In a recent paper we have analyzed the Spinor Theory of Gravity (STG) which is based on the intimate relation between Fermi (weak) interaction and gravity. We presented the hypothesis that the effect of matter upon the metric that represents gravitational interaction in General Relativity is an effective one. This lead us to consider gravitation to be the result of the interaction of two neutral spinorial fields (G-neutrinos) $\Psi_g$ and $\Omega_g$ with all kinds of matter and energy through the generation of such effective metric. In other words, the universal metric that represents gravitational interaction in the framework of General Relativity is constructed with the weak currents associated to $\Psi_g$ and $\Omega_g$. In the first paper we have shown that when only one spinor exists, the effective metric of a static and spherically symmetric configuration is identical to the Schwarzschild geometry of GR. In the present paper we go one step further and consider the case in which the field $\Psi_g$ has a self-interaction. The solution of a static and spherically symmetric configuration is distinct from the previous one. This new solution presents another horizon that we compare with the case of Schwarzschild.
I. INTRODUCTION

The main ideas of the spinor theory of gravity (STG) \[1\] can be synthesized in the following:

1. Gravity is represented by an effective metric that is constructed in terms of physical fields;

2. To implement this proposal we assume that in the flat Minkowski space-time there exists two fundamental spinorial fields, the G-neutrinos \( \Psi_g \) and \( \Omega_g \). Thus all operations concerning tensorial properties related to these fields are controlled by the Minkowski metric \( \gamma_{\mu \nu} \) in an arbitrary coordinate system;

3. These fields satisfy the Dirac-Heisenberg equation of motion. The covariant derivative of these spinors is controlled by a scalar field \( H \).

4. \( \Psi_g \) and \( \Omega_g \) interact via Fermi process;

5. Both fields couple with all forms of matter in an universal way that is represented by a modification of the metric of space-time, according to the main ideas of General Relativity (GR);

6. The effective metric \( g_{\mu \nu} \) does not have a dynamics by its own but inherits the dynamics of the fundamental fields \( \Psi_g \) and \( \Omega_g \).

II. THE GRAVITATIONAL NEUTRINOS \( \Psi_g \) AND \( \Omega_g \)

The STG assume the hypothesis that \( \Psi_g \) and \( \Omega_g \) generate an effective metric which is the one dealing in GR. In other words, the proposal of GR that gravity deals with metric’s modification of space-time has a substratum that we identify with these G-neutrinos.

The dynamics of \( \Psi_g \) and \( \Omega_g \) is given by the Dirac equation plus two kinds of interaction: a weak current-current vectorial form \( (\overline{\Psi}_g \gamma_\mu (1 - \gamma_5) \Omega_g)(\overline{\Omega}_g \gamma^\mu (1 - \gamma_5) \Psi_g) \) and a scalar-scalar of the form \( (\overline{\Psi}_g \Omega_g)(\overline{\Omega}_g \Psi_g) + (\overline{\Psi}_g \gamma_5 \Omega_g)(\overline{\Omega}_g \gamma_5 \Psi_g) \) besides self-interaction terms of the same form. In the present work we continue the analysis where only one of the fields exists and consider the scalar-scalar self-interaction similar to Heisenberg’s dynamics which is provided by the Lagrangian \( A^2 + B^2 = J_{\mu} J^\mu \) \[2, 3\], that is

\[
 i\gamma^\mu \nabla_\mu \Psi_g - 2s (A + iB \gamma^5) \Psi_g = 0 \tag{1}
\]

where

\[
 \nabla_\mu \Psi = \partial_\mu \Psi - \Gamma_\mu \Psi \tag{2}
\]

the constant \( s \) has the dimension of (length)^2 and the quantities \( A \) and \( B \) are given by

\[
 A \equiv \overline{\Psi} \Psi, \quad B \equiv i\overline{\Psi} \gamma^5 \Psi. \tag{3}
\]
We use the system of units such that $\hbar = c = 1$. Thus the dimensionality of Fermi constant and Newton constant are the same and equal to $L^2$. The internal connection contains beyond the conventional Fock-Ivanenko term an additional one driven by a scalar field, that is

$$\Gamma_\mu = \Gamma_{FI}^\mu + U_\mu \tag{4}$$

The origin of this term, which guarantees the Riemannian nature of the metric, was described in the previous paper. The Fock-Ivanenko connection is

$$\Gamma_{FI}^\mu = -\frac{1}{8} \left( \gamma^\nu \partial_\mu \gamma_\nu - \partial_\mu \gamma_\nu \gamma^\nu - \Gamma_\alpha^{\mu \nu} (\gamma^\nu \gamma_\alpha - \gamma_\alpha \gamma^\nu) \right) \tag{5}.$$

We remind the form of the covariant derivative of the $\gamma$’s

$$\gamma_{\nu;\mu} = \partial_\mu \gamma_\nu - \Gamma_\alpha^{\mu \nu} \gamma_\alpha + \gamma_\nu \Gamma_\mu - \Gamma_\mu \gamma_\nu \tag{6}$$

The metric $\gamma_{\mu \nu}$ is defined from the fundamental objects, by

$$\gamma_{\mu \nu} = \frac{1}{2} (\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu),$$

which is a multiple of the identity of the Clifford algebra.

In the case we assume $\gamma_{\nu;\mu} = 0$ a rather simple calculation yields the Fock-Ivanenko connection eq. (5). However, the Riemannian condition allows for a more general expression for the internal connection (see eq. 4) where $U_\mu$ besides to be a vector is an arbitrary object from the associated Clifford algebra. The simplest cases that we deal with here is provided by the formula

$$U_\mu = \frac{1}{4} \gamma_\mu \gamma^\alpha H_{\alpha} \tag{7}$$

where $\Box H = 0$ and a comma means simple derivative.

III. FERMI INTERACTION AND GRAVITY

The weak interaction deals with three kinds of neutrinos. In the gravitational process we deal with two fundamental distinct kind of massless spinorial fields. We analyze the consequences of assuming the hypothesis that $\Psi_g$ and $\Omega_g$ interacts universally with all matter through the modification of the geometry of the space-time according to the main principle of General Relativity.

In agreement with the framework suggested by Stuckelberg’s remark we define the gravitational metric $g_{\mu \nu}$ in terms of the vectors $\Delta_\mu$ and $\Pi_\mu$ constructed as a combination of the weak current $\Psi_g \gamma^\mu (1 - \gamma_5) \Psi_g$ and $\Omega_g \gamma^\mu (1 - \gamma_5) \Omega_g$, that is, we set

$$g_{\mu \nu} = \eta_{\mu \nu} - \kappa h_{\mu \nu} \tag{8}$$

where

$$h_{\mu \nu} = \Delta_\mu \Delta_\nu + \Pi_\mu \Pi_\nu + \Delta_\mu \Pi_\nu + \Delta_\nu \Pi_\mu. \tag{9}$$
Let us remind that this is an exact form. It is not an approximation. By dimensionality argument we set the null-vectors defined in terms of the vector and axial currents of the spinorial fields, that is, the vectors $\Delta_\mu$ and $\Pi_\mu$ of dimensionality $L^{-1}$:

$$\Delta_\mu = (J_\mu - I_\mu) \left( \frac{g_w}{j^2} \right)^{1/4}.$$  

(10)

$$\Pi_\mu = (j_\mu - i_\mu) \left( \frac{g_w}{j^2} \right)^{1/4}.$$  

(11)

where the currents $j_\mu$ and $i_\mu$ are defined in the same way as for the case of $\Psi_g$, that is $j_\mu = \Omega g \gamma_\mu \gamma_5 \Omega g$, $i_\mu = \Omega g \gamma_\mu \gamma_5 \gamma_0 \Omega g$, $j^2 = j^\alpha j^\beta \eta_{\alpha\beta}$ and $J^2 = J^\alpha J^\alpha$.

In the expression of the effective metric eq. (8) we use the analogy of the current-current interaction and substitute $g_w$ by $\kappa$ according to [1]. We then have

$$h_{\mu\nu} h^\kappa_\lambda = 2 \Delta_\nu \Pi^\nu h_{\mu\lambda}$$

which allows to write the inverse contra-variant metric as the exact form

$$g^{\mu\nu} = \eta^{\mu\nu} + \frac{\kappa}{(1 - 2\kappa \Delta_\alpha \Pi^\alpha)} h^{\mu\nu}.$$  

(12)

Let us analyze the case in which only one field is non-null.

**IV. THE GRAVITATIONAL FIELD OF A SPHERICALLY SYMMETRIC AND STATIC CONFIGURATION**

The effect of the interaction of the G-neutrino with all kind of matter is what we call gravity. Following the path open by General Relativity this is realized by the substitution of the flat background metric into the curved one controlled by $\Psi_g$ and $\Omega_g$. In the previous paper we have solved the problem of the gravitational field of a static and spherically symmetric configuration in the case only one field $\Psi_g$ was excited obeying the linear equation. Here we consider the case in which there is a self-interaction according to [1]. We set the background Minkowski metric, where the G-neutrino lives, in the $(t, r, \theta, \varphi)$ coordinate system

$$ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2.$$  

(13)

We write the corresponding $\gamma_\mu$ in terms of the Euclidean ones $\tilde{\gamma}_\mu$ that is

$$\gamma_0 = \tilde{\gamma}_0,$$

$$\gamma_1 = \tilde{\gamma}_1,$$

$$\gamma_2 = r \tilde{\gamma}_2,$$

$$\gamma_3 = r \sin \theta \tilde{\gamma}_3.$$
In [1] we exhibited our convention on the constant gamma’s. The unique non-identically zero Fock-Ivanenko coefficients are given by

\[ \Gamma^{FI}_2 = -\frac{1}{2} \tilde{\gamma}_1 \tilde{\gamma}_2, \]

\[ \Gamma^{FI}_3 = -\frac{1}{2} \cos \theta \tilde{\gamma}_2 \tilde{\gamma}_3 - \frac{1}{2} \sin \theta \tilde{\gamma}_1 \tilde{\gamma}_3. \]

For the extra term of the internal connection, we use eq. (7) with \( H = \ln \sqrt{r \sin \theta} \). We set the G-neutrinos as

\[ \Psi_g = f(r) \Psi^0, \]

\[ \Omega_g = f(r) \Omega^0, \]

with \( \Psi^0 \) and \( \Omega^0 \) constant spinors.

Let us set for the constant spinors the decomposition in terms of bi-spinors and write

\[ \Psi^o = \begin{pmatrix} \phi \\ \eta \end{pmatrix} \]

Besides, we set \( \sigma_1 \phi = \phi \), and \( \sigma_1 \eta = \eta \), that is

\[ \phi = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}, \]

\[ \eta = \begin{pmatrix} \beta \\ \beta \end{pmatrix}. \]

This choice yields that the unique non-identically null currents are \( J^\mu, I^\mu \) for \( \mu = 0 \) or \( \mu = 1 \), which is necessary for the compatibility of the spherical symmetry, once it follows that \( \Delta_2 = \Delta_3 = 0 \).

The dynamics provided by the Dirac-Heisenberg’s equation take the form

\[ i \left( f' + \frac{f}{2r} \right) \beta + \bar{f} f \left( M \alpha - N \beta \right) = 0 \] (14)

\[ i \left( f' + \frac{f}{2r} \right) \alpha - \bar{f} \bar{f} \left( M \beta - N \alpha \right) = 0 \] (15)

where \( M = 4 \, s \left( \alpha \alpha^* - \beta \beta^* \right) \) and \( N = 4 \, s \left( \alpha^* \beta - \beta^* \alpha \right) \) with \( f' = df/dr \).
Then it follows
\[ f' + \frac{f}{2r} + \lambda f^3 = 0 \] (16)
where \( \lambda \) is a constant related to the constant spinor \( \Psi^0 \). The solution for real \( \lambda \) is given by
\[ f = \frac{1}{\sqrt{a_0 r - 2 \lambda r \log r}} \] (17)

Using these results into equation (16) the effective gravitational metric takes the form
\[
ds^2 = \left(1 - \frac{1}{S}\right) dt^2 - \left(1 + \frac{1}{S}\right) dr^2 + \frac{2}{S} dt \, dr - r^2 d\theta^2 - r^2 \sin^2 \theta \, d\phi^2,
\] (18)
where \( S = r \left(a_0 - 2 \lambda \log r\right) \). Note that when the interaction vanishes (\( \lambda = 0 \)) the gravitational metric coincides with the Schwarzschild metric, which led to set \( a_0 = 1/2\kappa m \).

Making a coordinate change in order to eliminate the cross-term we set for the new time \( T \) the relation
\[ dt = dT - \frac{1}{S - 1} \, dr \]
to obtain the final expression
\[
ds^2 = \left(1 - \frac{1}{S}\right) dT^2 - \left(1 - \frac{1}{S}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta \, d\phi^2.
\] (19)
The horizon occurs for values of coordinate \( r = R_H \) given by
\[ \log R_H = \frac{1}{2\lambda} \left[ \frac{1}{r_H} - \frac{1}{R_H} \right] \]
where \( r_H = 2\kappa m \) is the Schwarzschild horizon.

A solution of this is given by
\[ R_H = -\frac{1}{2\lambda} \frac{1}{W(z)} \]
where
\[ z = -\frac{1}{2\lambda} e^{-1/2\lambda r_H} \]
\( W(z) \) is Lambert function defined by
\[ z = W(z) e^{W(z)} \].

We can then compare the values of the horizon \( R_H \) of STG and \( r_H \) for GR. Making an expansion of Lambert function to first order we can approximate
\[ R_H \approx 1 + \frac{1}{2\lambda r_H} \].

It then follows that, for positive values of \( \lambda \) we obtain
- For \( 0 < r_H < 1 + \sqrt{1 + 2/\lambda} \) then \( R_H > r_H \);
- \( r_H > 1 + \sqrt{1 + 2/\lambda} \) then \( R_H < r_H \).
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[5] The general case of equation (10) can be written \( \lambda = a + ib \). Then the solution for the metric is

\[
S \equiv \frac{1}{f^2} = r \left[ \left( \frac{1}{r_H} \right)^2 - 4 \frac{a}{r_H} \log r + 4(a^2 - b^2) \log^2 r \right]^{1/2}.
\]