X \). Moreover, for all \( i \in I \) and all \( x, y \in X_i \) we have
\[
d_g(f_i(x), f_i(y)) = d'(g(x), g(y)) = d'(f'_i(x), f'_i(y)) \leq d_i(x, y)
\]

due to nonexpansiveness of \( f'_i: (X_i, d_i) \overset{\rightarrow}{\rightarrow} (X', d') \). Thus all \( f_i \) are nonexpansive if seen as functions \( (X_i, d_i) \overset{\rightarrow}{\rightarrow} (X, d_g) \) and we have \( d_g \in M_X \). Using this we observe that for all \( x, y \in X \) we have
\[
d'(g(x), g(y)) = d_g(x, y) \leq \sup_{d_X \in M_X} d_X(x, y) = d(x, y)
\]

which shows that \( g \) is a nonexpansive function \( (X', d') \overset{\rightarrow}{\rightarrow} (X, d) \). We conclude that \( (f_i: (X, d_i) \overset{\rightarrow}{\rightarrow} (X, d))_{i \in I} \) is a colimit co-cone from \( D \).

**Proposition P.2.3.** Let \( \overline{F}: \text{PMet} \to \text{PMet} \) be a lifting of \( F: \text{Set} \to \text{Set} \). For a pseudometric space \( (X, d) \) let \( d^F \) denote the pseudometric on \( FX \) which we obtain by applying \( \overline{F} \) to \( (X, d) \). Then \( \overline{F} \) is monotone on pseudometrics in the following sense: If we have two pseudometrics \( d_1 \leq d_2 \) on a common set \( X \) we also have \( d_1^F \leq d_2^F \).

**Proof.** Since \( d_1 \leq d_2 \) the identity function on the set \( X \) can be regarded as a nonexpansive function \( f: (X, d_2) \overset{\rightarrow}{\rightarrow} (X, d_1) \) because we have for all \( x, y \in X \) that \( d_1(f(x), f(y)) = d_2(x, y) = d_2(x, y) \). By functoriality of \( \overline{F} \) also \( \overline{F} f: (FX, d_2^F) \overset{\rightarrow}{\rightarrow} (FX, d_1^F) \) is nonexpansive, i.e. for all \( t_1, t_2 \in FX \) we have \( d_1^F(FU f(t_1), FU f(t_2)) \leq d_2^F(t_1, t_2) \) and moreover \( d_1^F(FU f(t_1), FU f(t_2)) = d_1^F(FD_{X_1}(t_1), FD_{X_2}(t_2)) = d_1^F(id_{FX}(t_1), id_{FX}(t_2)) = d_1^F(t_1, t_2) \) and thus \( d_1^F \leq d_2^F \).
P.3. Lifting Functors to Pseudometric Spaces à la Kantorovich

We show that the following definition is sound and has the claimed properties.

\begin{definition}[Kantorovich Pseudometric & Kantorovich Lifting] Let \( F : \text{Set} \rightarrow \text{Set} \) be a functor with an evaluation function \( \text{ev}_F \). For every pseudometric space \((X, d)\) the Kantorovich pseudometric on \( FX \) is the function \( d^F : FX \times FX \rightarrow [0, \top] \), where for all \( t_1, t_2 \in FX \):

\[
d^F(t_1, t_2) := \sup \{ d_e(\tilde{F}f(t_1), \tilde{F}f(t_2)) \mid f : (X, d) \downarrow ([0, \top], d_e) \}.
\]

The Kantorovich lifting of the functor \( F \) is the functor \( \overline{F} : \text{PMet} \rightarrow \text{PMet} \) defined as \( \overline{F}(X, d) = (FX, d^F) \) and \( \overline{F}f = Ff \).

\begin{proposition} The Kantorovich pseudometric is a pseudometric on \( FX \).
\end{proposition}

\begin{proof} Reflexivity and symmetry are an immediate consequence of the fact that \( d_e \) is a (pseudo)metric. In order to show the triangle inequality let \( t_1, t_2, t_3 \in FX \). Then we have

\[
d^F(t_1, t_2) + d^F(t_1, t_2) = \sup_{f : (X, d) \downarrow ([0, \top], d_e)} d_e(\tilde{F}f(t_1), \tilde{F}f(t_2)) + \sup_{f : (X, d) \downarrow ([0, \top], d_e)} d_e(\tilde{F}f(t_2), \tilde{F}f(t_3)) \geq \sup_{f : (X, d) \downarrow ([0, \top], d_e)} (d_e(\tilde{F}f(t_1), \tilde{F}f(t_2)) + d_e(\tilde{F}f(t_2), \tilde{F}f(t_3))) = d^F(t_1, t_3)
\]

where the second inequality follows from the fact that \( d_e \) is a (pseudo)metric.
\end{proof}

\begin{proposition} The Kantorovich lifting \( \overline{F} \) is a functor on pseudometric spaces.
\end{proposition}

\begin{proof} \( \overline{F} \) preserves identities and composition of arrows because \( F \) does. Moreover, it preserves nonexpansive functions: Let \( f : (X, d_X) \downarrow (Y, d_Y) \) be nonexpansive and \( t_1, t_2 \in FX \), then

\[
d_Y^{\overline{F}}(Ff(t_1), Ff(t_2)) = \sup_{g : (Y, d_Y) \downarrow ([0, \top], d_e)} d_e(\tilde{F}(g \circ f)(t_1), \tilde{F}(g \circ f)(t_2)) \leq \sup_{h : (X, d_X) \downarrow ([0, \top], d_e)} d_e(\tilde{F}(h)(t_1), \tilde{F}(h)(t_2)) = d_X^{\overline{F}}(t_1, t_2)
\]

due to the fact that, of course, the composition \( (g \circ f) : (X, d_X) \downarrow ([0, \top], d_e) \) is nonexpansive.
\end{proof}

\begin{proposition} The Kantorovich lifting \( \overline{F} \) of a functor \( F \) preserves isometries.
\end{proposition}

\begin{proof} Let \( f : (X, d_X) \downarrow (Y, d_Y) \) be an isometry, i.e. a function such that \( d_Y \circ (f \times f) = d_X \). Since the Kantorovich lifting \( \overline{F} \) is a functor on pseudometric spaces, we already know that \( \overline{F}f \) is nonexpansive, i.e. we know that \( d_Y^{\overline{F}} \circ (Ff \times Ff) \leq d_X^{\overline{F}} \) thus we only have to show the opposite inequality. We will do that by constructing for every nonexpansive function \( g : (X, d_X) \downarrow ([0, \top], d_e) \) a nonexpansive function \( h : (Y, d_Y) \downarrow ([0, \top], d_e) \) such that
\end{proof}
We construct $h$ via Functor Lifting.

For the case $(x,g) \in X \times Y$ must be an equality. For elements of Lemma P.2.1, yields $d_y$ then there would be $h$ using nonexpansiveness of $y$ because $y \in Fh(x)$. Therefore we have equality $d_y(\tilde{F}h(Ff(x)), \tilde{F}h(Ff(x))) = d_y(\tilde{F}g(t_1), \tilde{F}g(t_2))$, because then we have

$$d_y^{\tilde{F}} \circ (Ff \times Ff)(t_1, t_2) = \sup \{d_y(\tilde{F}h(Ff)(t_1), \tilde{F}h(Ff)(t_2)) \mid h : (Y, d_Y) \to ([0, \tau], d_e)\} \geq \sup \{d_y(\tilde{F}g(t_1), \tilde{F}g(t_2)) \mid g : (X, d_X) \to ([0, \tau], d_e)\} = d_y^{\tilde{F}}(t_1, t_2).$$

We construct $h$ as follows: For each $y \in f[X]$ we choose a fixed $x_y \in f^{-1}\{y\}$ and define

$$h(y) := \begin{cases} g(x_y), & y \in f[X] \\ \inf_{y' \in f[X]} h(y') + d_Y(y', y), & \text{else.} \end{cases}$$

Let us first verify that this definition is independent of our choice of the $x_y$. Given $x_1, x_2 \in X$ with $f(x_1) = f(x_2) = y$ we get $d_X(x_1, x_2) = d_Y(f(x_1), f(x_2)) = d_Y(y, y) = 0$ using the fact that $f$ is an isometry. Thus by nonexpansiveness of $g$ we necessarily have $d_e(g(x_1), g(x_2)) \leq d_X(x_1, x_2) = 0$ and because $d_e$ is a metric this yields $g(x_1) = g(x_2)$. With the same reasoning we obtain $(h \circ f)(x) = h(f(x)) = g(f(x)) = g(x)$ for all $x \in X$ and therefore the desired equality

$$d_e(\tilde{F}h(Ff(t_1)), \tilde{F}h(Ff(t_2))) = d_e(\tilde{F}(h \circ f)(t_1), \tilde{F}(f \circ f)(t_2)) = d_e(\tilde{F}g(t_1), \tilde{F}g(t_2)).$$

It remains to show that $h$ is nonexpansive, which we will do by distinguishing three cases.

1. Let $y_1, y_2 \in f[X]$, then there are $x_1, x_2 \in X$ with $f(x_i) = y_i$. We calculate

$$d_e(h(y_1), h(y_2)) = d_e(g(x_1), g(x_2)) \leq d_X(x_1, x_2) = d_Y(f(x_1), f(x_2)) = d_Y(y_1, y_2)$$

using nonexpansiveness of $g$ and the fact that $f$ is an isometry.

2. Let without loss of generality $y_1 \in f[X]$ (so there is $x_1 \in X$ with $f(x_1) = y_1$) and $y_2 \in Y \setminus f[X]$, then by Lemma P.2.1 we have the equivalence:

$$d_e(h(y_1), h(y_2)) \leq d_Y(y_1, y_2) \iff h(y_1) \leq h(y_2) + d_Y(y_1, y_2) \land h(y_2) \leq h(y_1) + d_Y(y_1, y_2)$$

We will show these inequalities separately. The second one is easy:

$$h(y_1) + d_Y(y_1, y_2) \geq \inf_{y' \in f[X]} (h(y') + d_Y(y', y_2)) = h(y_2)$$

because $y_1 \in f[X]$. For the first one we calculate

$$h(y_2) + d_Y(y_1, y_2) = \inf_{y' \in f[X]} \left( h(y') + d_Y(y', y_2) \right) + d_Y(y_1, y_2)$$

$$= \inf_{y' \in f[X]} \left( h(y') + d_Y(y', y_2) + d_Y(y_1, y_1) \right)$$

$$\geq \inf_{y' \in f[X]} \left( h(y') + d_Y(y', y_1) \right) = h(y_1)$$

using symmetry and the triangle inequality for $d_Y$. Observe that the last equality is not true by definition because $y_1 \in f[X]$. Certainly we have $h(y_1) = h(y_1) + 0 = h(y_1) + d_Y(y_1, y_1) \geq \inf_{y' \in f[X]} (h(y') + d_Y(y', y_1))$. Assume this inequality was strict, then there would be $y' \in f[X]$ such that $h(y_1) > h(y') + d_Y(y', y_1)$ which, using Lemma P.2.1 yields $d_e(h(y_1), h(y')) > d_Y(y_1, y')$. This contradicts nonexpansiveness of $h$ for elements of $f[X]$. Thus our assumption must have been wrong and the inequality must be an equality.

For the case $y_1 \in Y \setminus f[X], y_2 \in f[X]$ we can simply use symmetry of $d_e$ and $d_Y$.\footnote{For the case $y_1 \in Y \setminus f[X], y_2 \in f[X]$ we can simply use symmetry of $d_e$ and $d_Y$.}
3. Let \( y_1, y_2 \in Y \setminus f[X] \). As in the previous case we use Lemma P.2.1, however, this time the two inequalities can be shown using exactly the same reasoning. Hence we only show the first one (which in turn is similar as in the prove above):

\[
    h(y_2) + d_Y(y_1, y_2) = \inf_{y' \in f[X]} \left( h(y') + d_Y(y', y_2) \right) + d_Y(y_1, y_2)
\]

\[
    = \inf_{y' \in f[X]} \left( h(y') + d_Y(y', y_2) + d_Y(y_2, y_1) \right)
\]

\[
    \geq \inf_{y' \in f[X]} \left( h(y') + d_Y(y', y_1) \right) = h(y_1).
\]

The main difference to the proof above is that the last equality now holds by definition because \( y_1 \notin f[X] \).

\[ \blacksquare \]

**Example 3.3** (Probability Distribution Functor). We take \( \mathbb{T} = 1 \) and the probability distribution functor \( D \) from Example 2.5. As evaluation function we take the expected value of the identity on \([0, 1]\), i.e. for any \( P \in D[0,1] \) we have \( ev(P) = \mathbb{E}_P[\text{id}_{[0,1]}] = \sum_{x \in [0,1]} x \cdot P(x) \) yielding \( \hat{D}g(P) = \mathbb{E}_P[g] = \sum_{x \in [0,1]} g(x) \cdot P(x) \) for all \( g: X \to [0,1] \).

For every pseudometric space \((X,d)\) we obtain the discrete Kantorovich pseudometric \( d^{1D}: DX \times DX \to [0,1] \), defined as, for all \( P_1, P_2 \in DX \), \( d^{1D}(P_1, P_2) = \sup\{\sum_{x \in X} f(x) \cdot (P_1(x) - P_2(x)) \mid f: (X,d) \xrightarrow{\downarrow} ([0,1],d_c)\} \).

**Proof.** We calculate \( \hat{D} \). Let \( g: X \to [0, \mathbb{T}] \) be a function and \( P \in DX \), then

\[
    \hat{D}g(P) = ev_D \circ Dg(P) = \mathbb{E}_{Dg(P)}[\text{id}_{[0,T]}] = \sum_{r \in [0,T]} r \cdot Dg(P)(r) = \sum_{r \in [0,T]} \left( r \cdot \sum_{x \in g^{-1}\{r\}} P(x) \right)
\]

\[
    = \sum_{r \in [0,T]} \sum_{x \in g^{-1}\{r\}} r \cdot P(x) = \sum_{r \in [0,T]} \sum_{x \in g^{-1}\{r\}} g(x) \cdot P(x) = \sum_{x \in X} g(x) \cdot P(x).
\]

With this calculation at hand we see, for \( f: (X,d) \xrightarrow{\downarrow} ([0,1],d_c) \) and \( P_1, P_2 \in DX \) that \( \hat{D}f(P_1) - \hat{D}f(P_2) = \sum_{x \in X} f(x) \cdot (P_1(x) - P_2(x)) \).

\[ \blacksquare \]

**P.4. Wasserstein Pseudometric and Kantorovich-Rubinstein Duality**

**Lemma 4.2** (Gluing Lemma). Let \( F: \text{Set} \to \text{Set} \) be a weak pullback preserving functor, \( X \) a set, \( t_1, t_2, t_3 \in FX \), \( t_{12} \in \Gamma_F(t_1, t_2) \), and \( t_{23} \in \Gamma_F(t_2, t_3) \) be couplings. Then there is a coupling \( t_{123} \in \Gamma_F(t_1, t_2, t_3) \) such that \( F(\langle \pi_1^3, \pi_2^3 \rangle)(t_{123}) = t_{12} \) and \( F(\langle \pi_2^3, \pi_3^3 \rangle)(t_{123}) = t_{23} \).

**Proof.** Consider the diagrams below. The left diagram is a pullback square: Given any set \( P \) along with functions \( p_1, p_2: P \to X \times X \) satisfying the condition \( \pi_2^3 \circ p_1 = \pi_2^3 \circ p_2 \) the unique mediating arrow \( u: P \to X \times X \times X \) is given by \( u = \langle \pi_1^3 \circ p_1, \pi_2^3 \circ p_1, \pi_2^3 \circ p_2 \rangle = \langle \pi_1^2 \circ p_1, \pi_1^2 \circ p_2 \rangle \).

Now consider the right diagram. Since \( F \) preserves weak pullbacks, the square in the middle of the right diagram is a weak pullback.
By the (weak) universality of the pullback and the fact that $F\pi_2^3(t_{12}) = t_2 = F\pi_2^3(t_{23})$ we obtain an element $t_{123} \in F(X \times X \times X)$ which satisfies the two equations of the lemma and moreover $F\pi_1^2(t_{23}) = F(\pi_1^2 \circ (\pi_1^3, \pi_2^3))(t_{23}) = F\pi_1^2 \circ F((\pi_1^3, \pi_2^3))(t_{23}) = F\pi_1^2(t_{12}) = t_1$ and analogously $F\pi_1^2(t_{23}) = t_2$, $F\pi_2^3(t_{23}) = t_3$ yielding $t_{123} \in \Gamma_F(t_1, t_2, t_3)$.

We recall the definition of well-behavedness for an evaluation function.

**Definition 4.3** (Well-Behaved Evaluation Function). Let $ev_F$ be an evaluation function for a functor $F \colon \text{Set} \to \text{Set}$. We call $ev_F$ well-behaved if it satisfies the following conditions:

1. $\tilde{F}$ is monotone, i.e., for $f, g \colon X \to [0, \top]$ with $f \leq g$, we have $\tilde{F}f \leq \tilde{F}g$.
2. For each $t \in F([0, \top]^2)$ it holds that $d_e(ev_F(t_1), ev_F(t_2)) \leq \tilde{F}d_e(t)$ for $t_i := F\pi_i(t)$.
3. $ev_F^{-1}(\{0\}) = F_i[\{0\}]$ where $i \colon \{0\} \to [0, \top]$ is the inclusion map.

Please note that Condition 2 above can be rephrased as $d_e(\tilde{F}\pi_1(t), \tilde{F}\pi_2(t)) \leq \tilde{F}d_e(t)$. In the following proofs we will often use this characterization.

**Example 4.4** (Finite Powerset Functor). Let $\top = \infty$. We take the finite powerset functor $\mathcal{P}_{fin}$ with evaluation function $\max \colon \mathcal{P}_{fin}([0, \infty]) \to [0, \infty]$ with $\max \emptyset = 0$. This evaluation function is well-behaved whereas $\min \colon \mathcal{P}_{fin}([0, \infty]) \to [0, \infty]$ is not well-behaved.

**Proof.** We first show that $\max \colon \mathcal{P}_{fin}([0, \infty]) \to [0, \infty]$ is well-behaved:

1. Let $f, g \colon X \to [0, \top]$ with $f \leq g$. Let $Y \in \mathcal{P}_{fin}X$, i.e., $Y \subseteq X$ and $Y$ finite. Then we have $\mathcal{P}_{fin}f(Y) = ev_{\mathcal{P}_{fin}}(\mathcal{P}_{fin}f(Y)) = \max f[Y] \leq \max g[Y] = ev_{\mathcal{P}_{fin}}(\mathcal{P}_{fin}g(Y)) = \mathcal{P}_{fin}g(Y)$.
2. For $T \subseteq [0, \top]^2$ we have $d_e[T] \geq d_e(\max \pi_1[T], \max \pi_2[T])$. This can be shown as follows: For $T = \emptyset$ this is obviously true. Otherwise let $m_i = \max \pi_i[T]$ and let $n_1, n_2 \in [0, \infty]$ such that $(m_1, n_2), (n_1, m_2) \in T$. It must hold that $m_1 \geq n_1$, $m_2 \geq n_2$. We distinguish the following cases: if $m_1 \geq m_2$, then $m_1 \geq m_2 \geq n_2$, so a simple case distinction $d_e(m_1, n_2) \geq d_e(m_1, m_2)$ and hence there is a pair in $T$ with distance larger than or equal to $d_e(m_1, m_2)$. If $m_2 \geq m_1$ we can conclude with an analogous argument.
3. We have $\mathcal{P}_{fin}i[\mathcal{P}_{fin}\{0\}] = \mathcal{P}_{fin}i[\{0\}] = \{i(\emptyset), i(\{0\})\} = \emptyset, \{0\} = \max^{-1}[\{0\}]$.

The function $\min \colon \mathcal{P}_{fin}([0, \infty]) \to [0, \infty]$ is not well-behaved. It does not satisfy Condition 2, nor Condition 3: $d_e(T) \geq d_e(\min \pi_1(T), \min \pi_2(T))$ fails for $T = \{(0, 1), (1, 1)\}$ and the set $\{0, 1\}$ is contained in the kernel.

**Proposition 4.6**. The Wasserstein pseudometric is a well-defined pseudometric on $FX$.

**Proof.** We check all three properties of a pseudometric separately.

**Reflexivity:** Let $t_1 \in FX$. To show reflexivity we will construct a coupling $t \in \Gamma_F(t_1, t_1)$ such that $\tilde{F}d(t) = 0$. In order to do that, let $\delta \colon X \to X^2, \delta(x) = (x, x)$ and define $t := F\delta(t_1)$. Then $F\pi_1(t) = F(\pi_1 \circ \delta)(t_1) = F(id_X)(t_1) = t_1$ and thus $t \in \Gamma_F(t_1, t_1)$. Since $d$ is reflexive, $d \circ \delta \colon X \to [0, \top]$ is the constant zero function. Let $i \colon \{0\} \to [0, \top], i(0) = 0$ and for any set $X$ let $!X \colon X \to \{0\}, !X(x) = 0$. Then also $i !X \colon X \to [0, \top]$ is the constant zero function and thus $d \circ \delta = i !X$. Using this we conclude that $\tilde{F}d(t) = \tilde{F}d(F\delta(t_1)) = \tilde{F}(d \circ \delta)(t_1) = \tilde{F}(i !X) = ev_F(F(i !F_X(t_1))) = 0$ where the last equality follows from the fact that $F !X(t_1) \in F\{0\}$ and $ev_F$ is well-behaved (Condition 3 of Definition 4.3).

\[^2\text{Explicitly: Consider } \{t_2\} \text{ with functions } p_1, p_2 \colon \{t_2\} \to F(X \times X) \text{ where } p_1(t_2) = t_{12} \text{ and } p_2(t_2) = t_{23}, \text{ then by the weak pullback property there is a (not necessarily unique) function } a \colon \{t_2\} \to F(X \times X \times X) \text{ satisfying } F((\pi_1^2, \pi_2^2) \circ u = p_1 \text{ and } F((\pi_1^2, \pi_2^2) \circ u = p_2. \text{ We simply define } t_{123} := u(t_2).\]
Symmetry: Let $t_1, t_2 \in FX, t_{12} \in \Gamma_F(t_1, t_2)$ and $\sigma := (\pi_2, \pi_1)$ be the swap map on $X \times X$, i.e. $\sigma: X \times X \to X \times X, \sigma(x_1, x_2) = (x_2, x_1)$. We define $t_{21} := F\sigma(t_{12}) \in FX$ and observe that it satisfies $F\pi_1(t_{21}) = F\pi_1(F\sigma(t_{12})) = F(\pi_1 \circ \sigma)(t_{12}) = F\pi_2(t_{12}) = t_2$ and analogously $F\pi_2(t_{21}) = t_1$, thus $t_{21} \in \Gamma_F(t_2, t_1)$. Moreover, due to symmetry of $d$ (i.e. $d \circ \sigma = d$), we obtain $\tilde{F}d(t_{21}) = ev_F(Fd(t_{21})) = ev_F(Fd(F\sigma(t_{12}))) = ev_F(F(d \circ \sigma)(t_{12})) = ev_F(Fd(t_{12})) = \tilde{F}d(t_{12})$ which yields the desired symmetry.

Triangle inequality: Using Lemma 2.3 we will show that for every $t_1 \in FX$ the function $d^{1F}_{t_1}(x, \_): (FX, d^{1F}) \to ([0, \infty], d_e)$ is nonexpansive, i.e. for all $t_2, t_3 \in FX$ we have $d_e(d^{1F}_{t_1}(t_2), d^{1F}_{t_1}(t_3)) \leq d^{1F}(t_2, t_3)$.

We start by observing that (also using Lemma 2.3) for all $x \in X$ the function $d(x, \_)$ is nonexpansive, i.e. $d_e \circ (d \times d) \circ \langle \langle \pi_1, \pi_2 \rangle, \langle \pi_1, \pi_3 \rangle \rangle \leq d \circ \langle \pi_2, \pi_3 \rangle$. Monotonicity of $\tilde{F}$ yields $\tilde{F}(d_e \circ (d \times d) \circ \langle \langle \pi_1, \pi_2 \rangle, \langle \pi_1, \pi_3 \rangle \rangle) \leq \tilde{F}(d \circ \langle \pi_2, \pi_3 \rangle)$. Let $t_1, t_2, t_3 \in FX$ and assume there are $t_{12} \in \Gamma_F(t_1, t_2)$ and $t_{23} \in \Gamma_F(t_2, t_3)$. By the Gluing Lemma we get a $t_{13} \in \Gamma_F(t_1, t_3)$ and observe that $t_{13} := F(\langle \langle \pi_1, \pi_3 \rangle \rangle)_{t_{123}}$ satisfies $t_{13} \in \Gamma_F(t_1, t_3)$. Plugging in $t_{123}$ in the inequality from above yields $\tilde{F}d_e(F(d \times d(t_{12}, t_{13})) \leq \tilde{F}d(t_{23})$. Using well-behavedness (Condition 2) of $ev_F$ on the left hand side we obtain the following, intermediary result:

$$\forall t_{12} \in \Gamma_F(t_1, t_2) \forall t_{23} \in \Gamma_F(t_2, t_3) \exists t_{13} \in \Gamma_F(t_1, t_3) : d_e(\tilde{F}d(t_{12}), \tilde{F}d(t_{13})) \leq \tilde{F}d(t_{23}).$$

Let $d_{ij} := d^{1F}(t_i, t_j)$. As explained in the beginning, we want to use Lemma 2.3 and thus we have to show that $d_e(d_{12}, d_{13}) \leq d_{23}$. This is obviously true for $d_{12} = d_{13}$. Without loss of generality we assume $d_{ij} < d_{13}$ and claim that for all $\varepsilon > 0$ we can find a coupling $t_{12} \in \Gamma_F(t_1, t_2)$ such that for all couplings $t_{13} \in \Gamma_F(t_1, t_3)$ we have

$$d_e(d_{12}, d_{13}) \leq \varepsilon + d_e(\tilde{F}d(t_{12}), \tilde{F}d(t_{13})).$$

Since the Wasserstein distance is defined as an infimum, we have $d_{13} \leq \tilde{F}d(t_{13})$ and for every $\varepsilon > 0$ we can pick a coupling $t_{12} \in \Gamma_F(t_1, t_2)$, such that $\tilde{F}d(t_{12}) \leq d_{12} + \varepsilon$. If $d_{13} = \infty$ we have $d_e(d_{12}, d_{13}) = \infty$ but also $\tilde{F}d(t_{13}) = \infty$ and $\tilde{F}d(t_{12}) \leq d_{12} + \varepsilon < \infty + \varepsilon = \infty$ and therefore $d_e(\tilde{F}d(t_{12}), \tilde{F}d(t_{13})) = \infty$ and thus (3) is valid. For $d_{13} < \infty$ we have

$$d_e(d_{12}, d_{13}) = d_{13} - d_{12} \leq \tilde{F}d(t_{13}) - (\tilde{F}d(t_{12}) - \varepsilon) = \varepsilon + (\tilde{F}d(t_{13}) - (\tilde{F}d(t_{12})) \leq \varepsilon + |\tilde{F}d(t_{13}) - \tilde{F}d(t_{12})| \leq \varepsilon + d_e(\tilde{F}d(t_{12}), \tilde{F}d(t_{13}))$$

where the last inequality is due to the fact that $\tilde{F}d(t_{12}) < \infty$. Hence we have established our claimed validity of (3). Using this, (2) and the fact that $- \varepsilon$ as above – given $\varepsilon > 0$ we have a coupling $t_{23}$ such that $\tilde{F}d(t_{23}) \leq d_{23} + \varepsilon$ we obtain

$$d_e(d_{12}, d_{13}) \leq \varepsilon + d_e(\tilde{F}d(t_{12}), \tilde{F}d(t_{13})) \leq \varepsilon + d_e(\tilde{F}d(t_{23})) \leq 2\varepsilon + d_{23}$$

which also proves $d_e(d_{12}, d_{13}) \leq d_{23}$. Indeed if $d_e(d_{12}, d_{13}) > d_{23}$ then we would have $d_e(d_{12}, d_{13}) = d_{23} + \varepsilon'$ and we just take $\varepsilon < \varepsilon'/2$ which yields the contradiction $d_e(d_{12}, d_{13}) \leq 2\varepsilon + d_{23} < \varepsilon' + d_{23} = d_e(d_{12}, d_{13})$.

So far we have established the triangle inequality for cases where there are couplings $t_{12} \in \Gamma_F(t_1, t_2)$ and $t_{23} \in \Gamma_F(t_2, t_3)$. If both do not exist we have for sure $d_e(d_{12}, d_{13}) \leq \infty = d_{23}$. Finally we observe that due to the Gluing Lemma it cannot be the case that just one out of three couplings does not exist as we could construct the third from the other two.

□

Proposition P.4.1. The Wasserstein lifting $\mathcal{F}$ is a functor on pseudometric spaces.
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Proof. \( \mathcal{F} \) preserves identities and composition of arrows because \( F \) does. Moreover, it preserves nonexpansive functions: Let \( f : (X, d_X) \xrightarrow{\sim} (Y, d_Y) \) be nonexpansive and \( t_1, t_2 \in FX \). Every \( t \in \Gamma_F(t_1, t_2) \) satisfies \( Ff(t) = Ff(F\pi_t(i)) = F(f \circ \pi_t(i)) = F(\pi_t \circ (f \times f))(t) = F\pi_t(F(x \times f)(t)) \). Hence we can calculate

\[
\begin{align*}
d^{\mathcal{F}}_X(t_1, t_2) &= \inf\{\widetilde{Fd}_X(t) \mid t \in \Gamma_F(t_1, t_2)\} \\
&\geq \inf\{\widetilde{Fd}_X(t) \mid t \in F(X \times X), \ F\pi_t(F(x \times f)(t)) = Ff(t)\} \\
&\geq \inf\{\widetilde{Fd}_Y(F(x \times f)(t)) \mid t \in F(X \times X), \ F\pi_t(F(x \times f)(t)) = Ff(t)\} \\
&\geq \inf\{\widetilde{Fd}_Y(t') \mid t' \in \Gamma_F(Ff(t_1), Ff(t_2))\} = d^{\mathcal{F}}_Y(Ff(t_1), Ff(t_2)).
\end{align*}
\]

In this calculation the inequality (4) is due to our initial observation. (5) holds because \( F \) preserves nonexpansive functions: Let \( \pi \) as desired. In this calculation the inequality is due to the fact that \( \tilde{F}d_X \geq \tilde{F}(d_Y \circ (f \times f)) = \tilde{F}d_Y \circ F(f \times f) \). The last inequality, (6), is due to the fact that there might be more couplings \( t' \) than those obtained via \( F(f \times f) \).

\[\blacksquare\]

**Proposition 4.7.** The Wasserstein lifting \( \mathcal{F} \) of a functor \( F \) preserves isometries.

Proof. Let \( f : (X, d_X) \xrightarrow{\sim} (Y, d_Y) \) be an isometry. Since \( \mathcal{F} \) is a functor, \( \tilde{F}f \) is nonexpansive, i.e. for all \( t_1, t_2 \in FX \) we have \( d^{\mathcal{F}}_X(t_1, t_2) \leq \inf_{t \in \Gamma_F(Ff(t_1), Ff(t_2))} \inf_{t' \in \Gamma_F(Ff(t_1), Ff(t_2))} \widetilde{Fd}_Y(\gamma(t)) \).

If \( \Gamma_F(Ff(t_1), Ff(t_2)) = \emptyset \) we have \( d^{\mathcal{F}}_X(Ff(t_1), Ff(t_2)) = \top \geq d^{\mathcal{F}}_X(t_1, t_2) \). Otherwise we will construct for each coupling \( t \in \Gamma_F(Ff(t_1), Ff(t_2)) \) a coupling \( \gamma(t) \in \Gamma_F(t_1, t_2) \) such that \( \tilde{F}d_X(\gamma(t)) = \tilde{F}d_Y(t) \) because then we have

\[
d^{\mathcal{F}}_X(t_1, t_2) = \inf_{t' \in \Gamma_F(t_1, t_2)} \tilde{F}d_X(t') \leq \inf_{t \in \Gamma_F(Ff(t_1), Ff(t_2))} \inf_{t' \in \Gamma_F(Ff(t_1), Ff(t_2))} \tilde{F}d_Y(\gamma(t)) = \inf_{t \in \Gamma_F(Ff(t_1), Ff(t_2))} \tilde{F}d_Y(t) = d^{\mathcal{F}}_Y(Ff(t_1), Ff(t_2)) \]

as desired. In this calculation the inequality is due to the fact that \( \gamma(t) \in \Gamma_F(t_1, t_2) \).

In order to construct \( \gamma : \Gamma_F(Ff(t_1), Ff(t_2)) \rightarrow \Gamma_F(t_1, t_2) \), we consider the diagram below.

\[
\begin{array}{ccc}
X & \xrightarrow{f \times \text{id}_X} & X \times X \\
\pi_1 \downarrow & & \downarrow \pi_2 \\
X \times Y & \xrightarrow{f \times \text{id}_Y} & Y \times Y \\
\pi_1 \downarrow & & \downarrow \pi_2 \\
Y & \xrightarrow{f} & X
\end{array}
\]

This diagram consists of pullbacks: it is easy to check that the diagram commutes. The unique mediating arrows are constructed as follows.

- For the lower left part let \( P \) be a set with \( p_1 : P \rightarrow X, \ p_2 : P \rightarrow Y \times Y \) such that \( f \circ p_1 = \pi_1 \circ p_2 \), then define \( u : P \rightarrow X \times Y \) as \( u := \langle p_1, p_2 \rangle \).
- Analogously, for the lower right part let \( P \) be a set with \( p_1 : P \rightarrow Y \times Y, \ p_2 : P \rightarrow X \) such that \( \pi_2 \circ p_1 = f \circ p_2 \), then define \( u : P \rightarrow X \times Y \) as \( u := \langle \pi_1 \circ p_1, p_2 \rangle \).

\[\text{Condition 1 of Definition 2.8} \quad \text{both } d_X \text{ and } d_Y \circ (f \times f) \text{ are functions with signature } X \times X \rightarrow [0, \top]\]
Finally, for the upper part let $P$ be a set with $p_1 : P \to X \times Y$, $p_2 : P \to Y \times X$ such that $(f \times \text{id}_Y) \circ p_1 = (\text{id}_X \times f) \circ p_2$, then define $u : P \to X \times X$ as $u := (\pi_1 \circ p_1, \pi_2 \circ p_2)$. We apply the weak pullback preserving functor $F$ to the diagram and obtain the following diagram which hence consists of three weak pullbacks.

Given a coupling $t \in \Gamma_F(F f(t_1), F f(t_2)) \subseteq F(Y \times Y)$ we know $F \pi_1(t) = F f(t_1) \in FY$.

Since the lower left square in the diagram is a weak pullback, we obtain an element $s_1 \in F(X \times Y)$ with $F \pi_1(s_1) = t_1$ and $F(f \times \text{id}_Y)(s_1) = t$. Similarly, from the lower right square, we obtain $s_2 \in F(Y \times X)$ with $F \pi_2(s_2) = t_2$ and $F(\text{id}_Y \times f)(s_2) = t$. Again by the weak pullback property we obtain our $\gamma(t) \in F(X \times X)$ with $F(\pi_1 \times f)(\gamma(t)) = s_1$, $F(f \times \pi_2)(\gamma(t)) = s_2$.

We convince ourselves that $\gamma(t)$ is indeed a coupling of $t_1$ and $t_2$: We have $F \pi_1(\gamma(t)) = F(\pi_1 \circ (f \times \text{id}_f))(\gamma(t)) = F \pi_1 \circ F(f \times \text{id}_f)(\gamma(t)) = F \pi_1(s_1) = t_1$ and analogously $F \pi_2(\gamma(t)) = F(\pi_2 \circ (\text{id}_X \times f))(\gamma(t)) = F \pi_2 \circ F(f \times \text{id}_X)(\gamma(t)) = F \pi_2(s_2) = t_2$.

Moreover, we have $F(f \times f)(\gamma(t)) = F((f \times \text{id}_f) \circ (\text{id}_X \times f))(\gamma(t)) = F(f \times \text{id}_f)(s_1) = t$ and thus $\tilde{F}d_Y(t) = \tilde{F}d_Y(F(f \times f)(\gamma(t))) = \tilde{F}(d_Y \circ (f \times f))(t) = \tilde{F}(d_X(t))$ as desired. Note that the last equality is due to the fact that $f$ is an isometry.

\begin{proposition} [Preservation of Metrics] \label{prop:preservation_of_metrics}
Let $(X, d)$ be a metric space and $F$ be a functor. If the infimum in Definition 4.5 is a minimum for all $t_1, t_2 \in FX$ where $d^{1F}(t_1, t_2) = 0$ then $d^{1F}$ is a metric, thus also $F(X, d) = (FX, d^{1F})$ is a metric space.
\end{proposition}

\begin{proof}
Let $(X, d)$ be a metric space and $t_1, t_2 \in FX$ with $d^{1F}(t_1, t_2) = 0$. We have to show that $t_1 = t_2$. Since $d$ is a metric its kernel is the set $\Delta_X = \{(x, x) \mid x \in X\}$. Hence the square on the left below is a pullback and adding the projections yields $\pi_1 \circ e = \pi_2 \circ e$ where $e : \Delta_X \to X \times X$ is the inclusion. Furthermore, due to Condition 3 of Definition 4.3 the square on the right is a weak pullback.

Since $F$ weakly preserves pullbacks, applying it to the first diagram yields a weak pullback. By combining this diagram with the right diagram from above we obtain the diagram below where the outer rectangle is again a weak pullback.

\begin{align*}
\begin{array}{c}
\Delta_X \\ \xymatrix{ & \{0\} \\
X \ar[r]^{\pi_1} \ar[u]_{e} & X \times X & \{0\} \\
X \ar[r]_{\pi_2} & \{0\} & \{0\} \\
& \{0\} \\
\end{array}
\end{align*}

\begin{align*}
\begin{array}{c}
F \{0\} \\
F[0, \top] \ar[r]^{ev_F} & [0, \top] \\
\end{array}
\end{align*}

\text{Since $F$ weakly preserves pullbacks, applying it to the first diagram yields a weak pullback. By combining this diagram with the right diagram from above we obtain the diagram below where the outer rectangle is again a weak pullback.}

\text{\footnote{\text{see the proof of Lemma 4.2 for the explicit constructions}}}
Let \( t \in F(X \times X) \) be the coupling that witnesses \( d^F(t_1, t_2) \), i.e., \( d^F(t_1, t_2) = \tilde{F}d(t) = 0 \).
Due to our assumption (infimum = minimum) such a coupling must exist if \( d^F(t_1, t_2) \neq \top \).
Now, since we have a weak pullback, we observe that there exists \( t' \in F \Delta_X \) with \( F \pi(t') = t \).
(Since \( F \pi \) is an embedding, \( t' \) and \( t \) actually coincide.) This implies that \( t_1 = F \pi_1(t) = F \pi_1(F \pi(t')) = F \pi_2(F \pi(t')) = F \pi_2(t) = t_2 \).

**Proposition P.4.2.** Let \( F : \text{Set} \to \text{Set} \) be a functor with a well-behaved evaluation function \( ev_F \) and \( (X, d) \) be a pseudometric space. For all \( t_1, t_2 \in FX \), all couplings \( t \in \Gamma_F(t_1, t_2) \) and all nonexpansive functions \( f : (X, d) \to ([0, \top], d_c) \) we have \( d_c(F\pi_1(t), F\pi_2(t)) \leq \tilde{F}d(t) \).

**Proof.** We have that \( d \geq d_c \circ (f \times f) \) since \( f \) is nonexpansive. Now, due to Condition 1 of Definition 2.8 we obtain \( \tilde{F}d \geq F(d_c \circ (f \times f)) = \tilde{F}d_c \circ F(f \times f) \).
Furthermore:
\[
\begin{align*}
d_c(\tilde{F}f(t_1), \tilde{F}f(t_2)) &= d_c(\tilde{F}(F\pi_1(t)), \tilde{F}(F\pi_2(t))) = d_c(\tilde{F}(f \circ \pi_1)(t), \tilde{F}(f \circ \pi_2)(t)) \\
&= d_c(\tilde{F}(\pi_1 \circ (f \times f))(t), \tilde{F}(\pi_2 \circ (f \times f))(t)) \\
&\leq \tilde{F}d_c(F(f \times f)(t)) = \tilde{F}(d_c \circ (f \times f))(t) \leq \tilde{F}d(t)
\end{align*}
\]
where the first inequality is due to Condition 2, the second due to the above observation.

**Proposition 4.9.** Let \( F \) be a weak pullback preserving functor with well-behaved evaluation function. Then for all pseudometric spaces \( (X, d) \) it holds that \( d^F \leq d^F \).

**Proof.** This is an immediate corollary of Proposition P.4.2.

**Example 4.10.** The functor of Example 3.3 preserves weak pullbacks and the evaluation function is well-behaved. We continue the example and take \( t_1 = (x_1, x_2), t_2 = (x_2, x_1) \).
The unique coupling \( t \in \Gamma_F(t_1, t_2) \) is \( t = (x_1, x_2, x_2, x_1) \). Using that \( d \) is a metric we conclude that \( d^F(t_1, t_2) = \tilde{F}d(t) = d(x_1, x_2) + d(x_2, x_1) = 2d(x_1, x_2) > 0 = d^F(t_1, t_2) \).

**Proof.** We prove that the evaluation function is well-behaved. Let \( f, g : X \to [0, \infty] \) be given with \( f \leq g \), then for all \( t = (a, b) \in FX \) we have \( \tilde{F}f(t) = f(a) + f(b) \leq g(a) + g(b) = \tilde{F}g(t) \).
For \( t = (r_1, r_2, r_3, r_4) \in F([0, \top]^2) \), then \( \tilde{F} \pi_1(t) = (r_1, r_3), \tilde{F} \pi_2(t) = (r_2, r_4) \) and thus
\[
d_c(r_1 + r_3, r_2 + r_4) = |r_1 + r_3 - r_2 - r_4| = |r_1 - r_2 + r_3 - r_4| \leq |r_1 - r_2| + |r_3 - r_4| = d_c(r_1, r_2) + d_c(r_3, r_4).
\]
Finally, \( ev_F^{-1}[\{0\}] = \{(0, 0)\} = (\times i)(\{0\} \times \{0\}) = F_1[F\{0\}] \).

**Example 4.12.** The functor \( D \) of Example 3.3 preserves weak pullbacks \( [Sok11] \) and the evaluation function is well-behaved. We recover the usual Wasserstein pseudometric \( d^D(P_1, P_2) = \inf \sum_{x_1, x_2 \in X} d(x_1, x_2) \cdot P(x_1, x_2) \mid P \in \Gamma_D(P_1, P_2) \) and the Kantorovich-Rubinstein duality \( [Vil09] \) from transportation theory for the discrete case.
We denote for any set $X$ the distances

$$d_e\left(\mathcal{D}\pi_1(P),\mathcal{D}\pi_2(P)\right) = |\mathbb{E}_P[\pi_1] - \mathbb{E}_P[\pi_2]| = \sum_{x_1,x_2 \in \{0,1\}} (x_1 - x_2) \cdot P(x_1, x_2).$$

We observe that for all $x \in X$, by $\delta_0^X \in \mathcal{D}X$ the Dirac distribution $\delta_0^X : X \to [0,1]$ with $\delta_0^X(0) = 1$ and $\delta_0^X(x) = 0$ for $x \in X \setminus \{0\}$. We observe that $\mathcal{D}\{0\} = \{\delta_0^{[0,1]}\}$ and thus we can easily see that also Condition 3 holds: $ev_{\mathcal{P}_n^{-1}[\{0\}]} = \{\delta_0^{[0,1]}\} = D_1[\mathcal{D}\{0\}].$

**Example 4.13** (Finite Powerset Functor & Hausdorff Pseudometric). Let $\mathcal{T} = \infty$, $F = \mathcal{P}_n$ with evaluation map $ev_{\mathcal{P}_n} : \mathcal{P}_n([0, \infty]) \to [0, \infty]$, $ev_{\mathcal{P}_n}(R) = \max R$ with $\max \emptyset = 0$ (as in Example 4.4). In this setting we obtain duality and both pseudometrics are equal to the Hausdorff pseudometric $d_H$ on $\mathcal{P}_n(X)$ which is defined as, for all $X_1, X_2 \in \mathcal{P}_n(X)$,

$$d_H(X_1, X_2) = \max \left\{ \min_{x_1 \in X_1} \min_{x_2 \in X_2} d(x_1, x_2), \max_{x_2 \in X_2} \min_{x_1 \in X_1} d(x_1, x_2) \right\}.$$

Note that the distance is $\infty$, if either $X_1$ or $X_2$ is empty.

**Proof.** We show that whenever $X_1, X_2$ are both non-empty there exists a coupling and a nonexpansive function that both witness the Hausdorff distance. Assume that the first value $\max_{x_1 \in X_1} \min_{x_2 \in X_2} d(x_1, x_2)$ is maximal and assume that $y_1 \in X_1$ is the element of $X_1$ for which the maximum is reached. Furthermore let $y_2 \in X_2$ the closest element in $X_2$, i.e., the element for which $d(y_1, y_2)$ is minimal. We know that for all $x_1 \in X_1$ there exists $x_2^{y_1}$ such that $d(x_1, x_2^{y_1}) \leq d(y_1, y_2)$ and for all $x_2 \in X_2$ there exists $x_1^{y_2}$ such that $d(x_1^{y_2}, x_2) \leq d(y_1, y_2)$. Specifically, $x_2^{y_1} = y_2$. We use the coupling $T \subseteq X \times X$ with

$$T = \{(x_1, x_2^{y_1}) \mid x_1 \in X_1\} \cup \{(x_1^{y_2}, x_2) \mid x_2 \in X_2\}.$$

Indeed we have $\mathcal{P}_n, \pi_1(T) = X_1$ and $\mathcal{P}_n d(T)$ contains all distances between the elements above, of which the distance $d(y_1, y_2) = d_H(X_1, X_2)$ is maximal. We now define a nonexpansive function $f : (X, d) \to ([0, \infty], d_e)$ as follows: $f(x) = \min_{x_2 \in X_2} d(x, x_2)$. It holds that $\max \mathcal{P}_n f(X_1) = \max f[X_1] = \max_{x_1 \in X_1} \min_{x_2 \in X_2} d(x_1, x_2) = d_H(X_1, X_2)$ and $\max \mathcal{P}_n f(X_2) = \max f[X_2] = 0$. Hence, the difference of both values is $d_H(X_1, X_2)$. It remains to show that $f$ is nonexpansive. Let $x, y \in X$ and let $x_2, y_2 \in X_2$ be elements for which the distances $d(x, x_2), d(y, y_2)$ are minimal. Hence

$$d(x, x_2) \leq d(x, y_2) \leq d(x, y) + d(y, y_2) \quad \text{and} \quad d(y, y_2) \leq d(y, x_2) \leq d(y, x) + d(x, x_2).$$

Now Lemma P.2.1 implies that $d(x, y) \geq d_e(d(x, x_2), d(y, y_2)) = d_e(f(x), f(y))$.

If $X_1 = X_2 = \emptyset$, we can use the coupling $T = \emptyset = \emptyset \times \emptyset$ and any function $f$. If, instead $X_1 = \emptyset, X_2 \neq \emptyset$, no coupling exists thus $d^{1F} = \infty$ and we can take the constant $\infty$-function to show that also $d^{1F} = \infty$ is attained.
P.5. **Lifting Multifunctors**

As addition to the main text we spell out the multifunctor definitions in details. As before, lifting is monotone on pseudometrics.

**Definition P.5.1 (Evaluation Function, Evaluation Functor, Well-Behaved).** An **evaluation function** for a functor \( F : \text{Set}^n \to \text{Set} \) is a function \( ev_F : F([0, \top], \ldots, [0, \top]) \to [0, \top] \).

Given \( ev_F \), we define the **evaluation functor** to be the functor \( \tilde{F} : (\text{Set}/[0, \top])^n \to \text{Set}/[0, \top] \) where \( \tilde{F}(g_1, \ldots, g_n) := ev_F \circ F(g_1, \ldots, g_n) \) for all \( g_i \in \text{Set}/[0, \top] \). On arrows \( \tilde{F} \) coincides with \( F \). We call \( ev_F \) well-behaved if it satisfies the following properties:

1. \( \tilde{F} \) is monotone: given \( f_i, g_i : X_i \to [0, \top] \) with \( f_i \leq g_i \) for all \( 1 \leq i \leq n \), we also have \( \tilde{F}(f_1, \ldots, f_n) \leq \tilde{F}(g_1, \ldots, g_n) \).
2. \( \forall t \in F([0, \top]^2, \ldots, [0, \top]^2) : d_e(\tilde{F}(\pi^1_1, \ldots, \pi^1_n)(t), \tilde{F}(\pi^2_1, \ldots, \pi^2_n)(t)) \leq \tilde{F}(d_e, \ldots, d_e)(t) \).
3. \( ev_F^{-1}(\{0\}) = F(i, \ldots, i)[F(\{0\}, \ldots, \{0\}] \) where \( i : \{0\} \to [0, \top] \) is the inclusion map.

The coupling definition for multifunctors is technically a bit more complicated than in the endofunctor setting but captures exactly the same idea as before.

**Definition P.5.2 (Coupling).** Let \( F : \text{Set}^n \to \text{Set} \) be a functor and \( m \in \mathbb{N} \). Given sets \( X_1, \ldots, X_n \) and elements \( t_j \in F(X_1, \ldots, X_n) \) for \( 1 \leq j \leq m \) we call an element \( t \in F(X^m_1, \ldots, X^m_n) \) such that \( F(\pi^m_{i,j}, \ldots, \pi^m_{n,j})(t) = t_j \) a **coupling** of the \( t_j \) (with respect to \( F \)) where \( \pi^m_{i,j} \) are the projections \( \pi^m_{i,j} : X^m_i \to X_i \). We write \( \Gamma_F(t_1, t_2, \ldots, t_m) \) for the set of all these couplings.

Note that the multifunctor approach is almost identical to the endofunctor approach and the only difference is that we start with \( n \) pseudometric spaces \( (X_1, d_1), \ldots, (X_n, d_n) \) instead of just one. Due to this we can straightforwardly adopt the proofs of the endofunctor cases to this setting. We will provide one exemplary calculation here. As in the endofunctor setting we have a gluing lemma.

**Lemma P.5.3 (Gluing Lemma for Multifunctors).** Let \( F : \text{Set}^n \to \text{Set} \) be a weak pullback preserving multifunctor, \( X_1, \ldots, X_n \) sets, \( t_1, t_2, t_3 \in F(X_1, \ldots, X_n) \), \( t_{12} \in \Gamma_F(t_1, t_2) \), and \( t_{23} \in \Gamma_F(t_2, t_3) \) be couplings. Then there is a coupling \( t_{123} \in \Gamma_F(t_1, t_2, t_3) \) such that \( F((\pi^3_{i,1}, \pi^3_{i,2}))(t_{123}) = t_{12} \) and \( F((\pi^3_{i,2}, \pi^3_{i,3}))(t_{123}) = t_{23} \).

**Proof.** Exactly as in the proof of the (endofunctor) Gluing Lemma, Lemma 4.2 we can see that the following is a pullback square for all \( 1 \leq i \leq n \).

\[
\begin{array}{ccc}
\langle \pi^3_{i,1}, \pi^3_{i,2} \rangle & X^3_i & \langle \pi^3_{i,2}, \pi^3_{i,3} \rangle \\
X^2_i & \pi^2_{i,2} & X^2_i \\
\pi^2_{i,1} & X_i & \pi^2_{i,1}
\end{array}
\]

Now consider the following diagram where we write \((\ldots, X_i, \ldots)\) instead of \((X_1, \ldots, X_n)\) for readability. Since \( F \) preserves weak pullbacks, the square in the middle of this diagram is a weak pullback.
With these definitions at hand it is now easy to generalize our approach to multifunctors.

**Kantorovich lifting**

The Kantorovich lifting for Proposition P.5.5.

The Wasserstein lifting for Proposition P.5.6.

By the (weak) universality of the pullback and the fact that

\[ F(..., \pi^3_{i,1}, \ldots) = F(..., \pi^3_{i,2}, \ldots) = t_2 = F(..., \pi^3_{i,1}, \ldots) \]

we obtain an element \( t_{123} \in F(..., X^3_i, \ldots) \) which satisfies the two equations of the lemma and moreover

\[ F(..., \pi^3_{i,1}, \ldots)(t_{123}) = F(..., \pi^2_{i,1}, \ldots) \circ (\pi^3_{i,1}, \pi^3_{i,2}) \circ (\pi^2_{i,1}, \pi^2_{i,2})(t_{123}) = F(..., \pi^3_{i,1}, \ldots)(t_{123}) = t_1 \]

and analogously \( F(..., \pi^3_{i,2}, \ldots)(t_{123}) = t_2, F(..., \pi^3_{i,3}, \ldots)(t_{123}) = t_3 \) so that indeed \( t_{123} \in \Gamma_F(t_1, t_2, t_3) \) is a coupling.

With these definitions at hand it is now easy to generalize our approach to multifunctors.

**Definition P.5.4 (Kantorovich/Wasserstein Pseudometric/Lifting).** Let \( F : \text{Set}^n \rightarrow \text{Set} \) be a functor with evaluation function \( ev_F \) and \( (X_1, d_1), \ldots, (X_n, d_n) \) be pseudometric spaces.

1. The **Kantorovich pseudometric** is \( d^{1F}_{1,...,n} : (F(X_1, \ldots, X_n))^2 \rightarrow [0, \top] \), where

\[ d^{1F}_{1,...,n}(t_1, t_2) := \sup_{f_i : (X_i, d_i) \xrightarrow{1} ([0, \top], d_e)} d_e\left( \tilde{F}(f_1, \ldots, f_n)(t_1), \tilde{F}(f_1, \ldots, f_n)(t_2) \right) \]

2. Whenever \( F \) preserves weak pullbacks and \( ev_F \) is a well-behaved evaluation function, the **Wasserstein pseudometric** is \( d^{1F}_{1,...,n} : (F(X_1, \ldots, X_n))^2 \rightarrow [0, \top] \), where

\[ d^{1F}_{1,...,n}(t_1, t_2) := \inf_{\tilde{F} \in \Gamma_F(t_1, t_2)} \tilde{F}(d_1, \ldots, d_n)(t) \]

The Kantorovich lifting \([\text{Wasserstein lifting}]\) of \( F \) is the functor \( \mathcal{F} : \text{PMet}^n \rightarrow \text{PMet} \).

\( \mathcal{F}(X_1, d_1), \ldots, (X_n, d_n) = (F(X_1, \ldots, X_n), d) \) with \( d = d^{1F}_{1,...,n}[d = d^{1F}_{1,...,n}] \), \( \mathcal{F}f = Ff \).

**Proposition P.5.5.** The Kantorovich lifting for \( F : \text{Set}^n \rightarrow \text{Set} \) is well defined, i.e.:

1. \( d^{1F}_{1,...,n} \) is a pseudometric on \( F(X_1, \ldots, X_n) \).
2. \( \mathcal{F} \) is an endofunctor on \( \text{PMet} \).

**Proof.** The proofs of Proposition P.3.1 and Proposition P.3.2 can be easily adapted.

**Proposition P.5.6.** The Wasserstein lifting for \( F : \text{Set}^n \rightarrow \text{Set} \) is well defined, i.e.:

1. \( d^{1F}_{1,...,n} \) is a pseudometric on \( F(X_1, \ldots, X_n) \).
2. \( \mathcal{F} \) is an endofunctor on \( \text{PMet} \).
Whenever the two pseudometrics coincide for a functor and an evaluation function, we say Kantorovich-Rubinstein duality holds and the supremum [infimum] of the Kantorovich pseudometrics is always a maximum [minimum]. For the first function $\rho_1$, we obtain the Wasserstein pseudometrics is always a maximum [minimum].

**Proof.**

1. We can state and prove a multifunctor equivalent of Proposition P.4.2 and as in that case our result is an immediate corollary.
2. Again, the same arguments as in Propositions 5.6 and 4.7
3. See Proposition 4.8

Whenever the two pseudometrics coincide for a functor and an evaluation function, we say that the Kantorovich-Rubinstein duality (for multifunctors) holds.

**Example 5.1 (Product Bifunctor).** For the product bifunctor $F: \text{Set}^2 \to \text{Set}$ where $F(X_1, X_2) = X_1 \times X_2$ and $F(f_1, f_2) = f_1 \times f_2$ we consider the evaluation function $\max: [0, \top]^2 \to [0, \top]$ and for fixed parameters $c_1, c_2 \in (0, 1]$ and $p \in \mathbb{N}$ the function $\rho: [0, \top]^2 \to [0, \top], \rho(x_1, x_2) = (c_1 x_1^p + c_2 x_2^p)^{1/p}$. These functions are well-behaved, the Kantorovich-Rubinstein duality holds and the supremum [infimum] of the Kantorovich [Wasserstein] pseudometrics is always a maximum [minimum]. For the first function we obtain the $\infty$-product pseudometric $\rho_1((x_1, x_2), (y_1, y_2)) = \max(d_1(x_1, y_1), d_2(x_2, y_2))$ and for the other function the weighted $p$-product pseudometric $\rho_p((x_1, x_2), (y_1, y_2)) = (c_1 d_1^p(x_1, y_1) + c_2 d_2^p(x_2, y_2))^{1/p}$.

**Proof.** $F$ preserves weak pullbacks: If we have two weak pullbacks in $\text{Set}$ as indicated in the left of the diagram below, then obviously also the right diagram is a weak pullback.

![Diagram](image)

We proceed by checking all three conditions for well-behavedness:

1. Let $f_1, f_2, g_1, g_2: X_i \to [0, \top]$ with $f_1 \leq g_1, f_2 \leq g_2$ then for any $x = (x_1, x_2) \in F(X_1, X_2) = X_1 \times X_2$ we see that in the case of the maximum we have

$$F(f_1, f_2)(x) = \max(f_1(x_1), f_2(x_2)) \leq \max(g_1(x_1), g_2(x_2)) = F(g_1, g_2)(x)$$

and for the second evaluation function, we also obtain

$$\tilde{F}(f_1, f_2)(x) = \left( c_1 \cdot f_1^p(x_1) + c_2 \cdot f_2^p(x_2) \right)^{1/p} \leq \left( c_1 \cdot g_1^p(x_1) + c_2 \cdot g_2^p(x_2) \right)^{1/p} = \tilde{F}(g_1, g_2)(x)$$

due to monotonicity of all involved functions since $c_1, c_2 > 0$. 

**Proof.** The proofs of Proposition 4.6 and Proposition P.4.1 can easily be adapted.
2. Let \( t := ((x_1, x_2), (y_1, y_2)) \in F([0, \top]^2, [0, \top]^2) = [0, \top]^2 \times [0, \top]^2 \). We have to show
\[ d_\epsilon(\tilde{F}(\pi_1, \pi_1)(t), \tilde{F}(\pi_2, \pi_2)(t)) \leq \tilde{F}(d_\epsilon, d_\epsilon)(t). \]
If we define \( z_i = ev_F(x_i, y_i) \) (where \( ev_F = \max \) or \( ev_F = \rho \) respectively) for \( i \in \{0, 1\} \) then
\[ d_\epsilon(\tilde{F}(\pi_1, \pi_1)(t), \tilde{F}(\pi_2, \pi_2)(t)) = d_\epsilon(z_1, z_2) \text{ and } \tilde{F}(d_\epsilon, d_\epsilon)(t) = ev_F(d_\epsilon(x_1, x_2), d_\epsilon(y_1, y_2)). \]
We thus have to show the inequality
\[ d_\epsilon(z_1, z_2) \leq ev_F(d_\epsilon(x_1, x_2), d_\epsilon(y_1, y_2)). \]  \( (7) \)

If \( z_1 = z_2 \) this is obviously true because \( d_\epsilon(z_1, z_2) = 0 \) and the rhs is non-negative. We
now assume \( z_1 > z_2 \) (the other case is symmetrical). For \( \infty = z_1 > z_2 \) the inequality holds because then \( x_1 = \infty \) or \( y_1 = \infty \) and \( x_2, y_2 < \infty \) (otherwise we would have \( z_2 = \infty \)) so both lhs and rhs are \( \infty \). Thus we can now restrict to \( \infty > z_1 > z_2 \) where necessarily also \( x_1, y_1, x_2, y_2 < \infty \) (otherwise we would have \( z_1 = \infty \) or \( z_2 = \infty \)). According to [Lemma P.2.1], the inequality \( (7) \) is equivalent to showing the two inequalities
\[ z_1 \leq z_2 + ev_F(d_\epsilon(x_1, x_2), d_\epsilon(y_1, y_2)) \quad \text{\text{and}} \quad z_2 \leq z_1 + ev_F(d_\epsilon(x_1, x_2), d_\epsilon(y_1, y_2)). \]

By our assumption \( (\infty > z_1 > z_2) \) the second of these inequalities is satisfied, so we just have to show the first. We do this separately for the different evaluation functions.

\[ \text{Suppose } ev_F = \max, \text{ then } z_1 = \max(x_1, y_i). \text{ If } z_1 = x_1 \text{ we have} \]
\[ z_2 + \max(d_\epsilon(x_1, x_2), d_\epsilon(y_1, y_2)) \geq z_2 + d_\epsilon(z_1, x_2) = z_2 + (z_1 - x_2) = z_1 + (z_2 - x_2) \geq z_1 \]
because \( z_2 = \max(x_2, y_2) \geq x_2 \) and therefore \( (z_2 - x_2) \geq 0 \). The same line of argument can be applied if \( z_1 = y_1 \).

\[ \text{Suppose } ev_F = \rho. \text{ We define} \]
\[ a_1 := c_1^{1/p} x_2, \quad a_2 := c_2^{1/p} y_2, \quad b_1 := c_1^{1/p} \cdot (x_1 - x_2), \quad b_2 := c_2^{1/p} \cdot (y_1 - y_2). \]
The Minkowski inequality tells us that
\[ (|a_1|^p + |a_2|^p)^{1/p} + (|b_1|^p + |b_2|^p)^{1/p} \geq (|a_1 + b_1|^p + |a_2 + b_2|^p)^{1/p}. \]

We calculate the different parts:
\[ (|a_1|^p + |a_2|^p)^{1/p} = (c_1 x_2^p + c_2 y_2^p)^{1/p} = z_2 \]
\[ (|b_1|^p + |b_2|^p)^{1/p} = (c_1 x_1 - x_2)^p + c_2 (y_1 - y_2)^p)^{1/p} = \rho(d_\epsilon(x_1, x_2), d_\epsilon(y_1, y_2)) \]
\[ |a_1 + b_1|^p = c_1^{1/p} x_2 + c_1^{1/p} \cdot (x_1 - x_2))^p = c_1 |x_1|^p = c_1 x_1^p \]
\[ |a_2 + b_2|^p = c_2^{1/p} y_2 + c_2^{1/p} \cdot (y_1 - y_2))^p = c_2 |y_1|^p = c_2 y_1^p \]
and thus the Minkowski inequality yields
\[ z_2 + \rho(d_\epsilon(x_1, x_2), d_\epsilon(y_1, y_2)) \geq (c_1 x_1^p + c_2 y_1^p)^{1/p} = z_1 \]
which concludes the proof.

3. Both evaluation functions satisfy Condition 3 of [Definition P.5.1] \( F(i, i)[F(\{0\}, \{0\})] = (i \times i)[\{0\} \times \{0\}] = \{(0, 0)\} \) and \( \max^{-1}[\{0\}] = \{(0, 0)\} = \rho^{-1}[\{0\}]. \)

We now prove (for both evaluation functions) that the product functor satisfies the Kantorovich-Rubinstein duality and simultaneously that the supremum (in the Kantorovich pseudometric) is a maximum and the infimum (of the Wasserstein pseudometric) is a minimum. Let \( (X_1, d_1), (X_1, d_2) \) be pseudometric spaces and let \( t_1 = (x_1, x_2), t_2 = \)
$$(y_1, y_2) \in F(X_1, X_2) = X_1 \times X_2$$ be given. We define $t := ((x_1, y_1), (x_2, y_2)) \in F(X_1^2, X_2^2)$ and observe that $F(\pi_1, \pi_1)(t) = t_1$, $F(\pi_2, \pi_2)(t) = t_2$ and thus $t \in \Gamma_F(t_1, t_2)$ is a coupling.

In the following we will construct nonexpansive functions $f_i : (X_i, d_i) \to ([0, \top], d_e)$ such that $d_e(\tilde{F}(f_1, f_2)(t_1), \tilde{F}(f_1, f_2)(t_2)) = \tilde{F}(d_1, d_2)(t)$. Due to Proposition P.5.7 we can then conclude that duality holds and both supremum and infimum are attained.

- Suppose $ev_F = \max$ then we have $\tilde{F}(d_1, d_2)(t) = \max(d_1(x_1, y_1), d_2(x_2, y_2))$ and assume wlog that $d_1(x_1, y_1)$ is the maximal element and define $f_1 := d_1(x_1, -)$, which is nonexpansive due to [Lemma 2.3] and $f_2$ is the constant zero-function which is obviously nonexpansive. Then we have:

$$d_e(\tilde{F}(f_1, f_2)(t_1), \tilde{F}(f_1, f_2)(t_2)) = d_e(\max(f_1(x_1), f_2(x_2))), \max(f_1(y_1), f_2(y_2))$$

$$= d_e(\max(f_1(x_1), 0), \max(f_1(y_1), 0)) = d_e(f_1(x_1), f_1(y_1)) = d_1(x_1, y_1)$$

$$= \max(d_1(x_1, y_1), d_2(x_2, y_2)) = \tilde{F}(d_1, d_2)(t)$$

The case where $d_2(x_2, y_2)$ is the maximal element is treated analogously.

- If $ev_F = \rho$ we define $f_1 := d_1(x_1, -)$ and $f_2 := d_2(x_2, -)$ (which are again nonexpansive by [Lemma 2.3]) and obtain

$$d_e(\tilde{F}(f_1, f_2)(t_1), \tilde{F}(f_1, f_2)(t_2))$$

$$= d_e\left(\left(c_1f_1^p(x_1) + c_2f_2^p(x_2)\right)^{1/p}, \left(c_1f_1^p(y_1) + c_2f_2^p(y_2)\right)^{1/p}\right)$$

$$= d_e\left(0, \left(c_1d_1^p(x_1, y_1) + c_2d_2^p(x_2, y_2)\right)^{1/p}\right)$$

$$= (c_1d_1^p(x_1, y_1) + c_2d_2^p(x_2, y_2))^{1/p} = \tilde{F}(d_1, d_2)(t)$$

which completes the proof. □

**Example 5.2 (Coproduct Bifunctor).** For the coproduct bifunctor $F : \text{Set}^2 \to \text{Set}$, where $F(X_1, X_2) = X_1 + X_2 = X_1 \times \{1\} \cup X_2 \times \{2\}$ and $F(f_1, f_2) = f_1 + f_2$ we take the evaluation function $ev_F : [0, \top] \times [0, \top] \to [0, \top]$, $ev_F(x, i) = x$. This function is well-behaved, the Kantorovich-Rubinovitch duality holds and the supremum of the Kantorovich pseudometric is always a maximum whereas the infimum of the Wasserstein pseudometric is a minimum if and only if any coupling of the two elements exists. We obtain the coproduct pseudometric $d_+$ where $d_+((x_1, i_1), (x_2, i_2))$ is equal to $d_i(x_1, x_2)$ if $i_1 = i_2 = i$ and equal to $\top$ otherwise.

**Proof.** $F$ preserves weak pullbacks: If we have two weak pullbacks in $\text{Set}$ as indicated in the left of the diagram below, then obviously also the right diagram is a weak pullback.

$$\begin{array}{c}
\begin{array}{ccc}
P_i & \overset{p_i^X}{\longrightarrow} & X_i \\
\downarrow{p_i^Y} & & \downarrow{f_i} \\
Y_i & \underset{g_i}{\longrightarrow} & Z_i
\end{array}
\end{array}$$

$$\begin{array}{c}
\begin{array}{ccc}
P_1 + P_2 & \overset{p_1^X + p_2^X}{\longrightarrow} & X_1 + X_2 \\
\downarrow{p_1^Y + p_2^Y} & & \downarrow{f_1 + f_2} \\
Y_1 + Y_2 & \underset{g_1 + g_2}{\longrightarrow} & Z_1 + Z_2
\end{array}
\end{array}$$

We now show that the evaluation function is well-behaved.

1. Let $f_1, f_2, g_1, g_2 : X \to [0, \top]$ with $f_1 \leq g_1, f_2 \leq g_2$ and $(z, i) \in F(X_1, X_2) = X_1 + X_2$. We have $Ff(z, i) = ev_F(F(f_1, f_2)(z, i)) = f_i(z) \leq g_i(z) = ev_F(F(g_1, g_2)(z, i)) = Fg(z, i)$. 

2. If $\pi_i : \text{Hom}(\text{Set}^2, \text{Set}) \to \text{Set}$ is the action of the first component, then $\pi_i(\text{id}) = \pi_i(F(f_1, f_2)(z, i)) = f_i(z)$ is the maximal and the minimal element $\pi_i(F(g_1, g_2)(z, i)) = g_i(z)$ is the maximal element. Thus, $\pi_i(\text{id}) = \pi_i(F(f_1, f_2)(z, i)) = f_i(z)$ is the maximal element and define $f_i := d_i(x_1, -)$, which is nonexpansive due to [Lemma 2.3] and $f_2$ is the constant zero-function which is obviously nonexpansive. Then we have:

$$d_e(\tilde{F}(f_1, f_2)(t_1), \tilde{F}(f_1, f_2)(t_2))$$

$$= d_e(\max(f_1(x_1), f_2(x_2))), \max(f_1(y_1), f_2(y_2))$$

$$= d_e(\max(f_1(x_1), 0), \max(f_1(y_1), 0)) = d_e(f_1(x_1), f_1(y_1)) = d_1(x_1, y_1)$$

$$= \max(d_1(x_1, y_1), d_2(x_2, y_2)) = \tilde{F}(d_1, d_2)(t)$$

The case where $d_2(x_2, y_2)$ is the maximal element is treated analogously.

- If $ev_F = \rho$ we define $f_1 := d_1(x_1, -)$ and $f_2 := d_2(x_2, -)$ (which are again nonexpansive by [Lemma 2.3]) and obtain

$$d_e(\tilde{F}(f_1, f_2)(t_1), \tilde{F}(f_1, f_2)(t_2))$$

$$= d_e\left(\left(c_1f_1^p(x_1) + c_2f_2^p(x_2)\right)^{1/p}, \left(c_1f_1^p(y_1) + c_2f_2^p(y_2)\right)^{1/p}\right)$$

$$= d_e\left(0, \left(c_1d_1^p(x_1, y_1) + c_2d_2^p(x_2, y_2)\right)^{1/p}\right)$$

$$= (c_1d_1^p(x_1, y_1) + c_2d_2^p(x_2, y_2))^{1/p} = \tilde{F}(d_1, d_2)(t)$$

which completes the proof. □
2. Let \( t = ((x, y), i) \in F([0, \top]^2, [0, \top]^2) = [0, \top]^2 \times \{0, 2\} \). Then we even obtain equality:
\[
\tilde{F}(d_e, d_e)(t) = ev_F(d_e(x, y), i) = d_e(x, y)
= d_e(ev_F(x, i), ev_F(y, i)) = d_e(\tilde{F}(\pi_1, \pi_1)(t), \tilde{F}(\pi_2, \pi_2)(t)).
\]

3. Let \( i: 0 \mapsto [0, \top] \) be the inclusion function. We have \( Fi[F(\{0\}, \{0\})] = (i + i)[\{0\} + \{0\}] = [0] \times [1, 2] = ev_F^{-1}([0]) \).

Now we show that the pair of functor and evaluation function \( ev_F \) satisfies the Kantorovich-Rubinstein duality and simultaneously that the supremum (in the Kantorovich pseudometric) is a maximum and the infimum (of the Wasserstein pseudometric) is a minimum iff there exists a coupling of the two given elements. Let \( (X_1, d_1), (X_2, d_2) \) be pseudometric spaces, \( t_1 = (z, i), t_2 = (z', i') \in F(X_1, X_2) = X_1 + X_2 \). Suppose \( i = i' \), when we define \( t = ((z, z'), i) \) and observe that \( F(\pi_1, \pi_1)((z, z'), i) = t_1, F(\pi_2, \pi_2)((z, z'), i) = t_2 \), thus \( t \in \Gamma_F(t_1, t_2) \). Furthermore \( \tilde{F}(d_1, d_2)(t) = d_i((z, z')) \). If \( i = i' = 1 \) we define \( f_1 := d_1(z, -) : (X_1, d_1) \downarrow (\{0\}, d_e) \) which is nonexpansive according to \( \text{Lemma 2.3} \) and consider an arbitrary nonexpansive function \( f_2 : (X_2, d_2) \downarrow (\{0\}, d_e) \) (e.g. the constant zero-function). Then we have:
\[
d_e(\tilde{F}(f_1, f_2)(t_1), \tilde{F}(f_1, f_2)(t_2)) = d_e(\tilde{F}(f_1, f_2)(z, 1), \tilde{F}(f_1, f_2)(z', 1)) = d_e(f_1(z), f_1(z'))
= d_e(0, d_1(z, z')) = d_1(z, z') = d_1(z, z')
\]

The case \( i = i' = 2 \) is analogous. In the case where \( i \neq i' \), there is no coupling that projects to \( (z, i) \) and \( (z', i') \), thus \( d_{1,2}(t_1, t_2) = \top \). We show that also \( d_{1,2}^\top(t_1, t_2) = \top \). We define \( f_1 \) to be the constant zero-function and \( f_2 \) the constant \( \top \)-function. We have:
\[
d_e(\tilde{F}(f_1, f_2)(t_1), \tilde{F}(f_1, f_2)(t_2)) = d_e(\tilde{F}(f_1, f_2)(z, i), \tilde{F}(f_1, f_2)(z', j))
= d_e(f_1(z), f_1(z')) = d_e(0, \top) = \top
\]
which completes the proof. 

\[\square\]

**P.6. Final Coalgebra & Coalgebraic Behavioral Pseudometrics**

**Theorem 6.1** Let \( F: \text{PMet} \to \text{PMet} \) be a lifting of a functor \( F: \text{Set} \to \text{Set} \) which has a final coalgebra \( \kappa: \Omega \to F\Omega \). For every ordinal \( i \) we construct a pseudometric \( d_i: \Omega \times \Omega \to [0, \top] \) as follows: \( d_0 := 0 \) is the zero pseudometric, \( d_{i+1} := d_i^F \circ (\kappa \times \kappa) \) for all ordinals \( i \) and \( d_j := \sup_{i<j} d_i \) for all limit ordinals \( j \). This sequence converges for some ordinal \( \theta \), i.e \( d_\theta = d_\theta^F \circ (\kappa \times \kappa) \). Moreover \( \kappa: (\Omega, d_\theta) \downarrow (\Omega, d_\theta^F) \) is the final \( F \)-coalgebra.

**Proof.** It can be easily shown that each of the \( d_i \) is a pseudometric, since the supremum of pseudometrics is again a pseudometric. Since \( d_\theta \) is a fixed-point, \( \kappa \) is an isometry and hence nonexpansive. Furthermore the chain converges once we reach an ordinal whose cardinality is larger than the cardinality of the lattice of metrics on \( \Omega \).

Let \( (X, d) \downarrow (\Omega, d_\theta) \) be any \( F \)-coalgebra, with underlying \( F \)-coalgebra \( \alpha: X \to FX \) in \( \text{Set} \). Since \( \kappa \) is the final \( F \)-coalgebra, there exists a unique function \( f: X \to \Omega \) such that \( \kappa \circ f = Ff \circ \alpha \). It is left to show that \( f \) is nonexpansive function \( (X, d) \downarrow (\Omega, d_\theta) \).

For each ordinal \( i \) we define a pseudometric \( e_i: X \times X \to [0, \top] \) as follows: \( e_0 \) is the constant zero-pseudometric, \( e_{i+1} := e_i^F \circ (\alpha \times \alpha) \) and \( e_j := \sup_{i<j} e_i \) if \( j \) is a limit ordinal. We show that \( e_i \leq d \): Obviously \( e_0 \leq d \) and furthermore \( e_{i+1} = e_i^F \circ (\alpha \times \alpha) \leq d_i^F \circ (\alpha \times \alpha) \leq d \) where the first inequality is due to the fact that the lifting preserves...
the order on pseudometrics and the second is nonexpansiveness of $\alpha$. If we take the limit $e_j = \sup_{i < j} e_i$, we know that $e_i \leq d$ for each $i < j$ and hence also $e_j \leq d$.

As an auxiliary step we will prove that all $f: (X, e_i) \rightarrow (\Omega, d_i)$ are nonexpansive. This holds for $i = 0$ since for all $x, y \in X$ we have $e_0(x, y) = 0 = d_0(f(x), f(y))$. For $i + 1$ we have

$$d_{i+1}(f(x), f(y)) = d_i^F((\kappa \circ f)(x), (\kappa \circ f)(y)) = d_i^F((F_f \circ \alpha)(x), (F_f \circ \alpha)(y)) \leq e_i^F(\alpha(x), \alpha(y)) = e_{i+1}(x, y).$$

The inequality above holds since if $f: (X, e_i) \rightarrow (\Omega, d_i)$ is a nonexpansive function also $F_f: (FX, e_i^F) \rightarrow (\Omega, d_i^F)$ is nonexpansive. Whenever $j$ is a limit ordinal we obtain:

$$d_j(f(x), f(y)) = \sup_{i < j} d_i(f(x), f(y)) \leq \sup_{i < j} e_i(x, y) = e_j(x, y).$$

Finally, we combine this result with the result from above to obtain the inequality $d_0(f(x), f(y)) \leq e_0(x, y) \leq d(x, y)$ which shows that $f: (X, d) \rightarrow (\Omega, d)$ is nonexpansive.

We next prove that the Wasserstein/Kantorovich liftings induced by the finite powerset functor and by the probability distribution functor with finite support are $\omega$-continuous.

**Proposition P.6.1 ($\omega$-continuity of the liftings of $\mathcal{P}_{fn}$ and $\mathcal{D}$).** Let $F$ be the finite powerset functor $\mathcal{P}_{fn}$ or the distribution functor $\mathcal{D}$ (with finite support). For any set $X$ the function $\lambda^F$ mapping a metric $d$ over $X$ to the metric $d^F$ over $FX$ is $\omega$-continuous, namely for any increasing chain of metrics $(d_i)_{i \in \mathbb{N}}$ over $X$, we have $(\sup_i d_i)^F = (\sup_i d_i)^F$.

**Proof.** Let $X$ be a fixed set. By Proposition P.2.3 we know that for any lifting, the function $\lambda^F$ is monotone, namely, whenever $d_1 \leq d_2$ it holds that $d_1^F \leq d_2^F$.

Given an increasing chain of metrics $(d_i)_{i \in \mathbb{N}}$ over $X$, simply by monotonicity of $\lambda^F$ we deduce that

$$\sup_i d_i^F \leq (\sup_i d_i)^F.$$ 

In fact, for any $i \in \mathbb{N}$, it holds that $d_i \leq \sup_i d_i$, hence $d_i^F \leq (\sup_i d_i)^F$ and therefore we conclude.

We next prove that for the Wasserstein/Kantorovich liftings of either $\mathcal{P}_{fn}$ or $\mathcal{D}$ also the converse inequality holds, and thus we obtain the result. We proceed separately for the two functors.

**Finite powerset.** Let us denote $d = \sup_i d_i$. We have to show that

$$d_{\mathcal{P}_{fn}} \leq \sup_i d_i^{\mathcal{P}_{fn}}.$$ 

Let $X_1, X_2 \in \mathcal{P}_{fn}(X)$ be finite subsets of $X$. Since $X_1$ and $X_2$ are finite and $d = \sup d_i$, for any $\varepsilon > 0$ we can find an $i \in \mathbb{N}$ such that for any $x_1 \in X_1, x_2 \in X_2$ and all $j \geq i$

$$d(x_1, x_2) - d_j(x_1, x_2) \leq \varepsilon.$$ 

According to the definition of the Wasserstein lifting $\mathcal{P}_{fn}$ we get for all $j \geq i$:

$$d^{\mathcal{P}_{fn}}(X_1, X_2) = \inf \left\{ \max_{(x_1, x_2) \in W} d(x_1, x_2) \mid W \in \Gamma_{\mathcal{P}_{fn}}(X_1, X_2) \right\} \leq \inf \left\{ \max_{(x_1, x_2) \in W} (d_j(x_1, x_2) + \varepsilon) \mid W \in \Gamma_{\mathcal{P}_{fn}}(X_1, X_2) \right\} = \inf \left\{ \max_{(x_1, x_2) \in W} d_j(x_1, x_2) \mid W \in \Gamma_{\mathcal{P}_{fn}}(X_1, X_2) \right\} + \varepsilon = d_j^{\mathcal{P}_{fn}}(X_1, X_2) + \varepsilon.$$
Therefore, \( d^D(X_1, X_2) \leq \sup_i d^D_i(X_1, X_2) + \varepsilon \). Given that \( \varepsilon \) can be arbitrarily small, we deduce that indeed \( d^D(X_1, X_2) \leq \sup_i d^D_i(X_1, X_2) \), as desired.

**Finitely supported distributions.** Let us denote \( d = \sup_i d_i \). We have to show that

\[
  d^D \leq \sup_i d_i^D
\]

Let \( P_1, P_2 \in D X \) and let \( X_1, X_2 \) be the corresponding finite supports of \( P_1 \) and \( P_2 \), namely \( X_i = \{ x \in X \mid P_i(X) > 0 \} \). As before, since \( X_1 \) and \( X_2 \) are finite and \( d = \sup d_i \), for any \( \varepsilon > 0 \) we can find an \( i \in \mathbb{N} \) such that for any \( x_1 \in X_1, x_2 \in X_2 \) and \( j \geq i \)

\[
  d(x_1, x_2) - d_j(x_1, x_2) \leq \varepsilon.
\]

Using the definition of the Wasserstein lifting of \( D \), we get for all \( j \geq i \):

\[
  d^D(X_1, X_2) = \inf \left\{ \sum_{x_1, x_2 \in X} d(x_1, x_2) \cdot P(x_1, x_2) \mid P \in \Gamma_D(P_1, P_2) \right\}
\]

since for any \((x_1, x_2) \in X \times X\), whenever \((x_1, x_2) \notin X_1 \times X_2\) necessarily \(P(x_1, x_2) = 0\)

\[
  = \inf \left\{ \sum_{(x_1, x_2) \in X_1 \times X_2} d(x_1, x_2) \cdot P(x_1, x_2) \mid P \in \Gamma_D(P_1, P_2) \right\}
\]

\[
  \leq \inf \left\{ \sum_{(x_1, x_2) \in X_1 \times X_2} (d_j(x_1, x_2) + \varepsilon) \cdot P(x_1, x_2) \mid P \in \Gamma_D(P_1, P_2) \right\}
\]

\[
  = \inf \left\{ \sum_{(x_1, x_2) \in X_1 \times X_2} d_j(x_1, x_2) \cdot P(x_1, x_2) + \varepsilon \cdot \sum_{x_1, x_2 \in X} P(x_1, x_2) \mid P \in \Gamma_D(P_1, P_2) \right\}
\]

\[
  = \inf \left\{ \sum_{(x_1, x_2) \in X_1 \times X_2} d_j(x_1, x_2) \cdot P(x_1, x_2) + \varepsilon \mid P \in \Gamma_D(P_1, P_2) \right\} + \varepsilon
\]

\[
  = \inf \left\{ \sum_{x_1, x_2 \in X} d_j(x_1, x_2) \cdot P(x_1, x_2) \mid P \in \Gamma_D(P_1, P_2) \right\} + \varepsilon = d^D_j(X_1, X_2) + \varepsilon
\]

Therefore, \( d^D(X_1, X_2) \leq \sup_i d^D_i(X_1, X_2) + \varepsilon \). Given that \( \varepsilon \) can be arbitrarily small, we deduce that indeed \( d^D(X_1, X_2) \leq \sup_i d^D_i(X_1, X_2) \), as desired.

**Theorem 6.2.** Let the chain of the \( d_i \) converge in \( \theta \) steps and \( F \) preserve isometries. Let furthermore \( \alpha : X \rightarrow FX \) be an arbitrary coalgebra. For all ordinals \( i \) we define a pseudometric \( e_i : X \times X \rightarrow [0, \infty) \) as follows: \( e_0 \) is the zero pseudometric, \( e_{i+1} = e_i^F \circ (\alpha \times \alpha) \) for all ordinals \( i \) and \( e_j = \sup_{i \leq j} e_i \) for all limit ordinals \( j \). Then we reach a fixed point after \( \zeta \leq \theta \) steps, i.e. \( e_{\zeta} = e_{\zeta}^F \circ (\alpha \times \alpha) \), such that \( bd = e_{\zeta} \).

**Proof.** The chain \( e_i \) of metrics is the same as the one constructed in the proof of Theorem 6.1. In this proof we have shown that all \( f : (X, e_i) \rightarrow (\Omega, d_i) \) are nonexpansive. Here we show that they are also isometries. For \( i = 0 \) this is true: for \( x, y \in X \) \( d_0(f(x), f(y)) = 0 =
We construct a series of metrics \( e_0(x, y) \). Now assume that \( f: (X, e_i) \xrightarrow{i} (\Omega, d_i) \) is an isometry, which implies (\( F \) preserves isometries) that \( Ff: (FX, e^F_i) \xrightarrow{i} (F\Omega, d^F_i) \) is an isometry. Hence for \( x, y \in X \) we have
\[
d_{i+1}(f(x), f(y)) = d^F_i(\kappa(f(x)), \kappa(f(y))) = d^F_i(Ff(\alpha(x)), Ff(\alpha(y))) = e^F_i(\alpha(x), \alpha(y)) = e_{i+1}(x, y)
\]
For a limit ordinal \( j \) we have
\[
d_j(f(x), f(y)) = \sup_{i < j} d_i(f(x), f(y)) = \sup_{i < j} e_i(x, y) = e_j(x, y)
\]
We know that \( d_\emptyset \) is a fixed-point, i.e. we have \( d_\emptyset = d^F_\emptyset \circ (\kappa \times \kappa) \). Then \( e_\emptyset \) must also be a fixed-point (\( e_\emptyset = e^F_\emptyset \circ (\alpha \times \alpha) \)), since:
\[
e_\emptyset(x, y) = d_\emptyset(f(x), f(y)) = d^F_\emptyset(\kappa(f(x)), \kappa(f(y))) = d^F_\emptyset(Ff(\alpha(x)), Ff(\alpha(y))) = e^F_\emptyset(\alpha(x), \alpha(y))
\]
using again the fact that \( Ff \) is an isometry. Hence \( \zeta \leq \theta \), i.e., the chain \( e_i \) might converge earlier and \( d_\kappa(x, y) = d_\emptyset(f(x), f(y)) = e_\emptyset(x, y) = e_\zeta(x, y). \)

\[\textbf{Theorem 6.5.}\] Let \( F: \text{PMet} \to \text{PMet} \) be a lifting of a functor \( F: \text{Set} \to \text{Set} \) which has a final coalgebra \( \kappa: \Omega \to F\Omega \). Assume that \( F \) preserves isometries and metrics, that the final chain for \( F \) converges and the chain of the \( d_i \) converges in \( \theta \) steps. Then \( d_\emptyset \) is a metric, i.e. for \( x, y \in \Omega \) we have \( d_\emptyset(x, y) = 0 \iff x = y \).

\[\textbf{Proof.}\] We construct a series of metrics \( e_i: W_i \times W_i \to [0, \top] \) for \( i \in \text{Ord} \), as follows: \( e_0 = 0 \): \( 1 \times 1 \to [0, \top] \) is the (unique!) zero metric on \( 1 \), \( e_{i+1} := e^F_i \) and if \( j \) is a limit ordinal we define \( e_j := \sup_{i < j} e_i \circ (p_{j,i} \times p_{j,i}) \). Since the functor preserves metrics \( e_{i+1} \) is a metric if \( e_i \) is. Given a limit ordinal \( j \) we can easily check that \( e_j \) is a pseudometric provided that all the \( e_i \) with \( i < j \) are pseudometrics. To see that \( e_j \) is also a metric when all \( e_i \) with \( i < j \) are metrics we proceed as follows: Suppose \( e_{j}(x, y) = 0 \) for some \( x, y \in W_j \), then we know that for all \( i \) we must have \( p_{j,i}(x) = p_{j,i}(y) \) because the \( e_i \) are metrics. Since the cone \( (W_j, p_{j,i}(W_i))_{i < j} \) is by definition a limit in \( \text{Set} \) we can now conclude that \( x = y \). This is due to the universal property of the limit: If \( x \neq y \) then for the cone \( \{(x, y) \xrightarrow{i} W_i\}_{i < j} \) the functional \( f_i(x) = p_{i,j}(x) = f_i(y) \) there would have to be a unique function \( u: \{x, y\} \to W_j \) satisfying \( p_{j,i} \circ u = f_i \). However, for example \( u, u': \{x, y\} \to W_j \) where \( u(x) = u(y) = x \) and \( u' = u'(y) = y \) are distinct functions satisfying this commutativity which is a contradiction to the uniqueness.

Using the metrics \( e_i \) we now consider the connection morphisms \( p^i := p_{\lambda,i}: \Omega \to W_i \) and proceed by showing that each \( p^i \) is an isometry (\( \Omega, d_i \xrightarrow{i} (W_i, e_i) \). By definition this holds for \( d_0 \) and \( e_0 \) (both are constantly zero). For \( i + 1 \) we recall that \( \kappa = p^{-1}_{\lambda+1,i} \). Hence we have by properties of the connection morphisms \( Fp^i = Fp_{\lambda,i} = p_{\lambda+1,i} \circ p_{\lambda+1,i+1} = p_{\lambda,i+1} \circ p_{\lambda+1,i} \circ p_{\lambda+1,i+1} = p^{i+1} \circ \kappa^{-1} \) and thus \( Fp^i \circ \kappa = p^{i+1} \). Since by the induction hypothesis, \( p^i: (\Omega, d_i) \xrightarrow{i} (W_i, e_i) \) is an isometry, the fact that isometries are preserved by \( F \) implies that \( Fp^i: (F\Omega, d^F_i) \xrightarrow{i} (F(W_i), d^F_i) \) is an isometry. Furthermore \( \kappa: (\Omega, d_{i+1}) \xrightarrow{i} (F\Omega, d^F_i) \) is an isometry. Hence also their composition \( p^{i+1} = Fp^i \circ \kappa \) is an isometry. For a limit ordinal \( j \) we calculate for \( x, y \in \Omega \)
\[
e_j(p^j(x), p^j(y)) = \sup_{i < j} e_i((p_{j,i} \circ p^j)(x), (p_{j,i} \circ p^j)(y)) = \sup_{i < j} e_i(p^j(x), p^j(y)) = \sup_{i < j} d_i(x, y) = d_j(x, y)
\]
and thus also $p^j : (\Omega, d_j) \rightarrow (W_j, e_j)$ is an isometry.

We assume that $\lambda \geq \theta$, otherwise set $\lambda = \theta$ (if the final chain converges in $\theta$ steps it also converges for all larger ordinals), thus $d_\lambda = d_\theta$. Now let $x, y \in \Omega$ with $d_\theta(x, y) = d_\lambda(x, y) = 0$. This implies $e_i(p^i(x), p^i(y)) = 0$ for all ordinals $i \leq \lambda$. Since all $e_i$ are metrics, we infer that $p^i(x) = p^i(y)$ for all ordinals $i$. With the same reasoning as above (where we proved that $e_j$ is a metric for limit ordinals $j$) this implies that $x$ and $y$ are equal. \qed