Thermal Width of the $\Upsilon$ at Large t’ Hooft Coupling

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Abstract

We use the AdS/CFT correspondence to show that the heavy quark (static) potential in a strongly-coupled plasma develops an imaginary part at finite temperature. Thus, deeply bound heavy quarkonia states acquire a small nonzero thermal width when the t’Hooft coupling $\lambda = g^2 N_c \gg 1$ and the number of colors $N_c \rightarrow \infty$. In the dual gravity description, this imaginary contribution comes from thermal fluctuations around the bottom of the classical sagging string in the bulk that connects the heavy quarks located at the boundary. We predict a strong suppression of $\Upsilon$’s in heavy-ion collisions and discuss how this may be used to estimate the initial temperature.

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The conjectured equivalence of strongly-coupled 4-dimensional $\mathcal{N} = 4$ Supersymmetric Yang-Mills (SYM) to type IIB string theory on $\text{AdS}_5 \otimes \text{S}^5$ \cite{1} has led to new insight into the strong coupling dynamics of large $N_c$ gauge theories at finite temperature. In fact, the Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence has been particularly useful to compute real time correlators of gauge invariant quantities in strongly-coupled plasmas such as 2-point functions of the energy-momentum tensor at finite temperature \cite{2}. For instance, it was shown that the shear viscosity to entropy density ratio satisfies $4\pi \eta/s \geq 1$ in all strongly-coupled gauge theories that possess a dual description in terms of supergravity \cite{3}. Here, we adopt strongly coupled $\mathcal{N} = 4$ SYM as a toy model for the deconfined, high temperature phase of QCD.

The dual description of the gauge theory at finite temperature involves a near-extremal black brane in the bulk, which leads to a 5-dimensional metric (in real time) given by

$$ds^2 = -G_{00}(U)dt^2 + G_{xx}(U)d\vec{x}^2 + G_{UU}(U)dU^2.$$  \hfill (1)

$G_{00}(U_h) = 0$ defines the location $U_h$ of the black brane horizon in the 5th coordinate, and the boundary is at $U \to \infty$.

The potential between fundamental static sources separated by a distance $L$ at large $t'$ Hooft coupling $\lambda$ in $\mathcal{N} = 4$ SYM was computed in \cite{4, 5} and shown to be proportional to $1/L$ (due to conformal invariance of the theory) and to $\sqrt{\lambda}$, which indicates that charges are partially screened even in vacuum \cite{4, 6}. In general, thermal effects are expected to reduce the binding energies of small states of very heavy quarks at high $T$. At strong coupling, thermal screening corrections appear at the same order in $\lambda$ as the vacuum potential \cite{7}, as opposed to Debye screening in weakly coupled quark-gluon plasmas \cite{8}. However, at distances $L < 1/T$ these corrections are suppressed by a factor of $(LT)^4$, which originates from the behavior of the dual bulk geometry near the black brane horizon. Thermal effects also diminish as $\eta/s$ increases \cite{9}.

In this Letter we show that at finite temperature the static potential in a strongly-coupled plasma develops an imaginary part due to fluctuations about the extremal configuration, which corresponds to a string connecting the fundamental sources at the boundary of the geometry. Such an imaginary part arises also in perturbative Quantum Chromodynamics (pQCD) at order $\sim g^4$ due to Landau damping of the static gluon exchanged by the heavy quark sources \cite{10}. At large $t'$ Hooft coupling, however, it appears already at the same order.
as the vacuum potential, i.e., at $O(\sqrt{\lambda})$. Therefore, energy levels in this potential are not sharp because they acquire a thermal width, $E = E_{\text{vac}} + \Delta E_T - i \Gamma$. The width $\Gamma$ is smaller than the vacuum energy $E_{\text{vac}}$ if $LT < 1$ (which is the relevant regime in the limit of very heavy quarks, $m_Q \to \infty$) and of the same order in both $\lambda$ and $LT$ as the shift $\Delta E_T$ of the real part of the potential due to thermal screening effects. We propose that the imaginary part of the potential mentioned above can be observed experimentally via the suppression of $\Upsilon$ to dilepton decays in heavy-ion versus $p+p$ collisions at RHIC and LHC.

The relevant operator for our discussion is the path-ordered Wilson loop defined as

$$W(C) = \frac{1}{N_c} \text{Tr} P e^{i \int A_\mu dx^\mu}$$

(2)

where $C$ denotes a closed loop in the boundary, $A_\mu$ is the non-Abelian gauge field, and the trace is over the fundamental representation of $SU(N_c)$. We consider a rectangular loop with one direction along the time coordinate $t$ and spatial extension $L$. In the asymptotic limit $t \to \infty$, the vacuum expectation value of the loop defines a static potential via $\langle W(C) \rangle \sim e^{-itV_Q Q(L)}$. This is what we call the “heavy quark potential”.

The expectation value of $W(C)$ can be calculated at strong t’Hooft coupling in non-Abelian plasmas at large $N_c$ that admit a weakly coupled dual gravity description according to AdS/CFT [1]. More specifically,

$$\langle W(C) \rangle_{\text{CFT}} = Z_{\text{string}}$$

(3)

where $Z_{\text{string}}$ is the full supersymmetric string generating functional, which is defined in a 10 dimensional background spacetime and includes a sum over all the string worldsheets whose boundary coincide with $C$. In the supergravity approximation $\lambda = g^2 N_c \gg 1$ and $N_c \to \infty$ and, in this case, an infinitely massive excitation in the fundamental representation of $SU(N_c)$ in the CFT is dual to a classical string in the bulk hanging down from a probe brane at infinity [4, 5]. Within this approximation $Z_{\text{string}} \sim e^{i S_{\text{NG}}}$ and the dynamics of the string is given by the classical Nambu-Goto (NG) action (we neglect the contribution from other background fields such as the dilaton)

$$S_{\text{NG}} = -\frac{1}{2\pi \alpha'} \int d^2 \sigma \sqrt{-\det h_{ab}}$$

(4)

where $h_{ab} = G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu$ ($a, b = 1, 2$), $G_{\mu\nu}$ is the background bulk metric, $\sigma^a = (\tau, \sigma)$ are the internal world sheet coordinates, and $X^\mu = X^\mu(\tau, \sigma)$ is the embedding of the string in
the 10-dimensional spacetime. For \( \mathcal{N} = 4 \) SYM, the configuration that minimizes the action is a U-shaped curve that connects the string endpoints at the boundary and has a minimum at some \( U_* \) in \( \text{AdS}_5 \) \[4, 5\].

The induced metric is

\[
\det h_{ab} = X'{}^2 \cdot \dot{X}^2 - (\dot{X} \cdot X')^2
\]

where \( X'{}^\mu(\tau, \sigma) = \partial_\sigma X^\mu(\tau, \sigma) \) and \( \dot{X}{}^\mu(\tau, \sigma) = \partial_\tau X^\mu(\tau, \sigma) \). We choose a gauge where the coordinates of the static string are \( X^\mu = (t, x, 0, 0, U(x)) \), where \( \tau = t \) and \( \sigma = x \). We neglect the string dynamics in the 5 dimensional compact space and perform the calculation in real time at the boundary. In fact, fixing the extremal configuration in such a way implies the \( t \to \infty \) limit; while a Wick rotation is of course still possible (by switching to Euclidean metric, which would still give a complex expectation value for the Wilson loop) one can no longer perform an analytic continuation to imaginary time where the expectation value should be real.

In this gauge,

\[
S_{\text{NG}} = -\frac{T}{2 \pi \alpha'} \int_{-L/2}^{L/2} dx \sqrt{U'^2 + V(U)}
\]

where \( T \to \infty \) is the total time interval. Note that we have assumed that \( G_{00} G_{UU} = 1 \) (which is in general valid when \( \lambda \to \infty \)) and defined \( V(U) \equiv G_{00} G_{xx} \), which satisfies \( V(U) \geq 0 \) for \( U \in [U_h, \infty) \). The equations of motion obtained from Eq. (6) determine the classical string profile \( U_c(x) \) as discussed in detail in refs. \[4, 5, 7, 9\]. The solution \( x = x(U_c) \) satisfies the following boundary condition

\[
\frac{L}{2} = \int_{U_*}^\infty dU \left\{ V(U) \left[ \frac{V(U)}{V(U_*)} - 1 \right] \right\}^{-1/2},
\]

which is used to obtain \( U_* = U_*(L, T) \). The expectation value of the Wilson loop is obtained by substituting the classical string solution \( U_c(x) \) into the action in Eq. (6). In general, when \( LT \ll 1 \) the dominant contribution to the potential comes from the extremal worldsheet configuration described above and the potential is computed as a series in \( LT \). Other configurations are expected to contribute significantly when \( LT > 1 \) \[11\]. However, \( L < 1/T \) is in fact the region of interest for bound states of very heavy quarks which have small radii.

The heavy quark potential in the vacuum of \( \mathcal{N} = 4 \) SYM has the following simple analytical form (after subtracting the self-energy contribution from the infinitely massive
which may be compared to the standard SU($N_c$) Coulomb potential at large $N_c$ corresponding to weak coupling:

$$V_{\text{Coul}}(L) = -\frac{1}{8\pi} g^2 N_c \frac{L}{L}.$$  

We now take into account thermal fluctuations around the classical solution $U_c(x)$. We shall show that the Wilson loop develops an imaginary part due to fluctuations near the bottom of the classical string configuration $U_*$. We consider long wavelength fluctuations of the string profile $U_c(x) \to U_c(x) + \delta U(x)$ (with $\delta U' \to 0$), which give the leading contribution to the string partition function in the supergravity approximation as follows

$$Z_{\text{string}} \sim \int D\mathbf{x} e^{i S_{\text{NG}}(\mathbf{x})} \sim \int D\delta U(x) e^{i S_{\text{NG}}(U_c + \delta U)}.$$  

When the bottom of the classical string is sufficiently close to the horizon (though still above it), the worldsheet fluctuations, $\delta U(x)$, near $x = 0$ (where $U'_c = 0$) can change the overall sign of the argument of the NG square root and generate an imaginary contribution to the action. In this case, both $U'^2$ and $V(U_c)$ are small and the NG square root cannot be expanded in powers of $\delta U$.

The integral over $\delta U(x)$ is performed by dividing the $x$ interval in $2N$ parts such that $x_N = L/2$, $x_{-N} = -L/2$, $x_j = j\Delta x$:

$$Z_{\text{string}} \sim \int d(\delta U_{-N}) \cdots d(\delta U_N) e^{-\frac{i T \Delta x}{2 \pi \alpha'}} \sum_j \sqrt{U'^2_j + V(U_j)}.$$  

Near $x = 0$ we expand $U_c(x_j) \simeq U_* + x_j^2 U''(0)/2$ and $U'^2_j + V(U_j) \simeq C_1 x_j^2 + C_2$, with

$$C_1 = \frac{1}{2} U''(0) \left(2U''(0) + V'_* + \delta U V''_* + \frac{1}{2} \delta U^2 V'''_* \right)$$

$$\simeq \frac{1}{2} U''(0) \left(2U''(0) + V'_* \right) \geq 0$$

$$C_2 = V_* + \delta U V'_* + \frac{1}{2} \delta U^2 V''_*. \tag{12}$$

Here, $V_* = V(U_*)$, $V'_* = V'(U_*)$ and so on. The imaginary part of the $Q\bar{Q}$ potential arises from the region of $\delta U$ where $C_2 < 0$.

We isolate the contribution to the path integral from $x = x_j$,

$$I_j \equiv \int_{\delta U_{j,\text{min}}}^{\delta U_{j,\text{max}}} d(\delta U_j) \exp \left\{ -i \frac{T \Delta x}{2 \pi \alpha'} \left( x_j^2 C_1 + C_2 \right)^{1/2} \right\}$$  

$$\left\{ -i \frac{T \Delta x}{2 \pi \alpha'} \left( x_j^2 C_1 + C_2 \right)^{1/2} \right\}$$
where $\delta U_{j\min,max} < 0$ are defined as the zeros of the argument of the square root. In this range of $\delta U_j$ the square root in the exponent develops an imaginary part. (The complement of the integration region in Eq. (13) provides a correction to the real part of the potential due to fluctuations about the extremal configuration that is not considered here.) Note that in this case $U_* + \delta U \sim U_h$, i.e., the bottom of the fluctuating string touches the horizon. This may be viewed in the dual gauge theory as a process analogous to Landau damping of the static color fields which bind the quarks together, leading to the formation of two unbound heavy quarks in the high-temperature plasma.

On the other hand, at lower temperatures on the order of the QCD crossover temperature and below, the dominant process should instead correspond to the breakup $Q\bar{Q} \rightarrow (Q\bar{q}) (\bar{Q}q)$ into color-singlet heavy-light bound states; $q$ stands for a light quark. Such tunnelling processes provide an exponentially small thermal width of deeply bound states [12]. It should be clear that the problem of quark tunneling cannot be solved rigorously since it involves genuinely non-perturbative QCD dynamics. However, the large mass of the heavy quark allows one to use the quasiclassical approximation [12]. For a calculation of $Q\bar{Q} \rightarrow (Q\bar{q}) (\bar{Q}q)$ via the AdS/CFT correspondence see Ref. [13]. Here, too, the temperature must be sufficiently low to allow for the formation of two new heavy-light quark bound states. In the gravity description, the final state corresponds to two strings connecting the Q-brane with the q-brane while in our approach they connect to the black-hole horizon.

The factor $1/\alpha'$ in the exponent implies that the leading order contribution is of order $\sqrt{\lambda}$. In the supergravity approximation $\lambda \gg 1$ and, thus, $I_j$ can be computed in the saddle-point approximation. This gives $\delta U = -V'_* / V''_*$ and so

\[
\exp \left\{ -i T V_{QQ} \right\} = \prod_j I_j \\
\sim \exp \left\{ - \frac{T}{2\pi \alpha'} \left\{ \int_{|x| < x_c} dx \sqrt{-x^2 C_1} - V_* + \frac{V'_*^2}{2V''_*} \right\} \right. \\
+ \left. i \int_{|x| > x_c} dx \sqrt{U_c^2 + V(U_c)} \right\}
\]

where $x_c = \sqrt{(-V_* + V''_*/2V'_*)/C_1}$ if this root is real, and $x_c = 0$ otherwise. The second contribution in the exponent gives the real part of the $Q\bar{Q}$ potential which we drop from now on (see refs. [4, 5, 7, 9]). Performing the integral over $|x| < x_c$ we find

\[
\text{Im } V_{QQ} = -\frac{1}{2\sqrt{2\alpha'}} \left[ \frac{V'_*}{2V''_*} - \frac{V_*}{V'_*} \right],
\]

(15)
where we used that \( U''_c(0) = V'_c/2 \). For \( \mathcal{N} = 4 \) SYM at large \( \lambda \) the potential entering the NG action is given by \( V(U) = (U^4 - U^4_h)/R^4 \), where \( R \) is the radius of AdS\(_5\) and \( U_h = \pi R^2 T \). This gives

\[
\text{Im } V_{QQ} = -\frac{\pi}{24\sqrt{2}} \sqrt{\lambda} T \frac{3\zeta^4 - 1}{\zeta},
\]

where \( \sqrt{\lambda} = R^2/\alpha' \) and \( \zeta \equiv U_h/U_\ast < 1 \). Note that the equation above applies only when \( \zeta > 3^{-1/4} \approx 0.76 \); otherwise \( \text{Im } V_{QQ} = 0 \) because the solution for \( x_c \) from above ceases to exist. Thus, in the vicinity of this point the width generated by the imaginary part of the potential (see below) is small compared to the binding energy.

The dependence of \( \zeta \) on \( L \) and \( T \) can be found from Eq. (7). At small \( LT \) we have \( LT = b \zeta \) with \( b = 2\Gamma(3/4)/\sqrt{\pi} \Gamma(1/4) \approx 0.38 \). On the other hand, when the bottom of the classical string comes too close to the horizon, \( \zeta \sim 0.85 \), the U-shaped configuration used here receives higher-order corrections [11] and cannot be used anymore. With \( \zeta \sim LT \), the imaginary part of the potential (16) is smaller than the dominant contribution to the real part, eq. (8), by a factor \( \sim (LT)^4 \). Here, we consider only temperatures such that thermal screening corrections to the real part of the potential are small; the bound state then probes the potential only in the region \( LT < 1 \).

The imaginary part in Eq. (16) shifts the Bohr energy level obtained with the Coulomb-like vacuum potential (8), \( E_0 \rightarrow E_0 - i\Gamma \); to first order,

\[
\Gamma_{QQ} \equiv -\langle \psi | \text{Im } V_{QQ} | \psi \rangle = -\frac{\pi \sqrt{\lambda}}{48\sqrt{2}} \frac{b}{a_0} \left[ 45 \left( \frac{a_0 T}{b} \right)^4 - 2 \right],
\]

where \( | \psi \rangle \) denotes the unperturbed Coulomb ground state wave function and \( a_0 = \Gamma(1/4)^4/2\pi^2 \sqrt{\lambda} m_Q \) is the Bohr radius. The width decreases with the quark mass and with the \( t' \) Hooft coupling, approximately as \( \Gamma_{QQ} \sim 1/\lambda m_Q^3 \); it increases rapidly with the temperature, \( \sim T^4 \). For \( m_Q = 4.7 \) GeV, \( T = 0.3 \) GeV, \( \sqrt{\lambda} = 3 \) [14] we obtain \( \Gamma_T \approx 48 \) MeV.

It is interesting to compare this result to the imaginary part of the heavy quark potential computed in pQCD: \( \text{Im } V_{QQ} \sim -\alpha_s^2 C_F N_c T(LT)^2 \log (LT)^{-1} \) [10]. This is smaller than \( \text{Re } V_{QQ} \sim \alpha_s C_F/L \) by one power of the coupling and three powers of \( LT \). With \( L \sim (\alpha_s C_F m_Q)^{-1} \), the width decreases less rapidly with the quark mass than for \( \mathcal{N} = 4 \) SYM at strong coupling. However, the numerical value of \( \Gamma_T \) is on the order of tens of MeV, similar to what we obtain here.
We suggest that the width computed above is accessible experimentally through the suppression of $\Upsilon \to \ell^+\ell^-$ processes in heavy-ion collisions at RHIC or LHC. Neglecting “regeneration” of bound states from $b$ and $\bar{b}$ quarks in the medium we can estimate the number of $\Upsilon$ mesons in the plasma at mid-rapidity, which have not decayed into unbound $b$ and $\bar{b}$ quarks up to time $t$ after the collision, from
\[
\frac{dN}{dt} = -\Gamma_\Upsilon(T(t)) N(t)
\]
\[
\rightarrow N(t) \approx N_0 \exp \left(-\int dt \Gamma_\Upsilon(t) \right).
\]
(18)

This solution assumes that $\Gamma_\Upsilon(T(t))$ is a slowly varying function of time. The initial number of $\Upsilon$ states may be estimated from the multiplicity in p+p collisions times the number of binary collisions at a given impact parameter: $N_0 \approx N_{\text{coll}} N_{\Upsilon}^{p+p}$. Thus, the integrated “nuclear modification factor” $R_{AA}$ for the process $\Upsilon \to \ell^+\ell^-$ is approximately given by $R_{AA}(\Upsilon \to \ell^+\ell^-) \approx \exp(-\bar{\Gamma}_\Upsilon t)$, where $\bar{\Gamma}$ denotes a suitable average of $\Gamma(T)$ over the lifetime of the quark-gluon plasma. Due to the strong temperature dependence of the width, this average is dominated by the early stage and thus we expect that $\bar{\Gamma}$ provides an estimate of the initial temperature in heavy-ion collisions via Eq. (17). For $t = 5 \text{ fm/c}$ and $\bar{\Gamma}_\Upsilon = 48 \text{ MeV}$ we obtain $R_{AA} \approx 0.3$. An experimental estimate for the thermal quarkonium decay rate $\bar{\Gamma}$ could be obtained once a statistically significant detection of the $\Upsilon \to \ell^+\ell^-$ process has been achieved [15].

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