Laurent phenomenon algebras and the discrete BKP equation

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Abstract
We construct the Laurent phenomenon algebras the cluster variables of which satisfy the discrete BKP equation, the discrete Sawada–Kotera equation and other difference equations obtained by its reduction. These Laurent phenomenon algebras are constructed from seeds with a generalization of mutation-period property. We show that a reduction of a seed corresponds to a reduction of a difference equation.

Keywords: discrete integrable system, Laurent phenomenon algebra, discrete BKP equation

1. Introduction
In this article, we deal with the Laurent phenomenon algebras introduced by Lam and Pylyavskyy [12]. The Laurent phenomenon algebra is a generalization of the cluster algebra introduced by Fomin and Zelevinsky [8]. The cluster algebra is a commutative ring associated with an \( N \times N \) integer skew-symmetric matrix \( B = (b_{ij}) \) and a \( N \)-tuple of (commutative) variables \( x = (x_1, x_2, \ldots, x_N) \). Each \( x_i \) is called a cluster variable and the pair \((x, B)\) is called a seed. The generators of the algebra are produced by algebraic procedures called ‘mutations’, which give rational transformations of the variables. The mutation of the seed \((x, B)\) at \( x_k \) is a transformation from \((x, B)\) to \((x', B')\). The mutation of the seed \((x, B)\) at \( x_k \) is defined as

\[
b_{ij}' = -b_{ij} \quad (i = k \text{ or } j = k),
\]

\[
b_{ij}' = b_{ij} + [-b_{i,k}]_+ b_{k,j} + b_{i,k} [b_{k,j}]_+ \quad (\text{otherwise}) \tag{1.1}
\]

and

\[
x_i' = x_i \quad (i \neq k),
\]

\[
x_k' = \frac{1}{x_k} \left( \prod_{j=1}^{N} x_j^{[b_{j,k}]_+} + \prod_{j=1}^{N} x_j^{-[b_{j,k}]_+} \right). \tag{1.2}
\]
where \([x]_+ = \max(x, 0)\). The most characteristic property of cluster algebras is the Laurent phenomenon [8]: for a initial seed \((x, B)\), all cluster variables obtained by a iteration of mutations from the initial seed can be expressed as Laurent polynomials of the cluster variables of the initial seed. It is known that some integrable difference equations (e.g. the Hirota–Miwa equation [3], the discrete KdV equation [1], the discrete Toda equation [2], Soms-4, and Somos-5 [4]) are described as rational transformations given by sequences of the mutations. Difference equations obtained by a cluster algebra are restricted to the form:

\[
(x_{n+1}^{m+1}x_{n+1}^{l+1}) = \text{product of two cluster variables binomial of cluster variables}. \tag{1.3}
\]

For example, the Hirota–Miwa equation

\[
x_{n+1}^{m+1}x_{n+1}^{l+1} = x_{n+1}^{m+1}x_{n+1}^{l+1} + x_n x_{n+1}^{m+1}x_{n+1}^{l+1}
\tag{1.4}
\]

has the form given by (1.3). However, the discrete BKP equation

\[
x_{n+1}^{m+1}x_{n+1}^{l+1} = x_{n+1}^{m+1}x_{n+1}^{l+1} + x_n x_{n+1}^{m+1}x_{n+1}^{l+1} + x_n x_{n+1}^{m+1}x_{n+1}^{l+1} + x_n x_{n+1}^{m+1}x_{n+1}^{l+1}
\tag{1.5}
\]

does not have the form given by (1.3).

The Laurent phenomenon algebra is a generalization of the cluster algebra, keeping the Laurent phenomenon property [12], where the rule of a skew symmetric matrix is replaced with a tuple of polynomials. In particular, the rational transformation (1.3) is generalized to

\[
(x_{n+1}^{m+1}x_{n+1}^{l+1}) = \text{product of two cluster variables polynomial of cluster variables}. \tag{1.6}
\]

It is known that the Somos-6 [4] and some related difference equations [12, 13] are described as rational transformations given by sequences of the mutations. In this paper, we construct the Laurent phenomenon algebras whose cluster variables satisfy the discrete BKP equation [3] (section 3), the discrete Sawada–Kotera equation [5], the Somos-7, and several other difference equations (section 4). Exchange polynomials of the Laurent phenomenon algebra which give the Somos-6, Somos-7, and related two-dimensional difference equations are obtained from reductions of exchange polynomials that give the discrete BKP equation (section 4).

2. Laurent phenomenon algebras

In this section, we briefly explain the notion of Laurent phenomenon algebra which we use in the following sections.

2.1. Definition of Laurent phenomenon algebra

Let \(x = (x_0, x_1, \ldots, x_{N-1})\) be an \(N\)-tuple of variables. Let \(F = (F_0, F_1, \ldots, F_{N-1})\) be an \(N\)-tuple of polynomials in \(\mathbb{Z}[x_0, x_1, \ldots, x_{N-1}]\). We assume that these polynomials satisfy the following conditions:

- (LP1) \(F_i \in \mathbb{Z}[x_0, x_1, \ldots, x_{N-1}]\) is irreducible and is not divisible by any \(x_j\).
- (LP2) \(F_i\) does not depend on \(x_i\).

Each \(x_i\) is called a cluster variable and each \(F_i\) is called an exchange polynomial. The pair \(t = (x, F)\) is called a seed. We will often express a seed as

\[
t = \{(x_0, F_0), (x_1, F_1), \ldots, (x_{N-1}, F_{N-1})\}. \tag{2.1}
\]

The number of cluster variables \(N\) in a seed \(t\) is called the rank of the seed \(t\) or the rank of the Laurent phenomenon algebra. For Laurent polynomials \(F, G \in \mathbb{Z}[x_0^+, x_1^+, \ldots, x_{N-1}^+]\) and a
cluster variable $x_i$, let $F|_{x_i \rightarrow G}$ be the rational function in which we substituted $x_i$ for $G$ in $F$. A mutation is a particular transformation of seeds.

**Definition 2.1** [12]. The mutation of the seed $t$ at $x_k$ is a transformation from $t$ to $t'$,

$$
\mu_k : t = \{ (x_0, F_0), (x_1, F_1), \ldots, (x_{N-1}, F_{N-1}) \} \mapsto t' = \{ (x'_0, F'_0), (x'_1, F'_1), \ldots, (x'_{N-1}, F'_{N-1}) \},
$$

that are defined via the following sequence of steps:

1. Let $\hat{F}_k \in \mathbb{Z}[x_0^\pm, x_1^\pm, \ldots, x_{k-1}^\pm, x_{k+1}^\pm, \ldots, x_{N-1}^\pm]$ be the unique Laurent polynomial satisfying following conditions:
   - There exist $a_0, a_1, \ldots, a_{N-1} \in \mathbb{Z}_{\leq 0}$ such that
     $$
     \hat{F}_k = x_0^{a_0} x_1^{a_1} \ldots x_{k-1}^{a_{k-1}} x_k^{a_k + 1} x_{k+1}^{a_{k+1}} \ldots x_{N-1}^{a_{N-1}} F_k.
     $$
   - For any $j \neq k$,
     $$
     \hat{F}_k|_{x_j \rightarrow F_j / x_j} \in \mathbb{Z}[x_0^\pm, x_1^\pm, \ldots, x_{j-1}^\pm, x_j^\pm, x_{j+1}^\pm, \ldots, x_{N-1}^\pm]
     $$
   and is not divisible by $F_j$.

2. New cluster variable $x_i'$ is defined as:
   $$
   x_i' = \begin{cases} 
   x_i & (i \neq k), \\
   \hat{F}_k / x_k, & (i = k).
   \end{cases}
   $$

3. If $F_i$ does not depend on $x_k$, then we define $F_i' = F_i$. In the following steps, we assume that $F_i$ depends on $x_k$.

4. We define the Laurent polynomial $G_i$ by
   $$
   G_i = F_i \bigg|_{x_i \rightarrow \hat{F}_k / x_k}.
   $$
   Note that we can show that if $F_i$ depends on $x_k$, then $a_k = 0$. Therefore, $\hat{F}_k|_{x_i \rightarrow 0}$ is well defined.

5. We define $H_i$ to be the result of removing all common factors in $\mathbb{Z}[x_0, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{N-1}]$ with $\hat{F}_k|_{x_i \rightarrow 0}$ from $G_i$.

6. Let $M_i$ be the unique Laurent monomial of $x_0^\pm, x_1^\pm, \ldots, x_{N-1}^\pm$ satisfying following conditions:
   - $M_i H_i \in \mathbb{Z}[x_0^\pm, x_1^\pm, \ldots, x_{N-1}^\pm]$.
   - $M_i H_i$ is not divisible by any $x_i'$.
   - $M_i H_i$ does not depend on $x_i'$.

   New exchange polynomial $F_i'$ is defined as $F_i' = M_i H_i$.
Proposition 2.2 [12]. For any $k \in \{0, 1, \ldots, N - 1\}$ and any seed $t$, it holds that $p_k^2(t) = t$.

Proposition 2.3 [12]. Suppose that we mutate at $x_i$. Then $F'_i$ depends on $x'_i$ if and only if $F_i$ depends on $x_i$.

Definition 2.4 [12]. Let us fix a seed $t = (x, F)$. This seed is called an initial seed. Let $X(t)$ be the set of all the cluster variables obtained by iterative mutations to the initial seed $t$. Let $A(t)$ be a Laurent phenomenon algebra defined as

$$ A(t) = \mathbb{Q}[x \in X(t)] \subset \mathbb{Q}(x). $$

(2.7)

Theorem 2.5 [12]. Let $t = (x, F)$ be a initial seed. If $x \in X(t)$, then $x \in \mathbb{Z}[x^\pm]$.

2.2. Period-1 seeds

For a polynomial

$$ f(x_0, x_1, \ldots, x_{N-1}) \in \mathbb{Z}[x_0, x_1, \ldots, x_{N-1}] $$

(2.8)

and for $j \geq 1$, we define

$$ u^j(f) = f(x_j, x_{j+1}, \ldots, x_{j+N-1}) \in \mathbb{Z}[x_j, x_{j+1}, \ldots, x_{j+N-1}]. $$

(2.9)

We also abbreviate $u := u^1$.

Definition 2.6 [13]. A seed

$$ t = \{ (x_0, F_0), (x_1, F_1), \ldots, (x_{N-1}, F_{N-1}) \} $$

(2.10)

is called a period-1 seed if

$$ \mu_0(t) = \{ (x_N, u(F_{N-1})), (x_0, u(F_0)), (x_2, u(F_1)), \ldots, (x_{N-1}, u(F_{N-2})) \} $$

(2.11)

holds, where $x_N = x'_0$ is the new cluster variable obtained by the mutation at $x_0$.

We assume that

$$ t_0 = \{ (x_0, F_0), (x_1, F_1), \ldots, (x_{N-1}, F_{N-1}) \} $$

(2.12)

is a period-1 seed. We inductively define the seed $t_n$ by $t_n = \mu_{n-1}(t_{n-1})$. The new cluster variable $x'_n$ in $\mu_n(t_n)$ is expressed as $x_{n+N}$. Hereafter, we write this process as $x_n \rightarrow x_{n+N}$. We have

$$ t_n = \{ (x_n, u^n(F_0)), (x_{n+1}, u^n(F_1)), \ldots, (x_{n+N-1}, u^n(F_{N-1})) \}. $$

(2.13)

If an exchange polynomial of a period-1 seed satisfies $F_0 = F_0$, then the cluster variables satisfy $x_{n+N} = u^n(F_0)$. In the following sections, all exchange polynomials satisfy $F_0 = F_0$. We put $F_0 = f(x_0, x_2, \ldots, x_{N-1})$. All cluster variables satisfy the difference equation

$$ x_{n+N} = f(x_{n+1}, x_{n+2}, \ldots, x_{n+N-1}). $$

(2.14)

Several results about a period-1 seed have already known.

- All the rank 2 or 3 period-1 seeds have been obtained [13].
- An example of a rank 6 period-1 seed has been obtained [12]. The cluster variables of this seed satisfy the difference equation
This difference equation is called Somos-6.

Several rank $N$ period-1 seeds have been obtained \[13\]. For example, the cluster variables of these seeds satisfy the following difference equations:

\[
x_{n+N} x_n = \prod_{i=1}^{N-1} x_{n+i}^{a_i} + 1 \quad (a_i \in \mathbb{Z}_{>0}),
\]

\[
x_{n+N} x_n = x_{n+1} x_{n+N-1} + A \sum_{i=1}^{N-1} x_{n+i} + B \quad (A, B \in \mathbb{Z}).
\]

## 3. Seed of the discrete BKP equation

### 3.1. Seed of the discrete BKP equation and its mutation

In this section, we consider the infinite rank seed. For $i \leq j$ ($i, j \in \mathbb{Z}$), we put \([i, j] = \{i, i+1, \ldots, j\}\). For a polynomial $P \in \mathbb{Z}[x_{m,i}^{|n, m, l}| n, m, l \in \mathbb{Z}]$, let $s_{j,k}^{i}(P)$ be the polynomial in which we substituted all variables $x_{m,i}^{|n, m, l}$ for $x_{m+i+j}^{m+n+k}$ in $P$. We take

\[
t_0 = \{(x_{i-j-k}^{l-k}, F_{i-j-k}^{l-k}) | i \in [0, 5], j, k \in \mathbb{Z}\}
\]

as an initial seed, where exchange polynomials are defined as

\[
F_{0}^{0,0} = x_{0}^{1,0} x_{1}^{0,0} + x_{0}^{1,0} x_{1}^{0,1} + x_{0}^{1,0} x_{1}^{0,1},
\]

\[
F_{1}^{0,0} = x_{0}^{1,0} x_{1}^{0,1} x_{2}^{0,0} + x_{0} x_{1}^{1,0} x_{2}^{0,0} + x_{0} x_{1}^{1,0} x_{2}^{0,1} + x_{0} x_{1}^{1,0} x_{2}^{0,1} + x_{0} x_{1}^{1,0} x_{2}^{0,1} x_{3} x_{4} x_{5}^{0,0},
\]

\[
F_{2}^{0,0} = x_{0}^{1,0} x_{1}^{0,1} x_{2}^{0,0} + x_{0} x_{1}^{1,0} x_{2}^{0,0} + x_{0} x_{1}^{1,0} x_{2}^{0,1} + x_{0} x_{1}^{1,0} x_{2}^{0,1} x_{3} x_{4} x_{5}^{0,0},
\]

\[
F_{3}^{0,0} = x_{0}^{1,0} x_{1}^{0,1} x_{2}^{0,0} + x_{0} x_{1}^{1,0} x_{2}^{0,0} + x_{0} x_{1}^{1,0} x_{2}^{0,1} + x_{0} x_{1}^{1,0} x_{2}^{0,1} x_{3} x_{4} x_{5}^{0,0},
\]

\[
F_{4}^{0,0} = x_{0}^{1,0} x_{1}^{0,1} x_{2}^{0,0} + x_{0} x_{1}^{1,0} x_{2}^{0,0} + x_{0} x_{1}^{1,0} x_{2}^{0,1} + x_{0} x_{1}^{1,0} x_{2}^{0,1} x_{3} x_{4} x_{5}^{0,0},
\]

\[
F_{5}^{0,0} = x_{0}^{1,0} x_{1}^{0,1} x_{2}^{0,0} + x_{0} x_{1}^{1,0} x_{2}^{0,0} + x_{0} x_{1}^{1,0} x_{2}^{0,1} + x_{0} x_{1}^{1,0} x_{2}^{0,1} x_{3} x_{4} x_{5}^{0,0},
\]

and

\[
F_{i-j-k}^{j-k} = s_{i-j-k}^{j-k}(F_{i}^{0,0}).
\]

For example, $F_{i-2j-k}^{0,0}$ is defined as

\[
F_{i-2j-k}^{0,0} = s_{i-2j-k}^{j-k}(F_{i}^{0,0}) = x_{0}^{0,0} x_{1}^{0,1} + x_{0}^{0,1} x_{1}^{0,0} + x_{0}^{1,0} x_{1}^{0,0}.
\]

We define $\bar{t} = t_{0}^{0,0}$ and the set of cluster variables $X_i$ as

\[
X_i = \{x_{i-j-k}^{l-k} | j, k \in \mathbb{Z}\}.
\]

A mutation $m_{i,j,k}^{l,k}$ $(i, j, k \in \mathbb{Z})$ denotes the mutation at $x_{i-j-k}^{l-k}$. We define the iteration of the mutations $n_{i}^{l}$ $(i \in \mathbb{Z}, n \in \mathbb{Z}_{\geq 0})$ by

\[
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\]

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\[
\nu^n_i = m_i^{-n-1,-1} \circ \ldots \circ m_i^{-1,-n+1} \circ m_i^{-0,-n} \circ m_i^{-1,-n+1} \circ \ldots \circ m_i^{-1,-n+1,0} \circ m_i^{-n+1,0} \\
\circ m_i^{-n+1,0} \circ \ldots \circ m_i^{-1,-n+1} \circ m_i^{-0,-n} \circ m_i^{-1,-n+1,0} \circ \ldots \circ m_i^{-1,-n+1,0} \circ m_i^{-n+1,0} \ (n \neq 0), \\
\nu^0_i = m_i^{0,0}.
\]

This \( \nu^n_i \) is the iteration of the mutations at each and all \( x_{j-k}^i \in X_i \) \((j + k = n)\) just once. We define the iteration of the mutations \( \tilde{\nu}_i \) \((i \in \mathbb{Z})\) by
\[
\tilde{\nu}_i = \ldots \circ \nu_i^2 \circ \nu_i^1 \circ \nu_i^0.
\]

This \( \tilde{\nu}_i \) is the iteration of the mutations at each and all \( x \in X_i \) just once. We also use the notation \( x \rightarrow y \) that means a new cluster variable \( x' \) is denoted by \( y \), i.e., \( y = x' \).

**Proposition 3.1.** We define the seed \( t_1 \) by \( t_1 = \tilde{\nu}_0(t_0) \). When we put \( x_{j-k}^i \rightarrow x_{j-k}^{i+1+k} \), it holds that
\[
t_1 = \{(x_{j-k}^{1-j-k+1+k}, \theta(F_{j-k}^{j-k})) \mid i \in [0, 5], j, k \in \mathbb{Z}\}.
\]

The new cluster variables \( x_{j-k}^{1-j-k+1+k} \) satisfy
\[
x_{j-k}^{1-j-k+1+k} = F_{j-k}^{j-k}.
\]

**Proof.** If \( F_{n,m,l} \) does not depend on \( x_{j-k}^i \), then it holds \( (F_{n,m,l})' = F_{n,m,l} \), when we mutate at \( x_{j-k}^i \in X_0 \). The dependence on \( x \in X_0 \) of the exchange polynomials is as follows:

- \( F_{j-k}^{j-k} \) does not depend on \( x \in X_0 \).
- \( F_{j-k}^{j-k} \) depends on only \( x_{j-k}^i \in X_0 \).
- \( F_{j-k}^{j-k} \) depends on only \( x_{j-k}^{i+1+j-k} \in X_0 \).
- \( F_{j-k}^{j-k} \) depends on only \( x_{j-k}^{i+1+j-k} \in X_0 \).
- \( F_{j-k}^{j-k} \) depends on only \( x_{j-k}^{i+1+j-k} \in X_0 \).
- \( F_{j-k}^{j-k} \) depends on only \( x_{j-k}^{i+1+j-k} \in X_0 \).

For each \( i \in [0, 5] \), we consider the change of \( (x_{j-k}^i, F_{j-k}^{j-k}) \) by the mutations of initial seed \( t_0 \) at each and all \( x \in X_0 \) just once.

- The case of \( i = 0 \)

By (3.10), the exchange polynomials \( F_{j-k}^{j-k} \) do not change by the mutations at \( x \in X_0 \). For each seed

\[
i = \{(x_{j-k}^{j-k}, F_{j-k}^{j-k}) \mid i \in [0, 5], j, k \in \mathbb{Z}\}
\]

obtained by the mutations at \( x \in X_0 \), if \( F_{j-k}^{j-k} \) is not divisible by \( \tilde{F}_{n,m,l} \) for any \( (n, m, l) \neq (-2j - k, -j - k, 0) \), then it holds that \( F_{j-k}^{j-k} = F_{j-k}^{j-k} \). In fact, all exchange polynomials which appear below satisfy \( \tilde{F}_{j-k}^{j-k} = F_{j-k}^{j-k} \). By the mutation at \( x_{j-k}^i \), the change of the cluster variable \( x_{j-k}^i \) is
\[
x_{j-k}^{i+1+j-k+1+k} = \tilde{F}_{j-k}^{j-k} x_{j-k}^{i+1+j-k+1+k} = F_{j-k}^{j-k} x_{j-k}^{i+1+j-k+1+k}.
\]
and other cluster variables \(x_{j-k}^{j',k'} \ ( (j', k') \neq (j, k) ) \) do not change, where 
\( \mu_{n}^{m,i} \ (n, m, i \in \mathbb{Z}) \) is the mutation at \( x_{n}^{m,i} \). Therefore, we have

\[
(x_{2-j}^{j-k}, \ F_{2-j}^{j-k}) \rightarrow (x_{2-j}^{1+j-k}, \ F_{2-j}^{1+j-k}) = (x_{6-2(j+1)+(k+1)}, \ \tilde{a} (F_{5-2(j+1)+(k+1)}) ) \tag{3.13}
\]

by the mutations of initial seed \( t_0 \) at each and all \( x \in X_0 \) just once.

- **The case of \( i = 1 \)**
  By (3.10), the exchange polynomials \( F_{1-2j}^{j-k} \) do not change until we execute the mutation at \( x_{2-j}^{j-k} \). Let \( \lambda_1 \) and \( \lambda_2 \) be the iteration of mutations before and after the mutation at \( x_{2-j}^{j-k} \) respectively. We have

\[
(F_{1-2j}^{j-k}, \ F_{1-2j}^{j-k}) \rightarrow (F_{1-2j}^{j-k}, \ F_{1-2j}^{j-k}). \tag{3.14}
\]

We mutate at \( x_{2-j}^{j-k} \) and calculate \((F_{1-2j}^{j-k})' \) from \( F_{1-2j}^{j-k} \) and \( F_{2-2j}^{j-k} \). It holds that

\[
(F_{1-2j}^{j-k}, \ F_{1-2j}^{j-k}) \rightarrow (F_{1-2j}^{j-k}, \ F_{1-2j}^{j-k}) \tag{3.15}
\]

\( \tilde{a}(F_{2-2j}^{j-k}) \) does not depend on any \( x \in X_0 \). We have

\[
\tilde{a}(F_{2-2j}^{j-k}) \rightarrow (F_{2-2j}^{j-k}). \tag{3.16}
\]

The cluster variable \( x_{2-j}^{j-k} \) does not change by the mutations at \( x \in X_0 \). Therefore, we have

\[
(x_{1-2j}^{j-k}, \ F_{1-2j}^{j-k}) \rightarrow (x_{1-2j}^{j-k}, \ F_{2-2j}^{j-k} \ (x \in X_0) \tag{3.17})
\]

by the mutations of the initial seed \( t_0 \) at each and all \( x \in X_0 \) just once.

- **The cases for \( i = 2, 4, 5 \)**
  The cases for \( i = 2, 4, 5 \) are the same as the case of \( i = 1 \). It holds that

\[
(x_{2-j}^{j-k}, \ F_{2-2j}^{j-k}) \rightarrow (x_{2-j}^{j-k}, \ F_{2-2j}^{j-k} \ (x \in X_0) \tag{3.18})
\]

by the mutations of initial seed \( t_0 \) at each and all \( x \in X_0 \) just once.

- **The case of \( i = 3 \)**
  By (3.10), the exchange polynomials \( F_{3-2j}^{j-k} \) do not change until we execute the mutation at \( x_{3-j}^{j-k-1} \) or \( x_{3-j}^{1+j-k-k} \). We consider two cases in which the order of the mutations at \( x_{3-j}^{j-k-1} \) and \( x_{3-j}^{1+j-k-k} \) is different from each other.

  - Suppose that we mutate at \( x_{3-j}^{j-k-1} \) before the mutation at \( x_{3-j}^{1+j-k-k} \). Let \( \lambda_1 \), \( \lambda_2 \), and \( \lambda_3 \) be the iteration of mutations before the mutation at \( x_{3-j}^{j-k-1} \), after the mutation at \( x_{3-j}^{1+j-k-k} \) and before the mutation at \( x_{3-j}^{1+j-k-k} \), and after the mutation at \( x_{3-j}^{1+j-k-k} \) respectively. We have

\[
(F_{3-j}^{j-k-1}, \ F_{3-j}^{j-k-1}, \ F_{2-j}^{j-k-1}) \rightarrow (F_{3-j}^{j-k-1}, \ F_{3-j}^{j-k-1}, \ F_{2-j}^{j-k-1}) \tag{3.19}
\]

We mutate at \( x_{3-j}^{j-k-1} \) and calculate \((F_{3-j}^{j-k-1})' \) from \( F_{3-j}^{j-k-1} \) and \( F_{3-j}^{j-k-1} \). Then we have

\[
(F_{3-j}^{j-k-1}, \ F_{3-j}^{j-k-1}, \ F_{2-j}^{j-k-1}) \rightarrow (F_{3-j}^{j-k-1})' = (F_{3-j}^{j-k-1}, \ F_{2-j}^{j-k-1}). \tag{3.20}
\]
\[ P_1 = \begin{pmatrix} x_1,0,1,0,0,0,0 \end{pmatrix} \begin{pmatrix} x_2,1,0,1,0,0,0 \end{pmatrix} + \begin{pmatrix} x_2,1,0,1,0,0,0 \end{pmatrix} \begin{pmatrix} x_3,0,1,0,0,0 \end{pmatrix} + \begin{pmatrix} x_2,0,0,1,0,1,0 \end{pmatrix} \begin{pmatrix} x_3,0,1,0,0,0 \end{pmatrix} + \begin{pmatrix} x_2,0,0,1,0,1,0 \end{pmatrix} \begin{pmatrix} x_3,0,1,0,0,0 \end{pmatrix} + \begin{pmatrix} x_2,0,0,1,0,1,0 \end{pmatrix} \begin{pmatrix} x_4,0,1,0,0,0 \end{pmatrix}. \] (3.21)

However, \( s_{j-k}^{1+j-k} (P_1) \) depends only on \( x_{\frac{1}{2}-\frac{1}{2}}^{1+j-k} \in X_0 \). Hence, \( s_{j-k}^{1+j-k} (P_1) \) does not change before the mutation at \( x_{\frac{1}{2}-\frac{1}{2}}^{1+j-k} \). We have

\[ (s_{j-k}^{1+j-k} (P_1), F_{j-k}^{1+j-k}) \xrightarrow{\lambda_1} (s_{j-k}^{1+j-k} (P_1), F_{j-k}^{1+j-k}). \] (3.22)

Mutating at \( x_{\frac{1}{2}-\frac{1}{2}}^{1+j-k} \) and calculating \( (s_{j-k}^{1+j-k} (P_1))' \) from \( s_{j-k}^{1+j-k} (P_1) \) and \( F_{j-k}^{1+j-k} \), we find

\[ (s_{j-k}^{1+j-k} (P_1), F_{j-k}^{1+j-k}) \xrightarrow{\mu_1} (F_{j-k}^{1+j-k}, F_{j-k}^{1+j-k}). \] (3.23)

Since \( \bar{u}(F_{j-k}^{1+j-k}) \) does not depend on any \( x \in X_0 \), we have

\[ \bar{u}(F_{j-k}^{1+j-k}) \xrightarrow{\lambda_1} \bar{u}(F_{j-k}^{1+j-k}). \] (3.24)

Suppose that we mutate at \( x_{\frac{1}{2}-\frac{1}{2}}^{1+j-k} \) before the mutation at \( x_{\frac{1}{2}-\frac{1}{2}}^{1+j-k} \). Let \( \lambda_1 \), \( \lambda_2 \), and \( \lambda_3 \) be the iteration of mutations before the mutation at \( x_{\frac{1}{2}-\frac{1}{2}}^{1+j-k} \), after the mutation at \( x_{\frac{1}{2}-\frac{1}{2}}^{1+j-k} \) and before the mutation at \( x_{\frac{1}{2}-\frac{1}{2}}^{1+j-k} \), and after the mutation at \( x_{\frac{1}{2}-\frac{1}{2}}^{1+j-k} \) respectively. We have

\[ (F_{j-k}^{1+j-k}, F_{j-k}^{1+j-k}, F_{j-k}^{1+j-k}) \xrightarrow{\lambda_1} (F_{j-k}^{1+j-k}, F_{j-k}^{1+j-k}, F_{j-k}^{1+j-k}). \] (3.25)

By mutation at \( x_{\frac{1}{2}-\frac{1}{2}}^{1+j-k} \) and calculation of \( (F_{j-k}^{1+j-k})' \) from \( F_{j-k}^{1+j-k} \) and \( F_{j-k}^{1+j-k} \), we find that

\[ (F_{j-k}^{1+j-k}, F_{j-k}^{1+j-k}, F_{j-k}^{1+j-k}) \xrightarrow{\mu_1} (\bar{u}(F_{j-k}^{1+j-k}), F_{j-k}^{1+j-k}). \] (3.26)

Since \( s_{j-k}^{1+j-k} (P_2) \) depends on only \( x_{\frac{1}{2}-\frac{1}{2}}^{1+j-k} \in X_0 \), \( s_{j-k}^{1+j-k} (P_2) \) does not change before the mutation at \( x_{\frac{1}{2}-\frac{1}{2}}^{1+j-k} \). We have

\[ (s_{j-k}^{1+j-k} (P_2), F_{j-k}^{1+j-k}, F_{j-k}^{1+j-k}) \xrightarrow{\lambda_1} (s_{j-k}^{1+j-k} (P_2), F_{j-k}^{1+j-k}, F_{j-k}^{1+j-k}). \] (3.27)

Mutation at \( x_{\frac{1}{2}-\frac{1}{2}}^{1+j-k} \) and calculation of \( (s_{j-k}^{1+j-k} (P_2))' \) from \( s_{j-k}^{1+j-k} (P_2) \) and \( F_{j-k}^{1+j-k} \) give

\[ (s_{j-k}^{1+j-k} (P_2), F_{j-k}^{1+j-k}, F_{j-k}^{1+j-k}) \xrightarrow{\mu_1} (\bar{u}(F_{j-k}^{1+j-k}), F_{j-k}^{1+j-k}). \] (3.28)

Since \( \bar{u}(F_{j-k}^{1+j-k}) \) does not depend on any \( x \in X_0 \), we have

\[ \bar{u}(F_{j-k}^{1+j-k}) \xrightarrow{\lambda_1} \bar{u}(F_{j-k}^{1+j-k}). \] (3.29)

The cluster variable \( x_{\frac{1}{2}-\frac{1}{2}}^{1+j-k} \) does not change by the mutations at \( x \in X_0 \) and we have

\[ (x_{\frac{1}{2}-\frac{1}{2}}^{1+j-k}, F_{j-k}^{1+j-k}) \xrightarrow{\lambda_1} (x_{\frac{1}{2}-\frac{1}{2}}^{1+j-k}, \bar{u}(F_{j-k}^{1+j-k})). \] (3.30)

by the mutations of initial seed \( t_0 \) at each and all \( x \in X_0 \) just once.

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Therefore, it holds that

\[
\begin{align*}
& t_0 = \{(x_{j}^{j-k,k}, F_{j}^{j-k,k}), (x_{j}^{j-k,k}, F_{j}^{j-k,k}) | i \in [1, 5], j, k \in \mathbb{Z}\} \\
\Rightarrow t_1 = \{(x_{j}^{j+2i-(k+1)(k+1)}, \tilde{u}(F_{j}^{j+2i-(k+1)(k+1)}), (x_{j}^{j-k,k}, \tilde{u}(F_{j}^{j-k,k})) | i \in [1, 5], j, k \in \mathbb{Z}\},
\end{align*}
\]

which does not depend on the order of mutations.

Note that the iteration of the mutations \(\tilde{\mu}_0\) does not depend on the order of the mutations.

**Theorem 3.2.** We defined the seed \(t_n\) by \(t_n = \tilde{\mu}_{n-1}(t_{n-1})\). When we put

\[
x_{n+1}^{j-k,k} = x_{n}^{j-k,k},
\]

it holds that

\[
t_n = \{(x_{n}^{j-k,k}, \tilde{u}(F_{j}^{j-k,k})) | i \in [0, 5], j, k \in \mathbb{Z}\}.
\]

All cluster variables \(x_{m,l}^{n}\) \((n, m, l \in \mathbb{Z})\) satisfy

\[
x_{n+1}^{m+1,l+1} = x_{n}^{m+1,l+1} x_{n+1}^{m,l+1} + x_{n+1}^{m+1,l} x_{n}^{m,l+1} + x_{n+1}^{m+1,l} x_{n}^{m+1,l}.
\]

**Proof.** Equations (3.33) and (3.34) are obtained by the shift of the suffixes in equations (3.8) and (3.9). Therefore, proof of theorem 3.2 is the same as that of proposition 3.1. Equation (3.34) is obtained by the definition of the new cluster variable

\[
x_{j}^{j-k,k} = \tilde{u}(F_{j}^{j-k,k})/x_{j}^{j-k,k}
\]

by the mutation of the seed \(t_n\) at \(x_{j}^{j-k,k}\).

Equation (3.34) is called the discrete BKP equation [3]. The discrete BKP equation is the relation among 8 point of figure 1. By proposition 3.1, it holds

\[
\tilde{\mu}_0(t_0) = \{(x_{i}^{j-k,k}, \tilde{u}(F_{i}^{j-k,k}) | i \in [0, 5], j, k \in \mathbb{Z}\}.
\]

The initial seed \(t_0\) satisfies the condition which is similar to the one in the definition 2.6. Therefore, the initial seed \(t_0\) is regarded as a generalization of period-1 seed. By theorem 2.5, all \(x_{m,l}^{n}\) are Laurent polynomials of the initial values in \(X_0\). This property is called Laurent phenomenon of a difference equation. The Laurent phenomenon of a difference equation implies that a solution of the initial value problem is always expressed by Laurent polynomials of the initial entries. It is known that the BKP equation, Somos-6, and Somos-7 have the Laurent phenomenon property [7].

![Figure 1. The discrete BKP equation.](image)
3.2. Another seed of the discrete BKP equation

Let us take

$$t_0 = \{ (x_{i-j-2k}^i, F_{i-j-2k}^i) | i \in [0, 6], j, k \in \mathbb{Z} \}$$

(3.37)

as an initial seed, where exchange polynomials are defined as

$$F_{0,0}^0 = x_0^{1,1} x_0^{0,0} + x_0^{1,0} x_0^{1,1} + x_0^{0,1} x_0^{1,0},$$
$$F_{0,0}^1 = x_0^{1,0} x_0^{1,1} x_2^{0,0} + x_0^{0,1} x_0^{1,0} x_2^{0,0} + x_0^{0,0} x_0^{1,0} x_2^{0,1} + x_0^{0,0} x_0^{1,1} x_2^{0,0},$$
$$F_{0,0}^2 = x_0^{1,0} x_0^{1,1} x_2^{0,0} + x_0^{0,1} x_0^{1,0} x_2^{0,0} + x_0^{0,0} x_0^{1,0} x_2^{0,1} + x_0^{0,0} x_0^{1,1} x_2^{0,0},$$
$$F_{0,0}^3 = x_0^{1,0} x_0^{1,1} x_2^{0,0} + x_0^{0,1} x_0^{1,0} x_2^{0,0} + x_0^{0,0} x_0^{1,0} x_2^{0,1} + x_0^{0,0} x_0^{1,1} x_2^{0,0},$$
$$F_{0,0}^4 = x_0^{1,0} x_0^{1,1} x_2^{0,0} + x_0^{0,1} x_0^{1,0} x_2^{0,0} + x_0^{0,0} x_0^{1,0} x_2^{0,1} + x_0^{0,0} x_0^{1,1} x_2^{0,0},$$
$$F_{0,0}^5 = x_0^{1,0} x_0^{1,1} x_2^{0,0} + x_0^{0,1} x_0^{1,0} x_2^{0,0} + x_0^{0,0} x_0^{1,0} x_2^{0,1} + x_0^{0,0} x_0^{1,1} x_2^{0,0},$$

and

$$F_{i-j-2k}^{j,k} = s_{i-j-2k}^{j,k} (F_{0,0}^0).$$

(3.39)

Now we define $\tilde{a} = s_0^{1,0}$ and the set of cluster variables $X_i$ as

$$X_i = \{ x_{i-j-2k}^{j,k} | j, k \in \mathbb{Z} \}.$$

(3.40)

A mutation $m_{i-j}^{j,k}$ ($i, j, k \in \mathbb{Z}$) denotes the mutation at $x_{i-j}^{j,k}$. We define the iteration of the mutations $\nu_i^n$ $(i \in \mathbb{Z}, n \in \mathbb{Z}_{\geq 0})$ by (3.6). This $\nu_i^n$ is the iteration of the mutations at each and all $x_{i-j}^{j,k} \in X_i$ ($j + k = n$) just once. We define the iteration of the mutations $\tilde{\nu}_i$ ($i \in \mathbb{Z}$) by (3.7). This $\tilde{\nu}_i$ is the iteration of the mutations at each and all $x \in X_i$ just once.

**Theorem 3.3.** We define the seed $t_n$ by $t_n = \tilde{\nu}_{n-1}(t_{n-1})$. When we put

$$x_{i-j}^{j,k} \to x_{i-j+1}^{j,k+1},$$

it holds that

$$t_n = \{ (x_{i-j}^{j,k}, F_{i-j}^{j,k}(F_{i-j}^{j,k})) | i \in [0, 6], j, k \in \mathbb{Z} \}.$$

(3.41)

All cluster variables $x_{i-j}^{m,n}$ $(n, m, l \in \mathbb{Z})$ satisfy the discrete BKP equation (3.34).

**Proof.** Proof of theorem 3.3 can be done in a similar manner to those of proposition 3.1 and theorem 3.2.

Note that the iteration of the mutations $\tilde{\nu}_{n-1}$ does not depend on the order of the mutations.

4. Several difference equations associated with reductions of the seed

In this section, we show that several seeds which give the difference equations obtained by imposing constraints on the BKP equation and discuss the relation to the reductions for initial seeds and polynomials in Laurent phenomenon algebra.
Figure 2. Two-dimensional difference equation (1).

4.1. Two-dimensional difference equation

We take

$$t_0 = \{ (x_{-2}^i, F_{-2}^i) | i \in [0, 5], j \in \mathbb{Z} \}$$

(4.1)

as an initial seed, where exchange polynomials are defined as

$$F_0^0 = x_1^2 x_1^0 + x_0^1 x_1^1 + (x_1^1)^2,$$

$$F_1^0 = x_0^1 x_2^0 + (x_1^1)^2 x_0^0 + x_0^0 x_1^1 + (x_1^1)^2,$$

$$F_0^1 = x_1^1 x_2^1 + x_1^0 x_3^0 + (x_2^1)^2 x_0^0 + x_0^0 x_1^0 + (x_1^0)^2 x_0^0 + x_0^0 x_1^0 x_3^{-1},$$

$$F_3^0 = x_4^{-1} x_1^1 x_1^0 + (x_3^{-1})^2 x_1^0 x_1^2 + x_0^0 x_1^0 x_3^{-1} + x_4^{-1} x_1^1 x_3^{-1} + x_1^0 x_3^{-1} x_1^2.$$

$$F_4^0 = x_5^{-1} x_1^1 + (x_4^{-1})^2 x_3^0 + x_5 x_4^{-1} x_2^{-1} + x_5^0 (x_3^{-1})^2,$$

$$F_2^0 = x_4^{-1} x_1^1 + x_4^{-1} x_3^{-1} + (x_4^{-1})^2,$$

and

$$F_{i-2j} = s_{i-2j}^j (F_0^i).$$

(4.3)

We define $\tilde{u} = x_1^0$ and the set of cluster variables $X_i$ by

$$X_i := \{ x_{-2j}^i | j \in \mathbb{Z} \}.$$  

(4.4)

A mutation $m_i^j (i, j \in \mathbb{Z})$ denotes the mutation at $x_{-2j}^i$. We define the iteration of the mutations $\tilde{\mu}_i$ ($i \in \mathbb{Z}$) by

$$\tilde{\mu}_i = \ldots \circ m_{i-2} \circ m_{i-1} \circ m_i^1 \circ m_1^0.$$  

(4.5)

This $\tilde{\mu}_i$ is the iteration of the mutations at each and all $x \in X_i$ just once.

Proposition 4.1. We define the seed $t_n$ by $t_n = \tilde{\mu}_{n+1} (t_{n+2})$. When we put $x_{n+1-2j} = x_{n+1-2j}^0$, it holds that

$$t_n = \{ (x_{n+1-2j}^i, \tilde{u}^n (F_{i-2j}^i)) | i \in [0, 5], j \in \mathbb{Z} \}.$$  

(4.6)

All cluster variables $x_{n}^m (n, m \in \mathbb{Z})$ satisfy

$$x_{n+1}^{m+1} x_n^m = x_{n+1}^m x_{n+1}^m + (x_{n+1}^m)^2 + x_{n+2}^m x_{n+1}^m.$$  

(4.7)

Two-dimensional equation (4.7) is the relation among 7 points shown in figure 2. Note that two-dimensional difference equation (4.7) is obtained from the discrete BKP equation (3.34) by imposing the reduction condition.
The initial seed (4.1) and the exchange polynomials (4.2) are obtained from the initial seed (3.1) and the exchange polynomials (3.2) by imposing the same condition (4.8). Therefore, the reduction of the seed corresponds to the reduction of the difference equation. Proof of proposition 4.1 is the same as those of proposition 3.1 and theorem 3.2. Note that the iteration of the mutations $\bar{\mu}_{n-1}$ does not depend on the order of the mutations.

### 4.2. Somos-6

We impose the reduction condition

$$x_n^{m+1} = x_{n+2}^m, \quad x_n^m := x_n^0$$

(4.9)

to the initial seed (4.1) and exchange polynomials (4.2). Then we obtain the seed and exchange polynomials

$$t_0 = \{(x_i, F_i) | i \in [0, 5]\}, \quad F_0 = x_5x_0 + x_2x_4 + (x_3)^2,$$

(4.10)

$$F_1 = (x_2)^2x_4 + (x_3)^2x_2 + x_0x_3x_5 + x_0(x_4)^2,$$

$$F_2 = (x_3)^2x_3 x_5 + (x_4)^2x_0x_3 + x_1x_3x_4 + (x_4x_5)^2 + (x_4)^3 x_5 x_4,$$

$$F_3 = (x_2)^2x_5x_0 + (x_4)^2x_5x_2 + x_4(x_2)^2x_1 + x_4x_5x_0x_1 + (x_4)^2x_0x_1,$$

$$F_4 = (x_3)^2x_4 + x_2x_3 + x_5x_2x_0 + x_5(x_1)^2,$$

$$F_5 = x_0x_4 + x_3x_1 + (x_2)^2.$$  

(4.11)

This seed is a period-1 seed. This seed has already been obtained in [12].

**Proposition 4.2** [12]. We define the seed $t_n$ by $t_n = \mu_{n-1}(t_{n-1})$. When we put $x_{n-1} \mapsto x_{n-5}$, it holds that

$$t_n = \{(x_{n+i}, u^e(F_i)) | i \in [0, 5]\}.$$  

(4.12)

All cluster variables $x_n (n \in \mathbb{Z})$ satisfy

$$x_{n+6}x_n = x_{n+5}x_{n+1} + (x_{n+3})^2 + x_{n+4}x_{n+2}.$$  

(4.13)

Difference equation (4.13) is called the Somos-6 [4]. The Somos-6 (4.13) is obtained from two-dimensional difference equation (4.7) by imposing the reduction condition (4.9).

### 4.3. The discrete Sawada–Kotera equation

We take

$$t_0 = \{(x_{i-4}^j, F_{i-4}^j) | i \in [0, 6], j \in \mathbb{Z}\}$$

(4.14)
as an initial seed, where exchange polynomials are defined as

\[
F_0^0 = x_2^1 x_1^0 + x_0^1 x_3^0 + x_2^0 x_1^1, \\
F_1^0 = x_0^1 x_2^0 + (x_2^0)^2 x_1^0 + x_0^0 x_1^1 x_4^0 + x_0^0 x_3^0 x_2^1, \\
F_2^0 = x_1^1 x_4^0 x_3^0 + (x_3^0)^2 x_0^0 x_1^1 + x_1^0 x_4^0 x_5^0 + x_1^0 x_0^0 x_2^5 + (x_4^0)^2 x_2^4 x_1^2, \\
F_3^0 = x_0^0 x_3^1 x_4^1 + (x_4^0)^2 x_1^0 + x_2^0 x_1^0 + (x_1^0)^2 x_3^0, \\
F_4^0 = x_5^1 x_2^0 x_4^0 x_5^0 + (x_3^0)^2 x_0^0 x_4^0 + x_5^0 x_2^0 x_4^5 + x_5^0 x_0^0 x_4^4 x_4^0 + (x_5^0)^2 x_5^0 x_4^0 x_1^1, \\
F_5^0 = x_6^1 x_3^0 x_4^1 + (x_4^0)^2 x_5^1 + x_0^0 x_5^1 x_2^0 + x_6 x_3 x_4^1, \\
F_6^0 = x_4^1 x_5^0 + x_6 x_1^0 + x_4 x_3^0 x_5^0. 
\]

(4.15)

and

\[
F_{i-4j} = s_{-4j}(F_i^0). 
\]

(4.16)

We define \( \bar{u} = x_1^0 \) and the set of cluster variables \( X_i \) as

\[
X_i = \{ x_{i-4j} \mid j \in \mathbb{Z} \}. 
\]

(4.17)

A mutation \( m_i^j \) \( (i, j \in \mathbb{Z}) \) denotes the mutation at \( x_{i-4j} \). We define the iteration of the mutations \( \bar{u}_i \) \( (i \in \mathbb{Z}) \) by (4.5). This \( \bar{u}_i \) is the iteration of the mutations at each and all \( x \in X_i \) just once.

**Proposition 4.3.** We define the seed \( t_n \) by \( t_n = \bar{u}_{n-1}(t_{n-1}) \). Putting \( x_{n-1-4j} = x_{n+2-4j} \) we find

\[
t_n = \{ (x_{n+i-4j}, \bar{u}^n(F_{i-4j})) \mid i \in [0, 6], j \in \mathbb{Z} \}. 
\]

(4.18)

All cluster variables \( x_n^m \) \( (n, m \in \mathbb{Z}) \) satisfy

\[
x_{n+3}^m x_n^m = x_{n+2}^m x_{n+1}^m + x_{n+1}^m x_{n+2}^m + x_{n+3}^m x_{n+1}^m. 
\]

(4.19)

Note that the iteration of the mutations \( \bar{u}_{n-1} \) does not depend on the order of the mutations. Equation (4.19) is called the discrete Sawada–Kotera equation [5]. The discrete Sawada–Kotera equation is the relation among 8 points shown in figure 3. Note that the discrete Sawada–Kotera equation (4.19) is obtained from the discrete BKP equation (3.34) by imposing the reduction condition

\[
x_{n}^{m,1+1} = x_{n+2}^{m,1}, \quad x_n^{m} := x_{n}^{m,0}. 
\]

(4.20)

The initial seed (4.14) and the exchange polynomials (4.15) are obtained from the initial seed (3.37) and the exchange polynomials (3.38) by imposing the same condition (4.20).

### 4.4. Somos-7

We take

\[
t_0 = \{ (x_i, F_i) \mid i \in [0, 6] \}
\]

(4.21)
as an initial seed, where exchange polynomials are defined as
\[
F_0 = x_0x_1 + x_4x_3 + x_2x_5 \\
F_1 = x_0x_3x_2 + (x_2)^2x_5 + x_0x_5x_4 + x_0x_3x_6 \\
F_2 = x_0x_4x_0x_3 + (x_3)^2x_0x_6 + x_0(x_4)^2x_3 + x_1x_0x_6x_5 + (x_5)^2x_4x_6 \\
F_3 = x_0x_5x_4 + (x_3)^2x_1 + x_2x_1x_6 + (x_2)^2x_3 \\
F_4 = x_1x_2x_6x_3 + (x_3)^2x_6x_0 + x_5(x_2)^2x_3 + x_5x_6x_0x_1 + (x_5)^2x_2x_0 \\
F_5 = x_2x_3x_4 + (x_4)^2x_1 + x_6x_1x_2 + x_6x_3x_0 \\
F_6 = x_0x_5 + x_2x_3 + x_4x_1.
\] (4.22)

**Proposition 4.4.** We define the seed \( t_n \) by \( t_n = \mu_{n-1}(t_{n-1}) \). We put \( x_{n-1} \to x_{n+6} \) and we have
\[
t_n = \{(x_{n+i}, u^i(F))\mid i \in [0, 6]\}.
\] (4.23)
All cluster variables \( x_n \) \((n \in \mathbb{Z})\) satisfy
\[
x_{n+7} = x_{n+6}x_{n+1} + x_{n+5}x_{n+2} + x_{n+3}x_{n+4}.
\] (4.24)

Difference equation (4.24) is called the Somos-7 [4]. Note that the Somos-7 (4.24) is obtained from the two-dimensional difference equation (4.19) by imposing the reduction condition
\[
x_n^{m+1} = x_n^m, \quad x_n^0 = x_n^0.
\] (4.25)
The initial seed (4.21) and the exchange polynomials (4.22) are obtained from the initial seed (4.14) and the exchange polynomials (4.15) by imposing the same condition (4.25).

### 5. Conclusion

We have shown that cluster variables can satisfy the discrete BKP equation, the two-dimensional difference equations of its reductions, and Somos-7, if we take appropriate initial seeds in Laurent phenomenon algebras. These initial seeds are obtained from reductions of the seed of the discrete BKP equation. It is known that cluster variables of suitable cluster algebras can satisfy the bilinear form of some \( q \)-discrete Painlevé equations [6], when the initial seed includes appropriate \( q \)-periodic quivers [11]. However, we have not obtained the \( q \)-discrete Painlevé equations of type \( A_1^{(1)} \) and \( E_6^{(1)} \) from cluster algebras. To clarify the relation between these equations and Laurent phenomenon algebras is one of the problems we wish to address in the future.

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