Formation of mathematical models of stationary dynamics of structures which include thin elastic plates with distributed inertial parameters

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Abstract. This paper is devoted to the problems of modeling stationary oscillatory processes in elastic systems that contain thin-walled plates. The described method allows us to include the mathematical formalization of thin-walled plates in oscillatory models of elastic structures with irregular boundaries of computational areas and heterogeneous boundary conditions. The formation of dynamic models of this kind is based on the use of discretization methods, in particular, finite element methods that allows approximating the initial models with distributed inertial and rigid parameters — discrete ones with concentration of parameters at certain points — nodes of a dynamic system. Obviously, such approximations are accompanied by errors, the estimation of which is extremely difficult to perform under the conditions of the mentioned heterogeneities and irregularities even in the interval variants. In the present work, the construction of elements of bending stationary vibrations of thin-walled plates under monoharmonic effects is proposed. Possessing all the properties of a finite element, the proposed element contains the values of the distributed masses as parameters. In contrast to the use of discretized parameters obtained by using the classical finite element, the proposed method eliminates discretization procedures and thus, eliminates the process of forming mathematical models from additional errors associated with discretization procedures. This element of mathematical modeling of the dynamics of bending is called a harmonic element (HaE). The proposed method is based on the development of dynamic compliance methods, previously used to simulate oscillations of beam systems with distributed parameters. The developed mathematical models of stationary oscillations of thin elastic plates with distributed inertial parameters make it possible to include them in discrete-continuous models containing infinite-dimensional flexural elements (plates and beams), material points, solids and concentrated elasticities. Thus, dynamic models of structures subjected to harmonic effects are represented by a set of harmonic elements (HaE), which allow harmonic matching of heterogeneous elements under different boundary conditions.

In the study of dynamic systems with a continuous distribution of stiffness and inertial parameters, as well as with an irregular distribution of boundary conditions, the boundaries of areas and types of structural elements, the expediency of discrete-continuous mathematical models use (DCM) arises [1-4]. The use of such models is caused by the need to formalize building structural elements, the ones with deliberately distributed inertial and stiffness parameters, as well as dynamic elements in the form of machines and process equipment, which are expedient to be considered as discrete elements. Often, such tasks arise when modeling the processes of dynamic interaction between supporting structures in constructions with discrete inclusions in the form of masses or solids describing vibroactive equipment. When solving simple systems containing one deformable element with discrete elements attached to it, difficulties can be quite easily overcome [5-9], but in practice most tasks require the use...
of mathematical models that take into account the properties of heterogeneity and irregularity [5, 10, 11]. Of course, the use of discrete dynamic models based, for example, on the application of FEM (finite element methods) simply solves the problems of the heterogeneity of boundary conditions and the irregularity of the area boundaries, since in this case the nodal stitching and the supporting methods allow the use of extremely simple approximation methods [12, 13]. However, the advantages of such simplicity are accompanied by the problems of estimating the sampling error and the “curse” of a large dimension of the formed models, which often require the use of special methods for storing and processing of high order matrices using the properties of their sparsity [14].

The simulation of vibroactive systems based on the development of the dynamic compliance method [4, 15] allows the use of both distributed and concentrated parameters as inertial values of the considered systems. At the same time, all properties of the finite element method [2, 3] are retained, that makes it possible to carry out the solution under various boundary conditions and irregularities in the distribution of physical and geometric parameters. Some similarity with finite elements (FE) allows us to talk about elements of finite size, but they are specialized for solving problems of dynamics and contain the frequency of action and distributed mass as parameters, in contrast to the classical versions of FE. Such elements are called harmonic elements (HaE).

The approach based on the analytical description of stationary oscillations parameters using HaE can significantly simplify the analysis and generalization of the solution results.

The physical prerequisites for the application of this method and harmonic elements are based on the fact that under the effects of a harmonic nature, an elastically deformable system changes to a stationary dynamic state characterized by a periodic change in stress and strain fields with a frequency of external influence. The parameters of this stationary dynamic process are determined by a particular solution [16, 17] of a system of inhomogeneous differential equations with the right-hand side in the form of harmonic functions. This solution provides sufficient information for the analysis of the oscillatory process, and is widely applicable in discrete versions in the design of vibration isolation devices for process equipment [8, 9].

Formation of DCM that include thin plates is carried out by decomposing the original dynamic system into elements for which an analytical basis is built for structurally acceptable variants of boundary conditions under monoharmonic effects, in the form of nodal displacements with unit amplitudes [2-4]. In this case alternate harmonic displacements of the links are made in the process of which a matrix of amplitudes of harmonic reactions is formed in the connections of the harmonic element. The process of forming such a matrix for a beam element is described in detail in [2, 3, 15]. The analytical expression that defines the oscillation amplitudes of the interstitial points of the beam is in one-to-one correspondence to the amplitude vector of generalized nodal displacements. This expression is called a forced oscillatory form of an infinite-dimensional element. Thus, an arbitrary vector of nodal displacements is associated with a certain vector of amplitudes of nodal dynamic reactions which allows the operation of forming an ensemble of elements at a formalized level - in the form of a system of linear resolving equations [1-3]. The nodal joining of solutions of the equations of the dynamic state of finite elements obtained in the form of amplitudes of harmonic reactions in imposed constraints, that takes place in the process, allows analytical (functional) representation of oscillatory forms.

Along with beams in building structures, flat flexural elements are also used (most often in the form of interfloor ceilings or work sites). Accordingly, it is of our interest to include infinite-dimensional flexible harmonic elements in the form of thin plates in mathematical models. Such elements, as well as the beam HaE, possessing the properties of flexible approximation of complex region boundaries and heterogeneous boundary conditions, could be included in previously developed discrete-continuous models (DCM). In contrast to the beam element, in this case for most variants of boundary conditions it is not possible to obtain exact analytical expressions for solutions [3-4].

We will show that the principle of dynamic compliance in combination with the method of harmonic scanning of bonds used to construct a beam harmonic element can be successfully applied in this case as well.
The forced bending vibrations of a thin plate will be considered. We introduce the following notation: \( w \) - oscillation frequency; \( h \) - plate thickness; \( m \) - evenly distributed mass per unit volume; \( E \) - modulus of deformation of isotropic material;

Differential equation of the transverse oscillations of the plate has the form [17-19]:

\[
\frac{\partial^4 W}{\partial x^4} + 2 \frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} + \frac{mh}{D} \frac{\partial^4 W}{\partial t^4} = 0, \tag{1}
\]

where \( D = \frac{Eh^3}{12(1-\mu^2)} \) - is cylindrical plate stiffness, \( W(x,y,t) \) - is a deflection, \( \mu \) - is Poisson's ratio, \( t \) - is a time parameter.

As in the beam element, we will consider so-called standing waves obtained under the condition of separation of time and space variables. To that end, we present the function \( W(x,y,t) \) as:

\[
W(x,y,t) = \sin(\omega t)g(x,y), \tag{2}
\]

Thus,

\[
D \frac{\partial^4 g}{\partial x^4} + 2 \frac{\partial^4 g}{\partial x^2 \partial y^2} + \frac{\partial^4 g}{\partial y^4} = g(x,y)\omega^2 \sin(\omega t). \tag{3}
\]

When \( \sin(\omega t) \neq 0 \)

\[
D \left( \frac{\partial^4 g}{\partial x^4} + 2 \frac{\partial^4 g}{\partial x^2 \partial y^2} + \frac{\partial^4 g}{\partial y^4} \right) = \omega^2 m \times h \times g(x,y). \tag{4}
\]

The exclusion of the time parameter leads to the solution of a static problem formulated in the parameters of space in the form of displacement amplitudes and dynamic reactions in the connections. Indeed, when cyclic effects of a harmonic nature take place, the system goes into a stationary dynamic state, which covers the overwhelming part of its service life. Therefore, direct calculation of the stationary state, without the analysis of the transition period is of our great interest.

Unlike beam harmonic elements [1,2], dynamic reactions in the nodes of flat elements cannot be determined analytically. For this purpose, we have to use approximation techniques similar to finite element ones. However, in contrast to the classical finite element models, this approach allows us to avoid discretization procedures for the inertial parameters of the plate. They remain distributed.

Direct use of classical methods of finite element construction turns out to be impossible due to the fact that the extracted equation (3) has an unknown variable in the right-hand side \( \omega^2 m \times h \times g(x,y) \) expressing the distributed inertial load when the plate surface reaches amplitude values, whereas in the construction of finite elements (FE) it is known and expressed by the dependence \( q(x,y) \) defining a given surface distributed load. The equation of the deformed state of the elementary part of the plate in this case is:

\[
D \left( \frac{\partial^4 W}{\partial x^4} + 2 \frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} \right) = q. \tag{4}
\]

The easiest way to analytically describe the movement of the surface of an element is to polynomially interpret movements as a function of \( g(x,y) \).

The number of coefficients of a polynomial is chosen equally to the number of nodal degrees of freedom of the element. Having performed the imposition of the required number of bonds, which provide the nodal kinematic immobility of the plate element under consideration and having carried out the numbering of the connections, we have the computational scheme of the element shown in (Fig. 1).
Figure 1. The design scheme of a thin-walled plate. 1,2,3,4 - numbers of linear constraints; 5,6,7,8 - numbers of angular constraints in planes parallel to ZoX; 9,10,11,12 - numbers of angular constraints in planes parallel to Z.

To construct a harmonic element of thin rectangular plates with hinged fastening, interpolating polynomials are formed for the basis of the boundary conditions determined by possible variants of alternate single displacements of the nodal connections:

\[ g(x, y) \approx \Delta(x, y) \cdot A_i, \]

where

\[ \Delta(x, y) = (1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^3y, xy^3), \]

\[ A_i = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12})^T (i=1..4) \]

The elements of vector \( A \) are determined by solving the system of equations (5):

\[
\nabla \cdot A = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 
\end{bmatrix},
\]

(5)

\( \nabla \) is the matrix of the node conditions of the displacement field which is constructed taking into account the hingedly supported edge (the sum of the bending and torsional moments are zero) and has the form:
To calculate the values of the sum of the bending and torsional moments at the nodal points, their expression is used according to the formulas:

\[
M_x = -D \left( \frac{\partial^2 g}{\partial x^2} + \mu \frac{\partial^2 g}{\partial y^2} \right),
\]

\[
M_y = -D \left( \frac{\partial^2 g}{\partial y^2} + \mu \frac{\partial^2 g}{\partial x^2} \right),
\]

\[
M_{xy} = -D(1 - \mu) \frac{\partial^2 g}{\partial x \partial y}.
\]

To obtain a solution, the system of equations (5) is presented in the form (6):

\[
\begin{bmatrix}
\nabla_{11} & \nabla_{12} \\
\nabla_{21} & \nabla_{22} \\
\end{bmatrix}
\begin{bmatrix}
A_1 \\
A_2 \\
\end{bmatrix}
=
\begin{bmatrix}
E \\
0 \\
\end{bmatrix}
\]

(6)

where \( \nabla_{11} \) - is the submatrix with dimension \( 4 \times 4 \),
\( \nabla_{12} \) - is the submatrix with dimension \( 4 \times 8 \),
\( \nabla_{21} \) - is the submatrix with dimension \( 8 \times 4 \),
\( \nabla_{22} \) - is the submatrix with dimension \( 8 \times 8 \).

System of equations (6) gives:

\[
\nabla_{22} A_2 = -\nabla_{21} A_1,
\]

Thus,

\[
A_2 = -\nabla_{22}^{-1} \nabla_{21} A_1.
\]

Accordingly

\[
\nabla_{11} A_1 - \nabla_{12} \nabla_{22}^{-1} \nabla_{21} A_1 = E.
\]
\[ A_I = \left( \nabla_{11} - \nabla_{12} \nabla_{22}^{-1} \nabla_{21} \right)^{-1}. \]

As a result, we have matrix \( A \), the columns of which are coefficients of polynomials \( g(x,y) \), interpolating functions of the displacement of the surface of the plate with the appropriate options for moving the nodal connections.

Having combined the plate bending deformation vector and the moment vector, taking into account the equilibrium conditions of the elementary plate section and using the reciprocity theorem \[11\], we form a matrix of dynamic stiffnesses (amplitudes of single dynamic reactions) \( R \), that allows approximating the amplitude states of stationary bending oscillations by means of the following nodal relations:

\[ R \cdot U = F, \]

where \( U \) is the amplitude vector of generalized nodal displacements, and \( F \) is the amplitude vector of nodal forces. Using the results obtained, we are able to form a model in the form of a system of resolving equations.

Thus, the simulation of stationary oscillatory processes is carried out without the application of discretization procedures using infinite-dimensional and discrete elements allowing for a flexible approximation of complex boundaries of areas and boundary conditions which are typical of ordinary finite elements.

The formation of the matrix of amplitudes of dynamic reactions of an ensemble of discrete and continual elements in the system of resolving equations of dynamic equilibrium is carried out by summing up the corresponding reactions in the process of combining displacements when ensembling elements.

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