Matrix Partitions of Split Graphs

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Abstract

Matrix partition problems generalize a number of natural graph partition problems, and have been studied for several standard graph classes. We prove that each matrix partition problem has only finitely many minimal obstructions for split graphs. Previously such a result was only known for the class of cographs. (In particular, there are matrix partition problems which have infinitely many minimal chordal obstructions.) We provide (close) upper and lower bounds on the maximum size of a minimal split obstruction. This shows for the first time that some matrices have exponential-sized minimal obstructions of any kind (not necessarily split graphs). We also discuss matrix partitions for bipartite and co-bipartite graphs.

1 Introduction

The approach to graph partition problems, proposed in [9, 2, 5], and used in this paper, is informed by the following distinction between different partition problems.

There are graph partition problems which may be solved in polynomial time and for which the set of minimal non-partitionable graphs is finite. The split graphs recognition problem is a well-known example [8]. On the other hand there are partition problems, such as the bipartition problem, which may be solved in polynomial time [10], but for which the set of minimal non-partitionable graphs is infinite (in the case of the bipartition problem, these are the odd cycles). Finally, there are numerous $NP$-complete graph partition problems, such as the $3$-colouring problem.

When discussing classes of partition problems, we will use patterns to describe the requirements of a partition. In particular, the patterns we examine specify partition problems in which the input graph’s vertices are to be partitioned into

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1 INTRODUCTION

independent sets, or cliques, or some combination of independent sets and cliques. Further, we might require that two parts of vertices in the partition be completely adjacent, or completely non-adjacent. Formally, we use matrices to describe these patterns.

Let $M$ be a symmetric $m \times m$ matrix over $0, 1, \ast$. An $M$-partition of a graph $G$ is a partition of the vertices of $G$ into parts $P_1, P_2, \ldots, P_m$ such that two distinct vertices in parts $P_i$ and $P_j$ (possibly with $i = j$) are adjacent if $M(i, j) = 1$, and nonadjacent if $M(i, j) = 0$. The entry $M(i, j) = \ast$ signifies no restriction.

Note that when $i = j$ these restrictions mean that part $P_i$ is either a clique, or an independent set, or is unrestricted, when $M(i, i)$ is 1, or 0, or $\ast$, respectively. Further, some of the parts may be empty. We may therefore assume that non of the diagonal entries of $M$ are asterisks or else the problem is trivial. For a fixed matrix $M$, the $M$-partition problem asks whether or not an input graph $G$ admits an $M$-partition.

If a graph $G$ fails to admit an $M$-partition, we say that $G$ is an $M$-obstruction. Further, if $G$ is an $M$-obstruction but deleting any vertex of $G$ results in an $M$-partitionable graph, then $G$ is a minimal $M$-obstruction.

Given a graph $G$ and lists $L(v) \subseteq \{1, \ldots, m\}$, with $v \in V(G)$, the list $M$-partition problem asks whether $G$ admits an $M$-partition respecting the lists. That is, an $M$-partition of $G$ such that, for every $v \in V(G)$, the vertex $v$ is placed in a part $P_i$ only if $i \in P_i$. Note that diagonal asterisks do not make the problem trivial when lists are involved. In this paper, we will focus on the non-list version, and will explicitly refer to the list version when it is discussed.

For any matrix $M$ in this paper, we assume that there are $k$ zero entries and $\ell$ one entries on $M$’s diagonal. By row and column permutations, we may further assume that $M(0, 0) = M(1, 1) = \ldots = M(k, k) = 0$ and $M(k + 1, k + 1) = \ldots = M(k + \ell, k + \ell) = 1$. Let $A$ be the submatrix of $M$ on rows $1, \ldots, k$ and columns $1, \ldots, k$; let $B$ be the submatrix of $M$ on rows $k + 1, \ldots, m$ and columns $k + 1, \ldots, k$; and let $C$ be the submatrix of $M$ on rows $1, \ldots, k$ and columns $k + 1, \ldots, m$. When $M$ has no diagonal asterisks, $k + \ell = m$, and we say that $M$ is in $(A, B, C)$-block form.

Feder et al. have shown that if there are asterisks in block $A$ or block $B$ of a matrix $M$, then there are infinitely many minimal $M$-obstructions [7]. Thus, when discussing general graphs, we must restrict our attention to matrices in which the only asterisk entries (if any) are in the block $C$. Such matrices are called friendly. Of these, for any $m \times m$ matrix $M$ containing no asterisk entries at all (i.e. having only entries in $\{0, 1\}$), it has been shown that the largest minimal $M$-obstruction is of size $(k + 1)(\ell + 1)$ [3].

Even when restricted to chordal graphs, there are matrices for which there are
infinitely many chordal minimal obstructions [1, 6]. One of these matrices and an infinite family of chordal minimal obstructions to this matrix, appear frequently in relation to other classes of graphs in this paper, and so are listed in Figure 1.1. The obstruction family in this figure is in fact a family of interval graphs, so that the matrix has infinitely many interval minimal obstructions. Nonetheless, for any matrix $M$, the $M$-partition problem restricted to interval graphs can be solved in polynomial time [11]. Note that the family in Figure 1.1 is not a family of split graphs, as each member contains $2K_2$ as an induced subgraph.

For general matrices $M$, all known upper bounds on the size of minimal obstructions to $M$-partition are exponential [3, 4, 9]; however, in none of these cases has it been shown that exponential-sized minimal obstructions to $M$-partition actually exist.

This paper is organized as follows: In Section 2, we show that for any $m \times m$ matrix $M$, a split minimal $M$-obstruction has at most $O(m^2 \cdot 2^m)$ vertices. This implies that any $M$-partition problem (without lists) is solvable in polynomial time when the input is restricted to split-graphs.

Section 3 exhibits, for a particular class of $m \times m$ matrices, a split minimal obstruction of size $\Omega(2^m)$, demonstrating that the exponential upper bound derived in Section 2 is nearly tight. As noted above, this means that the class of split graph obstructions is the first class with finite minimal obstructions known to contain exponentially large obstructions.

In section 4, we discuss graphs that admit other types of partitions, such as bipartite graphs and co-bipartite graphs. It is shown that for these classes also there are only finitely many minimal obstructions for any matrix $M$. These graph classes (including the class of split graphs) have a natural common generalization, namely graphs whose vertex set may be partitioned into $k$ independent sets and $\ell$ cliques, sometimes called $(k, \ell)$-graphs. Split graphs are $(1, 1)$-graphs, bipartite graphs are $(2, 0)$-graphs, and co-bipartite graphs are $(0, 2)$-graphs. By contrast we show that when $k + \ell \geq 3$, there is a matrix $M$ with infinitely many minimal $(k, \ell)$-graph obstructions. When $k \geq 2$, there are infinitely many minimal $(k, \ell)$-graph obstructions that are chordal.
2 Matrix Partitions of Split Graphs

In this section we prove the following theorem.

**Theorem 2.1.** If $M$ is a matrix with no diagonal asterisks, and $k \geq \ell$, then there are finitely many split minimal $M$-obstructions.

A set of vertices $H \subseteq V(G)$ is said to be homogeneous in $G$ if the vertices of $V(G) - H$ can be partitioned into two sets, $S_1$ and $S_2$ such that every vertex of $S_1$ is adjacent to every vertex of $H$, and no vertex of $S_2$ is adjacent to a vertex of $H$. The proof of Theorem 2.1 relies on the existence of large homogenous sets in $M$-partitionable split graphs.

**Proposition 2.2.** Let $A$ be a $k \times k$ matrix whose diagonal entries are all zero. Let $G_A$ be a split graph that admits an $A$-partition. Then every part $P$ of an $A$-partition of $G_A$ contains a homogeneous set in $G_A$ of size at least $\frac{|P| - 1}{2^k - 1}$.

**Proof.** Suppose the parts of the $A$-partition of $G_A$ are $P_1, \ldots, P_k$. Let $C \cup I$ be a partition of $V(G_A)$ into a clique $C$ and independent set $I$. Note that for $1 \leq i \leq k$, we have that $|P_i \cap C| \leq 1$, since each $P_i$ is an independent set. Now, the vertices in the set $P_1 \cap I$ are non-adjacent to all but at most $k - 1$ vertices, one in each $P_i \cap C$, for $2 \leq i \leq k$ (see Figure 2.1). Assume without loss of generality that $|P_1 \cap C| = 1$ and let $u_i \in P_i \cap C$, for $2 \leq i \leq k$. As each $u_i$ is either adjacent to at least half of the vertices of $P_1 \cap I$, or non-adjacent to at least half of the vertices of $P_1 \cap I$, a homogeneous set of size at least $\frac{|P_1| - 1}{2^k - 1}$ can be found in $|P_1|$. Since this argument may be repeated for any other part in the partition, we have the desired conclusion.

![Figure 2.1](image.png)

**Proposition 2.3.** Let $B$ be an $\ell \times \ell$ matrix whose diagonal entries are all 1. Let $G_B$ be a split graph that admits a $B$-partition. Then every part $P$ of a $B$-partition of $G_B$ contains a homogeneous set in $G_B$ of size at least $\frac{|P| - 1}{2^{\ell - 1}}$.

**Proof.** The result follows from Proposition 2.2, since $G_B$ admits a $B$-partition if and only if $\overline{G_B}$ admits a $\overline{B}$-partition, and the complement of a split graph is a split graph.
We also require the following observation.

**Fact 2.4.** Let $M$ be an $(A, B, C)$-block matrix and let $G$ be a split graph. If $C$ has an asterisk entry, then $G$ admits an $M$-partition.

**Proof.** If $C$ has an asterisk, then $M$ contains the matrix $([0, 1])$ as a principal submatrix. Thus $G$ admits this partition by definition of split graphs, since every other part may be empty. 

**Proof of Theorem 2.1.** Let $M$ be an $m \times m$ matrix, with $k$ diagonal 0s and $\ell$ diagonal 1s. Assume without loss of generality that $k \geq \ell$. We show that the number of vertices in a split minimal $M$-obstruction is at most 

$$2^{k-1}(k + \ell)(2k + 3) + 1 \in O(k^2 \cdot 2^k)$$

Suppose for contradiction that $G$ is a minimal $M$ obstruction with at least $2^{k-1}(k + \ell)(2k + 3) + 2$ vertices. By Fact 2.4, we may assume that the submatrix $C$ has no asterisks. Pick an arbitrary vertex $v$ and consider a partition of the graph $G - v$ on at least $2^{k-1}(k + \ell)(2k + 3) + 1$ vertices. As there are $k + \ell$ parts in the partition, by the pigeonhole principle there is a part, call it $P$, of size at least $2^{k-1}(2k + 3) + 1$. This part $P$ is either an independent set or a clique, and each of these cases will be considered separately below. Either way, by Propositions 2.2 and 2.3, $P$ contains a homogeneous set in $A$ or $B$ (depending on whether $P$ is an independent set or a clique) of size at least $\frac{|P| - 1}{2^{k-1}} \geq 2k + 3$. Since $C$ has no asterisks, this set is homogeneous in $G$. Thus $G - v$ has a homogeneous set of size at least $2k + 3$, and so $G$ has a homogeneous set $H$ of size at least $k + 2$, since by the pigeonhole principle at least $k + 2$ of the vertices of $P$ agree on $v$. Now let $w \in H$, consider a partition of $G - w$, and recall that $P$ is either an independent set or a clique.

**Case 1.** If $P$ is an independent set, then so is $H$; hence, there are at least $k + 1$ independent vertices in $G - w$. As there are $\ell \leq k$ clique parts in the partition of $G - w$, and no two independent vertices of $H$ may be placed in the same clique part, at least one vertex $w' \in H - \{w\}$ must be placed in an independent part $P'$. Since $w$ is not adjacent to $w'$ and both vertices belong to $H$, $w$ can be added to $P'$, contradicting the minimality of $G$.

**Case 2.** If $P$ is a clique then $H - w$ is a clique of size at least $k + 1$, and so in the partition of $G - w$, at least one vertex of $H - w$ falls in a clique part $P'$. As in Case 1, $w$ can be added to $P'$, contradicting minimality.

Since every matrix $M$ has finitely many split minimal obstructions, there is an obvious polynomial time algorithm for the $M$-partition problem. However, a
more efficient algorithm is described in what follows.

A matrix $M$ is crossed if each non-asterisk entry in its block $C$ belongs to a row or column in $C$ of non-asterisk entries. It has been shown that if $M$ is a crossed matrix, then the list $M$-partition problem for chordal graphs can be solved in polynomial time [1]. Since split graphs are chordal, the same result applies for split graphs, and we can use this to solve the $M$-partition problem for split graphs in polynomial time, bearing in mind that by Fact 2.4, we may assume that the block $C$ has no asterisks and so $M$ is crossed.

**Theorem 2.5.** If $G$ is a split graph and $M$ is any matrix, then the $M$-partition problem for $G$ can be solved in time $O(n^k)$. 

When dealing with the $M$-partition problem with lists, it is shown in [1] that there is a matrix $M$ for which the list $M$-partition problem is $NP$-complete, even when restricted to chordal graphs. In fact, the graphs constructed in that reduction are split graphs so that this list $M$-partition problem remains $NP$-complete even for split graphs.

### 3 A Special Class of Matrices

As seen in Section 2, for any $m \times m$ matrix $M$, there is an exponential upper bound on the size of a largest split minimal $M$-obstruction. In this section we show a family of matrices for which this bound is nearly tight.

For $k, t \in \mathbb{N}$, with $1 \leq t \leq k - 1$, let $M_{k,t}$ be a $k \times k$ matrix with diagonal entries all zero, $t$ ones in row $k$, symmetrically, $t$ ones in column $k$ and asterisks everywhere else. By permuting the rows and columns of $M_{k,t}$ we assume without loss of generality that the one entries of row $k$ are in columns $k - t, ..., k - 1$ and symmetrically, that the one entries of column $k$ are in rows $k - t, ..., k - 1$. See Figure 3.1 for some examples.

**Theorem 3.1.** There exist $k, t \in \mathbb{N}$ such that for the matrix $M_{k,t}$, the size of the largest split minimal $M$-obstruction is $\Omega(2^{k-1})$.

**Proof.** We choose values of $k$ and $t$ so that the matrix $M_{k,t}$ has a split minimal obstruction of size at least

$$\left(\pi \cdot \frac{k - 1}{2}\right)^{-\frac{1}{2}} \cdot 2^{k-1} + 2k - 1$$

Choose $k = 2n + 1$ and $t = n$ for some $n \in \mathbb{N}$, so that the matrix $M_{k,t}$ has $2n + 1$ parts. Place ones in row $2n + 1$ and columns $n, n + 1, ..., 2n$ as well as in columns $2n + 1$ and rows $n, n + 1, ..., 2n$. Let $P$ denote the part in row and column $2n + 1$. 


and designate the $n$ parts that have a one to $P$ as restricted parts, $R_1, \ldots, R_n$ and the remaining $n$ parts as unrestricted parts, $U_1, \ldots, U_n$. See Figure 3.2.

The minimal obstruction $G$, depicted in Figure 3.2, has a special vertex $a$, and $2n$ vertices forming a clique $B$, that are all adjacent to $a$ (so that $B \cup \{a\}$ is a clique of size $2n + 1$). Further, $G$ has another $2n$ vertices forming an independent set $B'$ such that for each $b \in B$ there is a $b' \in B'$ that is not adjacent to $b$ but is adjacent to every other vertex of $B \cup \{a\}$. Call $b$ and $b'$ mates. Finally, $G$ has an independent set $S$ of size $\binom{2n}{n}$ such that for every subset $\tilde{B}$ of $B$ of size $n$, there is exactly one vertex $s \in S$ adjacent to exactly the vertices of $\tilde{B}$. Note that $G$ is a split graph since $B \cup \{a\}$ is a clique and $B' \cup S$ is an independent set, as seen in Figure 3.3.
To see that $G$ is indeed an obstruction, suppose otherwise, and note that $B \cup \{a\}$ is a clique of size $2n + 1$, so each of its vertices must be placed in a different part. Since each vertex of $B$ has a mate in $B'$ that is adjacent to $a$ and all of the other vertices in $B$, all parts other than the part containing $a$ have size at least two in any $M_{k,t}$-partition of $G$. Thus only the part containing $a$ may be a singleton. Further $P$ must be the only singleton part, otherwise all of the restricted parts must be singletons, since $G$ contains no induced $C_4$. Therefore $a \in P$. Now whichever $n$ vertices of $B$ are placed in the unrestricted parts, as in Figure 3.4, there is a vertex $s \in S$ adjacent to exactly these vertices, and so must be placed into one of the restricted parts. But as $s$ is not adjacent to $a$, it cannot be placed in a restricted part, and $s$ can’t be added to $P$; hence, $G$ is not $M_{k,t}$-partitionable.

Figure 3.3 – A split partition for $G$.

Figure 3.4 – An attempt to partition $G$. 

\[ \text{Figure 3.3 – A split partition for } G. \]

\[ \text{Figure 3.4 – An attempt to partition } G. \]
To argue that $G$ is a minimal obstruction, we show that removing a vertex from one of $S, B, B', \text{ or } \{a\}$ allows a partition for the resulting graph:

(i) For $s \in S$ partition $G - s$ as follows: map $a$ to $P$, place each $b \in B$, together with its mate $b' \in B'$, in some part, taking care that neighbours of the missing $s$ are placed in unrestricted parts. Now each remaining vertex of $S$ has an unrestricted part to go to.

(ii) We consider $b \in B$ together with its mate $b' \in B'$. For $G - b$, place $a$ in $P$, place $b$’s mate $b'$ in an unrestricted part $P_{b'}$, and place all of $S$ and all of $B'$ in $P_{b'}$. This is possible since $B' \cup S$ is an independent set. place the remaining $2n - 1$ vertices of $B$ in the remaining $2n - 1$ parts arbitrarily. To partition $G - b'$, place $b$ in $P$, map $a$ together with all of the vertices of $S$ in an unrestricted part $P_a$, and place each other pair of mates $v, v'$ from $B$ and $B'$ into a part, different from $P$ and $P_a$.

(iii) Finally, $G - a$ can be partitioned using the restricted and unrestricted parts only, not placing anything in $P$. Place each $b$ and its mate $b'$ into a part. Each $s \in S$ is only forbidden from $n$ out of the $2n$ parts and so can be placed somewhere.

Now $G$ has $2k - 1 + \binom{k-1}{2} = 4n + 1 + \binom{2n}{n}$ vertices, and using Stirling’s approximation, we get

$$
\frac{2^{k-1}}{\sqrt{\pi \frac{k-1}{2}}} \leq \frac{2^{2n}}{\sqrt{\pi n}} \leq \frac{2^{2n}}{\sqrt{\pi n}} \left(1 - \frac{c}{n}\right) = \frac{2^{k-1}}{\sqrt{\pi \frac{k-1}{2}}} \left(1 - \frac{2c}{k-1}\right), \text{ where } \frac{1}{9} < c < \frac{1}{8}
$$

Therefore $G$ is of size exponential in $k$. \qed

4 Generalized Split Graphs

Recall that split graphs can be viewed as a special case of $(k, \ell)$-graphs - those graphs whose vertices can be partitioned into $k$ independent sets and $\ell$ cliques. (Thus split graphs are the $(1, 1)$-graphs.)

In this section, we focus on $(k, \ell)$-graphs other than the $(1, 1)$-graphs. We begin with $(2, 0)$- and $(0, 2)$-graphs, and then discuss other $(k, \ell)$-graphs. Recall that the $(2, 0)$-graphs are the bipartite graphs, while the $(0, 2)$-graphs are the co-bipartite graphs. As it turns out, there are finitely many bipartite or co-bipartite minimal obstructions, for any matrix $M$.

**Theorem 4.1.** For any $m \times m$ matrix $M$, there are finitely many bipartite minimal obstructions and finitely many co-bipartite minimal obstructions.
To prove Theorem 4.1 we use an approach similar in nature to that used Section 2. Starting with bipartite graphs, note that we may assume that the matrix \((0 \ 0)\) is not a principal submatrix of the matrix \(M\), or else the problem would be trivial.

**Proposition 4.2.** Let \(M\) be an \((A, B, C)\)-block matrix, with \(A\) of size \(k \times k\) and \(B\) of size \(\ell \times \ell\). Suppose the block \(A\) has no asterisk entries. If \(G\) is an \(M\)-partitionable bipartite graph, then any part \(P\) of \(A\) in an \(M\)-partition of \(G\) contains a homogeneous set of size at least \(|P|/2\).

**Proof.** Fix a bipartition of \(G\) and let \(P\) be a part of \(A\) in an \(M\)-partition of \(G\). We argue that \(P\) has the desired size. As \(A\) has no asterisks, the vertices of \(P\) all have the same adjacency relation to vertices in other parts of \(A\). Now let \(P'\) be some part of \(B\). Since \(G\) is bipartite, \(P'\) can have at most two vertices, one from each part of the bipartition of \(G\). Let these vertices be \(x\) and \(y\). By the pigeonhole principle, \(x\) is either adjacent to, or non-adjacent to, at least half of the vertices of \(P\). Suppose without loss of generality, that \(x\) is adjacent to at least half of the vertices of \(P\). Call these vertices \(P_x\). Applying the pigeonhole principle again, this time to the vertex \(y\), we have that \(y\) is either adjacent to, or non-adjacent to, at least half of the vertices of \(P_x\). Let the larger of these two sets be \(P_{xy}\), and note that \(P_{xy} \geq |P|/2\). Now there are \(\ell - 1\) clique parts other than \(P'\), each of size at most two. Inductively, we obtain a homogeneous set in \(P\) of size at least \(|P|/2\).

Theorem 4.1 now follows for bipartite graphs. The proof for co-bipartite graphs follows by complementation.

**Proof of Theorem 4.1.** As discussed above, we assume that \(A\) contains no asterisk entries. We show that any bipartite minimal obstruction is of size at most

\[2^{2\ell}(k + \ell)(2\ell + 3)\]

Suppose otherwise, and let \(G\) be a minimal obstruction with at least \(2^{2\ell}(k + \ell)(2\ell + 3) + 1\) vertices. For an arbitrary vertex \(v\), the graph \(G-v\) is \(M\)-partitionable, and so some part \(P\) in an \(M\)-partition of \(G-v\) contains at least \(2^{2\ell}(2\ell + 3)\) vertices. Since \(2^{2\ell}(2\ell + 3) \geq 3\) for \(\ell \geq 0\), and no clique part of \(M\) may contain more than two vertices, \(P\) must be an independent set. Thus by Proposition 4.2, \(P\) contains a homogeneous set of size at least \(|P|/2 \geq 2\ell + 3\). By the pigeonhole principle, \(G\) has an homogeneous set \(H\) of size at least \(\ell + 2\). Note that \(H\) is an independent set. Let \(h \in H\), and consider a partition of \(G-h\). As there are only \(\ell\) cliques and \(\ell + 1\) vertices in \(H-h\), there must be a part \(P'\) of \(A\) that contains a vertex \(h'\) of \(H-h\). But since \(H\) is an independent set, and \(h\) has the same neighbourhood as \(h'\), we may add \(h\) to \(P'\) to obtain a partition of \(G\), a contradiction.
We now consider \((k, \ell)-\text{graphs for values of } k \text{ and } \ell \text{ that satisfy } k + \ell \geq 3\). For convenience, let \((k, \ell)\) denote the set of \((k, \ell)-\text{graphs. The family of graphs depicted in Figure 1.1 is an infinite family of chordal minimal obstructions to the matrix } M_{3,1}[6].\) We define the family more precisely as follows.

For \(t \geq 3\), let \(G(t)\) be the graph consisting of an even path on \(2t\) vertices, and an additional vertex \(u\). \(u\) is adjacent to all vertices of the path, except the endpoints. Note that each \(G(t)\) is chordal.

**Theorem 4.3.** If \(k, \ell \in \mathbb{N}\) such that \(k + \ell \geq 3\), then there exists a matrix \(M\) that has infinitely many \((k, \ell)\)-minimal obstructions.

**Proof.** Note that for any \(t \geq 3\), \(G(t)\) is 3-colourable, and \(G(t)\) is partitionable into a bipartite graph and a clique. That is, \(G(t) \in (3,0) \cap (2,1)\). Therefore, for the matrix \(M_{3,1}\), there are infinitely many (chordal) minimal \((2,1) \cap (3,0)\) obstructions. By complementation, for any \(t \geq 3\), the graph \(\overline{G(t)}\) is in \((1,2) \cap (0,3)\), providing infinitely many (chordal) \((1,2) \cap (0,3)\) obstructions for the matrix \(\overline{M_{3,1}}\).

Now if \(k \leq 1\), then since \(k + \ell \geq 3\), it must be that \(\ell \geq 2\), and so the family \(\{G(t)|t \geq 3\}\) is a family of \((k, \ell)\)-minimal obstructions for \(\overline{M_{3,1}}\). On the other hand, if \(k \geq 2\), then the family \(\{G(t)|t \geq 3\}\) is a family of \((k, \ell)\)-minimal (chordal) obstructions for the matrix \(M_{3,1}\).

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\begin{proof}
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