Derived categories of small toric Calabi-Yau 3-folds and curve counting invariants

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Abstract

We first construct a derived equivalence between a small crepant resolution of an affine toric Calabi-Yau 3-fold and a certain quiver with a superpotential. Under this derived equivalence we establish a wall-crossing formula for the generating function of the counting invariants of perverse coherent sheaves. As an application we provide some equations on Donaldson-Thomas, Pandharipande-Thomas and Szendroi’s invariants. Finally, we show that moduli spaces associated with a quiver given by successive mutations are realized as the moduli spaces associated the original quiver by changing the stability conditions.

Introduction

This is a subsequent paper of [NN]. We study variants of Donaldson-Thomas (DT in short) invariants on small crepant resolutions of affine toric Calabi-Yau varieties.

The original Donaldson-Thomas invariants of a Calabi-Yau 3-fold $Y$ are defined by virtual counting of moduli spaces of ideal sheaves $I_Z$ of 1-dimensional closed subschemes $Z \subset Y$ ([Tho00], [Beh]). These are conjecturally equivalent to Gromov-Witten invariants after normalizing the contribution of 0-dimensional sheaves ([MNOP06]).

A variant has been introduced Pandharipande and Thomas (PT in short) as virtual counting of moduli spaces of stable coherent systems ([PT]). They conjectured these invariants also coincide with DT invariants after suitable normalization and mentioned that the coincidence should be recognized as a wall-crossing phenomenon. Here, a coherent system is a pair of a coherent sheaf and a morphism to it from the structure sheaf, which is first introduced by Le Potier in his study on moduli problems ([LP93]). Note that an ideal sheaf $I_Z$ is the kernel of the canonical surjections from the structure sheaf $O_Y$ to the structure sheaf $O_Z$. So in this sense DT invariants also count coherent systems.

On the other hand, a variety sometimes has a derived equivalence with a non-commutative algebra. A typical example is a noncommutative crepant resolution of a Calabi-Yau 3-fold introduced by Michel Van den Bergh ([VdB04], [VdB]). In the case of [VdB04], the Abelian category of modules of the noncommutative crepant resolution corresponds to the Abelian category of perverse coherent sheaves in the sense of Tom Bridgeland ([Bri02]). Recently, Balazs Szendroi proposed to study counting invariants of ideals of such noncommutative algebras.
He call these invariants noncommutative Donaldson-Thomas (NCDT in short) invariants. He originally studied on the conifold, but his definition works in more general settings ([Youn], [MR]).

Inspired by his work, Hiraku Nakajima and the author introduced perverse coherent systems (pairs of a perverse coherent sheaf and a morphism to it from the structure sheaf) and study their moduli spaces and counting invariants ([NN]). This attempt seems successful since

- we can describe explicitly a space of stability parameters with a chamber structure, and
- at certain chambers, the moduli spaces in DT, PT and NCDT theory are recovered.

Moreover, in the conifold case, we established the wall-crossing formula for the generating functions of counting invariants of perverse coherent systems and provide some equations on DT, PT and NCDT invariants. The chamber structure and the wall-crossing formula formally look very similar to the counter parts for moduli spaces of perverse coherent sheaves on the blow-up of a complex surface studied earlier by Nakajima and Yoshioka ([NYa], [NYb]).

The purpose of this paper is to show the wall-crossing formula (Theorem 2.18) for general small crepant resolutions of toric Calabi-Yau 3-folds. Here we say a crepant resolutions of affine toric Calabi-Yau 3-fold is small when the dimensions of the fibers are less than 2. In such cases, the lattice polygon in \( \mathbb{R}^2 \) corresponding to the affine toric Calabi-Yau 3-fold does not have any lattice points in its interior. Such lattice polygons are classified up to equivalence into the following two cases:

- trapezoids with heights 1, or
- the right isosceles triangle with with length 2 isosceles edges.

In this paper we study the first case. Our argument works for the second case as well.

In §1, we construct derived equivalences between small crepant resolutions of affine toric Calabi-Yau 3-folds and certain quivers with superpotentials. In §1.1, using toric geometry, we construct tilting vector bundles given by Van den Bergh ([VdB04]) explicitly. Then, we review Ishii and Ueda's construction of crepant resolutions as moduli spaces of representations of certain quivers with superpotentials ([IU]) in §1.2. In §1.3, we show the tautological vector bundles on the moduli spaces coincide with the tilting bundles given in §1.1. Using such moduli theoretic description, we calculate the endomorphism algebras of the tilting bundles in §1.4.

The argument in §2 is basically parallel to [NN]. In our case, the fiber on the origin of the affine toric variety is the type \( A \) configuration of \((-1, -1)\)- or \((0, -2)\)-curves. A wall in the space of stability parameters is a hypersurface which is perpendicular to a root vector of the root system of type \( \hat{A} \). Stability parameters in chambers adjacent to the wall corresponding to the imaginary root realize DT theory and PT theory ([NN, §2]). Note that the story is completely parallel to that of type \( \hat{A} \) quiver varieties (of rank 1), which are the moduli spaces of framed representations of type \( \hat{A} \) preprojective algebras ([Nak94], Nak98, Nak01]). Quiver varieties associated with a stability parameter in a chamber
adjacent to the imaginary wall realize Hilbert schemes of points on the minimal resolution of the Kleinian singularities of type $A$, whose exceptional fiber is the type $A$ configurations of $(-2)$-curves \cite{Nak99, Kuz}.

Our main result is the wall-crossing formula for the generating functions of the Euler characteristics of the moduli spaces (Theorem 2.18). The contribution of a wall depends on the information of self-extensions of stable objects on the wall. Note that in the conifold case (NN) every wall has a single stable object on it and every stable object has a trivial self-extension. Computations of self-extensions are done in §2.6.

Note that the sets of torus fixed points on the moduli spaces in DT, PT and NCDT theory are isolated, and we can show that DT, PT and NCDT invariants coincide with the Euler characteristics of the moduli spaces. In particular, the wall-crossing formula provides a product expansion formula of the generating functions of PT invariants. The indices in this formula are nothing but the BPS state counts $n_{g,\beta}$ \cite{GV, HST01, Tod} in the sense of Pandharipande-Thomas \cite{PTa, §3}. Although an algorithm to extract Gopakumar-Vafa invariants of our toric Calabi-Yau 3-folds from the topological vertex expression is known \cite{IKP}, the explicit formula in this paper is new as far as the author knows.

In §3, we provide alternative descriptions of the moduli spaces. Given a quiver with a superpotential $A = (Q, \omega)$, we can mutate it at a vertex $k$ to provide a new quiver with a superpotential $\mu_k(A) = (\mu_k(Q), \mu_k(\omega))$. For a generic stability parameter $\zeta$, we can associate a sequence $k_1, \ldots, k_r$ of vertices and the moduli space of $\zeta$-stable $A$-modules is isomorphic to the moduli space of cyclic modules over the quiver with the potential $\mu_{k_r} \circ \cdots \circ \mu_{k_1}(A)$. As an application, we show that for a stability parameter ”between DT and NCDT” the set of torus fixed points on the moduli space is isolated.

As in \cite{NN}, our formula does not cover the wall corresponding to the DT-PT conjecture. We can provide the wall-crossing formula for this wall applying Joyce’s formula \cite{Joy}. In §4 we make some observations on how the virtual counting version of the wall-crossing formula would be induced from the perspective of the recent work of Kontsevich and Soibelman \cite{KS}. In fact, the wall-crossing formula (and hence the invariants) coincides with the Euler characteristic version up to sign. The author learned from Tom Bridgeland that he and Balazs Szendroi reproved the Young’s product formula \cite{Youn} for NCDT invariants of the conifold. Their idea looks quite similar to our observation.

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\footnote{The author was informed on this reference by Yukiko Konishi.}

\footnote{Yukinobu Toda informed me that it is possible to prove the (Euler characteristic version of) DT-PT correspondence conjecture for arbitrary projective Calabi-Yau 3-folds using Joyce’s formula.}
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1 Derived equivalences

1.1 tilting generators

Let $N_0 > 0$ and $N_1 \geq 0$ be integers such that $N_0 \geq N_1$ and set $N = N_0 + N_1$. We set

$$I = \{1, \ldots, N - 1\},$$
$$\hat{I} = \{0, 1, \ldots, N - 1\},$$
$$\tilde{I} = \left\{\frac{1}{2}, \frac{3}{2}, \ldots, N - \frac{1}{2}\right\},$$
$$\tilde{Z} = \left\{n + \frac{1}{2} \mid n \in \mathbb{Z}\right\}.$$

For $l \in \mathbb{Z}$ and $j \in \tilde{Z}$, let $\underline{l} \in \hat{I}$ and $\underline{j} \in \tilde{I}$ be the elements such that $l - \underline{l} \equiv j - \underline{j} \equiv 0$ modulo $N$.

We denote by $\Gamma$ the quadrilateral (or the triangle in the cases $N_1 = 0$) in $\mathbb{R}^2 = \{(x, y)\}$ with vertices $(0, 0)$, $(0, 1)$, $(N_0, 0)$ and $(N_1, 1)$. Let $M^\vee \simeq \mathbb{Z}^3$ be the lattice with basis $\{x^\vee, y^\vee, z^\vee\}$, and we identify the plane

$$\{(x, y, 1)\} \subset M^\vee_\mathbb{R} := M^\vee \otimes \mathbb{R}$$

with the one where the quadrilateral $\Gamma$ is. Let $M$ be the dual lattice of $M^\vee$. We denote the cone of $\Gamma$ in $M^\vee_\mathbb{R}$ by $\Delta$ and consider the semigroup

$$S_\Delta = \Delta^\vee \cap M := \{u \in M \mid \langle u, v \rangle \geq 0 \ (\forall v \in \Delta)\}.$$

Let $R = R_\Gamma := \mathbb{C}[S_\Delta]$ be the group algebra and $X = X_\Gamma := \text{Spce}(R_\Gamma)$ the 3-dimensional affine toric Calabi-Yau variety corresponding to $\Delta$.

Let $\{x, y, z\} \subset M$ be the dual basis. The semigroup is generated by

$$X := x,$$
$$Y := -x - (N_0 - N_1)y + N_0z,$$
$$Z := y,$$
$$W := -y + z,$$

and they have a unique relation $X + Y = N_0Z + N_1W$. So we have

$$R \simeq \mathbb{C}[X, Y, Z, W]/(XY - Z^{N_0}W^{N_1}).$$

A partition $\sigma$ of $\Gamma$ is a pair of functions $\sigma_x : \tilde{I} \to \tilde{Z}$ and $\sigma_y : \tilde{I} \to \{0, 1\}$ such that

- $\sigma(i) := (\sigma_x(i), \sigma_y(i))$ gives a permutation of the set

$$\left\{\left(\frac{1}{2}, 0\right), \left(\frac{3}{2}, 0\right), \ldots, \left(N_0 - \frac{1}{2}, 0\right), \left(\frac{1}{2}, 1\right), \left(\frac{3}{2}, 1\right), \ldots, \left(N_1 - \frac{1}{2}, 1\right)\right\},$$
• if \( i < j \) and \( \sigma_y(i) = \sigma_y(j) \) then \( \sigma_x(i) > \sigma_x(j) \).

Giving a partition \( \sigma \) of \( \Gamma \) is equivalent dividing \( \Gamma \) into \( N \)-tuples of triangles \( \{T_i\}_{i \in \sigma} \) with area \( 1/2 \) so that \( T_i \) has \( (\sigma_x(i) \pm 1/2, \sigma_y(i)) \) as its vertices. Let \( \Gamma_\sigma \) be the corresponding diagram, \( \Delta_\sigma \) be the fun and \( f_\sigma: Y_\sigma \to X \) be the crepant resolution of \( X \).

We denote by \( D_{\varepsilon,x} \) \( (\varepsilon = 0,1 \text{ and } 0 \leq k \leq N_\varepsilon) \) the divisor of \( Y_\sigma \) corresponding to the lattice point \((x, \varepsilon)\) in the diagram \( \Gamma_\sigma \). Note that any torus equivariant divisor is described as a linear combination of \( D_{\varepsilon,x} \)'s. For a torus equivariant divisor \( D \) let \( D(\varepsilon, x) \) denote its coefficient of \( D_{\varepsilon,x} \). The support function \( \psi_D \) of \( D \) is the piecewise linear function on \( |\Delta_\sigma| \) such that \( \psi_D((x, \varepsilon, 1)) = -D(\varepsilon, x) \) and such that \( \psi_D \) is linear on each cone of \( \Delta_\sigma \). We sometimes denote the restriction of \( \psi_D \) on the plane \( \{z = 1\} \) by \( \psi_D \) as well.

**Definition 1.1.** For \( i \in I \) and \( k \in I \) we define effective divisors \( E_i^\pm \) and \( F_k^\pm \) by

\[
E_i^+ = \sum_{j=\sigma(i)+\frac{1}{2}}^{N_{\sigma(i)}} D_{\sigma(i),j}, \quad F_k^+ = \sum_{i=\frac{k}{h}}^{k-\frac{1}{h}} E_i^+, \\
E_i^- = \sum_{j=0}^{\sigma(i)-\frac{1}{2}} D_{\sigma(i),j}, \quad F_k^- = \sum_{i=k+\frac{1}{h}}^{N-\frac{1}{h}} E_i^-.
\]

**Example 1.2.** Let us consider as an example the case \( N_0 = 4, N_1 = 2 \) and

\[
(\sigma(i))_{i \in I} = \left( \left( \frac{7}{2}, 0 \right), \left( \frac{3}{2}, 1 \right), \left( \frac{5}{2}, 0 \right), \left( \frac{3}{2}, 0 \right), \left( \frac{1}{2}, 1 \right), \left( \frac{1}{2}, 0 \right) \right).
\]

We show the corresponding diagram \( \Gamma_\sigma \) in Figure 1. The divisors are given as follows:

\[
E_\frac{1}{2}^+ := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad F_1^+ := \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \\
E_{\frac{7}{2}}^+ := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad F_2^+ := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
E_\frac{7}{2}^- := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad F_3^- := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
E_{\frac{11}{2}}^+ := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad F_4^+ := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \\
E_{\frac{11}{2}}^- := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad F_5^- := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \\
E_{\frac{17}{2}}^+ := \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad F_6^+ := \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \\
E_{\frac{17}{2}}^- := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \quad F_6^- := \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix}.
\]

Here the \((\varepsilon, x)\)-th matrix element represent the coefficient of the divisor \( D_{\varepsilon,x} \).

**Lemma 1.3.** (1) \( \mathcal{O}_{Y_\sigma}(E_i^+ + E_i^-) \simeq \mathcal{O}_{Y_\sigma} \),
(2) $\mathcal{O}_{Y_\sigma}(F_k^+) \simeq \mathcal{O}_{Y_\sigma}(F_k^-)$.

Proof. We have

$$
\psi_{E_i^+ + E_i^-} = \begin{cases}
  y - z & (\sigma_x(i) = 0), \\
  -y & (\sigma_x(i) = 1),
\end{cases}
$$

so the equation (1) follows. Now, for the equation (2) it is enough to show that the deviser

$$
F_N^+ := \sum_{i=\frac{1}{2}}^{N-\frac{1}{2}} E_i^+
$$

gives the trivial bundle. In fact, we have $\psi_{F_N^+} = -x$. \hfill \square

We denote the line bundle $\mathcal{O}_{Y_\sigma}(F_k^+) \simeq \mathcal{O}_{Y_\sigma}(F_k^-)$ on $Y_\sigma$ by $L_k$. We set

$$
F^\pm = \sum_{k=1}^{N-1} F_k^\pm, \quad L = \bigotimes_{k=1}^{N-1} L_k.
$$

Example 1.4. In the case Example 1.2,

$$
F^+ = \begin{bmatrix}
 0 & 1 & 5 \\
 0 & 0 & 2 & 5 & 10
\end{bmatrix}.
$$

Lemma 1.5. For $i \in \tilde{I}$ we have

$$
F^+ \left( \sigma_y(i), \sigma_x(i) + \frac{1}{2} \right) - F^+ \left( \sigma_y(i), \sigma_x(i) - \frac{1}{2} \right) = F^+ \left( \sigma_y(i+1), \sigma_x(i+1) + \frac{1}{2} \right) - F^+ \left( \sigma_y(i+1), \sigma_x(i+1) - \frac{1}{2} \right) + 1.
$$

Proof. First, note that

$$
E_j^+ \left( \sigma_y(i), \sigma_x(i) + \frac{1}{2} \right) - E_j^+ \left( \sigma_y(i), \sigma_x(i) + \frac{1}{2} \right) = \delta_{i,j}.
$$

So we have

$$
F_k^+ \left( \sigma_y(i), \sigma_x(i) + \frac{1}{2} \right) - F_k^+ \left( \sigma_y(i), \sigma_x(i) + \frac{1}{2} \right) = \begin{cases}
  0 & (k < i), \\
  1 & (k > i),
\end{cases}
$$

and so

$$
F^+ \left( \sigma_y(i), \sigma_x(i) + \frac{1}{2} \right) - F^+ \left( \sigma_y(i), \sigma_x(i) - \frac{1}{2} \right) = N - i - \frac{1}{2}.
$$

Thus the claim follows. \hfill \square
Proposition 1.6. The line bundles $L$ is generated by its global sections.

Proof. It is enough to prove that the support function $\psi_{F^+}$ is upper convex (FT98 §3.4). It is enough to prove that $\psi_{F^+}$ is upper convex on $T_{k-rac{1}{2}} \cap T_{k+rac{1}{2}}$, for any $k \in I$. We denote the edge which is the intersection of $T_{k-rac{1}{2}}$ and $T_{k+rac{1}{2}}$ by $l_k$. The configurations of $l_k, T_{k-rac{1}{2}}$ and $T_{k+rac{1}{2}}$ are classified into the following two cases:

1. The union of $T_{k-rac{1}{2}}$ and $T_{k+rac{1}{2}}$ is a parallelogram and $l_k$ is its diagonal. In this case, the point $(\sigma_x(k + \frac{1}{2}) + \frac{1}{2}, \sigma_y(k + \frac{1}{2}))$ is the intersection of $l_k$ and $l_{k-1}$, the point $(\sigma_x(k - \frac{1}{2}) + \frac{1}{2}, \sigma_y(k - \frac{1}{2}))$ is the other end of $l_{k-1}$.

2. The union of $T_{k-rac{1}{2}}$ and $T_{k+rac{1}{2}}$ is a triangle and $l_k$ is its median line. In this case, the point $(\sigma_x(k + \frac{1}{2}) + \frac{1}{2}, \sigma_y(k + \frac{1}{2}))$ is the middle point and $(\sigma_x(k - \frac{1}{2}) + \frac{1}{2}, \sigma_y(k - \frac{1}{2})) = (\sigma_y(k + \frac{1}{2}) + \frac{1}{2}, \sigma_y(k + \frac{1}{2}))$.

In both cases it follows from Lemma 1.5 that $\psi_{F^+}$ is upper convex on $T_{k-rac{1}{2}} \cap T_{k+rac{1}{2}}$.

Given a divisor $D$ the space of global sections of the line bundle $O_{Y_x}(D)$ is described as follows:

$$H^0(Y_x, O_{Y_x}(D)) \simeq \bigoplus_{u \in S^0_\Delta(D)} \mathbb{C} \cdot e_u,$$

where

$$S^0_\Delta(D) := \{ u \in M \mid \langle u, v \rangle \geq \psi_D(v) \ (\forall v \in |\Delta|) \}.$$

For $u \in M$ we define

$$Z_D(u) := \{ v \in |\Delta| \mid \langle u, v \rangle \geq \psi_D(v) \}.$$

Then the cohomology of the line bundle $O_{Y_x}(D)$ is given as follows (FT98 §3.5):

$$H^k(Y_x, O_{Y_x}(D)) \simeq \bigoplus_{u \in M} H^k(|\Delta| \setminus |\Delta| \setminus Z_D(u); \mathbb{C}).$$

We have the exact sequence of relative cohomologies

$$0 \rightarrow H^0(|\Delta| \setminus |\Delta| \setminus Z_D(u)) \rightarrow H^0(|\Delta|) \rightarrow i_* H^0(|\Delta| \setminus Z_D(u)) \rightarrow H^1(|\Delta| \setminus Z_D(u)) \rightarrow H^2(|\Delta| \setminus Z_D(u)) \rightarrow \cdots$$

Note that $H^0(|\Delta|) = \mathbb{C}$, $H^1(|\Delta|) = 0$ and if $|\Delta| \setminus Z_D(u)$ is not empty then $i_*$ does not vanish. We define

$$Z_D^1(u) := \{ v \in |\Delta| \cap \{ z = 1 \} \mid \langle u, v \rangle < \psi_D(v) \},$$

then $|\Delta| \setminus Z_D(u)$ is homeomorphic to $Z_D^1(u) \times \mathbb{R}$.

Now, in our situation it follows from the convexity of $\psi_{-F}$ that the number of connected components of $Z_D^\circ(u)$ is at most 2. Let us denote

$$S^1_\Delta(-F) := \{ u \in M \mid Z_D^\circ(u) \text{ has two connected components} \}.$$
Then we have
\[ H^1(Y_\sigma, \mathcal{O}_{Y_\sigma}(-F)) \simeq \bigoplus_{u \in S^1_\Delta(-F)} \mathbb{C} \cdot f_u \]  \hspace{1cm} (5)
and the \( R \)-module structure is given by
\[ e_{u'} \cdot f_u = \begin{cases} f_{u+u'} & (u+u' \in S^1_\Delta(-F)), \\ 0 & (u+u' \notin S^1_\Delta(-F)) \end{cases} \]
for \( u \in S^1_\Delta(-F) \) and \( u' \in S_\Delta \).

For \( i \in I \), let \( t^i_F \in M \) the element such that \( \langle t^i_F, \ast \rangle \equiv \psi_F \) on the triangle \( T_i \). Note that \( t^i_F \in S^1_\Delta(-F) \) for \( i \in \left\{ \frac{3}{2}, \ldots, N - \frac{3}{2} \right\} \).

**Proposition 1.7.** The set \( \left\{ f_{t^i_F} \mid i \in \left\{ \frac{3}{2}, \ldots, N - \frac{3}{2} \right\} \right\} \) is a set of generators of \( H^1(Y_\sigma, \mathcal{O}_{Y_\sigma}(-F)) \) as an \( R \)-module.

**Proof.** It is enough to check that for any \( u \in S^1_\Delta(-F) \) there exist \( i \in \left\{ \frac{3}{2}, \ldots, N - \frac{3}{2} \right\} \) and \( u' \in S_\Delta \) such that \( u = t^i_F + u' \). For \( \varepsilon \in \{0,1\} \) we put
\[ m_\varepsilon := \max_{0 \leq j \leq N_\sigma} \langle (u, (j, \varepsilon, 1)) - F(\varepsilon, j) \rangle \geq 0. \]

Let \( u' \in S_\Delta \) be the element such that \( \langle u', (x, \varepsilon, 1) \rangle = m_\varepsilon \) for any \( x \) and \( \varepsilon \). Note that \( Z^+_{\varepsilon,F}(u) \) has two connected components, \( Z^+_{\varepsilon,F}(u) \subset Z^+_{\varepsilon,F}(u-u') \) and there exist \( x_\varepsilon \)'s such that \( \langle u-u', (x_\varepsilon, \varepsilon, 1) \rangle - F(\varepsilon, j) = 0 \). Thus \( Z^+_{\varepsilon,F}(u-u') \) also has two connected components, that is, \( u-u' \in S^1_\Delta(-F) \). The function \( \psi_{-F} - \langle u-u', \ast \rangle \) is upper convex and does not take negative values on \( \Gamma \). So if for some \( x_\varepsilon \)'s we have \( F(\varepsilon, j) - \langle u-u', (x_\varepsilon, \varepsilon, 1) \rangle = 0 \), then \( (x_0, 0) \) and \( (x_1, 1) \) should be the end points of some edge in \( \Gamma_\sigma \). Since \( u-u' \in S^1_\Delta(-F) \), \( (y_0, y_1) \) can not be neither \((0,0)\) nor \((N_0, N_1)\). So \((0, y_0)\) and \((1, y_1)\) is the end points of an edge \( l_k \) for some \( k \in I \). By Lemma 1.3, \( \psi_{-F} \) coincides with \( \langle u-u', \ast \rangle \) on either \( T_k \) or \( T_{k+1} \) since otherwise \( \psi_{-F} - \langle u-u', \ast \rangle \) takes negative values on \( \Gamma \). Thus claim follows. \( \square \)

For a divisor \( D \) and an effective divisor \( E \), let \( 1_{D,E} \) be the canonical inclusion \( \mathcal{O}_{Y_\sigma}(D) \hookrightarrow \mathcal{O}_{Y_\sigma}(D+E) \).

Let us denote effective divisors
\[ G_i^+ = \sum_{k=1}^{i+\frac{1}{2}} F_k^+, \quad G_i^- = \sum_{k=1}^{N-1} F_k^- \]

Note that \( G_i^+ + G_i^- \) is linearly equivalent to \( F^+ \) by lemma 1.3 and
\[ \left( \psi_{G_i^+} + \psi_{G_i^-} \right) \bigg|_{T_i} \equiv 0. \]
Hence we have
\[ \psi_{-F} - \langle t^i_F, \ast \rangle \equiv \psi_{G_i^+} + \psi_{G_i^-}. \]
Let us consider the following sequence:
\[ 0 \rightarrow \mathcal{O}_{Y_\sigma} \rightarrow \mathcal{O}_{Y_\sigma}(G_i^+) \oplus \mathcal{O}_{Y_\sigma}(G_i^-) \rightarrow \mathcal{O}_{Y_\sigma}(G_i^+ + G_i^-) \rightarrow 0. \]
Here the first map is given by \( 1_{0,G_i^+} \oplus (-1_{0,G_i^-}) \) and the second map is given by \( \left( 1_{G_i^+, G_i^-} \right) + \left( 1_{G_i^-, G_i^+} \right) \).
Proposition 1.8. The above sequence is exact and corresponds to the element $f_{i_k} \in H^1(Y_\sigma, \mathcal{O}_{Y_\sigma}(F))$.

Proof. Let $U_j \simeq \mathbb{C}^3 = \text{Spec}(\mathbb{C}[x_j, y_j, z_j])$ be the affine chart corresponding to the triangle $T_j$. It is enough to show that the sequence is exact on each $U_j$. Let $P_a \ (a = x, y, z)$ be the vertices of $T_j$. Then we have

- for $j < i$, $\psi_{G_i^+}(P_a) = \psi_{G_i^- + G_j^-}(P_a) = (d_j, a)$ and $\psi_{G_i^-}(P_a) = 0$,
- for $j = i$, $\psi_{G_i^+}(P_a) = \psi_{G_i^- + G_j^-}(P_a) = \psi_{G_i^-}(P_a) = 0$,
- for $j < i$, $\psi_{G_i^-}(P_a) = P_a \ (a = x, y, z)$ and $\psi_{G_i^+}(P_a) = 0$.

The sequence in the claim is restricted on $U_j$ to the following sequence of $\mathbb{C}[x_j, y_j, z_j]$-modules:

$$0 \to (0, 0, 0) \oplus (0, 0, 0) \oplus (d_j, d_j, d_j) \to (d_j, d_j, d_j) \to 0.$$

Here $(0, 0, 0)$ (resp. $(d_j, d_j, d_j)$) is spanned by

$$\{x^ay^bz^c \mid a, b, c \geq 0\} \quad (\text{resp. } \{x^ay^bz^c \mid a \geq -d_j, b \geq -d_j, c \geq -d_j\})$$

as a vector space and $1$ is the map which maps $x^ay^bz^c$ to $x^ay^bz^c$. We can verify this is exact. The corresponding element in $H^1(Y_\sigma, \mathcal{O}_{Y_\sigma}(F))$ can be checked by the Čech argument in [Ful93, §3.5].

For $k \in I$ and $i \in \left\{\frac{3}{2}, \ldots, N - \frac{3}{2}\right\}$ we define divisors

$$F_k^\Delta := F_k^+ - F_k^- \quad H_i := \sum_{k=1}^{\frac{i}{2}} F_k^\Delta \quad I_k := H_{k-\frac{1}{2}} + F_k^+,$$

and the exact sequence

$$0 \to \bigoplus_{i \in \left\{\frac{3}{2}, \ldots, N - \frac{3}{2}\right\}} \mathcal{O}_{Y_\sigma}(H_i) \to \bigoplus_{k \in I} \mathcal{O}_{Y_\sigma}(I_k) \to \mathcal{O}_{Y_\sigma}(F+) \to 0.$$

The first map the sum of compositions of the maps

$$\left(1_{H_i, F_i^-} \oplus -1_{H_i, F_i^+}\right) : \mathcal{O}_{Y_\sigma}(H_i) \to \mathcal{O}_{Y_\sigma}(I_i - \frac{1}{2}) \oplus \mathcal{O}_{Y_\sigma}(I_i + \frac{1}{2})$$

and the canonical inclusions

$$\mathcal{O}_{Y_\sigma}(I_i - \frac{1}{2}) \oplus \mathcal{O}_{Y_\sigma}(I_i + \frac{1}{2}) \to \bigoplus_{k \in I} \mathcal{O}_{Y_\sigma}(I_k).$$

The second map is the sum of $1_{I_k, F_i^+ - I_k}$’s.

Proposition 1.9. The above sequence gives the universal extension corresponding to the set $\{f_{i_k} \mid i \in \left\{\frac{3}{2}, \ldots, N - \frac{3}{2}\right\}\}$ of generators of $H^1(Y_\sigma, \mathcal{O}_{Y_\sigma}(F))$ as $H^0(Y_\sigma, \mathcal{O}_{Y_\sigma})$-module.
Proof. For $i \in \left\{ \frac{1}{2}, \ldots, N - \frac{3}{2} \right\}$ we can check the sequence

$$0 \to \mathcal{O}_{Y_s}(H_1) \to \left( \oplus_k \mathcal{O}_{Y_s}(I_k) \right) / \left( \oplus_{j \neq i} \mathcal{O}_{Y_s}(H_j) \right) \to \mathcal{O}_{Y_s}(F^+),$$

is isomorphic to the exact sequence in Proposition [1.8]

Now we have the following theorem:

**Theorem 1.10.** The direct sum $\mathcal{O}_{Y_s} \oplus \bigoplus_{k \in \mathbb{I}} L_k$ is a projective generator of $\text{Per}^{-1}(Y/X)$.

**Proof.** This claim follows from [VdB04, Proposition 3.2.5] and Proposition [1.9]

1.2 Crepant resolutions as moduli spaces

We will associate $\sigma$ with a quiver with superpotential $A_\sigma = (Q_\sigma, \omega_\sigma)$. The set of vertices of the quiver $Q_\sigma$ is $\hat{I}$, which is identified with $\mathbb{Z}/N\mathbb{Z}$. The set of edges of the quiver $Q_\sigma$ is given by

$$H := \left( \coprod_{i \in \hat{I}} h_i^+ \right) \sqcup \left( \coprod_{i \in \hat{I}} h_i^- \right) \sqcup \left( \prod_{k \in \hat{I}_r} r_k \right).$$

Here $h_i^+$ (resp. $h_i^-$) is an edge from $i - \frac{1}{2}$ to $i + \frac{1}{2}$ (resp. from $i + \frac{1}{2}$ to $i - \frac{1}{2}$), $r_k$ is an edge from $k$ to itself and

$$\hat{I}_r := \left\{ k \in \hat{I} \mid \sigma_y(k - \frac{1}{2}) = \sigma_y(k + \frac{1}{2}) \right\}.$$

The relation is given as follows:

- $h_i^+ \circ r_{i-\frac{1}{2}} = r_{i+\frac{1}{2}} \circ h_i^+$ and $r_{i-\frac{1}{2}} \circ h_i^- = h_i^- \circ r_{i+\frac{1}{2}}$ for $i \in \hat{I}$ such that $i - \frac{1}{2}, i + \frac{1}{2} \in \hat{I}$.

- $h_i^+ \circ r_{i-\frac{1}{2}} = h_{i-1}^- \circ h_{i+1}^+ \circ h_i^+$ and $r_{i-\frac{1}{2}} \circ h_i^- = h_i^- \circ h_{i+1}^- \circ h_i^+$ for $i \in \hat{I}$ such that $i - \frac{1}{2} \in \hat{I}_r, i + \frac{1}{2} \notin \hat{I}_r$.

- $h_i^+ \circ h_{i-1}^- \circ h_{i-1}^- = r_{i+\frac{1}{2}} \circ h_i^+$ and $h_{i-1}^- \circ h_i^- \circ h_{i+1}^- = h_i^- \circ r_{i+\frac{1}{2}}$ for $i \in \hat{I}$ such that $i - \frac{1}{2} \notin \hat{I}_r, i + \frac{1}{2} \in \hat{I}_r$.

- $h_{i-\frac{1}{2}} \circ h_{i+\frac{1}{2}} \circ h_{i-\frac{1}{2}} \circ h_{i+\frac{1}{2}} = h_{i+1}^- \circ h_{i+1}^+ \circ h_i^+$ and $h_{i-\frac{1}{2}} \circ h_{i+\frac{1}{2}} \circ h_i^- \circ h_{i+\frac{1}{2}} = h_i^- \circ h_{i+1}^- \circ h_{i+1}^+$ for $i \in \hat{I}$ such that $i - \frac{1}{2}, i + \frac{1}{2} \notin \hat{I}_r$.

- $h_{i-\frac{1}{2}} \circ h_{i-\frac{1}{2}} = h_{i+\frac{1}{2}} \circ h_{i+\frac{1}{2}}$ for $k \in \hat{I}_r$.

This quiver is derived from the following bipartite graph on a 2-dimensional torus. Let $S$ be the union of infinite number of rhombi with edge length 1 as in Figure [2] which is located so that the centers of the rhombi are on a line parallel to the $x$-axis in $\mathbb{R}^2$ and $H$ be the union of infinite number of hexagons with edge length 1 as in Figure [3] which is located so that the centers of the hexagons are in a line parallel to the $x$-axis in $\mathbb{R}^2$. We make the sequence $\mathbb{Z} \to \{ S, H \}$ which maps $l$ to $S$ (resp. $H$) if $l$ module $N$ is not in $\hat{I}_r$ (resp. is in $\hat{I}_r$) and cover
Figure 2: $S$

Figure 3: $H$

Figure 4: $P_\sigma$ in case Example 1.2
the whole plane $\mathbb{R}^2$ by arranging $S$’s and $H$’s according to this sequence (see Figure 3). We regard this as a graph on the 2-dimensional torus $\mathbb{R}^2/\Lambda$, where $\Lambda$ is the lattice generated by $(\sqrt{3}, 0)$ and $(N_0 - N_1, (N_0 - N_1)\sqrt{3} + N_1)$. We can colored the vertices of this graph by black or white so that each edge connect a black vertex and a white one. Let $P_\sigma$ denote this bipartite graph on the torus. For each edge $h^\nu$ in $P_\sigma$, we make its dual edge $h$ directed so that we see the black end of $h^\nu$ on our right hand side when we cross $h^\nu$ along $h$ in the given direction. The resulting quiver coincides with $Q_\sigma$. For each vertex $q$ of $P_\sigma$, the composition of all arrows in $Q_\sigma$ corresponding to edges in $P_\sigma$ with $q$ as their ends gives a superpotential. We define $\omega_\sigma$ as the sum of all such superpotentials.

Put $\delta = (1, \ldots, 1) = \mathbb{Z}^I$ and take a stability parameter $\theta \in \text{Hom}(\mathbb{Z}^I, \mathbb{R}) \simeq \mathbb{R}^I$ so that $\theta(\delta) = 0$.

**Definition 1.11.** An $A_\sigma$-module $V$ is $\theta$-(semi)stable if for any nonzero submodule $0 \neq S \subset V$ we have $\theta(\text{dim} S) \leq 0$.

By [Kin94] such a stability condition coincides with a stability condition in geometric invariant theory and we can define the moduli space $\mathcal{M}_\sigma^\theta(\delta)$ of $\theta$-semistable $A_\sigma$-modules $V$ such that $\text{dim} V = \delta$. A stability parameter $\theta$ is said to be generic if $\theta$-semistability and $\theta$-stability are equivalent to each other.

**Theorem 1.12 ([IU] Theorem 6.4]).** For a generic stability parameter $\theta$, the moduli space $\mathcal{M}_\sigma^\theta(\delta)$ is a crepant resolution of $X$.

**Proof.** We can verify easily that the assumptions in [IU] Theorem 6.4 are satisfied. We can also check that the convex hull of height changes (see [IU] §2) is equivalent to $\Delta$ (see the description of divisors before Theorem 1.14 for example).

Let $T \subset \mathcal{M}_\sigma(\delta)$ be the open subset consisting of representations $t$ such that $t(h^+_i) \neq 0$ for any $i \in I$ and $t(r_k) \neq 0$ for any $k \in I_r$. Then $T$ is the 3-dimensional torus acting on $\mathcal{M}_\sigma(\delta)$ by edge-wise multiplications.

### 1.3 Description at a specific parameter

Let $\theta_0$ be a stability parameter so that $\theta_0(\delta) = 0$ and $(\theta_0)_k < 0$ for any $k \neq 0$.

For $i \in I$ let $p_i \in \mathcal{M}_{\theta_0}(\delta)$ be the representation such that

$$p_i(h^+_j) = 1 \quad (j < i), \quad 0 \quad (j \geq i), \quad p_i(h^-_j) = 1 \quad (j > i), \quad 0 \quad (j \leq i), \quad p_i(r_k) = 0.$$ 

This is fixed by the torus action. We can take a coordinate $(x_i, y_i, z_i)$ on the neighborhood $U_i$ of $p_i$ in $\mathcal{M}_{\theta_0}(\delta)$ such that the representation $v[x_i, y_i, z_i]$ with the coordinate $(x_i, y_i, z_i)$ is given by

$$v[x_i, y_i, z_i](h^+_j) = 1 \quad (j < i),$$

$$v[x_i, y_i, z_i](h^-_j) = 1 \quad (j > i),$$

$$v[x_i, y_i, z_i](h^+_i) = x,$$

$$v[x_i, y_i, z_i](h^-_i) = y,$$

$$v[x_i, y_i, z_i](q^-_{i-\frac{1}{2}}) = v[x_i, y_i, z_i](q^+_{i-\frac{1}{2}}) = z,$$
where
\[ q^+_l = \begin{cases} r_l & (l \in \hat{I}_r), \\ h^+_{l+\frac{1}{2}} \circ h^+_{l+\frac{1}{2}} & (l \notin \hat{I}_r). \end{cases} \]

For \( k \in I \) let \( c^+_k \in \mathfrak{M}_{\theta_0}^\sigma(\delta) \) be the representation such that
\[ c^+_k(h^+_j) = \begin{cases} 1 & (j < k), \\ 0 & (j > k), \end{cases} \]
\[ c^+_k(h^-_j) = \begin{cases} 1 & (j > k), \\ 0 & (j < k), \end{cases} \]
\[ c^+_k(r_l) = 0, \]
and \( C_k \) be the closure of the orbit of \( c^+_k \) with respect to the torus action. This is isomorphic to \( \mathbb{P}^1 \), contained in \( U_{k-\frac{1}{2}} \cap U_{k+\frac{1}{2}} \) and
\[ C_k|_{U_{k-\frac{1}{2}}} = \{ y_{k-\frac{1}{2}} = z_{k-\frac{1}{2}} = 0 \}, \quad C_k|_{U_{k+\frac{1}{2}}} = \{ x_{k+\frac{1}{2}} = z_{k+\frac{1}{2}} = 0 \}. \]

Note that the coordinate transformation is given by
\[ (x_{k+\frac{1}{2}}, y_{k+\frac{1}{2}}, z_{k+\frac{1}{2}}) \mapsto \begin{cases} (x_{k-\frac{1}{2}}, y_{k-\frac{1}{2}}, x^{-1}_{k-\frac{1}{2}}, y^{-1}_{k-\frac{1}{2}}, z_{k-\frac{1}{2}}) & (k \in \hat{I}_r), \\ (x_{k+\frac{1}{2}}, y_{k+\frac{1}{2}}, x^{-1}_{k+\frac{1}{2}}, y^{-1}_{k+\frac{1}{2}}, z_{k+\frac{1}{2}}) & (k \notin \hat{I}_r), \end{cases} \]

Hence \( C_k \) is a \((0, -2)\)-curve if \( k \in \hat{I}_r \), and a \((-1, -1)\)-curve if \( k \notin \hat{I}_r \).

For a crepant resolution of \( X \) the configuration of \((0, -2)\)-curves and \((-1, -1)\)-curves determines its toric diagrams. So we have the following proposition:

**Proposition 1.13.** The crepant resolution \( \mathfrak{M}_{\theta_0}^\sigma(\delta) \) is isomorphic to \( Y_\sigma \). For \( i \in \hat{I} \) the triangle \( T_i \) corresponds to the fixed point \( p_i \) and for \( k \in I \) the edge \( l_k \) corresponds to the curve \( C_k \).

For \( \varepsilon = 0 \) or 1 and \( 0 \leq x \leq N_\varepsilon \) let \( d^1_{\varepsilon,x} \in \mathfrak{M}_{\theta_0}^\sigma(\delta) \) be the representation such that
\[ d^1_{\varepsilon,x}(h^+_j) = \begin{cases} 1 & (\sigma_y(j) \neq \varepsilon \text{ or } \sigma_x(i) > x), \\ 0 & \text{(otherwise)}, \end{cases} \]
\[ d^1_{\varepsilon,x}(h^-_j) = \begin{cases} 1 & (\sigma_y(j) \neq \varepsilon \text{ or } \sigma_x(i) < x), \\ 0 & \text{(otherwise)}, \end{cases} \]
\[ d^1_{\varepsilon,x}(r_k) = \begin{cases} 1 & (\sigma_y(k + \frac{1}{2}) = \varepsilon), \\ 0 & \text{(otherwise)}, \end{cases} \]

Then the closure of the orbit of \( d^1_{\varepsilon,x} \) with respect to the torus action coincides with the divisor \( D_{\varepsilon,x} \).

We take the tautological line bundles \( L^\sigma_{\theta_0}(k \in \hat{I}) \) on \( \mathfrak{M}_{\theta_0}^\sigma(\delta) \) corresponding to the vertex \( k \) of the quiver \( Q_\sigma \) such that \( L^\sigma_{\theta_0} \cong \mathcal{O}_{\mathfrak{M}_{\theta_0}^\sigma(\delta)} \). We have the tautological section of the line bundle \( \text{Hom}(L^\sigma_{\theta_0}, L^\sigma_{\theta_0}) \) corresponding to the
Figure 5: Universal representations on $\mathfrak{m}_{\theta_0}^7(\delta)$ in case Example 1.2
edge $h_i^\pm$. From the description above, its divisor coincides with $E_i^\pm$ defined in §1. Hence we have

$$L_k^\sigma,\theta_0 \simeq \mathcal{O}(\sum_{i=1}^{k-\frac{1}{2}} E_i^+) \simeq L_k,$$

where $L_k$ is defined just after §3. In summary, we have the following theorem:

**Theorem 1.14.** The tautological vector bundle $L_k^\sigma,\theta_0 = \bigoplus_{k \in I} L_k^{\sigma,\theta_0}$ is a projective generator of $\text{Per}(Y_\sigma/X)$.

In particular, we have the following equivalence:

$$D^b(\text{Coh}(Y_\sigma)) \simeq D^b(\text{mod}(\text{End}_{Y_\sigma}(L_k^\sigma,\theta_0))).$$

**1.4 Computation of the endomorphism algebra**

In this subsection, we denote $L_k^\sigma,\theta_0$ and $L_k^\sigma$ simply by $L_k^\sigma$ and $L_k$ respectively.

Since $L_k^\sigma$ is the tautological bundle on the moduli space of $A_\sigma$-modules, we have the tautological map

$$\phi: A_\sigma \to \text{End}_{Y_\sigma}(L_k^\sigma).$$

**Proposition 1.15.** Let $e_k \in A_\sigma$ be the idempotent corresponding to the vertex $k$. Then the restriction of the tautological map

$$\phi_k: e_k A e_k \to \text{End}_{Y_\sigma}(L_k^\sigma) \simeq H^0(Y_\sigma, \mathcal{O}_{Y_\sigma}) \simeq \mathbb{C}[X, Y, Z, W]/(XY - Z^N W^N).$$

is bijective.

**Proof.** We have the tautological section of the line bundle $\mathcal{E}_{Y_\sigma}(L_k^\sigma) \simeq \mathcal{O}_{Y_\sigma}$ corresponding to the path

$$X_k := h_{k-\frac{1}{2}}^+ \circ \cdots \circ h_{N-\frac{1}{2}}^+ \circ h_{k-\frac{1}{2}}^+ \circ \cdots \circ h_{k+\frac{1}{2}}^+ \circ h_{N+\frac{1}{2}}^+.$$

Its divisor is

$$F_N^+ = \sum_{i=\frac{1}{2}}^{N-\frac{1}{2}} E_i^+$$

and as we have seen in Lemma §1.3 we have $-\psi F_N^+ = x$. So $\phi_k(X_k)$ coincides with $X$ up to scalar multiplication (see the defining equation (1) of $X$ in §1.1).

Similarly, we put

$$Y_k := h_{k-\frac{1}{2}}^- \circ \cdots \circ h_{N-\frac{1}{2}}^- \circ h_{k-\frac{1}{2}}^- \circ \cdots \circ h_{k+\frac{1}{2}}^- \circ h_{N+\frac{1}{2}}^-,$$

$$Z_k := \begin{cases} h_{k+\frac{1}{2}}^+ \circ h_{k+\frac{1}{2}}^+ & (\sigma_y(k + \frac{1}{2}) = 1), \\ h_{k-\frac{1}{2}}^- \circ h_{k-\frac{1}{2}}^- & (\sigma_y(k - \frac{1}{2}) = 1), \\ r_k & (\text{otherwise}), \end{cases}$$

$$W_k := \begin{cases} h_{k+\frac{1}{2}}^- \circ h_{k+\frac{1}{2}}^- & (\sigma_y(k + \frac{1}{2}) = 0), \\ h_{k-\frac{1}{2}}^+ \circ h_{k-\frac{1}{2}}^+ & (\sigma_y(k - \frac{1}{2}) = 0), \\ r_k & (\text{otherwise}). \end{cases}$$
Then \(\phi_k(Y_k), \phi_k(Z_k)\) and \(\phi_k(W_k)\) respectively coincide with \(Y, Z\) and \(W\) up to scalar multiplications. Hence \(\phi_k\) is surjective. To show the injectiveness, it is enough to check that \(X_k, Y_k, Z_k\) and \(W_k\)

- generate \(e_k A e_k\),
- commute with each other, and
- satisfy the relation \(X_k Y_k = Z_k W_k\).

These follow from the lemma below.

**Lemma 1.16.** For a path \(P\) in \(Q_\sigma\) starting from \(k \in I\) let \(P_{\text{red}}\) be a minimal length path which is "homotopic" to \(P\). Namely \(P_{\text{red}}\) is the composition of \(h_i^+\)'s or \(h_i^-\)'s and is homotopic to \(P\) when we regard \(Q_\sigma\) as \(S^1\) and a path in \(Q_\sigma\) as a map from the interval \([0,1]\) to \(S^1\). We set

\[
\begin{align*}
  r_k(P) &= (\text{the number of } r_k \text{ appearing in } P), \\
  h_i(P) &= (\text{the number of } h_i^+ \text{ appearing in } P) - (\text{the number of } h_i^- \text{ appearing in } P_{\text{red}}) \\
  &= (\text{the number of } h_i^- \text{ appearing in } P) - (\text{the number of } h_i^- \text{ appearing in } P_{\text{red}}).
\end{align*}
\]

and

\[
\begin{align*}
  z(P) &= \sum_{i \in I, \sigma_i(I \frac{1}{2}) = 1} r_k(P) + \sum_{i \in I, \sigma_i(I \frac{1}{2}) = 1} h_i(P), \\
  w(P) &= \sum_{i \in I, \sigma_i(I \frac{1}{2}) = 0} r_k(P) + \sum_{i \in I, \sigma_i(I \frac{1}{2}) = 1} h_i(P).
\end{align*}
\]

Then we have

\[P = P_{\text{red}} \circ (Z_k)^{z(P)} \circ (W_k)^{w(P)}\]

**Proof.** From the relations of \(A_\sigma\) we can verify directly that for any path \(P\) from \(k\) to \(k'\) we have

\[Z_k' \circ P = P \circ Z_k, \quad W_k' \circ P = P \circ W_k\]

in \(A_\sigma\).

For a path \(P\) in \(Q_\sigma\) from \(k\) we have a expression

\[P = P' \circ Z_{k'} \circ P'' \quad \text{(or } P' \circ W_{k'} \circ P'')\]

for some paths \(P', P''\) in \(Q_\sigma\) and \(k' \in I\) unless \(P = P_{\text{red}}\). This case we have

\[P = P' \circ P'' \circ Z_k \quad \text{(or } P \circ P'' \circ W_k)\]

in \(A_\sigma\) by the equation (6). Then apply the same procedure for \(P' \circ P''\) unless it is reduced. By the induction with respect to the lengths of paths, we can verify the claim.

For \(k \neq k' \in I\) let \(X_{k,k'}\) (resp. \(Y_{k,k'}\)) be the minimal length path from \(k\) to \(k'\) which is a composition of \(h_i^+\)'s (resp. \(h_i^-\)'s). Then we have the following basis of \(e_k A_\sigma e_k\):

\[
\{(X_{k'}^n \circ X_{k,k'} \circ (Z_k)^m \circ (W_k)^l)\}_{n,m,l \geq 0} \sqcup \{(Y_{k'}^n \circ Y_{k,k'} \circ (Z_k)^m \circ (W_k)^l)\}_{n,m,l \geq 0}.
\]
We have

\[
X_{k',k} \circ (X_{k'})^n \circ X_{k,k'} \circ (Z_k)^m \circ (W_k)^l = (X_k)^{n+1} \circ (Z_k)^m \circ (W_k)^l,
\]

\[
X_{k,k'} \circ (Y_{k'})^n \circ Y_{k,k'} \circ (Z_k)^m \circ (W_k)^l = (Y_k) \circ (Z_k)^{m+m'} \circ (W_k)^l + l' ,
\]

where \( m' \) and \( l' \) is nonnegative integer such that \( X_{k,k'} \circ Y_{k,k'} = (Z_k)^{m'} \circ (W_k)^{l'} \).

In particular, we have

**Lemma 1.17.** The map

\[
X_{k,k'} \circ - : e_{k'} A_\sigma e_k \rightarrow e_k A_\sigma e_k
\]

is injective.

**Proposition 1.18.** For \( k \neq k' \in I \) the restriction of the tautological map

\[
\phi_{k,k'} : e_{k'} A e_k \rightarrow \text{Hom}_\sigma(L_k^\sigma, L_{k'}^\sigma)
\]

is bijective.

**Proof.** The injectiveness follows from Lemma 1.17 and the injectiveness of \( \phi_k \).

For the surjectiveness, it is enough to check that \( \text{Hom}_\sigma(L_k, L_{k'}) \) is generated by the image of \( X_{k,k'} \) and \( Y_{k,k'} \) as an \( R \)-module. Let \( F_{k,k'}^+ \) (resp. \( F_{k,k'}^- \)) be the sum of \( E_i^+ \)'s (resp. \( E_i^- \)'s) corresponding to \( X_{k,k'} \) (resp. \( Y_{k,k'} \)). Note that \( \psi_{F_{k,k'}^+ - F_{k,k'}^-} \) is the unique element in \( M \) such that \( \psi_{F_{k,k'}^+ - F_{k,k'}^-}(N_{\varepsilon, \varepsilon}, 1) = \psi_{F_{k,k'}^+ - F_{k,k'}^-}(N_{\varepsilon, \varepsilon}, 1) \) and such that \( \psi_{F_{k,k'}^+ - F_{k,k'}^-}(1, 0, 0) = -1 \).

Let \( u \in S_{\Delta}(F_{k,k'}^+) \) be an element such that \( \langle u, v \rangle < 0 \) for some \( v \in Q \). It is enough to check there exists \( u' \in S_{\Delta} \) such that \( u = \psi_{F_{k,k'}^+ - F_{k,k'}^-} + u' \).

Since \( \langle u, (0, 0, 1) \rangle, \langle u, (0, 1, 1) \rangle \geq 0 \) and \( \langle u, v \rangle < 0 \) for some \( v \in Q \), we have \( \langle u, (1, 0, 0) \rangle < 0 \). Let \( u' \in S_{\Delta} \) be the element such that

\[
\langle u', (1, 0, 0) \rangle = \langle u, (1, 0, 0) \rangle + 1
\]

\[
\langle u', (N_{\varepsilon, \varepsilon}, 1) \rangle = \langle u, (N_{\varepsilon, \varepsilon}, 1) \rangle + F_{k,k'}^+(\varepsilon, N_{\varepsilon}).
\]

It follows from the characterization of \( \psi_{F_{k,k'}^+ - F_{k,k'}^-} \) above that \( u-u' = \psi_{F_{k,k'}^+ - F_{k,k'}^-} \).

In summary, we have the following theorem:

**Theorem 1.19.** The tautological homomorphism

\[
A_\sigma \rightarrow \text{End}_\sigma(L^\sigma).
\]

is isomorphism.

## 2 Counting invariants

From now on, we denote \( Q_\sigma \) and \( A_\sigma \) simply by \( Q \) and \( A \). Let \( Q_0 \) and \( Q_1 \) denote the sets of vertices and edges of the quiver \( Q \) respectively, and \( A\text{-mod} \) denote the category of finite dimensional \( A \)-modules.
2.1 Koszul resolution

We set a grading on the path algebra $\mathbb{C}Q$ such that
\[
\text{deg}(e_k) = 0, \quad \text{deg}(h_i^{\pm}) = 1, \quad \text{deg}(r_k) = 2.
\]

The superpotential $\omega$ is homogeneous of degree 4 with respect to this grading. So the quiver with superpotential $A = (Q, \omega)$ is graded 3-dimensional Calabi-Yau algebra.

We denote the subalgebra $\mathbb{C}Q_0 = \bigoplus_{k \in \mathbb{Q}} \mathbb{C}e_k$ of $A$ by $S$. For an $S$-module $T$ we define an $A$-bimodule $F_T$ by
\[
F_T = A \otimes S T \otimes S A.
\]

For $k, k' \in I$ let $T_{k,k'}$ denote the 1-dimensional $S$-module given by
\[
e_l \cdot 1 = \delta_{k,l}, \quad 1 \cdot e_l = \delta_{k',l},
\]
and we set $F_k := F_{T_{k,k}}$, $F_{k,k'} := F_{T_{k,k'}}$.

Note that an element of $F_{k,k'}$ is described as a linear combination of $\{p \otimes 1 \otimes q\} p \in A e_k, q \in e_k A$.

For a quiver with superpotential $A$, the Koszul complex of $A$ is the following complex of $A$-bimodules:
\[
0 \to \bigoplus_{k \in Q_0} F_k \xrightarrow{d_3} \bigoplus_{a \in Q_1} F_{\text{out}(a),\text{in}(a)} \xrightarrow{d_2} \bigoplus_{b \in Q_1} F_{\text{in}(b),\text{out}(b)} \xrightarrow{d_1} \bigoplus_{k \in Q_0} F_k \xrightarrow{m} A \to 0.
\]

Here the maps $m$, $d_1$, $d_3$ are given by
\[
m(p \otimes 1 \otimes q) = pq \quad (p \in Ae_k, q \in e_k A),
\]
\[
d_1(p \otimes 1 \otimes q) = (pb \otimes 1 \otimes q) - (p \otimes 1 \otimes bq) \quad (p \in Ae_{\text{in}(b)}, q \in e_{\text{out}(b)} A),
\]
\[
d_3(p \otimes 1 \otimes q) = \left( \bigoplus_{a: \text{in}(a) = k} pa \otimes 1 \otimes q \right) - \left( \bigoplus_{a: \text{out}(a) = k} p \otimes 1 \otimes aq \right) \quad (p \in Ae_k, q \in e_k A).
\]

The map $d_2$ is defined as follows: Let $c$ be a cycle in the quiver $Q$. We define the map $\partial_{c:a,b} : F_{\text{out}(a),\text{in}(a)} \to F_{\text{in}(b),\text{out}(b)}$ by
\[
\partial_{c:a,b}(p \otimes 1 \otimes q) = \sum_{r \in e_{\text{in}(a)} A e_{\text{out}(b)}, s \in e_{\text{in}(a)} A e_{\text{out}(a)}, \text{arr} = c} ps \otimes 1 \otimes rq.
\]

Then $d_2 = \partial_\omega$ is defined as the linear combination of $\partial_c$’s.

Since $A$ is graded 3-dimensional Calabi-Yau algebra, the Koszul complex is exact ([Boc08, Theorem 4.3]).

2.2 new quiver

We make a new quiver $\hat{Q}$ by adding one vertex $\infty$ and one arrow from the vertex $\infty$ to the vertex 0 to the original quiver $Q$. The original superpotential
\( \omega \) gives the superpotential on the new quiver \( \tilde{Q} \) as well. We set \( \tilde{A} := (\tilde{Q}, \omega) \) and denote the category of finite dimensional \( \tilde{A} \)-modules by \( \tilde{A} \)-fmod. Giving a finite dimensional \( A \)-module \( \tilde{V} \) is equivalent to giving a pair \( (V, W, i) \) of a finite dimensional \( A \)-module \( V \), a finite dimensional vector space \( W \) at the vertex \( \infty \) and a linear map \( i : W \to V_0 \). Let \( \iota : \tilde{A} \)-fmod \( \to A \)-fmod be the forgetting functor mapping \( (V, W, i) \) to \( V \).

We also consider the following Koszul type complex \( \tilde{A} \)-bimodules as well:

\[
0 \to \bigoplus_{\tilde{k} \in \tilde{Q}_0} \tilde{F}_{\tilde{k}} \xrightarrow{d_3} \bigoplus_{a \in \tilde{Q}_1} \tilde{F}_{\text{out}(a), \text{in}(a)} \xrightarrow{d_2} \bigoplus_{b \in \tilde{Q}_1} \tilde{F}_{\text{in}(b), \text{out}(b)} \xrightarrow{d_1} \bigoplus_{\tilde{k} \in \tilde{Q}_0} \tilde{F}_{\tilde{k}} \xrightarrow{\tilde{m}} \tilde{A} \to 0,
\]

where \( \tilde{F}_{i}, \tilde{F}_{i'}, \tilde{d}, \) and \( \tilde{m} \) are defined in the same way. This is also exact. The exactness at the last three terms is equivalent to the definition of generators and relations of the algebra \( \tilde{A} \). The exactness at the first two terms is derived from that of the exactness of the Koszul complex of \( A \).

**Proposition 2.1.** For \( E, F \in \tilde{A} \)-fmod we have

\[
\text{hom}_{\tilde{A}}(E, F) - \text{ext}^1_{\tilde{A}}(E, F) + \text{ext}^1_{\tilde{A}}(F, E) - \text{hom}_{\tilde{A}}(F, E) = \dim E_\infty \cdot \dim F_0 - \dim E_0 \cdot \dim F_\infty.
\]

**Proof.** Note that we can compute \( \text{Ext}_{\tilde{A}}^\bullet(E, F) \) (resp. \( \text{Ext}_{\tilde{A}}^\bullet(\iota(E), \iota(F)) \)) using the Koszul resolution. Let \( d^3_{E}(E, F) \) (resp. \( d^3_{F}(E, F) \)) denote the derivation in the complex derived from the Koszul resolution. Then we have

\[
\text{hom}_{\tilde{A}}(E, F) - \text{ext}^1_{\tilde{A}}(E, F) = \sum_{k \in \tilde{Q}_0} (\dim E_k \cdot \dim F_k) - \sum_{a \in \tilde{Q}_1} (\dim E_{\text{out}(a)} \cdot \dim F_{\text{in}(a)}) + \text{rank} \left( d^3_{E}(E, F) \right).
\]

Note that

\[
\text{rank} \left( d^3_{E}(E, F) \right) = \text{rank} \left( d^3_{F}(E, F) \right) = \text{rank} \left( d^3_{F}(E, F) \right) = \text{rank} \left( d^3_{F}(E, E) \right),
\]

where the second equation comes from the self-duality of the Koszul complex of \( A \). Hence we have

\[
\text{hom}_{\tilde{A}}(E, F) - \text{ext}^1_{\tilde{A}}(E, F) + \text{ext}^1_{\tilde{A}}(F, E) - \text{hom}_{\tilde{A}}(F, E) = \sum_{h \in \tilde{Q}_1} (\dim E_{\text{out}(h)} \cdot \dim F_{\text{in}(h)}) - \dim E_0 \cdot \dim F_\infty.
\]

Here the last equation follows from the fact that for any \( i, j \in I \)

\( \sharp(\text{arrows from } i \text{ to } j) = \sharp(\text{arrows from } j \text{ to } i). \)

**Remark 2.2.** From the Koszul resolution of \( A \), the last equation in the proof is equivalent to the vanishing of the Euler form on \( A \)-fmod. This is equivalent to the vanishing of the Euler form on \( \text{Coh}_c(Y) \). The vanishing on \( \text{Coh}_c(Y) \) follows from Hirzebruch-Riemann-Roch theorem.
2.3 Counting invariants

Let \( \zeta \in \mathbb{R}^{\hat{Q}_0} \) be a pair of real numbers. For a finite dimensional \( \hat{A} \)-module \( \hat{V} \) we set

\[
\theta_{\zeta}(\hat{V}) = \frac{\sum_{k \in \hat{Q}_0} \zeta_k \cdot \dim \hat{V}_k}{\sum_{k \in \hat{Q}_0} \dim \hat{V}_k}.
\]

**Definition 2.3.** A finite dimensional \( \hat{A} \)-module \( \hat{V} \) is said to be \( \theta_{\zeta} \)-(semi)stable if we have

\[
\theta_{\zeta}(\hat{V}') (\leq) \theta_{\zeta}(\hat{V})
\]

for any nonzero proper \( \hat{A} \)-submodule \( \hat{V}' \).

Here we adapt the convention for the short-hand notation. The above means two assertions: semistable if we have ‘\( \leq \)’, and stable if we have ‘\( < \)’.

**Remark 2.4.** (1) These stability conditions coincide with ones in geometric invariant theory ([Kin94]).

(2) Given a real number \( c \) let \( \zeta' \in \mathbb{R}^{\hat{Q}_0} \) be the pair of real numbers \( (\zeta_k + c)_{k \in \hat{Q}_0} \).

Then we have

\[
\theta_{\zeta'}(\hat{V}) = \theta_{\zeta}(\hat{V}) + c.
\]

Hence \( \theta_{\zeta'} \)-(semi)stability and \( \theta_{\zeta} \)-(semi)stability are equivalent.

**Theorem 2.5** ([Rud97]). Let a stability parameter \( \zeta \in \mathbb{R}^{\hat{Q}_0} \) be fixed.

(1) A finite dimensional \( \hat{A} \)-module \( \hat{V} \in \hat{A} \)-fmod has the Harder-Narasimhan filtration:

\[
\hat{V} = \hat{V}_0 \supset \hat{V}_1 \supset \cdots \supset \hat{V}_k \supset \hat{V}_{k+1} = 0
\]

such that \( \hat{V}_i/\hat{V}_{i+1} \) is \( \theta_{\zeta} \)-semistable for \( i = 0, 1, \ldots, k \) and

\[
\theta_{\zeta}(\hat{V}_0/\hat{V}_1) < \theta_{\zeta}(\hat{V}_1/\hat{V}_2) < \cdots < \theta_{\zeta}(\hat{V}_k/\hat{V}_{k+1}).
\]

(2) A finite dimensional \( \theta_{\zeta} \)-stable \( \hat{A} \)-module \( \hat{V} \in \hat{A} \)-fmod has a Jordan-Hölder filtration:

\[
\hat{V} = \hat{V}_0 \supset \hat{V}_1 \supset \cdots \supset \hat{V}_k \supset \hat{V}_{k+1} = 0
\]

such that \( \hat{V}_i/\hat{V}_{i+1} \) is \( \theta_{\zeta} \)-stable for \( i = 0, 1, \ldots, k \) and

\[
\theta_{\zeta}(\hat{V}_0/\hat{V}_1) = \theta_{\zeta}(\hat{V}_1/\hat{V}_2) = \cdots = \theta_{\zeta}(\hat{V}_k/\hat{V}_{k+1}).
\]

We sometimes denote an \( \hat{A} \)-module \((V, C, i)\) with 1-dimensional vector space at the vertex \( \infty \) simply by \((V, i)\).

**Definition 2.6.** (1) Given \( \zeta \in \mathbb{R}^{\hat{Q}_0} \), we take \( \hat{\zeta} \in \mathbb{R}^{\hat{Q}_0} \) such that \( \hat{\zeta}_k = \zeta_k \) for \( k \in \hat{Q}_0 \). A finite dimensional \( A \)-module \( V \) said to be \( \zeta \)-(semi)stable if the \( \hat{A} \)-module \((V, 0, 0)\) is \( \theta_{\zeta} \)-(semi)stable. The definition does not depend on the choice of \( \hat{\zeta}_\infty \).
(2) Given \((V,i) \in \tilde{A}\text{-mod}\) and \(\zeta \in \mathbb{R}^{Q_0}\), we define \(\tilde{\zeta} \in \mathbb{R}^{Q_0}\) by \(\tilde{\zeta}_k = \zeta_k\) for \(k \in Q_0\) and
\[
\tilde{\zeta}_\infty = -\zeta \cdot \dim V.
\]

We say \((V,i)\) is \(\zeta\)-(semi)stable if it is \(\theta_{\tilde{\zeta}}\)-(semi)stable.

**Lemma 2.7.** An \(\tilde{A}\)-module \((V,i)\) is \(\zeta\)-(semi)stable if the following conditions are satisfied:

(A) for any nonzero \(A\)-submodule \(0 \neq S \subsetneq V\), we have
\[
\zeta \cdot \dim S \leq 0,
\]

(B) for any proper \(A\)-submodule \(T \subsetneq V\) such that \(\im(i) \subset T_0\), we have
\[
\zeta \cdot \dim T \leq \zeta \cdot \dim V.
\]

For \(\zeta \in \mathbb{R}^{Q_0}\) and \(v = (v_k) \in (\mathbb{Z}_{\geq 0})^{Q_0}\), let \(\mathcal{M}_\zeta(v)\) (resp. \(\mathcal{M}_\zeta^+(v)\)) denote the moduli space of \(\zeta\)-semistable (resp. \(\zeta\)-stable) \(\tilde{A}\)-modules \((V,i)\) such that \(\dim V = v\). They are constructed using geometric invariant theory ([Kin94]).

A stability parameter \(\zeta \in \mathbb{R}^{Q_0}\) is said to be generic if \(\zeta\)-semistability and \(\zeta\)-stability are equivalent to each other. Since the defining relation of \(A\) is derived from the derivation of the superpotential, the moduli space \(\mathcal{M}_\zeta(v)\) has a symmetric perfect obstruction theory ([Sze Theorem 1.3.1]). By the result of [Beh] a constructible \(\mathbb{Z}\)-valued function \(\nu\) is defined on the moduli space \(\mathcal{M}_\zeta(v)\).

We define the counting invariants
\[
D^\text{eu}_\zeta(v) := \chi(\mathcal{M}_\zeta(v)), \quad D_\zeta(v) := \sum_{n \in \mathbb{Z}} n \cdot \chi(\nu^{-1}(n))
\]
where \(\chi(-)\) denote topological Euler numbers. We encode them into the generating functions
\[
Z^\text{eu}_\zeta(q) := \sum_{v \in (\mathbb{Z}_{\geq 0})^{Q_0}} D^\text{eu}_\zeta(v) \cdot q^v, \quad Z_\zeta(q) := \sum_{v \in (\mathbb{Z}_{\geq 0})^{Q_0}} D_\zeta(v) \cdot q^v
\]
where \(q^v = \prod_{k \in Q_0} q_k^{v_k}\) and \(q_k\)'s are formal variables.

The 3-dimensional torus \(T\) acts on \(Y\), and so it also acts on \(\mathcal{M}_\zeta(v)\). Assume that the set of \(T\)-fixed points \(\mathcal{M}_\zeta(v)^T\) is isolated and finite (see [Sze 2.5, 3.4]). By the argument in the proof of [Sze Lemma 2.5.2, Corollary 2.5.3 and Theorem 2.7.1], the contribution of a \(T\)-fixed point \(P \in \mathcal{M}_\zeta(v)^T\) is
\[
(-1)^{v_0 + \sum_{a \in Q_1} v_{\im(a)} v_{\im(a)} - \sum_{k \in Q_0} v_k^2} = (-1)^{\sum_{k \in \hat{I}_r} v_k + \sum_{k \in I} v_k},
\]
and so we have
\[
Z_\zeta(q) = \sum_{n \in \mathbb{Z}} (-1)^{\sum_{k \in I_r} v_k + \sum_{k \in I} v_k} |\mathcal{M}_\zeta(v)^T| \cdot q^n = Z^\text{eu}_\zeta(p)
\]
where the new formal variables \(p\) are given by
\[
p_k = \begin{cases} 
q_k & k \neq 0, k \in \hat{I}_r, \text{ or, } k = 0, k \neq \hat{I}_r, \\
-q_k & k \neq 0, k \neq \hat{I}_r, \text{ or, } k = 0, k \in \hat{I}_r.
\end{cases}
\]
2.4 Wall-crossing formula

The set of non-generic stability parameters is the union of the hyperplanes in $\mathbb{R}^{Q_0}$. Each hyperplane is called a wall and each connected component of the set of generic parameters is called a chamber. The moduli space $\mathcal{M}_\zeta(v)$ does not change as long as $\zeta$ moves in a chamber.

Let $\zeta^\circ = (\zeta_k)_{k \in Q_0}$ be a stability parameter on a single wall and set $\zeta^\pm = (\zeta_k \pm \varepsilon)_{k \in Q_0}$ for sufficiently small $\varepsilon > 0$.

**Proposition 2.8.** (1) Let $\tilde{V}' = (V', C, i')$ be a $\zeta^+$-stable $\tilde{A}$-module. Then we have an exact sequence

$$0 \to \tilde{V} \to \tilde{V}' \to \tilde{V}'' \to 0,$$

where $\tilde{V} = (V, C, i)$ is a $\zeta^\circ$-stable $\tilde{A}$-module, $\tilde{V}'' = (V'', 0, 0)$ and $\tilde{V}''$ is a $\zeta^\circ$-semistable $A$-module. The isomorphism class of $\tilde{V}$ and $\tilde{V}''$ are determined uniquely. In particular, assume $\tilde{V}' = (V', C, i')$ is $T$-invariant then $\tilde{V}$ and $\tilde{V}''$ are also $T$-invariant.

(2) Let $\tilde{V}' = (V', C, i')$ be a $\zeta^-$-stable $\tilde{A}$-module. Then we have an exact sequence

$$0 \to \tilde{V}'' \to \tilde{V}' \to \tilde{V} \to 0,$$

where $\tilde{V} = (V, C, i)$ is a $\zeta^\circ$-stable $\tilde{A}$-module, $\tilde{V}'' = (V'', 0, 0)$ and $\tilde{V}''$ is a $\zeta^\circ$-semistable $A$-module. The isomorphism class of $\tilde{V}$ and $\tilde{V}''$ are determined uniquely. In particular, assume $\tilde{V}' = (V', C, i')$ is $T$-invariant then $\tilde{V}$ and $\tilde{V}''$ are also $T$-invariant.

**Proof.** We take $\zeta^\circ \in \mathbb{R}^{Q_0}$ as in Definition 2.6. Let

$$\tilde{V}' = \tilde{V}_0 \supset \cdots \supset \tilde{V}_M \supset \tilde{V}_{M+1} = 0$$

be a Jordan-Hölder filtration of $\tilde{V}'$ with respect to the $\theta_{\zeta^\circ}$-stability. Since $\dim \tilde{V}_{\infty}' = 1$, there is an integer $0 \leq m \leq M$ such that $\dim(\tilde{V}_m/\tilde{V}_{m+1})_{\infty} = 1$ and $\dim(\tilde{V}_m'/\tilde{V}_{m'+1})_{\infty} = 0$ for any $m' \neq m$. Then for $m' \neq m$ we have

$$\zeta^+ \cdot \dim(\tilde{V}_m'/\tilde{V}_{m'+1})_{1} = \varepsilon \cdot \sum_{k \in I}(\tilde{V}_m'/\tilde{V}_{m'+1})_{k} > 0.$$

From the $\zeta^+$-stability of $\tilde{V}'$, we have $m = M$. Put $\tilde{V} = \tilde{V}_M$ and $\tilde{V}'' = \tilde{V}/\tilde{V}_M$, we have the required sequence. Note that this sequence gives a part of the Harder-Narasimhan filtration of $\tilde{V}'$ with respect to the $\theta_{\zeta^\circ}$-stability. Assume $\tilde{V}'$ is $T$-invariant, then it follows from the uniqueness of the Harder-Narasimhan filtration that $\tilde{V}$ and $\tilde{V}''$ are $T$-invariant.

We can verify the claim of (2) similarly.

We identify $\mathbb{Z}^{Q_0}$ with the root lattice of affine Lie algebra of type $A_N$ and denote the set of positive root vectors by $\Lambda^+$. 

**Proposition 2.9.** Assume $C$ is a $\zeta$-stable $A$-module for some $\zeta \in \mathbb{R}^{Q_0}$. Then $\dim C \in \Lambda^+$. Moreover, given a positive real root $\alpha \in \Lambda^+$ and a stability parameter $\zeta^\circ$ such that $\zeta^\circ$ is on the wall $W_\alpha$ but not on any other wall $W_{\alpha'}$ ($\alpha' \neq \alpha$), then we have the unique $\zeta^\circ$-stable $T$-invariant $A$-module $C$ such that $\dim C = \alpha$.
Proof. Note that we have the natural homomorphism

\[ R \to \bigoplus_{k \in I} \text{End}_Y(L_k) \to \text{End}_Y(\oplus L_k) \cong A, \]

where the first one is the diagonal embedding. The image of this map is central subalgebra of \( A \). Any \( A \)-module has the \( R \)-module structure given by this homomorphism. Any finite dimensional \( A \)-module is supported on finite number of points on \( \text{Spec}(R) = X \) and so any finite dimensional \( A \)-module \( C \) which is \( \zeta \)-stable for some \( \zeta \) is supported on a maximal ideal \( I \subset R \). Any nonzero element of \( I \) induces an \( A \)-module automorphism on \( C \) by the multiplication. Since \( C \) is \( \zeta \)-stable, any \( A \)-module automorphism on \( C \) is either zero or isomorphic. But this cannot be isomorphic, because \( C \) is supported on \( I \). Hence we have \( I \cdot C = 0 \).

Suppose that \( I \) corresponds to a nonsingular point on \( X \) and that \( C \) is \( \zeta \)-stable and \( I \cdot C = 0 \), then we have \( \dim(C) = (1, \ldots, 1) \). The singular points on \( X \) is classified as follows:

- the unique \( T \)-invariant \( (X, Y, Z, W) \),
- \( (X, Y, Z - a, W) \) \( (a \neq 0) \) or
- \( (X, Y, Z, W - b) \) \( (b \neq 0) \).

An \( A \)-module \( C \) such that \( (X, Y, Z, W) \cdot C = 0 \) is a module over the preprojective algebra of type \( \hat{A}_N \). The dimension of the moduli space of representations of the preprojective algebra of type \( \hat{A}_N \) is \( vCv \), where \( C \) is the Cartan matrix of type \( \hat{A}_N \) \( [\text{Nak94}] \). If there exists a \( \zeta \)-stable \( A \)-module \( C \) with \( \dim C = v \), then \( vCv \) is nonnegative, which is the definition of the root vectors. The uniqueness follows from the irreducibility of the moduli space \( [\text{CB02}] \). Let \( C \) be an \( A \)-module such that \( (X, Y, Z - a, W) \cdot C = 0 \) \( (a \neq 0) \). For \( k \in I \) such that \( \sigma_y(k + \frac{1}{2}) = 1, h^-_{k+\frac{1}{2}} \) and \( h^+_{k+\frac{1}{2}} \) give isomorphisms between \( C_k \) and \( C_{k+1} \). For \( k \in I \) such that \( \sigma_y(k + \frac{1}{2}) = 0 \), we have

\[ h^+_{k+\frac{1}{2}} \circ h^-_{k+\frac{1}{2}} = 0, \quad h^-_{k+\frac{1}{2}} \circ h^+_{k+\frac{1}{2}} = 0. \]

Thus the category of \( A \)-modules \( C \) such that \( (X, Y, Z - a, W) \cdot C = 0 \) \( (a \neq 0) \) is equivalent to the category of modules over the preprojective algebra of type \( \hat{A}_{N_0} \) with \( h \circ \bar{h} = 0 \) for any edge \( h \). Under this equivalence, a root vector of the root system of type \( \hat{A}_{N_0} \) corresponds to a root vector of the root system of type \( \hat{A}_N \). Hence for a \( \zeta \)-stable \( A \)-module \( C \) such that \( (X, Y, Z - a, W) \cdot C = 0 \) its dimension vector is a root vector of the root system of type \( \hat{A}_N \). Similarly, for a \( \zeta \)-stable \( A \)-module \( C \) such that \( (X, Y, Z, W - b) \cdot C = 0 \) its dimension vector is a root vector of the root system of type \( \hat{A}_N \).

Remark 2.10. Note that the fiber over the point corresponds to the maximal ideal \((X, Y, Z - a, W)\) (resp. \((X, Y, Z, W - b)\)) is the \( A_{N_0} \) (resp. \( A_{N_1} \)) configuration of \( \mathbb{P}^1 \)'s.

Corollary 2.11. The set of nongeneric parameters is the union of the hyperplanes \( W_\alpha := \{ \zeta \in \mathbb{Z}^{Q_0} \mid \zeta \cdot \alpha = 0 \} \) \( (\alpha \in \Lambda^+) \).
Take a positive real root $\alpha \in \Lambda^{\text{re.}+}$ and a parameter $\zeta^o \in \mathbb{R}^{Q_0}$ which is on $W_\alpha$ but not on any other wall. Let $C$ be the unique $T$-invariant $\zeta^o$-stable $\tilde{A}$-module such that $\zeta^o \cdot \dim C$. We set $\zeta^\pm = (\zeta_k \pm \varepsilon)$ for sufficiently small $\varepsilon$. We fix these notations throughout this subsection.

**Proposition 2.12** ([NN, Proposition 3.7]). For a $\zeta^o$-stable $\tilde{A}$-module $\tilde{V} = (V, i)$ we have

$$\text{ext}^1_A(C, \tilde{V}) - \text{ext}^1_A(\tilde{V}, C) = \dim C_0.$$ 

**Proof.** Since $C$ and $\tilde{V}$ are $\zeta^o$-stable and not isomorphic each other, we have $\text{hom}_{\tilde{A}}(C, \tilde{V}) = \text{hom}_{\tilde{A}}(\tilde{V}, C) = 0$. So the claim follows form Proposition 2.1. \qed

**Proposition 2.13.**

$$\text{ext}^1_A(C, C) = \begin{cases} 0 & \text{if } \sum_{k \notin I_r} \alpha_k \text{ is odd}, \\ 1 & \text{if } \sum_{k \notin I_r} \alpha_k \text{ is even}. \end{cases}$$

We prove this proposition in §2.6.

Let $\alpha \in \Lambda^{\text{re.}+}$ be a positive real root such that $\sum_{k \notin I_r} \alpha_k$ is odd. In such cases, wall-crossing formulas are given in [NN]:

**Theorem 2.14** ([NN, Theorem 3.9]).

$$Z^{eu}_{\zeta^-}(q) = (1 + q^\alpha)^{\alpha_0} \cdot Z^{eu}_{\zeta^+}(q).$$

**Remark 2.15.** To be precise we should modify the argument in [NN] a little, since the stable objects on the wall are not unique, while so a re the $T$-invariant stable objects. See the argument after Proposition 2.17.

We will study the case $\sum_{k \notin I_r} \alpha_k$ is even. For a positive integer $m$, we have the unique indecomposable $A$-module $C_m$ which is described as $m - 1$ times successive extensions of $C$’s.

**Proposition 2.16.** (1) Let $\tilde{V}' = (V', C, i')$ be a $T$-invariant $\zeta^+$-stable $\tilde{A}$-module. Then we have an exact sequence

$$0 \to \tilde{V} \to \tilde{V}' \to \bigoplus_{m' \geq 1} (C_{m'})^\oplus n_{m'} \to 0,$$

where $\tilde{V} = (V, C, i)$ is a $T$-invariant $\zeta^o$-stable $\tilde{A}$-module. The integers $n_{m'}$ and isomorphism class of $\tilde{V}$ are determined uniquely and satisfy

$$\text{hom}(\tilde{V}', C_m) = \sum_{m' \geq 1} n_{m'} \cdot \min(m', m).$$

Moreover, the composition of the maps

$$C^{N_m} \hookrightarrow \text{Hom}_{\tilde{A}}(C_m, \bigoplus_{m' \geq 1} (C_{m'})^\oplus n_{m'}) \longrightarrow \text{Ext}^1_A(C, \tilde{V})$$

is injective. Here $N_m = \sum_{m' \geq m} n_{m'}$ and the first map is induced by inclusions $C_m \hookrightarrow C_{m'}$ ($m' \geq m$). The second map is given by composing the inclusion $C \hookrightarrow C_m$ and $\tilde{V}' \in \text{Ext}^1_A(\bigoplus_{m' \geq 1} (C_{m'})^\oplus n_{m'}, \tilde{V})$. 

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Let \( \tilde{V}' = (V', C, i') \) be a \( T \)-invariant \( \zeta^- \)-stable \( \tilde{A} \)-module. Then we have an exact sequence
\[
0 \to \bigoplus_{m' \geq 1} (C_{m'})^{\oplus n_{m'}} \to \tilde{V}' \to \tilde{V} \to 0,
\]
where \( \tilde{V} = (V, C, i) \) is a \( T \)-invariant \( \zeta^0 \)-stable \( \tilde{A} \)-module. The integers \( n_{m'} \) and isomorphism class of \( \tilde{V} \) are determined uniquely and satisfy
\[
\text{hom}(C_{m}, \tilde{V}') = \sum_{m' \geq 1} n_{m'} \cdot \min(m', m).
\]
Moreover, the composition of the maps
\[
\mathbb{C}^{N_m} \hookrightarrow \text{Hom}_{\tilde{A}}(\bigoplus_{m' \geq 1} (C_{m'})^{\oplus n_{m'}}, C_m) \to \text{Ext}^1_{\tilde{A}}(\tilde{V}, C)
\]
is injective. Here \( N_m = \sum_{m' \geq m} n_{m'} \) and the first map is induced by surjections \( C_{m'} \to C_m \) for \( m' \geq m \). The second map is given by composing the surjection \( C_{m'} \to C_m \) with \( \tilde{V}' \in \text{Ext}^1_{\tilde{A}}(\tilde{V}, \bigoplus_{m' \geq 1} (C_{m'})^{\oplus n_{m'}}) \).

Proof. The existence of the sequences follow from Proposition 2.8, Proposition 2.9 and Proposition 2.13. Since \( \text{hom}(\tilde{V}, C_{m'}) = 0 \), we have \( \text{hom}(\tilde{V}', C_{m'}) = \text{hom}(\bigoplus_{m' \geq 1} (C_{m'})^{\oplus n_{m'}}, \tilde{V}') \) from the long exact sequences. The middle equations follow from \( \text{hom}(C_{m'}, C_{m'}) = \min(m', m) \). The compositions of the maps are injective since otherwise \( \tilde{V}' \) has \( C \) as its direct summand.

Given a sequence \( (N_m)_{m \in \mathbb{Z}} \) such that \( N_{m''} = N_{m''+1} = \cdots < N \) for some \( m'' \), let \( Fl((N_k); N) \) be the flag variety
\[
\{ 0 \subseteq W_1 \subseteq \cdots \subseteq W_{m''} \subseteq C_N | \dim W_m = N_m \}.
\]
We can verify the following claim as well:

**Proposition 2.17.** (1) Let \( \tilde{V} = (V, i) \) be a \( T \)-invariant \( \zeta^0 \)-stable \( \tilde{A} \)-module. For an element
\[
(W_k) \in Fl((N_k); \text{ext}^1_{\tilde{A}}(C, \tilde{V}))^T,
\]
let \( \tilde{V}' \) denote the \( \tilde{A} \)-module given by the universal extension
\[
0 \to \tilde{V} \to \tilde{V}' \to \bigoplus_{m \geq 1} (C_m)^{\oplus n_m} \to 0,
\]
such that the image of the composition map in Proposition 2.16 coincides with \( W_k \). Then \( \tilde{V}' \) is \( T \)-invariant and \( \zeta^+ \)-stable.

(2) Let \( \tilde{V} = (V, i) \) be a \( T \)-invariant \( \zeta^0 \)-stable \( \tilde{A} \)-module. For an element
\[
(W_k) \in Fl((N_k); \text{ext}^1_{\tilde{A}}(\tilde{V}, C))^T,
\]
let \( \tilde{V}' \) denote the \( \tilde{A} \)-module given by the universal extension
\[
0 \to \bigoplus_{m \geq 1} (C_m)^{\oplus n_m} \to \tilde{V}' \to \tilde{V} \to 0
\]
such that the image of the composition map in Proposition 2.16 coincides with \( W_k \). Then \( \tilde{V}' \) is \( T \)-invariant and \( \zeta^- \)-stable.
Hereafter we denote the set of $T$-fixed points on $X$ by $TX$. Let $R(T)$ be the representation ring of $T$. For a nonnegative integer $N$ (resp. $N \in R(T)$), let $\mathcal{M}_\xi^s(v)_N$ (resp. $\mathcal{M}_\xi^s(v)_{N'}$) denote the subscheme of $\mathcal{M}_\xi^s(v)$ consisting of closed points $V$ such that $\text{ext}^1(C, V) = N$ (resp. $\text{Ext}^1(C, V) = N$) as $T$-modules. Let $\mathcal{M}_{\xi^+}(v'_{(n_m)})$ denote the subscheme of $\mathcal{M}_{\xi^+}(v')$ consisting of closed points $V'$ such that $\text{hom}(V', C_m) = \sum_{m' \geq 1} n_m \cdot \min(m', m)$ We have the canonical morphism $\mathcal{M}_{\xi^+}(v')_{(n_m)} \to \mathcal{M}_{\xi^+}(v')$ where $v = v' - \sum m n_m \cdot \dim C$ such that a closed point $V' \in \mathcal{M}_{\xi^+}(v')_{(n_m)}$ is mapped to the closed point $\tilde{V} \in \mathcal{M}_{\xi^+}(v')$ appeared in the exact sequence

$$0 \to \tilde{V} \to V' \oplus_{m' \geq 1} (C_{m'}) \oplus n_{m'} \to 0.$$ 

Let $\mathcal{M}_{\xi^+}(v')_{(n_m), N}$ (resp. $\mathcal{M}_{\xi^+}(v')_{(n_m), N'}$) denote the inverse image of $\mathcal{M}_{\xi^+}(v')_N$ (resp. $\mathcal{M}_{\xi^+}(v')_N$) with respect to the above morphism. Similarly, we define $\mathcal{M}_{\xi^+}(v')_N$, $\mathcal{M}_{\xi^+}(v')_{(n_m), N}$, $\mathcal{M}_{\xi^+}(v')_{(n_m), N}$ and $\mathcal{M}_{\xi^+}(v')_{(n_m), N}$.

By Proposition 2.16 and Proposition 2.17 the natural map

$$\mathcal{M}_{\xi^+}(v')_{(n_m), N} \to \mathcal{M}_{\xi^+}(v')_N$$

is a fibration. So we have

$$\sum_{v'} \chi(\mathcal{M}_{\xi^+}(v')) \cdot q^v = \sum_{v', (n_m), N} \chi(\mathcal{M}_{\xi^+}(v')_{(n_m), N}) \cdot q^v$$

$$= \sum_{v', (n_m), N} \chi(Fl((N_m); N)) \cdot \chi(\mathcal{M}_{\xi^+}(v')_N) \cdot q^{v + \sum m n_m \cdot \dim(C)}$$

$$= \sum_{v', N} \left( \sum_{(n_m)} \chi(Fl((n_m); N)) \cdot q^{m n_m \cdot \dim(C)} \right) \chi(\mathcal{M}_{\xi^+}(v')_N) \cdot q^v$$

$$= \sum_{v', N} \left( 1 - q^{\dim(C)} \right)^{-N} \chi(\mathcal{M}_{\xi^+}(v')_N) \cdot q^v.$$ 

Similarly we have

$$\sum_{v'} \chi(\mathcal{M}_{\xi^-}(v')) \cdot q^v = \sum_{v', N} \left( 1 - q^{\dim(C)} \right)^{-N} \chi(\mathcal{M}_{\xi^+}(v')_N) \cdot q^v.$$ 

By Proposition 2.12 we have $\mathcal{M}_{\xi}(v)_N = \mathcal{M}_{\xi}(v)^{N + \dim C_0}$. Hence we have

$$\sum_{v'} \chi(\mathcal{M}_{\xi^-}(v')) \cdot q^v = \sum_{v', N} \left( 1 - q^{\dim(C)} \right)^{-N} \chi(\mathcal{M}_{\xi^+}(v')_N) \cdot q^v$$

$$= \sum_{v', N} \left( 1 - q^{\dim(C)} \right)^{-N} \chi(\mathcal{M}_{\xi^+}(v')_N - \dim C_0) \cdot q^v$$

$$= \sum_{v', N} \left( 1 - q^{\dim(C)} \right)^{-N - \dim C_0} \chi(\mathcal{M}_{\xi^+}(v')_N) \cdot q^v$$

$$= \left( 1 - q^{\dim(C)} \right)^{-\dim C_0} \sum_{v'} \chi(\mathcal{M}_{\xi^+}(v')) \cdot q^v.$$ 

In summary, we have the following wall-crossing formula:
Theorem 2.18. We have
\[ Z_{\xi}^{\text{eu}}(q) = (1 - \varepsilon(\alpha)q^{\alpha})^{-\varepsilon(\alpha)\alpha_0} \cdot Z_{\zeta}^{\text{eu}}(q), \]
where
\[ \varepsilon(\alpha) = (-1)^{\sum_{k \in I} \alpha_k}. \]

2.5 DT, PT and NCDT

Note that the set \{ [C_k] \}_{k \in I} form a basis of \( H_2(Y; Z) \). We identify \( H_2(Y) \) with \( Z' \). For \( n \in \mathbb{Z}_{\geq 0} \) and \( \beta \in H_2(Y) \), let \( I_n(Y, \beta) \) denote the moduli space of ideal sheaves \( I_Z \) of one dimensional subschemes \( O_Z \subset O_Y \) such that \( \chi(O_Z) = n \) and \( |Z| = \beta \). Note that the set of torus fixed points \( I_n(Y, \beta)^T \) is isolated and finite ([MOP06]). We define the Donaldson-Thomas invariants \( I_{n, \beta} \) from \( I_n(Y, \beta) \) using Behrend’s function as is [2.3] ([Tho00], [Beh]), and their generating function by
\[ Z_{\text{DT}}(Y; q, t) := \sum_{n, \beta} I_{n, \beta} \cdot q^n t^\beta \]
where \( t^\beta = \prod_{k \in I} t_k^\beta_k \) and \( t_k (k \in I) \) is a formal variable.

Let \( P_n(Y, \beta) \) denote the moduli space of stable pairs \((F, s)\) such that \( \chi(F) = n \) and \( [\text{supp}(F)] = \beta \). The set of torus fixed points \( I_n(Y, \beta)^T \) is also isolated and finite ([PTb]). We define the Pandharipande-Thomas invariants \( P_{n, \beta} \) and their generating function \( Z_{\text{PT}}(Y; q, t) \) similarly ([PTa]).

We set \( \zeta^0 = (-N + 1, 1, \ldots, 1) \), \( \zeta^\pm = (-N + 1 \pm \varepsilon, 1, \ldots, 1) \) and \( q = q_0 \cdot q_1 \cdot \ldots \cdot q_{N-1} \). By the result in [NN] §2 and the fact that the sets of torus fixed points are isolated and finite, we have the following theorem:

Proposition 2.19.
\[ Z_{\text{DT}}(Y; q, q_1^{-1}, \ldots, q_{N-1}^{-1}) = Z_{\zeta^0}(q), \quad Z_{\text{PT}}(Y; q, q_1^{-1}, \ldots, q_{N-1}^{-1}) = Z_{\zeta^0}(q). \]

Let \( \zeta_{\text{triv}} \) be a parameter such that \( (\zeta_{\text{triv}})_k > 0 \) for any \( k \). Note that \( \mathcal{M}_{\zeta_{\text{triv}}}(v) \) is empty unless \( v = 0 \) and so \( Z_{\zeta_{\text{triv}}}(q) = 1 \). Let \( \zeta_{cyclic} \) be a parameter such that \( (\zeta_{cyclic})_k < 0 \) for any \( k \). The invariants \( D_{\zeta_{cyclic}}(v) \) are the non-commutative Donaldson-Thomas invariants defined in [Sze]. Note that the set of torus fixed points \( \mathcal{M}_{\zeta_{cyclic}}(v)^T \) is isolated and finite. We denote their generating function \( Z_{\zeta_{cyclic}}(q) \) by \( Z_{\text{NCDT}}(q) \).

We divide the set of positive real roots into the following two parts:
\[ \Lambda^{re. +} = \{ \alpha \in \Lambda^{re.} | \pm \zeta^0 \cdot \alpha < 0 \}. \]

Applying the wall-crossing formula and comparing the equations (7) in (2.3) we obtain the following relations between generating functions:

Theorem 2.20.
\[ Z_{\text{NCDT}}(q) = \prod_{\alpha \in \Lambda^{re. +}} (1 + (-1)^{\alpha_0} q^{\alpha})^{\varepsilon(\alpha)\alpha_0} \cdot \prod_{\alpha \in \Lambda^{re. +}} (1 + (-1)^{\alpha_0} q^{\alpha})^{\varepsilon(\alpha)\alpha_0} \cdot Z_{\text{DT}}(Y; q, q_1^{-1}, \ldots, q_{N-1}^{-1}), \]
\[ Z_{\text{PT}}(Y; q, q_1^{-1}, \ldots, q_{N-1}^{-1}) = \prod_{\alpha \in \Lambda^{re. +}} (1 + (-1)^{\alpha_0} q^{\alpha})^{\varepsilon(\alpha)\alpha_0} \cdot \prod_{\alpha \in \Lambda^{re. +}} (1 + (-1)^{\alpha_0} q^{\alpha})^{\varepsilon(\alpha)\alpha_0}. \]
Remark 2.21. For the case $N_1 = 0$, the formula on $Z_{NCDT}$ and $Z_{DT}$ have been given in [Youb].

We define the sets of positive real roots of the finite root system by

$$\Lambda^\text{fin}+ = \{ \alpha_{[a,b]} := \alpha_a + \cdots + \alpha_b \mid 0 < a \leq b < N \},$$

then we have

$$\Lambda^\text{re}+ = \{ \alpha + n\delta \mid \alpha \in \Lambda^\text{fin}+, \ n \geq 0 \}.$$

Note that for $\alpha \in \Lambda$ we have $\varepsilon(\alpha + n\delta) = \varepsilon(\alpha)$. Let

$$M(x,q) = \prod_{n=1}^{\infty} (1 - xq^n)^{-n}$$

be the MacMahon function.

Corollary 2.22.

$$Z_{PT}(Y; q, q_1, \ldots, q_{N-1}) = \prod_{0 < a \leq b < N} M(q[a,b], -q)^{\varepsilon(\alpha_{[a,b]})},$$

where $q[a,b] = q_a \cdots q_b$.

The root lattice of the finite root system is identified with $H_2(Y)$ so that $\alpha_k$ corresponds to $[C_k]$. The corollary claims the Gopakumar-Vafa BPS state counts in genus $g$ and class $\alpha$ defined in [PTa §3.4] is given by

$$n_{g,\alpha} = \begin{cases} -\varepsilon(\alpha_{[a,b]}) & \alpha = \alpha_{[a,b]}, \ g = 0, \\ 0 & \text{otherwise.} \end{cases}$$

2.6 Appendix

In this subsection, we prove Proposition 2.13. We can take integers $n_0$, $n_1$ and a basis $\{v_n\}_{n_0 \leq n \leq n_1}$ of $C$ such that

$$C_k = \bigoplus_{n_0 \leq n \leq n_1, n \equiv k \ (\text{mod } N)} C v_n,$$

$$(h_i^+(v_n), h_i^-(v_{n+1})) = (v_{n+1}, 0), \text{ or } (0, v_n) \quad (i - \frac{1}{2} = n),$$

$$h_i^- (v_{n_0}) = 0, \quad (i - \frac{1}{2} \equiv n_0),$$

$$h_i^+ (v_{n_1}) = 0, \quad (i - \frac{1}{2} \equiv n_1),$$

$$r_k(v_n) = 0, \quad (k \equiv n, k \in \hat{I}_c).$$

Lemma 2.23. Assume that we have $d \in \mathbb{Z}$ and $j < j' \in \hat{Z}$ such that $n_0 \leq j + \frac{1}{2}, j + dN + \frac{1}{2}$ and $j' - \frac{1}{2}, j' + dN - \frac{1}{2} \leq n_1$. If we have $v_{j+\frac{1}{2}} \notin \text{im}(h_{j'}^+), v_{j+1+\frac{1}{2}+dN} \in \ker(h_{j'}^-), v_{j'-\frac{1}{2}} \notin \text{im}(h_{j'}^-)$ and $v_{j+\frac{1}{2}+dN} \in \ker(h_{j'}^+)$, then $d = 0$ and $\{j, j'\} = \{n_0, n_1\}$.  

Proof. Note that $\bigoplus_{j < n < j'} \mathbb{C}v_n$ is a nonzero proper $A$-submodule of $C$ and $\bigoplus_{j + dN < n < j' + dN} \mathbb{C}v_n$ is a nonzero proper $A$-quotient module of $C$. By the $\theta$-stability of $C$ we have
\[ \sum_{j < n < j'} \theta_n < 0, \quad \sum_{j + dN < n < j' + dN} \theta_n > 0. \]
This is a contradiction. \hfill $\square$

Using the Koszul complex, we can compute $\text{Ext}^1_A(C, C)$ as the cohomology of the following complex:
\[ \bigoplus_{a \in Q_1} \text{Hom}(C_{\text{out}(a)}, C_{\text{in}(a)}) \xrightarrow{d_2} \bigoplus_{b \in Q_1} \text{Hom}(C_{\text{in}(b)}, C_{\text{out}(b)}) \xrightarrow{d_1} \bigoplus_{k \in Q_0} \text{Hom}(C_k, C_k). \]
For $d \in \mathbb{Z}$ and $j \in \mathbb{Z}$ such that both $j$ and $j - \frac{1}{2} + dN$ (resp. $j + \frac{1}{2} + dN$) are in the interval $[n_0, n_1]$ we define an element
\[ \alpha_{j,d}^+ \in \text{Hom}(C_{\text{in}(h_{\frac{j}{2}}^+)}, C_{\text{out}(k_{\frac{j}{2}}^+)}) \]
by
\[ \alpha_{j,d}^+(v_n) = \delta_{n,j + \frac{1}{2}} v_{j + \frac{1}{2} + dN}, \]
and for $l \in \mathbb{Z}$ such that both $l$ and $l + dN$ are in the interval $[n_0, n_1]$ we define an element
\[ \beta_{l,d} \in \text{Hom}(C_{\text{in}(r_{\frac{l}{2}})}, C_{\text{out}(r_{\frac{l}{2}})}) \]
by
\[ \beta_{l,d}(v_n) = \delta_{n,l} v_{l + dN}. \]
We define a set $J_0$ by
\[ J_0 = \{ n \in \mathbb{Z} \mid n_0 \leq n \leq n_1, n \in \hat{I}_r \} \cup \{ j \in \mathbb{Z} \mid n_0 < j < n_1 \}. \]
We define a set $J_d$ for $d \in \mathbb{Z}_{>0}$ such that $dN \leq n_1 - n_0$ by
\[ \{ n \in \mathbb{Z} \mid n_0 \leq n \leq n_1 - dN, n \in \hat{I}_r \} \cup \{ j \in \mathbb{Z} \mid n_0 < j < n_1 - dN \} \cup \{ n_0 - 1/2 \} \]
if $h_{(n_0-1/2)}^+(v_{n_0-1+dN}) = 0$, and
\[ \{ n \in \mathbb{Z} \mid n_0 \leq n \leq n_1 - dN, n \in \hat{I}_r \} \cup \{ j \in \mathbb{Z} \mid n_0 < j < n_1 - dN \} \cup \{ n_1 + 1/2 \} \]
if $v_{n_1+1-dN} \notin \text{im}(h_{(n_0+1/2)}^+)$. We also define a set $J_d$ for $d \in \mathbb{Z}_{<0}$ such that
\[ \{ n \in \mathbb{Z} \mid n_0 \leq n \leq n_1 - dN, n \in \hat{I}_r \} \cup \{ j \in \mathbb{Z} \mid n_0 < j < n_1 - dN \} \cup \{ n_1 + 1/2 \} \]
if $h_{(n_1+1/2)}^-(v_{n_1+1+dN}) = 0$, and
\[ \{ n \in \mathbb{Z} \mid n_0 \leq n \leq n_1 - dN, n \in \hat{I}_r \} \cup \{ j \in \mathbb{Z} \mid n_0 < j < n_1 - dN \} \cup \{ n_0 - 1/2 - dN \} \]
if $v_{n_0-1-dN} \notin \text{im}(h_{(n_0-1/2)}^-)$. For $d \in \mathbb{Z}$ and $j \in \mathbb{Z} \cap J_d$ we take an element $\beta_{j,d}$ in the kernel of $d_1$ as follows:
Lemma 2.24. The subspace of \( \ker(d_1) \) spanned by \( \{\beta_{a,d}\}_{a \in H} \) is contained in \( \im(d_2) \).

Proof. For \( a \in J_d \) we define an element \( \gamma_{a,d} \) by

- \( \beta_{a,d} \) if \( a \in \mathbb{Z} \),

- \( \bar{\alpha}_{a,d}^+ \in \Hom\left( C_{\text{out}}(h_a^-), C_{\text{in}}(h_a^-) \right) \) if \( a \in \mathbb{Z} \) and \( h_a^-(v_{a+1/2}) = h_a^+(v_{a+1/2}) = 0 \),

- \( \bar{\alpha}_{a,d}^- \in \Hom\left( C_{\text{out}}(h_a^-), C_{\text{in}}(h_a^-) \right) \) if \( a \in \mathbb{Z} \) and \( h_a^-(v_{a+1/2}) = h_a^+(v_{a+1/2}) = 0 \),

where \( \bar{\alpha}_{a,d}^\pm \) is given by

\[
\bar{\alpha}_{a,d}^\pm(v_n) = \delta_{n,a\pm1/2}v_{j+dN\pm1/2}.
\]

For \( a \in J_d \) such that \( a^+, a^- \in J_d \) we have

\[ d_2(\gamma_{a,d}) = \beta_{a^-,d} + \beta_{a^+,d}. \]

Let \( a_{\text{min}} \) and \( a_{\text{max}} \) be the maximal and minimal elements in \( H \). We can verify that either \( \beta_{a_{\text{min}},d} \in \im(d_2) \) or \( \beta_{a_{\text{max}},d} \in \im(d_2) \) holds by case-by-case argument. For example, if \( a_{\text{min}} \in \mathbb{Z} \) and \( h_{a_{\text{min}}}^-(v_{a_{\text{min}}+1/2}) = v_{a_{\text{min}}+1} \) then we have

\[ d_2(\bar{\alpha}_{a_{\text{min}}}^+,d) = \beta_{a_{\text{min}},d}.\]
Proposition 2.25. If \(|J_0|\) is even, then \(J_0\) is contained in \(\text{im}(d_2)\). If \(|J_0|\) is odd, then we have \(\dim(J_0/\text{im}(d_2) \cap J_0) = 1\).

Proof. Let \(a_{\text{min}}\) and \(a_{\text{max}}\) be the maximal and minimal elements in \(J_0\). For \(a \in J_0\) we have

\[
    d_2(\gamma_{a,0}) = \begin{cases} 
        \beta_{a_{\text{min}},d} & \text{if } a = a_{\text{min}}, \\
        \beta_{a_{\text{max}},d} & \text{if } a = a_{\text{max}}, \\
        \beta_{a,-d} + \beta_{a,+d} & \text{otherwise.}
    \end{cases}
\]

Thus the claim follows.

Now we finish the proof of Proposition 2.13.

3 Mutations and stabilities

3.1 mutations

In §1.2 we associate a quiver with a potential \(A := A_{\tau} \) with a map \(\tau: \hat{I} \to \{\pm 1\}\), where we use 1 and \(-1\) instead of \(H\) and \(S\) respectively. For \(k \in \hat{I}\), let \(\mu_k(\tau): \hat{I} \to \{\pm 1\}\) be the map given by

\[
    \mu_k(\tau)(l) = \begin{cases} 
        \tau(k)\tau(l) & (l = k \pm 1), \\
        \tau(l) & \text{(otherwise)},
    \end{cases}
\]

and let \(\mu_k(A)\) denote the quiver with the potential \(A_{\mu_k(\tau)}\).

Let \(P_k\) be the projective \(A\)-module associated with the vertex \(k \in \hat{I}\) and we set \(P := \bigoplus_k P_k (= A)\). We define the new \(A\)-module

\[
    P'_k := \text{coker}(P_k \to P_{k-1} \oplus P_{k+1})
\]

where \(P_k \to P_{k\pm 1}\) be the maps given by composing the arrows from \(k \pm 1\) to \(k\).

Proposition 3.1. (1) The object \(\mu_k(P) = \bigoplus_{l \neq k} P_l \oplus P'_k\) is a tilting generator in \(D^b(A\text{-mod})\).

(2) \(\text{End}_A(\mu_k(P)) \simeq \mu_k(A)\).

Proof. We provide just a sketch of the proof.

For the claim (1), it is clear that \(\mu_k(P)\) is a generator. Vanishing of the higher extensions from \(P_l (l \neq k)\) to \(P'_k\) follows from the surjectiveness of the map

\[
    \text{Hom}(P_l, P_{k-1}) \oplus \text{Hom}(P_l, P_{k+1}) \to \text{Hom}(P_l, P_k).
\]

We can verify the other conditions as well.

For the claim (2), we can construct the isomorphism explicitly. Let \(H^\pm_k\) and \(R_k\) denote the arrows of the mutated quiver \(\mu_k(A)\), where \(h^\pm_k\) and \(r_k\) are the arrows of the original quiver \(A\). Given \(k \in \hat{I}_{\tau}\), we associate the map induced by

\[
    (h^+_{k+\frac{1}{2}} \circ h^-_{k-\frac{1}{2}}) + (h^+_{k+\frac{1}{2}} \circ h^-_{k+\frac{1}{2}}): P_{k-1} \oplus P_{k+1} \to P_{k+1}
\]

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with the arrow \( H^+_{k+\frac{1}{2}} \), the map induced by
\[
0 \oplus \text{id}_{P_{k+1}} : P_{k+1} \to P_{k-1} \oplus P_{k+1}.
\]
with the arrow \( H^-_{k+\frac{1}{2}} \) and the map induced by
\[
P_k \xrightarrow{r_k} P_{k-1} \oplus P_{k+1} \xrightarrow{(R(k-1) \ 0 \ R(k+1))} P_{k-1} \oplus P_{k+1}.
\]
with the arrow \( R_k \). Here
\[
R(k \pm 1) = \begin{cases} r_{k \pm 1} & (k \pm 1 \in \hat{I}_r), \\ h^-_{k \pm \frac{1}{2}} \circ h^+_{k \pm \frac{1}{2}} & (k \pm 1 \notin \hat{I}_r). \end{cases}
\]
We can associate maps with the other arrows and verify that the construction actually provides an isomorphism.

In particular, we have the derived equivalence
\[
\Psi_k : D^b(A\text{-mod}) \to D^b((\mu_k(A))\text{-mod}).
\]

### 3.2 the affine Weyl group action

For \( k \in \hat{I} \) we define the map \( \mu_k : \mathbb{Z}^\hat{I} \to \mathbb{Z}^\hat{I} \) by
\[
(\mu_k(v))_l = \begin{cases} v_{k-1} - v_k + v_{k+1} & l = k, \\ v_l & \text{otherwise} \end{cases}
\]
for \( v \in \mathbb{Z}^{\hat{I}} \). We also define \( \mu_k : \mathbb{R}^\hat{I} \to \mathbb{R}^\hat{I} \) by
\[
(\mu_k(\zeta))_l = \begin{cases} -\zeta_k & l = k, \\ \zeta_{k+1} + \zeta_k & l = k \pm 1, \\ \zeta_l & \text{otherwise} \end{cases}
\]
for \( \zeta \in \mathbb{R}^{\hat{I}} \). Note that
\[
v \cdot \zeta = \mu_k(v) \cdot \mu_k(\zeta)
\]
for any \( v \) and \( \zeta \).

**Proposition 3.2.** Let \( \alpha \in \mathbb{Z}^\hat{I} \) be a positive root and \( \zeta \) be a parameter such that \( \alpha \cdot \zeta = 0 \), \( \zeta_k < 0 \) and such that \( \zeta \) is not on any other wall. Given a \( \zeta \)-stable \( A \)-module \( C \) with \( \dim C = \alpha \) and \( \mathcal{I} \cdot C = 0 \) for a maximal ideal \( \mathcal{I} \subset R \). Then \( \Psi_k(C) \) is the unique \( \mu_k(\zeta) \)-stable \( \mu_k(A) \)-module with \( \dim(\Psi_k(C)) = \mu_k(\alpha) \) such that \( \mathcal{I} \cdot \Psi_k(C) = 0 \).

**Proof.** Since \( C \) is \( \zeta \)-stable and \( \zeta_k < 0 \), the map
\[
C_{k-1} \oplus C_{k+1} \to C_k
\]
is surjective. Hence $\Psi_k(C)$ is a $\mu_k(A)$-module. Suppose we have an exact sequence

$$0 \rightarrow V \rightarrow \Psi_k(C) \rightarrow V' \rightarrow 0$$

of nontrivial $\mu_k(A)$-modules. Let $H^*_A(-)$ denote the cohomology with respect to the natural $t$-structure of $D^b(A\text{-mod})$. Note that for a $\mu_k(A)$-module $V$, $H^*_A(\Psi_k^{-1}(V))$ is concentrated on the degree 0 and $-1$ and $(H^1_A(\Psi_k^{-1}(V)))_l = 0$ for $l \neq k$. By the long exact sequence, we have $H_A^{-1}(\Psi_k^{-1}(V)) = 0$ and the following exact sequence

$$0 \rightarrow H_A^{-1}(\Psi_k^{-1}(V')) \rightarrow \Psi_k^{-1}(V) \rightarrow C \rightarrow H_A^0(\Psi_k^{-1}(V')) \rightarrow 0$$

of $A$-modules. Then we have

$$\mu_k(\zeta) \cdot \dim V = \zeta \cdot \dim \Psi_k^{-1}(V) \leq 0$$

where the first inequality follows from the $\zeta$-stability of $C$ and the second one follows from the assumption $\zeta_k < 0$.

In the rest of this section, we fix a parameter $\zeta$ such that $\sum \zeta_i < 0$ and such that $\zeta + d \cdot 1$ is not on an intersection of two walls. See Remark 3.9 for the case $\sum \zeta_i > 0$. Then we get the sequence $C_0, \ldots, C_r$ of chambers such that

- $\zeta - d \cdot 1 \in \bigcup C_s$ for any $d \geq 0$,
- for any $C_s$, there exists some $d \geq 0$ such that $\zeta - d \cdot 1 \in C_s$, and
- suppose $\zeta - d \cdot 1 \in C_s, \zeta - d' \cdot 1 \in C_s'$ and $s < s'$, then $d > d'$.

We also define the sequence $k_1, \ldots, k_r$ of elements in $\hat{I}$ such that

$$\overline{C}_{s-1} \cap \overline{C}_s \subset W_{\alpha^s} \quad (\alpha^s = \mu_{k_{s-1}} \circ \cdots \circ \mu_{k_1}(\alpha_{k_1})),$$

where $\alpha_k$ denote the simple root vector. We denote $\mu_s := \mu_{k_1} \circ \cdots \circ \mu_{k_s}$, $\Psi_s := \Psi_k \circ \cdots \circ \Psi_{k_1}$ and $\Psi_k := \Psi_r$.

**Corollary 3.3.** (1) Let $C$ be a finite dimensional $\theta_{\zeta}$-stable $A$-module. Suppose $\dim(C) \cdot \zeta < 0$, then $\Psi_{\zeta}(C) \in (\mu_{\zeta}(A))\text{-mod}$. Suppose $\dim(C) \cdot \zeta > 0$, then $\Psi_{\zeta}(C) \in (\mu_{\zeta}(A))\text{-mod}[1]$.

(2) Let $C'$ be a finite dimensional $\theta_{\mu_{\zeta}(\zeta)}$-stable $\mu_{\zeta}(A)$-module. Take $d \in \mathbb{R}$ such that $\mu_{\zeta}(\zeta) + d \cdot 1 \in \mu_{\zeta}(C'^{+})$ where $C'^{+} := \{\zeta \mid \zeta > 0\}$. Suppose $\dim(C') \cdot (\mu_{\zeta}(\zeta) + d \cdot 1) > 0$, then $\Psi_{\zeta}^{-1}(C') \in A\text{-mod}$. Suppose $\dim(C') \cdot (\mu_{\zeta}(\zeta) + d \cdot 1) < 0$, then $\Psi_{\zeta}^{-1}(C') \in A\text{-mod}[−1]$.

### 3.3 mutations and change of stability parameters

We set $\mathcal{P} := A\text{-fmod}$ and denote by $\mathcal{P}_\zeta$ the image of the Abelian category $\mu_{\zeta}(A)\text{-fmod}$ under the equivalence $\Psi_{\zeta}^{-1}$. Let $\overline{\mathcal{P}}$ (resp. $\overline{\mathcal{P}}_\zeta$) be the category consisting of pairs $(V, W, s)$, where $V \in \mathcal{P}$ (resp. $\in \mathcal{P}_\zeta$), $W$ is a finite dimensional vector space and $s: P_0 \otimes W \rightarrow V$. An object $(V, W, s)$ with 1-dimensional $W$ is simply written $(F, s)$. Note that $\overline{\mathcal{P}}$ is equivalent to $A\text{-fmod}$.
Definition 3.4. For $\xi \in \mathbb{R}^I$, we say an object $(V, s) \in P_\zeta$ is $(\xi, P_\zeta)$-(semi)stable if the following conditions are satisfied:

(A) for any nonzero subobject $0 \neq S \subseteq V$ in $P_\zeta$, we have
\[ \xi \cdot \dim S \leq 0, \]

(B) for any proper subobject $T \subset V$ in $P_\zeta$ which factors through, we have
\[ \xi \cdot \dim T \leq \xi \cdot \dim V. \]

From now on, the $\zeta$-(semi)stability for objects in $P \cong \tilde{A}$-fmod is written as the “$(\zeta, P)$-(semi)stability”. We set $\xi_{\text{cyclic}} := \mu_\zeta(\zeta)$. Note that $(\xi_{\text{cyclic}})$ is $0$ for any $l \in I$. The following four claims follow from Corollary 3.3 by the same argument as in [NN, §4.3]:

Lemma 3.5. (see [NN, Lemma 4.2]) Let $(F, s)$ be a $(\zeta, P)$-stable object, then $F \in P_\zeta$.

Proposition 3.6. (see [NN, Proposition 4.3]) Let $(F, s)$ be a $(\zeta, P)$-stable object, then $(F, s)$ is $(\xi_{\text{cyclic}}, P_\zeta)$-stable.

Lemma 3.7. (see [NN, Lemma 4.5]) Let $(F, s)$ be a $(\xi_{\text{cyclic}}, P_\zeta)$-stable object, then $F \in P$.

Proposition 3.8. (see [NN, Proposition 4.6]) Let $(F, s)$ be a $(\xi_{\text{cyclic}}, P_\zeta)$-stable object, then $(F, s)$ is $(\zeta, P)$-stable.

Remark 3.9. (1) For a parameter $\zeta$ such that $\sum \zeta_i > 0$, we can apply all the argument after a slight modification: first, the information in which chamber $\zeta + d \cdot 1$ $(d \geq 0)$ is contained defines the sequence $k_1, \ldots, k_r$. We define $\mu_\zeta$ and $\Psi_\zeta$ in the same way and denote by $P_\zeta$ the image of $(\mu_\zeta(A))$-fmod under the equivalence $\Psi_\zeta^{-1}[-1]$.

(2) The Abelian category $A$-fmod and the function
\[ Z_{\text{cyclic}} : K(A\text{-fmod}) \cong \mathbb{Z}^I \to \mathbb{C} \]
given by
\[ Z_{\text{cyclic}}(v) := (1 \cdot v) + (-\zeta_{\text{cyclic}} \cdot v)\sqrt{-1} \]
provide a Bridgeland stability condition $(Z_{\text{cyclic}}, \{P(\phi)\}_{\phi \in \mathbb{R}})$, where $P(\phi)$ is the full subcategory of $D^b(A\text{-fmod})$ of semistable objects with phase $\phi$. For $\phi \in \mathbb{R}$, let $P_{[\phi, \phi+1]}$ denote the full subcategory of objects such that all the factors in their Hardar-Narashimhan filtrations are in
\[ \bigcup_{\phi \leq \phi' < \phi + 1} P(\phi'). \]

Then we have
\[ P_{[0, 1]} = A\text{-fmod}, \quad P_{[\phi(d), \phi(d)+1]} = \mu_\zeta(A)\text{-fmod} \]
where $\phi(d)$ is determined by the condition $0 \leq \phi(d) < 1/2$ and $\tan(\phi(d)\pi) = d$. This is the reason why our argument works.
Let $A = (Q, \omega)$ be a quiver with a potential which is 3-dimensional Calabi-Yau and does not have any cycles of length 1 or 2. For a vertex $k$, we can mutate $A$ at $k$ to get a new one $\mu_k(A)$ and we have a derived equivalence

$$\Psi_k : D^b(\text{mod}(A)) \to D^b(\text{mod}(\mu_k(A))).$$

Let $\zeta \in \mathbb{R}^{Q_0}$ be a parameter. Assume that there exist a sequence $k_1, \ldots, k_s$ satisfying the conditions in §3.2. We define the category $P_\zeta$ in the same way. Then the same statement as Proposition 3.6 and 3.8 hold.

### 3.4 the tilting generator is a vector bundle

In this subsection, we assume that $N \cdot \zeta_k < \sum \zeta_l$ for any $k \neq 0$. Let $\nu_i (i \in \tilde{I})$ be real numbers such that

$$\nu_{k+\frac{1}{2}} - \nu_{k-\frac{1}{2}} = N \cdot \zeta_k - \sum \zeta_l$$

for any $k \in \tilde{I}$. We denote by $\pi_k : \mathbb{R}^\tilde{I} \to \mathbb{R}^\tilde{I}$ the permutation of the $(k-1/2)$-th and $(k+1/2)$-th elements and set $\pi_s := \pi_{k_s} \circ \cdots \circ \pi_{k_1}$. Note that

$$(\pi_s(\nu))_{k_s+\frac{1}{2}} - (\pi_s(\nu))_{k_s-\frac{1}{2}} = N \cdot (\mu_s(\zeta))_{k_s} - \sum \zeta_l.$$  

By the definition of the sequence $(k_s)$, we have $(\mu_{s-1}(\zeta))_{k_s} > 0$. Recall that we assume $\sum \zeta_l < 0$. Thus we have

$$(\pi_{s-1}(\nu))_{k_s+\frac{1}{2}} < (\pi_{s-1}(\nu))_{k_s-\frac{1}{2}}.$$  

Let $\nu_s : \tilde{I} \to \tilde{I}$ be the permutation such that the sequence $((\pi_s(\nu))_{\nu_s(i)})_{i \in \tilde{I}}$ is increasing and set $i := \nu_i$.

**Proposition 3.10.** Under the derived equivalence given in §4, the direct summand $(\mu_\zeta(P))_k$ is a line bundle $\mathcal{L}_\zeta^k$ on $Y$. Moreover, it is the following map that is associated with the arrow $H_1^\zeta$:

$$\mathcal{L}_\zeta^k \mapsto \mathcal{L}_\zeta^k \otimes \mathcal{O}_Y (E_{\nu_s(i)}) \simeq \mathcal{L}_\zeta^k \otimes \mathcal{O}_{E_{\nu_s(i)}}.$$

**Proof.** We prove the claim by induction.

By the argument in the proof of Proposition 1.8, we can verify that the sequence

$$0 \to \mathcal{O} \to \mathcal{O}(E^+_i) \oplus \mathcal{O}(E^-_j) \to \mathcal{O}(E^+_i + E^-_j) \to 0$$

is exact if $i < j$. Thus we have

$$(\mu_s(P))_{k_s} \simeq P_{k_s} \otimes \mathcal{O}(E^+_{s, (k_s-1/2)} + E^-_{s, (k_s+1/2)}).$$

The second half of the claim can be verified by the explicit description of the endomorphism algebra in the proof of Proposition 3.1. 

**Corollary 3.11.**

$$H^1(Y, (\mathcal{L}_\zeta^k)^{-1}) = 0.$$
Proof. By the argument in the proof of Proposition 3.10, \( L_\zeta_k \) is associated with the divisor which is described as a sum of divisors of the form \( E_i^+ + E_j^- \) (\( i < j \)). We can verify the claim using the description (5) of \( H^1 \) of a line bundle on \( Y \).

**Proposition 3.12.** The set of the torus fixed points \( \mathcal{M}_\zeta(v)^T \) is isolated.

**Proof.** By Corollary 3.11 we have \( \mathcal{O}_Y \in \mathcal{P}_\zeta \). Let \( P_\zeta \) be the \( \mu_k(A) \)-module corresponding to \( \mathcal{O}_Y \). By Proposition 3.6 and Proposition 3.8, the moduli space \( \mathcal{M}_\zeta(v) \) parametrizes finite dimensional quotient \( \mu_k(A) \)-modules \( V' \) of \( P_\zeta \) with \( \dim V' = \mu_k(v) \). Note that

\[
(P_\zeta)_k = H^0(Y, (L_\zeta_k)^{-1})
\]

and the \( T \)-weight decomposition of \( H^0 \) of a line bundle on \( Y \) is multiplicity free. Hence the claim follows.

**4 Remarks**

In this section, we make some observations on how Kontsevich-Soibelman’s wall-crossing formula (9) (they also call the formula (9) by ”Factorization Property”) would be applied in our setting.

First, we will review the work of Kontsevich-Soibelman ([KS]) very briefly. The core of their work is the construction of the algebra homomorphism from the ”motivic Hall algebra” to the ”quantum torus”. For an \( A_\infty \)-category \( \mathcal{C} \), the motivic Hall algebra \( H(\mathcal{C}) \) is, roughly speaking, the space of motives over the moduli \( \mathbb{O}b(\mathcal{C}) \) of all objects in \( \mathcal{C} \), with the product derived from the same diagram as the Ringel-Hall product. The quantum torus is a deformation of a polynomial ring described explicitly in the terms of the numerical datum of \( \mathcal{C} \). The homomorphism is given by taking, so to say, weighted Euler characteristics with respect to the motivic weight, where the motivic weight is defined using motivic Milnor fiber of the potentials coming from the \( A_\infty \)-structure. The formula (9) is the translation of the Harder-Narasimhan property under this homomorphism.

**Remark 4.1.** In the original Donaldson-Thomas invariants defined using symmetric obstruction theory, we adapt the Behrend’s function as a weight (see [2.3]). It is expected what, after taking the ”quasi-classical limit” as \( q \to 1 \), the motivic weight would coincide with the Behrend’s one. In [KS], the proof of the claim for some special situations and some evidences of the claim for more general situations are provided.

Now, we will explain the statement of ”Factorization Property”, restricting to our situation. We set \( \Lambda := \mathbb{Z}^{\mathcal{Q}_0} \) and define the skew symmetric bilinear form \( \langle -, - \rangle : \Lambda \times \Lambda \to \mathbb{Z} \) by

\[
\langle (e_i), (f_i) \rangle := e_\infty \cdot f_0 - e_0 \cdot f_\infty.
\]

Let \( Z \in \text{Hom}(\Lambda, \mathbb{C}) \) be a homomorphism such that \( \text{Im}(Z(\Lambda^\perp)) > 0 \) where \( \Lambda^\perp = \mathbb{Z}^{\mathcal{Q}_0}_{\geq 0} \).
Let $D^\mu$ be a certain ring of motives including the inverting motives $L^{-1}, [GL(n)]^{-1}$ $(n \geq 1)$ and the formal symbol $L^{1/2}$, where $L$ is the motive of the affine line. We have the homomorphism of rings $\phi: D^\mu \to Q(q^{1/2})$ mapping $L^{1/2}$ to $q^{1/2}$. We define the quantum torus $R_{\Lambda, q}$ as the $Q(q^{1/2})$-algebra generated by $x_\gamma$ $(\gamma \in \Lambda)$ with the relation

$$x_\gamma x_\mu = q^{\frac{1}{2}(\gamma, \mu)} x_{\gamma + \mu}.$$  

For a strict sector $V$ in the upper half plane, let $C^Z_V$ denote the category of $\tilde{A}$-modules which can be described as subsequent extensions of $Z$-semistable objects $E$ such that $Z(E) \in V$. Note that $C^Z_V$ does not change when $Z$ moves in $\text{Hom}(\Lambda, \mathbb{C})$, unless the values of $Z$ of semistable objects get close to the boundary $\partial V$. We define an element $A^Z_{V, q} \in R_{V, q}$ by "weighted" counting of objects in $C^Z_V$. Informally speaking,

$$A^Z_{V, q} := \sum_{E \in \text{Isom}(C^Z_V)} \phi \left( \frac{w(E)}{\text{Aut}(E)} \right) \cdot x_{\dim(E)} \in R_{V, q},$$

where $w(E) \in D^\mu$ is defined by the motivic Milnor fiber of the potential of the $A_\infty$-algebra algebra $\text{Ext}^*(E, E)$.

Assume that $V$ is decomposed into a disjoint union $V = V_1 \sqcup V_2$ in the clockwise order. Then the "Factorization Property" in [KS] claims that

$$A^Z_{V, q} = A^Z_{V_1, q} \cdot A^Z_{V_2, q}. \tag{9}$$

The key fact is, as we mentioned above, the existence of the algebra homomorphism from the motivic Hall algebra to $R_{V, q}$. Although the category of perverce coherent systems is not Calabi-Yau, Proposition 2.1 would assure the existence of the algebra homomorphism in our case. We can define an element $A^Z_{V, \text{mot}}$ in the motivic Hall algebra and the equation $A^Z_{V, \text{mot}} = A^Z_{V_1, \text{mot}} \cdot A^Z_{V_2, \text{mot}}$ follows from the Harder-Narashimhan property. Now the element $A^Z_{V, q}$ is the image of $A^Z_{V, \text{mot}}$ under the algebra homomorphism.

Now we end up with reviewing and begin to explain how to apply (9) in our setting. Since we are interested in $A$-modules $V$ with $V_\infty \simeq \mathbb{C}$, we will work on the quotient algebra

$$R'_{\Lambda, q} := R_{\Lambda, q}/(x_e | e_\infty \geq 2).$$

Consider the wall in $\text{Hom}(\Lambda, \mathbb{C})$ such that $e \in \Lambda^+$ with $e_\infty = 0$ and $f \in \Lambda^+$ with $f_\infty = 1$ are send on a same half line in the upper half plane. Assume $e \in \Lambda$ is primitive (i.e. $\{e_i\}_{i \in Q_0}$ are coprime to each other) and $f - e \notin \Lambda^+.$
Let $\Lambda_0 \in \Lambda$ be the sublattice generated by $e$ and $f$, $l_k$ be the half line passing through $k \cdot e + f$ and $l_\infty$ be the half line passing through $e$. Take $\mathbb{Z}^+$ and $\mathbb{Z}^-$ from the opposite side of the wall so that $l_1, l_2, \ldots, l_\infty$ are mapped on the upper half plane in the clockwise (resp. anticlockwise) way by $\mathbb{Z}^+$ (resp. $\mathbb{Z}^-$). The "Factorization Property" claims

$$A_{l_1}^{\mathbb{Z}^+} \cdot A_{l_2}^{\mathbb{Z}^+} \cdots A_{l_\infty}^{\mathbb{Z}^+} = A_{l_\infty}^{\mathbb{Z}^-} \cdots A_{l_2}^{\mathbb{Z}^-} \cdot A_{l_1}^{\mathbb{Z}^-}$$

in $\mathcal{R}_{\Lambda,q}$. We denote

$$\prod_k A_{l_k}^{\mathbb{Z}^+} := A_{l_1}^{\mathbb{Z}^+} \cdot A_{l_2}^{\mathbb{Z}^+} \cdots, \quad \prod_k A_{l_k}^{\mathbb{Z}^-} := \cdots A_{l_2}^{\mathbb{Z}^-} \cdot A_{l_1}^{\mathbb{Z}^-}.$$

Note that $A_{l_\infty}^{\mathbb{Z}^+} = A_{l_\infty}^{\mathbb{Z}^-}$ and we denote this by $A_{l_\infty}^{\mathbb{Z}}$. Then the above equation is described as following:

$$\prod_k A_{l_k}^{\mathbb{Z}^+} = A_{l_\infty}^{\mathbb{Z}} \cdot \left( \prod_k A_{l_k}^{\mathbb{Z}^-} \right) \cdot A_{l_\infty}^{\mathbb{Z}^-} \cdot 1.$$

An element $A$ of $\mathcal{R}_{\Lambda,q}^\prime$ can be uniquely described in the following form:

$$A = \sum_{e; e_\infty = 0} (a_e(q) \cdot x_e) + x_\infty \cdot \sum_{e; e_\infty = 0} (b_e(q) \cdot x_e).$$

We denote $\sum_{e; e_\infty = 0} (b_e(q) \cdot x_e)$ by $A^{x_\infty}$. Then

$$(q - 1) \cdot \left( \prod_k A_{l_k}^{\mathbb{Z}^+} \right)^{x_\infty} \bigg|_{q = 1}$$

makes sense and would coincide with the generating function of virtual counting of the moduli spaces we study in $\S 2$. Note that we have

$$\langle k \cdot e + f, e \rangle = e_0.$$
and so
\[ x_{m \cdot e} \cdot x_{k \cdot e + f} = q^{m \cdot e_0} x_{k \cdot e + f} \cdot x_{m \cdot e}. \]

Identify \( A_{l_{\infty}}^2 \in \mathcal{R}'_{\Lambda, q} \) with the polynomial in \( x_e \), then we have
\[
\prod_k A_{l_k}^2 = A_{l_{\infty}}^2 (x_e) \cdot \left( \prod_k A_{k}^2 (x_e) \right)^{-1} = \left( \prod_k A_{k}^2 \right) \cdot A_{l_{\infty}}^2 (q^{e_0} x_e) \cdot A_{l_{\infty}}^2 (x_e)^{-1}
\]
in \( \mathcal{R}'_{\Lambda, q} \). Now, if we can compute \( A_{l_{\infty}}^2 (q^{e_0} x_e) \cdot A_{l_{\infty}}^2 (x_e)^{-1} \big|_{q=1} \), we get wall-crossing formulas.

Here again, observations in \([KS]\) will help us. We put \( t := x_e \). Assume that we have the unique simple object \( E \) on \( l_{\infty} \) and \( \dim E = e \). Let \( B_E \) be the algebra generated by \( \text{Ext}^1(E, E) \) with relations defined from the potential \( W_E \). Then we have
\[
A_{l_{\infty}}^2 (qt) = A_{l_{\infty}}^2 (t) \cdot f(t) \quad (10)
\]
where \( f(t) \) is obtained by counting pairs of cyclic \( B_E \)-modules and their cyclic vectors. Applying this formula repeatedly we have
\[
A_{l_{\infty}}^2 (q^{e_0} t) \cdot A_{l_{\infty}}^2 (x_e)^{-1} \big|_{q=1} = (f(t)|_{q=1})^{e_0}.
\]

**Example 4.2.**

1. Assume \( \text{ext}^1(E, E) = 0 \). The algebra \( B_E \) is trivial. Hence we have \( f(t)|_{q=1} = 1 + t \). This corresponds to the formula in Theorem 2.14.

2. Assume \( \text{ext}^1(E, E) = 1 \) and \( B_E \simeq \mathbb{C}[z] \). In this case we have \( f(t)|_{q=1} = (1 - t)^{-1} \). This corresponds to the formula in Theorem 2.13.

3. Let us consider the wall corresponding to the imaginary root. The set of simple objects on this wall is \( \{ O_y \mid y \in Y \} \). By the same argument as they show the above equation (10) in \([KS]\), we would have
\[
A_{l_{\infty}}^2 (qt) = A_{l_{\infty}}^2 (t) \cdot f(t)
\]
where \( f(t) \) is obtained by counting 0-dimensional closed subscheme of \( Y \). By the results of \([MNOP06]\) and \([BF]\) we have
\[
f(t)|_{q=1} = M(-t)^{\varepsilon(Y)}
\]
where
\[
M(t) := \prod_{n=1}^{\infty} (1 - t^n)^{-n}
\]
is the MacMahon function. This provides DT-PT correspondence in our situation.


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