Subluminal stochastic gravitational waves in pulsar-timing arrays and astrometry

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The detection of a stochastic background of low-frequency gravitational waves by pulsar-timing and astrometric surveys will enable tests of gravitational theories beyond general relativity. These theories generally permit gravitational waves with non-Einsteinian polarization modes, which may propagate slower than the speed of light. We derive the angular correlation patterns of observables relevant for pulsar timing arrays and astrometry that arise from a background of subluminal gravitational waves with scalar, vector, or tensor polarizations. We find that the pulsar timing observables for the scalar longitudinal mode, which diverge with source distance in the luminal limit, are finite in the subluminal case. In addition, we apply our results to $f(R)$ gravity, which contains a massive scalar degree of freedom in addition to the standard transverse-traceless modes. The scalar mode in this $f(R)$ theory is a linear combination of the scalar-longitudinal and scalar-transverse modes, exciting only the monopole and dipole for pulsar timing arrays and only the dipole for astrometric surveys.

I. INTRODUCTION

There are world-wide efforts to detect a stochastic background of gravitational waves (GWs) using pulsar timing arrays (PTAs) [1–6]. Pulsars emit light at highly regular intervals, and the presence of GWs modifies the expected pulse arrival times at Earth. For a background GW, the pulse arrival times from different pulsars are correlated across the sky; in particular, a stochastic background produces an angular correlation given by the the Hellings-Downs curve [7]. The wave can also induce a shift in the apparent position of stars, which may be observed in astrometric surveys [8, 9]. Similar to PTAs, the stellar shift exhibits particular angular correlations from a stochastic GW background [10].

The GWs in general relativity (GR) arise from the transverse-traceless tensor modes of the metric perturbation. If, however, GR is modified, there may be additional propagating degrees of freedom from the scalar and vector modes, leading to GW polarization states beyond the standard two. As a result, the angular correlation of pulse arrival times for PTAs or stellar positions for astrometry has a different functional form that depends on the GW polarization and the relative amplitudes of the polarization states. There are generically six polarizations states, which we classify as follows: 2 transverse-traceless tensor modes, 2 vector modes, a scalar-longitudinal mode (SL), and a scalar-transverse mode (ST). Previous studies have calculated the normalized, individual contributions to the angular correlation function due to all six polarizations for PTA [11–13] and astrometry observables [13–16]. The relative amount each mode contributes depends on the particular theory. While most of these previous calculations assume all GW modes propagate at the speed of light, there are scenarios in which certain modes may experience subluminal propagation.

In this paper, we derive the auto- and cross-correlation patterns for PTA and astrometry observables due to subluminal GW polarization modes under the total-angular-momentum (TAM) formalism [17], following the methods outlined in Ref. [13]. We then investigate a particular form of $f(R)$ gravity that contains a single additional degree of freedom in the form of a massive scalar field. The speed of propagation then depends on the mass of the field, and the associated GW polarization is comprised of a superposition of the SL and ST modes.

Although observations from LIGO constrain the propagation speed of the transverse-traceless modes to between $1 - 3 \times 10^{-15}$ and $1 + 7 \times 10^{-16}$ times the speed of light [18], it may be possible for an alternative theory of gravity to possess a frequency-dependent propagation speed; therefore, we consider subluminal propagation of these modes for completeness. The same argument holds true for scalar and vector modes. To date, LIGO has found no evidence for non-Einsteinian polarizations [19, 20]; however, even if future studies were able to constrain the velocities of these modes, GWs at frequencies below LIGO’s range of sensitivity could avoid these bounds.

We note that recent work has derived the astrometric angular correlation functions and power spectra for nonluminal GW propagation [16]. Our results for subluminal GWs numerically agree with those in Ref. [16], which were derived using different methods described in Ref. [14].

The outline of this paper is as follows. In Sec. II, we calculate the general expressions for the PTA and astrometric response to a subluminal GW with non-standard polarizations. We then turn to $f(R)$ gravity as a concrete example in Sec. III and relate the scalar degree of freedom to a particular combination of ST and SL modes. We conclude in Sec. IV.

II. POWER SPECTRA

The geodesic of observed light emanating from a source may be altered by a GW passing between the source and Earth. A stochastic background of GWs is expected to
induce particular angular correlation patterns for PTA and astrometry observables. In PTAs, the GW affects the light travel time from a pulsar and thus affects the observed time of arrival. We choose, however, to work with the relative shift in the pulse arrival frequency \( z(t, \hat{n}) \), rather than the arrival time, to make connections to previous literature. We note that this change in the observable impacts the time domain information, but not the angular response of the signal \([13]\). We may express the shift as an expansion in spherical harmonics,

\[
z(t, \hat{n}) = \sum_{\ell,m} z_{\ell m}(t) Y_{\ell m}(\hat{n}),
\] (1)

for a pulsar located in the \( \hat{n} \) direction at time \( t \). In astrometry, the GW affects the apparent location of stars. The resulting shift in position may be expanded as

\[
\delta^a(t, \hat{n}) = \sum_{\ell,m} \left[ E_{\ell m}(t) Y_{\ell m}^{E,a}(\hat{n}) + B_{\ell m}(t) Y_{\ell m}^{B,a}(\hat{n}) \right],
\] (2)

where \( Y_{\ell m}^{E,a} \) and \( Y_{\ell m}^{B,a} \) are vector spherical harmonics \([17]\) and \( a \) is an abstract index labeling the components of a vector on the celestial sphere.

The correlation functions and power spectra for these observables are derived in Ref. \([13]\) using the TAM formalism \([17]\), under the assumption that all polarization modes of the GWs propagate at the speed of light with the GW frequency equaling its wave number, \( \omega = k \) (with \( c = 1 \)). Here, we consider the more general case of a GW of polarization \( \alpha \) with a dispersion relation \( \omega_\alpha(k) \). For massive gravity models, where the propagating mode behaves like a particle of mass \( m_\alpha \), this dispersion relation is given by \( \omega_\alpha^2(k) = k^2 + m_\alpha^2 \), which we assume for the remainder of the paper. We define the phase velocity \( v_{\phi,\alpha} \equiv \omega_\alpha/k \) and the group velocity \( v_{\alpha} \equiv d\omega_\alpha/dk = k/\omega_\alpha \); thus, while the group velocity is subluminal, the phase velocity is superluminal.

We expand the metric perturbation as

\[
h_{ab}(t, x) = \int \frac{k^2 dk}{(2\pi)^3} 4\pi i^\ell h_{\ell m}(k) \Psi_{(\ell m)ab}^{\alpha,k}(x)e^{-i\omega_\alpha(k)t},
\] (3)

for a single TAM wave \( \Psi_{(\ell m)ab}^{\alpha,k} \) with amplitude \( h_{\ell m}^{\alpha} \). Using this expansion, we write the power spectra from a stochastic GW background as \([13]\)

\[
C_{\ell}^{X'X,\alpha} \propto 32\pi^2 F_{\ell}^{X,\alpha} \left( F_{\ell}^{X',\alpha} \right)^*,
\] (4)

corresponding to the PTA and astrometry observables \( X, X' \in \{ z,E,B \} \). In this expression, we have omitted a factor that encompasses time domain information, including the dependence on the GW frequency and the cadence of the observation. As we show in the following subsections, the projection factors \( F_{\ell}^{X,\alpha} \) depend on the phase velocity and thus cannot be factored out from the integral over \( k \) in Eq. (3), unlike the case for luminal GWs. Therefore, the actual power spectrum receives contributions from a range of velocities, determined by the window function used for observation. For the purposes of this work, we assume the window function is narrow so that Eq. (4) holds; our results may be applied to the full expression of Eq. (4) in Ref. \([13]\) for more general cases. In the following subsections, we derive the expressions for the projection factors, in close parallel with Ref. \([13]\).

### A. Pulsar Timing Arrays

The fractional shift in the observed pulse frequency of a pulsar due to a metric perturbation \( h_{ab} \) is

\[
z(t, \hat{n}) = -\frac{1}{2} n^a n^b \int_e^{r_s} dt' \ h_{ab,0}(t', (t-t')\hat{n}),
\] (5)

where \( \hat{n} \) is the direction of the pulsar in the sky, \( t \) is the observation time of a pulse, and \( r_s \) is the distance to the pulsar. The time derivative acts only on the explicit time dependence in Eq. (3). Additionally, in the TAM formalism,

\[
n^a n^b \Psi_{(\ell m)ab}^{\alpha,k}(x) = -R_{\ell}^{L,\alpha}(kr)Y_{\ell m}(\hat{n}),
\] (6)

where \( Y_{\ell m} \) are spherical harmonics and \( R_{\ell}^{L,\alpha} \) are radial functions, given in Appendix A. Thus, the shift in pulse frequency becomes

\[
z(t, \hat{n}) = 4\pi i^\ell \int \frac{k^2 dk}{(2\pi)^3} h_{\ell m}(k) F_{\ell}^{z,\alpha} Y_{\ell m}(\hat{n}) e^{-i\omega_\alpha(k)t}.
\] (7)

The projection factor \( F_{\ell}^{z,\alpha} \) is analogous to that in Ref. \([13]\) for luminal GWs and is given by

\[
F_{\ell}^{z,\alpha} = -\frac{i v_{\phi,\alpha}}{2} \int_0^\infty dx \ R_{\ell}^{L,\alpha}(x) e^{ixv_{\phi,\alpha}},
\] (8)

where we have taken the distant-source limit, \( kr_s \to \infty \).

### B. Astrometry

The astrometric deflection due to a metric perturbation \( h_{ab} \) is

\[
\delta^a(t, \hat{n}) = \Pi^{ac} n^b \left\{ -\frac{1}{2} h_{bc}(t, 0) + \frac{1}{r_s} \int_0^{r_s} dr \left[ h_{bc}(t-r, r\hat{n}) - \frac{r_s-r}{2} n^d \partial_c h_{bd}(t-r, r\hat{n}) \right] \right\},
\] (9)
where $\Pi_{ab}(\hat{n}) = \eta_{ab} - \hat{n}_a \hat{n}_b$ projects onto the plane orthogonal to $\hat{n}$. Expanding the metric perturbation in terms of TAM waves, we obtain

$$\delta^a(\hat{n}, t) = \sum_{\ell, m} \sum_{\alpha} 4\pi \ell! \int \frac{k^2 \, dk}{(2\pi)^3} k^\alpha h_{\ell m}(k) \left[ F_{E, \alpha}^{E, \ell} Y_{\ell m}(\hat{n}) + F_{B, \alpha}^{B, \ell} Y_{\ell m}(\hat{n}) \right] e^{-i\omega_\alpha(k)t},$$

where

$$F_{E, \alpha}^{E, \ell} = -\frac{1}{2} R_{E, \alpha}^{E, \ell}(0) + \int_0^\infty dx \, R_{E, \alpha}^{E, \ell}(x) - \frac{1}{2} \ell (\ell + 1) R_{E, \alpha}^{E, \ell} \frac{1}{x} e^{ixv_{ph, \alpha}}$$

$$F_{B, \alpha}^{B, \ell} = \int_0^\infty dx \, R_{B, \alpha}^{B, \ell}(x) \frac{1}{x} e^{ixv_{ph, \alpha}}$$

in the distant-source limit.

### C. Power spectra

We derive the analytic expressions for the projection factors $F_{E, \alpha}^{E, \ell}$, $F_{E, \alpha}^{E, \ell}$, and $F_{B, \alpha}^{B, \ell}$ in Appendix A and summarize the results in Table I. We show the resulting power spectra $C_{\ell}^{zz}$, $C_{\ell}^{EE}$, $C_{\ell}^{BB}$, and $C_{\ell}^{zE}$ as functions of the multipole $\ell$ for each GW polarization in Figs. 1, 2, 3, and 4, respectively. For each case, we compare the power spectra at group velocities $v \in \{0.01, 0.4, 0.8, 0.9\}$ to the power spectra at $v = 1$ from Ref. [13]. The $C_{\ell}^{zz}$ and $C_{\ell}^{zE}$ spectra, however, diverge for SL modes in the $kr \to \infty$ limit for $v = 1$, due to the light ray surfing the GW; therefore, we show $v = 0.999$ rather than $v = 1$, since there is no surfing for subluminal GWs and the projection factor $F_{E,\alpha}^{E,\ell,SL}$ is finite. Note that we have dropped the polarization subscript $\alpha$ on $v_{\alpha}$ for notational simplicity, with the understanding that each GW polarization mode may propagate with its own distinct frequency.

We normalize all power spectra by their quadrupole contribution. For $v = 1$, the power spectra either have just one or two dominant contributions at low multipoles, while higher multipoles are either suppressed by factors of at least $\sim \ell^2$ or vanish altogether for the ST mode (see Table 1 of Ref. [13]). As the ST mode has no quadrupole to set the normalization, we omit the $\ell = 1$ curve (formed by the monopole and dipole) from Fig. 1 and the single $\ell = 1$ point (from the dipole) from Figs. 2 and 4.

For subluminal propagation, all power spectra become increasingly dominated by the quadrupole (and the monopole, for the scalar modes of $C_{\ell}^{zz}$) as $v$ decreases. We investigate this behavior in the $v \to 0$ limit in Appendix A, but we may also understand it using the plane-wave basis for GWs. The pulsar frequency shift and angular deflection for luminal GWs are

$$z(\hat{n}) = \frac{n^a n^b h_{ab}}{2(1 + \hat{k} \cdot \hat{n})},$$

$$\delta^a(\hat{n}) = \frac{(n^a + k^a) n^b h_{bc} - \frac{1}{2} n^b h_{ab}}{2(1 + \hat{k} \cdot \hat{n})},$$

where $\hat{k}$ denotes the direction of GW propagation [10]. The factors of $1 + \hat{k} \cdot \hat{n}$ are derived by assuming the wave propagates at the speed of light; if we instead have waves propagating with $v < 1$, these factors become $1 + \nu \hat{k} \cdot \hat{n}$, which go to unity as $v \to 0$. Then $z(\hat{n})$ and $\delta^a(\hat{n})$ reduce to projections of the GW polarizations onto the sky, which for the tensor and vector modes are quadrupolar in form, and for the scalar modes have both monopole and quadrupole components.

Notably, this trend holds true for the ST mode, for which power at $\ell > 1$ no longer vanishes. There is a stark contrast between the power spectra $C_{\ell}^{zz}$ of the ST mode with $v = 1$ (which is nonzero for only $\ell = 1$ and $\ell = 0$), $v \lesssim 1$ (which has contributions from all multipoles), and $v \ll 1$ (which is strongly peaked at $\ell = 2$ and $\ell = 0$).

While the change in the shape of the power spectrum for a given polarization mode is pronounced, it may be challenging to disentangle which modes contribute to an overall signal. In particular, for small enough values of $v$, the significant drop in power for $\ell \neq 2$ across all spectra renders them effectively degenerate, with the exception of $C_{\ell}^{zz}$ for the scalar modes, which have large monopole contributions. Moreover, the shapes of the spectra of different modes can look similar by adjusting the value of $v$, which is further complicated by the fact that the observed power spectra for subluminal GWs should have contributions from a range of velocities, corresponding to the range of observed frequencies, as discussed earlier in this section. Regardless, the presence of any monopole or dipole contributions in PTA and astrometric measurements (barring systematic uncertainties) would be a clear indicator of physics beyond GR.

Although we may assume the GW stochastic background consists mostly, if not all, of standard GR tensor
modes, we can also consider the possibility of a dominant subluminal mode that could generate a similar response to that expected from GR. We first note that the power spectrum for standard tensor modes is dominated by the quadrupole. While the power spectra for modes with $v \ll 1$ are also dominant at $\ell = 2$, they have a much steeper drop off at higher $\ell$. We can instead attempt to match the GR tensor mode power spectrum more closely by considering $v \lesssim 1$, softening the drop in power at large $\ell$, but at the expense of the monopole and/or dipole contribution for non-tensor modes being more prominent. Therefore, measurements at both small and large angular separations $\Theta$ (to probe large and small $\ell$, respectively) can help discriminate GR tensor modes from other possibilities.

We further demonstrate the possible similarities (or lack thereof) for PTAs by comparing the vector and tensor mode correlation functions

$$C^{zz}(\Theta) = \sum_\ell \frac{2\ell + 1}{4\pi} C^{zz}_\ell P_\ell(\cos \Theta),$$

where $P_\ell$ are Legendre polynomials, at different velocities to the standard Hellings-Downs curve in Fig. 5. While the vector correlation functions resemble the Hellings-Downs curve, the difference in amplitude compared to the Hellings-Downs curve is greater at small angles than at large angles; therefore, any rescaling of the amplitudes [curves in Fig. 5 are normalized such that $C^{zz}(90^\circ) = -1$] cannot match the Hellings-Downs curve at both small and large angles. Moreover, the vector correlation functions appear slightly shifted towards larger $\Theta$ compared to the tensor modes, making measurements at many different angular separations $\Theta$ important for discrimination power. The NANOGrav 11-year dataset, for instance, has the most sensitivity between $30^\circ$ and $60^\circ$ but has very few pulsar pairs at wide separations, above $90^\circ$ [5]. Without more wide-angle pairs, low multipoles (important for non-tensor modes) cannot be well studied or constrained.

Finally, we caution that for very small $v$, the distant-source limit assumed throughout this work ceases to be a reasonable approximation. For GWs of a given frequency $f$, the corresponding GW wavelength is $\lambda = v_{ph}/f = 1/(fv)$; thus, decreasing $v$ increases $\lambda$. If the wavelength of the GW is comparable to or larger than the separation between the Earth and the source or be-
FIG. 2. The $C_{\ell}^{EE}$ power spectra for the scalar, vector, and tensor GW polarization modes at various values of the group velocity $v$, as indicated in the legend of the lower-right panel. The spectra are normalized to $C_2^{EE}$. The $v = 1$ line for the ST mode is not shown, since it has contributions from $\ell = 1$ only.

FIG. 3. The $C_{\ell}^{BB}$ power spectra for the vector and tensor GW polarization modes at various values of the group velocity $v$, as indicated in the legend of the right panel. The spectra are normalized to $C_2^{BB}$. The scalar polarizations do not generate $B$-mode deflections.

tween source pairs, the distant-source limit is no longer valid [21]. For GWs of frequency $f \sim \text{yr}^{-1}$, there are $\sim 3 \times 10^3 v$ GW wavelengths between Earth and a source located a distance $r_s \sim \text{kpc}$ away. If two sources are located the same distance away, there are $\sim 10^4 v/\ell$ GW wavelengths between them for multipole $\ell$. Thus, for the case of $v = 0.01$, the distant-source limit is expected to break down for $\ell \gtrsim 10$. 

III. GRAVITATIONAL WAVE POLARIZATIONS IN $f(R)$ GRAVITY

As an example of subluminal GW propagation, we consider $f(R)$ gravity, where $f(R)$ is a function of the Ricci scalar $R$. The corresponding action is

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} f(R) ,$$

where $G$ is the gravitational constant and $g$ is the trace of the metric $g^{\mu\nu}$. The field equations (from varying the metric) and their trace are given by

$$0 = f'(R) R_{\mu\nu} - \frac{1}{2} R f(R) g_{\mu\nu} + (g_{\mu\nu} \Box - \nabla_{\mu} \nabla_{\nu}) f'(R) ,$$

$$0 = f'(R) R - 2f(R) + 3 \Box f'(R) ,$$

respectively, where $\Box \equiv g^{\mu\nu} \nabla_{\mu} \nabla_{\nu}$ and $R_{\mu\nu}$ is the Ricci tensor. For the case of standard GR, $f(R) = R$.

Let us assume $f(R)$ is well-behaved in order to Taylor expand around the static vacuum value $R = 0$,\(^2\) treating $R$ as a perturbation. To work at linear order in the expansion, we require that \(^{[22]}\)

$$f(0) + f'(0) R \gg \frac{1}{n!} f^{(n)}(0) R^n , \quad (18)$$

$$f'(0) + f''(0) R \gg \frac{1}{n!} f^{(n+1)}(0) R^n , \quad (19)$$

for all higher-order terms with $n > 1$. We focus on $f(R)$ models that contain $R$-dependent contributions beyond GR, such that $f(0) = 0$ and $f'(0) \neq 0$. Thus, expanding $f(R)$ to linear order in $R$, Eq. (17) becomes

$$0 = m^2 \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) + \frac{1}{3} (g_{\mu\nu} \Box - \nabla_{\mu} \nabla_{\nu}) R ,$$

$$0 = (\Box - m^2) R , \quad (20)$$

with

$$m^2 \equiv \frac{f'(0)}{3f''(0)} , \quad (21)$$

around a finite value $R_0$ \(^{[22]}\).

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\(^2\) For the case of $1/R^n$ gravity, where $n > 0$, we would expand...
The field equation for \( \phi \) is \( f''(\varphi) = R \) if \( f''(\varphi) \neq 0 \), and we recover the action of Eq. (16).

Let us now investigate how this scalar degree of freedom decomposes into the ST and SL GW polarizations. One can show that in synchronous gauge, the scalar perturbation can be written as

\[
h^{\text{scalar}}_{ab} = 2\alpha R \left( \delta_{ab} - \frac{k_a k_b}{\omega^2} \right),
\]

(22)

where \( \alpha = 1/6m^2 \), which can be interpreted as the parameter that appears in \( f(R) \) gravity is equivalent to a scalar-tensor theory of gravity [23, 24], in which a massive scalar field \( \varphi \) constitutes a single additional degree of freedom beyond GR [25]. This connection is clear from writing the action as \( S \sim \int d^4x \sqrt{-g} f(\varphi) + (R - \varphi) f'(\varphi) \).

The field equation for \( \varphi \) is \( f''(\varphi) \neq 0 \), and we recover the action of Eq. (16).

We can calculate the geodesic deviation, which is a gauge-invariant quantity in linearized gravity. If we write the metric for a scalar GW as

\[
h^{\text{scalar}}_{ab} = \left( \begin{array}{cc} \varepsilon_{ab} + \frac{1-v^2}{\sqrt{2}} \varepsilon_{ab}^S \end{array} \right) e^{i(kz - \omega t)},
\]

(25)

then the geodesic deviation equation gives

\[
\begin{align*}
\ddot{x} &= -\frac{\omega^2}{2} e^{i(kz - \omega t)} x, \\
\ddot{y} &= -\frac{\omega^2}{2} e^{i(kz - \omega t)} y, \\
\ddot{z} &= -\frac{m^2}{2} e^{i(kz - \omega t)} z.
\end{align*}
\]

(26-28)

This result is consistent with that in Ref. [25], where the geodesic deviation was calculated using a different choice of gauge, and in Ref. [26], which used a gauge-invariant method.

The analysis in this section has thus far been in terms of plane waves, rather than TAM waves. We translate between the two bases using [17]

\[
\varepsilon_{ab}^s(k) e^{i k \cdot x} = 4\pi \sum_{\alpha,\ell,m} i^\ell B_{(\ell, m)}^a(\hat{k}) \Psi_{(\ell m)ab}^S(\mathbf{x}),
\]

(29)

where \( \varepsilon_{ab}^s(k) \) are the polarization tensors for plane waves of polarization \( s \), and the coefficients \( B_{(\ell, m)}^a \) are given by

\[
B_{(\ell, m)}^a = \varepsilon_{ab}^s(k) Y^a_{(\ell m)ab}(\hat{k}).
\]

(30)

In the case of the \( f(R) \) scalar mode, the plane waves only project onto the ST and SL spherical harmonics. Then, Eq. (29) allows us to rewrite Eq. (25) as

\[
h^{\text{scalar}}_{ab} = 4\pi \sum_{\ell, m} i^\ell Y_{(\ell m)}(\hat{k}) \times \left[ \sqrt{2} \Psi_{(\ell m)ab}^{ST}(\mathbf{x}) + (1 - v^2) \Psi_{(\ell m)ab}^{SL}(\mathbf{x}) \right].
\]

(31)

From Table I in Appendix A, we note that

\[
\begin{align*}
F_{\ell}^{E,ST} &= -\frac{1-v^2}{\sqrt{2}} F_{\ell}^{E,SL}, & \text{for } \ell &\geq 2, \\
F_{\ell}^{E,ST} &= -\frac{1-v^2}{\sqrt{2}} F_{\ell}^{E,SL}, & \text{for } \ell &\geq 2,
\end{align*}
\]

(32-33)

and thus the monopole and dipole are the only non-vanishing moments for the \( C_{zz}^{zz} \) and \( C_{EE}^{zz} \) power spectra for the scalar mode. The \( C_{EE}^{zz} \) spectrum, however, does not experience the same cancellations and receives contributions from all multipoles.

IV. CONCLUSIONS

We have derived the power spectra for the induced time delay in pulsar-timing surveys and the induced stellar shifts in astrometry from a stochastic background of...
subluminal GWs. In the limit that the GW velocity approaches the speed of light, we recover the results presented in Ref. [13]. We have treated each GW polarization independently; however, a particular theory of modified gravity may relate the amplitudes between certain modes. As an example, we have considered \( f(R) \) gravity, which gives rise to a single massive scalar mode that is a linear combination of ST and SL modes. The relative contribution of ST and SL is set by the group velocity \( v \) of the GW. We find that this new mode only excites the monopole and dipole.

Previous studies of the angular correlations for non-Einsteinian polarizations focused on the case of luminal GW propagation. As there are, however, gravitational theories that contain a massive degree of freedom, our work provides the foundation for considering subluminal theories that contain a massive degree of freedom, our work provides the foundation for considering subluminal GW propagation. As there are, however, gravitational theories that contain a massive degree of freedom, our work provides the foundation for considering subluminal GW propagation. As there are, however, gravitational theories that contain a massive degree of freedom, our work provides the foundation for considering subluminal GW propagation. As there are, however, gravitational theories that contain a massive degree of freedom, our work provides the foundation for considering subluminal GW propagation. As there are, however, gravitational theories that contain a massive degree of freedom, our work provides the foundation for considering subluminal GW propagation.

\begin{align*}
R_{L,SL}^E(x) &= x j''_L(x) \\
R_{S,ST}^E(x) &= -\frac{1}{\sqrt{2}} \left[ R_{L,SL}^E(x) + j_L(x) \right] \\
R_{L,VE}^E(x) &= -\sqrt{2\ell(\ell+1)} \frac{d}{dx} \left[ j^{\prime}_L(x) \right] \\
R_{L,TE}^E(x) &= -\sqrt{2\ell(\ell+1)} X_1 + \sqrt{2\ell(\ell+1)} X_2 \\
R_{E,SL}^E(x) &= \frac{x}{2} \frac{d}{dx} R_{L,SL}^E(x) + 2 \sqrt{\ell(\ell+1)} R_{L,SL}^E(x) \\
R_{E,ST}^E(x) &= -\frac{1}{\sqrt{2}} R_{L,SL}^E(x) \\
R_{E,VE}^E(x) &= -\frac{x}{2} \frac{d}{dx} R_{L,VE}^E(x) + 2 \sqrt{\ell(\ell+1)} R_{L,VE}^E(x) + \frac{x}{2\sqrt{2}} j'_L(x) \\
R_{E,TE}^E(x) &= -\frac{x}{2} \frac{d}{dx} R_{L,TE}^E(x) + 2 \sqrt{\ell(\ell+1)} R_{L,TE}^E(x) + \frac{N_x}{2\sqrt{\ell(\ell+1)}} j_L(x) \\
R_{B,VB}^B(x) &= \frac{ix}{2\sqrt{\ell(\ell+1)}} R_{L,VE}^E(x) \\
R_{B,TB}^B(x) &= \frac{ix}{2\sqrt{\ell(\ell+1)}} R_{L,TE}^E(x),
\end{align*}

where \( j_L(x) \) is the spherical Bessel function of the first kind, \( N_x \equiv \sqrt{\ell(\ell+1)[2\ell(n-2)]} \), and functions with unlisted combinations of \( \{ L, E, B \} \) are zero. Note that we have used the differential equation for the spherical Bessel function

\[ x^2 j''_L(x) + 2x j'_L(x) + \left[ x^2 - \ell(\ell+1) \right] j_L(x) = 0 \]  

(A1)

to recast \( R_{L,SL}^E \) from its form given in Refs. [13, 17]. The simple relations between the radial functions of the ST and SL modes allow for the projection factors for ST to be easily obtained from those for SL. Additionally, \( F_{B,VB}^E \) and \( F_{B,TB}^E \) are easily obtained from \( F_{L,VE}^E \) and \( F_{L,TE}^E \), respectively. The somewhat complicated relations for \( R_{E,\alpha}^E \) are particularly useful to simplify the integrand in Eq. (11).

Plugging these radial functions into the equations for the projection factors, we simplify expression by integrating by parts any function with an explicit derivative. All boundary terms are proportional to \( j_L(x)/x^n \) or \( j'_L(x)/x^n \) for \( n \geq 0 \), which vanish at \( x \rightarrow \infty \); terms evaluated at \( x \rightarrow 0 \) are determined by the limiting behavior \( j_L(x) \rightarrow x^{2-\ell(n+1)/2} \sqrt{\Gamma(\ell+3/2)} \). The remaining terms in the projection factors are all proportional to

\[ I^{(n)}(v) = \int_0^{\infty} \frac{j_L(x)}{x^n} e^{ixvh} \, dx \]  

(A2)
with $v = 1/v_{\text{ph}}$. For instance, we determine $F_{\ell}^{z,\text{SL}}$ by integrating $j_{\ell}^{\mu}(x)$ by parts twice, leaving a term proportional to $I_{\ell}^{(0)}$; meanwhile, the boundary terms at $x \to 0$ involve $j_{\ell} \to \delta_{\ell 0}$ and $j_{\ell}' \to \delta_{\ell 1}/3$. We summarize our results in Table I.

Up to this point, all derivations and the results in Table I hold for generic $v$. If we fix $v = 1$, Eq. (A2) becomes

$$I_{\ell}^{(n)}(v = 1) = i^{\ell+1-n} 2^{n-1} \frac{(\ell-n)!}{(\ell+n)!} (n-1)!$$

(A3)

for $\ell + 1 > n > 0$. Table I then matches the analogous table in our previous work [13]. Note that Eq. (A3) diverges for $n = 0$, corresponding to the divergence of $F_{\ell}^{z,\text{SL}}$ discussed in Ref. [13]. Other terms in Table I involving $n = 0$ should be dropped due to $(1 - v^2)$ prefactors in order to recover previous results.

For the case of subluminal GWs, we have

$$I_{\ell}^{(n)}(v < 1) = \left(\frac{iv}{2}\right)^{\ell+1-n} \sqrt{\pi} \frac{\Gamma(\ell+1-n)}{2^n} \frac{\Gamma(\ell+2n)}{\Gamma(\ell+\frac{3}{2})} \times 2F_1 \left(\frac{\ell+1-n}{2}, \frac{\ell+2-n}{2}, \frac{\ell+3}{2}, v^2 \right)$$

(A4)

for $\ell + 1 > n \geq 0$, where $2F_1$ is Gauss’s hypergeometric function. We have verified numerically that our results for the power spectra and correlation functions agree with those in Ref. [16].

As $v \to 0$, the hypergeometric function in Eq. (A4) approaches 1 at leading order and $I_{\ell}^{(n)}(v) \sim v^{\ell+1-n}$. From Table I, all terms for $F_{\ell}^{E,\alpha}$ and $F_{\ell}^{z,\alpha}$ that involve $I_{\ell}^{(n)}(v)$ thus have a $v^{\ell-2}$ dependence in this limit; for $F_{\ell}^{B,\text{TB}}$ and $F_{\ell}^{B,\text{ST}}$, the velocity dependence of the $I_{\ell}^{(n)}(v)$ terms is $v^{\ell-1}$. Therefore, the projection factors for all polarizations with $\ell > 2$ are more suppressed compared to $\ell = 2$ for smaller $v$, and $F_2^{E,\alpha}$ and $F_2^{z,\alpha}$ approach constant values. As expected from the discussion in Sec. II C, all of our power spectra feature a dominate quadrupole, seen in Figs. 1-4.

In order to consider the $v \to 0$ limit for projection factors with $\ell = 0$ or 1, we must account for cancellations between the $I_{\ell}^{(n)}(v)$ terms and any $\delta_{00}$ or $\delta_{11}$ terms and use the expansion $2F_1(a, b, c, v^2) \rightarrow 1 + (ab/c)v^2$. For all relevant (i.e., scalar and vector) projection factors, the leading order contribution for $\ell = 1$ scales with one additional power of $v$ compared to the $\ell = 2$ case, resulting in the suppression of the dipole with respect to the quadrupole in Figs. 1-4. Finally, $F_{\ell}^{z,\text{ST}}$ and $F_{\ell}^{z,\text{SL}}$ approach constant values for $\ell = 0$, with $F_{0}^{z,\text{ST}}/F_{0}^{z,\text{ST}} \rightarrow 5$ and $F_{0}^{z,\text{SL}}/F_{2}^{z,\text{SL}} \rightarrow -5/2$. Therefore, the $C_{\ell}^{zz}$ power spectra for the ST and SL modes exhibit monopole contributions that are factors of 25 and 25/4 larger than the quadrupole, respectively, as observed in Fig. 1.

Although we do not discuss the possibility of superluminal propagation in this work, we can apply Table I to such a scenario, writing Eq. (A2) as

$$I_{\ell}^{(n)}(v > 1) = 2^{-(n+1)} \sqrt{\pi} \left\{ \frac{\Gamma(\ell+1-n)}{2^n} \frac{\Gamma(\ell+2n)}{\Gamma(\ell+\frac{3}{2})} 2F_1 \left(\frac{\ell+n}{2}, \frac{\ell+1-n}{2}, 1, v^2 \right) \right\}$$

for $\ell + 1 > n \geq 0$. The superluminal results in Ref. [16] are numeric, so we numerically integrate Eq. (A2) with a finite upper limit of integration in order to compare. Our results for the superluminal case do not align with those presented in Ref. [16]: we find that our power spectra follow the same general trend of power at large $\ell$ becoming less suppressed as $v$ increases, but without the oscillatory behavior. While we have not confirmed the source of the discrepancy, we regard subluminal propagation to be the more compelling physics case and leave the superluminal scenario for future consideration.

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**Appendix B: Synchronous gauge**

To determine the degrees of freedom in $f(R)$ gravity from the field equations, we start with a generic metric perturbation with components

$$h_{00} = -2\Phi \ ,$$

$$h_{0a} = w_a \ ,$$

$$h_{ab} = 2s_{ab} - 2\Psi \delta_{ab} \ ,$$

(B1)

where $\Psi = -\frac{1}{6} \delta_{ab} h_{ab}$ is proportional to the trace and $s_{ab} = \frac{1}{2} \left( h_{ab} - \frac{1}{3} \delta_{cd} h_{cd}\delta_{ab} \right)$ is traceless. From Eq. (20), we already know that this theory has a propagating scalar degree of freedom, $R$, so we will eventually rewrite some of the above components in terms of $R$. 
The Ricci tensor for this metric is

\[
R_{ab} = \nabla^2 \Phi + \partial_\alpha \partial_\beta c^\alpha + 3\partial_\beta \nabla^2 \Psi ,
\]

\[
R_{ab} = -\frac{1}{2} \nabla^2 w_a + \frac{1}{2} \partial_\alpha \partial_\beta c^\alpha + 2\partial_\alpha \partial_\beta \Psi + \partial_\alpha \partial_\beta s^\alpha_b ,
\]

\[
R_{ab} = -\partial_\alpha \partial_\beta (\Phi - \Psi) - \partial_\alpha \partial_\beta (a^a w_b) + \Box \Psi \delta_{ab} - \Box s_{ab} + 2\partial_\alpha \partial_\beta s^\alpha_b ,
\] (B2)

and the Ricci scalar is therefore

\[
R = -R_{00} + R_{aa} = -2\nabla^2 \Phi - 6\partial_\alpha \nabla^2 \Psi + 4\nabla^2 \Psi - 2\partial_\alpha \partial_\beta w^\alpha + 2\partial_\alpha \partial_\beta s^{ac}.
\] (B3)

Substituting these results into the field equations, we find for the various components:

\[
00 : \quad 0 = m^2 \left( \nabla^2 \Phi + \partial_\alpha \partial_\beta c^\alpha + 3\partial_\beta \nabla^2 \Psi + \frac{1}{3} R \right) + \frac{1}{6} (m^2 - \nabla^2) R ,
\] (B4)

\[
0a : \quad 0 = m^2 \left( \nabla^2 w_a + \frac{1}{2} \partial_\alpha \partial_\beta w^\alpha + 2\partial_\alpha \partial_\beta \Psi + \partial_\alpha \partial_\beta s^\alpha_b \right) - \frac{1}{6} \partial_\beta \partial_a R ,
\] (B5)

\[
ab : \quad 0 = m^2 \left[ -\partial_\alpha \partial_\beta (\Phi - \Psi) - \partial_\alpha \partial_\beta (a^a w_b) + \Box \Psi \delta_{ab} - \Box s_{ab} + 2\partial_\alpha \partial_\beta s^\alpha_b - \frac{1}{3} \delta_{ab} R \right] - \frac{1}{6} \partial_\alpha \partial_\beta R .
\] (B6)

The first equation has no time derivatives of \( \Phi \) and the second has no time derivatives of \( w_a \), so these two fields do not represent propagating degrees of freedom—they can be written purely in terms of other fields. We work in the synchronous gauge by setting \( \Phi = w^a = 0 \). The field equations simplify to

\[
00 : \quad 0 = m^2 \left( 3\partial_\beta \nabla^2 \Psi + \frac{1}{3} R \right) + \frac{1}{6} (m^2 - \nabla^2) R ,
\] (B7)

\[
0a : \quad 0 = m^2 \left( 2\partial_\alpha \partial_\beta \Psi + \partial_\alpha \partial_\beta s^\alpha_b \right) - \frac{1}{6} \partial_\beta \partial_a R ,
\] (B8)

\[
ab : \quad 0 = m^2 \left[ \partial_\alpha \partial_\beta \Psi + \Box \Psi \delta_{ab} - \Box s_{ab} + 2\partial_\alpha \partial_\beta s^\alpha_b - \frac{1}{3} \delta_{ab} R \right] - \frac{1}{6} \partial_\alpha \partial_\beta R .
\] (B9)

The 00 equation allows us to write \( \Psi \) purely in terms of \( R \). Since \( R \) satisfies a wave equation, let us assume
\[ \partial_t R = i k_a R \quad \text{and} \quad \partial_b R = -i \omega R, \quad \text{where} \quad \omega^2 - k^2 = m^2. \]

If we set the time-independent integration constants to zero, the 00 field equation can be integrated to give
\[ \Psi = \frac{3 \omega^2 - 2 k^2}{18m^2 \omega^2} R. \tag{B10} \]

Similarly, integrating the \( ab \) equation with respect to time, we find
\[ \partial_b s_{ab} = \partial_a \left( \frac{1}{6m^2} R - 2 \Psi \right) \propto R. \tag{B11} \]

Thus, \( \Psi \) and the divergence of \( s_{ab} \) are both related to the same single degree of freedom: the scalar \( R \).

The only equation we have not yet studied is the \( ab \) equation. Substituting \( \Psi \) and \( \partial_b s_{ab} \) with their relations to \( R \), we find the \( ab \) equation simplifies to
\[ \square s_{ab} = \frac{6k_a k_b - (2k^2 + 3 \omega^2) \delta_{ab}}{18 \omega^2} R. \tag{B12} \]

Then, since \( R = \frac{\square}{m^2} R \), we can rewrite this as
\[ \square \left[ s_{ab} + \left( \frac{2k^2 + 3 \omega^2}{3 \omega^2} \right) \delta_{ab} - \frac{k_a k_b}{\omega^2} \alpha R \right] = 0, \tag{B13} \]

where \( \alpha = 1/6m^2 \).

The full metric perturbation is \( h_{ab} = 2s_{ab} - 2\Psi \delta_{ab} \). If the tensor degrees of freedom are of the form \( s_{ab} + C_{ab} R \), where \( C_{ab} \) denotes the coefficient of \( R \) in Eq. (B13), the remaining scalar perturbation becomes
\[ h_{ab}^{\text{scalar}} = \Psi \delta_{ab} + C_{ab} R = 2\alpha R \left( \delta_{ab} - \frac{k_a k_b}{\omega^2} \right). \tag{B14} \]

Thus, after fixing the synchronous gauge, we have shown that our remaining degrees of freedom are a massless scalar \( R \) and two tensor degrees of freedom that satisfy a massless wave equation.

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