Formal BFV-Type Representation of Path-Integral for Dynamical System with Second Class Constraints

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Abstract

It is shown that the phase space of a dynamical system subject to second class constraints can be extended by ghost variables in such a way that some formal analogies of the Ω-charge and the unitarizing Hamiltonian can be constructed. Then BFV-type path integral representation for the generating functional of Green’s functions is written and shown to coincide with the standard one.

It is known that an embedding of second class constraints into the Hamiltonian BFV-quantization scheme can be done in several different ways: by application of the conversion methods [1, 2], by passing to holomorphic representation of the constraints [3, 4], and within the framework of the unified constrained dynamics [5]. Unfortunately, some restrictions on a structure of the initial constraint system must be imposed in each of these schemes. By this reason, a direct application of the developed methods in a manifestly Poincaré covariant fashion turns out to be problematic for some concrete models (see [3, 7] and references therein). In particular, for those cases the problem of constructing a manifestly covariant and really calculable expression for the generating functional of Green’s functions seems to have no a fully satisfactory solution in the path integral quantization context [8–10]. The purpose of this letter is to demonstrate that for a dynamical system subject to second class constraints $G_\alpha \approx 0$, some formal

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quantities analogous to the Ω-charge and the unitarizing Hamiltonian can be constructed. Then the standard expression for the path integral 

\[ Z = \int dz \det^{1/2}(G_\alpha, G_\beta) \delta(G) \exp i \int d\tau (p_A q^A - H_0) \]  

(1)
can be rewritten in BFV-like form (see Eqs. (21), (22) below). It allows to hope that this formal construction will be useful in attempts to treat quantization of second class constraints on the same footing as that of first class ones.

Consider a dynamical system with phase space variables \( z^A \equiv (q^A, p_A) \), Hamiltonian \( H_0(z) \) and second class constraints \( G_\alpha(z) \approx 0 \):

\[
S_H = \int d\tau (p_A \dot{q}^A - H_0 + \lambda^\alpha G_\alpha), \\
\{G_\alpha, G_\beta\}_{PB} = \Delta_{\alpha\beta}(z), \quad \det \Delta \neq 0, \\
\{G_\alpha, H_0\}_{PB} = V^{\beta}_\alpha(z)G_\beta. 
\]  

(2)

For definiteness, all the phase space coordinates and constraints are supposed to be even: \( \varepsilon(z^A) = \varepsilon(G_\alpha) = 0 \).

A constructing of the Ω-charge. To get the quantity similar to the Ω-charge, with property \( \{\Omega, \Omega\}_{PB} = 0 \), let me extend the initial phase space in the following way:

\[
(q^A, p_A), \lambda^\alpha, (C^\alpha, P_\alpha), D^\alpha, (\alpha, \beta), (\eta, \nu) 
\]  

(3)

where the brackets contain canonically conjugate variables, while \( \lambda^\alpha \) and \( D^\alpha \) are assumed to be dynamically passive. The Grassmann parity, ghost numbers and nonvanishing Poisson brackets of the variables are

\[
\varepsilon(C^\alpha) = \varepsilon(P_\alpha) = \varepsilon(\alpha) = \varepsilon(\beta) = \varepsilon(D^\alpha) = 1, \\
\varepsilon(\eta) = \varepsilon(\nu) = 0; \\
gh C^\alpha = -gh P_\alpha = gh \beta = -gh \alpha = gh \eta = -gh \nu = gh D^\alpha = +1; \\
\{q^A, p_B\} = \delta^A_B; \quad \{C^\alpha, P_\beta\} = -\delta^\alpha_\beta, \quad \{\alpha, \beta\} = -1, \quad \{\eta, \nu\} = 1. 
\]  

(4)

Note that in contrast to conversion methods, no additional variables for conversion of second class constraints into effective first class ones has been introduced. Instead of this, the Ω-charge will be constructed directly in terms of the initial constraints.
The following notation for the expansion terms of a phase space function $X(C^\alpha, \ldots) = \sum_{n=0}^\infty X_n$ in powers of the ghost parameters $C^\alpha$

$$X_n \equiv C^{\alpha_1} \ldots C^{\alpha_n} X_{n,\alpha_1 \ldots \alpha_n},$$
$$X[n] \equiv \sum_{k=0}^n C^{\alpha_1} \ldots C^{\alpha_k} X_{k,\alpha_1 \ldots \alpha_k},$$

(5)

will be used, and analogously for function $X(\alpha, C^\alpha, \ldots) \equiv \sum_{n=0}^\infty (X_{n,0} + \alpha X_{n,1})$.

Introducing now the function $\Omega[1] = C^\alpha G_\alpha$ one has: $\{\Omega[1], \Omega[1]\} = C^\alpha C^\beta \Delta_{\alpha\beta}(z) \equiv W$. Then, by virtue of the Jacobi identity, the quantity

$$\Omega = \Omega[1] + \frac{1}{\sqrt{2}} \alpha W + \frac{1}{\sqrt{2}} \beta$$

(6)

has the needed property $\{\Omega, \Omega\} = 0$ (compare Eq. (6) with that of Refs. [3, 4], where for the case of constraint system in the holomorphic representation a similar construction for the $\Omega$-charge was considered and its cogomologies were investigated).

A constructing of the unitarizing Hamiltonian. The Hamiltonian $H$ is sought as a solution of the equation

$$\{\Omega, H\} = 0,$$

(7)

subject to conditions

$$H|_{\alpha=\beta=C=P=0} = H_0, \quad \varepsilon(H) = 0, \quad \text{gh} \ H = 0.$$  

(8)

The solution will be sought in a form of series expansion in powers of the ghost parameters

$$H = \sum_{n=0}^\infty (H_{n,0} + \alpha H_{n,1}),$$

$$H_{n,0} = C^{\alpha_1} \ldots C^{\alpha_n} (H_{n,0,\alpha_1 \ldots \alpha_n}^{\beta_1 \ldots \beta_n} P_{\beta_1} \ldots P_{\beta_n} +$$
$$+ \beta \hat{H}_{n,0,\alpha_1 \ldots \alpha_n}^{\beta_1 \ldots \beta_n+1} P_{\beta_1} \ldots P_{\beta_{n+1}}),$$

$$H_{n,1} = C^{\alpha_1} \ldots C^{\alpha_n} (H_{n,1,\alpha_1 \ldots \alpha_n}^{\beta_1 \ldots \beta_n-1} P_{\beta_1} \ldots P_{\beta_{n-1}} +$$
$$+ \beta \hat{H}_{n,1,\alpha_1 \ldots \alpha_n}^{\beta_1 \ldots \beta_n} P_{\beta_1} \ldots P_{\beta_n}),$$

(9)

where the structure functions $H_{n,i}$ can be determined step by step after the substitution of Eq. (9) into Eq. (7).
Let me prove the existence theorem for the solution of Eqs. (7) and (8), inductively in powers of \((C)^n\). It is not difficult to find a solution in the lowest orders \(n = 1\) and \(n = 2\):

\[
H_{[2]} = H_0 - C^\alpha V_\beta P_\beta - \sqrt{2}\alpha C^\alpha C^\beta \{G_\alpha, V_\beta \} P_\gamma, \tag{10}
\]

then

\[
\{\Omega, H_{[2]}\}_2 = 0, \quad \text{or equivalently} \quad \{\Omega, H_{[2]}\} = X_3 + \ldots. \tag{11}
\]

Now, suppose that a solution of the problem is known in \((C)^n\) order, so that \(H_{[n]}\) is obtained, and

\[
\{\Omega, H_{[n]}\} = X_{n+1} + \ldots. \tag{12}
\]

In order to find \(H_{n+1}\), note that \((C)^{n+1}\) order in Eq. (7) has the following structure:

\[
\{\Omega, H\}_n = -(\delta + 1 \frac{1}{\sqrt{2}} \frac{\partial}{\partial \alpha}) H_{n+1} + X_{n+1} = 0,
\]

\[
X_{n+1} \equiv \{\Omega, H_{[n]}\}_{n+1}, \tag{13}
\]

where \(\delta \equiv G_\alpha \frac{\partial}{\partial P_\alpha}\) is an analog of the Koszul–Tate differential for second class constraints. Inserting the expansions \(H_{n+1} = H_{n+1,0} + \alpha H_{n+1,1}\), \(X_{n+1} = X_{n+1,0} + \alpha X_{n+1,1}\) into Eq. (13), one gets that the following two equations must be fulfilled separately

\[
\delta H_{n+1,1} = -X_{n+1,1}, \tag{14}
\]

\[
\delta H_{n+1,0} = X_{n+1,0} - \frac{1}{\sqrt{2}} H_{n+1,1}. \tag{15}
\]

To investigate these equations one can use the well known properties of the \(\delta\)-operator [12, 13]: Let the initial constraints are of the form \(G_\alpha = p_\alpha - f_\alpha(q^A, p_i)\), where \(p_\alpha\) are part of momenta \(p_A\) (which always can be done). Then any regular solution of the equation \(\delta X = 0\) obeying \(X|_{P=0} = 0\) is that \(X = \delta K\). Further, a necessary and sufficient condition for the existence of solutions to the inhomogeneous equation \(\delta X = Y\) with the unknown \(X\) is that \(\delta Y = 0\).

To be convinced that \(\delta \) [r.h.s. of Eqs. (14), (15)] = 0, note that the \((n + 1)\)-th order of the Jacobi identity \(\{\Omega, \{\Omega, H_{[n]}\}\}\) = 0 leads to the
equation
\[
\{\Omega, \{\Omega, H[n]\}\}\}_{n+1} = G_\alpha\{C^\alpha, X_{n+1}\} + \frac{1}{\sqrt{2}}\{\beta, X_{n+1}\} =
\]
\[
= - \left(\delta + \frac{1}{\sqrt{2}}\frac{\partial}{\partial \alpha}\right) X_{n+1} = -\delta X_{n+1,0} + \alpha \delta X_{n+1,1} - \frac{1}{\sqrt{2}}X_{n+1,1} = 0, \quad (16)
\]
or equivalently
\[
\delta X_{n+1,1} = 0, \quad (17)
\]
\[
\delta X_{n+1,0} = -\frac{1}{\sqrt{2}}X_{n+1,1}. \quad (18)
\]

From Eq. (17) it follows that there exists a solution \(H_{n+1,1}\) of Eq. (14). Assuming it is found and substituting it into Eq. (15), one gets, by virtue of Eqs. (14), (18): \(\delta[\text{r.h.s. of Eq. (15)}] = 0\), which proves the existence of a solution of Eq. (15) too. In the result, the solution of the problem (7), (8) has been found in the \((C)^{n+1}\) order.

To conclude the subsection, let me discuss an ambiguity in determining the Hamiltonian \(H\), which has two sources:

(i) By construction \(H_{n+1}\) is found from equation of the form \(\delta X = Y\), solution of which is not unique: given a solution \(X\) the quantity \(X + \delta K\) is a solution also.

(ii) In the initial formulation, one could start from equivalent constraints \(G'_\alpha\) instead of \(G_\alpha\): \(G_\alpha = d_\alpha^\beta G'_\beta\), det \(d \neq 0\) (note that I have used the properties of the \(\delta\)-operator, which were formulated for the constraint system of a special structure only).

It can be shown that the operator \(\tilde{\delta} \equiv \left(\delta + \frac{1}{\sqrt{2}}\frac{\partial}{\partial \alpha}\right)\), which first appeared in Eq. (13), has the same properties: \(\tilde{\delta}X = 0 \Rightarrow X = \tilde{\delta}K, K = K_0 + \alpha K_1\), as the \(\delta\)-operator has. Using this fact, one can repeat the standard reasoning [13] for describing the ambiguity in determining \(H\), and the final answer looks as follows: let \(H = H_0 + \ldots\) and \(\Omega\) be a solution of Eq. (7), where the \(\Omega\) is given in the form (6). Then any another solution \(H' = H_0 + \ldots\) and \(\Omega'\) (in particular \(\Omega'\) may be constructed with the help of equivalent constraints: \(\Omega' = C^\alpha G'_\alpha + \ldots\)) is related to the initial one as follows:
\[
H' = e^{\hat{X}}(H + \{\Omega, Y\}), \quad \Omega' = e^{\hat{X}}\Omega;
\]
\[
\hat{X}A \equiv \{X, A\}, \quad (19)
\]
with some \(X\) and \(Y\) obeying \(\varepsilon(X) = 0, \text{gh } X = 0, \varepsilon(Y) = 1, \text{gh } Y = -1\).
BFV-type representation for the path integral. Since within the framework of the presented construction part of variables (namely $\lambda^{\alpha}$ and $D^{\alpha}$) are dynamically passive, the following trivial extension of the BFV-theorem [14, 15] will be suitable for building the generating functional of Green’s functions: Let the extended phase space variables are splitted onto two groups: $Z = (Q^{A}, P_{A}; \omega^{a})$ where $\omega^{a}$ are dynamically passive. Let $\{\Omega, \Omega\} = 0$, $\{\Omega, H\} = 0$, where the $\Omega$-charge is independent on $\omega^{a}$. Then the expression

$$Z_{\Psi} = \int dZ \exp i \int_{\tau_{1}}^{\tau_{2}} d\tau [P_{A}\dot{Q}^{A} - H + \{\Psi, \Omega\}]$$

(20)
is independent on change of $\Psi$ provided the standard choice of the BRST-invariant boundary conditions [12] on $Q^{A}, P_{A}$-variables has been done.

This proposition can be proved along the same lines as the standard one.

For the case under consideration one has

$$Z_{\Psi} = \int dZ \exp i \int d\tau [p_{A}\dot{q}^{A} + P_{\alpha}\dot{C}^{\alpha} + \beta \dot{\alpha} + \eta \dot{\nu} - H + \{\Psi, \Omega\}]$$;

(21)

$$\Omega = C^{\alpha}G_{\alpha} + \frac{\alpha}{\sqrt{2}}C^{\alpha}C^{\beta} \Delta_{\alpha\beta} + \frac{1}{\sqrt{2}}\beta,$$

$$H = H_{0} - C^{\alpha}V_{\alpha}^{\beta}P_{\beta} - \sqrt{2}\alpha C^{\alpha}C^{\beta}\{G_{\alpha}, V_{\beta}\}P_{\gamma} +$$

$$+O(C^{3}P^{3}, C^{3}P^{4}\beta, \alpha C^{3}P^{2}, \alpha C^{3}P^{3}).$$

(22)

Now, let me demonstrate that $Z_{\Psi}$ in Eq. (21) exactly coincides with the standard expression of generating functional for a dynamical system with second class constraints that has been written in Eq. (1). The “gauge fixing fermion” of the form

$$\Psi = -\frac{1}{\varepsilon}\lambda^{\alpha}P_{\alpha} - \sqrt{2}\beta - \frac{1}{\varepsilon}\alpha P_{\alpha}D^{\alpha},$$

(23)

where $\varepsilon$ is some numerical parameter, turns out to be suitable for this aim. Substituting Eqs. (22), (23) into Eq. (21) and making use of the following displacement of integration variables

$$\alpha \rightarrow \varepsilon \alpha, \quad \beta \rightarrow \frac{1}{\varepsilon}\beta;$$

$$P_{\alpha} \rightarrow \varepsilon P_{\alpha}, \quad \lambda^{\alpha} \rightarrow \varepsilon \lambda^{\alpha}$$

(24)
with the unit Jacobian, one gets

\[ Z_\Psi = \int dZ \exp i \int d\tau [p_A q^A + \mathcal{P}_\alpha (\frac{1}{\sqrt{2}} D^\alpha + \varepsilon \dot{C}^{\alpha} - \varepsilon C^{\beta} V^{\beta \alpha}) + \beta \dot{\alpha} + \eta \dot{\nu} - H_0 + O(\varepsilon^2 \alpha C^2 P) + (\lambda^\alpha - \alpha D^\alpha) G^\alpha + C^{\alpha} C^{\beta} \Delta_{\alpha \beta} - \sqrt{2} \varepsilon \alpha \lambda C^{\beta} \Delta_{\alpha \beta}] \tag{25} \]

Since \( Z_\Psi \) is independent on a change of \( \Psi \) and, as a consequence, on a change of \( \varepsilon \), let me pass to the limit \( \varepsilon \to 0 \). After that, by making use of the displacement \( \lambda^\alpha \to \lambda^\alpha + \alpha D^\alpha \) with the unit Jacobian, one obtains the expression for \( Z_\Psi \), in which the integrations over \( \mathcal{P}_\alpha, D^\alpha \) can be immediately performed. The integrations over \( \alpha, \beta, \eta, \nu \) variables are performed by transition on discrete lattice, and after the necessary regularization \[^{14}\], one obtains

\[ \int d\alpha d\beta d\eta d\nu \exp i \int d\tau (\beta \dot{\alpha} + \eta \dot{\nu}) = \lim_{\varphi \to 0} \int d\alpha(0) d\nu(0) \exp i(\varphi \alpha - \varphi^2 \pi \nu^2) = 1. \tag{26} \]

(Thus an introduction of the ghosts \( \eta, \nu \) technically yields the correct balance between the divergent integral contributions.) The resulting expression for \( Z_\Psi \) is simply coincides with Eq. (1), as has been stated.

Note that the generating functional \( Z_\Psi \) is independent on the natural arbitrariness in constructing of \( H \) and \( \Omega \) being described by Eq. (19). It can be proved following the same arguments as in Ref. 13.

To conclude, in this article for a dynamical system subject to second class constraints some specific extension of the initial phase space by ghost variables has been suggested. It allows to construct formal BFV-type representation (21), (22) for the path integral in a similar framework as for the case of first class constraints. The only unusual property of the construction is that the ghost number of the “gauge fixing fermion” (23) is not fixed.

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