INDEX AND NULLITY OF THE GAUSS MAP OF THE
COSTA-HOFFMAN-MEEKS SURFACES

FILIPPO MORABITO

Abstract. The aim of this work is to extend the results of S. Nayatani about the index and the nullity of the Gauss map of the Costa-Hoffman-Meeks surfaces for values of the genus bigger than 37. That allows us to state that these minimal surfaces are non degenerate for all the values of the genus in the sense of the definition of J. Pérez and A. Ros.

Introduction

In the years 80’s and 90’s the study of the index of minimal surfaces in Euclidean space has been quite active. D. Fischer-Colbrie in [4], R. Gulliver and H. B. Lawson in [5] proved independently that a complete minimal surface \( M \) in \( \mathbb{R}^3 \) with Gauss map \( G \) has finite index if and only if it has finite total curvature. D. Fischer-Colbrie also observed that if \( M \) has finite total curvature its index coincides with the index of an operator \( L_{\bar{G}} \) (that is the number of its negative eigenvalues) associated to the extended Gauss map \( \bar{G} \) of \( \bar{M} \), the compactification of \( M \). Moreover \( N(\bar{G}) \), the null space of \( L_{\bar{G}} \), if restricted to \( M \) consists of the bounded solutions of the Jacobi equation. The nullity, \( \text{Nul}(\bar{G}) \), that is the dimension of \( N(\bar{G}) \), and the index are invariants of \( \bar{G} \) because they are independent of the choice of the conformal metric on \( M \).

The computation of the index and of the nullity of the Gauss map of the Costa surface and of the Costa-Hoffman-Meeks surface of genus \( g = 2, \ldots, 37 \) appeared respectively in the works [10] and [9] of S. Nayatani. The aim of this work is to extend his results to the case where \( g \geq 38 \).

In [10] he studied the index and the nullity of the operator \( L_G \) associated to an arbitrary holomorphic map \( G : \Sigma \to S^2 \), where \( \Sigma \) is a compact Riemann surface. He considered a deformation \( G_t : \Sigma \to S^2, t \in (0, +\infty) \), with \( G_1 = G \) (see equation (2)) and gave lower and upper bounds for the index of \( G_t \), \( \text{Ind}(G_t) \), and its nullity, \( \text{Nul}(G_t) \), for \( t \) near to 0 and \( +\infty \) and \( t = 1 \). The computation of the index and the nullity in the case of the Costa surface is based on the fact that the Gauss map of this surface is a deformation for a particular value of \( t \) of the map \( G \) defined by \( \pi \circ G = 1/\wp' \), that is its stereographic projection is equal to the inverse of the derivative of the Weierstrass \( \wp \)-function for a unit
square lattice. S. Nayatani computed \( \text{Ind}(G_t) \) and \( \text{Nul}(G_t) \) for \( t \in (0, +\infty) \), where \( G \) is the map defined above. So the result concerning the Costa surface follows as a simple consequence from that. He obtained that for this surface the index and the nullity are equal respectively to 5 and 4.

In [9] S. Nayatani extended the last result treating the case of the Costa-Hoffman-Meeks surface of genus \( g \), \( M_g \), but only for \( 2 \leq g \leq 37 \). He obtained that the index is equal to \( 2g + 3 \) and the nullity is equal to 4. Here we will show that these results continue to hold also for \( g \geq 38 \).

J. P´erez and A. Ros in [12] introduced a notion of non degenerate minimal surface in terms of the Jacobi functions having logarithmic growth at the ends of the surface. As consequence of the works [9] and [10], the Costa-Hoffman-Meeks surface was known to be non degenerate but only for \( 1 \leq g \leq 37 \).

The result of S. Nayatani about the nullity of the Gauss map of the Costa-Hoffman-Meeks surface is essential for the construction due to L. Hauswirth and F. Pacard [6] of a family of minimal surfaces with two limit ends asymptotic to half Riemann minimal surfaces and of genus \( g \) with \( 1 \leq g \leq 37 \). Their construction is based on a gluing procedure which involves the Costa-Hoffman-Meeks surface of genus \( g \) and two half Riemann minimal surfaces. In particular the authors needed show the existence of a family of minimal surfaces close to the Costa-Hoffman-Meeks surface, invariant under the action of the symmetry with respect to the vertical plane \( x_2 = 0 \), having one horizontal end asymptotic to the plane \( x_3 = 0 \) and having the upper and the lower end asymptotic (up to translation) respectively to the upper and the lower end of the standard catenoid whose axis of revolution is directed by the vector \( \sin \theta e_1 + \cos \theta e_3 \), \( \theta \leq \theta_0 \) with \( \theta_0 \) sufficiently small. That was obtained by Schäuder fixed point theorem and using the fact that the nullity of the Gauss map of the surface is equal to 4. In [9] the authors refer to this last result as a non degeneracy property of the Costa-Hoffman-Meeks surface. It is necessary to remark that here the choice of working with symmetric deformations of the surface with respect to the plane \( x_2 = 0 \), has a key role. Because of the restriction on the value of the genus which affects the result of S. Nayatani, it was not possible to prove the existence of this family of minimal surfaces for higher values of the genus.

So one of the consequences of our work is the proof of the non degeneracy of the Costa-Hoffman-Meeks surface for \( g \geq 1 \) in the sense of the definition given in [12] and also, only in a symmetric setting, in [6]. So we can state that the family of examples constructed by L. Hauswirth and F. Pacard exists for all the values of the genus.

Summarizing we will prove the following theorems.

**Theorem 1.** For \( 1 \leq g < +\infty \) the index of the Gauss map of \( M_g \) is equal to \( 2g + 3 \).
Theorem 2. For $1 \leq g < +\infty$ the null space of the Jacobi operator of $M_g$ has dimension equal to 4.

Using the definition of non degeneracy given in [12], we can also rephrase this last result giving the following statement.

Corollary 3. The surface $M_g$ is non degenerate for $1 \leq g < \infty$.

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1. Preliminaries

Let $M$ be a complete oriented minimal surface in $\mathbb{R}^3$. The Jacobi operator of $M$ is

$$L = -\Delta + 2K$$

where $\Delta$ is the Laplace-Beltrami operator and $K$ is the Gauss curvature. Moreover we suppose that $M$ has finite total curvature. Then $M$ is conformally equivalent to a compact Riemann surface with finitely many punctures and the Gauss map $G : M \to S^2$ extends to the compactified surface holomorphically. So in the following we will pay attention to a generic compact Riemann surface, denoted by $\Sigma$ and $G : \Sigma \to S^2$ a non constant holomorphic map, where $S^2$ is the unit sphere in $\mathbb{R}^3$ endowed with the complex structure induced by the stereographic projection from the north pole (denoted by $\pi$). We fix a conformal metric $ds^2$ on $\Sigma$ and consider the operator $L_G = -\Delta + |dG|^2$, acting on functions on $\Sigma$.

We denote by $N(G)$ the kernel of $L_G$. We define $\text{Nul}(G)$, the nullity of $G$, as the dimension of $N(G)$. Since $L(G) = \{a \cdot G \mid a \in \mathbb{R}^3\}$ is a three dimensional subspace of $N(G)$, then $\text{Nul}(G) \geq 3$. We denote the index of $G$, that is the number of negative eigenvalues of $L_G$, by $\text{Ind}(G)$. The index and the nullity are invariants of the map $G$: they are independent of the metric on the surface $\Sigma$. So we can consider on $\Sigma$ the metric induced by $G$ from $S^2$.

N. Ejiri and M. Kotani in [3] and S. Montiel and A. Ros in [8] proved that a non linear element of $N(G)$ is expressed as the support function of a complete branched minimal surface with planar ends whose extended Gauss map is $G$. In the following we will review briefly some results contained in [8] used by S. Nayatani in [10].
We will use some definitions and concepts of algebraic geometry. They are recalled in subsection 5.1.

Let \( \gamma \) be the meromorphic function defined by \( \pi \circ G \). Let \( p_j \) and \( r_i \) be respectively the poles and the branch points of \( \gamma \). We denote by \( P(G) = \sum_{j=1}^{\nu} n_j p_j \), \( S(G) = \sum_{i=1}^{\mu} m_i r_i \) respectively the polar and ramification divisor of \( \gamma \). Here \( n_j, m_i \) denote, respectively, the multiplicity of the pole \( p_j \) and the multiplicity with which \( \gamma \) takes its value at \( r_i \). We define on the surface \( \Sigma \) the divisor

\[
D(G) = S(G) - 2P(G)
\]

and introduce the vector space \( \tilde{H}(G) \) (see [8], theorem 4)

\[
\tilde{H}(G) = \left\{ \omega \in H^{0,1}(k_\Sigma + D(G)) \mid \text{Res}_{r_i} \omega = 0, 1 \leq i \leq \mu, \right. \\
\left. \text{Re} \int_\alpha (1 - \gamma^2, i(1 + \gamma^2), 2\gamma) \omega = 0, \forall \alpha \in H_1(\Sigma, \mathbb{Z}) \right\},
\]

where \( k_\Sigma \) is a canonical divisor of \( \Sigma \) and \( H_1(\Sigma, \mathbb{Z}) \) is the first group of homology of \( \Sigma \). Suppose that the divisor \( D \) has an expression of the form \( \sum n_j v_j - \sum m_i u_i \), with \( n_j, m_i \in \mathbb{N} \). An element of \( H^{0,1}(D) \) can be expressed as \( f dz \), where \( f \) is a meromorphic function on \( \Sigma \) with poles of order not bigger than \( n_j \) at \( v_i \) and zeroes of order not smaller than \( m_i \) at \( u_i \). Equivalently, if \( g dz \), where \( g \) is a meromorphic function, is the differential form associated with the divisor \( D \), the product \( fg \) must be holomorphic.

For \( \omega \in \tilde{H}(G) \), let \( X(\omega) : \Sigma \setminus \{r_1, \ldots, r_\mu\} \to \mathbb{R}^3 \) be the conformal immersion defined by

\[
X(\omega)(p) = \text{Re} \int_0^p (1 - \gamma^2, i(1 + \gamma^2), 2\gamma) \omega.
\]

Then \( X(\omega) \cdot G \), the support function of \( X(\omega) \), extends over the ramification points \( r_1, \ldots, r_\mu \) smoothly and thus gives an element of \( N(G) \). Conversely, every element of \( N(G) \) is obtained in this way. In fact the map

\[
i : \tilde{H}(G) \to N(G)/L(G) \\
\omega \mapsto [X(\omega) \cdot G]
\]

is an isomorphism. This result, used in association with the Weierstrass representation formula, gives a description of the space \( N(G) \). To obtain the dimension of \( N(G) \) it is sufficient to compute the dimension of \( \tilde{H}(G) \). Since the dimension of \( L(G) \) is equal to 3, then \( \text{Nul}(G) = 3 + \dim \tilde{H}(G) \).

We denote by \( A_t \) a one parameter family \( (0 < t < +\infty) \) of conformal diffeomorphisms of the sphere \( S^2 \) defined by

\[
\pi \circ A_t \circ \pi^{-1} w = tw, \quad w \in \mathbb{C} \cup \{\infty\}.
\]
We define for $0 < t < \infty$

\[(2)\quad G_t = A_t \circ G.\]

S. Nayatani in [10] gave lower and upper bounds for the index and, applying the method recalled above, for the nullity of $G_t$, $t \in (0, \infty)$, a deformation of an arbitrary holomorphic map $G : \Sigma \to S^2$, where $\Sigma$ is a compact Riemann surface. In the same work, choosing appropriately the map $G$ and the surface $\Sigma$, he computed the index and the nullity for the Gauss map of the Costa surface. In fact the extended Gauss map of this surface is a deformation of $G$ for a particular value of $t$. We describe briefly the principal steps to get this result.

Firstly it is necessary to study the vector space $\bar{H}(G_t)$. A differential $\omega \in H^{0,1}(k_\Sigma + D(G))$ with null residue at the ramification points, is an element of $\bar{H}(G_t)$ if and only if the pair $(t\gamma, \omega)$ defines a branched minimal surface by the Weierstrass representation. If one sets $\gamma = 1/\wp'$ then there exist only two values of $t$, denoted by $t' < t''$, for which the condition above is verified and moreover $\dim H(G_t) = 1$. In other words, thanks to the characterization of the non linear elements of $N(G_t)$ by the isomorphism described by (7), if $t = t', t''$, $\text{Nul}(G_t) = 4$. As for the index, if $t = t', t''$ then $\text{Ind}(G_t) = 5$. Since $G_{t''}$ is the extended Gauss map of the Costa surface, one can state:

**Theorem 4.** Let $\bar{G}$ be the extended Gauss map of the Costa surface. Then $\text{Nul}(\bar{G}) = 4$, $\text{Ind}(\bar{G}) = 5$.

The same author in [9] treated the more difficult case of the Costa-Hoffman-Meeks surfaces of genus $2 \leq g \leq 37$ by a slightly different method. That is the subject of next section.

**2. The case of the Costa-Hoffman-Meeks surface of genus smaller than 38**

In this section we expose some of the background details at the base of section 3 of the work [9]. S. Nayatani provided them to us in [11].

We denote by $M_g$ the Costa-Hoffman-Meeks surface of genus $g$. Let $\Sigma_g$ be the compact Riemann surface

\[(3)\quad \Sigma_g = \{(z, w) \in (C \cup \{\infty\})^2 \mid w^{g+1} = z^g(z^2 - 1)\}

and let $Q_0 = (0, 0), P_+ = (1, 0), P_- = (-1, 0), P_\infty = (\infty, \infty)$. It is known that $M_g = \Sigma_g \setminus \{P_+, P_-, P_\infty\}$.

The following result describes the properties of symmetry of $M_g$ and $\Sigma_g$. 
Lemma 5. ([7]) Consider the conformal mappings of \((\mathbb{C} \cup \{\infty\})^2\):
\[
\kappa(z, w) = (z, \bar{w}) \quad \lambda(z, w) = (-z, \rho w),
\]
where \(\rho = e^{i\pi} \in \mathbb{R}\). The map \(\kappa\) is of order 2 and \(\lambda\) is of order \(2g + 2\). The group generated by \(\kappa\) and \(\lambda\) is the dihedral group \(D_{2g+2}\). This group of conformal diffeomorphisms leaves \(M_g\) invariant, fixes both \(Q_0\) and \(P_\infty\) and extend to \(\Sigma_g\). Also \(\kappa\) fixes the points \(P_\pm\) while \(\lambda\) interchanges them.

We set \(\gamma(w) = w\). Let \(G : \Sigma_g \to S^2\) be the holomorphic map defined by
\[
\pi \circ G(z, w) = \gamma(w).
\]
We denote by \(r_i, i = 1, \ldots, \mu\), the ramification points of \(\gamma\) and by \(R(G)\) the ramification divisor \(\sum_{i=1}^\mu r_i\). Theorem 5 of [8] shows that the space \(N(G)/L(G)\), that we have introduced in previous section, is also isomorphic to a space of meromorphic quadratic differentials. This alternative description of \(N(G)/L(G)\) that we present in the following, was adopted by S. Nayatani in [9]. We start defining the vector spaces \(\hat{H}(G)\) and \(H(G)\).

\[
\hat{H}(G) = \left\{ \sigma \in H^{0,2}(2k_\Sigma + R(G)) \mid \text{Res}_{r_i} \frac{\sigma}{d\gamma} = 0, i = 1, \ldots, \mu \right\},
\]
\[
H(G) = \left\{ \sigma \in \hat{H}(G) \mid \text{Re} \int_\alpha (1 - \gamma^2, i(1 + \gamma^2), 2\gamma) \frac{\sigma}{d\gamma} = 0, \forall \alpha \in H_1(\Sigma, \mathbb{Z}) \right\},
\]
where \(k_\Sigma\) is a canonical divisor of \(\Sigma\). We remark that the elements of \(H^{0,2}(2k_\Sigma + R(G))\) are quadratic differentials (see subsection 5.1). Since hereafter we will work only with quadratic differentials, we can set \(H^0(\cdot) = H^{0,2}(\cdot)\) to simplify the notation. If we suppose that the divisor \(2k_\Sigma + R(G)\) has an expression of the form \(\sum n_j v_j - \sum m_i u_i\), with \(n_j, m_i \in \mathbb{N}\), an element of \(H^0(2k_\Sigma + R(G))\) can be expressed as \(f(z \bar{z})^2\), where \(f\) is a meromorphic function on \(\Sigma\) with poles of order not bigger than \(n_j\) at \(v_i\) and zeroes of order not smaller than \(m_i\) at \(u_i\). Equivalently, if \(g(z \bar{z})^2\), where \(g\) is a meromorphic function, is the differential form associated with the divisor \(2k_\Sigma + R(G)\), the product \(fg\) must be holomorphic.

For \(\sigma \in H(G)\), let \(X(\sigma) : \Sigma \setminus \{r_1, \ldots, r_\mu\} \to \mathbb{R}^3\) be the conformal immersion defined by
\[
X(\sigma)(p) = \text{Re} \int^p (1 - \gamma^2, i(1 + \gamma^2), 2\gamma) \frac{\sigma}{d\gamma}.
\]
Then \(X(\sigma) \cdot G\), the support function of \(X(\sigma)\), extends over the ramification points \(r_1, \ldots, r_\mu\) smoothly and thus gives an element of \(N(G)\). Conversely, every element of \(N(G)\) is obtained in this way. In fact the map
\[
i : H(G) \to N(G)/L(G)
\]
\[
\sigma \to [X(\sigma) \cdot G]
\]
is an isomorphism. So to obtain the dimension of $N(G)$ it is sufficient to compute the dimension of $H(G)$. We recall that the dimension of $L(G)$ is equal to 3, so $\text{Nul}(G) = 3 + \dim H(G)$.

Since the extended Gauss map of the Costa-Hoffman-Meeks surfaces is a deformation in the sense of the definition (2) of the map $G$, we need to study the space $H(G_t)$. From (6) and (2) it is clear that $\hat{H}(G) = \hat{H}(G_t)$ and

$$H(G_t) = \left\{ \sigma \in \hat{H}(G_t) \mid \text{Re} \int_{\alpha} (1 - t^2 \gamma^2, i(1 + t^2 \gamma^2), 2t \gamma) \frac{\sigma}{d\gamma} = 0, \forall \alpha \in H_1(\Sigma_g, \mathbb{Z}) \right\}.$$  

Long computations ([11], see subsection 5.2 for some details) show that a basis of the differentials of the form $\sigma/d\gamma$, where $\sigma \in \hat{H}(G_t)$, and whose residue at the ramification points of $\gamma(w) = w$ is zero, is formed by

$$\omega_k^{(1)} = \frac{z^{k-1} dw}{w^k}, \quad \text{with} \quad k = 1, \ldots, g - 1,$$

$$\omega_k^{(2)} = \frac{(k - 2) z^2 - k A^2}{(z^2 - A^2)^2} \left( \frac{z}{w} \right)^{k-1} \frac{dz}{w}, \quad \text{with} \quad k = 0, \ldots, g,$$

$$\omega_k^{(3)} = \frac{(k - 2) z^2 - k A^2}{w(z^2 - A^2)^2} \left( \frac{z}{w} \right)^{k-1} \frac{dz}{w}, \quad \text{with} \quad k = 0, \ldots, g - 1,$$

where $A = \sqrt{\frac{g}{g+2}}$.

Now we put attention to the space $H(G_t)$. We recall that we are interested in the computation of its dimension. By the definition of $H(G_t)$, a differential $\sigma \in \hat{H}(G_t)$ belongs to $H(G_t)$ if and only if $\forall \alpha \in H_1(\Sigma_g, \mathbb{Z})$ the differential form $\omega = \frac{\sigma}{d\gamma} = \frac{\sigma}{dw}$ satisfies

$$\int_{\alpha} \omega = t^2 \int_{\alpha} \gamma^2(\omega),$$

$$\text{Re} \int_{\alpha} \gamma(\omega) = 0.$$  

If these two conditions are satisfied then $(\gamma, w)$ are the Weierstrass data of a branched minimal surface. Of course, it is sufficient to impose that these equations are satisfied when $\alpha$ varies between the elements of a basis of $H_1(\Sigma_g, \mathbb{Z})$. The convenient basis of $H_1(\Sigma_g, \mathbb{Z})$ is constructed as follows. Let $\beta(s) = \frac{1}{2} + e^{2\pi s}$, $0 \leq s \leq 1$. Let $\tilde{\beta}(s) = (\beta(s), w(\beta(s)))$ be a lift of $\beta$ to $\Sigma_g$ such that, for example, $\tilde{\beta}(0) = (\frac{1}{2}, w(0))$, with $w(0) \in \mathbb{R}$. As stated in lemma 5, the group of conformal diffeomorphisms of $\Sigma_g$ is isomorphic to the dihedral group $D_{2g+2}$. The collection $\{ \lambda_l \circ \tilde{\beta}, l = 0, \ldots, 2g - 1 \}$, where $\lambda$ is the generator of $D_{2g+2}$ of order $2g + 2$, is a basis of $H_1(\Sigma_g, \mathbb{Z})$ (see [7]).
Now we must impose (8) and (9) for \( \alpha = \lambda^l \circ \tilde{\beta} \), with \( l = 0, \ldots, 2g - 1 \). To do that we collapse \( \beta \) to the unit interval. In other terms we deform continuously \( \beta \) in such a way the limit curve is the union of two line segments lying on the real line. We set

\[
\omega = \sum_{0}^{g-1} c_k^{(1)} \omega_k^{(1)} + \sum_{0}^{g} c_k^{(2)} \omega_k^{(2)} + \sum_{0}^{g-1} c_k^{(3)} \omega_k^{(3)},
\]

where \( c_k^{(i)} \in \mathbb{C} \).

Taking into account these assumptions, it is possible to show that the equation (8), if the genus \( g \) is 2, is equivalent to the following system of four equations (see subsection 5.3)

\[
\begin{cases}
  f_0 = -t^2 \bar{h}_0 \\
  f_1 = 0 \\
  p_1 = -t^2 \bar{q}_1 \\
  p_2 = -t^2 \bar{q}_0.
\end{cases}
\]

If \( g \geq 3 \) there are the following additional \( 2g - 4 \) equations to consider

\[
\begin{cases}
  f_k = -t^2 \bar{q}_{g-k+2} \\
  p_{g-k+2} = -t^2 \bar{q}_k
\end{cases}
\]

where \( k = 2, \ldots, g - 1 \) and

\[
\begin{align*}
  f_0 &= \frac{(g+2)^2}{2(g+1)} c_0^{(3)} \sin \left( \frac{\pi}{g+1} \right) K_0, \\
  f_k &= \left( -c_k^{(1)} + \frac{(g+2)(g+2+k)}{2(g+1)} c_k^{(3)} \right) \sin \left( \frac{(k+1)\pi}{g+1} \right) K_k, \quad k = 1, \ldots, g - 1, \\
  h_0 &= \frac{(g+2)^2}{2(g+1)} c_0^{(3)} \sin \left( -\frac{\pi}{g+1} \right) J_0, \\
  h_k &= \left( c_k^{(1)} + \frac{(g+2)(g+2-k)}{2(g+1)} c_k^{(3)} \right) \sin \left( \frac{(k-1)\pi}{g+1} \right) J_k, \quad k = 2, \ldots, g - 1, \\
  p_k &= -\frac{(g+2)k}{2(g+1)} c_k^{(2)} \sin \left( \frac{k\pi}{g+1} \right) I_k, \quad k = 1, \ldots, g, \\
  q_k &= \frac{(g+2)(2g+4-k)}{2(g+1)} c_k^{(2)} \sin \left( \frac{(k-2)\pi}{g+1} \right) L_k, \quad k = 0, 1, 3, \ldots, g,
\end{align*}
\]

and

\[
I_m = \frac{g+1}{m} \frac{\Gamma \left( 1 + \frac{m}{2(g+1)} \right) \Gamma \left( 1 - \frac{m}{g+1} \right) - \frac{m}{2(g+1)}}{\Gamma \left( 1 - \frac{m}{g+1} \right)},
\]
\[ J_m = \frac{g + 1}{g - m + 2} \frac{\Gamma\left(\frac{1}{2} + \frac{m-1}{2(g+1)}\right) \Gamma\left(1 - \frac{m-1}{g+1}\right)}{\Gamma\left(\frac{1}{2} - \frac{m-1}{2(g+1)}\right)}, \]
\[ K_m = J_{m+2}, \]
\[ L_m = \frac{m - 2}{2g - m + 4} I_{m-2}. \]

The equation (9) if the genus \( g \) is 2, is equivalent to the following system of two equations (see subsection 5.3)
\[
\begin{cases}
  d_1 = 0 \\
  e_2 = \bar{e}_0.
\end{cases}
\]

If \( g \geq 3 \) there are the following additional \( g - 2 \) equations to consider
\[
d_k = \bar{e}_{g-k+2}
\]
where \( k = 2, \ldots, g - 1 \), and
\[
d_k = \left( c^{(1)}_k - \frac{k(g+2)}{2(g+1)} c^{(3)}_k \right) \sin\left( \frac{k\pi}{g+1} \right) I_k, \quad k = 1, \ldots, g - 1,
\]
\[
e_k = \frac{(g+2)(g+2-k)}{2(g+1)} c^{(2)}_k \sin\left( \frac{(k-1)\pi}{g+1} \right) J_k, \quad k = 0, 2, \ldots, g.
\]

We are looking for the values of \( t \) such that the previous systems have non trivial solutions in terms of \( c^{(j)}_i \). Only for these special values of \( t \) it holds \( \dim H(G_t) > 0 \) or equivalently \( \text{Nul}(G_t) > 3 \).

We start with the analysis of the system (10). This system admits non trivial solutions if and only if \( t \) takes three values denoted by \( t_1, t_2, t_3 \). Obviously they are functions of \( g \).

If we set \( s = \frac{1}{g+1} \) then we can write
\[
t_1 = \sqrt{\frac{K_0}{J_0}} = \sqrt{1 - s^2} \frac{\sqrt{\Gamma(1-s) \Gamma(1-s/2)}}{\sqrt{\Gamma(1+s) \Gamma(1+s/2)}},
\]
\[
t_2 = \sqrt{\frac{I_1}{(2g+3)L_1}} = \frac{\sqrt{\Gamma(1-s) \Gamma(1+s/2)}}{\sqrt{\Gamma(1+s) \Gamma(1-s/2)}},
\]
\[
t_3 = \sqrt{\frac{I_2J_0}{gL_0K_0}} = \frac{2}{1-s} \sqrt{\frac{\Gamma(1+s)}{\Gamma(1-s)}} \sqrt{\frac{\Gamma(1-2s) \Gamma(3/2-s/2)}{\Gamma(1+2s) \Gamma(1/2+s/2)}}.
\]
We recall that if \( g \geq 3 \) there are other equations to consider. They are
\[
\begin{align*}
  f_k &= -t^2 q_{g-k+2} \\
  p_{g-k+2} &= -t^2 h_k \\
  d_k &= \bar{e}_{g-k+2}
\end{align*}
\]
where \( k = 2, \ldots, g - 1 \). Thanks to the particular structure of the equations, it is possible
to study separately for each set of three equations the existence of solutions. Each set
of three equations admits non trivial solutions if and only if the following matrix has
determinant equal to zero
\[
\begin{pmatrix}
  -K_k & (g + 2 + k)K_k & (g + 2 + k)t^2 L_{g-k+2} \\
  t^2 J_k & (g + 2 - k) t^2 J_k & (g + 2 - k) I_{g-k+2} \\
  I_k & -k I_k & -k J_{g-k+2}
\end{pmatrix}.
\]
After the change of variable \( l = g - k + 1 \) so that \( 2 \leq l \leq g - 1 \), it is possible to show
that the determinant is
\[(14)\]
\[- (g + 2)(at^4 + bt^2 + c),\]
with
\[
\begin{align*}
  a &= (2g - l + 3) I_{g-l+1} J_{g-l+1} L_{l+1} \\
  b &= -2(g - l + 1) I_{l+1} J_{g-l+1} K_{g-l+1} \\
  c &= (l + 1) I_{g-l+1} I_{l+1} K_{g-l+1}.
\end{align*}
\]
We are interested in finding the positive values of \( t \) such that
\[(15)\]
\[at^4 + bt^2 + c = 0.\]
To simplify the notation we introduce the following three functions
\[
\begin{align*}
  F(v) &= \left( \frac{\Gamma \left( \frac{1}{2} + \frac{v}{2} \right)}{\Gamma \left( \frac{1}{2} - \frac{v}{2} \right)} \right)^2 \frac{\Gamma(1 - v)}{\Gamma(1 + v)}, \\
  I(v) &= \left( \frac{\Gamma \left( 1 - \frac{v}{2} \right)}{\Gamma \left( 1 + \frac{v}{2} \right)} \right)^2 \frac{\Gamma(1 + v)}{\Gamma(1 - v)}, \\
  L(v) &= \left( \frac{\Gamma \left( 1 + \frac{v}{2} \right)}{\Gamma \left( 1 - \frac{v}{2} \right)} \right)^2 \frac{\Gamma(1 - v)}{\Gamma(1 + v)} = \frac{1}{I(v)}.
\end{align*}
\]
The discriminant \( b^2 - 4ac \) of the equation \[(15)\], seen like an equation of degree two in
the variable \( t^2 \), is negative if and only if \( X = b^2 / 4ac < 1 \). It is possible to show that
\[(16)\]
\[X = \frac{l^2}{l^2 - 1} F^2 \left( \frac{l}{g+1} \right) I \left( \frac{l-1}{g+1} \right) I \left( \frac{l+1}{g+1} \right).\]
S. Nayatani showed that if \( 2 \leq g \leq 37 \), then \( X < 1 \) and as consequence the equation
\[(15)\] has not any solution since its discriminant is negative. Then \( \dim H(G_t) > 0 \) only for
\( t = t_1, t_2, t_3 \). Summarizing we can state (see [9] for other details):
Theorem 6. If $2 \leq g \leq 37$ and $t \in (0, +\infty)$, then

$$\text{Nul}(G_t) = \begin{cases} 
4 & \text{if } t = t_1, t_2 \\
5 & \text{if } t = t_3 \\
3 & \text{elsewhere}.
\end{cases}$$

Since the extended Gauss map of the Costa-Hoffman-Meeks surfaces is exactly $G_{t_2}$, it is possible to state that the null space of the Jacobi operator of $M_g$ has dimension equal to 4 for $2 \leq g \leq 37$.

Other values of $t$ for which $\text{Nul}(G_t) > 3$ are admitted only if $g \geq 38$. In [9] S. Nayatani conjectured these values were bigger than $t_3$. The proof of the conjecture and its consequences will be showed in sections 3 and 4.

3. The case $g \geq 38$

S.Nayatani proved that $X$ is a decreasing function in the variables $l,$

$$x = \frac{l}{g+1}, \quad y = \frac{l+1}{g+1}, \quad z = \frac{l-1}{g+1}$$

with $2 \leq l \leq g - 1$. We recall that we have set $s = \frac{1}{g+1}$. We know that for $l = 2$ and $g = 37$ the discriminant of the equation (15) is negative. For these values of $l$ and $g$ the variables $x, y, z, s$ are respectively equal to $x_{\text{max}} = 2s_{\text{max}}, y_{\text{max}} = 3s_{\text{max}}, z_{\text{max}} = s_{\text{max}} = 1/38$. Then we will study the solutions of (15) for $i \in [0, i_{\text{max}}]$ (we call admissible values the values in these intervals ) where $i$ denotes $x, y, z, s$, because for bigger values of the variables the discriminant continues to be negative and so the equation (15) does not admit solutions.

All the solutions of (15), that we denote by $t_{\pm}(l, g)$, satisfy $t_{\pm}^2(l, g) = T_1 \pm T_2$, with

(17) $$T_1 = \frac{l}{l-1} F(x) I(z)$$

and

(18) $$T_2 = \sqrt{\left(\frac{l}{l-1}\right)^2 F^2(x) I^2(z) - \frac{l+1}{l-1} L(y) I(z)}.$$

We will prove that, for all the values of $l$ and $g$, such that $0 \leq \frac{l}{g+1} \leq x_{\text{max}} = \frac{2}{38}$, with $2 \leq l \leq g - 1$ and $g \geq 38$, such that $T_2$ is a real number, it holds

(19) $$t_{3}^2(s) < t_{\pm}^2(l, g).$$

We need study the behaviour of the functions $F, I, L, F^2, I^2$ that appear in (17) and (18). This aim is pursued by the use of zero order series of these functions.
The Mac-Laurin series of the functions $F(x), G(z), L(y), F^2(x), I^2(z)$ for admissible values of $x, y, z$ are

\begin{equation}
F(x) = 1 + R_F(d_1)x, \quad I(z) = 1 + R_I(d_2)z, \quad L(y) = 1 + R_L(d_3)y,
\end{equation}

\begin{equation}
F^2(x) = 1 + R_{F^2}(c_1)x, \quad I^2(x) = 1 + R_{I^2}(c_2)x,
\end{equation}

where $c_i, d_i \in (0, 1)$. So we can write

\begin{align*}
F(x)I(z) &= 1 + R_{FI}(x,z), \quad F^2(x)I^2(z) = 1 + R_{F^2I^2}(x,z), \quad L(y)I(z) = 1 + R_{LI}(y,z),
\end{align*}

with

\begin{align*}
R_{FI}(x,z) &= R_F(d_1)x + R_I(d_2)z + R_F(d_1)R_I(d_2)xz, \\
R_{F^2I^2}(x,z) &= R_{F^2}(c_1)x + R_{I^2}(c_2)z + R_{F^2}(c_1)R_{I^2}(c_2)xz, \\
R_{LI}(y,z) &= R_L(d_3)y + R_I(d_2)z + R_I(d_2)R_L(d_3)zy.
\end{align*}

In the following $\psi(x)$ denotes the digamma function. It is related to $\Gamma(x)$, the gamma function, by

$$
\psi(x) = \frac{d}{dx}(\ln \Gamma(x)).
$$

For the properties of these special functions we will refer to [1].

The following proposition gives useful properties of the functions just introduced.

**Proposition 7.** If $x \in [0, x_{max}], z \in [0, z_{max}]$ and $y \in [0, y_{max}]$, the following assertions hold:

1. $R_F(x) < 0$
2. $R_I(z) \leq 0$
3. $R_L(y) \geq 0$
4. $(R_F)_x(x) > 0$
5. $\min (R_I)_z(z) = -0.095 \ldots$
6. $R_{FI}(x,z) \geq Cx$ with $C = -4 \ln 2$
7. $R_{LI}(y,z) \geq 0$
8. $R_{I^2}(z) \leq 0$
9. $W(x) = R_{F^2}(x) < 0$
10. $W'_x(x) > 0$, so $R_{F^2}(x)$ is an increasing function
11. $W''_{xx}(x) < 0$
12. $W''_{xxx}(x) > 0$
13. If we set $Y(x) = xW(x)$, then $Y'_x(x) < 0$
14. $Y''_{xx}(x) > 0$
15. $Y'''_{xxx}(x) < 0$.

**Proof.**
(1) $R_F(x) = F'_x(x) = F(x)\Psi_F(x)$, where

$$\Psi_F(x) = -\psi(1 - x) - \psi(1 + x) + \psi\left(\frac{1}{2} - \frac{x}{2}\right) + \psi\left(\frac{1}{2} + \frac{x}{2}\right).$$

We observe that

$$\Psi_F(x) = 2 \sum_{k=0}^{\infty} \frac{1}{(2k)!} \left( \frac{1}{2^{2k}} \psi^{(2k)}(1) \left(\frac{1}{2}\right) - \psi^{(2k)}(1) \right) x^{2k}.$$ 

Since $\Psi_F(0) = 2\psi\left(\frac{1}{2}\right) - 2\psi(1) = -4\ln 2$, $\psi^{(2k)}(1) < 0$ and $\psi^{(2k)}\left(\frac{1}{2}\right) = (2^{2k+1} - 1)\psi^{(2k)}(1) < 0$, if $k \geq 1$ (see formulas 6.4.2 and 6.4.4 of [1]), we can conclude that $\Psi_F(x) < 0$ and it is a decreasing function. Since $F(x) > 0$ then $R_F(x) < 0$ and $F(x)$ is a decreasing function.

(2) $R_I(z) = I'_z(z) = I(z)\Psi_I(z)$, where

$$\Psi_I(z) = \psi(1 - z) + \psi(1 + z) - \psi\left(1 - \frac{z}{2}\right) - \psi\left(1 + \frac{z}{2}\right).$$

We observe that

$$\Psi_I(z) = 2 \sum_{k=1}^{\infty} \frac{1}{(2k)!} \psi^{(2k)}(1) \left(1 - \frac{1}{2^{2k}}\right) z^{2k}.$$ 

Since $\psi^{(2k)}(1) < 0$ for $k \geq 1$ then $\Psi_I(z) \leq 0$ and it is a decreasing function. Since $I(z) > 0$ then $R_I(z) \leq 0$.

(3) $R_L(y) = L'_y(y) = L(y)\Psi_L(y)$, where $\Psi_L(y) = -\Psi_I(y)$. Then $\Psi_L(y) \geq 0$ and it is an increasing function. Since $L(y) = 1/I(y) > 0$, then $R_L(y) \geq 0$.

(4) The derivative of $R_F$ is $F''_{xx}(x) = F(x)\Psi_F^2(x) + (\Psi_F')_x(x)$. Since $\Psi_F(x) < 0$ and it is a decreasing function, $\Psi_F^2(x) > 0$ and increasing. It holds $\Psi_F^2(x) \geq \Psi_F^2(0) = 16\ln^2 2$.

$$\left(\Psi_F\right)'_x(x) = 2 \sum_{k=1}^{\infty} \frac{1}{(2k-1)!} \left( \frac{1}{2^{2k}} \psi^{(2k)}(1) \left(\frac{1}{2}\right) - \psi^{(2k)}(1) \right) x^{2k-1}.$$ 

All the coefficients of the series are negative (see the point [1]) so $(\Psi_F)'_x(x) \leq 0$ and it is a decreasing function. In particular $(\Psi_F)_x(x) \geq (\Psi_F)_x(x_{max}) = -0.19 \cdots$. Since $F(x) > 0$ and it is a decreasing function we can conclude that

$$F''_{xx}(x) \geq F(x_{max})(\Psi_F^2(0) + (\Psi_F)'_x(x_{max})) = 6.4 \cdots.$$ 

(5) The derivative of $R_I$ is $I''_{zz}(z) = I(z)(\Psi_I^2(z) + (\Psi_I)'_z(z))$. Since $\Psi_I(z) \leq 0$ and it is a decreasing function (see the point [2]), $\Psi_I^2(z) \geq 0$ and increasing. It holds
\[ \Psi_1^2(z) \leq \Psi_1^2(z_{\max}) = 1.5 \cdot \cdots \cdot 10^{-6}. \]

\[ (\Psi_I)_z'(z) = 2 \sum_{k=1}^{\infty} \frac{1}{(2k-1)!} \psi^{(2k)}(1) \left( 1 - \frac{1}{2^{2k}} \right) z^{2k-1}. \]

All the coefficients of the series are negative so \((\Psi_I)_z'(z) \leq 0\) and it is a decreasing function. In particular \((\Psi_I)_z'(z) \geq (\Psi_I)_z'(z_{\max}) = -0.095 \cdots \). Since \(I(z) > 0\) and it is a decreasing function we can conclude that \(I''_{zz} \geq I(z_{\max})(\Psi_1^2(0) + (\Psi_I)_z'(z_{\max})) = -0.095 \cdots \).

(6) Since \(R_F < 0\) and \(R_t \leq 0\), it holds that
\[ R_{FI}(x, z) \geq R_F(d_1 x) + R_I(d_2 z), \]
where \(d_i \in (0, 1)\). The point [4] implies that \(R_F\) is an increasing function and we have computed the positive minimum (that we denote by \(m\)) value of its derivative. Thanks to the point [5] we have \(m > |n|\), where \(n\) denotes the negative minimum value of the derivative of \(R_f\). Now we observe that
\[ R_F(d_1 x) + R_I(d_2 z) \geq (R_F(0) + mx)x + (R_I(0) + nz)z \geq R_F(0)x + R_I(0)z = Cx. \]
To obtain this chain of inequalities we used the fact that \(m + n > 0\) and \(x \geq z\). Then \(R_{FI} \geq Cx\).

(7) We recall that \(R_{LI}(y, z) = L(y)I(z) - 1\), \(L(t) = 1/I(t)\) and
\[ y = \frac{l + 1}{g + 1} > \frac{l - 1}{g + 1} = z. \]
We want to prove that \(L(y)I(z) - 1 \geq 0\) or equivalently \(L(y) \geq 1/I(z)\). But thanks to the point [3] we have
\[ L(y) \geq L(z) = \frac{1}{I(z)}. \]

(8) \(R_{I^2}(z) = (I^2)_z'(z) = 2I^2(z)\Psi_I(z)\). From the proof of the point [2] \(\Psi_I(z) \leq 0\) and it is a decreasing function. Since \(2I^2(z) > 0\), then also \(R_{I^2}(z) \leq 0\).

(9) \(W(x) = (F^2)_z'(x) = 2F^2(x)\Psi_F(x)\). In the point [7] we have observed that \(\Psi_F(x)\) is a negative and decreasing function. Since \(2F^2(x) > 0\), then also \(W(x)\) is a negative function.

(10) \(W'(x) = F^2(4\Psi_F^2(x) + 2(\Psi_F)_z'(x))\). Since \(\Psi_F(x) < 0\) and it is a decreasing function, \(\Psi_F^2(x)\) is a positive and increasing function. In the proof of the point [8] we observed that \((\Psi_F)_z'(x) \leq 0\) and it is a decreasing function. Since \(2(\Psi_F)_z'(x_{\max}) = -0.38 \cdots\) and \(4\Psi_F^2(x) \geq 4\Psi_F^2(0) = 64\ln^2 2 = 30.74 \cdots\), we can conclude that
The explicit expression of $W''(x)$ is

$$W''(x) = \frac{1}{2} F^2(x) \left( 16 \Psi_F^2(x) + 24 \Psi_F(x) (\Psi_F)'_x(x) + 4(\Psi_F)''_{xx}(x) \right).$$

In the proof of the point 11 we observed that $\Psi_F(x)$ is a negative and decreasing function. So $16 \Psi_F^2(x) \leq 16 \Psi_F^2(0) = -1024 \ln^3 2 = -341 \cdots$. Thanks to the proof of the point 11 we know that $(\Psi_F)'_x(x) \leq 0$ and it is a decreasing function. In particular $0 \geq (\Psi_F)''_x(x) \geq (\Psi_F)''_x(x_{\text{max}}) = -0.19 \cdots$. We can conclude that

$$24 \Psi_F(x)(\Psi_F)'_x(x) \leq 24 \Psi_F(x_{\text{max}}) \Psi_F(x_{\text{max}}) = 12. \cdots.$$

As for the last summand, it is negative. In fact

$$(\Psi_F)''_{xx}(x) = 2 \sum_{k=1}^{\infty} \frac{1}{(2k-2)!} \left( \frac{1}{2} \psi^{(2k)}(x) - \psi^{(2k)}(1) \right) x^{2k-2}.$$

Since all the coefficients of the series are negative, we get

$$4(\Psi_F)''_{xx}(x) \leq 4(\Psi_F)''_{xx}(0) = -12 \zeta(3) = -14.4 \cdots,$$

where $\zeta(\cdot)$ denotes the Riemann zeta function. We can conclude that

$$16 \Psi_F^2(x) + 24 \Psi_F(x)(\Psi_F)'_x(x) + 4(\Psi_F)''_{xx}(x) \leq 16 \Psi_F^2(0) + 24 \Psi_F(x_{\text{max}})(\Psi_F)'_x(x_{\text{max}}) + 4(\Psi_F)''_{xx}(0) = -342.7 \cdots.$$

That assures $W''(x) < 0$.

(12) The explicit expression of $W'''_{xxx}$ is

$$W'''_{xxx} = \frac{1}{4} F^2(x) \left( 64 \Psi_F^4 + 192 \Psi_F^2(\Psi_F)'_x + 48((\Psi_F)'_x)^2 + 64 \Psi_F(\Psi_F)''_{xx} + 8(\Psi_F)'''_{xxx} \right).$$

We start observing that, since $\Psi_F$ is a negative decreasing function,

$$64 \Psi_F^4(x) \geq 64 \Psi_F^4(0) = 64(4 \ln 2)^4 = 3782 \cdots.$$

Since $(\Psi_F)'_x(x)$ is a not positive and decreasing function (point 10), then $192 \Psi_F^2(\Psi_F)'_x$ enjoys the same property. In particular

$$192 \Psi_F^2(\Psi_F)'_x \geq 192 \Psi_F^2(x_{\text{max}})(\Psi_F)'_x(x_{\text{max}}) = -282 \cdots.$$

From the previous observations it follows that $64 \Psi_F(\Psi_F)''_{xx} \geq 0$, $48((\Psi_F)'_x)^2 \geq 0$ and they are increasing functions.

As for the last summand which appears in the expression of $W'''_{xxx}$, we observe that

$$(\Psi_F)'''_{xxx} = 2 \sum_{k=1}^{\infty} \frac{1}{(2k-1)!} \left( \frac{1}{2} \psi^{(2k+2)} - \psi^{(2k+2)}(1) \right) x^{2k-1}.$$
It is a not positive and decreasing function. So we can write

\[ 8(\Psi_F)'_{xxx}(x) \geq 8(\Psi_F)''_{xxx}(x_{\text{max}}) = -19.9 \ldots. \]

We can conclude \( W''_{xxx}(x) > 0 \). Furthermore from our observations it follows that

\[ W''_{xxx}(x) \leq (16\Psi_F'(x_{\text{max}}) + 24((\Psi_F')_x)^2(x_{\text{max}}) + 16\Psi_F(x_{\text{max}})(\Psi_F)''_{xxx}(x_{\text{max}})) < C_W \]

with \( C_W = 1125 \).

(13) It holds that \( Y'(x) = W(x) + x W'(x) \). From the points \([9] [10] [11]\) we know that \( W'(x) \) is a negative increasing function and \( W''(x) \) is positive and decreasing for \( x \in [0, x_{\text{max}}] \). So we can write \( W'(x) \leq W'(x_{\text{max}}) = -4.1 \ldots \) and \( W''(0) = 64 \ln^2 2 = 30.7 \ldots \). Then \( Y'(x) \leq W(x_{\text{max}}) + x_{\text{max}}W'(0) < 0 \).

(14) It holds that \( Y''_{xx}(x) = 2W'(x) + x W''(x) \). From the points \([10] [11] [12]\) we know that \( W'(x) \) is a positive decreasing function and \( W''_{xx}(x) \) is negative and increasing. So we can write \( W'(x) \geq W'(x_{\text{max}}) = 22 \ldots \) and \( W''(x) \geq W''_{xx}(0) = -64 \ln^3 4 - 6\zeta(3) = -177 \ldots \). Then \( Y''_{xx}(x) \geq 2W'(x_{\text{max}}) + x_{\text{max}}W''_{xx}(0) > 0 \).

(15) It holds that \( Y'''_{xxx}(x) = 3W''_{xx}(x) + x W'''_{xxx}(x) \). From the points \([11] [12]\) we know that \( W''_{xx}(x) \) is a negative increasing function and \( 0 < W'''_{xxx}(x) < C_W \). Then \( Y'''_{xxx}(x) \leq 3W''_{xx}(x_{\text{max}}) + x_{\text{max}}C_W < 0 \).

\[ \square \]

**Proposition 8.** For all the values of \( l, x, y, z \) for which \( T_2(l, x, y, z) \) is real, it holds that

\[ T_2(l, x, y, z) \leq \frac{1 + C l^2 x}{l - 1}, \]

where \( C = -4 \ln 2 \).

**Proof.** The expression of \( T_2 \) is given by \([18]\). We rewrite it in the following way

\[ T_2 = \frac{1}{l - 1} \sqrt{l^2 F^2(x) T^2(z) - (l^2 - 1)L(y)I(\zeta_z)}. \]

We start studying the case of \( T_2 \) non zero. If \( 1 + \bar{R}(x, y, z, l) \) is the Mac-Laurin series of the function under the square root then we can write

\[ T_2 = \frac{1}{l - 1} \sqrt{1 + \bar{R}(x, y, z, l)}, \]

where \( \bar{R}(x, y, z, l) = l^2 (R_{F^2}(c_1 x) x (1 + R_{I^2}(c_2 z) z) + R_{I^2}(c_2 z) z) - (l^2 - 1) R_{L^1}(y, z), \) and \( c_1, c_2 \in (0, 1) \). Thanks to the points \([7] [8] [9] [10]\) of proposition \([7]\) we know that \( R_{L^1}(y, z) \geq 0, R_{I^2}(x) \leq 0 \) and that \( R_{F^2}(x) \) is a negative increasing function, so \( R_{F^2}(c_1 x) \leq R_{F^2}(x) \). We can conclude that, if we set

\[ R(x, z, l) = l^2 R_{F^2}(x) x (1 + R_{I^2}(c_2 z) z), \]
then
\[ \bar{R}(x, y, z, l) \leq l^2 R_F^2(c_1x)(1 + R_I^2(c_2z)z) \leq R(x, z, l), \]
\[ T_2 = \frac{1}{l-1} \sqrt{1 + \bar{R}(x, y, z, l)} \leq \frac{1}{l-1} \sqrt{1 + R(x, z, l)}. \]

We know that
\[ \sqrt{1 + f(x)} = \sqrt{1 + f(0)} + \frac{f'(t)}{2\sqrt{1 + f(t)}} |_{t=\epsilon x}, \]
where \( \epsilon \in (0, 1) \). If we apply this result to the function \( f(x) = R(x, z, l) \), we get
\[ T_2 \leq \frac{1}{l-1} \left( \sqrt{1 + R(0, z, l)} + \frac{R'(t, z, l)}{2\sqrt{1 + R(t, z, l)}} |_{t=\epsilon x} \right), \]
where \( \epsilon \in (0, 1) \). We observe that \( R(0, z, l) = 0 \). Then
\[ T_2 \leq \frac{1}{l-1} \left( 1 + \frac{R'(t, z, l)}{2\sqrt{1 + R(t, z, l)}} |_{t=\epsilon x} \right). \]

The proof will be completed after having proved the following result. \( \square \)

**Proposition 9.** Under the same hypotheses of proposition 8
\[ \frac{R'_t(t, z, l)}{2\sqrt{1 + R(t, z, l)}} \leq C l^2, \]
where \( C = -4 \ln 2 \).

**Proof.** We set \( H(z, l) = l^2 (1 + R_I^2(c_2z)z) \leq l^2 \) and \( Y(t) = R_F^2(t)t \). From the expression of \( R(t, z, l) = H(z, l)Y(t) \), it follows that \( R'_t(t, z, l) = H(z, l)Y'_t(t) \). Furthermore we can write
\[ \frac{R'_t(t, z, l)}{2\sqrt{1 + R(t, z, l)}} = \frac{H(z, l)Y'_t(t)}{2\sqrt{1 + H(z, l)Y(t)}}. \]

We know from proposition 7 that \( Y(t) \leq 0 \) and \( Y'_t(t) < 0 \), then \( R'_t(t, z, l) = H(z, l)Y'_t(t) \geq l^2 Y'_t(t) \), and
\[ -\frac{1}{2\sqrt{1 + R(t, z, l)}} \geq -\frac{1}{2\sqrt{1 + l^2 Y}}. \]

We can conclude that
\[ \frac{R'_t(t, z, l)}{2\sqrt{1 + R(t, z, l)}} \geq \frac{l^2 Y'_t(t)}{2\sqrt{1 + l^2 Y(t)}}. \]

We shall show that the function on the right side is increasing with respect to the variable \( t \). The derivative with respect to the variable \( t \) of this function is
\[ D(t, l) = -\frac{l^2 Y'' \sqrt{1 + l^2 Y} - l^2 (Y')^2}{2\sqrt{1 + l^2 Y}}. \]
We want to study the sign of $D(t, l)$. We start observing that $1 + l^2 Y \geq 1 + R > 0$. So it is sufficient to prove that the quantity

$$E(t, l) = 2Y'''(1 + l^2 Y) - l^2(Y')^2$$

is always not positive. It holds that

$$Y'(t) = R_{F^2}(t) + t(R_{F^2})'(t)$$

and

$$Y''(t) = 2(R_{F^2})'(t) + t(R_{F^2})''(t).$$

Then $Y(0) = 0$, $Y'(0) = R_{F^2}(0) = 2C$ and $Y''(0) = 2(R_{F^2})'(0) = 8\Psi_F(0)^2 = 8C^2$. Furthermore we observe that $l \geq 2$. So

$$E(0, l) = 16C^2 - 4l^2 C^2 \leq 0$$

and the equality holds if $l = 2$. The next step is to show that $E'(t, l) \leq 0$. It is possible to find the following equality

$$E'(t, l) = Y'''(1 + l^2 Y)$$

Observing that $1 + l^2 Y > 0$ and $Y''' < 0$ (see the point $15$ of proposition $7$), we can conclude that $D(t, l) \geq 0$ (the equality holding if $t = 0$). We have showed that

$$- \frac{l^2 Y'(t)}{2\sqrt{1 + l^2 Y(t)}}$$

is a non decreasing function. It gets the minimum for $t = 0$ and its value is $-Cl^2$. Then

$$- \frac{R'(t, z, l)}{2\sqrt{1 + R(t, z, l)}} \geq -Cl^2,$$

and the proof of proposition $9$ is completed.

To achieve the proof of proposition $8$ we need show that the statement continues to hold also for values of $l, x, y, z$ for which $T_2 = 0$. To get this aim it is sufficient to observe that we can extend the result obtained under the hypothesis $T_2 > 0$ for continuity.

As for the first summand which appears in the expression of $t_2^-$, that is $T_1$, the following result holds.

**Proposition 10.** For all the admissible values of $x, z$, it holds that

$$T_1 \geq \frac{l}{l - 1}(1 + Cx)$$

where $C = -4 \ln 2$. 
Proof. We recall that
\[ T_1 = \frac{l}{l-1} F(x) I(z) = \frac{l}{l-1} (1 + R_{F_I}(x,z)). \]
Thanks to the point 6 of proposition 7 we have \( R_{F_I}(x,z) \geq C x \). Then the result is immediate. \qed

The following result gives the estimate of \( t_2^- \).

**Proposition 11.** For all the values of \( x, y, z \) for which \( t_2^- \in \mathbb{R} \), it holds
\[ t_2^- \geq 1 - Clx, \]
where \( C = -4 \ln 2 \).

We recall that \( t_2^- = T_1 - T_2 \). Thanks to propositions 8 and 10 we get
\[ t_2^- \geq \frac{l}{l-1} (1 + Cx) + \frac{1}{l-1} (-1 - Cl^2 x) =
1 + \left( \frac{Cl}{l-1} - \frac{Cl^2}{l-1} \right) x = 1 + \left( \frac{-Cl}{l-1} (l-1) \right) x = 1 - Clx. \]
\qed

Now we turn our attention to the function \( t_3^- \). We recall that \( s_{\text{max}} = \frac{1}{38} \).

**Proposition 12.** For \( s \in [0, s_{\text{max}}] \)
\[ t_3^2(s) \leq 1 + \frac{7}{2}s. \]

**Proof.** We recall that
\[ t_3^2(s) = T(s) = \frac{4}{(1-s)^2} \left( \frac{\Gamma(1+s)}{\Gamma(1-s)} \right)^3 \frac{\Gamma(1-2s)}{\Gamma(1+2s)} \left( \frac{\Gamma(3/2 - s/2)}{\Gamma(1/2 + s/2)} \right)^2. \]
It holds that
\[ T_s'(s) = \frac{1}{(1-s)^2} T(s) B(s), \]
where
\[ B(s) = 2 + (1-s) (-2 \psi(1-2s) - 2 \psi(1+2s) + 3 \psi(1-s) + 3 \psi(1+s) -
- \psi\left( \frac{3}{2} - \frac{s}{2} \right) - \psi\left( \frac{1}{2} + \frac{s}{2} \right) \). \]
To complete the proof we need the following result.

**Proposition 13.** If \( s \in [0, s_{\text{max}}] \) then \( 1 < B(s) < 3 \).
Proof. We observe that for \( s \in [0, s_{\text{max}}] \)
\[
0 < \psi \left( \frac{3}{2} - \frac{s}{2} \right) < \psi \left( \frac{3}{2} \right) = 0.036 \cdots, \quad \frac{3}{2} < -\psi \left( \frac{1}{2} + \frac{s}{2} \right) < -\psi \left( \frac{1}{2} \right) < 2.
\]
We can conclude that
\[
1 < -\psi \left( \frac{1}{2} + \frac{s}{2} \right) - \psi \left( \frac{3}{2} - \frac{s}{2} \right) < 2.
\]
Furthermore
\[
\psi(1 - s) + \psi(1 + s) = 2 \sum_{k \geq 0} \frac{\psi^{(2k)}(1)}{(2k)!} s^{2k},
\]
from which it follows that
\[
D(s) = -2\psi(1 - 2s) - 2\psi(1 + 2s) + 3\psi(1 - s) + 3\psi(1 + s) = 2 \sum_{k \geq 0} \frac{\psi^{(2k)}(1)}{(2k)!} s^{2k}(3 - 2^{2k+1}).
\]
If \( k \geq 1 \) then \( 3 - 2^{2k+1} < 0 \) and \( \psi^{(2k)}(1) < 0 \) (see formula 6.4.2 of [1]) then
\[
2\psi(1) = -2\gamma_{EM} = D(0) \leq D(s) \leq D(s_{\text{max}}) = -1.146 \cdots,
\]
where \( \gamma_{EM} = 0.577 \cdots \) is the Euler-Mascheroni constant. So
\[
1 < B(s) \leq 2 + (1 - s)(2 + D(s_{\text{max}})) < 4 + D(s_{\text{max}}) < 3.
\]

Since \( B(s) > 0 \) then \( T(s) \) is an increasing function and we can deduce that
\[
T'(s) = \frac{1}{1 - s} T(s) B(s) \leq \frac{3}{1 - s_{\text{max}}} T(s_{\text{max}}) < 7/2.
\]
The Mac-Laurin series of order zero of \( T(s) \) is \( 1 + T'(cs)s \), where \( c \in (0,1) \). So it is immediate to conclude that
\[
T(s) \leq 1 + \frac{7}{2}s.
\]

The following proposition shows that the eventual solutions \( t_+(l,g) \geq t_-(l,g) \) of the equation (15) are always bigger than \( t_3 \).

Proposition 14. \( t_3(l \mathcal{g} + 1) < t_-(l,g) \) for \( g \geq 1 \) and \( 2 \leq l \leq g - 1 \) such that \( t_-(l,g) \in \mathbb{R} \).

Proof. From our observations, it is sufficient to show that \( t^2_3(s) < t^2_-(l,g) \) holds for \( g \geq 38 \). Propositions 11 and 12 assure that
\[
t^2_\geq \geq 1 - Clx, \quad t^2_3(s) \leq 1 + \frac{7}{2}s.
\]
We recall that \( x = ls \) and \( 2 \leq l \leq g - 1 \). Then the result is obvious.
4. The index and the nullity of the Costa-Hoffman-Meeks surfaces

We start recalling some results described in previous sections. We denoted by $G_t$, $t \in (0, +\infty)$, a deformation of the map $G$ defined by (5). Thanks to theorem $6$, $\text{Null}(G_t) > 3$ only if $t$ assumes special values. If $2 \leq g \leq 37$ these values are $t_1, t_2, t_3$. If $g \geq 38$ there are additional values. They are the positive solutions of the equation (15). We denoted them by $t_{\pm}(l, g)$, where $2 \leq l \leq g - 1$, and for definition $t_+ \geq t_-$. In previous section we have proved that the inequality $t_3(s) < t_-(l, g)$ holds. S. Nayatani showed in [9] that $t_3 > t_2$ for $g \geq 2$. We can conclude that no one of the $t_{\pm}$ can be equal to $t_2$. As consequence $\text{Null}(G_{t_2})$ continues to be equal to 4 also for $g \geq 38$, because $\text{dim } H(G_{t_2})$ is equal to 1 for all $g \geq 2$.

We recall that $M_g$ denotes the Costa-Hoffman-Meeks surface of genus $g$. Since the extended Gauss map of $M_g$ is exactly $G_{t_2}$, and taking into account the result of S. Nayatani about the Costa surface (theorem 4) showed in [10], we have proved theorem 2.

Now we turn our attention to the results relative to the index of the map $G_t$. We recall that $\Sigma_g$ denotes the compactification of $M_g$. S. Nayatani proved in [9] the following result.

**Theorem 15.** Let $G : \Sigma_g \to S^2$ be the holomorphic map defined by (5). If $2 \leq g \leq 37$, then

$$\text{Ind}(G_t) = \begin{cases} 
2g + 3 & \text{if } t \leq t_1, t_2 \leq t < t_3, t > t_3, \\
2g + 4 & \text{if } t_1 < t < t_2, \\
2g + 2 & \text{if } t = t_3.
\end{cases}$$

For $t = t_1, t_2, t_3$ we have $\text{Null}(G_t) > 3$, that is the kernel of $L_{G_t}$ contains at least one non linear element. The eigenvalue associated to this function is zero. The proof of theorem 15 is based on the analysis of the behaviour of these null eigenvalues under a variation of the value of $t$. Let’s suppose that $t \neq t_1, t_2, t_3$ but remaining in a neighbourhood of one of these values. For example we choose $t_1$. Then the eigenvalue $E$ that before the variation was associated to a non linear element of $N(G_{t_1})$, is no more equal to zero. To compute the index, it was necessary to understand which is the sign assumed by $E$, respectively for $t > t_1$ and $t < t_1$. Similar considerations are applicable to the eigenvalues associated with $t_2$ and $t_3$. See [9] for the details.

If $g \geq 38$, we have just proved that the other values for which $\text{Null}(G_t) > 3$ are bigger than $t_3$. The presence of these additional values $t_\pm$ does not influence the value of $\text{Ind}(G_t)$ if $t \leq t_3$. In other terms theorem 15 continues to hold for $g \geq 38$ if we consider $0 < t \leq t_3$. Taking into account also the result of S. Nayatani about the Costa surface ($g = 1$) showed in [10], we have proved theorem 1.

5. Appendix

This section contains some additional details of the computations made by S. Nayatani.
5.1. **Divisors and Riemann-Roch theorem.** Here we introduce some definitions and concepts of the algebraic geometry. See for example [2].

Let $\Sigma_g$ be a compact Riemann surface of genus $g$. A divisor on $\Sigma_g$ is a finite formal sum of integer multiples of points of $\Sigma_g$,

$$D = \sum_{x \in \Sigma_g} n_x x, \quad n_x \in \mathbb{Z}, n_x = 0 \text{ for almost all } x.$$ 

The set of the divisors on $\Sigma_g$ is denoted by $\text{Div}(\Sigma_g)$. The degree of a divisor is the integer $\deg(D) = \sum n_x$.

Let $\mathbb{C}(\Sigma_g)$ be the field of the meromorphic functions on $\Sigma_g$ and let $\mathbb{C}(\Sigma_g)^*$ be its multiplicative group of nonzero elements. Every $f \in \mathbb{C}(\Sigma_g)^*$ has a divisor $\text{div}(f) = \sum \nu_x(f)x$, where $\nu_x(f)$ denotes the order of $f$ at $x$.

Let $\omega$ be a nonzero meromorphic differential $n$-form on $\Sigma_g$. Then $\omega$ has a local representation $\omega_x = f_x(z)(dz)^n$ about each point $x$ of $\Sigma_g$, where $z$ is the local coordinate about $x$ and $f_x(z) \in \mathbb{C}(\Sigma_g)^*$. So we can define in a natural way $\nu_x(\omega) = \nu_0(f_x)$ and also associate a divisor with a differential form:

$$\text{div}(\omega) = \sum \nu_x(\omega)x.$$ 

A canonical divisor on $\Sigma_g$ is a divisor of the form $\text{div}(\omega)$ where $\omega$ is a nonzero meromorphic differential form.

Let $D \in \text{div}(\Sigma_g)$. We denote by $H^{0,n}(D)$ the vector space of the meromorphic differential $n$-forms $\omega$ such that

$$\text{div}(\omega) + D \geq 0.$$ 

In other terms, if $D = \text{div}(\eta)$, with $\eta$ differential form with local representation $\eta_x = g_x(z)(dz)^n$, then the elements of $H^{0,n}(D)$ are the differential forms $\omega$ having a local representation $\omega_x = f_x(z)(dz)^n$ with $f_x \in \mathbb{C}(\Sigma_g)$ vanishing to high enough order to make the product $fg$ holomorphic. We set $\dim H^{0,n}(D) = \ell(D)$.

We are ready to state the following result.

**Theorem 16** (Riemann-Roch). Let $\Sigma_g$ be a compact Riemann surface of genus $g$. Let $k\Sigma_g$ be a canonical divisor on $\Sigma$. Then for any divisor $D \in \text{Div}(\Sigma_g)$,

$$\ell(D) = \deg(D) - g + 1 + \ell(k\Sigma_g - D).$$ 

The next result gives information about the canonical divisor and a simpler version of Riemann-Roch theorem for divisors of large enough order.
Corollary 17. Let \( \Sigma_g, g, D, k_{\Sigma_g} \) as above.
- \( \deg(k_{\Sigma_g}) = 2g - 2 \),
- If \( \deg(D) > 2g - 2 \) then \( \ell(k_{\Sigma_g} - D) = 0 \). Equivalently \( \ell(D) = \deg(D) - g + 1 \).

5.2. The determination of a basis of differential forms with null residue at the ramification points. The ramification points (or branch points) of \( \gamma(w) = w \) are the zeroes of

\[
\frac{dw}{dz} = \frac{g + 2}{g + 1} \frac{z^{g-1}(z^2 - A^2)}{w^g} = \frac{g + 2}{g + 1} \frac{z^{g-1}(z^2 - A^2)}{(z^2 - 1)^{g+1}},
\]

with \( A = \sqrt{\frac{g}{g+2}} \), where \( g \) denotes the genus, the pole of \( \gamma \) and the origin of \( \mathbb{C}^2 \). That is \( Q_0 = (0,0), P_\infty = (\infty, \infty), P_m = (A, B_m) \) and \( S_m = (-A, C_m) \) for \( m = 0, \ldots, g \), where \( B_m, C_m \) denote, respectively, the \( m \)-th complex value of \( A^g(1-A^2)^{g-1} \) and \( A^{g-1}(-A)^g(1-A^2)^{g-1} \).

We have set \( P_\pm = (\pm 1, 0) \). We recall that

\[
\hat{H}(G) = \left\{ \sigma \in H^0(2k_{\Sigma_g} + R(G)) \mid \Res_{r_i} \frac{\sigma}{dw} = 0, i = 1, \ldots, \mu \right\},
\]

where \( k_{\Sigma_g} \) is a canonical divisor of \( \Sigma_g \) and \( R(G) = \sum_i r_i \) is the ramification divisor of \( G \). In our case it is given by \( R(G) = Q_0 + P_\infty + \sum_{m=0}^g (P_m + S_m) \). Furthermore it holds \( \hat{H}(G) = \hat{H}(G_i) \).

As for the canonical divisor \( k_{\Sigma_g} \), we consider \( k_{\Sigma_g} = (g - 1)P_+ + (g - 1)P_- \). We observe that \( \deg(k_{\Sigma_g}) = 2g - 2 \) like stated by corollary 17.

To study the space \( \hat{H}(G_i) \) we need understand which are the elements of the space \( H^0(2k_{\Sigma_g} + R(G)) \). Taking into account the definitions of \( k_{\Sigma_g} \) and \( R(G) \), then \( 2k_{\Sigma_g} + R(G) = 2(g - 1)P_+ + 2(g - 1)P_- + Q_0 + P_\infty + \sum_{m=0}^g P_m + \sum_{m=0}^g S_m \). Among the quadratic differentials \( \sigma \) that are in \( H^0(2k_{\Sigma_g} + R(G)) \), we consider the ones having one of the following forms:

\[
z^k w^j \left( \frac{dz}{w} \right)^2,
\]

\[
z^k w^j \frac{1}{z \pm A} \left( \frac{dz}{w} \right)^2.
\]

In fact from the definition of \( H^0 \), it follows that the quadratic differentials to consider can have a pole of order 0 (differentials of type (22)) or of order 1 (differentials of type (23)) at \( P_m \) and \( S_m \) for \( k = 0, \ldots, g \). We will determine separately which are the differential forms of type (22) and (23) belonging to \( H^0(2k_{\Sigma_g} + R(G)) \). To select the differential forms of type (23) it is convenient to introduce an auxiliary divisor.

\[
D = Q_0 + (g+2)P_\infty + 2(g-1)P_+ + 2(g-1)P_-.
\]
Actually to determine the differential forms of type \((23)\) which belong to \(H^0(2\kappa_{\Sigma^g} + R(G))\) is equivalent to look for the differential forms of type \((22)\) which are in \(H^0(D)\). We observe that the elements of the vector space \(H^0(D)\) after the multiplication by the factor \(z \pm A\) are elements of \(H^0(2\kappa_{\Sigma^g} + R(G))\). It is necessary to remark that to obtain a basis of \(H^0(2\kappa_{\Sigma^g} + R(G))\), we will not take into account the differentials of \(H^0(D)\) as described above. Otherwise the number of the founded differential forms would exceed the dimension of \(H^0(2\kappa_{\Sigma^g} + R(G))\), that we can compute as follows. We observe that \(\text{deg}(2\kappa_{\Sigma^g} + R(G)) = 6g\). Then thanks to corollary 17 we conclude that \(\dim H^0(2\kappa_{\Sigma^g} + R(G)) = 5g + 1\). So the basis we are looking for counts \(5g + 1\) elements. From the observations made above we can deduce that among the forms of type \((22)\), we will consider the ones which satisfy the following conditions

\[
\begin{cases}
k(g + 1) + jg \geq -1, \\
j \geq -2(g - 1), \\
-k(g + 1) - j(g + 2) \geq -(g + 2).
\end{cases}
\]

These relations assure that a differential form \(\omega\) of type \(z^k w^j \left(\frac{dz}{w}\right)^2\), satisfies \(\text{div}(\omega) + 2\kappa_{\Sigma^g} + R(G) \geq 0\). These differentials can be classified in three families. Each family is characterized by particular values of \(j\) and \(k\). That is

1. \(j = -g + 1, \ldots, 0, 1\) and \(k = -j\),
2. \(j = 2 - 2g, \ldots, -g\) and \(k = -j\),
3. \(j = 2 - 2g, \ldots, -g\) and \(k = -j - 1\).

As for the forms of type \((23)\) we shall consider only the ones which satisfy

\[
\begin{cases}
k(g + 1) + jg \geq -1, \\
j \geq -2(g - 1), \\
-k(g + 1) - j(g + 2) \geq -(g + 2).
\end{cases}
\]

These relations assure that a differential form of type \(z^k w^j \frac{1}{z \pm A} \left(\frac{dz}{w}\right)^2\), satisfies \(\text{div}(\omega) + D \geq 0\). We obtain that \(j = -g + 1, \ldots, 0, 1\) and \(k = -j + 1\).

Since we are looking for a basis of a vector space we can replace each couple of differentials \(\frac{f}{z - A} \left(\frac{dz}{w}\right)^2, \frac{f}{z + A} \left(\frac{dz}{w}\right)^2\) by an appropriate linear combination. We observe that

\[
\frac{1}{z - A} \pm \frac{1}{z + A} = \left\{ \begin{array}{ll}
\eta_1 = \frac{z}{z^2 - A^2} \\
\eta_2 = \frac{z^2 - 1}{z^2 - A^2}.
\end{array} \right.
\]

So in the following we will work with the forms \(f\eta_1 \left(\frac{dz}{w}\right)^2\) and \(f\eta_2 \left(\frac{dz}{w}\right)^2\), where \(f = z^k w^j\) as described above.

The \(5g + 1\) quadratic differentials we have found forms a basis of \(\hat{H}(G_t)\). The last step is to divide each elements of this basis by \(dw\). After simple algebraic manipulations, we
obtain the following \(5g + 1\) differential 1-forms:

\[
\begin{align*}
\omega_k &= \frac{z^k}{w^{k-1}} \frac{dz}{(z^2 - A^2)^2} \quad \text{for} \quad k = -1, 0, \ldots, g - 1, \\
\omega_k &= \frac{z^k}{w^{k+1}} \frac{dz}{(z^2 - A^2)^2} \quad \text{for} \quad k = -1, 0, \ldots, g - 1, \\
\omega_k &= \frac{z^k}{w^{k+1}} \frac{dz}{(z^2 - A^2)^2} \quad \text{for} \quad k = 1, \ldots, g - 1, \\
\omega_k &= \frac{z^{k+1}}{w^{k+1}} \frac{dz}{(z^2 - A^2)^2} \quad \text{for} \quad k = 1, \ldots, g - 1.
\end{align*}
\]  

(24)

Now it is necessary to select the 1-forms having residue equal to zero at the points \(Q_0, P_m\) and \(S_m\) with \(m = 0, \ldots, g\). Thanks to the properties of symmetry of the surface it is sufficient to verify the null residue condition at the points \(Q_0, P_1 = (A, e^{\frac{2\pi i}{g+1}} \sqrt[2g+1]{A^g}(A^2 - 1))\).

In fact from the coordinates of the points \(P_m\) and \(S_m\), we can deduce that for each \(Q \in \{P_m, S_m, m = 0, \ldots, g\}\) there exists \(n \in \{0, \ldots, 2g+1\}\) such that \(Q = \lambda^n(P_1)\), where \(\lambda\) is the conformal diffeomorphism described in lemma [5]. So we can state that the residue of an arbitrary form \(\omega\) at the point \(Q\) is related to the residue at \(P_1\) by

\[\text{Res}_Q \omega = \text{Res}_{P_1} (\lambda^{-1})^n \omega.\]

Applying this result to the differential forms of the list (24) and using the the definition (4) of \(\lambda\), it is easy to obtain that \(\text{Res}_Q \omega\) is equal to \(\text{Res}_{P_1} \omega\) times a power of \(\pm \rho\). So if \(\text{Res}_{P_1} \omega = 0\) then \(\text{Res}_Q \omega = 0\).

5.3. The equations equivalent to the condition of existence of a branched minimal surface. Let \(\omega_1\) and \(\omega_2\) two meromorphic differential forms on \(\Sigma_g\). We write \(\omega_1 \sim \omega_2\) if there exists a meromorphic function \(f\) on \(\Sigma_g\) such that \(\omega_2 = \omega_1 + df\). It is possible to prove that:

\[
\begin{align*}
\omega_k^{(2)} &= -\frac{k(g+2)}{2(g+1)} \frac{z^{k-1}}{w^k} dz \quad \text{for} \quad k = 0, \ldots, g, \\
\omega_k^{(3)} &= -\frac{(g+2)(g+k+2)}{2(g+1)} \frac{z^{k-1}}{w^{k+1}} dz \quad \text{for} \quad k = 0, \ldots, g - 1.
\end{align*}
\]
Using these relations we get:

\[
\int_{\bar{\beta}} \omega^{(1)}_k = -2i \sin \frac{(k+1)\pi}{g+1} K_k, \quad \int_{\bar{\beta}} \omega^{(2)}_k = -\frac{(g+2)k}{2(g+1)} 2i \sin \frac{k\pi}{g+1} I_k, \\
\int_{\bar{\beta}} \omega^{(3)}_k = \frac{(g+2)(g+2-k)}{2(g+1)} 2i \sin \frac{(k+1)\pi}{g+1} K_k,
\]

\[
\int_{\bar{\beta}} \gamma \omega^{(1)}_k = 2i \sin \frac{k\pi}{g+1} I_k, \quad \int_{\bar{\beta}} \gamma \omega^{(2)}_k = \frac{(g+2)(g+2-k)}{2(g+1)} 2i \sin \frac{(k-1)\pi}{g+1} J_k, \\
\int_{\bar{\beta}} \gamma \omega^{(3)}_k = -\frac{(g+2)k}{2(g+1)} 2i \sin \frac{k\pi}{g+1} L_k,
\]

\[
\int_{\bar{\beta}} \gamma^2 \omega^{(1)}_k = 2i \sin \frac{(k-1)\pi}{g+1} J_k, \quad \int_{\bar{\beta}} \gamma^2 \omega^{(2)}_k = \frac{(g+2)(2g+4-k)}{2(g+1)} 2i \sin \frac{(k-2)\pi}{g+1} L_k, \\
\int_{\bar{\beta}} \gamma^2 \omega^{(3)}_k = \frac{(g+2)(g+2-k)}{2(g+1)} 2i \sin \frac{(k-1)\pi}{g+1} J_k.
\]

We recall that we must impose that \(\omega = \sum_{0}^{g-1} c^{(1)}_k \omega^{(1)}_k + \sum_{0}^{g-1} c^{(2)}_k \omega^{(2)}_k + \sum_{0}^{g-1} c^{(3)}_k \omega^{(3)}_k\), where \(c^{(i)}_k \in \mathbb{C}\), satisfies

\[
\int_{\alpha} \omega = t^2 \int_{\alpha} \gamma^2(w)\omega, \quad \text{Re} \int_{\alpha} \gamma(w)\omega = 0
\]

for \(\alpha = \lambda^l \circ \bar{\beta}\) for \(l = 0, \ldots, 2g-1\). Now it is convient to introduce some additional notation.

Let

\[
\mathcal{L} = \begin{bmatrix} R_{\theta} & 0 \\ 0 & 1 \end{bmatrix}
\]

where \(R_{\theta}\) is the rotation in the plane by \(\theta = g\pi/(g+1)\).

If we denote \(\Phi(\omega) = (1 - \gamma^2, i(1 + \gamma^2), 2\gamma)\omega\), then it is possible to prove

\[
\int_{\lambda^l \circ \bar{\beta}} \Phi(\omega) = \int_{\bar{\beta}} \lambda^* \Phi(\omega).
\]

Since the differential form \(\omega\) is linear combination of \(\omega^{(j)}_k\), \(j = 1, 2, 3\), it is convenient to remark that:

\[
\lambda^* \Phi(\omega^{(1)}_k) = (-1)^k \rho^{-k} \mathcal{L} \Phi(\omega^{(1)}_k),
\]

\[
\lambda^* \Phi(\omega^{(2)}_k) = (-1)^k \rho^{-k+1} \mathcal{L} \Phi(\omega^{(2)}_k),
\]

\[
\lambda^* \Phi(\omega^{(3)}_k) = (-1)^k \rho^{-k} \mathcal{L} \Phi(\omega^{(3)}_k),
\]

where \(\rho = e^{i \frac{2\pi}{g}}\). Then the equations

\[
\text{Re} \int_{\lambda^l \circ \bar{\beta}} (1 - t^2 \gamma^2, i(1 + t^2 \gamma^2))\omega = 0, \quad \text{for} \quad l = 0, \ldots, 2g-1,
\]
are equivalent to:

\[
\text{Im} \left[ \sum_{k=0}^{g-1} \left\{ (-1)^k \rho^{k-1} \right\}^l f_k + \sum_{k=1}^g \left\{ (-1)^k \rho^{-(k-1)} \right\}^l p_k \right] =
\]

\[
t^2 \text{Im} \left[ \sum_{k=0, k \neq 1}^{g-1} \left\{ (-1)^k \rho^{k-1} \right\}^l h_k + \sum_{k=0, k \neq 2}^g \left\{ (-1)^k \rho^{-(k-1)} \right\}^l q_k \right],
\]

\[
\text{Re} \left[ \sum_{k=0}^{g-1} \left\{ (-1)^k \rho^{k-1} \right\}^l f_k + \sum_{k=1}^g \left\{ (-1)^k \rho^{-(k-1)} \right\}^l p_k \right] =
\]

\[
-t^2 \text{Re} \left[ \sum_{k=0, k \neq 1}^{g-1} \left\{ (-1)^k \rho^{k-1} \right\}^l h_k + \sum_{k=0, k \neq 2}^g \left\{ (-1)^k \rho^{-(k-1)} \right\}^l q_k \right],
\]

\(l = 0, \ldots, 2g - 1\). These last equations can be arranged as in the systems (10) and (11). The equations

\[
\text{Re} \int_{\lambda \circ \beta} 2t \gamma \omega = 0, \quad \text{for} \quad l = 0, \ldots, 2g - 1,
\]

are equivalent to:

\[
\text{Im} \left[ \sum_{k=1}^{g-1} \left\{ (-1)^k \rho^{k-1} \right\}^l e_k + \sum_{k=0, k \neq 1}^g \left\{ (-1)^k \rho^{-(k-1)} \right\}^l e_k \right] = 0,
\]

\(l = 0, \ldots, 2g - 1\). These last equations can be arranged as in the systems (12) and (13).

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Université Paris-Est, Laboratoire d’Analyse et Mathématiques Appliquées, 5 blvd Descartes, 77454 Champs-sur-Marne, FRANCE
E-mail address: filippo.morabito@univ-mlv.fr

Università Roma Tre, Dipartimento di Matematica, Largo S. L. Murialdo 1, 00146 Roma, ITALIA
E-mail address: morabito@mat.uniroma3.it