Abstract

For the Weber problem of construction of the minimal cost planar weighted network connecting four terminals with two extra facilities, the solution by radicals is proposed. The conditions for existence of the network in the assumed topology and the explicit formulae for coordinates of the facilities are presented. The obtained results are utilized for investigation of the network dynamics under variation of parameters. Extension of the results to the general Weber problem is also discussed.

Keywords: Multifacility location problem, Weber problem

1. Introduction

The classical Weber or generalized Fermat-Torricelli problem is stated as that of finding the point (facility, junction) \( W = (x_*, y_*) \) that minimizes the sum of weighted distances from itself to \( n \geq 3 \) fixed points (terminals) \( \{P_j = (x_j, y_j)\}_{j=1}^n \) in the Euclidean plane:

\[
\min_{W \in \mathbb{R}^2} \sum_{j=1}^n m_j |WP_j|.
\] (1.1)

Hereinafter \( |\cdot| \) stands for the Euclidean distance and the weights \( \{m_j\}_{j=1}^n \) are assumed to be positive real numbers.

The treatment of the problem in the case \( n = 3 \) terminals was first undertaken in 1872 by Launhardt [7] whose interest stemmed from the evident relation to the Economic Geography problem nowadays known as Optimal Facility Location. For instance, one can be interested in minimizing transportation costs for a plant manufacturing one ton of the final product from \( \{m_j\}_{j=1}^n \) tons of distinct raw materials located at corresponding \( \{P_j\}_{j=1}^n \).

Further development of the problem was carried out in 1909 by Alfred Weber. First, he suggested a different economic interpretation for the three-terminal problem. Let \( P_3 \) be a place of consumption of \( m_3 \) tons of a product produced from two different types of raw materials: \( m_1 \) tons of the first type located at \( P_1 \) and \( m_2 \) tons of the second type located at \( P_2 \), let \( m_3 < m_1 + m_2 \). Where is the...
optimal location of the production? In the course of the economic background, Weber formulated the following extension of the problem to the case of 4 terminals\(^1\):

“Let us take a simple case, an enterprise with three material deposits and one which is capable of being split, technologically speaking, into two stages. In the first stage two materials are combined into a half-finished product; in the second stage this half-finished product is combined with the third material into the final product... Let us suppose that possible location of the split production would be in \(W_1\) and \(W_2\); \(W_1\) for the first stage and \(W_2\) for the second stage. What will be the result if the splitting occurs?”\(^{[14]}\)

Mathematically the stated problem can be formulated as that of finding the points \(W_1 = (x_1, y_1)\) and \(W_2 = (x_2, y_2)\) which yield

\[
\min \{F(W_1, W_2)\} \quad \text{where} \quad F(W_1, W_2) = m_1|W_1P_1| + m_2|W_1P_2| + m_3|W_2P_3| + m_4|W_2P_4| + m|W_1W_2| \quad (1.2)
\]

and the weights \(\{m_j\}_{j=1}^4, m\) are treated as given positive real numbers.

The general Multifacility Weber problem is stated as that of location of the given number \(\ell \geq 2\) of the facility points (or, simply, facilities) \(\{W_i\}_{i=1}^\ell\) in \(\mathbb{R}^d\) connected to the terminals \(\{P_j\}_{j=1}^n \subset \mathbb{R}^d\) that solve the optimization problem

\[
\min \{\sum_{j=1}^n m_{ij}|W_iP_j| + \sum_{k=1}^{\ell-1} \sum_{i=k+1}^\ell \tilde{m}_{ik}|W_iW_k|\} ; \quad (1.3)
\]

here some of the weights \(m_{ij}\) and \(\tilde{m}_{ik}\) might be zero. We will refer to this value as to the minimal cost of the network. This problem can be considered as a natural generalization of the celebrated Steiner minimal tree problem aimed at construction of the network of minimal length linking the given terminals. Dozens of papers are devoted to the Weber problem, its ramifications and applications; we refer to [5, 8, 15] for the reviews. The majority of them are concerned with the problem statement where the objective function (1.3) is free of the inter-facilities connections, i.e. all the weights \(\tilde{m}_{ik}\) are zero. This problem is known as the Multisource Weber problem or the \(p\)-median problem\(^2\).

The present paper is focused on solution to the Multifacility Weber problem. The mainstream approach in the treatment of this nonlinear optimization problem is the one based on reducing it to an appropriate iterative numerical procedure. For instance, the unifacility version of the problem (1.1) can be resolved via the modified Weiszfeld algorithm. The main obstacle of this approach consists in the fact that the objective (or cost) function of the Weber problem is non-differentiable at terminal points, and the iterative procedure might diverge if any of the facilities happens to lie close to a terminal (or, in case of the multifacility problem, if two facilities are about to collide).

\(^1\)In the following citation we change the original notation of the points.
\(^2\)With \(p\) standing for the number of facilities.
The present paper is devoted to an alternative approach for the problem, namely an analytical one. We are looking for the conditions for existence of the network and the explicit expressions for the facility coordinates in terms of the problem parameters, i.e. terminal coordinates and weights. This approach has been originated in the recent papers [11] and [12] where the unifacility Weber problem for three terminals and the (full) Steiner minimal tree problem for four terminals had been solved by radicals. Within the framework of this approach, we will focus here on solution to the planar multifacility Weber problem for the case of $n = 4$ terminals and $\ell = 2$ facilities (i.e. the problem (1.2)), and also for the case of $n = 5$ terminals and $\ell = 3$ facilities.

Our analytical treatment stems from geometric solution to the problem originated by Georg Pick and published in the Mathematical Appendix of Weber’s book [14]. Pick’s solution, which we trace in Section 3, can be interpreted as a counterpart of the algorithm worked out by Gergonne in 1810 (and rediscovered by Melzak in 1961) for solution of the Steiner minimal tree problem for four terminals. Nevertheless, Pick did not provide any proof of validity for his algorithm. We also failed to find any references to Pick’s solution in subsequent papers on the subject. In the conference paper [13] the present authors have announced without a proof the claim that the Weber problem (1.2) is solvable by radicals. In a simplified version (and with an extra assumption missed in [13]), this statement is now proved in Section 4. The deduced formulae for the facilities coordinates approve analytically Pick’s considerations. In addition, the conditions for the existence of the desired configuration of the network are provided.

In the case of the problem involving variable parameters, analytics provides one with a unique opportunity to evaluate their influence on the solution. In particular, it gives the means to determine the bifurcation values for these parameters, i.e. those responsible for the degeneracy of the network topology. We discuss these issues in Section 5 via investigation of the facilities dynamics under variation of the terminals location or the value of the involved weights. We also prove here that, in the case of existence, the optimal bifacility network has its cost lower than the unifacility one.

In Section 6, we briefly discuss an opportunity for extension of the results to the case of $n \geq 5$ terminals and $\ell \geq 3$ facilities. This extension is based on the reduction of the problem to a similar one with $n - 1$ terminals and $\ell - 1$ facilities via a replacement of a pair of terminals by a suitable auxiliary phantom terminal. This trick is just a counterpart of the one utilized in Melzak’s algorithm for Steiner tree construction.

2. Unifacility case

2.1. Three terminals

We first outline via an example the geometric approach to the problem given in the paper by Launhardt [7].
Example 2.1. Find the optimal position of the facility $W$ to the problem (1.1) where

\[
\begin{align*}
P_1 &= (1, 5) & P_2 &= (2, 1) & P_3 &= (7, 2) \\
m_1 &= 3 & m_2 &= 2 & m_3 &= 3
\end{align*}
\]

Solution. First find the point $Q_1$ lying on the opposite side of the line $P_1P_2$ with respect to the point $P_3$ and such that

\[
|P_1Q_1| = \frac{m_2}{m_3}|P_1P_2|, \quad |P_2Q_1| = \frac{m_1}{m_3}|P_1P_2|.
\]

This condition means that the triangle $P_1P_2Q_1$ is similar to the so-called weight triangle of the problem, i.e. the triangle composed of the sides formally coinciding with the values of the weights $m_1, m_2, m_3$. We will further denote this triangle by \{m_1, m_2, m_3\} (Fig. 1 (a)).

Next, draw the circle $C_1$ circumscribing $P_1P_2Q_1$. Finally draw the line through $Q_1$ and $P_3$. The intersection point of this line with $C_1$ is the position of the optimal facility $W$. The corresponding (minimal) cost is equal to $m_1|P_1Q_1|$.

The feasibility of the suggested algorithm evidently depends on the condition for the existence of the weight triangle, i.e. $m_3 < m_1 + m_2, m_1 < m_2 + m_3, m_2 < m_1 + m_3$. However, even under this assumption, the locus of the point $W$ might lie outside of the triangle $P_1P_2P_3$, and, in this case, the obtained solution contradicts the common sense.

This geometrical solution can be developed to an analytical one [11].

Theorem 2.1. Denote by $\alpha_1, \alpha_2, \alpha_3$ the angles of the triangle $P_1P_2P_3$, while by $\beta_1, \beta_2, \beta_3$ the angles of the weight triangle (according with a rule similar to that displayed in Fig. 1 (a)). The necessary and sufficient condition for the existence of solution to the problem

\[
\min_{W\in\mathbb{R}^2} (m_1|WP_1| + m_2|WP_2| + m_3|WP_3|)
\]

(2.1)
is that of the following system of inequalities

\[ \{ \cos \alpha_j + \cos \beta_j > 0 \}^3_{j=1}. \]

Under this condition, the coordinates of the optimal facility \( W = (x_*, y_*) \) are given by the formulae

\[ x_* = \frac{K_1K_2K_3}{2|\mathcal{S}|\sqrt{k}d} \left( \frac{x_1}{K_1} + \frac{x_2}{K_2} + \frac{x_3}{K_3} \right), \quad y_* = \frac{K_1K_2K_3}{2|\mathcal{S}|\sqrt{k}d} \left( \frac{y_1}{K_1} + \frac{y_2}{K_2} + \frac{y_3}{K_3} \right) \]

with the cost of the optimal network

\[ \mathcal{E} = \sqrt{d}. \]

Here

\[ d = \frac{1}{\sqrt{k}} (m_1^2K_1 + m_2^2K_2 + m_3^2K_3) \]

\[ = |\mathcal{S}|\sqrt{k} + \frac{1}{2} \left[ m_1^2(r_{12}^2 + r_{13}^2 - r_{23}^2) + m_2^2(r_{23}^2 + r_{12}^2 - r_{13}^2) + m_3^2(r_{13}^2 + r_{23}^2 - r_{12}^2) \right], \]

\[ r_{ij} = |P_iP_j| = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \quad \text{for} \quad \{i, j\} \subseteq \{1, 2, 3\}, \]

\[ \mathcal{S} = x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_3y_2 - x_2y_1, \]

\[ k = (m_1 + m_2 + m_3)(-m_1 + m_2 + m_3)(m_1 - m_2 + m_3)(m_1 + m_2 - m_3), \]  \tag{2.2}

and

\[
\begin{cases}
K_1 = (r_{12}^2 + r_{13}^2 - r_{23}^2)\sqrt{k}/2 + (m_2^2 + m_3^2 - m_1^2)|\mathcal{S}|, \\
K_2 = (r_{23}^2 + r_{12}^2 - r_{13}^2)\sqrt{k}/2 + (m_1^2 + m_3^2 - m_2^2)|\mathcal{S}|, \\
K_3 = (r_{13}^2 + r_{23}^2 - r_{12}^2)\sqrt{k}/2 + (m_1^2 + m_2^2 - m_3^2)|\mathcal{S}|.
\end{cases}
\]

The proof consists in formal verification of the equalities

\[ m_1 \frac{x_* - x_1}{|WP_1|} + m_2 \frac{x_* - x_2}{|WP_2|} + m_3 \frac{x_* - x_3}{|WP_3|} = 0, \] \tag{2.3}

\[ m_1 \frac{y_* - y_1}{|WP_1|} + m_2 \frac{y_* - y_2}{|WP_2|} + m_3 \frac{y_* - y_3}{|WP_3|} = 0, \] \tag{2.4}

providing the stationary points of the objective function \( \sum_{j=1}^3 m_j |P_jW| \).

The theorem states that the three-terminal Weber problem is solvable by radicals. Geometric meaning of the constants appeared in this theorem is as follows: \( \frac{1}{2} |\mathcal{S}| \) equals the area of the triangle \( P_1P_2P_3 \) while \( \frac{1}{2} \sqrt{k} \) equals (due to Heron’s formula) the area of the weight triangle.

We now formulate two technical results to be exploited later. They can be proved via formal application of Theorem 2.1.

**Theorem 2.2.** If the facility \( W \) is the solution to the problem (2.1) for some configuration

\[
\{ \begin{array}{|c|c|c|} \hline P_1 & P_2 & P_3 \\ \hline m_1 & m_2 & m_3 \\ \hline \end{array} \}
\]

then this facility remains unchanged for the configuration

\[
\{ \begin{array}{|c|c|c|} \hline P_1 & P_2 & \bar{P}_3 \\ \hline m_1 & m_2 & m_3 \\ \hline \end{array} \}
\]

with any position of the terminal \( \bar{P}_3 \) in the half-line \( WP_3 \).
**Theorem 2.3.** For any position of the terminal $P_3$, the facility $W$ lies in the arc of the circle $C_1$ passing through the points $P_1, P_2$ and

$$Q_1 = \left(\frac{1}{2}(x_1 + x_2) + \frac{m_1^2 - m_3^2}{2m_3^2}(x_1 - x_2) - \sqrt{k}(y_1 - y_2), \right.$$

$$\left.\frac{1}{2}(y_1 + y_2) + \frac{m_1^2 - m_3^2}{2m_3^2}(y_1 - y_2) + \frac{\sqrt{k}(x_1 - x_2)}{2m_3^2}\right).$$

(2.5)

Its center is at

$$\left(\frac{1}{2}(x_1 + x_2) - \frac{m_1^2 + m_2^2 - m_3^2}{2\sqrt{k}}(y_1 - y_2), \frac{1}{2}(y_1 + y_2) + \frac{m_1^2 + m_2^2 - m_3^2}{2\sqrt{k}}(x_1 - x_2)\right)$$

while its radius equals $m_1m_2|P_1P_2|/\sqrt{k}$.

**Theorem 2.4.** Let the terminals $P_1, P_2, P_3$ be counted counterclockwise and the conditions of Theorem 2.1 be satisfied. Set

$$\mathcal{S}_1 = \begin{vmatrix} 1 & 1 & 1 \\ x & x_2 & x_3 \\ y & y_2 & y_3 \end{vmatrix}, \quad \mathcal{S}_2 = \begin{vmatrix} 1 & 1 & 1 \\ x & x_1 & x_3 \\ y & y_1 & y_3 \end{vmatrix}, \quad \mathcal{S}_3 = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x \\ y_1 & y_2 & y \end{vmatrix}.$$  \hspace{1cm} (2.6)

For any value of the weight $m_3$, the optimal facility $W$ lies in the arc of the algebraic curve of the 4th degree given by the equation

$$m_1^2\mathcal{S}_1^2[(x - x_2)^2 + (y - y_2)^2] = m_2^2\mathcal{S}_1^2[(x - x_1)^2 + (y - y_1)^2].$$ \hspace{1cm} (2.7)

**Proof.** If the conditions of Theorem 2.1 are fulfilled then the coordinates of the optimal facility $W = (x_*, y_*)$ satisfy the system (2.3)–(2.4). Treating this system as linear with respect to $m_1, m_2, m_3$, one arrives at the following relation

$$m_1 : m_2 : m_3 = |WP_1|\mathcal{S}_1 : |WP_2|\mathcal{S}_2 : |WP_3|\mathcal{S}_3.$$  

Example 2.2. For the configuration

$$\begin{cases}
P_1 = (1, 5) & P_2 = (2, 1) & P_3 = (7, 2) \\
\quad m_1 = 3 & \quad m_2 = 2 & \quad m_3
\end{cases},$$

find the locus of the facility $W$ under variation of the weight $m_3$.  

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Fig. 2: Example 2.2. Optimal facility location under variation of the weight \( m_3 \)

**Solution.** Equation (2.7) takes the form

\[
7(11x + 8y)(x + 4y)(x^2 + y^2) - (2122x^3 + 4942x^2y + 3950xy^2 + 3092y^3)
\]
\[+17242x^2 + 20480xy + 14713y^2 - 48666x - 33942y + 48069 = 0.
\]

The curve is depicted in Fig. 2, with its only branch giving the position of \( W \) displayed in blue boldface. The values of \( m_3 \) corresponding to this branch lie within the interval

\[
\left[ \frac{1}{85}(\sqrt{48365} - 12\sqrt{85}), \frac{1}{65}\sqrt{54925 + 3510\sqrt{130}} \right] \approx [1.285716, 4.740488]
\]

with its ends corresponding to collision of the optimal facility with \( P_1 \) or \( P_3 \). \( \square \)

2.2. **Four terminals**

**Assumption 1.** Hereinafter we will treat the case where the terminals \( \{P_j\}_{j=1}^{4} \), while counted counterclockwise, compose a convex quadrilateral \( P_1P_2P_3P_4 \).

Stationary points of the function \( \sum_{j=1}^{4} m_j|WP_j| \) are given by the system of equations

\[
\sum_{j=1}^{4} \frac{m_j(x - x_j)}{|WP_j|} = 0, \quad \sum_{j=1}^{4} \frac{m_j(y - y_j)}{|WP_j|} = 0.
\] (2.8)

Though this system is not an algebraic one with respect to \( x, y \), it can be reduced to this form via successive squaring of every equation. This permits one to apply the procedure of elimination of a variable via computation of the **resultant**. Thereby, the problem of finding the coordinates of the facility \( W \) can be reduced to that of resolving a univariate algebraic equation [10]. Unfortunately, for the considered case, this equation is generically of degree 10 (v. Example 5.4 below) and is not solvable by radicals [1].

Nevertheless, for some special configurations, such a solution exists. The following theorem generalizes the Fagnano’s result corresponding to the equal-weighted configurations (\( \{m_j = 1\}_{j=1}^{4} \)).
Theorem 2.5. Let Assumption 1 be fulfilled for the configuration \( \{ P_1, P_2, P_3, P_4 \} \). For any values of the weights \( m_1 \) and \( m_2 \), the position of the facility \( W \) providing the solution to the problem (1.1) is at the point of intersection of the quadrilateral \( P_1P_2P_3P_4 \) diagonals.

**Proof** consists in formal substitution of the coordinates

\[
x_* = \frac{(x_1 - x_3)(x_2y_4 - y_2x_4) - (x_2 - x_4)(y_3x_1 - y_1x_3)}{(x_3 - x_1)(y_2 - y_4) - (x_2 - x_4)(y_3 - y_1)},
\]
\[
y_* = \frac{(y_1 - y_3)(x_2y_4 - y_2x_4) - (y_2 - y_4)(y_3x_1 - y_1x_3)}{(x_3 - x_1)(y_2 - y_4) - (x_2 - x_4)(y_3 - y_1)},
\]

into the equations (2.8). \( \square \)

An analytical approach is also effective for establishing the dynamics of the optimal facility location under variation of parameters. The following result is a counterpart of Theorem 2.4.

Theorem 2.6. Let Assumption 1 be fulfilled. Let \( \{ S_j \}_{j=1}^3 \) be defined by (2.6). Set

\[
S_4 = \begin{vmatrix} 1 & 1 & 1 \\ x & x_3 & x_4 \\ y & y_3 & y_4 \end{vmatrix}.
\]

For any value of the weight \( m_3 \), the optimal facility \( W \) lies in the arc of the algebraic curve of the 12th degree given by the equation

\[
\frac{m_1^4S_2^4}{|WP_1|^4} + \frac{m_2^4S_1^4}{|WP_2|^4} + \frac{m_4^4S_4^4}{|WP_4|^4} - 2\frac{m_1^2m_2^2S_1^2S_2^2}{|WP_1|^2|WP_2|^2} - 2\frac{m_1^2m_4^2S_1^2S_4^2}{|WP_1|^2|WP_4|^2} - 2\frac{m_2^2m_4^2S_2^2S_4^2}{|WP_2|^2|WP_4|^2} = 0. \tag{2.9}
\]

**Proof** is based on an idea similar to that used in the proof of Theorem 2.4, i.e. one should eliminate the variable \( m_3 \) from the system (2.8) treated as a linear one with respect to the weights \( \{ m_j \}_{j=1}^4 \). \( \square \)

Example 2.3. For the configuration

\[
\begin{cases} P_1 = (1, 5) & P_2 = (2, 1) & P_3 = (7, 2) & P_4 = (6, 7) \\ m_1 = 3 & m_2 = 2 & m_3 & m_4 = 7/2 \end{cases},
\]

find the locus of the facility \( W \) under variation of the weight \( m_3 \).

**Solution.** The complete expression (2.9) is rather cumbersome and we restrict ourselves here with the presentation of its terms of the highest and the lowest degree.
The picture of the curve in the vicinity of the quadrilateral $P_1P_2P_3P_4$ is given in Fig. 3 with its only branch containing positions of the facility $W$ displayed in blue boldface. The values of $m_3$ corresponding to this branch lie within the interval

$$ \left[ \frac{-696\sqrt{85} - 952\sqrt{145} + \sqrt{-3621085 + 8696520\sqrt{493} + 17069 + 1326\sqrt{130}}}{4930}, \frac{\sqrt{17069 + 1326\sqrt{130}}}{26} \right] \approx [-0.834783, 6.900360] $$

with its ends corresponding to collision of the optimal facility with $P_1$ or $P_3$. A sample point $W \approx (3.451796, 4.701666)$ marked in Fig. 3 matches the value $m_3 \approx 1.394215$.

3. Bifacility Case: Geometry

First of all, we introduce the geometric observations given by Georg Pick in the Mathematical Appendix of Weber’s book [14]. We illustrate his algorithm with the following example.

**Example 3.1.** Find the optimal position for the facilities $W_1$ and $W_2$ for the problem (1.2) where

$$ \begin{cases} P_1 = (1, 5) & m_1 = 3 \\ P_2 = (2, 1) & m_2 = 2 \\ P_3 = (7, 2) & m_3 = 3 \\ P_4 = (6, 7) & m_4 = 4 \end{cases} \quad m = 4 \begin{array}{c} \text{where} \\ m \end{array} $$

**Solution.** First find the point $Q_1$ lying on the opposite side of the line $P_1P_2$ with respect to the point $P_3$ and such that

$$ |P_1Q_1| = \frac{m_2}{m} |P_1P_2|, \quad |P_2Q_1| = \frac{m_1}{m} |P_1P_2|. \quad (3.1) $$
The exact coordinates of this point are given by (2.5) where the substitution $m_3 \to m$ is made. Find then the second point $Q_2$ with the similar property with respect to the points $P_3$ and $P_4$ (Fig. 4):

$$|P_3Q_2| = \frac{m_4}{m}|P_3P_4|, \quad |P_4Q_2| = \frac{m_3}{m}|P_3P_4|.$$ 

Fig. 4: Example 3.1. Construction of the points $Q_1$ and $Q_2$.

Next, draw the circle $C_1$ circumscribing $P_1P_2Q_1$ and $C_2$ circumscribing $P_3P_4Q_2$. Finally draw the line through $Q_1$ and $Q_2$ (Fig. 5).

Fig. 5: Example 3.1. Pick’s construction of the Weber network
The intersection points of this line with $C_1$ and $C_2$ are the position of the optimal facilities $W_1$ and $W_2$ for the network with the corresponding (minimal) cost equal to $m|Q_1Q_2|$. □

The suggested geometric construction just illustrated via an example, in general case is subject to several extra assumptions. First of all, the point $Q_1$ exists and generates the triangle $P_1P_2Q_1$ iff the values of the weights $m, m_1, m_2$ satisfy the restrictions

$$m < m_1 + m_2, \ m_1 < m + m_2, \ m_2 < m + m_1,$$

i.e. it is possible to construct a weight triangle $\{m, m_1, m_2\}$. Similar restrictions are to be imposed onto the weights $m, m_3, m_4$

$$m < m_3 + m_4, \ m_3 < m + m_4, \ m_4 < m + m_3.$$ (3.3)

The relations (3.1) then mean that the triangle $P_1P_2Q_1$ is similar to the weight triangle $\{m, m_1, m_2\}$.

Secondly, even if both weight triangles exist, the segment $Q_1Q_2$ might not cross either of the circles $C_1$ or $C_2$ or both in the points lying inside the quadrilateral $P_1P_2P_3P_4$.

**Example 3.2.** For the configuration

$$\begin{cases} P_1 = (2, 4) & P_2 = (1, 1) & P_3 = (6, 2) & P_4 = (5, 5) \ P_1 = (2, 4) & P_2 = (1, 1) & P_3 = (6, 2) & P_4 = (5, 5) \ m = 4 & m = 4 & m = 4 & m = 4 \ \end{cases},$$

one has $Q_1 \approx (0.42263, 4.10912), \ Q_2 \approx (6.26863, 5.9437)$ with the points $W_1 \approx (0.83340, 4.23803)$ and $W_2 \approx (6.05325, 5.87612)$ lying outside the quadrilateral $P_1P_2P_3P_4$ (Fig. 6).
Our next aim is now to establish the conditions for the feasibility of Pick’s construction and to find the exact coordinates of the facilities.

4. Bifacility Case: Analytics

**Assumption 2.** We will assume the weights of the problem to satisfy the restrictions \(3.2\) and \(3.3\). From this follows that the values

\[
\begin{align*}
    k_{12} & = (m + m_1 + m_2)(m - m_1 + m_2)(m + m_1 - m_2)(-m + m_1 + m_2), \\
    k_{34} & = (m + m_3 + m_4)(m - m_3 + m_4)(m + m_3 - m_4)(-m + m_3 + m_4)
\end{align*}
\]  

(4.1)  

(4.2)

are positive. Additionally we assume the fulfillment of the following inequalities:

\[
\begin{align*}
    (m^2 - m_1^2 + m_3^2)/\sqrt{k_{12}} + (m^2 - m_1^2 + m_4^2)/\sqrt{k_{34}} & > 0, \\
    (m^2 + m_1^2 - m_3^2)/\sqrt{k_{12}} + (m^2 + m_1^2 - m_4^2)/\sqrt{k_{34}} & > 0.
\end{align*}
\]  

(4.3)  

(4.4)

The geometric sense of the latter restrictions will be clarified below.

**Theorem 4.1.** Let Assumptions 1 and 2 be fulfilled. Set

\[
\begin{align*}
    \tau_1 & = \sqrt{k_{12}} \left[ \sqrt{k_{34}}(x_4 - x_3) - (m^2 + m_3^2 - m_4^2)y_3 - (m^2 - m_3^2 + m_4^2)y_4 \right] \\
    & + 2m^2\sqrt{k_{12}}y_2 + k_{12}(x_1 - x_2) + (m^2 + m_1^2 - m_2^2)\sqrt{k_{34}}(y_3 - y_4) \\
    & + (m^2 + m_1^2 - m_2^2)x_1 + (m^2 - m_1^2 + m_2^2)x_2 - (m^2 + m_3^2 - m_4^2)x_3 - (m^2 - m_3^2 + m_4^2)x_4,
\end{align*}
\]

\[
\begin{align*}
    \tau_2 & = -\sqrt{k_{12}} \left[ \sqrt{k_{34}}(x_4 - x_3) - (m^2 + m_3^2 - m_4^2)y_3 - (m^2 - m_3^2 + m_4^2)y_4 \right] \\
    & - 2m^2\sqrt{k_{12}}y_1 - k_{12}(x_1 - x_2) + (m^2 - m_1^2 + m_2^2)\sqrt{k_{34}}(y_3 - y_4) \\
    & + (m^2 + m_1^2 - m_2^2)x_1 + (m^2 - m_1^2 + m_2^2)x_2 - (m^2 + m_3^2 - m_4^2)x_3 - (m^2 - m_3^2 + m_4^2)x_4,
\end{align*}
\]

\[
\begin{align*}
    \eta_1 & = \frac{1}{\sqrt{k_{12}}} \left[ (m^2 - m_1^2 - m_2^2)\tau_1 - 2m_1^2\tau_2 \right], \\
    \eta_2 & = \frac{1}{\sqrt{k_{12}}} \left[ 2m_2^2\tau_1 - (m^2 - m_1^2 - m_2^2)\tau_2 \right]
\end{align*}
\]

and set the values for \(\tau_3, \tau_4, \eta_3, \eta_4\) via the formulae obtained by the cyclic substitution for subscripts

\[
\begin{pmatrix}
    1 & 2 & 3 & 4 \\
    3 & 4 & 1 & 2
\end{pmatrix}
\]

in the above expressions for \(\tau_1, \tau_2, \eta_1, \eta_2\) correspondingly.

If all the values

\[
\begin{align*}
    \delta_1 & = \eta_2(x_1 - x_2) + \tau_2(y_2 - y_1), \\
    \delta_2 & = \eta_1(x_1 - x_2) + \tau_1(y_2 - y_1), \\
    \delta_3 & = \eta_4(x_3 - x_4) + \tau_4(y_4 - y_3), \\
    \delta_4 & = \eta_3(x_3 - x_4) + \tau_3(y_4 - y_3)
\end{align*}
\]  

(4.5)  

(4.6)
\[
\delta = -\frac{\delta_1 (m^2 + m_1^2 - m_2^2)}{\sqrt{k_{12}}} - \frac{\delta_3 (m^2 + m_3^2 - m_4^2)}{\sqrt{k_{34}}} + (\eta_1 + \eta_2) (y_1 - y_3) + (\tau_1 + \tau_2) (x_1 - x_3) \quad (4.7)
\]
are positive then there exists a pair of points \( W_1 \) and \( W_2 \) lying inside \( P_1P_2P_3P_4 \) that provides the global minimum value for the function (1.2). The coordinates of the optimal facility \( W_1 \) are as follows:
\[
x_* = x_1 - \frac{2 \delta_1 m^2 \tau_1}{\sqrt{k_{12}} [(\eta_1 + \eta_2)^2 + (\tau_1 + \tau_2)^2]}, \quad (4.8)
\]
\[
y_* = y_1 - \frac{2 \delta_1 m^2 \eta_1}{\sqrt{k_{12}} [(\eta_1 + \eta_2)^2 + (\tau_1 + \tau_2)^2]}, \quad (4.9)
\]
while those of \( W_2 \):
\[
x** = x_3 - \frac{2 \delta_3 m^2 \tau_3}{\sqrt{k_{34}} [(\eta_1 + \eta_2)^2 + (\tau_1 + \tau_2)^2]}, \quad (4.10)
\]
\[
y** = y_3 - \frac{2 \delta_3 m^2 \eta_3}{\sqrt{k_{34}} [(\eta_1 + \eta_2)^2 + (\tau_1 + \tau_2)^2]}.
\]
The corresponding minimum value of the function (1.2) (i.e. the cost of the optimal network) equals
\[
\mathcal{C} = \frac{\sqrt{(\eta_1 + \eta_2)^2 + (\tau_1 + \tau_2)^2}}{4m^3}. \quad (4.12)
\]

**Proof.** For brevity, we will use the following notation for the expression that appears nearly in any deduction of the proof:
\[
\Delta = [(\eta_1 + \eta_2)^2 + (\tau_1 + \tau_2)^2]. \quad (4.13)
\]

\[\textbf{(I)}\] We first present some directly verified relations between the values \( \tau \)-s , \( \eta \)-s and \( \delta \)-s.
\[
\tau_1 = \frac{1}{2 m^2} \left[ \sqrt{k_{12}} (\eta_1 + \eta_2) + (m^2 + m_1^2 - m_2^2)(\tau_1 + \tau_2) \right], \quad (4.14)
\]
\[
\tau_2 = \frac{1}{2 m^2} \left[ -\sqrt{k_{12}} (\eta_1 + \eta_2) + (m^2 - m_1^2 + m_2^2)(\tau_1 + \tau_2) \right], \quad (4.15)
\]
\[
\tau_3 = \frac{1}{2 m^2} \left[ -\sqrt{k_{34}} (\eta_1 + \eta_2) - (m^2 + m_3^2 - m_4^2)(\tau_1 + \tau_2) \right],
\]
\[
\tau_4 = \frac{1}{2 m^2} \left[ \sqrt{k_{34}} (\eta_1 + \eta_2) - (m^2 - m_3^2 + m_4^2)(\tau_1 + \tau_2) \right],
\]
\[
\tau_1 + \tau_2 + \tau_3 + \tau_4 = 0, \quad \eta_1 + \eta_2 + \eta_3 + \eta_4 = 0, \quad (4.16)
\]
\[
\sum_{j=1}^{4} (x_j \tau_j + y_j \eta_j) = \frac{\Delta}{4m^4}; \quad (4.17)
\]
\[ \tau_1^2 + \eta_1^2 = \frac{m_1^2}{m_2^2} \Delta, \quad (4.18) \]
\[ \tau_1 \eta_2 - \tau_2 \eta_1 = \frac{\sqrt{k_{12}}}{2m^2} \Delta, \quad (4.19) \]
\[ \tau_2 \eta_3 - \tau_3 \eta_2 = \frac{\sqrt{k_{12}k_{34}}}{4m^4} \left[ \frac{m_2^2 - m_1^2 + m_2^2}{\sqrt{k_{12}}} + \frac{m_2^2 - m_4^2 + m_3^2}{\sqrt{k_{34}}} \right] \Delta, \quad (4.20) \]
\[ \delta_1 + \delta_3 = (x_1 - x_3)(\eta_1 + \eta_2) - (y_1 - y_3)(\tau_1 + \tau_2), \quad (4.21) \]
\[ 2\delta_2 m_2^2 = (m^2 - m_1^2 - m_3^2) \delta_1 - \sqrt{k_{12}} [(y_1 - y_2) \eta_2 + (x_1 - x_2) \tau_2], \quad (4.22) \]
\[ 2\delta_4 m_4^2 = (m^2 - m_3^2 - m_4^2) \delta_3 - \sqrt{k_{34}} [(y_3 - y_4) \eta_4 + (x_3 - x_4) \tau_4]. \quad (4.23) \]

(II) Consider the system of equations for determining stationary points of the objective function (1.2):
\[
\frac{\partial F}{\partial x_*} = m_1 \frac{x_* - x_1}{|W_1P_1|} + m_2 \frac{x_* - x_2}{|W_1P_2|} + m \frac{x_* - x_{**}}{|W_1W_2|} = 0, \quad (4.24) \\
\frac{\partial F}{\partial y_*} = m_1 \frac{y_* - y_1}{|W_1P_1|} + m_2 \frac{y_* - y_2}{|W_1P_2|} + m \frac{y_* - y_{**}}{|W_1W_2|} = 0, \quad (4.25) \\
\frac{\partial F}{\partial x_{**}} = m_3 \frac{x_{**} - x_3}{|W_2P_3|} + m_4 \frac{x_{**} - x_4}{|W_2P_4|} + m \frac{x_{**} - x_*}{|W_2W_1|} = 0, \quad (4.26) \\
\frac{\partial F}{\partial y_{**}} = m_3 \frac{y_{**} - y_3}{|W_2P_3|} + m_4 \frac{y_{**} - y_4}{|W_2P_4|} + m \frac{y_{**} - y_*}{|W_2W_1|} = 0. \quad (4.27) 
\]

Let us verify the validity of (4.24). First establish the alternative representations for the coordinates (4.8) and (4.9):
\[ x_* = x_2 - \frac{2m^2 \delta_2 \tau_2}{\sqrt{k_{12}\Delta}}, \quad (4.28) \]
\[ y_* = y_2 - \frac{2m^2 \delta_2 \eta_2}{\sqrt{k_{12}\Delta}}. \quad (4.29) \]
Indeed, the difference of the right-hand sides of (4.8) and (4.28) equals
\[ x_1 - x_2 - \frac{2m^2 (\delta_1 \tau_1 - \delta_2 \tau_2)}{\sqrt{k_{12}\Delta}} \]
and the numerator of the involved fraction can be transformed into
\[ \tau_1 \eta_2 (x_1 - x_2) + \tau_1 \tau_2 (y_2 - y_1) - \tau_2 \eta_1 (x_1 - x_2) - \tau_2 \tau_1 (y_2 - y_1) \]

\[ = 2m^2 (x_1 - x_2)^2 (\tau_1 \eta_2 - \tau_2 \eta_1) \quad (4.19) = (x_1 - x_2) \sqrt{k_{12}\Delta}. \]
The equivalence of (4.29) and (4.9) can be demonstrated in a similar manner. Now express the segment lengths:
\[ |W_1P_1| = \sqrt{(x_1 - x_4)^2 + (y_1 - y_4)^2} \quad (4.8),(4.9) = \frac{2\delta_1 m^2}{\sqrt{k_{12}\Delta}} \sqrt{\tau_1^2 + \eta_1^2} = \frac{2m m_1}{\sqrt{k_{12}}} \frac{\delta_1}{\sqrt{\Delta}}. \quad (4.30) \]
and, similarly,

$$|W_1 P_2| \stackrel{(4.10),(4.11)}{=} \frac{2 m m_2}{\sqrt{k_{12}}} \frac{\delta_3}{\sqrt{\Delta}}. \quad (4.31)$$

With the aid of relations (4.8), (4.28), (4.30) and (4.31) one can represent the first two terms in the left-hand side of the equality (4.24) as

$$m_1 \frac{x_* - x_1}{|W_1 P_1|} + m_2 \frac{x_* - x_2}{|W_1 P_2|} = -\frac{m}{\sqrt{\Delta}}(\tau_1 + \tau_2). \quad (4.32)$$

The third summand in the equality (4.24) needs more laborious manipulations. We first transform its numerator:

$$x_* - x_{**} \stackrel{(4.10),(4.14)}{=} x_1 - x_3 + \frac{2 m^2}{\Delta} \left[ \frac{\delta_3 \tau_3}{\sqrt{k_{34}}} - \frac{\delta_1 \tau_1}{\sqrt{k_{12}}} \right].$$

Now write down the following modification:

$$2 m^2 \left[ \frac{\delta_3 \tau_3}{\sqrt{k_{34}}} - \frac{\delta_1 \tau_1}{\sqrt{k_{12}}} \right] \stackrel{(4.14),(4.15)}{=} \frac{-(\eta_1 + \eta_2) - m^2 + m_3^2 - m_3^2}{\sqrt{k_{34}}} (\tau_1 + \tau_2) \delta_3 - \frac{(\eta_1 + \eta_2) + m^2 + m_3^2 - m_3^2}{\sqrt{k_{12}}} (\tau_1 + \tau_2) \delta_1

= - \left[ (\eta_1 + \eta_2)(\delta_1 + \delta_3) + (\tau_1 + \tau_2) \left\{ \frac{\delta_1 (m^2 + m_3^2 - m_3^2)}{\sqrt{k_{12}}} + \frac{\delta_3 (m^2 + m_3^2 - m_3^2)}{\sqrt{k_{34}}} \right\} \right]

\stackrel{(4.7)}{=} - \left[ (\eta_1 + \eta_2)(\delta_1 + \delta_3) + (\tau_1 + \tau_2) \left\{ -\delta + (\eta_1 + \eta_2) (y_1 - y_3) + (\tau_1 + \tau_2) (x_1 - x_3) \right\} \right]

= \delta (\tau_1 + \tau_2) - (\eta_1 + \eta_2) [\delta_1 + \delta_3 + (\tau_1 + \tau_2)(y_1 - y_3) - (\eta_1 + \eta_2)(x_1 - x_3)] - \Delta (x_1 - x_3)

\stackrel{(4.21)}{=} \delta (\tau_1 + \tau_2) - \Delta (x_1 - x_3).

Finally,

$$x_* - x_{**} = x_1 - x_3 + \frac{\delta (\tau_1 + \tau_2) - \Delta (x_1 - x_3)}{\Delta} = \frac{\delta (\tau_1 + \tau_2)}{\Delta}. \quad (4.33)$$

Similarly the following equality can be deduced:

$$y_* - y_{**} = \frac{\delta (\eta_1 + \eta_2)}{\Delta}, \quad (4.34)$$

and both formulae yield

$$|W_1 W_2| = \sqrt{(x_* - x_{**})^2 + (y_* - y_{**})^2} = \frac{\delta}{\sqrt{\Delta}}. \quad (4.35)$$

Therefore, the last summand of equality (4.24) takes the form

$$m \frac{x_* - x_{**}}{|W_1 W_2|} = m \frac{\delta (\tau_1 + \tau_2) \sqrt{\Delta}}{\delta \Delta} = m \frac{\tau_1 + \tau_2}{\sqrt{\Delta}}. \quad (4.36)$$

Summation this with (4.32) yields 0 and this completes the proof of (4.24).

The validity of the remaining equalities (4.25)–(4.27) can be established in a similar way.
(III) We now deduce the formula (4.12) for the network cost. With the aid of the formulae (4.30), (4.31), (4.35) and their counterparts for the segment lengths |W_2P_3| and |W_2P_4|, one gets

\[ m_1|W_1P_1| + m_2|W_1P_2| + m_3|W_2P_3| + m_4|W_2P_4| + m|W_1W_2| \]

\[ = \frac{2m}{\sqrt{\Delta}} \left( \frac{m_1^2 \delta_1}{\sqrt{k_{12}}} + \frac{m_2^2 \delta_2}{\sqrt{k_{12}}} + \frac{m_3^2 \delta_3}{\sqrt{k_{34}}} + \frac{m_4^2 \delta_4}{\sqrt{k_{34}}} + \delta \right) \]

\[ \overset{(4.7)}{=} \frac{2m}{\sqrt{\Delta}} \left\{ \frac{\delta_1}{2\sqrt{k_{12}}} \left( m_2^2 + m_2^2 + m_1^2 \right) + \frac{\delta_3}{2\sqrt{k_{34}}} \left( -m_2^2 + m_3^2 + m_4^2 \right) + \frac{m_2^2 \delta_2}{\sqrt{k_{12}}} + \frac{m_4^2 \delta_4}{\sqrt{k_{34}}} \right\} + \frac{1}{2} \left( \eta_1 + \eta_2 \right) (y_1 - y_3) + \frac{1}{2} \left( \tau_1 + \tau_2 \right) (x_1 - x_3) \right\} \]

\[ \overset{(4.22),(4.23)}{=} \frac{2m}{\sqrt{\Delta}} \left\{ -\frac{1}{2} (y_1 - y_2) \eta_2 - \frac{1}{2} (x_1 - x_2) \tau_2 - \frac{1}{2} (y_3 - y_4) \eta_4 - \frac{1}{2} (x_3 - x_4) \tau_4 \right\} + \frac{1}{2} \left( \eta_1 + \eta_2 \right) (y_1 - y_3) + \frac{1}{2} \left( \tau_1 + \tau_2 \right) (x_1 - x_3) \right\} \]

\[ = \frac{m}{\sqrt{\Delta}} \left\{ y_1 \eta_1 + y_2 \eta_2 + y_4 \eta_4 + x_1 \tau_1 + x_2 \tau_2 + x_4 \tau_4 - x_3 (\tau_1 + \tau_2 + \tau_4) - y_3 (\eta_1 + \eta_2 + \eta_4) \right\} \]

\[ \overset{(4.16)}{=} \frac{m}{\sqrt{\Delta}} \sum_{j=1}^{4} (x_j \tau_j + y_j \eta_j) = \frac{\sqrt{\Delta}}{4m^3}. \]

(IV) If the facilities W_1 and W_2 provide the solution to the problem (1.2), they should lie inside the quadrilateral P_1P_2P_3P_4 [6]. Let us verify this condition checking the triangles P_1P_2W_1, P_2W_2W_1 and P_2P_3W_1 are oriented counterclockwise. Indeed,

\[
\begin{vmatrix}
1 & 1 & 1 \\
x_1 & x_2 & x_* \\
y_1 & y_2 & y_*
\end{vmatrix}
= \begin{vmatrix}
1 & 0 & 0 \\
x_1 & x_2 - x_1 & x_* - x_1 \\
y_1 & y_2 - y_1 & y_* - y_1
\end{vmatrix}
\overset{(4.8),(4.9)}{=} \begin{vmatrix}
1 & 1 & 1 \\
x_2 & x_* & x_* \\
y_2 & y_* & y_*
\end{vmatrix}
= \begin{vmatrix}
1 & 0 & 0 \\
x_2 & x_* - x_1 & x_* - x_1 \\
y_2 & y_* - y_1 & y_* - y_2
\end{vmatrix}
\overset{(4.33),(4.34)}{=} \begin{vmatrix}
\frac{2 \delta \delta_2 m^2}{\sqrt{k_{12} \Delta^2}} \tau_1 + \tau_2 & \tau_2 \\
\eta_1 + \eta_2 & \eta_2
\end{vmatrix}
\overset{(4.36)}{=} \begin{vmatrix}
2 \delta \delta_1 m^2 \tau_1 + \tau_2 \\
2 \delta_1 \delta_2 m^2 \frac{\sqrt{k_{12} \Delta^2}}{\tau_1 + \tau_2}
\end{vmatrix}
\overset{(4.37)}{=} \frac{2 \delta \delta_1 m^2}{2m^2} = \frac{\delta \delta_2}{\Delta}.
\]

Due to assumptions of positivity of all the deltas, both determinants are positive. In order to prove positivity of the determinant
let us extract it from the alternative computation of the determinant (4.36).

\[
\begin{pmatrix}
1 & 1 & 1 \\
x_2 & x_{**} & x_* \\
y_2 & y_{**} & y_*
\end{pmatrix}
= \begin{pmatrix}
1 & 1 & 1 \\
x_2 & x_3 & x_* \\
y_2 & y_3 & y_*
\end{pmatrix} + \begin{pmatrix}
1 & 0 & 0 \\
x_1 & x_{**} - x_3 & x_* - x_2 \\
y_1 & y_{**} - y_3 & y_* - y_2
\end{pmatrix}.
\]

Therefore, the determinant (4.37) equals

\[
\delta \frac{\Delta}{\Delta} - \frac{2m^4\delta_2\delta_3}{\sqrt{k_{12}k_{34}\Delta^2}} (\tau_3\eta_2 - \tau_2\eta_3) = \frac{\delta \Delta}{\Delta} + \frac{2m^4\delta_2\delta_3}{\sqrt{k_{12}k_{34}}} \left[ \frac{m^2 - m_1^2 + m_2^2}{\sqrt{k_{12}}} + \frac{m^2 - m_4^2 + m_3^2}{\sqrt{k_{34}}} \right]
\]

and it is positive due to the assumption (4.3).

(V) We finally prove that the formulae (4.8)–(4.11) furnish the minimal value for the function (1.2). For this aim, represent the Hessian of this function

\[
\mathcal{H}(F) = \begin{bmatrix}
\partial^2 F / \partial x^2 & \partial^2 F / \partial x \partial y_* & \partial^2 F / \partial x \partial y_{**} & \partial^2 F / \partial x \partial y_{**} & \partial^2 F / \partial x \partial y_{**} \\
* & \partial^2 F / \partial y_*^2 & \partial^2 F / \partial y_* \partial x_* & \partial^2 F / \partial y_* \partial x_{**} & \partial^2 F / \partial y_* \partial y_{**} \\
* & * & \partial^2 F / \partial x_{**}^2 & \partial^2 F / \partial x_{**} \partial y_{**} \\
* & * & * & \partial^2 F / \partial y_{**}^2
\end{bmatrix}
\]

(4.38)
as a product

\[
= \mathcal{M} \cdot \mathcal{M}^\top
\]

where

\[
\mathcal{M} = \begin{bmatrix}
\sqrt{m_2(y_* - y_1)} & \sqrt{m_2(y_* - y_2)} & \sqrt{m_2(y_* - y_{**})} & 0 & 0 \\
-\sqrt{m_2(x_* - x_1)} & -\sqrt{m_2(x_* - x_2)} & -\sqrt{m_2(x_* - x_{**})} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}_{4 \times 5}
\]

and \(\mathcal{M}^\top\) stands for transposition. Therefore, Hessian (4.38) can be interpreted as the Gramian of the rows of the matrix \(\mathcal{M}\). The minor of the latter obtained by deleting the third its column equals

\[
\frac{\sqrt{m_1 m_2 m_3 m_4}}{|W_1 P_1| \cdot |W_1 P_2| \cdot |W_2 P_3| \cdot |W_2 P_4|^{3/2}} \begin{vmatrix}
y_* - y_1 & y_* - y_2 & y_* - y_3 & y_* - y_4 \\
x_* - x_* & x_* - x_2 & x_* - x_3 & x_* - x_4
\end{vmatrix}
\]

and, under the assumptions of the theorem, is nonzero for any choice of the points \(W_1\) and \(W_2\) inside the quadrilateral \(P_1 P_2 P_3 P_4\). Consequently, the rank of the matrix \(\mathcal{M}\) equals 4, its rows are linearly independent, and their Gramian is a positive definite matrix. From the Convex Optimization theory [9, 2], it follows that the function (1.2) is strictly convex inside the convex (due to Assumption 1) domain given as the Cartesian product \(P_1 P_2 P_3 P_4 \times P_1 P_2 P_3 P_4\). Therefore the solution of the system (4.24) – (4.27) provides the global minimum value for this function. \(\square\)
Remark. In [4], it is proved that the function
\[ \tilde{F}(W_1, W_2) = m_1|W_1P_1| + m_2|W_1P_2| + m_3|W_2P_3| + m_4|W_2P_4| \]
is neither convex nor concave if treated as a function of variables \(x_*, y_*, x**, y**\) and \(\{m_j\}_{j=1}^4\). This result should be distinguished from that claimed in the part \((V)\) of the proof of Theorem 4.1: the objective function (1.2) contains an extra term and the weights are not treated as variables.

The result of Theorem 4.1 claims that the bifacility Weber problem for four terminals is solvable by radicals, and thus we get a natural extension of the three-terminal problem solution given in Theorem 2.1. An additional correlation between these two results can be watched, namely that the denominators of all the formulae for the facilities coordinates contain the explicit expression for the cost of the corresponding network. It looks like every facility “knows” the cost of the network which this point is a part of.

Example 4.1. Find the exact coordinates of the facilities \(W_1\) and \(W_2\) for Example 3.1.

Solution. The conditions of Theorem 4.1 are fulfilled: the values \(\{\delta_j\}_{j=1}^4\) and \(\delta\) are positive. Formulae (4.8)–(4.12) then give the coordinates for the facilities
\[
W_1 = \left( \frac{2266800 + 772027\sqrt{15} + 453352\sqrt{33} + 246177\sqrt{55}}{48(22049 + 2085\sqrt{15} + 945\sqrt{33} + 2559\sqrt{55})}, \frac{1379951 + 201984\sqrt{15} + 97279\sqrt{33} + 154368\sqrt{55}}{16(22049 + 2085\sqrt{15} + 945\sqrt{33} + 2559\sqrt{55})} \right)
\approx (3.701271, 4.430843);
\]
\[
W_2 = \left( \frac{188467345 + 18613485\sqrt{15} + 7149825\sqrt{33} + 20949207\sqrt{55}}{1760(22049 + 2085\sqrt{15} + 945\sqrt{33} + 2559\sqrt{55})}, \frac{188346565 + 19265895\sqrt{15} + 20525157\sqrt{33} + 7187445\sqrt{55}}{1760(22049 + 2085\sqrt{15} + 945\sqrt{33} + 2559\sqrt{55})} \right)
\approx (4.761622, 4.756175)
\]
with the cost of the network
\[
C = \frac{1}{8}\sqrt{44098 + 4170\sqrt{15} + 5118\sqrt{55} + 1890\sqrt{33}} \approx 41.280608.
\]

One can now verify directly correctness of Pick’s geometric solution from Section 3:

Corollary. Under the conditions of the theorem, the facilities \(W_1, W_2\) and the points \(Q_1, Q_2\) are collinear. The cost of the network equals \(m|Q_1Q_2|\).

We outline briefly the meaning of the assumptions from Theorem 4.1. First, we concern the values (4.1) and (4.2). The values \(\frac{1}{4}\sqrt{k_{12}}\) and \(\frac{1}{4}\sqrt{k_{34}}\) equal the areas of the weight triangles introduced in Section 3. Next, due to the law of cosines, one has
\[
(m^2 - m_1^2 + m_2^2)/\sqrt{k_{12}} = \cot \beta_1, (m^2 - m_2^2 + m_3^2)/\sqrt{k_{34}} = \cot \beta_4
\]
where $\beta_1$ and $\beta_4$ are the angles of the corresponding weight triangles (Fig. 1 (b)). Therefore the condition (4.3) is equivalent to the fact that the angle of the weight quadrilateral, as illustrated in Fig. 1 (b), is less than $\pi$. Together with the condition (4.4) this implies that the weight quadrilateral is convex. This condition is stated in Theorem 4.1 as a sufficient one for the existence of solution to the bifacility Weber problem. As yet we have failed to prove that it is stiff enough to be a necessary one.

Positivity of all the values (4.5)-(4.6) guarantees the non-collision of the facilities $W_1$ and $W_2$ with the terminals $\{P_j\}_{j=1}^4$. Finally, due to the equality (4.35), the condition (4.7) guarantees the non-collision of the facilities $W_1$ and $W_2$, i.e. the non-degeneracy of the network with two assumed facilities. It is possible to deduce some alternative representation for this value, say more “symmetric” with respect to the involved parameters. For instance, the following equality

\[
\delta = \frac{1}{x_1 - x_2, y_1 - y_2, x_3 - x_4, y_3 - y_4} \begin{vmatrix} 1 & x_1 & y_1 & -\delta_1(m^2 + m_1^2 - m_2^2) / \sqrt{k_{12}} \\ 1 & x_2 & y_2 & -\delta_2(m^2 - m_1^2 + m_2^2) / \sqrt{k_{12}} \\ 1 & x_3 & y_3 & \delta_3(m^2 + m_3^2 - m_4^2) / \sqrt{k_{34}} \\ 1 & x_4 & y_4 & \delta_4(m^2 - m_3^2 + m_4^2) / \sqrt{k_{34}} \end{vmatrix}
\]

is valid provided that the edges $P_1P_2$ and $P_3P_4$ are non-parallel.

The more detailed representation is as follows:

\[
\delta = 4m^4 [(x_1 - x_3)(x_2 - x_4) + (y_1 - y_3)(y_2 - y_4)] \\
+ \frac{2}{\sqrt{k_{34}}} [(m_2^2 - m_1^2)k_{34} - m^2(m^2 - m_1^2 + m_2^2)(m^2 - m_3^2 - m_4^2)] \cdot \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ x_2 & x_3 & x_4 & y_2 & y_3 \\ y_1 & y_3 & y_4 \end{vmatrix} \\
+ \frac{2}{\sqrt{k_{34}}} [(m_2^2 - m_1^2)k_{34} - m^2(m^2 + m_1^2 - m_2^2)(m^2 - m_3^2 - m_4^2)] \cdot \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ x_1 & x_3 & x_4 & y_1 & y_3 \\ y_1 & y_3 & y_4 \end{vmatrix} \\
+ \frac{2}{\sqrt{k_{12}}} [(m_1^2 - m_3^2)k_{12} - m^2(m^2 + m_3^2 - m_4^2)(m^2 - m_1^2 - m_2^2)] \cdot \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & y_1 & y_2 \\ y_1 & y_2 & y_3 \end{vmatrix} \\
+ \frac{2}{\sqrt{k_{12}}} [(m_3^2 - m_1^2)k_{12} - m^2(m^2 - m_3^2 + m_4^2)(m^2 - m_1^2 - m_2^2)] \cdot \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_4 & y_1 & y_2 \\ y_1 & y_2 & y_4 \end{vmatrix} \\
+ \frac{1}{\sqrt{k_{12}k_{34}}} \left[ \left( \sqrt{k_{12}} - \sqrt{k_{34}} \right)^2 m^4 - \left( \sqrt{k_{34}(m^2 - m_1^2)} + \sqrt{k_{12}(m^2 - m_4^2)} \right)^2 \right] \\
\times [(x_4 - x_3)(x_2 - x_1) + (y_4 - y_3)(y_2 - y_1)].
\]

This representation permits one to relate the general Weber problem to its important particular case.
Corollary. For the equal weighted case \( \{m_j = 1\}_{j=1}^4 \), \( m = 1 \), the expression for \( \delta \) can be represented in the form

\[
\delta = \frac{8}{\sqrt{3}} [x_3 - x_1, y_3 - y_1] \cdot \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix} \cdot \begin{bmatrix} x_4 - x_2 \\ y_4 - y_2 \end{bmatrix}.
\]

This value is positive iff the angle between the diagonal \( \overrightarrow{P_1P_3} \) of the quadrilateral and the other diagonal \( \overrightarrow{P_2P_4} \) turned through by \( \pi/6 \) clockwise is acute. Equivalently, if we denote by \( \psi \) the angle between the diagonal vectors \( \overrightarrow{P_1P_3} \) and \( \overrightarrow{P_2P_4} \) then \( \delta \) is positive iff \( \psi < \pi/2 + \pi/6 = 2\pi/3 \). This confirms the known condition for the existence of a full Steiner tree for the terminals \( \{P_j\}_{j=1}^4 \), i.e. the points \( S_1 \) and \( S_2 \) providing the solution to the problem

\[
\min_{\{S_1,S_2\} \subset \mathbb{R}^2} \{ |P_1S_1| + |P_2S_1| + |P_3S_2| + |P_4S_2| + |S_1S_2| \}.
\]

Formulae (4.8)-(4.11) yield then the coordinates of these (Steiner) points with the length of the minimal tree equal to

\[
\mathcal{C} = \frac{1}{2} \sqrt{A^2 + B^2}
\]

where

\[
A = \sqrt{3}(x_1 - x_2 - x_3 + x_4) + (y_1 + y_2 - y_3 - y_4),
B = (x_1 + x_2 - x_3 - x_4) + \sqrt{3}(-y_1 + y_2 + y_3 - y_4).
\]

5. Solution Analysis

Though the analytical solution obtained in the previous section looks cumbersome in comparison with elegance of the geometrical one described in Section 3, it possesses two undeniable advantages over the latter. First, it provides one with a unique opportunity to analyze the dynamics of the network under variation of the parameters of the configuration and to find the bifurcation values for these parameters, i.e. those responsible for the topology degeneracy. The second benefit is a wonderful occasion for replacing the formal proofs of some statements below (Theorems 5.2, 5.3 and 5.4) with the words “...via direct substitution of the formulae (4.8)-(4.11)”.

We first treat the case where the coordinates of a terminal are variated.
The following result is an evident counterpart of Theorem 2.2.

**Theorem 5.1.** If the facilities $W_1$ and $W_2$ give the solution to the problem (1.2) for some configuration \( \{ P_1, P_2, P_3, P_4 \} \) then these facilities remain unchanged for the configuration \( \{ P_1, P_2, \tilde{P}_3, P_4 \} \) with any position of the terminal $\tilde{P}_3$ in the half-line $W_2P_3$.

**Example 5.1.** For the configuration
\[
\begin{align*}
P_1 &= (1, 5) & P_2 &= (2, 1) & P_3 &= (6, 7) & P_4 &= (6, 7) \\
m_1 &= 3 & m_2 &= 2 & m_3 &= 3 & m_4 &= 4
\end{align*}
\]
find the loci of the facilities $W_1$ and $W_2$ under variation of the terminal $P_3$ moving somehow from the starting position at $(9, 2)$ towards $P_2$.

**Solution.** It turns out that when $P_3$ wanders, the facility $W_1$ moves along the arc of the circle $C_1$ introduced in Theorem 2.3 (with the replacement of $m_3$ by $m$). It is given by the equality
\[
\left( x - \frac{3}{2} + \frac{2}{\sqrt{15}} \right)^2 + \left( y - \frac{3 + 1}{30} \right)^2 = \frac{68}{15}.
\]
At the same time, the facility $W_2$ drifts along the circle $C_3$ passing through the points $Q_1, P_4$ and $Q_3$ (Fig. 7). Here $Q_3$ is constructed in the manner analogous to $Q_1$, i.e. the triangle $Q_1P_4Q_3$ should be similar to the weight triangle \( \{ m, m_4, m_3 \} \). Its coordinates can be obtained from Theorem 2.3:
\[
Q_3 = \left( \frac{235 - 12\sqrt{15} - 36\sqrt{55} - 5\sqrt{33}}{64}, \frac{1020 - 9\sqrt{15} + 149\sqrt{55} + 60\sqrt{33}}{192} \right).
\]
\[ C_3 \text{ is given as } (x - X_3)^2 + (y - Y_3)^2 = r_3^2 \]

where
\[
X_3 = \frac{235}{64} - \frac{207}{800} \sqrt{55} - \frac{23}{104} \sqrt{33} - \frac{3}{16} \sqrt{15} \approx 1.013521, \\
Y_3 = \frac{85}{16} + \frac{8477}{10560} \sqrt{55} + \frac{43}{176} \sqrt{33} - \frac{3}{64} \sqrt{15} \approx 8.288416, \\
r_3 = \sqrt{\frac{32}{\sqrt{15}} + \frac{1808}{99}} \approx 5.150241. \\
\]

The trajectory of \( P_3 \) does not influence those of \( W_1 \) and \( W_2 \), i.e. both facilities do not leave the corresponding arcs for any drift of \( P_3 \) until the latter swashes the line \( \mathcal{L} = Q_3W \). At this moment, \( W_1 \) collides with \( W_2 \) in the point
\[
W = \left( \frac{867494143740435 + 114770004066285 \sqrt{\frac{\sqrt{33} + 14973708000030 \sqrt{\sqrt{55} + 1926585102850 \sqrt{\sqrt{15}}}}{145001643291980}}}{581098602680450 + 10154769229801 \sqrt{\sqrt{55} + 1926585102850 \sqrt{\sqrt{15}}}} \right) \approx (3.936925, 4.048287)
\]

which stands for the second point of intersection of the circles \( C_1 \) and \( C_3 \), and yields a solution to the unifacility Weber problem (1.1) for the configuration \( \{ P_1, P_2, P_3, P_4 \} \) (due to Theorem 5.1, location of \( W \) is invariant for any position of \( P_3 \) in \( \mathcal{L} \)). The equation for \( \mathcal{L} \) is as follows:
\[
y = \frac{2872083714 - 841888053 \sqrt{\frac{\sqrt{55} - 546765794 \sqrt{\sqrt{33} + 411553980 \sqrt{\sqrt{55} + 1926850969306 \sqrt{\sqrt{15}}}}}{145001643291980}}}{31025058} - \frac{114568440 + 1171692495 \sqrt{\sqrt{55} + 694024390 \sqrt{\sqrt{33} - 319467193 \sqrt{\sqrt{55}}}}}{620501106} x + 10.104975 \\
\approx -1.538431 x + 10.104975. \\
\]

When \( P_3 \) crosses the line \( \mathcal{L} \), the solution to the bifacility Weber problem (1.2) does not exist (while the unifacility counterpart (1.1) still possesses a solution).

\[ \square \]

**Theorem 5.2.** For any position of the terminal \( P_3 \), the facility \( W_1 \) lies in the arc of the circle \( C_1 \) passing through the points \( P_1, P_2 \) and \( Q_1 = (q_{1x}, q_{1y}) \) given by the formula (2.5) where substitution \( m_3 \to m \) is made. At the same time, the facility \( W_2 \) lies in the arc of the circle \( C_3 \) passing through the points \( Q_1 = (q_{1x}, q_{1y}), P_4 \) and \( Q_3 \). Here \( Q_3 \) is given by (2.5) where substitution \( x_1 \to q_{1x}, y_1 \to q_{1y}, x_2 \to x_4, y_2 \to y_4, m_1 \to m, m_2 \to m_4 \) is applied to.

The scenario for the facilities behaviour in Example 5.1 looks similar to the equal weighted case (the Steiner problem) [12], whereas the problem statement of the next results is of a completely novel nature.

**Theorem 5.3.** Let the circle \( C_1 \) and the point \( Q_1 = (q_{1x}, q_{1y}) \) be defined as in Theorem 5.2. For any value of the weight \( m_3 \), the optimal facility \( W_1 \) lies in the arc of the circle \( C_1 \). At the same time, the facility \( W_2 \) lies in the arc of the 4th degree algebraic curve passing through the points \( P_3, P_4 \) and \( Q_1 \).
It is given by the equation
\[
 m^2 \begin{vmatrix} 1 & 1 & 1 \\ x & q_{1x} & x_3 \\ y & q_{1y} & y_3 \end{vmatrix}^2 = m_4^2 \begin{vmatrix} 1 & 1 & 1 \\ x & x_3 & x_4 \\ y & y_3 & y_4 \end{vmatrix}^2 \begin{pmatrix} (x - x_4)^2 + (y - y_4)^2 \\ (x - q_{1x})^2 + (y - q_{1y})^2 \end{pmatrix}. \quad (5.2)
\]

Example 5.2. For the configuration
\[
\begin{align*}
P_1 &= (1, 5) & P_2 &= (2, 1) & P_3 &= (7, 2) & P_4 &= (6, 7) \\
m_1 &= 3 & m_2 &= 2 & m_3 &= 7/2 & m_4 &= 4
\end{align*}
\]
find the loci of the facilities \( W_1 \) and \( W_2 \) under variation of the weight \( m_3 \) within the interval [1.1, 6.5].

**Solution.** The facility \( W_1 \) moves along the arc of the circle \( C_1 \) given by (5.1). The trajectory of \( W_2 \) is now a branch of the curve (5.2) (Fig. 8) which we present here by its terms of the highest and the lowest degree
\[
(x^2 + y^2)[7609x - (12924 + 597\sqrt{15})y][12243x + (13556 + 977\sqrt{15})y]
+ \cdots - 143901885100 - 9445715749\sqrt{15} = 0.
\]
It crosses that of \( W_1 \) when \( m_3 \) coincides with the zero of the equation \( \delta(m_3) = 0 \). The latter can be reduced to an algebraic one of the 8th degree. It happens to be even one in \( m_3 \) and, in principle, can be solved by radicals. We restrict ourselves here with a numerical approximation of this zero, namely \( m_3 = m_{3,1} \approx 1.394215 \). The intersection point \( W \approx (3.451796, 4.701666) \) yields a solution to the unifacility Weber problem for the configuration \( \{ P_1 \mid m_1 \} \mid P_2 \mid m_2 \mid P_3 \mid m_3 \mid P_4 \mid m_4 \} \) (v. solution of Example 2.3). In principle, the coordinates of \( W \) can also be expressed by radicals since the algebraic equations for their determination are of the 4th degree. When \( m_3 \) diminishes further from \( m_{3,1} \), the locus of \( W \) follows the curve found in the solution of Example 2.3 and displayed in Fig. 3.

Fig. 8: Example 5.2. Dynamics of the facilities \( W_1 \) and \( W_2 \) under variation of the weight \( m_3 \).
If the configuration of the previous example is slightly modified, the scenario for the network degeneracy varies.

**Example 5.3.** For the configuration

\[
\begin{align*}
P_1 &= (1, 5) & m_1 &= 3 \\ P_2 &= (2, 1) & m_2 &= 2 \\ P_3 &= (7, 2) & m_3 &= m_4 = 4
\end{align*}
\]

find the loci of the facilities \(W_1\) and \(W_2\) under variation of the weight \(m_3\) within the interval \([0, 7]\).

**Solution.** The trajectory of \(W_1\) remains the same as in two previous examples. As for the facility \(W_2\), this time the curve (5.2) splits into the two algebraic curves: the cubic

\[
(x^2 + y^2)(44809x + 3183y\sqrt{15} + 45696y) + (8589\sqrt{15} - 826932)x^2 - (33588\sqrt{15} + 850046)xy - (34053\sqrt{15} + 793288)y^2
\]

\[
+ (5231493 - 142194\sqrt{15})x + (5081366 + 177165\sqrt{15})y - 12657634 + 489213\sqrt{15} = 0
\]

and the line

\[
3843x - (5184 + 283\sqrt{15})y - 13230 + 1981\sqrt{15} = 0.
\]

(5.3)

With \(m_3\) decreasing from \(m_3 = 3\) to 0, the facility \(W_1\) moves towards \(P_1\) while \(W_2\) moves towards \(P_4\) along the cubic. These drifts tend to the points

\[
W_{1,0} = \left(\frac{332836}{253885} + \frac{305663}{761655}\sqrt{15}, \frac{788632}{253885} + \frac{129498}{253885}\sqrt{15}\right) \approx (2.865254, 5.081732),
\]

\[
W_{2,0} = \left(\frac{13871603}{3249728} + \frac{977199}{16248640}\sqrt{15}, \frac{18257483}{3249728} + \frac{1950153}{16248640}\sqrt{15}\right) \approx (4.501465, 6.082990)
\]

correspondingly. It turns out that these points and \(P_4\) lie in the line (5.3), and \(W_{1,0}\) is the solution to the unifacility Weber problem for the configuration \(\{P_1 \ m_1 \ P_2 \ m_2 \ P_4 \ m\}\).

With \(m_3\) increasing from \(m_3 = 3\) to 0, the facility \(W_1\) moves to \(P_2\) while \(W_2\) moves to \(P_3\). Which terminal is reached faster? Due to (4.30), the answer depends on the relative position of the zeros of equations \(\delta_2(m_3) = 0\) and \(\delta_3(m_3) = 0\). Via two successive squaring, both equations can be reduced to an algebraic form. The zero of \(\delta_2(m_3) = 0\) closest to \(m_3 = 3\) is that of

\[
48 841 m_3^4 - 3 283 618 m_3^2 + 19 616 041 = 0
\]

namely \(m_{3,2} \approx 7.784831\). The zero of \(\delta_3(m_3) = 0\) closest to \(m_3 = 3\) is that of

\[
48 767 485 m_3^8 - 6 242 238 080 m_3^6 + 275 224 054 560 m_3^4 - 4 830 235 904 000 m_3^2 + 28 564 663 646 464 = 0
\]

namely \(m_{3,3} \approx 6.607846\). Therefore, one should first expect the collision of \(W_2\) with \(P_3\) at \(m = m_{3,3}\) (Fig. 9).

\(\square\)
We finally treat the case of the variation of the parameter directly responsible for the inter-facilities connection.

**Example 5.4.** For the configuration

\[
\begin{align*}
    P_1 &= (1, 5) & P_2 &= (2, 1) & P_3 &= (7, 2) & P_4 &= (6, 7) \\
    m_1 &= 3 & m_2 &= 2 & m_3 &= 3 & m_4 &= 4
\end{align*}
\]

find the loci of the facilities \( W_1 \) and \( W_2 \) under variation of the weight \( m \) within the interval \([2, 4.8]\).

**Solution.** When the weight \( m \) increases starting from \( m = 4 \), the facilities \( W_1 \) and \( W_2 \) approach each other along the curves given in a parametric form as \((x_*(m), y_*(m))\) and \((x_{**}(m), y_{**}(m))\) correspondingly (Fig. 10). Using the resultant computation techniques, one can eliminate the parameter \( m \) and obtain the representation for both curves in an implicit form \( \Phi(x, y) = 0 \) with a polynomial \( \Phi(x, y) \). We failed to deduce a general form for \( \Phi(x, y) \) for an arbitrary configuration (i.e., the counterpart of formula (5.2)). As for the configuration of the present example, the trajectory of \( W_1 \) follows the branch of the 12th degree curve.
Due to (4.35), the trajectories of $W_1$ and $W_2$ meet when $m$ coincides with a zero of the equation $\delta(m) = 0$. The latter can be reduced to an algebraic one

$$24505 m^{20} - 3675750 m^{18} + 214114901 m^{16} - 6100395704 m^{14} + 88231771774 m^{12} - 596555669836 m^{10} + 1454634503494 m^8 - 2224914338408 m^6 + 13361952747497 m^4 - 3693301029102 m^2 + 25596924755077 = 0$$

with a (closest to $m = 4$) zero $m_{0,1} \approx 4.326092$. The collision point $W$ has its coordinates $(x_*, y_*)$ satisfying the 10th degree algebraic equations. Thus, for instance, $x_*$ is a zero of the (irreducible over $\mathbb{Z}$) equation

$$26172883257245641 x^{10} - 293131285793078989 x^9 + 13307558344313669247 x^8 - 96515684969701735656 x^7 + 29971992700274443198 x^6 - 615788986911179454876 x^5 - 10080764503138660399742 x^4 + 47075811782361663673544 x^3 - 125596018219466046236391 x^2 + 194932824845656435067806 x - 136815745438565609528929 = 0,$$ and $x_* \approx 4.537574$. The point $W \approx (4.537574, 4.565962)$ yields the solution to the unifacility Weber problem (1.1) for the configuration \( \{(P_1, m_1), (P_2, m_2), (P_3, m_3), (P_4, m_4)\} \). This scenario demonstrates a paradoxical phenomenon: the weight $m$ increase forces the facilities to a collision, i.e. to a network configuration where its influence disappears completely.
When $m$ decreases from $m = 4$, the facility $W_1$ moves towards $P_1$ while $W_2$ moves towards $P_4$. The first drift is faster than the second one: $W_1$ approaches $P_1$ when $m$ coincides with a zero of the equation $\delta_1(m) = 0$. The latter can be reduced to an algebraic one

$$377145 m^{12} - 15186678 m^{10} + 245711056 m^8 - 1983425640 m^6 + 8079368573 m^4 - 14857953930 m^2 + 8631109474 = 0$$

with a zero $m_{0,1} \approx 3.145546$.

Let us finally watch the dynamics of the cost (4.12) of the optimal network when $m$ increases.

In Fig. 11 one may notice that maximum value of $\mathcal{C}(m)$ is attained at the zero $m_{0,1} \approx 4.326092$ of $\delta(m)$. 

\[\square\]
Theorem 5.4. In the case of existence of the bifacility network, it is less costly than the unifacility one.

Proof. If the cost (4.12) is considered as the function of the configuration parameters then the following identities are valid:

\[
\frac{\partial C^2}{\partial m_1} = \frac{m_1 \delta_1}{m^2 \sqrt{k_{12}}}, \quad \frac{\partial C^2}{\partial m_2} = \frac{m_2 \delta_2}{m^2 \sqrt{k_{12}}}, \quad \frac{\partial C^2}{\partial m_3} = \frac{m_3 \delta_3}{m^2 \sqrt{k_{34}}}, \quad \frac{\partial C^2}{\partial m_4} = \frac{m_4 \delta_4}{m^2 \sqrt{k_{34}}} \quad \text{and} \quad \frac{\partial C^2}{\partial m} = \frac{\delta}{2m^3}.
\]

The last one results in

\[
\frac{\partial C}{\partial m} = \frac{\delta}{4m^3 C}.
\]

Therefore for any specialization of the weights \(\{m_j\}_{j=1}^4\), the function \(C(m)\) increases to its maximal value at the positive zero of \(\delta(m)\). □

Compared with the previous examples, in the solution of Example 5.4 one cannot expect the coordinates of the point \(W\) to be expressed by radicals since the degrees of the resulted equations exceed 4. Although this correlates somehow with the result by Bajaj [1] that the unifacility Weber problem for the case of \(n > 3\) terminals is generically not solvable by radicals, further investigation of solvability by radicals in the general case should be carried out.

The empirical data obtained in the present section allows one to conclude that there are two possible ways of changing the bifacility topology of the optimal network to the unifacility one under variation of a configuration parameter. Collision of two facilities results in the appearance of a single facility with a valency equal to 4. Collision of a facility with a terminal or the scenario similar to that outlined in Example 5.3 keeps the valency of the remaining facility equal to 3.

6. Five Terminals

In order to extend an analytical approach developed in Section 4 to the multifacility Weber problem, we first demonstrate here an alternative approach for solution of the four-terminal problem (1.2). It is based on the reduction of this problem to the pair of the three-terminal Weber problems. We will utilize abbreviations \(\{4t2f\}\) and \(\{3t1f\}\) for the corresponding problems.

Assume that solution for the \(\{4t2f\}\)-Weber problem (1.2) exists. Then the system of equations (4.24)-(4.27) providing the coordinates of the facilities could be split into two subsystems. Comparing equations (4.24) and (4.25) with (2.3) and (2.4) permits one to claim that the optimal facility \(W_1\) coincides with its counterpart for the \(\{3t1f\}\)-Weber problem for the configuration \(\{(P_1 P_2 W_2 m_1 m_2 m)\}\). A similar statement is also valid for the facility \(W_2\), i.e. it is the solution to the Weber problem for the configuration \(\{(P_3 P_4 W_1 m_3 m_4 m)\}\). From this point of view, it looks like the four-terminal Weber problem can be reduced to the pair of the three-terminal ones. However, this reduction should be modified since the loci of the facilities \(W_2\) or \(W_1\) remain still undetermined. The result of Theorem 2.3 permits one to replace these facilities by those with known positions.
Theorem 6.1. If the solution to the \{4t2f\}-Weber problem (1.2) exists then the facility \( W_2 \) coincides with the solution to the \{3t1f\}-Weber problem for the configuration \( \left\{ \frac{P_3}{m_3} \left| \frac{P_4}{m_4} \right| Q_1 \frac{m}{m} \right\} \). Here \( Q_1 \) is the point defined by (2.5) with the substitution \( m_3 \to m \). A similar statement is valid for the terminal \( W_1 \): it coincides with the solution to the \{3t1f\}-Weber problem for the configuration \( \left\{ \frac{P_1}{m_1} \left| \frac{P_2}{m_2} \right| Q_2 \frac{m}{m} \right\} \) where the coordinates for \( Q_2 \) are obtained via (2.5) where the substitution for the indices \( 1 \to 3, 2 \to 4 \) is made together with \( m_3 \to m \).

This theorem claims that the four-terminal Weber problem can be solved by its reduction to the three-terminal counterpart via a formal replacement of a pair of the real terminals, say \( P_3 \) and \( P_4 \), by a single \textit{phantom} terminal \( Q_2 \). This reduction algorithm is similar to that used for construction of the Steiner minimal tree (firstly introduced by Gergonne as early as in 1810, and 150 years later rediscovered by Melzak [3]). The approach can be evidently extended to the general case of \( n \geq 5 \) terminals as will be clarified by the following example.

Example 6.1. Find the coordinates of the facilities \( W_1, W_2, W_3 \) that minimize the cost

\[
m_1|P_1W_1| + m_2|P_2W_1| + m_3|P_3W_2| + m_4|P_4W_2| + m_5|P_5W_3| + m_{1,3}|W_1W_3| + m_{2,3}|W_2W_3| \tag{6.1}
\]

for the following configuration:

\[
\begin{align*}
P_1 &= (1, 6) & P_2 &= (5, 1) & P_3 &= (11, 1) & P_4 &= (15, 3) & P_5 &= (7, 11) \\
m_1 &= 10 & m_2 &= 9 & m_3 &= 8 & m_4 &= 7 & m_5 &= 13 \\
& & & & & & m_{1,3} = 10 & m_{2,3} = 12
\end{align*}
\]

Solution. (I) To reduce the problem to the \{4t2f\}-case, replace a pair of the terminals \( P_1 \) and \( P_2 \) by the point \( Q_1 \) defined by the formula (2.5) where the substitution \( m_3 \to m_{1,3} \) is made.

\[
Q_1 = \left( -\frac{9}{40} \sqrt{319} + \frac{131}{50}, -\frac{9}{50} \sqrt{319} + \frac{159}{40} \right) \approx (-1.398628, 0.760097).
\]

(II) Solve the \{4t2f\}-problem for the configuration \( \left\{ \frac{P_5}{m_5} \left| \frac{P_3}{m_3} \right| \frac{P_4}{m_4} \frac{\tilde{m}_{1,3}}{\tilde{m}_{1,3}} \right\} \) via formulae (4.8)–(4.11) and obtain the coordinates for the facilities

\[
W_2 \approx (10.441211, 3.084533) \quad \text{and} \quad W_3 \approx (7.191843, 5.899268).
\]

(III) Return \( P_1 \) and \( P_2 \) instead of \( Q_1 \) and solve the \{3t1f\}-Weber problem for the configuration \( \left\{ \frac{P_1}{m_1} \left| \frac{P_2}{m_2} \right| \frac{W_3}{\tilde{m}_{1,3}} \right\} \) by the formulae of Theorem 2.1: \( W_1 \approx (4.750727, 4.438893) \) (Fig. 12). We emphasize, that the coordinates of the facilities can be expressed by radicals similar to the following expression for the cost of the network

\[
\mathcal{E} = \frac{\sqrt{10}}{80} \left( 4158 \sqrt{87087} + 773402 \sqrt{231} + 271890 \sqrt{319} + 247470 \sqrt{143} + 326403 \sqrt{609} \right. \\
\left. + 104181 \sqrt{273} - 4455 \sqrt{377} + 15216515 \right)^{1/2} \approx 267.229644.
\]
The reduction procedure illuminated in the previous example, in the general case should be
accompanied by the conditions similar to those from Theorem 4.1.

We conclude this section with formulation of two problems for further research. The first one,
for simplicity, is given in terms of the last example:

Find the pair of the weights \((\tilde{m}_{1,3}, \tilde{m}_{2,3})\) with the minimal possible sum \(\tilde{m}_{1,3} + \tilde{m}_{2,3}\) such that the
corresponding optimal network contains a single facility.

The second problem consists in proving (or disproving) of the following
Conjecture. The \(\{n \text{ terminals} \ell \text{ facilities}\}\)-Weber problem (1.3) is solvable by radicals if
\(\ell = n - 2\) and the valency of every facility in the network equals 3.

7. Conclusions

We provide an analytical solution to the bifacility Weber problem (1.2) approving thereby the
geometric solution by G.Pick. We also formulate the conditions for the existence of the network
in a prescribed topology and analyze the potential scenarios of its degeneracy under variation of
parameters.

Several problems for further investigations are mentioned in Sections 5 and 6. One extra problem
concerns the treatment of distance depending functions like \(F_L(P) = \sum_{j=1}^{n} m_j |PP_j|^L\) with different
exponents \(L \in \mathbb{Q} \setminus 0\). The choice \(L = -1\) corresponds to Newton or Coulomb potential. It turns out
that the stationary point sets of all the functions \( \{F_L\} \) can be treated in the universal manner [10]. We hope to discuss these issues in the foregoing papers.

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