THE GENERALISED mKdV EQUATIONS FOR LEVEL $-3$ OF $\hat{sl}_2$

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November 20, 2018

Abstract

A certain generalisation of the hierarchy of mKdV equations (modified KdV), which forms an integrable system, is studied here. This generalisation is based on a Lax operator associated to the equations, with principal components of degrees between $-3$ and $0$. The results are the following ones: 1) an isomorphism between the space of jets of the system and a quotient of $Sl_2(\mathbb{C}((t)))$; 2) the fact that the monodromy matrixes of the Lax operators have, moreover, Poisson brackets given by the trigonometric $r$-matrix; 3) a definition of the action of screening operators on the densities; 4) an identification of the intersection of the kernel with the integrals of motion.

AMS classification 35Q53, 58F07.

Keywords Generalised mKdV equations, Kac-Moody algebras, proalgebraic manifolds, geometric quotient, Poisson brackets, screening operators, $r$-matrix, Lax operator, dressing, cohomology of bicomplex.
1 Introduction

1.1 Foreword

It is treated here of certain mathematical equations, from the point of vue of algebra, as far as the jets spaces of some differential equations presents an isomorphism with a quotient of proalgebraic groups. So here is an algebraic treatment of analytical data. The equations are ones of modified Korteweg-de Vries, or mKdV, which are generalised for level $-3$ of $\hat{sl}_2$.

The Lax operator of the usual mKdV theory has terms of principal degrees $-1$ and 0 in the Kac-Moody algebra $\hat{sl}_2$. The authors of [5] and of [10] have studied analogous equations, where the Lax operator is a sum of components of principal degrees between $-2n-1$ and 0; in the present paper, the case of the algebra $\hat{sl}_2$, $n = 1$, is studied. The goal of the study is the classical treatment of the system, in the sense of a family of Poisson commutativ integrals of motion.

The four following points are treated:

1) First it is showed (theorem 2.1) an isomorphism of proalgebraic structures between, on the one hand, a differential ring given by the variables of the generalised mKdV equations and on the other hand, the coordinates over a double quotient. The work is to be bound with the results of [9].

2) The authors of [5] and of [10] have showed Poisson brackets such that the generalised mKdV equations are hamiltonian systems. The Poisson brackets of the monodromy matrixes are calculated. The result (proposition 4.1) is analogous with [7]: the Poisson brackets are given by the trigonometric $r$-matrix associated with $\hat{sl}_2$.

3) The screening operators $Q_0$ and $Q_1$ are defined by the Poisson brackets of the monodomy matrix, and they can be viewed as vector fields over the variety of the jets. The result of the theorem 5.2 is the identification with the action of $n_+$ over $N_+/A_+$ following from theorem 2.1. The result is analogous with [8].

4) $Q_0$ and $Q_1$ are screening operators over the density spaces. In the theorem 7.4 the intersection of the kernels is identified with the space of the integral of motions. It is done by mean of a complex formed from $Q_0$ and $Q_1$ and a resolution of Bernstein-Gelfand-Gelfand (BGG) type.

The problem of the quantification of the system will be studied later.

1.2 The presentation of the mKdV equations

The generalised mKdV equations are studied here in a matrix form.
The usual mKdV equation is the following one:

$$\frac{\partial u}{\partial t} - 2u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0,$$

with:

$$u(x, t),$$

a function with real values. It is possible to consider the equation in the following form:

$$\mathcal{L} = \partial_x + p_{-1} + u(x, t)h_0,$$

with the matrixes:

$$p_{-1} = \begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix}, \tilde{p}_{-1} = \begin{pmatrix} 0 & \lambda \\ -1 & 0 \end{pmatrix}, h_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

taking a formal parameter $\lambda$, so that the equation becomes:

$$\frac{\partial \mathcal{L}}{\partial t} = [\mathcal{L}, A],$$

with $A$, the following matrix:

$$A = p_{-3} + uh_{-2} + u^2p_{-1} + ux\tilde{p}_{-1} + (u_{xx} - u^2)h_0,$$

$$p_{-3} = \begin{pmatrix} 0 & \lambda^2 \\ \lambda & 0 \end{pmatrix}, h_{-2} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}.$$

The following points are studied. First it is possible to construct commuting flows putting the Lax operator $\mathcal{L}$ in a conjugated form:

$$\mathcal{L} = \partial_x + L = K(\partial_x + p_{-1} + d_1p_1 + d_3p_3 + \ldots)K^{-1},$$

with functions $d_{2i+1}$ and:

$$p_{-2k+1} = \begin{pmatrix} 0 & \lambda^k \\ \lambda^{k-1} & 0 \end{pmatrix},$$

the flows are then:

$$\partial_t \mathcal{L} = [\mathcal{L}, A_n],$$

with:

$$A_n = [Kp_{-2n-1}K^{-1}]_-. $$
Indeed, a matrix which commutes with $\mathcal{L}$ furnishes a flow which preserves the form of $\mathcal{L}$, considering the degrees of the terms of the bracket. The following equation is considered:

$$[p_{-2n-1}, \partial_x + \sum_{i \geq 0} d_{2i-1} p_{2i-1}] = 0,$$

This identity is conjugated to obtain $A_n$. So, a geometrical interpretation follows by mean of an isomorphism with a quotient $N_+/A_+$; there is then an identification of differential rings.

A structure of Poisson brackets over the jets is given in a synthetic way in the form:

$$\{u(x), u(y)\} = \frac{1}{2} [\partial_x - \partial_y] \delta_{x,y},$$

which gives brackets for polynomial in the $u$ and derivativs, using the commutation of the bracket with the differential and the fact that the bracket is a derivation of the functions, when one of the sides is fixed.

It is possible to generalise the form of the operator $\mathcal{L}$ as studied by [5] and [10], taking an operator with level $-2n - 1$ of $\hat{sl}_2$, in the form:

$$\mathcal{L} = \partial_x + p_{-2n-1} + L,$$

$L$, a matrix which depends on the levels $0, -1, -2, ..., -2n$.

Here the case of $-2n - 1 = -3$ is treated:

$$L = H_{-2} h_{-2} + E_{-1} e_{-1} + F_{-1} f_{-1} + H_0 h_0.$$

1.3 Acknowledgments

I thank greatly B.Enriquez for his help in this work, during a stay in the Forschung-Institut für Mathematik (FIM) of Zürich.

2 The isomorphism of varieties

2.1 Recalls of the affine algebras

Let a Cartan matrix $(a_{ij})$ be, of dimension $n$ and rank $l$, it is possible to associate it a Kac-Moody algebra $\mathfrak{g}$, called affine or of infinite dimension. It is given by a matrix $(a_{ij})$ and realised by a triplet $(\mathfrak{h}, \Pi, \Pi^v)$, with $\mathfrak{h}$, a complex vector space, $\Pi = \langle \alpha_i \rangle$, in the dual space $\mathfrak{h}^*$ and $\Pi^v = \langle \alpha_i^v \rangle$ in $\mathfrak{h}$,

$$a_{ii} = 2, a_{ij} \leq 0 (\in \mathbb{Z}_-), a_{ij} = 0 \Rightarrow a_{ji} = 0.$$
Π, Π are linearly independant,

\[
< \alpha_i, \alpha_j^v > = a_{ij}, < \alpha_i, \alpha_j^v > = a_{ij}, n - l = \dim(h) - n,
\]

and the algebra is constructed by the generators \(e_i, f_i, h\), with the relations:

\[
[e_i, f_j] = \delta_{ij} \alpha_i^v, [h, h'] = 0, [h, e_j] = \langle \alpha_j, h \rangle e_j,
\]

\[
[h, f_j] = - \langle \alpha_j, h \rangle f_j, ad(e_i)^{-a_{ij}+1}(e_j) = 0, \quad ad(f_i)^{-a_{ji}+1}(f_j) = 0.
\]

Then, there is a decomposition:

\[
g = n_- \oplus h \oplus n_+,
\]

where \(n_+\) and \(n_-\) are the sub-algebras constructed by the \(e_i\) on the one hand and the \(f_j\), on the other hand.

**Notation 1**.

\[
b_- = n_- \oplus h,
\]

and:

\[
b_+ = h \oplus n_+.
\]

\(\tilde{sl}_2\) corresponds to the following Cartan matrix:

\[
\begin{pmatrix}
2 & -2 \\
-2 & 2
\end{pmatrix}.
\]

**Notation 2**:

\[
h_{-2} = \begin{pmatrix}
\lambda & 0 \\
0 & -\lambda
\end{pmatrix}, e_{-1} = \begin{pmatrix}
0 & \lambda \\
0 & 0
\end{pmatrix}, f_{-1} = \begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix},
\]

\[
h_0 = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}, p_{-3} = \begin{pmatrix}
0 & \lambda^2 \\
\lambda & 0
\end{pmatrix}, p_{-2k+1} = \begin{pmatrix}
0 & \lambda^k \\
\lambda^{k-1} & 0
\end{pmatrix},
\]

\[
\tilde{p}_{-2k+1} = \begin{pmatrix}
0 & -\lambda^k \\
\lambda^{k-1} & 0
\end{pmatrix}, h_{2n} = \begin{pmatrix}
\lambda^{-n} & 0 \\
0 & -\lambda^{-n}
\end{pmatrix},
\]

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The set of the $2 \times 2$ matrixes with coefficients in the ring $\mathbb{C}[[\lambda^{-1}]]$, with 1 determinant and $val(m_{12}) \geq 1$ is noted $N_+$. The set of matrices, exponential of the sub-algebra generated by the $p_{2k+1}$ is noted $A_+$. 

2.2 The isomorphism of differential rings

In this section, the following theorem is treated:

**Theorem 2.1**. It exists an isomorphism of differential rings between the one of the jets of the variables of the generalised mKdV equations, with the canonical differential and the one of the coordinates over the quotient $N_+/A_+$, with the differential given by the right action of:

$$p_{-3} + d_{-1}p_{-1},$$

over $N_+/A_+$.

First is established an isomorphism of rings, and then are identified the differentials.

2.3 Determination of a conjugation

In the generalised mKdV equations, the operator $\mathcal{L}$ is put in a conjugated form and the unicity of the conjugation is showed, [4] provided that a constraint is imposed over the matrix of passage.

2.4 The Lax operator in a conjugated form

The Lax operator of the generalised mKdV equations is:

$$\mathcal{L} = \partial_z + p_{-3} + H_{-2}h_{-2} + E_{-1}e_{-1} + F_{-1}f_{-1} + H_0h_0. \quad (1)$$

With $z \in \mathbb{R}$ and $E_{-1}, F_{-1}, H_{-2}, H_0$, some smooth functions of $z$. 

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Definition 2.1. For a matrix of rank $n$, $A = (a_{i,j})$, an antitrace is defined (sum of the terms over the antidiagonal):

$$\tau(A) = \sum_{i=0}^{n-1} a_{n-i,i+1}.$$ 

A conjugated form of the operator $L$ in the generalised mKdV equations is then presented.

Proposition 2.1. The Lax operator has the following conjugated form:

$$L = M(\partial_z + p_{-3} + d_{-1}p_{-1} +$$

$$d_1p_1 + \sum_{i \geq -1} d_{2i+1}p_{2i+1})M^{-1}. \quad (2)$$

The $d_{2i+1}$ are polynomials over the variables $E_{-1}, F_{-1}, H_0, H_{-2}$ and derivatives, and the matrix $M$ belongs to $M_2(\pi_0[[\lambda^{-1}]])$, the $2 \times 2$ matrices, with coefficients in the ring $\pi_0[[\lambda^{-1}]]$, is uniquely defined, with 1 determinant, $val(m_{12}) \geq 1$, and with $\tilde{\tau}(M) = 0$.

2.5 Appendix

There is a property for the adjoint action $ad(p_{-3}))$, linear application of $\mathfrak{sl}_2((\lambda))$, a decomposition in a direct sum:

$$\mathfrak{sl}_2((\lambda)) = Ker(ad(p_{-3})) \oplus Im(ad(p_{-3})).$$

$Ker(ad(p_{-3}))$ is defined as the Lie algebra whose base is formed by $p_{2k+1}$, and $Im(ad(p_{-3}))$, the $h_{2i}$ and the $\tilde{p}_{2i+1}$. $ad(p_{-3})$ is an isomorphism over its image.

2.6 Isomorphism of differential rings

An isomorphism is showed between $\pi_0^c = \pi_0/[d_{-1} = c]$ and the ring of coordinates over the quotient $N_+/A_+$, $N_+$ is the group associated with positiv nilpotents of the algebra of infinite dimension $\mathfrak{sl}_2((\lambda))$, $A_+$ is the canonical commutativ sub-group.

---

1This set is noted $\pi_0$.
2This set is noted $N_+(\pi_0)$.
The differential ring $\pi_0 = \mathbb{C}[Jets]$ is defined as the ring of polynomials given by the variables,

$$X_k = \{ E_{-1;k}, F_{-1;k}, H_{0;k}, H_{-2;k}; k \geq 0 \},$$

and the differential given by the derivation $\partial$.

$$\pi_0 = \mathbb{C}[Jets] = \mathbb{C}[E_{-1;k}, F_{-1;k}, H_{0;k}, H_{-2;k}, k \in \mathbb{N}].$$

For $c$ a complex number, the differential ring $\pi_0^c$ is defined as:

$$\pi_0^c = \mathbb{C}[Jets_c] = \mathbb{C}[E_{-1;k}, F_{-1;k}, H_{0;k}, H_{-2;k}; k \in \mathbb{N}] / (d_{-1} - c, d_{-1}^{(k)}),$$

it is clear that $\partial$ induces a derivation over $\mathbb{C}[Jets_c]$.

The elements of the rings $\mathbb{C}[Jets]$ and $\mathbb{C}[Jets_c]$ are respectively functions over the spaces:

$$Jets = \{(E_{-1;k}, F_{-1;k}, H_{0;k}, H_{-2;k}) \in (\mathbb{C})^4, k \in \mathbb{N} \},$$

and:

$$Jets_c = \{(E_{-1;k}, F_{-1;k}, H_{0;k}, H_{-2;k}) \in (\mathbb{C})^4, k \in \mathbb{N} \mid (d_{-1} - c)^{(k)} = 0 \}.$$

**Definition 2.2.**

$$J = \{ M \in M_2(\mathbb{C}((\lambda^{-1}))) / \det(M) = 1, \tilde{\tau}(M) = 0 \},$$

the $2 \times 2$ matrices, with 1 determinant and $\tilde{\tau} = 0$; these conditions being algebraic, an algebraic sub-variety is obtained.

**Lemma 2.1.** A natural application from $J$ to $N_+/A_+$ is an isomorphism of proalgebraic varieties.

**Corollary 2.1.** So:

$$N_+ = J.A_+.$$
Theorem 2.2. For all complex number $c$, the decomposition which associates to all jet of functions $E_{-1}, F_{-1}, H_{-2}, H_0$, an element $M$ of $N_+$, induces an isomorphism of proalgebraic varieties between $Jets_c$ and $N_+/A_+$:

$$\mathbb{C}[Jets_c] \cong \mathbb{C}[N_+/A_+] .$$

Corollary 2.2. The application of $Jets$ toward:

$$N_+/A_+ \times \mathbb{C}^\infty,$$

associating to a jet the couple formed by the class of $n_+$ and the family $(d_{-1}^{(k)})_{k \geq 0}$, defines an isomorphism of differential rings between $(\mathbb{C}[Jets], \partial)$ and:

$$\mathbb{C}[N_+/A_+] \otimes \mathbb{C}[x_0, x_1, ...],$$

with the differential:

$$r(p_{-3} + x_0p_{-1}) \otimes 1 + 1 \otimes (\sum_{i \geq 0} x_{i+1}\partial/\partial x_i),$$

$r$ is the right regular action.

An isomorphism between the rings of infinite dimension, the jets of variables in the generalised mKdV equations and the coordinates of a quotient has been showed.

2.7 The differential for the quotient

In the generalised mKdV equations, an identification of the differential for the quotient is here showed. The differential over the jets is the derivation of the variables.

Definition 2.3. Let $O$ be, the matrixes of zero trace, of $-\lambda$ determinant, and with positiv terms in $\lambda^{-1}$ and with $\lambda$ too:

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix},$$

with the determinant of the matrix $-\lambda$, which gives:

$$a^2 + bc = \lambda.$$
Lemma 2.2. An identification between $N_+/A_+$ and $O$ is given by the arrow:

$$N_+/A_+ \rightarrow O,$$

$$M \mapsto V = M_{p-1}M^{-1}.$$ 

The right action of the $p_{2i+1}$ over $O$ is:

$$[(\lambda^{-i-1}V)_-, V].$$

Proposition 2.2. The differential of the ring of jets of the variables is given by the right action over the quotient $N_+/A_+$, of:

$$p_{-3} + d_{-1}p_{-1}.$$ 

Formula:

$$(\lambda V)_{-} + c(V)_{-} = p_{-3} + L,$$

avec:

$$c = (E_{-1} + F_{-1} + H_{2-2})/2.$$ 

Doing the study of the generalised mKdV equations, an isomorphism of proalgebraic structures has been showed and, too, an identification of two differential rings, on the one hand, the ring of the jets and derivatives and in the other hand, the ring of the coordinates over a quotient with differential given by a right action. The generalised mKdV equations have so been studied in link with the affine algebra $\mathfrak{sl}_2((\lambda))$. So, an isomorphism of differential rings has been established.

3 The structures of Poisson brackets

3.1 The Poisson brackets over the variables

The Poisson brackets for the generalised mKdV equations have been defined by the authors of [5] in the following way:

$$\mathcal{R} = 1/2[\mathcal{P}_\alpha - \mathcal{P}_\beta],$$
the half difference of the projection, with $\alpha$ and $\beta$, the spaces of the affine algebra $\hat{sl}_2$ corresponding with $b_-$ and $n_+$. The Poisson brackets are:

$$\{f, g\}_\mathcal{R}(L) = \int_{-\infty}^{+\infty} dx \langle L, [(\delta f / \delta L), (\delta g / \delta L)]_\mathcal{R} \rangle -$$

$$\langle \mathcal{R}(\delta f / \delta L), \partial_x (\delta g / \delta L) \rangle - \langle \mathcal{R}(\delta f / \delta L), \partial_x \mathcal{R}(\delta g / \delta L) \rangle.$$

the bilinear form in the affine algebra for two elements $u$ and $v$ of $\hat{sl}_2$,

$$\mathcal{L} = \partial_x + L,$$

the Lax operator for the generalised mKdV equations,

$$f, g,$$

two functionals over the space of jets of the variables in the generalised mKdV equations, (as for example $\int_{-\infty}^{+\infty} E_{-1}(x) dx$, or more generally, the integral of a polynomial in the jets of the variables). It is a way to give Poisson brackets over the jets of the variables of the generalised mKdV equations. So the brackets have been obtained, it should be too possible to give them in an equivalent way over the jets of variables, or over the monodromy matrix.

### 3.2 The structures of Poisson brackets over the jets

The brackets of the variables in the generalised mKdV equations are given in the following way:

**Proposition 3.1**. Over the jets of the variables, the Poisson brackets are:

$$\{H_0(x), H_0(y)\} = \frac{1}{2} \partial_x \delta_{x,y},$$

$$\{H_0(x), E_{-1}(y)\} = 0, \{H_0(x), F_{-1}(y)\} = 0, \{H_0(x), H_{-2}(y)\} = 0,$$

$$\{E_{-1}(x), E_{-1}(y)\} = 0, \{F_{-1}(x), F_{-1}(y)\} = 0,$$

3Take variables with compact support.
\[
\{E_{-1}(x), F_{-1}(y)\} = -4H_{-2}(x)\delta_{x,y}, \\
\{E_{-1}(x), H_{-2}(y)\} = 2\delta_{x,y}, \{F_{-1}(x), H_{-2}(y)\} = -2\delta_{x,y}, \\
\{H_{-2}(x), H_{-2}(y)\} = 0.
\]

There is moreover associativity of the Poisson brackets in the variables.

### 3.3 \(d_{-1}\) central element

**Proposition 3.2**

\[d_{-1}(x) = 1/2[E_{-1}(x) + F_{-1}(x) + H_{-2}^2(x)],\]

is a central element for the Poisson bracket.

### 4 The Poisson brackets of the monodromy matrix

The matrix for the monodromy is \(P \exp \int_a^b L(x)dx\), with \(L = p_{-3} + H_0h_0 + H_{-2}h_{-2} + E_{-1}e_{-1} + F_{-1}f_{-1}\). There is so a way to found Poisson brackets of two functions, \(f, g\), at the level of the monodromy matrix from the synthetic formula. The following bracket has to be computed:

\[
\{P \exp \int_a^b L(x)dx \otimes P \exp \int_a^b L(y)dy\}.
\]

Let \(r_{\text{trigo}}\) be:

\[
r_{\text{trigo}} = [\lambda + \mu]/[\lambda - \mu]t + e \otimes f - f \otimes e,
\]

\[
t = 1/2h \otimes h + e \otimes f + f \otimes e.
\]

**Proposition 4.1**. At the level of the monodromy matrixes, the Poisson brackets are given by the following formula:

\[
\{P \exp \int_a^b L(x)dx \otimes P \exp \int_a^b L(y)dy\} =
\]

\[
[r_{\text{trigo}}, [P \exp \int_a^b L(t)dt] \otimes [P \exp \int_a^b L(t)dt]].
\]
5 The action of $Q_0$ and of $Q_1$ over the jets of the variables generalised mKdV

The screening operators are the following integrals:

$$
Q_0 = \int_{\mathbb{R}} E_{-1}(u) e^{2\int_{-\infty}^{u} H_0(v) dv} du,
$$

$$
Q_1 = \int_{\mathbb{R}} F_{-1}(u) e^{-2\int_{-\infty}^{u} H_0(v) dv} du.
$$

5.1 The action of $\bar{Q}_0$ and $\bar{Q}_1$ over the variables

In a heuristic way, the action of $\bar{Q}_0$ and $\bar{Q}_1$ over the polynomials in the jets is the following:

$$
\bar{Q}_0(P)(u) = e^{2 \int_{-\infty}^{u} H_0(v) dv} \{Q_0, P\}(u),
$$

with $P$, a polynomial in the jets of the variables of the generalised mKdV equations. The actions of $\bar{Q}_0$ and $\bar{Q}_1$ can be written down as vector fields:

$$
\bar{Q}_0 = \sum_{n \in \mathbb{N}} a_n \partial / \partial H_0^{(n)} + b_n \partial / \partial E_{-1}^{(n)} + c_n \partial / \partial F_{-1}^{(n)} + d_n \partial / \partial H_{-2}^{(n)}.
$$

Notation 3 .

$$
\begin{align*}
    h_0^{[n]} &= \frac{H_0^{(n)}}{n!}, \\
    e_{-1}^{[n]} &= \frac{E_{-1}^{(n)}}{n!}, \\
    f_{-1}^{[n]} &= \frac{F_{-1}^{(n)}}{n!}, \\
    h_{-2}^{[n]} &= \frac{H_{-2}^{(n)}}{n!}.
\end{align*}
$$

Proposition 5.1 . The action of $\bar{Q}_0$ is given by the following series:

$$
\sum_{n \in \mathbb{N}} a_n t^n = -\left(\sum_{n \in \mathbb{N}} e_{-1}^{[n]} t^n\right)\left(e^{-2 \sum_{n \in \mathbb{N}} h_0^{[n]} t^{n+1}/(n+1)}\right),
$$

$$
\sum_{n \in \mathbb{N}} b_n t^n = 0,
$$

$$
\sum_{n \in \mathbb{N}} c_n t^n = 4\left(\sum_{n \in \mathbb{N}} h_{-2}^{[n]} t^n\right)\left(e^{-2 \sum_{n \in \mathbb{N}} h_0^{[n]} t^{n+1}/(n+1)}\right),
$$

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\[
\sum_{n \in \mathbb{N}} d_n t^n = -2e^2 \sum_{n \in \mathbb{N}} h_0^{[n]} t^{n+1/(n+1)}.
\]

**Theorem 5.1.** The series for \( \bar{Q}_1 \) are:

\[
\sum_{n \in \mathbb{N}} d'_n = (\sum_{n \in \mathbb{N}} e_{-2}^{[n]} t^n) (\sum_{n \in \mathbb{N}} h_0^{[n]} t^{n+1/(n+1)}),
\]

\[
\sum_{n \in \mathbb{N}} b'_n = -4(\sum_{n \in \mathbb{N}} h_{-2}^{[n]} t^n) (\sum_{n \in \mathbb{N}} h_0^{[n]} t^{n+1/(n+1)}),
\]

\[
\sum_{n \in \mathbb{N}} c'_n = 0,
\]

\[
\sum_{n \in \mathbb{N}} d'_n = 2e^2 \sum_{n \in \mathbb{N}} h_0^{[n]} t^{n+1/(n+1)}.
\]

### 5.2 The action of the screening operators

Here is a result over the action of the \( \bar{Q}_i \), which is identified with the one of \( n_+ \) over the quotient, according to the following theorem:

**Theorem 5.2.** The action of \( \bar{Q}_0 \) and \( \bar{Q}_1 \) is defined by the Poisson brackets. With the isomorphism, the action is identified as being the one of \( n_+ \) over the coordinates of the quotient \( \mathbb{C}[N_+/A_+] \).

First the action of the \( Q_i \) is identified over the space \( \mathcal{O} \) (matrices of \(-\lambda\) determinant and zero trace). Then, it is showed a lemma concerning the inclusion of the ring \( R_i \) in the kernel of the operator \( \bar{Q}_i \), next an other lemma over the coefficients of \( V_i \), and last the theorem follows.

### 6 Identification of the action of \( \bar{Q}_i \) over the \( V_i \)

In the generalised mKdV equations, the action of the \( \bar{Q}_0 \) and \( \bar{Q}_1 \) is identified with help of a conjugation by the operators:

\[
\mathcal{L}^{(0)} = \partial_x + L^{(0)}, \quad \mathcal{L}^{(1)} = \partial_x + L^{(1)},
\]
two operators defined as being:

\[ L^{(0)} = p_{-3} + E_{-1}^{(0)}e_{-1} + F_{-1}^{(0)}f_{-1} + H_{0}^{(0)}h_{0} + F_{1}^{(0)}f_{1}, \]

\[ L^{(1)} = p_{-3} + E_{-1}^{(1)}e_{-1} + F_{-1}^{(1)}f_{-1} + H_{0}^{(1)}h_{0} + E_{1}^{(1)}e_{1}, \]

these operators are conjugable with \( L \), the Lax operator of the generalised mKdV equations.

\[ \mathcal{L}^{(0)} = n_{0}(\partial_{x} + L)n_{0}^{-1}, \mathcal{L}^{(1)} = n_{1}(\partial_{x} + L)n_{1}^{-1}, \]

with:

\[ L = p_{-3} + H_{-2}h_{-2} + E_{-1}e_{-1} + F_{-1}f_{-1} + H_{0}h_{0}. \]

The matrices of conjugation are the following ones:

\[ n_{1} = \exp(-H_{-2}e_{1}), n_{0} = \exp(H_{-2}f_{1}). \]

There is then an identification of the action over \( O \); the action of \( \tilde{Q}_{0} \) and \( \tilde{Q}_{1} \) is identified over the matrices \( V \) in \( O \) (the matrices of 1 determinant and zero trace).

\[ V^{(0)} = (n_{0}n_{+})p_{-1}(n_{0}n_{+})^{-1}, V^{(1)} = (n_{1}n_{+})p_{-1}(n_{1}n_{+})^{-1}. \]

### 6.1 Inclusion of \( R_{i} \) in \( \text{Ker} \tilde{Q}_{i} \)

\( R_{0} \) et \( R_{1} \) are defined as being the differential rings generated by the coefficients of \( \mathcal{L}^{(0)} \) and of \( \mathcal{L}^{(1)} \).

**Lemma 6.1.** \( R_{i} \) is contained in \( \text{Ker} \tilde{Q}_{i} \).

### 6.2 Lemma for the coefficients of \( V_{i} \)

Then, for \( i = 0,1 \), the coefficients of \( V_{i} \) are in \( R_{i} \). To see it, the following result must be showed:

**Lemma 6.2.** The following conjugations hold:

\[ \mathcal{L}_{i} = K_{i}(\partial_{x} + p_{-3} + \sum_{j \geq 0} d_{2j-1}^{(i)}p_{2j-1})K_{i}^{-1}, \tag{3} \]
where the $K_i$ are matrices of $\text{Sl}_2(R_i(\lambda))$ and the $d^{(i)}_{2j-1}$ belong to $R_i$.

The fact that the coefficients of $V_i$ are in $R_i$ can be showed then in the following way. By definition of the $\mathcal{L}_i$:

$$\mathcal{L}_i = (n_in_+)(\partial + \sum_{j \geq 0} d^{(i)}_{2j-1} p_{2j-1} + 1)(n_in_+)^{-1}.$$ 

Comparing the formula with (3), it is possible to deduce that the classes of $K_i$ and of $n_in_+$ in $N_+/A_+$ are the same. It follows the equality of $K_ip_{-1}K_i^{-1}$ and of $V^{(i)}$. From the lemma 6.2, the coefficients of $K_i$ are in $R_i$, which implies that the coefficients of $V^{(i)}$ are in $R_i$ too.

### 6.3 The action of $\bar{Q}_0$ and $\bar{Q}_1$

**Theorem 6.1 .** $\bar{Q}_0$ and $\bar{Q}_1$ define vector fields over the space $\text{Jets}_{c}$, which, by the isomorphism between the jets and the coordinates over $N_+/A_+$, are identifiable with the regular action of the generators $2f_{-1}$ and $2e_{-1}$ of $n_+$ over $N_+/A_+$.

### 7 The intersection of the kernels of $\tilde{Q}_i$

The intersections of the kernels of the operators $\tilde{Q}_i$, defined just above, are here identified with the integrals of motion given by the $d^{2i+1}_i$. To obtain the result, a resolution is used, of Bernstein-Gelfand-Gelfand (BGG) type of the modulus $\pi_0$ in the jets over which $n_+$ acts.

### 7.1 Recalls of cohomology

The cohomology for the algebra $n_+$ with coefficients in the modulus $M$ is defined by:

$$H^\ast(n_+, M) = \text{Hom}(R^\ast, M),$$

with $R^\ast$, a resolution of $n_+$. It is possible to find one by mean of the following complex:

$$C^\ast(M) = \{ \Psi : \Lambda^\ast n_+ \rightarrow M \},$$

which has the differential:

$$\partial \Psi(x_1, x_2, \ldots, x_{n+1}) =$$

$$\sum_{i<j} (-1)^{i+j+1} \Psi([x_i, x_j], x_1, x_2, \ldots, \tilde{x}_i, \ldots, \tilde{x}_j, \ldots, x_{n+1}) +$$

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\[ + \sum_{i=1}^{n+1} x_i \Psi(x_1, x_2, \ldots, \bar{x}_i, \ldots, x_{n+1}). \]

It is possible to verify:
\[ \partial_{n,n+1} \circ \partial_{n-1,n} = 0, \]
\[ H^0(n_+, M) = \{ m, n.m = m, \forall n \in n_+ \}, \]
\[ H^1(n_+, M) = Z^1(n_+, M)/B^1(n_+, M), \]
\[ Z^1(n_+, M) = \{ \Psi : n_+ \to M, \Psi([x, y]) = x\Psi(y) - y\Psi(x) \}, \]
\[ B^1(n_+, M) = \{ \Psi : n_+ \to M, \exists m \in M, \Psi(x) = x.m \}. \]

### 7.2 The BGG resolution

The general BGG resolution, for a modulus \( M \) over the algebra \( n_+ \), is the following complex:
\[ \mathcal{F}^*(M), \mathcal{F}^0 = M, \mathcal{F}^1 = \mathcal{F}^2 = \ldots = M^{\otimes 2}, \]
with the differentials \( d_{01}, d_{12}, d_{i,i+1}, \)
\[ d_{01} = e_0 \oplus e_1, \]
\[ d_{12}|_{M \times 0} = (e_0^3 - 3e_0^2e_1 + 3e_0e_1e_0 + e_1^2), \]
\[ d_{12}|_{0 \times M} = (e_0e_1^2 - 3e_1e_0e_1 + 3e_1^2e_0, e_1^3), \]
\[ d_{i,i+1}|_{M \times 0} = (e_0^{2i+1}, p_i), d_{i,i+1}|_{0 \times M} = (q_i, e_1^{2i+1}), \]
with \( p_i, q_i \in U n_+ \), so that the applications form a complex.

**Theorem 7.1** [3]. The cohomology of the complex is:
\[ H^*(n_+, M). \]
7.3 Lemma over the kernel of the operators $\tilde{Q}_i$

In the case of the modulus $\pi_0 = \mathbb{C}[Jets]$, and $\pi_0^c = \mathbb{C}[Jets_c]$, with action of $\mathfrak{n}_+$, the $\pi_n$ are defined as being the modulus $\pi_0$, with a differential $\partial + 2nH_0$, and the $\tilde{\pi}_n = \pi_n/\text{Im}(\partial_n)$. The operators $\tilde{Q}_i$ and $\bar{Q}_i$, act over them, by mean of the commutation lemma:

$$\tilde{Q}_0 \circ \partial_n = \partial_{n-1} \circ \bar{Q}_0,$$

$$\tilde{Q}_1 \circ \partial_n = \partial_{n+1} \circ \bar{Q}_1.$$

Then, the bicomplex $B^*(\tilde{\pi}_0, \delta)$, ($\delta$, the differential) is considered:

$$\begin{array}{c}
\mathcal{F}^* \\ \mathcal{F}^0 \\ \mathcal{F}^1 \\ \vdots \\ \mathcal{F}^n \\
\xrightarrow{\delta} \\
\xrightarrow{\delta} \\
\xrightarrow{\delta} \\
\xrightarrow{\delta} \\
\xrightarrow{\delta}
\end{array},$$

$$\begin{array}{c}
\mathcal{F}^* \\ \mathcal{F}^0 \\ \mathcal{F}^1 \\ \vdots \\ \mathcal{F}^n \\
\xrightarrow{\partial_n} \\
\xrightarrow{\partial_n} \\
\xrightarrow{\partial_n} \\
\xrightarrow{\partial_n} \\
\xrightarrow{\partial_n}
\end{array}.$$

Lemma 7.1.

$$\cap_{i=0,1} \text{Ker}(\tilde{Q}_i) = H^1(B^*(\tilde{\pi}_0, \delta)).$$

7.4 The kernels of $\tilde{Q}_i$, integrals of motion

A definition of a graduation over the space of the jets and the coordinates of the algebraic quotient is given. Over $\mathbb{C}[\mathfrak{N}_+/\mathfrak{A}_+]$, the following graduation is defined: for $V = \oplus_i V_i$ a $\mathfrak{n}_+$-modulus of graduation compatible with the principal one of $\mathfrak{n}_+$, and $V^*$, its dual; it is graduated by $(V^*)_i = (V_{-i})^*$. For $v$ in $V$, annulated by $\mathfrak{a}_+$, $\Psi$ in $V^*$ and $n$ in $\mathfrak{n}_+$, it is defined:
\[ n \mapsto \langle \Psi, nv \rangle \in \mathbb{C}[N_+/A_+], \]

the degree of the function defined equal to \( i - j \). It defines a structure of graduated ring over \( \mathbb{C}[N_+/A_+] \). A natural graduation over the cohomology \( H^i(\mathfrak{n}_+, \mathbb{C}[N_+/A_+]) \) can be deduced then. The isomorphism of the Shapiro lemma is compatible with the graduation. By the isomorphism with \( \mathcal{O} \), the coefficients of the matrix \( V_i \) are of degree \(-i - 1\); it can be showed by application of the above definition to the adjoint representation of \( \mathfrak{n}_+ \). On the other hand, it is demonstrated by recurrence that the \( d_{2i+1} \) are of degree \( 2i + 1 \). Let the degree over \( \mathbb{C}[\text{Jets}] \) be:

\[
\begin{align*}
\text{deg}(H_{-2}) &= -1, \ 
\text{deg}(E_{-1}) = \text{deg}(F_{-1}) = -2, \\
\text{deg}(H_0) &= -3, \ 
\text{deg}(\partial) = -3.
\end{align*}
\]

The differential ring \( (\mathbb{C}[d_{-1}^{(i)}], \partial_0) \) has the degree defined by:

\[
\begin{align*}
\text{deg}(d_{-1}) &= -2, \ 
\text{deg}(\partial_0) = -3,
\end{align*}
\]

and the ring \( \mathbb{C}[N_+/A_+] \otimes \mathbb{C}[d_{-1}^{(i)}] \), the degree defined as the product degree with the one of \( 7.4 \).

**Lemma 7.2.** The isomorphism:

\[
\mathbb{C}[\text{Jets}] \cong \mathbb{C}[N_+/A_+] \otimes \mathbb{C}[d_{-1}^{(i)}],
\]

established in corollary 2.2 preserves the degrees.

The following theorem concerns then the case \( \pi_0 \) and \( \pi_0^c \) too:

**Theorem 7.2.** The intersection of the kernels of the operators \( \tilde{Q}_i \) is the vector space generated by the integrals of the \( d_{2i+1} \):

\[
\int d_{2i+1}(x)dx,
\]

in the case of \( \pi_0^c \), and in the case of \( \pi_0 \), too by the functionals:

\[
\int P(d_{-1}, d'_{-1}, ..., d_{-1}^{(n)}, ...) (x)dx.
\]

The \( d_{2i+1} \) are moreover elements which are in involution in the algebra of the jets fitted with the Poisson brackets.

---

\[ ^4 \text{The } d_{2i+1} \text{ have been defined in the proposition 2.1 for the dressing of Drinfeld-Sokolov.} \]
Proposition 7.1. The $\oint d_{2i+1}$, which are in $\mathbb{C}[\pi_0^0]$, are in involution for the Poisson brackets.
References

[1] V. V. Bazhanov, S. L. Lukyanov and A. B. Zamolodchikov, *Integrable quantum field theories in finite volume: excited state energies*, preprint hep-th/9607099.

[2] J.N.Bernstein, I.M.Gelfand and S.I.Gelfand, dans *Representations of Lie groups*, ed. I.Gelfand, Wiley, New-York 1975, pp. 21-24.

[3] L.A.Dickey, *Why the general Zakharov-Shabat equations form a hierarchy?*, preprint hep-th/9305105. Comm. Math. Phys. **163**, 509-522 (1994).

[4] V.G.Drinfeld and V.V.Sokolov, *Equations of Kortweg-de Vries type and simple Lie algebras*, in Sov.Math.Dokl. 23, 457-462 (1981); *The Lie algebras and the equations of KdV type*, in J.Sov.Math. 30, 1975-2036 (1985).

[5] F.Delduc, L.Feher and L.Gallot, *Nonstandard Drinfeld-Sokolov reduction*, preprint solv-int/970802.

[6] 1. B.Enriquez and E.Frenkel, *Equivalence of two approaches to the mKdV hierarchies*, Comm. Math. Phys. **185**, 211-230 (1997).
2. B.Enriquez, *Nilpotent action on the KdV variables and 2-dimensional Drinfeld-Sokolov reduction*, Theor.Math.Phys. **98**, 256-258 (1994), preprint hep-th/9311161.

[7] L.D.Faddeev and L.A.Takhtajan, *Hamiltonian Methods in Soliton Equations*, Springer-Verlag 1992.

[8] B.Feigin and E.Frenkel, *Integrals of motion and quantum groups*, preprint hep-th/9310022v3.

[9] B.Feigin and E.Frenkel, *Kac-Moody groups and integrability of solitons equations*, Inventiones math. **120**, 379-408 (1995), preprint hep-th/9311171.

[10] 1. J.-L. Gervais and M.V. Saveliev, Phys.Lett. B **286**, 271-72 (1992).
   2. J.-L. Gervais and M.V. Saveliev, *Higher Grading Generalisations of the Toda Systems*, preprint hep-th/9505047.

[11] V.G.Kac, *Infinite Dimensional Lie Algebras*, third ed., Cambridge University Press 1990.

[12] S.P.Novikov, *Solitons & Geometry*, Cambridge University Press 1994.