Normal homogeneous Finsler spaces

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Abstract

In this paper, we study normal homogeneous Finsler spaces. We first define the notion of a normal homogeneous Finsler space, using the method of isometric submersion of Finsler metrics. Then we study the geometric properties. In particular, we establish a technique to reduce the classification of normal homogeneous Finsler spaces of positive flag curvature to an algebraic problem. The main result of this paper is a classification of positively curved normal homogeneous Finsler spaces. It turns out that a coset space $G/H$ admits a positively curved normal homogeneous Finsler metric if and only if it admits a positively curved normal homogeneous Riemannian metric. We will also give a complete description of the coset spaces admitting non-Riemannian positively curved normal homogeneous Finsler spaces.

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Résumé

Dans cet article, nous étudions les espaces de Finsler homogènes normales. Nous définissons d’abord la notion d’un espace homogène Finsler normale, en utilisant la méthode de l’immersion isométrique de métriques de Finsler. Ensuite, nous étudions les propriétés géométriques. En particulier, nous établissons une technique pour réduire la classification des espaces de Finsler homogènes normales de courbure du pavillon positif à un problème algébrique. Le résultat principal de cet article est une classification des espaces à courbure positive de Finsler homogènes normales. Il se trouve que d’un espace de coset $G/H$ admet une métrique homogène normale courbure positive Finsler si et seulement si il admet une courbure positive normale homogène métrique riemannienne. Nous allons aussi donner une description complète des espaces de coset admettant non-riemanniennes espaces à courbure positive de Finsler homogènes normales.

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1 Introduction

The goal of this paper is to extend the study of normal homogeneous Riemannian manifolds to normal homogeneous Finsler spaces. In Riemannian geometry, normal homogeneous manifolds play very important roles in many topics. For example, the first examples of Riemannian manifolds of positive sectional curvature, which are not isometric to a rank one symmetric manifold, are normal; see Berger’s paper [5]. This result motivated the study of homogeneous Riemannian manifolds with positive curvature and eventually led to a complete classification; see [11 3 29]. Meanwhile, normal homogeneous Riemannian manifolds provide many new examples with fine properties. For example, Wang and Ziller studied normal homogeneous Einstein manifolds and found many new examples of Einstein metrics; See [30]. The notion of normal homogeneous Riemannian manifolds has been generalized to several classes of special homogeneous Riemannian manifolds, such as δ-homogeneous metrics; see for example [7].

We now recall the notion of a normal homogeneous Riemannian manifold. Let $G$ be a (connected) compact Lie group and $H$ a closed subgroup of $G$. Suppose $⟨·,·⟩$ is a bi-invariant inner product on the Lie algebra $g$ of $G$ and

$$g = h + m$$

is the orthogonal decomposition, where $h$ is the Lie algebra of $H$ and $m$ is the orthogonal complement subspace of $h$ in $g$ with respect to $⟨·,·⟩$. Then $m$ is invariant under the adjoint action of $H$. The restriction of $⟨·,·⟩$ to $m$, which is an inner product on $m$, is also $\text{Ad}(H)$-invariant. This restricted inner product induces a $G$-invariant Riemannian metric on $G/H$ (see [22]). A homogeneous Riemannian metric of this type is called normal.

This definition can not be directly generalized to the Finslerian case. In fact, even if there exists a bi-invariant Finsler metric on the Lie group $G$, it does not give a decomposition of the Lie algebra as in (1.1), since in a Finsler space we have no notion of orthogonality. To get a natural generalization of the normality in Finsler geometry, we need the notion of isometric submersion for Finsler metrics. Note that the bi-invariant inner product on $g$ defines a bi-invariant Riemannian metric on $G$. The normal Riemannian metric defined above can also be uniquely determined under the requirement that the projection $\pi : G \rightarrow G/H$ is a submersion with respect to this bi-invariant Riemannian metric. On the other hand, in Finsler geometry the theory of isometric submersions has been established in [26], and this method can be used to define the normality in Finsler geometry.

After clarifying the definition of the normality in Finsler geometry, we study some fundamental geometric properties of normal Finsler spaces. Then we consider normal homogeneous Finsler spaces with positive flag curvature. This leads to a classification theorem:

**Theorem 1.1** Let $G$ be a connected Lie group and $H$ a closed subgroup of $G$. Suppose the coset space $G/H$ admits a normal homogeneous Finsler space of positive flag curvature. Then up to equivalence (defined in Section 5) $G/H$ must be one in the following list.


Riemannian symmetric coset spaces of rank 1, i.e., $S^{n-1} = \text{SO}(n)/\text{SO}(n-1)$, $\mathbb{C}P^{n-1} = \text{SU}(n)/\text{SU}(n-1) \times \text{U}(1)$, $\mathbb{H}P^{n-1} = \text{Sp}(n)/\text{Sp}(n-1)\text{Sp}(1)$ and $\mathbb{O}P^2 = F_4/\text{Spin}(9)$.

Other normal homogeneous Finsler spheres, i.e., $\text{SU}(n)/\text{SU}(n-1)$, $\text{U}(n)/\text{U}(n-1)$, $\text{Sp}(n)/\text{Sp}(n-1)$, $\text{Sp}(n)S^1/\text{Sp}(n-1)S^1$, $\text{Sp}(n)\text{Sp}(1)/\text{Sp}(n-1)\text{Sp}(1)$, $G_2/\text{SU}(3)$, $\text{Spin}(7)/G_2$, and $\text{Spin}(9)/\text{Spin}(7)$.

Exceptional ones, i.e., $\text{SU}(3) \times \text{SO}(3)/U^*2$, $\text{Sp}(2)/\text{SU}(2)$ and $\text{SU}(5)/\text{Sp}(2)S^1$.

Any of the above coset spaces admit normal homogeneous Riemannian metrics with positive sectional curvature. The coset space $\text{SU}(3) \times \text{SO}(3)/U^*2$ was found to admit invariant normal Riemannian metrics by Wilking in [31], and all the other spaces were found by Berger in [5].

This classification, combined with Berger’s classification of normal homogeneous Riemannian manifolds of positive sectional curvature [5], gives the following theorem.

**Theorem 1.2** Let $G$ be a connected compact Lie group and $H$ a closed subgroup of $G$. Then there exists a $G$-invariant normal homogeneous Finsler metric on $G/H$ with positive flag curvature if and only if there exists a normal homogeneous Riemannian metric on $G/H$ with positive sectional curvature.

It is well known that if $G/H$ is a rank one Riemannian symmetric coset space, then any $G$-invariant Finsler metric must be Riemannian (see [12]). Hence a rank one Riemannian coset space does not admit any non-Riemannian normal homogeneous Finsler metric metric with positive curvature. It is therefore interesting to find out which coset space in the list of Theorem [1.1] admits non-Riemannian normal homogeneous Finsler metrics with positive curvature. This is completely settled by the following theorem.

**Theorem 1.3** Among the coset spaces in the list in Theorem [1.1], any invariant normal Finsler metric with positive flag curvature on the symmetric spaces of rank 1, the homogeneous spheres $S^3 = \text{SU}(2)/\text{SU}(1) = \text{Sp}(1)/\text{Sp}(0)$, $S^6 = G_2/\text{SU}(3)$, and $S^7 = \text{Spin}(7)/G_2$ must be Riemannian. On the other hand, any of the other spaces admits non-Riemannian normal homogeneous Finsler metrics with positive flag curvature.

In Section 2, we present some preliminaries on Finsler geometry. In Section 3, we define the notion of normal homogeneous Finsler space, using the method of isometric submersion in Finsler geometry. Moreover, we study the fundamental geometric properties of normal homogeneous spaces and prove that any normal homogeneous Finsler space has vanishing S-curvature and non-negative flag curvature. Sections 4, 5 and 6 are devoted to classifying all normal homogeneous spaces of positively flag curvature. This also gives a proof of Theorems [1.1] and [1.2]. Finally, in Section 7, we complete the proof of Theorem [1.3].
2 Preliminaries

2.1 Minkowski norms and Finsler metrics

Let $V$ be a real vector space of dimension $n$. A Minkowski norm on $V$ is a continuous function $F : V \to [0, +\infty)$ satisfying the following conditions:

1. $F$ is positive and smooth on $V \setminus \{0\}$;
2. $F(\lambda y) = \lambda F(y)$ for any $\lambda > 0$;
3. With respect to any linear coordinates $y = y^i e_i$, the Hessian matrix

$$
(g_{ij}(y)) = \left(\frac{1}{2}[F^2]_{y^iy^j}(y)\right)
$$

is positive definite at any $y \neq 0$.

The Hessian matrix $(g_{ij}(y))$ and its inverse $(g^{ij}(y))$ of a Minkowski norm can be used to raise or lower down indices of tensors on the vector space.

Given any $y \neq 0$, the Hessian matrix $(g_{ij}(y))$ defines an inner product $\langle \cdot, \cdot \rangle_y$ (or $\langle \cdot, \cdot \rangle^F_y$ when $F$ needs to be specified) on $V$ by $\langle u, v \rangle_y = g_{ij}(y)u^i v^j$, where $u = u^i e_i$ and $v = v^i e_i$. This inner product can also be written as

$$
\langle u, v \rangle_y = \frac{1}{2} \frac{\partial^2}{\partial s \partial t} [F^2(y + su + tv)]|_{s=t=0},
$$

which is independent of the choice of the linear coordinates.

A Finsler metric $F$ on a smooth manifold $M$ is a continuous function $F : TM \to [0, +\infty)$ such that it is positive and smooth on the slit tangent bundle $TM \setminus 0$, and its restriction to each tangent space is a Minkowski norm. We generally say that $(M, F)$ is a Finsler manifold or a Finsler space.

Important examples of Finsler metrics include Riemannian metrics, Randers metrics, $(\alpha, \beta)$-metrics, etc. Riemannian metrics are a special class of Finsler metrics whose Hessian matrices at each point only depends on $x \in M$ rather than $y \in T_x M$. A Riemannian metric can also be defined as a global smooth section $g_{ij} dx^i dx^j$ of $\text{Sym}^2(T^*M)$. Randers metrics are the simplest class of non-Riemannian metrics in Finsler geometry. They are defined by $F = \alpha + \beta$, where $\alpha$ is a Riemannian metric and $\beta$ is a 1-form. Randers metrics can be naturally generalized as $(\alpha, \beta)$-metrics which have the form $F = \alpha \phi(\beta/\alpha)$, where $\phi$ is a positive smooth function $\phi$ (see [9]). In recent years, there have been a lot of work on Randers metrics and $(\alpha, \beta)$-metrics.

2.2 Geodesic spray, geodesics and S-curvature

Now we recall some relevant notations and terminologies. Let $(M, F)$ be a Finsler space and $(x^1, x^2, \cdots, x^n)$ be a local coordinate system on an open subset $U$ in $M$. In the following, we usually use the standard local coordinates $(x^i, y^j)$ for the open subset $TU$ in the tangent bundle $TM$, where $y = y^j \partial_{x^j} \in T_x M$. 


The geodesic spray is a smooth vector field $G$ globally defined on the slit tangent bundle $TM \setminus 0$. On a standard local coordinate system, $G$ can be given by

$$G = y^i \partial_{x^i} - 2G^i \partial_{y^i},$$

where

$$G^i = \frac{1}{4} \theta^i ([F^2]_{x^k y^l} y^k - [F^2]_{x^i}).$$

A curve $c(t)$ on $M$ is called a geodesic if $(c(t), \dot{c}(t))$ is an integral curve of $G$. On a standard local coordinate system, a geodesic $c(t) = (c^i(t))$ can be characterized as a curve satisfying the partial differential equations

$$\ddot{c}^i(t) + 2G^i(c(t), \dot{c}(t)) = 0.$$  

It is well known that the geodesic spray is tangent to the indicatrix bundle in $TM$. So $F(\dot{c}(t))$ is a constant function when $c(t)$ is a geodesic. Therefore we only need to consider geodesics of nonzero constant speed.

Z.Shen defines the following important non-Riemannian curvature in terms of the geodesic spray, which is now generally called S-curvature.

Let $(M, F)$ be a Finsler space and $(x^1, \cdots, x^n, y^1, \cdots, y^n)$ a standard local coordinate system. The Busemann-Hausdorff volume form can be defined as $dV_{BH} = \sigma(x) dx^1 \cdots dx^n$, in which

$$\sigma(x) = \frac{\omega_n}{\text{Vol}\{ (y^i) \in \mathbb{R}^n | F(x, y^i \partial_{x^i}) < 1 \}},$$

where Vol denotes the volume of a subset with respect to the standard Euclidian metric on $\mathbb{R}^n$, and $\omega_n = \text{Vol}(B_n(1))$. It is easily seen that the Busemann-Hausdorff form is globally defined and does not depend on the specific coordinate system. On the other hand, although the coefficient function $\sigma(x)$ is only locally defined and depends on the choice of local coordinates $x = (x^i)$, the distortion function

$$\tau(x, y) = \ln \frac{\sqrt{\det(g_{ij}(x, y))}}{\sigma(x)}$$

on $TM \setminus 0$ is independent of the local coordinates and globally defined. The S-curvature $S(x, y)$ on $TM \setminus 0$ is defined as the derivative of $\tau(x, y)$ in the direction of the geodesic spray $G(x, y)$.

### 2.3 Riemannian curvature, flag curvature and totally geodesic submanifolds

On a Finsler manifold, we have a similar notion of curvature as in the Riemannian case, which is called the Riemannian curvature. It can be defined either by the Jacobi field or the structure equation for the curvature of the Chern connection.

On a standard local coordinate system the Riemannian curvature can be given as a family of linear maps $R_y$ (or $R^k_y$ when the metric needs to be specified), where $y \in T_x M \setminus 0$, such that $R_y = R^l_k(y) \partial_{x^l} \otimes dx^k : T_x M \to T_x M$, here

$$R^l_k(y) = 2 \partial_{x^k} G^i - y^j \partial^2_{x^j y^k} G^i + 2G^j \partial^2_{y^j y^k} G^i - \partial_{y^j} G^l \partial_{y^k} G^l.$$
The Riemannian curvature $R_y$ is self-adjoint with respect to $\langle \cdot, \cdot \rangle_y$.

Using the Riemannian curvature, one can generalize the sectional curvature to Finsler geometry, which is called the flag curvature. Let $y$ be a nonzero tangent vector in $T_xM$ and $P$ a 2-dimensional subspace (a tangent plane) containing $y$. The pair $(y, P)$ is called a flag and $y$ is called the flag pole. If $P$ is linearly spanned by $y$ and $v$, then the flag curvature of the flag for $(y, P)$ is defined by

$$K(x, y, P) = \frac{\langle Ryv, v \rangle_y}{\langle y, y \rangle_y \langle v, v \rangle_y - \langle y, v \rangle_y^2}. \quad (2.8)$$

The flag curvature may also be denoted as $K(x, y, y \wedge v)$, or $K^F(\cdot, \cdot, \cdot)$ when the metric needs to be specified. It is obvious that it does not depend on the choice of the nonzero tangent vector $v$ in $P$.

It is an important observation of Z. Shen that the Riemannian curvature and flag curvature have a very close relation with relevant curvatures in Riemannian geometry. In fact, let $Y$ be a tangent field on an open set $U \subset M$ which is nowhere vanishing. Then the Hessian matrices $(g_{ij}(Y(x)))$ define a smooth Riemannian metric $g_Y$ on $U$. We say that $Y$ is a geodesic field on an open subset $U \subset M$, if the integration curves of $Y$ are geodesics of nonzero constant speed. Now we have the following theorem.

**Theorem 2.1 (Shen [9])** Let $Y$ be a geodesic field on an open set $U \subset M$ and suppose for $x \in U$, we have $y = Y(x) \neq 0$. Then the Riemannian curvature $R^F_y$ of $F$ coincides with the Riemannian curvature $R^{g_Y}_y$ of the Riemannian metric $g_Y$.

It follows from the above theorem that, if $P$ is a 2-dimensional tangent plane in $T_xM$ containing $y$, then

$$K^F(x, y, P) = K^{g_Y}(x, P).$$

A submanifold $N$ of a Finsler space $(M, F)$ can be naturally endowed a submanifold metric, denoted as $F|_N$. At each point $x \in N$, the Minkowski norm $F|_N(x, \cdot)$ is just the restriction of the Minkowski norm $F(x, \cdot)$ to $T_xN$. We say that $(N, F|_N)$ is a **Finsler submanifold** or a **Finsler subspace**.

A Finsler submanifold $(N, F|_N)$ of $(M, F)$ is called **totally geodesic** if any geodesic of $(N, F|_N)$ is also a geodesic of $(M, F)$. On a standard local coordinate system $(x^i, y^j)$ such that $N$ is locally defined by $x^{k+1} = \cdots = x^n = 0$, the totally geodesic condition can be equivalently given as

$$G^i(x, y) = 0, \quad k < i \leq n, x \in N, y \in T_xN.$$  

A direct calculation shows that in this case the Riemannian curvature $R^F|_N_y : T_xN \to T_xN$ of $(N, F|_N)$ is just the restriction of the Riemannian curvature $R^F_y$ of $(M, F)$, where $y$ is a nonzero tangent vector of $N$ at $x \in N$. Therefore we have

**Proposition 2.2** Let $(N, F|_N)$ be a totally geodesic submanifold of $(M, F)$. Then for any $x \in N$, $y \in T_xN \setminus 0$, and a tangent plane $P \subset T_xN$, we have

$$K^{F|_N}(x, y, P) = K^F(x, y, P). \quad (2.9)$$
2.4 Homogeneous Finsler spaces

A Finsler manifold \((M, F)\) is called homogeneous if the full group \(I(M, F)\) of isometries acts transitively on \(M\). It is shown in [14] that \(G = I(M, F)\) is a Lie transformation group. Let \(H\) be the isotropy subgroup at a point \(o \in M\). Then \(M\) is diffeomorphic to the smooth coset space \(G/H\). The tangent space \(T_oM\) can be naturally identified with the quotient space \(m = g/h\) through the natural \(\text{Ad}(H)\)-actions (the isotropic representation). In many occasions, \(m\) can be regarded as a subspace of \(g\), if we have a direct decomposition \(g = h + m\) of \(g\) into the sum of invariant subspaces of the isotropic representation of \(H\).

Similarly, for any closed subgroup \(G\) of \(I(M, F)\) which acts transitively on \(M\), we have a presentation \(M = G/H\), where \(H\) is the isotropic subgroup of \(G\) at a fixed point. Sometimes, we may also drop the requirement that \(G \subset I(M, F)\) (or \(I_0(M, F)\) when \(G\) is connected), and only assume that \(G\) acts on \((M, F)\) isometrically. For the same manifold, different homogeneous presentations may reveal different geometric properties of the manifold. The most typical example is the nine classes of homogeneous spheres [8]. But sometimes they do not affect the (local) geometric properties of manifolds. So, to study homogeneous Finsler geometry, it is convenient to introduce an equivalence relation among different homogeneous spaces. We will make this equivalence relation precise for normal homogeneous spaces below.

By the homogeneity, the geometric quantities such as curvatures of a homogeneous Finsler space can be reduced to an \(\text{Ad}(H)\)-invariant vectors in certain tensor algebra of \(m\). On the other hand, the homogeneous metric \(F\) itself is completely determined by an \(\text{Ad}(H)\)-invariant Minkowski norm on \(m\); see [12].

2.5 Submersion and subduced metric

Before defining normal homogeneous spaces in Finsler geometry, we first briefly review the theory of Finslerian submersions. For details we refer to [26].

A linear map \(\pi : (V_1, F_1) \to (V_2, F_2)\) between two Minkowski spaces is called an \textit{isometric submersion} (or simply \textit{submersion}), if it maps the unit ball \(\{y \in V_1 | F_1(y) \leq 1\}\) in \(V_1\) onto the unit ball \(\{y \in V_2 | F_2(y) \leq 1\}\) in \(V_2\). It is obvious that a submersion map \(\pi\) must be surjective, and that the Minkowski norm \(F_2\) on \(V_2\) is uniquely determined by the following equality:

\[
F_2(w) = \inf\{F_1(v) | \pi(v) = w\}. \tag{2.10}
\]

Given a Minkowski space \((V_1, F_1)\) and a surjective linear map \(\pi : V_1 \to V_2\), there exists a unique Minkowski norm \(F_2\) on \(V_2\) such that \(\pi\) is a submersion. We usually say that \(F_2\) is the subduced norm through \(\pi\); see [26].

Now we clarify the relationship between the Hessian matrices of \(F_1\) and \(F_2\). For this we need the notion of horizontal lift of vectors. Given a nonzero vector \(w\) in \(V_2\), the infimum in (2.10) can be reached at a unique vector \(v \in V_1\). We call \(v\) the horizontal lift of \(w\) with respect to the submersion \(\pi\). Obviously the horizontal lift of 0 is the zero vector. The horizontal lift \(v\) can also be determined by

\[
\langle w, \ker \pi \rangle_v = 0. \tag{2.11}
\]

Now the Hessian matrix of \(F_2\) at \(w\) is determined by the following proposition.
Proposition 2.3 Let $\pi : (V_1, F_1) \rightarrow (V_2, F_2)$ be a submersion between Minkowski spaces. Assume that $v$ is the horizontal lift of the nonzero vector $w$ in $V_2$. Then 

$\pi : (V_1, \langle \cdot , \cdot \rangle_{F_1}^V) \rightarrow (V_2, \langle \cdot , \cdot \rangle_{F_2}^V)$ is also a submersion between Euclidean spaces.

A submersion between two Finsler spaces $(M_1, F_1)$ and $(M_2, F_2)$ is a surjective map $\rho$ such that for any $x \in M_1$, the tangent map $d\rho|_x : (T_x M_1, F_1) \rightarrow (T_{\rho(x)} M_2, F_2)$ is a submersion between Minkowski spaces. The horizontal lift of a tangent vector field can be similarly defined for a submersion between Finsler spaces. Then the corresponding integration curves define the horizontal lift of smooth curves. Horizontal lift provides a one-to-one correspondence between the geodesics on $M_2$ and the horizontal geodesics on $M_1$, so the horizontal lift of a geodesic field is also a geodesic field. Now Theorem 2.1, Proposition 2.3 and the curvature formula for Riemannian submersions give the following theorem in [26].

Theorem 2.4 Let $\rho : (M_1, F_1) \rightarrow (M_2, F_2)$ be a submersion. Assume that $x_2 = \rho(x_1)$, and $y_2, v_2 \in T_{x_2} M_2$ are linearly independent. Let $y_1$ be the horizontal lift of $y_2$, and $v_1$ the horizontal lift of $v_2$ for the induced submersion $\rho_* : (T_{x_1} M_1, \langle \cdot , \cdot \rangle_{F_1}^V) \rightarrow (T_{x_2} M_2, \langle \cdot , \cdot \rangle_{F_2}^V)$. Then we have

$$K_{F_1}^F(x_1, y_1, y_1 \land v_1) \leq K_{F_2}^F(x_2, y_2, y_2 \land v_2).$$ (2.12)

Submersions can be similarly defined for vector spaces with singular Minkowski norms and smooth manifolds with singular Finsler metrics. In the level of norms in linear spaces, the singular Minkowski norm of the domain space can still induce a unique singular Minkowski norm of the target space. But in general horizontal lift can not be uniquely defined, so it does not share most of the properties for the submersion between smooth Minkowski norms or Finsler manifolds.

3 Normal homogeneous Finsler spaces

In this section, we will define the notion of normal homogeneous Finsler spaces and study their fundamental geometric properties. Unless otherwise stated, Lie groups and smooth manifolds will always be assumed to be connected.

3.1 Definition of normal homogeneous spaces

Let $G$ be a connected Lie group with a bi-invariant Finsler metric $\tilde{F}$, and $H$ a closed subgroup of $G$. Denote the Lie algebras of $G$ and $H$ as $g$ and $h$, respectively. Let $\rho$ be the natural projection from $G$ to $M = G/H$.

The existence of a bi-invariant Finsler metric on $G$ implies that the universal covering of $G$ is a product of a compact simply connected Lie group and a Euclidean space. Thus there exists a bi-invariant inner product on $g$. Fixing a bi-invariant inner product on $g$, we then have the orthogonal decomposition $g = h + m$ such that $[h, m] \subset m$. Moreover, the linear space $m$ can be identified with the tangent space of $M$ at the point $o = \rho(e)$.

The following lemma gives the existence of the subduced metric in this situation.
Lemma 3.1 Keep all the notations as above. There exists a uniquely defined $G$-invariant metric $F$ on $M$ such that for any $g \in G$, the tangent map $\rho_\ast : (T_g G, \bar{F}(g, \cdot)) \to (T_{\pi(g)} M, F(\pi(g), \cdot))$ is a submersion.

**Proof.** The tangent map $\rho_\ast : T_e G \to T_o M$ defines a unique subduced Minkowski norm $F$ on $m = T_o M$ from $\bar{F}(e, \cdot)$. Since $\bar{F}(e, \cdot)$ is Ad($H$)-invariant, $F$ is also Ad($H$)-invariant. Then there exists a (unique) $G$-invariant Finsler metric on $M$ whose restriction to $T_o M$ is equal to $F$ (see [12]). For simplicity, we denote this Finsler metric as $F$. Now given $g \in G$, the tangent map $\rho_\ast |_{T_o G} = g_\ast \circ \pi_\ast |_{T_o G} \circ (L_{g^{-1}})_\ast$ is a submersion between the Minkowski spaces $(T_g G, \bar{F}(g, \cdot))$ and $(T_{\pi(g)} M, F(\pi(g), \cdot))$, since $\rho_\ast |_{T_o G}$ is a submersion, and $\bar{F}$ is bi-invariant. ■

The $G$-invariant Finsler metric $F$ defined in Lemma 3.1 will be called the normal homogeneous metric induced by $F$. We will also call $(M, F)$ the normal homogeneous space induced by $\rho : G \to M = G/H$ and $\bar{F}$.

This definition is a natural generalization of the Riemannian normal homogeneous space. Although in most cases the normal homogeneity is only referred to a compact coset space $M = G/H$ of a compact Lie group $G$, the theory can be applied to some special non-compact coset spaces with very minor technical adjustment. Note that if $G$ is a compact simple Lie group, then there exists a unique bi-invariant Riemannian metric on $G$ (up to homotheties). However, there usually exists infinitely many bi-invariant Finsler metrics on a compact Lie group (up to homotheties), even for a compact simple Lie group. This implies that normal homogeneous Finsler metrics are not related to the group $G$ as closely as in the Riemannian case. Hence the problem of this paper is much more difficult than the same problem in the Riemannian case.

We end this subsection with the following easy but useful observation on normal metrics on coset spaces of products of Lie groups.

Let $\bar{F}_1$ be a bi-invariant Finsler metric on $G_1 = G_2 \times G_3$. Then it induces a bi-invariant normal homogeneous metric $\bar{F}_2$ on $G_2$ through the projection from $G_1$ to its $G_2$-factor. For any closed subgroup $H_2 \in G_2$, denote $H_1 = H_2 \times G_3$. Then $\bar{F}_1$ and $\bar{F}_2$ define isometric normal homogeneous metrics $F_1$ and $F_2$ respectively on $M_1 = G_1/H_1 = G_2/H_2 = M_2$. Any normal homogeneous metric $F_2$ on $G_2/H_2$ can be obtained in this way. The identity map gives a many-to-one correspondence between the normal homogeneous Finsler metrics on $G_1/H_1$ and the normal homogeneous Finsler metrics on $G_2/H_2$.

### 3.2 The flag curvature and S-curvature

Now we consider the fundamental geometric properties of normal homogeneous Finsler spaces. It turns out that this class of spaces behave very well. We first prove

**Proposition 3.2** A normal homogeneous space has vanishing S-curvature and non-negative flag curvature.

**Proof.** Let $(M, F)$ be a normal homogeneous Finsler space induced by the projection $\rho : G \to M = G/H$ and the bi-invariant metric $\bar{F}$ on $G$. Then $\bar{F}$ is a Berwald metric (see [12]), i.e., its Chern connections coincide with the Levi-Civita connection of a Riemannian metric. So the flag curvature of $(G, \bar{F})$ is non-negative. Moreover, it is
also known that any left or right invariant field on \( G \) is a geodesic field. Now the statement for the flag curvature follows from this observation and Theorem 2.4.

Now we prove that \((M, F)\) has vanishing S-curvature. On a local standard coordinate system, the S-curvature \( S(x, y) \) is the derivative of the distortion function

\[
\tau(x, y) = \ln \frac{\sqrt{\det(g_{ij}(x, y))}}{\sigma(x)} \tag{3.13}
\]

in the direction of the geodesic spray \( G(x, y) \). By the homogeneity, we only need to prove that \( S(o, y) = 0 \) for any nonzero tangent vector \( y \in m = T_o M \). Let \( \bar{y} \in T_o G = g \) be the horizontal lift of \( y \). The one-parameter subgroup \( \bar{c}(t) = \exp(t \bar{y}) \) is a horizontal geodesic which is the horizontal lift of the geodesic \( c(t) \) at \( o \in M \) in the direction of \( y \). The vector \( \bar{y} \in g \) defines a right invariant vector field on \( G \) which induces a Killing vector field \( Y \) on \( M \). The integration curve of \( Y \) at \( o \) coincides with the geodesic \( c(t) \). So along \( (c(t), \dot{c}(t)) \), the distortion function \( \tau \) is a constant function. Therefore \( S(o, y) \) is equal to 0.

As S-curvature has some mysterious relationship with other curvatures in Finsler geometry, Proposition 3.2 may be useful in the study of other curvatures of normal homogeneous spaces.

Normal homogeneous Finsler spaces provide us with a large class of complete Finsler manifolds of non-negative flag curvature. Thus it is important to find out the condition for a normal homogeneous Finsler space to have strictly positive flag curvature. For simplicity, a Finsler space with strictly positive flag curvature will be simply called positively curved. Recall that positively curved normal homogeneous Riemannian manifolds have been classified by Berger in [5], and this classification produces several new examples of positively curved homogeneous Riemannian manifolds besides the rank one symmetric spaces. To find all possible (connected) coset spaces admitting positively curved normal homogeneous Finsler metrics, we will need more accurate results on the flag curvature, which will be discussed in the next subsection.

### 3.3 Flat splitting subalgebras and vanishing of flag curvature

Let \( F \) be a normal homogeneous Finsler metric on \( M = G/H \) induced by the projection \( \rho : g \rightarrow h \) and a bi-invariant Finsler metric \( \bar{F} \) on \( G \). We keep all the notations as above.

Let \( g = h + m \) be the orthogonal decomposition with respect to a fixed bi-invariant inner product on \( g \). A commutative subalgebra \( \mathfrak{s} \) of \( g \) is called a flat splitting subalgebra with respect to the pair \((g, h)\) (or with respect to the decomposition \( g = h + m \)) if the following conditions are satisfied

1. \( \mathfrak{s} \) is the intersection of a family of Cartan subalgebras \( t_a, a \in \mathcal{I}, \) i.e., \( \mathfrak{s} = \bigcap_{a \in \mathcal{I}} t_a; \)
2. \( \mathfrak{s} \) is splitting, that is, \( \mathfrak{s} = (\mathfrak{s} \cap h) + (\mathfrak{s} \cap m); \)
3. \( \dim \mathfrak{s} \cap m > 1.\)

If furthermore \( \mathfrak{s} \) is a Cartan subalgebra of \( g \), then it is called a flat splitting Cartan subalgebra (FSCS for simplicity).
The following theorem is the main technique to reduce the classification of connected coset spaces admitting positively curved normal homogeneous metrics to an algebraic problem.

**Theorem 3.3** Keep all the notations as above and assume that the normal homogeneous Finsler metric $F$ on $M = G/H$ is positively curved. Then for any closed subgroup $K$ of $G$ containing the identity component $H_0$ of $H$ and the corresponding orthogonal decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ with respect to the same fixed bi-invariant inner product, there does not exist any FSCS or flat splitting subalgebra of $\mathfrak{g}$ with respect to the pair $(\mathfrak{g}, \mathfrak{k})$.

In particular, if $\dim K < \dim G$, then $G/K$ admits positively curved normal homogeneous Finsler metrics. Moreover, there does not exist any FSCS or flat splitting subalgebra of $\mathfrak{g}$ with respect to the decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$.

Theorem 3.3 makes sense for the term “flat” in these newly defined notations. Compared to the classification of positively curved normal homogeneous Riemannian spaces, there exists a more intrinsic connection between the condition that there is flat splitting subalgebra or FSCS, and the condition that there is a commuting linearly independent pair in $\mathfrak{m}$ (we will simply refer to this as Berger’s condition because it first appears in Berger’s classification work). In all the cases we will consider in this paper, FSCS or flat splitting subalgebra can be constructed using the Berger’s condition. But we can not find a proof for this fact without a case by case discussion.

For any $K$ in Theorem 3.3, there is a natural projection $\pi : G/H_0 \to G/K$. With respect to the normal homogeneous Finsler metric on $G/K$ induced by the same bi-invariant Finsler metric $\bar{F}$ as for $M = G/H$, this projection is also a submersion. We assert that the normal homogeneous space $G/H_0$ induced by the same metric $\bar{F}$ is also positively curved. In fact, By Theorem 2.4, if $\dim G/K > 1$, i.e., if $\dim G - \dim K > 1$, then $G/K$ is positively curved with the normal homogeneous Finsler metric induced by the same $\bar{F}$ on $G$. On the other hand, the case $\dim G - \dim K = 1$ can not happen, since otherwise the positively curved manifold $G/H_0$ must be compact, hence $G/K$ must be a circle. Then $G/H_0$ has an infinite fundamental group, which is a contradiction. This observation reduces the proof of Theorem 3.3 to the case $K = H$.

### 3.4 Proof of Theorem 3.3

As remarked in the above subsection, we only need to prove Theorem 3.3 for $K = H$.

Assume conversely that there is a flat splitting subalgebra $\mathfrak{s} = \bigcap_{a \in \mathcal{I}} \mathfrak{t}_a$ for the family of Cartan subalgebras $\mathfrak{t}_a, a \in \mathcal{I}$. It is obvious that the set $\mathcal{I}$ can be chosen to be finite. For each $a \in \mathcal{I}$, there is a a closed torus $T_a$ with $\text{Lie}(T_a) = \mathfrak{t}_a$. Then the identity component $S$ of $\bigcap_{a \in \mathcal{I}} T_a$ is a closed connected subgroup of $G$ with $\text{Lie}(S) = \mathfrak{s}$. The orbit $S \cdot o$ is a closed torus in $M$ with $\dim S \cdot o > 1$. Then the Finsler metric $F|_{S \cdot o}$ on the submanifold $S \cdot o$ is locally Minkowski, which has constant 0 flag curvature.

We now prove that $(S \cdot o, F|_{S \cdot o})$ is a totally geodesic submanifold in $(M, F)$. We need the following lemma.

**Lemma 3.4** Let $\rho_* : (\mathfrak{g}, \langle \cdot, \cdot \rangle^F) \to (\mathfrak{m}, \langle \cdot, \cdot \rangle^F)$ be a submersion. Then the horizontal lift of any vector of $\mathfrak{s} \cap \mathfrak{m}$ must be contained in $\mathfrak{s}$. 

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Proof. Since $\bar{F}(e, \cdot)$ is Ad($G$)-invariant, for any $a \in I$, and at any $y \in t_a \setminus \{0\}$, the derivative of $\bar{F}$ vanishes in the directions of $t_a^\perp$ (with respect to the chosen bi-invariant inner product on $\mathfrak{g}$). So at any nonzero vector $y \in \mathfrak{s} = \bigcap_{a \in I} t_a$, $\bar{F}$ vanishes in all the directions of $s^\perp = \sum_{a \in I} t_a^\perp$. Now we consider the restriction of the Minkowski norm $\bar{F}(e, \cdot)$ to $s$. For any nonzero vector $y$ in $s$, there exists a unique $y' \in y + s \cap h \subset s$, such that the derivative of $\bar{F}(e, \cdot)$ vanishes in the directions of $s \cap h$. Then at $y'$, the derivative of $\bar{F}(e, \cdot)$ vanishes in all the directions of $s \cap h + s^\perp$, since $s$ is splitting, and

$$h = s \cap h + s^\perp \cap h \subset s \cap h + s^\perp. \quad (3.14)$$

Therefore $y'$ is the horizontal lift of $y$ with respect to the submersion $\rho_* : (\mathfrak{g}, \bar{F}(e, \cdot)) \to (\mathfrak{m}, F(o, \cdot))$. $lacksquare$

For any nonzero tangent vector $y \in T_o(S \cdot o) = s \cap \mathfrak{m}$, the curve $\exp(ty) \cdot o$ is the geodesic of $(S \cdot o, F|_{S \cdot o})$ with initial vector $y$. With respect to the projection $\rho$, it is covered by $\exp(ty') \cdot o$, which is a horizontal geodesic of $(G, \bar{F})$. So $\exp(ty) \cdot o$ is also a geodesic of $(M, F)$. This proves that $(S \cdot o, F|_{S \cdot o})$ is totally geodesic in $(M, F)$. Now by Proposition 2.2 the flag curvature of $(M, F)$ for any tangent plane of $S \cdot o$ (which does exist due to the dimension condition) vanishes. This implies that $(M, F)$ is not positively curved, which is a contradiction. This completes the proof of Theorem 3.3.

Remark 3.5 Theorem 3.3 can also be proven using a flag curvature formula in [33] for Finslerian submersions.

4 Classification of positively curved normal homogeneous spaces

In this section we will develop the techniques to classify normal homogeneous Finsler spaces with positive flag curvature.

4.1 Equivalence between normal homogeneous spaces

As flag curvature is only relevant to the local geometric properties of a Finsler space, we will not distinguish normal homogeneous spaces $M_1 = G_1/H_1$ and $M_2 = G_2/H_2$ which are induced by the bi-invariant Finsler metrics $\bar{F}_1$ on $G_1$, respectively, in the following cases:

(1) $G_2$ is a covering group of $G_1$ with $\bar{F}_2$ induced from $\bar{F}_1$, and $H_2$ has the same identity component as $H_1$.

(2) $G_1 = G_2 \times G_3$, $H_1 = H_2 \times G_3$, and there is a bi-invariant metric $\bar{F}_3$ on $G_3$ and a real function $f$ on $\mathbb{R}^2$ which is positively homogeneous of degree 1, such that $\bar{F}_1 = f(\bar{F}_2, \bar{F}_3)$.

(3) There is an isomorphism $\phi$ between $G_1$ and $G_2$ which maps $H_1$ onto $H_2$, and $\phi^*(\bar{F}_2) = \bar{F}_1$. 

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We will call two normal homogeneous spaces $M_1 = G_1/H_1$ and $M_2 = G_2/H_2$ equivalent, if there are a finite sequence of normal homogeneous spaces connecting $M_1$ and $M_2$, such that each adjacent pair satisfies the conditions in one of the above three cases.

This equivalence provides a lot of convenience for us. Since we only need to study the classification of positively curved Finsler spaces up to equivalence, we can assume that the Lie group $G$ for the normal homogeneous space $M = G/H$ to be simply connected (which means that $G$ may have an Euclidean factor and may be non-compact) and the subgroup $H$ to be connected. Moreover, in some special cases, we can also reduce $G$ and $H$ to their appropriate subgroups, or replace $H$ with its image under an automorphism of $G$. Sometimes this method can give the normal homogeneous space the most standard and recognizable presentation.

4.2 Some general classification results

Keep all the notations as above. We call the dimension of the Cartan subalgebra in $\mathfrak{g}$ the rank of $G$ (or $\mathfrak{g}$), denoted as $\text{rk} G$ (or $\text{rk} \mathfrak{g}$). We will also call the maximal dimension of the commutative subalgebras in $\mathfrak{m}$ the rank of $M$.

The following two propositions are direct corollaries of Theorem 3.3.

**Proposition 4.1** Let $M = G/H$ be a positively curved normal homogeneous space. Then either $\text{rk} G = \text{rk} H$ or $\text{rk} G = \text{rk} H + 1$.

**Proposition 4.2** Let $M = G/H$ be a positively curved symmetric normal homogeneous space. Then $M$ is positively curved if and only if it is of rank 1.

Recall that the normal homogeneous space $G/H$ is symmetric if and only if in the decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, we have $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}$. If $G/H$ is a positively curved normal homogeneous space with $\text{rk} G > \text{rk} H + 1$, then we can construct a splitting Cartan subalgebra of $\mathfrak{g}$ from any Cartan subalgebra of $\mathfrak{h}$, which must be a FSCS by the assumption $\text{rk} G > \text{rk} H + 1$. This is a contradiction to Theorem 3.3. In [33], we present another proof for the same conclusion where we only require $M$ to be a positive curved homogeneous Finsler space.

Now we prove Proposition 4.2. In fact, otherwise there exists a maximal commutative subalgebra $\mathfrak{a}$ in $\mathfrak{m}$ with $\dim \mathfrak{a} > 1$. Then the symmetric condition implies that any Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$ is a FSCS, which is a contradiction. Proposition 4.2 can also be proven by arguing that $(M, F)$ is Berwald with the same connection and Riemannian curvature as a Riemannian normal homogeneous space; see [12].

The positively curved normal homogeneous Finsler spaces with $\text{rk} G = \text{rk} H$ and $\text{rk} G = \text{rk} H + 1$ will be discussed in the next two sections.

4.3 Some notations and techniques

We now summarize some basic facts on compact Lie groups from [20], and develop some techniques for our classification.

Assume that the normal homogeneous space $M = G/H$ is connected, simply connected and positively curved, $G$ is connected and simply connected and $H$ is connected.
Fix an orthogonal decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ with respect to a given bi-invariant inner product on $\mathfrak{g}$.

Fix a Cartan subalgebra $\mathfrak{t}$ such that $\mathfrak{t} \cap \mathfrak{h}$ has maximal dimension, i.e., $\mathfrak{t} \subset \mathfrak{h}$ if $\text{rk}\mathfrak{G} = \text{rk}\mathfrak{H}$, or $\mathfrak{t} \cap \mathfrak{h}$ is a subspace of $\mathfrak{t}$ with codimension 1 if $\text{rk}\mathfrak{G} = \text{rk}\mathfrak{H} + 1$. In the following, unless otherwise stated, the notations (e.g., roots, root systems, root planes, etc.) will be defined with respect to $\mathfrak{t}$ when concerning $\mathfrak{g}$ or its subalgebras containing $\mathfrak{t}$, or to $\mathfrak{t} \cap \mathfrak{h}$ when concerning $\mathfrak{h}$.

With respect to $\mathfrak{t}$, we have a decomposition

$$\mathfrak{g} = \mathfrak{t} + \sum_{\alpha \in \Delta^+_t} \mathfrak{g}_{\pm\alpha}. \quad (4.15)$$

The 2-dimensional subspace $\mathfrak{g}_{\pm\alpha}$ is called the root plane of $\mathfrak{g}$. Through the given bi-invariant inner product, any root $\alpha$ can be naturally identified with a vector in $\mathfrak{t}$, which generates $[\mathfrak{g}_{\pm\alpha}, \mathfrak{g}_{\pm\alpha}]$. The bracket relation between different root planes is given by the following well known formula,

$$[\mathfrak{g}_{\pm\alpha}, \mathfrak{g}_{\pm\beta}] = \mathfrak{g}_{\pm(\alpha+\beta)} + \mathfrak{g}_{\pm(\alpha-\beta)}, \quad (4.16)$$

where $\mathfrak{g}_{\pm\alpha}$ and $\mathfrak{g}_{\pm\beta}$ are different root planes, i.e. $\alpha$ and $\beta$ are linearly independent, and each term of the right side can be 0 when the corresponding vectors are not roots of $\mathfrak{g}$.

We now assert that if the right side of (4.16) is 2-dimensional, then for any nonzero vector $v$ of $\mathfrak{g}_{\pm\alpha}$, $\text{ad}(v)$ maps $\mathfrak{g}_{\pm\beta}$ isomorphically onto the right side. In fact, assume that $\alpha + \beta$ is a root but $\alpha - \beta$ is not. Then for any $u \in (\alpha + \beta)^\perp$ which is not perpendicular to either $\alpha$ or $\beta$, we have

$$[[u, v], \mathfrak{g}_{\pm\beta}] = [v, [u, \mathfrak{g}_{\pm\beta}]] = [v, \mathfrak{g}_{\pm\beta}]. \quad (4.17)$$

As $\mathfrak{g}_{\pm\alpha}$ is linearly generated by $v$ and $[u, v]$, $[v, \mathfrak{g}_{\pm\beta}] = [\mathfrak{g}_{\pm\alpha}, \mathfrak{g}_{\pm\beta}]$ is two dimensional, that is, $\text{ad}(v)$ is an isomorphism from $\mathfrak{g}_{\pm\beta}$ onto $\mathfrak{g}_{\pm(\alpha+\beta)}$.

The above discussion can be summarized as the following lemma.

**Lemma 4.3** Keep all the notations as above.

1. For any root $\alpha$ of $\mathfrak{g}$, $[\mathfrak{g}_{\pm\alpha}, \mathfrak{g}_{\pm\alpha}]$ is the line in $\mathfrak{t}$ spanned by the vector which is dual to $\alpha$ with respect to the chosen bi-invariant inner product.

2. Let $\alpha$ and $\beta$ be two linearly independent roots of $\mathfrak{g}$. If neither $\alpha + \beta$ nor $\alpha - \beta$ is a root of $\mathfrak{g}$, then $[\mathfrak{g}_{\pm\alpha}, \mathfrak{g}_{\pm\beta}]$ is equal to 0; If $\alpha + \beta$ (resp. $\alpha - \beta$) is a root, but the other is not, then $[\mathfrak{g}_{\pm\alpha}, \mathfrak{g}_{\pm\beta}]$ is equal to the 2-dimensional root plane $\mathfrak{g}_{\pm(\alpha+\beta)}$ (resp. $\mathfrak{g}_{\pm(\alpha-\beta)}$); If both $\alpha \pm \beta$ are roots of $\mathfrak{g}$, then $[\mathfrak{g}_{\pm\alpha}, \mathfrak{g}_{\pm\beta}]$ is equal to the 4-dimensional sum $\mathfrak{g}_{\pm(\alpha+\beta)} + \mathfrak{g}_{\pm(\alpha-\beta)}$.

3. In (2), if $[\mathfrak{g}_{\pm\alpha}, \mathfrak{g}_{\pm\beta}]$ is 2-dimensional, then for any nonzero vector $v \in \mathfrak{g}_{\pm\alpha}$, the linear map $\text{ad}(v)$ is an isomorphism from $\mathfrak{g}_{\pm\beta}$ onto $[\mathfrak{g}_{\pm\alpha}, \mathfrak{g}_{\pm\beta}]$.

When $\text{rk}\mathfrak{G} = \text{rk}\mathfrak{H}$, the roots and root planes of $\mathfrak{h}$ are also those of $\mathfrak{g}$. Any root plane $\mathfrak{g}_{\pm\alpha}$ of $\mathfrak{g}$ is either contained in $\mathfrak{h}$ (when $\alpha$ is a root of $\mathfrak{h}$), or contained in $\mathfrak{m}$ (when $\alpha$ is not a root of $\mathfrak{h}$). For simplicity, in the later case we will say that $\alpha$ is a root of $\mathfrak{m}$. 

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When \( \text{rk} G = \text{rk} H + 1 \), the roots of \( \mathfrak{h} \) can be naturally identified with vectors on \( t \cap \mathfrak{h} \) through the restriction of the given bi-invariant inner product to \( t \cap \mathfrak{h} \). Let \( \text{pr} \) be the orthogonal projection from \( t \) to \( t \cap \mathfrak{h} \). Then for any root \( \alpha' \in \Delta_h^+ \), the root plane \( \mathfrak{h}_{\pm \alpha'} \) is contained in

\[
\mathfrak{g}_{\pm \alpha'} = \sum_{\text{pr}(\alpha) = \alpha'} \mathfrak{g}_{\pm \alpha}.
\]  

(4.18)

In particular, if \( \text{pr}(\alpha) \) is not a root of \( \mathfrak{h} \) for the root \( \alpha \) of \( \mathfrak{g} \), then \( \mathfrak{g}_{\pm \alpha} \subset \mathfrak{m} \). In this case, we say that \( \pm \alpha \) are roots of \( \mathfrak{m} \).

The following lemmas will be used repeatedly in our later discussion.

**Lemma 4.4** Assume that \( M = G/H \) is a positively curved normal homogeneous Finsler space with \( \text{rk} G = \text{rk} H \).

1. Let \( \alpha \) and \( \beta \) be two roots of \( \mathfrak{m} \) with \( \alpha \neq \pm \beta \). Then either \( \alpha + \beta \) or \( \alpha - \beta \) is a root of \( \mathfrak{g} \).

2. Let \( \alpha \) and \( \beta \) be two roots of \( \mathfrak{m} \) with angle \( \frac{\pi}{3} \) or \( \frac{2\pi}{3} \), and suppose \( \alpha \) and \( \beta \) do not belong to a \( G_2 \)-factor of \( \mathfrak{g} \). Then either \( \alpha + \beta \) or \( \alpha - \beta \) is a root of \( \mathfrak{h} \).

**Proof.**

(1) If \( \alpha \neq \pm \beta \) and \( \alpha \pm \beta \) are not roots of \( \mathfrak{g} \), then for any nonzero vectors \( v_1 \in \mathfrak{g}_{\pm \alpha} \) and \( v_2 \in \mathfrak{g}_{\pm \beta} \), we have \( [v_1, v_2] = 0 \). Let \( \alpha^\perp \), resp. \( \beta^\perp \), be the orthogonal complement of \( \alpha \), resp. \( \beta \), in \( t \) with respect to the chosen bi-invariant inner product. Then the linear span of \( v_1, v_2 \) and \( \alpha^\perp \cap \beta^\perp \) is a FSCS, which is a contradiction to Theorem [3.3].

(2) Suppose conversely that \( \pm \alpha \pm \beta \) are roots of \( \mathfrak{m} \). Then the subalgebra \( \mathfrak{g}' = \mathbb{R} \alpha + \mathbb{R} \beta + \sum_{a,b \in \mathbb{Z}} \mathfrak{g}_{\pm (a \alpha + b \beta)} \) of \( \mathfrak{g} \) is isomorphic to \( \mathfrak{su}(3) \), by an isomorphism \( l \) which maps \( \mathfrak{g}' \cap \mathfrak{h} = \mathbb{R} \alpha + \mathbb{R} \beta \) to the diagonal matrices. Now we consider the matrices

\[
u = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}, \quad v = \begin{pmatrix} 0 & -\sqrt{-1} & -\sqrt{-1} \\ \sqrt{-1} & 0 & \sqrt{-1} \\ -\sqrt{-1} & \sqrt{-1} & 0 \end{pmatrix}
\] in \( \mathfrak{su}(3) \).

Then \( l^{-1}(u) \) and \( l^{-1}(v) \) are linearly independent and commutative. Thus the linear span of \( l^{-1}(u), l^{-1}(v) \) and \( \alpha^\perp \cap \beta^\perp \) is a FSCS. This is a contradiction to Theorem [3.3].

**Lemma 4.5** Let \( M = G/H \) be a positively curved normal homogeneous space with \( \text{rk} G = \text{rk} H + 1 \).

1. If there is a root \( \alpha \) of \( \mathfrak{g} \) perpendicular to \( t \cap \mathfrak{h} \), then \( \mathfrak{g}_{\pm \alpha} \subset \mathfrak{m} \).

2. Any root \( \alpha \in t \cap \mathfrak{h} \) of \( \mathfrak{g} \) is a root of \( \mathfrak{h} \).

3. Let \( \alpha \) and \( \beta \) be two roots of \( \mathfrak{m} \) with angle \( \frac{\pi}{3} \) or \( \frac{2\pi}{3} \). Suppose \( \mathfrak{t} \cap \mathfrak{h}^\perp \subset \mathbb{R} \alpha + \mathbb{R} \beta \) and \( \alpha \) and \( \beta \) do not belong to a \( G_2 \)-factor of \( \mathfrak{g} \). Then either \( \text{pr}(\alpha + \beta) \) or \( \text{pr}(\alpha - \beta) \) is a root of \( \mathfrak{h} \).

**Proof.**

(1) This follows directly from the above observations and the fact that \( \text{pr}(\alpha) = 0 \), which is not a root of \( \mathfrak{h} \).
(2) If $\alpha = \text{pr}(\alpha)$ is not a root of $\mathfrak{h}$, then $\mathfrak{g}_{\pm \alpha} \subset \mathfrak{m}$. Let $v$ be any nonzero vector in $\mathfrak{g}_{\pm \alpha}$. Then the linear span of $v$ and the orthogonal complement $\alpha^\perp$ of $\alpha$ in $t$ is a FSCS, which is a contradiction to Theorem 5.3.

(3) Notice that if the one dimensional space $t \cap \mathfrak{h}^\perp$ is contained in $\mathbb{R}\alpha + \mathbb{R}\beta$, then $\alpha^\perp \cap \beta^\perp$ is a subspace of $t \cap \mathfrak{h}$ of codimension one. Now the assertion can be proved similarly as (2) of Lemma 4.3. \hfill \blacksquare

5 Positively curved normal homogeneous spaces $M = G/H$

with $\text{rk}G = \text{rk}H$

In this section, we assume that $(M, F)$ is a positively curved connected normal homogeneous space induced by the projection $\rho : G \to M = G/H$ with $\text{rk}G = \text{rk}H$ and a bi-invariant Finsler metric $\bar{F}$ on $G$. We will also assume that $G$ is connected simply connected, and $H$ is connected.

We first make a simple observation. Suppose $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \cdots \oplus \mathfrak{g}_m \oplus \mathbb{R}^l$ is the decomposition of $\mathfrak{g}$ into the direct sum of simple ideals and the Euclidean factor. Then $\mathfrak{h}$ has a similar decomposition which only differs from the above one in exactly one simple factor, for example, $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_m \oplus \mathbb{R}^l$, where $\mathfrak{h}_1$ is a subalgebra of $\mathfrak{g}_1$ with a strictly lower dimension. To prove this statement, we assume conversely that $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3 \oplus \cdots \oplus \mathfrak{h}_m \oplus \mathbb{R}^l$ where $\dim \mathfrak{h}_i < \dim \mathfrak{g}_i$ for $i = 1, 2$. Then there exists a nonzero vector $v_i$ from a root plane $\mathfrak{g}_{\pm \alpha_i}$ in $m \cap \mathfrak{g}_i$, for $i = 1, 2$. Thus $v_1, v_2$, and the orthogonal complement $\alpha_1^\perp \cap \alpha_2^\perp$ in $t$ span a FSCS, which is a contradiction. This means that we have the direct sum decompositions,

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_m \oplus \mathbb{R}^l, \quad (5.19)$$

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_m \oplus \mathbb{R}^l, \quad (5.20)$$

where $\mathfrak{h}_1$ is a subalgebra of the simple compact $\mathfrak{g}_1$ with $\text{rk}\mathfrak{h}_1 = \text{rk}\mathfrak{g}_1$. Let $G_1$ and $H_1$ be the closed connected subgroups with Lie algebras $\mathfrak{g}_1$ and $\mathfrak{h}_1$ respectively. Let $\bar{F}_1$ be the restriction of $\bar{F}$ to $G_1$. Then $(M, F)$ is equivalent to the normal homogeneous space $(M, F_1)$ induced by the projection $\rho : G_1 \to M = G_1/H_1$ and $\bar{F}_1$. So, to classify the positively curved normal homogeneous spaces in this case, up to equivalence, we can further assume that $G$ is a compact simple Lie group, i.e., $\mathfrak{g}$ is one of the $A_n, B_n$ with $n > 1$, $C_n$ with $n > 2$, $D_n$ with $n > 3$, $E_6, E_7, E_8, F_4$ or $G_2$.

Now we start a case by case study. In the following, we will presuppose that the coset space $G/H$ be endowed with an invariant normal Finsler metric of positive flag curvature. If a contradiction arises, then we can conclude the corresponding coset space does not admit any positively curved normal homogeneous Finsler metric. At the final part of the discussion of each case, we will summarize the conclusion.

The case $\mathfrak{g} = A_n$. The root system $\Delta_\mathfrak{g}$ can be identified with the subset

$$\{\pm(e_i - e_j) \mid 1 \leq i < j \leq n + 1\} \quad (5.21)$$

in $\mathbb{R}^{n+1}$ with the standard orthonormal basis $\{e_1, \ldots, e_{n+1}\}$. The subalgebra $\mathfrak{h}$ is then a direct sum of the simple compact subalgebras of type $A$ or $\mathbb{R}$ (with abelian Lie bracket). More precisely, there is a decomposition of the set $\{1, \ldots, n + 1\}$ into a non-overlapping
union $\cup_{a \in A} S_a$, such that for different $a$ and $b$ in $A$, $\pm(e_i - e_j)$ are not roots of $\mathfrak{h}$ when $i \in S_a$ and $j \in S_b$, and for any $a$ in $A$, $\pm(e_i - e_j)$ are roots of $\mathfrak{h}$ when $i, j \in S_a$ and $i \neq j$.

Upon the conjugation of a Weyl group action, we can assume that each $S_a$ is a segment of continuous integers. When $S_a$ contains more than one element, it corresponds to a subalgebra of type $A$ in the direct sum decomposition of $\mathfrak{g}$, otherwise it corresponds to a subalgebra isomorphic to $\mathbb{R}$.

If $n > 2$ and $\mathfrak{h}$ is not of the type $A_{n-1} \oplus \mathbb{R}$, then upon the conjugation of the Weyl group action, $\alpha = e_1 - e_{n-1}$ and $\beta = e_2 - e_n$ are roots of $\mathfrak{m}$, and $\alpha \pm \beta$ are not roots of $\mathfrak{g}$. However, this is a contradiction to (1) of Lemma 4.4.

If $n = 2$, i.e., $\mathfrak{g} = \mathfrak{su}(3)$, and $\mathfrak{h}$ is not of the type $A_1 \oplus \mathbb{R}$, then upon the conjugation of the Weyl group action, $\mathfrak{h}$ is a Cartan subalgebra $\mathbb{R} \oplus \mathbb{R}$, and all the roots belong to $\mathfrak{m}$. This is a contradiction to (2) of Lemma 4.4.

In summarizing, in this case the only possibility is that $\mathfrak{h} = A_{n-1} \oplus \mathbb{R}$ and $(M, F)$ is equivalent to the normal homogeneous complex projective space $SU(n+1)/S(U(n) \times U(1))$. Note that this is a symmetric coset space of rank one which does admit positively curved normal Finsler (in fact Riemannian) metrics.

The case $\mathfrak{g} = B_n$, $n > 1$. The root system can be identified with the subset

$$\{\pm e_i | 1 \leq i \leq n\} \cup \{\pm e_i \pm e_j | 1 \leq i < j \leq n\}$$

in $\mathbb{R}^n$ with the standard orthonormal basis $\{e_1, \ldots, e_n\}$. The subalgebra $\mathfrak{h}$ is a direct sum of subalgebras of type $A$, $B$, $D$ or $\mathbb{R}$. To be precise, there is a decomposition of the set $\{1, \ldots, n\}$ into a non-overlapping union $\cup_{a \in A} S_a$, such that for different $a$ and $b$ in $A$, $\pm e_i \pm e_j$ are not roots of $\mathfrak{h}$ when $i \in S_a$ and $j \in S_b$, and for any $a$ in $A$, one of the following holds:

1. For any $i \in S_a$, $\pm e_i$ are roots of $\mathfrak{h}$, and for any $i, j \in S_a$ with $i \neq j$, $\pm e_i \pm e_j$ are roots of $\mathfrak{h}$;
2. For any $i, j \in S_a$ with $i \neq j$, $\pm e_i \pm e_j$ are roots of $\mathfrak{h}$, but $\pm e_i$s are not roots of $\mathfrak{h}$;
3. For any $i, j \in S_a$ with $i \neq j$, either $\pm(e_i + e_j)$ or $\pm(e_i - e_j)$, but not both, are roots of $\mathfrak{h}$, and $\pm e_i$s are not roots of $\mathfrak{h}$;
4. $S_a = \{i\}$ and $\pm e_i$ are roots of $\mathfrak{g}$;
5. $S_a = \{i\}$ and there is no root for $S_a$.

Upon the Weyl group action, we can assume that each $S_a$ is a segment of continuous integers, and for $S_a$ in (3), $\pm(e_i - e_j)$ are roots of $\mathfrak{h}$ but $\pm(e_i + e_j)$ are not. Then each $S_a$ corresponds to a subalgebra of type $B$, $D$, $A \oplus \mathbb{R}$ (with a one dimensional center), $A_1$ or $\mathbb{R}$.

If $n > 3$ and $\mathfrak{h}$ is not of the type $B_{n-1} \oplus \mathbb{R}$, $D_{n-1} \oplus \mathbb{R}$ or $D_n$, then we can take $\alpha = e_1 + e_{n-1}$ and $\beta = e_2 + e_n$ from roots of $\mathfrak{m}$. Then $\alpha \pm \beta$ are not roots of $\mathfrak{g}$, which is a contradiction to (1) of Lemma 4.4. If $\mathfrak{h}$ is $B_{n-1} \oplus \mathbb{R}$ or $D_{n-1} \oplus \mathbb{R}$ (here, without losing generality, we assume that $\mathbb{R}$ corresponds to $\{n\}$, and we only need $n > 2$), then
we can take \( \alpha = e_1 + e_n \) and \( \beta = e_1 - e_n \). This is also a contradiction to (1) of Lemma 4.4.

If \( n = 3 \) and \( \mathfrak{h} \) has an \( A_1 \) or a \( \mathbb{R} \) factor from (4) or (5) of the list in it, then we can also use \( \alpha = e_1 + e_n \) and \( \beta = e_1 - e_n \) to deduce a contradiction. If \( \mathfrak{h} = A_2 \oplus \mathbb{R} \) from (3) of the list, then we can take \( \alpha = e_1 + e_2 \) and \( \beta = e_3 \) to deduce a contradiction to (1) of Lemma 4.4. Thus \( \mathfrak{h} \) can only be \( D_3 \).

If \( n = 2 \) and \( \mathfrak{h} \) have a \( A_1 \) or \( \mathbb{R} \) factor from (4) or (5) of the list, then we can take \( \alpha = e_1 + e_2 \) and \( \beta = e_1 - e_2 \) to deduce a contradiction to (1) of Lemma 4.4. Now besides \( D_2 \), \( \mathfrak{h} \) can also be \( A_1 \oplus \mathbb{R} \) from (3) of the list, where \((M, F)\) is equivalent to a homogeneous complex projective space \( CP^3 = Sp(2)/Sp(1)S^1 \).

In summarizing, in this case there are only two possibilities, namely, \((M, F)\) is equivalent to a normal homogeneous sphere \( S^{2n} = SO(2n+1)/SO(2n) \), or a homogeneous complex projective space \( CP^3 = Sp(2)/Sp(1)S^1 \). Note that the first one is a symmetric coset space of rank one, and the second one appears in Berger’s list, hence they both admit positively curved normal Finsler (in fact Riemannian) metrics.

**The case** \( g = C_n, \ n > 2 \). The root system of \( g \) can be identified with the subset
\[
\{ \pm 2e_i | 1 \leq i \leq n \} \cup \{ \pm e_i \pm e_j | 1 \leq i < j \leq n \} \tag{5.23}
\]
in \( \mathbb{R}^n \) with the standard orthonormal basis \( \{ e_1, \ldots, e_n \} \). The subalgebra \( \mathfrak{h} \) is a direct sum of subalgebras of type \( C, A \) and \( \mathbb{R} \). To be precise, there is a decomposition of the set \( \{ 1, \ldots, n \} \) into a non-overlapping union \( \bigcup_{a \in A} S_a \), such that for different \( a \) and \( b \) in \( A \), \( \pm e_i \pm e_j \) are not roots of \( \mathfrak{h} \) when \( i \in S_a \) and \( j \in S_b \), and for any \( a \) in \( A \), one of the following holds:

1. For any \( i \in S_a, \pm 2e_i \) are roots of \( \mathfrak{h} \), and for any \( i, j \in S_a \) with \( i \neq j \), \( \pm e_i \pm e_j \) are roots of \( \mathfrak{h} \);
2. For any \( i, j \in S_a \) with \( i \neq j \), either \( \pm (e_i + e_j) \) or \( \pm (e_i - e_j) \), but not both, are roots of \( \mathfrak{h} \), and \( \pm 2e_i \) are not roots of \( \mathfrak{h} \);
3. \( S_a = \{ i \} \) and \( \pm 2e_i \) are roots of \( \mathfrak{h} \);
4. \( S_a = \{ i \} \) and there are no roots for \( S_a \).

Upon the Weyl group action, we can assume that each \( S_a \) is a segment of continuous integers, and for any \( S_a \) in (2), \( \pm (e_i - e_j) \) are roots of \( \mathfrak{h} \) but \( \pm (e_i + e_j) \) are not. Then each \( S_a \) corresponds to a subalgebra of type \( C, A \oplus \mathbb{R} \) (with a one dimensional center), \( A_1 \) or \( \mathbb{R} \).

If \( n > 3 \) and \( \mathfrak{h} \) is not \( C_{n-1} \oplus \mathbb{R} \) or \( C_{n-1} \oplus A_1 \), then considering \( \alpha = e_1 + e_{n-1} \) and \( \beta = e_2 + e_n \) we get a contradiction to (1) of Lemma 4.4.

Suppose \( n = 3 \) and \( \mathfrak{h} = A_1 \oplus A_1 \oplus \mathbb{R} \), where the \( \mathbb{R} \) factor is from (4). Then we can assume that \( \mathbb{R} \) corresponds to \{3\}. Considering \( \alpha = e_1 + e_2 \) and \( \beta = 2e_3 \), we get a contradiction to (1) of Lemma 4.4. If \( \mathfrak{h} = A_1 \oplus (A_1 \oplus \mathbb{R}) \), where the factor \( A_1 \oplus \mathbb{R} \) is from (2) of the list and assumed to correspond to \{2, 3\}, then we can still use \( \alpha = e_1 + e_2 \) and \( \beta = 2e_3 \) to deduce a contradiction. Moreover, if \( n = 3 \) and \( \mathfrak{h} = A_1 \oplus A_1 \oplus A_1 \), then considering the roots \( \pm (e_1 - e_2), \pm (e_2 - e_3) \) and \( \pm (e_1 - e_3) \) in \( m \), we get a contradiction to (2) of Lemma 4.4.
In summarizing, in this case the only possibility is that $\mathfrak{h} = C_{n-1} \oplus \mathbb{R}$ or $C_{n-1} \oplus A_1$. Thus $(M,F)$ is equivalent to the normal homogeneous complex projective space $\text{Sp}(n)/\text{Sp}(n-1)S^1$ or the normal homogeneous quaternionic projective space $\text{Sp}(n)/\text{Sp}(n-1)\text{Sp}(1)$. It is well known that both the above spaces admit positively curved normal Finsler (in fact Riemannian) metrics.

**The case $\mathfrak{g} = D_n, n > 3$.** The root system of $\mathfrak{g}$ can be identified with the subset

$$\{\pm e_i \pm e_j | 1 \leq i < j \leq n\}$$

in $\mathbb{R}^n$ with the standard orthonormal basis $\{e_1, \ldots, e_n\}$. The subalgebra $\mathfrak{h}$ is a direct sum of subalgebras of type $D$, $A$ and $\mathbb{R}$. To be precise, there is a decomposition of the set $\{1, \ldots, n\}$ into a non-overlapping union $\bigcup_{a \in \mathcal{A}} S_a$, such that for different $a$ and $b$ in $\mathcal{A}$, $\pm e_i \pm e_j$ are not roots of $\mathfrak{h}$ when $i \in S_a$ and $j \in S_b$, and for any $a$ in $\mathcal{A}$, one of the following holds:

1. For any $i, j \in S_a$ with $i \neq j$, $\pm e_i \pm e_j$ are roots of $\mathfrak{h}$;
2. For any $i, j \in S_a$ with $i \neq j$, either $\pm(e_i + e_j)$ or $\pm(e_i - e_j)$, but not both, are roots of $\mathfrak{h}$;
3. $S_a = \{i\}$ and there is no root for $S_a$.

Upon the automorphisms of $D_n$ induced by the Weyl group action of $B_n$, we can assume that each $S_a$ is a segment of continuous integers, and for any $S_a$ of the second case, $\pm (e_i - e_j)$ are roots of $\mathfrak{h}$ but $\pm (e_i + e_j)$ are not. Then each $S_a$ corresponds to a subalgebra of type $D$, $A \oplus \mathbb{R}$ (with a one dimensional center) or $\mathbb{R}$.

If $\mathfrak{h}$ is not $D_{n-1} \oplus \mathbb{R}$, then upon the conjugation of the Weyl group action, we can select $\alpha = e_1 + e_{n-1}$ and $\beta = e_2 + e_n$ and deduce a contradiction to (1) of Lemma 4.4.

If $\mathfrak{h} = D_{n-1} \oplus \mathbb{R}$ (here we can assume that $\mathbb{R}$ corresponds to $\{n\}$), then we can select $\alpha = e_1 + e_n$ and $\beta = e_1 - e_n$ and deduce a contradiction to (1) of Lemma 4.4.

In conclusion, in this case, there does not exist any positively curved normal homogeneous Finsler space.

**The case $\mathfrak{g} = E_6$, $E_7$ or $E_8$.** First note that for any two different root planes $\mathfrak{g}_{\pm \alpha}$ and $\mathfrak{g}_{\pm \beta}$ in $\mathfrak{m}$, the angle between $\alpha$ and $\beta$ can not be $\pi/2$, otherwise $\alpha \pm \beta$ are not roots of $\mathfrak{g}$ and we can deduce a contradiction to (1) of Lemma 4.4. If the angle is $\pi/3$, then $\pm (\alpha + \beta)$ are not roots of $\mathfrak{g}$ and $\mathfrak{g}_{\pm \alpha, \mathfrak{g}_{\pm \beta}} = \mathfrak{g}_{\pm (\alpha - \beta)}$ is a root plane. Then $\mathfrak{g}_{\pm (\alpha - \beta)}$ must be a root plane in $\mathfrak{h}$, otherwise we can get a contradiction to (2) of Lemma 4.4.

Moreover, for any root $\alpha$ of $\mathfrak{m}$, we have $[\mathfrak{g}_{\pm \alpha}, \mathfrak{g}_{\pm \alpha}] \subset t \subset \mathfrak{h}$.

Consequently we have $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. By Proposition 4.2, $(M,F)$ is a symmetric normal homogeneous space of rank 1, which is impossible (see [20]). In fact, this argument is also valid for the cases of $A_n$ and $D_n$. In conclusion, in this case there does not exist any positively curved normal homogeneous Finsler spaces.

**The case $\mathfrak{g} = F_4$.** The root system can be identified with the subset

$$\{\pm e_i | 1 \leq i \leq 4\} \cup \{\pm e_i \pm e_j | 1 \leq i < j \leq 4\} \cup \{\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\}$$

(5.25)
in $\mathbb{R}^4$ with the standard orthonormal basis $\{e_1, e_2, e_3, e_4\}$. If there are two short roots $\alpha$ and $\beta$ of $m$ with $\alpha \neq \beta$, then using the Weyl group action (which changes $h$ by a conjugation, without changing the equivalent class of the normal homogeneous space $M$) we can assume that $\alpha = \frac{1}{2}(e_1 + e_2 + e_3 + e_4)$. By (1) of Lemma 4.4, all long roots of the form $\pm (e_i - e_j)$ belong to $h$. Taking into account the root $\beta$, we conclude that there are more long roots belonging to $h$ by (1) of Lemma 4.4. Thus all long roots belong to $h$. There are only two choices left for $h$, namely, $h = B_4$ if it has a short root, or $D_4$ if it has no short root. If $h = D_4$, then the plane spanned by $e_1$ and $\frac{1}{2}(e_1 + e_2 + e_3 + e_4)$ contains exactly 6 roots which all belong to $m$, i.e. $\pm e_1, \pm \frac{1}{2}(e_1 + e_2 + e_3 + e_4)$ and $\pm \frac{1}{2}(-e_1 + e_2 + e_3 + e_4)$ which gives a root system of $A_2$. But then we can deduce a contradiction to (2) of Lemma 4.4 by setting $\alpha = e_1$ and $\beta = \frac{1}{2}(e_1 + e_2 + e_3 + e_4)$. If $h = B_4$, then by a suitable conjugation of the Weyl group action of $g$, we can assume that all roots $\pm e_1$ belong to $h$, and all other roots $\pm \frac{1}{2}(-e_1 + e_2 + e_3 + e_4)$ belong to $m$. Then it is easy to see that $(M, F)$ is equivalent to the symmetric normal homogeneous Cayley plane $F_4/\text{Spin}(9)$. If $m$ contains at most one pair of short roots $\pm \alpha$ of $g$, then it is easily seen that $h = g$, which is a contradiction.

In conclusion, in this case the only possibility is that $(M, F)$ is equivalent to the symmetric normal homogeneous Cayley plane $F_4/\text{Spin}(9)$. Since it is a symmetric coset space of rank one, there does exist positively curved normal Finsler (Riemannian) metric on it.

**The case $g = G_2$.** In this case it is easily seen that $h = A_2, A_1 \oplus A_1, A_1 \oplus \mathbb{R}$, or $\mathbb{R} \oplus \mathbb{R}$. By (1) of Lemma 4.4, $m$ can not have a pair of roots $\alpha$ and $\beta$ with angle $\pi/2$. Then $h$ must be $A_2$. Hence the only possibility is that $(M, F)$ is equivalent to the normal homogeneous sphere $S^6 = G_2/\text{SU}(3)$. This coset space appears in Berger’s list and it admits positively curved normal Finsler (Riemannian) metrics.

The above discussion can be summarized as the following proposition.

**Proposition 5.1** Let $(M, F)$ be a positively curved normal homogeneous space, with $M = G/H$ and $\text{rk} G = \text{rk} H$. Then we have

(1) In the sense of equivalent coset spaces, $(M, F)$ must be equivalent to one of the coset spaces $\mathbb{C}P^n = \text{SU}(n + 1)/\text{U}(n)$ with $n \geq 1$, $\text{SU}(2n + 1)/\text{SO}(2n)$ with $n > 1$, $\mathbb{H}P^n = \text{Sp}(n + 1)/\text{Sp}(n)$ with $n > 1$, $\mathbb{C}P^{2n + 1} = \text{Sp}(n + 1)/\text{Sp}(n)S^1$ with $n \geq 1$, or $S^6 = G_2/\text{SU}(3)$.

(2) In the Lie algebra level, under the assumption that $g$ is simple compact, the pair $(g, h)$ must be one of $(A_n, A_{n-1} \oplus \mathbb{R})$ with $n \geq 1$, $(B_n, D_n)$ with $n > 1$ (notice that $B_2 = C_2$ and $D_2 = A_1 \oplus A_1$), $(C_n, C_{n-1} \oplus A_1)$ with $n > 2$, $(C_n, C_{n-1} \oplus \mathbb{R})$ with $n > 1$ (notice that $(C_2, A_1 \oplus \mathbb{R}) = (B_2, A_1 \oplus \mathbb{R})$, or $(G_2, A_2)$).

**Remark 5.2** In [33], we (with two other co-authors) have classified all positively curved homogeneous Finsler spaces satisfying the condition $\text{rk} G = \text{rk} H$, and get exactly the same list as in [29]. Without the normal condition, we only have (1) of Lemma 4.4. The homogeneous spaces $\text{SU}(3)/T^2$, $\text{Sp}(3)/\text{Sp}(1) \times \text{Sp}(1)$ and $F_4/\text{Spin}(8)$, which can be endowed with positively curved homogeneous non-Riemannian metrics, can not be endowed with any positively curved normal homogeneous metrics by (2) of Lemma 4.4.
6 Positively curved normal homogeneous spaces \( M = G/H \) with \( \mathrm{rk} G = \mathrm{rk} H + 1 \)

In the above section, we have determined positively curved normal homogeneous Finsler spaces \( G/H \) where \( G \) and \( H \) have the same rank. In this section, we consider the other case. We first give the theme for the study.

6.1 The theme for the study

In this section, we assume that \((M, F)\) is a connected positively curved normal homogeneous space induced by the projection \( \rho : G \to M = G/H \) with \( \mathrm{rk} G = \mathrm{rk} H + 1 \) and a bi-invariant Finsler metric \( \bar{F} \) on \( G \). We will also assume that \( G \) is connected and simply connected, and \( H \) is connected. We keep the notations of the above sections.

In the following, we will frequently consider the subalgebra \( \mathfrak{k} \) of \( \mathfrak{g} \) generated by \( \mathfrak{h} \) and \( \mathfrak{t} \). Since \( \mathfrak{k} \) contains the Cartan subalgebra \( \mathfrak{t} \), there is a connected closed subgroup \( K \) of \( G \) such that \( \text{Lie}(K) = \mathfrak{k} \). If \( \dim \mathfrak{k} < \dim \mathfrak{g} \), then the discussion in Theorem 3.3 shows that the corresponding coset space \( G/K \) admits positively curved normal homogeneous Finsler metrics. Thus \( G/K \) must be one of the coset spaces in the list of Proposition 5.1.

It is obvious that we only need to consider the following three cases:

I Each root plane of \( \mathfrak{h} \) is a root plane of \( \mathfrak{g} \);

II There is a root plane \( \mathfrak{h}_{\pm \alpha'} \) of \( \mathfrak{h} \) which is not a root plane of \( \mathfrak{g} \), and there are two roots \( \alpha \) and \( \beta \) of \( \mathfrak{g} \) belonging to different simple components of \( \mathfrak{g} \) such that \( \text{pr}(\alpha) = \text{pr}(\beta) = \alpha' \).

III There is a root plane \( \mathfrak{h}_{\pm \alpha'} \) of \( \mathfrak{h} \) which is not a root plane of \( \mathfrak{g} \), and there are two roots \( \alpha \) and \( \beta \) of \( \mathfrak{g} \) belonging to the same simple component such that \( \text{pr}(\alpha) = \text{pr}(\beta) = \alpha' \).

Now we start a case by case study. As in Section 5, if a contradiction arises, then we conclude that the coset space under consideration does not admit any positively curved normal Finsler metric.

Case I. We assume that each root plane of \( \mathfrak{h} \) is a root plane of \( \mathfrak{g} \). Then the subalgebra \( \mathfrak{k} \) has the same root planes as \( \mathfrak{h} \). Moreover, \( \mathfrak{h} \) has the same semi-simple components as \( \mathfrak{t} \). If \( \mathfrak{k} = \mathfrak{g} \), then \( \dim M = 1 \), which is a contradiction. If \( \mathfrak{k} \neq \mathfrak{g} \), then by Theorem 3.3 we can apply the discussion in Section 5 to the positively curved normal homogeneous space \( G/K \). So we have the following direct sum decompositions

\[
\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_m \oplus \mathbb{R}^l, \quad \text{(6.26)}
\]

\[
\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_m \oplus \mathbb{R}^l, \quad \text{(6.27)}
\]

where the pair \((\mathfrak{g}_1, \mathfrak{k}_1)\) can only be \((A_n, A_{n-1} \oplus \mathbb{R}), (B_n, D_n), (C_n, C_{n-1} \oplus A_1), (C_n, C_{n-1} \oplus \mathbb{R}), (F_4, B_4)\) or \((G_2, A_2)\) (see Proposition 5.1). Note that the cases that \((\mathfrak{g}_1, \mathfrak{k}_1) = (B_n, D_n), (C_n, C_{n-1} \oplus A_1), (F_4, B_4)\) and \((G_2, A_2)\) are impossible. Otherwise we will have

\[
\mathfrak{h} = \mathfrak{k}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_m \oplus \mathbb{R}^{l-1}, \quad \text{(6.28)}
\]
Choose an arbitrary nonzero vector $v$ in a root plane $g_{\pm\alpha}$ in $m$. Then $v$ and $\alpha^\perp$ in $t$ span a FSCS, which is a contradiction to Theorem 3.3. For the remaining two cases $(g_1, t_1) = (A_n, A_{n-1} \oplus \mathbb{R})$ and $(g_1, t_1) = (C_n, C_{n-1} \oplus \mathbb{R})$, the corresponding $M = G/H$ is a $S^1$-bundle over the complex projective space $G/K$, which is equivalent to its universal covering, i.e., a normal homogeneous sphere $SU(n+1)/SU(n)$, $U(n+1)/U(n)$, $Sp(n)/Sp(n-1)$, or $Sp(n)S^1/Sp(n-1)S^1$. These spaces all appear in Berger’s list, so they all admit positively curved normal Finsler (Riemannian) metrics.

**Case II.** We assume that there is a root plane $h_{\pm\alpha'}$ which is not a root plane of $g$, and there are two roots $\alpha$ and $\beta$ of $g$ belonging to different simple components of $g$, say $g_1$ and $g_2$, such that $pr(\alpha) = pr(\beta) = \alpha'$. Then $g_{\pm\alpha'} = (g_1)_{\pm\alpha} + (g_2)_{\pm\beta}$, and all other root planes of $h$ are also root planes of $g$. We claim that $h_{\pm\alpha'} \cap (g_1)_{\pm\alpha} = h_{\pm\alpha'} \cap (g_2)_{\pm\beta} = 0$. In fact, otherwise the $Ad(exp(t \cap h))$ actions will make $h_{\pm\alpha'}$ a root plane of $g$.

Now we have the direct sum decompositions

$$g = g_1 \oplus g_2 \oplus g_3 \oplus \cdots \oplus g_m \oplus \mathbb{R}^l, \quad (6.29)$$

$$h = h' \oplus g_3 \oplus \cdots \oplus g_m \oplus \mathbb{R}^l, \quad (6.30)$$

where $h'$ is a subalgebra of $g_1 \oplus g_2$. We now prove that, if there is any other root $\alpha''$ of $g_1$, such that $(g_1)_{\pm\alpha''}$ is a root plane of $h'$, then $[(g_1)_{\pm\alpha''}, (g_1)_{\pm\alpha}] = 0$, from which we can also see $\alpha$ and $\alpha''$ must be orthogonal to each other. In fact, otherwise we have

$$[(g_1)_{\pm\alpha''}, h_{\pm\alpha}] = [(g_1)_{\pm\alpha''}, (g_1)_{\pm\alpha}] \neq 0.$$ 

Then

$$(g_1)_{\pm\alpha} \subset [(g_1)_{\pm\alpha''}, (g_1)_{\pm\alpha}] \subset h',$$

which is a contradiction. Similarly, if there is any other root $\beta''$ of $g_2$ such that $(g_2)_{\pm\beta''}$ is a root plane of $h'$, then $[(g_2)_{\pm\beta''}, (g_2)_{\pm\beta}] = 0$, and $\beta''$ is orthogonal to $\beta$.

The above argument shows that, if there exists any root of $h'$ other than $\pm\alpha'$, then the subalgebra $t$ must be the direct sum of three subalgebras in $g_1 \oplus g_2$. This implies that $t \neq g$. According to the above arguments, we have direct sum decompositions:

$$g = (g_1 \oplus g_2) \oplus g_3 \oplus \cdots \oplus g_m \oplus \mathbb{R}^l, \quad (6.31)$$

$$t = (h'' \oplus A_1 \oplus A_1) \oplus t_3 \oplus \cdots \oplus t_n \oplus \mathbb{R}^l, \quad (6.32)$$

$$h = (h'' \oplus A_1) \oplus t_3 \oplus \cdots \oplus t_n \oplus \mathbb{R}^l, \quad (6.33)$$

where each $t_i$ is a subalgebra of $g_i$. Moreover, in the decomposition of $t$, $h'' \oplus A_1 \oplus A_1$ is a subalgebra of $g_1 \oplus g_2$, and the two $A_1$-factors correspond to the roots $\alpha$ and $\beta$, respectively. Furthermore, in the decomposition of $h$, the $A_1$-factor is a diagonal subalgebra of $A_1 \oplus A_1$ in $t$, corresponding to the root $\alpha'$. Applying the same arguments in Section 5 to $G/K$, we can assume, up to a proper reorder, that $t_i = g_i$ for $i > 2$, and $g_2 = A_1$ or $h''$.

By Proposition 5.1, there are only very limited choices for $(g_1, g_2)$ and $h''$. More precisely, if $g_2 = A_1$ and $h'' = \mathbb{R}$, then we have $g_1 = A_2$, and $(M, F)$ is equivalent to the normal homogeneous space $SU(3) \times SO(3)/U^*(2)$ constructed in 3.1. If $g_2 = A_1$ and $h'' = A_1$, then we have $g_1 = C_2$, and $(M, F)$ is equivalent to the homogeneous sphere $S^7 = Sp(2)Sp(1)/Sp(1)Sp(1)$. Furthermore, if $g_2 = A_1$ and $h'' = C_{n-1}$ with
Assume that Lemma 6.1 and Lie algebra. But then the matrix \( n > 2 \), then we have \( g_1 = C_n \), and \((M, F)\) is equivalent to the homogeneous sphere \( S^{4n-1} = \text{Sp}(n)\text{Sp}(1)/\text{Sp}(n-1)\text{Sp}(1) \). If \( g_2 = h'' \), then \( g_1 \) must be \( C_2 \), and we have \( S^7 = \text{Sp}(2)\text{Sp}(1)/\text{Sp}(1)\text{Sp}(1) \) as well.

If there does not exist any root other than \( \pm \alpha' \) for \( h' \), then \((6.31)-(6.33)\) are still valid for \( h'' = 0 \). If \( f \neq g \), then by the discussion in Section 5 and Proposition 5.1, the pair \( \{g_1, g_2\} \) must be \( \{0, C_2\} \), which is a contradiction with our assumption. So \( g_1 = g_2 = A_1 \), and \((M, F)\) is equivalent to the symmetric coset space \( S^3 = \text{SO}(4)/\text{SO}(3) \).

**Case III.** We assume that there is a root plane \( h_{\pm \alpha'} \) of \( h \) which is not a root plane of \( g \), and there are two roots \( \alpha \) and \( \beta \) of \( g \) belonging to the same simple component, say \( g_1 \), such that \( \text{pr}(\alpha) = \text{pr}(\beta) = \alpha' \). From our description for \( h \) we have the direct sum decompositions

\[
\begin{align*}
g &= g_1 \oplus g_2 \oplus \cdots \oplus g_m \oplus \mathbb{R}^l, \\
h &= h_1 \oplus h_2 \oplus \cdots \oplus h_m \oplus \mathbb{R}^l,
\end{align*}
\]

(6.34)

(6.35)

where \( h_i \) is a subalgebra of the simple factor \( g_i \) with \( \text{rk} h_i = \text{rk} g_i \), \( \forall i \). We assert that \( h_i = g_i \) for any \( i > 1 \). In fact, otherwise we can find a root plane \( g_{\pm \gamma} \) in \( m \) from some \( g_i, i > 1 \). Then for any nonzero vector \( v \in g_{\pm \gamma} \), the linear span of \( v \) and the orthogonal complement \( \gamma^\perp \) in \( t \) is a FSCS. This is a contradiction to Theorem 3.3. Now let \( H_1 \) be the connected subgroup of \( G_1 \) with \( \text{Lie}(H_1) = h_1 \). Then \((M, F)\) is equivalent to a normal homogeneous space \( G_1/H_1 \). Thus we can assume that \( g \) is a simple compact Lie algebra.

There are some common cases which can be excluded by the following lemmas.

**Lemma 6.1** Assume that \( G \) is a compact simple Lie group which is not \( G_2 \), and \( M = G/H \) is a positively curved normal homogeneous space in Case III. Keep all the above notations. Then the angle between \( \alpha \) and \( \beta \) can not be \( \frac{\pi}{3} \) or \( \frac{2\pi}{3} \).

**Proof.** The condition that \( g \) is not \( G_2 \) and the angle between \( \alpha \) and \( \beta \) is \( \frac{\pi}{3} \) or \( \frac{2\pi}{3} \) indicates that the subalgebra algebra \( g' = \mathbb{R}\alpha + \mathbb{R}\beta + \sum_{m, n \in \mathbb{Z}} g_{\pm (m\alpha + n\beta)} \) is isomorphic to \( A_2 \), and \( h \cap g' = \mathbb{R}\alpha' + h_{\pm \alpha'} \) is isomorphic to \( A_1 \). If the angle between \( \alpha \) and \( \beta \) is \( \frac{\pi}{3} \), then we can naturally present \( h \) by matrices in \( \mathfrak{su}(3) \), i.e., \( h \cap g' \) is linearly spanned by the following three nonzero matrices:

\[
u = \begin{pmatrix}
\sqrt{-1} & 0 & 0 \\
0 & \sqrt{-1} & 0 \\
0 & 0 & -2\sqrt{-1}
\end{pmatrix} \in \mathbb{R}\alpha',
\]

\[
v = \begin{pmatrix}
0 & 0 & a \\
0 & 0 & \bar{b} \\
-a & -\bar{b} & 0
\end{pmatrix} \in h_{\alpha'},
\]

and

\[
w = \frac{1}{3}[u, v] = \begin{pmatrix}
0 & 0 & \sqrt{-1}a \\
0 & 0 & \sqrt{-1}b \\
\sqrt{-1}a & \sqrt{-1}b & 0
\end{pmatrix} \in h_{\alpha'}.
\]

But then the matrix

\[
\frac{1}{2}[v, w] = \begin{pmatrix}
\sqrt{-1}|a|^2 & \sqrt{-1}ab & 0 \\
\sqrt{-1}ab & \sqrt{-1}|b|^2 & 0 \\
0 & 0 & \sqrt{-1}|a|^2 - \sqrt{-1}|b|^2
\end{pmatrix}
\]

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can belong to \( h \cap g' \) only when \( a = b = 0 \), which is a contradiction.

The proof for the case that the angle between \( \alpha \) and \( \beta \) is \( \frac{2\pi}{3} \) is similar. \( \blacksquare \)

Now we continue the case by case study for Case III.

### 6.2 The exceptional cases

**The case** \( g = G_2 \). The root system of \( G_2 \) can be identified with the subset

\[
\{(\pm \sqrt{3}, 0), (\pm \frac{\sqrt{3}}{2}, \pm \frac{3}{2}), (0, \pm 1), (\pm \frac{\sqrt{3}}{2}, \pm \frac{1}{2})\}
\]

in \( \mathbb{R}^2 \) with the standard inner product. Since \( h = A_1 \), all roots of \( g \) outside the subset \( pr^{-1}(\pm \alpha') \) belong to \( m \). Up to the Weyl group actions, there are only very few choices of \( \alpha \) and \( \beta \), and in each case, one can easily find a perpendicular pair of roots \( \gamma_1 \) and \( \gamma_2 \) from \( m \). Then any nonzero vectors \( v_1 \in g_{\pm \gamma_1} \) and \( v_2 \in g_{\pm \gamma_2} \) span a FSCS, which is a contradiction. In summarizing, the normal homogeneous Finsler space \((M, F)\) can not be positively curved in this case.

**The case** \( g = F_4 \). The standard presentation for the root system is given in \( (5.26) \).

Upon the Weyl group action, we only need to consider the following subcases:

**Subcase 1.** The angle between \( \alpha \) and \( \beta \) is \( \pi/4 \). In this case, we can assume that \( \alpha = e_1 + e_2 \) and \( \beta = e_2 \). Then \( \alpha' = e_2 \) is a root of \( h \). By (1) of Lemma \( 4.5 \), \( \pm e_1 \) are roots of \( m \). The length of the vector \( pr(e_1 + e_2 + \frac{1}{2}e_3 + \frac{1}{2}e_4) \) is \( \sqrt{2} \) and it is not orthogonal to \( \alpha' \). So it is not a root of \( h \). Thus \( \pm(\frac{1}{2}e_1 + \frac{1}{2}e_2 + \frac{1}{2}e_3 + \frac{1}{2}e_4) \) are roots of \( m \). Then one can deduce a contradiction by applying (3) of Lemma \( 4.5 \) to \( e_1 \) and \( \pm(\frac{1}{2}e_1 + \frac{1}{2}e_2 + \frac{1}{2}e_3 + \frac{1}{2}e_4) \).

**Subcase 2.** The angle between \( \alpha \) and \( \beta \) is \( \frac{\pi}{2} \). Using suitable Weyl group actions one can reduce the problem to the following cases:

1. \( \alpha \) and \( \beta \) are long roots. Without losing generality, we can assume that \( \alpha = e_1 + e_2 \), \( \beta = e_2 - e_1 \);
2. \( \alpha \) and \( \beta \) are not of the same length. Without losing generality, we can assume that \( \alpha = e_1 + e_2 \) and \( \beta = -e_3 \);
3. \( \alpha \) and \( \beta \) are short roots. Without losing generality, we can assume that \( \alpha = e_1 \) and \( \beta = e_2 \).

Notice that (1) is covered in the the case of the last subcase. We will discuss the rest situations in the following.

First consider the case \( \alpha = e_1 + e_2 \) and \( \beta = -e_3 \). Then \( \alpha' = \frac{1}{3}e_1 + \frac{1}{3}e_2 - \frac{2}{3}e_3 \) is a root of \( h \) with length \( \sqrt{\frac{2}{3}} \), and \( g_{\pm \alpha'} = g_{\pm(e_1 + e_2)} + g_{\pm e_3} \). By (1) of Lemma \( 4.5 \), \( \pm e_1 \) are roots of \( h \), and \( g_{\pm e_4} = g_{\pm e_4} \). Thus \( g_{\pm e_4} \subset h \). Now the vector \( pr(-e_3 + e_4) \) is not orthogonal to \( \alpha' \), and have length \( \sqrt{\frac{2}{3}} \). So it is not a root of \( h \), and \( g_{\pm(e_4 - e_3)} \subset m \). Therefore we have

\[
g_{\pm e_4} = [g_{\pm e_4}, g_{\pm(e_4 - e_3)}] \subset m, \tag{6.37}
\]

and \( h_{\pm \alpha} = g_{\pm (e_1 + e_2)} \), which is a contradiction with the assumption.
Now we assume that $\alpha = e_1$ and $\beta = e_2$. Then $\alpha' = \frac{1}{2}(e_1 + e_2)$ is a root of $\mathfrak{h}$. By (2) of Lemma 4.3, $e_1 + e_2 = 2\alpha'$ is also a root of $\mathfrak{h}$, which is a contradiction.

**Subcase 3.** The angle between $\alpha$ and $\beta$ is $\frac{2\pi}{3}$. Without losing generality, we assume that $\alpha = e_1 + e_2$ and $\beta = -e_2$. Then $\alpha' = \frac{2}{3}e_1 - \frac{1}{3}e_2$ is a root of $\mathfrak{h}$ with length $\frac{\sqrt{2}}{\sqrt{3}}$. There are only two roots of $\mathfrak{g}$ in $\text{pr}^{-1}(\alpha')$, i.e., $\mathfrak{h}_{\pm\alpha'} \subset \mathfrak{g}_{\pm\alpha'} = \mathfrak{g}_{\pm(e_1+e_2)} + \mathfrak{g}_{\pm e_2}$.

By (2) of Lemma 4.3, $e_3$ is a root of $\mathfrak{h}$. It is obvious that $e_3$ is the only root of $\mathfrak{g}$ in $\text{pr}^{-1}(e_3)$, i.e., $\mathfrak{g}_{\pm e_3} = \mathfrak{h}_{\pm e_3}$. The vector $\text{pr}(e_2 + e_3)$ is not a root of $\mathfrak{h}$, since it is not orthogonal to $\alpha'$ and its length is $\frac{\sqrt{2}}{\sqrt{3}}$. So $e_2 + e_3$ is a root of $\mathfrak{m}$, i.e., $\mathfrak{g}_{\pm(e_2+e_3)} \subset \mathfrak{m}$. So we have

$$\mathfrak{g}_{\pm e_2} \subset [\mathfrak{g}_{\pm(e_2+e_3)}, \mathfrak{g}_{\pm e_3}] \subset \mathfrak{m}. \quad (6.38)$$

Then $\mathfrak{h}_{\pm\alpha'}$ must be the root plane $\mathfrak{g}_{\pm(e_1+e_2)}$, which is a contradiction to our assumption.

To summarize, the normal homogeneous Finsler space $(M, F)$ can not be positively curved in this case.

**The case $\mathfrak{g} = E_6$.** The root system can be identified with the subset

$$\{\pm e_1 \pm e_j | 1 \leq i < j < 6\} \cup \{-\frac{1}{2}e_1 \pm \cdots \pm \frac{1}{2}e_5 \pm \frac{\sqrt{3}}{2}e_6, \text{ with odd plus signs}\} \quad (6.39)$$

in $\mathbb{R}^6$ with the standard orthonormal basis $\{e_1, \ldots, e_6\}$. We only need to consider the case that the angle between $\alpha$ and $\beta$ is $\frac{\pi}{2}$. Using suitable Weyl group actions, we can assume that $\alpha = e_1 + e_2$ and $\beta = -e_1 + e_2$, or $\alpha = e_1 + e_2$ and $\beta = e_3 + e_4$. However, it is easy to see that for $E_6$, we have more automorphisms for the root system, which can reduce the discussion the case that $\alpha = e_1 + e_2$ and $\beta = e_2 - e_1$. Then the unit vector $\alpha' = e_2$ is a root of $\mathfrak{h}$. By calculating the orthogonal projections of all the roots of $\mathfrak{g}$ in $\mathfrak{t} \cap \mathfrak{h}$, one easily sees that none of the vectors $\text{pr}(\pm \frac{1}{2}e_1 \pm \cdots \pm \frac{1}{2}e_5 \pm \frac{\sqrt{3}}{2}e_6)$ is a root of $\mathfrak{h}$. Thus the roots of the subalgebra $\mathfrak{t}$ are those of $D_5$, i.e., $\mathfrak{t} \neq \mathfrak{g}$. By the discussion in Section 5, $G/K$ admits positively curved normal homogeneous Finsler metrics, which is impossible by Proposition 5.1.

To summarize, the normal homogeneous Finsler space $(M, F)$ can not be positively curved in this case.

**The case $\mathfrak{g} = E_7$.** The root system can be identified with the subset

$$\{\pm e_i \pm e_j, \forall 1 \leq i < j \leq 6; \pm \sqrt{2}e_7\}$$

$$\cup \{-\frac{1}{2}e_1 \pm \cdots \pm \frac{1}{2}e_6 \pm \frac{\sqrt{2}}{2}e_7, \text{ where the number of } + \frac{1}{2} \text{'s is even}\} \quad (6.40)$$

in $\mathbb{R}^7$ with the standard orthonormal basis $\{e_1, \ldots, e_7\}$. We only need to consider the case that the angle between $\alpha$ and $\beta$ is $\frac{\pi}{2}$. Using suitable Weyl group actions we can reduce the discussion to two situations that $\alpha = e_1 + e_2$ and $\beta = e_2 - e_1$, or $\alpha = e_1 + e_2$ and $\beta = e_3 - e_4$.

**Subcase 1.** Assume that $\alpha = e_1 + e_2$, $\beta = e_2 - e_1$. Then $\alpha' = e_2$ is a root of $\mathfrak{h}$. By (2) of Lemma 4.3, $\pm \sqrt{2}e_7$ are roots of $\mathfrak{h}$, and obviously $\mathfrak{h}_{\pm \sqrt{2}e_7} = \mathfrak{g}_{\pm \sqrt{2}e_7}$. On the other hand, none of the vectors among $\text{pr}(\pm \frac{1}{2}e_1 \pm \cdots \pm \frac{1}{2}e_6 \pm \frac{\sqrt{2}}{2}e_7)$ is orthogonal to $\alpha'$ or has the proper length. So except $\pm \sqrt{2}e_7$, all roots of $\mathfrak{h}$ are linear combinations of
the first six $e_i$s. Then the subalgebra $\mathfrak{t}$ is a subalgebra of $D_6 \oplus A_1$, which is a proper subalgebra of $\mathfrak{g}$. By the discussion in Section 5, $G/K$ admits positively curved normal homogeneous Finsler metrics, which is impossible by Proposition 5.1.

**Subcase 2.** Assume that $\alpha = e_1 + e_2$ and $\beta = -e_3 - e_4$. Then the unit vector $\alpha' = \frac{1}{5}(e_1 + e_2 - e_3 - e_4)$ is a root of $\mathfrak{h}$. Thus none of the vectors $\text{pr}(\pm e_i \pm e_j)$, with $1 \leq i \leq 4 < j < 7$, or $\text{pr}(\pm (\frac{1}{2}e_1 + \frac{1}{2}e_2 + \frac{1}{2}e_3 + \frac{1}{2}e_4 - e_i) \pm (\frac{1}{2}e_5 - \frac{1}{2}e_6) \pm \frac{\sqrt{2}}{2}e_7)$, with $1 \leq i \leq 4$, is orthogonal to $\alpha'$. Moreover, they all have length $\frac{\sqrt{2}}{2}$. Thus they are not roots of $\mathfrak{h}$. By (2) of Lemma 4.5, $\pm(e_5 - e_6)$ are roots of $\mathfrak{h}$. It is obvious that $\mathfrak{g}(e_5 - e_6) = \mathfrak{g}(e_5 - e_6)$. Now by the above discussion, for any root $\gamma' \neq \pm(e_5 - e_6)$ of $\mathfrak{h}$, and any root plane $\mathfrak{g}_{\pm \gamma}$ in $\mathfrak{g}(e_5 - e_6)$ is a root of $h$, and $\mathfrak{g}_{\pm \gamma} = \sum_{\gamma, \gamma' \neq \gamma} \alpha = e_5 - e_6$, and $\{\mathfrak{g}_{\pm \gamma}, \mathfrak{g}(e_5 - e_6)\} = 0$. Then the subalgebra $\mathfrak{t}$ is contained in the subalgebra $\mathfrak{t}'$ of $\mathfrak{g}$ generated by $\mathfrak{t}$, and the root subspaces $\mathfrak{g}_{\pm \gamma}$ among the subspaces $\mathfrak{g}_{\pm \gamma'}$. Thus $\mathfrak{t}'$ has a three dimensional ideal (an $A_1$) spanned by $\mathfrak{g}(e_5 - e_6)$ and $e_5 - e_6 \in \mathfrak{t}$. In particular, $\mathfrak{t} \subset \mathfrak{t}' \neq \mathfrak{g}$. By the discussion in Section 5, $G/K$ admits positively curved normal homogeneous Finsler metrics, which is impossible by Proposition 5.1.

To summarize, the normal homogeneous Finsler space $(M, F)$ is not positively curved in this case.

**The case $\mathfrak{g} = E_8$.** The root system can be identified with the subset

$$\{\pm e_i \pm e_j | 1 \leq i < j \leq 8\} \cup \{\pm \frac{1}{2}e_1 \pm \cdots \pm \frac{1}{2}e_8 \text{ with even number of plus signs}\} \quad (6.1)$$

in $\mathbb{R}^8$ with the standard orthonormal basis $\{e_1, \ldots, e_8\}$. We only need to consider the case that the angle between $\alpha$ and $\beta$ is $\frac{\pi}{4}$. Using suitable Weyl group actions we can reduce the discussion to the two situations that $\alpha = e_1 + e_2$ and $\beta = e_2 - e_1$, or $\alpha = e_1 + e_2$ and $\beta = -e_3 - e_4$.

**Subcase 1.** Assume that $\alpha = e_1 + e_2$ and $\beta = e_2 - e_1$. Then the unit vector $\alpha' = e_2$ is a root of $\mathfrak{h}$. Note that none of the vectors $\text{pr}(\pm \frac{1}{2}e_1 \pm \cdots \pm \frac{1}{2}e_8)$ is a root of $\mathfrak{h}$, since they have the same length $\frac{\sqrt{2}}{2}$, and are not orthogonal to $\alpha'$. Thus $\mathfrak{t}$ is contained in the subalgebra $\mathfrak{t}'$ generated by $\mathfrak{t}$ and all root planes in the root subspaces $\mathfrak{g}_{\pm \gamma}$. Therefore $\mathfrak{t}'$ is contained in a subalgebra generated by $\mathfrak{t}$ and the root subspaces $\mathfrak{g}_{\pm(e_1 - e_2)}$, which is isomorphic to $D_8$. In particular, $\mathfrak{t} \neq \mathfrak{g}$. By the discussion in Section 5, $G/K$ admits positively curved normal homogeneous Finsler metrics, which is impossible by Proposition 5.1.

**Subcase 2.** Assume that $\alpha = e_1 + e_2$ and $\beta = -e_3 - e_4$. Then the unit vector $\alpha' = \frac{1}{3}(e_1 + e_2 - e_3 - e_4)$ is a root of $\mathfrak{h}$. Note that none of the vectors $\text{pr}(\pm e_i \pm e_j)$ with $1 \leq i \leq 4 < j \leq 8$ and $\text{pr}(\pm e_1 \pm \cdots \pm e_8)$, where there are odd number of plus signs in the first four terms and the rest four terms, is a root of $\mathfrak{h}$, since they are not orthogonal to $\alpha'$ and they have the same improper length $\frac{\sqrt{2}}{2}$. So the root planes of the corresponding roots of $\mathfrak{g}$ are all contained in $\mathfrak{m}$. Now a direct calculation shows that the orthogonal complement of the sum of these root planes is in fact the subalgebra $\mathfrak{t}'$ linearly spanned by $\mathfrak{t}$, the root planes of $\mathfrak{g}$ for the roots $\pm e_i \pm e_j$, $1 \leq i < j \leq 4$ or $5 \leq i < j \leq 8$, and $\frac{1}{3}(\pm e_1 \pm \cdots \pm e_8)$, where there are even numbers of plus signs in the first four terms and in the rest four terms. This subalgebra $\mathfrak{t}'$ is another $D_8$ in $\mathfrak{g}$.
It contains $\mathfrak{h}$ and $\mathfrak{t}$, hence it contains $\mathfrak{t}$, which is not equal to $\mathfrak{g}$. By the discussion in Section 5. G/K admits positively curved normal homogeneous Finsler metrics, which is impossible by Proposition 5.1.

To summarize, the normal Finsler space $(M,F)$ cannot be positively curved in this case.

### 6.3 The $A$ and $D$ cases

We will use the standard presentation of the root systems of classical types given in Section 5.

**The case $\mathfrak{g} = D_n$, $n > 3$.** We only need to consider the case that the angle between $\alpha$ and $\beta$ is $\frac{\pi}{2}$. Using suitable automorphism of $\mathfrak{g}$ we can reduce the problem to the two situations that $\alpha = e_1 + e_2$ and $\beta = e_2 - e_1$, or $\alpha = e_1 + e_2$ and $\beta = -e_3 - e_4$. Note that the last case can appear only when $n > 4$.

**Subcase 1.** Assume that $\alpha = e_1 + e_2$ and $\beta = e_2 - e_1$. Then $\alpha' = e_2$ is a root of $\mathfrak{h}$ with length 1. We assert that the subalgebra $\mathfrak{h}$ must be isomorphic to $B_{n-1}$. To see this, we need to determine its root systems. By (2) of Lemma 4.5, all the roots $\pm e_i \pm e_j$ of $\mathfrak{g}$, with $1 < i < j \leq n$, are also roots of $\mathfrak{h}$. It is obvious that $\mathfrak{h}_{\pm e_i \pm e_j} = \mathfrak{g}_{\pm e_i \pm e_j} = \mathfrak{h}_{\pm e_i \pm e_j}$, when $1 < i < j \leq n$. By Lemma 4.3, for any nonzero vector $v \in \mathfrak{g}_{\pm (e_2 - e_1)}$ with $i > 2$, $\text{ad}(v)$ maps $\mathfrak{g}_{\pm e_2}$ isomorphically onto $\sum_{\varepsilon = \pm 1} \mathfrak{g}_{\pm (e_2 + \varepsilon e_1)}$, and it preserves the decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Thus for each $i \geq 2$, $e_i$ is a root of $\mathfrak{h}$, and $\mathfrak{h}_{\pm e_i} = \sum_{\varepsilon = \pm 1} \mathfrak{g}_{\pm (e_i + \varepsilon e_1)}$. This argument has exhausted all possible roots of $\mathfrak{h}$ from the projections of roots of $\mathfrak{g}$. Hence the root system of $\mathfrak{h}$ is $B_{n-1}$ (see [5.22]).

From the above arguments, we see that $\mathfrak{h}$ is totally determined by the choice of $\mathfrak{h}_{\pm e_2}$ in $\mathfrak{g}' = \mathfrak{Re}_1 + \mathfrak{Re}_2 + \mathfrak{g}_{\pm (e_1 + e_2)} + \mathfrak{g}_{\pm (e_2 - e_1)}$, which is isomorphic to $A_1 \oplus A_1$, where the first (resp. second) $A_1$ factor is algebraically generated by $\mathfrak{g}_{\pm (e_1 + e_2)}$ (resp. $\mathfrak{g}_{\pm (e_2 - e_1)}$). Now we will prove a lemma showing that a suitable $\text{Ad}(\exp(\mathfrak{Re}_1 + \mathfrak{Re}_2))$-action, which preserves $\mathfrak{t}$ and all the roots, gives an isomorphism between any of the two possible subalgebras $\mathfrak{h}$.

We first give a definition. For a compact Lie algebra of type $A_1$ endowed with a bi-invariant metric, we call an orthogonal basis $\{u_1, u_2, u_3\}$ standard, if $u_1, u_2, u_3$ have the same length, and they satisfying the condition $[u_i, u_j] = u_k$ for $(i,j,k) = (1,2,3)$, $(2,3,1)$ or $(3,1,2)$. The length $c$ of the vectors in a standard basis is a constant which only depends on the the scale of the bi-invariant inner product. In fact, the bracket of any two orthogonal vectors with length $c$ is also a vector with a length $c$.

**Lemma 6.2** Let $\mathfrak{g}' = \mathfrak{g}_1 \oplus \mathfrak{g}_2 = A_1 \oplus A_1$ be endowed with a bi-invariant inner product. Assume that $\mathfrak{t}'$ is a Cartan subalgebra, and $\mathfrak{h}'$ and $\mathfrak{h}''$ are subalgebras isomorphic to $A_1$ satisfying the following conditions:

1. $\mathfrak{h}' \cap \mathfrak{t}' = \mathfrak{h}'' \cap \mathfrak{t}'$ is one dimensional;

2. $\mathfrak{h}' \cap \mathfrak{g}_i = \mathfrak{h}'' \cap \mathfrak{g}_i = 0$, $i = 1, 2$. 

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calculation indicates that it also maps a real number $t$ length is standard basis must be equivalent to the symmetric sphere with Proposition 5.1. Then we freely choose any vector $u_2$ of length $c_1$ from $t^\perp \cap g_1$ and then set $u_3 = [u_1, u_2]$. By (2) and (3) in the lemma, we can find a vector of $h$ which belongs to $u_2 + g_2 \cap t^\perp$. Then its $g_2$-factor is not 0, which can be positively scaled to have length $c_2$, and we can set this vector to be $v_2$. Finally, we set $v_3 = [v_1, v_2]$. Now the subalgebra $h'$ is linearly spanned by $u_1 + av_1$, $u_2 + bv_2$, and their bracket $u_3 + abv_3$, where $a$ is a fixed nonzero constant and $b > 0$. However, as a subalgebra, it must satisfy $[u_2 + bv_2, u_3 + abv_3] = u_1 + av_1$, hence $b = 1$. We can similarly construct another standard basis $\{v_1', v_2', v_3'\}$ for $h''$. Notice that $v_1' = v_1$. It is easy to see that there exists a real number $t$ such that $Ad(exp(tv_1))$ maps $v_2$ to $v_2'$ and $v_3$ to $v_3'$. Then the above calculation indicates that it also maps $h'$ to $h''$. 

\textbf{Proof.} Let $c_1$ and $c_2$ be the length of standard basis vectors for $g_1$ and $g_2$, respectively. We can choose standard bases $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ for $g_1$ and $g_2$ as follows. First, we choose vectors $u_1$ and $v_1$ from $t \cap g_1$ and $t \cap g_2$ with lengths $c_1$ and $c_2$, respectively. Then we freely choose any vector $u_2$ of length $c_1$ from $t^\perp \cap g_1$ and then set $u_3 = [u_1, u_2]$. By (2) and (3) in the lemma, we can find a vector of $h$ which belongs to $u_2 + g_2 \cap t^\perp$. Then its $g_2$-factor is not 0, which can be positively scaled to have length $c_2$, and we can set this vector to be $v_2$. Finally, we set $v_3 = [v_1, v_2]$. Now the subalgebra $h'$ is linearly spanned by $u_1 + av_1$, $u_2 + bv_2$, and their bracket $u_3 + abv_3$, where $a$ is a fixed nonzero constant and $b > 0$. However, as a subalgebra, it must satisfy $[u_2 + bv_2, u_3 + abv_3] = u_1 + av_1$, hence $b = 1$. We can similarly construct another standard basis $\{v_1', v_2', v_3'\}$ for $h''$. Notice that $v_1' = v_1$. It is easy to see that there exists a real number $t$ such that $Ad(exp(tv_1))$ maps $v_2$ to $v_2'$ and $v_3$ to $v_3'$. Then the above calculation indicates that it also maps $h'$ to $h''$. 

In conclusion, in this case $(M, F)$ must be equivalent to a symmetric sphere $S^{2n-1} = SO(2n)/SO(2n-1)$, with $g = D_n$ and $h = B_{n-1}$. The argument is also valid for $g = D_3 = A_3$.

\textbf{Subcase 2.} Assume that $n > 4$, and $\alpha = e_1 + e_2$, $\beta = -e_3 - e_4$. Then $\alpha' = e_1 + e_2 - e_3 - e_4$ is a root of $h$. None of the vectors $pr(\pm e_i \pm e_j)$ with $1 \leq i \leq 4 < j \leq n$ is orthogonal to $\alpha'$ and they all have the same length $\sqrt{2}$. By similar arguments as before, any root of the subalgebra $t$ is either a linear combination of the elements $e_i$ with $i \leq 4$, or a linear combination of the elements $e_i$ with $i > 4$. Thus $t \neq g$ and $G/K$ have positive curved normal homogeneous Finsler metrics. But this is a contradiction with Proposition 5.1.

To summarize, in this case, the positively curved normal homogeneous space $(M, F)$ must be equivalent to the symmetric sphere $S^{2n-1} = SO(2n)/SO(2n-1)$, where $n > 3$.

\textbf{The case} $g = A_n$. We only need to consider the case that the angle between $\alpha$ and $\beta$ is $\arcsin \frac{\sqrt{2}}{2}$ (hence $n > 2$). By suitable Weyl group actions, we can assume that $\alpha = e_1 - e_2$ and $\beta = e_3 - e_4$. Then the unit vector $\alpha' = \frac{1}{\sqrt{2}}(e_1 - e_2) + \frac{1}{\sqrt{2}}(e_3 + e_4)$ is a root of $h$.

If $n > 4$, then none of the vectors $pr(\pm (e_i - e_j))$, with $1 \leq i \leq 4 < j \leq n + 1$, is orthogonal to $h$, and they all have the same length $\sqrt{2}$. Thus none of the above vectors is a root of $h$. On the other hand, a root of $h$ is either a linear combination of the vectors $e_i$ with $i \leq 4$, or a linear combination of the vectors $e_j$ with $j > 4$. The same assertion also holds for any root of the subalgebra $t$. Hence $t$ is contained in a subalgebra of the type $A_3 \oplus A_{n-4} \oplus \mathbb{R} \subseteq g$. But the pair $(g, t)$ is not covered in (2) of Proposition 5.1. This is a contradiction. Thus in this case, the corresponding normal homogeneous Finsler space $(M, F)$ can not be positively curved.

If $n = 3$, then $g$ can also be viewed as $D_3$, with $\alpha = e_1 + e_2$ and $\beta = e_1 - e_2$. We have shown in the above that the positively curved normal homogeneous Finsler space $(M, F)$ is equivalent to the symmetric sphere $S^5 = SO(6)/SO(5)$.

Suppose $n = 4$. Let $g'$ be the subalgebra $A_3$ corresponding to the first four roots $e_i$, $1 \leq i \leq 4$. By the arguments for $n > 4$, one easily sees that all the roots of $t$, which is generated by $h$ and $t$, must belong to $g'$. Then by Proposition 5.1 we have $t = g' \oplus \mathbb{R}$.
By similar arguments as in Subcase 2 for $D_n$, we can see $\mathfrak{h} = \mathfrak{h}' \oplus \mathbb{R}$, in which $\mathfrak{h}'$ is a subalgebra of $\mathfrak{g}'$ isomorphic to $B_2 = C_2$ with the following roots
\[ \{ \pm(e_1 - e_4), \pm(e_2 - e_3), \pm(e_1 - 2e_2 + e_3 - \frac{1}{2}e_4), \pm(e_1 + 2e_2 - e_3 - \frac{1}{2}e_4) \}. \]

Moreover, up to inner isomorphisms of $\mathfrak{g}'$, the subalgebra $\mathfrak{h}'$ is uniquely determined. Berger [5] has proved that, in this case, $\text{SU}(5)/\text{Sp}(2)S^1$ admits a positively curved normal homogeneous Riemannian metric. It is generally called a Berger’s space. Another Berger’s space $\text{Sp}(2)/\text{SU}(2)$ will appear later.

To summarize, in this case, $(M, F)$ is positively curved if and only if it is equivalent to the symmetric normal homogeneous sphere $S^5 = \text{SO}(6)/\text{SO}(5)$, or the Berger’s space $\text{SU}(5)/\text{Sp}(2)S^1$.

### 6.4 The $B$ and $C$ cases

We keep all the notations as above and use the standard presentations for the root systems as in the last section.

**The case $\mathfrak{g} = B_n$, $n > 1$.** Using suitable Weyl group actions we can reduce the the consideration to the following cases.

**Subcase 1.** The angle between $\alpha$ and $\beta$ is $\frac{\pi}{4}$. In this case, we can assume that $\alpha = e_1 + e_2$ and $\beta = e_2$. Then $\alpha' = e_2$ is a root of $\mathfrak{h}$ with $\mathfrak{h}_{\pm e_2} = \mathfrak{g}_{\pm (e_2 - e_1)} + \mathfrak{g}_{\pm e_2} + \mathfrak{g}_{\pm (e_2 + e_1)}$. Denote $\mathfrak{g}' = \mathbb{R}e_1 + \mathbb{R}e_2 + \sum_{a, b} \mathfrak{g}_{\pm (ae_1 + be_2)}$ and $\mathfrak{g}'' = \mathbb{R}e_1 + \mathfrak{g}_{\pm e_1}$. They are Lie algebras $B_2$ and $A_1$ respectively. We can use real matrices in $\mathfrak{so}(5)$ to give a basis of $\mathfrak{h} \cap \mathfrak{g}'$, i.e.,

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{bmatrix} \in \mathbb{R}e_2,
\begin{bmatrix}
0 & 0 & 0 & -a & -a' \\
0 & 0 & 0 & -b & -b' \\
0 & 0 & 0 & -c & -c' \\
a' & b' & c' & 0 & 0
\end{bmatrix} \in \mathfrak{h}_{\pm e_2}
\]

and

\[
\begin{bmatrix}
0 & 0 & 0 & a' & -a \\
0 & 0 & 0 & b' & -b \\
0 & 0 & 0 & c' & -c \\
-a' & -b' & -c' & 0 & 0 \\
a & b & c & 0 & 0
\end{bmatrix} \in \mathfrak{h}_{\pm e_2}.
\]

Since $\mathfrak{h} \cap \mathfrak{g}'$ is a Lie subalgebra, we have $[v, w] \in \mathfrak{h} \cap \mathfrak{g}'$. A direct calculation then shows that $(a, b, c)$ and $(a', b', c')$ are linearly dependent vectors. Using a suitable element $l \in \text{Ad}(\exp A_1)$, which is represented by a conjugation by a matrix of the form $\text{diag}(P, I)$, where $P \in \text{SO}(3)$, and $I$ is the $2 \times 2$ identity matrix, we can make $b = c = b' = c' = 0$ in $v$. Let $u' \in \mathfrak{g}_{\pm (e_2 + e_1)}$ and $v' \in \mathfrak{g}_{\pm (e_2 - e_1)}$ be a pair of nonzero vectors. Then $l^{-1}(u')$ and $l^{-1}(v')$ are vectors of $\mathfrak{m} \cap \mathfrak{g}'$. Since $[u', v'] = 0$, we have $[l^{-1}(u'), l^{-1}(v')] = 0$. Now $l^{-1}(u')$ and $l^{-1}(v')$, together with all the elements $e_i$ with $i > 2$, span a FCS. This is a contradiction.

To summarize, there is no positively curved normal homogeneous Finsler space in this subcase.
In the following subcases we assume that the angle between \( \alpha \) and \( \beta \) is \( \frac{\pi}{2} \). Then using suitable Weyl group actions one can reduce the discussion to one of the following situations:

1. \( \alpha \) and \( \beta \) are long roots. In this case, we can assume that either \( \alpha = e_1 + e_2 \) and \( \beta = e_2 - e_1 \), or \( \alpha = e_1 + e_2 \) and \( \beta = -e_3 - e_4 \);

2. \( \alpha \) and \( \beta \) are not of the same length. Then we can assume \( \alpha = e_1 + e_2 \) and \( \beta = -e_3 \);

3. \( \alpha \) and \( \beta \) are short roots. Then we can assume that \( \alpha = e_1 \) and \( \beta = e_2 \).

We now consider these situations case by case.

**Subcase 2.** The situation that \( \alpha = e_1 + e_2 \) and \( \beta = e_2 - e_1 \) has been covered by the discussion in the last subcase, and in this case there does not exist any positively curved normal homogeneous spaces.

Now we assume that \( \alpha = e_1 + e_2 \) and \( \beta = -e_3 - e_4 \). Then the unit vector \( \alpha' = \frac{1}{2}e_1 + \frac{1}{2}e_2 - \frac{1}{2}e_3 - \frac{1}{2}e_4 \) is a root of \( \mathfrak{h} \). If \( n > 4 \), then none of the vectors \( \pm e_i \pm e_j \) with \( 1 \leq i < j \leq n \) is orthogonal to \( \alpha' \) and each of them has the same length \( \sqrt{3} \). Therefore none of the above roots is a root of \( \mathfrak{h} \). The subalgebra \( \mathfrak{h} \) is then contained in the subalgebra \( B_4 \oplus B_{n-4} \) or \( B_4 \oplus A_1 \). The coset space \( G/K \) can admit positively curved normal homogeneous Finsler metrics. This is a contradiction with Proposition 5.1. Hence in this case the normal homogeneous space cannot be positively curved.

If \( n = 4 \), then by (2) of Lemma 4.5 all the roots \( \pm(e_i - e_j) \) of \( \mathfrak{g} \), with \( 1 \leq i < j \leq 4 \), are roots of \( \mathfrak{h} \), and we have \( \mathfrak{h}(e_i - e_j) = \mathfrak{g}(e_i - e_j) = \hat{\mathfrak{g}}(e_i - e_j) \) for \( 1 \leq i < j \leq 4 \). For any \( v \in \mathfrak{g}(e_2 - e_3) \), \( \text{ad}(v) \) maps \( \hat{\mathfrak{g}}(1/2(e_1 + e_2 - e_3 - e_4)) = \mathfrak{g}(e_1 + e_2) + \mathfrak{g}(e_3 + e_4) \) isomorphically onto \( \mathfrak{g}(e_1 + e_3) + \mathfrak{g}(e_2 + e_4) \), which preserves the decomposition \( \mathfrak{g} = \mathfrak{h} + \mathfrak{m} \). So \( \pm \frac{1}{2}(e_1 + e_3 - e_2 - e_4) \) is also a root of \( \mathfrak{h} \), and \( \hat{\mathfrak{g}}(1/4(e_1 + e_3 - e_2 - e_4)) = \mathfrak{g}(e_1 + e_3) + \mathfrak{g}(e_2 + e_4) \). The same assertion also holds for \( \pm \frac{1}{2}(e_1 + e_4 - e_2 - e_3) \). On the other hand, none of the vectors \( \text{pr}(\pm e_i) \) is orthogonal to the unit root \( \alpha' \) of \( \hat{\mathfrak{h}} \), and they all have the same length \( \frac{\sqrt{7}}{\sqrt{3}} \). Thus they are not roots of \( \hat{\mathfrak{h}} \). In the above, we have determined all the roots of \( \hat{\mathfrak{h}} \) and showed that \( \hat{\mathfrak{h}} = B_3 \).

There is a known example in this subcase, which provide the homogeneous sphere \( S^{15} = \text{Spin}(9)/\text{Spin}(7) \). Up to equivalence, this is the only normal coset space in this subcase. The subalgebra \( \hat{\mathfrak{h}} \) is totally determined by the choice of \( \mathfrak{h}(e_1 + e_3 - e_2 - e_4) \). Consider the subalgebra \( \mathfrak{g}' = \mathbb{R}(e_1 + e_2) + \mathbb{R}(e_3 + e_4) + \mathfrak{g}(e_1 + e_2) + \mathfrak{g}(e_3 + e_4) \) which is an \( A_1 \oplus A_1 \) containing \( \hat{\mathfrak{g}}(e_1 + e_2 - e_3 - e_4) \). By Lemma 6.2, \( \hat{\mathfrak{h}} \cap \mathfrak{g}' \), as well as \( \mathfrak{h}(e_1 + e_2 - e_3 - e_4) \), is uniquely determined up to inner isomorphisms \( \text{Ad}(\exp(t(e_3 + e_4))) \). Note that in the paper [28], the authors present the Spin(9)-invariant Riemannian metric on \( S^{15} \) as a family of Riemannian metrics \( g_t \) (up to homothety), and show that the Riemannian metric has positive curvature if and only if \( 0 < t < \frac{4}{7} \). The normal Riemannian metric is the metric corresponding to \( t = \frac{4}{7} \).

Therefore, in this case, \( (M, F) \) is equivalent to the normal homogeneous sphere \( S^{15} = \text{Spin}(9)/\text{Spin}(7) \).

**Subcase 3.** We assume that \( \alpha = e_1 + e_2 \) and \( \beta = -e_3 \). Then \( \alpha' = \frac{1}{3}e_1 + \frac{1}{3}e_2 - \frac{2}{3}e_3 \) is a root of \( \mathfrak{h} \) with length \( \frac{\sqrt{7}}{\sqrt{3}} \). If \( n > 3 \), then none of the vectors \( \text{pr}(\pm e_i \pm e_j) \), with
$1 \leq i \leq 3 < j \leq n$, is orthogonal to $\alpha'$, and they all have the same length $\sqrt{\frac{5}{3}}$.

Therefore none of the above roots is a root of $\mathfrak{h}$. So any root of the subalgebra $\mathfrak{t}$ is either a linear combinations of $e_1$, $e_2$ and $e_3$, or a linear combinations of the elements $e_i$ with $i > 3$. In particular, $\mathfrak{t} \neq \mathfrak{g}$, and the corresponding coset space $G/K$ admits positively curved normal homogeneous Finsler metrics. By Proposition 5.1 it must be isomorphic to $D_n$, which is a contradiction. So the corresponding normal homogeneous Finsler space cannot be positively curved in this case.

If $n = 3$, then $\mathfrak{h} = G_2$. In fact, by (2) of Lemma 4.5, $\pm(e_i - e_j)$ for $1 \leq i < j \leq 3$ are long roots of $\mathfrak{h}$, with $\mathfrak{h}_{\pm(e_i - e_j)} = \mathfrak{g}_{\pm(e_i - e_j)}$. Then for any nonzero vector $v \in \mathfrak{g}_{\pm(e_2 - e_3)}$, the linear map $\text{ad}(v)$ sends $\hat{\mathfrak{g}}_{\pm\alpha'} = \mathfrak{g}_{\pm(e_1 + e_3)}$ isomorphically onto $\mathfrak{g}_{\pm(e_1 + e_3)} + \mathfrak{g}_{\pm e_2}$, and preserves the decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Thus $\pm(\frac{1}{3}e_1 - \frac{2}{3}e_2 + \frac{1}{3}e_3)$ are roots of $\mathfrak{h}$, with $\hat{\mathfrak{g}}_{\pm(e_1 - 2e_2 + e_3)} = \mathfrak{g}_{e_2} + \mathfrak{g}_{e_1 + e_3}$. The same assertion also holds for $\pm(-\frac{2}{3}e_1 + \frac{1}{3}e_2 + \frac{1}{3}e_3)$. We have found all six long roots and six short roots for $\mathfrak{h}$ and they exhaust all possible roots of $\mathfrak{h}$. This implies that $\mathfrak{h} = G_2$.

The subalgebra $\mathfrak{h}$ is totally determined by the choice of $\mathfrak{h}_{\pm\alpha'} \subset \hat{\mathfrak{g}}_{\pm\alpha'}$. Denote $\mathfrak{g}' = \mathbb{R}(e_1 + e_2) + \mathbb{R}e_3 + \mathfrak{g}_{\pm(e_1 + e_2)} + \mathfrak{g}_{\pm e_3}$, which is a subalgebra of the type $A_1 \oplus A_1$ containing $\hat{\mathfrak{g}}_{\pm\alpha'}$. By Lemma 6.2, we have $\mathfrak{h} \cap \mathfrak{g}'$. Meanwhile, $\mathfrak{h}_{\pm\alpha'}$ is uniquely determined up to inner isomorphisms. So up to equivalence, the only positively curved normal homogeneous Finsler space in this subcase is the homogeneous sphere $S^7 = \text{Spin}(7)/G_2$.

Subcase 4. We assume that $\alpha = e_1$ and $\beta = e_2$. Then $\alpha' = \frac{1}{2}e_1 + \frac{1}{2}e_2$ is a root of $\mathfrak{h}$. But by (2) of Lemma 4.5 $e_1 + e_2 = 2\alpha'$ is also a root of $\mathfrak{h}$. This is a contradiction. So there does not exist positively curved normal homogeneous space in this subcase.

Subcase 5. There is one more subcase left where the angle between $\alpha$ and $\beta$ is $\frac{3\pi}{4}$. We can assume that $\alpha = e_1 + e_2$ and $\beta = -e_2$. Then $\alpha' = \frac{1}{2}e_1 - \frac{1}{2}e_2$ is a root of $\mathfrak{h}$ with length $\frac{1}{\sqrt{3}}$. If $n > 2$, then none of the vectors $\text{pr}(\pm e_i \pm e_j)$, with $1 \leq i \leq 2 < j \leq n$, is orthogonal to $\alpha'$ and each of them has length $\frac{\sqrt{3}}{\sqrt{5}}$ or $\frac{\sqrt{7}}{\sqrt{5}}$. Thus none of these vectors can be a root of $\mathfrak{h}$. On the other hand, each root of the subalgebra $\mathfrak{t}$ is either a linear combination of $e_1$ and $e_2$, or a linear combination of the elements $e_i$, with $i > 2$. So $\mathfrak{t}$ is contained in $B_2 \oplus B_{n-2}$ or $B_2 \oplus A_1$. In particular, $\mathfrak{t} \neq \mathfrak{g}$ and the corresponding coset space $G/K$ admits positively curved normal homogeneous metrics. However, according to Proposition 5.1, $G/K$ can not be positively curved normal homogeneous, which is a contradiction. This implies that in this subcase $(M, F)$ cannot be positively curved.

If $n = 2$, then any linearly independent commuting pair in $\mathfrak{m}$ span a FSCS. Thus in the decomposition of the Lie algebra of a positively curved normal homogeneous Finsler space, the subspace $\mathfrak{m}$ can not have a commuting pair. However, this implies exactly that the corresponding normal homogeneous Riemannian metric is positive curved. Up to equivalence, there is only one such space, that is, the coset space $\text{Sp}(2)/SU(2)$ found by Berger in [5].

To summarize, in this case, the positively curved normal homogeneous space $(M, F)$ is equivalent to $S^7 = \text{Spin}(7)/G_2$, $S^{15} = \text{Spin}(9)/\text{Spin}(7)$, or the Berger’s space $\text{Sp}(2)/SU(2)$.

The case $\mathfrak{g} = C_n$, $n > 2$. Upon the Weyl group action, we only need to consider the following subcases.

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Subcase 1. The angle between $\alpha$ and $\beta$ is $\frac{3\pi}{4}$. We can assume that $\alpha = 2e_1$ and $\beta = e_1 + e_2$. Then $\alpha' = e_1 + e_2$ is a root of $\mathfrak{h}$ with length $\sqrt{2}$. By (1) of Lemma 4.5 we have $g_{\pm(e_1 - e_2)} \subset \mathfrak{m}$. Now none of the vectors $pr(\pm(e_1 - e_3))$ is orthogonal to $\alpha'$, and each has length $\frac{\sqrt{2}}{\sqrt{3}}$. Thus none of them is a root of $\mathfrak{h}$, and we have $g_{\pm(e_1 - e_3)}, g_{\pm(e_2 - e_3)} \subset \mathfrak{m}$. Then we can apply (3) of Lemma 4.5 to $\pm(e_1 - e_2)$, $\pm(e_2 - e_3)$ and $\pm(e_1 - e_3)$ to get a contradiction. Thus the corresponding normal space cannot be positively curved in this subcase.

In the following subcases we consider the situation that the angle between $\alpha$ and $\beta$ is $\frac{3\pi}{4}$. Using suitable Weyl group actions, one can reduce the discussion to the following cases:

1. $\alpha$ and $\beta$ are long roots, and we can assume that $\alpha = 2e_1$ and $\beta = 2e_2$;

2. $\alpha$ and $\beta$ are not of the same length, and we can assume that $\alpha = e_1 + e_2$ and $\beta = -2e_3$; and when $\alpha$ and $\beta$ are short roots, we assume that either
   
   3. $\alpha = e_1 + e_2$ and $\beta = e_2 - e_1$; or
   
   4. $\alpha = e_1 + e_2$ and $\beta = -e_3 - e_4$.

Notice that (1) has been covered by the discussions in the previous subcase. Now we discuss the other cases in the following three subcases.

Subcase 2. Assume that $\alpha = e_1 + e_2$ and $\beta = -2e_3$. Then $\alpha' = \frac{2}{3}(e_1 + e_2 - e_3)$ is a root of $\mathfrak{h}$ with length $\frac{2}{\sqrt{3}}$. By (2) of Lemma 4.5, $\pm(e_1 - e_2)$ are roots of $\mathfrak{h}$. Now none of the vectors $pr(\pm 2e_i), i = 1, 2$, is orthogonal to $\alpha'$, and each of them has length $\frac{\sqrt{3}}{\sqrt{2}}$. Thus none of these vectors is a root of $\mathfrak{h}$, i.e., $g_{\pm 2e_i} \subset \mathfrak{m}$ for $i = 1, 2$. Take any nonzero $u \in g_{\pm 2e_1}$, $v \in g_{\pm 2e_2}$, and two linearly independent $w_1$ and $w_2$ from $g_{\pm 2e_3}$. Then there are two Cartan subalgebras, $t_1$ spanned by $u$, $v$, $w_1$ and all the elements $e_i$ with $i > 3$; and $t_2$ with the same linear generators except that $w_1$ is changed by $w_2$. Then $s = t_1 \cap t_2$ is a flat splitting subalgebra. This is a contradiction to Theorem 3.3.

Subcase 3. We assume that $\alpha = e_1 + e_2$ and $\beta = e_2 - e_1$. Then $\alpha' = e_2$ is a root of $\mathfrak{h}$. By (2) of Lemma 4.5 the root $2e_2 = 2\alpha' \in t \cap \mathfrak{h}$ of $\mathfrak{g}$ is also a root of $\mathfrak{h}$. This is a contradiction.

Subcase 4. We assume that $\alpha = e_1 + e_2$ and $\beta = -e_3 - e_4$. Then the unit vector $\alpha' = \frac{1}{2}e_1 + \frac{1}{2}e_2 - \frac{1}{2}e_3 - \frac{1}{2}e_4$ is a root of $\mathfrak{h}$. By (2) of Lemma 4.5, $e_2 - e_3$ is root of $\mathfrak{h}$ with length $\sqrt{2}$. This implies that $\alpha'$ cannot be the root of a simple factor of $\mathfrak{h}$ isomorphic to $G_2$. Now none of the vectors $pr(\pm 2e_i)$ with $1 \leq i \leq 4$ is orthogonal to $\alpha'$, and they all have the same length $\frac{\sqrt{3}}{\sqrt{2}}$. So they are not roots of $\mathfrak{h}$, and $g_{\pm e_i} \subset \mathfrak{m}$ for $1 \leq i \leq 4$. Let $v_i$ be a nonzero vector in $g_{\pm e_i}$. Then the elements $v_i$ with $1 \leq i \leq 4$, and the elements $e_i$ with $4 < i \leq n$, span a FSCS. This is a contradiction to Theorem 3.3 Therefore in this subcase the normal homogeneous space cannot be positively curved.

Subcase 5. Now in the last subcase we consider the situation that the angle between $\alpha$ and $\beta$ is $\frac{3\pi}{4}$. Obviously we can assume that $\alpha = e_1 + e_2$ and $\beta = -2e_1$. Then $\alpha' = -\frac{1}{3}e_1 + \frac{2}{3}e_2$ is a root of $\mathfrak{h}$ with length $\frac{\sqrt{3}}{\sqrt{2}}$. Now none of the vectors $pr(\pm e_i \pm e_j)$,
with $1 \leq i \leq 2 < j \leq n$, is orthogonal to $\alpha'$, and each has a length $\sqrt{11}/\sqrt{10}$ or $\sqrt{19}/\sqrt{10}$. Thus none of these vectors is a root of $h$. Similarly, $pr(\pm 2e_2)$ are not roots of $h$. On the other hand, any root of the subalgebra $\mathfrak{f}$ is either a linear combination of $e_1$ and $e_2$, or a linear combination of the elements $e_i$ with $i > 2$. Thus $\mathfrak{f}$ is contained in $C_2 \oplus C_{n-2}$ for $n > 3$, or contained in $C_2 \oplus A_1$ for $n = 3$. If $n > 3$, then Proposition 5.1 indicates that $G/K$ can not admit positively curved normal homogeneous Finsler metrics, which is a contradiction.

If $n = 3$, then we have $\mathfrak{g} = C_3$ and $\mathfrak{h} = A_1 \oplus A_1$. Consider a linearly independent commuting pair $u \in \mathfrak{g}_{\pm 2e_2}$ and $v \in \mathfrak{g}_{\pm (e_1 - e_3)}$ in $\mathfrak{m}$. From the centralizer of $s = R\mathfrak{u} + R\mathfrak{v} \subset \mathfrak{m}$ in $\mathfrak{g}$, we can find nonzero vector $w_1 = e_1 + e_3$, and another nonzero vector $w_2$ from $\mathfrak{g}_{\pm 2e_1 + \mathfrak{g}_{\pm 2e_3}}$. Then the vectors $u, v$ and $w_1$ span a Cartan subalgebra $t_1$, and with $w_1$ changed to $w_2$, we have another $t_2$. So $s = t_1 \cap t_2$ is a flat splitting subalgebra of $\mathfrak{g}$. This is a contradiction to Theorem 6.3.

To summarize, there is no positively curved normal homogeneous Finsler space in this subcase.

Now we complete the proof of Theorem 1.1 and Theorem 1.2. Up to equivalence, we have found all smooth coset spaces which may admit positively curved normal homogeneous Finsler metrics, as listed in Theorem 1.1. They are exactly those in Berger’s classification work [5] (plus [31]), which does admit positive normal homogeneous Riemannian metrics (which are just a special class of normal homogeneous Finsler metrics). This finishes the proof of Theorems 1.1 and 1.2.

7 Proof of Theorem 1.3

In this section we give a proof of Theorem 1.3. It will be completed through a case by case study on the coset spaces appearing in Theorem 1.1.

The coset spaces in (1) of Theorem 1.1 are Riemannian symmetric spaces of rank one. For these spaces, the isotropy group acts transitively on the unit sphere $S$ of $\mathfrak{m}$ with respect to any $\text{Ad}(G)$-invariant inner product of $\mathfrak{g}$ (which is unique up to homotheties). So the corresponding normal homogeneous Finsler metrics must be Riemannian.

On the other hand, in (2) of Theorem 1.1 the above assertion for the isotropic action is also valid for the coset spaces $G_2/SU(3)$ and $S^3 = SU(2)/SU(1) = \text{Sp}(1)$, $S^6 = G_2/SU(3)$, and $S^7 = \text{Spin}(7)/G_2$. Therefore the normal homogeneous Finsler metrics on the above coset spaces must also be Riemannian. This proves the first statement of Theorem 1.3.

Now we turn to the second statement. In the following we use the term “the other spaces” to indicate the spaces in the lists of Theorem 1.1 which does not fall into the spaces considered in the above. To complete the proof, we introduce the Condition (R) which is vital for our consideration. Let $G/H$ be a coset spaces of a compact Lie group $G$ and $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be the reductive decomposition with respect to a normal Riemannian metric. Denote the unit sphere of $\mathfrak{m}$ with respect to the induced inner product by $S$.

Condition (R): the unit sphere $S$ in $\mathfrak{m}$ is contained in a single $\text{Ad}(G)$-orbit in $\mathfrak{g}$.

We now give a useful lemma. Recall that a Riemannian homogeneous manifold $G/H$ is called isotropic, if the action of $H$ on the unit sphere $S$ is transitive. Note that
an isotropic normal homogeneous Riemannian manifold must satisfy the condition (R), but it is unknown whether the converse is true.

Lemma 7.1 Let $G/H$ be a coset space of a compact Lie group $G$ with a normal homogeneous Riemannian metric $Q$. If $G/H$ satisfies the condition (R), then any normal homogeneous Finsler metric on $G/H$ must be a positive multiple of $Q$, hence must be Riemannian. On the other hand, if $G/H$ does not satisfy the condition (R) and the Riemannian metric $Q$ has positive sectional curvature, then there exist non-Riemannian normal homogeneous Finsler metrics on $G/H$ with positive flag curvature.

Proof. The first assertion is obvious, since if $G/H$ satisfies the condition (R), then the restriction of the induced Minkowski norm of any normal homogeneous Finsler metric in $S$ must be a constant multiple of the Euclidean norm of $Q$.

Now we prove the second assertion. By the assumption, there exist two points $x_1, x_2 \in S$, such that $\text{Ad}(G) \cdot x_1 \cap \text{Ad}(G) \cdot x_2 = \emptyset$. By the parity of unit of smooth manifolds, there exists a smooth nonnegative function $f_1$ on the unit sphere $S'$ of $g$ with respect to the inner product induced by $Q$, such that $f_1$ is equal to 1 on $\text{Ad}(G) \cdot x_2$ and the support of $f_1$ is contained in an open neighborhood $U$ of $\text{Ad}(G) \cdot x_2$ with $U \cap \text{Ad}(G) \cdot x_1 = \emptyset$. Now using the bi-invariant Haar measure of the Lie group $G$, we can define a smooth nonnegative function $f$ on $S'$ which is invariant under the adjoint action of $G$ such that $f(\text{Ad}(G) \cdot x_2) = 1$ and there is an open subset of $U_1$ containing $\text{Ad}(G) \cdot x_2$, which is invariant under the adjoint action of $G$, such that $U_1 \cap \text{Ad}(G) \cdot x_1 = \emptyset$ and $f(x) = 0$, for any $x \notin U_1$. Now for any positive number $\varepsilon$, we define a real function on $g \setminus \{0\}$ by

$$\bar{F}_\varepsilon(x) = \sqrt{\langle x, x \rangle + \varepsilon|x|^2f(x/|x|)}.$$

Then it is easy to check that for sufficiently small $\varepsilon$, $\bar{F}_\varepsilon$ is a positive definite Minkowski norm on $g$ (see [11]). Since $f$ is invariant under the adjoint action of $G$, $\bar{F}_\varepsilon$ is bi-invariant. On the other hand, when $\varepsilon$ is sufficiently close to 0, the indicatrix of the Minkowski norm $\bar{F}_\varepsilon$ coincides with $S'$ around $x_1$, but differs with $S'$ around $x_2$. Thus the corresponding normal homogeneous metric $\bar{F}_\varepsilon$ on $G/H$ are non-Riemannian when $\varepsilon$ is positive and sufficiently small. By the assumption, $\bar{F}_0$ has positive sectional curvature, so for $\varepsilon$ sufficiently close to 0, $F_\varepsilon$ has positive flag curvature. ■

We now return to the proof of the there. We first assert that for any of the other spaces such that the group $G$ in $M = G/H$ is not simple, Condition (R) is not satisfied. In fact, we can re-scale the bi-invariant inner product differently on each simple factor or Euclidean factor to get a family of induced inner product on $m$ which are not the same up to scalar multiplications. Therefore for any nonzero vector $x \in m$, there exists a bi-invariant inner product $\langle \cdot, \cdot \rangle_1$ on $g$, such that the unit sphere $S_1$ satisfies the conditions that $S_1 \cap \text{Ad}(G) \cdot x \neq \emptyset$ and $S_1 \not\subset \text{Ad}(G) \cdot x$. This proves the assertion.

Now we consider the other spaces in Theorem 1.1 with simple Lie group $G$. We need some calculation. For any Lie algebra $g$ of the Lie groups, there is a canonical way to present the elements in the Lie algebra $g$ by real or complex matrices. Let $v \in g$. Then we define the eigenvalue sequence of $v$ to be the sequence of all its eigenvalues of the corresponding matrix, canonically ordered (notice that all the eigenvalues belong to the imaginary line), and viewed as a vector. It is easy to see that, if Condition
(R) is satisfied, then the eigenvalue sequences for each pair of vectors in \( \mathfrak{m} \) must be linearly dependent to each other. Therefore, if we can find a family of elements in \( \mathfrak{m} \) such that some of the eigenvalue sequences of them are linearly independent, then the corresponding coset space does not satisfy Condition (R). In the following, we will construct such a family \( v(t) \) of vectors in \( \mathfrak{m}, \ t \in \mathbb{R} \), presented canonically by matrices.

First we consider \( M = SU(n)/SU(n-1) \) with \( n > 2 \). With \( \mathfrak{g} \) identified with the matrix algebra \( \mathfrak{su}(n) \), and \( \mathfrak{h} \) identified with \( \mathfrak{su}(n-1) \subset \mathfrak{su}(n) \), corresponding to the left up block, we have a family of vectors
\[
v(t) = \begin{pmatrix}
-\sqrt{-1} & 0 & \cdots & 0 & 0 \\
0 & \ddots & \ddots & \vdots & \vdots \\
\vdots & \ddots & -\sqrt{-1} & 0 & 0 \\
0 & \cdots & 0 & -\sqrt{-1} & t \\
0 & \cdots & 0 & -t & (n-1)\sqrt{-1}
\end{pmatrix} \in \mathfrak{m}, \ t \in \mathbb{R}.
\]

Next consider \( M = Sp(n)/Sp(n-1) \) with \( n > 1 \). With \( \mathfrak{g} \) identified with \( \mathfrak{sp}(n) \subset \mathfrak{su}(2n) \), such that any quaternion number \( a + bi + cj + dk \) is identified with the matrix
\[
\begin{pmatrix}
a + b\sqrt{-1} & c + d\sqrt{-1} \\
-c + d\sqrt{-1} & a - b\sqrt{-1}
\end{pmatrix},
\]
and \( \mathfrak{h} \) identified with subalgebra for the left up block, we have a family of vectors
\[
v(t) = \begin{pmatrix}
0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & \cdots & 0 & 0 & 0 & t \\
\vdots & \ddots & 0 & 0 & -t & 0 & 0 \\
0 & \cdots & 0 & t & \sqrt{-1} & 0 & 0 \\
0 & \cdots & 0 & -t & 0 & 0 & -\sqrt{-1}
\end{pmatrix} \in \mathfrak{m}, \ t \in \mathbb{R}.
\]

Now we consider \( M = \text{Spin}(9)/\text{Spin}(7) \). Identifying \( \mathfrak{g} \) with the real matrix algebra \( \mathfrak{so}(9) \), and applying our discussion about the root system of \( \mathfrak{h} \) for this subcase, we have a family of vectors
\[
v(t) = \begin{pmatrix}
0 & t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-t & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0
\end{pmatrix} \in \mathfrak{m}, \ t \in \mathbb{R},
\]
where the right down eight by eight block is from \( \mathbb{R}(e_1 + e_2 + e_3 + e_4) \in \mathfrak{m} \), and the rest terms is from \( \mathfrak{g}_{\pm e_1} \in \mathfrak{m} \).
Let us consider the space $M = \text{Sp}(2)/\text{SU}(2)$. Using the matrix presentation given in [31], we have a family of vectors

$$v(t) = \begin{pmatrix} i & t j \\ t j & -3i \end{pmatrix} = \begin{pmatrix} \sqrt{-1} & 0 & 0 & t \\ 0 & -\sqrt{-1} & -t & 0 \\ 0 & t & -3\sqrt{-1} & 0 \\ -t & 0 & 0 & 3\sqrt{-1} \end{pmatrix} \in \mathfrak{m}, \quad t \in \mathbb{R}.$$

Finally, consider the space $M = \text{SU}(5)/\text{Sp}(2)S^1$. Using the matrix presentation given in [31], and identifying

$$\begin{pmatrix} a_1 + b_1i + c_1j + d_1k \\ a_2 + b_2i + c_2j + d_2k \\ a_3 + b_3i + c_3j + d_3k \\ a_4 + b_4i + c_4j + d_4k \end{pmatrix} \in \mathfrak{sp}(2),$$

with

$$\begin{pmatrix} a_1 + b_1\sqrt{-1} & c_1 + d_1\sqrt{-1} & a_2 + b_2\sqrt{-1} & c_2 + d_2\sqrt{-1} \\ -c_1 + d_1\sqrt{-1} & a_1 - b_1\sqrt{-1} & -c_2 + d_2\sqrt{-1} & a_2 - b_2\sqrt{-1} \\ a_3 + b_3\sqrt{-1} & c_3 + d_3\sqrt{-1} & a_4 + b_4\sqrt{-1} & c_4 + d_4\sqrt{-1} \\ -c_3 + d_3\sqrt{-1} & a_3 - b_3\sqrt{-1} & -c_4 + d_4\sqrt{-1} & a_4 - b_4\sqrt{-1} \end{pmatrix}$$

at the left up corner of $\mathfrak{su}(5)$, we have a family of vectors

$$v(t) = \begin{pmatrix} \sqrt{-1} & 0 & 0 & 0 \\ 0 & \sqrt{-1} & 0 & 0 \\ 0 & 0 & -\sqrt{-1} & 0 \\ 0 & 0 & 0 & -t \end{pmatrix} \in \mathfrak{m}, \quad t \in \mathbb{R}.$$

Now a direct calculation shows that, in each of the above cases, the eigenvalue sequence of the element $v(0)$ is linearly independent to the eigenvalue sequence of $v(t)$ for $t \neq 0$ and $|t|$ sufficiently small. Therefore the spaces in these cases do not satisfy Condition (R).

The proof of Theorem 1.3 is now completed.

**Remark 7.2** For those non-Riemannian normal homogeneous spaces listed in Theorem 1.1, the properties of their isotropic representations (i.e., the $\text{ad}h$ actions on $\mathfrak{m}$) can sometimes give us some more information on the types of invariant Finsler metrics on those spaces. For example, on the homogeneous spheres $\text{SU}(n)/\text{SU}(n-1)$, $\text{U}(n)/\text{U}(n-1)$, $\text{Sp}(n)/\text{Sp}(n-1)$ and $\text{Sp}(n)S^1/\text{Sp}(n-1)S^1$, any invariant Finsler metric must be an $(\alpha, \beta)$-metric. On $\text{Sp}(n)\text{Sp}(1)/\text{Sp}(n-1)\text{Sp}(1)$, any invariant Finsler metric must be an $(\alpha_1, \alpha_2)$-metric (see [32]).

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