NEW FUNCTIONAL EQUATIONS OF FINITE MULTIPLE POLYLOGARITHMS

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Abstract. We give a finite analogue of the well-known formula \( \underset{n}{\underbrace{\text{Li}_1, \ldots, \text{Li}_1}}(t) = \frac{1}{n!} \text{Li}_1(t)^n \) of multiple polylogarithms for any positive integer \( n \) by using the shuffle relation of finite multiple polylogarithms of Ono–Yamamoto type. Unlike the usual case, the terms regarded as error terms appear in this formula. As a corollary, we obtain “\( t \leftrightarrow 1 - t \)” type new functional equations of finite multiple polylogarithms of Ono–Yamamoto type and Sakugawa-Seki type.

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1. Introduction

In this article, we give a finite analogue of the well-known formula \( \underset{n}{\underbrace{\text{Li}_1, \ldots, \text{Li}_1}}(t) = \frac{1}{n!} \text{Li}_1(t)^n \) of multiple polylogarithms for any positive integer \( n \) by using the shuffle relation of finite multiple polylogarithm (FMP) of Ono–Yamamoto type (OY-type). As a corollary, we obtain “\( t \leftrightarrow 1 - t \)” type new functional equations of FMPs of OY-type. In addition, by using the known relation between FMPs of OY-type and Sakugawa–Seki type (SS-type), we also obtain new functional equations of FMPs of SS-type, which seem to be difficult obtained only by using Sakugawa–Seki’s results [SS].

First, we review of the history of FMPs. Recently, the author and Yamamoto [OY] introduced finite multiple polylogarithms (FMPs) \( \mathcal{L}^{\text{OY}}_{A,k}(t) \) as an element of the \( \mathbb{Q} \)-algebra \( A_{\mathbb{Z}[t]} := \)

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\( \left( \prod_p (\mathbb{Z}/p\mathbb{Z})[t] \right) / \left( \bigoplus_p (\mathbb{Z}/p\mathbb{Z})[t] \right) \), where \( p \) runs through all the rational primes. Thus, an element of \( \mathcal{A}_\mathbb{Z}[t] \) is represented by a family \( (f_p)_p \) of polynomials \( f_p \in (\mathbb{Z}/p\mathbb{Z})[t] \), and two families \( (f_p)_p \) and \( (g_p)_p \) represent the same element of \( \mathcal{A}_\mathbb{Z}[t] \) if and only if \( f_p = g_p \) for all but finitely many primes \( p \). We denote such an element of \( \mathcal{A}_\mathbb{Z}[t] \) simply by \( t \) if there is no fear of confusion. For example, we denote an element \( (t \mod p)_p \) of \( \mathcal{A}_\mathbb{Z}[t] \) by \( t \).

Note that the idea considering several objects depending on a fixed prime \( p \) in some adelic rings is due to Kaneko–Zagier’s theory of finite multiple zeta values [KZ]. Also note that the \( \mathbb{Q} \)-algebra \( \mathcal{A}_\mathbb{Z}[t] \) is denoted by \( \mathcal{B} \) in [OY]. The symbol \( \mathcal{A}_\mathbb{Z}[t] \) is Sakugawa–Seki’s notation in [SS].

**Definition 1.1** ([OY Definition 1.2]). For a positive integer \( r \) and an index \( k = (k_1, \ldots, k_r) \in (\mathbb{Z}_{\geq 1})^r \), we define a finite multiple polylogarithm of Ono–Yamamoto type (OY-type for short) by

\[
\mathcal{L}_A^{\text{OY}}(t) := \sum'_{0 < l_1, \ldots, l_r < p} \frac{t^{l_1+\cdots+l_r}}{l_1^{k_1} (l_1 + l_2)^{k_2} \cdots (l_1 + \cdots + l_r)^{k_r} \mod p}
\]

as an element of \( \mathcal{A}_\mathbb{Z}[t] \). Here, \( \sum' \) denotes the sum of fractions whose denominators are prime to \( p \).

We respectively call integers \( r \) and \( k_1 + \cdots + k_r \) the depth and weight of \( k = (k_1, \ldots, k_r) \) and we denote the weight of index of \( k \) by \( \text{wt}(k) \).

One of the reasons why we introduce FMPs of OY-type was to establish a finite analogue of the shuffle relation for multiple polylogarithms.

On the other hand, Sakugawa and Seki introduced another type of FMPs in [SS]. We call their FMPs SS-type in this article. One of their motivations of introducing their FMPs was to establish functional equations of their FMPs.

More recently, Seki put a question about functional equations of FMPs of OY-type with the index \( \{1\}^n := (1, \ldots, 1) \) for a positive integer \( n \). The results of Kontsevich [K, (A)] and Elbaz-Vincent and Gangl [EG, PROPOSITION 5.9 (1)] say that \( \mathcal{L}_A^\text{OY}(t) = \mathcal{L}_A^\text{OY}(1-t) \) holds. Furthermore, by using functional equations of FMPs of SS-type and the fact that FMPs of OY-type can be written in terms of FMPs of SS-type [SS Proposition 3.26], Seki [Se Theorem 14.6] proved the equality \( \mathcal{L}_A^\text{OY}_{\{1\}^2}(t) = \mathcal{L}_A^\text{OY}_{\{1\}^2}(1-t) \).

In this article, we give an answer to Seki’s question, that is, we give functional equations between \( \mathcal{L}_A^\text{OY}_{\{1\}^n}(t) \) and \( \mathcal{L}_A^\text{OY}_{\{1\}^n}(1-t) \) for a positive integer \( n \), which contain Seki’s result as the case of \( n = 2 \). In order to state our main theorem, we recall the definition of a variant \( \zeta_A^{(i)}(k) \) of finite multiple zeta values (FMZVs) \( \zeta_A(k) \).
Definition 1.2 ([OY, Definition 2.1]). For an index $k = (k_1, \ldots, k_r)$ and $1 \leq i \leq r$, we define a variant of FMZVs as an element of $A := (\prod_r \mathbb{Z}/p\mathbb{Z}) / (\bigoplus_r \mathbb{Z}/p\mathbb{Z})$ by

$$\zeta^{(i)}_A(k) := \sum_{0 \leq l_1, \ldots, l_r < p, (i-1)p < l_1 + \cdots + l_r < ip} \frac{1}{l_1^{k_1}(l_1 + l_2)^{k_2} \cdots (l_1 + \cdots + l_r)^{k_r}} \mod p.$$ 

Note that $\zeta^{(1)}_A(k)$ coincides with the usual FMZV $\zeta_A(k)$ defined by

$$\zeta_A(k) = \sum_{0 < n_1 < \cdots < n_r < p} \frac{1}{n_1 \cdots n_r^{k_r}} \mod p$$

and we see that $\zeta^{(r)}_A(k) = (-1)^{\text{wt}(k)} \zeta_A(k)$.

The main theorem of this article is the following.

Theorem 1.3. For a positive integer $n$, we define two elements

(1) 

$$f_n(t) := \sum_{k=0}^{n-2} \left( \sum_{i=1}^{n-k-1} \zeta^{(i)}_A(\{1\}^{n-k-2}, 2) t^{ip} \right) \mathcal{L}^{OY}_{A,\{1\}^k}(t)$$

and

(2) 

$$g_n(t) := \sum_{k=0}^{n-2} \left( \sum_{i=1}^{n-k-2} \zeta^{(i)}_A(\{1\}^{n-k-2}) t^{ip} \right) \mathcal{L}^{OY}_{A,\{2,\{1\}^k}(t)$$

of $A_{\mathbb{Z}[t]}$. Here, we understand that these elements are equal to 0 if the sums are empty. Then, we have

(3) 

$$\mathcal{L}^{OY}_{A,\{1\}^n}(t) = \frac{1}{n!} \mathcal{L}^{OY}_{A,\{1\}^1}(t)^n + \frac{1}{n!} \sum_{k=1}^{n} (k-1)! (f_k(t) + g_k(t)) \mathcal{L}^{OY}_{A,\{1\}^1}(t)^{n-k}.$$ 

We remark that this equality can be regarded as an finite analogue of the well-known formula $\text{Li}_{\{1\}^n}(t) = \frac{1}{n!} \text{Li}_1(t)^n$, where $\text{Li}_k(t) := \sum_{0 < n_1 < \cdots < n_r} \frac{t^{n_r}}{n_1 \cdots n_r^{k_r}}$ is the (one variable) multiple polylogarithm.

As a corollary of our main theorem and the equality $\mathcal{L}^{OY}_{A,\{1\}^1}(t) = \mathcal{L}^{OY}_{A,\{1\}^1}(1-t)$, we obtain “$t \leftrightarrow 1-t$” type functional equations of FMPs of OY-type.

Corollary 1.4. For a positive integer $n$, set

$$\mathcal{L}_{A,n}(t) := \mathcal{L}^{OY}_{A,\{1\}^n}(t) - \frac{1}{n!} \sum_{k=1}^{n} (k-1)! (f_k(t) + g_k(t)) \mathcal{L}^{OY}_{A,\{1\}^1}(t)^{n-k}.$$ 

Then we obtain

$$\mathcal{L}_{A,n}(t) = \mathcal{L}_{A,n}(1-t).$$

Proof. By Theorem 1.3, we have $\mathcal{L}_{A,n}(t) = \frac{1}{n!} \mathcal{L}^{OY}_{A,\{1\}^1}(t)^n$. Therefore, the assertion holds by the result of Elbaz-Vincent and Gangl [EG, PROPOSITION 5.9 (1)].
The contents of this article is as follows. In Section 2, we prove Theorem [3.3] by using the shuffle relation of FMPs of OY-type. In Section 3, by using our main theorem, we give examples of functional equations of FMPs of OY-type. In the final section, by using the relation between FMPs of OY-type and SS-type, we also give functional equations of FMPs of SS-type, which seem to be difficult to obtain only using the results of [SS].

2. Proof of the main theorem

In this section, we prove the main theorem by using the shuffle relation of FMPs of OY-type, which was proved by the author and S. Yamamoto in [OY].

First, we explicitly calculate the shuffle relation of \(L^OY_{A,\{1\}^{n-1}}(t)\) and \(L^OY_{A,1}(t)\) for a positive integer \(n\).

**Lemma 2.1.** For a positive integer \(n\), we have
\[
L^OY_{A,\{1\}^{n-1}}(t)L^OY_{A,1}(t) = nL^OY_{A,\{1\}^n}(t) - f_n(t) - g_n(t).
\]

Recall the definitions of \(f_n(t)\) and \(g_n(t)\). See (I) and (2).

**Proof.** For \(0 \leq k \leq n - 1\), set
\[
F_k(t) := L^OY_A(\{1\}^{n-k-1}, (1), \{1\}^k; t).
\]

Here, for indices \(\lambda = (\lambda_1, \ldots, \lambda_a), \mu = (\mu_1, \ldots, \mu_b)\) and \(\nu = (\nu_1, \ldots, \nu_c)\) \((a, b, c \in \mathbb{Z}_{\geq 0})\), \(L^OY_A(\lambda, \mu, \nu; t)\) is the FMP of type \((\lambda, \mu, \nu)\) [OY Definition 3.1] defined by
\[
L^OY_A(\lambda, \mu, \nu; t) := \sum_{0 < l_1, \ldots, l_a < p} \prod_{x=1}^{a} L_x \prod_{y=1}^{b} M_y^{\mu_y} \prod_{z=1}^{c} (L_a + M_b + N_z)^{\nu_z}
\]
where \(L_x := l_1 + \cdots + l_x, M_y := m_1 + \cdots + m_y\) and \(N_z := n_1 + \cdots + n_z\). By [OY] Remark 3.2, we have \(F_0(t) = L^OY_{A,\{1\}^{n-1}}(t)L^OY_{A,1}(t)\) and \(F_{n-1}(t) = L^OY_{A,\{1\}^n}(t)\). By using [OY] Proposition 3.7 in the case \(\lambda := \{1\}^{n-k-1}, \mu := \{1\}^k\) \((0 \leq k \leq n - 2)\), we obtain
\[
(4) \quad F_k(t) = F_{k+1}(t) + L^OY_{A,\{1\}^n}(t) - \left( \sum_{i=1}^{n-k-1} s_A^{(i)}(\{1\}^{n-k-2}) t^{ip} \right) L^OY_{A,\{1\}^k}(t)
\]
\[
- \left( \sum_{i=1}^{n-k-2} s_A^{(i)}(\{1\}^{n-k-2}) t^{ip} \right) L^OY_{A,\{2,\{1\}^k\}}(t).
\]

Therefore, the statement holds from taking the telescoping sum of (4).

**Proof of Theorem 2.3** We prove the statement by the induction on \(n \geq 1\). Note that the statement for \(n = 1\) holds by [K, (A)] or [EG, PROPOSITION 5.9]. For \(n \geq 2\), assume that the
Lemma 3.1

By Lemma 2.1, the product of the left hand side of (5) and statement holds for n = 1:

$$L_{A_i}^{OY,n-1}(t) = \frac{1}{(n-1)!} L_{A_1}^{OY,n-1}(t) + \frac{1}{(n-1)!} \sum_{k=1}^{n-1} (k-1)!(f_k(t) + g_k(t)) L_{A_i}^{OY}(t)^{n-k-1}.\]

By Lemma 2.1 the product of the left hand side of (5) and $L_{A_1}^{OY}(t)$ coincides with

$$L_{A_i}^{OY,n-1}(t) L_{A_1}^{OY}(t) = n L_{A_i}^{OY}(t) - f_n(t) - g_n(t).$$

On the other hand, the product of the right hand side of (5) and for example

$$A_{1}^{2} \text{ and } 2.$$  

The last equality holds since 1 - $t^p$ (1 - $t^p$) holds in $(\mathbb{Z}/p\mathbb{Z})[t]$ for all primes $p$. On the other hand, we have $g_3(t) = \zeta_A(1,2) t^p L_{A_2}^{OY}(t) = 0$ by Lemma 3.1. Therefore, Theorem [OY] says the equality

$$L_{A_i}^{OY,1,1}(t) = \frac{1}{3!} L_{A_1}^{OY}(t)^3 + \frac{1}{3} \zeta_A(1,2) t^p (1 - t)^p.$$
Moreover, since $f_3(t) = f_3(1-t)$ by (9), we see that Corollary 1.3 says that $\mathcal{L}^{OY}_{\mathcal{A}_1(1)}(t) = \mathcal{L}^{OY}_{\mathcal{A}_1(1)}(1-t)$.

(iii) Furthermore, we consider the case of $n = 4$. By an easy calculation, we obtain

\begin{equation}
\begin{aligned}
f_4(t) &= \zeta_A(1, 1, 2) t^p + \zeta_A^{(2)}(1, 1, 2) t^{2p} + \zeta_A^{(3)}(1, 1, 2) t^{3p} \\
&\quad + (\zeta_A(1, 2) t^p + \zeta_A^{(2)}(1, 2) t^{2p}) \mathcal{L}^{OY}_{\mathcal{A}_1(t)} + \zeta_A(2) t^p \mathcal{L}^{OY}_{\mathcal{A}_1(1)}(t) \\
&= \zeta_A(1, 1, 2) t^p (1 - t^2p) + \zeta_A^{(2)}(1, 1, 2) t^{2p} + f_3(t) \mathcal{L}^{OY}_{\mathcal{A}_1(t)}.
\end{aligned}
\end{equation}

By [OY] Example 2.6, (ii)], $\zeta_A^{(2)}(1, 1, 2)$ is a sum of FMZVs of weight 4, we see that $f_4(t) = f_3(t) \mathcal{L}^{OY}_{\mathcal{A}_1(t)}$ by Lemma 3.2. On the other hand, since $\zeta_A(1, 1) = \zeta_A^{(2)}(1, 1) = 0$ by Lemma 3.1, we have

$$g_4(t) = (\zeta_A(1, 1) t^p + \zeta_A^{(2)}(1, 1) t^{2p}) \mathcal{L}^{OY}_{\mathcal{A}_1(t)} + \zeta_A(1) t^p \mathcal{L}^{OY}_{\mathcal{A}_1(1)}(t) = 0.$$ 

Therefore, we obtain

$$\mathcal{L}^{OY}_{\mathcal{A}_1(1)}(t) = \frac{1}{24} \mathcal{L}^{OY}_{\mathcal{A}_1(1)}(t)^4 + \frac{1}{3} \zeta_A(1, 2) t^p (1 - t^p) \mathcal{L}^{OY}_{\mathcal{A}_1(1)}(t).$$

Moreover, since $f_4(t) = f_4(1-t)$ by (9), we have $\mathcal{L}^{OY}_{\mathcal{A}_1(1)}(t) = \mathcal{L}^{OY}_{\mathcal{A}_1(1)}(1-t)$.

(iv) Finally, consider the case of $n = 5$. In this case, it is difficult to expect the "$t \leftrightarrow 1-t$" type relation of $\mathcal{L}^{OY}_{\mathcal{A}_1(n)}(t)$ for $n \geq 5$.

First, by [OY] Example 2.6 and Lemma 3.1, we have $\zeta_A(1, 1, 1) = -\zeta_A^{(3)}(1, 1, 1) = 0$ and $\zeta_A(1, 1, 1) = 4 \zeta_A(1, 1, 1) + \zeta_A(2, 1) + \zeta_A(1, 2) = 0$. Therefore, we obtain

$$g_5(t) = (\zeta_A(1, 1, 1) t^p + \zeta_A^{(2)}(1, 1, 1) t^{2p} + \zeta_A^{(3)}(1, 1, 1) t^{3p}) \mathcal{L}^{OY}_{\mathcal{A}_1(t)}(t) \\
+ (\zeta_A(1, 1, 1) t^p + \zeta_A^{(2)}(1, 1, 1) t^{2p}) \mathcal{L}^{OY}_{\mathcal{A}_1(2,1)}(t) + \zeta_A(1) t^p \mathcal{L}^{OY}_{\mathcal{A}_1(2,1)}(t) = 0.$$

Thus, Theorem 1.3 says that

$$\mathcal{L}^{OY}_{\mathcal{A}_1(1)}(t) = \frac{1}{5!} \mathcal{L}^{OY}_{\mathcal{A}_1(1)}(t)^5 + \frac{21 f_3(t) \mathcal{L}^{OY}_{\mathcal{A}_1(1)}(t)^2 + 3 f_4(t) \mathcal{L}^{OY}_{\mathcal{A}_1(1)}(t) + 4 f_5(t)}{5!}$$

$$= \frac{1}{5!} \mathcal{L}^{OY}_{\mathcal{A}_1(1)}(t)^5 + \frac{1}{15} f_3(t) \mathcal{L}^{OY}_{\mathcal{A}_1(1)}(t)^2 + \frac{1}{5} f_5(t).$$

Since $f_5(t) \mathcal{L}^{OY}_{\mathcal{A}_1(1)}(t)^2 = f_5(1-t) \mathcal{L}^{OY}_{\mathcal{A}_1(1)}(1-t)^2$, we have

$$\mathcal{L}^{OY}_{\mathcal{A}_1(1)^5}(t) - \frac{1}{5} f_5(t) = \mathcal{L}^{OY}_{\mathcal{A}_1(1)^5}(1-t) - \frac{1}{5} f_5(1-t).$$
Next, we have
\[ f_5(t) = \zeta_A^{(1)}(1,1,1,2)t^p + \zeta_A^{(2)}(1,1,1,2)t^{2p} + \zeta_A^{(3)}(1,1,1,2)t^{3p} + \zeta_A^{(4)}(1,1,1,2)t^{4p} \]
\[ + \left( \zeta_A^{(1)}(1,1,2)t^p + \zeta_A^{(2)}(1,1,2)t^{2p} + \zeta_A^{(3)}(1,1,2)t^{3p} \right) L_{A,1}^{\text{OY}}(t) \]
\[ = \zeta_A(1,1,1,2)(t^p - t^{4p}) + \zeta_A^{(2)}(1,1,1,2)(t^{2p} - t^{3p}) + f_3(t)L_{A,1}^{\text{OY}}(t)^2 \]
\[ = \zeta_A(1,1,1,2)t^p(1 - t^p)(1 + t^p + t^{2p}) + \zeta_A^{(2)}(1,1,1,2)t^{2p}(1 - t^p) + f_3(t)L_{A,1}^{\text{OY}}(t)^2. \]
Therefore, we see that
\[ L_{A,(1)^5}^{\text{OY}}(t) - L_{A,(1)^5}^{\text{OY}}(1 - t) = \frac{f_5(t) - f_5(1 - t)}{5} \]
\[ = \frac{2\zeta_A(1,1,1,2) + \zeta_A^{(2)}(1,1,1,2)}{10} p(1 - t^p)(2t^p - 1). \]
By [OY] Example 2.6 (2) and [Sa] Table 2, we see that \( \zeta_A(1,1,1,2) = B_{p-5} \) and \( \zeta_A^{(2)}(1,1,1,2) = 0. \) Here, we set \( B_{p-5} := (B_{p-5} \bmod p) \in A. \) Therefore, we obtain
\[ L_{A,(1)^5}^{\text{OY}}(t) - L_{A,(1)^5}^{\text{OY}}(1 - t) = \frac{B_{p-5}}{5} p(1 - t^p)(2t^p - 1). \]
Thus, since it is conjectured that \( B_{p-5} \) does not vanish in \( A \) (for example, see [Z] Conjecture 2.1), we see that \( L_{A,(1)^5}^{\text{OY}}(t) \neq L_{A,(1)^5}^{\text{OY}}(1 - t). \)

4. Functional equations of finite multiple polylogarithms of Sakugawa-Seki type

We end this article with new functional equations of FMPs of SS-type.

**Definition 4.1 (SS Definition 3.8).** Let \( r \) be a positive integer and \( k = (k_1, \ldots, k_r) \) an index. Then we define finite harmonic multiple polylogarithms and 1-variable finite multiple polylogarithms as follows:
\[ L_{A,k}^r(t_1, \ldots, t_r) = \sum_{0 < n_1 < \cdots < n_r < p} \frac{t_1^{n_1} \cdots t_r^{n_r}}{n_1^{k_1} \cdots n_r^{k_r}} \mod p \in A_{\mathbb{Z}[t]}; \]
\[ L_{A,k}(t) := L_{A,k}^r([1]^{r-1}, t) \in A_{\mathbb{Z}[t]}, \quad L_{A,k}(t, \{1\}^{r-1}) \in A_{\mathbb{Z}[t]}. \]
Here, for an \( r \)-tuple of variables \( t := (t_1, \ldots, t_r) \), we set \( A_{\mathbb{Z}[t]} := (\prod_p (\mathbb{Z}/p\mathbb{Z})[t]) / (\bigoplus_p (\mathbb{Z}/p\mathbb{Z})[t]). \)

Now we prepare the following notation to describe the relation between the FMPs of OY-type and SS-type (cf. [OY] Section 2). First, for a positive integer \( r \), set
\[ [r] := \{1, 2, \ldots, r\} \]
and
\[ \Phi_r := \bigsqcup_{s=1}^{r} \Phi_{r,s}, \quad \Phi_{r,s} := \{ \phi : [r] \rightarrow [s] : \text{surjective} \mid \phi(a) \neq \phi(a + 1) \text{ for all } a \in [r - 1] \}. \]
Next, for $\phi \in \Phi_{r,s}$, set $s_\phi := s$. Furthermore, for $\phi \in \Phi_r$ and $1 \leq i \leq r$, we define an integer $\delta_\phi(i)$ by
\[\delta_\phi(i) := \# \{a \in [i - 1] \mid \phi(a) > \phi(a + 1) \} \quad (1 \leq i \leq r).\]
Finally, a map $\beta : \Phi_r \to \{r\}$ is defined by $\beta(\phi) := \delta_\phi(r) + 1$ and we set $\Phi^i_r := \beta^{-1}(i)$.

**Proposition 4.2**: For an index $k = (k_1, \ldots, k_r)$, we have
\[L^\text{OY}_{A,k}(t) = \sum_{i=1}^{r} t^{(i-1)p} \sum_{\phi \in \Phi^i_r} L^\text{A}(\sum_{\phi(j)=1} k_j, \ldots, \sum_{\phi(j)=s_\phi} k_j) \{1\}^{\phi(r)-1}, t, \{1\}^{s_\phi-\phi(r)}.\]

By Proposition 4.2, our main theorem gives functional equations of FMPs of SS-type. It seems very difficult to obtain our functional equations of FMPs of SS-type only using Sakugawa-Seki's theory and without using the shuffle relation of FMPs of OY-type. We describe only two functional equations of FMPs of SS-type which are obtained from that of FMPs of OY-type with indices $\{1\}^3$ and $\{1\}^4$.

**Corollary 4.3.** We have
\[(1 + t^p)L_{A,1}^3(t) + t^p(1 + t^p)\tilde{L}_{A,1}^3(t) + 2t^pL_{A,1}^3(1, t, 1) + t^pL_{A,2,1}(1, t) + t^p\tilde{L}_{A,1,2}(t) = (2 - t^p)L_{A,1}^3(1 - t) + (2 - t^p)(1 - t^p)\tilde{L}_{A,1}^3(1 - t) + 2(1 - t^p)L_{A,1}^3(1, 1 - t, 1) + (1 - t^p)L_{A,1,2}(1 - t) + (1 - t^p)\tilde{L}_{A,2,1}(1 - t).\]

**Corollary 4.4.** We have
\[(1 + 4t^p + t^2p)L_{A,1}^4(t) + 2t^p(2 + t^p)L_{A,1}^4(1, 1, 1, 1) + 2t^p(1 + 2t^p)L_{A,1}^4(1, 1, 1, 1) + t^p(1 + 4t^p + t^2p)\tilde{L}_{A,1}^4(t) + t^p(1 + 3t^p)\tilde{L}_{A,2,1,1}(t) + 2t^p(1 + t^p)L_{A,1,2,1}(1, t, 1, 1) + t^p(3 + t^p)L_{A,1,1,2}(t) + t^p(L_{A,2,1,1}(1, 1, 1) + L_{A,1,2,1}(1, 1, 1) + L_{A,2,2}(t)) + t^2p(L_{A,1,2,1}(1, 1, 1) + L_{A,1,1,2}(1, 1, 1) + \tilde{L}_{A,1,2,1}(1, 1) + \tilde{L}_{A,2,2}(t)) = (6 - 6t^p + t^2p)L_{A,1}^4(1 - t) + 2(1 - t^p)(3 - t^p)L_{A,1}^4(1, 1, 1, 1, 1 - t) + 2(1 - t^p)(3 - 2t^p)L_{A,1}^4(1, 1, 1, 1) + (1 - t^p)(6 - 6t^p + t^2p)\tilde{L}_{A,1}^4(1 - t) + (1 - t^p)(4 - 3t^p)\tilde{L}_{A,2,1,1}(1 - t) + 2(1 - t^p)(2 - t^p)L_{A,1,2,1}(1, 1, 1 - t, 1) + (1 - t^p)(4 - t^p)L_{A,1,1,2}(1 - t) + (1 - t^p)L_{A,2,1,1}(1 - t) + (1 - t^p)L_{A,1,2,1}(1, 1, 1, 1) + L_{A,2,1,1}(1 - t) + L_{A,2,2}(1 - t) + (1 - t^p)^2(\tilde{L}_{A,1,1,2}^4(1 - t) + L_{A,1,2,1}^4(1, 1, 1, 1)).\]
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References

[EG] P. Elbaz-Vincent and H. Gangl, On poly(ana)log I, Comp. Math. 130 (2002) 161–210.

[H] M. E. Hoffman, Quasi-symmetric functions and mod p multiple harmonic sums, Kyushu J. Math. 69 (2015) 345–366.

[KZ] M. Kaneko and D. Zagier, Finite multiple zeta values, in preparation.

[K] M. Kontsevich, The 1+1/2 logarithm, appendix to [EG], Comp. Math. 130 (2002) 211–214.

[OY] M. Ono and S. Yamamoto, Shuffle product of finite multiple polylogarithms, manuscripta mathematica 152 (2017) 153–166.

[Sa] S. Saito, Numerical tables of finite multiple zeta values, to appear in RIMS Kōkyūroku Bessatsu.

[SS] K. Sakugawa and S. Seki, On functional equations of finite multiple polylogarithms, Journal of Algebra 469 (2017) 323–357.

[Se] S. Seki, Finite multiple polylogarithms, doctoral dissertation.

[Z] J. Zhao, Mod p structure of alternating and non-alternating multiple harmonic sums. J. Théor. Nombres Bordeaux 23(1) (2011), 299–308.

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