On a method for constructing the Lax pairs for integrable models via a quadratic ansatz

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Abstract

A method for constructing the Lax pairs for nonlinear integrable models is suggested. First we look for a nonlinear invariant manifold of the linearization of the given equation. Examples show that such an invariant manifold does exist and can effectively be found. Actually, it is defined by a quadratic form. As a result we get a nonlinear Lax pair consisting of the linearized equation and the invariant manifold. Our second step consists of finding an appropriate change of the variables to linearize the found nonlinear Lax pair. The desired change of the variables is again defined by a quadratic form. The method is illustrated by the well-known KdV equation, the modified Volterra chain and a less studied coupled lattice connected to the affine Lie algebra $A_1^{(1)}$. New Lax pairs are found. The formal asymptotic expansions for their eigenfunctions are constructed around the singular values of the spectral parameter. By applying the method of the formal diagonalization to these Lax pairs, the infinite series of the local conservation laws are obtained for the corresponding nonlinear models.

Keywords: Lax pair, higher symmetry, invariant manifold, conservation law, integrability, modified Volterra chain, KdV type equations

1. Introduction

The Linearization of nonlinear equations at the vicinity of their arbitrary solutions is often used in integrability theory. For instance, any evolutionary-type symmetry

\[ u_\tau = g(x, t, u, u_1, \ldots, u_l), \quad u_j = D_j^x u \]  

(1.1)
of the partial differential equation
\[ u_t = f(u, u_1, \ldots, u_k) \] (1.2)
is effectively found as a solution \( U = g(x, t, u, u_1, \ldots, u_l) \) of the linearized equation
\[ U_t = \left( \frac{\partial f}{\partial u} + \frac{\partial f}{\partial u_1} D_x + \frac{\partial f}{\partial u_2} D_x^2 + \cdots + \frac{\partial f}{\partial u_k} D_x^k \right) U. \] (1.3)

Here, \( D_x \) is the operator of the total differentiation with respect to \( x \). The present article is devoted to further development of the approach to the problem of constructing the Lax pairs for integrable equations discussed in our recent article \([1]\) with Poptsova. In \([1]\), we suggested a method for constructing the recursion operator via linear invariant manifolds of their linearized equations. This provides a direct tool for finding the recursion operators and might be useful, for instance, in the case when the Lax pair of the equation is not known.

Recall the necessary definitions. Consider a surface in the space of the dynamical variables labeled by uppercase letters \( \{U_j\}_{j=0}^\infty \), defined by an equation of the form
\[ H(U, U_1, \ldots, U_m; u, u_1, \ldots, u_m) = 0. \] (1.4)
Here, the dynamical variables labeled by lowercase letters \( u, u_1, u_2, \ldots \) are considered as parameters. The surface (1.4) is referred to as an invariant manifold of the linearized equation (1.3) if the following condition
\[ D_t H \bigg|_{(1.2)-(1.4)} = 0 \] holds everywhere on the surface (1.4) identically for all values of the variables \( \{u_j\} \). In a similar way, the invariant manifolds are defined for the hyperbolic type PDE and the discrete models. In our opinion, the following two classes of invariant manifolds are worth attention:

- \( H \) is a linear form \( H = \sum_{j=0}^m \alpha_j(u, u_1, \ldots) U_j; \)
- \( H \) is a quadratic form \( H = \sum_{i,j=0}^s \alpha_{ij}(u, u_1, \ldots) U_i U_j. \)

The former kind of invariant surface is connected to the recursion operators, while the latter can be used for finding the Lax pairs. Let us illustrate these two observations by the well-known KdV equation
\[ u_t = u_3 + uu_1. \] (1.6)
The linearized equation of (1.6)
\[ U_t = U_3 + uU_1 + u_1 U \] (1.7)
Admits a linear invariant manifold of order \( m = 3 \), i.e. corresponding to the linear form \( H \)
\[ H := U_3 - \frac{u_2}{u_1} U_2 + \left( \frac{2u}{3} + \lambda \right) U_1 - \left( \frac{2uu_2}{3u_1} - u_1 + \frac{u_2}{u_1} \lambda \right) U = 0 \] (1.8)
where \( \lambda \in \mathbb{C} \) is an arbitrary parameter. Equation (1.8) can be rewritten as follows:
\[ RU = \lambda U, \] (1.9)
where \( R \) coincides with the well-known recursion operator for the KdV equation: \( R = D_x^2 + \frac{2u}{3} + \frac{u}{u_1} D_x^{-1}. \) Actually, a couple of the equations (1.7) and (1.9) can be regarded as a Lax pair for the equation (1.6). However this Lax pair differs from the classical one, where instead of the third order equation we have the second order one. Actually, the system of equations (1.7) and (1.9) is closely connected to the well-known system of equations, which the ‘squares’ of the eigenfunctions of the classical Lax pairs satisfy. Such kinds of Lax pairs...
were considered earlier in [2, 3]. Note that equation (1.7) does not admit any linear invariant manifold of the order \(m = 2\). In the present article, we observe that it is possible to assign the second order invariant manifold defined by a quadratic form to (1.6). Let us first slightly simplify equation (1.7) with the potentiation \(U = W_\chi\), which leads to

\[
W_t = W_\chi + aW_\chi.
\]

(1.10)

In section 2 we show that equation (1.10) admits the second order quadratic invariant manifold of the form

\[
W_2W - \frac{1}{2}W_\chi^2 + \left(\frac{u}{3} - \lambda\right)W_\chi^2 = 0.
\]

(1.11)

Note that formula (1.11) appears in the frame of the finite-gap integration (see, for instance, formula (16) in [4]). It can be verified that equations (1.10) and (1.11) constitute a nonlinear Lax pair for the KdV equation (1.6). The point transformation \(W = \varphi^2\) reduces (1.10) and (1.11) to the usual Lax pair. Remark that earlier in [5] it was observed that the ‘square’ of the eigenfunction of the Lax pair of the sine-Gordon equation solves the linearization of this equation.

The article is organized as follows. In section 2 we explain the method of constructing the Lax pairs with examples of the KdV equation, the Volterra lattice and a coupled lattice related to the affine Lie algebra \(A_1^{(1)}\). Then in section 3, we apply the method to two equations of the KdV type found in [6], the Lax pair of which was not known before. In the last section 4, we demonstrate that the found systems are true Lax pairs since they admit formal eigenfunctions, which allow one to produce the infinite series of the conservation laws.

2. Illustrative examples of the application of the algorithm to seek the Lax pair

In this section, we explain with some examples the algorithm of constructing the Lax pairs via quadratic forms.

2.1. Korteweg–de Vries equation

As the first example, we consider the KdV equation (1.6). For technical reasons we have to simplify the linearized equation (1.7) with the following differential substitution \(U = W_\chi\) by keeping the variables \(u, u_t, u_{xt}, ...\) unchanged. Then, obviously (1.7) is transformed into

\[
W_t = W_{\chi\chi} + uW_\chi.
\]

(2.1)

Observe that the coefficients of (2.1) depend on \(u\) only, while the coefficients of (1.7) depend on \(u, u_t\). Therefore, we can look for the invariant manifold of (2.1) depending on the parameter \(u\) only:

\[
W_{\chi\chi} = F(W, W_\chi, u).
\]

(2.2)

Here, we do not impose any additional assumptions on equation (2.2). The unknown \(F\) is found from the following equation, derived as the consistency condition of the equations (2.1) and (2.2)

\[
\frac{d}{dr}F(W, W_\chi, u) = \left. \frac{d^2}{dx^2}(W_{\chi\chi} + uW_\chi) \right|_{(1.6),(2.1),(2.2)} = 0.
\]

(2.3)
After evaluating the derivatives and simplifying them due to equations (1.6), (2.1) and (2.2) we get:

\[
3F_{uu}u_{x}u_{xx} + (3W_{x}F_{Wx} + W_{x} + 3F_{W,a}F)u_{xx} + (3F_{Wuu}W_{x} + 3F_{W,uu}F + 3F_{W,a}F_{a})u_{x}^{2} \\
+ F_{uu}u_{x}^{2} + (3F_{Wx}F_{Wx} + 3F_{W,x}W_{x}^{2} - F_{Wx}W_{x} + 3W_{x}F_{W,x}F_{x} + 6W_{x}F_{W,x}W_{x}F \\
+ 3W_{x}F_{W,x}F_{a} + 3F_{W,a}F + 2F + 3F_{W,W,F}F_{a} + 3F_{W,W,a}F^{2})u_{x} + F_{W,WW}W_{x}^{3} \\
+ F_{W,x}W_{x}F_{3} + 3F_{W,x}W_{x}F^{2} + 3W_{x}F_{W,x}F_{x}F + 3W_{x}F_{W,x}W_{x}F + 3F_{W,x}F^{2}F_{W,x} \\
+ 3W_{x}^{2}F_{W,W}F + 3W_{x}^{2}F_{W,W}F_{x} + 3W_{x}F_{W,W}F + 3W_{x}F_{W,W,W}F^{2} = 0. \tag{2.4}
\]

By collecting the coefficients before the product \(u_{x}u_{xx}\) we find the following equation:

\[
\frac{\partial^{2}}{\partial u^{2}}F(W, W_{x}, u) = 0
\]

which implies

\[
F(W, W_{x}, u) = F_{1}(W, W_{x})u + F_{2}(W, W_{x}). \tag{2.5}
\]

By means of (2.5) the coefficient before \(u_{x}u_{x}\) in (2.4) gives

\[
F_{1}(W, W_{x}) - \frac{\partial}{\partial W_{x}}F_{1}(W, W_{x}) = 0.
\]

Since the relation \(F_{1}(W, W_{x}) = 0\) leads to a contradiction, we put \(F_{1}(W, W_{x}) = F_{1}(W)\). Now, we continue the investigation of equation (2.4) by using the specifications of \(F\) obtained above. By collecting the coefficients before \(u_{x}\) we find

\[
3 \frac{\partial}{\partial W}F_{1}(W) + 1 = 0
\]

and therefore

\[
F_{1}(W) = \frac{1}{3}W + C_{1}. \tag{2.6}
\]

Let us compare the coefficients of the term \(uu_{x}\):

\[
\frac{\partial^{2}}{\partial W_{x}^{2}}F_{2}(W_{x}, W) = \frac{1}{W - 3C_{1}}.
\]

The latter gives rise to

\[
F_{2}(W_{x}, W) = \frac{W_{x}^{2}}{2(W - 3C_{1})} + F_{3}(W)W_{x} + F_{4}(W). \tag{2.7}
\]

Let us substitute the specifications (2.6) and (2.7) again into (2.4) and collect the coefficients before \(u_{x}\):

\[
\frac{\partial}{\partial W}F_{3}(W) + \frac{1}{W - 3C_{1}}F_{3}(W) = 0.
\]

The solution to this equation is

\[
F_{3}(W) = \frac{C_{2}}{W - 3C_{1}}. \tag{2.8}
\]
The coefficient before $uW_t$ gives $C_2 = 0$. The coefficient before $u$ generates the second order ODE for the last unknown function $F_4(W)$, which is nothing else but the Euler equation:

$$\frac{d^2}{dW^2}F_4(W) + \frac{1}{W - 3C_1} \frac{d}{dW}F_4(W) - \frac{1}{(W - 3C_1)^2}F_4(W) = 0.$$ 

The equation is immediately solved

$$F_4(W) = \frac{C_3}{W - 3C_1} + C_4(W - 3C_1). \quad (2.9)$$

Summarizing the reasonings above we find

$$F(W, W_t, u) = (C_1 - \frac{1}{3} W)u + \frac{W^2}{2W - 6C_1} + \frac{C_3}{W - 3C_1} + C_4(W - 3C_1). \quad (2.10)$$

The constant parameter $C_1$ is easily removed by the transformation $W \rightarrow \tilde{W} = W - 3C_1$, hence we can take $C_1 = 0$. Then, by putting $C_3 = 0$ and $\lambda := C_4$ we get $F = -\frac{1}{3} uW + \lambda W + \frac{W^2}{2W}$. Then equations (2.1) and (2.2) can be written as follows:

$$\begin{cases}
W_{xx} = \left(\lambda - \frac{1}{3} u\right) W + \frac{W^2}{2W - 6C_1}, \\
W_t = \left(2\lambda + \frac{1}{3} u\right) W_x - \frac{1}{4} uW.
\end{cases} \quad (2.11)$$

Now it is evident that the invariant surface (2.2) is defined by the quadratic form $H = W_{xx}W - \left(\lambda - \frac{1}{3} u\right) W^2 - \frac{1}{2} W^2$. Note that the point transformation $W = \varphi^2$ reduces (2.11) to the usual Lax pair for the KdV equation (see [7])

$$\begin{cases}
\varphi_{xx} = \left(\frac{1}{2} \lambda - \frac{1}{3} u\right) \varphi, \\
\varphi_t = \left(2\lambda + \frac{1}{3} u\right) \varphi_x - \frac{1}{2} u\varphi.
\end{cases} \quad (2.12)$$

### 2.2. Modified Volterra chain

As the second illustrative example we take the well-known modified Volterra chain

$$p_t = -p^2(p_1 - p_{-1}). \quad (2.13)$$

Here the sought function $p$ depends on the discrete $n$ and the continuous $t$: $p = p_n(t)$. For simplicity, we omit $n$ and use an abbreviated notation by writing $p_n = p$, $p_{n+1} = p_1$, $p_{n-1} = p_{-1}$ and so on.

The linearization of equation (2.13)

$$p_t = -p^2(P_1 - P_{-1}) - 2p(p_1 - p_{-1})P \quad (2.14)$$

contains an explicit dependance on the dynamical variables $p_1, p, p_{-1}$. In order to simplify the further necessary computations, it is reasonable to find a more convenient form of the linearized equation (2.14). By applying the substitution $P = -p^2(U_1 - U_{-1})$ we reduce it to the following form

$$U_t = -p^2(U_1 - U_{-1}), \quad (2.15)$$

where the r.h.s. depends only on $p, U_1, U_{-1}$. Initiated by this circumstance, we look for the invariant manifold of (2.15) in the form
The unknown $F$ is found from the defining equation
\[
\frac{d}{dt} U_1 - D_n U_t \bigg|_{(2.13),(2.15),(2.16)} = 0. \tag{2.17}
\]
To exclude $U_{\pm 2}$ we use the relations
\[
U_2 = F(F(U, U_{-1}, p), U, p_1) \quad \text{and} \quad U_{-2} = G(U, U_{-1}, p_{-1}) \tag{2.18}
\]
where $G$ is found as the ‘inverse’ of $F$:
\[
U_{-1} = G(U_1, U, p). \tag{2.19}
\]
By evaluating the derivatives, shifting the arguments and substituting the expressions for all of the involved variables in terms of the dynamical ones in (2.17), we obtain a comparatively simple equation for the unknowns $F$ and $G$
\[
F_{U_{-1}}(U, U_{-1}, p) \left( G(U, U_{-1}, p_{-1}) - U \right) p_{-1}^2 + F_p(U, U_{-1}, p) p^2 p_{-1} - F_p(U, U_{-1}, p) p^2 p_1 \\
+ \left( F(F(U, U_{-1}, p), U, p_1) - U \right) p_1^2 - F_U(U, U_{-1}, p) p^2 \left( F(U, U_{-1}) - U_{-1} \right) = 0. \tag{2.20}
\]
By applying the operator $\partial^3 \partial p_{-1}$ to (2.20) we find
\[
F_{U_{-1}}(U, U_{-1}, p) \left( p_{-1}^2 \frac{\partial^3}{\partial p_{-1}^3} G(U, U_{-1}, p_{-1}) \right) \\
+ 6 p_{-1} \frac{\partial^2}{\partial p_{-1}^2} G(U, U_{-1}, p_{-1}) + 6 \frac{\partial}{\partial p_{-1}} G(U, U_{-1}, p_{-1}) \right) = 0. \tag{2.21}
\]
Since the first factor does not vanish, then (2.21) implies the Euler equation for $G$, which is easily solved:
\[
G(U, U_{-1}, p_{-1}) = G_1(U, U_{-1}) + \frac{G_2(U, U_{-1})}{p_{-1}} + \frac{G_3(U, U_{-1})}{p_{-1}}. 
\]
Substitute the latter into (2.20) and afterward differentiate it twice with respect to $p_{-1}$ and find
\[
G_1(U, U_{-1}) = U. 
\]
Now we differentiate (2.20) three times with respect to $p_1$ and again obtain the Euler equation solution, which is of the form
\[
F(U, U_{-1}, p) = F_1(U, U_{-1}) + \frac{F_2(U, U_{-1})}{p} + \frac{F_3(U, U_{-1})}{p^2}. 
\]
By applying $\partial^2 \partial p_1$ to (2.20) we find
\[
F_1(U, U_{-1}) = U_{-1}. 
\]
Then we substitute the simplification into (2.20) and differentiate the equation obtained with respect to $p_{-1}$. As a result, we get the following three relations:
\begin{align}
(1) \quad & \frac{\partial F_3(U, U_{-1})}{\partial U_{-1}} G_3(U, U_{-1}) = 0, \\
(2) \quad & G_3(U, U_{-1}) = F_2(U, U_{-1}), \\
(3) \quad & F_3(U, U_{-1}) = \frac{1}{2} \frac{\partial F_2(U, U_{-1})}{\partial U_{-1}} G_3(U, U_{-1}).
\end{align}

Let us concentrate on the first one. It yields either \( \frac{\partial F_3(U, U_{-1})}{\partial U_{-1}} = 0 \) or \( G_3(U, U_{-1}) = 0 \). The latter case implies \( F_2(U, U_{-1}) = F_3(U, U_{-1}) = 0 \) due to (2) and (3) and leads to a negative result. The former case gives:

\begin{align}
(1) \quad & F_3(U, U_{-1}) = F_3(U), \\
(2) \quad & G_3(U, U_{-1}) = F_2(U, U_{-1}), \\
(3) \quad & F_3(U, U_{-1}) = \sqrt{4F_3(U)U_{-1} + F_4(U)}.
\end{align}

Due to these relations we can specify \( F \) and \( G \):

\begin{align}
U_1 = F(U, U_{-1}, p) = U_{-1} + \frac{F_3(U)}{p} + \frac{\sqrt{4F_3(U)U_{-1} + F_4(U)} - U}{p}, \tag{2.22}
\end{align}

\begin{align}
U_{-2} = G(U, U_{-1}, p_{-1}) = U + \frac{G_2(U, U_{-1})}{p_{-1}^2} + \frac{\sqrt{4F_3(U)U_{-1} + F_4(U)} - U}{p_{-1}}. \tag{2.23}
\end{align}

Express \( U_{-1} \) from the equation (2.22) and then find \( U_{-2} \) by applying the operator \( D_n^{-1} \):

\begin{align}
U_{-2} = U + \frac{F_3(U_{-1})}{p_{-1}^2} + \frac{\sqrt{4F_3(U_{-1})U + F_4(U_{-1})}}{p_{-1}}. \tag{2.24}
\end{align}

A comparison of (2.23) with (2.24) leads to the equations:

\begin{align}
(1) \quad & G_3(U, U_{-1}) = F_3(U_{-1}), \\
(2) \quad & 4F_3(U)U_{-1} + F_4(U) = 4F_3(U_{-1})U + F_4(U_{-1}). \tag{2.25}
\end{align}

Equation (2.25) convinces us that the functions \( F_3, F_4 \) are linear:

\( F_3(U) = c_1 U + c_2, \quad F_4(U) = 4c_2 U + c_4. \)

Let us summarize the reasonings above and obtain the desired manifold

\begin{align}
U_1 = U_{-1} + \frac{c_1 U + c_2}{p^2} + \frac{\sqrt{4(c_2 U + c_2)U_{-1} + 4c_2 U + c_4}}{p}.
\end{align}

By setting \( c_2 = c_4 = 0, c_1 = \lambda \) we get

\begin{align}
U_1 = U_{-1} + \lambda \frac{U}{p^2} + \frac{\lambda}{p} \sqrt{\lambda U U_{-1}}, \tag{2.26}
\end{align}

Since we have a square root, then we change the variables in such a way \( U = \varphi^2 \) in order to get rid of the root:

\begin{align}
\varphi_1^2 = \varphi_{-1}^2 + \frac{\lambda}{p^2} \varphi^2 + \frac{2\lambda}{p} \varphi \varphi_{-1}, \quad \varepsilon^2 = 1. \tag{2.27}
\end{align}

By taking the square root from both sides in (2.27) we find
\[ \varphi_1 = \frac{\sqrt{\lambda}}{p} \varphi + \varepsilon \varphi_{-1}. \]  
(2.28)

By applying the same change of the variables \( U = \varphi^2 \) to the equation (2.15) we bring it to the form
\[ \varphi_t = -\frac{1}{2} \lambda \varphi + \varepsilon \sqrt{\lambda} \varphi_{-1}. \]  
(2.29)

The equation system (2.28) and (2.29) coincides with the well-known Lax pair for the modified Volterra lattice.

2.3. An example of the application of the algorithm to a system of equations

The algorithm above can be applied to higher order integrable equations and systems as well. Below it is illustrated with the coupled lattice
\[
\begin{align*}
    u_t &= \frac{1}{v_1} + u^2 v, \\
    v_t &= -\frac{1}{u_{-1}} - uv^2 
\end{align*}
\]  
(2.30)

found in [8].

Evidently, the linearization of (2.30) is a system of the form
\[
\begin{align*}
    U_t &= -\frac{1}{v_1} V_1 + 2uvU + u^2 V, \\
    V_t &= \frac{1}{u_{-1}} U_{-1} - v^2 U - 2uvV. 
\end{align*}
\]  
(2.31)

Now our goal is to find appropriate invariant manifolds for the linearized system (2.31). It is easily checked that the simplest invariant manifold is defined by the equations
\[
\begin{align*}
    U_1 &= \frac{1}{uv_1} U - \frac{c}{v_1}, \\
    V_1 &= u^2 v_1^2 V - cuv_1^2. 
\end{align*}
\]  
(2.32)

Observe that \( \rho = \log uv_1 \) is a conserved density for the system (2.30).

As the next step we look for an invariant manifold of the form
\[
\begin{align*}
    U_1 &= F(U, V, u, v, u_1, v_1), \\
    V_1 &= G(U, V, u, v, u_1, v_1). 
\end{align*}
\]  
(2.33)

It can be proved that the linearized system (2.31) admits only the following kinds of invariant manifolds:
\[
\begin{align*}
    U_1 &= \frac{(1 + \xi v_1)}{uv_1} U + \xi u^2 V - \frac{1 + \xi v_1}{v_1} \sqrt{4\xi UV + c}, \\
    V_1 &= \xi v_1^2 U + u^2 v_1^2 V - uv_1^2 \sqrt{4\xi UV + c}. 
\end{align*}
\]  
(2.34)

Note that by taking \( \xi = 0 \) in (2.34) we arrive at the simplest invariant manifold (2.32). By comparing (2.31) and (2.34) we find the time evolution of \( U, V \):
\[
\begin{align*}
    U_t &= (2uv - \xi) U + u \sqrt{4\xi UV + c}, \\
    V_t &= (\xi - 2uv) V + v \sqrt{4\xi UV + c}. 
\end{align*}
\]  
(2.35)

It can be verified by a direct computation that the system (2.34) and (2.35) is consistent, if and only if the functions \( u, v \) solve the system (2.30), and therefore (2.34) and (2.35) define a nonlinear Lax pair for the system (2.30). In order to derive the usual form of the system we set \( c = 0 \), \( U = \varphi^2 \), \( V = \xi \psi^2 \). Omitting the calculations we give only the answer.
\[
\begin{align*}
\varphi_1 &= -\frac{1+\xi uv_1}{\eta_1} \varphi + \xi u \psi, \\
\psi_1 &= v_1 \varphi - uv_1 \psi,
\end{align*}
\]
(2.36)

\[
\begin{align*}
\varphi_t &= (uv - \frac{1}{2} \xi) \varphi + \xi u \psi, \\
\psi_t &= v \varphi + \left(\frac{1}{2} \xi - uv\right) \psi.
\end{align*}
\]
(2.37)

Let us set \( \xi = 0 \) in the Lax pair and get a first order linear system \( \varphi_1 = -\frac{1}{uv_1} \varphi \) and \( \varphi_t = uv \varphi \), which generates the following local conservation law \( D_t \log uv_1 = -(D_{n1} - 1)uv \). Actually, the Lax pair (2.36) and (2.37) found earlier in [9] allows one to find an infinite series of the local conservation laws.

3. New Lax pairs

In this section, we seek the Lax pairs for the following two KdV-type integrable models:

\( u_t = u_{xxx} + \frac{1}{2} u^3 - \frac{3}{2} u_x \sin^2 u \) (3.1)

\( u_t = u_{xxx} - \frac{3u_x u_{xx}^2}{2(1 + u_x^2)} + \frac{1}{2} u_x^3 \) (3.2)

found in [6] as a result of the symmetry classification (see also [10]). The connection of these equations to the integrable hyperbolic-type equations is discussed in [11]. The recursion operators of (3.1) and (3.2) have been found in [1]. In the present article, the Lax pairs of them are found via quadratic invariant manifolds of their linearizations.

3.1. Equation (3.1)

Let us begin with equation (3.1). First we evaluate the linearized equation

\[
U_t = U_{xxx} + \frac{3}{2} (u_x^3 - \sin^2 u) U_x - 3u_x \sin u \cos u U.
\]
(3.3)

The coefficients of equation (3.3) explicitly depend on \( u \) and \( u_x \). By direct computations, we proved that there is no differential substitution that converts equation (3.3) into a more simple linear equation with the coefficients depending only on \( u \) or only on \( u_x \).

Below, we give several examples of the invariant manifolds for equation (3.3):

\[
\begin{align*}
(1) \quad &U_x = (\sin u) \sqrt{U^2 + c}; \\
(2) \quad &U_x = (\cos u) \sqrt{c - U^2}; \\
(3) \quad &U_{xx} = (\cot u) u_x U_x + (\sin^2 u) U; \\
(4) \quad &U_{xx} = -\tan u u_x U_x - (\cos^2 u) U;
\end{align*}
\]
(3.4)

which, however, do not fit for the construction of the true Lax pairs\(^3\).

In order to find the true Lax pair we look for a quadratic form \( H = \sum_{ij=0}^{2} \alpha_{ij} U_i U_j \), such that the surface

\[
H = 0
\]
(3.5)

\(^3\)Case (4) has independently been found by Garifullin (private communication).
defines an invariant manifold for the equation (3.5). Since the coefficients in (3.3) depend on \(u, u_x\) we assume that the coefficients \(\alpha_{ij}\) of the quadratic form \(H\) depend on the same variables. For the sake of convenience we rewrite equation (3.5) in the form solved with respect to \(U_{xx}\):

\[
U_{xx} = F(U, U_x, u, u_x)
\]

where \(F = aU_x + bU + R\) and \(R = \sqrt{cUU_x + rU_x^2} + sU^2\). The functions \(a, b, \ldots\) and \(s\) depend on \(u, u_x\).

**Remark 1.** The degenerate case, when \(F\) is a rational function of \(U, U_x\), leads to the linear invariant manifolds (3) and (4) in (3.4).

Evidently, the consistency condition of equations (3.3) and (3.6) gives rise to the following equation:

\[
D_rF - D_sU_r|_{(3.1),(3.3),(3.5)} = 0.
\]

By doing all of the differentiations in (3.7) and then expressing the appeared variables through the dynamical ones \(U, U_x, u, u_x, \ldots\), we obtain an overdetermined system of the equations, which the sought function \(F\) should satisfy. Here, we give it in an abbreviated form

\[
F_{u,a}u_{xx}u_{xxx} + (u_xF_{u_x} + U_xF_{u_x} + FF_{U_{xx}} + u_{xx}F_{u_{xx}} - \sin u \cos u U)u_{xxx} = G(U, U_x, u, u_x, u_{xx}).
\]

(3.8)

By gathering the coefficients at the product \(u_{xx}u_{xxx}\) we obtain the equation \(F_{u,a} = 0\). Due to the representation \(F = a(u, u_x)U_x + b(u, u_x)U + \sqrt{c(u, u_x)UU_x + r(u, u_x)U_x^2} + s(u, u_x)U^2\) the equation implies

\[
a_{u,a}U_x + b_{u,a}U + \frac{1}{4R^2}(2rr_{u,a} - r_x^2)U_x^2 + 2r_{u,a}r_{u_x} + cr_{u,a}U_x^2U_x
\]

\[
+2(c_{u,a} - c_{u_x} + cr_{u,a})U_x^2U_x + 2(s_{u,a} - c_{u_x} + cs_{u,a})U_x^2U_x + (2s_{u,a} - s_{u_x} + s_{u,a})U^4 = 0.
\]

(3.9)

Since the variables \(U, U_x\) and \(R\) are assumed to be independent, we compare the coefficients at these variables as well as their products. As a result, we find the following equations:

1. \(a_{u,a} = 0\);
2. \(b_{u,a} = 0\);
3. \(2rr_{a,a} - r_x^2 = 0\);
4. \(2s_{a,a} - s_x^2 = 0\);
5. \(rc_{a,a} - c_{a_x} + cr_{a,a} = 0\);
6. \(sc_{a,a} - c_{a_x} + cs_{a,a} = 0\);
7. \(2sr_{a,a} + 2c_{a_x} - c_{a_x}^2 + 2rs_{a,a} - 2s_{a_x} = 0\).

(3.10)

The first four of these equations are easily integrated in quadratures

1. \(a(u, u_x) = a_1(u)u_x + a_2(u)\);
2. \(b(u, u_x) = b_1(u)u_x + b_2(u)\);
3. \(r(u, u_x) = \frac{1}{4}(r_1(u)u_x + r_2(u))^2\);
4. \(s(u, u_x) = \frac{1}{4}(s_1(u)u_x + s_2(u))^2\).

10
Let us study the other three of them (5)–(7). By virtue of the formula (3'), one can easily solve equation (5). Depending on \( r_1(u) \) there are two possibilities here:

\[
(5a) \ c(u, u_s) = c_1(u) u_x + c_2(u) \quad \text{for} \quad r_1(u) = 0;
\]

\[
(5b) \ c(u, u_s) = c_3(u) \left( u_x + \frac{r_2(u)}{r_1(u)} \right)^2 + c_4(u) \left( u_x + \frac{r_2(u)}{r_1(u)} \right) \quad \text{for} \quad r_1(u) \neq 0.
\]

Let us first concentrate on case (5a). Then, due to (4') equation (6) yields

\[
(a) \ s_1(u) = 0 \quad \text{or} \quad (b) \ c_1(u)s_2(u) = s_1(u)c_2(u).
\]

In case (a) due to equation (7) we get \( c_1(u) = 0 \). Therefore, the sought function \( F \) is of the form

\[
F(U, U_x, u, u_s) = (a_1(u) u_x + a_2(u)) U_x + (b_1(u) u_x + b_2(u)) U
\]

\[
+ \frac{1}{2} \left( u_x + \frac{r_2(u)}{r_1(u)} \right) \sqrt{r_1^2(u) U_x^2 + 4c_2(u) U_x U + s_1^2(u) U^2}.
\]

(3.12)

In case (b) from the same equation (7) we obtain \( c_1(u) = \frac{1}{2} \varepsilon s_1(u) r_2(u), \varepsilon = \pm 1 \). Then

\[
F(U, U_x, u, u_s) = (a_1(u) u_x + a_2(u) + \varepsilon r_2(u)) U_x
\]

\[
+ (b_1(u) u_x + s_1(u) u_x + b_2(u) + s_2(u)) U.
\]

(3.13)

Turn to case (5b). Now from equation (6) due to (4') we find

\[
(a) \ r_1(u)s_2(u) = s_1(u)r_2(u);
\]

\[
(b) \ s_1(u)c_3(u) r_2(u) + s_1(u)c_4(u) r_1(u) - r_1(u)s_2(u) c_3(u) = 0.
\]

(3.14)

In case (a) in (3.14) equation (7) gives \( c_4(u) = 0 \), hence

\[
F(U, U_x, u, u_s) = (a_1(u) u_x + a_2(u)) U_x + (b_1(u) u_x + b_2(u)) U
\]

\[
+ \frac{1}{2} \left( u_x + \frac{r_2(u)}{r_1(u)} \right) \sqrt{r_1^2(u) U_x^2 + 4c_3(u) U_x U + s_1^2(u) U^2}.
\]

(3.15)

In case (b) in (3.14) due to equation (7) the following two conditions should satisfy:

\[
c_4(u) = \frac{1}{2} \varepsilon (r_1(u)s_2(u) - s_1(u)r_2(u)) \quad \text{and} \quad c_3(u) = \frac{1}{2} \varepsilon r_1(u)s_3(u).
\]

Then \( F \) is given by

\[
F(U, U_x, u, u_s) = (a_1(u) u_x + a_2(u) - \frac{1}{2} (r_1(u) u_x + r_2(u))) U_x
\]

\[
+ (b_1(u) u_x + b_2(u) + \frac{1}{2} (s_1(u) u_x + s_2(u))) U.
\]

(3.16)

Summarising the reasonings above, we have found four possible forms of the sought function \( F \), two of which are linear with respect to \( U \) and \( U_x \) and hence coincide with that given by (3.4) above, while the other two contain the square root and need further investigation.

Let us show that the case (3.12) is contradictory. We substitute (3.12) into equation (3.8) and compare the coefficients at the independent variables \( u_x, u_{xx}, u_{xxx}, U, U_x, U_{xx}, \) and \( R \). As a result, we get a set of the equations for finding the unknown coefficients \( a_1(u), a_2(u), b_1(u), b_2(u) \). It turns out that some of these equations contradict each other. For example, the coefficients at \( u_{xxx}R \)
and \( u_{xxx} U \) give rise to equations \( a_1(u) = 0 \) and \( a_1(u)b_2(u) - \sin u \cos u = 0 \), which contradict each other.

Let us study the last possibility (3.15). We have two different cases: \( r_2(u) \neq 0 \) and \( r_2(u) = 0 \). Assume first that \( r_2(u) \neq 0 \). Substitute the ansatz (3.15) into equation (3.8) and then split down (3.8) into a set of the equations by collecting the coefficients before the independent variables \( u_{xxx}, u_x, u, U, \) and \( R \) and their products. In the first stage, we will use the following four of them:

(i) \( r_2(u)r_1^2(u)c_3(u)(r_1(u)s_1(u) - 2c_3(u))(r_1(u)s_1(u) + 2c_3(u)) = 0; \)
(ii) \( a_1(u)r_1^2(u)r_2(u) + 2c_3(u)b_2(u)r_1(u) = 0; \)
(iii) \( 2a_1(u)b_2(u)r_1(u) + c_3(u)r_2(u) - 2r_1(u) \sin u \cos u = 0; \)
(iv) \( 2r_1(u)c_3(u)a_2(u) + r_1^2(u)b_2(u) + 4a_1(u)c_3(u)r_2(u) + r_1(u)s_1^2(u) = 0. \)  
(3.17)

As it evidently follows from equation (i) there are only two possibilities

(1) \( c_3(u) = 0; \)
(2) \( 2c_3(u) = \varepsilon r_1(u)s_1(u), \quad \varepsilon = \pm 1. \)  
(3.18)

Let us begin with the first one in (3.18). Then, due to (ii) we have an alternative \( a_1(u) = 0 \) or \( s_1(u) = 0 \). However, due to equation (iii) it should be \( a_1(u) \neq 0 \) and \( b_2(u) \neq 0 \), hence \( s_1(u) \) must vanish. But then (iv) implies \( b_2(u) = 0 \), in contradiction with the previous sentence.

Now we consider case (2) in (3.18). It follows immediately from (2) that in \( R \) under the square root, we have the complete square of a linear combination of the variables \( U \) and \( U_x \), and hence the function \( F \) is reduced to the previously investigated linear form. Therefore, the case \( r_2(u) \neq 0 \) in (3.15) does not lead to a nonlinear invariant manifold.

Let us consider the last possibility of finding a nonlinear \( F \) by assuming \( r_2(u) = 0 \):

\[
F(U, U_x, u, u_x) = (a_1(u)u_x + a_2(u))U_x + (b_1(u)u_x + b_2(u))U + \frac{1}{2} u_x \sqrt{r_1^2 U_x^2 + 4c_3(u)U_x^2 + s_1^2(u)U^2}. 
\]
(3.19)

We substitute the ansatz (3.19) into equation (3.8) and derive a set of necessary equations. Begin with the following ones:

(1) \( a_1(u)b_2(u) - \sin u \cos u = 0; \)
(2) \( c_3(u)b_2(u) = 0; \)
(3) \( 2c_3(u) + r_1^2(u)a_2(u) = 0; \)
(4) \( a_1(u)a_2(u) + b_1(u) = 0. \)  
(3.20)

From equations (1) and (2) it evidently follows that \( c_3(u) = 0 \). Afterward, from (3) and (4) we find \( a_2(u) = b_1(u) = 0 \). Due to these specifications, from the set of equations mentioned above we get

\[
b_2'(u) + a_1(u)b_2(u) - 3 \sin u \cos u = 0. \]  
(3.21)

By comparing equation (3.21) and the first equation in (3.20) we obtain

\[
b_2(u) = \sin^2 u + k, \quad k = \text{const}. \]  
(3.22)

Then the first equation in (3.20) allows us to determine \( a_1(u) \):
We rewrite equation (3.8) by virtue of the found specifications and take from it the coefficient before $u_{xxx}u_x U^2$. This allows us to find

$$r_1(u) = \frac{k_1}{2 \sin^2 u + 2k}, \quad k_1 = \text{const.}$$

By using the coefficients before $u_{xxx}u_x U^2$ in (3.8) we obtain

$$s_1(u) = \frac{k_2}{\sqrt{2 \sin^2 u + 2k}}, \quad k_2 = \text{const.}$$

Continuing this way, and collecting the coefficients before $u_{xxx}u_x U_x R$ and $u_{xxx}U_x$, we can establish the relations between the constant parameters:

$$k_2^2 + 16(k^2 + k) = 0 \quad \text{and} \quad k_2^2 - 8(k^2 + k) = 0.$$ 

Now we summarize all of the reasonings above and give the final form of the invariant sought manifold (3.6):

$$U_{xx} = \frac{u_x \sin u \cos u}{\sin^2 u + \lambda} U_x + (\sin^2 u + \lambda)U + \frac{\sqrt{(\lambda + 1)u_x}}{\sin^2 u + \lambda} \sqrt{(\sin^2 u + \lambda)U^2 - U_x^2},$$

where $\lambda := k$. Evidently, for $\lambda = 0$ and $\lambda = -1$ we get the linear invariant surfaces (3) and respectively (4), as discussed at the beginning of the section (see (3.4)). Now we have to find a change of the variables linearizing equation (3.26). We express the variables $U$ and $U_x$ as some quadratic forms of the new variables $\varphi$ and $\psi$, chosen in such a way that the function under the square root is a complete square of a combination of the variables $\varphi$ and $\psi$. Assume that

$$U = \alpha_1 \varphi^2 + 2\beta_1 \varphi \psi + \gamma_1 \psi^2, \quad U_x = \alpha_2 \varphi^2 + 2\beta_2 \varphi \psi + \gamma_2 \psi^2.$$  

Actually, we are introducing the vector valued function $\Phi = (\varphi, \psi)^T$, satisfying a linear equation, which is just the first equation of the sought Lax pair

$$\Phi_x = A(u, u_x, \ldots, \lambda) \Phi.$$  

Since equation (3.28) is defined up to a linear transformation $\Phi \rightarrow Z \Phi$, by applying the linear transformation we can bring one of the quadratic forms (3.27) (say the first one) to the canonical form $U = \varphi^2 + \psi^2$. It is easily checked that the expression $zU^2 - U_x^2$ with $z = \lambda + \sin^2 u$ is a complete square, if and only if the following three conditions hold:

1. $\alpha_2 \beta_2 = 0$,  
2. $\beta_2 \gamma_2 = 0$,  
3. $(z - 2\beta_2^2 - \alpha_2 \gamma_2)^2 = (z - \alpha_2^2)(z - \gamma_2^2).$

Here we have two choices: (a) $\beta_2 = 0$ and (b) $\alpha_2 = 0, \gamma_2 = 0$. Case (a) is not acceptable since it implies $\alpha_2 = \gamma_2$ and then the map $(\varphi, \psi) \rightarrow (U, U_x)$ defined by (3.27) has the degenerate Jacobian. In case (b) the third equation yields $z = \beta_2^2$. Thus, an appropriate change of variables is as follows:

$$U = \varphi^2 + \psi^2, \quad U_x = 2\sqrt{\sin^2 u + \lambda \varphi \psi}.$$  

Then equation (3.26) is reduced to a system of linear equations of the following form
The change of the variables:
\[ U = \varphi^2 + \psi^2, \quad U_x = \frac{2\sqrt{1 + u_x^2}}{\sqrt{\lambda}} \varphi \psi \]
(3.35)
brings the system (3.33) and (3.34) to a Lax pair for (3.2):
\[
\begin{align*}
\varphi_x &= \frac{1}{2\sqrt{\lambda}} \left( \sqrt{1 + u_x^2} - u_x \sqrt{1 + \lambda} \right) \psi, \\
\psi_x &= \frac{1}{2\sqrt{\lambda}} \left( \sqrt{1 + u_x^2} + u_x \sqrt{1 + \lambda} \right) \varphi,
\end{align*}
\]  
(3.36)

\[
\varphi_t = \frac{1}{4\lambda^2} \left( (1 + u_x^2)^2 (\lambda u_x^2 + 2) + 2 u_x u_{xxx} \lambda (1 + u_x^2) - u_x^2 \lambda (3 u_x^2 + 1) \right) \psi + \frac{1}{2} \sqrt{\frac{\lambda}{1 + u_x^2}} \varphi,
\]
\[
\psi_t = \frac{1}{4\lambda^2} \left( (1 + u_x^2)^2 (\lambda u_x^2 + 2) + 2 u_x u_{xxx} \lambda (1 + u_x^2) - u_x^2 \lambda (3 u_x^2 + 1) \right) \varphi - \frac{1}{2} \sqrt{\frac{\lambda}{1 + u_x^2}} \psi.
\]  
(3.37)

4. Formal diagonalization of the found Lax pairs and the local conservation laws

In this section we find formal asymptotic expansions for the eigenfunctions of the Lax pairs (3.30), (3.31) and (3.36), (3.37) around the singular points of the spectral parameter \( \lambda \). From these asymptotic expansions, we deduce the conservation laws for equations (3.1) and (3.2). It is generally accepted that the Lax pair, which produces the infinite series of the conserved densities, is certainly the true Lax pair.

Let us begin with the system (3.30) and (3.31). By the linear transformation \( \Phi = \tilde{T} Y \) where \( \Phi = (\varphi, \psi)^T \) and \( \tilde{T} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \) we reduce (3.30) and (3.31) to the following form:
\[
Y_t = AY, \quad Y_x = GY.
\]  
(4.1)

Here, the matrices \( A, G \) are given by
\[
A = \begin{pmatrix} \frac{1}{2} \sqrt{\sin^2 u + \lambda} - \frac{1}{2} u_x \sqrt{\lambda (\lambda + 1)} \\ u_x \sqrt{\lambda (\lambda + 1)} - \frac{1}{2} \sqrt{\sin^2 u + \lambda} \end{pmatrix},
\]
\[
G = \begin{pmatrix} \frac{1}{2} u_x \sin u \cos u + (\sin^2 u + \lambda) (u_x^2 - \sin^2 u + 2\lambda) \sqrt{\sin^2 u + \lambda} - \frac{\lambda (\lambda + 1)}{4 \sqrt{\sin^2 u + \lambda}} \left( 2u_{xxx} + 2 u_x u_{xx} (u_x^2 - \sin^2 u + 2\lambda) \right) \\ \sqrt{\lambda (\lambda + 1)} \left( 2u_{xxx} + 2 u_x u_{xx} (u_x^2 - \sin^2 u + 2\lambda) \right) - 2 u_{xx} \left( \frac{1}{4} 2u_x \sin u \cos u + (\sin^2 u + \lambda) (u_x^2 - \sin^2 u + 2\lambda) \right) \right) \\ \frac{\lambda (\lambda + 1)}{4 \sqrt{\sin^2 u + \lambda}} \left( 2u_{xxx} + 2 u_x u_{xx} (u_x^2 - \sin^2 u + 2\lambda) \right) - \frac{1}{2} u_x \sqrt{\sin^2 u + \lambda} \left( \frac{1}{4} 2u_x \sin u \cos u + (\sin^2 u + \lambda) (u_x^2 - \sin^2 u + 2\lambda) \right) \right) \\ \frac{\lambda (\lambda + 1)}{4 \sqrt{\sin^2 u + \lambda}} \left( 2u_{xxx} + 2 u_x u_{xx} (u_x^2 - \sin^2 u + 2\lambda) \right) - \frac{1}{2} u_x \sqrt{\sin^2 u + \lambda} \left( \frac{1}{4} 2u_x \sin u \cos u + (\sin^2 u + \lambda) (u_x^2 - \sin^2 u + 2\lambda) \right) \right) \\ \frac{\lambda (\lambda + 1)}{4 \sqrt{\sin^2 u + \lambda}} \left( 2u_{xxx} + 2 u_x u_{xx} (u_x^2 - \sin^2 u + 2\lambda) \right) - \frac{1}{2} u_x \sqrt{\sin^2 u + \lambda} \left( \frac{1}{4} 2u_x \sin u \cos u + (\sin^2 u + \lambda) (u_x^2 - \sin^2 u + 2\lambda) \right) \right) 
\]  
(4.2)

Matrix \( A \) has singularities at the points \( \lambda = 0, \lambda = -1 \) and \( \lambda = \infty \). Note that the first two singularities are not essential, since they are removed by the linear conjugation \( Y = \Omega \tilde{Y} \), where \( \Omega = \text{diag}(\sqrt{\lambda (\lambda + 1)}, 1) \) is a diagonal matrix.

Let us expand \( A \) around \( \lambda = \infty \):
\[
A = \sum_{j=0}^{\infty} A_j \lambda^{-j/2}, \quad A_{-1} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]  
(4.2)
According to the general theory (see [12–14]), we look for a formal change of the variables
\[ \Psi = T Y \]
transforming (4.1) to the form
\[ \Psi_x = h \Psi, \quad \Psi_t = S \Psi, \]
(4.3)
where \( T, h \) and \( S \) are the formal power series:
\[ T = \sum_{j=0}^{\infty} T_j \lambda^{-j/2}, \quad h = \sum_{j=-1}^{\infty} h_j \lambda^{-j/2}, \quad S = \sum_{j=-3}^{\infty} S_j \lambda^{-j/2}. \]
(4.4)
The matrices \( h_j \) are assumed to be diagonal. Setting \( T_0 = 1 \) and assuming that for \( \forall i \geq 1 \) all diagonal entries of \( T_i \) vanish, we find the coefficients of the series \( T \) and \( h \) from the equation \( T_s = AT - Th \). By comparing the coefficients in
\[ \sum_{j=1}^{\infty} D_s(T_j) \lambda^{-j/2} = \sum_{j=0}^{\infty} A_j \lambda^{-j/2} \sum_{j=0}^{\infty} T_j \lambda^{-j/2} - \sum_{j=0}^{\infty} T_j \lambda^{-j/2} \sum_{j=0}^{\infty} h_j \lambda^{-j/2} \]
we obtain sequences of the equations for defining \( T_j, h_j \):
\[ \begin{align*}
    h_{-1} &= A_{-1}, \\
    [A_{-1}, T_1] - h_0 &= -A_0, \\
    [A_{-1}, T_2] - h_1 &= D_s(T_1) - A_0 T_1 - A_1 + T_1 h_0.
\end{align*} \]
(4.6)
The system of equation (4.6) is consecutively solved. Omitting the computations, we give only the answers
\[ h = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \lambda^{\frac{1}{2}} + \left( -\frac{1}{2} \left( u_x^2 - \sin^2 u \right) \right) \lambda^{-\frac{1}{2}} \]
\[ + \begin{pmatrix} \frac{1}{2} u_x^2 u_{xx} \\ 0 \end{pmatrix} \lambda^{-1} + \begin{pmatrix} h_{3,11} \\ 0 \end{pmatrix} \lambda^{-\frac{3}{2}} + \ldots, \]
where \( h_{3,11} = \frac{5}{8} u_x^2 \sin^2 u - \frac{1}{4} u_x^2 - \frac{1}{16} u_x^2 - \frac{1}{4} u_x u_{xxx} - \frac{1}{16} \sin^4 u, \)
\[ T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{2} u_x \\ \frac{1}{2} u_x & 0 \end{pmatrix} \lambda^{\frac{1}{2}} + \begin{pmatrix} 0 & \frac{1}{2} u_{xx} \\ \frac{1}{2} u_{xx} & 0 \end{pmatrix} \lambda^{-1} \]
\[ + \begin{pmatrix} \frac{1}{2} u_{xxx} + \frac{1}{4} u_x + \frac{1}{8} u_x^3 \sin^2 u \\ \frac{1}{2} u_{xxx} + \frac{1}{4} u_x + \frac{1}{8} u_x^3 - \frac{1}{4} u_x \sin^2 u \end{pmatrix} \lambda^{-\frac{3}{2}} + \ldots. \]
For known \( T \) and \( h \) the series \( S \) is defined as follows:
\[ S = T^{-1} GT - T^{-1} T_t = \sum_{j=-3}^{\infty} S_j \lambda^{-j/2} \]
(4.7)
due to the expansion of \( G \) around \( \lambda = \infty \): \( G = \sum_{j=-3}^{\infty} G_j \lambda^{-j/2} \). It can be verified that the coefficients of the \( S \) series are diagonal:
\[ S = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \lambda^{\frac{1}{2}} + \begin{pmatrix} S_{1,11} & 0 \\ 0 & -S_{1,11} \end{pmatrix} \lambda^{-\frac{1}{2}} \]
\[ + \begin{pmatrix} S_{2,11} \\ 0 \end{pmatrix} \lambda^{-1} + \begin{pmatrix} S_{3,11} \\ 0 \end{pmatrix} \lambda^{-\frac{3}{2}} + \ldots. \]
where

\[ S_{1,11} = \frac{1}{16} u_x^2 (14 \sin^2 u - 3 u_x^2 - 4) - \frac{1}{2} u_x u_{xxx} + \frac{1}{4} u_x (u_{xx} + \sin 2u) - \frac{3}{16} \sin^4 u, \]
\[ S_{2,11} = \frac{1}{4} u_u u_{xxx} - \frac{3}{8} u_x u_{xx} \sin^2 u + \frac{3}{8} u_x^2 u_{xx} - \frac{3}{8} u_x^3 \sin 2u, \]
\[ S_{3,11} = -\frac{1}{4} u_x u_{xxxx} + \frac{1}{4} u_x u_{xxx} - \frac{1}{4} u_x^2 + \frac{1}{8} (13 \sin^2 u - 4 - 5 u_x^2) u_x u_{xxx} \]
\[ + \frac{1}{8} (2 - 3 u_x^2 - 5 \sin^2 u) u_x^2 + \frac{1}{16} \sin^6 u + \frac{1}{8} (7 u_x^2 - 5 \sin^2 u) u_x \sin 2u \]
\[ - \frac{1}{16} u_x^6 + \frac{1}{4} (3 \cos^2 u - \frac{1}{8} \sin^2 u - 1) u_x^4 - \frac{1}{16} (23 \sin^2 u - 12) u_x^2 \sin^2 u. \]

The consistency condition of the system (4.3)

\[ D_h = D_S \]

(4.8)

shows that \( h \) and \( S \) are generating functions for the local conservation laws. Equation (4.8) generates an infinite series of the conservation laws for the equation (3.1). We give two of them in an explicit form

\[ D_t (u_x^2 - \sin^2 u) = D_t \left( 2 u_x u_{xxx} - u_x^2 - u_x \sin 2u + \frac{3}{4} u_x^4 + u_x^2 - \frac{7}{2} u_x^2 \sin^2 u + \frac{3}{4} \sin^4 u \right), \]
\[ D_t (4u_x^2 + 10u_x^2 \sin^2 u - 4u_x^2 - u_x^4 - \sin^4 u) \]
\[ = D_t \left( 8u_x u_{xxxx} - 4u_x^2 - 4u_x (u_x^2 - 5 \sin^2 u + 2) u_x + \sin^6 u \right) \]
\[ + 4 (3u_x^2 - 4 \sin^2 u + 1) u_x^4 - 2 \sin 2u (\sin^2 u + 5u_x^2) u_x \]
\[ + (11 \sin^2 u - 4 - u_x^4) u_x^4 - (23 \sin^2 u - 12) u_x^2 \sin^2 u. \]

In a similar way, we can investigate the Lax pair (3.36) and (3.37). Here, we give only two local conservation laws of the equation (3.2), evaluated by the same method of the formal series

\[ D_t \left( \frac{u_x}{u_x \sqrt{u_x^2 + 1}} + 1 \right) = D_t \left( \frac{u_x}{u_x \sqrt{u_x^2 + 1}} + \frac{3(u_x^2 - 1)u_x^3}{(u_x^2 + 1)^{3/2}} + \frac{3u_x u_{xx}}{2 \sqrt{u_x^2 + 1}} \right), \]
\[ D_t \left( \frac{u_x^2 - u_x^4}{u_x^2 + 1} \right) = D_t \left( \frac{4u_x u_{xx}}{(1 + u_x^2)^2} + 2u_x \right) u_{xx} + \frac{(9u_x^2 - 11)u_x^4}{4(1 + u_x^2)} + \frac{(11u_x^2 + 2)u_x^4}{2(1 + u_x^2)} - \frac{3}{4} \sin^4 u. \]

5. Conclusions

The problem of seeking the Lax pair for integrable equations has been studied by many researchers (see, for instance [7, 15–22]). In this article, we discussed a direct algorithm for constructing the Lax pairs of some given integrable equations. The essence of the algorithm is in computing the quadratic form, which defines an invariant manifold for the linearized equation. Let us briefly comment on the main steps of the method:
• linearize the given equation and simplify, if possible, the linearized equation by a properly chosen linear substitution;
• find a quadratic form, consistent with the linearized equation. As a result one finds a nonlinear Lax pair for the given equation;
• look for a transformation reducing the obtained nonlinear Lax pair to a linear one. Usually, this transformation is defined by another quadratic form.

We illustrated the efficiency of the algorithm by applying it to the well-studied models (1.6), (2.13) and (2.30). By using the algorithm we also obtained the Lax pairs for equations (3.1) and (3.2). In the last section 4, by constructing formal eigenfunctions and the conservation laws we demonstrated that the found Lax pairs are not fake.

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