Relativistic accelerating electromagnetic waves

Shahen Hacyan

Instituto de Física, Universidad Nacional Autónoma de México, Apartado Postal 20-364, México D F, 01000, Mexico
E-mail: hacyan@fisica.unam.mx

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Abstract
The general form of the exact solutions of Maxwell’s equations in Rindler coordinates is obtained. These solutions represent electromagnetic waves that preserve their shapes in a relativistic uniformly accelerating frame. In the non-relativistic limit, the Airy beam turns out to be an exceptional solution with physically realizable parameters. A similarity with Bessel beams is pointed out.

Keywords: electromagnetic waves, Airy beams, special relativity

1. Introduction
In 1979, Berry and Balazs [1] obtained a solution of the Schrödinger equation describing a non-spreading accelerating wavepacket in terms of the Airy function. More recently, the Airy beam, an equivalent solution of the paraxial equation of optics, has received a considerable amount of attention from both a theoretical and an experimental point of view (see, e.g., [2–4]).

Previous works on accelerating waves have been restricted to the non-relativistic case. The aim of the present work is to elucidate the mathematical nature of the Airy beam, within the context of an exact relativistic formulation, and to characterize it as a singular solution of the Maxwell equations. For this purpose, the most general exact solution of the Maxwell equations is obtained in Rindler coordinates, which are the natural coordinates associated with uniform acceleration in special relativity. The resulting solution represents an electromagnetic wave that is shape invariant in a uniformly accelerating frame and thus appears as an accelerating wave in an inertial frame. In general, such waves correspond to physical conditions that are not realizable in a laboratory, but it turns out that the Airy beam is a singular case that can be constructed with realistic parameters in the non-relativistic limit.

The paper is organized as follows. The Maxwell equations are solved in section 2 in Rindler coordinates and the two fundamental modes of the field are identified. The general solution, invariant under space–time hyperbolic rotations, is obtained and the explicit forms of the energy density and the Poynting vector are worked out. The relation with the Airy beam in the non-relativistic limit is shown in section 3 and a discussion of the results is given in section 4.

2. Maxwell equations
The Rindler metric is obtained with a change from Cartesian $(ct, x, y, z)$ to Rindler coordinates $(cT, x, y, Z)$ defined as

$$
ct = Z \sinh(aT), \quad z = Z \cosh(aT),
$$

(2.1)

where $a$ is a parameter with dimensions of frequency (or acceleration divided by $c$). The metric thus takes the form

$$
ds^2 = -a^2 Z^2 dT^2 + dx^2 + dy^2 + dZ^2.
$$

(2.2)

The electromagnetic field tensor $F_{\alpha\beta}$ follows from a vector potential $A_{\alpha}$ such that

$$
F_{\alpha\beta} = \partial_{\alpha} A_{\beta} - \partial_{\beta} A_{\alpha}
$$

(2.3)

and is assumed to satisfy the source-free Maxwell equations

$$
\frac{\partial}{\partial x^\mu} (ZF^{\alpha\mu}) = 0.
$$

(2.4)

The two modes of the electromagnetic field in Rindler coordinates can be represented by the transverse electric ($E$) and transverse magnetic ($M$) vector potentials,

$$
A_{\mu E} = (0; \partial_2 U^E, -\partial_1 U^E, 0)
$$

(2.5)
and
\[ A^M = \left( aZ \partial_0 U^M, 0, 0, \frac{1}{aZ} \partial_3 U^M \right), \] (2.6)
given in terms of two potentials \( U^E \) and \( U^M \). The total electromagnetic field turns out to be
\[ F_{12} = -\partial_1 U^E - \partial_2 U^E, \]
\[ F_{23} = \frac{1}{aZ} \partial_3 U^M + \partial_3 U^E, \]
\[ F_{31} = -\frac{1}{aZ} \partial_1 U^M + \partial_3 U^E, \]
\[ F_{01} = -aZ \partial_3 U^M + \partial_3 U^E, \]
\[ F_{02} = -aZ \partial_3 U^M - \partial_3 U^E, \]
\[ F_{03} = -\partial_3 (aZ \partial_0 U^M) + \frac{1}{aZ} \partial_3 U^E. \] (2.7)

As can be checked by direct substitution, the complete Maxwell equations are satisfied if
\[ aZ \partial_0 U - a^2 \partial_2 z^2 V_2^2 U - a^2 Z \partial_0 (Z \partial_0 U) = 0, \] (2.8)
for both potentials \( U^E \) and \( U^M \). Here \( V_2^2 = \partial_1^2 + \partial_2^2 \).

This last equation admits a solution by separation of variables:
\[ U(x^\mu) = e^{-\Omega T + ik_\perp x^\perp + ik_\parallel y} U(Z), \] (2.9)
where \( \Omega \) is the frequency as measured in the accelerated frame and \( U(Z) \) satisfies the equation
\[ Z^2 \frac{d^2}{dZ^2} U + Z \frac{d}{dZ} U + \left( \frac{\Omega}{a} \right)^2 - k_\perp^2 Z^2 U = 0 \] (2.10)
for both modes, with \( k_\parallel^2 \equiv k_\perp^2 + k_\parallel^2 \). The solution is
\[ U(Z) = C_{(E,M)} K_{\Omega/\omega}(k_{\parallel} Z), \] (2.11)
where \( K_{\Omega/\omega}(k_{\parallel} Z) \) is the modified Bessel function of the third kind with imaginary argument and \( C_{(E,M)} \) are constants. The other linearly independent solution is \( L_{\Omega/\omega}(k_{\parallel} Z) \), but it diverges at \( Z \to \infty \) [5].

The electromagnetic tensor has the following forms in Rindler coordinates. For the transverse electric mode
\[ F_\perp^\mu = C_E e^{-\Omega T + ik_\perp x^\perp + ik_\parallel y} \frac{c^2}{(aZ)^2} \left( \begin{array}{ccc} 0 & -e^{i\Omega Z} k_\parallel & 0 \\ -e^{-i\Omega Z} k_\parallel & 0 & k_\perp (aZ)^2 k_{\parallel} \end{array} \right) \times \left( \begin{array}{ccc} 0 & -e^{-i\Omega Z} k_\parallel & 0 \\ -e^{i\Omega Z} k_\parallel & 0 & k_\perp (aZ)^2 k_{\parallel} \\ 0 & ik_\perp k_{\parallel} (aZ)^2 k_{\parallel} & 0 \end{array} \right), \] (2.12)
and for the transverse magnetic mode
\[ F_\perp^M = C_M e^{-\Omega T + ik_\perp x^\perp + ik_\parallel y} \frac{c}{aZ} \left( \begin{array}{ccc} 0 & -ik_\perp k_{\parallel} K' & -k_\perp^2 K \\ -ik_\perp k_{\parallel} K' & 0 & -e^{-i\Omega Z} k_\parallel K \\ k_\parallel^2 K & -e^{-i\Omega Z} k_\parallel K & -e^{-i\Omega Z} k_\parallel K \\ k_\perp^2 K & e^{-i\Omega Z} k_\parallel K & 0 \end{array} \right), \] (2.13)
where \( K \equiv K_{\Omega/\omega}(k_{\parallel} Z) \) and \( K' \) is the derivative of \( K \) with respect to its argument.

### 2.1. Minkowski coordinates

The electric and magnetic fields in the inertial frame are \( E_a = F_{a0} \) and \( B_a = \frac{1}{2} e_{abc} F_{bc} \) in Minkowski coordinates. It then follows that
\[ E = e^{-i\Omega T + ik_\perp x^\perp + ik_\parallel y} \left[ -C_E G_{k_{\parallel}} \times e_\perp + C_M (F_{k_{\parallel}} - k_\perp^2 K e_\perp) \right], \] (2.14)
\[ B = e^{-i\Omega T + ik_\perp x^\perp + ik_\parallel y} \left[ -C_E (F_{k_{\parallel}} - k_\perp^2 K e_\perp) + C_M G_{k_{\parallel}} \times e_\perp \right], \] (2.15)
where
\[ F \equiv \frac{\Omega}{aZ} \sinh(aT) K - ik_\perp \cosh(aT) K', \] (2.16)
\[ G \equiv \frac{\Omega}{aZ} \cosh(aT) K - ik_\perp \sinh(aT) K'. \] (2.17)

In these formulas, \( T \) and \( Z \) must be interpreted as functions of \( t \) and \( z \) given by equations (2.1). In the accelerating frame, the electric and magnetic field have a simpler form; it is enough to set \( T = 0 \) in the above formulas.

From the above expressions, it can be seen that a linear polarization corresponds to either \( C_M = 0 \) or \( C_E = 0 \).

It also follows from the above equations that the invariants of the electromagnetic field are
\[ E \cdot B^* + c.c. = -(C_E C_M^* + C_M C_E^*) k_{\parallel}^4 F(k_{\parallel} Z), \] (2.18)
and
\[ E \cdot E^* - B \cdot B^* = (|C_E|^2 - |C_M|^2) k_{\parallel}^4 F(k_{\parallel} Z), \] (2.19)
where
\[ F(k_{\parallel} Z) \equiv (K')^2 + \left[ 1 - \left( \frac{\Omega}{ek_\parallel Z} \right)^2 \right] K^2. \] (2.20)

These two invariants vanish if \( C_E \pm iC_M = 0 \), a condition that can be interpreted as describing a circularly polarized wave.

For the energy density \( W \) and Poynting vector \( S \), it follows that
\[ W \equiv \frac{1}{2} (E \cdot E^* + B \cdot B^*) = \frac{1}{2} (|C_E|^2 + |C_M|^2) k_{\parallel}^4 F(k_{\parallel} Z + K^2), \] (2.21)
and
\[ S \equiv \frac{1}{2} (E \times B^* + c.c.) \equiv k_{\parallel}^4 \cosh(\alpha T) \times \left( \begin{array}{ccc} (|C_E|^2 + |C_M|^2) k_{\parallel}^4 [1/2 \sinh(\alpha T) G(k_{\parallel} Z)] e_\perp + \left( \frac{\Omega}{ek_\parallel Z} \right)^2 \times K^2 e_\perp \right. \] \[ \left. + i(C_M C_E^* - C_E C_M^*) K K^\perp e_\perp \times e_\perp \right), \] (2.22)
where
\[ G(k_{\parallel} Z) \equiv (K')^2 + \left( \frac{\Omega}{ek_\parallel Z} \right)^2 K^2. \] (2.23)
and \( e_\perp = k_{\perp} / k_{\parallel} \) is the unit vector in the direction of \( k_{\perp} \).

The total energy integrated over the whole Rindler wedge, \( z > 0 \), is unbounded due to the divergence of the modified Bessel function at the horizon \( z = \pm \infty \). This is related to the nonphysical behavior of the field in that limit, since it would correspond to a section of the electromagnetic field with
infinite acceleration. Nevertheless, it is possible to integrate from a certain value of $Z$, say $e$, to $Z \to \infty$ and obtain a finite value of the energy. According to equation (A.7) in the appendix,

$$\int_{e}^{\infty} \frac{W \, dz}{\left[ (|C_E|^2 + |C_M|^2)^{1/2} \right]} \frac{\pi k^4 \Omega}{\epsilon \sinh(\pi \Omega/a)} = \frac{\pi k^4 \Omega}{\epsilon \sinh(\pi \Omega/a)}.$$

the integration being taken over the three-dimensional space $t = 0 = T$ and $z > 0$. Quite generally, $\Omega/a > 1$ for realistic values of frequency and acceleration, say $\Omega \sim 10^{15}$ s$^{-1}$ and $ac \sim 10$ m s$^{-2}$. Consequently, the following asymptotic expressions for the associated Bessel function can be used [5–7]:

$$K_{\Omega/a}(k_{\perp}Z) \approx \frac{\pi}{2} \left( \frac{a}{\Omega} \tan u \right)^{1/2} \exp \left\{ -\frac{\Omega}{a} (u + \cot u) \left[ 1 + O(a/\Omega) \right] \right\}$$

if $\Omega/a < k_{\perp}Z$, with $\sin u \equiv \Omega/ak_{\perp}Z$, and

$$K_{\Omega/a}(k_{\perp}Z) \approx \sqrt{2\pi} \left( \frac{\Omega}{a} \tanh v \right)^{-1/2} e^{-\pi \Omega/2a} \cos \left[ \frac{\Omega}{a} (v - \tanh v) \right] \left[ 1 + O(a/\Omega) \right]$$

if $\Omega/a > k_{\perp}Z$, with $\cosh v \equiv \Omega/ak_{\perp}Z$.

Thus we have an exponentially decaying potential for $k_{\perp}Z > \Omega/a$ and an oscillating one for $k_{\perp}Z < \Omega/a$. However, it is important to note that these asymptotic expressions are not valid if $\Omega/a \equiv k_{\perp}Z$, in which case a different approximation should be used, as seen in section 3.

### 3. Non-relativistic limit

The non-relativistic limit of the above solutions can be obtained by defining first an acceleration parameter $g \equiv ac$ and then performing a coordinate shift

$$z \to z + \frac{c^2}{g} t.$$

from where it follows that

$$Z \approx \frac{c^2}{g} t + z - \frac{1}{2} g t^2.$$

and

$$\sinh(aT) \approx \frac{g}{c} t, \quad \cosh(aT) = 1 + O[(gt/c)^2],$$

and of course $T \approx t$, thus identifying $\Omega$ as the instantaneous frequency in an inertial frame.

As a next step, one can use the approximations (2.25) and (2.26) for the Bessel function with

$$u \approx \sin^{-1} \left( \frac{\Omega}{k_{\perp}c} \right) = \frac{\Omega g}{c^2 \sqrt{(k_{\perp}c)^2 - \Omega^2}} \zeta,$$

and

$$\theta^2 \approx 2g \zeta/c^2$$

and

$$\zeta_1 \equiv \left( \frac{c}{2 \Omega^2 g} \right)^{1/3}.$$

It is a noteworthy fact that for typical laboratory values of the physical parameters, such as $\Omega \sim 5 \times 10^{14}$ Hz and $g \sim 10$ m s$^{-2}$, this length scale turns out to be of order $\zeta_1 \sim 10$ m, which corresponds to a macroscopic length (compare with $\zeta_0$ above).

Furthermore,

$$K'_{\Omega/a}(k_{\perp}Z) \approx \pi (2a/\Omega)^{1/3} e^{-\pi \Omega/2a} \frac{\Omega}{c} \zeta,$$

and thus $K'/K \sim (a/\Omega)^{1/3}$.

The electric and magnetic fields, given by equations (2.14) and (2.15), reduce in a first approximation, at $t = 0 = T$, to

$$E \propto |C_E|e_\perp \times e_\perp + |C_M|e_\perp \times e_\perp,$$

and

$$B \propto [-C_E e_\perp + C_M e_\perp \times e_\perp].$$

where $e_\perp = k_{\perp}/k_{\perp}$, neglecting terms of order $(a/\Omega)^{1/3}$ and higher; this is also the form of the electromagnetic field in an accelerating non-inertial frame. The Poynting vector in the same inertial frame is in the transverse direction and is given by

$$\mathbf{E} \times \mathbf{B} \propto (|C_E|^2 + |C_M|^2) \frac{\Omega}{c} \zeta_0 e_\perp.$$

For the energy density, it follows directly from equations (2.21) and (2.23) that (also at $t = 0 = T$)

$$W = (|C_E|^2 + |C_M|^2) k_{\perp}^2 \mathcal{E}(\zeta/\zeta_0),$$

where now $\mathcal{E} = \mathcal{G} + K^2$. Thus, from the approximate form of $K$ given by equation (3.7).
\[ \mathcal{E}(\xi/\xi_0) = 2\pi^2 \left( \frac{2\alpha}{\Omega} \right)^{2/3} e^{-\pi\Omega/\alpha} [A^2(\xi/\xi_0) + O((\alpha/\Omega)^{1/3})]. \]

To a good approximation, if \( \alpha/\Omega \ll 1 \), the energy density is just proportional to the squared Airy function.

4. Discussion

The general vectorial solution of Maxwell’s equations in Rindler coordinates was obtained. From the expression of the Poynting vector in the inertial frame, given by equation (2.22), it can be seen that this solution describes a shape-invariant light beam accelerating along the (positive) \( z \) axis with a turning point at time \( t = 0 \). The wavevector of the beam has a non-zero component perpendicular to the averaged direction of propagation, which is typical of structured light beams (for instance, Bessel beams [8]).

The obtained solution is given in terms of a modified Bessel function of imaginary order and is a linear superposition of the two polarization modes of the electromagnetic field. It should be noticed, however, that the total energy of the relativistic accelerating field is not bounded due to its divergence at the Rindler horizon \( z = \pm \epsilon \). This fact is not unexpected since the energy of an electromagnetic field integrated over the whole space is usually divergent in many idealized situations, including plane waves, and it is necessary to constrain the field to a finite region of space. Moreover, as shown by Boulware [9], singularities at the horizon also occur in at least one other case: the Bessel beam also exhibits a longitudinal component seems counterintuitive, but it occurs in at least one other case: the Bessel beam also exhibits a Poynting vector that is purely transverse in a very particular reference frame, as shown in [8] where this limiting case was called ‘antiparaxial’. The results here obtained suggest that the Airy beam (unlike the Bessel beam) is physically realizable in the ‘antiparaxial’ limit of optics.

Appendix

The modified Bessel function of the third kind is defined in general as

\[ K_\nu(x) = \int_0^{\infty} e^{-x \cosh u} \cosh(\mu u) \, du. \]  (A.1)

For purely imaginary order, \( K_{ia}(x) \) has the following asymptotic behaviors (see, e.g. Dunster [5]): near \( x = 0 \)

\[ K_{ia}(x) = \left( \frac{-\pi}{\alpha \sinh(\pi x)} \right)^{1/2} [\sin(\alpha \ln(x/2)) - \text{Arg}(\Gamma(1+i\nu)) + O(x^2)], \]  (A.2)

and for \( x \to \infty \)

\[ K_{ia}(x) = \left( \frac{\pi}{2x} \right)^{1/2} e^{-x} [1 + O(1/x)]. \]  (A.3)

The energy density is proportional to the function \( \mathcal{E}(x) = \mathcal{E}_0(x) + K_{ia}^2(x) \), where \( \mathcal{E}_0 \) is defined by (2.23). The recurrence relation

\[ K_{ia} = i\frac{\alpha}{x} K_{ia} = -K_{ia+1} \]

implies

\[ \mathcal{E}(x) = K_{ia-1}(x) K_{ia+1}(x) + K_{ia}^2(x). \]

Using Nicholson’s formula ([10], section 13.72)

\[ K_\nu(x) K_{\nu'}(x) = 2 \int_0^{\infty} K_{\nu+\nu'}(2x \cosh t) \cosh(\mu \pm \nu)t \, dt, \]  (A.4)

it follows that

\[ \mathcal{E}(x) = 2 \int_0^{\infty} [K_0(2x \cosh t) + K_2(2x \cosh t)] \cos(2\alpha t) \, dt, \]  (A.5)

and since \( K_0 + K_2 = -2K_1 \),

\[ I \equiv \int_{-\infty}^{\infty} \mathcal{E}(x) \, dx = \int_0^{\infty} \frac{\cos(2\alpha t)}{\cosh t} K_1(2e \cosh t) \, dt. \]  (A.6)

Since \( K_1(2z) \approx z^{-1} \) for \( z \ll 1 \), we finally obtain

\[ I \approx \frac{\pi \alpha}{\epsilon \sinh(\pi \alpha)} \]  (A.7)

as an approximation valid for \( \epsilon \ll 1 \).

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