Tachyon mass, c-function and Counting localized degrees of freedom.

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Abstract

We discuss the localized tachyon condensation in the non-supersymmetric orbifold theories by taking the cosmological constant as the measure of degrees of freedom (d.o.f). We first show asymptotic density of state is not a proper quantity to count the 'localized' d.o.f. We then show that localized d.o.f lead us a c-function given by the lightest tachyon mass, which turns out to be the same as the tachyon potential recently suggested by Dabholkar and Vafa. We also argue that delocalized d.o.f also encode information on the process of localized tachyon condensation in the g-function, based on the fact that the global geometry of the orbifolds is completely determined by the local geometry around the fixed points. For type II, both c- and g-function respect the stability of the supersymmetric models and both allow all the process suggested by Adams, Polchinski and Silverstein.
1 Introduction and Summary

Recently there has been some interests on tachyon condensation in closed string theories [1, 2, 3, 4, 5]. Closed string tachyons indicate the decay of the spacetime itself and full dynamical understanding is still lacking. To simplify the problem and to utilize the experience from the open string case [3] Adams, Polchinski, and Silverstein (APS) [2] studied the cases where the tachyons are localized at the tip of the orbifolds. They argued that the tachyon condensation causes the orbifolds to decay and the tip of the orbifolds smooth out. They supported this conjecture by D-brane probes in sub-stringy regime and by general relativity analysis beyond string scale. According to the analysis of APS, the tachyon condensation induces cascade of phase transitions until all the tachyons disappear and supersymmetry restored. So each step of cascade involve a RG flow along which IR is less tachyonic, less singular, and more supersymmetric than UV.

Having acquired the general tendency of localized tachyon condensation from the analysis of APS, it is most desired to have a c-function [7] which decreases along the possible RG-flows and summarize the possible decay modes. In this context, Dabholkar and Vafa (DV) [8], utilizing the worldsheet $N = 2$ supersymmetry, have proposed a closed string tachyon action describing the real time tachyon dynamics, where generalized c-function [9] was adopted as the tachyon potential. In the same context, Harvey, Kutasov, Martinec and Moore (HKMM) [10] suggested another criteria based on counting the asymptotic density of states (ADOS), which suggested that not all examples of APS is consistent with their analysis. In Ref. [11], we proposed a c-function of RG flow that is analogous but different from that suggested in [10]. The proposal in [8, 11] allows all of the process of APS, while that in [10] does not. Since different proposals, which do not agree fully, are suggested, it is an urgent matter to resolve this issue.

In this paper, we discuss this issue by counting the degrees of freedom (DOF) from slightly different point of view. First we will show that, for the problem at hand, the cosmological constant (CC) is a good measure of degree of freedom even in the presence of the spacetime fermions. We will then show that while the cosmological constant and the asymptotic density of states (ADOS) yield the same central charge for the total DOF, they give different results for the localized DOF. After showing that the latter (ADOS) yields the bulk degree of freedom smaller than the localized degree of freedom, which is counter intuitive, we will suggest that we have to use the former (CC).

We conclude that what measure the localized degree of freedom is the minimal twisted tachyon mass, which turns out to be equal to a c-function’s value at conformal points. The
result does not depend on whether we use total partition function or its bosonic part.

The minimal tachyon mass can be shown to be equal to the deficit angle of the orbifold geometry. It is also equal to the tachyon potential suggested by Dabholkar and Vafa [8]. The proposal that the absolute value of minimal tachyon mass as the c-function of the RG flow means that the theory becomes less tachyonic along the phase transition. It is certainly a generalized c-theorem, proof of which, however, is not an issue here.

Although we mainly discuss the localized degree of freedom, the bulk degree of freedom also encodes some information of the orbifold geometry, because local geometry around the tip completely fix the global geometry for the orbifolds. It turns out that considering the delocalized degree of freedom leads us g-function; $g_{cl}$ for type 0 as given in [10] and $g_{cl}^{II}$ as given in [11]. So emerging picture is following: localized DOF gives a c-function that is given by minimal twisted tachyon mass, while the delocalized DOF leads us to g-theorems.

Our analysis seems to support the picture that the cosmological constant itself, although divergent due to the tachyons, can play the role of the tachyon potential in closed string theories. This is in a close analogy with the open string phenomena where world-sheet (sphere) partition function is the spacetime action. In the presence of the worldsheet supersymmetry, there is no potential coming from the sphere amplitude for the closed string tachyon [12], therefore the 1-loop result should be the leading order tachyon potential. This picture predicts that IR is more supersymmetric, less tachyonic and less singular than UV, which is precisely what the analysis of Adams, Polchinski and Silverstein suggests. We end the paper by possible future problems.

2 Cosmological constant v.s Asymptotic density of states

We first consider cosmological constant, namely, the string vacuum amplitude on torus,

$$Z = \int_F \frac{d^2 \tau}{\tau_2} Z(\tau),$$  \hspace{1cm} (2.1)

where $F$ represent a fundamental domains of torus moduli, $|\tau| > 1, |\tau_1| < \frac{1}{2}, \tau = \tau_1 + i\tau_2$. Notice that our fundamental domain does not contain the $\tau_2 \to 0$ region. $Z(\tau)$ is the torus partition function.

On the other hand, the general partition function can be written as follows:

$$Z(\tau) = \text{Tr} \exp(-\tau_2 \pi \alpha' M^2 + 2\pi i \tau_1 (L_0 - \bar{L}_0)),$$  \hspace{1cm} (2.2)
where $M^2$ is the mass operator. Therefore in the presence of tachyons, the dominant divergent part comes from the $\tau_2 \to \infty$ region. In type 0 case, the bulk tachyon (tachyon in untwisted sector) gives the dominant contribution. In type II theory, it is projected out so the dominant contribution comes from the lightest twisted tachyon. Although the integral diverges, we are interested in the degree of divergence, $c_{\text{eff}}$, which precisely is the information of degree of freedom. To do this, we can simply approximate the integral by the value of $Z(\tau_2)$ at $\tau_2 = \infty$, $\tau_1 = 0$ multiplied by the area of fundamental domain, $\int_F d^2\tau/\tau_2^2 = \frac{4}{3}$. One can do slightly better by averaging over $\tau_1$;

$$
\int_F \frac{d^2\tau}{\tau_2^2} Z(\tau) = \frac{\pi}{3} \lim_{\tau_2 \to \infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 Z(\tau). \tag{2.3}
$$

This is nothing but a version of the theorem [13] by Kutasov and Seiberg in the presence of tachyon. For practical calculation, we can simply set $\tau_1 = 0$ with the level matching condition properly imposed. See Eq.(2.2). So, if there are tachyons, the cosmological constant can be simply related to the low temperature limit of the partition function;

$$
Z = \lim_{\tau_2 \to \infty} \frac{\pi}{3} Z(\tau_2 = \infty) = \lim_{\tau_2 \to \infty} \exp(\pi \alpha'|M^2 \min|\tau_2). \tag{2.4}
$$

Here, $M^2 \min$ is minimal mass of the twisted tachyons and we neglected power of $\tau_2^{-3}$ as well as constant factors in front of the exponential divergence.

In a general conformal field theory, the partition function in the low temperature limit gives the central charge by

$$
Z(\tau_2 \to \infty, \tau_1 = 0) \sim g q^{-c_{\text{eff}}/12}, \tag{2.5}
$$

where $q = \exp(-2\pi \tau_2)$, $g$ is a co-efficient of the leading term and

$$
c_{\text{eff}} := c - 24\Delta_{\min} \tag{2.6}
$$

with $\Delta_{\min} = \min\frac{1}{2}(\Delta + \bar{\Delta})$. From Eqs.(2.2) and (2.5), the lightest tachyon mass naturally gives an effective central charge

$$
c_{\text{eff}} = 6|\min \alpha' M^2|. \tag{2.7}
$$

Therefore the precise definition of the quantity we are interested in can be given as

$$
c_{\text{eff}} := \frac{6}{\pi} \lim_{\tau_2 \to \infty} \frac{1}{\tau_2} \log Z(\tau). \tag{2.8}
$$

$^1\tau_2$ is the inverse temperature $\beta$
Since $c_{\text{eff}}$ is the measure of degree of freedom, so is the cosmological constant. On the other hand, there is well known measure of degrees of freedom which is a high energy density of states. It comes from the high temperature behaviour of the partition function;

$$Z(\tau_2 \to 0) \sim \tilde{q}^{-c_{\text{eff}}/12}, \quad (2.9)$$

where $\tilde{q} = \exp(-2\pi/\tau_2)$. When we deal with modular invariant (total) partition functions, two $c_{\text{eff}}$’s appearing in Eq.(2.5) and Eq.(2.9) are the same, of course. However, for the localized degree of freedom states, two are different as we will show shortly.

In our case, $Z(\tau)$ is the partition function of the CFT representing lightcone string theory in 10 dimension. If the bulk tachyon is present, Eq.(2.7) correctly identifies the transverse dimension 8 out of central charge; $c = c_{\text{eff}} = 6 \times | -2 | = 8 \times 3/2$ from the tachyon spectrum.

So far, we have seen that, in the presence of tachyons, the cosmological constant or the low temperature limit of the partition function can a good measure of degree of freedom. However, if there are spacetime fermions, it is more subtle. For example, in the supersymmetric theory, the GSO projected partition function $Z = Z_B - Z_F$ is identically zero. Then when it correctly count the degree of freedom? Looking at the eq. (2.5) the answer is obvious: As far as the dominant piece, $q^{-c_{\text{eff}}/12}$, is not cancelled, it is a good measure of degrees of freedom.

3 Counting localized degree of freedom

Now we discuss on the splitting the degree of freedoms into localized and delocalized ones. In ref. [10], it is suggested that localized degree of freedom is coming from the partition function of the twisted fields, and the delocalized (or bulk) degree of freedom is coming from the untwisted part;

$$Z(\tau) = Z_{\text{un}} + Z_{\text{tw}}. \quad (3.1)$$

Since the twisted field is fixed at the tip of the orbifolds, this is very reasonable suggestion. They proceeded to count the localized degree of freedom by looking at the ADOS coming from the high temperature behaviour of the twisted partition function, $Z_{\text{tw}}(\tau_2 \to 0)$.

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2If we consider the conformal field theory on the orbifold $\mathbb{C}^4/\mathbb{Z}_N$ rather than the full string partition function in 10 dimension, its partition function is above $Z(\tau)$ without $|\eta|^{-18} \sim q^{-9/12}$. So the central charge 3 could have been simply read off from the bulk tachyon. But there is no meaning of "mass spectrum", so is the word "tachyon". This is partly our motive to deal with string partition function rather than the relevant orbifold CFT itself.
3.1 type 0: with bulk tachyon

If we look at the high temperature limit, we get the result of HKMM [10]:

\[ Z_{tw}(\tau_2 \to 0) \sim g_{cl} q^{-1}, \]  

(3.2)

where \(-1\) comes from the bulk tachyon spectrum \(\frac{1}{2} \alpha' M^2 = -1\). It gives us \(c = 12\) suggesting that it is measuring bulk degrees of freedom, while we want to measure localized ones when we use \(Z_{tw}\). Notice that while \(Z_{tw}(\tau_2)\) does not contain any bulk tachyon spectrum, the bulk tachyon spectrum is introduced by the modular transformation. Now, if we look at the delocalized degree of freedom, its high temperature limit gives

\[ Z_{un}(\tau_2 \to 0) \sim q^{\frac{1}{2} \alpha' M_{\text{min}}^2}, \]  

(3.3)

where \(\frac{1}{2} \alpha' M_{\text{min}}^2\) is the minimal mass of the twisted sector. Since

\[ \left| \frac{1}{2} \alpha' M_{\text{min}}^2 \right| = 1 - \frac{1}{N} < 1, \]  

(3.4)

for \(C^1/Z_N\) model, the localized degree of freedom is bigger than the bulk degree of freedom (DOF), which is counter-intuitive. Bulk and local DOF are interchanged. The reason of bulk-local interchange is obviously due to the modular transformation. So, when we use modular non-invariant object as a indicator of physical quantity, we need care.

On the other hand, if we use the cosmological constant, there is no modular transformation involved and we get intuitive results:

\[ Z_{tw}(\tau_2 \to \infty) \sim q^{\frac{1}{2} \alpha' M_{\text{min}}^2}, \]  

(3.5)

and

\[ Z_{un}(\tau_2 \to \infty) \sim g_{cl} q^{-1}. \]  

(3.6)

Using the cosmological constant as the measure of degree of freedom is also consistent with following intuitive picture: the bulk tachyon counts the bulk degree of freedom while the twisted tachyon counts the localized degree of freedom. So, the quantity measuring the localized quantity is the minimal tachyon mass. We will evaluate it explicitly in later section.

3.2 type II: no bulk tachyon

Now we apply this result to type II case, where bulk tachyon is projected out by chiral GSO and the dominant contribution comes from the lightest tachyon. As we have shown before, we
can use the total partition function, since the dominant contribution is coming from twisted sector and that term is not cancelled out. Furthermore we get the same value of $c_{eff}$ whether we use the full GSO projected partition function or just its bosonic part. The reason is very simple: when we conceptually write the total partition function as the difference of bosonic and fermionic contributions,

$$Z^{II} = Z^{II}_B - Z^{II}_F,$$

(3.7)

neither $Z^{II}_B$ nor $Z^{II}_F$ should contain the bulk tachyon; it is simply not in the spectrum of the theory. Therefore both quantities $Z^{II}_{tw}, Z^{II}_{B,tw}$ give the same localized degree of freedom given by

$$c_{eff} = 6|\alpha'M^2_{min}|,$$

(3.8)

which we will evaluate shortly.

Now we meet interesting question for type II: Since there is no bulk tachyon, calculating the delocalized degree of freedom

$$Z_{un}(\tau_2 \to \infty) \sim g^{II}_{cl} q^0$$

(3.9)

shows that there is no bulk degree of freedom ($c_{eff} = 0$). At first looking this looks puzzling. However, it is not so surprising by taking the analogy of the minimal models, where we get $c < 1$ minimal models by taking out the degrees of freedom from the Hilbert space of the $c = 1$ models. Along this line of thinking, spacetime supersymmetric theory is the case where all the degree of freedom is eliminated from 2 dimensional point of view. From this point of view, even in the supersymmetric case, where $Z = 0$, $Z$ still correctly counts the central charge of the relevant 2d CFT. This is also consistent with the c-theorem in the non-compact space: the derivative of the central charge must be zero, hence the bulk degree of freedom should not change along the RG-flow, as pointed out by HKMM [10]. It is 0 in type II and 12 in type 0. These values do not change under the local tachyon condensation process. Therefore the transition between the type 0 and type II is impossible from this point of view.

4 Minimal twisted tachyon mass as a $c$-function

In the previous sections, we argued that the $c$-function for the RG-flow in localized tachyon condensation is the minimal tachyon mass. We now explicitly evaluate the tachyon masses of string theory on $\mathbb{R}^{7,1} \times \mathbb{C}/\mathbb{Z}_N$. 

4.1 Type II orbifolds

Our starting point is the following 1-loop string vacuum amplitude of type II orbifold model:

\[ Z^{II} = \int_F \frac{(d\tau)^2}{4\tau_2} (4\pi^2\alpha'\tau_2)^{-4} Z^{II}(\tau), \quad (4.1) \]

where \[ Z^{II}(\tau) = \sum_{l,m=0}^{N-1} Z_{l,m}(\tau) \]

\[ = \sum_{l,m=0}^{N-1} \frac{|\theta_3(\nu_{lm}|\tau)\theta_3(\tau)^3 - \theta_2(\nu_{lm}|\tau)\theta_2(\tau)^3 - \theta_4(\nu_{lm}|\tau)\theta_4(\tau)^3|^2}{4N|\eta(\tau)|^{18}|\theta_1(\nu_{lm}|\tau)|^2}, \quad (4.2) \]

where \( \nu_{lm} = \frac{k}{N}(l - m\tau) \). The dominant contribution comes from the tachyons in the \( \tau_2 \to \infty \) limit. \( Z^{II}_{tw}(\tau) \), \( Z^{II}_{un}(\tau) \) are defined by

\[ Z^{II}_{tw}(\tau) = \sum_{m=1}^{N-1} \sum_{l=0}^{N-1} Z_{l,m}(\tau), \quad Z^{II}_{un}(\tau) = \sum_{l=0}^{N-1} Z_{l,0}(\tau). \quad (4.3) \]

The low temperature limit of the partition function \( Z^{II}_{tw}(\tau) \) can be easily evaluated to be

\[ Z^{II}_{tw}(\tau_2 \to \infty) = \sum_{m=1}^{N-1} q^{\frac{1}{2}\alpha'M^2} \sim \exp \left( \tau_2|\min_{m=1}^{N-1} \pi\alpha'M^2| \right), \quad (4.4) \]

where

\[ \frac{1}{2}\alpha'M^2 = \begin{cases} \{ km \} & \text{if } \left[ \frac{km}{N} \right] \in 2\mathbb{Z}, \\ \left[ \frac{km}{N} \right] - 1 & \text{if } \left[ \frac{km}{N} \right] \in 2\mathbb{Z} + 1, \end{cases} \]

\[ (4.5) \]

where we used the notation \( \{ x \} \) for the fractional part of \( x \) and \( [x] \) for the integer part of \( x \).

The minimal mass of the twisted tachyon can be easily seen to be

\[ |\min_{m=1}^{N-1} \alpha'M^2| = 2 \left( 1 - \frac{1}{N} \right). \quad (4.6) \]

Notice that it is independent of \( k \) as long as \( N, k \) are co-prime to each other. It is easy to prove above statement using the basic lemma of the number theory. From eq.(2.7) we get
\( c_{\text{eff}} = 12 \left( 1 - \frac{1}{N} \right) \). Notice that this value is proportional to the deficit angle, \( \delta = 2\pi(1 - 1/N) \) of the orbifold geometry.

In a recent paper [3], Dabholkar and Vafa (DV) suggested that the tachyon potential should be given by the maximum value of the axial vector charge \( Q_5 = F_L + F_R \), i.e. \( V(t, \bar{t}) = \max |Q^5| \). For \( \mathbb{C}/\mathbb{Z}_N \) case, one has \( Q_5 = 1 - \frac{1}{N} \). It is also well known that with \( N=2 \) worldsheet supersymmetry, \( Q_5 \) plays the role of the effective central charge, \( c_{\text{eff}} = 12 \times \max |Q^5| \). Therefore, we can identify the potential as the minimal tachyon mass;

\[
V(t, \bar{t}) = \frac{1}{2} |\min \alpha' M^2|. 
\] (4.7)

Quite some time ago, Cecotti and Vafa [4], proposed that \( Q^5 \) is the generalized \( c \)-function for \( d=2, \ N=2 \) SCFT. They called this as algebraic \( c \)-function and partially proved the theorem near critical points. What DV did is to adopt this \( c \)-function as the tachyon potential. So, we now can say that the minimal tachyon mass is the \( c \)-function of the RG-flow. From now on we use the tachyon potential and \( c \)-function as the same words.

In \( \mathbb{C}^2/\mathbb{Z}_N \) model, the tachyon spectrum again can be calculated by taking the \( \tau_2 \to \infty \) limit of the partition function:

\[
Z_{\text{tw}}^{II}(\tau) = \sum_{l,m=0}^{N-1} \frac{|\theta_1(\nu_{l,m}^1 | \tau)\theta_1(\nu_{l,m}^2 | \tau)|^2 - \theta_2(\nu_{l,m}^1 | \tau)\theta_2(\nu_{l,m}^2 | \tau)|^2 - \theta_4(\nu_{l,m}^1 | \tau)\theta_4(\nu_{l,m}^2 | \tau)|^2|^2}{4N |\eta(\tau)|^{12}|\theta_1(\nu_{l,m}^1 | \tau)|^2}. 
\] (4.8)

The low temperature limit of \( Z_{\text{tw}}^{II}(\tau) \) is

\[
Z_{\text{tw}}^{II}(\tau_2 \to \infty) = \sum_{m=1}^{N-1} \exp(-\pi \tau_2 \alpha' M^2), \tag{4.9}
\]

where,

\[
\frac{1}{2} \alpha' M^2 = - \left\lfloor \frac{k_1 m}{N} \right\rfloor - \left\lfloor \frac{k_2 m}{N} \right\rfloor, \quad \text{if} \quad \left\lceil \frac{k_1 m}{N} \right\rceil + \left\lfloor \frac{k_2 m}{N} \right\rfloor \in 2\mathbb{Z},
\]

\[
= - \left\lfloor \frac{k_1 m}{N} \right\rfloor + \left\lfloor \frac{k_2 m}{N} \right\rfloor - 1, \quad \text{if} \quad \left\lceil \frac{k_1 m}{N} \right\rceil + \left\lfloor \frac{k_2 m}{N} \right\rfloor \in 2\mathbb{Z} + 1, \tag{4.10}
\]

here again \( \{x\} \) is the fractional part and \( [x] \) is the integer part of \( x \). Notice that the identification of the minimal mass as the tachyon potential

\[
V(N, k_1, k_2) = -\min_{m=1}^{N-1} \frac{1}{2} \alpha' M^2(N, k_2, m), \tag{4.11}
\]
still holds and it agrees with that of DV for $C^2/Z_N$ also. Notice that the minimal mass vanishes along the supersymmetric line $k_1 = k_2$. The minimal mass of the twisted tachyon is not simple in terms of rational function of $k_1, k_2$ and $N$. For $k_2 = 2p + 1, k_1 = 1$, however, we find

$$V(N, 1, k_2) = (1 - 1/k_2)(1 - r/N), \quad if \quad N \equiv r \mod k_2, \quad r = 1, 2, ..., p$$

$$\quad = (1 - 1/k_2)(1 - r(1 + 1/p)/N), \quad if \quad N \equiv -r \mod k_2,$$

(4.12)

For $k_1 = 2$, some of the $V(N, 2, k_2)$ can be shown to be

$$V(N, 2, 4p) = 1 - (2p + 1)/N, \quad if \quad N \equiv 1 \mod 4p,$$

$$V(N, 2, 4p + 2) = (1 - k_1/k_2)(1 - 1/N), \quad if \quad N \equiv 1 \mod 4p + 2.$$

(4.13)

Other cases where both $k_1, k_2$ are bigger than 2, are more complicated. Using these formula, it is easy to show that all of the process studied in APS are allowed by the potential. In figure 1, we give a contour plot of the potential in $k_1, k_2$ plane. The darker the color, the lower the potential.

![Contour plot of type II's tachyon potential](image)

Figure 1: Contour plot of type II’s tachyon potential $V_{II}(N, k_1, k_2) = \frac{1}{2} |\min \alpha'M^2|$ in $(k_1, k_2)$ plane. Only $k_1 > 0, k_2 > 0$ is shown for $N = 23$. It has a valley along the line $k_1 = k_2$, which guarantees the stability of the supersymmetric models.

The proposal that the minimal twisted tachyon mass as the c-function of the RG flow means the theory becomes less tachyonic along the phase transition. It is certainly a generalized c-theorem: $c_{eff}$ decreases along the RG-flow, which still need more rigorous proof. Since the $c_{eff}$
represent the most dominant term of the cosmological constant, decrease in tachyon mass is equivalent to the decrease in the cosmological constant. Therefore all these argument seems to support the speculation that in the absence of the tachyon potential coming from sphere amplitude, torus partition function is the dominant contribution to the effective potential for the tachyon of closed superstring theory.

4.2 Type 0

We now work out the c-function for the type 0 orbifolds by calculating minimal mass of localized tachyons. First for $C^1/Z_N$ model, the partition function is

$$Z^0(\tau) = \sum_{l,m=0}^{N-1} |\theta_3(\nu_{lm}|\tau)\theta_3(\tau)^3|^2 + |\theta_2(\nu_{lm}|\tau)\theta_2(\tau)^3|^2 + |\theta_4(\nu_{lm}|\tau)\theta_4(\tau)^3|^2, \quad (4.14)$$

with $k = odd$ and $(N,k)$ co-prime. The low temperature limit of the twisted partition function $Z^0_{tw}(\tau)$ can be easily evaluated to be

$$Z^0_{tw}(\tau_2 \rightarrow \infty) = \sum_{m=1}^{N-1} q^{1/2} \alpha' M^2$$

where

$$\frac{1}{2} \alpha' M^2 = \min \left( -\left\{ \frac{km}{N} \right\}, \left\{ \frac{km}{N} \right\} - 1 \right). \quad (4.16)$$

So the localized degrees of freedom is counted by the dominant twisted tachyon term $q^{1/2} \alpha' M^2$, from which we identify the c-function for localized tachyon for type 0;

$$V(N,k) = \min_{m=1}^{N-1} \frac{1}{2} \alpha' M^2(N,k,m) = \max_{m=1}^{N-1} \max \left( \left\{ \frac{km}{N} \right\}, 1 - \left\{ \frac{km}{N} \right\} \right). \quad (4.17)$$

This looks slightly different from type II result. But in fact it gives identical result;

$$V(N,k) = 1 - \frac{1}{N}. \quad (4.18)$$

Notice that $g_{cl} q^{-1}$, which is coming from the bulk tachyon, is not present in $Z_{tw}$ in the low temperature limit. It is contained in $Z_{un}$, counting the delocalized degrees of freedom. So, we expect that $g_{cl}$ can encode the information for the process of the localized tachyon condensation, due to the equivalence of the of the local and global geometry of the orbifold as mentioned before. This is discussed in separate publication [11].
Now we turn to type 0, $C^2/Z_N$ case. The partition function is given by

$$Z^0(\tau) = \sum_{l,m=0}^{N-1} \left| \theta_3(\nu^1_{l,m}|\tau)\theta_3(\nu^2_{l,m}|\tau)\theta_3(\tau)^2 \right| + \left| \theta_2(\nu^1_{l,m}|\tau)\theta_2(\nu^2_{l,m}|\tau)\theta_2(\tau)^2 \right| + \left| \theta_4(\nu^1_{l,m}|\tau)\theta_4(\nu^2_{l,m}|\tau)\theta_4(\tau)^2 \right|^2.$$  

(4.19)

with $k_1 + k_2$ odd, gcd($k_1, k_2, N$) = 1.

The low temperature limit of the partition function $Z^0_{tw}(\tau)$ is

$$Z^0_{tw}(\tau_2 \to \infty) = \sum_{m=1}^{N-1} q^{-\frac{1}{2}12}\alpha'M^2,$$  

(4.20)

where

$$\frac{1}{2}\alpha'M^2 = \min \left( -\left\{ \frac{k_1m}{N} \right\} - \left\{ \frac{k_2m}{N} \right\}, -\left\{ \frac{k_1m}{N} \right\} + \left\{ \frac{k_2m}{N} \right\} - 1 \right).$$  

(4.21)

Again the c-function is given by

$$V(N, k_1, k_2) = \left| \min_{m=1}^{N-1} \frac{1}{2}\alpha'M^2(N, k_1, k_2, m) \right|.$$  

(4.22)

Instead of evaluating this in terms of rational functions, we give a result of numerical analysis summarized in the contour plot in figure 2.

## 5 Summary and Discussion

In this paper, we use cosmological constant to count the localized degree of freedom, and got following very intuitive results: The bulk tachyon mass counts the bulk degree of freedom while the lowest twisted tachyon mass counts the localized degree of freedom. One can even refine the statement by saying that in each twisted sector, the lowest tachyon mass in each sector counts the degree of freedom in that sector. We also evaluated the lowest tachyon mass in the type 0 and type II orbifold models. We evaluated these minimal masses for $C^1/Z_N$, $C^2/Z_N$ both for type 0 and type II.

Our results suggest that the cosmological constant (1-loop string amplitude) can play the role of tachyon potential. According to it, the RG flow along the direction of more SUSY, less tachyonic, less singular, which is precisely what APS analysis has shown. This is a close analogue of open string result\[17\], where effective action in 10 dimension is partition function at sphere. As shown by Tseytlin\[12\] there is no tachyon potential coming from sphere. So naturally torus amplitude can play the leading tachyon potential. In this approach it is necessary to related the localized nature of the tachyon potential and the divergence of the potential.
Finally, we discuss future problems. Obviously, the most demanding problem is actually to prove the generalized c-theorem: tachyon becomes less tachyonic along the RG-flow along the line of original Zamolochikov’s\cite{7} for the localized degree of freedom. It should be clarified explicitly why the $c_{eff}$ for the localized degree of freedom can be changed while the central charge for the bulk degree of freedom can not. In this paper, we mainly discussed the localized degree of freedom and got a c-function given by the minimal tachyon mass. However, the bulk degrees of freedom also encodes some information on the precession of localized condensation since local geometry around the fixed points completely fixes the global geometry for the orbifolds. In fact, by considering the delocalized degree of freedom, we arrive at the quantity $g_{cl}$ for type 0 appeared in \cite{10} and $g_{cl}^{II}$ discussed in \cite{11} as a c-function. Then the detailed relation between these different criteria (c-functions v.s g-functions) are desired. We will come back to this issue in future publication\cite{18}.

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