 REPRESENTATIONS OF PRINCIPAL $W$-ALGEBRA FOR THE SUPeralgebra $Q(n)$

ELENA POLETAЕVA AND VERA SERGANOVA

ABSTRACT. We classify irreducible representations of finite $W$-algebra of the queer Lie superalgebra $Q(n)$ associated with the principal nilpotent coadjoint orbits.

1. INTRODUCTION

In the classical case a finite $W_e$-algebra is a quantization of the Slodowy slice to the adjoint orbit of a nilpotent element $e$ of a semisimple Lie algebra $g$. Finite-dimensional simple $W_e$-modules are used for classification of primitive ideals of $U(g)$, [7, 8, 9].

In the supercase the theory of the primitive ideals is even more complicated, [3]. It is interesting to generalize Losev’s result to the supercase. One step in this direction is to study representations of finite $W$-algebras for a Lie superalgebra $g$. In the case when $g = gl(m|n)$ and $e$ is the even principal nilpotent, Brown, Brundan and Goodwin classified irreducible representation of $W_e$ and explored the connection with the category $O$ for $g$ using coinvariants functor, [1, 2].

In this paper, we study representations of finite $W$-algebras for the Lie superalgebra $Q(n)$ associated with the principal even nilpotent coadjoint orbit. Note that in this case the Cartan subalgebra $h$ of $g = Q(n)$ is not abelian and contains a non-trivial odd part. By our previous results ([12]), we realize $W$ as a subalgebra of the universal enveloping algebra $U(h)$. The main result of the paper is a classification of simple $W$-modules (they are all finite-dimensional by [12]). The technique we use is completely different from one used in [2] due to the lack of triangular decomposition of $W$ in our case. Instead we can describe the restriction of simple $U(h)$-modules to $W$ and prove that any simple $W$-module occurs as a constituent of this restriction.

Note that our results should have applications to classification of simple modules for super Yangians of type $Q$. We also plan in a subsequent paper to study the coinvariants functor from the category $O$ for $Q(n)$ to the category of $W$-modules.
2. Notations and preliminary results

We work in the category of super vector spaces over $\mathbb{C}$. All tensor products are over $\mathbb{C}$ unless specified otherwise. By $\Pi$ we denote the functor of parity switch $\Pi(X) = X \otimes \mathbb{C}^{[0]}$.

Recall that if $X$ is a simple finite-dimensional $\mathcal{A}$-module for some associative superalgebra $\mathcal{A}$, then $\text{End}_{\mathcal{A}}(X) = \mathbb{C}$ or $\text{End}_{\mathcal{A}}(X) = \mathbb{C}[\epsilon]/(\epsilon^2 - 1)$, where the odd element $\epsilon$ provides an $\mathcal{A}$ isomorphism $X \to \Pi(X)$. We say that $X$ is of M-type in the former case and of Q-type in the latter (see [6, 4]).

If $X$ and $Y$ are two simple modules over associative superalgebras $\mathcal{A}$ and $\mathcal{B}$, we define the $\mathcal{A} \otimes \mathcal{B}$-module $X \boxtimes Y$ as the usual tensor product if at least one of $X$, $Y$ is of M-type and the tensor product over $\mathbb{C}[\epsilon]$ if both $X$ and $Y$ are of Q-type.

In this paper we consider the Lie superalgebra $\mathfrak{g} = Q(n)$ defined as follows (see [5]). Equip $\mathbb{C}^{n|n}$ with the odd operator $\zeta$ such that $\zeta^2 = -\text{Id}$. Then $Q(n)$ is the centralizer of $\zeta$ in the Lie superalgebra $\mathfrak{gl}(n|n)$. It is easy to see that $Q(n)$ consists of matrices of the form $\begin{pmatrix} A & B \\ B & A \end{pmatrix}$ where $A, B$ are $n \times n$-matrices. We fix the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ to be the set of matrices with diagonal $A$ and $B$. By $\mathfrak{n}^+$ (respectively, $\mathfrak{n}^-$) we denote the nilpotent subalgebras consisting of matrices with strictly upper triangular (respectively, low triangular) $A$ and $B$. The Lie superalgebra $\mathfrak{g}$ has the triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ and we set $\mathfrak{b} = \mathfrak{n}^+ \oplus \mathfrak{h}$.

Denote by $W$ the finite $W$-algebra associated with a principal even nilpotent element $\varphi$ in the coadjoint representation of $Q(n)$. Let us recall the definition (see [14]). Let $\{e_{i,j}, f_{i,j} \mid i, j = 1, \ldots, n\}$ denote the basis consisting of elementary even and odd matrices. Choose $\varphi \in \mathfrak{g}^*$ such that

$$\varphi(f_{i,j}) = 0, \quad \varphi(e_{i,j}) = \delta_{i,j+1}.$$ 

Let $I_\varphi$ be the left ideal in $U(\mathfrak{g})$ generated by $x - \varphi(x)$ for all $x \in \mathfrak{n}^-$. Let $\pi : U(\mathfrak{g}) \to U(\mathfrak{g})/I_\varphi$ be the natural projection. Then

$$W = \{\pi(y) \in U(\mathfrak{g})/I_\varphi \mid \text{ad}(x)y \in I_\varphi \text{ for all } x \in \mathfrak{n}^\text{-}\}.$$ 

Using identification of $U(\mathfrak{g})/I_\varphi$ with the Whittaker module $U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} \mathbb{C}_\varphi \simeq U(\mathfrak{b}) \otimes \mathbb{C}$ we can consider $W$ as a subalgebra of $U(\mathfrak{b})$. The natural projection $\vartheta : U(\mathfrak{b}) \to U(\mathfrak{h})$ with the kernel $\mathfrak{n}^+U(\mathfrak{b})$ is called the Harish-Chandra homomorphism. It is proven in [12] that the restriction of $\vartheta$ to $W$ is injective.

The center of $U(\mathfrak{g})$ is described in [16]. Set

$$\xi_i := (-1)^{i+1} f_{i,i}, \quad x_i := \xi_i^2 = e_{i,i},$$

then

$$U(\mathfrak{b}) \simeq \mathbb{C}[\xi_1, \ldots, \xi_n]/(\xi_i \xi_j + \xi_j \xi_i)_{i < j \leq n}.$$ 

1There is a unique open orbit in the nilpotent cone of the coadjoint representation, elements of this orbit are called principal.
The center of $U(\mathfrak{h})$ coincides with $\mathbb{C}[x_1, \ldots, x_n]$ and the image of the center of $U(\mathfrak{g})$ under the Harish-Chandra homomorphism is generated by the polynomials $p_{2k+1} = x_1^{2k+1} + \cdots + x_n^{2k+1}$ for all $k \in \mathbb{N}$. These polynomials are called $Q$-symmetric polynomials.

In [12] we proved that the center $Z$ of $W$ coincides with the image of the center of $U(\mathfrak{g})$ and hence can be also identified with the ring of $Q$-symmetric polynomials.

3. The structure of $W$-algebra

Using Harish-Chandra homomorphism we realize $W$ as a subalgebra in $U(\mathfrak{h})$. It is shown in [12] that $W$ has $n$ even generators $z_0, \ldots, z_{n-1}$ and $n$ odd generators $\phi_0, \ldots, \phi_{n-1}$ defined as follows. For $k \geq 0$ we set

\begin{equation}
\phi_0 := \sum_{i=1}^{n} \xi_i, \quad \phi_k := T^k(\phi_0),
\end{equation}

where the matrix of $T$ in the standard basis $\xi_1, \ldots, \xi_n$ has 0 on the diagonal and

\begin{equation}
t_{ij} := \begin{cases} x_j & \text{if } i < j, \\ -x_j & \text{if } i > j. \end{cases}
\end{equation}

For odd $k \leq n - 1$ we define

\begin{equation}
z_k := [ \sum_{i_1 \geq i_2 \geq \cdots \geq i_k+1} (x_{i_1} + (-1)^k \xi_{i_1}) \cdots (x_{i_k} - \xi_{i_k})(x_{i_{k+1}} + \xi_{i_{k+1}})]_{\text{even}},
\end{equation}

and for even $k \geq 0$ we set

\begin{equation}
z_k := \frac{1}{2}[\phi_0, \phi_k].
\end{equation}

Let $W_0 \subset W$ be the subalgebra generated by $z_0, \ldots, z_{n-1}$. By [12] Theorem 6.6, $W_0$ is isomorphic to the polynomial algebra $\mathbb{C}[z_0, \ldots, z_{n-1}]$. Furthermore there are the following relations

\begin{equation}
[\phi_i, \phi_j] = \begin{cases} (-1)^i z_{i+j} & \text{if } i + j \text{ is even} \\ 0 & \text{if } i + j \text{ is odd} \end{cases}
\end{equation}

Define the $\mathbb{Z}$-grading on $U(\mathfrak{h})$ by setting the degree of $\xi_i$ to be 1. It induces the filtration on $W$, for every $y \in W$ we denote by $\bar{y}$ the term of the highest degree.

Note that for even $k$, we have $z_k = \bar{z}_k$. Moreover, $z_k$ is in the image under the Harish-Chandra map of the center of the universal enveloping algebra $U(Q(n))$. Therefore by [16] $z_{2p}$ is a $Q$-symmetric polynomial in $\mathbb{C}[x_1, \ldots, x_n]$ of degree $2p + 1$. For example,

$$z_0 = x_1 + \cdots + x_n, \quad z_2 = \frac{1}{3} \left( (x_1^3 + \cdots + x_n^3) - (x_1 + \cdots + x_n)^3 \right).$$
For odd \( k \) the leading term is given by the complete symmetric polynomial

\[
\bar{z}_k = \sum_{i_1 \geq i_2 \geq \ldots \geq i_{k+1}} x_{i_1} \cdots x_{i_{k+1}}.
\]

**Lemma 3.1.**

1. \( \text{gr } W_0 \) is isomorphic to the algebra of symmetric polynomials \( \mathbb{C}[x_1, \ldots, x_n]^{S_n} = \mathbb{C}[\bar{z}_0, \ldots, \bar{z}_{n-1}] \) and the degree of \( \bar{z}_k \) is \( 2k + 2 \);
2. \( U(\mathfrak{h}) \) is a free right \( W_0 \)-module of rank \( 2^n n! \).

**Proof.** Since \( \bar{z}_0, \ldots, \bar{z}_{n-1} \) are algebraically independent generators of \( \mathbb{C}[x_1, \ldots, x_n]^{S_n} \) we obtain (1).

It is well-known fact that \( \mathbb{C}[x_1, \ldots, x_n] \) is a free \( \mathbb{C}[x_1, \ldots, x_n]^{S_n} \)-module of rank \( n! \), see, for example, [17] Chapter 4. Since \( U(\mathfrak{h}) \) is a free \( \mathbb{C}[x_1, \ldots, x_n] \)-module of rank \( 2^n \) we get that \( U(\mathfrak{h}) \) is a free \( \mathbb{C}[x_1, \ldots, x_n]^{S_n} \)-module of rank \( m = 2^n n! \). Let us choose a homogeneous basis \( b_1, \ldots, b_m \) of \( U(\mathfrak{h}) \) over \( \mathbb{C}[x_1, \ldots, x_n]^{S_n} \). We claim that it is a basis of \( U(\mathfrak{h}) \) as a right module over \( W_0 \). Indeed, let us prove first the linear independence. Suppose

\[
\sum_{j=1}^{m} b_j y_j = 0
\]

for some \( y_j \in W_0 \). Let \( k = \max\{ \deg y_j + \deg b_j | j = 1, \ldots, m \} \). If \( J = \{ j | \deg y_j + \deg b_j = k \} \) we have \( \sum_{j \in J} b_j y_j = 0 \). By above this implies \( y_j = 0 \) for all \( j \in J \) and we obtain all \( y_j = 0 \). On the other hand, it follows easily by induction on degree that \( U(\mathfrak{h}) = \sum_{j=1}^{m} b_j W_0 \). The proof of (2) is complete. \( \square \)

Consider \( U(\mathfrak{h}) \) as a free \( U(\mathfrak{h}_0) \)-module and let \( W_1 \) denote the free \( U(\mathfrak{h}_0) \)-submodule generated by \( \xi_1, \ldots, \xi_n \). Then \( W_1 \) is equipped with \( U(\mathfrak{h}_0) \)-valued symmetric bilinear form \( B(x, y) = [x, y] \).

**Lemma 3.2.** Let \( p(x_1, \ldots, x_n) := \prod_{i<j} (x_i + x_j) \) and \( \Gamma \) denotes the Gramm matrix \( B(\phi_i, \phi_j) \). Then \( \det \Gamma = cp^2 x_1 \cdots x_n \), where \( c \) is a non-zero constant.

**Proof.** Recall that \( \phi_k = T^k \phi_0 \). Since the matrix of the form \( B \) in the basis \( \xi_1, \ldots, \xi_n \) is the diagonal matrix \( C = \text{diag}(x_1, \ldots, x_n) \), then \( \Gamma = Y^t CY \), where \( Y \) is the square matrix such that \( \phi_i = \sum_{j=1}^{n} y_j \xi_j \). Hence \( \det \Gamma = x_1 \cdots x_n \det Y^2 \). Since \( B(\phi_i, \phi_j) \) is a symmetric polynomial in \( x_1, \ldots, x_n \), the determinant of \( \Gamma \) is also a symmetric polynomial. The degree of this polynomial is \( n^2 \). Therefore it suffices to prove that \( (x_1 + x_2)^2 \) divides \( \det \Gamma \), or equivalently \( x_1 + x_2 \) divides \( \det Y \). In other words, we have to show that if \( x_1 = -x_2 \), then \( \phi_0, \ldots, \phi_{n-1} \) are linearly dependent. Indeed, one can easily see from the form of \( T \) that the first and the second coordinates of \( T^k \phi_0 \) coincide, hence \( \phi_0, T \phi_0, \ldots, T^{n-1} \phi_0 \) are linearly dependent. \( \square \)
We also will use another generators in $W$ introduced in [13], Corollary 5.15:

\begin{equation}
(3.6) \quad u_k(0) := \left[ \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} (x_{i_1} + (-1)^{k+1} \xi_{i_1}) \cdots (x_{i_{k-1}} - \xi_{i_{k-1}})(x_{i_k} + \xi_{i_k}) \right]_{\text{even}},
\end{equation}

\begin{equation}
(3.7) \quad u_k(1) := \left[ \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} (x_{i_1} + (-1)^{k+1} \xi_{i_1}) \cdots (x_{i_{k-1}} - \xi_{i_{k-1}})(x_{i_k} + \xi_{i_k}) \right]_{\text{odd}}.
\end{equation}

For convenience we assume $u_k(0) = u_k(1) = 0$ for $k > n$.

Let $i + j = n$. We have the natural embedding of the Lie superalgebras $Q(i) \oplus Q(j) \hookrightarrow Q(n)$. If $\mathfrak{h}_r$ denotes the Cartan subalgebra of $Q(r)$, the above embedding induces the isomorphism

\begin{equation}
(3.8) \quad U(\mathfrak{h}) \simeq U(\mathfrak{h}_i) \otimes U(\mathfrak{h}_j).
\end{equation}

The following lemma implies that we have also the embedding of $W$-algebras.

**Lemma 3.3.** Let $i + j = n$. Then $W$ is a subalgebra in the tensor product $W^i \otimes W^j$, where $W^r \subset U(\mathfrak{h}_r)$ denotes the $W$-algebra for $Q(r)$.

**Proof.** Introduce generators in $W^i$ and $W^j$:

\begin{equation}
(3.9) \quad u_k^+(0) := \left[ \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq i} (x_{i_1} + (-1)^{k+1} \xi_{i_1}) \cdots (x_{i_{k-1}} - \xi_{i_{k-1}})(x_{i_k} + \xi_{i_k}) \right]_{\text{even}},
\end{equation}

\begin{equation}
(3.10) \quad u_k^-(1) := \left[ \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq j} (x_{i_1} + (-1)^{k+1} \xi_{i_1}) \cdots (x_{i_{k-1}} - \xi_{i_{k-1}})(x_{i_k} + \xi_{i_k}) \right]_{\text{odd}}.
\end{equation}

Then for $d, e, f \in \mathbb{Z}/2\mathbb{Z}$ we have

\begin{equation}
(3.11) \quad u_k(d) = \sum_{e+f=d, a+b=k} (-1)^{eb}u_a^+(e)u_b^-(f).
\end{equation}

Here we assume $u_0^+(0) = 1$ and $u_0^-(1) = 0$. \hfill $\square$

**Corollary 3.4.** If $i_1 + \ldots + i_p = n$, then $W$ is a subalgebra in $W^{i_1} \otimes \ldots \otimes W^{i_p}$.

4. IRREDUCIBLE REPRESENTATIONS OF $W$

4.1. **Representations of $U(\mathfrak{h})$.** Let $s = (s_1, \ldots, s_n) \in \mathbb{C}^n$. We call $s$ regular if $s_i \neq 0$ for all $i \leq n$ and typical if $s_i + s_j \neq 0$ for all $i \neq j \leq n$.

It follows from the representation theory of Clifford algebras that all irreducible representations of $U(\mathfrak{h})$ up to change of parity can be parameterized by $s \in \mathbb{C}^n$. Indeed, let $M$ be an irreducible representation of $U(\mathfrak{h})$. By Schur’s lemma every $x_i$ acts on $M$ as a scalar operator $s_i \text{Id}$. Let $I_s$ denote the ideal in $U(\mathfrak{h})$ generated by
If $s_i = 0$, then the quotient algebra $U(\mathfrak{h})/I_s$ is isomorphic to the Clifford superalgebra $C_s$ associated with the quadratic form:

$$B_s(\xi_i, \xi_j) = \delta_{ij}s_i.$$ 

Then $M$ is a simple $C_s$-module.

The radical $R_s$ of $C_s$ is generated by the kernel of the form $B_s$. Let $m(s)$ be the number of non-zero coordinates of $s$, then $C_s/R_s$ is isomorphic to the matrix superalgebra $M(2^{m-1}|2^{m-1})$ for even $m$ and to the superalgebra $M(2^{m-1})\otimes \mathbb{C}[e]/(e^2 - 1)$ for odd $m$.

Therefore $C_s$ has one (up to isomorphism) simple $\mathbb{Z}_2$-graded module $V(s)$ of type $Q$ for odd $m(s)$, and two simple modules $V(s)$ and $IIV(s)$ of type $M$ for even $m(s)$ (see [10]). In the case when $s$ is regular, the form $B_s$ is non-degenerate and the dimension of $V(s)$ equals $2^k$, where $k = \lceil n/2 \rceil$. In general, $\dim V(s) = 2^{[m(s)/2]}$.

Consider the embedding $Q(p) \oplus Q(q) \hookrightarrow Q(n)$ for $p+q = n$ and the isomorphism (3.7). It induces an isomorphism of $U(\mathfrak{h})$-modules

$$(4.1) \quad V(s) \simeq V(s_1, \ldots, s_p) \boxtimes V(s_{p+1}, \ldots, s_n).$$

4.2. Restriction from $U(\mathfrak{h})$ to $W$. We denote by the same symbol $V(s)$ the restriction to $W$ of the $U(\mathfrak{h})$-module $V(s)$.

**Proposition 4.1.** Let $S$ be a simple $W$-module. Then $S$ is a simple constituent of $V(s)$ for some $s \in \mathbb{C}^n$.

**Proof.** Since $W_0$ is commutative and $S$ is finite-dimensional, there exists one dimensional $W_0$-submodule $\mathbb{C}_\nu \subset S$ with character $\nu$. Therefore $S$ is a quotient of $\text{Ind}_{W_0}W_0$ $\mathbb{C}_\nu$. On the other hand, the embedding $W \hookrightarrow U(\mathfrak{h})$ induces the embedding $\text{Ind}_{W_0}W_0 \mathbb{C}_\nu \hookrightarrow \text{Ind}_{W_0}U(\mathfrak{h}) \mathbb{C}_\nu$. Thus, $S$ is a simple constituent of $\text{Res}_W \text{Ind}_{W_0}U(\mathfrak{h}) \mathbb{C}_\nu$. By Lemma 3.1, $\text{Ind}_{W_0}U(\mathfrak{h}) \mathbb{C}_\nu$ is finite-dimensional, and hence has simple $U(\mathfrak{h})$-constituents isomorphic to $V(s)$ for some $s$. Hence $S$ must appear as a simple $W$-constituent of some $V(s)$. $\square$

4.3. Typical representations.

**Theorem 4.2.** If $s$ is typical, then $V(s)$ is a simple $W$-module.

**Proof.** First, we assume that $s$ is regular, i.e. $s_i \neq 0$ for all $i = 1, \ldots, n$. The specialization $x_i \mapsto s_i$ induces an injective homomorphism $\theta_s : W/(I_s \cap W) \hookrightarrow C_s$ and a specialization of the quadratic form $B \mapsto B_s$. By Lemma 3.2 $\det \Gamma(s) \neq 0$. Therefore $B_s$ is non-degenerate and $\theta_s$ is an isomorphism. Thus, $V(s)$ remains irreducible when restricted to $W$.

If $s$ is typical non-degenerate, there is exactly one $i$ such that $s_i = 0$. Let $s' = (s_1, \ldots, s_i - 1, s_{i+1}, \ldots, s_n)$. Note that $(\theta_s(\xi_i))$ is a nilpotent ideal of $C_s$ and hence $\xi_i$ acts by zero on $V(s)$. Then $V(s)$ is a simple module over the quotient $C_s' \cong C_s/(\theta_s(\xi_i))$.

\[2\text{We consider Clifford algebras as superalgebras with the natural } \mathbb{Z}_2\text{-grading.}\]
Recall $Y$ from the proof of Lemma 3.2 and let $Y'$ denote the minor of $Y$ obtained by removing the $i$-th column and the $i$-th row. Then

$$
\phi_k = \sum_{j \neq i} y_{kj} \xi_j \mod (\xi_i).
$$

Hence $\theta_s(\phi_0), \ldots, \theta_s(\phi_{n-1})$ generate $C_s' \cong C_s/(\theta_s(\xi_i))$ and the statement follows from the regular case for $n - 1$.

4.4. **Simple $W$-modules for $n = 2$.** Let $n = 2$, then by Theorem 4.2 $V(s)$ is simple as $W$-module if $s_1 \neq -s_2$. The action of $U(\mathfrak{h})$ in $V(s_1, s_2)$ is given by the following formulas in a suitable basis:

$$
\xi_1 \mapsto \begin{pmatrix} 0 & \sqrt{s_1} \\ \sqrt{s_1} & 0 \end{pmatrix}, \quad \xi_2 \mapsto \begin{pmatrix} 0 & \sqrt{s_2} \\ -\sqrt{s_2} & 0 \end{pmatrix}.
$$

Note that $W$ is generated by $\phi_0, \phi_1, z_0$ and $z_1'$, where $z_1' := u_2(0)$. Using

$$
\phi_0 = \xi_1 + \xi_2, \quad \phi_1 = x_2 \xi_1 - x_1 \xi_2, \quad z_0 = x_1 + x_2, \quad z_1' = x_1 x_2 - \xi_1 \xi_2
$$

we obtain the following formulas for the generators of $W$:

\begin{align}
(4.2) \quad & \phi_0 \mapsto \begin{pmatrix} 0 & \sqrt{s_1} + \sqrt{s_2} \\ \sqrt{s_1} - \sqrt{s_2} & 0 \end{pmatrix}, \quad \phi_1 \mapsto \begin{pmatrix} \sqrt{s_1 s_2} & \sqrt{s_2} - \sqrt{s_1} \\ \sqrt{s_2} + \sqrt{s_1} & 0 \end{pmatrix}, \\
(4.3) \quad & z_0 \mapsto (s_1 + s_2) \text{Id}, \quad z_1' \mapsto \begin{pmatrix} s_1 s_2 + \sqrt{s_1 s_2} & 0 \\ 0 & s_1 s_2 - \sqrt{s_1 s_2} \end{pmatrix}.
\end{align}

Assume that $s_1 = -s_2$. If $s_1, s_2 = 0$ then $V(s)$ is isomorphic to $\mathbb{C} \oplus \Pi \mathbb{C}$, where $\mathbb{C}$ is the trivial module. If $s_1 \neq 0$, we choose $\sqrt{s_1}, \sqrt{s_2}$ so that $\sqrt{s_2} = \sqrt{s_1} i$. Note that the choice of sign controls the choice of the parity of $V(s)$. The following exact sequence easily follows from (4.2) and (4.3):

\begin{align}
(4.4) \quad & 0 \to \Pi \Gamma_{-x^t + s_1} \to V(s) \to \Gamma_{-x^t - s_1} \to 0,
\end{align}

where $\Gamma_1$ is the simple module of dimension $(1|0)$ on which $\phi_0, \phi_1$ and $z_0$ act by zero and $z_1'$ acts by the scalar $t$. The sequence splits only in the case $s_1 = 0$, when $\Gamma_0 \cong \mathbb{C}$ is trivial. Thus, using Proposition 4.1, Theorem 4.2 and (4.4) we obtain

**Lemma 4.3.** If $n = 2$, then every simple $W$-module is isomorphic to one of the following

1. $V(s_1, s_2)$ or $\Pi V(s_1, s_2)$ for $s_1 \neq -s_2, s_1, s_2 \neq 0$;
2. $V(s, 0)$ if $s \neq 0$;
3. $\Gamma_t$ or $\Pi \Gamma_t$. 

4.5. Invariance under permutations.

**Theorem 4.4.** Let $s' = \sigma(s)$ for some permutation of coordinates.

1. If $s$ is typical, then $V(s)$ is isomorphic to $V(s')$ as a $W$-module.
2. If $s$ is arbitrary, then $[V(s)] = [V(s')]$ or $[\Pi V(s')]$, where $[X]$ denotes the class of $X$ in the Grothendieck group.

**Proof.** First, we will prove the statement for $n = 2$. Assume first that $s_2 \neq -s_1$. In this case $V(s_1, s_2)$ is a $(1|1)$-dimensional simple $W$-module.

Let

$$D = \begin{pmatrix} \sqrt{s_2} + \sqrt{s_1} & 0 \\ 0 & \sqrt{s_1} + \sqrt{s_2} \end{pmatrix}.$$ 

Then by direct computation we have

$$D\phi_0 D^{-1} = \begin{pmatrix} 0 & \sqrt{s_2} + \sqrt{s_1} \\ \sqrt{s_2} - \sqrt{s_1} & 0 \end{pmatrix}$$

and

$$D\phi_1 D^{-1} = \sqrt{s_1 s_2} \begin{pmatrix} 0 & \sqrt{s_1} - \sqrt{s_2} \\ \sqrt{s_2} + \sqrt{s_1} & 0 \end{pmatrix}.$$ 

Therefore $D$ defines an isomorphism between $V(s_1, s_2)$ and $V(s_2, s_1)$.

Now consider the case $s_1 = -s_2$. Then the structure of $V(s_1, -s_1)$ is given by the sequence (4.4). Let $V(s'_2) = V(-s_1, s_1)$, then analogously we have the exact sequence

$$0 \rightarrow \Pi\Gamma_{-s_1^2} \rightarrow V(s'_2) \rightarrow \Gamma_{-s_1^2 + s_1} \rightarrow 0.$$ 

The statement (2) now follows directly from comparison of (4.4) and (4.5). Now we will prove the statement for all $n$. Note that it suffices to consider the case of the adjacent transposition $\sigma = (i, i + 1)$.

The embedding of $Q(i-1) \oplus Q(2) \oplus Q(n-i-1)$ into $Q(n)$ provides the isomorphism

$$U(\mathfrak{h}) \simeq U(\mathfrak{h}^-) \otimes U(\mathfrak{h}^0) \otimes U(\mathfrak{h}^+),$$ 

where $\mathfrak{h}^-$, $\mathfrak{h}^0$ and $\mathfrak{h}^+$ are the Cartan subalgebras of $Q(i-1)$, $Q(2)$ and $Q(n-i-1)$ respectively. Using twice the isomorphism (4.4) we obtain the following isomorphism of $U(\mathfrak{h})$-modules

$$V(s) \simeq (V(s_1, \ldots, s_{i-1}) \boxtimes V(s_i, s_{i+1})) \boxtimes V(s_{i+2}, \ldots, s_n).$$

Suppose that $s_i \neq -s_{i+1}$. Let $D_{i,i+1} = 1 \otimes D \otimes 1$. By Corollary 3.3 we have that $W$ is a subalgebra in $W^{i-1} \otimes W^2 \otimes W^{n-i-1}$ and hence $D_{i,i+1}$ defines an isomorphism of $W$-modules $V(s)$ and $V(s')$.

If $s_i = -s_{i+1}$, then the statement follows from (4.4) and (4.5). This completes the proof of the theorem. \qed
4.6. **Construction of simple $W$-modules.** Now we give a general construction of a simple $W$-module. Let $r, p, q \in \mathbb{N}$ and $r + 2p + q = n$, $t = (t_1, \ldots, t_p) \in \mathbb{C}^p$, $t_1, \ldots, t_p \neq 0$, and $\lambda = (\lambda_1, \ldots, \lambda_q) \in \mathbb{C}^q$, $\lambda_1, \ldots, \lambda_q \neq 0$, such that $\lambda_i + \lambda_j \neq 0$ for any $1 \leq i \neq j \leq q$. Recall that by Corollary 3.3 we have an embedding $W \hookrightarrow W^r \otimes (W^2)^{\otimes p} \otimes W^q$. Set

$$S(t, \lambda) := \mathbb{C} \otimes \Gamma_{t_1} \otimes \cdots \otimes \Gamma_{t_p} \otimes V(\lambda),$$

where the first term $\mathbb{C}$ in the tensor product denotes the trivial $W^r$-module. For $q = 0$ we use the notation $S(t, 0)$.

**Lemma 4.5.** All $u_k(1)$ act by zero on $S(t, 0)$. The action of $u_k(0)$ is given by the formula

$$u_k(0) = \begin{cases} 0 & \text{for odd } k, \text{ and for } k > 2p, \\ \sigma_{k}(t_1, \ldots, t_p) & \text{for even } k, \end{cases}$$

where $\sigma_a$ denote the elementary symmetric polynomials, $0 \leq a \leq p$.

**Proof.** The first assertion is trivial. We prove the second assertion by induction on $p$. For $p = 1$ it is a consequence of the definition of $\Gamma_t$ for $Q(2)$. For $p > 1$ we consider the embedding $Q(n - 2) \oplus Q(2) \hookrightarrow Q(n)$. The formula (3.10) degenerates to

$$u_k(0) = u_k^+(0) \otimes 1 + u_{k-1}^+(0) \otimes z_0 + u_{k-2}^+(0) \otimes z_1'.$$

As $z_0$ acts by zero on $\Gamma_{t_p}$, the statement now follows from the obvious identity

$$\sigma_k(t_1, \ldots, t_p) = \sigma_k(t_1, \ldots, t_{p-1}) + t_p \sigma_{k-1}(t_1, \ldots, t_{p-1}).$$

\[\square\]

**Theorem 4.6.**

1. $S(t, \lambda)$ is a simple $W$-module;
2. Every simple $W$-module is isomorphic to $S(t, \lambda)$ up to change of parity.

**Remark 4.7.** By construction, if $q$ is odd then $S(t, \lambda)$ is of type $Q$ and of dimension $2^{2+1}$. If $q$ is even then $S(t, \lambda)$ is of type $M$ and of dimension $2^{1+2}$. Let $u_k^-(d), d \in \mathbb{Z}/2\mathbb{Z}, 1 \leq k \leq n$ be as in (3.9) where indices are taken in the interval $[n - q + 1, n]$. If $q = 0$ we set $u_k^-(0) = 1$ and $u_k^-(1) = 0$. Using Lemma 4.5 and formula (3.10) we can easily write the action of $u_k^-(d)$ in $S(t, \lambda)$ in terms of $u_k^-(d)$ after identifying $S(t, \lambda)$ with $V(\lambda)$:

$$(4.6) \quad u_k(d) = \sum_{2a + j = k} \sigma_a(t_1, \ldots, t_p) u_j^-(d),$$

From these formulas we see that $u_k^-(d)$ and $u_k(d)$ generate the same subalgebra in $\text{End}_\mathbb{C}(V(\lambda))$. By Theorem 4.2 this proves irreducibility of $S(t, \lambda)$.

To show (2) we use Proposition 4.1. Every simple $W$-module is a subquotient of $V(s)$. By Theorem 4.4 (2) we may assume that $s_1 = \cdots = s_r = 0, s_i \neq 0$ for $i > r$, $s_{r+1} = -s_{r+2}, \ldots, s_{r+2p-1} = -s_{r+2p}$. We can compute $W^r \otimes (W^2)^{\otimes p} \otimes W^q$-simple
constituents of $V(s)$. They are $S(t, \lambda)$ (up to change of parity) with $t_j = -s_{r+2j}^2 \pm s_{r+2j}$ and $\lambda_i = s_{r+2p+i}$. By (1) $S(t, \lambda)$ remains simple when restricted to $W$. Hence the statement.

4.7. **Central characters.** Recall that the center of $U(Q(n))$ coincides with the center $Z$ of $W$, see Section 2. Every $s$ defines the central character $\chi_s : Z \rightarrow \mathbb{C}$. Furthermore, Theorem 4.6 (2) implies that every simple $W$-module admits central character $\chi_s$ for some $s$. For every $s = (s_1, \ldots, s_n)$ we define the core $c(s) = (s_{i_1}, \ldots, s_{i_m})$ as a subsequence obtained from $s$ by removing all $s_j = 0$ and all pairs $(s_i, s_j)$ such that $s_i + s_j = 0$. Up to a permutation this result does not depend on the order of removing. Thus, the core is well defined up to permutation. We call $m$ the length of the core. The notion of core is very useful for describing the blocks in the category of finite-dimensional $Q(n)$-modules, see [11] and [15].

**Example 4.8.** Let $s = (1, 0, 3, -1, -1)$, then $c(s) = (3, -1)$.

The following is a reformulation of the central character description in [16].

**Lemma 4.9.** Let $s, s' \in \mathbb{C}^n$. Then $\chi_s = \chi_{s'}$ if and only if $s$ and $s'$ have the same core (up to permutation).

It follows from Lemma 4.9 that the core depends only on the central character $\chi_s$, we denote it $c(\chi)$. By Theorem 4.4 we obtain the following.

**Corollary 4.10.** Let $\chi : Z \rightarrow \mathbb{C}$ be a central character with core $c(\chi)$ of length $m$. Then $W^m$-module $V(c(\chi))$ is well-defined. From now on we denote it by $V(\chi)$ and call it the core representation.

The category $W \mod$ of finite dimensional $W$-modules decomposes into direct sum $\bigoplus W^\chi \mod$, where $W^\chi \mod$ is the full subcategory of modules admitting generalized central character $\chi$.

**Lemma 4.11.** A simple $W$-module $S$ belongs to $W^\chi \mod$ if and only if it is isomorphic to $S(t, \lambda)$ with $\lambda = c(\chi)$.

*Proof.* We have to compute the central character of $S(t, \lambda)$. For a $Q$-symmetric polynomial $p_k = x_1^{2k+1} + \cdots + x_n^{2k+1}$ we have $p_k(t, \lambda) = \lambda_1^{2k+1} + \cdots + \lambda_q^{2k+1}$. Since $p_k$ generate the center of $W$ the statement follows.

**Proposition 4.12.** Two simple modules $S(t, \lambda)$ and $S(t', \lambda')$ are isomorphic if and only if $t' = \sigma(t)$ and $\lambda' = \tau(\lambda)$ for some $\sigma \in S_p$ and $\tau \in S_q$.

*Proof.* First, (4.6) and Theorem 4.4 imply the “if” statement. To prove the “only if” statement, assume that $S(t, \lambda)$ and $S(t', \lambda')$ are isomorphic. Then these modules admit the same central character. Therefore by Lemma 4.11 $\lambda' = \tau(\lambda)$ for some $\tau \in S_q$. Hence without loss of generality we may assume that $\lambda' = \lambda$. Denote by $tr x$ and $tr' x$ the trace of $x \in W$ in $S(t, \lambda)$ and $S(t', \lambda)$ respectively. Then we must have $tr u_k(0) = tr' u_k(0)$. 


Using the formula (4.6) we get

\[ \text{tr} u_k(0) = \sum_{2a+j=k} \sigma_a(t_1, \ldots, t_p) \text{tr}_{V(\lambda)} u_j^-(0), \]

\[ \text{tr'} u_k(0) = \sum_{2a+j=k} \sigma_a(t'_1, \ldots, t'_p) \text{tr}_{V(\lambda)} u_j^-(0). \]

Let \( b_j := \text{tr}_{V(\lambda)} u_j^-(0) \). Then the above implies

\[ \sigma_a(t_1, \ldots, t_p)b_0 + \sigma_{a-1}(t_1, \ldots, t_p)b_2 + \cdots + \sigma_0(t_1, \ldots, t_p)b_{2a} = \]

\[ \sigma_a(t'_1, \ldots, t'_p)b_0 + \sigma_{a-1}(t'_1, \ldots, t'_p)b_2 + \cdots + \sigma_0(t'_1, \ldots, t'_p)b_{2a}, \]

where we assume \( b_i = 0 \) for \( i > q \). Since \( b_0 = \text{dim} V(\lambda) \neq 0 \) the above equations imply \( \sigma_a(t_1, \ldots, t_p) = \sigma_a(t'_1, \ldots, t'_p) \) for all \( a = 1, \ldots, p \). Therefore \( t' = \sigma(t) \) for some \( \sigma \in S_p \). \( \square \)

We denote by \( \mathcal{P}^l \) the subcategory of \( W^l \)-modules which admit trivial generalized central character.

**Lemma 4.13.** Let \( \chi : Z \to \mathbb{C} \) be a central character with core \( c(\chi) \) of length \( m \). Then the functor \( W^{n-m} - \text{mod} \to W - \text{mod} \) defined by \( F(M) = \text{Res}_W (M \otimes V(\chi)) \) restricts to the functor \( \Phi : \mathcal{P}^{n-m} \to W^\chi - \text{mod} \). Furthermore, \( \Phi \) is an exact functor which sends a simple object to a simple object.

**Proof.** The first assertion is immediate consequence of Lemma 4.11 and the second follows from the construction of \( S(t, \lambda) \). \( \square \)

**Conjecture 4.14.** The functor \( \Phi : \mathcal{P}^{n-m} \to W^\chi - \text{mod} \) defines an equivalence of categories.

**Acknowledgments**

This work was supported by a grant from the Simons Foundation (#354874, Elena Poletaeva) and the NSF grant (DMS-1701532, Vera Serganova). We would like to thank V. G. Kac for useful comments.

**References**

1. J. Brown, J. Brundan, S. Goodwin, Principal \( W \)-algebras for \( GL(m|n) \), *Algebra Numb. Theory* 7 (2013), 1849–1882.
2. J. Brundan, S. Goodwin, Whittaker coinvariants for \( GL(m|n) \), arXiv:1612.08152v2.
3. K. Coulembier, I. Musson, The primitive spectrum for \( gl(m|n) \), *Tohoku Math. J.* (2) 70 (2018), no. 2, 225–266.
4. S.-J. Cheng W. Wang, Dualities and representations of Lie superalgebras. Graduate Studies in Math., *Amer. Math. Soc.* 144 (2012), Providence, RI.
5. V. G. Kac, Lie superalgebras, *Adv. Math.* 26 (1977) 8–96.

\[ ^3 \text{We consider here the usual exterior tensor product in contrast with } \boxtimes \]
6. V. G. Kac, Representations of classical Lie superalgebras, *Lecture Notes in Math.* 676 (1978) 597–626.
7. I. Losev, Finite $W$-algebras, *Proceedings of the International Congress of Mathematicians.* Volume III, 1281–1307, Hindustan Book Agency, New Delhi, 2010. arXiv:1003.5811v1.
8. I. Losev, Quantized symplectic actions and $W$-algebras, *J. Amer. Math. Soc.* 23 (2010) 35–59.
9. I. Losev, Finite-dimensional representations of $W$-algebras, *Duke Math. J.* 159 (2011), 99–143.
10. E. Meinrenken, Clifford algebras and Lie theory, *Surveys in Mathematics* 58, Springer, Heidelberg, 2013.
11. I. Penkov, Characters of typical irreducible finite-dimensional $q(n)$-modules, *Funktional. Anal. i Prilozhen.* 20 (1986), 37–45.
12. E. Poletaeva, V. Serganova, On Kostant’s theorem for the Lie superalgebra $Q(n)$. *Adv. Math.* 300 (2016), 320–359. arXiv:1403.3866v1.
13. E. Poletaeva, V. Serganova, On the finite $W$-algebra for the Lie superalgebra $Q(n)$ in the non-regular case. *J. Math. Phys.* 58 (2017), no. 11, 111701. arXiv:1705.10200
14. A. Premet, Special transverse slices and their enveloping algebras, *Adv. Math.* 170 (2002) 1–55.
15. V. Serganova, Finite-dimensional representations of algebraic supergroups, *Proceedings of ICM*, V. 1. 603–632, Kyung Moon Sa, Seoul 2014.
16. A. Sergeev, The centre of enveloping algebra for Lie superalgebra $Q(n,\mathbb{C})$, *Lett. Math. Phys.* 7 (1983) 177–179.
17. T. A. Springer, Invariant theory. *Lecture Notes in Math.*, 585, (1977).

School of Mathematical and Statistical Sciences, University of Texas Rio Grande Valley, Edinburg, TX 78539

E-mail address: elena.poletaeva@utrgv.edu

Dept. of Mathematics, University of California at Berkeley, Berkeley, CA 94720

E-mail address: serganov@math.berkeley.edu