THE GINZBURG-LANDAU THEORY AND THE SURFACE ENERGY OF A COLOUR SUPERCONDUCTOR

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Abstract

We apply the Ginzburg-Landau theory to the colour superconducting phase of a lump of dense quark matter. We calculate the surface energy of a domain wall separating the normal phase from the super phase with the bulk equilibrium maintained by a critical external magnetic field. Because of the symmetry of the problem, we are able to simplify the Ginzburg-Landau equations and express them in terms of two components of the di-quark condensate and one component of the gauge potential. The equations also contain two dimensionless parameters: the Ginzburg-Landau parameter $\kappa$ and $\rho$. The main result of this paper is a set of inequalities obeyed by the critical value of the Ginzburg-Landau parameter—the value of $\kappa$ for which the surface energy changes sign—and its derivative with respect to $\rho$. In addition we prove a number of inequalities of the functional dependence of the surface energy on the parameters of the problem and obtain a numerical solution of the Ginzburg-Landau equations. Finally a criterion for the types of colour superconductivity (type I or type II) is established in the weak coupling approximation.

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1. Introduction.

The Ginzburg-Landau theory provides a powerful tool for exploring systems with inhomogeneous order parameters near the critical temperature. For instance, its application to the surface energy of a normal-superconducting interface led to the discovery of the two types of superconductors. When an external magnetic field whose magnitude is less than a critical value $H_c$, is applied to type I superconductors, the field is expelled from the interior of the sample (the Meissner effect). As the magnitude of the magnetic field increases above $H_c$, superconductivity disappears completely and the normal phase is restored. In the case of type II superconductors, the sample behaves in a similar manner as long as the external magnetic field remains below a lower critical magnitude, $H_{c1}$. A qualitatively different behaviour emerges when the applied magnetic field lies in the range $H_{c1} < H_c < H_{c2}$, where it becomes energetically favourable for the magnetic field to penetrate the sample in the form of quantized flux lines called vortices. Beyond the upper critical magnitude of the field, the vortices are too dense to maintain the condensate, and the normal phase is once again restored.

The criterion that determines whether a superconductor is of type I or type II is the Ginzburg-Landau parameter. It is defined as the ratio of the penetration depth of the magnetic field $\delta$ over the coherence length $\xi$, the distance over which changes in the order parameter occur, i.e. $\kappa = \frac{\delta}{\xi}$. These characteristic lengths imply that fluctuations in the magnitude of the condensate decay as $e^{-\sqrt{2}\frac{\xi}{\delta}}$ while the magnetic field inside the superconductor falls off as $e^{-\frac{\xi}{\delta}}$. In their original paper, Ginzburg and Landau [1] analyzed the energy of the interface between a normal and a superconducting phase, kept in equilibrium in the bulk by an external magnetic field at the critical value. They found analytically that the surface energy vanishes at

$$\kappa = \kappa_c = \frac{1}{\sqrt{2}} \approx 0.707. \quad (1.1)$$

The physical meaning of this critical value was clarified further by Abrikosov [2]. It represents the demarcation line between type I ($\kappa < \frac{1}{\sqrt{2}}$) and type II ($\kappa > \frac{1}{\sqrt{2}}$) superconductors.

In the present paper we shall carry out a similar analysis of the Ginzburg-Landau theory for colour superconductors. Colour superconductivity is essentially the quark analog of BCS superconductivity [3], [4], [5], [6]. Because of attractive quark-quark interactions in QCD, the Fermi sphere of quarks becomes unstable against the formation of Cooper pairs and the system becomes superconducting at sufficiently low temperatures. In this paper we shall consider a domain wall separating the normal and colour superconducting phases in equilibrium under the influence of an external magnetic field and calculate the surface free energy per unit area. The Ginzburg-Landau equations for this system are considerably more complicated than the analogous equations for an ordinary superconductor [7], [8],...
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[9], [10], [11], because of the non-abelian nature of the problem. By using symmetry arguments, we were able to simplify the problem of the domain wall and express the G-L equations in terms of one gauge potential and two components of the di-quark condensate. When expressed in terms of dimensionless quantities, the G-L equations contains two dimensionless parameters: the usual Ginzburg-Landau parameter \( \kappa \) and another parameter \( \rho \). Weak coupling calculations fix the value of \( \rho \) to be equal to \(-\frac{1}{2}\). Although we were unable to derive an analytical result analogous to (1.1) we proved the following inequalities for the critical G-L parameter and its derivative with respect to \( \rho \):

\[
\kappa_c(\rho) \leq \frac{1}{\sqrt{2}}, \quad \frac{d\kappa_c}{d\rho} \geq 0. \tag{1.2}
\]

We also derived a number of inequalities of the functional dependence of the surface energy on the parameters of the problem and obtained a numerical solution for \( \kappa_c(\rho) \). For weak coupling we found that

\[
\kappa_c \simeq 0.589, \tag{1.3}
\]

in contrast to (1.1).

In section 2 of this paper, we shall review the Ginzburg-Landau theory of colour superconductivity. The analytical and numerical solution of the domain wall problem will be presented in section 3. In the final section of the paper we shall summarize our results and in the appendix we shall discuss the validity of the Ginzburg-Landau theory in the presence of the fluctuations of the gauge field.

2. The Ginzburg-Landau Theory of Colour Superconductivity.

The symmetry group of QCD in the chiral limit is

\[
SU(3)_c \times SU(3)_{fR} \times SU(3)_{fL} \times U(1)_B, \tag{2.1}
\]

where the subscript \( c \) denotes colour, the subscript \( f_R(f_L) \) the right(left)-hand flavour and \( B \) the baryon number. The electromagnetic gauge group \( U(1)_{em} \) is not a separate symmetry group, but a subgroup of \( SU(3)_{fR} \times SU(3)_{fL} \). The dominant pairing channel consists of two quarks of the same helicity and the corresponding order parameters in a colour superconducting quark matter will be denoted by \( \Psi_R \) and \( \Psi_L \). These condensates transform into each other under space inversions. Neglecting the parity violating processess,
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the Ginzburg-Landau free energy functional which is consistent with the symmetry group (2.1) reads

\[
\Gamma = \int d^3 \vec{r} \left\{ \frac{1}{4} F_{ij}^l F_{ij}^l + \frac{1}{2} (\vec{\nabla} \times \vec{A})^2 + \frac{1}{2} \text{Tr}[(\vec{D} \Psi_R)\dagger(\vec{D} \Psi_R) + (\vec{D} \Psi_L)\dagger(\vec{D} \Psi_L)] \\
+ \frac{1}{2} a \text{Tr}[\Psi_R\dagger \Psi_R + \Psi_L\dagger \Psi_L] + \frac{1}{4} b \text{Tr}[(\Psi_R\dagger \Psi_R)^2 + (\Psi_L\dagger \Psi_L)^2] \\
+ \frac{1}{4} b' [(\text{Tr} \Psi_R\dagger \Psi_R)^2 + (\text{Tr} \Psi_L\dagger \Psi_L)^2] \\
+ \frac{1}{2} c \text{Tr}(\Psi_R\dagger \Psi_R) \text{Tr}(\Psi_L\dagger \Psi_L) \right\} 
\]

(2.2)

where the gauge covariant derivative of the di-quark condensate \( \Psi \) reads

\[
(\vec{D} \Psi_{R(L)})_{f_1 f_2}^{c_1 c_2} = \vec{\nabla} \Psi_{R(L)}^{c_1 c_2} - ig \vec{A}^{c_1 c_2 c_1' c_2'} \Psi_{R(L)}^{c_1 c_2} - ig \vec{A}^{c_1 c_2 c_1' c_2'} \Psi_{R(L)}^{c_1 c_2} \\
- ie (q_{f_1} + q_{f_2}) \vec{A} \Psi_{R(L)}^{c_1 c_2}
\]

(2.3)

where \( \vec{A} = \vec{A}^l T^l \) denotes the classical vector potential of the \( SU(3) \) colour gauge field, \( \vec{A} \) the electromagnetic field, and \( eq_f \) is the electric charge of the \( f \)-th flavour quark. The \( SU(3) \) generator \( T^l \) is in its fundamental representation.

At weak coupling, the parameters are calculable from either the perturbative one-gluon exchange interaction of QCD or from the Nambu-Jona-Lasinio effective action. Both approaches lead to the same expression [7], [8]:

\[
a = \frac{48 \pi^2}{7 \zeta(3)} k_B^2 T_c (T - T_c), \\
b = \frac{576 \pi^4}{7 \zeta(3)} \left( \frac{k_B T_c}{\mu} \right)^2, \\
b' = c = 0, 
\]

(2.4)

where \( \mu \) is the chemical potential and \( T_c \) the transition temperature.

Since in this paper we address the domain wall problem- the interface between a normal and a superconducting phase in an external magnetic field- the boundary conditions select the even parity sector of (2.2)

\[
\Psi_R = \Psi_L \equiv \Psi. 
\]

(2.5)

The one-gluon exchange process that dominates the di-quark interaction at ultrahigh chemical potential is attractive for quarks within the colour antisymmetric channel. Thus the Cooper pairs realise the colour antisymmetric representation to the leading order of the QCD running coupling constant at \( T = 0 \) [12], [13], or to the leading order of \( (1 - \frac{T}{T_c}) \)
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near $T_c$ [8]. Assuming such a pairing pattern persists even at moderately high chemical potential, the di-quark condensate $\Psi$ can be expressed in terms of a $3 \times 3$ complex matrix, $\Phi$

$$\Psi^{e_1e_2}_{f_1f_2} = e^{e_1e_2}e_{f_1f_2}f \Phi^c.$$  \hspace{1cm} (2.6)

The gauge covariant derivative of $\Phi$ can be written as

$$(\vec{D}\Phi)^c_f = \vec{\nabla}\Phi^c_f - ig\vec{A} \Phi^c_f - ieQ_f \vec{A}\Phi^c_f$$ \hspace{1cm} (2.7)

where

$$\vec{A} = \vec{A}^l T^l, \quad \Phi = \Phi_0 + \Phi^l T^l \hspace{1cm} (2.8)$$

with $T^l = -T^{l*}$ being the generator of the $\bar{3}$ representation, $Q_f = q_{f_1} + q_{f_2}$ and $f_{f_1f_2}$ represent a cyclic permutation of $1, 2, 3$. Arranging the flavour index in the conventional order of $u, d, s$ we find the diagonal electric charge matrix

$$Q = -\text{diag}(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}) = -\frac{2}{\sqrt{3}} \vec{T}^8. \hspace{1cm} (2.9)$$

Here we have adapted an expression of $T^l$ that differs from the standard one by a cyclic permutation of rows (columns), in which $\vec{T}^8 = \frac{1}{2\sqrt{3}} \text{diag}(2, -1, -1)$. The equation (2.7) takes the matrix form

$$\vec{D}\Phi = \vec{\nabla}\Phi - ig\vec{A}\Phi + i\frac{2}{\sqrt{3}}e\vec{A}\Phi \vec{T}^8, \hspace{1cm} (2.10)$$

and the Ginzburg-Landau free energy (2.2) becomes

$$\Gamma = \int d^3\vec{r} \left[ \frac{1}{4} F_{ij}^l F_{ij}^l + \frac{1}{2}(\vec{\nabla} \times \vec{A})^2 + 4\text{tr}(\vec{D}\Phi)\dagger(\vec{D}\Phi) + 4\text{tr}\Phi\dagger\Phi + b_1\text{tr}(\Phi\dagger\Phi)^2 + b_2(\text{tr}\Phi\dagger\Phi)^2 \right]$$ \hspace{1cm} (2.11)

where the parameters $b_1$ and $b_2$ are related to the parameters of equation (2.2) in the following manner

$$b_1 = b, \quad b_2 = b + 8b' + 8c. \hspace{1cm} (2.12)$$

First, let us review the case with a homogeneous condensate, i.e. $\vec{A} = \vec{A} = \vec{\nabla}\Phi = 0$, for which the Ginzburg-Landau free energy (2.11) becomes

$$\Gamma = \int d^3\vec{r} \left[ 4\text{tr}\Phi\dagger\Phi + b_1\text{tr}(\Phi\dagger\Phi)^2 + b_2(\text{tr}\Phi\dagger\Phi)^2 \right]. \hspace{1cm} (2.13)$$

We distinguish the following three regions of the parameter space $b_1 - b_2$, following the treatment of [7]

1) $b_1 > 0$ and $b_1 + 3b_2 > 0$: The minimum free energy corresponds to the colour-flavour locked condensate [4],

$$\Phi = \phi_0 U \hspace{1cm} (2.14)$$
where
\[ \phi_0 = \sqrt{-\frac{2a}{b_1 + 3b_2}} \]  
(2.15)

and \( U \) is a unitary matrix.\(^\dagger\) Consequently, we find that
\[ \Gamma_{\text{min}} = -\Omega \frac{12a^2}{b_1 + 3b_2}. \]  
(2.16)

2) \( b_1 < 0 \) but \( b_1 + b_2 > 0 \): In this case the colour-flavour locked condensate (2.14) becomes a saddle point of the free energy, which is nevertheless bounded from below. The minimum corresponds to an isoscalar condensate, given by
\[ \Phi = \text{diag}(\phi'_0 e^{i\alpha}, 0, 0). \]  
(2.17)

where \( \phi'_0 = \sqrt{-\frac{2a}{b_1 + b_2}} \). We find then that
\[ \Gamma_{\text{min}} = -\Omega \frac{4a^2}{b_1 + 3b_2}. \]  
(2.18)

3) For \( b_1 \) and \( b_2 \) outside the region specified by 1) and 2), the free energy is no longer bounded from below. Higher powers of the order parameter have to be included and the superconducting transition becomes first order. Since case 2) is mathematically identical to a metallic superconductor, we shall not address it in this paper.

In order to identify the various characteristic lengths of the system, we consider fluctuations about the homogeneous condensate (2.14). We parametrize the order parameter by
\[ \Phi = \phi_0 + \frac{1}{\sqrt{6}} \left( \frac{X + iY}{2} + \bar{T}^l \left( \frac{X_l + iY_l}{2} \right) \right) \]  
(2.19)

with \( X \)'s and \( Y \)'s real, and form linear combinations of the ordinary electromagnetic gauge potential with the eighth component of the colour gauge potential [14],
\[ \vec{V} = \vec{A}_8 \cos \theta + \vec{A}_8 \sin \theta \]
\[ \vec{V} = -\vec{A}_8 \sin \theta + \vec{A}_8 \cos \theta, \]  
(2.20)

where the “Weinberg angle” is given by
\[ \tan \theta = -\frac{2e}{\sqrt{3}g}, \]  
(2.21)

\(^\dagger\) It is very important to maintain this general form of the CFL condensate, since its nontrivial winding onto the gauge group, \( SU(3)_c \times U(1)_{\text{em}} \) gives rise to vortex filaments.
Next we expand (2.11) to quadratic order in $X$’s $Y$’s $\vec{A}$’s and $A$. We find
\[
\Gamma = \Gamma_{\text{min}} + \int d^3\vec{r} \left[ \frac{1}{2} (\vec{\nabla} \times \vec{V})^2 + \text{tr}[ (\vec{\nabla} \times \vec{W})^2 + m_W^2 \vec{W}^2 ] + \frac{1}{2} [ (\vec{\nabla} \times \vec{Z})^2 + m_Z^2 \vec{Z}^2 ] \\
+ \frac{1}{2} (\vec{\nabla} X)^2 + m_H^2 X^2] + \frac{1}{2} [ (\vec{\nabla} X') (\vec{\nabla} X') + m_H^2 X' X'] + \frac{1}{2} (\vec{\nabla} Y)^2 \right],
\tag{2.22}
\]
where $\vec{W} = \sum_{l=1}^{7} \bar{T}^l \vec{W}^l$ and
\[
\vec{W}^l = \vec{A}^l - \frac{1}{m_W} \vec{V} Y^l \quad \text{for} \quad l = 1, 2, \ldots 7
\tag{2.23}
\]
The masses of the excitations provide us with the relevant length scales which are the coherence lengths, $\xi$ and $\xi'$ defined by:
\[
m_H^2 = (b_1 + 3b_2) \phi_0^2 = \frac{2}{\xi^2}, \quad m_H^2 = b_1 \phi_0^2 = \frac{2}{\xi'^2},
\tag{2.24}
\]
that indicate the distances over which the di-quark condensate varies and the magnetic penetration depths, $\delta$ and $\delta'$ by
\[
m_Z^2 = 4g^2 \phi_0^2 \sec^2 \theta = \frac{1}{\delta^2}, \quad m_W^2 = m_Z^2 \cos^2 \theta = \frac{1}{\delta'^2}.
\tag{2.25}
\]
The excitations that violate (2.5) correspond to the Goldstone bosons associated with chiral symmetry breaking, and the $\eta$ particle that becomes massive through the anomaly.

In the rest of this section, we shall determine the critical magnetic field of a homogeneous colour superconductor. For simplicity, the diamagnetic response of the quarks in the normal phase will be neglected. In the presence of an external magnetic field, the thermodynamic function to be minimized is the Gibbs free energy $\tilde{\Gamma}$, which is a Legendre transformation of the Helmholtz free energy $\Gamma$, i.e.
\[
\tilde{\Gamma} = \Gamma - \vec{H} \cdot \int d^3\vec{r} \vec{\nabla} \times \vec{A}.
\tag{2.26}
\]
Following the decomposition (2.20), we write
\[
\tilde{\Gamma} = \tilde{\Gamma}_1 + \tilde{\Gamma}_2
\tag{2.27}
\]
with
\[
\tilde{\Gamma}_1 = \int d^3\vec{r} \left[ \frac{1}{4} \sum_{i=1}^{7} F_{ij} F_{ij} + \frac{1}{2} (\vec{\nabla} \times \vec{V})^2 + 4\text{tr}(\bar{D} \Phi) \bar{D} \Phi + 4\text{tr} \Phi \bar{\Phi} + b_1 \text{tr}(\Phi \bar{\Phi})^2 \\
+ b_2 (\text{tr} \Phi \bar{\Phi})^2 - \vec{H} \cdot \vec{\nabla} \times \vec{V} \sin \theta \right]
\tag{2.28}
\]
\[
\tilde{\Gamma}_2 = \int d^3\vec{r} \left[ \frac{1}{2} (\vec{\nabla} \times \vec{V})^2 - \vec{H} \cdot \vec{\nabla} \times \vec{V} \cos \theta \right]
\]
The minimization of $\tilde{\Gamma}_2$ gives rise to
\[ \vec{\nabla} \times \vec{V} = \vec{H} \cos \theta, \] (2.29)
and no distinction will be made between the super-phase and the normal phase. In the remaining of the section we shall concentrate solely on $\tilde{\Gamma}_1$.

A homogeneous super-phase is characterized by a perfect Meissner effect, $\vec{\nabla} \times \vec{V} = 0$ and the CFL condensate $\Phi = \phi_0$, which gives rise to the minimum of $\tilde{\Gamma}$, $\tilde{\Gamma}_s = \Gamma_{\text{min}}$, given by (2.16). $\tilde{\Gamma}_1$ for a homogeneous normal phase is minimized by $\vec{\nabla} \times \vec{V} = \vec{H} \sin \theta$ and the minimum reads $\tilde{\Gamma}_n = -\Omega \frac{1}{2} H^2 \sin^2 \theta$. The critical magnetic field $H = H_c$ is determined from the equilibrium condition
\[ \tilde{\Gamma}_n = \tilde{\Gamma}_s = \Gamma_{\text{min}} = -\Omega \frac{12a^2}{b_1 + 3b_2} \] (2.30),
which leads to
\[ H_c = 2 \sqrt{\frac{6a^2}{b_1 + 3b_2}} |\csc \theta|. \] (2.31)
Substituting the parameters (2.4) which were calculated using weak coupling approximation into (2.30) yields
\[ H_c = 4 \sqrt{\frac{3}{14\zeta(3)}} \mu^2 \left( \frac{k_B T_c}{\mu} \right) (1 - \frac{T}{T_c}) |\csc \theta|. \] (2.32)
Extrapolating this expression to a realistic value for the quark chemical potential, for example $\mu = 400\text{MeV}$, and using the one-loop running coupling constant
\[ g^2 = \frac{12\pi^2}{(\frac{N_c}{2} - N_f) \ln \frac{4\Lambda}{\mu}} \] (2.33)
with $N_c = N_f = 3$ and $\Lambda = 200\text{MeV}$, we find
\[ \theta \simeq -5.6^\circ \] (2.34)
and
\[ H_c \simeq 1.47 \times 10^{20} \left( \frac{k_B T_c}{\mu} \right) (1 - \frac{T_c}{T}) \text{ Gauss}, \] (2.35)
which is much higher than the typical magnetic field inside a neutron star. This estimation of course depends on the validity of the extrapolation of the weak coupling formulas to realistic values of the chemical potential.
3. The Domain Wall Problem.

Let’s consider an interface between the normal phase and the colour superconducting phase, for example the $y-z$ plane. The equilibrium in the bulk is maintained by a uniform external magnetic field of critical magnitude, $\vec{H} = H_c \hat{\zeta}$. All quantities then in both phases will depend solely on $x$. It is evident from symmetry that the gauge field lies in one plane, let this be the $x-y$ plane, so that $\vec{V} = V(x) \hat{y}$ and $\nabla \times \vec{V} = \frac{4V}{dx} \hat{\zeta}$. The boundary conditions on the Ginzburg-Landau equations in the problem that we are considering (corresponding to the normal and colour superconducting phases as $x \rightarrow -\infty$ and $x \rightarrow \infty$) are

$$
\phi \mapsto 0, \quad \chi \mapsto 0, \quad \frac{dV}{dx} \mapsto H_c \sin \theta \quad \text{at} \quad x \rightarrow -\infty
$$

$$
\phi \mapsto \phi_0, \quad \chi \mapsto \sqrt{2} \phi_0, \quad \frac{dV}{dx} \mapsto 0 \quad \text{at} \quad x \rightarrow \infty
$$

The surface energy $\sigma$ is defined as

$$
\sigma = \frac{\tilde{\Gamma}_1 - \tilde{\Gamma}_s}{\text{Area of } yz \text{ plane}} \quad (3.2)
$$

where the Gibbs free energy of a homogeneous superphase, $\tilde{\Gamma}_s$ is given by (2.30), and it is equal to that of the normal phase in the presence of $H = H_c$.

The minimization of $\tilde{\Gamma}_1$ generates a set of coupled nonlinear differential equations-the Ginzburg-Landau equations-subject to the boundary conditions (3.1). Now we divide the field variables in equation (2.8) into two groups, the first group contains $\Phi_0, \Phi^8, \vec{A}^8, \vec{A}$, while the second one includes $\Phi^l, \vec{A}^l$ with $l = 1, 2, \cdots 7$. A closer inspection of the structure of $\tilde{D} \Phi$ reveals the absence of terms in the free energy (2.2) that are linear in the variables of the second group. Therefore a solution of the equations of motion exists in which only the fields of the first group acquire non-zero values. Since this ansatz implements the maximum symmetry allowed by the boundary conditions, we expect that it includes the solution that minimizes the free energy of the domain wall.

By setting $\Phi^l = \vec{A}^l = 0$ for $l = 1, 2, \cdots 7$ and transforming the remaining variables

$$
\phi = \Phi_0 + \frac{1}{\sqrt{3}} \Phi^8, \quad \chi = \sqrt{2} (\Phi_0 - \frac{1}{2\sqrt{3}} \Phi^8), \quad (3.3)
$$

we arrive at the following expression for the Ginzburg-Landau free energy functional (2.2)

$$
\tilde{\Gamma}_1 = \int d^3 \vec{r} \left[ \frac{1}{2} (\nabla \times \vec{V})^2 + 4 \left| (\nabla - i \frac{g}{\sqrt{3}} \sec \theta \vec{V}) \phi \right|^2 + 4 \left| (\nabla + i \frac{g}{2\sqrt{3}} \sec \theta \vec{V}) \chi \right|^2 
+ 4a(\left| \phi \right|^2 + \left| \chi \right|^2) + b_1(\left| \phi \right|^4 + \frac{1}{2} \left| \chi \right|^4) + b_2(\left| \phi \right|^2 + \left| \chi \right|^2)^2 \right] - H_c \hat{\zeta} \cdot (\nabla \times \vec{V}) \sin \theta \right].
$$

(3.4)
The condition for the colour-flavour locking (that the $3 \times 3$ matrix $\Phi$ is proportional to a unitary matrix) amounts to $|\chi|^2 = 2|\phi|^2$. Consequently, we shall use instead of the variable $x$ and the functions $V$, $\phi$ and $\chi$ the dimensionless quantities

$$s = \delta x, \quad \phi = \phi_0 u, \quad \chi = \sqrt{2}\phi_0 v, \quad V = -\frac{\sqrt{-3a}}{g} A \cos \theta.$$  \hfill (3.5)

The boundary conditions in terms of the new dimensionless quantities become

$$u \mapsto 0, \quad v \mapsto 0, \quad A' \mapsto 1 \quad \text{at} \quad s \mapsto -\infty \quad (3.6)$$

$$u \mapsto 1, \quad v \mapsto 1, \quad A' \mapsto 0 \quad \text{at} \quad s \mapsto \infty$$

where prime indicates differentiation with respect to $s$. The surface energy per unit area in terms of the dimensionless quantities becomes

$$\sigma = \frac{6a^2}{b} \int_{-\infty}^{\infty} ds \left\{ \frac{1}{2}(A' - 1)^2 + \frac{1}{3\kappa^2}(u'^2 + 2v'^2) + \frac{1}{6} A^2(2u^2 + v^2) - \frac{1}{3}(u^2 + 2v^2) \right. \right.$$  

$$+ \left. \frac{1}{18}(2u'^4 + 2u^2v'^2 + 5v'^4) + \frac{1}{18\rho}(u'^2 - v'^2)^2 \right\}$$  \hfill (3.7)

where

$$\kappa = \frac{\delta}{\xi}$$  \hfill (3.8)

is the Ginzburg-Landau parameter, $b = \frac{1}{4}(b_1 + 3b_2)$, and

$$\rho = \frac{b_1 - 3b_2}{b_1 + 3b_2}$$  \hfill (3.9)

is another dimensionless parameter. The weak coupling calculation fixes $\rho$ to be equal to $-\frac{1}{2}$. The Ginzburg-Landau equations which determine the profile of the colour condensate $u$, $v$ and the magnetic field in the colour superconductor are determined by minimizing the surface energy with respect to functions $u$, $v$ and $A$

$$-A'' + \frac{1}{3} A(2u^2 + v^2) = 0,$$  

$$-\frac{1}{\kappa^2}u'' + (A^2 - 1)u + \frac{1}{3}(2u^2 + v^2)u + \frac{1}{3\rho}(u^2 - v^2)u = 0,$$  \hfill (3.10)  

$$-\frac{1}{\kappa^2}v'' + \frac{1}{4}(A^2 - 4)v + \frac{1}{6}(u^2 + 5v^2)v - \frac{1}{6\rho}(u^2 - v^2)v = 0.$$  

In the presence of a non-zero $A$, as it is required by the boundary conditions, this set of equations does not admit a solution in which $u = v$ everywhere. Stated differently, in the presence of an inhomogeneity and a non-zero gauge potential the unlocked condensate—the octet—has to acquire a non-zero value somewhere. This situation is analogous to the Ginzburg-Landau theory of a cuprate superconductor [15], where the $s$-wave condensate
becomes non-zero in the vicinity of a vortex filament while the \( d \)-wave condensate dominates in the bulk. It is easily verified that equations (3.10) have the first integral

\[
\frac{1}{2} A'^2 + \frac{1}{3\kappa^2} (u'^2 + 2v'^2) - \frac{1}{6} A^2 (2u'^2 + v'^2) + \frac{1}{3} (u^2 + 2v^2) - \frac{1}{18} (2u^4 + 2u^2v^2 + 5v^4) - \frac{1}{18} \rho (u^2 - v^2)^2 = \frac{1}{2},
\]

which implies, according to the boundary conditions (3.6), that

\[
A(\infty) = 0. \tag{3.12}
\]

Let \( \sigma_{\min} \) be the minimum surface energy. It is obtained from the substitution of the field variables \( u, v, A \) that satisfy the GL equations (3.10) into (3.7), and is a function of the parameters \( \kappa \) and \( \rho \). We proceed now to prove the following lemma

**Lemma:**

\[
\left( \frac{\partial \sigma_{\min}}{\partial \kappa} \right)_{\rho} \leq 0; \tag{3.13}
\]

\[
\left( \frac{\partial \sigma_{\min}}{\partial \rho} \right)_{\kappa} \geq 0; \tag{3.14}
\]

**Proof:**

According to (3.7),

\[
\left( \frac{\partial \sigma_{\min}}{\partial \kappa} \right)_{\rho} = -\frac{4a^2}{b\kappa^3} \int_{-\infty}^{\infty} ds (u'^2 + 2v'^2) \\
+ \int_{-\infty}^{\infty} ds \left[ \frac{\delta \sigma_{\min}}{\delta A(s)} \left( \frac{\partial A(s)}{\partial \kappa} \right)_{\rho} + \frac{\delta \sigma_{\min}}{\delta u(s)} \left( \frac{\partial u(s)}{\partial \kappa} \right)_{\rho} + \frac{\delta \sigma_{\min}}{\delta v(s)} \left( \frac{\partial v(s)}{\partial \kappa} \right)_{\rho} \right] \tag{3.15}
\]

The second term vanishes because of the equation of motion (3.10) and the boundary conditions (3.6), (3.12). Consequently, we find that

\[
\left( \frac{\partial \sigma_{\min}}{\partial \kappa} \right)_{\rho} = -\frac{4a^2}{b\kappa^3} \int_{-\infty}^{\infty} ds (u'^2 + 2v'^2) \leq 0. \tag{3.16}
\]

Similarly

\[
\left( \frac{\partial \sigma_{\min}}{\partial \rho} \right)_{\kappa} = \frac{a^2}{3b} \int_{-\infty}^{\infty} ds (u^2 - v^2)^2 \geq 0. \tag{3.17}
\]

The lemma is then proved.
Furthermore let’s denote the critical value of the Ginzburg-Landau parameter for which \( \sigma_{\text{min}} \) vanishes by \( \kappa_c(\rho) \). We shall prove then the following theorem

**Theorem:**

\[
\kappa_c(\rho) \leq \frac{1}{\sqrt{2}}
\]

(3.18)

and

\[
\frac{d\kappa_c}{d\rho} \geq 0.
\]

(3.19)

**Proof:**

Consider a special field configuration \( u = v \), which satisfies the boundary conditions (3.6) but not the Ginzburg-Landau equations (3.10) with \( A \neq 0 \). Therefore

\[
\sigma_{\text{min}} \leq \frac{6a^2}{b} \int_{-\infty}^{\infty} ds \left[ \frac{1}{2} (A' - 1)^2 + \frac{1}{\kappa^2} u'^2 + \frac{1}{2} A^2 u^2 - u^2 + \frac{1}{2} u^4 \right] \equiv \bar{\sigma}
\]

(3.20)

The minimization of \( \bar{\sigma} \) with respect to \( u \) and \( A \) yields the set of differential equations,

\[
\begin{align*}
-A'' + Vu^2 &= 0, \\
-\frac{1}{\kappa^2} u'' + \frac{1}{2} (A^2 - 2)u + u^3 &= 0,
\end{align*}
\]

(3.21)

subject to the boundary conditions (3.6). We recognize that (3.20) and (3.21) correspond exactly to the domain wall problem for an ordinary superconductor analyzed by Ginzburg and Landau in their original work. It follows then that \( \bar{\sigma} = 0 \) at \( \kappa = \frac{1}{\sqrt{2}} \). On the other hand, this field configuration does not satisfy the equations of motion (3.7) and therefore \( \sigma_{\text{min}} \leq 0 \) at \( \kappa = \frac{1}{\sqrt{2}} \). Following (3.13) of the lemma we establish the first equation of the theorem.

The second equation of the theorem follows from the identity

\[
\left( \frac{\partial \kappa}{\partial \rho} \right)_{\sigma_{\text{min}}} = -\left( \frac{\partial \sigma_{\text{min}}}{\partial \rho} \right)_\kappa \left( \frac{\partial \sigma_{\text{min}}}{\partial \kappa} \right)_\rho
\]

(3.22)

and the lemma. The theorem is then proved.

The solution to the Ginzburg-Landau equations (3.10) and the critical value of the Ginzburg-Landau parameter \( \kappa \) for various values of \( \rho \in (-1, \infty) \) were also determined numerically. The continuous range \( s \in (-\infty, \infty) \) was replaced by a finite lattice with

\[
s_n = -L + n\epsilon
\]

(3.23)
The Ginzburg-Landau theory and the surface energy . . .

where \( n = 0, 1, 2, \ldots, N+1 \) and \( (N+1)\epsilon = 2L \). A good approximation amounts to \( \epsilon \) and \( L \) that satisfy \( \epsilon \ll \min(1, 1/\kappa) \) and \( L \gg \max(1, 1/\kappa) \). The field variables \( u_n, v_n, \) and \( A_n \) are assigned to each site and their derivatives can be approximated by \( u'_n = (u_{n+1} - u_n)/\epsilon, \) \( v'_n = (v_{n+1} - v_n)/\epsilon \), and \( A'_n = (A_{n+1} - A_n)/\epsilon \). The surface energy is then written

\[
\sigma = 6a^2 \left\{ \epsilon \sum_{n=0}^{N} \frac{1}{2} (A'_n - 1)^2 + \frac{1}{3\kappa^2} (u'^2_n + 2v'^2_n) \right\} + \sum_{n=0}^{N+1} \epsilon_n \left\{ \frac{1}{6} A_n^2 (2u_n^2 + v_n^2) - \frac{1}{3} (u_n^2 + 2v_n^2) + \frac{1}{18} (2u_n^4 + 2u_n^2 v_n^2 + 5v_n^4) + \frac{1}{18} \rho (u_n^2 - v_n^2)^2 \right\} \right\}
\]

with \( \epsilon_0 = \epsilon_{N+1} = \frac{1}{2} \epsilon \) and \( \epsilon_n = \epsilon \) for \( n = 1, \ldots, N \). The multivariable function (3.24) is then minimized by iteration subject to the conditions \( u_0 = v_0 = 0, u_{N+1} = v_{N+1} = 1, A_0 = 1 \) and \( A_{N+1} = 0 \).

The critical GL parameter, \( \kappa_c \) versus \( \rho \) is plotted in Fig. 1. The critical value of GL parameter in the weak coupling approximation \( (\rho = -\frac{1}{2}) \), is \( \kappa_c = 0.589 \). The two components of the di-quark condensate, the colour-flavour locked and the colour-flavour octet, and the corresponding magnetic field, that correspond to the solution with \( \rho = -\frac{1}{2} \) and \( \kappa = \kappa_c \) are plotted in Fig. 2. The numerical results are consistent with the inequalities (3.18) and (3.19) that we derived analytically.

Substituting the weak coupling expression of the parameters, the coherence length reads

\[
\xi^2 = \frac{7\zeta(3)}{48\pi^2} \left( \frac{1}{k_BT_c} \right)^2 \frac{T_c}{T_c - T}, \tag{3.25}
\]

and the penetration depth

\[
\delta^2 = \frac{3\pi}{2\alpha_s \mu^2} \frac{T_c}{T_c - T} \cos^2 \theta \tag{3.26}
\]

with \( \alpha_s = \frac{g^2}{4\pi} \). The Ginzburg-Landau parameter is given then by

\[
\kappa = \frac{\delta^2}{\xi^2} = \frac{72\pi^3}{7\zeta(3)\alpha_s} \frac{k_BT_c \cos \theta}{\mu} \geq \frac{16.3 k_BT_c}{\sqrt{\alpha_s} \mu}. \tag{3.27}
\]

If we consider a realistic value for the chemical potential, for example \( \mu = 400\text{MeV} \), it follows from (1.3) that the colour superconductor will be of type I if \( k_BT_c < 14\text{MeV} \) and of type II otherwise. Furthermore if we extrapolate the weak coupling formula for \( T_c \) [16], [17], [18], [19], [20], which is valid at asymptotic densities, to \( \mu = 400\text{MeV} \) we find \( k_BT_c = 3.5\text{MeV} \) and the colour superconductor is of type I. Of course if nonperturbative effects raise \( T_c \) significantly, for example by one order of magnitude, the colour superconductor could be of type II.
4. Concluding Remarks.

In this paper we have applied the Ginzburg-Landau theory to the problem of a domain wall which separates the normal phase from the colour superconducting phase of dense quark matter. The main purpose of this work was to establish a criterion that determines whether the colour superconductor is type I or type II with respect to an external magnetic field. We initially derived the Ginzburg-Landau equations of motion for the gauge fields and the order parameter by minimizing the surface energy of the interface between the two phases. The condition that the surface energy is a minimum is equivalent to the requirement that the normal and the colour superconducting phases are in stable equilibrium with each other. This equilibrium is maintained by an external magnetic field. Using symmetry arguments we were able to simplify the original set of equations and rewrite them in terms of only two components of the order parameter and one gauge potential. The equations also contain two dimensionless parameters: the Ginzburg-Landau parameter \( \kappa \) and \( \rho \). Therefore the problem is more complicated than the problem of the metallic superconductor which was addressed by Ginzburg and Landau. Although we were unable to derive an analytical result analogous to the Ginzburg-Landau criterion (1.1), we derived several rigorous inequalities about the critical parameter \( \kappa_c \), the value for which the surface energy vanishes, and its derivative with respect to \( \rho \). Those analytical results are supported by a numerical solution of the Ginzburg-Landau equations. By extrapolating the weak coupling approximation formulas for the parameters to realistic values of the chemical potential, for example \( \mu = 400\text{MeV} \), we find that the colour superconductor is of type I (type II) if the transition temperature is below (above) 14MeV.

The value of our work is mainly theoretical. It addresses the broken \( U(1) \) gauge field onto which the electromagnetic field has a small projection. Although we have been referring to the typical chemical potential of a neutron star, the implications of our work might be quite limited [14]. First of all, there has not been any evidence yet that colour superconductivity is realised in the core of a neutron star. Even if this is the case, the critical magnetic field (2.35) is too high for the distinction between type I and type II to be observed unless the pairing force is strong enough to reduce the lower critical field by several orders of magnitude. In addition the small mixing angle (2.34) and the dependence of the magnetic response on the thickness of the crust of the CSC core make it difficult to observe the partial magnetic Meissner effect and the appearance of vortex filaments.

Because of the strength of the QCD coupling and the ultra-relativistic Fermi sea, the fluctuations of the gauge field could be more significant than those in a non-relativistic superconductor [21]. These fluctuations contribute to the Ginzburg-Landau free energy functional an energy density

\[
\Delta \Gamma \sim k_B T_c \delta^{-3}.
\] (4.1)
As the condensate energy (2.16) is proportional to \((1 - \frac{T}{T_c})^2\) while \(\delta \sim \sqrt{1 - \frac{T}{T_c}}\), the contribution from the fluctuations will eventually dominate as \(T\) approaches \(T_c\) and modify the nature of the phase transition.

The fluctuation energy is estimated in the appendix within the framework of the Ginzburg-Landau theory, where we demonstrate that the inclusion of this energy modifies the phase transition from second order to first order. The parameter

\[
t = \frac{T}{T_c} - 1
\]  

which describes the deviation of the first order transition temperature \(T\) from the second order one \(T_c\), is used as a measure of the importance of the fluctuations. We find that for realistic values of the quark chemical potential, \(\mu = 400\text{MeV}\) and \(k_B T_c \geq 13\text{MeV}\), \(t \leq 0.1\). In that case, the Ginzburg-Landau free energy functional (2.2) represents a reasonable approximation.

For \(t \sim 1\) the approximation employed to derive the Ginzburg-Landau free energy (2.2) with (2.4) from QCD or from the Nambu-Jona-Lasinio effective action breaks down. Higher powers of the order parameter have to be included. It would be very interesting to develop a systematic weak coupling approximation in this case and we hope to be able to report progress in the future.

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6. Appendix.

In this appendix we shall estimate the effect of the fluctuations of the gauge field. A systematic evaluation of the contribution from the fluctuations to the free energy requires a resummation of a set of ring diagrams of QCD at finite temperature. Here we shall follow the treatment of Bailin and Love [21] and estimate the fluctuation terms within the framework of the Ginzburg-Landau theory. Due to the Debye screening the contributions of
the longitudinal components of the gauge potentials are suppressed, and only the transverse components contribute in the static limit. We expect that our simple-minded estimation captures the fluctuation terms to the leading order of the condensate.

Assuming that the small fluctuations are governed by the quadratic form of the Ginzburg-Landau free energy functional, the shift of the free energy due to the Meissner effect is given by

$$\Delta \Gamma = -k_B T \ln \int [dW][dZ] e^{-S} \int [dW][dZ] e^{-S_0}$$

(6.1)

where $\vec{W}$ and $\vec{Z}$ represent the transverse degrees of freedom and

$$S = \frac{\beta}{2} \int d^3 \vec{r} [ (\vec{\nabla} \times \vec{W})^2 + m_W^2 \vec{W}^2 + (\vec{\nabla} \times \vec{Z})^2 + m_Z^2 \vec{Z}^2 ]$$

$$S_0 = \frac{\beta}{2} \int d^3 \vec{r} [ (\vec{\nabla} \times \vec{W})^2 + (\vec{\nabla} \times \vec{Z})^2 ]$$

(6.2)

Completing the gaussian integral, we find that

$$\Delta \Gamma = k_B T \int \frac{d^3 \vec{k}}{(2\pi)^3} \left[ 7 \ln \left( 1 + \frac{m_W^2}{k^2} \right) + \ln \left( 1 + \frac{m_Z^2}{k^2} \right) \right]$$

$$= -\frac{k_B T C}{6\pi} (7\delta'^{-3} + \delta^{-3}) = -\frac{4g^3}{3\pi} (7 + \sec^2 \theta) k_B T_c |\phi|^3.$$ 

(6.3)

In the previous calculation, dimensional regularization was employed to eliminate ultraviolet divergences. By including this term to the homogeneous Ginzburg-Landau free energy with $\Phi = \phi$, we find

$$\Gamma = 12a|\phi|^2 + 12b|\phi|^4 - \frac{4g^3}{3\pi} (7 + \sec^2 \theta) k_B T_c |\phi|^3.$$ 

(6.4)

The presence of the cubic term will induce a first-order phase transition at $\bar{T} > T_c$. The temperature is determined by the condition that the free energy curve $\Gamma$ vs. $|\phi|$ becomes tangent to the $\phi$-axis at some $\phi \neq 0$, i.e.

$$a = \frac{16\pi \alpha^3}{81b} (7 + \sec^2 \theta)^2 (k_B T_C)^2.$$ 

(6.5)

Substituting the weak coupling expression for $a$ and $b$ we obtain that

$$t \equiv \frac{\bar{T}}{T_C} - 1 = \frac{49\epsilon^2(3)(7 + \sec^2 \theta)^2 \alpha^3}{139968\pi^5} \left( \frac{\mu}{k_B T_C} \right)^2.$$ 

(6.6)

With $\mu = 400\text{MeV}$, and $k_B T_C > 13\text{MeV}$, $t < 0.1$. As a result the fluctuations are not significant and the Ginzburg-Landau energy functional (2.2) represents a reasonable approximation.
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Figure 1. The critical Ginzburg-Landau parameter $\kappa_c$ vs. the parameter $\rho$.

Figure 2. The solutions to the Ginzburg-Landau equations (3.10) for $\rho = \frac{-1}{2}$ and $\kappa = 0.589$. The solid line represents the colour-flavour locked component, $\frac{1}{3}(u + 2v)$, the dashed line represents the colour-flavour octet component, $\frac{2}{\sqrt{3}}(v - u)$ and the dotted line the colour-magnetic field, $A'$. 