CONDITIONAL STABILITY OF MULTI-SOLITONS FOR THE 1D NLKG EQUATION WITH DOUBLE POWER NONLINEARITY

XU YUAN

ABSTRACT. We consider the one-dimensional nonlinear Klein-Gordon equation with double power focusing-defocusing nonlinearity
\[ \partial_t^2 u - \partial_x^2 u + u - |u|^{p-1} u + |u|^{q-1} u = 0, \quad \text{on } \mathbb{R} \times \mathbb{R}, \]
with \(1 < q < p < \infty\). The main result concerns the stability of the sum of several solitary waves with different speeds in the energy space \(H^1(\mathbb{R}) \times L^2(\mathbb{R})\), up to the natural instabilities. The proof involves techniques developed by Martel, Merle and Tsai [12, 13] for the generalized KdV and NLS equations. In particular, we rely on an energy method and virial type estimates.

1. Introduction

1.1. Main result. We consider the one-dimensional nonlinear Klein-Gordon equation with double power nonlinearity
\[ \begin{cases}
\partial_t^2 u - \partial_x^2 u + u - |u|^{p-1} u + |u|^{q-1} u = 0, & (t, x) \in [0, \infty) \times \mathbb{R}, \\
u_{t=0} = u_0 \in H^1, & \partial_t u_{t=0} = u_1 \in L^2, 
\end{cases} \]
(1.1)
where \(1 < q < p < \infty\). This equation also rewrites as a first order system in time for the function \(\vec{u} = (u_1, u_2)\),
\[ \begin{cases}
\partial_t u_1 = u_2, \\
\partial_t u_2 = \partial_x^2 u_1 - u_1 + f(u_1),
\end{cases} \]
(1.2)
where \(f(u_1) = |u_1|^{p-1} u_1 - |u_1|^{q-1} u_1\). Recall that the Cauchy problem for equation (1.2) is locally well-posed in the energy space \(H^1 \times L^2\). See e.g. [7]. Denote \(F(u_1) = \frac{1}{p+1} |u_1|^{p+1} - \frac{1}{q+1} |u_1|^{q+1}\). For any \(H^1 \times L^2\) solution \(\vec{u} = (u_1, u_2)\) of (1.2), the energy \(E(\vec{u})\) and momentum \(I(\vec{u})\) are conserved, where
\[ E(\vec{u}) = \int_{\mathbb{R}} \left\{ (\partial_x u_1)^2 + \phi_1^2 + \phi_2^2 - 2F(u_1) \right\} \, dx, \quad I(\vec{u}) = 2 \int_{\mathbb{R}} (\partial_x u_1) u_2 \, dx. \]
Denote by \(Q\) the ground state, which is the unique positive even solution of the equation
\[ Q'' - Q + f(Q) = 0 \quad \text{on } \mathbb{R}. \]
The existence and properties of this solution are studied in [7, Section 6] (see also Remark 1.4). It is well-known that \(Q, Q', Q''\) have exponential decay at infinity: there exists \(\theta > 0\) such that
\[ Q(x) + |Q'(x)| + |Q''(x)| \lesssim e^{-\theta|x|}. \]
(1.3)
The ground state generates the stationary solution \(\vec{Q} = (Q, 0)\) of (1.2). Using the Lorentz transformation on \(Q\), one obtains traveling solitary waves or solitons: for \(\ell \in \mathbb{R}\), with \(-1 < \ell < 1\), let
\[ Q_\ell(x) = Q \left( \frac{x}{\sqrt{1 - \ell^2}} \right), \quad \vec{Q}_\ell = \begin{pmatrix} Q_\ell \\ -\ell \partial_x Q_\ell \end{pmatrix} \]
then \(\vec{u}(t, x) = \vec{Q}_\ell(x - \ell t)\) is a solution of (1.2).
It is well-known that the operator
\[ L = -\partial_y^2 + 1 - f'(Q) \]
appearing after linearization of equation (1.2) around \( \bar{Q} = (Q,0) \), has a unique negative eigenvalue \( -\nu_0^2 (\nu_0 > 0) \), with corresponding smooth even eigenfunction \( Y \). Set
\[ \bar{Y}^+ = \begin{pmatrix} Y \\ \nu_0 Y \end{pmatrix} \quad \text{and} \quad \bar{Z}^+ = \begin{pmatrix} \nu_0 Y \\ Y \end{pmatrix}. \]
From explicit computations, the function \( \bar{u}^+(t,x) = \exp(\nu_0 t)\bar{Y}^+(x) \) is solution of the linearized system
\[ \begin{cases} \partial_t u_1 = u_2 \\ \partial_t u_2 = -Lu_1. \end{cases} \]
Since \( \nu_0 > 0 \), the solution \( \bar{u}^+ \) illustrates the (one-dimensional) exponential instability of the solitary wave \( \bar{Q} \) in positive time. An equivalent formulation of instability is obtained by observing that for any solution \( \bar{u} \) of (1.2), it holds
\[ \frac{d}{dt} a^+ = \nu_0 a^+ \quad \text{where} \quad a^+(t) = \left( \bar{u}(t), \bar{Z}^+_t \right)_{L^2}. \]
More generally, for \( -1 < \ell < 1 \), set
\[ Y_\ell = Y \left( \frac{x}{\sqrt{1 - \ell^2}} \right) \quad \text{and} \quad Z^+_\ell = \left( (\ell \partial_x Y_\ell + \frac{\nu_0}{\sqrt{1 - \ell^2}} Y_\ell) e^{-\nu_0 \ell x}, \frac{\nu_0}{\sqrt{1 - \ell^2}} Y_\ell e^{-\nu_0 \ell x} \right). \]
The main purpose of this article is to study the conditional stability of multi-solitons with different speeds for (1.1). More precisely, the main result is the following.

**Theorem 1.1.** Let \( N \geq 2 \). For all \( n \in \{1, \cdots, N\} \), let \( \sigma_n = \pm 1, -1 < \ell_n < 1 \) with \( -1 < \ell_1 < \ell_2 < \cdots < \ell_N < 1 \). There exist \( L_0 > 0, C_0 > 0, \gamma_0 > 0 \) and \( \delta_0 > 0 \) such that the following is true. Let \( \bar{\varepsilon} \in H^1 \times L^2 \) and \( y_1^0 < \cdots < y_N^0 \) be such that there exist \( L > L_0 \) and \( 0 < \delta < \delta_0 \) with
\[ \| \bar{\varepsilon} \|_{H^1 \times L^2} < \delta \quad \text{and} \quad y_{n+1}^0 - y_n^0 > L \quad \text{for all} \quad n = 1, \cdots, N - 1. \]
Then, there exist \( h^+ = (h_n^+)_{n \in \{1, \cdots, N\}} \) satisfying
\[ \sum_{n=1}^N |h_n^+| \leq C_0 (\delta + e^{-\gamma_0 L}), \]
such that the solution \( \bar{u} = (u_1, u_2) \) of (1.2) with initial data
\[ \bar{u}_0 = \sum_{n=1}^N (\sigma_n \bar{Q}_{\ell_n} + h_n^+ \bar{Z}_{\ell_n}^+) (\cdot - y_n^0) + \bar{\varepsilon} \]
is globally defined in \( H^1 \times L^2 \) for \( t \geq 0 \) and, for all \( t \geq 0 \),
\[ \left\| \bar{u}(t) - \sum_{n=1}^N \sigma_n \bar{Q}_{\ell_n} (\cdot - y_n(t)) \right\|_{H^1 \times L^2} \leq C_0 (\delta + e^{-\gamma_0 L}), \]
where \( y_1(t), \cdots, y_N(t) \) are \( C^1 \) functions satisfying, for all \( n = 1, \cdots, N, t \geq 0 \),
\[ |y_n(0) - y_n^0| \leq C_0 (\delta + e^{-\gamma_0 L}), \quad |y_n(t) - t_n| \leq C_0^2 (\delta + e^{-\gamma_0 L}). \]

**Remark 1.2.** As remarked before, each soliton \( \bar{Q}_{\ell_n} \) has exactly one exponential instability direction, which for a given perturbation \( \bar{\varepsilon} \), requires the choice of the \( N \) parameters \( (\ell_n^+)_{n \in \{1, \cdots, N\}} \) to control it.
Historically, multi-solitons were studied extensively for integrable equations, mainly for the Korteweg-de Vries equation and cubic nonlinear Schrödinger equation in dimension one. In the nonintegrable cases, for dispersive and wave equations, the first result concerning stability and asymptotic stability of multi-soliton solutions was given by Perelman [15], following Buslaev and Perelman [2] (single soliton case) for the nonlinear Schrödinger equation (NLS). We refer to [12, 13] for results on the stability and asymptotic stability of multi-solitons solutions (or sums of several solitons) for the generalized Korteweg-de Vries equation (gKdV) and (NLS) equation, that inspired the present work. See also [8, 14] for the derivative (NLS) equation.

Such stability results are closely related to the existence of asymptotic pure multi-solitons for non-integrable dispersive and wave equations which have been established in several previous works, for both stable and unstable solitons, see [3, 5, 9, 10, 11, 16] for (gKdV), (NLS), and the energy critical wave equation. For the nonlinear Klein-Gordon equation (1.2), the existence of asymptotic multi-solitons was established by Côte and Muñoz [6].

Our original motivation for studying multi-solitons problems for (1.1) was to provide the first statement of (conditional) stability of sums of solitons for a wave-type equation. Observe that the Lorentz transform, used to propagate solitons with different speeds has rather different properties than the Galilean transform for (NLS) or the natural propagation phenomenon related to the solitons of (gKdV). It was an interesting challenge to extend the methods of [12, 13] to this case. Second, the double power nonlinearity appears as a typical nonlinearity in dispersive equations, especially for the (NLS) equation.

Remark 1.3. The double power focusing-defocusing nonlinearity such as in (1.1) makes the nonlinearity defocusing for small value of $u$, which is important in our proof. In the next remark, we give explicitly more general conditions on the nonlinearity. This issue is already present in [13] for (NLS) though the condition imposed on the nonlinearity is weaker.

Remark 1.4. Let $f$ be a real-valued $C^{1,\alpha}$ function and $F$ be the standard integral

$$F(s) = \int_{0}^{s} f(\sigma)d\sigma \quad \text{for } s \in \mathbb{R}.$$  

Theorem 1.1 can be extended to any nonlinearity $f$ satisfying

(i) $f$ is odd, and $f(0) = f'(0) = 0$.

(ii) There exists a smallest $s_0 > 0$ such that $F(s_0) - \frac{1}{2}s_0^2 = 0$, and $f(s_0) - s_0 > 0$.

(iii) There exists $r_0 > 0$ such that for all $s \in (-r_0, r_0)$, $sf(s) - 2F(s) \leq 0$.

Conditions (i) and (ii) are related to the existence and uniqueness condition (6.2) in [1]. Condition (iii) ensures that the nonlinearity is defocusing near 0. See the previous remark.

This paper is organized as follows. Section 2 introduces technical tools involved in a dynamical approach to the $N$-soliton problem for (1.1): estimates of the nonlinear interactions between solitons, decomposition by modulation and parameter estimates. Energy estimates and monotonicity properties are proved in Section 3. Finally, Theorem 1.1 is proved in Section 4.

1.2. Notation. We denote $(\cdot, \cdot)_{L^2}$ the $L^2$ scalar product for real-valued functions $u, v \in L^2$,

$$(u, v)_{L^2} := \int_{\mathbb{R}} u(x)v(x)dx.$$
Lemma 2.1. \( L^Q \) is self-adjoint, its continuous spectrum is \([1, +\infty)\), its kernel is spanned by \( Q' \) and it has a unique negative eigenvalue \(-\nu_0^2\) (\(\nu_0 > 0\)), with corresponding smooth radial eigenfunction \( Y \). Moreover, on \( \mathbb{R} \),
\[
|Y^{(\alpha)}(x)| \lesssim e^{-\sqrt{1+\nu_0^2}|x|} \quad \text{for any} \ \alpha \in \mathbb{N}.
\]

(ii) Coercivity property of \( L \). There exists \( \nu > 0 \) such that, for all \( v \in H^1 \),
\[
(Lv, v)_{L^2} \geq \nu \|v\|^2_{H^1} - \nu^{-1}(\langle v, Q' \rangle^2_{L^2} + \langle v, Y \rangle^2_{L^2}).
\]

(iii) Coercivity property of \( L_\ell \). There exists \( \nu > 0 \) such that, for all \( v \in H^1 \),
\[
(L_\ell v, v)_{L^2} \geq \nu \|v\|^2_{H^1} - \nu^{-1}(\langle v, \partial_x Q_\ell \rangle^2_{L^2} + \langle v, Y_\ell \rangle^2_{L^2}).
\]

Second, we define
\[
\mathcal{H}_\ell = \begin{pmatrix}
-\partial_x^2 + 1 - f'(Q_\ell) & -\ell \partial_x \\
-\ell \partial_x & 1
\end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]
and
\[
\tilde{Z}_\ell^0 = \begin{pmatrix} \partial_x Q_\ell \\ -\ell \partial_x^2 Q_\ell \end{pmatrix}, \quad \tilde{Z}_\ell^\pm = \begin{pmatrix} (\partial_x Y_\ell \pm \frac{\nu_0}{\sqrt{1-\ell^2}} Y_\ell e^{\pm \nu_0 x} \sqrt{1-\ell^2}) \\ Y_\ell e^{\pm \nu_0 x} \sqrt{1-\ell^2} \end{pmatrix}.
\]

We recall the following technical facts.

**Lemma 2.2 (\cite{6}).** (i) Properties of \( \mathcal{H}_\ell \) and \( \mathcal{H}_\ell J \). It holds
\[
\mathcal{H}_\ell \tilde{Z}_\ell^0 = 0, \quad (\tilde{Z}_\ell^0, \tilde{Z}_\ell^\pm)_{L^2} = 0 \quad \text{and} \quad \mathcal{H}_\ell J(\tilde{Z}_\ell^\pm) = \mp \nu_0 (1-\ell^2)^{1/2} \tilde{Z}_\ell^\pm. \tag{2.1}
\]

(ii) Coercivity property of \( \mathcal{H}_\ell \). There exists \( \nu > 0 \) such that, for all \( \tilde{v} = (v, z) \in H^1 \times L^2 \),
\[
(\mathcal{H}_\ell \tilde{v}, \tilde{v})_{L^2} \geq \nu \|\tilde{v}\|^2_{H^1} - \nu^{-1}(\langle \tilde{v}, \tilde{Z}_\ell^0 \rangle^2_{L^2} + \langle \tilde{v}, \tilde{Z}_\ell^+ \rangle^2_{L^2} + \langle \tilde{v}, \tilde{Z}_\ell^- \rangle^2_{L^2}). \tag{2.2}
\]

**Proof.** See the proof of Lemma 2 and Proposition 2 in [6]. \( \square \)
2.2. Decomposition of the solution around \( N \) solitary waves. We recall general results on solutions of (1.2) that are close to the sum of \( N \geq 2 \) decoupled solitary waves. For any \( n \in \{1, \cdots, N\} \), let \( \sigma_n = \pm 1 \) and \( t \mapsto y_n(t) \in \mathbb{R} \) be \( C^1 \)-functions such that

\[
y_{n+1} - y_n \gtrsim 1 \quad \text{for any } n = 1, \cdots, N - 1.
\] (2.3)

For \( n \in \{1, \cdots, N\} \), define

\[
Q_n = \sigma_n Q_{\ell_n} (\cdot - y_n), \quad \tilde{Q}_n = \left( \begin{array}{c} Q_n \\ -\ell_n \partial_x Q_n \end{array} \right).
\]

Similarly,

\[
\tilde{Z}_n = \sigma_n \tilde{Z}_{\ell_n} (\cdot - y_n), \quad \tilde{Z}_n^\pm = \sigma_n \tilde{Z}_{\ell_n}^\pm (\cdot - y_n).
\]

We recall a decomposition result for solutions of (1.2).

**Lemma 2.3.** There exist \( L_0 > 0 \) and \( 0 < \delta_0 \ll 1 \) such that if \( \bar{u} = (u_1, u_2) \) is a solution of (1.2) on \([0, T_0]\), where \( T_0 > 0 \), such that for all \( t \in [0, T_0] \)

\[
\inf_{z_n + 1 - z_n > L_0} \| \bar{u}(t) - \sum_{n=1}^{N} \sigma_n \tilde{Q}_{\ell_n} (\cdot - z_n)\|_{L^2} < \delta_0.
\] (2.4)

then there exist \( C^1 \)-functions \( y = (y_n)_{n \in \{1, \cdots, N\}} \) on \([0, T_0]\) such that, \( \bar{\varphi} \) being defined by

\[
\bar{\varphi} = \left( \begin{array}{c} \varphi_1 \\ \varphi_2 \end{array} \right), \quad \bar{u} = \sum_{n=1}^{N} \tilde{Q}_n + \bar{\varphi},
\] (2.5)

satisfies

\[
(\bar{\varphi}, \tilde{Z}_n)_{L^2} = 0, \quad \text{for } n = 1, \cdots, N,
\] (2.6)

and

\[
\|\bar{\varphi}\|_{L^2} \lesssim \delta_0, \quad y_{n+1} - y_n \gtrsim \frac{3}{4} L_0 \quad \text{for } n = 1, \cdots, N - 1.
\] (2.7)

**Proof.** The proof of the decomposition lemma relies on a standard argument based on the Implicit function Theorem (see e.g. Lemma 3 in [4]) and we omit it. \( \square \)

Set

\[
\bar{U} = \left( \begin{array}{c} U_1 \\ U_2 \end{array} \right) = \sum_{n=1}^{N} \tilde{Q}_n \quad \text{and} \quad G = f(U_1) - \sum_{n=1}^{N} f(Q_n).
\]

**Lemma 2.4** (Equation of \( \bar{\varphi} \)). The function \( \bar{\varphi} \) satisfies

\[
\begin{cases}
\partial_t \varphi_1 = \varphi_2 + \text{Mod}_1, \\
\partial_t \varphi_2 = \partial_x^2 \varphi_1 - \varphi_1 + f(U_1 + \varphi_1) - f(U_1) + G + \text{Mod}_2,
\end{cases}
\] (2.8)

where

\[
\text{Mod}_1 = \sum_{n=1}^{N} (y_n - \ell_n) \partial_x Q_n, \quad \text{Mod}_2 = - \sum_{n=1}^{N} (y_n - \ell_n) \ell_n \partial_x^2 Q_n.
\] (2.9)

**Proof.** First, from the definition of \( \bar{\varphi} = (\varphi_1, \varphi_2) \) in (2.3),

\[
\partial_t \varphi_1 = \partial_t u_1 - \partial_t u_1 = \varphi_2 + U_2 - \partial_t U_1 = \varphi_2 + \sum_{n=1}^{N} (y_n - \ell_n) \partial_x Q_n.
\]

Second, using (1.2),

\[
\partial_t \varphi_2 = \partial_t u_2 - \partial_t U_2 = \partial_x^2 u_1 - u_1 + f(u_1) - \sum_{n=1}^{N} \ell_n \partial_x^2 Q_n.
\]
We observe from (2.5) and \(-1 - \ell_n^2\partial_{x}^2 Q_n + Q_n - f(Q_n) = 0\),
\[
\partial_{x}^2 u_1 - u_1 + f(u_1) = \partial_{x}^2 \varphi_1 - \varphi_1 + f(U_1 + \varphi_1) - \sum_{n=1}^{N} f(Q_n) + \sum_{n=1}^{N} \ell_n^2 \partial_{x}^2 Q_n.
\]

Therefore, from the definition of \(G\) and \(\text{Mod}_2\), we obtain the second line of (2.8). \(\square\)

First, we derive some preliminary estimates associated to the equation of \(\varphi\) and the nonlinear interaction term \(G\) from Taylor’s formula. Fix \(\gamma_0 = \frac{1}{100} \min (1, \theta) \times \min \left((1 - \ell_1^2)^{-\frac{1}{2}}, \cdots, (1 - \ell_n^2)^{-\frac{1}{2}}\right)\)
\[
\times \min \left(1, \frac{q-1}{2}, q-2\right) \times \min (\ell_1, \ell_2 - \ell_1, \cdots, \ell_N - \ell_{N-1}) > 0,
\]
where \(\theta\) is defined in (1.3).

**Lemma 2.5.** Assume (2.3), for any \(n, n' \in \{1, \cdots, N\}, n \neq n'\), the following estimates hold. For \(n \neq n'\),
\[
\int_{\mathbb{R}} \left(|Q_n Q_{n'}| + |\partial_{x} Q_n \partial_{y} Q_{n'}| + |\partial_{x}^2 Q_n \partial_{x}^2 Q_{n'}|\right) \, dx \lesssim e^{-3\gamma_0 |y_n - y_{n'}|}, \quad (2.10)
\]
and
\[
\int_{\mathbb{R}} \left|F(U_1) - \sum_{n=1}^{N} F(Q_n)\right| \, dx \lesssim \sum_{n=1}^{N-1} e^{-3\gamma_0 (y_{n+1} - y_n)}, \quad (2.11)
\]
\[
\|G\|_{L^2} + \|f'(U_1) - \sum_{n=1}^{N} f'(Q_n)\|_{L^2} \lesssim \sum_{n=1}^{N-1} e^{-3\gamma_0 (y_{n+1} - y_n)}. \quad (2.12)
\]

**Proof.** Proof of (2.10). First, by change of variable, for \(n \neq n'\),
\[
\int_{\mathbb{R}} |Q_n Q_{n'}| \, dx = (1 - \ell_n^2)^{\frac{1}{2}} \int_{\mathbb{R}} Q(x) Q \left(\frac{(y_n - y_{n'}) + (1 - \ell_n^2)^{\frac{1}{2}} x}{(1 - \ell_{n'}^2)^{\frac{1}{2}}}\right) \, dx = H_1 + H_2,
\]
where
\[
H_1 = (1 - \ell_n^2)^{\frac{1}{2}} \int_{I_1} Q(x) Q \left(\frac{(y_n - y_{n'}) + (1 - \ell_n^2)^{\frac{1}{2}} x}{(1 - \ell_{n'}^2)^{\frac{1}{2}}}\right) \, dx,
\]
\[
H_2 = (1 - \ell_n^2)^{\frac{1}{2}} \int_{I_2} Q(x) Q \left(\frac{(y_n - y_{n'}) + (1 - \ell_n^2)^{\frac{1}{2}} x}{(1 - \ell_{n'}^2)^{\frac{1}{2}}}\right) \, dx,
\]
and
\[
I_1 = \left\{ x \in \mathbb{R} : \left|(1 - \ell_n^2)^{\frac{1}{2}} x\right| \leq \frac{1}{2} |y_n - y_{n'}| \right\},
\]
\[
I_2 = \left\{ x \in \mathbb{R} : \left|(1 - \ell_n^2)^{\frac{1}{2}} x\right| \geq \frac{1}{2} |y_n - y_{n'}| \right\}.
\]
From the decay properties of \(Q\) and the definition of \(\gamma_0\), we obtain
\[
H_1 \lesssim e^{-3\gamma_0 |y_n - y_{n'}|} \int_{I_1} Q(x) \, dx \lesssim e^{-3\gamma_0 |y_n - y_{n'}|},
\]
\[
H_2 \lesssim e^{-3\gamma_0 |y_n - y_{n'}|} \int_{I_2} Q \left(\frac{(y_n - y_{n'}) + (1 - \ell_n^2)^{\frac{1}{2}} x}{(1 - \ell_{n'}^2)^{\frac{1}{2}}}\right) \, dx \lesssim e^{-3\gamma_0 |y_n - y_{n'}|}.
\]
This proves estimate (2.10) for \(Q_n Q_{n'}\). The proof of (2.10) for \(\partial_{x} Q_n \partial_{y} Q_{n'}\) and \(\partial_{x}^2 Q_n \partial_{x}^2 Q_{n'}\) follows from similar arguments and it is omitted.
Proof of (2.11). From Taylor expansion and $1 < q < p < \infty$, we infer
\[
|F(U_1) - \sum_{n=1}^{N} F(Q_n)| \lesssim \sum_{n \neq n'} |Q_n|^q |Q_{n'}| + \sum_{n \neq n'} |Q_n|^p |Q_{n'}| \lesssim \sum_{n \neq n'} |Q_n||Q_{n'}|.
\]
Therefore, we conclude (2.11) from (2.10).

Proof of (2.12). First, from Taylor expansion and $1 < q < p < \infty$, we infer
\[
|f'(U_1) - \sum_{n=1}^{N} f'(Q_n)| \lesssim \begin{cases} 
\sum_{n \neq n'} |Q_n|^q |Q_{n'}| & \text{when } 2 < q < \infty, \\
\sum_{n \neq n'} |Q_n|^p |Q_{n'}|^p & \text{when } 1 < q \leq 2.
\end{cases}
\]
Therefore, using the similar argument as in the proof of (2.11), we obtain (2.12) for $f'(U_1) - \sum_{n=1}^{N} f'(Q_n)$. The proof of (2.12) for $G$ follows from similar arguments and it is omitted. \(\square\)

Second, we derive the control of $y$ from the orthogonality conditions (2.10).

**Lemma 2.6** (Control of $y$). It holds
\[
\sum_{n=1}^{N} \left| \bar{y}_n - \ell_n \right| \lesssim \|\bar{\varphi}\|_{H} + \sum_{n=1}^{N-1} e^{-2\gamma_0(y_{n+1} - y_n)}.
\] (2.13)

**Proof.** First, we rewrite the equation of $\bar{\varphi} = (\varphi_1, \varphi_2)$ as
\[
\partial_t \bar{\varphi} = \mathcal{L} \bar{\varphi} + \tilde{\text{Mod}} + \tilde{G} + \tilde{R}_1 + \tilde{R}_2,
\] (2.14)
where
\[
\mathcal{L} = \begin{pmatrix} \partial_x^2 - 1 + \sum_{n=1}^{N} f'(Q_n) & 1 \\ 0 & 0 \end{pmatrix}, \quad \tilde{\text{Mod}} = \begin{pmatrix} \text{Mod}_1 \\ \text{Mod}_2 \end{pmatrix},
\]
\[
\tilde{G} = \begin{pmatrix} 0 \\ G \end{pmatrix}, \quad \tilde{R}_1 = \begin{pmatrix} 0 \\ R_1 \end{pmatrix} = \begin{pmatrix} f(U_1 + \varphi_1) - f(U_1) - f'(U_1)\varphi_1 \\ 0 \end{pmatrix},
\]
and
\[
\tilde{R}_2 = \begin{pmatrix} 0 \\ R_2 \end{pmatrix} = \begin{pmatrix} 0 \\ f'(U_1) - \sum_{n=1}^{N} f'(Q_n) \end{pmatrix} \varphi_1.
\]
Second, from the orthogonality conditions (2.10),
\[
0 = \frac{d}{dt} \langle \bar{\varphi}, \bar{\varphi}_n \rangle_{L^2} = \left( \partial_t \bar{\varphi}, \bar{\varphi}_n \right)_{L^2} + \left( \bar{\varphi}, \partial_t \bar{\varphi}_n \right)_{L^2}.
\]
Thus, using (2.11),
\[
0 = \left( \mathcal{L} \bar{\varphi}, \bar{\varphi}_n \right)_{L^2} + \left( \tilde{R}_1, \bar{\varphi}_n \right)_{L^2} + \left( \tilde{R}_2, \bar{\varphi}_n \right)_{L^2} + \left( \tilde{G}, \bar{\varphi}_n \right)_{L^2} + \left( \tilde{\text{Mod}}, \bar{\varphi}_n \right)_{L^2} - (-\ell_n - \Delta) \left( \bar{\varphi}, \partial_x \bar{\varphi}_n \right)_{L^2}.
\]
Since $(-1 - \ell_n - \Delta) \partial_x^2 + f'(Q_n) \partial_x Q_n = 0$, the first term is
\[
\left( \mathcal{L} \bar{\varphi}, \bar{\varphi}_n \right)_{L^2} = -\ell_n \sum_{n \neq n'} (\varphi_1, f'(Q_n)\partial_x Q_n)_{L^2} + (\varphi_2, \partial_x Q_n)_{L^2} = O(\|\bar{\varphi}\|_{H}).
\]
Next, by Taylor expansion (as $1 < q < p < \infty$), we infer
\[
R_1 = f(U_1 + \varphi_1) - f(U_1) - f'(U_1)\varphi_1 = O(|\varphi_1|^2 + |\varphi_1|^q + |\varphi_1|^p).
\]
and by the Sobolev embedding theorem, we obtain
\[
\left| \left( \tilde{R}_1, \bar{\varphi}_n \right)_{L^2} \right| \lesssim \|\bar{\varphi}\|_{H}^2 + \|\bar{\varphi}\|_{H}^q + \|\bar{\varphi}\|_{H}^p.
\]
Using the Cauchy-Schwartz inequality and (2.12), we obtain
\[
\left| \left( \vec{R}_n, \vec{Z}_n^0 \right)_{L^2} \right| \lesssim \| \vec{\varphi} \|_H \| f'(U_1) - \sum_{n=1}^N f'(Q_n) \|_{L^2} \lesssim \| \vec{\varphi} \|_H^2 + \sum_{n=1}^{N-1} e^{-4\gamma_0(y_{n+1}-y_n)}.
\]
Then, using again (2.12), we have
\[
\left| \left( \vec{G}, \vec{Z}_n^0 \right)_{L^2} \right| \lesssim \| \vec{G} \|_{L^2} \lesssim \sum_{n=1}^{N-1} e^{-2\gamma_0(y_{n+1}-y_n)}
\]
Next, using the expression of \( \vec{M}_n \), we have
\[
\left( \vec{M}_n, \vec{Z}_n^0 \right)_{L^2} = (\hat{y}_n - \ell_n) \left( \vec{Z}_n^0, \vec{Z}_n^0 \right)_{L^2} + \sum_{n' \neq n} (\hat{y}_{n'} - \ell_{n'}) \left( \vec{Z}_{n'}^0, \vec{Z}_n^0 \right)_{L^2}.
\]
Moreover, from (2.10),
\[
\sum_{n' \neq n} (\hat{y}_{n'} - \ell_{n'}) \left( \vec{Z}_{n'}^0, \vec{Z}_n^0 \right)_{L^2} = O \left( \left( \sum_{n' \neq n} |\hat{y}_{n'} - \ell_{n'}| \right) \left( \sum_{n=1}^{N-1} e^{-2\gamma_0(y_{n+1}-y_n)} \right) \right).
\]
Last, by the Cauchy-Schwartz inequality,
\[
\left| (\hat{y}_n - \ell_n) \left( \vec{\varphi}, \partial_\ell \vec{Z}_n^0 \right)_{L^2} \right| + |\ell_n \left( \vec{\varphi}, \partial_\ell \vec{Z}_n^0 \right)_{L^2}| \lesssim \| \vec{\varphi} \|_H (1 + |\hat{y}_n - \ell_n|).
\]
Gathering above estimates, from the orthogonality condition \( \left( \vec{\varphi}, \vec{Z}_n^0 \right)_{L^2} = 0 \), we obtain
\[
\left| \hat{y}_n - \ell_n \right| \lesssim \| \vec{\varphi} \|_H + \sum_{n=1}^{N-1} e^{-2\gamma_0(y_{n+1}-y_n)}
\]
\[
+ O \left( \left( \sum_{n'=1}^N |\hat{y}_{n'} - \ell_{n'}| \right) \left( \sum_{n'=1}^{N-1} e^{-2\gamma_0(y_{n'+1}-y_{n'})} + \| \vec{\varphi} \|_H \right) \right).
\]
Similarly, using the other orthogonality conditions,
\[
\sum_{n=1}^N |\hat{y}_n - \ell_n| \lesssim \| \vec{\varphi} \|_H + \sum_{n=1}^{N-1} e^{-2\gamma_0(y_{n+1}-y_n)}
\]
\[
+ O \left( \left( \sum_{n'=1}^N |\hat{y}_{n'} - \ell_{n'}| \right) \left( \sum_{n'=1}^{N-1} e^{-2\gamma_0(y_{n'+1}-y_{n'})} + \| \vec{\varphi} \|_H \right) \right),
\]
which implies (2.13).

Last, we consider the equation of the unstable directions.

**Lemma 2.7 (Unstable direction).** Let \( a_n^\pm = (\vec{\varphi}, \vec{Z}_n^\pm)_{L^2} \). It holds
\[
\left| \frac{d}{dt} a_n^\pm + \alpha_n a_n^\pm \right| \lesssim \| \vec{\varphi} \|_H^2 + \| \vec{\varphi} \|_H^2 + \sum_{n=1}^{N-1} e^{-3\gamma_0(y_{n+1}-y_n)},
\]
(2.15)
where \( \alpha_n = \nu_0(1 - \ell_n^2)^{\frac{1}{2}} \).

**Proof.** Using (2.14), we calculate,
\[
\frac{d}{dt} a_n^\pm = \left( \partial_\ell \vec{\varphi}, \vec{Z}_n^\pm \right)_{L^2} + \left( \vec{\varphi}, \partial_\ell \vec{Z}_n^\pm \right)_{L^2}
\]
\[
= \left( \ell \vec{\varphi}, \vec{Z}_n^\pm \right)_{L^2} - \ell_1 \left( \vec{\varphi}, \partial_\ell \vec{Z}_n^\pm \right)_{L^2} + \left( \vec{G}, \vec{Z}_n^\pm \right)_{L^2} + \left( \vec{R}_n^1, \vec{Z}_n^\pm \right)_{L^2} + \left( \vec{R}_n^2, \vec{Z}_n^\pm \right)_{L^2}
\]
\[
+ \left( \vec{M}_n, \vec{Z}_n^\pm \right)_{L^2} - (\hat{y}_n - \ell_1) \left( \vec{\varphi}, \partial_\ell \vec{Z}_n^\pm \right)_{L^2}.
\]
Observe that,
\[
\left( \tilde{\vec{C}} \vec{\varphi}, \tilde{Z}_1^\pm \right)_{L^2} - \ell_1 \left( \vec{\varphi}, \partial_t \tilde{Z}_1^\pm \right)_{L^2} = \left( \vec{\varphi}, \left( H_{\ell_1} J \tilde{Z}_1^\pm \right) \cdot \left( -y_1 \right) \right)_{L^2} + \sum_{n=2}^{N} (\varphi_0, f'(Q_n)Z_1^\pm)_{L^2}
\]
\[
= \pm \alpha_1 a_1^\pm + \sum_{n=2}^{N} (\varphi_0, f'(Q_n)Z_1^\pm)_{L^2}.
\]
where
\[
Z_1^\pm = \left( Y_{\ell_1} e^{\frac{\pm t}{\sqrt{1-\xi^2}}} \right) \cdot (-y_1).
\]
By the decay properties of \(Z_n^\pm\) and similar argument as (2.10) in Lemma 2.5,
\[
\left| \sum_{n=2}^{N} (\varphi_0, f'(Q_n)Z_1^\pm)_{L^2} \right| \lesssim \left( \sum_{n=1}^{N-1} e^{-3\gamma_0(y_{n+1}-y_n)} \right) \left\| \varphi_0 \right\|_{L^2} \lesssim \left\| \varphi_0 \right\|_{L^2}^2 + \sum_{n=1}^{N-1} e^{-4\gamma_0(y_{n+1}-y_n)}.
\]
Next, from (2.12), Taylor’s formula and Sobolev embedding theorem,
\[
\left| \left( \tilde{\vec{R}}_1, \tilde{Z}_1^\pm \right)_{L^2} \right| \lesssim \int_{\mathbb{R}} \left| \left( \varphi_0^2 + |\varphi_0|^9 + |\varphi_0|^p \right) \right| dx \lesssim \left\| \varphi_0 \right\|_{H^2}^2 + \left\| \varphi_0 \right\|_{H^2}^2 + \left\| \varphi_0 \right\|_{H^2}^2,
\]
and
\[
\left| \left( \tilde{\vec{R}}_2, \tilde{Z}_1^\pm \right)_{L^2} \right| \lesssim \left| f'(U_1) - \sum_{n=1}^{N-1} f'(Q_n) \right| \left\| \varphi_0 \right\|_{L^2} \lesssim \left\| \varphi_0 \right\|_{L^2}^2 + \sum_{n=1}^{N-1} e^{-6\gamma_0(y_{n+1}-y_n)},
\]
Moreover, from the decay properties of \(Z_1^\pm\), the similar argument of (2.10) and concerning the term with Mod,
\[
\left( \text{Mod,} \; \tilde{Z}_1^\pm \right)_{L^2} = \sum_{n=2}^{N} (\tilde{y}_n - \ell_n) \left( \tilde{Z}_n^0, \tilde{Z}_1^\pm \right)_{L^2} = O \left( \left\| \varphi_0 \right\|_{H^2}^2 + \sum_{n=1}^{N-1} e^{-6\gamma_0(y_{n+1}-y_n)} \right).
\]
Finally, from (2.13),
\[
\left| \left( \tilde{y}_1 - \ell_1 \right) \left( \vec{\varphi}, \partial_t \tilde{Z}_1^\pm \right)_{L^2} \right| \lesssim \left\| \varphi \right\|_{H^2}^2 + \sum_{n=1}^{N-1} e^{-6\gamma_0(y_{n+1}-y_n)}.
\]
Gathering above estimates and proceeding similarly for \(a_n^\pm\) in (2.14), we obtain
\[
3. \text{ Monotonicity property for the 1D Klein-Gordon equation}
\]
3.1. **Bootstrap setting.** We introduce the following bootstrap estimates: for \(C_0\) to be chosen later,
\[
\left\| \varphi(t) \right\|_{H^2} \leq C_0 \left( \delta + e^{-\gamma_0 L} \right), \quad \min_{n}(y_{n+1} - y_n) \geq (1 - C_0^{-1})L + 2\gamma_0 t,
\]
\[
\sum_{n=1}^{N} |a_n^\pm(t)|^2 \leq C_0^2 \left( \delta^2 + e^{-2\gamma_0 L} \right), \quad \sum_{n=1}^{N} |a_n^\pm(t)|^2 \leq C_0^2 \left( \delta^2 + e^{-2\gamma_0 L} \right), \quad (3.1)
\]
For \(u_0\) satisfies (2.4), set
\[
T_* (\bar{u}_0) = \sup \{ t \in [0, \infty); \bar{u} \text{ satisfies (2.4) and (3.1) holds on } [0, t] \},
\]
where \(\bar{u}\) is the solution of (2.2) with the initial data \(\bar{u}_0\).
3.2. Monotonicity property. First, we choose suitable cutoff functions. Let \( \chi(x) \) be a \( C^1 \)-function such that
\[
\chi' \geq 0, \quad \chi(x) = 0, \quad \text{for } x \leq -1, \quad \chi(x) = 1, \quad \text{for } x > 1.
\]
Set
\[
\beta_n = \frac{\ell_{n-1} + \ell_n}{2}, \quad \tilde{y}_n = \frac{y_{n-1}(0) + y_n(0)}{2},
\]
and
\[
\chi_1 = 1, \quad \chi_{N+1} = 0, \quad \chi_n(t, x) = \chi \left( \frac{x - \beta_n t - \tilde{y}_n}{(t + a)^\alpha} \right)
\]
for \( n = 2, \ldots, N \), where \( \alpha \) and \( a \) are chosen so that
\[
\frac{1}{2} < \alpha < \frac{4}{7} \quad \text{and} \quad a = \left( \frac{L}{10} \right) \cdot
\]
Let
\[
\psi_n(t, x) = \chi_n(t, x) - \chi_{n+1}(t, x) \quad \text{for } n = 1, \ldots, N.
\]
Note that \( \psi_n \equiv 1 \) around the solitary wave \( Q_n \), and \( \psi_n \equiv 0 \) around the solitary waves \( n' \) for \( n' \neq n \). Moreover
\[
\sum_{n=1}^{N} \psi_n = 1, \quad \text{and} \quad \chi_n = \sum_{n'=n}^{N} \psi_{n'}, \quad \text{for } n = 1, \ldots, N. \tag{3.3}
\]
Set
\[
\Omega_n = \{ x \in \mathbb{R} : |x - \beta_n t - \tilde{y}_n| < (t + a)^\alpha \}
\]
Note that, from the definition of \( \chi_n \) and \( \Omega_n \), we have the following estimates,
\[
\partial_x \chi_n = \left( \partial_x \chi \right) \left( \frac{x - \beta_n t - \tilde{y}_n}{(t + a)^\alpha} \right) \frac{1}{(t + a)^\alpha} = O \left( \frac{1}{(t + a)^\alpha} \right), \tag{3.4}
\]
\[
|\partial^2_x \chi_n| + |\partial_t \chi_n| \lesssim \frac{1}{(t + a)^{2\alpha}} 1_{\Omega_n}, \quad |\partial^3_x \chi_n| \lesssim \frac{1}{(t + a)^{3\alpha}} 1_{\Omega_n}. \tag{3.5}
\]
Second, let
\[
c_1 = \ell_1, \quad c_2 = \ell_2 - \ell_1 - 1 - \beta_2 \ell_2, \quad c_n = \left( \frac{\ell_n - \ell_{n-1}}{1 - \beta_n \ell_n} \right) \prod_{n'=2}^{n-1} \left( \frac{1 - \beta_n \ell_{n'-1}}{1 - \beta_n \ell_n} \right), \tag{3.6}
\]
for \( n = 3, \ldots, N \). Denote
\[
\bar{c}_1 = 1 \quad \text{and} \quad \bar{c}_n = 1 + \sum_{n'=2}^{n} c_{n'} \beta_{n'} \quad \text{for } n = 2, \ldots, N. \tag{3.7}
\]
By direct computation, we obtain the following Lemma.

**Lemma 3.1.** For \( n = 2, \ldots, N \), we have
\[
\sum_{n'=1}^{n} c_{n'} = \bar{c}_n \ell_n, \quad \text{and} \quad \bar{c}_n = \prod_{n'=2}^{n} \left( \frac{1 - \beta_n \ell_{n'-1}}{1 - \beta_n \ell_n} \right). \tag{3.8}
\]

**Proof.** We prove (3.8) by induction.

**Step 1.** For \( n = 2 \). By direct computation,
\[
\bar{c}_2 = 1 + c_2 \beta_2 = 1 + \frac{\beta_2 \ell_2 - \beta_2 \ell_1}{1 - \beta_2 \ell_2} = 1 + \frac{1 - \beta_2 \ell_1}{1 - \beta_2 \ell_2} = \ell_1 + \frac{\ell_2 - \ell_1}{1 - \beta_2 \ell_2} = \ell_1 + \frac{\ell_1}{1 - \beta_2 \ell_2} = \ell_1 + \frac{\ell_1}{1 - \beta_2 \ell_2} = \ell_2 \ell_2,
\]
which implies (3.8) for \( n = 2 \).
Step 2. We assume that (3.8) is true for $n = k$. Now, we prove that also true for $n = k + 1$. From the definition of $c_n$ for $n \geq 3$, (3.8) is true for $n = k$, we obtain

$$
\tilde{c}_{k+1} = \prod_{n'=2}^{k} \left( 1 - \frac{\beta_{n'} \ell_{n'-1}}{1 - \beta_{n'} \ell_{n'}} \right) + \left( \frac{\ell_{k+1} - \ell_k}{1 - \beta_{k+1} \ell_{k+1}} \right) \prod_{n'=2}^{k} \left( 1 - \frac{\beta_{n'} \ell_{n'-1}}{1 - \beta_{n'} \ell_{n'}} \right)
$$

$$
= \left( 1 + \frac{\beta_{k+1}(\ell_{k+1} - \ell_k)}{1 - \beta_{k+1} \ell_{k+1}} \right) \prod_{n'=2}^{k+1} \left( 1 - \frac{\beta_{n'} \ell_{n'-1}}{1 - \beta_{n'} \ell_{n'}} \right) = \tilde{c}_{k+1} \ell_{k+1} + c_{k+1}
$$

and

$$
\sum_{n'=1}^{k+1} c_{n'} = \tilde{c}_k \ell_k + c_{k+1}
$$

Therefore, (3.8) is also true for $n = k + 1$. By induction argument, we have proved (3.8) for $n = 2, \cdots, N$.

Third, we introduce the following modified virial elements

$$
\mathcal{J}_n(\bar{u}) = \mathcal{I}_n(\bar{u}) + \beta_n E_n(\bar{u}) + \beta_n F_n(\bar{u}).
$$

(3.9)

where

$$
\mathcal{I}_n(\bar{u}) = 2 \int_{\mathbb{R}} \left( \chi_n \partial_x u_1 + \frac{1 - \beta_n^2}{2} (\partial_x \chi_n) u_1 \right) u_2 dx,
$$

(3.10)

$$
E_n(\bar{u}) = \int_{\mathbb{R}} \left( (\partial_x u_1)^2 + u_1^2 + u_2^2 - 2F(u_1) \right) \chi_n dx,
$$

(3.11)

$$
F_n(\bar{u}) = - \frac{a}{(t + a)^{1 - a}} \int_{\mathbb{R}} (u_1 u_2) \left( \frac{x - \beta_n t - \beta_n^0}{(t + a)^a} \right) \partial_x \chi_n dx
$$

for $n = 2, \cdots, N$.

Last, we set

$$
\mathcal{E}(\bar{u}) = E(\bar{u}) + c_1 \mathcal{I}(\bar{u}) + \sum_{n=2}^{N} c_n \mathcal{J}_n(\bar{u})
$$

where we recall that

$$
E(\bar{u}) = \int_{\mathbb{R}} \left( (\partial_x u_1)^2 + u_1^2 + u_2^2 - 2F(u_1) \right) dx,
$$

$$
\mathcal{I}(\bar{u}) = 2 \int_{\mathbb{R}} (\partial_x u_1) u_2 dx.
$$

By expanding $\bar{u}(t) = \sum_{n=1}^{N} \bar{Q}_n(t) + \bar{\varphi}(t)$, we obtain the following formula.

Lemma 3.2. The following holds,

$$
\mathcal{E}(\bar{u}) = \sum_{n=1}^{N} \tilde{c}_n (1 - \ell_n^0)^4 E(\bar{Q}) + \sum_{n=1}^{N} \tilde{c}_n H_n(\bar{\varphi}, \bar{\varphi}) + O \left( \frac{||\bar{\varphi}||_H^4 + ||\bar{\varphi}||_H^2 + ||\bar{\varphi}||_H^{2+1}}{L} + e^{-3\gamma_n(L + \gamma_0 t)} \right)
$$

(3.12)

where

$$
H_n(\bar{\varphi}, \bar{\varphi}) = \int_{\mathbb{R}} \left( (\partial_x \varphi_1)^2 + \varphi_1^2 + \varphi_2^2 + 2\ell_n (\partial_x \varphi_1) \varphi_2 - f'(Q_n) \varphi_2^2 \right) \bar{\psi}_n dx.
$$

for $n = 1, \cdots, N$. 

Proof. Step 1. Expansion of $E(\vec{u})$. We prove the following estimate

$$E(\vec{u}) = \sum_{n=1}^{N} (1 - \ell_n^2) \frac{1}{2} E(\vec{Q}) + 2 \sum_{n=1}^{N} \int_{\mathbb{R}} (\ell_n \partial_x Q_n) (\ell_n \partial_x Q_n + \ell_n \partial_x \varphi_1 - \varphi_2) \, dx$$

$$+ \int_{\mathbb{R}} (\partial_x \varphi_1^2 + \varphi_1^2 + \varphi_2^2 - \sum_{n=1}^{N} f'(Q_n) \varphi_1^2) \, dx$$

$$+ O \left( \|\vec{\varphi}\|_{L^3}^3 + \|\vec{\varphi}\|_{H^1}^{q+1} + e^{-3\gamma_0(L+\gamma_0 t)} \right).$$

(3.13)

First, using the decomposition (2.13), the definition of $G$, the equation $-(1 - \ell_n^2) \partial_x^2 Q_n + Q_n - f(Q_n) = 0$ and integration by parts, we find

$$E(\vec{u}) = E(\vec{U}) + 2 \sum_{n=1}^{N} \int_{\mathbb{R}} (\ell_n \partial_x Q_n) (\ell_n \partial_x \varphi_1 - \varphi_2) \, dx$$

$$+ \int_{\mathbb{R}} ((\partial_x \varphi_1)^2 + \varphi_1^2 + \varphi_2^2 - \sum_{n=1}^{N} f'(Q_n) \varphi_1^2) \, dx + \tilde{E}_1 + \tilde{E}_2 + \tilde{E}_3,$$

where

$$\tilde{E}_1 = -2 \int_{\mathbb{R}} (F(U_1 + \varphi_1) - F(U_1) - f(U_1) \varphi_1 - \frac{1}{2} f'(U_1) \varphi_1^2) \, dx$$

$$\tilde{E}_2 = -2 \int_{\mathbb{R}} G \varphi_1 \, dx \quad \text{and} \quad \tilde{E}_3 = - \int_{\mathbb{R}} (f'(U_1) - \sum_{n=1}^{N} f'(Q_n)) \varphi_1^2 \, dx.$$

Note that by the Taylor formula,

$$F(U_1 + \varphi_1) - F(U_1) - f(U_1) \varphi_1 - \frac{1}{2} f'(U_1) \varphi_1^2 = O \left( |\varphi_1|^3 + |\varphi_1|^q + |\varphi_1|^{q+1} \right).$$

Therefore, using Sobolev embedding and $1 < q < p < \infty$,

$$|\tilde{E}_1| \lesssim \int_{\mathbb{R}} (|\varphi_1|^{q+1} + |\varphi_1|^q + |\varphi_1|^3) \, dx \lesssim \|\varphi_1\|_{H^q}^3 + \|\varphi_1\|_{H^q}^{q+1}.$$

By (2.12),

$$|\tilde{E}_2| \lesssim \|G\|_{L^2} \|\varphi_1\|_{L^2} \lesssim \|\varphi_1\|_{L^2}^3 + e^{-4\gamma_0(L+\gamma_0 t)},$$

and then (3.13) and Sobolev embedding,

$$|\tilde{E}_3| \lesssim \|f'(U_1) - \sum_{n=1}^{N} f'(Q_n)\|_{L^2} \|\varphi_1\|_{H^q}^2 \lesssim \|\vec{\varphi}\|_{H^1}^3 + e^{-4\gamma_0(L+\gamma_0 t)}.$$

Second, by direct computation, $-(1 - \ell_n^2) \partial_x^2 Q_n + Q_n - f(Q_n) = 0$,

$$E(\vec{U}) = \sum_{n=1}^{N} (1 - |\ell_n|^2) \frac{1}{2} E(\vec{Q}) + 2 \sum_{n=1}^{N} \int_{\mathbb{R}} (\ell_n \partial_x Q_n)^2 \, dx$$

$$+ \sum_{n \neq n'} \int_{\mathbb{R}} [(1 + \ell_n \ell_{n'}) (\partial_x Q_n) (\partial_x Q_{n'}) + Q_n Q_{n'}] \, dx$$

$$- 2 \int_{\mathbb{R}} (F(U_1) - \sum_{n=1}^{N} F(Q_n)) \, dx.$$

Moreover, using (2.10), (2.11) and (3.1),

$$\sum_{n \neq n'} \int_{\mathbb{R}} [\partial_x Q_n \partial_x Q_{n'} + |Q_n Q_{n'}|] \, dx + \int_{\mathbb{R}} |F(U_1) - \sum_{n=1}^{N} F(Q_n)| \, dx \lesssim e^{-3\gamma_0(L+\gamma_0 t)}.$$
We see that (3.13) follows from above estimates.

**Step 2.** Expansion of $\mathcal{I}(\vec{u})$. We claim

$$
\mathcal{I}(\vec{u}) = -2 \sum_{n=1}^{N} \int_{\mathbb{R}} (\partial_x Q_n) (\ell_n \partial_x Q_n + \ell_n \partial_x \varphi_1 - \varphi_2) \, dx \\
+ 2 \int_{\mathbb{R}} (\partial_x \varphi_1) \varphi_2 \, dx + O \left( e^{-3\gamma_0 (L+\gamma_0 t)} \right). 
$$

(3.14)

By direct computation and (2.5)

$$
\mathcal{I}(\vec{u}) = -2 \sum_{n=1}^{N} \int_{\mathbb{R}} (\partial_x Q_n) (\ell_n \partial_x Q_n + \ell_n \partial_x \varphi_1 - \varphi_2) \, dx \\
+ 2 \int_{\mathbb{R}} (\partial_x \varphi_1) \varphi_2 \, dx - 2 \sum_{n \neq n'} \ell_n \int_{\mathbb{R}} (\partial_x Q_n) (\partial_x Q_{n'}) \, dx.
$$

From (2.10) and (3.1), we obtain (3.14).

**Step 3.** Expansion of $\mathcal{I}_n(\vec{u})$. We claim

$$
\mathcal{I}_n(\vec{u}) = -2 \sum_{n'=n}^{N} \int_{\mathbb{R}} (\partial_x Q_{n'}) (\ell_n \partial_x Q_{n'} + \ell_n \partial_x \varphi_1 - \varphi_2) \, dx \\
+ 2 \int_{\mathbb{R}} \chi_n (\partial_x \varphi_1) \varphi_2 \, dx + O \left( \|\varphi\|_{L}^3 + \|\varphi\|_{H}^3 + e^{-3\gamma_0 (L+\gamma_0 t)} \right). 
$$

(3.15)

We decompose,

$$
\mathcal{I}_n(\vec{u}) = \mathcal{I}^1_n(\vec{u}) + (1 - \beta_n^2) \mathcal{I}^2_n(\vec{u}),
$$

where

$$
\mathcal{I}^1_n(\vec{u}) = 2 \int_{\mathbb{R}} (\chi_n \partial_x u_1) u_2 \, dx, \quad \mathcal{I}^2_n(\vec{u}) = \int_{\mathbb{R}} (\partial_x \chi_n) u_1 u_2 \, dx.
$$

Estimate on $\mathcal{I}^1_n$. We claim

$$
\mathcal{I}^1_n(\vec{u}) = -2 \sum_{n'=n}^{N} \int_{\mathbb{R}} (\partial_x Q_{n'}) (\ell_n \partial_x \varphi_1 + \ell_n \partial_x Q_{n'} - \varphi_2) \, dx \\
+ 2 \int_{\mathbb{R}} \chi_n (\partial_x \varphi_1) \varphi_2 \, dx + \mathcal{I}^1_{n+1} + \mathcal{I}^1_{n+2} + \mathcal{I}^1_{n+3},
$$

(3.16)

By direct computation and (2.5),

$$
\mathcal{I}^1_n(\vec{u}) = -2 \sum_{n'=n}^{N} \int_{\mathbb{R}} (\partial_x Q_{n'}) (\ell_n \partial_x \varphi_1 + \ell_n \partial_x Q_{n'} - \varphi_2) \, dx \\
+ 2 \int_{\mathbb{R}} \chi_n (\partial_x \varphi_1) \varphi_2 \, dx + \mathcal{I}^{1,1}_n + \mathcal{I}^{1,2}_n + \mathcal{I}^{1,3}_n,
$$

where

$$
\mathcal{I}^{1,1}_n = -2 \sum_{n' \neq n'} \ell_n' \int_{\mathbb{R}} \chi_n (\partial_x Q_{n'}) (\partial_x Q_{n'}) \, dx,
$$

$$
\mathcal{I}^{1,2}_n = -2 \sum_{n'=1}^{n-1} \int_{\mathbb{R}} (\chi_n \partial_x Q_{n'}) (\ell_n \partial_x \varphi_1 + \ell_n \partial_x Q_{n'} - \varphi_2) \, dx,
$$

$$
\mathcal{I}^{1,3}_n = -2 \sum_{n'=n}^{N} \int_{\mathbb{R}} (\chi_n - 1) (\partial_x Q_{n'}) (\ell_n \partial_x \varphi_1 + \ell_n \partial_x Q_{n'} - \varphi_2) \, dx.
$$
From (2.10) and (3.1),
\[
|T_n^{1,1}| \lesssim \sum_{n' \neq n} \int |\partial_x Q_n''| |\partial_x Q_{n'}| dx \lesssim e^{-3\gamma_0(L+\gamma_0 t)}.
\]

By the decay properties of $Q$, the definition of $\chi_n$ and (3.1),
\[
|T_n^{1,2}| \lesssim \sum_{n'=1}^{n-1} \|\chi_n \partial_x Q_{n'}\|_{L^2} (\|\varphi\|_H + \|\partial_x Q_{n'}\|_{L^2}) \lesssim \|\varphi\|_H^3 + e^{-3\gamma_0(L+\gamma_0 t)}
\]
\[
|T_n^{1,3}| \lesssim \sum_{n'=n}^{N} \|\chi_n - 1\|_{L^2} (\|\varphi\|_H + \|\partial_x Q_{n'}\|_{L^2}) \lesssim \|\varphi\|_H^3 + e^{-3\gamma_0(L+\gamma_0 t)}.
\]

We see that (3.10) follows from above estimates.

**Estimate on $T_n^2$.** We decompose
\[
T_n^2(\bar{u}) = T_n^{2,1} + T_n^{2,2} + T_n^{2,3},
\]
where
\[
T_n^{2,1} = -\sum_{n',n''=1}^{N} \ell_{n'} \int \partial_x Q_{n'}(\partial_x Q_{n''}) dx,
\]
\[
T_n^{2,2} = \sum_{n'=1}^{N} \int (\partial_x \chi_n)(-\ell_{n'}(\partial_x Q_{n''})\varphi_1 + Q_{n''}\varphi_2) dx,
\]
\[
T_n^{2,3} = \int (\partial_x \chi_n)\varphi_1\varphi_2 dx.
\]

Note that, taking $L$ large enough, for any $x \in \Omega_n$,
\[
|x - \ell_{n'} t - y_{n'}^0| \geq |(\ell_{n'} - \beta_{n'})| t + |y_{n'}^0 - \bar{y}_{n'}^0| - (t + a)^\alpha \geq 10\gamma_0 t + \frac{L}{10}
\]
(3.17) for $1 \leq n' \leq N$. Therefore, from the decay properties of $Q$, (3.4) and Cauchy-Schwartz inequality,
\[
|T_n^{2,1}| \lesssim \sum_{n',n''=1}^{N} \int |\partial_x Q_{n''}| |Q_{n''}| dx \lesssim e^{-3\gamma_0(L+\gamma_0 t)}
\]
\[
|T_n^{2,2}| \lesssim \sum_{n'=1}^{N} \|\partial_x Q_{n'}\|_{L^2} + \|Q_{n'}\|_{L^2} \|\varphi\|_H \lesssim \|\varphi\|_H^3 + e^{-3\gamma_0(L+\gamma_0 t)}
\]

Moreover, from the definition of $a$,
\[
|T_n^{2,3}| \lesssim \frac{\|\varphi\|_H^2}{(t+a)^\alpha} \lesssim \frac{\|\varphi\|_H^2}{L}.
\]

From above estimates, we conclude,
\[
|T_n^2| \lesssim \frac{\|\varphi\|_H^2}{L} + \|\varphi\|_H^3 + e^{-3\gamma_0(L+\gamma_0 t)}.
\]

We see that (3.15) follows from (3.10) and (3.18).

**Step 4.** Expansion of $E_n(\bar{u})$. We claim
\[
E_n(\bar{u}) = \sum_{n'=n}^{N} \left(1 - \ell_{n'}^2\right) E(\bar{Q}) + \sum_{n'=n}^{N} \left(\ell_{n'} \partial_x Q_{n'}(\ell_{n'} \partial_x Q_{n'} + \ell_{n'} \partial_x \varphi_1 - \varphi_2) dxight.
\]
\[
+ \int \left((\partial_x \varphi_1)^2 + \varphi_1^2 + \varphi_2^2 - \sum_{n'=n}^{N} f'(Q_{n'}) \varphi_2^2\right) \chi_n dx
\]
\[
+ O \left(\|\varphi\|_H^3 + \|\varphi\|_H^4 + e^{-3\gamma_0(L+\gamma_0 t)}\right).
\]
(3.19)
First, from (2.5), integration by parts and an elementary computation,
\[
E_n(\bar{u}) = E_n(\bar{U}) + 2 \sum_{n' = n}^{N} \int_{\mathbb{R}} (\ell_n' \partial_x Q_{n'}) (\ell_n' \partial_x \varphi_1 - \varphi_2) \, dx
\]
\[
+ \int_{\mathbb{R}} \left( (\partial_x \varphi_1)^2 + \varphi_1^2 + \varphi_2^2 - \sum_{n' = n}^{N} f'(Q_{n'}) \varphi_1^2 \right) \chi_n \, dx
\]
\[
+ \tilde{E}_n^1 + \tilde{E}_n^2 + \tilde{E}_n^3 + \tilde{E}_n^4 + \tilde{E}_n^5 + E_n^6,
\]
where
\[
\tilde{E}_n^1 = -2 \int_{\mathbb{R}} \left( F(U_1 + \varphi_1) - F(U_1) - f(U_1) \varphi_1 - \frac{1}{2} f''(U_1) \varphi_1^2 \right) \chi_n \, dx,
\]
\[
\tilde{E}_n^2 = - \int_{\mathbb{R}} (f'(U_1) - \sum_{n' = n}^{N} f'(Q_{n'})) \varphi_1^2 \chi_n \, dx,
\]
\[
\tilde{E}_n^3 = -2 \int_{\mathbb{R}} G \varphi_1 \chi_n \, dx,
\]
\[
\tilde{E}_n^4 = -2 \sum_{n = 1}^{N} (1 - \ell_n^2) \int_{\mathbb{R}} (\partial_x Q_{n'}) (\partial_x \chi_n) \varphi_1 \, dx,
\]
\[
\tilde{E}_n^5 = 2 \sum_{n = 1}^{N} \int_{\mathbb{R}} (\ell_n' \partial_x Q_{n'}) (\ell_n' \partial_x \varphi_1 - \varphi_2) \chi_n \, dx,
\]
\[
\tilde{E}_n^6 = 2 \sum_{n = 1}^{N} \int_{\mathbb{R}} (\ell_n' \partial_x Q_{n'}) (\ell_n' \partial_x \varphi_1 - \varphi_2) (\chi_n - 1) \, dx.
\]
Using the similar argument as in step 1, we obtain
\[
|\tilde{E}_n^1| + |\tilde{E}_n^2| + |\tilde{E}_n^3| \lesssim \|\varphi\|_H^3 + \|\varphi\|_H^4 + e^{-3\gamma_0(L+\gamma_0 t)}.
\]
Next, by the decay properties of $Q$ and the definition of $\chi_n$,
\[
|\tilde{E}_n^4| \lesssim \|\varphi\|_H \left( \sum_{n = 1}^{N} \|\partial_x Q_{n'} \chi_n\|_{L^2} \right) \lesssim \|\varphi\|_H^4 + e^{-3\gamma_0(L+\gamma_0 t)},
\]
\[
|\tilde{E}_n^5| \lesssim \|\varphi\|_H \left( \sum_{n = 1}^{N} \|\partial_x Q_{n'} \chi_n\|_{L^2} \right) \lesssim \|\varphi\|_H^4 + e^{-3\gamma_0(L+\gamma_0 t)},
\]
\[
|\tilde{E}_n^6| \lesssim \|\varphi\|_H \left( \sum_{n = 1}^{N} \|\chi_n - 1\|_{L^2} \right) \lesssim \|\varphi\|_H^4 + e^{-3\gamma_0(L+\gamma_0 t)}.
\]
Second, by direct computation,
\[
E_n(\bar{U}) = \sum_{n = 1}^{N} \left( 1 - \ell_n^2 \right) \frac{1}{2} E(Q) + 2 \sum_{n = 1}^{N} \int_{\mathbb{R}} (\ell_n' \partial_x Q_{n'})^2 \, dx
\]
\[
+ \tilde{E}_n^7 + \tilde{E}_n^8 + \tilde{E}_n^9 + \tilde{E}_n^{10},
\]
where
\[
\tilde{E}_n^7 = -2 \int_{\mathbb{R}} \left( F(U_1) - \sum_{n' = 1}^{N} f'(Q_{n'}) \right) \chi_n \, dx,
\]
\[
\tilde{E}_n^8 = \sum_{n = 1}^{N} \int_{\mathbb{R}} \left( \sum_{n' = 1}^{N} ((1 + \ell_n^2) (\partial_x Q_{n'})^2 + Q_{n'}^2 - 2F(Q_{n'})) \right) \chi_n \, dx,
\]
\[
\tilde{E}_n^9 = \sum_{n = 1}^{N} \int_{\mathbb{R}} \left( \sum_{n' = 1}^{N} ((1 + \ell_n^2) (\partial_x Q_{n'})^2 + Q_{n'}^2 - 2F(Q_{n'})) \chi_n \right) \, dx.
\]
\[
\tilde{E}_{n}^{10} = \sum_{n' \neq n''} \int_{\mathbb{R}} \left( (1 + \ell_{n'}) \partial_{x} Q_{n'} \partial_{x} Q_{n''} + Q_{n'} Q_{n''} \right) \chi_{n} \, dx.
\]

By (2.12) and (3.1),
\[
\left| \tilde{E}_{n}^{7} \right| \lesssim \int_{\mathbb{R}} \left| F(U_{1}) - \sum_{n' = 1}^{N} F(Q_{n'}) \right| \, dx \lesssim e^{-3\gamma_{0}(L + \gamma_{0}t)}.
\]

From the decay properties of \(Q\), we obtain
\[
\left| \tilde{E}_{n}^{8} \right| \lesssim \sum_{n = 1}^{N} \int_{\mathbb{R}} \left( (\partial_{x} Q_{n'})^{2} + Q_{n'}^{2} + |Q_{n'}|^{p+1} + |Q_{n'}|^{q+1} \right) \chi_{n} \, dx \lesssim e^{-3\gamma_{0}(L + \gamma_{0}t)},
\]
\[
\left| \tilde{E}_{n}^{9} \right| \lesssim \sum_{n = 1}^{N} \int_{\mathbb{R}} \left( (\partial_{x} Q_{n'})^{2} + Q_{n'}^{2} + |Q_{n'}|^{p+1} + |Q_{n'}|^{q+1} \right) \chi_{n} - 1 \, dx \lesssim e^{-3\gamma_{0}(L + \gamma_{0}t)}.
\]

Last, using again (2.10) and (3.1), we have
\[
\left| \tilde{E}_{n}^{10} \right| \lesssim \sum_{n' \neq n''} \int_{\mathbb{R}} \left| (\partial_{x} Q_{n'}) \partial_{x} Q_{n''} + Q_{n'} Q_{n''} \right| \, dx \lesssim e^{-3\gamma_{0}(L + \gamma_{0}t)}.
\]

We see that (3.19) follows from above estimates.

**Step 5. Estimate of \(F_{n}(\vec{u})\).** We claim
\[
|F_{n}(\vec{u})| \lesssim \frac{\|\vec{\varphi}\|_{L}^{2}}{L} + \|\vec{\varphi}\|_{H}^{3} + e^{-3\gamma_{0}(L + \gamma_{0}t)}.
\]

From (2.5), we decompose
\[
F_{n}(\vec{u}) = F_{n}^{1} + F_{n}^{2} + F_{n}^{3},
\]
where
\[
F_{n}^{1} = - \frac{\alpha}{(t + a)^{1-\alpha}} \int_{\mathbb{R}} \left( \varphi_{1}\varphi_{2} \left( \frac{x - \beta_{n}t - \bar{g}_{n}^0}{(t + a)^{\alpha}} \right) \partial_{x} \chi_{n} \right) \, dx,
\]
\[
F_{n}^{2} = \frac{\alpha}{(t + a)^{1-\alpha}} \sum_{N' = 1}^{N} \ell_{n'} \int_{\mathbb{R}} \left( Q_{n'} \partial_{x} Q_{n''} \left( \frac{x - \beta_{n}t - \bar{g}_{n}^0}{(t + a)^{\alpha}} \right) \partial_{x} \chi_{n} \right) \, dx,
\]
\[
F_{n}^{3} = - \frac{\alpha}{(t + a)^{1-\alpha}} \sum_{N' = 1}^{N} \ell_{n'} \int_{\mathbb{R}} \left( \partial_{x} Q_{n'} - \ell_{n'} \varphi_{1} \partial_{x} Q_{n''} \left( \frac{x - \beta_{n}t - \bar{g}_{n}^0}{(t + a)^{\alpha}} \right) \partial_{x} \chi_{n} \right) \, dx.
\]

First, from (3.3),
\[
|F_{n}^{1}| \lesssim \frac{1}{(t + a)} \int_{\mathbb{R}} \left( |\varphi_{1}| |\varphi_{2}| \right) \, dx \lesssim \frac{\|\vec{\varphi}\|_{H}^{2}}{L}.
\]

Next, using again the decay properties of \(Q\), (3.4) and the Cauchy-Schwarz inequality,
\[
|F_{n}^{2}| \lesssim \frac{1}{(t + a)} \sum_{n' = 1}^{N} \int_{\mathbb{R}} \left( |Q_{n'}| |\partial_{x} Q_{n''}| \right) \, dx \lesssim e^{-3\gamma_{0}(L + \gamma_{0}t)},
\]
\[
|F_{n}^{3}| \lesssim \frac{1}{(t + a)} \sum_{n' = 1}^{N} \int_{\mathbb{R}} \left( |\varphi_{2} Q_{n'}| + |\varphi_{1} \partial_{x} Q_{n''}| \right) \, dx \lesssim \|\vec{\varphi}\|_{H}^{3} + e^{-3\gamma_{0}(L + \gamma_{0}t)}.
\]

Gathering above estimates, we obtain (3.20).

**Step 6. Conclude.** First, we claim
\[
E(\vec{u}) + \sum_{n = 2}^{N} c_{n} \beta_{n} E_{n}(\vec{u}) = E_{1} + E_{2} + E_{3} + O \left( \|\vec{\varphi}\|_{H}^{3} + \|\vec{\varphi}\|_{H}^{q+1} \right) e^{-3\gamma_{0}(L + \gamma_{0}t)}.
\]

(3.21)
where
\[ E_1 = \sum_{n=1}^{N} \tilde{c}_n (1 - \ell_n^2)^{\frac{1}{2}} E(\tilde{Q}), \]
\[ E_2 = 2 \sum_{n=1}^{N} \tilde{c}_n \ell_n \int_{\mathbb{R}} (\partial_x Q_n)(\ell_n \partial_x Q_n + \ell_n \partial_x \varphi_1 - \varphi_2) \, dx, \]
\[ E_3 = \sum_{n=1}^{N} \tilde{c}_n \int_{\mathbb{R}} ((\partial_x \varphi_1)^2 + \varphi_1^2 + \varphi_2^2 - f'(Q_n) \varphi_1^2) \psi_n \, dx. \]

Indeed, from (3.13) and (3.19) and direct computation
\[
E(\tilde{u}) + \sum_{n=2}^{N} c_n \beta_n E_n(\tilde{u})
\]
\[ = \sum_{n=1}^{N} \tilde{c}_n (1 - \ell_n^2)^{\frac{1}{2}} E(\tilde{Q}) - \sum_{n=1}^{N} \left[ \int_{\mathbb{R}} (1 + \sum_{n'=2}^{n} c_{n'/\beta_n' \chi_{n'}}) f'(Q_n) \varphi_1^2 \, dx \right] + 2 \sum_{n=1}^{N} \tilde{c}_n \ell_n \int_{\mathbb{R}} (\partial_x Q_n)(\ell_n \partial_x Q_n + \ell_n \partial_x \varphi_1 - \varphi_2) \, dx \]
\[ + \int_{\mathbb{R}} \left( 1 + \sum_{n=1}^{N} c_n \beta_n \chi_n \right) ((\partial_x \varphi_1)^2 + \varphi_1^2 + \varphi_2^2) \, dx + O \left( \| \varphi \|_{H^1}^3 + \| \varphi \|_{H^1}^{\gamma+1} + e^{-3\gamma_0(L+\gamma t)} \right). \]

Observe that, from (3.33) and the definition of \( \tilde{c}_n \),
\[ \int_{\mathbb{R}} \left( 1 + \sum_{n=1}^{N} c_n \beta_n \chi_n \right) ((\partial_x \varphi_1)^2 + \varphi_1^2 + \varphi_2^2) \, dx = \sum_{n=1}^{N} \tilde{c}_n \int_{\mathbb{R}} ((\partial_x \varphi_1)^2 + \varphi_1^2 + \varphi_2^2) \psi_n \, dx, \]
and
\[ \sum_{n=1}^{N} \left[ \int_{\mathbb{R}} (1 + \sum_{n'=2}^{n} c_{n'/\beta_n' \chi_{n'}}) f'(Q_n) \varphi_1^2 \, dx \right] \]
\[ = \sum_{n=1}^{N} \tilde{c}_n \int_{\mathbb{R}} f'(Q_n) \varphi_1^2 \psi_n \, dx + \sum_{n=1}^{N} \sum_{n'/n} \left[ \sum_{k=2}^{(n')^+} (c_k \beta_k) \int_{\mathbb{R}} f'(Q_n) \varphi_1^2 \psi_{n'} \, dx \right], \]
where \( (n')^+ = \min(n' - 1, n) \). Note that, by the decay properties of \( Q \) and the Cauchy-Schwarz inequality, for any \( n \neq n' \),
\[ \left| \sum_{k=2}^{(n')^+} (c_k \beta_k) \right| \int_{\mathbb{R}} f'(Q_n) \varphi_1^2 \psi_{n'} \, dx \leq \| \varphi \|_{H^1}^3 + e^{-3\gamma_0(L+\gamma t)}. \]

We see that (3.21) follows from combining these identities and estimates.

Second, we claim
\[ c_1 I + \sum_{n=2}^{N} c_n I_n \]
\[ = -2 \sum_{n=1}^{N} \left[ \left( \sum_{n'=1}^{n} c_{n'} \right) \int_{\mathbb{R}} (\partial_x Q_n)(\ell_n \partial_x Q_n + \ell_n \partial_x \varphi_1 - \varphi_2) \psi_n \, dx \right] + 2 \sum_{n=1}^{N} \left[ \left( \sum_{n'=1}^{n} c_{n'} \right) \int_{\mathbb{R}} (\partial_x \varphi_1) \varphi_2 \psi_n \, dx \right] + O \left( \frac{\| \varphi \|_{L^2}^2}{L} + \| \varphi \|_{H^1}^3 + e^{-3\gamma_0(L+\gamma t)} \right). \]
(3.22)
Indeed, from (3.14) and (3.15),
\[ c_1 \mathcal{I} + \sum_{n=2}^{N} c_n \mathcal{I}_n = -2 \sum_{n=1}^{N} \int_{\mathbb{R}} \left( 1 + \sum_{n'=2}^{N} c_{n'} \chi_{n'} \right) (\partial_x Q_n)(\ell_n \partial_x Q_n + \ell_n \partial_x \varphi_1 - \varphi_2) \, dx \]
\[ + 2 \int_{\mathbb{R}} \left( c_1 + \sum_{n=2}^{N} c_n \chi_n \right) (\partial_x \varphi_1) \varphi_2 \, dx + O \left( \frac{\| \varphi \|^2_H}{L} + \| \varphi \|^3_H + e^{-3\gamma_0(L+\gamma_0 t)} \right) . \]
Using again (3.3), we obtain (3.22).

Next, using the standard localized argument, we obtain the following coercivity result.

**Lemma 3.3 (Coercivity).** There exist $0 < \mu \ll 1$ such that
\[ \mu \| \varphi \|^2_H - \mu^{-1} \left( \sum_{n=1}^{N} (a_n^-)^2 + \sum_{n=1}^{N} (\alpha_n^+)^2 \right) + O(e^{-4\gamma_0(L+\gamma_0 t)}) \leq \sum_{n=1}^{N} c_n H_n(\varphi, \varphi). \] (3.23)

**Proof.** First, from (3.3), we have $c_n$ is positive for $n = 1, \ldots, N$. Second, we obtain the localized coercivity of $H_n(\varphi, \varphi)$ from the coercivity property (2.2) around one solitary wave with the orthogonality properties (3.6). Last, we conclude the coercivity (3.23) by an elementary localization argument. See e.g. [13, Appendix B].

Last, we prove the following almost monotonicity property for $\mathcal{J}_n$ by virial argument.

**Lemma 3.4.** There exist $C_1$ such that for any $n = 1, \ldots, N, t \in [0, T_n]$,
\[ \mathcal{J}_n(t) - \mathcal{J}_n(0) \leq \frac{C_1}{L^{2\alpha-2}} \sup_{s \in [0, t]} \| \varphi(s) \|^2_H + C_1 e^{-3\gamma_0 L}. \] (3.24)

**Proof.** Step 1. Time variation of $\mathcal{I}_n$. We claim
\[ \frac{d}{dt} \mathcal{I}_n = -2 \int_{\mathbb{R}} (\partial_x u_1 + \beta_n u_2)^2 \partial_x \chi_n \, dx + (1 - \beta_n^2) \int_{\mathbb{R}} (u_1 f(u_1) - 2F(u_1)) \partial_x \chi_n \, dx \]
\[ + \beta_n^2 \int_{\mathbb{R}} (\partial_x u_1)^2 + u_1^2 + u_2^2 \partial_x \chi_n \, dx + 2 \beta_n \int_{\mathbb{R}} (\partial_x u_1) u_2 \partial_x \chi_n \, dx \]
\[ - 2\beta_n^2 \int_{\mathbb{R}} F(u_1) \partial_x \chi_n \, dx - \frac{\alpha}{(t+a)^{1-\alpha}} \int_{\mathbb{R}} (2(\partial_x u_1) u_2) \left( \frac{x - \beta_n t - y_n^0}{(t+a)\alpha} \right) \partial_x \chi_n \, dx \]
\[ + O \left( \frac{1}{(t+a)^{2\alpha}} \| \varphi \|^2_H + e^{-3\gamma_0(L+\gamma_0 t)} \right) . \] (3.25)

First, using (1.2) and integrating by parts
\[ \frac{d}{dt} \int_{\mathbb{R}} 2 (\chi_n \partial_x u_1) u_2 \, dx = - \int_{\mathbb{R}} (\partial_x u_1)^2 - u_1^2 + u_2^2 + 2F(u_1) \partial_x \chi_n \, dx \]
\[ + 2 \int_{\mathbb{R}} (\partial_x \chi_n)(\partial_x u_1) u_2 \, dx. \]

Observe that
\[ \partial_t \chi_n = -\beta_n \partial_x \chi_n - \frac{\alpha}{(t+a)^{1-\alpha}} \left( \frac{x - \beta_n t - y_n^0}{(t+a)\alpha} \right) \partial_x \chi_n. \] (3.26)
Therefore,

\[
\frac{d}{dt} \int_{\mathbb{R}} 2(\chi_n \partial_x u_1) u_2 \, dx = \int_{\mathbb{R}} (\partial_x u_1)^2 - u_1^2 + u_2^2 + 2\beta_n (\partial_x u_1) u_2 \partial_x \chi_n \, dx \\
- \frac{\alpha}{(t + \alpha)^{1-\alpha}} \int_{\mathbb{R}} (2(\partial_x u_1) u_2) \left( \frac{x - \beta_n t - y_0}{(t + \alpha)^{1-\alpha}} \right) \partial_x \chi_n \, dx \\
- 2 \int_{\mathbb{R}} F(u_1) \partial_x \chi_n \, dx
\]

and integrating by parts,

\[
\frac{d}{dt} \int_{\mathbb{R}} u_1 u_2 \partial_x \chi_n \, dx = \int_{\mathbb{R}} - (\partial_x u_1)^2 - u_1^2 + u_2^2 + u_1 f(u_1) \partial_x \chi_n \, dx \\
+ \int_{\mathbb{R}} (u_1 u_2) \partial_x \chi_n \, dx + \frac{1}{2} \int_{\mathbb{R}} u_1^2 \partial_x^2 \chi_n \, dx.
\]

From (2.20), (3.30) and the decay properties of \(Q\),

\[
\left| \int_{\mathbb{R}} (u_1 u_2) \partial_x \chi_n \, dx \right| \lesssim \frac{1}{(t + \alpha)^{2\alpha}} \int_{\mathbb{R}} \left( \sum_{n' = 1}^{N} (Q_{n'})^2 + \sum_{n' = 1}^{N} (\partial_x Q_{n'})^2 + \varphi_1^2 + \varphi_2^2 \right) \Omega_n \, dx \\
\lesssim \frac{1}{(t + \alpha)^{2\alpha}} \| \nabla \|^2_{\mathcal{H}} + e^{-\gamma_0 (L + \gamma_0 t)},
\]

\[
\left| \int_{\mathbb{R}} u_1^2 \partial_x^2 \chi_n \, dx \right| \lesssim \frac{1}{(t + \alpha)^{2\alpha}} \int_{\mathbb{R}} \left( \sum_{n' = 1}^{N} (Q_{n'})^2 + \sum_{n' = 1}^{N} (\partial_x Q_{n'})^2 + \varphi_1^2 \right) \Omega_n \, dx \\
\lesssim \frac{1}{(t + \alpha)^{2\alpha}} \| \nabla \|^2_{\mathcal{H}} + e^{-\gamma_0 (L + \gamma_0 t)}.
\]

It follows that,

\[
\frac{d}{dt} \int_{\mathbb{R}} u_1 u_2 \partial_x \chi_n \, dx = \int_{\mathbb{R}} - (\partial_x u_1)^2 - u_1^2 + u_2^2 + u_1 f(u_1) \partial_x \chi_n \, dx \\
+ O\left( \frac{1}{(t + \alpha)^{2\alpha}} \| \nabla \|^2_{\mathcal{H}} + e^{-\gamma_0 (L + \gamma_0 t)} \right) \tag{3.28}
\]

Gathering estimates \(3.27\) and \(3.28\), we obtain \(3.25\).

**Step 2.** Time variation of \(E_n\). We claim

\[
\frac{d}{dt} E_n = - 2 \int_{\mathbb{R}} ((\partial_x u_1) u_2) \partial_x \chi_n \, dx - \beta_n \int_{\mathbb{R}} ((\partial_x u_1)^2 + u_1^2 + u_2^2) \partial_x \chi_n \, dx \\
+ 2\beta_n \int_{\mathbb{R}} F(u_1) \partial_x \chi_n \, dx + \frac{2\alpha}{(t + \alpha)^{1-\alpha}} \int_{\mathbb{R}} F(u_1) \left( \frac{x - \beta_n t - y_0}{(t + \alpha)^{1-\alpha}} \right) \partial_x \chi_n \, dx \\
- \frac{\alpha}{(t + \alpha)^{1-\alpha}} \int_{\mathbb{R}} ((\partial_x u_1)^2 + u_1^2 + u_2^2) \left( \frac{x - \beta_n t - y_0}{(t + \alpha)^{1-\alpha}} \right) \partial_x \chi_n \, dx.
\]

First, from (1.9),

\[
\partial_t ((\partial_x u_1)^2 + u_1^2 + u_2^2 - 2F(u_1)) = 2\partial_x (u_2 \partial_x u_1) = 2(\partial_x u_1) \partial_x u_2 + 2(\partial_x^2 u_1) u_2.
\]

Therefore, by integration by parts,

\[
\int_{\mathbb{R}} (\partial_t ((\partial_x u_1)^2 + u_1^2 + u_2^2 - 2F(u_1))) \chi_n \, dx = - 2 \int_{\mathbb{R}} ((\partial_x u_1) u_2) \partial_x \chi_n \, dx.
\]
Second, from (3.20),
\[
\int_{\mathbb{R}} \left( (\partial_x u_1)^2 + u_1^2 + u_2^2 - 2F(u_1) \right) \partial_x \chi_n \, dx \\
= -\beta_n \int_{\mathbb{R}} \left( (\partial_x u_1)^2 + u_1^2 + u_2^2 - 2F(u_1) \right) \partial_x \chi_n \, dx \\
- \frac{\alpha}{(t+a)^{1-\alpha}} \int_{\mathbb{R}} \left( (\partial_x u_1)^2 + u_1^2 + u_2^2 - 2F(u_1) \right) \left( x - \beta_n t - y_n^0 \right) \frac{1}{(t+a)^\alpha} \partial_x \chi_n \, dx.
\]

Gathering these identities, we obtain (3.29).

Step 3. Time variation of $F_n$. We claim
\[
\frac{d}{dt} F_n = -\frac{\alpha}{(t+a)^{1-\alpha}} \int_{\mathbb{R}} \left[ - (\partial_x u_1)^2 - u_1^2 + u_2^2 + u_1 f(u_1) \right] \left( x - \beta_n t - y_n^0 \right) \frac{1}{(t+a)^\alpha} \partial_x \chi_n \, dx \\
+ O \left( \frac{1}{(t+a)^{1+\alpha}} \| \mathcal{F} \|_H^2 + e^{-3\gamma t} \right).
\]

(3.30)

First, from (122),
\[
\partial_t (u_1 u_2) = u_2^2 - u_1^2 + u_1 f(u_1) + u_1 (\partial_x^2 u_1).
\]

Therefore, by integration by parts,
\[
\int_{\mathbb{R}} \left[ \partial_t (u_1 u_2) \right] \left( x - \beta_n t - y_n^0 \right) \frac{1}{(t+a)^\alpha} \partial_x \chi_n \, dx \\
= \frac{1}{(t+a)^{1-\alpha}} \int_{\mathbb{R}} \left( (\partial_x u_1)^2 - u_1^2 + u_2^2 + u_1 f(u_1) \right) \left( x - \beta_n t - y_n^0 \right) \frac{1}{(t+a)^\alpha} \partial_x \chi_n \, dx \\
+ \frac{1}{t+a} \int_{\mathbb{R}} u_1^2 \partial_x^2 \chi_n \, dx + \frac{1}{2(t+a)^{1-\alpha}} \int_{\mathbb{R}} u_1^2 \left( x - \beta_n t - y_n^0 \right) \frac{1}{(t+a)^\alpha} \partial_x \chi_n \, dx.
\]

Observe that, from (3.5)
\[
\left| \frac{1}{t+a} \int_{\mathbb{R}} u_1^2 \partial_x^2 \chi_n \, dx \right| \lesssim \frac{1}{(t+a)^{1+\alpha}} \int_{\mathbb{R}} \left( \sum_{n=1}^N Q_n^2 + \varphi_n^2 \right) 1_{\Omega_n} \, dx \\
\lesssim \frac{1}{(t+a)^{2\alpha}} \| \mathcal{F} \|_H^2 + e^{-3\gamma t}.
\]

and
\[
\left| \frac{1}{2(t+a)^{1-\alpha}} \int_{\mathbb{R}} u_1^2 \left( x - \beta_n t - y_n^0 \right) \frac{1}{(t+a)^\alpha} \partial_x \chi_n \, dx \right| \\
\lesssim \frac{1}{(t+a)^{2\alpha}} \int_{\mathbb{R}} \sum_{n=1}^N Q_n^2 + \varphi_n^2 \, dx \lesssim \frac{1}{(t+a)^{2\alpha}} \| \mathcal{F} \|_H^2 + e^{-3\gamma t}.
\]

It follows that
\[
\int_{\mathbb{R}} \left[ \partial_t (u_1 u_2) \right] \left( x - \beta_n t - y_n^0 \right) \frac{1}{(t+a)^\alpha} \partial_x \chi_n \, dx \\
= \frac{1}{(t+a)^{1-\alpha}} \int_{\mathbb{R}} \left( (\partial_x u_1)^2 - u_1^2 + u_2^2 + u_1 f(u_1) \right) \left( x - \beta_n t - y_n^0 \right) \frac{1}{(t+a)^\alpha} \partial_x \chi_n \, dx \\
+ O \left( \frac{1}{(t+a)^{2\alpha}} \| \mathcal{F} \|_H^2 + e^{-3\gamma t} \right).
\]

(3.31)
Indeed, by the Cauchy-Schwarz inequality,

\[
\partial_t \left( \frac{x - \beta_n t - y_n^0}{t + a} \right) \partial_x \chi_n = -\frac{\beta_n}{(t + a)} \partial_x \chi_n - \frac{1}{(t + a)^{2-\alpha}} \left( \frac{x - \beta_n t - y_n^0}{(t + a)\alpha} \right) \partial_x \chi_n
\]

\[
-\frac{\beta_n}{(t + a)^{2-\alpha}} \left( \frac{x - \beta_n t - y_n^0}{(t + a)\alpha} \right)^2 \partial_x^2 \chi_n
\]

Therefore, from (3.4) and (3.5),

\[
\text{We see that (3.30) follows from (3.31) and (3.32).}
\]

It follows that,

\[
\left| \partial_t \left( \frac{x - \beta_n t - y_n^0}{t + a} \right) \partial_x \chi_n \right| \lesssim \frac{1}{(t + a)^{1+\alpha}} 1_{\Omega_n}.
\]

We see that (3.30) follows from (3.31) and (3.32).

**Step 4.** Conclude. Note that from (3.25), (3.29) and (3.30),

\[
\frac{d}{dt} F_n - \frac{d}{dt} F_n + \partial_n \frac{d}{dt} F_n + \partial_n \frac{d}{dt} F_n = F_1 + F_2 + O \left( \frac{1}{(t + a)^{2\alpha}} \| \varphi \|_{L^2}^2 + e^{-3\gamma(L+\gamma t)} \right).
\]

where

\[
F_1 = -2 \int_R \partial_x u_1 + \beta_n u_2 \right)^2 \partial_x \chi_n \, dx
\]

\[
- \frac{\alpha}{(t + a)^{1+\alpha}} \int_R 2u_2 (\partial_x u_1 + \beta_n u_2) \left( \frac{x - \beta_n t - y_n^0}{(t + a)\alpha} \right) \partial_x \chi_n \, dx,
\]

and

\[
F_2 = \int_R u_1 f(u_1) - 2F(u_1) \right) \left[ 1 - \beta_n^2 - \frac{\alpha \beta_n}{(t + a)^{1+\alpha}} \left( \frac{x - \beta_n t - y_n^0}{(t + a)\alpha} \right) \right] \partial_x \chi_n \, dx.
\]

Estimates of $F_1$. We claim

\[
F_1 \leq \frac{1}{(t + a)^{2\alpha}} \| \varphi \|_{L^2}^2 + e^{-3\gamma(L+\gamma t)}.
\]

Indeed, by the Cauchy-Schwarz inequality,

\[
\frac{\alpha}{(t + a)^{1+\alpha}} \int_R 2u_2 (\partial_x u_1 + \beta_n u_2) \left( \frac{x - \beta_n t - y_n^0}{(t + a)\alpha} \right) \partial_x \chi_n \, dx
\]

\[
\lesssim \frac{1}{(t + a)^{2-\alpha}} \int_R u_2^2 \partial_x \chi_n \, dx + \frac{1}{(t + a)^{1+\alpha}} \int_R (\partial_x u_1 + \beta_n u_2)^2 \partial_x \chi_n \, dx.
\]

Moreover, using (3.33), the decay properties of $Q$,

\[
\left| \frac{1}{(t + a)^{2-\alpha}} \int_R u_2^2 \partial_x \chi_n \, dx \right| \lesssim \frac{1}{(t + a)^{2-\alpha}} \int_R \left( \sum_{n'=1}^N (\partial_x Q_{n'})^2 + \varphi_2^2 \right) 1_{\Omega_n} \, dx
\]

\[
\lesssim \frac{1}{(t + a)^{2-\alpha}} \| \varphi \|_{L^2}^2 + e^{-4\gamma(L+\gamma t)}.
\]
Thus, from $\frac{1}{4} < \alpha < \frac{2}{7}$ and taking $L$ large enough,
\[
\mathcal{F}_1 \leq -2 \int_R (\partial_x u_1 + \beta_n u_2) \partial_x \chi_n dx + \frac{1}{L^2} \int_R (\partial_x u_1 + \beta_n u_2)^2 \partial_x \chi_n dx + \frac{1}{(t + a)^{2\alpha}} \|\tilde{\varphi}\|^2_H + e^{-3\gamma_0 (L + \gamma_0 t)},
\]
which implies (3.33).

Estimates of $\mathcal{F}_2$. We claim
\[
\mathcal{F}_2 \leq 0.
\]
First, observe that, for $L$ large enough,
\[
1 - \beta_n^2 - \frac{\alpha \beta_n}{(t + a)^{2\alpha}} \left( \frac{x - \beta_n t - y_n^0}{(t + a)^\alpha} \right) \geq 1 - \beta_n^2 - \frac{10}{L} \left| \frac{\alpha \beta_n}{L} \right| > 0
\]
Second, from the decay properties of $Q$ and $\tilde{\varphi}_n$, for $x \in \Omega_n$,
\[
|u_1(t, x)| \leq \sum_{n=1}^N |Q_n(t, x)| \Omega_n + |\varphi_1(t, x)| \lesssim e^{-3\gamma_0 (L + \gamma_0 t)} + \|\tilde{\varphi}\|^2_H \lesssim e^{-\gamma_0 L + \delta}.
\]
Therefore, for $L$ large enough and $\delta$ small enough, we obtain for $x \in \Omega_n$,
\[
u_1 f(u_1) - 2F(u_1) = -\frac{q - 1}{q + 1} |u_1|^{q+1} + \frac{p - 1}{p + 1} |u_1|^{p+1} \leq 0,
\]
since $1 < q < p < \infty$. We conclude (3.34) from above estimates.

Gathering estimates (3.33) and (3.34), we obtain
\[
\frac{d}{dt} \mathcal{J}_n(t) \leq \frac{1}{(t + a)^{2\alpha}} \|\tilde{\varphi}(t)\|^2_H + e^{-3\gamma_0 (L + \gamma_0 t)}.
\]
Integrating on $[0, t]$ for any $t \in [0, T^*)$, we obtain
\[
\mathcal{J}_n(t) - \mathcal{J}_n(0) \leq \frac{1}{L^{2\alpha - 1}} \max_{s \in [0, t]} \|\tilde{\varphi}(s)\|^2_H + e^{-\gamma_0 L},
\]
which implies (3.34).

\section*{4. Proof of Theorem 1.1}

In this section, we prove Theorem 1.1 using a bootstrap argument. We start with a technical result that will allow us to adjust the initial value with $N$ free parameters.

**Lemma 4.1** (Adjusting the initial unstable modes). Let $N \geq 2$. For $n \in \{1, \ldots, N\}$, let $\sigma_n = \pm 1$ and $-1 < \ell_n < 1$ with $-1 < \ell_1 < \ell_2 < \cdots < \ell_N < 1$. There exist $L_0 \gg 1$ and $0 < \delta_0 \ll 1$ such that the following is true. Let $y^0 = (y_n^0)_{n \in \{1, \ldots, N\}} \in \mathbb{R}^N$ be such that
\[
L = \min(y_{n+1}^0 - y_n^0, n = 1, \ldots, N - 1) > L_0,
\]
and $\tilde{\varepsilon} \in H^1 \times L^2$, $\alpha^+ = (a_n^+)_{n \in \{1, \ldots, N\}} \in \mathbb{R}^N$ be such that
\[
\|\tilde{\varepsilon}\| < \delta < \delta_0 \quad \text{and} \quad \alpha^+ \in \tilde{B}_{\mathbb{R}^N}(r) \quad \text{where} \quad r = C_0 \left( \delta^2 + e^{-2\gamma_0 L} \right)^{\frac{1}{2}},
\]
$C_0$ is defined in the bootstrap (3.1) and to be taken large enough. Then, there exist $h^+ = (h_n^+)_{n \in \{1, \ldots, N\}}$ and $\tilde{y}^0 = (\tilde{y}_n^0)_{n \in \{1, \ldots, N\}}$ satisfying
\[
\sum_{n=1}^N (|h_n^+| + |\tilde{y}_n^0 - y_n^0|) \leq C_0 \left( \delta + e^{-\gamma_0 L} \right)
\]
such that the initial value defined by
\[
\tilde{u}_0 = \sum_{n=1}^N \left( \sigma_n Q_{\ell_n} + h_n^+ \tilde{Z}_{\ell_n}^+ \right) (- y_n^0) + \tilde{\varepsilon}
\]
Moreover, the initial data \( \vec{u}_0 \) satisfies for all \( n = 1, \cdots, N \),

\[ \left( \vec{v}(0), Z_{\ell_1}^0(-\vec{y}_n^0) \right)_{L^2} = 0, \quad a_n^+(0) = \left( \vec{v}(0), \tilde{Z}_{\ell_1}^+(\cdot - \vec{y}_n^0) \right)_{L^2} = a_n^+. \]  

Moreover, the initial data \( \vec{u}_0 \) is modulated in the sense of Lemma \( \text{[1.2]} \) with \( y_n(0) = \vec{y}_n^0 \), for all \( n = 1, \cdots, N \).

Proof. Let

\[ \Gamma_0 = (0, y_1^0, \cdots, y_N^0) \in \mathbb{R}^{2N}, \quad \Gamma = (h_1^+, \cdots, h_N^+, \vec{y}_1^0, \cdots, \vec{y}_N^0) \in \mathbb{R}^{2N}. \]

Consider the map

\[ \Psi : \quad X \to \mathbb{R}^{2N} \]

\[ (\vec{z}, \vec{a}^+, \vec{\Gamma}) \mapsto (\Psi_n^a, \cdot, \Psi_N^0) \]

where \( X = (H^1 \times L^2) \times \mathbb{R}^N \times \mathbb{R}^{2N} \), and for \( n = 1, \cdots, N \),

\[ \Psi_n^a = \sum_{n' = 1}^{N} \sigma_{n'} \left( \left( \vec{Q}_{\ell_n}(-y_{n'}^0) - \vec{Q}_{\ell_n}(-\vec{y}_{n'}^0) \right), \tilde{Z}_{\ell_n}^+(\cdot - \vec{y}_{n'}^0) \right)_{L^2} + \sum_{n' = 1}^{N} h_{n'}^+ \left( \tilde{Z}_{\ell_n}^+(\cdot - y_{n'}^0), \tilde{Z}_{\ell_n}^+(\cdot - \vec{y}_{n'}^0) \right)_{L^2} - a_n^+. \]

\[ \Psi_n^0 = \sum_{n' = 1}^{N} \sigma_{n'} \left( \left( \vec{Q}_{\ell_n}(-y_{n'}^0) - \vec{Q}_{\ell_n}(-\vec{y}_{n'}^0) \right), \tilde{Z}_{\ell_n}^0(\cdot - \vec{y}_{n'}^0) \right)_{L^2} + \sum_{n' = 1}^{N} h_{n'}^+ \left( \tilde{Z}_{\ell_n}^0(\cdot - y_{n'}^0), \tilde{Z}_{\ell_n}^0(\cdot - \vec{y}_{n'}^0) \right)_{L^2} + \left( \vec{z}, \tilde{Z}_{\ell_n}^0(\cdot - \vec{y}_{n'}^0) \right)_{L^2}. \]

From \( \text{[1.2]} \),

\[ \vec{v}(0) = \sum_{n = 1}^{N} \sigma_n \left( \vec{Q}_{\ell_n}(-y_n^0) - \vec{Q}_{\ell_n}(-\vec{y}_n^0) \right) + \sum_{n = 1}^{N} h_n^+ \tilde{Z}_{\ell_n}(\cdot - y_n^0) + \vec{z}, \]

and thus the set of conditions in \( \text{[1.3]} \) is equivalent to \( \Psi (\vec{z}, \vec{a}^+, \vec{\Gamma}) = 0 \in \mathbb{R}^{2N} \). We solve this nonlinear system by the Implicit Function Theorem. First, it is easy to check that

\[ \Psi(\vec{0}, 0, \Gamma_0) = 0. \]

Second, by direct computation and integration by parts,

\[ D_{\vec{\Gamma}} \Psi(\vec{z}, \vec{a}^+, \vec{\Gamma}) = \begin{pmatrix} A & C \\ B & D \end{pmatrix}, \]

where

\[ A = \left( \left( \tilde{Z}_{\ell_n}^+(\cdot - y_n^0), \tilde{Z}_{\ell_n}^+(\cdot - \vec{y}_n^0) \right)_{L^2} \right)_{n,n' \in \{1, \cdots, N\}} + O \left( \| \vec{z} \|_{\mathcal{H}} + \| \vec{\Gamma} - \Gamma_0 \| \right), \]

\[ B = \left( \left( \tilde{Z}_{\ell_n}^0(\cdot - y_n^0), \tilde{Z}_{\ell_n}^0(\cdot - y_n^0) \right)_{L^2} \right)_{n,n' \in \{1, \cdots, N\}} + O \left( \| \vec{z} \|_{\mathcal{H}} + \| \vec{\Gamma} - \Gamma_0 \| \right), \]

\[ C = \left( \left( \tilde{Z}_{\ell_n}^+(\cdot - y_n^0), \tilde{Z}_{\ell_n}^+(\cdot - y_n^0) \right)_{L^2} \right)_{n,n' \in \{1, \cdots, N\}} + O \left( \| \vec{z} \|_{\mathcal{H}} + \| \vec{\Gamma} - \Gamma_0 \| \right), \]

\[ D = \left( \left( \tilde{Z}_{\ell_n}^0(\cdot - y_n^0), \tilde{Z}_{\ell_n}^0(\cdot - y_n^0) \right)_{L^2} \right)_{n,n' \in \{1, \cdots, N\}} + O \left( \| \vec{z} \|_{\mathcal{H}} + \| \vec{\Gamma} - \Gamma_0 \| \right). \]
Moreover, from \( \left( \tilde{Z}_l^0, \tilde{Z}_l^+ \right) \equiv 0 \) and (2.10), we obtain
\[
B = O \left( \|\bar{\varepsilon}\|_H + |\Gamma - \Gamma^0| + e^{-3\gamma L} \right),
\]
\[
C = O \left( \|\bar{\varepsilon}\|_H + |\Gamma - \Gamma^0| + e^{-3\gamma L} \right),
\]
\[
A = \text{diag} \left( \left( \tilde{Z}_l^0, \tilde{Z}_l^+ \right) \right), \quad O \left( \|\bar{\varepsilon}\|_H + |\Gamma - \Gamma^0| + e^{-3\gamma L} \right),
\]
\[
D = \text{diag} \left( \left( \tilde{Z}_l^0, \tilde{Z}_l^0 \right) \right), \quad O \left( \|\bar{\varepsilon}\|_H + |\Gamma - \Gamma^0| + e^{-3\gamma L} \right).
\]

Thus, \( D \Psi(\vec{0}, 0, \Gamma^0) \) is an invertible matrix for \( L > L_0 \) large enough, with a lower bound uniform around \( \left( \vec{0}, 0, \Gamma^0 \right) \). Therefore, by the uniform variant of the implicit function theorem, there exist \( 0 < \delta_1 \ll 1 \) and \( 0 < \delta_2 \ll 1 \) (independent with the choose of \( y^0 = (y^0_n)_{n \in \{1, \ldots, N\}} \)) and continuous map
\[
\Pi : B_{H^1 \times L^2}(\vec{0}, \delta_1) \times B_{\mathbb{R}^N}(0, \delta_1) \to B_{\mathbb{R}^N}(\Gamma^0, \delta_2),
\]
such that for all \( \bar{\varepsilon} \in B_{H^1 \times L^2}(\vec{0}, \delta_1) \), \( a^+ \in B_{\mathbb{R}^N}(0, \delta_1) \) and \( \Gamma \in B_{\mathbb{R}^N}(\Gamma^0, \delta_2) \),
\[
\Psi(\bar{\varepsilon}, a^+, \Gamma) = 0 \quad \text{if and only if} \quad \Gamma = \Pi(\bar{\varepsilon}, a^+).
\]
Moreover, taking \( 0 < \delta < \delta_0 \ll 1 \) small enough, \( C_0 \) and \( L > L_0 \) large enough,
\[
|\Gamma - \Gamma^0| \lesssim \|\bar{\varepsilon}\|_{H^1 \times L^2} + \|a^+\| \lesssim \delta + C_0^\frac{1}{n+1} \left( \delta^2 + e^{-2\gamma L} \right)^\frac{n}{2},
\]
which implies (4.1). The proof of Lemma 4.1 is complete.

Now, we start the proof of Theorem 1.1

**Proof of Theorem 1.1** Let \( \bar{\varepsilon} \in H^1 \times L^2 \) and \( y^0 = (y^0_n)_{n \in \{1, \ldots, N\}} \) as in the statement of the theorem. For all \( a^+(0) = a^+ = (a^+_1, \ldots, a^+_N) \in B_{\mathbb{R}^N}(r) \), we consider the solution \( \vec{u} = (u_1, u_2) \) with the initial data as defined in Lemma 4.1
\[
\vec{u}_0 = \sum_{n=1}^N \left( \sigma_n \vec{Q}_{\ell_n} + h_n^+ \vec{Z}_n \right) \vec{y}_n^0 + \tilde{\varepsilon},
\]
\[
\vec{u}_0 = \sum_{n=1}^N \left( \sigma_n \vec{Q}_{\ell_n} + h_n^+ \vec{Z}_n \right) \vec{y}_n^0 + \tilde{\varepsilon} = \sum_{n=1}^N \sigma_n \vec{Q}_{\ell_n} (\vec{y}_n^0 - \vec{y}_n^0) + \tilde{\varepsilon}.
\]
Note that, for the proof of Theorem 1.1, we just need to prove the existence of \( \vec{u}_0 \) such that \( T_n(\vec{u}_0) = \infty \). We start closing estimates except for the instable modes. Last, we will prove the existence of suitable parameters \( a^+ = (a^+_1, \ldots, a^+_N) \) by contradiction and a topology argument.

**Step 1.** Closing the estimates in \( \bar{\varepsilon} \). First, from (4.1) and (4.3),
\[
\|\tilde{\varepsilon}\|_H \lesssim \sum_{n=1}^N \left( |h_n^+| + |y_n^0 - y_n^0| \right) \lesssim C_0^\frac{1}{n+1} \left( \delta + e^{-2\gamma L} \right).
\]

Second, from (3.24) and the energy \( E(\vec{u}(t)) \) and momentum \( \mathcal{I}(\vec{u}(t)) \) are conserved,
\[
E(\vec{u}(t)) - E(\vec{u}(0)) \leq C_1 \left( \sum_{n=2}^N c_n \right) \left( \frac{1}{L^{2\alpha - 1}} \sup_{s \in [0, t]} \|\tilde{\varepsilon}(s)\|_H^2 + e^{-3\gamma(L + \gamma t)} \right).
\]
Note that, from (3.12) and (5.1) and the choose of initial data,
\[
\sum_{n=1}^N \tilde{c}_n \mathcal{H}_n(\bar{\varepsilon}(t), \bar{\varepsilon}(t)) = E(\vec{u}(t)) - E(\vec{u}(0)) + O \left( \frac{\|\tilde{\varepsilon}\|_H^2}{L} + \|\tilde{\varepsilon}(t)\|_H^{q_0 + 1} + e^{-3\gamma(L + \gamma t)} + \|\tilde{\varepsilon}(t)\|_H^2 \right).
\]
where \( q_0 = \min(2, q) \). Therefore, using (3.1), (3.23) and (4.1), we obtain

\[
\mu \| \tilde{\varphi}(t) \|_{L^2}^2 \leq \sum_{n=1}^{N} c_n H_n(\tilde{\varphi}(t), \tilde{\varphi}(t)) + \mu^{-1} \left( \sum_{n=1}^{N} (a_n^+(t))^2 + \sum_{n=1}^{N} (a_n^-(t))^2 \right) + e^{-3\gamma_0 L} \\
\leq \frac{1}{L^{2\alpha-1}} \sup_{s \in [0,t]} \| \tilde{\varphi}(s) \|_{L^2}^2 + \frac{1}{L} \| \tilde{\varphi}(t) \|_{L^2}^2 + \| \tilde{\varphi}(t) \|_{H^\gamma}^2 + C_0 \left( \delta^2 + e^{-2\gamma_0 L} \right) \\
+ C_0^2 \left( \delta^2 + e^{-2\gamma_0 L} \right) + \| \tilde{\varphi}(0) \|_{L^2}^2 \lesssim \left( \frac{C_0^2}{L^{2\alpha-1}} + \frac{C_0^2}{L} + C_0^2 \right) \left( \delta^2 + e^{-2\gamma_0 L} \right),
\]

which strictly improves the estimate on \( \tilde{\varphi} \) in (3.1) for taking \( L, C_0 \) large enough and \( \delta \) small enough.

**Step 2.** Closing the estimates in \( y = (y_n)_{n \in \{1, \ldots, N\}} \). Note that, from (2.13) and (3.1), for \( n = 1, \ldots, N - 1 \),

\[
y_{n+1} - y_n = \ell_{n+1} - \ell_n + O \left( \delta + e^{-\gamma_0 L} \right) \geq 5\gamma_0,
\]

for \( \delta \) small enough and \( L \) large enough. Integrating on \([0,t]\) and using (4.1), we obtain

\[
y_{n+1}(t) - y_n(t) \geq 5\gamma_0 t + y_0^{n+1} - y_0^n + (\tilde{y}_{n+1} - \tilde{y}_n) - (\tilde{y}_n - y_0^n) \geq 5\gamma_0 t + (1 - C_0^{-1}) L,
\]

which strictly improves the estimate on \( y = (y_n)_{n \in \{1, \ldots, N\}} \) in (3.1).

**Step 3.** Closing the estimates in \( a^- = (a_n^-)_{n \in \{1, \ldots, N\}} \). Note that, from (4.13),

\[
\sum_{n=1}^{N} (a_n^-(0))^2 \lesssim \| \tilde{\varphi}(0) \|_{H^\gamma}^2 \lesssim C_0^{3/2} \left( \delta + e^{-\gamma_0 L} \right).
\]

By direct computation, (2.13) and (3.1), for \( n = 1, \ldots, N \),

\[
\frac{d}{dt} \left( e^{2\alpha_n t} (a_n^-(t))^2 \right) = 2 e^{2\alpha_n t} a_n^-(t) \left( \frac{d}{dt} a_n^-(t) + \alpha_n a_n^-(t) \right) = e^{2\alpha_n t} O \left( |a_n^-|^3 + \| \tilde{\varphi} \|_{H^\gamma}^2 + \sum_{n'=1}^{N} e^{-4(y_n^+ - y_{n'}^-)} \right) = e^{2\alpha_n t} O \left( C_0^3 \left( \delta^2 + e^{-3\gamma_0 L} \right) \right).
\]

Integrating on \([0,t]\), for any \( t \in [0, T_*] \) and any \( n = 1, \ldots, N \), we obtain

\[
(a_n^-(0))^2 - e^{-2\alpha_n t} (a_n^-(0))^2 \lesssim C_0^3 e^{-2\alpha_n t} \int_0^t e^{2\alpha_n s} \left( \delta^2 + e^{-3\gamma_0 L} \right) ds \lesssim C_0^3 \left( \delta^3 + e^{-3\gamma_0 L} \right),
\]

which strictly improves the estimate on \( a^- = (a_n^-)_{n \in \{1, \ldots, N\}} \) in (3.1).

**Step 4.** Final argument on the unstable parameters. Let

\[
b(t) = \sum_{n=1}^{N} (a_n^+(t))^2 \quad \text{and} \quad \bar{a}_n = \min_{n} a_n.
\]

Observe that for any time \( t \in [0, T_*] \) where it holds \( b(t) = C_0^3 \left( \delta^2 + e^{-2\gamma_0 L} \right) \), the following transversality property holds

\[
\frac{db}{dt}(t) = 2 \sum_{n=1}^{N} \alpha_n (a_n^+(t))^2 + O \left( \sum_{n=1}^{N} |a_n^-|^3 + \| \tilde{\varphi} \|_{H^\gamma}^2 + e^{-3\gamma_0 L} \right) \\
\geq 2\alpha C_0^3 \left( \delta^2 + e^{-2\gamma_0 L} \right) - \delta^2 e^{-\gamma_0 L} > 0,
\]

for \( \delta \) small enough and \( L \) large enough. This transversality relation is enough to justify the existence of at least a couple \( a^+(0) = (a_1^+(0), \ldots, a_N^+(0)) \in B_{2N}^0(r) \) such that \( T_* = \infty \) where \( r = C_0^3 \left( \delta^2 + e^{-2\gamma_0 L} \right)^{1/2} \).
The proof is by contradiction, we assume that for all $a^+(0) \in \bar{B}_R^N(r)$, it holds $T_* < \infty$. Then, a construction follows from the following discussion (see for instance more details in [5] and [6, Section 3.1]).

**Continuity of $T_*$**. The above transversality condition (4.6) implies that the map

$$a^+(0) \in \bar{B}_R^N(r) \mapsto T_* \in [0, \infty)$$

is continuous and

$$T_* = 0 \quad \text{for} \quad a^+(0) \in S^N_R(r).$$

**Construction of a retraction**. We define

$$M : \bar{B}_R^N(r) \mapsto S^N_R(r)$$

$$a^+(0) \mapsto a^+(T_*).$$

From what precedes, $M$ is continuous. Moreover, $M$ restricted to $S^N_R(r)$ is the identity. The existence of such a map is contradictory with the non-retraction theorem for continuous maps from the ball to the sphere.

**Step 5.** Conclude. At this point, we have proved the existence of $a^+(0) \in \bar{B}_R^N(r)$, associated with a global solution $\vec{u} = (u_1, u_2)$ of (1.2) with initial data defined in Lemma 4.1, which also satisfies (3.1) for all $t \in [0, \infty)$. The proof of Theorem 1.1 is complete. 

\[\square\]

**References**

[1] H. Berestycki and P.-L. Lions, Nonlinear scalar field equations. I. Existence of a ground state. *Arch. Rational Mech. Anal.* **82**, (1983) 313–345.

[2] V. S. Buslaev and G. S. Perelman, On the stability of solitary waves for nonlinear Schrödinger equations. *Nonlinear evolution equations*, 75–98, Amer. Math. Soc. Transl. Ser. 2, **164**, Adv. Math. Sci., 22, Amer. Math. Soc., Providence, RI, 1995.

[3] V. Combet, Multi-soliton solutions for the supercritical gKdV equations. *Comm. Partial Differential Equations* **36** (2011), no. 3, 380–419.

[4] R. Côte and Y. Martel, Multi-travelling waves for the nonlinear Klein-Gordon equation. *Trans. Amer. Math. Soc.* **370** (2018), no. 10, 7461–7487.

[5] R. Côte and F. Merle, Construction of multi-soliton solutions for the $L^2$-supercritical gKdV and NLS equations, *Rev. Mat. Iberoamericana* **27** (2011), 273-302.

[6] R. Côte and C. Muñoz, Multi-solitons for nonlinear Klein-Gordon equations. *Forum of Mathematics, Sigma* **2** (2014).

[7] J. Ginibre and G. Velo, The global Cauchy problem for the nonlinear Klein-Gordon equation. *Math. Z.* **189** (1985), no. 4, 487—505.

[8] S. Le Coz and Y. Wu, Stability of multisolitons for the derivative nonlinear Schrödinger equation. *Int. Math. Res. Not. IMRN* 2018, no. 13, 4120–4170.

[9] Y. Martel, Asymptotic $N$-soliton-like solutions of the subcritical and critical generalized Korteweg-de Vries equations. *Amer. J. Math.* **127** (2005), 1103–1140.

[10] Y. Martel and F. Merle, Multi-solitary waves for nonlinear Schrödinger equations, *Annales de l’IHP (C) Non Linear Analysis* **23** (2006), 849–864.

[11] Y. Martel and F. Merle, Construction of multi-solitons for the energy-critical wave equation in dimension 5. *Arch. Ration. Mech. Anal.* **222** (2016), no. 3, 1113–1160.

[12] Y. Martel, F. Merle and T.-P. Tsai, Stability and asymptotic stability in the energy space of the sum of $N$ solitons for subcritical gKdV equations. *Comm. Math. Phys.* **231** (2002), 347-373.

[13] Y. Martel, F. Merle and T.-P. Tsai, Stability in $H^1$ of the sum of $K$ solitary waves for some nonlinear Schrödinger equations. *Duke Math. J.* **133** (2006), no. 3, 405–466.

[14] C. Miao, X. Tang and G. Xu, Stability of the traveling waves for the derivative Schrödinger equation in the energy space. *Calc. Var. Partial Differential Equations* **56** (2017), no. 2, Art. 45, 48 pp.

[15] G. Perelman, Some results on the scattering of weakly interacting solitons for nonlinear Schrödinger equations. *Spectral Theory, Microlocal Analysis, Singular Manifolds*, 78-137, Math. Top., 14, Adv. Partial Differential Equations, Akademie Verlag, Berlin, 1997.

[16] X. Yuan, On multi-solitons for the energy-critical wave equation in dimension 5. *Nonlinearity* **32** (2019), no. 12, 5017–5048.
