COMPOSITION OPERATORS ON THE DISCRETE ANALOGUE
OF GENERALIZED HARDY SPACE ON HOMOGENOUS TREES

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Abstract. In this paper, we study the basic properties such as boundedness and
compactness of composition operators on discrete analogue of generalized Hardy
space defined on a homogeneous rooted tree. Also, we compute the operator norm
of composition operator when inducing symbol is automorphism of a homogenous
tree.

1. Introduction

Let Ω be a nonempty set and X be a complex Banach space of complex valued
functions defined on Ω. For a self map φ of Ω, the composition operator $C_\phi$ induced
by the symbol φ is defined as

$$C_\phi(f) = g \quad \text{where} \quad g(x) = f(\phi(x)) \quad \text{for all} \quad x \in \Omega \quad \text{and} \quad f \in X.$$ 

In the classical case, Ω is the unit disk $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ and the choices
for X are analytic functions spaces, eg. the Hardy space $H^p$, the Bergman space $A^p$,
the Bloch space $\mathcal{B}$, etc. The study of composition operators on various analytic
function spaces defined on $\mathbb{D}$ is well known. There are excellent books on composition
operators, see [10, 12, 13] and the references therein. The approach in the first two
books [10, 12] are function theoretic whereas [13] deals in measure theoretic point
of view. Also, there a number of articles dealing with composition operators on
different transform spaces, see for example [1, 2, 7]. In this article, Ω will be a
homogeneous rooted tree and X the discrete analogue of generalized Hardy space
introduced in [11].

In the recent years, there has been a great interest in studying operator theory on
discrete structure such as graphs, in particular on an infinite tree graph [3, 4, 5, 8, 9].
In [6], Colonna et al. studied composition operators on Lipschitz functions on a
tree with edge counting metric to the complex plane with Euclidean metric, which
is a discrete analogue of Bloch space, because Bloch space is also consisting of
only Lipschitz functions on the unit disk under Hyperbolic metric to the complex
plane with Euclidean metric. In [11], the present authors defined discrete analogue
($T_p$) of generalized Hardy spaces on homogeneous rooted tree. In the same article
multiplication operators on $T_p$ spaces are studied.

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spaces.
In this article, we deal with the study of composition operators on $\mathbb{T}_p$ spaces. We refer to Section 2 for the definitions of homogeneous rooted tree and $\mathbb{T}_p$ spaces. In Section 3, we consider the boundedness of composition operators on $\mathbb{T}_p$ spaces and some of its consequences including norm estimates. In Section 4, we consider the compactness of composition operators on $\mathbb{T}_p$ spaces and derive equivalent conditions for compactness. Finally, in Section 5, we present three examples to show the following: there are self maps of $T$ which do not induce bounded composition operator on $\mathbb{T}_p$; there exists a bounded composition operator on $\mathbb{T}_p$ which is not compact; there are unbounded self maps of $T$ which induces compact composition operators on $\mathbb{T}_p$ for the case of $(q+1)$-homogeneous trees with $q \geq 2$.

2. Preliminaries and Lemmas

Let $G = (V, E)$ be a graph such that $E \subseteq V \times V$, where the elements of the sets $V$ and $E$ are called vertices and edges of the graph $G$, respectively. We shall not always distinguish between a graph and its vertex set and so, we may write $x \in G$ (rather than $x \in V$) and by a function defined on a graph, we mean a function defined on its vertices. Two vertices $x, y \in G$ are said to be neighbours (denoted by $x \sim y$) if $(x, y) \in E$. If all the vertices of $G$ have the same number $k$ of neighbours, then the graph is said to be $k$-homogeneous or $k$-regular graph. A finite path is a nonempty subgraph $P = (V, E)$ of the form $V = \{x_0, x_1, \ldots, x_k\}$ and $E = \{(x_0, x_1), (x_1, x_2), \ldots, (x_{k-1}, x_k)\}$, where $x_i$’s are distinct. In this case, we call $P$ be a path between $x_0$ and $x_k$. If $P$ is a path between $x_0$ and $x_k$ ($k \geq 2$), then $P$ with an additional edge $(x_n, x_0)$ is called a cycle. A nonempty graph $G$ is called connected if for any two of its vertices, there is a path between them. A connected graph without cycles is called a tree. Thus, any two vertices of a tree are linked by a unique path. The distance between any two vertex of a tree is the number of edges in the unique path connecting them. Sometimes it is convenient to consider one vertex of a tree as special; such a vertex is then called the root of this tree. A tree $T$ with fixed root $o$ is called a rooted tree. If $G$ is a rooted tree with root $o$, then $|v|$ denotes the distance between the root $o$ and the vertex $v$. Further, the parent (denoted by $v^-$) of a vertex $v$, which is not a root, is the unique vertex $w \in G$ such that $w \sim v$ and $|w| = |v| - 1$. In this case, $v$ is called child of $w$. For basic issues regarding graph theory, one can refer standard texts on this subject.

Throughout the paper, unless otherwise stated explicitly, $T$ denotes a homogeneous rooted tree (hence infinite graph), $\phi$ denotes a self map of $T$, $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

For $p \in (0, \infty]$, the Hardy space $H^p$ consists of all those analytic functions $f : \mathbb{D} \to \mathbb{C}$ such that $\|f\|_p < \infty$, where

$$\|f\|_p = \sup_{0 \leq r < 1} M_p(r, f)$$
and

\[ M_p(r, f) = \begin{cases} \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{\frac{1}{p}} & \text{if } p \in (0, \infty) \\ \sup_{|z|=r} |f(z)| & \text{if } p = \infty. \end{cases} \]

The generalized Hardy space \( H^p \) is defined similarly, upon replacing analytic functions by measurable functions.

As in [11], in a \((q + 1)\)-homogeneous tree \( T \) rooted at \( o \), we define

\[ \|f\|_p := \sup_{n \in \mathbb{N}_0} M_p(n, f), \]

where \( M_p(0, f) := |f(o)| \) and for every \( n \in \mathbb{N} \),

\[ M_p(n, f) := \begin{cases} \left( \frac{1}{(q + 1)q^{n-1}} \sum_{|v|=n} |f(v)|^p \right)^{\frac{1}{p}} & \text{if } p \in (0, \infty) \\ \max_{|v|=n} |f(v)| & \text{if } p = \infty. \end{cases} \]

The discrete analogue of the generalized Hardy space, denoted by \( T_{q,p} \), is then defined by

\[ T_{q,p} := \{ f : T \to \mathbb{C} \mid \|f\|_p < \infty \} \]

for every \( p \in (0, \infty] \). For the sake of simplicity, we shall write \( T_{q,p} \) as \( T_p \). Throughout the discussion, \( \| \cdot \| \) denotes \( \| \cdot \|_p \) in \( T_p \) spaces. The following results proved by the present authors [11] are needed for our present investigation.

**Lemma A.** For \( 1 \leq p \leq \infty \), \( \| \cdot \|_p \) induces a Banach space structure on the space \( T_p \).

**Lemma B.** (Growth Estimate) Let \( T \) be a \((q + 1)\)-homogeneous tree rooted at \( o \) and \( 0 < p < \infty \). Then, for \( v \in T \), we have the following: If \( f \in T_p \), then

\[ |f(v)| \leq \{q + 1\}^{\frac{n-1}{q}} \|f\|_p. \]

**Lemma C.** Norm convergence in \( T_p \) implies pointwise convergence. That is,

\[ \lim_{n \to \infty} \|f_n - f\| = 0 \Rightarrow \lim_{n \to \infty} f_n(v) = f(v) \text{ for each } v \in T. \]

3. **Bounded Composition Operators**

A linear operator \( A \) from a normed linear space \( X \) to a normed linear space \( Y \) is said to be bounded if the operator norm \( \|A\| = \sup\{\|Ax\|_Y : \|x\|_X = 1\} \) is finite.

Before we proceed to discuss our results, it is appropriate to recall some basic results about bounded composition operators in the classical setting. For example (see [10, Corollary 3.7]), every analytic self map \( \phi \) of \( \mathbb{D} \) induces bounded composition operator \( C_\phi \) on \( H^p \), \( 1 \leq p < \infty \). Moreover,

\[ \|C_\phi\|^p \leq \frac{1 + |\phi(0)|}{1 - |\phi(0)|}. \]
It is also known that (see [10, Theorem 3.8]) equality holds in (3.1) for every inner function of $D$ (for example, for every automorphism of $D$). For the case $p = \infty$, it is easy to see that $\|C_\phi\| = 1$ for every analytic self map $\phi$ of $D$.

Now, for our setting, we let $\phi$ be a self map of $(q + 1)$-homogeneous rooted tree $T$. For $n \in \mathbb{N}_0$ and $w \in T$, let $N_\phi(n, w)$ denote the number of pre-images of $w$ for $\phi$ in $|v| = n$. That is $N_\phi(n, w)$ is the number of elements in $\{\phi^{-1}(w)\} \cap \{|v| = n\}$. For $w \in T$, we define the weight function $W$ as follows:

\[
W(w) := \begin{cases} 
(q + 1)^{|w| - 1} & \text{if } w \in T \setminus \{o\} \\
1 & \text{if } w = o.
\end{cases}
\]

Let $|D_n|$ denote the number of vertices with $|v| = n$. Thus,

\[
|D_n| = \begin{cases} 
(q + 1)^n - 1 & \text{if } n \in \mathbb{N} \\
1 & \text{if } n = 0.
\end{cases}
\]

**Theorem 1.** Every self map $\phi$ of $T$ induces bounded composition operator on $\mathbb{T}_\infty$ with $\|C_\phi\| = 1$.

**Proof.** For each $f \in \mathbb{T}_\infty$ and every self map $\phi$ of $T$, we have

\[
\|C_\phi(f)\|_\infty = \|f \circ \phi\|_\infty = \sup_{w \in \phi(T)} |f(w)| \leq \|f\|_\infty.
\]

Thus, $C_\phi$ is bounded on $\mathbb{T}_\infty$. It is easy to see that $\|\chi_{\{v\}} \circ \phi\|_\infty = 1$ for each $v \in T$, where $\chi_{\{v\}}$ denotes the characteristic function on $\{v\}$. It gives that $\|C_\phi\| = 1$. \qed

In order to study the boundedness of the composition operators on $\mathbb{T}_p$ for $1 \leq p < \infty$, it is convenient to deal with the case $q = 1$ and $q \geq 2$ independently. First, we begin with the case $q = 1$.

**Theorem 2.** For every self map $\phi$ of 2-homogeneous tree $T$, $C_\phi$ is bounded on $\mathbb{T}_p$ with $\|C_\phi\|^p \leq 2$, $1 \leq p < \infty$.

**Proof.** By the growth estimate (Lemma B) for 2-homogeneous trees, it follows that $|f(v)|^p \leq 2\|f\|^p$ for all $v \in T$ and $f \in \mathbb{T}_p$. So,

\[
M_p^p(0, C_\phi f) = |f(\phi(o))|^p \leq 2\|f\|^p.
\]

Since $|D_n| = 2$ for all $n \in \mathbb{N}$,

\[
M_p^p(n, C_\phi f) = \frac{1}{|D_n|^p} \sum_{|v| = n} |f(\phi(v))|^p \leq \frac{2\|f\|^p + 2\|f\|^p}{2} = 2\|f\|^p,
\]

showing that $\|C_\phi(f)\|^p \leq 2\|f\|^p$ and the result follows. \qed

**Theorem 3.** If $T$ is a $(q + 1)$-homogeneous tree with $q \geq 2$ such that

\[
\sup_{n \in \mathbb{N}} \left( \sum_{|v| = n} q^{\phi(v) - n} \right) < \infty,
\]

then $C_\phi$ is bounded on $\mathbb{T}_p$, $1 \leq p < \infty$. 

\[ (3.3) \]
Proof. For \( n \in \mathbb{N}, \ w \in T \) and \( f \in \mathbb{T}_p \), by definition and Lemma B on growth estimate, we have

\[
M_p(n, C_\phi f) \leq \frac{1}{|D_n|} \sum_{|v|=n} (q+1)q^{(|\phi(v)|-1)\|f\|_p} = \sum_{|v|=n} q^{(|\phi(v)|-n)\|f\|_p}.
\]

Moreover, \( M_p(0, C_\phi f) = |f(\phi(o))|^p \leq (q+1)q^{(|\phi(o)|-1)\|f\|_p} \) and thus,

\[
\|C_\phi f\|_p \leq \max \left\{ (q+1)q^{(|\phi(o)|-1)}, \sup_{n \in \mathbb{N}} \left( \sum_{|v|=n} q^{(|\phi(v)|-n)} \right) \right\} \|f\|_p
\]

showing that \( C_\phi \) is bounded on \( \mathbb{T}_p \).

\[\square\]

**Theorem 4.** Let \( T \) be a \((q+1)\)-homogeneous tree and \( 1 \leq p < \infty \). If \( C_\phi \) is bounded on \( \mathbb{T}_p \), then

\[
\sup_{w \in T} \sup_{n \in \mathbb{N}_0} \left\{ \frac{W(w)}{|D_n|} N_\phi(n, w) \right\} \leq \|C_\phi\|^p.
\]

Proof. For each \( w \in T \), define \( f_w = \{W(w)\chi_{\{w\}}\}^{\frac{1}{p}} \), where \( W \) is defined in (3.2). It is easy to verify that for every \( w \in T \), \( M_p(n, f_w) = 1 \) when \( n = |w| \) and 0 otherwise. This observation gives that \( \|f_w\| = 1 \) for all \( w \in T \). Now, for each fixed \( w \in T \), we have for \( n \in \mathbb{N}_0 \),

\[
M_p(n, C_\phi f_w) = \frac{1}{|D_n|} \sum_{|v|=n} W(w)\chi_{\{w\}}(\phi(v))
= \frac{1}{|D_n|} \sum_{\phi(v)=w} W(w) = \frac{W(w)}{|D_n|} N_\phi(n, w)
\]

which yields that

\[
\|C_\phi f_w\|^p = \sup_{n \in \mathbb{N}_0} \left\{ \frac{W(w)}{|D_n|} N_\phi(n, w) \right\}.
\]

Consequently,

\[
\|C_\phi\|^p = \sup_{\|f\|=1} \|C_\phi(f)\|^p \geq \sup_{w \in T} \|C_\phi(f_w)\|^p = \sup_{w \in T} \sup_{n \in \mathbb{N}_0} \left\{ \frac{W(w)}{|D_n|} N_\phi(n, w) \right\}
\]

and the desired conclusion follows. \[\square\]

**Corollary 1.** If \( C_\phi \) is bounded on \( \mathbb{T}_p \), then

\[
\sup \left\{ q^{|w|-n} N_\phi(n, w) : w \in T \setminus \{o\}, n \in \mathbb{N} \right\}
\]

is finite.
In particular, every $\phi f$ and $n \leq \|f\|$, the desired result follows by Theorem 4.

**Proof.** Let $w = \phi(o)$ and, as before, consider $f_w = \{W(w)\chi_{\{w\}}\}^\frac{1}{p}$. Now, we observe that

\[
\|f_w\| = 1 \quad \text{and} \quad M_p^p(0, C\phi f_w) = |f_w(\phi(o))|\|f_w(\phi(o))\| = (q + 1)q|w|^{p-1} \geq (q + 1)q|w|^{p-1}
\]

which shows that $\|C\phi(f_w)\|^p \geq (q + 1)q|w|^{p-1}$ and the proof follows.

**Corollary 3.** If $\phi$ does not fix the root, i.e. $\phi(o) \neq o$, then

\[
\|C\phi\|^p \geq (q + 1)q|\phi(o)|^{p-1}.
\]

**Proof.** Setting $q = 1$ in Corollary 3 and Theorem 2 gives $\|C\phi\|^p \geq 2$ and $\|C\phi\|^p \leq 2$, respectively.

A self map $\phi$ of $T$ is called an automorphism of $T$, denoted as $\phi \in \text{Aut}(T)$, if $\phi$ is bijective and any two vertices $v$, $w$ are neighbours ($v \sim w$) if and only if $\phi(v) \sim \phi(w)$. Now we will compute the norm of the composition operator $C\phi$ when the inducing symbol $\phi$ is an automorphism of $T$.

**Theorem 5.** Let $T$ be a $(q + 1)$-homogeneous tree and consider $C\phi$ on $\mathbb{T}_p$, where $1 \leq p < \infty$ and $\phi \in \text{Aut}(T)$. Then we have

(i) $\|C\phi\| = 1$ if $\phi(o) = o$

(ii) $\|C\phi\|^p = (q + 1)q|\phi(o)|^{p-1}$ if $\phi(o) \neq o$.

In particular, every $\phi \in \text{Aut}(T)$ induces bounded composition operator $C\phi$ on $\mathbb{T}_p$.

**Proof.** Let $D_n = \{v \in T : |v| = n\}$ and consider the case $\phi(o) = o$. Then, for each $n$, $\phi$ is a bijective map from $D_n$ to $D_n$ (since $\phi \in \text{Aut}(T)$ and $\phi(o) = o$). For $n \in \mathbb{N}_0$ and $f \in \mathbb{T}_p$, we thus have

\[
M_p^p(n, C\phi f) = \frac{1}{|D_n|} \sum_{|\phi(v)|=n} |f(\phi(v))|^p = M_p^p(n, f).
\]
Taking supremum on both sides, we get \( \|C_\phi(f)\| = \|f\| \) which proves the first part.

Next, we consider the case \( \phi(o) \neq o \). The result is obviously true for \( q = 1 \), by Corollary 4. Thus, it suffices to prove the theorem for \((q+1)\)-homogeneous tree with \( q \geq 2 \). Let \( k = |\phi(o)| \). Since \( \phi \in \text{Aut}(T) \), is easy to see that

\[
M^n_p(0, C_\phi f) = |f(\phi(0))|^p \leq (q+1)q^{k-1}\|f\|^p.
\]

For the remaining part of the proof, we need to deal with the cases \( n = m (1 \leq m \leq k-1) \), \( n = k \), and \( n \geq k+1 \) separately. We begin with

\[
M^n_p(m, C_\phi f)
\]

\[
= \frac{1}{(q+1)q^{m-1}} \sum_{|v|=m} |f(\phi(v))|^p
\]

\[
\leq \frac{1}{(q+1)q^{m-1}} \left\{ \sum_{|v|=k+m} |f(v)|^p + \sum_{|v|=k+m-2} |f(v)|^p + \cdots + \sum_{|v|=k-m} |f(v)|^p \right\}
\]

\[
\leq \frac{1}{(q+1)q^{m-1}} \left\{ (q+1)q^{k+m-1} + (q+1)q^{k+m-3} + \cdots + (q+1)q^{k-m-1} \right\} \|f\|^p
\]

\[
= \left\{ q^k + q^{k-2} + \cdots + q^{k-2m} \right\} \|f\|^p
\]

\[
\leq (q+1)q^{k-1}\|f\|^p
\]

showing that \( M^n_p(n, C_\phi f) \leq (q+1)q^{k-1}\|f\|^p \) for \( n = 1, 2, \ldots, k-1 \). Next, for \( n = k \), we find that

\[
M^n_p(k, C_\phi f)
\]

\[
= \frac{1}{(q+1)q^{k-1}} \sum_{|v|=k} |f(\phi(v))|^p
\]

\[
\leq \frac{1}{(q+1)q^{k-1}} \left\{ \sum_{|v|=2k} |f(v)|^p + \sum_{|v|=2k-2} |f(v)|^p + \cdots + \sum_{|v|=2} |f(v)|^p + |f(o)|^p \right\}
\]

\[
\leq \frac{1}{(q+1)q^{k-1}} \left\{ (q+1)q^{2k-1} + (q+1)q^{2k-3} + \cdots + (q+1)q + 1 \right\} \|f\|^p
\]

\[
= \left\{ q^k + q^{k-2} + \cdots + q^{2-k} + \frac{1}{(q+1)q^{k-1}} \right\} \|f\|^p
\]

\[
\leq \left\{ q^k + q^{k-2} + \cdots + q^{2-k} + q^{1-k} \right\} \|f\|^p
\]

\[
\leq (q+1)q^{k-1}\|f\|^p.
\]

\[

\begin{array}{|c|c|c|}
\hline
\text{Domain} & \text{Range of } \phi \text{ contained in} & \text{Number of circles} \\
\hline
D_0 & D_k & 1 \\
D_m & D_{k+m}, D_{k+m-2}, \cdots, D_{k-m} & m+1 \\
(1 \leq m \leq k-1) & & \\
D_k & D_{2k}, D_{2k-2}, \cdots, D_2, D_0 & k+1 \\
D_{k+m+1} & D_{2k+m+1}, D_{2k+m-1}, \cdots, D_{2m+1} & k+1 \\
\hline
\end{array}
\]
Finally, for each $m \in \mathbb{N}_0$,
\[
M_p^p(m + k + 1, C_\phi f) = \frac{1}{(q + 1)q^{m+k}} \sum_{|v|=m+k+1} |f(\phi(v))|^p \\
\leq \frac{1}{(q + 1)q^{m+k}} \left\{ \sum_{|v|=m+2k+1} |f(v)|^p + \sum_{|v|=m+2k-1} |f(v)|^p + \cdots + \sum_{|v|=2m+1} |f(v)|^p \right\} \\
\leq \frac{1}{(q + 1)q^{m+k}} \left\{ (q + 1)q^{m+2k} + (q + 1)q^{m+2k-2} + \cdots + (q + 1)q^m \right\} \|f\|^p \\
= \{q^k + q^{k-2} + \cdots + q^{-k}\} \|f\|^p \\
\leq (q + 1)q^{k-1}\|f\|^p.
\]
The above discussion implies that
\[
M_p^p(n, C_\phi f) \leq (q + 1)q^{k-1}\|f\|^p \text{ for all } n \in \mathbb{N}_0
\]
and thus, $\|C_\phi\|_p^p \leq (q + 1)q^{\phi(0)-1}$. Other way inequality follows from Corollary 3 and the proof is complete.

\section{Compact Composition Operators}

A bounded linear operator $A$ from a normed linear space $X$ to a normed linear space $Y$ is said to be a compact operator if the image of closed unit ball $\{Ax : \|x\|_X \leq 1\}$ has compact closure in $Y$.

In the classical case, for an analytic self map $\phi$ of $\mathbb{D}$, the following statements are equivalent (see [12, Section 2.7 and Compactness Theorem, Chapter 10]):

(a) $C_\phi$ is compact on $H^p$ for $1 \leq p < \infty$.

(b) $C_\phi$ is compact on $H^2$.

(c) $\lim_{|w| \to 1} \frac{N_\phi(w)}{\log \frac{1}{|w|}} = 0$, where $N_\phi$ is the Nevanlinna counting function of $\phi$.

Also, $C_\phi$ is compact on $H^\infty$ if and only if $\sup \{|\phi(z)| : z \in \mathbb{D}\} < 1$ (see [12, Problem 10, Chapter 2]).

For the discrete setting, we now consider the compactness of composition operators on $\mathbb{T}_p$ spaces. A self map $\phi$ of $(q + 1)$-homogeneous tree $T$ is called a bounded map if $\sup \{|\phi(v)| : v \in T\}$ is finite.

\textbf{Theorem 6.} Every bounded self map $\phi$ of $T$ induces compact composition operator on $\mathbb{T}_p$ for $1 \leq p \leq \infty$.

\textbf{Proof.} Suppose $\phi$ is a bounded self map of a $(q + 1)$-homogeneous tree $T$. Then $\text{Range}(\phi)$ is finite set, say, $\text{Range}(\phi) = \{v_1, v_2, \ldots, v_k\}$. For each $1 \leq i \leq k$, denote by $E_i$ for the pre-image of $v_i$ under $\phi$. If $\phi(v) = v_i$, then $f \circ \phi(v) = f(v_i)$ so that
\[
f \circ \phi = f(v_1)\chi_{E_1} + f(v_2)\chi_{E_2} + \cdots + f(v_k)\chi_{E_k}
\]
and \( \text{Range}(C_{\phi}) = \text{span}\{\chi_{E_1}, \chi_{E_2}, \ldots, \chi_{E_k}\} \). Thus, \( C_{\phi} \) is a finite rank operator and hence it is compact.

**Theorem 7.** If \( \phi \) is a self map of \((q + 1)\)-homogeneous tree \( T \), then the following are equivalent:

(a) \( C_{\phi} \) is compact on \( \mathbb{T}_p \) for \( 1 \leq p \leq \infty \).

(b) \( \|C_{\phi}f_n\| \to 0 \) as \( n \to \infty \) whenever bounded sequence of functions \( \{f_n\} \) that converges to 0 pointwise.

**Proof.** (a) \( \Rightarrow \) (b): Assume that \( C_{\phi} \) is compact on \( \mathbb{T}_p \) and \( \{f_n\} \) is a bounded sequence in \( \mathbb{T}_p \) that converges to 0 pointwise. Suppose on the contrary that \( \|C_{\phi}(f_n)\| \not\to 0 \) as \( n \to \infty \) and an \( \epsilon > 0 \) such that \( \|C_{\phi}(f_n)\| \geq \epsilon \) for all \( j \). Denote \( \{f_{n_j}\} \) by \( \{g_j\} \). Since \( C_{\phi} \) is compact, there is a subsequence \( \{g_{j_k}\} \) of \( \{g_j\} \) such that \( \{C_{\phi}(g_{j_k})\} \) converges to some function, say, \( g \). It follows that \( \{C_{\phi}(g_{j_k})\} \) converges to \( g \) pointwise and \( g \equiv 0 \) implying that \( \{C_{\phi}(g_{j_k})\} \) converges to 0 which is a contradiction to \( \|C_{\phi}(g_j)\| \geq \epsilon \) for all \( j \). Hence, \( \|C_{\phi}(f_n)\| \to 0 \) as \( n \to \infty \).

(b) \( \Rightarrow \) (a): Conversely, suppose that case (b) holds. First let us consider the case \( 1 \leq p < \infty \). Let \( \{g_n\} \) be a sequence in unit ball of \( \mathbb{T}_p \). By Lemma B, for each \( v \in T \), the sequence \( \{g_n(v)\} \) is bounded. By the diagonalization process, there is a subsequence \( \{g_{n_m}\} \) of \( \{g_n\} \) that converges pointwise to \( g \) (say). We see that, for each \( m \in \mathbb{N}_0 \),

\[
M_p^p(m, g) = \lim_{n \to \infty} \frac{1}{(q + 1)q^{m-1}} \sum_{|v|=m} |g_{n_m}(v)|^p \leq \lim \sup \|g_{n_m}\|^p \leq 1
\]

showing that \( g \in \mathbb{T}_p \) with \( \|g\| \leq 1 \). Consequently, if \( f_n = g_{n_m} - g \), then \( \{f_n\} \) converges to 0 pointwise and \( \|f_n\| \leq 2 \). By the assumption (b), \( \|C_{\phi}f_n\| \to 0 \) as \( n \to \infty \) and thus, \( \{C_{\phi}(g_{n_m})\} \) converges to \( C_{\phi}(g) \). Hence \( C_{\phi} \) is compact on \( \mathbb{T}_p \).

The proof for the case \( p = \infty \) is similar to the above.

**Remark 1.** Since edge counting metric on \( T \) induces discrete topology, compact sets are only sets having finitely many elements. Thus convergence uniformly on compact subsets of \( T \) is equivalent to pointwise convergence. In view of this remark, Theorem 7 is a discrete analog of weak convergence theorem (see [12, section 2.4, p. 29]) in the classical case.

**Corollary 5.** Let \( \phi \) be a self map of \( T \). Then \( C_{\phi} \) is compact on \( \mathbb{T}_\infty \) if and only if \( \phi \) is a bounded self map of \( T \).

**Proof.** If \( \phi \) is a bounded self map of \( T \), then \( C_{\phi} \) is compact, by Theorem 6. Conversely, suppose \( \phi \) is not a bounded map. Then, there exists a sequence of vertices \( \{v_k\} \) of \( T \) such that \( \phi(v_k) = w_k \) and \( |w_k| \to \infty \) as \( k \to \infty \). Take \( f_k = \chi_{\{w_k\}} \) for each \( k \in \mathbb{N} \). Then, \( \|f_k\|_\infty = 1 \) for each \( k \) and \( \{f_k\} \) converges to 0 pointwise. Since \( C_{\phi} \) is compact, \( \|C_{\phi}(f_k)\|_\infty \to 0 \) as \( k \to \infty \), by Theorem 7. This is not possible, because \( \|C_{\phi}(f_k)\|_\infty = 1 \) for each \( k \in \mathbb{N} \), which can be observed from the definition of \( f_k \). Hence \( \phi \) should be a bounded map. \( \square \)
Corollary 6. Let $T$ be a $(q+1)$-homogeneous tree and $1 \leq p < \infty$. If $C_{\phi}$ is compact on $\mathbb{T}_p$, then
\[ \sup_{n \in \mathbb{N}_0} \left\{ q^{\|w\|^{-n}} N_{\phi}(n, w) \right\} \to 0 \quad \text{as} \quad |w| \to \infty. \]

Proof. As in the earlier situations, for each $w \in T \setminus \{o\}$, we let $f_w = \{W(w)\chi(w)\}_{n=0}^\infty$. Then, $\|f_w\| = 1$ for all $w$ and, since $f_w(v) = 0$ whenever $|w| > n = |v|$, $\{f_w\}$ converges to 0 pointwise. Since $C_{\phi}$ is compact, $\|C_{\phi}(f_w)\| \to 0$ as $|w| \to \infty$. However, we have already shown that
\[ \|C_{\phi}f_w\|^p = \sup_{n \in \mathbb{N}_0} \left\{ \frac{W(w)}{|D_n|} N_{\phi}(n, w) \right\} = \sup_{n \in \mathbb{N}_0} \left\{ q^{\|w\|^{-n}} N_{\phi}(n, w) \right\} \]
and the desired conclusion follows. $\square$

Remark 2. For 2-homogeneous trees, Corollary 6 takes a simpler form: If $C_{\phi}$ is compact on $\mathbb{T}_p$, then
\[ \sup_{n \in \mathbb{N}_0} \{N_{\phi}(n, w)\} \to 0 \quad \text{as} \quad |w| \to \infty. \]
This remark is helpful in the proof of Corollary 8.

Corollary 7. If $C_{\phi}$ is compact on $\mathbb{T}_p$, then $|v| - |\phi(v)| \to \infty$ as $|v| \to \infty$.

Proof. We will prove this result by contradiction. Suppose that $|v| - |\phi(v)| \not\to \infty$ as $|v| \to \infty$. Then there exists a sequence of vertices $\{v_k\}$ and an $M > 0$ such that $|v_k| - |\phi(v_k)| \leq M$ for all $k$ which implies that $|\phi(v_k)| \to \infty$ as $k \to \infty$. Since $N_{\phi}(|v_k|, \phi(v_k)) \geq 1$ for all $k$, where $N_{\phi}(n, w)$ is defined as in Section 3, we obtain that $N_{\phi}(|v_k|, \phi(v_k))q^{\|\phi(v_k)|-|v_k|} \geq q^{-M}$ which yields that
\[ \sup_{n \in \mathbb{N}_0} \left\{ N_{\phi}(n, \phi(v_k))q^{\|\phi(v_k)|-|v_k|} \right\} \geq q^{-M} \quad \text{for all} \quad k \]
and thus,
\[ \sup_{n \in \mathbb{N}_0} \left\{ q^{\|w\|^{-n}} N_{\phi}(n, w) \right\} \not\to 0 \quad \text{as} \quad |w| \to \infty \]
which gives that $C_{\phi}$ is not compact, by Corollary 6. This contradiction completes the proof. $\square$

Corollary 8. Let $T$ be a 2-homogeneous tree. Then $C_{\phi}$ is compact on $\mathbb{T}_p$ if and only if $\phi$ is a bounded self map of $T$.

Proof. Since every bounded self map $\phi$ of $T$ induces compact composition operator on $\mathbb{T}_p$, one way implication is hold. For the proof of the converse part, we suppose that $\phi$ is not bounded. Then the range contains an infinite set, say $\{w_1, w_2, \ldots\}$. For each $k$, choose $v_k \in T$ such that $\phi(v_k) = w_k$. This gives $N_{\phi}(|v_k|, w_k) \geq 1$ and thus $\sup_{n \in \mathbb{N}_0} N_{\phi}(n, w_k) \geq 1$ for all $k$. It follows that $\sup_{n \in \mathbb{N}_0} \{N_{\phi}(n, w)\} \not\to 0$ as $|w| \to \infty$ and hence, $C_{\phi}$ cannot be compact. $\square$
Remark 3. It is worth to recall from [12, Chapter 3, p. 37] that if a “big-oh” condition describes a class of bounded operators, then the corresponding “little-oh” condition picks out the subclass of compact operators”. We have already shown that if \( \sum_{|v|=n} q^{\phi(v)} = O(q^n) \) then \( C_\phi \) is bounded on \( \mathbb{T}_p \). So it is natural to ask whether \( \sum_{|v|=n} q^{\phi(v)} = o(q^n) \) guarantees the compactness of \( C_\phi \) on \( \mathbb{T}_p \). Indeed, the answer is yes. Clearly the later observation is not useful because no self map \( \phi \) of \( T \) satisfies this condition. This is because \( \sum_{|v|=n} q^{\phi(v)} \geq \sum_{|v|=n} q^0 = (q+1)q^{n-1} \) and thus, \( \sum_{|v|=n} q^{\phi(v)} = o(q^n) \) is cannot be possible.

5. Examples

Example 1. For each \( n \in \mathbb{N}_0 \), choose the vertex \( v_n \) such that \( v_n \in D_n = \{v \in T : |v| = n\} \). Define \( \phi_1(v) = v_n \) if \( |v| = n \). Now, for \( q \neq 1 \), consider the function \( f \) defined by

\[
 f(v) = \begin{cases} 
((q + 1)q^{n-1})^{\frac{1}{p}} & \text{if } v = v_n \text{ for some } n \in \mathbb{N} \\
0 & \text{elsewhere.} 
\end{cases}
\]

Then \( f \in \mathbb{T}_p, \|f\| = 1 \) and for each \( m \in \mathbb{N} \), we see that

\[
 M^p_p(m, C_{\phi_1} f) = \frac{1}{(q + 1)q^{m-1}} \sum_{|v|=m} |f(v_m)|^p = (q+1)q^{m-1}
\]

showing that \( \|C_{\phi_1} f\| = \sup_{m \in \mathbb{N}_0} M_p(m, C_{\phi_1} f) \) which is not finite for \( q \geq 2 \). This example shows that there are self maps of \( T \) which do not induce bounded composition operator on \( \mathbb{T}_p \) unlike the case of Hardy spaces on the unit disk.

Example 2. Consider the following self map \( \phi_2 \) of \( T \) defined by

\[
 \phi_2(v) = \begin{cases} 
o & \text{if } v = o \\
v^- & \text{otherwise.}
\end{cases}
\]

where \( v^- \) denotes the parent of \( v \). Then it follow easily that

\[
 M^p_p(0, C_{\phi_2} f) = M^p_p(0, f) \quad \text{and} \quad M^p_p(1, C_{\phi_2} f) = \frac{1}{(q + 1)} \sum_{|v|=1} |f(0)|^p = M^p_p(0, f).
\]

Finally, for \( n \geq 2 \), we have

\[
 M^p_p(n, C_{\phi_2} f) = \frac{1}{(q + 1)q^{n-1}} \sum_{|v|=n} |f(v^-)|^p
\]

\[
 = \frac{q}{(q + 1)q^{n-1}} \sum_{|w|=n-1} |f(w)|^p = M^p_p(n-1, f)
\]

and thus,

\[
 \|C_{\phi_2} f\| = \sup_{m \in \mathbb{N}_0} M_p(m, C_{\phi_2} f) = \sup_{m \in \mathbb{N}_0} M_p(m, f) = \|f\|
\]

showing that \( C_{\phi_2} \) is bounded on \( \mathbb{T}_p \). On the other hand, since \( |v| - |\phi_2(v)| = 1 \) for all \( |v| \geq 1 \), we have \( |v| - |\phi_2(v)| \nrightarrow \infty \) as \( |v| \rightarrow \infty \). Hence \( C_{\phi_2} \) is not compact, by
Corollary 7. This is an example of bounded composition operator on $T_p$ which is not compact.

**Remark 4.** Let $\phi_3$ be a map on $T$ such that $\phi_3$ maps every vertex into any one of its child. Then, as in the case of $C_{\phi_2}$, it is easy to see that $C_{\phi_3}$ is bounded but not compact. Moreover, it can be seen that, for each $n \in \mathbb{N}$, $(C_{\phi_2})^n$ and $(C_{\phi_3})^n$ are also bounded but not compact.

**Example 3.** For each $n \in \mathbb{N}_0$, choose a vertex $v_n$ such that $|v_n| = n$. Define a self map $\phi_4$ by

$$\phi_4(v) = \begin{cases} v_k & \text{if } v = v_{2k} \text{ for some } k \in \mathbb{N} \\ o & \text{otherwise.} \end{cases}$$

Then we obtain that

$$M_p^n(0, C_{\phi_4}f) = |f(\phi_4(o))|^p = |f(o)|^p = M_p^n(0, f).$$

Next, for an odd natural number $n$, we see that

$$M_p^n(n, C_{\phi_4}f) = \frac{1}{(q+1)q^{n-1}} \sum_{|v|=n} |f(o)|^p = |f(o)|^p.$$

Finally, for an even natural number, say $n = 2k$, for some $k \in \mathbb{N}$, we find that

$$M_p^n(n, C_{\phi_4}f) = \frac{1}{(q+1)q^{n-1}} \left\{ \sum_{|v|=n, v \neq v_{2k}} |f(\phi_4(v))|^p + |f(\phi_4(v_{2k}))|^p \right\} \leq |f(o)|^p + \frac{|f(v_k)|^p}{(q+1)q^{n-1}}.$$

Thus, by Lemma B, we have

$$\|C_{\phi_4} f\|^p \leq |f(o)|^p + \sup_{k \in \mathbb{N}} \left\{ \frac{f(v_k)}{(q+1)q^{2k-1}} \right\} \leq 2\|f\|^p$$

which shows that $C_{\phi_4}$ is bounded on $T_p$.

Suppose now that $T$ is a $(q+1)$-homogeneous tree with $q \geq 2$. Let $\{f_n\}$ be a sequence in the unit ball of $T_p$ which converges to 0 pointwise. Note that

$$\left\{ \frac{f(v_k)}{(q+1)q^{2k-1}} \right\} \leq \frac{1}{q^k},$$

by Lemma B. We now claim that $\|C_{\phi_4} f_n\|^p \to 0$ as $n \to \infty$.

Let $\epsilon > 0$ be given. Then there exists a natural number $N_1$ such that $q^{-k} < \epsilon/2$ for all $k \geq N_1$. Consider the set $S = \{v_1, v_2, \ldots, v_{N_1}\}$. Since $\{f_n\}$ converges to 0 pointwise, we can choose a natural number $N > N_1$ such that $|f_n(o)|^p < \epsilon/2$ and $|f_n(v)|^p < \epsilon/2$ for all $v \in S$ and for all $n \geq N$. Thus,

$$\|C_{\phi_4} f_n\|^p \leq |f_n(o)|^p + \sup \left\{ \epsilon, \frac{1}{q^{N_1}}, \frac{1}{q^{N_1+1}}, \ldots \right\} \leq \epsilon$$

for all $n \geq N$,

which gives that $\|C_{\phi_4} f_n\|^p \to 0$ as $n \to \infty$ and hence $C_{\phi_4}$ is compact on $T_p$. This example shows that there are unbounded self maps of $T$ which induces compact composition operators on $T_p$ for the case of $(q+1)$-homogeneous trees with $q \geq 2$. 

We conclude the paper with a comparison in the case of $q = 1$ and $q > 1$. For 2-homogeneous trees, circle of radius $n$ has 2 vertices for all $n \in \mathbb{N}$ whereas in the case of $(q + 1)$-homogeneous trees with $q \geq 2$, circle of radius $n$ has $(q + 1)q^{n-1}$ vertices. Due to this basic fact, we can expect a difference in operator theoretic point of view. The following table explains how composition operators on $T_p$ for $1 \leq p < \infty$ differ in both the cases.

| 2-homogeneous tree | $(q + 1)$-homogeneous tree with $q \geq 2$ |
|---------------------|------------------------------------------|
| Every self map $\phi$ of $T$ induces bounded composition operators on $T_p$. | There are self maps of $T$ which induces unbounded composition operators on $T_p$. |
| Only bounded self map of $T$ induces compact composition operators on $T_p$. | There are unbounded self maps of $T$ which induces compact operators on $T_p$. |

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