On the Interpolation of Analytic Maps

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The interpolation theorem for nonlinear maps was essentially used in our paper [11], as well, as in the papers [2] and [3]. The papers dealt with nonlinear analytic maps associated with inverse Sturm-Liouville problems. The required estimates for these maps firstly were obtained for integer indices, and then for all intermediate values by nonlinear interpolation. Working on the paper [1] we were not aware on nonlinear interpolation theorems, but assuming analyticity as additional assumption we succeeded to get the following result.

**Theorem 1.** Let \((E_0, E_1)\) and \((H_0, H_1)\) be a pair of Banach spaces, such that \(E_1\) is densely embedded into \(E_0\) and \(\|x\|_{E_0} \leq \|x\|_{E_1}\), while \(H_1\) is densely and continuously embedded into \(H_0\). Denote by \(E_\theta = [E_0, E_1]_\theta\) and \(H_\theta = [H_0, H_1]_\theta\) the spaces obtained by the method of complex interpolation for \(\theta \in [0, 1]\). Denote also by \(B_\theta(0, R)\) the ball of radius \(R\) centered at zero in the space \(E_\theta\). Let \(\Phi\) be an analytic map acting from the ball \(B_\theta(0, R)\) into the space \(H_\theta\), such that

\[
\|\Phi(x)\|_{H_\theta} \leq C_0(R)\|x\|_{E_0}, \quad x \in B_\theta(0, R).
\]

Let \(\Phi\) maps also the ball \(B_1(0, R)\) into \(H_1\), and

\[
\|\Phi(x)\|_{H_1} \leq C_1(R)\|x\|_{E_1}, \quad x \in B_1(0, R).
\]

Then for all \(\theta \in [0, 1]\) the map \(\Phi\) maps the ball \(B_\theta(0, r)\) in the space \(E_\theta\) of radius \(r \leq (0, R)\) into \(H_\theta\), and

\[
\|\Phi(x)\|_{H_\theta} \leq C_1^{1-\theta}C_0^{\theta} \frac{R}{R-r}\|x\|_{E_\theta}, \quad x \in B_\theta(0, r).
\]

This theorem is sufficient to complete the proof of the main Theorem 6.1 of the paper [1] (see §6 of this paper). However, soon we found a theorem in the paper of Tartar [2], which we assumed to be more general rather then the above result (up to constants having no essential role in the estimates). We used Tartar theorem in the following form.

**Theorem (Tartar [5]).** Let \((E_0, E_1)\) and \((H_0, H_1)\) be two pairs of Banach spaces with dense and continuous embeddings \(E_1 \hookrightarrow E_0, H_1 \hookrightarrow H_0\). Let \(\Phi\) be a nonlinear map from \(E_0\) into \(E_1\), which maps \(E_1\) into \(H_1\) and satisfies the following conditions: There are positive increasing functions \(C_0(R)\) and \(C_1(R)\), such that

\[
\|\Phi(\sigma) - \Phi(\tilde{\sigma})\|_{H_\theta} \leq C_0(R)\|\sigma - \tilde{\sigma}\|_{E_0}, \quad \text{if } \max\{\|\sigma\|_{E_0}, \|\tilde{\sigma}\|_{E_0}\} \leq R,
\]

\[
\|\Phi(\sigma)\|_{H_1} \leq C_1(R)\|\sigma\|_{E_1}.
\]

Then \(\Phi\) maps the intermediate spaces \([E_0, E_1]_\theta\) into \([H_0, H_1]_\theta\) for all \(0 \leq \theta \leq 1\), and

\[
\|\Phi(\sigma)\|_{H_\theta} \leq C_\theta(R)\|\sigma\|_{E_\theta},
\]

where \(C_\theta(R)\) is an increasing function on \(R\).

We understood this theorem not correctly. We assumed that the assertion of Theorem is valid provided that estimate (2) holds for \(\max\{\|\sigma\|_{E_0}, \|\tilde{\sigma}\|_{E_0}\} \leq R\), while estimate (3) holds for \(\|\sigma\|_{E_1} \leq R\). However, it is assumed in Tartar Theorem that estimate (3) also holds for \(\|\sigma\|_{E_0} \leq R\). The last assumption is too strong and in such a form Tartar Theorem becomes
useless for our purposes. Prof. T. Kappeler paid our attention to this mistake and we are are very grateful to him for this note. At the same time we informed him that we had an independent approach to nonlinear interpolation and possessed the proof of the above Theorem 1 where the required estimate was obtained (in contrast to Tartar Theorem) under additional assumption of analyticity.

The goal of this paper is to present the proof of Theorem 1. We note that the definition of analytic maps in Banach spaces and basic facts related to this topic can be found in the book [4]. The basic facts on the interpolation in Banach spaces can be found in books [6] and [7], for example.

First we prove the following lemma.

**Lemma.** Let \((E_0, E_1)\) and \((H_0, H_1)\) be two pairs of Banach spaces with dense and continuous embeddings \(E_1 \hookrightarrow E_0\) and \(H_1 \hookrightarrow H_0\). Let \(E_\theta = [E_0, E_1]_\theta\) and \(H_\theta = [H_0, H_1]_\theta\) be intermediate spaces obtained by the method of complex interpolation for \(\theta \in [0, 1]\). Assume that \(\Phi\) is a homogeneous map of degree \(k\), i.e. \(\Phi(\lambda x) = \lambda^k \Phi(x)\), moreover, \(\Phi\) maps \(E_\theta\) into \(H_\theta\) and \(E_1\) into \(H_1\). Assume also that

\[
\|\Phi(x)\|_{H_\theta} \leq M_0 \|x\|_{E_\theta}^k
\]

and

\[
\|\Phi(x)\|_{H_1} \leq M_1 \|x\|_{E_1}^k.
\]

Then \(\Phi\) maps the space \(E_\theta\) into \(H_\theta\) and

\[
\|\Phi(x)\|_{H_\theta} \leq M_0^{1-\theta} M_1^\theta \|x\|_{E_\theta}^k.
\]

**Proof.** First, let us remind some facts from the theory of complex interpolation in Banach spaces. Let \(f\) be a bounded analytic function in the strip \(\text{Re} \ z \in (0, 1)\) with values in the space \(E_0\) and continuous in the closed strip \(\text{Re} \ z \in [0, 1]\). Moreover, assume that the values \(f(it)\) for \(t \in \mathbb{R}\) lie in \(E_0\), but the values \(f(1+it)\) for \(t \in \mathbb{R}\) lie in \(E_1\). Let also the functions \(f(it)\) and \(f(1+it)\) be continuous in the spaces \(E_0\) and \(E_1\), respectively, and tend to zero in the corresponding norms as \(|t| \to \infty\). Denote by \(F = F(E_0, E_1)\) the space of such functions endowed with the norm

\[
\|f\|_F := \max\{\max_{t \in \mathbb{R}} |f(it)|_{E_0}, \max_{t \in \mathbb{R}} |f(1+it)|_{E_1}\}.
\]

Then the space \(F\) is complete and the interpolation space \(E_\theta\) is defined by the set \(E_\theta = \{f(\theta) : f \in F\}\) endowed with the norm

\[
\|x\|_{E_\theta} := \inf\{\|f\|_F : f \in F, f(\theta) = x\}.
\]

Now, let us start proving lemma. For any function \(f \in F\) define the function \(g_f = g(z) := M_0^{-1} M_1^{-\theta} \Phi(f(z))\) and notice that \(g\) is analytic in the strip \(\text{Re} \ z \in (0, 1)\) and continuous in the closed strip \(\text{Re} \ z \in [0, 1]\) as the function with values in \(H_0\). For \(z = it\) we have \(\|g(it)\|_{H_\theta} \leq \|f(it)\|_{E_\theta}\), therefore \(\|g(it)\|_{H_\theta} \to 0\) as \(|t| \to \infty\). For \(z = 1+it\) it takes the values in the space \(H_1\), and due to the inequality \(\|g(1+it)\|_{H_1} \leq \|f(1+it)\|_{E_1}\), we have \(\|g(1+it)\|_{H_1} \to 0\) as \(|t| \to \infty\). This implies that \(g \in F(H_0, H_1)\) and \(\|g\|_{F(H_0, H_1)} \leq \|f\|_{F(E_0, E_1)}^k\). For given \(x \in E_\theta\) and small \(\varepsilon > 0\) take a function \(f \in F(E_0, E_1)\), such that

\[
\|x\|_{E_\theta} \leq \|f\|_{F(E_0, E_1)} \leq \|x\|_{E_\theta} + \varepsilon.
\]

Then \(\Phi(x) = \Phi(f(\theta)) = M_0^{1-\theta} M_1^\theta g_f(\theta)\), hence

\[
\|\Phi(x)\|_{H_\theta} \leq M_0^{1-\theta} M_1^\theta \|g_f\|_{F(H_0, H_1)} \leq M_0^{1-\theta} M_1^\theta \|f\|_{F(E_0, E_1)}^k \leq M_0^{1-\theta} M_1^\theta \left(\|x\|_{E_\theta} + \varepsilon\right)^k.
\]
This gives the desired result, since the number $\varepsilon > 0$ can be chosen arbitrary. Lemma is proved.

Now, let us pass to the proof of Theorem 1. Represent the map $\Phi$ in the ball $B_0(0, R)$ as the series

$$\Phi(h) = \sum_{n=0}^{\infty} P_n(h)$$

(see [1], for example). Here $P_n$ is a homogeneous map of the degree $n$, acting from the space $E_0$ into $H_0$ and defined by the Cauchy formula

$$P_0 = 0, \quad P_n(h) = \frac{1}{2\pi i} \int_{|\xi| = \rho} \frac{\Phi(\xi h)}{\xi^{n+1}} d\xi,$$

where the integral does not depend on $\rho \in (0, R/\|h\|)$. Tending $\rho \to R/\|h\|$ we obtain the estimate $\|P_n(h)\|_{H_0} \leq C_0(R) R^{1-n} \|h\|_{E_0}^n$. The same expansion is valid for the map $\Phi$ in the ball $B_1(0, R)$. By assumption of Theorem this ball is embedded into the ball $B_0(0, R)$, therefore the maps $P_n$ in the space $E_1$ are the restrictions onto $E_1$ of the maps $P_n$ in the space $E_0$. Hence, we can apply the same arguments and obtain the estimate $\|P_n(h)\|_{H_1} \leq C_1(R) R^{1-n} \|h\|_{E_1}^n$. By virtue of Lemma 2 $P_n$ maps elements $h \in E_\theta$ into the space $H_\theta$, moreover,

$$\|P_n(h)\|_{H_\theta} \leq C_0(R)^{1-\theta} C_1(R)^\theta R^{1-n} \|h\|_{E_\theta}^n.$$

Then the map $\Phi$ defined in the ball $B_\theta(0, r), r < R$, by the series

$$\Phi(h) = \sum_{n=0}^{\infty} P_n(h)$$

maps this ball into the space $H_\theta$, and

$$\|\Phi(h)\|_{H_\theta} \leq C_0(R)^{1-\theta} C_1(R)^\theta \sum_{n=1}^{\infty} R^{1-n} \|h\|_{E_\theta}^n \leq C_0(R)^{1-\theta} C_1(R)^\theta \|h\|_{E_\theta} \frac{R}{R - \|h\|_{E_\theta}} \leq C_0^{1-\theta} C_1^\theta \frac{R}{R - r} \|h\|_{E_\theta}.$$

This ends the proof of Theorem 1.

**Remark.** The estimate $\|x\|_{E_0} \leq \|x\|_{E_1}$ in Theorem 1 can be replaced by the estimate $\|x\|_{E_0} \leq C \|x\|_{E_1}$, i.e. by the assumption that the embedding $E_1 \hookrightarrow E_0$ is continuous. Assuming the last assumption one can achieve the first estimate by passing in one of the spaces to equivalent norm. However, the constants in [1] have to be changed if we return to the original norm.

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