TRACE ASYMPTOTICS FORMULA FOR THE SCHRÖDINGER OPERATORS WITH CONSTANT MAGNETIC FIELDS.

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Abstract. In this paper, we consider the 2D-Schrödinger operator with constant magnetic field

\[ H(V) = (D_x - By)^2 + D_y^2 + V_h(x, y), \]

where \( V \) tends to zero at infinity and \( h \) is a small positive parameter. We will be concerned with two cases: the semi-classical limit regime \( V_h(x, y) = V(hx, hy) \), and the large coupling constant limit case \( V_h(x, y) = h^{-\delta} V(x, y) \). We obtain a complete asymptotic expansion in powers of \( h^2 \) of \( \text{tr}(\Phi(H(V), h)) \), where \( \Phi(\cdot, h) \in C^\infty_0(\mathbb{R}; \mathbb{R}) \). We also give a Weyl type asymptotics formula with optimal remainder estimate of the counting function of eigenvalues of \( H(V) \).

1. Introduction

Let \( H_0 = (D_x - By)^2 + D_y^2 \) be the 2D-Schrödinger operator with constant magnetic field \( B > 0 \). Here \( D_\nu = \frac{1}{i} \partial_\nu \). It is well known that the operator \( H_0 \) is essentially self-adjoint on \( C^\infty_0(\mathbb{R}^2) \) and its spectrum consists of eigenvalues of infinite multiplicity (called Landau levels, see, e.g., [1]). We denote by \( \sigma(H_0) \) (resp. \( \sigma_{\text{ess}}(H_0) \)) the spectrum (resp. the essential spectrum) of the operator \( H_0 \). Then,

\[ \sigma(H_0) = \sigma_{\text{ess}}(H_0) = \bigcup_{n=0}^{\infty} \{(2n + 1)B\}. \]

Let \( V \in C^\infty(\mathbb{R}^2; \mathbb{R}) \) and assume that \( V \) is bounded with all its derivatives and satisfies

\[ \lim_{|(x, y)| \to \infty} V(x, y) = 0. \]

We now consider the perturbed Schrödinger operator

\[ H(V) = H_0 + V_h(x, y), \]

where \( V_h \) is a potential depending on a semi-classical parameter \( h > 0 \), and is of the form \( V_h(x, y) = V(hx, hy) \) or \( V_h(x, y) = h^{-\delta} V(x, y) \), \( (\delta > 0) \). Using the Kato-Rellich theorem and the Weyl criterion, one sees that \( H(V) \) is essentially self-adjoint on \( C^\infty_0(\mathbb{R}^2) \) and

\[ \sigma_{\text{ess}}(H(V)) = \sigma_{\text{ess}}(H_0) = \bigcup_{n=0}^{\infty} \{(2n + 1)B\}. \]

The spectral properties of the 2D-Schrödinger operator with constant magnetic field \( H(V) \) have been intensively studied in the last ten years. In the case of perturbations, the Landau levels \( \Lambda_n = (2n + 1)B \) become accumulation points of the eigenvalues and the asymptotics of the function counting the number of the eigenvalues lying in a neighborhood of \( \Lambda_n \) have been

2010 Mathematics Subject Classification. 81Q10, 35J10, 35P20, 35C20, 47F05.
Key words and phrases. Magnetic Schrödinger operators, asymptotic trace formula, eigenvalues distribution.
examined by many authors in different aspects. For recent results, the reader may consult [19, 8, 20, 15, 14, 10, 2] and the references therein.

The asymptotics with precise remainder estimate for the counting spectral function of the operator $H(h) := H_0 + V(hx, hy)$ have been obtained by V. Ivrii [13]. In fact, he constructs a micro-local canonical form for $H(h)$, which leads to the sharp remainder estimates.

However, there are only a few works treating the case of the large coupling constant limit (i.e., $V_h(x, y) = h - \delta V(x, y)$) (see [16, 17, 18]). In this case, the asymptotic behavior of the counting spectral function depends both on the sign of the perturbation and on its decay properties at infinity. In [18], G. Raikov obtained only the main asymptotic term of the counting spectral function as $h \searrow 0$.

The method used in [18] is of variational nature. By this method one can find the main term in the asymptotics of the counting spectral function with a weaker assumption on the perturbation $V$. However, it is quite difficult to establish with these techniques an asymptotic formula involving sharp remainder estimates.

For both the semi-classical and large coupling constant limit, we give a complete asymptotic expansion of the trace of $\Phi(H(V), h)$ in powers of $h^2$. We also establish a Weyl-type asymptotic formula with optimal remainder estimate for the counting function of eigenvalues of $H(V)$. The remainder estimate in Corollary 2.3 and Corollary 2.6 is $O(1)$, so it is better than in the standard case (without magnetic field, see e.g., [9]).

To prove our results, we show that the spectral study of $H(h)$ near some energy level $z$ can be reduced to the study of an $h^2$-pseudo-differential operator $E^{-\frac{1}{2}}(z)$ called the effective Hamiltonian. Our results are still true for the case of dimension $2d$ with $d \geq 1$. For the transparency of the presentation, we shall mainly be concerned with the two-dimensional case.

The paper is organized as follows: In the next section we state the assumptions and the results precisely, and we give an outline of the proofs. In Section 3 we reduce the spectral study of $H(V)$ to the one of a system of $h^2$-pseudo-differential operators $E_{-\frac{1}{2}}(z)$. In Section 4 we establish a trace formula involving the effective Hamiltonian $E_{-\frac{1}{2}}(z)$, and we prove the results concerning the semi-classical case. Finally, Section 5 is devoted to the proofs of the results concerning the large coupling constant limit case.

### 2. Formulations of main results

#### 2.1. Semi-classical case.

In this section we will be concerned with the semi-classical magnetic Schrödinger operator

$$H(h) = H_0 + V(hx, hy),$$

where $V$ satisfies (1.1). By choosing $B = \text{constant}$, we may actually assume that $B = 1$.

Fix two real numbers $a$ and $b$ such that $[a, b] \subset \mathbb{R} \setminus \sigma_{\text{ess}}(H(h))$. We define

$$l_0 := \min \left\{ q \in \mathbb{N}; V^{-1}(\lfloor a - (2q + 1), b - (2q + 1)\rfloor) \neq \emptyset \right\},$$

$$l := \sup \left\{ q \in \mathbb{N}; V^{-1}(\lfloor a - (2q + 1), b - (2q + 1)\rfloor) \neq \emptyset \right\} .$$

We will give an asymptotic expansion in powers of $h^2$ of $\text{tr}(f(H(h), h))$ in the two following cases:
Theorem 2.2. Fix $\mu \in X = \mathbb{R}$. In the sequel we shall say that $\lambda \in X$.

In addition to the hypotheses of Theorem 2.1 suppose that $\theta C \in C_0^\infty((1,1); \mathbb{R})$.

Corollary 2.3. In addition to the hypotheses of Theorem 2.1 suppose that $a$ and $b$ are not critical values of $((2j + 1) + V)$ for all $j = l_0, ..., l$. Let $\mathcal{N}_h([a, b])$ be the number of eigenvalues of $H(h)$ in the interval $[a, b]$ counted with their multiplicities. Then we have

\begin{align*}
\mathcal{N}_h([a, b]) &= h^{-2}C_0 + \mathcal{O}(1), h \searrow 0,
\end{align*}

where

\begin{align*}
C_0 &= \frac{1}{2\pi} \sum_{j = l_0}^l \text{Vol} \left( \frac{V^{-1}(a - (2j + 1), b - (2j + 1))}{h} \right).
\end{align*}
2.2. Large coupling constant limit case. We apply the above results to the Schrödinger operator with constant magnetic field in the large coupling constant limit case. More precisely, consider
\begin{equation}
H_\lambda = (D_x - y)^2 + D_y^2 + \lambda V(x, y).
\end{equation}
Here $\lambda$ is a large constant, and the electric potential $V$ is assumed to be strictly positive. Let $X := (x, y) \in \mathbb{R}^2$. We suppose in addition that for all $N \in \mathbb{N}$,
\begin{equation}
V(X) = \sum_{j=0}^{N-1} \omega_{2j} \left( \frac{X}{|X|} \right) |X|^{-2-2j} + r_{2N}(X), \quad \text{for } |X| \geq 1,
\end{equation}
where
- $\omega_0 \in C^\infty(\mathbb{S}^1; (0, +\infty))$, $\omega_{2j} \in C^\infty(\mathbb{S}^1; \mathbb{R})$, $j \geq 1$. Here $\mathbb{S}^1$ denotes the unit circle.
- $\delta$ is some positive constant,
- $|\partial_X^\delta r_{2N}(X)| \leq C_\beta (1 + |X|)^{-|\beta| - 2N}, \quad \forall \beta \in \mathbb{N}^2$.

Since $V$ is positive, it follows that $\sigma(H_\lambda) \subset [1, +\infty)$. Fix two real numbers $a$ and $b$ such that $a > 1$ and $[a, b] \subset \mathbb{R} \setminus \sigma_{\text{ess}}(H_\lambda)$. Since $\sigma_{\text{ess}}(H_\lambda) = \bigcup_{j=0}^{\infty} \{2j + 1\}$, there exists $q \in \mathbb{N}$ such that $2q + 1 < a < b < 2q + 3$. The following results are consequences of Theorem 2.1

Theorem 2.4. Assume (2.9), and let $f \in C_0^\infty((a, b); \mathbb{R})$. There exists a sequence of real numbers $(b_j(f))_{j \in \mathbb{N}}$, such that
\begin{equation}
\text{tr}(f(H_\lambda)) \sim \lambda^{\frac{q}{2}} \sum_{k=0}^{\infty} b_k(f)\lambda^{-\frac{2k}{q}}, \quad \lambda \nearrow +\infty
\end{equation}
where
\begin{equation}
b_0(f) = \frac{1}{2\pi\delta} \int_0^{2\pi} (\omega_0(\cos \theta, \sin \theta))^\frac{q}{2} d\theta \sum_{j=0}^{q} \int f(u)(u - (2j + 1))^{-1 - \frac{q}{2}} du.
\end{equation}

Theorem 2.5. Let $f \in C_0^\infty((a - \epsilon, b + \epsilon); \mathbb{R})$ and $\theta \in C_0^\infty\left((-\frac{1}{2}, \frac{1}{2}); \mathbb{R}\right)$, with $\theta = 1$ near 0. Then there exist $\epsilon > 0$, $C > 0$ and a functional sequence $c_j \in C^\infty(\mathbb{R}; \mathbb{R})$, $j \in \mathbb{N}$, such that for all $M, N \in \mathbb{N}$, we have
\begin{equation}
\text{tr}\left(f(H_\lambda)^\frac{1}{2} (t - H_\lambda)\right) \sim \lambda^{\frac{q}{2}} \sum_{k=0}^{M} c_k(t)\lambda^{-\frac{2k}{q}} + O\left(\lambda^{-\frac{2M}{q}(t)^{N}}\right)
\end{equation}
uniformly in $t \in \mathbb{R}$, where
\begin{equation}
c_0(t) = \frac{1}{2\pi} f(t) \sum_{j=0}^{q} \int_{X \in \mathbb{R}^2; 2j + 1 + W(X) = t} \frac{dS_t}{|\nabla X W(X)|}.
\end{equation}

Here $W(X) = \omega_0(|X|)|X|^{-\delta}$.

Corollary 2.6. Let $\mathcal{N}_\lambda([a, b])$ be the number of eigenvalues of $H_\lambda$ in the interval $[a, b]$ counted with their multiplicities. We have
\begin{equation}
\mathcal{N}_\lambda([a, b]) = \lambda^{\frac{q}{2}} D_0 + O(1), \quad \lambda \nearrow +\infty,
\end{equation}
where
\[ D_0 = \frac{1}{4\pi} \sum_{j=0}^{q} \left( (a - 2j - 1)^{-\frac{3}{2}} - (b - 2j - 1)^{-\frac{3}{2}} \right) \int_0^{2\pi} (\omega_0(\cos \theta, \sin \theta))^{\frac{3}{2}} d\theta. \]

2.3. Outline of the proofs. The purpose of this subsection is to provide a broad outline of the proofs. By a change of variable on the phase space, the operator \( H(h) \) is unitarily equivalent to
\[ P(h) := P_0 + V^w(h) = -\frac{\partial^2}{\partial y^2} + y^2 + V^w(x + hD_y, hy + h^2D_x), X = (x,y) \in \mathbb{R}^2. \]

Let \( \Pi = 1_{[c,d]}(P_0) \) be the spectral projector of the harmonic oscillator on the interval \([c,d] = [a - \|V\|_{L^\infty(\mathbb{R}^2)}, b + \|V\|_{L^\infty(\mathbb{R}^2)}]\). Using the explicit expression of \( \Pi \) we will reduce the spectral study of \((P - z)\) for \( z \in [a, b] + i[-1, 1] \) to the study of a system of \( h^2 \)-pseudo-differential operator, \( E_{\rightarrow +}(z) \) depending only on \( x \) (see Remark 3.7 and Corollary 3.9). In particular, modulo \( O(h^\infty) \), we are reduced to proving Theorem 2.1 and Theorem 2.2 for a system of \( h^2 \)-pseudo-differential operator (see Proposition 1.1). Thus, (2.2) and (2.4) follows easily from Theorem 1.8 in [7] (see also [8]). Corollary 2.3 is a simple consequence of Theorem 2.1 Theorem 2.2 and a Tauberian-argument.

To deal with the large coupling constant limit case, we note that for all \( M > 0 \) and \( \lambda \) large enough, we have
\[ \{(x,y,\eta,\xi) \in \mathbb{R}^4; |(x,y)| < M, (\xi - y)^2 + \eta^2 + \lambda V(x,y) \in [a, b]\} = \emptyset. \]
Thus, on the symbolic level, only the behavior of \( V(x,y) \) at infinity contributes to the asymptotic behavior of the left hand sides of (2.10) and (2.12). Since, for \(|X| \) large enough, \( \lambda V(X) = \varphi_0(X) + \varphi_2(hX)h^2 + \cdots + \varphi_{2j}(hX)h^{2j} + \cdots \) with \( h = \lambda^{-\frac{1}{2}} \) and \( \varphi_0(X) = \omega_0(\frac{X}{\lambda})|X|^{-\frac{3}{2}}, \) Theorem 2.4 (resp. Theorem 2.5) follows from Theorem 2.1 (resp. Theorem 2.2).

3. The effective Hamiltonian

3.1. Classes of symbols. Let \( M_n(\mathbb{C}) \) be the space of complex square matrices of order \( n \). We recall the standard class of semi-classical matrix-valued symbols on \( T^*\mathbb{R}^d = \mathbb{R}^{2d} \):
\[ S^m(\mathbb{R}^{2d}; M_n(\mathbb{C})) = \left\{ a \in C^\infty(\mathbb{R}^{2d} \times (0,1]; M_n(\mathbb{C})); \|\partial_2^\alpha \partial_\xi^\beta a(x,\xi;h)\|_{M_n(\mathbb{C})} \leq C_{\alpha,\beta} h^{-m} \right\}. \]
We note that the symbols are tempered as \( h \searrow 0 \). The more general class \( S^m(\mathbb{R}^{2d}; M_n(\mathbb{C})) \), where the right hand side in the above estimate is replaced by \( C_{\alpha,\beta} h^{-m-\delta(|\alpha|+|\beta|)} \), has nice quantization properties as long as \( 0 \leq \delta \leq \frac{1}{2} \) (we refer to [111 Chapter 7]).

For \( h \)-dependent symbol \( a \in S^m(\mathbb{R}^{2d}; M_n(\mathbb{C})) \), we say that \( a \) has an asymptotic expansion in powers of \( h \) in \( S^m(\mathbb{R}^{2d}; M_n(\mathbb{C})) \) and we write
\[ a \sim \sum_{j \geq 0} a_j h^j, \]
if there exists a sequence of symbols \( a_j(x,\xi) \in S^m(\mathbb{R}^{2d}; M_n(\mathbb{C})) \) such that for all \( N \in \mathbb{N} \), we have
\[ a - \sum_{j=0}^{N} a_j h^j \in S^m_{N-1}(\mathbb{R}^{2d}; M_n(\mathbb{C})). \]
In the special case when \( m = \delta = 0 \) (resp. \( m = \delta = 0, n = 1 \)), we will write \( S^0(\mathbb{R}^{2d}; M_n(\mathbb{C})) \) (resp. \( S^0(\mathbb{R}^{2d}) \)) instead of \( S^0(\mathbb{R}^{2d}; M_n(\mathbb{C})) \) (resp. \( S^0(\mathbb{R}^{2d}; M_1(\mathbb{C})). \)

We will use the standard Weyl quantization of symbols. More precisely, if \( a \in S^m_\delta(\mathbb{R}^{2d}; M_n(\mathbb{C})), \) then \( a^w(x, hD_x; h) \) is the operator defined by

\[
a^w(x, hD_x; h)u(x) = (2\pi h)^{-d} \int e^{i(x-y) \cdot a \left( \frac{x + y}{2} \right)} u(y) dy d\xi, \ u \in S(\mathbb{R}^d; \mathbb{C}^n).
\]

In order to prove our main results, we shall recall some well-known results

**Proposition 3.1. (Composition formula)** Let \( a_i \in S^m_\delta(\mathbb{R}^{2d}; M_n(\mathbb{C})), i = 1, 2, \delta \in [0, \frac{1}{2}). \) Then \( b^w(y, hD_y; h) = a_2^w(y, hD_y) \circ a_1^w(y, hD_y) \) is an \( h \)-pseudo-differential operator, and

\[
b(y, \eta; h) \sim \sum_{j=0}^{\infty} b_j(y, \eta) h^j, \text{ in } S^{m_1+m_2}_\delta(\mathbb{R}^{2d}; M_n(\mathbb{C})).
\]

**Proposition 3.2. (Beals characterization)** Let \( A = A_h : S(\mathbb{R}^d; \mathbb{C}^n) \to S'(\mathbb{R}^d; \mathbb{C}^n), 0 < h \leq 1. \) The following two statements are equivalent:

1. \( A = a^w(x, hD_x; h), \) for some \( a = a(x, \xi; h) \in S^0(\mathbb{R}^{2d}; M_n(\mathbb{C})). \)
2. For every \( N \in \mathbb{N} \) and for every sequence \( l_1(x, \xi), \ldots, l_N(x, \xi) \) of linear forms on \( \mathbb{R}^{2d}, \) the operator \( \text{ad}^N_{l}(x, hD_x) \circ \cdots \circ \text{ad}^N_{l}(x, hD_x)A_h \) belongs to \( \mathcal{L}(L^2, L^2) \) and is of norm \( O(h^N) \) in that space. Here, \( \text{ad}_A B := [A, B] = AB - BA. \)

**Proposition 3.3. (\( L^2 \)-boundedness)** Let \( a = a(x, \xi; h) \in S^0_\delta(\mathbb{R}^{2d}; M_n(\mathbb{C})), 0 \leq \delta \leq 1/2. \) Then \( a^w(x, hD_x; h) \) is bounded : \( L^2(\mathbb{R}^d; \mathbb{C}^n) \to L^2(\mathbb{R}^d; \mathbb{C}^n), \) and there is a constant \( C \) independent of \( h \) such that for \( 0 < h \leq 1; \)

\[
\|a^w(x, hD_x; h)\| \leq C.
\]

**3.2. Reduction to a semi-classical problem.** Here, we shall make use of a strong field reduction onto the \( j \)th eigenfunction of the harmonic oscillator, \( j = l_0 \cdots l, \) and a well-posed Grushin problem for \( H(h). \) We show that the spectral study of \( H(h) \) near some energy level \( z \) can be reduced to the study of an \( h^2 \)-pseudo-differential operator \( E_{-+}(z) \) called the effective Hamiltonian. Without any loss of generality we may assume that \( l_0 = 1. \)

**Lemma 3.4.** There exists a unitary operator \( \tilde{W} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2) \) such that

\[
P(h) = \tilde{W} H(h) \tilde{W}^*
\]

where \( P(h) := P_0 + V^w(h), \) \( P_0 := -\frac{\partial^2}{\partial y^2} + y^2 \) and \( V^w(h) := V^w(x + hD_y, hy + h^2 D_x). \)

**Proof.** The linear symplectic mapping

\[
\tilde{S} : \mathbb{R}^4 \to \mathbb{R}^4 \text{ given by } (x, y, \xi, \eta) \mapsto \left( \frac{1}{h} x + \eta, y + h \xi, h \xi, \eta \right),
\]

maps the Weyl symbol of the operator \( H(h) \) into the Weyl symbol of the operator \( P(h). \) By Theorem A.2 in [11, Chapter 7], there exists a unitary operator \( \tilde{W} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2) \) associated to \( \tilde{S} \) such that \( P(h) = \tilde{W} H(h) \tilde{W}^*. \) \( \square \)
Introduce the operator \( R_j^- : L^2(\mathbb{R}_x) \to L^2(\mathbb{R}_{x,y}) \) by
\[
(R_j^- v)(x, y) = \phi_j(y)v(x),
\]
where \( \phi_j \) is the \( j \)th normalized eigenfunction of the harmonic oscillator. Further, the operator \( R_j^+ : L^2(\mathbb{R}_{x,y}) \to L^2(\mathbb{R}_x) \) is defined by
\[
(R_j^+ u)(x) = \int \phi_j(y)u(x,y)dy.
\]
Notice that \( R_j^+ \) is the adjoint of \( R_j^- \). An easy computation shows that \( R_j^+ R_j^- = I_{L^2(\mathbb{R}_x)} \) and \( R_j^- R_j^+ = \Pi_j \), where
\[
\Pi_j : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2), \quad v(x,y) \mapsto \int v(x,t)\phi_j(t)dt\phi_j(y).
\]
Define \( \Pi = \sum_{j=1}^l \Pi_j \).

**Lemma 3.5.** Let \( \Omega := \{ z \in \mathbb{C}; \text{Re} z \in [a,b], \ |\text{Im} z| < 1 \} \). The operator
\[
(I - \Pi)P(h)(I - \Pi) - z : (I - \Pi)L^2(\mathbb{R}^2) \to (I - \Pi)L^2(\mathbb{R}^2)
\]
is uniformly invertible for \( z \in \Omega \).

**Proof.** It follows from the definition of \( \Pi \) that \( \sigma((I - \Pi)P_0(I - \Pi)) = \bigcup_{k \in \mathbb{N}\setminus\{1,\ldots,l\}} \{ 2k + 1 \} \).
Hence
\[
\sigma((I - \Pi)P(h)(I - \Pi)) \subset \bigcup_{k \in \mathbb{N}\setminus\{1,\ldots,l\}} [2k + 1 - \| V \|_{L^\infty(\mathbb{R}^2)}, 2k + 1 + \| V \|_{L^\infty(\mathbb{R}^2)}],
\]
which implies
\[
\sigma((I - \Pi)P(h)(I - \Pi)) \cap [a,b] = \emptyset.
\]
Consequently,
\[
\| (I - \Pi)P(h)(I - \Pi) - z \| \geq \text{dist} ([a,b], \sigma((I - \Pi)P(h)(I - \Pi))) > 0
\]
uniformly for \( z \in \Omega \). Thus, we obtain
\[
(I - \Pi)P(h)(I - \Pi) - z : (I - \Pi)L^2(\mathbb{R}^2) \to (I - \Pi)L^2(\mathbb{R}^2)
\]
is uniformly invertible for \( z \in \Omega \). \( \square \)

For \( z \in \Omega \), we put
\[
\mathcal{P}(z) = \begin{pmatrix}
(P(h) - z) & R_1^- & \ldots & R_l^- \\
R_1^+ & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
R_l^+ & \ldots & \ddots & 0 \\
0 & \ldots & \ldots & 0
\end{pmatrix}
\]
and
\[
\tilde{\mathcal{E}}(z) = \begin{pmatrix}
R(z) & R_1^- & \ldots & R_l^- \\
R_1^+ & A_1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
R_l^+ & \ldots & \ddots & 0 \\
0 & \ldots & \ldots & A_l
\end{pmatrix},
\]
where \( A_j = z-(2j+1)-R_j^+V^w(h)R_j^- \), \( j = 1,\ldots,l \) and \( R(z) = ((I - \Pi)P(h)(I - \Pi) - z)^{-1}(I - \Pi) \).

Let \( \mathcal{E}_1(z) := \mathcal{P}(z)\tilde{\mathcal{E}}(z) = (a_{k,j})_{k,j=1}^{l+1} \). In the next step we will compute explicitly \( a_{k,j} \).
Using the fact that \( \Pi R(z) = 0 \) as well as the fact that \( \Pi \) commutes with \( P_0 \), we deduce that \( (P(h) - z)R(z) = (I - \Pi) + [\Pi, V^w(h)]R(z) \). Consequently,

\[
(3.1) \quad a_{1,1} = (P(h) - z)R(z) + \sum_{j=1}^{1} R_j^{-1} R_j^{+} = I + [\Pi, V^w(h)]R(z).
\]

Next, from the definition of \( A_1 \) and the fact that \( P_0 R_j^{-1} = 3R_j^{-} \) (we recall that \( l_0 = 1 \)), one has

\[
a_{1,2} = (P(h) - z)R_1^{-} + R_1^{-} A_1
\]

\[
= -(z - 3)R_1^{-} + V^w(h)R_1^{-} + R_1^{-} (z - 3) - \Pi_1 V^w(h)R_1^{-}
\]

\[
= V^w(h)R_1^{-} - \Pi_1 V^w(h)R_1^{-}
\]

\[
= [V^w(h), \Pi_1]R_1^{-}.
\]

Similarly, \( a_{1,j} = [V^w(h), \Pi_{j-1}]R_{j-1}^{-}, j \geq 3. \)

Since \( R_j^{+}(1 - \Pi_1) = R_j^{+} - R_j^{+} R_1^{-} R_j^{+} = 0 \) and \( R_j^{+} \Pi_j = R_j^{+} R_j^{-} R_j^{+} = 0 \) for \( j \neq 1 \), it follows that \( a_{2,1} = R_j^{+} R_1^{-} = 0 \). Evidently, \( a_{2,2} = R_j^{+} R_j^{-} = I_{L^2(\mathbb{R})} \) and \( a_{2,j} = R_j^{+} R_j^{-} = 0 \) for \( j \geq 3. \)

The same arguments as above show that \( a_{k,j} = \delta_{j,k} I_{L^2(\mathbb{R})} \) for all \( k \geq 3. \) Summing up we have proved

\[
\mathcal{E}_1(z) = \mathcal{P}(z) \tilde{E}(z) = \begin{pmatrix}
(I + [\Pi, V^w(h)]R(z) & [V^w(h), \Pi_1]R_1^{-} & \cdots & [V^w(h), \Pi_1]R_1^{-} \\
0 & I_{L^2(\mathbb{R})} & \cdots & 0 \\
. & . & \cdots & . \\
. & . & \cdots & . \\
. & . & \cdots & . \\
0 & 0 & \cdots & I_{L^2(\mathbb{R})}
\end{pmatrix},
\]

Let \( f_j \in C_0^\infty(\mathbb{R}), \, f_j \equiv 1 \) near \( 2j + 1 \) and \( \text{supp } f_j \subset [2j, 2j + 2] \). By the spectral theorem we have \( \Pi_j = f_j(D_y^2 + y^2) \). On the other hand, the functional calculus of pseudo-differential operators shows that \( \Pi_j = f_j(D_y^2 + y^2) = B^w(y, D_y) \) with \( B(y, \eta) = O((\eta - \infty(\eta))^{-\infty}) \).

The composition formula of pseudo-differential operators (Proposition 3.1) gives

\[
(V^w(h), \Pi_j) = \sum_{k=1}^{N} b^w(x, h^2 D_x)c^w(y, D_y)h^k + O(h^{N+1}), \, \forall N \in \mathbb{N},
\]

where \( b_{k,j}, c_{k,j} \in S^0(\mathbb{R}^2) \). This together with the Calderon-Vaillancourt theorem (Proposition 3.8) yields \( [V^w(h), \Pi_j] = O(h) \) in \( \mathcal{L}(L^2(\mathbb{R}^2)) \). Therefore, for \( h \) is sufficiently small, \( \mathcal{E}_1(z) \) is uniformly invertible for \( z \in \Omega \), and

\[
\mathcal{E}_1(z)^{-1} = \begin{pmatrix}
a(z) & -a(z)[V^w(h), \Pi_1]R_1^{-} & \cdots & -a(z)[V^w(h), \Pi_1]R_1^{-} \\
0 & I_{L^2(\mathbb{R})} & \cdots & 0 \\
. & . & \cdots & . \\
. & . & \cdots & . \\
. & . & \cdots & . \\
0 & 0 & \cdots & I_{L^2(\mathbb{R})}
\end{pmatrix},
\]

where \( a(z) = (I + [\Pi, V^w(h)]R(z))^{-1} \). Using the explicit expressions of \( \tilde{E}(z) \) and \( \mathcal{E}_1(z)^{-1} \), we get
Proposition 3.8. In powers of $h$, symbol of the operator $E$ that they are holomorphic in $z$.

Theorem 3.6. Thus, we have proved the following theorem.

From now on, we write $E$ inverse are $h$ pseudo-differential operators with bounded symbols. Moreover, there exist $R_j^\pm a(z)$, $R_\pm$ are independent of $z$ (see [8, 12]):

(3.3) $(z - P(h))^{-1} = -E(z) + E_+(z)(E_{++}(z))^{-1}E_-(z)$, $z \in \rho(P(h))$,

(3.4) $\partial_z E_{++}(z) = E_-(z)E_+(z)$.

In what follows, the explicit formulae for $E(z)$ and $E_{\pm}(z)$ are not needed. We just indicate that they are holomorphic in $z$. In the remainder of this section, we will prove that the symbol of the operator $E_{++}(z)$ is in $S^0(\mathbb{R}^2; M_l(\mathbb{C}))$, and has a complete asymptotic expansion in powers of $h$. Moreover, we will give explicitly the principal term.

Proposition 3.8. For $1 \leq k, j \leq l$, the operators $R_j^+ V^w(h) R_j^-$ and $R_k^+ a(z)[V^w(h), \Pi_j] R_j^-$ are $h^2$-pseudo-differential operators with bounded symbols. Moreover, there exist $v_{j,n}, b_{k,j,n} \in S^0(\mathbb{R}^2)$, $n = 1, 2, ..$, such that

(3.5) $R_j^+ V^w(h) R_j^- = \sum_{n=0}^{N} h^{2n} v_{j,n}(x, h^2 D_x) + O \left(h^{2(N+1)}\right)$,

(3.6) $R_k^+ a(z)[V^w(h), \Pi_j] R_j^- = \sum_{n=1}^{N} h^w b_{k,j,n}(x, h^2 D_x, z) h^n + O \left(h^{N+1}\right)$, for $k \neq j$,

(3.7) $R_j^+ a(z)[V^w(h), \Pi_j] R_j^- = \sum_{n=1}^{N} h^w b_{j,j,2n}(x, h^2 D_x, z) h^{2n} + O \left(h^{2(N+1)}\right)$, $\forall N \in \mathbb{N}$.
Here
\[ v_{j,0}(x, \xi) = V(x, \xi), \quad j = 1, \ldots, l. \]

Proof. The proofs of (3.5), (3.6) and (3.7) are quite similar, and are based on the Beal’s characterization of \( h^2 \)-pseudo-differential operators (see Proposition 3.2). We give only the main ideas of the proof of (3.5) and we refer to [8, 11, 12] for more details. Let \( Q \) denote the left hand side of (3.5). Let \( l^w(x, h^2D_x) \) be as in Proposition 3.2. Using the fact that \( R_j^\pm \) commutes with \( l^w(x, h^2D_x) \) as well as the fact that \( V^w(h) \) is an \( h^2 \)-pseudo-differential operator on \( x \), we deduce from Proposition 3.2 that \( Q = q^w(x, h^2D_x; h) \), with \( q \in S^0(\mathbb{R}^2) \).

On the other hand, writing
\[ V^w(h) = V^w(x, h^2D_x) + hD_y \left( \frac{\partial V}{\partial x} \right)^w (x, h^2D_x) + hy \left( \frac{\partial V}{\partial y} \right)^w (x, h^2D_x) + \cdots, \]
and using Proposition 3.2 we see that \( q(x, \xi; h) \) has an asymptotic expansion in powers of \( h \).

Notice that the odd powers of \( h \) in (3.5) and (3.7) disappear, due to the special properties of the eigenfunctions of the harmonic oscillator (i.e., \( \int_{\mathbb{R}} y^{2j+1} |\phi_j(y)|^2 dy = \int_{\mathbb{R}} \phi_j(y) \partial_y^{2j+1} \phi_j(y) dy = 0 \)). Finally, since \( R_j^+ R_j^- = I_{L^2(\mathbb{R})} \), it follows from (3.8) that \( v_{j,0}(x, \xi) = V(x, \xi) \).

\[ \square \]

Let \( e_{-+}(x, \xi, z, h) \) denote the symbol of \( E_{-+}(z) \). The following corollary follows from the above proposition and the definition of \( E_{-+}(z) \).

**Corollary 3.9.** We have
\[ e_{-+}(x, \xi, z, h) \sim \sum_{j=0}^{\infty} e_{-+}^j(x, \xi, z) h^j, \quad \text{in } S^0(\mathbb{R}^2; M_l(\mathbb{C})), \]
with
\[ e_{-+}^j(x, \xi, z) = \left( (z - (2j + 1) - V(x, \xi)) \delta_{i,j} \right)_{1 \leq i, j \leq l}. \]

**4. Proof of Theorem 2.1 and Theorem 2.3**

**4.1. Trace formulae.** Let \( f \in C_0^\infty((a, b); \mathbb{R}) \), where \( (a, b) \subset \mathbb{R} \setminus \sigma_{ess}(P(h)) \), and let \( \theta \in C_0^\infty(\mathbb{R}; \mathbb{R}) \). Set
\[ \Sigma_j([a, b]) = \{ (x, \xi) \in \mathbb{R}^2; \ 2j + 1 + V(x, \xi) \in [a, b] \} , \quad j = 1, \ldots, l \]
and
\[ \Sigma_{[a, b]} = \bigcup_{j=1}^{l} \Sigma_j([a, b]). \]

Let \( \tilde{f} \in C_0^\infty((a, b) + i[-1, 1]) \) be an almost analytic extension of \( f \), i.e., \( \tilde{f} = f \) on \( \mathbb{R} \) and \( \overline{\partial}_z \tilde{f} \) vanishes on \( \mathbb{R} \) to infinite order, i.e. \( \overline{\partial}_z \tilde{f}(z) = O_N(|\text{Im} \ z|^N) \) for all \( N \in \mathbb{N} \). Then the functional calculus due to Helffer-Sjöstrand (see e.g. [11] Chapter 8) yields
\[ f(P(h)) = -\frac{1}{\pi} \int \overline{\partial}_z \tilde{f}(z)(z - P(h))^{-1} L(dz), \]
where \( L(dz) \) is the 

(4.3) \[ f(P(h))\hat{\theta}_{hz}(t-P(h)) = -\frac{1}{\pi} \int \overline{\partial}_z f(z)\hat{\theta}_{hz}(t-z)(z-P(h))^{-1}L(dz). \]

Here \(L(dz) = dx dy\) is the Lebesgue measure on the complex plane \(\mathbb{C} \sim \mathbb{R}^2_{x,y}\). In the last equality we have used the fact that \(\overline{f(z)\hat{\theta}_{hz}(t-z)}\) is an almost analytic extension of \(f(x)\hat{\theta}_{hz}(t-x)\), since \(z \mapsto \hat{\theta}_{hz}(t-z)\) is analytic.

**Proposition 4.1.** For \(h\) small enough, we have

(4.4) \[ \text{tr}(f(P(h))) = \text{tr}\left(-\frac{1}{\pi} \int \overline{\partial}_z f(z)(E_{-+}(z))^{-1}\partial_z E_{-+}(z)L(dz)\chi^w(x,h^2D_x)\right) + O(h^\infty), \]

(4.5) \[ \text{tr}\left(-\frac{1}{\pi} \int \overline{\partial}_z f(z)\hat{\theta}_{hz}(t-z)(E_{-+}(z))^{-1}\partial_z E_{-+}(z)L(dz)\chi^w(x,h^2D_x)\right) + O(h^\infty), \]

where \(\chi \in \mathcal{C}_0^\infty(\mathbb{R}^2; \mathbb{R})\) is equal to one in a neighbourhood of \(\Sigma_{[a,b]}\).

**Proof.** Replacing \((z-P(h))^{-1}\) in (4.2) by the right hand side of (3.3), and using the fact that \(E(z)\) is holomorphic in \(z\), we obtain

(4.6) \[ f(P(h)) = -\frac{1}{\pi} \int \overline{\partial}_z f(z)E_{+}(z)(E_{-+}(z))^{-1}E_{-}(z)L(dz). \]

Let \(\tilde{V} \in S^0(\mathbb{R}^2)\) be a real-valued function coinciding with \(V\) for large \((x,y)\), and having the property that

(4.7) \[ |z - (2j + 1) - \tilde{V}(x,y)| > c > 0, \quad j = 1, 2, ..., l, \]

uniformly in \(z \in \text{supp } f\), and \((x,y) \in \mathbb{R}^2\). We recall that for \(z \in \text{supp } f\), \(\text{Re } z \in (a,b) \subset \mathbb{R} \setminus \sigma_{\text{ess}}(H(h)) = \mathbb{R} \setminus \bigcup_{k=0}^\infty (2k+1)\). Then (4.7) holds for \(\tilde{V} \in S^0(\mathbb{R}^2)\) with \(\|\tilde{V}\|\) small enough.

Set \(\tilde{E}_{-+}(z) := E_{-+}(z) + \left(V^w(x,h^2D_x) - \tilde{V}^w(x,h^2D_x)\right)I_l\), and let \(\tilde{e}(x,\xi, z)\) be the principal symbol of \(\tilde{E}_{-+}(z)\). Here \(I_l\) denotes the unit matrix of order \(l\). It follows from (4.7) that \(|\det \tilde{e}(x,\xi, z)| > \epsilon^l\). Then for sufficiently small \(h > 0\), the operator \(\tilde{E}_{-+}(z)\) is elliptic, and \(\tilde{E}_{-+}(z)^{-1}\) is well defined and holomorphic for \(z\) in some fixed complex neighbourhood of \(\text{supp } f\), (see chapter 7 of [11]). Hence, by an integration by parts, we get

\[-\frac{1}{\pi} \int \overline{\partial}_z f(z)E_{+}(z)\tilde{E}_{-+}(z)^{-1}E_{-}(z)L(dz) = 0.\]

Combining this with (4.6) and using the resolvent identity for \(\text{Im } z \neq 0\)

\[ E_{-+}(z)^{-1} = \tilde{E}_{-+}(z)^{-1} + E_{-+}(z)^{-1}(\tilde{E}_{-+}(z) - E_{-+}(z))\tilde{E}_{-+}(z)^{-1}, \]

we obtain

(4.8) \[ \text{tr } (f(P(h))) = -\frac{1}{\pi} \text{tr}\left(\int \overline{\partial}_z f(z)E_{+}(z)E_{-+}(z)^{-1}((\tilde{E}_{-+}(z) - E_{-+}(z))\tilde{E}_{-+}(z)^{-1}E_{-}(z)L(dz)\right). \]

Since the symbol of \(E_{-+}(z) - \tilde{E}_{-+}(z)\) is \((\tilde{V} - V)I_l\) belonging to \(C_0^\infty(\mathbb{R}^2; M_l(\mathbb{C}))\), we have \(E_{-+}(z) - \tilde{E}_{-+}(z)\) is a trace class operator. It is then clear that we can permute integration and the operator “tr” in the right hand side of (4.8).
Using the property of cyclic invariance of the trace, and applying (3.4) we get
\[
\text{tr} \left( E_+(z)E_+(z)^{-1}(\tilde{E}_+(z) - E_+(z))\tilde{E}_+(z)^{-1}E_-(z) \right) = \\
\text{tr} \left( E_+(z)^{-1}(\tilde{E}_+(z) - E_+(z))\tilde{E}_+(z)^{-1}\partial_z E_+(z) \right).
\]
Let \( \chi \in C_0^\infty(\mathbb{R}^2) \) be equal to 1 in a neighbourhood of \( \text{supp} (\tilde{V} - V) \). From the composition formula for two \( h^2 - \Psi \) DOs with Weyl symbols (see Proposition 3.1), we see that all the derivatives of the symbol of the operator \( (E_+(z) - \tilde{E}_+(z))(\tilde{E}_+(z)^{-1}\partial_z E_+(z)(1 - \chi^w(x, h^2D_x)) \) are \( O(h^2N(\langle x, \xi \rangle)^{-N}) \) for every \( N \in \mathbb{N} \). The trace class-norm of this expression is therefore \( O(h^\infty) \), and consequently
\[
(4.9) \quad \text{tr}(E_+(z)E_+(z)^{-1}(\tilde{E}_+(z) - E_+(z))\tilde{E}_+(z)^{-1}E_-(z)) = \\
\text{tr}(E_+(z)^{-1}(\tilde{E}_+(z) - E_+(z))\tilde{E}_+(z)^{-1}\partial_z E_+(z)\chi^w(x, h^2D_x)) + O(h^\infty|\text{Im}z|^{-1}).
\]
Here we recall from (4.3) that \( E_+(z)^{-1} = O(|\text{Im}z|^{-1}) \).

Inserting (4.9) into (4.8), and using the fact that \( \tilde{E}_+(z)^{-1}\partial_z E_+(z) \) is holomorphic in \( z \) we obtain (4.4). The proof of (4.5) is similar.

Trace formulas involving effective Hamiltonian like (4.4) and (4.5) were studied in [7, 8]. Applying Theorem 1.8 in [7] to the left hand side of (4.4), we obtain
\[
(4.10) \quad \text{tr}(f(P(h))) \sim \sum_{j=0}^{\infty} \beta_j h^{j-2}, \quad (h \searrow 0).
\]
To use Theorem 1.8 in [7] we make the following definition.

**Definition 4.2.** We say that \( p(x, \xi) \in S^0(\mathbb{R}^2; M_{\mathbb{C}}) \), is micro-hyperbolic at \((x_0, \xi_0)\) in the direction \( T \in \mathbb{R}^2 \), if there are constants \( C_0, C_1, C_2 > 0 \) such that
\[
(\langle dp(x, \xi), T \rangle \omega, \omega) \geq \frac{1}{C_0}||\omega||^2 - C_1||p(x, \xi)\omega||^2.
\]
for all \( (x, \xi) \in \mathbb{R}^2 \) with \( ||(x, \xi) - (x_0, \xi_0)|| \leq \frac{1}{C_2} \) and all \( \omega \in \mathcal{C}' \).

The assumption of Theorem 2.2 implies that the principal symbol \( e^0_{\theta}(x, \xi, z) \) of \( E_+(z) \) is micro-hyperbolic at every point \((x_0, \xi_0) \in \Sigma_{\mu} \) for \( \mu \in \mathbb{R}^2 \) and \( \xi \in S_{\mathbb{C}} \). Thus, according to Theorem 1.8 in [7] there exists \( C > 0 \) large enough and \( \epsilon > 0 \) small such that for \( f \in C_0^\infty(\mathbb{R}^2, \mathbb{R}) \), \( \theta \in C_0^\infty(\mathbb{R}^2, \mathbb{R}) \), and \( \mu \in \mathbb{R} \) we have:
\[
(4.11) \quad \text{tr} \left( f(P(h))\tilde{\partial}_{h^2}(t - P(h)) \right) \sim \sum_{j=0}^{\infty} \gamma_j(t) h^{j-2}, \quad (h \searrow 0),
\]
with \( \gamma_0(t) = c_0(t) \).

By observing that the \( h \)-pseudo-differential calculus can be extended to \( h \leq 0 \), we have
\[
\left| h^2\text{tr} \left( f(P(h))\tilde{\partial}_{h^2}(t - P(h)) \right) - \sum_{0 \leq j \leq N} \gamma_j(t) h^j \right| \leq C_N|h|^{N+1}, \quad h \in [-h_N, h_N]\setminus\{0\}.
\]
In what follows, we choose $\theta$

Proof. Since $\psi M$ Set

Now Corollary 4.3 follows from (2.4).

Corollary 4.3. There is $x,y$ By the change of variable $(4.13)$ $tr \left( (4.12)$ $a$ their multiplicity and lying in the interval $(4.2.$ Theorem 2.1 and Theorem 2.2.

operators then $tr(A) = tr(B)$. Consequently, $\gamma_{2j+1} = \beta_{2j+1} = 0$. This ends the proof of Theorem[2.1] and Theorem[2.2].

4.2. Proof of Corollary [2.3] Pick $\sigma > 0$ small enough. Let $\phi_1 \in C_0^\infty((a, -\sigma, a+\sigma); [0, 1])$, $\phi_2 \in C_0^\infty((a, +\sigma, b - \sigma); [0, 1])$, $\phi_3 \in C_0^\infty((b - \sigma, b + \sigma); [0, 1])$ satisfy $\phi_1 + \phi_2 + \phi_3 = 1$ on $(a - \sigma, b + \sigma)$. Let $\gamma_0(h) \leq \gamma_1(h) \leq \cdots \leq \gamma_N(h)$ be the eigenvalues of $H(h)$ counted with their multiplicity and lying in the interval $(a - \sigma, b + \sigma)$. We have

$$N_h(a,b) = \sum_{a \leq \gamma_j(h) \leq b} (\phi_1 + \phi_2 + \phi_3)(\gamma_j(h)) = \sum_{a \leq \gamma_j(h)} \phi_1(\gamma_j(h)) + \sum_{a \leq \gamma_j(h)} \phi_2(\gamma_j(h)) + \sum_{a \leq \gamma_j(h)} \phi_3(\gamma_j(h)) = \sum_{a \leq \gamma_j(h)} \phi_1(\gamma_j(h)) + \mbox{tr}(\phi_2(H(h))) + \sum_{a \leq \gamma_j(h)} \phi_3(\gamma_j(h)).$$

According to Theorem[2.1] we have

$$\mbox{tr} (\phi_m(H(h))) = \frac{1}{2\pi h^2} \sum_{j=1}^l \int_{\mathbb{R}^2} \phi_m((2j + 1) + V(X))dX + O(1), \quad m = 1, 2, 3.$$ 

Set $M(\tau, h) := \sum_{\gamma_j(h) \leq \tau} \phi_3(\gamma_j(h))$. Evidently, in the sense of distribution, we have

$$M(\tau) := M'(\tau, h) = \sum_{j} \delta(\tau - \gamma_j(h))\phi_3(\gamma_j(h)).$$

In what follows, we choose $\theta \in C_0^\infty((-\frac{1}{C}, \frac{1}{C}); [0, 1])$, $(C > 0$ large enough) such that $\theta(0) = 1$, $\dot{\theta}(t) \geq 0, t \in \mathbb{R}, \dot{\theta}(t) \geq \epsilon_0, t \in [-\delta_0, \delta_0]$ for some $\delta_0 > 0, \epsilon_0 > 0$.

Corollary 4.3. There is $C_0 > 0$, such that, for all $(\lambda, h) \in \mathbb{R} \times (0, h_0)$, we have:

$$|M(\lambda + \delta_0 h^2, h) - M(\lambda - \delta_0 h^2, h)| \leq C_0.$$ 

Proof. Since $\phi_3 \geq 0$, it follows from the construction of $\theta$ that

$$\frac{\epsilon_0}{h^2} \sum_{\lambda - \delta_0 h^2 \leq \gamma_j(h) \leq \lambda + \delta_0 h^2} \phi_3(\gamma_j(h)) \leq \sum_{|\lambda - \gamma_j(h)| < \delta_0 h^2} \ddot{h}^2(\lambda - \gamma_j(h)) \phi_3(\gamma_j(h)) \leq \ddot{h}^2 \sum_{j} (\lambda - \gamma_j(h)) \phi_3(\gamma_j(h)) = \ddot{h}^2 \ast M(\lambda) = \mbox{tr} \left( \phi_3(H(h)) \ddot{h}^2(\lambda - H(h)) \right).$$ 

Now Corollary[4.3] follows from (2.4).
According to Corollary 4.3, we have
\[(4.15) \quad \int \left( \frac{\tau - \lambda}{h^2} \right)^{-2} \mathcal{M}(\tau)d\tau = \sum_{k \in \mathbb{Z}} \int \left\{ \delta_{0k} \leq \frac{\tau}{h} \leq \delta_{0(k+1)} \right\} \left( \frac{\tau - \lambda}{h^2} \right)^{-2} \mathcal{M}(\tau)d\tau \leq C_0 \left( \sum_{k \in \mathbb{Z}} \langle \delta_{0k} \rangle^{-2} \right).\]

On the other hand, since \( \tilde{\theta} \in \mathcal{S}(\mathbb{R}) \) and \( \theta(0) = 1 \), there exists \( C_1 > 0 \) such that:
\[
\left| \int_{-\infty}^{\lambda} \tilde{\theta}h^2(\tau - y)dy - 1_{(-\infty,\lambda)}(\tau) \right| = \left| \int_{-\infty}^{+\infty} \tilde{\theta}(y)dy - 1_{(-\infty,\lambda)}(\tau) \right| \leq C_1 \left( \frac{\tau - \lambda}{h^2} \right)^{-2},
\]
uniformly in \( \tau \in \mathbb{R} \) and \( h \in (0, h_0) \). Consequently,
\[(4.16) \quad \left| \int_{-\infty}^{\lambda} \tilde{\theta}h^2 * \mathcal{M}(\tau)d\tau - \int_{-\infty}^{\lambda} \mathcal{M}(\tau)d\tau \right| \leq C_1 \int \left( \frac{\tau - \lambda}{h^2} \right)^{-2} \mathcal{M}(\tau)d\tau.
\]
Putting together (4.14), (4.15) and (4.16), we get
\[(4.17) \quad \int_{-\infty}^{\lambda} \tilde{\theta}h^2 * \mathcal{M}(\tau)d\tau = M(\lambda, h) + O(1).
\]
Note that \( \tilde{\theta}h^2 * \mathcal{M}(\tau) = \text{tr} \left( \phi_3(H(h))\tilde{\theta}h^2(\tau - H(h)) \right) \). As a consequence of (2.4), (2.5) and (4.17) we obtain
\[(4.18) \quad M(\lambda, h) = h^{-2}m(\lambda) + O(1),\]
where
\[(4.19) \quad m(\lambda) = \int_{-\infty}^{\lambda} c_0(\tau)d\tau = \frac{1}{2\pi} \sum_{j=1}^{l} \int_{\{X \in \mathbb{R}^2; (2j+1) + V(X) \leq \lambda\}} \phi_3((2j + 1) + V(X))dX.
\]
Here we have used the fact that if \( E \) is not a critical value of \( V(X) \), then
\[
\frac{\partial}{\partial E} \left( \int_{\{X \in \mathbb{R}^2; V(X) \leq E\}} \phi(V(X))dX \right) = \phi(E) \int_{S_E} \frac{dS_E}{|\nabla X|},
\]
where \( S_E = V^{-1}(E) \) (see [22, Lemma V-9]).

Applying (2.2), (1.18) and (1.19) to \( \phi_1 \) and writing:
\[
\sum_{a \leq \gamma_j(h)} \phi_1(\gamma_j(h)) = \sum_{\gamma_j(h) < a} \phi_1(\gamma_j(h)) + \sum_{\gamma_j(h) = a} \phi_1(\gamma_j(h)),
\]
we get
\[(4.20) \quad \sum_{a \leq \gamma_j(h)} \phi_1(\mu_j(h)) = h^{-2}m_1(a) + O(1),\]
with
\[(4.21) \quad m_1(a) = \frac{1}{2\pi} \sum_{j=1}^{l} \int_{\{X \in \mathbb{R}^2; (2j+1) + V(X) \geq a\}} \phi_1((2j + 1) + V(X))dX.
\]
Now Corollary 2.3 results from (4.13), (4.14), (4.18), (4.19), (4.20) and (4.21).
5. Proof of Theorem 2.4 and Theorem 2.6

As we have noticed in the outline of the proofs, we will construct a potential
\[ \varphi(X; h) = \varphi_0(X) + \varphi_2(X)h^2 + \cdots + \varphi_{2j}(X)h^{2j} + \cdots, \]
such that for all \( f \in C_0^\infty((a, b); \mathbb{R}) \) and \( \theta \in C_0^\infty(\mathbb{R}; \mathbb{R}) \), we have
\[
(5.1) \quad \text{tr}(f(H_\lambda)) = \text{tr}(f(Q)) + \mathcal{O}(h^\infty),
\]
\[
(5.2) \quad \text{tr}\left(f(H_\lambda)\hat{\lambda}^{-\frac{1}{2}}(t - H_\lambda)\right) = \text{tr}\left(f(Q)\hat{\lambda}^{-\frac{1}{2}}(t - Q)\right) + \mathcal{O}(h^\infty),
\]
where \( Q := H_0 + \varphi(hX; h) \) and \( h = \lambda^{-\frac{1}{2}} \). By observing that Theorem 2.1, Theorem 2.2 and Corollary 2.3 remain true when we replace \( H(h) = H_0 + V(hX) \) by \( Q \), Theorem 2.4, Theorem 2.5 and Corollary 2.6 follow from (5.1) and (5.2). The remainder of this paper is devoted to the proof of (5.1) and (5.2).

5.1. Construction of reference operator \( Q \). Set \( h = \lambda^{-\frac{1}{2}} \). For \( M > 0 \), put
\[
(5.3) \quad \Omega_M(h) = \{ X \in \mathbb{R}^2; h^{-\delta}V(X) > M \}.
\]
Since \( \omega_0 > 0 \) and continuous on the unit circle, there exist two positive constants \( C_1 \) and \( C_2 \) such that \( C_1 < (\min_{S^1} \omega_0)^{1/\delta} \leq (\max_{S^1} \omega_0)^{1/\delta} < C_2 \).

According to the hypothesis (2.9), there exists \( h_0 > 0 \) such that
\[
B(0, C_1 M^{-1/\delta} h^{-1}) \subset \Omega_M(h) \subset B(0, C_2 M^{-1/\delta} h^{-1}), \quad \text{for all } 0 < h \leq h_0.
\]
Here \( B(0, r) \) denotes the ball of center 0 and radius \( r \).

Let \( \chi \in C_0^\infty(B(0, C_1 M^{-1/\delta}); [0, 1]) \) satisfying \( \chi = 1 \) near zero. Set
\begin{itemize}
  \item \( \varphi(X; h) := (1 - \chi(X))h^{-\delta}V(X) + M\chi(X), \)
  \item \( W_h(X) := h^{-\delta}V(X) - \varphi(hX; h) = \chi(hX)(h^{-\delta}V(X) - M). \)
\end{itemize}

By the construction of \( \varphi(\cdot; h) \) and \( W_h \), we have
\[
(5.4) \quad |\partial_X^\alpha \varphi(X; h)| \leq C_\alpha, \quad \text{uniformly for } h \in (0, h_0],
\]
\[
(5.5) \quad \varphi(hX; h) > \frac{M}{2} \quad \text{for } X \in \Omega_M(h),
\]
\[
(5.6) \quad \text{supp} W_h \subset B(0, C_1 M^{-1/\delta} h^{-1}) \subset \Omega_M(h).
\]
On the other hand, it follows from (2.9) that for all \( N \in \mathbb{N} \), there exist \( \varphi_0, \ldots, \varphi_{2N}, K_{2N+2}(\cdot; h) \in C^\infty(\mathbb{R}^2; \mathbb{R}) \), uniformly bounded with respect to \( h \in (0, h_0] \) together with their derivatives such that:
\[
(5.7) \quad \varphi(X; h) = \sum_{j=0}^N \varphi_{2j}(X)h^{2j} + h^{2N+2}K_{2N+2}(X; h)
\]
with
\[
\varphi_0(X) = (1 - \chi(X))\omega_0 \left( \frac{X}{|X|} \right) |X|^{-\delta} + M\chi(X).
\]
In fact, if \( X \in \text{supp} \chi \) then \( \omega_0 \left( \frac{X}{|X|} \right) |X|^{-\delta} > C_\delta |X|^{-\delta} > M \), which implies that \( \varphi_0(X) \geq (1 - \chi(X))M + M\chi(X) = M \) for all \( X \in \text{supp} \chi \). Consequently, we have
Lemma 5.1. If \( \varphi_0(X) < M \) then \( \varphi_0(X) = \omega_0 \left( \frac{X}{|X|} \right) |X|^{-\delta} \).

Let \( \psi \in C^\infty(\mathbb{R}; [\frac{M}{2}, +\infty)) \) satisfying \( \psi(t) = t \) for all \( t \geq \frac{M}{2} \). We define \( F_1(X; h) := \psi(\varphi(hX; h)) \) and \( F_2(X; h) := \psi(h^{-\delta}V(X)) \).

Let \( \mathcal{U} \) be a small complex neighborhood of \([a, b]\). From now on, we choose \( M > a + b \) large enough such that

\[
F_j(X; h) - \text{Re} z \geq \frac{M}{4}, \quad j = 1, 2,
\]

uniformly for \( z \in \mathcal{U} \). This choice of \( M \) implies that:

- If \( 2j + 1 + \varphi(X; h) \in [a, b] \) then \( \varphi_0(X) < M \) for all \( h \in (0, h_0] \),
- The function defined by \( z \mapsto (z - H_{F_j})^{-1} \) is holomorphic from \( \mathcal{U} \) to \( \mathcal{L}(L^2(\mathbb{R}^2)) \), where \( H_{F_j} := H_0 + F_j(X; h) \), \( j = 1, 2 \).

Moreover, it follows from (5.4) that \( \partial_X F_j(X; h) = O_\alpha(h^{-\delta}) \).

Finally, (5.5) shows that \( \partial_X F_j(X; h) = O_\alpha(h^{-\delta}) \).

Lemma 5.2. Let \( \tilde{\chi} \in C_0^\infty(\mathbb{R}^2) \). For \( z \in \mathcal{U} \), the operators \( \tilde{\chi}(hX)(z - H_{F_j})^{-1} \), \( j = 1, 2 \), belong to the class of Hilbert-Schmidt operators. Moreover

\[
\|\tilde{\chi}(hX)(z - H_{F_j})^{-1}\|_{\text{HS}} = O(h^{-3-\delta}).
\]

Here we denote by \( \|\cdot\|_{\text{HS}} \) the Hilbert-Schmidt norm of operators.

Proof. We prove (5.10) for \( j = 1 \). The case \( j = 2 \) is treated in the same way.

Using the resolvent equation, one has

\[
(z - H_{F_1})^{-1} = \left( z - \frac{M}{6} - H_0 \right)^{-1} + \left( z - \frac{M}{6} - H_0 \right)^{-1} \left( F_1(X; h) - \frac{M}{6} \right) (z - H_{F_1})^{-1}.
\]

On the other hand, the operator \( (z - \frac{M}{6} - H_0)^{-1} \) was shown to be an integral operator with integral kernel \( K_0(X, Y, z) \) satisfying \( |K_0(X, Y, z)| \leq C e^{-\frac{1}{3}|X - Y|^2} \) uniformly for \( z \in \mathcal{U} \) (see [4] Formula 2.17]). Let \( K_1(X, Y, z) \) be the integral kernel of \( \tilde{\chi}(hX)(z - \frac{M}{6} - H_0)^{-1} \). Then \( K_1(X, Y, z) = \tilde{\chi}(hX)K_0(X, Y, z) \).

Let \( \langle X \rangle = (1 + |X|^2)^{\frac{1}{2}} \), \( X \in \mathbb{R}^2 \). Since \( \tilde{\chi} \in C_0^\infty(\mathbb{R}^2) \), one has \( \tilde{\chi}(hX)h^3(X)^3 \) is uniformly bounded for \( h > 0 \). Combining this with the fact that \( \frac{1}{h^3(X)^3} e^{-\frac{1}{3}|X - Y|^2} \in L^2(\mathbb{R}^4) \), we obtain

\[
\|K_1(X, Y, z)\|_{L^2(\mathbb{R}^4)} = \left\| \tilde{\chi}(hX)h^3(X)^3 \frac{1}{h^3(X)^3} K_0(X, Y, z) \right\|_{L^2(\mathbb{R}^4)} = O(h^{-3}).
\]
It shows that \( \tilde{\chi}(hX)(z - \frac{M}{6} - H_0)^{-1} \) is a Hilbert-Schmidt operator and

\[
(5.13) \quad \left\| \tilde{\chi}(hX) \left( z - \frac{M}{6} - H_0 \right)^{-1} \right\|_{HS} = \| K_1(X, Y; z) \|_{L^2(\mathbb{R}^4)} = O(h^{-3}).
\]

Consequently, (5.11) and (5.13) imply that

\[
\| \tilde{\chi}(hX)(z - H_{F_1})^{-1} \|_{HS} \leq \left\| \tilde{\chi}(hX) \left( z - \frac{M}{6} - H_0 \right)^{-1} \right\|_{HS}
\]

\[
+ \left\| \tilde{\chi}(hX) \left( z - \frac{M}{6} - H_0 \right)^{-1} \right\|_{HS} \| F_1(X; h) - \frac{M}{6} \|_{L^\infty(\mathbb{R}^2)} \| (z - H_{F_1})^{-1} \|
\]

\[= O(h^{-3-\delta}),
\]

where we have used \( F_1(X; h) = O(h^{-\delta}). \)

\[
\square
\]

**Lemma 5.3.** For \( z \in \mathcal{U}, \) the operator

\[
W_h(X)(H_{F_1} - z)^{-1}(\varphi(hX; h) - F_1(X; h))
\]

belongs to the class of Hilbert-Schmidt operators. Moreover,

\[
(5.14) \quad \| W_h(X)(H_{F_1} - z)^{-1}(\varphi(hX; h) - F_1(X; h)) \|_{HS} = O(h^\infty).
\]

**Proof.** Let \( H_{F_1}^0 := -\Delta + F_1(X; h). \) We denote by \( G(X, Y; z) \) (resp. \( G_0(X, Y; \text{Re}z) \)) the Green function of \( (H_{F_1} - z)^{-1} \) (resp. \( (H_{F_1}^0 - \text{Re}z)^{-1} \)).

From the functional calculus, one has

\[
(H_{F_1} - z)^{-1} = \int_0^\infty e^{tz} e^{-tH_{F_1}} dt,
\]

\[
(H_{F_1}^0 - \text{Re}z)^{-1} = \int_0^\infty e^{t\text{Re}z} e^{-tH_{F_1}^0} dt.
\]

For \( t \geq 0, \) the Kato inequality (see [5] Formula 1.8) implies that

\[
(5.15) \quad \| e^{-tH_{F_1}} u \| \leq e^{-tH_{F_1}^0} \| u \| \text{ (pointwise), } u \in L^2(\mathbb{R}^2).
\]

Then (5.15) and (5.16) yield

\[
(5.17) \quad \| (H_{F_1} - z)^{-1} u \| \leq (H_{F_1}^0 - \text{Re}z)^{-1} \| u \| \text{ (pointwise), } u \in L^2(\mathbb{R}^2).
\]

Consequently, applying [3] Theorem 10 we have \( |G(X, Y; z)| \leq G_0(X, Y; \text{Re}z) \) for a.e. \( X, Y \in \mathbb{R}^2. \) From this, one obtains

\[
(5.18) \quad \| W_h(X)G(X, Y; z)(\varphi(hY; h) - F_1(Y; h)) \| \leq \| W_h(X)G_0(X, Y; \text{Re}z)(\varphi(hY; h) - F_1(Y; h)) \|
\]

for a.e. \( X, Y \in \mathbb{R}^2. \)

On the other hand, using (5.9) M. Dimassi proved that (see [6] Proposition 3.3)

\[
(5.19) \quad \| W_h(X)G_0(X, Y; \text{Re}z)(\varphi(hY; h) - F_1(Y; h)) \|_{L^4(\mathbb{R}^4)} = O(h^\infty).
\]

Thus, (5.18) and (5.19) give

\[
(5.20) \quad \| W_h(X)G(X, Y; z)(\varphi(hY; h) - F_1(Y; h)) \|_{L^4(\mathbb{R}^4)} = O(h^\infty).
\]
The estimate (5.20) shows that the operator $W_h(X)(H_{F_1} - z)^{-1}(\varphi(hX; h) - F_1(X; h))$ is Hilbert-Schmidt and
\begin{equation}
\|W_h(X)(H_{F_1} - z)^{-1}(\varphi(hX; h) - F_1(X; h))\|_{HS} = O(h^\infty).
\end{equation}

By using the same arguments as in Lemma 5.3 we also obtain

**Lemma 5.4.** For $z \in \mathcal{U}$, the operator
\[ W_h(X)(H_{F_2} - z)^{-1}(h^{-\delta}V(X) - F_2(X; h)) \]
belongs to the class of Hilbert-Schmidt operators and
\[ \|W_h(X)(H_{F_2} - z)^{-1}(h^{-\delta}V(X) - F_2(X; h))\|_{HS} = O(h^\infty). \]

Let $Q := H_0 + \varphi(hX; h)$. For $z \in \mathcal{U}$, $\text{Im} z \neq 0$, put
\begin{equation}
G(z) = (z - H_\lambda)^{-1} - (z - Q)^{-1} - (z - H_{F_2})^{-1}W_h(z - H_{F_1})^{-1}.
\end{equation}

**Proposition 5.5.** The operator $G(z)$ is of trace class and satisfies the following estimate:
\begin{equation}
\|G(z)\|_{tr} = O(h^\infty|\text{Im} z|^{-2}),
\end{equation}
uniformly for $z \in \mathcal{U}$ with $\text{Im} z \neq 0$.

**Proof.** It follows from the resolvent equation that
\begin{equation}
(z - H_\lambda)^{-1} - (z - Q)^{-1} = (z - H_\lambda)^{-1}W_h(z - Q)^{-1}.
\end{equation}
On the other hand, one has
\begin{equation}
(z - H_\lambda)^{-1} = (z - H_{F_2})^{-1} + (z - H_\lambda)^{-1}(h^{-\delta}V(X) - F_2(X; h))(z - H_{F_2})^{-1}
\end{equation}
and
\begin{equation}
(z - Q)^{-1} = (z - H_{F_1})^{-1} + (z - H_{F_1})^{-1}(\varphi(hX; h) - F_1(X; h))(z - Q)^{-1}.
\end{equation}
Substituting (5.25) and (5.26) into the right hand side of (5.24), one gets
\begin{align*}
G(z) &= (z - H_{F_2})^{-1}W_h(z - H_{F_1})^{-1}(\varphi(hX; h) - F_1(X; h))(z - Q)^{-1} \\
&+ (z - H_\lambda)^{-1}(h^{-\delta}V(X) - F_2(X; h))(z - H_{F_2})^{-1}W_h(z - H_{F_1})^{-1} \\
&+ (z - H_\lambda)^{-1}(h^{-\delta}V(X) - F_2(X; h))(z - H_{F_2})^{-1}W_h(z - H_{F_1})^{-1} \times
\quad (\varphi(hX; h) - F_1(X; h))(z - Q)^{-1} \\
&=: A(z) + B(z) + C(z).
\end{align*}
Next we choose $\tilde{\chi} \in C_0^\infty(\mathbb{R}^2)$ such that $\tilde{\chi}(hX)W_h(X) = W_h(X)$. It follows from Lemma 5.2 and Lemma 5.3 that
\[ \|A(z)\|_{tr} \leq \|(z - H_{F_2})^{-1}\tilde{\chi}(hX)\|_{HS}\|W_h(z - H_{F_1})^{-1}(\varphi(hX; h) - F_1(X; h))\|_{HS}\|(z - Q)^{-1}\| = O(h^\infty|\text{Im} z|^{-1}). \]
Here we have used the fact that $\|(z - Q)^{-1}\| = O(|\text{Im} z|^{-1})$. Similarly, we also obtain $\|B(z)\|_{tr} = O(h^\infty|\text{Im} z|^{-1})$ and $\|C(z)\|_{tr} = O(h^\infty|\text{Im} z|^{-2})$. Thus,
\[ \|G(z)\|_{tr} \leq \|A(z)\|_{tr} + \|B(z)\|_{tr} + \|C(z)\|_{tr} = O(h^\infty|\text{Im} z|^{-2}). \]
5.2. Proof of (5.1) and Theorem 2.4. Let $f \in C_0^\infty((a, b); \mathbb{R})$ and let $\tilde{f} \in C_0^\infty(U)$ be an almost analytic extension of $f$. From the Helffer- Sjöstrand formula and (5.22), we get

$$f(H_\lambda) - f(Q)$$

(5.27)

$$= -\frac{1}{\pi} \int \overline{\partial}_z \tilde{f}(z) [(z - H_\lambda)^{-1} - (z - Q)^{-1}] L(dz)$$

$$= -\frac{1}{\pi} \int \overline{\partial}_z \tilde{f}(z) [(z - H_{F_2})^{-1}W_h(z - H_{F_1})^{-1} + G(z)] L(dz).$$

Notice that $(z - H_{F_2})^{-1}W_h(z - H_{F_1})^{-1}$ is holomorphic in $z \in U$, then

(5.28)

$$-\frac{1}{\pi} \int \overline{\partial}_z \tilde{f}(z) (z - H_{F_2})^{-1}W_h(z - H_{F_1})^{-1} L(dz) = 0.$$

Thus, (5.27) and (5.28) follow that

(5.29)

$$f(H_\lambda) - f(Q) = -\frac{1}{\pi} \int \overline{\partial}_z \tilde{f}(z) G(z) L(dz),$$

which together with (5.23) yields (5.1).

Applying Theorem 2.4 to the operator $Q$ and using (5.1) we obtain (2.10) with

$$b_0(f) = \sum_{j=0}^q \frac{1}{2\pi} \int f(2j + 1 + \varphi_0(X)) dX$$

According to Lemma 5.1, one has

$$2j + 1 + \varphi_0(X) \in [a, b] \iff \varphi_0(X) = \omega_0 \left( \frac{X}{|X|} \right) |X|^{-\delta}, j = 0, ..., q.$$

Thus, after a change of variable in the integral we get

$$b_0(f) = \frac{1}{2\pi\delta} \int_0^{2\pi} (\omega_0(\cos \theta, \sin \theta))^\frac{\hat{q}}{2} d\theta \sum_{j=0}^q \int f(u)(u - (2j + 1))^{-1 - \frac{\hat{q}}{2}} du.$$

We recall that $\text{supp} f \subset ]a, b[$, with $2q + 1 < a < b < 2q + 3$. This ends the proof of Theorem 2.4.

5.3. Proof of (5.2) and Theorem 2.5. The proof of (5.2) is a slight modification of (5.1). For that, let $\phi \in C_0^\infty((-2, 2); [0, 1])$ such that $\phi = 1$ on $[-1, 1]$. Put $\phi_h(z) = \phi \left( \frac{|\text{Im} z|}{h} \right)$, then $\tilde{f}(z) \phi_h(z)$ is also an almost analytic extension of $f$. Applying again the Helffer-Sjöstrand formula, we get

$$f(H_\lambda) \tilde{\phi} \frac{1}{2\pi} \int (t - H_\lambda) - f(Q) \tilde{\phi}_h^2(t - Q)$$

(5.30)

$$= -\frac{1}{\pi} \int \overline{\partial}_z (\tilde{f} \phi_h)(z) \tilde{\phi}_h^2(t - z) [(z - H_\lambda)^{-1} - (z - Q)^{-1}] L(dz)$$

$$= -\frac{1}{\pi} \int \overline{\partial}_z (\tilde{f} \phi_h)(z) \tilde{\phi}_h^2(t - z) [ (z - H_{F_2})^{-1}W_h(z - H_{F_1})^{-1} + G(z)] L(dz)$$

$$= -\frac{1}{\pi} \int \overline{\partial}_z (\tilde{f} \phi_h)(z) \tilde{\phi}_h^2(t - z) G(z) L(dz),$$
where in the last equality we have used the fact that \((z - H_{F_1})^{-1} W_h(z - H_{F_1})^{-1}\) is holomorphic in \(z \in \mathcal{U}\).

According to the Paley-Wiener theorem (see e.g. [21 Theorem IX.11]) the function \(\bar{\theta}_{h^2}(t - z)\) is analytic with respect to \(z\) and satisfies the following estimate

\[
\bar{\theta}_{h^2}(t - z) = \mathcal{O}\left(\frac{1}{h^2} \exp\left(\frac{|\text{Im} z|}{C h^2}\right)\right).
\]

Combining this with the fact that \(\overline{\partial_z (\bar{f} \phi_h)}(z) = \mathcal{O}(|\text{Im} z|)\phi_h(z) + \mathcal{O}\left(\frac{1}{h^2}\right) 1_{[h^2, 2h^2]}(|\text{Im} z|)\), and using Proposition 5.5 we get

\[
|\overline{\partial_z (\bar{f} \phi_h)}(z)\bar{\theta}_{h^2}(t - z) G(z)|_{\text{tr}} = \mathcal{O}(h^\infty).
\]

This together with (5.30) ends the proof of (5.2).

By observing that \(X.\nabla_X \left(\omega_0 \left(\frac{X}{|X|}\right)\right) = 0\), we have

\[
X.\nabla_X \left(\omega_0 \left(\frac{X}{|X|}\right) |X|^{-\delta}\right) = -\delta \omega_0 \left(\frac{X}{|X|}\right) |X|^{-\delta}.
\]

Then, since \(\omega_0 > 0\), we obtain \(\nabla_X \omega_0 \left(\frac{X}{|X|}\right) |X|^{-\delta} \neq 0\) for \(X \in \mathbb{R}^2 \setminus \{0\}\). It implies that the functions \(2j + 1 + \varphi_0(X), j = 1, \ldots, q,\) do not have any critical values in the interval \([a, b]\).

Consequently, Theorem 2.5 follows from (5.2) and Theorem 2.2.

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