TWISTED CONJUGACY CLASSES IN RESIDUALLY FINITE GROUPS

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Abstract. We prove for residually finite groups the following long standing conjecture: the number of twisted conjugacy classes \((g \sim h\phi(h^{-1}))\) of an automorphism \(\phi\) of a finitely generated group is equal (if it is finite) to the number of finite dimensional irreducible unitary representations being invariant for the dual of \(\phi\).

Also, we prove that any finitely generated residually finite non-amenable group has the \(R_\infty\) property (any automorphism has infinitely many twisted conjugacy classes). This gives a lot of new examples and covers many known classes of such groups.

1. Introduction

The following two interrelated problems are among the principal ones in the theory of twisted conjugacy (Reidemeister) classes in infinite discrete groups. The first one is the 20-years-old conjecture on existence of an appropriate twisted Burnside-Frobenius theory (TBFT), i.e. identification of the number \(R(\phi)\) of Reidemeister classes and the number of fixed points of the induced homeomorphism \(\hat{\phi}\) on an appropriate dual object (supposing \(R(\phi) < \infty\)). The second one is the problem to outline the class of \(R_\infty\) groups (that is \(R(\phi) = \infty\) for any \(\phi\)).

In this paper important advances in both problems are obtained. Namely, first, it is proved that TBFT holds for finitely generated residually finite groups (we take the finite-dimensional part of the unitary dual as the dual space). Secondly, it is discovered that finitely generated residually finite non-amenable groups are \(R_\infty\)-groups. Several supplementary results of independent interest are obtained.

The interest in twisted conjugacy relations has its origins, in particular, in the Nielsen-Reidemeister fixed point theory (see, e.g. [66, 27]), in Selberg theory (see, eg. [101, 1]), and Algebraic Geometry (see, e.g. [57]). In representation theory twisted conjugacy probably occurs first in Gantmacher’s paper [47] (see, e.g [102, 88]).

The problem of determining which classes of groups have the \(R_\infty\) property is an area of active research initiated in [29]. In some situations the \(R_\infty\) property (for fundamental groups) has direct topological consequences. For example, using this property in [54] for any \(n \geq 4\) a compact nilmanifold \(M\), \(\dim M = n\), is constructed, such that any homeomorphism \(f : M \to M\) is homotopic (for \(n \geq 5\) isotopic) to a fixed point free map. In [20] it is

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shown that an infra-nilmanifold that admits an Anosov diffeomorphism cannot have the $R_\infty$ property since the Reidemeister number of an Anosov diffeomorphism on an infra-nilmanifold is always finite. Also, as we will see below it can be used to detect the trivial knot.

The Reidemeister class of $g \in G$ we will denote by $\{g\}_\phi$: $\{g\}_\phi := \{xg\phi(x^{-1}) \in G \mid x \in G\}$. A $\phi$-class function is any function on $G$, which is constant on Reidemeister classes, or, equivalently, which is invariant under the twisted action $g \mapsto xg\phi(x^{-1})$ of $G$ on itself.

**Definition 1.1.** Denote by $\widehat{G}$ the set of equivalence classes of unitary irreducible representations of $G$ and by $\widehat{G}_f$ its part corresponding to finite-dimensional representations. The class of $\rho$ in $\widehat{G}$ we will denote by $[\rho]$. An automorphism $\phi$ of $G$ induces a bijection $\hat{\phi}: \widehat{G} \to \hat{G}$ by the formula $[\hat{\phi}(\rho)] := [\rho \circ \phi]$.

The following (first) main theorem is proved as Theorem 4.1 below.

**Theorem A.** Let $\phi: G \to G$ be an automorphism of a residually finite finitely generated group $G$ with $R(\phi) < \infty$. Then $R(\phi)$ is equal to the number of $\hat{\phi}$-fixed points on $G_f$.

This theorem covers the known case of almost polycyclic groups [33] and contains f.g. metabelian groups and restricted wreath products. Some known examples of $\phi$ with $R(\phi) < \infty$ are listed in Section 8.

In the case of $|G| < \infty$ and $\phi = \text{Id}$ it becomes the celebrated Burnside-Frobenius theorem, which says that the number of conjugacy classes is equal to the number of equivalence classes of irreducible unitary representations of $G$. In [29] it was discovered that this statement remains true for any automorphism $\phi$ of any finite group $G$. Indeed, $R(\phi)$ is equal to the dimension of the space of $\hat{\phi}$-class functions on this group. Hence, by Peter-Weyl theorem, $R(\phi)$ is identified with the sum of dimensions $d_\rho$ of twisted invariant elements of End$(H_\rho)$, where $\rho$ runs over $\widehat{G}$, and the space of a representation $\rho$ is denoted by $H_\rho$. By the Schur lemma, $d_\rho = 1$, if $\rho$ is a fixed point of $\hat{\phi}$, and is zero otherwise. Hence, $R(\phi)$ coincides with the number of fixed points of $\hat{\phi}$.

Theorem A has important dynamical consequences (see [27, 32] for an extended discussion). In particular, suppose, $R(\hat{\phi}^n) < \infty$, $n \geq 1$, in Theorem A. Then we have for all $n$ the following Gauss congruences for the Reidemeister numbers, which are important for the theory of the Reidemeister zeta function:

$$\sum_{d|n} \mu(d) \cdot R(\phi^{n/d}) \equiv 0 \mod n,$$

where $\mu(d), d \in \mathbb{N}$, is the M"{o}bius function. Indeed, apply the M"{o}bius’ inversion formula to the following evident consequence of Theorem A: $R(\phi^n) = \# \text{Fix}(\hat{\phi}^n|_{G_f}) = \sum_{d|n} P_d$, where $P_n$ denote the number of periodic points of $\hat{\phi}$ on $G_f$ of least period $n$. We obtain (1) with $P_n$ on the right. But $P_n$ is always divisible by $n$, because $P_n$ is exactly $n$ times the number of orbits of length $n$.

Formula (1) was previously known in the special case of almost polycyclic groups (see [32, 33], where more detail can be found). The history of the Gauss congruences for integers can be found in [111]. Gauss congruences for Lefschetz numbers of iterations of a continuous map were proved by A. Dold [23].

The second main theorem (proved as Corollary 5.6 below) is as follows.

**Theorem B.** Any non-amenable residually finite finitely generated group is an $R_\infty$ group. (It is enforced in Prop. 5.8).
This gives a lot of new examples of groups with $R_\infty$-property. (A list of known cases of $R_\infty$ and non-$R_\infty$ groups can be find in Section 8, to not overload the Introduction.) Among the new classes we would like to emphasize the following important ones.

1) Irreducible infinite non-affine finitely generated Coxeter groups. Indeed, they are linear [106], hence residually finite [75]. They contain a subgroup which maps onto a non-amenable group [76], thus they are non-amenable.

2) Lattices (finitely generated cocompact discrete subgroups) $G$ in direct products $G_1 \times \cdots \times G_m$ of linear groups over fields $k_1, \ldots, k_m$ of zero characteristic, where at least one of $G_i$, say $G_1$, is a non-compact semi-simple Lie group. Indeed, $G_1$ is non-amenable by [44] (see [6, p. 432]), hence $G_1 \times \cdots \times G_m$ is non-amenable, and $G$ is non-amenable (see, e.g. [6, Corollary G.3.8]). By the same argument as in the previous item, $G$ is residually finite. This class contains $S$-arithmetic lattices, in particular groups from (7), Sect. 8.

3) Fundamental groups of compact 3-dimensional manifolds and knot groups with few exceptions. They are finitely generated and residually finite by [61] and Perelman’s proof of Thurston’s geometrization conjecture [104]. More precisely, the JSJ decomposition (see, e.g. [60, 98]) gives parts, which are either Seifert manifolds, or hyperbolic. The fundamental group is a graph product of fundamental groups of these parts. Thus, by Theorem B, the group under consideration is $R_\infty$ group if it is non-amenable. Such groups are characterized geometrically (via bounded cohomology [64]) in [43, 42].

In particular, via existence/non-existence of $R_\infty$ property for knot groups we can detect unknot. Indeed, since the trivial knot’s group is abelian, it is not $R_\infty$, while the other ones have infinite-dimensional second bounded cohomology [43, 42]. Thus, they are non-amenable and $R_\infty$ groups by Theorem B.

4) Several important classes are among non-linear groups:

(a) Periodic Golod-Shafarevich groups are finitely generated residually finite non-amenable (having quotients with property (T)) [24] (see also [25]). A linear non-amenable group contains $F_2$ (Tits alternative, [109, pp. 145–146]). Thus, periodic Golod-Shafarevich groups are non-linear.

(b) Similarly, finitely generated infinite torsion groups with positive rank gradient from [89] or positive power $p$-deficiency from [96] (see also [25, Sect. 9]) are $R_\infty$ groups.

(c) $\text{Aut}(F_n), n \geq 3,$ is finitely generated [86], residually finite (Baumslag, see [5, Sect. 2]), non-amenable and non-linear [40]. The same is true for $\text{Out}(F_n), n \geq 4$ (see e.g. [107]).

Also, the theorem covers iterated semidirect products $F_{d_1} \rtimes \cdots \rtimes F_{d_k}$ of free groups and its finite extensions (poly-fg-free-by-finite groups), because they are residually finite (see, e.g. [80, Theorem 7, p. 29]) and evidently non-amenable (having a non-amenable subgroup) and finitely generated. This class contains Artin’s pure braid groups, the fundamental group of the complement of any affine fiber-type hyperplane arrangement and some others, see e.g. [14, 79].

On the other hand, the new result covers (up to the conjecture about the residually finiteness) the most extended known class, non-elementary Gromov hyperbolic groups (1) in Sect. 8.

Some other new classes are discussed in Section 8.
In Section 3 we establish important inequalities, relating the number of fixed points of \( \phi \) and \( R(\phi) \). We prove Theorem A in Section 4 and Theorem B in Section 5. In Section 6 we introduce (ir)rational representations and describe their connection with twisted conjugacy classes. In Section 7 we introduce and investigate a new class of groups: \( S_{\infty} \) groups.

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2. Preliminary Considerations

We start from the following estimation (supposed to be known to E. Landau (cf. [71]) and R. Brauer (cf. [12])) (cf. [84]).

**Lemma 2.1.** Let \( x_1 \leq x_2 \leq \ldots \leq x_n \) be positive integers such that \( \sum_{i=1}^{n} (x_i)^{-1} = 1 \). Then \( x_n \leq n^{2n-1} \).

**Proof.** One has
\[
\frac{n - r}{x_{r+1}} \geq \sum_{i=r+1}^{n} \frac{1}{x_i} = 1 - \sum_{i=1}^{r} \frac{1}{x_i} = \frac{y_r}{x_1 x_2 \ldots x_r} \geq \frac{1}{x_1 x_2 \ldots x_r},
\]

(3)
\[
x_1 x_2 \ldots x_r \geq (n - r)x_1 x_2 \ldots x_r \geq x_{r+1}.
\]

By induction, (2) and (3) imply the statement. \( \square \)

With some additional efforts the estimation can be improved (cf. e.g. [84]) but we do not need this. Let us note that, since \( \log_2 n < n \), one has
\[
n^{n-1} < n^n = 2^{n \log_2 n} < 2^{n^2}.
\]

Denote the stabilizers related to the twisted action of \( G \) on itself by
\[
St_{\phi}^{tw}(g, h) := \{ k \in \Gamma \mid kg\phi(k^{-1}) = h \}, \quad St_{\phi}^{tw}(g) := St_{\phi}^{tw}(g, g).
\]

In particular,
\[
St_{\phi}^{tw}(e) = \{ k \in \Gamma \mid k\phi(k^{-1}) = e \} = C_G(\phi)
\]
(fixed elements of \( \phi \)). Evidently, \( St_{\phi}^{tw}(g, h) = \emptyset \) if \( h \not\in \{ g \}_\phi \). Otherwise
\[
St_{\phi}^{tw}(g, sg\phi(s^{-1})) = \{ k \in \Gamma \mid kg\phi(k^{-1}) = sg\phi(s^{-1}) \} = \{ k \in \Gamma \mid s^{-1}kg\phi(k^{-1}s) = g \}.
\]
i.e. \( St^t_w(g, sg\phi(s^{-1})) = s \cdot St^t_w(g) \) is a coset of this group. Thus
\[
|St^t_w(g, sg\phi(s^{-1}))| = |St^t_w(g)|.
\]

An evident consequence of the above statement is the following lemma (see [65, Lemma 4]).

**Lemma 2.2.** Let \(|G| < \infty\), \(r = R(\phi)\), \(\phi : G \to G\). Then \(|C_G(\phi)| \leq r^{r-1}\).

**Proof.** Let \(\{g_i\}_\phi\) be distinct Reidemeister classes, \(i = 1, \ldots, r\), \(g_1 = e\). Then
\[
|G| = \sum_{i=1}^r \#\{g_i\}_\phi = \sum_{i=1}^r \frac{|G|}{|St^t_w(g_i)|}.
\]
Dividing by \(|G|\) and applying Lemma 2.1 we obtain
\[
|C_G(\phi)| = |St^t_w(e)| \leq \max_i |St^t_w(g_i)| \leq r^{r-1}.
\]

The following fact is well known.

**Lemma 2.3.** Let \(G\) be finitely generated, and \(H' \subset G\) its subgroup of finite index. Then there is a characteristic subgroup \(H \subset G\) of finite index, \(H \subset H'\).

**Proof.** Since \(G\) is finitely generated, there is only finitely many subgroups of the same index as \(H'\) (see [59], [70, § 38]). Let \(H\) be their intersection. Then \(H\) is characteristic, in particular normal, and of finite index.

Let us denote by \(\tau_g : G \to G\) the automorphism \(\tau_g(\tilde{g}) = g\tilde{g}g^{-1}\) for \(g \in G\). Its restriction to a normal subgroup we will denote by \(\tau_g\) as well.

**Lemma 2.4.** \(\{g\}_\phi k = \{g k\}_{\tau_{k^{-1}} \circ \phi}\).

**Proof.** Let \(g' = fg \phi(f^{-1})\) be \(\phi\)-conjugate to \(g\). Then
\[
g'k = fg \phi(f^{-1})k = fg k \phi(f^{-1})k = fg (g k) (\tau_{k^{-1}} \circ \phi)(f^{-1}).
\]
Conversely, if \(g'\) is \((\tau_{k^{-1}} \circ \phi)\)-conjugate to \(g\), then
\[
g'k^{-1} = fg (\tau_{k^{-1}} \circ \phi)(f^{-1})k^{-1} = fg k^{-1} \phi(f^{-1}).
\]
Hence a shift maps \(\phi\)-conjugacy classes onto classes related to another automorphism.

**Corollary 2.5.** \(R(\phi) = R(\tau_g \circ \phi)\).

If \(H \subset G\) is a \(\phi\)-invariant and normal, the quotient map induces an epimorphism of Reidemeister classes, in particular,
\[
R_G(\phi) \geq R_{G/H}(\overline{\phi})
\]
for the induced automorphism \(\overline{\phi} : G/H \to G/H\) (see [48, 49, 50] and [33, Sect. 3] for details and refinements).

**Definition 2.6.** A group \(G\) is called **residually finite** if for any finite set \(K \subset G\) there exists a normal group \(H\) of finite index such that \(H \cap K = \emptyset\). Taking \(K\) formed by \(g_i^{-1}g_j\) for some finite set \(K_0 = \{g_1, \ldots, g_s\}\) one obtains an epimorphism \(G \to G/H\) onto a finite group, which is injective on \(K_0\).
Remark 2.7. If a residually finite group $G$ is finitely generated, then groups $H$ in the
definition can be supposed to be characteristic (by Lemma 2.3). We will remind now some facts from Representation Theory and Harmonic Analysis (see
[22] and [6] for an effective introduction). (Left) regular representation $\lambda_G$ is the unitary
representation of $G$ on $\ell^2(G)$ by left translations. The completion $C^*_\lambda(G)$ of $\ell^1(G)$ by
the norm of $B(\ell^2(G))$ is called reduced group $C^*$-algebra of $G$. The completion $C^*(G)$ of $\ell^1(G)$ by
the norm of all unitary representations is called (full) group $C^*$-algebra of $G$. The algebra
$C^*_\lambda(G)$ is a quotient of $C^*(G)$.

Non-degenerate representations of $C^*(G)$ are exactly unitary representations of $G$, in
particular, $\hat{C}(G) = \hat{\hat{G}}$. For a representation $\rho$ of $G$ we denote by $C^*\rho$ the corresponding
representation of $C^*(G)$, and by $C^*\text{Ker}\rho$ the kernel of $C^*\rho$. One introduces on $\hat{\hat{G}}$ the
Jacobson-Fell or hull-kernel topology defining the closure of a set $X$ by the following formula
\[
\overline{X} = \{[\rho] : C^*\text{Ker}\rho \supseteq \bigcup_{[\pi] \in X} C^*\text{Ker}\pi \}.
\]
This topology can be described in terms of weak containment: a representation $\rho$ is weakly
contained in representation $\pi$ (we write $\rho \prec \pi$) if diagonal matrix coefficients of $\rho$ can be
approximated by linear combinations of diagonal matrix coefficients of $\pi$ uniformly on finite
sets. Here a matrix coefficient of a representation $\rho$ on a Hilbert space $H$ is the function
$g \mapsto \langle \rho(g)\xi, \eta \rangle$ on $G$ for some fixed $\xi, \eta \in H$, and a diagonal one corresponds to $\xi = \eta$. Then
$C^*\text{Ker}\pi \subset C^*\text{Ker}\rho$ if and only if $\rho \prec \pi$. Since
\[
C^*\text{Ker}(\rho_1 \oplus \cdots \oplus \rho_m) = \cap_{i=1}^m C^*\text{Ker}\rho_i,
\]
(7)
\[
\rho_1 \oplus \cdots \oplus \rho_m \prec \pi \text{ if } \rho_i \prec \pi, \quad i = 1, \ldots, m.
\]
An amenable group may be characterized in several equivalent ways (see e.g. [6]), in
particular:

- There exists an invariant mean on $\ell^\infty(G)$.
- $1_G \prec \lambda_G$, where $1_G$ is the trivial 1-dimensional representation.
- $C^*(G) = C^*_\lambda(G)$.

A group $G$ has property $(T)$ if the trivial 1-dimensional representation $1_G$ is an isolated
point of $\hat{\hat{G}}$ and is $C^*$-simple if $\rho \prec \lambda$ implies $\lambda \prec \rho$ for any $\rho$ [6, 19].

Definition 2.8. A function $f : G \to \mathbb{C}$ is called positively definite function if for any finite
collection $\{g_1, \ldots, g_s\} \subset G$ the matrix $\|f(g_i^{-1}g_j)\|$ is non-negatively definite.

Definition 2.9. Fourier-Stieltjes algebra $B(G)$ can be defined in three equivalent ways:

1. $B(G)$ is formed by finite linear combinations of positively definite functions;
2. $B(G)$ is formed by all matrix coefficients of all unitary representations of $G$;
3. $B(G) = C^*(G)'$, i.e. it is the Banach space dual to $C^*(G)$.

Remark 2.10. Matrix coefficients of distinct finite-dimensional representations are linear
independent (see [16, Corollary (27.13)]) for an algebraic argument). Another argument uses
the fact that these representations are disjoint having maximal ideals as kernels. Passing to
the weak containment interpretation gives the linear independence.

Let us remind now some facts from [32, 33].
Lemma 2.11. Let \( \rho \) be a finite dimensional irreducible representation of \( G \) on \( V_\rho \), and \( \phi : G \to G \) is an automorphism.

1. There exists a twisted invariant function \( \omega : G \to \mathbb{C} \) being a matrix coefficient of \( \rho \) if and only if \( \hat{\phi}[\rho] = [\rho] \).

2. In this case such \( \omega \) is unique up to scaling.

3. The space \( K_\rho := \{ b \in \text{End} V_\rho \mid b = a - \rho(g)ab(\phi(g^{-1})) \text{ for some } g \in G \text{ and some } a \in \text{End} V_\rho \} \) has codimension 1 if \( \hat{\phi}[\rho] = [\rho] \) and coincides with \( \text{End} V_\rho \) otherwise.

4. If we have several distinct \( \phi \)-fixed classes, then the correspondent twisted invariant functions are independent.

Proof. Let us sketch a proof, the details can be found in [32, 33, 105]. Matrix coefficients of a finite dimensional representation \( \rho \) arise from functionals on \( \text{End} V_\rho \) and can be written as \( g \mapsto \text{Trace}(a\rho(g)) \) for some matrix \( a \in \text{End} V_\rho \). Since the equality

\[
0 = \text{Trace}(ab) - \text{Trace}(a\rho(h)b(\phi(h^{-1}))) = \text{Trace}((a - \rho(\phi(h^{-1}))a\rho(h))b)
\]

for any \( b \) and \( h \) implies \( \rho(\phi(h))a = a\rho(h) \), the above matrix coefficient is twisted invariant if and only if \( a \) is an intertwining operator between \( \rho \) and \( \phi \circ \rho \). This gives 1), and Schur’s lemma gives 2). Now, 3) follows immediately from 1) and 2), because \( K_\rho \) is evidently the intersection of kernels of all twisted invariant functionals on \( \text{End} V_\rho \). Finally, 4) follows from Remark 2.10.

\[
\square
\]

3. Fixed points and Reidemeister numbers

Theorem 3.1 (an extraction from [65]). Let \( \Gamma \) be a finitely generated residually finite group. If \( C_\Gamma(\phi) = \text{Fix}(\phi) \) is infinite, then \( R(\phi) = \infty \).

Proof. By Lemma 2.2 and inequality (4) for any automorphism \( \psi \) of a finite group \( G \)

\[
(8) \quad \sqrt{\log_2 |C_G(\psi)|} \leq R_G(\psi).
\]

Let \( \{x_1, x_2, \ldots\} = C_\Gamma(\phi) \). Then for every \( n \) we can find (see Remark 2.7) a characteristic subgroup \( \Gamma_n \) of finite index in \( \Gamma \) such that the quotient map \( p_n : \Gamma \to \Gamma/\Gamma_n =: G_n \) is injective on \( \{x_1, \ldots, x_n\} \). Let \( \phi_n : G_n \to G_n \) be the induced automorphism. Then \( \{p_n(x_1), \ldots, p_n(x_n)\} \subset C_{G_n}(\phi_n) \), hence (6) and (8) imply

\[
R_\Gamma(\phi) \geq R_{G_n}(\phi_n) \geq \sqrt{\log_2 |C_{G_n}(\phi_n)|} \geq \sqrt{\log_2 n}.
\]

Since \( n \) was arbitrary, we are done.

\[
\square
\]

Corollary 3.2. Suppose, \( \Gamma \) is a torsion-free finitely generated residually finite group. If \( C_\Gamma(\phi) > 1 \), then \( R(\phi) = \infty \).

Proof. Indeed, in this case \( \Gamma \) has an infinite subgroup \( \cong \mathbb{Z} \) formed by fixed points.

\[
\square
\]

Theorem 3.3. Let \( \Gamma \) be as in Theorem 3.1 and \( R(\phi) < \infty \). Then \( |St_{\phi}(g, sg(\phi(s^{-1})))| < \infty \) for any \( g, s \in \Gamma \).

Proof. By (5) we need to estimate \( |St_{\phi}(g)| \) only. Under the left translation by \( g^{-1} \) the Reidemeister class \( \{g\}_\phi \) will be moved to the class \( \{e\}_{\tau_{g^{-1}} \phi} \) (see Lemma 2.4). Thus

\[
|St_{\phi}(g)| = |St_{\tau_{g^{-1}} \phi}(e)| = |C_\Gamma(\tau_{g^{-1}} \circ \phi)|.
\]

If it is infinite, by Theorem 3.1 and Corollary 2.5 we have \( \infty = R(\tau_{g^{-1}} \circ \phi) = R(\phi) \). A contradiction.

\[
\square
\]
Corollary 3.4. Let $\Gamma$ and $\phi$ be as in Theorem 3.3 and $|\Gamma| = \infty$. Then all Reidemeister classes of $\phi$ are infinite.

Proof. Indeed, $|\Gamma| = |\{g\} \phi| \cdot |\text{St}_{\phi}(g)|$. \hfill \Box

4. Twisted Burnside-Frobenius theorem

Theorem 4.1. Let $R(\phi) < \infty$ for an automorphism $\phi$ of a finitely generated residually finite group $G$. Then $R(\phi)$ is equal to the number of finite dimensional fixed points of $\hat{\phi}$.

Proof. $R(\phi)$ equals the dimension of the space of twisted invariant elements of $\ell^\infty(G)$, i.e., functionals on $\ell^1(G)$ such that their kernels contain the closure $K_1$ in $\ell^1(G)$ of the space of elements of the form $b - g[b]$, $g[b](x) := b(gx\phi(g^{-1}))$.

Since $R(\phi) < \infty$, $\text{codim} K_1 = R(\phi)$, and $K_1$ has a Banach space complement of dimension $R(\phi)$. We can take it in a way such that it has a base $a_i \in \mathbb{C}[G]$, $i = 1, \ldots, R(\phi)$, i.e., all $a_i$'s have a finite support. Let $p : G \to F = G/H$ be an epimorphism on a finite group $F$ such that it distinguishes all elements from the union of (finite) supports of $a_i$ and $H$ is characteristic (see Remark 2.7). The image of $\ell^1(G)$ under the induced homomorphism $p_1$ is $\ell^1(F) = \mathbb{C}[F]$. Also $K_1$ maps epimorphically onto the space $\mathcal{K}_p$ of elements $\beta - p(g)[\beta] = p_1(b) - p(g)p_1(b) = p_1(b - g[b]) = \mathbb{C}[F]$. Thus, $\{p_1(a_i)\}$ form a basis of a complement to $K_p$ in $\mathbb{C}[F]$.

Decompose this (finite dimensional) algebra $\mathbb{C}[F]$ into a direct sum of matrix algebras, i.e., decompose the image $p_1(\ell^1(G))$ (in fact, $p_1(C^\ast(G))$) into blocks:

$$J : \mathbb{C}[F] \cong \bigoplus_{i=1}^N \text{End} V_i,$$

where $\rho_i$ ($i = 1, \ldots, N$) are some irreducible representations on $V_i$. Since this decomposition is in fact associated with the decomposition of the left regular representation of $\mathcal{K}_p$,

$$\lambda_p \cong \bigoplus_{i=1}^N V_i \otimes V_i^*,$$

these $\rho_i$'s are pairwise non-equivalent [99, I, §2]. Let $K_i$ be formed by $x - \rho_i(g)[x]$ in $\text{End} V_i$. Since $J$ is an algebra isomorphism,

$$R(\phi) = \text{codim} K_1 = \sum_i \text{codim} K_i.$$

The last one is $1$ if $\hat{\phi}(\rho_i) = \rho_i$ (in this case intertwining operators give a linear complement to $K_i$) and $0$ otherwise by Lemma 2.11. Thus, $R(\phi) \leq$ the number of finite dimensional fixed points of $\hat{\phi}$.

The inverse estimation follows from Lemma 2.11. \hfill \Box

Remark 4.2. We use here an approach from [105]) with $\ell^1(G)$ instead of $C^\ast(G)$.

Remark 4.3. As it is known (see e.g. [94, 110]) f.g. residually finite groups ($\infty$ is closed in the profinite topology) form a strictly larger class then conjugacy separable groups (each conjugacy class is closed). Our argument is not applicable to a “proof” of a wrong fact (coincidence of these classes). Indeed, we have no map to identify invariant elements from $\ell^\infty(G)$ with a complement to $K_1$. So, our argument uses in a crucial way the finiteness of $R(\phi)$, while the number of usual conjugacy classes is typically infinite. We have an infinite number of independent characters of finite dimensional irreducible representations for a residually finite group and can separate infinitely many ordinary conjugacy classes, but not necessary all of them.
Corollary 4.4. In the situation of Theorem 4.1 φ-class functions are in $B(G)$, Fourier-Stieltjes algebra of $G$ (c.f. the discussion in [105]).

A group $G$ is φ-conjugacy separable, if there is a homomorphism onto finite group $G/H$ with a φ-invariant normal subgroup $H$ inducing a bijection of twisted conjugacy classes.

Corollary 4.5. In the situation of Theorem 4.1 $G$ is φ-conjugacy separable.

One can slightly modify the proof of Theorem 4.1 to obtain the following statement.

Proposition 4.6. Suppose, $R(\phi) = \infty$ for an automorphism φ of a finitely generated residually finite group $G$. Then the number of finite dimensional fixed points of $\hat{\phi}$ is infinite.

Proof. Let $M$ be an arbitrary positive integer. Consider a subspace $L \subset \ell^1(G)$ such that $K_1 \subset L$ and $\text{codim}(L) = m$. Choose a complement to $L$ with a base $\{a_1, \ldots, a_m\}$ of finitely supported functions and corresponding $F$ and $H$ as in the proof of Theorem 4.1. In the same way $K_1$ maps onto $K_p$ and the images of $a_i$ form a part of a basis of a complement to $K_p$ in $\mathbb{C}[F]$. We obtain $m \leq$ the number of finite dimensional fixed points of $\hat{\phi}$. Since $m$ is arbitrary, we obtain the statement. □

5. AMENABILITY, TWISTED INNER REPRESENTATION, AND $R_\infty$ PROPERTY

Our argument here partially follows [68].

Definition 5.1. Denote by $\gamma_G^\phi$ the twisted inner representation of $G$ on $\ell^2(G)$, i.e.

$$\gamma_G^\phi(x)(f)(g) = f(xg\phi(x^{-1})), \quad x, g \in G, \quad f \in \ell^2(G).$$

Denote $C_\phi(a) := St^w_\phi(a), a \in G$. Evidently, $\gamma_G^\phi$ decomposes into a direct sum of representations $\gamma_a^\phi$ being restrictions of $\gamma_G^\phi$ onto $\{a\}_\phi$ (i.e. on $\ell^2(\{a\}_\phi)$).

Lemma 5.2. The representation $\gamma_a^\phi$ is equivalent to the induced representation $\text{ind}_{C_\phi(a)}^{G} 1_{C_\phi(a)}$.

Proof. Indeed, this induced representation $T$ can be realized on $\ell^2(C_\phi(a) \setminus G)$ by the following action $[T(g)(f)](x) = f(xg), x \in C_\phi(a) \setminus G, g \in G$, where $C_\phi(a) \setminus G$ is the space of left cosets by $C_\phi(a)$. Let us identify $C_\phi(a) \setminus G$ with $\{a\}_\phi$ by $i(C_\phi(a) \cdot g) = \gamma_G^\phi(g)(a)$. Evidently, this map is well defined and gives a unitary isomorphism

$$I : \ell^2(\{a\}_\phi) \to \ell^2(C_\phi(a) \setminus G), \quad I(f)(x) := f(i(x)).$$

Then

$$[I \circ \gamma_G^\phi(g)(f)](x) = [\gamma_G^\phi(g)(f)](i(x)) = f(gha\phi((gh)^{-1})), \quad x = C_\phi(a) \cdot h,$$

$$[T(g) \circ I(f)](x) = I(f)(xg) = f(i(xg)) = f(\gamma_G^\phi(hg)(a)) = f(gha\phi((gh)^{-1})).$$

Thus, $I$ is an intertwining unitary. □

Theorem 5.3. Suppose, $|C_\phi(a)| < \infty$, for any $a \in G$. Then $\gamma_G^\phi$ is weakly contained in the regular representation $\lambda_G$.

Proof. The characteristic functions $\chi_{C_\phi(a)}, a \in G$, are positively definite functions associated to $\lambda_G$, because they are finite sums of translations of $\delta_e$. Hence, $\text{ind}_{C_\phi(a)}^{G} 1_{C_\phi(a)} < \lambda_G$ (cf. [6, E.4.4]). By Lemma 5.2 and the decomposition of $\gamma_G^\phi$ we obtain $\gamma_G^\phi \prec \lambda_G$. □

Theorem 5.4. Suppose, $G$ is a finitely generated residually finite group and $R(\phi) < \infty$ for some $\phi : G \to G$. Then $G$ is amenable.
Proof. In this case (see Theorem 3.3 above) \( |C_\phi(a)| < \infty \), for any \( a \in G \). Thus, by Theorem 5.3 \( \gamma^\phi_G < \lambda_G \). So, it is sufficient to verify that \( 1_G < \gamma^\phi_G \), i.e. \( C^* \text{Ker} \gamma^\phi_G \subset C^* \text{Ker} 1_G \).

Suppose, \( f = \sum_{g \in G} f^g \delta_g \in C^* \text{Ker} \gamma^\phi_G \), i.e.

\[
\sum_{g \in G} f^g \gamma^\phi_G(g) \delta_h = \sum_{g \in G} f^g L_g R_{\phi(g^{-1})} \delta_h = \sum_{g \in G} f^g \delta_{gh \phi(g^{-1})} = 0 \quad \text{for any} \ h \in G.
\]

Since \( gh \phi(g^{-1}) = g_1 h \phi(g_1^{-1}) \) if and only if \( g_1, g_2 \in St^{tw}_\phi(h, x) \) for some \( x \in \{h\}_\phi \), (9) is equivalent to

\[
\sum_{g \in St^{tw}_\phi(h, x)} f^g = 0, \quad \text{for any} \ h \in G \text{ and } x \in \{h\}_\phi.
\]

Take any \( h \), e.g. \( h = e \). Then \( St^{tw}_\phi(e, e) = C_\phi(e) \) is a finite subgroup of \( G \) and the other \( St^{tw}_\phi(h, x) \) form the set of all cosets with the respect to this subgroup. This implies \( \sum_{g \in G} f^g = 0 \) (for any reasonable convergence, because the subgroup is finite). In particular, \( f \in C^* \text{Ker} 1_G \) and \( 1_G < \gamma^\phi_G < \lambda_G \). \( \square \)

Remark 5.5. Main results of [65] illustrate this theorem.

Corollary 5.6. Any non-amenable residually finite finitely generated group is an \( R_\infty \) group.

Remark 5.7. In fact we have used only the estimation \( |C_\phi(a)| < \infty \), for any \( a \in G \). This can occur not only when the group is residually finite. The following case is especially important for applications.

Suppose, we have an exact sequence

\[
0 \to F \to G \xrightarrow{p} H = G/F \to 0, \quad |F| = m < \infty,
\]

\( G \) is finitely generated, \( H \) is residually finite and non-amenable, \( F \) not necessary characteristic, in particular, \( \phi \)-invariant (otherwise we have the \( R_\infty \) property with the help of epimorphism of mapping of Reidemeister classes). As in Theorem 3.1, suppose \( \{x_1, x_2, \ldots \} = C_G(\phi) \) is infinite. Let \( H_n \) be a characteristic subgroup of finite index in \( H \) such that the quotient map \( p_n : H \to H/H_n =: F_n \) is injective on \( \{p(x_1), \ldots, p(x_n)\} \). Then \( G_n := \text{Ker}(p_n \circ p) \) is a subgroup of \( G \) of index \( |F_n| \). As above, we can assume \( G_n \) to be characteristic, since \( G \) is finitely generated. Denote \( s := \#\{p(x_1), \ldots, p(x_n)\} \). Thus, \( s \geq n/m \). Then

\[
R_G(\phi) \geq R_{G_n}(\phi_n) \geq \sqrt{\log_2 |C_{F_n}(\phi_n)|} \geq \sqrt{\log_2 s} \geq \sqrt{\log_2 n - \log_2 m}.
\]

Since \( n \) was arbitrary, we have \( R_G(\phi) = \infty \). Thus, if \( R_G(\phi) < \infty \), then \( C_G(\phi) < \infty \). Hence, \( C_\phi(a) < \infty \), and by Remark 5.7 we obtain the following statement.

Proposition 5.8. Suppose, we have (11), where \( G \) is finitely generated, \( H \) is residually finite and non-amenable, \( |F| < \infty \). Then \( G \) is an \( R_\infty \)-group.

Another large class of examples arises from

Proposition 5.9. Let \( G \) be a normal finitely generated subgroup of a residually finite \( C^* \)-simple group. Then \( G \) is an \( R_\infty \)-group.

Proof. Since \( G \) is non-amenable (see e.g. [19, Prop. 3]), \( G \) enter conditions of Corollary 5.6. \( \square \)
6. Rational points

In this section we show that not every finite-dimensional representation can be fixed by \( \hat{\phi} \) if \( R(\hat{\phi}) < \infty \).

**Definition 6.1.** Let \( [\rho] \in \hat{G}_f \), \( g \mapsto T_g \) be a (class of a) finite-dimensional representation. We say that \( \rho \) is *rational* if the number of distinct \( T_g \)'s is finite, and *irrational* otherwise.

**Remark 6.2.** Evidently, \( \rho \) is rational if and only if it can be factorized through a homomorphism \( G \to F \) on a finite group.

As a consequence of the proof of Theorem 4.1 (see also Corollary 4.5) we obtain the following statement.

**Theorem 6.3.** Suppose, \( G \) is a finitely generated residually finite group, and \( \phi \) is its automorphism with \( R(\phi) < \infty \). Then no irrational representation is fixed by \( \hat{\phi} \).

**Theorem 6.4.** Let \( G \) be a finitely generated group and for an automorphism \( \phi \) at least one of the following two conditions holds:

1). There exist infinitely many finite-dimensional representation classes in \( \hat{G} \) fixed by \( \hat{\phi} \).

2). There exists an irrational representation \( \rho \) fixed by \( \hat{\phi} \).

Then \( R(\phi) = \infty \).

In particular, if we have one of these conditions for every automorphism \( \phi \), then \( G \) has \( R_\infty \) property.

**Proof.** 1) This follows immediately from Lemma 2.11.

2) Suppose that \( f_\rho(g) = \text{Trace}(a_\rho(g)) \) (see Lemma 2.11) takes finitely many values and will arrive to a contradiction. Indeed, \( f_\rho \) is a non-trivial matrix coefficient. Hence, (see, e.g. [69]) its left translations generate a finite-dimensional representation, which is equivalent to a direct sum of several copies of \( \rho \). The space \( W \) of this representation has a basis \( L_{g_1}f_\rho, \ldots, L_{g_k}f_\rho \). Thus, all functions from \( W \) take only finitely many values (with level sets of the form \( \cap_j g_i U_j \), where \( U_j \) are the level sets of \( f_\rho \)). Taking unions of these sets (if necessary) we can form a finite partition \( G = V_1 \sqcup \cdots \sqcup V_m \) such that elements of \( W \) are constant on the elements of the partition and for each pair \( V_i \neq V_j \) there exists a function from \( W \) taking distinct values on them. Thus any left translation maps \( V_i \) onto each other and the representation \( G \) on \( W \) factorizes through (a subgroup of) the permutation group on \( m \) elements, i.e. a finite group. The same is true for its subrepresentation \( \rho \), thus it is rational. A contradiction.\(^1\)

\[ \square \]

7. Isogredience classes

**Definition 7.1.** (see [73]) Suppose, \( \Phi \in \text{Out}(G) := \text{Aut}(G)/\text{Inn}(G) \). We say that \( \alpha, \beta \in \Phi \) are *isogredient* (or *similar*) if \( \beta = \tau_h \circ \alpha \circ \tau_h^{-1} \) for some \( h \in G \).

Let \( S(\Phi) \) be the set of isogredience classes of \( \Phi \). If \( \Phi = \text{Id}_G \), then above \( \alpha \) and \( \beta \) are inner, say \( \alpha = \tau_g, \beta = \tau_s \). Since elements of center \( Z(G) \) give trivial inner automorphisms, we may suppose \( g, s \in G/Z(G) \). Then the equivalence relation takes the form \( \tau_s = \tau_{gh^{-1}} \), i.e., \( s \) and \( g \) are conjugate in \( G/Z(G) \). Thus, \( S(\text{Id}) \) is the set of conjugacy classes of \( G/Z(G) \).

Denote by \( S(\Phi) \) the cardinality of \( S(\Phi) \).

---

\(^1\)Probably this argument is known, but we have not found an appropriate reference.
For a topological motivation of the above definition of the isogredience suppose that \( G \cong \pi_1(X) \), \( X \) is a compact space, and \( \Phi \) is induced by a continuous map \( f : X \to X \). Let \( p : \tilde{X} \to X \) be the universal covering of \( X \), \( \tilde{f} : \tilde{X} \to \tilde{X} \) a lifting of \( f \), i.e. \( p \circ \tilde{f} = f \circ p \). Two liftings \( \tilde{f} \) and \( \tilde{f}' \) are called isogredient or conjugate if there is a \( \gamma \in G \) such that \( \tilde{f}' = \gamma \circ \tilde{f} \circ \gamma^{-1} \). Different lifting may have very different properties. Nielsen observed [85] that conjugate lifting of homeomorphism of surface have similar dynamical properties. This led him to the definition of the isogredience of liftings in this case. Later Reidemeister [93] and Wecken [108] succeeded in generalizing the theory to continuous maps of compact polyhedra.

The set of isogredience classes of automorphisms representing a given outer automorphism and the notion of index \( \text{Ind}(\Phi) \) defined via the set of isogredience classes are strongly related to important structural properties of \( \Phi \) (see [46]), for example in another (with respect to Bestvina–Handel [8]) proof of the Scott conjecture [45].

One of the main results of [73] is that for any non-elementary hyperbolic group and any \( \Phi \) the set \( S(\Phi) \) is infinite, i.e., \( S(\Phi) = \infty \). We will extend this result. First, we introduce an appropriate definition.

**Definition 7.2.** A group \( G \) is an \( S_\infty \)-group if for any \( \Phi \) the set \( S(\Phi) \) is infinite, i.e., \( S(\Phi) = \infty \).

Thus, the above result from [73] says: any non-elementary hyperbolic group is an \( S_\infty \)-group. On the other hand, finite and Abelian groups are evidently non \( S_\infty \)-groups.

Now, let us generalize the above calculation for \( \Phi = \text{Id} \) to a general \( \Phi \) (see [73, p. 512]). Two representatives of \( \Phi \) have form \( \tau_s \circ \alpha, \tau_q \circ \alpha \), with some \( s, q \in G \). They are isogredient if and only if

\[
\tau_q \circ \alpha = \tau_q \circ \tau_s \circ \alpha \circ \tau_g^{-1} = \tau_g \circ \tau_s \circ \tau_{\alpha(g^{-1})} \circ \alpha, \\
\tau_q = \tau_{gsa(g^{-1})}, \quad q = gsa(g^{-1})c, \quad c \in Z(G).
\]

So, the following statement is proved.

**Lemma 7.3.** \( S(\Phi) = R_{G/Z(G)}(\alpha) \), where \( \alpha \) is any representative of \( \Phi \) and \( \overline{\alpha} \) is induced by \( \alpha \) on \( G/Z(G) \).

Since \( Z(G) \) is a characteristic subgroup, we obtain from Lemma 7.3 and epimorphy (see before (6)) the following statement (in one direction it was discussed in [50, Remark 2.1]).

**Theorem 7.4.** Suppose, \( |Z(G)| < \infty \). Then \( G \) is an \( R_\infty \)-group if and only if \( G \) is an \( S_\infty \)-group.

**Remark 7.5.** Of course, this argument is applicable to an individual \( \Phi \) as well.

Theorem 7.4 and Corollary 5.6 imply

**Corollary 7.6.** Suppose, \( G \) is a finitely generated non-amenabale residually finite group with \( |Z(G)| < \infty \). Then \( G \) is an \( S_\infty \)-group.

**Corollary 7.7.** Suppose, \( G \) is a finitely generated \( C^* \)-simple residually finite group. Then \( G \) is an \( S_\infty \)-group.

**Proof.** In this case the center is trivial and the group is non-amenable (see e.g. [19]). \( \square \)

Now we can give a more advanced example of a non \( S_\infty \)-group. Namely, consider Osin’s group [90]. This is a non-residually finite exponential growth group with two conjugacy classes. Since it is simple, it is not \( S_\infty \) by Theorem 7.4.
8. Examples

8.1. Known examples of $R_\infty$ groups. It was shown by various authors that the following groups have the $R_\infty$-property:

1. non-elementary Gromov hyperbolic groups [28, 73]; relatively hyperbolic groups [35];
2. Baumslag-Solitar groups $BS(m,n)$ except for $BS(1,1)$ [36], generalized Baumslag-Solitar groups, that is, finitely generated groups which act on a tree with all edge and vertex stabilizers infinite cyclic [72]; the solvable generalization $\Gamma$ of $BS(1,n)$ given by the short exact sequence $1 \rightarrow \mathbb{Z}[\frac{1}{m}] \rightarrow \Gamma \rightarrow \mathbb{Z}^k \rightarrow 1$, as well as any group quasi-isometric to $\Gamma$ [103];
3. a wide class of saturated weakly branch groups (including the Grigorchuk group [55] and the Gupta-Sidki group [58]) [31], Thompson’s group $F$ [10]; generalized Thompson’s groups $F_n$, $0$ and their finite direct products [52];
4. symplectic groups $Sp(2n,\mathbb{Z})$, the mapping class groups $\text{Mod}_S$ of a compact oriented surface $S$ with genus $g$ and $p$ boundary components, $3g+p-4>0$, and the full braid groups $B_n(S)$ on $n>3$ strings of a compact surface $S$ in the cases where $S$ is either the compact disk $D$, or the sphere $S^2$ [37], some classes of Artin groups of infinite type [67];
5. extensions of $SL(n,\mathbb{Z})$, $PSL(n,\mathbb{Z})$, $GL(n,\mathbb{Z})$, $PGL(n,\mathbb{Z})$, $Sp(2n,\mathbb{Z})$, $PSp(2n,\mathbb{Z})$, $n>1$, by a countable abelian group, and normal subgroup of $SL(n,\mathbb{Z})$, $n>2$, not contained in the centre [81];
6. $GL(n,K)$ and $SL(n,K)$ if $n>2$ and $K$ is an infinite integral domain with trivial group of automorphisms, or $K$ is an integral domain, which has a zero characteristic and for which $\text{Aut}(K)$ is torsion [83];
7. irreducible lattice in a connected semi simple Lie group $G$ with finite center and real rank at least 2 [82];
8. some metabelian groups of the form $\mathbb{Q}^n \rtimes \mathbb{Z}$ and $\mathbb{Z}[1/p]^n \rtimes \mathbb{Z}$ [38]; lamplighter groups $\mathbb{Z}_n \wr \mathbb{Z}$ if and only if $2|n$ or $3|n$ [53]; free nilpotent group $N_{rc}$ of rank $r=2$ and class $c \geq 9$ [54], $N_{rc}$ of rank $r=2$ or $r=3$ and class $c \geq 4r$, or rank $r \geq 4$ and class $c \geq 2r$, any group $N_{2t}$ for $c \geq 4$, every free solvable group $S_{2t}$ of rank $2$ and class $t \geq 2$ (in particular the free metabelian group $M_2 = S_{22}$ of rank 2), any free solvable group $S_{rt}$ of rank $r \geq 2$ and class $t$ big enough [95]; some crystallographic groups [21, 74].

8.2. Coverage by Theorem B. Coverage of items (1) and (7) is discussed in Introduction. Non-elementary (residually finite) relatively hyperbolic without finite normal subgroups (in particular, hyperbolic without torsion) groups are $C^*$-simple [2, 18]. Thus their f.g. normal subgroups are $R_\infty$ groups by Proposition 5.9.

This observation can be extended to the following large class introduced in [17]. Suppose that a residually finite finitely generated group $G$ contains a non-degenerate hyperbolically embedded subgroup [17]. Then it is non-amenable [17, Theorem 8.5(b)], hence, $R_\infty$ by Theorem B.\footnote{D.Osin has communicated to us that he has another proof that groups with non-degenerate hyperbolically embedded subgroup are $R_\infty$ groups.} If $G$ enters conditions of [17, Therem 8.11], for example, if $G$ has no nontrivial finite normal subgroups, the $R_\infty$ property holds for any its finitely generated residually finite subgroup by Proposition 5.9.

Items (2), (3), (8) are not covered.
Most part of groups from item (5) is covered by Theorem B. Linear groups are residually finite (using \( \mathbb{Z} \to \mathbb{Z}/k\mathbb{Z} \) on elements). Taking the quotient by the center commutes with this map. Thus, projective linear groups are residually finite as well. They are finitely generated and non-amenable. Indeed, all of them have the free group \( F_2 \) as a subgroup (for linear groups this is a direct fact and the center evidently does not meet this \( F_2 \), hence for the projective as well).

Groups from (4) are covered by Theorem B. For the symplectic group it is explained above. Braid groups \( B_n = B_n(D^2) \) are residually finite finitely generated. Indeed, one can use either Artin’s representation in \( \text{Aut}(F_n) \) (see [92, 2.19–2.20]), or to use the exact sequence \( B_n \leftarrow N_{1,n} \leftarrow \cdots \leftarrow N_{n,n} = e \) with all quotients being free groups except of the first one which is the (finite) symmetric group (see, e.g. [77]). This sequence implies non-amenability of \( B_n \) for \( n \geq 3 \).

Mapping class groups for compact oriented surfaces are residually finite [56] and are non-amenable for \( 3g + p - 4 > 0 \). Indeed, these groups act on hyperbolic space [78, 11] in such a way that [7] is applicable. Some other cases can be added after an individual analysis. The case of \( B_n(S^2) \) follows from Proposition 5.8 applied to

\[
1 \to \mathbb{Z}_2 \to B_n(S^2) \to \text{Mod}_{S^2_n} \to 1,
\]

where \( S^2_n \) is the sphere with \( n \) boundary components (see [97]).

Under the supposition of f.g. the most part of groups from (6) is covered by Theorem B being residually finite (by Mal’cev’s theorem) and containing \( F_2 \subset SL(2, \mathbb{Z}) \) for characteristic 0. For a finite characteristic one can find an embedded free product \( \mathbb{Z}_m \ast \mathbb{Z}_n \) (in the cases covered by [15, Theorem 10.3]). For characteristic \( \geq 3 \) one has \( m - 1 \geq 2 \), \( n - 1 \geq 2 \), and \( \mathbb{Z}_m \ast \mathbb{Z}_n \) is a Powers group, in particular, non-amenable (see [19, Corollary 12]).

The new classes, related to this item are pure braid groups (as it is explained in Introduction) and (full) mapping class groups for compact non-orientable surfaces (with some low-genus exclusions). Indeed, let \( N \) be a compact non-orientable surface, such that its orienting cover \( \tilde{S} \) is neither a sphere with \( \leq 4 \) boundary circles, nor a torus with \( \leq 2 \) boundary circles, nor a closed surface of genus 2. Then \( \text{Mod}_S \) is \( C^* \)-simple [13] and residually finite. Hence, \( \text{Mod}_N \) is non-amenable [19, Prop. 3] and residually finite being a subgroup of \( \text{Mod}_S \) (see, e.g. [3, Sect. 2.1]).

Another new class is braid groups \( B_n(\Sigma) \) and pure braid groups \( P_n(\Sigma) \) of a compact connected oriented surface \( \Sigma \) (for \( n \geq 2 \)). Remind, that \( P_n(\Sigma) \) is defined as the fundamental group of \( F_n(\Sigma) := \{ (x_1, \ldots, x_n) \in \Sigma^n \mid x_i \neq x_j \text{ if } i \neq j \} \). The symmetric group \( S_n \) acts on \( F_n(\Sigma) \) by permutation of coordinates, and the fundamental group of the orbit space is the braid group \( B_n(\Sigma) \). We obtain a regular \( n! \) fold covering and the exact sequence

\[
1 \to P_n(\Sigma) \to B_n(\Sigma) \to S_n \to 1.
\]

Thus, Theorem B is applicable to \( B_n(\Sigma) \) if and only if it is applicable to \( P_n(\Sigma) \). We have ([9], see also [91]) an epimorphism \( P_m(\Sigma) \to \pi_1(\Sigma)^m \). If \( \Sigma \) has genus \( \geq 1 \) and a border, then \( \pi_1(\Sigma) \) is free non-abelian (homotopically 1-dimensional complex with at least 2 loops). Thus \( P_n(\Sigma) \) is non-amenable in this case. For a closed surface (still different from \( S^2 \) and \( \mathbb{R}P^2 \)) one can consider the short exact sequence from [26]

\[
1 \to \pi_1(\Sigma \setminus \{ x_1, \ldots, x_{n-1} \}, x_n) \to P_n(\Sigma) \to P_{n-1}(\Sigma) \to 1
\]

to prove non-amenability for \( n \geq 3 \) (now a free group is a subgroup). Non-amenability, as well as the residual finiteness in the case of a closed surface can be obtained from [51,
Theorem 5] (the residual finiteness can be also obtained by [4]). The residual finiteness for surfaces with boundary follows inductively from [51, Theorem 6] and [80, Theorem 7(b)]. The several remaining cases should be treated individually.

Artin’s groups $A_{\Gamma}$ (for a graph $\Gamma$) are naturally included in the exact sequence

$$0 \to PA_{\Gamma} \to A_{\Gamma} \to W_{\Gamma} \to 0,$$

where $PA_{\Gamma}$ is the corresponding pure Artin’s group and $W_{\Gamma}$ is the corresponding Coxeter group. Evidently, $A_{\Gamma}$ is non-amenable (containing the free group). If it is residually finite (e.g. if it is of finite (or spherical) type (i.e. $|W_{\Gamma}| < \infty$), or $W_{\Gamma}$ is free abelian, see [80, Theorem 7(c)]), we can apply Theorem B. Also, $A_{\Gamma}$ has the $R_{\infty}$ property, if $PA_{\Gamma}$ is a characteristic subgroup and $W_{\Gamma}$ is neither finite nor almost abelian, because (as we have proved above) in this case $W_{\Gamma}$ has the $R_{\infty}$ property. Actually, it is proved that $PA_{\Gamma}$ is characteristic for the finite type case [41]. To compare this with [67] is a separate problem.

8.3. Known examples of $R(\phi) < \infty$, Reidemeister spectrum, and Theorem A. Now we would like to list some cases, where Theorem A is applicable.

Following [95], define the Reidemeister spectrum of $G$ as

$$Spec(G) := \{ k \in \mathbb{N} \cup \infty | R(\phi) = k \text{ for some } \phi \in Aut(G) \}.$$ 

In particular, $R_{\infty} \Leftrightarrow Spec(G) = \{ \infty \}$.

It is easy to see that $Spec(\mathbb{Z}) = \{ 2 \} \cup \{ \infty \}$, and, for $n \geq 2$, the spectrum is full, i.e. $Spec(\mathbb{Z}^n) = \mathbb{N} \cup \{ \infty \}$. For free nilpotent groups we have the following: $Spec(N_{22}) = 2\mathbb{N} \cup \{ \infty \}$ ($N_{22}$ is the discrete Heisenberg group) [63, 30, 95], $Spec(N_{23}) = \{ 2k^2 | k \in \mathbb{N} \} \cup \{ \infty \}$ [95] and $Spec(N_{32}) = \{ 2n - 1 | n \in \mathbb{N} \} \cup \{ 4n | n \in \mathbb{N} \} \cup \{ \infty \}$ [95].

Metabelian non-polycyclic groups have quite interesting Reidemeister spectrum [38]: for example, if $\theta(1)$ is of the form $\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}$, then we have the following cases:

a) If $r = s = \pm 1$ then $Spec(\mathbb{Z}[1/p]^2 \rtimes_{\theta} \mathbb{Z}) = \{ 2n | n \in \mathbb{N}, (n, p) = 1 \} \cup \{ \infty \}$ where $(n, p)$ denote the greatest common divisor of $n$ and $p$.

b) If $r = s = \pm 1$ then $Spec(\mathbb{Z}[1/p]^2 \rtimes_{\theta} \mathbb{Z}) = \{ 2p^l | p^l + 1, 4p^l | l, k > 0 \} \cup \{ \infty \}$.

c) If $rs = 1$ and $|r| \neq 1$ then $Spec(\mathbb{Z}[1/p]^2 \rtimes_{\theta} \mathbb{Z}) = \{ 2(p^l + 1), 4l | l > 0 \} \cup \{ \infty \}$.

d) If either $r$ or $s$ does not have module equal to one, and $rs \neq 1$ then $Spec(\mathbb{Z}[1/p]^2 \rtimes_{\theta} \mathbb{Z}) = \{ \infty \}$.

An interesting case was studied in [34], where the Reidemeister number and the number of fixed points of $\hat{\phi}$ where compared directly. In this example $G$ is a semidirect product of $\mathbb{Z}^2$ and $\mathbb{Z}$ by Anosov automorphism defined by the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. $G$ is solvable and of exponential growth. The automorphism $\phi$ is defined by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on $\mathbb{Z}^2$ and as $-\text{Id}$ on $\mathbb{Z}$.

Now we will show examples of some groups with extreme properties, where $R(\phi) < \infty$ but Theorem A is not only unapplicable, but the statement is wrong. Namely, Osin’s group [90] has exactly 2 ordinary conjugacy classes, thus it is not residually finite. Hence, by Mal’cev’s theorem it is not linear. Thus, it has no finite-dimensional representations without kernel. Since it is simple, there is no representations with non-trivial kernel (only the trivial 1-dimensional representation). Thus, $R(\text{Id}) = 2$, but the (fixed) finite-dimensional representation is unique. A similar argument works for Ivanov’s group [87, Theorem 41.2] and some HNN-extensions [62, 100] (see [39] for details).
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