Primordial vorticity and gradient expansion

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Abstract
The evolution equations of the vorticities of the electrons, ions and photons in a pre-decoupling plasma are derived, in a fully inhomogeneous geometry, by combining the general relativistic gradient expansion and the drift approximation within the Adler–Misner–Deser decomposition. The vorticity transfer between the different species is discussed in this novel framework and a set of general conservation laws, connecting the vorticities of the three-component plasma with the magnetic field intensity, is derived. After demonstrating that a source of large-scale vorticity resides in the spatial gradients of the geometry and of the electromagnetic sources, the total vorticity is estimated to lowest order in the spatial gradients and by enforcing the validity of the momentum constraint. By acknowledging the current bounds on the tensor to scalar ratio in the (minimal) tensor extension of the LCDM paradigm, the maximal comoving magnetic field induced by the total vorticity turns out to be, at most, of the order of $10^{-37}$ G over the typical comoving scales ranging between 1 and 10 Mpc. While the obtained results seem to be irrelevant for seeding a reasonable galactic dynamo action, they demonstrate how the proposed fully inhomogeneous treatment can be used for the systematic scrutiny of pre-decoupling plasmas beyond the conventional perturbative expansions.

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1. Primordial vorticities?

The observed Universe might originate from a strongly coupled electromagnetic plasma existing prior to photon decoupling where the angular momentum transfer between ions, electrons and photons in an expanding spacetime geometry leads to the formation of large-scale vortices as speculated by various authors including, with slightly different perspectives, Hoyle [1], Harrison [2–4], Mishustin and Ruzmaikin [5], Ozernoy and Chernin [6–8] and others (see also [9]). The primordial vorticity, if present in the pre-decoupling plasma, might
lead eventually to the formation of large-scale magnetic fields possibly relevant for galactic magnetogenesis.

The physical description of the angular momentum exchange between ions, electrons and photons can be realized by appropriately translating the evolution equations describing ionized gases \([10, 11]\) to an expanding geometry supplemented by its own relativistic fluctuations \([12, 13]\). In the context of the \(\Lambda\)CDM paradigm\(^4\), it is both reasonable and justified to assume that the background geometry is conformally flat and that its inhomogeneities stem from the relativistic fluctuations of the spatial curvature described either in gauge-invariant terms or in an appropriate gauge. The latter assumption rests exactly on the absence of large-scale vorticity which is assumed to be vanishing at least within the current observational precision.

A gross argument could suggest that the vorticity must be negligible for \(\Lambda\)CDM initial conditions, since it is the curl of a velocity. Thanks to the momentum constraint (connecting the first derivatives of the linearized fluctuations of the geometry to the peculiar velocities), the total velocity field is subleading when compared with the density contrasts or with the curvature perturbation for typical scales larger than the Hubble radius and in the case of the conventional adiabatic initial conditions postulated in the vanilla \(\Lambda\)CDM scenario. The latter argument suggests that the treatment of large-scale vorticity assumes, more or less tacitly, a correct treatment of the spatial gradients. To transform this incomplete observation in a more rigorous approach it is necessary to introduce a description of the vorticity which does not rely on the purported smallness or largeness of the gravitational fluctuations. It is rather desirable to describe the angular momentum exchange between ions, electrons and photons in a gravitating plasma which is also fully inhomogeneous. By fully inhomogeneous plasma we mean the situation where not only the concentrations of charged and neutral species depend, in an arbitrary manner, upon the spatial coordinates but where the geometry as well as the electromagnetic fields are not homogeneous. It has recently been argued \([14]\) that such a description can be rather effective for the analysis of a wide range of phenomena including the physics of pre-decoupling plasmas. In this paper, the results of \([14]\) will be first extended and then applied to a concrete situation with the purpose of obtaining an explicit set of equations describing the evolution of the vorticities of the various species of the plasma. The proposal of \([14]\) is built on the fully inhomogeneous description of the geometry in terms of the Adler–Misner–Deser (ADM) variables \([15, 16]\) which are customarily exploited for the implementation of the general relativistic gradient expansion \([17–22]\). The second key ingredient of \([14]\) is the fully inhomogeneous description of cold plasmas in flat space which is the starting point of the analysis of nonlinear effects in kinetic theory and in magnetohydrodynamics (see, e.g., \([23–25]\)). Consequently, the vorticity exchange between ions, electrons and photons can be analyzed in gravitating plasmas with the help of an expansion scheme which involves not only the gradients of the geometry, but also the gradients of the electromagnetic sources, by combining the general relativistic gradient expansion and the drift approximation (sometimes dubbed guiding center approximation) typical of cold plasmas.

It is appropriate to stress that there exist two sorts of gradient expansions which are often employed in different areas of physics, i.e. the gradient expansion typical of plasma physics (holding on a fixed Minkowski background) and the general relativistic gradient expansion (holding on a fixed electromagnetic background). If the geometry is a dynamical quantity, the gradients of electric and of the magnetic fields must be treated consistently with the gradients of the geometry. In a flat-space plasma to zeroth-order in the expansion, only the time derivative of the magnetic fields are kept. To first-order, the spatial derivatives of

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\(^4\) The acronym \(\Lambda\)CDM (where \(\Lambda\) denotes the dark energy component and CDM stands for cold dark matter) and the terminology concordance paradigm will be used interchangeably.
the magnetic field can then be used as an input to deduce the electric fields. The first derivatives of the electric fields (obtained to first order) can be used to deduce the second spatial derivatives of the magnetic fields and so on. If we need to combine the general relativistic gradient expansion with the drift approximation, the essential step is the generalization of the two-fluid description and of the usual MHD reduction to the situation where the fully inhomogeneous geometry is parametrized in the standard ADM form. This analysis will lead automatically to the correct dynamical variables whose explicit form can be compared, for instance, with the corresponding variables deduced in the context of conformally flat geometries which are usually assumed in more conventional perturbative expansions.

The approach pursued in this paper reproduces, in the conformally flat limit, the conventional treatment which will be made more precise in section 2. The evolution of the vorticity in gravitating plasmas which are also fully inhomogeneous will be discussed in section 3. In sections 4 and 5, the total vorticity of the geometry will be computed within the gradient expansion and estimated in the framework of the ΛCDM paradigm. The maximal magnetic field induced by the total vorticity will be computed in section 6. Section 7 contains our concluding remarks. In appendix A some useful complements have been included to make the paper self-contained while in appendix B useful details on the calculations of correlation functions of multiple fields in real space have been included for the technical benefit of the interested readers.

2. Vorticities in conventional perturbative expansions

The treatment proposed here differs slightly from the one of [2–5] for three reasons: (i) the conformal time coordinate is preferred to the cosmic time; (ii) the relativistic fluctuations of the geometry are included in the longitudinal gauge; (iii) the three-fluid, two-fluid and one-fluid descriptions are discussed more explicitly within the appropriate temperature ranges where they are applicable.

The conformal flatness of the geometry does not imply the invariance of the system under the Weyl rescaling of the metric. Such a potential symmetry is broken by the masses of the electrons and ions which are crucial in the large-scale evolution of the vorticity. As stressed in the introduction, it is appropriate to start the discussion from the more standard perturbative approach to the vorticity in the case of a conformally flat background geometry supplemented by its relativistic fluctuations. The results obtained in this framework can be generalized to include spatial curvature in the framework of the standard Friedmann–Robertson–Walker models but we will stick to the vanilla ΛCDM scenario where the background geometry is assumed to be spatially flat. Consider first the case of a conformally flat background geometry characterized by a metric tensor $g_{\mu\nu} = a^2(\tau)\eta_{\mu\nu}$ and supplemented by the corresponding relativistic fluctuations which we write in the longitudinal gauge

$$\delta_s g_{00}(\vec{x}, \tau) = 2 a^2(\tau) \phi(\vec{x}, \tau), \quad \delta_s g_{ij}(\vec{x}, \tau) = 2 a^2(\tau) \psi(\vec{x}, \tau) \delta_{ij};$$

(2.1)

note that $\delta_s$ describes a metric perturbation which preserves the scalar nature of the fluctuation since, in the ΛCDM paradigm, the dominant source of inhomogeneity comes from the scalar modes of the geometry. By defining the comoving electromagnetic fields $\vec{E}$ and $\vec{B}$ as well as the comoving concentrations of electrons and ions (i.e. $n_e$ and $n_i$)

$$\vec{E}(\vec{x}, \tau) = a^2(\tau) \vec{\tilde{E}}(\vec{x}, \tau), \quad \vec{B}(\vec{x}, \tau) = a^2(\tau) \vec{\tilde{B}}(\vec{x}, \tau),$$

$$n_i(\vec{x}, \tau) = a^2(\tau) \tilde{n}_i(\vec{x}, \tau), \quad n_e(\vec{x}, \tau) = a^2(\tau) \tilde{n}_e(\vec{x}, \tau);$$

(2.2)

Maxwell’s equations read

$$\nabla \cdot \vec{E} = 4\pi e(n_i - n_e), \quad \nabla \cdot \vec{B} = 0,$$

(2.3)
In equations (2.2), (2.3) and (2.4), all the fields are appropriately rescaled so that the resulting equations are formally equivalent to the ones of the flat spacetime. The peculiar velocities of the ions, electrons and photons obey the following set of equations:

\[
\partial_t \vec{v}_e + \mathcal{H} \vec{v}_e = -\frac{en_e}{\rho_e a^4} [\vec{E} + \vec{v}_e \times \vec{B}] - \vec{\nabla} \phi + \frac{4}{3} \frac{\rho_e}{\rho_0} a \Gamma_{\gamma e} (\vec{v}_e - \vec{v}_0) + a \Gamma_{\varepsilon i} (\vec{v}_i - \vec{v}_e),
\]

(2.5)

\[
\partial_t \vec{v}_i + \mathcal{H} \vec{v}_i = \frac{en_i}{\rho_i a^4} [\vec{E} + \vec{v}_i \times \vec{B}] - \vec{\nabla} \phi + \frac{4}{3} \frac{\rho_i}{\rho_0} a \Gamma_{\gamma i} (\vec{v}_i - \vec{v}_0) + a \Gamma_{\varepsilon e} (\vec{v}_e - \vec{v}_i),
\]

(2.6)

\[
\partial_t \vec{v}_\gamma + \mathcal{H} \vec{v}_\gamma = -\frac{1}{\rho_0} \vec{\nabla} \phi - \vec{\nabla} \phi + a \Gamma_{\gamma 1} (\vec{v}_1 - \vec{v}_\gamma) + a \Gamma_{\gamma e} (\vec{v}_e - \vec{v}_\gamma).
\]

(2.7)

In equations (2.5)–(2.7), the relativistic fluctuations of the geometry are included from the very beginning in terms of the longitudinal gauge variables of equation (2.1); the electron–photon, electron–ion and ion–photon rates of momentum exchange appearing in equations (2.5)–(2.7) are given by

\[
\Gamma_{\gamma e} = \tilde{n}_e \sigma_{\gamma e}, \quad \Gamma_{\gamma i} = \tilde{n}_i \sigma_{\gamma i}, \quad \sigma_{\gamma e} = \frac{8}{3} \pi \left( \frac{e^4}{m_e^2} \right)^2, \quad \sigma_{\gamma i} = \frac{8}{3} \pi \left( \frac{e^4}{m_i^2} \right)^2,
\]

(2.8)

\[
\Gamma_{\varepsilon i} = \tilde{n}_e \sqrt{\frac{T}{m_e}} \sigma_{\varepsilon i} = \Gamma_{\varepsilon i}, \quad \sigma_{\varepsilon i} = \frac{e^4}{T^2} \ln \Lambda_C, \quad \Lambda_C = \frac{3}{2e^4} \sqrt{\frac{T^4}{\tilde{n}_e \pi}}
\]

(2.9)

Note that in equations (2.8) and (2.9), \( T \) and \( \tilde{n} \) are, respectively, physical temperatures and physical concentrations. If the rates and the cross sections would be consistently expressed in terms of comoving temperatures \( \tilde{T} = a T \) and comoving concentrations \( \tilde{n} = a^3 \tilde{n} \), the corresponding rates will inherit a scale factor for each mass. For instance, \( a \Gamma_{\varepsilon i} \) becomes \( \tilde{n}_e \sqrt{\tilde{T}/(m_e a)} \left( e^4/\tilde{T}^2 \right) \ln \Lambda_C \), if comoving temperature and concentrations are used.

Let us then define the vorticities associated with the peculiar velocities of the various species

\[
\vec{\omega}_e (\vec{x}, \tau) = \vec{\nabla} \times \vec{v}_e, \quad \vec{\omega}_i (\vec{x}, \tau) = \vec{\nabla} \times \vec{v}_i, \quad \vec{\omega}_\gamma (\vec{x}, \tau) = \vec{\nabla} \times \vec{v}_\gamma,
\]

(2.10)

and their corresponding three-divergences:

\[
\theta_e (\vec{x}, \tau) = \vec{\nabla} \cdot \vec{v}_e, \quad \theta_i (\vec{x}, \tau) = \vec{\nabla} \cdot \vec{v}_i, \quad \theta_\gamma (\vec{x}, \tau) = \vec{\nabla} \cdot \vec{v}_\gamma.
\]

(2.11)

The evolution equations of the vorticities and of the divergences can be obtained by taking, respectively, the curl and the divergence of equations (2.5)–(2.7) and by using equations (2.3) and (2.4). To simplify the obtained expressions, it is useful to introduce the total comoving charge density and the comoving current density:

\[
\rho_0 = e (n_e - n_i), \quad \vec{J} = e (n_i \vec{v}_i - n_e \vec{v}_e).
\]

(2.12)

Thus, the evolution of the vorticities and of the divergences of the electrons are, respectively,

\[
\partial_t \vec{\omega}_e + \mathcal{H} \vec{\omega}_e = -\frac{en_e}{\rho_e a^4} [\partial_t \vec{B} + (\vec{v}_e \cdot \vec{\nabla}) \vec{B} + \phi \vec{B} - (\vec{\nabla} \phi) \vec{v}_e]
\]

\[
+ \frac{4}{3} \frac{\rho_e}{\rho_0} a \Gamma_{\gamma e} (\vec{\omega}_\gamma - \vec{\omega}_e) + a \Gamma_{\varepsilon i} (\vec{\omega}_i - \vec{\omega}_e),
\]

(2.13)

5 As usual \( \mathcal{H} = \partial_\tau \ln a \) and its relation with the Hubble rate is simply \( \mathcal{H} = a H \).

6 Note that \( T \) denotes the temperature and \( \Lambda_C \) is the Coulomb logarithm [10, 11].
\[ \partial_t \rho_e + \mathcal{H} \rho_e = -\frac{e n_e}{\rho_e a^3} [4 \pi \rho_h \bar{B} - 4 \pi \bar{v}_e \cdot \bar{J} - \bar{v}_e \cdot \partial_t \bar{E}] - \nabla^2 \phi + \frac{4 \rho_e}{3 \rho_c} a \Gamma_{\gamma e} (\theta_e - \theta_c) + a \Gamma_{\gamma i} (\theta_i - \theta_c). \]  

(2.14)

Conversely, the vorticity and the three-divergence of the ions evolve as

\[ \partial_t \bar{\omega}_i + \mathcal{H} \bar{\omega}_i = -\frac{en_i}{\rho_i a^3} [4 \pi \rho_h \bar{B} + (\bar{v}_i \cdot \nabla) \bar{B} + \partial_t \bar{B} - (\bar{B} \cdot \nabla) \bar{v}_i] + \frac{4 \rho_e}{3 \rho_i} a \Gamma_{\gamma i} (\bar{\omega}_i - \bar{\omega}_e) + a \Gamma_{\gamma i} (\bar{\omega}_e - \bar{\omega}_i), \]

(2.15)

\[ \partial_t \rho_i + \mathcal{H} \rho_i = \frac{en_i}{\rho_i a^3} [4 \pi \rho_h \bar{B} + (\bar{v}_i \cdot \nabla) \bar{B} + \partial_t \bar{B} - (\bar{B} \cdot \nabla) \bar{v}_i] - \nabla^2 \phi + \frac{4 \rho_e}{3 \rho_i} a \Gamma_{\gamma i} (\theta_i - \theta_e) + a \Gamma_{\gamma i} (\theta_e - \theta_i). \]

(2.16)

Finally, the evolution equations for the photons are given by

\[ \partial_t \bar{\omega}_\gamma = a \Gamma_{\gamma r} (\bar{\omega}_\gamma - \bar{\omega}_r) + a \Gamma_{\gamma e} (\bar{\omega}_e - \bar{\omega}_\gamma), \]

(2.17)

\[ \partial_t \theta_r = -\frac{1}{3} \nabla^2 \delta_r - \nabla^2 \phi + a \Gamma_{\gamma i} (\theta_i - \theta_r) + a \Gamma_{\gamma e} (\theta_e - \theta_r). \]

(2.18)

The system described by the set of equations deduced so far will be considered as globally neutral. In particular, prior to photon decoupling, the electron and ion (comoving) concentrations have a common value \( n_0 \), i.e. \( n_e = n_i = n_0 \) where

\[ n_0 = n_{b0} \eta_{b0}, \quad \eta_{b0} = 6.177 \times 10^{-10} \left( \frac{h_0^2 \Omega_{b0}}{0.02258} \right) \left( \frac{2.725 \text{ K}}{T_{\gamma 0}} \right)^3, \]

(2.19)

and \( T_{\gamma 0} \) is the present value of the CMB temperature determining the concentration of the photons; \( \Omega_{b0} \) is the present value of the critical fraction of baryons, while \( h_0 \) is the Hubble constant in units of 100 km(Mpc × s)\(^{-1}\). The system of equations (2.13)–(2.18) is coupled with the evolution of the density contrasts of the electrons, ions and photons (i.e. \( \delta_e, \delta_i \), and \( \delta_\gamma \)):

\[ \delta'_e = -\theta_e + 3 \psi' - \frac{en_e}{\rho_e a^3} \bar{E} \cdot \bar{v}_e, \quad \delta'_i = -\theta_i + 3 \psi' + \frac{en_i}{\rho_i a^3} \bar{E} \cdot \bar{v}_i, \]

(2.20)

\[ \delta'_\gamma = 4 \psi' - \frac{4}{3} \theta_\gamma. \]

(2.21)

Finally the metric fluctuations, the density contrasts and the divergences of the peculiar velocities are both determined and constrained by the perturbed Einstein equations (see, e.g. equations (2.43)–(2.46) in the first article of [12]). Concerning the system of equations (2.13)–(2.18), two comments are in order.

- Equations (2.13)–(2.14) (as well as equations (2.15)–(2.16)) couple together the evolution of the vorticities, the evolution of the divergences and the gradients of the magnetic field; while in the linearized approximation the spatial gradients are simply neglected, in the forthcoming sections the evolution of the vorticity will be studied to a given order in the spatial gradients.

- The electron and ion masses break the Weyl rescaling of the whole system of equations; this aspect can be appreciated by noting that the prefactor appearing in front of the square brackets on the right-hand side of equations (2.13)–(2.14) and equations (2.15)–(2.16) is, respectively, \( e/(m_e a) \) and \( e/(m_i a) \).

7 If not otherwise stated the pivotal values of the cosmological parameters will be the ones determined from the WMAP 7 yr data alone in the light of the \( \Lambda \)CDM paradigm.
Equations (2.13)–(2.18) have three different scales of vorticity exchange: the photon–ion, the photon–electron and the electron–ion rates whose respective magnitude determines the subleading terms and the different dynamical regimes. By taking the ratios of the two rates appearing on the right-hand side of equations (2.13) and (2.15), the following two dimensionless ratios can be constructed:

\[
\frac{3 \rho_e \Gamma_{ei}}{4 \rho_i \Gamma_{ye}} = \frac{135 \xi (3)}{16 \pi^5} \left( \frac{T}{m_e} \right)^{-5/2} \eta_{b0} \ln \Lambda_C \equiv \left( \frac{T}{T_{ei}} \right)^{-5/2},
\]

(2.22)

\[
\frac{3 \rho_e \Gamma_{ei}}{4 \rho_i \Gamma_{yi}} = \left( \frac{m_i}{m_e} \right)^2 \left( \frac{T}{T_{yi}} \right)^{-5/2} \equiv \left( \frac{T}{T_{yi}} \right)^{-5/2},
\]

(2.23)

where \( \xi (3) = 1.202 \) and the ion mass has been estimated through the proton mass; the effective temperatures \( T_{ei} \) and \( T_{yi} \) introduced in the second equality of equations (2.22) and (2.23) are defined as

\[
T_{ei} = m_e \Lambda^{2/5} \eta_{b0}^{2/5}, \quad T_{yi} = m_i m_e^{4/5} N^{2/5} \eta_{b0}^{2/5}, \quad N' = \frac{270 \xi (3)}{32 \pi^5} \ln \Lambda_C.
\]

(2.24)

In explicit terms and for the fiducial set of cosmological parameters determined on the basis of the WMAP 7 yr data alone in the light of the \( \Lambda \)CDM scenario [26, 27],

\[
T_{ei} = 88.6 \left( \frac{h_0^2 \Omega_{b0}}{0.022 58} \right)^{2/5} \text{eV}, \quad T_{yi} = 36.08 \left( \frac{h_0^2 \Omega_{b0}}{0.022 58} \right)^{2/5} \text{keV}.
\]

(2.25)

On the basis of equation (2.25), there are three different dynamical regimes. When \( T > T_{yi} \) the ion–photon and the electron–photon rates dominate against the Coulomb rate: in this regime the photons, electrons and ions are all coupled together and form a unique physical fluid with the same effective velocity. When \( T_{ei} < T < T_{yi} \), the Coulomb rate dominates against the Coulomb rate which is anyway larger than the ion–photon rate. Finally for \( T < T_{ei} \), the Coulomb rate is always dominant which means that the ion–electron fluid represents a unique entity characterized by a single velocity which is customarily referred to as the baryon velocity. The effective temperatures \( T_{ei} \) and \( T_{yi} \) determine the hierarchies between the different rates and should not be confused with the kinetic temperatures of the electrons and of the ions which coincide approximately with the photon temperature \( T_{\gamma} \approx T_e \approx T_i \). For instance after matter-radiation equality \((T_e - T_{\gamma})/T_{\gamma} \approx \mathcal{O}(H/T_{\gamma})\) and \((T_i - T_e)/T_{\gamma} \approx \mathcal{O}(H/T_{ei})\) where \( H \) is the standard Hubble rate at the corresponding epoch.

Depending on the range of temperatures, the effective evolution equations for the vorticities will change. In the regime \( T > T_{yi} \), the Coulomb rate can be neglected in comparison with the Thomson rates and the vorticities of photons, electrons and ions approximately coincide. For \( T_{ei} < T < T_{yi} \), the Ohm law can be easily obtained from equation (2.5) and it is given by

\[
\vec{E} + \vec{v}_e \times \vec{B} = \frac{\vec{J}}{\sigma} + \frac{4 \rho_e}{3 \rho_i} \frac{m_i}{e} a_i^2 \frac{\Gamma_{ye}}{\Gamma_{ei}} (\vec{v}_e - \vec{v}_i),
\]

(2.26)

where it has been used that the baryon density \( \rho_i = (m_i + m_e)\tilde{n}_0 \) coincides approximately with the ion density in the globally neutral case and that \( n_0 = a^3 \tilde{n}_0 \); furthermore, in equation (2.26), \( \sigma \) denotes the electric conductivity \[13\]

\[
\sigma = \frac{\omega_{pe}^2}{4 \pi a_i \Gamma_{ei}}, \quad \omega_{pe} = \sqrt{\frac{4 \pi e^2 n_e}{m_e a}}.
\]

(2.27)

Note that \( \rho_i \) must simplify when taking the ratio of the two rates in equation (2.15).
expressed in terms of the Coulomb rate and in terms of the electron plasma frequency
\( \omega_{pe} \). By taking the curl of both sides of equation (2.26), the following relation can be easily derived:
\[
\vec{\nabla} \times \vec{E} + \vec{\nabla} \times (\vec{v}_e \times \vec{B}) = \frac{\vec{\nabla} \times \vec{J}}{\sigma} + \frac{4}{3} \frac{\rho_v}{\rho_b} \frac{m_i}{e} a^2 \Gamma_{ve} (\vec{\omega}_v - \vec{\omega}_e).
\] (2.28)

Recalling now equations (2.3) and (2.4), equation (2.28) becomes
\[
\frac{\partial \vec{B}}{\partial \tau} = \vec{\nabla} \times (\vec{v}_e \times \vec{B}) + \frac{\gamma^2 \vec{B}}{4 \pi \sigma} - \frac{4}{3} \frac{\rho_v}{\rho_b} a^2 \frac{m_i}{e} \Gamma_{ve} (\vec{\omega}_v - \vec{\omega}_e).
\] (2.29)

In the same regime, the evolution equations for the vorticities of the ions and of the photons are, up to spatial gradients,
\[
\partial_\tau \vec{\omega}_i + \mathcal{H} \vec{\omega}_i = -\frac{e \nu_i}{\rho_i a^2} \partial_\tau \vec{B},
\] (2.30)
\[
\partial_\tau \vec{\omega}_\gamma = a \Gamma_{ve} (\vec{\omega}_e - \vec{\omega}_\gamma).
\] (2.31)

By eliminating the electron–photon rate between equations (2.30) and (2.31) and by neglecting the spatial gradients in equation (2.29), the following pair of approximate conservation laws can be obtained:
\[
\partial_\tau \left( \vec{\omega}_i + \frac{e}{m_i} \vec{B} \right) = 0,
\] (2.32)
\[
\partial_\tau \left( \frac{e}{m_i} \vec{B} - \frac{a}{R_b} \vec{\omega}_\gamma \right) = 0,
\] (2.33)

where the ratio \( R_b \) is given by
\[
R_b = \frac{3 \rho_b}{4 \rho_v} = 30.36 \left( \frac{10^3}{z} \right) h_\text{p}^2 \Omega_{z0}.
\] (2.34)

By further combining the relations of equations (2.32) and (2.33), the vorticity of the photons can be directly related to the vorticity of the ions since \( \partial_\tau \left[ R_b \vec{\omega}_i + \vec{\omega}_\gamma \right] = 0 \). By assuming that at a given time \( \tau_i \), the primordial value of the vorticity in the electron photon system is \( \vec{\omega}_i \) and that \( \vec{B}(\tau_i) = 0 \), we shall have that
\[
a_\tau \vec{\omega}_i(\tau_i) + \frac{4}{3} \frac{\rho_v(\tau_i)}{\rho_i(\tau_i)} a_\tau \vec{\omega}_\gamma(\tau_i) = \vec{\omega}_i.
\] (2.35)

Thus, the solution of equations (2.32) and (2.33) with the initial condition (2.35) can be written as
\[
\vec{\omega}_i(\vec{x}, \tau) = -\frac{e}{m_i} \frac{\vec{B}(\vec{x}, \tau)}{a(\tau)} + \frac{a_\tau}{a(\tau)} \vec{\omega}_i,
\] (2.36)
\[
\vec{\omega}_\gamma(\vec{x}, \tau) = \frac{R_b(\tau)}{a(\tau)} [\vec{\omega}_i - a(\tau) \vec{\omega}_i(\vec{x}, \tau)].
\] (2.37)

The approximate conservation laws of equations (2.32)–(2.33) can also be phrased in terms of the physical vorticities \( \Omega_X(\vec{x}, \tau) = a(\tau) \vec{\omega}_X(\vec{x}, \tau) \) where \( X \) denotes a generic subscript\(^10\).

For typical temperatures \( T < T_{e\gamma} \), the electrons and the ions are more strongly coupled than the electrons and the photons. This means that the effective evolution can be described in

\(^9\) The electron plasma frequency of equation (2.27) must not be confused with the vorticity.

\(^{10}\) Note that while \( \vec{\omega}_i \) is related to \( \vec{B} \), the physical vorticity \( \Omega_X \) is directly proportional to \( \vec{B} \). For instance, in the treatment of [2–4], the use of the physical vorticity and of the physical magnetic field is preferred.
terms of the one-fluid magnetohydrodynamical (MHD in what follows) equations where on top of the total current \( \vec{J} \), the center of mass vorticity of the electron–ion system is introduced:

\[
\vec{\omega}_b = \frac{m_e \vec{\omega}_e + m_i \vec{\omega}_i}{m_e + m_i}
\]  

Equation (2.13) (multiplied by \( m_e \)) and equation (2.15) (multiplied by \( m_i \)) can therefore be summed up with the result that

\[
\partial_t \vec{\omega}_b + \mathcal{H} \vec{\omega}_b = \frac{\vec{\nabla} \times (\vec{J} \times \vec{B})}{a^4 \rho_b} + \frac{\epsilon'}{R_b} (\vec{\omega}_\gamma - \vec{\omega}_b),
\]  

The evolution equation for the total current can be obtained from the difference of equations (2.5) and (2.6). Since the interaction rates are typically much larger than the expansion rates, the Ohm equation can be simplified and becomes

\[
\vec{E} + \vec{v}_b \times \vec{B} = \vec{J} + \frac{4 \rho_\gamma}{3 \rho_b} a^2 \Gamma_{\gamma e} (\vec{v}_\gamma - \vec{v}_b),
\]  

where \( \vec{v}_b \) is the baryon velocity related to the baryon vorticity as \( \vec{\omega}_b = \vec{\nabla} \times \vec{v}_b \). The similarity of equations (2.28) and (2.39) should not be misunderstood: while equation (2.28) follows from the right-hand side of equation (2.5), equation (2.39) follows by taking the difference of equation (2.6) (multiplied by \( n_i \)) and of equation (2.5) (multiplied by \( n_i \)). The expression obtained by means of the latter difference is rather lengthy and can be found in its full generality in [13] (see, in particular, equations (7) and (10)). Here the expression has been simplified by neglecting higher orders in \((m_e/m_i)\). The effective set of evolution equations can then be written, in this regime, as

\[
\partial_t \vec{\omega}_b + \mathcal{H} \vec{\omega}_b = \frac{\vec{\nabla} \times (\vec{J} \times \vec{B})}{a^4 \rho_b} + \frac{\epsilon'}{R_b} (\vec{\omega}_\gamma - \vec{\omega}_b),
\]

\[
\partial_t \vec{B} = \vec{\nabla} \times (\vec{v}_b \times \vec{B}) + \frac{\nabla^2 \vec{B}}{4 \pi \sigma} + \frac{m_e a}{e R_b} \epsilon' (\vec{\omega}_b - \vec{\omega}_\gamma),
\]

\[
\partial_t \vec{\omega}_\gamma = \epsilon' (\vec{\omega}_b - \vec{\omega}_\gamma),
\]

where \( \epsilon' = a \Gamma_{\gamma e} \) is the differential optical depth where, as usual, the contribution of the ions has been neglected. In the tight coupling limit, equations (4.21), (4.22) and (4.23) imply that \( \vec{\omega}_b = \vec{\omega}_\gamma \) while \( \vec{\omega}_b \) obeys

\[
\partial_t \vec{\omega}_b + \frac{\mathcal{H} R_b}{R_b + 1} \vec{\omega}_b = R_b \left( \frac{\vec{\nabla} \times (\vec{J} \times \vec{B})}{a^2 (R_b + 1)} \right)
\]

In analogy with what has been done before, the conservation laws can be derived by combining equations (4.21) and (4.22):

\[
\partial_t \left( \vec{B} + \frac{m_e}{e} a \vec{\omega}_b \right) = \vec{\nabla} \times (\vec{v}_b \times \vec{B}) + \frac{\nabla^2 \vec{B}}{4 \pi \sigma} + \frac{m_e}{e} \frac{\vec{\nabla} \times (\vec{J} \times \vec{B})}{a^2 \rho_b}.
\]  

From equations (4.22) and (4.23) and by neglecting the spatial gradients, it also follows that

\[
\partial_t \left( \frac{\vec{B}}{R_b} - \frac{m_e}{e} a \vec{\omega}_\gamma \right) = 0.
\]  

Equations (4.25) and (4.26) are separately valid, but, taken together and in the limit of tight baryon–photon coupling, they imply that the magnetic field must be zero when the tight coupling is exact (i.e. \( \vec{\omega}_b = \vec{\omega}_\gamma \)). In spite of the various physical regimes encountered in the analysis of the evolution of the vorticity, the key point is to find a suitable source of large-scale vorticity which could be converted in some way into a large-scale magnetic field [28] (see also
The conversion can not only occur prior to matter-radiation equality but also after [5] in the regime where, as explained, the baryon-photon coupling becomes weak. Indeed, equations (2.32) and (2.45) have the same dynamical content when the spatial gradients are neglected and the only difference involves the coupling to the photons.

There have been, through the years, suggestions involving primordial turbulence (see the interesting accounts of [31]), cosmic strings with small scale structure (see, e.g. [32–34]). Since matter flow in baryonic wakes is turbulent, velocity gradients will be induced in the flow by the small-scale wiggles of the string producing ultimately the vorticity. Dynamical friction between cosmic strings and matter may provide a further source of vorticity [33].

There have also been studies trying to generate large-scale magnetic fields in the context of superconducting cosmic strings (see, for instance, [34] and references therein). The possible generation of large-scale magnetic fields prior to hydrogen recombination has been discussed in [35–37] (see also [38]). The vorticity required in order to produce the magnetic fields is generated, according to [35], by the photon diffusion at second order in the temperature fluctuations. In a similar perspective, Hogan [37] got less optimistic estimates which, according to [35, 36], should be attributed to different approximation schemes employed in the analysis.

Along this perspective, various analyses discussed higher-order effects using the conventional perturbative expansion in the presence of the relativistic fluctuations of the geometry [39]. In this paper, as already mentioned, we will follow a different route since we intend to use the gradient expansion for a direct estimate of the vorticity.

3. Vorticity evolution in gradient expansion

The conservation laws derived in section 2 hold under the hypothesis that the spatial gradients are neglected in the evolution equations of the vorticity. The logic of the gradient expansion [17–22] can be combined with the tenets of the drift approximation [23–25] in the context of the ADM decomposition [15, 16]. It will be shown hereunder that the resulting formalism [14] provides a more general description of the angular momentum transfer between the various species of the plasma. Consider therefore the standard ADM decomposition where the shift vectors are set to zero but the lapse function kept arbitrary, i.e. $g_{00}(\vec{x}, \tau) = N^2(\vec{x}, \tau)$ and $g_{ij}(\vec{x}, \tau) = -\gamma_{ij}(\vec{x}, \tau)$. In this case, the Maxwell equations can be written as

$$\vec{\partial} \cdot \vec{E} = 4\pi e[n_i - n_e], \quad \vec{\partial} \cdot \vec{B} = 0,$$

(3.1)

$$\vec{\partial}_i \vec{B} + \vec{\partial} \times \vec{E} = 4\pi e[n_i \vec{v}_i - n_e \vec{v}_e] + \vec{\partial}_i \vec{E},$$

(3.2)

where the rescaled electric and magnetic fields are given by

$$E^i(\vec{x}, \tau) = \left(\frac{\sqrt{\gamma}}{N}\right)_{(\vec{x}, \tau)} E^i(\vec{x}, \tau), \quad B^i(\vec{x}, \tau) = \left(\frac{\sqrt{\gamma}}{N}\right)_{(\vec{x}, \tau)} B^i(\vec{x}, \tau).$$

(3.3)

in equation (3.3), the subscripts specify that the rescaling is spacetime dependent. The rescaled concentrations are

$$n_i(\vec{x}, \tau) = \sqrt{\gamma} \tilde{n}_i(\vec{x}, \tau), \quad n_e(\vec{x}, \tau) = \sqrt{\gamma} \tilde{n}_e(\vec{x}, \tau).$$

(3.4)

The shorthand notation\(^{11}\) employed in equations (3.1)–(3.3) implies that for a generic vector $A^i$,

$$\vec{\partial} \cdot \vec{A} \equiv \partial_i A^i, \quad (\vec{\partial} \times \vec{A})^i = \partial_j [N \gamma^{ik} \gamma^{jm} \tilde{n}_{mk} A^m].$$

(3.5)

11 Note that the operators introduced in equations (3.1)–(3.3) are the generalized curl, divergence and gradient operators; they reduce to the conventional curl, divergence and gradient operators in the conformally flat limit.
In appendix A, some relevant complements on this formalism have been collected to avoid a digression from the main line of arguments contained in this section. Two relevant aspects must anyway be born in mind:

• in the conformally flat limit (i.e. $N(\tilde{x}, \tau) \rightarrow a(\tau)$ and $\gamma_{ij}(\tilde{x}, \tau) \rightarrow a^2(\tau)\delta_{ij}$) equations (3.1) and (3.2) reproduce exactly equations (2.2) and (2.3);

• the same comment holds for all the other fields (i.e. comoving or physical) involved in the fully inhomogeneous description.

Using the generalized curl operator of equation (3.5), the vorticity of the ions of the electrons and of the photons can be written as

$$\omega_i^e = \partial_j (\Lambda_m^{ij} v_m^e), \quad \omega_i^\gamma = \partial_j (\Lambda_m^{ij} v_m^\gamma), \quad \omega_i^\gamma = \partial_j (\Lambda_m^{ij} v_m^\gamma),$$

(3.6)

where $\Lambda_{ij}$ is the extrinsic curvature (see appendix A) while $\Lambda_{m}^{ij}$ and $\Lambda_{m}^{ij}$ are defined as

$$\Lambda_{m}^{ij} = N \gamma^{jk} \gamma^{jm} n_{\text{mank}}, \quad \Lambda_{m}^{ij} = 2N^2 [K^{ik} \gamma^{jm} + K^{jm} \gamma^{ik}] n_{\text{mank}}.$$  

(3.7)

Using equations (3.6)–(3.7) as well as the evolution equations of the velocities (see equations (A.8)–(A.9)), the evolution for the vorticity of the electrons and of the ions can be written, respectively, as

$$\partial_t \omega_i^e + \left( NK - \frac{\partial_t N}{N} \right) \omega_i^e - G_i^k \omega_k^e - \mathcal{F}_i^e = - \frac{\epsilon n_e N^2}{\rho_e \sqrt{\gamma}} \left[ (\tilde{\varphi} \times \tilde{E})_i + [\tilde{\varphi} \times (\tilde{v}_e \times \tilde{B})]_i \right],$$

(3.8)

$$+N \Gamma_{\text{e}i}(\omega_i^e - \omega_i^e) + \frac{4}{3} \frac{\rho_e}{\rho_i} N \Gamma_{\text{e}i} (\omega_i^e - \omega_i^e).$$

(3.9)

Similarly, from equation (A.10) the evolution equation for the vorticity of the photons can be written as

$$\partial_t \omega_i^\gamma + \left[ 4 \frac{NK - \partial_t N}{N} \right] \omega_i^\gamma - G_i^k \omega_k^\gamma - \mathcal{F}_i^\gamma = N \Gamma_{\text{e}c}(\omega_i^\gamma - \omega_i^\gamma) + N \Gamma_{\text{e}i} (\omega_i^\gamma - \omega_i^\gamma).$$

(3.10)

The quantities $\mathcal{F}_i^e$, $\mathcal{F}_i^\gamma$ and $\mathcal{F}_i^\gamma$ appearing in equations (3.8), (3.9) and (3.10) are of the same order of the other terms appearing in the equations and they are defined as

$$\mathcal{F}_i^e = \partial_j \left( \Lambda_m^{ij} v_m^e \right) + \frac{4}{3} N \Gamma_{\text{e}c} \partial_j \left( \frac{\rho_e}{\rho_i} \right) \Lambda_m^{ij} (v_m^e - v_m^e),$$

(3.11)

$$+ \partial_j G_m^{ij} \Lambda_m^{ij} v_m^e - N \partial_j K \Lambda_m^{ij} v_m^e - \partial_j \left( \frac{\epsilon n_e N^2}{\rho_e \sqrt{\gamma}} \right) \Lambda_m^{ij} [E^m + (\tilde{v}_e \times \tilde{B})^m],$$

$$\mathcal{F}_i^\gamma = \partial_j \left( \Lambda_m^{ij} v_m^\gamma \right) + \frac{4}{3} N \Gamma_{\text{e}i} \partial_j \left( \frac{\rho_e}{\rho_i} \right) \Lambda_m^{ij} (v_m^\gamma - v_m^\gamma),$$

(3.12)

$$+ \partial_j G_m^{ij} \Lambda_m^{ij} v_m^\gamma - N \partial_j K \Lambda_m^{ij} v_m^\gamma + \partial_j \left( \frac{\epsilon n_e N^2}{\rho_e \sqrt{\gamma}} \right) \Lambda_m^{ij} [E^m + (\tilde{v}_e \times \tilde{B})^m].$$

12 Recall that $n_{\text{mank}} = \sqrt{\gamma} s_{\text{mank}}$, and that $\gamma^{jk} = \epsilon^{km} / \sqrt{\gamma}$.

13 We will focus, without loss of generality, on the situation where the lapse function is homogeneous, i.e. $N(\tilde{x}, \tau) = N(\tau)$; in this case, the already lengthy expressions will be more manageable since the spatial derivatives of the lapse function will vanish.
\( F^i_j = \partial_j (\bar{K}^{ij}_k v^k) + \Lambda^i_j v^j \partial_i G^k_k - \frac{4}{3} N \partial_j K \Lambda^i_j v^j - \frac{N^2}{4} \partial_j \left\{ \frac{\Lambda^{ij}_j}{\rho} \partial_j [\rho \gamma^m] \right\}. \)  

(3.13)

The generalized scalar and vector products appearing in equations (3.11), (3.12) and (3.13) are defined as

\[ \vec{F} \cdot \vec{G} = \gamma_{mn} F^m G^n, \quad (\vec{F} \times \vec{G})^k = \frac{\gamma_{mn}}{N} F^m G^n t^i e^k. \]  

(3.14)

and coincide with the ordinary scalar and vector products in the conformally flat limit introduced after equation (3.5). The velocity fields appearing in equations (3.8) and (3.9) are all subjected to the fully inhomogeneous form of the momentum constraint implying, from equation (A.6),

\[ \frac{1}{N} (\nabla_i K - \nabla_i K^k) = \ell^2_p (p + \rho) u^i u_i, \quad u^0 = \frac{1}{N} \sqrt{1 + u^2}, \]  

(3.15)

where \( u^2 = u^i u^i \) and \( u^0 \) can also be defined in terms of the total velocity field \( v^i \) which turns out to be the weighted sum of the velocity fields of the electrically charged and of the electrically neutral species, i.e.

\[ (p + \rho) v^i = \sum_a (p_a + \rho_a) v^i_a = \rho_v v^i_v + \frac{4}{3} \rho_v v^i_v + \frac{4}{3} \rho_v v^i_v + \rho_v v^i_v, \]  

(3.16)

where the contribution of the cold dark matter particles and the massless neutrinos has also been added. The explicit connection between \( u^i, u^i \) and \( v^i \) is given by

\[ u^0 = \frac{\cosh \gamma}{N}, \quad u^i = \frac{v^i}{N} \cosh \gamma, \quad \cosh \gamma = \frac{1}{\sqrt{1 - v^2/N^2}}, \]  

(3.17)

where \( v^2 = v^i v^i \). In terms of \( v^i \) and \( v^2 \), the momentum constraint of equation (3.15) can also be written as

\[ \ell^2_p (p + \rho) \frac{v^i}{N} = \left( 1 - \frac{v^2}{N^2} \right) \nabla_i (K_i - K^k_i). \]  

(3.18)

All the discussion of section 2 can be generalized to the fully inhomogeneous case and we will be particularly interested in the generalization of the conservation laws determining the angular momentum exchange between the various species. Consider then the situation where the electron–photon rate dominates against the Coulomb rate. In this case, the fully inhomogeneous form of the Ohm law reads

\[ -E^k - (\bar{v}_e \times \bar{B})^k + \frac{j^k}{\sigma} + \frac{4}{3} e \frac{\rho_v}{\rho_b} m_1 \gamma_{cy} \frac{\sqrt{\gamma}}{N} (\bar{v}_e v^k_e - \bar{v}_e v^k_e) = 0. \]  

(3.19)

By taking the generalized curl of equation (3.19) (see equation (3.5)), the following equation can be obtained:

\[ - \partial \times \bar{E} - \partial \times (\bar{v}_e \times \bar{B}) + \partial \times (\bar{I}/\sigma) + \frac{4}{3} e \frac{\rho_v}{\rho_b} m_1 \gamma_{cy} \frac{\sqrt{\gamma}}{N} (\bar{\omega}_e - \bar{\omega}_e) \]

\[ - \frac{4}{3} e \frac{m_1}{\sqrt{\gamma}} N^2 (\bar{v}_e v^k_e - \bar{v}_e v^k_e) \times \partial \left[ \gamma_{cy} \frac{\sqrt{\gamma}}{N} \frac{\rho_v}{\rho_b} \right] = 0, \]

(3.20)

where, consistently with equation (3.14), the last term on the left-hand side is defined in terms of the generalized vector product and it vanishes exactly in the conformally flat limit. By assuming, as physically plausible prior to decoupling, that the conductivity is homogeneous, equations (3.1) and (3.2) can be used inside equation (3.20) and the final equation will then be

\[ \partial v_e \bar{B} = \frac{1}{4 \pi \sigma} \bar{\omega} \times (\bar{v}_e \times \bar{B}) \]

\[ + \frac{4}{3} m_1 (\bar{v}_e - \bar{v}_e) \times \partial \left[ \gamma_{cy} \frac{\sqrt{\gamma}}{N} \frac{\rho_v}{\rho_b} \right]. \]

(3.21)
Equation (3.21) reduces, in the conformally flat limit, to equation (2.29). The same logic can be applied in all the other derivations and the obtained result expanded to first order in the spatial gradients with the result that the generalized system for the evolution of the vorticities reads

$$\partial_\tau \omega^k_i = \left( NK + 2 \frac{\partial_i N}{N} \right) \omega^k_i - \frac{e\tilde{n}_i}{\rho_b \sqrt{\gamma}} N^2 \partial_\tau B^k, \quad (3.22)$$

$$\partial_\tau B^k = -\frac{4}{3e} \Gamma_{\gamma} \frac{\rho_e}{\rho_b} m_1 \sqrt{\gamma} \left( \omega^k_\gamma - \omega^k_b \right), \quad (3.23)$$

$$\partial_\tau \omega^k_\gamma = \frac{2}{3} NK + 2 \frac{\partial_i N}{N} \omega^k_\gamma + N \Gamma_{\gamma} \left( \omega^k_\gamma - \omega^k_b \right). \quad (3.24)$$

Equations (3.22), (3.23) and (3.24) reduce, respectively, to equations (2.29), (2.30) and (2.31) in the conformally flat limit. Equations (3.22), (3.23) and (3.24) apply in the situation where the magnetic fields are initially zero and do not contribute to the extrinsic curvature so that in the conformally flat limit, with equation (2.25)–(2.29) and (2.30)–(2.31) do not have a source term. In principle, we could just determine the source terms depending on the gradients; in equation (3.26), $R_b(\vec{x}, \tau_1)$ is a constant in time (but not in space) and come from the inhomogeneous generalization of $R_b$:

$$R_b(\vec{x}, \tau) = \frac{3}{4} \frac{\rho_b(\vec{x}, \tau)}{\rho_\gamma(\vec{x}, \tau)} = R_b(\vec{x}, \tau_1) \gamma^{1/6}.$$  

The evolution of the vorticity of the baryons as well as the tight coupling between the baryons and the photons can be discussed in full analogy with the considerations already developed above in the case of the electron–photon coupling. The inhomogeneous generalization of the Ohm law when the Coulomb scattering dominates against both the electron–photon and the ion–photon coupling has been derived in [14] (see equation (3.34)). To leading order in the gradient expansion, the evolution of the baryon vorticity can be written as

$$\partial_\tau \omega^k_b = \left( NK + 2 \frac{\partial_i N}{N} \right) \omega^k_b + \epsilon' \left( \omega^k_\gamma - \omega^k_b \right), \quad (3.28)$$

$$\partial_\tau B^k = -\frac{m_b}{e} \frac{\epsilon'}{R_b} \sqrt{\gamma} \left( \omega^k_\gamma - \omega^k_b \right), \quad (3.29)$$

where $\epsilon' = N \Gamma_{\gamma}$ is the inhomogeneous generalization of the optical depth. By eliminating $\epsilon'$ between equations (3.28) and (3.29), the equation

$$\partial_\tau \left[ \frac{\sqrt{\gamma}}{N^2} \omega^k_b + \frac{e\tilde{n}_i}{\rho_b} B^k \right] = 0 \quad (3.30)$$

is readily obtained. Note that equation (3.30) coincides, up to spatial gradients and in the conformally flat limit, with equation (2.45).

To zeroth order in the gradient expansion, the conservation of the vorticity is exact and equations (3.25) and (3.26) do not have a source term. In principle, we could just determine
iteratively the solution order by order and then compute the vorticity. Instead of expanding the equations and then demanding the consistency of all the vorticities with the vorticity computed from the momentum constraint, it is more productive to expand in spatial gradients the momentum constraint itself and derive the total vorticity. Of course in this procedure, there is a loss of information but a gain in accuracy. The total vorticity is better determined but the vorticity of the individual species is not accessible. The magnetic field estimated in section 6 is the one obtained by attributing the total vorticity to the ions. In this sense, what has been estimated, as explained in sections 4 and 5, is the maximal magnetic field induced by the total vorticity.

4. Maximal vorticity induced by the geometry

In this paper, the expansion is organized not in terms of the relative magnitude of the gravitational and electromagnetic fluctuations but in terms of the number of gradients carried by each order of the expansion. From the momentum constraint (see equation (3.18)), the total velocity field can be written, formally, as

\[ v^i = -\frac{NS}{2S^2}[1 - \sqrt{1 + 4S^2}] \simeq \frac{NS}{2S^2} - O(\epsilon^3) + O(\epsilon^4), \tag{4.1} \]

where the orders of the expansion appearing in equation (4.1) are defined by the number of gradients. From equations (4.1) and (3.6)–(3.7), the total vorticity can be written as

\[ \omega^i_{\text{tot}} = \partial_j \left\{ N \Lambda^j_m S^m \left[ 1 - S^2 + O(\epsilon^3) \right] \right\}. \tag{4.3} \]

To implement the gradient expansion, let us parametrize the geometry as

\[ \gamma_{ij}(\vec{x}, \tau) = a^2(\tau) [\alpha_{ij}(\vec{x}) + \beta_{ij}(\vec{x}, \tau)], \quad \gamma_{ij}(\vec{x}, \tau) = 1/a^2(\tau) \left[ \alpha_{ij}(\vec{x}) - \beta_{ij}(\vec{x}, \tau) \right], \tag{4.4} \]

and keep the lapse function homogeneous, i.e. \( N(\tau) = a(\tau) \); \( \alpha_{ij}(\vec{x}) \) does not contain any spatial gradient while \( \beta_{ij}(\vec{x}, \tau) \) contains at least one spatial gradient. The extrinsic curvature becomes

\[ K^j_i = -\left( \frac{\mathcal{H}}{a} \delta^j_i + \frac{1}{2} \frac{\partial^j_{\tau}}{a} \right), \quad K^k_i = -\frac{1}{a^3} \left[ \mathcal{H}(\alpha^k_i - \beta^k_i) + \frac{1}{2} \partial^k_{\tau} \beta^i_k \right]. \tag{4.5} \]

Furthermore, we also have that the spatial Christoffel are

\[ \Gamma^k_{ia} = \partial_a \ln \sqrt{\gamma} = \frac{1}{2} \left( \frac{\partial_a \alpha}{\alpha} + \partial_i \beta \right), \tag{4.6} \]

\[ \Gamma^m_{ab} = \frac{1}{2} \left[ \alpha^{mn} \lambda_{nab} + \alpha^{mn} \Sigma_{nab} - \beta^{mn} \lambda_{nab} \right], \tag{4.7} \]

where \( \lambda_{nab} \) and \( \Sigma_{nab} \)

\[ \lambda_{nab} = - \partial_n \alpha_{ab} + \partial_b \alpha_{na} + \partial_a \alpha_{bn}, \tag{4.8} \]

\[ \Sigma_{nab} = - \partial_n \beta_{ab} + \partial_b \beta_{na} + \partial_a \beta_{bn}. \tag{4.9} \]

The relevant term appearing in the momentum constraint then becomes

\[ \nabla_k (K^{km} - K\gamma^{km}) = \nabla_k K^{km} + \frac{\alpha^{km} \partial_i \partial_j \beta}{2a^2}. \tag{4.10} \]
The previous expression can also be recast in a more handy form:
\[
\nabla_4 K^{km} = -\frac{1}{a^3} \left[ \frac{\mathcal{H}}{2\alpha} (\partial_\alpha \alpha) \alpha^{am} + \mathcal{H} \partial_\alpha \alpha^{km} + \mathcal{H} \frac{\alpha^{mi} \alpha^{ab}}{2} \lambda_{iab} \right] \\
+ \frac{1}{a^3} \left[ -\partial_\alpha \partial_\beta \beta^{km} - \frac{1}{2} \partial_\tau \beta^{ab} \alpha^{mi} \lambda_{iab} - \left( \frac{\partial_\alpha \alpha}{2\alpha} \right) \partial_\tau \beta^{am} \right] \\
- \mathcal{H} \partial_\alpha \beta \alpha^{km} + \mathcal{H} \left( \frac{\partial_\alpha \alpha}{\alpha} \right) \beta^{am} + \mathcal{H} \partial_\alpha \beta^{km} - \mathcal{H} \alpha^{mi} \alpha^{ab} \lambda_{iab} \\
+ \mathcal{H} \left( \alpha^{mi} \beta^{ab} + \alpha^{ab} \beta^{mi} \right) \lambda_{iab} \right].
\] (4.11)

Equation (4.10) can therefore be written as
\[
\nabla_4 (K^{km} - \nabla K^{km}) = -\frac{\mathcal{H}}{a^3} \left[ \frac{1}{2\alpha} (\partial_\alpha \alpha) \alpha^{am} + \partial_\alpha \alpha^{km} + \mathcal{H} \frac{\alpha^{mi} \alpha^{ab}}{2} \lambda_{iab} \right] \\
+ \frac{1}{a^3} \left[ -\partial_\alpha \partial_\beta \beta^{km} - \frac{1}{2} \partial_\tau \beta^{ab} \alpha^{mi} \lambda_{iab} - \left( \frac{\partial_\alpha \alpha}{2\alpha} \right) \partial_\tau \beta^{am} \right] \\
+ \mathcal{H} \left[ 2 \partial_\alpha \beta^{km} - \partial_\alpha \beta^{am} + \left( \frac{\partial_\alpha \alpha}{\alpha} \right) \beta^{am} - \alpha^{mi} \alpha^{ab} \lambda_{iab} \right] \\
+ \left( \alpha^{mi} \beta^{ab} + \alpha^{ab} \beta^{mi} \right) \lambda_{iab} \right].
\] (4.12)

The previous expression can also be recast in a more handy form:
\[
\nabla_4 (K^{km} - \nabla K^{km}) = -\frac{\mathcal{H}}{a^3} \left[ Z^m (\alpha) + \frac{1}{2a^3} \left[ T^m_2 (\alpha, \beta) - T^m_2 (\alpha, \beta) + \mathcal{H} T^m_3 (\alpha, \beta) \right] \right].
\] (4.13)

where the three functionals of \(\alpha_j (\vec{x})\) and \(\beta_j (\vec{x}, \tau)\) are defined as
\[
Z^m (\alpha) = \frac{1}{2a} \partial_\alpha \alpha^{am} + \partial_\alpha \alpha^{km} + \frac{\alpha^{mi} \alpha^{ab}}{2} \lambda_{iab},
\] (4.14)
\[
T^m_2 (\alpha, \beta) = \alpha^{am} \partial_\alpha \beta - \partial_\alpha \partial_\beta \beta^{am},
\] (4.15)
\[
T^m_3 (\alpha, \beta) = 2 \partial_\alpha \beta^{am} - (\partial_\alpha \beta) \alpha^{am} + \frac{\partial_\alpha \alpha}{\alpha} \beta^{am} + \lambda_{iab} (\alpha^{am} \beta^{ab} + \alpha^{ab} \beta^{am}) - \alpha^{mi} \alpha^{ab} \lambda_{iab},
\] (4.16)

With the result of equation (4.13), we can compute the first relevant part of the final expression, namely
\[
N^2 \gamma^{ij} \gamma^{in} \eta_{mn} \nabla_4 (K^{km} - \nabla K^{km}) = \sqrt{\alpha} \left[ \frac{1}{a} \left( 1 + \frac{\beta}{2} \right) \right] \left[ -\mathcal{H} \alpha^{kj} \alpha^{in} \nabla Z^m (\alpha) \epsilon_{kmn} \right. \\
+ \mathcal{H} \left( \alpha^{kj} \beta^{in} + \alpha^{in} \beta^{kj} \right) Z^m (\alpha) \epsilon_{kmn} + \frac{\alpha^{kj} \alpha^{in}}{2} \epsilon_{kmn} \left[ T^m_1 (\alpha, \beta) - T^m_2 (\alpha, \beta) \right] \\
+ \mathcal{H} T^m_3 (\alpha, \beta) \right].
\] (4.18)

Recalling that furthermore\(^{14}\)
\[
\tilde{e}^2_1 (p + \rho) a_1^2 = \frac{3H_1^2 (1 + w)}{a^{(\alpha^{(w+1)/2})} (1 + \beta/2)^{\alpha^{(w+1)/2}}} \left( \frac{a_1}{a} \right)^{3w+1}, \quad \tilde{e}^2_1 a_1^2 = 3H_1^2.
\] (4.19)

\(^{14}\) We will assume that \(w\), the dominant barotropic index of the fluid sources, is constant.
Putting all the various parts of the calculation together we have that, from equation (4.3),
\[ \omega^i = \partial_j A^j, \quad A^j = \frac{N^2 \gamma^{\ell j}}{\ell \hat{p}} (p + \rho) \nabla_n \left( K^{mn} - \gamma^{am} K \right); \] (4.20)
then, the quantity \( A^j \) becomes
\[ A^j(\alpha, \beta) = \frac{a^{(w+2) / 2}}{3H_2^2 (w + 1)(w + 2) \alpha} \left\{ -\mathcal{H}_{\alpha} \alpha^m \gamma^{m(\alpha)} \epsilon_{kmn} \right. \]
\[ \left. + \mathcal{H} \left[ \alpha \beta^m + \alpha^m \beta^j \right] Z^m(\alpha) \epsilon_{kmn} \right. \]
\[ \left. + \frac{\alpha \epsilon_{kmn} \left[ T^m_1(\alpha, \beta) - T^m_2(\alpha, \beta) + \mathcal{H} T^m(\alpha, \beta) \right] \right. \]
\[ \left. - \frac{\mathcal{H}}{2} (w + 2) \beta \alpha^{kmn} \gamma^{m(\alpha)} \epsilon_{kmn} \right\}. \] (4.21)

The first line on the right-hand side of equation (4.21) does not contain any spatial gradient and it is therefore \( O(\alpha) \). The remaining part of the expression on the right-hand side of the relation reported in equation (4.21) is instead \( O(\beta) \). Sticking to the situation treated in this paper, the explicit form of \( \beta_{ij}(\vec{x}, \tau) \) can be determined in terms of \( \alpha_{ij}(\vec{x}) \) by solving the remaining Einstein equations written in terms of the ADM decomposition [15, 16]. For this purpose, equations (A.5) and (A.7) can be written, respectively, as
\[ \partial_\tau K - N \tau K^2 = \frac{a \epsilon^2}{2} (p + \rho), \] (4.22)
\[ \partial_\tau K^i_j - N K^i_j - N r^i_j = \frac{a \epsilon^2}{2} (p - \rho) \delta^i_j. \] (4.23)
Using equations (4.4) and (4.5) into equations (4.22), the following pair of conditions are obtained:
\[ \partial_\tau \frac{\partial \beta}{2a} + \frac{\mathcal{H}}{\alpha} \partial_\tau \beta = \frac{a \epsilon^2}{2} (3p^{(1)} + \rho^{(1)}), \] (4.24)
\[ \partial_\tau \mathcal{H} = -\frac{a^2 \epsilon^2}{2} (p^{(0)} + 3p^{(0)}). \] (4.25)

To obtain equations (4.24) and (4.25), the total pressure and the total energy density have been separated as
\[ p(\vec{x}, \tau) = p^{(0)}(\tau) + p^{(1)}(\vec{x}, \tau), \quad \rho(\vec{x}, \tau) = \rho^{(0)}(\tau) + \rho^{(1)}(\vec{x}, \tau), \] (4.26)
where \( p^{(1)}(\vec{x}, \tau) \) and \( \rho^{(1)}(\vec{x}, \tau) \) vanish in the conformally flat limit. Using equations (4.4) and (4.5) into equations (4.23), two further equations are obtained and they are
\[ \partial_\tau \left( \frac{\partial \beta}{2a} \right) + \mathcal{H} \frac{\partial \beta}{2a} \beta^i_j + \frac{3 \mathcal{H}}{2a} \partial_\tau \beta^i_j + a r^i_j = -\frac{a \epsilon^2}{2} (p^{(1)} - \rho^{(1)}) \delta^i_j, \] (4.27)
\[ \partial_\tau \mathcal{H} + 2 \mathcal{H} \epsilon^2 = -\frac{a \epsilon^2}{2} (p^{(0)} - \rho^{(0)}). \] (4.28)
Solving equations (4.25) and (4.28) under the hypothesis of the constant barotropic index (already assumed in equation (4.19)), \( p^{(1)} \) and \( \rho^{(1)} \) can be eliminated between equations (4.24) and (4.27) and it turns out that \( \beta_{ij}(\vec{x}, \tau) \) obeys the following evolution equation:
\[ \partial_\tau^2 \beta^i_j + 2 \mathcal{H} \partial_\tau \beta^i_j + \epsilon^2 \left( \frac{1 - w}{1 + 3w} \partial^2 \beta^i_j + 2 \frac{1 + w}{1 + 3w} \mathcal{H} \partial_\tau \beta^i_j \right) + 2a^2 r^i_j = 0. \] (4.29)
By solving equation (4.29), the explicit form of \( \beta_{ij} \) can be written in a separable form as

\[
\beta_{ij}^\alpha(\vec{x}, \tau) = g(\tau)\mu_{ij}^\alpha(\vec{x})
\]

where

\[
g(\tau) = a^{2w+1},
\]

\[
\mu_{ij}^\alpha(\vec{x}) = -\frac{4}{H_0^2(3w + 5)(3w + 1)}\left[ P_i^\alpha(\vec{x}) + \frac{3w^2 - 6w - 5}{4(9w + 5)} P_j^\alpha(\vec{x})\delta_{ij}^\alpha \right].
\]

Note that \( P_i^\alpha(\vec{x}) = \rho_i^\alpha(\vec{x}, \tau)a^2(\tau) \) accounts for the intrinsic curvature computed from \( \alpha_{ij}(\vec{x}) \). In equations (4.22) and (4.23), the contribution of the velocity fields and of the magnetic fields has been neglected because they are subleading to \( \mathcal{O}(\beta) \). In the following two sections, we will therefore present the full estimate of the vorticity to first order in the gradient expansion. If needed the first order result, together with equations (4.30) and (4.31), can be used to estimate the vorticity to higher order.

Equations (4.30) and (4.31) constitute the basis for the next-to-leading calculation in the gradient expansion and must be regarded as the completion of equations (4.20) and (4.21). To leading order in the gradient expansion, \( A^{ij} = A^{ij}(\alpha) \). This will be the basis of the analysis of section 5. To next-to-leading order, \( A^{ij} = A^{ij}(\alpha, \beta) \). As will be clear from the results of section 5, the leading order result will depend on the product of three tensor power spectra, i.e. as we say for short three tensor insertions. Equation (4.30) and (4.31) show that the next-to-leading result is proportional to \( \beta_{ij} \) which has a computable time-dependent part and an inhomogeneous part, i.e. \( \mu_{ij} \). But since \( \mu_{ij} \) contains \( \rho_i^\alpha \) (i.e. the spatial curvature computed from the preceding order in the expansion), it will be proportional, again, to three tensor insertions and it will contain, by construction, two supplementary spatial gradients. The general structure of the next-to-leading correction to the vorticity will be briefly outlined at the end of section 5.

### 5. Vorticity to first order in the gradient expansion

The simplest parametrization of \( \alpha_{ij}(\vec{x}) \) which does not contain spatial gradients can be written as

\[
\alpha_{ij}(\vec{x}) = e^{-2\Psi(\vec{x})}\delta_{ij}, \quad \alpha = \det \alpha_{ij} = e^{-6\Psi(\vec{x})}.
\]

In this case, it is easy to show that \( Z^m(\alpha) = 0 \) and therefore the first order in the gradient expansion vanishes identically. In the \( \Lambda \)CDM scenario, the scalar mode appearing in equation (5.1) leads to \( |\Psi(\vec{x})| \ll 1 \) and therefore, in practice, \( \alpha_{ij}(\vec{x}) \) is accurately estimated by \( \delta_{ij} - 2\Psi(\vec{x})\delta_{ij} \). To have a \( Z^m(\alpha) \neq 0 \), the contribution of the tensor modes must be included and \( \alpha_{ij}(\vec{x}) \) will then be given by

\[
\alpha_{ij}(\vec{x}) = [\delta_{ij} + h_{ij}(\vec{x})], \quad \alpha^{ij}(\vec{x}) = \left[ \delta^{ij} - h^{ij} + h^k_i h^k_j \right], \quad \sqrt{\alpha} = \left[ 1 - \frac{1}{2} h^k_i h^k_j \right],
\]

where \( h_{ij} \) is divergenceless and traceless, i.e. \( \delta h^{ij} = h^i_j = 0 \). It must be borne in mind that the scalar and the tensor modes, in the \( \Lambda \)CDM scenario and in its tensor extension, are defined in terms of the conventional perturbative expansion. As a consequence of the latter statement, the informations on the spatial inhomogeneities of the model are not specified by assigning the analog \( \alpha_{ij}(\vec{x}) \) (or \( \gamma_{ij}(\vec{x}, \tau) \) to a given order in the spatial gradients). In contrast, as is more natural, the scalar and tensor modes of the geometry are specified by assigning the corresponding power spectra at a given pivot scale. To evaluate the appropriate correlators defining the vorticity, we will first need to obtain the fluctuations in real space (as opposed to Fourier space). Therefore, as we will show in the present and in the following section, the idea will be first to compute the fluctuations in real space and then to use the obtained result for the determination of the correlators defining the vorticity. This procedure will circumvent the
calculation of complicated convolutions and will also be perfectly suitable for the applications described in section 6. Then using equation (4.14), we have that
\[ Z^n(\alpha) = \partial_\alpha \epsilon_{nm} + \epsilon_{mn} \partial_\alpha \epsilon_{mn} = H^{\alpha \beta} \partial_\beta h_{\alpha \beta} + h^{\alpha \beta} \partial_\beta h_{\alpha \beta}. \]  
(5.3)

From equation (4.21), the tensor \( A^{i j}(\alpha, \beta) \) can be computed to lowest order (i.e. by setting \( \beta = 0 \)) and the result will therefore be written, using equation (5.3), as
\[ A^{i j}(\alpha) = \left( \frac{a}{\alpha_h} \right)^{3u+1} \epsilon^{\alpha \beta \gamma \delta} \left[ H^{\gamma \delta} \partial_\gamma h_{\alpha \beta} + h^{\gamma \delta} \partial_\gamma h_{\alpha \beta} \right] + O(\epsilon^2). \]
(5.4)

Finally, the total vorticity can be derived directly from equation (4.20)
\[ \omega_i^{\text{tot}} = -\mathcal{L}(\tau, w) \epsilon^{\alpha \beta \gamma \delta} \left[ H^{\gamma \delta} \partial_\gamma h_{\alpha \beta} + h^{\gamma \delta} \partial_\gamma h_{\alpha \beta} \right] + O(\epsilon^2), \]
(5.5)

Equations (5.4) and (5.5) are the main result of section 4 and give us an estimate of the total vorticity. Then, equation (3.25) (see also (3.26)) is used to connect the vorticity of the ions to the magnetic field to lowest order (i.e. in the absence of the gradients).

To give an explicit estimate of the primordial vorticity, the relevant cosmological parameters will be taken to be the ones determined on the basis of the WMAP 7 yr data alone [26, 27]. In the \( \Lambda \)CDM paradigm, the sole source of curvature inhomogeneities is represented by the standard adiabatic mode whose associated power spectrum is assigned at the comoving pivot scale \( k_0 = 0.002 \text{Mpc}^{-1} \) with characteristic amplitude \( A_R \)
\[ \langle R(\vec{k}, \tau) R(\vec{p}, \tau) \rangle = \frac{2 \pi^2}{k^3} P_R(k) \delta^{(3)}(\vec{k} + \vec{p}), \quad P_R(k) = A_R \left( \frac{k}{k_p} \right)^{n_s - 1}, \]
(5.6)

where \( n_s \) denotes the spectral index associated with the fluctuations of the spatial curvature. According to the WMAP 7 yr data alone analyzed in the light of the \( \Lambda \)CDM paradigm and without tensors modes [26, 27], the determinations of \( A_R \) and of \( n_s \), lead, respectively, to \( A_R = (2.43 \pm 0.11) \times 10^{-9} \) and to \( n_s = 0.963 \pm 0.014 \). The standard \( \Lambda \)CDM scenario, sometimes dubbed vanilla \( \Lambda \)CDM, is defined by six pivotal parameters whose specific values are, in the absence of tensor modes\(^{15}\)
\[ (\Omega_b, \Omega_c, \Omega_{de}, h_0, n_s, \epsilon_0) \equiv (0.0449, 0.222, 0.734, 0.710, 0.963, 0.088). \]
(5.7)

To estimate the correlation functions associated with equations (5.4) and (5.5), it is mandatory to know in detail the numerical value of the correlation function of the tensor modes of the geometry which have not been detected so far but whose specific upper limits will determine the maximal magnetic field obtainable from the vorticity of the geometry. The tensor modes of the geometry are described in terms of a rotationally and parity invariant two-point function
\[ \langle h_{ij}(\vec{x}, \tau) h_{ij}(\vec{y}, \tau) \rangle = \int \frac{dk}{k} P_T(k, \tau) \sin kr \frac{kr}{k^2}, \]
(5.8)

where the tensor power spectrum at the generic time \( \tau \) is given by the product of the appropriate transfer function multiplied by the primordial spectrum
\[ P_T(k, \tau) = M(k, k_{eq}, \tau) \overline{P}_T(k), \quad \overline{P}_T(k) = A_T \left( \frac{k}{k_p} \right)^{n_T}; \]
(5.9)

\(^{15}\)Following the standard notations (slightly modified to avoid possible clashes with previously defined variables), \( \Omega_b, \Omega_c, \Omega_{de} \) denote, respectively, the present critical fractions of the baryons, of the dark matter, of the dark energy; \( h_0 \) is the Hubble constant in units of 100 km(s Mpc\(^{-1}\))\(^{-1} \), \( n_s \) is the scalar spectral index while \( \epsilon_0 \) denotes the optical depth at recombination.
note that $A_T$ is the amplitude of the tensor power spectrum and $n_T$ is the tensor spectral index. The transfer function $\mathcal{M}(k, k_{eq}, \tau)$ can be computed under several approximations depending upon the required accuracy. The transfer function for the amplitude of the tensor modes can be numerically computed by solving the evolution of the tensor fluctuations across the matter-radiation equality and the result is [40, 41]

$$\mathcal{M}(k, k_{eq}, \tau) = \frac{9}{4} j_1^2(k_{eq}) \left[ 1 + c_1 \left( \frac{k}{k_{eq}} \right) + c_2 \left( \frac{k}{k_{eq}} \right)^2 \right] \left[ 1 + c_1 \left( \frac{k}{k_{eq}} \right) + c_2 \left( \frac{k}{k_{eq}} \right)^2 \right].$$

(5.10)

where\footnote{The analysis of [42] gave $c_1 = 1.34$ and $c_2 = 2.50$ which is fully compatible with the results of [40, 41].} according to [40, 41], $c_1 = 1.26$ and $c_2 = 2.68$. In equation (5.10) $j_1(y) = (\sin y/y^2 - \cos y/y)$ is the spherical Bessel function of first kind which is related to the approximate solution of the evolution equations for the tensor mode functions whenever the solutions are computed deep in the matter-dominated phase (i.e. $a(\tau) \approx \tau^3$). Instead of working directly with $A_T$, it is often preferred to introduce the quantity customarily called $r_T$ denoting the ratio between the tensor and the scalar amplitude at the pivot scale $k_p$:

$$r_T = \frac{A_T}{A_R} = \frac{\overline{P}_T(k_p)}{\overline{P}_R(k_p)}.$$

(5.11)

In principle, $n_T$ can be taken to be independent of $r_T$ and this possibility will also be contemplated in the present discussion. At the same time, if the scalar and the tensor modes are both of inflationary origin, $n_T$ is related to $r_T$ and to the slow-roll parameter $\epsilon$ which measure the rate of decrease of the Hubble parameter during the conventional inflationary stage of expansion:

$$n_T = -\frac{r_T}{8} = -2\epsilon, \quad \epsilon = -\frac{\dot{H}}{H^2};$$

(5.12)

the overdot denotes the usual derivative with respect to the cosmic time coordinate; in equation (5.12), the spectral index is frequency independent but there exist situations where more general possibilities can be contemplated such as, for instance,

$$n_T = -2\epsilon + \frac{\alpha_T}{2} \ln (k/k_p), \quad \alpha_T = \frac{r_T}{8} \left( n_s - 1 + \frac{r_T}{8} \right).$$

(5.13)

If $\alpha_T = 0$ the tensor spectral index $n_T$ does not depend upon the frequency and this is the case which is, somehow, endorsed when introducing gravitational waves in the minimal tensor extension of the $\Lambda$CDM. If a tensor component is allowed in the analysis of the WMAP 7 yr data alone, the relevant cosmological parameters are determined to be

$$\Omega_b, \Omega_c, \Omega_{de}, h_0, n_s, \epsilon_{eq}) = (0.0430, 0.200, 0.757, 0.735, 0.982, 0.091).$$

(5.14)

In the case of equation (5.7), the amplitude of the scalar modes is $A_R = (2.43 \pm 0.11) \times 10^{-9}$ while in the case of equation (5.14) the corresponding values of $A_R$ and of $r_T$ are given by

$$A_R = (2.28 \pm 0.15) \times 10^{-9}, \quad r_T < 0.36,$$

(5.15)

to 95% confidence level. To avoid confusion, it is appropriate to spend a word of care on the figures implied by equations (5.14) and (5.15) which have been used in the numeric analysis just for the sake of accuracy. The qualitative features of the effects discussed here do not change if, for instance, one would endorse the parameters drawn from the comparison of the minimal tensor extension of the $\Lambda$CDM with the WMAP 5 yr data release [44, 45], implying,
for instance, $A_\Omega = 2.1 \pm 2.2 \times 10^{-9}$, $n_e = 0.984$ and $r_T < 0.65$ (95% confidence level). Similar orders of magnitude can be also obtained from even older releases [46, 47].

It is now appropriate to complete the comments on the range of validity of the gradient expansion in the concrete case of the problem at hand. In the same way as it is difficult to discuss in general and abstract terms the range of validity of the standard perturbative expansion, it is equally difficult to know when and where the gradient expansion breaks down. The rationale for the latter statement is that the role of the gradients becomes more or less pronounced depending on the barotropic index $w$, as already stressed at the end of section 4. It is instead easier to focus on the specific case addressed in our analysis. For immediate convenience, let us parametrize the correction to the leading order result of equation (5.4); using the results of equations (4.20)–(4.21) and of equations (4.30) and (4.31) to next-to-leading order that $A^j$ will depend both on $\alpha$ and $\beta$ (see also the discussion at the end of section 4), i.e.

$$\mathcal{K}^j(\alpha, \beta) \simeq A^j(\alpha) \left[ 1 + \left( \frac{a}{a_1} \right)^{3w+1} \frac{1}{\mathcal{R}_1} Q^j(\alpha, \beta) \right],$$  \quad (5.16)

where we have denoted with $\mathcal{K}^j$ the leading order result.

The term labeled by $Q^j(\alpha, \beta)$ will be made more explicit by considering the results of equation (4.21). While an explicit calculation of the whole next-to-leading contribution is beyond the scope of the present discussion, it is useful to remark that the order of magnitude of $Q^j$ can be easily obtained from the results already reported in section 4. In short, the counting goes as follows. The term $Q^j$ contains, at least, one $\mu^j$ (see equations (4.20)–(4.21)). One $\mu^j$ carries three tensor insertions and two spatial gradients; at the end of section 6, we will deal with a more explicit estimate of the correction of equation (5.16). However, already at an intuitive level equations (4.21) and (5.16) show that between equality and decoupling the correction will certainly remain smaller than 1 because of the presence of the tensor insertions contained in $\alpha_{ij}$ whose second spatial derivatives determine $\beta_{ij}$, as established in equations (4.30) and (4.31). Note that the prefactor in equation (5.16) comes from the $g(t)$ of equations (4.30) and (4.31). While it is clear that specific numerical factors can appear in the second term in the squared bracket of equation (5.16), we are here interested in the parametric dependence on the scale factor and on the spatial gradients.

6. Magnetic field induced by the total vorticity

The total vorticity derived in the previous sections is larger than the vorticity of the ions. Therefore, the total magnetic field derived on the basis of $\omega_{\text{tot}}$ is larger than the one derived on the basis of the ion contribution. Of course this statement holds in an averaged sense since what matters is not the vorticity itself but rather its two-point function which will be explicitly computed in the present section. Using equation (5.5), the maximal obtainable magnetic field will be the one given by equations (3.25)–(3.26) or (3.30) where the total vorticity induced by the geometry is given by equation (5.5):

$$B_{\text{max}}^j(\tilde{x}, \tau) = -\frac{\rho_i \sqrt{g}}{e N^2 n_i} \alpha_{\text{tot}}(\tilde{x}, \tau),$$  \quad (6.1)

which can also be written, by explicitly keeping track of the number of gradients, as

$$B_{\text{max}}^j(\tilde{x}, \tau) = \left\{ \xi(\tau, w) e^{m_j} \partial_t \left[ n h_m h_0^2 \partial_t \partial_m + n h_m h_0^2 \partial_m n_0^2 \right] + \mathcal{O}(\epsilon^3) \right\} \alpha(\tau) [1 + \mathcal{O}(\epsilon^2)].$$  \quad (6.2)

The prefactor appearing in equation (6.1) has been estimated, in equation (6.2), by recalling that to lowest order in the gradient expansion

$$\partial_t \rho_i = NK \rho_i, \quad \partial_t n_i = NK n_i,$$  \quad (6.3)
implying that \( \rho_i \) and \( \bar{n}_i \) scale in the same way with \( \sqrt{r} \) since \( NK = -\bar{a}_r \ln \sqrt{r} \). But then, from equations (4.30), (4.31) and (5.2):

\[
\frac{\rho_i \sqrt{r}}{N^2 \bar{n}_i} = a(\tau)[1 + \mathcal{O}(\alpha^2)], \tag{6.4}
\]

where the first correction is \( \mathcal{O}(\alpha^2) \) and depends on \( \beta \) (see, e.g., equations (4.30) and (4.31)) but it will be immaterial for the present ends. From now on, the subscripts will be dropped but it will always be understood that we are referring here to the total vorticity and to the maximally achievable magnetic field. As a consequence of equation (6.1), the correlation function of the magnetic field can be related to the correlation function of the vorticity. To estimate the correlation of the vorticity and to obtain an explicit expression, the key point is to reduce the six-point function of the tensor modes to the product of two-point functions. For this purpose, it is not sufficient to consider the trace of the two-point function introduced in equation (5.8) but it is rather necessary to proceed with the full tensorial structure of the correlator whose general parity and rotationally invariant form will be denoted as

\[
G_{ijmn}(r) = \langle h_{ij}(\vec{x}, \tau) h_{mn}(\vec{y}, \tau) \rangle, \tag{6.5}
\]

where \( G_{ijmn}(r) \) is the only function of \( r = |\vec{r}| \) where \( \vec{r} = \vec{x} - \vec{y} \). Since both \( h_{ij}(\vec{x}, \tau) \) and \( h_{mn}(\vec{y}, \tau) \) are transverse and traceless, \( G_{ijmn}(r) \) will have to share the same properties. In particular, \( G_{ijmn}(r) \) must be symmetric for \( i \to j, m \to n, (ij) \to (mn) \) and satisfy the following properties:

\[
\frac{\partial}{\partial r'} G_{ijmn} = 0, \quad G_{ijmn} = G_{jimn} = 0 \tag{6.6}
\]

\[
\text{Tr}[G_{ijmn}] = G_{ijij} = \int \frac{dk}{k} P_r(k) \sin kr \frac{kr}{k}. \tag{6.7}
\]

The properties of equations (6.6) and (6.7) are a reflection of the divergenceless and traceless nature of \( h_{ij}(\vec{x}, \tau) \) while the requirement on the trace follows from the consistency with equation (5.8). The general form of \( G_{ijmn} \) can therefore be written as

\[
G_{ijmn}(r) = (\delta_{im} \delta_{jn} + \delta_{in} \delta_{jm}) G_1(r) + (\delta_{ij} \delta_{mn} G_2(r) + (\delta_{ij} r_m r_n + \delta_{mn} r_i r_j) G_3(r)
\]

\[
+ (\delta_{jm} r_m r_n + \delta_{jm} r_m r_n + \delta_{in} r_m r_n) G_4(r) + r_i r_j r_m r_n G_5(r), \tag{6.8}
\]

where the various independent functions appearing in equation (6.8) are determined in appendix B. The methods used to analyze the real-space correlators are the ones exploited in usual applications of statistical fluid mechanics [48, 49]. To evaluate, equation (6.2) can then proceed as follows. Using equation (5.5), the explicit form of the correlator of the vorticity becomes

\[
\langle \omega^i(\vec{x}, \tau) \omega^j(\vec{y}, \tau) \rangle = L^2(\tau, w) \epsilon^{lnmi} \epsilon^{km'i'} \frac{\partial^2}{\partial y^l \partial y'^{l'}} \frac{\partial^2}{\partial x^m \partial x'^{m'}} T_{mb'm'}, \tag{6.9}
\]

By definition \( \langle \omega^2(r) \rangle = \langle \omega^i(\vec{x}, \tau) \omega^i(\vec{y}, \tau) \rangle \) and by recalling the notations of appendix B, we will have that\(^\text{17}\)

\[
\langle \omega^2(r) \rangle = L^2(\tau, w) \epsilon^{lnmi} \epsilon^{km'i'} \frac{\partial^2}{\partial r^l \partial r'^{l'}} \frac{\partial^2}{\partial r^m \partial r'^{m'}} T_{mb'mb}, \tag{6.10}
\]

\(\text{17 Recall that } \vec{r} = \vec{x} - \vec{y}, \text{ i.e. } r = |\vec{r}| = |\vec{x} - \vec{y}|.\)
where the quantity $\mathcal{T}_{m'b'\text{mb}}$ is a function of $r$; the explicit form of $\mathcal{T}_{m'b'\text{mb}}$ is given in appendix B in terms of the two-point functions $G_{ijmm}$. Furthermore, $\mathcal{T}_{m'b'\text{mb}}$ can be written, in general terms, as

$$
\mathcal{T}_{m'b'\text{mb}} = T_1(r) (\delta_{m'b'}\delta_{\text{mb}} + \delta_{mm'}\delta_{bb'}) + T_2(r) \delta_{m'b'} \delta_{\text{mb}} + T_3(r) (\delta_{m'b'} \rho_mF_b + \delta_{mb'} \rho_bF_m) + T_4(r) (\delta_{m'b'} \rho_mF_b' + \delta_{mb'} \rho_bF_m')
$$

$$
+ \delta_{mm'} \rho_mF_b' + \delta_{bb'} \rho_bF_m') + r_mF_b'r_mF_bT_6(r).
$$

(6.11)

By using the results of appendix B, the explicit values of the five $T_i(r)$ can be expressed in terms of the two-point functions $G_{ijmm}$ (see equation (B.22) and (B.23)–(B.27)). There are two physically complementary regimes where the primordial vorticity and hence the magnetic field can be evaluated. Comoving lengths $r_G$ defined between 1 and 100 Mpc are smaller than the Hubble radius at equality since

$$
r_{\text{eq}} = \frac{2(\sqrt{2} - 1) \sqrt{\Omega_{\text{R0}}}}{H_0} = 119.397 \left(\frac{h_0^2 \Omega_{\text{M0}}}{0.134}\right)^{-1} \left(\frac{h_0^2 \Omega_{\text{R0}}}{4.15 \times 10^{-5}}\right)^{1/2} \text{Mpc},
$$

(6.12)

where $H_0$ is the present value of the Hubble rate, $\Omega_{\text{M0}}$ is the present value of the critical fraction in matter and $\Omega_{\text{R0}}$ is the present value of the critical fraction in radiation. The pivot length $r_p = 500$ Mpc at which the tensor amplitudes are assigned is such that $r_G < r_{\text{eq}} < r_p$. Therefore, after matter-radiation equality and, in particular, at photon decoupling, the correlation function of the magnetic field can be estimated as

$$
\langle B^2(r) \rangle = 6.348 \times 10^{-7} \left(\frac{r}{0.32}\right)^3 \left(\frac{A_T}{2.43 \times 10^{-9}}\right)^3 \left(\frac{z_{\text{dec}} + 1}{1089.2}\right)^2 \times \left(\frac{h_0^2 \Omega_{\text{M0}}}{0.134}\right)^6 \left(\frac{h_0^2 \Omega_{\text{R0}}}{4.15 \times 10^{-5}}\right)^{-6} C(n_T, r) G^2,
$$

(6.13)

in units of $G^2 \equiv \text{Gauss}^2$ and where the constants $c(n_T)$ and $d(n_T)$ are given by

$$
c(n_T) = -2(n_T - 1)(n_T - 3)[2n_T(n_T - 6) + 19] \cos^3 \left(\frac{n_T \pi}{2}\right) \Gamma^3(n_T - 5),
$$

$$
d(n_T) = -36 + 14n_T(n_T - 4) \frac{4n_T(n_T^2 - 6n_T + 8)^2}{5n_T^2(n_T^2 - 4n_T + 8)}. \quad (6.14)
$$

The typical values of $n_T$ are negative and $\mathcal{O}(10^{-2})$. Indeed, assume, consistently with equation (5.15), that $r_T \sim 0.32$. Then, according to equation (5.12), $n_T \sim -0.04$ and $\epsilon \sim 0.02$. Concerning the results of equations (6.13) and (6.14), few comments are in order:

- the prefactor $\mathcal{L}(r, w)$ is estimated in the hypothesis $w = 0$, $a_1 = a_{\text{eq}}$ and $\mathcal{H}_1 = \mathcal{H}_{\text{eq}}$ since we ought to estimate the field prior to photon decoupling;
- recalling that $\mathcal{H} = aH$ the value of the Hubble rate at the equality time can be estimated as

$$
H_{\text{eq}} = \sqrt{2 \Omega_{\text{M0}} H_0 \left(\frac{a_0}{a_{\text{eq}}}\right)^{3/2}} = 1.65 \times 10^{-56} \left(\frac{h_0^2 \Omega_{\text{M0}}}{0.134}\right)^2 M_P;
$$

(6.15)

- the result of equation (6.13) holds for comoving scales $r < r_p = 500$ Mpc (which are the ones relevant for the gravitational collapse of the protogalaxy) and it is not sensitive to the variation of $r$ provided $n_T$ is nearly scale-invariant;

18 The quantity $r_T$ (denoting, in section 5, the tensor to scalar ratio) must not be confused with $r_G$ and $r_{\text{eq}}$ which denote specific values of the radial coordinate.

19 Recall that because of relation (5.12), $n_T < 0$ and $\epsilon > 0$. 

21
• if \( r_T \simeq 0.32 \), then \( N_{\text{rot}} \simeq -0.04 \); from equation (6.14), \( C(r_G, -0.04) \simeq 0.07 \) while for \( r = 100 r_G \), we have that \( C(100 r_G, -0.04) \simeq -0.01 \).

By thus approximating \( C(n_T, r) \simeq C(1) \) in the range \( r = 1–100 \) Mpc and for \( 0.2 < r_T < 0.3 \), we get the following value for \( B_{\text{max}} = \sqrt{|B^2(r)|} \):

\[
B_{\text{max}} = 2.519 \times 10^{-38} \left( \frac{r_T}{0.32} \right)^{3/2} \left( \frac{A_R}{2.43 \times 10^{-9}} \right)^{3/2} \left( \frac{z_{\text{dec}} + 1}{1089.2} \right) \times \left( \frac{h_G^2 \Omega_M}{0.134} \right)^3 \left( \frac{r_0 M_0}{4.15 \times 10^{-5}} \right)^{-3} \text{G.} \tag{6.16}
\]

The result of equation (6.16) does not seem to be even remotely relevant for galactic magnetogenesis or for cluster magnetogenesis. In spite of the intricacy and of the ramification of the galactic dynamo hypothesis, it is useful to compare equation (6.16) with the minimal requirements stemming from what we would call optimal or ideal dynamo, namely a process where the kinetic energy of the protogalaxy is converted into magnetic energy with maximal efficiency. Let us denote with \( N_{\text{rot}} \) the number of (effective) rotations performed by the galaxy since gravitational collapse and with \( \rho_a \) and \( \rho_b \) the matter density after and before gravitational collapse.

The typical rotation period of a spiral galaxy is of the order of \( 3 \times 10^8 \) years which should be compared with \( 10^{10} \) years, i.e. the approximate age of the galaxy. The maximal number of rotations performed by the galaxy since its origin is then of the order of \( N_{\text{rot}} \sim 30 \). Under the hypothesis that the kinetic energy of the plasma is transferred to the magnetic energy with maximal efficiency, the protogalactic field will be amplified by one efold during each rotation. The effective number of efolds is however always smaller than 30 for various reasons. Typically it can happen that the dynamo quenches prematurely because some of the higher wavenumbers of the magnetic field become critical (i.e. comparable with the kinetic energy of the plasma) before the smaller ones. Other sources of quenching have been recently discussed in the literature (see, for an introduction to this topic, section 4.2 of [50] and references therein). There is also another source of amplification of the primordial magnetic field and it has to do with compressional amplification. At the time of the gravitational collapse of the protogalaxy the conductivity of the plasma was sufficiently high to justify the neglect of nonlinear corrections in the equations expressing the conservation of the magnetic flux and of the magnetic helicity. The conservation of the magnetic flux implies that during the gravitational collapse, the magnetic field should undergo compressional amplification, i.e. the same kind of mechanism which is believed to be the source of the large magnetic fields of the pulsars. Taking into account the two previous observations, the estimate of equation (6.16) must be compared with the bound

\[
B_{\text{bound}} \simeq 3 \times 10^3 e^{-N_{\text{rot}}} \left( \frac{\rho_b}{\rho_a} \right)^{2/3} \text{nG} \tag{6.17}
\]

in nG units. Even assuming \( N_{\text{rot}} = 30, \rho_a \simeq 10^{-24} \text{g cm}^{-3} \) and \( \rho_b \simeq 10^{-29} \text{g cm}^{-3} \), the minimal value of \( B_{\text{bound}} \) is \( \mathcal{O}(10^{-25}) \text{G} \). Clearly, by comparing equation (6.16) with equation (6.17), \( B_{\text{max}} \ll B_{\text{bound}} \).

Going then to cluster magnetogenesis, the typical scale of the gravitational collapse of a cluster is larger (roughly by one order of magnitude) than the scale of gravitational collapse of the protogalaxy. The mean mass density within the Abell radius \( \simeq 1.5 h_0^{-1} \text{Mpc} \) is roughly \( 10^9 \) times larger than the critical density since clusters are formed from peaks in the density field. Moreover, clusters rotate much less than galaxies even if it is somehow difficult to disentangle, observationally, the global (coherent) rotation of the cluster from the rotation curves of the
constituent galaxies. By assuming, for instance, $N_{\text{tot}} = 5$, a density gradient of $10^3$ and 500 nG as the final field, equation (6.17) demands an initial seed of the order $0.15$ nG.

Another application of the results obtained in the previous sections can be the estimate of the magnetic field induced by the total vorticity for scales which are larger than the Hubble radius prior to matter–radiation equality. To conduct this estimate, the explicit form of the correlators will change. First of all in the pre-factor $\mathcal{C}(r, w)$, we will choose $w = 1/3$ and $a_1 = a_0$ and $H_1 = H_t$ with $H_t \simeq 10^{-5} M_{\odot}$. Thus, for typical length scales larger than the Hubble radius at equality and for typical times of the order of the equality time, the analog of equation (6.13) can be written as

$$\langle B^2(r) \rangle = 2.915 \times 10^{-70} \left( \frac{r_T}{0.32} \right)^3 \left( \frac{A_R}{2.43 \times 10^{-9}} \right)^3 \left( \frac{z_{\text{dec}} + 1}{1089.2} \right)^2 \left( \frac{h_0^2\Omega_{\text{MB}}}{0.134} \right)^{-4} \mathcal{C}(n_T, r) G^2,$$

$$\mathcal{C}(n_T, r) = \tilde{c}(n_T) \left( \frac{r}{r_p} \right)^{-4 - 3n_T} + \tilde{d}(n_T),$$

(6.18)

where the numerical constants $\tilde{c}(n_T)$ and $\tilde{d}(n_T)$ are given by

$$\tilde{c}(n_T) = -2n_T(n_T + 1)[2n_T(n_T + 2) + 3]\cos^3 \left( \frac{n_T\pi}{2} \right) \Gamma^4(n_T - 1),$$

$$\tilde{d}(n_T) = -2(7n_T^2 + 28n_T + 18)$$

$$45n_T^2(n_T + 2)^2(n_T + 4).$$

(6.19)

Equation (6.18) holds under the assumption $r < r_p$ which means, in practice, that it applies only for a narrow range of scales $120 \text{ Mpc} < r < 500 \text{ Mpc}$. If $r \simeq 250 \text{ Mpc}$, then $r/r_p = 0.5$ and $\mathcal{C}(n_T, r) \simeq \mathcal{O}(163)$ and

$$B_{\text{max}} = 4.3 \times 10^{-39} \left( \frac{r_T}{0.32} \right)^{3/2} \left( \frac{A_R}{2.43 \times 10^{-9}} \right)^{3/2} \left( \frac{h_0^2\Omega_{\text{MB}}}{0.134} \right)^{-2} G.$$

(6.20)

It is finally appropriate to present a more explicit estimate of the correction factor appearing in equation (5.16). The correction to the vorticity of equation (5.5) is proportional to the following numerical factor:

$$A_T^{3/2} \left( \frac{a}{a_{\text{eq}}} \right) \left( \frac{r_{\text{eq}}}{r_p} \right)^2 \ll 1,$$

(6.21)

where, as before, $A_T = r_T A_R$. The explicit form of the correction given in equation (6.21) can be obtained from equation (5.16) by setting $a_1 = a_{\text{eq}}$, $H_1 \simeq H_{\text{eq}}$ and by taking into account that the typical inhomogeneity scale is of the order of $r_p$ as proven in the present section by considering the explicit form of the correlators\textsuperscript{20} in the present section by considering the explicit form of the correlators\textsuperscript{20}. We neglected terms containing $n_T$ which could make the result even smaller. The results of equations (5.16) and (6.21) show that a perturbative description of the vorticity within the gradient expansion is rather accurate. The growth of the scale factor is not a source of concern given the overall smallness of the term. While equation (6.21) is just an order of magnitude estimate, we do not attempt here a higher-order computation of the vorticity which would be technically too demanding. In spite of this, the result of equation (6.21) seems to be rather robust on the basis of the results reported in sections 4 and 5.

\textsuperscript{20} We recall that $r_p$ is, in practice, the physical scale related to the pivot scale $k_p$ which appears in the scalar and tensor power spectra. This scale is given by $k_p = 0.002 \text{ Mpc}^{-1}$. So roughly speaking $r_p \simeq k_p^{-1}$ and $r_{\text{eq}} \simeq 120 \text{ Mpc}$ (see also equation (6.12)).
7. Concluding remarks

The idea explored in this paper has been to compute the vorticity by employing a recently devised framework for the treatment of fully inhomogeneous plasmas which are also gravitating. The latter description brings a new perspective to the study of the evolution of the vorticity exchange in the electron–ion–photon system without postulating the customary separation between a (preferably conformally flat) background geometry and its relativistic fluctuations. A set of general conservation laws has been derived on the basis of the fully inhomogeneous equations in different temperature regimes depending on the hierarchies between the exchange rate of the vorticity between electrons, ions and photons. After expanding the Einstein equations as well as the vorticity equations to a given order in the spatial gradients, the total vorticity has then been estimated to lowest order in the gradient expansion. Insofar as the problem discussed in this paper demands a simultaneous treatment of the gradients of the electromagnetic fields and of the geometry in a globally neutral plasma, it is indeed a demonstration of the use of gradient expansion for the analysis of primordial plasmas.

The maximal comoving magnetic field induced in the $\Lambda$CDM paradigm depends upon the tensor to scalar ratio and it is, at most, of the order of $10^{-37}$ G over the typical comoving scales ranging between 1 and 10 Mpc. The obtained results are irrelevant for seeding a reasonable galactic dynamo action and they demonstrate how the proposed fully inhomogeneous treatment can be used for a systematic scrutiny of pre-decoupling plasmas beyond the conventional perturbative expansions. The estimate of the primordial vorticity induced in the $\Lambda$CDM scenario can also turn out to be relevant in related contexts such as the ones contemplated by nonconventional paradigms of galaxy formation.

Appendix A. Gradient expansion and pre-decoupling physics

In this appendix, we will recap the essentials of the fully inhomogeneous description of pre-decoupling plasmas already introduced in equations (3.1)–(3.5). We will follow here the formalism developed in [14] and describe the fully inhomogeneous geometry in terms of the ADM decomposition [15, 16]:

$$
\begin{align*}
g_{00} &= N^2 - N_i N^i, \\
g_{ij} &= -\gamma_{ij}, \\
g_{0i} &= -N_i,
\end{align*}
$$

In the ADM variables, the extrinsic curvature $K_{ij}$ and the spatial components of the Ricci tensor $r_{ij}$ become

$$
K_{ij} = \frac{1}{2N} \left[ -\partial_t \gamma_{ij} + (3)^{(3)} \nabla_j N_i + (3)^{(3)} \nabla_i N_j \right],
$$

$$
r_{ij} = \partial_m^{(3)} \Gamma_{ij}^m - \partial_j^{(3)} \Gamma_{im}^m + (3)^{(3)} \Gamma_{ij}^n (3)^{(3)} \Gamma^m_{mn} (3)^{(3)} \Gamma^n_{jm}.
$$

Defining $T_{\mu\nu}$ as the total energy–momentum tensor of the fluid sources, the contracted form of the Einstein equations reads

$$
R_{\mu\nu} = \frac{\ell^2}{\rho} \left( T_{\mu\nu} - \frac{T}{2} \delta_{\mu\nu} \right),
$$

\[ T = g^{\mu\nu} T_{\mu\nu} = T_{\mu\mu}. \]

As in the bulk of the paper we will now focus on the situation where the shift vectors vanish and the lapse function is homogeneous but time dependent (i.e. $N(\vec{x}, \tau) = N(\tau)$). The $(00)$, $(ij)$ and $(0i)$ components of equation (A.4) then become

$$
\partial_t K - N \text{Tr} K^2 + \nabla^2 N = \mathcal{N} \ell^2 \left\{ \frac{3p + \rho}{2} + (p + \rho) u^2 \right\},
$$
\[ \nabla_i K_i - \nabla_t K^t_i = N \dot{\varphi} u^0 \cdot u_i (p + \rho), \]  
\[ \partial_t v^i - NK v^i - N \tilde{\gamma} v^i \nabla_i / \nabla^t = \epsilon_0^2 N \left[ \frac{p - \rho}{2} 	ilde{\delta}_t^i - (p + \rho) u_i u^t \right], \]  
where \( u^2 = u^i u^t \gamma_{ij} \). The electron and ion velocities appearing in equations (3.1) and (3.2) reduce in the conformally flat case (i.e. \( N(t) \rightarrow a(t) \) and \( \gamma_{ij}(t, \tau) \rightarrow a^2(\tau) \delta_{ij} \)) to the velocity fields appearing in equations (2.3), (2.13) and (2.15). In the fully inhomogeneous case, the evolution equations for the velocities of the electrons, ions and photons can be written, respectively, as

\[ \partial_t v_e^i + N \tilde{\gamma}^e N - G^e_j v_e^j = \frac{e n_e N^2}{\rho_e \sqrt{\gamma}} \left[ E^k + (\vec{v}_e \times \vec{B}^k) \right] \]

\[ + N \Gamma_{e_i} (v_e^k - v_e^k) + \frac{4 \rho_e}{3 \rho_e} N \Gamma_{e_t} (v_e^k - v_e^k), \]  

\[ \partial_t v_i^i + N \tilde{\gamma}^i N - G^i_j v_i^j = \frac{e n_i N^2}{\rho_i \sqrt{\gamma}} \left[ E^k + (\vec{v}_i \times \vec{B}^k) \right] \]

\[ + N \frac{\rho_i}{\rho_i} \Gamma_{ij} (v_i^k - v_i^k) + \frac{4 \rho_i}{3 \rho_i} N \Gamma_{ij} (v_i^k - v_i^k), \]  

\[ \partial_t v_p^k + N \tilde{\gamma}^k N - G_p^k v_p^k = \frac{G_p^k}{N} \left[ N \Gamma_p (v_p^k - v_p^k) + N \Gamma_p (v_i^k - v_i^k) \right], \]

where

\[ G_p^k = \left[ \frac{\partial_t N}{N} \tilde{\delta}_i^k + 2 N \kappa_k^i \right]. \]  

As in the conformally flat case, the evolution equations of the electrons and of the ions can be combined by defining the center of mass velocity of the electron–ion system \( v_c^i = (m_e v_e^i + m_i v_i^i) / (m_e + m_i) \) so that the effective evolution equations for the baryon–lepton–photon fluid become

\[ \partial_t \rho_e = \frac{4}{3} K \kappa \rho_e - \frac{4}{3} N \partial_t \left( \frac{\rho_e}{N} v_e^k \right), \]  

\[ \partial_t v_e^k = G_e^k v_e^k - \tilde{\gamma}^e N + \frac{J^k \times \vec{B}^k}{\gamma \rho_e N \left( 1 + m_e / m_i \right)} + \frac{4 \rho_e}{3 \rho_e} N \Gamma_p (v_p^k - v_e^k), \]  

\[ \partial_t v_p^k = \left[ G_p^k \left[ \frac{4}{3} \delta_i^k \right] v_i^k - \frac{N^2}{4 \rho_p} \partial_m (\rho_p \gamma^{mk}) - N \tilde{\gamma}^k v_e^k \right]. \]

Appendix B. Some relevant correlators

The correlator appearing in equation (6.9), i.e.

\[ T_{mbi} = \langle [h_{ab} h_{ag} h_{ap}] [h_{ai} b_{aj} b_{aq}] \rangle, \]  

25
must be computed in terms of the corresponding two-point functions in real space (see equation (6.5)). The general form of the two-point function in real space has already been mentioned in equation (6.8) and the functions \( G_i(r) \) (with \( i = 1, \ldots, 5 \)) are given by

\[
G_1(r) = F_1(r) + \frac{2}{r} \frac{\partial F_3}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial F_1}{\partial r} \right),
\]

\[\text{(B.2)}\]

\[
G_2(r) = \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial F_1}{\partial r} \right) - F_1(r) - \frac{2}{r} \frac{\partial F_2}{\partial r},
\]

\[\text{(B.3)}\]

\[
G_3(r) = \frac{1}{r} \frac{\partial}{\partial r} \left[ \frac{1}{r} \frac{\partial F_1}{\partial r} \right] - \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial F_2}{\partial r} \right),
\]

\[\text{(B.4)}\]

\[
G_4(r) = \frac{1}{r} \frac{\partial}{\partial r} \left[ \frac{1}{r} \frac{\partial F_1}{\partial r} \right] + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial F_2}{\partial r} \right),
\]

\[\text{(B.5)}\]

\[
G_5(r) = \frac{1}{r} \frac{\partial}{\partial r} \left[ \frac{1}{r} \frac{\partial F_1}{\partial r} \right],
\]

\[\text{(B.6)}\]

where \( F_1(r), F_2(r) \) and \( F_3(r) \) are fully determined once the power spectrum is known and are defined as

\[
F_1(r) = \frac{1}{4} \int \frac{dk}{k} P_\varphi(k, \tau) \sin kr, \quad F_2(r) = \frac{1}{4} \int \frac{dk}{k^3} P_\varphi(k, \tau) \sin kr, \quad F_3(r) = \frac{1}{4} \int \frac{dk}{k^5} P_\varphi(k, \tau) \sin kr.
\]

Using equation (B.7) inside equations (B.2)–(B.6), we have that

\[
G_1(r) = \frac{1}{4} \int \frac{dk}{k} P_\varphi(k, \tau) \left[ \left( 1 - \frac{1}{k^2 r^2} \right) j_0(kr) + \left( \frac{3}{k^3 r^3} - 2 \right) j_1(kr) \right],
\]

\[\text{(B.8)}\]

\[
G_2(r) = \frac{1}{4} \int \frac{dk}{k} P_\varphi(k, \tau) \left[ \left( 2 + \frac{3}{k^2 r^2} \right) j_1(kr) - \left( 1 + \frac{1}{k^2 r^2} \right) j_0(kr) \right],
\]

\[\text{(B.9)}\]

\[
G_3(r) = \frac{1}{4} \int k dk P_\varphi(k, \tau) \left[ j_0(kr) \left( 1 + \frac{5}{k^2 r^2} \right) - j_1(kr) \left( 2 + \frac{15}{k^2 r^2} \right) \right],
\]

\[\text{(B.10)}\]

\[
G_4(r) = \frac{1}{4} \int k dk P_\varphi(k, \tau) \left[ j_0(kr) \left( -1 + \frac{5}{k^2 r^2} \right) + j_1(kr) \left( 4 - \frac{15}{k^2 r^2} \right) \right],
\]

\[\text{(B.11)}\]

\[
G_5(r) = \frac{1}{4} \int k^2 dk P_\varphi(k, \tau) \left[ j_0(kr) \left( 1 - \frac{35}{k^2 r^2} \right) - \frac{5 j_1(kr)}{k^3 r^3} \left( 2 - \frac{21}{k^2 r^2} \right) \right],
\]

\[\text{(B.12)}\]

where \( j_0(kr) \) and \( j_1(kr) \) are spherical Bessel functions of zeroth and first order [51, 52]:

\[
j_0(kr) = \sin kr, \quad j_1(kr) = \sin kr - \cos kr.
\]

\[\text{(B.13)}\]

It is useful to compare the two different asymptotic limits of the various \( G_i(r) \), i.e. for \( kr < 1 \) and for \( kr > 1 \). In the limit \( kr > 1 \), we have that

\[
G_1(r) \to \frac{1}{4} \int \frac{dk}{k} P_\varphi(k, \tau) j_0(kr), \quad G_2(r) \to -G_1(r),
\]

\[\text{(B.14)}\]

\[
G_3(r) \to \frac{1}{4} \int k dk P_\varphi(k, \tau) j_0(kr) \left( 1 + \frac{5}{k^2 r^2} \right), \quad G_4(r) \to -G_3(r).
\]

\[\text{(B.15)}\]
Conversely, in the limit $kr < 1$, equations (B.8)–(B.12) imply that

$$G_1(r) = \frac{1}{10} \int \frac{dk}{k} P_T(k, \tau) \left[ 1 - \frac{11}{42} k^2 r^2 \right],$$

\(\text{(B.17)}\)

$$G_2(r) = -\frac{1}{15} \int \frac{dk}{k} P_T(k, \tau) \left[ 1 - \frac{5k^2 r^2}{14} \right],$$

\(\text{(B.18)}\)

$$G_3(r) = -\frac{2}{105} \int k \, dk P_T(k, \tau) \left[ 1 - \frac{5}{72} k^2 r^2 \right],$$

\(\text{(B.19)}\)

$$G_4(r) = \frac{1}{70} \int k \, dk P_T(k, \tau) \left[ 1 - \frac{2k^2 r^2}{27} \right],$$

\(\text{(B.20)}\)

$$G_5(r) = \frac{1}{3780} \int k^3 \, dk P_T(k, \tau) \left[ 1 - \frac{k^2 r^2}{22} \right].$$

\(\text{(B.21)}\)

The explicit form of the two-point function $G_{\mu\nu}$ implies that the six-point function appearing in the correlator of the vorticity can be expressed as

$$T_{\mu\nu\mu'n'} = \sum_{i=1}^{5} T^{(i)}_{\mu\nu\mu'n'},$$

\(\text{(B.22)}\)

where the five distinct contributions correspond to

$$T^{(1)}_{\mu\nu\mu'n'} = \bar{G}_{\mu\nu b} [G_{\mu b a} \bar{G}_{a' q' q'} + G_{\mu a} \bar{G}_{a b'} q' q' + G_{\mu q q'} \bar{G}_{a' q' a'},]$$

\(\text{(B.23)}\)

$$T^{(2)}_{\mu\nu\mu'n'} = \bar{G}_{\mu b q} [G_{\mu b a} \bar{G}_{a q' q'} + G_{\mu q} \bar{G}_{a b'} q' q' + G_{\mu q q'} \bar{G}_{a' q' a'},]$$

\(\text{(B.24)}\)

$$T^{(3)}_{\mu\nu\mu'n'} = \bar{G}_{\mu b q} [G_{\mu b a} \bar{G}_{a q' q'} + G_{\mu a} \bar{G}_{a b'} q' q' + G_{\mu q q'} \bar{G}_{a' q' a'},]$$

\(\text{(B.25)}\)

$$T^{(4)}_{\mu\nu\mu'n'} = \bar{G}_{\mu a q} [G_{\mu a b} \bar{G}_{a b' q'} + G_{\mu a q} \bar{G}_{a b'} q' q' + G_{\mu q q'} \bar{G}_{a' q' a'},]$$

\(\text{(B.26)}\)

$$T^{(5)}_{\mu\nu\mu'n'} = \bar{G}_{\mu a q} [G_{\mu a b} \bar{G}_{a b' q'} + G_{\mu a q} \bar{G}_{a b'} q' q' + G_{\mu q q'} \bar{G}_{a' q' a'},]$$

\(\text{(B.27)}\)

The overline signifies that the corresponding correlator is evaluated in the limit $r \to 0$. According to equations (B.17)–(B.21) this limit is non-singular. Note finally that in terms of $T_{\mu\nu\mu'n'}$ the correlation function of the magnetic field can also be written, with shorthand notation, as

$$\langle B^2(r) \rangle = J(\tau, w) e^{jim} e^{j' m'i} \partial_\tau^2 \partial_w^2 \bar{T}_{\mu\nu\mu'n'}$$

$$J(\tau, w) = \frac{m^2}{\alpha \sin^{-1} \frac{\eta^2 \alpha^2}{9 H_0^2 (w + 1)^2} \left( \frac{a}{a_1} \right)^{\delta w + \theta}}.$$  

\(\text{(B.28)}\)

The real space approach is more effective and convenient for an explicit estimate of the vorticity and the idea is therefore to express the correlation functions in real space, take the appropriate derivatives and then expand the result in the desired limit. Denoting with $R = r/r_p$ and with $x = k/k_p$, the integrals over $k$ appearing in equations (B.8)–(B.12) can be computed explicitly by changing variable and by using the following pair of relations [53]:

$$\int_1^\infty x^{n-m} \sin xR \, dx = \frac{R}{m-n-2} F_3 \left[ a_1, b_1, b_2, -\frac{R^2}{4} \right]$$

$$+ R^{m-n-1} \cos \left[ \frac{\pi \left( m-n \right)}{2} \right] \Gamma \left[ 1 - m + n \right].$$

\(\text{(B.29)}\)
\[
\int_1^\infty x^{n-m} \cos xR \, dx = \frac{1}{m-n-2} \, iF_2 \left[ a_1, b_1, b_2; -\frac{R^2}{4} \right] + R^{m-n-1} \sin \left( \frac{\pi (m-n)}{2} \right) \Gamma[1-m+n],
\]
where \( n < m \) (i.e. \( n - m \) is negative). In equations (B.29) and (B.30), \( pF_q[a_1, \ldots, a_p; b_1, \ldots, b_q; z] \) denotes the generalized hypergeometric function of argument \( z \); in the case of equations (B.29) and (B.30), \( p = 1, q = 2 \) and
\[
a_1 = 1 + \frac{n-m}{2}, \quad b_1 = \frac{3}{2}, \quad b_2 = 2 + \frac{n-m}{2},
\]
\[
\tilde{a}_1 = a_1 - \frac{1}{2}, \quad \tilde{b}_1 = b_1 - 1, \quad \tilde{b}_2 = b_2 - \frac{1}{2}.
\]

The integrals are taken from 1 to infinity since the integral over \( k \) starts from \( k_p \), implying that the lower limit of integration in \( x \) is 1. Equations (B.29) and (B.30) can be used to derive the real space form of the correlator of equation (6.5). Using equations (B.29) and (B.30), the explicit form of equations (B.8)–(B.12) can be derived and inserted into equation (6.8) whose explicit form determines the real-space expression of the two-point functions of the vorticities through equations (B.23)–(B.27). After taking the appropriate derivatives, the obtained result can be expanded in the wanted limits (e.g. \( R \gg 1 \) or \( R \ll 1 \)). The explicit real-space expressions of equations (B.8)–(B.12) are typically rather lengthy but they are conceptually straightforward. This is why they will be omitted and only an example will be given. Even if the scales interesting for section 6 will be the ones close to the galactic scale, consider, for instance, the expression of \( G_1(R) \) in the opposite limit, i.e. comoving scales much larger than \( r_{eq} \). In this case, the expressions simplify since \( \mathcal{M}(k, k_{eq}, \tau) \to 1 \). Therefore, using equations (B.29) and (B.30), equation (B.8) becomes
\[
G_1(R) = \frac{A_T}{4} \left\{ \frac{1}{n_T} \, iF_2 \left[ \frac{n_T}{2}, \frac{3}{2}, \frac{n_T}{2}; -\frac{R^2}{4} \right] + \frac{3}{4(n_T - 4)R^2} \left[ iF_2 \left[ -2 + \frac{n_T}{2}, \frac{1}{2}, -1 + \frac{n_T}{2}; -\frac{R^2}{4} \right] - iF_2 \left[ -2 + \frac{n_T}{2}, \frac{3}{2}, -1 + \frac{n_T}{2}; -\frac{R^2}{4} \right] \right] \right. \\
\left. + \frac{1}{4(n_T - 2)R^2} \left[ 3iF_2 \left[ -1 + \frac{n_T}{2}, \frac{3}{2}, -\frac{R^2}{4} \right] - 2iF_2 \left[ -1 + \frac{n_T}{2}, \frac{1}{2}, \frac{n_T}{2}; -\frac{R^2}{4} \right] \right] \right. \\
\left. - \frac{1}{4Rr_T} \cos \left( \pi n_T \right) \left[ \Gamma[n_T - 4] + \Gamma[n_T - 2] + 3\Gamma[n_T - 5] \right] + 3\Gamma[n_T - 3] + \Gamma[n_T - 1] \right\}.
\]

To conclude this appendix, let us show that the expression of \( G_{ijmn} \) given in equation (6.5) coincides with the result directly obtainable in the particular case where the tensor modes of the geometry are quantized in terms of gravitons. When \( \hat{h}_{ij}(\bar{x}, \tau) \) is a field operator its expression can be written as [40, 41]
\[
\hat{h}_{ij}(\bar{x}, \tau) = \frac{\sqrt{2} \kappa}{(2\pi)^{3/2}a(\tau)} \int d^3k e^{i(k_{ij}(\bar{z}) \cdot \bar{x})} \hat{a}_{\bar{e}_i, \bar{e}_j}(\bar{z}, \tau) e^{-i\bar{z} \cdot \bar{x}} + \hat{a}_{\bar{e}_i, \bar{e}_j}^\dagger(\bar{z}, \tau) e^{i\bar{z} \cdot \bar{x}},
\]
(34)
which also implies that, using the properties of the creation and annihilation operators,

\[ G_{ijmn}(\tau) = \langle \hat{h}_{ij}(\vec{x}, \tau) \hat{h}^\dagger_{mn}(\vec{y}, \tau) \rangle = \int \frac{dk}{k} \mathcal{P}_T(k, \tau) \mathcal{Q}_{ijmn}(\vec{k}) j_0(kr), \quad \text{(B.35)} \]

where

\[ \mathcal{P}_T(k, \tau) = 4t_p^2 \frac{k^3}{\pi n^2} |f_k(\tau)|^2, \quad \text{(B.36)} \]

\[ \mathcal{Q}_{ijmn} = \frac{1}{4} \sum_\lambda \epsilon_{ij}^{(\lambda)} \epsilon_{mn}^{(\lambda)} = \frac{1}{4} [P_{ij}(\hat{k})P_{mn}(\hat{k}) + P_{ij}(\hat{k})P_{mn}(\hat{k}) - P_{ij}(\hat{k})P_{mn}(\hat{k})]; \quad \text{(B.37)} \]

\[ P_{ij}(\hat{k}) = (\delta_{ij} - \hat{k}_i \hat{k}_j) \quad \text{with} \quad \hat{k}^i = k^i/|\hat{k}|. \]

In equation (B.35), it has been used that

\[ \langle 0 | \hat{a}^\dagger_{\vec{p}, \mu} \hat{a}^\dagger_{\vec{p}', \lambda} | 0 \rangle = \delta^{(3)}(\vec{k} - \vec{p}) \delta_{\mu, \lambda}. \quad \text{(B.38)} \]

Furthermore, to derive equation (B.37), the explicit form of the two tensor polarizations can be written, in explicit terms, as

\[ \epsilon_{ij}^{(\oplus)}(\hat{k}) = (\hat{a}_i \hat{a}_j - \hat{b}_i \hat{b}_j), \quad \epsilon_{ij}^{(\otimes)}(\hat{k}) = (\hat{a}_i \hat{b}_j + \hat{b}_i \hat{a}_j). \quad \text{(B.39)} \]

where \( \hat{a}, \hat{b}, \) and \( \hat{k} \) are three mutually orthogonal unit vectors.

References

[1] Hoyle F 1958 Proc. Solvay Conf. La structure et l’evolution de l’Univers (Brussels, June 1958) ed R Stoop p 59
[2] Harrison E R 1967 Phys. Rev. Lett. 18 1011
[3] Harrison E R 1968 Phys. Rev. 167 1170
[4] Harrison E R 1969 Astrophys. Lett. 3 133
[5] Mishustin I and Ruzmaikin A 1972 Sov. Phys.—JETP 34 233
[6] Ozernoy L M and Chermin A D 1969 Sov. Astron. Astron. J. 12 901
[7] Ozernoy L M and Chermin A D 1972 Sov. Astron. Astron. J. 15 923
[8] Ozernoy L M 1971 Astron. Z. 48 1160
[9] Peebles P J E 1993 Principles of Physical Cosmology (Princeton, NJ: Princeton University Press) p 555
[10] Spitzer L 1962 Physics of Fully Ionized Plasmas (New York: Wiley)
[11] Krall N A and Trivelpiece A W 1986 Principles of Plasma Physics (San Francisco, CA: San Francisco Press)
[12] Giovannini M 2009 Phys. Rev. D 79 103007
[13] Giovannini M 2009 Phys. Rev. D 79 121302
[14] Giovannini M and Rezaei Z 2011 Phys. Rev. D 83 083519 (arXiv:1102.3572 [astro-ph.CO])
[15] Arnowitt R, Deser S and Misner C W 1960 Phys. Rev. 117 1595
[16] Arnowitt R and Deser S 1959 Phys. Rev. 113 745
[17] Khalatnikov I M and Lifshitz E M 1970 Phys. Rev. Lett. 24 76
[18] Belinskii V A and Khalatnikov I M 1973 Sov. Phys.—JETP 36 591
[19] Deruelle N and Tomita K 1994 Phys. Rev. D 50 7216
[20] Deruelle N and Goldwirth D 1995 Phys. Rev. D 51 1563
[21] Tanaka Y and Sasaki M 2007 Prog. Theor. Phys. 117 633
[22] Tanaka Y and Sasaki M 2007 Prog. Theor. Phys. 118 455
[23] Hwang J-c and Noh H 2005 Phys. Rev. D 72 044012
[24] Noh H and Hwang J-c 2008 Phys. Rev. D 77 123533
[25] Bonin et al 2011 Astrophys. J. 727 22
[26] Salopek D S and Bond J R 1991 Phys. Rev. D 43 1005
[27] Salopek D S and Bond J R 1990 Phys. Rev. D 42 3936-3962
[28] Rigopoulos G I and Shellard E P S 2003 Phys. Rev. D 68 123518
[23] Lifshitz E M and Pitaevskii L P 1980 Physical Kinetics (Oxford: Pergamon)
[24] Alfén H and Fälthammer C-G 1963 Cosmical Electrodynamics 2nd edn (Oxford: Clarendon)
[25] Biskamp D 1994 Non-Linear Magnetohydrodynamics (Cambridge: Cambridge University Press)
[26] Bennett C L et al 2011 Astrophys. J. Suppl. 192 17
Jarosik N et al 2011 Astrophys. J. Suppl. 192 14
[27] Weiland J L et al 2011 Astrophys. J. Suppl. 192 19
Larson D et al 2011 Astrophys. J. Suppl. 192 16
Gold B et al 2011 Astrophys. J. Suppl. 192 15
Komatsu E et al 2011 Astrophys. J. Suppl. 192 18
[28] Giovannini M 2004 Int. J. Mod. Phys. D 13 391
[29] Barrow J D, Maartens R and Tsagas C G 2007 Phys. Rep. 449 131
[30] Barrow J D and Tsagas C G 2011 arXiv:1101.2390 [astro-ph.CO]
[31] Barrow J 1977 Mon. Not. R. Astron. Soc. 178 625
Barrow J 1977 Mon. Not. R. Astron. Soc. 179 47
[32] Vachaspati T and Vilenkin A 1991 Phys. Rev. Lett. 67 1057
[33] Avelino P and Shellard P 1995 Phys. Rev. D 51 3946
[34] Dimopoulos K 1998 Phys. Rev. D 57 4629
[35] Bereziani Z and Dolgov A D 2004 Astropart. Phys. 21 59
[36] Dolgov A D 2003 Magnetic Fields in Cosmology arXiv:astro-ph/0306443
[37] Hogan C 2000 arXiv:astro-ph/0005380
[38] Campanelli L, Dolgov A D, Giannotti M and Villante F L 2004 Astrophys. J. 616 1
[39] Hortua H J, Castaneda L and Tejeiro J M 2011 arXiv:1104.0701 [astro-ph.CO]
[40] Giovannini M 2008 Phys. Lett. B 668 44
Giovannini M 2009 Class. Quantum Grav. 26 045004
Giovannini M 2010 Phys. Rev. D 82 083523
[41] Giovannini M 2010 PMC Phys. A 4 1
[42] Zhao W and Zhang Y 2006 Phys. Rev. D 74 043503
Zhang Y, Zhao W, Xia T and Yuan Y 2006 Phys. Rev. D 74 083006
[43] Turner M S, White M J and Lidsey J E 1993 Phys. Rev. D 48 4613
[44] Hinshaw G et al (WMAP Collaboration) 2009 Astrophys. J. Suppl. 180 225–45
Dunkley J et al (WMAP Collaboration) 2009 Astrophys. J. Suppl. 180 306
[45] Gold B et al (WMAP Collaboration) 2009 Astrophys. J. Suppl. 180 265–82
Komatsu E et al (WMAP Collaboration) 2009 Astrophys. J. Suppl. 180 330
Nolta M R et al (WMAP Collaboration) 2009 Astrophys. J. Suppl. 180 296
[46] Spergel D N et al (WMAP Collaboration) 2007 Astrophys. J. Suppl. 170 377
Page L et al (WMAP Collaboration) 2007 Astrophys. J. Suppl. 170 335
[47] Spergel D N et al (WMAP Collaboration) 2003 Astrophys. J. Suppl. 148 175
Peiris H V et al (WMAP Collaboration) 2003 Astrophys. J. Suppl. 148 213
Bennett C L et al (WMAP Collaboration) 2003 Astrophys. J. Suppl. 148 1
[48] Monin A S and Yaglom A M 1975 Statistical Fluid Mechanics: Mechanics of Turbulence (Cambridge, MA: MIT Press)
[49] Lumley J L 1970 Stochastic Tools in Turbulence (New York: Dover)
[50] Giovannini M 2006 Class. Quantum Grav. 23 R1
[51] Abramowitz M and Stegun I A 1972 Handbook of Mathematical Functions (New York: Dover)
[52] Erdelyi A, Magnus W, Obetthetter F and Tricomi F 1953 Higher Transcendental Functions (New York: McGraw-Hill)
[53] Gradshteyn I S and Ryzhik I M 2000 Table of Integrals, Series and Products (San Diego, CA: Academic)