FINITENESS AND DUALITY FOR THE COHOMOLOGY OF PRISOMATIC CRYSTALS

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Abstract. Let $(A, I)$ be a bounded prism, and $X$ be a smooth $p$-adic formal scheme over $\text{Spf}(A/I)$. We consider the notion of crystals on Bhatt–Scholze’s prismatic site $(X/A)_{\Delta}$ of $X$ relative to $A$. We prove that if $X$ is proper over $\text{Spf}(A/I)$ of relative dimension $n$, then the cohomology of a prismatic crystal is a perfect complex of $A$-modules with tor-amplitude in degrees $[0, 2n]$. We also establish a Poincaré duality for the reduced prismatic crystals, i.e. the crystals over the reduced structural sheaf of $(X/A)_{\Delta}$. The key ingredient is an explicit local description of reduced prismatic crystals in terms of Higgs modules.

0. Introduction

In a recent groundbreaking work [8], Bhatt and Scholze introduced the prismatic site for $p$-adic formal schemes (with $p$ a fixed prime), and they studied the cohomology of the natural structural sheaf on the prismatic site, called the prismatic cohomology. This new cohomology theory seems to occupy a central role in the study of cohomological properties of $p$-adic formal schemes, since it is naturally related to various previously known $p$-adic cohomology theories, and thus provides new insight on the $p$-adic comparison theorems with integral coefficients. For instance, it gives a natural site theoretic construction of the $A_{inf}$-cohomology which was previously constructed by Bhatt–Morrow–Scholze [6] in an ad-hoc way using the magical $L_\eta$-functor.

As analogues of classical crystalline crystals, there is a natural notion of crystals on the prismatic site. Indeed, such objects have already been considered in some special cases by many authors such as Anschütz–Le Bras [1], Gros–Le Strum–Quirós [10], Li [11] and Morrow–Tsuji [12]. In this article, we will consider the cohomology of prismatic crystals on rather general prismatic sites and prove some finiteness and duality theorems for such crystals.

Let us explain the main results of this article in more detail. Let $(A, I)$ be a bounded prism, and $X$ be a smooth $p$-adic formal scheme over $A/I$. We denote by $(X/A)_{\Delta}$ Bhatt–Scholze’s prismatic site of $X$ relative to $A$ [8, Def. 4.1]. The category $(X/A)_{\Delta}$ consists of bounded prisms $(B, J)$ over $(A, I)$ together with a structural map $\text{Spf}(B/J) \to X$, and coverings in $(X/A)_{\Delta}$ are $(p, I)$-completely faithfully flat maps of such bounded prisms. We have the structural sheaf $\mathcal{O}_{\Delta}$ (resp. the reduced structural sheaf $\overline{\mathcal{O}}_{\Delta}$) which sends each object $(B, J)$ to $B$ (resp. to $B/J$). An $\mathcal{O}_{\Delta}$-crystal (resp. an $\overline{\mathcal{O}}_{\Delta}$-crystal) is a $(p, I)$-completely flat and derived $(p, I)$-complete $\mathcal{O}_{\Delta}$-module (resp. $\overline{\mathcal{O}}_{\Delta}$-module) that satisfies similar properties as classical crystals on crystalline sites (cf. Def. 2.3). Here, we insist to impose the flatness condition in order to avoid some technical difficulties in faithfully flat descent for derived $(I, p)$-complete modules (cf. Prop. 1.9).

The first main result of this article is a finiteness theorem for $\mathcal{O}_{\Delta}$-crystals (Theorem 2.9), which claims that, if $X$ is proper and smooth of relative dimension $n$ over $A/I$ and $F$ is an $\mathcal{O}_{\Delta}$-crystal locally free of finite rank, then $R\Gamma((X/A)_{\Delta}, F)$ is a perfect complex of $A$-modules with tor-amplitude in degree $[0, 2n]$. Moreover, the formation of $R\Gamma((X/A)_{\Delta}, F)$ commutes with arbitrary base change in $A$. This finiteness theorem for $\mathcal{O}_{\Delta}$-crystals is a consequence of a similar result for $\overline{\mathcal{O}}_{\Delta}$-crystals. Actually, if $\mathcal{E}$ is an $\overline{\mathcal{O}}_{\Delta}$-crystal locally free of finite rank, we will show in Theorem 2.8 that the derived
We are thus reduced to the study of $\mathcal{O}_\Delta$-crystals. Since the problem is local for the étale topology of $X$, we may assume that $X = \text{Spf}(R)$ is affine such that $R$ is $p$-completely étale over the convergent power series ring $A/I(T_1, \ldots, T_n)$. In this case, we can give a rather explicit description of $\mathcal{O}_\Delta$-crystals in terms of Higgs modules. More precisely, after choosing a smooth lift $\tilde{R}$ over $A$ of $R$ together with a $\delta$-structure on $\tilde{R}$ compatible with that on $A$, we can show that there exists an equivalence between the category of $\mathcal{O}_\Delta$-crystals and that of topologically quasi-nilpotent Higgs modules over $R$ (cf. Theorem 4.12); furthermore, the cohomology of an $\mathcal{O}_\Delta$-crystal is computed by the de Rham complex of its associated Higgs module (cf. Theorem 4.14). From this description, our finiteness theorem for $\mathcal{O}_\Delta$-crystals follows easily.

As another application of the local description of $\mathcal{O}_\Delta$-crystals, we establish also in Theorem 5.3 a Poincaré duality for the cohomology of $\mathcal{O}_\Delta$-crystals. This can be viewed as a combination of the duality for de Rham complexes of Higgs modules and the classical Grothendieck–Serre duality. If one can construct a trace map for the prismatic cohomology of proper and smooth formal schemes, our results imply also a Poincaré duality for $\mathcal{O}_\Delta$-crystals as well (cf. Remark 5.4).

The organization of this article is as follows. In Section 1, we prove some preliminary results in commutative algebra. The main result of this section is a descent result on derived $I$-complete and $I$-completely flat modules (cf. Proposition 1.9). In Section 2, we discuss the notion of prismatic crystals and state the main finiteness theorems. Section 3 is devoted to the local study of prismatic crystals. When $X = \text{Spf}(R)$ is affine equipped with a lifting $\tilde{R}$ over $A$ together with a $\delta$-structure, we show that the category of prismatic crystals is equivalent to that of modules over $\tilde{R}$ equipped with a certain stratification (cf. Proposition 3.8). In Section 4, we work in the local situation of affine formal schemes equipped with étale local coordinates, and we related $\mathcal{O}_\Delta$-crystals to Higgs modules as mentioned above. Then we finish the proof of finiteness theorems at the end of Section 4. Finally, in Section 5 we prove a Poincaré duality for $\mathcal{O}_\Delta$.

It should be pointed out that Frobenius structures and filtrations on prismatic crystals are ignored in this article, even though these aspects should play important roles in many applications.

The results of this article was announced in a conference in honor of Luc Illusie in June 2021. At this conference, I learned that similar results in this article were also obtained independently by Ogus [13] for crystalline prisms and Bhatt–Lurie [5] for absolute prismatic crystals.

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0.1. Notation. Let $A$ be a commutative ring. We denote by $\text{Mod}(A)$ the abelian category of $A$-modules, and by $D(A)$ the derived category of $\text{Mod}(A)$. If $M$ is an $A$-module and $f \in A$ is an element, we denote by $M[f]$ the kernel of the multiplication by $f$ on $M$. We put also $M[f^\infty] = \bigcup_{n \geq 1} M[f^n]$.

Let $J \subset A$ be a PD-ideal. For $x \in J$ and an integer $n \geq 1$, we denote by $x^{[n]}$ the $n$-th divided power of $x$.

0.2. Sign Conventions. We will use the following conventions on the signs of complexes:

- For a naïve double complex $K^{\bullet, \bullet}$, its associated simple complex $\overline{s}(K^{\bullet, \bullet})$ has $n$-th differential given by

$$
\sum_{i+j=n} (d_1^{i,j} + (-1)^i d_2^{i,j}) : \bigoplus_{i+j=n} K^{i,j} \to \bigoplus_{a+b=n+1} K^{a,b},
$$
where \( d_1^{i,j} : K^{i,j} \to K^{i+1,j} \) and \( d_2^{i,j} : K^{i,j} \to K^{i,j+1} \) are the two differentials of \( K^{\bullet,\bullet} \) satisfying \( d_1^{i,j+1} \circ d_2^{i,j} = d_1^{i+1,j} \circ d_2^{i,j} \).

- For a complex \( K \) and an integer \( n \in \mathbb{Z} \), the \( i \)-th differential of the shift \( K[n] \) is obtained by multiplying \((-1)^n\) with the \((i + n)\)-th differential of \( K \).
- For two complexes \( K \) and \( L \), their tensor product \( K \otimes L \) is the simple complex attached to the naïve double complex with \((i,j)\)-component given by \( K^i \otimes L^j \) (whenever such a tensor product is well defined). The hom-complex \( \text{Hom}^\bullet(K, L) \) is defined as the simple complex with \( n \)-th term
  \[
  \text{Hom}^n(K, L) = \prod_{i+j=n} \text{Hom}(K^{-j}, L^i)
  \]
  and \( n \)-th differential
  \[
  d(f) = d_L \circ f - (-1)^n f \circ d_K
  \]

1. Preliminaries in commutative algebra

We recall first some general facts on derived completion, for which the main reference is [14, Tag 091N].

We consider a pair \((A, J)\), where \( A \) is a commutative ring, and \( J \subset A \) is an ideal. A complex \( K \) of \( A \)-modules is called derived \( J \)-complete, if \( R \text{Hom}(A_f, K) = 0 \) for all \( f \in J \), where \( A_f = A[\frac{1}{f}] \); and an \( A \)-module \( M \) is called derived \( J \)-complete if \( M[0] \) is derived \( J \)-complete. Then \( \hat{K} \) is derived \( J \)-complete if and only if so are \( H^q(K) \) for all \( q \in \mathbb{Z} \). The derived \( J \)-complete \( A \)-modules form an abelian full subcategory of \( \text{Mod}(A) \) that is stable under kernels, cokernels, images and extensions. Any classically \( J \)-adically complete \( A \)-module is derived \( J \)-complete, but the converse is not necessarily true. Let \( D_{\text{comp}}(A) \) be the full subcategory consisting of \( D(A) \) consisting of derived \( J \)-complete objects.

Assume from now on that \( J \) is finitely generated. Then the natural inclusion functor \( D_{\text{comp}}(A) \to D(A) \) admits a left adjoint \( K \mapsto \hat{K} \), called the derived \( J \)-completion. An explicit construction of \( \hat{K} \) is given as follows. Write \( J = (f_1, \ldots, f_r) \). Let \( \text{Kos}(A; f_1, \ldots, f_r) \) be the homological Koszul complex sitting in degrees \([-r, 0]\):

\[
\bigwedge^r A^{\oplus r} \to \bigwedge^r A^{\oplus r} \to \cdots \to A^{\oplus r} \xrightarrow{(f_1, \ldots, f_r)} A.
\]

For an integer \( n \geq 1 \), we have a transition map of complexes

\[
\text{Kos}(A; f_1^{n+1}, \ldots, f_r^{n+1}) \to \text{Kos}(A; f_1^n, \ldots, f_r^n)
\]
given by the multiplication by \( f_1, \ldots, f_m \) on a basis element

\[
e_{i_1} \wedge \cdots \wedge e_{i_m} \in \text{Kos}^{-m}(A; f_1^{n+1}, \ldots, f_r^{n+1}) = \bigwedge^m A^{\oplus r}.
\]

Then for an object \( K \in D(A) \), we have

\[
(1.0.1) \quad \hat{K} = R \text{lim} \left( K \otimes_A^{\mathbb{L}} \text{Kos}(A; f_1^n, \ldots, f_r^n) \right).
\]

Note that, in general, the canonical map \( \hat{K} \to R \text{lim}_n (K \otimes_A^{\mathbb{L}} A/J^n) \) is not an isomorphism in \( D(A) \). However, we will see later (Prop. [15]) that this indeed holds in an important special case.

**Lemma 1.1.** For an object \( K \) of \( D(A) \) and an integer \( n \geq 1 \), the canonical maps

\[
K \otimes^{\mathbb{L}} \text{Kos}(A; f_1^n, \ldots, f_r^n) \xrightarrow{\sim} \hat{K} \otimes^{\mathbb{L}} \text{Kos}(A; f_1^n, \ldots, f_r^n),
\]

\[
K \otimes_A^{\mathbb{L}} A/J^n \xrightarrow{\sim} \hat{K} \otimes_A^{\mathbb{L}} A/J^n
\]

are isomorphisms.
Lemma 1.2. Let $A \to B$ be a $J$-completely faithfully flat map of rings.

1. An object $K$ of $D(A)$ is $J$-completely flat (resp. $J$-completely faithfully flat) if and only if so is $K := (K \otimes^L_A B)^\wedge$.

2. Let $\phi : M \to N$ be a morphism in $D(A)$ with $M, N$ derived $J$-complete. Then $\phi$ is a quasi-isomorphism if and only if so is $\phi_B : M_B \to N_B$.

Proof. (1) By Lemma [1.1] we have

\[ K_B \otimes^L_B B/JB = K \otimes^L_A B/JB = (K \otimes^L_A A/J) \otimes^L_{A/J} B/JB. \]

Statement (1) follows immediately from the usual fpqc-descent of modules via the faithfully flat map $A/J \to B/JB$.

(2) Pick a distinguished triangle $M \xrightarrow{\phi} N \to K \to M[1]$ in $D(A)$, from which we deduce a distinguished triangle in $D(B)$: $M_B \xrightarrow{\phi_B} N_B \to K_B \to M_B[1]$. We need to prove that $K = 0$ if and only if $K_B = 0$. In view of (1.2.1) and the faithful flatness of $B/JB$ over $A/J$, one has $K \otimes^L_A A/J = 0$ if and only if $K_B \otimes^L_B B/JB = 0$. Then we conclude by the derived Nakayama Lemma [14] [Tag 0G1U].

The following definition is motivated by the notion of bounded prism [8] Def. 3.2].

Definition 1.3. A pair $(A, J)$ is of reduced prismatic type if $J = (f)$ for some $f \in A$, $A$ is derived $J$-complete and has bounded $f^\infty$-torsion, i.e. there exists an integer $c \geq 0$ such that $A[f^c] = A[f^\infty]$.

A pair $(A, J)$ is of prismatic type if $A$ is derived $J$-complete, where $J \subset A$ is an ideal of the form $J = (I, f)$ such that

- $I \subset A$ is a finitely generated ideal that defines a Cartier divisor of Spec($A$),
- $I^r$ is a principal ideal for some integer $r \geq 1$,
- and $f \in A$ is a nonzero divisor such that $A/I$ has bounded $f^\infty$-torsion.

Remark 1.4. By [8] Lemma 3.6], a bounded prism is a pair of prismatic type in the sense above. If $(A, J)$ is a pair of prismatic type with $J = (I, f)$ as in the definition, then $(A, J_r)$ with $J_r = (I^r, f)$ is also a pair of prismatic type. Moreover, as $J^r \subset J_r \subset J$, the derived completion and the classical adic completion with respect to $J$ agree with those with respect to $J_r$ (see [14] Tag 091S]). This allows us to reduce many arguments below to the case of principal $I$, i.e. $J = (\xi, f)$ for two nonzero divisors $\xi, f \in A$ such that $A/(\xi)$ has bounded $f^\infty$-torsion.

Proposition 1.5. Let $(A, J)$ be a pair of (reduced) prismatic type.

1. For an object $K \in D(A)$, the canonical map

\[ \hat{K} \xrightarrow{\sim} R\lim_n (K \otimes^L_A A/J^n) \]

Proof.
deduced from the universal property of $\hat{K}$ is an isomorphism.

(2) If $M$ is a $J$-completely (faithfully) flat complex of $A$-modules, then $\hat{M}$ is concentrated in degree 0 and a classically $J$-adically complete $A$-module such that $\hat{M}/J^n\hat{M}$ is (faithfully) flat over $A/J^n$. In particular, $A$ is classically $J$-complete.

(3) Conversely, if $N$ is a classically $J$-adically complete $A$-module such that $N/J^nN$ is (faithfully) flat over $A/J^n$, then $N$ is a $J$-completely (faithfully) flat and derived $J$-complete object in $D(A)$.

Proof. For (1), we will treat only the case when $(A, J)$ is a pair of prismatic type, the arguments for the case of reduced prismatic type being similar and much simpler. As explained in Remark 1.4, we may assume that $J = (\xi, f)$ such that $\xi, f \in A$ are nonzero divisor and $A/\xi A$ has bounded $f^\infty$-torsion.

By dévissage, the assumption that $A/\xi A$ has bounded $f^\infty$-torsion implies that so does $A/\xi^n A$, which in turn implies that the pro-object \{Kos($A/\xi^n A$; $f^m$) : $m \geq 1\}$ is isomorphic to \{$(A/\xi^n, f^m)$ : $m \geq 1\}$ by [14, Tag 091X]. Hence, we get

$$\hat{K} = R \lim_n R \lim_m (K \otimes^L_A \text{Kos}(A; \xi^n, f^m)).$$

Since $\xi$ is a nonzero divisor, we have a quasi-isomorphism

$$\text{Kos}(A; \xi^n, f^m) \cong \text{Kos}(A/\xi^n; f^m).$$

By dévissage, the assumption that $A/\xi A$ has bounded $f^\infty$-torsion implies that so does $A/\xi^n A$, which in turn implies that the pro-object \{Kos($A/\xi^n A$; $f^m$) : $m \geq 1\}$ is isomorphic to \{$(A/\xi^n, f^m)$ : $m \geq 1\}$ by [14, Tag 091X]. Hence, we get

$$\hat{K} = R \lim_n R \lim_m (K \otimes^L_A \text{Kos}(A; \xi^n, f^m))$$
$$\cong R \lim_n R \lim_m (K \otimes^L_A \text{Kos}(A/\xi^n; f^m))$$
$$\cong R \lim_n R \lim_m (K \otimes^L_A (A/\xi^n, f^m))$$
$$\cong R \lim_n (K \otimes^L_A (A/J^n),$$

where the last step follows from a cofinality argument.

Statement (2) follows from the same argument as [8, Lemma 3.7(2)].

For (3), we consider first the case when $(A, J)$ is a pair of reduced prismatic type. We write thus $J = (f)$ with $f \in A$ such that $A$ has bounded $f^\infty$-torsion. By the same proof as [7, Lemma 4.7] (with $p$ replaced by $f$), it suffices to show that $N$ has bounded $f^\infty$-torsion. By assumption, there exists an integer $c \geq 1$ such that $A[f^\infty] = A[f^c]$. Put $A' = A/A[f^c]$, which is $f$-torsion free. Put $N' = N \otimes_A A'$. Then by the exact sequence

$$A[f^c] \otimes_A N \to N \to N' \to 0,$$

it is enough to show that $N'$ is $f$-torsion free. We prove first $N'$ is $f$-adically complete. For any integer $n \geq 1$, let $A'_n = A'/f^n A'$, $N_n = N/f^n N$ and $N'_n = N'/f^n N'$. As $A'$ is $f$-torsion free, we have an exact sequence

$$0 \to A[f^c] \to A_n \to A'_n \to 0$$

for each $n \geq c$. Tensoring with $N_n$ over $A_n$, we get

$$0 \to A[f^c] \otimes_{A_n} N_n \to N_n \to N'_n \to 0$$

where the left exactness follows from the flatness of $N_n$ over $A_n$. Note that $A[f^c] \otimes_{A_n} N_n \cong A[f^c] \otimes_A N$ is independent of $n$ for $n \geq c$. Taking inverse limits, one gets thus

$$0 \to A[f^c] \otimes_A N \to N \to \varprojlim_n N'_n \to 0,$$
and hence $N' = \lim_{\to - n} N'_n$ is complete. Now note that the $f$-torsion freeness of $A'$ implies an exact sequence

$$0 \to A'_{n-1} \xrightarrow{x f} A'_n \to A'_1 \to 0.$$  

Note that the flatness of $N/f^n N$ over $A/f^n A$ implies the flatness of $N'_n$ over $A'_n$. Tensoring with $N'_n$, we get

$$0 \to N'_{n-1} \xrightarrow{x f} N'_n \to N'_1 \to 0.$$  

Now let $x \in N'$ with $fx = 0$. We get then $x \in f^{n-1} N'$ for any integer $n \geq 1$, and thus $x = 0$ by the completeness of $N'$. This finishes the proof of (3) in the case of reduced prismatic type.

Assume now that $(A, J)$ is of prismatic type. By Remark 1.4, we may assume that $J = (\xi, f)$ such that $\xi, f \in A$ are nonzero divisor and $A/\xi A$ has bounded $f^\infty$-torsion. Let $c \geq 0$ be an integer such that $(A/\xi A)[f^\infty] = (A/\xi A)[f^c]$. For any integer $n \geq 1$, we put $A_n = A/(f^n, \xi^n)$ and $N_n = N \otimes_A A_n$.

Let $x_0 \in A_n[\xi]$ with a lift $x \in A$. Then one has $\xi x = \xi^n y + f^n z$ for some $y, z \in A$. By our choice of $c$, there exists $z_1 \in A$ such that $f^c z = \xi z_1$. If $n \geq c$, one gets thus $\xi x = \xi^n y + f^{n-c} \xi z_1$ and hence $x = \xi^{n-1} y + f^{n-c} z_1$ since $\xi$ is a nonzero divisor. It follows that $A_n[\xi] \subset \xi^{n-1} A_n + f^{n-c} A_n$ for all $n \geq c$. Since $N_n$ is flat over $A_n$ by assumption, one has

$$N_n[\xi] = A_n[\xi] \otimes_A N_n \subset (\xi^{n-1} A_n + f^{n-c} A_n) \otimes_A N_n = \xi^{n-1} N_n + f^{n-c} N_n$$

for all $n \geq c$. Now consider the exact sequence

$$0 \to N_n[\xi] \to N_n \xrightarrow{x \xi} N_n \to N/(\xi, f^n N) \to 0$$

for all $n \geq 1$. Note that the canonical map $N_{n+c-1}[\xi] \to N_n[\xi]$ is zero, hence the inverse system $(N_n[\xi])_n$ is essentially null. Taking inverse limits, one gets a short exact sequence

$$0 \to N \xrightarrow{x \xi} N \xrightarrow{\lim_{\to n}} N/(\xi, f^n N) \to 0.$$  

It follows that $N$ is $\xi$-torsion free and $N/\xi N$ is $f$-adically complete. Note that $N \otimes^L_A A/\xi A = N/\xi N$, and hence

$$N \otimes^L_A A/J = (N \otimes^L_A A/\xi A) \otimes^L_{A/\xi A} A/J = (N/\xi N) \otimes^L_{A/\xi A} A/J.$$  

The statement (3) then follows immediately the case of reduced prismatic type applied to $N/\xi N$.

Let $\mathbf{Mod}_J(A)$ denote the subcategory of $\mathbf{Mod}(A)$ consisting of derived $J$-complete $A$-modules. Let $M, N$ be objects of $\mathbf{Mod}_J(A)$. Then $\text{Hom}_A(M, N)$ is also derived $J$-complete by [14, Tag 0A6E]. We put

$$(1.5.1) \quad M \hat{\otimes}_A N := H^0((M \otimes^L_A N)^\wedge).$$  

Note that the functor $M \mapsto M \hat{\otimes}_A N$ is right exact (cf. [14, Tag 0AAJ]) and we have (cf. [9, Appendix])

$$\text{Hom}_A(L \hat{\otimes}_A M, N) = \text{Hom}_A(L, \text{Hom}_A(M, N))$$

for all objects $L, M, N \in \mathbf{Mod}_J(A)$. There is a canonical surjection from $M \hat{\otimes}_A N$ to the classical $J$-adic completion of $M \otimes_A N$, which is in general not an isomorphism. Let $\mathbf{FMod}_J(A)$ be the full subcategory of $\mathbf{Mod}_J(A)$ consisting of $J$-completely flat $A$-modules.

Let $\phi : A \to B$ be a derived $J$-complete $A$-algebra. We denote by

$$(1.5.2) \quad \phi^* : \mathbf{Mod}_J(A) \to \mathbf{Mod}_{J_B}(B)$$

the base change functor $M \mapsto M \hat{\otimes}_A B$. Then $\phi^*$ is the left adjoint of the functor of restriction of scalars $\mathbf{Mod}_{J_B}(B) \to \mathbf{Mod}_J(A)$.
Lemma 1.6. Let $\phi : A \to B$ and $\psi : B \to C$ be morphisms of derived $J$-complete $A$-algebras. Then for any $M \in \text{Mod}^J_A(A)$, there is a canonical isomorphism
\[
\psi^* \phi^*(M) \simeq (\psi \circ \phi)^*(M).
\]

Proof. Consider the distinguished triangle
\[
\tau_{\leq -1}((M \otimes_B^L C)^{\wedge}) \to (M \otimes_B^L C)^{\wedge} \to \phi^*(M) = H^0((M \otimes_B^L C)^{\wedge}) \to .
\]
Taking the derived tensor product with $C$ and derived $J$-completion, one gets
\[
\tau_{\leq -1}((M \otimes_B^L C)^{\wedge}) \to ((M \otimes_B^L C)^{\wedge}) \to (\phi^*(M) \otimes_B^L C)^{\wedge} \to .
\]
If we write $J = (f_1, \ldots, f_r)$, then one has
\[
((M \otimes_B^L C)^{\wedge}) = R \lim_n \left( (M \otimes_B^L C) \otimes B \Kos(C; f_1^n, \ldots, f_r^n) \right)
\]
\[
\simeq R \lim_n \left( (M \otimes_B^L \Kos(B; f_1^n, \ldots, f_r^n) \otimes B) \right)
\]
\[
\simeq R \lim_n \left( M \otimes_B^L \Kos(C; f_1^n, \ldots, f_r^n) \right)
\]
\[
= (M \otimes_B^L C)^{\wedge},
\]
where the second isomorphism uses Lemma 1.1. One deduces then
\[
H^0((\tau_{\leq -1}((M \otimes_B^L C)^{\wedge}) \to (\psi \circ \phi)^*(M) \to \psi^* \phi^*(M) \to H^1((\tau_{\leq -1}((M \otimes_B^L C)^{\wedge}) \to
\]
Since $(\tau_{\leq -1}((M \otimes_B^L C)^{\wedge})$ is concentrated in degree $\leq -1$ by [11, Tag 0AAJ], the Lemma follows immediately.

Lemma 1.7. Let $(A, J)$ be a pair of prismatic type (resp. of reduced prismatic type), and $\phi : A \to B$ be an $A$-algebra such that $(B, JB)$ is also of prismatic type (resp. of reduced prismatic type).

1. If $M, N \in \text{FMod}^J_A(A)$, then $(M \otimes^L_J N)^{\wedge} \cong M \otimes^J_A N$ is $J$-completely flat and it coincides with the classical $J$-adic completion of $M \otimes^J_A N$.

2. The base change functor $\phi^*$ sends $\text{FMod}^J_A(A)$ to $\text{FMod}^J_B(B)$.

Proof. (1) By Proposition 1.3 we have
\[
(M \otimes^L_J N)^{\wedge} = R \lim_n (M \otimes_B^L N \otimes_B^L A/J^n) = \lim_n (M \otimes_B A N) / J^n (M \otimes_B A N),
\]
where the last equality uses the $J$-complete flatness of $M \otimes_B^L N$. The Lemma follows immediately.

(2) Let $M$ be an object of $\text{FMod}^J_A(A)$. Then $(M \otimes_B^L B)^{\wedge}$ is $JB$-completely flat. By Proposition 1.3(2), we have $M \otimes_A B = (M \otimes_B^L B)^{\wedge}$.

Let $B^{\otimes^\bullet}$ be the Čech nerve of $\phi : A \to B$ in the category of $J$-completely flat $A$-algebras, i.e. $B^{\otimes^\bullet}$ is the cosimplicial object of $J$-completely flat $A$-algebras with its $n$-component for $n \geq 0$ given by
\[
B^{\otimes^n} := (B \otimes^\wedge A)^{(n+1)}.
\]
In particular, for any integer $i$ with $0 \leq i \leq n$, one has a map of $A$-algebras $\delta^n_i : B^{\otimes(n-1)} \to B^{\otimes^n}$ given by
\[
\delta^n_i : b_0 \otimes \cdots \otimes b_{n-1} \mapsto b_0 \otimes \cdots \otimes b_{i-1} \otimes b_i \otimes \cdots \otimes b_{n-1}.
\]
We have thus a diagram of morphisms of derived $J$-complete $A$-algebras:
\[
\begin{array}{ccc}
B & \xrightarrow{\delta^0_i} & B \oplus \Delta_A B \\
\delta^1_i & \downarrow & \Rightarrow \downarrow \delta^1_i \\
B \otimes_A B & \xrightarrow{\delta^2_i} & B \otimes_A B \otimes_A B,
\end{array}
\]
Let $\mu : B \otimes_A B \to B$ denote the canonical diagonal surjection.
**Definition 1.8.** A descent pair relative to \( \phi : A \to B \) consists of
- a derived \( JB \)-complete \( B \)-module \( M \),
- and an isomorphism of \( B \hat{\otimes}_A B \)-modules
  \[ \varepsilon : \delta^{1,*}_0(M) = M \hat{\otimes}_A B \xrightarrow{\sim} \delta^{1,*}_1(M) = B \hat{\otimes}_A M, \]
called a descent datum on \( M \), such that
  \[ \mu^*(\varepsilon) = \text{id}_M, \quad \delta^{1,*}_1(\varepsilon) = \delta^{2,*}_0(\varepsilon) \circ \delta^{2,*}_0(\varepsilon). \]
A morphism of descent pairs \( f : (M_1, \varepsilon_1) \to (M_2, \varepsilon_2) \) is a morphism of \( B \)-modules \( f : M_1 \to M_2 \) such that \( \varepsilon_2 \circ \delta^{1,*}_1(f) = \delta^{1,*}_0(f) \circ \varepsilon_1 \). We denote by \( \text{Desc}_{B/A}^{\wedge} \) the category of descent pairs relative to \( \phi \).

For an object \( N \in \text{Mod}^\wedge_J(A) \), there is a canonical isomorphism
\[ \varepsilon_N : \delta^{1,*}_0(N \hat{\otimes}_A B) \simeq N \hat{\otimes}_A (B \hat{\otimes}_A B) \simeq \delta^{1,*}_1(N \otimes A B) \]
which clearly satisfies the axiom for a descent pair. We get thus a functor
\[ \phi^\wedge : \text{Mod}^\wedge_J(A) \to \text{Desc}_{B/A}^{\wedge}, \quad N \mapsto (N \hat{\otimes}_A B, \varepsilon_N). \]
A descent pair relative to \( \phi \) is called effective if it lies in the essential image of \( \phi^\wedge \).

**Proposition 1.9.** Let \( (A, J) \) be a pair of (reduced) prismatic type, and \( \phi : A \to B \) a \( J \)-completely faithfully flat derived \( J \)-complete \( A \)-algebra.

1. Let \( M \in \text{FMod}^\wedge_J(A) \), and \( \mathbf{s}(M \hat{\otimes}_A B^{\wedge \bullet}) \) denote the complex associated to the cosimplicial object \( M \hat{\otimes}_A B^{\wedge \bullet} \). Then the canonical map
   \[ M \xrightarrow{\sim} \mathbf{s}(M \hat{\otimes}_A B^{\wedge \bullet}) \]
is a quasi-isomorphism. In particular, one has an equalizer diagram of \( A \)-modules
   \[ M \longrightarrow M \hat{\otimes}_A B \xrightarrow{\sim} M \hat{\otimes}_A B \hat{\otimes}_A B. \]

2. Let \( \text{FDesc}_{B/A}^{\wedge} \) be the subcategory of \( \text{Desc}_{B/A}^{\wedge} \) consisting of objects \( (M, \theta) \) with \( M \) an \( JB \)-completely flat \( B \)-module. Then the functor \( \phi^\wedge \) induces an equivalence of categories
   \[ \text{FMod}^\wedge_J(A) \xrightarrow{\sim} \text{FDesc}_{B/A}^{\wedge}. \]

**Proof.** As usual, we treat here only the case of prismatic type, and that of reduced prismatic is similar and much simpler.

For (1), it suffices to prove the statement for \( M = A \), the general case being obtained by applying \( M \otimes^L_A \cdot \) and taking derived \( J \)-completion. By Lemma 1.2(2), it suffices to prove that \( B \xrightarrow{\sim} (\mathbf{s}(B^{\wedge \bullet}) \otimes^L_A B)^\wedge \) is a quasi-isomorphism. But note that
\[ (\mathbf{s}(B^{\wedge \bullet}) \otimes^L_A B)^\wedge \simeq \mathbf{s}(B^{\wedge \bullet} \hat{\otimes}_A B), \]
where \( B^{\wedge \bullet} \hat{\otimes}_A B \) is nothing but the \( \check{\text{C}} \text{ech} \) nerve of \( \delta^1_B : B \to B \hat{\otimes}_A B \), which admits a right inverse given by the multiplication map \( B \hat{\otimes}_A B \to B \). It follows from [13, Tag 019Z] that \( B^{\wedge \bullet} \hat{\otimes}_A B \) is homotopy equivalent to the constant cosimplicial object \( B \). Hence the canonical map \( B \xrightarrow{\sim} \mathbf{s}(B^{\wedge \bullet} \hat{\otimes}_A B) \) is also a homotopy equivalence of complexes.

For (2), the full faithfulness of the restriction of \( \phi^\wedge \) to \( \text{FMod}^\wedge_J(A) \) follows easily from (1). It remains to prove the essential surjectivity. Let \( (M, \varepsilon) \) be an object of \( \text{FDesc}_{B/A}^{\wedge} \). For any integer \( n \geq 1 \), let \( M_n := M/J^n B \) and
\[ \varepsilon_n : \delta^{1,*}_0(M_n) = M_n \otimes_{A/J^n} B/J^n B \xrightarrow{\sim} \delta^{1,*}_1(M_n) = B/J^n B \hat{\otimes}_{A/J^n} M_n. \]
be the reduction modulo \( J^n(B \otimes_A B) \) of \( \varepsilon \). Then \((M_n, \varepsilon_n)\) is a classical descent datum relative to \( A/J^n \to B/J^n \). By the classical fpqc-descent, the descent datum \((M_n, \varepsilon_n)\) comes from a flat \( A/J^n \)-module \( N_n \). It is clear that \( N_{n+1} \otimes_{A/J^{n+1}} A/J^n \simeq N_n \). We put \( N = \lim_{\longrightarrow} N_n \). By [13][Tag 05B8], \( N \) is \( J \)-adically complete and \( N_n = N/J^nN \) for all \( n \geq 1 \). By Proposition [13][Tag 05B8], \( N \) is \( J \)-completely flat. By Proposition [13][Tag 05B8], we have

\[
(N \otimes_A B) / B / J^n B = (N \otimes_A B)^\wedge_B B / J^n B \simeq N_n \otimes_{A/J^n} B / J^n B \simeq M_n.
\]

Passing to the limit, we get thus an isomorphism of descent data \((N \otimes_A B, \varepsilon_N) \simeq (M, \varepsilon)\).

\[\square\]

**Remark 1.10.** As pointed by the anonymous referee, if \((A, J)\) is a pair of (reduced) prismatic type, Proposition [13][Tag 05B8] and [13][Tag 05B8] actually imply that \( N \mapsto (N \otimes_A A/J^n)_{n \geq 1} \) establishes an equivalence of categories between \( \text{FMod}_p(A) \) and the category of inverse systems of \( A \)-modules \((N_n)_{n \geq 1}\) such that each \( N_n \) is flat over \( A/J^n \) and \( N_{n+1} \otimes_{A/J^{n+1}} A/J^n \simeq N_n \).

### 2. Prismatic Crystals and Finiteness Theorems

In this section, we fix a prime number \( p \). All the rings are supposed to be \( \mathbb{Z}_{(p)} \)-algebras.

#### 2.1. Prismatic site.

Let \((A, I)\) be a bounded prism in the sense of [8][Def. 3.2], and \( X \) be a \( p \)-adic formal scheme over \( \text{Spf}(A/I) \). We recall first the prismatic site of \( X \) relative to \( A \), denoted by \((X/A)_\Delta\), introduced in [8][Def. 4.1]:

- The underlying category of \((X/A)_\Delta\) is the opposite of bounded prisms \((B, IB)\) over \((A, I)\) together with a map of \( p \)-adic formal schemes \( \text{Spf}(B/IB) \to X \) over \( A/I \); the notion of morphism in \((X/A)_\Delta\) is the obvious one. We shall often denote such an object by \((\text{Spf}(B) \leftarrow \text{Spf}(B/IB) \to X)\);
  - or by \((B \to B/IB \leftarrow R)\) if \( X = \text{Spf}(R) \) is affine.
- A map \((\text{Spf}(C) \leftarrow \text{Spf}(C/IC) \to X) \to (\text{Spf}(B) \leftarrow \text{Spf}(B/IB) \to X)\)
  - in \((X/A)_\Delta\) is a flat cover if the underlying map of \( \delta \)-\( A \)-algebras \( B \to C \) is \((p, I) \)-completely faithfully flat.

Following Grothendieck, we denote by \((X/A)^\wedge_\Delta\) the associated topos. From Proposition [13][Tag 05B8] it follows that there is a canonical embedding \((X/A)^\wedge_\Delta \to (X/A)_{\Delta}^\wedge\).

**Remark 2.2.** Let \( f : V := (\text{Spf}(C) \leftarrow \text{Spf}(C/IC) \to X) \to U := (\text{Spf}(B) \leftarrow \text{Spf}(B/IB) \to X) \) and \( g : W := (\text{Spf}(D) \leftarrow \text{Spf}(D/ID) \to X) \to U \) be two morphisms in \((X/A)_\Delta\). Assume that one of \( f \) and \( g \), say \( g \), is \((p, I) \)-completely flat (i.e., the underlying map of \( \delta \)-\( A \)-algebras \( B \to D \) is \((p, I) \)-completely flat). Then \( E := (C \otimes_B D)^\wedge \) concentrated in degree 0 and it is a \((p, I) \)-completely flat \( \delta \)-\( C \)-algebra (cf. Lemma [13][Tag 117]); moreover, there exists a canonical map

\[
\text{Spf}(E/IE) = \text{Spf}(C/IC) \times_{\text{Spf}(B/IB)} \text{Spf}(D/ID) \to \text{Spf}(B/IB) \to X.
\]

Then the object \((\text{Spf}(E) \leftarrow \text{Spf}(E/IE) \to X)\) gives the fibre product \( V \times_U W \).

We denote by \( \mathcal{O}_{(X/A)_\Delta} \) (resp. \( \mathcal{O}_{(X/A)_\Delta} = \mathcal{O}_{(X/A)_\Delta} \)) the structural sheaf (resp. the reduced structural sheaf) on \((X/A)_\Delta\) which sends an object \((B, J)\) in \((X/A)_\Delta\) to \( B \) (resp. to \( B/J \)). If no confusion arises, we will simply write \( \mathcal{O}_{(X/A)_\Delta} \) and \( \mathcal{O}_{(X/A)_\Delta} \) for \( \mathcal{O}_{(X/A)_\Delta} \) and \( \mathcal{O}_{(X/A)_\Delta} \).

**Definition 2.3.** An \( \mathcal{O}_{(X/A)_\Delta} \)-**crystal** (resp. an \( \mathcal{O}_{(X/A)_\Delta} \)-**crystal**) on \((X/A)_\Delta\) is a sheaf of \( \mathcal{O}_{(X/A)_\Delta} \)-modules \( \mathcal{F} \) on \((X/A)_\Delta\) such that
• for each object \((\text{Spf}(B) \leftarrow \text{Spf}(B/IB) \rightarrow X)\) of \((X/A)_\Delta\), the evaluation

\[ F_B := F(\text{Spf}(B) \leftarrow \text{Spf}(B/IB) \rightarrow X) \]

is a derived \((p,I)\)-complete and \((p,I)\)-completely flat \(B\)-module (resp. a derived \(p\)-complete and \(p\)-completely flat \(B/I\)-module),

• for any morphism

\[(\text{Spf}(C) \leftarrow \text{Spf}(C/IC) \rightarrow X) \xrightarrow{f} (\text{Spf}(B) \leftarrow \text{Spf}(B/IB) \rightarrow X),\]

in \((X/A)_\Delta\) the canonical linearized transition map

\[ c_f(F) : f^*(F_B) := F_B \otimes_B C \rightarrow F_C \]

(resp. \(c_f(F) : \tilde{f}^*(F_B) := F_B \otimes_{B/IB} C/IC \rightarrow F_C\)) is an isomorphism, where \(\otimes\) is the completed tensor product for the ideal \((p,I)\) (resp. for the ideal \((p)\)) defined in [1.5.1].

An \(\mathcal{O}_\Delta\)-crystal (resp. an \(\mathcal{O}_\Delta\)-crystal) \(F\) is called locally free of finite rank if \(F_B\) is a locally free \(B\)-module (resp. \(B/IB\)-module) of finite rank for each object \((\text{Spf}(B) \leftarrow \text{Spf}(B/IB) \rightarrow X)\).

We denote by \(\text{CR}((X/A)_\Delta, \mathcal{O}_\Delta)\) (resp. \(\text{CR}((X/A)_\Delta, \mathcal{O}_\Delta)\)) the category of \(\mathcal{O}_\Delta\)-crystals (resp. \(\mathcal{O}_\Delta\)-crystals).

**Lemma 2.4.** The functor \(F \mapsto (\{F_B\}, \{c_f(F)\})\) induces an equivalence of \(\text{CR}((X/A)_\Delta, \mathcal{O}_\Delta)\) and the category of the data \((\{M_B\}, \{c_f\})\), where

- \(\{M_B\}\) is the collection of derived \((p,I)\)-complete and \((p,I)\)-completely flat \(B\)-modules \(M_B\) corresponding to each object \((\text{Spf}(B) \leftarrow \text{Spf}(B/IB) \rightarrow X)\) of \((X/A)_\Delta\);
- \(\{c_f\}\) is the collection of isomorphisms of \(C\)-modules

\[ c_f : C \otimes_B M_B \xrightarrow{\sim} M_C \]

for each morphism \(f : (\text{Spf}(C) \leftarrow \text{Spf}(C/IC) \rightarrow X) \rightarrow (\text{Spf}(B) \leftarrow \text{Spf}(B/IB) \rightarrow X)\) in \((X/A)_\Delta\) such that

- \(c_f\) is the identity map if \(f\) is the identity of an object of \((X/A)_\Delta\);
- the cocycle condition

\[ c_{fg} = c_g \circ g^*(c_f) \]

is satisfied for a composition of morphisms.

**Proof.** It is sufficient to construct a quasi-inverse to the evaluation functor \(F \mapsto (\{F_B\}, \{c_f(F)\})\). Given a datum \((\{M_B\}, \{c_f\})\) as above, the cocycle condition guarantees that

\[ \mathcal{M} : (\text{Spf}(B) \leftarrow \text{Spf}(B/IB) \rightarrow X) \mapsto M_B \]

together with transition map

\[ \mathcal{M}_f : M_B \rightarrow f^*(M_B) = M_B \otimes_B C \xrightarrow{c_f} M_C \]

for each morphism \(f : (\text{Spf}(C) \leftarrow \text{Spf}(C/IC) \rightarrow X) \rightarrow (\text{Spf}(B) \leftarrow \text{Spf}(B/IB) \rightarrow X)\) in \((X/A)_\Delta\) indeed defines a presheaf on \((X/A)_\Delta\). Then Proposition [1.9.1] implies that \(\mathcal{M}\) is indeed a sheaf. The fact that \(\mathcal{M}\) is an \(\mathcal{O}_\Delta\)-crystal follows immediately from the conditions on \(\{c_f\}\).

**Remark 2.5.** There is an obvious analogue of this Lemma for \(\mathcal{O}_\Delta\)-crystals by requiring \(c_f : C/IC \otimes_{B/IB} M_B \rightarrow M_C\) to be an isomorphism for each morphism \(f\) as above.

**Example 2.6.** Assume that \((A,I)\) is a perfect prism, and \(X = \text{Spf}(R)\) with \(R\) quasi-regular semiperfectoid and \(A/I \rightarrow R\) is surjective. Then the prismatic cohomology \(\Delta_{R/A} := R^\Gamma((X/A)_\Delta, \mathcal{O}_\Delta)\) is concentrated in degree 0 and \(\Delta_{R/A, I\Delta_{R/A}}\) becomes the final object in \((X/A)_\Delta\) (see [1 \S 3.4]). Therefore, \(\text{CR}((X/A)_\Delta, \mathcal{O}_\Delta)\) is equivalent to the category of derived \(I\)-complete and completely \(I\)-flat \(\Delta_{R/A}\)-modules.
2.7. Functoriality. Let \((A, I) \to (A', I')\) be a morphism of bounded prisms. Let

\[
\begin{array}{ccc}
Y & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
\text{Spf}(A'/I') & \longrightarrow & \text{Spf}(A/I)
\end{array}
\]

be a commutative diagram of \(p\)-adic formal schemes. Then \(f\) induces a morphism of sites

\[f^\#: (Y/A')_\Delta \to (X/A)_\Delta\]

given by

\[(\text{Spf}(B') \leftarrow \text{Spf}(B'/J') \to Y) \mapsto (\text{Spf}(B') \leftarrow \text{Spf}(B'/J') \to Y \xrightarrow{f} X).\]

It is easy to see that \(f^\#\) is cocontinuous, hence it induces a morphism of topoi:

\[f^\#: (f^{-1}_\Delta, f_{\Delta, *}) : (Y/A')^\_\Delta \to (X/A)^\_\Delta.\]

For a sheaf \(\mathcal{F}\) on \((X/A)^\_\Delta\), its inverse image is given by

\[(f^{-1}_\Delta \mathcal{F})(B', J') = \mathcal{F}(f^\#(B', J')).\]

For a sheaf \(\mathcal{E}\) on \((Y/A')^\_\Delta\), its direct image \(f_{\Delta, *}(\mathcal{E})\) is described as follows. For an object \((B, J)\) in \((X/A)^\_\Delta\), let \(Y_B = Y \times_X \text{Spf}(B/J)\) and \(B' = B \hat{\otimes}_A A'\). Assume that \((B', IB')\) is a bounded prism (which is the case if either \(A'\) or \(B\) is \((p, I)\)-completely flat over \(A\)). Then the direct image of a sheaf \(\mathcal{E}\) on \((Y/A')^\_\Delta\) is given by

\[(f_{\#}(\mathcal{E}))^\#(B, J) = \Gamma((Y_B/B')^\_\Delta, \mathcal{E}|_{(Y_B/B')^\_\Delta})\]

where \(\mathcal{E}|_{(Y_B/B')^\_\Delta}\) is the pullback of \(\mathcal{E}\) to \((Y_B/B')^\_\Delta\).

For an \(\mathcal{O}_{(X/A)^\_\Delta}\)-module \(\mathcal{F}\), we put

\[f^\#_\Delta \mathcal{F} := f^{-1}_\Delta \mathcal{F} \hat{\otimes}_\Delta f^{-1}_\Delta \mathcal{O}_{(X/A)^\_\Delta} \mathcal{O}_{(Y/A')^\_\Delta}.\]

Then the functor \(f^\#_\Delta\) induces the pullback map for prismatic crystals:

\[f^\#_\Delta : \text{CR}((X/A)^\_\Delta, \mathcal{O}_{(X/A)^\_\Delta}) \to \text{CR}((Y/A')^\_\Delta, \mathcal{O}_{(Y/A')^\_\Delta}).\]

Similarly, the formula

\[f^\#_\Delta : \mathcal{F} \mapsto f^{-1}_\Delta \mathcal{F} \hat{\otimes}_\Delta f^{-1}_\Delta \mathcal{O}_{(X/A)^\_\Delta} \mathcal{O}_{(Y/A')^\_\Delta}\]

defines a pullback functor

\[\bar{f}^\#_\Delta : \text{CR}((X/A)^\_\Delta, \mathcal{O}_{(X/A)^\_\Delta}) \to \text{CR}((Y/A')^\_\Delta, \mathcal{O}_{(Y/A')^\_\Delta}).\]

We can now state the main results of this article. Let \(X_{\acute{E}t}\) be the big étale site of \(X\) consisting of all \(p\)-adic formal schemes over \(X\). Let \(\nu_{X/A} : (X/A)^\_\Delta \to X_{\acute{E}t}\) be the canonical projection. For any sheaf \(\mathcal{F}\) on \((X/A)^\_\Delta\) and any object \(U\) of \(X_{\acute{E}t}\), we have

\[(\nu_{X/A, *\Delta} \mathcal{F})(U) = \Gamma((U/A)^\_\Delta, \mathcal{F}_U),\]

where \(\mathcal{F}_U\) is the inverse image of \(\mathcal{F}\) under the natural map of topos \((U/A)^\_\Delta \to (X/A)^\_\Delta\).

**Theorem 2.8.** Let \(X\) be a smooth \(p\)-adic formal scheme over \(A/I\) of relative dimension \(n\). Let \(\mathcal{E}\) be an \(\mathcal{O}_{(X/A)^\_\Delta}\)-crystal locally free of finite rank on \((X/A)^\_\Delta\). Then the following statements hold:

1. \(R\nu_{X/A, *}^\Delta(\mathcal{E})\) is a perfect complex of \(\mathcal{O}_X\)-modules with tor-amplitude in \([0, n]\).
(2) Let \((A, I) \to (A', I')\) be a morphism of bounded prisms. Consider the cartesian diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
\text{Spf}(A'/I') & \longrightarrow & \text{Spf}(A/I).
\end{array}
\]

The canonical base change map

\[
f^{-1}R\nu_{X/A,*}(E) \otimes^{L}_{f^{-1}\mathcal{O}_X} \mathcal{O}_{X'} \xrightarrow{\sim} R\nu_{X'/A',*}(f^*_{\Delta}E)
\]

is an isomorphism.

The proof of this Theorem will be given in Subsection [4.19] after some local preparations. For the moment, one can deduce immediately from Theorem [2.8] the following finiteness result on the cohomology of an \(\mathcal{O}_{\Delta}\)-crystal.

**Theorem 2.9.** Let \(X\) be a proper and smooth \(p\)-adic formal scheme over \(\text{Spf}(A/I)\) of relative dimension \(n\). Let \(\mathcal{F}\) be an \(\mathcal{O}_{\Delta}\)-crystal locally free of finite rank on \((X/A)_{\Delta}\). Then \(R\Gamma((X/A)_{\Delta}, \mathcal{F})\) is a perfect complex of \(\mathcal{A}\)-modules with tor-amplitude in \([0, 2n]\). Moreover, if \((A, I) \to (A', I')\) is a morphism of \(p\)-torsion free bounded prisms that induces the cartesian diagram (2.8.1), then the canonical base change map

\[
R\Gamma((X/A)_{\Delta}, \mathcal{F}) \otimes^{L}_{A} A' \xrightarrow{\sim} R\Gamma((X'/A')_{\Delta}, f^*_{\Delta} \mathcal{F})
\]

is an isomorphism.

**Proof.** Applying Theorem [2.8](1) to \(\mathcal{F} := \mathcal{F}/I\mathcal{F}\), we see that \(R\nu_{X/A,*}(\mathcal{F})\) is a perfect complex of \(\mathcal{O}_{X}\)-modules with perfect amplitude in \([0, n]\). Since \(X\) is assumed to be proper and smooth, it follows that

\[
R\Gamma((X/A)_{\Delta}, \mathcal{F}) \otimes^{L}_{A} A/I \xrightarrow{\sim} R\Gamma((X/A)_{\Delta}, \mathcal{F}) \cong R\Gamma(X_{\text{et}}, R\nu_{X/A,*}(\mathcal{F}))
\]

is a perfect complex of \(A/I\)-modules with perfect amplitude \([0, n]\). We then conclude by [14, Tag 07LU] that \(R\Gamma((X/A)_{\Delta}, \mathcal{F})\) is a perfect complex with tor-amplitude in \([0, 2n]\).

For the second part of the Theorem, according to the derived Nakayama Lemma [14, Tag 0G1U], it suffices to show that

\[
R\Gamma((X/A)_{\Delta}, \mathcal{F}) \otimes^{L}_{A/I} A'/I' \xrightarrow{\sim} R\Gamma((X'/A')_{\Delta}, f^*_{\Delta} \mathcal{F}) \otimes^{L}_{A} A'/I'
\]

is an isomorphism. But this follows from the following sequence of canonical isomorphisms:

\[
R\Gamma((X/A)_{\Delta}, \mathcal{F}) \otimes^{L}_{A/I} A'/I' \cong R\Gamma(X_{\text{et}}, R\nu_{X/A,*}(\mathcal{F})) \otimes^{L}_{A/I} A'/I'
\]

\[
\cong R\Gamma(X_{\text{et}}, f^{-1}R\nu_{X/A,*}(\mathcal{F}) \otimes^{L}_{f^{-1}\mathcal{O}_X} \mathcal{O}_{X'})
\]

\[
\cong R\Gamma((X'/A')_{\Delta}, f^*_{\Delta} \mathcal{F})
\]

where the second isomorphism is the projection formula for coherent cohomology, and the third one is Theorem [2.8](2).

It is standard to deduce from Theorem 2.9 some finiteness results in the relative case.

**Theorem 2.10.** Let \(f : Y \to X\) be a proper and smooth morphism of \(p\)-adic formal schemes over \(\text{Spf}(A/I)\) of relative dimension \(n\), and \(f_{\Delta} : (Y/A)_{\Delta} \to (X/A)_{\Delta}\) be the associated morphism of topoi. Let \(\mathcal{F}\) be an \(\mathcal{O}_{\Delta}\)-crystal locally free of finite rank on \((Y/A)_{\Delta}\). Then \(Rf_{\Delta,*}(\mathcal{F})\) is a perfect complex of \(\mathcal{O}_{\Delta}\)-crystals on \((X/A)_{\Delta}\) with tor-amplitude in degree \([0, 2n]\) in the following sense:
• for an object \((\text{Spf}(B) \leftarrow \text{Spf}(B/IB) \to X)\) in \((X/A)_\Delta\), its evaluation \(Rf_{\Delta,*}(\mathcal{F})_B\) is a perfect complex of \(B\)-module with tor-amplitude in degree \([0, 2n]\);
• for any morphism \(\alpha : (\text{Spf}(C) \leftarrow \text{Spf}(C/IC) \to X) \to (\text{Spf}(B) \leftarrow \text{Spf}(B/IB) \to X)\) in \((X/A)_\Delta\), the natural base change map
\[
(Rf_{\Delta,*}(\mathcal{F})_B \otimes_B C)^\wedge \to Rf_{\Delta,*}(\mathcal{F})_C
\]
is an isomorphism.

**Proof.** For an object \((\text{Spf}(B) \leftarrow \text{Spf}(B/IB) \to X)\), we put \(Y_B = Y \times_X \text{Spf}(B/IB)\). Then it follows from \([8, \text{Lemma } 2.17]\) that \(Rf_{\Delta,*}(\mathcal{F})_B \cong R\Gamma((Y_B/B)_\Delta, \mathcal{F}|_{(Y_B/B)_\Delta})\). This theorem follows immediately from Theorem \([2.9]\). \(\square\)

### 3. Local description of prismatic crystals

In this section, we fix a bounded prism \((A, I)\). Let \(X = \text{Spf}(R)\) be an affine smooth \(p\)-adic formal scheme over \(\text{Spf}(A/I)\) of relative dimension \(n \geq 0\). We will make the following assumption:

**Assumption 3.1.** The \(A/I\)-algebra \(R\) admits a lift to a derived \((p, I)\)-complete \(\delta\)-\(A\)-algebra \(\widetilde{R}\) that is formally smooth over \(A\).

We fix such a lift \(\widetilde{R}\), and let \(\widetilde{X} := (\widetilde{R} \rightarrow \widetilde{R}/I\widetilde{R} \cong R)\) denote the corresponding object in \((X/A)_\Delta\).

**Lemma 3.2.** Under Assumption \([3.1]\), for any object \((B \rightarrow B/IB \leftarrow R)\) in \((X/A)_\Delta\), the product of \((B \rightarrow B/IB \leftarrow R)\) and \(\widetilde{X}\) in \((X/A)_\Delta\) exists. Moreover, if we denote this product by \((\widetilde{B} \rightarrow \widetilde{B}/I\widetilde{B} \leftarrow R)\), then \(\widetilde{B}\) is \((p, I)\)-completely faithfully flat over \(B\). In particular, \(\widetilde{X}\) is a cover of the final object of \((X/A)^\wedge_\Delta\).

**Proof.** Let \(C := (B \otimes_A^L \tilde{R})^\wedge\) be the derived \((p, I)\)-completion of \(B \otimes_A^L \tilde{R}\). Then \(C\) is \((p, I)\)-completely faithfully flat over \(B\). Hence by Proposition \([1.3.2]\), it is concentrated in degree 0 and coincides with the classical \((p, I)\)-adic completion of \(B \otimes_A \tilde{R}\). By \([8, \text{Lemma } 2.17]\), there is a unique \(\delta\)-structure on \(C\) compatible with the natural product \(\delta\)-structure on \(B \otimes_A \tilde{R}\). Consider the surjection \(C \rightarrow B/IB \otimes_{A/I} R \rightarrow B/IB\), and denote its kernel by \(J\). As \(R\) is formally smooth over \(A/I\) of relative dimension \(n\), \(J\) is locally generated by \(I\) and a regular sequence of length \(n\) relative to \(B\).

Applying \([8, \text{Lemma } 3.13]\), we get the prismatic envelope \(\tilde{B} := C\{\frac{1}{I}\}^\wedge\) which is \((p, I)\)-completely faithfully flat over \(B\), and commutes with base change in \((B, IB)\). The fact that \((\tilde{B} \rightarrow \tilde{B}/IB \leftarrow R)\) is the product of \((B \rightarrow B/IB \leftarrow R)\) and \(\tilde{X}\) in \((X/A)_\Delta\) follows easily from the universal property of the prismatic envelope.

The second part of the Lemma follows immediately from the following general fact: if \(\mathcal{C}\) is a topos, an object \(U \in \mathcal{C}\) is a cover of the final object of \(\mathcal{C}\) if and only if for any object \(V \in \mathcal{C}\) there exists a cover \(W \rightarrow V\) such that \(W\) admits a morphism to \(U\). \(\square\)

#### 3.3. Simplicial object.

For each integer \(m \geq 0\), let \(\tilde{X}(m)\) be the \((m + 1)\)-fold self-product of the object \(\tilde{X}\) in \((X/A)_\Delta\). By the proof of Lemma \([3.2]\), \(\tilde{X}(m)\) is explicitly given by as follows. Let \(\tilde{J}(m)\) be the kernel of the canonical surjection
\[
\tilde{R}^{\otimes (m+1)} := \underbrace{\tilde{R} \otimes_A \ldots \otimes_A \tilde{R}}_{(m+1)\text{ copies}} \rightarrow R.
\]

Since \(R\) is formally smooth over \(A/I\), \(\tilde{J}(m)\) is locally generated by \(I\) together with a \((p, I)\)-completely regular sequence relative to \(A\). We let
\[
\tilde{R}(m) := \tilde{R}^{\otimes (m+1)}\{\frac{\tilde{J}(m)}{I}\}^\wedge
\]
be the prismatic envelope by the construction of \[3\] Lemma 3.13, and put \( R(m) := \tilde{R}(m)/I\tilde{R}(m) \). Then we have \( \tilde{X}(m) = (\tilde{R}(m) \to \tilde{R}(m)/I \tilde{R} \leftarrow R) \). The \((\tilde{R}(m), I\tilde{R}(m))\)'s form naturally a cosimplicial object of bounded prisms \( (\tilde{R}(\bullet), I\tilde{R}(\bullet)) \) over \((A, I)\). For an integer \( i \) with \( 0 \leq i \leq m \), let

\[
\delta^m_i : \tilde{R}(m-1) \to \tilde{R}(m)
\]

denote the map of \( \delta\text{-}A\)-algebras corresponding to the strictly increasing map of simplices \([m-1] := \{0, 1, \ldots, m-1\} \to [m] \) that skips \( i \), i.e. \( \delta^m_i \) is induced by the map \( \tilde{R}^\otimes_{m} \to \tilde{R}^\otimes_{(m+1)} \) given by

\[
b_0 \otimes \cdots \otimes b_{m-1} \mapsto b_0 \otimes \cdots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \cdots \otimes b_{m-1}.
\]

In particular, we have a diagram of \( \delta\text{-}A\)-algebras:

\[
\begin{array}{ccc}
R & \xrightarrow{\delta^1_0} & \tilde{R}(1) \\
\delta^1_1 & & \xrightarrow{\delta^2_1} \tilde{R}(2)
\end{array}
\]

Let \( \mu : \tilde{R}(1) \to \tilde{R} \) denote the diagonal surjection induced by \( \tilde{R} \otimes_A \tilde{R} \to \tilde{R} \) given as \( a \otimes b \mapsto ab \).

**Remark 3.4.** Note that, for each integer \( m \geq 1 \), one has an isomorphism of objects in \((X/A)_{\dagger}\):

\[
\tilde{X}(m) \cong \tilde{X}(1) \times_{\tilde{X}} \cdots \times_{\tilde{X}} \tilde{X}(1),
\]

where each \( \tilde{X}(1) = \tilde{X} \times_{(X/A)_{\dagger}} \tilde{X} \) maps to \( \tilde{X} \) via the first projection. This implies that

\[
\tilde{R}(m) \cong \tilde{R}(1) \otimes_{\tilde{R}} \tilde{R}(1) \otimes_{\tilde{R}} \cdots \otimes_{\tilde{R}} \tilde{R}(1)
\]

where each \( \tilde{R}(1) \) is viewed as an \( \tilde{R} \)-algebra via \( \delta^1_1 \).

**Definition 3.5.** Let \( M \) be a derived \((p, I)\)-complete and \((p, I)\)-completely flat \( \tilde{R} \)-module. A stratification of \( M \) over \( \tilde{R}(1) \) is an isomorphism of \( \tilde{R}(1) \)-modules:

\[
\epsilon : \delta^1_{\ast}^{\dagger}(M) \xrightarrow{\sim} \delta^1_{\ast}^{\dagger}(M)
\]

where \( \delta^1_{\ast}^{\dagger} \) is the base change functor \([1,5.2] \), such that the cocycle condition is satisfied:

\[
\mu^*(\epsilon) = \text{id}_M, \quad \delta^2_{\ast}^{\dagger}(\epsilon) = \delta^2_{\ast}^{\dagger}(\epsilon) \circ \delta^2_{\ast}^{\dagger}(\epsilon).
\]

We denote by \( \text{Strat}(\tilde{R}, \tilde{R}(1)) \) the category of derived \((p, I)\)-complete and \((p, I)\)-completely flat \( \tilde{R} \)-modules together with a stratification over \( \tilde{R}(1) \).

**Remark 3.6.** The condition \( \mu^*(\epsilon) = \text{id}_M \) is actually redundant, but we choose to keep it for convenience. Indeed, by applying the functor \((\text{id} \otimes \mu)_{\ast} \) to the cocycle condition \( \delta^2_{\ast}^{\dagger}(\epsilon) = \delta^2_{\ast}^{\dagger}(\epsilon) \circ \delta^2_{\ast}^{\dagger}(\epsilon) \), one gets

\[
\epsilon = \epsilon \circ \delta^1_{\ast}^{\dagger}(\mu^*(\epsilon))
\]

and hence \( \delta^1_{\ast}^{\dagger}(\mu^*(\epsilon)) = \text{id}_{\delta^1_{\ast}^{\dagger}(M)} \). As \( \delta^1_0 : \tilde{R} \to \tilde{R}(1) \) is \((p, I)\)-completely faithfully flat, one deduces from Proposition \([1,9] \) that \( \mu^*(\epsilon) = \text{id}_M \).

We have also a variant of Definition 3.3 and Proposition 3.8 for \( \overline{O}_{\Delta} \)-crystals. In general, if \( f \) is morphism of objects over \( A \) (e.g. \( A\)-modules, \( A\)-algebras or \( A\)-formal schemes, ...), we denote by \( \tilde{f} : B/I\overline{B} \to C/I\overline{C} \) its reduction modulo \( I \).
Definition 3.7. Let $M$ be a derived $p$-complete and $p$-completely flat $R$-module. A stratification of $M$ over $R(1) = \tilde{R}(1)/\mathcal{I}\tilde{R}(1)$ is an isomorphism of $R(1)$-modules

$$\epsilon : \delta^{\delta M}_0(M) \xrightarrow{\sim} \delta^{\delta M}_1(M)$$

such that $\mu^*(\epsilon) = \text{id}_M$ and $\delta^{\delta M}_2(\epsilon) = \delta^{\delta M}_2(\epsilon) \circ \delta^{\delta M}_0(\epsilon)$.

We denote by $\text{Strat}(R, R(1))$ the category of derived $p$-complete and $p$-completely flat $R$-modules together with a stratification over $R(1)$.

Let $\mathcal{F}$ be an $\mathcal{O}_\Delta$-crystal on $(X/A)_\Delta$, and $\mathcal{F}(\tilde{X})$ be its value on $\tilde{X}$. The crystal condition gives rise to a canonical isomorphism

$$\epsilon_\mathcal{F} : \delta^{\delta \mathcal{F}}_0(\mathcal{F}(\tilde{X})) \xrightarrow{\epsilon^{\delta \mathcal{F}}(\mathcal{F})} \mathcal{F}(\tilde{X}(1)) \xrightarrow{\epsilon^{\delta \mathcal{F}}^{-1}(\mathcal{F})} \delta^{\delta \mathcal{F}}_1(\mathcal{F}(\tilde{X}))$$

which makes $(\mathcal{F}(\tilde{X}), \epsilon_\mathcal{F})$ an object of $\text{Strat}(\tilde{R}(1))$. We get thus an evaluation functor $\text{ev}_{\tilde{X}} : \text{CR}((X/A)_\Delta, \mathcal{O}_\Delta) \rightarrow \text{Strat}(\tilde{R}, \tilde{R}(1))$ sending $\mathcal{F}$ to $(\mathcal{F}(\tilde{X}), \epsilon_\mathcal{F})$. Similarly, we have also an evaluation functor $\text{ev}_{\tilde{X}} : \text{CR}((X/A)_\Delta, \overline{\mathcal{O}}_\Delta) \rightarrow \text{Strat}(R, R(1))$. for $\overline{\mathcal{O}}_\Delta$-crystals.

Proposition 3.8. Under the notation above, the functors $\text{ev}_{\tilde{X}}$ and $\text{ev}_{\tilde{X}}$ are both equivalences of categories.

Proof. We will prove only the statement for $\text{ev}_{\tilde{X}}$, and the case for $\text{ev}_{\tilde{X}}$ is similar. We shall construct a functor quasi-inverse to $\text{ev}_{\tilde{X}}$. Let $(M, \epsilon)$ be an object of $\text{Strat}(R, R(1))$. We need to associate an $\mathcal{O}_\Delta$-crystal $M_\Delta$ to $(M, \epsilon)$.

Let $(B \rightarrow B/IB \leftarrow R)$ be an object in $(X/A)_\Delta$, and $(\tilde{B} \rightarrow \tilde{B}/\tilde{I}\tilde{B} \leftarrow R)$ be the product of $(B \rightarrow B/IB \leftarrow R)$ and $\tilde{X}$ given by Lemma 3.2. By the universal property of prismatic envelopes, the canonical map of bounded prisms $p_B : (\tilde{R}, \tilde{R}) \rightarrow (\tilde{B}, \tilde{B})$ induces a commutative diagram of $2$-truncated cosimplicial $\delta$-$A$-algebras:

$$\begin{array}{ccc}
\tilde{R} & \xrightarrow{\epsilon} & \tilde{R}(1) \\
\downarrow{p_B} & & \downarrow{p_B(1)} \\
\tilde{B} & \xrightarrow{\epsilon} & \tilde{B} \otimes_B \tilde{B}
\end{array}$$

Applying the functor $p_B^*$ to $(M, \epsilon)$, one gets a descent pair (Def. 1.8) $(p_B^*M, p_B^*(1)\epsilon)$ relative to the $(p, I)$-completely faithfully flat map $B \rightarrow \tilde{B}$ such that $p_B^*M$ is $(p, I)$-completely flat over $\tilde{B}$. By Proposition 1.9(2), there exists a derived $(p, IB)$-complete and $(p, IB)$-completely flat $B$-module $M_B$ such that $M_B \otimes_B \tilde{B} \simeq p_B^*(M)$.

Let $f : (C \rightarrow C/IC \leftarrow R) \rightarrow (B \rightarrow B/IB \leftarrow R)$ be a morphism in $(X/A)_\Delta$, and $(\tilde{C} \rightarrow \tilde{C}/\tilde{I}\tilde{C} \leftarrow R)$ be the product of $(C \rightarrow C/IC \leftarrow R)$ with $\tilde{X}$. We need to check that there exists a transition isomorphism

$$c_f : f^*(M_B) = M_B \otimes_B C \xrightarrow{\sim} M_C$$

satisfying the natural cocycle condition for a composition of morphisms as in Lemma 2.4. Denote by $\tilde{f} : (\tilde{B}, \tilde{IB}) \rightarrow (\tilde{C}, \tilde{IC})$ the map of bounded prisms induced by $f$. Then one has $p_C = \tilde{f} \circ p_B$. By functoriality, one has a composition of isomorphisms

$$\tilde{c}_f : M_B \otimes_B C \otimes_C \tilde{C} = M_B \otimes_B \tilde{B} \otimes_B \tilde{C} \simeq \tilde{f}^*p_B^*(M) = p_C^*(M) \simeq M_C \otimes_C \tilde{C};$$
Lemma 3.12. Let \( n \) where the \( \Gamma((X/A)_{\Delta}) \). Then under Assumption 3.1, \( f \circ g : (D \to D/ID \leftarrow R) \xrightarrow{g} (C \to C/IC \leftarrow R) \xrightarrow{f} (B \to B/IB \leftarrow R) \) in \( (X/A)_{\Delta} \), one has the cocycle condition \( c_{fg} = c_g \circ g^*(c_f) \). Indeed, this can be easily checked after base change to \( \tilde{D} \) by functoriality, and we conclude by faithfully flat descent (Proposition 1.9).

Now by Lemma 2.4 the data \( \{ M_B \}, \{ c_f \} \) is equivalent to an \( \mathcal{O}_{\Delta} \)-crystal \( M_{\bar{\Delta}} \) on \( (X/A)_{\Delta} \). The construction \( (M, \epsilon) \mapsto M_{\bar{\Delta}} \) is clearly functorial, which gives a functor that is easily checked to be a quasi-inverse of \( \text{ev}_{\tilde{\chi}} \).}

\[ \square \]

Remark 3.9. When \((A, I)\) is a bounded prism over \((\mathbb{Z}_q[[q-1]], [p])\) (with \( \delta(q) = 0 \) and \( [p]_{q^{-1}} = q^{-1} \)), Prop. 3.8 was obtained by [10, Cor. 6.7] and [12, Chap. 3]. An analogue for \( q \)-crystalline crystals was proved in [9, Thm. 1.3.3].

We can use the simplicial object \( \tilde{\chi}(\bullet) \) to compute the cohomology of an \( \mathcal{O}_{\Delta} \)-crystals or an \( \mathcal{O}_{\Delta} \)-crystal. For an abelian sheaf \( \mathcal{F} \) on \((X/A)_{\Delta} \), the Čech–Alexander complex \( \hat{\mathcal{C}}A(\tilde{\chi}(\bullet), \mathcal{F}) \) is defined as the simple complex associated to the cosimplicial abelian group \( \mathcal{F}(\tilde{\chi}(\bullet)) \), i.e. one has

\[ \hat{\mathcal{C}}A(\tilde{\chi}(\bullet), \mathcal{F}) = \left( \mathcal{F}(\tilde{\chi}) \to \cdots \to \mathcal{F}(\tilde{\chi}(n)) \xrightarrow{d_n^{n+1}} \mathcal{F}(\tilde{\chi}(n+1)) \to \cdots \right) \]

with \( d^n = \sum_{i=0}^{n+1} (-1)^i \text{pr}^{n+1}_i \mathcal{F}(\tilde{\chi}(n)) \to \mathcal{F}(\tilde{\chi}(n+1)) \) is the map induced by the projection \( \text{pr}^{n+1}_i : \tilde{\chi}(n+1) \to \tilde{\chi}(n) \) that corresponds to \( \delta_i^{n+1} : R(n) \to R(n+1) \). If \( \mathcal{F} \) is an \( \mathcal{O}_{\Delta} \)-crystal, the projection \( \text{pr}^0 : \tilde{\chi}(n) \to \tilde{\chi} \) to the 0-th copy induces an isomorphism

\[ c_{\text{pr}^0} : \mathcal{F}(\tilde{\chi}) \otimes_{\tilde{R}} \tilde{R}(n) \cong \mathcal{F}(\tilde{\chi}(n)). \]

Then \( \hat{\mathcal{C}}A(\tilde{\chi}(\bullet), \mathcal{F}) \) is isomorphic to

\[ \mathcal{F}(\tilde{\chi}) \to \mathcal{F}(\tilde{\chi}) \otimes_{\tilde{R}} \tilde{R}(1) \to \mathcal{F}(\tilde{\chi}) \otimes_{\tilde{R}} \tilde{R}(2) \to \cdots \]

with the \( n \)-th differential given by

\[ d^n = \sum_{i=0}^{n+1} (-1)^i \text{pr}^{n+1}_i \circ \text{pr}^{n+1}_i \circ c_{\text{pr}^0} : \mathcal{F}(\tilde{\chi}) \otimes_{\tilde{R}} \tilde{R}(n) \to \mathcal{F}(\tilde{\chi}) \otimes_{\tilde{R}} \tilde{R}(n+1). \]

Proposition 3.10 (Čech–Alexander complex). Let \( \mathcal{F} \) be an \( \mathcal{O}_{\Delta} \)-crystal or an \( \mathcal{O}_{\Delta} \)-crystal on \((X/A)_{\Delta} \). Then under Assumption 3.7 \( R(\tilde{\chi}(X/A)_{\Delta}, \mathcal{F}) \) is computed by the Čech–Alexander complex \( \hat{\mathcal{C}}A(\tilde{\chi}(\bullet), \mathcal{F}) \).

Remark 3.11. If \( \mathcal{F} \) is an \( \mathcal{O}_{\Delta} \)-crystal, its Čech–Alexander complex \( \hat{\mathcal{C}}A(\tilde{\chi}(\bullet), \mathcal{F}) \) is isomorphic to

\[ \mathcal{F}(\tilde{\chi}) \to \mathcal{F}(\tilde{\chi}) \otimes_{\tilde{R}} R(1) \to \mathcal{F}(\tilde{\chi}) \otimes_{\tilde{R}} R(2) \to \cdots \]

where the \( n \)-th differential is given by a similar formula as above.

For the proof of this proposition, we need the following

Lemma 3.12. Let \( \mathcal{F} \) be an \( \mathcal{O}_{\Delta} \)-crystal or \( \mathcal{O}_{\Delta} \)-crystal on \((X/A)_{\Delta} \). Then for any object \( U = (B \to B/IB \leftarrow R) \) of \((X/A)_{\Delta} \) and any integer \( q > 0 \), we have

\[ H^q(U, \mathcal{F}) = 0. \]
By Lemma 3.12, we have \( H \) we conclude by [3, Exposé V, Cor. 4.3] that its higher cohomology groups vanish. Then we conclude by \([3\), Exposé V, Cor. 4.3\] that \( H^q(U, F) = 0 \) for all \( q > 0 \).

Proof of Prop. 3.10. Since \( \tilde{X} \) is a cover of the final object of the topos \( (X/A)_{\tilde{\Delta}} \) (Lemma 3.2), we have a spectral sequence

\[
E_1^{i,j} = H^j(\tilde{X}(i), F) \Rightarrow H^{i+j}((X/A)_{\tilde{\Delta}}, F).
\]

By Lemma 3.12, we have \( H^j(\tilde{X}(i), F) = 0 \) for \( j > 0 \). Hence \( R\Gamma((X/A)_{\tilde{\Delta}}, F) \) is computed by the Čech-Alexandre complex \( \Gamma(\tilde{X}(i), F) \), which is isomorphic to \( \text{(3.11.1)} \) by the crystal property of \( F \).

Corollary 3.13 (weak base change). Let \( X \) be a smooth \( p \)-adic formal scheme over \( \text{Spf}(A/I) \) (without assuming Assumption 3.7). Let \( F \) be an \( \mathcal{O}_{\tilde{\Delta}} \)-crystal (resp. an \( \mathcal{O}_{\tilde{\Delta}} \)-crystal) on \( (X/A)_{\tilde{\Delta}} \). Let \( (A, I) \rightarrow (A', I') \) be a morphism of bounded prisms of finite tor-dimension, \( X' = X \times_{\text{Spf}(A/I)} \text{Spf}(A'/I') \) and \( F' \) be the pullback of \( F \) to \( (X'/A')_{\tilde{\Delta}} \). Then the natural completed base change map

\[
(\nu_{X/A,*}(F) \otimes_A^n A')^{\wedge} \xrightarrow{\sim} \nu_{X'/A',*}(F')
\]

(resp. \( (\nu_{X/A,*}(F) \otimes_{A/I}^L A'/I')^{\wedge} \xrightarrow{\sim} \nu_{X'/A',*}(F') \))

is an isomorphism.

Proof. The problem is clearly local for the étale topology of \( X \). Up to étale localization, we may impose thus Assumption 3.1. In this case, the statement follows immediately from Prop. 3.10 and the fact that the formation of Čech–Alexander complex \( \text{(3.11.1)} \) commutes with the base change \( A \rightarrow A' \).

Remark 3.14. After establishing Theorem 4.14, we will see that Corollary 3.13 holds without the assumption that \( A \rightarrow A' \) is of finite tor-dimension (cf. Theorem 2.8(2)).

4. Prismatic Crystals and Higgs Fields

In this section, we keep the notation of Section 3. We will restrict ourselves to the following special case of Assumption 3.1.

Situation 4.1. We assume that \( X = \text{Spf}(R) \) admits an \((p, I)\)-completely étale map to \( \text{Spf}(A/I(T_n)) \), where \( A/I(T_n) := A/I(T_1, \ldots, T_n) \) denotes the convergent power series ring over \( A/I \) in \( n \) variables (for the \( p \)-adic topology). Then by deformation theory, there exists a unique derived \((p, I)\)-complete and \((p, I)\)-completely étale \( A(T_n) \)-algebra \( \tilde{R} \) which makes the following diagram cocartesian:

\[
\begin{array}{ccc}
\tilde{R} & \rightarrow & R \\
\uparrow & & \uparrow \\
A(T_n) & \rightarrow & A/I(T_n)
\end{array}
\]

We choose a \( \delta \)-structure on \( A(T_n) \) extending that on \( A \). Then by \([8\), Lemma 2.18\], it extends uniquely to a \( \delta \)-structure on \( \tilde{R} \). In particular, Assumption 3.1 is satisfied. For technical reasons, we suppose that one of the following assumptions is satisfied:

1. \((A, I)\) is a crystalline prism, i.e. \( I = (p) \);
2. there exists a map of bounded prisms \( (A_0, I_0) \rightarrow (A, I) \) such that
   - the Frobenius map on \( A_0/p \) is flat.
• $A_0/I_0$ is $p$-torsion free;
• the $\delta$-structure on $A(T_n)$ descends to a $\delta$-structure on $A_0(T_n)$.

For instance, if $I = (d)$ is principal and we take the $\delta$-structure with $\delta(T_i) = 0$, then assumption (2) is satisfied with $(A_0, I_0) = (\mathbb{Z}_p \{ d, \delta(d)^{-1} \}, (d))$, the $(d, p)$-completed universal oriented prism.

We will give an explicit description of $\overline{\mathcal{O}}_A$-crystals on $(X/A)_A$ in terms of Higgs fields, and compute the cohomology of $\overline{\mathcal{O}}_A$-crystals via the de Rham cohomology of its associated Higgs field. Recall the cosimplicial object $\tilde{R}(\bullet)$ defined in (3.3). We give first a more transparent description of $\tilde{R}(1)$. Consider the canonical diagonal surjection

$$A(T_n) \otimes_A A(T_n) \to A/I(T_n),$$

and denote its kernel by $J_n$. Note that we have an isomorphism

$$A(T_n) \otimes_A A(T_n) \simeq A(T_n)/\langle \xi_1, \ldots, \xi_n \rangle,$$

given by $T_i \otimes 1 \mapsto T_i$ and $1 \otimes T_i \mapsto T_i + \xi_i$ so that $\xi_i$ corresponds to $1 \otimes T_i - T_i \otimes 1$ for $1 \leq i \leq n$. Via this isomorphism, the ideal $J_n$ corresponds to $(I, \xi_1, \ldots, \xi_n)$. Recall that we have chosen a $\delta$-structure on $A(T_n)$ compatible with that on $A$. We equip $A(T_n)/\langle \xi_1, \ldots, \xi_n \rangle$ with the $\delta$-structure that corresponds to the canonical induced tensor $\delta$-structure on $A(T_n) \otimes _A A(T_n)$: explicitly, if $\delta(T_i) = f_i(T) \in A(T_n)$, we have

$$(\delta(T_i))^p = \sum_{j=1}^{p-1} \frac{1}{p} \binom{p}{j} \xi_i^j T_i^{p-j} + f_i(T + \xi_i) - f_i(T) \in \langle \xi_1, \ldots, \xi_n \rangle$$

Applying the construction of [8, Lemma 3.13], we get the prismatic envelope

$$(A(T_n) \otimes_A A(T_n))/J_n \simeq A(T_n)/\langle \hat{\xi}_1, \ldots, \hat{\xi}_n \rangle,$$

which is $(p, I)$-completely faithfully flat over $A(T_n)$. The formation of $A(T_n)/\langle \hat{\xi}_1, \ldots, \hat{\xi}_n \rangle$ commutes with base change in $(A(T_n), IA(T_n))$ in the following sense: If $(C, IC)$ is a bounded prism over $(A(T_n), IA(T_n))$ and we extend the $\delta$-structure on $C$ to $C(\xi_1, \ldots, \xi_n)$ with $\delta(\xi_i)$ given by the image of $(4.1.1)$ in $C(\xi_1, \ldots, \xi_n)$, then

$$C(\hat{\xi}_1, \ldots, \hat{\xi}_n) := \hat{C}(\hat{\xi}_1, \ldots, \hat{\xi}_n) \simeq \hat{C}(\hat{\xi}_1, \ldots, \hat{\xi}_n)$$

is nothing but the prismatic envelope of $C(\xi_1, \ldots, \xi_n)$ with respect to the ideal $(I, \xi_1, \ldots, \xi_n)$. Note that the surjection $C(\xi_1, \cdots, \xi_n) \to C$ sending all $\xi_i$ to $0$ induces a canonical surjection of bounded prisms:

$$(4.1.3) \quad C(\hat{\xi}_1, \ldots, \hat{\xi}_n) \to C.$$

By (4.1.2) and the functoriality of prismatic envelope, there exists a canonical map of $\delta$-$A$-algebras

$$(4.1.4) \quad A(T_n)/\langle \hat{\xi}_1, \ldots, \hat{\xi}_n \rangle \to \tilde{R}(1).$$

Recall the morphism $\delta^1_1 : \tilde{R} \to \tilde{R}$ induced by $\tilde{R} \to \tilde{R} \otimes_A \tilde{R}$: $b \mapsto b \otimes 1$ (see (3.3.1)). Taking the tensor product of $\delta^1_1$ with (4.1.4), one gets a map

$$\eta : \tilde{R}(\xi_1, \ldots, \xi_n) = \tilde{R} \otimes_A A(T_n)/\langle \xi_1, \ldots, \xi_n \rangle \to \tilde{R}(1)$$

which induces a map of prisms over $(A, I)$. 18
Lemma 4.2. The morphism \( \eta \) induces an isomorphisms of prisms
\[
(\tilde{R}\{\frac{\xi_1}{T}, \ldots, \frac{\xi_n}{T}\}, I\tilde{R}\{\frac{\xi_1}{T}, \ldots, \frac{\xi_n}{T}\}) \overset{\sim}{\rightarrow} (\tilde{R}(1), I\tilde{R}(1)).
\]

Proof. We need to construct an inverse to \( \eta \). Consider the following diagram
\[\begin{align*}
A(T_n) \ar[r]^-{i_2} & \tilde{R} \\
T_i \ar[r]_-{\delta_i} & \tilde{R}\{\frac{\xi_1}{T}, \ldots, \frac{\xi_n}{T}\} \ar[r] & \tilde{R}\{\frac{\xi_1}{T}, \ldots, \frac{\xi_n}{T}\}/I\tilde{R}\{\frac{\xi_1}{T}, \ldots, \frac{\xi_n}{T}\},
\end{align*}\]
where the right vertical map is the composed canonical map:
\[
\tilde{R} \to \tilde{R}/I\tilde{R} \to \tilde{R}\{\frac{\xi_1}{T}, \ldots, \frac{\xi_n}{T}\}/I\tilde{R}\{\frac{\xi_1}{T}, \ldots, \frac{\xi_n}{T}\}.
\]
It is clear that the square of the above diagram is commutative. Since \( A(T_n) \to \tilde{R} \) is \((p, I)\)-completely \( \acute{e} \)tale and \( \tilde{R}\{\frac{\xi_1}{T}, \ldots, \frac{\xi_n}{T}\} \) is \( I \)-adically complete, there exists a unique map \( i_2 : \tilde{R} \to \tilde{R}\{\frac{\xi_1}{T}, \ldots, \frac{\xi_n}{T}\} \) as the dotted arrow that makes all triangles in the diagram commute. Moreover, since the left vertical arrow is a map of \( \delta \)-\( A \)-algebras and the \( \delta \)-structure on \( \tilde{R} \) is uniquely determined by its restriction to \( A(T_n) \) (cf. [8 Lemma 2.18]), it follows that \( i_2 \) is a map of \( \delta \)-\( A \)-algebras. Consider the morphism
\[
f : \tilde{R} \otimes_A \tilde{R} \to \tilde{R}\{\frac{\xi_1}{T}, \ldots, \frac{\xi_n}{T}\}
\]
given by the tensor product of the natural inclusion \( i_1 : \tilde{R} \to \tilde{R}\{\frac{\xi_1}{T}, \ldots, \frac{\xi_n}{T}\} \) and \( i_2 \). By the commutativity of the diagram (4.2.1), \( i_1 \) and \( i_2 \) agree after post-composition with the natural surjection
\[
\tilde{R}\{\frac{\xi_1}{T}, \ldots, \frac{\xi_n}{T}\} \to \tilde{R}\{\frac{\xi_1}{T}, \ldots, \frac{\xi_n}{T}\}/I\tilde{R}\{\frac{\xi_1}{T}, \ldots, \frac{\xi_n}{T}\}.
\]
Therefore, the composed map
\[
\tilde{R} \otimes_A \tilde{R} \to \tilde{R}\{\frac{\xi_1}{T}, \ldots, \frac{\xi_n}{T}\} \to \tilde{R}\{\frac{\xi_1}{T}, \ldots, \frac{\xi_n}{T}\}/I\tilde{R}\{\frac{\xi_1}{T}, \ldots, \frac{\xi_n}{T}\}
\]
factors though \( \tilde{R} \otimes_A \tilde{R} \to \tilde{R} \), i.e. \( f \) sends \( \tilde{J}(1) \subset \tilde{R} \otimes_A \tilde{R} \) to \( I\tilde{R}\{\frac{\xi_1}{T}, \ldots, \frac{\xi_n}{T}\} \). By the universal property of the prismatic envelope, it induces a morphism of prisms over \( (A, I) \):
\[
\eta' : (\tilde{R}(1), I\tilde{R}(1)) \to (\tilde{R}\{\frac{\xi_1}{T}, \ldots, \frac{\xi_n}{T}\}, I\tilde{R}\{\frac{\xi_1}{T}, \ldots, \frac{\xi_n}{T}\}).
\]
It remains to check that \( \eta \) and \( \eta' \) are inverse of each other. If we regard \( \tilde{R}\{\frac{\xi_1}{T}, \ldots, \frac{\xi_n}{T}\} \) and \( \tilde{R}(1) \) as \( \tilde{R} \)-algebras via \( i_1 \) and \( \delta_1 \) respectively, then both \( \eta \) and \( \eta' \) are maps of \( \tilde{R} \)-algebras. In order to show that \( \eta' \circ \eta = \text{id} \), it suffices to see that \( \eta' \circ \eta(\xi_i) = \xi_i \). For this, one can reduce to the case \( \tilde{R} = A(T_n) \), which follows from (4.1.2). To check \( \eta \circ \eta' = \text{id} \), it is enough, by the universal property of \( (\tilde{R}(1), I\tilde{R}(1)) \), to show that \( \eta \circ \eta' \circ \delta_1^i = \delta_1^i \) for \( i = 0, 1 \) where \( \delta_0^1 \circ \delta_1^0 : \tilde{R} \to \tilde{R} \) are the two natural maps defined in (3.3.1). For \( i = 1 \), this is evident. For \( i = 0 \), by noting that \( i_2 = \eta' \circ \delta_0^1 \) by the definition of \( \eta' \), we are reduced to proving \( \eta \circ i_2 = \delta_0^1 \). By the commutative diagram (4.2.1), both sides agree after restricting to \( A(T_n) \) or after reducing modulo \( I \). We conclude by the fact that \( A(T_n) \to \tilde{R} \) is \((p, I)\)-completely \( \acute{e} \)tale. \( \square \)

From now on, we will use Lemma 4.2 to identify \( \tilde{R}(1) \) with \( \tilde{R}\{\frac{\xi_1}{T}, \ldots, \frac{\xi_n}{T}\} \). Let \( \tilde{K} \subset \tilde{R}(1) \) be the kernel of the surjection map
\[
\tilde{R}(1) = \tilde{R}\{\frac{\xi_1}{T}, \ldots, \frac{\xi_n}{T}\} \to \tilde{R},
\]
defined in (4.1.3). The following technical Lemma will play an important role in our later discussion.

**Lemma 4.3.** Let \( \phi \) denote the Frobenius map on \( \tilde{R}(1) \). Then for any \( x \in \tilde{K} \), we have \( \phi(x) \in I\tilde{R}(1) \).

**Proof.** Let \( A' \) be a \((p,I)\)-completely faithfully flat map of \( \delta \)-\( A \)-algebra such that \( IA' \) is principal. Put \( \tilde{R}' = \tilde{R} \otimes_A A' \). Then one has \( \tilde{R}(1) \otimes_A A' \cong \tilde{R}'(1) \). As \( \tilde{R}(1)/I\tilde{R}(1) \to \tilde{R}'(1)/I\tilde{R}'(1) \) is injective, it suffices to show that \( \phi(x) \in I\tilde{R}'(1) \) for all \( x \) in the kernel of the canonical surjection map \( \tilde{R}(1) \to \tilde{R}' \).

Therefore, up to a base change in \((A,I)\), we may assume that \( I \) is generated by a distinguished element \( d \in A \). Then we have

\[
\tilde{R}(1) \cong \tilde{R}\{\frac{x_1}{d}, \ldots, \frac{x_n}{d}\}^\wedge = \left( \tilde{R}\{x_1, \ldots, x_n\} \{X_1, \ldots, X_n\}/(x_i - dX_i : 1 \leqslant i \leqslant n)_{d} \right)^\wedge
\]

where \( \tilde{R}\{x_1, \ldots, x_n\} \{X_1, \ldots, X_n\} \) is the free \( \delta \)-algebra over \( \tilde{R}\{x_1, \ldots, x_n\} \) in \( n \)-variables, and \( (x_i - dX_i : 1 \leqslant i \leqslant n)_{d} \) is the ideal generated by \( \delta^r(x_i - dX_i) \) for all \( r \geqslant 0 \) and \( 1 \leqslant i \leqslant n \). Let \( x_i \) denote the image of \( X_i \) in \( \tilde{R}(1) \). Then \( \tilde{K} \subseteq \tilde{R}(1) \) is the closure of the ideal generated by \( \delta^r(x_i) \) with \( r \geqslant 0 \) and \( 1 \leqslant i \leqslant n \). Therefore, in order to prove the Lemma, it suffices to see \( \phi(\delta^r(x_i)) \in I\tilde{R}(1) \) for all \( r \geqslant 0 \) and \( 1 \leqslant i \leqslant n \).

Let \( J_r \subseteq \tilde{R}(1) \) denote the closed ideal generated by \( d^r \delta^j(x_k) \) with \( 0 \leqslant j \leqslant r \) and \( 1 \leqslant k \leqslant n \). We claim that for every \( r \geqslant 0 \) and \( 1 \leqslant i \leqslant n \), there exists a \( b_{i,r+1} \in \tilde{R}(1) \) such that

\[
\phi(\delta^r(x_i)) = \delta^r(\phi(x_i)) = \delta^r(x_i)^p + p\delta^r+1(x_i) \equiv b_{i,r+1}d^{r+1}\delta^{r+1}(x_i) \mod J_r.
\]

This claim implies immediately \( \phi(\delta^r(x_i)) \in I\tilde{R}(1) \). To show the claim, we proceed by induction on \( r \geqslant 0 \). For \( 1 \leqslant i \leqslant n \), we have

\[
\delta(x_i) = \delta(dx_i) = d^r\delta(x_i) + \phi(x_i)\delta(d).
\]

In view of (4.1.1) and \( \xi_j = dx_j \), we have \( \delta(\xi_i) \in J_0 \). As \( d \) is distinguished, we have \( \delta(d) \in A^\times \) and

\[
\phi(x_i) \equiv -\frac{1}{\delta(d)}\delta^r(dx_i) \mod J_0.
\]

This proves the claim for \( r = 0 \). Suppose now that the statement holds for all integers \( \leqslant r \) so that there exist \( b_{i,j}^{r,k} \in \tilde{R}(1) \) and \( b_{i,r+1} \in \tilde{R}(1)^\times \) such that

\[
\delta^r(\phi(x_i)) = \sum_{j=0}^r \sum_{k=0}^n b_{i,j}^{r,k}d^j\delta^j(x_k) + b_{i,r+1}d^{r+1}\delta^{r+1}(x_k).
\]

Applying \( \delta \), we get

\[
\delta^{r+1}(\phi(x_i)) = \delta \left( \sum_{j=0}^r \sum_{k=0}^n b_{i,j}^{r,k}d^j\delta^j(x_k) + b_{i,r+1}d^{r+1}\delta^{r+1}(x_k) \right)
\]

\[
\equiv \sum_{j=0}^r \sum_{k=0}^n \delta(b_{i,j}^{r,k}d^j\delta^j(x_k)) + \delta(b_{i,r+1}d^{r+1}\delta^{r+1}(x_k)) \mod J_{r+1}.
\]

Note that

\[
\delta(b_{i,j}^{r,k}d^j\delta^j(x_k)) = \delta(b_{i,j}^{r,k}d^j\delta^j(x_k)) + (b_{i,j}^{r,k})^pd^{r+1}\delta^{r+1}(x_k),
\]

and

\[
\delta(\delta^j(x_k)) = \delta^j(\phi(x_k)) \in J_r \subseteq J_{r+1} \text{ by the induction hypothesis. We have also}
\]

\[
\delta(b_{i,r+1}d^{r+1}\delta^{r+1}(x_k)) = \delta(b_{i,r+1}d^{r+1}\delta^{r+1}(x_k)) + b_{i,r+1}^pd^{r+2}\delta^{r+2}(x_k).
\]

It follows that

\[
(1 - \delta(b_{i,r+1}d^{r+1}))\delta^{r+1}(\phi(x_i)) \equiv b_{i,r+1}^pd^{r+2}\delta^{r+2}(x_k) \mod J_{r+1}.
\]
Now by induction on \( m \geq 1 \), it is easy to see that \( \delta(bd^m) \in (dp, p)^{m-1} \) for any \( b \in \bar{R}(1) \). It follows that \( \delta(b_i r_{+1} d^p r_{+1}) \) is topologically nilpotent and \( 1 - \delta(b_i r_{+1} d^p r_{+1}) \) is invertible as \( \bar{R}(1) \) is \((p,d)\)-complete. We conclude that

\[
\delta^{r+1}(\phi(x_i)) \equiv b_i r_{+2} d^{p+2} \delta^{r+2}(x_i) \mod \mathcal{J}_{r+1}
\]

with \( b_i r_{+2} = (1 - \delta(b_i r_{+1} d^p r_{+1}))^{-1} b_i r_{+1} \in \bar{R}(1)^\times \). This finishes the induction process and hence the proof of Lemma 4.3 is finished. \( \square \)

4.4. A divided power algebra. Recall that \( \bar{J}(1) \) is the kernel of the canonical diagonal surjection \( \bar{R} \otimes_A \bar{R} \to \bar{R} \). Since \( \bar{R} \) is \((p, I)\)-completely étale over \( A(T_n)_0 \), the module of continuous differential 1-forms of \( \bar{R} \) relative to \( A \)

\[
\Omega^1_{\bar{R}/A} := \bar{J}(1)/\bar{J}(1)^2
\]

is free of rank \( n \) over \( \bar{R} \) with basis \( (dT_i : 1 \leq i \leq n) \). Let \( \text{Bl}_\mathcal{J}(\bar{R} \otimes_A \bar{R}) \) denote the blow-up of \( \text{Spec}(\bar{R} \otimes_A \bar{R}) \) along the ideal \( \mathcal{J} := (I, \bar{J}(1)) \). Let \( \tilde{U} \) be the \((p, I)\)-adic completion of the affine open \( \text{Spec}(\bar{R} \otimes_A \bar{R}) \), where the inverse image of \( I \) generates that of \( \mathcal{J} \), and put \( U := \tilde{U} \otimes_A A/I \).

Note that \( \mathcal{J}\tilde{R}(1) \) is generated by \( I \). The universal property of blow-up implies that there exists a morphism of formal schemes \( \text{Spf}(\bar{R}(1)) \to \tilde{U} \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
\text{Spf}(R(1)) & \to & \text{Spf}(\bar{R}(1)) \\
\downarrow & & \downarrow \\
U & \to & \tilde{U} \\
\downarrow & & \downarrow \\
\text{Spf}(R) & \to & \text{Spf}(\bar{R} \otimes \bar{R}).
\end{array}
\]

Here, the bottom horizontal arrow is the diagonal closed embedding defined by the ideal \( \mathcal{J} \), and the upper square is cartesian.

We can give a more transparent description of \( U \). Put \( \Omega^1_R := \Omega^1_{\bar{R}/A} \otimes_A A/I \). In general, for an \( A/I \)-module \( M \) and an integer \( i \geq 0 \), we put \( M\{i\} = M \otimes_{A/I} (I/I^2)^{\otimes i} \) and \( M\{-i\} = M \otimes_{A/I} (I/I^2)^{\vee, \otimes i} \).

Lemma 4.5. Let \( R[J^1{-1}] \) denote the \( p \)-adic completion of the symmetric power algebra of the locally free \( R \)-module \( \Omega^1_R \{ -1 \} \). Then one has

\[
U \cong \text{Spf}(R(J^1{-1}))
\]

Proof. Since \( \mathcal{J} = (I, \bar{J}(1)) \) is locally generated by a regular sequence, \( \mathcal{J}/\mathcal{J}^2 \) is a \( R \)-module locally free of rank \( n + 1 \), and the exceptional divisor of \( \text{Bl}_\mathcal{J}(\bar{R} \otimes \bar{R}) \) is isomorphic to the projective space \( \mathbb{P}(\mathcal{J}/\mathcal{J}^2) := \text{Proj}(R[\mathcal{J}/\mathcal{J}^2]) \) over \( \text{Spec}(R) \), where \( R[\mathcal{J}/\mathcal{J}^2] \) denotes the symmetric power algebra of \( \mathcal{J}/\mathcal{J}^2 \) over \( R \). Then \( U \) is identified with the \( p \)-adic completion of the affine open subscheme of \( \mathbb{P}(\mathcal{J}/\mathcal{J}^2) \) where the linear function given by \( IR \subset \mathcal{J}/\mathcal{J}^2 \) is invertible. Therefore, the coordinate ring of \( U \) is canonically isomorphic to the \( p \)-adic completion of the symmetric power algebra of the \( R \)-module

\[
\text{Hom}_R(IR, \bar{J}(1)/\bar{J}(1)^2 \otimes_A A/I) \cong \Omega^1_R \{ -1 \}.
\]

\( \square \)
Let $\mathcal{I}^+ := \Omega_R^1(-1)R(\Omega_R^1(-1))$ be the kernel of the canonical surjection $R(\Omega_R^1(-1)) \to R$. Let $(R^{PD}(1), K)$ denote the $p$-adic completion of the PD-envelope with respect to $\mathcal{I}^+$, and $K[2] \subset R^{PD}(1)$ be the closed ideal generated by elements $x[n]$ for $x \in K$ and $n \geq 2$. Then the canonical map $\Omega_R^1(-1) \subset \mathcal{I}^+ \to K$ induces an isomorphism

$$\Omega_R^1(-1) \cong K/K[2].$$

As $dT_i \in \Omega_R^1$ is nothing but the image of $\xi_i = 1 \otimes T_i - T_i \otimes 1$, it is convenient to write

$$R(\Omega_R^1(-1)) = R(\frac{\xi_1}{T}, \cdots, \frac{\xi_n}{T}), \quad R^{PD}(1) = R(\frac{\xi_1}{T}, \cdots, \frac{\xi_n}{T})^{PD, \wedge}.$$ 

This notation makes sense, because if $I = (d)$ is principal, then $R(\Omega_R^1(-1))$ and $R^{PD}(1)$ are respectively the convergent power series ring and the $p$-completed divided power polynomial ring in $n$-variables $\{\frac{\xi_i}{d} : 1 \leq i \leq n\}$ over $R$.

4.6. We now compare $R(1) = \tilde{R}(1)/I\tilde{R}(1)$ with $R^{PD}(1)$. Recall that $\tilde{K} \subset \tilde{R}(1)$ is the canonical projection $\tilde{R}(1) \cong \tilde{R}(\frac{\xi_1}{T}, \cdots, \frac{\xi_n}{T})^{\wedge} \to \tilde{R}$. For $x \in \tilde{K}$, we put $\gamma_p(x) := -\frac{\delta(x)}{(p-1)!} \in \tilde{K}$. We claim that

1. $p! \gamma_p(x) = x^p \mod I\tilde{R}(1)$ for $x \in \tilde{K};$
2. $\gamma_p(ax) \equiv a^p \gamma_p(x) \mod I\tilde{R}(1)$ for any $a \in \tilde{R}(1)$ and $x \in \tilde{K};$
3. $\gamma_p(x + y) \equiv \gamma_p(x) + \gamma_p(y) + \sum_{i=1}^{p-1} \frac{1}{p!} \frac{1}{n^i} x^iy^{p-i} \mod I\tilde{R}(1)$ for $x, y \in \tilde{K}.$

Indeed, Lemma 4.3 says that

$$\phi(x) = x^p + p\delta(x) \in I\tilde{R}(1)$$

for all $x \in \tilde{K}$, which implies immediately statement (1). Statement (2) follows from the computation:

$$\gamma_p(ax) = a^p \gamma_p(x) + \gamma_p(a) \phi(x) \equiv a^p \gamma_p(x) \mod I\tilde{R}(1),$$

and (3) is a direct consequence of the additive property of $\delta$.

Let $K'$ denote the image of $\tilde{K}$ in $R(1) = \tilde{R}(1)/I\tilde{R}(1)$. For $\bar{x} \in K'$, we denote by $\gamma_p(\bar{x}) \in K'$ the image of $\gamma_p(x)$ for an lift $x \in \tilde{K}$ of $\bar{x}$. As $\tilde{K} \cap I\tilde{R}(1) = I\tilde{K}$, it is easy to see that $\gamma_p(\bar{x})$ does not depend on the choice of $x$. By the properties of $\gamma_p$ and [EGA4-1], there exists a unique PD-structure on the ideal $K' \subset R(1)$ such that the $p$-th divided power function is given by $\gamma_p : K' \to K'$. Note that the morphism $\text{Spf}(R(1)) \to U$ in [4.4.1] gives a map of $R$-algebras

$$R(\Omega_R^1(-1)) \to R(1)$$

that sends the ideal $\mathcal{I}^+$ to $K'$. By the universal property of the PD-envelope, such a map extends uniquely to a morphism of PD-algebras over $R$

$$\psi : R^{PD}(1) \to R(1)$$

sending $K'$ to $K$.

**Proposition 4.7.** Suppose that we are in Situation 4.4. Then $\psi$ is an isomorphism of PD-algebras over $R$ that identifies $K'$ with $K$. Moreover, via this isomorphism, the two maps $\delta_{1}^{i} : R \to R(1)$ with $i = 0, 1$, which are reduction mod $I$ of (3.3.1), are both identified with the natural inclusion $R \to R^{PD}(1)$.

**Proof.** We have to show that $\psi$ is an isomorphism. It suffices to do this after a $(p, I)$-completely faithfully flat base change in $(A, I)$. As the formation of $\psi$ commutes with such a base change, up to changing notation, we may assume that $I = (d)$ is principal. In this case, we have

$$\tilde{R}(1) \cong \tilde{R}(\frac{\xi_1}{d^{2}}, \cdots, \frac{\xi_n}{d})^{\wedge},$$
and \( R^{PD}(1) \) is the \( p \)-completed free divided power polynomial ring in the variables \( \{ \xi^i_p : 1 \leq i \leq n \} \). Since \( R(1) = \tilde{R}(1)/\tilde{R}(1) \) is topologically generated by the image of \( \xi^i_p \) and the iterations of their \( \gamma_p \)'s, we see that \( \psi \) is surjective. It remains to see that \( \psi \) is injective.

We consider first the special case \( (d) = (p) \), i.e. \((A,I) = (A,(p))\) is a crystalline prism. Let \( \tilde{R}(\xi^1_p, \ldots, \xi^n_p) \) be the convergent power series ring over \( \tilde{R} \) in the variables \( \{ \xi^i_p : 1 \leq i \leq n \} \), and \( \tilde{R}(\xi^1_p, \ldots, \xi^n_p)^{PD,\wedge} \) be the corresponding \( p \)-completed divided power polynomial ring. One considers the following Frobenius structure on \( \tilde{R}(\xi^1_p, \ldots, \xi^n_p) \) given by

\[
\phi(\xi^i_p) = \frac{\phi(\xi^i_p)}{p} = \frac{\xi^i_p}{p} + \delta(\xi^i_p) = p^{i-1}(\xi^i_p)^p + \delta(\xi^i_p),
\]

with \( \delta(\xi^i_p) \) given by (4.1.1). Note that \( \delta(\xi^i_p) \) is divisible by \( p \) in \( \tilde{R}(\xi^1_p, \ldots, \xi^n_p) \) so that \( \phi(\xi^i_p) \in p\tilde{R}(\xi^1_p, \ldots, \xi^n_p) \). It follows that such a lift of Frobenius, or equivalently such a \( \delta \)-structure, extends uniquely to \( \tilde{R}(\xi^1_p, \ldots, \xi^n_p)^{PD,\wedge} \): Indeed, if \( \phi(\xi^i_p) = py_i \), then we put

\[
\phi((\xi^i_p)^{[n]}) = \frac{p^n y_i^n}{n!},
\]

which defines a unique Frobenius structure on \( \tilde{R}(\xi^1_p, \ldots, \xi^n_p)^{PD,\wedge} \). Here, \((\xi^i_p)^{[n]}\) denotes the \( n \)-th divided power of \( \xi^i_p \). By the universal property of the prismatic envelope \( R(1) \cong \tilde{R}(\xi^1_p, \ldots, \xi^n_p)^{\wedge} \), there exists a unique map of \( \delta \)-\( \tilde{R} \)-algebras \( \tilde{R}(1) \to \tilde{R}(\xi^1_p, \ldots, \xi^n_p)^{PD,\wedge} \). Reducing modulo \( p \), one gets a map of divided power \( R \)-algebras

\[
R(1) = \tilde{R}(1)/p\tilde{R}(1) \to R^{PD}(1) = R(\xi^1_d, \ldots, \xi^n_d)^{PD,\wedge}
\]

which is easily seen to be the inverse of \( \psi \).

We consider now the general case of \( d \). First, we note that the formation of \( \psi \) commutes with the \( \acute{e}tale \) localization in \( R \). Therefore, one can reduce to the case \( R = A(\mathbb{T}_n) \). By our assumptions in Situation [4.1], there exists a map of bounded prisms \( (A_0,I_0) \to (A,I) \) such that the \( \delta \)-structure on \( A(\mathbb{T}_n) \) descends to \( A_0(\mathbb{T}_n) \) and

(a) the Frobenius map on \( A_0/p \) is flat;

(b) \( A_0/I_0 \) is \( p \)-torsion free.

Since the formation of \( \psi \) commutes with base change in \( (A_0,I_0) \), it suffices to prove the statement for \((A_0,I_0)\). Up to changing notation, we may assume that \((A,I)\) satisfies conditions (a) and (b).

Consider the ring \( B = A\{\phi(d)^{\wedge}\} \), the \( p \)-completed \( \delta \)-\( A \)-algebra obtained by freely adjoining \( \phi(d)^{\wedge}_p \) to \( A \). By [8, Cor. 2.38], \( B \) is identified with the \( p \)-completed PD-envelope of \( A \) with respect to \( (d) \). Let \( \alpha : A \to B \) denote the composite of the canonical map \( A \to B \) with the Frobenius \( \phi : A \to A \). Then \( \alpha \) induces a map of bounded prisms \( (A,I) \to (B,(p)) \) since \( \alpha(I) \subset (p) \). Reducing mod \( p \), we get a factorization

\[
\bar{\alpha} : A/p \to A/(p,I) \xrightarrow{\phi} A/(p,I^p) \xrightarrow{\iota} B/p,
\]

where the first arrow is the canonical reduction, \( \phi \) is the Frobenius, and \( \iota \) is induced by the canonical map \( A \to B \). Note that \( \phi \) is faithfully flat by condition (a). We claim that \( \iota \) is also faithfully flat as well. Note that \( B/p \) is the PD-envelope of \( A/(p) \) with respect to the ideal \( (d) \). Since condition (b) implies that \( d \in A/(p) \) is nonzero divisor, it follows from [14, Tag 07HC] that

\[
B/p = A/p \{x\}^{PD}/(x-d),
\]

where \( A/p \{x\}^{PD} \) is the free divided power ring in one variable over \( A/p \). We have an isomorphism

\[
A/p \{x\}^{PD} \cong A/p[x_0,x_1,\cdots]/(x_i^p : i \geq 0)
\]
where $x_i$ is the image $x^{[p^i]}$. Hence, we get
\[ B/p \cong A/(p, d^p)[x_1, x_2, \ldots]/(x_i^p : i \geq 1) \]
which is clearly faithfully flat over $A/(p, d^p)$.

Put $\tilde{\beta} = \iota \circ \phi : A/(p, I) \rightarrow B/p$. Then $\tilde{\beta}$ is faithfully flat, because so are both $\iota$ and $\phi$. Let $\bar{\psi}$ be the reduction of $\psi$ modulo $p$, and $\bar{\psi}_B = \psi \circ 1 : R^{PD}(1) \otimes_A B \rightarrow \bar{R}(1) \otimes_A B$ be the base change of $\psi$ via $\alpha : A \rightarrow B$. Then one has a commutative diagram:
\[
\begin{array}{ccc}
R/p\{\xi_1^p, \ldots, \xi_n^p\} & \xrightarrow{\bar{\psi}} & \bar{\tilde{R}}(1)/(p, I)\tilde{R}(1) \\
\downarrow & & \downarrow \bar{\psi}_B = \psi \circ 1 \\
(R/p \otimes_{A/(p, I)} \beta B/p)\{\xi_1^p, \ldots, \xi_n^p\} & \xrightarrow{(\bar{\tilde{R}}(1)/(p, I)\tilde{R}(1)) \otimes_{A/(p, I), \beta} B/p,}
\end{array}
\]
where the vertical maps are natural inclusions induced by $\tilde{\beta}$, and the bottom map $\bar{\psi}_B$ is reduction of $\psi_B$. As the formation of $\psi$ commutes with base change in $(A, I)$, the previous discussion in the case of $(d) = (p)$ implies that $\psi_B$ is injective (hence an isomorphism), it follows that $\bar{\psi}$ is also injective. Let $M$ denote the kernel of $\bar{\psi}$. Since both the source and the target of $\psi$ are flat over $A/I$, it follows that $M/pM = \ker(\bar{\psi}) = (0)$. As $M$ is separate for the $p$-adic topology, it follows that $M = (0)$. This finishes the proof of Proposition 4.7.

\[ \square \]

Recall that one has a map of cosimplicial objects $\tilde{R}^{\otimes(\bullet + 1)} \rightarrow \tilde{R}(\bullet)$ (Subsection 3.3). For integers $m \geq 1$, $1 \leq i \leq n$ and $0 \leq j \leq m$, let
\[ T_{i,j} := 1 \otimes \cdots \otimes 1 \otimes T_i \otimes 1 \otimes \cdots \otimes 1 \in \tilde{R}^{\otimes(m+1)} \]
with $T_i$ sitting at $j$-th place, and $\xi_{i,j}$ be the image of $T_{i,j} - T_{i,j-1}$ in $\tilde{R}(m)$ for $j \geq 1$. In view of Lemma 4.2 and Remark 3.4, if we view $\tilde{R}(m)$ as an $\tilde{R}$-algebra via $\delta_m^1 \circ \cdots \circ \delta_1^1$ (which corresponds to the projection $X(m) \rightarrow X$ to the 0-th copy of $X$), one has an isomorphism of $\tilde{R}$-algebras:
\[ \tilde{R}(m) \cong \tilde{R}\{\frac{\xi_{i,j}}{m} : 1 \leq i \leq n, 1 \leq j \leq m\}^{\wedge} \]
We put
\[ R^{PD}(m) := \left(R^{PD}(1) \otimes_R R^{PD}(1) \otimes_R \cdots \otimes_R R^{PD}(1)\right)^m \]
Similarly as (4.5.2), we write
\[ R^{PD}(m) = R\{\frac{\xi_{i,j}}{m} : 1 \leq i \leq n, 1 \leq j \leq m\}^{PD, \wedge} \]
in the coordinates $\xi_{i,j}$. By Remark 3.3, the isomorphism $\psi : R^{PD}(1) \cong R(1)$ induces isomorphisms
\[ \psi_m : R^{PD}(m) \cong R(1) \otimes_R \cdots \otimes_R R(1) \]
for all integers $m \geq 1$. We will always use $\psi_m$ to identify $R^{PD}(m)$ with $R(m)$. If $I = (d)$ is principal, then the cosimplicial map $\delta_k^m : R(m-1) \rightarrow R(m)$ with $0 \leq k \leq m$ (which is the reduction mod $I$ of (3.3.1)) is compatible with the PD-structure and determined by
\[ \delta_k^m : \frac{\xi_{i,j}}{d} \mapsto \begin{cases} 
\frac{\xi_{i,j}}{d} & \text{if } k < j, \\
\frac{\xi_{i,j}}{d} + \frac{\xi_{i,j+1}}{d} & \text{if } k = j, \\
\frac{\xi_{i,j}}{d} & \text{if } k > j,
\end{cases} \]
4.8. Higgs modules. By a Higgs module over \( R \), we mean a \( p \)-completely flat and derived \( p \)-complete \( R \)-module \( M \) together with a \( R \)-linear map

\[
\theta : M \to M \otimes_R \Omega^1_R \{-1\}
\]
such that the induced map \( \theta \wedge \theta : M \to M \otimes \Omega^2_R \{-2\} \) vanishes. Denote by \( TR := \text{Hom}_R(\Omega^1_R, R) \) the tangent bundle of \( R \). Then \( \theta \) induces a map of \( R \)-modules

\[
\varphi_\theta : TR\{1\} = TR \otimes_{A/I} I/I^2 \to \text{End}_R(M).
\]
The condition \( \theta \wedge \theta = 0 \) is equivalent to saying that two endomorphisms in the image of \( \varphi_\theta \) commute with each other. Indeed, this claim can be checked after a \((p, I)\)-completely faithfully flat base change in \( (A, I) \) so that we may assume that \( I = (d) \) is principal. Then we have \( \Omega^1_R \{-1\} = \oplus_{i=1}^n R \frac{dT_i}{d} \), and we can write \( \theta = \sum_{i=1}^n \theta_i \frac{dT_i}{d} \) with \( \theta_i \in \text{End}_R(M) \). Then the condition \( \theta \wedge \theta = 0 \) is equivalent to \( \theta_i \theta_j = \theta_j \theta_i \) for all \( i, j \). It is clear that \( \{\theta_i, 1 \leq i \leq n\} \) generate the image of \( \varphi_\theta \), and the claim follows.

Let \( \text{Sym}(TR\{1\}) := \bigoplus_{m \geq 0} \text{Sym}^m(TR\{1\}) \) be the symmetric algebra of \( TR\{1\} \) over \( R \). Then \( \varphi_\theta \) extends to a homomorphism of \( R \)-algebras:

\[
\varphi_\theta^* : \text{Sym}(TR\{1\}) \to \text{End}_R(M).
\]

**Definition 4.9.** A Higgs module \((M, \theta)\) is **topologically quasi-nilpotent** if for each \( x \in M \), the submodule \( \varphi_\theta^*(\text{Sym}^m(T\{1\}) \cdot x) \) of \( M \) tends to 0 as \( m \to +\infty \), i.e. for any integer \( c > 0 \) there exists \( N > 0 \) such that \( \varphi_\theta^*(\text{Sym}^m(T\{1\}) \cdot x) \subset p^cM \) for all \( m \geq N \). Let \( \text{Higgs}^\wedge(R) \) denote the category of topologically quasi-nilpotent Higgs modules over \( R \).

If \( I = (d) \) is principal and \( \theta = \sum_{i=1}^n \theta_i \frac{dT_i}{d} \) as above, then \((M, \theta)\) is topologically quasi-nilpotent if and only if \( \theta^m(x) \) tends to 0 as \( \|m\| := \sum_{i=1}^n m_i \to +\infty \) for all \( x \in M \) and \( m \in \mathbb{N}^n \), where we put

\[
\theta^m = \prod_{i=1}^n \theta_i^{m_i} \in \text{End}_R(M),
\]
with \( \theta^m = \text{id} \) for \( m = (0, \ldots, 0) \).

4.10. **Stratification and Higgs modules.** Let \((M, \epsilon)\) be an object of \( \text{Strat}(R, R(1)) \) (Definition 3.7). Note that we have an isomorphism:

\[
\text{Hom}_{R(1)}(M \widehat{\otimes}_R R(1), M \widehat{\otimes}_R R(1)) \cong \text{Hom}_R(M, M \widehat{\otimes}_R R(1)) \cong \text{Hom}_R(M, M \widehat{\otimes}_R R^{PD}(1))
\]

where the last isomorphism is Proposition 4.7. When no confusions arise, we will still denote by \( \epsilon \) the image of \( \epsilon \) in \( \text{Hom}_R(M, M \widehat{\otimes}_R R^{PD}(1)) \).

Let

\[
\iota : M \to M \widehat{\otimes}_R R^{PD}(1)
\]
be the natural inclusion induced by the structural map \( R \to R^{PD}(1) \). By the definition of stratification, we have \((\epsilon - \iota)(M) \subset M \widehat{\otimes}_R K \). We denote by

\[
\theta_\epsilon : M \to M \widehat{\otimes}_R K[K^{[2]}] \cong M \widehat{\otimes}_R \Omega^1_R \{-1\}
\]
the reduction of \( \epsilon - \iota \) modulo \( K^{[2]} \), where the second isomorphism is (4.5.1).

**Proposition 4.11.** The functor \((M, \epsilon) \mapsto (M, \theta_\epsilon)\) establishes an equivalence of categories

\[
\text{Strat}(R, R(1)) \xrightarrow{(\sim)} \text{Higgs}^\wedge(R).
\]
Proof. We prove first that, for any object \((M, \epsilon)\) of \(\text{Strat}(R, R(1))\), the attached object \((M, \theta_\epsilon)\) is indeed a topologically quasi-nilpotent Higgs module over \(R\). Up to base change to a \((\mathfrak{p}, I)\)-completely faithfully flat \(\delta\)-\(A\)-algebra, we may assume that \(I = (d)\) is principal for a distinguished element \(d\). We write as usual that
\[
\theta_\epsilon = \sum_{i=1}^{n} \theta_{\epsilon, i} \frac{dT_i}{d}.
\]

By Proposition 4.7 we have
\[
M \overset{\otimes}{\otimes} R(1) \cong M \overset{\otimes}{\otimes} R^{PD}(1) = \left\{ \sum_{m \in \mathbb{N}^n} x_m \left( \frac{\xi \cdot i}{d} \right)^{[m]} : x_m \in M, x_m \to 0 \text{ as } \|m\| \to \infty \right\}
\]
where we put
\[
\left( \frac{\xi \cdot i}{d} \right)^{[m]} = \prod_{i=1}^{n} \left( \frac{\xi_{i, d}}{d} \right)^{[m_i]}
\]
For \(x \in M\), we write
\[
\epsilon(x) = \sum_{m \in \mathbb{N}^n} \Theta_m(x) \left( \frac{\xi \cdot i}{d} \right)^{[m]} \in M \overset{\otimes}{\otimes} R^{PD}(1)
\]
for some \(\Theta_m \in \text{End}_R(M)\) with \(\Theta_m = \text{id}_M\) if \(m = (0, \ldots, 0)\). By the definition of \(\theta_\epsilon\), it is clear that \(\theta_{\epsilon, i} = \Theta_{\xi_i}\) with \(\xi_i \in \mathbb{N}^n\) the element with the \(i\)-th component equal to 1 and others components equal to 0. To see that \((M, \theta_\epsilon)\) is an object of \(\text{Higgs}^\wedge(R)\), it suffices to show that
\begin{equation}
(4.11.1) \quad \theta_{\epsilon, i} \theta_{\epsilon, j} = \theta_{\epsilon, j} \theta_{\epsilon, i}, \quad \Theta_m = \prod_{i=1}^{n} \theta_{\epsilon, i}^{m_i} = \theta_{\epsilon}^m.
\end{equation}

According to (4.7.1) and (4.7.2), we have \(R(2) \cong R^{PD}(2) = R\{ \frac{\xi_{i, j}}{d} : 1 \leq i \leq n, 1 \leq j \leq 2 \}\) and the maps \(\delta_i : R(1) \to R(2)\) with \(j = 0, 1, 2\) are given by
\[
\delta_0^2 \left( \frac{\xi_{i, 2}}{d} \right) = \frac{\xi_{i, 2}}{d}, \quad \delta_1^2 \left( \frac{\xi_{i, 1}}{d} \right) = \frac{\xi_{i, 1}}{d} + \frac{\xi_{i, 2}}{d}, \quad \delta_2^2 \left( \frac{\xi_{i, 1}}{d} \right) = \frac{\xi_{i, 1}}{d},
\]
for all \(1 \leq i \leq n\). Then for all \(x \in M\), we have
\begin{equation}
(4.11.2) \quad \delta_1^2(x) = \sum_{m \in \mathbb{N}^n} \Theta_m(x) \left( \frac{\xi_1 \cdot i}{d} + \frac{\xi_2 \cdot i}{d} \right)^{[m]} = \sum_{m_1, m_2 \in \mathbb{N}^n} \Theta_{m_1 + m_2}(x) \left( \frac{\xi_1 \cdot i}{d} \right)^{[m_1]} \left( \frac{\xi_2 \cdot i}{d} \right)^{[m_2]},
\end{equation}
\[
\delta_2^2(\epsilon(x)) = \sum_{m_1, m_2 \in \mathbb{N}^n} \Theta_{m_1}(\Theta_{m_2}(x)) \left( \frac{\xi_1 \cdot i}{d} \right)^{[m_1]} \left( \frac{\xi_2 \cdot i}{d} \right)^{[m_2]}.
\]
Then the cocycle condition \(\delta_1^2(x) = \delta_2^2(\epsilon(x))\) is equivalent to saying that
\[
\Theta_{m_1 + m_2} = \Theta_{m_1} \circ \Theta_{m_2},
\]
for all \(m_1, m_2 \in \mathbb{N}^n\), from which (4.11.1) follows immediately.

To finish the proof of the Proposition, it suffices to construct a functor quasi-inverse to \((M, \epsilon) \mapsto (M, \theta_\epsilon)\). Let \((M, \theta)\) be an object of \(\text{Higgs}^\wedge(R)\). We need to construct a stratification \(\epsilon\) on \(M\) over
By the usual descent argument, we may reduce the problem to the case when \( I = (d) \) is principal. For any \( x \in M \), we put

\[
\epsilon(x) := \sum_{m \in \mathbb{N}^n} \theta^m(x) \left( \frac{\xi}{d} \right)^{|m|} \in M \otimes_R R^{PD}(1) \cong M \otimes_R R(1),
\]

which is well defined by the topological quasi-nilpotence of \((M, \theta)\). This defines an element

\[
\epsilon \in \text{Hom}_R(M, M \otimes_R R(1)) \cong \text{End}_{R(1)}(M \otimes_R R(1)).
\]

We prove first that \( \epsilon \) is an isomorphism of \( M \otimes_R R(1) \). Let \( \epsilon' \) be the endomorphism of \( M \otimes_R R(1) \) defined by

\[
\epsilon'(x) = \sum_{m \in \mathbb{N}^n} (-1)^{|m|} \theta^m(x) \left( \frac{\xi}{d} \right)^{|m|} \forall x \in M.
\]

We claim that \( \epsilon' \) is the inverse of \( \epsilon \). Indeed, we have

\[
\epsilon'(\epsilon(x)) = \epsilon' \left( \sum_{m \in \mathbb{N}^n} \theta^m(x) \left( \frac{\xi}{d} \right)^{|m|} \right) = \sum_{m \in \mathbb{N}^n} \sum_{m' \in \mathbb{N}^n} (-1)^{|m'|} \theta^{m'}(\theta^m(x)) \left( \frac{\xi}{d} \right)^{|m|} \left( \frac{\xi}{d} \right)^{|m'|} = \sum_{m \in \mathbb{N}^n} \theta^m(x) \sum_{m' \in \mathbb{N}^n} (-1)^{|m'|} \left( \frac{\xi}{d} \right)^{|m'|} \left( \frac{\xi}{d} \right)^{|s-m|}.
\]

Here, the notation \( m' \leq s \) in the last equality means \( m'_i \leq s_i \) for all \( 1 \leq i \leq n \). Note that if \( s_i > 0 \) for some \( i \), then we have

\[
\sum_{m'_i=0}^{s_i} (-1)^{m'_i} \left( \frac{\xi_i}{d} \right)^{m'_i} \left( \frac{\xi_i}{d} \right)^{s_i-m'_i} = \sum_{m'_i=0}^{s_i} \left( \frac{\xi_i}{d} - \frac{\xi_i}{d} \right)^{s_i} = 0.
\]

Therefore, the contribution of the terms with \( |s| > 0 \) to \( (4.11.4) \) is zero, and we deduce that \( \epsilon'(\epsilon(x)) = x \), i.e. \( \epsilon' \circ \epsilon = \text{id} \). A similar computation shows as well that \( \epsilon \circ \epsilon' = \text{id} \). It remains to check that \( \epsilon \) verifies the cocycle condition \( \delta^2_{\epsilon}(\epsilon) = \delta^2_{\epsilon}(\epsilon) \circ \delta^2_{\epsilon}(\epsilon) \). By the computation \( (4.11.2) \), this follows immediately from the obvious fact that \( \theta^{m} + \theta^{m'} = \theta^{m} \circ \theta^{m'} \). This construction gives a functor \( \text{Higgs}^\wedge(R) \to \text{Strat}(R, R(1)) \), which is easily seen to be a quasi-inverse to \((M, \epsilon) \mapsto (M, \theta)\).

Combining now Propositions 3.8 and 4.11, we obtain the following

**Theorem 4.12.** Suppose that we are in Situation 4.1. Then we have a sequence of equivalence of categories

\[
\text{CR}((X/A)_\Delta, \overline{\mathcal{O}}_\Delta) \xrightarrow{\tau_X} \text{Strat}(R, R(1)) \xrightarrow{\text{Prop. 4.11}} \text{Higgs}^\wedge(R).
\]

**Remark 4.13.** (1) It is natural to expect that there still exists an equivalence of categories between \( \text{CR}((X/A)_\Delta, \overline{\mathcal{O}}_\Delta) \) and \( \text{Higgs}^\wedge(R) \) under the more general assumption 3.1.

(2) Gros–Le Strum–Quirós [10, Cor. 6.6] and Morrow–Tsujii [12, Thm. 3.2] give similar descriptions of \( \mathcal{O}_\Delta \)-crystals when \((A, I)\) is a bounded prism over \((\mathbb{Z}_p[[q-1]], ([p]_q))\) with \([p]_q = \frac{q^p-1}{q-1}\). In these special cases, Theorem 4.12 is compatible with their results after modulo \( I \).

We now describe the cohomology of an \( \overline{\mathcal{O}}_\Delta \)-crystal in terms of the de Rham complex of its associated Higgs module.
Theorem 4.14. Suppose that we are in Situation 4.1. Let \( \mathcal{E} \) be an \( \mathcal{O}_\Delta \)-crystal, and \((\mathcal{E}_X, \theta)\) be its associated object of \( \text{Higgs}^\wedge(R) \) via Theorem 4.12. Then \( \text{R} \nu_{X/A,*}(\mathcal{E}) \) is computed by the de Rham complex of \((\mathcal{E}_X, \theta)\):

\[
\text{DR}^\bullet(\mathcal{E}_X, \theta) = (\mathcal{E}_X \xrightarrow{\theta} \mathcal{E}_X \otimes_R \Omega^1_{R(-1)} \xrightarrow{\theta} \cdots \xrightarrow{\theta} \mathcal{E}_X \otimes_R \Omega^n_{R(-n)})
\]

We prove Theorem 4.14 by following the strategy of [1]. For an integer \( m \geq 0 \), we have an isomorphism (4.7.1)

\[
R(m) = \tilde{R}(m)/I\tilde{R}(m) \cong R^{PD}(m) = R\{\frac{\xi_{i,j}}{I} : 1 \leq i \leq n, 1 \leq j \leq m\}^{PD,\wedge},
\]

where \( R^{PD}(m) \) is a certain twisted divided power polynomial ring over \( R \) in \( mn \)-variables. The kernel \( K(m) \) of the canonical surjection \( R(m) \to R \) has a divided power structure. Let \( R^{1}(R(m)/R) \) denote the module of continuous differential 1-forms relative to \( R \) compatible with the divided power structure, i.e. \( dx^r = dx^{r-1} \) for all integer \( r \geq 1 \) and \( x \in K(m) \). If \( I = (d) \) is principal, this is a locally free \( R(m) \)-module with basis \( \{\frac{dx^k}{d} : 1 \leq k \leq n, 1 \leq l \leq m\} \). Here, \( \frac{dx^k}{d} \) should be rather understood as the differential of the variable \( \frac{\xi_{k,l}}{d} \in R(m) \), because the natural image of \( \xi_{k,l} = T_{k,l} - T_{k,l-1} \in \tilde{R}(m) \) in \( R(m) \) is zero. For an integer \( j \geq 1 \), we put \( \Omega^j_{R(m)/R} = \wedge^j \Omega^1_{R(m)/R} \), where the wedge product is over \( R(m) \).

We put

\[
\Omega^1_{R(m)} := \Omega^1_{R}\{-1\} \otimes_R R(m) \bigoplus \Omega^1_{R(m)/R}
\]

which is noncanonically isomorphic to the module of continuous differential 1-forms of \( R(m) \) with respect to \( A/I \) compatible with the divided power structure. For each map of simplices \( f : [m] \to [m'] \), we will construct a map

\[
f_* : \Omega^1_{R(m)} \to \Omega^1_{R(m')}
\]

such that \( (g \circ f)_* = g_* \circ f_* \) for any two composable maps of simplices. We assume first that \( I = (d) \) is principal. Then \( \Omega^1_{R(m)} \) is a free \( R(m) \)-module with a basis given by

\[
\{\frac{dT_k}{d} = dT_k \otimes d^{-1} : 1 \leq k \leq n\} \cup \{\frac{d\xi_{k,l}}{d} : 1 \leq k \leq n, 1 \leq l \leq m\}
\]

By the discussion after the proof of Proposition 4.7, \( \xi_{k,l} \) should be understood as the image of the variable \( \frac{\xi_{k,l}}{d} \in R(m) \), because the natural image of \( T_{k,l} - T_{k,l-1} \in \tilde{R}(m) \) in \( R(m) \) is zero. For each map of simplices \( f : [m] \to [m'] \), we define

\[
f_* : \Omega^1_{R(m)} \to \Omega^1_{R(m')}, \quad \frac{dT_{k,l}}{d} \mapsto \frac{dT_{k,f(l)}}{d}.
\]

Such a definition is independent of the choice of the generator \( d \) of \( I \). If \( I \) is not necessarily principal, we take a \( (p, I) \)-completely faithfully flat \( \delta \)-\( A' \)-algebra \( A' \) such that \( I' = IA \) is principal, and make the construction of \( f_* \) after base change to \( A' \). An usual descent argument with Proposition 1.9 allows us to descend the morphism \( f_* \) to \( A \). It is also direct to see that \( (g \circ f)_* = g_* \circ f_* \) for two composable maps of simplices. This defines a cosimplicial module \( \Omega^1_{R(\bullet)} \) over the cosimplicial ring \( R(\bullet) \). Taking wedges, we get a cosimplicial \( R(\bullet) \)-module \( \Omega^j_{R(\bullet)} \) for each integer \( j \geq 1 \).

Lemma 4.15. For each integer \( j \geq 1 \), the cosimplicial object \( \Omega^j_{R(\bullet)} \) is homotopic to zero.
Proof. For \( j = 1 \), the proof is the same as [4] Lemma 2.15. Indeed, the cosimplicial object \( \Omega^j_{R(\bullet)} \) is a direct sum of \( n \) copies of cosimplicial objects in [4] Example 2.16. Therefore, it is homotopic to zero. The general case follows since being homotopic to zero is stable under the application of the functor \( \wedge^j \) termwise. \( \square \)

Note that

\[
\Omega^j_{R(m)} = \bigoplus_{k=0}^{j} \Omega^k_{R}(-k) \otimes_R \Omega^{j-k}_{R(m)/R}.
\]

Let \( d_R : \Omega^j_{R(m)} \to \Omega^{j+1}_{R(m)} \) be the \( R \)-linear map given by

\[
d_R \left( \sum_{k=0}^{j} \omega_k \otimes \eta_{j-k} \right) = \sum_{k=0}^{j} (-1)^k \omega_k \otimes d\eta_{j-k}
\]

for \( \omega_k \in \Omega^k_{R}\{-k\} \) and \( \eta_{j-k} \in \Omega^{j-k}_{R(m)/R} \). In other words, \((\Omega^*_R(m), d_R)\) is the simple complex attached to the tensor product of

\[
\begin{array}{c}
R \to \Omega^1_R{-1} \to \cdots \to \Omega^n_R{-n}
\end{array}
\]

with the relative de Rham complex \((\Omega^*_R(m)/R, d)\). One checks easily that \( d_R : \Omega^j_{R(\bullet)} \to \Omega^{j+1}_{R(\bullet)} \) is a map of cosimplicial objects, i.e. for any map of simplices \( f : [m] \to [m'] \), \( d_R \) commutes with \( f_* : \Omega^j_{R(m)} \to \Omega^j_{R(m')} \).

Let \( \tilde{X}(\bullet) \) be the simplicial object in \((X/A)_\Delta\) defined in §3.3. For an integer \( i \geq 0 \), let \( \mathcal{E}_{\tilde{X}(i)} \) be the evaluation of \( \mathcal{E} \) at \( \tilde{X}(i) \). As \( \mathcal{E} \) is an \( \Omega_\Delta \)-crystal, we have an isomorphism

\[
c_{\text{pr}_0}(\mathcal{E}) : \mathcal{E}_{\tilde{X}(i)} \overset{\sim}{\to} \mathcal{E}_{\tilde{X}} \otimes_R R(i)
\]

induced by the 0-th projection \( \text{pr}_0 : \tilde{X}(i) \to \tilde{X} \). In the sequels, we will always use \( c_{\text{pr}_0}(\mathcal{E}) \) to identify these two modules. For integers \( i, j \geq 0 \), we put

\[
\mathcal{C}^{i,j} = \mathcal{E}_{\tilde{X}(i)} \otimes_R \Omega^j_{R(i)} \cong \mathcal{E}_{\tilde{X}} \otimes_R \Omega^j_{R(i)}.
\]

Then for each integer \( j \geq 0 \), \( \mathcal{C}^{\bullet,j} = \mathcal{E}_{\tilde{X}(\bullet)} \otimes_R \Omega^j_{R(\bullet)} \) has a natural structure of cosimplicial \( R(\bullet) \)-modules. Let \((\mathcal{C}^{i,j}, d_{i,j}^{\bullet})\) be the chain complex associated to the cosimplicial object \( \mathcal{C}^{i,j} \), i.e. if \( \delta_{k,s}^{i+1} \) is the unique injective order preserving map \( \delta_{k,s}^{i+1} : [i] := \{0, 1, \ldots, i\} \to [i+1] \) that skips \( k \), we have

\[
d_{i,j}^1 = \sum_{k=0}^{i+1} (-1)^k \delta_{k,s}^{i+1} : \mathcal{C}^{i,j} \to \mathcal{C}^{i+1,j}.
\]

We also define \( d_{i,j}^{2} : \mathcal{C}^{i,j} \to \mathcal{C}^{i,j+1} \) by

\[
d_{i,j}^{2}(x \otimes \omega) = \theta(x) \wedge \omega + x \otimes d_R \omega
\]

for \( x \in \mathcal{E}_{\tilde{X}} \) and \( \omega \in \Omega^1_{R(i)} \). It is easy to see that, for each \( i \geq 0 \), \((\mathcal{C}^{i,\bullet}, d_{i,\bullet}^{\bullet})\) is the simple complex associated to the tensor product of \( \text{DR}^\bullet(\mathcal{E}_{\tilde{X}(i)}, \theta) \) with the relative de Rham complex \( \Omega^\bullet_{R(i)/R} \) (see [4, Lemma 2.2] for the convention).

**Lemma 4.16.** For each integer \( j \geq 0 \), \( d_{2,j}^{\bullet} : \mathcal{C}^{\bullet,j} \to \mathcal{C}^{\bullet,j+1} \) is a morphism of cosimplicial objects. In particular, \((\mathcal{C}^{\bullet,\bullet}, d_{1,\bullet}^{\bullet}, d_{2,\bullet}^{\bullet})\) is a naïve double complex.
Proof. For integers \( i, k \geq 0 \) with \( k \leq i \), let \( \sigma_{k}^{i+1} : [i+1] \to [i] \) be the surjective order preserving map of simplices with \( (\sigma_{k}^{i+1})^{-1}(k) = \{k, k+1\} \). It suffices to prove that \( \sigma_{k,*}^{i} d_{2}^{i+1,j} = d_{2}^{i+1,j} \sigma_{k,*}^{i} \) and \( d_{2}^{i+1,j} \delta_{k,*}^{i+1} = \delta_{k,*}^{i+1} d_{2}^{i+1,j} \).

For \( y \otimes \eta \in \mathcal{E}_{X} \otimes R \Omega_{R(i+1)}^{j} = \mathcal{E}_{R}^{i+1,j} \), we have

\[
\sigma_{k,*}^{i} (y \otimes \eta) = y \otimes \sigma_{k,*}^{i} (\eta).
\]

Then the commutation of \( d_{2} \) with \( \sigma_{k,*}^{i} \) follows from the fact that \( d_{R} : \Omega_{R(\bullet)}^{j} \to \Omega_{R(\bullet)}^{j+1} \) is a morphism of cosimplicial objects.

It remains to check \( d_{2}^{i+1,j} \delta_{k,*}^{i+1} = \delta_{k,*}^{i+1} d_{2}^{i+1,j} \). As usual, it suffices to check this equality after base change to a \( (p, I) \)-completely faithfully flat \( \delta \)-\( A \)-algebra \( A' \) with \( IA' \) principal. Therefore, up to changing the notation, we may assume that \( I = (d) \) is principal. For \( x \otimes \omega \in \mathcal{E}_{X} \otimes R \Omega_{R(i)}^{j} \), which is identified with \( \mathcal{E}_{X(i)} \otimes R(i) \Omega_{R(i)}^{j} \) via \( \mathcal{E}_{X} \otimes R \Omega_{R(i)}^{j} \) via \( (4.15.2) \), we have

\[
\delta_{k,*}^{i+1} (x \otimes \omega) = \begin{cases} 
 x \otimes \delta_{k,*}^{i+1} (\omega), & \text{if } k \geq 1; \\
 \text{pr}_{0,1,*}(\epsilon)(x) \otimes \delta_{k,*}^{i+1} (\omega), & \text{if } k = 0.
\end{cases}
\]

Here, \( \text{pr}_{0,1,*} : R(1) \to R(i+1) \) is the map induced by the natural inclusion of simplices \([1] \to [i+1]\). By the formula \( (4.11.3) \), we get

\[
\text{pr}_{0,1,*}(\epsilon)(x) = \sum_{m \in \mathbb{N}^{n}} \theta_{m} (x) \otimes \left( \frac{\epsilon_{\bullet,1}}{d} \right)^{[m]}.
\]

Assume first that \( k \geq 1 \). Then \( d_{2}^{i+1,j} \delta_{k,*}^{i+1} = \delta_{k,*}^{i+1} d_{2}^{i+1,j} \) amounts to the commutation of \( d_{R} \) with \( \delta_{k,*}^{i+1} \).

Consider now the case \( k = 0 \). Then for \( x \otimes \omega \in \mathcal{E}_{X} \otimes R \Omega_{R(i)}^{j} \), we have

\[
d_{2}^{i+1,j} (\delta_{0,*}^{i+1} (x \otimes \omega)) = \sum_{m \in \mathbb{N}^{n}} d_{2}^{i+1,j} \left( \theta_{m} (x) \otimes \left( \frac{\epsilon_{\bullet,1}}{d} \right)^{[m]} \delta_{0,*}^{i+1} (\omega) \right)
\]

\[
= \sum_{m \in \mathbb{N}^{n}} \sum_{l=1}^{n} \theta_{l} (\theta_{m} (x)) \otimes \left( \frac{\epsilon_{l,1}}{d} \right)^{[m-e_{l}]} \left( \frac{d \xi_{1}}{d} \right) \delta_{0,*}^{i+1} (\omega) \]

\[
+ \sum_{m \in \mathbb{N}^{n}, |m| \geq 1} \sum_{l=1}^{n} \theta_{l} (\theta_{m} (x)) \otimes \left( \frac{\epsilon_{l,1}}{d} \right)^{[m]} \left( \frac{d \xi_{1}}{d} + \frac{d \xi_{l,1}}{d} \right) \delta_{0,*}^{i+1} (\omega) + \text{pr}_{0,1,*}(\epsilon)(x) \otimes d_{R}(\delta_{0,*}^{i+1} \omega).
\]

Here, \( e_{l} \in \mathbb{N}^{n} \) is the element with 1 at \( l \)-th component and 0 at other components. On the other hand, we have

\[
\delta_{0,*}^{i+1} (d_{2}^{i,j} (x \otimes \omega)) = \delta_{0,*}^{i+1} \left( \sum_{l=1}^{n} \theta_{l} (x) \otimes \frac{d \xi_{l,1}}{d} \delta_{0,*}^{i+1} (\omega) \right)
\]

\[
= \sum_{m \in \mathbb{N}^{n}} \sum_{l=1}^{n} \theta_{m} (\theta_{l} (x)) \otimes \delta_{0,*}^{i+1} (\frac{d \xi_{l,1}}{d}) \delta_{0,*}^{i+1} (\omega) + \text{pr}_{0,1,*}(\epsilon)(x) \otimes \delta_{0,*}^{i+1} (d_{R} \omega).
\]

Now \( \delta_{0,*}^{i+1} (x \otimes \omega) = \delta_{0,*}^{i+1} (d_{2}^{i,j} (x \otimes \omega)) \) follows from the fact that \( \delta_{0,*}^{i+1} (\frac{d \xi_{l,1}}{d}) = \frac{d \xi_{l,1}}{d} + \frac{d \xi_{l,1}}{d} \) and that \( \delta_{0,*}^{i+1} \) commutes with \( d_{R} \). \qed
Lemma 4.17. For an integer \( i \geq 0 \), any morphism of simplices \( \delta : [0] \to [i] \) induces a quasi-isomorphism of complexes

\[
\text{DR}^\bullet (E_{\tilde X}, \theta) = (\mathcal{C}^0, d_2^{i}) \xrightarrow{\sim} (\mathcal{C}^i, d_2^i).
\]

Proof. Note first that \((\mathcal{C}^i, d_2^i)\) is the simple complex attached to the tensor product of \(\text{DR}^\bullet (E_{\tilde X}, \theta)\) with the relative de Rham complex \((\Omega_{R(i)/R}^\bullet, d)\); in particular, one has \(\text{DR}^\bullet (E_{\tilde X}, \theta) = (\mathcal{C}^0, d_2^0)\).

By Lemma 4.16 \(d_2^j\) is a morphism of cosimplicial \(R(\bullet)\)-modules for each fixed \( j \geq 0 \). If we let \( j \) vary, any morphism of simplices \( \delta : [0] \to [i] \) induces a map of complexes of complexes \( \delta_* : (\mathcal{C}^0, d_2^0) \to (\mathcal{C}^i, d_2^i) \). We have to see that this is a quasi-isomorphism.

First, we claim that the natural map of complexes \( R \xrightarrow{\sim} \Omega_{R(i)/R}^\bullet \) is quasi-isomorphism. Indeed, this can be checked after a \((p, I)\)-completely faithfully flat base change in \((A, I)\). Up to making such a base change, we may assume that \( I = (d) \) is principal. Then \( R(i) \) is a free divided power polynomial ring over \( R \) in \( ni \)-variables, from which the claim follows. As a consequence, the natural projections \( \Omega_{R(i)}^j \to \Omega_{R}^j \{ -j \} \) in the decomposition (4.15.1) for all \( j \) give a quasi-isomorphism

\[
(\mathcal{C}^i, d_2^i) \xrightarrow{\sim} \text{DR}^\bullet (E_{\tilde X}, \theta).
\]

Since the composition of \( \delta_* \) with the above contraction map is the identity on \( \text{DR}^\bullet (E_{\tilde X}, \theta) \), the Lemma follows immediately.

End of the proof of Theorem 4.14. Note that \( \mathcal{C}^{*, 0} \) is by definition the Čech–Alexander complex \( \check{\mathcal{C}} \check{A}(\tilde{X}(\bullet), E) \) (cf. (3.11.1)), and \( \text{DR}^\bullet (E_{\tilde X}, \theta) = (\mathcal{C}^0, d_2^0) \). By Proposition 3.10 \( R\nu_{X/A,*}(E) \) is computed by \( \mathcal{C}^{*, 0} \). To finish the proof, it suffices to show that both \( \mathcal{C}^{*, 0} \) and \( \mathcal{C}^{0, *} \) are both quasi-isomorphic to \( s(\mathcal{C}^{*, *}) \), the simple complex associated to \( \mathcal{C}^{*, *} \). Recall that the map of complexes \( d_1^i : \mathcal{C}^i \to \mathcal{C}^{i+1} \) is given by \( \sum_{k=0}^{i+1} (-1)^k \delta_{k,i}^{i+1} \). By Lemma 4.17 \( \mathcal{C}^{i, *}, \mathcal{C}^{i+1, *} \) are both quasi-isomorphic to \( \text{DR}^\bullet (E_{\tilde X}, \theta) \) and each \( \delta_{k,i}^{i+1} \) corresponds to the identity map of \( \text{DR}^\bullet (E_{\tilde X}, \theta) \). Therefore, via these quasi-isomorphisms, \( d_1^i \) corresponds to the identity map of \( \text{DR}^\bullet (E_{\tilde X}, \theta) \) if \( i \) is odd, and to the zero map if \( i \) is even. It follows that \( s(\mathcal{C}^{*, *}) \) is quasi-isomorphic to

\[
(\text{DR}^\bullet (E_{\tilde X}, \theta) \xrightarrow{0} \text{DR}^\bullet (E_{\tilde X}, \theta) \xrightarrow{id} \text{DR}^\bullet (E_{\tilde X}, \theta) \xrightarrow{0} \cdots) \cong \text{DR}^\bullet (E_{\tilde X}, \theta) = \mathcal{C}^{0, *}.
\]

On the other hand, by Lemma 4.15 the complex \( \mathcal{C}^{*, j} = \mathcal{C}_{\tilde X}(\bullet) \otimes_{R(\bullet)} \Omega_{R(\bullet)}^j \) is homotopic to zero for all integers \( j \geq 1 \) since being homotopic to zero is stable under tensoring with another cosimplicial module. Hence, the natural map

\[
\mathcal{C}^{*, 0} \xrightarrow{\sim} s(\mathcal{C}^{*, *})
\]

is a quasi-isomorphism. This finishes the proof of Theorem 4.14.

Remark 4.18. It is clear that the trivial \( \mathcal{O}_{\tilde A} \)-crystal \( E = \mathcal{O}_{\tilde A} \) corresponds to the trivial Higgs module \( (\mathcal{O}_X, 0) \). Thus Theorem 4.14 implies that

\[
R\nu_{X/A,*}(\mathcal{O}_{\tilde A}) = \bigoplus_{i=0}^n \Omega_R^i \{-i\}.
\]

From this, it is easy to deduce Bhatt–Scholze’s Hodge–Tate comparison theorem [3, Theorem 4.11] for general smooth \( p \)-adic formal scheme over \( \text{Spf}(A/I) \).
4.19. **End of the proof of** [2.8] First, we note that the perfectness of \( R\nu_{X/A,n}(E) \) can be checked after base change to a \((p,I)\)-completely faithfully flat \( \delta \)-\(A\)-algebra \( A' \) with \( IA' \) principal. Indeed, as \( R\nu_{X/A,n}(E) \) is derived \( p\)-complete, by [14 Tag 09AW], it suffices to show that \( R\nu_{X/A,n}(E) \otimes_{A/I}^{L} A/(I,p^n) \) is a perfect \( \mathcal{O}_X \otimes A A/(I,p^n) \)-module for each integer \( n \geq 1 \). Then according to [14 Tag 0687], the perfectness of \( R\nu_{X/A,n}(E) \otimes_{A/I}^{L} A/(I,p^n) \) can be checked after the faithfully flat base change \( A/(I,p^n) \to A'/(I,p^n) \). Therefore, we are reducing to showing that \( (R\nu_{X/A,n}(E) \otimes_{A/I}^{L} A')^\wedge \) is a perfect \( \mathcal{O}_{X'} \)-module, where \( X' = X \times_{\text{Spf}(A/I)} \text{Spf}(A'/IA') \). By Corollary 3.13 one has a canonical isomorphism \( (R\nu_{X/A,n}(E) \otimes_{A/I}^{L} A')^\wedge \cong R\nu_{X'/A',n}(E') \), with \( E' \) the pull-back of \( E \) to \( (X'/A')_{\Delta} \). Up to performing such a base change, we may assume that \( I \) is principal.

Moreover, statements (1) and (2) are both local for the étale topology of \( X \). Up to étale localization, we may assume that \( X = \text{Spf}(R) \) is affine and satisfies the assumptions in Situation 4.1.

Then statement 4.1.4(1) follows immediately from Theorem 4.1.4.

For (2), let \( \widetilde{R} \) be a lift of \( R \) as in Situation 4.1. For a morphism of bounded prisms \((A, I) \to (A', I')\), put \( \widetilde{R}' = \widetilde{R} \otimes_{A} A', \widetilde{R}' = \widetilde{R}'/I'\widetilde{R}' \) and \( X' = \text{Spf}(R') \). If \( E \) corresponds to the Higgs module \((E_\hat{X}, \theta)\), then \( f_\Delta^! \mathcal{E} \) corresponds to \((E_\hat{X} \otimes_{R} R', \theta \otimes 1)\). By Theorem 4.1.4 the canonical base change map \( f^{-1}R\nu_{X/A,n}(E) \otimes_{f^{-1}\mathcal{O}_X}^{L} \mathcal{O}_{X'} \to R\nu_{X'/A',n}(f^!_\Delta \mathcal{E}) \) is identified in the derived category of \( R'\)-modules with the base change map

\[
\text{DR}^\bullet(E_\hat{X}, \theta) \otimes_R R' \xrightarrow{\sim} \text{DR}^\bullet(E_\hat{X} \otimes_{R} R', \theta')
\]

which is clearly an isomorphism of complexes.

5. **Poincaré Duality**

We fix a bounded prism \((A, I)\), and a smooth \( p\)-adic formal scheme of relative dimension \( n \) over \( \text{Spf}(A/I) \).

5.1. Let \( \mathbf{CR}((X/A)_{\Delta}, \mathcal{O}_{\Delta})^{fr} \) be the category of \( \mathcal{O}_{\Delta} \)-crystals locally free of finite rank. We have the natural notions of tensor product and internal hom in \( \mathbf{CR}((X/A)_{\Delta}, \mathcal{O}_{\Delta})^{fr} \): If \( E_1, E_2 \) are two objects of \( \mathbf{CR}((X/A)_{\Delta}, \mathcal{O}_{\Delta})^{fr} \), their tensor product \( E_1 \otimes E_2 \) is the \( \mathcal{O}_{\Delta} \)-crystal such that

\[
(E_1 \otimes E_2)(U) = E_1(U) \otimes_B E_2(U)
\]

for every object \( U = (\text{Spf}(B) \leftarrow \text{Spf}(B/IB) \to X) \) of \( (X/A)_{\Delta} \), and \( \underline{\text{hom}}(E_1, E_2) \) is the \( \mathcal{O}_{\Delta} \)-crystal such that

\[
\underline{\text{hom}}(E_1, E_2)(U) = \text{Hom}_B(E_1(U), E_2(U)).
\]

Assume that \( X = \text{Spf}(R) \) satisfies the assumptions in Situation 4.1. Let \((M_i, \theta_i)\) with \( i = 1, 2 \) be the object of Higgs \((R)\) corresponding to \( E_i \) via the equivalence of categories in Theorem 4.1.2 Then \( E_1 \otimes E_2 \) corresponds to the Higgs module \((M_1 \otimes_R M_2, \theta_1 \otimes 1 + 1 \otimes \theta_2)\), and \( \underline{\text{hom}}(E_1, E_2) \) corresponds to \((\text{Hom}_R(M_1, M_2), \theta)\) such that

\[
\theta(f)(x) = \theta_2(f(x)) - (f \otimes 1)(\theta_1(x)) \in M_2 \otimes_R \Omega^1_R \{-1\}
\]

for all \( x \in M_1 \).

5.2. **Duality pairing.** Let \( E \) be an object of \( \mathbf{CR}((X/A)_{\Delta}, \mathcal{O}_{\Delta})^{fr} \). We put \( E^\vee := \underline{\text{hom}}(E, \mathcal{O}_{\Delta}) \) and

\[
E^\{i\} := E \otimes_{A/I} (I/I^2)^{\otimes i}
\]

for all integers \( i \). Then the cup product induces a pairing

\[
(5.2.1) \quad R\nu_{X/A,n}(E^\{n\}) \otimes_{\mathcal{O}_X} R\nu_{X/A,n}(E) \to R\nu_{X/A,n}(E \otimes E^\{n\}) \to R\nu_{X/A,n}(\mathcal{O}_{\Delta} \{n\}) \to \Omega^\bullet_X \{-n\},
\]

32
where the last map is induced by the Hodge–Tate comparison isomorphism [8, Theorem 4.11]

\[ R^n \nu_{X/A,*}(O_\Delta) \cong \Omega^n_X \{ -n \}. \]

If moreover \( X \) is proper over \( \text{Spf}(A/I) \), then one has similarly a pairing of perfect complexes of \( A/I \)-modules:

\[
\begin{align*}
(5.2.2) \quad & R\Gamma((X/A)_\Delta^!, \mathcal{E}^\vee \{ n \}) \otimes_{A/I}^L R\Gamma((X/A)_\Delta^!, \mathcal{E}) \to R\Gamma((X/A)_\Delta^!, \mathcal{O}_\Delta \{ n \}) \to A/I[-2n],
\end{align*}
\]

where the last map is induced by the Grothendieck–Serre trace map

\[
(5.2.3) \quad H^{2n}((X/A)_\Delta^!, \mathcal{O}_\Delta \{ n \}) \cong H^n(X_{\text{ét}}, R^n \nu_{X/A,*}(\mathcal{O}_\Delta \{ n \})) = H^n(X_{\text{ét}}, \Omega^n_X) \to A/I.
\]

**Theorem 5.3.** Under the notation above, the following statements hold:

1. The pairing \((5.2.1)\) induces an isomorphism of perfect complexes of \( O_X \)-modules:

\[
R\nu_{X/A,*}(\mathcal{E}^\vee \{ n \}) \sim \text{RHom}_{O_X}(R\nu_{X/A,*}(\mathcal{E}), \Omega^n_X)[-n].
\]

2. If moreover \( X \) is proper over \( \text{Spf}(A/I) \), the pairing \((5.2.2)\) induces an isomorphism of perfect complexes of \( (A/I) \)-modules

\[
R\Gamma((X/A)_\Delta^!, \mathcal{E}^\vee \{ n \}) \cong \text{RHom}_{A/I}(R\Gamma((X/A)_\Delta^!, \mathcal{E}), A/I)[-2n]
\]

**Proof.** It is clear that statement (2) is an immediate consequence of (1) and the classical Grothendieck–Serre duality. Since statement (1) is local for the étale topology of \( X \), up to étale localization we may assume thus that \( X = \text{Spf}(R) \) satisfies the assumptions in Situation [4.1]. Let \((M, \theta)\) be the object of \( \mathcal{H}iggs^\vee(R) \) corresponding to \( \mathcal{E} \) by Theorem [4.2], and \((M^\vee \{ n \}, \theta^\vee)\) be the object corresponding to \( \mathcal{E}^\vee \{ n \} \). Then we have \( M^\vee \{ n \} = \text{Hom}_R(M, R\{ n \}) \) and \( \theta^\vee \) is given by

\[
\theta^\vee(f)(x) = -(f \otimes 1)(\theta(x)) \in \Omega^1_R\{ n-1 \},
\]

for all \( x \in M \) and \( f \in M^\vee \{ n \} \). By Theorem [4.1] the pairing \((5.2.1)\) is represented by the pairing of complexes

\[
\text{DR}^\bullet (M^\vee \{ n \}, \theta^\vee) \otimes_R \text{DR}^\bullet (M, \theta) \to \Omega^n_R[-n]
\]

given by

\[
\langle f \otimes \omega_i, x \otimes \eta_j \rangle = \begin{cases} 0 & \text{if } i + j \neq n, \\ f(x) \omega_i \wedge \eta_j & \text{if } i + j = n, \end{cases}
\]

for \( x \in M \), \( f \in M^\vee \{ n \} \), \( \omega_i \in \Omega^i_R\{ -i \} \) and \( \eta_j \in \Omega^j_R\{ -j \} \). Now, it is straightforward to verify that, with the sign convention [0.2], such a pairing induces an isomorphism of complexes

\[
\text{DR}^\bullet (M^\vee \{ n \}, \theta^\vee) \cong \text{Hom}(\text{DR}^\bullet (M, \theta), \Omega^n_R)[-n].
\]

This finishes the proof of (1). \( \square \)

**Remark 5.4.** Assume that \( X \) is proper and smooth of relative dimension \( n \) over \( \text{Spf}(A/I) \). If one can construct a trace map

\[
\text{Tr}_X : H^{2n}((X/A)_\Delta^!, \mathcal{O}_\Delta \{ n \}) \to A
\]

which reduces to the classical trace map \((5.2.3)\) when modulo \( I \) (so that \( \text{Tr}_X \) itself is an isomorphism if \( X \) is geometrically connected), then Theorem [5.3] implies a similar duality for \( \mathcal{O}_\Delta \)-crystals.
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