Classification of the spaces $C^*_p(X)$ within the Borel-Wadge hierarchy for a projective space $X$

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Abstract

We study the complexity of the space $C^*_p(X)$ of bounded continuous functions with the topology of pointwise convergence. We are allowed to use descriptive set theoretical methods, since for a separable metrizable space $X$, the measurable space of Borel sets in $C^*_p(X)$ (and also in the space $C_p(X)$ of all continuous functions) is known to be isomorphic to a subspace of a standard Borel space. It was proved by A. Andretta and A. Marcone that if $X$ is a $\sigma$-compact metrizable space, then the measurable spaces $C_p(X)$ and $C^*_p(X)$ are standard Borel and if $X$ is a metrizable analytic space which is not $\sigma$-compact then the spaces of continuous functions are Borel-$\Pi^1_1$-complete. They also determined under the assumption of projective determinacy (PD) the complexity of $C_p(X)$ for any projective space $X$ and asked whether a similar result holds for $C^*_p(X)$.

We provide a positive answer, i.e. assuming PD we prove, that if $n \geq 2$ and if $X$ is a separable metrizable space which is in $\Sigma^1_n$ but not in $\Sigma^1_{n-1}$ then the measurable space $C^*_p(X)$ is Borel-$\Pi^1_n$-complete. This completes under the assumption of PD the classification of Borel-Wadge complexity of $C^*_p(X)$ for $X$ projective.
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1. Introduction

First of all, we recall the needed terminology. Most of the definitions in this introductory section are taken from [1] (and [2]). In Definitions 1 and 2 we recall the Wadge hierarchy.

**Definition 1.** Let $X$ and $Y$ be topological spaces and let $A$, $B$ be subsets of $X$, $Y$, respectively. We say that $A$ is Wadge reducible to $B$ and write $(A,X) \leq_W (B,Y)$ (or simply $A \leq_W B$ if the spaces $X$ and $Y$ are understood) if there exists a continuous map $f : X \to Y$ (called a Wadge reduction of $A$ to $B$) such that $A = f^{-1}(B)$.

**Definition 2.** Let $\Gamma$ be a class of sets in Polish spaces. Let $X$ be a Polish space and $A$ be a subset of $X$. We say that $A$ is $\Gamma$-hard in $X$ if for any zero-dimensional Polish space $Y$ and any $B \in \Gamma(Y)$, we have $(B,Y) \leq_W (A,X)$. If, moreover, $A \in \Gamma(X)$, we say that $A$ is $\Gamma$-complete in $X$.

In this paper, we are mostly interested in the Borel-Wadge hierarchy, where the notions of topology and continuity are replaced by notions of $\sigma$-algebra and measurability. This is recalled in Definitions 3, 4 and 5.

**Definition 3.** Let $X$ and $Y$ be measurable spaces and let $A$, $B$ be subsets of $X$, $Y$, respectively. We say that $A$ is Borel-Wadge reducible to $B$ and write $(A,X) \leq_B (B,Y)$ (or simply $A \leq_B B$ if the spaces $X$ and $Y$ are understood) if there exists a measurable map $f : X \to Y$ (called a Borel-Wadge reduction of $A$ to $B$) such that $A = f^{-1}(B)$.

**Definition 4.** A measurable space $(X,S)$ is called a standard Borel space if there is a Polish space $(Y,\tau)$ and an isomorphism (of measurable spaces) $f$ of
\((X, \mathcal{S})\) onto \((Y, \mathcal{B}(\tau))\) where \(\mathcal{B}(\tau)\) is the Borel \(\sigma\)-algebra on \(Y\) generated by the topology \(\tau\).

**Definition 5.** Let \(\Gamma\) be a class of sets in standard Borel spaces. Let \(X\) be a standard Borel space and \(A\) be a subset of \(X\). We say that \(A\) is Borel-\(\Gamma\)-hard in \(X\) if for any standard Borel space \(Y\) and any \(B \in \Gamma(Y)\), we have 

\((B, Y) \leq_B (A, X)\). If, moreover, \(A \in \Gamma(X)\), we say that \(A\) is Borel-\(\Gamma\)-complete in \(X\).

The projective classes \(\Sigma^1_n\) or \(\Pi^1_n\), \(n \in \mathbb{N}\), (see e.g. [1, 37.A]) are most often considered as classes of sets in Polish spaces but they can also be considered as classes of sets in standard Borel spaces due to Definition 5.

**Definition 6.** Let \(\Gamma\) be one of the classes \(\Sigma^1_n\) or \(\Pi^1_n\), \(n \in \mathbb{N}\), of sets in Polish spaces. Let \(X\) be a standard Borel space and let \(A\) be a subset of \(X\). We say that \(A \in \Gamma(X)\) if for some Polish space \(Y\) and some Borel isomorphism \(f: X \to Y\) (or equivalently, for any Polish space \(Y\) and any Borel isomorphism \(f: X \to Y\), we have 

\(f(A) \in \Gamma(Y)\).

**Remark 7.** Suppose that \(\Gamma\) is one of the classes \(\Sigma^1_n\) or \(\Pi^1_n\), \(n \in \mathbb{N}\). Let \((X, \tau)\) be a Polish space and \(A\) be a subset of \(X\) which is \(\Gamma\)-hard in \(X\) (resp. \(\Gamma\)-complete in \(X\)). If we consider \(A\) as a subset of the standard Borel space \((X, \mathcal{B}(X))\) (i.e., of \(X\) endowed with the Borel \(\sigma\)-algebra generated by \(\tau\)) then \(A\) is also Borel-\(\Gamma\)-hard in \(X\) (resp. Borel-\(\Gamma\)-complete in \(X\)). This easily follows from the fact that any standard Borel space \((Y, \mathcal{S})\) can be endowed with a zero-dimensional Polish topology \(\nu\) such that the Borel \(\sigma\)-algebra generated by \(\nu\) equals to \(\mathcal{S}\) (see e.g. [1, Exercise 13.5]).

Definition 6 enables us to describe the projective degree of any separable metrizable topological space.

**Definition 8.** Let \(\Gamma\) be one of the classes \(\Sigma^1_n\) or \(\Pi^1_n\), \(n \in \mathbb{N}\), of sets in Polish spaces and let \(X\) be a separable metrizable space. We say that \(X\) is in \(\Gamma\) if for some Polish space \(Y\) and some homeomorphism \(f\) of \(X\) into \(Y\) (or equivalently,
for any Polish space \(Y\) and any homeomorphism \(f\) of \(X\) into \(Y\), we have \(f(X) \in \Gamma(Y)\).

We say that a separable metrizable space is projective if it is in \(\Sigma^1_n\) for some \(n \in \mathbb{N}\) (or equivalently, if it is in \(\Pi^1_n\) for some \(n \in \mathbb{N}\)).

Similarly, due to Definition 9, we can describe the projective degree of an arbitrary measurable space which can be embedded into a standard Borel space.

Unlike the previous definitions, we have not found any explanation of the correctness of this definition so we provide at least a short explanation here. Let \(\Gamma\) be one of the classes \(\Sigma^1_n\) or \(\Pi^1_n\), \(n \in \mathbb{N}\), of sets in standard Borel spaces. Suppose that \(X\) is a measurable space, \(Y, Z\) are standard Borel spaces, \(f\) is an isomorphism (of measurable spaces) of \(X\) into \(Y\) and \(g\) is an isomorphism (of measurable spaces) of \(X\) into \(Z\). Then \(f(X)\) is in \(\Gamma(Y)\) (resp. Borel-\(\Gamma\)-hard in \(Y\) or Borel-\(\Gamma\)-complete in \(Y\)) if and only if \(g(X)\) is in \(\Gamma(Z)\) (resp. Borel-\(\Gamma\)-hard in \(Z\) or Borel-\(\Gamma\)-complete in \(Z\)). Indeed, by [1, Exercise 12.3], there are Borel sets \(A \subseteq Y, B \subseteq Z\) with \(f(X) \subseteq A, g(X) \subseteq B\) and a Borel isomorphism \(\Phi: A \to B\) extending the Borel isomorphism \(g \circ f^{-1}\) of \(f(X)\) and \(g(X)\).

\[
\begin{array}{ccc}
X & \xrightarrow{f} & f(X) \subseteq A \subseteq Y \\
\downarrow & \searrow & \downarrow \Phi \\
g(X) \subseteq B \subseteq Z
\end{array}
\]

From this, the conclusion easily follows and we can formulate the definition.

**Definition 9.** Let \(\Gamma\) be one of the classes \(\Sigma^1_n\) or \(\Pi^1_n\), \(n \in \mathbb{N}\), of sets in standard Borel spaces and let \(X\) be a measurable space. We say that \(X\) is in \(\Gamma\) (resp. Borel-\(\Gamma\)-hard or Borel-\(\Gamma\)-complete) if for some standard Borel space \(Y\) and some isomorphism (of measurable spaces) \(f\) of \(X\) into \(Y\) (or equivalently, for any standard Borel space \(Y\) and any isomorphism (of measurable spaces) \(f\) of \(X\) into \(Y\)), the set \(f(X)\) is in \(\Gamma(Y)\) (resp. Borel-\(\Gamma\)-hard in \(Y\) or Borel-\(\Gamma\)-complete in \(Y\)).

Let \(X\) be a separable metrizable topological space. We denote by \(C_p(X)\) (resp. \(C^*_p(X)\)) the measurable space of all real continuous (resp. all real bounded
continuous) functions on $X$ equipped with the Borel $\sigma$-algebra generated by the topology of pointwise convergence on $X$. The Borel-Wadge degree of the measurable spaces $C_p(X)$ and $C_p^*(X)$ was already studied e.g. in \cite{3, 4, 5, 2}, all the relevant results being summarized in \cite{2}. By these results, it is already known that if $X$ is $\sigma$-compact then both $C_p(X)$ and $C_p^*(X)$ are standard Borel spaces. And if $X$ is in $\Sigma^1_1$ but not $\sigma$-compact then both $C_p(X)$ and $C_p^*(X)$ are Borel-$\Pi^1_1$-complete (see \cite{2, Corollary 3.4}). This completely classifies the Borel-Wadge degree of these spaces for $X$ in $\Sigma^1_1$. Now suppose that $X$ is projective but not in $\Sigma^1_1$. Then both $C_p(X)$ and $C_p^*(X)$ are in $\Pi^1_n$ where $n \geq 2$ is the first such that $X$ is in $\Sigma^1_n$ (see \cite{2, Lemma 2.3 and the last paragraph in Section 2}). The precise Borel-Wadge degree of $C_p(X)$ in this case was also studied in \cite{2} under the additional assumption of projective determinacy (henceforth denoted by PD). The principle of PD states that every infinite game $G(\mathbb{N}, X)$ with a projective payoff set $X \subseteq \mathbb{N}^\mathbb{N}$ is determined (for more detailed information, we refer to \cite{1}). It was shown in \cite{2, Theorem 4.3} that if $n \geq 2$ is the first such that $X$ is in $\Sigma^1_n$ then $C_p(X)$ is Borel-$\Pi^1_n$-complete (under PD). So the Borel-Wadge degree of $C_p(X)$ for $X$ projective is also completely classified (under PD). But the proof of \cite{2, Theorem 4.3} uses unbounded functions in an essential way, and so the Borel-Wadge degree of $C_p^*(X)$ for $X$ projective remained unresolved. Instead, the following question was posed in \cite{2}.

**Question 10 (\cite{2, Problem 4.4}).** Assume PD. Let $X$ be a separable metrizable projective space which is not in $\Sigma^1_1$. Let $n \geq 2$ be the first such that $X$ is in $\Sigma^1_n$. Is the measurable space $C_p^*(X)$ Borel-$\Pi^1_n$-complete?

In this paper, we positively answer this question by proving the following main theorem.

**Theorem 11.** Assume PD. Let $X$ be a separable metrizable projective space which is not in $\Sigma^1_1$. Let $n \geq 2$ be the first such that $X$ is in $\Sigma^1_n$. Then the measurable space $C_p^*(X)$ is Borel-$\Pi^1_n$-complete.
Under PD, this concludes the classification of the Borel-Wadge degree of $C^*_p(X)$ for $X$ projective. As in [2], we use PD to know that if $n \in \mathbb{N}$, $X$ is a Polish space and $A$ is a subset of $X$ which is not in $\Pi^1_n$, then $A$ is $\Sigma^1_n$-hard in $X$. If $X$ is zero-dimensional, this follows (under PD) from an easy analogy of [1, Theorem 21.14 (Wadge’s Lemma)]. The general case can be reduced to the previous one by [1, Exercise 13.5].

One of the main ideas of the proof of Theorem 11 is the same as in [2], i.e. providing a Borel-Wadge reduction of the $\Pi^1_n$-hard subset $K(W \setminus X)$ of $K(W)$ (where $W = Z \setminus D$, $D$ is a countable dense subset of $X$ and $Z$ is a metric completion of $X$) to $C^*_p(X)$. This is done almost in the same way as in [2, proof of Theorem 4.3] in the particular case of $X$ being nowhere locally compact. The only refinement is hidden in the fact that in this case, the completion $Z$ of $X$ can be chosen to be a Peano continuum due to [6, Corollary 7]. The general case is then reduced to this particular one by using Proposition 14 which seems to be very intuitive but not trivial.

2. Proof of the main theorem

By the well known Tietze extension theorem, every real continuous function on a closed subspace $H$ of a metric space $X$ can be extended to a real continuous function on $X$. We will need the following version of this theorem since it provides a simple formula for the extension. This version of the Tietze theorem is due to F. Riesz and was published in Kerékjártó’s book [7] but probably the most accessible source is [8].

**Theorem 12.** Let $(X,d)$ be a metric space and $H$ be a closed subset of $X$. Let $f : H \to [1,2]$ be continuous. Then the function $F : X \to \mathbb{R}$ defined by

$$F(x) = \begin{cases} f(x) & \text{if } x \in H \\ \inf \left\{ f(h) \frac{d(x,h)}{d(x,H)} : h \in H \right\} & \text{if } x \in X \setminus H \end{cases}$$

is continuous with values in the interval $[1,2]$. 

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Remark 13. Assume the hypothesis and notation of Theorem 12. Let $D$ be an arbitrary dense subset of $H$. Then for every $x \in X \setminus H$, we clearly have

$$F(x) = \inf \left\{ \frac{f(a)d(x,a)}{d(x,H)} : a \in D \right\}.$$ 

For a separable metrizable space $X$ and a countable dense subset $D$ of $X$, we put

$$\tilde{C}^*_p(X,D) = \{(r_a)_{a \in D} \in \mathbb{R}^D : \exists f \in C^*_p(X) \forall a \in D f(a) = r_a \}.$$ 

Let $\phi: C^*_p(X) \to \mathbb{R}^D$ be defined by $\phi(f) = (f(a))_{a \in D}$, $f \in C^*_p(X)$. It was shown in [2, Lemma 2.2 and the last paragraph of Section 2] that $\phi$ is an isomorphism of the measurable space $C^*_p(X)$ and the measurable subspace $\tilde{C}^*_p(X,D)$ of the standard Borel space $\mathbb{R}^D$ (which is equipped with the Borel $\sigma$-algebra generated by the product topology). So we only need to examine the Borel-Wadge degree of the set $\tilde{C}^*_p(X,D)$ in $\mathbb{R}^D$. Observe also that the countable dense subset $D$ of $X$ can be chosen arbitrarily.

Proposition 14. Let $X$ be a separable metrizable space and $H$ be a closed subspace of $X$. Let $D$, $E$ be countable dense subsets of $H$, $X \setminus H$ respectively (so that $D \cup E$ is countable dense in $X$). Then $(\tilde{C}^*_p(H,D),\mathbb{R}^D) \leq_B (\tilde{C}^*_p(X,D \cup E),\mathbb{R}^{D\cup E})$.

Proof. We may suppose that the cardinality of $X$ is infinite since the other case is trivial. In two steps, we will show that

$$(\tilde{C}^*_p(H,D),\mathbb{R}^D) \leq_B (\tilde{C}^*_p(H,D) \cap [1,2]^D,\mathbb{R}^D) \leq_B (\tilde{C}^*_p(X,D \cup E),\mathbb{R}^{D\cup E}).$$

(i) $(\tilde{C}^*_p(H,D),\mathbb{R}^D) \leq_B (\tilde{C}^*_p(H,D) \cap [1,2]^D,\mathbb{R}^D)$

For $n \in \mathbb{N}$, let $\psi_n$ be a homeomorphism of the interval $[-n,n]$ onto $[1,2]$. We define $\phi: \mathbb{R}^D \to \mathbb{R}^D$ by

$$\phi((r_a)_{a \in D}) = \begin{cases} 
(\psi_n(r_a))_{a \in D} & \text{if } (r_a)_{a \in D} \in [-n,n]^D \\
(r_a)_{a \in D} & \text{if } (r_a)_{a \in D} \in \mathbb{R}^D \setminus \bigcup_{n \in \mathbb{N}} [-n,n]^D.
\end{cases}$$
Then \( \phi \) is obviously Borel measurable. We will show that \( \phi \) is the required Borel-Wadge reduction of \( \hat{C}_p^* (H, D) \) to \( \hat{C}_p^* (H, D) \cap [1, 2]^D \).

Suppose that \( (r_a)_{a \in D} \in \hat{C}_p^* (H, D) \) and let \( n \in \mathbb{N} \) be the first such that \( (r_a)_{a \in D} \) is bounded by \( n \). Then there is \( f \in C_p^*(X) \) with values in the interval \([-n, n]\) which extends \( (r_a)_{a \in D} \) from \( D \) to \( X \). But then \( \psi_n \circ f \) is a continuous function with values in the interval \([1, 2]\) extending \( \phi((r_a)_{a \in D}) = (\psi_n(r_a))_{a \in D} \) from \( D \) to \( X \), and so \( \phi((r_a)_{a \in D}) \in \hat{C}_p^* (H, D) \cap [1, 2]^D \).

Now suppose that \( (r_a)_{a \in D} \in \mathbb{R}^D \setminus \hat{C}_p^* (H, D) \). If \( (r_a)_{a \in D} \) is bounded then it has no continuous extension from \( D \) to \( X \). Let \( n \in \mathbb{N} \) be the first such that \( (r_a)_{a \in D} \) is bounded by \( n \), then neither \( \phi((r_a)_{a \in D}) = (\psi_n(r_a))_{a \in D} \) has a continuous extension from \( D \) to \( X \), and so \( \phi((r_a)_{a \in D}) \in \mathbb{R}^D \setminus \hat{C}_p^* (H, D) \subseteq \mathbb{R}^D \setminus \left( \hat{C}_p^* (H, D) \cap [1, 2]^D \right) \). And if \( (r_a)_{a \in D} \) is unbounded then \( \phi((r_a)_{a \in D}) = (r_a)_{a \in D} \) is unbounded, too. So again, \( \phi((r_a)_{a \in D}) \in \mathbb{R}^D \setminus \left( \hat{C}_p^* (H, D) \cap [1, 2]^D \right) \).

(ii) \( \hat{C}_p^*(H, D) \cap [1, 2]^D, \mathbb{R}^D \leq_B (\hat{C}_p^*(X, D \cup E), \mathbb{R}^{D_{X \cup E}}) \)

Let \( d \) be a compatible metric on \( X \). We fix arbitrary unbounded \((s_b)_{b \in D \cup E} \in \mathbb{R}^{D_{X \cup E}} \) (this is possible since \( D \cup E \) is clearly infinite) and define \( \phi : \mathbb{R}^D \to \mathbb{R}^{D_{X \cup E}} \) by

\[
\phi((r_a)_{a \in D})(b) = \begin{cases} 
    s_b & \text{if } (r_a)_{a \in D} \notin [1, 2]^D \text{ (and } b \in D \cup E), \\
    r_b & \text{if } (r_a)_{a \in D} \in [1, 2]^D \text{ and } b \in D, \\
    \inf \left\{ r_a \frac{d(b, a)}{d(b_0, H)} : a \in D \right\} & \text{if } (r_a)_{a \in D} \in [1, 2]^D \text{ and } b \in E.
\end{cases}
\]

We will show that \( \phi \) is the required Borel-Wadge reduction of \( \hat{C}_p^*(H, D) \cap [1, 2]^D \) to \( \hat{C}_p^*(X, D \cup E) \).

For fixed \( a_0 \in D \) and \( b_0 \in E \), the function \( (r_a)_{a \in D} \mapsto r_{a_0} \frac{d(b_0, a_0)}{d(b_0, H)} \in \mathbb{R} \) is continuous (since \( \frac{d(b_0, a_0)}{d(b_0, H)} \) is constant). So for every \( b_0 \in E \), the function \( (r_a)_{a \in D} \mapsto \inf \left\{ r_a \frac{d(b_0, a)}{d(b_0, H)} : a \in D \right\} \in \mathbb{R} \) is upper semicontinuous (since it is an infimum of continuous functions). It immediately follows that \( \phi \) is Borel measurable.

Suppose that \( (r_a)_{a \in D} \in \mathbb{R}^D \setminus \left( \hat{C}_p^*(H, D) \cap [1, 2]^D \right) \). If \( (r_a)_{a \in D} \notin [1, 2]^D \) then \( \phi((r_a)_{a \in D}) \) is unbounded and so \( \phi((r_a)_{a \in D}) \in \mathbb{R}^{D_{X \cup E}} \setminus \hat{C}_p^*(X, D \cup E) \). Otherwise,
\((r_a)_{a \in D} \in [1, 2]^D \setminus \tilde{C}_p(H, D)\). Then \((r_a)_{a \in D}\) has no bounded continuous extension from \(D\) to \(H\) and so \(\phi((r_a)_{a \in D})\) clearly has no bounded continuous extension from \(D \cup E\) to \(X\), in other words again \(\phi((r_a)_{a \in D}) \in \mathbb{R}^{D \cup E} \setminus \tilde{C}_p^*(X, D \cup E)\).

Now suppose that \((r_a)_{a \in D} \in \tilde{C}_p^*(H, D) \cap [1, 2]^D\). Then there is \(f \in C_p^*(X)\) with values in \([1, 2]\) such that \(f(a) = r_a\) for every \(a \in D\). By Theorem 12 (used on such \(f\)) and Remark 13, there is \(F \in C_p^*(X)\) with values in \([1, 2]\) such that \(F(b) = \phi((r_a)_{a \in D})(b)\) for every \(b \in D \cup E\), and so \(\phi((r_a)_{a \in D}) \in \tilde{C}_p^*(X, D \cup E)\).

\(\square\)

In the following, by a continuum, we mean a nonempty, compact, connected, metrizable topological space. A Peano continuum is a locally connected continuum. A topological space is nowhere locally compact if no point has a compact neighborhood. One of the tools we will need is the following theorem from \([6]\) (nowhere locally compact spaces are called just nowhere compact in \([6]\)).

**Theorem 15** (\([6, \text{Corollary 7}]\)). Every separable metrizable nowhere locally compact space has a compactification which is a Peano continuum.

Now we are ready for the proof of Theorem 11.

**Proof of Theorem 11.** By \([2, \text{Lemma 2.3 and the last paragraph in Section 2}]\), the measurable space \(C_p^*(X)\) is in \(\Pi^1\) so we only need to show that it is Borel-\(\Pi^1\)-hard.

Suppose first that \(X\) is nowhere locally compact. Then by Theorem 14, \(X\) has a compactification \(Z\) which is a Peano continuum. Let \(d\) be a compatible metric on \(Z\) and let \(D\) be a countable dense subset of \(X\) (then \(D\) is also dense in \(Z\)). We put \(W = Z \setminus D\) (so that \(W\) is a Polish space when equipped with the topology inherited from \(Z\)) and \(Y = Z \setminus X\). Since \(X\) is not \(\Sigma^1_{n-1}\) in \(Z\), neither is \(X \setminus D\) (because \(D\) is countable and thus \(\Sigma^0_2\) in \(Z\)). So \(X \setminus D\) is not \(\Sigma^1_{n-1}\) neither in \(W\). It follows that \(Y = W \setminus (X \setminus D)\) is not \(\Pi^1_{n-1}\) in \(W\). By PD, \(Y\) is \(\Sigma^1_{n-1}\)-hard in \(W\). By this and \([2, \text{Lemma 4.2}]\) (in fact, this lemma was first proved in \([4, \text{Lemma 1}]\) but formulated only under some more restrictive assumptions on the space \(W\)), it follows that \(\mathcal{K}(Y)\) (i.e., the set of all compact
subsets of $Y$) is $\Pi^1_n$-hard in $\mathcal{K}(W)$ (i.e., the Polish space of all compact subsets of $W$ equipped with the Vietoris topology). So, due to Remark [7] it suffices to show that $(\mathcal{K}(Y), \mathcal{K}(W)) \leq_W (\hat{C}_p^*(X, D), \mathbb{R}^D)$.

We define $\phi: \mathcal{K}(W) \rightarrow \mathbb{R}^D$ by

$$
\phi(K) = \left( \sin \left( \frac{1}{d(a, K)} \right) \right)_{a \in D}, \quad K \in \mathcal{K}(W).
$$

This is a correct definition since for every $K \in \mathcal{K}(W)$, we have $K \cap D = \emptyset$ and so $d(a, K) > 0$ for every $a \in D$. The map $\phi$ is obviously continuous. We will show that $\phi$ is the required Wadge reduction of $\mathcal{K}(Y)$ to $\hat{C}_p^*(X, D)$. If $K \in \mathcal{K}(Y)$ then $K \cap X = \emptyset$ and so the function $f: X \rightarrow \mathbb{R}$ defined by $f(x) = \sin \left( \frac{1}{d(x, K)} \right)$, $x \in X$, is a bounded continuous extension of $\phi(K)$ from $D$ to $X$. So we have $\phi(K) \in \hat{C}_p^*(X, D)$ whenever $K \in \mathcal{K}(Y)$. Now suppose that $K \in \mathcal{K}(W) \setminus \mathcal{K}(Y)$.

Then we can find some $x \in K \cap X$. Since $Z$ is a Peano continuum, it is locally connected at $x$. So for every $n \in \mathbb{N}$, there is an open connected subset $U_n$ of $Z$ such that $x \in U_n \subseteq \{ z \in Z: d(z, x) < \frac{1}{n} \}$. Let us fix $n \in \mathbb{N}$ for now. The function $\psi: Z \rightarrow \mathbb{R}$ defined by $\psi(z) = d(z, K)$, $z \in Z$, is continuous, and so the image $\psi(U_n)$ of the set $U_n$ is a connected subset of $\mathbb{R}$. Since $x \in K \cap U_n$, we have $0 \in \psi(U_n)$. And since $D$ intersects $U_n$ (it is dense in $Z$) and $D \cap K = \emptyset$, the function $\psi$ also attains some positive values in $U_n$. So there is $k_n \in \mathbb{N}$ such that the connected set $\psi(U_n)$ contains the interval $[0, \frac{1}{2k_n \pi})$. Then we can find $z_n \in U_n$ such that

$$
\psi(z_n) = \begin{cases} 
\frac{1}{2k_n \pi + \frac{\pi}{2}} & \text{if } n \text{ is odd}, \\
\frac{1}{2k_n \pi + \frac{3\pi}{2}} & \text{if } n \text{ is even}.
\end{cases}
$$

In this way, we obtain a sequence $(z_n)_{n \in \mathbb{N}}$ in $Z$ such that for every $n \in \mathbb{N}$, we have $z_n \in U_n$, $\psi(z_n) > 0$ and

$$
\sin \left( \frac{1}{\psi(z_n)} \right) = \begin{cases} 
1 & \text{if } n \text{ is odd}, \\
-1 & \text{if } n \text{ is even}.
\end{cases}
$$

The function $z \in Z \setminus K \mapsto \sin \left( \frac{1}{\psi(z)} \right)$ is continuous and $D \cup \{ z_n: n \in \mathbb{N} \}$ is a subset of its domain $Z \setminus K$, so by the density of $D$ in $Z$, we can find a sequence
\((a_n)_{n \in \mathbb{N}}\) in \(D\) such that for every \(n \in \mathbb{N}\), we have \(a_n \in U_n\) and
\[
\phi(K)(a_n) = \sin \left( \frac{1}{\psi(a_n)} \right) \begin{cases} 
> \frac{1}{2} & \text{if } n \text{ is odd}, \\
< -\frac{1}{2} & \text{if } n \text{ is even}.
\end{cases}
\]
Clearly \(\lim_{n \to \infty} a_n = x\) but \(\lim_{n \to \infty} \phi(K)(a_n)\) does not exist. So \(\phi(K)\) cannot be continuously extended from \(D\) to \(D \cup \{x\}\). It follows that \(\phi(K) \in \mathbb{R}^D \setminus \tilde{C}_p^{*}(X, D)\) whenever \(K \in \mathcal{K}(W) \setminus \mathcal{K}(Y)\). This completes the proof for \(X\) being nowhere locally compact.

Now suppose that \(X\) is arbitrary. We will construct a decreasing (with respect to inclusion) transfinite sequence \((X_\alpha)_{\alpha<\omega_1}\) of closed subspaces of \(X\) in the following way. We start by \(X_0 = X\). Now suppose that for some \(\alpha < \omega_1\), we already have \(X_\beta\) for every \(\beta < \alpha\). If \(\alpha = \beta + 1\) for some ordinal \(\beta\), we put \(X_\alpha = X_\beta \setminus \bigcup \{V \subseteq X_\beta : V \text{ is open and relatively compact in } X_\beta\}\). And if \(\alpha\) is a limit ordinal, we put \(X_\alpha = \bigcap_{\beta < \alpha} X_\beta\). Since \(X\) is second countable, there is some \(\alpha < \omega_1\) such that \(X_{\alpha+1} = X_\alpha\). Let \(\alpha_0\) be the least such \(\alpha\), then \(X_{\alpha_0}\) is clearly nowhere locally compact. Moreover, it easily follows by the hereditary Lindelöfness of \(X\) that \(X \setminus X_{\alpha_0}\) is contained in a \(\sigma\)-compact subset of \(X\), and so the topological space \(X_{\alpha_0}\) is clearly in \(\Sigma^1_n\) but not in \(\Sigma^1_{n-1}\). By the previous step, the measurable space \(C_p^*\) of \(X_{\alpha_0}\) is Borel-\(\Pi^1_1\)-complete. Let \(D\), \(E\) be countable dense subsets of \(X_{\alpha_0}\), \(X \setminus X_{\alpha_0}\) respectively. Since \(X_{\alpha_0}\) is a closed subspace of \(X\), we have \((\tilde{C}_p^*(X_{\alpha_0}, D), \mathbb{R}^D) \leq_B (\tilde{C}_p^*(X, D \cup E), \mathbb{R}^{D \cup E})\) by Proposition [13]. But as it was already explained, the measurable spaces \(C_p^*(X_{\alpha_0}), C_p^*(X)\) are isomorphic to the subspaces \(\tilde{C}_p^*(X_{\alpha_0}, D), \tilde{C}_p^*(X, D \cup E)\) of the standard Borel spaces \(\mathbb{R}^D, \mathbb{R}^{D \cup E}\) respectively, and so the conclusion immediately follows. □

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