Exceptional circles of radial potentials

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Received 26 October 2012
Published 22 March 2013
Online at stacks.iop.org/IP/29/045004

Abstract

A nonlinear scattering transform is studied for the two-dimensional Schrödinger equation at zero energy with a radial potential. Explicit examples are presented, both theoretically and computationally, of potentials with nontrivial singularities in the scattering transform. The singularities arise from non-uniqueness of the complex geometric optics solutions that define the scattering transform. The values of the complex spectral parameter at which the singularities appear are called \textit{exceptional points}. The singularity formation is closely related to the fact that potentials of conductivity type are ‘critical’ in the sense of Murata.

1. Introduction

We study singularities of the scattering transform at zero energy for two-dimensional Schrödinger operators with radial and compactly supported potentials. We will present what is, to our knowledge, the first example of an infinite-dimensional family of potentials for which the singularities of the scattering transform can be computed explicitly (see note added in proof). The singularities, occurring at so-called \textit{exceptional points} of the potential, arise from non-uniqueness of complex geometric optics (CGO) solutions of the Schrödinger equation. We also present the numerical computations which inspired this work and display the formation of singularities under perturbation of a potential of conductivity type.

To motivate our main results, let us define a class of potentials that plays a central role in our work. Here and in what follows, we set \(z = x + iy\) and denote by \(f(z)\) a smooth function of \(x\) and \(y\) which need not be an analytic function.

\textbf{Definition 1.1.} A compactly supported real-valued potential \(q \in C_0^\infty(\mathbb{R}^2)\) is of conductivity type if \(q = \psi^{-1}(\Delta \psi)\) for some real-valued \(\psi \in C^\infty(\mathbb{R}^2)\) satisfying \(\psi(z) \geq c > 0\) for all \(z\) in a bounded set \(\Omega \subset \mathbb{R}^2\) and \(\psi(z) \equiv 1\) for all \(z \in \mathbb{R}^2 \setminus \Omega\).

Note that the positive function \(\psi\) above solves \((-\Delta + q) \psi = 0\). This terminology arose when Schrödinger scattering theory was used to analyze the inverse conductivity problem in Nachman [16], and was also needed in [12, 17]. In those works \(q\) is not necessarily compactly...
supported, but a condition implying \( \lim_{|z| \to \infty} \psi(z) = 1 \) is crucial. In appendix C, we show that the associated Schrödinger operator has no eigenvalues, and that the function \( \psi \) representing \( q \) is unique.

Recall that the inverse conductivity problem of Calderón [5] consists in reconstructing the conductivity \( \sigma \) of a conducting body \( \Omega \subset \mathbb{R}^2 \) from the Dirichlet to Neumann map, defined as follows. Let \( f \in H^{1/2}(\partial \Omega) \) and let \( u \in H^1(\Omega) \) solve

\[
\nabla \cdot (\sigma \nabla u) = 0 \\
u|_{\partial \Omega} = f.
\]

This solution is unique if \( \sigma \in L^\infty(\Omega) \) is real-valued and strictly positive. The Dirichlet-to-Neumann map is the map \( \Lambda_\sigma : H^{1/2}(\partial \Omega) \to H^{-1/2}(\partial \Omega) \) given by

\[
\Lambda_\sigma f = \sigma \frac{\partial u}{\partial \nu}|_{\partial \Omega}.
\]

Calderón’s problem is to reconstruct the function \( \sigma \) from knowledge of the map \( \Lambda_\sigma \). In dimension two, Calderón’s problem in its original form was solved in [3].

Nachman [16] exploited the fact that \( \psi = \sigma^{1/2} u \) solves the Schrödinger equation \((-\Delta + q) \psi = 0\) where \( q = \sigma^{-1/2} \Delta (\sigma^{1/2}) \). The Schrödinger problem \((-\Delta + q) \psi = 0\) also has a Dirichlet-to-Neumann map \( \Lambda_q \) which can be determined from \( \Lambda_\sigma \). Using \( \Lambda_\sigma \), Nachman was able to construct CGO solutions to the Schrödinger equation, use these solutions to compute the scattering transform of \( q \), and reconstruct \( \sigma \) by inverting the scattering transform. The first numerical computations of scattering transforms were published in [19, 20].

The theory of the scattering transform has been worked out in detail for potentials of conductivity type (see [11] and references therein), but far less is known about general classes. Grinevich and Novikov [7, part I of supplement 1] commented that Schrödinger potentials with well-behaved scattering transforms are “not in general position,” and Nachman proved under minimal hypotheses that the scattering transform is regular if and only if the potential is of conductivity type. Below we show how work of Murata [15] implies that the set of conductivity type potentials is unstable under \( C_0^\infty \) perturbations and is therefore unstable in any reasonable function space!

Let us define the CGO solutions and scattering transform for a potential \( q \in C_0^\infty(\mathbb{R}^2) \). To define the CGO solutions, let \( k \in \mathbb{C} \), set \( z = x + iy \) and write

\[
kz = (k_1 + i k_2) (x + iy)
\]

(complex multiplication). The CGO solutions \( \psi(z, k) \) solve

\[
(-\Delta + q) \psi = 0,
\]

\[
\lim_{|z| \to \infty} e^{-ikz} \psi(z, k) = 1.
\]

These solutions were introduced in scattering theory by Faddeev (see note added in proof). The set of nonzero \( k \in \mathbb{C} \) for which (1.2) does not have a unique solution is called the exceptional set \( \mathcal{E} \). Existence of exceptional points in dimensions three and higher have been discussed by Nachman in [13]. If the exceptional set is empty, the solutions \( \psi(z, k) \) form a smooth family, and the scattering transform of \( q \) is given by

\[
t(k) = \int_{\mathbb{C}} e^{ikz} q(z) \psi(z, k) \, dz.
\]

Here and in what follows, \( dz \) and \( dk \) denote Lebesgue measure on the plane, and the integration is taken over the plane unless otherwise indicated. The behavior of \( t \) at \( k = 0 \) plays a special role which we will elucidate in what follows.

Nachman showed (under rather less stringent regularity assumptions than \( q \in C_0^\infty(\mathbb{R}^2) \)) that \( q \) is of conductivity type if and only if:
(i) \( E \) is empty, and
(ii) \(|t(k)| \leq C|k|^{\epsilon}\) for \(|k|\) small and some \(\epsilon > 0\).

(see [16], theorem 3). Until this time it was not clear how the scattering transform behaved for potentials outside this limited class. Our purpose here is to construct and analyze examples for which the singularities of \(t(k)\) can be computed explicitly. In our examples, \(t(k)\) is well-defined except on a circle in the complex \(k\)-plane, and we can compute the singularity explicitly.

We will study the scattering transform for families of radial potentials \(q_\lambda\) defined as follows. Denote by \(B_1\) the open unit disc in \(\mathbb{R}^2\) centered at 0, so that \(\partial B_1 = S^1\) regarded as an embedded manifold in \(\mathbb{R}^2\). Suppose that \(\sigma \in C^\infty(\mathbb{R}^2)\) is a real-valued radial function satisfying \(\sigma(z) > 0\) for all \(z \in \mathbb{R}^2\) and \(\sigma - 1 \in C^\infty_0(B_1)\). Then \(q_0 := \sigma^{-1/2} \Delta \sigma^{1/2} \in C^\infty_0(B_1)\) is a radial potential of conductivity type. For a nonnegative radial function \(w \in C^\infty_0(B_1)\) not identically zero, set

\[ q_\lambda = q_0 + \lambda w. \tag{1.4} \]

We show in appendix A that, for a radial potential, \(t(k)\) is a real-valued and radial function. Our main result is:

**Theorem 1.2.** Denote by \(t_\lambda\) the scattering transform of \(q_\lambda\). For small \(\lambda \neq 0\),

\[ t(k) = -\frac{2\pi}{\log|k|} + \mathcal{O}(\log|k|)^{-2} \]

as \(k \to 0\). Moreover:

1. For \(\lambda > 0\) sufficiently small, the exceptional set \(E\) is empty, and the scattering transform \(t_\lambda\) is \(C^\infty\) away from \(k = 0\).
2. For \(\lambda < 0\) sufficiently small and a unique \(r(\lambda) > 0\), the exceptional set \(E\) is a circle \(C_\lambda\) of radius \(r(\lambda)\) about the origin, and the function \(t_\lambda\) is \(C^\infty\) on \(\mathbb{R}^2 \setminus \{C_\lambda \cup \{0\}\}\), while

\[ \lim_{|k| \to r(\lambda)} |t_\lambda(k)| = \infty. \]

The radius \(r(\lambda)\) obeys the formula

\[ r(\lambda) \sim \exp \left( -\gamma + \frac{1 + \mathcal{O}(\lambda))}{\mu(\lambda)} \right) \quad \text{as } \lambda \uparrow 0, \tag{1.5} \]

where \(\gamma\) is Euler’s constant, and \(\mu(\lambda)\) is the eigenvalue of the Dirichlet-to-Neumann operator for \(q_\lambda\) corresponding to the constant functions on \(S^1\).

Theorem 1.2 shows that \(\lambda = 0\) is an ‘essential singularity’ for the map \(\lambda \mapsto t_\lambda\). The cases \(\lambda = 0\), \(\lambda < 0\), and \(\lambda > 0\) may be characterized in the following way. We recall from Murata [15] (see also Gesztesy and Zhao [6]) that a Schrödinger operator \(-\Delta + q\) is called

- **subcritical** if \(-\Delta + q\) has a positive Green’s function,
- **critical** if \(-\Delta + q\) does not have a positive Green’s function, but the quadratic form

\[ q(v, v) = \int_C \left( |(\nabla v)(z)|^2 + q(z)|v(z)|^2 \right) \, dz \]

on \(C^\infty_0(\mathbb{R}^2) \times C^\infty_0(\mathbb{R}^2)\) is nonnegative, and
- **supercritical** if the quadratic form \(q\) is not nonnegative.

Appendix B below shows how [15] implies the following. First of all, the conductivity-type potential \(q_0\) is critical. Furthermore, taking \(\lambda < 0\) in (1.4) gives a supercritical potential \(q_\lambda\) which cannot be of conductivity type since there is no positive solution of \((-\Delta + q) \psi = 0\). Finally, taking \(\lambda > 0\) in (1.4) gives a subcritical potential \(q_\lambda\) allowing a unique positive solution \(\psi\) of \((-\Delta + q) \psi = 0\). However, this \(\psi(z)\) grows logarithmically in \(|z|\) and does not satisfy
\[ \lim_{|z| \to \infty} \psi(z) = 1, \] so \( q \) is not of conductivity type in the sense of definition 1.1. Therefore, the class of conductivity-type potentials is not stable under perturbations of \( \lambda \).

The scattering transform \( T : q \to t \) plays an important role not only in Nachman’s solution of the inverse conductivity problem, but also in the solution of the Novikov–Veselov (NV) equation by the method of inverse scattering (see [12] and [17] for details and further references). Thus, a clear understanding of the singularities of \( T \) is important both for Schrödinger inverse scattering and for understanding the dynamics of the NV equation and other completely integrable equations in the NV hierarchy.

It follows from Perry [17] that the NV equation with initial data \( q_0 \) has a global solution. On the other hand, Taimanov and Tsarev [22–25] have used the Moutard transformation to construct initial data \( q \) for the NV so that the solutions blow up in finite time. These potentials have \( L^2 \) eigenvalues at zero energy, and hence are not of conductivity type (see proposition C.1). We conjecture the following dichotomy for the Cauchy problem for the NV equation:

- If the initial data is critical or slightly subcritical, the solution exists globally in time,
- In all other cases, the solution blows up in finite time.

We hope to return to this question in a subsequent paper.

We close this introduction by sketching the proof of theorem 1.2 and summarizing the contents of this paper. To analyze the singularities of the scattering transform \( t \lambda \) for the family (1.4), we recall the reduction of (i) the problem (1.2) and (ii) the map (1.3) respectively to (i) a boundary integral equation of Fredholm type (see (1.9)), and (ii) a boundary integral (see (1.12)). We refer the reader to [16], theorem 5 and its proof in section 7 for a complete discussion.

In order to state the reduction, we first define the Dirichlet-to-Neumann map, denoted \( \Lambda_q \), for the Dirichlet problem

\[ (-\Delta + q) u = 0 \text{ in } B_1 \]
\[ u|_{S^1} = f. \]

If zero is not an eigenvalue of the operator \( -\Delta + q \) with Dirichlet boundary conditions on \( B_1 \), the problem (1.6) has a unique solution \( u \) for given \( f \in H^{1/2}(S^1) \), and we set

\[ \Lambda_q f = \frac{\partial u}{\partial v} \bigg|_{S^1} \]

where \( \partial / \partial v \) denotes differentiation with respect to the outward normal on \( S^1 \). We let \( \langle \cdot, \Lambda_q \cdot \rangle \) denote the corresponding bilinear form:

\[ \langle g, \Lambda_q f \rangle = \int_D (\nabla v \cdot \nabla u + q v u) \, dz \quad \left( = \int_{S^1} v \frac{\partial u}{\partial v} \, dS \right), \]

where \( u \) solves (1.6) and \( v \in H^1(\Omega) \) with \( v|_{S^1} = g \). We will denote by \( \Lambda_0 \) the Dirichlet-to-Neumann operator for (1.6) with \( q = 0 \). It is known that \( \Lambda_q : H^{1/2}(\partial \Omega) \to H^{-1/2}(\partial \Omega) \). If \( \psi \) denotes the restriction of the unique solution of (1.2) to \( S^1 \), then

\[ \psi|_{S^1} = e^{ikz} - T \psi \]

where \( T : H^{1/2}(S^1) \to H^{1/2}(S^1) \) is the compact operator

\[ T \psi = S_k(\Lambda_q - \Lambda_0) \psi. \]

Here, the operator \( S_k \) is an integral operator

\[ (S_k \psi)(z) = \int_{S^1} G_k(z - z') \psi(z') \, dS(z'), \]
$G_k(\cdot)$ is Faddeev’s Green’s function (see (2.4) in what follows), and $dS$ is arc length measure on $S^1$. The formula

$$t(k) = \int_{S^1} e^{ikz} [(\Lambda_q - \Lambda_0)\psi](z, k) \, dS(z)$$

computes the scattering transform in terms of the boundary data $\psi$ that solve (1.9).

To prove theorem 1.2, we will study the family of operators $T_{k,\lambda}$ corresponding to (1.10) with $q$ given by (1.4). We will show that the resolvent $(I + T_{k,\lambda})^{-1}$ has a rank-one singularity if $\lambda < 0$ for $|k| = r(\lambda)$, where the asymptotics of $r(\lambda)$ are given by (1.5). We will then use (1.12) to show that this rank-one singularity leads to a singularity in $t(k)$ on the circle of radius $r(\lambda)$.

The structure of this paper is as follows. In section 2 we recall some basic facts needed to analyze (1.9) and the associated operators. In section 3 we extract the rank-one singularity of the resolvent $(I + T)^{-1}$ and show the existence of a ‘circle of singularities’ for $t$. Finally, in section 4, we present numerical computations of $t$, and compare the results to the analysis in section 3.

### 2. Preliminaries

First, we recall that $H^{1/2}(S^1)$ can be defined in terms of the Fourier basis $\{\varphi_n\}_{n=-\infty}^{\infty}$ of $L^2(S^1)$ given by

$$\varphi_n(\theta) = \frac{e^{in\theta}}{\sqrt{2\pi}} \quad (2.1)$$

in the following way. A function $f \in L^2(S^1)$ belongs to $H^{1/2}(S^1)$ if $f = \sum_n a_n \varphi_n$ and

$$\|f\|_{H^{1/2}(S^1)}^2 = \sum_{n=-\infty}^{\infty} (1 + |n|) |a_n|^2$$

(2.2) is finite. If we denote by $P$ the projection

$$(P\psi)(z) = \frac{1}{2\pi} \int_{S^1} \psi(w) \, dS(w),$$

(2.3)

and let $Q = I - P$, then $P$ and $Q$ are orthogonal projections in $H^{1/2}(S^1)$. We denote by $H^{-1/2}(S^1)$ the topological dual of $H^{1/2}(S^1)$ and by $(\cdot, \cdot)$ the inner product associated to the norm (2.2).

Next, we recall some basic facts about the operators $S_k$ and $\Lambda_q$ that appear in the boundary integral equation (1.9).

First of all, we need some properties of the integral operator $S_k$. A reference for this material is the thesis of Siltanen [19]. For $k \in \mathbb{C}$, the operator $S_k$ in (1.11) is defined in terms of Faddeev’s Green’s function (setting $z = x + iy$)

$$G_k(z) = e^{ikz} g_k(z)$$

(2.4)

where $kz$ is given by (1.1) and

$$g_k(z) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{e^{i\xi \cdot z}}{\xi (\xi + 2k)} \, d\xi_1 d\xi_2.$$  

(2.5)

Here $\xi \cdot z = \xi_1 x + \xi_2 y$. In the denominator of (2.5), $\xi = \xi_1 + i\xi_2$. The function $g_k$ is the convolution kernel for Green’s function for the operator $-\frac{1}{4} (\overline{\partial} (\partial + k))^{-1}$. The factor $e^{ikz}$ normalizes $G_k$ to be a fundamental solution for $\Delta = 4\overline{\partial} \partial$. Thus

$$H_k(z) = G_k(z) - G_0(z)$$

(2.6)

is smooth and harmonic, where

$$G_0(z) = -\frac{1}{2\pi} \log |z|$$

is the normalized fundamental solution for $-\Delta$. The integral operator $S_k$ is a bounded operator from $H^{-1/2}(S^1)$ to $H^{1/2}(S^1)$ (see, for example, lemma 7.1 of Nachman [16]), but we will need
a finer description. The following lemma is a simple consequence of [19], theorem 3.2 and following discussion), and we omit the proof.

**Lemma 2.1.** The formula

\[ H_k(z) = H_1(kz) - \frac{1}{2\pi} \log |k|, \]

holds. Here \( H_1(\cdot) \) is real-valued, smooth, and harmonic, and

\[ H_1(0) = -\frac{\gamma}{2\pi}, \]

where \( \gamma \) is Euler’s constant.

From the lemma, we immediately conclude:

**Lemma 2.2.** The decomposition

\[ S_k = S_0 + \mathcal{H}_k - (\log |k|) \mathcal{P} \]

holds, where

\[ (\mathcal{H}_k \psi)(z) = \int_{S^1} H_1(k(z - z'))\psi(z') \, dS(z'). \]

**Remark 2.3.** By straightforward computation, \( S_0 \mathcal{P} = 0 \) so that

\[ (S_k \mathcal{P} \psi)(z) = [2\pi H_1(kz) - \log |k|] (\mathcal{P} \psi)(z) \]

where we have used the mean value property for harmonic functions, while

\[ S_k Q = S_0 Q + \mathcal{H}_k Q. \]

We will use this decomposition to analyze the singularities of the operator \( T_{k,\lambda} \) as \( \lambda \uparrow 0 \).

Next, we need some simple properties of the Dirichlet-to-Neumann map and the integral operator \( T \) in the presence of radial symmetry. First, if \( q \) is radial, the problem (1.6) can be solved by Fourier analysis on the circle using the basis (2.1). We set, for \( z = re^{i\theta} \),

\[ \psi(z, k) = \sum_{n=-\infty}^{\infty} \psi_n(r) \varphi_n(\theta). \]

To compute the Dirichlet-to-Neumann map, we solve the problem

\[ (-\Delta + q)\psi = 0 \text{ in } B_1 \]

\[ \psi|_{S^1} = \phi_n \]

Writing \( \psi = \psi_n(r)\varphi_n(\theta) \) we have

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi_n}{\partial r} \right) + \left( \frac{n^2}{r^2} + q(r) \right) \psi_n = 0 \]

(2.12)

\[ \psi_n(1) = 1 \]

It follows that

\[ \Lambda_q \psi_n = \mu_n(q) \psi_n \]

(2.13)

where

\[ \mu_n(q) = \psi_n'(1). \]

Thus \( \Lambda_q \) has a complete set of orthonormal eigenfunctions and real eigenvalues, and hence commutes with complex conjugation. Furthermore, it is easy to see that

\[ \mu_{-n} = \mu_n \in \mathbb{R}. \]

(2.14)

Let

\[ \mu(\lambda) := \mu_0(q_\lambda), \]

(2.15)

where \( q_\lambda \) is given by (1.4). The following fact is crucial.

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Lemma 2.4. Suppose that \( \sigma \in C^\infty(\mathbb{R}^2) \) is a real-valued radial function satisfying \( \sigma(z) > 0 \) for all \( z \in \mathbb{R}^2 \) and \( \sigma - 1 \in C_0^\infty(B_1) \). Denote \( q_0 := \sigma^{-1/2} \Delta \sigma^{1/2} \in C_0^\infty(B_1) \). Let \( w \in C_0^\infty(B_1) \) be a nonzero, nonnegative radial function, and define \( q_\lambda \) by (1.4). Then \( \mu(0) = 0 \) and \( \mu'(0) > 0 \).

Proof. The function \( \sigma^{1/2} \) is the unique solution to \((-\Delta + q_0) \phi = 0 \) with \( \phi|_{\Sigma^1} = 1 \), and since \( \sigma^{1/2} \) is constant in a neighborhood of the boundary we have \( \frac{\partial}{\partial n} \sigma^{1/2}|_{\Sigma^1} = 0 \). It follows that \( \mu(0) = 0 \). The fact that \( \mu(\lambda) \) is continuously differentiable in \( \lambda \) follows from the fact that the unique solution to the problem (2.12) with \( q \) given by (1.4) depends analytically on \( \lambda \).

To compute \( \mu'(0) \), let \( \psi_\lambda \) solve the Dirichlet problem

\[
\begin{align*}
(-\Delta + q_\lambda) \psi_\lambda &= 0, \\
\psi_\lambda|_{\Sigma^1} &= 1,
\end{align*}
\]

and let \( \hat{\psi}_\lambda = \partial \psi_\lambda / \partial \lambda \). Since \( 2\pi \mu(\lambda) = (1, \Lambda_{q_\lambda}, 1) \), we calculate \( \mu'(0) \) by taking the derivative of this pairing:

\[
2\pi \mu'(\lambda) = \frac{d}{d\lambda} (1, \Lambda_{q_\lambda}, 1)
= \frac{d}{d\lambda} \int_D (|\nabla \psi_\lambda|^2 + q_\lambda |\psi_\lambda|^2) \, dz
= \int_D 2\nabla \hat{\psi}_\lambda \cdot \nabla \psi_\lambda + 2q_\lambda \hat{\psi}_\lambda \psi_\lambda + w |\psi_\lambda|^2 \, dz
= 2 \langle \psi_\lambda|_{\Sigma^1}, \Lambda_{q_\lambda}, 1 \rangle + \int_D w |\psi_\lambda|^2 \, dz.
\]

We have \( \hat{\psi}_\lambda|_{\Sigma^1} = 0 \), so \( 2\pi \mu'(0) = \int_D w |\psi_0|^2 \, dz > 0 \).

\[\Box\]

3. Singularities of CGO solutions and scattering transform

In this section we compute the singularities of solutions to the boundary integral equation (1.9). Our computation is based on the following decomposition of the resolvent \((I + T_{k,\lambda})^{-1}\) as a bounded operator from \(H^{1/2}(S^1)\) to itself. We note that, since \( q_0 \) is of conductivity type, it follows from theorem 5 in [16] that the operator \((I + T_{k,0})\) has a bounded inverse for all \( k \in C \).

Lemma 3.1. Suppose that \( \sigma \in C^\infty(\mathbb{R}^2) \) is a real-valued radial function satisfying \( \sigma(z) > 0 \) for all \( z \in \mathbb{R}^2 \) and \( \sigma - 1 \in C_0^\infty(B_1) \). Denote \( q_0 := \sigma^{-1/2} \Delta \sigma^{1/2} \in C_0^\infty(B_1) \). Let \( w \in C_0^\infty(B_1) \) be a nonnegative and nonzero function, and define \( q_\lambda \) by (1.4). Suppose that \( T_{k,\lambda} \) is given by (1.10) with \( q = q_\lambda \). Then, for all \( |\lambda| \) sufficiently small, there is a rank-one operator \( F = F(k, \lambda) \) and a bounded operator \( R = R(k, \lambda) \) so that

\[
(I + T_{k,\lambda})^{-1} = (I + T_{k,0} + R)^{-1} (I + \mu F)^{-1}
\]

where \( \mu = \mu(\lambda) \) is defined in (2.15),

\[
\sup_{|k| \leq K_0} \| R \| \leq C(K_0) |\lambda|
\]

for a positive constant \( C \) depending on \( q_0 \) and \( w \), and

\[
F = S_k P (I + T_{k,0} + R)^{-1}.
\]

The operators \( R \) and \( F \) commute with complex conjugation.
Proof. Let

\[ R = S_k (\Lambda_{q_0} - \Lambda_{q_0}) Q. \]

Since \( Q \) commutes with \( \Lambda_{q_0} \) and \( \Lambda_0 \), it follows that

\[ R = (S_k Q) (\Lambda_{q_0} - \Lambda_{q_0}) Q. \]

From remark 2.3, we have

\[ S_k Q = S_k Q + \mathcal{H}_k Q \]

so that \( \sup_{|k| \leq K_0} \| S_k Q \|_{H^{-1/2}(S^1) \rightarrow H^{1/2}(S^1)} \) is finite. We then use the fact that

\[ \| \Lambda_{q_0} - \Lambda_{q_0} \|_{H^{-1/2}(S^1) \rightarrow H^{1/2}(S^1)} \leq C |\lambda| \]

to conclude that (3.2) holds. Write

\[ T_{k, \lambda} = \mu S_k P + S_k Q (\Lambda_{q_0} - \Lambda_{q_0}) + S_k Q (\Lambda_{q_0} - \Lambda_0) \]

and use the fact that \( T_{k, 0} = S_k Q (\Lambda_{q_0} - \Lambda_0) \) (since \( \Lambda_{q_0} P = \Lambda_0 P = 0 \)) to conclude that (3.1) holds. The statement about complex conjugation follows from the fact that the projections \( P \) and \( Q \), the operators \( \Lambda_{q_0} \), and the operator \( S_k \) have the same property.

Remark 3.2. The fact that \( T_{k, 0} = S_k Q (\Lambda_{q_0} - \Lambda_0) \) implies that \( \sup_{|k| \leq K_0} \| T_{k, 0} \| \) is bounded, and hence that \( \sup_{|k| \leq K_0} \| (I + T_{k, 0})^{-1} \| \) is bounded since \( k \mapsto T_{k, 0} \) is a bounded continuous operator-valued function for \( |k| \leq K_0 \) and \( (I + T_{k, 0})^{-1} \) is known to exist for all \( k \) by [16].

The effect of lemma 3.1 is to focus attention on the rank-one operator \( F \). If we write

\[ F \psi = (\varphi, \psi) \chi \]

a short computation shows that

\[ (I + \mu F)^{-1} = I - \frac{(\varphi, \cdot) \chi}{1 + \mu (\varphi, \chi)} \]

while

\[ D(k, \lambda) := \det (I + \mu F) = 1 + \mu (\varphi, \chi). \]

In our case we have, with \( \varphi_0 \) defined in (2.1),

\[ \varphi = (I + T_{k, 0}^* + R^*)^{-1} \varphi_0, \quad (3.3) \]

\[ \chi = [2\pi H_1 (k \cdot) + \log (|k|)] \varphi_0. \quad (3.4) \]

We can prove:

Lemma 3.3. For any \( K_0 > 0 \) and all \( |k| \leq K_0 \) and \( \lambda \) sufficiently small, the function \( D(k, \lambda) \) is real-valued, smooth in \( \lambda \) and \( k \) for \( k \neq 0 \), and radial in \( k \). Moreover, \( D(k, \lambda) \) obeys the small-\( |\lambda| \) asymptotics

\[ D(k, \lambda) = 1 + \mu (\lambda) (2\pi h - \log |k|) + \mathcal{O} (|\lambda| |k|). \quad (3.5) \]

Proof. From (3.3)–(3.4) we compute \( (\varphi, \chi) = F_1 + F_2 \), where

\[ F_1 (k, \lambda) = (2\pi h - \log |k|)(\varphi_0, (I + T_{k, 0}^* + R)^{-1} \varphi_0), \quad (3.6) \]

\[ F_2 (k, \lambda) = ((I + T_{k, 0}^* + R)^{-1} \varphi_0, 2\pi H (k \cdot) \varphi_0). \quad (3.7) \]
In (3.6), \( h = H_1(0) = -\gamma / 2\pi \), and, in (3.7), \( H(z) = H_1(z) - H_1(0) \). Since
\[
(I + T_{k,0} + R)^{-1} \varphi_0 = \varphi_0,
\]
we have
\[
F_1(k, \lambda) = \mu(\lambda)(2\pi h - \log |k|)
\]
which is obviously radial in \( k \). To see that \( F_2(k, \lambda) \) is radial in \( k \), we note that if
\[
C_k = T_{k,0} + R
\]
and \( U_\varphi \) is the unitary operator on \( H^{1/2}(\mathbb{S}^1) \) given by \( (U_\varphi f)(e^{i\theta}) = \mu(\lambda) f(e^{i(\theta + \phi)}) \), we have
\[
U_\varphi C_k = C_{\varphi k} U_\varphi
\]
and
\[
U_\varphi H_k = H_{\varphi k} U_\varphi.
\]
The second identity implies that \( H_1(kz) \varphi_0 = H_{\varphi k} \varphi_0 \) satisfies \( U_{\varphi} (H_1(kz) \varphi_0) = H_k(e^{i\theta}kz \varphi_0) \). Using these identities and the fact that \( U_{\varphi} \varphi_0 = \varphi_0 \) we easily deduce that
\[
F_2(e^{i\theta}k, \lambda) = F_2(k, \lambda)
\]
so that \( F_2 \) is radial in \( k \). Finally,
\[
|F_2(k, \lambda)| \leq C \|(I + T_{k,0} + R^*)^{-1} \sup_{|\alpha| \leq 1} |H(k \cdot)| \leq C |k|,
\]
which gives the required asymptotics since \( |\mu(\lambda)| \leq C |\lambda| \) for small \( |\lambda| \). The reality of \( D(k, \lambda) \) follows from the fact that \( \varphi_0 \) and \( H(k \cdot) \) are real-valued functions and that the operators \( T_{k,0} \) and \( R \) commute with complex conjugation. Smooth dependence on the parameters \( k \) and \( \lambda \) is a consequence of the smooth dependence of \( T_{k,0} \) on these parameters. \( \square \)

**Remark 3.4.** It follows from the lemma that for \( |k| \leq K_0 \) and \( \lambda \) sufficiently close to zero and \( k \neq 0 \),
\[
|\nabla D(k, \lambda)| = \frac{|\mu(\lambda)|}{|k|}(1 + O(|\lambda|))
\]
is nonzero, so that the zero set
\[
Z_\lambda = \{ k \in \mathbb{R}^2 : k \neq 0, D(k, \lambda) = 0 \}
\]
is locally a smooth curve. (A similar remark is given by Tsai in [26, page 779]; however, that result involves a smallness assumption on the potential and does not apply to conductivity-type potentials.) By radial symmetry, \( Z_\lambda \) is actually a circle of radius \( r_\lambda \) depending on \( \lambda \). We easily deduce from (3.5) and lemma 2.4 that
\[
r_\lambda \sim \exp \left( -\gamma + \frac{1 + O(\lambda)}{\mu(0) \lambda} \right).
\]
(3.8)

**Corollary 3.5.** For \( \lambda > 0 \) small, the exceptional set \( \mathcal{E} \) is empty.

**Proof.** We see from (3.8) that for fixed \( K_0 \) and small positive \( \lambda, r_\lambda > K_0 \) and thus there are no singularities for \( |k| < K_0 \). From theorem 2.1 of [21], if \( |k| \geq C\|(1 + |z|^2)^{1/2}q_\lambda\|_{L^\infty} \) then the modified CGO solutions \( e^{-ikz}U(\zeta, k) \), exist, are unique, and are locally integrable. The function \( q_\lambda \) has support in \( |z| < 1 \), therefore choosing \( K_0 = \sup_{0 \leq \lambda \leq 1} C\|q_\lambda\|_{L^\infty} \) and using (1.3), there are no singularities for \( |k| \geq K_0 \).

The purpose of the following lemma is to show that, even when there are no exceptional points of \( t_\lambda(k) \), this does not violate Nachman's result [16, theorem 3] because \( t_\lambda(k) \) does not satisfy the small \( k \) decay requirement.
Lemma 3.6. If \((I + T_{k,\lambda})^{-1}\) exists and is bounded for small \(k \neq 0\) and \(\mu(\lambda) \neq 0\) then
\[
t_k(k) = -\frac{2\pi}{\log |k|} + O((\log |k|)^{-2})
\]

Proof. We expand \(e^{ikz}\) and \(e^{i\bar{k}z}\) for small \(k\) as \(1 + O(|k|)\) and then calculate \((I + T_{k,\lambda})^{-1}\) explicitly.

\[
t_k(k) = \int_{S^1} e^{i\bar{k}z}(\Lambda_{\psi} - \Lambda_0)(I + T_{k,\lambda})^{-1}e^{ikz} dS
\]
\[
= \int_{S^1} (1 + O(k))(\Lambda_{\psi} - \Lambda_0)(I + T_{k,\lambda})^{-1}(1 + O(k)) dS
\]
\[
= \int_{S^1} (\Lambda_{\psi} - \Lambda_0)(I + T_{k,\lambda})^{-1} dS + O(k)
\]
(3.9)

Expanding \((I + T_{k,\lambda})1\) using (2.8) gives us
\[
(I + T_{k,\lambda})1 = [I + (S_0 + \mathcal{H}_k - (\log |k|) P)(\Lambda_{\psi} - \Lambda_0)]1
\]
\[
= 1 + \mu(\lambda)(2\pi h - \log |k|),
\]
and so \((I + T_{k,\lambda})^{-1}1 = 1/[1 + \mu(\lambda)(2\pi h - \log |k|)]\). Applying this to (3.9) gives
\[
t_k(k) = \frac{2\pi\mu(\lambda)}{1 + \mu(\lambda)(2\pi h - \log |k|)} + O(k) = -\frac{2\pi}{\log |k|} + O((\log |k|)^{-2}).
\]

For small \(\lambda \neq 0\), we see the assumptions of the lemma are satisfied. Remark (3.2) gives us that \((I + T_{k,\lambda})^{-1}\) is bounded, and lemma (2.4) gives us that \(\mu(\lambda) \neq 0\). Therefore we have that
\[
|t_k(k)| = \left|\frac{-2\pi}{\log |k|} + O((\log |k|)^{-2})\right| > c |k|^{\epsilon}
\]
for all \(c, \epsilon > 0\) and \(k\) small, which shows that \(t_k(k)\) does not decay fast enough for Nachman’s theorem 3 to apply.

Next, we show:

Lemma 3.7. There is a \(\lambda_0 > 0\) so that, for all \(\lambda\) with \(0 < -\lambda < \lambda_0\) and any \(k_c \in Z\),
\[
\lim_{k \to k_c, k \notin Z_c} |t_k(k)| = \infty.
\]

Proof. From the formula
\[
\psi(z, k) = (I + T_{k,\lambda})^{-1}(e^{ik(z)})
\]
we compute
\[
\psi(z, k) = [(I + T_{k,0} + R)^{-1}(I + \mu F)^{-1}(e^{ik(z)})](z)
\]
\[
= \left[(I + T_{k,0} + R)^{-1}(I - \frac{\mu(\lambda)}{D(k, \lambda)}(\psi, e^{ik(z)})\chi(\phi))\right](z)
\]
\[
= \frac{\mu(\lambda)(\psi, e^{ik(z)})}{D(k, \lambda)}[(I + T_{k,0} + R)^{-1}\chi(z) + R(z, k)]
\]
(3.10)
where \(R(z, k)\) is regular in \(z, k\). If \(\lambda < 0\) and \(k_c \in Z\), the zero set of \(D(k, \lambda)\), we have
\[
\lim_{k \to k_c} \left|\frac{1}{D(k, \lambda)}\right| = \infty.
\]
Formula (1.12) and the fact that \( \mu(\lambda) \) is nonzero for small nonzero \( \lambda \) imply that, to show
\[
\lim_{k \to k_0} |t_\lambda(k)| = \infty,
\]
it suffices to show that
\[
\liminf_{k \to k_0} |\langle \varphi, e^{ikz} \rangle| > 0 \quad (3.11)
\]
and
\[
\liminf_{k \to k_0} \left| \int_{S^1} e^{iz}\{(A_q - A_0)(I + T_{k,0} + R)^{-1} \chi\}dS(z) \right| > 0 \quad (3.12)
\]
where \( \varphi \) and \( \chi \) are given respectively by (3.3) and (3.4). Now
\[
\varphi = (I + T_{k,0}^* + R)^{-1}\varphi_0 = (I + T_{k,0}^*)^{-1}\varphi_0 + O(\lambda).
\]
Writing
\[
(I + T_{k,0}^*)^{-1}\varphi_0 = \varphi_0 - (I + T_{k,0}^*)^{-1}T^*_{k,0}\varphi_0,
\]
using the fact that
\[
T^*_{k,0}\varphi_0 = (A_{q_0} - A_0)QS\varphi_0 = (A_{q_0} - A_0)Q(H(k))
\]
(see (2.8)) satisfies
\[
\|T^*_{k,0}\varphi_0\| \leq C|k|,
\]
and using the fact that \( |k_c| \sim C e^{\epsilon/\lambda} \), we conclude that, for \( \lambda < 0 \) small, \( \varphi = \varphi_0 + O(\lambda) \), that \( e^{ikz} \sim 1 + O(e^{\epsilon/\lambda}) \) (\( \epsilon \) is a positive constant), and hence that (3.11) holds.

To prove (3.12), first note that
\[
\chi = (H(k) + \log|k|)\varphi_0
\]
\[
\sim -\frac{1 + O(\lambda)}{\mu(\lambda)}\varphi_0 + O(e^{\epsilon/\lambda})
\]
since \( H \) is smooth and \( |k| \sim e^{\epsilon/\lambda} \). Next, note that \((I + T_{k,0} + R)^{-1}\varphi_0 = \varphi_0 \) so that finally
\[
\left| \int_{S^1} e^{iz}\{(A_q - A_0)(I + T_{k,0} + R)^{-1} \chi\}dS(z) \right| \geq \int_{S^1} e^{iz}\varphi_0dS(z) + O(e^{\epsilon/\lambda})
\]
which shows that (3.12) holds. This proves the lemma. 

**Proof of theorem 1.2.** We have already shown that, for all sufficiently small negative \( \lambda \), the scattering transform \( t_\lambda \) is singular on a circle of radius \( r_\lambda \) with the asymptotic behavior (3.8), and for small positive \( \lambda \) the set \( \mathcal{E} \) is empty by corollary 3.5. The behavior of \( t_\lambda(k) \) near \( k = 0 \) is given by lemma 3.6. It remains to show that \( t_\lambda \) is smooth elsewhere. This follows from the fact that \( D(k, \lambda) \) is smooth and nonzero away from the singular circle, the formula (3.10), and the explicit formula for \( t_\lambda \).
4. Computational methods

4.1. Evaluating eigenvalues of DN maps

Given a radial potential \( q \), we wish to compute numerically the eigenvalues \( \mu_n(q) \) defined in (2.13). The straightforward approach would be this: use the finite element method (FEM) to solve the Dirichlet problem (1.6) with \( f = \varphi_n \). Then evaluate \( \Lambda q \varphi \) directly from (1.7) by numerical differentiation of the FEM solution. We will actually compute \( \mu_0(q) \) in this way, but for \( \mu_n(q) \) with \( n \neq 0 \) we can avoid the instability of numerical differentiation as explained below.

Consider the Neumann problem

\[
(-\Delta + q)u = 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu} \bigg|_{\partial \Omega} = g, \tag{4.1}
\]

where the mean value of \( g \in H^{-1/2}(\partial \Omega) \) is zero. Define the Neumann-to-Dirichlet map by

\[
R_q g := u|_{\partial \Omega},
\]

where the condition \( \int_{\partial \Omega} u|_{\partial \Omega} \, ds = 0 \) makes the solution unique. The functions \( \varphi_n \) are eigenfunctions for the ND map: \( R_q \varphi_n = \nu_n(q) \varphi_n \).

Taking \( f = \varphi_n \) with \( n \neq 0 \) in (1.6) results in

\[
(-\Delta + q)u_n = 0 \text{ in } \Omega, \quad u_n|_{\partial \Omega} = \varphi_n, \tag{4.2}
\]

and we know that

\[
\frac{\partial u_n}{\partial \nu} \bigg|_{\partial \Omega} = \Lambda q \varphi_n = \mu_n(q) \varphi_n.
\]

Consider the Neumann problem

\[
(-\Delta + q)u_n = 0 \text{ in } \Omega, \quad \frac{\partial u_n}{\partial \nu} \bigg|_{\partial \Omega} = \mu_n(q) \varphi_n. \tag{4.3}
\]

Now since the solution \( u_n \) is the same in (4.2) and (4.3) we see that \( R_q(\mu_n(q) \varphi_n) = \varphi_n \). If \( \mu_n(q) \neq 0 \), by linearity we get the following connection between the eigenvalues of the DN and ND maps:

\[
\nu_n(q) \varphi_n = R_q \varphi_n = \frac{1}{\mu_n(q)} \varphi_n. \tag{4.4}
\]

Equation (4.4) provides us with a stable way to compute \( \mu_n(q) \) using the FEM for the solution of (4.1), since there is no numerical differentiation involved in the evaluation of \( \nu_n(q) \).

4.2. Solution of the boundary integral equation

We explain how to solve equation (1.9) approximately by numerical computation. We follow the method described in [10]. The trick is to write the integral equation approximately as a matrix equation on the truncated Fourier series domain.

Choose \( N > 0 \). We represent a function \( f \in H^s(\partial \Omega) \) approximately by the truncated Fourier series vector

\[
\hat{f} := \begin{bmatrix} \hat{f}(-N) \\ \hat{f}(-N+1) \\ \vdots \\ \hat{f}(0) \\ \vdots \\ \hat{f}(N-1) \\ \hat{f}(N) \end{bmatrix},
\]
where the Fourier coefficients are defined for \(-N \leq n \leq N\) by

\[
\hat{f}(n) := \langle f, \varphi_n \rangle = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(\theta) e^{-i n \theta} d\theta.
\]

By standard Fourier series theory we get for well-behaved \(f\)

\[
f(\theta) \approx \sum_{n=-N}^{N} \hat{f}(n) \varphi_n(\theta).
\]

Our goal is to approximate the operator \(T = S_{L} (\Lambda_q - \Lambda_0)\) using a matrix acting on the truncated Fourier basis.

We know analytically that \(\Lambda_0 \varphi_n = |n| \varphi_n\), so the \((2N + 1) \times (2N + 1)\) matrix representing the operator \(\Lambda_0\) is

\[
\mathbf{I}_0 := \text{diag} \{N, N - 1, \ldots, 2, 1, 0, 1, 2, \ldots, N - 1, N\}.
\]

Likewise, the \(\text{dn}\) map \(\Lambda_q\) can be represented by a diagonal matrix containing the eigenvalues \(\mu_k(q)\) defined in (2.13):

\[
\mathbf{I}_q := \text{diag} \{\mu_N(q), \mu_{N-1}(q), \ldots, \mu_1(q), \mu_0(q), \mu_1(q), \ldots, \mu_{N-1}(q), \mu_N(q)\}.
\]

Note that in (4.6) we made use of the symmetry (2.14).

By lemma 2.2 we can write \(S_L = S_0 + \mathcal{H}_k - (\log |k|) P\). In our case of \(\Omega\) being the unit disk, the standard single layer operator \(S_0\) has the matrix

\[
S_0 = \frac{1}{2} \text{diag} \left[ \frac{1}{N}, \frac{1}{N-1}, \ldots, \frac{1}{2}, 1, 0, \frac{1}{2}, \ldots, \frac{1}{N-1}, 1 \right].
\]

Furthermore, the projection operator \(P\) defined in (2.3) can be represented by

\[
P = \text{diag} \{0, \ldots, 0, 1, 0, \ldots, 0\}.
\]

It remains to find a matrix \(\mathbf{H}_k\) for the operator \(\mathcal{H}_k\). We define the elements of \(\mathbf{H}_k = [H_k(m, n)]\) by

\[
H_k(m, n) := \langle \mathcal{H}_k \varphi_n, \varphi_m \rangle = \frac{1}{2\pi} \int_0^{2\pi} (\mathcal{H}_k e^{i \theta}) e^{-i m \theta} d\theta.
\]

Here \(m \in \{-N, \ldots, N\}\) is the row index and \(n \in \{-N, \ldots, N\}\) is the column index. The function \(\mathcal{H}_k e^{i \theta}\) can be evaluated numerically by applying a quadrature rule to the integral in (2.9). For this we need to be able to compute point values of \(H_k(z)\). By (2.7) and (2.6) and (2.4) we can write

\[
H_k(z) = H_1(k z) - \frac{\log |k|}{2\pi} = e^{iz} g_1(k z) - G_0(k z) - \frac{\log |k|}{2\pi}.
\]

Now the evaluation of \(H_k(z)\) is reduced to computing Faddeev’s fundamental solution \(g_1(z)\), since everything else is explicit in the right hand side of (4.8). Following [4, (3.10)], that can be done simply using formula

\[
g_1(z) = \frac{1}{4\pi} e^{-iz} \text{Re}(\text{Ei}(iz)),
\]

where \(\text{Ei}\) stands for the exponential-integral special function whose implementation is readily available in mathematical software packages. As explained in [19], one can avoid evaluating the functions \(g_1(z)\) and \(G_0(z)\) in (4.8) near the singularity at \(z = 0\) by calculating the harmonic function \(H_k(z)\) first on a circle \(|z| = R\) enclosing the evaluation domain, and then using the classical Poisson kernel to calculate \(H_k(z)\) for \(|z| < R\).
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Figure 1. Test function \( w(z) = w(|z|) \) defined by formula (5.1) with \( R_1 = 0.8 \) and \( R_2 = 0.9 \).

Figure 2. Eigenvalues \( \mu(\lambda) = \mu_0(q_\lambda) \) corresponding to the first example potential. Left: plot with full parameter range \(-35 \leq \lambda \leq 35\). Right: detail of the left plot. Note that \( \mu(0) = 0 \) and \( \mu'(0) > 0 \) as predicted by lemma 2.4. Also, note that Dirichlet eigenvalues of the potential \( q_\lambda \) in the unit disc cause singularities in \( \mu(\lambda) \).

Approximate solution of (1.9) is now given in the frequency domain by

\[
\widehat{\psi}|_{\partial \Omega} = [I + S_i (L_q - L_0)]^{-1}(e^{ik\zeta}|_{\partial \Omega}) = [I + (S_0 + H_k - (\log |k|)P)(L_q - L_0)]^{-1}(e^{ik\zeta}|_{\partial \Omega}),
\]

(4.10)

where we used the decomposition (2.8), and \( e^{ik\zeta}|_{\partial \Omega} \) stands for the Fourier expansion of \( e^{ik\zeta} \), calculated as follows. Write \( z = e^{i\theta} \) and compute as in [9, section 2] to get

\[
e^{ik\zeta} = \sum_{n=-\infty}^{\infty} a_n(k) e^{in\theta}, \quad a_n(k) = \begin{cases} \frac{(ik)^n}{n!}, & n \geq 0, \\ 0, & n < 0. \end{cases}
\]
Figure 3. Scattering transform corresponding to the first example. The horizontal axis is the parameter $\lambda$ in the definition $q_\lambda(z) = \lambda w(z)$ of the potential. The vertical axis is $|k|$. There are curves along which a singular jump 'from $-\infty$ to $+\infty$' appears, indicated by arrows. The $k$ values at those curves are exceptional points. See figure 4 for further illustration of the singularities.

The vector $\hat{\psi}|_{\partial\Omega}$ thus takes the explicit form

$$
\hat{\psi}|_{\partial\Omega} = \sqrt{2\pi} \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1 \\
ik \\
-k^2/2 \\
\vdots \\
(ik)^N/N!
\end{bmatrix},
$$

Recall now the infinite-precision formula

$$
t(k) = \int_{\partial\Omega} e^{ik\zeta}(\Lambda_q - \Lambda_0)\psi(\cdot, k)ds.
$$

Once the Fourier coefficient vector $\hat{\psi}|_{\partial\Omega}$ is solved from (4.10), set

$$
\hat{\varphi} = (\mathbf{L}_q - \mathbf{L}_0)\hat{\psi}|_{\partial\Omega}
$$

and define a function $g : \partial\Omega \rightarrow \mathbb{C}$ using the truncated Fourier series inversion:

$$
g(\theta) = \sum_{n=-N}^{N} \hat{\varphi}(n) \varphi_n(\theta).
$$
Then the scattering transform can be computed approximately using the formula
\[ t(k) \approx \int_0^{2\pi} e^{i\lambda \exp(-i\theta) g(\theta)} d\theta. \] (4.11)

The approximation in (4.11) is most accurate for \( k \) near zero.

Of course, the accuracy in (4.11) can be improved also by increasing \( N \), but there is a limit to that. With reasonable computational resources it is possible to compute \( \mu_n(q) \) accurately enough up to \( |n| \leq N = 16 \), but computation for \( |n| > 16 \) quickly becomes an extremely hard problem. The reason is that the difference \( |\mu_n(q) - |n|| \) becomes exponentially small.
as $|n|$ grows. We remark that one can achieve higher accuracy by computing the difference $|\mu_n(q) - |n||$ directly analogously to the approach in [8], but in this work there was no need for that.

5. Computational results

We will study two examples numerically: the simple case $q_\lambda = \lambda w$ in section 5.2 and a more complicated case $\tilde{q}_\lambda = \tilde{q}_0 + \lambda w$ in section 5.3. Here $\tilde{q}_0$ is a nontrivial conductivity-type potential.

5.1. Definition of a test function $w$

We define a radial $C^2_0$ function $w(z) = w(|z|)$ as follows. Take two radii $0 < R_1 < R_2 < 1$ and define $w(|z|)$ in three pieces:

$$w(|z|) = \begin{cases} 
1 & \text{for } 0 \leq |z| \leq R_1, \\
\rho(|z|) & \text{for } R_1 < |z| < R_2, \\
0 & \text{for } R_2 \leq |z| \leq 1. 
\end{cases}$$

(5.1)

The polynomial $p$ in (5.1) is constructed as follows. Note that the polynomial $\tilde{p}(t) = 1 - 10t^3 + 15t^4 - 6t^5$ is smooth in the interval $0 \leq t \leq 1$ and satisfies $p''(0) = p'(0) = 0 = p'(1) = p''(1)$. Set for $R_1 \leq t \leq R_2$,

$$p(t) = \tilde{p} \left( \frac{t - R_1}{R_2 - R_1} \right).$$

The test function defined above is twice continuously differentiable instead of infinitely smooth as in the theoretical part above. However, the discrepancy is not essential in this numerical work. See figure 1 for a plot of the test function $w$.

5.2. First example: zero potential at $\lambda = 0$

We set $q_\lambda = \lambda w$ with the radial test function defined by (5.1) with $R_1 = 0.8$ and $R_1 = 0.9$. 

![Figure 5](image-url)
Our aim is to compute the radial scattering transform \( t_{\lambda}(|k|) \) for \( 0 < |k| \leq 3.5 \) and for the parameter \( \lambda \) ranging in a suitable interval. In practice we choose the following finite set of \( k \)-values:

\[
k = 0.01, 0.02, 0.03, \ldots, 3.49, 3.50.
\] (5.2)

Note that the \( k \)-grid is bounded away from zero by a significant gap of size \( 10^{-2} \). Furthermore, we consider the following choices of parameter \( \lambda \):

\[
\lambda = -35.00, -34.95, -34.90, \ldots, 34.90, 34.95, 35.00.
\] (5.3)
Scattering transform $t_0(k)$

Profile of scattering transform

Figure 7. Left: mesh plot of the rotationally symmetric scattering transform $t_0(k) = t_0(|k|)$ corresponding to the initial potential $q_0$ shown in figure 6. Right: profile plot of $t_0(|k|)$.

Figure 8. Eigenvalues $\mu(\lambda) = \mu_0(\tilde{q}_\lambda)$ corresponding to the second example potential. Left: plot with full parameter range $-35 \leq \lambda \leq 35$. Right: detail of the left plot. Note that $\mu(0) = 0$ and $\mu'(0) > 0$ as predicted by lemma 2.4. Also, note that Dirichlet eigenvalues of the potential $\tilde{q}_\lambda$ in the unit disc cause singularities in $\mu(\lambda)$. Compare to figure 2.

We start the numerical computations by constructing the matrices (4.7) for each $k$-value listed in (5.2). We take $N = 12$, so each $H_k$ has size $25 \times 25$. These matrices need to be computed only once for a given $k$, so we can reuse the matrices in our second example below.

Next we use the methods described in section 4.1 to compute the eigenvalues of the DN map corresponding to each potential $q_\lambda$ with $\lambda$ ranging as in (5.3). We construct a finite element mesh for the unit disc with 131 585 nodes and 262 144 triangles. We use Matlab’s PDE toolbox to solve the Neumann problem (4.1) with $g = \phi_n$ for $1, \ldots, N$, and get accurate approximations to the eigenvalues $\nu_1, \ldots, \nu_N$ of the ND map. By (2.14) and (4.4) we see that we know all eigenvalues $\mu_n(q_\lambda)$ of the DN maps except for $\mu_0(q_\lambda)$. We use FEM to solve Dirichlet problems of the form (1.6) where $q = q_\lambda$, and $f = 1$; this way we get good approximations to $\mu_0(q_\lambda)$. See figure 2 for plots of the eigenvalue $\mu_0(q_\lambda)$ as function of the parameter $\lambda$. 

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Figure 9. Scattering transform corresponding to the second example. The horizontal axis is the parameter $\lambda$ in the definition $\tilde{q}_\lambda = \tilde{q}_0 + \lambda w$ of the potential. The vertical axis is $|k|$. There are curves along which a singular jump 'from $-\infty$ to $+\infty$' appears, indicated by arrows. The $k$ values at those curves are exceptional points. Compare to figure 3.

Figure 10. Comparison of numerical results and the asymptotic formula (1.5) for the radius of the exceptional circle. This plot is for example 2. Left: detail from figure 9 with parameters ranging in the rectangle $-4 \leq \lambda \leq 0$ and $0.001 \leq |k| \leq 0.6$. Right: the asymptotic function $r(\lambda)$ given by theorem 1.2. For ease of comparison, we also show in the background the pixel image from the left but with a lighter colormap. The asymptotic formula matches the computational result very closely in the interval $-2 \leq \lambda \leq 0$. Compare to figure 5.

Now we have all the ingredients for solving the matrix equation (4.10) and evaluating the scattering transform by (4.11). We used Matlab, a parallelization middleware solution provided
No exceptional points

$R(\lambda)$

$|k|$

$-\lambda_0$ $0$ $\lambda_0$

$\lambda$

Figure 11. Conceptual diagram illustrating the current understanding of exceptional points in dimension two. The underlying grayscale image shows the scattering transform of figure 3 corresponding to example 1. The potential $q_0$ is of conductivity type, and therefore by Nachman [16] there are no exceptional points for $\lambda = 0$. Theorem 1.2 above implies that there is a $\lambda_0 > 0$ such that (i) for $0 < \lambda < \lambda_0$ the potential $q_\lambda$ has no nonzero exceptional points, and (ii) for $-\lambda_0 < \lambda < 0$ the complex numbers satisfying $|k| = r(\lambda)$ are the only nonzero exceptional points for $q_\lambda$. Furthermore, the Neumann series argument in the proof of [16, theorem 1.1] shows that there exists a radius $R(\lambda)$, depending on $\|q_\lambda\|_{L^p(\mathbb{R}^2)}$, such that $q_\lambda$ does not have exceptional points satisfying $|k| > R(\lambda)$. Note that the number $\lambda_0$ and the radius $R(\lambda)$ shown above are not in scale.

5.3. Second example: nontrivial conductivity-type potential at $\lambda = 0$

Here we define $q_\lambda = \tilde{q}_0 + \lambda w$, where the test function $w$ is defined by formula (5.1) and plotted in figure 1. The potential $\tilde{q}_0$ corresponding to $\lambda = 0$ is defined by $\tilde{q}_0 = \sigma^{-1/2} \Delta \sigma^{1/2}$. See figure 6 for plots of the conductivity $\sigma$ and the potential $\tilde{q}_0$. Figure 7 shows the profile of the non-singular scattering transform of $\tilde{q}_0$.

We compute the eigenvalues of the DN map corresponding to each potential $q_\lambda$, similarly than in example 1. See figure 8 for plots of the eigenvalue $\mu(\lambda) = \mu_0(\tilde{q}_0)$ as function of the parameter $\lambda$.

We compute the scattering transform similarly to example 1 above. See figure 9 for a plot of the scattering transform profiles as a grayscale plot. Further, see figure 10 for a comparison of numerical results and the asymptotic formula (1.5) for the radius of the exceptional circle.
6. Discussion

We study zero-energy exceptional points of radial, compactly supported Schrödinger potentials in dimension two. Our work was inspired by preliminary numerical experiments showing the emergence of singularities in the scattering transforms of potentials constructed by subtracting a test function from a radial conductivity-type potential.

We prove new results for radial, real-valued potentials of the form \( q_\lambda = q + \lambda w \), where \( q \) is of conductivity type, the test function \( w \) is non-negative, and \( \lambda \in \mathbb{R} \). It turns out that for small positive \( \lambda \) there are no exceptional points and for small negative \( \lambda \) there is exactly one exceptional circle whose radius we can determine asymptotically. In our two computational examples the asymptotic formula is quite accurate in the rather substantial interval \( -2 \leq \lambda \leq 0 \), see figures 5 and 10. See figure 11 for a diagram illustrating all currently known theoretical facts about two-dimensional exceptional points at zero energy.

Our numerical computations raise some further theoretical questions as well. Figures 3 and 9 suggest that for negative \( \lambda \) far away from zero there are several exceptional circles. Also, for large positive \( \lambda \) some exceptional points appear, but it is unclear where in the \((\lambda, |k|)\) plane they originate. Furthermore, there are some quite complicated features in the parameter interval \( -23 \leq \lambda \leq -18.5 \) that are very different for our two examples. See figure 12 for a zoom-in.

Acknowledgments

The authors thank Fritz Gesztesy for pointing out [6] and for helpful correspondence. Two of us (PAP and SS) thank the Isaac Newton Institute for hospitality during part of the time this work was carried out. The work of SS was supported in part by the Academy of Finland (Finnish Centre of Excellence in Inverse Problems Research 2006–2011 and 2012–2017, decision numbers 213476 and 250215). Funding from Teknologiateollisuus r.y. was used to cover Microsoft Azure cloud computing resources. Also, we thank Techila Ltd for providing technical support and discounts for parallelization middleware solutions used in the
computations. MM and PAP thank the National Science Foundation for support under grant DMS-1208778.

Appendix A. Radial symmetry of scattering transforms

Let $q_\lambda$ be a potential of the form \eqref{1.4}. If $k \in \mathbb{C} \setminus 0$ is not an exceptional point of $q_\lambda$, then uniqueness of the CGO solution $\psi(z, k)$ shows that all $k' \in \mathbb{C}$ with $|k'| = |k|$ are non-exceptional for $q_\lambda$ as well. Furthermore, we can argue as in \cite[section 4.1]{12} and find out that the scattering transform satisfies $t_\lambda(k) = t_\lambda(|k|)$ and $t_\lambda(k) = t_\lambda(k)$.

Also, we know from remark 3.4 that exceptional points appear on a set of measure zero (circles centered at the origin). So, for illustrating $t_\lambda(k)$ it is enough to display real-valued profiles of $t_\lambda(|k|)$ evaluated at $|k| > 0$, as is done in figures 3 and 9.

Appendix B. Conductivity-type potentials and criticality

Let $q(z) = q(|z|)$ be a radial potential of conductivity type in the sense of definition 1.1. Then we can write $q = \psi^{-1}(\Delta \psi)$ with some smooth and strictly positive function $\psi$ for which $\psi^{-1}$ is compactly supported. Set

$$q_\lambda = q + \lambda w,$$

with $w \in C_0^\infty(\mathbb{R}^2)$ a radial, nonnegative test function which is not identically zero.

First of all, \cite[theorem 3.1(iii)]{15} implies that $q_0$ is critical. To see this, note that $g_0(|z|) = \psi(|z|)$ is the unique solution of \cite[equation (3.3)]{15} with $j = 0$ and $n = 2$ satisfying the asymptotic condition $g_0(|z|) = 1 + o(1)$ as $|z| \to \infty$. Then the integral in \cite[formula (3.5)]{15} diverges.

It follows from \cite[theorem 2.4(i)]{15} that the potential $q_\lambda$ is supercritical for all $\lambda < 0$. In that case we know from \cite{1, 14, 2} that there is no positive solution of $(-\Delta + q_\lambda) \psi = 0$, so $q_\lambda$ is not of conductivity type.

It follows from \cite[theorem 2.5(i)]{15} that the potential $q_\lambda$ is subcritical for all $\lambda > 0$. Furthermore, by \cite[theorem 5.6(i)]{15} the unique positive solution of $(-\Delta + q_\lambda) \psi = 0$ has asymptotics $c \log |z| + O(1)$ with a positive constant $c$ as $|z| \to \infty$. Thus $q_\lambda$ is not of conductivity type because that would require a finite and constant limit $\lim_{|z| \to \infty} \psi(z)$.

Appendix C. Spectral properties of conductivity-type potentials

The purpose of this appendix is to analyze the spectrum of the self-adjoint operator $H = -\Delta + q$ when $q$ is a potential of conductivity type.

Proposition C.1. Suppose that $q_0 \in C_0^\infty(\mathbb{R}^2)$ is a potential of conductivity type, and let $H = -\Delta + q_0$. Then $H$ has no $L^2$-eigenvalues.

Proof. By standard arguments (see, for example, theorem XIII.56 in \cite{18}), the equation $H \psi = \lambda \psi$ has no $L^2$ solutions for any $\lambda > 0$, while the positivity of the quadratic form associated to $H$ shows that there can be no eigenvalues with $\lambda < 0$. It remains to show that, also, there are no solutions of $H \psi = 0$ that vanish at infinity, hence no $L^2$-eigenvalues at zero energy. This is an immediate consequence of arguments in Nachman \cite{16} but we reproduce them for the reader’s convenience.
Suppose that $h \in H^1(\mathbb{R}^2)$ is a weak solution of $Lh = 0$. Without loss we may assume that $h$ is real-valued. By elliptic regularity, $h$ is a bounded, $C^\infty$ function. Let

$$v = h\bar{\psi}_0 - \psi_0 \partial h.$$ 

Note that, as $\psi_0 \in L^\infty$, $\partial \psi_0 \in C^\infty_0$, and $h \in H^1$, it follows that $v \in L^2$. A straightforward computation using the facts that $4\partial \bar{\psi}_0 = q\psi_0$ and $4\partial \bar{h} = qh$ shows $\nabla v = av - \bar{v}a$ where $a = \bar{\psi}_0/\psi_0$. Note that $a \in C^\infty_0$. By a standard vanishing theorem for generalized analytic functions (see for example [27]), we conclude that $v = 0$, and hence that $\bar{\psi}_0/\psi_0 = 0$, i.e., $h/\psi_0$ is antiholomorphic. Since $h$ vanishes at infinity while $\psi_0$ is bounded below, we conclude that $h = c\psi_0$ for a constant $c$, and hence, $h = 0$ since $\psi_0(z) \to 1$ as $|z| \to \infty$.

**Remark C.2.** The argument above shows that a conductivity-type potential is represented by a unique normalized positive solution $\psi_0$.

Next, we consider spectral properties of $-\Delta + q$ on a bounded domain containing the support of $q$. Recall that $H^1(\Omega)$ is the completion of $C^\infty_0(\Omega)$ in the $H^1$-norm.

**Proposition C.3.** Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with smooth boundary, so chosen that $\text{supp} q_0$ is strictly contained in $\Omega$. Denote by $H_\Omega$ the operator $-\Delta + q_0$ with Dirichlet conditions on $\partial \Omega$. Then $0$ is not a Dirichlet eigenvalue of $H_\Omega$.

**Proof.** Suppose that $\psi \in H^1_0(\Omega)$ satisfies $H_\Omega \psi = 0$. Extend $\psi$ to a function $\chi$ in $H^1(\mathbb{R}^2)$ by setting $\chi(z) = 0$ for $z \in \mathbb{R}^2 \setminus \Omega$. The distribution gradient of $\chi$ is given by $(\nabla \chi)(z) = (\nabla \psi)(z)$ for $z \in \Omega$, and $(\nabla \chi)(z) = 0$ otherwise. Letting $q$ be the quadratic form of $-\Delta + q_0$ on $H^1(\mathbb{R}^2)$, we have

$$q(\chi, \chi) = 0$$

so that, for any $\psi \in C^\infty_0(\mathbb{R}^2)$, the function

$$F(t) = q(\chi + t\psi, \chi + t\psi)$$

has an absolute minimum at $t = 0$. Since $F'(0) = 0$ we recover $\text{Re}q(\psi, \psi) = 0$ for any such $\psi$. It follows that $q(\psi, \psi) = 0$ for all $\psi \in C^\infty_0(\mathbb{R}^2)$, hence $\chi$ is a weak solution of $H \chi = 0$ which vanishes identically outside $\Omega$. We can now use unique continuation arguments (see, for example, [18], theorem XIII.57) to show that $\chi = 0$, hence $\psi = 0$.

**Note added in proof.** R. G. Novikov kindly pointed out his interesting paper with Grinevich (see [28]) which predates our work and constructs families of complex geometric optics solutions to the Schrödinger equation at fixed energy $E$ whose scattering transforms exhibit singular contours. In their case, the potential is a delta-function potential, and the explicit formulas for the complex geometric optics solutions and scattering transform, specialized to energy $E = 0$, show similar singularities. These potentials may be regarded as a perturbation of the zero potential, which is of conductivity type. The examples of Grinevich and Novikov complement and illuminate the phenomena described here. Professor Novikov also pointed out the paper of Faddeev [29] which introduced exponentially growing solutions (complex geometric optics solutions) into Schrödinger scattering theory.

**References**

[1] Allegretto W 1974 On the equivalence of two types of oscillation for elliptic operators Pac. J. Math. 55 319–28
[2] Allegretto W 1981 Positive solutions of elliptic operators in unbounded domains J. Math. Anal. Appl. 84 372–80
[3] Astala K and Päivärinta L 2006 Calderon’s inverse conductivity problem in the plane Ann. Math. 163 265–99
[4] Boiti M, Leon J P, Manna M and Pempinelli F 1987 On a spectral transform of a KdV-like equation related to the Schrödinger operator in the plane Inverse Problems 3 25–36
[5] Calderón A 1980 On an inverse boundary value problem Seminar on Numerical Analysis and Its Applications to Continuum Physics, Coleções Anais Soc. Brasil. Mat. 12 65–73
[6] Gesztesy F and Zhao Z 1995 On positive solutions of critical Schrödinger operators in two dimensions J. Funct. Anal. \textbf{127} 235–56

[7] Grinevich P G and Novikov S P 1988 A two-dimensional inverse scattering problem for negative energies, and generalized analytic functions: I. Energies lower than the ground state Funktsional. Anal. i Prilozhen. \textbf{22} 23–33

Grinevich P G and Novikov S P 1988 A two-dimensional inverse scattering problem for negative energies, and generalized analytic functions: I. Energies lower than the ground state Funktsional. Anal. i Prilozhen. \textbf{22} 96

Grinevich P G and Novikov S P 1988 Funkt. Anal. Appl. \textbf{22} 19–27 (Engl. transl.)

[8] Hanke M, Hartmann L, Hyvönen N and Schweickert E 2012 Convex source support in three dimensions BIT Numer. Math. \textbf{52} 45–63

[9] Isaacson D and Cheney M 1991 Effects of measurement precision and finite numbers of electrodes on linear impedance imaging algorithms SIAM J. Appl. Math. \textbf{15} 1705–31

[10] Knudsen K, Lassas M, Mueller J L and Siltanen S 2009 Regularized D-bar method for the inverse conductivity problem Inverse Problems Imaging \textbf{3} 599–624

[11] Lassas M, Mueller J L and Siltanen S 2007 Mapping properties of the nonlinear Fourier transform in dimension two Commun. Partial Differ. Equas \textbf{32} 591–610

[12] Lassas M, Mueller J L, Siltanen S and Stahel A 2012 The Novikov–Veselov equation and the inverse scattering method, part I: analysis Physica D \textbf{241} 1322–35

[13] Lavine R B and Nachman A I 1987 On the inverse scattering transform for the n-dimensional Schrödinger operator Topics in Soliton Theory and Exactly Solvable Nonlinear Equations (Singapore: World Scientific) pp 33–44

[14] Moss W F and John P 1978 Positive solutions of elliptic equations Pac. J. Math. \textbf{75} 219–26

[15] Minoru M 1986 Structure of positive solutions to \((-\Delta + V)\psi = 0\) in \(\mathbb{R}^n\) Duke Math. J. \textbf{53} 869–943

[16] Nachman A I 1996 Global uniqueness for a two-dimensional inverse boundary value problem Ann. Math. \textbf{143} 71–96

[17] Perry P A 2012 Miura maps and inverse scattering for the Novikov–Veselov equation arXiv:1201.2385v1

[18] Reed M and Simon B 1978 \textit{Methods of Modern Mathematical Physics. IV. Analysis of Operators} (New York: Academic)

[19] Siltanen S Electrical impedance tomography and Faddeev’s Green functions Ann. Acad. Sci. Fenn. Math. Dissertations \textbf{121} (Available in postscript form at \url{www.siltanen-research.net/publications.html})

[20] Siltanen S, Mueller J and Isaacson D 2000 An implementation of the reconstruction algorithm of A Nachman for the 2D inverse conductivity problem Inverse Problems Imaging \textbf{16} 681–99

Siltanen S, Mueller J and Isaacson D 2001 Inverse Problems \textbf{17} 1561–3 (erratum)

[21] Sylvester J and Uhlmann G 1986 A uniqueness theorem for an inverse boundary problem in electrical prospection Commun. Pure Appl. Math \textbf{39} 92–112

[22] Taimanov I A and Tsarev S P 2007 Two-dimensional Schrödinger operators with rapidly decaying rational potential and multidimensional \(L^2\)-kernel Uspekhi Mat. Nauk \textbf{62} 217–8 (in Russian)

Taimanov I A and Tsarev S P 2007 Russ. Math. Surv. \textbf{62} 631–3 (Engl. transl.)

[23] Taimanov I A and Tsarev S P 2008 Blowing up solutions of the Veselov–Novikov equation Dokl. Akad. Nauk \textbf{420} 744–5 (in Russian)

Taimanov I A and Tsarev S P 2008 Dokl. Math. \textbf{77} 467–8 (Engl. transl.)

[24] Taimanov I A and Tsarev S P 2008 Two-dimensional rational solitons constructed by means of the Moutard transformations, and their decay Teor. Mat. Fiz. \textbf{157} 188–207 (in Russian)

Taimanov I A and Tsarev S P 2008 Theor. Math. Phys. \textbf{157} 1525–41 (Engl. transl.)

[25] Taimanov I A and Tsarev S P 2010 On the Moutard transformation and its applications to spectral theory and soliton equations Sovrem. Mat. Fundam. Napravl. \textbf{35} 101–17 (in Russian)

Taimanov I A and Tsarev S P 2010 Dokl. Akad. Nauk SSSR \textbf{165} 514–7 (in Russian)

[26] Tsai T 1993 The Schrödinger operator in the plane Inverse Problems \textbf{9} 763–87

[27] Vekua I N 1962 Generalized Analytic Functions (Reading, MA: Addison-Wesley)

[28] Grinevich P G and Novikov R G 2012 Faddeev eigenfunctions for point potentials in two dimensions Phys. Lett. A \textbf{376} 1102–6

[29] Faddeev L D 1965 Growing solutions of the Schrödinger equation Dokl. Akad. Nauk SSSR \textbf{165} 514–7 (in Russian)

Faddeev L D 1965 Sov. Phys. Dokl. \textbf{10} 1033–5 (Engl. translation)