Non-universal scaling in a model of information transmission and herd behavior

Dafang Zheng,\textsuperscript{1,2} P. M. Hui,\textsuperscript{2} and N. F. Johnson\textsuperscript{3}

\textsuperscript{1} Department of Applied Physics, South China University of Technology, Guangzhou 510641, P.R. China
\textsuperscript{2} Department of Physics, The Chinese University of Hong Kong, Shatin, New Territories, Hong Kong
\textsuperscript{3} Department of Physics, University of Oxford, Clarendon Laboratory, Oxford OX1 3PU, U.K.

Abstract

We present a generalized dynamical model describing the sharing of information, and corresponding herd behavior, in a population based on the recent model proposed by Eguiluz and Zimmermann [Phys. Rev. Lett. 85, 5659 (2000)]. By introducing a size-dependent probability for dissociation of a cluster, we show that the exponent characterizing the distribution of cluster sizes becomes model-dependent and non-universal. The resulting system, which provides a simplified model of a financial market, yields power law behavior with an easily tunable exponent.

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I. INTRODUCTION

There has been increasing interest in the study of systems of interacting agents. Such systems are useful for the study of global behavior in many situations of practical importance \[1\]. In the newly established area of econophysics \[2,3\], for example, multi-agent models have been intensively studied \[4,5\]. Agents in a population (e.g. a market) often do not act independently. The collective behavior of clusters of agents, also referred to as crowds, in which there is efficient information and opinion sharing among the agents, is an important factor in both real and simulated markets \[6\]. These crowds are dynamic in nature in that there is a continual process of crowd formation and dissociation within a competing population.

A simple model for stochastic opinion cluster formation and information dispersal, has recently been proposed and studied by Eguíluz and Zimmermann \[7\]. It is a dynamical model (henceforth referred to as the EZ model) in which there is a continual grouping and re-grouping of agents to form clusters. A cluster or crowd of agents act together (either buying or selling) and then dissolve after the transaction has occurred. When a cluster of agents decides not to trade, i.e. an inactive state, it may combine with another cluster of agents to form a bigger cluster. Detailed numerical studies \[7\] and mean field analysis \[8\] revealed several interesting features of the model. For example, it was observed that this simple model of herd behavior could lead to a fat-tail distribution of price returns similar to that observed in real markets. In addition, the cluster size distribution $n_s$ shows a scaling behavior of the form $n_s \sim s^{-5/2}$ for a range of cluster size $s$, followed by an exponential cutoff \[8\]. The EZ model represents a generalization of the static percolation-type model of Cont and Bouchard \[9\] in which herd formation is described by random connection between agents, and the cluster size distribution is found to follow the same scaling behavior. Several variations on the model have been proposed and studied. These variations include the spreading of opinion to multiple clusters \[8\] and inhomogeneous dissociation of clusters \[10\]. Interestingly, it was found that the values of the exponent characterizing the cluster size
distribution seem to be robust against these variations, i.e. these values remain unchanged for the different variations proposed so far. We note that similar scaling behavior has been found in the size distribution of businesses [11,12], although the value of the exponent is different.

Here we introduce a generalized version of the EZ model in which a cluster of agents of size $s_i$ will dissolve with a probability $s_i^{-\delta}$ if a transaction is made, and will combine with a cluster of size $s_j$ with a probability $s_i^{-\delta} s_j^{-\delta}$ if a transaction is not made. The exponent becomes non-universal and tunable with values depending on the parameter $\delta$. An analytic mean-field theory is presented which provides a quantitative explanation of the numerical results. The EZ model is recovered as a limiting case of our model with $\delta = 0$.

The plan of the paper is as follows. We define our generalized model in Sec. II and present the numerical results on the cluster size distribution. Section III provides a mean field analysis and identifies the scaling exponent. The analytic results are compared with numerical simulations. The resulting distribution of price-returns is presented in Sec.IV together with a discussion on possible extensions of our model.

II. THE MODEL

We consider a model with a total of $N$ agents. Following Ref. [7], a cluster or crowd is a group of agents who can exchange information efficiently and thus have a common opinion. These agents make the same decision at a given moment in time. Initially, all the agents are isolated, i.e. each agent belongs to a cluster of size unity. As time evolves, an agent belongs to a cluster of a certain size. At each timestep an agent, say the $i$-th one, is chosen at random. Let $s_i$ be the size of the cluster to which the chosen agent belongs. Since the agents within a cluster have a common opinion, all agents in such a cluster tend to imitate each other and hence act together. With probability $a$ the agent, and hence the whole cluster, decides to make a transaction, e.g. to buy or to sell with equal probability. After the transaction, the cluster is then broken up into isolated agents with a probability $s_i^{-\delta}$.
with $0 \leq \delta < 1$. With probability $(1 - a)$ the agents decide not to make a transaction, i.e. they wait and try to gather more information. The other agents in the cluster follow. In this case, another agent $j$ is chosen at random. The two clusters of sizes $s_i$ and $s_j$ then either combine to form a bigger cluster with probability $s_i^{-\delta}s_j^{-\delta}$, or the two clusters remain separate with probability $(1 - s_i^{-\delta}s_j^{-\delta})$. Here $a$ can be treated as a parameter reflecting the investment rate showing how frequent a transaction is made. Our model thus represents a generalization of the basic EZ model to the case in which a cluster of agents may stay together to form a group after making a transaction. The probability of dissociation $s_i^{-\delta}$ implies that larger clusters have a larger tendency to remain grouped while smaller clusters are easier to break up. For the special case of $\delta = 0$, our model reduces to the EZ model.

In the EZ model ($\delta = 0$), clusters of agents break up after a transaction. Here, our model includes a dissociation probability depending on the cluster size - this feature may be invoked to mimic practical aspects of a financial market, such as the effect of news arrival. Imagine one of the agents in a cluster of size $s_i$ receives some external news with probability $a$ at a given timestep. This external news suggests that he, and hence the other members of his cluster, should immediately trade (buy or sell). Since the news is external, the crowd act together in this one moment, leaving the cluster with a finite probability of subsequently dissociating. Suppose that they sense, e.g. from the resulting price-movement, that they are a member of a large crowd of like-minded agents: in practice many traders like to feel part of a larger crowd for reassurance. We therefore assume that the crowd breaks up with a size-dependent probability $p(s_i)$, where $p(s_i) = 1$ for $s_i = 1$ and $p(s_i)$ decreases monotonically as $s_i$ increases. By contrast, with probability $(1 - a)$ there is no news arrival from outside. The agent in the chosen cluster, uncertain about whether to buy or sell, makes contact with an agent in another cluster of size $s_j$. The agents share information and come up with a new opinion. Each of them then separately tries to persuade the other members of his cluster of the new opinion. With probability $p(s_i)$ ($p(s_j)$) the opinion of cluster $i$ ($j$) changes to the new opinion. Thus, the two clusters combine with probability $p(s_i)p(s_j)$. It turns out that this particular form of the two combined modifications to the EZ model, can be treated
using our mean field analysis. As a specific example, our numerical simulations are carried out for the case in which \( p(s) \sim s^{-\delta} \).

Let \( n_s \) be the number of clusters of size \( s \). Figure 1 shows the results of numerical simulations on the cluster size distribution in the steady state, for various values of the parameter \( \delta \). The results are obtained for a system with \( N = 10^4 \) and \( a = 0.3 \). Averages are taken over a time window of \( 10^6 \) time steps after the transient behavior has disappeared, together with a configuration average over 100 different runs with different initial conditions. The \( \delta = 0 \) results give the features in the EZ model. For a range of \( s \), \( n_s \sim s^{-\beta} \) with \( \beta = 5/2 \). Deviation from the scaling behavior sets in at a value of \( s \) depending on the value of the parameter \( a \). These features are consistent with previous numerical and analytical studies. For \( 0 \leq \delta < 1 \), it is observed that the size distribution \( n_s \) still scales with \( s \) in a range of \( s \) as in the EZ model. However, the exponent becomes model dependent and hence non-universal. The data shows that the exponent \( \delta \) is consistent with the behavior \( n_s \sim s^{-\beta(\delta)} \), where \( \beta(\delta) = 5/2 - \delta \). A mean field analysis can be used to extract this scaling behavior, as will be described in the next section.

It is interesting to note that several attempts have been made to modify the EZ model. These extensions include, for example, the study by d’Hulst and Rodgers on democracy versus dictatorship by incorporating an inhomogeneous investment rate in the population and also allowing rumor to spread to multiple clusters in one time step after a chosen cluster decides not to make a transaction. All the extensions proposed so far give \( \beta = 5/2 \), hence the value seems to be robust. The present model incorporates a size-dependent dissociation probability of a cluster after a transaction and leads to a tunable and model-dependent \( \beta(\delta) \). Thus our model actually gives a set of models with different values of \( \beta \), similar to the case of changing a system from one universality class to another in problems in critical phenomena. In fact, the situation is reminiscent of the non-universal exponent of conductivity in continuum percolation. In percolation problems, it is known that the effective conductivity for a system consisting of insulators and conductors exhibits the scaling behavior \( \sigma_c \sim (p - p_c)^t \) near the percolation threshold \( p_c \). The exponent \( t \) is universal.
in that its value depends only on the dimension of the system, regardless of other details, e.g. lattice type. However, if the conductances $\sigma$ of the conductors follow a distribution of the form $P(\sigma) \sim \sigma^{-\delta}$ with $0 < \sigma < 1$, the $t$-exponent \[14\] and other related properties \[15,17\] become non-universal with exponents taking on a value depending on $\delta$. It should be noted that it is not so surprising to see a connection between percolation and model for herd behavior. In the model of Cont and Bouchard \[9\], the EZ model \[7\] and their variations \[18\], an agent could be connected to any one of $(N-1)$ other agents to form a cluster. These models hence represent a problem of connectivity in high dimensions. Several other percolation type models \[19,20\] have also been proposed to explain the phenomena observed in real markets.

III. MEAN FIELD ANALYSIS

The cluster size distribution in the EZ model can be studied via a mean field analysis \[8\]. The treatment can be extended to the present model to extract the scaling behavior of $n_s$, though the algebra is more complicated. Let $n_s(t)$ be the number of clusters of size $s$ at time $t$. At a certain time, $n_s(t)$ changes as a result of the collective action of the members of the cluster containing the chosen agent. A master equation can thus be written down:

$$N \frac{\partial n_s}{\partial t} = -as^{1-\delta}n_s + \frac{(1-a)}{N} \sum_{r=1}^{s-1} r^{1-\delta}n_r(s-r)^{1-\delta}n_{s-r} - \frac{2(1-a)s^{1-\delta}n_s}{N} \sum_{r=1}^{\infty} r^{1-\delta}n_r$$ (1)

for $s > 1$. Each of the terms on the right hand side of Eq.(1) represents the consequence of a possible action of the agent. The first term describes the dissociation of a cluster of size $s$ after a transaction is made. The second term represents coagulation of two clusters to form a cluster of size $s$. The third term represents the combination of a cluster of size $s$ with another cluster. For clusters of size unity ($s = 1$), we have

$$N \frac{\partial n_1}{\partial t} = a \sum_{r=2}^{\infty} r^{-\delta}n_r - \frac{2(1-a)n_1}{N} \sum_{r=1}^{\infty} r^{-\delta}n_r.$$ (2)

Here, the first term comes from the dissociation of any clusters into isolated agents and the second term describes the combination of a cluster of size unity with another cluster. In the
steady state, \( \frac{\partial n_s}{\partial t} = 0 \), we have

\[
s^{1-\delta} n_s = A \sum_{r=1}^{x-1} r^{1-\delta} (s-r)^{1-\delta} n_r n_{s-r} \tag{3}
\]

for \( s > 1 \), and

\[
n_1 = B \sum_{r=2}^{\infty} r^{2-\delta} n_s \tag{4}
\]

for \( s = 1 \), where

\[
A = \frac{1 - a}{Na + 2(1 - a) \sum_{r=1}^{\infty} r^{1-\delta} n_r} \tag{5}
\]

and

\[
B = \frac{Na}{2(1 - a) \sum_{r=1}^{\infty} r^{1-\delta} n_r}. \tag{6}
\]

The aim here is to extract the scaling behavior. Invoking a generating function approach \cite{21}, we let

\[
G(\omega) = \sum_{r=0}^{\infty} r^{1-\delta} n_r e^{-\omega r} = g(\omega) + n_1 e^{-\omega}, \tag{7}
\]

where \( g(\omega) = \sum_{r=2}^{\infty} r^{1-\delta} n_r e^{-\omega r} \). It follows from Eq.(3) that the function \( g(\omega) \) satisfies the equation

\[
g^2(\omega) + (2n_1 e^{-\omega} - \frac{1}{A}) g(\omega) + n_1^2 e^{-2\omega} = 0. \tag{8}
\]

Note that \( A \) can be expressed in terms of \( n_1 \) and \( g(0) \), and

\[
g(\omega) = \frac{1}{4A} (1 - \sqrt{1 - 4n_1 A e^{-\omega}})^2. \tag{9}
\]

The number of clusters of size \( s \) can be found formally by

\[
n_s = \frac{1}{s^{1-\delta} s!} \frac{\partial^s G}{\partial z^s} \bigg|_{z=0}, \tag{10}
\]

where \( z = e^{-\omega} \). The resulting expression for \( n_s \) is

\[
n_s = \frac{(2s - 2)!(1 - a)^{s-1}(\sum_{r=1}^{\infty} r^{1-\delta} n_r)^s[(1 - a) \sum_{r=1}^{\infty} r^{1-\delta} n_r + Na]^s}{(s!)^2 s^{2-\delta} [Na + 2(1 - a) \sum_{r=1}^{\infty} r^{1-\delta} n_r]^{2s-1}}. \tag{11}
\]
Invoking Sterling’s formula yields
\[ n_s \sim N \left[ \frac{4(1-a)[(1-a) + \sum_{r=1}^{Na} r^{1-\delta}n_r]}{[\sum_{r=1}^{Na} r^{1-\delta}n_r + 2(1-a)]^2} \right] s^{-\left(\frac{5}{2} - \delta\right)}. \]  

(12)

For \( \delta = 0 \), the sum \( \sum_{r=1}^{\infty} r^{1-\delta} n_r = N \) and the previous results of Refs. [7,8] are recovered. For \( \delta \neq 0 \), it is difficult to solve for \( n_s \). Since the summations in Eqs. (11) and (12) give a number, our result shows that \( n_s \sim s^{-\beta(\delta)} \) with \( \beta(\delta) = \frac{5}{2} - \delta \) for the present model. This \( \delta \)-dependent exponent is also indicated in Fig. 1 (lines are a guide to the eye). We note that the scaling behavior is masked by the behavior of the term in the squared brackets in Eq. (12) for large values of \( s \), similar to the situation for the EZ model [8].

IV. DISCUSSION

Eguíluz and Zimmermann [7] applied their model to study the distribution of price returns. A price can be generated according to
\[ P(t+1) = P(t) \exp(s'/\lambda), \]

(13)

where \( \lambda \) is a parameter for the liquidity of the market. The price return \( R(t) = \ln P(t) - \ln P(t - 1) \) is defined to be the relative number of agents buying or selling at a time with \( s' = s \) for a cluster of agents deciding to buy, and \( s' = -s \) for a cluster deciding to sell at a given timestep. Numerical results for the EZ model showed that the distribution of returns \( P(R) \sim R^{-\alpha} \) with \( \alpha = 3/2 \). We have carried out similar calculations for our model. Figure 2 shows the price return distributions for different values of \( \delta \) on a log-log scales. As for the cluster size distribution, the exponent \( \alpha \) is now non-universal and takes on the value \( 3/2 - \delta \), which is also the value of \( \beta(\delta) - 1 \) [7].

In summary, we have proposed and studied the cluster size distribution, and the price return distribution, of a generalized version of the EZ model. Our model is a dynamical model for herd behavior and information sharing in a population. By introducing a probability for dissociation of a cluster depending on its size, the exponent characterizing the cluster size
distribution takes on a model-dependent non-universal value. Our model thus provides a simple way for tuning the power law behavior. Several extensions are immediately possible. Our particular choice of the form of the probability for dissociation of clusters allows us to tune the exponent within a range of unity. Changing the functional form of the probability slightly may alter the range of values in which the exponent can be tuned. A positive and negative value of $\alpha$ in our model may lead to different features in the cluster size distribution. Our model can also be extended to study the size distribution of businesses. A model similar to the EZ model has already been proposed in this context [13], however the scaling behavior seems to be non-universal for data from different countries [11,12]. The present model thus provides a possible extension to cope with this observed non-universality.

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**FIGURE CAPTIONS**

Figure 1: The size distribution \( n_s/n_1 \) as a function of the size \( s \) on a log-log scale for different values of \( \delta \) obtained by numerical simulations (symbols). The values of \( \delta \) used in the calculations are: \( \delta = 0, 0.25, 0.50, 0.75 \). The solid lines are a guide to the eye corresponding to exponents \( \beta = -2.5, -2.25, -2.0, -1.75 \) respectively.

Figure 2: The distribution of price returns \( P(R)/P(1) \) as a function of \( R \) on a log-log scale for different values of \( \delta \) (symbols). The values of \( \delta \) used in the calculations are: \( \delta = 0, 0.25, 0.50, 0.75 \). The solid lines are a guide to the eye corresponding to exponents \( \alpha = -1.5, -1.25, -1.0, -0.75 \) respectively.
Figure 2