Event-triggered scheduling for pinning networks of coupled dynamical systems under stochastically fast switching

Yujuan Han1 | Wenlian Lu2,3,4 | Tianping Chen5

1 College of Information Engineering, Shanghai Maritime University, Shanghai, China
2 School of Mathematical Sciences, Fudan University, Shanghai, China
3 Shanghai Center for Mathematical Sciences, Fudan University, Shanghai, China
4 Shanghai Key Laboratory for Contemporary Applied Mathematics, Shanghai, China
5 School of Computer Sciences and Mathematical Sciences, Fudan University, Shanghai, China

Correspondence
Wenlian Lu, School of Mathematical Sciences, Fudan University, Shanghai 200433, China.
Email: wenlian@fudan.edu.cn

Funding information
National Natural Science Foundation of China,
Grant/Award Numbers: 61703271, 62072111, 51879156, 91630314

Abstract
This paper studies the stability of linearly coupled dynamical systems with feedback pinning algorithms. Here, both the coupling matrix and the set of pinned-nodes are time-varying, induced by stochastic processes. Event-triggered rules are employed in both diffusion coupling and feedback pinning terms, which can reduce the actuation and communication loads. Two event-triggered rules are proposed and it is proved that if the system with time-average couplings and pinning gains is stable and the switching of coupling matrices and pinned nodes is sufficiently fast, the proposed event-triggered strategies can stabilize the system. Moreover, Zeno behaviour can be excluded for all nodes. Numerical examples of networks of mobile agents are presented to illustrate the theoretical results.

1 | INTRODUCTION

In recent years, the study of complex dynamical networks has attracted broad attentions due to its potential applications in various fields [1–4]. Among them, the synchronization problem is a hot research topic and has been extensively investigated [5, 6]. In dynamical networks, synchronization means that all nodes approach to a uniform dynamical behaviour and are generally assured by the couplings among nodes and/or external distributed and cooperative control. Many control strategies are taken into account to steer the synchronization dynamics to a desired trajectory. Among them, pinning control is an effective scheme, which was introduced in [7] to study the coupled map lattice and was then extended to complex networks by applying some local feedback controllers only to a fraction of nodes. Pinning control strategies for different models have been proposed and investigated. For example, pinning controllability of complex networks was studied in [8], pinning synchronization of complex dynamical networks was investigated in [9, 10], pinning complex dynamical networks via a single controller was considered in [11], and an adaptive pinning strategy for fuzzy coupled neural networks was proposed in [12].

In recent decades, a number of researchers have suggested that the event-based control algorithms can reduce communication and computation loads in networked systems while maintaining control performance [13–21]. Dimarogonas et al. [13] investigated centralized and distributed formulation of event-driven strategies for consensus of multi-agent systems. Hu et al. [14] addressed output consensus of heterogeneous linear multi-agent systems via event-triggered and self-triggered control. In [16, 17], consensus problem of second-order multi-agent systems with event-triggered control was investigated. Event-triggered control was also applied to pinning synchronization problem of complex networks [18–21]. Lu et al. [18] studied pinning synchronization of linearly coupled dynamical systems with time-varying coupling matrix and pinned node set induced by a Markov chain. Furthermore, pinning synchronization problem of Markovian switching complex networks...
with partly unknown transition rates was studied in [19]. Liu et al. [20] investigated cluster pinning synchronization problem for complex dynamical networks with switching signal characterized by average dwell-time constraint. In [21], the pinning quasi-synchronization problem of Markovian switching heterogeneous networks was studied. In [22], a novel memory sampled-data control scheme was proposed to ensure the synchronization of semi-Markov jumping complex networks. In these works, the network topology is assumed to be static or slowly switching.

However, in many biological and engineering networks, the on–off interactions among nodes lead to a stochastically fast switching system [23–26], and in networks of mobile agents on–off interactions among nodes lead to a stochastically fast synchronization of semi-Markov jumping complex networks. In our previous work [29], pinning synchronization problem of fast Markovian switching complex networks was investigated, and then the event-triggered control was employed to the fast Markovian switching system in [30]. However, the Markovian switching is restrictive because in practice switching rules can be non-Markovian, leading to a more general switching rule with boarder applicability. Moreover, the proposed event-triggered strategies in [30] cannot rule out Zeno behaviour for some cases. This motivates the present study.

This paper works on the event-triggered scheduling for complex dynamical networks under stochastically fast switching. Sufficient conditions are given to guarantee the pinning synchronization of coupled dynamical systems under fast switching couplings and event-triggered communications. The main contributions and novelties can be illustrated as follows.

First, the concerned complex dynamical networks under stochastically fast switching can be employed to describe a broader class of practical stochastic systems. Second, two novel event-triggered rules are proposed to stabilize the switched complex dynamical networks. Under the proposed event-triggered rules, the interevent interval will be greater than some positive number and Zeno behaviour will be excluded. Third, criteria for synchronization of coupled dynamical systems under fast switching couplings and event-triggered communications are derived. In comparison with the existing literature, we do not suppose the stability of every subsystem, that is, subsystems among switching can be unstable.

This paper is organized as follows. In Section 2, the underlying problem is formulated. In Section 3, we propose the event-triggered schemes of diffusion configuration and pinning terms to pin the coupled systems to a homogenous pre-assigned trajectory of the uncoupled node system. Theoretical analysis is given in this section. Simulations are given in Section 4 to verify the theoretical results. In Section 5, we discuss future research directions and conclude the paper.

**Notations:** Denote the identity matrix of $m$ dimensions by $I_m$. $A_{ij}$ denotes the $(i, j)$th element of matrix $A$ and $A^T$ the transpose of $A$. For a square matrix $A$, $A^{sym} = (A + A^T)/2$ denotes its symmetry part; $A > 0 (\geq 0)$ denotes that $A$ is positive (semi-) definite and $A < 0 (\leq 0)$ denotes $A$ is negative (semi-) definite; $\lambda(A)$ and $\lambda(A)$ are the largest and smallest eigenvalues in module, respectively. For a symmetric matrix $B$, denote its $i$-th largest eigenvalue by $\lambda_i(B)$. $\|A\|$ denotes the matrix norm of $A$ induced by the vector norm $\|\cdot\|$. For a matrix $A$, $\|A\|_\infty = \max_i \sum_j |A_{ij}|$. In particular, without special notes, $L_2$-vector norm is used in this paper and denote it by $\|\cdot\|_2 = \|x\|_2 = \sum_i |x_i|^2$. The symbol $\otimes$ represents the Kronecker product.

## 2 PROBLEM FORMULATION

Consider linearly coupled ordinary differential equations (LCODEs) as follows:

$$\dot{x}_i(t) = f(x_i(t), t) + \sum_{j=1}^m L_{ij}(t) \Gamma [x_j(t) - x_i(t)], \quad i = 1, ..., m,$$

where $x_i(t) \in \mathbb{R}^n$ denotes the state vector of node $i$; the continuous map $f(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ denotes the node dynamics if there are no couplings; $\epsilon$ is the uniform coupling strength at each node; $L_{ij}(t) \geq 0$ for $i, j = 1, ..., m, i \neq j$, denote the coupling coefficients, and $L_{ii}(t) = - \sum_{j \neq i} L_{ij}(t)$; $\Gamma = [\Gamma_k]_{k=1}^m \in \mathbb{R}^{n \times n}$ is the inner configuration matrix with $\Gamma_k \neq 0$ if two nodes are connected by the $k$-th and $l$-th state components, respectively.

The desired trajectory $s(t)$ satisfies:

$$\dot{s} = f(s(t), t), \quad s(0) = s_0,$$

where $s(t)$ can be an equilibrium, a periodic orbit or even a chaotic orbit. The pinning controlled network is described as follow:

$$\dot{x}_i(t) = f(x_i(t), t) + \sum_{j=1}^m L_{ij}(t) \Gamma [x_j(t) - x_i(t)]$$

$$- \alpha D_i(t) \Gamma [x_i(t) - s(t)], \quad i = 1, ..., m, \quad (1)$$

where $D_i(t) = 1$ if note $i$ is pinned at time $t$, otherwise $D_i(t) = 0$; $\alpha$ is the pinning strength gain over the coupling strength. Denote $D(t) = \text{diag}[D_1(t), ..., D_m(t)]$.

This paper supposes the network topology and pinned node set are switching with respect to time and the switching rate is finite in any time period $[0, t]$, namely, that only finite switches occur in any finite time interval. Hence, the solution of (1) exists
for the interval $[0, +\infty)$ and is unique. Moreover, we assume $L(t)$ and $D(t)$ satisfy the following assumption.

**Assumption 1.** For matrices $L(t)$ and $D(t)$, there exist two average matrices $\bar{L}$ and $\bar{D}$ such that

$$
\bar{L} = \lim_{\Delta \to \infty} \int_{t}^{t+\Delta} L(s) ds, \quad \bar{D} = \lim_{\Delta \to \infty} \int_{t}^{t+\Delta} D(s) ds,
$$

(2)

where the limits hold uniformly with respect to $t$.

Here the convergence of limits in Assumption 1 is uniform in the sense that there are two strictly continuous and two strictly decreasing functions $\mu_i(t) : [0, \infty) \to [0, \infty)$ satisfying $\lim_{t \to \infty} \mu_i(t) = 0$, $i = 1, 2$, such that for any $t \geq 0$, $\| \int_{t}^{t+\Delta} L(s) ds / \Delta - \bar{L} \| \leq \mu_1(\Delta)$ and $\| \int_{t}^{t+\Delta} D(s) ds / \Delta - \bar{D} \| \leq \mu_2(\Delta)$ [31–33]. There are many switching rules of time-varying coupling topologies and pinned node sets satisfy Assumption 1, including periodic switching, Markovian switching and the switching subject to the Cox process [34].

Under Assumption 1, it was proven in [29] that the switching system (1) is stable if the time-average system is stable and the network topology and pinned node sets switch sufficiently fast. Motivated by the setup in [31, 33, 35] to describe the fast switching network, we consider the coupling matrix and the pinned node matrix of the form $L(t/\varepsilon)$ and $D(t/\varepsilon)$, where parameter $\varepsilon > 0$ determines the switching speed of the network topology and pinned node sets.

The Lipschitz and/or the QUAD condition are widely used in the synchronization literature [5, 6, 8–11] to assure the non-increasing of some proposed Lyapunov function. Throughout this paper, we assume the node dynamics $f(x, t)$ satisfies the following assumption.

**Assumption 2.** $f(x, t)$ is uniformly Lipschitz with constant $L_f$, that is, $\| f(x, t) - f(y, t) \| \leq L_f \| x - y \|$ holds for all $x, y \in \mathbb{R}^n$, $t \geq 0$.

In fact, we do not need the uniformly Lipschitz condition hold for all $x, y \in \mathbb{R}^n$ but for a region $\Lambda \subset \mathbb{R}^n$ which contains the global attractors of the coupling systems.

**Definition 1.** The continuous function $f(x, t) : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}^n$ is said to satisfy $f(x, t) \in \text{QUAD}(G, \alpha \Gamma, \beta)$, if there exist a positive definite matrix $G \in \mathbb{R}^{n \times n}$ and constants $\alpha \in \mathbb{R}, \beta > 0$, such that

$$(x - y)^T G \left[ f(x, t) - f(y, t) - \alpha \Gamma (x - y) \right] \leq -\beta (x - y)^T G (x - y)$$

holds for all $x, y \in \mathbb{R}^n$.

The QUAD condition implies that linear state feedback with suitable control gains can stabilize the system. Consider the system:

$$\dot{x}(t) = f(x(t), t) - \alpha \Gamma [x(t) - s(t)] \quad \text{with} \quad \dot{s}(t) = f(s(t), t),$$

we can find that if $f(x, t) \in \text{QUAD}(G, \alpha \Gamma, \beta)$, then a sufficiently large $\alpha$ can guarantee that $\lim_{t \to \infty} \| x(t) - s(t) \| = 0$. It should be noted that the QUAD condition can be satisfied by many well-known systems, such as the Lorenz system, the Chen system, the Chua’s circuit and so on.

It can be verified that if $f(x, t)$ is Lipschitz with constant $L_f$, then $f(x, t) \in \text{QUAD}(G, \alpha \Gamma, \beta)$ with $\alpha > 0$ and

$$\beta = \frac{\alpha \lambda (GT + \Gamma^T G)}{2\lambda (G)}$$

(3)

For more discussions on the relations among QUAD, Lipschitz and contracting conditions, we refer readers to the work [36].

This study employs an event-triggered strategy to pin the network to the desired trajectory. Each node updates the diffusion coupling term and pinning control term (if pinned) at its latest event-triggered time point based on some criteria, which gives

$$\dot{x}_i(t) = f(x_i(t), t) + \sum_{j=1}^{m} L_{ij}(t/\varepsilon) \Gamma [x_j(t) - x_i(t)] - \alpha D_{ij}(t/\varepsilon) \Gamma [x_j(t) - s(t)],$$

(4)

Here, $i, j, k \in \mathbb{N}$ are the event-triggered time points for node $i$. At triggering time $t_k$, node $i$ collects the state information of its neighbours and the target (if pinned), then updates its coupling and pinning control (if pinned) terms accordingly. This kind of updating rule is called the pull-based one [37] and adopted in many works [14, 18, 38, 39]. The triggered event is defined based on the neighbours’, the target trajectory’s and its own states with some prescribed rule to be defined later.

Here, the diffusion coupling term of a node will remain unchanged until its next triggering time, even if one of its neighbours is triggered in this period. This could reduce computation loads of the network. However, every node needs to acquire its neighbours’ states and the target state (if pinned) at every triggering time. To realize this, one can equip each node with an embedded microprocessor, which is in charge of information collection, computation and controller actuation [13]. The microprocessor monitors the triggering condition. It will update the controller and actuate the latest controller to the node once an event is triggered.

In the following, we study the stability of trajectory $s(t)$ of system (4).

**Definition 2.** System (4) is said to be stable at $s(t)$ in the mean square sense, if

$$\lim_{t \to +\infty} \mathbb{E} \left[ \| x_i(t) - s(t) \|^2 \right] = 0, \quad i = 1, ..., m.$$
3 MAIN RESULTS

3.1 Stability analysis

Theorem 1. Suppose that there exist a diagonal matrix \( P > 0 \), a matrix \( G > 0 \) and positive constants \( \alpha, \beta, \epsilon, \kappa \) such that:

1. \( f(x, t) \in \text{QUAD}(G, \alpha \Gamma, \beta) \);
2. \([GP]^\infty \geq 0\);
3. \( \{P(\alpha L_m + cI - \alpha D)\}^\infty \leq 0 \).

Let \( \lambda = \lambda(P \otimes G), \lambda = \lambda(P \otimes G) \),

\[
e_i(t) = \sum_j L_{ij} \Gamma [x_j(t_e^j) - x_j(t_e^j) - x_j(t) + \xi_j(t)]
- \kappa D \Gamma [x_i(t^e) - s(t^e_i) - x_i(t) + s(t)],
\]

\[t \in [t^e_i, t^e_{i+1}].\] \hspace{1cm} (5)

and

\[
e_i(t) = \sum_j L_{ij} (t/\epsilon) \Gamma [x_j(t^e^j) - x_j(t^e^j) - x_j(t) + \xi_j(t)]
- \|\kappa D(t/\epsilon) \Gamma [x_i(t^e) - s(t^e_i) - x_i(t) + s(t)],
\]

\[t \in [t^e_i, t^e_{i+1}].\] \hspace{1cm} (6)

Then there exist \( \epsilon^* > 0 \) such that for any \( 0 < \epsilon \leq \epsilon^* \), system (4) is stable at \( s(t) \) in the mean square sense, under either of the following two updating rules:

1. set \( t^e_{i+1} \) as the next triggering time point by the rule

\[\tau_k^i = \inf \left\{ t - t_k^i : \|e_i(t)\| > \frac{\beta_1 \Lambda}{\alpha} \|x_i(t) - s(t)\| \right\}, \hspace{1cm} \text{(7)}\]

\[t^e_{i+1} = t^e_i + \min \{\max\{\tau_k, T_i\}, T_i\}; \hspace{1cm} \text{(8)}\]

2. set \( t^e_{i+1} \) as the next triggering time point by the rule

\[\tau_k^i = \inf \left\{ t - t_k^i : \|e_i(t)\| > \frac{\beta_1 \Lambda}{\alpha} \|x_i(t) - s(t)\| \right\}, \hspace{1cm} \text{(9)}\]

\[t^e_{i+1} = t^e_i + \max\{\tau_k, T_i\}; \hspace{1cm} \text{(10)}\]

where \( \beta < \beta_1, T \geq T_i \) can be any positive constants and

\[T_1 = \frac{1}{\epsilon_1} \ln \left( \frac{\eta}{\max\{\tau_2, \tau_1\} + 1} \right), \quad T_2 = \frac{1}{\epsilon_1} \ln \left( \frac{\eta}{\epsilon_2} + 1 \right), \hspace{1cm} \text{(11)}\]

\[\hat{\tau}_1 = L_{ii} + \epsilon \max_i \|L_{ii} - \kappa D_{ii}\| \otimes \Gamma \| + \eta \sqrt{m}, \hspace{1cm} \text{(12)}\]

\[\hat{\tau}_2 = \sqrt{2} \left( \sqrt{m - 1 \max_i L_{ij} + \max_i |L_{ii} - \kappa D_{ii}|} \right) ||\Gamma||, \hspace{1cm} \text{(13)}\]

\[\hat{\tau}_2 = \sqrt{2} \left( \sqrt{m - 1 \max_i L_{ij} + \max_i |L_{ii} - \kappa D_{ii}|} \right) \|\Gamma\|. \hspace{1cm} \text{(14)}\]

Moreover, Zeno behaviour can be excluded for each node.

Proof of Theorem 1 is deferred to Appendix A.1.

Remark 1. Zeno behaviour refers to the phenomenon that an infinite number of events happens in finite time period. It can be excluded if the interevent interval is greater than some positive number. Inspired by [14, 38], two positive constants \( T_1 \) and \( T_2 \) are incorporated in rule (8) and (10), respectively, such that the interevent interval is lower bounded by \( T_1 \) or \( T_2 \). Specifically, for event-triggered rule (7), (8), \( t^e_{i+1} - t^e_i \geq T_1 \) holds for \( \forall i, \forall k \), while for event-triggered rule (9), (10), \( t^e_{i+1} - t^e_i \geq T_2 \) holds for \( \forall i, \forall k \).

For the average coupling matrix \( \bar{L} \), define its underlying graph by \( \mathcal{G}(\bar{L}) \) in such a way that there exists a link from the \( i \)-th vertex to the \( j \)-th one if and only if \( \bar{L}_{ij} > 0 \). For the average pinned matrix \( \bar{D} \), we denote the pinned node set by \( P(\bar{D}) \) and define that \( j \in P(\bar{D}) \) if and only if \( \bar{D}_{ij} > 0 \). Then Condition 3 in the above Theorem can be obtained by the following remark.

Remark 2. If the pinned node set \( P(\bar{D}) \) can access all other nodes in the graph \( \mathcal{G}(\bar{L}) \), there exist positive constants \( \alpha, \beta, \kappa \) and a positive diagonal matrix \( P \) such that \( \{P(\alpha L_m + cI - \alpha D)\}^\infty \leq 0 \), see [29] for the detailed proof.

In [10], it was proved that if the pinned node set can access all other nodes in the graph and the QUAD condition is satisfied, then the dynamics of nodes can be steered to the desired trajectory. Therefore, Conditions 1–3 in Theorem 1 implies that the system with the average coupling matrix and the average pinning gains is stable at \( s(t) \). In other words, Theorem 1 indicates that if the system with the average coupling matrix and average pinning gains is stable and the switching is sufficiently fast, switched system could be stabilized under the event-triggered schemes.

Remark 3. In Theorem 1, triggering rule (7), (8) is based on error \( e_i(t), i = 1, \ldots, m \) in (5), which depends on the coupling coefficients and pinned node sets in average, while triggering rule (9), (10) is based on error \( \hat{e}_i(t), i = 1, \ldots, m \) in (6), depending on the instantaneous coupling coefficients and pinned node sets. Different from triggering rules in [30], positive constants \( T_1 \) and \( T_2 \) are incorporated into the triggering rules.
3.2 The switching rule follows an ergodic Cox process

The Cox process $\sigma_j$ is a combination of two processes: $\{\xi^\sigma, n \in \mathbb{N}\}$ as a discrete adapted process with respect to filtration $(\Omega, F_n)$, $n = 1, 2, \ldots$, embedded in $(\Omega, F)$, and the switching point process $\{t_n, n \in \mathbb{N}\}$ [34]. In detail, let $N_{(t_1, t_2)}$ be the counting number of the switches in $[t_1, t_2]$ and $N(t) = N_{(0,t)}$ with $N(0) = 1$. All switches occur independently of each other and the switching rate $\lambda(t)$ follows a stochastic process depending on the adapted process $\{\xi^\sigma, n \in \mathbb{N}\}$: $\lambda(t) = \mu(\xi^\sigma(t))$, where $\mu(\cdot)$ is a positive measurable function with respect to $(\Omega, F_N(t))$ with upper bound $\mu_{\infty}$, that is, $\mu(\xi) \leq \mu_{\infty}$ for all $\xi \in \Omega$. Namely, for $\{\xi^\sigma, n \in \mathbb{N}\}$, $P(N_{(t_1, t_2)} \geq 1) = \mu(\xi^\sigma(t_1)) \leq \mu_{\infty} t \to 0$ and $P(N_{(t_1, t_2)} \geq 2) = o(b)$ as $b \to 0$. Thus, the point process $N(t)$ is also an adapted process, measurable on some filtration, denoted by $\mathcal{F}$. The process $\mathcal{F}$ is right-continuous and defined as

$$\sigma_j = \xi^\sigma, \quad t \in [t_n, t_{n+1}).$$

Thus $\{\sigma_j, t \in \mathbb{R}^+\}$ is a well-defined adapted process with respect to a joint filtration $(\Omega, \mathcal{F}, \mathcal{F}_N(t))$. Denote the state space of the adapted process $\{\xi^\sigma, n \in \mathbb{N}\}$ by $\mathcal{S}$. An ergodic Cox process $\{\sigma_j, t \in \mathbb{R}^+\}$ satisfies that:

$$\lim_{\Delta \to \infty} \frac{1}{\Delta} \int_{t_n}^{t_{n+1}} \left[ \pi(j_i) - \pi(j) \right] ds = 0$$

holds for some probability distribution $\pi(j_i), i \in \mathcal{S}$ and any $t$ with probability 1. One can check that (2) hold for $L(\sigma_j)$ and $D(\sigma_j)$ with $L = \sum_i \pi(j) L_j$ and $D = \sum_i \pi(j) D_j$.

Here, we suppose the switching of coupling topologies and pinned node sets follow an ergodic Cox process $\sigma_j/\epsilon$, where $\epsilon$ is the parameter that affects the switching rate of coupling topologies and pinned node sets. Then we have the following corollary.

**Corollary 1.** Suppose that condition items 1)-3) in Theorem 1 hold and $\sigma_j$ is an ergodic Cox process that satisfies:

1) the switching rate $\lambda(t) \in [\mu_0, \mu_{\infty}]$, for all $t$ and some positive constants $\mu_0, \mu_{\infty}$;
2) $\max (\pi(n) - \pi) \leq C_0 \delta^\sigma$ holds for some $C_0 > 0$ and $\delta < 1$, where $\pi_j(n) = P(\xi^\sigma = j)$.

Take a constant $\beta' \in (0, \beta)$ with $\beta$ given in (3) and consider system (4) with switching process $\sigma_j/\epsilon$, we have that

(1) under updating rule (7), (8), the system is stable at $s(t)$ in the mean square sense if

$$\epsilon \leq \min \left\{ \frac{3\mu_1}{2\mu_1 + 3\mu_1^2}\left( \frac{4\mu_1^2}{3\mu_1^2(\beta - \beta')^2} \right), \frac{3\mu_1(\mu_0 + 3\mu_1)^2(K_1 + K_2 + K_3) \max\{8, \exp(\rho_1 T)\}}{8} \right\};$$

(15)

(2) under updating rule (9), (10), the system is stable at $s(t)$ in the mean square sense if

$$\epsilon \leq \min \left\{ \frac{3\mu_1}{2\mu_1 + 3\mu_1^2}\left( \frac{4\mu_1^2}{3\mu_1^2(\beta - \beta')^2} \right), \frac{3\mu_1(\mu_0 + 3\mu_1)^2(K_1 + K_2 + K_3) \max\{8, \exp(\rho_1 T)\}}{8} \right\};$$

(16)

where $T \geq T_1$ is any positive constant and

$$K_1 = \frac{2}{\Delta} \left[ (\epsilon + 1) \max_j \alpha(A_i) \frac{\beta^2}{(A_i)^T (A_i)^T} + L_j^2 + c m \eta^2 \right]$$

$$+ 2 \max_j \left[ \left( A_i \left[ (L_j - \kappa D(j)) \otimes I \right] \right) \frac{c m \eta^2}{\Delta} \right]$$

(17)

$$K_2 = K_1 + c m \max_j \left\| R_j \right\| \frac{\alpha(GGT^T G)}{\Delta}$$

$$\times \left[ \left( \frac{L_j^2}{\alpha(G)} + \frac{c m \eta^2 \| P \|_{\infty}}{\Delta} \right) \right]$$

$$+ \max \alpha \left[ \left( S_j^T P^2 S_j \right) \otimes I \right] \frac{c m \max_j \left\| R_j \right\|_{\infty}}{\Delta},$$

(18)

$$K_3 = \left[ (1 + c) \| P \|_{\infty} + 2 \alpha \left( \frac{G^T G}{\Delta} \right) \right]$$

$$\times \left[ \left( \frac{L_j^2}{\alpha(G)} + \frac{c m \eta^2 \| P \|_{\infty}}{\Delta} \right) \right]$$

$$+ \frac{2 c m \max_j \left\{ \left( \frac{P S_j \otimes G T} {\infty} \right) \right\}}{\alpha(G)},$$

$$R(j) = L(j) - I - \kappa D(j) + \kappa D,$$

$$A(j) = \left( P S_j \otimes G T \right),$$

$$S_j = L(j) - \kappa D(j).$$

(19)

Proof of corollary 1 is deferred to Appendix A.2.

**Remark 4.** In the above updating rules, continuous-time monitoring is needed to calculate the next triggering time. To get rid of continuous monitoring, the discrete-time monitoring strategy can be derived analog to the procedures given in our previous work [18, 39]. However, it should be pointed out that as a pay-off for the small cost of discrete monitoring, triggering events will happen more frequently than those in continuous-time monitoring.
4 | NUMERICAL SIMULATIONS

We employ theoretical results to the “random waypoint” (RWP) model [40], which is widely used in the performance evaluation of protocols of ad hoc networks. The RWP model contains (among some other technical assumptions) the following conditions: realistic movements of agents, a sensible transmission range, and limited buffering storage spaces. In detail, we consider \( m \) agents moving in the planar space \( \Lambda \subset \mathbb{R}^2 \) according to the RWP model. The agent moves towards a randomly selected target with a random velocity. After approaching the target, the agent waits for a time interval of random length and then continues the process. The motion of each agent is stochastically independent of the others and of time. Two agents are considered to be coupled at time \( t \) with adjacent coefficient \( 1 \) if the distance between them is less than a given interaction radius \( r > 0 \). Then according to the positions of agents in the given area \( \Lambda \), an undirected graph can be structured. And the coupling matrix can be assigned by the spatial distribution of the agents at each time. Denote \( S_1 = \{ L(1), \ldots, L(N_1) \} \) all possible topologies of the \( m \) agents and \( L(\sigma_i) \) the coupling matrix of the network at time \( t \), here \( \sigma_i : \mathbb{R}^+ \to \{1, \ldots, N_1\} \). Hence, \( L(\sigma_i) \) is a homogeneous continuous time Markov chain with finite state space \( S_1 \) [41].

For networks with time-varying links, implementing a selective pinning is difficult, since the controller must move with the pinned agents. Inspired by [27], the reference model is equipped with all non-diagonal elements being identical, and the diagonal \( \sigma_i \). Hence, \( L(\sigma_i) \) is a higher dimensional homogeneous Markov chain. For convenient, we suppose \( \sigma_i = \eta_i \) in the following.

The node distribution of RWP model was studied in [42], which proved that the node distribution is ergodic and the stationary distribution has strictly positive probability everywhere in the region. That is, every pair of agents has a positive probability to be coupled and every agent has a positive probability to be pinned, with the stationary distribution. Therefore, the expected coupling matrix \( L \) corresponds to a complete graph with all non-diagonal elements being identical, and the diagonals of \( D \) are all positive and identical.

In the following examples, we pick 50 independent agents moving in the square area \( \Lambda = [0, 1000] \times [0, 1000] \) and the control region \( \Lambda_0 = [400, 600] \times [400, 600] \). The initial location of each agent is randomly chosen in \( \Lambda \). The velocity of movement follows a uniform distribution in \([1000, 2000]\) (m/sec). After approaching the target, the agent waits for a random time period following the uniform distribution in \([0.2, 0.5]\) (s). Take the coupling radius \( r = 100 \) (m). The average coupling matrices \( L \) and \( D \) are numerically calculated by taking average over 50 runs of the simulator for \( 10^7 \) iterations via abandoning the initial \( 10^6 \) iterations: \( I = [I_{ij}]_{500 \times 500} \) with \( I_{ij} = L = 0.0332 \) for all \( i \neq j \), \( D_{ii} = D = 0.0632 \) for all \( i \).

A dynamical system is associated to each agent, here we consider a three-dimensional neural network as the uncoupled node dynamics [43]:

\[
\frac{dx}{dt} = -Dx + Tg(x)
\]

with \( x = (x_1, x_2, x_3)^T \in \mathbb{R}^3 \).

\[
T = \begin{bmatrix}
1.2500 & -3.200 & -3.200 \\
-3.200 & 1.100 & -4.400 \\
-3.200 & 4.400 & 1.000 \\
\end{bmatrix}
\]

\[
D = \frac{1}{2}I_3, \quad g(x) = (g(x_1), g(x_2), g(x_3))^T, \quad \text{and} \quad g(\alpha) = (|\alpha + 1| - |\alpha - 1|)/10.
\]

This system has a double-scrolling chaotic attractor with initial value \( x_1(0) = x_2(0) = x_3(0) = 0.1000 \). Noting that \( f(x) = -Dx + Tg(x) \) has 9 Jacobin matrices and \( \chi = 1.3 \) is the upper bound of the spectral norm of these Jacobin matrices of \( f \). We pick \( \alpha = 2, G = \Gamma = I_m \) and \( \beta^* = 0.65 < \beta = \alpha - 0.5 - \chi^2/2 \). Pick \( \chi = 10 \), we get \( \min Re\lambda(-\tilde{L} + \chi \tilde{D}) = 0.632 \). Noting that if \( \epsilon > \frac{\alpha}{\min Re\lambda(-\tilde{L} + \chi \tilde{D})} \approx 3.1646 \), all eigenvalues of \( \alpha I_m + \tilde{L} + \beta \tilde{D} \) have negative real part, which implies that there exists a matrix \( P > 0 \) such that \( P(\alpha I_m + \tilde{L} - \beta \tilde{D})^{\sigma m} \leq 0 \). Here, we pick \( \epsilon = 3.2 \). Then the coefficient in the triggering rules is \( \frac{\beta \lambda}{\alpha} = 0.2 \). Therefore, all assumptions in Theorem 1 hold.

Figure 1 shows the dynamics of \( V(t) \) under triggering rules (7), (8) and (9), (10), which implies that the coupled system under either of these two triggering rules is stable. In comparison, the dynamics of the linearly coupled system with continuous updating is also plotted.

Furthermore, randomly picking 10 nodes, their triggering time slots under rules (7), (8) and (9), (10) are plotted in Figure 2.
As shown in Figure 2, the events of updating rule based on the instantaneous couplings and pinning gains are more than the rule based on average ones. As a trade-off, the performance of the updating rule (9), (10) in terms of convergence rate of \( V(\ell) \) is higher than the updating rule (7), (8), as shown in Figure 1.

5 DISCUSSIONS AND CONCLUSIONS

This study employs the event-triggered configurations and pinning control to stabilize linearly coupled dynamical systems with fast switching in both coupling matrix and pinned node set, towards reducing communication and computation loads. The event-triggered rules were proved to perform well and can exclude Zeno behaviours. Simulations were given to verify these theoretical results for networks of mobile agents.

The present study has several limitations. First, we assume the state information of the complex network is available. This assumption has been widely made in the literature. This assumption is reasonable because obtaining the state information is a research problem that is orthogonal to the present study.

Recently, various state estimation methods for complex dynamical networks under different constraints have been developed. For example, delay-compensation-based state estimation for networks with communication delays, fading observations and dynamical bias disturbances was considered in [44], event-triggered state estimation was investigated for networks with bounded distributed delay [45] and networks with randomly switching topologies and multiple missing measurements [46]. It is an interesting future work to study the event-triggered state estimation for complex dynamical networks under stochastically fast switching.

Second, we apply the pinning control strategy with local feedback controllers. Additionally, we assume the feedback gains of the controllers and the coupling strength are constant. Future work will focus on developing effective control strategies for fast switching complex dynamical networks, including sliding mode control and adaptive pinning control strategy.

Third, time-delays and non-linearity are inevitable for complex dynamical networks. Therefore, it is a significant future work to explore event-triggered strategies for switched complex dynamical networks with constraints, such as quantization or transmission delays.

ACKNOWLEDGEMENTS

This work is jointly supported by the National Natural Science Foundation of China under Grant Nos. 61703271, 62072111 and 51879156, and the Key Program of the National Science Foundation of China, No. 91630314.

ORCID

Yejuan Han https://orcid.org/0000-0002-1395-4247
Wenlian Lu https://orcid.org/0000-0003-1880-6240
Tianping Chen https://orcid.org/0000-0001-6117-6673

REFERENCES

1. Strogatz, S.H.: Exploring complex networks. Nature 410, 268–276 (2001)
2. Boccaletti, S., et al.: Complex networks: structure and dynamics. Phys. Rep. 424, 175–308 (2006)
3. Liu, Y., Wang, Z., Liang, J.: Synchronization and state estimation for discrete-time complex networks with distributed delays. IEEE Trans. Syst. Man Cybern. Part B 38(5), 1314–1325 (2008)
4. Dong, H., et al.: Variance-constrained state estimation for complex networks with randomly varying topologies. IEEE Trans. Neural Netw. Learn. Syst. 29(7), 2757–2768 (2018)
5. Wu, C.W., Chua, L.O.: Synchronization in an array of linearly coupled dynamical systems. IEEE Trans. Circuits Syst. I 42, 430–447 (1995)
6. Cao, J., Li, P., Wang, W.: Global synchronization in arrays of delayed neural networks with constant and delayed coupling. Phys. Lett. A 353, 318–325 (2006)
7. Grigoriev, R.O., Cross, M.C., Schuster, H.G.: Pinning control of spatiotemporal chaos. Phys. Rev. Lett. 79(15), 2795–2798 (1997)
8. Porfiri, M., Bernardo, D.M.: Criteria for global pinning-controllability of complex networks. Automatica 44, 3100–3106 (2008)
9. Yu, W., Chen, G., Lu, J.: On pinning synchronization of complex dynamical networks. Automatica 45, 429–435 (2009)
10. Lu, W., Li, X., Rong, Z.: Global stabilization of complex networks with digraph topologies via a local pinning algorithm. Automatica 46(1), 116–121 (2010)
11. Chen, T., Liu, X., Lu, W.: Pinning complex networks by a single controller. IEEE Trans. Circuits Syst. I 54, 1317–1326 (2007)
12. Wang, J. et al.: Fuzzy-model-based \( H_\infty \) pinning synchronization for coupled neural networks subject to reaction-diffusion. IEEE Trans. Fuzzy Syst., early access, (2020), https://doi.org/10.1109/TFUZZ.2020.3036697.
13. Dimarogonas, D.V., Frazzoli, E., Johansson, K.H.: Distributed event-triggered control for multi-agent systems. IEEE Trans. Automat. Contr. 57, 1291–1297 (2012)
14. Hu, W., Liu, L., Feng, G.: Output consensus of heterogeneous linear multi-agent systems by distributed event-triggered/self-triggered strategy. IEEE Trans. Cybern. 47(9), 1914–1924 (2017)
15. Yu, Y., Wu, Z.: Distributed adaptive event-triggered fault-tolerant synchronization for multiagent systems, IEEE Trans. Ind. Electron. 68(2), 1537–1547 (2021)
16. Xie D., et al.: Event-triggered consensus control for second-order multi-agent systems. IET Control Theory Appl. 9(5), 667–680 (2015)
17. Zhao, M., et al.: Event-triggered communication for leader-following consensus of second-order multiagent systems. IEEE Trans. Cybern. 48(6), 1888–1897 (2018)
18. Lu, W., Han, Y., Chen, T: Pinning networks of coupled dynamical systems with Markovian switching couplings and event-triggered diffusions. J. Franklin I. 352, 3526–3545 (2015)
19. Dong, H., Zhou, J., Xiao, M.: Centralized/decentralized event-triggered pinning synchronization of stochastic coupled networks with noise and incomplete transitional rate. Neural Netw. 121, 10–20 (2020)
20. Liu, L., et al.: Dynamic event-triggered approach for cluster synchronization of complex dynamical networks with switching via pinning control. Neurocomputing 340, 32–41 (2019)
21. Liu, X., et al.: Quasi-synchronization of heterogeneous networks with a generalized Markovian topology and event-triggered communication. IEEE Trans. Cybern. 50(10), 4200–4213 (2020)
22. Liu, Y.A. et al.: Extended dissipative synchronization for semi-markov jump complex dynamic networks via memory sampled-data control scheme. J. Franklin I. 357, 10900–10920 (2020).
23. Porfiri, M., et al.: Random talk: random walk and synchronizability in a moving neighborhood network. Physica D 224, 102–113 (2006)
24. Belykh, I., Belykh, V.N., Hasler, M.: Blinking model and synchronization in coupled dynamical systems via event-triggered diffusions. IEEE Trans. Neural Netw. 21, 928–935 (2010)
25. Skufca, J.D., Bollt, E.M.: Communication and synchronization in, disconnected networks with dynamic topology: moving neighborhood networks. Math. Biosci. Eng. 1, 347–59 (2004)
26. Frasca, M., et al.: Spatial pinning control. Phys. Rev. Lett. 108(20), 204102 (2012)
27. Wang, Y., et al.: Network-based passive estimation for switched complex dynamical networks under stochastically fast switching. IEEE Trans. Circuits Syst. I 58(3), 576–583 (2011)
28. Fan, Y., et al.: Self-triggered consensus for multi-agent systems with zeno-free triggers. IEEE Trans. Autom. Control 60(10), 2779–2784 (2015)
29. Han, Y., Lu, W., Chen, T.: Event-triggered stabilization of coupled dynamical systems with fast Markovian switching. In: 14th International Conference on Control Automation Robotics & Vision, pp. 1–6. IEEE, Piscataway, NJ (2016)
30. Liu, Y., et al.: Event-triggered partial-nodes-based state estimation for coupled dynamical systems under stochastically fast switching. IEEE Trans. Cybern., early access, 2021, https://doi.org/10.1109/TCYB.2020.3043283
31. Han, Y., Lu W., Chen T.: Event-triggered partial-nodes-based state estimation for delayed complex networks with bounded distributed delays. IEEE Trans. Syst., Man, Cybern., Syst. 49(6), 1088–1098 (2019)
32. Han, Y., et al.: Event-triggered recursive state estimation for dynamical networks under randomly switching topologies and multiple missing measurements. Automatica 115, 108908 (2020)
33. Luo, C., et al.: Stochastic Differential Equations with Markovian Switching. Imperial College Press, London (2006)
34. Chamberlain, T., Miao, C.: A poison process model for activity forecasting. In: IEEE International Conference on Image Processing, pp. 3339–3343. IEEE, Piscataway, NJ (2016)
35. Christensen, S., Irle, A.: Convergence of switching diffusions. Stochastic Process. Appl. 126(11), 3217–3240 (2016)
36. Delellis, P., Bernardo, M. D., Russo, G.: On QUAD, Lipschitz, and contraction vector fields for consensus and synchronization of networks. IEEE Trans. Circuits Syst. I 58(3), 576–583 (2011)
37. Wali, L., Lu, W., Chen, T.: Consensus analysis of networks with time-varying topology and event-triggered diffusions. Neural Netw. 71, 196–205 (2015)
38. Fan, Y., et al.: Self-triggered consensus for multi-agent systems with zeno-free triggers. IEEE Trans. Autom. Control 60(10), 2779–2784 (2015)
39. Han, Y., Lu, W., Chen, T.: Dynamic event-triggered scheduling for pinning networks of coupled dynamical systems under stochastically fast switching. IET Control Theory Appl. 2021;15:1673–1685. https://doi.org/10.1049/cth2.12151

APPENDIX A

To prove Theorem 1, we need the following lemma. Let \( \hat{x}_i(t) = x_i(t) - s_i(t) \) and \( \hat{x}(t) = [\hat{x}_1(t)^T, ... , \hat{x}_m(t)^T]^T \).

**Lemma 1.** For system (4), the measurements \( e_i(t) \) and \( \hat{e}_i(t) \) defined in (5) and (6) satisfy that

\[
\|\hat{e}_i(t)\| \leq \eta \|\hat{x}(t)\|
\]

holds for any \( i \) and \( t \in [t_k^{(i)}, t_k^{(i)} + T_2] \) with \( \eta, T_2 \) defined in (11) and (12);

\[
\|e_i(t)\| \leq \eta \|\hat{x}(t)\|
\]

holds for any \( i \) and \( t \in [t_k^{(i)}, t_k^{(i)} + T_1] \) with \( \eta, T_1 \) given in (11) and (12).

**Proof.**

The dynamics of \( \hat{x}_i(t) \) satisfies:

\[
\dot{\hat{x}}_i(t) = f(\hat{x}_i(t)) + s_i(t), t = f(s_i(t), t) + e_i(t)
\]

\[
+ \epsilon \sum_{j=1}^m L_{ij} (f(\hat{x}_i(t), t)^T \Gamma (\hat{x}_j(t) - \hat{x}_i(t))
\]

\[
- \gamma D_i (e_i(t))^T \hat{x}_i(t) + \hat{e}_i(t).
\]

Let \( F(s(t), t) = [(f(s_1(t), t))^T, ... , (f(s_m(t), t))^T]^T, F_x(s(t), t) = [(f_x(s_1(t), t))^T, ... , (f_x(s_m(t), t))^T]^T \), \( \bar{e}(t) = [\hat{e}_1(t)^T, ... , \hat{e}_m(t)^T]^T \), and \( D(\bar{e}_i) = \text{diag}(D_1(\bar{e}_i), ... , D_m(\bar{e}_i)) \). Then, applying the Lipschitz assumption of \( f \) and the assumption \( \|\hat{e}_i(t)\| \leq \eta \|\hat{x}(t)\| \) on system (A1), we have that

\[
\|\hat{e}_i(t)\| \leq \|L_i + \epsilon \max_{j \neq i} \|L(j) - \gamma D(j)\| \otimes \Gamma + \sigma \|\hat{x}(t)\|
\]

\[
\triangleq \bar{e}_i(t) \|\hat{x}(t)\|.
\]

Noting that \( L(t/\epsilon) \) and \( D(t/\epsilon) \) are right continuous, the right-hand Dini derivative of \( \bar{e}_i(t) \) is

\[
D^+ \bar{e}_i(t) = \sum_{j} L_{ij} (t/\epsilon)^T \Gamma (\hat{x}_j(t) - \hat{x}_i(t)) + \gamma D_i(t/\epsilon)^T \hat{x}_i(t).
\]
By estimations (A2) and (A3), we have

\[
\| D^+ \hat{\varepsilon}_i(t) \| \leq \sqrt{2} \left( \sqrt{m - 1} \max_{i,j,k} |L_{ij}(k)| + \max_{i,j,k} |L_{ij}(k) - \kappa D_{ij}(k)| \right) \| \Gamma \| \cdot \| \hat{\xi}(t) \| \\
\leq \hat{\varepsilon}_2 \| \hat{\xi}(t) \|,
\]

where \( \hat{\varepsilon}_2 \) is given in (13). Then it follows that

\[
D^+ \frac{\| \hat{\varepsilon}_i(t) \|}{\| \hat{\xi}(t) \|} = \frac{\tilde{\varepsilon}_i(t)^T D^+ \tilde{\varepsilon}_i(t)}{\| \tilde{\varepsilon}_i(t) \| \cdot \| \hat{\xi}(t) \|} \leq \hat{\varepsilon}_2 \left( \frac{\| \tilde{\varepsilon}_i(t) \|}{\| \hat{\xi}(t) \|} \right)^2.
\]

The solution of differential equation \( \dot{\xi}(t) = \tilde{\varepsilon}_i[\xi(t) + \xi_0(t)] \) with initial \( \xi(0) = \xi_0 \) has the form \( \xi(t) = \xi_0 + \xi_0(t) \exp(\tilde{\varepsilon}_2 - \tilde{\varepsilon}_2 \tau). \)

Then for any \( i \) and any \( t \in [t_i^0, t_i^0 + T_2] \), \( \| \tilde{\varepsilon}_i(t) \| \leq \tilde{\varepsilon}_2 \exp(\tilde{\varepsilon}_2 \tau) \leq \tilde{\varepsilon}_2 \exp(\tilde{\varepsilon}_2 \tau) \leq \tilde{\varepsilon}_2 \exp(\tilde{\varepsilon}_2 \tau) \).

In other words, \( \| \tilde{\varepsilon}_i(t) \| \leq \tilde{\varepsilon}_2 \| \hat{\xi}(t) \| \) holds for any \( t \in [t_i^0, t_i^0 + T_2] \).

Secondly, notice that

\[
\tilde{\varepsilon}_i(t) = \sum_j L_{ij} \hat{\xi}_j(t) - \hat{\xi}_j(t) + \kappa D_{ij} \hat{\xi}_j(t).
\]

From (A2), (A4) and \( \| \tilde{\varepsilon}_i(t) \| \leq \tilde{\varepsilon}_2 \| \hat{\xi}(t) \| \), we have that for any \( t \in [t_i^0, t_i^0 + T_2] \),

\[
\| \tilde{\varepsilon}_i(t) \| \leq \tilde{\varepsilon}_2 \| \hat{\xi}(t) \|,
\]

where \( \tilde{\varepsilon}_2 \) is given in (14). Then it follows that for any \( t \in [t_i^0, t_i^0 + T_2] \),

\[
\frac{d}{dt} \frac{\| \tilde{\varepsilon}_i(t) \|}{\| \hat{\xi}(t) \|} \leq \frac{\tilde{\varepsilon}_i(t)^T \tilde{\varepsilon}_i(t)}{\| \tilde{\varepsilon}_i(t) \| \cdot \| \hat{\xi}(t) \|} \leq \tilde{\varepsilon}_2 \left( \frac{\| \tilde{\varepsilon}_i(t) \|}{\| \hat{\xi}(t) \|} \right)^2.
\]

A.1 Proof of Theorem 1

**Proof.** We only prove that system (4) is stable in mean square sense under the triggering rule (7) and (8). The same procedure can be adapted to obtain the stability of system under the triggering rule (9) and (10).

Denote \( V(t) = \frac{1}{2} \hat{\xi}(t)^T (P \otimes G) \hat{\xi}(t) \). By (A1) and Dynkin’s formula [47], we have that for any \( \Delta > 0 \),

\[
E \left[ V(t + \Delta) | \hat{\xi}(t) \right] - V(t)
\]

\[
= E \left[ \int_t^{t+\Delta} \hat{\xi}(\tau)^T (P \otimes G) \hat{\xi}(\tau) d\tau \right] | \hat{\xi}(t)
\]

\[
= E \left[ V_1(t, \Delta) | \hat{\xi}(t) \right] + E \left[ V_2(t, \Delta) | \hat{\xi}(t) \right] + E \left[ V_3(t, \Delta) | \hat{\xi}(t) \right]
\]

with

\[
V_1(t, \Delta) = \int_t^{t+\Delta} \hat{\xi}(\tau)^T (P \otimes G) \{ F(\hat{\xi} \tau) - F(\xi, \tau) \} d\tau,
\]

\[
V_2(t, \Delta) = \int_t^{t+\Delta} \hat{\xi}(\tau)^T \left( \{ P \otimes G \} \alpha L_m + \alpha L(\hat{\xi} \tau) \right) d\tau,
\]

\[
V_3(t, \Delta) = \int_t^{t+\Delta} \hat{\xi}(\tau)^T (P \otimes G) \kappa D(\hat{\xi} \tau) d\tau.
\]

In the following, we estimate the terms \( E[V_i(t, \Delta) | \hat{\xi}(t)] \), \( i = 1, 2, 3 \), respectively.

First, from the assumption \( f \in \text{QUAD}(G, \alpha \Gamma, \beta) \), we have

\[
E \left[ V_1(t, \Delta) | \hat{\xi}(t) \right] \leq -2 \beta \cdot E \left[ \int_t^{t+\Delta} V(\tau) d\tau \right] | \hat{\xi}(t) \right] \leq 0.
\]

Second, denote

\[
\zeta(\varepsilon, \Delta) = \frac{1}{\Delta} \max \left\{ \| E \left[ \int_t^{t+\Delta} (L(\hat{\xi} \tau) - L) d\hat{\xi}(\tau) \right] \|_\infty \right\}.
\]

Similarly to the analysis of \( \| \tilde{\varepsilon}_i(t) \| \leq \tilde{\varepsilon}_2 \| \hat{\xi}(t) \| \) to increase to 0 from 0 to \( \eta \), we have that for any \( t \in [t_i^0, t_i^0 + T_2] \),

\[
\| \tilde{\varepsilon}_i(t) \| \leq \tilde{\varepsilon}_2 \| \hat{\xi}(t) \|.
\]

By the limits (2), we have \( \lim_{\varepsilon \to 0} \zeta(\varepsilon, \Delta) = 0 \). Write \( V_2(t, \Delta) \) as

\[
V_2(t, \Delta) = \int_t^{t+\Delta} \hat{\xi}(\tau)^T \left( P \otimes G \right) \{ \alpha L_m + \alpha L - \alpha D(\hat{\xi} \tau) \} d\tau.
\]
where \( A(t) \) and \( K_1 \) are given in (A9) and (17), respectively. By (A8)–(A12) and \( \Phi_1(t, \Delta) \leq 0 \), we have

\[
E[V_2(t, \Delta)|\hat{x}(t)] 
\leq M_1 \Delta \cdot \hat{\xi}(\epsilon, \Delta) V'(t)
\]

\[
+ K_1 \cdot E \left[ \Delta \cdot \int_t^{t+\Delta} V(\tau)d\tau | \hat{x}(t) \right].
\]

(A13)

Third, noting the event-triggered rule is defined based on error state \( e(t) \), we have

\[
V_3(t, \Delta) = e \int_t^{t+\Delta} \hat{x}(\tau)^T (P \otimes G)e(\tau)d\tau
\]

\[
+ e \int_t^{t+\Delta} \hat{x}(\tau)^T [(P \otimes G)e(\tau) - e(\tau)]d\tau.
\]

(A14)

Note that for \( 0 < \beta' < \beta \),

\[
\hat{c}(t)^T (P \otimes G)e(t)
\]

\[
\leq \frac{e}{2\nu} \hat{x}(t)^T (P^2 \otimes G^2)\hat{x}(t) + \frac{c_0}{2} e(t)^T e(t)
\]

\[
\leq 2\beta' V'(t) + \left[ \frac{\alpha^2}{2\nu} - \frac{\beta^2}{\beta'} \right] \hat{x}(t)^T \hat{x}(t) + \frac{c_0}{2} e(t)^T e(t).
\]

(A15)

holds for any \( \nu > 0 \). If \( \tau'_k \) in rule (7) satisfies \( \tau'_k \geq T_1, \| e(t) \| \leq \frac{\beta^2}{\dot{\lambda}^2} \| \hat{x}(t) \| \) holds for \( t \in [t'_{k-1}, t'_k) \); Otherwise, \( \| e(t) \| \leq \eta \| \hat{x}(t) \| \) holds according to Lemma 1. Then, by taking

\[
\eta = \frac{\beta^2}{\sqrt{\max} \nu},
\]

we get that under triggering rule (7) and (8),

\[
e(t)^T e(t) \leq \frac{\beta'^2 \lambda^2}{c^2 \lambda^2} \hat{x}(t)^T \hat{x}(t).
\]

(A16)

Note that

\[
\max_{\nu} \frac{1}{\nu} \left[ \frac{\beta'^2 \lambda^2}{c^2 \lambda^2} - \frac{\alpha^2}{2\nu} \right] = \frac{\beta'^2 \lambda^2}{c^2 \lambda^2}
\]

(A17)

and the maximum is reached when \( \nu = \frac{\alpha^2}{\beta'^2 \lambda^2} \). Hence, by letting \( \nu = \frac{\alpha^2}{\beta'^2 \lambda^2} \), inequality (A16) implies that

\[
e(t)^T e(t) \leq \frac{2}{\nu} \left[ \frac{\beta'^2 \lambda^2}{c^2 \lambda^2} - \frac{\alpha^2}{2\nu} \right] \hat{x}(t)^T \hat{x}(t).
\]

(A18)
Then substituting the above inequality into (A15), we have
\[
\hat{\varepsilon}(t)^T (P \otimes G) \varepsilon(t) \leq 2\beta' V(t).
\]
This means
\[
E[\Psi_1(t, \Delta) \hat{\xi}(t)] \leq 2\beta' \cdot E \left[ \int_I^{I+\Delta} V(\tau) d\tau \hat{\xi}(t) \right]. \tag{A19}
\]
That is to say, the assumption of \( \varepsilon(t) \) guarantees (A19). Similarly, we can derive
\[
E[\Psi_2(t, \Delta) \hat{\xi}(t)] \leq M_2 \cdot \Delta \cdot \zeta(\varepsilon, \Delta) V(t) + K_2 \cdot E \left[ \Delta \cdot \int_I^{I+\Delta} V(\tau) d\tau \hat{\xi}(t) \right] + \left[ M_3 \cdot \Delta \cdot \zeta(\varepsilon, \Delta) + K_3 \Delta^2 \right] E \left[ \max_{j} \max_{t_{i,j}(t)} V'(t_{i,j}(t)) \right], \tag{A20}
\]
where \( K_2, K_3 \) are given in (18), (19), and
\[
M_2 = M_1 + \frac{c(m + \kappa) \overline{A}(G)}{\overline{\Delta}(G)}, \quad M_3 = \frac{c(m + \kappa) \| P \|_\infty}{\overline{\Delta}}.
\]
By (A14)–(A20), we have \( V(\varepsilon, \Delta) \) satisfies:
\[
E[V(\varepsilon, \Delta) \hat{\xi}(t)] \leq (2\beta' + K_2 \Delta) E \left[ \int_I^{I+\Delta} V(\tau) d\tau \hat{\xi}(t) \right] + M_2 \cdot \Delta \cdot \zeta(\varepsilon, \Delta) V(t) + \left[ M_3 \cdot \Delta \cdot \zeta(\varepsilon, \Delta) + K_3 \Delta^2 \right] E \left[ \max_{j} \max_{t_{i,j}(t)} V'(t_{i,j}(t)) \right]. \tag{A21}
\]
To sum up, by (A5), (A6), (A13), and (A21), we have
\[
E[V(\varepsilon, \Delta) \hat{\xi}(t)] - V(t) \leq [-2(\beta - \beta')] \Delta \cdot \hat{M}_1 \Delta \cdot \zeta(\varepsilon, \Delta) + \hat{M}_1 \Delta \cdot \zeta(\varepsilon, \Delta) + o(\Delta)] V(t), \tag{A22}
\]
First, by \( f \in QUAD(G, \alpha \Gamma, \beta) \), the Cauchy–Schwarz inequality and Lemma 1, the derivative of \( V(t) \) along the system (4) satisfies:
\[
|V'(t)| \leq |\hat{\xi}(t)^T (P \otimes G) \{ F(\xi, t) - F(\xi, t) + [cL(\sigma_j) - \alpha D(\sigma_j)] \otimes \Gamma \} \hat{\xi}(t) + \hat{\varepsilon}(t)| \leq \rho_1 V(t),
\]
where \( \rho_1 \) is defined in (20). Thus, for any \( \tau > t \), we have
\[
V(t)e^{-\rho_1(\tau-t)} \leq V(t) \leq V(t)e^{-\rho_1(\tau-t)}, \tag{A23}
\]
and
\[
\Delta e^{-\rho_1 \Delta} V(t) \leq E \left[ \int_I^{I+\Delta} V(\tau) d\tau \right] \leq \Delta \cdot e^{-\rho_1 \Delta} V(t). \tag{A24}
\]
Let \( V(t_{i,j}(t)) = \max_{t \in [t_{i,j}(t)]} V(t_{i,j}(t)) \). The assumption that every inter-event interval is less than some fixed constant \( T \) implies \( t_{i,j}(t) \in [t - T, t + T] \). By inequality (A23), we have that
\[
\max_{j} \max_{t_{i,j}(t)} V'(t_{i,j}(t)) \leq \max_{t \in [t + \Delta]} \{ e^{\rho_1 T}, \rho_1 \Delta \} V(t). \tag{A25}
\]
Denote \( \epsilon(\Delta) \) the infinite small quality with respect to \( \Delta \), that is, \( \lim_{\Delta \to 0} \frac{d\Delta}{\Delta} = 0 \). Then substituting (A24) and (A25) into (A22) gives
\[
E[V(\varepsilon + \Delta) \hat{\xi}(t)] - V(t) \leq [-2(\beta - \beta')] \Delta \cdot \hat{M}_1 \Delta \cdot \zeta(\varepsilon, \Delta) + o(\Delta)] V(t), \tag{A26}
\]
where \( \hat{M}_1 = M_1 + M_2 + M_3 \max \{ \exp(\rho_1 T), \exp(\rho_1 \Delta) \} \). By (2), we have \( \lim_{\Delta \to 0} \zeta(\varepsilon, \Delta) = 0 \), which implies that sufficiently fast switching in any \( \Delta \)-length time interval leads to a sufficiently small \( \zeta(\varepsilon, \Delta) \). For any \( \Delta \) satisfying \( (\beta - \beta') \Delta \cdot e^{-\rho_1 \Delta} < 1 \), and \( (\beta - \beta') \Delta \cdot e^{-\rho_1 \Delta} - o(\Delta) > 0 \), there exists \( \epsilon^* \), such that if \( \epsilon < \epsilon^* \),
\[
\zeta(\varepsilon, \Delta) \leq \left( (\beta - \beta') \Delta \cdot e^{-\rho_1 \Delta} - o(\Delta) \right) \Delta \tag{A27}
\]
Substituting (A27) to (A26), we could obtain
\[
E[V(\varepsilon + \Delta) \hat{\xi}(t)] - V(t) \leq - (\beta - \beta') \Delta \cdot e^{-\rho_1 \Delta} V(t). \tag{A28}
\]
Denote \( \gamma = (\beta - \beta') \Delta \cdot e^{-\rho_1 \Delta} \), hence, \( 0 < \gamma < 1 \). Then taking expectation on both sides of (A28), we have
\[
E[V(t + \Delta)] \leq (1 - \gamma) E[V(t)]. \tag{A29}
\]
Iterating (A29) gives that for any integer $p$,

$$
\mathbb{E}[V(t + p\Delta)] \leq (1 - \gamma)^p \mathbb{E}[V(t)].
$$

Noticing that $\hat{x}(t)^T \hat{x}(t) \leq V(t) \leq \Delta \hat{x}(t)^T \hat{x}(t)$, we obtain

$$
\mathbb{E}[\hat{x}(t + p\Delta)^T \hat{x}(t + p\Delta)] \leq \frac{\Delta}{\lambda} (1 - \gamma)^p \mathbb{E}[\hat{x}(t)^T \hat{x}(t)]
$$

which implies $\{\hat{x}(t + p\Delta)\}_{p=1}^\infty$ converges to zero in the mean square sense. From the arbitrariness of $t$, we have $\hat{x}(t)$ converges to zero in the mean square sense. This completes the proof of the first part.

From the triggering rule (8), we get that the inter-event interval for each node is larger than $\min\{T_1, T\}$, meaning that Zeno behaviour is ruled out for each agent.

### A.2 Proof of Corollary 1

**Proof.** Let $\{\tau_i\}_{i \in \mathbb{N}}$ denote the switching time sequence of $\sigma_{i/k}$ and $\Delta_k = \tau_{k+1} - \tau_k$. Then from Lemma 2 in [30], we have that

$$
\|\mathbb{E}\left[\int_{\tau_k}^{\tau_{k+1}} (L(\sigma_{i/k}) - L) \, dt \right] \|_\infty \leq M_0 \delta^k,
$$

$$
\|\mathbb{E}\left[\int_{\tau_k}^{\tau_{k+1}} (D(\sigma_{i/k}) - D) \, dt \right] \|_\infty \leq M_0 \delta^k,
$$

where $M_0 = 2NC_0 \max\{\max \|L(\|)\|_\infty, \max \|D(\|\|)\|_\infty\} \mu_e / \mu_0^2$.

Follow the proof of Theorem 1, we could derive that

$$
\mathbb{E}[V(\tau_{k+1})|\tau(\tau_k)] - V(\tau_k)
\leq -2(\beta - \beta') \mathbb{E}\left[\int_{\tau_k}^{\tau_{k+1}} V(\tau)\, d\tau \right] \hat{x}(\tau_k) + (K_1 + K_2) \mathbb{E}\left[\Delta_k \int_{\tau_k}^{\tau_{k+1}} V(\tau)\, d\tau \right] \hat{x}(\tau_k)
+ (M_1 + M_2)M_0 \delta^k V(\tau_k)
+ \mathbb{E}\left[(M_3M_0 \delta^k + K_3 \Delta_k^2) \max_{j \in [1, \tau_{k+1}]} \max_{j \in [\tau_{k+1}, \tau_{k+1}]} V(\tau_j) \right] \hat{x}(\tau_k)
\leq -2(\beta - \beta') \mathbb{E}[\Delta_k \cdot e^{-\rho_1 \Delta_k}] V(\tau_k) + (K_1 + K_2) \mathbb{E}[\Delta_k^2 \cdot e^{-\rho_1 \Delta_k}] V(\tau_k)
+ K_3 \mathbb{E}[\Delta_k^2 \cdot e^{-\rho_1 \Delta_k}] \max\{\rho_1^T, \rho_1^\Delta_k\} \max_{j \in [1, \tau_{k+1}]} \max_{j \in [\tau_{k+1}, \tau_{k+1}]} V(\tau_j)
+ \mathbb{E}\left[(M_1 + M_2 + M_3) \max\{\rho_1^T, \rho_1^\Delta_k\} \right] M_0 \delta^k V(\tau_k).
$$

Observe that the switching rate of process $\sigma_{i/k}$ is $\lambda(t) / \epsilon$. Then according to the result in [48], the probability density function of $\Delta_k$ is $p(t) = \lambda(t) / \epsilon \exp(-\int_0^t \lambda(\tau) / \epsilon \, d\tau)$. From the boundedness of $\lambda(t)$, that is, $\mu_0 \leq \lambda(t) \leq \mu_1$, we have

$$
\frac{\mu_e \mu_1}{\mu_1} \leq \mathbb{E}[\Delta_k] \leq \frac{\mu_e \mu_0}{\mu_1 + \rho_1 \epsilon^2},
$$

$$
\mathbb{E}[\Delta_k^2 \cdot e^{-\rho_1 \Delta_k}] \leq \frac{2\mu_e^2}{(\mu_0 - \rho_1 \epsilon^3)},
$$

$$
\mathbb{E}[\Delta_k^2 \cdot e^{-\rho_1 \Delta_k}] \leq \frac{2\mu_e^2}{(\mu_0 - 2\rho_1 \epsilon^3)}.
$$

From $\delta < 1$, we have that for any constant $0 < \omega < 1$, there exists $N(\omega)$ such that for any $k > N(\omega)$,

$$
\mathbb{E}\left[(M_1 + M_2 + M_3) \max\{\rho_1^T, \rho_1^\Delta_k\} \right] M_0 \delta^k \leq \omega.
$$

If $\epsilon$ satisfies assumption (15), we have

$$
\rho_1 \epsilon \leq \frac{1}{3} \mu_0, \quad \frac{\beta - \beta'}{4} \mu_1 \epsilon \mu_0^2 < 1
$$

and

$$
(K_1 + K_2) \frac{2\mu_e^2}{(\mu_0 - \rho_1 \epsilon^3)} + K_3 \max\left\{\frac{2\mu_e^2}{(\mu_0 - \rho_1 \epsilon^3)}, \frac{2\mu_e^2}{(\mu_0 - 2\rho_1 \epsilon^3)}\right\}
\leq \frac{7(\beta - \beta')}{4} \mu_1 \epsilon \mu_0^2 / (\mu_1 + \rho_1 \epsilon^2).
$$

Therefore, applying (A31)–(A35) to (A30), we have

$$
\mathbb{E}[V(\tau_{k+1})|\tau(\tau_k)] - V(\tau_k)
\leq -\left[\frac{\beta - \beta'}{4} \mu_1 \epsilon \mu_0^2 / (\mu_1 + \rho_1 \epsilon^2) - \omega\right] V(\tau_k).
$$

Taking expectations on both sides of the above inequality, we get that

$$
\mathbb{E}[V(\tau_{k+1})] \leq 1 - \left[\frac{\beta - \beta'}{4} \mu_1 \epsilon \mu_0^2 / (\mu_1 + \rho_1 \epsilon^2) + \omega\right] \mathbb{E}[V(\tau_k)].
$$

From the arbitrariness of $\omega$, we can take $0 < \omega < (\beta - \beta') \mu_0 \epsilon / [4(\mu_1 + \rho_1 \epsilon^2)]$, then we have that $\lim_{k \to \infty} \mathbb{E}[V(\tau_k)] = 0$. On the other hand, from (A23), we get that for any $t \in [\tau_k, \tau_{k+1})$, $\mathbb{E}[V(t)] \leq \mathbb{E}[V(\tau_k)] e^{-\rho_1 \epsilon \mu_1 / \mu_0^2}.$
which means \( \lim_{t \to \infty} \mathbb{E}[V(t)] = 0 \). This completes the proof of part (1).

Next, we consider the stability of system (4) under event-triggered rule (9),(10). Following the definition and notations given in the proof of Theorem 1, we have that \( V_1(t, \Delta) \) and \( V_2(t, \Delta) \) satisfy estimation (A6) and (A13). As for the term \( V_3(t, \Delta) \), according to Lemma 1, the event-triggered rule (9),(10) implies that \( c \hat{x}(t)^T (P \otimes G) \hat{x}(t) \leq 2\beta' V(t) \) and

\[
\mathbb{E}[V_3(t, \Delta) | \hat{x}(t)] \leq 2\beta' \int_t^{t+\Delta} V(\tau) d\tau.
\]

Taking \( t = \tau_k, \Delta = \Delta_k \), we have that

\[
\mathbb{E}[V'(\tau_{k+1}) | x(\tau_k)] - V'(\tau_k) \leq -2(\beta - \beta') \mathbb{E} \left[ \int_{\tau_k}^{\tau_{k+1}} V'(\tau) d\tau \right] \hat{x}(\tau_k) + K_1 \Delta_k \int_{\tau_k}^{\tau_{k+1}} V'(\tau) d\tau \hat{x}(\tau_k) + M_1 M_0 \delta^k V(\tau_k) \leq -2(\beta - \beta') \frac{\mu_0 \varepsilon}{(\mu_1 + \rho_1 \varepsilon)^2} + K_1 \frac{2\mu_1 \varepsilon^2}{(\mu_0 - \rho_1 \varepsilon)^3} + M_1 M_0 \delta^k \]

Notice that if \( \varepsilon \) satisfies assumption (16), then (A36) holds for system with event-triggered rule (9),(10), which completes the proof.