HEAT KERNEL ASYMPTOTICS, LOCAL INDEX THEOREM AND TRACE INTEGRALS FOR CR MANIFOLDS WITH $S^1$ ACTION

JIH-HSIN CHENG, CHIN-YU HSIAO, AND I-HSUN TSAI

Abstract. Among those transversally elliptic operators initiated by Atiyah and Singer, Kohn’s $\Box_b$ operator on CR manifolds with $S^1$ action is a natural one of geometric significance for complex analysts. Our first main result establishes an asymptotic expansion for the heat kernel of such an operator with values in its Fourier components, which involves an unprecedented contribution in terms of a distance function from lower dimensional strata of the $S^1$-action. Our second main result computes a local index density, in terms of tangential characteristic forms, on such manifolds including Sasakian manifolds of interest in String Theory, by showing that certain non-trivial contributions from strata in the heat kernel expansion will eventually cancel out by applying Getzler’s rescaling technique to off-diagonal estimates. This leads to a local result which can be thought of as a type of local index theorem on these CR manifolds. As applications of our CR index theorem we can prove a CR version of Grauert-Riemenschneider criterion, and produce many CR functions on a weakly pseudoconvex CR manifold with transversal $S^1$ action and many CR sections on some class of CR manifolds, answering (on this class of manifolds) some long-standing questions in several complex variables and CR geometry. We give examples of these CR manifolds, some of which arise from Brieskorn manifolds. Moreover in some cases, without use of equivariant cohomology method nor keeping contributions arising from lower dimensional strata as done in previous works, we can reinterpret Kawasaki’s Hirzebruch-Riemann-Roch formula for a complex orbifold with an orbifold holomorphic line bundle, as an index theorem obtained by a single integral over a smooth CR manifold which is essentially the circle bundle of this line bundle. By contrast, if one computes the trace integral (instead of supertrace as in the case of index theorems) of our heat kernel, then the contributions arising from the stratification of the $S^1$ action necessarily occur in a nontrivial way. Some explicit expressions about these corrections are obtained in this paper.

In short, besides certain applications our paper treats three major topics: i) an asymptotic expansion of a (transversal) heat kernel related to Kohn Laplacian (Theorem 1.3); ii) a formulation of a local CR index theorem for the case of locally free $S^1$ action (Corollary 1.13); iii) study of this heat kernel trace integral in terms of some explicit invariants as reflections upon the geometry of the $S^1$ stratification inside the CR manifold (Theorems 1.14, 7.19 and 7.23). Among the three topics, the first topic is foundational. The third topic focuses on the role of the Gaussian part of the heat kernel (which is boiled down to a Dirac type delta function on the $S^1$ stratification) while the second topic does mainly on the non-Gaussian part. Jointly, the three topics explore and integrate the separate aspects of this class of CR manifolds in our study.

Contents

1. Introduction and statement of the results
1.1. Introduction and Motivation
1.2. Main theorems
1.3. Applications
1.4. Kawasaki’s Hirzebruch-Riemann-Roch and Grauert-Riemenschneider criterion for orbifold line bundles
1.5. Examples
1.6. Proof of Theorem 1.2
1.7. The idea of the proofs of Theorem 1.3, Theorem 1.10 and Corollary 1.13
HEAT KERNEL ASYMPTOTICS, LOCAL INDEX THEOREM AND TRACE INTEGRALS FOR CR MANIFOLDS WITH $S^1$ ACTION

2. Preliminaries
2.1. Some standard notations
2.2. Set up and terminology
2.3. Tangential de Rham cohomology group, Tangential Chern character and Tangential Todd class
2.4. BRT trivializations and rigid geometric objects
3. A Hodge theory for $\Delta^{(q)}_{b,m}$
4. Modified Kohn Laplacian (Spin$^c$ Kohn Laplacian)
5. Asymptotic expansions for the heat kernels of the modified Kohn Laplacians
5.1. Heat kernels of the modified Kodaira Laplacians on BRT trivializations
5.2. Heat kernels of the modified Kohn Laplacians (Spin$^c$ Kohn Laplacians)

Part II: Proofs of main theorems
6. Proofs of Theorems 1.3 and 1.10
7. Trace integrals and proof of Theorem 1.14
7.1. A setup, including a comparison with recent developments
7.2. Local angular integral
7.3. Global angular integral
7.4. Patching up angular integrals over $X$; proof for the simple type
7.5. Types for $S^1$ stratifications; proof for the general type

References

1. INTRODUCTION AND STATEMENT OF THE RESULTS

1.1. Introduction and Motivation. Let $(X, T^{1,0} X)$ be a compact (with no boundary) CR manifold of dimension $2n + 1$ and let $\overline{\partial}_b : \Omega^{0,q}(X) \to \Omega^{0,q+1}(X)$ be the tangential Cauchy-Riemann operator. For a smooth function $u$, we say $u$ is CR if $\overline{\partial}_b u = 0$. In CR geometry, it is crucial to be able to produce many CR functions. When $X$ is strongly pseudoconvex and the dimension of $X$ is greater than or equal to five, it is well-known that the space of global smooth CR functions $H^0_{b}(X)$ is infinite dimensional. When $X$ is weakly pseudoconvex or the Levi form of $X$ has negative eigenvalues, the space of global CR functions could be trivial. In general, it is very difficult to determine when the space $H^0_{b}(X)$ is large.

A clue to the above phenomenon arises from the following. By $\overline{\partial}_b^2 = 0$, one has the $\overline{\partial}_b$-complex: $\cdots \to \Omega^{0,q-1}(X) \to \Omega^{0,q}(X) \to \Omega^{0,q+1}(X) \to \cdots$ and the Kohn-Rossi cohomology group: $H^j_{b}(X) := \ker \overline{\partial}_b : \Omega^{0,q}(X) \to \Omega^{0,q+1}(X) / \text{im} \overline{\partial}_b : \Omega^{0,q-1}(X) \to \Omega^{0,q}(X)$. As in complex geometry, to understand the space $H^0_{b}(X)$, one could try to establish, in the CR case, a Hirzebruch-Riemann-Roch theorem for $\sum_{j=0}^{n} (-1)^j \dim H^j_{b}(X)$, an analogue of the Euler characteristic, and to prove vanishing theorems for $H^j_{b}(X)$, $j \geq 1$. The first difficulty with
such an approach lies in the fact that $\dim H^j_b(X)$ could be infinite for some $j$. Let’s say more about this in the following.

If $X$ is strongly pseudoconvex and of dimension $\geq 5$, it is well-known that $\overline{\partial}_b : \Omega^{0,0}(X) \to \Omega^{0,1}(X)$ is not hypoelliptic and for $q \geq 1$, $\overline{\partial}_b : \Omega^{0,q}(X) \to \Omega^{0,q+1}(X)$ is hypoelliptic so that $\dim H^j_b(X) = \infty$ and $\dim H^q_b(X) < \infty$ for $q \geq 1$. In other cases if the Levi form of $X$ has exactly one negative, $n-1$ positive eigenvalues on $X$ and $n > 3$, it is well-known that $\dim H^1_b(X) = \infty$, $\dim H^{q-1}_b(X) = \infty$ and $\dim H^q_b(X) < \infty$, for $q \notin \{1, n-1\}$. With these possibly infinite dimensional spaces, we have the trouble with defining the Euler characteristic $\sum_{j=0}^n (-1)^j \dim H^j_b(X)$ properly.

Another line of thought lies in the fact that the space $H^q_b(X)$ is related to the Kohn Laplacian $\Box_b^{(q)} = \overline{\partial}_b \partial_b + \partial_b \overline{\partial}_b^* : \Omega^{0,q}(X) \to \Omega^{0,q}(X)$. One can try to define the associated heat operator $e^{-t\Box_b^{(q)}}$ by using spectral theory and then the small $t$ behavior of $e^{-t\Box_b^{(q)}}$ is closely related to the dimension of $H^q_b(X)$. Unfortunately without any Levi curvature assumption, $\Box_b^{(q)}$ is not hypoelliptic in general and it is unclear how to determine the small $t$ behavior of $e^{-t\Box_b^{(q)}}$. Even if $\Box_b^{(q)}$ is hypoelliptic, it is still difficult to calculate the local density.

We are led to ask the following question.

**Question 1.1.** Can we establish some kind of heat kernel asymptotic expansions for Kohn Laplacian and obtain a CR Hirzebruch-Riemann-Roch theorem (not necessarily the usual ones) on some class of CR manifolds?

It turns out that the class of CR manifolds with $S^1$ action is a natural choice for the above question. On this class of CR manifolds, the geometrical significance of Kohn’s $\Box_b$, operator in connection with transversally elliptic operators initiated by Atiyah and Singer [1], has been mentioned in the seminal work of Folland and Kohn ([33], p.113). Three dimensional (strongly pseudoconvex) CR manifolds with $S^1$ action have been intensively studied back to 1990s in relation to the CR embeddability problem. We refer the reader to the works [24] and [47] of Charles Epstein and Laszlo Lempert respectively. (see more comments on this in Section 1.3). Another related paper is about CR structure on Seifert manifolds by Kamishima and Tsuboi [43] (cf. Remark 1.15). In our present paper the CR manifold with $S^1$ action is not restricted to the three dimensional case.

To motivate our approach, let’s first look at the case which can be reduced to complex geometry. Consider a compact complex manifold $M$ of dimension $n$ and let $(L, h^L) \to M$ be a holomorphic line bundle over $M$, where $h^L$ denotes a Hermitian fiber metric of $L$. Let $(L^*, h^{L^*}) \to M$ be the dual bundle of $(L, h^L)$ and put $X = \{v \in L^*; |v|_{h^{L^*}}^2 = 1\}$. We call $X$ the circle bundle of $(L^*, h^{L^*})$. It is clear that $X$ is a compact CR manifold of dimension $2n + 1$. Clearly $X$ is equipped with a natural (globally free) $S^1$ action (by acting on the circular fiber).

Let $T \in C^\infty(X, TX)$ be the real vector field induced by the $S^1$ action, that is, $Tu = \frac{\partial}{\partial \theta}(u(e^{-i\theta} \circ x)))|_{\theta=0}, u \in C^\infty(X)$. This $S^1$ action is $CR$ and transversal, i.e. $[T, C^\infty(X, T^{1,0}X)] \subset C^\infty(X, T^{1,0}X)$ and $\langle CT(x) \circ T^1 \rangle X \oplus T^{0,1}_x X = CT_x X$ respectively. For each $m \in \mathbb{Z}$ and $q = 0, 1, 2, \ldots, n$, put

$$\Omega^{0,q}_m(X) := \{u \in \Omega^{0,q}(X); Tu = -imu\}$$

$$= \{u \in \Omega^{0,q}(X); u(e^{-i\theta} \circ x) = e^{-im\theta}u(x), \forall \theta \in [0, 2\pi]\}.$$

Since $\partial_b T = T \overline{\partial}_b$, we have $\overline{\partial}_b = \partial_b : \Omega^{0,q}_m(X) \to \Omega^{0,q+1}_m(X)$. We consider the cohomology group:

$$H^q_{b,m}(X) := \frac{\text{Ker} \partial_{b,m} : \Omega^{0,q}_m(X) \to \Omega^{0,q+1}_m(X)}{\text{Im} \partial_{b,m} : \Omega^{0,q-1}_m(X) \to \Omega^{0,q}_m(X)},$$

and call it the $m$-th $S^1$ Fourier component of the Kohn-Rossi cohomology group.

The following result can be viewed as the starting point of this paper. Note $\Omega^{0,q}(M, L^m)$ denotes the space of smooth sections of $(0,q)$ forms on $M$ with values in $L^m$ ($m$-th power of $L$) and $H^q(M, L^m)$ the $q$-th $\overline{\partial}$-Dolbeault cohomology group with values in $L^m$. 
Theorem 1.2. For every \( q = 0, 1, 2, \ldots, n \), and every \( m \in \mathbb{Z} \), there is a bijective map \( A_{m}^{(q)} : \Omega_{m}^{0,q}(X) \to \Omega_{m}^{0,q}(M, L^{m}) \) such that \( A_{m}^{(q+1)} \circ \partial_{b,m} = A_{m}^{(q)} \circ \partial_{b,m} \) on \( \Omega_{m}^{0,q}(X) \). Hence, \( \Omega_{m}^{0,q}(X) \cong \Omega_{m}^{0,q}(M, L^{m}) \) and \( H^{q}_{\partial_{b,m}}(X) \cong H^{q}(M, L^{m}) \). In particular \( \dim H^{q}_{\partial_{b,m}}(X) < \infty \) and \( \sum_{j=0}^{n} (-1)^{j} \dim H^{j}_{\partial_{b,m}}(X) = \sum_{j=0}^{n} (-1)^{j} \dim H^{j}(M, L^{m}) \).

Theorem 1.2 is probably known to the experts. As a precise reference is not easily available (see, however, Folland and Kohn [33] p.113), we will give a proof of Theorem 1.2 in Section 1.6 for the convenience of the reader.

In this paper by Kodaira Laplacian \( \square_{b,m}^{(q)} \) on \( L^{m} \)-valued \( (0, q) \) forms (on \( M \)) associated with the \( \overline{\partial} \) operator, a term we borrow from the work of Ma and Marinescu [48]. Let \( e^{-\square_{b,m}^{(q)}} \) be the associated heat operator. It is well-known that \( e^{-\square_{b,m}^{(q)}} \) admits an asymptotic expansion as \( t \to 0^{+} \). Consider \( B_{m}(t) := (A_{m}^{(q)})^{-1} \circ e^{-\square_{b,m}^{(q)}} \circ A_{m}^{(q)} \). (See (1.18) in Theorem 1.3 for details and for our first main result.) It should be emphasized that no pseudoconvexity condition is assumed. Rather, our asymptotic expansion involves an unprecedented contribution in terms of a distance function \( \hat{d} \) acting on \( \Omega_{b}^{0,q}(X) \).

One goal of this work is to establish an asymptotic expansion of the form similar to (1.2). However, to define those Laplacians above an appropriate choice of metrics is needed (for adjoint of an operator) so that \( A_{m}^{(q)} \) of Theorem 1.2 also preserves the chosen metrics. With this set up it is fundamental that

\[
(1.1) \quad e^{-\square_{b,m}^{(q)}} = \left( (A_{m}^{(q)})^{-1} \circ e^{-\square_{b,m}^{(q)}} \circ A_{m}^{(q)} \right) \circ Q_{m} = B_{m}(t) \circ Q_{m} = Q_{m} \circ B_{m}(t) \circ Q_{m},
\]

where \( Q_{m} : \Omega_{m}^{0,q}(X) \to \Omega_{m}^{0,q}(X) \) is the orthogonal projection. Hence the asymptotic expansion of \( e^{-\square_{b,m}^{(q)}} \) and (1.1) lead to an asymptotic expansion

\[
(1.2) \quad e^{-\square_{b,m}^{(q)}}(x, x) \sim t^{-n}a_{n}^{(q)}(x) + t^{-n+1}a_{n-1}^{(q)}(x) + \cdots.
\]

One goal of this work is to establish a formula similar to (1.2) (which is however not exactly of this form) on any CR manifold with \( S^{1} \) action. More precisely, due to the assumption that the \( S^{1} \) action is only locally free, it turns out that \( e^{-\square_{b,m}^{(q)}}(x, x) \) cannot have the standard asymptotic expansion as (1.2). Rather, our asymptotic expansion involves an unprecedented contribution in terms of a distance function from lower dimensional strata of the \( S^{1} \) action. (See (1.18) in Theorem 1.3 for details and for our first main result.) It should be emphasized that no pseudoconvexity condition is assumed.

Roughly speaking, on the regular part of \( X \) we have

\[
(1.3) \quad e^{-\square_{b,m}^{(q)}}(x, x) \sim t^{-n}a_{n}^{(q)}(x) + t^{-n+1}a_{n-1}^{(q)}(x) + \cdots \mod O\left(t^{-n}e^{-\frac{\hat{d}^{2}(x, X_{a_{n}})}{1}}\right).
\]

On the whole \( X \) we have, however,

\[
(1.4) \quad e^{-\square_{b,m}^{(q)}}(x, x) \sim t^{-n}A_{n}^{(q)}(t, x) + t^{-n+1}A_{n-1}^{(q)}(t, x) + \cdots.
\]

The difference between (1.4) and (1.3) lies in that \( A_{s}^{(q)}(t, x) \) in (1.4) cannot be \( t \)-independent for all \( s \) and are not canonically determined (by our method) while \( a_{s}^{(q)}(x) \) in (1.3) are \( t \)-independent for all \( s \) and are canonically determined. This \( t \)-dependence nature so introduced presents a great distinction between our asymptotic expansion and those in the previous literature, and this distinct point of departure appears to have a big influence on the formulation and proof of the relevant index theorems and trace integrals. See Section 7 for more comments.

In addition to the introduction of a distance function \( \hat{d} \) in (1.3) our generalization has another feature, which is pertinent to the third topic of this paper, as follows. A heat kernel result for orbifolds obtained in 2008 by Dryden, Gordon, Greenwald and Webb for the case of the Laplacian on functions (see (1.30) and [19]) and independently by Richardson ([56]) seems to suggest that integrating (1.3) over \( X \) is basically a power series in \( t^{\frac{1}{2}} \). See (1.30) for more. To see such a possible connection, one
considers \( X \) as a fiber space over \( X/S^1 \) which is then an orbifold, and presumes boldly an analogy with “(1.2) for the orbifold case”. Then by the above result [19], integrating (1.3) over \( X \) might give an asymptotic expansion which is a power series at most in the fractional power \( t^{\frac{1}{2}} \) of \( t \) (cf. Theorem 1.14) (while for the case where the \( S^1 \) action is globally free, such as in the circle bundle above, the asymptotic expansion is expressed in the integral power of \( t \)). However, our further study shows that the coefficients of \( t^j \) for \( j \) being half-integral necessarily vanish in our present case (irrespective of the local or global freeness of the \( S^1 \) action). Despite that there is no nontrivial fractional power in the \( t \)-expansion, the corrections/contributions associated with the stratification of the locally free \( S^1 \) action do arise nontrivially in a proper sense. Some explicit computations about these extra terms are worked out in the main result of the final section (Section 7) regarded as the third topic of this paper.

As far as the asymptotic expansion is concerned, we remark that the approach of using Kodaira Laplacian on \( M \) (downstairs) as done above is no longer applicable to the general CR case, as the contribution of a distance function on \( X \) involved in our expansion cannot be easily forseen by use of objects in the space downstairs (an orbifold in general). (However, for trace integrals on invariant functions, cf. Section 7, like \( \sum_m e^{-\lambda_m} \) denoted by \( I(t) \) in certain Riemannian cases, \( I(t) \) has been studied asymptotically with the help of the underlying/quotient manifold/orbifold, cf. [56, p. 2316-2317]. See also Proposition 5.1, Remark 5.3.) We must work on the entire \( X \) from scratch with the operator being only transversally elliptic (on \( X \)). (See HRR theorem below for another instance of this idea.) Furthermore, as we make no assumption on (strong) pseudoconvexity of \( X \), this renders the techniques usually useful in this direction by previous work (e.g. [3]) hardly adequate in our case. Our current approach is essentially independent of the previous methods. This technicality partly accounts for the length of the present paper (see Section 1.7 for an outline of proof and Section 7 for a comparison with the previous work).

We expect that the coefficients \( a^{(q)}_j(x) \) in (1.3) are related to some geometric quantities. For \( q = 0 \), function case with strong pseudoconvexity, we refer the reader to the paper of Beals, Greiner, and Stanton [3]. In this regard, Chern-Moser invariants (see [16]) or Tanaka-Webster invariants (see [58] or [60]) should be used to express these coefficients. In our present situation (without assumptions on pseudoconvexity) it is however more natural to use geometric quantities adapted to the \( S^1 \) invariance property, so that a notion of tangential curvature arises (with the associated tangential characteristic forms, cf. Section 2.3) and enters into the coefficients of our asymptotic expansion. It essentially comes back to the Tanaka-Webster curvature in the strongly pseudoconvex case (cf. Remark 1.9).

The mathematics (existence, asymptotics etc.) of equivariant/transversal heat kernels in the Riemannian situation (including that of Riemannian foliations) have been studied in recent years and last decades. For a comparison between these developments and our results, we postpone the survey, together with that of trace integrals, until Section 7.

Back to the special case of the circle bundle \( X \) over a compact complex manifold \( M \), the Hirzebruch-Riemann-Roch Theorem or Atiyah-Singer index Theorem, together with Theorem 1.2, tells us that

\[
\sum_{j=0}^{n} (-1)^j \dim H^j_{b,m}(X) = \sum_{j=0}^{n} (-1)^j \dim H^j(M, L^m) = \int_M Td(T^{1.0}M)ch(L^m),
\]

in terms of standard characteristic classes on \( M \). Let’s reformulate (1.5) in geometric terms on \( X \) rather than on \( M \):

\[
\sum_{j=0}^{n} (-1)^j \dim H^j_{b,m}(X) = \frac{1}{2\pi} \int_X Td_h(T^{1.0}X) \wedge e^{-m \frac{d\omega_0}{\pi}} \wedge \omega_0,
\]

where \( Td_h(T^{1.0}X) \) denotes the tangential Todd class of \( T^{1.0}X \) and \( e^{-m \frac{d\omega_0}{\pi}} \) denotes the Chern polynomial of the Levi curvature and \( \omega_0 \) is the uniquely determined global real 1-form (see Section 2.2 and Section 2.3 for the precise definitions).
Our second main result turns to any (abstract) CR manifold $X$ with (locally free) $S^1$ action (but with no assumption on pseudoconvexity); we see that the above Euler characteristic has an index interpretation related to $\partial_b + \partial_b^*$ on $X$ (see (3.12) and (3.13)). We are able to establish (1.6) (cf. Corollary 1.13) on such $X$ based on our asymptotic expansion for the heat kernel $e^{-\frac{1}{t} \Box (q)} b,m(x,x)$ and a type of McKean-Singer formula on $X$ (see Corollaries 4.8 and 5.15).

As an application to complex (orbifold) geometry, it is worth noting a comparison between the present result and a result of Kawasaki on Hirzebruch-Riemann-Roch theorem (HRR theorem for short) on a complex orbifold $N$ ((45)) (which plays the role as our $M$ above). Our formula simplifies in the sense that in the original formula of Kawasaki, his part involving the dependence on the (lower dimensional) strata of the orbifold $M$ entirely drops out here, at least for the class of orbifolds and orbifold line bundles that fit into our assumption (see Theorem 1.27, remarks following it and Subsection 1.5.2 for examples). In our view this simplification does not appear obvious at all within the original approach of Kawasaki because by his approach the contributions from the (lower dimensional) strata of the orbifold cannot be avoided (unless it is proved to be vanishing) even if the total space of the (orbifold) circle bundle is smooth. Conceptually speaking one may attribute such a simplification to one’s working on the entire (smooth) $X$ rather than on the downstairs $M$ (as Kawasaki), a strategy already employed for the asymptotic expansion above and proving useful again in this context of (CR) index theorem. We remark that the vanishing of the contribution of strata also occurs in a related context studied by these works [54], [32] (see also discussions after Theorem 1.27).

In short our second main result (Theorem 1.10) computes a local index density in terms of tangential characteristic forms, which is to show that certain non-trivial contributions (cf. $t^{-n} e^{-\frac{1}{t} \Box (q)} b,m(x,x)$ of (1.3)) in the heat kernel expansion will eventually cancel out in the index density computation. We can do this by applying Getzler’s rescaling technique to the off-diagonal estimate (not needed in the classical index theorems). As, to the best of our knowledge, an appropriate term for such a result about the local density hasn’t appeared in the literature yet, we shall follow the classical cases and call it a local index theorem on these CR manifolds (Corollary 1.13), including Sasakian manifolds of interest in String Theory.

With reference to the questions in the beginning of this Introduction, for further application of our results to CR geometry it is important to produce many CR functions or CR sections. Namely we hope to know when $H^0_b(X,E)$ or $H^0_{b,m}(X,E)$ is large (see Questions 1.17, 1.18 and 1.22 in Section 1.3). Progress towards this circle of questions seems limited (Section 1.3). We can now develop a tool for tackling some of these questions. The idea here is to combine our version of CR index theorem with a sort of vanishing theorem for higher cohomology groups, which is intimately related to a version of Grauert-Riemenschneider criterion adapted to the CR case. This methodology turns out to be effective for those CR manifolds studied in this paper.

In Section 1.3 we apply our CR index theorem to prove a CR version of Grauert-Riemenschneider criterion, and produce many CR functions on a weakly pseudoconvex CR manifold with transversal $S^1$ action and many CR sections on some class of CR manifolds, which give answers to some long-standing questions in several complex variables and CR geometry. In Section 1.5 we provide an abundance of examples of those CR manifolds studied in the present paper, some of which arise from Brieskorn manifolds (generalized Hopf manifolds).

There is another index theory of geometric significance, developed by Charles Epstein. He studied the so called relative index of a pair of embeddable CR structures through their Szegő projectors in a series of papers (see [25], [26], [27], [28] and [29]). On the other hand, Erik van Erp derived an index formula for subelliptic operators on a contact manifold (see [30], [31]). Moreover, recent work of Paradan and Vergne [54] gave an expression for the index of transversally elliptic operators which is an integral of compactly supported equivariant form on the cotangent bundle; see also Fitzpatrick [32] for
related directions. Brüning, Kamber and Richardson [11, 12] computed the index of an equivariant transversally elliptic operator as a sum of integrals over blow ups of the strata of the group action.

Finally it is natural to ask for a generalization from the action of $S^1$ to that of other Lie groups or even to foliations (cf. Subsection 7.1 for references). As we will discuss in Section 7, the asymptotic expansion in the form (1.4) as indicated there a sort of remedy for (1.3) by involving a “distance function” $\hat{d}$ seems to be best illustrated in the $S^1$ case. It appears also conceivable that these features shall be preserved for generalization in a certain (as yet unknown) way. This paper may be presented or read in token of a prototype for further study into much more complicated, diversified situations.

1.2. Main theorems. We shall now formulate the main results. We refer to Section 2.2 and Section 2.3 for some notations and terminologies used here. After the background material, we will discuss in the sequel i) asymptotic expansions, ii) a local index theorem and iii) trace integrals.

1.2.1. Background. Let $(X, T^{1,0}X)$ be a compact connected CR manifold with a transversal CR locally free $S^1$ action $e^{-i\theta}$, where $T^{1,0}X$ is a CR structure of $X$. $X$ is of dimension $2n + 1$ throughout this paper.

Let $T \in C^\infty(X, TX)$ be the real vector field induced by the $S^1$ action and let $\omega_0 \in C^\infty(X, T^*X)$ be the global real one form determined by $\langle \omega_0, T \rangle = 1$, $\langle \omega_0, u \rangle = 0$, for every $u \in T^{1,0}X \oplus T^{0,1}X$.

Associated with the $S^1$ action of $X$ it is natural to consider various geometric objects admitting an $S^1$ action. In the following, to streamline the exposition we shall freely use the notion of rigid objects: “rigid bundles”, “rigid metrics” etc., and refer to Definitions 2.3, 2.4 and 2.5 for the precise meanings. (See also the work of Baouendi-Rothschild-Treves [4, Definition II.2] for a similar use of this term.) It suffices to say here that this notion of rigid objects is nothing but an equivalent way (by using metric) to consider objects (originally defined without assumption on metric) which admit (compatible) $S^1$ actions (or $S^1$ invariance, subject to the proper context) provided one starts with a CR manifold with an $S^1$ action (cf. Theorem 2.11).

Henceforth let $E$ be a rigid CR vector bundle over $X$, equipped with a rigid Hermitian metric $\langle \cdot, \cdot \rangle_E$. We note that $T^{1,0}X$ is known to be a rigid complex vector bundle (see the work of Baouendi-Rothschild-Treves [4]) with a rigid Hermitian metric $\langle \cdot, \cdot \rangle$ satisfying extra properties (not specified here, cf. (2.5)). Let $\langle \cdot, \cdot \rangle_E$ be the Hermitian metric on $T^{1,0}X \otimes E$ induced by those on $E$ and $\mathbb{C}TX$. Denoting by $dv_X = dv_X(x)$ the volume form on $X$ induced by the Hermitian metric $\langle \cdot, \cdot \rangle$ on $\mathbb{C}TX$ we get the natural global $L^2$ inner product $\langle \cdot, \cdot \rangle$ on $\Omega^{0,\bullet}(X, E)$.

As remarked in Introduction, for $u \in \Omega^{0,\bullet}(X, E), Tu \in \Omega^{0,\bullet}(X, E)$ is defined and $T\overline{\partial}_b = \overline{\partial}_b T$. For $m \in \mathbb{Z}$, put

$$\Omega^0_m(X, E) := \{ u \in \Omega^{0,\bullet}(X, E); Tu = -i mu \}$$

$$= \{ u \in \Omega^{0,\bullet}(X, E); (e^{-i\theta})^*u = e^{-im\theta}u, \quad \forall \theta \in [0, 2\pi[ \} ,$$

where $(e^{-i\theta})^*$ denotes the pull-back by the map $e^{-i\theta} : X \to X$ of $S^1$ action.

Write $L^2(X, T^{1,0}X \otimes E)$ (resp. $L^2_m(X, T^{1,0}X \otimes E)$) for the $L^2$-completion of $\Omega^{0,\bullet}(X, E)$ (resp. $\Omega^0_m(X, E)$) with respect to $\langle \cdot, \cdot \rangle_E$.

By $T\overline{\partial}_b = \overline{\partial}_b T$ one defines $\overline{\partial}_{b,m} : \Omega^0_m(X, E) \to \Omega^0_m(X, E)$ as the restriction of $\overline{\partial}_b$ on $\Omega^0_m$. Write

$$\overline{\partial}_b : \Omega^{0,\bullet}(X, E) \to \Omega^{1,\bullet}(X, E), \quad \text{resp.} \quad \overline{\partial}_{b,m} : \Omega^0_m(X, E) \to \Omega^0_m(X, E)$$

for the formal adjoint of $\overline{\partial}_b$ (under $\langle \cdot, \cdot \rangle_E$, $\langle \cdot, \cdot \rangle_E$ are rigid, one sees

\begin{equation}
T\overline{\partial}_b = \overline{\partial}_b^* T \quad \text{on} \quad \Omega^{0,\bullet}(X, E),
\end{equation}

$$\overline{\partial}_{b,m} = \overline{\partial}_{b,m}^* : \Omega^0_m(X, E) \to \Omega^0_m(X, E), \quad \forall m \in \mathbb{Z}.$$
Let $A_m : \Omega^0_m(X,E) \rightarrow \Omega^0_m(X,E)$ be a certain smooth zeroth order operator with $TA_m = A_m T$ and $A_m : \Omega^0_m(\bar{X},E) \rightarrow \Omega^0_m(\bar{X},E)$ (arising from a CR version of Spin$^c$ Dirac operator, cf. Definition 4.3).

Put
\[
\tilde{D}_{b,m} := \partial_{b,m} + \partial_{b,m}^* + A_m : \Omega^0(X,E) \rightarrow \Omega^0(X,E)
\]
and let
\[
\tilde{D}_{b,m}^* : \Omega^0(X,E) \rightarrow \Omega^0(X,E)
\]
be the formal adjoint of $\tilde{D}_{b,m}$ (with respect to $(\cdot|\cdot)_E$).

We have $\Box_{b,m}$, given by
\[
\Box_{b,m} := \tilde{D}_{b,m} \tilde{D}_{b,m} : \Omega^0(X,E) \rightarrow \Omega^0(X,E)
\]
which denotes the $m$-th modified Kohn Laplacian, thought of as Spin$^c$ Kohn Laplacian (cf. Definition 4.3 and the paragraph below it). We extend $\Box_{b,m}$ by
\[
\Box_{b,m} : \text{Dom} \; \Box_{b,m} \subset L^2_m(X,T^{0\cdot}X \otimes E) \rightarrow L^2_m(X,T^{0\cdot}X \otimes E),
\]
with $\text{Dom} \; \Box_{b,m} := \{ u \in L^2_m(X,T^{0\cdot}X \otimes E); \Box_{b,m} u \in L^2_m(X,T^{0\cdot}X \otimes E) \}$ in which $\Box_{b,m} u$ is defined in the sense of distribution.

We will show in Section 3 that $\Box_{b,m}$ is self-adjoint, Spec$\Box_{b,m}$ is a discrete subset of $[0,\infty]$ and for $\nu \in \text{Spec} \Box_{b,m}$, $\nu$ is an eigenvalue of $\Box_{b,m}$ with finite multiplicities $d_\nu < \infty$. Let $\{ f^1_1, \ldots, f^d_1 \}$ be an orthonormal frame for the eigenspace of $\Box_{b,m}$ with eigenvalue $\nu$. The (smooth) heat kernel $e^{-\Box_{b,m}(x,y)}$ can be given by
\[
e^{-\Box_{b,m}(x,y)} = \sum_{\nu \in \text{Spec} \Box_{b,m}} \sum_{j=1}^{d_\nu} e^{-\nu t} f^\nu_j(x) \wedge (f^\nu_j(y))^\dagger,
\]
where $f^\nu_j(x) \wedge (f^\nu_j(y))^\dagger$ denotes the linear map:
\[
f^\nu_j(x) \wedge (f^\nu_j(y))^\dagger : T^{0\cdot}_x X \otimes E_x \rightarrow T^{0\cdot}_x X \otimes E_x,
\]
\[
u(y) \in T^{0\cdot}_y X \otimes E_y \rightarrow \langle u(y) \mid f^\nu_j(y) \rangle E \in T^{0\cdot}_x X \otimes E_x.
\]

Let $e_1(x), \ldots, e_d(x)$ be an orthonormal frame of $T^{0\cdot}X \otimes E_x$, and $A \in \text{End}(T^{0\cdot}_x X \otimes E_x)$. Put $\text{Tr}^{(q)} A := \sum_{j=1}^d \langle Af_j \mid f_j \rangle E$ and set
\[
\text{Tr} A := \sum_{j=0}^n \text{Tr}^{(j)} A,
\]
\[
\text{STr} A := \sum_{j=0}^n (-1)^j \text{Tr}^{(j)} A.
\]

Let $\nabla^{TX}$ be the Levi-Civita connection on $TX$ (with respect to $(\cdot|\cdot)$). Then $T^{1\cdot}X$ is equipped with a connection $\nabla^{T^{1\cdot}X} := P_{T^{1\cdot}X} \nabla^{TX}$ where $P_{T^{1\cdot}X}$ be the projection from $\mathbb{C}TX$ onto $T^{1\cdot}X$.

Let $\nabla^E$ be the connection on $E$ induced by $(\cdot|\cdot)_E$ (see Theorem 2.12). Let $\text{TD}_b (\nabla^{T^{1\cdot}X}, T^{1\cdot}X)$ denote the representative of the tangential Todd class of $T^{1\cdot}X$ and $\text{ch}_b(\nabla^E, E)$ the representative of the tangential Chern character of $E$ (see Section 2.3 for tangential classes).

In what follows we aim to define a distance function $d$ which plays an important role (for the asymptotic expansion) in this paper. For $x \in X$, we say that the period of $x$ is $\frac{2\pi \ell}{\ell}$, $\ell \in \mathbb{N}$ provided that
\[e^{-i\theta} \circ x \neq x \text{ for every } 0 < \theta < \frac{2\pi}{p} \text{ and } e^{-i2\pi} \circ x = x.\] Put, for each \(\ell \in \mathbb{N}\),

\begin{equation}
X_\ell = \{ x \in X; \text{ the period of } x \text{ is } \frac{2\pi}{\ell} \}
\end{equation}

and let \(p = \min \{\ell \in \mathbb{N}; X_\ell \neq \emptyset\}\). We call \(X_p = X_{p_1}\) the principal stratum. It is well-known that if \(X\) is connected, then \(X_p\) is an open and dense subset of \(X\) (see Proposition 1.24 in Meinrenken [50] and Duistermaat-Heckman [23]). Assume \(X = X_{p_1} \cup X_{p_2} \cup \cdots \cup X_{p_k}, p = p_1 < p_2 < \cdots < p_k\). Put \(X_{\text{sing}} = X_{\text{sing}}^1 := \bigcup_{j=2}^k X_{p_j}, X_{\text{sing}}^r := \bigcup_{j=r+1}^k X_{p_j}, k \geq 1, r \geq 1\). Set \(X_{\text{sing}}^k := \emptyset\). Note \(p_1 | p_j\) for \(1 \leq j \leq k\) (cf. Remark 1.16).

Let \(d(\cdot, \cdot)\) denotes the standard Riemannian distance with respect to the given Hermitian distance. Take \(\zeta\)

\begin{equation}
0 < \zeta < \inf \left\{ \frac{2\pi}{pk}, \frac{2\pi}{pr} - \frac{2\pi}{pr+1}; r = 1, \ldots, k - 1 \right\}.
\end{equation}

Set, for \(x \in X\) and \(r = 1, 2, \ldots, k\),

\begin{equation}
\hat{d}_\zeta(x, X_{\text{sing}}^r) := \inf \left\{ d(x, e^{-i\theta} x); \zeta \leq \theta \leq \frac{2\pi}{pr} - \zeta \right\}.
\end{equation}

This notation reflects the fact that \(\hat{d}_\zeta(x, X_{\text{sing}}^r)\) is equivalent to the ordinary distance \(d(x, X_{\text{sing}}^r)\) (see below). Note by definition \(\hat{d}_\zeta(x, X_{\text{sing}}^k) = \hat{d}_\zeta(x, \emptyset) = 0\) for all \(x \in X\). We remark that for any \(0 < \zeta, \zeta_1\) satisfying (1.15), \(\hat{d}_\zeta(x, X_{\text{sing}}^r)\) and \(d_{\zeta_1}(x, X_{\text{sing}}^r)\) are equivalent (as far as the estimate in Theorem 1.3 below is concerned). We shall denote \(d(x, X_{\text{sing}}^r) := \hat{d}_\zeta(x, X_{\text{sing}}^r)\).

Remark that, by examining the definition \(d(x, X_{\text{sing}}^r) = 0\) if and only if \(x \in X_{\text{sing}}^r\). Further, for \(\varepsilon > 0\) there is a \(\delta > 0\) such that \(\hat{d}(x, X_{\text{sing}}^r) \geq \delta\) provided \(x \in X\) satisfies (the ordinary distance) \(d(x, X_{\text{sing}}^r) \geq \varepsilon\). It is thus convenient to think of \(\hat{d}(x, X_{\text{sing}}^r)\) as a distance function from \(x\) to \(X_{\text{sing}}^r\).

Indeed in Theorem 6.7 for a strongly pseudoconvex \(X\) there is a constant \(C \geq 1\) such that

\[\frac{1}{C} d(x, X_{\text{sing}}^r) \leq \hat{d}(x, X_{\text{sing}}^r) \leq C d(x, X_{\text{sing}}^r), \forall x \in X.\]

**1.2.2. Asymptotic expansion of the heat kernel \(e^{-t\Box_{M,\alpha}}(x, x)\).** With the distance function \(\hat{d}\) we state the first main result of this paper (see Section 6 for a proof).

**Theorem 1.3.** Suppose \((X, T^{1,0}X)\) is a compact, connected CR manifold (of dimension \(2n + 1\)) with a transversal CR locally free \(S^1\) action. With the notations above, there exist \(a_s(t, x) (= a_{s, m}(t, x)) \in C^\infty(\mathbb{R}_+ \times X, \text{End}(T^* X \otimes E))\) with \(|a_s(t, x)| \leq C\) on \(\mathbb{R}_+ \times X\) where \(C > 0\) is independent of \(t, s = n, n - 1, \ldots\), such that

\begin{equation}
e^{-t\Box_{M,\alpha}}(x, x) \sim t^{-n} a_n(t, x) + t^{-n+1} a_{n-1}(t, x) + \cdots \text{ as } t \to 0^+.
\end{equation}

(See Definition 5.4 for “\(\sim\).”)

Moreover, there exist \(a_s(x) (= a_{s, m}(x)) \in C^\infty(X, \text{End}(T^* X \otimes E)), s = n, n - 1, \ldots\), satisfying the following property. Given any differential operator \(P_t : C^\infty(X, T^* X \otimes E) \to C^\infty(X, T^* X \otimes E)\) of order \(\ell \in \mathbb{N}_0\), there exist \(\varepsilon_0 > 0\) and \(C_0 > 0\) such that

\begin{equation}
\left| P_t \left( a_s(x) - \left( \sum_{s=1}^{p_r} e^{2\pi(i-1)/pr} m_{ij} a_s(x) \right) \right) \right| \leq C_0 t^{-\ell/2} e^{-\varepsilon_0 d(x, X_{\text{sing}}^r)^2}, \forall t \in \mathbb{R}_+, \forall x \in X_{p_r},
\end{equation}

\(r = 1, \ldots, k.\)

The following is immediate from the proof of Theorem 1.3.
Corollary 1.4. Suppose \((X, T^{1,0} X)\) is a compact, connected CR manifold with a transversal CR locally free \(S^1\) action. With the notations above, for any \(r = 1, \ldots, k\), any differential operator \(P_r : C^\infty(X, T^{*,*} X \otimes E) \to C^\infty(X, T^{*,*} X \otimes E)\) of order \(\ell \in \mathbb{N}_0\), every \(N_0 \in \mathbb{N}\) with \(N_0 \geq N_0(n)\) for some \(N_0(n)\), there are \(\varepsilon_0 > 0\), \(\delta > 0\) and \(C_{N_0} > 0\) such that

\[
\left| P_r \left( e^{-t\tilde{h}_{b,m}}(x, x) - \sum_{s=1}^{p_r} e^{2s(x-1)} \sum_{j=0}^{N_0} t^{-n+j} \alpha_{n-j}(x) \right) \right| \\
\leq C_{N_0} \left( t^{-n+N_0+1-\frac{r}{2}} + t^{-n-\frac{r}{2}} e^{-\frac{\varepsilon_0|x|^2}{4}} \right), \forall x \in X_{p_r}, \forall 0 < t < \delta.
\]

Remark 1.5. Let \(\sum_{s=1}^{p_r} e^{2(x-1)} \sum_{j=0}^{N_0} t^{-n+j} \alpha_{n-j}(x) = 0\) if \(p_r \parallel m\).

Remark 1.6. We shall now see that if one wants an asymptotic expansion of \(e^{-t\tilde{h}_{b,m}}(x, x)\) to be valid around each \(x \in X\) (cf. Definition 5.4), then (1.17) is basically optimal (i.e. in general, \(a_s(x, t, y)\) cannot be \(t\)-independent for all \(s\)). For \(U \subset X_{p_1}\), a tradition-like formula (assuming \(p_1 = 1\) for simplicity)

\[
e^{-t\tilde{h}_{b,m}}(x, x) \sim C(t^{-n} \alpha_n(x) + t^{-n+1} \alpha_{n-1}(x) + \cdots)
\]

is valid for \(x \in U\) and \(C = 1\) (as follows from (1.18) for \(l = 0\)) whereas for \(x \in X_{p_r}, r \geq 2\), an asymptotic expansion (for \(p_r \parallel m\)) with \(C = p_r\) is valid around an open subset (\(\exists x\)) of the stratum \(X_{p_r}\). Since \(e^{-t\tilde{h}_{b,m}}(x, y)\) is going to be a well defined smooth kernel, it is easily seen that those functions \(\alpha_{n+r}(x)\) \((s = n, n-1, \cdots)\) satisfying Theorem 1.3 are unique (if they exist). We notice that \(a_s(x, t, x)\) in (1.17) are not canonically defined by our method which is subject to choice of BRT trivializations, cf. (5.40) and Subsection 2.4.) In short, the above suggests that an asymptotic expansion of the form as (1.20) can only be true in the piecewise sense with respect to strata. See also Subsection 7.1.

To confirm this, one uses Theorem 1.14 (see Theorems 7.19 and 7.23 for a more precise version) by noting

\[
\int_X \text{Tr} \alpha_s^+(x) d\nu_X(x) = S_{1,s}^+ \text{ in Theorem 7.19 (which is taking the “even” part of the Laplacian.)}
\]

Hence one can interpret the trace integral result (obtained by integrating \(\text{Tr} e^{-t\tilde{h}_{b,m}}(x, x)\) over \(X\)) as one that gives extra nonzero correction terms, cf. the second line in (1.32) or the third line in (7.50).

It follows that if there exists a global asymptotic expansion (not just in the piecewise sense) such as (1.17), then not all of \(a_s(x, t, x)\) can be independent of \(t\). Otherwise, if all \(a_s(x, t, x)\) are independent of \(t\), it would be of the form (1.20) globally by assumption (\(C = 1\) if \(p_1 = 1\)), so by integrating the trace over \(X\), there would be no correction terms as discussed above. To say more, \(e^{-t\tilde{h}_{b,m}}(x, x)\) cannot have any asymptotic expansion of the form

\[
t^{m_1} \beta_{m_1}(x) + t^{m_2} \beta_{m_2}(x) + \cdots \text{ (globally) } m_1 < m_2, \cdots \in \mathbb{R}, \beta_{m_1}(x), \beta_{m_2}(x), \cdots \text{ continuous functions on } X.
\]

Otherwise by equating it to (1.17), each \(a_s(x, t, x)\) would be rendered independent of \(t\), absurd as just remarked (see the next remark for argument independent of Theorems 1.14, 7.19).

The next remark shows that \(a_s(x, t, x)\) for the particular \(s = n\) must be dependent on \(t\) (nontrivially). This part will not use Theorem 1.14.

Remark 1.7. In the above remark a certain discontinuity in the form (1.20) for, say \(x \in X_{p_1}\) and \(x \in X_{p_2}\) seems to appear. We shall now explore it. If the (Gaussian-like) term to the right of (1.18) is examined, it arises from a precise integral (see (6.8)). To show that this integral is generally nontrivial, regardless of whether our estimate given by (1.18) is a fine or crude one, we are actually going to show that the term for \(s = n\) in (1.18)

\[
a_n(t, x) - \sum_{s=1}^{p_r} e^{2s(x-1)} \sum_{j=0}^{N_0} t^{-n+j} \alpha_{n-j}(x)
\]
is nontrivial. For the sake of illustration we assume that $X = X_1 \cup X_2$, that is $p_1 = 1$ and $p_2 = 2$, and take $m$ to be an even number. For $x \in X_1$, by (1.18) (for $l=0$) and $p_1 = 1$, we see that $a_n(t, x) = \alpha_n(x) + r_n(t, x)$

$$|r_n(t, x)| \lesssim e^{-\frac{c_0 d(x, x_{\text{sing}})^2}{t}}. \tag{1.21}$$

As our $\alpha_n(x)$ essentially arises from a local Kodaira Laplacian (see (6.1), similar to discussion after Theorem 1.2), it is well known that $\alpha_n(x)$, as the coefficient of the leading term (in the $t$-expansion of the heat kernel for Kodaira Laplacian), is constant in $x$ with $\text{Tr} \alpha_n > 0$ (cf. [36, Lemma 4.1.4 and Section 4.4]). By continuity ($a_s$ and $\alpha_s$ being globally continuous functions)

$$a_n(t, x) = \alpha_n(x) + r_n(t, x) \tag{1.22}$$

remains true on $X_2$. For $x_0 \in X_2$, the estimate of (1.18) is given by (with $p_2 = 2$ and discussion after (1.16) for $d(x_0, 0) > 0$)

$$a_n(t, x_0) = 2\alpha_n(x_0) + O(t^\infty). \tag{1.23}$$

By (1.22) and (1.23) it follows $r_n(t, x_0) = \alpha_n(x_0) + O(t^\infty)$ so $r_n(t, x) \approx \alpha_n(x)$ around $x_0$ as $t \to 0$, giving $|r_n(t, x)| \geq \epsilon > 0$ nearby $x_0$ for some constant $\epsilon$ independent of $x$ and $t$. But this would be absurd by (1.21) if $r_n$ were independent of $t$ (taking $x \in X_1, \neq x_0$ near $x_0$ so that $|r_n(t, x)| \geq \epsilon$ and letting $t \to 0$ in (1.21)). Hence $a_n(t, x)$ cannot be independent of $t$ either, as desired.

**Remark 1.8.** To discuss the estimate (1.19), let’s take $\hat{d}$ in (1.19) to be $d$ for convenience (as remarked previously $\hat{d}$ is equivalent to the ordinary distance function $d$ at least in the strongly pseudoconvex case, cf. Theorem 6.7). Take $P_l = \text{id}$ (so $l = 0$). The term to the rightmost of (1.19) appears as a Gaussian-like term. As $t \to 0$, this term tends to a sort of Dirac delta function supported along the strata $X_{\text{sing}}^r$ (with an extra singular factor $t^{-\frac{n-1}{2}}$, $a = \dim X_{\text{sing}}^r$). This may conceptually explain the piecewise continuity nature just discussed in Remarks 1.6 and 1.7 if the asymptotic expansion is to be expressed in something, without $t$-dependence, such as $\alpha_s(x)$. Conversely, the estimate as (1.19) involving a type of Dirac delta function is conceptually reasonable under the piecewise continuity phenomenon in terms of $\alpha_s(x)$. For more about this, some quantitative information may be available by Theorems 1.14, 7.19 and 7.23.

**Remark 1.9.** We make a short comment on the coefficients $a_s(t, x)$ or $\alpha_s(x)$ in (1.18) (the difference between $a_s(t, x)$ and $p_1 \alpha_s(x)$ at a given $x \in X_{p_1}$) is $O(t^\infty)$ by (1.18); this is partly explained conceptually right below. For the standard (elliptic) case (of Dirac type) it is well-known that the coefficients of a heat kernel along the diagonal (by taking trace) are expressible in terms of the curvature and its covariant derivatives (e.g. [36]). In our transversally elliptic case (without bundle $E$ for simplicity) if $S^1$ action is globally free, it follows from the standard case above (cf. (1.1)-(1.2)) that these coefficients of the (transversal) heat kernel are expressible in terms of the tangential curvature (and its derivatives) (cf. Section 2.3). In the locally free case the same results can be achieved in view of the proof of Theorem 1.3, which basically arises from a procedure of patching and successive approximations based on the local (transversal) heat kernels that give the asymptotic approximations of the final (transversal) heat kernel (see Section 1.7 for details of an outline). Since the local kernels can be so expressed as just said (at least on the principal stratum), it follows from the asymptotic approximation (e.g. Theorems 2.23 and 2.30 of [5] or Theorem 5.14 in our case) that the same (expression in tangential curvature and its derivatives) can be said for the global kernel (on the principal stratum then followed by continuous extension of this global kernel on $X$). It is also of interest to consider the integral version of these coefficients, which is the topic of Section 7 of this paper.

**1.2.3. A local index theorem for CR manifolds with $S^1$ action.** Here we discuss issues related to the index theorem we will prove. We recall that the term to the left of the inequality in (1.18) is basically nontrivial by Remark 1.7. In our formulation of index theorems, the contribution arising from such a
Theorem 1.10. Suppose \((X, T^{1,0} X)\) is a compact, connected \(CR\) manifold with a transversal \(CR\) locally free \(S^1\) action. With the notations above, if \(\square_{b,m}\) is the \(Spin^c\) Kohn Laplacian (see (4.12)), then for \(r = 1, \ldots, k\) and every \(N_0 \in \mathbb{N}\) with \(N_0 \geq N_0(n)\) for some \(N_0(n)\), there are \(\varepsilon_0 > 0, \delta > 0\) and \(C_{N_0} > 0\) such that (\(STr\) denoting supertrace, cf. (1.13))

\[
STr \ e^{-t\square_{b,m}}(x, x) - \left( \sum_{s=1}^{p_r} e^{2\pi s^{-1} m s} \sum_{j=0}^{N_0} t^{-n+j} |STr \alpha_{n-j}(x)| \right) 
\leq C_{N_0} \left( t^{-n+N_0} + e^{-\varepsilon_0|x|^2+|X^\bot|^2} \right), \quad \forall 0 < t < \delta, \quad \forall x \in X_{p_r},
\]

and

\[
\sum_{\ell=0}^{n} t^{-\ell} |STr \alpha_{\ell}(x)| \|dv_X(x)\|
= \frac{1}{2\pi} \left[ T_{ch} \left( (\nabla^{T^{1,0} X}, T^{1,0} X) \right) \Lambda ch_0 \left( (\nabla^{E}, E) \right) \Lambda e^{-m \frac{dv}{\pi}} \wedge \omega_0 \right]_{2n+1}(x),
\]

where \([\ldots]_{2n+1}\) to the right denotes the part of \((2n+1)\)-form.

As \(Spin^c\) objects can be simplified in the Kähler case, so can the \(Spin^c\) Kohn Laplacian in the \(CR\) Kähler case, to which we turn now.

Definition 1.11. We say that \(X\) is \(CR\) Kähler if there is a closed form \(\Theta \in C^\infty(X, T^{*1,1} X)\) such that \(\Theta(Z, \overline{Z}) > 0\), for all \(Z \in C^\infty(X, T^{1,0} X)\). We call \(\Theta\) a \(CR\) Kähler form on \(X\).

When \(X\) is a strongly pseudoconvex \(CR\) manifold with a transversal \(CR\) locally free \(S^1\) action, the closed form \(d\omega^0\) satisfies \(d\omega^0(Z, \overline{Z}) > 0\), for all \(Z \in C^\infty(X, T^{1,0} X)\). Hence \(X\) is \(CR\) Kähler.

A quasi-regular Sasakian manifold is also a \(CR\) Kähler manifold. We recall that for a compact smooth manifold \(X\) of \(\dim X = 2n + 1, n \geq 1\), the triple \((X, g, \alpha)\) where \(g\) is a Riemannian metric and \(\alpha\) is a real \(1\)-form is called a Sasakian manifold if the cone \(C(X) = \{(x, t) \in X \times \mathbb{R}_{>0}\}\) is a Kähler manifold with complex structure \(J\) and Kähler form \(t^2 dx \wedge d\alpha\) where \(t^2 g + dt \wedge d\alpha\) is compatible with the metric \(t^2 g + dt \wedge d\alpha\). As a consequence, \(X\) is a compact strongly pseudoconvex \(CR\) manifold and the Reeb vector field \(\xi\), defined by \(\alpha(\xi) = g(\xi, \cdot)\), induces a transversal \(CR\) \(\mathbb{R}\) action on \(X\). If the orbits of this \(\mathbb{R}\) action are compact, the Sasakian structure is called quasi-regular. In this case, the Reeb vector field generates a locally free transversal \(CR\) \(S^1\) action on \(X\). We can thus identify a compact quasi-regular Sasakian manifold with a compact strongly pseudoconvex \(CR\) manifold \((X, T^{1,0} X)\) equipped with a transversal \(CR\) locally free \(S^1\) action such that the induced vector field of the \(S^1\) action coincides with the Reeb vector field on \(X\) (see [51], [52]).

Let \(X\) be a \(CR\) Kähler manifold with a transversal \(CR\) locally free \(S^1\) action. If \(\cdot, \cdot\) is induced by a \(CR\) Kähler form on \(X\), then \(\square_{b,m}\) is equal to the \(Spin^c\) Kohn Laplacian. By Theorem 1.10, we immediately obtain a version of \(local\ index\ theorem\) on \(CR\) Kähler manifolds with transversal \(CR\) locally free \(S^1\) action (which include the compact quasi-regular Sasakian manifolds as a special case by above). These results are discussed below.

For a proof of the following, see the beginning of Subsection 1.7.4 and the discussion leading to Proposition 5.8):

Corollary 1.12. (\(CR\) Kähler case of Theorem 1.10) Suppose \((X, T^{1,0} X)\) is a compact, connected \(CR\) Kähler manifold with a transversal \(CR\) locally free \(S^1\) action and assume that \(\cdot, \cdot\) is induced by a \(CR\)
Kähler form on $X$. With the notations above, for $r = 1, \ldots, k$ and every $N_0 \in \mathbb{N}$ with $N_0 \geq N_0(n)$ for some $N_0(n)$, there are $\varepsilon_0 > 0$, $\delta > 0$ and $C_{N_0} > 0$ such that

$$
|\text{STr} \, e^{-t\overline{\Delta}_{b,m}} (x,x) - \sum_{s=1}^{p_r} e^{\frac{2\pi(s-1-k)}{p_r}} \sum_{j=0}^{N_0} t^{-n+j} \text{STr} \, \alpha_{n-j}(x)| 
\leq C_{N_0} \left( t^{-n+N_0+1} + e^{-\frac{\varepsilon_0 d(x,X_{\text{sing}})^2}{t}} \right), \quad \forall 0 < t < \delta, \ \forall x \in X_{p_r},
$$

(1.26)

and

$$
\sum_{\ell=0}^{n} t^{-\ell} \text{STr} \, \alpha_{\ell}(x) dv_X(x) 
= \frac{1}{2\pi} \left[ Td_b \left( \nabla^{T^{1,0}X}, T^{1,0}X \right) \wedge \ch_b \left( \nabla^E, E \right) \wedge e^{-\frac{d_0}{\pi} \text{Tr} \, \omega_0} \right]_{2n+1} (x).
$$

(1.27)

We are in a position to state an index theorem (including a local index theorem in the CR Kähler case). Recall $\overline{\partial}_{b,m} := \overline{\partial}_b : \Omega^{0,q}_m(X,E) \to \Omega^{0,q+1}_m(X,E), m \in \mathbb{Z}$, and a $\overline{\partial}_{b,m}$-complex:

$$
\overline{\partial}_{b,m} : \cdots \to \Omega^{0,q-1}_m(X,E) \to \Omega^{0,q}_m(X,E) \to \Omega^{0,q+1}_m(X,E) \to \cdots.
$$

The $q$-th $\overline{\partial}_{b,m}$ Kohn-Rossi cohomology group (regarded as the $m$-th Fourier component of the ordinary $q$-th Kohn-Rossi cohomology group) is

$$
H^q_{b,m}(X,E) := \frac{\text{Ker} \, \overline{\partial}_{b,m} : \Omega^{0,q}_m(X,E) \to \Omega^{0,q+1}_m(X,E)}{\text{Im} \, \overline{\partial}_{b,m} : \Omega^{0,q-1}_m(X,E) \to \Omega^{0,q}_m(X,E)}.
$$

We will prove in Theorem 3.7 that there holds $\dim H^q_{b,m}(X,E) < \infty$ (for each $m \in \mathbb{Z}$ and $q = 0, 1, 2, \ldots, n$) without any Levi curvature assumption.

In Corollary 4.8 (see also Remark 4.9) we have a McKean-Singer type formula in our CR case: for every $t > 0$,

$$
\sum_{j=0}^{n} (-1)^j \dim H^j_{b,m}(X,E) = \int_X \text{STr} \, e^{-t\overline{\Delta}_{b,m}} (x,x) dv_X.
$$

(1.28)

Combining (1.28), (1.24) and (1.25) and noting $e^{-\frac{\varepsilon_0 d(x,X_{\text{sing}})^2}{t}}$ is bounded by 1 and rapidly decays to 0 for $x$ in the principal stratum as $t \to 0$, we conclude the following form of an index theorem on our CR manifolds (see Section 2.3 for the precise meanings of $Td_b \left( T^{1,0}X \right)$ and $\ch_b \left( E \right)$ below):

**Corollary 1.13.** (CR Index Theorem, cf. Corollary 6.5) Suppose $(X,T^{1,0}X)$ is a compact, connected CR manifold with a transversal CR locally free $S^1$ action. Then

$$
\sum_{j=0}^{n} (-1)^j \dim H^j_{b,m}(X,E)
= \left( \sum_{s=1}^{p} e^{\frac{2\pi(s-1-k)}{p}} \right) \frac{1}{2\pi} \int_X Td_b \left( T^{1,0}X \right) \wedge \ch_b \left( E \right) \wedge e^{-\frac{d_0}{\pi} \text{Tr} \, \omega_0} \wedge \omega_0,
$$

(1.29)

where $Td_b \left( T^{1,0}X \right)$ denotes the tangential Todd class of $T^{1,0}X$ and $\ch_b \left( E \right)$ denotes the tangential Chern character of $E$.

For a connection with other works on index theorems by different formulations and methods, we refer to comments that come after Theorem 1.27 and to the sixth paragraph in Subsection 7.1.
1.2.4. Trace integrals in terms of geometry of the \( S^1 \) stratification. This is the third and the last topic of this paper. For some applications (e.g. see a natural connection with Remark 1.6), one studies the asymptotic behavior of \( \int_X \text{Tr} e^{-L_{h,m}}(x, x) dv_X(x) \). More comments on the historical respect come in the beginning of Section 7. Suppose \( M \) is an orbifold of (real) dimension \( k \) and \( H(t, x, x) \) is the associated heat kernel on the diagonal for the standard Laplacian on \( M \). It is known in 2008 [19] (for Laplacian on functions; see also Richardson [56, Theorem 3.5]) that

\[
(1.30) \quad \int_M H(t, x, x) dv_X(x) \sim \sum_{a \in \mathbb{R}} (t^{-\frac{a}{2}} a_k + t^{-\frac{a}{2} + \frac{1}{2}} a_{k-1} + t^{-\frac{a}{2} + \frac{3}{2}} a_{k-2} + \cdots)
\]

where \( a \in \mathbb{R} \) is independent of \( t, s, k, k-\frac{1}{2}, k-1, \ldots \). A novelty is that apart from the overall \( t^{-\frac{k}{2}} \) the expansion is a power series in \( t^\frac{1}{2} \).

By a strategy partly in connection with the proof of Theorem 1.3, we obtain an expansion of the trace integral similar to (1.30) in spirit. We find that in our case, the expansion is a power series still in integral power of \( t \). However, there appear various corrections (depending on \( m \)) supported on each stratum (cf. (7.49) and (7.54)) in contrast to the expansion in the globally free case (of \( S^1 \) action).

More precisely, we have (see Theorems 7.19 and 7.23 for more information and proof):

**Theorem 1.14.** (cf. Theorems 7.19, 7.23) With notations in Theorem 1.3 and assumption that the \( S^1 \) action is locally free but not globally free, let \( e \) be the number (which is even) defined to be the minimum of the (real) codimensions of connected components \( M \) of \( X_{pt} \) for all \( \ell \geq 2 \). For \( s = n, n-1, \ldots \), we have

\[
(1.31) \quad \int_X \text{Tr} a_{s,m}(t, x) dv_X(x) \sim q_{s,0} + t q_{s,1} + t^2 q_{s,2} + \cdots \quad \text{as } t \to 0^+,
\]

where \( a_{s,m}(t, x) = a_s(t, x) \) is as in (1.17) and \( q_{s, j} \in \mathbb{R} \) is independent of \( t \) (dependent on \( m \) though), \( j = 0, 1, 2, \ldots \). Similarly, as \( t \to 0^+ \),

\[
(1.32) \quad \int_X e^{-L_{h,m}}(x, x) dv_X(x) \sim \sum_{s=1}^{p_1} e^{\frac{2\pi i(s-1)}{p_1}} (t^{-n} c_n + t^{-n+1} c_{n-1} + t^{-n+2} c_{n-2} + \cdots)
\]

\[
+ t^{-n+\frac{e}{2}} e^{-\frac{n}{2}} + O(t^{-n+\frac{e}{2}+1}).
\]

These coefficients satisfy the following. For an \( \ell \geq 2 \), write \( \{M_{\ell,\gamma}\}_{\gamma} \) (possibly empty for some \( \ell \)) for those connected components \( M_{\ell,\gamma} \) of \( X_{pt} \) with the codimension \( \text{codim} M_{\ell,\gamma} = e \). Set \( S_{\ell,\gamma, s, m} = \int_{M_{\ell,\gamma}} \text{Tr} a_{s,m} dv_{M_{\ell,\gamma}} \) where \( a_{s,m}(\ell,\gamma) \) is as in (1.18) and the numerical factor

\[
(1.33) \quad D_{\ell,m} = (\sqrt{\pi})^e \sum_{c,h \in \mathbb{N}, (h,e)=1} \frac{e^{-i\frac{2\pi h}{e}}} {e^{\frac{i\pi h}{e} - 1}} (0 \text{ if } p_1|m).
\]

i) \( q_{s,1} = q_{s,2} = \cdots = q_{s, e-1} = 0 \), \( q_{s,0} = \sum_{s=1}^{p_1} e^{\frac{2\pi i(s-1)}{p_1}} \int_X \text{Tr} a_{s,m} dv_X \) (s = n, n - 1, n - 2, \ldots).

ii) \( q_{s, e} \) is (a finite sum) of the form \( \sum_{c,h \in \mathbb{N}, (h,e)=1} \frac{e^{-i\frac{2\pi h}{e}}} {e^{\frac{i\pi h}{e} - 1}} (0 \text{ if } p_1|m) \).

iii) \( c_s = \sum_{s=1}^{p_1} e^{\frac{2\pi i(s-1)}{p_1}} \int_X \text{Tr} a_{s,m} dv_X \) (s = n, n - 1, n - 2, \ldots).

iv) \( \tilde{c}_{n-\frac{e}{2}} = (2\pi)^{-\frac{e}{2}+1} \sum_{c,h \in \mathbb{N}, (h,e)=1} \frac{e^{-i\frac{2\pi h}{e}}} {e^{\frac{i\pi h}{e} - 1}} (0 \text{ if } p_1|m) \).

The Laplacian in the work [19] is limited to the Laplacian acting on functions while ours above is not. We remark that in [56, Theorem 3.5] the nontrivial fractional power in \( t^\frac{1}{2} \) does occur. This is however due partly to a fixed point set of codimension 1 under a reflection isometry (loc. cit., p. 2315). In our CR case, all of the various fixed point submanifolds are of even (real) codimension, cf. i) of Remark 7.21 or [56, p. 2324]. See Section 7 for a comparison of these methods and results.

It will be of interest to study the geometrical significance of the various coefficients in (1.31) and (1.32) as usually studied in the standard heat kernel case. Explicit expressions for more in this regard are available by our treatment, e.g. (7.49), (7.54) and Theorems 7.19, 7.23.
Remark that the above results essentially deal with the Gaussian part of the heat kernel, which behaves as a Dirac type delta function supported on (each) stratum. By contrast, the CR local index theorem as Corollary 1.13 is derived by exploring the non-Gaussian parts of the heat kernel such as the off-diagonal estimate in Theorem 5.9. In spirit, the two approaches are complementary to each other in the present paper, and jointly enhance the understanding of heat kernels for this special class of CR manifolds.

Two more remarks go as follows.

Remark 1.15. We note that the topological obstruction exists for a CR manifold to admit a transversal CR $S^1$ action. For instance, a compact strictly pseudoconvex CR 3-manifold must have even first Betti number if admitting a transversal CR $S^1$ action. The reason is that such a manifold must be pseudohermitian torsion free (see [46]), and this vanishing pseudohermitian torsion implies even first Betti number as shown by Alan Weinstein (see the Appendix in [15]). In this paper, we only consider the $S^1$ action that is transversal and locally free. Here are two examples:

Example I: $X = \left\{ (z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_1^2 + z_2|^4 + |z_2^2 + z_3|^6 = 1 \right\}$. Then $X$ admits a transversal CR locally free $S^1$ action: $e^{-i\theta} \circ (z_1, z_2, z_3) = (e^{-i\theta}z_1, e^{-2i\theta}z_2, e^{-6i\theta}z_3)$. It is clear that this $S^1$ action is not globally free.

Example II: Let $X$ be a compact orientable Seifert 3-manifold. Kamishima and Tsuboi [43] proved that $X$ is a compact CR manifold with a transversal CR locally free $S^1$ action. $X$ is $S^1$-fibered over a possibly singular base (an orbifold).

In Section 1.5, we collect more examples.

Remark 1.16. The $S^1$ action might admit a reduction to a simpler one as $\text{Hom}(S^1, S^1) \neq \text{id}$. Recall $p_1 = p < p_2 < \cdots < p_k$, associated with periods of $X$ under the given $S^1$ action $(e^{-i\theta}, x) \to e^{-i\theta} \circ x$. Then $p_1 = p$ divides each $p_j$, $j > 1$. For, the isotropy subgroup $\mathbb{Z}_p = (\mathbb{Z}/p\mathbb{Z}) \subset S^1$ acts trivially on the principal stratum, which is dense and open, hence on the whole $X$ by continuity. The isotropy subgroups $\mathbb{Z}_{p_j}$, $j = 2, \ldots, k$, on any other stratum must contain $\mathbb{Z}_p$, giving $\frac{p_j}{p} \in \mathbb{N}$.

One renormalizes the given $S^1$ action by the new $S^1$ action satisfying $p_1 = 1$. More precisely, define

$S^1 \times X \to X$,

$(e^{-i\theta}, x) \to e^{-i\theta} \circ x := e^{-i\theta} \circ x$.

The new $S^1$-action $(e^{-i\theta}, \circ)$ has $p_1 = 1$. Let $\tilde{\omega}_0$ be the global real one form with respect to $(e^{-i\theta}, \circ)$ and let $\tilde{H}^q_{b,m}(X, E)$ be the corresponding cohomology group with respect to $(e^{-i\theta}, \circ)$. One sees

$$\tilde{\omega}_0 = p\omega_0,$$

$$\tilde{H}^q_{b,m}(X, E) = H^q_{b,pm}(X, E), \ \forall m \in \mathbb{Z}, \ \forall q = 0, 1, 2, \ldots, n.$$

Examining (1.34) and Corollary 1.13 yields that the index formulas in both cases can be transformed to each other.

1.3. Applications.

1.3.1. Applications in CR geometry. In CR geometry, it has been an important issue to produce many CR functions or CR sections. Put

$$H^0_b(X, E) = \left\{ u \in C^\infty(X, E) ; \bar{\partial}_b u = 0 \right\}.$$

The following belongs to one of the standard questions in this respect.

Question 1.17. Let $X$ be a compact weakly pseudoconvex CR manifold. When is the space $H^0_b(X, E)$ large? (Pseudoconvex CR manifolds will be briefly reviewed following Definition 2.2.)
In [47] Lempert proved that a three dimensional compact strongly pseudoconvex CR manifold $X$ with a transversal CR locally free $S^1$ action can be CR embedded into $\mathbb{C}^N$. In [24] Epstein proved that a three dimensionalcompact strongly pseudoconvex CR manifold $X$ with a transversal CR globally free $S^1$ action can be embedded into $\mathbb{C}^N$ by the positive Fourier components.

The embeddability of $X$ by positive Fourier coefficients is related to the behavior of the $S^1$ action on $X$. For example, suppose for $f_1, \ldots, f_d \in H^0_{b,m}(X)$ and $g_1, \ldots, g_{h_1} \in H^0_{b,1}(X)$ the map

$$
\Phi_{m,l} : x \in X \to (f_1(x), \ldots, f_d(x), g_1(x), \ldots, g_{h_1}(x)) \in \mathbb{C}^{d+h_1}
$$

is a CR embedding. Then, the $S^1$ action on $X$ naturally induces an $S^1$ action on $\Phi_{m,l}(X)$, given by the following:

$$(1.35) \quad e^{-i\theta} \circ (z_1, \ldots, z_d, z_{d+1}, \ldots, z_{d+h_1}) = (e^{-im\theta}z_1, \ldots, e^{-im\theta}z_d, e^{-il\theta}z_{d+1}, \ldots, e^{-il\theta}z_{d+h_1}).$$

In short, under a CR embedding by positive Fourier components, one can describe the $S^1$ action explicitly. Conversely, to study the embedding theorem of those CR manifolds by positive Fourier components, it becomes important to know

**Question 1.18.** When is $\dim H^0_{b,m}(X,E) \approx m^n$ for $m$ large?

We shall answer, combining our index theorems with some vanishing theorems (see below), Question 1.17 and Question 1.18 for CR manifolds with transversal CR locally free $S^1$ action.

Firstly it follows from Corollary 1.13 (by extracting the leading coefficient of the term $m^n$)

**Corollary 1.19.** In the same assumption as in Corollary 1.13, one has

$$
\sum_{j=0}^{n} (-1)^j \dim H^j_{b,m}(X,E) = r(\sum_{s=1}^{p} e^{\frac{2\pi(s-1)m_i}{p}} \frac{m^n}{n!(2\pi)^{n+1}} \int_X (-d\omega_0)^n \wedge \omega_0 + O(m^{n-1}),
$$

where $r$ denotes the complex rank of the vector bundle $E$.

For a vanishing theorem we can repeat the proof of Theorem 2.1 in [42] with minor change and get

**Proposition 1.20.** In the same assumption as in Corollary 1.13, suppose further that $X$ is weakly pseudoconvex. Then, for $m \gg 1 \dim H^j_{b,m}(X,E) = o(m^n)$, for every $j = 1, 2, \ldots, n$.

Combining Corollary 1.19 and Proposition 1.20 one has

**Corollary 1.21.** In the same assumption as in Proposition 1.20 (with $X$ being weakly pseudoconvex). One has, for $m \gg 1$,

$$
\dim H^0_{b,m}(X,E) = r(\sum_{s=1}^{p} e^{\frac{2\pi(s-1)m_i}{p}} \frac{m^n}{n!(2\pi)^{n+1}} \int_X (-d\omega_0)^n \wedge \omega_0 + o(m^n),
$$

where $r$ denotes the complex rank of the vector bundle $E$. In particular, if the Levi form is strongly pseudoconvex at some point of $X$, then $\dim H^0_{b,pm}(X) \approx m^n$ for $m \gg 1$, and hence $\dim H^0_{b,pm}(X,E) = \infty$.

These results have provided answers to Question 1.17 and Question 1.18 (for our class of CR manifolds).

For another application, it is of great interest in CR geometry to study whether and when a CR manifold $X$ can be CR embedded into a complex space. It is a classical theorem of L. Boutet de Monvel [7] which asserts that $X$ can be globally CR embedded into $\mathbb{C}^N$ for some $N \in \mathbb{N}$ provided that $X$ is compact (with no boundary), strongly pseudoconvex, and of dimension greater than or equal to five.
When $X$ is not strongly pseudoconvex, the space of global CR functions could even be trivial. As many interesting examples live in the projective space (e.g. the quadric $\{[z] \in \mathbb{C}P^{N-1}: |z_1|^2 + \ldots + |z_q|^2 - |z_{q+1}|^2 - \ldots - |z_N|^2 = 0\}$), it is natural to consider a setting analogous to the Kodaira embedding theorem and ask if $X$ can be embedded into the projective space by means of CR sections of a CR line bundle $L \to X$ or its $k$-th power $L^k$.

For a study into the above question it is natural to seek the case where the dimension of the space $H^0_b(X, L^k)$ of CR sections of $L^k$ is large as $k \to \infty$ (so one may hopefully find many CR sections to carry out the embedding). In this regard the following question is asked by Henkin and Marinescu [49, p.47-48].

**Question 1.22.** When is $\dim H^0_b(X, L^k) \approx k^{n+1}$ for $k$ large?

Assume that $L$ is a rigid CR line bundle with a rigid Hermitian fiber metric $h^L$ (i.e. $L$ a CR line bundle admitting a compatible $S^1$ action, cf. the beginning of Section 1.2). Let $\mathcal{R}^L \in \Omega^2(X)$ be the curvature of $L$ associated to $h^L$. For a local trivializing ($S^1$-invariant) section $s$ of $L$, $|s(x)|^2 = e^{-2\phi(x)}$ with $T\phi = 0$. Then $\mathcal{R}^L = 2\partial\bar{\partial}\phi \in \Omega^2(X)$. $(L^k, h^{L^k})$ denotes the $k$-th power of $(L, h^L)$.

With Corollary 1.13 one can show

**Proposition 1.23.** With the notations above, for $k$ large we have

$$
\sum_{m \in \mathbb{Z}, |m| \leq k} \sum_{j=0}^n (-1)^j \dim H^j_{b,m}(X, L^k \otimes E) = r(2\pi)^{-n-1} \frac{1}{n!} k^{n+1} \int_X (iR^L_x - md\omega_0(x))^n \wedge \omega_0(x)ds + o(k^{n+1}),
$$

where $\delta > 0$ and $r$ denotes the complex rank of the vector bundle $E$.

**Proof.** By Corollary 1.13 one can check

$$
\sum_{m \in \mathbb{Z}, |m| \leq k} \sum_{j=0}^n (-1)^j \dim H^j_{b,m}(X, L^k \otimes E)
$$

$$
= r(2\pi)^{-n-1} \sum_{m \in \mathbb{Z}, |m| \leq k} \left( \sum_{s=1}^p e^{2\pi i (s-1) m_i} \frac{1}{n!} \int_X (ik\mathcal{R}_x^L - md\omega_0(x))^n \wedge \omega_0(x) + o(k^n) \right)
$$

$$
= r(2\pi)^{-n-1} \sum_{m \in \mathbb{Z}, |m| \leq k} \left( \sum_{s=1}^p e^{2\pi i (s-1) m_i} \frac{k^n}{n!} \int_X (i\mathcal{R}_x^L - \frac{m}{k} d\omega_0(x))^n \wedge \omega_0(x) + o(k^n) \right).
$$

Note $\sum_{s=1}^p e^{2\pi i (s-1) m_i} = p$ if $p|m$, and 0 otherwise. By this and (1.38) we get

$$
\sum_{m \in \mathbb{Z}, |m| \leq k} \sum_{j=0}^n (-1)^j \dim H^j_{b,m}(X, L^k \otimes E) = r(2\pi)^{-n-1} \sum_{\ell \in \mathbb{Z}, |\ell| \leq k} \frac{p}{n!} k^n \int_X (i\mathcal{R}_x^L - \frac{\ell}{k} d\omega_0(x))^n \wedge \omega_0(x) + o(k^n).
$$

It is clear that the (Riemann) sum $\sum_{\ell \in \mathbb{Z}, |\ell| \leq k} \frac{p}{k} \int_X (i\mathcal{R}_x^L - \frac{\ell}{k} d\omega_0(x))^n \wedge \omega_0(x)$ converges to

$$
\int_X \int_{[-\delta, \delta]} (i\mathcal{R}_x^L - sd\omega_0(x))^n \wedge \omega_0(x)ds
$$
as \( k \to \infty \). Hence
\[
\sum_{\ell \in \mathbb{Z}, |p\ell| \leq k\delta} \left\{ \frac{p}{n!} k^n \int_X (iR^L_x - \frac{p\ell}{k} d\omega_0(x))^n \wedge \omega_0(x) \right\} + o(k^n) \\
= \left\{ \frac{1}{n!} k^{n+1} \int_X \int_{[-\delta,\delta]} (iR^L_x - sd\omega_0(x))^n \wedge \omega_0(x) ds \right\} + o(k^{n+1}).
\]
(1.40) Combining (1.40) with (1.39) we have (1.37). \( \square \)

The following two results may be viewed as a companion of the Grauert-Riemenschneider criterion in the CR case (with \( S^1 \) action). To start with

**Definition 1.24.** We say that \((L, h_L)\) is positive at \( p \in X \) if the curvature \( R^L_p \) is a positive Hermitian quadratic form over \( T^1_{1,0} \). We say that \((L, h_L)\) is semipositive if for any \( x \in X \) there exists a constant \( \delta > 0 \) such that \( R^L_x - sd\omega_0(x) \) is a semipositive Hermitian quadratic form over \( T^1_{1,0} \) for any \(|s| < \delta\).

We can repeat the proof of Theorem 1.24 in [41] with minor change and get

**Proposition 1.25.** (Asymptotical vanishing) Assume that \((L, h_L)\) is a semi-positive CR line bundle over \( X \). Then, for \( \delta > 0 \), \( \delta \) small, we have
\[
\sum_{m \in \mathbb{Z}, |m| \leq k\delta} \dim H^j_{b,m}(X, L^k \otimes E) = o(k^{n+1}), \quad j = 1, 2, \ldots, n.
\]
(1.41) Combining Proposition 1.23 and Proposition 1.25, we get

**Corollary 1.26.** (Bigness) Assume that \((L, h_L)\) is semi-positive. Then, for \( \delta > 0 \), \( \delta \) small, we have
\[
\sum_{m \in \mathbb{Z}, |m| \leq k\delta} \dim H^0_{b,m}(X, L^k \otimes E) = r(2\pi)^{-n-1} \frac{1}{n!} k^{n+1} \int_X \int_{[-\delta,\delta]} (iR^L_x - sd\omega_0(x))^n \wedge \omega_0(x) ds + o(k^{n+1}),
\]
where \( r \) denotes the complex rank of the vector bundle \( E \). In particular, if \((L, h_L)\) is positive at some point of \( X \), then
\[
\dim H^0_{b,m}(X, L^k \otimes E) \approx k^{n+1}.
\]

The above result yields an answer to Question 1.22 in the case pertinent to our class of CR manifolds.

### 1.4. Kawasaki’s Hirzebruch-Riemann-Roch and Grauert-Riemenschneider criterion for orbifold line bundles.

There is a link between our CR result and a complex geometry result of Kawasaki on Hirzebruch-Riemann-Roch formula over complex orbifolds [45]. Compared to Kawasaki’s, we get a simpler Hirzebruch-Riemann-Roch formula for some class of orbifold line bundles using our second main result Corollary 1.13. Moreover, from Corollary 1.21 it follows a Grauert-Riemenschneider criterion for orbifold line bundles.

To the aim we shortly review the orbifold geometry and also set up notations. Let \( M \) be a manifold and \( G \) a compact Lie group. Assume that \( M \) admits a \( G \)-action:
\[
G \times M \to M, \\
(g, x) \to g \circ x.
\]
We suppose that the action \( G \) on \( M \) is locally free, that is, for every point \( x \in M \), the stabilizer group \( G_x = \{ g \in G; g \circ x = x \} \) of \( x \) is a finite subgroup of \( G \). In this case the quotient space
\[
M/G
\]
(1.42)
Theorem 1.27. With the notations above, recall that we work with assumptions that \( M \) is connected and \( \text{Tot} (L^*)/G \) is smooth. Then (for every \( m \in \mathbb{Z} \))

\[
\sum_{j=0}^{n} (-1)^j \dim H^j(M/G, L^m/G) = \left( \sum_{s=1}^{p} e^{2\pi i (s-1) \frac{m}{p}} \right) \frac{1}{2\pi} \int_X \text{Td}_b (T^{1,0}X) \land e^{-m \frac{d\omega_0}{2\pi}} \land \omega_0.
\]
To compare this result with that of Kawasaki ([45]) we assume \( p = 1 \) for simplicity. Note \( X \) is smooth (yet \( M/G \) could be singular). The above integral (1.47) reduces to an integral over the principal stratum of \( M/G \) (by integrating \( w_0 \) along the fiber \( S^1 \), which gives 1). It is thus the same as to say that the contributions from the lower dimensional strata sum up to zero. As remarked in Introduction a vanishing result as such is not readily available in the formula of [45]. Note that the notion of “orbifold” in [45] is slightly more general than that of Satake (on which the present section is based). As this generality does no real harm to the reasoning above, we omit the details in this regard.

The above result on the vanishing of the contributions from strata may also be reflected in the index formula of the works [54], [32] which study the index of transversally elliptic operators on a smooth compact manifold with the action of a compact Lie group \( G \), by using the framework of equivariant cohomology theory. A remarkable point is that they define the index as a generalized function (on \( G \)) (also discussed in Atiyah [1], p.9-17). In fact it is not difficult to verify that for the case of the \( S^1 \) action, our present \( m \)-th index is basically the \( m \)-th Fourier component of the corresponding index (in the sense of generalized functions) of theirs (for the case \( g = \text{id} \in G \) in [32], [54]).

The consistency of our result with those works above helps to shape our own view towards the asymptotic expansion of a (transversal) heat kernel conceived in this subject.

For examples that satisfy Theorem 1.27 we refer to Section 1.5. There, we construct, among others, an orbifold holomorphic line bundle over a singular complex manifold such that the assumptions of Theorem 1.27 and Corollary 1.28 are fulfilled (see Corollary 1.30 and Subsection 1.5.2 below). Indeed there are ample examples in this respect.

As promised in the beginning of this section, we obtain now a Grauert-Riemenschneider criterion for orbifold line bundles, upon combining Corollary 1.21 and (1.45).

**Corollary 1.28.** With the notations and assumptions above, suppose that there is a \( G \)-invariant Hermitian fiber metric \( h^L \) on \( L \) such that the associated curvature \( R^L \) is semipositive and positive at some point of \( M \). Then \( \dim H^0(M/G, L^mG) \approx m^n \) for \( m \gg 1 \) and \( p \) as in (1.46).

1.5. **Examples.** In this subsection, some examples of CR manifolds with locally free \( S^1 \) action (including those fitting Theorem 1.27 above) are collected.

We first review the construction of generalized Hopf manifolds introduced by Brieskorn and Van de Ven [9].

1.5.1. **Generalized Hopf manifolds.** Let \( a = (a_1, \ldots, a_{n+2}) \in \mathbb{N}_+^{n+2} \), let \( z = (z_1, \ldots, z_{n+2}) \) be the standard coordinates of \( \mathbb{C}^{n+2} \) and let \( M(a) \) be the affine algebraic variety given by the equation

\[
\sum_{j=1}^{n+2} z_j^{a_j} = 0.
\]

If some \( a_j = 1 \), the variety \( M(a) \) is non-singular. Otherwise \( M(a) \) has exactly one singular point, namely \( 0 = (0, \ldots, 0) \). Put \( \tilde{M}(a) := M(a) - \{0\} \). Now we define a holomorphic \( \mathbb{C} \)-action on \( \tilde{M}(a) \) by

\[
t \circ (z_1, \ldots, z_{n+2}) = (e^{\frac{t}{\pi a_1}} z_1, \ldots, e^{\frac{t}{\pi a_{n+2}} z_{n+2}}), \quad t \in \mathbb{C}, \quad (z_1, \ldots, z_{n+2}) \in \tilde{M}(a).
\]

It is easy to see that the \( \mathbb{Z} \)-action on \( \tilde{M}(a) \) is globally free. The equivalence class of \( (z_1, \ldots, z_{n+2}) \in \mathbb{C}^{n+2} \) with respect to the \( \mathbb{Z} \)-action is denoted by \( (z_1, \ldots, z_{n+2}) + \mathbb{Z} \) and hence

\[
H(a) := \tilde{M}(a)/\mathbb{Z} = \left\{ (z_1, \ldots, z_{n+2}) + \mathbb{Z}; \; (z_1, \ldots, z_{n+2}) \in \tilde{M}(a) \right\}
\]

is a compact complex manifold of complex dimension \( n+1 \). We call \( H(a) \) a (generalized) Hopf manifold.

Let \( \Gamma_a \) be the discrete subgroup of \( \mathbb{C} \), generated by 1 and \( 2\pi \alpha i \), where \( \alpha \) is the least common multiple of \( a_1, a_2, \ldots, a_{n+2} \). Consider the complex 1-torus \( T_a = \mathbb{C}/\Gamma_a \). \( H(a) \) admits a natural \( T_a \)-action. Put
$V(a) := H(a)/T_a$. By Holmann [34], $V(a)$ is a complex orbifold. Let $\pi_a : H(a) \to V(a)$ be the natural projection.

The following is well-known (see the discussion before Proposition 4 in [9]).

**Theorem 1.29.** Let $p = (z_1, \ldots, z_{n+2}) + Z \in H(a)$. Assume that there are exactly $k$ coordinates $z_{j_1}, \ldots, z_{j_k}$ all different from zero, $k \geq 2$. Then, $V(a)$ is non-singular at $\pi_a(p)$ if and only if

$$
\left[\frac{a_1, \ldots, a_{n+2}}{\ell \notin \{j_1, \ldots, j_k\}} \right] = \prod_{\ell \notin \{j_1, \ldots, j_k\}} \left(\frac{a_1, \ldots, a_{n+2}}{3}\right) \cdot \left(\frac{a_1, \ldots, a_{n+2}}{a_{k-1}, a_{k+1}, \ldots, a_{n+2}}\right).
$$

where $[m_1, \ldots, m_d]$ denotes the least common multiple of $m_1, \ldots, m_d \in \mathbb{N}$.

It follows readily

**Corollary 1.30.** Assume $n \geq 2$ and $(a_1, a_2, \ldots, a_{n+2}) = (4b_1, 4b_2, 2b_3, 2b_4, \ldots, 2b_{n+2})$, where $b_j \in \mathbb{Z}$, $b_j$ is odd, $j = 1, \ldots, n+2$. Let $p = (0, 0, 1, i, 0, 0, \ldots, 0) + Z \in H(a)$. Then, $V(a)$ is singular at $\pi_a(p)$.

The ideas in the next two (sub)subsections are heavily based on Theorem 1.29 and Corollary 1.30.

### 1.5.2. Smooth orbifold circle bundle over a singular orbifold

Put

$$X := \{ (z_1, \ldots, z_{n+2}) \in \mathbb{C}^{n+2}, \ z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+2}^{a_{n+2}} = 0, \ |z_1|^{2a_1} + |z_2|^{2a_2} + \cdots + |z_{n+2}|^{2a_{n+2}} = 1 \}.
$$

(1.48)

It can be checked that $X$ is a compact weakly pseudoconvex CR manifold of dimension $2n + 1$ with CR structure $T^{1,0}X := T^{1,0} \mathbb{C}^{n+2} \cap CTX$, where $T^{1,0} \mathbb{C}^{n+2}$ denotes the standard complex structure on $\mathbb{C}^{n+2}$.

Let $\alpha$ be the least common multiple of $a_1, \ldots, a_{n+2}$. Consider the following $S^1$ action on $X$:

$$S^1 \times X \to X, \quad e^{-i\theta} \circ (z_1, \ldots, z_{n+2}) \to (e^{-i\frac{\alpha}{a_1} \theta} z_1, \ldots, e^{-i\frac{\alpha}{a_{n+2}} \theta} z_{n+2}).
$$

(1.49)

One sees that the $S^1$ action is well-defined, locally free, CR and transversal. Moreover one has that the quotient $X/S^1$ is equal to $V(a)$, $a = (a_1, a_2, \ldots, a_{n+2})$. Hence, $X/S^1$ is a complex orbifold.

One sees, by using Corollary 1.30, that the above $X/S^1$ is singular if $n \geq 2$ and $(a_1, \ldots, a_{n+2}) = (4b_1, 4b_2, 2b_3, 2b_4, \ldots, 2b_{n+2})$, where $b_j \in \mathbb{Z}$, $b_j$ is odd, $j = 1, 2, \ldots, n+2$.

We now show that $(X, T^{1,0}X)$ is CR-isomorphic to the (orbifold) circle bundle associated with an orbifold line bundle over $X/S^1 = V(a)$.

To see this and to construct the circle bundle in the first place, let $L = (\overline{\mathbb{M}(a)} \times \mathbb{C})/\equiv$, where $(z_1, \ldots, z_{n+2}, \lambda) \equiv (\tilde{z}_1, \ldots, \tilde{z}_{n+2}, \tilde{\lambda})$ if

$$
\tilde{z}_j = e^m z_j, \quad j = 1, \ldots, n+2,
\tilde{\lambda} = e^{\alpha m} \lambda,
$$

where $m \in \mathbb{Z}$. We can check that $\equiv$ is an equivalence relation and $L$ is a holomorphic line bundle over $H(a)$. The equivalence class of $(z_1, \ldots, z_{n+2}, \lambda) \in \overline{\mathbb{M}(a)} \times \mathbb{C}$ is denoted by $[(z_1, \ldots, z_{n+2}, \lambda)]$. The complex 1-torus $T_a$ action on $L$ is given by the following:

$$T_a \times L \to L,
$$

(1.50)

$$
(t + i\theta) \circ [(z_1, \ldots, z_{n+2}, \lambda)] \to [(e^{\frac{t+\theta}{\alpha} z_1}, \ldots, e^{\frac{t+\theta}{\alpha} z_{n+2}}, e^{-i\frac{\theta}{\alpha} \lambda})],
$$

where $\alpha$ is the least common multiple of $a_1, \ldots, a_{n+2}$. One has that the torus action (1.50) is well-defined and $L/T_a$ is an orbifold line bundle over $H(a)/T_a = V(a)$. 

Let \( \tau : L \to L/T_a \) be the natural projection and for \( [(z_1, \ldots, z_{n+2}, \lambda)] \in L \), we write \( \tau([(z_1, \ldots, z_{n+2}, \lambda)]) = \{z_1, \ldots, z_{n+2}, \lambda + T_a\} \). One sees that the pointwise norm
\[
|[(z_1, \ldots, z_{n+2}, \lambda)] + T_a|^2_{h_{L/T_a}} := |\lambda|^2 \left( |z_1|^{2a_1} + |z_2|^{2a_2} + |z_3|^{2a_3} + \cdots + |z_{2n+2}|^{2a_{2n+2}} \right)^{-1}
\]
is well-defined as a Hermitian metric on \( L/T_a \). The (orbifold) circle bundle \( C(L/T_a) \) with respect to \( (L/T_a, h_{L/T_a}) \) is given by
\[
C(L/T_a) := \left\{ v \in L/T_a; \ |v|^2_{h_{L/T_a}} = 1 \right\}
\]
(1.51)
where \( T^{1,0}C(L/T_a) \) denotes the complex structure on \( L/T_a \). Moreover, the orbifold line bundle \( L/T_a \to V(a) \) satisfies a similar situation as in Theorem 1.27 (i.e. the space \( X/S^1 = V(a) \) here as the \( M/G \) there, is singular and \( C(L/T_a) \) as a (orbifold) circle bundle over \( M/G \) is smooth).

We are ready to give an CR isomorphism of \( X \) and the (orbifold) circle bundle \( C(L/T_a) \). Note \( C(L/T_a) \) admits a nature \( S^1 \) action:
\[
e^{-i\theta} \circ \left( [(z_1, \ldots, z_{n+2}, \lambda)] + T_a \right) = \left( [z_1, \ldots, z_{n+2}, e^{-i\theta} \lambda] \right) + T_a,
\]

Let \( \Phi : C(L/T_a) \to X \) be the smooth map defined as follows. For every \( [(z_1, \ldots, z_{n+2}, \lambda)] + T_a \in C(L/T_a) \), there is a unique \( (\hat{z}_1, \ldots, \hat{z}_{n+2}) \in X \) such that
\[
\left( [z_1, \ldots, z_{n+2}, \lambda] + T_a \right) = \left( [\hat{z}_1, \ldots, \hat{z}_{n+2}, 1] \right) + T_a
\]
Then, \( \Phi\left( [(z_1, \ldots, z_{n+2}, \lambda)] + T_a \right) := (\hat{z}_1, \ldots, \hat{z}_{n+2}) \in X \). It can be checked that \( \Phi \) is a CR embedding, globally one to one, onto and the inverse \( \Phi^{-1} : X \to C(L/T_a) \) is also a CR embedding. Moreover \( e^{-i\theta} \circ \Phi(x) = \Phi(e^{-i\theta} \circ x) \), \( \forall x \in C(L/T_a) \). We conclude \( \Phi \) is a CR isomorphism.

1.5.3. Family, non-pseudoconvex cases and deformations. In the notation of Subsection 1.5.1 we assume \( a_1 = 1 \), so
\[
M(a) = \left\{ (z_1, \ldots, z_{n+2}) \in \mathbb{C}^{n+2}; z_1 = -z_2^2 - \cdots - z_{n+2}^2 \right\}
\]
Fix a \( q = 2, 3, \ldots, n+1 \). Put, for \( t \in \mathbb{C} \),
\[
X_{q,t} := \left\{ (z_1, \ldots, z_{n+2}) + Z \in H(a); -|z_2^{a_2} + t z_3^{a_3}|^2 - |z_3|^{2a_3} - \cdots - |z_q|^{2a_q} + |z_{q+1}|^{2a_{q+1}} + \cdots + |z_{n+2}|^{2a_{n+2}} = 0 \right\}
\]
(1.52)
One can check that for each \( t \), \( X_{q,t} \) is a compact CR manifold of dimension \( 2n + 1 \) with CR structure \( T^{1,0}X_{q,t} := T^{1,0}H(a) \cap CT X_{q,t} \), where \( T^{1,0}H(a) \) denotes the natural complex structure inherited by \( M(a) \). Note \( X_{q,t_1} \) is diffeomorphic to \( X_{q,t_2} \) for \( t_1, t_2 \in \mathbb{C} \) since they can be connected through a (smooth) family of compact manifolds.

Let \( \tilde{a} \) be the least common multiple of \( a_1, \ldots, a_q \). Consider the following \( S^1 \) action on \( X_{q,t} \):
\[
S^1 \times X_{q,t} \to X_{q,t},
\]
(1.53)
\[
e^{-i\theta} \circ \left( (z_1, \ldots, z_{n+2}) + Z \right)
\]
\[
\to \left( e^{-i\tilde{a} \theta} (-z_2^{a_2} - \cdots - z_q^{a_q}) - z_{q+1}^{a_{q+1}} - \cdots - z_{n+2}^{a_{n+2}}, e^{-i\frac{\tilde{a}}{2} \theta} z_2, \ldots, e^{-i\frac{\tilde{a}}{2} \theta} z_q, z_{q+1}, \ldots, z_{n+2} \right) + Z.
\]
One sees that the \( S^1 \) action is well-defined, locally free, CR and transversal. This is an example for a family of CR manifolds admitting a transversal CR locally free \( S^1 \) action.

Moreover these CR manifolds \( X_{q,t} \) are not pseudoconvex.
Now we consider certain CR deformations of a compact CR manifold $X$ with a transversal CR locally free $S^1$ action. Let $F(x) \in C^\infty(X)$ with $TF = 0$ ($T$ the global real vector field induced by the $S^1$ action). Let $Z_1, \ldots, Z_n \in C^\infty(X, T^{1,0}X)$ be a basis for $T^{1,0}X$. Put

\begin{equation}
H^{1,0}X := \{Z_j + Z_j(F)T; j = 1, 2, \ldots, n\}.
\end{equation}

One can check that $H^{1,0}X$ is a CR structure and the $S^1$ action is locally free, CR and transversal with respect to this new CR structure $H^{1,0}X$ (see (2.10) via the BRT construction).

To see how “new” this CR structure $H^{1,0}X$ is, let’s take $X$ to be a circle bundle associated with a holomorphic line bundle $(L, || \cdot ||)$ over a compact complex manifold $M$. Consider a change of metric $|| \cdot || \rightarrow e^{-2f}|| \cdot ||$ on $L$ and the circle bundle $\tilde{X}$ thus induced by this new metric. By using the formula (1.56) below one sees that $H^{1,0}X$ of (1.54) for $F = -if$ is equivalent to $T^{1,0}\tilde{X}$. But is $(X, T^{1,0}X)$ CR equivalent to $(\tilde{X}, T^{1,0}\tilde{X})$? The answer is in general no. For instance, spherical CR structures on a certain topological type of $X$ can be obtained by using special metrics on $L$ (cf. [17]). Hence an arbitrary perturbation of the bundle metric, say by the multiplier $e^{-2f}$, would bring $X$ out of the spherical category. Note that the moduli space of spherical CR structures in [17] is finite dimensional. It follows that for $F$ a purely imaginary function on $X$, the CR structure $H^{1,0}X$ is in general not CR equivalent to $T^{1,0}X$.

If, however, $F$ is a real function, it is easily seen that the change $(z, \theta) \rightarrow (z, \theta + F)$ is globally defined, hence it gives a diffeomorphism $\phi$ of $X$. One sees $\phi_*(T^{1,0}X) = H^{1,0}X$, cf. (1.56) below. So in this case the CR structure $H^{1,0}X$ is equivalent to the original one.

1.6. **Proof of Theorem 1.2.** Notations as in Theorem 1.2 let $s$ be a local trivializing section of $L$ defined on some open set $U$ of $M$, $|s|^2_L = e^{-2\phi}$. Let $z = (z_1, \ldots, z_n)$ be holomorphic coordinates on $U$. We identify $U$ with an open set of $\mathbb{C}^n$ and have the local diffeomorphism:

\begin{equation}
\tau : U \times [-\varepsilon_0, \varepsilon_0] \rightarrow X, \ (z, \theta) \mapsto e^{-\phi(z)} s^*(z) e^{-i\theta}, \ 0 < \varepsilon_0 \leq \pi.
\end{equation}

Put $D = U \times [-\varepsilon_0, \varepsilon_0]$ as a canonical coordinate patch with $(z, \theta)$ canonical coordinates (with respect to the trivialization $s$) such that $T = \frac{\partial}{\partial \theta}$ (recall $T$ is the global real vector field induced by the $S^1$ action). Moreover one has

\begin{equation}
T^{1,0}X = \left\{ \frac{\partial}{\partial z_j} - i \frac{\partial \phi}{\partial z_j}(z) \frac{\partial}{\partial \theta}; \ j = 1, 2, \ldots, n \right\},
\end{equation}

and

\begin{equation}
T^{0,1}X = \left\{ \frac{\partial}{\partial \overline{z}_j} + i \frac{\partial \phi}{\partial \overline{z}_j}(z) \frac{\partial}{\partial \theta}; \ j = 1, 2, \ldots, n \right\},
\end{equation}

and

\begin{equation}
T^{s,1,0}X = \{dz_j; \ j = 1, 2, \ldots, n\}, \ T^{s,0,1}X = \{d\overline{z}_j; \ j = 1, 2, \ldots, n\}.
\end{equation}

See also Theorem 2.9 and proof of Proposition 4.2 for similar formulas in the general case of $S^1$ action.

Let $f(z) \in \Omega^{0,q}(D)$. By (1.57) we may identify $f$ with an element in $\Omega^{0,q}(U)$.

The key object in our proof is the map $A^q_m : \Omega^0_m(X) \rightarrow \Omega^{0,q}(M, L^m)$, to be defined as follows. Let $u \in \Omega^0_m(X)$. We can write $u(z, \theta) = e^{-im\theta} \hat{u}(z)$ (on $D$) for some $\hat{u}(z) \in \Omega^{0,q}(U)$. Then, on $U \subset M$, we define

\begin{equation}
A^q_m u := s^m(z) e^{im\phi(z)} \hat{u}(z) \in \Omega^{0,q}(U, L^m).
\end{equation}

We need to check the following.

i) $A^q_m$ in (1.58) is well-defined, hence gives rise to a global element $A^q_m u \in \Omega^{0,q}(M, L^m)$.

ii) It satisfies the commutativity $\overline{\partial} A^q_m = A^{q+1}_m \overline{\partial}$ (thus induces a map on respective cohomologies).
To check i) let $s$ and $s_1$ be local trivializing sections of $L$ on an open set $U$. Let $(z, \theta) \in \mathbb{C}^n \times \mathbb{R}$ and $(z, \eta) \in \mathbb{C}^n \times \mathbb{R}$ be canonical coordinates of $D$ with respect to $s$ and $s_1$ respectively ($D$ as the above). Set $|s|^2 h_{LL} = e^{-2\phi}$ and $|s_1|^2 h_{LL} = e^{-2\phi_1}$. We write (on $D$)

\begin{align}
(1.59) \\
&u = e^{-im\theta} \hat{u}(z) \\
&u = e^{-im\eta} \hat{u}_1(z).
\end{align}

To check i) amounts to the following

\begin{align}
(1.60) \\
& s^m(z)e^{m\phi(z)} \hat{u}(z) = s_1^m(z)e^{m\phi_1(z)} \hat{u}_1(z), \quad \forall z \in U.
\end{align}

Let $s_1 = gs$ for $g$ a unit on $U$. To find relations between $\phi$ and $\phi_1$, $\hat{u}$ and $\hat{u}_1$ in terms of $g$,

\[ |s_1|^2 h_{LL} = e^{-2\phi_1} = |g|^2 |s|^2 h_{LL} = e^{2\log|g|-2\phi}, \]

giving

\begin{align}
(1.61) \\
& \phi_1 = \phi - \log |g|.
\end{align}

For $\hat{u}$ and $\hat{u}_1$, we first claim the following ($\tau$ in (1.55) for $(z, \theta)$ and $\tau_1$ the similar one for $(z, \eta)$)

\begin{align}
(1.62) \\
& \text{If } \tau(z, \theta) = \tau_1(z, \eta), \text{ then } e^{-i\theta} \left( \frac{\phi(z)}{g(z)} \right)^{\frac{1}{2}} = e^{-i\eta} \text{ (with a certain branch of the square root)}.
\end{align}

**Proof of the claim** (1.62). Combining (1.55) and (1.61) one sees

\begin{align}
(1.63) \\
& \tau(z, \theta) = s^*(z)e^{-i\theta-\phi(z)} = s_1^*(z)g(z)e^{-i\theta-\phi(z)} \\
& = s_1^*(z)g(z)e^{-i\theta-\phi_1(z)-\log|g(z)|} \\
& = s_1^*(z)\left( \frac{g(z)}{g(z)} \right)^{\frac{1}{2}} e^{-i\theta-\phi_1(z)}.
\end{align}

The condition $\tau(z, \theta) = \tau_1(z, \eta)$ is the same as to say, by (1.55),

\begin{align}
(1.64) \\
& s^*(z)e^{-i\theta-\phi(z)} = s_1^*(z)e^{-i\eta-\phi_1(z)}.
\end{align}

By (1.63) and (1.64) we deduce that $\left( \frac{g(z)}{g(z)} \right)^{\frac{1}{2}} e^{-i\theta} = e^{-i\eta}$, as claimed. \hfill \Box

Now that the relations (1.61) and (1.62) have been found, the (1.60) follows by using (1.59).

Hence $A^{(q)}_m : \Omega^{0,q}_m(X) \rightarrow \Omega^{0,q}(M, L^m)$ is well-defined, proving i) above.

Moreover it is easily checked that $A^{(q)}_m$ is bijective. We omit the detail.

To prove ii) that $\bar{\partial}A^{(q)}_m = A^{(q+1)}_m \bar{\partial}_b$, by (1.56) and (1.57) one sees (on $D$)

\begin{align}
(1.65) \\
& \bar{\partial}_b u = \bar{\partial}_b (e^{-im\theta} \hat{u}) = \sum_{j=1}^n e^{-im\theta} dz_j \wedge (\frac{\partial \hat{u}}{\partial z_j}(z) + m \frac{\partial \phi}{\partial z_j}(z) \hat{u}(z)).
\end{align}

Hence (1.65) and (1.58) yield

\begin{align}
(1.66) \\
& A^{(q+1)}_m(\bar{\partial}_b u) = s^m(z)e^{m\phi(z)} \sum_{j=1}^n dz_j \wedge (\frac{\partial \hat{u}}{\partial z_j}(z) + m \frac{\partial \phi}{\partial z_j}(z) \hat{u}(z)) \\
& = s^m(z) \bar{\partial}(e^{m\phi(z)} \hat{u}(z)) \quad \text{on } U,
\end{align}

giving $\bar{\partial}A^{(q)}_m = A^{(q+1)}_m \bar{\partial}_b$. Theorem 1.2 follows.

**Remark 1.31.** The map $A^{(q)}_m$ does not depend on the metrics of the manifolds $X$ and $M$. In later sections we study the Kohn Laplacian and Kodaira Laplacian on $X$ and $M$ respectively, and try to establish a link between the two Laplacians (with the aim at the Kohn’s). In this regard we need equip $X$ and $M$ with appropriate metrics so that $A^{(q)}_m$ thus defined is also compatible with these metrics. Note a localization of this (metrical) construction (cf. Proposition 5.1) paves the way for our subsequent plan in this work.
Some difficulties (and ways out) for a straightforward generalization of the proof for this special case (globally free $S^1$ action) will be discussed in the subsection below.

1.7. The idea of the proofs of Theorem 1.3, Theorem 1.10 and Corollary 1.13. We will give an outline of main ideas of some proofs. For the proof of Theorem 1.14, some ideas are outlined in the beginning of Section 7. We refer to Section 2.2 and Section 2.3 for notations and terminologies used here. The main technical tool of our method lies in a construction of a heat kernel for the Kohn Laplacian associated to the $m$-th $S^1$ Fourier component.

1.7.1. Global difficulties. For simplicity we assume that $X$ is CR Kähler (cf. Definition 1.11) without $E$ and $(\cdot | \cdot)$ is induced by a CR Kähler form $\Theta$ on $X$. Write $\overline{\partial}_b^* \overline{\partial}_b$ for the adjoint of $\overline{\partial}_b$ with respect to $(\cdot | \cdot)$ and $\overline{\partial}_{b,m}^* = \overline{\partial}_b^* : \Omega_m^{0,q}(X) \to \Omega_m^{0,q}(X)$ with $\Omega_m^{0,+}(X)$ and $\Omega_m^{0,-}(X)$ denoting forms of even and odd degree. Consider

$$D_{b,m}^\pm := \overline{\partial}_{b,m} + \overline{\partial}_{b,m}^* : \Omega_m^{0,\pm}(X) \to \Omega_m^{0,\pm}(X), \quad m \in \mathbb{Z}$$

and let $\Box_{b,m}^\pm := D_{b,m}^\pm D_{b,m}^\pm : \Omega_m^{0,\pm}(X) \to \Omega_m^{0,\pm}(X)$ ($\Box_{b,m}^\pm := D_{b,m}^\pm D_{b,m}^\pm$ similarly).

Extending $\Box_{b,m}^\pm$ and $\Box_{b,m}$ to $L^2_{m,+}(X)$ and $L^2_{m,-}(X)$ ($L^2$-completion), respectively in the standard way, we will show in Theorem 3.5 that $\text{Spec} \Box_{b,m}^\pm$ are discrete subsets of $[0, \infty]$ and $\text{Spec} \Box_{b,m}^\pm$ consist of eigenvalues of $\Box_{b,m}^\pm$.

For $\nu \in \text{Spec} \Box_{b,m}^\pm$, let $\{f_{\nu}^\pm, \ldots, f_{\nu}^\pm\}$ be an orthonormal frame for the eigenspace of $\Box_{b,m}^\pm$ with eigenvalue $\nu$. Write $T^{a,0}\bullet X = \oplus_{0 \leq q \leq n} T^{a,q}X$. $e^{-t\Box_{b,m}^\pm}(x,y) : T_y^{0,+}X \to T_x^{0,+}X$, said to be a heat kernel, is given by (cf. (1.12))

$$e^{-t\Box_{b,m}^\pm}(x,y) = \sum_{\nu \in \text{Spec} \Box_{b,m}^\pm} \sum_{j=1}^{d_{\nu}} e^{-t\nu} f_{\nu}^\pm(x) \wedge (f_{\nu}^\pm(y))^\dagger.$$ 

(Similarly we can define $e^{-t\Box_{b,m}^\pm}(x,y)$.)

We will show in Corollary 4.8 (see also Remark 4.9) that we have a CR McKean-Singer type formula: for $t > 0$,

$$\sum_{j=0}^{n}{(-1)^j \dim H_j^{b,m}(X)} = \int_X \left( \text{Tr} e^{-t\Box_{b,m}^\pm}(x,x) - \text{Tr} e^{-t\Box_{b,m}^\pm}(x,x) \right) dv_X.$$ 

By this formula the proof of our index theorem (cf. Corollary 1.13) is reduced to determining the small $t$ behavior of the function $\left( \text{Tr} e^{-t\Box_{b,m}^\pm}(x,x) - \text{Tr} e^{-t\Box_{b,m}^\pm}(x,x) \right)$.

With the kernel $e^{-t\Box_{b,m}^\pm}(x,y)$ there is associated an operator denoted by $e^{-\Box_{b,m}^\pm} : \Omega^{0,+}(X) \to \Omega^{0,+}_m(X)$. Note the domain is set to be the full space $\Omega^{0,+}(X)$. From (1.67) it follows that the kernel satisfies a heat equation which is expressed in the following operator form

$$\frac{\partial e^{-t\Box_{b,m}^\pm}}{\partial t} + \Box_{b,m}^\pm e^{-t\Box_{b,m}^\pm} = 0$$

and

$$e^{-t\Box_{b,m}^\pm}|_{t=0} = Q_{b,m}^\pm,$$

where $Q_{b,m}^\pm : L^{2,+}_m(X) \to L^{2,+}_m(X)$ is the orthogonal projection.

The main difficulty lies in that the initial condition (1.70) is a projection operator rather than an identity operator because we are dealing with part of the $L^2$ space (i.e. the $m$-th eigenspaces) rather than the whole $L^2$ space (as in the usual case). In a similar vein, let us quote in a paper of Richardson [55, p. 358]: “A point of difficulty that often arises in this area of research is that the space...is not the set of all sections of any vector bundle, and therefore the usual theory of elliptic operators and heat
kernels does not apply directly...”. The condition (1.70) eventually leads to the result that the heat kernels $e^{-t□^+_b,m}(x,y)$ do not have the standard expansions (as usually seen).

For a better understanding let’s assume that $X$ is a (orbifold) circle bundle of an orbifold line bundle $L$ over a Kähler orbifold $M$ (see Section 1.5 for specific examples). As in Theorem 1.2 (see Section 1.6), one sees bijective maps

$$A^\pm_m : \Omega^b_m(X) \to \Omega^b_m(M, L^m)$$

such that $A^-_m \overline{\partial}_b = \overline{D} A^+_m$. Let $\Box^+_m$ be the Kodaira Laplacian with values in $T^{*0,1/2} \otimes L^m$ and let $e^{-t\Box^+_m}$ be the associated heat operator. Consider $B_m(t) := (A^+_m)^{-1} \circ e^{-t\Box^+_m} \circ A^+_m$. $A^+_m$ are metric-independent (on a given $X$). To get a link between $\Box^+_b,m$ and $\Box^+_m$ it requires, however, a compatible choice of metrics on $X$ and $M$. With this done, one checks that $B'_m(t) + \Box^+_{b,m} B_m(t) = 0$ and $B_m(0) = I$ on $\Omega^0_m(X)$.

But $B_m(t)$ is not the heat operator $e^{-t\Box^+_b,m}$. A trivial reason is that $B_m(t)$ is defined on $\Omega^0_m(X)$ while $e^{-t\Box^+_b,m}$ is on the whole $\Omega^1_m(X)$. In fact one has

$$e^{-t\Box^+_b,m} = ((A^+_m)^{-1} \circ e^{-t\Box^+_m} \circ A^+_m) \circ Q^+_m = B_m(t) \circ Q^+_m = Q^+_m \circ B_m(t) \circ Q^+_m).$$

Let $B_m(t,x,y)$ be the distribution kernel of $B_m(t)$. To emphasize the role played by $Q^+_m$ in our construction, it is illuminating to note the following (cf. (1.71), (2.2) and (4.17))

$$e^{-t\Box^+_b,m}(x,y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} B_m(t,x,e^{-iu} \circ y)e^{-imu} du.$$  

For $x \in X_p$ (the principal stratum), from (1.72) and the much better known kernel $e^{-t\Box^+_m}$ (on the principal stratum of $M$) it follows

$$e^{-t\Box^+_b,m}(x,x) \sim t^{-n} a^+_m(x) + t^{-(n-1)} a^{+1}_{m-1}(x) + \cdots.$$  

However for $x \notin X_p$, by lack of the asymptotic expansion of $B_m(t)$ (or $e^{-t\Box^+_m}$ on low dimensional strata of $M$) it is unclear how one can understand the asymptotic behavior of $e^{-t\Box^+_b,m}(x,x)$ by means of (1.72). This presents a major deviation from proof of the globally free case (as Theorem 1.2, cf. (1.1), (1.2)).

To see more clearly the discrepancy between the two cases (locally free and globally free) we note that the expansion (1.73) converges only locally uniformly on $X_p$, due to a nontrivial contribution involving a “distance function” (see Subsection 1.7.3 for more). In fact the expansion of the form (1.73) which is usually seen, cannot hold here (globally on $X$) (cf. Remark 1.6).

It is thus not immediate for one to arrive at a detailed understanding of the (transversal) heat kernel by only using the global argument. Even in the (smooth) orbifold circle bundle case, to understand the asymptotic behavior of the heat operator $e^{-t\Box^+_b,m}$ we will still need to work directly on the CR manifold $X$ instead of $M$.

In this paper we give a construction which is independent of the use of orbifold geometry and is more adapted to CR geometry as our CR manifold $X$ is not assumed to be an orbifold circle bundle of a complex orbifold. Because of the failure of the global argument as just said, we are now led to work on it locally. The framework for this is BRT trivialization (Section 2.4) which is first treated by Baouendi, Rothschild and Treves [4] in a more general context.

1.7.2. Transition to local situation. Let $B := (D, (z, \theta), \varphi)$ be a BRT trivialization (see Theorem 2.9). We write $D = U \times] - \varepsilon, \varepsilon[$, where $\varepsilon > 0$ and $U$ is an open set of $\mathbb{C}^n$. Let $L \to U$ be a trivial line bundle with a non-trivial Hermitian fiber metric $|1|^2_{h^L} = e^{-2\varphi}$ (where $\varphi \in C^\infty(D, \mathbb{R})$ is as in Theorem 2.9) and $(L^m, h^{L^m}) \to U$ be the $m$-th power of $(L, h^L)$. $\Theta$ (cf. Definition 1.11, recalling $X$ is CR Kähler as assumed for the moment) induces a Kähler form $\Theta_U$ on the complex manifold $U$. Let $(\cdot, \cdot)$ be the Hermitian metric on $CTU$ (associated with $\Theta_U$), inducing together with $h^{L^m}$ the $L^2$ inner product $(\cdot, \cdot)_m$ on $\Omega^{b,m}(U, L^m)$.
Let $\overline{\partial}^{*,m} : \Omega^{0,q+1}(U, L^m) \to \Omega^{0,q}(U, L^m)$ be the formal adjoint of $\overline{\partial}$ with respect to $(\cdot, \cdot)_m$. Put, as the case of $D_{b,m}$ and $\Box_{b,m}$, $D_{b,m}^{\pm} := \overline{\partial} + \overline{\partial}^* : \Omega^{0,q}(U, L^m) \to \Omega^{0,q}(U, L^m)$ and $\Box_{b,m}^{\pm} := D_{b,m}^2 + D_{b,m}^{\pm} : \Omega^{0,q}(U, L^m) \to \Omega^{0,q}(U, L^m)$.

In Proposition 5.1, by the above choice of metrics in forming Laplacians on two different spaces ($D$ and $U$), we can provide a link between these Laplacians, asserting that

\begin{equation}
(1.74)
\end{equation}

\[ e^{-m\varphi} \Box_{b,m}^{\pm} e^{m\varphi \overline{\partial}^*} = e^{im\theta} \Box_{b,m}^{\pm} (u) \]

where as before, $u \in \Omega^{0,\pm}(X)$ can be written (on $D$) as $u(z, \theta) = e^{-im\theta} \overline{u}(z)$ for some $\overline{u}(z) \in \Omega^{0,\pm}(U, L^m) \subset \Omega^{0,\pm}(D, L^m)$.

Write $x = (z, \theta)$, $y = (w, \eta)$ (on $D$). With (1.74) one expects that the heat kernel $e^{-t \Box_{b,m}^{\pm}(x, y)}$ locally (on $D$) should be

\begin{equation}
(1.75)
\end{equation}

\[ e^{-m\varphi(z)-im\theta} e^{-t \Box_{b,m}^{\pm}(x, y)} e^{m\varphi(y)+im\eta}. \]

Thus one obtains local heat kernels on these BRT charts.

We would like to patch them up. Assume that $X = D_1 \cup D_2 \cup \cdots \cup D_N$ (where $D_j$ in a BRT trivialization $B_j := (D_j, (z, \theta), \varphi_j)$ with $D_j = U_j \times [\delta_j, \delta_j + 1]$ and $\delta_j > 0$, $\tilde{\delta}_j > 0$, $U_j$ is an open set in $\mathbb{C}^n$).

Let $\chi_j, \tilde{\chi}_j \in C_0^\infty(D_j)$ ($j = 1, 2, \ldots, N$). Put

\begin{equation}
(1.76)
\end{equation}

\[ A_m(t) = \sum_{j=1}^N \chi_j(x) \left( e^{-m\varphi_j(z)-im\theta} e^{-t \Box_{b,m}^{\pm}(z, w)} e^{m\varphi_j(w)+im\eta} \right) \tilde{\chi}_j(y), \]

\[ P_m(t) = A_m(t) \circ Q_m. \]

It is hoped that $P_m(0) = Q_m^+$ and $P_m'(t) + \Box_{b,m}^{\pm} P_m(t)$ is small as $t \to 0^+$ for certain $\chi_j, \tilde{\chi}_j$. This is related to asymptotic heat kernel. But as we will see, this standard patch-up construction does not quite work out in our case.

In short, we will see that in the locally free case the nice (pointwise) relation (1.74) between Kodaira and Kohn Laplacians does not quite carry over to the global objects: heat kernels, whose mutual relation is to be seen below by more delicate analysis relevant to the presence of strata beyond the principal stratum.

1.7.3. Local difficulties. A necessary condition for $P_m(0) = Q_m^+$ is (cf. Lemma 5.10)

\begin{equation}
(1.77)
\end{equation}

\[ \sum_{j=1}^N \chi_j(x) \int_{-\pi}^{\pi} \tilde{\chi}_j(w, \eta)|_{w=z} d\eta = 1. \]

For the cut-off functions $\chi_j, \tilde{\chi}_j$ above, a reasonable choice (adapted to BRT trivializations) is the following (for $j = 1, 2, \ldots, N$):

i) $\chi_j(z, \theta) \in C^\infty_0(D_j)$ with $\sum_{j=1}^N \chi_j = 1$ on $X$;

ii) $\tau_j(z) \in C^\infty_0(U_j)$ with $\tau_j(z) = 1$ if $(z, \theta) \in \text{Supp } \chi_j$,

iii) $\sigma_j \in C^\infty_0([\delta_j, \delta_j])$ with $\int_{\delta_j}^{\delta_j} \sigma_j(\eta) d\eta = 1$.

Set $\tilde{\chi}_j(y) \equiv \tau_j(w) \sigma_j(\eta)$. Then $\chi_j(x), \tilde{\chi}_j(y)$ satisfy (1.77).

One can check $P_m(0) = Q_m^+$ and a little more work shows

\begin{equation}
(1.78)
\end{equation}

\[ P'_m(t) + \Box_{b,m}^{\pm} P_m(t) = R_m(t) \circ Q_m \]

where for some $k$,

\begin{equation}
(1.79)
\end{equation}

\[ R_m(t) = \sum_{j=1}^N \sum_{\ell=1}^k L_{\ell,j} \left( \chi_j(x) e^{-m\varphi_j(z)-im\theta} \right) \left( e^{-t \Box_{b,m}^{\pm}(z, w)} e^{m\varphi_j(z)+im\eta} \tilde{\chi}_j(y), \right). \]
$L_{\ell,j}$ is a partial differential operator of order $\geq 1$ and $\leq 2$ (for all $\ell, j$) and $P_{\ell,j}$ is a partial differential operator of order 1 acting on $x$ (for all $\ell, j$). Since $|e^{-t\Box_{B_j,m}}(z, w)| \sim \frac{1}{t^n} e^{-\frac{|z-w|^2}{t}}$, there could be terms of the form, say

$$(1.80) \quad P_{\ell,j}(e^{-t\Box_{B_j,m}}(z, w)) \sim \frac{1}{t^n} e^{-\frac{|z-w|^2}{t}}.$$  

To require $P'_m(t) + \Box_{b,m} P_m(t)$ to be small (as $t \to 0^+$) we need (by substituting (1.80) into (1.79) to get singular terms in powers of $\frac{1}{t}$ smooth out):

$$(1.81) \quad L_{\ell,j}(\chi_j(x) e^{-m\varphi_j(z) - im\theta} + im\eta \tilde{\chi}_j(y) = 0 \text{ if } z \text{ is close to } w \ (|z - w| \lesssim \sqrt{t}).$$

Since $\chi_j$ may not be constant on $\text{Supp } \tilde{\chi}_j$ (for some $j$), it is hard for (1.81) to hold. Despite that in the usual (elliptic) case a construction of the heat kernel using cut-off functions as above is available, in view that a distance function will appear in our asymptotic expression (cf. (1.85) below) it is unclear whether this type of standard construction can be immediately carried out in our case.

It turns out that upon transferring to an adjoint version of the original equation one may bypass the aforementioned difficulty (cf. Lemmas 5.10, 5.11), to which we turn now.

For $j = 1, 2, \ldots, N$ there exists $A_{B_j,+}(t, z, w) \in C^\infty(\mathbb{R}_+ \times U_j \times U_j, T^{a_0}U \boxtimes (T^{a_0}U)^*)$, cf. Theorem 5.6, regarded as an adjoint heat kernel, such that

$$(1.82) \quad \lim_{t \to 0^+} A_{B_j,+}(t) = I \text{ in } \mathcal{D}'(U, T^{a_0}U),$$

$$A_{B_j,+}(t)u + A_{B_j,+}(t)(\Box_{B_j,m} u) = 0, \; \forall u \in \Omega_{0+}(U), \; \forall t > 0,$$

and $A_{B_j,+}(t, z, w)$ admits an asymptotic expansion as $t \to 0^+$ (see (5.19)). Put

$$(1.83) \quad H_j(t, x, y) = \chi_j(x) e^{-m\varphi_j(z) - im\theta} A_{B_j,+}(t, z, w) e^{m\varphi_j(w) + im\eta \tilde{\chi}_j(y)}.$$

Also set

$$(1.84) \quad \Gamma(t) := \sum_{j=1}^N H_j(t) \circ Q_m^+ : \Omega_{0+}(X) \to \Omega_{0+}(X).$$

By using the adjoint equation, we can avoid the difficulty mentioned in (1.81) so that $\Gamma(t)$ gives an asymptotic (adjoint) heat kernel (see that below (1.76)). To get back to the kernel of the original equation, we can now start with the adjoint of $\Gamma(t)$. By carrying out the (standard) method of successive approximation, we can reach the global kernel of the adjoint of $e^{-t\Box_{b,m}}$ (Section 5.2). This yields the kernel of $e^{-t\Box_{b,m}}$ since $e^{-t\Box_{b,m}}$ is self-adjoint. More precisely we can prove that (see Theorem 5.14 and Theorem 6.1)

$$(1.85) \quad \left\| e^{-t\Box_{b,m}}(x, y) - \Gamma(t, x, x) \right\|_{C^0(X \times X)} \leq e^{-\frac{\rho d(x, X_{\text{sing}})^2}{t}}, \; \forall t \in (0, \varepsilon_2),$$

where $\alpha_s^+(x) \in C^\infty(X, \text{End} (T^{a_0}X \otimes E))$, $s = n, n-1, \ldots, \varepsilon_0, \varepsilon_1, \varepsilon_2 > 0$ some constants and $\rho$ a sort of “distance function” (discussed above Theorem 1.3).

The appearance of this distance function $\rho$ may be attributed to the use of projection $Q_m^+$ in (1.84) (which picks up the $m$-th Fourier component; see (5.40) and (6.8)). See below for more about this point. By the first inequality in (1.85) one obtains the (same) asymptotic expansion

$$(1.86) \quad e^{-t\Box_{b,m}}(x, x) \sim \left(\sum_{s=1}^p e^{2s(x-1)} \int_{\mathbb{R}^n} \sum_{j=0}^\infty t^{-n-j} \alpha_{n-j}(x) \ mod O \left(\frac{t^{-n} e^{-\rho d(x, X_{\text{sing}})^2}}{t}\right)\right)$$
on $X_p$. Similar results hold for $e^{-t\Box_{b,m}^-}(x,x)$.

The terms involved in $O\left(t^{-n}e^{-\frac{\rho(x,X_{\text{sing}})}{t}}\right)$ of (1.86) are singular (due to $t^{-n}$ as $x \to X_{\text{sing}}$). Only upon taking the supertrace can these terms be (partially) cancelled ($t^{-n}$ dropping out). That is, for $x \in X_p$

\[
\text{Tr } e^{-t\Box_{b,m}^-}(x,x) - \text{Tr } e^{-t\Box_{b,m}^+}(x,x)
\]

(1.87)

\[
\sim \left(\sum_{s=1}^{p} e^{\frac{2\pi(s-1)mi}{p}}\right) \sum_{j=0}^{\infty} t^{-n+j} \left(\text{Tr } \alpha_{n-j}^+(x) - \text{Tr } \alpha_{n-j}^-(x)\right) \mod O\left(e^{-\frac{\rho(x,X_{\text{sing}})^2}{t}}\right).
\]

To see this conceptually, let’s take, for instance, (1.71) and (1.72) in which along the diagonal (i.e. setting $x = y$ to the left of (1.72)), the off-diagonal contribution (in the term to the right of the same equation) still enters nontrivially (unseen in the usual elliptic case) due to the projection $Q_m^+$.

To get estimates on these off-diagonal terms our argument (cf. Theorem 5.9) is based on the rescaling technique of Getzler and on a supertrace identity in Berenzin integral (cf. Prop. 3.21 of [5]), which combine to give the needed (partial) cancellation.

From (1.68), (1.86) and (1.87) it follows

(1.88) \[
\sum_{j=0}^{n} (-1)^j \dim H^j_{b,m}(X,E) \left(\sum_{s=1}^{p} e^{\frac{2\pi(s-1)mi}{p}}\right) \lim_{t \to 0^+} \int_X \sum_{\ell=0}^{n} t^{-\ell} \left(\text{Tr } \alpha_{\ell}^+(x) - \text{Tr } \alpha_{\ell}^-(x)\right) dv_X(x).
\]

Remark that we have had a (transversal) heat kernel which is put in the disguise of the spectral geometry (1.67), (4.15). To our knowledge no argument in the literature claims that (in the transversally elliptic case) the spectral heat kernel shall have the asymptotic estimates as (1.86). The somewhat lengthy part of our reconstruction of the (transversal) heat kernel (beyond its spectral realization) becomes indispensable as far as our purpose is concerned.

1.7.4. Completion by evaluating local density and by using Spin$^c$ structure. As above we first treat the case that $X$ is CR Kähler (Definition 1.11). In view of (1.88), to complete the proof of our index theorem (cf. Corollary 1.13) amounts to understanding the small $t$ behavior of the local density

\[
\sum_{\ell=0}^{n} t^{-\ell} \left(\text{Tr } \alpha_{\ell}^+(x) - \text{Tr } \alpha_{\ell}^-(x)\right).
\]

Let’s be back to the local situation. Fix $x_0 \in X_p$. Let $B_j = (D_j, (z,\theta), \varphi_j)$ ($j = 1,2,\ldots,N$) be BRT trivializations as before. Assume that $x_0 \in D_j$ and $x_0 = (z_j,0) \in U_j \subset D_j$.

As our heat kernel (on $X$) is related to the local heat kernel (on $U_j$), one sees (for some $N_0(n) \geq n$)

\[
\sum_{\ell=0}^{n} t^{-\ell} \left(\text{Tr } \alpha_{\ell}^+(x_0) - \text{Tr } \alpha_{\ell}^-(x_0)\right)
\]

(1.89)

\[
= \frac{1}{2\pi} \sum_{j=1}^{N} \chi_j(x_0) \left(\text{Tr } A_{B_j,+}(t,z_j,z_j) - \text{Tr } A_{B_j,-}(t,z_j,z_j)\right) + O(t),
\]

where $A_{B_j,+}(t,z,w)$ is as in (1.82).

By borrowing the rescaling technique in [5] and [22] we can show (in a fairly standard manner, cf. Theorem 5.8 or the second half of this section) that for each $j = 1,2,\ldots,N$,

\[
\left(\text{Tr } A_{B_j,+}(t,z,z) - \text{Tr } A_{B_j,-}(t,z,z)\right) dv_{U_j}(z)
\]

(1.90)

\[
= [\text{Td } (\nabla^{T^{1,0}U_j}, T^{1,0}U_j) \wedge \text{ch } (\nabla^{L_m}, L_m)]_{2n}(z) + O(t), \; \forall z \in U_j,
\]

$(dv_{U_j}$ the induced volume form on $U_j$) where $\text{Td } (\nabla^{T^{1,0}U_j}, T^{1,0}U_j)$ and $\text{ch } (\nabla^{L_m}, L_m)$ denote the representatives of the Todd class of $T^{1,0}U_j$ and the Chern character of $L_m$, respectively.
A novelty here is Section 2.3 in which we will introduce tangential characteristic classes, tangential Chern character and tangential Todd class on CR manifolds with \( S^1 \) action, so that

\[
\left[ \text{Td} \left( \nabla^{T,0}U_j, T^{1,0}U_j \right) \wedge \text{ch} \left( \nabla L^m, L^m \right) \right]_{2n}(z_j) = \left[ \text{Td}_b \left( \nabla^{T,0}X, T^{1,0}X \right) \wedge e^{-m \frac{d\omega_0}{dt}} \wedge \omega_0 \right]_{2n+1}(x_0),
\]

where \( \text{Td}_b \left( \nabla^{T,0}X, T^{1,0}X \right) \) denotes the representative of the tangential Todd class of \( T^{1,0}X \) (associated with the given Hermitian metric (2.9)). From (1.89), (1.90) and (1.91) it follows

\[
\sum_{t=0}^{n} t^{-\ell} \left( \text{Tr} \, \alpha_\ell^+(x) - \text{Tr} \, \alpha_\ell^-(x) \right) dv_X(x)
\]

(1.92)

\[
= \frac{1}{2\pi} \left[ \text{Td}_b \left( \nabla^{T,0}X, T^{1,0}X \right) \wedge e^{-m \frac{d\omega_0}{dt}} \wedge \omega_0 \right]_{2n+1}(x) + O(t), \quad \forall x \in X_p.
\]

(The \( O(t) \) term to the rightmost of (1.92) actually vanishes by using (5.20).)

Combining (1.92) and (1.88) we get our index theorem (cf. Corollary 1.13) when \( X \) is CR Kähler. When \( X \) is not CR Kähler, we still have (1.85), (1.86) and (1.88). The ensuing obstacle is more or less known:

i) the rescaling technique does not quite work well as the local operator \( \Box_{B,J}^+ \) in (1.82) is not going to be of Dirac type (in a strict sense);

ii) it is obscure to understand the small \( t \) behavior of \( A_{B,J,+}(t, z, z) \) in this case;

iii) (1.90) is not even true in general.

To overcome this difficulty in the CR case, we will give the classical (yet non-Kähler) case and introduce some kind of CR Spin\(^c\) Dirac operator on CR manifolds with \( S^1 \) action:

\[
\bar{D}_{b,m} = \bar{\partial}_b + \overline{\partial}_b + \text{zeroth order term}
\]

with modified/Spin\(^c\) Kohn Laplacians \( \tilde{\Box}_{b,m}^+ \) defined as \( \tilde{D}_{b,m}^+ \overline{D}_{b,m}^+ - \overline{D}_{b,m} D_{b,m}^+ = \tilde{D}_{b,m} D_{b,m}^+ \).

A word of caution is in order. The above adaptation of the idea of Spin\(^c\) structure to our CR case is not altogether straightforward. Locally \( X \) is realized as a (portion of a) circle bundle over a small piece of complex manifold (via BRT charts), so presumably there could arise a problem of patching up when this global Spin\(^c\) operator is to be formed. See Proposition 4.2 for more.

We will show in Theorem 4.7 the homotopy invariance for the index of \( \overline{\partial}_b + \partial_b \), and in Corollary 4.8 a McKean-Singer formula for the modified Kohn Laplacians: for \( t > 0 \),

\[
\sum_{j=0}^{n} (-1)^j \dim H^j_{b,m}(X, E) = \int_X \left( \text{Tr} \, e^{-\overline{\partial}_{b,m}^+} - \text{Tr} \, e^{-\overline{\partial}_{b,m}^-} \right) dv_X(x).
\]

(1.93)

For \( u \in \Omega^{0,\pm}_{m}(X) \) we can write (on \( D \)) \( u(z, \theta) = e^{-im\theta} \tilde{u}(z) \) for some \( \tilde{u}(z) \in \Omega^{0,\pm}(U, L^m) \) with \( D \) in a BRT trivialization \( B := (D, (z, \theta), \varphi) \).

A fundamental relation that we will show in Proposition 5.1, based on Proposition 4.2, is that

\[
e^{-m\varphi} \overline{\Box}_{B,m}^\pm(e^{m\varphi}u) = e^{im\theta} \overline{\Box}_{b,m}^\pm(u)
\]

(1.94)

where \( \overline{\Box}_{B,m}^\pm = D_{B,m}^+ D_{B,m} \) and \( \Omega^{0,\pm}(U, L^m) \to \Omega^{0,\pm}(U, L^m) \) and \( D_{B,m} : \Omega^{0,\pm}(U, L^m) \to \Omega^{0,\pm}(U, L^m) \) the (ordinary) Spin\(^c\) Dirac operator (cf. Definition 4.1) with respect to the Chern connection on \( L^m \) (induced by \( hL^m \)) and the Clifford connection on \( \Lambda(T^{*,0,1}U) \) (induced by the given Hermitian metric on \( \Lambda(T^{*,0,1}U) \)).

It is conceivable that \( X \) with the CR structure and \( X/S^1 = M \) with the complex structure (if defined) are linked in some way (as Theorem 1.2). To say more, the result (1.94) asserts a fundamental fact that not only complex/CR geometrically can the two spaces be linked, but metrically in the sense of Laplacians they also can. This link is important for our Spin\(^c\) approach to the CR case to be possible.
In the remaining let’s give an outline with the CR Spin\(^c\) Dirac operator when \(X\) is not CR Kähler. Although the following ingredients mostly parallel those in the preceding Subsection 1.7.3, the success of this method relies on, among others, the Spin\(^c\) structure and the associated Clifford connection. For that reason and for the sake of clarity, we prefer to put down the precise formulas despite the great similarity in expressions as above.

As (1.82), there exists (modified) \(\tilde{A}_{B,j,+}(t, z, w)\) such that

\[
\lim_{t \to 0^+} \tilde{A}_{B,j,+}(t) = I \quad \text{in} \quad \mathcal{D}'(U_j, T^*0^0 + U_j),
\]

\[
\tilde{A}'_{B,j,+}(t) u + \tilde{A}_{B,j,+}(t)(\bar{\square}_{B,j,m} u) = 0, \quad \forall u \in \Omega^{0,0+}_0(U_j), \quad \forall t > 0,
\]

and \(\tilde{A}_{B,j,+}(t, z, w)\) admits an asymptotic expansion as \(t \to 0^+\) (see (5.19)). Put

\[
\tilde{H}_j(t, x, y) = \chi_j(x)e^{-m\varphi_j(z) - im\theta}\tilde{A}_{B,j,+}(t, z, w)e^{m\varphi_j(w) + im\eta_j\chi_j(y)},
\]

\[
\tilde{\Gamma}(t) = \sum_{j=1}^N \tilde{H}_j(t) \circ \mathcal{Q}_m.
\]

Similar to (1.85) and (1.86) in Subsection 1.7.3, one has

\[
\left\|e^{-t\bar{\square}_{b,m}^+}(x, y) - \tilde{\Gamma}(t, x, y)\right\|_{C^0(X \times X)} \leq e^{-\frac{t}{\epsilon}}, \quad \forall t \in (0, \epsilon_2)
\]

and

\[
\tilde{\Gamma}(t, x, x) \sim \left(\sum_{s=1}^P e^{\frac{2\pi(s-1)}{p} m_i}\right) \sum_{j=0}^{\infty} t^{-n+j} \tilde{\alpha}_{n-j}^+(x) \mod O\left(t^{-n}e^{-\frac{\sqrt{(x, x)}^2}{t}}\right), \quad \forall x \in X_p,
\]

with some constants \(\epsilon_0, \epsilon_1, \epsilon_2 > 0\), giving

\[
e^{-t\bar{\square}_{b,m}^+(x, x)} \sim \left(\sum_{s=1}^P e^{\frac{2\pi(s-1)}{p} m_i}\right) \sum_{j=0}^{\infty} t^{-n+j} \left(\text{Tr} \tilde{\alpha}_{n-j}^+(x) - \text{Tr} \tilde{\alpha}_{n-j}^-(x)\right) \mod O\left(t^{-n}e^{-\frac{\sqrt{(x, x)}^2}{t}}\right),
\]

on \(X_p\). Similar results hold for \(e^{-t\bar{\square}_{b,m}^-(x, x)}\).

The novelty here is analogous to (1.87). By taking supertrace we can improve the estimates in (1.99) (see Theorem 6.4) so that \(t^{-n}\) is removed:

\[
\text{Tr} e^{-t\bar{\square}_{b,m}^+(x, x)} - \text{Tr} e^{-t\bar{\square}_{b,m}^-(x, x)} \\
\sim \left(\sum_{s=1}^P e^{\frac{2\pi(s-1)}{p} m_i}\right) \sum_{j=0}^{\infty} t^{-n+j} \left(\text{Tr} \tilde{\alpha}_{n-j}^+(x) - \text{Tr} \tilde{\alpha}_{n-j}^-(x)\right) \mod O\left(e^{-\frac{\sqrt{(x, x)}^2}{t}}\right),
\]

for \(x \in X_p\). Hence (1.93) and (1.100) give

\[
\sum_{j=0}^{n} (-1)^j \dim H_{b,j}^2(X, E) = \left(\sum_{s=1}^P e^{\frac{2\pi(s-1)}{p} m_i}\right) \lim_{t \to 0^+} \int_X \sum_{j=0}^{n} t^{-j} \left(\text{Tr} \tilde{\alpha}_{n-j}^+(x) - \text{Tr} \tilde{\alpha}_{n-j}^-(x)\right) dv_X(x).
\]

A key advantage of introducing our CR Spin\(^c\) Dirac operator is basically that Lichnerowicz formulas hold for \(\bar{\square}_{B,m}^+\) (and \(\bar{\square}_{B,m}^-\)). This enables us to apply the rescaling technique (this part of rescaling is essentially the same as in classical cases, cf. [5] and [22]) and to obtain that for each \(j = 1, 2, \ldots, N\),

\[
\left(\text{Tr} \tilde{A}_{B,j,+}(z, z) - \text{Tr} \tilde{A}_{B,j,-}(z, z)\right) dv_{U_j}(z)
\]

\[
= [T d (\nabla^{1,0} U_j, T^{1,0} U_j) \wedge (\nabla^{L^m, L^m})]_{2n}(z) + O(t), \quad \forall z \in U_j.
\]
Rewriting (1.102) in tangential forms, one has
\[
\sum_{t=0}^{n} t^{-\ell} \left( \text{Tr} \tilde{\alpha}^+_t(x) - \text{Tr} \tilde{\alpha}^-_t(x) \right) d\nu_X(x)
\]
(1.103)
where
\[
= \frac{1}{2\pi} \left[ Td_{b,m} \left( \nabla^{T^{1,0}X}, T^{1,0}X \right) \wedge e^{-m \frac{d\omega}{\pi}} \wedge \omega_0 \right]_{2n+1}(x) + O(t)
\]
for \( t > 0 \) and \( x \in X_p \).

Theorem 1.3, Theorem 1.10 and Corollary 1.13 follows from (1.99), (1.100), (1.101) and (1.103).

The layout of this paper is as follows. In Section 2.1 and Section 2.2, we collect some notations, definitions, terminologies and statements we use throughout. In Section 2.3, we introduce the tangential de Rham cohomology group, tangential Chern character and tangential Todd class on CR manifolds with \( S^1 \) action. In Section 2.4, we prove a classical result of Baouendi-Rothschild-Treves [4] which plays an important role in our construction of the heat kernel. We also prove that for a rigid vector bundle \( F \) over \( X \) there exist rigid Hermitian metric and rigid connection on \( F \). In Section 3, we establish a Hodge theory for Kohn Laplacian in the \( L^2 \) space of the \( m \)-th \( S^1 \) Fourier component. In Section 4, we introduce our CR \( \text{Spin}^c \) Dirac operator \( D_{b,m} \), modified/\( \text{Spin}^c \) Kohn Laplacians \( \tilde{\square}^\pm_{b,m} \) and prove (1.93). In Section 5, we construct approximate heat kernels for the operators \( e^{-\tilde{\square}^\pm_{b,m}} \) and prove that \( e^{-\tilde{\square}^\pm_{b,m}}(x, y) \) admit asymptotic expansions in the sense as (1.97). In Section 6, we prove (1.98), (1.100), (1.103) and finish the proofs of Theorem 1.3, Theorem 1.10 and Corollary 1.13. In Section 7 we prove Theorem 1.14.

**Part I: Preparatory foundations**

2. **Preliminaries**

2.1. **Some standard notations.** We use the following notations: \( \mathbb{N} = \{1, 2, \ldots\} \), \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), \( \mathbb{R} \) is the set of real numbers, \( \mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\} \). For a multiindex \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n \) we set \( |\alpha| = \alpha_1 + \ldots + \alpha_n \). For \( x = (x_1, \ldots, x_n) \) we write
\[
x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad \partial_{x_j} = \frac{\partial}{\partial x_j}, \quad \partial^\alpha = \partial^\alpha_1 \cdots \partial^\alpha_n = \frac{\partial^{|\alpha|}}{\partial x^\alpha},
\]
\[
D_{x_j} = \frac{1}{i} \partial_{x_j}, \quad D^\alpha = D^\alpha_1 \cdots D^\alpha_n, \quad D_x = \frac{1}{i} \partial_x.
\]

Let \( z = (z_1, \ldots, z_n) \), \( z_j = x_{2j-1} + i x_{2j}, j = 1, \ldots, n \), be coordinates of \( \mathbb{C}^n \). We write
\[
z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}, \quad \bar{z}^\alpha = \bar{z}_1^{\alpha_1} \cdots \bar{z}_n^{\alpha_n},
\]
\[
\partial_{z_j} = \frac{\partial}{\partial z_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_{2j-1}} - i \frac{\partial}{\partial x_{2j}} \right), \quad \partial_{\bar{z}_j} = \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_{2j-1}} + i \frac{\partial}{\partial x_{2j}} \right),
\]
\[
\partial^\alpha = \partial_{z_1}^{\alpha_1} \cdots \partial_{z_n}^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial z^\alpha}, \quad \partial^\bar{z} = \partial_{\bar{z}_1}^{\alpha_1} \cdots \partial_{\bar{z}_n}^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial \bar{z}^\alpha}.
\]

Let \( X \) be a \( C^\infty \) orientable paracompact manifold. We denote the tangent and cotangent bundle of \( X \) by \( TX \) and \( T^*X \) respectively, and the complexified tangent and cotangent bundle by \( CTX \) and \( CT^*X \). We write \( \langle \cdot, \cdot \rangle \) to denote the pointwise pairing between \( T^*X \) and \( TX \) and extend \( \langle \cdot, \cdot \rangle \) bilinearly to \( CT^*X \times CTX \).

Let \( E, F \) be \( C^\infty \) vector bundles over \( X \). We write \( F \otimes E^* \) for the vector bundle over \( X \times X \) with fiber over \((x, y) \in X \times X\) consisting of linear maps from \( E_y \) to \( F_x \).

Let \( Y \subset X \) be an open subset. The spaces of smooth sections and distribution sections of \( E \) over \( Y \) will be denoted by \( C^\infty(Y, E) \) and \( \mathcal{D}'(Y, E) \) respectively. Let \( \mathcal{D}'(Y, E) \) be the subspace of \( \mathcal{D}'(Y, E) \)
whose elements are of compact support in $Y$. For $m \in \mathbb{R}$, we let $H^m(Y, E)$ denote the Sobolev space of order $m$ for sections of $E$ over $Y$. Put

$$H^m_{\text{loc}}(Y, E) = \{ u \in \mathcal{D}'(Y, E); \varphi u \in H^m(Y, E), \varphi \in C^\infty_0(Y) \},$$

$$H^m_{\text{comp}}(Y, E) = H^m_{\text{loc}}(Y, E) \cap \mathcal{E}'(Y, E).$$

2.2. Set up and terminology. Let $(X, T^{1,0}X)$ be a compact CR manifold of dimension $2n + 1$, $n \geq 1$, where $T^{1,0}X$ is a CR structure of $X$. That is $T^{1,0}X$ is a subbundle of rank $n$ of the complexified tangent bundle $CTX$, satisfying $T^{1,0}X \cap T^{0,1}X = \{ 0 \}$, where $T^{0,1}X = \overline{T^{1,0}X}$, and $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$, $\mathcal{V} = C^\infty(X, T^{1,0}X)$.

We assume that $X$ admits an $S^1$ action: $S^1 \times X \to X$. We write $e^{-i\theta}$ to denote the $S^1$ action. Let $T \in C^\infty(X, TX)$ be the global real vector field induced by the $S^1$ action given by $(Tu)(x) = \frac{\partial}{\partial \theta} (u(e^{-i\theta} \circ x)) |_{\theta=0}$ for $u \in C^\infty(X)$.

**Definition 2.1.** We say that the $S^1$ action $e^{-i\theta}$ is CR if $[T, C^\infty(X, T^{1,0}X)] \subset C^\infty(X, T^{1,0}X)$ and the $S^1$ action is transversal if for each $x \in X$, $\mathcal{C}T(x) \oplus T_x^{1,0}X \oplus T_x^{0,1}X = \mathcal{C}T_xX$. Moreover, we say that the $S^1$ action is locally free if $T \neq 0$ everywhere.

We assume throughout that $(X, T^{1,0}X)$ is a compact CR manifold with a transversal CR locally free $S^1$ action $e^{-i\theta}$ with $T$ the global vector field induced by the $S^1$ action. Let $\omega_0 \in C^\infty(X, T^*X)$ be the global real one form determined by $\langle \omega_0, u \rangle = 0$ for all $u \in T^{1,0}X \oplus T^{0,1}X$, and $\langle \omega_0, T \rangle = 1$.

**Definition 2.2.** For $p \in X$, the Levi form $\mathcal{L}_p$ is the Hermitian quadratic form on $T_p^{1,0}X$ given by $\mathcal{L}_p(U, \overline{V}) = -\frac{1}{2} \{ d\omega_0(p)(U, \overline{\nabla}), U \wedge \overline{V} \}$, $U, \overline{V} \in T_p^{1,0}X$.

If the Levi form $\mathcal{L}_p$ is semi-positive definite (resp. positive definite), we say that $X$ is weakly pseudoconvex (resp. strongly pseudoconvex) at $p$. If the Levi form is semi-positive definite (resp. positive definite) at every point of $X$, we say that $X$ is weakly pseudoconvex (resp. strongly pseudoconvex).

Denote by $T^{s1,0}X$ and $T^{s0,1}X$ the dual bundles of $T^{1,0}X$ and $T^{0,1}X$ respectively. Define the vector bundle of $(p, q)$ forms by $T^{s,p,q}X = \Lambda^p(T^{s1,0}X) \wedge \Lambda^q(T^{s0,1}X)$.

Let $D \subset X$ be an open subset and $E$ be a complex vector bundle over $D$. Denote by $\Omega^{p,q}(D, E)$ (resp. $\Omega^{p,q}(D)$) the space of smooth sections of $T^{s,p,q}X \otimes E$ (resp. $T^{s,p,q}X$) over $D$ and by $\Omega^{p,q}_0(D, E)$ (resp. $\Omega^{p,q}_0(D)$) those elements of compact support in $D$.

Put

$$T^{0,0,*}X := \oplus_{j \in \{0, \ldots, n\}} T^{s0,j}X,$$

$$T^{0,0,+}X := \oplus_{j \in \{0, \ldots, n\}, j \text{ is even}} T^{s0,j}X,$$

$$T^{0,0,-}X := \oplus_{j \in \{0, \ldots, n\}, j \text{ is odd}} T^{s0,j}X.$$

Put $\Omega^{0,*}(X, E)$, $\Omega^{0,+}(X, E)$ and $\Omega^{0,-}(X, E)$ in a similar way as above.

Fix $\theta_0 \in [-\pi, \pi]$. Let $(e^{-i\theta_0})^* : \Lambda^r(\mathcal{C}T^*X) \to \Lambda^r(\mathcal{C}T^*X)$ be the pull-back map, $(e^{-i\theta_0})^* : T^{s,p,q}_x X \to T^{s,p,q}_{e^{-i\theta_0}x} X$. Define for $u \in \Omega^{p,q}(X)$

$$Tu := \frac{\partial}{\partial \theta} ((e^{-i\theta})^* u) |_{\theta=0} \in \Omega^{p,q}(X).$$

(See also (2.13).)

We shall write $u(e^{-i\theta} \circ x) := (e^{-i\theta})^* u(x)$ for $u \in \Omega^{p,q}(X)$. Clearly

$$u(x) = \sum_{m \in \mathbb{Z}} \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{-i\theta} \circ x) e^{im\theta} d\theta.$$

Let $\partial_b : \Omega^{0,q}(X) \to \Omega^{0,q+1}(X)$ be the tangential Cauchy-Riemann operator. From the CR property of the $S^1$ action it follows that (see also (2.14))

$$T\partial_b = \partial_b T \text{ on } \Omega^{0,q}(X).$$
Naturally associated with the $S^1$ action are the so-called rigid objects. See also [4] for a similar use of this term (cf. Definition II.2 of loc.cit.).

Definition 2.3. Let $D \subset X$ be an open set and $u \in C^\infty(D)$. We say that $u$ is rigid if $Tu = 0$, $u$ is Cauchy-Riemann (CR for short) if $\partial_b u = 0$ and $u$ is a rigid CR function if $\partial_b u = 0$ and $Tu = 0$.

Definition 2.4. Let $F$ be a complex vector bundle of rank $r$ over $X$. We say that $F$ is rigid (resp. CR) if $X$ can be covered by open subsets $U_j$ with trivializing frames $\{f_j^1, f_j^2, \ldots, f_j^r\}$ such that the corresponding transition functions are rigid (resp. CR) (in the sense of the preceding definition). In this case the frames $\{f_j^1, f_j^2, \ldots, f_j^r\}$ are called rigid frames (resp. CR frames).

Let $F$ be a rigid complex vector bundle over $X$ in the sense of Definition 2.4.

Definition 2.5. Let $\langle \cdot | \cdot \rangle_F$ be a Hermitian metric on $F$. We say that $\langle \cdot | \cdot \rangle_F$ is a rigid Hermitian metric if for every rigid local frames $\{f_1, \ldots, f_r\}$ of $F$, we have $T(f_j | f_k) = 0$, for $j, k = 1, 2, \ldots, r$.

The condition of being rigid is not a severe restriction as far as the $S^1$ action is concerned. See Theorems 2.11 and 2.12 which we shall prove within the framework of BRT trivializations in the next section.

Henceforth let $E$ be a rigid CR vector bundle over $X$. Write $\overline{\partial}_b : \Omega^0,q(X, E) \to \Omega^{0,q+1}(X, E)$ for the tangential Cauchy-Riemann operator. Since $E$ is rigid, we can define $Tu$ for $u \in \Omega^{0,q}(X, E)$ (cf. Theorem 2.11) and have

$$T\overline{\partial}_b = \partial_b T \text{ on } \Omega^{0,q}(X, E).$$

For $m \in \mathbb{Z}$, let

$$\Omega^{0,q}_m(X, E) := \{u \in \Omega^{0,q}(X, E); Tu = -imu\}$$

and put $\Omega^{0,q}_m(X, E)$, $\Omega^{0,q}_m(X, E)$ and $\Omega^{0,q}_m(X, E)$ in a similar way as above.

Put $\overline{\partial}_{b,m} := \overline{\partial}_b : \Omega^0,q(X, E) \to \Omega^{0,q+1}(X, E)$ with a $\overline{\partial}_{b,m}$-complex:

$$\overline{\partial}_{b,m} : \cdots \to \Omega^0,q-1(X, E) \to \Omega^0,q(X, E) \to \Omega^{0,q+1}(X, E) \to \cdots.$$

Define

$$H^q_{b,m}(X, E) := \operatorname{Ker} \overline{\partial}_{b,m} : \Omega^0,q(X, E) \to \Omega^{0,q+1}(X, E) = \operatorname{Im} \overline{\partial}_{b,m} : \Omega^{0,q-1}(X, E) \to \Omega^0,q(X, E).$$

It is instructive to think of $H^q_{b,m}(X, E)$ as the $m$-th $S^1$ Fourier component of the $q$-th $\overline{\partial}_b$ Kohn-Rossi cohomology group.

We will prove in Theorem 3.7 that $\dim H^q_{b,m}(X, E) < \infty$, for $m \in \mathbb{Z}$ and $q = 0, 1, 2, \ldots, n$.

We take a rigid Hermitian metric $\langle \cdot | \cdot \rangle_E$ on $E$ (in the sense of Definition 2.5), and a rigid Hermitian metric $\langle \cdot | \cdot \rangle$ on $\mathcal{C}TX$ such that

$$\mu \perp \langle T^{1,0} X \perp T^{0,1} X \rangle, \quad \langle T | T \rangle = 1$$

and $T^{1,0} X \perp T^{0,1} X$. (This is always possible; see Theorem 2.11 and Theorem 9.2 in [40].)

The Hermitian metric $\langle \cdot | \cdot \rangle$ on $\mathcal{C}TX$ induces by duality a Hermitian metric on $\mathcal{C}TX$ and on the bundles of $(0, q)$ forms $T^{0,q}X$ $(q = 0, 1 \cdots, n)$, to be denoted by $\langle \cdot | \cdot \rangle$ too. A Hermitian metric denoted by $\langle \cdot | \cdot \rangle_E$ on $T^{0,0}X \otimes E$ is induced by those on $T^{0,0}X$ and $E$. Let the linear map $A(x, y) \in (T^\ast \ast X \otimes E) \otimes (T^\ast \ast X \otimes E) \ast(x,y)$. We write $|A(x, y)|$ to denote the natural matrix norm of $A(x, y)$ induced by $\langle \cdot | \cdot \rangle_E$.

We denote by $dv_X = dv_X(x)$ the induced volume form, and form the global $L^2$ inner products $\langle \cdot | \cdot \rangle_E$ and $\langle \cdot | \cdot \rangle$ on $\Omega^{0,q}(X, E)$ and $\Omega^{0,q}(X)$ respectively, with $L^2$-completion $L^2(X, T^{0,q}X \otimes E)$ and $L^2(X, T^{0,q}X)$.

Similar notation applies to $L^2_{b}(X, T^{0,q}X \otimes E)$ and $L^2_{b}(X, T^{0,q}X)$ (the completions of $\Omega^{0,q}_m(X, E)$ and $\Omega^{0,q}_m(X)$ with respect to $\langle \cdot | \cdot \rangle_E$ and $\langle \cdot | \cdot \rangle$).

Put $L^2(X, T^{0,0}X \otimes E), L^2_{b}(X, E)$ and $L^2_{b}(X, E)$ in a similar way as above, and $L^2_m(X, T^{0,0}X \otimes E), L^2_{b,m}(X, E)$ and $L^2_{b,m}(X, E)$ too.
2.3. Tangential de Rham cohomology group, Tangential Chern character and Tangential Todd class. In this section it is convenient to put \( \Omega^*_0(X) = \{ u \in \oplus_{p+q=r} \Omega^{p,q}(X); Tu = 0 \} \) for \( r = 0, 1, 2, \ldots , 2n \) (without the danger of confusion with \( \Omega^{p,q}_0 \) in the preceding section) and set \( \Omega^*_0(X) = \oplus_{r=0}^{2n} \Omega^*_0(X) \). Since \( Td = d T \) (see (2.3)), we have \( d \)-complex:

\[
d : \cdots \to \Omega^{-1}_0(X) \to \Omega^0_0(X) \to \Omega^1_0(X) \to \cdots
\]

Define the \( r \)-th tangential de Rham cohomology group:

\[
\mathcal{H}^r_{b,0}(X) := \frac{\ker d : \Omega^r_0(X) \to \Omega^{r+1}_0(X)}{\text{Im} d : \Omega^{r-1}_0(X) \to \Omega^r_0(X)}.
\]

Put \( \mathcal{H}^{*}_0(X) = \oplus_{r=0}^{2n} \mathcal{H}^r_{b,0}(X) \).

Let a complex vector bundle \( F \) over \( X \) of rank \( r \) be rigid as in Definition 2.4. We will show in Theorem 2.12 that there exists a connection \( \nabla \) on \( F \) such that for any rigid local frame \( f = (f_1, f_2, \ldots , f_r) \) of \( F \) on an open set \( D \subset X \), the connection matrix \( \Theta(\nabla, f) = (\Theta_{j,k})_{j,k=1}^r \) satisfies

\[
\theta_{j,k} \in \Omega^1_0(D),
\]

for \( j, k = 1, \ldots , r \). We call \( \nabla \) as such a rigid connection on \( F \). Let \( \Theta(\nabla, F) \in C^\infty(X, \Lambda^2(CT^*X) \otimes \text{End}(F)) \) be the associated tangential curvature.

Let \( h(z) = \sum_{j=0}^{\infty} a_j z^j \) be a real power series on \( z \in \mathbb{C} \). Set

\[
H(\Theta(\nabla, F)) = \text{Tr} \left( h\left( \frac{i}{2\pi} \Theta(\nabla, F) \right) \right).
\]

It is clear that \( H(\Theta(\nabla, F)) \in \Omega^*_0(X) \).

The following is well-known (see Theorem B.5.1 in Ma-Marinescu [48]).

**Theorem 2.6.** \( H(\Theta(\nabla, F)) \) is a closed differential form.

That the tangential de Rham cohomology class

\[
[H(\Theta(\nabla, F))] \in \mathcal{H}^{*}_0(X)
\]

does not depend on the choice of rigid connections \( \nabla \) is given by

**Theorem 2.7.** Let \( \nabla' \) be another rigid connection on \( F \). Then, \( H(\Theta(\nabla, F)) - H(\Theta(\nabla', F)) = dA \), for some \( A \in \Omega^*_0(X) \).

**Proof.** The idea of the proof is standard. For each \( t \in [0, 1] \), put \( \nabla_t = (1 - t)\nabla + t\nabla' \) which is a rigid connection on \( F \). Set

\[
Q_t = \frac{i}{2\pi} \text{Tr} \left( \frac{\partial \Theta}{\partial t} \left( \frac{i}{2\pi} \Theta(\nabla_t, F) \right) \right).
\]

Since \( \nabla_t \) is rigid, it is easily seen that

\[
(2.6) \quad Q_t \in \Omega^*_0(X).
\]

It is well-known that (see Remark B.5.2 in Ma-Marinescu [48])

\[
(2.7) \quad H(\Theta(\nabla, F)) - H(\Theta(\nabla', F)) = d \int_0^1 Q_t dt.
\]

From (2.6) and (2.7), the theorem follows. \( \square \)

For \( h(z) = e^z \) put

\[
(2.8) \quad \text{ch}_b(\nabla, F) := H(\Theta(\nabla, F)) \in \Omega^*_0(X),
\]

and for \( h(z) = \log(\frac{z}{1-z}) \) set

\[
(2.9) \quad \text{Td}_b(\nabla, F) := e^{H(\Theta(\nabla, F))} \in \Omega^*_0(X).
\]

We can now introduce tangential Todd class and tangential Chern character.
Definition 2.8. The tangential Chern character of $F$ is given by

$$ch_b(F) := [ch_b(\nabla, F)] \in H^*_b(X)$$

and the tangential Todd class of $F$ is given by

$$Td_b(F) = [Td_b(\nabla, F)] \in H^*_b(X).$$

Baouendi-Rothschild-Treves [4] proved that $T^{1,0}X$ is a rigid complex vector bundle over $X$ (cf. the first part of Theorem 2.11 below). The tangential Todd class of $T^{1,0}X$ and tangential Chern character of $T^{1,0}X$ are thus well defined.

The tangential Chern classes can be defined similarly. Put

$$\det(\frac{\Theta(\nabla, F)}{2\pi} t + I) = \sum_{j=0}^{r} \hat{c}_j(\nabla, F)t^j.$$

Thus $\hat{c}_j(\nabla, F) \in \Omega^2_0(D)$. By the matrix identity $\det A = e^{Tr(\log A)}$ and taking $h(z) = \log(1 + z)$, one sees $\hat{c}_j(\nabla, F)$ is a closed differential form on $X$ and its tangential de Rham cohomology class $[\hat{c}_j(\nabla, F)] \in H^*_b(X)$ is independent of the choice of rigid connections $\nabla$. Put

$$\hat{c}_j(F) = [\hat{c}_j(\nabla, F)] \in H^2_0(X).$$

We call $\hat{c}_j(F)$ the $j$-th tangential Chern class of $F$, and $\hat{c}(F) = 1 + \sum_{j=1}^{r} \hat{c}_j(F) \in H^*_b(X)$ the tangential total Chern class of $F$.

2.4. BRT trivializations and rigid geometric objects. In this paper, much of our strategy is heavily based on the following result thanks to Baouendi-Rothschild-Treves [4, Proposition 1.2]. Note in the following, $Z_j$ corresponds to $\tilde{T}_j$ in their proposition. Some geometrical significance related to a certain circle bundle structure will be discussed in the proof of Proposition 4.2.

Theorem 2.9. For every point $x_0 \in X$ there exist local coordinates $x = (x_1, \cdots, x_{2n+1}) = (z, \theta) = (z_1, \cdots, z_n, \theta)$, $z_j = x_{2j-1} + ix_{2j}, j = 1, \cdots, n, x_{2n+1} = \theta$, defined in some small neighborhood $D = \{ (z, \theta) : |z| < \delta, -\varepsilon_0 < \theta < \varepsilon_0 \}$ of $x_0$, $\delta > 0$, $0 < \varepsilon_0 < \pi$, such that $(z(x_0), \theta(x_0)) = (0, 0)$ and

$$Z_j = \frac{\partial}{\partial z_j} - i \frac{\partial \phi}{\partial z_j}(z) \frac{\partial}{\partial \theta}, \quad j = 1, \cdots, n$$

where $Z_j(x)$, $j = 1, \cdots, n$, form a basis of $T^{1,0}_xX$ for each $x \in D$ and $\varphi(z) \in C^\infty(D, \mathbb{R})$ is independent of $\theta$. We summarize these data by the notation $(D, (z, \theta, \varphi))$.

Furthermore, let $(D, (z, \theta, \varphi))$ and $(\tilde{D}, (w, \eta, \tilde{\varphi}))$ be two such data on $D$. Then the coordinate transformation between them (on $D \cap \tilde{D}$) can be given such that if

$$w = (w_1, \ldots, w_n) = H(z) = (H_1(z), \ldots, H_n(z))$$

then

$$H_j(z) \in C^\infty(|z| < \delta), \quad \overline{\partial}H_j(z) = 0, \quad \forall j$$

$$\eta = \theta + \arg g(z) \quad (\mod 2\pi) \quad \text{where} \quad \arg g(z) = \text{Im} \log g(z)$$

$$\tilde{\varphi}(H(z), \overline{H(z)}) = \varphi(z, \overline{z}) + \log |g(z)|$$

for some nowhere vanishing holomorphic function $g(z)$ on $|z| < \delta$.

Remark 2.10. The relation between $\tilde{\varphi}$ and $\varphi$ in (2.11) is a corrected version of a similar formula in [4, the line below (1.31)]. See the proof of Proposition 4.2 for a derivation.

There exist examples that $H$ is not necessarily one to one. Nevertheless, it can be shown that after shrinking $D$ and $\tilde{D}$ properly, it is one to one, hence a biholomorphism.

We call the above triple $(D, (z, \theta, \varphi))$ a BRT trivialization. Note for $(z, \theta) \in D$ and $-\pi < \alpha < \pi$, $e^{-i\alpha} \circ (z, \theta) = (z, \theta + \alpha)$ if $\{e^{-i\alpha} \circ (z, \theta)\}_{0 \leq \alpha \leq 1} \subset D$. 
By using BRT trivializations some operations simplify, as follows. Under the BRT triple \((D, (z, \theta), \varphi)\) it is clear that
\[
\{d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}, 1 \leq j_1 < \cdots < j_q \leq n\}
\]
is a basis for \(T^* x^0 q X\) for every \(x \in D\). For \(u \in \Omega^0 q(X)\), on \(D\) we write
\[
u = \sum_{j_1 < \cdots < j_q} u_{j_1 \cdots j_q} d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}.
\]
(2.12)
Recall \(T\) is the vector field associated with the \(S^1\) action. We have
\[
Tu = \sum_{j_1 < \cdots < j_q} (Tu_{j_1 \cdots j_q}) d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}
\]
(2.13)
and \(Tu\) is independent of the choice of BRT trivializations.

For \(\partial_b\) on the BRT triple \((D, (z, \theta), \varphi)\) we have
\[
\partial_b = \sum_{j=1}^n d\overline{z}_j \wedge (\frac{\partial}{\partial \overline{z}_j} + i \frac{\partial \varphi}{\partial \overline{z}_j}(z) \frac{\partial}{\partial \theta}).
\]
(2.14)
The rigid objects (discussed in the preceding section) are natural geometric objects pertinent to the \(S^1\) action. In the following \(X\) is again a compact connected CR manifold with a transversally CR locally free \(S^1\) action.

**Theorem 2.11.** Suppose \(F\) is a complex vector bundle over \(X\) (not necessarily a CR bundle) and admits an \(S^1\) action compatible with that on \(X\). Then \(F\) is actually a rigid vector bundle (in the sense of Definition 2.4). Moreover there is a rigid Hermitian metric \((\cdot | \cdot)_F\) on \(F\). Conversely if \(F\) is a rigid vector bundle, then \(F\) admits a compatible \(S^1\) action.

**Proof.** We first work on the existence of a rigid Hermitian metric (assuming \(F\) is rigid). Fix \(p \in X\) and let \((D, (z, \theta), \varphi)\) be a BRT trivialization around \(p\) such that \((z(p), \theta(p)) = (0, 0)\), \((z, \theta) \in \{z \in \mathbb{C}^{n-1} : |z| < \delta\} \times \{\theta \in \mathbb{R} : |\theta| < \delta\}\) for some \(\delta > 0\). Put
\[
A := \{\lambda \in [-\pi, \pi] : \text{there is a local rigid trivializing frame (l.r.t. frame for short) defined on}
\{e^{-i\theta} \circ (z, 0) : |z| < \epsilon, \theta \in [\pi - \lambda, \pi + \lambda]\}
\text{for some } 0 < \epsilon < \delta\}.
\]
Clearly \(A\) is a non-empty open set in \([-\pi, \pi]\). We claim \(A = [-\pi, \pi]\). (Remark that the l.r.t. frame above is closely related to the canonical basis in [4, Definition I.3 without (I.29a)] when \(E\) is \(T^1 q X\).)

It suffices to prove \(A\) is closed. Let \(\lambda_0\) be a limit point of \(A\). For some small \(\epsilon_1 > 0\), there is a l.r.t. frame \(f = (f_1, \ldots, f_r)\) defined on \(\{e^{-i\theta} \circ (z, 0) : |z| < \epsilon_1, \lambda_0 - \epsilon_1 < \theta < \lambda_0 + \epsilon_1\}\). By assumption \(\lambda_0 \in A\) there exists a l.r.t. frame \(\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_r)\) defined around \(\{e^{-i\theta} \circ (z, 0)\}\) in which \(|z| < \epsilon_2, \theta \in [-\pi, \lambda_0 - \frac{\epsilon_1}{2}]\) for some \(\epsilon_2 > 0\). Now \(\tilde{f} = g\tilde{f}\) on
\[
\left\{e^{-i\theta} \circ (z, 0) : |z| < \epsilon_0, \theta \in (\lambda_0 - \epsilon_1, \lambda_0 - \frac{\epsilon_1}{2})\right\},
\epsilon_0 = \min \{\epsilon_1, \epsilon_2\}
\]
for some rigid \(r \times r\) matrix \(g\).

We now patch up the frames. Put, for \(\theta \in [-\pi, \lambda_0 - \frac{\epsilon_1}{2}]\), \(f = \tilde{f}\) (on \(\{e^{-i\theta} \circ (z, 0)\}\)) and for \(\theta \in [\lambda_0 - \frac{\epsilon_1}{2}, \lambda_0 + \epsilon_1]\), \(f = g\tilde{f}\) because \(g\) is independent of \(\theta\). By \(\tilde{f} = g\tilde{f}\) on the overlapping, \(f\) is well-defined as a l.r.t. frame on
\[
\left\{e^{-i\theta} \circ (z, 0) : |z| < \epsilon_0, \theta \in [-\pi, \lambda_0 + \epsilon_1]\right\}.
\]
extending \(\theta = \lambda_0\). Thus \(A\) is closed as desired.

By the discussion above we can actually find local rigid trivializations \(W_1, \ldots, W_N\) such that \(X = \bigcup_{j=1}^N W_j\) and each \(W_j \supset \bigcup_{-\pi \leq \theta \leq \pi} e^{-i\theta} W_j\) (i.e. \(W_j\) is \(S^1\) invariant). Take any Hermitian metric \((\cdot | \cdot)_F\) on \(F\). Let \((\cdot | \cdot)_F\) be the Hermitian metric on \(F\) defined as follows. For each \(j = 1, 2, \ldots, N\), let
$h^j_1, \ldots, h^j_r$ be local rigid trivializing frames on $W_j$. Put $\langle h^j_s(x) | h^j_t(x) \rangle_F = \frac{1}{2\pi} \int_\pi \langle h^j_s(e^{-i\theta} \circ x), h^j_t(e^{-i\theta} \circ x) \rangle_F du$, $s, t = 1, 2, \ldots, r$. One sees that $\langle \cdot | \cdot \rangle_F$ is well-defined as a rigid Hermitian metric on $F$.

By examining the above reasoning we have also proved that if $F$ is rigid, then it admits a natural $S^1$ action (by declaring the l.r.t. frames as $S^1$ invariant frames) compatible with that on $X$.

For the reverse direction if $F$ admits a compatible $S^1$ action, by using BRT trivializations one can construct $S^1$ invariant local frames. These invariant local frames can be easily verified to be local rigid frames, or equivalently the transition functions between them are annihilated by $T$ due to the $S^1$ invariant property. Hence $F$ is rigid by definition.

We shall now prove

**Theorem 2.12.** Assume the complex vector bundle $F$ is rigid (on $X$). There exists a rigid connection on $F$. And if $F$ is equipped with a rigid Hermitian metric, there exists a rigid connection compatible with this Hermitian metric. Suppose $F$ is furthermore CR (and rigid) equipped with a rigid Hermitian metric $h$. Then there exists a unique rigid connection $\nabla^F$ compatible with $h$ such that $\nabla^F$ induces the Chern connection on $F|_{\theta=0}$ with respect to $h|_{\theta=0}$ (on any given BRT chart) and that the $S^1$ invariant sections are its parallel sections along $S^1$ orbits of $X$.

**Proof.** Let $\nabla$ be a connection on $F$. For any $g \in S^1$ considering $g^{-1}\nabla$ on $g^{-1}F$ which is $F$ by using Theorem 2.11 and summing over $g$ (in analogy with the construction of a rigid Hermitian metric above), one obtains a rigid connection on $F$. Suppose $F$ has a rigid Hermitian metric $\langle \cdot | \cdot \rangle_F$ and a connection $\nabla$ compatible with $\langle \cdot | \cdot \rangle_F$. One readily sees that the rigid connection resulting from the preceding procedure of sumation, is still compatible with $\langle \cdot | \cdot \rangle_F$. For the last statement, note that i) given a rigid CR bundle $F$ with a rigid Hermitian metric $h$ and any BRT chart $D_j = U_j \times ] \varepsilon, \varepsilon[ \subset (F, h)$ can descend to $U_j$ as a holomorphic vector bundle with the inherited metric, and ii) for a holomorphic vector bundle with a Hermitian metric, the Chern connection is canonically defined. Combining i) with ii) and using the $S^1$ invariant local frames (cf. proof of Theorem 2.11), one can construct a canonical connection $\nabla^F$ on $U_j$. Then by using (2.10), (2.11) and the canonical property of the Chern connection, one sees that these $\nabla^F$ patch up to form a global connection $\nabla^F$ on $X$, satisfying the property as stated in the proposition.

### 3. A Hodge theory for $\Box^{(q)}_{b,m}$

A Hodge decomposition theorem for $\overline{\partial}_b$ on pseudoconvex CR manifolds has been well developed. See [14, Section 9.4] for a nice presentation in some respects; see also [58]. Our goal of this section is to develop an analogous theory for $\Box^{(q)}_{b,m}$ on CR manifolds with transversal CR locally free $S^1$ action (irrespective of pseudoconvexity). Much of what follows appears to parallel the corresponding part of Hodge theory in complex geometry.

Besides the relevance to the index theorem on CR manifolds, the present theory has an application to our proof of homotopy invariance of index (Theorem 4.7).

As before, $X$ is a compact CR manifold with a transversal CR locally free $S^1$ action. Let $\overline{\partial}_{b}^*: \Omega^{0,q+1}(X, E) \to \Omega^{0,q}(X, E)$ ($q = 0, 1, 2, \ldots, n$) be the formal adjoint of $\overline{\partial}_b$ with respect to $\langle \cdot | \cdot \rangle_E$. Put

$$\Box^{(q)}_{b,m} := \partial_{b,m} \overline{\partial}_{b}^* + \overline{\partial}_{b,m} \partial_{b}^* : \Omega^{0,q}(X, E) \to \Omega^{0,q}(X, E).$$

$T$ is the vector field on $X$ induced by the $S^1$ action, $T\overline{\partial}_b = \partial_b T$ and $\overline{\partial}_{b,m} := \partial_{b,m} \langle \cdot | \cdot \rangle_{\Omega^m_q(X, E)} : \Omega^m_q(X, E) \to \Omega^m_{q+1}(X, E)$ on eigenspaces of the $S^1$ acton ($\forall m \in \mathbb{Z}$).

Recall $\langle \cdot | \cdot \rangle_E$ is rigid. One sees $T\overline{\partial}_b = \overline{\partial}_b T$ so that $\overline{\partial}_b|_{\Omega^{0,q+1}_m(X, E)} : \Omega^{0,q+1}_m(X, E) \to \Omega^{0,q}_m(X, E)$ is the same as the formal adjoint $\overline{\partial}_{m}^*$ of $\overline{\partial}_m$. Form

$$\Box^{(q)}_{b,m} = \partial_{b,m} \overline{\partial}_{b,m}^* + \overline{\partial}_{b,m} \partial_{b,m}^* : \Omega^m_q(X, E) \to \Omega^m_{q}(X, E).$$

We have $\Box^{(q)}_{b,m} = \Box^{(q)}_{b,m|_{\Omega^{0,q}_m(X, E)}}$.

On a general compact CR manifold, there is a fundamental result that follows from Kohn’s $L^2$ estimates. (See [14, Theorem 8.4.2]). Adapting it to our present situation, we can state the result as follows.
Theorem 3.1. For every \( s \in \mathbb{N}_0 \) there is a constant \( C_s > 0 \) such that
\[
\|u\|_{s+1} \leq C_s \left( \left\| \Box_b^{(q)} u \right\|_s + \|Tu\|_s + \|u\|_s \right), \quad \forall u \in \Omega^{0,q}(X,E),
\]
where \( \|\cdot\|_s \) denotes the usual Sobolev norm of order \( s \) on \( X \).

Theorem 3.1 restricted to \( \Omega^{0,q}_{b,m}(X,E) \) yields

Corollary 3.2. Fix \( m \in \mathbb{Z} \). For every \( s \in \mathbb{N}_0 \) there is a constant \( C_s > 0 \) such that
\[
\|u\|_{s+1} \leq C_s \left( \left\| \Box_b^{(q)} u \right\|_s + \|Tu\|_s + \|u\|_s \right), \quad \forall u \in \Omega^{0,q}_{b,m}(X,E).
\]

This suggests that a good regularity theory might exist on our \( X \). Observe that \( \Box_b^{(q)} - T^2 \) is elliptic on \( X \) while \( \Box_b^{(q)} \) is not, which on \( \Omega^{0,q}_{b,m}(X,E) \) is \( \Box_b^{(q)} + m^2 \). In fact, without using the above theorems all of the following results are essentially proven by standard results in elliptic theory.

Write \( \text{Dom} \Box_b^{(q)} := \{ u \in L^2_m(X,T^{0,q}X \otimes E) ; \Box_b^{(q)} u \in L^2_m(X,T^{0,q}X \otimes E) \} \) where \( \Box_b^{(q)} u \) is defined in the sense of distribution. \( \Box_b^{(q)} \) is extended by
\[
(3.1) \quad \Box_b^{(q)} : \text{Dom} \Box_b^{(q)} \subset L^2_m(X,T^{0,q}X \otimes E) \to L^2_m(X,T^{0,q}X \otimes E).
\]

Lemma 3.3. We have \( \text{Dom} \Box_b^{(q)} = L^2_m(X,T^{0,q}X \otimes E) \cap H^2(X,T^{0,q}X \otimes E) \).

Proof. For the inclusion put \( v = \Box_b^{(q)} u \in L^2_m(X,T^{0,q}X \otimes E) \). Then \( (\Box_b^{(q)} - T^2)u = v + m^2 u \in L^2_m(X,T^{0,q}X \otimes E) \). Since \( (\Box_b^{(q)} - T^2) \) is elliptic, we conclude \( u \in H^2(X,T^{0,q}X \otimes E) \). The reverse inclusion is clear. \( \square \)

Lemma 3.4. \( \Box_b^{(q)} : \text{Dom} \Box_b^{(q)} \subset L^2_m(X,T^{0,q}X \otimes E) \to L^2_m(X,T^{0,q}X \otimes E) \) is self-adjoint.

Proof. Since the similar extension of \( \Box_b^{(q)} \) on \( L^2(X,T^{0,q}X \otimes E) \) is self-adjoint and its restriction to (an invariant subspace) \( L^2_m(X,T^{0,q}X \otimes E) \) gives \( \Box_b^{(q)} \), \( \Box_b^{(q)} \) is also self-adjoint. \( \square \)

Let \( \text{Spec} \Box_b^{(q)} \subset [0,\infty) \) denote the spectrum of \( \Box_b^{(q)} \) (Davies [18]).

Proposition 3.5. \( \text{Spec} \Box_b^{(q)} \) is a discrete subset of \( [0,\infty) \). For any \( \nu \in \text{Spec} \Box_b^{(q)} \), \( \nu \) is an eigenvalue of \( \Box_b^{(q)} \) and the eigenspace
\[
\mathcal{E}_{\nu}(X,E) := \{ u \in \text{Dom} \Box_b^{(q)} ; \Box_b^{(q)} u = \nu u \}
\]
is finite dimensional with \( \mathcal{E}_{\nu}(X,E) \subset \Omega^{0,q}_{b,m}(X,E) \).

Proof. \( \Box_b^{(q)} - T^2 \equiv \Delta \) is a second order elliptic operator. By standard elliptic theory, \( \Delta \) and hence \( \Delta + m^2 \), satisfy the statement of the proposition on the (invariant) subspace \( L^2_m(X,T^{0,q}X \otimes E) \subset \Omega^{0,q}_{b,m}(X,E) \). On it \( T^2 \) acts as \( -m^2 \), the proposition follows for \( \Box_b^{(q)} \) which is \( \Delta + m^2 \) on \( L^2_m(X,T^{0,q}X \otimes E) \). \( \square \)

A role analogous to the Green’s operator in the ordinary Hodge theory is given as follows. Let
\[
N^{(q)}_m : L^2_m(X,T^{0,q}X \otimes E) \to \text{Dom} \Box_b^{(q)}
\]
be the partial inverse of \( \Box_b^{(q)} \) and let
\[
\Pi^{(q)}_m : L^2_m(X,T^{0,q}X \otimes E) \to \text{Ker} \Box^{(q)}_b
\]
be the orthogonal projection. We have
\[
(3.2) \quad \Box_b^{(q)} N^{(q)}_m + \Pi^{(q)}_m = I \quad \text{on} \quad L^2_m(X,T^{0,q}X \otimes E), \quad N^{(q)}_m \Box_b^{(q)} + \Pi^{(q)}_m = I \quad \text{on} \quad \text{Dom} \Box_b^{(q)}.
\]
**Lemma 3.6.** We have $N_{m}^{(q)} : \Omega_{m}^{0,q}(X, E) \to \Omega_{m}^{0,q}(X, E)$.

**Proof.** A slight variant of the standard argument applies as $\square_{b}^{(q)}$ is almost elliptic. Let $u \in \Omega_{m}^{0,q}(X, E)$ and put $N_{m}^{(q)}u = v \in L_{m}^{2}(X, T^{*0,q}X \otimes E)$. By (3.2), $(I - \Pi_{m}^{(q)})u = \square_{b,m}^{(q)}u$, giving

$$
(\square_{b,m}^{(q)} - T^{2})v = (I - \Pi_{m}^{(q)})u + m^{2}v.
$$

By Proposition 3.5, $\ker \square_{b,m}^{(q)}$ consists of smooth sections, so $\Pi_{m}^{(q)}u$ is smooth and

$$
(I - \Pi_{m}^{(q)})u \in \Omega_{m}^{0,q}(X, E).
$$

By combining (3.3) and (3.4) and noting $\square_{b}^{(q)} - T^{2}$ is elliptic, the standard technique in elliptic regularity applies to give $v \in \Omega_{m}^{0,q}(X, E)$. □

The following is a version of “harmonic realization” of cohomology.

**Theorem 3.7.** For every $q \in \{0, 1, 2, \ldots, n\}$ and every $m \in \mathbb{Z}$, we have

$$
\ker \square_{b,m}^{(q)} = \mathcal{L}_{m,0}^{q}(X, E) \cong H_{b,m}^{q}(X, E).
$$

As a consequence $\dim H_{b,m}^{q}(X, E) < \infty$ by Proposition 3.5.

**Proof.** The argument is mostly standard (although $\square_{b}^{(q)}$ is not elliptic). Consider the map

$$
\tau_{m}^{q} : \ker \overline{\partial}_{b,m} \cap \Omega_{m}^{0,q}(X, E) \to \ker \square_{b,m}^{(q)},
$$

$$
u \mapsto \Pi_{m}^{(q)}u.
$$

Clearly $\tau_{m}^{q}$ is surjective. Put $M_{m}^{q} := \{\overline{\partial}_{b,m}u; \ u \in \Omega_{m}^{0,q-1}(X, E)\}$. The theorem follows if one shows

$$
\ker \tau_{m}^{q} = M_{m}^{q}.
$$

It is easily seen $M_{m}^{q} \subset \ker \tau_{m}^{q}$ since $M_{m}^{q} \perp \ker \square_{b,m}^{(q)}$. For the reverse let $u \in \ker \tau_{m}^{q}$ so $\Pi_{m}^{(q)}u = 0$. From (3.2) we have

$$
u = \square_{b,m}^{(q)}N_{m}^{(q)}u + \Pi_{m}^{(q)}u
$$

$$
= (\overline{\partial}_{b} \overline{\partial}_{b}^{*} + \overline{\partial}_{b}^{*} \overline{\partial}_{b})N_{m}^{(q)}u.
$$

We claim that

$$
(\overline{\partial}_{b} \overline{\partial}_{b}^{*}N_{m}^{(q)}u)_{E} = 0.
$$

One sees, by using $\overline{\partial}_{b}^{2} = 0$ for the first equality below,

$$
(\overline{\partial}_{b} \overline{\partial}_{b}^{*}N_{m}^{(q)}u | \overline{\partial}_{b} \overline{\partial}_{b}^{*}N_{m}^{(q)}u)_{E} = (\overline{\partial}_{b} \overline{\partial}_{b}^{*}(\Pi_{m}^{(q)}N_{m}^{(q)}u) | N_{m}^{(q)}u)_{E} = (\overline{\partial}_{b} \overline{\partial}_{b}(I - \Pi_{m}^{(q)})u | N_{m}^{(q)}u)_{E}
$$

$$
= (\overline{\partial}_{b} \overline{\partial}_{b}u | N_{m}^{(q)}u)_{E}
$$

which is zero because $u \in \ker \overline{\partial}_{b,m}$ by (3.6), giving the claim (3.9). By (3.8) and (3.9),

$$
u = \overline{\partial}_{b} \overline{\partial}_{b}^{*}N_{m}^{(q)}u,
$$

with $\overline{\partial}_{b}N_{m}^{(q)}u \in \Omega_{m}^{0,q-1}(X, E)$ by Lemma 3.6. By (3.11), $u \in M_{m}^{q}$, yielding the desired inclusion $\ker \tau_{m}^{q} \subset M_{m}^{q}$. Hence (3.7). □
Let $D_{b,m} := \overline{\partial}_b + \overline{\partial}_b^j : \Omega^0_{\mathfrak{m}}(X, E) \to \Omega^{0}_{\mathfrak{m}}(X, E)$ with extension $D_{b,m}$

$D_{b,m} : \text{Dom} \ D_{b,m} (\subset L^2_m(X, E)) \to L^2_m(X, E),$

$\text{Dom} \ D_{b,m} = \Big\{ u \in L^2_m(X, E); \text{distribution} \ D_{b,m}u \in L^2_m(X, E) \Big\}$. The Hilbert space adjoint of $D_{b,m}$ with respect to $(\cdot, \cdot)_E$ is given by $D^*_b,m : \text{Dom} \ D^*_b,m (\subset L^2_m(X, E)) \to L^2_m(X, E)$.

Combining Proposition 3.5 and Theorem 3.7, one can verify (as in standard Hodge theory)

**Theorem 3.8.** In the notation above

$$\text{Ker} \ D_{b,m} = \bigoplus_{q \in \{0,1,\ldots,n\}} \text{Ker} \square_{b,m}^{(q)} (\subset \Omega^0_{\mathfrak{m}}(X, E)),$$

$$\text{Ker} \ D^*_b,m = \bigoplus_{q \in \{0,1,\ldots,n\}} \text{Ker} \square_{b,m}^{(q)} (\subset \Omega^0_{\mathfrak{m}}(X, E)).$$

Put $\text{ind} \ D_{b,m} := \dim \text{Ker} \ D_{b,m} - \dim \text{Ker} \ D^*_b,m$. Hence, together with Theorem 3.7,

$$\sum_{j=0}^n (-1)^j \dim H^j_{b,m}(X, E) = \text{ind} \ D_{b,m}.$$

## 4. Modified Kohn Laplacian (Spin$^c$ Kohn Laplacian)

We are prepared by Theorem 3.8 above to see that to calculate $\sum_{j=0}^n (-1)^j \dim H^j_{b,m}(X, E)$ is the same as to calculate the index $\text{ind} \ D_{b,m}$. To do so effectively we need to modify the Dirac type operator $D_{b,m}$ hence the standard Kohn Laplacian because the modified versions $\tilde{D}_{b,m}, \tilde{\square}_{b,m}$ will have a manageable heat kernel that suits our purpose better for the CR non-Kähler case (cf. Remark 4.9).

Lastly we shall give an argument for the homotopy invariance, and obtain $\text{ind} \ D_{b,m} = \text{ind} \tilde{D}_{b,m}$.

The main idea here is borrowed from that of classical cases. But as the CR manifold $X$ is not assumed to be a (orbifold) circle bundle globally, there could arise the problem of patching (from local constructions to the global one). Part of the technicality in the beginning of this section lies in a careful treatment in this regard.

We recall some basics of Clifford connection and Spin$^c$ Dirac operator. For more details we refer to Chapter 1 in [48] and [22].

Let $B := (D, (z, \theta), \varphi)$ be a BRT trivialization with $D = U \times] - \varepsilon, \varepsilon[$ where $\varepsilon > 0$ and $U$ is an open set of $\mathbb{C}^n$. Using $\varphi$ in $B$, we let $(\cdot, \cdot)$ be the Hermitian metric on $CTU$ induced by that on $D$

$$\langle \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k} \rangle = \langle \frac{\partial}{\partial \bar{z}_j} - i \frac{\partial \varphi_j}{\partial z_j} \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \bar{z}_k} - i \frac{\partial \varphi_k}{\partial z_k} \frac{\partial}{\partial \theta} \rangle, \quad j, k = 1, 2, \ldots, n$$

(cf. Theorem 2.9). By (2.10) and Theorem 2.9, the above metric is actually intrinsically defined. The $(\cdot, \cdot)$ induces Hermitian metrics on $T^{1,0}U$ still denoted by $(\cdot, \cdot)$ and a Riemannian metric $g_{TU}$ on $TU$.

For any $v \in TU$ with decomposition $v = v^{(1,0)} + v^{(0,1)} \in T^{1,0}U \oplus T^{0,1}U$, let $\overline{v}^{(1,0),*} \in T^{0,1}U$ be the metric dual of $v^{(1,0)}$ with respect to $(\cdot, \cdot)$. That is, $\overline{v}^{(1,0),*}(u) = (v^{(1,0)}, \pi) (u)$ for all $u \in T^{0,1}U$.

The Clifford action $v$ on $\Lambda T^{0,1}U := \bigoplus_{\alpha=0}^{1} T^{0,\alpha}U$ is defined by

$$c(v)(\cdot) = \sqrt{2}(\overline{v}^{(1,0),*} \wedge \cdot - i_{v^{(0,1)}}(\cdot))$$

(where $\wedge$ and $i$ denote the exterior and interior product respectively).

Let $\{w_j\}_{j=1}^n$ be a local orthonormal frame of $T^{1,0}U$ with respect to $(\cdot, \cdot)$ with dual frame $\{w^j\}_{j=1}^n$.

Write

$$e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \overline{w}_j)$$

and

$$e_{2j} = \frac{i}{\sqrt{2}}(w_j - \overline{w}_j), \quad j = 1, 2, \ldots, n,$$
for an orthonormal frame of $TU$. Let $\nabla^{TU}$ be the Levi-Civita connection on $TU$ (with respect to $g^{TU}$), and $\nabla^{\det}$ be the Chern connection on the determinant line bundle $\det(T^{1,0}U)$ (with $\langle \cdot, \cdot \rangle$), with connection forms $\Gamma^{TU}$ and $\Gamma^{\det}$ associated to the frames $\{e_j\}_{j=1}^{2n}$ and $w_1 \wedge \cdots \wedge w_n$. That is,

\begin{equation}
\nabla^{TU}_{e_j} e_\ell = \Gamma^{TU}(e_j)e_\ell, \quad j, \ell = 1, 2, \ldots, 2n,
\end{equation}

\begin{equation}
\nabla^{\det}(w_1 \wedge \cdots \wedge w_n) = \Gamma^{\det} w_1 \wedge \cdots \wedge w_n.
\end{equation}

The Clifford connection $\nabla^{Cl}$ on $\Lambda(T^{*0,1}U)$ is defined for the frame

\[
\{ \overline{w}^{j_1} \wedge \cdots \wedge \overline{w}^{j_q}; \quad 1 \leq j_1 < \cdots < j_q \leq n \}
\]

by the local formula

\begin{equation}
\nabla^{Cl} = d + \frac{1}{4} \sum_{j, \ell = 1}^{2n} (\Gamma^{TU} e_j, e_\ell)c(e_j)c(e_\ell) + \frac{1}{2} \Gamma^{\det}.
\end{equation}

In general a Levi-Civita connection $\nabla$ cannot be compatible with the complex structure unless a certain extra condition is imposed such as Kähler condition on the metric. Or else one takes the orthogonal projection $P_{T^{1,0}X} \nabla$ to produce a connection on $T^{1,0}X$. One key point above is that the Clifford connection $\nabla^{Cl}$ (regardless of Kähler condition nor orthogonal projection) defines a Hermitian connection (connection compatible with the underlying Hermitian metric) on $\Lambda(T^{*0,1}U)$ (see Proposition 1.3.1 in [48]).

Let’s be back to the CR case. In the same notation as before, $\Omega^{0,q}(U, E)$ denotes the space of $(0, q)$ forms on $U$ with values in $E$, $\Omega^{0,+}(U, E)$ the even part and $\Omega^{0,-}(U, E)$ the odd part of $\Omega^{0,*}(U, E)$ etc.

Assume $X$ is equipped with a CR bundle $E$ which is rigid. Being rigid $E$ can descend as a holomorphic vector bundle over $U$. We may assume that $E$ is (holomorphically) trivial on $U$ (possibly after shrinking $U$). A rigid Hermitian (fiber) metric $\langle \cdot | \cdot \rangle$ descends to a Hermitian (fiber) metric $\langle \cdot | \cdot \rangle_E$ on $E$ over $U$. Let $\nabla^E$ be the Chern connection on $E$ associated with $\langle \cdot | \cdot \rangle_E$ (over $U$).

We still denote by $\nabla^{Cl}$ the connection on $\Lambda(T^{*0,1}U) \otimes E$ induced by $\nabla^{Cl}$ and $\nabla^E$.

**Definition 4.1.** The Spin$^c$ Dirac operator $D_B$ is defined by

\begin{equation}
D_B = \frac{1}{\sqrt{2}} \sum_{j=1}^{2n} c(e_j)\nabla_{e_j}^{Cl} : \Omega^{0,*}(U, E) \to \Omega^{0,*}(U, E).
\end{equation}

It is well-known that $D_B$ is formally self-adjoint (see Proposition 1.3.1 and equation (1.3.1) in [48]) and $D_B : \Omega^{0,\pm}(U, E) \to \Omega^{0,\mp}(U, E)$.

Write $\overline{\nabla}^* : \Omega^{0,q+1}(U, E) \to \Omega^{0,q}(U, E)$ for the adjoint of $\overline{\nabla} : \Omega^{0,q}(U, E) \to \Omega^{0,q+1}(U, E)$ with respect to the $L^2$ inner product on $\Omega^{0,q}(U, E)$ induced by $\langle \cdot | \cdot \rangle$ and $\langle \cdot | \cdot \rangle_E$ ($q = 0, 1, 2, \ldots, n - 1$). Then, by Theorem 1.4.5 in [48]

\begin{equation}
D_B = \overline{\nabla}^* + A_B : \Omega^{0,\pm}(U, E) \to \Omega^{0,\mp}(U, E)
\end{equation}

where $A_B : \Omega^{0,\pm}(U, E) \to \Omega^{0,\mp}(U, E)$ is a smooth zeroth order operator (and $A_B = A_B(z)$, independent of $\theta$). Note that $A_B$ as an operator $\Omega^{0,*}(U, E) \to \Omega^{0,*}(U, E)$ is self-adjoint because both $D_B$ and $\overline{\nabla}^* + A_B$ are so.

The following is instrumental in forming a global operator from local ones, whose proof is based on canonical coordinates of BRT trivializations. Note for $u \in \Omega^{0,q}(D, E)$ ($q = 0, 1, 2, \ldots, n$) with $u = u(z)$, i.e. $u$ is independent of $\theta$, we may identify such $u$ with an element in $\Omega^{0,q}(U, E)$ by using (2.12) (and vice versa).

**Proposition 4.2.** Let $B = (D, (z, \theta), \varphi)$ and $\tilde{B} = (D, (w, \eta), \tilde{\varphi})$ be two BRT trivializations with $D = U \times [\varepsilon, \varepsilon[ \times \text{for } \varepsilon > 0 \text{ and an open set } U \text{ of } \mathbb{C}^n$. Let $A_B, \tilde{A}_B : \Omega^{0,\pm}(U, E) \to \Omega^{0,\mp}(U, E)$ be the operators
given by (4.7). Fix an \( m \in \mathbb{Z} \). For \( u \in \Omega_{m}^{0,\pm}(X, E) \) we can write \( u = u|_{D} = e^{-im\theta}v(z) = e^{-in\eta}v(w) \) for some \( v(z), \tilde{v}(w) \in \Omega^{0,\pm}(U, E) \). Then
\[
(4.8) \quad e^{-im\theta}A_{B}(v(z)) = e^{-in\eta}A_{\tilde{B}}(\tilde{v}(w)) \quad \text{on } D.
\]

**Proof.** Although we shall only use part of the coordinate transformation of BRT trivializations (2.11)
\[
(4.9) \quad w = H(z), \quad \overline{\partial}H(z) = 0; \quad \eta = \theta + \arg g(z); \quad \varphi(H(z), \overline{H(z)}) = \varphi(z, \overline{z}) + \log|g(z)|,
\]
let’s give a geometrical interpretation of how the above can be obtained for an independent interest. This complements the treatment of [4]. See Subsection 5.1 (cf. Remark 5.2) for its use in the construction of a modified Kodaira Laplacian.

To see (4.9), we are going to realize \( D \) (possibly after shrinking it) as (part of) the total space of a circle bundle associated with a trivial holomorphic line bundle \( L \) over a complex manifold \( U \subset \mathbb{C}^{n} \).

More precisely suppose \( L \) is equipped with a Hermitian metric such that a local basis \( \{ z, \lambda \} \subset \mathbb{C}^{n+1}; |\lambda|^{2}e^{2\phi} = 1 \) is the circle bundle inside the \( L^{*} \). Write \( \rho = |\lambda|^{2}e^{2\phi} - 1 \) and \( \lambda|_{Y} = e^{-\phi - i\theta} \). One has \( T^{0,1}Y = (\text{Ker } \overline{\partial}_{\rho}) \cap T^{0,1}\mathbb{C}^{n+1} \). In terms of \( (z, \theta) \) coordinates on \( Y \), one has \( T^{0,1}Y = \{ \partial_{\overline{\theta}}, \partial_{\theta} \} \) \( z = 1, 2, \ldots, n \) (and \( T^{1,0}Y = \overline{T^{0,1}Y} \) because the RHS is checked to be contained in \( \text{Ker } \overline{\partial}_{\rho} \) and it has the correct dimension). In view of Theorem 2.9 by taking the above \( \phi \) to be \( \varphi \) of \( Z_{j} \) in (2.10) and mapping \( (z, \theta) \) of \( D \) to \( (z, e^{-\phi(z)-i\theta}) \) of \( Y \), it will be seen that \( D \) is realized in this way as a portion of \( Y \).

For this realization, we compare the above with [4, (0.1) or Theorem I.2] and set up the holomorphic transformation in coordinates: setting our \( \varphi, \lambda, \theta \) and \( z \) (on \( U \)) above to be \( \phi, e^{iw}, -s_{1} \) and \( z \) respectively in [4, (0.1) and Theorem I.2)]. Then our \( Z_{j} \) in (2.10) corresponds to \( L_{j} \) in [4, Proposition I.2]. One sees that the ambient complex space of [4] is (locally) biholomorphic to the above \( L^{*} \). For the next purpose, let us give an intrinsic formulation of this complex space from another point of view. Let \( \mathbb{R}_{>0} \) be the set of positive real numbers. Consider \( D \times \mathbb{R}_{>0} \) and equip it with the complex structure \( J \) defined as follows. \( J|_{TU} \) is set to be the complex structure of \( U \); it suffices to define \( J \) on \( ] - \varepsilon, \varepsilon [ \times \mathbb{R}_{>0} \) with coordinates \( s, \tau \) : \( J(\partial/\partial s) = -\tau \partial/\partial \tau, J(\partial/\partial \tau) = \partial/\partial s \). Note this definition of \( J \) is independent of choice of BRT trivializations (since \( Z_{j} \subset T^{1,0}Y \) identified with \( T^{1,0}U \), gives a local basis of \( T^{1,0}D \); see Theorem 2.9, and \( \{ z \} \times -\varepsilon, \varepsilon \) for each \( z \in U \) is (part of) an \( S^{1} \) orbit in \( X \)). \( J \) is seen to be (equivalent to) the complex structure on \( L^{*} \) (with \( (z, s, r) \in D \times \mathbb{R}_{>0} \) and \( (z, re^{is}) \in L^{*} \) in correspondence), hence an integrable complex structure.

Let now \( B = (D, (w, \eta), \tilde{\varphi}) \) be any other BRT chart. Correspondingly we will denote the associated objects by the same notation as in \( B \) but topped with a tilde. Note that in \( \tilde{B} \) the set defined by \( w = w_{0} \) for a fixed \( w_{0} \) is part of an \( S^{1} \) orbit in \( X \); the same can be said with \( B \). Conversely, any \( S^{1} \) orbit of \( X \) is described (locally) by the \( \theta \) parameter in any BRT charts with \( z \)-coordinates being fixed. By using \( (D \times \mathbb{R}_{>0}, J) \) above, one has \( L^{*} \cong \tilde{L}^{*} \) (locally) by a biholomorphism \( F \) that preserves respective fibers (since these are \( S^{1} \) orbits, described by \( \theta \) parameters in each chart and hence must be in correspondence via \( D \times \mathbb{R}_{>0} \) by the property of BRT charts as just remarked). Further, one sees that \( F \) restricted to fibers has to be linear; hence \( F \) is a bundle isomorphism. Geometrically this picture is essentially the same as a local change of holomorphic coordinates on the base manifold \( U \) and a change of a local basis of \( L^{*} \) by \( \tilde{e}^{*}(z) = g(z)e^{*}(z) \) with \( ||e^{*}||_{2}^{2} = e^{2\tilde{\varphi}}, ||\tilde{e}^{*}||_{2}^{2} = e^{2\varphi} \) for some nowhere vanishing holomorphic function \( g(z) \) on \( U \). The above transformation formula (4.9) easily follows from this concrete realization.

Using the above transformation (4.9), one claims
\[
(4.10) \quad D_{B} = D_{\tilde{B}} \quad \text{on } \Omega^{0,\pm}(U, E),
A_{B} = A_{\tilde{B}} \quad \text{on } \Omega^{0,\pm}(U, E).
\]

To see this for the case without \( E \), note \( D \) is just realized as (part of) the total space of a circle bundle of a holomorphic line bundle. Clearly \( \overline{\partial}_{U} + \overline{\partial}_{\tilde{U}} \) does not depend on choice of holomorphic
coordinates on $U$; that is, $\partial U + \overline{\partial} U$ is an intrinsic object (cf. the Hermitian metric used for $\partial U$ is intrinsic, (4.1)). The same idea can be applied to $D_B$ which is defined above as an intrinsic object too. Therefore (4.10) holds with the change of coordinates in (4.9) (using only $w = H(z)$). Now with $E$ resumed, the reasoning is basically unchanged. Hence the claim (4.10).

By (4.9) $v(\zeta) = e^{-imG(\zeta)}\tilde{v}(w)$ ($G(z) = \arg g(z)$), hence by (4.10)

$$e^{-im\theta}A_B(v(\zeta)) = e^{-im\theta} A_B(e^{-imG(\zeta)}\tilde{v}(w)) = e^{-im\theta} A_B(e^{-imG(z)}\tilde{v}(w)) = e^{-im\theta} A_B(\tilde{v}(w)),$$

proving the Proposition.

We are now ready to introduce a global operator:

**Definition 4.3.** For every $m \in \mathbb{Z}$, let $A_m : \Omega^0_{m,\pm}(X,E) \to \Omega^0_{m,\mp}(X,E)$ be the linear operator defined as follows. Let $u \in \Omega^0_{m,\pm}(X,E)$. Then, $v := A_m u$ is an element in $\Omega^0_{m,\mp}(X,E)$ such that for every BRT trivialization $B := (D, (z, \theta, \varphi); (D = U \times] - \varepsilon, \varepsilon[; \varepsilon > 0), U$ an open set in $\mathbb{C}^n$) we have $v|_D = e^{-im\theta}A_B(\tilde{u}(z))$ where $u = e^{-im\theta}\tilde{u}(z)$ on $D$ for some $\tilde{u} \in \Omega^0_{m,\pm}(U,E)$ and $A_B$ is given in (4.7).

In view of Proposition 4.2, Definition 4.3 is well-defined.

We are now in a position to define the modified Kohn Laplacian ($\text{Spin}^c$ Kohn Laplacian) including a type of CR $\text{Spin}^c$ Dirac operator $\tilde{D}_{b,m}$. One goal of this part is to express the index of $\tilde{D}_{b,m}$ in an integral form of the heat kernel density (cf. Proposition 4.6).

The treatment below mostly follows traditional cases except the use of the projection operator $Q^\perp_{m}$ together with its explicit expression in integral (see (4.16) and (4.17)).

By using $A_m$ in Definition 4.3 we consider

$$\tilde{D}_{b,m} = \partial_b + \overline{\partial}_b + A_m : \Omega^0_{m,\pm}(X,E) \to \Omega^0_{m,\mp}(X,E),$$

with the formal adjoint $\tilde{D}^*_b$ on $\Omega^0_{m,\pm}(X,E)$.

We remark that $\tilde{D}^*_b \equiv \tilde{D}^*_{b,m}$ on $\Omega^0_{m,\pm}(X,E)$. For, by (2.5) the $L^2$ inner product on $\Omega^0_{m,\pm}(D,E)$ is clearly $2\varepsilon(\cdot, \cdot)$ with the $L^2$ inner product $(\cdot, \cdot)$ on $\Omega^0_{m,\pm}(U,E)$. Now that $A_B$ is self-adjoint on $\Omega^0_{m,\pm}(U,E)$ as aforementioned, it follows that $A_m$ is self-adjoint on $\Omega^0_{m,\pm}(X,E)$. That $\tilde{D}_{b,m}$ is self-adjoint follows as $\partial_b + \overline{\partial}_b$ is also self-adjoint.

The modified/Spin$^c$ Kohn Laplacian is given by

$$\tilde{\Box}_{b,m} := \tilde{D}_{b,m}^* \tilde{D}_{b,m} : \Omega^0_{m,\pm}(X,E) \to \Omega^0_{m,\mp}(X,E),$$

$$\tilde{\Box}^+_{b,m} := \tilde{D}_{b,m}^* \tilde{D}_{b,m} = \tilde{\Box}_{b,m} = \tilde{\Box}^0_{b,m} : \Omega^0_{m,\mp}(X,E) \to \Omega^0_{m,\pm}(X,E).$$

We extend $\tilde{\Box}^+_{b,m}$ and $\tilde{\Box}^-_{b,m}$ by

$$\tilde{\Box}^\pm_{b,m} : \text{Dom} \tilde{\Box}^\pm_{b,m} (\subset L^2_{m,\pm}(X,E)) \to L^2_{m,\mp}(X,E),$$

where $\text{Dom} \tilde{\Box}^\pm_{b,m} := \{ u \in L^2_{m,\pm}(X,E); \text{distribution} \tilde{\Box}^\pm_{b,m} u \in L^2_{m,\mp}(X,E) \}$.

Clearly $\tilde{\Box}_{b,m} - T^2$ (where $T$ is the real vector field induced by the $S^1$ action) is (the restriction of) an elliptic operator on $X$ since $(\partial_b + \overline{\partial}_b)^2 - T^2$ is.

Put as usual, the Sobolev spaces (cf. Subsection 2.1) $H^{s,\pm}(X,E), H^{s,-}(X,E)$ the even and odd part of $H^{s}(X,T^*\otimes E)$. In the same vein as Lemma 3.3 and Lemma 3.4 one has

$$\text{Dom} \tilde{\Box}^\pm_{b,m} = L^2_{m,\pm}(X,E) \cap H^2_{m,\mp}(X,E),$$

$\tilde{\Box}^+_{b,m}$ and $\tilde{\Box}^-_{b,m}$ are self-adjoint.

Further, for the spectrum $\text{Spec} \tilde{\Box}^+_{b,m} \subset [0, \infty]$ (resp. $\text{Spec} \tilde{\Box}^-_{b,m} \subset [0, \infty]$) one has (similar to Proposition 3.5).
Proposition 4.4. Spec $\tilde{\Sigma}^+_{b,m}$ is a discrete subset of $[0, \infty[$. For any $\mu \in \text{Spec } \tilde{\Sigma}^+_{b,m}$, $\mu$ is an eigenvalue of $\tilde{\Sigma}^+_{b,m}$ and the eigenspace

$$\tilde{E}^+_{m,\nu}(X, E) := \{ u \in \text{Dom } \tilde{\Sigma}^+_{b,m}; \tilde{\Sigma}^+_{b,m} u = \nu u \}$$

is finite dimensional with $\tilde{E}^+_{m,\nu}(X, E) \subset \Omega^0_{m,\nu}(X, E)$. Similar results hold for the case of $\tilde{\Sigma}^-_{b,m}$.

The following can be proved by standard argument.

Lemma 4.5. We have $\text{Spec } \tilde{\Sigma}^+_{b,m} \cap [0, \infty[ = \text{Spec } \tilde{\Sigma}^-_{b,m} \cap [0, \infty[$, and for every $0 \neq \mu \in \text{Spec } \tilde{\Sigma}^+_{b,m}$, $\dim \tilde{E}^+_{m,\mu}(X, E) = \dim \tilde{E}^-_{m,\mu}(X, E)$.

We are going to introduce a McKean-Singer type formula (Corollary 4.8). Let $F$ be a complex vector bundle over $X$ of rank $r$ with a Hermitian metric $(\cdot | \cdot)_F$. Let $A(x, y) \in C^\infty(X \times X, F \boxtimes F^*)$. For every $u \in C^\infty(X, F)$, $\int_X A(x, y)u(y)dv_X(y) \in C^\infty(X, F)$ is defined in a fairly standard manner.

Much of what follows parallels the classical cases except that $Q^+_{m,\nu}$ is introduced in our case. For $\nu \in \text{Spec } \tilde{\Sigma}^+_{b,m}$ let $P^\pm_{m,\nu} : L^2(\Omega^0_{m,\nu}(X, E)) \rightarrow \tilde{E}^+_{m,\nu}(X, E)$ be the orthogonal projections (with respect to $(\cdot | \cdot)_E$), and $P^\pm_{m,\nu}(x, y) \in C^\infty(X \times X, (T^{*0,\pm}X \otimes E) \boxtimes (T^{*0,\pm}X \otimes E^*))$ the distribution kernels of $P^\pm_{m,\nu}$.

The heat kernels of $\tilde{\Sigma}^+_{b,m}$ and $\tilde{\Sigma}^-_{b,m}$ are given by

$$(4.15) e^{-\tilde{\Sigma}^\pm_{b,m}(x, y)} = P^\pm_{m,0}(x, y) + \sum_{\nu \in \text{Spec } \tilde{\Sigma}^\pm_{b,m}, \nu > 0} e^{-\nu t} P^\pm_{m,\nu}(x, y)$$

with the associated continuous operators $e^{-\tilde{\Sigma}^\pm_{b,m}} : \Omega^0_{m,\nu}(X, E) \rightarrow \Omega^0_{m,\nu}(X, E) \subset \Omega^0_{m,\nu}(X, E)$. $e^{-\tilde{\Sigma}^\pm_{b,m}}$ is self-adjoint on $\Omega^0_{m,\nu}(X, E)$.

Remark that the heat kernels (4.15) are smooth. For, the eigenfunctions involved (in the equivalent form as (1.67)) are still eigenfunctions of $(\tilde{\Sigma}^\pm_{b,m} - T^2)$ hence eigenfunctions of an elliptic operator. In the elliptic case, one has the Gårding type inequality which estimates the various Sobolev norms of the eigenfunctions, and hence mainly by Sobolev embeddings, gives eventually the smoothness of the heat kernels (cf. [36, Lemmas 1.6.3 and 1.6.5]).

An important operator is given by the orthogonal projection

$$(4.16) Q^\pm_{m} : L^2(\Omega_{m,\nu}(X, E)) \rightarrow L^2(\Omega_{m,\nu}(X, E))$$

(for the $m$-th Fourier component). Fourier analysis with (2.2) gives

$$(4.17) Q^\pm_{m} u = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{-i\theta} \circ x)e^{i\theta} d\theta, \forall u \in \Omega^0_{m,\nu}(X, E)$$

(where $u(e^{-i\theta} \circ x)$ stands for the pull back $(e^{-i\theta})^* u$, cf. (2.2)). The explicit expression (4.17) turns out to be crucial to many (unconventional) estimates later.

It is fairly standard (note $Q^+_m$ in the second line below) to obtain by (4.15)

$$(4.18) \left( \frac{\partial}{\partial t} + \tilde{\Sigma}^\pm_{b,m} \right)(e^{-\tilde{\Sigma}^\pm_{b,m}} u) = 0, \forall u \in \Omega^0_{m,\nu}(X, E), \forall t > 0,$$

$$\lim_{t \rightarrow 0^+} (e^{-\tilde{\Sigma}^\pm_{b,m}} u) = Q^\pm_{m} u, \forall u \in \Omega^0_{m,\nu}(X, E).$$

For $\nu \in \text{Spec } \tilde{\Sigma}^+_{b,m}$, let $\{ f^\nu_1, \ldots, f^\nu_{d_\nu} \}$ be an orthonormal basis for $\tilde{E}^+_{m,\nu}(X, E)$. Define

$$(4.19) \text{Tr } P^\pm_{m,\nu}(x, x) := \sum_{j=1}^{d_\nu} | f^\nu_j(x) |^2 \in C^\infty(X)$$

which is equal to $\text{Tr } P^\pm_{m,\nu}(x, x) = \sum_{j=1}^{d_\nu} ( P^\pm_{m,\nu}(x, x) e_j(x) | e_j(x) )_E$ where $\{ e_j(x) \}_j$ is any orthonormal basis of $T^{*0,\pm}_x X \otimes E_x$. Define $\text{Tr } P^\pm_{m,\nu}(x, x)$ similarly.
Clearly $d^\pm_{\nu} = \int_X \text{Tr} P^\pm_{m,\nu}(x, x)dv_X(x)$.

Put $\text{Tr} e^{-t\Delta^\pm_{b,m}(x, x)} := \text{Tr} P^\pm_{m,0}(x, x) + \sum_{\nu \in \text{Spec} \Delta^\pm_{b,m,\nu} > 0} e^{-\nu t} \text{Tr} P^\pm_{m,\nu}(x, x)$, so for $t > 0$

\begin{equation}
(4.20) \quad \int_X \text{Tr} e^{-t\Delta^\pm_{b,m}(x, x)}dv_X(x) = \dim \tilde{\mathcal{E}}^\pm_{m,0}(X, E) + \sum_{\nu \in \text{Spec} \Delta^\pm_{b,m,\nu} > 0} e^{-\nu t} \dim \tilde{\mathcal{E}}^\pm_{m,\nu}(X, E).
\end{equation}

Combining Lemma 4.5 and (4.20) gives

\begin{equation}
(4.21) \quad \int_X \left( \text{Tr} e^{-t\Delta^\pm_{b,m}(x, x)} - \text{Tr} e^{-t\Delta^\pm_{b,m}(x, x)} \right)dv_X(x) = \dim \text{Ker} \tilde{\Delta}^\pm_{b,m} - \dim \text{Ker} \tilde{\Delta}^\pm_{b,m}.
\end{equation}

As in Theorem 3.8 one has

\begin{equation}
(4.22) \quad \text{Ker} \tilde{D}_{b,m} = \text{Ker} \tilde{\Delta}^+_m, \text{Ker} \tilde{D}^*_m = \text{Ker} \tilde{\Delta}^-_m \subset \Omega^0_{m,\nu}(X, E) \quad \text{Ker} \tilde{D}^*_{b,m} = \text{Ker} \tilde{\Delta}^-_{b,m} \subset \Omega^0_{m,\nu}(X, E).
\end{equation}

Put $\text{ind} \tilde{D}_{b,m} := \dim \text{Ker} \tilde{D}_{b,m} - \dim \text{Ker} \tilde{D}^*_{b,m}$, We express the index (by (4.22) and (4.21)) as

**Proposition 4.6.** For every $t > 0$, we have

\begin{equation}
(4.23) \quad \text{ind} \tilde{D}_{b,m} = \int_X \left( \text{Tr} e^{-t\Delta^\pm_{b,m}(x, x)} - \text{Tr} e^{-t\Delta^\pm_{b,m}(x, x)} \right)dv_X(x).
\end{equation}

The invariance of the index is expressed by the following (some aspects on $\text{ind} D_{b,m}$ refer to Theorem 3.8).

**Theorem 4.7.** (Homotopy invariance) We have $\text{ind} D_{b,m} = \text{ind} \tilde{D}_{b,m}$.

To summarize (with Theorem 4.7, Proposition 4.6 and (3.13)) we have a McKean-Singer formula (cf. Corollary 5.15 for McKean-Singer (II)).

**Corollary 4.8.** (McKean-Singer (I)) Fix $m \in \mathbb{Z}$. For $t > 0$, we have

\begin{equation}
(4.24) \quad \sum_{j=0}^{n} (-1)^j \dim H^j_{b,m}(X, E) = \int_X \left( \text{Tr} e^{-t\Delta^\pm_{b,m}(x, x)} - \text{Tr} e^{-t\Delta^\pm_{b,m}(x, x)} \right)dv_X(x).
\end{equation}

**Remark 4.9.** To compare with the original Kohn Laplacian, a similar formula (as Corollary 4.8)

\begin{equation}
\sum_{j=0}^{n} (-1)^j \dim H^j_{b,m}(X, E) = \int_X \left( \text{Tr} e^{-t\Delta^\pm_{b,m}(x, x)} - \text{Tr} e^{-t\Delta^\pm_{b,m}(x, x)} \right)dv_X(x)
\end{equation}

holds. When $X$ is not CR Kähler, it is obscure, by the experience from classical cases, to calculate the density $\text{Tr} e^{-t\Delta^\pm_{b,m}(x, x)} - \text{Tr} e^{-t\Delta^\pm_{b,m}(x, x)}$ with the original Kohn Laplacian. The introduction of the modified Kohn Laplacians replacing $\Delta^\pm_{b,m}$ by $\tilde{\Delta}^\pm_{b,m}$ is expected to facilitate this calculation. But because of the unconventional asymptotic expansion of $e^{-t\Delta^\pm_{b,m}(x, x)}$ some novelty beyond the classical cases shows up (as mentioned in Introduction). It should be noted that when $X$ is CR Kähler, $\tilde{\Delta}^\pm_{b,m} = \Delta^\pm_{b,m}$.

To prove Theorem 4.7 in the remaining of this section, observe that it is nothing but a statement of homotopy invariance of the index. For, with $A_m$ a global operator (see Definition 4.3), putting $L_t = \tilde{D}_{b,m} + \tilde{D}^*_{b,m} + tA_m : \Omega^0_{m,\nu}(X, E) \rightarrow \Omega^0_{m,\nu}(X, E)$ for $t \in [0, 1]$, gives the homotopy between $L_0 = D_{b,m}$ and $L_1 = \tilde{D}_{b,m}$.

Remark that there have been proofs for results of this type; for instance, see [5] using heat kernel method and [2] using functional analysis method (both not exactly formulated in the above form though). To make it accessible to a wider readership, we include a (comparatively) self-contained and short proof. It is amusing to note that the Hodge theory in Section 3 is useful at certain points of our proof.

Some preparations are in order. We extend $L_t$ by setting

$\text{Dom} L_t = \{ u \in L^2_m(X, E); \text{distribution} L_t u \in L^2_m(X, E) \}$
so that \( L_t : \text{Dom} \ L_t (\subset L^{2,\infty}_m (X, E)) \to L^{2,\infty}_m (X, E) \). Write \( L_t^* \) for the Hilbert space adjoint of \( L_t \).

Let \( H^{1,\infty}_m (X, E) \) be the completion of \( \Omega^{0,\infty}_m (X, E) \) with respect to the Hermitian inner product
\[
Q(u, v) = (u | v)_E + (\overline{\partial}_b u | \overline{\partial}_b v)_E + (\overline{\partial}_b^* u | \overline{\partial}_b^* v)_E.
\]

Clearly \( H^{1,\infty}_m (X, E) \subset \text{Dom} \ L_t, \forall t \in \mathbb{R} \). One can show that \( H^{1,\infty}_m (X, E) = \text{Dom} \ L_t \). Let \( t = 0 \). Assume \( L_0 f = u \) with \( f, u \in L^{2,\infty}_m \) and also assume \( f \perp \text{Ker} \overline{\square}^+_b \) since for any smooth \( g, f + g \in H^{1,\infty}_m \) iff \( f \in H^{1,\infty}_m \). Using the partial inverse \( N_m \) in (3.2) of our Hodge theory in Section 3, we have \( L_0 N_m f = N_m L_0 f = N_m u \) since \( L_0 \) commutes with \( N_m \) as in the ordinary Hodge theory. Now \( L_0 L_0 N_m f = \overline{\square}^+_b N_m f = f \) by (3.2), one has \( f = L_0 N_m u \). But \( N_m u \) increases the (Sobolev) order of regularity of \( u \) by 2 and then \( L_0^* N_m u \) decreases by 1, the regularity of \( f \) is of order 1. By localization, with a partition of unity, on an open subset \( D \) in place of \( X \) and by the formula (2.14) in BRT charts \( D = U \times ] - \delta, \delta[ \) for \( \overline{\partial}_b \), it follows from the standard Gårding’s inequality (e.g. [37, p. 93]) that the above \( Q(\cdot, \cdot) \) is equivalent to the Sobolev norm of order one (on the \( m \)-th component). Hence \( f \in H^{1,\infty}_m \). For \( t \neq 0 \), since \( L_t = L_0 + t A_m \) with \( A_m \) a smooth zeroth order operator, it follows \( \text{Dom} \ L_t = \text{Dom} \ L_0 \).

Consider \( H_0 := H^{1,\infty}_m (X, E) \oplus \text{Ker} L_0^* \) and \( H_1 = L^{2,\infty}_m (X, E) \oplus \text{Ker} L_0 \). Let \( \langle \cdot, \cdot \rangle_{H_0} \) and \( \langle \cdot, \cdot \rangle_{H_1} \) be inner products on \( H_0 \) and \( H_1 \), respectively, given by
\[
\langle (f_1, g_1), (f_2, g_2) \rangle_{H_0} = Q(f_1, f_2) + (g_1 | g_2)_E, \quad \langle (\overline{f}_1, \overline{g}_1), (\overline{f}_2, \overline{g}_2) \rangle_{H_1} = (\overline{f}_1 | \overline{f}_2)_E + (\overline{g}_1 | \overline{g}_2)_E.
\]

Let \( P_{\text{Ker} L_0} \) denote the orthogonal projection onto \( \text{Ker} L_0 \) with respect to \( \langle \cdot, \cdot \rangle_E \).

Let \( A_t : H_0 \to H_1 \) be the (continuous) linear map defined as follows. For \( (u, v) \in H_0 \),
\[
A_t (u, v) = (L_t u + v, P_{\text{Ker} L_0} u) \in H_1.
\]

**Lemma 4.10.** There is an \( r > 0 \) such that \( A_t : H_0 \to H_1 \) is invertible, for every \( 0 \leq t \leq r \).

**Proof.** We first claim that
\[
A_0 \text{ is invertible.}
\]

If \( A_0 (u, v) = 0 \) for some \( (u, v) \in H_0 \), then
\[
\begin{align*}
&i) \ L_0 u = -v \in \text{Ker} L_0^*, \quad ii) \ P_{\text{Ker} L_0} u = 0.
\end{align*}
\]

By (4.26)
\[
(L_0 u | L_0 v)_E = -(L_0 u | v)_E = -(u | L_0^* v)_E = 0,
\]
giving \( u \in \text{Ker} L_0 \). Hence by ii) of (4.26) we obtain \( u = 0 \), giving also \( v = 0 \) by i) of (4.26). We have proved that \( A_0 \) is injective.

We shall now prove that \( A_0 \) is surjective. Let \( (a, b) \in H_1 \). First we note \( L_0 : \text{Dom} \ L_0 \to L^{2,\infty}_m (X, E) \) has an \( L^2 \) closed range, so
\[
a = L_0 \alpha + \beta, \quad \alpha \in H^{1,\infty}_m (X, E), \quad \alpha \perp \text{Ker} L_0, \quad \beta \in (\text{Rang} L_0)^\perp = \text{Ker} L_0^*.
\]

Another way to see (4.28) is to use \( \overline{\square}^+_b N_m + \Pi_m = I \) (on \( L_m \)) of (3.2) (for the “−” case) of Hodge theory in Section 3, and obtain \( a = L_0 L_0^* N_m a + \Pi_m a \) where \( \Pi_m a \in \text{Ker} L_0^* \) (cf. Theorem 3.8) and \( L_0^* N_m a \in H^{1,\infty}_m (X, E) \) as mentioned above this lemma. In either way, by (4.28) one sees
\[
A_0 (\alpha + b, \beta) = (L_0 (\alpha + b) + \beta, P_{\text{Ker} L_0} (\alpha + b)) = (L_0 \alpha + \beta, b) = (a, b).
\]

Thus \( A_0 \) is surjective. The claim (4.25) follows.

Let \( A_0^{-1} : H_1 \to H_0 \) be the inverse of \( A_0 \). It follows from open mapping theorem that \( A_0^{-1} \) is continuous.
To finish the proof the following argument based on geometric series is standard. Write \( A_t = A_0 + R_t \), where \( R_t : H_0 \to H_1 \) is continuous and there is a constant \( c > 0 \) such that \( \| R_t u \|_{H_1} \leq ct \| u \|_{H_0} \), for \( u \in H_0 \). Put
\[
H_t = I - A_0^{-1} R_t + (A_0^{-1} R_t)^2 - (A_0^{-1} R_t)^3 + \cdots ,
\]
\[
\widetilde{H}_t = I - R_t A_0^{-1} + (R_t A_0^{-1})^2 - (R_t A_0^{-1})^3 + \cdots .
\]
Since \( A_0^{-1} \) is continuous, \( H_t : H_0 \to H_0 \) and \( \widetilde{H}_t : H_1 \to H_1 \) are well-defined as continuous maps for small \( t \geq 0 \). Moreover \( A_t \circ (H_t \circ A_0^{-1}) = I \) (on \( H_1 \)) and \( (A_0^{-1} \circ \widetilde{H}_t) \circ A_t = I \) (on \( H_0 \)), giving right and left inverses of \( A_t \) for small \( t \geq 0 \). Hence the lemma.

For \( t \in [0,1] \) write \( L_t : \text{Dom} L_t^* (\subset L_m^{2-}(X,E)) \to L_m^{2+}(X,E) \) for the adjoint of \( L_t \) with respect to \( (\cdot|\cdot)_E \). Similar to \( L_0 \) and \( L_1 \), one has \( \dim \ker L_t < \infty \) and \( \dim \ker L_t^* < \infty \) (with \( \ker L_t \subset \Omega_m^{0+}(X,E), \ker L_t^* \subset \Omega_m^{0-}(X,E) \)).

Put \( \ind L_t := \dim \ker L_t - \dim \ker L_t^* \).

**Lemma 4.11.** There is a \( r_0 > 0 \) such that \( \ind L_t = \ind L_0 \), for every \( 0 \leq t \leq r_0 \).

**Proof.** Let \( r > 0 \) be as in Lemma 4.10. We first show that
\[
\text{ind } L_0 \leq \text{ind } L_t, \quad \forall 0 \leq t \leq r.
\]

Fix \( 0 \leq t \leq r \). We define
\[
B : \ker L_t^* \oplus \ker L_0 \to \ker L_t \oplus \ker L_0^*
\]
as follows. Let \( (a,b) \in \ker L_t^* \oplus \ker L_0 \). By Lemma 4.10,
\[
A_t : H_m^{1+}(X,E) \oplus \ker L_0 \to L_m^{2-}(X,E) \oplus \ker L_0
\]
is invertible. There is a unique \( (u,v) \in H_m^{1+}(X,E) \oplus \ker L_0^* \) such that \( A_t(u,v) = (a,b) \). Let \( P_{\ker L_t} : L_m^{2+}(X,E) \to \ker L_t \) be the orthogonal projection with respect to \( (\cdot|\cdot)_E \). Then the above map \( B \) is defined by
\[
B(a,b) := (P_{\ker L_t} u, v) \in \ker L_t \oplus \ker L_0^*.
\]

We claim that \( B \) is injective. If so, then
\[
\dim \ker L_t^* + \dim \ker L_0 \leq \dim \ker L_t + \dim \ker L_0^*.
\]
i.e. \( \dim \ker L_0 - \dim \ker L_0^* \leq \dim \ker L_t - \dim \ker L_t^* \), yielding the desired (4.29).

For the claim that \( B \) is injective, if \( B(a,b) = (0,0) \) for some \( (a,b) \in \ker L_t^* \oplus \ker L_0 \), write \( (u,v) \in H_m^{1+}(X,E) \oplus \ker L_0^* \) such that \( A_t u = (a,b) \). As \( (0,0) = B(a,b) = (P_{\ker L_t} u, v) \), \( P_{\ker L_t} u = 0 \) and \( v = 0 \). Using the definition of \( A_t \), one has
\[
A_t(u,v) = A_t(u,0) = (L_t u, P_{\ker L_0} u) = (a,b) \in \ker L_t^* \oplus \ker L_0
\]
to give \( a = L_t u \in \ker L_t^* \), hence \( a|a)_E = (a|L_t u)_E = (L_t^* a | u)_E = 0 \) gives \( L_t u = a = 0 \) so that \( u \in \ker L_t \), i.e. \( u = P_{\ker L_t} u \) by definition. It follows that \( u = 0 \) since \( P_{\ker L_t} u = 0 \) as just seen. With \( u = 0 \) and (4.31) one sees \( (a,b) = (L_t u, P_{\ker L_0} u) = (0,0) \), giving the injectivity of \( B \).

By the same argument, \( \ind L_t^* \leq \ind L_t \) for small \( t \). By \( \ind L_t^* = -\ind L_t, \ind L_0 \geq \ind L_t \) holds. This and (4.29) prove the lemma.

**Proof of Theorem 4.7.** Let
\[
I_0 := \{ r \in [0,1] : \text{there is an } \varepsilon > 0 \text{ such that } \ind L_t = \ind L_0, \forall t \in (r - \varepsilon, r + \varepsilon) \cap [0,1] \}.
\]
\( I_0 \neq \emptyset \) is open by Lemma 4.11. Around a limit point \( r_\infty \) of \( I_0 \), by the same type of argument in the proof of Lemma 4.11 and Lemma 4.10 (replacing \( t = 0 \) by \( t = r_\infty \) in \( H_0, H_1 \) and \( A_0 \)), one finds \( \ind L_t = \ind L_{r_\infty} \) for \( t \in (r_\infty - \varepsilon_0, r_\infty + \varepsilon_0) \) with some \( \varepsilon_0 > 0 \). This implies \( I_0 \) is closed in \([0,1] \), so \( I_0 = [0,1] \).
5. Asymptotic expansions for the heat kernels of the modified Kohn Laplacians

In view of the McKean-Singer formula (Corollary 4.8), one of the goals is to calculate the local density (i.e., the term to the right of (4.24)). It consists in obtaining an asymptotic expansion for the heat kernel of the modified Kohn Laplacian (Spin^c Kohn Laplacian), to which we base our approach on two main steps. While the first step is motivated by the globally free case (see Theorem 1.2), it will be replaced by a local version within the framework of BRT trivializations (Section 2.4). A crucial off-diagonal estimate is going to be done in this subsection (cf. Theorem 5.9). In the second step we use the adjoint version of the heat equation to construct a global heat kernel with an asymptotic expansion related to local kernels.

5.1. Heat kernels of the modified Kodaira Laplacians on BRT trivializations.

This subsection is motivated by the globally free case (cf. Theorem 1.2). Here the emphasis is made on the localization of the argument including the Spin^c structure (which is needed for explicit local formulas of the heat kernel density). An important heat kernel estimate, termed as off-diagonal estimate, will be established in Theorem 5.9.

It is worth remarking that in the statement and proof of Theorem 1.2, we make no use of harmonic theory. In the locally free case, by contrast, it will be an important step to relate the (modified) Kohn Laplacian to (modified) Kodaira Laplacian (see discussion after that theorem). Since these Laplacians are defined via certain adjoints, suitable matching of metrics involved in both Laplacians must be done as an essential step.

We will use the same notations as in Section 4. Let \( B := (D, (z, \theta), \varphi) \) be a BRT trivialization (with \( D = U \times ]-\varepsilon, \varepsilon[ \), \( \varepsilon > 0 \) and \( U \) an open subset of \( \mathbb{C}^n \), cf. Subsection 2.4). For \( x \in D \) write \( z = z(x) \) and \( \theta = \theta(x) \). Since \( E \) is rigid and CR, equipped with a rigid Hermitian (fiber) metric \( \langle \cdot | \cdot \rangle_E \), (as in Section 4) \( E \) descends as a (holomorphically trivial) vector bundle over \( U \) (possibly after shrinking \( U \)) and \( \langle \cdot | \cdot \rangle_E \) as a Hermitian (fiber) metric on \( E \) over \( U \).

Let \( L \to U \) be a trivial (complex) line bundle with a non-trivial Hermitian fiber metric \( |1|_L^2 = e^{-2\varphi} \) (\( \varphi \) as in the above BRT triple \( B \)). Write \( (L^m, h^{L^m}) \to U \) for the \( m \)-th power of \( (L, h^L) \). Let \( \Omega^{0,q}(U, E \otimes L^m) \) be the space of \((0,q)\) forms on \( U \) with values in \( E \otimes L^m \) \((q = 0, 1, 2, \ldots, n)\). As usual, \( \Omega^{0,+}(U, E \otimes L^m) \) and \( \Omega^{0,-}(U, E \otimes L^m) \) denote forms of even and odd degree.

To start with the matching of the metrics we let \( \langle \cdot | \cdot \rangle \) be the Hermitian metric on \( CTU \) given by (cf. (4.1))

\[
\langle \partial_j \partial_k \rangle = \langle \partial_j \partial_k \rangle - i \frac{\partial^2 \varphi}{\partial z_j \partial \theta} - \frac{\partial^2 \varphi}{\partial z_k \partial \theta}, \quad j, k = 1, 2, \ldots, n.
\]

\( \langle \cdot | \cdot \rangle \) induces Hermitian metrics on \( T^{*0,q}U \) (bundle of \((0,q)\) forms on \( U \)), denoted also by \( \langle \cdot | \cdot \rangle \). These metrics induce Hermitian metrics on \( T^{*0,q}U \otimes E \), still denoted by \( \langle \cdot | \cdot \rangle_E \).

Let \( (\cdot, \cdot) \) be the \( L^2 \) inner product on \( \Omega^{0,q}(U, E) \) induced by \( \langle \cdot | \cdot \rangle \), \( \langle \cdot | \cdot \rangle_E \), and similarly \( (\cdot, \cdot)_m \) the \( L^2 \) inner product on \( \Omega^{0,q}(U, E \otimes L^m) \) induced by \( (\cdot, \cdot) \), \( (\cdot | \cdot)_E \) and \( h^{L^m} \).

Let \( \overline{\partial}_{L^m} : \Omega^{0,q}(U, E \otimes L^m) \to \Omega^{0,q+1}(U, E \otimes L^m) \), \((q = 0, 1, 2, \ldots, n - 1)\), be the Cauchy-Riemann operator. Let

\[
\overline{\partial}^*_{L^m} : \Omega^{0,q+1}(U, E \otimes L^m) \to \Omega^{0,q}(U, E \otimes L^m)
\]

be the formal adjoint of \( \overline{\partial}_{L^m} \) with respect to \( (\cdot, \cdot)_m \).

An essential operator that enters our picture is the following one (of Dirac type).

\[
D_{B,m} := D_{B,m}^+ + D_{B,m}^* + A_B : \Omega^{0,+}(U, E \otimes L^m) \to \Omega^{0,-}(U, E \otimes L^m)
\]

where \( A_B : \Omega^{0,+}(U, E \otimes L^m) \to \Omega^{0,-}(U, E \otimes L^m) \) is as in (4.7) (replacing \( E \) there by \( E \otimes L^m \) here) and

\[
D_{B,m}^+ : \Omega^{0,-}(U, E \otimes L^m) \to \Omega^{0,+}(U, E \otimes L^m)
\]
the formal adjoint of $D_{B,m}$ with respect to $(\cdot, \cdot)_m$. (Note $D_{B,m}$ on the full $\Omega^0_{m,+} = \Omega^{0,+} \oplus \Omega^{0,-}$ is self-adjoint; see the line below (4.6). But we prefer to use the above $D^*_{B,m}$ in the present context.) Note also $L$ with the metric $h^L$ depends on the choice of a BRT trivialization. However, $A_B$ is indeed an intrinsic object; we refer to Remark 5.2 in this regard.

One has the modified/Spin$^c$ Kodaira Laplacian:

\[(5.4) \quad \boxtimes_{B,m}^\pm := D^*_{B,m} D_{B,m} : \Omega^{0,+}(U, E \otimes L^m) \to \Omega^{0,+}(U, E \otimes L^m).\]

One may define $\boxtimes_{B,m}^- : \Omega^{0,-}(U, E \otimes L^m) \to \Omega^{0,-}(U, E \otimes L^m)$ analogously (by starting with $D^*_{B,m}$ or $D^*_{B,m}$).

The following fact appears fundamental in itself. It is instrumental to our construction of a heat kernel (cf. (5.44)) (See, however, remarks after its proof).

**Proposition 5.1.** In notations above let $u \in \Omega^0_{m,+}(X, E)$. On $D$ we can write $u(z, \theta) = e^{-im\theta} \tilde{u}(z)$ for some $\tilde{u}(z) \in \Omega^{0,+}(U, E)$. Recall the modified Kohn Laplacian $\boxtimes_{b,m}$ in (4.12). We write $s$ for the local basis $1^m$ of $L^m$. Then

\[(5.5) \quad e^{-m\varphi} \boxtimes_{B,m}^\pm (e^{m\varphi} \tilde{u} \otimes s) = (e^{im\theta} \boxtimes_{b,m}^\pm (u)) \otimes s.\]

*Proof.* One may work out this result by explicit computations. The following gives a somewhat conceptual proof. The idea is that one continues to match the objects on $U$ and on $D (\subset X)$. (In this way it turns out that no explicit computations of these Laplacians in local coordinates are needed.)

We define $\chi (= \chi_q) : \Omega^0 q(U, E) \to \Omega^0 q(U, E \otimes L^m)$ ($q = 0, 1, 2, \ldots, n$) by $\tilde{v}(z) \to \tilde{v}(z) e^{m\varphi(z)} \otimes s(z)$ for $\tilde{v} \in \Omega^{0,q}(U, E)$. Note $\chi$ preserves the (pointwise) norms. Equivalently $\chi(e^{-m\varphi} \tilde{v}) = \tilde{v} \otimes s$.

We define $\delta \tilde{v} = \delta \tilde{v} + m(\delta \varphi) \wedge \tilde{v}$ for $\tilde{v} \in \Omega^{0,q}(U, E)$ where $\delta : \Omega^{0,q}(U, E) \to \Omega^{0,q+1}(U, E)$. One may verify

\[(5.7) \quad \overline{\partial}_L^m \circ \chi = \chi \circ \delta \quad \text{on} \quad \Omega^0 q(U, E).\]

Indeed, by $\chi(e^{-m\varphi} \overline{\partial} \tilde{u}) = \overline{\partial} \tilde{u} \otimes s = \overline{\partial}_L^m (\tilde{u} \otimes s)$, one sees the term to the left of (5.7): $\overline{\partial}_L^m \circ \chi(e^{-m\varphi} \overline{\partial} \tilde{u}) = \chi(e^{-m\varphi} \overline{\partial} \tilde{u})$. Further, by using definition of $\delta$ one computes $e^{m\varphi} \delta(e^{-m\varphi} \tilde{u}) = \overline{\partial} \tilde{u}$. Then $\chi(e^{-m\varphi} \overline{\partial} \tilde{u}) = \chi(e^{-m\varphi} (e^{m\varphi} \delta(e^{-m\varphi} \tilde{u})))$ which is $\chi(\delta(e^{-m\varphi} \tilde{u}))$, giving the term to the right of (5.7) and proving (5.7).

Since $\chi$ is norm-preserving, we have also

\[(5.8) \quad \overline{\partial}_L^* \circ \chi = \chi \circ \delta^* \]

between respective adjoints. Combining (5.7) and (5.8) gives for $\overline{\partial}_{B,m} = (\overline{\partial}_L^m + \overline{\partial}_L^m)^2$ and $\Delta \equiv (\delta^* + \delta)^2$

\[(5.9) \quad \overline{\partial}_{B,m} \circ \chi = \chi \circ \Delta.\]

By (2.14) for $\overline{\partial}_b$, one computes, for $g = e^{-im\theta} \tilde{g} \in \Omega^0_{m+1}(U \times \varepsilon, \varepsilon, E) = \Omega^0_{m+1}(D, E)$

\[(5.10) \quad e^{im\theta} \overline{\partial}_b(e^{-im\theta} \tilde{g}(z)) = \delta(\tilde{g}(z)).\]

Write the map $\chi_1 : \Omega^0_{m+1}(D, E) \to \Omega^0_{m+1}(U, E)$ for $\chi_1(g) = \chi_1(e^{-im\theta} \tilde{g}) = \tilde{g}$, equivalently, $\chi_1(g) = e^{im\theta} g$.

Note $\chi_1$ preserves the respective (pointwise) norms (cf. (5.1)).

By (5.10) one sees (with $\overline{\partial}_{b,m} = \overline{\partial}_b |_{\Omega^0_{m+1}}$)

\[(5.11) \quad \chi_1 \circ \overline{\partial}_{b,m} = \delta \circ \chi_1 \quad \text{on} \quad \Omega^0_{m+1}(D, E).\]
By (2.5) the $L^2$ inner product on $\Omega^0_{m,q}(D, E)$ is clearly $2\varepsilon(\cdot, \cdot)$ with the $L^2$ inner product $(\cdot, \cdot)$ on $\Omega^0_{m,q}(U, E)$. Thus, in the same way as (5.9) by using (5.11) we have for $\Box_{b,m} \equiv (\bar{\partial}_{b,m} + \bar{\partial}_{b,m})^2$ (and $\Delta \equiv (\partial^* + \partial)^2$ as above) 

\[(5.12) \quad \chi_1 \circ \Box_{b,m} = \Delta \circ \chi_1 \quad \text{on} \quad \Omega^0_{m,q}(D, E).\]

Combining (5.12) and (5.9) yields

\[(5.13) \quad \Box_{b,m} = (\chi_1)^{-1} \circ \Box_{B,m} \circ (\chi_1)\]

By $\chi_1(e^{-im\theta} \bar{u}) = e^{m\varphi} \bar{u} \otimes s$ and $(\chi_1)^{-1}(\bar{u} \otimes s) = e^{-im\theta} e^{-m\varphi} \bar{u}$, one obtains

\[(5.14) \quad (\Box_{b,m} u) \otimes s = e^{-im\theta} e^{-m\varphi}(\Box_{B,m} e^{m\varphi}(\bar{u} \otimes s)) \quad \text{for} \quad u = e^{-im\theta} \bar{u} \in \Omega^0_{m,q}(D, E),\]

giving $e^{-m\varphi} \Box_{B,m}(e^{m\varphi} \bar{u}) = e^{im\theta} \Box_{b,m}(u)$ in notation similar to (5.6).

For modified Laplacians, from the definition of the zeroth order operator $A_m : \Omega^0_{m,+}(X, E) \rightarrow \Omega^0_{m,-}(X, E)$ (see Definition 4.3), it is clear that (in notation similar to (5.6))

\[(5.15) \quad e^{-m\varphi} A_B(e^{m\varphi} \bar{u}) = e^{im\theta} A_m(u).\]

In a way similar to (5.14) it follows by using (5.15) that

\[e^{-m\varphi} D_{B,m}(e^{m\varphi} \bar{u}) = e^{im\theta} D_{b,m}(u)\]

hence easily that

\[e^{-m\varphi} \Box_{B,m}(e^{m\varphi} \bar{u}) = e^{im\theta} \Box_{b,m}(u)\]

proving the proposition. \qed

Remark that one might be led by Proposition 5.1 to reduce the study of Kohn $\Box_{b,m}$ to that of Kodaira $\Box_{B,m}$. Indeed such a reduction works quite well in the globally free case (see discussion following Theorem 1.2 in Introduction). In the locally free case (of $S^1$ action), however, a naive thought of using the Kodaira Laplacian and its associated (local) heat kernels for a better understanding of the heat kernel in Kohn’s case is not directly accessible (see remarks following proof of Theorem 5.14). Namely the associated heat kernels of the two Laplacians cannot be easily linked as (5.5) seems to suggest. This reflects the fact that the associated heat kernels, rather than Laplacians themselves, are objects which are more global in nature. More in this regard will be pursued in the coming Subsection 5.2 and Section 6.

Remark 5.2. The definition of $A_B$ in (5.2) depends on a BRT triple, and the same can be said with Proposition 5.1. To see that $A_B$ has an intrinsic meaning, one uses the transformation of BRT coordinates as shown in the proof of Proposition 4.2. The geometrical construction given there shows that locally $X$ is part of a circle bundle inside the $L^*$ (with metric induced by that of $L$) over $U$, and the quantities such as $\varphi$, $z$ and $\theta$ in a BRT triple are associated with geometric ones as metric for a local basis (of $L$), coordinates on the base $U$ and (part of) a holomorphic coordinate on fibers (of $L^*$) respectively. The transformation in these quantities with another choice of a BRT chart is nothing more than a change of holomorphic coordinates of the same line bundle. It follows that $A_B$ is intrinsic in a proper sense. A similar explanation can be given to Proposition 5.1 too (although we do not strictly need this intrinsic property in what follows).

Remark 5.3. In the case of certain Riemannian foliations, it is known that the Laplacian downstairs and Laplacian upstairs (in a suitable generalized sense) can be related in spirit similar to that in our proposition above. See [56, p. 2310-2311].

As remarked in Subsection 1.7.3, to suit our purpose we will actually be considering adjoint heat equation and adjoint heat kernel first.

To proceed further, some notations are in order. Let $M$ be a $C^\infty$ orientable paracompact manifold with a vector bundle $F$ over it.
Definition 5.4. Let $A(t, x) \in C^\infty(\mathbb{R}_+ \times M, F)$. We write

$$A(t, x) \sim b_k(x, t) + t^{k+1} b_{k+1}(x, t) + t^{k+2} b_{k+2}(x, t) + \cdots \text{ as } t \to 0^+,$$

for $k \in \mathbb{Z}$, provided that for every compact set $K \Subset M$, every $\ell, M_0 \in \mathbb{N}_0$ with $M_0 \geq M_0(m)$ for some $M_0(m)$ (independent of $t$) such that

$$\left| A(t, x) - \sum_{j=0}^{M_0} t^{k+j} b_{k+j}(x, t) \right|_{C^\ell(K)} \leq C_{t, K, M_0} t^{M_0-M_1(m, \ell)}, \quad \forall 0 < t < \varepsilon_0.$$

Remark 5.5. In the important case of the heat kernel $p_t(x, y)$ of a generalized Laplacian on a compact Riemannian manifold $B$ of dimension $\beta$, $M = B \times B$ is of dimension $m = 2\beta$ and $k = -\beta/2$. One can take $M_0(m) = \lceil \frac{\beta}{2} \rceil + 1$ and $M_1(m, \ell) = \frac{\beta + \ell}{2}$. See [5, Theorem 2.30]. In this case $b_s(t, x)$ for all $s$ can be taken to be independent of $t$.

The novelty above is that $b_s$ could be nontrivially dependent on $t$ (in contrast to the conventional case of an asymptotic expansion for heat kernels).

Let $T^{a_0, +} U$ and $T^{a_0, -} U$ denote forms of even degree and odd degree in $T^{a_0} U$, respectively as before. If $T(z, w) \in (T^{a_0} + U \otimes E) \boxtimes (T^{a_0} + U \otimes E)^*|(z, w)$, write $[T(z, w)]$ for the standard pointwise matrix norm of $T(z, w)$ induced by $\langle \cdot, \cdot \rangle$ and $\langle \cdot \rangle_E$.

Suppose $G(t, z, w) \in C^\infty(\mathbb{R}_+ \times U \times U, (T^{a_0} + U \otimes E) \boxtimes (T^{a_0} + U \otimes E)^*)$. As usual, we denote $G(t) : \Omega^{a_0, +}_0(U, E) \to \Omega^{a_0, +}(U, E)$ (resp. $G'(t)$) the continuous operator associated with the kernel $G(t, z, w)$ (resp. $\frac{DG(t, z, w)}{dt}$) ($\Omega^{a_0, +}(U, E)$ denotes elements of compact support in $U$).

We are now ready to consider the heat operators associated with $\Box_{B, m}^+$ and $\Box_{B, m}^-$ in an adjoint version. By using the Dirichlet heat kernel construction (see [35] or [13]) we can obtain the theorem stated in the following form.

Proposition 5.6. There exists an $A_{B_+, m}(t, z, w) =: A_{B_+, m}(t, z, w) \in C^\infty(\mathbb{R}_+ \times U \times U, (T^{a_0} + U \otimes E) \boxtimes (T^{a_0} + U \otimes E)^*)$ such that

$$\lim_{t \to 0^+} A_{B_+, m}(t) = I \text{ in } \mathcal{D}'(U, T^{a_0} + U \otimes E),$$

$$A_{B_+, m}'(t) u + A_{B_+, m}(t) (\Box_{B, m}^+ u) = 0, \quad \forall u \in \Omega^{a_0, +}_0(U, E), \quad \forall t > 0,$$

and $A_{B_+, m}(t, z, w)$ satisfies the following: (I) For every compact set $K \Subset U$ and every $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{N}_0^n$, every $\gamma \in \mathbb{N}_0^n$, there are constants $C_{\gamma, \alpha_1, \alpha_2, \beta_1, \beta_2, K} > 0$, $\varepsilon_0 > 0$ and $P \in \mathbb{N}$ independent of $t$ such that

$$\left| \partial_t^\gamma \partial_{z_1}^{\alpha_1} \partial_{z_2}^{\alpha_2} \partial_{w_1}^{\beta_1} \partial_{w_2}^{\beta_2} A_{B_+, m}(t, z, w) \right| \leq C_{\gamma, \alpha_1, \alpha_2, \beta_1, \beta_2, K} t^{-P} e^{-\varepsilon_0 \frac{|z-w|^2}{t}}, \quad \forall (t, z, w) \in \mathbb{R}_+ \times K \times K.$$

(II) Let $g \in \Omega^{a_0, +}_0(U, E)$. For every $\alpha_1, \alpha_2 \in \mathbb{N}_0^n$ and every compact set $K \Subset U$, there is a $C_{\alpha_1, \alpha_2, K} > 0$ independent of $t$ such that

$$\sup \left\{ |\partial_{z_1}^{\alpha_1} \partial_{z_2}^{\alpha_2} (A_{B_+, m}(t) g)(z) | ; z \in K \right\} \leq C_{\alpha_1, \alpha_2, K} \sum_{\beta_1, \beta_2 \in \mathbb{N}_0^n, |\beta_1| + |\beta_2| \leq |\alpha_1| + |\alpha_2|} \sup \left\{ |\partial_{z_1}^{\beta_1} \partial_{z_2}^{\beta_2} (g)(z) | ; z \in U \right\}.$$

(III) $A_{B_+, m}(t, z, w)$ admits an asymptotic expansion in the following sense (see Definition 5.4 for $K_{B_+, m}(t, z, w)$)

$$A_{B_+, m}(t, z, w) = e^{-\frac{b_{z,w}(t,x)}{1}} K_{B_+, m}(t, z, w),$$

for some $K_{B_+, m}(t, z, w)$ as $t \to 0^+$, $b_{t}^{+}(z, w) = b_{t}^{+}(z, w) \in C^\infty(\mathbb{R}_+ \times U \times U, (T^{a_0} + U \otimes E) \boxtimes (T^{a_0} + U \otimes E)^*)$, $s = n, n-1, n-2, \ldots$, as $t \to 0^+$.
where \( h_+(z, w) \in C^\infty(U \times U, \mathbb{R}_+) \) with \( h_+(z, z) = 0 \) for every \( z \in U \) and for every compact set \( K \subseteq U \), there is a constant \( C_K > 1 \) such that \( \frac{1}{C_K} |z - w|^2 \leq h_+(z, w) \leq C_K |z - w|^2 \).

In (5.16) with \( \Box_{B,m} \) in place of \( \Box_{B,m} \), corresponding statements (with \( A_{B,-}, K_{B,-} \) etc.) for \( \Box_{B,m} \) hold true as well.

**Remark 5.7.** One may use a (weaker) version in the sense of asymptotic heat kernel (cf. [22], p. 96 for an informal explanation) with the property that this kernel (without uniqueness) satisfies (5.17)-(5.19) (so that it admits the same asymptotic expansion as the above kernel) and also satisfies (5.16) asymptotically (cf. Lemma 5.11 with (5.43)). Although the asymptotic heat kernels are not unique, the “formal” heat kernel given (in above notation) by this form

\[
e^{-\frac{h_+(z,w)}{4t}} (t^{-n}b_h^+_n(z,w) + t^{-n+1}b_h^+_{n-1}(z,w) + \cdots + b_h^+_0(z,w) + t b_h^+_1(z,w) + \cdots)
\]

is unique.

We are interested in calculating \( \text{Tr} b_h^+_s(z, z) - \text{Tr} b_h^-_s(z, z) \) \((s = n, n - 1, \ldots, 0)\) (where \( \text{Tr} b_h^+_s(z, z) = \sum_j (b_h^+_s(z, z)e_j | e_j)_E \) for any orthonormal frame \( e_j \) of \( T_z^{\text{SU}(1,1)} \otimes E_2 \)). The idea relies on Lichnerowicz formulas for (modified/Spin\(^c\)) Kodaira Laplacians \( \Box_{B,m} \) and \( \Box_{B,m} \) (cf. Theorem 1.3.5 and Theorem 1.4.5 in [48]) so that the (by now standard) rescaling technique in [5] and [22] can apply well.

To state the result precisely, we introduce some notations. Let \( \nabla^{TU} \) be the Levi-Civita connection on \( \mathcal{C}TU \) with respect to \( \langle \cdot, \cdot \rangle \). Let \( P_{T^1,0,U} \) be the natural projection from \( \mathcal{C}TU \) onto \( T^1,0,U \). \( \nabla^{T^1,0,U} := P_{T^1,0,U} \nabla^{TU} \) is a connection on \( T^1,0,U \). Let \( \nabla^{E \otimes L^m} \) be the (Chern) connection on \( E \otimes L^m \to U \) (induced by \( \langle \cdot, \cdot \rangle_E \) and \( h^m \), see Theorem 2.12). Let \( \Theta(\nabla^{T^1,0,U}, T^1,0,U) \in C^\infty(U, \Lambda^2(\Omega^\ast U) \otimes \text{End}(T^1,0,U)) \) and \( \Theta(\nabla^{E \otimes L^m}, E \otimes L^m) \in C^\infty(U, \Lambda^2(\Omega^\ast U) \otimes \text{End}(E \otimes L^m)) \) be the associated curvatures. As in complex geometry, put

\[
\text{Td} \left( \nabla^{T^1,0,U}, T^1,0,U \right) = e^{\text{Tr}(h(z) \Theta(\nabla^{T^1,0,U}, T^1,0,U))}, \quad h(z) = \log \left( \frac{z}{1 - e^{-z}} \right),
\]

\[
\text{ch} \left( \nabla^{E \otimes L^m}, E \otimes L^m \right) = \text{Tr} \left( \tilde{h}(\frac{i}{2\pi} \Theta(\nabla^{E \otimes L^m}, E \otimes L^m)) \right), \quad \tilde{h}(z) = e^z.
\]

Then the above calculation leads to the following.

\[
(\text{Tr} b_h^+_s(z, z) - \text{Tr} b_h^-_s(z, z)) = 0, \quad s = n, n - 1, \ldots, 1,
\]

\[
(\text{Tr} b_h^+_0(z, z) - \text{Tr} b_h^-_0(z, z)) \text{d}v_U(z) = \left[ \text{Td} \left( \nabla^{T^1,0,U}, T^1,0,U \right) \wedge \text{ch} \left( \nabla^{E \otimes L^m}, E \otimes L^m \right) \right]_{2n}(z), \quad \forall z \in U,
\]

where \([\cdot, \cdot]_{2n}\) denotes the \(2n\)-form part.

As the calculation to be performed here is almost entirely the same as in the standard case, we omit the detail.

Let \( \nabla^{TX} \) be the Levi-Civita connection on \( TX \) with respect to \( \langle \cdot, \cdot \rangle \) and \( \nabla^E \) the connection on \( E \) associated with \( \langle \cdot, \cdot \rangle_E \) (cf. Theorem 2.12). In similar notation as above \( \nabla^{T^1,0,X} := P_{T^1,0,X} \nabla^{TX} \) is a connection on \( T^1,0,X \).

Since \( \nabla^{T^1,0,X} \) and \( \nabla^E \) are rigid, in view of compatibility of metrics (and connections) in (5.1) and Theorem 2.12, one sees that \( \forall (z, \theta) \in D \) (for \( \omega_\theta \)) see lines below Definition 2.1):

\[
\text{Td} \left( \nabla^{T^1,0,U}, T^1,0,U \right) = \text{Td}_b \left( \nabla^{T^1,0,X}, T^1,0,X \right)(z, \theta)
\]

\[
\text{ch} \left( \nabla^{E \otimes L^m}, E \otimes L^m \right)(z) = \left( \text{ch}_b \left( \nabla^E, E \right) \wedge e^{-m \frac{d\theta}{2\pi}} \right)(z, \theta)
\]

and

\[
\left[ \text{Td} \left( \nabla^{T^1,0,U}, T^1,0,U \right) \wedge \text{ch} \left( \nabla^{E \otimes L^m}, E \otimes L^m \right) \right]_{2n}(z) \wedge d\theta = \left[ \text{Td}_b \left( \nabla^{T^1,0,X}, T^1,0,X \right) \wedge \text{ch}_b \left( \nabla^E, E \right) \wedge e^{-m \frac{d\theta}{2\pi}} \wedge \omega_0 \right]_{2n+1}(z, \theta).
\]
To sum up we arrive at the following (by (5.20), (5.22) and $du_T \wedge d\theta = du_X$ on $D$ cf. (2.5))

**Proposition 5.8.** With the notations above, we have

$$
\left( \text{Tr} \ b^+_0(z, z) - \text{Tr} \ b^-_0(z, z) \right) du_X(z, \theta) = \left[ Tdb \left( \nabla^{T^{1,0}X}, T^{1,0}X \right) \wedge chb \left( \nabla^E, E \right) \wedge e^{-m \frac{du}{{\omega_0}}} \wedge \omega_0 \right]_{2n+1} \left( z, \theta \right), \ \forall (z, \theta) \in D.
$$

(5.23)

To state the final technical result of this subsection, we first identify $T^{a,0}_z U \otimes E_{z_2}$ with $T^{a,0}_z U \otimes E_{z_1}$ (by parallel transport along geodesics joining $z_1, z_2 \in U$), so we can identify $T \in (T^{a,0}_0 U \otimes E) \otimes (T^{a,0}_0 U \otimes E)^* \mid (z_1, z_2)$ with an element in $\text{End} \left( T^{a,0}_1 U \otimes E_{z_1} \right)$. With this identification, write

$$
\text{Tr}_t T := \sum_{j=1}^{d} \langle Te_j \mid e_j \rangle E,
$$

(5.24)

where $e_1, \ldots, e_d$ is an orthonormal frame of $T^{a,0}_1 U \otimes E_{z_1}$.

In the proof of Theorem 1.10 (see Theorem 6.4), somewhat surprisingly, as deviated from the classical case, we need to estimate the off-diagonal terms $\text{Tr}_t b^+_s(z, w) - \text{Tr}_t b^-_s(z, w)$ for each $s$. For this, the following can be considered as another application of the rescaling technique (and an identity in Berenzin integral as usual).

**Theorem 5.9.** (Off-diagonal estimate) With the notations above, we have

$$
\text{Tr}_t b^+_s(z, w) - \text{Tr}_t b^-_s(z, w) = O(|z - w|^{2s}) \text{ locally uniformly on } U \times U, \ s = n, n - 1, \ldots, 1.
$$

(5.25)

**Proof.** Recall $E$ is (holomorphically) trivial on $U$. Let $e_1, \ldots, e_{2n}$ be an orthonormal basis for $T^0_0 U$. For $f \in T^0_0 U$, let $c(f) \in \text{End} \left( T^0_0 U \right)$ be the natural Clifford action of $f$ (see (4.2) or [5]). As usual, for every strictly increasing multi-index $J = (j_1, \ldots, j_q)$ we set $|J| := q$, $e_J := e_{j_1} \wedge \ldots \wedge e_{j_q}$ and $c(e_J) = c(e_{j_1}) \cdots c(e_{j_q})$. For $T \in \text{End} \left( T^0_0 U \right)$, we can always write $T = \sum'_{|J| \leq 2n} c(e_J) T_J$ ($T_J \in \mathbb{C}$), where $\sum'$ denotes the summation over strictly increasing multiindices. For $k \leq 2n$, we put

$$
[T]_k := \sum'_{|J| = k} T_J e_J \quad (\in CT^k_0 U).
$$

(5.26)

and a similar expression for $[T]$ (without the subscript $k$). We identify $T$ with $[T]$ without the danger of confusion. We say that $\text{ord} \ T \leq k$ if $T_J = 0$, for all $|J| > k$, and $\text{ord} \ T = k$ if $\text{ord} \ T \leq k$ and $[T]_k \neq 0$.

A crucial result for our need here is an identity in Berenzin integral (see [5, Proposition 3.21, Definition 3.4 and (1.28)]) which asserts that if $\text{ord} \ T < 2n$ then $\text{STr} \ T = 0$ (see (1.13) for the definition of supertrace there) and

$$
\text{STr} \ T = (-2i)^{2n} \text{STr} \ T_J \ c(e_J), \ J_J = (1, 2, \ldots, 2n).
$$

(5.27)

Recall the identification $T^0_0 U \cong T^a_0 U$ just mentioned above the theorem, so that a smooth function $F(x) \in (T^a_0 U) \otimes (T^a_0 U)^* \mid (0, x)$ is identified with a function $x \to F(x) \in \text{End} \left( T^0_0 U \right)$, giving a Taylor expansion

$$
F(x) = \sum_{\alpha \in \mathbb{N}^n_0, |\alpha| \leq P} x^\alpha F_{\alpha} + O(|x|^{P+1}), \ F_{\alpha} \in \text{End} \left( T^0_0 U \right).
$$

We are ready to apply Getzler’s rescaling technique to off-diagonal estimates. Consider $A_B(t, x, y) = A_B(t, z, w) \oplus A_B(t, z, w)$ (cf. Proposition 5.6) and let $\chi \in C^\infty_0 (U)$ with $\chi = 1$ near $z = 0$. Let

$$
r(u, t, x) := \sum_{k=1}^{2n} u^{-\frac{k}{2} + n} \chi(\sqrt{u} x) A_B(ut, 0, \sqrt{u} x).k
$$

(5.28)
Note that $A_B$ is actually identified with $[A_B]$ similar to the case of $T$ above, so that the $k$-form part ($k > n$) of (5.28) makes sense.

It is well-known that (see [5]) $\lim_{u \to 0} r(u, l, x) = g(t, x) \in C^\infty(R_+ \times C^n, C^T \times C^n)$ in $C^\infty$-topology locally uniformly ($C^T \times C^n = \oplus_{k=0}^{\infty} \Lambda^k C^T \times C^n$). In particular, $\lim_{u \to 0} r(u, 1, x) = g(1, x) \in C^\infty$-topology locally uniformly, for their $2n$-form parts.

Let $b_s(z, w) := b^+_s(z, w) + b^-_s(z, w)$, $s = n, n - 1, \ldots$ (cf. (5.19)). One sees

(5.29) $r(u, 1, x)_{2n} = e^{-\frac{h_s(u, \sqrt{ux})}{u}} \chi(\sqrt{ux}) \left( u^{-n}[b_n(0, \sqrt{ux})]_{2n} + u^{-n+1}[b_{n-1}(0, \sqrt{ux})]_{2n} + \cdots \right)$.

Since $\lim_{u \to 0} e^{-\frac{h_s(u, \sqrt{ux})}{u}}$ converges to a smooth function in $C^\infty$-topology locally uniformly on $C^n$ (see (5.19) in Proposition 5.6), we deduce that

(5.30) $\lim_{u \to 0} \left( u^{-n}[b_n(0, \sqrt{ux})]_{2n} + u^{-n+1}[b_{n-1}(0, \sqrt{ux})]_{2n} + \cdots \right) = \hat{g}(x) \in C^\infty(C^n, C^T \times C^n)$

in $C^\infty$-topology locally uniformly. Fix $P \gg 1$. Write

(5.31) $\hat{g}(x) = \sum_{\alpha \in \mathbb{N}_0^n, |\alpha| \leq P} \hat{g}_\alpha x^\alpha + O(|x|^{P+1}).$

and for each $s, n - 1, \ldots$,

(5.32) $b_s(0, x) = \sum_{\alpha \in \mathbb{N}_0^n, |\alpha| \leq P} b_{s, \alpha} x^\alpha + O(|x|^{P+1}).$

Hence

(5.33) $[b_s(0, \sqrt{ux})]_{2n} = \sum_{\alpha \in \mathbb{N}_0^n, |\alpha| \leq P} u^{\frac{|\alpha|}{2}} [b_{s, \alpha}]_{2n} x^\alpha + u^{\frac{p+1}{2}} O(|x|^{P+1}),$ $s, n - 1, \ldots$

and from (5.30), (5.31) and (5.33) it follows that for every $\alpha \in \mathbb{N}_0^n$

(5.34) $\lim_{u \to 0} \left( u^{-n+\frac{|\alpha|}{2}} [b_{n, \alpha}]_{2n} + u^{-n+1+\frac{|\alpha|}{2}} [b_{n-1, \alpha}]_{2n} + \cdots \right) = \hat{g}_\alpha.$

With (5.34) we conclude

(5.35) $[b_{n, \alpha}]_{2n} = 0, \ \forall |\alpha| < 2n.$

Combining (5.35) and (5.32), we see

$$\text{Tr}_0 b^+_s(0, w) - \text{Tr}_0 b^-_s(0, w) = O(|w|^{2n}).$$

We can repeat the method above for the second leading term, and deduce similarly

$$\text{Tr}_0 b^+_s(0, w) - \text{Tr}_0 b^-_s(0, w) = O(|w|^{2n}), \ s = n - 1, n - 2, \ldots, 1.$$ 

The theorem follows. \qed

5.2. Heat kernels of the modified Kohn Laplacians (Spin$^c$ Kohn Laplacians). Based on Proposition 5.1, one is tempted to patch up the local heat kernels of the modified Kodaira Laplacian constructed in Proposition 5.6 to form a global heat kernel for the modified Kohn Laplacian. This is no problem in the globally free case (of the $S^1$ action). In the locally free case, however, some delicate points arise as the relation of the two Laplacians given in the above proposition is, by nature, a local property, whereas the heat kernels are global objects. See discussions after proof of Proposition 5.1 and of Theorem 5.14 with (5.54) for more.

As remarked in Subsection 1.7.3, if we use the adjoint version of the original heat equation, it becomes more effective to go over the desired process of patching up. It is worth noting that an important role, mostly unseen traditionally, is played by the projection $Q^+_m$ in our situation.

Assume $X = D_1 \cup D_2 \cup \cdots \cup D_N$ with each $B_j := (D_j, (z, \theta), \varphi_j)$ a BRT trivialization. A slightly more complicated set up is as follows. Write for each $j$, $D_j = U_j \times [-2\delta_j, 2\delta_j] \subset C^n \times \mathbb{R}$, $\delta_j > 0$, 

\[ \]
Slightly tediously, we put \( \tilde{\delta}_j > 0, \ U_j = \{ z \in \mathbb{C}^n ; |z| < \gamma_j \} \). Put \( \tilde{D}_j = \tilde{U}_j \times ] - \frac{\delta_j}{2}, \frac{\delta_j}{2} [ \), \( \tilde{U}_j = \{ z \in \mathbb{C}^n ; |z| < \frac{\gamma_j}{2} \} \). We suppose \( X = \tilde{D}_1 \cup \tilde{D}_2 \cup \cdots \cup \tilde{D}_N \).

Here are some cut-off functions; the choice is adapted to BRT trivializations.

i) \( \chi_j(x) \in C^\infty_0(\tilde{D}_j) \) with \( \sum_{j=1}^N \chi_j = 1 \) on \( X \). Put

\[
A_j = \left\{ z \in \tilde{U}_j ; \text{ there is a } \theta \in ] - \frac{\delta_j}{2}, \frac{\delta_j}{2} [ \text{ such that } \chi_j(z, \theta) \neq 0 \right\}.
\]

ii) \( \tau_j(z) \in C^\infty_0(\tilde{U}_j) \) with \( \tau_j \equiv 1 \) on some neighborhood \( W_j \) of \( A_j \).

iii) \( \sigma_j \in C^\infty_0(] - \frac{\delta_j}{2}, \frac{\delta_j}{2} [ \) with \( \int_{\delta_j/2}^{\delta_j/2} \sigma_j(\theta) d\theta = 1 \).

iv) \( \tilde{\sigma}_j \in C^\infty_0(] - \delta_j, \delta_j [ \) such that \( \tilde{\sigma}_j = 1 \) on some neighbourhood of \( \text{Supp} \sigma_j \) and \( \tilde{\sigma}_j(\theta) = 1 \) if \( (z, \theta) \in \text{Supp} \chi_j \).

Write \( x = (z, \theta), \ y = (w, \eta) \in \mathbb{C}^n \times \mathbb{R} \). We are going to lift many objects in the preceding subsection defined on \( U_j \) to the ones defined on \( \tilde{D}_j \) via the above cut-off functions.

Let \( A_{B_j,+}(t, z, w), K_{B_j,+}, h_{j,+}(z, w) \) and \( b_{j,s}^+(s = n, n-1, \ldots) \) be as in Proposition 5.6 and (5.19).

Slightly tediously, we put

\[
H_j(t, x, y) = H_{j,+}(t, x, y) = \chi_j(x)e^{-m\varphi_j(z) - im\theta} A_{B_j,+}(t, z, w)e^{m\varphi_j(w) + im\eta} \tau_j(w)\sigma_j(\eta)
\]

\[
G_j(t, x, y) = G_{j,+}(t, x, y) = \chi_j(x)e^{-m\varphi_j(z) - im\theta} A_{B_j,+}(t, z, w)e^{m\varphi_j(w) + im\eta} \tau_j(w).
\]

and (the last two equations are from (5.19))

\[
K_{j,+}(t, x, y) = K_{j,+}(t, x, y) = \chi_j(x)e^{-m\varphi_j(z) - im\theta} K_{B_j,+}(t, z, w)e^{m\varphi_j(w) + im\eta} \tau_j(w)\sigma_j(\eta)
\]

\[
\hat{h}_{j,+}(x, y) = \tilde{\sigma}_j(\theta) h_{j,+}(z, w) \tilde{\sigma}_j(\eta) \in C^\infty_0(\tilde{D}_j), \ x = (z, \theta), \ y = (w, \eta)
\]

\[
b_{j,s}^+(x, y) = \chi_j(x)e^{-m\varphi_j(z) - im\theta} b_{j,s}^+(z, w)e^{m\varphi_j(w) + im\eta} \tau_j(w)\sigma_j(\eta), \ s = n, n-1, \ldots
\]

\[
\tilde{b}_{j,s}^+(x, y) = \chi_j(x)e^{-m\varphi_j(z) - im\theta} \tilde{b}_{j,s}^+(z, w)e^{m\varphi_j(w) + im\eta} \tau_j(w), \ s = n, n-1, \ldots
\]

\[
A_{B_j,+}(t, z, w) = e^{-h_{j,+}(z, w)/2} K_{B_j,+}(t, z, w)
\]

\[
K_{B_j,+}(t, z, w) \sim t^{-n}\tilde{b}_{j,n}^+(z, w) + t^{-n+1}\tilde{b}_{j,n-1}^+(z, w) + \cdots + b_{j,0}^+(z, w) + t b_{j,-1}^+(z, w) + \cdots \text{ as } t \to 0^+.
\]

Remark that these expressions, apart from cut-off functions, are mainly motivated by the formulas (5.5), (5.6) of Proposition 5.1. Let \( H_j(t) \) be the continuous operator associated with \( H_j(t, x, y) \), for which we put down the expression for later use (cf. (5.42) and (5.44))

\[
H_j(t) : \Omega^{0,+}(X, E) \to \Omega^{0,+}(X, E),
\]

\[
u \to \int_{\Omega^{0,+}(X, E)} \chi_j(x)e^{-m\varphi_j(z) - im\theta} A_{B_j,+}(t, z, w)e^{m\varphi_j(w) + im\eta} \tau_j(w)\sigma_j(\eta) u(y) d\nu_X(y).
\]

Consider the patched up kernel (recall \( Q_m \) is the projection on the \( m \)-th Fourier component, cf. (4.16))

\[
\Gamma(t) := \sum_{j=1}^N H_j(t) \circ Q_m : \Omega^{0,+}(X, E) \to \Omega^{0,+}(X, E)
\]

and let \( \Gamma(t, x, y) \in C^\infty(\mathbb{R}_+ \times X \times X, (T^{s,0,0} X \otimes E) \boxtimes (T^{s,0,0} X \otimes E)^\ast) \) be the distribution kernel of \( \Gamma(t) \).
For an explicit expression, one sees (using $Q_m$ of (4.17)) that

\[
\Gamma(t, x, y) = \frac{1}{2\pi} \sum_{j=1}^{N} \int_{-\pi}^{\pi} H_j(t, x, e^{-iu} \circ y)e^{-imu} \, du
\]

(5.40)

\[
= \frac{1}{2\pi} \sum_{j=1}^{N} \int_{-\pi}^{\pi} e^{-\frac{\beta_{j,+}(x, e^{-iu} \circ y)}{t}} \hat{K}_j(t, x, e^{-iu} \circ y)e^{-imu} \, du
\]

\[
\sim t^{-n}a^+_1(t, x, y) + t^{-n+1}a^+_2(t, x, y) + \cdots \text{ as } t \to 0^+,
\]

where we have written

\[
a^+_s(t, x, y) = a^+_s(t, x, y) \quad (b^+_s = \hat{b}^+_s)
\]

(5.41)

\[
= \frac{1}{2\pi} \sum_{j=1}^{N} \int_{-\pi}^{\pi} e^{-\frac{\beta_{j,+}(x, e^{-iu} \circ y)}{t}} \hat{b}^+_s(x, e^{-iu} \circ y)e^{-imu} \, du, \quad s = n, n-1, n-2, \ldots.
\]

For the initial condition of $\Gamma(t, x, y)$, one has the following.

**Lemma 5.10.**

\[
\lim_{t \to 0^+} \Gamma(t)u = Q_m u \quad \text{(on $\mathcal{D}'(X, T^{*0,+}X \otimes E)$)}
\]

for $u \in \Omega^{0,+}(X, E)$.

**Proof.** For $u \in \Omega^{0,+}(X, E)$, $Q_m u \in \Omega^{0,+}(X, E)$, $Q_m u|_{D_j}$ can be expressed as $e^{-imu}v_j(w)$ for some $v_j(w) \in \Omega^{0,+}(U_j, E)$. With (5.38) we find (note $A_{B_j,\tau}(t) = I$ as $t \to 0$ and $dv_{D_j} = dv_{U_j}d\eta$ by (2.5))

\[
\lim_{t \to 0^+} H_j(t)Q_m u = \lim_{t \to 0^+} \int \chi_j(x)e^{-m\varphi_j(z) - im\theta} A_{B_j,\tau}(t, z, w)e^{m\varphi_j(w) + im\eta\tau_j(w)}\sigma_j(\eta)e^{-imu}v_j(w)dv_{U_j}(w)d\eta
\]

(5.42)

\[
\quad = \lim_{t \to 0^+} \int \chi_j(x)e^{-m\varphi_j(z) - im\theta} A_{B_j,\tau}(t, z, w)e^{m\varphi_j(w)\tau_j(w)}v_j(w)dv_{U_j}(w)
\]

\[
\quad = \chi_j(x)e^{-m\varphi_j(z) - im\theta} e^{m\varphi_j(z)\tau_j(z)}v_j(z)
\]

\[
= \chi_j e^{-im\theta} v_j = \chi_j Q_m u.
\]

With the above, the lemma follows from (5.39) and $\sum_j \chi_j = 1$. \hspace{1cm} \square

$\Gamma(t)$ satisfies an adjoint type heat equation asymptotically in the following sense (cf. [22, p. 96]).

**Lemma 5.11.** We consider $\Box^{+}_{b,m} \circ Q_m$ still denoted by $\Box^{+}_{b,m}$. $\Gamma(t, x, y)$ satisfies

\[
\Gamma'(t)u + \Gamma(t)\Box^{+}_{b,m}u = R(t)u, \quad \forall u \in \Omega^{0,+}(X, E),
\]

where $R(t) : \Omega^{0,+}(X, E) \to \Omega^{0,+}(X, E)$ is the continuous operator with distribution kernel $R(t, x, y)$ ($\in C_{\infty}(\mathbb{R}_+ \times X \times X, (T^{*0,+}X \otimes E) \otimes (T^{*0,+}X \otimes E)^*)$) which satisfies the following. For every $\ell \in \mathbb{N}_0$, there exists an $\varepsilon_0 > 0$, $C_\ell > 0$ independent of $t$ such that

\[
\|R(t, x, y)\|_{C^\ell(X \times X)} \leq C_\ell e^{-\frac{t}{\varepsilon_0}}, \quad \forall t \in \mathbb{R}_+.
\]

(5.43)
Proof. As in the preceding lemma let \( u \in \Omega^{0,+}_{m}(X, E) \) and write \( u = e^{-im\eta}v_j(w) \) for some \( v_j(w) \in \Omega^{0,+}(U_j, E) \) on \( D_j \). By this, (5.5), (5.16) and (5.38) it is a bit tedious but straightforward to compute

\[
H'_j(t)u + H_j(t)\tilde{b}_{b,m}u
= \int \chi_j(x)e^{-m\varphi_j(z)-im\theta}A_{j,+}(t, z, w)e^{m\varphi_j(w)+im\eta}\tau_j(w)\sigma_j(\eta)(u(y))dy_X(y)
\]

for some \( S_j(t, x, w) \in C^n(\mathbb{R}_+ \times D_j \times U_j, (T^{\ast 0,+}X \otimes E) \otimes (T^{\ast 0,+}X \otimes E)^\ast) \).

Note \( \tau_j(z) = 1 \) for \( (z, \theta) \) in some small neighborhood of \( \text{Supp} \chi_j \). One sees that \( S_j(t, x, w) = 0 \) if \( (x, w) \in \text{some small neighborhood of} \ (z, z) \). Hence by using (5.17) for (5.44) (on \( |z - w| \) away from zero), we conclude that for every \( \ell \in \mathbb{N}_0 \), there is an \( \varepsilon > 0 \) independent of \( t \) such that

\[
\|S_j(t, x, w)\|_{C^\ell(X \times X)} \leq C_\ell e^{-\varepsilon t}, \quad \forall t \in \mathbb{R}_+.
\]

Put \( \tilde{R}(t, x, y) := \sum_{j=1}^N \chi_j S_j(t, x, w)e^{im\eta}\sigma_j(\eta) \in C^\infty(\mathbb{R}_+ \times X \times X, (T^{\ast 0,+}X \otimes E) \otimes (T^{\ast 0,+}X \otimes E)^\ast) \)

and set

\[
R(t, x, y) = \frac{1}{2\pi} \sum_{j=1}^N \int_0^\pi \tilde{R}(t, x, e^{-iu}x, y)e^{-im\eta}du.
\]

Let \( R(t) : \Omega^{0,+}(X, E) \to \Omega^{0,+}(X, E) \) be the continuous operator with distribution kernel \( R(t, x, y) \). Note that \( R(t) = R(t) \circ Q_m \) (cf. (5.40) and (5.39)). By (5.45) \( R(t, x, y) \) satisfies (5.43) and by (5.44) one sees \( \Gamma^\ast(t)u + \Gamma(t)\tilde{b}_{b,m}u = R(t)u, \ \forall u \in \Omega^{0,+}(X, E) \). The lemma follows.

To get back to the original heat equation from its adjoint version, it suffices to take the adjoints \( \Gamma^\ast(t) \) of \( \Gamma(t) \) and \( R^\ast(t) \) of \( R(t) \) (with respect to \( (\cdot, \cdot)_E \)) because \( \tilde{b}_{b,m}^\ast \) is self-adjoint. Hence combining Lemma 5.10 and Lemma 5.11 one obtains the following (asymptotic) heat kernel.

**Theorem 5.12.** With the notations above, we have

\[
\lim_{t \to \infty} \Gamma^\ast(t)u = Q_m u \quad \text{on} \quad \mathcal{D}'(X, T^{\ast 0,+}X \otimes E)
\]

for \( u \in \Omega^{0,+}(X, T^{\ast 0,+}X \otimes E) \) and \( (\tilde{b}_{b,m}^\ast \circ Q_m \text{ still denoted by } \tilde{b}_{b,m}^+ \text{ below, which is self-adjoint}) \)

\[
\frac{\partial \Gamma^\ast(t)}{\partial t}u + \tilde{b}_{b,m}^+ \Gamma^\ast(t)u = R^\ast(t)u, \quad \forall u \in \Omega^{0,+}(X, E)
\]

where \( R^\ast(t) \) is the continuous operator with the distribution kernel \( R^\ast(t, x, y) \) satisfying the estimate similar to Lemma 5.11.
Based on the above theorem, one way to solving our heat equation resorts to the method of successive approximation. This part of reasoning is basically standard. But because of the important role played by $Q^m_n$ in the final result (cf. McKean-Singer (III) in Corollary 5.15), for the convenience of the reader we sketch some details and refer the full details to, e.g. [5, Section 2.4].

To start with, suppose $A(t), B(t)$ and $C(t) : \Omega^{0,+}(X, E) \to \Omega^{0,+}(X, E)$ are continuous operators with distribution kernels $A(t, x, y), B(t, x, y)$ and $C(t, x, y) \in C^\infty(\mathbb{R}_+ \times X \times X, (T^{s_0,+}X \otimes E) \boxtimes (T^{s_0,+}X \otimes E)^*)$. Define the (continuous) operator $(A^2_\pi B)(t) : \Omega^{0,+}(X, E) \to \Omega^{0,+}(X, E)$ with distribution kernel

\[
(A^2_\pi B)(t, x, y) := \int_0^t \int_X A(t - s, x, z)B(s, z, y)dv_X(z)ds
\]

(\in C^\infty(\mathbb{R}_+ \times X \times X, (T^{s_0,+}X \otimes E) \boxtimes (T^{s_0,+}X \otimes E)^*)

It is standard that $((A^2_\pi B)^\pi C)(t) = (A^2_\pi (B^\pi C))(t)$, denoted in common by

$(A^2_\pi B^\pi C)(t)$.

(The generalization to more operators is similar.)

The method of successive approximation results in a solution (which is actually unique by Theorem 5.14 below) to our heat equation, as follows.

**Proposition 5.13.** i) (Existence) Fix $\ell \in \mathbb{N}$, $\ell \geq 2$. There is an $\epsilon > 0$ such that the sequence

\[
\Lambda(t) := \Gamma^*(t) - (\Gamma^* R^\pi)(t) + (\Gamma^* R^\pi R^\pi)(t) - \cdots
\]

converges in $C^{\ell}((0, \epsilon) \times X \times X, (T^{s_0,+}X \otimes E) \boxtimes (T^{s_0,+}X \otimes E)^*)$ and

\[
\Lambda(t) : \Omega^{0,+}(X, E) \to C^{\ell}(X, T^{s_0,+}X \otimes E) \bigcup L^{2,\epsilon}_m(X, E),
\]

\[
\lim_{t \to 0^+} \Lambda(t)u = Q_mu \quad \text{on } \mathcal{D}'(X, T^{s_0,+}X \otimes E), \quad \forall u \in \Omega^{0,+}(X, E),
\]

\[
\Lambda'(t)u + \hat{\mathcal{C}}_{b,m}^\pi \Lambda(t)u = 0, \quad \forall u \in \Omega^{0,+}(X, E).
\]

ii) (Approximation) Write $\Lambda(t, x, y), (\in C^{\ell}((0, \epsilon) \times X \times X, (T^{s_0,+}X \otimes E) \boxtimes (T^{s_0,+}X \otimes E)^*))$ for the distribution kernel of $\Lambda(t)$. Then there exists an $\epsilon_0 > 0$ independent of $t$ such that

\[
\|\Lambda(t, x, y) - \Gamma^*(t, x, y)\|_{C^{\ell}(X \times X)} \leq e^{-\epsilon_0/2}, \quad \forall t \in (0, \epsilon_0).
\]

With $\hat{\mathcal{C}}_{b,m}^\pi$ in place of $\mathcal{C}_{b,m}^\pi$ in (5.48), the corresponding statements for $\hat{\mathcal{C}}_{b,m}^\pi$ (with $\Lambda_-, \Gamma_-$ etc.) hold true as well.

**Proof.** We sketch a proof of ii) and comment on i). For notational convenience we set $Z = R^\pi$, $Z^2 = R^\pi R^\pi$, $Z^3 = R^\pi R^\pi R^\pi$ etc. as defined in (5.46) with $\| \cdot \|_\ell$ as the $C^{\ell}$-norm on $X \times X$. By using (5.43) (for $R^\pi$) one sees that there are $1 > \delta_0, \delta_1 > 0$ such that for all $t \in (0, \delta_0),

\[
\|Z\|_{\ell} \leq \frac{1}{2} e^{-\frac{\delta_0}{\ell}}, \quad \|Z^2\|_{\ell} \leq \frac{1}{2^2} e^{-\frac{\delta_1}{\ell}}, \cdots.
\]

Similarly from the estimate of $\Gamma^*(t)$ (see (5.17)) with the above (5.50) we conclude that for all $t \in (0, \delta_0),

\[
\|\Gamma^*Z\|_{\ell} \leq \frac{C_1}{2} e^{-\frac{\delta_0}{\ell}}, \quad \|\Gamma^*Z^2\|_{\ell} \leq \frac{C_1}{2^2} e^{-\frac{\delta_1}{\ell}}, \cdots
\]

where $C_1 > 0$ is some constant. Hence the sequence (5.47) converges (in $C^{\ell}((0, \epsilon) \times X \times X, (T^{s_0,+}X \otimes E) \boxtimes (T^{s_0,+}X \otimes E)^*)$) and (5.49) holds.

It takes slightly more work to verify (5.48) in i). Let $q^k(t, x, y)$ be the $(k+1)$-th term in (5.47). One verifies directly by computation of the convolution that

\[
\partial_t q^k(t, x, y) + \hat{\mathcal{C}}_{b,m}^\pi q^k(t, x, y) = Z^k(t, x, y) + Z^{k+1}(t, x, y)
\]
Definition 5.4). Since \( \Lambda(t, x, y) \) is the alternating sum of these \( g^k \), by the good estimates (5.50) and (5.51), one interchanges the order of the action of \((\partial_t + \Box_{b,m}^+)\) on \( \Lambda(t, x, y) \) with the summation. By telescoping with (5.52), one finds that the heat equation (5.48) of \( i \) is satisfied (cf. [5, Theorem 2.23]).

The uniqueness part of the above theorem is included in the following. (Note \( e^{-t\Box_{b,m}^+} (x, y) \) is as in (4.15).)

Theorem 5.14. i) (Uniqueness) We have \( e^{-t\Box_{b,m}^+} (x, y) = \Lambda(t, x, y) \in C^\ell((0, \epsilon) \times X \times X, (T^{a,0,+} X \otimes E) \boxtimes (T^{a,0,+} X \otimes E)^*) \). Hence by (5.49), for every \( \ell \in \mathbb{N}_0 \) there exist \( \epsilon_0 > 0 \) and \( \epsilon > 0 \) (independent of \( t \)) such that

\[
\left\| e^{-t\Box_{b,m}^+} (x, y) - \Gamma(t, x, y) \right\|_{C^\ell(X \times X)} \leq e^{-\frac{\epsilon t}{2}}, \quad \forall t \in (0, \epsilon).
\]

As a consequence \( e^{-t\Box_{b,m}^+} (x, y) \) and \( \Gamma(t, x, y) \) are the same in the sense of asymptotic expansion (as defined in Definition 5.4).

ii) (Asymptotic expansion) More explicitly one has (cf. (5.40))

\[
e^{-t\Box_{b,m}^+} (x, y) \sim t^{-n} a^+_x (t, x, y) + t^{-n+1} a^+_{x-1} (t, x, y) + \cdots + a^+_0 (t, x, y) + t a^+_1 (t, x, y) + \cdots \quad \text{as } t \to 0^+,
\]

\[
a^+_x (t, x, y) = \frac{1}{2\pi} \sum_{j=1}^N \int_{-\pi}^{\pi} e^{-j(x, y)_{\text{Re}} \cdot \tilde{b}_j^+(s, e^{-iu} \cdot y)} e^{-\imath y u} du
\]

\( (\in C^\infty(\mathbb{R}_+ \times X \times X, (T^{a,0,+} X \otimes E) \boxtimes (T^{a,0,+} X \otimes E)^*) ) \), \( s = n, n - 1, n - 2, \ldots \).

Similar statements hold for the case of \( e^{-t\Box_{b,m}^+} (x, y) \) as well.

Proof. The argument for the uniqueness part is standard. To sketch it, there is the following trick (cf. [5, Lemma 2.16]) in which we shall use heat equations for both kernels (cf. (4.18) and (5.48)). For \( 0 < t < \epsilon \) (\( \epsilon \) as in Proposition 5.13) and \( f, g \in \Omega^{b,0} (X, E) \)

\[
0 = \int_0^t \frac{\partial}{\partial s} \left( (\Lambda(t-s) f \| e^{-s\Box_{b,m}^+} g) \right) ds = (Q_m f \| e^{-t\Box_{b,m}^+} g) - (\Lambda(t) f \| g) E = (f \| e^{-t\Box_{b,m}^+} g) - (\Lambda(t) f \| g) E = (e^{-t\Box_{b,m}^+} f \| g) - (\Lambda(t) f \| g) E,
\]

proving that \( e^{-t\Box_{b,m}^+} (x, y) = \Lambda(t, x, y) \).

The estimates (5.53) and (5.54) follow from (5.49) and (5.40) (\( e^{-t\Box_{b,m}^+} \) is self-adjoint). We remark that by Proposition 5.1 it was tempting to speculate that the heat kernel for (modified/\( \text{Spin}^c \)) Kohn Laplacian might be (at least asymptotically) the (local) heat kernel for (modified/\( \text{Spin}^c \)) Kodaira Laplacian. This is however too much to be true as suggested by the above Theorem 5.14 because the asymptotic expansion of (modified) Kohn Laplacian involves a nontrivial \( t \)-dependence in \( a_s (t, x, y) \) (cf. Remark 1.6 and Remark 1.7).

We are ready to establish a link between our index and the heat kernel density of (modified) Kodaira Laplacian. For \( a^+_x (t, x, x) \) in (5.54), define \( \text{Tr} a^+_x (t, x, x) = \sum_{s=1}^d \langle a^+_x (t, x, x), e_s (x), e_s (x) \rangle_E \) as usual, where \( \{ e_s (x) \} \) is an orthonormal frame (of \( T_x^{a,0,+} X \otimes E_x \)). (Similar notation and definition apply to the case of \( a^-_x (t, x, x). \)

To sum up from Corollary 4.8 and (5.54), there is a second form of McKean-Singer type formula for the index in our case (cf. Corollary 4.8 for the first form).
Corollary 5.15. (McKean-Singer (II)) We have

\[
\sum_{j=0}^{n} (-1)^j \dim H^j_{b,m}(X, E) = \lim_{t \to 0^+} \int_X \sum_{j=0}^{n} t^{-j} \left( \text{Tr} a^+_t(t, x, x) - \text{Tr} a^-_t(t, x, x) \right) dv_X(x).
\]

By this result we are now reduced to computing \( a_s(t, x, x) \) in the following section.

Part II: Proofs of main theorems

6. PROOFS OF THEOREMS 1.3 AND 1.10

We are in a position to prove the main results of this paper. A new ingredient is the notion of “distance function” \( \hat{d} \) (see (1.16) for the definition and Theorem 6.7 for its property). This function naturally appears when we compute \( a_s(t, x, x) \) in the form of an integral (5.54). In the remaining part of this section we prove that this “distance function” is equivalent to the ordinary distance function at least in the strongly pseudoconvex case (Theorem 6.7).

Theorem 1.3 is proved in Theorem 6.1, Remark 6.2, Corollary 6.3 together with Theorem 5.14; Theorem 1.10 proved in Theorem 6.4 and in (6.15).

In the same notations as before recall that \( X_{pt} = \{ x \in X; \text{the period of } x \text{ is } \frac{2\pi}{p} \}, 1 \leq \ell \leq k \) with \( p_1 | p_\ell \) (all \( \ell \)) and \( p = p_1 \). \( X_p \) is open and dense in \( X \). See the discussion preceding Theorem 1.3 for more detail.

Let \( G_j(t, x, y) \) be as in (5.36). (Notations set up in (5.36)–(5.41) will be useful in what follows.) By the construction of \( G_j(t, x, y) \), it is clear that

\[
\frac{1}{2\pi} \sum_{j=1}^{N} G_j(t, x, x) \sim t^{-n} \alpha^+_n(x) + t^{-n+1} \alpha^+_{n-1}(x) + \cdots \text{ as } t \to 0^+,
\]

\[
\alpha^+_s(x) = \frac{1}{2\pi} \sum_{j=1}^{N} \tilde{\beta}^+_j(x, y)|_{y=x} \in C^\infty(X, \text{End}(T^{s,0,0} X \otimes E)), \ s = n, n-1, \ldots.
\]

\( \alpha^+_s(x) \) are independent of choice of BRT trivialization charts (in view of Remark 5.7). It is perhaps instructive to think of these as the data of the asymptotic expansion associated with the “underlying Kodaira Laplacian” (cf. loc. cit. and Proposition 5.1) regardless of the existence of a genuine “underlying space”.

Recall the asymptotic expansions of \( \Gamma(t, x, y) \) and \( e^{-C_{b,m}^t}(x, y) \) (they coincide by Theorem 5.14), in which we have \( a^+_s(t, x, y) \) \( (\in C^\infty(\mathbb{R}_+ \times X \times X, (T^{s,0,0} X \otimes E) \otimes (T^{s,0,0} X \otimes E)^*)) \), \( s = n, n-1, \ldots, \) cf. (5.54) or (5.40). By the construction, \( \Gamma(t, x, y) \) and \( a_s(t, x, y) \) of (5.54) depend on the choice of BRT charts. (The authors do not know whether there exists a canonical choice of \( a_s(t, x, y) \) in this respect.)

We are now ready to give a proof of the following.

**Theorem 6.1.** (cf. Theorem 1.3) For every \( N_0 \in \mathbb{N} \) with \( N_0 \geq N_0(n) \) for some \( N_0(n) \), there exist \( \varepsilon_0 > 0 \), \( \delta > 0 \) and \( C_{N_0} > 0 \) such that

\[
\sum_{j=0}^{N_0} t^{-n+j} a^+_{n-j}(t, x, x) - \left( \sum_{s=1}^{p_r} e^{2\pi (x-1) \cdot mi} \right) \sum_{j=0}^{N_0} t^{-n+j} \alpha^+_{n-j}(x)
\]

\[
\leq C_{N_0} \left( t^{-n+N_0+1} + t^{-n} e^{-\varepsilon_0 d(x, X_{pt})^2} \right), \ \forall x \in X_{p_r} \ (r = 1, \ldots, k), \ \forall 0 < t < \delta.
\]
Proof. For simplicity, we only prove Theorem 6.1 for \( r = 1 \). The proof for \( r > 1 \) is similar.

As in the beginning of Section 5.2, there are BRT trivializations \( B_j := (D_j, (z, \theta), \varphi_j), j = 1, \ldots, N \).

We write
\[
D_j = U_j \times [2\delta_j, 2\delta_j[, \quad \hat{D}_j = \hat{U}_j \times [-\frac{\delta_j}{2}, \frac{\delta_j}{2}] \quad (\subset \mathbb{C}^n \times \mathbb{R}),
\]
with \( U_j = \{ z \in \mathbb{C}^n; \| z \| < \gamma_j \}, \hat{U}_j = \{ z \in \mathbb{C}^n; \| z \| < \gamma_j/2 \} \) for some \( \delta_j > 0, \hat{\delta}_j > 0, \gamma_j > 0 \).

Assume \( X = \hat{D}_1 \cup \cdots \cup \hat{D}_N \). In the following we let \( \delta_j = \hat{\delta}_j = \zeta \) (all \( j \)), \( \zeta \) satisfy (1.15) with \( 4|\zeta| < \frac{2\pi}{p} \).

It is easily verified that there is an \( \hat{\varepsilon}_0 > 0 \) such that (\( d(\cdot, \cdot) \) the ordinary distance function on \( X \))
\[
\hat{\varepsilon}_0 d((z_1, \theta_1), (z_2, \theta_1)) \leq |z_1 - z_2| \leq \frac{1}{\hat{\varepsilon}_0} d((z_1, \theta_1), (z_2, \theta_1)), \quad \forall (z_1, \theta_1), (z_2, \theta_1) \in D_j,
\]
\[
\hat{\varepsilon}_0 d((z_1, \theta_1), (z_2, \theta_1))^2 \leq h_{j,+}(z_1, z_2) \leq \frac{1}{\hat{\varepsilon}_0} d((z_1, \theta_1), (z_2, \theta_1))^2, \quad \forall (z_1, \theta_1), (z_2, \theta_1) \in D_j,
\]
where \( h_{j,+}(z, w) \) is as in (5.19).

Recall the modified distance \( \hat{d} \) which is defined in (1.16). We are going to compare \( \hat{d} \) with (6.3).

Fix \( x_0 \in X_p \). Suppose \( x_0 \in \hat{D}_j \) for some \( j = 1, \ldots, N \) and also suppose \( x_0 = (z, 0) \) on \( D_j \).

Some crucial remarks are in order.

i) For \( 0 \leq |u| \leq 2\zeta \) the action of \( e^{-iu} \) on \( x_0 \) is only moving along the “angle” direction (due to the assumption that a BRT trivialization \( D_j \) is valid here), i.e. the \( z \)-coordinates of \( x_0 \) and \( e^{-iu} \circ x_0 \) are the same.

ii) Let a \( u_0 \in [2\zeta, \frac{2\pi}{p} - 2\zeta] \) be given. Assume that the action by \( e^{-iu_0} \) on \( x_0 \) still belongs to \( \hat{D}_j \) with a coordinate \( e^{-iu_0} \circ x_0 = (\hat{z}, \hat{\theta}) \). Then it could happen that \( \hat{z} \neq z \) because the orbit \( \{ e^{-iu} \circ x_0 \} \) for \( 2\zeta \leq v \leq \frac{2\pi}{p} - 2\zeta \) may partly lie outside of \( D_j \). We will show (6.4) below that indeed \( \hat{z} \neq z \) in this case.

Remark that the above ii) is basically the reason responsible for why the contribution of our distance function \( \hat{d} \) enters, as seen shorty. The question about whether the condition \( e^{-iu_0} \circ x_0 \in D_j \) of ii) is vacuous or not will be discussed below (equivalent to whether \( J \) below is an empty set or not).

We shall now formulate the above ii) more precisely. If \( x_0 \in \hat{D}_j \), we claim the following.

\[
\text{Suppose } e^{-i\theta} \circ x_0 = (\hat{z}, \hat{\theta}) \text{ also belongs to } \hat{D}_j \text{ for some } \theta \in [2\zeta, \frac{2\pi}{p} - 2\zeta].
\]

Then \( |z - \hat{z}| \geq \hat{\varepsilon}_0 \hat{d}(x_0, X_{\text{sing}}) (> 0) \).

proof of claim. By \( (\hat{z}, \hat{\theta}) \in \hat{D}_j \) one has \( e^{i\hat{\theta}} \circ (\hat{z}, \hat{\theta}) = (\hat{z}, 0) \) equivalently \( e^{i\hat{\theta}} \circ (\hat{z}, 0) = (\hat{z}, \hat{\theta}) \) (by the above i) as \( |\hat{\theta}| \leq \frac{\zeta}{2} \) here). One sees (by (6.3) and isometry of \( S^1 \) action for the first inequality below)
\[
|\hat{z} - z| \geq \hat{\varepsilon}_0 d(e^{i\hat{\theta}} \circ (\hat{z}, \hat{\theta}), e^{i\hat{\theta}} \circ (z, \hat{\theta})) = \hat{\varepsilon}_0 d(e^{i\hat{\theta}} \circ (e^{-i\theta} \circ x_0), x_0)
\]
\[
\geq \hat{\varepsilon}_0 \inf \left\{ d(e^{-iu} \circ e^{-i\theta} \circ x_0, x_0); |u| \leq \frac{\zeta}{2} \right\}
\]
\[
\geq \hat{\varepsilon}_0 \inf \left\{ d(e^{-i\hat{\theta}} \circ x_0, x_0); \zeta \leq \hat{\theta} \leq \frac{2\pi}{p} - \zeta \right\}
\]
\[
= \hat{\varepsilon}_0 \hat{d}(x_0, X_{\text{sing}})
\]
(see (1.16) for the definition of \( \hat{d} \), as claimed.

Remark that a sharp result in this direction (6.4) is proved in Lemma 7.5.

We continue with the proof of the theorem. We need to estimate \( \Gamma(t) = \sum_{j=1}^{N} H_j(t) \circ Q_m \) for the first summation to the left of (6.2). By definition (see (5.40)) this is in turn to estimate
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} H_j(t, x_0, e^{-iu} \circ x_0) e^{-imu} du \quad (6.6)
\]
and sum over $j = 1, \ldots, N$.

We first assume that in (6.6), $x_0 = (z, 0)$ in $\hat{D}_j$ and $x_0 \notin \hat{D}_k$ for any other $k \neq j$.

To work on (6.6) we shall divide $[-\pi, \pi]$ in (6.6) into two types.

The first type is to estimate $\frac{1}{2\pi} \int_{-2\zeta}^{2\zeta} H_j(t, x_0, e^{-iu} \circ x_0)e^{-imu}du$. Note if $u \in [-2\zeta, 2\zeta]$ and $e^{-iu} \circ x_0 = (z_u, \theta_u)$, then $(z_u, \theta_u) = (z, u)$ (by ii) above (6.4)), i.e. $e^{-iu} \circ (z, 0) = (z, u)$. Hence, by (5.36) the factor $e^{imu}$ in $H_j$ is going to be $e^{imu}$ and this is cancelling off the term $e^{-imu}$ in the integral (6.6). By (5.19), $h_+(z, z_u) = h_+(z, z) = 0$ (also by (6.3)) and the factor $e^{-\frac{i}{t}}$ of $A_{B_j,+}$ in $H_j$ of (5.36) becomes 1. Finally we note for $\sigma_j$ in $H_j$ of (5.36), $\int \sigma_j(\theta)d\theta = 1, I = [-\frac{\zeta}{2}, \frac{\zeta}{2}]$.

To sum up, by (5.36) and (5.19) one obtains the following ($x_0 \notin \hat{D}_k$ for $k \neq j$)

$$\frac{1}{2\pi} \int_{-2\zeta}^{2\zeta} H_j(t, x_0, e^{-iu} \circ x_0)e^{-imu}du$$

$$\sim \left( t^{-n} \alpha^+_n(x_0) + t^{-(n-1)} \alpha^+_{n-1}(x_0) + \cdots \right) as t \to 0^+,$$

where $\alpha^+_n(x)$, $s = n, n-1, \ldots$, are as in (6.1).

For the second type suppose $u \in [2\zeta, \frac{2\pi}{p} - 2\zeta]$. Note the action by $e^{-iu}$ on $x_0$ may change the $z$ coordinate of $x_0$ by ii) above (6.4). We let $J$ be the subset of those $u \in [2\zeta, \frac{2\pi}{p} - 2\zeta] \equiv E$ that $e^{-iu} \circ x_0 = (z_u, \theta_u)$ belongs to $\hat{D}_j$ (then $z_u \neq z$ by (6.4)), and $J'$ be the complement of $J$ in $E$. One finds, for some $\varepsilon_0 > 0, \delta > 0$ and $C_0 > 0$ (independent of $j, x_0$),

$$\left| \frac{1}{2\pi} \int_{u \in [2\zeta, \frac{2\pi}{p} - 2\zeta] \setminus J} H_j(t, x_0, e^{-iu} \circ x_0)e^{-imu}du \right|$$

$$\leq \frac{1}{2\pi} \int_{u \in J} \left| H_j(t, x_0, e^{-iu} \circ x_0)e^{-imu} \right| du + \frac{1}{2\pi} \int_{u \in J'} \left| H_j(t, x_0, e^{-iu} \circ x_0)e^{-imu} \right| du$$

$$\leq C_0 t^{-n} e^{-\frac{\varepsilon_0 \|\xi\| \cdot X_{\text{sing}}}{t}}, \forall 0 < t < \delta$$

where the integral over $J'$ vanishes because the cut-off function $\sigma_j$ in $H_j$ of (5.36) gives $\sigma_j(\eta(y)) = 0$ for $y = e^{-iu} \circ x_0 \notin \hat{D}_j$ as $\sigma_j = 0$ outside $\hat{D}_j$ (see lines above (5.36)), and the second inequality arises from applying (6.3) and (6.4) to $h_+(z, z_u)$ in $H_j$ (see (5.36) and (5.19)).

Is $J$ an empty set? We remark that the top term in (6.8) is in general nonzero (by combining (6.7) and Remark 1.7 for $p = 1$). Hence $J \neq \emptyset$ in general. There is a geometrical way to see the claim that for some open subset $V$ of $X$, if $x_0 \in V$, then $J \neq \emptyset$. For simplicity assume $X = X_1 \cup X_2$, i.e. $p_1 = 1$ and $p_2 = 2$. Choose $y \in X_2$. Let $g = e^{-i\frac{2\pi}{p}} \in S^1$. Fix a neighborhood $U \subset \hat{D}_j$ of $y$ in $X$. Since $g \circ y = y$, by continuity argument there are neighborhoods $N_1, N_2$ of $y$, in $X, S^1$ respectively such that the action $h \circ x \in U$ if $(h, x) \in N_2 \times N_1$. Choose $N_1 \subset \hat{D}_j, N_2$ small and set $V \equiv N_1 \setminus X_2$. It follows that for these $x_0 \in V$, $J \neq \emptyset$ since $N_2 \subset J$. This result also accounts for the necessity of the remark ii) above (6.4) and hence that of a certain extra contribution (e.g. $d$) in estimates (6.8).

Suppose $p (= p_1) = 1$. Then (6.7) and (6.8), Definition 5.4 for $\sim$ and Remark 5.5 (by noting $\dim X = 2n + 1, \dim U = 2n, M_0(m) = n + 1, m = 4n = 2\beta, M_1(m, \ell) = n$ for $\ell = 0$) immediately lead to

$$\left| \Gamma(t, x_0, x_0) - \sum_{j=0}^{N_0} t^{-n+j} \alpha^+_n(x_0) \right|$$

$$\leq C N_0 \left( t^{-n+N_0} + C_0 t^{-n} e^{-\frac{\varepsilon_0 \|\xi\| \cdot X_{\text{sing}}}{t}} \right), \quad N_0 \geq N_0(n), \forall 0 < t < \delta.$$
Suppose \( p > 1 \). Then one has the extra \( p - 1 \) sectors in \([-\pi, \pi] \) (obtained by shifting the above first sector \( s = 1 \) by a common \((s - 1) \frac{2\pi}{p}\)):

\[
(s - 1) \frac{2\pi}{p} - 2\zeta, (s - 1) \frac{2\pi}{p} + 2\zeta, \quad (s - 1) \frac{2\pi}{p} + 2\zeta, (s - 1) \frac{2\pi}{p} - 2\zeta, \quad s = 1, \ldots, p
\]

(6.10)

\( (s = p + 1 \) identified with \( s = 1 \) over which the integrals correspond to types I (6.7) and II (6.8) respectively. One may check without difficulty that the version of the claim (6.4) and (6.5) holds true as well in this modified case; iv) By the preceding ii) and iii), the results (6.4) and (6.5). One can check that the reasoning in (6.5) remains basically unchanged, and the conclusion of (6.5) holds true as well. On each of these sectors, a simple (linear) change of variable for \( u \), which is to bring the intervals of the integration on these sectors back to those in (6.7) and (6.8), produces the extra numerical factor in sum (by \( e^{-imu} du \) in (6.6)):

\[
\sum_{s=1}^{p} e^{2\pi i s - 2im t} \text{as expressed in (6.2)}.
\]

Finally, note that we have assumed \( x_0 = (z, 0) \in \tilde{D}_j \). In this case the above argument appears symmetrical in writing. This (assumption) is however not essential. Since we shall also adopt a similar assumption in Section 7, we give an outline about the asymmetrical way (i.e. \( x_0 = (z, \theta) \neq (0, \theta) \)) of the argument. By going over the same process, one sees the following. i) If \( x = (z, v) \), with \( 0 < v \leq \frac{\zeta}{2} \), the intervals in (6.7), (6.8) shall be replaced by \([-2\zeta - v, 2\zeta - v], [2\zeta - v, \frac{2\pi}{p} - 2\zeta - v] \) (thought of as translated by a common \(-v\)) with the new integrals denoted by (6.7)', (6.8)', respectively; ii) \([-2\zeta - v, 2\zeta - v] \supseteq [-\zeta, \zeta] \) hence \( \sigma(v) \) is still 1 in (6.7)'; iii) In the proof of claim (6.5), \( e^{ij} \) should be replaced by \( e^{\gamma} \) with \( \gamma = \eta - v, (\tilde{z}, 0) \) by \( (\tilde{z}, v) \) and \( \theta \in [2\zeta, \frac{2\pi}{p} - 2\zeta] \) by \( \theta \in [2\zeta - v, \frac{2\pi}{p} - 2\zeta - v] \) throughout (6.4) and (6.5). One can check that the the reasoning in (6.5) remains basically unchanged, and the conclusion of (6.5) holds true as well in this modified case; iv) By the preceding ii) and iii), the results corresponding to (6.7)' and (6.8)' hold true. Hence the asymmetrical way follows.

We have also assumed \( x_0 \neq \tilde{D}_b \) for \( k \neq j \). This condition is unimportant if we take the preceding asymmetrical way into account (for \( x_0 \neq \tilde{D}_b \) in the general case), and note that there is a hidden partition of unity in \( \{H_j\}_{j=1,\ldots,N} \).

For an alternative to the above, it is to use the kernel \( e^{-t\xi_{b,m}^+}(x, y) \) in place of \( \Gamma(t, x, y) \) and \( a_+^s(t, x, y) \). An advantage is that \( e^{-t\xi_{b,m}^+} \) is independent of BRT charts, so that for a given point \( x_0 \) we can take a covering of \( X \) by convenient BRT charts for the previous special conditions to be satisfied (e.g. \( x_0 = (z, 0), x_0 \in \tilde{D}_j \) for exactly one \( j \) etc.). By the asymptotic property between \( e^{-t\xi_{b,m}^+}(x, y) \) and \( \Gamma(t, x, y) \) (5.53), this also leads to Theorem 6.1.

**Remark 6.2.** For the relation between \( a_+^s(t, x, y)|_{y=x} = \alpha_s(x) \) as stated in (1.18) of Theorem 1.3, the method of the above proof works. By using (setting \( y = x \) below)

\[
a_+^s(t, x, y) = \frac{1}{2\pi} \sum_{j=1}^{N} \int_{-\pi}^{\pi} e^{-\frac{\beta_{j,s}(x, e^{-iu} y)}{t}} \xi_{b,m}^+(x, e^{-iu} y) e^{-imu} du
\]

(see (5.54)) with the same reasoning as (6.7), (6.8) and (6.10), one obtains (1.18) for \( P_\ell = \text{id} \). For \( \ell > 0 \), (1.18) follows by noting

\[
(6.12) \quad \partial_x e^{-\frac{\tau^2}{t}} = -2t^{-\frac{1}{2}}(\frac{x^2}{t})^{1/2}e^{-\frac{\tau^2}{t}} = O(t^{-\frac{1}{2}})
\]

applied to (6.11) and by noting Remark 5.5 as in (6.9). Remark that if one extracts the corresponding coefficients of \( t^{-s} \) in (6.2) of Theorem 6.1 and uses the result (6.2), the estimate appears to be \( e^{-\frac{\tau^{2}}{t} + O(t^\infty)} \) which is slightly weaker than above (due to \( O(t^\infty) \)).

For similar estimates with regard to \( C^t \) topology we have the following (cf. (6.12) and Remark 5.5).
Corollary 6.3. In the same notation as above, for any differential operator $P_t : C^\infty(X, T^{\alpha,0} X \otimes E) \to C^\infty(X, T^{\alpha,0} X \otimes E)$ of order $\ell \in \mathbb{N}$ and every $N_0 \geq N_0(n)$ for some $N_0(n)$,

\[
P_t \left( \sum_{j=0}^{N_0} t^{-n+j} a^+_{n-j}(t, x, x) - \left( \sum_{s=1}^{p_r} e^{2\pi(s-1) t^{1/2}} \right) \sum_{j=0}^{N_0} t^{-n+j} a^+_{n-j}(x) \right)
\]

(6.13)

\[
\leq C_{N_0} \left( t^{-n+N_0+1 - \frac{\ell}{2}} + t^{-n - \frac{\ell}{2}} e^{-\delta_0(x,X^\prime_{\text{sing}})^2} \right), \quad \forall 0 < t < \delta, \quad \forall x \in X_{p_r}
\]

for some $\varepsilon_0 > 0$, $\delta > 0$ and $C_{N_0} > 0$ independent of $x$.

Note the singular behavior $t^{-n}$ (in the term to the rightmost of (6.2)). So the estimate (6.2) is not directly applicable to the proof of our local index theorem. That is, computation involving $a_s$ cannot be automatically reduced to computation involving $a_s$ as soon as $x \ (\in X_{p_1})$ approaches $X_{\text{sing}}$. Intuitively $t^{-n} e^{-\delta_0(x,X^\prime_{\text{sing}})^2}$ goes to a kind of Dirac delta function (along $X_{\text{sing}}$) as $t \to 0$ (apart from a factor of the form $\frac{1}{t^\beta}$, some $\beta > 0$). So after integrating (6.2) over $X$, a nonzero contribution due to this term could appear or even blow up as $t \to 0$. A more precise analysis along this line will be taken up in the study of trace integrals in Section 7.

Fortunately, the abovementioned singular behavior can be removed ($t^{-n}$ dropping out completely) after taking the supertrace, so that the index density for our need does exist. (However, as far as the full kernel is concerned, a certain estimate such as that in Theorem 6.1 is unavoidable as discussed in Remark 1.8).

We shall now take up this improvement on (6.2) under supertrace. We formulate it as follows, whose proof is heavily based on the off-diagonal estimate obtained in Theorem 5.9.

Theorem 6.4. (cf. Theorem 1.10) With the notations above, for every $N_0 \in \mathbb{N}$, $N_0 \geq N_0(n)$ for some $N_0(n)$, there exist $\varepsilon_0 > 0$, $\delta > 0$ and $C_{N_0} > 0$ such that

\[
\left| \text{Tr} e^{-t^{\frac{\varepsilon_0}{b,m}}(x,x)} - \text{Tr} e^{-t^{\frac{\varepsilon_0}{b,m}}(x,x)} - \left( \sum_{s=1}^{p_r} e^{2\pi(s-1) t^{1/2}} \right) \sum_{j=0}^{N_0} t^{-n+j} \left( \text{Tr} a^+_{n-j}(x) - \text{Tr} a^+_{n-j}(x) \right) \right|
\]

(6.14)

\[
\leq C_{N_0} \left( t^{-n+N_0+1} + e^{-\delta_0(x,X^\prime_{\text{sing}})^2} \right), \quad \forall 0 < t < \delta, \quad \forall x \in X_{p_r},
\]

The implication of Theorem 6.4 yields a link between the two identities arising from Corollary 5.15 and Proposition 5.8 together with (5.20):

\[
\lim_{t \to 0^+} \int_X \sum_{\ell=0}^{n} t^{-\ell} \left( \text{Tr} a^+_{\ell}(t, x, x) - \text{Tr} a^-_{\ell}(t, x, x) \right) d\nu_X(x) = \sum_{j=0}^{n} (-1)^j \dim H^j_{b,m}(X, E)
\]

(6.15)

\[
= \frac{1}{2\pi} \int_X Td_b(\nabla^{1,0} X, T^{1,0} X) \wedge \text{ch}_b(\nabla^E, E) \wedge e^{-n^0 \frac{dw_0}{2\pi}} \wedge \omega_0(x).
\]

It follows that the two in (6.15) are equal because in (6.14), $e^{-\delta_0(x,X^\prime_{\text{sing}})^2} (\leq 1) \to 0$ in $L^1$ by Lebesgue’s dominated convergence theorem as $t \to 0^+$ on $X_p$. We arrive now at an index theorem for our class of CR manifolds.
Corollary 6.5. (cf. Corollary 1.13)

\[
\sum_{j=0}^{n} (-1)^j \dim H^j_{b,m}(X,E)
\]

(6.16)

\[
= \left( \sum_{s=1}^{p} e^{2\pi s^{-1} m_1} \right) \frac{1}{2\pi} \int_X \left[ \text{Td}_b(\nabla^{1,0}X, T^{1,0}X) \wedge \text{ch}_b(\nabla E, E) \wedge e^{-m_\omega} \wedge \omega_0 \right]_{2n+1} (x)
\]

where \([\cdots]_{2n+1}\) denotes the \((2n+1)\)-form part.

We turn now to the proof of Theorem 6.4.

Proof of Theorem 6.4. For simplicity, we only prove Theorem 6.4 for \(r = 1\). The proof for \(r > 1\) is similar. Adopting the same notations as in the proof of Theorem 6.1 (e.g. \(B_j, D_j, \tilde{D}_j \cdots\)), we shall follow a similar line of thought as in Theorem 6.1.

Fix \(x_0 \in X_p\). As \(e^{-t\Box_{b,m}}(x,y)\) and \(\Gamma(t,x,y)\) are asymptotically the same (Theorem 5.14), we also break the desired estimate at \(x = x_0\) into two types of integrals corresponding to (6.7) and (6.8).

One integral is over \(I = [-2\zeta, 2\zeta]\) and the other over \(I'\), the complement of \(I\) in \([-\pi, \pi]\). The first type gives rise to the first term to the right of (6.14) almost the same way as (6.7).

The key of this proof lies in the second type which corresponds to (6.8). It is estimated over \(I'\), as in (6.17) below. (Here we rewrite \(H_t\) in a convenient form, in terms of \(\hat{h}_{j,+}, \hat{K}_{j,+}\) of (3.7), reminiscent of an analogous relation \(A_{B,+} = e^{-\frac{\hat{h}_+}{t}}K_{B,+}\) in (5.19).)

(6.17)

\[
\frac{1}{2\pi} \sum_{j=1}^{N} \int_{u \in I'} e^{-\frac{\hat{h}_{j,+}(0, e^{-iu}x_0)}{t}} \left( \text{Tr} \hat{K}_{j,+}(t,x_0, e^{-iu} \circ x_0) - \text{Tr} \hat{K}_{j,-}(t,x_0, e^{-iu} \circ x_0) \right) du
\]

We shall now show that there exist \(\varepsilon_0 > 0\) and \(C > 0\) (independent of \(x_0\)) such that (6.17) is bounded above by

(6.18)

\[
Ce^{-\frac{\varepsilon_0 d(x_0, X_{\text{sing}})}{t}}
\]

for small \(t \in \mathbb{R}_+\).

To see this we first note that for \(k \geq 0\),

(6.19)

\[
e^{-t^2} e^{\frac{2^2}{t}} \leq C_{k,t} e^{-t^2}
\]

for some constant \(C_{k,t}\) independent of \(x\) and \(t > 0\). Write \(x_0 = (z, \theta)\) and \(e^{-iu} \circ x_0 = (z_u, \theta_u)\) in BRT coordinates. Since \(\hat{h}_{j,+}(x_0, e^{-iu} \circ x_0)\) is essentially \(h_+(z, z_u) \approx |z - z_u|^2\), we have

(6.20)

\[
e^{-\frac{\hat{h}(x_0, e^{-iu}x_0)}{t}} \leq e^{-2c_1 \frac{|z - z_u|^2}{t}} \leq e^{-c_1 \frac{\varepsilon_0 d(x_0, X_{\text{sing}})}{t}}
\]

for some constant \(c_1 > 0\) by using (6.4) for \(\hat{d}\). By using the off-diagonal estimate of Theorem 5.9 and by (5.19), (5.37) for linking \(\bullet\) with \(K_\bullet\), one obtains the following estimate from (6.20)

(6.21)

\[
\left| e^{-\frac{\hat{h}_{j,+}(x_0, e^{-iu}x_0)}{t}} \text{Tr} \hat{K}_{j,+}(t,x_0, e^{-iu} \circ x_0) - \text{Tr} \hat{K}_{j,-}(t,x_0, e^{-iu} \circ x_0) \right| \leq e^{-c_1 \frac{\varepsilon_0 d(x_0, X_{\text{sing}})}{t}} \sum_{k=0}^{n} \text{constants} \cdot e^{-c_1 \frac{|z - z_u|^2}{t}} \left( \frac{|z - z_u|^{2k}}{t^k} + O(t) \right)
\]

for (6.17). Now one readily obtains the bound (6.18) from (6.21) and (6.19).

Combining the above estimates for integrals of the first type and second type (6.17), we obtain (6.14) in the way similar to (6.9) (with \(t^{-n}\) dropping out of \(t^{-n}e^{-\frac{\varepsilon_0 d^2}{t}}\)). \(\square\)
In the remaining part of this section we give a geometric meaning for \( \hat{d}(x, X_{\text{sing}}) \) (when \( X \) is strongly pseudoconvex). To this aim it is useful to use another equivalent form of the function \( \hat{d} \), as follows (without any pseudoconvexity condition on \( X \)).

**Lemma 6.6.** There exists a small constant \( \varepsilon_0 > 0 \) (satisfying (1.15) at least) with the following property. Fix an \( \varepsilon \) with \( 0 < \varepsilon \leq \varepsilon_0 \). For \( x \in X \) define another “distance function” \( \hat{d}_2 \) by (for a fixed \( \ell \))

\[
\hat{d}_2(x, X_{\text{sing}}^{\ell-1}) = \inf \left\{ d(x, e^{-i\theta} \circ x) ; \frac{2\pi}{p\ell} - \varepsilon \leq \theta \leq \frac{2\pi}{p\ell} + \varepsilon \right\}
\]

\( (X_{\text{sing}}^{\ell-1} = X_{p\ell} \cup X_{p\ell+1} \cdots) \). Then \( \hat{d}_2(x, X_{\text{sing}}^{\ell-1}) \) is equivalent to \( \hat{d}(x, X_{\text{sing}}^{\ell-1}) \). (Namely \( \frac{1}{C_{\ell,\varepsilon}} \hat{d}_2 \leq \hat{d} \leq C_{\ell,\varepsilon} \hat{d}_2 \) for some constant \( C_{\ell,\varepsilon} \) independent of \( x \)).

We postpone the proof of the lemma until after Theorem 6.7.

For technical reasons we impose a pseudoconvex condition on \( X \) in the following although the same result is expected to hold without this condition.

**Theorem 6.7.** With the notations above, assume that \( X \) is strongly pseudoconvex. Then there is a constant \( C \geq 1 \) such that

\[
\frac{1}{C} d(x, X_{\text{sing}}) \leq \hat{d}(x, X_{\text{sing}}) \leq Cd(x, X_{\text{sing}}), \quad \forall x \in X.
\]

**Proof.** For simplicity, we assume that \( X = X_1 \cup X_2 \), i.e. \( p_1 = 1, p_2 = 2 \), so that \( X_{\text{sing}} = X_{\text{sing}}^1 = X_2 \) \((r = 1)\) by definition. For the general case, the proof is essentially the same. By Lemma 6.6, for every (small and fixed) \( \varepsilon > 0 \) we have

\[
(6.22) \quad \hat{d}(x, X_{\text{sing}}) \approx \inf \left\{ d(x, e^{-i\theta} \circ x) ; \pi - \varepsilon \leq \theta \leq \pi + \varepsilon \right\}.
\]

Since \( X \) is strongly pseudoconvex, it is well-known that (see [39]) there exists a CR embedding:

\[
\Phi : X \to \mathbb{C}^N,
\]

\[
x \to (f_1(x), \ldots, f_N(x))
\]

with \( f_j \in H^0_{p,m_j}(X) \) for some \( m_j \in \mathbb{N} \) \((j = 1, \ldots, N)\).

We assume that \( m_1, \ldots, m_s \) are odd numbers and \( m_{s+1}, \ldots, m_N \) are even numbers. By \( p_1 = 1 \) and \( p_2 = 2 \) one sees that (cf. (1.35))

\[
(6.24) \quad x \in X_{\text{sing}} \text{ if and only if } f_1(x) = \cdots = f_s(x) = 0
\]

so that

\[
(6.25) \quad d(x, X_{\text{sing}})^2 \approx \sum_{j=1}^{s} |f_j(x)|^2, \quad \forall x \in X.
\]

Now, by using the embedding theorem (6.23) (together with (1.35)) we have

\[
(6.26) \quad d(x, e^{-i\pi} \circ x)^2 \approx \sum_{j=1}^{N} |f_j(x) - f_j(e^{-i\pi} \circ x)|^2 = 4 \sum_{j=1}^{s} |f_j(x)|^2 \approx d(x, X_{\text{sing}})^2
\]

and hence for every \( \pi - \varepsilon \leq \theta \leq \pi + \varepsilon \) \((\varepsilon > 0 \text{ small})\)

\[
(6.27) \quad d(x, e^{-i\theta} \circ x)^2 \approx \sum_{j=1}^{N} \left| f_j(x) - f_j(e^{-i\theta} \circ x) \right|^2 \geq \sum_{j=1}^{s} \left| (1 - e^{-im_j\theta}) f_j(x) \right|^2 \approx \sum_{j=1}^{s} |f_j(x)|^2 \approx d(x, X_{\text{sing}})^2.
\]

By (6.27) we conclude that

\[
(6.28) \quad \inf \left\{ d(x, e^{-i\theta} \circ x) ; \pi - \varepsilon \leq \theta \leq \pi + \varepsilon \right\} \approx d(x, X_{\text{sing}}).
\]

Combining (6.22) and (6.28) we have proved the theorem. \( \square \)
We give now:

proof of Lemma 6.6. In the following we write \( d(x) = \hat{d}(x, \lambda_{1}^{\ell-1}) \) and \( \hat{d}_{2}(x) = \hat{d}_{2}(x, \lambda_{1}^{\ell-1}) \) for a fixed \( \ell \). For an illustration we assume \( X = X_{1} \bigcup X_{2} \), i.e. \( p_{1} = 1, p_{2} = 2 \), and \( x \in X_{1} \) \( (\hat{d} = \hat{d} = 0 \) for \( x \in X_{2} \) \). Write \( I \) for the complement of \( I' \equiv ]\pi - \varepsilon, \pi + \varepsilon[ \) in \([\xi_1, 2\pi - \xi_2] \equiv K \) (where \( \xi_1 \) satisfies (1.15) and \( \varepsilon > 0 \) a small constant to be specified, cf. the line above (6.35)). By definition \( \hat{d}_{2} \geq \hat{d} = \hat{d}_{2} \) \( (\hat{d}_{2} \) is to take \( \inf \) over \( I' \) while \( \hat{d} \) over \( K \), and \( I' \subset K \)).

It remains to see \( \hat{d}_{2} \leq C\hat{d} \) for some \( C \). Put

\[
\hat{d}_{2}(x) = \inf_{\theta \in \mathbb{S}} \{ d(x, e^{-i\theta} \circ x) \}
\]

for a set \( \mathbb{S} \). We claim that there exists a \( c, 1 > c > 0 \),

\[
(6.29) \quad f_{I}(x) \geq c
\]

for all \( x \in X_{1} \). Indeed for each \( x \in X \) and for any \( \theta \in I = [\xi_1, \pi - \varepsilon] \bigcup [\pi + \varepsilon, 2\pi - \xi_2] \) one sees \( x \neq e^{-i\theta} \circ x \). So (6.29) follows by a compactness argument. Let \( M \geq 1 \) be an upper bound of \( \hat{d}_{2} \). We claim

\[
(6.30) \quad \hat{d}_{2}(x) \leq \frac{M}{c} \hat{d}(x), \quad x \in X.
\]

Note \( \hat{d}(x) = f_{K}(x) = \min\{ f_{I}(x), f_{I'}(x) \} \) and \( \hat{d}_{2} = f_{I'} \). Suppose \( f_{K}(y) < f_{I}(y) \). Then \( f_{K}(y) = f_{I'}(y) \), i.e. \( \hat{d}(y) = \hat{d}_{2}(y) \) and (6.30) holds for these \( y \) (as \( \frac{M}{c} > 1 \)). If \( f_{K}(y) \geq f_{I}(y) \) (for some \( y \in X_{1} \)), then \( f_{K}(y) = f_{I}(y) \), giving \( \hat{d}(y) \geq c \) by (6.29). For these \( y \), (6.30) still holds. In any case we have proved (6.30) for \( x \in X_{1} \), hence for \( x \in X \) \( (\hat{d} = \hat{d}_{2} = 0 \) at \( x \in X_{2} \)).

For another illustration, in the same notation as above except that say, \( X = X_{1} \bigcup X_{2} \bigcup X_{4} \) (i.e. \( p_{3} = 4 \)). We are going to prove the lemma for the case \( \ell = 2 \) (with \( x \in X_{1} \), as \( \hat{d} = \hat{d}_{2} = 0 \) at \( x \notin X_{1} \) for \( \ell = 2 \)).

With the above \( I, I' \) and \( K \), let \( J \) be the complement of \( J' \equiv ]\pi - \varepsilon, \pi + \varepsilon[ \bigcup ]3\pi - \varepsilon, 3\pi + \varepsilon[ \) \( \in I \).

It follows, similarly as (6.29), that there exists a \( c_{2} \), \( 1 > c_{2} > 0 \) such that

\[
(6.31) \quad f_{J}(x) \geq c_{2}, \quad \forall x \in X.
\]

Let \( \{ W_{a} \}_{a} \) be the set of connected components of \( X_{4} \). Each \( y \in X_{4} \) is a fixed point of the subgroup \( Z_{4} = \{ 1, e^{i\frac{\pi}{2}}, e^{i\pi}, e^{i\frac{3\pi}{2}} \} \) of \( S^{1} \); write \( \lambda_{1, \alpha}(g) \) for all the eigenvalues of the isotropy (and isometric) action of \( g \in Z_{4} \) on \( T_{y}X \) for \( y \in W_{a} \). All of them are independent of the choice of \( y \in W_{a} \). Let

\[
C_{M} = \max_{1 \neq g \in Z_{4}, \lambda_{1, \alpha}(g) \neq 1} \{ |\lambda_{1, \alpha}(g) - 1| \} > 0; \quad c_{m} = \min_{1 \neq g \in Z_{4}, \lambda_{1, \alpha}(g) \neq 1} \{ |\lambda_{1, \alpha}(g) - 1| \} > 0.
\]

Let \( B = \{ x \in X; \hat{d}_{2}(x) \geq (\frac{C_{M}}{c_{m}} + 1) M \hat{d}(x) > 0 \} \) \( (M \geq 1 \) as above). Clearly \( B \subset X_{1} \) (zero distance for \( x \in X_{2} \bigcup X_{4} \)). We claim that

\[
(6.32) \quad \overline{B} \cap X_{4} = \emptyset.
\]

To see (6.32) suppose otherwise. Let \( y_{n} \in B \) and \( y_{n} \to y \in X_{4} \) as \( n \to \infty \). Observe that \( f_{K}(y_{n}) \neq f_{I'}(y_{n}) \) for all \( n \) because the equality \( \hat{d}(y_{n}) = \hat{d}_{2}(y_{n}) \) (note \( f_{K} = \hat{d} \) and \( f_{I'} = \hat{d}_{2} \) ) clearly contradicts the definition of \( B \) with \( y_{n} \in B \). By \( K = I' \bigcup J' \bigcup J \), we are left with two possibilities for a \( y_{n} \)

\[
(6.33) \quad \begin{cases} \quad i) \quad f_{K}(y_{n}) = f_{I'}(y_{n}) \\ \quad ii) \quad f_{K}(y_{n}) = f_{I}(y_{n}) \end{cases}
\]

Suppose i). By examining the isotropy (and isometric) action of \( Z_{4} \) at \( y \in X_{4} \), one sees that both ratios below

\[
\frac{d(y_{n}, e^{i\pi} \circ y_{n})}{d(y_{n}, e^{i\frac{\pi}{2}} \circ y_{n})}, \quad \frac{d(y_{n}, e^{i\pi} \circ y_{n})}{d(y_{n}, e^{i\frac{3\pi}{2}} \circ y_{n})}
\]
are bounded above by \( \frac{c_M}{c_m} + \frac{1}{2} \) as \( n \gg 1 \). Since \( I' \) and \( J' \) are \( \varepsilon \)-neighborhoods around \( \pi \) and \( \left\{ \frac{\pi}{2}, \frac{3\pi}{2} \right\} \) respectively, by choosing a sufficiently small \( \varepsilon \) (say \( \varepsilon \leq \varepsilon_0 \)) one sees from (6.34)

(6.35)

\[
\frac{f_{I'}(y_n)}{f_{J'}(y_n)} \leq \frac{c_M}{c_m} + \frac{1}{2}, \quad n \gg 1
\]

We claim that this contradicts \( y_n \in B \). Note \( f_K = \hat{d} \) and \( f_{I'} = \hat{d}_2 \) so that the assumption i) \( f_K(y_n) = f_{I'}(y_n) \) amounts to \( \hat{d}(y_n) = f_{I'}(y_n) \) and (6.35) gives

(6.36)

\[
\frac{d_2(y_n)}{d(y_n)} \leq \frac{c_M}{c_m} + \frac{1}{2}, \quad n \gg 1.
\]

By \( y_n \in B \), (6.36) contradicts the definition of \( B \).

Suppose ii) of (6.33). By (6.31), \( f_J(x) \geq c_2 \) for all \( x \in X \) hence by \( f_K = \hat{d} \) and ii) of (6.33), one obtains \( \hat{d}(y_n) \geq c_2 \), giving \( \hat{d}_2(y_n) \geq (\frac{c_M}{c_m} + 1)M \) by using \( y_n \in B \), which is absurd since \( \hat{d}_2 \leq M \) by assumption. The claim (6.32) is proved by contradictions in i) and ii) of (6.33).

Granting the claim (6.32) we have \( \overline{B} \subset X_1 \cup X_2 \) (which is open in \( X \) ). Since for \( \theta \in I \) and \( x \in X_1 \cup X_2 \) (in particular for \( x \in \overline{B} \)) \( x \neq e^{-i\theta} \circ x \), by compactness there exists a \( c_3, 1 > c_3 > 0 \) satisfying (as in (6.29))

(6.37)

\[
f_I(x) \geq c_3
\]

for all \( x \in \overline{B} \). One asserts that

(6.38)

\[
d_2(x) \leq (\frac{c_M}{c_m} + 1) \frac{M}{c_2 c_3} d(x), \quad \forall x \in \overline{B}.
\]

The argument is similar. By \( f_K = \min\{f_I, f_{I'}\} \), a) \( f_K(x) = f_{I'}(x) \) or b) \( f_K(x) = f_I(x) \). a) If \( f_K(x) = f_{I'}(x) \), then by \( f_K = \hat{d} \) and \( f_{I'} = \hat{d}_2 \), \( d(x) = d_2(x) \); b) if \( f_K(x) = f_I(x) \), then by (6.37) and \( f_K = \hat{d} \), \( d(x) \geq 1 \) (for \( x \in \overline{B} \)). In both cases a) and b), (6.38) holds (by \( M \geq 1 \) an upper bound of \( \hat{d}_2 \) and \( \frac{1}{c_2} \leq \frac{1}{c_3} > 1 \)).

Finally, Since the same inequality of (6.38) holds for all \( x \) outside \( B \) by definition of \( B \) (with \( \hat{d}_2 = \hat{d} = 0 \) for \( x \in X_2 \cup X_3 \)), the equivalence between \( d \) and \( \hat{d}_2 \) (for all \( x \in X \)) is proved.

The proof for the general case clearly flows from the similar pattern as above (although tedious). We shall omit the detail. \( \square \)

7. Trace integrals and Proof of Theorem 1.14

7.1. A setup, including a comparison with recent developments. There is a vast literature about heat kernels on manifolds. A comparison between the previous results and those of ours in the present paper shall now be discussed before we proceed further. A concise account of the (ordinary) heat kernel in diversified aspects is given in Richardson [56] and references therein. A generalization of the heat kernel to orbit spaces of a group \( \Gamma \) (of isometries) acting on a manifold \( M \) dates back to the seminal work of H. Donnelly in late ’70s [20], [21]. Among others, Donnelly calculated the asymptotic expansion of the trace of the ordinary heat kernel on \( M \) restricted to \( \Gamma \)-invariant functions (here \( \Gamma \)-action is assumed to be properly discontinuous on \( M \)). Brüning and Heintze in ’84 [10] studied the equivariant trace with \( \Gamma \) replaced by a compact group \( G \) of isometries (including the trace restricted to \( G \)-invariant eigenfunctions). A similar study (of trace) into the orbifold case has been made recently in [56] and [19]. In all of these works the asymptotic expansion of the (ordinary) heat kernel is more or less regarded as known. The questions or techniques come down partly to that used in Donnelly [20] where the contributions to the trace integral are shown to be essentially supported on the fixed point set of the group action.

In a closely related direction some authors consider the case of Riemannian foliations. In this regard, if the orbits of a group acting by isometries are of the same dimension, this forms an example of a Riemannian foliation. For a Riemannian foliation, one is usually restricted to the space of basic
functions} which are constant on leaves of the foliation. Similar ideas apply to give basic forms. The basic Laplacian and basic heat kernel \( K_B(t, x, y) \) can then be defined. Over decades there has been much study into the existence part of the basic heat kernel \( K_B(t, x, y) \), which is finally proven in great generality by E. Park and K. Richardson in '96 [53]. Another proof on the existence is found in '98 [55], which gives a specific formula for \( K_B(t, x, y) \) and allows them to obtain an asymptotic formula for \( K_B(t, x, x) \). We denote the trace integral (on basic functions) by \( \text{Tr} e^{-t\triangle_B} \) (which is \( \sum_m e^{-t\lambda_m} \) for certain eigenvalues with multiplicities). In [56] and [55] the trace integral is also denoted by \( K_B(t) \) which will be avoided here due to a possible confusion. We shall dwell upon this important point after the next paragraph.

Let’s first pause for a moment for comparison. For the part of the trace integral, the basic technique based on Donnelly is also employed here so that the extra contributions, if exist, are expected to be supported on the (lower dimensional) strata. One of our features, however, is Lemma 7.5 which leads to a precise information about the Gaussian-like term of the heat kernel and facilitates our ensuing asymptotic expansion (of the trace integral) in explicit expressions essentially based only on the data given by the ordinary (Kodaira) heat kernel (hence computable in a sense, cf. Remarks 7.24, 1.9). In the process we also need to sum over the group elements (Subsections 7.2, 7.3) and patch up these local sums over \( X \) (Subsections 7.4, 7.5). For the part of the asymptotic expansion, our present heat kernel by its very definition is similar to the \( K_B \) above. Yet objects beyond the basic forms, allowing a generalization in the equivariant sense, indexed by \( m(\in \mathbb{Z}) \) in our notation (with \( m = 0 \) corresponding to the case for \( K_B \)), with bundle-values, are considered here. Since we allow CR nonKähler case, suitable Spin\(^c\) structure in our CR version need be devised and equipped here in order for the rescaling technique of Getzler and our discovery of the off-diagonal estimate (Theorem 5.9) to go through. In this regard, it is not obvious at all (to us) whether the existence theory in the Riemannian case as above can be directly applied to our case. Indeed, besides the need of the Spin\(^c\) structure, our proof of the heat kernel is heavily based on the feature of the group action on CR manifolds, encoded by the BRT trivialization (Subsection 2.4), through the use of the adjoint version of the original equation (Subsection 1.7). Above all, it lies in the following how our approach distinguishes itself from those of others.

Notably, a seeming inconsistency could occur. That is, a discovery in the works [56] and [55] reveals that the so obtained asymptotic expansion for \( K_B(t, x, x) \) there cannot be integrated (over \( x \)) to give the asymptotics of the trace (integral). This perhaps takes one by surprise. See p. 2304 of [56] and Remark in p. 379 of [55]. Despite this, the work [56] manages to prove an asymptotic expansion for the trace integral (on basic functions) by using the work [10] (rather than by integrating the asymptotics of \( K_B(t, x, x) \) obtained therein). In this way, some nontrivial logarithmic terms are to appear unless they are proved to be vanishing. A conjecture has thus been introduced by K. Richardson in ’10 [56, Conjecture 2.5] to the effect that in the Riemannian setting as above, for the (special) case of the isometric group action on a compact manifold, the logarithmic terms in the asymptotic expansion of the trace integral \( \text{Tr} e^{-t\triangle_B} \) must vanish and under a mild assumption (on orientation), there shall be no fractional powers in \( t \) (except possibly an overall fractional power in \( t \)). It is worth mentioning that the works [55] and [56] discuss a number of interesting examples pertinent to the aforementioned peculiar phenomenon. Despite that the seeming inconsistency is consistent with examples by explicit computations, it remains conceptually unclear how this phenomenon comes about.

Our present work affirms the above conjecture of Richardson (with extension to the \( S^1 \)-equivariant case) in the special case of CR manifolds studied here (see Theorems 7.19, 1.14). One key point for all of this lies in (1.4) with \( t \)-dependent coefficients in \( t \) powers, which is regarded as the asymptotic expansion one shall be dealing with in this paper, rather than a classical looking one (1.3) (which is similar in nature to those proposed and studied in [55], [56]). See also our Remarks 1.6, 1.7 and 1.8, which are closely related to the above singular behavior of a classical formulation of asymptotic expansion. Put simply, the formulation (1.3) of an asymptotic expansion leads to certain discontinuities...
of the \(t\)-coefficients along the strata (cf. [55, (4.7)] for a concrete example). A remedy for (1.3) by (1.4) is mainly made via the introduction of a “distance function” (see Theorem 1.3). Eventually, in this work we can restore the trace integral as the integration of our (unconventional) asymptotic expansion of the relevant heat kernel (see Definition 5.4 for the meaning of our asymptotic expansion). Thus, our trace integral and our asymptotic expansion of the heat kernel jointly clarify (with our class of manifolds) the somewhat undesirable phenomenon which is as mentioned above.

To go from the trace integral to the index theorem (thought of as a supertrace integral) is usually not immediate. To the knowledge of the authors, the argument for the proof of index theorems by using trace integrals remains unclarified (cf. Remark 7.25). Completely new ideas might be required; see [11], [12] for very interesting ideas. In the present paper, we couldn’t make our understanding of the (transversal) heat kernel (for our class of CR manifolds) complete without employing the rescaling technique of Getzler and the off-diagonal estimate (Theorem 5.9) adapted to our setting. These results explore in depth the non-Gaussian terms of our (transversal) heat kernel, in contrast to the Gaussian-like term explored in the trace integral here. With these two parts together, our approach studies the meaningful separate aspects of the heat kernel in a unified manner, hence results in an (local) index like term explored in the trace integral here. With these two parts together, our approach studies the technique of Getzler and the off-diagonal estimate (Theorem 5.9) adapted to our setting. These results see [11], [12] for very interesting ideas. In the present paper, we couldn’t make our understanding of using trace integrals remains unclarified (cf. Remark 7.25). Completely new ideas might be required; not immediate. To the knowledge of the authors, the argument for the proof of index theorems by of manifolds) the somewhat undesirable phenomenon which is as mentioned above.

Thus, our trace integral and our asymptotic expansion of the relevant heat kernel (see Definition 5.4 for the meaning of our asymptotic expansion). This work we can restore the trace integral as the integration of our (unconventional) asymptotic expansion (1.4) is mainly made via the introduction of a “distance function” (see Theorem 1.3). Eventually, in of the \(t\)-coefficients along the strata (cf. Theorem 6.1) leaves us the freedom of choosing convenient BRT charts to work out some computations. The salient fact that \(e^{-\ell_\gamma(t,x,y)}\) is an intrinsic object (yet not directly computable), thus is independent of choice of BRT trivializations, is essential to giving intrinsic meanings to some BRT-dependent computations (cf. the contrast between Propositions 7.15 and 7.17 on \(\eta_s\)-terms). This conceptual understanding turns out to be crucial to our final result. The extension of the previous notion “type” to the \(S^1\) stratification is the last conceptual step for the completion of the proof.

As before, \(X = \text{dim } X = 2n + 1\) is a compact connected CR manifold with a transversal CR locally free \(S^1\) action. To proceed with the proof of Theorem 1.14, assume \(X = \bigcup_{\gamma = 1}^{N} \bigcup_{t=1}^{p} X_{p(t)}(\gamma)\) as a disjoint union of (connected) submanifolds \(X_{p(t)}(\gamma)\) being the fixed point set of an isometry \(e^{-\frac{2\pi i}{p}\gamma}\), is a submanifold (possibly disconnected).

Write \(\ell_{(\gamma)}\) for the (real) codimension of \(X_{p(t)}\). When there is no danger of confusion, we may drop \(\gamma\) and write \(\ell\) for \(\ell_{(\gamma)}\). Recall \(X_{\text{sing}}^{\frac{\ell}{n}} = X_p \bigcup X_{p(t+1)}\).

We follow the notations in Subsection 5.2 and the beginning of the last section. Thus \(B_j := (D_j, (z, \theta, \varphi_j)) \) \((j = 1, 2, \ldots, N)\) with \(D_j = U_j \times ] - 2\delta, 2\delta_j]\), \(U_j = \{ z \in \mathbb{C}^n; |z| < \gamma_j \}\) and similarly...
\(\tilde{D}_j = \tilde{U}_j \times -\frac{\delta_j - \delta_j^2}{2}, \quad \tilde{U}_j = \{ z \in \mathbb{C}^n; |z| < \frac{\zeta}{2}\}\). We let \(\delta_j = \tilde{\delta}_j = \zeta, \quad j = 1, 2, \ldots, N\) and assume \(X = \tilde{D}_1 \cup \cdots \cup \tilde{D}_N\). As before, we assume that \(\zeta > 0\) satisfies (1.15).

### 7.2. Local angular integral.

Recall \(\hat{h}_{j+}(x, y), \quad \hat{b}_{j,s}^+(x, y)\) of (5.37) (to be given below); \(a^+(t, x, y)\) involves a certain integral over \([0, 2\pi]\) (cf. (5.41)), \(s = n, n - 1, \ldots\). One key step is the following local version. That is, the (trace) integral of the form

\[
I = I^{(j)}(p, g(x)) \equiv \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_X g(x) e^{-\hat{h}_{j+}(x, e^{-iu}x)} \text{Tr} \hat{b}_{j,s}^+(x, e^{-iu}x) e^{-imu} dx du.
\]

The trace “Tr” here is actually well defined despite a slight abuse of notation about \(\hat{b}_{j,s}^+(x, e^{-iu}x)\) at the second variable (see the line above (2.2)).

Recall the expressions in (5.37) (to be used in what follows):

\[
\begin{align*}
\hat{h}_{j+}(x, y) &= \hat{\delta}_j(\theta) \hat{h}_{j+}(z, w) \hat{\delta}_j(\eta) \in C_0^\infty(D_j), \quad x = (z, \theta), \quad y = (w, \eta) \\
\hat{b}_{j,s}^+(x, y) &= \chi_j(x) e^{-m\varphi_j(z)} - im\hat{h}_{j,s}(z, w) e^{m\varphi_j(w)} + imn\tau_j(w) \sigma_j(\eta), \quad s = n, n - 1, \ldots
\end{align*}
\]

with suitable cut-off functions \(\chi_j, \tau_j, \sigma_j\) and \(\hat{\delta}_j\) defined there.

There will be cases for the result (7.1). We need some preparations and notations.

For \(I = I(p, g)\) of (7.1), take a point \(x_0 \in \text{Supp} \ g \cap X_p\), then \(x_0 \in X_{p(\gamma_\ell)}\) for a \(\gamma_\ell = 1, \ldots, s_\ell\). Locally at \(x_0\) there are higher dimensional strata

\[
X_{p_1(\gamma_1)} = X \supseteq X_{p_2(\gamma_2)} \supseteq \cdots \supseteq X_{p_\ell(\gamma_\ell)} \supseteq X_{p_{\ell+1}(\gamma_{\ell+1})} = X_{p_{\ell+2}}
\]

passing through \(x_0\) where \(i_1 = 1 < i_2 < \ldots < i_\ell < i_{\ell+1} = \ell, \quad \ell \in \{1, 2, \ldots, \ell - 1, \ell\}\). Here (to be useful later) \(p_1, p_2, \ldots, p_\ell|p_\ell\) (by Remark 1.16 similarly). We say

**Definition 7.1.** i) The type \(\tau(I)\) of \(I(p, g)\) is \(\tau(I) := (i_1(\gamma_{i_1}), i_2(\gamma_{i_2}), \ldots, i_\ell(\gamma_{i_\ell}), i_{\ell+1}(\gamma_{i_{\ell+1}}))\) where \(i_1 = \gamma_{i_1} = 1\) and \(i_{\ell+1} = \ell\) always. The length \(l(\tau(I))\) of the type is \(f + 1\). \(I(p, g)\) is said to be of *simple type* if in \(\tau(I)\), \((i_1, i_2, \ldots, i_{\ell+1}) = (1, 2, \ldots, \ell - 1, \ell)\).

ii) Two given types

\[
\tau(I(p_1, g_1)) = (i_1(\gamma_{i_1}), i_2(\gamma_{i_2}), \ldots, i_{\ell+1}(\gamma_{i_{\ell+1}}))
\]

\[
\tau(I(p_2, g_2)) = (j_1(\gamma_{j_1}), j_2(\gamma_{j_2}), \ldots, j_{\ell+1}(\gamma_{j_{\ell+1}}))
\]

are said to be in the same *class* provided a) \(f_1 = f_2 := f, \quad \ell_1 = \ell_2, \quad i_1 = j_1, i_2 = j_2, \ldots, i_\ell = j_\ell\) and b) the codimensions of the corresponding strata coincide: \(e(i_1(\gamma_{i_1})), e(i_2(\gamma_{i_2})), e(i_1(\gamma_{i_1})), e(j_1(\gamma_{j_1})), e(j_2(\gamma_{j_2})) = e(j_2(\gamma_{j_2})), \ldots, e(j_{\ell+1}(\gamma_{j_{\ell+1}})) = e(j_{\ell+1}(\gamma_{j_{\ell+1}}))\).

iii) As above \(I = I(p, g)\), suppose \(\text{Supp} \ g \cap X_{p, \ell} = \emptyset\), equivalently \(\text{Supp} \ g \subseteq \bigcup_{q=1}^{\ell} X_{p, \ell-q}\). We say \(\tau(I)\) is of *trivial type*.

Remark that \(g(x)\) will be chosen to be of very small support and the local nature of \(I, \tau(I)\) will be obvious. Namely, in this case \(\tau(I)\) is independent of choice of \(x_0 \in \text{Supp} \ g \cap X_{p, \ell}\). In the final subsection, the notion of “type” will be naturally extended to each connected submanifold \(X_{p(\gamma_\ell)}\) in the strata. By this, the influence of the geometry of the \(S^1\) stratification on the heat kernel trace integral will become more evident.

Most numerical results in what follows will only depend on the equivalence classes of types. But for the sake of notational convenience, we assume \(I\) to be of simple type or trivial type in the proposition below. The modification to the general type is basically only complicated in notation and will be treated later.

**Proposition 7.2.** Suppose \(x_0 \in \tilde{D}_j\). Then there exist a neighborhood \(\tilde{\Omega} (\subset \tilde{D}_j)\) of \(x_0\) and an \(\varepsilon > 0\) (depending on \(x_0\)) such that for every \(\Omega \subset \tilde{\Omega}\), every \(g(x) \in C_0^\infty(\Omega)\) we have the following for \(I\) of (7.1) with any \(\varepsilon \leq \varepsilon\). Note \(I\) is assumed to be of simple type (if not of trivial type) as said prior to the
proposition. (In the following, ii) and Case a) of iii) are basically of trivial type; i) and Case b) of iii) are of simple type.)

i) \( \ell = 1 \) \((p = p_1)\). For \( x = (z, v) \in \mathcal{D}_j \), write \( z(x) = z \) and \( \theta(x) = v \).

\[
I = e^{-\frac{2\pi i m}{\ell}} \int_{-\varepsilon}^{\varepsilon} \int_X g(x) \chi_j(x) \text{Tr} b^+_j(x, z, z) \tau_j(z) \sigma_j(v + \psi) dv_X(x) dv.
\]

In particular, \( I \) is a constant independent of \( t \). (Note it is \( b^+_{j, s} \) instead of \( \hat{b}^+_{j, s} \) here; the same can be said with (7.3) below.)

ii) Suppose \( e^{-\frac{2\pi i}{\ell}} \circ x_0 \notin \mathcal{D}_j \) (here \( \ell = 2, 3, \ldots, k \)). Then \( I = 0 \).

iii) Suppose \( e^{-\frac{2\pi i}{\ell}} \circ x_0 \in \mathcal{D}_j \) (here \( \ell = 2, 3, \ldots, k \)).

Case a) \( x_0 \in \bigcup_{q=1}^{q=\ell-1} X_\nu_q \). Then \( I \sim O(t^\infty) \) as \( t \to 0^+ \).

Case b) \( x_0 \notin \bigcup_{q=1}^{q=\ell-1} X_\nu_q \) and \( x_0 \in \overline{X}_\nu \subset \overline{X}_p \). Take local coordinates \((e_\ell = e(\gamma_\ell) = \gamma_1, \ldots, \gamma_{n+1})\) defined on \( \Omega \) such that

\[
\overline{X}_p \cap \Omega = \{ y \in \Omega; y_1 = \cdots = y_{n+1} = 0 \}.
\]

Assume (possibly after shrinking \( \Omega \) about \( x_0 \)) \( \Omega = \bigcup_{j \in \{1, \ldots, k\}} (X_{p_j(\gamma_j)} \cap \Omega) \) (for some \( \gamma_j = 1, \ldots, s_j \)) which is seen to be (by assumption of simple type)

\[
(\overline{X}_{p_j(\gamma_j)} \cap \Omega) \cup (X_{p_j(\gamma_j-\eta_j-\epsilon_j)} \cap \Omega).
\]

Write \( e_{\ell-\eta+1} - e_{\ell-\eta} \) for the codimension of \( X_{p_j(\gamma_j-\eta_j+1)} \) in \( X_{p_j(\eta_j)} \) where \( p_\mu = p_{p_j(\gamma_j)} \) for \( \mu = \ell - \eta + 1 \) and \( \mu = \ell - \eta \) respectively. If \( y = (z, \theta) \) (in BRT coordinates), write \( z(y) \) for \( z \) and if \( y = (0, Y) \), write \( Y \) for \( (0, 0, Y) \) and \( z(Y) \) for \( z(y) \). Similar notation for \( \theta(Y) \) etc.

Then \((e_\ell = e(\gamma_\ell))\)

\[
I = b^{(j)}_{s, \frac{\ell}{2}} t^{e_\ell} + b^{(j)}_{s, \frac{\ell+1}{2}} t^{e_{\ell+1}} + \ldots
\]

where the first coefficient \( b^{(j)}_{s, \frac{\ell}{2}} \) is given by

\[
b^{(j)}_{s, \frac{\ell}{2}} = \pi \frac{\epsilon}{\ell} e^{-\frac{2\pi i m}{\ell}} \prod_{q=1}^{q=\ell-1} \left| e_{\ell-q} \nu_q - 1 \right|^{-e_{\ell-q+1}-e_{\ell-q}} \times
\]

\[
\frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} \int_{X_{p_j(\gamma_j)}} \frac{g(Y) \chi_j(Y) \text{Tr} b^+_j(z(Y), z(Y)) \tau_j(z(Y)) \sigma_j(\theta(Y) + u) dv_X}{(x, z, z)} du.
\]

In particular, for \( s = n \) (cf. \( \text{dim} X = 2n+1 \)), (7.3) for \( b^{(j)}_{n, \frac{\ell}{2}} \) simplifies by using \( \text{Tr} b^+_j(z, z) \equiv (2\pi)^{-n} \).

Proof. Write \( x_0 = (z_0, \theta_0) \). For simplicity, assume \( \theta_0 = 0 \) without loss of generality (cf. the last three paragraphs of the proof of Theorem 6.1 for a similar situation). Note that the existence of \( \Omega \) and \( \varepsilon \) in the statement above will be obvious from the proof below and we shall not refer to them explicitly.

To see i), we note that \( e^{-\frac{2\pi i}{p}} = id \) \((p = p_1)\) on \( X \) (because it is so on \( X_p \) by definition which is dense (and open) in \( X \)). For \( x = (z, v) \) lying in the BRT neighborhood \( \mathcal{D}_j \) and \( u = \frac{2\pi i}{p} \pm \varepsilon \) such that \( e^{-iu} \circ x = e^{\pm i\varepsilon} \circ x \) lies in \( D_j \) \((\mathcal{D}_j) \), one has \( e^{\pm i\varepsilon} \circ x = (z, v + \varepsilon) \) by construction of BRT charts \( D_j \). In this case \( e^{-\frac{2\pi i}{p} x} \) \((-\varepsilon < \varepsilon < \varepsilon) \) for all \( \hat{h}_{j,s}(x, e^{-iu} \circ x) = 0 \) from \( \hat{h}_{j,s}(z, z) = 0 \) from (7.2). The same reasoning applies to \( \text{Tr} b^+_j(x, e^{-iu} \circ x) \) to reach \( \text{Tr} b^+_j(x, z, z) \). Now choose a neighborhood \( \Omega \subset \mathcal{D}_j \) of \( x_0 \) then a small \( \varepsilon > 0 \) (depending on \( x_0 \)) such that \( e^{\pm i\varepsilon} \circ x \) lies in \( D_j \) for \( x \in \Omega \). As \( g \in C^\infty(\Omega) \), we can apply the above argument for these \( x \) by making \( \psi = u - \frac{2\pi}{p} \) \((|\psi| \leq \varepsilon) \) so that \( e^{-iu} \circ x = e^{-i\psi} \circ x = (z, v + \psi) \). In
(7.2) one thus has $\theta = v, w = z$ and $\eta = v + \psi$. By these remarks, i) of the proposition follows from (7.1) and (7.2).

For ii) of the proposition, with $x_0 \in \tilde{D}_j$ yet $e^{-\frac{2\pi}{p\ell}} \circ x_0 \notin \tilde{D}_j$, by continuity of $S^1$ action there exist a neighborhood $\Omega (\subset \tilde{D}_j)$ of $x_0$ and an $\varepsilon > 0$ such that $e^{-iu} \circ x \notin \tilde{D}_j$ for $x \in \Omega$ and $u \in [\frac{2\pi}{p\ell}, -\varepsilon, \frac{2\pi}{p\ell} + \varepsilon]$. Hence for these $x \in \Omega$, $\tilde{b}_{j,+}(x, e^{-\frac{2\pi}{p\ell}} \circ x) = 0$ by a cut-off function $\tau_j$ (of compact support in $\tilde{U}_j \subset \tilde{D}_j$) involved in $\tilde{b}_{j,+}$ (see (7.2)), giving $I = 0$ in (7.1).

For case a) of iii), the assumption gives $x_0 \in X_{p\ell-q}$, $q \in \{1, 2, \ldots, \ell - 1\}$. Further, by assumption $e^{-\frac{2\pi}{p\ell}} \circ x_0 \in \tilde{D}_j$ we write $e^{-\frac{2\pi}{p\ell}} \circ x_0 = (\tilde{z}_0, \tilde{\theta}_0)$ with $|\tilde{\theta}_0| < \frac{\zeta}{2}$. We claim $\tilde{z}_0 \neq z_0$. The line of argument is slightly different from that in (6.4). Suppose $\tilde{z}_0 = z_0$. Then by $e^{i\tilde{\theta}_0} \circ (e^{-\frac{2\pi}{p\ell}} \circ x_0) = e^{i\theta_0} \circ (\tilde{z}_0, \tilde{\theta}_0) = (\tilde{z}_0, 0) = (z_0, 0) = x_0$ (recall $\theta_0 = 0$ in the beginning of the proof). Hence,

$$2\pi \frac{p\ell}{p\ell - q} - \tilde{\theta}_0 = m \frac{2\pi}{p\ell - q}, \quad m \in \mathbb{Z}$$

by assumption $x_0 \in X_{p\ell-q}$. But $|\tilde{\theta}_0| < \frac{\zeta}{2}$ and $\zeta$ is assumed to satisfy (1.15), so the above equality is absurd, proving the claim $\tilde{z}_0 \neq z_0$ by contradiction.

Now that $\tilde{z}_0 \neq z_0$, there exists a neighborhood $\Omega$ of $x_0$ and an $\varepsilon > 0$ (dependent on $x_0$) such that for $x \in \Omega$ and $\theta \in [\frac{2\pi}{p\ell} - \varepsilon, \frac{2\pi}{p\ell} + \varepsilon]$, writing $e^{-i\theta} \circ x = (\tilde{z}, \tilde{\theta})$ and $x = (z, \theta)$ one has $|\tilde{z} - z| \geq \frac{1}{2} |\tilde{z}_0 - z_0| \equiv \delta$ by using continuity of $S^1$ action at $x = x_0$ and $\theta = \frac{2\pi}{p\ell}$. From the property of $\hat{h}_{j,+}(x, y)$ (which is essentially $|z - w|^2$, cf. (5.19) and (5.37)) one sees that $I$ of (7.1) gives

$$\frac{1}{2\pi} \int_{\frac{2\pi}{p\ell} + \varepsilon}^{\frac{2\pi}{p\ell} - \varepsilon} \int_X g(x)e^{-\frac{\hat{h}_{j,+}(x, e^{-iu} x)}{t}} \text{Tr} \hat{b}_{j,s}(x, e^{-iu} x) e^{-imu} dv_X(x) du = O(t^{\infty}), \quad t \to 0^+$$

(for $g \in C_0^\infty(\Omega)$) simply because the exponential term in (7.5) decays rapidly if $|\tilde{z} - z| \geq \delta$ here, proving case a) of iii) of the proposition.

To prove case b) of iii), we first give an estimate under an additional assumption that $X$ is strongly pseudoconvex, then we will drop this assumption and carry out some refined computations to complete the proof.

Since $x_0$ is a fixed point of $e^{-\frac{2\pi}{p\ell}}$ by assumption, by continuity of $S^1$ action there exist an open subset $\Omega$ of $x_0$ and a small constant $0 < \varepsilon < \frac{\zeta}{2}$ such that $e^{-i\theta} \circ x \in \tilde{D}_j$ for $x \in \Omega$ and $\theta \in [\frac{2\pi}{p\ell} - \varepsilon, \frac{2\pi}{p\ell} + \varepsilon]$. We assume that $\Omega$ is small, say contained in the BRT chart $\tilde{D}_j$, and satisfies the local coordinates of case b) of iii). For $x = (z, \theta) \in \Omega$ and $\theta \in [\frac{2\pi}{p\ell} - \varepsilon, \frac{2\pi}{p\ell} + \varepsilon]$, write $e^{-i\theta} \circ x = (\tilde{z}, \tilde{\theta}) \in \tilde{D}_j$.

We claim that there exists a positive continuous function $f_1(x)$ such that

$$h_{j,+}(z, \tilde{z}) \geq f_1(x) d(x, X_{p\ell})^2, \quad \forall x \in \Omega$$

where $h_{j,+}$ is as in (7.2) (cf. (5.19)). (Here is the only place where we use the assumption $X$ is strongly pseudoconvex.)

Granting the claim (7.6), with the local coordinates in iii), suppose $y(x_0) = Y(x_0) = 0$. Rewrite the quotient $\frac{d(y, X_{p\ell})^2}{(|y_1|^2 + \cdots + |y_{e_r}|^2)}$ as

$$d(y, X_{p\ell})^2 = f_2(y)(|y_1|^2 + \cdots + |y_{e_r}|^2), \quad \forall y \in \Omega$$

where $f_2(y)$ is a positive continuous function.
With (7.6) and (7.7), we estimate \( \hat{h}_{j,+} \) below and have the following (see (7.2) or (5.37) and note \( g(x) \in C_0^\infty(\Omega), \Omega \) small).

\[
I = \frac{1}{2\pi} \int_{2^{\pi}} \int_{2^{\pi}} g(x)e^{-\frac{h_{j,+}(x,e^{-iu}\circ x)}{t}} \text{Tr} \hat{I}_{j,+}(x, e^{-iu} \circ x)e^{-imu}dv_X(x)du \\
\leq \frac{1}{2\pi} \int_{2^{\pi}} \int_{2^{\pi}} g(x)e^{-\frac{f_1(x)d(x,X_p\ell)}{t}} \text{Tr} \hat{I}_{j,+}(x, e^{-iu} \circ x)e^{-imu}dv_X(x)du \\
= \frac{1}{2\pi} \int_{2^{\pi}} \int_{2^{\pi}} g(y)e^{-\frac{f_1(y)}{t}2(y)|y|^2+\cdots} \text{Tr} \hat{I}_{j,+}(y, e^{-iu} \circ y)e^{-imu}dv_X(y)du \\
\sim c_{s,j} e^{\frac{|y|}{2}} + c_{s,j+1} e^{\frac{|y|}{2}} + \cdots \text{as } t \to 0^+,
\]

(7.8)

where the last step is obtained by a change-of-variable (rescaling \( y_t = \sqrt{t}y_t, \ i = 1, \ldots, e_\ell, e_\ell \geq 1 \) as \( \ell \geq 2 \)) and \( c_{s,j} \in \mathbb{R} \) is independent of \( t \) (\( k = e_s, e_{s+1} \), \( \ldots \)).

We are left with the proof of the claim (7.6). Part of the argument echoes that for (6.4). We first estimate \( |z - \bar{z}|^2 \). Without the danger of confusion we omit “\( \circ \)” in what follows. By \( (z, 0) = e^{iv}x \) and \( (\bar{z}, 0) = e^{iv}(e^{-i\theta}x) \), \( |z - \bar{z}|^2 \) is equivalent to \( d(e^{iv}x, e^{iv}(e^{-i\theta}x)) \) (cf. (6.3)) which is the same as \( d(x, e^{-iv}(e^{-i\theta}x)) \). As \( (z, \bar{z}), (z, v) \in D_j \), one has \( \bar{z}, v \leq \frac{\zeta}{2} \). By choosing \( \zeta, \varepsilon \) to be (much) less than the \( \varepsilon_0 \) of Lemma 6.6, one sees \( d(x, e^{-iv}(e^{-i\theta}x)) \geq d_2(x, X_{\text{sing}}^{\ell-1}) \) of Lemma 6.6. By the same lemma

\[
d_2(x, X_{\text{sing}}^{\ell-1}) \text{ is equivalent to } d(x, X_{\text{sing}}^{\ell-1}).
\]

(7.9)

As \( |z - \bar{z}|^2 \) is also equivalent to \( h_{j,+}(z, \bar{z}) \) of (7.6) (cf. (6.3)) and (7.9) is equivalent to \( d(x, X_{\text{sing}}^{\ell-1}) \) by Theorem 6.7 with our assumption \( X \) is strongly pseudoconvex, we have now shown

\[
h_{j,+}(z, \bar{z}) \geq c \cdot d(x, X_{\text{sing}}^{\ell-1})^2
\]

(7.10)

for some constant \( c > 0 \). In view that \( X_{\text{sing}}^{\ell-1} = X_p\ell \cup X_{p\ell+1} \cdots \) and the assumption that \( x_0 \in X_{p\ell} \), one sees, possibly after shrinking \( \Omega \), \( d(x, X_{\text{sing}}^{\ell-1}) = d(x, X_{p\ell}) = d(x, X_p\ell) \). Hence we have reached (7.6) from (7.10), as desired.

Following (7.8) we shall now make some accurate computations for case b) of iii) of this proposition.

Henceforward we do not assume that \( X \) is strongly pseudoconvex; we will not use Lemma 6.6 and Theorem 6.7 as used above.

Write \( \frac{2\pi}{p_\ell} = \omega \) and \( u = \psi + \omega \). In coordinates of case b) of iii), in view of (7.8) one sees to identify, among others,

\[
\lim_{t \to 0^+} \frac{\hat{h}_{j,+}((\sqrt{\ell}y, Y), e^{-iu}(\sqrt{\ell}y, Y))}{t}
\]

where we have rescaled \( \sqrt{\ell} \to \sqrt{\ell} \) and we omit “\( \circ \)” for the \( e^{-iu} \in S^1 \) action.

Since the fixed point set of an isometry is totally geodesic, we assume \( Y = 0 \) to be a system of geodesic coordinates at \( Y = 0 \) of \( X_{p\ell} \), as \( (\sqrt{\ell}y, Y) \) the geodesic coordinates at \( (0, 0) \) of \( X \). We choose \( Y = 0 \) in (7.11) to simplify the notation. Expressed in BRT coordinates, \( (\sqrt{\ell}y, 0) = (z_0, v_0) \) and \( e^{-i\omega}(\sqrt{\ell}y, 0) = (z_1, v_1) \) (by continuity of \( S^1 \) action, for \( t \) small \( e^{-i\omega}(\sqrt{\ell}y, 0) \in D_j \) since \( e^{-i\omega}(0, 0) = (0, 0) \)).

One sees \( e^{-iu}(\sqrt{\ell}y, 0) = e^{-iu}(z_1, v_1) = (z_1, v_1 + \psi) \) for \( \psi \leq e \). By (7.2) one sees

\[
\hat{h}_{j,+}((\sqrt{\ell}y, 0), e^{-iu}(\sqrt{\ell}y, 0)) \text{ is now reduced to } \hat{h}_{j,+}(z_0, z_1)
\]

(7.12)

which is independent of \( \psi \) for \( u \) in the \( \varepsilon \)-neighborhood of \( \omega \). Namely \( \hat{h}_{j,+}((\sqrt{\ell}y, 0), e^{-iu}(\sqrt{\ell}y, 0)) \equiv \hat{h}_{j,+}((\sqrt{\ell}y, 0), e^{-iu}(\sqrt{\ell}y, 0)) \) for \( u \) in the \( \varepsilon \)-neighborhood of \( \omega \).

Now \( x_0 = (0, 0) \) is a fixed point of \( e^{-i\omega} \) (\( \omega = \frac{2\pi}{p_\ell} \)). One can see that \( T_{x_0}X \) under the isotropy action induced by \( e^{-i\omega} \) decomposes as an orthogonal direct sum of eigenspaces (where \( N(S/M) \) denotes the
normal bundle of a submanifold $S$ in an ambient manifold $M$ with $N_p(S/M)$ the fiber of $N(S/M)$ at $p$, and $\bar{X}_\mu = \bar{X}_{\mu(\gamma)}$ for some $\gamma_\mu = 1, 2, \cdots, s_\mu$; $\mu = 1, 2, \cdots, \ell$.

Moreover, suppose the general case where $\beta = ||\cdot||$ similarly.

Sublemma 7.4.

\[ T_{x_0}X_{p_\mu}, N_{x_0}(\bar{X}_{p_t}/\bar{X}_{p_t-1}), N_{x_0}(\bar{X}_{p_t-1}/\bar{X}_{p_t-2}), \ldots, N_{x_0}(\bar{X}_{p_2}/\bar{X}_{p_1}) \]

associated with eigenvalues

\[ (7.13) \quad 1, e^{i\omega p_\ell-1}, e^{i\omega p_\ell-2}, \ldots, e^{i\omega p_1} \]

respectively.

For instance, assume $\ell = 2$ and take $g = e^{2\pi i}$. Set $q = \frac{p_2}{p_1} (\in \mathbb{N})$. On $N_{x_0}(\bar{X}_{p_2}/\bar{X}_{p_1})$, $g \neq \text{id}$ and $g^q = \text{id}$. Hence $v \in N_{x_0}(\bar{X}_{p_2}/\bar{X}_{p_1})$ is rotated by the angle $\frac{2\pi}{q}$ which is $\omega p_1$.

The goal in what follows is to prove the claim that for $q = 1, \ldots, \ell - 1,$

\[ (7.14) \quad \lim_{t \to 0^+} \frac{\hat{h}_{j,+}((\sqrt{t}\hat{y}, 0), e^{-i\omega}(\sqrt{t}\hat{y}, 0))}{t} = |e^{i\omega p_{\ell-q}} - 1|^2 ||\hat{y}||^2 \quad \text{for } \hat{y} \in N_{x_0}(\bar{X}_{\ell+1}/\bar{X}_{\ell-q}) \]

or equivalently, in the notation above (see (7.12))

\[ (7.15) \quad \lim_{t \to 0^+} \frac{h_{j,+}(z_0, z_1)}{t} = |e^{i\omega p_{\ell-q}} - 1|^2 ||\hat{y}||^2 \quad \text{(for } \hat{y} \in N_{x_0}(\bar{X}_{\ell+1}/\bar{X}_{\ell-q})) \]

where $|| \cdot ||$ denotes the norm with respect to the metric tensor of $X$ at $x_0$.

Our proof of claim (7.14) is based on the following sequence of lemmas.

Lemma 7.3. In the notation above, for $\hat{y} \in N_{x_0}(\bar{X}_{\ell+1}/\bar{X}_{\ell-q})$ we have

\[ \lim_{t \to 0^+} \frac{d_X^2((\sqrt{t}\hat{y}, 0), e^{-i\omega}(\sqrt{t}\hat{y}, 0))}{t} = |e^{i\omega p_{\ell-q}} - 1|^2 ||\hat{y}||^2 \]

where $d_X$ denotes the distance on $X$.

Proof. On $T(0,0)X$ the action induced by $e^{-i\omega}$ rotates the tangent vector $\hat{y}$ by the angle $\omega p_{\ell-q}$. Hence the lemma follows from the well known fact that in a Riemannian manifold $(M,g)$, if $a, b$ in $M$ are the images of $A, B$ in $T_pM$ by the exponential map at $p \in M$, then as $(a,b) \to (p,p)$

\[ (7.16) \quad \lim_{||A-B|| \to 0} \frac{d_M(a,b)}{||A-B||} \to 1 \]

where $|| \cdot ||$ is $g$ at $p$ (cf. [38, Proposition 9.10]).

\[ \square \]

Sublemma 7.4. Suppose $N$ is a Riemannian submanifold of a Riemannian manifold $M$. Then the respective distance functions on $M$ and on $N$ are infinitesimally the same. More precisely, suppose in $N$, $p_n \neq q_n$ for all $n \in \mathbb{N}$, and $p_n, q_n \to p$ as $n \to \infty$ for a given point $p \in N$. Then $\lim_{n \to \infty} \frac{d_M(p_n,q_n)}{d_N(p_n,q_n)} = 1$.

Moreover, suppose $t_{n,M}$ (resp. $t_{n,N}$) in $T_{p_n} M$ are the unit tangent vectors along which the minimal geodesics in $M$ (resp. $N$) join $p_n$ and $q_n$. Then $\lim_{n \to \infty} (t_{n,M} - t_{n,N}) = 0$.

Proof. Suppose the special case $p_n = p$ for all $n$. Let $\gamma_n$ be a geodesic (with unit speed) of $N$ joining $p$ and $q_n$, and $\beta_n = \exp_p^{-1}(\gamma_n) \subset T_p M$. Write $l_n(t)$ for the length of (part of) $\beta_n$ (with the parameter going from 0 to $t$) measured with the metric $g_{ij} = 1 + O(|x|^2)$ in geodesic coordinates (at $p$). Write $||v||$ for the Euclidean norm of a vector $v \in T_p M$ expressed in geodesic coordinates. Given a curve $\beta(t) \subset T_p M$, $\beta(0) = p$, $\dot{\beta}(0) \neq 0$, one sees the length function $l(t) = \int_0^t \sqrt{\langle \dot{\beta}(t), \dot{\beta}(t) \rangle_{g_{ij}}} dt$ satisfies

\[ \frac{||\beta(t)|| - l(t)}{||\beta(t)||} = O(t) \leq C t \quad \text{for a (locally bounded) quantity } C \quad \text{which depends only, apart from } \beta, \text{ on the local geometry at } p \quad \text{(uniformly).} \]

Clearly this implies the lemma if $q_n$ is assumed to approach $p$ along a given geodesic $\gamma$ of $N$. If $q_n$ approaches $p$ along different geodesics $\gamma_n$, since these geodesics can be uniformly controlled by the local geometry around $p$, the same results hold as well. For the general case where $p_n$ are different, the similar argument using the control by local geometry implies

\[ \frac{d_M(p_n,q_n) - d_N(p_n,q_n)}{d_M(p_n,q_n)} \leq C(d_M(p_n,q_n)). \]

The assertion about the unit tangent vectors can be proved similarly. \[ \square \]
The third lemma (as our main lemma) is as follows. (This lemma can be viewed as a sharp version of the important claim (6.4) in the proof of Theorem 6.1, which bears upon the reason why our distance function \( d \) arises.)

**Lemma 7.5.** In the previous notation, write \( p = (z(p), \theta(p)) \) and \( q = (z(q), \theta(q)) \) in \((z, \theta)\) coordinates on the BRT chart \( \Delta_j = \tilde{U}_j \times [\frac{j}{2}, \frac{j+1}{2}] \). We omit the subscript \( j \) in what follows. Let \( S = S^1 \circ p \) be the \( S^1 \)-orbit of \( p \) and \( N(p(S/X)) \) be the normal bundle of \( S \) in \( X \) identified with the orthogonal complement of \( TS \) in \( TX \). Suppose \( p_n \neq q_n \) for all \( n \) and \( p_n, q_n \to p \in X \) as \( n \to \infty \) such that \( D_n = \exp^{-1}_{X,p}(p_n), \) \( A_n = \exp^{-1}_{X,p}(q_n) \in N_p(S/X) \). In the case where \( p_n \neq p \) and \( q_n \neq p \) (all \( n \)), suppose the angle at \( p \) given by the vectors \( D_n \) and \( A_n \) are bounded away from 0 as \( n \to \infty \). Then

\[
\lim_{n \to \infty} \frac{d_X(p_n, q_n)}{d_U(z(p_n),z(q_n))} = 1. 
\]

In particular \( z(p_n) \neq z(q_n) \) for \( n \) large.

**Proof.** We think of \( U = \tilde{U}_j \) as an embedded submanifold of \( X \) with \( \theta = 0 \). As in the sublemma above, we first assume \( p_n = p \) for all \( n \). By applying the \( S^1 \) action we assume \( p = (z(p),0) \in U \), hence \( N_p(S/X) = T_pU \) by the construction of our rigid metric on \( X \) (cf. (2.5) and (4.1)). Fix an \( n \) and set \( q = q_n, q' = e^{i\theta(q)} \circ q = (z(q),0) \in U \). Put \( A = \exp^{-1}_{X,p}(q), B = \exp^{-1}_{X,p}(q') \in T_pX, \) so \( \|A\| = d_X(p,q), \|B\| = d_X(p,q') \) (where \( T_pX \) is equipped with the Euclidean metric \( \|\cdot\| \) in geodesic coordinates). We are going to prove, as \( n \to \infty \),

\[
\lim_{n \to \infty} \frac{\|A\|}{\|B\|} = 1. 
\]

One has \( d_X(p,q')/d_U(p,q') \to 1 \) by Sublemma 7.4. Hence, to prove (7.17) for this special case \( p_n = p \) is the same as to prove \( d_X(p,q)/d_X(p,q') \to 1 \) which is (7.18) above.

To see (7.18) (hence (7.17)), we first argue (7.19) below. Let \( L \subset T_pX \) be the line determined by \( A, B \), i.e. \( L = \{A + t(B-A); t \in \mathbb{R}\} \). Then

\[
\text{as } n \text{ large.}
\]

Note \( A \in N_p(S/X) = T_jU \) from the condition of the lemma, and \( B \) is nearly lying on \( T_pU \) (with a small angle between \( B \) and \( T_pU \)) by using Sublemma 7.4 on tangents. Let \( \Gamma_1 = \{e^{i\theta} \circ q \in [0,\theta(q)] \in X \} \) joining \( q \) and \( q' \) and \( \Gamma = \exp^{-1}_{X,p} \Gamma_1 \subset T_pX \) joining \( A \) and \( B \). Recall that the vector field \( T \) induced by the \( S^1 \) action is orthogonal to \( U \) by our rigid metric ((2.5) and (4.1)), hence \( T \perp \Gamma_1 \perp T_pU \).

On the other hand, by construction \( T_pS \perp T_pU \) and \( \Gamma_1 \simeq S \) (as \( n \gg 1 \)), one sees by \( A \in T_pU \) that \( T_A \Gamma \perp T_pU \) approximately (as vector subspaces in \( T_pX \)). In sum, if \( q, q' \) are close to \( p \) (so \( T_pU \) close to \( T_pU \)), then \( T_B \Gamma \perp T_pU, T_A \Gamma \perp T_pU \) approximately; for this we write \( \Gamma \perp A, B \) approximately. Pulling back the \( S^1 \) foliation locally around \( p \) via the \( \exp_{X,p} \) the same way as \( \Gamma \) obtained by \( \Gamma_1 \), there is a foliation \( F \) around \( p \) in \( T_pX \) in which (part of) \( \Gamma \) lies as a leaf. Write \( p \in \Gamma_0 \subset \exp_{X,p}^{-1}(S) \in F \). As \( n \gg 1 \), the line \( L \) determined by \( A, B \in \Gamma \) tends to the tangent line \( = T_pS \) to \( \Gamma_0 \) at \( p \) (since the leaf \( \Gamma_0 \) of \( F \) tends to the leaf \( \Gamma_0 \)). Hence by using the uniform continuity for \( F \) around \( p \), \( L \) is close to lines \( \tilde{L} \) tangent to leaves \( \tilde{\Gamma} \) of \( F \) if \( \Gamma \) is nearby \( \Gamma_0 \) and \( \tilde{L} \) nearby \( T_pS \). In particular, \( L \) is close to the tangent lines \( T_A \Gamma, T_B \Gamma \) as \( n \gg 1 \). Since \( \Gamma \perp A, B \) approximately as just shown, i.e. \( T_A \Gamma \perp A, T_B \Gamma \perp B \) approximately, it yields \( L \perp A, B \) approximately \((n \gg 1)\), proving (7.19).

For \( q' \) close to \( p \), by simple Euclidean geometry (on \( T_pX \), \( ||A - B|| \) is rather small in comparison to \( ||A|| \) and \( ||B|| \) by using \( L \perp A, B \) approximately (7.19), i.e. \( ||A - B|| = o(||A||), o(||B||) \)). By using law of cosines, one can obtain (7.18), yielding the special case \( p_n = p \) of the lemma. As this step appears crucial and will be instrumental to the general case, we prefer to supply some details as follows.

Take a triangle with vertices \( T_i \) \((i = 1, 2, 3)\), angles \( \alpha_i \) at \( T_i \) and \( \delta_i \) the length of the side facing \( T_i \). Suppose \( \alpha_2 \leq \alpha_3 \) and both \( \approx \frac{\pi}{2} \). Set \( \alpha_2 = \frac{\pi}{2} - \alpha, 0 < \alpha < 1 \). Let \( D \) sit on the line \( L_1 \) determined by \( T_2 \) and \( T_3 \) such that the line \( L_2 \) determined by \( T_1 \) and \( D \) is perpendicular to \( L_1 \). Assume first that \( D \) sits
between $T_2$ and $T_3$. Then $\delta_1 = \delta_3 \sin \theta_1 + \delta_2 \sin \theta_2$ where $\theta_1, \theta_2$ (with $\theta_1 + \theta_2 = \alpha_1$) are angles given by $L_2$ and the two sides at $T_1$. Thus $\frac{\delta_1}{\delta_3} \leq 2 \sin \alpha_1 \left( \frac{\alpha_2}{\alpha_3} \leq 1 \right)$ by $\alpha_2 \leq \alpha_3$. If $D$ sits outside the triangle, then $\delta_1 = \delta_3 \sin \alpha - \delta_2 \sin \theta_3$, $\theta_3 = \alpha - \alpha_1$, so $\frac{\delta_1}{\delta_3} \leq \sin \alpha$. One obtains $\frac{\delta_1}{\delta_3} \to 0$ if both $\alpha_1 \to 0$ and $\alpha \to 0$. By $\delta_1^2 = \delta_2^2 + \delta_3^2 - 2\delta_2 \delta_3 \cos \alpha_1$, one has

\[
(1 - \frac{\delta_3}{\delta_3})^2 = \left( \frac{\delta_1}{\delta_3} \right)^2 - 2 \frac{\delta_2}{\delta_3} \left( 1 - \cos \alpha_1 \right) \leq \left( \frac{\delta_1}{\delta_3} \right)^2 \leq \sin \alpha + 2 \sin \alpha_1
\]

giving $\frac{\delta_1}{\delta_3} \to 1$ if both $\alpha_2, \alpha_3 \approx \frac{\pi}{2}$ (hence $\alpha, \alpha_1 \approx 0$). As said, this yields (7.18).

We draw some consequences in order for the general case. If $\alpha_1 \to 0$ (by $\alpha_2, \alpha_3 \to \pi/2$), then the two sides at $T_1$ are close to each other, i.e. $\lim (A/||A|| = B/||B||) = 0$ (as $q \to p$). One sees that if $C = \exp_{U,p}^1(q') \in T_p U$, then by using (7.18) and Sublemma 7.4 on tangents via $B$,

\[(7.20) \quad \alpha) \lim ||A/||A|| = 1, \quad \beta) \lim (A/||A|| - C/||C||) = 0.
\]

We are ready to prove the general case $p_n \neq p$. Write $D = \exp_{X,p}^{-1}(p_n),$ $F = \exp_{U,p}^{-1}(p_n')$ in the same way as $A = \exp_{X,p}^1(q_n),$ $C = \exp_{U,p}^{-1}(q_n')$ above. With $D, F$ in place of $A, C$ in (7.20) one has the same results for $D, F$:

\[(7.21) \quad a) \quad \lim ||D/||D|| = 1, \quad b) \quad \lim (D/||D|| - F/||F||) = 0.
\]

In view of (7.16) one has $||D - A||/d_X(p_n, q_n) \to 0, ||C - F||/d_U(p_n', q_n') \to 0$. Hence to prove (7.17), i.e. $d_X(p_n, q_n)/d_U(p_n', q_n') \to 1$, is the same as to show $\lim ||D - A||/||D - F|| = 1$. This is intuitively clear by (7.20), (7.21) (which alludes to $A \approx C$ and $D \approx F$) provided that the angle given by the two vectors $D$ and $A$ at $p$ (hence by $F$ and $C$ at $p$, cf. b) of (7.20) and (7.21)) is not approaching zero. This is precisely the condition given in the lemma. For the rigor of this argument one may use law of cosines without difficulty. Hence the lemma follows.

\[\text{proof of claim (7.15). By combining Lemma 7.5 and Lemma 7.3 we can finish the proof of the claim (7.15) provided that } h_{j}(z_1, z_2) = d_{L_{j}}(z_1, z_2). \text{ But this is a standard fact for the heat kernels of Dirac and Laplacian type (see [5, Theorem 2.29]); see also the famous result of S. R. S. Varadhan[59] for a generalization in this regard.}\]

\[\text{proof of Proposition 7.2 resumed.}\]

We are now ready to prove case b) of iii) of Proposition 7.2. To work on the integral $I$ of (7.1) we are going to refine the computation contained in (7.8). Indeed, case b) of iii) can be obtained if one notes the following $\alpha) - \epsilon)$ (part of them similar to the proof of i) of this proposition):

\[
\alpha) \int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}; \quad \beta) \text{ using (7.14) (with } q = 1, 2, \ldots, \ell - 1 \text{ for } e^{-\frac{h_{j}}{b_{j}}}}, \quad \gamma) \text{ change of variable } u = \psi + \frac{2\pi}{\rho} \text{ in (7.1)}; \quad \delta) \text{ in (7.1), by rescaling } \hat{y} = \sqrt{\hat{\eta}}, \hat{b}_{j,s}((\sqrt{\hat{\eta}}, Y), e^{-i\omega} ((\sqrt{\hat{\eta}}, Y) \text{ replacing } \hat{b}_{j,s}(x, e^{-i\omega} o, x) \text{ tends to } \hat{b}_{j,s}^+(0, Y), e^{-i\omega} o (0, Y)) = \hat{b}_{j,s}^+(0, Y), e^{-i\omega} o (0, Y)) \text{ because } (0, Y) \in \bar{X}_{p_{\ell}}. \text{ But } |\psi| \text{ being small } (\leq \epsilon), e^{-i\omega} o (0, Y) \text{ does not change the } z((0, Y)) \text{ coordinate in } \hat{D}_{j}, \text{ giving } \hat{b}_{j,s}^+(0, Y), e^{-i\omega} o (0, Y)) = \hat{b}_{j,s}^+(z((0, Y))), e^{-i\omega} o (0, Y)) \text{ (up to cut-off functions)}; \quad \epsilon) \text{ as } t \to 0, \sigma_{j}(\eta) = \sigma_{j}(\theta(e^{-i\omega} o (0, Y))) = \sigma_{j}(\theta(e^{-i\omega} o (0, Y))) = \sigma_{j}(\theta(Y)) + \psi. \text{ With } \eta = \theta + \psi \text{ and } u = \psi + 2\pi/\rho \text{ in (7.1) and (7.2), a cancellation occurs for the three exponentials there; eventually a numerical factor } e^{-2\pi m/\rho} \text{ is pulled out. And instead of } \text{Tr } \hat{b}_{j,s}^+ \text{ in } I \text{ of (7.1), we are reduced to } \chi_{j} \text{Tr } \hat{b}_{j,s}^+ \text{ (no “hat” on } \hat{b}_{j,s}^+ \text{ here) as put down in this proposition.}\]

The formula for the coefficient $b_{j,s}^{(j)}$ of (7.3) follows from $\alpha) - \epsilon$ above.
Finally, for $s = n$, it is well-known that (dropping $j$ here) $\text{Tr} b_{1 n}^+(z, z)$ in the integral (7.3) being the leading coefficient term in the asymptotic expansion of the ($\text{Spin}^c$) Kodaira heat kernel, is constant in $z$ and equals $(4\pi)^{-n} \cdot (\text{rk}(\mathcal{A}''(U))) = (2\pi)^{-n}$ (cf. Theorem 4.4.1). [36, (a) of Theorem 2.4.1]. \hfill \Box

7.3. Global angular integral. To work out the global version (i.e. the integration on $[0, 2\pi]$) it is natural to consider not only (an $\varepsilon$-neighborhood of) $2\pi/p_\ell$ but also all their multiples $s2\pi/p_\ell$, $s \in \mathbb{N}, s \leq p_\ell$. The analysis will thus partly depend on whether $s2\pi/p_\ell = s'2\pi/p_{\ell'}$ for some $s, s', p_\ell, p_{\ell'}$ or not. One needs a systematic control of the behavior in this regard. Further, since the result will appear as a sum over these $\varepsilon$-neighborhoods, to organize this sum in a manageable way is also desirable. We shall now mainly deal with these issues in this subsection.

There are minor duplication and perhaps discrepancy in notation between this subsection and the preceding subsections, but it would have appeared cumbersome if we had set up this generality right in the preceding subsection (as the resulting proof would have become much less illuminating).

Recall the $S^1$-period of $X$ denoted by $\frac{2\pi}{p_\ell} > \frac{2\pi}{p_{\ell'}} > \ldots > \frac{2\pi}{p_k}$ with $p_1|p_{\ell}$, $1 \leq \ell \leq k$ (cf. Remark 1.16) and the stratum $X_{p_\ell}$ (the set of points of period $\frac{2\pi}{p_\ell}$) is a disjoint union of connected submanifolds $\bigcup_{1 \leq \ell \leq s} X_{p_\ell(p_\gamma)}$.

**Definition 7.6.** Fix a smooth function $g(x) \neq 0$ on $X$. We say that $g$ is of small support if the following conditions are satisfied.

\begin{enumerate}
  \item $\text{Supp } g \subset X_{p_1(p_\gamma_1)} \cup X_{p_2(p_\gamma_2)} \cup \ldots \cup X_{p_{i}(p_{\gamma_i})} \cup X_{p_{i+1}(p_{\gamma_{i+1}})}$, $1 \leq i \leq i_2 < \ldots < i_{t+1} \leq k$ (7.22)
  \item $\text{Supp } g \cap X_{p_{s}(p_\gamma_s)} \neq \emptyset$, $\forall s, 1 \leq s \leq t+1$
  \item $\overline{X}_{p_1(p_\gamma_1)} \supset \overline{X}_{p_2(p_\gamma_2)} \supset \ldots \supset \overline{X}_{p_{i}(p_{\gamma_i})} \supset \overline{X}_{p_{i+1}(p_{\gamma_{i+1}})}$.
\end{enumerate}

Obviously, given any $x_0 \in X$ there exists a neighborhood $\Omega \ni x_0$ such that every nontrivial $g(x) \in C_0^\infty(\Omega)$ is of small support in the sense above.

**Definition 7.7.** Let $g(x)$ be a smooth function on $X$ of small support in the sense above, (7.22). Let $c \in \mathbb{N}$. We define a number $i(c, g) = i(c)$ associated with $c$ and $g$ as follows.

\begin{enumerate}
  \item $i(c, g) := \ell \geq 2$ if the following is satisfied a) $c|p_\ell$, $\ell = i_2$ for some $s$, $2 \leq s \leq t + 1$ and b) $c \not{|} p_{s'}$ for all $s' < s$.
  \item $i(c, g) := 1$ if $c|p_1$ ($p_1 = p_{i_1}$). This is independent of $g$.
  \item $i(c, g) := \infty$ if $c \not{|} p_{i_s}$ for each $s$ with $1 \leq s \leq t + 1$.
\end{enumerate}

It is easily seen that $p_{i_s}|p_{i_s}$ for each $i_s$ with $i(c) \leq i_s \leq i_{t+1}$ if $i(c) \neq \infty$. Indeed, $p_{i_1}|p_{i_2}| \ldots |p_{i_{t+1}}$ (cf. Remark 1.16 for a similar case).

By the above definition, one sees

**Lemma 7.8.** Let $x \in \text{Supp } g$, $i(c) \neq \infty$ and $h \in \mathbb{N}$ with $(h, c) = 1$. It holds $e^{-\frac{2\pi h}{c} x} \circ x = x$ if and only if $x \in \overline{X}_{p_{i(c)}}$.

Let $h \in \mathbb{N}$ with $(h, c) = 1$ and $h < c$. We consider the integral similar to (7.1) for $i(c) \neq \infty$:

\begin{equation}(7.23) I = I^{(j)}(p_{i(c)}, g(x), \frac{h}{c}) = \frac{1}{2\pi} \int_{\frac{2\pi h}{c} - e}^{\frac{2\pi h}{c} + e} \int_X g(x) e^{-\frac{b_{j,x}(x, c, -e^{-i\alpha})}{i} x} \text{Tr} b_{j,x}(x, e^{-iu} x) e^{-im\alpha} dv_X(x) du.\end{equation}

The above extends to the case $i(c) = \infty$ simply by formally setting $I^{(j)}(p_{i(c)}=\infty, g(x), \frac{h}{c})$ to be the integral to the right of (7.23).

**Definition 7.9.** i) Set $i(c)$ of Definition 7.7 to be $\ell$. Assume $\ell \neq \infty$. Define the type $\tau(I(p_{i(c)}, g(x), \frac{h}{c}))$ to be $\tau(I(p_{i(c)}, g(x)))$ where $\tau(I(p_{i(c)}, g))$ is given in i) of Definition 7.1. ii) The notions of simple type and class are defined similarly. (Note there is no trivial type for $\ell \neq \infty$.) iii) If $i(c) = \infty$, then $\text{Supp } g \cap X_a = \emptyset$ where $X_a$ is the fixed point set of $a = e^{-2\pi/c} \in S^1$ (whether $X_a$ is empty or not depends on $c$). In this case we define it to be the trivial type (cf. iii) of Definition 7.1).
Remark 7.10. The notion of type concerns only the local stratification (of the $S^1$ action) at $x_0$ around which $\Omega \supset \text{Supp }g$ is a small neighborhood. Thus, with a small open subset $\Omega \subset X$ one may associate the type $\tau(\Omega)$ without referring to any kind of integral. In the fourth subsection, we will basically adopt this viewpoint for our purpose.

First, one examines the case $i(c, g) = \infty$, which turns out to be inessential.

Lemma 7.11. Let $c \in \mathbb{N}$. There exists a (finite) covering of BRT trivializations on $X$ with the following property. For any smooth function $g$ of small compact support (cf. Definition 7.6), suppose $i(c, g) = \infty$ and $x_0$ with $g(x_0) \neq 0$, is given. Then there exist a small open set $\Omega \ni x_0$ and a small $\varepsilon > 0$ such that for any $\chi \in C^\infty_0(\Omega)$, if one replaces $g$ by $\chi g$ in the integral $I$ of (7.23), this $I$ with any $0 < \varepsilon \leq \varepsilon$, equals 0, or $O(t^\infty)$ as $t \to 0^+$.

Proof. By the definition of $i(c, g) = \infty$, if the fixed point set $X_\alpha$ of $a = e^{-i2\pi/c}$ is empty, one chooses a covering of BRT trivializations $\{D_j\}_{j}$ (cf. lines above Subsection 7.2) such that $D_j \cap a^h(D_j) = \emptyset$ for all $j$ and $h$ with $(h, c) = 1$. In this case the remaining argument by using the continuity of the $S^1$ action, is almost the same as ii) of Proposition 7.2, yielding $I = 0$. If $X_\alpha$ is not empty, one chooses any (finite) covering of BRT trivializations (such as the one given prior to Subsection 7.2). Then it may occur the extra case $a^h \circ x_0 \in D_j$ for some $D_j \ni x_0$ if the choice of $g$ is such that $x_0$ is very near $X_\alpha$. In this case the remaining argument is essentially similar to Case a) of iii) of Proposition 7.2, giving rise to $I = O(t^\infty)$ as $t \to 0^+$. \hfill \Box

To compute $I$ of (7.23), we assume the simple type condition for $I$ (when it is not of trivial type) as given in Definition 7.9. Combining Lemma 7.11 we have the following corollary, as a generalization of Proposition 7.2.

Corollary 7.12. Notations and the simple type condition as above. Assume that the covering by BRT trivializations satisfies Lemma 7.11. Let $c \in \mathbb{N}$, $x_0 \in X$, $\Omega \ni x_0$ an open subset and $g \in C^\infty_0(\Omega)$ (of small support as above). Then the $\varepsilon > 0$ (in I) and $\Omega$ can be chosen to satisfy the following. a) The same results (for computing $I = I^{(j)}(p_i(c), g(x), \frac{1}{c})$ of (7.23)) hold true as in Proposition 7.2 provided that one adopts the replacement of $e^{-i2\pi/p_i} e^{-i2\pi/c}$ by $e^{-i2\pi/c}$ in i), ii) and iii) of the statement, and $\ell$ (not the one in $\frac{2\pi}{p_i}$) by $i(c)$ in Cases a), b) of Case b) throughout (so $p_i(e^{\ell}) \to p_i(e^{\ell}+1) \to p_i(e^{\ell}+q) \to p_i(e^{\ell}+q+1)$ and $\gamma_\ell \to \gamma_i(\ell)$ of Case b) of Case b) of Case b)), Note that after the replacement, $i(c) = \infty$ in Case a) and $i(c) \neq \infty$ in Case b) of Case b) of Case b).

b) More generally, for $h \in \mathbb{N}$, $(h, c) = 1$ and $h < c$, with the replacement $\frac{2\pi}{p_i} \to \frac{2\pi}{p_i} h$ (and $\ell$ not the one in $\frac{2\pi}{p_i}$) by $i(c)$ in Cases a), b) of iii)), the same results (for computing $I = I^{(j)}(p_i(c), g(x), \frac{h}{c})$ of (7.23)) hold true as well.

proof of Corollary 7.12. One sees that with the replacement of $\frac{2\pi}{p_i}$ by $\frac{2\pi}{p_i} h$ or $\frac{2\pi}{p_i} h$, the condition on $c$ (in Definition 7.7) renders the argument in proof of Proposition 7.2 essentially unchanged. For instance, with (7.4) replaced by $\frac{2\pi}{p_i} h - \theta_0 = m \frac{2\pi}{p_i},$ taking $\zeta$ smaller does the job. Further, with substitution of $\ell$ (not in $\frac{2\pi}{p_i}$) by $i(c)$, the distinct eigenvalues $\{1, e^{i \frac{2\pi}{p_i} p_{-1}}, e^{i \frac{2\pi}{p_i} p_{-2}}, \ldots, e^{i \frac{2\pi}{p_i} p_{1}}\}$ of the isotropy action (of $e^{-i2\pi/c}$ at $x_0$) (cf. (7.13)) are changed to $\{1, e^{i \frac{2\pi}{p_i} p_{-1}}, e^{i \frac{2\pi}{p_i} p_{-2}}, \ldots, e^{i \frac{2\pi}{p_i} p_{1}}\}$ (of $e^{-i2\pi/c}$ at $x_0$) (which remain distinct). \hfill \Box

Let $c \in \mathbb{N}$ and $g$ a smooth function on $X$ of small support as above, with $i(c) (= i(c, g))$ in Definition 7.7. We are going to associate certain numerical factors. For a contrast, we will give them for cases of the simple type and the general type separately (cf. Definitions 7.1 and 7.9).
For the simple type, the numerical factor $d_c = d_{c,g,m}$ is set to be
\[ (7.24) \]
\[
i) \quad i(c) = 1
\]
if \( c > 1 \), \( d_c = d_{c,g,m} := \sum_{h \in \mathbb{N}, h < c(h,c)=1} e^{-\frac{2\pi i}{c} hm} \); if \( c = 1 \), \( d_c := 1 \).

\[
n) \quad \infty > i(c) \geq 2
\]
\[
d_c := d_{c,g,m} := \left(\sqrt{\pi}\right)^{c(e)} \sum_{h \in \mathbb{N}, h < c(h,c)=1} \frac{e^{-\frac{2\pi h}{c} m}}{\prod_{q=1}^{i(c)-1} \left| e^{\frac{2\pi h}{c} t_q} - 1 \right|^{e_{q-1}(c) - e_q(c)}}
\]

\[
ni) \quad i(c) = \infty
\]
\[d_c := d_{c,g,m} := 1.\]

**Remark 7.13.** For the general type, the $d_{c,g,m}$ for $i(c) \geq 2$ should be modified as follows.

In notation of Definition 7.9, let it be given \( \tau(I(p_i,g,h,\frac{1}{2})) = (i_1(\gamma_{i_1}), i_2(\gamma_{i_2}), \ldots, i_f(\gamma_{i_f}), i_{f+1}(\gamma_{i_{f+1}})) \) where \( i_1 = \gamma_{i_1} = 1, i_{f+1} = i(c) \neq \infty \), say. (In the previous Definition 7.1, \( i_{f+1} = \ell \). Here we have \( i_{f+1} = i(c) \).)

One sees that the eigenvalues of the isotropy action of \( e^{-\frac{2\pi i}{c} h} \) (at \( x_0 \in X_{p_i(g,\gamma_{i(c)})} \)) are:
\[ (7.25) \]
\[
e^{\frac{2\pi h}{c} p_{i_1}}, e^{\frac{2\pi h}{c} p_{i_2}}, \ldots, e^{\frac{2\pi h}{c} p_{i_f}}, 1
\]
(because by writing \( e^{-\frac{2\pi i}{c} h} = \left( e^{-\frac{2\pi}{c} r} \right)^j \) if \( h/c = j/p_i \) with \( \ell = i(c) \), the eigenvalues are \( \lambda^j \) where \( \lambda = e^{i\omega p_{i_1}}, e^{i\omega p_{i_2}}, \ldots, \omega = \frac{2\pi}{p_i} \), cf. (7.13)) with multiplicity (where \( e_{i_1}(\gamma_{i_1}) = 0 \))
\[ (7.26) \]
\[
e_{i_2(\gamma_{i_2})}, \ldots, e_{i_f(\gamma_{i_f})} - e_{i_1(\gamma_{i_1})}, e_{i(c)(\gamma_{i(c)})} - e_{i_1(\gamma_{i_1})}, \dim X - e_{i(c)(\gamma_{i(c)})}.
\]

Write, for \( 1 \leq r \leq f \),
\[ (7.27) \]
\[
\lambda_r := e^{\frac{2\pi h}{c} p_{i_r}}, \quad m_r := e_{i_{r+1}(\gamma_{i_{r+1}})} - e_{i_r(\gamma_{i_r})} \quad (\text{with } i_{f+1} := i(c)).
\]

**Numerical factors** $d_c$ **for the general type.** Given \( \tau(I) = (i_1(\gamma_{i_1}), i_2(\gamma_{i_2}), \ldots, i_f(\gamma_{i_f}), i_{f+1}(\gamma_{i_{f+1}})) \) of general type, \( I = I(p_i,g,h,\frac{1}{2}) \) with \( i(c) = \ell \), the numerical factors $d_c$ similar to (7.24) are defined as follows.

If \( c \) is of \( i(c) = 1 \) or \( \infty \), then $d_c = d_{c,g,m,l}$ is the same as $d_c$ in i), iii) of (7.24). For $c$ with \( \infty > i(c) \geq 2 \),
\[ (7.28) \]
\[
\infty > i(c) = i(c,g) \geq 2, \quad d_c := d_{c,g,m,l} = d_{c,g,m,\tau(l)} = d_{c,g,m,\tau(\Omega)}
\]
\[
:= \left(\sqrt{\pi}\right)^{c(e)(\gamma_{i(c)})} \sum_{h \in \mathbb{N}, h < c(h,c)=1} \frac{e^{-\frac{2\pi h}{c} m}}{\prod_{r=1}^{i(c)-1} |\lambda_r - 1|^{m_r}}
\]
where we recall \( e_{i(c)(\gamma_{i(c)})} = \text{codim } X_{p_i(g,\gamma_{i(c)})} \) and we refer $\tau(\Omega)$ to Remark 7.10.

Note that the $\Omega \supset \text{Supp } g$ is chosen to be small enough so that there is no mixing of types. The factor $d_c$ of (7.28) is well defined.

We turn now to the global version (over the entire $[0,2\pi)$) of the integral $I$ (of (7.1) or (7.23)).

Let
\[ (7.29) \]
\[
J_s = J_s^j = J_{s,n_i^j}(g(x)) = \frac{1}{2\pi} \int_X g(x)^{-\frac{1}{2} b_{j,x}(x,c-e^{-iu})} \text{Tr} b_{j,x}^+(x, e^{-iu} \circ x) e^{-iu} dv_X(x)
\]
where $g(x) \in C_0^\infty(\Omega)$ with $\Omega$ satisfying Proposition 7.2. We assume $g(x)$ is of small support in the sense of Definition 7.6.

We are going to compute \( \int_0^{2\pi} J_s du \), a $[0,2\pi]$-integral version of the integral $I$ in (7.1).
To this aim, the main tool is the corollary below which is a reformulation of Corollary 7.12. But prior to this, let’s write, for $x_0 \in X$, $g(x) \in C_0^\infty(\Omega)$ with $\Omega$ a small neighborhood at $x_0$,

\[(7.30)\]

\[
\ell \geq 1, \quad \ell_{(\gamma),s} = \ell_{X,P(\gamma),s,m}^{(j)}(g(x)) \quad (z(Y) = z(x) \text{ below}; \quad b_{j,s}^+ = b_{j,s,m}^+ \text{ without “hat” over it})
\]

\[
= \frac{1}{2\pi} \int_{-\epsilon}^\epsilon \int_{X_{P(\gamma)}} g(Y)\chi_j(Y)\text{Tr} b_{j,s}^+(z(Y), z(Y))\tau_j(z(Y))\sigma_j(\theta(Y) + u)\nu_{X,P(\gamma)}(Y)du \quad \text{or } \sim O(t^\infty) \quad (\text{as } t \to 0^+).
\]

The proof of the previous results in the case of simple type remains basically unchanged for the case of general type. With the numerical factor $d_e$ introduced in (7.24) and (7.28), one sees the following.

**Corollary 7.14.** Notations as above with the general type $\tau(1)$ allowed (cf. Definitions 7.1 and 7.9). Assume that the covering by BRT trivializations $\{\hat{D}_j\}_j$ satisfies Lemma 7.11. Let $c \in \mathbb{N}$ and $x_0 \in X$. For an $\epsilon$, write $\lambda_c := \bigcup_{\epsilon \in \mathbb{N}, \epsilon < c, (\epsilon, c) = 1} 2\epsilon^2 - \epsilon, 2\epsilon^2 + \epsilon$ for $c > 1$ and $\lambda_1 := -\epsilon, \epsilon$ for $c = 1$. Write $\Omega \subset X$ for an open subset with $x_0 \in \Omega$. Then the $\epsilon > 0$ and $\Omega$ can be chosen to satisfy the following. (Recall that with respect to $\{\hat{D}_j\}_j$ we write $J_s^{(j)} = J_s^{(j)}(g)$ for any given $g \in C_0^\infty(\Omega)$ of small support in the sense of Definition 7.6.)

i) Suppose $x_0 \in X_{P_1}$. Then

\[
\text{if } i(c) = 1, \quad \int_{\lambda_c} J_s^{(j)}du = d_eI_{X,s,s}^{(j)};
\]

\[
\text{if } i(c) \geq 2 \; (\text{giving } i(c) = \infty \text{ here}), \quad \int_{\lambda_c} J_s^{(j)}du = 0 \; \text{or } \sim O(t^\infty) \; (\text{as } t \to 0^+).
\]

The $\ell \geq 2$ in the following ii) and iii) is such that $x_0 \in X_{P_1}$ (so $x_0 \in X_{P(\gamma),j}$ for some $\gamma$).

ii) Assume $e^{-\frac{2\epsilon^2}{c}} \cdot \chi_j(0) \notin \hat{D}_j$ (giving $i(c) = \infty$). Then $\int_{\lambda_c} J_s^{(j)}du = 0$.

iii) Assume $e^{-\frac{2\epsilon^2}{c}} \cdot \chi_j(x_0) \in \hat{D}_j$. Then (as $t \to 0^+$)

\[
\text{if } i(c) \geq \ell + 1 \; (\text{giving } i(c) = \infty \text{ here}), \quad \int_{\lambda_c} J_s^{(j)}du = 0 \; \text{or } \sim O(t^\infty);
\]

\[
\text{if } 2 \leq i(c) \leq \ell, \quad \int_{\lambda_c} J_s^{(j)}du \sim d_eI_{(c)(\tau(1),s)}^{(j)}(\sqrt{t}^{-\tau(1,1)}(c) + O(\sqrt{t}^{\tau(1,1)}(c)) + 1);
\]

\[
\text{if } i(c) = 1, \quad \int_{\lambda_c} J_s^{(j)}du = d_eI_{X,s,s}^{(j)};
\]

where we note $d_e = d_{c,g,m,T}(\Omega)$ cf. (7.28), also written as $d_{c,g,m,l}$ with $I = I_{(c)(\tau(1),s)}^{(j)}$.

We are now ready to compute $j_0^{2\pi} J_sdu$.

Let $\lambda_c$ be as in Corollary 7.14 (with a given $g(x)$ of small support).

\[(7.31)\]

\[
\Lambda = \Lambda_1 \cup \Lambda_2 \quad \text{with } \Lambda_1 = \bigcup_{\epsilon \in S_1} \lambda_c, \quad \Lambda_2 = \bigcup_{\epsilon \in S_2} \lambda_c; \quad N = [0, 2\pi] \setminus \Lambda.
\]

By using (7.31) and corollaries above one has the following.

**Proposition 7.15.** Suppose we are given any $\delta > 0$ and an $s = n, n - 1, \ldots$. Then there exists a (finite) covering $\{\hat{D}_j\}_j$ of $X$ by BRT charts that satisfy Lemma 7.11 and the following. Suppose $x_0 \in X$ and $\Omega$ a neighborhood of $x_0$ with $\Omega \subset \hat{D}_j$ for every $\hat{D}_j \ni x_0$. Then there exist an $\epsilon > 0$ and a choice of $\Omega$ above.
such that for \( g(x) \in C_0^\infty(\Omega) \) (of small support in the sense of Definition 7.6), writing \( J_s^{(j)} = J_{s,m}^{(j)}(g(x)) \) (for any \( D_j \ni x_0 \), \( s = n, n-1, \ldots \) as above) one has the following.

i) \( \int_{\Lambda_1} J_s^{(j)} \, du = (\sum_{q=1}^{p_1} e^{-\frac{i2\pi q m}{p_1}}) J_{X,s}^{(j)}. \)

ii) Case a) If \( x_0 \in X_{p_1} \), \( \int_{\Lambda_2} J_s^{(j)} \, du \sim 0 \) or \( O(t^\infty) \) (as \( t \to 0^+ \)).

Case b) If \( x_0 \in X_{p_i} \) with \( \ell \geq 2 \),
\[
\int_{\Lambda_2} J_s^{(j)} \, du \sim \left( \sum_{2 \leq i(\ell) \leq \ell} (d_c^{i(\ell)}(\gamma_{i(\ell)}), s) \sqrt{t^{\epsilon_i(\ell)}(\gamma_{i(\ell)})} + O(\sqrt{t^{\epsilon_i(\ell)}(\gamma_{i(\ell)})+1}) + O(t^\infty) \right) \quad (\text{as } t \to 0^+)
\]
where \( d_c = d_{c,g,m,I}, I = I^{(j)}_m(\gamma_{i(\ell)}), s) \), cf. (7.28) and (7.30).

iii) For the part \( \int_N J_s^{(j)}(g) \, du \equiv \eta_s^{(j)}(g) \), the estimate holds \( |\eta_s^{(j)}(g)| < \delta \cdot \int_X |g(x)|dv_X(x) \).

Proof: To see i), by i) and iii) of Corollary 7.14 it suffices to note \( \sum_{c,i(\ell)=1} d_c = \sum_{q=1}^{p_1} e^{-\frac{i2\pi q m}{p_1}} \) by (7.24). Case a) of ii) follows directly from i) of loc. cit. whereas Case b) from iii) of loc. cit. by writing \( \int_{\Lambda_2} J_s \, du = \sum_{2 \leq i(\ell) \leq \ell} \int_{\Lambda_c} J_s \, du + \sum_{i(\ell) \geq \ell+1} \int_{\Lambda_c} J_s \, du. \)

To see iii), note the following. First, the action by \( e^{-i\theta} \) with \( \theta \in N \) is fixed point free on \( X \). Each point in \( X \) has a distinguished neighborhood \( \Omega \) such that if \( x \in \Omega \) and \( \theta \in N \), \( e^{-i\theta} \circ x \in \Omega \) by using continuity of the \( S^1 \) action. By compactness, we can now assume (reset) a (finite) covering of BRT charts consisting of these \( \Omega \) and satisfying Lemma 7.11 (as there are only finitely many \( c \) here). Given \( x_0 \in X \), we take a small neighborhood \( \Omega \subset D_j \) for every \( D_j \ni x_0 \). One sees that by the cut-off function \( \tau_j \) involved in the integrand of \( J_s^{(j)}, J_s^{(j)} \equiv 0 \) identically (since \( \tau_j(e^{-i\theta} \circ x) = 0 \) if \( e^{-i\theta} \circ x \notin D_j \)), cf. (5.37), which holds for \( x \in \Omega = D_j \) and \( \theta \in N \).

However, a word of warning is in order. Given the preceding covering of certain BRT charts, our integral \( J_s^{(j)} \) shall be computed with respect to this new covering (because as said above, the covering has been “reset”) which we just did. However, it is not necessarily true that the original \( \varepsilon \) (of Corollary 7.14 used above) remains altogether applicable in the new setting. Namely, under the new BRT covering we need to choose an \( \varepsilon_1 \), possibly smaller than \( \varepsilon \), to ensure that the original argument of the proof (of Proposition 7.2) go through well. To see what the above means, let’s make the dependence on the parameter \( \varepsilon \) explicit and denote by \( \Lambda(\varepsilon), N(\varepsilon) \) etc. The \( N \) in the last paragraph will be denoted by \( N(\varepsilon) \). Only with the replacement by \( \Lambda(1), \Lambda(2), \Lambda(3) \subset \Lambda(1), \Lambda(2), \Lambda(3) \) respectively), \( N(1) \cap N(\varepsilon(1)) \) and the new BRT covering throughout the present proposition, can we obtain the corresponding \( \varepsilon_1 \)-versions of i) and ii) of this proposition. And iii) will correspondingly be replaced by \( \int_{N(\varepsilon)} J_s^{(j)} \, du \). But as just shown in the last paragraph, \( \int_{N(\varepsilon)} J_s^{(j)} \, du = 0 \), it follows \( \int_{N(\varepsilon)} J_s^{(j)} \, du = \int_{N(\varepsilon)} N(\varepsilon) J_s^{(j)} \, du. \) Now the measure of \( N(\varepsilon) \setminus N(\varepsilon) \) is controlled by \( C \cdot (\varepsilon - \varepsilon_1) \) for a fixed constant \( C \); the integrand of \( J_s^{(j)} \) can be bounded in a way independent of \( \varepsilon, \varepsilon_1 \) and the BRT covering. By choosing a sufficiently small \( \varepsilon \) beforehand, iii) (for \( \varepsilon_1 \)-version) follows. We have proved the proposition with \( \varepsilon_1 \) (in place of the previous \( \varepsilon \)).

\[ \square \]

7.4. Patching up angular integrals over \( X \); proof for the simple type. We are going to study the main issue
\[ \int_X \text{Tr} a_s^+(t, x, x)dv_X(x) \]
where we recall (by (5.41))
\[ a_s^+(t, x, y) = a_{s,m}^+(t, x, y), \quad \hat{b}_{s,m}^+ = \hat{b}_{s,m}^+ \]
\[ = \frac{1}{2\pi} \sum_{j=1}^{N} \int_{-\pi}^{\pi} e^{-i\hat{b}_{j,s}^+(x, e^{-iu}y) t} \hat{b}_{j,s}^+(x, e^{-iu}y)e^{-imu}du, \quad s = n, n-1, n-2, \ldots. \]
For this, one would like to patch up those integrals $\int_0^{2\pi} J_s^{(j)}(x) du$ of the last subsection over $j$. However, $\alpha^+_{s}(t, x, y)$ is not canonically defined by our method and is in fact dependent on the choice of BRT charts. A direct study of it appears inefficient (unless one sticks to a fixed covering of BRT charts).

It turns out to be more effective if instead, one studies its equivalence (cf. (5.54) in the asymptotic sense):

\begin{equation}
(7.34) \quad \int_X \operatorname{Tr} e^{-t\Xi^+_{b,m}(x,x)} dv_X(x)
\end{equation}

in which $e^{-t\Xi^+_{b,m}(x,y)}$ is of course independent of choice of BRT charts.

Suppose a $\delta > 0$ and an $s = n, n - 1, \ldots$ are given. Assume that the BRT covering $\{\hat{D}_j\}_j$ satisfies Proposition 7.15 in which by using compactness, one can find a (finite) covering $\{\Omega_\alpha\}_\alpha$ of $X$, $\Omega_\alpha \subseteq \hat{D}_j$ if $\hat{D}_j \cap \Omega_\alpha \neq \emptyset$, and an $\varepsilon > 0$ such that the conclusion i), ii) and iii) of that proposition hold with each of these $\Omega_\alpha$ and this $\varepsilon$. As indicated in Proposition 7.2, whenever necessary, one can shrink the size of $\Omega_\alpha$ without changing $\varepsilon$. For $\rho = \xi$, we assume (possibly after shrinking $\Omega_\alpha$ and using compactness) for each $\alpha, j$, and for some (possibly big) $m > 1$,

\begin{equation}
(7.35) \quad \theta\text{-coordinates of } \Omega_\alpha, \hat{D}_j \text{ lie inside of } [-\rho, \rho], [-m\rho, m\rho] \text{ respectively.}
\end{equation}

Let $\{g_\alpha(x)\}_\alpha$ be a partition of unity subordinate to this covering (i.e. $\operatorname{Supp} g_\alpha \subseteq \Omega_\alpha$). We further assume each $g_\alpha$ is of small support in the sense of Definition 7.6. One sees that as $t \to 0^+$

\begin{equation}
(7.36) \quad \int_X g_\alpha(x) \operatorname{Tr} e^{-t\Xi^+_{b,m}(x,x)} dv_X(x) \sim \sum_j \sum_{s=n,n-1,\ldots} t^{-s} \int_0^{2\pi} J_s^{(j)}(g_\alpha(x)) du
\end{equation}

where the term to the right is computed with respect to any given BRT covering of $X$, including but not restricted to, the previous $\{\hat{D}_j\}_j$. Hence at each stage of the computation we may choose convenient BRT charts for the need (as far as the asymptotic expansion as $t \to 0^+$ is concerned).

By Proposition 7.15, (7.36) is reduced to computing $I_{hs}(g_\alpha)$ (see (7.30)) (for a fixed $g_\alpha$).

Henceforth, in the following we fix an (arbitrarily given) $\alpha$. As aforementioned, we are free to reset the BRT charts $\{\hat{D}_j\}_j$ (with certain cut-off functions). To do so, we make the following definition for convenience.

**Definition 7.16.** Fix an $x_0 \in X$. $\{\hat{D}_j\}_j (\hat{D}_j \subseteq D_j$ etc. notations as in the beginning of this section) a (finite) covering of $X$, is said to be a covering by distinguished BRT charts at $x_0$ provided that $x_0 \in \hat{D}_j$ for some $j$ and $x_0 \notin \hat{D}_k$ for all $k \neq j$.

Now, we can further assume that for the above fixed $\alpha$ and for an $x \in \Omega_\alpha$, $\{\hat{D}_j\}_j$ is distinguished at $x$ in the sense of Definition 7.16. In fact we can assume a little more that $\Omega_\alpha \subseteq \hat{D}_{j_0}$ and $\Omega_\alpha \cap \hat{D}_k = \emptyset$ for $k \neq j_0$; namely $\{\hat{D}_j\}_j$ is distinguished at $x$ for each $x \in \Omega_\alpha$. Also, we assume that (7.35) is satisfied.

We shall now choose the cut-off function $\sigma_{j_0}$, in notation of (7.30), that satisfies (see lines above (5.36))

\begin{equation}
(7.37) \quad \int_{-\rho}^{\rho} \sigma_{j_0}(u) du = \int_{-\rho}^{\rho} \sigma_{j_0}(u) du = 1, \quad \operatorname{Supp} \sigma_{j_0} \subseteq ]-\rho, \rho[\end{equation}

and choose $\chi_{j_0} \equiv 1$, so $\tau_{j_0} \equiv 1$, on $\overline{\Omega_\alpha}$ (see loc. cit.).

With the above setup, some simplifications for (7.36) occur. Firstly,

\begin{equation}
(7.38) \quad \int_X g_\alpha(x) \operatorname{Tr} e^{-t\Xi^+_{b,m}(x,x)} dv_X(x) \sim \sum_{s=n,n-1,\ldots} t^{-s} \int_0^{2\pi} J_s^{(j_0)}(g_\alpha(x)) du.
\end{equation}

We are reduced, by Proposition 7.15, to computing the integrals in (7.30).
Secondly, in notation of (7.30) there is an angular integral
\[
\int_{-\varepsilon}^{\varepsilon} \sigma_{j_0}(\theta(Y) + u) du.
\]
For a fixed \( Y \in \Omega_\alpha, \theta(Y) \in [-\rho, \rho] \) by (7.35), and for \( u \) going through \([-\varepsilon, \varepsilon] = [-2\rho, 2\rho]\) of (7.39), one sees that \( \theta(Y) + u \) covers \([-\rho, \rho]\), it follows from (7.37) that the angular integral (7.39) is 1.

Thirdly, by the above condition on \( \chi_{j_0} \) and \( \tau_{j_0} \) one obtains, with (7.39) \( \equiv 1 \), the following for (7.30).
\[
I_{j_0}^+((x), s, n, \alpha, \delta, \tau_{j_0}^*) (g_\alpha(x)) = \frac{1}{2\pi} \int_{X_{\rho(\gamma^\ell)}} g_\alpha(Y) \alpha_{s,m}^+(Y, Y) dv_X^{\rho(\gamma^\ell)}(Y)
\]
\((\ell \geq 1; \ s = n, n - 1, \ldots).\)

Finally, recall \( \alpha_s^+(x) \) as in our main result Theorem 1.3 (cf. Theorem 6.1) defined in (6.1) which is independent of choice of BRT charts and cut-off functions (cf. Remarks 1.6 and 5.7). Indeed one sees, for \( x \in \Omega_\alpha, \)
\[
\frac{1}{2\pi} b_{j_0,s}^+(z(x), z(x)) = \alpha_s^+(x) (= \alpha_{s,m}(x))
\]
by (6.2) and the choice of distinguished BRT charts here.

In sum, since the above applies to each \( \Omega_\alpha \) in the covering \( \{\Omega_\alpha\}_\alpha \), by (7.40) and (7.41) we have reached the following invariant expressions (independent of choice of BRT coverings)
\[
\int_X g_\alpha(x) \text{Tr} e^{-tC_{b,m}}(x, x) dv_X(x) \sim \sum_{s=n,n-1,...} t^{-s} A_s(g_\alpha) \quad \text{where } A_s(g_\alpha) \text{ is given by}
\]
\[
A_s(g_\alpha) = \left( \sum_{q=1}^{p_1} e^{-d_{c,s,m}t} \right) S_{\ell,s}^+(g_\alpha) + \sum_{2 \leq i(c) \leq \ell} (d_c S_{i,c}^+(\gamma_{i,c}), s) (g_\alpha) \sqrt{t^i(c)(\gamma_{i,c})} + O(\sqrt{t^{i(c)(\gamma_{i,c})+1}})
\]
\[
+ \eta_s(g_\alpha) \quad \text{where } i(c) = i(c, g_\alpha), d_c = d_{c,g_\alpha,m,I}, I = S_{\ell,c}^+(\gamma_{i,c}), s) (g_\alpha) \text{ by (7.30) and (7.28), and } \eta_s(g_\alpha) \text{ (which equals } \eta_s^{(j_0)}(g_\alpha) \text{ in notation of Proposition 7.15 and distinguished BRT charts at } x_0 \text{ satisfies the estimate}
\]
\[
|\eta_s(g_\alpha)| \leq \delta \cdot \int_X g_\alpha dv_X
\]
for \( s = n, n - 1, n - 2, \ldots, -m. \)
In the remaining of this subsection, to streamline the argument we assume the simplest case that

1. Each $X_{p_1}$, $1 \leq \ell \leq k$, is connected
2. $X = \overline{X}_{p_1} \supseteq \overline{X}_{p_2} \supseteq \overline{X}_{p_3} \ldots \supseteq \overline{X}_{p_k}$.

(We postpone the general case to the next subsection.) One sees $p_1 | p_2 | \ldots | p_k$.

Hence all types reduce to simple types (cf. Definition 7.9).

In this case, the subscript $\gamma_i$ in $\ell(\gamma_i)$ will henceforth be dropped throughout the remaining of this subsection.

It will take a bit more work to sum (7.44) over $\alpha$. The numerical factor $d_c$ (cf. (7.24)) in this simplified case satisfies the following. For smooth functions $c, g$ of small support (Definition 7.6), if $i(c, g) = i(c, g') (\leq \infty)$, then $d_{c,g,m} = d_{c,g',m}$. It is useful to set, for $g = g_\alpha$ with $\chi(\alpha) \geq 1$ in ii) below (\chi(\alpha) as in (7.43)),

\begin{equation}
\tag{7.45}
i) D_{1,g} (= D_{1,g,m}) \equiv \sum_{c, i(c,g)=1} d_{c,g,m} = \sum_{q=1}^{p_1} \sum_{i(c,g)=1}^\infty \alpha^{\frac{\mu}{\nu}} e^{-\frac{\|\alpha\|}{\nu}}; \quad ii) (k \geq \ell \geq 2) \quad D_{\ell,g} (= D_{\ell,g,m}) \equiv \sum_{\alpha, i(c,g)=\ell} d_{c,g,m}.
\end{equation}

Suppose $\alpha, \beta \in \bigcup_{\ell \geq 1} \chi^{-1}(\ell)$. As said above, one sees $D_{\ell,\alpha} = D_{\ell,\beta}$ for $1 \leq \ell \leq k$ (because $i(c, g_\alpha) = \ell$ if and only if $i(c, g_\beta) = \ell$ here). We write

Definition 7.18. $D_\ell (= D_{\ell,m}) := D_{\ell,\alpha} (= D_{\ell,\beta,m})$ for any $\alpha$ with $\chi(\alpha) \geq 1$, $1 \leq \ell \leq k$.

By using (7.44) of Proposition 7.17 and Definition 7.18, one sees, for $\alpha \in \chi^{-1}(\ell)$,

\begin{equation}
\tag{7.46}
\alpha \in \chi^{-1}(\ell), \quad \int_X g_\alpha(x) \operatorname{Tr} e^{-E_{b,m}^+(x,x)} dv_X(x) \sim \sum_{\alpha, n, n-1,} t^{-s} x
\end{equation}

\begin{equation}
\begin{aligned}
&\eta_s(g_\alpha) + D_{1, S_{1,s}^+}(g_\alpha) + \sum_{c, i(c)=2} (d_c S_{2,s}^+ g_\alpha \sqrt{t}^{2} + O(\sqrt{t}^{2+1})) + \ldots + \sum_{c, i(c)=\ell} (d_c S_{\ell,s}^+ g_\alpha \sqrt{t}^{2} + O(\sqrt{t}^{2+1})) \\
= &\sum_{s} t^{-s} \left( \eta_s(g_\alpha) + D_{1,S_{1,s}^+}(g_\alpha) + (D_2 S_{2,s}^+ g_\alpha \sqrt{t}^{2} + O(\sqrt{t}^{2+1})) + \ldots + (D_\ell S_{\ell,s}^+ g_\alpha \sqrt{t}^{2} + O(\sqrt{t}^{2+1})) \right).
\end{aligned}
\end{equation}

Combining (7.43) and (7.46) yields the following as $t \to 0^+$ (where $\{\alpha\}_\alpha = \chi^{-1}(1) \cup \chi^{-1}(2) \cup \chi^{-1}(3) \ldots$):

\begin{equation}
\tag{7.47}
\int_X \operatorname{Tr} e^{-E_{b,m}^+(x,x)} dv_X(x) \left( = \sum_\alpha \int_X g_\alpha(x) \operatorname{Tr} e^{-E_{b,m}^+(x,x)} dv_X(x) \right) \\
\sim \sum_{s=n, n-1,} t^{-s} \left( \sum_{\alpha \in \chi^{-1}(1)} D_{1,S_{1,s}^+}(g_\alpha) + \left( \sum_{\alpha \in \chi^{-1}(2)} D_2 S_{2,s}^+ g_\alpha \sqrt{t}^{2} + O(\sqrt{t}^{2+1})) \right) + \ldots + \sum_{\alpha \in \chi^{-1}(3)} \eta_s(g_\alpha) \right).
\end{equation}
We rearrange (7.47) as (only keeping terms in leading order)
\[
\sum_{s} t^{-s} \times \left( \sum_{\alpha} \eta_{s}(g_{\alpha}) + \left( \sum_{\alpha \in \chi^{-1}(1)} D_{1}S_{1,s}^{+}(g_{\alpha}) + \sum_{\alpha \in \chi^{-1}(2)} D_{1}S_{1,s}^{+}(g_{\alpha}) + \sum_{\alpha \in \chi^{-1}(3)} D_{1}S_{1,s}^{+}(g_{\alpha}) + \ldots \right) \right) \\
+ \left( \sum_{\alpha \in \chi^{-1}(2)} D_{2}S_{2,s}^{+}(g_{\alpha}) \sqrt{t^{2}} + \ldots \right) + \left( \sum_{\alpha \in \chi^{-1}(3)} D_{2}S_{2,s}^{+}(g_{\alpha}) \sqrt{t^{2}} + \ldots \right) + \left( \sum_{\alpha \in \chi^{-1}(4)} \ldots + \ldots \right) \\
+ \left( \sum_{\alpha \in \chi^{-1}(3)} D_{3}S_{3,s}^{+}(g_{\alpha}) \sqrt{t^{3}} + \ldots \right) + \left( \sum_{\alpha \in \chi^{-1}(4)} D_{3}S_{3,s}^{+}(g_{\alpha}) \sqrt{t^{3}} + \ldots \right) + \left( \sum_{\alpha \in \chi^{-1}(5)} \ldots + \ldots \right)
\]
which equals by (7.42) and \( S_{t,s}^{+}(g_{\alpha}) = 0 \) for \( \alpha \in \chi^{-1}(\ell') \) with \( \ell' < \ell \),
\[
\sum_{s=n-1} t^{-s} \times \left( \sum_{\alpha} \eta_{s}(g_{\alpha}) + D_{1}S_{1,s}^{+} + D_{2}S_{2,s}^{+} \sqrt{t^{2}} + O(\sqrt{t^{2+1}}) \right) \\
+ \left( D_{3}S_{3,s}^{+} \sqrt{t^{3}} + O(\sqrt{t^{3+1}}) \right) + \ldots \right).
\]
For \( s = n \), we have \( S_{t,n}^{+} = \frac{1}{(2\pi)^{-n}} \text{vol}(X_{p_{l}}) \) (see (iii) of Proposition 7.2).
For the term given by the sum \( \sum_{\alpha} \eta_{s}(g_{\alpha}) \) in (7.49), by Proposition 7.17 we obtain \( \sum_{\alpha} \eta_{s}(g_{\alpha}) \leq \delta \cdot \text{vol}(X) \) (as \( \sum_{\alpha} \int_{X} g_{\alpha} dv_{X} = \text{vol}(X) \)), \( s = n, n-1, \ldots, m \), where \( \delta > 0 \) and \( m \in \mathbb{N} \) are arbitrarily prescribed. By the definition of asymptotic expansion (cf. Definition 5.4) and the fact that (7.49) has been an asymptotic expansion, one sees the term \( \sum_{\alpha} \eta_{s}(g_{\alpha}) \) becomes immaterial to the exact form of the asymptotic expansion.
Further, the asymptotic expansion of \( \int_{X} \text{Tr} a_{s}^{+}(t,x,x) dv_{X}(x) \) of (7.32) basically follows from that of
\[
\int_{X} \text{Tr} e^{-\frac{1}{4}S_{s,m}^{+}} dv_{X}(x).
\]
We have now proved (part of) the main result of this section.

**Theorem 7.19.** (cf. Theorem 1.14) Suppose \( X = X_{p_{1}} \supseteq X_{p_{2}} \supseteq \cdots \supseteq X_{p_{k}} = X_{p_{k}} \) with each stratum \( X_{p_{l}} \) a connected submanifold. Let \( a_{s}^{+}(t,x,y) = a_{s,m}^{+}(t,x,y) \), \( s = n, n-1, \ldots, m \), be as in (5.54). Write \( e_{2} \) for the (real) codimension of \( X_{p_{l}} \) (which is an even number, cf. Remark 7.21 below). (Recall the numerical factors \( D_{1,m} \) as given in Definition 7.18 and the integrals \( S_{t,s}^{+} = S_{t,s,m}^{+} \) in (7.42) with subscripts simplified in the present case.) Then the following holds.
1) As \( t \to 0^{+} \),
\[
\int_{X} \text{Tr} e^{-\frac{1}{4}S_{s,m}^{+}} dv_{X}(x) \\
\sim D_{1,m} ((2\pi)^{-1}(2\pi)^{-n} \text{vol}(X) + t^{-n+1} S_{1,n-1}^{+} + t^{-n+2} S_{1,n-2}^{+} + \ldots) \\
+ (2\pi)^{-1}(n+1) D_{2,m} \text{vol}(X_{p_{2}}) t^{-n+\frac{e_{2}}{2}} + O(t^{-n+\frac{e_{2}+1}{2}}).
\]
In particular, by \( \sum_{q=1}^{m} e^{-\frac{2\pi}{n} q m} = p_{1} \) for \( p_{1} \mid m \) and 0 otherwise, one has \( D_{1,m} = p_{1} \) if \( p_{1} \mid m \). If \( p_{2} \mid m \) (thus \( p_{1} \mid m \) too), then \( D_{1,m} = D_{2,m} > 0 \).
2) In the asymptotic expansion (7.50), all the coefficients of \( t^{j} \) for \( j \) being half-integer, vanish.
3) As a consequence of (7.49) and (ii)
\[
\int_{X} \text{Tr} a_{s,m}^{+}(t,x,x) dv_{X}(x) \sim D_{1,m} S_{1,s}^{+} + D_{2,m} S_{2,s}^{+} t^{\frac{e_{2}}{2}} + O(t^{\frac{e_{2}+1}{2}}), \text{ as } t \to 0^{+}.
\]
The similar results hold true for the case \( \int_{X} \text{Tr} e^{-\frac{1}{4}S_{s,m}^{+}} dv_{X}(x) \) and \( \int_{X} \text{Tr} a_{s,m}^{+}(t,x,x) dv_{X}(x) \) as well.

**Proof.** It remains to prove ii) of the theorem. Recall the last two paragraphs of the proof of Proposition 7.2, especially the item \( \delta \) there. In the present case, by scaling \( (y \to \sqrt{t} y) \) and using (7.14), it
of \((7.1)\) into the geometry of the stratification of the \(S\) just as above, one repeats the pattern similarly. The process continues.

\[ a) \quad \hat{b}_{j,s}^+((\sqrt{t}\tilde{y}, Y), e^{-iu} \circ (\sqrt{t}\tilde{y}, Y)) \]
\[ b) \quad dv_X(x) \]

for a fixed \(u\).

Write \(g_u(x) = e^{-iu} \circ x\) and \(\hat{b}_{j,s}^+((\sqrt{t}\tilde{y}, Y), e^{-iu} \circ (\sqrt{t}\tilde{y}, Y)) = (\hat{b}_{j,s} \circ (id, g_u))((\sqrt{t}\tilde{y}, Y), (\sqrt{t}\tilde{y}, Y))\). By \(\delta\) mentioned above, \(g_u(0, Y)\) is only away from \((0, Y)\) by a small difference \((\leq \varepsilon)\) in their \(\theta\)-coordinates, hence by continuity, \(g_u((\sqrt{t}\tilde{y}, Y))\) lies in an \(O(2\varepsilon)\)-small neighborhood of \((0, Y)\) (as \(t \to 0^+\)), giving that the Taylor expansion of \(\hat{b}_{j,s}(x, y), x = (\sqrt{t}\tilde{y}, Y), y = e^{-iu} \circ x\) around \(x = y = (0, Y) \equiv Y \equiv 0\) can be done in terms of integral powers of \(\sqrt{t}\tilde{y}\) where \(\tilde{y}\) is in \(y\). Hence the coefficients of the \(t^j\) for \(j\) being half-integral must involve an odd power of some variable \(\tilde{y}_i\) in \(\tilde{y}\). Since \(\tilde{y}\) sits in an even dimensional space (cf. i) of Remark 7.21 below), \(dv_X(x)\) is of integral power in \(t\). With a), b) of (7.52), by using i) the claim (7.14), ii) \(\int_{\hat{1}/\sqrt{t}} e^{-x^j\hat{y}_i^j} dy_i = 0\) for an odd number \(n\) and iii) for a polynomial \(P(x)\), \(\int_{\hat{1}/\sqrt{t}} e^{-x^j\hat{y}_i^j} P(x) dx \sim O(t^\infty)\) (as \(t \to 0^+\)), our assertion about the asymptotic expansion in ii) of the theorem follows.

\[ \square \]

Remark 7.20. One may think of the second line in (7.50) as the main terms which remind one of the close relation between the Kodaira Laplacian and Kohn Laplacian (cf. Proposition 5.1). However, one key point in this paper is the idea that if the \(S^1\) action is locally free (but not globally free) on \(X\), then this relation cannot be altogether extended to their heat kernels. In this regard the correction terms exist, and consist in the third line of (7.50) linked up with the higher strata of the (locally free) \(S^1\) action beyond the principal stratum.

Remark 7.21. i) We argue \(e_\ell\) is even. \(X_{p\ell}\) is \(S^1\) invariant; \(TX_{p\ell} = (\mathbb{R}T^1 \cap TX_{p\ell}) \oplus RT|_{X_{p\ell}}\) where \(RT\) is the line subbundle of \(TX\) generated by \(\partial/\partial\theta\) such that \(RT \oplus RT^1 = TX\). Thus, in a BRT chart \(U \times [\varepsilon, \varepsilon]\), one has \((RT^1 \cap TX_{p\ell})|_{U \cap X_{p\ell}} \equiv E \subset TU\) \((TU \subset RT^1\) by (2.5) and (4.1)). Write \(g = e^{i2\pi/p\ell}(\in S^1)\) which is CR and an isometry with the fixed point set \(X_{p\ell}\). By \(dg \circ J = J \circ dg = J\) on \(E\) with \(J\) the complex structure of \(TU\), \(E\) is invariant under \(J\). It follows \(E\) is of even dimension, so \(X_{p\ell}\) is of odd dimension, i.e. \(e_\ell\) is even. ii) For \(1 \leq \ell \leq k\) we can write \(X_{p\ell} \to M_{p\ell}\) for a complex manifold \(M_{p\ell}\), as an \(S^1\) fiber bundle. The quantities \(\alpha_{\ell}^j(x)\) by the construction (see (7.40)-(7.42)) is \(S^1\) invariant hence descend to \(M_{p\ell}\). One sees \(S_{N_{X_{p\ell}}}^e(\equiv S_0^{p\ell}) = \frac{2\pi}{p\ell} S_{M_{p\ell}}^e(\equiv S_0^{p\ell})\) \((\equiv S_{M_{p\ell}}^e), \text{Tr} \alpha_{\ell}^j dv_{M_{p\ell}}\)). Here, the metric on \(M_{p\ell}\) (cf. \(dv_{M_{p\ell}}\)) is defined in a way similar to that given in (4.1). This suggests a question of how the heat kernels (for the locally free \(S^1\) action) of the present paper may be connected with (certain suitably defined) heat kernels in the orbifold base \(X/S^1\). In a certain Riemannian setting, some work in a similar direction has been done (cf. [56, Theorems 3.5, 3.6]).

7.5. Types for \(S^1\) stratifications; proof for the general type. Lastly, to modify the above reasoning to the case beyond the simple type is essentially not difficult. Suppose, say, \(X_{p\ell}\) has connected components \(Y_i\) and the simple type condition is assumed along each component \(Y_i\). Then, clearly the above argument applies to the individual \(Y_i\) and the result is just to sum up over \(i\). Without the simple type condition on \(Y_i\), say, inside some \(Y_i\) the next stratum \(X_{p\ell}\) has seated several components \(Z_j\) or some components \(Z_j\) are seated even outside of each \(Y_i\). Then by localization argument along each \(Z_j\) etc. just as above, one repeats the pattern similarly. The process continues.

We are now motivated to transplant the notion of “type”, “class” in Definition 7.1 for the integral \(I\) of (7.1) into the geometry of the stratification of the \(S^1\) action.

For a connected component \(X_{p\ell}(\gamma) \subset X_{p\ell}, \gamma = 1, \ldots, s_\epsilon\), contained in the higher dimensional connected components of the strata

\[(X =) X_{p\ell}(\gamma_1) \supset X_{p\ell}(\gamma_2) \cdots \supset X_{p\ell}(\gamma_j) \supset X_{p\ell}(\gamma_{j+1}) = X_{p\ell}(\gamma_k)\]
where \( i_1 = 1 < i_2 < \ldots < i_f < i_{f+1} = \ell \in \{1, 2, \ldots, \ell - 1, \ell\} \), we define its type \( \tau(X_{P_{\ell}(\gamma_\ell)}) \) by

(7.53)

\[ \tau(X_{P_{\ell}(\gamma_\ell)}) = \tau(\ell(\gamma_\ell)) := (i_1(\gamma_i), i_2(\gamma_i), \ldots, i_f(\gamma_i), i_{f+1}(\gamma_{i_{f+1}})), \quad i_1 = \gamma_{i_1} = s_1 = 1; i_{f+1} = \ell. \]

One has \( p_{i_1} | p_{i_2} | \ldots | p_{i_{f+1}} \) (cf. Remark 1.16 for a similar case).

The notions such as simple type, class and length \( l(\tau) \) are defined similarly, cf. Definition 7.1. (No definition of trivial type is given here. See iii) of Definition 7.22 below in which \( i(c, [\tau]) = \infty \) corresponds to the trivial type, cf. iii) of Definition 7.9.)

Recall if \( M \subset \mathcal{N} \) is a finite disjoint union of submanifolds \( M_j \), then the dimension of \( M \) is \( \max_j \{ \dim \mathcal{M}_j \} \) and the codimension of \( M \) is \( \dim \mathcal{N} - \dim \mathcal{M} \).

The following definition, which is bit tedious yet bears a great similarity as previously, is set up for the immediate use in the general situation.

**Definition 7.22.** i) Write \( \nu_\ell = \{ [\tau]; \tau = \tau(X_{P_{\ell}(\gamma_\ell)}), \gamma_\ell = 1, 2, \ldots, s_\ell \} \) for the set of equivalence classes of types \( \tau = \tau(X_{P_{\ell}(\gamma_\ell)}) \) of connected components \( X_{P_{\ell}(\gamma_\ell)} \) in \( X_{P_{\ell}} \).

ii) Write (similar to (7.42), (7.41))

\[
S_{+}(X_{P_{\ell}(\gamma_\ell)}, \nu_\ell) = S_{+}(X_{P_{\ell}(\gamma_\ell)}, s_m, m) = \int_{X_{P_{\ell}(\gamma_\ell)}} \text{Tr} \alpha_s^+(Y, X_{P_{\ell}(\gamma_\ell)}(Y)) \quad (\alpha_s^+ = \alpha_s^+) \]

associated with \( X_{P_{\ell}(\gamma_\ell)} \).

iii) Let \( [\tau] = \{ [i_1(\gamma_i), i_2(\gamma_i), \ldots, i_f(\gamma_i), i_{f+1}(\gamma_{i_{f+1}})] \} \) be given. If \( c | p_{i_1} \), define \( i(c, [\tau]) = 1 \). (Hence it is independent of \( [\tau] \).) If \( c \not| p_{i_1} \) and \( c | p_{i_{f+1}} \), \( \ell = i_s \) for some \( s, 2 \leq s \leq f + 1 \), such that \( c \not| i_s \) for all \( s' < s \). Then \( i(c, [\tau]) = \ell \geq 2 \). If \( c \not| p_{i_s} \) for \( 1 \leq s \leq t + 1 \), \( i(c, [\tau]) = \infty \). We may write \( i(c, [\tau]) = i(c, [\tau]). \)

iv) For \( i(c, [\tau]) \geq 2 \), define the numerical factors \( d_{c, m, [\tau]} \) correspondingly as in (7.28). For \( i(c) = 1 \), define \( d_{c, m, [\tau]} = d_c \) (which is independent of \( \tau \)) as in (7.24).

v) For a given \( [\tau] \), if \( i(c) = 1 \), then define the weight factors \( D_1 \) as in (7.45) (which is independent of \( \tau \)) and if \( i(c) \neq 1 \), \( \infty \), define \( D_{1, [\tau]} = D_{m, [\tau]} \equiv \sum_{c, i(c) = \ell} d_{c, m, [\tau]} \).

vi) Write \( e_{i_1, [\tau]} = e_{i_1}(\gamma_{i_1}) \) with \( \bar{\tau}(\ell(\gamma_\ell)) = (i_1(\gamma_{i_1}), i_2(\gamma_{i_1}), \ldots, i_{f+1}(\gamma_{i_{f+1}})) \) for the codimension of \( X_{P_{\ell}(\gamma_{i_1})} \). Obviously, \( e_{i_1, [\tau]} < e_{i_2, [\tau]} < \cdots \). For \( [\tau] \in \nu_\ell \), write \( e_{[\tau]} = e_{i_{f+1}(\gamma_{i_{f+1}})} \), i.e. \( e_{\ell(\gamma_\ell)} \).

vii) Write \( e = \min_{2 \leq \ell \leq k} e_{i_1, [\tau]} \) (\( = \min_{2 \leq \ell \leq k} e_{i_1, [\tau]} \) by vi) above) and for \( \ell \geq 2 \),

\[ \hat{\nu}_\ell = \{ [\hat{\tau}] \in \nu_\ell; \hat{\tau} = (1, \ell(\gamma_\ell)) \} \]

of length two such that \( e_{\ell(\gamma_\ell)} = e \), i.e. \( e_{[\hat{\tau}]} = e \) \( \subset \nu_\ell \).

Of course, it is not ruled out that for some values of \( \ell, \hat{\nu}_\ell \) could be an empty set. Intuitively, one thinks of \( e \) as the minimal codimension among those connected components \( X_{P_{\ell}(\gamma_\ell)} \) such that if \( X_{P_{\ell}} \supseteq X_{P_{\ell}(\gamma_\ell)} \), then \( X_{P_{\ell}} = X_{P_{1}} \) the principal stratum.

viii) For a fixed \( [\kappa] \in \nu_\ell \) in i), write (see (7.53) for \( \tau(\ell(\gamma_{i_1})) = \tau(X_{P_{\ell}(\gamma_{i_1})}) \)),

\[
Z_{[\kappa], s} = \sum_{1 \leq \gamma_{i_1} \leq \gamma_{i_{f+1}}} \sum_{1 \leq \gamma_{i_1} \leq \gamma_{i_{f+1}}} S_{+}(X_{P_{\ell}(\gamma_{i_1})})^s.
\]

For the case of general type, we can obtain results corresponding to (7.47), (7.48) and (7.49) (yet complicated in expressions here). We are content with summarizing the final result as follows.

\[
\int_X \text{Tr} e^{-\ell}_{\ell m} f(x, x) d\nu_X(x) \sim \sum_{s = m, n, n - 1, \ldots} t^{-s} \times \left( D_1 S_{+}^1 + \sum_{[\kappa_1] \in \nu_1} (D_2_{[\kappa_2]} Z_{[\kappa_2], s} \sqrt{\nu_{[\kappa_2]}^{[\kappa_2]} + O(\sqrt{\nu_{[\kappa_2]}^{[\kappa_2]+1}})} + \sum_{[\kappa_3] \in \nu_3} (D_3_{[\kappa_3]} Z_{[\kappa_3], s} \sqrt{\nu_{[\kappa_3]}^{[\kappa_3]} + O(\sqrt{\nu_{[\kappa_3]}^{[\kappa_3]+1}})} + \ldots \right).
\]
The following main result of this subsection parallels Theorem 7.19 in the last subsection. By comparison, to collect the coefficients for the next leading order in \( t \) or \( \sqrt{t} \) in (7.54) here, we have a slightly more complicated summation (regarded as part of corrections as indicated in Remark 7.20) in the formula below. Note that the conversion to the stated form of Theorem 1.14 is nothing but a direct consequence of an examination (slightly tedious) of the various definitions here.

**Theorem 7.23.** (cf. Theorem 1.14) Notations as in Theorem 7.19 without assuming the conditions of connectedness and simple-type there. The weight factors \( D_{\ell,m,|\tau|} \), the integrals \( S^+_{\ell(\gamma_\ell),s} = S^+_{\ell(\gamma_\ell),s,m} \), \( e_\ell \), \( \tilde{e}_\ell \) etc. are just given above. One has the following.

i) As \( t \to 0^+ \),

\[
\int_X \text{Tr} e^{-\sqrt{t}\ell h,m}(x,x)dv_X(x) \\
\sim D_{1,m}((2\pi)^{-1}(2\pi t)^{-n}\text{vol}(X) + t^{-n+1}S^+_{1,n-1} + t^{-n+2}S^+_{1,n-2} + \ldots) + O(t^{-n+\frac{e_\ell+1}{2}})
\]  

(7.55)

(where recall \( Z_{[\tau]_n} = (2\pi)^{-n+1}\sum_{\gamma_\ell}^{\text{codim }X_{\tau(\gamma_\ell)}=e} \text{vol}(X_{\tau(\gamma_\ell)}) > 0 \) in the locally free case of the \( S^1 \) action). If \( p_1|n \) (thus \( p_1|m \) too), then \( D_{1,m}, D_{\ell,m,|\tau|} > 0 \).

ii) In the asymptotic expansion (7.55), all the coefficients of \( t^j \) for \( j \) being half-integral, vanish.

iii) As a consequence of (7.54) and ii),

\[
\int_X \text{Tr} a^{+}_{s,m}(t,x,x)dv_X(x) \sim D_{1,m}S^+_{1,s} + t^{\tilde{e}+1} \sum_{[\tau]_n} D_{\ell,m,|\tau|} Z_{[\tau]_n} + O(t^{\tilde{e}+1})
\]  

(7.56)

(where \( Z_{[\tau],s} = \sum_{\text{codim }X_{\tau(\gamma_\ell)}=e} S^+_{\ell(\gamma_\ell),s} \)).

The similar results hold for \( \int_X \text{Tr} e^{-\sqrt{t}\ell h,m}(x,x)dv_X(x) \) and \( \int_X \text{Tr} a^{-}_{s,m}(t,x,x)dv_X(x) \).

**Remark 7.24.** The quantities involved above are computable in the sense that they are basically reduced to those involved in the (ordinary) Kodaira heat kernel by ii) of Definition 7.22, cf. Remark 1.9.

**Remark 7.25.** It is not obvious how one can compute the supertrace integral, hence our index theorem 6.4, Corollary 6.5 solely by techniques similar to those derived in (7.55), partly because here we are not using the off-diagonal estimate of Theorem 5.9 which is partly based on a cancellation result in Berenzin integral (see the proof of that theorem). These results (estimate and cancellation) appear to lie beyond what the geometry of the stratifications can reveal as done in this Section 7.

**Acknowledgements.** The first named author would like to thank the Ministry of Science and Technology of Taiwan for the support of the project: 104-2115-M-001-011-MY2. The second named author was partially supported by the Ministry of Science and Technology of Taiwan for the project: 104-2628-M-001-003-MY2 and the Golden-Jade fellowship of Kenda Foundation. We would also like to thank Professor Paul Yang for his interest and discussion in this work. The second named author would like to express his gratitude to Professor Rung-Tzung Huang for useful discussion in this work.

**References**

[1] M.-F. Atiyah, *Elliptic operators and compact groups*, Lecture Notes in Mathematics 401, Berlin: Springer-Verlag 1974.

[2] B. Booss and D.D. Bleecker, *Topology and analysis: the Atiyah-Singer index formula and gauge theoretic physics*, translated by D.D. Bleecker and A. Mader, 1985 Springer-Verlag New York Inc.

[3] R. Beals, P.C. Greiner, and N.K. Stanton, *The heat equation on a CR manifold*, J. Diff. Geom. 20 (1984), 343-387.
[39] H. Herrmann, C.-Y. Hsiao and X. Li, Szegö kernel expansion and embedding of Sasakian manifolds, arXiv:1512.03952[math.CV].
[40] C.-Y. Hsiao, Szegö kernel asymptotics for high power of CR line bundles and Kodaira embedding theorems on CR manifolds, 116 pages, preprint available at arXiv:1401.6647, to appear in Mem. of the Amer. Math. Soc..
[41] C.-Y. Hsiao and X. Li, Szegö kernel asymptotics and Morse inequalities on CR manifolds with $S^1$ action, preprint available at arXiv:1502.02365.
[42] C.-Y. Hsiao and X. Li, Morse inequalities for Fourier components of Kohn-Rossi cohomology of CR manifolds with $S^1$-action, Math. Zeit., DOI: 10.1007/s00209-016-1661-6.
[43] Y. Kamishima and T. Tsuboi, CR-structure on Seifert manifolds, Invent. Math. 104 (1991), 149-164.
[44] T. Kawasaki, The signature theorem for V-manifolds, Topology 17 (1978), 75-83.
[45] T. Kawasaki, The Riemann-Roch theorem for complex V-manifolds, Osaka J. Math. 16 (1979), 151-159.
[46] J.-M. Lee, Pseudo-Einstein structures on CR manifolds, Amer. J. Math. 110 (1988), 157-178.
[47] L. Lempert, On three dimensional Cauchy-Riemann manifolds, J. Amer. Math. Soc. 5 (1992), 1-50.
[48] X. Ma and G. Marinescu, Holomorphic Morse inequalities and Bergman kernels, Progress in Math. 254, Birkhäuser, Basel 2007, 422 pp.
[49] G. Marinescu, Asymptotic Morse Inequalities for Pseudoconcave Manifolds, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 23 (1996), 27-55.
[50] E. Meinrenken, Group actions on manifolds, Lecture Notes, University of Toronto, Spring 2003.
[51] L. Ornea and M. Verbitsky, Embeddings of compact Sasakian manifolds, Mathematical Research Letters 14 (2007), 703-710.
[52] L. Ornea and M. Verbitsky, Sasakian structures on CR-manifolds, Geometriae Dedicata 125 (2007), 159-173.
[53] E. Park and K. Richardson, The basic Laplacian of a Riemannian foliation, Amer. J. Math. 118 (1996), 1249-1275.
[54] P.-E. Paradan and M. Vergne, Index of transversally elliptic operators, Astérisque No. 328 (2009), 297-338.
[55] K. Richardson, The asymptotics of heat kernels on Riemannian foliations, GAFA, Geom. Funct. Anal. 8 (1998), 356-401.
[56] K. Richardson, Traces of heat operators on Riemannian foliations, Trans. Amer. Math. Soc. 362 (2010), 2301-2337.
[57] I. Satake, On a generalization of the notion of manifold, Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 359-363.
[58] N. Tanaka, A differential geometric study on strongly pseudo-convex manifolds, Kinokuniya Book Store Co., Ltd., Kyoto 1975.
[59] S. R. S. Varadhan, On the behavior of the fundamental solution of the heat equation with variable coefficients, Communications on pure and applied math. 20 (1967), 431-455.
[60] S.M. Webster, Pseudohermitian structures on a real hypersurface, J. Diff. Geom. 13 (1978), 25-41.