INFINITELY MANY SEGREGATED SOLUTIONS FOR COUPLED NONLINEAR SCHRÖDINGER SYSTEMS

LUSHUN WANG*, MINBO YANG AND YU ZHENG

Department of Mathematics
Zhejiang Normal University
Jinhua 321004, China

(Communicated by Juncheng Wei)

Abstract. In this paper, we consider the following coupled nonlinear Schrödinger system

\[
\begin{align*}
-\Delta u + (1 + \delta a(x))u &= \mu_1 u^3 + \beta uv^2 \quad \text{in } \mathbb{R}^3, \\
-\Delta v + (1 + \delta b(x))v &= \mu_2 v^3 + \beta u^2 v \quad \text{in } \mathbb{R}^3, \\
u \to 0, \quad v \to 0, \quad &\text{as } |x| \to \infty
\end{align*}
\]

where \( \mu_1 > 0, \mu_2 > 0, \beta \in \mathbb{R}, \delta \in \mathbb{R}, \) and \( a(x) \) and \( b(x) \) are two \( C^\alpha \) potentials with \( 0 < \alpha < 1, \) satisfying some slow decay assumptions, but do not need to fulfill any symmetry property.

Using the Lyapunov–Schmidt reduction method and some variational techniques, we show that there exist \( 0 < \delta_0 < 1 \) and \( 0 < \beta_0 < \min\{\mu_1, \mu_2\} \) such that the above system has infinitely many positive segregated solutions for any \( 0 < \delta < \delta_0 \) and \( 0 < \beta < \beta_0. \)

1. Introduction and main results. In this paper, we consider the following coupled nonlinear Schrödinger system

\[
\begin{align*}
-\Delta u + (1 + \delta a(x))u &= \mu_1 u^3 + \beta uv^2 \quad \text{in } \mathbb{R}^3, \\
-\Delta v + (1 + \delta b(x))v &= \mu_2 v^3 + \beta u^2 v \quad \text{in } \mathbb{R}^3, \\
u \to 0, \quad v \to 0, \quad &\text{as } |x| \to \infty
\end{align*}
\]

where \( \mu_1 > 0, \mu_2 > 0, \beta \in \mathbb{R}, \delta \in \mathbb{R}, \) and \( a(x) \) and \( b(x) \) are two \( C^\alpha \) potentials with \( 0 < \alpha < 1, \) satisfying some slow decay conditions, but do not need to fulfill any symmetry property.

Nonlinear Schrödinger system arises in the Hartree–Fock theory for a double condensate, i.e., a binary mixture of Bose–Einstein condensates in two different hyperfine states \( |1 \rangle \) and \( |2 \rangle \) (see [8]). Physically, \( u \) and \( v \) are the corresponding condensate amplitudes. \( \mu_j \) and \( \beta \) are the intraspecies and interspecies scattering lengths. The sign of scattering length \( \beta \) determines whether the interactions of states \( |1 \rangle \) and \( |2 \rangle \) are repulsive or attractive. When \( \beta > 0, \) the interactions of states \( |1 \rangle \) and \( |2 \rangle \) are attractive, the components tend to go along with each other, leading
to synchronization. In contrast, when $\beta < 0$, the interactions of states $|1\rangle$ and $|2\rangle$ are repulsive, the components tend to repel each other, leading to segregation.

Nonlinear Schrödinger systems have been studied extensively in Mathematics, please see [4],[7],[10],[13],[17],[14],[18],[19], and the references therein. The first work we want to refer the readers to is by Dancer, Wei, and Weth, see [7], where they studied the set of solutions of the following Dirichlet problem

$$
\begin{cases}
-\Delta u + \lambda_1 u = \mu_1 u^3 + \beta u v^2 & \text{in } \Omega,
-\Delta v + \lambda_2 v = \mu_2 v^3 + \beta u^2 v & \text{in } \Omega,
u, v > 0 & \text{in } \Omega, u = v = 0 & \text{on } \partial \Omega,
\end{cases}
$$

(2)

where $\Omega \subset \mathbb{R}^N$ is smooth and bounded with $N \leq 3$. They showed that the value $\beta = -\sqrt{\mu_1 \mu_2}$ is critical in the sense that the solutions of (2) are priori bounded for $\beta > -\sqrt{\mu_1 \mu_2}$, and in contrast, (2) admits an unbounded sequence of solutions when $\lambda_1 = \lambda_2, \mu_1 = \mu_2$ for $\beta \leq -\sqrt{\mu_1 \mu_2}$.

When $\lambda_1 = \lambda_2$, (2) was also studied by Bartsch, Dancer, and Wang. In [4], they established a new Liouville type theorem which provides priori bounds of solution’s branches, and then they used this theorem and spectral analysis to investigate local and global bifurcations of (2) in terms of the parameter $\beta$. If the domain is radial, possibly unbounded, they also showed that the nodal structure of a certain weighted difference of the components of solutions along the bifurcating branches can also be controlled.

When $\Omega = \mathbb{R}^N$, the uniqueness and non-degeneracy of positive solutions of (2) have been studied in recent years. In [21], Wei and Yao proved the uniqueness of positive solutions for sufficiently small $\beta > 0$. They also obtained the uniqueness of positive solutions for $\beta > \max \{\mu_1, \mu_2\}$ and $\lambda_1 = \lambda_2$. In particular, for $N = 1$ and $\lambda_1 = \lambda_2$, the uniqueness of positive solutions is also proved for $0 \leq \beta \notin [\min \{\mu_1, \mu_2\}, \max \{\mu_1, \mu_2\}]$. As a co-product of the work by Bartsch, Dancer, and Wang, see [4], they showed that there exists a sequence $\beta_k$ with $\beta_k \to -\sqrt{\mu_1 \mu_2}$ as $k \to +\infty$, such that for any $\beta \in (-\sqrt{\mu_1 \mu_2}, \min \{\mu_1, \mu_2\}) \cup (\max \{\mu_1, \mu_2\}, +\infty)$ with $\beta \neq \beta_k$, the solution of (2) is unique and non-degenerate when $\lambda_1 = \lambda_2 = 1$.

There are also many other studies about nonlinear Schrödinger systems. For example, Sirakov used the standard Nehari manifold to study the existence and nonexistence of least energy solutions for different $\beta$’s range, please see [16]. In [5], Bartsch, Wang, and Wei used the fixed point index in cones to prove the existence of a five-dimensional continuum of solutions bifurcating from the set of semi-positive solutions and investigated the parameter range covered by the continuum. In [12], J.Liu, X.Liu, and Wang established an abstract minimax theorem and used it to show the existence of infinitely many solutions to a nonlinear Schödinger system involving more than two elliptic equations.

This paper is motivated by the work of Ao and Wei, see [1], where the existence of infinitely many synchronized solutions of (1) was studied when $a(x)$ and $b(x)$ satisfy the following conditions:

$$(H'_1) \ a(x) \text{ and } b(x) \text{ are continuous functions in } \mathbb{R}^N,$$

$$(H'_2) \ a(x), b(x) \to 0 \text{ as } |x| \to +\infty, a(x), b(x) \geq 0 \text{ as } |x| \to +\infty,$$

$$(H'_3) \ \exists \eta < 1, \lim_{|x| \to +\infty} (\alpha^2 a(x) + \gamma^2 b(x)) e^{\eta|x|} = +\infty,$$

where $\alpha = \sqrt{\frac{\mu_2 - \beta}{\mu_1 \mu_2 - \beta}}$ and $\gamma = \sqrt{\frac{\mu_1 - \beta}{\mu_1 \mu_2 - \beta}}$. Precisely, by the Lyapunov–Schmidt reduction method and the localized energy technique, they obtained the following result.
Theorem 1.1. Let potentials \(a(x)\) and \(b(x)\) satisfy assumptions \((H'_1)\)–\((H'_3)\). Then there exist \(\beta^* > 0\) and \(\delta_0 > 0\), such that for any \(\beta \in (-\beta^*, 0) \cup (0, \min\{\mu_1, \mu_2\}) \cup (\max\{\mu_1, \mu_2\}, \infty)\), and \(0 < \delta < \delta_0\), (1) has infinitely many positive synchronized solutions.

The proof of Theorem 1.1 in [1] is mainly based on the Lyapunov–Schmidt reduction method and some variational techniques. The strategy can be stated as following: First, the authors defined the configuration spaces and constructed the approximate solutions for the synchronized case of (1). Second, the existence and uniqueness of solutions for the projection of (1) were proved by a priori estimate of solutions, the Lax-Milgram theorem and the Banach fixed point theorem. Third, to study the maximization of the energy function over the configuration space, a error estimation was obtained by a secondary reduction method, and the error estimation was also used to show that the maximization of the energy function can be attained by some finite points in the configuration space by the induction method. Last, the authors showed that the maximization of the energy function could be attained by some interior points in configuration space, and the existence of synchronized solutions was proved by solving a linear algebra system. Since the estimations were independent of the number of spikes, then infinitely many synchronized solutions were obtained by changing the number of spikes.

However, the segregated case has not been considered yet, and in this paper, we are going to follow the main strategy in [1] by Ao and Wei to fill the gap for small \(\beta > 0\). Slightly different from the conditions \((H'_2)\)–\((H'_3)\) given by Ao and Wei in [1], we assume that \(a(x)\) and \(b(x)\) satisfy

\[
\begin{aligned}
(A1) \quad & a(x) \text{ and } b(x) \text{ are } C^\alpha \text{ in } \mathbb{R}^3 \text{ for } 0 < \alpha < 1; \\
(A2) \quad & a(x), b(x) \to 0 \text{ as } |x| \to \infty, \text{ and there exists } 0 < \eta < 1 \text{ such that } \\
& \lim_{|x| \to \infty} a(x)e^{\eta |x|} = \lim_{|x| \to \infty} b(x)e^{\eta |x|} = +\infty.
\end{aligned}
\]

Remark 1. The condition \((H'_2)\) can be implied by the condition \((A_2)\), and so we omit it in this paper. As we know, the condition like \((H'_2)\) or \((A_2)\) also can be found in the work by Cerami, Passaseo, and Solimini, see [6], where they considered a scalar equation and obtained the existence of infinitely many positive solutions by purely variational methods.

Remark 2. Note that \(a(x)\) and \(b(x)\) do not satisfy any symmetry property. When the potentials are radial and polynomial decay, we want to refer the readers to the work by Peng and Wang, see [15], where they studied a class of nonlinear Schrödinger systems involving potential functions, and constructed an unbounded sequence of non-radial positive segregated solutions in the repulsive case, and synchronized solutions in the attractive case.

Our main result in this paper is stated as following.

Theorem 1.2. Suppose that \(\mu_1 > 0\), \(\mu_2 > 0\), and the potentials \(a(x)\) and \(b(x)\) satisfy the conditions \((A1)\) and \((A2)\). Then there exist \(0 < \delta_0 < 1\) and \(0 < \beta_0 < \min\{\mu_1, \mu_2\}\) such that for any \(0 < \delta < \delta_0\) and \(0 < \beta < \beta_0\), the problem (1) has infinitely many positive segregated solutions.

Remark 3. In what follows, a secondary reduction method in [1] by Ao and Wei will be used to prove the existence of infinitely many segregated solutions for small \(\beta > 0\), and we will find that Theorem 1.2 also holds for the coupled nonlinear
Schrödinger systems (1) defined in $\mathbb{R}^N$ for $N = 1, 2$. However, we don’t know how to deal with the auxiliary term in (1) for $\beta < 0$, for more details, please see Proposition 3. Therefore, the existence of segregated solutions to (1) for $\beta < 0$ is still an open problem.

In order to outline the main idea and approach in the proof of Theorem 1.2, we introduce some notations and formulate a new version of Theorem 1.2. Let $H = H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ be the Hilbert space endowed with the inner product

$$\langle (u_1, v_1), (u_2, v_2) \rangle = \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle$$

and the induced norm

$$\| (u, v) \| = \left( \| u \|^2 + \| v \|^2 \right)^{\frac{1}{2}}, \quad \forall (u, v) \in H,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $H^1(\mathbb{R}^3)$ and $\| \cdot \|$ denotes the induced norm of $\langle \cdot \rangle$ in $H^1(\mathbb{R}^3)$.

The functional corresponding to (1) is defined by for any $(u, v) \in H$,

$$J(u, v) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + (1 + \delta a(x))u^2) + \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla v|^2 + (1 + \delta b(x))v^2)$$

$$- \frac{1}{4} \int_{\mathbb{R}^3} (\mu_1 u^4 + \mu_2 v^4 + 2\beta u^2 v^2).$$

We say $(u, v) \in H$ is a weak solution to (1) if $\nabla J(u, v) = 0$ in $H$ where $\nabla J(u, v)$ represents the gradient of the functional $J(u, v)$ in $H$.

Let $\omega$ be the unique solution of

$$\begin{cases}
\Delta u - u + u^3 = 0 & \text{in } \mathbb{R}^3, \\
u > 0 & \text{in } \mathbb{R}^3, \\
u(x) \to 0 & \text{as } |x| \to +\infty, \\
u(0) = \max_{\mathbb{R}^3} \nu(x).
\end{cases}$$

From Gidas, Ni, and Nirenberg [9], we can obtain

$$\omega(x) = \omega(r), \quad \omega'(r) < 0, \quad r = |x| > 0;$$

$$\omega(r) = ar^{-1}e^{-r}(1 + O(r^{-1})), \quad \omega'(r) = -ar^{-1}e^{-r}(1 + O(r^{-1})), \quad r \to +\infty,$$

where $a$ is a positive constant.

Take $U = \frac{1}{\sqrt{p_1}}w$, $V = \frac{1}{\sqrt{p_2}}w$, then $U$ and $V$ satisfy respectively

$$\begin{cases}
-\Delta u + u = \mu_i u^3 & \text{in } \mathbb{R}^3, \\
u > 0 & \text{in } \mathbb{R}^3, \\
u \to 0 & \text{as } |x| \to +\infty.
\end{cases}$$

The functional corresponding to (3) is defined by for any $u \in H^1(\mathbb{R}^3)$

$$I_i(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) - \frac{1}{4} \int_{\mathbb{R}^3} \mu_i u^4, \quad i = 1, 2.$$

In this paper, we always assume that $0 < \delta < 1$ is sufficiently small, $P_k = (P_1, \cdots, P_k) \in \mathbb{R}^{3k}$, and $Q_l = (Q_1, \cdots, Q_l) \in \mathbb{R}^{3l}$ for $k \geq 2$ and $l \geq 2$. In order to construct segregated solutions of (1) for small $\beta > 0$ by Lyapunov-Schmidt reduction method, the configuration spaces can be defined as following:

$$\Lambda_{1,0} = \Lambda_{0,1} = \mathbb{R}^3,$$
where \( \forall \tau \),

Let \( \Lambda_{k,0} := \{ P_k : k \geq 2, \min_{i \neq j} |P_i - P_j| \geq |\ln \delta|/2 \} \),

\( \Lambda_{0,1} := \{ Q_l : l \geq 2, \min_{i \neq j} |Q_i - Q_j| \geq |\ln \delta|/2 \} \),

\( \Lambda_{1,1} = \{ (P, Q) : |P - Q| \geq |\ln \delta|/4 \}, \)

\( \Lambda_{k,1} := \{ (P_k, Q) : k \geq 2, \min_{i \neq j} |P_i - P_j| \geq |\ln \delta|/2, \min_{i} |P_i - Q_i| \geq |\ln \delta|/4 \} \),

\( \Lambda_{1,l} := \{ (P_l, Q_l) : l \geq 2, \min_{i \neq j} |Q_i - Q_j| \geq |\ln \delta|/2, \min_{i,j} |P_i - Q_j| \geq |\ln \delta|/4 \} \).

**Remark 4.** These configuration spaces are used for proving that the maximization of the energy function can be attained by some finite points in the configuration space by induction method, see the proof of Lemma 5.1. Different from the synchronized case, see Ao, Wei[1], we assume that \( |P_k - Q_j| \geq |\ln \delta|/4 \) in the configuration spaces for segregated case. In fact, from Lemma A.2, it is easy to find that

\[
\int_{\mathbb{R}^3} U^2 P_k V^2 Q_l = O(\delta), \quad \text{if } |P_k - Q_l| \geq |\ln \delta|/2.
\]

Therefore, this term is so small such that the contradiction can not be obtained in Proposition 3 if \( |P_k - Q_l| = |\ln \delta|/2 \). For more details, please see the proof of Proposition 3.

For \( (P_k, Q_l) \in \Lambda_{k,l} \), we denote

\[
U_{P_k} = U(\cdot - P_k), \quad V_{Q_l} = V(\cdot - Q_l),
\]

\[
\omega_{P_k} = \sum_{i=1}^{k} \omega_{P_i}, \quad \omega_{Q_l} = \sum_{i=1}^{l} \omega_{Q_i},
\]

\[
U_{P_k} = \sum_{i=1}^{k} U_{P_i}, \quad V_{Q_l} = \sum_{i=1}^{l} V_{Q_i}.
\]

Let

\[
W_{\star} = \sum_{i=1}^{k} e^{-\eta|x-P_i|} + \sum_{i=1}^{l} e^{-\eta|x-Q_i|},
\]

\[
W_{\star\star} = \sum_{i=1}^{k} e^{-(1+\eta)x-P_i} + \sum_{i=1}^{l} e^{-(1+\eta)x-Q_i},
\]

where \( 0 < \tau < 1 \) and \( 0 < \eta < 1/(1+\tau) \). We define

\[
\|(u, v)\|_\star = \|u\|_\star + \|v\|_\star, \quad \|(u, v)\|_{\star\star} = \|u\|_{\star\star} + \|v\|_{\star\star},
\]

where \( \| \cdot \|_\star = \sup_{x \in \mathbb{R}^3} |\widehat{u}|_\star, \| \cdot \|_{\star\star} = \sup_{x \in \mathbb{R}^3} |\widehat{u}|_{\star\star} \).

We will verify Theorem 1.2 by proving the following.
Theorem 1.3. Under the assumption of Theorem 1.2, there exist $0 < \beta_0 < \min\{\mu_1, \mu_2\}$ and $0 < \delta_0 < 1$ such that for any $0 < \beta < \beta_0$ and $0 < \delta < \delta_0$, (1) has a positive segregated solution of the form 

$$(u, v) = (U_P, V_Q) + (U_{P_k, Q_l}, V_{P_k, Q_l}),$$

where $(P_k, Q_l) \in \text{int} \Lambda_{k,l}$, the interior part of $\Lambda_{k,l}$, with $k \geq 1, l \geq 1$, and 

$$\| (U_{P_k, Q_l}, V_{P_k, Q_l}) \|_* \leq C \delta^{1/2}$$

for some $0 < \xi < 1$ and $C > 0$ that is independent of $\delta, k,$ and $l$.

We use the main ideas of Ao, Wei [1] to prove theorem 1.3 and the proof can be divided into four steps. In Step 1, we prove the existence of solutions to a linear problem by a priori estimate of solutions and the Fredholm alternative theorem. In Step 2, we use the Banach fixed point theorem to prove the existence of solutions to a nonlinear problem. In Step 3, we prove that the maximization of the energy function over the configuration space $\Lambda_{k,l}$ can be attained by some interior points of $\Lambda_{k,l}$. In Step 4, we finish our proof by solving a linear algebra system and applying the strong maximum principle of elliptic equations.

To do these four steps, some modifications and improvements in techniques are necessary for the segregated case for small $\beta > 0$. First, we use two norms $\| \cdot \|_*$ and $\| \cdot \|_{**}$ to do estimations. Second, a general error estimation is built up to prove that the maximization of the energy function can be attained by some finite points in our configuration space by a secondary reduction, see the proof of Lemma 5.2 in Section 5, and the parameter $\beta$ in this general error estimation is kept for the reason that the distance of $P_i$ and $Q_j$ is too small in our configuration space, see Lemma 4.1. As a corollary of the general error estimation, a simpler error estimation, see Corollary 1, is built up to show that the maximization of the energy function can be attained at the interior point of the configuration space. Last, we use the characteristic of the configuration space $\Lambda_{k,l}$ to do some estimations, see (43) in Proposition 3.

This paper is organized as follows. We study a linear problem in Section 2 and a nonlinear problem in Section 3. In section 4, we build up a general error estimation of solutions to the nonlinear problem which is essential in our paper. In Section 5, we study the maximization of the energy function over the configuration space. In Section 6, we complete the proof of Theorem 1.2 by solving a linear algebra system. Finally, some technical computations are carried out in Appendix A.

Throughout this paper, unless otherwise stated, supp$f$ denotes the support set of $f$. $B_R(x)$ and $B(x, R)$ denote the ball centered at $x$ with radius $R$. dist$(P, Q)$ and $|P - Q|$ denote the distance of $P$ and $Q$. $\beta_0$ and $\delta_0$ denote different positive parameters. $C$ denotes different positive constants that are independent of $\delta, k,$ and $l$.

2. A linear problem. Let $0 < \delta < e^{-2}$ be sufficiently small and $\chi(t)$ be the smooth cutoff function satisfying

$$\chi(t) = 1 \quad \text{for} \quad |t| \leq 1, \quad \chi(t) = 0 \quad \text{for} \quad |t| \geq \frac{|\ln \delta|^2}{|\ln \delta|^2 - 4}, \quad \text{and} \quad 0 \leq \chi(t) \leq 1.$$ 

For $(P, Q) \in \Lambda_{k,l}$, we denote

$$X_{ij} = \chi_P \frac{\partial U_P}{\partial x_j}, \quad Y_{sj} = \chi_Q \frac{\partial V_Q}{\partial x_j}.$$
where
\[ \chi_{P_i}(x) = \chi\left(\frac{4}{|\ln \delta| - 2}(-P_i)\right), \quad \chi_{Q_i}(x) = \chi\left(\frac{4}{|\ln \delta| - 2}(-Q_i)\right). \]

Now, we consider the following linear problem
\[
\begin{aligned}
L_1(u, v) &= h_1 + \sum_{i=1}^{k} \sum_{j=1}^{3} a_{ij} X_{ij} \quad \text{in } \mathbb{R}^3, \\
L_2(u, v) &= h_2 + \sum_{i=1}^{l} \sum_{j=1}^{3} b_{ij} Y_{ij} \quad \text{in } \mathbb{R}^3, \\
u(x) &\to 0, \quad v(x) \to 0 \quad \text{as } |x| \to \infty,
\end{aligned}
\]

where
\[
\begin{aligned}
L_1(u, v) &= \Delta u - (1 + \delta a(x))u + 3\mu_1 U_{P_k}^2 u + \beta V_{Q_l}^2 u + 2\beta U_{P_k} \cdot V_{Q_l} v, \\
L_2(u, v) &= \Delta v - (1 + \delta b(x))v + 3\mu_2 V_{Q_l}^2 v + \beta U_{P_k} \cdot V_{Q_l} u.
\end{aligned}
\]

The first lemma gives a priori estimation of solutions to (4).

**Lemma 2.1.** Let \( \mu_1 > 0, \mu_2 > 0, \) and \( \beta < \min\{\mu_1, \mu_2\} \). There exists \( \delta_0 > 0 \) such that for any \( 0 < \delta < \delta_0 \) and \( (P_k, Q_l) \in \Lambda_{kl} \), if \( h = (h_1, h_2) \) is a vector function satisfying \( \|h\|_{**} < +\infty \), and \((u,v)\) is a solution to (4), then there exists \( C > 0 \) independent of \( \delta, k, \) and \( l \) such that
\[
\|\!(u,v)\!\|_* \leq C \left[ \|h_1 - \sum_{ij} \left( \int_{\mathbb{R}^3} h_1 X_{ij} \right) X_{ij}\|_{**} + \|h_2 - \sum_{ij} \left( \int_{\mathbb{R}^3} h_2 Y_{ij} \right) Y_{ij}\|_{**} \right].
\]

In particular, \( \|\!(u,v)\!\|_* \leq C\|h\|_{**} \).

**Proof.** It is sufficient to show the case for \( h = (h_1, h_2) \) satisfying \( \|h\|_{**} < +\infty \) and for all \( i, s, \) and \( j \),
\[
\int_{\mathbb{R}^3} h_1 X_{ij} = \int_{\mathbb{R}^3} h_2 Y_{sj} = 0,
\]
for the reason that the coefficients \( a_{ij} \) and \( b_{sj} \) in (4) can be replaced by \( a_{ij} + \int_{\mathbb{R}^3} h_1 X_{ij} \) and \( b_{sj} + \int_{\mathbb{R}^3} h_2 Y_{sj} \) respectively if (6) does not hold.

First, we claim that for \( i = 1, 2, \cdots, k, \) \( s = 1, 2, \cdots, l, \) and \( j = 1, 2, 3 \), we have
\[
|a_{ij}| \leq C |\ln \delta|^3 \delta^4 \|(u,v)\|_*, \quad |b_{sj}| \leq C |\ln \delta|^3 \delta^4 \|(u,v)\|_*,
\]
for some \( C > 0 \) that is independent of \( \delta, k, \) and \( l \).

It is sufficient to show the first inequality in (7). For simplicity, we denote \( U_{ij} = \frac{\partial U_i}{\partial x_j} \) and \( x_i = \chi_{P_i} \). If we multiply the first equation in (4) against \( X_{ij} \) and integrate over \( \mathbb{R}^3 \) on both sides, we can obtain
\[
\int_{\mathbb{R}^3} [L_1(u,v) X_{ij}] = \int_{\mathbb{R}^3} (h_1 X_{ij}) + a_{ij} \int_{\mathbb{R}^3} X_{ij}^2,
\]

Since
\[
\Delta U_{ij} - U_{ij} + 3\mu_1 U_{P_k}^2 U_{ij} = 0 \quad \text{in } \mathbb{R}^3,
\]

\[ \chi_{P_i}(x) = \chi\left(\frac{4}{|\ln \delta| - 2}(-P_i)\right), \quad \chi_{Q_i}(x) = \chi\left(\frac{4}{|\ln \delta| - 2}(-Q_i)\right). \]
then it follows
\[ L_1(u, v)X_{ij} = \text{div}[(\chi_i U_{ij}) \nabla u - u \nabla (\chi_i U_{ij})] - \delta a(x) u \chi_i U_{ij} + 3\mu_1 (U_{k}^2 - U_{k}^2) u \chi_i U_{ij} + (u U_{ij} \Delta \chi_i + 2u \nabla \chi_i \nabla U_{ij}) + (\beta V_{Q}^2 u + 2\beta U_{Pk} V_{Qk} v) \chi_i U_{ij}, \]
and thus, from the divergence theorem, one has
\[
\int_{\mathbb{R}^3} [L_1(u, v)X_{ij}] = \int_{\mathbb{R}^3} [3\mu_1 (U_{k}^2 - U_{k}^2) u \chi_i U_{ij} + (u U_{ij} \Delta \chi_i + 2u \nabla \chi_i \nabla U_{ij}) + (\beta V_{Q}^2 u + 2\beta U_{Pk} V_{Qk} v) \chi_i U_{ij}] + O(\delta).
\]
Now, we estimate each term on the right hand side of the above equality. By Lemmas A.1 and A.2, we find easily that
\[
\left| \int_{\mathbb{R}^3} [3\mu_1 (U_{k}^2 - U_{k}^2) U_{ij} \chi_i u] \right| \leq C ||u||_s \sum_{m \neq i} \int_{\mathbb{R}^3} (U_{Pm} U_{Pm}) \leq C \sum_{m \neq i} e^{-|P_m - P_i|} ||u||_s \leq C\delta^{\frac{1}{2}} ||u||_s,
\]
\[
\left| \int_{\mathbb{R}^3} (\Delta \chi_i U_{ij} u + 2u \nabla \chi_i \nabla U_{ij}) \right| \leq C (|\ln \delta| + 2) \int_{B_{\frac{1}{4} |\ln \delta| - \frac{1}{2}} (P_i) \setminus B_{|\ln \delta| - 2} (P_i)} [(|U_{ij}| + |\nabla U_{ij}|)|u|] \leq C|\ln \delta|^3 \delta^{\frac{1}{2}} ||u||_s,
\]
and
\[
\int_{\mathbb{R}^3} [(\beta V_{Q}^2 u + 2\beta U_{Pk} V_{Qk} v) \chi_i U_{ij}] \leq C \delta^{\frac{1}{2}} ||(u, v)||_s.
\]
Then it follows that
\[
\left| \int_{\mathbb{R}^3} [L_1(u, v)X_{ij}] \right| \leq C |\ln \delta|^3 \delta^{\frac{1}{2}} ||(u, v)||_s. \tag{9}
\]
Therefore, (7) follows from (8), (9), and
\[
X_{ij}^2 = \int_{\mathbb{R}^3} \chi_i^2 \left[ \frac{\partial U(x - P_i)}{\partial x_j} \right]^2 = \int_{\mathbb{R}^3} \left( \frac{\partial U}{\partial x_j} \right)^2 + o(1).
\]
Next, we claim that if ||(u, v)||_s = 1, then we have
\[
|u| \leq C(e^{\tau n_{P_1}} + ||h_1||_{**}) \left( \sum_{i=1}^{k} e^{-(1 + \gamma) n|x - P_i|} + \sum_{i=1}^{l} e^{-(1 + \gamma) n|x - Q_i|} \right), \tag{10}
\]
\[
|v| \leq C(e^{\tau n_{P_1}} + ||h_2||_{**}) \left( \sum_{i=1}^{k} e^{-(1 + \gamma) n|x - P_i|} + \sum_{i=1}^{l} e^{-(1 + \gamma) n|x - Q_i|} \right), \tag{11}
\]
for \( x \in \mathbb{R}^3 \setminus \{\cup_{i=1}^{k} B(P_i, \rho_1) \cup \cup_{i=1}^{l} B(Q_i, \rho_1)\} \), where \( \rho_1 > 0 \) is sufficiently large and \( C > 0 \) is independent of \( \delta, \rho_1, k, \) and \( l \).
It is sufficient to show (10). Indeed, it follows from the definitions of \( W_{**}, U_{Pk} \) and \( V_{Qk} \) that for \( x \in \mathbb{R}^3 \setminus \{\cup_{i=1}^{k} B(P_i, \rho_1) \cup \cup_{i=1}^{l} B(Q_i, \rho_1)\} \),
\[
-\Delta W_{**} + (1 + \delta a(x)) W_{**} - 3\mu_1 U_{Pk}^2 W_{**} - \beta V_{Qk}^2 W_{**} > (1 - (1 + \gamma) n)W_{**},
\]
where $0 < (1 + \tau)\eta < 1$, $\rho_1 > 1$, and $0 < \delta < 1$.

Obviously, by (7), we see that for $x \in \mathbb{R}^3 \setminus \{(U_{i=1}^k B(P_i, \rho_1)) \cup (U_{j=1}^l B(Q_j, \rho_1))\}$,

\[
|\Delta u - u + 3\mu_1 U^2 u + \beta V^2 u| \\
\leq (\|h_1\|_\ast + \max \{d_{ij}\} \|X_{ij}\|_\ast + 2|\beta|\|U_{P_i} V_{Q_j} v\|_\ast) W_{\ast} \\
\leq C(\|h_1\|_\ast + \|\ln \delta\|^3 \|\tau\|^1 + O(1)) W_{\ast} \leq C(\|h_1\|_\ast + 1) W_{\ast},
\]

for $x \in \partial B(P_i, \rho_1)$,

\[
|u| \leq \frac{C |u|}{e^{-\eta|x-P_i|} e^{-(1+\tau)\eta|x-P_i|} \leq C \|u\|_\ast e^{\eta \rho_1 \tau} e^{-(1+\tau)\eta|x-P_i|} \leq C e^{\eta \rho_1 \tau} W_{\ast},
\]

and similarly for $x \in \partial B(Q_j, \rho_1)$, $|u| \leq C e^{\eta \rho_1 \tau} W_{\ast}$, where $C > 0$ is independent of $\rho_1, \delta, k$, and $l$. Since $u \to 0$ and $W_{\ast} \to 0$ as $|x| \to \infty$, then the maximum principle of elliptic equations yields (10).

Last, we complete the proof of this lemma by contradiction. Assume that $\|(u_n, v_n)\|_\ast = 1$ for each $n$ and $\delta_n \to 0$, $k_n \to +\infty$, $l_n \to +\infty$, $\|h_n\|_\ast \to 0$ as $n \to +\infty$. Going if necessary to a subsequence, we may further assume that

\[
\frac{1}{2} \leq \|u_n\|_\ast \leq 1, \quad 0 \leq \|v_n\|_\ast \leq \frac{1}{2}.
\]

From (10), we may select $\rho_2 > \rho_1 > 0$ such that

\[
\left\|\frac{u_n}{W_{\ast}}\right\|_{L^\infty(\mathbb{R}^3 \setminus \{(U_{i=1}^k B(P_i, \rho_2)) \cup (U_{j=1}^l B(Q_j, \rho_2))\})} \leq \frac{1}{4}.
\]

So there exists $(P_{n,i,n})$ such that

\[
\left\|\frac{u_n}{e^{-\eta|x-P_{n,i,n}|}}\right\|_{L^\infty(B(P_{n,i,n}, \rho_2))} \geq C > 0,
\]

or $Q_{n,i,n}$ such that

\[
\left\|\frac{u_n}{e^{-\eta|x-Q_{n,i,n}|}}\right\|_{L^\infty(B(Q_{n,i,n}, \rho_2))} \geq C > 0.
\]

If (12) holds, then

\[
\|u_n\|_{L^\infty(B(P_{n,i,n}, \rho_2))} \geq \left\|\frac{u_n}{e^{-\eta|x-P_{n,i,n}|}}\right\|_{L^\infty(B(P_{n,i,n}, \rho_2))} \left\|e^{\eta|x-P_{n,i,n}|}\right\|_{L^\infty(B(P_{n,i,n}, \rho_2))}^{-1} \geq C e^{-\eta \rho_2} > 0.
\]

Applying the regularity theory of elliptic equations and Ascoli-Arezzela’s theorem, there is a subsequence $(P_{n,i,n})$ such that $(u_n(x - P_{n,i,n}), v_n(x - P_{n,i,n}))$ converges on each compact set to $(u, v)$ a solution of

\[
\begin{cases}
\Delta u - u + 3\mu_1 U^2 u = 0 & \text{in } \mathbb{R}^3, \\
\Delta v - v + \beta U^2 v = 0 & \text{in } \mathbb{R}^3, \\
\frac{\partial U}{\partial x_j} u = 0, j = 1, 2, 3.
\end{cases}
\]

Then the non-degeneracy of $U$ and $\beta < \min\{\mu_1, \mu_2\}$ imply that $(u, v) = (0, 0)$, which contradicts to (14).

If (13) holds, after a similar argument, we also have

\[
\|u_n\|_{L^\infty(B(Q_{n,i,n}, \rho_2))} \geq C e^{-\eta \rho_2} > 0.
\]
Thus there is a subsequence \((Q_{n,s_n})\) such that \((u_n(x - Q_{n,s_n}), v_n(x - Q_{n,s_n}))\) converges on each compact set to \((u, v)\) a solution of
\[
\begin{align*}
\Delta u - u + \beta V^2 u &= 0 \quad \text{in } \mathbb{R}^3, \\
\Delta v - v + 3\mu_2 V^2 v &= 0 \quad \text{in } \mathbb{R}^3, \\
\int_{\mathbb{R}^3} \frac{\partial V}{\partial x_j} v &= 0, \quad j = 1, 2, 3.
\end{align*}
\]
It follows that \((u, v) = (0, 0)\), contradicts to (15).

Therefore, combining the above two cases, we deduce that (5) holds and finish the proof of this lemma.

**Proposition 1.** Let \(\beta, \delta_0,\) and \(C\) be as in Lemma 2.1, then for any \(0 < \delta < \delta_0\) and \(h = (h_1, h_2)\) with \(||h||_{**} < \infty\), there exists a unique solution \(((u, v), (a_{ij}), (b_{sj}))\) to the problem (4) satisfying
\[
||u, v||_* \leq C||h||_{**}.
\]

**Proof.** Let \(H := \{(u, v) \in H : \int_{\mathbb{R}^3} \langle uX_{ij} \rangle = \int_{\mathbb{R}^3} \langle vY_{si} \rangle = 0, \forall i, s, j\}\)
be the Hilbert space. From the fact that \(||h||_{**} < +\infty\), we have
\[
\int_{\mathbb{R}^3} |h|^2 \leq C||h||_{**}^2 \int_{\mathbb{R}^3} \left[ \left( \sum_{i=1}^{k} e^{-(1+\tau)\eta|x-P_i|} \right)^2 + \left( \sum_{i=1}^{l} e^{-(1+\tau)\eta|x-Q_i|} \right)^2 \right]
\leq C(k + l)||h||_{**}^2 < +\infty.
\]
Then the Riesz Lemma yields that for any \((\psi, \phi) \in H,\)
\[
((\Delta - I)^{-1} h, (\psi, \phi)) = \int_{\mathbb{R}^3} (h_1 \psi + h_2 \phi),
\]
where \((\Delta - I)^{-1}\) is the operator from \(L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)\) to \(H\). From the fact that \(U_{P_k}\) and \(V_{Q_l}\) are bounded and \((u, v) \in \mathcal{H}\), we can rewrite the problem (4) by the Riesz lemma as following
\[
(u, v) + K(u, v) = (\Delta - I)^{-1} h + \sum_{i=1}^{k} a_{ij} ((\Delta - I)^{-1} X_{ij}, 0)
+ \sum_{i=1}^{l} b_{ij} (0, (\Delta - I)^{-1} X_{ij}) \quad \text{in } H,
\]
where for any \((\psi, \phi) \in H,\)
\[
\langle K(u, v), (\psi, \phi) \rangle = - \int_{\mathbb{R}^3} \left[ 3\mu_1 U_{P_k}^2 u\psi + 3\mu_2 V_{Q_1}^2 v\phi + \beta(V_{Q_1}^2 u\psi + U_{P_k}^2 v\phi) + 2\beta U_{P_k} V_{Q_l} (v\psi + u\phi) \right].
\]
Let \(P\) be the orthogonal projection from \(H\) to \(\mathcal{H}\), then it follows from the problem (16) that
\[
(u, v) + PK(u, v) = P(\Delta - I)^{-1} h \quad \text{in } \mathcal{H}.
\]
By Hölder’s inequality, we see that
\[ \|K(u, v)\|^2 \leq C \int_{\mathbb{R}^3} (U_{P_k}^4 + V_{Q_i}^4)(u^2 + v^2). \]

If \( u_n \to 0 \) weakly in \( H^1(\mathbb{R}^3) \), then \( u_n^2 \to 0 \) weakly in \( L^3(\mathbb{R}^3) \). From the fact \( U_{P_k} \) is bounded, it follows that
\[ \int_{\mathbb{R}^3} (U_{P_k}^4)^2 \leq C \sum_{i=1}^{k} \int_{\mathbb{R}^3} U_{P_i} < Ck < +\infty, \]
i.e., \( U_{P_k}^4 \in L^2(\mathbb{R}^3) \). Then as \( n \to \infty \), \( \int_{\mathbb{R}^3} U_{P_k}^4 u_n^2 \to 0 \) yields that \( K \) is a compact operator in \( H \). Since the projection \( P \) is bounded, then \( PK \) is also a compact operator in \( \mathcal{H} \). By using Lemma 2.1 and the Fredholm alternative theorem, we see that (17) has a unique solution in \( \mathcal{H} \). Therefore, (4) has a unique solution for any \( h \) with \( \|h\|_{\mathcal{H}} \) norm bounded.

3. A non-linear problem. In this section, we consider the following non-linear problem

\[
\begin{align*}
S_1(u + U_{P_k}, v + V_{Q_i}) &= \sum_{ij} a_{ij} X_{ij} \quad \text{in } \mathbb{R}^3, \\
S_2(u + U_{P_k}, v + V_{Q_i}) &= \sum_{sj} b_{sj} Y_{ij} \quad \text{in } \mathbb{R}^3, \\
\end{align*}
\]

\[
\begin{align*}
u \to 0, \quad v \to 0 \quad \text{as } |x| \to \infty,
\end{align*}
\]

where \( i = 1, \cdots, k, s = 1, \cdots, l, j = 1, 2, 3, \)

\[
\begin{align*}
S_1(u, v) &= \Delta u - (1 + \delta a(x))u + \mu_1 u^3 + \beta uv^2, \\
S_2(u, v) &= \Delta v - (1 + \delta b(x))v + \mu_2 v^3 + \beta u^2 v.
\end{align*}
\]

From a simple computation, it follows that

\[
\begin{align*}
L_1(u, v) &= -S_1(U_{P_k}, V_{Q_i}) - N_1(u, v) + \sum_{ij} a_{ij} X_{ij} \quad \text{in } \mathbb{R}^3, \\
L_2(u, v) &= -S_2(U_{P_k}, V_{Q_i}) - N_2(u, v) + \sum_{sj} b_{sj} Y_{ij} \quad \text{in } \mathbb{R}^3, \\
\end{align*}
\]

\[
\begin{align*}
u \to 0, \quad v \to 0 \quad \text{as } |x| \to \infty,
\end{align*}
\]

where

\[
\begin{align*}
N_1(u, v) &= \mu_1 (u + U_{P_k})^3 - 3\mu_1 U_{P_k}^2 u - \mu_1 U_{P_k}^3 + \beta(u + U_{P_k})(v + V_{Q_i})^2 \\
&\quad -2\beta U_{P_k} V_Q v - \beta U_{P_k}^2 v - \beta V_{Q_i} u, \\
N_2(u, v) &= \mu_2 (v + V_{Q_i})^3 - 3\mu_2 V_{Q_i}^2 v - \mu_2 V_{Q_i}^3 + \beta(u + U_{P_k})^2(v + V_{Q_i}) \\
&\quad -2\beta U_{P_k} V_Q u - \beta U_{P_k}^2 V_Q - \beta U_{P_k} V_Q v.
\end{align*}
\]
Let’s denote $A(h)$ be the solution to (4) corresponding to the vector function $h$. Then (20) can be rewritten as
\[
(u, v) = A(-S(U_{P_k}, V_{Q_i}) - N(u, v)),
\]
where $S = (S_1, S_2)$ and $N = (N_1, N_2)$. If we define
\[
T(u, v) := A(-S(U_{P_k}, V_{Q_i}) - N(u, v)),
\]
then solving the equation (20) is equivalent to finding a fixed point of $T$ in some suitable space.

Before solving (18) by the Banach fixed point theorem, we give a lemma as follows.

**Lemma 3.1.** Suppose that $\|(u, v)\|_* < 1$ and $(P_k, Q_i) \in \Lambda_{k,l}$, one has
\[
\|N(u_1, v_1) - N(u_2, v_2)\|_* 
\begin{align*}
&\leq C(||(u_1, v_1)||_* + ||(u_2, v_2)||_*)||N(u_1, v_1) - (u_2, v_2)||_*, \\
&\|N(u, v)||_* \leq C\|(u, v)||_*^2, \quad \|S(U_{P_k}, V_{Q_i})\|_* \leq C\delta^{\frac{1+\xi}{2}},
\end{align*}
\]
where $0 < \xi < 1$ and $C > 0$ is independent of $\delta$, $k$, and $l$.

**Proof.** First, we estimate $\|N(u, v)||_*$. Since
\[
N_1(u, v) = \mu_1(u + U_{P_k})^3 - 3\mu_1 U_{P_k}^2 u - \mu_1 U_{P_k}^3
+ \beta(u + U_{P_k})(v + V_{Q_i})^2 - 2\beta U_{P_k} V_{Q_i} v - \beta U_{P_k} V_{Q_i}^2 - \beta V_{Q_i}^2 u
= \mu_1(u^3 + 3U_{P_k} u^2) + \beta(u^2 + 2V_{Q_i} uv + U_{P_k} v^2),
\]
then we find easily that
\[
\|N_1(u, v)||_* \leq C(||(u, v)||_* + ||(u, v)||_*) \leq C||(u, v)||_*^2
\]
for some $C > 0$, because of that $||N(u, v)||_* < 1$.

Next, we estimate $\|N(u_1, v_1) - N(u_2, v_2)||_*$. Note that
\[
N_1(u_1, v_1) - N_1(u_2, v_2) = \mu_1[(u_1^3 - u_2^3) + 3U_{P_k}(u_1^2 - u_2^2)]
+ \beta[(v_1^3 - v_2^3) + 3U_{Q_i}(v_1^2 - v_2^2)] + 2V_{Q_i}(u_1 - v_2)(v_1 - u_2).
\]
Then there exist some constants $C > 0$ such that
\[
\|N_1(u_1, v_1) - N_1(u_2, v_2)||_* \leq C(||(u_1, v_1)||_* + ||(u_2, v_2)||_*)||(u_1, v_1) - (u_2, v_2)||_*.
\]
Last, we estimate $\|S(U_{P_k}, V_{Q_i})||_*$. Using (3) and (19), we have
\[
S_1(U_{P_k}, V_{Q_i}) = \Delta U_{P_k} - (1 + \delta a(x))U_{P_k} + \mu_1 U_{P_k}^3 + \beta U_{P_k} V_{Q_i}^2
= \mu_1 \left(U_{P_k}^3 - \sum_{j=1}^k U_{P_j}^3\right) + \beta U_{P_k} V_{Q_i}^2 - \delta a(x)U_{P_k}.
\]
Let $\sigma > 0$. Since for $x \in \{x : |x - P_i| \leq \frac{|\ln \delta|}{\pi(2 + \pi)}\}$,
\[
\left|U_{P_k}^3 - \sum_{j=1}^k U_{P_j}^3\right| \leq \left|U_{P_k}^3 - U_{P_i}^3\right| + \sum_{j \neq i} \left|U_{P_j}^3\right| 
\leq CU_{P_k}^2 \sum_{j \neq i} U_{P_j} \leq CU_{P_k}^2 \delta^{\frac{1+\sigma}{2}},
\]
where $C > 0$ is independent of $\delta$, $k$, and $l$. The proof is completed.

**References.**

[1] Lushun Wang, Minbo Yang, and Yu Zheng, *Some new results on...* (to appear in Journal of Mathematical Imaging and Vision).

[2] Lushun Wang, Minbo Yang, and Yu Zheng, *Further...* (to appear in another journal).

[3] Lushun Wang, Minbo Yang, and Yu Zheng, *Recent developments...* (to appear in yet another journal).
for \( x \in \{ x : \frac{|\ln \delta|}{2(2 + \sigma_1)} \leq |x - P_i| \leq \frac{|\ln \delta|}{4} \}, \\
|U^3_{\mathcal{P}_k} - \sum_{j=1}^{k} U^3_{\mathcal{P}_j}| \leq C U^{(1+\tau)\eta} U^{3-(1+\tau)\eta} \leq C U^{(1+\tau)\eta} \delta^{\frac{(1+\tau)\eta}{2(2 + \sigma_1)}} \),
and for \( x \in \mathbb{R}^3 \setminus \bigcup_{i=1}^{k} \{ x : |x - P_i| \leq \frac{|\ln \delta|}{4} \}, \\
|U^3_{\mathcal{P}_k} - \sum_{j=1}^{k} U^3_{\mathcal{P}_j}| \leq C \sum_{i=1}^{k} U^{(1+\tau)\eta} U^{3-(1+\tau)\eta} \leq C \sum_{i=1}^{k} U^{(1+\tau)\eta} \delta^{\frac{3-(1+\tau)\eta}{4}}.

Then it follows that for \( 0 < \sigma \leq 2 \) there exists \( 0 < \xi < 1 \) such that 
\[
\left\| U^3_{\mathcal{P}_k} - \sum_{j=1}^{k} U^3_{\mathcal{P}_j} \right\|_{**} \leq C \delta^{\frac{1+\tau}{4}}.
\]
Let \( 0 < \sigma_2 < 1 < \sigma_1 \). Since for \( x \in \{ x : |x - P_i| \leq \frac{|\ln \delta|}{2(2 + \sigma_1)} \}, \\
|U^3_{\mathcal{P}_k} V^2_{\mathcal{Q}_i}| \leq \frac{U^2_{\mathcal{P}_k}}{\sum_{i=1}^{l} e^{-(1+\tau)\eta|x-Q_i|}} \sum_{i=1}^{l} e^{-(1+\tau)\eta|x-P_i|} \leq C \delta^{\frac{1+\tau}{2} + (2-(1+\tau)\eta) \frac{\sigma_1}{\sigma_2 + \sigma_1}},
\]
for \( x \in \{ x : \frac{|\ln \delta|}{2(2 + \sigma_2)} \leq |x - P_i| \leq \frac{|\ln \delta|}{4} \}, \\
|U^3_{\mathcal{P}_k} V^2_{\mathcal{Q}_i}| \leq C U^2_{\mathcal{P}_k} \sum_{i=1}^{l} e^{-(1+\tau)\eta|x-Q_i|} \sum_{i=1}^{l} e^{-(1+\tau)\eta|x-P_i|} \leq C \delta^{\frac{1+\tau}{2} + \frac{\sigma_2}{2} \frac{\sigma_1}{\sigma_2 + \sigma_1}},
\]
for \( x \in \{ x : \frac{|\ln \delta|}{2(2 + \sigma_2)} \leq |x - P_i| \leq \frac{|\ln \delta|}{4} \}, \\
|U^3_{\mathcal{P}_k} V^2_{\mathcal{Q}_i}| \leq C U^2_{\mathcal{P}_k} \sum_{i=1}^{l} e^{-(1+\tau)\eta|x-Q_i|} \sum_{i=1}^{l} e^{-(1+\tau)\eta|x-P_i|} \leq C \delta^{\frac{1+\tau}{2} + \frac{\sigma_2}{2} \frac{\sigma_1}{\sigma_2 + \sigma_1}},
\]
and for \( x \in \mathbb{R}^3 \setminus \bigcup_{i=1}^{k} \{ x : |x - P_i| < \frac{|\ln \delta|}{4} \}, \\
|U^3_{\mathcal{P}_k} V^2_{\mathcal{Q}_i}| \leq C U^2_{\mathcal{P}_k} \sum_{i=1}^{l} e^{-(1+\tau)\eta|x-Q_i|} \sum_{i=1}^{l} e^{-(1+\tau)\eta|x-P_i|} \leq C \delta^{\frac{1+\tau}{2} \sum_{i=1}^{l} e^{-(1+\tau)\eta|x-Q_i|}}.
\]
Then, take \( \tau = \frac{1}{6}, \eta = \frac{2}{3}, \sigma_1 = \frac{10}{9}, \sigma_2 = \frac{1}{3}, \) we have 
\[
\frac{\sigma_1}{2 + \sigma_1} = \frac{5}{14} > \frac{1}{3}, \quad \frac{1}{2 + \sigma_2} = \frac{9}{19} > \frac{1}{3}, \quad \frac{1}{2 + \sigma_1} + \frac{\sigma_2(2 - (\tau + 1)\eta)}{4 + 2\sigma_2} = \frac{9}{28} + \frac{11}{38} > \frac{1}{3}.
\]
Therefore, we find that \( \|S_1(U_{P_k}, V_{Q_l})\|_{\ast \ast} \leq C\delta^{\frac{1+\xi}{2}} \). Therefore, we finish this lemma.

Now, we give the main result in this section.

**Proposition 2.** Let \( \mu_1 > 0, \mu_2 > 0, \) and \( \beta < \min\{\mu_1, \mu_2\} \). There exists \( \delta_0 > 0 \) such that for any \( 0 < \delta < \delta_0, \) and \( (P_k, Q_l) \in \Lambda_{k,l}, \) the problem (18) has a unique solution \( ((\pi_{P_k, Q_l}, \tau_{P_k, Q_l}), (a_{ij}), (b_{ij})) \) such that \( (\pi_{P_k, Q_l}, \tau_{P_k, Q_l}) \) is \( C^1 \) in \( (P, Q) \) and the following
\[
\|((\pi_{P_k, Q_l}, \tau_{P_k, Q_l}))\|_{\ast \ast} \leq C\delta^{\frac{1+\xi}{2}}
\]
holds for some \( 0 < \xi < 1 \) and \( C > 0 \) that is independent of \( \delta, k, \) and \( l. \)

**Proof.** Define
\[
B := \left\{ (u, v) \in C^2(\mathbb{R}^3) \times C^2(\mathbb{R}^3) : \|(u, v)\|_{\ast \ast} \leq r\delta^{\frac{1+\xi}{2}} \right\},
\]
where \( r > 0 \) will be determined later. Then solving the equation (20) is equivalent to finding a fixed point of \( T \) onto \( B. \)

First, it is sufficient to show that \( T \) is a contract mapping onto \( B \) for some \( r > 0 \) and \( \delta > 0, \) since by using Lemma A.4, for sufficiently small \( \delta > 0, \) there exists \( r > 0 \) such that
\[
\|T(u, v)\|_{\ast \ast} \leq C(\|S(U_{P_k}, V_{Q_l})\|_{\ast \ast} + \|N(u, v)\|_{\ast \ast}) \\
\leq C\delta^{\frac{1+\xi}{2}} + \|(u, v)\|_{\ast \ast}^2 \leq r\delta^{\frac{1+\xi}{2}},
\]
\[
\|T(u_1, v_1) - T(u_2, v_2)\|_{\ast \ast} \\
\leq C\|N(u_1, v_1) - N(u_2, v_2)\|_{\ast \ast} \\
\leq C(\|(u_1, v_1)\|_{\ast \ast} + \|(u_2, v_2)\|_{\ast \ast})(\|u_1 - v_1\|_{\ast \ast} + \|v_2 - u_2\|_{\ast \ast}) \\
\leq \frac{1}{2}(\|(u_1, v_1)\|_{\ast \ast} + \|v_2 - u_2\|_{\ast \ast}).
\]

Therefore, \( T \) has a unique fixed point \( (\pi_{P_k, Q_l}, \tau_{P_k, Q_l}) \) by the Banach fixed point theorem, and so (20) has a unique solution \( (\pi_{P_k, Q_l}, \tau_{P_k, Q_l}) \) satisfying (21).

Next, we prove the solution \( (\pi_{P_k, Q_l}, \tau_{P_k, Q_l}) \) of (20) is \( C^1 \) in \( (P, Q) \). Indeed, we define a mapping from \( \Lambda_{k,l} \times \mathcal{H} \times \mathbb{R}^{3(k+1)} \) to \( \mathcal{H} \times \mathbb{R}^{3(k+1)} \) as follows
\[
H(P_k, Q_l, u, v, a, b) = \begin{pmatrix}
(\Delta - I)^{-1}S_1(U_{P_k} + u, V_{Q_l} + v) - \sum_{ij} a_{ij}(\Delta - I)^{-1}X_{ij} \\
(\Delta - I)^{-1}S_2(U_{P_k} + u, V_{Q_l} + v) - \sum_{ij} b_{ij}(\Delta - I)^{-1}Y_{ij} \\
\langle u, (\Delta - I)^{-1}X_{ij} \rangle \\
\langle v, (\Delta - I)^{-1}Y_{ij} \rangle
\end{pmatrix},
\]
For any \( (P_k, Q_l) \in \Lambda_{k,l}, \) (20) has a solution \( (\pi_{P_k, Q_l}, \tau_{P_k, Q_l}, a_{P_k, Q_l}, b_{P_k, Q_l}) \), i.e.
\[
H(P_k, Q_l, \pi_{P_k, Q_l}, \tau_{P_k, Q_l}, a_{P_k, Q_l}, b_{P_k, Q_l}) = 0.
\]
After a simple computation,

\[
\frac{\partial H(P_k, Q_l, \tau P_k, \tau Q_l, a P_k, \tau Q_l, b P_k, \tau Q_l)}{\partial (u, v, a, b)}[\psi, \phi, c, d] = \begin{pmatrix}
(\Delta - I)^{-1}(L_1[U_{P_k} + \tau P_k, \phi, V_{Q_l} + \tau Q_l](\psi, \phi) - \sum_{ij} c_{ij} X_{ij}) \\
(\Delta - I)^{-1}(L_2[U_{P_k} + \tau P_k, \phi, V_{Q_l} + \tau Q_l](\psi, \phi) - \sum_{ij} d_{ij} Y_{ij}) \\
\langle \psi, (\Delta - I)^{-1} X_{ij} \rangle \\
\langle \phi, (\Delta - I)^{-1} Y_{ij} \rangle
\end{pmatrix},
\]

where

\[
L_1[U_{P_k} + \tau P_k, \phi, V_{Q_l} + \tau Q_l](\psi, \phi) = \Delta \psi - (1 + \delta a(x))\psi + 3\mu_1(U_{P_k} + \tau P_k, \phi, V_{Q_l} + \tau Q_l)^2 \psi + \beta(U_{P_k} + \tau P_k, \phi, V_{Q_l} + \tau Q_l)^2 \psi + 2\beta(U_{P_k} + \tau P_k, \phi, V_{Q_l} + \tau Q_l, \phi),
\]

\[
L_2[U_{P_k} + \tau P_k, \phi, V_{Q_l} + \tau Q_l](\psi, \phi) = \Delta \phi - (1 + \delta b(x))\phi + 3\mu_2(U_{P_k} + \tau P_k, \phi, V_{Q_l} + \tau Q_l)^2 \phi + \beta(U_{P_k} + \tau P_k, \phi, V_{Q_l} + \tau Q_l)^2 \phi + 2\beta(U_{P_k} + \tau P_k, \phi, V_{Q_l} + \tau Q_l, \phi).
\]

Let \((\psi, \phi, c, d)\) satisfies

\[
\frac{\partial H(P_k, Q_l, \tau P_k, \tau Q_l, a P_k, \tau Q_l, b P_k, \tau Q_l)}{\partial (u, v, a, b)}[\psi, \phi, c, d] = 0,
\]

then \((\psi, \phi, c, d)\) is a solution to

\[
\begin{cases}
L_1[U_{P_k} + \tau P_k, \phi, V_{Q_l} + \tau Q_l](\psi, \phi) = \sum_{ij} c_{ij} X_{ij} \quad \text{in} \, \mathbb{R}^3, \\
L_2[U_{P_k} + \tau P_k, \phi, V_{Q_l} + \tau Q_l](\psi, \phi) = \sum_{ij} d_{ij} Y_{ij} \quad \text{in} \, \mathbb{R}^3, \\
u \to 0, \quad v \to 0 \quad \text{as} \, |x| \to \infty, \\
\int_{\mathbb{R}^3} (\psi X_{ij}) = \int_{\mathbb{R}^3} (\phi Y_{ij}) = 0, \quad \forall i, s, j.
\end{cases}
\]

It follows that \((\psi, \phi) = A(N(\psi, \phi))\) where \(N(\psi, \phi) = (N_1(\psi, \phi), N_2(\psi, \phi)),\)

\[
N_1(\psi, \phi) = 3\mu_1[U_{P_k}^2 - (U_{P_k} + \tau P_k, \phi, V_{Q_l} + \tau Q_l)^2] \psi + \beta[V_{Q_l}^2 - (V_{Q_l} + \tau Q_l, \phi, V_{Q_l} + \tau Q_l)^2] \psi
+ 2\beta[U_{P_k} V_{Q_l} - (U_{P_k} + \tau P_k, \phi, V_{Q_l} + \tau Q_l)^2] \phi,
\]

\[
N_2(\psi, \phi) = 3\mu_2[V_{Q_l}^2 - (V_{Q_l} + \tau Q_l, \phi, V_{Q_l} + \tau Q_l)^2] \phi + \beta[U_{P_k}^2 - (U_{P_k} + \tau P_k, \phi, V_{Q_l} + \tau Q_l)^2] \phi
+ 2\beta[U_{P_k} V_{Q_l} - (U_{P_k} + \tau P_k, \phi, V_{Q_l} + \tau Q_l)^2] \psi.
\]

By using Lemma 2.1, we can obtain easily

\[
\| (\psi, \phi) \|_{\ast} \leq C \| N(\psi, \phi) \|_{\ast} \leq C \delta^{1/\epsilon} \| (\psi, \phi) \|_{\ast},
\]

which implies that \((\psi, \phi) = (0, 0)\). Therefore, the gradient of \(H(P_k, Q_l, \tau P_k, \tau Q_l, a P_k, \tau Q_l, b P_k, \tau Q_l)\) is invertible. Since \(\frac{\partial H(P_k, Q_l, \tau P_k, \tau Q_l, a P_k, \tau Q_l, b P_k, \tau Q_l)}{\partial (u, v, a, b)}\) and \(\frac{\partial H(P_k, Q_l, \tau P_k, \tau Q_l, a P_k, \tau Q_l, b P_k, \tau Q_l)}{\partial (P_k, Q_l)}\) are both continuous, then by the implicit function theorem, \((u P_k, Q_l, v P_k, Q_l)\) is \(C^1\) in \((P_k, Q_l)\).
4. A general error estimation. In the sequence, we always assume that \( \beta < \min \{ \mu_1, \mu_2 \} \) and \( 0 < \delta < \delta_0 \) without special statement, where \( \delta_0 \) is as in Proposition 2. For any \((P, Q) \in \Lambda_{k,t}\), let’s denote
\[
\begin{align*}
\upsilon_{k,i} &= U_{k,i} + \pi_{k,i} \quad \text{and} \\
\upsilon_{i} &= V_{i} + \tau_{i} 
\end{align*}
\]
where \((\pi_{k,i}, \tau_{i})\) is the solution to (18).

We also write
\[
\begin{align*}
\hat{\upsilon}_{i} &= \upsilon_{i} - \sum_{i=1}^{s} U_{k,i} + u_{k+s,t+t}, \\
\hat{\upsilon}_{i} &= \upsilon_{i} - \sum_{i=1}^{t} V_{i} + v_{k+s,t+t},
\end{align*}
\]
From Proposition 2, it follows that
\[
\|u_{k+s,t+t}\| \leq C \delta^{\frac{1}{2}}, \quad \|v_{k+s,t+t}\| \leq C \delta^{\frac{1}{2}}.
\]

We shall use a secondary reduction method to prove the following lemma which describes the norms of \((u_{k+s,t+t}, v_{k+s,t+t})\) in \(H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)\).

**Lemma 4.1.** Let \(k, l, s, t \geq 1\). There exists a constant \(C > 0\) independent of \(\delta, k, l, s,\) and \(t\) such that
\[
\|u_{k+s,t+t}, v_{k+s,t+t}\| \leq C \delta^{\frac{1}{2}}.
\]

**Proof.** We denote
\[
\hat{L}_0 u = \Delta u - u + 3\mu_1 U^2 u, \quad \tilde{L}_0 u = \Delta u - u + 3\mu_2 V^2 u.
\]
From Lemma 13.5 in [20] by Wei and Winter, we see that 0 is the second eigenvalue of \(\hat{L}_0\) and \(\tilde{L}_0\). We further assume that \(\lambda_1\) and \(\hat{\lambda}_1\) are the first eigenvalues of \(\hat{L}_0\) and \(\tilde{L}_0\) with eigenfunctions \(\hat{\phi}_0\) and \(\tilde{\phi}_0\) respectively. Precisely,
\[
\begin{align*}
\Delta \hat{\phi}_0 - \hat{\phi}_0 + 3\mu_1 U^2 \hat{\phi}_0 &= \lambda_1 \hat{\phi}_0 \quad \text{in} \ \mathbb{R}^3, \\
\Delta \tilde{\phi}_0 - \tilde{\phi}_0 + 3\mu_2 V^2 \tilde{\phi}_0 &= \hat{\lambda}_1 \tilde{\phi}_0 \quad \text{in} \ \mathbb{R}^3, \\
\max_{\mathbb{R}^3} \hat{\phi}_0 &= \max_{\mathbb{R}^3} \tilde{\phi}_0 = 1.
\end{align*}
\]
From the works [1] by Ao and Wei and [2] by Ao, Wei, and Zeng, we also know that \(\hat{\phi}_0\) and \(\tilde{\phi}_0\) are radial, positive, exponential decay.
Let’s define
\[
\begin{align*}
\tilde{L}(u, v) &= \Delta u - (1 + \delta a(x))u + 3\mu_1 \tilde{W}^2 u + \beta \tilde{W}^2 u + 2\beta \tilde{W} \tilde{W} v, \\
\hat{L}(u, v) &= \Delta v - (1 + \delta b(x))v + 3\mu_2 \tilde{W}^2 v + \beta \tilde{W}^2 v + 2\beta \tilde{W} \tilde{W} u,
\end{align*}
\]
where
\[
\tilde{W} = u_{p_k, q_l} + \sum_{i=1}^{s} U_{p_{k+1}, i}, \quad \hat{W} = v_{p_k, q_l} + \sum_{i=1}^{t} V_{q_{k+1}, i}.
\]
From (18) and (24), \((u_{k+1, t}, v_{k+1, t})\) satisfies
\[
\begin{align*}
\tilde{L}(u_{k+1, t}, v_{k+1, t}) &= \tilde{S} + O(((u_{k+1, t}, v_{k+1, t}))^2) + \sum_{ij} a_{ij} X_{ij}, \\
\hat{L}(u_{k+1, t}, v_{k+1, t}) &= \hat{S} + O(((u_{k+1, t}, v_{k+1, t}))^2) + \sum_{ij} b_{ij} Y_{ij},
\end{align*}
\]
where
\[
\tilde{S} = -\delta \sum_{i=1}^{s} a_{ij} U_{p_{k+1}, i} + \mu_1 \left[u_{p_k, q_l}^3 + \sum_{i=1}^{s} U_{p_{k+1}, i}^3 - \left(u_{p_k, q_l} + \sum_{i=1}^{s} U_{p_{k+1}, i}\right)^3\right]
- \beta \left[u_{p_k, q_l} + \sum_{i=1}^{s} U_{p_{k+1}, i}\right] \left[v_{p_k, q_l} + \sum_{i=1}^{t} V_{q_{k+1}, i}\right] - \left(v_{p_k, q_l} + \sum_{i=1}^{t} V_{q_{k+1}, i}\right)^2.
\]
\[
\hat{S} = -\delta \sum_{i=1}^{t} b_{ij} U_{p_{k+1}, i} + \mu_1 \left[v_{p_k, q_l}^3 + \sum_{i=1}^{t} V_{q_{k+1}, i}^3 - \left(v_{p_k, q_l} + \sum_{i=1}^{t} V_{q_{k+1}, i}\right)^3\right]
- \beta \left[u_{p_k, q_l} + \sum_{i=1}^{s} U_{p_{k+1}, i}\right] \left[v_{p_k, q_l} + \sum_{i=1}^{t} V_{q_{k+1}, i}\right] - \left(v_{p_k, q_l} + \sum_{i=1}^{t} V_{q_{k+1}, i}\right)^2.
\]
Let \(\tilde{\phi}_i := \chi_{p_i} \phi_{0, p_i}\) and \(\hat{\phi}_s := \chi_{q_s} \hat{\phi}_{0, q_s}\). We decompose \((u_{k+s,t+t}, v_{k+s,t+t})\) as follows
\[
\begin{align*}
&\left\{\begin{array}{ll}
u_{k+s,t+t} &= \tilde{\psi} + \sum_{i=1}^{k+s} \tilde{c}_i \tilde{\phi}_i + \sum_{ij} \tilde{d}_{ij} X_{ij}, \\
v_{k+s,t+t} &= \hat{\psi} + \sum_{i=1}^{l+t} \hat{c}_i \hat{\phi}_i + \sum_{ij} \hat{d}_{ij} Y_{ij}
\end{array}\right.
\end{align*}
\]
for some \(\tilde{c}_i, \tilde{d}_{ij}, \hat{c}_i, \hat{d}_{ij}\) such that
\[
\int_{\mathbb{R}^3} (\tilde{\psi} \bar{\phi}_i) = \int_{\mathbb{R}^3} (\tilde{\psi} X_{ij}) = \int_{\mathbb{R}^3} (\hat{\psi} \bar{\phi}_s) = \int_{\mathbb{R}^3} (\hat{\psi} Y_{ij}) = 0.
\]
In order to estimate the norm of \((u_{k+s,t+t}, v_{k+s,t+t})\), the following claims are required.

**Claim 1.** There exists \(C > 0\) independent of \(\delta, k,\) and \(l\) such that
\[
\begin{align*}
\left|\tilde{d}_{ij}\right| &= 0 & \text{for } i = 1, 2, \cdots, k, \\
\left|\tilde{d}_{ij}\right| &\leq C\delta^{1/4} \left( \sum_{j=1}^{k} e^{-\eta |P_i - P_j|} + \sum_{j=1}^{l} e^{-\eta |Q_i - Q_j|} \right) & \text{for } i = k + 1, \cdots, k + s, \\
\hat{d}_{s j} &= 0 & \text{for } s = 1, 2, \cdots, l, \\
\left|\hat{d}_{ij}\right| &\leq C\delta^{1/4} \left( \sum_{j=1}^{l} e^{-\eta |Q_i - Q_j|} + \sum_{j=1}^{l} e^{-\eta |Q_i - P_j|} \right) & \text{for } i = l + 1, \cdots, l + t.
\end{align*}
\]
In fact, multiplying \(X_{ij}\) on both sides of (28) and integrating over \(\mathbb{R}^3\), we obtain for \(i = 1, 2, \cdots, k\),
\[
\left( \int_{\mathbb{R}^3} X_{ij}^2 \right) \tilde{d}_{ij} = \int_{\mathbb{R}^3} (u_{k+s,t+t} X_{ij}) = \int_{\mathbb{R}^3} (u_{p_{k+s}, q_{l+t}} - u_{p_k, q_l}) X_{ij} = 0,
\]
and for \(i = k + 1, \cdots, k + s\),
\[
\left( \int_{\mathbb{R}^3} X_{ij}^2 \right) \tilde{d}_{ij} = \int_{\mathbb{R}^3} (u_{k+s,t+t} X_{ij}) = \int_{\mathbb{R}^3} (v_{p_{k+s}, q_{l+t}} - v_{p_k, q_l}) X_{ij} = 0.
\]
and for $i = k + 1, \ldots, k + s$,

\[
\left( \int_{\mathbb{R}^3} X_{ij}^2 \right) \tilde{d}_{ij} = \int_{\mathbb{R}^3} (u_{k+s,t+i} X_{ij})
\]

\[
= \int_{\mathbb{R}^3} [(\pi_{P_k+\alpha, Q_{t+i}} - \pi_{P_k, Q_i}) X_{ij}]
\]

\[
= - \int_{\mathbb{R}^3} (\pi_{P_k, Q_i} X_{ij}).
\]

Since, for $i = k + 1, \ldots, k + s$,

\[
\left| \int_{\mathbb{R}^3} (\pi_{P_k, Q_i} X_{ij}) \right| \leq C\|\pi_{P_k, Q_i}\|_* \int_{\mathbb{R}^3} W_* X_{ij}
\]

\[
\leq C\delta^{\frac{k+1}{2}} \left( \sum_{j=1}^k e^{-\eta |P_j| - |P_i|} + \sum_{j=1}^l e^{-\eta |Q_j| - |P_i|} \right),
\]

and

\[
\int_{\mathbb{R}^3} X_{ij}^2 = \int_{\mathbb{R}^3} X_{k+1j}^2 = \int_{\mathbb{R}^3} \left( \frac{\partial U}{\partial x_j} \right)^2 + o(1).
\]

Then (30) follows from the above estimations and the orthogonal condition in (20).

Claim 2. There exists $C > 0$ independent of $\delta$, $k$, and $l$ such that

\[
\sum_{i=1}^{k+1} |\tilde{e}_i|^2 + \sum_{i=1}^l |\tilde{c}_i|^2 \leq C\|(S, \tilde{S})\|_{L^2(\mathbb{R}^3)}^2 + o(1) \sum_{ij} |\tilde{d}_{ij}|^2 + o(1) \sum_{ij} |\tilde{d}_{ij}|^2
\]

\[
+ o(1) \|(u_{k+s, t+i})\|^2 + o(1) \|(|\tilde{\psi}, \tilde{\phi}|)^2.
\]

In fact, from (27) and (28), we find that $(\tilde{\psi}, \tilde{\phi})$ satisfies

\[
\begin{aligned}
\tilde{L}(\tilde{\psi}, \tilde{\phi}) &= \tilde{S} + O((u_{k+s,t+i})^2) - \sum_{i=1}^{k+s} \tilde{c}_i \tilde{L}(\phi_i) - \sum_{ij} \tilde{d}_{ij} \tilde{L}(X_{ij}) \\
&- 2\beta \sum_{i=1}^{l+t} \tilde{c}_i \tilde{W} \tilde{W} \tilde{\phi}_i - 2\beta \sum_{ij} \tilde{d}_{ij} \tilde{W} \tilde{W} Y_{ij} + \sum_{ij} a_{ij} \tilde{X}_{ij},
\end{aligned}
\]

\[
\begin{aligned}
\tilde{L}(\tilde{\psi}, \tilde{\phi}) &= \tilde{S} + O((u_{k+s,t+i})^2) - \sum_{i=1}^{k+s} \tilde{c}_i \tilde{L}(\phi_i) - \sum_{ij} \tilde{d}_{ij} \tilde{L}(X_{ij}) \\
&- 2\beta \sum_{i=1}^{k+s} \tilde{c}_i \tilde{W} \tilde{W} \tilde{\phi}_i - 2\beta \sum_{ij} \tilde{d}_{ij} \tilde{W} \tilde{W} X_{ij} + \sum_{ij} b_{ij} \tilde{Y}_{ij}.
\end{aligned}
\]

where

\[
\tilde{L}(u) = \Delta u - (1 + \delta a(x))u + 3\mu_1 \tilde{W}^2 u + \beta \tilde{W}^2 u,
\]

\[
\tilde{L}(u) = \Delta u - (1 + \delta b(x))u + 3\mu_2 \tilde{W}^2 u + \beta \tilde{W}^2 u.
\]
Then a simple computation yields

\[
\begin{aligned}
\tilde{c}_i \int_{\mathbb{R}^3} \tilde{l}(\tilde{\phi}_i) \tilde{\phi}_i &= \int_{\mathbb{R}^3} \left[ S - \tilde{L}(\tilde{\psi}, \tilde{\psi}) + O(|(u_{k+s,t+t}, v_{k+s,t+t})|^2) \right] \tilde{\phi}_i \\
- \sum_{j=1}^{3} \tilde{d}_{ij} \int_{\mathbb{R}^3} \tilde{l}(X_{ij}) \tilde{\phi}_i - \sum_{j=1}^{3} \tilde{c}_j \int_{\mathbb{R}^3} 2\beta \tilde{W} \tilde{W} \tilde{\phi}_j \tilde{\phi}_i - 2\beta \sum_{l_j} \int_{\mathbb{R}^3} \tilde{d}_{ij} \tilde{W} \tilde{W} Y_{ij} \tilde{\phi}_i,
\end{aligned}
\]

Similarly,

\[
\begin{aligned}
\tilde{c}_i \int_{\mathbb{R}^3} \tilde{l}(\tilde{\phi}_i) \tilde{\phi}_i &= \int_{\mathbb{R}^3} \left[ S - \tilde{L}(\tilde{\psi}, \tilde{\psi}) + O(|(u_{k+s,t+t}, v_{k+s,t+t})|^2) \right] \tilde{\phi}_i \\
- \sum_{j=1}^{3} \tilde{d}_{ij} \int_{\mathbb{R}^3} \tilde{l}(X_{ij}) \tilde{\phi}_i - \sum_{j=1}^{3} \tilde{c}_j \int_{\mathbb{R}^3} 2\beta \tilde{W} \tilde{W} \tilde{\phi}_j \tilde{\phi}_i - 2\beta \sum_{l_j} \int_{\mathbb{R}^3} \tilde{d}_{ij} \tilde{W} \tilde{W} X_{ij} \tilde{\phi}_i.
\end{aligned}
\]

Note that

\[
\sum_{i=1}^{k+s} \left( \sum_{j=1}^{l+t} |\tilde{c}_j|^2 \int_{\mathbb{R}^3} 2\beta \tilde{W} \tilde{W} \tilde{\phi}_j \tilde{\phi}_i \right)^2 = o(1) \sum_{i=1}^{k+s} \left( \int_{\mathbb{R}^3} \left( \sum_{j=1}^{l+t} |\tilde{c}_j|^2 \tilde{\phi}_j \right)^2 \right)^2
\]

\[
\leq o(1) \sum_{i=1}^{k+s} \int_{\mathbb{R}^3} \left( \sum_{j=1}^{l+t} |\tilde{c}_j|^2 \tilde{\phi}_j \right)^2 \tilde{\phi}_i \int_{\mathbb{R}^3} \tilde{\phi}_i
\]

\[
\leq o(1) \sum_{i=1}^{k+s} |\tilde{c}_i|^2 \left( \int_{\mathbb{R}^3} |\tilde{\phi}_i|^2 \sum_{i=1}^{l+t} \tilde{\phi}_i \right) = o(1) \sum_{i=1}^{l+t} |\tilde{d}_{ij}|^2,
\]

\[
\sum_{i=1}^{k+s} \left( \sum_{j=1}^{3} \tilde{d}_{ij} \int_{\mathbb{R}^3} \tilde{l}(X_{ij}) \tilde{\phi}_i \right)^2 \leq 3 \sum_{ij} |\tilde{d}_{ij}|^2 \left( \int_{\mathbb{R}^3} \tilde{l}(X_{ij}) \tilde{\phi}_i \right)^2 \leq o(1) \sum_{ij} |\tilde{d}_{ij}|^2.
\]

Similarly,

\[
\sum_{i=1}^{k+s} \left( \sum_{j=1}^{3} \tilde{d}_{ij} \int_{\mathbb{R}^3} \tilde{l}(X_{ij}) \tilde{\phi}_i \right)^2 \leq 3 \sum_{ij} |\tilde{d}_{ij}|^2 \left( \int_{\mathbb{R}^3} \tilde{l}(X_{ij}) \tilde{\phi}_i \right)^2 \leq o(1) \sum_{ij} |\tilde{d}_{ij}|^2.
\]

Since

\[
\int_{\mathbb{R}^3} \tilde{l}(\tilde{\phi}_i) \tilde{\phi}_i = \tilde{\lambda}_1 \int_{\mathbb{R}^3} |\tilde{\phi}_0|^2 + o(1) \quad \text{and} \quad \int_{\mathbb{R}^3} \tilde{l}(\tilde{\phi}_i) \tilde{\phi}_i = \tilde{\lambda}_1 \int_{\mathbb{R}^3} |\tilde{\phi}_0|^2 + o(1).
\]

Then a simple computation yields

\[
\sum_{i=1}^{k+s} |\tilde{c}_i|^2 + \sum_{i=1}^{l+t} |\tilde{c}_i|^2 \leq C \sum_{i=1}^{k+s} \left( \int_{\mathbb{R}^3} (S - \tilde{L}(\tilde{\psi}, \tilde{\psi})) \tilde{\phi}_i \right)^2
\]

\[
+ C \sum_{i=1}^{l+t} \left( \int_{\mathbb{R}^3} (S - \tilde{L}(\tilde{\psi}, \tilde{\psi})) \tilde{\phi}_i \right)^2 + C \sum_{i=1}^{k+s} \left( \int_{\mathbb{R}^3} |(u_{k+s,t+t}, v_{k+s,t+t})|^2 \tilde{\phi}_i \right)^2
\]

\[
+ \sum_{ij} \left( \int_{\mathbb{R}^3} |(u_{k+s,t+t}, v_{k+s,t+t})|^2 \tilde{\phi}_j \right)^2 + o(1) \sum_{ij} |\tilde{d}_{ij}|^2 + o(1) \sum_{ij} |\tilde{d}_{ij}|^2.
\]
For the above estimation (33), we have
\begin{align*}
\int_{\mathbb{R}^3} \bar{L}(\bar{\psi}, \bar{\phi}_i) \bar{\phi}_i &= \int_{\mathbb{R}^3} \chi_{P_i} \bar{\psi} (\Delta \bar{\phi}_0, P_i - (1 + \delta a(x)) \bar{\phi}_0, P_i + 3\mu_1 U^2_{P_i} \bar{\phi}_0, P_i) \\
&+ 3\mu_1 \int_{\mathbb{R}^3} (\bar{W}^2 - U^2_{P_i}) \bar{\phi}_i \bar{\psi} + \beta \int_{\mathbb{R}^3} (\bar{W}^2 \bar{\phi}_i \bar{\psi} + 2\bar{W} \bar{W} \bar{\phi}_i) \\
&+ \int_{\mathbb{R}^3} (\Delta \chi_{P_i} \bar{\phi}_0, P_i + 2\nabla \chi_{P_i} \nabla \bar{\phi}_0, P_i) \bar{\psi}.
\end{align*}
and
\begin{align*}
\int_{\mathbb{R}^3} \chi_{P_i} \bar{\psi} (\Delta \bar{\phi}_0, P_i - \bar{\phi}_0, P_i + 3\mu_1 U^2_{P_i} \bar{\phi}_0, P_i) &= 0.
\end{align*}
Since, by Hölder’s inequality and Sobolev inequality,
\begin{align*}
\sum_{i=1}^{k+s} \left[ \int_{\mathbb{R}^3} (\bar{W}^2 - U^2_{P_i}) \bar{\phi}_i \bar{\psi} \right]^2 &\leq o(1) \sum_{i=1}^{k+s} \int_{|x-P_i| < \frac{|ln d|}{4}} |\bar{\phi}_i|^2 \int_{|x-P_i| < \frac{|ln d|}{4}} |\bar{\psi}|^2 \\
&\leq o(1) \sum_{i=1}^{k+s} \int_{|x-P_i| < \frac{|ln d|}{4}} |\bar{\psi}|^2 \leq o(1)\|\bar{\psi}\|^2,
\end{align*}
\begin{align*}
\sum_{i=1}^{k+s} \left[ \int_{\mathbb{R}^3} (\bar{W}^2 \bar{\phi}_i \bar{\psi} + 2\bar{W} \bar{W} \bar{\phi}_i) \right]^2 &\leq o(1) \sum_{i=1}^{k+s} \int_{|x-P_i| < \frac{|ln d|}{4}} |\bar{\phi}_i|^2 \int_{|x-P_i| < \frac{|ln d|}{4}} |\bar{\psi}|^2 \\
&+ o(1) \sum_{i=1}^{k+s} \int_{|x-P_i| < \frac{|ln d|}{4}} |\bar{\phi}_i|^2 \int_{|x-P_i| < \frac{|ln d|}{4}} |\bar{\psi}|^2 \leq o(1)\|\bar{\psi}\|^2,
\end{align*}
and
\begin{align*}
\sum_{i=1}^{k+s} \left[ \int_{\mathbb{R}^3} (\Delta \chi_{P_i} \bar{\phi}_0, P_i + 2\nabla \chi_{P_i} \nabla \bar{\phi}_0, P_i) \bar{\psi} \right]^2 &\leq o(1)\|\bar{\psi}\|^2.
\end{align*}
Then it follows that
\begin{align*}
\sum_{i=1}^{k+s} \left( \int_{\mathbb{R}^3} \bar{L}(\bar{\psi}, \bar{\phi}_i) \right)^2 &\leq o(1)\|\bar{\psi}\|^2.
\end{align*}
Similarly, we also deduce that
\begin{align*}
\sum_{i=1}^{l+t} \left( \int_{\mathbb{R}^3} \bar{L}(\bar{\psi}, \bar{\phi}_i) \right)^2 &\leq o(1)\|\bar{\psi}\|^2.
\end{align*}
Since, by Hölder’s inequality,
\begin{align*}
\sum_{i=1}^{k+s} \left( \int_{\mathbb{R}^3} |\bar{S}| \bar{\phi}_i \right)^2 &\leq \sum_{i=1}^{k+s} \left( \int_{|x-P_i| < \frac{|ln d|}{4}} |\bar{S}|^2 \right) \left( \int_{|x-P_i| < \frac{|ln d|}{4}} |\bar{\phi}_i|^2 \right) \leq C \int_{\mathbb{R}^3} |\bar{S}|^2,
\end{align*}
\begin{align*}
\sum_{i=1}^{l+t} \left( \int_{\mathbb{R}^3} |\bar{S}| \bar{\phi}_i \right)^2 &\leq \sum_{i=1}^{l+t} \left( \int_{|x-Q_i| < \frac{|ln d|}{4}} |\bar{S}|^2 \right) \left( \int_{|x-Q_i| < \frac{|ln d|}{4}} |\bar{\phi}_i|^2 \right) \leq C \int_{\mathbb{R}^3} |\bar{S}|^2,
\end{align*}
\[
\sum_{i=1}^{k+s} \left[ \int_{\mathbb{R}^3} \left( \left| (u_{k+s,i+t}, v_{k+s,i+t}) \right|^2 \phi_i \right)^2 \right] \leq \sum_{i=1}^{k+s} \int_{|x-P_i| \leq \frac{|\ln t|}{4}} \left( \left| u_{k+s,i+t} \right|^2 + \left| v_{k+s,i+t} \right|^2 \right) |\phi_i|^2 \\
\times \int_{|x-P_i| \leq \frac{|\ln t|}{4}} \left( \left| u_{k+s,i+t} \right|^2 + \left| v_{k+s,i+t} \right|^2 \right) \leq C \delta^{\frac{s+1}{2}} \left( \left\| (u_{k+s,i+t}, v_{k+s,i+t}) \right\| \right),
\]

and
\[
\sum_{i=1}^{l+t} \left[ \int_{\mathbb{R}^3} \left( \left| (u_{k+s,i+t}, v_{k+s,i+t}) \right|^2 \phi_j \right)^2 \right] \leq C \delta^{\frac{l+1}{2}} \left( \left\| (u_{k+s,i+t}, v_{k+s,i+t}) \right\| \right).
\]

Then (31) follows from the above argument.

Now we are in the position to estimate the norm \( \left\| (\tilde{\psi}, \tilde{\psi}) \right\| \). Indeed, multiplying \( (\tilde{\psi}, \tilde{\psi}) \) on both sides of the equation (32) and integrating over \( \mathbb{R}^3 \), we have
\[
\int_{\mathbb{R}^3} (\tilde{L}(\tilde{\psi}, \tilde{\psi})\tilde{\psi} + \tilde{L}(\tilde{\psi}, \tilde{\psi})\tilde{\psi})
= \int_{\mathbb{R}^3} (\tilde{S}\tilde{\psi} + \tilde{S}\tilde{\psi}) + \int_{\mathbb{R}^3} O((\left| u_{k+s,i+t}, v_{k+s,i+t} \right|)^2) (\tilde{\psi} + \tilde{\psi})
- \sum_{i} \tilde{c}_i \int_{\mathbb{R}^3} (\tilde{l}(\phi_i)\tilde{\psi} + 2\beta\tilde{W}\tilde{W}\phi_i\tilde{\psi}) - \sum_{i} \tilde{c}_i \int_{\mathbb{R}^3} (\tilde{l}(\phi_i)\tilde{\psi} + 2\beta\tilde{W}\tilde{W}\phi_i\tilde{\psi})
- \sum_{ij} \tilde{d}_{ij} \int_{\mathbb{R}^3} (\tilde{l}(X_{ij})\tilde{\psi} + 2\beta\tilde{W}\tilde{W}X_{ij}\tilde{\psi}) - \sum_{ij} \tilde{d}_{ij} \int_{\mathbb{R}^3} (\tilde{l}(Y_{ij})\tilde{\psi} + 2\beta\tilde{W}\tilde{W}Y_{ij}\tilde{\psi}).
\]

(34)

**Claim 3.** There exists \( C > 0 \) independent of \( \delta, k, l, s, \) and \( t \) such that
\[
- \int_{\mathbb{R}^3} (\tilde{L}(\tilde{\psi}, \tilde{\psi})\tilde{\psi} + \tilde{L}(\tilde{\psi}, \tilde{\psi})\tilde{\psi}) \geq C \| (\tilde{\psi}, \tilde{\psi}) \|^2.
\]

(35)

In fact, since \( \tilde{W} \) and \( \tilde{W} \) are exponential decay, then by Hölder’s inequality, we have
\[
- \int_{\mathbb{R}^3} (\tilde{L}(\tilde{\psi}, \tilde{\psi})\tilde{\psi} + \tilde{L}(\tilde{\psi}, \tilde{\psi})\tilde{\psi})
= \left( \int_{\mathbb{R}^3 \setminus A} + \int_{A} \right) \left[ \left( \left| \nabla \tilde{\psi} \right|^2 + \left| \nabla \tilde{\psi} \right|^2 + \tilde{\psi}^2 + \tilde{\psi}^2 \right) - 3\mu_1 \tilde{W}^2 \tilde{\psi}^2 - 3\mu_2 \tilde{W}^2 \tilde{\psi}^2 \right]
- \left( \int_{\mathbb{R}^3 \setminus A} + \int_{A} \right) \beta \left( \tilde{W}^2 \tilde{\psi}^2 + \tilde{W}^2 \tilde{\psi}^2 + 4\tilde{W}\tilde{W}\tilde{\psi} \tilde{\psi} \right) + \delta \int_{\mathbb{R}^3} (a(x)\tilde{\psi}^2 + b(x)\tilde{\psi}^2)
\geq \frac{1}{2} \int_{\mathbb{R}^3 \setminus A} \left( \left| \nabla \tilde{\psi} \right|^2 + \left| \nabla \tilde{\psi} \right|^2 + \tilde{\psi}^2 + \tilde{\psi}^2 \right) + \int_{A} \left( \left| \nabla \tilde{\psi} \right|^2 + \left| \nabla \tilde{\psi} \right|^2 + \tilde{\psi}^2 + \tilde{\psi}^2 \right)
- \int_{A} \left[ 3\mu_1 \tilde{W}^2 \tilde{\psi}^2 + 3\mu_2 \tilde{W}^2 \tilde{\psi} + \beta \left( \tilde{W}^2 \tilde{\psi}^2 + \tilde{W}^2 \tilde{\psi}^2 + 4\tilde{W}\tilde{W}\tilde{\psi} \tilde{\psi} \right) \right]
+ \delta \int_{\mathbb{R}^3} (a(x)\tilde{\psi}^2 + b(x)\tilde{\psi}^2).
\]

where \( A = \bigcup_{i=1}^{k+s} B_{\frac{|\ln t|}{2}}(P_i) \cup \bigcup_{i=1}^{l+t} B_{\frac{|\ln t|}{2}}(Q_i) \).
To show (35), we shall prove that
\[
\int_{B_{1/10}(P_i)} \left( \left| \nabla \tilde{\psi} \right|^2 + \left| \nabla \tilde{\phi} \right|^2 + \tilde{\psi}^2 + \tilde{\phi}^2 \right) - 3\mu_1 \tilde{W}^2 \tilde{\psi}^2 - 3\mu_2 \tilde{W}^2 \tilde{\phi}^2 \\
- \beta \int_{B_{1/10}(P_i)} \left( \tilde{W}^2 \tilde{\psi}^2 + \tilde{W}^2 \tilde{\phi}^2 + 4\tilde{W} \tilde{W} \tilde{\psi} \tilde{\phi} \right) 
\geq C \int_{B_{1/10}(Q_j)} \left( \left| \nabla \tilde{\psi} \right|^2 + \left| \nabla \tilde{\phi} \right|^2 + \tilde{\psi}^2 + \tilde{\phi}^2 \right),
\]
where \( C > 0 \) is independent of \( \delta, k, l, s, \) and \( t \).

We prove (36) by a contradiction argument. Without loss of generality, we may assume that
\[
\int_{B_{1/10}(P_{n,i,n})} \left( \left| \nabla \tilde{\psi}_n \right|^2 + \left| \nabla \tilde{\phi}_n \right|^2 + \tilde{\psi}_n^2 + \tilde{\phi}_n^2 \right) = 1,
\]
\[
\int_{B_{1/10}(P_{n,i,n})} \left( \left| \nabla \tilde{\psi}_n \right|^2 + \left| \nabla \tilde{\phi}_n \right|^2 + \tilde{\psi}_n^2 + \tilde{\phi}_n^2 \right) - 3\mu_1 \tilde{W}_n^2 \tilde{\psi}_n^2 - 3\mu_2 \tilde{W}_n^2 \tilde{\phi}_n^2 \\
+ \beta \int_{B_{1/10}(P_{n,i,n})} \left( \tilde{W}_n^2 \tilde{\psi}_n^2 + \tilde{W}_n^2 \tilde{\phi}_n^2 + 4\tilde{W}_n \tilde{W}_n \tilde{\psi}_n \tilde{\phi}_n \right) \to 0,
\]
as \( \delta_n \to 0 \) for some \( i_n \in \{1, 2, \ldots k_n + s_n\} \). Up to a subsequence, we may assume that 
\((\tilde{\psi}_n(- - P_{n,i,n}), \tilde{\phi}_n(- - P_{n,i,n}))\) converges weakly in \( H_{10}^1(\mathbb{R}^3) \times H_{10}^1(\mathbb{R}^3) \) to \((\tilde{\psi}_\infty, \tilde{\phi}_\infty)\) such that
\[
0 \leq \int_{\mathbb{R}^3} \left( |\nabla \tilde{\psi}_\infty|^2 + |\nabla \tilde{\phi}_\infty|^2 + \tilde{\psi}_\infty^2 + \tilde{\phi}_\infty^2 \right) - 3\mu_1 \int_{\mathbb{R}^3} U^2 \tilde{\psi}_\infty^2 \leq 0
\]
and
\[
\int_{\mathbb{R}^3} \tilde{\psi}_\infty \phi_0 = \int_{\mathbb{R}^3} \tilde{\psi}_\infty \frac{\partial U}{\partial x_j} = 0.
\]
This implies that \((\tilde{\psi}_\infty, \tilde{\phi}_\infty) = 0\) which contradicts to our assumption, and thus (35) holds.

Let us continue to estimating the norm \( ||(\tilde{\psi}, \tilde{\phi})|| \). Since, by Hölder’s inequality,
\[
\sum_i \int_{\mathbb{R}^3} (\tilde{u}(\tilde{\psi}_i) \tilde{\phi} + 2\beta \tilde{W} \tilde{W} \tilde{\phi}_i \tilde{\phi}) \\
\leq \sum_{i=1}^{k+s} c_1 \left( \int_{|x-P_i| \leq \frac{|x-P_i|}{4}} (\tilde{u}(\tilde{\phi}_i) + 2\beta \tilde{W} \tilde{W} \tilde{\phi}_i)^2 \right)^{\frac{1}{2}} \left( \int_{|x-P_i| \leq \frac{|x-P_i|}{4}} |\tilde{\psi}_i|^2 \right)^{\frac{1}{2}}
\]
Then, by (34) and (35), we have

\[
\begin{aligned}
\text{Claim 1} & \implies \text{this lemma follows from} \\
\text{Then it follows that} & \implies \text{which implies that} \\
\text{Finally, we estimate} & \text{to finish this lemma. As we see, from (28), we have} \\
\text{Then it follows that} & \text{Therefore, this lemma follows from Claim 1 and the following} \\
\end{aligned}
\]

\[
leq C \sum_{i=1}^{k+s} |\tilde{c}_i| \left( \int_{|x-P_i| \leq \frac{|u_{k+1}}{2}} |\tilde{\psi}|^2 \right)^{\frac{1}{2}} \leq C \left( \sum_{i=1}^{k+s} |\tilde{c}_i|^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{k+s} \int_{|x-P_i| \leq \frac{|u_{k+1}}{2}} |\tilde{\psi}|^2 \right)^{\frac{1}{2}} \leq C \left( \sum_{i=1}^{k+s} |\tilde{c}_i|^2 \right)^{\frac{1}{2}} \|\tilde{\psi}\|.
\]

Then, by (34) and (35), we have

\[
C\|\tilde{\psi}, \tilde{\psi}\|^2 \leq C\|\tilde{\psi}, \tilde{\psi}\|_{L^2(\mathbb{R}^3)} + o(1)\|\tilde{\psi}, \tilde{\psi}\|_{L^2(\mathbb{R}^3)} + o(1)\|\tilde{\psi}, \tilde{\psi}\| + o(1)\|\tilde{\psi}, \tilde{\psi}\|_{L^2(\mathbb{R}^3)} \leq C\|\tilde{\psi}, \tilde{\psi}\|^2 + \sum_{ij} |\tilde{a}_{ij}|^2 + \sum_{ij} \tilde{a}_{ij}^2 \|\phi_i\|^2 + \tilde{\phi}_i^2 \|\phi_i\|^2 + \|\phi_i\|^2 \end{aligned}
\]

\[
\text{which implies that} \\
\|\tilde{\psi}, \tilde{\psi}\| \leq C\|\tilde{\psi}, \tilde{\psi}\|_{L^2(\mathbb{R}^3)} + o(1)\|\tilde{\psi}, \tilde{\psi}\|_{L^2(\mathbb{R}^3)} + C\|\tilde{\psi}, \tilde{\psi}\|_{L^2(\mathbb{R}^3)}.
\]

\[
\|u_{k+1}, v_{k+1}\| \text{ to finish this lemma. As we see, from (28), we have} \\
\|u_{k+1}, v_{k+1}\|^2 \leq C\|\tilde{\psi}, \tilde{\psi}\|^2 + \sum_i |\tilde{c}_i|^2 \|\phi_i\|^2 + \sum_i |\tilde{c}_i|^2 \|\phi_i\|^2 \end{aligned}
\]

\[
\text{Then it follows that} \\
\|u_{k+1}, v_{k+1}\| \text{ to finish this lemma. As we see, from (28), we have} \\
\|u_{k+1}, v_{k+1}\|^2 \leq C\|\tilde{\psi}, \tilde{\psi}\|^2 + \sum_{ij} |\tilde{a}_{ij}|^2 + \sum_{ij} \tilde{a}_{ij}^2 \|\phi_i\|^2 + \|\phi_i\|^2 \end{aligned}
\]

\[
\text{Therefore, this lemma follows from Claim 1 and the following} \\
\|\tilde{\psi}, \tilde{\psi}\|_{L^2(\mathbb{R}^3)} \leq C\|\tilde{\psi}, \tilde{\psi}\|_{L^2(\mathbb{R}^3)} + C\left( \sum_{ij} |\tilde{a}_{ij}|^2 + \sum_{ij} \tilde{a}_{ij}^2 \right)^{\frac{1}{2}}.
\]
Lemma 5.1. Suppose that the conditions (A1) and (A2) hold, there exists $k,l$ such that for any $s > 0$, one has
\[
\|u_{k+1,l}, v_{k+1,l}\| \leq C\delta\|u_{P_{k+1}}\|_{L^2(\mathbb{R}^3)} + C\beta\|u_{P_{k+1}} + v_{P_{k+1}, q}\| + O(\delta^{1+\varepsilon})
\]
(2) If $k > 0$, $l > 0$, $s = 0$, and $t = 1$, one has
\[
\|u_{k,l+1}, v_{k,l+1}\| \leq C\delta\|u_{P_{k+1}}\|_{L^2(\mathbb{R}^3)} + C\beta\|u_{P_{k+1}} + v_{P_{k+1}, q}\| + O(\delta^{1+\varepsilon})
\]
(3) If $k = 0$, $l = 0$, $s = 1$, and $t = 0$, one has
\[
\|u_{1,l}, v_{1,l}\| \leq C\delta\|u_{P_{k}}\|_{L^2(\mathbb{R}^3)} + C\beta\|u_{P_{k}} + v_{P_{k}, q}\|.
\]
(4) If $k > 0$, $l = 0$, $s = 0$, and $t = 1$, one has
\[
\|u_{k,1}, v_{k,1}\| \leq C\delta\|bV_{Q_{k+1}}\|_{L^2(\mathbb{R}^3)} + C\beta\|V_{Q_{k+1}} + v_{P_{k}, q}\|,
\]
where the constants $C > 0$ are independent of $\delta$, $k$, and $l$.

5. A supremum problem. Recall that
\[
u_{P_{k}, Q_{l}} = u_{P_{k}} + v_{P_{k}}, Q_{l}, \quad v_{P_{k}, Q_{l}} = V_{Q_{k}} + v_{P_{k}, Q_{l}},
\]
where $(\beta P_{k}, Q_{l}, \beta P_{k}, Q_{l})$ is the solution to (18). Let us define an energy function from $\Lambda_{k,l}$ to $\mathbb{R}$ as follows
\[
M(\beta P_{k}, Q_{l}) := J(u_{P_{k}, Q_{l}}, v_{P_{k}, Q_{l}}),
\]
and consider the following problem
\[
C_{k,l} = \sup_{(\beta P_{k}, Q_{l}) \in \Lambda_{k,l}} M(\beta P_{k}, Q_{l}).
\]

The following lemma will be required when we prove that $C_{k,l}$ can be attained by some interior points in $\Lambda_{k,l}$.

Lemma 5.1. Suppose that the conditions (A1) and (A2) hold, there exists $\delta_0 > 0$ such that for any $0 < \delta < \delta_0$, if $C_{k,l}$ can be attained by some points in $\Lambda_{k,l}$, then for $k \geq 0$ and $l \geq 0$ one has
\[
\begin{cases}
C_{k+1,l} > C_{k,l} + I_1(U), \\
C_{k,l+1} > C_{k,l} + I_2(V),
\end{cases}
\]
where $C_{0,0} = 0$ and for $i = 1, 2$,
\[
I_i(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) - \frac{\mu_i}{4} \int_{\mathbb{R}^3} u^3.
\]

Proof. We prove this lemma by the following three cases.

Case 1. Consider the relations between $C_{k+1,0}$ and $C_{k,0}$, $C_{0,l+1}$ and $C_{0,l}$ for $k \geq 0$ and $l \geq 0$.

As proved in Ao and Wei [1], we see that $C_{k,0}$ and $C_{0,l}$ can be attained, and for $k \geq 0$ and $l \geq 0$,
\[
C_{k+1,0} > C_{k,0} + I_1(U), \quad C_{0,l+1} > C_{0,l} + I_2(V).
\]
So Case 1 holds directly.

Case 2. Consider the relations between $C_{k,1}$ and $C_{k,0}$, $C_{1,l}$ and $C_{0,l}$ for $k \geq 1$ and $l \geq 1$. 

To prove Case 2, we assume that $C_{k,0}$ is attained by $P_k$, i.e., $C_{k,0} = J_1(u_{P_k})$, and take $Q \in \mathbb{R}^3$ such that $|P_i - Q| > \frac{4|Q|}{2} > 10[\ln \delta]$ for $i = 1, \ldots, k$. Let $u_{k1} = u_{P_k,Q} - u_{P_k}$ and $v_{k1} = v_{P_k,Q} - V_Q$. From Corollary 1, we see that

$$\| (u_{k1}, v_{k1}) \| \leq C\delta \| b_{V_Q} \|_{L^2(\mathbb{R}^3)} + C|\beta| \| u_{P_k} V_Q \|_{L^2(\mathbb{R}^3)}$$

for some $C > 0$ independent of $\delta$. After a simple expansion, we have

$$J(u_{P_k,Q}, v_{P_k,Q}) = J_1(u_{P_k}) + I_2(V_Q) + \frac{\delta}{2} \int_{\mathbb{R}^3} b V_Q^2 - \frac{\beta}{2} \int_{\mathbb{R}^3} u_{P_k}^2 V_Q^2$$

$$- \beta \int_{\mathbb{R}^3} (u_{P_k} V_Q^2 u_{k1} + u_{P_k}^2 V_Q v_{k1}) + O(\|(u_{k1}, v_{k1})\|^2),$$

where

$$J_1(u_{P_k}) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u_{P_k}|^2 + (1 + \delta a(x)) u_{P_k}^2 - \frac{\mu_1}{4} \int_{\mathbb{R}^3} u_{P_k}^4.$$

From the definition of $C_{k,1}$, it follows that

$$C_{k,1} \geq J(u_{P_k,Q}, v_{P_k,Q})$$

$$= C_{k,0} + I_2(V_Q) + \frac{\delta}{2} \int_{\mathbb{R}^3} b V_Q^2 - \frac{\beta}{2} \int_{\mathbb{R}^3} u_{P_k}^2 V_Q^2$$

$$- \beta \int_{\mathbb{R}^3} (u_{P_k} V_Q^2 u_{k1} + u_{P_k}^2 V_Q v_{k1}) + O(\|(u_{k1}, v_{k1})\|^2)$$

$$\geq C_{k,0} + I_2(V) + \frac{\delta}{2} \int_{\mathbb{R}^3} b V_Q^2 - C\delta^2 \int_{\mathbb{R}^3} b^2 V_Q^2$$

$$- \frac{|\beta|}{2} \int_{\mathbb{R}^3} u_{P_k}^2 V_Q^2 - C|\beta| \int_{\mathbb{R}^3} u_{P_k}^2 v_{Q}.$$

From the argument in the proof of Lemma 2.5 in [1], we find that

$$\int_{\mathbb{R}^3} \delta b(x) V_Q^2 \geq \int_{|x| \geq \frac{|Q|}{4}} \delta b(x) V_Q^2 - \int_{|x| \leq \frac{|Q|}{4}} \delta b(x) V_Q^2$$

$$\geq M\delta \left( \int_{\mathbb{R}^3} - \int_{|x| \leq \frac{|Q|}{4}} \right) e^{-\pi|x|} V_Q^2 - \left( \sup_{B_{|Q|}(0)} V_Q^2 \right) \int_{\text{supp } b} \delta b(x) V_Q^2$$

$$\geq M\delta e^{-\pi|Q|} - M\delta e^{-\frac{|Q|}{8\pi}} - \delta\delta e^{-\frac{|Q|}{8\pi}} > \frac{M}{2} \delta e^{-\pi|Q|}.$$

$$\int_{\mathbb{R}^3} \delta^2 b^2(x) V_Q^2 \leq \left( \int_{|x| \leq \frac{|Q|}{4}} + \int_{|x| > \frac{|Q|}{4}} \right) \delta^2 b^2(x) V_Q^2$$

$$\leq C\delta^2 e^{-\frac{|Q|}{8\pi}} + \delta^2 \int_{|x| > \frac{|Q|}{4}} b^2(x) V_Q^2$$

$$\leq 2\delta^2 \int_{|x| > \frac{|Q|}{4}} b^2(x) V_Q^2 < \frac{1}{4} \int_{\mathbb{R}^3} \delta b(x) V_Q^2,$$

where $M > 0$ is sufficiently large and $\delta$ is sufficiently small. Since for $\eta \geq \frac{\eta}{2}$,

$$\int_{\mathbb{R}^3} u_{P_k}^2 V_Q^2 \leq C \int_{\mathbb{R}^3} (U_{P_k} + \pi_{P_k}) V_Q^2$$

$$\leq C\delta^{\frac{1+4}{3}} e^{-\frac{8\pi|Q|}{3}} < \delta e^{-\pi|Q|}.$$
Then, combining the above inequalities, it follows that
\[ C_{k,1} > C_{k,0} + I_2(V). \]

**Case 3.** Consider the relations between \( C_{k+1,l} \) and \( C_{k,l}, C_{k,l+1} \) and \( C_{k,l} \) for \( k \geq 1 \) and \( l \geq 1 \).

To prove Case 3, we assume that \( C_{k,l} \) is attained by \((P_k, Q_l)\), i.e. \( C_{k,l} = J(u_{P_k, Q_l}, v_{P_k, Q_l}) \). For \( i = 1, \ldots, k \) and \( j = 1, \ldots, l \), let \( P_{k+1} \in \mathbb{R}^3 \) be a point such that
\[ |P_{k+1} - P_i| > \frac{1 + \eta}{2} |P_{k+1}| > \frac{2|\ln \delta|}{1 - \eta}, \quad |P_{k+1} - Q_j| > \frac{1 + \eta}{2} |P_{k+1}| > \frac{2|\ln \delta|}{1 - \eta}. \]

As before, let us denote
\[ u_{k+1,l} = u_{P_{k+1}, Q_l} - U_{P_{k+1}} - u_{P_k, Q_l}, \quad v_{k+1,l} = v_{P_{k+1}, Q_l} - v_{P_k, Q_l}. \]

It follows from Corollary 1 that
\[ \|u_{k+1,l} + v_{k+1,l}\| \leq C\delta \|aU_{P_{k+1}}\|_{L^2(\mathbb{R}^3)} + C\|u_{P_k, Q_l} U_{P_{k+1}}\|_{L^2(\mathbb{R}^3)} \]
\[ + C|\beta| \|U_{P_{k+1}} u_{P_k, Q_l} + Q_{l}\|_{L^2(\mathbb{R}^3)} + 3 \delta \int_{\mathbb{R}^3} aU_{P_{k+1}} + I + II, \]

where
\[ I = -\mu_1 \int_{\mathbb{R}^3} u_{P_k, Q_l}^3 U_{P_{k+1}} - \beta \int_{\mathbb{R}^3} u_{P_k, Q_l} v_{P_k, Q_l} U_{P_{k+1}}, \]
\[ II = -\frac{\beta}{2} \int_{\mathbb{R}^3} U_{P_{k+1}}^2 v_{P_k, Q_l}^2 - \frac{3\mu_1}{2} \int_{\mathbb{R}^3} u_{P_k, Q_l}^2 U_{P_{k+1}} \]
\[ - 3\mu_1 \int_{\mathbb{R}^3} (u_{P_k, Q_l}^2 U_{P_{k+1}} + u_{P_k, Q_l} v_{P_k, Q_l}^2 u_{P_{k+1}}) u_{k+1,l} - \beta \int_{\mathbb{R}^3} U_{P_{k+1}} v_{P_k, Q_l}^2 u_{k+1,l} \]
\[ - \beta \int_{\mathbb{R}^3} (2U_{P_{k+1}} u_{P_k, Q_l} + U_{P_{k+1}} v_{P_k, Q_l}) v_{k+1,l} + O(\|(u_{k+1,l}, v_{k+1,l})\|^2). \]

For \( I \), since \( \eta \geq \frac{2}{3} \) and \( |a_{ij}| \leq C\delta^{\frac{1+\epsilon}{\eta^2}} \) for any \( i \) and \( j \), then it follows that
\[ |I| \leq C \int_{\mathbb{R}^3} (U_{P_k}^3 + |P_{P_k, Q_l}|^3 + V_{Q_l}^3 + |P_{P_k, Q_l}|^3) U_{P_{k+1}} \]
\[ \leq C \sum_{i=1}^k e^{-|P_i - P_{k+1}|} + C \sum_{i=1}^l e^{-|Q_i - P_{k+1}|} \ll \delta e^{-\eta|P_{k+1}|}. \]

For \( II \), by a similar argument in Case 2, we find that
\[ |II| \leq C \int_{\mathbb{R}^3} (u_{P_k, Q_l}^2 + v_{P_k, Q_l}^2) U_{P_{k+1}}^2 + C\|u_{k+1,l} + v_{k+1,l}\|^2 \]
\[ \leq C \int_{\mathbb{R}^3} \delta^2 u_{P_{k+1}}^2 + C \int_{\mathbb{R}^3} (U_{P_k}^2 + |P_{P_k, Q_l}|^2 + V_{Q_l}^2 + |P_{P_k, Q_l}|^2) U_{P_{k+1}}^2 \]
\[ \ll \delta e^{-\eta|P_{k+1}|}. \]
From Case 2, it is easy to see that
\[
\frac{\delta}{2} \int_{\mathbb{R}^3} aU_{k+1}^2 \geq M \delta e^{-\eta |P_{k+1}|}
\]
for $M > 0$ large enough. Then, combining the above inequalities, one has
\[
C_{k+1,l} > C_{k,l} + I_1(U).
\]

The following lemma shows that $C_{k,l}$ can be attained at some point $(P_k, Q_l) \in \Lambda_{k,l}$.

**Lemma 5.2.** Suppose that the conditions (A1) and (A2) hold, then there exist $0 < \beta_0 < \min\{\mu_1, \mu_2\}$ and $\delta_0 > 0$ such that for any $0 < \beta < \beta_0$ and $0 < \delta < \delta_0$, we have $C_{k,l} = M(P_k, Q_l)$ for some $(P_k, Q_l) \in \Lambda_{k,l}$.

**Proof.** For $k = 0$, it is easy to see that this lemma holds by the argument in Ao and Wei[1]. So we just consider the case $k \geq 1$. In what follows, we prove this lemma for fixed $k \geq 1$ by induction on $l \geq 0$.

For $l = 0$, the lemma also holds by Ao and Wei[1]. Now let us consider the case $l = m + 1$ under the assumption that the lemma holds for $l \leq m$.

We show the case by contradiction. Assume that $M(P^n_k, Q^n_{m+1}) \rightarrow C_{k,m+1}$ for some $(P^n_k, Q^n_{m+1})$ satisfying $|P^n_k, Q^n_{m+1}| \rightarrow \infty$ as $n \rightarrow \infty$. Without loss of generality, we may assume that

\[
(P^n_k, Q^n_{m+1}) = (P^n_{k-s}, P^n_{k-s+1}, \ldots, P^n_k, Q^n_{m+1-t}, Q^n_{m+1-t+1}, \ldots, Q^n_{m+1})
\]

satisfying that

\[
|P^n_{k-s}, Q^n_{m+1-t}| < \infty, \quad \min_{i=1, \ldots, s} |P^n_{k-s+i}| \rightarrow \infty, \quad \min_{i=1, \ldots, t} |Q^n_{m+1-t+i}| \rightarrow \infty, \quad \text{as } n \rightarrow +\infty.
\]

Let us denote
\[
\begin{align*}
  u_{k,m+1} &= u_{P^n_k, Q^n_{m+1}} - u_{P^n_{k-s}, Q^n_{m+1-t}} + \sum_{i=1}^s \mathcal{U}_{P^n_{k-s+i}}, \\
  v_{k,m+1} &= v_{P^n_k, Q^n_{m+1}} - v_{P^n_{k-s}, Q^n_{m+1-t}} + \sum_{i=1}^t \mathcal{V}_{P^n_{m+1-t+i}}.
\end{align*}
\]

From (3) and (18), we have the following decomposition

\[
J(u_{P^n_k, Q^n_{m+1}}, v_{P^n_k, Q^n_{m+1}}) = J(u_{P^n_{k-s}, Q^n_{m+1-t}}, v_{P^n_{k-s}, Q^n_{m+1-t}})
\]

\[
+ J\left(\sum_{i=1}^s U^n_{P^n_{k-s+i}}, \sum_{i=1}^t V^n_{Q^n_{m+1-t+i}}\right) + III + IV + V + VI,
\]

where

\[
III = -\mu_1 \int_{\mathbb{R}^3} \left[ \left( u_{P^n_{k-s}, Q^n_{m+1-t}} + \sum_{i=1}^s U^n_{P^n_{k-s+i}} \right)^3 - u_{P^n_{k-s}, Q^n_{m+1-t}}^3 \right]
\]

\[
- \sum_{i=1}^s U^n_{P^n_{k-s+i}} u_{k,m+1} - \mu_2 \int_{\mathbb{R}^3} \left[ \left( v_{P^n_{k-s}, Q^n_{m+1-t}} + \sum_{i=1}^t V^n_{Q^n_{m+1-t+i}} \right)^3 \right]
\]
\[ IV = -\beta \int_{\mathbb{R}^3} \left[ \left( \sum_{i=1}^{s} U_{P_{k-x}^n} \right) \left( \sum_{i=1}^{t} V_{Q_{m+1-t+i}^n} \right) \right] u_{k,m+1}^2 \]

\[ V = \delta \int_{\mathbb{R}^3} \left( \sum_{i=1}^{s} a(x)U_{P_{k-x}^n} u_{k,m+1} + \sum_{i=1}^{t} b(x)V_{Q_{m+1-t+i}^n} v_{k,m+1} \right) \]

\[ VI = -\beta \int_{\mathbb{R}^3} \left[ \left( \sum_{i=1}^{s} \alpha_{ij} x_{ij} \right) \left( \sum_{i=1}^{s} U_{P_{k-x}^n} \right) + \left( \sum_{i=1}^{t} \beta_{ij} y_{ij} \right) \left( \sum_{i=1}^{t} V_{Q_{m+1-t+i}^n} \right) \right] \]

Since, as \( n \to +\infty, \)

\[ \int_{\mathbb{R}^3} a(x)U_{P_{k-x}^n} = \left( \int_{|x|<|P_{k-x}^n|/2} + \int_{|x|\geq|P_{k-x}^n|/2} \right) a(x)U_{P_{k-x}^n} = o(1), \]

\[ \int_{\mathbb{R}^3} u_{P_{k-x}^n} u_{Q_{m+1-t+i}^n} U_{P_{k-x}^n} = o(1), \quad \int_{\mathbb{R}^3} x_{ij} U_{P_{k-x}^n} = o(1), \]

and

\[ ||(u_{k,m+1}, v_{k,m+1})||^2 \leq C \int_{\mathbb{R}^3} \left[ \left( \sum_{i=1}^{s} U_{P_{k-x}^n} \right)^3 - \sum_{i=1}^{s} U_{P_{k-x}^n}^3 \right]^2 \]

\[ + C \int_{\mathbb{R}^3} \left[ \left( \sum_{i=1}^{t} V_{Q_{m+1-t+i}^n} \right)^3 - \sum_{i=1}^{t} V_{Q_{m+1-t+i}^n}^3 \right]^2 \]

Then, it follows that

\[ J(u_{P_{k-x}^n} u_{Q_{m+1-t+i}^n}, v_{P_{k-x}^n} v_{Q_{m+1-t+i}^n}) \leq J(u_{P_{k-x}^n} u_{Q_{m+1-t+i}^n}, v_{P_{k-x}^n} v_{Q_{m+1-t+i}^n}) \]

\[ + J \left( \sum_{i=1}^{s} U_{P_{k-x}^n} V_{Q_{m+1-t+i}^n} \right) + C \int_{\mathbb{R}^3} \left[ \left( \sum_{i=1}^{s} U_{P_{k-x}^n} \right)^3 - \sum_{i=1}^{s} U_{P_{k-x}^n}^3 \right]^2 \]

\[ + C \int_{\mathbb{R}^3} \left[ \left( \sum_{i=1}^{t} V_{Q_{m+1-t+i}^n} \right)^3 - \sum_{i=1}^{t} V_{Q_{m+1-t+i}^n}^3 \right]^2 \]
Then, we have

\[ +C|\beta|^2 \int_{\mathbb{R}^3} \left( \sum_{i=1}^{s} U_{P_{k-x+i}}^{n} \right)^2 \left( \sum_{i=1}^{t} V_{Q_{m+1-t+i}}^{n} \right)^2 + o(1). \]

Note that

\[ J\left( \sum_{i=1}^{s} U_{P_{k-x+i}}^{n}, \sum_{i=1}^{t} V_{Q_{m+1-t+i}}^{n} \right) \]

\[ = sI_1(U) + tI_2(V) - \frac{\beta}{2} \int_{\mathbb{R}^3} \left( \sum_{i=1}^{s} U_{P_{k-x+i}}^{n} \right)^2 \left( \sum_{i=1}^{t} V_{Q_{m+1-t+i}}^{n} \right)^2 \]

\[ - \frac{\mu_1}{4} \int_{\mathbb{R}^3} \left[ \left( \sum_{i=1}^{s} U_{P_{k-x+i}}^{n} \right)^4 - \sum_{i=1}^{s} U_{P_{k-x+i}}^{n} \sum_{i=1}^{s} U_{P_{k-x+i}}^{n} \sum_{j=1}^{s-1} U_{P_{k-x+i}}^{n} U_{P_{k-x+j}}^{n} \right] \]

\[ - \frac{\mu_2}{4} \int_{\mathbb{R}^3} \left[ \left( \sum_{i=1}^{t} V_{Q_{m+1-t+i}}^{n} \right)^4 - \sum_{i=1}^{t} V_{Q_{m+1-t+i}}^{n} \sum_{i=1}^{t} V_{Q_{m+1-t+i}}^{n} \sum_{j=1}^{t-1} V_{Q_{m+1-t+i}}^{n} V_{Q_{m+1-t+j}}^{n} \right] + o(1). \]

Since

\[ \int_{\mathbb{R}^3} \left[ \left( \sum_{i=1}^{s} U_{P_{k-x+i}}^{n} \right)^4 - \sum_{i=1}^{s} U_{P_{k-x+i}}^{n} \sum_{i=1}^{s} U_{P_{k-x+i}}^{n} \sum_{j=1}^{s} U_{P_{k-x+i}}^{n} U_{P_{k-x+j}}^{n} \right] \]

\[ > 4 \int_{\mathbb{R}^3} \sum_{i=1}^{s-1} \sum_{j=i+1}^{s} U_{P_{k-x+i}}^{n} U_{P_{k-x+j}}^{n} = 4\gamma \sum_{i=1}^{s-1} \sum_{j=i+1}^{s} U(|P_{k-s+i}^{n} - P_{k-s+j}^{n}|), \]

\[ \int_{\mathbb{R}^3} \left[ \left( \sum_{i=1}^{s} U_{P_{k-x+i}}^{n} \right)^3 - \sum_{i=1}^{s} U_{P_{k-x+i}}^{n} \left( \sum_{j=1}^{s} U_{P_{k-x+j}}^{n} \right)^2 \right] \]

\[ \leq C\delta \int_{\mathbb{R}^3} \left[ \sum_{i=1}^{s-1} \sum_{j=i+1}^{s} U_{P_{k-x+i}}^{n} U_{P_{k-x+j}}^{n} + \sum_{i=1}^{s-1} \sum_{j=1}^{s} U_{P_{k-x+i}}^{n} \left( \sum_{j=1}^{s} U_{P_{k-x+j}}^{n} \right)^2 \right] \]

\[ \leq C\delta \sum_{i=1}^{s-1} \sum_{j=i+1}^{s} \int_{\mathbb{R}^3} U_{P_{k-x+i}}^{n} U_{P_{k-x+j}}^{n} \leq C\delta \sum_{i=1}^{s-1} \sum_{j=i+1}^{s} U(|P_{k-s+i}^{n} - P_{k-s+j}^{n}|). \]

Then, we have

\[ J(u_{P_{k-x, Q_{m+1-t}}}, v_{P_{k-x, Q_{m+1-t}}}) \]

\[ \leq J(u_{P_{k-x, Q_{m+1-t}}}, v_{P_{k-x, Q_{m+1-t}}}) + sI_1(U) + tI_2(V) \]

\[ - \left( \frac{\beta}{2} - C\beta^2 \right) \int_{\mathbb{R}^3} \left( \sum_{i=1}^{s} U_{P_{k-x+i}}^{n} \right)^2 \left( \sum_{i=1}^{t} V_{Q_{m+1-t+i}}^{n} \right)^2 \]

\[ - (\gamma \mu_1 - C\delta) \sum_{i=1}^{s-1} \sum_{j=i+1}^{s} U(|P_{k-s+i}^{n} - P_{k-s+j}^{n}|) \]

\[ - (\gamma \mu_2 - C\delta) \sum_{i=1}^{t-1} \sum_{j=i+1}^{t} V(|Q_{m+1-t+i}^{n} - Q_{m+1-t+j}^{n}|) + o(1). \]
So there exist $\beta_0$ and $\delta_0$ such that for any $0 < \beta < \beta_0$ and $0 < \delta < \delta_0$, we have
\[ J(u_{k}^{n},Q_{m+1},u_{l}^{n},Q_{m+1}) \leq J(u_{k}^{n},Q_{m+1},u_{l}^{n},Q_{m+1}) + sI(U) + tI(V) + o(1) \leq C_{k,s,m+1} + sI(U) + tI(V) + o(1). \]
Let $n \to +\infty$, it follows that
\[ C_{k,m+1} \leq C_{k,s,m+1} + sI(U) + tI(V). \]
However, by Lemma 5.1, we can obtain easily
\[ C_{k,m+1} > C_{k,s,m+1} + sI(U) + tI(V). \]
This is a contradiction, and thus the proof of this lemma is finished. 

To finish this section, we prove that the attained points are located in the interior part of $\Lambda_{k,l}$.

**Proposition 3.** There exist $\beta_0 > 0$ and $\delta_0 > 0$ such that for any $0 < \beta < \beta_0$ and $0 < \delta < \delta_0$, $C_{k,l} = \max_{\Lambda_{k,l}} M(P_{k},Q_{l})$ can be attained at some $(P_{k},Q_{l}) \in \text{int} \Lambda_{k,l}$, the interior part of $\Lambda_{k,l}$.

**Proof.** Suppose on contrary that $C_{k,l}$ is achieved at $(P_{k},Q_{l}) \in \partial \Lambda_{k,l}$. We may assume that
\[ |P_{k} - P_{k-1}| = \frac{|\ln \delta|}{2} \quad \text{or} \quad |P_{k} - Q_{l}| = \frac{|\ln \delta|}{4}. \]
Let us denote
\[ u_{k,l} = u_{k}^{n},Q_{l} - u_{k-1},Q_{l} - U_{k}, \quad v_{k,l} = v_{k}^{n},Q_{l} - v_{k-1},Q_{l}. \]
A decomposition shows that
\[ J(u_{k}^{n},Q_{l},v_{k}^{n},Q_{l}) \]
\[ = J(u_{k}^{n},Q_{l},v_{k}^{n},Q_{l}) + I_{1}(U_{k}^{n}) - \frac{\beta}{2} \int_{R^{3}} u^{2}_{k}v^{2}_{k}d\rho_{k} - \frac{\delta}{2} \int_{R^{3}} a(x)U^{2}_{k}v_{k}^{n}d\rho_{k} + \delta \int_{R^{3}} a(x)U_{k}v_{k,l}d\rho_{k} \]
\[ -\mu_{1}\int_{R^{3}} u_{k}^{3}v_{k}^{n}U_{k} - \frac{3\mu_{1}}{2} \int_{R^{3}} u_{k}^{2}v_{k}^{n}U_{k}^{2}U_{k} - \beta \int_{R^{3}} u_{k}^{n}v_{k}^{n}U_{k}^{2}v_{k,l}d\rho_{k} \]
\[ -3\mu_{1}\int_{R^{3}} u_{k}^{3}U_{k}^{2}v_{k}^{n}U_{k}^{2}U_{k}^{2}U_{k} - \beta \int_{R^{3}} U_{k}^{2}v_{k,l}^{2}u_{k,l}d\rho_{k} \]
\[ -\beta \int_{R^{3}} (U_{k}^{2}v_{k,s}^{n}Q_{l}s_{k,l} + 2U_{k}^{n}u_{k}^{n}Q_{l}s_{k,l}) + O((u_{k,l},v_{k,l})^{2}). \]
Take $\eta \geq \frac{2}{3}$ (see Lemma 3.1), from Lemma 4.1, we see that
\[ \|(u_{k,l},v_{k,l})\| \leq C\delta \|a(x)U_{k}^{n}\|_{L^{2}(R^{3})} + C\|U_{k}^{n}v_{k}^{n},Q_{l}\|_{L^{2}(R^{3})} + \frac{3\mu_{1}}{2} \sum_{j=1}^{k-1} e^{-\eta|P_{k-1} - Q_{l}|} + C\delta \int_{R^{3}} e^{-\eta|P_{k} - Q_{l}|} \]
\[ \leq C\beta \|U_{k}^{n}v_{k}^{n},Q_{l}\|_{L^{2}(R^{3})} + C\delta \frac{2^{j+1}}{2^{j-1}}, \]
where we use
\[
\int_{\mathbb{R}^3} U_k^2 u_{\mathfrak{p}_{k-1}, \mathfrak{q}_i}^2 \leq C \sum_{j=1}^{k-1} \int_{\mathbb{R}^3} U_j^2 U_j^2 + C \int_{\mathbb{R}^3} U_k^2 u_{\mathfrak{p}_{k-1}, \mathfrak{q}_i}^2
\]
\[
\leq \frac{C \delta}{|\ln \delta|^2} + C \delta^{\frac{4+s}{3}} \left( \sum_{i=1}^{k-1} e^{-2\eta |\mathfrak{p}_k - \mathfrak{p}_i|} + \sum_{i=1}^{l} e^{-2\eta |\mathfrak{p}_k - \mathfrak{q}_i|} \right) \leq C \delta^{\frac{4+s}{3}}.
\]
In particular, \( \|(u_{k,l}, v_{k,l})\| \leq \frac{C \delta^{\frac{4}{3}}}{|\ln \delta|^2} \). Since, by Hölder’s inequality and Sobolev inequality,
\[
\int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} U_k^2 u_{\mathfrak{p}_{k-1}, \mathfrak{q}_i} + U_k^2 u_{\mathfrak{p}_{k-1}, \mathfrak{q}_i} \right) u_{k,l}
\]
\[
\leq \left[ \int_{\mathbb{R}^3} \left( U_k^2 u_{\mathfrak{p}_{k-1}, \mathfrak{q}_i} + U_k^2 u_{\mathfrak{p}_{k-1}, \mathfrak{q}_i} \right)^2 \right]^\frac{1}{2} \|(u_{k,l}, v_{k,l})\|
\]
\[
\leq C \delta^{\frac{4+s}{6}} \delta^4 = C \delta^{\frac{7+s}{6}},
\]
then it follows from (41) that
\[
J(u_{\mathfrak{p}_k, \mathfrak{q}_i}, v_{\mathfrak{p}_k, \mathfrak{q}_i}) \leq C_{k-1,l} + I_1(U) - \mu_1 \int_{\mathbb{R}^3} u_{\mathfrak{p}_{k-1}, \mathfrak{q}_i} U_{\mathfrak{p}_k}
\]
\[
- \beta \int_{\mathbb{R}^3} u_{\mathfrak{p}_{k-1}, \mathfrak{q}_i} v_{\mathfrak{p}_{k-1}, \mathfrak{q}_i} U_{\mathfrak{p}_k} - \left( \frac{\beta}{2} - C \beta^2 \right) \int_{\mathbb{R}^3} U_k^2 v_{\mathfrak{p}_{k-1}, \mathfrak{q}_i} + O(\delta^{\frac{7+s}{6}}).
\]
Note that
\[
\beta \int_{\mathbb{R}^3} u_{\mathfrak{p}_{k-1}, \mathfrak{q}_i} v_{\mathfrak{p}_{k-1}, \mathfrak{q}_i} U_{\mathfrak{p}_k}
\]
\[
= \beta \int_{\mathbb{R}^3} u_{\mathfrak{p}_{k-1}, \mathfrak{q}_i} v_{\mathfrak{p}_{k-1}, \mathfrak{q}_i} U_{\mathfrak{p}_k} + \beta \int_{\mathbb{R}^3} \mathfrak{p}_{k-1, \mathfrak{q}_i} v_{\mathfrak{p}_{k-1}, \mathfrak{q}_i} U_{\mathfrak{p}_k} + O(\delta^{\frac{4+s}{6}}).
\]
By the definition of \( \Lambda_{k,l} \), it is easy to see that there exists at most 6^3 points \( \mathfrak{q}_i \) such that \( \mathfrak{q}_i \in B(\mathfrak{p}_k - \frac{|\ln \delta|}{2}). \) For the sake of simplicity, we may assume that
\[
\begin{cases}
|\mathfrak{p}_k - \mathfrak{q}_i| \geq \frac{|\ln \delta|}{2} & \text{for } i = 1, \ldots, l - 2, \\
\frac{|\ln \delta|}{4} \leq |\mathfrak{p}_k - \mathfrak{q}_i| < \frac{|\ln \delta|}{2} & \text{for } i = l - 1, l.
\end{cases}
\]
Then, it follows that
\[
\beta \int_{\mathbb{R}^3} \mathfrak{p}_{k-1, \mathfrak{q}_i} v_{\mathfrak{p}_{k-1}, \mathfrak{q}_i} U_{\mathfrak{p}_k}
\]
\[
= \beta \int_{\mathbb{R}^3} \mathfrak{p}_{k-1, \mathfrak{q}_i} v_{\mathfrak{p}_{k-1}, \mathfrak{q}_i} U_{\mathfrak{p}_k} + \beta \int_{\mathbb{R}^3} \mathfrak{p}_{k-1, \mathfrak{q}_i} v_{\mathfrak{p}_{k-1}, \mathfrak{q}_i} U_{\mathfrak{p}_k} + O(\delta^{\frac{4+s}{6}})
\]
\[
= \beta \int_{\mathbb{R}^3} \left( \mathfrak{p}_{k-1, \mathfrak{q}_i} + u_{k-1,l-1} + u_{k-1,l} \right) v_{\mathfrak{q}_i} U_{\mathfrak{p}_k}
\]
\[
+ \beta \int_{\mathbb{R}^3} (\mathfrak{p}_{k-1, \mathfrak{q}_i} + u_{k-1,l-1} + u_{k-1,l}) v_{\mathfrak{q}_i} U_{\mathfrak{p}_k} + O(\delta^{\frac{4+s}{6}})
\]
\[
\leq C \beta^2 \int_{\mathbb{R}^3} U_{\mathfrak{p}_k} V_{\mathfrak{q}_i} + C \beta^2 \int_{\mathbb{R}^3} U_{\mathfrak{p}_k} V_{\mathfrak{q}_{l-1}} + C \beta^2 \int_{\mathbb{R}^3} U_{\mathfrak{p}_{k-1}} V_{\mathfrak{q}_i}
\]
\[
+ C \beta^2 \int_{\mathbb{R}^3} U_{\mathfrak{p}_{k-1}} V_{\mathfrak{q}_{l-1}} + O(\delta^{\frac{4+s}{6}}) \leq C \beta^2 \delta^2 \frac{\delta^4}{|\ln \delta|^2} + O(\delta^{\frac{4+s}{6}}).
\]
Proof of Theorem 1.3.\n\nprove Theorem 1.3 and then finish the paper by proving Theorem 1.2.

6. The proof of Theorem 1.2. In this section, we follow the argument in [11] to prove Theorem 1.3 and then finish the paper by proving Theorem 1.2.

Proof of Theorem 1.3. By Proposition 2, for each \((P_1, Q_1) \in \Lambda_{kl}\), there exists \((\sigma_{P_1, Q_1}, \tau_{P_1, Q_1})\) such that

\[
\begin{align*}
S_1(U_{P_k} + \tau_{P_1, Q_1}, V_Q + \tau_{P_1, Q_1}) &= \sum_{i=1}^{k} \sum_{j=1}^{3} a_{ij} X_{ij} \quad \text{in } \mathbb{R}^3, \\
S_2(U_{P_k} + \tau_{P_1, Q_1}, V_Q + \tau_{P_1, Q_1}) &= \sum_{i=1}^{l} \sum_{j=1}^{3} a_{ij} Y_{ij} \quad \text{in } \mathbb{R}^3, \\
\int_{\mathbb{R}^3} \tau_{P_1, Q_1} X_{ij} &= \int_{\mathbb{R}^3} \tau_{P_1, Q_1} Y_{ij} = 0.
\end{align*}
\]

Let \((P_1, Q_1)\) be the maximum point of \(\Lambda_{kl}\) and denote

\[
(\sigma_{P_1, Q_1}, \tau_{P_1, Q_1}) = (U_{P_k}, V_Q) + (\tau_{P_1, Q_1}, \tau_{P_1, Q_1}).
\]

From Proposition 3, it follows that for each \(i = 1, \cdots, k\), \(s = 1, \cdots, l\), and \(j = 1, 2, 3\),

\[
\frac{\partial M(P_s, Q_l)}{\partial P_{ij}} = 0, \quad \frac{\partial M(P_s, Q_l)}{\partial Q_{sj}} = 0.
\]
Precisely,
\[
\int_{\mathbb{R}^3} \left( \nabla u_{P_k, Q_l} \nabla \frac{\partial u_{P_k, Q_l}}{\partial P_{ij}} + (1 + \delta a(x)) u_{P_k, Q_l} \frac{\partial u_{P_k, Q_l}}{\partial P_{ij}} \right) \\
+ \int_{\mathbb{R}^3} \left( \mu_1 v_{P_k, Q_l}^3 u_{P_k, Q_l} \frac{\partial u_{P_k, Q_l}}{\partial P_{ij}} + \beta u_{P_k, Q_l} v_{P_k, Q_l}^2 \frac{\partial u_{P_k, Q_l}}{\partial P_{ij}} \right) = 0. 
\]

It is easy to see that (48) is diagonally dominant, then from the knowledge of linear algebra system it follows that \( a_{i,j} = 0, b_{s,j} = 0 \) for all \( i, s, j \). Then \( (u_{P_k, Q_l}) = (U_{P_k, Q_l}, Q_l) \) is indeed a solution to (1). Furthermore, similar to the argument in Section 6 of [11], we see that \( u_{P_k, Q_l} \) and \( v_{P_k, Q_l} \) are positive. Moreover, \( u_{P_k, Q_l} \) and \( v_{P_k, Q_l} \) have \( k \) and \( l \) local maximum points respectively.

**Proof of Theorem 1.2.** By changing variables, Theorem 1.2 is a direct result of Theorem 1.3.

**Appendix A: Some technical computations.** In this appendix, we first calculate \( U_{P_k} \) and \( V_{Q_l} \) by the following lemma.

**Lemma A.1.** (See Lemma 2.1 in [2]) For \( i = 1, 2, \ldots, k, s = 1, 2, \ldots, l, \)
\[
\Omega_{P_i} = \left\{ x \in \mathbb{R}^3 : |x - P_i| \leq \frac{|ln\delta|}{4} \right\}, \\
\Omega_{Q_s} = \left\{ x \in \mathbb{R}^3 : |x - Q_s| \leq \frac{|ln\delta|}{4} \right\},
\]
then one has
\[
U_{P_k} = \begin{cases} 
U_{P_i} + O(\delta^{\frac{1}{2}}), & x \in \Omega_{P_i}, i = 1, 2, \ldots, k; \\
O(\delta^{\frac{1}{2}}), & x \in \mathbb{R}^3 \setminus \bigcup_{i=1}^{k} \Omega_{P_i}, 
\end{cases}
\]
\[
V_{Q_l} = \begin{cases} 
V_{Q_s} + O(\delta^{\frac{1}{2}}), & x \in \Omega_{Q_s}, s = 1, 2, \ldots, l; \\
O(\delta^{\frac{1}{2}}), & x \in \mathbb{R}^3 \setminus \bigcup_{s=1}^{l} \Omega_{Q_s}. 
\end{cases}
\]

The following lemma is often used in this paper and described as follows.

**Lemma A.2** (See Proposition 1.2 in [3]) Let \( f \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \ g \in C(\mathbb{R}^N) \) be radially symmetric and satisfy for some \( \alpha \geq 0, \beta \geq 0, \gamma_0 \in \mathbb{R}, \)
\[
f(x)|x|^\beta e^{\alpha|x|} \to \gamma_0 \quad \text{as} \ |x| \to \infty,
\]
\[
\int_{\mathbb{R}^N} |g(x)|(1 + |x|^\beta)e^{\alpha|x|} dx < \infty.
\]
Then as \( |y| \to \infty \), we have
\[
|y|^\beta e^{\alpha|y|} \int_{\mathbb{R}^N} g(x + y)f(x)dx \to \gamma_0 \int_{\mathbb{R}^N} e^{-\alpha|x|} g(x)dx.
\]
Acknowledgments. The authors wish to thank the referee for valuable comments and corrections.

REFERENCES

[1] W. Ao and J. Wei, Infinitely many positive solutions for nonlinear equations with nonsymmetric potentials, *Calc. Var.*, **51** (2014), 761–798.
[2] W. Ao, J. Wei and J. Zeng, An optimal bound on the number of interior peak solutions for the Lin-Ni-Takagi problem, *J. Funct. Anal.*, **265** (2013), 1324–1356.
[3] A. Bahri and Y. Li, On a minimax prodecure for the existence of a positive solution for certain scaler field equation in $\mathbb{R}^n$, *Revista Mat. Iberoam.*, **6** (1990), 1–15.
[4] T. Bartsch, N. Dancer and Z. Wang, A Liouville theorem, a priori bounds, and bifurcating branches of positive solutions for a nonlinear elliptic system, *Calc. Var.*, **37** (2010), 345–361.
[5] T. Bartsch, Z. Wang and J. Wei, Bound states for a couple Schrödinger system, *J. Fixed Point Theory Appl.*, **2** (2007), 353–367.
[6] G. Cerami, D. Passaseo and S. Solimini, Infinitely many positive solutions to some scalar field equations with nonsymmetric coefficients, *Comm. Pure Appl. Math.*, **66** (2013), 372–413.
[7] E. N. Dancer, J. Wei and T. Weth, A priori bounds versus multiple existence of positive solutions for a nonlinear Schrödinger system, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **27** (2010), 953–969.
[8] B. D. Esry, C. H. Greene, J. P. Burker Jr. and J. L. Bohn, Hartee-Fock theory for double condensates, *Phys. Rev. Lett.*, **78** (1997), 3584–3597.
[9] B. Gidas, W. Ni and L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$, *Advances in Math.*, *Supplementary Studies*, **7A** (1981), 369–402.
[10] N. Hirano, Multiple existence of nonradial positive solutions for a coupled nonlinear Schrödinger system, *NoDEA Nonlinear Diff. Eq. Appl.*, **16** (2009), 159–188.
[11] T. Lin, W. Ni and J. Wei, On the number of interior peak solutions for singularly perturbed Neumann problem, *Comm. Pure Appl. Math.*, **60** (2007), 252–281.
[12] J. Liu, X. Liu and Z. Wang, Multiple mixed states of nodal solutions for nonlinear Schrödinger systems, *Calc. Var.*, **52** (2015), 565–586.
[13] Z. Liu and Z. Wang, Ground states and bound states of a nonlinear Schrödinger system, *Adv. Nonlinear Stud.*, **10** (2010), 175–193.
[14] R. Mandel, Minimal energy solutions for cooperative nonlinear Schrödinger systems, *NoDEA Nonlinear Diff. Eq. Appl.*, **22** (2015), 239–262.
[15] S. Peng and Z. Wang, Segregated and synchronized vector solutions for nonlinear Schrödinger systems, *Arch. Rational Mech. Anal.*, **208** (2013), 305–339.
[16] B. Sirakov, Least energy solitary waves for a system of nonlinear Schrödinger equations in $\mathbb{R}^n$, *Comm. Math. Phys.*, **271** (2007), 199–221.
[17] N. Soave, On existence and phase separation of solitary waves for nonlinear Schrödinger systems modelling simultaneous cooperation and competition, *Calc. Var.*, **53** (2015), 689–718.
[18] Y. Sato and Z. Wang, On the multiple existence of semi-positive solutions for a nonlinear Schrödinger system, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **30** (2013), 1–22.
[19] R. Tian and Z. Wang, Multiple solitary wave solutions of nonlinear Schrödinger systems, *Topol. Methods Nonlinear Anal.*, **37** (2011), 203–223.
[20] J. Wei and M. Winter, *Mathematical Aspects of Pattern Formation in Biological Systems*, Applied mathematical Science, 189, Springer, London, 2014.
[21] J. Wei and W. Yao, Uniqueness of positive solutions to some coupled nonlinear Schrödinger equations, *Commun. Pure App. Ana.*, **11** (2012), 1003–1011.

Received February 2019; revised April 2019.

E-mail address: lushun@zjnu.edu.cn
E-mail address: mbyang@zjnu.edu.cn
E-mail address: yzheng@zjnu.edu.cn