## Appendix A

### Overview of the Contents of the Course and Related Reading-strategy Videos

| Title                                      | Content                                                                 | Reading-strategy video                                                                 |
|--------------------------------------------|------------------------------------------------------------------------|---------------------------------------------------------------------------------------|
| **1. Congruence and isometries of the Euclidian plane** | Definition of congruence. Definition of the isometries of the Euclidian plane. | Understanding mathematical concepts by examples and counterexamples                     |
| **2. Isometries and symmetry**             | Characteristics of isometries. Invariance and fixpoints of isometries. Definitions of symmetry and symmetry group. | Understanding mathematical concepts and rules by case-by-case analysis                   |
| **3. Compositions of isometries***        | Compositions of isometries. Compositions of two reflections (including proof). Three-reflections-theorem (including proof). | Understanding mathematical statements and argumentation while-reading                   |
| **4. Frieze groups**                      | Definition of frieze patterns. Seven frieze groups.                     |                                                                                       |
| **5. Platonic tessellations**             | Definition of (Platonic) tessellation. Number of Platonic tessellations (including proof). |                                                                                       |
| **6. Archimedean tessellations**          | Definition of Archimedean tessellation. Number of Archimedean tessellations (including proof). |                                                                                       |
| **7. (Platonic and) Archimedean solids**  | Definition of Platonic and Archimedean solids. Characteristics of Archimedean soli’s: Euler’s polyhedral formula. |                                                                                       |
| **8. Existence and completeness of (Platonic and) Archimedean solids** | Proofing the existence and total number of Archimedean solids. Construction of Archimedean solids from Platonic solids. | Visualization I (based on using provided materials, e.g., polygons are visualizations in the text) |
| **9. Dual-Archimedean solids**            | Definition of dual-Archimedean solids.                                  | Visualization II (based on building 3d-models or projections of 3d-models in GeoGebra) |
| **10. Symmetry of (dual-) Archimedean solids** | Symmetry of Archimedean solids and their dedicated dual-Archimedean solids. | Visualization III (based on operating with given 3d-models)                           |

* focus in this paper
### Appendix B

#### Sample of the Script

| Theorem: | The product of two reflections $S_g$ (first mapping) and $S_h$ (second mapping) is: |
|----------|----------------------------------------------------------------------------------------------------------------------------------|
| (i)     | the translation $V_{2\cdot a}$ (with length $2\cdot a$) (with the identity as a special case, if $g = h$), if $g$ and $h$ are parallel and have the (oriented) perpendicular $a$ (vector with length $a\geq0$) from $g$ to $h$. |
| (ii)    | the rotation $D_{Z;2\cdot w}$ (with the "point reflection" as a special case for $w = 90^\circ$), if $g$ and $h$ are not parallel, but have a well-defined intersection point $Z$ and enclose the (oriented) angle with measure $w \neq 0^\circ$ (between $90^\circ$ exclusive and $90^\circ$ inclusive) from $g$ to $h$. |
| (iii)   | the identity, if $g = h$ (special case of (i) with $a$ as the zero vector, i.e. $a = 0$; can also be treated like (ii) with $g = h$, i.e. $w = 0^\circ$ and some center $Z$). |
| (iv)    | If you reverse the order of the first and the second mapping, then $V_{2\cdot a}$ or $D_{Z;2\cdot w}$. |

**Understanding mathematical statements and argumentations while-reading.**

### Conclusions:

(i) **In general, a composition of two reflections is not commutative.** In only two cases $S_h \cdot S_g = S_g \cdot S_h$: If $g = h$ it is the identity, and if $g$ and $h$ are orthogonal, then the composition is both times a rotation by $180^\circ$ (=point reflection), and indeed one order equals a rotation by $180^\circ$, the other by $-180^\circ$, which is the same.

The composition of three or more reflections is associative, that means $S_i \cdot (S_h \cdot S_g) = (S_i \cdot S_h) \cdot S_g$.

(ii) Let $g$ and $h$ be **two parallel straight lines** with the (oriented) perpendicular $a$ from $g$ to $h$. Let $k$ and $l$ be two more parallel lines, also with the (oriented) perpendicular $a$ from $k$ to $l$ (so $k$ and $l$ are parallel to $g$ and $h$). Then $S_l \cdot S_k$ is equal to the translation $V_{2\cdot a}$ as $S_h \cdot S_g$, so $S_l \cdot S_k = S_h \cdot S_g$.

Conversely, if you have a translation $V_b$, you can substitute it by the product of two reflections $S_h \cdot S_g$. The two axes need to be chosen as follows: They have to be perpendicular to $b$, that means in particular parallel to each other; and the **length of the perpendicular from $g$ to $h$ must be $\frac{b}{2}$**. — If these conditions are met, then the axes can be **anywhere in the plane**; — both reflections, performed in the correct order, result in the given translation.

(iii) Let $g$ and $h$ be **two non-parallel straight lines** with the intersection $Z$ and measure $w$ of the (oriented) angle from $g$ to $h$ ( $-90^\circ < w < 90^\circ$, $w \neq 0^\circ$ ). Let $k$ and $l$ be **two other non-parallel lines**, also with the intersection $Z$ and the same measure $w$ of the (oriented) angle from $k$ to $l$. Then $S_l \cdot S_k$ is equal to the rotation $D_{Z;2\cdot w}$ as $S_h \cdot S_g$, so $S_l \cdot S_k = S_h \cdot S_g$.

Conversely, if you have a rotation $D_{Y,u}$, you can substitute it by the product of two reflections $S_h \cdot S_g$. Besides the case $u=0^\circ$, the two axes need to be chosen as follows: Both axes have to pass through $Y$, and the measure of the angle from $g$ to $h$ has to be $\frac{u}{2}$ (in particular, it needs to have the same algebraic sign as $u$). — If these conditions are met, then the axes can **lie in any direction** through $Y$; — both reflections, performed in the correct order, result in the given rotation.
(vi) A product $S_n \cdot S_{n-1} \cdot ... \cdot S_1$ of $n$ reflections $S_1$, $S_2$, ..., $S_n$ ($n>3$) is equal to the product $T_m \cdot T_{m-1} \cdot ... \cdot T_1$ of $m$ (in generally other) reflections $T_1$, $T_2$, ..., $T_m$ with $m = n - 2$. This decrement by 2 reflections can be continued as long as there are more than 3, and finally every product of $n$ reflections is equal to a product of at most three reflections (for even $n$ of at most two) reflections.

The statement in conclusion (vi) is a special case of the three-reflection theorem we already know; it is now proved again constructively for the case of 4 reflections.

We justify this statement using a configuration of 4 reflections as an example. For the sake of clarity, we do not designate the lines of reflection with lower case letters this time, but number the reflections according to the formulation in the statement above. Therefore, in this case the axes of reflections are designated with the numbers 1-4. $S_1$ denotes the first reflection at axis with the number 1 which is performed in the composition of the four reflections.

Compared to the first image of Fig. I.3.14, $S_1$ and $S_2$ are replaced by $S_1'$ and $S_2'$ in the second image, preserving intersection point A and angular measure $u$, in a way that axis $2'$ passes through intersection point B of axes 3 and 4. In the third image, $S_3$ and $S_4$ are replaced by $S_3'$ and $S_4'$, preserving intersection point B and angular measure, in a way that axes $3'$ and $2'$ are identical.

**Abb. I.3.14: Product of 4 reflections is replaced by a product of 2 reflections.**

Then

$$S_4 \cdot (S_3 \cdot (S_2 \cdot S_1)) = (S_4 \cdot S_3) \cdot (S_2 \cdot S_1) = (S_4 \cdot S_3)(S_2 \cdot S_1) =$$

$$S_2 \cdot ((S_3 \cdot S_2) \cdot S_1') = (S_2 \cdot 1) \cdot S_1' = S_2 \cdot S_1',$$

—and this is a rotation (fourth image) (in special cases a translation). While the measure of the rotation angle is relatively easy to identify, namely as sum $u+v$ of the angle measures $u$ and $v$ of the two rotations $S_4 \cdot S_3$ and $S_2 \cdot S_1$, the determination of the center of rotation is more complicated. Here, one has initially the product of two rotations $D_{A2\cdot u}$ and $D_{B2\cdot v}$ and obtains as a result the rotation $D_{C2 \cdot (u+v)}$. — For our purposes, it is essential to have succeeded in writing a product of 4 reflections as an **equal** product of 2 reflections.
Appendix C

Overview of the RSV “Comprehending mathematical argumentation”

| Time (min:sec) | Operation | Strategy | Aim | Content | Quote from the video |
|---------------|-----------|----------|-----|---------|----------------------|
| 0:09 – 0:24   | Presenting the learning intentions | Learning intentions (Fisher & Frey, 2016) | Making the goal of the RSV transparent to students | Goals of the RSV |
| 0:25 – 1:52   | Presenting the structure of mathematical statement. Highlighting premises and their indicators in green and conclusions and their indicators in red, exemplified by if-then structures | Understand the theorem statement (Weber, 2015) | Providing knowledge about the structure of mathematical statements in order to better understand them | The product of two reflections in non-parallel lines but intersection point Z | “Mathematical statements are implications, that is, logical compositions, between two statements.” |
| 1:52 – 3:15   | Highlighting premises and their indicators in green and conclusions and their indicators in red, exemplified by alternative phraseologies | Understand the theorem statement (Weber, 2015) | Providing knowledge about the structure of mathematical statements in order to better understand them | The product of two reflections in non-parallel lines but intersection point Z | “This if-then structure is not always expressed linguistically in this way. Alternative phraseologies are ‘provided certain conditions hold, then a certain statement is or is valid.’ Sometimes the order of the statements is also reversed. In the sentence about the |
| Time (min:sec) | Operation | Strategy | Aim | Content | Quote from the video | Screenshot |
|---------------|-----------|----------|-----|---------|----------------------|------------|
| **3:16 – 5:25** | Highlighting premises in green, conclusions in red, and associated indicators in yellow by reading a script section sentence by sentence | Understand the theorem statement (Weber, 2015) | Applying provided knowledge about the structure of mathematical statements to the script in order to foster understanding | Conclusions of the theorem of the product of two reflections in parallel lines | “There [in the script] we find the phrase ‘let there be’ as a marker for a premise…. ‘Then’ follows the statement.” | ![Screenshot](image1.png) |
| **5:26 – 7:09** | Exemplifying the mathematical theorem by developing a geometrical construction of the situation based on key statements of the theorem | Using examples to make sense of statements within the proof (Weber, 2015). Exploring the content using multiple representations (Shanahan et al., 2011) | Comprehending the mathematical statement by developing a geometrical model of the situation using examples | Conclusion of the theorem about the product of two reflections in parallel lines | “We need two parallel straight lines g and h with the oriented perpendicular a from g to h as well as two further parallel straight lines k and l also with the oriented perpendicular a from k to l.” | ![Screenshot](image2.png) |
| **7:10 – 8:25** | Rereading | Understand the theorem statement, put a strong emphasis on accuracy | Comprehending mathematical statements while reading | The product of two reflections in parallel lines f and g is a translation | “This statement is completely independent of the position of the two straight lines g and h in the plane…. The | ![Screenshot](image3.png) |
| Time (min:sec) | Operation | Strategy | Aim | Content | Quote from the video | Screenshot |
|---------------|-----------|----------|-----|---------|----------------------|------------|
| 8:26 – 9:07   | Highlighting premises and their indicators in green and conclusions and their indicators in red | Understand the theorem statement (Weber, 2015) | Analysis of the structure of mathematical statements | Justification of the reduction strategy using the example of reflections in four straight lines | “Given a product of $n$ reflections, that product can be replaced by another product of reflections, where the number of reflections is less by two.” | ![Screenshot](image1.png) |
| 9:08 – 12:26  | Asking why with every illustration of the linking reflections and comparing it to a former mathematical theorem | Partitioning the proof into parts (Weber, 2015) | Making connections (Berger, 2019a) | Justification of the reduction strategy using the example of reflections in four straight lines | “To understand the rationale, we need to justify each step by asking ourselves ‘Why is this true?’ … We find the answer above [in the script].” | ![Screenshot](image2.png) |
| 12:27 – 16:02 | Asking oneself why at every equal sign of the chain of equations and comparing it to former illustrations | Making connections (Berger, 2019a) | Justifying and comprehending mathematical statements while reading | Justification of the reduction strategy using the example of reflections in four straight lines | “What have we achieved now through this action? Let’s look at this in the individual steps of the chain of equations…. Overall, we have accounted for every single step through previous content of the script.” | ![Screenshot](image3.png) |