Buckling of a clamped strip-like beam with a linear pre-stress distribution

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Funding information
Austrian Research Promotion Agency, Grant/Award Number: 861493

A thin linear elastic strip is clamped at both ends and subjected to a linear stress distribution across its width. We use Kirchhoff beam theory to study this problem. If displacements out of the strip's own plane are prohibited, the straight configuration remains stable as long as the compression is not too high. With the three-dimensional spatial description of the rod theory, we find possible buckling modes even in the case of average tensile stresses in the beam. Comparison with shell and beam finite elements shows excellent agreement with the analytical investigation, also with respect to the supercritical behavior.

KEYWORDS
incremental equations, lateral-torsional buckling, nonlinear rod model, pre-stress, spatial rod model, super-critical behavior

1 | INTRODUCTION

In technical applications like steel belt drives or rolling mills, thin long strips of material are transported along their elongated direction (called the axial one). These strips may have small imperfections due to their production or operation, which cause a waviness of the rolling stock [1]. Even a folding of such a moving strip is possible. The planar deformation of the strip in its own plane within a rolling mill has been studied in [2] with usage of non-Lagrangian shell finite elements. There, an eigen strain distribution was considered, which varied both in the axial direction as well as in the lateral direction. The resulting stress distribution can cause buckling of the strip. A great number of publications can be found for static solutions of clamped beams and shells in the two-dimensional case. Often, instead of initial strains or stresses, a temperature distribution is applied. In [3], thermal buckling of functionally graded beams is investigated. In [4] also plates and shells can be found. The thermal post-buckling of beams on an elastic foundation can be found in [5]. All these articles presume that the deformation of the beam and its post-buckling configuration stay in one plane. If three-dimensional deformations and rotations are allowed, additional buckling modes occur, which can be more critical. A first attempt can be found in [6], but there the erroneous assumption was used that the buckled configuration causes no additional reactions at the clamped boundaries. This led to incorrect buckling loads and mode shapes. Buckling and post-buckling of a simply supported beam loaded with a constant moment can be found in [7]. A related analysis was shown in [8], where a thin strip, which has already lost stability about the axis of smaller bending stiffness, was distorted by a small intrinsic curvature causing three-dimensional configurations of the beam.

In this article we study the problem of buckling of a clamped thin strip due to pre-stresses which cause a resulting moment and normal force. At first, the kinematics of a Kirchhoff beam is presented which leads to equilibrium equations as well as constitutive equations. Due to the thinness of the cross-section, the Wagner effect [9], which couples normal stresses and twist
in a nonlinear way, has to be included. Linearizing the three-dimensional equations of equilibrium of a Kirchhoff beam near the straight pre-stressed state according to the procedure, shown in [10], we search for non-trivial solutions of the resulting homogeneous system of differential equations. Finally we find the regions of stability loss as function of the resulting force and moment in the unperturbed state. To our knowledge, the closed-form solution for the critical combination of parameters seems to be a novel result. The post-buckling behavior for the important case of a pre-stress without resulting force was observed analytically too. For short strips we consider restrained warping, which slightly improves the solution of long beams as well. The results are compared with beam finite elements, presented in [11]. These $C^1$-continuous elements are unshearable but extensible. The formulation of curvatures needs derivatives of the rotations of the beam. The precise definition of these rotations and their interpolation within an element is important to avoid undesired impact of rigid body motions on the strain energy, see [12] and [13]. Problems with large rotations - for example the bending of a straight beam into a helix - need a complex treatment, see [14] for a detailed view on the geometrically exact beam theory. In that article several Kirchhoff-Love beam finite elements were compared with each other and with Simo-Reissner beams, which take shear into account as well. In the present problem, the centerline of the strip will remain in the vicinity of its straight initial configuration, which allows for simpler computation of the curvatures. Then a constant reference vector can be defined, which together with the actual tangent vector of the centerline, defines the local basis vector as well as the twist of each point of the beam, see [10] and [15]. Beside the beam finite elements, also Kirchhoff-Love shell finite elements, which were used in [16] and [17] show remarkable agreement with the analytical solutions.

2 | THEORY OF KIRCHHOFF RODS FOR STRIP-LIKE BEAMS

We will briefly introduce the equations of beams in the form, suggested by Eliseev [18] and used in [10], [8]. The material particles of the beam form a spatial line $\mathbf{r}(s)$, at which each particle is identified with a material coordinate $s$. These material points have three rotational and three translational degrees of freedom. The rotational degrees of freedom influence the orientation of the local basis $\mathbf{e}_k$. Considering unshearable beams, we align one of the local basis vectors with the tangent vector of the centerline $\mathbf{r}'(s) = \partial \mathbf{r} / \partial s$, that is

$$ e_3 = t = r' / |r'|. $$

The beam is inextensible, hence $|r'| = 1$. The vector of twist and curvature $\Omega$ defines the rate of change of the local basis of the beam and is defined by

$$ e'_k = \Omega \times e_k, \quad \Omega = \sum_{k=1}^{3} \Omega_k e_k := \Omega_k e_k = \frac{1}{2} e_k \times e'_k. $$

The undeformed reference state, which is pre-stressed in the considered case, is described with the undeformed axis $\mathbf{r}_0(s)$, with the initial orientation of the local basis $e_{k,0}(s)$ and with

$$ |r'_0| = 1, \quad M_0 = M_{k,0} e_{k,0}, \quad Q_0 = Q_{k,0} e_{k,0}. $$

The pre-stresses are combined to the eigen stress resultants, namely the moment $M_0$ and the force $Q_0 = Q_0 t$. The strain measures $\kappa_1$ and $\kappa_2$ for bending as well as $\kappa_3$ for torsion can be found as

$$ \kappa = \kappa_k e_k = \Omega_k e_k. $$

They are equal to $\Omega$ because the beam in its initial configuration is straight and untwisted. The internal forces $Q$ and moments $M$ have to fulfil the equations of equilibrium

$$ Q' + q = 0, $$

$$ M' + r' \times Q + m = 0, $$

with external distributed forces $q$ as well as distributed moments $m$, which are counted per unit material length of the beam. The constitutive relations follow from the principle of virtual work

$$ M_k = \frac{\partial \Pi}{\partial \kappa_k}. $$
These relations will result in decoupled constitutive equations for \( M_1, M_2 \) and \( M_3 \) according to the internal energy per unit length

\[
\Pi = \frac{1}{2} \kappa_i a_{ij} \kappa_j + M_k \kappa_k
\]  

(8)

with the tensor of stiffness for bending and torsion \( a = a_{ij} e_i e_j \).

3 | COUPLING OF TORSION AND NORMAL FORCES

Torsion and bending are decoupled for our flat strip. With \( e_1 \) and \( e_2 \) being the main axes of inertia of the cross section the stiffness tensor \( a \) has the form

\[
a = a_1 e_1 e_1 + a_2 e_2 e_2 + a_t t t,
\]

(9)

with the bending stiffness values \( a_1 \) and \( a_2 \). The torsional stiffness \( a_t \) can be obtained by solving the classical Saint-Venant problem of pure torsion. For the present case of a strip-like beam one higher order term which couples tension and torsion has to be considered in the internal energy. This nonlinearity, sometimes called the Wagner effect, describes the (additional) stretching of fibers at some distance from the rotation axis \( t \) of a twisted strip. We find this term by considering a straight beam, which is pre-stressed with a force \( Q_0 \) and twisted by a small twist rate \( \kappa_3 = \varphi' \), with the rotation angle \( \varphi(s) \) and evaluate the resulting Green-Lagrange strains, see [10] for details. The beam axis remains straight at all deformations. Then, the position vector \( r_3 \) of each particle of the three-dimensional body and its derivative with respect to \( s \) are

\[
\begin{align*}
r_3(s, \xi, \eta) &= r(s) + \varphi(s) t \times x(\xi, \eta), \\
r_3'(s, \xi, \eta) &= t + \varphi'(s) t \times x(\xi, \eta).
\end{align*}
\]

(10)

(11)

\( \xi \) and \( \eta \) are the material coordinates in the cross section and \( x = \xi e_1 + \eta e_2 \), see Figure 1. The axial component of the Green-Lagrangian strain \( \varepsilon^{GL} \), depending on \( s, \eta \) and \( \xi \), reads

\[
\varepsilon^{GL} = \frac{1}{2} (r_3' \cdot r_3' - 1) = \frac{1}{2} (1 + \varphi'^2 (t \times x) \cdot (t \times x) - 1).
\]

(12)
The cross section is orthogonal to the tangent, hence the coefficient at \( \varphi'^2 \) is \((t \times x) \cdot (t \times x) = x \cdot x\). The entire axial strain component is geometrically nonlinear and reads

\[
\varepsilon^GL = \frac{1}{2} x \cdot x \varphi'^2.
\]

(13)

The principle of virtual work, with \( \kappa_3 = \varphi' \) says that

\[
\delta \Pi = \int_A \sigma \cdot \delta \varepsilon \, dA = \int_A \sigma_3(x) \delta \varepsilon^GL \, dA + a_i \varphi' \delta \varphi' = \frac{\partial \Pi}{\partial \varphi'} \delta \varphi',
\]

(14)

with the area of the cross section \( A \), the tensors \( \sigma \) and \( \varepsilon \) and the longitudinal stress component \( \sigma_3 \). The contribution of torsional shear stresses is found from linear beam theory. Finally, the variation of the internal energy is found to be

\[
\delta \Pi = \int_A \left( \frac{Q_0}{A} + \frac{E}{2} x \cdot x \varphi'^2 \right) (x \cdot x \varphi' \delta \varphi') \, dA + a_i \varphi' \delta \varphi' \]

\[
= \left( a_i + \frac{Q_0}{A} \right) \int_A x \cdot x \, dA \varphi' \delta \varphi' + \cdots
\]

(15)

(16)

For a detailed presentation of the omitted term of higher order, see [19]. There, also a possible elongation of the axis was considered. See [10] for a comparison with a nonlinear shell theory. The integral term represents the polar moment

\[
J_p = \int_A x \cdot x \, dA
\]

(17)

of the cross section, hence we obtain the constitutive equation

\[
M_i = \left( a_i + \frac{Q_0}{A} J_p \right) \varphi'.
\]

(18)

For thin open cross sections, the torsional rigidity is far smaller than \( J_p \), hence even for small axial forces this nonlinear term can become significant. As \( Q_0 \) remains approximately constant during buckling we use a constant corrected torsional stiffness

\[
\tilde{a}_i = a_i + \frac{J_p}{A} Q_0,
\]

(19)

instead of the original \( a_i \).

4 | INCREMENTAL EQUATION OF THE CLAMPED STRIP

We consider a beam with a flat rectangular cross section with length \( L \) along the \( z \)-axis. The beam is clamped at both ends at coordinates \( z = \pm L/2 \), see Figure 1. The stiffness values of the beam, namely the ones for bending \( a_x \) and \( a_y \) and the torsion stiffness \( a_t \) according to the coordinate system \( \{xyz\} \) are

\[
a_x = E \frac{w h^3}{12}, \quad a_y = E \frac{w^3 h}{12}, \quad a_t = \frac{E}{6(1+\nu)} w h^3,
\]

(20)

with width \( w \) and thickness \( h \) of the rectangular cross section, Young’s modulus \( E \) and Poisson ratio \( \nu \) of the isotropic beam material. We consider a linear distribution of pre-stresses according to

\[
\tau(x) = \frac{Q_0}{A} + \frac{M_0}{a_y} E x
\]

(21)

with the eigen moment \( M_0 \) and the tension force \( Q_0 \) of the rod. If the right end would be free, it will just deform according to Figure 1 left with \( R = a_y/M_0 \). Due to the clamping condition it may deform as seen in Figure 1 right, with displacement
components in lateral direction \(e_x\), vertical direction \(e_y\) and different rotations. The cross section remains fixed in the local basis, see Figure 1. In the picture, the deformed configuration with zero eigen moment is shown as dashed line. The equations of equilibrium (5), (6) allow with the clamping conditions and \(\mathbf{q} = 0, \mathbf{m} = 0\) for the trivial solution

\[
\mathbf{r}' = e_z. \tag{22}
\]

The tangential force \(Q_t = Q \cdot e_z\), which is the only component of the force in the non-buckled configuration, is

\[
Q_t = Q_0 \tag{23}
\]

The moments are found to be

\[
\mathbf{M} = \mathbf{a} \cdot \kappa = M_0 e_y. \tag{24}
\]

As we are searching for the buckling loads of a conservative system we use Euler’s approach to stability. When a bifurcation takes place there exist adjacent equilibria next to the original one. Hence, we search for non-trivial solutions in the vicinity of the straight equilibrium state. Similar to [20], where buckling of a closed ring was considered, we use the incremental equations of equilibrium in the linearized form

\[
Q' + Q = 0, \quad M' + r' \times Q + u' \times Q + m' = 0, \tag{25}
\]

where ()' denotes an increment and the displacements are denoted by \(\mathbf{u} = \mathbf{r}'\). The increments of both \(Q'\) and \(m'\) are zero. Further we use the constitutive equation of the increments of the moments

\[
\mathbf{M}' = \theta \mathbf{M} + \mathbf{a} \cdot \theta', \tag{26}
\]

with the small rotations \(\theta = \theta_x e_x + \theta_y e_y + \theta_z e_z\) defining the increments of the local basis \(e_i' = \theta \times e_i\). The unshearability as well as the inextensibility constraint couple displacements \(\mathbf{u}\) and rotations with

\[
\mathbf{u}' = \theta \times e_z. \tag{27}
\]

The incremental force \(Q = Q_x e_x + Q_y e_y + Q_z e_z\) is constant and it can be seen that due to the cross products in (25), \(Q_z\) does not enter the incremental equations of equilibrium. Computing the remaining terms and projecting them on \(e_x, e_y\) and \(e_z\), we obtain

\[
a_x \theta_x'' - M_0 \theta_x' - Q_0 \theta_x - Q_y = 0, \tag{28}
\]

\[
a_y \theta_y'' - Q_0 \theta_y + Q_x = 0, \tag{29}
\]

\[
\ddot{\theta}_z + M_0 \theta_z' = 0. \tag{30}
\]

These are three homogeneous differential equations for the small rotations \(\theta_i\). The buckling modes can be found with the boundary conditions

\[
\theta_x(-L/2) = \theta_x(L/2) = \theta_y(-L/2) = \theta_y(L/2) = \theta_z(-L/2) = \theta_z(L/2) = 0, \tag{31}
\]

and with the additional constraints for vanishing displacements from (27) at the ends

\[
\int_{-L/2}^{L/2} \theta_x ds = 0, \quad \int_{-L/2}^{L/2} \theta_y ds = 0. \tag{32}
\]

Other boundary conditions of displacements and rotations and their derivatives at \(\pm L/2\), like simply supported beams can be observed as well. The state prior to buckling may already be deformed dependent on which rotations are allowed at the ends. One has to note, that other nonlinear terms might be necessary in the post-buckling analysis in Section 5. Thus, the study of looped
belt drives will need the consideration of the contact between pulleys and belt and is postponed for future work. Equation (29) with the relevant boundary conditions decouples from the other equations and it can be seen that there are non-trivial solutions possible if the compression force equals the critical value

$$Q_{\text{crit}}^0 = -4a_y \pi^2 / L^2.$$  \hfill (33)

This is the Eulerian buckling load for bending about the $y$-axis. The corresponding buckling mode is symmetric with $Q_x = 0$. The remaining equations can be summarized to one single differential equation. Therefore, we differentiate (28) and use (30) to get

$$\theta_{xx}''' + \frac{M_0^2 - \bar{a}_x Q_0}{\bar{a}_x a_x} \theta_x' = 0, \quad \theta_x(-L/2) = \theta_x(L/2) = 0, \quad \int_{-L/2}^{L/2} \theta_x ds = 0.$$ \hfill (34)

The general solution to this linear differential equation is

$$\theta_x = A + B \cos \omega s + C \sin \omega s, \quad \omega^2 = \frac{M_0^2 - \bar{a}_x Q_0}{\bar{a}_x a_x}.$$ \hfill (35)

From the two boundary conditions at $\pm L/2$ we obtain

$$A + B \cos \frac{L \omega}{2} + C \sin \frac{L \omega}{2} = 0,$$ \hfill (36)

$$A + B \cos \frac{-L \omega}{2} + C \sin \frac{-L \omega}{2} = 0,$$ \hfill (37)

from which follows

$$C \sin \frac{L \omega}{2} = 0, \quad A = -B \cos \frac{L \omega}{2}.$$ \hfill (38)

Next, we use the integral condition and get

$$\int_{-L/2}^{L/2} \theta_x ds = B \left( \frac{2}{\omega} \sin \frac{\omega L}{2} - L \cos \frac{\omega L}{2} \right) = 0.$$ \hfill (39)

Note that due to the symmetric boundaries of the integral $C \sin \omega s$ has no contribution. With the abbreviation $\alpha = \omega L/2$, we summarize the search for non-trivial solutions to

$$B = 0, \quad C \neq 0 \quad \text{if} \quad \sin \alpha = 0$$ \hfill (40)

$$B \neq 0, \quad C = 0 \quad \text{if} \quad \tan \alpha = \alpha$$ \hfill (41)

$$B = 0, \quad C = 0 \quad \text{otherwise}$$

The smallest value of $\alpha$, for which non-trivial solutions are possible is $\alpha_1 = \pi$ and therefore, $\omega_1 = 2\pi/L$. The corresponding mode with $C \neq 0$ is symmetric with respect to the displacement $u_y$. For this mode, $Q_y$ and $Q_x$ are zero. In the rest of the paper we restrain to this symmetric eigen mode with $Q_y = Q$. The eigen modes of $\theta_x$ and $\theta_z$ are determined up to the constant $C$:

$$\theta_x = C \sin(\omega_1 s),$$ \hfill (42)

$$\theta_z = -\frac{C L}{M_0 \omega_1} (a_y \omega_1^2 + Q_0) (\cos(\omega_1 s) + 1) = \frac{\theta_x}{2} (\cos(\omega_1 s) + 1)$$ \hfill (43)

It is worth to mention, that the derivative of $\theta_z$ is zero at both clamped ends. Although we have demanded it solely at the center line, each other material line of the beam has a zero slope at the boundaries too. Thus, the obtained buckling mode fulfills
the same boundary conditions as a clamped shell. The critical combinations of \( Q_0 \) and \( M_0 \) are determined by the equation
\[
\frac{M_0^2 - \tilde{a}_i Q_0}{\tilde{a}_i} = \frac{M_0^2}{\tilde{a}_i} - Q_0 = a_x \left( \frac{2\pi}{L} \right)^2.
\] (44)

With (19), this novel result can be written as
\[
M_0^2 = \left( \frac{4a_x \pi^2}{L^2} + Q_0 \right) \left( a_i + \frac{J}{A} Q_0 \right) \] (45)

As soon as the compression force reaches the Eulerian buckling force we can see that the critical moment is reduced to zero. Theoretically, there is also a second root of the equation when \( \tilde{a}_i = 0 \), but at significantly higher compression forces. Further, we can find a critical moment \( M_0^* \) for \( Q_0 = 0 \)
\[
M_0^* = \sqrt{a_i a_x} \frac{2\pi}{L}.
\] (46)

Note that due to \( Q_0 = 0 \), the Wagner effect does not appear in this case. Hence, \( \tilde{a}_i = a_i \). Alongside with temperature gradients, an important source of the pre-moment would be an initial curvature \( \Omega_{y,0} \), which we omitted in the kinematic relation (4) for simplicity. With \( M_0^2 = (a_i \Omega_{y,0})^2 \), the critical natural curvature of the rod is
\[
\Omega_{y,0}^* = \frac{\sqrt{a_i a_x}}{a_y} \left( \frac{2\pi}{L} \right) = \sqrt{\frac{2}{1 + \nu} \left( \frac{h}{w} \right)^2 \frac{2\pi}{L}}.\] (47)

We can see, that the aspect ratio \( h/w \) of the cross section quadratically influences the critical natural curvature. Below we will define a non-dimensional pre-moment, in which you will easily recognize the corresponding natural curvature.

5 POST-BUCKLING ANALYSIS

The very beginning of the post-buckling behavior of the aforementioned case with \( Q_0 = 0 \) and therefore \( \tilde{a}_i = a_i \) is studied. There, \( M_0 \) just exceeded the critical value \( M_0^* \). The linear solution is augmented with essential nonlinear terms. In the present example, this is the elongation \( \varepsilon = |r'| - 1 \). From the symmetry of the linear buckling mode we conclude that \( Q = Q e_z = \text{const} \) and therefore \( \varepsilon = Q / b = \text{const} \). \( b = EA \) is the tensile stiffness of the beam. In the linear analysis we found that \( \theta_y = 0 \). Hence, we assume that in the nonlinear analysis \( \theta_y \ll \theta_x \) is still valid as long as we stay in the vicinity of the buckling point. The distance between the ends of the stretched and rotated center line of the beam must still be \( L \):
\[
\frac{1}{L/2} \int_{-L/2}^{L/2} (1 + \varepsilon) \cos \theta_x ds \approx \frac{1}{L/2} \int_{-L/2}^{L/2} (1 + \varepsilon) \left( 1 - \frac{\theta_x^2}{2} \right) ds = L.
\] (48)

The principal term is exactly fulfilled. We neglect the highest order term \( \varepsilon \theta_x^2 \) and obtain
\[
\varepsilon = \frac{1}{L} \int_{-L/2}^{L/2} \frac{\theta_x^2}{2} ds.
\] (49)

We augment the incremental Equation (28) for \( Q_0 = 0 \) with a term that contains the incremental stretch and obtain
\[
a_x \theta_x'' + M_0 \theta_z' = b \varepsilon \theta_x,\] (50)
\[
a_i \theta_z'' - M_0 \theta_x' = 0.
\]

The nonlinear term was also rigorously derived by Lacarbonara for the plane bending of an axially restrained beam [21]. Further terms, like nonlinear terms regarding the curvatures are not considered as the tensile stiffness is far larger than that for bending
and torsion. We combine the equations and get with \( u' = -\theta_x \)

\[
a_x u'''' + \frac{M_0^2}{a_t} u''' = bu'' \int_{-L/2}^{L/2} \frac{u_y'^2}{2} \, ds. \tag{51}
\]

We seek a solution of the form

\[
u_y = -\int_{-L/2}^{s} \theta_x \, ds = \frac{u_y(0)}{2 \pi} \left( \cos \frac{2\pi s}{L} + 1 \right), \tag{52}
\]

with unknown amplitude \( u_{y0} \). This function fulfils the boundary conditions \( u_y(-L/2) = u_y(L/2) = u'(-L/2) = u'(L/2) = 0 \) at both ends of the strip. Comparison of the coefficients gives the amplitude of the post-buckling solution of the vertical displacement

\[
u_y = \pm \frac{2}{\pi \sqrt{a_t b}} \sqrt{(LM_0)^2 - 4a_t a_x \pi^2} \text{ and } u_{y0} = 0. \tag{53}
\]

With \( \theta_{z0} = u_{y0} M_0/a_t \) from (30), also the rotation of the middle of the beam is found. We are still lacking a description of the lateral displacement, which is identical to zero according to the linear analysis \( u'_x = \theta_y = 0 \). We remember, that the linear displacements \( u \) result from a linear approximation of

\[
u'_x = P \cdot r' - r' \approx \theta \times r'. \tag{54}
\]

As we see above, the linear term of \( u'_x \) is zero. Without a strict mathematical proof, we can imagine, that \( u'_x \) can be obtained by a rotation of \( u'_y \) about the axis \( e_z \). Hence, the nonlinear displacement in \( x \)-direction can be described with

\[
u'_x = e_x \cdot \theta_x e_z \times u'_y e_y = -u'_y \theta_x = \theta_x \theta_z, \tag{55}
\]

which we integrate from \(-L/2\) to 0.

\[
u_{x0} = \int_{-L/2}^{0} \theta_x \theta_z \, ds = -\theta_x u_y \bigg|_{-L/2}^{0} + \int_{-L/2}^{0} u'_y \theta_z \, ds \tag{56}
\]

The functions \( u_y \) and \( \theta_z \) have similar form. Hence, with \( u_y(s) = u_{y0}/\theta_{z0} \theta_z(s) \) we obtain

\[
u_{x0} = -u_{x0} \theta_{z0} + \frac{u_{y0}}{\theta_{z0}} \int_{-L/2}^{0} \theta_z \theta'_z \, ds. \tag{57}
\]

Finally, we obtain a relation between \( u_x \), \( u_y \) and \( \theta_z \) in the middle of the beam

\[
u_{x0} = -\frac{u_{x0} \theta_{z0}}{2}, \tag{58}
\]

where \( u_{x0} \) and \( \theta_{z0} \) are known from above. The computation of this second component of the displacement indicates progress of the previous work, initiated in [7]. It should be noted, that even in cases where \( u_{x0}, u_{y0} \) and \( \theta_{z0} \) cannot be computed analytically, this equation seems to hold.

6 | PLATE BUCKLING ANALYSIS

We treat the buckling behavior with shell respectively plate equations too. This will serve as a reference solution for the beam results. As shown in [10], the incremental equation for the deflection \( u_y(z, x) \) of an initially stressed plane plate is

\[
\nabla \cdot (\mathbf{r}_0 \cdot \nabla u_y) - (D_1 + D_2) \Delta u_y = 0, \quad D_1 = \frac{vEh^3}{12(1-v^2)}, \quad D_2 = \frac{Eh^3}{12(1+v)}. \tag{59}
\]
FIGURE 2  Stability diagram of an elongated plate with $h = 10^{-3} \text{m}$, $w = 0.05 \text{m}$, $L = 1 \text{m}$, $\nu = 0$, computed with beam theory with $a$, (dashed line), beam theory with $\tilde{a}$, (solid line) and plate finite elements (dashed and thick).

As we are interested in buckling of the plate, there is no increment of external forces. Existence of a non-trivial solution to this homogeneous problem indicates buckling. We consider a rectangular plate in the plane $xz$, hence, the Nabla operator is $\nabla = e_x \partial / \partial x + e_z \partial / \partial z$ and the Laplace operator is $\Delta = \partial^2 / \partial x^2 + \partial^2 / \partial z^2$. The kinematic boundary conditions are

$$ u_y(-L/2, x) = u_y(L/2, x) = \frac{\partial u_y(-L/2, x)}{\partial z} = \frac{\partial u_y(L/2, x)}{\partial z} = 0. \quad (60) $$

At $u_y(z, \pm w/2)$, the dynamic boundary conditions of vanishing moments and forces have to be fulfilled. Finite elements do not need to fulfill them. Corresponding to the above beam analysis we define the pre-stress of the plate as

$$ \tau_0 = h(Q_0/A + xEM_0/a_y)e_z e_z. \quad (61) $$

Note that, in contrast to beams, there is a difference between applying the eigen stress $\tau_0$, a linear distribution of eigen strains or a corresponding temperature distribution. Exact solutions of (59) can hardly be found. However, the boundary value problem is equivalent to the search for a stationary value of a functional $U$ defined as

$$ U = \frac{1}{2} \int_{-L/2}^{L/2} \int_{-w/2}^{w/2} \left[ D_2 \kappa \cdot \kappa + D_1 \text{Tr} \kappa^2 + Eh(Q_0/b + xM_0/a_y) \left( \frac{\partial u_y}{\partial z} \right)^2 \right] dx dz, \quad (62) $$

with $\kappa = -\nabla u_y$. The trace $\text{Tr} \kappa$ is only important if $\nu \neq 0$. The functional $U$ is used to create a stiffness matrix $K$ of linear plate finite elements with purely the deflection $u_y$ and its derivatives chosen as nodal degrees of freedom. Critical pairs of $M_0$ and $Q_0$ are found for $\det K = 0$. Note that the plate finite elements are only capable of finding the bifurcation point. Post-buckling configurations have been computed with a non-linear shell formulation as introduced in [10] and [16]. These four-node Kirchhoff-Love shell elements with twelve nodal degrees of freedom and $C^1$ interelement continuity were used in [17] to verify some results of thin-walled beam theory.

7 | NUMERICAL EXAMPLES

The theory presented above is now confirmed with some numerical examples. As a first example we consider a beam with parameters $L = 1 \text{m}$, $h = 10^{-3} \text{m}$, $w = 0.05 \text{m}$, $\nu = 0$. We can see, that the aspect ratio of the cross section $w/h = 50$ is larger than the beam aspect ratio $L/w = 20$. Therefore, we assume a significant influence of torsion-tension coupling. Below we use the dimensionless form of eigen moment and force

$$ a = \frac{M_0 L}{a_y}, \quad \beta = \frac{Q_0}{b}. \quad (63) $$

In Figure 2, we can see the stability diagram of the structure. The white domain shows, where the straight configuration of the strip is stable according to the plate finite elements and the gray domain shows where it is not. The critical dimensionless moment for absence of pre-force is $a^* = 0.00355431$. Because it is independent from a correction of $a_y$, the stability curve of the corrected beam formulation (thin line) and of the original one (dotted line) have to cross at $\beta = 0$, an effect, which can be
FIGURE 3 Post-buckling of a beam with $\beta = 0$, analytic expression (dotted) versus beam finite elements (solid line).

FIGURE 4 Post-buckling behavior of an elongated plate with $h = 10^{-3} \text{m}$, $w = 0.05 \text{m}$, $L = 1 \text{m}$, $\nu = 0$, $\beta = 10^{-4}$: beam FE solution with $a_t$ (dashed line), beam FE solution with $\tilde{a}_t$ (solid line), shell FE solution (thick and dashed), critical values of $\alpha$ (dotted).

seen better below in Figure 5. These lines also have to cross at $\alpha = 0$, which corresponds to the Eulerian buckling mode. At increasing $\beta$, the torsional stiffness and therefore the stable regime grows. It can be seen, that the beam solution shows excellent agreement with the plate finite elements (thick dashed line).

The post-buckling solutions with $\beta = 0$ for increasing $\alpha$ is shown in Figure 3. The buckled finite element solution is found with beam elements without Wagner effect by adding an imperfection in the form of a small gravitational term acting along $-e_y$. Plotting the deflection $u_{y0}$ (in meters), rotation $\theta_{z0}$ and the lateral displacement $u_{x0}$ (in meters) of the middle of the beam for increasing $\alpha$, we can see, that the analytic approach describes the post-buckling behavior quite well.

In Figure 4 we can see the bifurcation diagrams of the problem including pre-tension. If again the kinematic quantities in the middle are compared, we see an excellent agreement of the critical values of

$$\alpha^{\text{crit}} = 0.0211242 \quad \text{with Wagner effect ($\tilde{a}_t$),}$$

$$\alpha^{\text{crit}} = 0.0199156 \quad \text{without Wagner effect ($a_t$),}$$

as well as of the post-buckling behavior of beam finite elements and shell finite elements for $\beta = 10^{-4}$. It can be seen, that pitchfork bifurcations of $\theta_{z}$ and $u_{y}$ just occur where the analytical formulas predict them and an approximately linear growth of the lateral displacement $u_{x}$ occurs there too. As mentioned above, the analytic Equation (58) holds whereas (53) for $u_{y0}$ as well as the corresponding equation for $\theta_{z0}$ fail. The relative error of (58) for the maximum value of the moment $\alpha = 0.025$ is

$$\left| \frac{u_{x0} + 0.5u_{y0}\theta_{z0}}{u_{x0}} \right| \approx 0.0045.$$  \hspace{1cm} (64)

After this detailed view on the beam-like strip, the second example shows a rather short strip with $L/w = 2$, $w = 0.05 \text{m}$, $h = 10^{-3} \text{m}$. In Figure 5 we see the stability diagram with and without Poisson effect. In the two pictures we can clearly see, that for $\beta < 0$, the stability region of the beam with correction is reduced compared to the original one. As mentioned before, the stability lines of the different beam formulations cross at points with $\beta = 0$ and at $\alpha = 0$. 

Astonishingly, the beam formulation without the Wagner term fits the plate solutions very well in the vicinity of the Eulerian buckling load, whereas for larger tensile stresses, the corrected version better follows the shell FE solution. This is not a coincident and can be observed for other parameter combinations as well. Hence, we can say, that in the vicinity of the Eulerian buckling force, it is more efficient to solve the beam problem with the original torsional rigidity. This empirical observation is supported by an analysis with account for the effect of restrained warping in the subsequent section.

8 RESTRAINED WARPING

Especially for short beams, restrained warping becomes significant. The warping function $\phi(x, y)$ of a thin rectangle can be found to be approximately that of a thin ellipse [22]:

$$\phi = -\frac{w^2 - h^2}{w^2 + h^2} xy \approx -xy.$$  \hfill (65)

In the theory of curved thin-walled cross sections, this term is usually neglected compared to other contributions to the warping function, see [23]. In the present example, those larger terms are zero. Hence, only this term affects the warping stiffness, which is obtained as

$$C_\phi = E \int_A \phi^2 dA = E \int_{-h/2}^{h/2} \int_{-w/2}^{w/2} x^2 y^2 dy dx = E \frac{w^3 h^3}{12} = \frac{a_y a_x}{b}. \hfill (66)$$

As shown in [22], [17] and [10], the restrained warping causes one additional term with higher order derivative $\theta_z'''$ in the equations of equilibrium (28) and (30), which become

$$a_x \theta_x'' - M_0 \theta_x' - Q_0 \theta_x = 0, \hfill (67)$$

$$- \frac{a_y a_x}{b} \theta_z''' + \tilde{a}_z \theta_z'' + M_0 \theta_z' = 0, \hfill (68)$$

with the additional boundary conditions $\theta_z'(-L/2) = \theta_z'(L/2) = 0$. Again, we look for nontrivial solutions of $\theta_z$ and $\theta_x$. We find $\theta_z$ in the form (43) with a suitable $\theta_x$ from (68) if the moment fulfills

$$M_0^2 = \left( Q_0 + a_x \left( \frac{2\pi}{L} \right)^2 \right) \left( a_x Q_0 + a_y \frac{a_x}{b} + \left( \frac{2\pi}{L} \right)^2 \frac{a_x a_y}{b} \right), \hfill (69)$$
which is necessary to fulfil (67). With \( Q_0^{\text{crit}} = -4\pi^2 a_x / L^2 \) being the critical Eulerian load for plane buckling and \( a_x \ll a_y \), we obtain

\[
M_0^2 = (Q_0 - Q_0^{\text{crit}}) \left( a_t + (Q_0 - Q_0^{\text{crit}}) \frac{a_x}{b} \right).
\]  

This new result perfectly explains the above empirical observation, based on Figure 5. If \( Q_0 \gg Q_0^{\text{crit}} \), we again obtain the solution with free warping (45), whereas for \( Q_0 = Q_0^{\text{crit}} \) we see that the right bracketed terms equal \( a_t \). This new expression for \( M_0 \) describes the stability curve for various pre-forces quite well (see Figure 6). Also for the originally considered elongated strip we find a slight increase in accuracy, especially when we compare \( \alpha^* \), see Table 1.

**9 | CONCLUSION**

In this paper we analyze buckling and supercritical behavior of a clamped beam in the three-dimensional space. With this additional freedom compared to a two-dimensional analysis, the beam is able to undergo buckling even in the case of a positive mean tension force. Correction terms of the torsional rigidity when the beam becomes a thin strip are derived. We find critical combinations of the pre-loading force and moment, which appears to be a novel result. For the example of pre-loading only with a moment, the supercritical behavior was studied analytically too. Comparison with shell and beam finite elements shows remarkable agreement for both, the buckling loads as well as the post-buckling behavior. In Figure 7, we visualize the supercritical shell solutions, which also demonstrate the possible appearance of higher buckling modes with growing pre-loading.
moment. For completeness, it has been shown that restrained warping is important to obtain good results, if the strip becomes rather short. Besides the theoretical aspect, there is also a practical relevance in this study. Belt drives with intrinsic curvature as small imperfection may show this kind of buckling behavior too, although the boundary conditions are different.

ACKNOWLEDGEMENTS

The authors wish to thank for the support of the Austrian Research Promotion Agency (FFG), project number 861493.

CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

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How to cite this article: C. Schmidrathner. Buckling of a clamped strip-like beam with a linear pre-stress distribution. Z Angew Math Mech. 2020;100:e201900336. https://doi.org/10.1002/zamm.201900336