Rational values of powers of trigonometric functions

Genki Shibukawa

MSC classes: 11J72, 11R18, 33B10

Abstract

We extend the theorem by Olmsted (1945) and Carlitz-Thomas (1963) on rational values of trigonometric functions to powers of trigonometric functions.

1 Introduction

Throughout the paper, we denote the ring of rational numbers by \( \mathbb{Q} \), the ring of real numbers by \( \mathbb{R} \), the set of positive rational numbers by \( \mathbb{Q}_{>0} \) and a \( m \)th root of unity by \( \zeta_m := e^{\frac{2\pi i}{m}} \). Olmsted [1] and Carlitz-Thomas [2] determined all rational values of trigonometric functions.

**Theorem 1** (Olmsted (1945), Carlitz-Thomas (1963)). If \( \theta \in \mathbb{Q} \), then the only possible rational values of the trigonometric functions are:

\[
\sin (\pi \theta), \cos (\pi \theta) = 0, \pm \frac{1}{2}, \pm 1; \tan (\pi \theta) = 0, \pm 1.
\]

By this Theorem [1] and well-known facts

\[
\cos (\pi \theta)^2 = \frac{1 + \cos (2\pi \theta)}{2}, \quad \tan (\pi \theta)^2 = \frac{1}{\cos (\pi \theta)^2} - 1,
\]

we have the following result immediately.

**Corollary 2.** If \( \theta \in \mathbb{Q} \) and \( \cos (\pi \theta)^2 \in \mathbb{Q} \), then the only possible values of the trigonometric functions are:

\[
\sin (\pi \theta), \cos (\pi \theta) = 0, \pm \frac{1}{2}, \pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{3}}, \pm 1; \tan (\pi \theta) = 0, \pm \frac{1}{\sqrt{3}}, \pm 1, \pm \sqrt{3}.
\]

In this note, we propose a generalization of Theorem [1] and Corollary [2].

**Theorem 3.** If \( N \geq 3 \) and \( \alpha \) is a positive rational number such that \( \alpha^{\frac{1}{N}}, \ldots, \alpha^{\frac{N-1}{N}} \notin \mathbb{Q} \) then for any positive integer \( m \) we have

\[
\sqrt[N]{\alpha} \notin \mathbb{Q}(\zeta_m).
\]

In particular, there is no \( \theta \in \mathbb{Q} \) such that \( \cos (\pi \theta), \cos (\pi \theta)^2, \ldots, \cos (\pi \theta)^{N-1} \notin \mathbb{Q} \) and \( \cos (\pi \theta)^N \in \mathbb{Q} \) (resp. \( \tan (\pi \theta), \tan (\pi \theta)^2, \ldots, \tan (\pi \theta)^{N-1} \notin \mathbb{Q} \) and \( \tan (\pi \theta)^N \in \mathbb{Q} \)).
Theorem 4. If there exists a positive integer \( n \) and \( \theta \in \mathbb{Q} \) such that \( \cos(\pi \theta)^n \in \mathbb{Q} \) (resp. \( \tan(\pi \theta)^n \in \mathbb{Q} \)), then the only possible values of the trigonometric functions are:

\[
\sin(\pi \theta), \cos(\pi \theta) = \begin{cases} 
0, \pm \frac{1}{2}, \pm 1 & (n : \text{odd}) \\
0, \pm \frac{1}{\sqrt{2}}, \pm \frac{1}{2}, \pm \sqrt{\frac{3}{2}} \pm 1 & (n : \text{even})
\end{cases}
\] (1)

resp.

\[
\tan(\pi \theta) = \begin{cases} 
0, \pm 1 & (n : \text{odd}) \\
0, \pm \frac{1}{\sqrt{3}}, \pm 1, \pm \sqrt{3} & (n : \text{even})
\end{cases}
\] (2)

2 Preliminaries

To prove Theorem 3 and Theorem 4, we list some fundamental facts of the cyclotomic fields and Kummer extension in this section. First we mention the Galois group \( \mathbb{Q}(\zeta_n) \) (see [3]).

Lemma 5. The degree of the cyclotomic extension \( \mathbb{Q}(\zeta_n) \) over \( \mathbb{Q} \) is

\[
[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \varphi(n) := |\{1 \leq a \leq n \mid \gcd(a,n) = 1\}|
\]

and its Galois group \( \text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \) is

\[
(\mathbb{Z}/n\mathbb{Z})^\times \simeq \text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \triangleright \mathbb{Q}(\zeta_n) \rightarrow \mathbb{Q}(\zeta_n)
\]

\[
c \in (\mathbb{Z}/n\mathbb{Z})^\times \rightarrow \tau_c \in \text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \triangleright \zeta_n \rightarrow \tau_c(\zeta_n) := \zeta_n^c
\]

In particular \( \text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \) is an abelian extension, and its subfields \( L \supset \mathbb{Q} \) are Galois and abelian extension over \( \mathbb{Q} \).

Under the following, let \( \alpha \) be a positive rational number such that

\[
\alpha^{\frac{1}{n}}, \ldots, \alpha^{\frac{n-1}{n}} \notin \mathbb{Q}
\]

and \( K := \mathbb{Q}(\sqrt[n]{\alpha}, \zeta_n) \).

Proposition 6. (1) The binomial polynomial \( x^n - \alpha \) is irreducible over \( \mathbb{Q} \) and \([\mathbb{Q}(\sqrt[n]{\alpha}) : \mathbb{Q}] = n\).

(2) For any \( n \geq 2 \), we have \( \sqrt[n]{\alpha} \notin \mathbb{Q}(\zeta_n) \).

Proof. (1) We consider the following decomposition of the binomial polynomial:

\[
x^n - \alpha = \prod_{i=1}^{n} (x - \sqrt[n]{\alpha} \zeta_n^i) = f_I(x) f_J(x)
\]

where \( I, J \) are subsets of \([n] := \{1, 2, \ldots, n\}\) such that

\[
[n] = I \cup J, \quad I \neq \emptyset, \quad J \neq \emptyset, \quad I \cap J = \emptyset
\]

and

\[
f_I(x) := \prod_{i \in I} (x - \sqrt[n]{\alpha} \zeta_n^i) = x^{|I|} + \cdots + (-1)^{|I|} \alpha^{|I|} \prod_{i \in I} \zeta_n^i \in \mathbb{Q}[x],
\]
If \( x^n - \alpha \) is reducible over \( \mathbb{Q} \), then \( f_I(x) \) and \( f_J(x) \) are rational coefficient polynomials. Hence, by the definition of \( \alpha \in \mathbb{Q}_{>0} \) and

\[
(-1)^m \alpha^{|J|} \prod_{j \in J} \zeta_j^j \in \mathbb{Q},
\]

the product

\[
\prod_{j \in J} \zeta_j^j
\]

is real. Thus we have

\[
\prod_{j \in J} \zeta_j^j = \prod_{j \in J} \zeta_n^{-j}
\]

and

\[
\prod_{j \in J} \zeta_n^j = \pm 1.
\]

From the assumption \( \alpha^{|J|} \notin \mathbb{Q} \), the constant term of \( f_J(x) \)

\[
(-1)^m \alpha^m \prod_{j \in J} \zeta_n^j = \pm (-1)^m \alpha^m
\]

is irrational. It is a contradiction.

(2) If \( \sqrt[n]{\alpha} \in \mathbb{Q}(\zeta_n) \), then we have the contradiction

\[
n = [\mathbb{Q}(\sqrt[n]{\alpha}) : \mathbb{Q}] \leq [\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right).
\]

\[\square\]

**Lemma 7.** (1) If \( n = p \) is an odd prime, then the binomial type polynomial \( x^p - \alpha \) is irreducible over \( \mathbb{Q}(\zeta_p) \) and \( [K : \mathbb{Q}(\zeta_p)] = p \). Its Galois group \( \text{Gal}(K/\mathbb{Q}) \) is

\[
\mathbb{Z}/p\mathbb{Z} \times (\mathbb{Z}/p\mathbb{Z})^\times \quad \simeq \quad \text{Gal}(K/\mathbb{Q}) \quad \subset \quad K \quad \rightarrow \quad K
\]

\[
(1, 1) \quad \mapsto \quad \sigma \quad \subset \quad \sqrt[p]{\alpha} \quad \mapsto \quad \sigma(\sqrt[p]{\alpha}) := \zeta_p \sqrt[p]{\alpha},
\]

\[
(0, c) \quad \mapsto \quad \tau_c \quad \subset \quad \sqrt[p]{\alpha} \quad \mapsto \quad \tau_c(\sqrt[p]{\alpha}) := \sqrt[p]{\alpha}
\]

\[
\zeta_p \quad \mapsto \quad \tau_c(\zeta_p) := \zeta_p^c
\]

In particular, \( \tau_c \sigma = \sigma^c \tau_c \) and for \( n \geq 3 \) the Galois group \( \text{Gal}(K/\mathbb{Q}) \) is non-abelian.

(2) For any \( n \geq 3 \), the Galois group \( \text{Gal}(K/\mathbb{Q}) \) is non-abelian.
Proof. (1) We consider the factorization \(x^p - \alpha = f_1(x)f_2(x)\) again. By \(\gcd(|I|, p) = 1\) and \(\gcd(|J|, p) = 1\), \(\mathbb{Q}(\alpha^{\frac{1}{p^i}})\) and \(\mathbb{Q}(\alpha^{\frac{1}{p^j}})\) contain \(\sqrt[p^i]{\alpha}\). Hence, from Proposition 6 (2), \(\alpha^{\frac{1}{p^i}}\) and \(\alpha^{\frac{1}{p^j}}\) are not contained in \(\mathbb{Q}(\zeta_p)\). Therefore the binomial polynomial \(x^p - \alpha\) is irreducible over \(\mathbb{Q}(\zeta_p)\) and \([K : \mathbb{Q}(\zeta_p)] = p\).

(2) When an odd prime \(p\) divides \(n\), \(K\) contains a non-abelian Galois extension \(\mathbb{Q}(\sqrt[p]{\alpha}, \zeta_p)\) over \(\mathbb{Q}\), so the Galois group \(\text{Gal}(K/\mathbb{Q})\) is non-abelian. If \(n = 2^m (m \geq 2)\), then \(K\) contains a non-abelian Galois extension \(\mathbb{Q}(\sqrt[2^m]{\alpha}, \zeta_4)\) over \(\mathbb{Q}\) and \(\text{Gal}(K/\mathbb{Q})\) is also non-abelian.

Remark 8. Lemma 7 (1) is not true in general. For example, when \(n = 8\) and \(\alpha = 2\) the polynomial \(x^8 - 2\) is reducible over \(\mathbb{Q}(\zeta_8)\) even though \(2^{\frac{1}{8}}, \ldots, 2^{\frac{7}{8}}\) are irrational. In fact
\[
x^8 - 2 = (x^4 - \sqrt{2})(x^4 + \sqrt{2}) = (x^4 - \zeta_8 - \zeta_8^{-1})(x^4 + \zeta_8 + \zeta_8^{-1}).
\]

3 Proof of Theorem 3

Assume there exists \(N \geq 3\), \(\alpha \in \mathbb{Q}_{>0}\) and a positive integer \(m\) such that \(\alpha^{\frac{1}{N}}, \ldots, \alpha^{\frac{N-1}{N}} \notin \mathbb{Q}\) and
\[
\sqrt[N]{\alpha} \in \mathbb{Q}(\zeta_m).
\]
Then
\[
\mathbb{Q}(\sqrt[N]{\alpha}) \subset \mathbb{Q}(\zeta_m).
\]
Although \(\mathbb{Q}(\sqrt[N]{\alpha})\) is not a Galois extension over \(\mathbb{Q}\), \(K\) is a Galois extension over \(\mathbb{Q}\) and
\[
K \subset \mathbb{Q}(\zeta_m, \zeta_N) \subset \mathbb{Q}(\zeta_{mN}).
\]
Further the field \(K\) is a subfield of \(\mathbb{Q}(\zeta_{mN})\) and the Galois group \(\text{Gal}(K/\mathbb{Q})\) is a normal subgroup of \(\text{Gal}(\mathbb{Q}(\zeta_{mN})/\mathbb{Q})\):
\[
\text{Gal}(K/\mathbb{Q}) \triangleleft \text{Gal}(\mathbb{Q}(\zeta_{mN})/\mathbb{Q}) \simeq (\mathbb{Z}/mN\mathbb{Z})^\times.
\]
However the Galois group \(\text{Gal}(K/\mathbb{Q})\) is non-abelian. It is a contradiction. Then for any positive integer \(m\),
\[
\sqrt[N]{\alpha} \notin \mathbb{Q}(\zeta_m). \quad (5)
\]

For the above \(N \geq 3\) and positive rational number \(\alpha \in \mathbb{Q}_{>0}\), assume there exists \(\theta \in \mathbb{Q}\) such that
\[
\cos(\pi \theta) = \sqrt[N]{\alpha}.
\]
By \(\theta \in \mathbb{Q}\), there exists a positive integer \(m\) such that
\[
\sqrt[N]{\alpha} = \cos(\pi \theta) \in \mathbb{Q}(\zeta_m).
\]
For \(N \geq 3\), it is contrary to (5). For \(\tan(\pi \theta)\) we prove similarly.
Remark 9. From Theorem $3$ if $n \geq 3$ then there is no cyclotomic field over $\mathbb{Q}$ containing $n$th root of a positive rational number $\alpha$ with $\sqrt[n]{\alpha} \notin \mathbb{Q}$. On the other hand, from Gauss sum’s formulas $[1]$

$$\sum_{k=0}^{m-1} \zeta_m^{k^2} = \frac{1 + \sqrt{-1}}{2} (1 + (-\sqrt{-1})^m) \sqrt{m} = \begin{cases} 
(1 + \sqrt{-1}) \sqrt{m} & (m \equiv 0 \mod 4) \\
\sqrt{m} & (m \equiv 1 \mod 4) \\
0 & (m \equiv 2 \mod 4) \\
\sqrt{-1} \sqrt{m} & (m \equiv 3 \mod 4) 
\end{cases},$$

$$\zeta_4 = \sqrt{-1}, \quad \zeta_8 + \zeta_8^{-1} = \sqrt{2},$$

for any $\alpha \in \mathbb{Q}$ there exists a positive integer $m$ such that $\sqrt[n]{\alpha} \in \mathbb{Q}(\zeta_m)$.

4 Proof of Theorem $4$

Since the proof of $[2]$ is similar to $[1]$, we only prove $[1]$. The cases of $n = 1$ and $n = 2$ are Theorem $[1]$ and Corollary $[2]$ respectively. For $n \geq 3$, the following three cases are possible:

1) $\cos (\pi \theta) \in \mathbb{Q}$ and $\cos (\pi \theta)^2 \in \mathbb{Q}$,
2) $\cos (\pi \theta) \notin \mathbb{Q}$ and $\cos (\pi \theta)^2 \in \mathbb{Q}$,
3) $\cos (\pi \theta) \notin \mathbb{Q}$ and $\cos (\pi \theta)^2 \notin \mathbb{Q}$.

When 1), from Theorem $[1]$ and Corollary $[2]$, the possible values of $\cos (\pi \theta)$ (or $\sin (\pi \theta)$) are $0, \pm \frac{1}{2}, \pm 1$. Similarly, for 2) the possible values of $\cos (\pi \theta)$ are $\pm \frac{1}{\sqrt{2}}, \pm \frac{\sqrt{3}}{2}$. Finally, the case of 3) is impossible from Theorem $[3]$ (1). Then we obtain the conclusion $[1]$.

Acknowledgement

We would like to thank Professor Takashi Taniguchi (Kobe University) for his comments on cyclotomic and Kummer extensions.

References

[1] B. C. Berndt, R. J. Evans and K. S. Williams: Gauss and Jacobi sums, (1998), John Wiley.

[2] L. Carlitz and J. M. Thomas: Rational tabulated values of trigonometric functions, Amer. Math. Monthly 69 (1962), 789–793.

[3] S. Lang: Algebra Revised Third Edition, GTM 211, 2002.

[4] J. M. H. Olmsted, Rational values of trigonometric functions, Amer. Math. Monthly 52-9 (1945) 507–508.
Department of Mathematics, Graduate School of Science, Kobe University, 1-1, Rokkodai, Nada-ku, Kobe, 657-8501, JAPAN
E-mail: g-shibukawa@math.kobe-u.ac.jp