A New Sparse and Robust Adaptive Lasso Estimator for the Independent Contamination Model

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Abstract—Many problems in signal processing require finding sparse solutions to under-determined, or ill-conditioned, linear systems of equations. When dealing with real-world data, the presence of outliers and impulsive noise must also be accounted for. In past decades, the vast majority of robust linear regression estimators has focused on robustness against rowwise contamination. Even so-called ‘high breakdown’ estimators rely on the assumption that a majority of rows of the regression matrix is not affected by outliers. Only very recently, the first cellwise robust regression estimation methods have been developed. In this paper, we define robust oracle properties, which an estimator must have in order to perform robust model selection for under-determined, or ill-conditioned linear regression models that are contaminated by cellwise outliers in the regression matrix. We propose and analyze a robustly weighted and adaptive Lasso type regularization term which takes into account cellwise outliers for model selection. The proposed regularization term is integrated into the objective function of the MM-estimator, which yields the proposed MM-Robust Weighted Adaptive Lasso (MM-RWAL), for which we prove that at least the weak robust oracle properties hold. A performance comparison to existing robust Lasso estimators is provided using Monte Carlo experiments. Further, the MM-RWAL is applied to determine the temporal releases of the European Tracer Experiment (ETEX) at the source location. This ill-conditioned linear inverse problem contains cellwise and rowwise outliers and is sparse both in the regression matrix and the parameter vector. The proposed RWAL penalty is not limited to the MM-estimator but can easily be integrated into the objective function of other robust estimators.

Index Terms—Sparse and Robust Estimation, Outlier, Lasso, Independent Contamination Model, Robust Oracle Properties, Atmospheric Emissions.

I. INTRODUCTION

Many of today’s signal processing problems can be formulated as a linear regression

\[ y = X\beta + u, \]  

where we assume that the regressor matrix \( X \in \mathbb{R}^{n \times p} \), the errors \( u \in \mathbb{R}^{n \times 1} \) and observations \( y \in \mathbb{R}^{n \times 1} \) are independent and identically distributed (iid) random variables, \( \beta \in \mathbb{R}^{p \times 1} \) are the unknown parameters of interest, and \( X \) and \( u \) are mutually independent.

The presence of outliers and impulsive noise has been reported in applications as diverse as wireless communication, ultrasonic systems, computer vision, electric power systems, automated detections of defects, biomedical signal analysis, genomics and the estimation of the temporal releases of a pollutant to the atmosphere. See \[1\], \[2\], \[3\], \[4\], \[5\], \[6\], \[7\], \[8\], \[9\] and references therein. Violations of the Gaussian assumption cause a drastic performance drop for the commonly used least-squares estimator (LSE) \[10\], \[11\], \[12\]

\[ \hat{\beta}_{LSE} = \arg \min_{\beta} \| y - x\beta \|_2^2. \]  

For decades, the vast majority of robust linear regression estimators has focused on robustness against ‘rowwise’ contamination. Under this so-called Tukey-Huber contamination model (THCM) \[11\], a small fraction of rows of \( X \) may be contaminated. Even ‘high-breakdown’ regression estimators, such as the LTS-, S-, MM-, and \( \tau \)-estimators \[1\], \[11\] rely on the THCM. In \[13\], Rousseau and Van den Bossche state that recently researchers have come to realize that the outlying rows paradigm is no longer sufficient for modern high-dimensional data sets. It often happens that most data cells (entries) in a row are regular and just a few of them are anomalous.

The case that independent cells of \( X \) are outliers is referred to as the independent contamination model (ICM) \[14\], \[15\], \[16\]. Only very recently, the first ‘cellwise robust’ regression estimation methods have been developed \[15\], \[16\]. The extension of existing estimators to other contamination models, such as the ICM, and even the development of completely new robust estimators is necessary to solve many real-world problems. For example, the estimation of the spatio-temporal emissions of a pollutant, given noisy observations, can be formulated as a linear inverse problem with the help of an atmospheric dispersion model \[6\]. The data of the European Tracer Experiment (ETEX) which was conducted in Monterfil, Brittany in 1994, where Perfluorocarbon (PFC) tracers were released into the atmosphere, for instance, contains both cellwise and rowwise outliers.

Additionally to the robustness considerations, atmospheric inverse problems, like many other problems in signal processing, require finding sparse solutions to under-determined, or ill-conditioned, linear systems of equations. For example, handling large datasets in terms of model interpretation, including the case where the number of explanatory variables \( p \) is larger than the sample size \( n \), requires penalized estimators, such as the classical least absolute shrinkage and selection operator (Lasso) \[17\]

\[ \hat{\beta}_{\text{Lasso}} = \arg \min_{\beta} \| y - x\beta \|_2^2 + \lambda \| \beta \|_1. \]  

with \( \lambda \in \mathbb{R}^+ \).

Many other regularizations have been proposed \[18\], \[19\], \[20\], \[21\], \[22\]. In this paper the focus lies on Lasso estimation, to select a robust and interpretable model in high
dimensional settings. Zou [23] showed that the Lasso variable selection can be inconsistent, so that the oracle properties do not hold and proposed the adaptive Lasso

$$\hat{\beta}_{\text{Lasso}}^{\text{ad}} = \arg \min_\beta \| y - x\beta \|^2 + \lambda \sum_{j=1}^{p} \hat{w}_j |\beta_j|,$$  

(4)

where $\hat{w}_j = 1/|\beta_j|^\gamma (\gamma > 0)$ are non-negative weights depending on $\beta$, which is a $\sqrt{n}$-consistent estimator of $\beta$.

Just like the LSE, the Lasso and the adaptive Lasso rely on the Gaussian noise assumption and are sensitive to outliers. In recent years, some robust and regularized approaches have been proposed that replace the penalized square objective function by a penalized bounded objective function [7], [9], [24], [25], [26]. These methods, however, again, rely on the THCM, and to date, no penalized robust regression method exists that can handle cellwise and rowwise outliers.

Original Contributions: First, we give a weak and strong definition of what we call the ‘robust oracle properties’. These are properties that estimators aiming at performing robust variable selection need to have. Next, we propose and analyze a robustly weighted and adaptive Lasso-type regularization term, which takes into account cellwise outliers for model selection. The proposed regularization term is integrated into the objective function of the MM-estimator, which yields the proposed MM-Robust Weighted Adaptive Lasso (MM-RWAL), for which we prove that at least the weak robust oracle properties hold. We would like to highlight, that the proposed Robust Weighted Adaptive Lasso penalty can easily be integrated into the objective function of other robust estimators. A performance comparison to existing robust Lasso estimators is provided using Monte Carlo experiments. Further, a challenging real-data application of estimating the sparse non-negative spatio-temporal emissions of a pollutant is considered, given noisy observations and an imprecisely estimated ill-conditioned and sparse dispersion model $X$. This example contains both cellwise and rowwise outliers.

Notation: Scalars are denoted by lowercase letters, e.g., $x$, column vectors by bold-faced lowercase letters, e.g. $\mathbf{x}$, matrices by bold-faced uppercase letters, e.g. $\mathbf{X}$, sets are denoted by calligraphic letters, e.g. $\mathcal{X}$ with associated cardinality $|\mathcal{X}|$. The $j$th column of a matrix $\mathbf{X}$ is denoted by $\mathbf{x}_j$ while $(\mathbf{x})_{i,j}$ denotes the vector that contains the entries $i$ to $j$ of vector $\mathbf{x}$. The $i$th element of vector $\mathbf{x}$ is denoted by $x_i$, $\mathbf{I}_p$ is the $p$-dimensional identity-matrix, $\mathbf{0}_p$ is the $p$-dimensional all-zeros vector and $\text{diag}(\mathbf{x})$ forms a matrix that contains the entries of $\mathbf{x}$ as its diagonal. $\hat{\beta}$ refers to the estimator (or estimate) of the parameter vector $\beta$, $(\cdot)^\top$ is the transpose operator. The derivative of a function $f$ with respect to its argument is abbreviated by $f'$. $P(X)$ is the probability of event $X$. $\text{Bin}(1, \epsilon)$ denotes the binomial distribution with one trial and a success probability of $\epsilon$. Convergence to the normal distribution with mean vector $\mu$ and covariance matrix $\Sigma$ is denoted by $\xrightarrow{d} \mathcal{N}(\mu, \Sigma)$.

Organization: Section II discusses the Tukey-Huber and Independent Contamination models and motivates the use of cellwise robust methods. Section III defines the robust oracle properties, introduces the proposed estimator and provides algorithms to compute the estimates. Section IV provides numerical experiments, while Section V contains a challenging real-data application of source estimation for an atmospheric inverse problem. Finally, Section VI concludes the paper with a brief outlook on future work.

II. TUKEY-HUBER AND INDEPENDENT CONTAMINATION MODEL

A. Tukey-Huber Contamination Model (THCM)

The THCM is based on the assumption that a majority $(1 - \epsilon)$ of the data points is not contaminated, while a minority $\epsilon$ is contaminated. The univariate THCM is formulated as

$$x = (1 - b)y + bz,$$  

(5)

where $b, y, z$ are mutually independent scalar random variables. Let $F'$ be the true distribution of the data and let $G$ be an unspecified contaminating distribution, from which the outliers are generated. Then, with $y \sim F$, $z \sim G$ and $b \sim \text{Bin}(1, \epsilon)$, the distribution of the observed variable $x$ becomes

$$H = (1 - \epsilon)F + \epsilon G.$$  

(6)

The multivariate THCM is defined by

$$\mathbf{x} = (\mathbf{I}_p - b\mathbf{I}_p)y + b\mathbf{I}_p\mathbf{z},$$  

(7)

where $y, \mathbf{x}, \mathbf{z}$ are $p$-dimensional random variables.

A highly valuable property of the THCM is that the percentage of contaminated rows in the data-matrix stays unchanged under affine transformations, that is, if the random vector $\mathbf{x}$ follows the THCM, then the affine transformed vector

$$\tilde{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}$$  

(8)

also follows the THCM. Thus, estimators designed for the THCM can be affine equivariant. Additionally, many important robustness concepts such as the influence function and the breakdown point are based on the THCM [10], [11], [12], [13]. However, the disadvantages of this contamination model, which occur especially in high dimensional settings, are alerting. First of all, the assumption that a major fraction $(1 - \epsilon)$ of the data points is outlier free, very unlikely holds in higher dimensions. A mathematical legitimation of this criticism will be given in Eq. (13) and the associated Fig. 2. Secondly, as illustrated in Fig. 1, we see that a few cellwise outliers in the THCM lead to flagging the whole corresponding row of $\mathbf{X}$ as outliers. Although much information is lost, such contamination may be handled by high breakdown point estimators in low dimensions. However, Fig. 1b illustrates that in cases where the number of predictors $p$ exceeds the number of rows $n$, it becomes more and more likely that a few highly contaminated predictors force the THCM-based estimators to flag all data points as outliers, which makes it impossible to draw any inferences from the data.
A. Under affine transformations. Let \( x \) for the ICM, which is referred to as 'outlier propagation'.

From these considerations, we calculate the probability of being contaminated. Notice that for independent Bernoulli random variables with success probability \( p < n \), the ICM reduces to the THCM.

\[ P_{\text{cont, row}} = 1 - (1 - \epsilon)^p. \]  

So, for any high breakdown point estimator, tuned to have the highest possible breakdown point of 50%, we obtain the inequality

\[ P_{\text{cont, row}} < 0.5 \]

\[ \Leftrightarrow 1 - (1 - \epsilon)^p < 0.5 \]

\[ \Leftrightarrow \epsilon < 1 - 0.5^p. \]

This means that the tolerable fraction of contamination \( \epsilon \) in every predictor - for simplicity, we assume that \( \epsilon \) is equal for every predictor - is bounded and depends on the dimension of the dataset. From this point of view, the probability of having only THCM-outliers in the data depends on the dimension of the predictor-matrix and the number of rows as follows:

\[ P(\text{''THCM} \rightarrow \text{ICM''}) = \left[ 1 \cdot \frac{n!}{(n - ne2)! \cdot (ne2)!} \right] \cdot \ldots \cdot \left[ \frac{n!}{(n - ne2)! \cdot (ne2)!} \right]^{-1} \]

Since the THCM assumes that \( \epsilon_1 = \ldots = \epsilon_p = \epsilon \), it follows that

\[ P(\text{''THCM} \rightarrow \text{ICM''}) = \left[ \frac{n!}{(n - ne)! \cdot (ne)!} \right]^{-p} \]

\[ = \left( \frac{n}{[ne]} \right)^{-p}. \]

Fig. 2 illustrates how rapidly the probability of having only THCM-outliers decreases for fixed \( n \) and \( \epsilon \). This, again supports the statement made in [13] that the outlying rows paradigm is no longer sufficient for modern high-dimensional data sets.

III. PROPOSED METHODS

A. Proposed Definitions of Robust Oracle Properties

Extending the ideas of [23] to the ICM, we next introduce what we call the 'robust oracle properties'. We propose a strong and a weak version of the robust oracle properties.

Definition III.1. (Weak Robust Oracle Properties)
Let \( \{x_1, x_2, \ldots, x_p\} \) be the set of predictors, the set of indices corresponding to the set of robust active predictors, and \( A^n := \{ j : \beta_j^n \neq 0 \} \) the set of indices corresponding to the set of predictors chosen by a Lasso type estimator to be active. Then, a Lasso type estimator needs to have the following properties to be a robust oracle estimator:
Intuitively speaking, the strong robust oracle properties hold for any Lasso type estimator, whose adaptive $\ell_1$-penalty term ensures the penalization of the predictors in an ascending order given by the predictor contamination order statistics

\[(\epsilon)_{1:p} \leq (\epsilon)_{2:p} \leq \ldots \leq (\epsilon)_{p:p},\]

(17)

while choosing the tuning parameter $\lambda$ in the $\ell_1$-penalty such that predictors enter the model until the breakdown point of the estimator is reached.

B. MM-Robust Weighted Adaptive Lasso (MM-RWAL)

In this section, we introduce a new method called the MM-Robust Weighted Adaptive Lasso (MM-RWAL). Recently, the MM-Lasso and adaptive MM-Lasso were introduced to robustify against outliers [26]. The objective function of the MM estimator is

\[
\hat{\beta}_{\text{MM}} = \arg \min_{\beta} \frac{1}{n} \sum_{i=1}^{n} \rho(\frac{r_i(\beta)}{s_n(r(\beta_i))} + \lambda_n \sum_{j=1}^{p} \hat{w}_j|\beta_j|.\]

(18)

Here, $\rho(\cdot)$ is a robustifying function (see, e.g. [10], [11], [12]), $r(\beta_i) = y - x\beta_i$ is the residual of an $S$-estimator whose estimates $\hat{\beta}_i$ have the property of minimizing a robust M-scale $s_n(r(\beta))$ that satisfies

\[
\frac{1}{n} \sum_{i=1}^{n} \rho\left(\frac{r_i(\beta)}{s_n}\right) = b,
\]

where $b$ is usually chosen such that consistency under the Gaussian distribution is obtained. For the MM-(adaptive) Lasso, (18) is extended by the penalty terms of (3) and (4), respectively [26].

The proposed MM-RWAL estimator minimizes an MM objective function to which a robust adaptive $\ell_1$-penalty term is added:

\[
\hat{\beta}_{\text{MM-RWAL}} = \arg \min_{\beta} \frac{1}{n} \sum_{i=1}^{n} \rho(\frac{r_i(\beta)}{s_n(r(\beta_i))} + \lambda_n \sum_{j=1}^{p} \hat{w}_j|\beta_j|.\]

(19)

Here, the MM-Lasso estimator [26] is used to calculate the weights according to $\hat{w}_j = 1/|z_j: \hat{\beta}_{\text{MM}}|$, with $z_j$ defined in [23].

To robustify the variable selection of the adaptive MM Lasso, we propose to incorporate a measure of outlyingness for each predictor. We use the Stahel Donoho Outlyingness (SDO) [27] and adjust it in a similar vein to the Adjusted Stahel Donoho Outlyingness of [23]. Let $B = \{b_1, b_2, \ldots, b_n\} \subset \mathbb{R}^p$ be a set of $n$ observations. Then, the robust Stahel Donoho outlyingness is given by

\[
r(b_i, x) = \max_{a \in S_p} \frac{a^Tb_i - \text{med}(a^T x)}{\text{mad}(a^T x)}, \quad i = 1, \ldots, n,
\]

(20)

where $S_p = \{a \in \mathbb{R}^p : \|a\|_2 = 1\}$ and $\text{med}(\cdot)$ and $\text{mad}(\cdot)$ denote the median and the median absolute deviation (mad). Since we assume that most rows flagged by the SDO as outliers are not outlying in all of their components, the SDO is extended by also taking into account the outlyingness of the predictors. The idea that has been introduced in [23] in a

**Definition III.2. (Strong Robust Oracle Properties)**

Let $\{x_1, x_2, \ldots, x_p\}$ be the set of predictors,

\[A := \{j : \beta_j \neq 0 \wedge \Pi_{k=1}^{k} (1 - (\epsilon)_{j:k}) < 0, k \in \{1, \ldots, p\}\}\]

(16)

the set of indices corresponding to the set of robust active predictors, and $A^n \subseteq \{j : \beta_{j:n} \neq 0\}$ the set of indices corresponding to the set of predictors chosen by a Lasso type estimator to be active. Then, a Lasso type estimator needs to have the following properties to be a robust oracle estimator:

1) Consistency in variable selection:

\[
\lim_{n \to \infty} P(A^n_n = A) = 1.
\]

2) Asymptotic normality:

\[
\sqrt{n}(\hat{\beta}_A - \beta_A) \overset{d}{\to} N(0, \Sigma^*), \text{ where } \Sigma^* \text{ is the covariance matrix, knowing the true subset model.}
\]
similar vein, is to adjust the SDO of every observation using the outlyingness of every single predictor. This gives us the Predictor Outlyingness (PO)

\[ c_j = \sum_{i=1}^{n} \frac{|x_{ij} - \text{med}(x_j)|}{\text{mad}(x_j)}, \quad j = 1, \ldots, p. \]  

(21)

Combining both, the SDO and the PO, we introduce an outlyingness-matrix, whose \((i,j)\)-th element is

\[ r_{ij} = \alpha z_i + (1 - \alpha) c_j, \quad i = 1, \ldots, n, \quad j = 1, \ldots, p. \]  

(22)

We chose the tuning parameter \(\alpha\) to be 0.5 throughout this paper, to equally weigh the SDO and the PO, in order to perform well in both contamination models, THCM and ICM. Finally, by applying a weight function \(w(\cdot)\), summing up the rows of the outlyingness matrix and dividing by its cell sum, we obtain the weights

\[ z_j = \frac{\sum_{i=1}^{p} \sum_{j=1}^{n} w(r_{ij})}{\sum_{i=1}^{n} \sum_{j=1}^{p} w(r_{ij})}, \quad j = 1, \ldots, p. \]  

(23)

where \(\sum_{j=1}^{p} z_j = p\).

In order to downweight cells, whose overall outlyingness exceeds a certain threshold, we choose \(w(\cdot)\) to be the Huber weight function,

\[ w(r) = \mathbb{I}(r \leq c) + \frac{c}{r} (c/r)^2 \mathbb{I}(r > c), \]  

(24)

with \(c = \min(\sqrt{x_{2p}^2(0.5)}, 4)\) as proposed in [28].

### C. Analysis of the Proposed Predictor Weights

In this section, we analyze the behavior of the proposed weights \(z_j\). For this purpose, we define two sets:

1. \(C_j = \{i : w(r_{ij}) < 1, \ i = 1, \ldots, n\}\) with cardinality \(|C_j| = \lfloor n \epsilon_j \rfloor\), is the set of indices corresponding to the contaminated cells in predictor \(j\).

2. \(E = \{l : \epsilon_l > 0, \ l = 1, \ldots, p\}\) with cardinality \(|E| = \gamma\) is the set of indices corresponding to the contaminated predictors, so \(\gamma\) is the number of contaminated predictors.

We will start with the most general formula that describes the behavior of our weights and introduce step by step assumptions, which simplify the equation.

Let \(\epsilon_j\) be the fraction of contamination of the \(j\)th predictor, and \(r_{ij}\) the magnitude of outlyingness of each cell in our designed outlyingness-matrix. Then, the weight of the \(j\)th predictor is given by

\[ z_j = \frac{p}{\sum_{i=1}^{p} \sum_{j=1}^{n} w(r_{ij})} \sum_{i=1}^{n} w(r_{ij}) \]  

(25)

\[ = \frac{p \cdot n (1 - \epsilon_j) + \sum_{i \in C_j} \left( \frac{c}{r_{ij}} \right)^2}{pm - n \sum_{i \in E} \epsilon_i + \sum_{i \in E} \sum_{l \in C_l} \left( \frac{c}{r_{il}} \right)^2}. \]  

(26)

Now, we assume that the magnitude of outlyingness is fixed, that is \(r_{ij} = r\) for all \(i, j\). This yields

\[ z_j = \frac{p \cdot \left[ 1 - \epsilon_j \left( 1 - \left( \frac{c}{r} \right)^2 \right) \right]}{p - \left( 1 - \left( \frac{c}{r} \right)^2 \right) \gamma \epsilon}. \]  

(27)

Eq. (27) provides additional insights that help us understand the behavior of our proposed weights. It shows us: the larger \(\epsilon_j\) the smaller \(z_j\). This leads to

**Proposition III.3.**

Let \(\epsilon_j \in E, r_{ij} = r\) for all \(i, j\) and \(\epsilon_1 \geq \epsilon_2 \geq \ldots \geq \epsilon_p\). Then, it follows that

\[ z_1 \leq z_2 \leq \ldots \leq z_p. \]

In this step, we assume that all contaminated predictors contain the same fraction of contamination, which results in

\[ z_j = \frac{p \cdot \left[ 1 - \epsilon \left( 1 - \left( \frac{c}{r} \right)^2 \right) \right]}{p - \left( 1 - \left( \frac{c}{r} \right)^2 \right) \gamma \epsilon}. \]  

(28)

To carry out a plausibility analysis for the derived formulas, we assume now that all \(p\)-predictors are contaminated (\(\gamma = p\)) with the same fraction of outliers \(\epsilon\). Intuitively, we expect that all predictors get the same weight and no one preferred over an other one. Since our weights are designed to sum up to \(p\), we expect each predictor \(j \in \{1, \ldots, p\}\) to receive the weight \(z_j = 1\). Applying these assumptions to Eq. (28), we obtain

\[ z_j = \frac{p - p \epsilon \left( 1 - \left( \frac{c}{r} \right)^2 \right)}{p - p \epsilon \left( 1 - \left( \frac{c}{r} \right)^2 \right)} = 1, \]  

(29)

which confirms our expectations.

The proof of the robust oracle properties in the next section requires the weights \(z_j\) to be smaller than one. Here, we will prove this property only for Eq. (28).

**Proposition III.4.**

If \(\epsilon_j > 0\), then \(z_j \leq 1\) for all \(j\).
We now formulate our assumptions as follows:

Theorem 2 in [29] and Theorem 5 in [26].

In this proof, we stick to the notation and structure of the asymptotic normality.

Definition III.1. All occurring $\rho(\cdot)$-functions are twice continuously differentiable and $\Psi = \rho'$.

A3. The second moments of $G$ exist.

A4. The estimator $\hat{\beta}_R := \hat{\beta}_{R,\text{MM}}$ is a $\sqrt{n}$-consistent estimator of $\beta_R := \beta_{R,\text{MM}}$, where $\beta_R$ complies with Definition III.1.

A5. $\hat{\beta}_2$ is $\sqrt{n}$-consistent.

We next have to show that $G_n(u_1, u_2) - G_n(u_1, 0_{p-s}) > 0$ holds under the given condition and when $\|u\| > 0$. Let

$$G_n(u_1, u_2) - G_n(u_1, 0_{p-s}) = D + E$$

with $D$ and $E$ being defined in Eq. (32).

With the Mean Value Theorem, we obtain

$$D = (0_s, u_2)^\top \frac{-1}{\sqrt{n}s_n(r(\beta_1))} \sum_{i=1}^n \Psi_i \left( \frac{r_i(\theta_n^*)}{s_n(r(\beta_1))} \right) x_i,$$

where $\theta_n^* = (\beta_{0,1} + u_1/\sqrt{n}, 1 - \alpha_n)u_2/\sqrt{n}$ and $\alpha_n \in [0, 1]$. 

D. Proof of the Robust Oracle Properties for the Proposed MM-RWAL

To prove that the MM-RWAL possesses at least the weak robust oracle properties, we will need some assumptions. Let $G$ denote the distribution of the rows $(x_{i1}, x_{i2}, \ldots, x_{ip})$ in $X$ and let $F$ be the distribution of the errors $u_i$, which results in the distribution $H$ of the data points $(x_{i1}, x_{i2}, \ldots, x_{ip}, y_i)$ to become

$$H(x, y) = G(x) F(y - x^\top \beta).$$

We now formulate our assumptions as follows:

A1. All occurring $\rho(\cdot)$-functions are twice continuously differentiable and $\Psi = \rho'$.

A2. The density $f$ of the error terms $u$ is an even and monotonically decreasing function of $|u|$.

A3. The second moments of $G$ exist.

A4. The estimator $\hat{\beta}_R := \hat{\beta}_{R,\text{MM}}$ is a $\sqrt{n}$-consistent estimator of $\beta_R := \beta_{R,\text{MM}}$, where $\beta_R$ complies with Definition III.1.

A5. $\hat{\beta}_2$ is $\sqrt{n}$-consistent.

Note, that we will explicitly prove only the consistency in variable selection here, because the weights $z_j$ do not affect the asymptotic normality.

Proof. (Consistency in Variable Selection)

In this proof, we stick to the notation and structure of Theorem 2 in [29] and Theorem 5 in [26].

With (A4.) and $\beta_0 = (\beta_{0,1}, \beta_{0,II})^\top$ being the true parameter vector corresponding to Definition III.1, where $I = A$ are the $s$ indices belonging to the robust active predictors and $II = \{1, \ldots, p\} \setminus A = A^C$ is the complementary set of $A$.

we know that, with arbitrarily high probability, there exists a constant $M_1 > 0$ with

$$\|\hat{\beta}_R - \beta_0\| < \frac{M_1}{\sqrt{n}}$$

(31)

Now, let

$$G_n(u_1, u_2) = \sum_{i=1}^n \rho \left( \frac{r_i(\beta_0, I) + u_1/\sqrt{n}, \beta_{0,II} + u_2/\sqrt{n}}{s_n(r(\beta_1))} \right)$$

\[ + \lambda_n \sum_{j=1}^{s} \left| \beta_{0,j} + u_{1,j}/\sqrt{n} \right| \frac{1}{| \beta_{2,j} |} \]

\[ + \lambda_n \sum_{j=s+1}^{p} \left| u_{2,j-s}/\sqrt{n} \right| \frac{1}{| z_j \cdot \beta_{2,j} |}. \]

We obtain $(\hat{\beta}_{R,1}, \hat{\beta}_{R,II})$ by minimizing $G_n(u_1, u_2)$, subject to $\|u_1\| + \|u_2\|^2 \leq M_2^2$ that we get from:

$$\|\hat{\beta}_R - \beta_0\| \leq \frac{M_1}{\sqrt{n}}$$

$$\iff \|\beta_{0,1} + u_1/\sqrt{n} + \beta_{0,II} + u_2/\sqrt{n} - \beta_0\| \leq \frac{M_1}{\sqrt{n}}$$

$$\iff \|u_1 + u_2/\sqrt{n}\| \leq \frac{M_1}{\sqrt{n}}$$

$$\iff \|u_1\|^2 + \|u_2\|^2 \leq M_2^2.$$
Applying the Mean Value Theorem a second time yields
\[
(\mathbf{0}_s, \mathbf{u}_2)^\top \frac{-1}{\sqrt{n}s_n(\mathbf{r}(\beta_1))} \sum_{i=1}^{n} \Psi_1 \left( \frac{r_i(\theta_n^*)}{s_n(\mathbf{r}(\beta_1))} \right) \mathbf{x}_i
\]
\[
+ \frac{1}{\sqrt{n}s_n^2(\mathbf{r}(\beta_1))}(\mathbf{0}_s, \mathbf{u}_2)^\top \sum_{i=1}^{n} \Psi_1' \left( \frac{r_i(\theta_n^*)}{s_n(\mathbf{r}(\beta_1))} \right) \mathbf{x}_i^T \mathbf{x}_i^\top \cdot (\mathbf{u}_1/\sqrt{n}, (1 - \alpha_n)\mathbf{u}_2/\sqrt{n})
\]
\[
= \frac{-1}{\sqrt{n}s_n(\mathbf{r}(\beta_1))}(\mathbf{0}_s, \mathbf{u}_2)^\top \sum_{i=1}^{n} \Psi_1 \left( \frac{r_i(\beta_0)}{s_n(\mathbf{r}(\beta_1))} \right) \mathbf{x}_i
\]
\[
+ \frac{1}{n s_n^2(\mathbf{r}(\beta_1))}(\mathbf{0}_s, \mathbf{u}_2)^\top \sum_{i=1}^{n} \Psi_1' \left( \frac{r_i(\beta_0)}{s_n(\mathbf{r}(\beta_1))} \right) \mathbf{x}_i x_i^\top \cdot (\mathbf{u}_1, (1 - \alpha_n)\mathbf{u}_2) = O_p(\|\mathbf{u}_2\|),
\]
where \(\|\theta_n - \beta_0\| \leq \|\theta_n - \beta_0\|\). The last equation follows from the assumptions (A1)-(A3), Lemma 1 in [26] and Lemma 5 in the Technical Report associated with [30].

Additionally, we have that \(E\) is stochastically bounded from below:
\[
E = \lambda_n \sum_{j=s+1}^{p} \frac{|u_{2j-s}|}{|z_j| \cdot \sqrt{n} |\beta_{2j}|} = \lambda_n \sum_{j=s+1}^{p} \frac{|u_{2j-s}|}{|z_j| \cdot \sqrt{n} |\beta_{2j}|} + \sum_{\{j: \beta_j=0 \land \epsilon_j > 0\}} |z_j| \cdot \sqrt{n} |\beta_{2j}|
\]
\[
= \lambda_n \left[ \sum_{\{j: \beta_j=0 \land \epsilon_j > 0\}} |u_{2j}| \cdot \sqrt{n} |\beta_{2j}| + \sum_{\{j: \beta_j=0 \land \epsilon_j > 0\}} |z_j| \cdot \sqrt{n} |\beta_{2j}| \right]
\]
\[
= \lambda_n \left[ \sum_{\{j: \beta_j=0 \land \epsilon_j > 0\}} |u_{2j}| \cdot \sqrt{n} |\beta_{2j}| + \sum_{\{j: \beta_j=0 \land \epsilon_j > 0\}} |z_j| \cdot \sqrt{n} |\beta_{2j}| \right]
\]
\[
\geq \lambda_n \cdot \left[ \sum_{\{j: \beta_j=0 \land \epsilon_j > 0\}} |u_{2j}| \cdot \sqrt{n} |\beta_{2j}| + \sum_{\{j: \beta_j=0 \land \epsilon_j > 0\}} |z_j| \cdot \sqrt{n} |\beta_{2j}| \right]
\]
\[
= \lambda_n \sum_{j=s+1}^{p} \frac{|u_{2j}|}{\sqrt{n} |\beta_{2j}|} + \lambda_n \sum_{j=s+1}^{p} \frac{|z_j|}{\sqrt{n} |\beta_{2j}|}
\]
\[
\geq \lambda_n \cdot \left[ \sum_{\{j: \beta_j=0 \land \epsilon_j > 0\}} |u_{2j}| \cdot \sqrt{n} |\beta_{2j}| + \sum_{\{j: \beta_j=0 \land \epsilon_j > 0\}} |z_j| \cdot \sqrt{n} |\beta_{2j}| \right]
\]
\[
= \lambda_n \sum_{j=s+1}^{p} \frac{|u_{2j}|}{\sqrt{n} |\beta_{2j}|} + \lambda_n \sum_{j=s+1}^{p} \frac{|z_j|}{\sqrt{n} |\beta_{2j}|}
\]

The above inequality follows from Proposition III.4 and the last equation follows with assumption (A5). Now, let \(M_2, M_3 > 0\) be some real numbers with \(M_3 \lambda_n > M_2\), then we have with arbitrarily high probability
\[
G_n(\mathbf{u}_1, \mathbf{u}_2) - G_n(\mathbf{u}_1, 0_{p-s}) > -M_2 \|\mathbf{u}_2\| + M_3 \lambda_n \|\mathbf{u}_2\|
\]
\[
= \|\mathbf{u}_2\| \cdot (-M_2 + M_3 \lambda_n) > 0
\]
and the proposition follows for sufficiently large \(n\).

\[\square\]

E. Computation of the Weights for MM-RWAL

The main problem of calculating the weights \(z_j\) is to compute the supremum in the SDO, because the cardinal number of \(S_p\) is infinite and the objective function is non-convex. Therefore, we need to apply a random search algorithm to obtain an approximation of the supremum. We chose to take a subsample from \(S_p\) by sampling from a \((p-1)\)-dimensional unit-hypersphere, as in [31]. We use the following algorithm [32] to obtain \(S_p\) in [20].

Algorithm III.5. (Uniform Sampling from a \(p\)-Dimensional Unit Hypersphere)

1) Generate \(p\) vectors with \(k\) entries
\[
\mathbf{x}_j = (x_{1j}, x_{2j}, \ldots, x_{kj})^\top, \quad j = 1, \ldots, p,
\]
where \(x_{ij} \sim \mathcal{N}(0,1)\).

2) Calculate \(k\) \(p\)-dimensional vectors
\[
a_i = \sum_{j=1}^{p} \frac{x_{ij}}{\sqrt{x_{1j}^2 + x_{2j}^2 + \ldots + x_{kj}^2}} \cdot e_j, \quad i = 1, \ldots, k,
\]
where \(e_j\) is the \(j\)th unit vector.

3) Set \(S_p := \{a_i \in \mathbb{R}^p : i \in \{1, \ldots, k\}\}\).
IV. Numerical Experiments

A. Simulation Setup

Two different Monte Carlo studies are conducted, to assess the performance of the proposed RWAD MM-Lasso.

Scenario 1: \(p > n\), correlated predictors, cellwise outliers

A setup with \(p = 50\) predictors and \(n = 30\) observations is considered. The regression parameters are defined by \(\beta_j = j/5\) \(j \in \{1, \ldots, 5\}\), while \(\beta_j = 0\) \(j \in \{6, \ldots, 50\}\). Correlated predictors \(x_j, j \in \{1, \ldots, p\}\) are generated by sampling from a multivariate zero mean Gaussian distribution with covariance matrix \(\Sigma_{ij} = 0.5^{i-j}, \forall i, j \in \{1, \ldots, p\}\). The errors \(u_i\) are zero mean i.i.d. Gaussian distributed with variance \(\sigma^2 = 0.5^2\). The responses \(y_i\) follow the linear model

\[ y_i = x_i^T \beta + u_i, \quad i = 1, \ldots, n. \]

To create cellwise outliers in \(X\), \(\epsilon = \{0\%, 10\%, 20\%, 30\%\}\) of the predictors \(x_j\) are contaminated. For the contaminated predictors, 30\% of the entries are independently and additively contaminated by samples drawn from the distribution \(x_{\text{cont}} \sim N(0, 100^2)\).

Scenario 2: \(p > n\), correlated predictors, cellwise outliers and additive outliers

The setup is identical to Scenario 1 but additionally, 5\% of the responses are additively contaminated by samples drawn from the distribution \(y_{\text{cont}} \sim N(0, 100^2)\).

B. Performance Measures

To assess the performance in terms of parameter estimation and model selection, we display the average mean squared error (MSE)

\[ \text{MSE}(\hat{\beta}) = \frac{1}{R} \sum_{r=1}^{R} \frac{1}{p} \sum_{j=1}^{p} (\beta_j - \hat{\beta}_j)^2, \]  

the average false positive rate (FPR)

\[ \text{FPR}(\hat{\beta}) = \frac{1}{R} \sum_{r=1}^{R} \frac{|\{j \in \{1, \ldots, p\} : \beta_j = 0 \land \hat{\beta}_j^{(r)} \neq 0\}|}{|\{j \in \{1, \ldots, p\} : \beta_j = 0\}|}, \]  

and the average false negative rate (FNR)

\[ \text{FNR}(\hat{\beta}) = \frac{1}{R} \sum_{r=1}^{R} \frac{|\{j \in \{1, \ldots, p\} : \beta_j \neq 0 \land \hat{\beta}_j^{(r)} = 0\}|}{|\{j \in \{1, \ldots, p\} : \beta_j \neq 0\}|}, \]  

where \(\hat{\beta}_j^{(r)}\) refers to the parameter estimate of the \(r\)th Monte Carlo experiment. All results represent averages over \(R = 100\) Monte Carlo simulations.

C. Benchmark Methods and Choice of Parameters

The performance of the MM-RWAL is compared to:

- OLS Lasso \[17\]
- OCD Lasso \[32\]
- M-Lasso and adaptive M-Lasso \[9\]
- MM-Lasso and adaptive MM-Lasso \[26\]
- Sparse LTS \[24\]

For all methods, we use a grid of \(N_\lambda = 1000\) candidate regularization parameters, which are equally spaced on the interval \((0, \lambda_{\text{max}}]\). Here, \(\lambda_{\text{max}}\) is the value that results in a Lasso estimate for which all regression parameters are equal to zero, i.e., \(\beta = 0\). \(\lambda_{\text{max}}\) is calculated by

\[ \lambda_{\text{max}} = \frac{2}{n \max_{j \in \{1, \ldots, p\}} y_j^T x_j}, \]  

using a robust correlation, as described in \[25\]. For all methods, we choose the penalty parameter \(\lambda\) that provides the lowest MSE so that the different methods are comparable, independent of the method that is used to select \(\lambda\).

For the OCD Lasso we use a threshold of \(c_{\text{huber}} = 1.215\) is used, while for the adaptive M-Lasso we use the bisquare \(\rho\)-function with a clipping of \(c_{\text{bisquare}} = 3.44\).

The initial estimate is the M-Lasso solution, as proposed in \[9\]. For the sparse LTS, we use a subsample proportion of 75\%. Further, 500 subsamples are used for the first two C-Steps and the best 10 subsample sets are kept to carry out the C-Steps until convergence \[25\]. For the MM-Lasso, the S-Ridge estimator serves as initial estimate, as described in \[24\]. For the MM-Lasso \[26\], we use a bisquare \(\rho\)-function with clipping constant \(c_{\text{bisquare}} = 3.44\). The adaptive MM-Lasso and MM-RWAL use the same bisquare function and are initialized with the the MM-Lasso estimate.

D. Simulation Results

Tables \[I\] and \[II\] display the estimation results for Scenarios 1 and Scenarios 2, respectively. Table \[III\] documents the average computation time for one Monte Carlo Run of Scenario 2 for the different methods, using an Intel® Core™ i7-4510U with 8 GB RAM. In both setups, the OLS Lasso breaks down even for low contamination in either the regressors or the responses and \(\text{FNR}(\hat{\beta}_{\text{OLS}}) > 0.95\) since the OLS Lasso selects a regularization parameter for which all coefficients are equal to zero. The MM-RWAL performs best in terms of robust model selection. In all experiments, it maximizes the probability of correctly finding the indices of the non-zero parameters, i.e., \(1 - (\text{FPR} + \text{FNR})\), and in most cases provides the best parameter estimation accuracy in terms of the MSE. A drawback is that it is computationally heavy, which can be contributed mainly to the MM-Lasso algorithm. Future work will investigate combining the RWAL with computationally more efficient estimators.

V. A REAL DATA EXAMPLE OF SOURCE ESTIMATION FOR AN ATMOSPHERIC INVERSE PROBLEM

Quantifying the emissions of a pollutant into the atmosphere is essential, for example, in the case of nuclear power plant accidents, volcano eruptions, or to track the releases of greenhouse gases. In this paper, we apply penalized robust estimation to determine the temporal releases of the particles of the European Tracer Experiment (ETEX) at the source location. During the ETEX experiment tracers (perfluorocarbons)
Table I

| Estimator       | $\epsilon = 0\%$ | $\epsilon = 10\%$ | $\epsilon = 20\%$ | $\epsilon = 30\%$ |
|-----------------|------------------|-------------------|-------------------|-------------------|
|                 | $n$-MSE | FPR | FNR | $n$-MSE | FPR | FNR | $n$-MSE | FPR | FNR | $n$-MSE | FPR | FNR |
| OLS Lasso [17]  | 0.14   | 0.11 | 0.74 | 1.88   | 0.019 | 0.95 | 2.8    | 0.019 | 0.97 | 2.76   | 0.018 | 0.97 |
| OCD Lasso [32]  | 0.15   | 0.11 | 0.078 | 1.09   | 0.034 | 0.71 | 1.43   | 0.06  | 0.40 | 0.92   | 0.09  | 0.40 |
| M-Lasso [9]     | 1.22   | 0.43 | 0.604 | 1.25   | 0.064 | 0.71 | 1.51   | 0.07  | 0.41 | 1.26   | 0.09  | 0.41 |
| ad. M-Lasso [9] | 0.15   | 0.066| 0.12  | 0.36   | 0.07  | 0.24 | 0.41   | 0.07  | 0.30 | 0.53   | 0.05  | 0.41 |
| MM-Lasso [26]   | 0.16   | 0.19 | 0.09  | 0.39   | 0.27  | 0.11 | 0.34   | 0.098 | 0.74 | 0.43   | 0.16  | 0.41 |
| ad. MM-Lasso [26]| 0.15   | 0.052| 0.16  | 0.34   | 0.28  | 0.04 | 0.073  | 0.34  | 0.61 | 0.084  | 0.46  | 0.16 |
| MM-RW AL        | 0.13   | 0.05 | 0.09  | 0.33   | 0.046 | 0.11 | 0.40   | 0.056 | 0.98 | 0.62   | 0.042 | 0.16 |
| sparse LTS [25] | 0.24   | 0.12 | 0.16  | 0.39   | 0.11  | 0.28 | 0.43   | 0.10  | 0.31 | 0.58   | 0.092 | 0.38 |

Table II

| Estimator       | $\epsilon = 0\%$ | $\epsilon = 10\%$ | $\epsilon = 20\%$ | $\epsilon = 30\%$ |
|-----------------|------------------|-------------------|-------------------|-------------------|
|                 | $n$-MSE | FPR | FNR | $n$-MSE | FPR | FNR | $n$-MSE | FPR | FNR | $n$-MSE | FPR | FNR |
| OLS Lasso [17]  | 2.24   | 0.013| 0.98 | 1.88   | 0.019 | 0.95 | 2.8    | 0.019 | 0.97 | 2.76   | 0.018 | 0.97 |
| OCD Lasso [32]  | 1.34   | 0.016| 0.98 | 1.83   | 0.022 | 0.37 | 0.016 | 0.97  | 2.69 | 0.029  | 0.95  | 0.95 |
| M-Lasso [9]     | 1.23   | 0.35 | 0.16 | 1.20   | 0.35  | 0.16 | 1.24   | 0.37  | 0.21 | 1.24   | 0.36  | 0.26 |
| ad. M-Lasso [9] | 0.46   | 0.031| 0.16 | 0.52   | 0.031 | 0.40 | 0.59   | 0.036 | 0.43 | 0.75   | 0.023 | 0.56 |
| MM-Lasso [26]   | 0.29   | 0.17 | 0.15 | 0.44   | 0.27  | 0.14 | 0.59   | 0.35  | 0.15 | 0.83   | 0.46  | 0.16 |
| ad. MM-Lasso [26]| 0.21   | 0.038| 0.22 | 0.29   | 0.052 | 0.28 | 0.45   | 0.056 | 0.38 | 0.65   | 0.067 | 0.48 |
| MM-RW AL        | 0.21   | 0.038| 0.15 | 0.29   | 0.042 | 0.14 | 0.44   | 0.035 | 0.15 | 0.63   | 0.036 | 0.16 |
| sparse LTS [25] | 0.23   | 0.11 | 0.17 | 0.32   | 0.10  | 0.22 | 0.45   | 0.097 | 0.32 | 0.56   | 0.097 | 0.41 |

Table III

| Estimator       | ACT [s] |
|-----------------|---------|
| OLS Lasso [17]  | 0.02    |
| OCD Lasso [32]  | 0.97    |
| M-Lasso [9]     | 95.00   |
| ad. M-Lasso [9] | 97.51   |
| MM-Lasso [26]   | 177.63  |
| ad. MM-Lasso [26]| 188.97  |
| MM-RW AL        | 191.54  |
| sparse LTS [25] | 266.79  |

were released into the atmosphere in Monterfil, Brittany in 1994. Hourly measurements were taken at 168 ground-level sampling stations in 17 European countries as illustrated in Fig. 3.

Atmospheric dispersion models, such as the Lagrangian Particle Dispersion Model (LPDM) allow to formulate the source estimation problem as a linear inverse problem according to Eq. (1) as follows. The regression matrix $X$ is
The ETEX data is sparse in $\beta$, since only 12 of the $p = 120$ unknown regression variables. Every regression parameter $\beta_j, j \in \{1, \ldots, p\}$ corresponds to the amount of PFC that is released by the source at time instant $j$. The sampling interval is one hour and 120 measurements are taken at each sampling station, resulting in $p = 120$ unknown regression variables. The responses $y = (y_1, \ldots, y_n)^\top = (\tilde{y}_1, \ldots, \tilde{y}_k, \ldots, \tilde{y}_K)^\top$ are a set of stacked observations of the $K = 168$ sensors.

The ETEX data is sparse in $\beta$, because only 12 of the regression parameters, i.e. 10%, are unequal to zero. The residuals $r = y - X\beta$, given the ground truth values of $\beta$, are non-Gaussian, as displayed by the histogram in Fig. 3. Additionally, the regression matrix $X$ contains outliers, as exemplified for the 68th predictor via the histogram in Fig. 3. Furthermore, $X$ is sparse, since most of the time, the PFC particles do not reach a sensor, resulting in a regression matrix, which is dominated by zero-valued cells.

A. Pre-Processing of the ETEX data

The following preprocessing steps are applied to the data:

1) Remove data points $(x_j, y_j)^\top$, where all entries of the predictor $x_j$ are equal to zero.

2) Normalize the data robustly using the mad and the median, 
$$y \leftarrow \frac{y - \text{med}(y)}{\text{mad}(y)}$$
$$x_j \leftarrow \frac{x_j - \text{med}(x_j)}{\text{mad}(x_j)}.$$ 

3) When applying the median or the mad to the predictors $x_j$, only use samples which are greater than zero.

4) Apply a robust PCA [33] and reconstruct $X$ using only $N_p$ principal components, such that the mad of the $N_p$ principal components corresponds to 90% of the total mad.

5) Further, since we know that the number of particles omitted by the source can only be positive $\beta_j \geq 0$, we impose a non-negativity constraint on the parameters $\beta_j$, $j \in \{1, \ldots, p\}$.

This leads to the positive Lasso.

B. The Positive MM-RWAL Estimator

**Definition V.1. (The Positive Lasso Estimator)**

$$\hat{\beta}_{\text{pos Lasso}} = \arg \min_\beta \|y - X\beta\|_2^2,$$

subject to $\|\beta\|_1 \leq t$ and $\beta_j \geq 0$,

$$j \in \{1, \ldots, p\}. \quad (39)$$

The positive Lasso can be calculated using a modified cyclic coordinate descent algorithm as follows.

**Algorithm V.2. (The Positive Lasso Estimator Using Cyclic Coordinate Descent)**

1) **Standardize the regressors so that** $\sum_i x_{ij} / n = 0$ and $\sum_j x_{ij}^2 = 1$.

2) **Initialize the regression parameters with an arbitrary value,** e.g., $\beta_j = 0$, $j \in \{1, \ldots, p\}$.

3) **Calculate the update of the $j$th regression parameter by keeping the parameters $k \neq j$ fixed. Start with $j = 1$ and end with $j = p$. If a regression parameter is negative replace it by zero**

$$\hat{\beta}_j = \max \left\{ S \left( \sum_{i=1}^n x_{ij}(y_i - \sum_{k \neq j} x_{ik}\beta_k), \lambda \right), 0 \right\}.$$ 

4) **Repeat step 3 until convergence.**

Here, $S(x, \lambda) = \text{sign}(x)|x| - \lambda)_+$ is the soft thresholding function with $(\cdot)_+ = \max\{\cdot, 0\}$.

1) The predictors contain an overwhelming number of components which are zero since $X$ is highly sparse. Robust estimates like the median or the mad will result in a value of zero if more than 50% of the entries of $x_j$ are zero. Obviously, taking only the positive components into account leads to estimates that are based on the distribution of the data that we are actually interested in.

2) The reason for applying a robust PCA is to provide for all algorithms a matrix that is not as badly conditioned as the original regression matrix. The value 90% has been empirically determined from a range of possible values between 85% - 99%.
The performance metrics are the mean squared error (MSE) the false positive rate (FPR) and the false negative rate (FNR). However, the OLS Lasso estimate still has poor model selection properties. The OCD Lasso estimates are dominated by outlying predictors rendering this estimator useless for the above real-data application. The MM-RW AL is the only estimator, which correctly detects all coefficients that are equal to zero, i.e. FNR=0. The MM-RW AL also by far outperforms its competitors in correctly finding the indices of the non-zero parameters, i.e. $1 - (\text{FPR} + \text{FNR}) = 0.947$.

### VI. Conclusion

The problem of finding sparse solutions to underdetermined, or ill-conditioned, linear regression problems that are contaminated by cellwise and rowwise outliers was investigated. We defined 'robust oracle properties' that are required to perform robust variable selection for such models. We introduced and analyzed a robustly weighted and adaptive Lasso type regularization term and integrated it into the objective function of the MM-estimator, resulting in the proposed $\text{MM-RW AL}$ for which we showed that at least the weak robust oracle properties hold. An algorithm to compute the weights was proposed and analyzed. The $\text{MM-RW AL}$ outperformed existing robust sparse estimators in numerical experiments and proved its usefulness in a real-data application of estimating the sparse non-negative spatio-temporal emissions of a pollutant, given noisy observations and an imprecisely estimated ill-conditioned and sparse dispersion model containing cellwise and rowwise outliers. In future work, the proposed RW AL penalty can easily be integrated into the objective function of other rowwise robust estimators to extend them to the cellwise contamination framework.

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Fig. 6. Estimated source emissions for the Lasso, Ad. M-Lasso, Ad. MM-Lasso and MM-RWAL.