The spherical multipole resonance probe: kinetic damping in its spectrum

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Abstract

The multipole resonance probe is one of the more recently developed measurement devices to measure plasma parameters like electron density and temperature based on the concept of active plasma resonance spectroscopy. The dynamical interaction between the probe and the plasma in an electrostatic, kinetic description can be modeled in an abstract notation based on functional analytic methods. These methods provide the opportunity to derive a general solution, which is given as the response function of the probe-plasma system. It is defined by the matrix elements of the resolvent of an appropriate dynamical operator. Based on the general solution a residual damping for vanishing pressure can be predicted and can only be explained by kinetic effects. Within this manuscript an explicit response function of the multipole resonance probe is derived. Therefore, the resolvent is determined by its algebraic representation based on an expansion in orthogonal basis functions. This allows us to compute an approximated response function and its corresponding spectra, which show additional damping due to kinetic effects.

Keywords: multipole resonance probe, active plasma resonance spectroscopy, kinetic damping, plasma diagnostic

1. Introduction

Active plasma resonance spectroscopy (APRS) denotes a concept of plasma diagnostics in which a radio frequency signal is coupled into a plasma via an electrical probe and the frequency dependent response is recorded. Based on a mathematical model, plasma parameters like electron density and/or electron temperature can be determined from detected resonances.

This concept was first investigated with the so-called resonance probe by Takayama, Ikegami, and Miyazaki [1] in the year 1960. Since then several different designs have been invented, analyzed, and characterized [2–34]. New interest in this concept has been gained during the last two decades, especially in the context of industrially compatible plasma diagnostics. For this purpose a specific class of APRS is sufficient: probes, which excite resonances below the electron plasma frequency \( \omega_p \). Thus, the dynamical interaction between the probe and the plasma can be modeled in electrostatic approximation and can be applied without calibration. Representatives of this class are, e.g., the impedance probe (IP) [35, 36], the plasma absorption probe (PAP) [37], and the multipole resonance probe (MRP) [38, 39], just to name a few. It could be shown, that the idealized MRP represents the optimal probe design [40].

These probes are intensively investigated, experimentally and theoretically. The theoretical works cover analytic and numerical methods and plasma models of different complexity. They have in common that they focus just on a particular probe design. However, it is also of interest to study generic features of such probes independently of any particular realization. Such an analysis, based on Hilbert space methods, is presented in [40]. In the description of the cold plasma model the main result was that, for any possible probe design, the spectral response function could be expressed as a matrix element of the resolvent of the dynamical operator.

The analysis of the study of [40] can be generalized in a full kinetic description and yields an abstract kinetic model of electrostatic resonances valid for all pressures [41]. The main result of this model is similar to the result of the fluid model, which is: for any possible probe design, the spectral response of the probe-plasma system can be expressed as a matrix
element of the resolvent of the dynamical operator. In addition, it was generally shown that resonances in the spectra of APRS probes exhibit a residual damping in the limit of vanishing pressure. This residual damping can only be explained by kinetic effects and not by Ohmic dissipation.

To study generic features of APRS probes Hilbert space methods are the optimal choice, because they allow an analysis of the probe-plasma system without any specification of the geometry. These methods can also be applied to determine explicit spectra for a specific probe design. This was done in the fluid dynamical description for certain geometries of the IP, the MRP, and the planar MRP, respectively [42, 43], to determine analytic solutions of the response function. It is impossible to derive an analytic solution of the response function in the kinetic description, but an efficient algorithm can be applied to determine an adequate approximation. This was presented in [44, 45] for the parallel electrode probe (PEP) and the IP. The spectra of these probes computed with this algorithm show kinetic damping as predicted by the general analysis.

However, kinetically calculated spectra of the MRP are not presented, yet. Within this manuscript the author follows the approximation algorithm presented in [44] to compute kinetic spectra of the MRP. These spectra will be compared to that of the IP [45]. It will be shown, that the kinetic determined resonances of these probes show a similar relation to each other, like the resonances determined with a fluid model, but of course with additional kinetic damping.

2. Model of the idealized multipole resonance probe

In figure 1 the idealized MRP is depicted. It consists of two half-spherical electrodes \( \mathcal{E}_{1/2} \) of radius \( R - d \), which are surrounded by a dielectric \( \mathcal{D} \) of thickness \( d \). Applying 180° phase shifted RF voltages \( U_{1/2} \) at the electrodes the plasma will be dynamically disturbed in the surrounding of the probe. This perturbed plasma \( \mathcal{P} \) united with the dielectric is defined as the influence region of the probe \( \mathcal{V} = \mathcal{P} \cup \mathcal{D} \). In the general kinetic model of [41] an interface \( \mathcal{F} \) between the perturbed and unperturbed plasma is defined, which is treated in this manuscript as a grounded spherical surface at a large distance \( R_\infty \) on the length scale of the Debye length \( \lambda_D \) (Theoretically it is located in an infinite distance).

To describe the dynamics of the probe-plasma system in \( \mathcal{P} \) the linearized and normalized Boltzmann equation in electrostatic approximation has to be applied. The geometry of the idealized MRP is symmetric with respect to a rotation axes in \( \mathcal{Z} \) direction, if the insulation of the electrodes is assumed to be in the \( x, y \) plane. Due to that, the 6-dimensional distribution function reduces to 5 dimensions. It depends on two variables in physical space (radial distance \( r \in [R, R_\infty] \) and polar angle \( \vartheta \in [0, \pi] \)) and three variables in velocity space (absolute value of the velocity \( v \in [0, \infty) \), projection angle \( \chi \in [0, \pi] \) of \( v \) to the \( r \) direction, and rotation angle \( \psi \in [0, 2\pi] \) of \( v \) around \( r \) ). The Boltzmann equation in these variables is given by

\[
\frac{\partial g}{\partial t} + v \left( \cos(\chi) \frac{\partial}{\partial r} + \frac{\sin(\chi) \cos(\psi)}{r} \frac{\partial}{\partial \vartheta} \right) \times \left( g - \Phi - \sum_{n=1}^{2} U_n \psi_n \right) \\
+ \frac{\partial^2 \Phi}{\partial r^2} \left( \frac{\partial g}{\partial r} + \sin(\chi) \frac{\partial g}{\partial \chi} \right) \\
- \sin(\chi) \frac{v}{r} \frac{\partial g}{\partial \chi} + \cot(\vartheta) \sin(\psi) \frac{\partial g}{\partial \psi} \\
= \frac{\nu_0}{4\pi} \int_{0}^{2\pi} \int_{0}^{\pi} g \sin(\chi) d\chi \ d\psi - \nu_{bg} \quad (1)
\]

\( g \) represents the perturbed distribution function of the electrons and is defined in \( \mathcal{P} \) with homogeneous boundary conditions at the surface of the probe \( g(R, \vartheta, \chi, \psi, t) = 0 \) and the outer grounded surface \( g(R_\infty, \vartheta, \chi, \psi, t) = 0 \). Only pure elastic collisions between electrons and the neutral background are taken into account and the collision frequency \( \nu_{bg} \) is assumed to be constant.

Due to the electrostatic approximation the force term is given by the electrical potential, which is separated in three parts. \( \Phi \) is called inner potential and is a linear functional of \( g \). It obeys the Poisson equation

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \varepsilon(r) \frac{\partial \Phi}{\partial r} \right) + \frac{\varepsilon(r)}{r^2 \sin(\vartheta)} \frac{\partial}{\partial \vartheta} \left( \sin(\vartheta) \frac{\partial \Phi}{\partial \vartheta} \right) = \left\{ \begin{array}{ll} 0, & r \in \mathcal{D} \\ \int_{\mathcal{F}} \int_{\mathcal{P}} w \ g \ d^2v, & r \in \mathcal{P} \end{array} \right. \\
\quad (2)
\]

with homogeneous boundary conditions \( \Phi(R - d, \vartheta, t) = \Phi(R_\infty, \vartheta, t) = 0 \). The dielectric constant \( \varepsilon(r) \) is defined as \( \varepsilon(r) = \varepsilon_0 \) in \( \mathcal{D} \) and \( \varepsilon(r) = 1 \) in \( \mathcal{P} \). \( w \) is a positive weighting function, which is defined as the negative derivative of the equilibrium distribution \( F(e) \) with respect to the total energy \( e = \frac{1}{2}v^2 - \tilde{\Phi} \) in equilibrium. If the equilibrium distribution is assumed to be Maxwellian, \( w \) equals to

\[
w(r, v) = \frac{1}{(2\pi)^2} e^{-\frac{v^2}{2}} e^{\varepsilon_0 + \Phi(r)}, \quad (3)
\]

where \( \tilde{\Phi} \) is the equilibrium potential as second part of the potential.
The third part of the potential is represented by the radio frequency excitation of the probe in terms of the electrode functions $\psi_{1/2}(r)$. They follow Laplace’s equation

$$\frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \psi_{1/2}^{(D)} \right) + \frac{1}{\sin(\vartheta)} \frac{\partial}{\partial \vartheta} \left( \sin(\vartheta) \frac{\partial}{\partial \vartheta} \psi_{1/2}^{(D)} \right) = 0$$

(4)

and fulfill the boundary conditions

$$\psi_{1/2}^{(D)}(R - d, \vartheta) = \begin{cases} 1, & \vartheta \in \left[ 0, \frac{\pi}{2} \right) \\ 0, & \vartheta \in \left[ \frac{\pi}{2}, \pi \right] \end{cases}$$

(5)

$$\psi_{1/2}^{(P)}(R - d, \vartheta) = \begin{cases} 1, & \vartheta \in \left[ 0, \frac{\pi}{2} \right) \\ 0, & \vartheta \in \left[ \frac{\pi}{2}, \pi \right] \end{cases}$$

(6)

$$\psi_{1/2}^{(D)}(R_\infty, \vartheta) = 0$$

(7)

and the transition conditions

$$\psi_{1/2}^{(D)}(R, \vartheta) = \psi_{1/2}^{(P)}(R, \vartheta) \quad \frac{\partial \psi_{1/2}^{(D)}}{\partial r} \bigg|_R - \frac{\partial \psi_{1/2}^{(P)}}{\partial r} \bigg|_R$$

(8)

### 3. Inner admittance in terms of functional analysis

Within the last section the model of the idealized MRP is defined. Applying the results of the general analysis presented in [41], the inner current $i_1$ at electrode $E_1$ can be written in a Hilbert space notation as

$$i_1 = \sum_{n=1}^{\infty} \langle e_1 | (i\omega - T_V - T_S)^{-1} e_n \rangle U_{e_n} = \sum_{n=1}^{\infty} Y_{i n} U_{e_n}.$$ 

(9)

The inner admittance $Y_{i n}$ is determined by the scalar product of the excitation and observation vectors

$$e_{1/2} = v \left( \cos(\chi) \frac{\partial}{\partial r} + \frac{\sin(\chi) \cos(\psi)}{r} \frac{\partial}{\partial \vartheta} \right) \psi_{1/2}$$

(10)

and the resolvent of the dynamical operator $T_V + T_S$

$$Y_{i n} = \langle e_1 | (i\omega - T_V - T_S)^{-1} e_n \rangle.$$ 

(11)

$T_V$ and $T_S$ are the Vlasov and the collision operator, respectively. They, applied to a dynamical state vector $g$, are defined as follows:

$$T_V g = v \left( \cos(\chi) \frac{\partial g}{\partial r} + \frac{\sin(\chi) \cos(\psi)}{r} \frac{\partial g}{\partial \vartheta} \right)$$

$$+ \frac{\partial \Phi}{\partial r} \left( \sin(\chi) \frac{\partial g}{\partial \vartheta} - \cos(\chi) \frac{\partial g}{\partial \vartheta} \right)$$

$$+ \sin(\chi) \frac{\partial g}{\partial \vartheta} + \cot(\vartheta) \sin(\psi) \frac{\partial g}{\partial \psi}$$

(12)

$$T_S g = \frac{v_0^2}{4\pi} \int_0^{2\pi} \int_0^{\pi} g(\sin(\chi)) d\chi d\psi - v_0 g.$$

(13)

The scalar product, which is applied in (11), is of particular importance for Hilbert space methods. To allow for a physical interpretation of the final results the scalar product has to be connected to the system dynamics. This connection is fulfilled by a scalar product motivated by the kinetic free energy $\mathcal{G}$, which is defined for two dynamical state vectors $g'$ and $g$ as follows

$$\langle g'|g \rangle = \langle g'|g \rangle_p + \langle g'|g \rangle_V$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} g'^* g \ e^{-\frac{i}{\omega} - \frac{\vartheta}{\vartheta}}$$

$$\times \sin(\chi) \sin(\vartheta) v^2 r^2 \ dv \ d\chi \ d\psi \ dr \ d\vartheta$$

$$+ 2\pi \int_0^{2\pi} \int_{-\infty}^{\infty} \varepsilon^2 \ \frac{\partial \Phi^*}{\partial r} \frac{\partial \Phi}{\partial r} + \frac{\partial \Phi^*}{\partial \vartheta} \frac{\partial \Phi}{\partial \vartheta} \ d\vartheta. $$

(14)

By means of a complete orthonormal basis $\{a\}$ in the Hilbert space $H_{i n}$ can be expanded based on the scalar product (14).

The corresponding completeness relation can be entered twice into equation (11) and yields

$$Y_{i n} = \sum_{a_1} \sum_{a_2} \langle a_1 | a_2 \rangle \sum_{a} \langle a | (i\omega - T_V - T_S)^{-1} a \rangle \langle a | e_n \rangle.$$ 

(15)

Such an expansion with an orthonormal and complete basis equates to a vector-matrix-vector multiplication in which the resolvent is represented by its algebraic equivalent. To determine the algebraic representation of the resolvent, the inverse of the algebraic representation of $i\omega - T_V - T_S$ has to be calculated [42].

### 4. Explicit expansion of the inner admittance

In the previous section the general expansion of $Y_{i n}$ is shown. As presented in [44], the expansion has to be truncated for computational purposes to a finite dimension

$$Y_{i n} = e_i^T \cdot (i\omega - \mathcal{T}_V - \mathcal{T}_S)^{-1} \cdot e_i.$$ 

(16)

The finite expansion is determined by the identity matrix $I$, the collision matrix $\mathcal{T}_S$, and the Vlasov matrix $\mathcal{T}_V$. $e_i$ are the explicitly expanded excitation vectors.

To determine the identity matrix, a complete orthonormal basis is needed. An appropriate basis function in velocity space is given by

$$g_{\phi}(\chi, \psi) = \pi^{\frac{1}{4}} \Lambda_{\phi}(v) P_{\nu}^{\ell}(\cos(\chi)) e^{i\psi}.$$ 

(17)

The complex exponential function $e^{i\psi}$ is orthogonal and complete on $\psi \in [0, 2\pi]$. Its expansion index $\mu \in [\ell, \bar{\ell}]$ is restricted to a certain interval. One limit can be defined by $\bar{\ell} = \lambda$ (A further limit of $\bar{\ell}$ will be given below), where $\lambda \in \mathbb{N}_0$ is the expansion index of $P_{\nu}^{\ell}(\cos(\chi))$. They are the normalized associated Legendre polynomials, which are orthonormal and complete on the interval $\chi \in [0, \pi]$

$$P_{\nu}^{\ell}(\chi) = \frac{2\lambda + 1}{\nu!} \left( \frac{\lambda + \mu}{\lambda + \mu} \right) \frac{1}{\nu} P_{\nu}^{\ell}(\cos(\chi)).$$

(18)

$\kappa \in \mathbb{N}_0$ is the expansion index of $\Lambda_{\phi}(v)$, which are based on the generalized Laguerre polynomials $L_{\nu}^{\ell + 1/2}(\nu^2)$. They are
an adequate choice on the interval $v \in [0, \infty)$, because the weighting function $w$ has an exponential part in the absolute value of the velocity $v$. By means of an additional factor $L_{n_2}^{-\lambda+\frac{1}{2}}(\nu v^2)$ become the orthonormal functions

$$
\Lambda_{l}^{v}(v) = \frac{\kappa!}{\sqrt{\Gamma(\kappa + \lambda + \frac{1}{2})}} \left(\frac{\nu^2}{2}\right)^{\frac{\kappa}{2}} L_{n_2}^{-\lambda+\frac{1}{2}}\left(\frac{\nu^2}{2}\right)
$$

(19)

The scalar product in (14) is given by two different parts, which makes it difficult to determine an orthogonal function in physical space. As in [45], the author focuses on the first part $\langle g' \mid g \rangle_{\mathcal{P}}$, which allows to define

$$
g_{r}^{kl}(r, \vartheta) = g^{k}(r) \Theta^{l}(\vartheta).
$$

(20)

$\Theta^{l}(\vartheta)$ is an orthonormal and complete function on the interval of the polar angle

$$
\Theta^{l}(\vartheta) = \frac{(2l+1)\Gamma(l+1)!}{\sqrt{2(l-\mu+1)}\Gamma(l+\mu+1)} (1 - \cos(\vartheta))^\frac{\mu}{2} \times (1 + \cos(\vartheta))^\frac{l+\mu}{2} P_{l}^{-(\mu)}(\cos(\vartheta)),
$$

(21)

which is also an eigenfunction of the Vlasov operator. Here, $P_{l}^{-(\mu)}(\cos(\vartheta))$ are the Jacobi polynomials with their expansion index $l \in \mathbb{N}_{0}$. By means of the normalization coefficient one can identify $|\mu| \leq l$, which yields $\hat{\mu} = \min(l, \lambda)$. Finally, an appropriate basis function on the interval of the radial distance can be defined as

$$
g^{k}(r) = \frac{\sin(k \pi r - R_{\infty} - R)}{2\pi e^{\Phi_{m}(R_{\infty} - R)}}
$$

with $k \in \mathbb{N}$. It is orthonormal and complete on the interval $[R, R_{\infty}]$ and fulfills the boundary conditions $g^{k}(R) = g^{k}(R_{\infty}) = 0$.

In summary,

$$
\delta_{\mu}^{\lambda}(r, \vartheta, v, \chi, \psi) = \delta^{kl}(r, \vartheta) g^{k}(v, \chi, \psi)
$$

(23)

is an orthonormal and complete basis function based on the plasma part of the scalar product $\langle g' \mid g \rangle_{\mathcal{P}}$. Indeed, this basis function is not orthonormal on the complete scalar product (14), but a non-diagonal basis matrix $\mathbb{B}$ can be determined and diagonalized afterwards. Therefore, the complete scalar product of two basis functions has to be computed, which requires the derivative of the inner potential $\Phi$ follows Poisson’s equation (2) in which the expanded distribution

$$
g(r, \vartheta, v, \chi, \psi, \omega) = \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \sum_{\lambda=-l}^{l} \sum_{\mu=-\mu}^{\mu} \sum_{\nu=0}^{\nu} \sum_{\kappa=0}^{\kappa} \hat{\mu} \hat{\lambda} \hat{\nu}
$$

\begin{align*}
&\times \delta^{kl}(r, \vartheta) g^{k}(v, \chi, \psi) \delta_{\mu}^{\lambda}(\omega) g_{\mu}^{\lambda}(r, \vartheta, v, \chi, \psi)
\end{align*}

(24)

is entered. Expanding also $\Phi$ as

$$
\Phi(r, \vartheta, \omega) = \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \sum_{\lambda=-l}^{l} \sum_{\mu=-\mu}^{\mu} \sum_{\nu=0}^{\nu} \sum_{\kappa=0}^{\kappa} \hat{\mu} \hat{\lambda} \hat{\nu}
$$

\begin{align*}
&\times \hat{\lambda} \hat{\nu} \Phi_{\mu}^{k}(\omega) \Phi_{\lambda}(r) \Theta_{l}(\vartheta)
\end{align*}

(25)

allows to reduce the Poisson equation only to the radial dependence. It is important to note, that $\Theta^{l}(\vartheta)$ collapses to the Legendre polynomial $P_{l}^{+(\mu)}(\cos(\vartheta))$, which can be utilized to simplify the expansion: the main result from MRP’s fluid model with an expansion in Legendre polynomials in the polar angle $\vartheta$ is, that its resonance behavior is dominated by the dipole mode [40]. In the limit from low pressure to higher pressure the resonance behavior has to be equal and thus, the dipole mode has to be dominant in the kinetic description, too. Due to that, the author takes only this mode into account and sets $l = 1$.

Based on the simplified expansion, the radial dependent derivative of the inner potential is given by

$$
\frac{\partial \Phi^{l}}{\partial r} = \frac{\delta_{\lambda,0} \delta_{\mu,0}}{r^{2}} \left\{ A^{(D)}_{k}, r \in \mathcal{D} \right\}
$$

\begin{align*}
&+ \mu \int_{r_{0}}^{r} r'^{2} e^{\delta_{\mu}(r') g^{k}(r') dr' r' \in \mathcal{P}^{\prime}}
\end{align*}

(26)

(\delta_{\lambda,0}, \delta_{\mu,0} and $\delta_{\lambda,0}$ are Kronecker deltas. The derivation is presented in appendix A). From equation (26) one can see, that the inner potential $\Phi_{l}$ of a basis function is zero for all $\kappa = 0, \lambda = 0, \mu = 0$. This result can be transferred to the inner potential $\Phi_{l}$, if $\kappa' = 0, \lambda' = 0, \mu' = 0$ and simplifies the second part of the scalar product to

$$
\langle \delta_{\mu}^{\lambda}(v, \psi) \mid \delta_{\mu}^{\lambda}(v, \psi) \rangle_{\nu} = \frac{2\pi}{R_{\infty}} \int_{R_{\infty}}^{R_{\infty}} \epsilon \left( \frac{\partial \Phi_{\mu}^{k}}{\partial r} \right)_{\nu}^{2} + 2 \Phi_{\mu}^{k} \Phi_{\mu}^{k}
$$

(27)

$\mathbb{B}^{(000)}$ is not zero only if $\lambda = \lambda' = \kappa = \kappa' = \mu = \mu' = 0$ and leads to the result of the full scalar product of two basis functions, which can be separated into two different parts

$$
\langle \delta_{\mu}^{\lambda}(v, \psi) \mid \delta_{\mu}^{\lambda}(v, \psi) \rangle_{\nu} = \delta_{\kappa, \kappa'}^{\lambda, \lambda'} \Phi_{\mu}^{k} + \epsilon_{\kappa, \kappa'}^{\lambda, \lambda'}
$$

(28)

$$
\langle \delta_{\mu}^{\lambda}(v, \psi) \mid \delta_{\mu}^{\lambda}(v, \psi) \rangle_{\nu} = \delta_{\kappa, \kappa'}^{\lambda, \lambda'} \Phi_{\mu}^{k} + \epsilon_{\kappa, \kappa'}^{\lambda, \lambda'}
$$

(29)

This result of the scalar product defines the elements of the basis matrix $\mathbb{B}$, which is a block diagonal matrix. It is almost diagonal, except the first matrix $\mathbb{B}^{(000)}$ if $\lambda = \lambda' = \kappa = \kappa' = \mu = \mu' = 0$. However, it can be diagonalized with a rotation matrix $\mathbb{C}$ to find the diagonal matrix $\mathbb{D}^{(000)} = \mathbb{C}^{(000)} \mathbb{B}^{(000)} \mathbb{C}^{(000)}$. Normalizing $\mathbb{D}^{(000)}$ with its inverse leads to the identity matrix $I^{(000)} = \mathbb{D}^{(000)} \mathbb{D}^{(000)^{-1}}$. Then, $\mathbb{B}$ becomes the pure identity matrix and is the equivalent of an expansion with an orthonormal and complete basis of functions.

Since an appropriate system of basis functions is defined, the operators have to be expanded with this system. The collision operator is easier to expand, which will be presented first. Applying it to the basis function and computing the scalar product leads to the matrix elements of the collision
\[
(g_{k_1^\varepsilon}^{\Gamma_1^{\lambda_1^{\mu_1}}} | T_{S g_{k_1^\varepsilon}^{\lambda_1^{\mu_1}}})_p = \nu_0 (\delta_{0\varepsilon} \delta_{\lambda_1\lambda}^{(n)} \delta_{\mu_1\mu}^{(n)} - \delta_{\lambda_1\lambda}^{(n)} \delta_{\mu_1\mu}^{(n)}) \delta_{\mu\mu'} \delta_{k_1 \varepsilon}^{(n)}.
\]

(30)

\(\mathbf{T}_S\) is a pure diagonal matrix and in addition with zero elements on the main diagonal if \(\lambda = \lambda' = 0\). Due to that, no correction by the diagonalization is needed.

The derivation of the Vlasov matrix \(\mathbf{T}_V\) is more complicated. Its elements are determined by the scalar product between the basis functions and the Vlasov operator \(\langle g_{k_1^\varepsilon}^{\Gamma_1^{\lambda_1^{\mu_1}}} | T_{V g_{k_1^\varepsilon}^{\lambda_1^{\mu_1}}} \rangle\). As shown in [44], \(\mathbf{T}_V\) is an anti-symmetric block matrix. Due to that only inner block matrices with the indices \(k = k' = \lambda = 0\), \(\lambda' = 1\) and \(\lambda = \lambda' = 0\), \(\lambda = 1\) have to be multiplied with the rotation matrices \(\mathbb{C}\) or \(\mathbb{C}^T\) to get the correct expanded Vlasov matrix. Detailed calculations to the Vlasov matrix can be found in appendix B.

Finally, the excitation vectors \(e_{1/2}\) have to be determined. They include the electrode functions \(\psi_{1/2}\), which are solved in appendix C. Due to the anti-symmetric excitation of the probe, only the dipole mode has to be taken into account and the elements of the excitation vectors are then defined by

\[
\langle g_{k_1^\varepsilon}^{\Gamma_1^{\lambda_1^{\mu_1}}} | e_{1/2} \rangle = 2\pi \int_{R^3} r^2 e^{i\varphi(r)} g^k \frac{\partial \psi_{1/2}}{\partial r} dr \delta_{\psi_0} \delta_{\lambda_1} \delta_{\varepsilon_0}.
\]

\[
-2\pi \int_{R^3} r e^{i\varphi(r)} g^k \psi_{1/2} dr (\delta_{\psi_1} + \delta_{\psi_1^{(n)}} - \delta_{\psi_0} \delta_{\lambda_1} \delta_{\varepsilon_0}) = \pm (e_1^{(n)} \delta_{\psi_0} - e_2^{(n)} (\delta_{\psi_1} + \delta_{\psi_1^{(n)}})) \delta_{\lambda_1} \delta_{\varepsilon_0}.
\]

(31)

By means of the Kronecker Deltas it is obvious that \(e_{1/2}\) have non-vanishing elements only for \(n = 0\) and \(\lambda' = 1\). Thus, the full vectors can be written as

\[
e_{1/2} = \pm (0, \mathbb{D}^{(100)} - \mathbb{C} \mathbb{e}_{(n)}, 0, \ldots)^T.
\]

(32)

Now, all elements of the expanded admittance \(Y_{\psi'}\) in (16) are determined and the expanded current at electrode \(e_1\) can be defined. The dipole excitation of the probe with \(U_1 = -U_2 = U\) yields also anti-symmetric excitation vectors \(e_1 = -e_2\), which can be utilized to simplify the current

\[
i = \sum_{a=1}^2 e_i^T \cdot (i \omega_0 - \mathbf{I}_V - \mathbf{I}_S)^{-1} \cdot e_{\psi'} U_{\psi'}
\]

\[
= 2 e_1^T \cdot (i \omega_0 - \mathbf{I}_V - \mathbf{I}_S)^{-1} \cdot e_{1} U = Y_{\text{MRP}} U.
\]

(33)

The half widths of the resonance peaks in the spectra of the MRP are broader than the half widths of the IP spectra and increase for higher collision frequencies (see figure 3, right). This indicates that higher modes are more strongly influenced by kinetic effects than lower modes. Here, the half widths of the MRP are determined by the right part of the resonance peak and then multiplied by a factor of two, due to the asymmetric peak shapes.

The half widths in the spectra represent the damping of the probe-plasma system. They are larger than determined by fluid models, which is caused by kinetic effects. Assuming \(\Delta \omega = \nu_0 + \nu_{\text{kin}}\), the kinetic damping can be determined as \(\nu_{\text{kin}} \in [0.531, 0.692, 0.87] \omega_p\).

5. Spectra of the spherical multipole resonance probe

Within the last section an explicit expansion of the inner admittance of the idealized spherical MRP is derived and can be used to compute approximated spectra. To compare the first calculated kinetic spectra of the MRP all parameters are taken from Buckley [46] and the spectra will be compared to the kinetic spectra of the spherical IP [45].

6. Conclusion

Within this manuscript a kinetic model of the spherical multipole resonance probe is presented, where its dynamical interaction with the surrounding plasma is given by the inner admittance of the probe-plasma system. This admittance is determined by the resolvent of the dynamical operator \(T_v + T_s\), which has to be expanded to allow for the computation of the corresponding
spectra. The expanded inner admittance of the MRP is derived by means of a complete basis in its spherical geometry and leads to the matrix representation of the dynamical operator. Truncating the expansion allows to approximate the inner admittance and thus to analyze the kinetic damping within its spectra.

To compare the approximated spectra of the MRP, the parameters in the calculations are taken from former computations of spectra of the IP [45] for the three different collision frequencies \( \nu_0 \in \{0.05, 0.15, 0.25\} \omega_p \). The resonance frequencies of the MRP, which excites a dipole mode, are about 0.2 \( \omega_p \) higher than the ones from the IP, which excites a monopole mode. Such a difference can also be observed in spectra determined by a fluid model [42].

In addition the half widths of the resonance peaks are determined. It is shown that the half widths of the MRP spectra are broader than the ones from the IP. This indicates that higher resonance modes are more strongly affected by kinetic effects than the monopole mode, which can be explained by the kinetic loss mechanism as described in [41]: the probe produces kinetic free energy, which is transported through the plasma and escapes at a large distance from the probe, where the probe cannot detect it anymore. This loss of kinetic free energy is recorded in the spectrum of the probe as damping. In other words, particles in the detection region gain energy by the probe and leave this region. Due to the fact that the electric field of the dipole mode decreases with \( r^{-3}\), the detection region is smaller than the detection region of the monopole mode, where the field decreases with \( r^{-2}\). It can be assumed, that the kinetic damping \( \nu_{kin} \) is proportional to the fraction of the thermal velocity of the electrons \( \nu_{th} \) and anti-proportional to the dimension of the detection region \( R_{det} \)

\[
\nu_{kin} \propto \frac{\nu_{th}}{R_{det}}.
\]

Thus, more particles, which gain energy by the probe, can more easily or quickly leave the detection region, which is recorded in the spectrum of the MRP as stronger kinetic damping.

In summary it is shown that the approximated spectra of the MRP based on the functional analytic approach show resonance frequencies and half widths as expected from former results of the impedance probe. The stronger kinetic damping is explainable by the decreased influence range of the dipole mode.

In further works the parameter of the real MRP will be applied to compare the results with measurements. Then a parameter study for different \( n_e \) and \( T_e \) is required to analyze their influence on \( \omega_r \) and \( \Delta \omega \) and generate at least a lookup table. Due to the proportionality \( \omega_r \propto \omega_p \) the electron density can be determined from a measured resonance frequency. The half width can be assumed as \( \Delta \omega = \nu_0(T_e) + \nu_{kin}(T_e) \), where both parts are dependent on the electron temperature. In the range of \( T_e \in \{1, 7\} \text{ eV} \) the elastic electron collision frequency, e.g. in argon, can be approximated by [48]

\[
\nu_0 = \frac{\rho_N}{T_N} e^{-31.3879 + 1.609 \ln(T_e) + 0.0618 \ln(T_e)^2 - 0.1171 \ln(T_e)^3} \text{ m}^3 \text{ s}^{-1}.
\]

\( \rho_N \) represents the pressure of the neutral background gas and \( T_N \) its temperature, which are known for a certain process. In addition, the parameter study will define the dimension of the detection region \( R_{det} \), that also \( \nu_{kin}(T_e) \) is known. This will allow to determine \( T_e \) from a measured half width.

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### Appendix A. Potential of a basis function

Entering the basis function (23) into Poisson’s equation (2) the integrals over the velocity space can be solved which yields...
In the scalar product the gradient of the potential $\Phi^e(r)$ can be written as

$$
\frac{\partial}{\partial r} \left( \frac{2}{r^2} \frac{\partial \Phi^e}{\partial r} \right) = -2 \Phi^e = \delta_{0,0} \delta_{0,0} \delta_{0,0} r \quad (r \in \mathcal{D}, \quad r \in \mathcal{P})
$$

(A1)

The potentials for the different regions (plasma $\mathcal{P}$ and dielectric $\mathcal{D}$) can be solved by integration and using the boundary conditions $\Phi^{\mathcal{P}}(R_\infty) = 0$ and $\Phi^{\mathcal{D}}(R - d) = 0$:

$$
\Phi^{\mathcal{P}}(r) = A^{\mathcal{P}}_k \left( r - \frac{R^3}{r^2} \right) \delta_{0,0} \delta_{0,0} \delta_{0,0}
$$

(A2)

$$
\frac{1}{3} \int_{R}^{r'} \left( \frac{r - r'}{r^2} \right) e^{G(r')} g^k \, dr' \delta_{0,0} \delta_{0,0} \delta_{0,0}
$$

$$
\frac{1}{3} \int_{R}^{R^3} \left( r^3 - R^3 \right) e^{G(r')} g^k \, dr' \delta_{0,0} \delta_{0,0} \delta_{0,0}
$$

(A3)

The constants $A^{\mathcal{P}}_k$ and $A^{\mathcal{D}}_k$ are determined by the transition conditions

$$
\varepsilon_p \left[ \frac{\partial \Phi^{\mathcal{P}}}{\partial r} \right]_k = \left[ \frac{\partial \Phi^{\mathcal{D}}}{\partial r} \right]_k
$$

(A4)

(A5)

In the scalar product also the derivative of the potential is needed and can be written as

$$
\frac{\partial}{\partial r} \Phi^{\mathcal{P}}(r) = A^{\mathcal{P}}_k \left( 1 + \frac{2R^3}{r^3} \right) \delta_{0,0} \delta_{0,0} \delta_{0,0}
$$

$$
\frac{1}{3} \int_{R}^{r'} \left( 1 + \frac{2R^3}{r^3} \right) e^{G(r')} g^k \, dr' \delta_{0,0} \delta_{0,0} \delta_{0,0}
$$

$$
\frac{1}{3} \int_{R}^{R^3} \left( r^3 - R^3 \right) e^{G(r')} g^k \, dr' \delta_{0,0} \delta_{0,0} \delta_{0,0}
$$

(A6)

$$
\frac{\partial}{\partial r} \Phi^{\mathcal{D}}(r) = A^{\mathcal{D}}_k \left( 1 + \frac{2(R - d)^3}{r^3} \right) \delta_{0,0} \delta_{0,0} \delta_{0,0}
$$

(A7)

Appendix B. Matrix elements of the Vlasov-Operator

The Vlasov operator applied to the basis function in [44] is shown that this gradient is given by the electron particle flux within the plasma $\mathcal{P}$ and vanishes within the dielectric $\mathcal{D}$. In the geometry of the MRP one finds

$$
\frac{\partial \Phi^{\mathcal{P}}}{\partial r} = -\frac{1}{(2\pi)^2} g_{\nu,\mu} e^{G(r)}
$$

$$
\times \int_{0}^{2\pi} \int_{0}^{\infty} e^{-\frac{1}{2} g_{\nu,\mu} v^2} \sin{\chi} \cos{\chi} \, dv \, d\psi.
$$

(B2)

$$
\frac{\partial \Phi^{\mathcal{P}}}{\partial \theta} = \frac{1}{(2\pi)^2} g_{\nu,\mu} e^{G(r)}
$$

$$
\times \int_{0}^{2\pi} \int_{0}^{\infty} e^{-\frac{1}{2} g_{\nu,\mu} v^2} \sin{\chi} \cos{\chi} \, dv \, d\psi.
$$

(B3)

Due to that, the elements of the Vlasov matrix are given by

$$
\langle \psi_{\nu,\mu} | T_{V} | \psi_{\sigma, \lambda} \rangle = \frac{V_{(1)}}{(2\pi)^2} \int_{0}^{\pi} \int_{0}^{2\pi} \int_{0}^{\infty} e^{-\frac{1}{2} g_{\nu,\mu} v^2} \sin{\chi} \cos{\chi} \, dv \, d\psi \, d\delta
$$

$$
- \int_{0}^{\pi} \int_{0}^{2\pi} \int_{0}^{\infty} e^{-\frac{1}{2} g_{\nu,\mu} v^2} \sin{\chi} \cos{\chi} \, dv \, d\psi \, d\delta
$$

$$
+ \frac{V_{(2)}}{(2\pi)^2} \int_{0}^{\pi} \int_{0}^{2\pi} \int_{0}^{\infty} e^{-\frac{1}{2} g_{\nu,\mu} v^2} \sin{\chi} \cos{\chi} \, dv \, d\psi \, d\delta
$$

$$
+ \frac{V_{(3)}}{(2\pi)^2} \int_{0}^{\pi} \int_{0}^{2\pi} \int_{0}^{\infty} e^{-\frac{1}{2} g_{\nu,\mu} v^2} \sin{\chi} \cos{\chi} \, dv \, d\psi \, d\delta
$$

$$
+ \frac{V_{(4)}}{(2\pi)^2} \int_{0}^{\pi} \int_{0}^{2\pi} \int_{0}^{\infty} e^{-\frac{1}{2} g_{\nu,\mu} v^2} \sin{\chi} \cos{\chi} \, dv \, d\psi \, d\delta
$$

(B4)
with

\[
V_{kk}^{(1)} = 2\pi \int_R^{\infty} r^2 g^k \frac{\partial \Phi_k(r)}{\partial r} g^k dr, \quad (B5)
\]

\[
V_{kk}^{(2)} = -2\pi \int_R^{\infty} r^2 g^k e^{\delta(r)} \frac{\partial g^k}{\partial r} dr, \quad (B6)
\]

\[
V_{kk}^{(3)} = 2\pi \int_R^{\infty} r g^k e^{\delta(r)} g^k dr, \quad (B7)
\]

\[
V_{kk}^{(4)} = 2\pi \int_R^{\infty} r^2 g^k e^{\delta(r)} \frac{\partial \Phi_k(r)}{\partial r} dr = -V_{kk}^{(5)}, \quad (B8)
\]

\[
V_{kk}^{(5)} = -2\pi \int_R^{\infty} r^2 \frac{\partial \Phi_k(r)}{\partial r} e^{\delta(r)} g^k dr \quad (B9)
\]

\[
V_{kk}^{(6)} = 2\pi \int_R^{\infty} r g^k e^{\delta(r)} \Phi_k(r) dr = -V_{kk}^{(7)}, \quad (B10)
\]

\[
V_{kk}^{(7)} = -2\pi \int_R^{\infty} r \Phi_k(r) e^{\delta(r)} g^k dr. \quad (B11)
\]

The integrals over the velocity space in (B4) can be solved analytically, but lead to long expressions. The integrals over the physical space in equations (B5) to (B11) usually have to be solved numerically, depending on the equilibrium potential \( \Phi(r) \).

The final Vlasov matrix \( \mathbf{V} \) is an anti-symmetric block matrix, where the inner blocks are given by the matrices of the physical space \( \mathbf{V}_{\ell}^{\ell'} \) over the indices \( k \) and \( k' \). Due to the anti-symmetry, only the block matrices at the positions with the indices \( \kappa = \kappa' = \lambda = 0 \), \( \lambda' = 1 \) and \( \kappa = \kappa' = \lambda = 0 \), \( \lambda = 1 \) have to be corrected for the complete orthonormal expansion. The correct block matrices at these positions are \( \mathbf{D}_{\ell}^{0000-1/2} \mathbf{V}^{\ell} \mathbf{D}_{\ell}^{0000-1/2} \) for \( \kappa = \kappa' = \lambda = 0 \), \( \lambda' = 1 \) and \( \mathbf{V}^{\ell} \mathbf{T} \mathbf{D}_{\ell}^{0000-1/2} \mathbf{V}^{\ell} \mathbf{D}_{\ell}^{0000-1/2} \) for \( \kappa = \kappa' = \lambda = 0 \), \( \lambda = 1 \). After this correction the complete Vlasov matrix can be computed as

\[
\mathbf{V} = \sum_{i=1}^{7} \mathbf{V}_{\ell}^{\ell'}, \quad (B12)
\]

**Appendix C. Solution of the electrode functions**

To determine the electrode function, which fulfill the Laplace equation in (4), \( \psi_{1/2} \) can be expanded in the Legendre polynomials \( \tilde{P}_l(\vartheta) \) and yields

\[
\frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi_{1/2}(R)}{\partial r} \right) - \frac{l(l + 1)}{r^2} \psi_{1/2}(R) = 0. \quad (C1)
\]

The solution of the radial dependent Laplace equation is given by

\[
\psi_{1/2}(R) = \alpha_{1/2}^{(D)} R^l + \beta_{1/2}^{(D)} R^{-l} + \gamma_{1/2}^{(D)}. \quad (C2)
\]

By means of the boundary condition \( \psi_{1/2}(R_\infty, \vartheta) = 0 \) and the transition conditions

\[
\frac{\partial}{\partial \vartheta} \psi_{1/2}(R) = \frac{\partial}{\partial \vartheta} \psi_{1/2}(R), \quad (C3)
\]

\[
\frac{\partial}{\partial \vartheta} \psi_{1/2}(R) = \frac{\partial}{\partial \vartheta} \psi_{1/2}(R), \quad (C4)
\]

six coefficients can be determined

\[
\alpha_{1/2}^{(D)} = \frac{(2l + 1)\delta_d}{l(\varepsilon_d - 1)R_{\infty}^{2l+1} - R_{\infty}^{2l+1}(\varepsilon_d + l + 1)} \beta_{1/2}^{(D)}, \quad (C5)
\]

\[
\beta_{1/2}^{(D)} = \frac{(2l + 1)\delta_d R_{\infty}^{2l+1}}{R_{\infty}^{2l+1}(\varepsilon_d + l + 1) - l(\varepsilon_d - 1)R_{\infty}^{2l+1}} \alpha_{1/2}^{(D)}, \quad (C6)
\]

\[
\beta_{1/2}^{(D)} = \frac{R_{\infty}^{2l+1}(\varepsilon_d + l + 1) - (l + 1)(\varepsilon_d - 1)R_{\infty}^{2l+1}}{l(\varepsilon_d - 1)R_{\infty}^{2l+1} - R_{\infty}^{2l+1}(\varepsilon_d + l + 1)} \beta_{1/2}^{(D)} \quad (C7)
\]

The expanded electrode functions can then be written as

\[
\psi_{1/2}(R, \vartheta) = \sum_{l=0}^{\infty} \beta_{1/2}^{(D)} \tilde{P}_l(\vartheta) \psi_{1/2}(R). \quad (C8)
\]

Two coefficients are undefined, yet, and have to be solved separately with the boundary conditions

\[
\psi_{1/2}(R - d, \vartheta) = \begin{cases} 
1, & \vartheta \in \left[0, \frac{\pi}{2}\right] \\
0, & \vartheta \in \left[\frac{\pi}{2}, \pi\right]
\end{cases} \quad (C9)
\]

\[
\psi_{2/2}(R - d, \vartheta) = \begin{cases} 
0, & \vartheta \in \left[0, \frac{\pi}{2}\right] \\
1, & \vartheta \in \left[\frac{\pi}{2}, \pi\right]
\end{cases} \quad (C10)
\]

Therefore the orthonormal relation of the Legendre polynomials can be utilized and yields

\[
\beta_{1/2}^{(D)} = \frac{RR_{\infty}(d - R)}{\sqrt{2}(\varepsilon_d d - (R_{\infty} - R)(d - R))} \delta_d + \frac{\pi}{2} \left[ \frac{R_{\infty}^{2l+1}((l + 1)\varepsilon_d + l) - (l + 1)(\varepsilon_d - 1) + \frac{l(\varepsilon_d - 1)\varepsilon_d^{2l+1} - (\varepsilon_d + l + 1)}{(l + 1)\varepsilon_d + l}}{(l + 1)\varepsilon_d + l + 1} \right] R_{\infty}^{2l+1} \quad (C11)
\]
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