Refined form of the paper on the canonical formalism of the $f(R)$-type gravity in terms of Lie derivatives

Y. Ezawa$^a$ and Y. Ohkuwa$^b$

$^a$Department of Physics, Ehime University, Matsuyama, 790-8577, Japan
$^b$Section of Mathematical Science, Department of Social Medicine, Faculty of Medicine, University of Miyazaki, Kiyotake, Miyazaki, 889-1692, Japan
Email: ezawa@sci.ehime-u.ac.jp, ohkuwa@med.miyazaki-u.ac.jp

Abstract

We refine the presentation of the previous paper of our group, Y. Ezawa et al., Class. and Quantum Grav. 23 (2006), 3205. In that paper, we proposed a canonical formalism of $f(R)$-type generalized gravity by using the Lie derivatives instead of the time derivatives. However, the use of the Lie derivatives was not sufficient. In this note, we make use of the Lie derivatives as far as possible, so that no time derivatives are used and the presentation is largely improved.

1 Introduction

Since the use of the $f(R)$-type gravity by Caroll et al. [1] to explain the discovered accelerated expansion of the universe [2], the theory has been attracting much attention and its various aspects and applications have been investigated [3]. However, its canonical formalism had not been so systematic. So in [4], our group proposed a formalism by generalizing the canonical formalism of Ostrogradski [5]. The generalization is necessary. As the scalar curvature $R$ depends on the time derivatives of the lapse function and shift vector, these variables have to obey the field equations, if the Ostrogradski’s method is directly applied. Then, only the solutions to these equations are allowed for these variables. This, however, is in conflict with general covariance since these variables specify the coordinate frame so should be taken arbitrarily. One of the ways to resolve this problem had been given by Buchbinder and Lyakhovich (BL method) [6]. However, the BL method has an undesirable property that, when the generalized coordinates are transformed, Hamiltonian is also transformed [4].

So in the previous paper [4], we proposed a canonical formalism of the $f(R)$-type gravity using the Lie derivatives instead of the time derivatives, which is a natural and economical generalization of the formalism of Ostrogradski, so remedies the property of the method by BL mentioned above. However the use of the Lie derivatives was not sufficient, i.e., Lie derivatives and time derivatives were used in a mixed way, so some expressions are complex. In this note, we refine the presentation of our previous paper by making use of the Lie derivatives as far as possible so that expressions are more concise.

2 Ostrogradski’s method

Before presenting the refined form of the previous paper, we briefly describe the method of Ostrogradski [7]. We consider a system with $N$ degrees of freedom, the generalized coordi-
nates of which will be denoted as \( q^i \) \((i = 1, 2, \ldots, N)\). The Lagrangian \( L \) is assumed to be defined in the \( N(n + 1) \)-dimensional velocity phase space, the coordinates of which are expressed as

\[
D^s q^i \quad (s = 0, 1, \ldots, n), \quad \text{with} \quad D \equiv \frac{d}{dt},
\]

and \( n \) is the order of the highest time derivative of the generalized coordinate \( q^i \), so that the Lagrangian \( L \) is expressed as \( L = L(D^s q^i) \). It is possible that \( n \) is different for different \( i \), but we do not think of this possibility for simplicity. Transition to the canonical formalism is given by the Ostrogradski transformation (map), which is the straightforward generalization of the Legendre transformation (map) for systems described without higher time derivatives.

The transformation is given in the following way. Consider the variation of the action \( S \)

\[
\delta S \equiv S[q^i + \delta q^i] - S[q^i] = \int_{t_1}^{t_2} \delta L \, dt.
\]

Here

\[
\delta L = \sum_{i=1}^{N} \sum_{s=0}^{n} \frac{\partial L}{\partial D^s q^i} \delta (D^s q^i).
\]

The first term in the second line gives the boundary terms in \( \delta S \). \( n \times N \) velocity variables \( D^s q^i \) \((s = 0, 1, \ldots, n - 1)\) are transformed (mapped) to the generalized coordinates of the phase space and are usually denoted as \( q^i_s \) (or \( Q^i_s \)) and are often called as the new generalized coordinates for \( s \geq 1 \) in the canonical formalism. The momenta canonically conjugate to them, which we will denote as \( p^i_s \), are defined as the coefficient of the variation of the \( D^s q^i \), which are transformed to the new generalized coordinate, in the boundary terms: that is

\[
p^i_s \equiv \sum_{r=1}^{n} (-1)^{r-s-1} D^{r-s-1} \left\{ \frac{\partial L}{\partial (D^r q^i)} \right\}, \quad (i = 1, 2, \ldots, N; \quad s = 0, 1, \ldots, n - 2),
\]

and the momentum conjugate to \( q_{n-1}^i \) is defined as

\[
p^i_{n-1} \equiv \frac{\partial L}{\partial q_{n-1}^i} = \frac{\partial L}{\partial (D^n q^i)}.
\]

Thus the phase space is \( 2nN \)-dimensional. However, the relations \( \dot{q}_{s}^i = q_{s+1}^i \) give \( N(n - 1) \) constraints when \( \dot{q}_{s}^i \) are written using the conjugate momenta, the dimension of the subspace spanned by independent coordinates is \( N(n + 1) \) as it should be. The Hamiltonian is defined similarly to the Legendre transformation:

\[
H \equiv \sum_{j=1}^{N} \sum_{s=0}^{n-1} p^s_j \dot{q}_s^j - L(q_0^i, q_1^i, \ldots, q_{n-1}^i; \dot{q}_{n-1}^i)
\]

where \( \dot{q}_{n-1}^i \) should be replaced by new generalized coordinates and momentum from (4). Canonical equations of motion are satisfied if the Euler-Lagrange equations are satisfied. Euler-Lagrange equations are given by setting the coefficients of \( \delta q^i \) to vanish in the second term of the second line of (2) for each \( i \). It is easily seen that (3)–(5) reduce to the Legendre transformation for \( n = 1 \). Finally we note that the boundary terms in the variation of the action vanish by requiring the vanishing of the variations of new generalized coordinates at the boundaries, which can be consistently imposed since the Euler-Lagrange equations are \( 2n \)-th order differential equations if \( L(D^s q^i) \) is non-linear in \( D^n q^i \).
3 Action of $f(R)$-type gravity

We start from the following action of the generalized gravity of $f(R)$-type;

$$S = S_G + S_M = \int d^4x \sqrt{-g} f(R) + S_M,$$

where $S_M$ is the action of matters. In other words, the Lagrangian density for gravity $\mathcal{L}_G$ is expressed as

$$\mathcal{L}_G = \sqrt{-g} f(R).$$

As the variables for gravity, we adopt the ADM variables[8]. Then, as noted in the introduction, the scalar curvature, $R$, is expressed in terms of the Lie derivatives instead of the time derivatives as follows:

$$R = h^{ij} \mathcal{L}_n^2 h_{ij} + \frac{1}{4} \left( h^{ij} \mathcal{L}_n h_{ij} \right)^2 - \frac{3}{4} h^{ik} h^{jl} \mathcal{L}_n h_{ij} \mathcal{L}_n h_{kl} + 3R - 2N^{-1} \Delta N,$$

where $h_{ij}$ is the metric of the hypersurface $\Sigma_t$ which has the normal vector field $n^\mu = N^{-1}(1, -N^i)$, $N$ is the lapse function and $N^i$ is the shift vector. $\mathcal{L}_n$ represents the Lie derivative along the normal vector field $n$. $3R$ is the scalar curvature of $\Sigma_t$. From (7) and (8), $\mathcal{L}_G$ depends on the ADM variables in the following way:

$$\mathcal{L}_G = \mathcal{L}_G(N, h_{ij}, \mathcal{L}_n h_{ij}, \mathcal{L}_n^2 h_{ij}).$$

4 Variation of the action

From the expression (9), variation of $\mathcal{L}_G$ is expressed as

$$\delta \mathcal{L}_G = \frac{\delta \mathcal{L}_G}{\delta N} \delta N + \frac{\delta \mathcal{L}_G}{\delta h_{ij}} \delta h_{ij} + \frac{\partial \mathcal{L}_G}{\partial \mathcal{L}_n h_{ij}} \delta \mathcal{L}_n h_{ij} + \frac{\partial \mathcal{L}_G}{\partial (\mathcal{L}_n^2 h_{ij})} \delta \mathcal{L}_n^2 h_{ij}.$$  

Here of the first two terms on the right-hand are not the partial derivatives but the functional derivatives since the scalar curvature $R$ depends on the derivatives of $N$ and $h_{ij}$ in $\Delta N$ and $3R$ as seen in (8). Actual calculation is made easier when concrete form (8) is used. Then we have

$$\delta \mathcal{L}_G = \delta \sqrt{h} N f(R) + \sqrt{h} \delta N f(R) + \sqrt{h} N f'(R) \delta R,$$

where

$$\begin{align*}
\delta \sqrt{h} &= \frac{1}{2} \sqrt{h} h^{ij} \delta h_{ij}, \\
\delta R &= h^{ij} \mathcal{L}_n^2 \delta h_{ij} + (h^{ij} K - 3K^{ij}) \mathcal{L}_n \delta h_{ij} + \left[ -h^{ik} h^{jl} (\mathcal{L}_n K) (\mathcal{L}_n h_{kl}) + 6K^{il} K^j_l \right] \delta h_{ij} \\
&\quad + \delta 3R + 2N^{-2} \Delta N \delta N - 2N^{-1} \Delta \delta N,
\end{align*}$$

where $K_{ij}$ is the extrinsic curvature defined as

$$Q_{ij} = \frac{1}{2} \mathcal{L}_n h_{ij} = K_{ij}.$$

Note that $\delta \mathcal{L}_n h_{ij} = \mathcal{L}_n \delta h_{ij}$, $\delta \mathcal{L}_n^2 h_{ij} = \mathcal{L}_n^2 \delta h_{ij}$ and also
\[
\begin{align*}
\sqrt{\hbar} N f'(R) \delta^3 R &= -\sqrt{\hbar} N f'(R) R^{ij} \delta h_{ij} \\
&+ \sqrt{\hbar} \left[ (N f'(R))^i j h^{kl} \Gamma^j_{kl} - (N f'(R))^i l h^{jk} \Gamma^j_{kl} \right] \delta h_{ij} \\
&+ \partial_k \left[ \sqrt{\hbar} \left\{ h^{i k} (N f'(R))^j i - h^{i j} (N f'(R))^j k \right\} \right] \delta h_{ij} \\
&+ \partial_i \left[ \sqrt{\hbar} N f'(R) \left( h^{k l} \delta \Gamma^i_{kl} - h^{i l} \delta \Gamma^k_{i l} \right) \right] \\
&- \partial_i \left[ \sqrt{\hbar} \left\{ h^{i j} (N f'(R))^k j \delta h_{k j} - (N f'(R))^i k h^{j i l} \right\} \right].
\end{align*}
\]
(14)

When we use (12) in (11) and apply the variational principle, partial integrations have to be done for terms including \( L_n \delta h_{ij} \) and \( L_n^2 \delta h_{ij} \). This is done by using a relation for a scalar field \( \Phi \):

\[
L_n(\sqrt{\hbar} N \Phi) = L_n(\sqrt{\hbar} N) \Phi + \sqrt{\hbar} N L_n \Phi, \quad L_n \Phi = n^\mu \partial_\mu \Phi.
\]
(15)

Then we have

\[
\sqrt{\hbar} N f'(R)(h^{ij} K - 3K^{ij}) L_n \delta h_{ij} = -\sqrt{\hbar} N (L_n + K) \left[ f'(R)(h^{ij} K - 3K^{ij}) \right] \delta h_{ij} + \partial_\mu \left[ n^\mu \sqrt{\hbar} N f'(R)(h^{ij} K - 3K^{ij}) \delta h_{ij} \right],
\]
(16)

and

\[
\sqrt{\hbar} N f'(R) h^{ij} L_n^2 \delta h_{ij} = \sqrt{\hbar} N \left[ K^2 f'(R) h^{ij} + L_n^2 (f'(r) h^{ij}) + K L_n (f'(R) h^{ij}) \right] \delta h_{ij} + \partial_\mu \left[ n^\mu \sqrt{\hbar} N \{ f'(R) h^{ij} \delta L_n h_{ij} - (L_n + K) (f'(R) h^{ij}) \delta h_{ij} \} \right].
\]
(17)

Using these relations, we have for \( \delta \mathcal{L}_G \) the following expression:

\[
\delta \mathcal{L}_G = \sqrt{\hbar} \left[ \Delta f'(R) + 2N^{-1} \Delta N f'(R) \right] \delta N + \partial_i \left[ \sqrt{\hbar} \left( f'(R) \nabla^i \delta N - \nabla^i f'(R) \delta N \right) \right] + \left[ f''(R) (L_n R)^2 h^{ij} + f''(R) \left( L_n^2 R h^{ij} - L_n K^{ij} \right) \right] + f'(R) \left( K K^{ij} - h^{ij} L_n K - h^{ik} h^{jl} L_n K_{kl} + 6K^{ik} K^{j l} + \frac{3 \delta^3 R}{\delta h_{ij}} \right) \delta h_{ij} + \partial_\mu \left[ n^\mu \sqrt{\hbar} N \{ f'(R) h^{ij} \delta L_n h_{ij} - (f''(R) L_n R h^{ij} + f'(R) K^{ij}) \delta h_{ij} \} \right].
\]
(18)

\section{New generalized coordinates and momenta canonically conjugate to them}

New generalized coordinates, denoted as \( Q_{ij} \), are taken, as in [6], to be (a half of) the Lie derivatives of the original generalized coordinates \( h_{ij} \) which is equal to the extrinsic curvature (13). Momenta canonically conjugate to the original and new generalized coordinates, \( p^{ij} \) and \( P^{ij} \) respectively, are defined to be the coefficient of their variations in the total time derivative terms in (17):

\[
\begin{align*}
p^{ij} &= -\sqrt{\hbar} \left[ L_n f'(R) h^{ij} + f'(R) Q^{ij} \right], \\
P^{ij} &= 2\sqrt{\hbar} f'(R) h^{ij},
\end{align*}
\]
(19)
where, of course, \( \mathcal{L}_n f'(R) \) is also expressed as \( f''(R) \mathcal{L}_n R \). Expressions of these equations that correspond to (10) read as follows \(^1\):

\[
p^{ij} = n^0 \frac{\partial \mathcal{L}_G}{\partial (\mathcal{L}_n h_{ij})} - \mathcal{L}_n \left( n^0 \frac{\partial \mathcal{L}_G}{\partial (\mathcal{L}_n^2 h_{ij})} \right), \quad p^{ij} = 2n^0 \frac{\partial \mathcal{L}_G}{\partial (\mathcal{L}_n^2 h_{ij})},
\]

(20a)

Or reversingly, we have

\[
\frac{\partial \mathcal{L}_G}{\partial (\mathcal{L}_n h_{ij})} = \frac{1}{n^0} \left( p^{ij} + \frac{1}{2} \mathcal{L}_n p^{ij} \right), \quad \frac{\partial \mathcal{L}_G}{\partial (\mathcal{L}_n^2 h_{ij})} = \frac{1}{2n^0} \mathcal{L}_n p^{ij}.
\]

(20b)

\section{Hamiltonian density}

Correspondence of each point on different \( \Sigma_t \) are given by a 1-parameter transformation along the timelike curve for which the vector field \( t^\mu \) is the tangent, so we have, e.g.,

\[
h_{ij}(x, t + \delta t) = h_{ij}(x, t) + \mathcal{L}_t h_{ij} \delta t.
\]

(21)

Actually, we have \( \mathcal{L}_t h_{ij} = \partial_0 h_{ij} \) in the coordinate frame we are using. Thus Hamiltonian density \( \mathcal{H}_G \) is defined to be

\[
\mathcal{H}_G \equiv p^{ij} \mathcal{L}_t h_{ij} + p^{ij} \mathcal{L}_t Q_{ij} - \mathcal{L}_G.
\]

(22)

\( \mathcal{H}_G \) has the following form:

\[
\mathcal{H}_G = N \mathcal{H}_0 + N^i \mathcal{H}_i + \text{divergent term},
\]

(23)

where, after a canonical transformation \((Q, P) \rightarrow (\bar{Q}, \bar{P}) \equiv (P, -Q)\), we have

\[
\begin{align*}
\mathcal{H}_0 &= \frac{2}{\sqrt{h}} p^{ij} p^i_{ij} - \frac{2}{d} d P + \frac{1}{2} Q \psi(Q/2\sqrt{h}) - \frac{d-3}{2d} Q P^2 - \frac{1}{2} \frac{3R}{\sqrt{h}} \\
\mathcal{H}_k &= 2p^i_{ij} p^{ij} - \frac{2}{d} d p_i + \frac{2}{d} (QP)_i
\end{align*}
\]

(24)

\section{Invariance of the Hamiltonian}

We consider the following transformations of the generalized coordinates \( h_{ij} \):

\[
h_{ij} \rightarrow \phi_{ij} \equiv F_{ij}(h_{kl}) \quad \text{or inversely} \quad h_{ij} \equiv G_{ij}(\phi_{kl}),
\]

(25)

and show that the Hamiltonian is invariant under this transformation. New generalized coordinates \( \Phi_{ij} \) are defined as in (13), i.e.,

\[
\Phi_{ij} \equiv \frac{1}{2} \mathcal{L}_n \phi_{ij}.
\]

(26)

\(^1\)If we use a scalar function \( \tilde{\mathcal{L}}_G \) defined as \( \mathcal{L}_G \equiv N \sqrt{h} \tilde{\mathcal{L}}_G \), factors \( n^0 \) disappear and we have

\[
p^{ij} = \sqrt{h} \frac{\partial \tilde{\mathcal{L}}_G}{\partial (\mathcal{L}_n h_{ij})} - \mathcal{L}_n \left( \sqrt{h} \frac{\partial \tilde{\mathcal{L}}_G}{\partial (\mathcal{L}_n^2 h_{ij})} \right), \quad p^{ij} = \sqrt{h} \frac{\partial \tilde{\mathcal{L}}_G}{\partial (\mathcal{L}_n^2 h_{ij})}.
\]
Hamiltonian density $\bar{\mathcal{H}}_G$ expressed in the transformed variables is defined to be

$$\bar{\mathcal{H}}_G \equiv \pi^{ij}\mathcal{L}_t\phi_{ij} + \Pi^{ij}\mathcal{L}_t\Phi_{ij} - \tilde{\mathcal{L}}_G(N, \phi_{ij}, \mathcal{L}_n\phi_{ij}, \mathcal{L}_n^2\phi_{ij}),$$

where $\pi^{ij}$ and $\Pi^{ij}$ are momenta canonically conjugate to $\phi_{ij}$ and $\Phi_{ij}$, respectively, and since

$$\mathcal{L}_n h_{ij} = \frac{\partial G_{ij}}{\partial \phi_{kl}} \mathcal{L}_n\phi_{kl}, \quad \mathcal{L}_n^2 h_{ij} = \mathcal{L}_n\left(\frac{\partial G_{ij}}{\partial \phi_{kl}}\right) \mathcal{L}_n\phi_{kl} + \frac{\partial G_{ij}}{\partial \phi_{kl}} \mathcal{L}_n^2\phi_{kl},$$

$\tilde{\mathcal{L}}_G$ is defined as

$$\tilde{\mathcal{L}}_G(N, \phi_{ij}, \mathcal{L}_n\phi_{ij}, \mathcal{L}_n^2\phi_{ij}) \equiv \mathcal{L}_G\left(N, G_{ij}(\phi_{kl}), \frac{\partial G_{ij}}{\partial \phi_{kl}} \mathcal{L}_n\phi_{kl}, \mathcal{L}_n\left(\frac{\partial G_{ij}}{\partial \phi_{kl}}\right) \mathcal{L}_n\phi_{kl} + \frac{\partial G_{ij}}{\partial \phi_{kl}} \mathcal{L}_n^2\phi_{kl}\right).$$

$\pi^{ij}$ and $\Pi^{ij}$ satisfy relations similar to (20a,b), and from these relations, we have

$$\pi^{ij} = p^{ij}\frac{\partial G_{ij}}{\partial \phi_{ij}}, \quad \Pi^{ij} = P^{ij}\frac{\partial G_{ij}}{\partial \phi_{ij}},$$

or inversely

$$p^{ij} = \pi^{ij}\frac{\partial F_{ij}}{\partial h_{ij}}, \quad P^{ij} = \Pi^{ij}\frac{\partial F_{ij}}{\partial h_{ij}}.$$  

With help of (30a,b), we have

$$p^{ij}\mathcal{L}_t h_{ij} = \pi^{ij}\frac{\partial F_{ij}}{\partial h_{ij}} \frac{\partial F_{ij}}{\partial \phi_{mn}} \mathcal{L}_t \phi_{mn} = \pi^{ij}\mathcal{L}_t \phi_{ij}.$$  

Similar relation holds between $P^{ij}$ and $\Pi^{ij}$, so we have

$$\mathcal{H}_G = \bar{\mathcal{H}}_G.$$  

It is noted that the transformation (25) includes the coordinate transformation on $\Sigma_t$.

8 Summary

We presented a canonical formalism of $f(R)$-type gravity in terms of the Lie derivatives by refining our previous paper[4]. The formalism is a natural and economical generalization of the Ostrogradski’s formalism. Generalization is necessary to assure the invariance of the theory under the general coordinate transformation.

References

[1] S. Caroll, V. Duvuri, M. Trodden and M. S. Turner, Phys. Rev. D70 (2004), 043528
[2] B. N. Reid et al., MNRAS 404 (2010), 60
  W. J. Percival et al., MNRAS 401 (2009), 2148
  M. Hicken et al., ApJ 700 (2009), 1097
  R. Kessler et al., ApJS 185 (2009), 32
  V. Vikhlini et al., ApJ 692 (2009), 1033
A. Mantz et al., *MNRAS* **406** (2010), 1759
A. G. Riess et al., *ApJ* **699** (2009), 539
S. H. Suyu et al., arXiv: 0910.2773 [astro-ph.CO]
R. Fadely et al., *ApJ* **711** (2009), 211
R. Masset et al., *ApJS* **172** (2007), 239
L. Fu et al., *Astron. Astrophys.* **479** (2009), 9
T. Schrabback et al., *Astron. Astrophys.* **516** (2009), 63

[3] See for example, T. P. Sotiriou and V. Faraoni, *Rev. Mod. Phys.* **82** (2008), 451
S. Nojiri and S. Odintsov, *Int. J. Geom. Meth. Phys.* **4** (2007), 115

[4] Y. Ezawa, H. Iwasaki, Y. Ohkuwa, S. Watanabe, N. Yamada and T. Yano, *Class. Quantum Grav.* **23** (2006), 3205

[5] M. Ostrogradski, *Mem. Acad. Sci. St. Petersberg* VI **4** (1850), 385

[6] I. L. Buchbinder and S. L. Lyakhovich, *Class. Quantum Grav.* **4** (1987), 1487

[7] Detailed description is given in, T. Kimura and T. Ohta, *Classical and Quantum Theory of Gravity* (in Japanese) (McGrowhill, Tokyo, 1989), and in, T. Kimura and R. Sugano, *Analytical Dynamics in Terms of Differential Forms: enlarged and revised version* (in Japanese) (Yoshioka, Kyoto, 1996). See also
Y. Saito, R. Sugano, T. Ohta and T. Kimura, *J. Math. Phys.* **30** (1989), 1122; ibid. **34** (1993), 3775

[8] R. Arnowitt, S. Deser and C. Misner, arXiv:gr-qc/0405109 gThe Dynamics of General Relativityh(2004)