Monte Carlo simulation of Tetraquark and Meson mixing in a dynamical model of the strong interaction

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Abstract.
We study the formation of a tetraquark state \((q - q - \bar{q} - \bar{q})\) from the presence of two initially uncorrelated \(q - \bar{q}\) states, using a Monte Carlo simulation based on a QCD-inspired model (String-Flip), incorporating an effective many-body potential between particles. In a first step we study a two body system linked by a linear potential and elaborate on how to build a variational wave function that can mimic the exact result and from general requirements we propose an ansatz for a four body system that can be bound either as two mesons or as a tetraquark. We present preliminary results of the implementation of these ideas and the procedure to evaluate the 4-body potential.

1. Introduction
The current understanding of the elementary particle physics phenomena is encoded in the Standard Model of Particle Physics (SM). Hadrons are described as composed of quarks; although the quarks carry a color charge, all known bound states are found to be colorless, they are also referred as color singlets. This phenomenological observation is one of the strongest guidelines to model hadrons.

Hadrons that cannot be matched by any quark-antiquark or three-quark configuration, like the so-called hybrid \((q)\)-antiquark\((\bar{q})\)-gluon), multiquark \((q)\)-antiquark\((q)\)-antiquark\((\bar{q})\)… or Multimeson \((M1,M2,...)\) states, may have been already observed. The multiquark \((q - q - \bar{q} - \bar{q})\) tetraquark state is the simplest of its kind and is under study theoretically and experimentally [1, 2, 4, 3]. The main theoretical difficulty faced to describe these states is that it is required to model the strong interaction in a low energy regime. Thus, an effective potential including the gross general properties of the strong interaction is usually invoked.

In the present work an effective model (String-Flip Model) is adopted to mimic the interaction of quarks as a many-body potential able to confine quarks within color-singlets, to study a system composed of 2 quarks and 2 anti-quarks. All the quarks (antiquarks) carry the same color (anticolor) charge.

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We perform a 3-D Monte Carlo simulation of the system including quantum correlations between particles to determine whether it is energetically more favorable to form a 4-body state (tetraquark or molecular) or 2 mesons, as a function of particle density.

2. Variational Wave Function

2.1. Two body system

Let us consider a system composed of quark and an antiquark of mass $m_1$ and $m_2$ respectively. The strong interaction between this pair can be represented by an effective linear potential $V[\vec{r}_i, \vec{r}_j] = k|\vec{r}_i - \vec{r}_j| = kr_{ij}$ where $k$ is an interaction constant. For a three-dimensional two-particle system (meson), one can analytically solve the corresponding Schrödinger equation for the radial part. The corresponding eigen-energies are given by

$$E_n = -|\xi_n| \left( \frac{k^2}{2\mu} \right)^{\frac{1}{3}} ; \quad n = 1, 2, \ldots$$

(1)

where $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass, and $\xi_n$ are the roots of the Airy function.

If we consider units in which $m_1 = m_2 = m = k = 1$ the corresponding energy for the ground state takes the following form: $E_0 = \xi_0 \sim 2.338$.

Previous studies have explored how to approach a variational wave function that can account for this energy [5] in the 1-Dim case. Following that result, we have explored a family of 3-D variational wave functions that can also approach this value. The general wave function takes the following

$$F_\lambda(r) = r^ae^{-\lambda r^b}$$

(2)

where $a$, $b$ and $\lambda$ are variational parameter and $r$ is the relative coordinate of the two-body system.

We have sampled $F_\lambda$ analytically, for different values of $a$ and $b$, and found the value of $\lambda$ that minimizes the energy; this particular value is labeled as $\lambda_0$. Table 1 exhibits the corresponding sets of optimal values. In Table 2 we show the corresponding minimal energy of the system. Our analysis shows that using $a = 1/2$ and $b = 1$, the wave function reproduces the exact minimum energy value within a 3% error, corresponding to an optimal variational parameter $\lambda_0 = 2^{1/3}$. Thus, these values will be taken as a reference when extending the discussion to larger systems.

2.2. Four-body system

Using the previous results we can now extend our study to a four-body system. To study such system, we need a model able to capture the properties of the strong interaction and of the
Table 2. Minimum energy $E(\lambda_0)$ for a given value of $a$ and $b$.

| $a$ | 1.2  | 1.1  | 1.0  | 0.9  | 0.8  |
|-----|------|------|------|------|------|
| 0.7 | 2.393| 2.387| 2.387| 2.395| 2.416|
| 0.6 | 2.381| 2.378| 2.382| 2.396| 2.422|
| 0.5 | 2.370| 2.372| 2.380| 2.400| 2.433|
| 0.4 | 2.362| 2.369| 2.383| 2.408| 2.450|
| 0.3 | 2.359| 2.370| 2.391| 2.423| 2.475|

constituent quarks. In particular, we are interested in capturing the expected behavior as a function of the quark density. At low particle-densities, where the distance between quarks is large, the model has to reproduce a system of hadrons, clustering the particles into groups of 2 or 4 bodies. In addition, at high densities the quarks must behave like a free Fermi gas, and the transition between these two limits must be dynamically generated. The String-Flip Model [6, 7] has these characteristics and was used to make this study.

Generalizing the previous result to include different configurations, other than the meson one, at different densities, we propose the following variational wave function, which reproduces the exact energy at low densities and becomes a Fermi Gas at high ones.

$$\Psi_\lambda = \Phi_{fg} \left( \left( \frac{r_{AB}}{\lambda_0} e^{-V} \right)^{1/2} \right)$$

(3)

where $\lambda$ is the single variational parameter, $V$ is a many-body potential, $r_{AB}$ is a product of relative distances for the interaction (see below), and $\Phi_{fg}$ is the Fermi Gas wave function given by a product of Slater determinants, one for each color-flavor combination of quarks, using the wave function of a particle in a box.

In order to recover the two-body result, we propose the following form for $r_{AB}$:

$$r_{AB} = \left( \frac{V^2 - \sum_{i=1}^{4} r_{ij}^2}{2} \right)$$

(4)

For a two meson system, the total potential $V$ is the sum of two linear potentials and the $r_{ij}$ distances are just the corresponding relative distances among the particles in a single meson. For a four-body system, the corresponding potential takes a more elaborated form and the $r_{ij}$ distances become relative distances among the links required to get the lowest potential energy. We expect this form of the variational wave function to be able to describe the departure from the two-mesons case into the four-body case.

Using such variational wave function, the expectation value of the Hamiltonian operator can be separated into three parts, a purely free kinetic part $(T_{fg})$, a kinetic part affected by the interaction $(W)$ and the potential energy $(V)$. This structure is useful to find the variational parameter that minimizes it, in the following form:

$$\langle H \rangle_\lambda = T_{fg} + \langle W \rangle_\lambda + \langle V \rangle_\lambda$$

$$\frac{\partial \langle H \rangle_\lambda}{\partial \lambda} = \frac{\partial E(\lambda)}{\partial \lambda} = 0$$

(5)

To numerically determine the expectation value we make use of Monte Carlo techniques, in particular we employ a Metropolis Algorithm. In Figure 1 we have plotted the energy of the two body system as a function of the variational parameter. The exact solution is shown as a solid
Figure 1. Energy comparisons between analytical and numerical results in the case of two isolated mesons.

line and the symbols correspond to the result of computing the energy using the Monte Carlo program. We can appreciate the accuracy that can be achieved to reproduce the exact result.

3. Many-Body Potential
The many-body potential is defined as the clustering of quarks into color-singlets objects that minimizes the potential energy. An optimal value for the potential energy of a 4-body configuration will be important to differentiate it from a meson system. In the case of four bodies, we can distinguish five possible configurations for the system: meson (two configurations), molecular meson (two configurations) and tetraquark (one configuration), as shown in Figure 3. Every line of interaction, or gluon flux tube, is simulated with an effective linear potential. In particular, a 4-body configuration requires two additional vector points \((\vec{k}, \vec{l})\) for an optimal potential energy, which need to be determined in a non trivial way, this is the case for both the tetraquark and molecular configurations.

4. Preliminary Results
The determination of the optimal potential is at the hart of the description of the system. We present here preliminary results on the extensive study made to improve the numerical estimation and the range of validity.

Let us take for definiteness the effective tetraquark potential (this can be easily generalized to the molecular case), which is given by

\[
V_{4Q-tetra} = \sum_{i=1}^{2} V[\vec{r}_{q_i}, \vec{k}] + \sum_{i=1}^{2} V[\vec{r}_{\bar{q}_i}, \vec{l}] + V[\vec{k}, \vec{l}] \tag{6}
\]
We compute these additional vector points using an additional algorithm that samples discrete values for the dimensional components of these vectors, within the confined space of the 4-body system. We chose the best possible configuration of $\vec{k}$ and $\vec{l}$ and then a second sampling can be made around these new values to optimize the potential energy.

The Nelder-Mead method were also considered to compute $\vec{k}$ and $\vec{l}$. This method uses the concept of a simplex (polytope of $N+1$ vertices in $N$ dimensions) to approximate a local optimum of a problem with $N$ variables, such as the many-body potential. This method was discarded because it was too sensitive to initial conditions and approached mostly a local minimum.

Figure 2. Possible configurations of the 4-particle system: meson, molecular meson, and tetraquark

Figure 3. General scan for vectors $\vec{k}$ and $\vec{l}$ for a 4-body potential. Starting with a scan within the system confinement, an individual scan for the vector points is made.
5. Perspectives

The following steps in our study is to compute the evolution of the variational parameter $\lambda$ as a function of the particle density $\rho$, considering several cases: i) same mass for all quarks, ii) one and two different masses. This procedure is implemented in each case by giving a value for $\rho$, which, for a fixed number of particles, means changing the size of the box where the quarks are sampled, we compute the energy of the system as a function of $\lambda$; the minimal energy determines the optimal value of $\lambda$ for the corresponding value of $\rho$. Given these optimal values we can characterize the tetraquark and molecular state and perform a simultaneous simulation of all possible configurations to determine favorable conditions for each one.

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