A SIMPLIFIED CONSTRUCTION OF THE LEBESGUE INTEGRAL

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Abstract. We present a modification of Riesz’s construction of the Lebesgue integral, leading directly to finite or infinite integrals, at the same time simplifying the proofs.

1. Introduction

Among the many approaches to the Lebesgue integral that of Riesz [8, 9, 11] is probably the shortest and most elementary. As Daniell’s abstract method [3], it is motivated by the research of weak sufficient conditions ensuring the relation \( \int f_n \, dx \to \int f \, dx \) for pointwise convergent sequences \( f_n \to f \). Based on two elementary lemmas concerning monotone sequences of step functions, the functions having a finite Lebesgue integral are constructed in two steps. It is completed by introducing measurable functions having infinite integrals.

This theory has regained popularity recently; see, e.g., Bear [1], Chae [2], Johnston [6], Roselli [12].

We modify this method by defining directly the functions having a finite or infinite integral. Technically we postpone the use of the crucial “Lemma B” of Riesz to the second stage of the construction. This leads to even shorter and more transparent proofs, and yields the Lebesgue measure at once as a byproduct. In our approach the fundamental lemmas have a symmetric use: just as Lemma A justifies the correctness of the first step of the extension of the integral, Lemma B plays the same role for the second step.

For clarity the theory is presented in Sections 2–6 for functions defined on the real line. In Sections 7–8 we extend the results to arbitrary measure spaces and we investigate product measures. In the final Section 9 we explain that various difficulties and counter-examples disappear if we return to Riesz’s natural definition of measurability and to Fréchet’s \( \sigma \)-ring framework. We omit in the main text some proofs that remain the same as in the classical setting; for the convenience of the reader they are recalled in an Appendix.

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2. INTEGRAL OF STEP FUNCTIONS

As usual, \( N \subset \mathbb{R} \) is called a null set if for each \( \varepsilon > 0 \) it has a countable cover by intervals of total length < \( \varepsilon \). We say that a property holds almost everywhere (a.e.) if it holds outside a null set.

We identify two functions if they are equal almost everywhere, i.e., we are working with equivalence classes of functions. Accordingly, we write \( f = g \), \( f \leq g \), \( f \geq g \), \( f_n \rightarrow f \), \( f_n \nearrow f \) or \( f_n \searrow f \) if these relations hold a.e. (Sometimes we keep the words “a.e.” for clarity.)

We denote by \( \chi_A \) the characteristic function of the set \( A \). By a step function we mean a function having a representation of the form

\[
\varphi = \sum_{i=1}^{m} c_i \chi_{I_i}
\]

with bounded intervals \( I_i \) and real numbers \( c_i \). We define its integral by the formula

\[
\int \varphi \, dx := \sum_{i=1}^{m} c_i |I_i|
\]

where \( |I_i| \) denotes the length of \( I_i \).

Proposition 2.1.
(i) The step functions form a vector lattice \( R_0 \).
(ii) \( \int \varphi \, dx \) does not depend on the particular representation of \( \varphi \).
(iii) The integral is a positive linear functional on \( R_0 \).

Proof. The proof is elementary and well-known if we consider individual functions and not equivalence classes. In the latter case (ii) follows from Borel’s theorem stating that the non-degenerate intervals are not null sets. \( \square \)

In the next two sections we extend the integral in two steps to larger function classes \( R_1 \) and \( R_2 \). We recall without proof two elementary results of Riesz: they will ensure the correctness of these extensions to \( R_1 \) and \( R_2 \), respectively.\footnote{For the convenience of the reader we present the missing classical proofs of Sections 2—8 in an appendix at the end of the paper.}

Lemma A. If \( (\varphi_n) \subset R_0 \) and \( \varphi_n(x) \searrow 0 \), then \( \int \varphi_n \, dx \searrow 0 \).

Lemma B. If \( (\varphi_n) \subset R_0 \), \( \varphi_n(x) \nearrow f \) and \( \sup \int \varphi_n \, dx < \infty \), then \( f \) is finite a.e.
Proposition 3.1 (i), (ii) below will show that the definition of the integral is correct.

Riesz used instead of $R_1$ the smaller class

$$C_1 := \{ f \in R_1 : \int f \, dx \text{ is finite} \}.$$  

The elements of $R_1$ may have infinite integrals, and they may even take the value $\infty$ on non-null sets. Nevertheless, we show that the usual properties of $C_1$ continue to hold in $R_1$ by essentially the same proofs.

**Proposition 3.1.**

(i) $\int f \, dx$ does not depend on the particular choice of $(\varphi_n)$.

(ii) The integral on $R_1$ is an extension of the integral on $R_0$.

(iii) If $f, g \in R_1$ and $f \leq g$, then $\int f \, dx \leq \int g \, dx$.

(iv) If $f, g \in R_1$ and $c$ is a nonnegative real number, then

\begin{equation}
(3.1) \quad cf, \ f + g, \ \min \{f, g\} \text{ and } \max \{f, g\}
\end{equation}

also belong to $R_1$, and

$$\int cf \, dx = c \int f \, dx, \quad \int f + g \, dx = \int f \, dx + \int g \, dx.$$  

**Proof.** (i) and (iii) follow from Lemma 3.2 below. (ii) follows from (i) by using constant sequences of step functions. For (iv) we observe that if $(\varphi_n), (\psi_n) \subset R_0$, $\varphi_n \nearrow f$ and $\psi_n \nearrow g$, then the sequences

$$(c\varphi_n), \ (\varphi_n + \psi_n), \ (\min \{\varphi_n, \psi_n\}) \text{ and } (\max \{\varphi_n, \psi_n\})$$

are non-decreasing, belong to $R_0$, and tend to the sequences in (3.1). The equalities follow from the linearity of the integral on $R_0$.

**Lemma 3.2.** Let $(\varphi_n), (\psi_n) \subset R_0$ and $\varphi_n \nearrow f, \psi_n \nearrow g$. If $f \leq g$, then

$$\lim \int \varphi_n \, dx \leq \lim \int \psi_n \, dx.$$  

**Proof.** It suffices to show for each fixed $m$ the inequality

$$\int \varphi_m \, dx \leq \lim_{n \to \infty} \int \psi_n \, dx.$$  

Since the left side is finite,\(^3\) this is equivalent to the inequality

$$\lim_{n \to \infty} \int \varphi_m - \psi_n \, dx \leq 0.$$  

Applying Lemma A to the sequence of functions

$$(\varphi_m - \psi_n)^+ := \max \{\varphi_m - \psi_n, 0\} \searrow 0 \text{ as } n \to \infty$$

\(^2\)We use the conventions $0 \cdot (\pm \infty) := 0$.

\(^3\)We did not need this observation in the classical framework.
we obtain the still stronger relation
\[
\lim_{n \to \infty} \int (\varphi_m - \psi_n)^+ \, dx \leq 0.
\]
\[\square\]

The class \( R_1 \) is stable for the process used in its definition:

**Proposition 3.3.** If \( (f_n) \subset R_1 \) and \( f_n \nearrow f \), then \( f \in R_1 \) and
\[
\int f_n \, dx \nearrow \int f \, dx.
\]

**Proof.** Fix for each \( n \) a sequence \((\varphi_{n,k}) \subset R_0\) satisfying \( \varphi_{n,k} \nearrow f_n \).

Then the formula
\[
\varphi_k := \sup_{n,i \leq k} \varphi_{n,i}
\]
defines a non-decreasing sequence of step functions.

For each fixed \( n \), we have \( \varphi_{n,k} \leq \varphi_k \leq f \) for all \( k \geq n \); letting \( k \to \infty \) we conclude that \( f_n \leq \lim \varphi_k \leq f \) for each \( n \). Since \( f_n \to f \) a.e., we conclude that \( \varphi_k \nearrow f \) a.e. Therefore \( f \in R_1 \) and \( \int \varphi_k \, dx \to \int f \, dx \).

Since \( \varphi_n \leq f_n \leq f \) for all \( n \), integrating and then letting \( n \to \infty \) we obtain that
\[
\lim \int f_n \, dx = \lim \int \varphi_n \, dx = \int f \, dx.
\]
\[\square\]

4. The \( R_2 \) Function Class and the Lebesgue Measure

It is natural to define \( \int -f \, dx := -\int f \, dx \) if \( f \in R_1 \). More generally, we write \( f_1 - f_2 \in R_2 \) if \( f_1, f_2 \in R_1 \) and \( \int f_1 \, dx - \int f_2 \, dx \) is well defined, and for \( f = f_1 - f_2 \in R_2 \) we set
\[
\int f \, dx := \int f_1 \, dx - \int f_2 \, dx.
\]

The function class \( R_2 \) is well defined: if \( f_1, f_2 \in R_1 \), then \( \int f_1 \, dx - \int f_2 \, dx \) is defined if and only if at least one of the two integrals is finite. Then at least one of the functions \( f_1, f_2 \) is finite a.e. by Lemma B, so that the function \( f_1 - f_2 \) is defined a.e.

Proposition 4.1 (i), (ii) below will show that the definition of the integral is also correct.

**Remark.** We may assume in the definition of \( R_2 \) that \( f_1, f_2 \geq 0 \): choose for \( i = 1, 2 \) a step function \( \varphi_i \) satisfying \( \varphi_i \leq f_i \), and change \( f_i \) to \( f_i - \max \{ \varphi_1, \varphi_2 \} \).

Riesz used instead of \( R_2 \) the smaller class
\[
C_2 := \left\{ f \in R_2 : \int f \, dx \text{ is finite} \right\}.
\]

The elements of \( R_2 \) may have infinite integrals, and they are not necessarily finite a.e. Nevertheless, most properties of the integral on \( C_2 \) remain valid on \( R_2 \).
Our first result readily follows from Proposition 3.1 for the original class $C_2$; for the extended class $R_2$ some new arguments are needed:

**Proposition 4.1.**

(i) $\int f \, dx$ does not depend on the particular choice of $f_1$ and $f_2$.

(ii) The integral on $R_2$ is an extension of the integral on $R_1$.

(iii) If $f, g \in R_2$ and $f \leq g$, then $\int f \, dx \leq \int g \, dx$.

(iv) If $f \in R_2$ and $c \in \mathbb{R}$, then $\int cf \, dx = c \int f \, dx$.

(v) If $f, g \in R_2$ and $\int f \, dx + \int g \, dx$ is well defined, then $f + g \in R_2$ and

$$\int f + g \, dx = \int f \, dx + \int g \, dx.$$ 

(vi) If $f, g \in R_2$, then $\max\{f, g\}$ and $\min\{f, g\}$ also belong to $R_2$.

**Proof.** For (i) and (iii) we have to show that if $f = f_1 - f_2$ and $g = g_1 - g_2$ as in the definition of $f, g \in R_2$, and $f \leq g$, then

$$\int f_1 \, dx - \int f_2 \, dx \leq \int g_1 \, dx - \int g_2 \, dx.$$ 

The inequality is obvious if $\int f_2 \, dx = \infty$. Henceforth we assume that $\int f_2 \, dx$ is finite.

If $\int g_2 \, dx$ is also finite, then our assumption $f_1 - f_2 \leq g_1 - g_2$ implies that $f_1 + g_2 \leq g_1 + f_2$ because $f_2$ and $g_2$ are finite a.e. by Lemma B, and then $\int f_1 \, dx + \int g_2 \, dx \leq \int g_1 \, dx + \int f_2 \, dx$ by Proposition 3.1 (iv). Since $\int g_1 \, dx$ and $\int g_2 \, dx$ are finite, hence (4.1) follows.

If $\int g_2 \, dx = \infty$, then choose a sequence $(\varphi_n) \subset R_0$ satisfying $\varphi_n \nearrow g_2$. We have $f_1 - f_2 \leq g_1 - \varphi_n$ for each $n$. Applying the preceding arguments with $\varphi_n$ in place of $g_2$ we get

$$\int f_1 \, dx - \int f_2 \, dx \leq \int g_1 \, dx - \int \varphi_n \, dx,$$

and (4.1) follows by letting $n \to \infty$.

(ii) follows from (i) by choosing $f_1 := f$ and $f_2 := 0$ if $f \in R_1$.

(iv) is obvious.

(v) Write $f = f_1 - f_2$ and $g = g_1 - g_2$ with $f_1, f_2, g_1, g_2 \in R_1$ as in the definition of $f, g \in R_2$. If both integrals $\int f_2 \, dx$ and $\int g_2 \, dx$ are finite, then Proposition 3.1 (iv) shows that $f_1 + g_1, f_2 + g_2 \in R_1$, and $\int f_2 + g_2 \, dx = \int f_2 \, dx + \int g_2 \, dx$ is finite. Therefore

$$f + g = (f_1 + g_1) - (f_2 + g_2)$$
belongs to $R_2$, and using Proposition 3.1 (iv) again we obtain that
\[
\int (f + g) \, dx = \int (f_1 + g_1) \, dx - \int (f_2 + g_2) \, dx
\]
\[
= \left( \int f_1 \, dx + \int g_1 \, dx \right) - \left( \int f_2 \, dx + \int g_2 \, dx \right)
\]
\[
= \left( \int f_1 \, dx - \int f_2 \, dx \right) + \left( \int g_1 \, dx - \int g_2 \, dx \right)
\]
\[
= \int f \, dx + \int g \, dx.
\]

If one of the integrals $\int f_2 \, dx$ and $\int g_2 \, dx$ is infinite, then both
integrals $\int f_1 \, dx$ and $\int g_1 \, dx$ are finite by the definition of $f \in R_2$ and
by our assumption that $\int f \, dx + \int g \, dx$ is well defined, and we may
repeat the above proof.

(vi) By symmetry we consider only the case of maximum. We have
to show that if $f = f_1 - f_2$ and $g = g_1 - g_2$ as in the definition of
$f, g \in R_2$, then $\max \{f_1 - f_2, g_1 - g_2\} \in R_2$. If $f_1, g_1 \in R_0$, then
\[
\max \{f_1 - f_2, g_1 - g_2\} = (f_1 + g_1) + \max \{-g_1 - f_2, -f_1 - g_2\}
\]
\[
= (f_1 + g_1) - \min \{g_1 + f_2, f_1 + g_2\}
\]
is the difference of two elements of $R_1$. Furthermore, $\int f_1 + g_1 \, dx$
is finite because $f_1 + g_1 \in R_0$, so that the difference belongs to $R_2$.

In the general case we choose two sequences $(\varphi_n), (\psi_n) \subset R_0$
satisfying $\varphi_n \nearrow f_1$ and $\psi_n \nearrow g_1$. We have $\max \{\varphi_n - f_2, \psi_n - g_2\} \in R_2$ by
the preceding arguments, and
\[
\max \{\varphi_n - f_2, \psi_n - g_2\} \nearrow \max \{f_1 - f_2, g_1 - g_2\}.
\]
Since
\[
\int \max \{\varphi_n - f_2, \psi_n - g_2\} \, dx > -\infty
\]
for all $n$, because at least one of the integrals $\int f_2 \, dx$ and $\int g_2 \, dx$ are
finite, we may conclude by applying Theorem 4.2 below.

The class $R_2$ has a similar invariance property as $R_1$:

**Theorem 4.2** (Generalized Beppo Levi theorem). Let $(f_n) \subset R_2$ and
$f_n \nearrow f$. If $\int f_n \, dx > -\infty$ for at least one $n$, then $f \in R_2$ and
\[
(4.2) \quad \int f_n \, dx \nearrow \int f \, dx.
\]

**Examples.** The functions
\[
f_n := -\chi_{(-\infty,0)} + \chi_{(0,n]} \quad \text{and} \quad f_n := -\chi_{(n,\infty)}
\]
show that the assumption $\lim \int f_n \, dx > -\infty$ cannot be omitted. In-
deed, in the first case $f = \text{sign} \notin R_2$; in the second case $f = 0 \in R_2$, but (4.2) fails.
The proof below is a simple adaptation (even a simplification) of the usual one for the smaller class $C_2$. Since this theorem is fundamental in the present construction, we give the proof here for the convenience of the reader.

Proof of Theorem 4.2. By omitting a finite number of initial terms we may assume that $\int f_n \, dx > -\infty$ for all $n$. Write $f_n = g_n - h_n$ with $g_n, h_n \in R_1$. Since $\int h_n \, dx$ is finite, there exists $\varphi_n \in R_0$ satisfying $\varphi_n \leq h_n$ and $\int h_n - \varphi_n \, dx < 2^{-n}$. Changing $g_n$ and $h_n$ to $g_n - \varphi_n$ and $h_n - \varphi_n$ we may assume that

$$f_n = g_n - h_n, \quad g_n, h_n \in R_1, \quad h_n \geq 0 \quad \text{and} \quad \int h_n \, dx \leq 2^{-n}$$

for $n = 1, 2, \ldots$. Finally, changing $g_n$ and $h_n$ by induction on $n = 2, 3, \ldots$ to

$$h_1 + \cdots + h_{n-1} + g_n \quad \text{and} \quad h_1 + \cdots + h_{n-1} + h_n$$

we may assume that

$$f_n = g_n - h_n, \quad g_n, h_n \in R_1, \quad h_n \geq 0 \quad \text{and} \quad \int h_n \, dx \leq 1$$

for $n = 1, 2, \ldots$.

Applying Proposition 3.3 to the non-decreasing sequences $(h_n)$ and $(g_n) = (f_n + h_n)$ we obtain that $h_n \nearrow h$ and $g_n \nearrow g$ with suitable functions $h, g \in R_1$, and $\int h \, dx \leq 1 < \infty$. Hence $f = g - h \in R_2$ and

$$\int f_n \, dx = \int g_n \, dx - \int h_n \, dx \rightarrow \int g \, dx - \int h \, dx = \int f \, dx. \quad \square$$

Now we may greatly generalize the length of intervals. We write $A \in \mathcal{M}$ if $\chi_A \in R_2$, and we set $\mu(A) := \int \chi_A \, dx$ in this case. Theorem 4.2 yields at once the

**Theorem 4.3.** $\mu$ is a $\sigma$-finite, complete measure on $\mathcal{M}$.

The following notions are used here. Given a family $\mathcal{A}$ of subsets of a set $X$ with $\emptyset \in \mathcal{A}$, by a measure on $\mathcal{A}$ we mean a nonnegative function $\mu : \mathcal{A} \to [0, \infty]$ such that $\mu(\emptyset) = 0$, and satisfying the $\sigma$-additivity relations

$$\mu(A) = \sum_{n=1}^{\infty} \mu(A_n)$$

whenever $A \in \mathcal{A}$ is the disjoint union of a sequence $(A_n) \subset \mathcal{A}$.

This measure is called $\sigma$-finite if each $A \in \mathcal{A}$ has a countable cover by sets of finite measure, and complete if all subsets of a set of measure zero also belong to $\mathcal{A}$.

We call the elements of $\mathcal{M}$ measurable sets.

**Remark.** We will describe the structure of $\mathcal{M}$ in Proposition 6.2 (iv) below.
5. The space $L^1$

We write $f \in L^1$ if $f \in R_2$ and $\int f \, dx$ is finite, i.e.,

$$L^1 := \left\{ f \in R_2 : \int f \, dx \text{ is finite} \right\}.$$

Each $f \in L^1$ is finite a.e. by Lemma B.

Observe that $L^1$ coincides with the class $C_2$ of Riesz. We are going to show that $L^1$ has a simple structure, and the integral on $L^1$ has some remarkable properties.

**Proposition 5.1.**

(i) $L^1$ is a vector lattice.

(ii) The integral is a positive linear functional on $L^1$.

**Proof.** (i) We show that if $f, g \in L^1$ and $c \in \mathbb{R}$ then $cf, f + g, \min \{f, g\}$ and $\max \{f, g\}$ also belong to $L^1$. Write $f = f_1 - f_2$ and $g = g_1 - g_2$ with $f_1, f_2, g_1, g_2 \in R_1$ having finite integrals. Then the claim follows from the representations

$$cf = cf_1 - cf_2 \quad \text{if} \quad c \geq 0,$$

$$cf = (-c)f_2 - (-c)f_1 \quad \text{if} \quad c < 0,$$

$$f + g = (f_1 + g_1) - (f_2 + g_2),$$

$$\min \{f, g\} = \min \{f_1 + g_2, g_1 + f_2\} - (f_2 + g_2),$$

$$\max \{f, g\} = \max \{f_1 + g_2, g_1 + f_2\} - (f_2 + g_2),$$

because by Proposition 3.1 (iv) the right side of each equality is the difference of two functions from $R_1$ having finite integrals.

(ii) The integral is linear by Proposition 5.1 (iv) and by the definition of the integral on $R_2$, and it is monotone by Proposition 4.1 (iii). $\square$

Theorem 4.2 yields at once the fundamental

**Theorem 5.2** (Beppo Levi). If $(f_n) \subset L^1$, $f_n \rightarrow f$ and $\sup \int f_n \, dx < \infty$, then $f \in L^1$ and

$$\int f_n \, dx \nearrow \int f \, dx.$$

**Corollary 5.3.**

(i) The formula $\|f\|_1 := \int |f| \, dx$ defines a norm on $L^1$.

(ii) A set $A$ is a null set $\iff A \in \mathcal{M}$ and $\mu(A) = 0$.

**Proof.** (i) The only non-trivial property is that if $\int |f| \, dx = 0$, then $f = 0$. This follows by applying Theorem 5.2 with $f_n := n |f|$. 

(ii) We have to show that $\chi_A = 0 \iff \int \chi_A \, dx = 0$. The implication $\implies$ follows from the definition of the integral. The converse implication follows from (i). $\square$
We recall that Theorem 5.2 also implies the following two important theorems:

**Theorem 5.4 (Fatou).** Let \((f_n) \subset L^1\) and \(f_n \to f\). If \(f_n \geq 0\) for all \(n\), and \(\lim \inf \int f_n \, dx < \infty\), then \(f \in L^1\) and 
\[
\int f \, dx \leq \lim \inf \int f_n \, dx.
\]

**Theorem 5.5 (Lebesgue).** Let \((f_n) \subset L^1\) and \(f_n \to f\). If there exists \(g \in L^1\) such that \(|f_n| \leq g\) for all \(n\), then \(f \in L^1\), and 
\[
\int f_n \, dx \to \int f \, dx.
\]

6. MEASURABLE FUNCTIONS

We may simplify the manipulation of integrable functions by introducing the notion of measurability. This will also allow us to precise the structure of the family \(M\) of measurable sets, and to generalize Fatou’s theorem for functions having infinite integrals.

Following Riesz we call a function \(f : \mathbb{R} \to \mathbb{R}\) measurable if there exists a sequence \((\varphi_n) \subset \mathbb{R}_0\) such that \(\varphi_n \to f\) a.e.

**Proposition 6.1.**

(i) \(f \in R_2\), then \(f\) is measurable.

(ii) If \(f\) and \(g\) are measurable, then \(|f|, fg, \max\{f, g\}\) and \(\min\{f, g\}\) are also measurable. Furthermore, \(f/g\) and \(f \pm g\) are also measurable whenever they are defined a.e.

**Proof.** (i) Write \(f = f_1 - f_2\) with \(f_1, f_2 \in R_1\). If \((\varphi_n), (\psi_n) \subset R_0\) and \(\varphi_n \nearrow f_1, \psi_n \nearrow f_2\), then \((\varphi_n - \psi_n) \subset R_0\), and \(\varphi_n - \psi_n \to f\).

(ii) If \((\varphi_n) \subset R_0\) and \(\varphi_n \to f\), then \(|\varphi_n| \subset R_0\) and \(|\varphi_n| \to |f|\). The proof of the other statements is analogous. \(\square\)

The sign function shows that the not every measurable function belongs to \(R_2\). In our next result we collect several useful statements, including a partial converse of Proposition 6.1 (i), the description of the family \(M\) of measurable sets, and a generalization of Fatou’s theorem for functions having infinite integrals. The proofs will rely on Lebesgue’s theorem.

We recall that by a \(\sigma\)-ring in \(X\) we mean a family \(M\) of subsets of \(X\) containing \(\emptyset\), the difference \(A \setminus B\) of any two sets \(A, B \in M\), and the union \(\bigcup A_n\) of any disjoint sequence \((A_n) \subset M\). Then we have \(\bigcup A_n \in M\) and \(\bigcap A_n \in M\) for any countable sequence \((A_n) \subset M\), even if it is not disjoint.

**Proposition 6.2.**

(i) If \(f\) is measurable, \(g \in L^1\) and \(|f| \leq g\), then \(f \in L^1\).

(ii) If \(f\) is measurable and nonnegative, then \(f \in R_2\).

(iii) The measurable sets form a \(\sigma\)-ring.
If \((f_n)\) is a sequence of measurable functions and \(f_n \to f\), then \(f\) is measurable.

Let \((f_n)\) be a sequence of nonnegative measurable functions. If \(f_n \to f\), then \(f\) is also a nonnegative measurable function, and

\[
\int f \, dx \leq \liminf \int f_n \, dx.
\]

Part (ii) and Proposition 6.1 (i) justify the terminology of Section 4: a set is measurable if and only if its characteristic function is measurable.

Proof. (i) If \((\varphi_n) \subset \mathbb{R}_0^+\) and \(\varphi_n \to f\), then the functions

\[ f_n := \text{med}\{ -g, \varphi_n, g \} \]

belong to the lattice \(L^1\), and \(f_n \to f\). Since \(|f_n| \leq g\) for all \(n\), we may conclude by applying Lebesgue’s theorem.

(ii) Choose a non-decreasing sequence \(A_1 \subset A_2 \subset \cdots\) of sets of finite measure such that \(f = 0\) outside their union. The functions

\[ f_n(x) := \min\{f(x), n\chi_{A_n}\} \]

belong to \(L^1\) by (i), and \(f_n \not\to f\). We conclude by applying Theorem 4.2.

(iii) We have \(\emptyset \in \mathcal{M}\) because \(\chi_{\emptyset} = 0 \in \mathbb{R}_0 \subset \mathbb{R}_2\). Theorem 4.2 yields that \(\bigcup A_n \in \mathcal{M}\) for any disjoint sequence \((A_n) \subset \mathcal{M}\).

It remains to show that if \(A, B \in \mathcal{M}\), then \(A \setminus B \in \mathcal{M}\). By (ii) it suffices to observe that if \(\chi_A, \chi_B\) are measurable, then \(\chi_{A \setminus B} = \chi_A - \chi_A\chi_B\) is also measurable.

(iv) Fix a positive function \(g \in L^1\), and set

\[ h_n := \frac{gf_n}{g + |f_n|} \quad \text{and} \quad h := \frac{gf}{g + |f|}. \]

Then \(h_n\) is measurable and \(|h_n| \leq g\), so that \(h_n \in L^1\) by (i). Since \(h_n \to h\), by Lebesgue’s theorem \(h \in L^1\), and hence \(h\) is measurable by (i). Since \(f\) and \(h\) have the same sign, \(|f|h = f|h|\), and hence

\[ f = \frac{gh}{g - |h|} \]

is also measurable by Proposition 6.1 (ii).

(v) \(f\) is a nonnegative measurable function by (iv), hence all integrals in (6.1) are defined by (ii). The relation (6.1) is obvious if the right side of (6.1) is finite; otherwise it follows from Fatou’s theorem. \(\square\)

\[\text{Here } \text{med}\{x, y, z\} \text{ denotes the middle number among } x, y \text{ and } z. \text{ For } x \leq z \text{ it is equal to } \max\{x, \min\{y, z\}\}.\]
7. Generalization to arbitrary measure spaces

The above construction of the Lebesgue integral may be greatly generalized as follows. Let \( \mu : \mathcal{P} \to \mathbb{R} \) be a finite measure on a semiring \( \mathcal{P} \) in a set \( X \), i.e., on a family of sets \( \mathcal{P} \subset 2^X \) having the following properties:

- \( \emptyset \in \mathcal{P} \);
- if \( P, Q \in \mathcal{P} \), then \( P \cap Q \in \mathcal{P} \);
- if \( P, Q \in \mathcal{P} \), then there is a finite disjoint sequence \( P_1, \ldots, P_n \) in \( \mathcal{P} \) such that \( P \setminus Q = P_1 \cup \cdots \cup P_n \).

Two simple examples are the length of bounded intervals in \( \mathbb{R} \) and the counting measure on the finite subsets of any given set \( X \): \( \mu(P) \) is the number of elements of \( P \).

We say that \( N \subset X \) is a null set if for each \( \varepsilon > 0 \) there exists a sequence \( (P_n) \subset \mathcal{P} \) satisfying \( N \subset \bigcup P_n \) and \( \sum \mu(P_n) < \varepsilon \). We identify two functions if they are equal a.e., i.e., outside a null set.

We denote by \( R_0 \) the vector space of step functions spanned by the characteristic functions of the sets \( P \in \mathcal{P} \), and we define the integral of step functions by the formula

\[
\int \left( \sum_{i=1}^n c_i \chi_{P_i} \right) d\mu := \sum_{i=1}^n c_i \mu(P_i).
\]

Now we may repeat the above construction of the integral and measure, and all theorems of Sections 2–6 remain valid. Moreover, only the proof of Proposition 6.2 (iv) has to be modified because there are measure spaces containing no positive measurable functions. It suffices to use a nonnegative function \( g \in L^1 \) satisfying

\[
g(x) = 0 \implies f_n(x) = 0 \quad \text{for all} \quad n,
\]

and setting \( h_n = h := 0 \) when \( g = 0 \). See, e.g., [11] or [7] for details.

8. The Fubini–Tonelli theorem

Given two finite measures \( \mu : \mathcal{P} \to \mathbb{R} \) and \( \nu : \mathcal{Q} \to \mathbb{R} \) where \( \mathcal{P} \) is a semiring in \( X \) and \( \mathcal{Q} \) is a semiring in \( Y \),

\[
\mathcal{P} \times \mathcal{Q} := \{ P \times Q : P \in \mathcal{P} \quad \text{and} \quad Q \in \mathcal{Q} \}
\]

is a semiring in the product space \( \times Y \), and the formula

\[
(\mu \times \nu)(P \times Q) := \mu(P) \nu(Q)
\]

defines a finite measure on \( \mathcal{P} \times \mathcal{Q} \).

We are going to describe the relationship between the three corresponding integrals. The notation \( R_i(X), R_i(X), R_i(X \times Y) \) will refer to the spaces \( R_i \) for the measures \( \mu, \nu \) and \( \mu \times \nu \), respectively, and we will write \( dx \), \( dy \) and \( dx \, dy \) instead of \( d\mu \), \( d\nu \) and \( d(\mu \times \nu) \).
Theorem 8.1 (Fubini–Tonelli). We have

\[ \int_{X \times Y} f(x, y) \, dx \, dy = \int_X \left( \int_Y f(x, y) \, dy \right) \, dx \]

whenever the left side is defined, i.e., \( f \in R_2(X \times Y) \).

We emphasize that the integrals in (8.1) may be infinite.

Remark. Under the same assumption we also have

\[ \int_{X \times Y} f(x, y) \, dx \, dy = \int_Y \left( \int_X f(x, y) \, dx \right) \, dy \]

by symmetry.

Proof. We may repeat the proof given in [11] for the special case \( f \in C_2(X \times Y) = L^1(X \times Y) \); it becomes even simpler because we do not have to check the boundedness of the sequences of integrals.

Example. We recall that the theorem does not hold under the weaker assumption that both successive integrals exist and are equal. For example, let \( \mu = \nu \) be the counting measure on \( X = Y := \mathbb{Z} \), and

\[ f(x, y) := \begin{cases} 
1 & \text{if } x = y + 1, \\
-1 & \text{if } x = y - 1, \\
0 & \text{otherwise}.
\end{cases} \]

Then

\[ \int_X \left( \int_Y f(x, y) \, dy \right) \, dx = \int_Y \left( \int_X f(x, y) \, dx \right) \, dy = 0, \]

but the integral \( \int_{X \times Y} f(x, y) \, dx \, dy \) is undefined.

9. \( \sigma \)-RING OR \( \sigma \)-ALGEBRA?

In this section we argue in favor of the measurability à la Riesz, and the \( \sigma \)-rings à la Fréchet [4], instead of \( \sigma \)-algebras, i.e., \( \sigma \)-rings containing the fundamental set \( X \).

The measurability notion adopted here differs from the one used in most modern texts. They coincide if \( X \) has a countable cover by sets of finite measure (or equivalently by sets belonging to \( \mathcal{P} \)), like the Lebesgue measure in \( \mathbb{R}^n \) and the probability measures.

Otherwise, like for the counting measure on an uncountable set \( X \), the present definition is more restrictive: for example the constant functions are not measurable.

In the latter case it is tempting to adopt a weaker definition, by calling a function \( f \) locally measurable if \( f \chi_P \) is measurable for all \( P \in \mathcal{P} \). Indeed, we may extend the integral to locally measurable functions \( f \) by setting \( \int f \, dx := \infty \) if \( f \) is nonnegative and non-measurable, and then setting \( \int f \, dx := \int f_+ \, dx - \int f_- \, dx \) whenever the right side is well defined. This extended integral is still monotone.
However, the Fubini–Tonelli theorem may fail for locally measurable functions:

**Example.** Let \( \mu \) be the zero measure and \( \nu \) the counting measure on the finite subsets of an uncountable set \( X \). Then the characteristic function \( f \) of the set

\[
D := \{(x, x) : x \in X\}
\]

is locally measurable for the product measure,

\[
\int_{X \times X} f(x, y) \, dx \, dy = \infty \quad \text{and} \quad \int_X \left( \int_X f(x, y) \, dx \right) \, dy = 0.
\]

Next we take a closer look of Theorem 4.3. It may be shown (see, e.g., [5], Chapter 13) that the measure \( \mu : M \to \mathbb{R} \) is the only possible extension of its restriction to the initial semiring \( P \).

If \( X \) is measurable, then \( M \) is not only a \( \sigma \)-ring, but also a \( \sigma \)-algebra. Otherwise we may extend \( \mu \) further to a \( \sigma \)-algebra \( \overline{M} \) by setting \( \overline{\mu}(A) := \int \chi(A) \, dx \) whenever the characteristic function of \( A \) is locally measurable. However, this extension is not unique in general:

**Example.** Consider the zero measure \( \mu \) on the semiring \( P \) of finite subsets of an uncountable set \( X \). Then \( \overline{M} = 2^X \), and

\[
\overline{\mu}(A) = \begin{cases} 
0 & \text{if } A \text{ is countable,} \\
\infty & \text{if } A \text{ is uncountable.}
\end{cases}
\]

But the zero measure on \( 2^X \) is also an extension of \( \mu \).

Moreover, the two measures already differ on the smallest \( \sigma \)-algebra \( \mathcal{A} \) containing \( P \), i.e., on the family of countable subsets and their complements. In fact, there are infinitely many other extensions of \( \mu \) to \( \mathcal{A} \): the formula

\[
\mu_\alpha(A) = \begin{cases} 
0 & \text{if } A \text{ is countable,} \\
\alpha & \text{if } X \setminus A \text{ is countable}
\end{cases}
\]

defines a different extension of \( \mu \) for each \( 0 \leq \alpha \leq \infty \).

## 10. Appendix

For the convenience of the reader we reproduce here some known proofs that were admitted in the text.

**Proof of Lemma A in \( \mathbb{R} \).** We may fix a compact interval \([a, b]\) and a number \( M > 0 \) such that \( \varphi_1 \), and hence all \( \varphi_n \) are bounded by \( M \) and vanish outside \([a, b]\).

For any fixed \( \varepsilon > 0 \) there exists a countable sequence of open intervals of total length \( < \varepsilon \) such that outside their union \( U \) all \( \varphi_n \) are continuous.
Then \( \varphi_n \to 0 \) uniformly on the compact set \([a,b] \setminus U\) by Dini’s theorem, so that \( 0 \leq \varphi_n < \varepsilon \) on this set for all sufficiently large \( n \). Then we also have

\[
0 \leq \int \varphi_n \, dx \leq M \varepsilon + \varepsilon(b-a),
\]

and the lemma follows by the arbitrariness of \( \varepsilon \).

In the proof of Lemma A for a general measure we may use the \( \sigma \)-additivity of the measure instead of the topological (compactness) arguments: see [11] or [7].

**Proof of Lemma B.** Changing \( \varphi_n \) to \( \varphi_n - \varphi_1 \) we may assume that the functions \( \varphi_n \) are nonnegative. Fix an upper bound \( A > 0 \) of the integrals \( \int \varphi_n \, dx \), and set

\[
E_\varepsilon := \left\{ x \in \mathbb{R} : f(x) > \frac{A}{\varepsilon} \right\}, \quad \varepsilon > 0.
\]

Since \( \{ x \in \mathbb{R} : f(x) = \infty \} \subset E_\varepsilon \), it suffices to show that each \( E_\varepsilon \) has a countable cover by intervals of total length \( \leq \varepsilon \).

Setting \( f_0 := 0 \) and

\[
E_{\varepsilon,n} := \left\{ x \in \mathbb{R} : f_n(x) > \frac{A}{\varepsilon} \geq f_{n-1}(x) \right\}, \quad n = 1, 2, \ldots,
\]

\( E_\varepsilon \) is the union of the disjoint sets \( E_{\varepsilon,n} \) by the monotonicity of \( (f_n) \). Since each \( E_{\varepsilon,n} \) is a finite union of disjoint intervals \( I_{n,1}, \ldots, I_{n,k_n} \), it remains to show that

\[
\sum_{n=1}^{\infty} \sum_{k=1}^{k_n} |I_{n,k}| \leq \varepsilon.
\]

By the definition of \( E_{\varepsilon,n} \) we have for each \( m = 1, 2, \ldots \) the inequality

\[
\frac{A}{\varepsilon} \sum_{n=1}^{m} \sum_{k=1}^{k_n} |I_{n,k}| \leq \sum_{n=1}^{m} \int_{E_{n}} \varphi_n \, dx \leq \int \varphi_m \, dx \leq A,
\]

and we conclude by letting \( m \to \infty \).

**Proof of Fatou’s theorem**[5.4]. Setting

\[
h_n := \inf \{ f_k : k \geq n \}
\]

we have \( h_n \not\supset f \). Since \( h_n \leq f_n \) and hence \( \sup \int h_n \, dx < \infty \) by our assumption \( \liminf \int f_n \, dx < \infty \), the relation \( f \in L^1 \) will follow by applying Theorem 5.2 if we show that \( (h_n) \subset L^1 \).

For each fixed \( n \) we set

\[
h_{n,m} := \inf \{ f_k : n \leq k \leq m \}, \quad m = n, n+1, \ldots,
\]

Then \( (h_{n,m})_{m=n}^{\infty} \subset L^1 \) by Proposition 5.1 (i). Since \( h_{n,m} \not\supset h_n \), and \( \int h_{n,m} \, dx \geq 0 \) for all \( m \), we may apply Theorem 5.2 to get \( h_n \in L^1 \), and hence \( h_n \in L^1 \).
Finally, applying Theorem 4.2 and using the relations $h_n \leq f_n$ we get the required estimate:

\[
\int f \, dx = \lim \int h_n \, dx = \lim \inf \int h_n \, dx \leq \lim \inf \int f_n \, dx. \quad \square
\]

**Proof of Lebesgue’s theorem 5.5.** Applying Fatou’s Theorem to the sequences $(g - f_n)$ and $(g + f_n)$ we obtain that $g - f, g + f \in L^1$ and

\[
\int g - f \, dx \leq \lim \inf \int g - f_n \, dx = \int g \, dx - \lim \sup \int f_n \, dx,
\]

\[
\int g + f \, dx \leq \lim \inf \int g + f_n \, dx = \int g \, dx + \lim \inf \int f_n \, dx.
\]

Since $L^1$ is a vector space, hence $f \in L^1$, and

\[
\lim \sup \int f_n \, dx \leq \int f \, dx \leq \lim \inf \int f_n \, dx. \quad \square
\]

Let us give finally the complete proof of the Fubini–Tonelli theorem. First we recall from [11] an elementary lemma clarifying the relationship between one- and two-dimensional null sets:

**Lemma C.** If $E$ is a null set in $X \times Y$, then the “sections”

\[
\{ y \in Y : (x, y) \in E \}
\]

of $E$ are null sets in $Y$ for almost every $x \in X$.

**Proof.** Fix a sequence of “rectangles” $R_n = P_n \times Q_n$ in $P \times Q$, covering each point of $E$ infinitely many times, and satisfying

\[
\sum_{n=1}^{\infty} (\mu \times \nu)(R_n) < \infty.
\]

We may get such a sequence by applying the definition of null sets with $\varepsilon = 2^{-n}$ for $n = 1, 2, \ldots$, and then combining the resulting covers into one sequence.

By the definition of the integral of step functions we have

\[
(\mu \times \nu)(R_n) = \int_{X \times Y} \chi_{R_n}(x, y) \, dx \, dy = \int_X \left( \int_Y \chi_{R_n}(x, y) \, dy \right) \, dx
\]

(there their common value is $\mu(P_n)\nu(Q_n)$), so that the series

\[
\sum_{n=1}^{\infty} \int_X \left( \int_Y \chi_{R_n}(x, y) \, dy \right) \, dx
\]

is convergent. Applying the Beppo Levi theorem we obtain that the series

\[
\sum_{n=1}^{\infty} \int_Y \chi_{R_n}(x, y) \, dy
\]
is convergent for a.e. \( x \in X \). If \( x_0 \) is such a point, then another application of the Beppo Levi theorem implies that the series 
\[
\sum_{n=1}^{\infty} \chi_{R_n}(x_0, y)
\]
is convergent for a.e. \( y \in Y \). If \( y_0 \) is such a point, then because at the points of \( E \) we have \( \sum \chi_{R_n} = \infty \).

\[\square\]

**Proof of Theorem 8.1** We have to show the following:

(10.1) \( f(x, \cdot) \in R_2(Y) \) for almost each \( x \in X \);

(10.2) \( \int_Y f(\cdot, y) \, dy \in R_2(X) \);

(10.3) the two sides of (10.1) are equal.

These properties obviously hold if \( f \) is the characteristic function of a “rectangle” \( P \times Q \). Taking linear combinations we see that they hold for all step functions \( f \in R_0(X \times Y) \) as well.

Next let \( f \in R_1(X \times Y) \). Choose a sequence \( (\varphi_n) \subset R_0(X \times Y) \) and a null set \( E \subset X \times Y \) such that 
\[
\varphi_n(x, y) \nearrow f(x, y) \quad \text{for each} \quad (x, y) \in (X \times Y) \setminus E;
\]
then

(10.4) \( \int_{X \times Y} \varphi_n(x, y) \, dx \, dy \nearrow \int_{X \times Y} f(x, y) \, dx \, dy \)

by the definition of the integral.

By Lemma C we may fix a set \( X_1 \subset X \) such that \( X \setminus X_1 \) is a null set in \( X \), and for each fixed \( x \in X_1 \), \( (\varphi_n(x, \cdot)) \subset R_0(Y) \) and 
\[
\varphi_n(x, \cdot) \nearrow f(x, \cdot) \quad \text{in} \quad Y,
\]
implying \( f(x, \cdot) \in R_1(Y) \) and the relation 
\[
\int_Y \varphi_n(x, y) \, dy \nearrow \int_Y f(x, y) \, dy \quad \text{for a.e.} \quad x \in X.
\]
Since \( (\int_Y \varphi_n(\cdot, y) \, dy) \subset R_0(X) \), this implies \( \int_Y f(\cdot, y) \, dy \in R_1(X) \) and the relation

(10.5) \( \int_X \left( \int_Y \varphi_n(x, y) \, dy \right) \, dx \nearrow \int_X \left( \int_Y f(x, y) \, dy \right) \, dx. \)

In particular, we have established (10.1) and (10.2). The equality (10.3) follows by observing that the left sides of (10.4) and (10.5) are equal, and that (10.3) is already known for step functions.

If \( f \in R_1(X \times Y) \) and \( \int_{X \times Y} f(x, y) \, dx \, dy < \infty \), then applying Lemma B we also see that \( f \) is finite a.e. in \( X \times Y \), and \( \int_Y f(x, y) \, dy \) is finite for every \( x \in X_1 \).

Finally, if \( f \in R_2(X \times Y) \), then writing \( f = f_1 - f_2 \) with \( f_1, f_2 \in R_1(X \times Y) \) we have the required equality for \( f_1 \) and \( f_2 \) in place of \( f \).
Since one of the integrals \( \int_{X \times Y} f_1(x, y) \, dx \, dy \) and \( \int_{X \times Y} f_2(x, y) \, dx \, dy \) is finite, by the preceding paragraph we may take the difference of these equalities to obtain the required identity for \( f \). \( \square \)

We end the appendix by recalling a not too well-known example of Weir [13, p. 43] (see also Johnston [6, pp. 54–55]). The strict inclusion \( R_1 \subsetneq R_2 \) follows from the lower boundedness of the elements of \( R_1 \). The converse does not hold: a nonnegative element of \( R_2 \) does not necessarily belong to \( R_1 \). To see this we enumerate the rational numbers in \((0, 1)\) into a sequence \((r_n)\) and we set

\[
S := (0, 1) \cap \left( \bigcup_{n=1}^{\infty} (r_n - 2^{-n-3}, r_n + 2^{-n-3}) \right).
\]

Then \( S \subset (0, 1) \) and \( 0 < \mu(S) < 1 \).

Now \( f := \chi_{(0,1)} \) and \( g := \chi_S \) belong to \( R_1 \), so that \( f - g \in R_2 \), and \( f - g \geq 0 \). However, \( f - g \notin R_1 \) because \( \int f - g \, dx > 0 \), and \( \varphi \leq 0 \) for every step function satisfying \( \varphi \leq f - g \).

REFERENCES

[1] H.S. Bear, *A Primer of Lebesgue Integration*, Second Edition, Academic Press, 2002.
[2] S.B. Chae, *Lebesgue Integration*, 2nd Edition, Springer, Universitext, 1995.
[3] P. Daniell, A general form of integral, Annals of Math. 19 (1917/18), 279–294.
[4] M. Fréchet, Sur l’intégrale d’une fonctionnelle étendue à un ensemble abstrait, Bull. Soc. Math. France 43 (1915), 248–265.
[5] P.R. Halmos, *Measure Theory*, D. Van Nostrand Co., Inc., Princeton, N.J., 1950.
[6] W. Johnston, *The Lebesgue Integral for Undergraduates*, MAA Press, 2015.
[7] V. Komornik, *Lectures on Functional Analysis and the Lebesgue Integral*, Universitext, Springer, London, 2016.
[8] F. Riesz, L’évolution de la notion d’intégrale depuis Lebesgue, Ann. Inst. Fourier 1 (1949), 29-42; see also [10] I, 327-340.
[9] F. Riesz, Les ensembles de mesure nulle et leur rôle dans l’analyse, Proceedings of the First Hungarian Mathematical Congress (1952), 214–224; see [10] I, 363–372.
[10] F. Riesz, *Oeuvres complètes I-II* (ed. Á. Császár), Akadémiai Kiadó, Budapest, 1960.
[11] F. Riesz, B. Sz.-Nagy, *Leçons d’analyse fonctionnelle*, Akadémiai Kiadó, Budapest, 1952. English translation: *Functional Analysis*, Dover, 1990.
[12] P. Roselli, The Riesz approach to the Lebesgue integral and complete function spaces, Real Anal. Exchange 27 (2001/02), no. 2, 635–660.
[13] A.J. Weir, *Lebesgue Integration and Measure*, Cambridge University Press, 1973.