Milan Tvrdý

Boundary value problems for generalized linear integrodifferential equations with left-continuous solutions

Časopis pro pěstování matematiky, Vol. 99 (1974), No. 2, 147–157

Persistent URL: http://dml.cz/dmlcz/117837

Terms of use:

© Institute of Mathematics AS CR, 1974

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.
BOUNDARY VALUE PROBLEMS FOR GENERALIZED LINEAR
INTEGRODIFFERENTIAL EQUATIONS WITH
LEFT-CONTINUOUS SOLUTIONS

MILAN TVRDÝ, Praha
(Received November 9, 1972)

Introduction. In [11] the boundary value problem

\[ \frac{dx}{d\tau} = D \left[ A(t) x + \int_a^b d_s G(t, s) x(s) + f(t) \right], \]
\[ \int_a^b dL(s) x(s) = l \quad (-\infty < a < b < \infty) \]

was treated. In particular, theorems analogous to the well-known Fredholm theorems from the theory of integral equations were proved. The first of the above equations is a generalized ordinary differential equation in the sense of J. Kurzweil [6]. An \( n \)-vector function \( x(t) \) is said to be its solution on \( [a, b] \) if for any \( t \in [a, b] \)

\[ x(t) = x(a) + \int_a^t dA(s) x(s) + \int_a^b d_s (G(t, s) - G(a, s)) x(s) + f(t) - f(a). \]

The principal assumption was that of the regularity of \( A(t) \) on \( [a, b] \) (i.e. \( A(a+) = A(a), \ A(b-) = A(b) \) and \( A(t+) + A(t-) = 2A(t) \) for all \( a < t < b \)). In this note the analogous investigation of the case of left-continuous \( A(t) \) is carried out. Before treating the general problem we also generalize the results of [11] concerning the two-point problem

\[ \frac{dx}{d\tau} = D [A(t) x + f(t)], \quad M x(a) + N x(b) = l. \]

For motivations of the study of such problems and for more detailed bibliography see [11]. Let us note furthermore that in [10] J. Taufert developed numerical methods for solving some boundary value problems (taken from the technical practise) which are very close to those we are going to study here. Some of them (or at least some
important special cases) even can be reduced to boundary value problems for generalized ordinary differential equations. For example, the following (interface) problem occurs in [10]. To find a vector function $x(t)$ piecewise absolutely continuous on $[a, b]$ such that

$$\begin{align*}
\dot{x} &= A(t) x + f(t) \quad \text{a.e. on } [a, b], \\
x(t_i^+) &= W_i x(t_i^-) + w_i \quad (i = 1, 2, \ldots, p), \\
M x(a) + N x(b) &= l.
\end{align*}$$

($A$ and $f$ are $L$-integrable on $[a, b]$, $M, N$ and $W_i$ are constant matrices, $w_i$ and $l$ are constant vectors, $a \leq t_1 < t_2 < \ldots < t_p \leq b$.) Putting

$$B(t) = \int_{a}^{t} A(s) \, ds \quad \text{for } a \leq t \leq t_1,$$

$$B(t) = \int_{a}^{t} A(s) \, ds + (W_1 - I) \quad \text{for } t_1 < t \leq t_2,$$

$$\ldots, B(t) = \int_{a}^{t} A(s) \, ds + (W_1 - I) + \ldots + (W_p - I) \quad \text{for } t_p < t \leq b$$

and

$$g(t) = \int_{a}^{t} f(s) \, ds \quad \text{for } a \leq t \leq t_1,$$

$$g(t) = \int_{a}^{t} f(s) \, ds + w_1 \quad \text{for } t_1 < t \leq t_2,$$

$$\ldots, g(t) = \int_{a}^{t} f(s) \, ds + w_1 + \ldots + w_p \quad \text{for } t_p < t \leq b,$$

we get an equivalent two-point boundary value problem

$$\frac{dx}{d\tau} = D[B(t) x + g(t)], \quad M x(a) + N x(b) = l.$$

Moreover, in [10] the transformation of problems with integral additional conditions to equivalent two-point boundary value problems was found independently of W. R. Jones [5].

1. **Preliminaries.** Let $-\infty < a < b < \infty$. The closed interval $a \leq t \leq b$ is denoted by $[a, b]$, its interior $a < t < b$ by $(a, b)$ and the corresponding half-open intervals by $[a, b)$ and $(a, b]$. Given a matrix $M$, $M'$ denotes its transpose, $I$ is the identity matrix, $O$ is the zero matrix. Vectors are generally considered as columns. Row vectors are denoted as transpositions of column vectors. The space of $n$-vector functions with bounded variation on $[a, b]$ is denoted by $\text{BV}_n$ and the $n$-dimensional Euclidean
space is denoted by \( \mathbb{R}^n \). Given a matrix function \( F(t) \) of bounded variation on \([a, b]\), we put \( F(a-) = F(a) \), \( F(b+) = F(b) \), \( \Delta^- F(t) = F(t) - F(t^-) \), \( \Delta^+ F(t) = F(t+) - F(t) \) and \( \Delta F(t) = F(t+) - F(t^-) \) for any \( t \in [a, b] \). The space of all functions \( F(t) \) of bounded variation on \([a, b]\) such that \( F(t+) = F(t--) = F(a) \) for all \( t \in [a, b] \) is denoted by \( N \). Integrals are always the Perron-Stieltjes ones. For our purpose the notion of the \( \sigma \)-Young integral (\( Y \)-integral), which is on the space of functions with bounded variation equivalent to the Perron-Stieltjes integral, is fully sufficient. (For the definition and basic properties of the \( Y \)-integral see [4] II 19.3. An exhaustive survey was given in [12], too.)

Without any special quotation we shall make use of the fundamental results concerning generalized linear differential equations

\[
\frac{dx}{dt} = D[A(t)x + f(t)],
\]

which were established by T. H. Hildebrandt [3] and Š. Schwabik [7]. (In particular, in [7] the case of left-continuous \( A(t) \) was intensively studied. For some details see also [9]. A detailed survey of these matters was given also in [11] and [12].)

2. Two-point problem. The subject of this section is the two-point boundary value problem (p)

\[
\begin{align*}
\frac{dx}{dt} &= D[A(t)x + f(t)] , \\
M \ x(a) + N \ x(b) &= l ,
\end{align*}
\]

where \( A(t) \) is an \( n \times n \)-matrix function of bounded variation on \([a, b]\), \( f \in \BV_n \), \( l \in \mathbb{R}^m \) and \( M, N \) are constant \( m \times n \)-matrices. We are seeking a function \( x \in \BV_n \) which is on \([a, b]\) a solution to the generalized linear differential equation (2.1) and fulfils the condition (2.2). (Let us notice that the equation (2.1) possesses only solutions with bounded variation on \([a, b]\), cf. [9].)

Any problem (p*) to find \( y \in \BV_n \) and \( \lambda \in \mathbb{R}^m \) such that \( y'(t) \) is on \([a, b]\) a solution to the generalized linear differential equation

\[
\frac{dy'}{dt} = D[-y' \ B(t)]
\]

and

\[
y'(a) = -\lambda' M , \quad y'(b) = \lambda' N
\]

with \( A - B \in N \) is said to be the adjoint of the problem (p).

Throughout the section we suppose that

\[
A - B \in N , \quad \Delta^+ B(t) \Delta^+ A(t) = \Delta^- B(t) \Delta^- A(t) \quad \text{on} \quad [a, b]
\]
and that at least one of the following conditions is satisfied

\[(2.6) \det (I - C A(t)) \det (I - C B(t)) \det (I + C A(t)) \neq 0 \text{ on } [a, b],\]
\[(2.7) \det (I - C A(t)) \det (I - C B(t)) \det (I + C B(t)) \neq 0 \text{ on } [a, b],\]
\[(2.8) \det (I + C A(t)) \det (I + C B(t)) \det (I - C B(t)) \neq 0 \text{ on } [a, b],\]
\[(2.9) \det (I + C A(t)) \det (I - C B(t)) \det (I - C A(t)) \neq 0 \text{ on } [a, b].\]

Without any loss of generality we may also assume \(A(a) = B(a) = 0\).

Let \(U(t, s)\) and \(V(t, s)\) denote the fundamental matrix solutions on \([a, b]\) to the equations (2.1) and (2.3), respectively. It means that

\[(2.10) \quad U(t, s) = I + \int_s^t [dA(\tau)] U(\tau, s)\]

for \(t, s \in [a, b]\), \(t \geq s\) if \(\det (I - C A(t)) \neq 0 \text{ on } [a, b]\) and for \(t \leq s\) if \(\det (I + C A(t)) \neq 0 \text{ on } [a, b]\). Furthermore,

\[(2.11) \quad V(t, s) = I + \int_s^t V(t, \sigma) dB(\sigma)\]

for \(t, s \in [a, b]\), \(t \geq s\) if \(\det (I + C B(t)) \neq 0 \text{ on } [a, b]\) and for \(t \leq s\) if \(\det (I + C B(t)) \neq 0 \text{ on } [a, b]\). The following lemma establishes the relation between the functions \(U\) and \(V\).

**2.1. Lemma.** Let \(A - B \in N\). If (2.6) or (2.7) holds, then for \(t, s \in [a, b]\), \(t \geq s\)

\[(2.12) \quad V(t, s) = U(t, s) + V(t, s) (A(s) - B(s)) - (A(t) - B(t)) U(t, s) +
                        + V(t, s) \Delta^+ B(s) \Delta^+ A(s) - \Delta^- B(t) \Delta^- A(t) U(t, s) +
                        + \sum_{s < \tau < t} (V(\tau, s) (\Delta^+ B(\tau) \Delta^+ A(\tau) - \Delta^- B(\tau) \Delta^- A(\tau)) U(\tau, s)).\]

If (2.8) or (2.9) holds, then for \(t, s \in [a, b]\), \(t \leq s\)

\[(2.13) \quad V(t, s) = U(t, s) + V(t, s) (A(s) - B(s)) - (A(t) - B(t)) U(t, s) +
                        + V(t, s) \Delta^- B(s) \Delta^- A(s) - \Delta^+ B(t) \Delta^+ A(t) U(t, s) +
                        + \sum_{s < \tau < t} (V(\tau, s) (\Delta^- B(\tau) \Delta^- A(\tau) - \Delta^+ B(\tau) \Delta^+ A(\tau)) U(\tau, s)).\]

**Proof.** Let e.g. (2.6) be satisfied. Let \(t, s \in [a, b]\), \(t \geq s\). By the substitution theorem for Y-integrals and by (2.10) and (2.11)

\[Q = \int_s^t [dV(t, \tau)] U(\tau, t) + \int_s^t V(t, \tau) dU(\tau, t) = \int_s^t V(t, \tau) [d(A(\tau) - B(\tau))] U(\tau, t) =
                        = V(t, s) (\Delta^+ A(s) - \Delta^+ B(s)) U(s, t) + (\Delta^- A(t) - \Delta^- B(t)) =
                        = -V(t, s) (A(s) - B(s)) + (A(t) - B(t)).\]
On the other hand, according to the integration-by-parts theorem

\[ Q = I - V(t, s) U(s, t) - \Delta^+_t V(t, s) \Delta^+_t U(s, t) + \Delta^-_t V(t, s) \Delta^-_t U(s, t) + \]

\[ + \sum_{s < \tau < t} (\Delta^+_t V(t, \tau) \Delta^-_t U(\tau, t) - \Delta^-_t V(t, \tau) \Delta^+_t U(\tau, t)), \]

where \( \Delta^+_t U(t, s) = U(t +, s) - U(t, s) \), \( \Delta^-_t U(t, s) = U(t, s+) - U(t, s) \). It follows readily from (2,10) and (2,11) and from the properties of the Y-integral as a special kind of the Kurzweil integral ([6], Theorem 1, 3, 6; cf. also [4], [7], [9] or [12]) that

\[ \Delta^+_t V(t, s) = -V(t, s) \Delta^+_t U(t, s), \quad \Delta^-_t U(t, s) = \Delta^+_t V(t, s), \]

\[ \Delta^-_t V(t, s) = -V(t, s) \Delta^-_t U(t, s), \quad \Delta^-_t U(t, s) = \Delta^-_t V(t, s). \]

Hence the relation (2,12) immediately follows.

The other cases can be treated similarly. If (2,8) or (2,9) holds, then instead of the expression \( Q \) we should handle the expression

\[ \int_s^t [d_s V(s, \tau)] U(\tau, s) + \int_s^t V(s, \tau) d_s U(\tau, s). \]

The variation-of-constants formula yields the following

2.2. Lemma. Let (2,5) hold. If (2,6) or (2,7) holds then an n-vector function \( x(t) \) is a solution of (p) iff it is on \([a, b]\) given by

\[ x(t) = U(t, a) c + f(t) - f(a) - \int_a^t [d_s U(t, s)] (f(s) - f(a)), \]

where \( c \in \mathbb{R}^n \) is a solution to the linear equation

\[ [M + NV(a, b)] c = l + M \left\{ -f(a) + V(a, b) f(b) - \int_a^b [d_s V(a, s)] f(s) \right\}. \]

If (2,8) or (2,9) holds, then any solution \( x(t) \) to (p) is of the form

\[ x(t) = U(t, b) c + f(t) - f(b) + \int_t^b [d_s U(t, s)] (f(s) - f(b)), \]

where

\[ [MV(a, b) + N] c = l + M \left\{ -f(a) + V(a, b) f(b) - \int_a^b [d_s V(a, s)] f(s) \right\}. \]

Proof. Putting the variation-of-constants formula (2,14) (valid if (2,6) or (2,7) holds) into (2,2) we get that \( x(t) \) is a solution to (p) iff

\[ [M + NU(b, a)] c = l + N \left\{ U(b, a) f(a) - f(b) + \int_a^b [d_s U(b, s)] f(s) \right\}. \]
Since by (2,12)
\[ V(b, s) = U(b, s) + V(b, s) (A(s) - B(s)) + V(b, s) A + A(s) \]
for \( s \in [a, b] \), \( W(s) = V(b, s) - U(b, s) \in N \) and \( V(b, a) = U(b, a) \). Thus
\[ \int_a^b [dV(b, s)] u(s) = \int_a^b [dU(b, s)] u(s) \]
for any \( u \in BV_n \). (In fact, \( W(a) = W(a+) = W(t+) = W(t-) = W(b-) = W(b) = O \)
for all \( t \in (a, b) \). The continuous part \( W_c \) of \( W \) vanishes everywhere on \([a, b]\). Hence
by the definition of the Y-integral
\[ \int_a^b [dW(s)] u(s) = \int_a^b [dW_c(s)] u(s) + \lambda^+ W(a) u(a) + \lambda^- W(b) u(b) + \]
\[ + \sum_{a<s<b} \Delta W(s) u(s) = O. \]

Hence our assertion follows. The cases (2,8) and (2,9) could be treated analogously.

2.3. Lemma. Let \( \det (I - \Delta^+ B(t)) \neq 0 \) on \([a, b]\) (or \( \det (I + \Delta^- B(t)) \neq 0 \) on \([a, b]\)). Then a couple \( (y(t), \lambda) \) is a solution to (p*) iff
\[ y'(t) = \lambda^+ N V(b, t) \]  
(2,18) \[ \lambda^+ [M + N V(b, a)] = O \]  
for \( (2,6)-(2,9) \) be satisfied. Then the problem (p) possesses a solution iff
\[ y'(b) f(b) - y'(a) f(a) - \int_a^b [dy'(s)] f(s) = \lambda'l \]
for any solution \( (y(t), \lambda) \) of (p*).

Proof follows immediately from 2,2 and 2,3.

2.5. Remark. The assumptions (2,5) are fulfilled e.g. if
(i) \( B(t) = A(t) \) on \([a, b]\) and \((\Delta^+ A(t))^2 = (\Delta^- A(t))^2 \) on \([a, b]\) (this case was studied in [11]),
(ii) \( B(a) = A(a), \) \( B(t) = A(t+) \) for \( t \in (a, b) \), \( B(b) = A(b) \) \( (B = A^*) \), \( (\Delta^+ A(a))^2 = (\Delta^- A(b))^2 = 0 \) and \( \Delta^+ A(t) \Delta^- A(t) = \Delta^- A(t) \Delta^+ A(t) \) on \((a, b)\).
In particular, the assumptions of this section are fulfilled if e.g.
a) \( A, f \) are left continuous on \([a, b]\) and right-continuous at \(a\), \( \det (I + \Delta^+ A(t)) \neq 0 \) on \([a, b]\) and \( B = A^* \), or
b) \( A, f \) are regular on \([a, b]\), \( \det (I - (\Delta^+ A(t))^2) \neq 0 \) on \([a, b]\) and \( B = A \), or
c) \( A, f \) are regular on \([a, b]\), \( (\Delta^+ A(t))^2 = O \) on \([a, b]\) and \( B = A \), or
d) \( A, f \) are right-continuous on \([a, b]\) and left-continuous at \(b\), \( \det (I - \Delta^- A(t)) \neq 0 \) on \([a, b]\) and \( B = A^* \).

(Only the cases b) and c) are included in [11].)

2.6. Remark. Of course, also Theorems 2,1, 2,3 and 2,4 of [11] are valid under the assumptions of this section.

3. General problem. The subject of this section is the boundary value problem

\begin{equation}
(P) \quad \text{to find } x \in \text{BV}_n \text{ which is on } [a, b] \text{ a solution to the equation}
\end{equation}

\begin{equation}
\frac{dx}{dt} = D \left[ A(t) x + C(t) x(a) + D(t) x(b) + \int_a^b [dG(t, s)] x(s) + f(t) \right]
\end{equation}

and fulfills the condition

\begin{equation}
M x(a) + N x(b) + \int_a^b [dL(s)] x(s) = l .
\end{equation}

We assume that

\begin{equation}
A(t), C(t), D(t) \text{ are } n \times n \text{-matrix functions of bounded variation on } [a, b],
\end{equation}

\( f \in \text{BV}_n \), \( G(t, s) \) is an \( n \times n \)-matrix function of strongly bounded variation on \([a, b] \times [a, b]\), \( L(t) \) is an \( m \times n \)-matrix function of bounded variation on \([a, b]\), \( M \) and \( N \) are constant \( m \times n \)-matrices and \( l \in \mathbb{R}_m \)

and

\begin{equation}
A, C, D, f \text{ and } G(\cdot, s) \text{ are for an arbitrary } s \in [a, b] \text{ left-continuous on } (a, b) \text{ and right-continuous at } a, \text{ while } \det (I + \Delta^+ A(t)) \neq 0 \text{ on } [a, b].
\end{equation}

(A matrix function \( K(t, s) \) is said to be of strongly bounded variation on \([a, b] \times [a, b]\) if it is of bounded two-dimensional Vitali variation \( v(K) \) on \([a, b] \times [a, b]\) and \( \text{var}_a^b K(a, \cdot) + \text{var}_a^b K(\cdot, a) < \infty \). For details see [4] III,4 and also [8] or [11].)

Without any loss of generality we may assume further that

\begin{equation}
G(t, \cdot) \text{ and } L \text{ are for any } t \in [a, b] \text{ right-continuous on } [a, b] \text{ and left-continuous at } b, \text{ while } G(a, \cdot) = O \text{ on } [a, b], \text{ } C(a) = D(a) = O, \text{ } L(a) = O \text{ and } f(a) = O.
\end{equation}

Let us put

\begin{equation}
B(t) = A(t+) \text{ on } [a, b] \text{ (} B(b) = A(b+) = A(b) \text{).}
\end{equation}
Then $B$ is right-continuous on $[a, b)$ and left-continuous at $b$, while $\Delta^+ B(t) = \Delta^- A(t)$ for $t \in [a, b]$. Evidently $A - B \in N$. Let $U$ and $V$ be again the fundamental matrix solutions to the equations

$$\frac{dx}{d\tau} = D[A(t)x] \quad \text{and} \quad \frac{dy}{d\tau} = D[-y \cdot B(t)],$$

respectively.

3.1. **Remark.** Let $t, s \in [a, b]$. Then by 2.1, $V(t, s) = U(t, s) - V(t, s) \Delta^+ A(s) + \Delta^+ A(t) U(t, s)$ or $(I + \Delta^+ A(t))^{-1} V(t, s) = U(t, s) (I + \Delta^+ A(s))^{-1}$. Since $U(t, s)$.

$(I + \Delta^+ A(s))^{-1} = U(t, s+)$ (cf. Theorem 2 in [9]) and analogously $(I + \Delta^- B(t))^{-1}$.

$V(t, s) = V(t-, s)$, we have

$$(3.7) \quad V(t-, s) = U(t, s+) \quad \text{for all} \quad t, s \in [a, b].$$

Furthermore,

$$(3.8) \quad V(b-, s) = V(b, s) \quad \text{and} \quad U(t, a+) = U(t, a) \quad \text{for all} \quad t, s \in [a, b].$$

(Hence $V(b, a) = U(b, a).$)

3.2. **Lemma.** Given $g \in BV_n$ right-continuous on $[a, b)$ and $c \in \mathbb{R}^n$, the unique solution $y(t)$ of

$$\frac{dy}{d\tau} = D[-y \cdot B(t) + g'(t)]$$

on $[a, b]$ such that $y(b) = c$ is on $[a, b]$ given by

$$y'(t) = cV(b, t) - \int_t^b [dg'(s)] V(s-, t).$$

(The proof is similar to that of the analogous assertion for the equation (2.1) given in [7].)

3.3. **Definition.** The problem $(P^*)$ of finding a couple $(y(t), \lambda) \in BV_n \times \mathbb{R}_m$ such that $y(t)$ is on $[a, b]$ a solution to

$$(3.9) \quad \frac{dy}{d\tau} = D \left[-y \cdot B(t) - \lambda' L(t) - \int_a^b y'(s) \, ds G(s, t) \right]$$

and

$$(3.10) \quad y'(a) + \lambda' M + \int_a^b y'(s) \, dC(s) = O,$$

$$y'(b) - \lambda' N - \int_a^b y'(s) \, dD(s) = O$$

is called the adjoint boundary value problem to the problem $(P)$. 

154
3.4. Remark. Any solution \(x(t)\) of (3,1) is left-continuous on \((a, b]\) and right-continuous at \(a\), while any solution \(y(t)\) of (3,9) is right-continuous on \([a, b)\) and left-continuous at \(b\).

3.5. Theorem. Let (3,3)—(3,6) hold. The given problem \((P)\) has a solution iff

\[ \int_a^b y'(s) \, df(s) = \lambda' I \]

for any solution \((y(t), \lambda)\) of its adjoint \((P^*)\).

Proof. Sufficiency. An \(n\)-vector function \(x \in \text{BV}_n\) is a solution of (3,1) on \([a, b]\) such that \(x(a) = c \in \mathbb{R}^n\) iff it is on \([a, b]\) given by

\[ x(t) = U(t, a) c + \int_a^t U(t, s+) \, dh(s) + \int_a^t U(t, s+) \, df(s), \]

where

\[ h(t) = C(t) x(a) + D(t) x(b) + \int_a^b [d_x G(t, s)] x(s) \]

has bounded variation on \([a, b]\), is left-continuous on \((a, b]\) and right-continuous at \(a\) and vanishes at \(a\) \((h \in \text{BV}_n)\). Putting (3,11) into (3,12) and (3,2) and making use of the Dirichlet formula we find that the given problem \((P)\) is equivalent to the system of equations for \((h(t), c) \in \text{BV}_n \times \mathbb{R}^n\)

\[ h(t) + P(t) c + \int_a^b K(t, s) \, dh(s) = u(t) \quad \text{on} \quad [a, b], \]

\[ Qc + \int_a^b R(s) \, dh(s) = v, \]

where for \(t, s \in [a, b]\)

\[ P(t) = - \left\{ C(t) + D(t) U(b, a) + \int_a^b [d_x G(t, \sigma)] U(\sigma, a) \right\}, \]

\[ R(t) = NU(b, t+) + \int_t^b [dL(\sigma)] U(\sigma, t+), \]

\[ K(t, s) = - \left\{ D(t) U(b, s+) + \int_s^b [d_x G(t, \sigma)] U(\sigma, s+) \right\}, \]

\[ u(t) = - \int_a^b K(t, \sigma) \, df(\sigma), \quad v = l - \int_a^b R(\sigma) \, df(\sigma). \]

(If a couple \((h(t), c)\) is a solution to (3,13) and if \(x(t)\) is given by (3,11), then \(x(t)\) is a solution to \((P)\). Conversely, if \(x(t)\) is a solution to \((P)\) and \(c = x(a)\) and \(h(t)\) is given by (3,12), then a couple \((h(t), c)\) is a solution to (3,13).) Obviously, \(P, R\) and \(u\) have
bounded variation on \([a, b]\), while \(P\), \(u\) and \(K(., s)\) are for any \(s \in [a, b]\) left-continuous on \((a, b]\) and right-continuous at \(a\) and vanishing at \(a\). Moreover, by Lemma 5.1 of \([11]\), \(K(t, s)\) is of strongly bounded variation on \([a, b]\) \(\times [a, b]\). Analogously as in Lemma 4.1 in \([11]\) we can show that the system (3.13) possesses a solution iff

\[
\int_{a}^{b} \chi'(s) \, du(s) + \gamma'v = 0
\]

for any solution \((\chi(t), \gamma) \in BV_n \times R_m\) of the system adjoint to (3.13)

\[
\chi'(t) + \frac{\gamma'}{\gamma'} R(t) + \int_{a}^{b} \chi'(s) \, d_{\gamma}K(s, t) = 0 \quad \text{on} \quad [a, b],
\]

\[
\gamma'Q + \int_{a}^{b} \chi'(s) \, dP(s) = 0.
\]

Inserting (3.14) into (3.15), we get by Proposition 2.4 of \([8]\)

\[
O = \gamma' l - \int_{a}^{b} \left\{ \gamma' R(t) + \int_{a}^{b} \chi'(s) \left[ d_{\gamma}K(s, t) \right] \right\} df(t)
\]

or by (3.16),

\[
O = \gamma' l + \int_{a}^{b} \chi'(t) \, df(t).
\]

Let \((\chi(t), \gamma)\) be an arbitrary solution of (3.16). We shall show that then the couple \((\chi(t), -\gamma)\) is a solution to (P*) and thus we shall complete the first part of the proof. Since by Prop. 2.4 of \([8]\)

\[
\int_{a}^{b} \chi'(s) \left[ d_{\gamma} \left\{ \gamma' \left[ d_{\gamma}G(s, \sigma) \right] U(\sigma, t+ \right\} = \int_{t}^{b} \left[ d_{\gamma} \int_{a}^{b} \chi'(s) \left[ d_{\gamma}G(s, \sigma) \right] \right] U(s, t+),
\]

we have according to (3.7) and (3.8)

\[
\chi'(t) = -\gamma' R(t) - \int_{a}^{b} \chi'(s) \, d_{\gamma}K(s, t) =
\]

\[
\left\{ -\gamma'N + \int_{a}^{b} \chi'(s) \, dD(s) \right\} V(b, t) - \int_{t}^{b} \left[ d_{\gamma} \left\{ \gamma' L(s) - \int_{a}^{b} \chi'(s) \left[ d_{\gamma}G(\sigma, s) \right] \right\} \right] V(s-, t)
\]

for \(t \in [a, b]\). Hence by 3.2 \(\chi(t)\) is a solution of (3,9) on \([a, b]\) such that

\[
\chi'(b) = -\gamma'N + \int_{a}^{b} \chi'(s) \, dD(s).
\]

Finally, by (3.16)\(_2\)

\[
\chi'(a) = -\gamma'NU(b, a) + \int_{a}^{b} \chi'(s) \, dD(s) \, U(b, a) - \gamma' \int_{a}^{b} \left[ dL(s) \right] U(s, a) +
\]

156
\[ + \int_{a}^{b} \chi'(s) \left[ d_s \int_{a}^{b} [d_{G(s, \sigma)} U(\sigma, a)] \right] = \gamma' M - \int_{a}^{b} \chi'(s) \, dC(s). \]

The second part of the proof (necessity) is identical with the second part of the proof of Theorem 5,1 in [11].

3.6. Remark. Analogously we could prove that also Theorems 5,2 and 5,3 from [11] are valid under the assumptions of this section.

Corrigendum to [11]. The lines 20110—11 and 2039 in [11] should read

"Let \( T : \mathcal{X} \to \mathcal{X} \) be a completely continuous operator and let the operator \( T' : \mathcal{Y} \to \mathcal{X}^* \) be completely continuous and such that \( y(Tx) = T'y(x) \) for all \( x \in X \) and \( y \in \mathcal{Y} \) and \( T'(\mathcal{Y}) \subseteq \mathcal{Y} \)..."

and

"Since by Theorems 3,1 and 3,2 of [14] \( T \) and \( T' \) are completely continuous and by Lemma 5,1 of [14] ...".

References

[1] Bryan R. N.: A nonhomogeneous linear differential system with interface conditions, Proc. A.M.S. 22 (1969), 270—276.
[2] Halanay A. and Moro A.: A boundary value problem and its adjoint, Ann. Mat. pura appl. 79 (1968), 399—412.
[3] Hildebrandt T. H.: On systems of linear differential-Stieltjes-integral equations, Ill. J. Math. 3 (1959), 352—373.
[4] Hildebrandt T. H.: Introduction to the Theory of Integration, Academic Press, New York, London, 1963.
[5] Jones W. R.: Differential systems with integral boundary conditions, Journ. Diff. Eq. 3 (1967), 191—202.
[6] Kurzweil J.: Generalized ordinary differential equations and continuous dependence on a parameter, Czech. Math. J. 7 (82) (1957), 418—449.
[7] Schwabik Št.: Verallgemeinerte lineare Differentialgleichungssysteme, Čas. pěst. mat. 96 (1971), 183—211.
[8] Schwabik Št.: On an integral operator in the space of functions with bounded variation, Čas. pěst. mat. 97 (1972), 297—330.
[9] Schwabik Št., Tordý M.: On generalized linear differential equations, Čas. pěst. mat. 98 (1973), 206—211.
[10] Tauer J.: Lösung der Randwertprobleme für Systeme von linearen Differentialgleichungen, Rozpravy ČSAV, Řada mat. a přir. věd, 83 (1973), 5, Academia, Praha.
[11] Tordý M.: Boundary value problems for linear generalized differential equations and their adjoints, Czech. Math. J. 23 (98), (1973), 183—217.
[12] Vejvoda O., Tordý M.: Existence of solutions to a linear integro-boundary-differential equation with additional conditions, Ann. Mat. pura appl. 89 (1971), 169—216.

Author’s address: 115 67 Praha 1, Žitná 25, (Matematický ústav ČSAV).