BONDAŁ-ORŁOŚ FULLY FAITHFULNESS CRITERION FOR
DELIGNE-MUMFORD STACKS

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Abstract. Suppose $F: \mathcal{D}(X) \to \mathcal{T}$ is an exact functor from the bounded derived category of coherent sheaves on a smooth projective variety $X$ to a triangulated category $\mathcal{T}$. If $F$ possesses a right adjoint, then the Bondal-Orlov criterion gives a simple way of determining if $F$ is fully faithful. We prove a natural extension of this theorem to the case when $X$ is a smooth and proper DM stack with projective coarse moduli space.

1. Introduction

1.1. Bondal-Orlov Criterion. Suppose $X$ is a smooth projective scheme over an algebraically closed field $k$ of characteristic zero and $F: \mathcal{D}(X) \to \mathcal{T}$ is an exact functor, with $\mathcal{T}$ a triangulated category. One is often interested in checking whether $F$ embeds $\mathcal{D}(X)$ as a full triangulated subcategory of $\mathcal{T}$. If $F$ admits a right adjoint $G$, then the following well-known Bondal-Orlov Criterion is the primary tool used, [BO95, Bri99].

Theorem 1.1. The functor $F$ is fully faithful if, and only if, it admits a right adjoint $G$, with $G \circ F$ of Fourier-Mukai type, and

- for any closed point $x \in X$, one has
  $$\text{Hom}_\mathcal{T}(F(O_x), F(O_x)) = k;$$
- for any pair of closed points $x, y \in X$ one has
  $$\text{Hom}_\mathcal{T}(F(O_x), F(O_y)[i]) = 0 \text{ unless } x = y \text{ and } 0 \leq i \leq \dim(X).$$

This theorem has been extended to the quasi-projective and gerby projective setting, to the case when $X$ is allowed to have some singularities, and to the case of positive characteristic [HRLMnSs09, Ss09, LM17, Căl02].

Recent interest in derived categories of Deligne-Mumford stacks warrants an investigation of a similar criterion for this category. In this article, we extend the Bondal-Orlov criterion to the class of smooth and proper Deligne-Mumford stacks with projective coarse moduli.

In the case of stacks the notion of a $k$-point has to be replaced by that of a generalized point, $(x, \xi)$, where $x : \text{Spec}(k) \to X$ is a morphism and $\xi$ is an irreducible representation of $\text{Aut}(x)$, see Proposition 2.2. These pairs are considered up to an isomorphism. For each generalized point, $(x, \xi)$ there is a natural coherent sheaf $O_{x,\xi}$ on $X$, which is an analog of the skyscraper sheaf (see Sec. 2.2).

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Theorem 1.2. Let $\mathcal{X}$ be a smooth and proper DM-stack with projective coarse moduli space over an algebraically closed field $k$ of characteristic zero. Suppose $F: \mathcal{D}(\mathcal{X}) \rightarrow \mathcal{T}$ is an exact functor with a right adjoint $G: \mathcal{T} \rightarrow \mathcal{D}(\mathcal{X})$ such that $G \circ F$ is of Fourier-Mukai type. Then $F$ is fully-faithful if and only if

- for each generalized point $(x, \xi)$ of $\mathcal{X}$, one has
  $\text{Hom}_\mathcal{T}(F(O_{x,\xi}), F(O_{x,\xi})) = k$;

- for each pair of generalized points $x, y$, one has
  $\text{Hom}_\mathcal{T}(F(O_{x,\xi}), F(O_{y,\eta}[i])) = 0$ unless $x \simeq y$ and $0 \leq i \leq \dim(\mathcal{X})$; and
  $\text{Hom}_\mathcal{T}(F(O_{x,\xi}), F(O_{y,\eta})) = 0$ unless $(x, \xi) \simeq (y, \eta)$.

Remark 1.1. Note that the natural dg-enhancement of $\mathcal{D}(\mathcal{X})$ is saturated. Thus, in the case when the triangulated category $\mathcal{T}$ admits a dg enhancement and $F$ lifts to the dg level, the conditions that $F$ admits a right adjoint and $G \circ F$ is of Fourier-Mukai type are automatic (see [Gen17, Theorem 1.3], [BFN08, Theorem 1.2]).

1.2. Outline of the paper. The proof will proceed similarly to the proof of the original Bondal-Orlov criterion in [Huy06, Section 7.1]. We collect the relevant background material in Section 2. The key technical idea is to use the trade-off between nontrivial generic stabilizer and “gerbyness”. This was observed by Bergh-Gorchinskiy-Larsen-Lunts in [BGLL17] in the form of an equivalence of the category of $G$-equivariant coherent sheaves corresponding to an ineffective action of a finite group $G$, with some “gerby” category. We call this BGLL equivalence and recall the details in Section 3. We complete the proof in Section 4.

Conventions. Throughout $k$ will be an algebraically closed field of characteristic zero. Unless otherwise stated, our stacks will be smooth and proper over $k$ with projective coarse moduli. All functors are assumed to be derived. The bounded derived category of coherent sheaves on $\mathcal{X}$ is denoted by $\mathcal{D}(\mathcal{X})$.

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2. Preliminaries on DM Stacks and Triangulated Categories

2.1. Serre Duality. Since our stacks are smooth and proper, the exotic inverse image functor

$$p^!:\mathcal{D}(\text{Spec}(k)) \rightarrow \mathcal{D}(\mathcal{X})$$

is defined, see [Nir08], and we set $\omega_{\mathcal{X}} = p^!O_{\text{Spec}(k)}$ to be the dualizing sheaf on $\mathcal{X}$. Moreover, the associated endofunctor $S: \mathcal{D}(\mathcal{X}) \rightarrow \mathcal{D}(\mathcal{X})$ given by

$$S(\mathcal{F}) = (\mathcal{F} \otimes \omega_{\mathcal{X}})[\dim(\mathcal{X})]$$

is a Serre functor for $\mathcal{D}(\mathcal{X})$. 
2.2. **Points.** By a closed point of \( X \), we mean a morphism \( x : \mathrm{Spec}(k) \to X \). Any closed point gives rise to a closed substack \( \iota_x : \text{BAut}(x) \to X \) called the *residual gerbe* at \( x \). Here, \( \text{Aut}(x) \) is the finite stabilizer group of \( x \) and \( \text{BAut}(x) \cong \text{pt}/\text{Aut}(x) \) is the classifying stack.

For any finite group \( G \), Maschke’s Theorem gives a completely orthogonal decomposition

\[
\mathcal{D}(BG) = \bigoplus_{\xi \in \text{Irr}(G)} \mathcal{D}(\text{Spec}(k)) \otimes \xi.
\]

For any closed point, \( x : \text{Spec}(k) \to X \), and irreducible representation, \( \xi \in \text{Irr}(\text{Aut}(x)) \), we denote by \( O_{x,\xi} \) the sheaf \( \iota_x^* (O_{\text{Spec}(k)} \otimes \xi) \). We will think of the pair \( (x, \xi) \) as a *generalized point* with structure sheaf \( O_{x,\xi} \).

2.3. **Fourier-Mukai Functors.** Let \( X, Y \) be smooth and proper DM stacks of finite type over \( k \) with generically trivial stabilizers. Any object \( P \in \mathcal{D}(X \times Y) \), determines an exact functor \( \Phi_P : \mathcal{D}(X) \to \mathcal{D}(Y) \) defined by the formula

\[
\Phi_P(E) = \pi_Y^*(\pi_X^*(E) \otimes P).
\]

We will say that an exact functor \( F : \mathcal{D}(X) \to \mathcal{D}(Y) \) is of *Fourier-Mukai type* or an *integral functor* if \( F \cong \Phi_P \) for some \( P \in \mathcal{D}(X \times Y) \). Since Serre duality holds in this setting, we have the standard formulas for the left and right adjoint. Namely, let us set

\[
P_L = P^\vee \otimes \pi_X^* \omega_X \quad \text{and} \quad P_R = P^\vee \otimes \pi_Y^* \omega_Y.
\]

**Proposition 2.1.** Let \( F = \Phi_P : \mathcal{D}(X) \to \mathcal{D}(Y) \). Then \( G = \Phi_{P_L} \) and \( H = \Phi_{P_R} \) are left and right adjoint functors to \( F \), respectively.

**Example 2.1.** Let \( X \) be a DM stack, then the diagonal object \( \Delta^* O_X \in \mathcal{D}(X \times X) \) is a kernel for the identity functor

\[
\Phi_{\Delta^* O_X} \cong \text{Id} : \mathcal{D}(X) \to \mathcal{D}(X).
\]

Note that \( \Delta : X \to X \times X \) is finite and so the argument in [Huy06, Example 5.4(i)] carries over exactly.

2.4. **Spanning classes.** Recall that a *spanning class* in a triangulated category \( \mathcal{T} \) is a subclass of objects \( \Omega \subset \mathcal{T} \) such that for all \( t \in \mathcal{T} \) we have:

\[
\text{Hom}_\mathcal{T}(\omega[i], t) = 0 \quad \text{for all } i \in \mathbb{Z} \quad \text{and for all } \omega \in \Omega \quad \text{implies } t = 0;
\]

\[
\text{Hom}_\mathcal{T}(t, \omega[i]) = 0 \quad \text{for all } i \in \mathbb{Z} \quad \text{and for all } \omega \in \Omega \quad \text{implies } t = 0.
\]

In the (quasi-)projective setting, the structure sheaves of closed points form a spanning class. We need an analogue in the stacky setting. The following proposition seems to be well known and follows analogously to [Huy06]. We include the proof for completeness and for lack of a suitable reference.

**Proposition 2.2.** The subclass of objects

\[
\Omega_{\text{pt}} = \{ O_{x,\xi} \mid x : \text{Spec}(k) \to X \text{ and } \xi \in \text{Irr}(\text{Aut}(x)) \}
\]

form a spanning class in \( \mathcal{D}(X) \).
Proof. By Serre duality, it suffices to show that if \( \mathcal{F} \in \mathcal{D}(\mathcal{X}) \) is not zero, then there exists a \( \mathcal{O}_{x,\xi} \) and \( i \in \mathbb{Z} \) such that
\[
\text{Hom}(\mathcal{F}, \mathcal{O}_{x,\xi}[i]) \neq 0
\]
Since \( \mathcal{F} \neq 0 \) and is bounded, there exists a maximal \( m \) such that \( m \)-th cohomology sheaf \( \mathcal{H}^m \) is nonzero. Now using the spectral sequence
\[
E_2^{r,q} = \text{Hom}(\mathcal{H}^{-r}, \mathcal{O}_{x,\xi}[p]) \Rightarrow \text{Hom}(\mathcal{F}, \mathcal{O}_{x,\xi}[p+q])
\]
we see that the differentials with source \( E_r^{0,-m} \) are zero for all \( r \geq 2 \) and, similarly to the non-stacky case, all the differentials with target \( E_\infty^{0,-m} \) are also trivial. Thus, \( E_\infty^{0,-m} = E_2^{0,-m} \). Since \( \mathcal{H}^m \) is a sheaf, there exists a residual gerbe \( \iota_x : B\text{Aut}(x) \to \mathcal{X} \) such that \( \iota_x^* \mathcal{H}^m \neq 0 \).

Since \( \iota_x^* \mathcal{H}^m \neq 0 \), there exists an irreducible representation \( \xi \) and a nonzero morphism \( \mathcal{H}^m \to \mathcal{O}_{\text{Spec}(k)} \otimes \xi \). Since
\[
E_\infty^{0,-m} = E_2^{0,-m} = \text{Hom}(\mathcal{H}^m, \mathcal{O}_{x,\xi}) \neq 0,
\]
we conclude \( \text{Hom}(\mathcal{F}, \mathcal{O}_{x,\xi}[-m]) \neq 0 \) as desired.

\[\square\]

Remark 2.1. The spanning class in Proposition 2.2 should be thought of as a refinement of the spanning class
\[
\Omega = \{ \mathcal{O}_Z | Z \text{ is a closed substack of } \mathcal{X} \text{ and } \pi(Z) \text{ is a closed point in } X \}
\]
where \( \pi : \mathcal{X} \to X \) is the coarse moduli, see [CT08].

Recall that spanning classes can be used to check fully-faithfulness of exact functors.

Proposition 2.3 ([Bri99]). Suppose \( F : \mathcal{T} \to \mathcal{T}' \) is an exact functor with a left and right adjoint. Then \( F \) is fully-faithful if and only if there exists a spanning class \( \Omega \subset \mathcal{T} \) such that \( \omega, \omega' \in \Omega \) and so that
\[
\text{Hom}_T(\omega, \omega'[i]) \to \text{Hom}_{T'}(F(\omega), F(\omega')[i])
\]
is an isomorphism for all \( i \).

2.5. Some Lemmas. We need to the following criterion for a complex to be a sheaf, flat over the base, as in [Bri99].

Lemma 2.1. Let \( \pi : \mathcal{S} \to \mathcal{T} \) be a morphism of DM stacks, and for each closed point \( t : \text{Spec}(k) \to \mathcal{T} \), let \( j_t : \mathcal{S}_t \to \mathcal{S} \) denote the inclusion of the fiber \( \mathcal{S}_t = \mathcal{S} \times_{\mathcal{T}} \text{Spec}(k) \). Let \( \mathcal{Q} \) be an object of \( \mathcal{D}(\mathcal{S}) \) such that for all \( t : \text{Spec}(k) \to \mathcal{T} \), the derived restriction \( j_t^!(\mathcal{Q}) \) is a sheaf on \( \mathcal{S}_t \). Then \( \mathcal{Q} \) is a sheaf on \( \mathcal{S} \), flat over \( \mathcal{T} \).

Proof. We remark that \( \mathcal{Q} \) is a sheaf, flat over \( \mathcal{T} \), if and only if the base change to an étale cover on the source and on the target is a sheaf, flat over the base. In this case, this is [Bri99] Lemma 4.3).

Specifically, pick an étale cover from a scheme \( p_T : \mathcal{T} \to \mathcal{T} \). Then the morphism \( t : \text{Spec}(k) \to \mathcal{T} \) lifts to \( t' : \text{Spec}(k) \to \mathcal{T} \). Set \( \mathcal{S}_T = \mathcal{S} \times_{\mathcal{T}} \mathcal{T} \) and so for any \( t \in \mathcal{T}(k) \), we can set \( \mathcal{S}_t \cong \mathcal{S}_T \times_{\mathcal{T}} \text{Spec}(k) \). Let \( p_S : \mathcal{S}_T \to \mathcal{S} \) be an étale cover of \( \mathcal{S}_T \)

\[\text{We are using that } k \text{ is algebraically closed here.}\]
and $S_t = S_T \times_T \text{Spec}(k)$. Thus we have the following diagram where all squares are Cartesian

$$
\begin{array}{ccc}
S_t & \longrightarrow & S_T \\
\downarrow & & \downarrow p_T \\
S_t & \stackrel{s}{\longrightarrow} & S_T & \stackrel{p_T'}{\longrightarrow} & S \\
\downarrow \pi'' & & \downarrow \pi' & & \downarrow \pi \\
\text{Spec}(k) & \stackrel{t'}{\longrightarrow} & T & \stackrel{p_T}{\longrightarrow} & T
\end{array}
$$

Thus $Q$ is a sheaf, flat over $T$ if and only if $Q' = (p_T' \circ p_S)^* Q$ is a sheaf, flat over $T$. The statement now follows from loc. cit. □

By the derived pullback of an object $F \in \mathcal{D}(X)$ to a generalized point $O_{x,\xi}$, we will mean the following. Take the derived restriction $\iota_x^* F \in \mathcal{D}(B \text{Aut}(x))$ and then use the decomposition in (1) to project $\iota_x^* F$ onto the $\xi$-isotypical component. We will abbreviate this as $\iota_{x,\xi}^* F$.

**Lemma 2.2.** Let $x \in \mathcal{D}(X)$ be a point, and $F \in \mathcal{D}(X)$. Suppose

$$\text{Hom}(F, \mathcal{O}_{y,\eta}[i]) = 0$$

for $i \in \mathbb{Z}$, all points $y \neq x$, and all $\eta \in \text{Irr}(\text{Aut}(y))$, and

$$\text{Hom}(F, \mathcal{O}_{x,\xi}[i]) = 0$$

for $i \notin [0, \dim(X)]$ and all $\xi \in \text{Irr}(\text{Aut}(x))$.

Then $F$ is a sheaf supported at $x$.

**Proof.** Let $\pi: U \rightarrow X$ be an étale cover. If $\pi^* F$ is a sheaf concentrated at $\pi^{-1}(x)$, then $F$ must also be a sheaf concentrated at $x$. But by the argument in [Huy06, Lemma 7.2] and our assumptions, the object $\pi^* F$ is a sheaf concentrated at $\pi^{-1}(x)$. □

### 3. Ineffective group actions and twisted sheaves

We will use the following description of $\mathcal{D}[X/G]$ from [BGLL17, Theorem 5.5(i)] in terms of sheaves twisted by a Brauer class.

#### 3.1. BGLL Equivalence with twisted sheaves

Suppose $G$ is a finite group and $X$ is a smooth quasi-projective $G$-variety. Let us denote by $N \subset G$ the kernel of the action so that $H = G/N$ acts effectively on $X$. In [BGLL17], the authors describe the category $\mathfrak{Coh}[X/G]$ in terms of twisted $H$-equivariant sheaves on $\text{Irr}(N) \times X$. We recall this now.

Let $V$ be any representation of $G$ and consider the algebra

$$A := \text{End}_N(V)^{op}$$

We will assume that $V$ is an $N$-generator, i.e., $V$ contains all irreducible representations of $N$. For example, $V = k[G]$ would work.

Let $Z$ be the center of the group algebra of $N$,

$$Z := Z(k[N]),$$

and let

$$\text{Irr}(N) := \text{Spec}(Z)$$
denote the scheme of irreducible representations (discrete under our assumptions). The group $H$ acts naturally on $\text{Irr}(N)$, and $A$ is an $H$-equivariant Azumaya algebra over $\text{Irr}(N)$ (via the natural embedding $Z \to A$). Hence, $A$ determines an $H$-equivariant Brauer class $\alpha \in \text{Br}^H(\text{Irr}(N))$.

We equip $\text{Irr}(N) \times X$ with the diagonal $H$-action, and denote by $\pi_1 : \text{Irr}(N) \times X \to \text{Irr}(N)$ and $\pi_2 : \text{Irr}(N) \times X \to X$ the natural ($H$-equivariant) projections. Let us consider the sheaf of algebras

$$A := A \otimes_k O_X \simeq \pi_2^*(\pi_1^* A)$$

on $X$, equipped with an $H$-equivariant structure.

Since $\pi_2$ is a finite morphism, we have an equivalence of categories

$$\pi_2^* : \text{Coh}^H(\text{Irr}(N) \times X, \pi_1^* A) \simeq \text{Coh}^H(\text{Irr}(N) \times X, \pi_1^* A) \xrightarrow{\sim} \text{Coh}^H(X, A).$$

Set $V := V \otimes_k O_X$ and define

$$\text{Hom}_N(V, -) : \text{Coh}^G(X) \to \text{Coh}^H(X, A).$$

**Theorem 3.1 (BGLL Equivalence).** There is an equivalence of categories

$$\text{Coh}^G(X) \simeq \text{Coh}^H(\text{Irr}(N) \times X, \pi_1^* A).$$

given by $\pi_2^{-1} \circ \text{Hom}_N(V, -)$.

Let us consider the stack quotient

$$X_N = [(\text{Irr}(N) \times X)/H]$$

which has trivial generic automorphism group. The $H$-equivariant class $\pi_1^* \alpha$ defines an element $\bar{\alpha}$ in the Brauer group $\text{Br}(X_N)$, so we can rewrite the above equivalence as

$$\text{Coh}^G(X) \simeq \text{Coh}(X_N, \bar{\alpha}).$$

3.2. **BGLL Equivalence and generalized points.** Let us assume in addition that $H$ acts freely on $X$, so that $X_N$ is the usual space (not a stack).

For each generalized point $(x, \xi)$, we set $\bar{x}$ to be the image in $X/H$. Then $(\xi, \bar{x}) \in X_N$ is a $k$-point of $X_N$. The corresponding skyscraper sheaf $O_{(\xi, \bar{x})}$ can be viewed as an $\bar{\alpha}$-twisted sheaf on $X_N$.

**Lemma 3.1.** Under the BGLL equivalence above, the structure sheaves of generalized points $O_{x, \xi}$ are mapped to the skyscraper sheaves $O_{(\xi, \bar{x})}$ viewed as twisted sheaves.

**Proof.** Recall that for a generalized point $(x, \xi)$, one has $O_{x, \xi} = \iota_{x*}(\xi)$. Thus,

$$\text{Hom}_N(V, O_{x, \xi}) \cong \iota_{x*} \text{Hom}_N(V, \xi) \cong \iota_{x*}(V \otimes \xi^{-1})^N \otimes \xi$$

where the rightmost $\xi$ is there to remember the $A$-action. The image under $\pi_1^{-1}$ will then be $O_{(\xi, \bar{x})}$. \qed

3.3. **BGLL Equivalence and Fourier-Mukai Functors.** Let $Q$ be a $G \times G$-equivariant sheaf on $X \times X$. Then $Q$ determines, under the BGLL equivalence, a twisted sheaf $Q'$ on $X_N \times X_N$.

**Lemma 3.2.** Suppose $Q$ is flat over $X$ via the first projection, then $Q'$ is flat over $X_N$ over the first projection.

**Proof.** We just need to check that the functor $\text{Hom}_N(V, -)$ preserves flatness as all of the other functors clearly do. But this is clear as $V$ is a vector bundle. \qed
The last ingredient is a generalization of Bridgeland’s Hilbert scheme argument. For a smooth quasiprojective scheme $S$, we denote by $\text{Hilb}_\ell(S)$ the preimage of $S^{(\ell)}$ in the Hilbert scheme of length $\ell$ finite subschemes, $\text{Hilb}_\ell(S)$, where $S$ is some compactification of $S$.

Let $(Y, \alpha)$ be a twisted smooth scheme and $\pi: U \to Y$ an étale cover trivializing $\alpha$.

**Lemma 3.3.** Suppose $Q$ is a coherent $\pi_\ast\alpha$-sheaf on $U \times Y$, for $\alpha \in \text{Br}(Y)$, which is flat over $U$. Suppose for each closed point $u \in U$, the following two conditions hold:

- $Q_u := Q \{u\} \times Y$ is concentrated at $\pi(u)$;
- $\text{Hom}(Q_u, \mathcal{O}_{\pi(u)}) = k$.

Then there exists an open subscheme $U'$ of $U$ such that the corresponding composite map $U' \to \text{Coh}(Y, \alpha) \xrightarrow{\pi^\ast} \text{Coh}(U)$ factors through a finite map to $\text{Hilb}_\ell(U)$, for some $\ell \geq 0$, where $\text{Coh}(Y, \alpha)$ is the stack of coherent $(Y, \alpha)$-twisted sheaves.

**Proof.** Since $Q_u$ is concentrated at $\pi(u)$, the support is a zero-dimensional subscheme of $Y$. As the étale topology is invariant under nilpotent extensions, $Q_u$ is an honest sheaf. Bridgeland’s original argument shows that $Q_u$ is the structure sheaf of a zero-dimensional subscheme.

Let $Q'$ denote the induced family on $U \times U$. That is, $Q'$ is the pullback of $Q$. Then for each $u \in U$, $Q'_u := Q' \{u\} \times U$ is the structure sheaf of a zero-dimensional subscheme with proper support over $U$. The local map $\mathcal{O}_U \to Q'_u$ extends to a section $H^0(U \times U, Q')$ which is surjective upon shrinking $U$ to a smaller open set. Then we have the commutative diagram:

\[
\begin{array}{ccc}
U & \xrightarrow{Q} & \text{Coh}(Y, \alpha) \xrightarrow{\pi^\ast} \text{Coh}(U) \\
& & \downarrow \\
& & \text{Hilb}_\ell(U)
\end{array}
\]

\[\square\]

4. **Proof of Theorem 1.2**

We will proceed similarly to [Huy06, Section 7.1]. We have already shown that generalized points $\mathcal{O}_{x, \xi}$ are spanning in Proposition 2.2. We just need to show that the natural homomorphisms

\[\text{Hom}_{\mathcal{D}(X)}(\mathcal{O}_{x, \xi}, \mathcal{O}_{y, \zeta}[i]) \to \text{Hom}_T(F(\mathcal{O}_{x, \xi}), F(\mathcal{O}_{y, \zeta}))[i]\]

are isomorphisms for all generalized points $\mathcal{O}_{x, \xi}, \mathcal{O}_{y, \zeta}$ and any integer $i \in \mathbb{Z}$. The proof will occupy the remainder of this section.

4.1. **Reduction to** $G(F(\mathcal{O}_{x, \xi})) \cong \mathcal{O}_{x, \xi}$. As in the original proof, to prove (2), we have to show the bijectivity of the map

\[\text{Hom}_{\mathcal{D}(X)}(\mathcal{O}_{x, \xi}, \mathcal{O}_{y, \zeta}[i]) \to \text{Hom}_T(GF(\mathcal{O}_{x, \xi}), \mathcal{O}_{y, \zeta}[i])\]

induced by the adjunction morphism $G \circ F \to \text{Id}_{\mathcal{D}(X)}$. 


If $GF(O_{x,\xi}) \cong O_{x,\xi}$, then either the adjunction morphism is zero or it is an isomorphism. But as in the original proof, it cannot be zero as

$$\text{Hom}_T(F(O_{x,\xi}), F(O_{x,\xi})) = k.$$ 

Thus, if we prove that $GF(O_{x,\xi}) \cong O_{x,\xi}$ then we can deduce that (2) is bijective.

4.2. **Reduction to injectivity of (2) for $i = 1$**. Fix a generalized point $O_{x,\xi}$ and suppose that the homomorphism in (2) is injective for $i = 1$.

By Lemma 2.2, $Q_{x,\xi} := G(F(O_{x,\xi}))$ is a sheaf supported at $x$. Since the adjunction map is not trivial, there is a surjection $\delta: Q_{x,\xi} \to O_{x,\xi}$. Indeed, it is not zero and $\xi$ is irreducible, so it is surjective. We need to show $\delta$ is bijective. There is a short exact sequence

$$0 \to \text{Ker}(\delta) \to Q_{x,\xi} \xrightarrow{\delta} O_{x,\xi} \to 0$$

where $\text{Ker}(\delta)$ is supported at $x$ as well.

To see $\text{Ker}(\delta) = 0$, it suffices to show $\text{Hom}(\text{Ker}(\delta), O_{x,\eta}) = 0$ for any $\eta \in \text{Irr}(\text{Aut}(x))$. But we have the identification $\text{Hom}(\text{Ker}(\delta), O_{x,\eta}) = \text{Ker}(\text{Hom}(O_{x,\xi}, O_{x,\eta}[1])) \to \text{Hom}(Q_{x,\xi}, O_{x,\eta}[1])$.

Thus, injectivity of (2) for $i = 1$ implies that $\text{ker}(\delta) = 0$, i.e., $Q_{x,\xi} \cong O_{x,\xi}$.

4.3. **Injectivity of (2) for $i = 1$ follows from generic injectivity for $i = 1$**. By assumption, $G \circ F$ is of Fourier-Mukai type given by some kernel $Q_x$. For any residual gerbe $\iota_x: G_x \to X$ the pullback $(\iota_x \times \text{id})^*(Q)$ is exactly

$$Q_x := G \circ F(O_x) = \bigoplus_{\xi} \xi^\vee \otimes Q_{x,\xi},$$

so it is a sheaf. Hence, by Lemma 2.1 $Q$ is flat over $\mathcal{X}$ (with respect to the first projection). Let $\varepsilon: Q \to \Delta_* O_{\mathcal{X}}$ be the adjunction morphism. This map is in fact surjective since $(\iota_x \times \text{id})^*(\varepsilon)$ is the surjective map

$$Q_x = \bigoplus_{\xi} \xi^\vee \otimes Q_{x,\xi} \to \bigoplus_{\xi} \xi^\vee \otimes O_{x,\xi} = O_x.$$ 

Thus, we have an exact sequence of coherent sheaves on $\mathcal{X} \times \mathcal{X}$

$$0 \to K \to Q \to \Delta_* O_{\mathcal{X}} \to 0.$$ 

It follows that $K$ is flat over $\mathcal{X}$ (via the first projection). If we assume injectivity of (2) for $i = 1$ and generic $x \in \mathcal{X}$ (and arbitrary $\xi$ and $\eta$), then as above we deduce that for generic point $x$, one has $(\iota_x \times \text{id})^* K = 0$. Since $K$ is flat over $\mathcal{X}$, it follows that $\mathcal{K} = 0$, and the adjunction morphism $\varepsilon$ is an isomorphism.

4.4. **Generic injectivity for $i = 1$**. We want to prove that for generic $x$ the natural maps (2) are injective for $i = 1$ (for all $\xi$ and $\eta$). This is equivalent to the injectivity of the natural map

$$(3) \quad \text{Ext}^1(O_{x,\xi}, O_x) \to \text{Ext}^1(Q_{x,\xi}, Q_x).$$

By [Kre09], there is a Zariski open substack $\mathcal{Y} \subset \mathcal{X}$ of the form $\mathcal{Y} \cong [Y/G]$, where $Y$ is a quasi-projective variety and $G$ is a finite group. Let $N$ be the kernel of the action and $H = G/N$. By shrinking $Y$, we can assume $H$ acts freely. Set the quotient map to be $\pi_Y: Y \to \bar{Y} = Y/H$. Denote also by $Q$ the sheaf $Q$ restricted to $\mathcal{Y} \times \mathcal{Y}$. 
By Theorem 3.1, there is an equivalence of categories between \( \mathcal{Coh}(\mathcal{Y}) \) and \( \alpha\)-twisted sheaves on \( \bar{Y}_N = \text{Irr}(N) \times \mathcal{Y} \), where \( \alpha \) is the corresponding Brauer class. Let \( Q' \) be the image of \( Q \) under the corresponding equivalence for the product. By Lemma 3.2, \( Q' \) is still flat over \( \pi_1 \). Let \( \pi: U \to \bar{Y}_N \) be an étale cover trivializing \( \alpha \), then the pullback of \( Q' \) to \( U \times \bar{Y}_N \) satisfies the conditions of Lemma 3.3. It follows that the corresponding map (abusively denoted by \( Q' \))

\[
Q': U \to \mathcal{Coh}(\bar{Y}_N, \alpha) \to \mathcal{Coh}(U)
\]

factors through a finite map to \( \text{Hilb}_\ell(U) \) (maybe after shrinking \( U \)). Thus \( Q' \) has generically injective tangent and so generically \( Q' \) defines an isomorphism:

\[
\text{Ext}^1_U(O_u, O_u) \cong T_uU \to T_{Q'_u} \text{Hilb}_\ell(U) \cong \text{Ext}^1_{\text{Hilb}_\ell(U)}(Q'_u, Q'_u).
\]

Finally, the following diagram is commutative:

\[
\begin{array}{ccc}
\text{Ext}^1_{\alpha}(O_{x,\xi}, O_{x,\xi}) & \xrightarrow{Q} & \text{Ext}^1_{\mathcal{Coh}(\alpha)}(Q_{x,\xi}, Q_{x,\xi}) \\
\text{Ext}^1_U(O_u, O_u) & \xrightarrow{Q'} & \text{Ext}^1_{\text{Hilb}_\ell(U)}(Q'_u, Q'_u) \\
\end{array}
\]

where \( u \) is such that \( \pi(u) = (x, \xi) \). This completes the proof as the two vertical arrows are generically isomorphisms.

**References**

[BFN08] David Ben-Zvi, John Francis, and David Nadler, *Integral Transforms and Drinfeld Centers in Derived Algebraic Geometry*, arXiv e-prints (2008), arXiv:0805.0157.

[BGGL17] Daniel Bergh, Sergey Gorchinskiy, Michael Larsen, and Valery Lunts, *Categorical measures for finite group actions*, arXiv e-prints (2017), arXiv:1709.00620.

[BO95] A. Bondal and D. Orlov, *Semiorthogonal decomposition for algebraic varieties*, eprint arXiv:alg-geom/9506012, June 1995.

[Bri99] Tom Bridgeland, *Equivalences of triangulated categories and Fourier-Mukai transforms*, Bull. London Math. Soc. 31 (1999), no. 1, 25–34. MR 1651025

[Câl02] Andrei Căldăraru, *Nonfine moduli spaces of sheaves on K3 surfaces*, Int. Math. Res. Not. (2002), no. 20, 1027–1056. MR 1902629

[CT08] Jiun-Cheng Chen and Hsian-Hua Tseng, *A note on derived McKay correspondence*, Math. Res. Lett. 15 (2008), no. 3, 435–445. MR 2407221

[Gen17] Francesco Genovese, *Adjunctions of quasi-functors between DG-categories*, Appl. Categ. Structures 25 (2017), no. 4, 625–657. MR 3669175

[HRLMnSdS09] Daniel Hernández Ruipérez, Ana Cristina LópezMartín, and Fernando Sancho de Salas, *Relative integral functors for singular fibrations and singular partners*, J. Eur. Math. Soc. (JEMS) 11 (2009), no. 3, 597–625. MR 2565443

[Huy06] D. Huybrechts, *Fourier-Mukai transforms in algebraic geometry*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, Oxford, 2006. MR 2241106

[Kre09] Andrew Kresch, *On the geometry of Deligne-Mumford stacks*, Algebraic geometry—Seattle 2005. Part 1, Proc. Sympos. Pure Math., vol. 80, Amer. Math. Soc., Providence, RI, 2009, pp. 259–271. MR 2483938

[LM17] A. C. López Martín, *Fully faithfulness criteria for quasi-projective schemes*, Collectanea Mathematica 68 (2017), no. 2, 219–227 (English).

[Nir08] Fabio Nironi, *Grothendieck Duality for Deligne-Mumford Stacks*, arXiv e-prints (2008), arXiv:0811.1955.

[SdS09] Fernando Sancho de Salas, *Koszul complexes and fully faithful integral functors*, Bull. Lond. Math. Soc. 41 (2009), no. 6, 1085–1094. MR 2575339
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